CONTEMPORARY INFINITESIMALIST THEORIES OF CONTINUA AND THEIR LATE 19TH- AND EARLY 20TH-CENTURY FORERUNNERS

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1. INTRODUCTION

In the decades bracketing the turn of the twentieth century the real number system was dubbed the arithmetic continuum because it was held that this number system is completely adequate for the analytic representation of all types of continuous phenomena. In accordance with this view, the geometric linear continuum is taken to be isomorphic with the arithmetic continuum, the axioms of geometry being so selected to insure this would be the case. In honor of Georg Cantor and Richard Dedekind, who first proposed this mathematico-philosophical thesis, the presumed correspondence between the two structures is sometimes called the Cantor-Dedekind axiom. Since that time, the Cantor-Dedekind philosophy has emerged as a pillar of standard mathematical philosophy that underlies the standard formulation of analysis, the standard analytic and synthetic theories of the geometrical linear continuum, and the standard axiomatic theories of continuous magnitude more generally.

Since its inception, however, there has never been a time at which the Cantor-Dedekind philosophy has either met with universal acceptance or has been without competitors. The period that has transpired since its emergence as the standard philosophy has been especially fruitful in this regard having witnessed the rise of a variety of constructivist and predicativist theories of real numbers and corresponding theories of analysis as well as the emergence of a number of alternative theories

Acknowledgement. The idea of writing this historical overview grew out of our (2007). We are grateful to the editors for providing us this opportunity to do so, and especially for tolerating our going well beyond the designated page limits. During the course of writing the paper we have profited from correspondence with C. Ward Henson and H. J. Keisler about nonstandard analysis, and with Marta Bunge, Paolo Giordano, Wolfgang Bertram and Mikhail Katz about SDG/SDT, IDG, TDC and NSDG respectively, and we are grateful to them all for their comments and helpful responses to our queries. We are especially grateful to Marta Bunge for permitting us to quote from our correspondence.
that make use of infinitesimals. Whereas the constructivist and predicativist theories have their roots in the early twentieth-century debates on the foundations of mathematics and were born from critiques of the Cantor-Dedekind theory, the infinitesimalist theories are intended to either provide intuitively satisfying (and, in some cases, historically rooted) alternatives to the Cantor-Dedekind conception that have the power to meet the needs of geometry, analysis or portions of differential geometry, or to situate the Cantor-Dedekind system of real numbers in a grander conception of an arithmetic continuum.

The purpose of this paper is to provide a historical overview of some of the contemporary infinitesimalist alternatives to the Cantor-Dedekind theory of continua. Among the theories we will consider are those that emerge from nonstandard analysis, nilpotent infinitesimalist approaches to portions of differential geometry and the theory of surreal numbers. Since these theories have roots in the algebraic, geometric and analytic infinitesimalist theories of the late nineteenth and early twentieth centuries, we will also provide overviews of the latter theories and some of their relations to the contemporary ones. We will find that the contemporary theories, while offering novel and possible alternative visions of continua, need not be (and in many cases are not) regarded as replacements for the Cantor-Dedekind theory and its corresponding theories of analysis and differential geometry, but rather as companion theories or methods to be situated in the toolkits of mathematicians, or, as in the case of the surreal numbers, as a panorama from within which one may view the classical arithmetic continuum as the Archimedean portion of a transfinite recursively unfolding maximal arithmetico-tree-theoretic continuum whose elements are individually

1 Among the infinitesimalist conceptions we will not consider is the one arising from Alain Connes’s noncommutative (differential) geometry. One of Connes’s original motivations for developing noncommutative geometry was to apply geometric ideas and concepts to spaces that are intractable when considered from the usual set-theoretic framework of Riemannian geometry. The principal feature of noncommutative geometry is a novel conception of geometric space, encoded within an algebra, in general noncommutative, instead of a set of points, the geometry in classical spaces being recovered when the algebra is commutative. In Connes’s framework the ordinary differential and integral calculus is replaced by a new calculus of infinitesimals—the quantized calculus—based on the operator formalism of quantum mechanics, and therein for each positive real $\alpha$, infinitesimals of order $\alpha$ (forming a two-sided ideal) are introduced in connection with compact operators on a Hilbert space. According to Connes, the framework that emerges from this perspective is “a geometric space which is neither a continuum nor a discrete space but a mixture of both” (Connes 1994, p. 30; also see, Connes 1998, 2006).
definable in terms of sets of standard set theory and whose first-order theory is that of the reals.

With the exception of §9 and a portion of §11, all of the contemporary theories discussed below can be formalized in \textit{Zermelo-Fraenkel set theory with Choice} (ZFC) occasionally supplemented with one or another familiar assumption about infinite cardinals. In §9, where sets and proper classes are required, the underlying set theory is \textit{von Neumann-Bernays-Gödel set theory with Global Choice} (NBG), where all proper classes have the “cardinality” of the proper class On of ordinals, and a sentence in the language of sets is true if and only if it is true in ZFC².

Since most of the earliest infinitesimalist theories of continua are based on non-Archimedean (totally) ordered fields, rings and abelian groups, we will begin with these and their historical roots. Following standard practice, an ordered abelian group, an ordered ring or an ordered field \(A\) is said to be Archimedean if for all \(x, y \in A\), where \(0 < x < y\), there is an \(n \in \mathbb{N}\) such that \(nx > y\)³.

²For NBG and its relation to ZFC, see (Smullyan and Fitting 2010).
³The concepts of ordered class, divisible ordered abelian group, ordered commutative ring with identity, ordered field and positive cone of an ordered abelian group play prominent roles in portions of the text. For the reader’s convenience, the definitions of these and some related concepts are collected below, where “ordered” is understood to mean “totally ordered”.

An ordered class is a structure \(\langle A, \leq \rangle\), where \(A\) is a class (a set or proper class) and \(\leq\) is a binary relation on \(A\) that satisfies the conditions: \(\forall xy(x \leq y \lor y \leq x)\); \(\forall xyz((y \leq z \land x \leq y) \to x \leq z)\); \(\forall xy((x \leq y \land y \leq x) \to x = y)\). If \(x \leq y\) and \(x \neq y\), we write \(x < y\) or \(y > x\). An ordered abelian group (written additively) is a structure \(\langle A, \leq, +, 0 \rangle\), where \(\langle A, \leq \rangle\) is an ordered class and + is commutative, associative binary operation on \(A\) satisfying the conditions: \(\forall x(x + 0 = x)\); \(\forall x \exists y(x + y = 0)\); \(\forall xyz(x \leq y \to (x + z \leq y + z))\). An ordered abelian group \(A\) is divisible, if for each \(x \in A\) and each positive integer \(n\), there is an \(a \in A\) such that \(na = x\). An ordered abelian group \(\langle A, \leq, 1 \rangle\) (written multiplicatively) is defined similarly using \(\cdot\) and 1 in place of + and 0, and divisibility is defined similarly by the condition: for each \(x \in A\) and each positive integer \(n\), there is an \(a \in A\) such that \(a^n = x\). An ordered commutative ring with identity (or with unity) is a structure \(\langle A, \leq, +, 0, 1 \rangle\), where \(\langle A, \leq, +, 0 \rangle\) is an ordered abelian group, \(0 \neq 1\), and \(\cdot\) is commutative, associative binary operation on \(A\) that satisfies the conditions: \(\forall x(x \cdot 1 = x)\); \(\forall xyz((x \cdot (y + z) = (x \cdot y) + (x \cdot z))\); \(\forall xy((0 \leq x \land 0 \leq y) \to 0 \leq xy)\). An ordered field is an ordered commutative ring with identity that satisfies the further condition: \(\forall x[x \neq 0 \to \exists y(x \cdot y = 1)]\). The ordered additive group of every ordered field is divisible. The positive cone of an ordered additive abelian group \(A\) is \(\{x \in A : x > 0\}\).

A nonzero element \(x\) of an ordered commutative ring \(A\) with identity is said to be a proper zero-divisor if for some nonzero \(y \in A\), \(xy = 0\). An ordered commutative ring with identity is said to be an ordered integral domain if it has no proper zero-divisor. In the case of an ordered integral domain the condition \(\forall xy((0 \leq x \land 0 \leq y) \to 0 = xy)\).
2. THE EMERGENCE OF NON-ARCHIMEDEAN SYSTEMS OF MAGNITUDES

Even before Cantor (1872) and Dedekind (1872) had published the modern theories of real numbers that would be employed to all but banish infinitesimals from late 19th- and pre-Robinsonian, 20th-century analysis, Johannes Thomae (1870) and Paul du Bois-Reymond (1870-71) were beginning the process that would in the years bracketing the turn of the century not only establish consistent and relatively sophisticated theories of infinitesimals in mainstream mathematics but make them the focal point of great interest and a mathematically profound and philosophically significant research program. One theory grew out of the pioneering investigations of non-Archimedean geometry of Giuseppe Veronese (1889, 1891, 1894), Tullio Levi-Civita (1892/93, 1898) and David Hilbert (1899), and led to the celebrated algebraico-set-theoretic work of Hans Hahn (1907). And another emerged from a parallel development of du Bois-Reymond’s (1870-71, 1875, 1877, 1882) groundbreaking work on the rates of growth of real functions and led in the same period to the famous works of G. H. Hardy (1910, 1912) and some less well known though important work of Felix Hausdorff (1907, 1909). Each of these research programs, which collectively gave rise to modern-day non-Archimedean mathematics, led to nonstandard theories of continua that have come to play important roles in the mathematics of our day, albeit not always in the guise of theories of continua.

Before turning to these research programs and the nonstandard theories of continua they gave rise to, we will first provide some historical background on the Archimedean axiom.

3. THE ARCHIMEDEAN AXIOM

In his historically important paper Zur Geometrie der Alten, insbesondere über ein Axiom des Archimedes (On the Geometry of the Greeks, in Particular, on the Axiom of Archimedes) Otto Stolz observed that:

\[ \forall x y ((0 < x \land 0 < y) \rightarrow 0 < x y) \]

It has often been noted that Euclid implicitly used the principle: *a magnitude can be so often multiplied that it exceeds any other of the same kind*.... Archimedes employed this principle as an explicit axiom in some of his works .... For brevity, we will therefore henceforth refer \( y \rightarrow 0 \leq xy \) can be replaced by \( \forall xy((0 < x \land 0 < y) \rightarrow 0 < xy) \). An ordered field is an ordered integral domain.
to this principle as the *Axiom of Archimedes*. To investigate whether or not this is a necessary proposition, requires us first to have agreement on a characterization of the concept of “magnitude” (1883, p. 504).

Such agreement was required for, as Stolz emphasized, the term “magnitude” occurs in Euclid’s *Elements*, but he nowhere explains the concept. In response to his query, Stolz provided an axiomatization for the type of ordered additive systems of line segments occurring in the *Elements*—an ordered, additive system of equivalence classes (of congruent line segments) constituting what we today call the *positive cone of a divisible (additively written) ordered abelian group*, and therewith established the following result: *whereas every such system of magnitudes that is continuous in the sense of Dedekind is Archimedean, there are such systems of magnitudes that are non-Archimedean*. To establish the existence of a latter such system he made use of an algebraic development of a fragment of du Bois-Reymond system of *orders of infinity* that emerged from the latter’s aforementioned work on the rates of growth of real functions (see §6). It was with these and related discoveries that Stolz (1883, 1884, 1885), Rodolfo Bettazzi (1890), Veronese (1889) and Otto Hölder (1901) laid the groundwork for the modern theory of magnitudes, the branch of late 19th- and early 20th-century mathematical philosophy that would, in the decades that followed, evolve into the theories of Archimedean and non-Archimedean ordered algebraic systems, theories in which talk of systems of magnitudes would be replaced by talk of ordered abelian groups, ordered fields, the positive cones of such structures, and so on.

The non-Archimedean systems of magnitudes studied in the just-mentioned works are additive structures that sometimes have modest multiplicative structures as well. Unlike the system of real numbers, however, none of them is an ordered field. Just as ordered fields of real numbers arose in conjunction with the study of Euclidean geometry, it was from the study of non-Archimedean geometry that non-Archimedean ordered fields emerged. It was also from non-Archimedean geometry that the first well-developed non-Archimedean theory of continua emerged.

4. VERONESE’S THEORY OF CONTINUA

Following Wallis’s and Newton’s incorporation of directed segments into Cartesian geometry, it became loosely understood that given a unit segment $AB$ of a line $L$ of a classical Euclidean space, the collection of

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4See (Ehrlich 2012, pp. 24-27) for Stolz’s construction.
directed segments of \( L \) emanating from \( A \), including the degenerate segment \( AA \) itself, constitutes an Archimedean ordered field with \( AA \) and \( AB \) the additive and multiplicative identities of the field and addition and multiplication of segments suitably defined. This idea was made precise by Veronese (1891, 1894) and Hilbert (1899) in their pioneering works on the foundations of geometry. Also emerging from these works, and inspired in part by the aforementioned work of Stolz, was the idea that it is possible to construct an axiomatization for the central theorems of Euclidean geometry that is independent of the Archimedean axiom and for which the aforementioned system of line segments in a model of the geometry continues to be an ordered field; however, in those models of the geometry in which the Archimedean axiom fails, the ordered fields in question are non-Archimedean ordered fields.

Following Veronese, two segments \( s \) and \( s' \) are said to be finite relative to one another if there are positive integers \( m \) and \( n \) such that \( s' \prec ms \) and \( s \prec ns' \), where \( a \prec b \) indicates that \( a \) is congruent to a proper subsegment of \( b \), i.e. a subsegment of \( b \) that is not identical to \( b \). In accordance with this terminology, a system \( S \) of segments may be said to be Archimedean if every pair of such segments is finite relative one another. If \( S \) is non-Archimedean, there are segments \( s \) and \( s' \) that are not finite relative to one another, in which case \( s \) is said to be infinitesimal relative to \( s' \) and \( s' \) is said to be infinite relative to \( s \), if \( s \prec s' \). Thus, \( s \) is infinitesimal relative to \( s' \) and \( s' \) is infinite relative to \( s \) just in case \( ns \prec s' \) for all positive integers \( n \). If \( S \) is the positive cone of a non-Archimedean ordered field, then for each segment \( s \) in \( S \) there are segments \( s' \) and \( s'' \) in \( S \) such that \( s \) is infinite relative \( s' \) and infinitesimal relative to \( s'' \).

Unlike the analytic constructions of Hilbert, Veronese’s construction of a non-Archimedean ordered field of line segments is clumsy and quite complicated, though to some extent the complexity is a by-product of what he is attempting to achieve. After all, Veronese is not merely attempting to construct a non-Archimedean ordered field of segments that is appropriate to the geometry in question, but moreover an ordered field of such segments that models his novel theory of non-Archimedean continua, and one that is synthetically constructed to boot.

Veronese had a distinctive, well-developed philosophy of geometry that underlay his philosophy of the continuum. For the sake of space, 5

For the remainder of this section, except when referring to an ordered field of segments, we employ the terms “segment” and “subsegment” without the modifiers “directed” or “degenerate” to refer to a nondegenerate line segment without a direction.
we simply note that in addition to holding that our conception of geometric continua should be grounded in our conception of magnitude rather than in number or in points, Veronese held that just as it is compatible with geometrical intuition that the parallel postulate fails, it is compatible with our geometrical intuition that continua need not be Archimedean. It was for these reasons that, unlike Cantor and Dedekind, Veronese sought a segment-based, synthetic theory of geometric continua that is independent of the Archimedean condition.

Despite the lack of elegance in its presentation and elements of obscurity in its formulation, the theory of rectilinear continua developed in Veronese’s *Fondamenti di Geometria* (1891) is a profound and relatively sophisticated scheme, several of whose central concepts and ideas permeate the 20th-century theory of ordered algebraic systems and through it nonstandard analysis, the theory of the rates of growth of functions and the theory of surreal numbers. For our purpose here, however, we limit our attention to its two continuity conditions, each of which, unlike the Dedekind continuity condition, is satisfiable by Archimedean as well as non-Archimedean ordered abelian groups and ordered fields. In fact, as Veronese (1889, 1891) and Levi-Civita (1898) collectively demonstrate, each of the conditions is equivalent to the Dedekind continuity condition if and only if the Archimedean axiom is assumed.

Veronese formulates his two continuity conditions as follows.

Relative Continuity Condition. Every segment $XX'$ whose ends vary in opposite directions and becomes indefinitely small contains an element outside the domains of variability of its ends (Veronese 1891, p. 128).

Absolute Continuity Condition. Every segment $XX'$ whose ends vary in opposite directions and becomes indefinitely small in the absolute sense contains an element outside the domains of variability of its ends (Veronese 1891, p. 150).

Veronese unpacks his continuity conditions in terms of a variable segment $XX'$ that is the difference $AX' - AX$ of a pair of subsegments of a segment $AB$, where $AX$ is a proper subsegment of $AX'$, henceforth written $AX \prec^* AX'$. While keeping $A$ fixed and preserving the condition that $AX \prec^* AX'$, $X$ is envisioned to increase in a strict monotonic fashion (without a greatest member) as $X'$ decreases in a

\footnote{While Veronese has a segment-based, rather than a point-set based, theory of continua, the notion of a point is a primitive of his geometric system.}
strict monotonic fashion (without a least member),

\[ A \rightarrow \overrightarrow{X} \rightarrow \overrightarrow{X'} \rightarrow B \]

subject to the following conditions that flesh out Veronese’s conceptions of becoming “indefinitely small” and “indefinitely small in the absolute sense”, respectively.

**Indefinitely Small.** For each segment \( s \) that is finite relative to an arbitrarily given unit segment which could be taken to be \( AB \), \( X \) and \( X' \) take on values \( X_s \) and \( X'_s \), respectively, where \( X_s X'_s \prec s \).

**Indefinitely Small in the Absolute Sense.** For each segment \( s \), \( X \) and \( X' \) take on values \( X_s \) and \( X'_s \), respectively, where \( X_s X'_s \prec s \).

