On complex deformations of Kähler-Ricci solitons

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Abstract

We obtain a formal obstruction, i.e. a necessary condition for the existence of polarised complex deformations of Kähler-Ricci solitons. This obstruction is expressed in terms of the harmonic part of the variation of the complex structure.

1 The obstruction result

Despite the remarkable work of Podesta-Spiro, [Po-Sp], not much is known on the existence of complex deformations of Kähler-Ricci solitons. In this paper, we provide an effective result on this topic. Namely, given any polarised family of complex deformations over a Kähler-Ricci soliton (polarised by the symplectic form of the initial Kähler-Ricci soliton), we can effectively establish a necessary condition for this family to exist.

Let \((X,J,g,\omega)\) be a Fano manifold with \(\omega = \text{Ric}_J(\Omega)\), where \(\Omega > 0\) is the unique volume form such that \(\int_X \Omega = 1\). (We denote by \(\text{Ric}_J(\Omega)\) the Chern-Ricci form associated to the volume form \(\Omega\)). We introduce the \(\Omega\)-divergence operator acting on vector fields \(\xi\) as

\[
\text{div}^\Omega \xi := \frac{d}{d\xi - \Omega}.
\]

It is well known (see [Fut]), that the Lie algebra of \(J\)-holomorphic vector fields \(H^0(X,T_{X,0}J)\) identifies with the space of complex valued functions

\[
A^\Omega_{g,J} := -\text{div}^\Omega H^0(X,T_{X,0}J) \subset C^\infty(\Omega,\mathbb{C})_0,
\]

where \(C^\infty_\Omega(X,\mathbb{C})_0\) is the space of smooth complex valued functions with vanishing integral with respect to \(\Omega\). We denote by \(H^{0,1}_{g,\Omega}(TX,J)\) the space of \(TX\)-valued \((0,1)\)-forms which are harmonic with respect to the Hodge-Witten Laplacian determined by the volume form \(\Omega\).

Assume now \((X,J,g,\omega)\) is a compact Kähler-Ricci soliton and consider the functions \(f := \log \frac{dV}{\Omega}\), \(F := f - \int_X f\Omega\). The solution of the variational stability problem in [Pal2] shows that the vanishing harmonic cone

\[
H^{0,1}_{g,\Omega}(TX,J)_0 := \left\{ A \in H^{0,1}_{g,\Omega}(TX,J) \mid \int_X |A|_g^2 F\Omega = 0 \right\},
\]
is relevant for the deformation theory of compact Kähler-Ricci solitons. In the Dancer-Wang Kähler-Ricci soliton case $H^0_{g,\Omega}(T_X,J) \neq \{0\}$, thanks to a result in [Ha-Mu].

For any $A \in H^0_{g,\Omega}(T_X,J)$ we define the $\mathbb{R}$-linear functional

$$
\Phi_A : A^\Omega_{g,J} \longrightarrow \mathbb{R},
$$

$$
\Phi_A (u) := \int_X \left[ 2 \operatorname{Re} u \langle \nabla^2_g f, A^2 \rangle_g - \langle J \nabla_g f - \nabla_g A, i \nabla \times A \rangle_g \right] \Omega.
$$

With these notations we can state our obstruction result.

**Theorem 1** Let $(X,J,g,\omega)$ be a compact Kähler-Ricci soliton, let $(J_t,\omega)_{t \in (-\epsilon,\epsilon)}$ be a smooth family of Kähler-Ricci solitons with $J_0 = J$ and let $A \in H^0_{g,\Omega}(T_X,J)$ be the harmonic part of the variation $J_0$. Then $A \in H^0_{g,\Omega}(T_X,J)_0$ and $\Phi_A = 0$.

The fact that $A \in H^0_{g,\Omega}(T_X,J)_0$ is a statement in our previous work [Pal2]. We will show also that for any $A \in H^0_{g,\Omega}(T_X,J)$ holds the identity

$$
\int_X |A|^2_g F\Omega = - \int_X \left[ 2 \langle \nabla^2_g f, A^2 \rangle_g - \langle J \nabla_g f - \nabla_g A, JA \rangle_g \right] \Omega,
$$

whose right-hand side shows some similarity with the integral $\Phi_A (u)$.

## 2 Properties of the first variation of Perelman’s $H$ map

We need to remind a few basic facts proved in [Pal2]. We first remind some of the notations in [Pal2]. Let $\Omega > 0$ be a smooth volume form over an oriented compact and connected Riemannian manifold $(X,g)$. We equip the set of smooth Riemannian metrics $\mathcal{M}$ over $X$ with the scalar product

$$(u,v) \longmapsto \int_X (u,v)_g \Omega,$$

for all $u,v \in L^2(X,S^2_g T^*_X)$. Let $P^*_g$ be the formal adjoint of some operator $P$ with respect to the metric $g$. We observe that the operator $P^*_g := e^f P^*_g (e^{-f} \bullet)$, with $f := \log \frac{dv_g}{dv}$, is the formal adjoint of $P$ with respect to the scalar product $(1)$. We define the real weighted Laplacian operator $\Delta^\Omega_g := \nabla^g \alpha \nabla_g$. We notice in particular the identity $\operatorname{div}^\Omega \nabla_g u = - \Delta^\Omega_g u$, for all functions $u$.