Given the satisfaction of these respective conditions, Veronese’s continuity conditions assert that there is a segment \( AY \) such that

\[ AX_s \prec^* AY \prec^* AX'_s \]

for all such values \( X_s \) of \( X \) and all such values \( X'_s \) of \( X' \).

Veronese refers to his first continuity condition as a relative continuity condition since it is concerned with families of segments that grow arbitrarily small subject to the proviso that they remain finite relative to a given unit segment. The absolute continuity condition, by contrast, is concerned with families of segments that grow arbitrarily small subject to the limits of the geometric space itself.

Veronese’s relative continuity condition ensures that if one limits oneself to the segments that are finite relative to an arbitrarily selected segment \( s \) and if one collects together into equivalence classes all such segments that differ from one another by amounts that are infinitesimal relative to \( s \), the resulting system of equivalence classes with order defined in the expected manner is isomorphic to the standard continuum. Moreover, if (as in Veronese’s geometry) the system of directed segments on a line emanating from a point is an ordered field, then if one takes the equivalence class containing \( s \) as the unit and defines addition and multiplication of the equivalence classes in the familiar geometrical fashion, the resulting system is isomorphic to the positive cone \( \mathbb{R}^+ \) of the ordered field of real numbers. That is, *Veronese’s non-Archimedean continuum is indistinguishable from the classical continuum when infinite and infinitesimal differences are ignored*.

In addition to a non-Archimedean Euclidean space, which is what we are considering above, Veronese considers a non-Archimedean elliptic space, where the the system of directed segments emanating from a point has a somewhat different structure. However, in the elliptic case, the import of his relativity continuity condition is much the same.
Unlike the segment $AY$ in the relative continuity condition, which is unique if and only if the Archimedean axiom holds, the segment $AY$ in the absolute continuity hypothesis is invariably unique, as Veronese was well aware. For this reason, Veronese’s absolute continuity condition may also be stated in the following algebraic form that more clearly highlights its relation to the continuity condition of Dedekind.

**Absolute Continuity: Algebraic Formulation.** Let $G$ be an ordered abelian group (or the positive cone thereof). If $(A, B)$ is a Dedekind cut of $G$ and if for each positive $\epsilon \in G$ there are elements $a$ of $A$ and $b$ of $B$ for which $b - a < \epsilon$, then either $A$ has a greatest member or $B$ has a least member, but not both.

It is a simple matter to show that in the Archimedean case, and only in the Archimedean case, Veronese’s metrical condition on cuts is invariably satisfied. Thus, for Veronese, unlike for Dedekind, continuous systems of magnitudes need not be completely devoid of Dedekind gaps, though they must be devoid of those Dedekind gaps that satisfy the metrical condition satisfied in the classical case. Veronese maintained that insofar as the intuitive conception of a continuum does not require the Archimedean axiom, it is his absolute continuity condition, rather than Dedekind’s continuity condition, that is intuitively more justifiable.

The algebraic and geometric formulations of Veronese’s absolute continuity condition were widely discussed during the first decade of the twentieth century by authors such as Hölder, A. Schoenflies, L. E. J. Brouwer, K. Vahlen, G. Vitali, F. Enriques and Hahn, before sinking into relative obscurity as the Cantor-Dedekind conception of the continuum solidified its status as the standard conception. However, it was resurrected (without reference to Veronese) by a number of authors including R. Baer (1929, 1970), L. W. Cohen and C. Goffman (1949), K. Hauschild (1966) and Dana Scott (1969a), who (along with others) carefully studied it as a completeness condition. Among the results that emerged from these investigations is that every nontrivial densely ordered abelian group (ordered field) admits an extension, unique up to isomorphism, to a least ordered abelian group (ordered field) that satisfies Veronese’s absolute continuity condition. Motivated by the work of Cohen and Goffman and especially Scott, and unaware of the Veronesean roots of the condition, Zakon (1969, p. 226) asked if nonstandard models of analysis are complete in the absolute sense of Veronese. In response it was found that some are (e.g. Keisler 1974; Keisler and Schmerl 1991; Jin and Keisler 1993) and others are not (e.g. 8For references, see (Ehrlich 2006, pp. 66-70).
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Kamo 1981, 1981a; Keisler and Schmerl 1991; Ozawa 1995), something we will return to in §8. Those that are are usually said to be Scott Complete. On the other hand, since (as can be readily shown) an ordered field is continuous in the relative sense of Veronese if and only if contains an isomorphic copy of \( \mathbb{R} \), every nonstandard model of analysis satisfies Veronese’s relative continuity condition. The same is also true of the ordered fields due to Hahn discussed in the following section, the first group of which drew inspiration from the Cantor-Dedekind continuum as well as the non-Archimedean continua of Veronese.

5. HAHN’S NON-ARCHIMEDEAN GENERALIZATIONS OF THE ARCHIMEDEAN ARITHMETIC CONTINUUM

Though Veronese’s construction of his non-Archimedean geometric continuum is synthetic, he represents the line segments that emerge from his construction using a loosely defined, complicated system of numbers consisting of finite and transfinite series of the form

\[ \infty_1^{y_1} r_1 + \infty_1^{y_2} r_2 + \infty_1^{y_3} r_3 + \cdots \]

where \( r_1, r_2, r_3, \ldots \) are real numbers, and \( \infty_1^{y_1}, \infty_1^{y_2}, \infty_1^{y_3}, \ldots \) is a sequence of units, each of which is infinitesimal relative to the preceding units, \( \infty_1 \) being the number (of “infinite order 1” (1891, p. 101)) introduced by Veronese to represent the infinitely large line segment whose existence is postulated by his “hypothesis on the existence of bounded infinitely large segments” (1891, p. 84). Veronese’s number system was provided an analytic foundation by Levi-Civita (1892–1893, 1898), who therewith provided the first analytic constructions of non-Archimedean ordered fields. Building on the work of Levi-Civita, Hahn (1907) constructed non-Archimedean ordered fields (and ordered abelian groups more generally) having properties that generalize the familiar continuity properties of Dedekind and Hilbert (Ehrlich 1995, 1997, 1997a), and he demonstrated (vis-à-vis his celebrated Hahn Embedding Theorem for ordered abelian groups\(^{10}\)) that his number systems provide a panorama of the finite, infinite and infinitesimal numbers that can enter into a non-Archimedean theory of continua based on the concept of an

\(^9\)Veronese’s absolute continuity condition has emerged as a standard concept in the theory of ordered algebraic systems, albeit usually under a variety of other names. For further references, see (Ehrlich 1997, p. 224).

\(^{10}\)For a historical account of Hahn’s embedding theorem and his momentous contribution to non-Archimedean mathematics more generally, see (Ehrlich 1995) which includes an extensive discussion with references of all the material in this section.
ordered field (Ehrlich 1995, 1997, 1997a). This idea was later brought into sharper focus when it was demonstrated that every ordered field can be embedded in a perspicuous fashion in a suitable Hahn field\footnote{This theorem, which extends Hahn’s embedding theorem for ordered abelian groups to ordered fields, has a complicated history that makes it difficult to attribute it to any single author. However, by the early 1950s, as a result of the work of Kaplansky (1942), it appears to have assumed the status of a “folk theorem” among knowledgeable field theorists, with numerous proofs published thereafter. For references and details, see (Ehrlich 1995).} the general construction of the latter of which is given as follows.

**Hahn Field 1** (Hahn 1907). Let $\mathbb{R}$ be the ordered field of real numbers and $G$ be a nontrivial ordered abelian group. The collection,

$$\mathbb{R}((t^G))$$

of all series

$$\sum_{\alpha<\beta} t^{g_\alpha} \cdot r_\alpha,$$

where $\beta$ is an ordinal, $\{g_\alpha : \alpha < \beta\}$ is a (possibly empty) strictly decreasing sequence of members of $G$ and $\{r_\alpha : \alpha < \beta\}$ is a sequence of members of $\mathbb{R} - \{0\}$ is a non-Archimedean ordered field when the order is defined lexicographically and sums and products are defined termwise, it being understood that $t^\gamma \cdot t^\kappa = t^{\gamma+\kappa}$.

Hahn further established

**Hahn Field 2** (Hahn 1907). If in the above construction one restricts the members of the universe to series indexed over all ordinals having cardinality less that an uncountable cardinal $\aleph_\alpha$, the resulting structure, which we will denote

$$\mathbb{R}((t^G))_\alpha$$

is likewise an ordered field.

While Hahn assumed $G$ to be a set, Hahn Field 2 continues to hold in NBG when $G$ is a proper class and the series are indexed over all ordinals $\alpha < \beta$, where $\beta$ is an arbitrary ordinal. Henceforth, we will assume that Hahn Field 2 is so extended to include such cases, the latter of which will be denoted

$$\mathbb{R}((t^G))_{\text{On}}.$$

The theories of Hahn groups and Hahn fields play critical roles in the theory of ordered algebraic systems and in the corresponding model theory thereof. In \S 9 we shall see how particular $\aleph_\alpha$-restricted Hahn fields from Hahn Field 2 shed light on some of the most distinguished
non-Archimedean continua of our day. In the remainder of this section we will introduce a number of related concepts that will be employed in the subsequent discussion and which shed further light on Hahn’s celebrated constructs.

Using absolute values, Veronese’s comparative notions of segments that are finite, infinite, and infinitesimal relative to one another were extended to the members of nontrivial ordered abelian groups by Levi-Civita (1898), and through the work of Levi-Civita and Hahn have become fixtures in post-nineteenth century mathematics, albeit under a variety of different rubrics. For example, following B. H. Neumann (1949, p. 205), instead of saying that the nonzero elements are finite relative to one another, they are now commonly said to be Archimedean equivalent to one another. Archimedean equivalence partitions the nonzero elements of an (additively written) ordered abelian group into disjoint classes called Archimedean classes.

Formally speaking, if \( a \) and \( b \) are nonzero members of an ordered abelian group \( G \), then \( a \) is said to be Archimedean equivalent to \( b \) if there are positive integers \( m \) and \( n \) such that \( m|b| > a \) and \( n|a| > b \); if \( a \) and \( b \) are not Archimedean equivalent, then \( a \) is said to be infinitesimal (in absolute value) relative to \( b \) and \( b \) is said to be infinite (in absolute value) relative to \( a \), if \( |a| < |b| \). In accordance with these conventions, 0 is infinitesimal (in absolute value) relative to every other member of \( G \). Moreover, if \( G \) is the additive group of an ordered field or of an ordered ring with a unit or identity more generally, the elements are simply said to be infinite (in absolute value) and infinitesimal (in absolute value), respectively, if they are infinite (in absolute value) relative to and infinitesimal (in absolute value) relative to the identity.

In virtue of the lexicographical ordering, every nonzero member of a Hahn field \( \mathbb{R}(t^G) \) is Archimedean equivalent to exactly one element of the form \( t^g \) (\( g \in G \)), the latter of which may be regarded as a canonical “unit” element of the Archimedean class. Two such elements \( x \) and \( y \) are Archimedean equivalent if and only if their zeroth exponents (\( g_0 \) in the statement of Hahn Field 1) are equal, and \( x \) is infinitesimal (in absolute value) relative to \( y \) if and only if the zeroth exponent of \( x \) is less than the zeroth exponent of \( y \). Thus, like the numbers representing Veronese’s non-Archimedean continuum, Hahn’s numbers are formal sums of terms, each being infinitesimal (in absolute value) relative to the preceding terms, where each term is a nonzero real multiple of the canonical unit of its Archimedean class. Furthermore, every nonzero \( x \in \mathbb{R}(t^G) \) is the sum of three components: the purely infinite part of \( x \), whose terms have positive exponents; the real part of \( x \), whose sole term has exponent 0; and the infinitesimal part of \( x \), whose terms
have negative exponents. The appellation “real part” is motivated by
the fact that \( \{ r^{t^0} : r \in \mathbb{R} \} \) is a canonical copy of the ordered field of
reals in \( \mathbb{R}((t^G)) \).

6. THE PANTACHIES OF DU BOIS-REYMOND AND HAUSDORFF

Although interest in the rates of growth of real functions is already
found in Euler’s *De infinitis infinitis gradibus tam infinite magnorum quam infinite parvorum* (On the infinite degrees of infinity of the in-
finity large and infinitely small) (1778), their systematic study was
first undertaken by Paul du Bois-Reymond (cf. 1870-71, 1875, 1877,
1882), under the rubric *Infinitārcalcūl* (infinitary calculus). And
while du-Bois-Reymond never attempted to employ this work to de-
velop a non-Archimedean theory of continua, he and others including
Poincaré (1893/1952, pp. 28-29) believed it provided an intimation of
the possibility of such a theory. Moreover, as we shall see in §8, there
are intriguing historical and conceptual relations between his theory
and one of the most important such theories of our day.

Though du Bois-Reymond’s contribution to the *Infinitārcalcūl* is
concerned solely with functions and is analytic by nature, it is in-
timately related to his ideas on quantity, in general, and the geo-
metric linear continuum, in particular. Unlike Veronese, who admit-
ted the possibility of an Archimedean geometric linear continuum, for
du Bois-Reymond the geometric linear continuum is necessarily non-
Archimedean. Indeed, on the basis of a misguided argument that
has been aptly characterized as “breathtaking” (Fisher 1981 p. 114),
du Bois-Reymond maintained that the infinite divisibility of the line
implies that “the unit segment decomposes into infinitely many sub-
segments of which none is finite” (1882, p. 72). Moreover, as in
Veronese’s non-Archimedean continuum, for every segment of du Bois-
Reymond’s continuum, there are segments that are infinitesimal rel-
ative to it. However, unlike Veronese’s non-Archimedean continuum,
du Bois-Reymond’s continuum of segments is not the positive cone of
an ordered field (or even of an ordered abelian group) since, accord-
ing to du Bois-Reymond, “two finite segments are equal when there
is no finite difference between them. [That is,] a finite quantity does
not change if an infinitely small quantity is added to it or taken away
from it” (1882, pp. 73-74), even when the infinitely small quantity is
nondegenerate.

\(^{12}\)For a complete list of du Bois-Reymond’s writings on his *Infinitārcalcūl* and a
survey of the contents thereof, see (Fisher 1981).
It was the comparison of quantities (of the same type) having different orders of magnitude that du Bois-Reymond took to be the primary object of his infinitary calculus (1882, pp. 75-66), but as was mentioned above, he only developed the theory for functions. In particular, du Bois-Reymond erects his *Infinitärcalcül* primarily on families of increasing functions from $\mathbb{R}^+ = \{ x \in \mathbb{R} : x > 0 \}$ to $\mathbb{R}^+$ such that for each function $f$ of a given family, $\lim_{x \to \infty} f(x) = +\infty$, and for each pair of functions $f$ and $g$ of the family, $0 \leq \lim_{x \to \infty} f(x)/g(x) \leq +\infty$. He assigns to each such function $f$ a so-called *infinity*, and defines an ordering on the infinities of such functions by stipulating that for each pair of such functions $f$ and $g$:

- $f(x)$ has an infinity greater than that of $g(x)$, if $\lim_{x \to \infty} f(x)/g(x) = +\infty$;
- $f(x)$ has an infinity equal to that of $g(x)$, if $\lim_{x \to \infty} f(x)/g(x) = a \in \mathbb{R}^+$;
- $f(x)$ has an infinity less than that of $g(x)$, if $\lim_{x \to \infty} f(x)/g(x) = 0$.

In accordance with this scheme, the infinities of the following functions

$\ldots, \ln(\ln x), \ln x, \ldots, x^{1/n}, \ldots, x^{1/3}, x^{1/2}, x, x^2, x^3, \ldots, x^n, \ldots, e^x, e^{e^x}, \ldots$

increase as we move from left to right. Moreover, as the comparative graphs of several of these functions illustrate (see Figure 1), given any two functions $f$ and $g$ having different infinities from a family of the just-said kind, $f(x)$ has a greater infinity than $g(x)$ if and only if $f(x) > g(x)$ for all $x >$ some $x_0$. Furthermore, since, for example, $x^2$ has a greater infinity than $x$ and

$$\lim_{x \to \infty} (x^2 + x)/x^2 = 1,$$

$x^2 + x$ has the same infinity as $x^2$, which illustrates for functions du Bois-Reymond’s corresponding idea for segments, that if a segment $s$ is infinitesimal relative to a segment $s'$, then the segment resulting from adding $s$ to $s'$ is equal in length to $s'$.

While du Bois-Reymond usually restricted his investigations to families of functions of the above said kind, on occasion he mistakenly assumed that each pair of increasing functions $f$ and $g$ from $\mathbb{R}^+$ to $\mathbb{R}^+$ for which $\lim_{x \to \infty} f(x) = +\infty$ and $\lim_{x \to \infty} g(x) = +\infty$ could be compared in the manner described above. This led him to postulate the existence

Stolz, who was a student of du Bois-Reymond (as well as Weierstrass), developed another number system (Stolz 1884) that modeled this absorptive aspect of du Bois-Reymond’s conception. For a discussion of this little-known system of “moments”, see (Ehrlich 2006, §3).
of an all-inclusive ordering of the infinities of such real functions—an \textit{infinitary pantachie}, as he called it (1882, p. 220).\textsuperscript{14} Such a pantachie, according to du Bois-Reymond, would provide a conception of a numerical linear continuum “denser” than that of Cantor and Dedekind.

Georg Cantor, however, who was an ardent opponent of infinitesimals and non-Cantorian infinities (cf. Ehrlich 2006) would have no part of this. Indeed, having demonstrated as Stolz (1879, p. 232) and Pincherle (1884, p. 742) had before him that there could be no such all-inclusive ordering of the infinities of such functions, Cantor proclaimed: “the ‘\textit{infinitary pantachie},’ of du Bois-Reymond, belongs in the wastebasket \textit{as nothing but paper numbers!}” (1895, p. 107). Felix Hausdorff, on the other hand, suggested, “[t]here is no reason to reject the entire theory because of the possibility of incomparable functions as G. Cantor has done” (1907, p. 107), and in its place undertook the study of \textit{maximally inclusive sets of pairwise comparable real functions},

\textsuperscript{14}Du Bois-Reymond explains that his adjective “pantachie” derives from the Greek words for “everywhere”.

\textbf{Figure 1.}
each of which, retaining du Bois-Reymond’s term, he calls an *infinitary pantachie* or a *pantachie* for short. This led him to his well-known investigation of $\eta_\alpha$-*orderings* (1907), and to the following less well-known theorem.

**Pantachie 1** (Hausdorff 1907, 1909). *Infinitary pantachies exist.* If $P$ is an infinitary pantachie, then $P$ is an $\eta_1$-*ordering* of power $2^{\aleph_0}$; in fact, $P$ is (up to isomorphism) the unique $\eta_1$-*ordering* of power $\aleph_1$, assuming (the Continuum Hypothesis) CH.

An ordered set $L$ is said to be an $\eta_\alpha$-*ordering*, if for all subsets $A$ and $B$ of $L$ of power $< \aleph_\alpha$, where $A < B$ (i.e. every member of $A$ precedes every member of $B$), there is a $y \in L$ such that $A < \{y\} < B$. The ordered field $\mathbb{R}$ of real numbers is an $\eta_0$-*ordering*, though not an $\eta_1$-*ordering*, and as such Hausdorff’s pantachies are in a precise sense more dense than $\mathbb{R}$, thereby lending precision to du Bois-Reymond’s intimation.

As was noted above, du Bois-Reymond further believed that pantachies exhibit numerical aspects, but he never undertook an arithmetization of them. Motivated by different considerations, however, Hausdorff did. Moreover, like his theory of pantachies, more generally, he did so in the following broader setting.

In addition to modifying du Bois-Reymond’s conception of an infinitary pantachie, Hausdorff redirected du Bois-Reymond’s investigation by investigating *numerical sequences* rather than continuous functions, though he showed that the desired results about the latter can be obtained as corollaries from results about the former. He also deleted the monotonicity assumption and replaced the infinitary rank ordering with the *final rank ordering* (which was illustrated above). That is, Hausdorff redirected du Bois-Reymond’s investigation to the study of subsets of the set $\mathcal{B}$ of all numerical sequences $A = (a_1, a_2, a_3, \ldots, a_n, \ldots)$ in which the $a_n$ are real numbers, and he defines the “final ordering” on $\mathcal{B}$ (and subsets thereof) by the conditions $A < B$ if eventually $a_n < b_n$, $A = B$ if eventually $a_n = b_n$, $A > B$ if eventually $a_n > b_n$, and $A \parallel B$ (i.e., $A$ is incomparable with $B$) in all other cases, where “eventually” means for all values of $n$ with the exception of a finite number, thus for all $n \geq$ some $n_0$ [1909 in (Plotkin 2005, p. 276)].

Hausdorff, who bases his theory on representative elements of the equivalences classes

15Strictly speaking, in his (1907), unlike his (1909), Hausdorff only considers numerical sequences in which the $a_n$ are positive real numbers. However, the proof of the above theorem carries over to the more general case. Moreover, in his (1907), unlike his later works, he uses the term “finally” instead of “eventually” in the definitions of $<$, $>$, $=$ and $\parallel$. 
of eventually equal numerical sequences rather than on the equivalence classes themselves, calls a subset $\mathcal{B}'$ of $\mathcal{B}$ totally ordered by the final order a pantachie if it is not properly contained in another subset of $\mathcal{B}$ totally ordered by the final order.

In his aforementioned investigation of 1907, Hausdorff first raised the question of the existence of a pantachie that is algebraically a field, but he only made partial headway in providing an answer. However, in 1909 he returned to the problem and provided a stunning positive answer. Indeed, beginning with the ordered set of numerical sequences of the form $(r, r, r, \ldots, r, \ldots)$ where $r$ is a rational number, and utilizing what appears to be the very first algebraic application of his maximal principle, Hausdorff proves the following little-known, remarkable result.

**Pantachie 2** (Hausdorff 1909). *There is a pantachie of numerical sequences that is an ordered field, whose field operations are given by*

\[
A + B = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n, \ldots), \\
A - B = (a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n, \ldots), \\
AB = (a_1b_1, a_2b_2, \ldots, a_nb_n, \ldots), \\
A/B = (a_1/b_1, a_2/b_2, \ldots, a_n/b_n, \ldots),
\]

*where $A + B$, $A - B$, $AB$ and $A/B$ are defined up to final equality. Any ordered field that is a pantachie is, in fact, real-closed.*

Writing before Artin and Schreier (1926), Hausdorff does not employ the term “real-closed”, though as Ehrlich (2012, p. 29) observed, Hausdorff proves enough to show that an ordered field that is a pantachie is real-closed. However, to fully appreciate the significance of Hausdorff’s result as well as the extent of its intimate relation to certain contemporary non-Archimedean continua, we require some background about real-closed fields.

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16This result had been all but forgotten until it was resurrected in (Plotkin 2005, pp. 271-301) and (Ehrlich 2012, p. 29).

17By a classical result of R. Baer (1927) and W. Krull (1932), the collection of Archimedean classes of an ordered field $A$ constitutes an ordered abelian group induced by the order and multiplication in $A$ (see, Ehrlich 1995, p. 186). Though we will not further pursue the matter here, we note that the system of orders of infinity associated with the members of a Hausdorff pantachie $\mathbb{H}_p$ is isomorphic to the positive cone of the ordered abelian group of Archimedean classes of $\mathbb{H}_p$, the latter being (up to isomorphism) the unique divisible ordered abelian group that is an $\eta_1$-ordering of power $\aleph_1$, assuming CH (e.g., (Ehrlich 1988)).
In their groundbreaking work on “real algebra” Emil Artin and Otto Schreier (1926) sought to characterize the algebraic content of a kind of ordered field of which the arithmetic continuum is paradigmatic. This led to their theory of real-closed fields, which is a jewel of twentieth-century mathematics. Following Artin and Schreier, an ordered field $K$ may be said to be real-closed if it admits no extension to a more inclusive ordered field that results from supplementing $K$ with solutions to polynomial equations with coefficients in $K$. Intuitively speaking, real-closed ordered fields are precisely those ordered fields having no “holes” that can be filled by algebraic means alone. Among a number of important properties of the arithmetic continuum they showed are satisfied by real-closed ordered fields more generally is the intermediate value theorem for polynomials of a single variable. In fact, as soon became clear, for an ordered field $K$, the satisfaction of the intermediate value theorem for polynomials of a single variable over $K$ is equivalent to $K$ being real-closed (cf. Tarski 1939; Warner 1965, pp. 492-493). Accordingly, the idea of a real-closed ordered field is equivalent for the case of polynomials of a single variable to satisfying what has traditionally been regarded as one of the quintessential properties of a continuous function—the formalization of the intuitive idea that a continuous curve connecting points on the opposite sides of a straightline intersects the given line. A similar equivalence was later established for the extreme value theorem for polynomials in a single variable (Gamboa 1987), another prototypical classical property of continuous functions. Shedding still more light on the relation between real-closed ordered fields and $\mathbb{R}$, Tarski (1948, 1959) demonstrated that real-closed ordered fields are precisely the ordered fields that are first-order indistinguishable from $\mathbb{R}$, or, to put this another way, they are precisely the ordered fields that satisfy the elementary (first-order) content of the Dedekind continuity axiom. For this reason they are sometimes called elementary continua. When Tarski’s result is combined with a classical result of Artin and Schreier (1926) on real-closed ordered fields, one obtains the critical fact that every ordered field is contained in a (to within isomorphism) smallest elementary continuum, the latter having the same cardinality of the given field.

While $\mathbb{R}$ is the best known elementary continuum, as is evident from the above it is not the only one. Some elementary continua, like $\mathbb{R}$, are Archimedean, though most are non-Archimedean, and among the latter many are extensions of $\mathbb{R}$. In the following section, we will discuss a class of real-closed extensions of $\mathbb{R}$ that are among the foremost
non-Archimedean continua of our day, and in the subsequent section we will draw attention to some of the interesting connections between them, Hausdorff’s pantachies, the surreals and some of the elementary continua that emerge from the constructions of Hahn. Unlike Veronese’s non-Archimedean continuum, which was motivated by generalizing the geometrical properties of the classical geometric continuum, those to which we now turn are motivated by the idea of providing a non-Archimedean treatment of the analytic properties of $\mathbb{R}$.

8. NONSTANDARD (ROBINSONIAN) CONTINUA

In the early 1960s Abraham Robinson (1961, 1966) made the momentous discovery that among the real-closed extensions of the reals there are number systems that can provide the basis for a consistent and entirely satisfactory nonstandard approach to analysis based on infinitesimals, a possibility that had been called into question by many since the latter decades of the nineteenth century. By analogy with Thoralf Skolem’s (1934) nonstandard model of arithmetic, a number system from which Robinson drew inspiration, Robinson called his number systems nonstandard models of analysis. These number systems, which are now more often called hyperreal number systems (Keisler 1976, 1994), may be characterized as follows: let $\langle \mathbb{R}, S : S \in \mathcal{F} \rangle$ be a relational structure where $\mathcal{F}$ is the set of all finitary relations defined on $\mathbb{R}$ (including all functions). Furthermore, let $^*\mathbb{R}$ be a proper extension of $\mathbb{R}$ and for each $n$-ary relation $S \in \mathcal{F}$ let $^*S$ be an $n$-ary relation on $^*\mathbb{R}$ that is an extension of $S$. The structure $\langle ^*\mathbb{R}, ^*\mathbb{R}, ^*S : S \in \mathcal{F} \rangle$ is said to be a hyperreal number system if it satisfies the transfer principle: every $n$-tuple of real numbers satisfies the same first-order formulas in $\langle \mathbb{R}, S : S \in \mathcal{F} \rangle$ as it satisfies in $\langle ^*\mathbb{R}, ^*\mathbb{R}, ^*S : S \in \mathcal{F} \rangle$.

The existence of hyperreal number systems is a consequence of the compactness theorem of first-order logic and there are a number of algebraic techniques that can be employed to construct such systems. The earliest and still one of the most commonly employed such techniques is the ultrapower construction (e.g. Keisler 1976, pp. 48-57; Goldblatt 1998, ch. 3), a construction that was introduced in full generality.

18There were some earlier successes in this direction, including the modest contributions of Levi-Civita (1898), Neder (1941, 1943) and Gesztesy (1958). The most successful of these is the theory of Schmieden and Laugwitz (1958; also see, Laugwitz 1983, 1992, 2001). Unlike Robinson, Schmieden and Laugwitz make use of a partially-ordered number system containing zero-divisors, with the result that many of the classical results need to be reformulated. As Laugwitz aptly acknowledged, Robinson’s system “was much more powerful when it came to applications in contemporary research” (2001, p. 128).
by Loś (1955), identified as a source of hyperreal number systems by Robinson (1961, p. 3) and popularized as such by Luxemburg (1962) and Stroyan and Luxemburg (1976). Not all hyperreal number systems can be obtained this way, however. By results of H. J. Keisler (1963, 1976, pp. 58-59), on the other hand, every hyperreal number system is isomorphic to a limit ultrapower.

Using the transfer principle, one can develop satisfactory nonstandard conceptions and treatments of all of the concepts and theorems of the calculus (e.g. Keisler 1976; Goldblatt 1998; Loeb 2000). For example, it follows from the transfer principle that:

A real-valued function \( f \) is continuous at \( a \in \mathbb{R} \) (in the standard sense) if and only if \( *f(x) \) is infinitely close to \( *f(a) \) whenever \( x \) is infinitely close to \( a \), for all \( x \in *\mathbb{R} \);

and on the basis of this one may prove the familiar classical results concerning the continuity of real-valued functions including the intermediate and extreme value theorems (e.g. Goldblatt 1998, pp. 79-80).

Of course modern analysis goes well beyond the traditional province of the calculus, dealing with arbitrary sets of reals, sets of sets of reals, sets of functions from sets of reals to sets of reals, and the like. For example, it is commonplace for analysts to prove theorems about the set of all continuous functions on the reals or about the set of all open subsets of the reals, sets to which the just-said transfer principle does not apply. To obtain nonstandard treatments of these aspects of analysis a more general transfer principle is required. For this purpose Robinson (1966) originally employed a type-theoretical version of higher order logic, but it proved to be unpopular. Since then, following Robinson and Zakon (1969), it has become most common to employ a transfer principle associated with a structure \( \langle V(*\mathbb{R}), V(\mathbb{R}), * \rangle \) that generalizes the corresponding hyperreal number system \( \langle *\mathbb{R}, \mathbb{R}, *S : S \in \mathcal{F} \rangle \). In this setup the transfer principle emerges from an elementary embedding * that relates the members of the superstructure \( V(\mathbb{R}) \) over \( \mathbb{R} \) with those of the superstructure \( V(*\mathbb{R}) \) over \( *\mathbb{R} \), where for any set of individuals \( X \), the superstructure \( V(X) \) over \( X \) is defined by \( V(X) = \bigcup_{n<\omega} V_n(X) \), where \( V_0(X) = X \) and \( V_{n+1}(X) = V_n(X) \cup \mathcal{P}(V_n(X)) \), \( \mathcal{P} \) being the

\[ ^{19} \text{Hewitt (1948) had already employed the ultrapower construction to obtain a non-Archimedean real-closed extension of the reals, but unlike Loś he did not establish the transfer principle that accrues from the construction, the latter being crucial to hyperreal number systems in the sense used above.} \]
power set operator. More specifically, the generalized transfer principle asserts that any property expressible in the language \{\ldots, \leq, \in\} with only bounded quantifier formulas holds in \(V(\mathbb{R})\) for an entity \(S\) if and only if it holds in \(V(\ast \mathbb{R})\) for the corresponding entity \(\ast S\).

Robinson and Zakon’s superstructure paper was presented at a conference on applications of model theory to algebra, analysis and probability held at Cal Tech in May of 1967. Luxemburg, who organized the conference, also presented a paper (1969) that has had a profound impact on the development of nonstandard analysis. In his paper, Luxemburg incorporated the conception of a \(\kappa\)-saturated model (developed by Morley and Vaught 1962 and Keisler 1964: see Chang and Keisler 1990) into the subject and applied it to the theory of topological spaces. Soon thereafter, Peter Loeb (1975; also see, Hurd and Loeb 1985, ch. IV), making critical use of Luxemburg’s ideas on saturation, introduced the Loeb measure and therewith (along with Anderson (1976), Henson (1979) and others) developed the nonstandard treatments of the classical theories of measure and integration due to Lebesgue. Since that time, the use of saturation assumptions in nonstandard analysis has become standard.

As the term “saturation” suggests, models satisfying saturation assumptions are rich in elements. The saturation assumptions most commonly employed make use of the critical notion of an internal set or an internal relation, where an entity \(b \in V(\ast \mathbb{R})\) is said to be internal if \(b \in \ast a\) for some \(a \in V(\mathbb{R}) - \mathbb{R}\). Roughly speaking, the internal sets are those sets of hyperreals that inherit the first-order properties of their real counterparts.

As the following formulation makes clear, saturation assumptions come in varying strengths.

**Saturation for \(V(\ast \mathbb{R})\).** \(V(\ast \mathbb{R})\) is said to be \(\kappa\)-saturated if every subset \(X\) of \(V(\ast \mathbb{R})\) consisting of \(< \kappa\) internal sets has a nonempty intersection whenever every finite subset of \(X\) has a nonempty intersection. If \(\kappa\) is the cardinality of \(\ast \mathbb{R}\), \(V(\ast \mathbb{R})\) is said to be saturated.

For many purposes, \(\omega_1\)-saturation is all the saturation of \(V(\ast \mathbb{R})\) a nonstandard analyst requires. However, as Henson and Keisler (1986)
demonstrated, even a reliance on $\omega_1$-saturation is of metamathematical significance. In particular, if, as mathematical practice seems to suggest, standard analysis can be formalized in a conservative extension of second-order arithmetic, then “nonstandard analysis (i.e. second order nonstandard arithmetic) with the $\omega_1$-saturation axiom scheme has the same strength as third order arithmetic” (1986, p. 377) and is therefore stronger than standard analysis as is typically practiced. Indeed, as Henson and Keisler put it:

This shows that in principle there are theorems which can be proved with nonstandard analysis but cannot be proved by the usual standard methods. The problem of finding a specific and mathematically natural example of such a theorem remains open. However, there are several results, particularly in probability theory, whose only known proofs are nonstandard arguments which depend on saturation principles; see, for example, the monograph (Keisler 1984). Experience suggests that it is easier to work with nonstandard objects at a lower level than with sets at a higher level. This underlies the success of nonstandard methods in discovering new results. To sum up, nonstandard analysis still takes place within ZFC, but in practice it uses a larger portion of full ZFC than is used in standard mathematical proofs. (1986, pp. 377-378)

While a theorem of nonstandard analysis expressible in standard mathematics that cannot be established by standard means has yet to be identified, the recognition of the value of nonstandard techniques for discovering new results, a value emphasized by Robinson (1974, p. IX) and Gödel (Robinson 1974, p. X), and reiterated by Henson, Keisler and others, appears to be growing. Noteworthy in this regard is the high-profile championing in both words and deeds of the use of nonstandard techniques for discovering new results by Terence Tao (e.g. Tao 2008, 2013, 2014), one of the most celebrated mathematicians of our day.

Unlike $\mathbb{R}$, the structures that may play the role of $\mathbb{R}$ in a hyperreal number system are far from being unique up to isomorphism. In addition to there being such systems satisfying varying types and strengths

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21 The result of Henson and Keisler stands in contrast with those of Harvey Friedman and Kreisel (Kreisel 1969), who established conservation results when no saturation assumption is assumed.
of saturation conditions, in an attempt to find hyperreal number systems that look more like \( \mathbb{R} \) some authors have sought and established the existence of hyperreal number systems that are continuous in the sense of Veronese (see §4) or that satisfy generalizations of the Bolzano-Weierstrass property (e.g. Keisler and Schmerl 1991), properties that are not universally satisfied by hyperreal number systems. From a purely mathematical point of view this absence of uniqueness causes no difficulty, and it has been argued that from the standpoint of varying applications can even be advantageous (e.g. Keisler 1994, p. 229; Jin 1992, 1997). On the other hand, if one takes \( \ast\mathbb{R} \) to be a model of the continuous straight line of geometry—something practitioners of nonstandard analysis tend not to do—the absence of uniqueness is a bit disconcerting.