Over a Fano manifold $(X,J,g,\omega)$, with $\omega = \operatorname{Ric}_J(\Omega)$, $\int_X \Omega = 1$, we define the linear operator $B^\Omega_{g,J}$ acting on smooth complex valued functions $u$ as $B^\Omega_{g,J} u := \operatorname{div}^\Omega (J \nabla_g u)$. This is a first order differential operator. Indeed

$$
B^\Omega_{g,J} u = \operatorname{Tr}_R (J \nabla^2_g u) - df \cdot J \nabla_g u
$$

$$
= g(\nabla_g u, J \nabla_g f),
$$
since $J$ is $g$-anti-symmetric. We define the weighted complex Laplacian operator
\[ \Delta_{g,J}^\Omega := \Delta_{g,J}^\Omega - i B_{g,J}^\Omega, \]
acting on smooth complex valued functions. We remind the identity $\Lambda_{g,J}^\Omega = \text{Ker}(\Delta_{g,J}^\Omega - 2\mathbb{I})$, (see \cite{Fut}).

We remind now that the $\Omega$-Bakry-Emery-Ricci tensor of the metric $g$ is defined by the formula
\[ \text{Ric}_g(\Omega) := \text{Ric}(g) + \nabla_g d \log \frac{dV_g}{\Omega}. \]
A Riemannian metric $g$ is called a $\Omega$-shrinking Ricci soliton if $g = \text{Ric}_g(\Omega)$. We define the following fundamental objects
\[ h \equiv h_{g,\Omega} := \text{Ric}_g(\Omega) - g, \]
\[ 2H \equiv 2H_{g,\Omega} := -\Delta_{g,J}^\Omega f + \text{Tr}_g h + 2f, \]
with $f := \log \frac{dV_g}{\Omega}$. We define also the normalised function $H := H - \int_X H \Omega$. We denote by $\mathcal{V}_1$ the space of smooth positive volume forms with unitary integral over $X$. For any $V \in T\mathcal{V}_1$, we define $V_\Omega^* := V/\Omega$.

We notice now that over a polarised Fano manifold $(X, \omega)$, $\omega \in 2\pi c_1(X)$, the space of $\omega$-compatible complex structures $J_\omega$ embeds naturally inside $\mathcal{M} \times \mathcal{V}_1$. For any $V \in T\mathcal{V}_1$, we define $V_\Omega^* := V/\Omega$.

We denote by $\mathcal{M}_\omega$ the tangent cone of $\mathcal{S}_\omega$ at an arbitrary point $(g, \Omega) \in \mathcal{S}_\omega$. This is by definition the union of all tangent vectors of $\mathcal{S}_\omega$ at the point $(g, \Omega)$. We denote that, (see for example \cite{Pal1}), the tangent cone $\text{TC}_{\mathcal{M}_\omega,g}$ of $\mathcal{M}_\omega$ at an arbitrary point $g \in \mathcal{M}_\omega$ satisfies the inclusion
\[ \text{TC}_{\mathcal{M}_\omega,g} \subseteq \mathbb{D}_{g,[0]}, \]
with
\[ \mathbb{D}_{g,[0]} := \{ v \in C^\infty (X, S^2_{\mathbb{R}} T_X^* \mid v = -J^* v J, \nabla_{T_X} v_g^* = 0 \}, \]
with $v_g^* := g^{-1} v$. It has been showed in \cite{Pal2} that for any $(g, \Omega) \in \mathcal{S}_\omega$ holds the inclusion
\[ \text{TC}_{\mathcal{S}_\omega,(g,\Omega)} \subseteq \mathbb{T}_{g,\Omega}, \]
with
\[ \mathbb{T}_{g,\Omega} := \left\{ (v, V) \in \mathbb{D}_{g,[0]} \times T\mathcal{V}_1 \mid 2dd^c_V V_\Omega^* = -d(\nabla_V^* v_g^* - \omega) \right\}. \]
(We will use the definition $2d^c_{J} := i(\overline{J}_J - \partial_J)$ in this paper). We remind (see \cite{Pal2}) that a point $(g, \Omega) \in \mathcal{S}_\omega$ is a Kähler-Ricci soliton if and only if $H_{g,\Omega} = 0$. Furthermore,
\[ 2H_{g,\Omega} = -(\Delta_{g,\cdot}^\Omega - 2\mathbb{I}) F \in \Lambda_{g,J}^{\Omega,\perp} \cap C^\infty (X, \mathbb{R})_0, \]
for all \((g, \Omega) \in \mathcal{S}_\omega\). The infinitesimal properties of the map \((g, \Omega) \in \mathcal{S}_\omega \mapsto \mathbf{L}_{g, \Omega}\) are explained in the next sub-section.

### 2.1 Triple splitting of the space \(T_{g, \Omega}^J\)

In \([\text{Pal2}]\), we introduce a pseudo-Riemannian metric \(G\) over \(\mathcal{M} \times \mathcal{V}_1\) which is positive defined over \(T_{g, \Omega}^J\) for any \((g, \Omega) \in \mathcal{S}_\omega\), with \(J := -\omega^{-1}g\). We denote by

\[
\Lambda_{g, J}^{\Omega, \perp} := [\ker(\Delta_{g, J}^{\Omega} - 2\mathbb{I}))^{\perp, \Omega} \subset C_\omega^\infty(X, \mathbb{C})_0, 
\]

the \(L^2\)-orthogonal space to \(\Lambda_{g, J}^{\Omega}\) inside \(C_\omega^\infty(X, \mathbb{C})_0\). By abuse of notations we will denote by \(G_{g, \Omega}\) the scalar product over \(\Lambda_{g, J}^{\Omega, \perp}\), induced by the isomorphism

\[
\eta : \Lambda_{g, J}^{\Omega, \perp} \oplus \mathcal{H}_{g, \Omega}^0(T_X, J) \rightarrow T_{g, \Omega}^J
\]

\[
(\psi, A) \mapsto \left( (\nabla_{g, J}\psi + A), -\frac{1}{2} \text{Re} \left[ (\Delta_{g, J}^{\Omega} - 2\mathbb{I})\psi \right] \right). 
\]

Explicitly (see \([\text{Pal2}]\)),

\[
G_{g, \Omega}(\varphi, \psi) = \frac{1}{2} \int_X \left[ (\Delta_{g, J}^{\Omega} - 2\mathbb{I})\varphi \cdot \psi + (\Delta_{g, J}^{\Omega} - 2\mathbb{I})\psi \cdot \varphi \right] \Omega 
+ \frac{1}{2} \int_X \text{Im} \left[ (\Delta_{g, J}^{\Omega} - 2\mathbb{I})\varphi \right] \text{Im} \left[ (\Delta_{g, J}^{\Omega} - 2\mathbb{I})\psi \right] \Omega.
\]

For any \((g, \Omega) \in \mathcal{S}_\omega\), we introduce in \([\text{Pal2}]\) the vector spaces

\[
E_{g, \Omega}^J := \left\{ u \in \Lambda_{g, J}^{\Omega, \perp} \mid (\Delta_{g, J}^{\Omega} - 2\mathbb{I}) u = (\Delta_{g, J}^{\Omega} - 2\mathbb{I}) u \right\},
\]

\[
\Phi_{g, \Omega}^J := \left( E_{g, \Omega}^J \right)^{\perp, \Omega} \cap \Lambda_{g, J}^{\Omega, \perp},
\]

and we denote by \([g, \Omega]_{\omega} := \text{Symp}^0(X, \omega) \cdot (g, \Omega) \subset \mathcal{S}_\omega\) the orbit of the point \((g, \Omega)\) under the action of the identity component of the group of smooth symplectomorphisms \(\text{Symp}^0(X, \omega)\) of \(X\). The map \(\eta\) restricts to a \(G\)-isometry

\[
\eta : \Phi_{g, \Omega}^J \rightarrow T_{[g, \Omega]_{\omega}, (g, \Omega)}.
\]

The positivity of the metric \(G_{g, \Omega}\) over \(\Lambda_{g, J}^{\Omega, \perp}\), combined with an elliptic argument (see \([\text{Pal2}]\)) implies the decomposition

\[
\Lambda_{g, J}^{\Omega, \perp} = \Phi_{g, \Omega}^J \oplus_G E_{g, \Omega}^J.
\]

Over a compact Kähler-Ricci soliton \((X, J, g, \omega)\), we introduce the operator

\[
P_{g, J}^{\Omega, \omega} := (\Delta_{g, J}^{\Omega} - 2\mathbb{I})(\Delta_{g, J}^{\Omega} - 2\mathbb{I}).
\]
This is a non-negative self-adjoint real elliptic operator with respect to the $L^2_{\Omega}$-hermitian product. The restriction of the differential of the map map $(g, \Omega) \in S_{\omega} \mapsto H_{g, \Omega}$ over the space $\Lambda^{\Omega, \perp}_{g, J}$, identifies, via the isomorphism $\eta$, with the map

$$D_{g, \Omega}H : \Lambda^{\Omega, \perp}_{g, J} \rightarrow \Lambda^{\Omega, \perp}_{g, J} \cap C^\infty \Omega(X, \mathbb{R})_0$$

$$\psi \mapsto \frac{1}{4} P_{g, J} \text{Re} \psi.$$  

This map restricts to an isomorphism

$$D_{g, \Omega}H : \mathbb{E}_{g, \Omega}^J \rightarrow \Lambda^{\Omega, \perp}_{g, J} \cap C^\infty \Omega(X, \mathbb{R})_0,$$

(see [Pa2] for the technical details), and

$$\Phi^{J}_{g, \Omega} = \text{Ker} D_{g, \Omega}H \cap \Lambda^{\Omega, \perp}_{g, J}.$$  