Still, as several nonstandard analysts including Robinson (1973, p. 130), Lindstrøm (1988, p. 82) and Keisler (1994, p. 229) emphasize, even \( \mathbb{R} \) is not as unique as one would like to think since its uniqueness up to isomorphism is relative to the underlying set theory. Indeed, as Dana Scott aptly observed:

\begin{quote}
Maybe we have to face the fact that there are many distinct theories of the continuum. (Scott 1969, p. 87)
\end{quote}

In particular, by retaining the construction of \( \ast\mathbb{R} \) and supplementing the set theory with additional axioms, one can change the second-order theory of the real line. This led Keisler (1994) to suggest that not only is ZFC not the appropriate underlying set theory for the hyperreal number system but also that set theory might have developed differently had it been developed with the hyperreal numbers rather than the real numbers in mind. According to Keisler, an appropriate set theory should have the power set operation to insure the unique existence of the real number system, and another operation which insures the unique existence of the pair consisting of the real and hyperreal number systems (1994, p. 230).

Motivated by the above, Keisler (1976, pp. 57-60) introduced the following saturation assumption for hyperreal number systems to secure categoricity. Unlike the saturation condition for \( * \), the saturation assumption for \( \ast\mathbb{R} \) does not appeal to the notion of an internal set.

**Saturation Axiom for \( \ast\mathbb{R} \).** Let \( S \) be a set of equations and inequalities involving real functions, hyperreal constants, and variables, such that \( S \) has a smaller cardinality than \( \ast\mathbb{R} \). If every finite subset of \( S \) has a hyperreal solution, then \( S \) has a hyperreal solution.
Saturated hyperreal number systems cannot be proved to exist in ZFC. However, in virtue of classical results from the theory of saturated models there is (up to isomorphism) a unique saturated hyperreal number system of power \( \kappa \) whenever either \( \kappa \) is (strongly) inaccessible and uncountable or the generalized continuum hypothesis holds at \( \kappa \) (i.e., \( \kappa = \aleph_{\alpha+1} = 2^{\aleph_\alpha} \) for some \( \alpha \)). So, for example, by supplementing ZFC with the assumption of the existence of an uncountable inaccessible cardinal, one can obtain uniqueness (up to isomorphism) by limiting attention to saturated hyperreal lines having the least such power (Keisler 1976, p. 60). Keisler (1976, p. 59) and others (e.g. Kanovei and Reeken 2004; Ehrlich 2012; Borovik, Jin and Katz 2012) have also noted that it is possible to obtain uniqueness (up to isomorphism) by employing a saturated model that is a proper class, something we will say more about in the following section. In addition, following Henson (1974) and Ross (1990), it is possible to introduce axioms which, while falling short of full saturation, imply varying degrees of saturation and guarantee uniqueness up to isomorphism in certain powers that provably exist in ZFC.

In the Preface to the Second Edition of his monograph *Non-Standard Analysis*, Robinson expressed the view that:

the application of non-standard analysis to a particular mathematical discipline is a matter of choice and that it is natural for the actual decision of an individual to depend on his early training. (Robinson 1974, ix)

Following a talk given by Robinson at the Institute for Advanced Study in Princeton in March of 1973, Kurt Gödel went further when he maintained:

There are good reasons to believe that non-standard analysis, in some version or other, will be the analysis of the future. (Robinson 1974, p. x)

Among the reasons offered by Gödel was that “non-standard analysis frequently simplifies substantially the proofs, not only of elementary theorems, but also of deep results” with the consequence that it can

22The aforementioned ultrapower construction of hyperreal number systems is another source of nonuniqueness since the construction depends on an arbitrary choice of a nonprincipal ultrafilter. Hyperreal number systems that arise via iterated ultrapower constructions are analogous sources of nonuniqueness. However, Kanovei and Shelah (2004) have shown that in certain important cases these sources of nonuniqueness are eliminable via the existence of definable hyperreal number systems.
“facilitate discovery” as witnessed by “the proof of the of invariant subspaces for compact operators”. (Robinson 1974, p. x)

“An even more convincing reason”, said Gödel, is that:

Arithmetic starts with the integers and succeeds by successively enlarging the number system by rational and negative numbers, irrational numbers, etc. But the next quite natural step after the reals, namely the introduction of infinitesimals, has simply been omitted. (Robinson 1973, p. x)

Roughly two decades later, motivated by considerations of simplicity and facility of discovery, H. J. Keisler more guardedly observed:

At the present time, the hyperreal number system is regarded as somewhat of a novelty. But because of its broad potential, it may eventually become part of the basic toolkit of mathematicians. This process will probably take a very long time, perhaps 50 to 100 years. (1994, p. 235)

As we mentioned above, nonstandard analysis has indeed made inroads among standard analysts and mathematicians more generally and has been employed to prove a wide range of new results. Given its growing visibility and the growing recognition of its heuristic virtues, there is some reason to believe that nonstandard analysis, together with its non-Archimedean continuum, may indeed become part of the basic toolkit of mathematicians in the timeframe Keisler suggested.

9. THE ABSOLUTE ARITHMETIC CONTINUUM: CONWAY’S SYSTEM OF SURREAL NUMBERS

In his monograph *On numbers and Games* (1976), J. H. Conway introduced a real-closed ordered field of *surreal numbers* containing the reals and the ordinals as well as a great many less familiar numbers including $-\omega$, $\omega/2$, $1/\omega$, $\sqrt{\omega}$, $e^\omega$, $\log \omega$ and $\sin (1/\omega)$ to name only a few, where $\omega$ is the least infinite ordinal. This particular real-closed field, which Conway calls $\text{No}$, is so remarkably inclusive that, subject to the proviso that numbers—construed here as members of ordered fields—be individually definable in terms of sets of NBG, it may be said to contain, “All Numbers Great and Small” (Conway 1976, p. 3). In this regard, $\text{No}$ bears much the same relation to ordered fields that $\mathbb{R}$ bears to Archimedean ordered fields.

The relation between the inclusiveness of $\mathbb{R}$ and $\text{No}$ may be made precise by the following result where an ordered field (Archimedean
ordered field) $A$ is said to be *homogeneous universal* if it is universal—every ordered field (Archimedean ordered field) whose universe is a set or a proper class of NBG can be embedded in $A$—and it is *homogeneous*—every isomorphism between subfields of $A$ whose universes are sets can be extended to an automorphism of $A$.

**Surreal 1** (Ehrlich 1988, 1989, 1992). Whereas $\mathbb{R}$ is (up to isomorphism) the unique homogeneous universal Archimedean ordered field, $\mathbb{N}o$ is (up to isomorphism) the unique homogeneous universal ordered field.

In the case of Archimedean ordered fields, universality and homogeneous universality are equivalent since there is one and only one embedding of an Archimedean ordered field into $\mathbb{R}$. In the non-Archimedean case, on the other hand, this is not so since there are ordered fields that are universal, but not homogeneous. The homogeneity of $\mathbb{N}o$ ensures that whenever subfields of $\mathbb{N}o$, whose universes are sets, are structurally indistinguishable when considered in isolation, they along with their surroundings in $\mathbb{N}o$ are structurally indistinguishable as well.

Since there is a multitude of real-closed ordered fields, it is natural to inquire if, like $\mathbb{R}$, it is possible to distinguish $\mathbb{N}o$ (to within isomorphism) from the remaining real-closed ordered fields by appealing solely to its order. The following definition, which is a straightforward extension for proper classes of Hausdorff’s conception of an $\eta_\alpha$-ordering (see §6), enables one to do just this.

**Definition** (Ehrlich 1987, 1988). An ordered class $\langle A, < \rangle$ is said to be an absolute linear continuum if for all subsets $L$ and $R$ of $A$ where $L < R$ there is a $y \in A$ such that $L < \{y\} < R$.

An absolute linear continuum $\langle A, < \rangle$ is both absolutely dense in the sense that for each pair of nonempty subsets $L$ and $R$ of $A$ where $L < R$, there is a $y \in A$ such that $L < \{y\} < R$, and absolutely extensive in the sense that given any (possibly empty) subset $X$ of $A$ there are members $a$ and $b$ of $A$ that are respectively smaller than and greater than every member of $X$. In fact, an ordered class is an absolute linear continuum just in case it has both of the just-stated properties. Accordingly, since every element of an ordered class must either lie between two of its nonempty subclasses or lie to the right or to the left of some (possibly empty) subclass, absolute linear continua are

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23The just-said concepts are straightforward adaptations for Archimedean structures and proper classes of conceptions from the classical theory of *homogeneous universal structures* due to Jónsson (1960) and Morley and Vaught (1962).
ordered classes having no order-theoretic limitations that are definable in terms of sets of standard set theory.

The following is the analog for absolute linear continua of Cantor’s (1895) classical order-theoretic characterization of \( \mathbb{R} \).

**Surreal 2** (Ehrlich 1988). \( \mathbb{No} \) is (up to isomorphism) the unique absolute linear continuum.

Unlike \( \mathbb{R} \), however, \( \mathbb{No} \) is not distinguished (up to isomorphism) from other ordered fields by its structure as an ordered class. Indeed, there are infinitely many pairwise nonisomorphic ordered fields that are absolute linear continua. What one can prove, however, is

**Surreal 3** (Ehrlich 1988, 1992). \( \mathbb{No} \) is (up to isomorphism) the unique real-closed ordered field that is an absolute linear continuum.

Thus, in virtue of Surreal 3, \( \mathbb{No} \) is not only devoid of set-theoretically defined order-theoretic limitations, it is devoid of algebraic limitations as well; moreover, to within isomorphism, it is the unique ordered field that is devoid of both types of limitations or “holes”, as they might more colloquially be called. That is, \( \mathbb{No} \) not only exhibits all possible algebraic and set-theoretically defined order-theoretic gradations consistent with its structure as an ordered field, it is to within isomorphism the unique such structure that does. It is ultimately this together with a number of related results (Ehrlich 1989, 1992) that underlies Ehrlich’s contention that whereas \( \mathbb{R} \) should be regarded as the arithmetic continuum (modulo the Archimedean axiom), \( \mathbb{No} \) may be regarded as an absolute arithmetic continuum (modulo NBG).

However, to fully appreciate the nature and significance of this absolute continuum, according to Ehrlich, one must appeal to its rich algebraico-tree-theoretic structure that emerges from the recursive clauses in terms of which it is defined. From the standpoint of Conway’s construction, this *simplicity hierarchical* or *s-hierarchical* structure, as Ehrlich calls it (1994, 2001, 2011, 2012), depends upon \( \mathbb{No} \)'s implicit structure as a *lexicographically ordered full binary tree* and arises from the fact that the sums and products of any two members of the tree are

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24 A historically important infinitesimalist theory of continua not touched on in this paper is that of C. S. Peirce (circa 1897, 1898, 1900). For a reconstruction (with arithmetization) of aspects of Peirce’s conception based on variations of Surreal 1, Surreal 2 and Surreal 3 above, see (Ehrlich 2010). For a discussion of Peirce’s theory more generally, see Matthew Moore’s contribution to this volume.

25 A tree \( \langle A, <_s \rangle \) is a partially ordered class such that for each \( x \in A \), the class \( \{ y \in A : y <_s x \} \) of predecessors of \( x \), written ‘\( \text{pr}_A(x) \)’, is a set well ordered by \( <_s \). A maximal subclass of \( A \) well ordered by \( <_s \) is called a branch of the tree. Two elements \( x \) and \( y \) of \( A \) are said to be *incomparable* if \( x \neq y \), \( x \not<_s y \) and \( y \not<_s x \).
the simplest possible elements of the tree consistent with $\mathbf{No}$’s structure as an ordered group and an ordered field, respectively, it being understood that $x$ is simpler than $y$ (written $x <_s y$) just in case $x$ is a predecessor of $y$ in the tree. In (Ehrlich 1994, 2001, 2002, 2012) the just-described simplicity hierarchy was brought to the fore and made a part of an algebraico-tree-theoretic definition of $\mathbf{No}$. Intuitively speaking, in accordance with this approach, $x$ is simpler than $y$ if one cannot construct $y$ until one has already constructed $x$.

Among the striking $s$-hierarchical features of $\mathbf{No}$ is that much as the surreal numbers emerge from the empty set of surreal numbers by means of a transfinite recursion that provides an unfolding of the entire spectrum of numbers great and small (modulo the aforementioned provisos), the recursive process of defining $\mathbf{No}$’s arithmetic in turn provides an unfolding of the entire spectrum of ordered fields in such a way

The tree-rank of $x \in A$, written $\rho_A(x)$, is the ordinal corresponding to the well-ordered set $\langle \rho_A(x), <_s \rangle$; the $\alpha$th level of $A$ is $\{ x \in A : \rho_A(x) = \alpha \}$; and a root of $A$ is a member of the zeroth level. If $x, y \in A$, then $y$ is said to be an immediate successor of $x$ if $x <_s y$ and $\rho_A(y) = \rho_A(x) + 1$; and if $(x_\alpha)_{\alpha < \beta}$ is a chain in $A$ (i.e., a subclass of $A$ totally ordered by $<_s$), then $y$ is said to be an immediate successor of the chain if $x_\alpha <_s y$ for all $\alpha < \beta$ and $\rho_A(y)$ is the least ordinal greater than the tree-ranks of the members of the chain. The length of a chain $(x_\alpha)_{\alpha < \beta}$ in $A$ is the ordinal $\beta$. A tree $\langle A, <_s \rangle$ is said to be binary if each member of $A$ has at most two immediate successors and every chain in $A$ of limit length (including the empty chain) has at most one immediate successor. If every member of $A$ has two immediate successors and every chain in $A$ of limit length (including the empty chain) has an immediate successor, then the binary tree is said to be full. Since a full binary tree has a level for each ordinal, the universe of a full binary tree is a proper class. A binary tree $\langle A, <_s \rangle$ together with a total ordering $<$ defined on $A$ may be said to be lexicographically ordered if for all $x, y \in A$, $x$ is incomparable with $y$ if and only if $x$ and $y$ have a common predecessor lying between them (Ehrlich 2001, p. 1234). A subtree $B$ of a tree $\langle A, <_s \rangle$ is said to be initial if for each $x \in B$, $\rho_B(x) = \rho_A(x)$.

In Conway’s development of $\mathbf{No}$, and most expositions thereof (e.g. Gonshor 1986; Alling 1987; Siegel 2013, ch. VIII), a surreal number $x$ is said to be simpler than a surreal number $y$ if $x$ is constructed at an earlier stage of recursion than $y$ (i.e. the birthday of $x$ is less than the birthday of $y$). In Ehrlich’s approach, which is growing more standard in research papers on the surreals, the tree-rank of a surreal plays the role of the birthday. While tree-ranks (or birthdays) remain of importance, it is the simpler than relation defined in terms of the predecessor relation that has emerged as the theoretically central notion. The results designated Surreal 4 and Surreal 6 below only begin to show the power of the simpler than relation so defined.

In the investigation of Conway’s class of games, which subsumes the surreal numbers, but where the corresponding tree structure is lacking, the birthday structure remains of central importance.
that an isomorphic copy of each such system either emerges as an initial subfield—a subfield that is an initial subtree of $\mathbb{No}$ (see Note 25)—or is contained in a theoretically distinguished instance of such a system that does. In particular:

**Surreal 4** (Ehrlich 2001, p. 1253). *Every real-closed ordered field is isomorphic to an initial subfield of $\mathbb{No}$.***

Another striking s–hierarchical feature of $\mathbb{No}$ is the following generalization of the Cantor Normal Form Theorem.

**Surreal 5** (Conway 1979, pp. 31-33). *Every surreal number can be assigned a canonical “proper name” (or normal form) that is a reflection of its characteristic s–hierarchical properties. These Conway names, as they are sometimes called, are expressed as formal sums of the form*

$$\sum_{\alpha < \beta} \omega^{y_{\alpha}} \cdot r_{\alpha}$$

*where $\beta$ is an ordinal, $(y_{\alpha})_{\alpha < \beta}$ is a strictly decreasing sequence of surreals, and $(r_{\alpha})_{\alpha < \beta}$ is a sequence of nonzero real numbers. Every such expression is in fact the Conway name of some surreal number, the Conway name of an ordinal being just its Cantor normal form i.e. the unique Cantorian sum*

$$\sum_{\alpha < m} \omega^{y_{\alpha}} \cdot n_{\alpha}$$

*equal to the given ordinal where $m$ is a natural number, $(y_{\alpha})_{\alpha < m}$ is a strictly decreasing series of ordinals, and $(n_{\alpha})_{\alpha < m}$ is a series of nonzero natural numbers.*

Figure 2 below offers a glimpse of the some of the early stages of the recursive unfolding of this s-hierarchical absolute continuum, where $\omega$ is the least infinite ordinal as well as the simplest positive infinite number, $-\omega$ is the additive inverse of $\omega$ as well as the simplest negative infinite number, $1/\omega$ is the multiplicative inverse of $\omega$ as well as the simplest positive infinitesimal number, and so on.$^{27}$

At present there are four basic ways of constructing the surreals: Conway’s inductive cut construction, which is a synthesis of the Dedekind cut construction and the von Neumann ordinal construction (Conway 1976); Conway’s sign-expansion construction (Conway 1976, 2001, p. 65), which was made popular by Gonshor (1986); Ehrlich’s inductive cut construction using Cuesta Dutari cuts (Ehrlich 1988, Alling

$^{27}$For a complete characterization of the recursive unfolding of $\langle \mathbb{No}, < \rangle$ in terms of Conway names of surreal numbers, see (Ehrlich 2011).
and Ehrlich 1986, 1987), which is a generalization of the Dedekind cut construction; and Ehrlich’s inductive generalization of the von Neumann ordinal construction (2002, 2012). In the latter construction each surreal number \( x \) emerges as a canonical partition \((L, R)\) of the set of all surreal numbers that are simpler than \( x \) in the full binary surreal number tree \( \langle \text{No}, <s \rangle \). Although \( L \) and \( R \) are defined independently of the total (lexicographic) ordering \(<\) that is defined on \( \langle \text{No}, <s \rangle \), they are found to coincide with the sets of all surreal numbers that are simpler than \( x \) and less than \( x \) and simpler than \( x \) and greater than \( x \), respectively, in the lexicographically ordered full binary surreal number tree \( \langle \text{No}, <, <s \rangle \). \( \text{No}'s \) ordinals (whose arithmetic in \( \text{No} \) is different than the familiar Cantorian arithmetic) are identified with the members of the “rightmost” branch of \( \langle \text{No}, <, <s \rangle \), i.e. the unique initial subtree of \( \text{No} \) that is a proper class satisfying the condition \( x < y \) whenever \( x <s y \). \( \text{No}'s \) system of reals is identified with the unique initial subtree of \( \text{No} \) that is a Dedekind complete ordered field, and, as is evident from Surreal 4 above, many of the the non-Archimedean number systems that have been the focus of our discussion likewise emerge as initial substructures of \( \text{No} \)-substructures of

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28 Ehrlich (1994, p. 242) has also provided an explicit definition of the surreals as they arise in the latter construction. In addition, working in the HoTT framework, M. Shulman (2013) has introduced an interesting variation on Conway’s original construction.
No that are initial subtrees of No. Indeed, in some cases, as we will see in Surreal 7, they emerge as highly distinguished initial substructures of No.

It is this algebraico-tree-theoretic s-hierarchical structure of No together with its remarkably inclusive nature that underwrites the theorem that is central to Ehrlich’s portrayal of No as a unifying, s-hierarchical absolute arithmetic continuum. To motivate the result, recall that besides being (up to isomorphism) the unique Dedekind complete ordered field, \(\mathbb{R}\) is (up to isomorphism) the unique universal and the unique maximal (or nonextensible) Archimedean ordered field, the condition of maximality or of nonextensibility being Hilbert’s (1900) classical continuity condition. Analogs of these results also hold for \(\mathbb{R}\) considered as an s-hierarchical ordered field. More importantly, however, as Surreal 6 below indicates, s-hierarchical analogs of these classical characterization theorems also hold for No.

Following Ehrlich (2001, p. 1236), \(\langle A, +, \cdot, 0, 1, <, <_s \rangle\) is said to be an s-hierarchical ordered field if \(\langle A, +, \cdot, 0, 1, < \rangle\) is an ordered field, \(\langle A, <, <_s \rangle\) is a lexicographically ordered binary tree and + and \(\cdot\) are defined in an analogous matter that Conway defines the field operations in No, thereby ensuring the sums and products of the members of \(A\) are the simplest possible members of \(A\) consistent with \(A\)'s structure as an ordered field (Ehrlich 1994, pp. 251-252, 2001, p. 1236, 2012, pp. 15-16). If \(A\) is an s-hierarchical ordered field, then \(A\) is said to be universal if and only if \(A\) is maximal if and only if \(A\) is complete if and only if \(A\) is isomorphic to No.

Mirroring the classical equivalences regarding Dedekind’s and Hilbert’s characterizations of the classical arithmetic continuum, we have

**Surreal 6** (Ehrlich 2001, p. 1239). Let \(A\) be an s-hierarchical ordered field. \(A\) is complete if and only if \(A\) is universal if and only if \(A\) is maximal if and only if \(A\) is isomorphic to No.

Whereas Surreal 1 and Surreal 3 may be said to characterize No as an absolute arithmetic continuum (modulo NGB), Surreal 6 may be said to characterize No as an absolute s-hierarchical arithmetic continuum (modulo NGB). Moreover, since every initial subfield of No is an s-hierarchical ordered field (Ehrlich 2001, p. 1236), it is evident from Surreal 5 that the universal component of this characterization
has real teeth, as one would expect of a purported absolute continuum. Moreover, as was intimated above, not only does every elementary continuum emerge as an initial subfield of $\mathbb{N}_o$, in some cases they emerge as distinguished initial substructures of $\mathbb{N}_o$. As apt illustrations, we note

**Surreal 7** (Ehrlich 1988, 2012). Let $\mathbb{N}_o(\lambda)$ be the class of surreal numbers having tree-rank $< \lambda$ and let $\alpha$ be an ordinal.

I. Assuming CH, $\mathbb{N}_o(\omega_1)$ (with $+, \cdot$, and $<$ defined à la Conway) is isomorphic to the underlying ordered field of the (unique up to isomorphism) saturated hyperreal number system of power $\aleph_1$.

II. Assuming the existence of an inaccessible cardinal, $\aleph_\alpha$ being the least, $\mathbb{N}_o(\omega_\alpha)$ (with $+, \cdot$, and $<$ defined à la Conway) is isomorphic to the underlying ordered field of the (unique up to isomorphism) saturated hyperreal number system of power $\aleph_\alpha$.

III. $\mathbb{N}_o$ is isomorphic to the underlying ordered field of the maximal hyperreal number system that exists in NBG.\(^{29}\)

It is worth noting that while nonstandard analysis can be carried out in $\mathbb{N}_o$ (and in suitable subfields thereof), $\mathbb{N}_o$ was neither motivated by, nor is presently well suited for, that purpose. As Conway observed, “$\mathbb{N}_o$ is really irrelevant to non-standard analysis” (1976, p. 44). Central to nonstandard analysis is the transfer theorem, and at present the only way to obtain the fruit of transfer in $\mathbb{N}_o$ would be by inducing it vis-à-vis an ordered field embedding of a given hyperreal number system. Hence, while cross-fertilization between the surreal and hyperreal number systems might in time lead to employing subfields of $\mathbb{N}_o$ as a vehicle for doing nonstandard analysis, at present there is little point in doing so.

As was mentioned above, assuming CH, an ordered field that is an infinitary pantachie in the sense of Hausdorff is isomorphic to the underlying ordered field of one of the most familiar non-Archimedean continua of our day. Indeed, this follows immediately from Part I of Surreal 7 and

\(^{29}\)There is a general result relating the underlying ordered fields of saturated hyperreal number systems to particular $\mathbb{N}_o(\lambda)$ (Ehrlich 1988), but since the ones commonly employed in the literature are those referred to in I and II, we have focused attention on them.
**Surreal 8** (Ehrlich 2012). Let $\mathbb{H}_p$ be an ordered field that is an infinitary pantachie in the sense of Hausdorff. Assuming CH, $\mathbb{N}o(\omega_1)$ (with $+$, $\cdot$, and $<$ defined à la Conway) is isomorphic to $\mathbb{H}_p$.

The most common way of constructing a saturated hyperreal number system of power $\aleph_1$ is as the reduced power of $\mathbb{R}$ over the index set $\mathbb{N}$ modulo a nonprincipal ultrafilter on $\mathbb{N}$ (e.g. Goldblatt 1998, ch. 3). Hausdorff’s construction, while similar to the just-said ultrapower construction, differs in that he uses the reduced power of $\mathbb{R}$ over the index set $\mathbb{N}$ modulo the filter of cofinite subsets of $\mathbb{N}$ (Plotkin 2005, p. 269). Not being a maximal filter, the filter of cofinite subsets of $\mathbb{N}$ is not an ultrafilter on $\mathbb{N}$ (e.g. Goldblatt 1998, p. 18). Despite this difference, it is truly remarkable that the byproduct of Hausdorff’s perfection of an idea rooted in du Bois-Reymond’s belief that an infinitesimally enriched alternative to the Cantor-Dedekind conception of the continuum could be obtained is isomorphic (modulo CH) to the underlying ordered field of what is perhaps the most familiar nonstandard model of analysis employed today.

In §5 it was observed that, by appealing to ideas of Hahn one may shed light on the structure of $\mathbb{N}o$ and on some of the most important non-Archimedean continua of our day more generally. Although there is much to be said about this, we will simply illustrate this with respect to the non-Archimedean continua referred to in Surreal 7. While the result embraces the subfields of $\mathbb{N}o$ relevant to Parts I and II of Surreal 7, unlike those results, which assume CH or the existence of an inaccessible cardinal, it is provable in NBG.

**Surreal 9** (Ehrlich 1988, 2012). 1. If $\aleph_\alpha$ is a regular cardinal, then $\mathbb{N}o(\omega_\alpha)$ (with $+$, $\cdot$, and $<$ defined à la Conway) is isomorphic to

$$\mathbb{R} \left( \left( t^{\mathbb{N}o(\omega_\alpha)} \right)_{\omega_\alpha} \right).$$

II. $\mathbb{N}o$ is isomorphic to

$$\mathbb{R} \left( \left( t^{\mathbb{N}o} \right)_{\omega_1} \right).$$

The surreal unification of systems of numbers great and small has been extended beyond the range of number systems referred to above.

\[\text{30Recently, Aschenbrenner, van den Dries and van der Hoeven (2019) have shown that assuming CH, the underlying ordered field of every maximal Hardy field is likewise isomorphic to } \mathbb{N}o(\omega_1) \text{ (with } +, \cdot, \text{ and } < \text{ defined à la Conway). At present, it is not known if without CH one can either show that } \mathbb{N}o(\omega_1) \text{ is isomorphic to the underlying ordered field of a nonstandard model of analysis, to an infinitary pantachie in the sense of Hausdorff, or to a maximal Hardy field. In each case, however, it is known that such a structure is isomorphic to an initial subfield of } \mathbb{N}o \text{ containing } \mathbb{N}o(\omega_1) \text{ (Ehrlich 2014, p. 29).}\]
through the work of van den Dries and Ehrlich (2001, forthcoming), Be-
rarducci and Mantova (2018, forthcoming), Fornasiero (2016), Ehrlich
and Kaplan (2018) and Aschenbrenner, van den Dries and van der
Hoeven (2019, forthcoming). With the exception of (Ehrlich and Ka-
plan 2018), which is concerned with ordered abelian groups and or-
dered domains, this work deals with ordered exponential fields and
ordered differential fields, making use of the exponential function on
$\mathbb{N}^*$ introduced by Martin Kruskal and developed by Harry Gonshor
(1987) and a derivation $\partial_{BM}$ on $\mathbb{N}^*$ due to Berarducci and Mantova
(2018). Rudiments of analysis on the surreals have also been devel-
oped by Alling (1987), Rubinstein-Salzedo and Swaminathan (2014),
and Costin, Ehrlich and Friedman (24 Aug 2015; also see, Costin and
Ehrlich 2016). Costin, Ehrlich and Friedman, in particular, have de-
veloped a theory of integration (and differentiation) that extends the
range of analysis from the reals to the surreals for a large subclass of
\textit{analyzable functions}. The analyzable functions, which constitutes a
broad generalization of the analytic functions and includes most stan-
dard functions that arise in applied analysis, was isolated by Jean
Écalle (1981-1985,1992,1993) in connection with his proof of Dulac’s
Conjecture, which is the best currently known result on Hilbert’s 16th
problem. Unlike nonstandard analysis, which provides an infinitesi-
malist approach to integration on the extended real number system
$\mathbb{R} \cup \{\pm \infty\}$, the theory of surreal integration deals with integrals whose
bounds and values need not be extended reals at all. For example, in
the surreal theory
\[ \int_0^{\omega} e^x dx = e^\omega - 1 = \omega^\omega - 1, \]
where $\omega$ is the least infinite ordinal and $e$ is the Kruskal-Gonshor ex-
ponential function on $\mathbb{N}^*$ that extends the familiar exponential function
on $\mathbb{R}$.

The just-mentioned works of Berarducci and Mantova, Aschenbren-
er, van den Dries and van der Hoeven, and Costin, Ehrlich and Fried-
man all make use of the system $T$ of \textit{transseries} or of a surreal isomor-
phic copy thereof. A typical transseries is a formal sum (of finite or
countable infinite length) of the form
\[ e^e - 3e^{x^2} + 5x^{1/2} - \log x + 1 + x^{-1} + x^{-2} + x^{-3} + \cdots + e^{-x} + x^{-1}e^{-x}, \]

\[ \text{For discussions of transseries, see (van der Hoeven 2006; Costin 2009; Edgar 2010; Aschenbrenner, van den Dries, and van der Hoeven 2017).} \]
where $x$ is an infinitely large variable. Transseries naturally arise as asymptotic solutions of differential or more general functional equations. The system $\mathbb{T}$, which has proven to be of considerable interest to analysts, model theorists, differential algebrists, computer algebrists, surrealists and theoretical physicists, was introduced by Dahn and Göring (1987), as an ordered exponential field, in connection with their work on Tarski’s problem of the decidability of the theory of real numbers with the exponential function (Tarski 1948, p. 45), and independently by Écalle (1992), as an ordered differential field, in connection with his work on analyzable functions. Building on ideas from Dahn and Göring (1987), van den Dries, Macintyre and Marker (2001) showed that $\mathbb{T}$ may be constructed as the union of an inductively defined chain of Hahn fields, and Aschenbrenner, van den Dries and van der Hoeven (forthcoming) subsequently showed that $\mathbb{T}$, so construed, has a natural embedding $\iota$ of ordered differential fields into $\langle \mathbb{No}, \partial_{BM} \rangle$. Making use of Berarducci and Mantova’s analysis of the image of $\iota$ (forthcoming, Theorem 4.11), Elliot Kaplan has shown that the image of $\iota$ is an initial subtree of $\mathbb{No}$.

Thus, as the material in §4 - §9 reveals, there are trails beginning with the work of Stolz, Veronese, Levi-Civita and du Bois-Reymond, extended further by Hahn, Hausdorff, Artin and Schreier, and further still by Robinson, Conway and a host of other nonstandard analysts and surrealists, collectively inspired by interrelated considerations regarding the arithmetic continuum and an array of generalizations thereof, that enjoy a harmonious integration in the absolute s-hierarchical arithmetic continuum of surreal numbers.

With respect to model theory, we note that Aschenbrenner, van den Dries, and van der Hoeven have been awarded the 2018 Karp Prize for their work (2017) on the model theory of transseries and asymptotic differential algebra.

An ordered exponential field is an ordered field $K$ together with an isomorphism $\exp_K$ from the additive group of $K$ onto the multiplicative group of $K$, and an ordered differential field is an ordered field $K$ together with a map $\partial : K \to K$ such that $\partial(a + b) = \partial(a) + \partial(b)$ and $\partial(ab) = \partial(a)b + a\partial(b)$ for all $a, b \in K$.

Kaplan’s result is announced in (Aschenbrenner, van den Dries and van der Hoeven 2019).

Unfortunately, for lack of space, we have not been able to take into account G. H. Hardy’s aforementioned development of the ideas of du Bois-Reymond, the subsequent development of the theory of Hardy fields, and the far-reaching theory of $H$-fields of Aschenbrenner, van den Dries, and van der Hoeven (2017).
10. HJELMSLEV’S NILPOTENT INFINITESIMALIST CONTINUUM

Heretofore we have been concerned primarily with infinitesimalist conceptions of continua that are either based upon, or can be modeled by, non-Archimedean ordered fields. However, not all infinitesimalist theories of continua have such a structure. In this and the subsequent section we will touch on the principal ones that do not.

The system of dual numbers, commonly denoted $\mathbb{R}[\epsilon]$, is a commutative ring with identity consisting of

$$\{r_0 + r_1 \epsilon : r_0, r_1 \in \mathbb{R}\}$$

with sums and products defined termwise, it being understood that $\epsilon \neq 0$ and $\epsilon^2 = 0$. Thus, for any two dual numbers $a_0 + a_1 \epsilon$ and $b_0 + b_1 \epsilon$

$$(a_0 + a_1 \epsilon) + (b_0 + b_1 \epsilon) = (a_0 + b_0) + (a_1 + b_1)\epsilon$$

$$(a_0 + a_1 \epsilon)(b_0 + b_1 \epsilon) = a_0 b_0 + (a_0 b_1 + a_1 b_0)\epsilon.$$

$\mathbb{R}[\epsilon]$ is not a field since its nonzero elements of the form $0 + a_1 \epsilon$ are not invertible, and all such elements are proper zero-divisors—nonzero elements $x$ such that for some nonzero $y$, $xy = 0$—which even precludes $\mathbb{R}[\epsilon]$ from being an integral domain. $\mathbb{R}[\epsilon]$ so defined is isomorphic to the quotient $\mathbb{R}[X]/(X^2)$ of the polynomial ring $\mathbb{R}[X]$ by the ideal $(X^2)$ generated by the polynomial $X^2$, which is also frequently referred to as the system of dual numbers and is likewise denoted $\mathbb{R}[\epsilon]$.

The element $\epsilon$ in $\mathbb{R}[\epsilon]$ is a nilpotent, an element $x$ such that $x^n = 0$ for some positive integer $n$. The least $n$ for which $x^n = 0$ is the index of nilpotency of $x$. The index of nilpotency of $\epsilon$ in $\mathbb{R}[\epsilon]$ is 2. Nonzero nilpotent elements of index $n$ have many applications in algebra, one of them being a convenient way of representing quantities up to infinitesimal order $n$. When nilpotents are thus interpreted, they are referred to as nilpotent infinitesimals. The number systems we have hitherto considered have no nonzero nilpotent elements.