Moreover, $\text{Ker} P_{g, J} \cap C^\infty \Omega(X, \mathbb{R})_0 = \{ \text{Re} u \mid u \in \Lambda^{\Omega}_{g, J} \} =: \text{Re} \Lambda^{\Omega}_{g, J}$ and

$$P_{g, J} C^\infty \Omega(X, \mathbb{R})_0 = \Lambda^{\Omega, \perp}_{g, J} \cap C^\infty \Omega(X, \mathbb{R})_0.$$  

In general for any $(g, \Omega) \in S_{\omega}$ Kähler-Ricci soliton holds the identity

$$\text{Ker} D_{g, \Omega}H \cap \mathbb{T}_{g, \Omega} = T_{[g, \Omega], (g, \Omega)} \oplus \mathbb{H}^{0, 1}_{g, \Omega}(T_X, J),$$

with $J := -\omega^{-1}g$. We finally notice that applying the finiteness theorem (see for example [Eb], proposition 6.6, page 26), to the real elliptic operator $P_{g, J} : C^\infty \Omega(X, \mathbb{R})_0 \rightarrow C^\infty \Omega(X, \mathbb{R})_0$, we deduce the $L^2_{\Omega}$-orthogonal decomposition

$$C^\infty \Omega(X, \mathbb{R})_0 = \left[ \Lambda^{\Omega, \perp}_{g, J} \cap C^\infty \Omega(X, \mathbb{R})_0 \right] \oplus \text{Re} \Lambda^{\Omega}_{g, J}. \tag{4}$$

**Remark 1** We denote by $\Lambda^{\Omega}_{g, R} := \text{Ker}_R(\Delta_g - 2I) \subset C^\infty \Omega(X, \mathbb{R})_0$, and by

$$\Lambda^{\Omega, \perp}_{g, R} := \left[ \text{Ker}_R(\Delta_g - 2I) \right]^{\perp}_{\Omega} \subset C^\infty \Omega(X, \mathbb{R})_0,$$

its $L^2_{\Omega}$-orthogonal inside $C^\infty \Omega(X, R)_0$. It is easy to see that the map

$$\chi : \Lambda^{\Omega, \perp}_{g, R} \cap C^\infty \Omega(X, \mathbb{R})_0 \rightarrow T_{[g, \Omega], (g, \Omega)},$$

$$u \mapsto \left( 2\omega \overrightarrow{\partial_T u}, \nabla_g u, (B_{g, J}^\Omega u) \Omega \right),$$

is an isomorphism. Thus, there exists an isomorphism map

$$\tau : \mathbb{O}_{g, \Omega} \rightarrow i\Lambda^{\Omega, \perp}_{g, R}$$

$$\theta \mapsto iu : \theta - iu \in \Lambda^{\Omega}_{g, J},$$

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3 Variation formulas for the $\Omega$-divergence operators

For any $u,v \in C^\infty(X, S^2 T^*_X)$ we define in [Pal2] the real valued 1-form

$$M_g(u, v)(\xi) := 2 \nabla_g v(e_k, u^*_g e_k, \xi) + \nabla_g u(\xi, v^*_g e_k, e_k),$$

for all $\xi \in T_X$. We show now the following important lemma.

**Lemma 1** The first variation of the operator valued map

$$(g, \Omega) \mapsto \nabla^*_g \Omega : C^\infty(X, S^2 T^*_X) \longrightarrow C^\infty(X, T^*_X),$$

in arbitrary directions $(v, V)$ is given by the formula

$$2 [(D_g \nabla^*_g)(v, V)] u = M_g(v, u) - 2u \cdot (\nabla^*_g v^*_g + \nabla_g V^*_g).$$

**Proof** We first differentiate the identity defining the covariant derivative of a symmetric 2-tensor $u$ in the direction $v$. We infer

$$\hat{\nabla}_g u(\xi, \eta, \mu) = -u \left( \hat{\nabla}_g(\xi, \eta, \mu) \right) - u \left( \eta, \hat{\nabla}_g(\xi, \mu) \right),$$

where $\hat{\nabla}_g := (D_g \nabla_\ast)(v)$. Using the variation formula for the Levi-Civita connection in [Bes], we obtain

$$2 \hat{\nabla}_g u(\xi, \eta, \mu) = -u \left( \hat{\nabla}_g(\xi, \eta, \mu) \right) - u \left( \eta, \hat{\nabla}_g(\xi, \mu) \right).$$

We transform the term

$$u \left( (\nabla_g v^*_g)\eta^\ast \right) = g \left( u^*_g (\nabla_g v^*_g)\eta^\ast \right) = g \left( (\nabla_g v^*_g)\eta^\ast \right) u^*_g \mu = g \left( \xi, \nabla_g v^*_g(\eta^\ast \right) u^*_g \mu \right) = \nabla_g v(u^*_g \mu, \eta, \xi).$$