$\mathbb{R}[\epsilon]$ was introduced by William Clifford (1873) and has been widely employed by mathematicians, physicists and engineers ever since, including W. Study (1903), J. Grünwald (1906), and C. Segre (1911), whose work initiated the study of geometrical spaces whose coordinates are $n$-tuples of elements of a ring that is neither a field nor a division ring more generally. However, it was Johannes Hjelmslev (1923) who first employed $\mathbb{R}[\epsilon]$ to develop a nilpotent-based infinitesimalist theory of the continuum. Unfortunately, like the aforementioned works of Veronese, Levi-Civita, Hahn, du Bois-Reymond and Hausdorff, his work is not as well known among historians and philosophers of mathematics as it deserves to be.
Hjelmslev regarded classical geometry as a crude approximation of the empirical world. In particular, inspired by the view of the pre-Socratic philosopher Protagoras, as recounted in Aristotle’s *Metaphysics* (III: 2, 998a), he held that the axiom that two straight lines always share at most one point is incompatible with perceptual experience, as is the assertion that a circle and a line tangent to it meet at a single point (1923, pp. 1-2). This led him to devise a finitary “geometry of reality” (1916) or a “natural geometry” (1923), as he later called it, whose subject is the lines and circles of perception constructed with real rulers and compasses. He further devised (1923, pp. 12-13) an abstraction of the latter, coordinated by \( \mathbb{R}[\epsilon] \), that is a prototype of what are today called *affine Hjelmslev geometries* (e.g. Lorimer 1985, pp. 87-88). In these geometries, which are coordinated by *affine Hjelmslev rings*, whereas for each pair of distinct points there is a line joining them, it need not be unique. Indeed, a pair of distinct points may lie on a pair of distinct lines; when this happens the points are said to be *neighboring points* and the lines are said to be *neighboring lines*, the two notions of neighbor being equivalence relations. *Remote points*, by contrast, are joined by a unique line and *remote lines* that intersect intersect in a unique point. In these geometries a circle and a line tangent to it intersect in a nondegenerate “infinitesimal segment” of neighboring points. Moreover, when neighboring points and neighboring lines of these geometries are equated (i.e. collected together into equivalence classes), they give rise to their more familiar affine geometric counterparts in which two points determine a line and intersecting lines as well as circles and lines tangent to them intersect in a unique point. Hjelmslev’s prototype, so conceived, reduces to standard Euclidean geometry over the reals.

Hjelmslev’s plane over \( \mathbb{R}[\epsilon] \), henceforth \( H \), consists of all ordered pairs \((A, B) \in \mathbb{R}[\epsilon] \times \mathbb{R}[\epsilon] \). As usual, a straight line of \( H \) is defined by a first-degree equation

\[
Ax + By + C = 0,
\]

where now, however, \( A, B, C \in \mathbb{R}[\epsilon] \) and \((A, B) \notin J \times J, J \) being \( \{re : r \in \mathbb{R}\} \). \( \mathbb{R}[\epsilon] \) admits a relational expansion to a non-Archimedean (totally) ordered ring, where the order \(<'\) is defined lexicographically by the condition: \( a_0 + a_1 \epsilon <' b_0 + b_1 \epsilon \) if \( a_0 < b_0 \) or \( a_0 = b_0 \) and \( a_1 < b_1 \). In virtue of the just-said ordering, the points on a line of \( H \) are themselves totally ordered.

In \( H \), the circle defined by the equation

\[
Ax^2 + By^2 = 1
\]
intersects the line defined by the equation

\[ y = 1 \]

in a segment consisting of all points \((r \epsilon, 1)\) where \(r \in \mathbb{R}\). In virtue of the order on \(\mathbb{R}[\epsilon]\), the points of the segment of intersection are isomorphic to \(\mathbb{R}\) considered as an ordered set.

Hjelmslev continued to develop these ideas in an influential six-part work under the rubric *general congruence theory* (1929-1949). In the third installment (1942, p. 48), he identified for each positive integer \(n\) a ring containing nilpotents having index of nilpotency \(n + 1\) over which one can define geometries analogous to \(H\). These are the familiar quotient rings \(\mathbb{R}[X]/(X^{n+1})\), the system \(\mathbb{R}[X]/(X^2)\) of dual numbers being the smallest, consisting of elements of the form

\[ r_0 + \ldots + r_n \epsilon^n \]

with \(r_0, \ldots, r_n \in \mathbb{R}\) and sums and products defined termwise, it being understood that \(\epsilon \neq 0\) and \(\epsilon^{n+1} = 0\).

Inspired by these ideas, the theory of Hjelmslev geometries and corresponding Hjelmslev rings was introduced by W. Klingenberg (1954, 1954a, 1955), and through subsequent work of Klingenberg (1956), Veldkamp (1981, 1985, 1987, 1995) and others these geometries and corresponding rings have undergone substantial generalization and variation, yielding, for example, geometries in which not all neighboring points are connected by a line. In this family of neighbor-based geometries, Hjelmslev’s idea that lines, lines and circles, and so on may intersect in infinitesimal segments characterized using nilpotent infinitesimals plays a central role. These ideas, which emerged from Hjelmslev’s nilpotent-based infinitesimalist conception of the geometrical continuum, have found application in other areas of mathematics as well, including algebra, algebraic geometry, differential geometry and, most recently, in a number of nilpotent-infinitesimalist approaches to

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36For English presentations of Klingenberg’s axiomatization, see (Dembrowski 1968, Appendix 7.2) and (Lorimer 1985), the latter of which contains detailed discussions of Hjelmslev planes and rings. For an extension of Klingenberg’s axiomatization to higher dimensional spaces, see (Kreuzer 1987, 1988). For an overview with references of Hjelmslev geometries, Hjelmslev rings and various generalizations of both, see (Veldkamp 1995). For F. Bachmann’s generalization of Hjelmslev planes and their incorporation into his geometry of reflections, see (Bachmann 1971, 1973, 1973a, 1989). In Bachmann’s treatment, a pair of points may either have more than one line or no line joining them, both conceptions he traces to Hjelmslev. Finally, for finite Hjelmslev planes, see (Drake and Jungnickel 1985).
those portions of differential geometry concerned with smooth (i.e. infinitely differentiable) manifolds and various generalizations thereof. It is to the latter that we now turn.

11. INFINITESIMALIST APPROACHES TO DIFFERENTIAL GEOMETRY OF SMOOTH MANIFOLDS AND THEIR UNDERLYING CONTINUA

Abraham Robinson begins the Elementary Differential Geometry section of his book on nonstandard analysis by observing that:

Quite early in the history of the Differential Calculus, infinitesimals were employed in the development of the theory of curves, and long after the classical $\epsilon, \delta$-method had displaced the naive use of infinitesimals in Analysis they survived in Differential Geometry and Physics. Even now there are many classical results in Differential Geometry which have never been established in any other way, the assumption being that somehow the rigorous but less intuitive, $\epsilon, \delta$-method would lead to the same result. So far as one can see without a complete check this assumption is usually correct. (1966, 1974, p. 83).

He then goes on to illustrate a point he had already made in earlier works, namely, using nonstandard analytic techniques

we may now justify the use of infinitesimals in all these problems [from differential geometry] directly. (1961, p. 437; also see 1963, p. 265)

It is therefore not surprising that influenced in part by Robinson’s placement of analysis on a logically sound infinitesimal foundation, logically sound infinitesimal foundations for portions of differential geometry would likewise be sought. At present there are at least four such approaches, at varying stages of development, all of which are concerned with smooth manifolds or generalizations thereof: Synthetic Differential Geometry (SDG), introduced by F. W. Lawvere in 1967 (see Note 41) and developed by Anders Kock (1981, 2010) and others; Infinitesimal Differential Geometry (IDG) developed by Paolo Giordano (2001, 2010, 2011; 2016 with Wu); Topological Differential Calculus (over General Base Fields and Rings) (TDC), introduced by Wolfgang Bertram (2008\textsuperscript{37} and Nonstandard Differential Geometry (NSDG), based on nonstandard analysis, the most recent version being that of

\textsuperscript{37}The appellation “Topological Differential Calculus,” which was requested by Bertram, comes from the title of his (2011).
Tahl Nowik and Mikhail Katz (2015) In simplest terms, differential geometry of smooth manifolds is concerned with the application of calculus to spaces that locally behave like some Euclidean space \( \mathbb{R}^n \). This includes Riemannian geometry, the branch of differential geometry that studies smooth manifolds with a Riemannian metric.

Whereas NSDG makes sole use invertible infinitesimals, SDG, IDG and TDC are developed using nilpotent infinitesimals. Moreover, unlike SDG, IDG and NSDG, which provide infinitesimal-based approaches to classical differential geometry of smooth manifolds (albeit sometimes within enriched frameworks), TDC seeks to provide a vast generalization of (the differential aspects of) classical differentiable geometry of smooth manifolds by providing a framework that is applicable to a wide range of underlying “model[s] of the continuum” (Bertram 2008, p. 1) including the ordered field of reals, systems of hyperreals, the ring of dual numbers and “good” topological fields and topological commutative unital rings more generally.

Since the underlying continua in NSDG are hyperreal number systems, which we have already touched on, and of the remaining three theories SDG is at present the most developed, we will direct the bulk of our brief remarks to it, its prehistory, and some of the important differences that exist between it and the other theories, the latter of which deserve more attention than space will allow.

At roughly the time Klingenberg was developing the theory of Hjelmslev rings, the system of dual numbers was undergoing a distinct though overlapping generalization and incorporated into differential geometry by André Weil (1953) to treat Charles Ehresmann’s (1951) theory of jets and prolongations, which are concerned with entities that generalize the notion of a tangent vector. In addition to Ehresmann’s now classic work, Weil was motivated by the desire to “return to the methods of Fermat in the first-order infinitesimal calculus” (1953, p. 111), methods that were introduced by Fermat to treat tangent constructions making use of nilpotent infinitesimals. Soon thereafter, Alexander Grothendieck (with the assistance of Jean Dieudonné) (1960, 1971), making use of the system of dual numbers (and generalizations

\[38\] For earlier treatments of portions of differential geometry based on nonstandard analysis, see (Stroyan and Luxemburg 1976; Stroyan 1977; Lutz and Goze 1981; Almeida, Neves and Stroyan 2014).

\[39\] By a topological ring (field), Bertram means a commutative unital ring (field) \( \mathbb{K} \) together with a topology (assumed to be Hausdorff) whose ring operations \( + : \mathbb{K} \times \mathbb{K} \to \mathbb{K} \) and \( \cdot : \mathbb{K} \times \mathbb{K} \to \mathbb{K} \) are continuous and for which the set \( \mathbb{K}^\times \) of invertible elements is open in \( \mathbb{K} \) and the inversion map \( i : \mathbb{K}^\times \to \mathbb{K} \) is continuous. Bertram says \( \mathbb{K} \) is good if in addition \( \mathbb{K}^\times \) is dense.
thereof), developed a nilpotent-infinitesimal-based algebraic treatment of differential calculus adequate for the needs of algebraic geometry, which remains a prominent fixture of the theory till this day. In Grothendieck’s approach to algebraic geometry, as in Hjelmslev’s treatment of the geometric continuum, curves meet their tangents at non-degenerate infinitesimal segments or arcs.

In May of 1967, utilizing an extension of Grothendieck’s just-said ideas and incorporating them into a topos-theoretic setting, F. W. Lawvere proposed a nilpotent-infinitesimal-based approach to differential geometry of smooth manifolds, in which Weil’s generalization of the dual number system—Weil algebras, as they are now called—would come to play a prominent role. Unlike Robinson, who was stimulated by Leibniz’s idea that the properties of infinitesimals should reflect the properties of the reals, Lawvere’s ideas more closely mirror the heuristic ideas of geometers (like Cartan, Hjelmslev, Klingenberg, Lie, Weil and Grothendieck) who envision a vector tangent to a surface at a point.

In classical algebraic geometry the rings that were studied are integral. In Serre’s subsequent treatment zero-divisors were permitted but there were no nonzero nilpotents. Grothendieck considered arbitrary commutative rings with identity, thereby permitting nilpotent infinitesimal elements. Differential properties of functions are of critical importance in algebraic geometry and this is what motivated Grothendieck’s nilpotent-infinitesimal-based algebraic approach to differential calculus for the subject. For an introduction to these ideas, see (Shafarevich 2005, §7: The Algebraic View of Infinitesimal Notions) and (Perren 2008, ch. 5 and Appendix B).

Nilpotent infinitesimals already appear on the second page of mathematics of the first installment of Grothendieck’s Éléments de géométrie algébrique I (1960, p. 12), and they are found throughout the subsequent installments published from 1961 through 1967. In the revised edition, a paragraph was added explicitly stating their importance for the theory (1971, p. 11). However, while it was Grothendieck who made the rigorous use of nilpotent-infinitesimalist ideas a mainstay of algebraic geometry, an informal use of infinitesimals in algebraic geometry is already found in important work of Enriques of the 1930s. For a fascinating mathematically-historical discussion of how Grothendieck’s nilpotent-infinitesimalist ideas can be employed to lend precision to Enriques’s ideas, see (Mumford 2011).

This took place as part of a series of lectures—summarized in (Lawvere 1979)—given at the University of Chicago. Among the central goals of Lawvere’s lectures was to employ category theory in a substantial manner in physical theorizing. Given the role played by differential geometry therein, the connection was natural. For remarks on the critical role of category theory, in general, and topos theory, in particular, in the historical development and formulation of SDG, see (Kock 1981; McLarty 1990, §5; Bunge, Gago, San Luis 2018).

For a general introduction to Weil algebras, see (Moerdijk and Reyes 1991, ch. I). For the present, we simply note that for each positive integer n, the Hjelmslev ring $\mathbb{R}[X]/(X^{n+1})$ is a simple example of a Weil algebra.
point as a tiny arc of a curve having the vector tangent to it. Building on Lawvere’s ideas, G. Wraith, G. E. Reyes, E. J. Dubuc, and A. Kock developed the basic ingredients of the theory, which were given a systematic treatment by Kock (1981) under the now familiar rubric “Synthetic Differential Geometry”.

SDG is developed in an axiomatic framework in which a line is modeled by a commutative ring $\mathbb{R}$ with unity containing a subset $D = \{d \in \mathbb{R} : d^2 = 0\}$ of nilpotent infinitesimals that satisfies the

**Kock-Lawvere axiom:** For every mapping $f : D \to \mathbb{R}$, there is a unique $b \in \mathbb{R}$, such that for all $d \in D$, $f(d) = f(0) + d \cdot b$.

Geometrically speaking, this axiom asserts that the graph of every function $f : D \to \mathbb{R}$ is a piece of the unique straight line through $(0, f(0))$ with slope $b$. It is a consequence of this assumption that in SDG a tangent vector to a curve $C$ at a point $p$ is a nondegenerate infinitesimal line segment around $p$ coincident with $C$.

For motivational purposes, one can employ the Kock-Lawvere axiom as stated above, where $\mathbb{R} = \mathbb{R}[\epsilon]$ (without the lexicographic total ordering). For the development of SDG, however, the Kock-Lawvere axiom is replaced by a more sophisticated axiom scheme, due to Kock (1981, p. 64), which is formulated for an arbitrary Weil algebra. The importance of Weil algebras for SDG was first recognized by G. E. Reyes and G. Wraith (1978), and subsequently used to great effect by E. J. Dubuc, who introduced and developed the critical idea of a well-adapted model of SDG and established the existence of such models making use of Weil algebras (1979, 1981, 1981a). Well-adapted models are of fundamental importance for SDG because they are precisely the models of SDG that make it possible to recover theorems of classical (i.e. standard $\epsilon, \delta$) differential geometry. That is, it is the existence of well-adapted models that makes the axioms of SDG relevant to classical mathematics.

Using the Kock-Lawvere axiom, one may define the *derivative* $f'(x)$ of any function $f : \mathbb{R} \to \mathbb{R}$ at $x \in \mathbb{R}$ to be $b$. More specifically, using the Kock-Lawvere axiom one may derive the following version of

**Taylor’s formula:** For any function $f : \mathbb{R} \to \mathbb{R}$ and any $x \in \mathbb{R}$,

$$f(x + d) = f(x) + d \cdot f'(x) \quad \forall d \in D.$$  

At first sight this appears to be impossible since it implies that all functions are smooth, which of course is not the case. Indeed, by a classical result of topology due to Banach (1931), even if we limit ourselves to the space of continuous functions, the set of functions having a derivative at a point is *meager* (negligible in a precise sense). However,
as authors of works on SDG hasten to add, the seeming catastrophic consequence of the Kock-Lawvere axiom does not materialize since the logic employed in SDG is intuitionistic as opposed to classical, and the impossibility cannot be established using intuitionistic logic, the latter of which results from classical logic by omitting the law of excluded middle.\footnote{The idea of introducing the derivative in the manner described above has roots that predate SDG. Neder (1941), for example, employed this approach (albeit, of course, not for all functions) in his attempt to develop a “Leibnizian differential calculus with actual infinitesimal magnitudes (differentials) of the first order” based on the system of dual numbers “that is free from contradiction” (p. 251). Indeed, according Neder: “Optimists may hope that one day the considerations presented here will be used to give a new (and by no means disparaged) justification for differential calculus” (p. 251). Moreover, according to Neder (p. 252), the basic idea was essentially in (Peterson 1898) and (Predella 1912). What Neder does not mention is that J. Petersen is J. Hjelmslev (see §10). Thus, Hjelmslev not only anticipated the idea of a nilpotent-infinitesimalist conception of a continuum based on the system of dual numbers, he also envisioned therein the rudiments of a corresponding theory of differentiation. The latter idea was further developed in (Gesztelyi 1958) and in the aforementioned work of Grothendeick. These ideas are also central to the theory of “automatic differentiation” over the system of dual numbers, which has a vast literature whose roots go back to the 1950s.}

Neither the Kock-Lawvere axiom nor Kock’s generalization thereof implies results about basic integration and to obtain such results one requires further axioms. In basic treatments, one usually introduces the following axiom due to Kock and Reyes (1981):

*Integration axiom:* For every function $f : [0, 1] \to \mathbb{R}$, there is a unique function $g : [0, 1] \to \mathbb{R}$, such that $g' = f$ and $g(0) = 0$.

In harmony with standard notation, for each $x$ in $[0, 1]$, the value of the hypothesized function $g$ is denoted

$$\int_0^x f(t)dt.$$  

It is worth noting, however, that the integral in SDG has a different character than it does in ordinary calculus, where it is defined as the limit of an infinite sum, not as an antiderivative. Thus far, however, no one knows how to treat infinite sums or limits in SDG.\footnote{Remarks of Kock (25 Sept 2017, p. 23) suggest that he does not preclude such a possibility down the road, albeit perhaps with modifications of SDG.}

Unlike the Kock-Lawvere axiom or Kock’s generalization thereof, the Integration axiom presupposes an order $\leq$ on the ring $\mathbb{R}$ to make sense of the notion of the interval $[0, 1]$. In particular, the order is assumed
to be a \textit{preorder} (a reflexive and transitive relation), and on the basis of this one defines

\[ [a, b] = \{ x \in \mathbb{R} : a \leq x \leq b \}. \]

It is further assumed that \( \leq \) satisfies conditions which ensures its compatibility with the unitary ring structure of \( \mathbb{R} \) and in addition it is assumed that

\[ (\ast) \quad \forall d \in D(0 \leq d \land d \leq 0). \]

It is important to note that \( \leq \) is not a total ordering, or even a partial ordering, since if it were assumed that \( \leq \) is antisymmetric (i.e. \( \forall xy((x \leq y \land y \leq x) \rightarrow x = y) \)), it would follow from (\( \ast \)) that \( D = \{0\} \). But this is not the case; in fact, it is a consequence of the Kock-Lawvere axiom that \( D \neq \{0\} \). On the other hand, since the underlying logic of SDG is intuitionistic, in SDG one cannot prove \( \exists d \in D(d \neq 0) \) from \( D \neq \{0\} \), as one might expect, and would be the case if the underlying logic were classical. In fact, since (by definition) \( a < b \) if and only if \( a \leq b \land a \neq b \), if one could prove \( \exists d \in D(d \neq 0) \), it would follow from (\( \ast \)) that \( \exists d \in D(d < 0 \land 0 < d) \), which is impossible (even in intuitionistic logic).

While the just-said axioms of SDG permit the development of swaths of basic calculus (Belair 1981; Belair and Reyes 1982; Bell 2008), they go just so far in the development of differential geometry of smooth manifolds. Whereas they provide the basis for developing many aspects of the theory concerned with infinitesimal behavior (Kock 2010), they provide little basis for the treatment of local finite behavior and are entirely silent on issues of a global nature. For example, they do not imply the existence and uniqueness of solutions of ordinary differential equations, permit the treatment integration of vector fields, or provide the means for the representability of germs of a smooth nature. As a result, over the years a variety of additional axioms have been proposed to make the study of SDG closer to what is ordinarily considered to be the analysis of differentiable functions of a real variable, early examples being those suggested by L. Belair (1981) and C. McLarty (1983). However, the most far reaching work in this direction was initiated by Marta Bunge and Dubuc (1987), carried further in (Bunge and Gago 1988; San Luis 1999), and recently integrated and extended under the appellation \textit{Synthetic Differential Topology} (Bunge, Gago, San Luis 2018). In this work (SDT), not only are the three just-mentioned issues addressed, but additional topics such as Morse theory

\[ 45 \forall xyz(x \leq y \rightarrow (x + z \leq y + z)); \forall xy((0 \leq x \land 0 \leq y) \rightarrow 0 \leq xy); 0 \leq 1 \land \neg(1 \leq 0). \]
of singularities and the theory of stable germs of smooth mappings are covered including a proof of Mather’s theorem.

As in Hjelmslev geometries, the notion of neighborhood and corresponding incidence structure is of fundamental importance in SDG (Kock 2003, 2010, 20 Sept 2017). For example, it is a consequence of the Kock-Lawvere axiom that in SDG, unlike in Euclidean geometry, there are pairs of points in the plane that are not connected by a unique straight line. However, unlike in Hjelmslev geometries, in SDG there are pairs of points in a plane that are not connected by any line at all. This arises in part from the fact that whereas the nilpotent infinitesimals in Hjelmslev geometries have “a quantitative (linear ordered) character,” those employed in SDG do not (Kock 2003, pp. 226-228). In this respect, the lines of SDG more closely resemble some of the lines of geometric spaces with neighbor relations investigated by Veldkamp (e.g. Veldkamp 1985). In these spaces, unlike those of Hjelmslev, the neighbor relation is not transitive. This permits a hierarchy of neighborhoods—first order, second order and so on for each positive integer $n$—as is found in the algebraic geometry of Grothendieck, which lent inspiration to SDG.

A space $X$ in SDG is said to be indecomposable if there are no disjoint nonempty subsets $U$ and $V$ of $X$ such that $U \cup V = X$ (Bell 2001, 2005, 2008, 2009). There are models of SDG in which a classical space $\mathbb{R}^n$ has a counterpart $X$ that is indecomposable if $X$ is connected. John Bell takes this to imply that the connected continua of SDG are true continua in something like the Anaxagoran sense (1995, p. 56, 2009). In this respect, they are also reminiscent of the unsplittable continuum of Brouwer; however, the similarity is not perfect and varies depending on the axioms adopted for SDG (Bell 2001, 2009).

Another respect in which SDG is similar to Brouwer’s theory is the failure of the intermediate value theorem in its underlying theory of analysis. In fact, in SDG, unlike in Brouwer’s system, the theorem even fails for some polynomials (Moerdijk and Reyes 1991, pp. 317-318), a failure that runs contrary to the thinking of Leibniz and Euler.

46For further discussion of the differences in the underlying continua of Hjelmslev’s geometry and SDG, see (Kock 2003); for a detailed discussion of the significance of the presence or absence of the transitivity of the neighbor relations, see (Veldkamp 1985, 1995); and for a penetrating discussion of neighborhoods in the context of SDG, see (Kock 25 Sept 2017).

47According to Bell (2001, p. 22, 2005, p. 302), the critical axiom is: $\forall xy (\neg x = y \implies x < y \lor y < x)$. Though not in (Kock 1981), this assertion is a consequence of axioms assumed in (Moerdijk and Reyes 1991) and (Bunge, Gago, San Luis 2018).
alone Bolzano, Cauchy, and Weierstrass. Accordingly, while SDG may in time provide a viable treatment of the smooth aspects of differential geometry, its underlying analysis may not be as well suited to provide a natural alternative for classical analysis, at least not if it hopes to mirror the latter’s most central ideas regarding continuity.

How then should SDG/SDT be viewed vis-à-vis classical differential geometry/topology and classical analysis? Not surprisingly, the answer one gets depends on who one asks. In a recent statement, Kock expresses his view thus:

I prefer not to think of SDG as a monolithic global theory, but as a method to be used locally, in situations where it provides insight and simplification of a notion, of a construction, or of an argument. The assumptions, or axioms that are needed, may be taken from the valuable treasure chest of real analysis. (Kock 25 Sept 2017, p. 24)

These remarks suggest that Kock does not see SDG so much as a theory intended to replace classical differential geometry, but rather as a methodological accompaniment that should be part of the classical differential geometer’s toolkit (to borrow a term from Keisler) much as classical differential geometry is part of the synthetic differential geometer’s toolkit, though the roles they would serve would be different.

In a private communication, Marta Bunge voices a somewhat similar sentiment, writing:

I agree with the view expressed by Anders in that SDG (or SDT) is not meant to replace classical differential geometry (or topology) but to be inspired by it and enhance it. Due to the rich structure of a topos and to the presence of infinitesimals, classical differential geometry (or topology) can be studied in a context that permits better conceptual simplification and unification in ways that often differ from those of its classical sources. I have stated this point of view in my paper “Toposes in Logic and Logic in Toposes” (1984) intended for philosophers and in the Introduction to SDT. However, I do view SDG (or SDT) as a theory and not just as a method. In it some developments are possible which are totally non classical some of which originating in the work of Jacques Penon (1981, 1985). It is as a theory that

Bunge is referring to the fact that in addition to the nilpotent infinitesimals of SDG, SDT employs what she calls “logical infinitesimals” (in particular $\Delta = \cdots 1$).
and therefore one can consider models of it, preferably well-adapted in that, by means of such models, themselves often inspired by classical differential geometry (or topology), one can recover classical results. The well adapted part is I think important and Archimedeaness of the ring of line type is inherent in it. (M. Bunge, Private Communication, November 26, 2017)

At present, in addition to five axioms for SDG, which are taken to include the Archimedean axiom, SDT is based on two basic axioms, which address the representability of germs of smooth mappings and the existence and uniqueness of solutions of ordinary differential equations, and four specialized postulates that are employed to treat different parts of the theory (Bunge, Gago, San Luis 2018, §§2.1, §6.1-§6.2). Bunge anticipates that additional axioms may be needed to further extend SDG/SDT and she has expressed confidence that the Dubuc topos (Dubuc 1981), which is the only known well-adapted model for SDT (Bunge, Gago, San Luis 2018, pp. 5, 182 and ch. 12), will be found to model them.

A reason that Kock offers for viewing SDG as a method rather than as a theory is because he believes that

full fledged analysis in axiomatic terms, incorporating SDG, quickly becomes overloaded with axioms, and is better developed as a descriptive theory, describing what actually holds in specific models.... (Kock 25 Sept 2017, p. 24)

¬¬{(0)} and also “logical (or Penon) opens” neither of which could be discussed in a classical setting.

As Bunge alludes to above, among the assumptions adopted in SDT is what contributors to SDG typically call the Archimedean condition, the assertion

\[
(A) \quad \forall x \in \exists n \in N(x < n).
\]

Perhaps it would be more appropriate to say (A) requires \( \mathbb{R} \) to be finitely bounded or weakly Archimedean, since there is a vast array of non-Archimedean ordered rings that satisfy this weak form of the Archimedean condition for rings, including the aforementioned ones of Hjelmslev. In fact, if \( F \) is a non-Archimedean ordered field, then the non-Archimedean subring of \( F \) consisting of all the finite and infinitesimal members of \( F \) satisfies (A).

As the last line of Bunge’s remark suggests, the assumption of the Archimedean axiom in SDT is motivated by the desire for a well-adapted model. Indeed, the main theorem of (Bunge and Dubuc 1986), developed in collaboration with André Joyal, “explains in part why the axiom is important if one wants to develop a theory (or imply a method) which does not go against classical mathematics but which incorporates it....” (M. Bunge, Private Communication, June 26, 2018)

Private communication.
Despite the methodological difference between Kock and Bunge, it is worth noting the difference between the attitudes expressed by these two leading architects of SDG/SDT and those of some of its champions, such as John Bell, who on occasion do not portray real analysis as a treasure chest from which to draw upon and seek partnership with, but rather as an edifice with paradoxical consequences of the axiom of choice and unwarranted applications of the law of excluded middle that need “jettisoning”, at least if one wishes to obtain “a faithful account of the truly continuous” (Bell 2008, p. 5, 2005, pp. 294-297, 2009).

As was noted above, it is the Kock-Lawvere axiom that underlies the incompatibility of SDG with classical logic (e.g. Kock 1981, pp. 2-5; Lavendhomme 1996, pp. 2-5). In particular, it is the contention that the axiom applies to every mapping \( f : D \to \mathbb{R} \) that runs afoul of the law of excluded middle. Accordingly, since the idea that a tangent vector to a curve intersects the curve in a nondegenerate infinitesimal line segment is entirely compatible with classical logic as is the employment of nilpotent infinitesimals for the representation of such intersecting segments, it is natural to inquire if it is possible to develop a variation of SDG in which the underlying logic is classical rather than intuitionistic. It is the idea of providing just such a variation of SDG that motivates Giordano’s IDG.

At the heart of the original formulation of IDG is the idea of replacing the Kock-Lawvere axiom by a theorem asserting: for each smooth function \( f : \mathbb{R} \to \mathbb{R} \) (that is pathwise Lipschitzian) there is a function \( \star f : \star \mathbb{R} \to \star \mathbb{R} \) extending \( f \), and a unique \( b \in \mathbb{R} \) such that

\[
\forall d \in D : \star f(d) = \star f(0) + d \cdot b,
\]

where \( \star \mathbb{R} \) is Giordano’s ring of Fermat reals that extends \( \mathbb{R} \), \( D = \{ d \in \star \mathbb{R} : d^2 = 0 \} \) and the functions \( \star f : \star \mathbb{R} \to \star \mathbb{R} \) are analogs for \( \star \mathbb{R} \) of the standard smooth functions \( f : \mathbb{R} \to \mathbb{R} \) (Giordano 2001).

Hellman (2006) and Shapiro (2014) offer philosophical discussions of SDG and its underlying analysis viewed as constructivist replacements for standard analysis and portions of standard differential geometry. We suspect that philosophical assessments of SDG/SDT and its underlying analysis from the more nuanced perspectives of Kock and Bunge, as expressed in the above quotations, also would be worthy of attention. For remarks on SDG from an author who “invites you to leave the overcrowded Cantor’s paradise” (p. 496), see (Bauer 2017), and for further details on Bell’s view, see his contribution to the present volume.

The universe of \( \star \mathbb{R} \) is introduced in three steps. One starts with the class \( \mathbb{R}_0[t] \) of “little-oh polynomials”, i.e. functions \( x : \mathbb{R}_{\geq 0} \to \mathbb{R} \) that can be written as

\[
x(t) = r + \sum_{i=1}^{k} a_i \cdot t^{\alpha_i} + o(t) \quad \text{as} \quad t \to 0^+,
\]

\[52\text{Hellman (2006) and Shapiro (2014) offer philosophical discussions of SDG and its underlying analysis viewed as constructivist replacements for standard analysis and portions of standard differential geometry. We suspect that philosophical assessments of SDG/SDT and its underlying analysis from the more nuanced perspectives of Kock and Bunge, as expressed in the above quotations, also would be worthy of attention. For remarks on SDG from an author who “invites you to leave the overcrowded Cantor’s paradise” (p. 496), see (Bauer 2017), and for further details on Bell’s view, see his contribution to the present volume.}\]
Thus, in addition to the tuples of reals contained in $f$, $\bullet f$ contains tuples of nonstandard Fermat reals. Each Fermat real number $x$ differs from a unique real number $^0x$ by an infinitesimal amount, and can be written in a unique form

$$x = ^0x + \sum_{i=1}^{k} \alpha_i \cdot dt_{a_i},$$

where $^0x$ is the standard part of $x$ (the unique real number closest to $x$), $k \in \mathbb{N}$, the $\alpha_i$s are nonzero real numbers and the $dt_{a_i}$s constitute a descending sequence of nilpotent infinitesimals (in the sense that $dt_{a_n}$ is infinitesimal relative to $dt_{a_m}$ whenever $1 \leq m < n \leq k$), it being understood that $x = ^0x$ if $k = 0$ (e.g. Giordano 2010, 2011 §11). Moreover, for each $n > 1$, there are nilpotent infinitesimals in $\bullet R$ having index of nilpotency $n$.

However, despite their overlapping motivations, there are notable differences between Giordano’s theorem and the Kock-Lawvere axiom. For example, in addition to specifying that the slope $b$ is a real number rather than a member of $\mathbb{R}$, the theorem refers to all standard smooth functions (that are pathwise Lipschitzian) as opposed to all functions $f : D \rightarrow \mathbb{R}$. Moreover, unlike $\mathbb{R}$ and the Weil algebras more generally employed in SDG, $\bullet \mathbb{R}$ is totally ordered. Accordingly, since the set $D$ in Giordano’s theorem contains nilpotent infinitesimals that are provably nonzero, the totally ordered commutative unitary ring $\bullet \mathbb{R}$ is necessarily non-Archimedean despite the fact that it satisfies the finitely bounded version of the Archimedean condition employed in SDG. Furthermore, whereas the product of any two members of $D$ is equal to 0 in IDG (Giordano 2010: Theorem 24, p. 172), this is not the case in SDG, the latter being a consequence of the Kock-Lawvere axiom (cf. Kock 1981, pp. 6, 15). In this respect, the nilpotent infinitesimals employed in IDG more closely resemble those employed in Hjelmslev geometries.

where the coefficients, powers and $r$ are reals, $k \in \mathbb{N}$ and $o$ is a variant of Landau’s little-o notation. Next, one defines equivalence between $x, y \in R_o[t]$ by the condition: $x \sim y$ if and only if $x(t) = y(t) + o(t)$ as $t \to 0^+$. Finally, $\bullet \mathbb{R}$ is taken to be the quotient set of $R_o[t]$ with respect to $\sim$.

For the order and ring operations on $\bullet \mathbb{R}$, as well as detailed discussions of its properties, see (Giordano 2010a; Giordano 2011a; Giordano and Kunzinger 2013).

In his (2010), Giordano suggests that due to the just-stated difference his nilpotent infinitesimals are more intuitively satisfying than those employed in SDG. In particular, he suggests that an intuitive picture of infinitesimal segments of lengths $h$ and $k$ where $h^2 = k^2 = 0$ and $h \cdot k \neq 0$ is not possible. However, he has since recanted and now more modestly maintains that in the end “it is a matter of taste about what approaches are felt as beautiful, manageable and in accordance with our philosophical approach to mathematics” (Giordano and Wu 2016, p. 898).
Like SDG, IDG is developed in a category-theoretic framework. However, unlike SDG, IDG as developed above does not lead to a Cartesian closed category of spaces. To address this, Giordano has replaced the just-described treatment based on pathwise Lipschitzian functions with one based on quasi-standard smooth functions. In their most basic form, these are functions $g : \mathbb{R} \rightarrow \mathbb{R}$ that locally can be written as $g(x) = \varphi(x, p)$, where the $p \in \mathbb{R}^n$ are nonstandard fixed parameters. The derivative of such a function is defined by appealing to what Giordano calls the Fermat-Reyes Theorem, which asserts: for each quasi-standard smooth function $g$ there is precisely one quasi-standard smooth function $r : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that

$$\forall(x, h) \in \mathbb{R}^2 : g(x + h) = g(x) + h \cdot r(x, h).$$

This permits one to define $g'(x) = r(x, 0)$ for all $x \in \mathbb{R}$. Integration for quasi-standard smooth functions is developed by proving the existence and uniqueness of primitives.