We deduce the variation formula

$$2 \hat{\nabla}_g u(\xi, \eta, \mu) = -u \left( \nabla_g(\xi, \eta, \mu) \right) - u \left( \eta, \nabla_g(\xi, \mu) \right).$$
Thus, using the fact that $u$ is symmetric we infer

$$2 \left( g^{-1} \nabla_g u \right) \mu = 2u \left( \nabla_g v^*_g, \mu \right) + \nabla_g v(u_g^*, \mu, e_k, e_k)$$

$$- u \left( \nabla_{g,e_k} v^*_g \mu + \nabla_{g,\mu} v^*_g e_k, e_k \right) + \nabla_g v(e_k, u_g^* e_k, \mu),$$

where $g^{-1} \in C^\infty(X, S^2 T_X)$ and $(e_k)_k$ is a $g$-orthonormed basis of $T_{X,p}$ which diagonalises $u$ at the point $p$. We observe however that the right hand side of the previous equality is independent of the choice of the $g$-orthonormed basis $(e_k)_k$ thanks to the intrinsic definition of trace. Simplifying, we deduce

$$2 \left( g^{-1} \nabla_g u \right) \mu = 2u \left( \nabla_g v^*_g, \mu \right) + \nabla_g v(u_g^*, \mu, e_k, e_k) - \nabla_g v(\mu, u_g^* e_k, e_k). \quad (6)$$

We can compute now the first variation of the expression

$$\nabla^*_g u = -g^{-1} \nabla_g u + \nabla_g f u,$$

with $f = f_{g, \Omega} := \log \frac{d\log g}{d\Omega}$. We observe the identity

$$[(D_{g, \Omega} \nabla^*_g) (v, V)] u = \left( u_g^* g^{-1} \right) - \nabla_g u - g^{-1} - \nabla_g u$$

$$+ [(D_{g, \Omega} \nabla^*_g f, *, *) (v, V)] - u.$$

Let $(e_k)_k$ be a $g$-orthonormed local frame of $T_X$ such that $\nabla_g e_k(p) = 0$, for some arbitrary point $p$. Using (6) and the variation formulas

$$\frac{d}{dt} \left( \nabla_{g,t} f_t \right) = \nabla_{g,t} \dot{f}_t - \dot{\nabla}_{g,t} f_t,$$

$$\dot{f}_t = \frac{1}{2} \text{Tr}_{g,t} \dot{\Omega}_t - \dot{\Omega}_t^* \quad (7)$$

we obtain the equalities at the point $p$,

$$2 \left[ (D_{g, \Omega} \nabla^*_g) (v, V) \right] u(\mu) = 2\nabla_g u(e_k, v_g^* e_k, \mu) - 2u \left( \nabla_g v^*_g, \mu \right)$$

$$- \nabla_g v(u_g^* \mu, e_k, e_k) + \nabla_g v(\mu, u_g^* e_k, e_k)$$

$$+ u \left( \nabla_g \left( \text{Tr}_g v - 2V_\Omega^* \right) - 2v_g^* \nabla_g f, \mu \right)$$

$$= 2\nabla_g u(e_k, v_g^* e_k, \mu) - 2u \left( \nabla_g v^*_g, \mu \right)$$

$$- \nabla_g v(u_g^* \mu, e_k, e_k) + \nabla_g v(\mu, u_g^* e_k, e_k)$$

$$+ u \left( \nabla_g \left( \text{Tr}_g v - 2V_\Omega^* \right), \mu \right)$$

$$= M_g(v, u)(\mu) - 2u \left( \nabla^*_g v_g^* + \nabla_g V_\Omega^*, \mu \right).$$

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thanks to the identity at the point \( p \),
\[
\nabla_g v(u_g^* \mu, e_k, e_k) = u(\nabla_g \text{Tr}_g v, \mu).
\]

In order to see this last fact, we observe the equalities
\[
\begin{align*}
\quad u(\nabla_g \text{Tr}_g v, \mu) &= g(u_g^* \nabla_g \text{Tr}_g v, \mu) \\
&= g(\nabla_g \text{Tr}_g v, u_g^* \mu) \\
&= (d\nabla_g \text{Tr}_g v)(u_g^* \mu) \\
&= (u_g^* \mu)_v(e_k, e_k) \\
&= \nabla_g v(u_g^* \mu, e_k, e_k),
\end{align*}
\]
at the point \( p \). We obtain the required variation formula. \( \square \)

In a similar way we compute the first variation formula for the operator \( \text{div}_g \Omega \) acting on 1-forms.

**Lemma 2** The first variation of the operator valued map
\[
(g, \Omega) \mapsto \text{div}_g \Omega : C^\infty(X, T^*_X) \rightarrow C^\infty(X, \mathbb{R}),
\]
in arbitrary directions \((v, V)\) is given by the formula
\[
\left[(D_{g, \Omega} \text{div}^*_g)(v, V)\right] \alpha = -\langle \nabla_g \alpha^*_g, v^*_g \rangle_g + 2\alpha \cdot \langle \nabla^*_g v^*_g + \nabla_g V^*_\Omega \rangle.
\]

We include the proof for readers convenience.