On the basis of this (or rather a treatment based on a more general class of quasi-standard smooth functions $f : S \rightarrow T$ where $S$ and $T$ are subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively), Giordano and Wu have constructed a calculus that is at least as strong as its counterpart in SDG (Giordano 2011; Giordano and Wu 2016, forthcoming), and which in some respects more closely resembles the standard calculus of smooth functions. For example, in their system, the intermediate value theorem holds. The quasi-standard smooth functions are also arrows of the Cartesian closed category of Fermat Spaces, which are essentially $\mathbb{R}$-enriched extensions of diffeological spaces, and the category of Fermat

For historical motivation as well as remarks regarding the impact this difference has on SDG, see (Kock 2003, p. 228). Also see (Hellman 2006, §4) for philosophical remarks bearing on the difference.

In SDG, one is assured of working in a Cartesian closed category since every topos is a Cartesian closed category. For an overview of the virtues of Cartesian closed categories and some of the roles they play in a variety of mathematical and physical theories, see (Giordano 2011a).

The class of quasi-standard smooth functions is a $\mathbb{R}$ analog of the class of quasi-standard functions employed by Robinson in the early days of nonstandard analysis (Robinson 1961, pp. 437-438, 1963/1965, pp. 266-268; Giordano 2011, p. 864; Giordano and Wu 2013, p. 890) to deal with the theory of distributions. It was subsequently replaced by the more versatile notion of an internal function (Robinson 1966, 1974, p. 42).

Since the latter decades of the twentieth century the needs of theoretical physics have repeatedly challenged the limitations of classical differential geometry of smooth manifolds. Diffeological spaces (which are defined on $\mathbb{R}$) are generalizations of smooth manifolds that have been motivated by those needs. Diffeology, which is a substantial extension of differential geometry, treats such matters. See
spaces together with its underlying calculus of quasi-standard smooth functions serves as the basis for the forthcoming development of IDG referred to in (Giordano and Wu 2016, pp. 889-890).

Echoing the attitudes expressed by Kock and Bunge regarding SDG, Giordano does not envision IDG and its underlying calculus as providing replacements for their classical $\epsilon, \delta$-counterparts, but rather as companions that provide new insights into the structure of the smooth world while lending precision to some of the nilpotent infinitesimalist techniques that have been employed on occasion by analysts, differential geometers, physicists and engineers since the time of Fermat.

As was alluded to above, Bertram’s TDC seeks to provide a generalization of the differential aspects of classical differentiable geometry of smooth manifolds by providing a nilpotent-infinitesimalist framework that is applicable to a wide range of models of the continuum. Underlying TDC is a general nilpotent-infinitesimalist approach to differential calculus developed earlier by Bertram, Glöcker and Neeb (2004). Since integral calculus does not appear to admit such a unified theory, Bertram maintains that integration theory for the various underlying continua needs to be developed in localized settings, adding assumptions of an analytic or a topological nature, according to the strength of the results one desires. Unlike SDG and IDG, TDC is developed in a set-theoretic (rather than a category-theoretic) framework, and like IDG the varying sorts of logical apparatus that are part and parcel of SDG (topos theory) and NSDG (model theory) are absent from TDC (Bertram 2008, p. 160). Of the three above mentioned nilpotent-infinitesimalist approaches to differential geometry, TDC is the most purely algebraic.

Among the motivations underlying TDC are considerations of simplicity. As Bertram puts it:

> The suggestion to use dual numbers in differential geometry is not new—one of the earliest steps in this direction was by A. Weil [1953]; one of the most recent is [IDG due to Giordano]. However, most of the proposed constructions [including SDG] are so complicated that one is discouraged to reiterate them. But this is what makes the dual number formalism [of TDC] so useful. (Bertram 2008, p. 3)

For the sake of space, we will limit our overview of TDC to a few snapshots of the theory presented by Bertram himself, beginning with (Iglesias-Zemmour 2013) for a comprehensive discussion of the history, foundations and physical motivation of the subject.
the following motivational observation, where \( \mathbb{K} \) is understood to be a good topological ring in the sense defined above (Note 39).

We define *manifolds* and *tangent bundles* in the classical way using charts and atlases...; then, intuitively, we may think of the tangent space \( T_p M \) of [a manifold] \( M \) at [point] \( p \) as a “(first order) infinitesimal neighborhood” of \( p \). This idea may be formalized by writing, with respect to some fixed chart of \( M \), a tangent vector at the point \( p \) in the form \( p + \epsilon v \), where \( \epsilon \) is a “very small” quantity (say, Planck’s constant), thus expressing that the tangent vector is “infinitesimally small” compared to elements of \( M \) (in a chart). The property of being “very small” can mathematically be expressed by requiring that \( \epsilon^2 \) is zero, compared with all “space quantities”. This suggests that, if \( M \) is a manifold modelled on a \( \mathbb{K} \)-module \( V \), then the tangent bundle \( [T M] \) should be modelled on the space \( V \oplus \epsilon V =: V \times V \), which is considered a module over the [good topological] ring \( \mathbb{K}[\epsilon] = \mathbb{K} \oplus \epsilon \mathbb{K} \) of *dual numbers* over \( \mathbb{K} \) (it is constructed from \( \mathbb{K} \) in a similar way as the complex numbers are constructed from \( \mathbb{R} \), replacing the condition \( i^2 = -1 \) by \( \epsilon^2 = 0 \)). All this would be really meaningful if \( T M \) were a manifold not only over \( \mathbb{K} \), but also over the extended ring \( \mathbb{K}[\epsilon] \). (2008, p. 2).

Bertram proves that this is indeed the case. More specifically, he shows that: (i) if \( M \) is a smooth manifold over \( \mathbb{K} \), then \( T M \) is, in a natural way, a manifold over \( \mathbb{K}[\epsilon] \); and (ii) if \( f : M \to N \) is smooth over \( \mathbb{K} \), then \( T f : T M \to T N \) is smooth over \( \mathbb{K}[\epsilon] \), i.e. the familiar tangent functor \( T \) of differential geometry can be regarded as a functor of scalar extension from \( \mathbb{K} \) to \( \mathbb{K}[\epsilon] \) in the category of smooth manifolds (2008: Theorem 6.2, p. 36).

Hence [says Bertram] one may use the “dual number unit” \( \epsilon \) when dealing with tangent bundles with the same right as the imaginary unit \( i \) when dealing with complex manifolds. The proof...is conceptual and allows [one] to understand why dual numbers naturally appear in this context....(2008, p. 2)

Indeed, according to Bertram:

in a way this result contains a new justification of infinitesimals as proposed by A. Weil in [1953], having
the advantage not to use the heavy logical or model-
theoretic machinery introduced in synthetic differential
geometry.... (Bertram 2006, p. 96)

Moreover, and quite importantly, by iterations of the above result one
gets much more.

[For instance, if $TM$ is the scalar extension of $M$ by $K[\epsilon_1]$, then the double tangent bundle $T^2M := T(TM)$
is simply a scalar extension of $M$ by the ring ...

$$K[\epsilon_1][\epsilon_2] \cong K \oplus \epsilon_1 K \oplus \epsilon_2 K \oplus \epsilon_1 \epsilon_2 K,$$

and so on for all higher tangent bundles $T^kM$. As a
matter of fact, most of the important notions of differ-
etial geometry deal, in one way or another, with the
second order tangent bundle $T^2M$ (e.g. Lie bracket,
 exterior derivative, connections) or with $T^3M$ (e.g. curvature). Therefore second and third order differential
geometry really is the central part of all differential ge-
ometry and finding a good notation concerning second
and third order tangent bundles becomes a necessity.
Most textbook, if at all $TTM$ is considered, use a com-
ponent notation in order to describe objects related to
this bundle. In this situation, the use of different sym-
bols $\epsilon_1, \epsilon_2, ...$ for the infinitesimal units of the various
scalar extensions is a great notational progress, com-
bining algebraic rigour and transparency. It becomes
clear that many structural features of $T^kM$ are simple
consequences of corresponding structural features of the
rings ... $K[\epsilon_1, ..., \epsilon_k]...[where \epsilon_i^2 = 0 and \epsilon_i \epsilon_j = \epsilon_j \epsilon_i$ for
all $1 \leq i, j, \leq k$]. (Bertram, 2008, p. 3)

It is worth emphasizing that while Bertram believes nilpotent in-
finitesimals do have an important role to play in smooth differential
geometry, they need not necessarily be viewed as elements of the un-
derlying continuum itself. As Bertram remarks:

Some readers may wish to avoid the use of rings which
are not fields, or even to stay in the context of the real
base field. In principle, all our results that do not di-
rectly involve dual numbers can be proved in a “purely
real” way, i.e., by interpreting $\epsilon$ just as a formal (and
very useful !) label....But in the end, just as the “imagi-
ary unit” $i$ got its well-deserved place in mathematics,
so will the “infinitesimal unit” $\epsilon$. (Bertram, 2008, p. 3)
While we have focused our attention in this section on the nilpotent-infinitesimalist approaches to smooth differential geometry, we would be remiss if we failed to at least mention the promise of NSDG. Formally speaking, in SDG tangent vectors are treated as infinitesimal displacements and vector fields are regarded as infinitesimal transformations. The aforementioned paper of Nowik and Katz presents a similar approach from the standpoint of nonstandard analysis. Though still in the early stages of its development, the theory of Nowik and Katz is already promising enough that in his review of their paper for *Mathematical Reviews*, H. Nishimura, a longstanding contributor to the development and application of SDG, wrote: “A serious comparison between synthetic differential geometry and nonstandard differential geometry might presumably be intriguing” (MR3457545). Building on Nishimura’s observation, we draw this section to a close by suggesting that a serious comparison of all four infinitesimalist approaches to smooth differential geometry touched upon in this section along with an examination of their corresponding underlying treatments of smooth continua would be very intriguing indeed.

12. INVERTIBLE AND NILPOTENT INFINITESIMALS: AFTERTHOUGHTS

As was earlier mentioned, Robinson expressed the opinion that using the techniques of nonstandard analysis one could justify the residual uses of infinitesimals in the writings of post δ, ε differential geometers and physicists. Since some of those uses were of nilpotent infinitesimals his words are in part misleading unless (as we suspect) he was suggesting that using the techniques of nonstandard analysis one can prove the classical results they established using infinitesimals, regardless of their nature. By the same token, proponents of SDG often say their use of nilpotent infinitesimals lends precision to the techniques employed by classical differential geometers such as Sophus Lie and Élie Cartan. However, as I. Moerdijk and G. Reyes emphasize in the Preface to their monograph on models of SDG (1991, p. v):

\[58\]

While outside the province of this paper, we note that there is an ongoing debate concerning whether, and if so, in what respects, nonstandard analysis is a useful lens to evaluate and interpret various infinitesimalist techniques employed by Leibniz, Euler, Cauchy and other pre-ε, δ analysts. The debate grew out of some early remarks of Robinson (1966, 1974, ch. x, 1967) and Lakatos (1978). For an introduction to the debate along with many references, see (Bos 1974, 2004; Laugwitz 1987, 1989; Fraser 2015; Borovik and Katz 2012; Katz and Sherry 2013; Blåszycyk, Katz and Sherry 2013; Bair *et al* 2017).
two kinds of infinitesimals were used by geometers like S. Lie and E. Cartan, namely invertible infinitesimals and nilpotent ones.

This, together with the occasional use of infinite elements by physicists, led Moerdijk and Reyes (1991, pp. 241-243, 285-286) to introduce models of the basic axioms of SDG having both sorts of infinitesimals as well as infinite elements and to issue the following proclamation (1991, p. 239):

[There are] at least two different kinds of infinitesimals that have appeared in the literature. On the one hand, there are the nilpotent, [used] in handling “infinitesimal” structures like jets, prolongations and connections...On the other hand, there are the invertible infinitesimals, which together with infinitely large integers, are used to analyze such notions as limits and convergence along the lines of non-Standard Analysis, as exemplified by Robinson’s book. Thus, these types of infinitesimals serve different (and complimentary purposes), and both should appear in a theory of infinitesimals worth its salt.

Thus far, however, models having both types of infinitesimals as well as infinite elements have been employed sparingly in SDG–originally by Moerdijk and Reyes in connection with the theory of distributions and the Dirac \( \delta \) function (1991, pp. 320-337) and later in connection with differential equations (Kennison 1999). Moreover, their use runs contrary to the stated Archimedean nature of SDT, and according to Kock, “the[ir] value...for synthetic reasoning in geometry is not evident” (1993, p. 354). Invertible infinitesimals along with their multiplicative inverses can also be added to IDG (Giordano 2001, p. 77), though thus far they have not. And, while it has been suggested that nilpotent infinitesimals could be added to nonstandard analysis (Giordano and Wu 2016, p. 896), that would require substantial modifications of the theory which would no longer give rise to an elementary extension of the reals and it is by no means clear the gains would outweigh the losses. Collaterally, as Robinson appears to have maintained, and further development of the work of Nowik, Katz and others may eventually show, it may be possible to develop an entirely adequate infinitesimalist approach to the smooth aspects of differential geometry without nilpotent infinitesimals at all.

On the other hand, like the models of Moerdijk and Reyes, systems containing both types of infinitesimals as well as infinite numbers fall within the framework of TDC (Bertram 2008, p. viii); and so, at
least from the standpoint of the purely differential aspects of smooth
differential geometry as characterized by TDC the transition to such
a system is seamless. Moreover, such systems have found application
in algebra and geometry since Zemmer (1953, p. 177; also see, Fuchs
1963, pp. 108-109) and Klingenberg (1954a) pioneered their use well
over half a century ago.

The question whether a theory of infinitesimals worth its salt re-
quires both types of infinitesimals, as Moerdijk and Reyes maintain,
is a fascinating one, as is the corresponding question regarding an in-
finitesimalist theory of continua built thereon. While support for Mo-
erdijk and Reyes’s contention has thus far been muted, whether with
time this will change, only time will tell.

13. CONCLUDING REMARKS

Since the discovery that a diagonal of a square is incommensurable
with its sides, the question of the possibility of bridging the gap between
the domains of discreteness and of continuity, or between arithmetic
and geometry, has been a central problem in the foundations of math-
ematics. Cantor and Dedekind of course believed they had bridged the
gap with the creation of their arithmetico-set-theoretic continuum of
real numbers, and it remains a central tenet of standard mathematical
philosophy that indeed they had. Nevertheless, Cantor was overly
optimistic when he suggested that his theory of the continuum, unlike
that of the ancients, had “been thought out ... with the clarity and
completeness ... required to exclude the possibility of different opinions
among [its] posterity” (Cantor 1883/1996, p. 903). After all, whereas
Cantor and Dedekind had succeeded in replacing the vague ancient
conception with a clear and precise arithmetico-set-theoretic concep-
tion that proved itself adequate for the needs of analysis, differential
geometry and the empirical sciences of their day, they could neither
free their theory of its logical, theoretical, and philosophical presup-
positions, nor preclude the possibility that other adequate conceptual
schemes, each self-consistent, could be devised offering alternative vi-
sions of the continuum.

However, it was critiques of the former and/or the realization of the
logical possibility of the latter that has given rise to a host of non-
Cantor-Dedekindean conceptions, including the recent infinitesimalist
conceptions canvassed above. In the latter cases, with the exception
of the surreals, the architects were motivated by the belief, or at least
the hope, that their respective theories are, or with time would be,
adequate for the needs of analysis or portions of differential geometry,
and the empirical sciences served thereby. Like their late nineteenth-
and early twentieth-century non-Archimedean geometric forerunners,
nonstandard analysis and the infinitesimalist approaches to portions of
differential geometry have drawn attention to the possibility of phys-
cical continua whose logical cogency, let alone physical possibility, had
long been in doubt. Whether empirical science will require such a
theory, as some, like Fenstad (1987, 1988), already contend, and oth-
ers, like Veronese (1909/1994, p. 180) and Hahn (1933/1980 p. 100),
would not rule out, only time will tell. On the other hand, while
showing no sign of displacing the Cantor-Dedekind theory and the
collateral theories based thereon, the infinitesimalist approaches have
performed, and continue to perform, important logical and philosoph-
ical service. Nonstandard analysis has also had substantial success in
shedding important light on, and establishing significant new results
in, various areas of analysis, theoretical physics and economics, and
SDG/SDT has provided important insights into the relation between
algebraic geometry, differential geometry and smooth spaces more gen-
ernally. Along with their theoretical accomplishments, the practition-
ers of nonstandard analysis and SDG/SDT maintain that their approaches
have heuristic and intuitive advantages over their standard counter-
parts. The authors of IDG and TDC likewise maintain such advan-
tages for their approaches, but without, they add, either the need for
model-theoretic apparatus or the abandonment of classical logic. Not
eschewing model-theoretic apparatus, NSDG seeks an extension of the
range of application of nonstandard analysis, which, in principle, could
provide an infinitesimalist approach to all of analysis and the smooth
portions of differential geometry, something that is beyond the scope
of SDG/SDT, IDG and TDC.

Of course, whether nonstandard analysis, SDG/SDT or any of the
other just-said infinitesimalist approaches, together with its corre-
sponding conception of the continuum, will eventually displace its stan-
dard counterpart or even become a widely employed implement in the
toolkits of mathematicians remains to be seen. Also remaining to be
seen is whether the s-hierarchical ordered field of surreal numbers will
come to be widely regarded as an absolute arithmetic continuum mod-
ulo NBG. However, regardless of how these questions are ultimately
answered, one cannot help but wonder whether or not F. W. Lawvere
was prophetic when he recently maintained: “Contrary to common
opinion, the question “what is the continuum?” does not have a final
answer, the immortal work of Dedekind notwithstanding” (2011, p.
249).
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Kyle,

Since I will be going over the exam tomorrow, I am willing to give you a make up tomorrow in my office (231S Lindley Hall) at 11:50.

Please let me know if you will be there.

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