**Proof** Let \( \alpha \) be a 1-form and let \( \xi, \eta \) be two smooth vector fields. Differentiating the identity
\[
\xi \cdot (\alpha \cdot \eta) = \nabla_{g, \xi} \alpha \cdot \eta + \alpha \cdot \nabla_{g, \xi} \eta,
\]
with respect to the variable \( g \) we obtain
\[
\begin{align*}
2\hat{\nabla}_g \alpha(\xi, \eta) &= -\alpha \cdot 2\hat{\nabla}_g(\xi, \eta) \\
&= -\alpha \cdot (\nabla_{g, \xi} v^*_g \cdot \eta + \nabla_{g, \eta} v^*_g \cdot \xi) + \nabla_g v(\alpha^*_g, \xi, \eta).
\end{align*}
\]
We notice indeed the equalities
\[
\alpha \cdot \left[(\nabla_{g, \bullet} v^*_g \cdot \eta)^T \cdot \xi\right] = g\left(\alpha^*_g, (\nabla_{g, \bullet} v^*_g \cdot \eta)^T \cdot \xi\right)
\]
\[
= g\left(\nabla_{g, \alpha^*_g} v^*_g \cdot \eta, \xi\right)
\]
\[
= \nabla_g v(\alpha^*_g, \xi, \eta).
\]
We deduce
\[ 2 \left( g^{-1} \nabla_g \alpha \right) = 2 \alpha \cdot \nabla^*_g v^* + \alpha^*_g \cdot \text{Tr}_g v \]
= \alpha \cdot \left( 2 \nabla^*_g v^* + \nabla_g \Tr_g v \right).

We can compute now the first variation of the expression
\[ \text{div}_g^\Omega \alpha = g^{-1} \nabla_g \alpha - \alpha \cdot \nabla_g f. \]

We observe the identities
\[ 2 \left[ (D_g, \Omega \text{div}_g^\bullet) (v, V) \right] \alpha = -2(v^*_g g^{-1}) \nabla_g \alpha + 2g^{-1} \nabla_g \alpha \]
\[ - 2\alpha \cdot \left[ (D_g, \Omega \nabla_g f, \bullet) (v, V) \right] \]
\[ = -2\nabla_g \alpha (e_k, v^*_g e_k) + \alpha \cdot \left( 2\nabla^*_g v^* + \nabla_g \Tr_g v \right) \]
\[ - \alpha \cdot \left( \nabla_g (\text{Tr}_g v - 2V^*_\Omega) - 2v^*_g \cdot \nabla_g f \right). \]

We infer the required variation formula.

We can compute now a first variation formula for the double divergence operator \( \text{div}_g \nabla^\alpha_g \). We observe first the trivial identity
\[ \left[ D_g, \Omega \left( \text{div}_g^\bullet, \nabla_g^\bullet \right) (v, V) \right] \]
\[ = \left[ (D_g, \Omega \text{div}_g^\bullet) (v, V) \right] \nabla^\alpha_g v \]
\[ + \text{div}_g^\Omega \left\{ \left[ (D_g, \Omega \nabla_g^\bullet) (v, V) \right] v \right\}, \]

and we explicit the last term:
\[ 2 \text{div}_g^\Omega \left\{ \left[ (D_g, \Omega \nabla_g^\bullet) (v, V) \right] v \right\} \]
\[ = e_l \left[ 2\nabla_g v (e_k, v^*_g e_k, e_l) + \nabla_g v (e_l, v^*_g e_k, e_k) - 2v (\nabla^*_g v^*_g + V^*_\Omega, e_l) \right] \]
\[ - 2\nabla_g v (e_k, v^*_g e_k, \nabla_g f) - \nabla_g v \left( \nabla_g f, v^*_g e_k, e_k \right) + 2v (\nabla^*_g v^*_g + V^*_\Omega, \nabla_g f). \]
Developing further we obtain

\[
2 \text{div}^\Omega_g \left\{ \left[ (D_{g,\Omega} \nabla^\ast) (v, V) \right] v \right\}
\]

\[
= 2g(\nabla_{g,e_l} \nabla_{g,e_k} v_g^* \cdot v_g^* e_k, e_l) + 2g(\nabla_{g,e_l} v_g^* \cdot \nabla_{g,e_l} v_g^* e_k, e_l)
\]

\[
+ \nabla^2_{g,e_l,e_l} v (v_g^* e_k, e_k) + g(\nabla_{g,e_l} v_g^* \cdot \nabla_{g,e_l} v_g^* e_k, e_k)
\]

\[
- 2\nabla_{g,e_l} v (e_l, \nabla^\ast \Omega_g v_g^* + V_{\Omega}^* ) - 2v \left( \nabla_{g,e_l} (\nabla^\ast \Omega_g v_g^* + V_{\Omega}^* ) , e_l \right)
\]

\[
- 2g(\nabla_{g,e_k} v_g^* \cdot v_g^* e_k, \nabla_{g,f} v_g^* ) - \nabla_{g,f} v (\nabla_{g,f} v_g^* , v_g^* e_k ) + 2v \left( \nabla^\ast \Omega_g v_g^* + V_{\Omega}^* , \nabla_{g,f} \right)
\]

\[
= 2g(v_g^* e_k, \nabla_{g,e_l} \nabla_{g,e_l} v_g^* e_l) + 2g(\nabla_{g,e_l} v_g^* e_k, \nabla_{g,e_l} v_g^* e_l)
\]

\[
- \Delta^\Omega_g v (v_g^* e_k, e_k) + g(\nabla_{g,e_l} v_g^* e_k, \nabla_{g,e_l} v_g^* e_k)
\]

\[
+ 2\nabla^\ast \Omega_g v \cdot (\nabla^\ast \Omega_g v_g^* + V_{\Omega}^* ) - 2g \left( \nabla_{g,e_l} (\nabla^\ast \Omega_g v_g^* + V_{\Omega}^* ) , v_g^* e_l \right)
\]

\[
- 2g(v_g^* e_k, \nabla_{g,e_k} v_g^* \cdot \nabla_{g,f} )
\]

If we set

\[
\widehat{\nabla_{g} v_{g}^*} (\xi, \eta) := \nabla_{g} v_{g}^* (\eta, \xi),
\]

then the last expression writes as

\[
2 \text{div}^\Omega_g \left\{ \left[ (D_{g,\Omega} \nabla^\ast) (v, V) \right] v \right\}
\]

\[
= 2g \left( v_{g}^* e_k, \nabla_{g,e_l} \nabla_{g,e_l} v_{g}^* (e_l, e_k) \right) + 2g \left( \nabla_{g} v_{g}^* (e_l, e_k), \nabla_{g} v_{g}^* (e_l, e_k) \right)
\]

\[
- g \left( \Delta^\Omega_g v_{g}^* \cdot v_{g}^* e_k, e_k \right) + |\nabla_{g} v_{g}^* |^2_g
\]

\[
+ 2\nabla_{g} v_{g}^* \cdot (\nabla^\ast \Omega_g v_{g}^* + V_{\Omega}^* ) - 2g \left( \nabla_{g} (\nabla^\ast \Omega_g v_{g}^* + V_{\Omega}^* ) , v_{g}^* \right)_g
\]

\[
- 2g \left( v_{g}^* e_k, \nabla_{g} v_{g}^* (\nabla_{g,f} e_k) \right)
\]
We infer the formula
\[
\text{div}_g \left\{ \left[ (D_{g,\Omega} \nabla^\ast) (v, V) \right] v \right\} = - \frac{1}{4} \Delta^\Omega_g |v|^2_g \\
- \left\langle \nabla^\ast v^g, v^*_g \right\rangle_g + \left\langle \nabla^\ast v^g, \nabla v^*_g \right\rangle_g \\
+ \nabla^\ast v^g \cdot (\nabla^\ast v^g + \nabla^\ast V^g_{\Omega}) \\
- \left\langle \nabla \left( \nabla^\ast v^g + \nabla^\ast V^g_{\Omega} \right), v^*_g \right\rangle_g.
\]

We obtain in conclusion the variation identity
\[
\left[ D_{g,\Omega} (\text{div}^\ast \nabla^\ast) (v, V) \right] v = - \frac{1}{4} \Delta^\Omega_g |v|^2_g \\
- \left\langle \nabla^\ast v^g, v^*_g \right\rangle_g + \left\langle \nabla^\ast v^g, \nabla v^*_g \right\rangle_g \\
+ 2 \nabla^\ast v^g \cdot (\nabla^\ast v^g + \nabla^\ast V^g_{\Omega}) \\
- \left\langle \nabla \left( 2 \nabla^\ast v^g + \nabla^\ast V^g_{\Omega} \right), v^*_g \right\rangle_g.
\] (9)

4 The second variation of Perelman’s $H$ map

Lemma 3 The Hessian form $\nabla_G DH(g, \Omega)$ of Perelman’s map 
$(g, \Omega) \in \mathcal{M} \times V_1 \mapsto H_{g,\Omega}$, with respect to the pseudo-Riemannian structure 
$G$ at the point $(g, \Omega) \in \mathcal{M} \times V_1$ in arbitrary directions $(v, V)$ is given by the expression
\[
2 \nabla_G DH(g, \Omega)(v, V; v, V) = - \frac{1}{2} \left( c^2_g v, v \right)_{\Omega} - \Delta^\Omega_g \left[ \frac{1}{4} |v|^2_g + (V^g_{\Omega})^2 \right] \\
+ \frac{1}{2} |v|^2_g + (V^g_{\Omega})^2 - \frac{1}{2} G_{g,\Omega} (v, V; v, V) \\
- 2 \left| \nabla^\ast v^g + \nabla^\ast V^g_{\Omega} \right|^2_g \\
+ \left\langle \nabla \left( 2 \nabla^\ast v^g + 3 \nabla^\ast V^g_{\Omega} \right), v^*_g \right\rangle_g \\
+ \left\langle \nabla^\ast v^g, \nabla^\ast v^g + 2 \nabla^\ast V^g_{\Omega} \right\rangle_g \\
+ V^g_{\Omega} \left( \text{div}^\Omega \nabla^\ast v^g + \left\langle v, h_{g,\Omega} \right\rangle \right).
\]
Using the first variation formula for $H$ expressions covariant derivative $\nabla_G$.

Using the identity (9) we obtain $g$ and with arbitrary speed $\langle \dot{\theta}, \dot{\Theta} \rangle$. We show in [Pal2] that the $G$-covariant derivative $\nabla_G$ of its speed, in the speed direction, is given by the expressions

$$\begin{align*}
(\theta_t, \Theta_t) &= \nabla_G (\dot{\theta}_t, \dot{\Theta}_t)(\dot{g}_t, \dot{\Omega}_t), \\
\theta_t &= \ddot{\theta}_t + \dot{\theta}_t \left( \dot{\Omega}_t^* - \dot{g}_t^* \right), \\
\Theta_t &= \ddot{\Theta}_t + \frac{1}{4} \left[ |\dot{\theta}_t|^2 - 2(\dot{\Omega}_t^*)^2 - G_{\theta, \Omega} (\dot{g}_t, \dot{\Omega}_t; \dot{g}_t, \dot{\Omega}_t) \right] \Omega_t.
\end{align*}$$

Then

$$\nabla_G DH(g_t, \Omega_t)(\dot{g}_t, \dot{\Omega}_t; \dot{g}_t, \dot{\Omega}_t) = \frac{d^2}{dt^2} H(g_t, \Omega_t) - D_{g_t, \Omega_t} H(\theta_t, \Theta_t).$$

Using the first variation formula for $H$ in [Pal2] we obtain the equalities

$$\begin{align*}
2\nabla_G DH(g_t, \Omega_t)(\dot{g}_t, \dot{\Omega}_t; \dot{g}_t, \dot{\Omega}_t)
&= \frac{d}{dt} \left[ (\Delta_{g_t|\theta_t}^t - 2\|)\dot{\Omega}_t^* - \text{div}_{\dot{\theta}_t} \left( \nabla_{\theta_t}^t \dot{g}_t + \dot{\theta}_t^* \right) - \langle \dot{g}_t, \dot{h}_t \rangle_{g_t} \right] \\
& \quad - (\Delta_{g_t|\theta_t}^t - 2\|)\Theta_t^* + \text{div}_{\dot{\theta}_t} \left( \nabla_{\theta_t}^t \theta_t + \dot{\theta}_t^* \right) + \langle \theta_t, \dot{h}_t \rangle_{g_t}.
\end{align*}$$

Using the identity (9) we obtain

$$\begin{align*}
2\nabla_G DH(g_t, \Omega_t)(\dot{g}_t, \dot{\Omega}_t; \dot{g}_t, \dot{\Omega}_t)
&= 2(\Delta_{g_t|\theta_t}^t - \|) \left( \frac{d}{dt} \dot{\Omega}_t^* - \dot{\Theta}_t^* \right) \\
& \quad + 2(\nabla_{\dot{\theta}_t} \dot{\bar{\Omega}}_t, \dot{g}_t)_{g_t} - 2(\nabla_{\dot{\theta}_t} \dot{\bar{\Omega}}_t^*, \dot{\bar{\Omega}}_t)_{g_t} \\
& \quad + \text{div}_{\dot{\theta}_t} \left( \nabla_{\dot{\theta}_t} \dot{\bar{\Omega}}_t \right) (\theta_t - \dot{g}_t) \\
& \quad + \frac{1}{4} \Delta_{g_t|\theta_t}^t |\dot{\theta}_t|^2_{g_t} + \langle \nabla_{\nabla_{\theta_t}^t} \nabla_{\dot{g}_t} \dot{\theta}_t, \dot{g}_t \rangle_{g_t} - \langle \nabla_{\dot{g}_t} \dot{\theta}_t, \nabla_{\theta_t} \dot{g}_t \rangle_{g_t} \\
& \quad - 2g_{\nabla_{\theta_t}^t} \dot{g}_t \left( \nabla_{\theta_t}^t \dot{g}_t^* + \nabla_{\theta_t} \dot{\theta}_t^* \right) - \langle \nabla_{\theta_t}^t \left( 2\nabla_{\theta_t}^t \dot{g}_t + \nabla_{\theta_t} \dot{\theta}_t \right) \dot{g}_t \rangle_{g_t} \\
& \quad + \text{Tr}_R \left[ \left( \theta_t^* - \frac{d}{dt} \dot{\theta}_t^* \right) \bar{h}_t^* - \dot{g}_t^* \left( \dot{h}_t^* - \dot{\theta}_t^* \right) \right].
\end{align*}$$
Rearranging the previous expression, we obtain

\[ 2 \nabla_G DH(g_t, \Omega_t)(\dot{g}_t, \dot{\Omega}_t) = 2(\Delta^\partial_{g_t} - \Pi) \left( \frac{d}{dt} \dot{\Omega}_t^* - \Theta_t^* \right) + \frac{1}{4} \Delta^\partial_{g_t} |\dot{g}_t|^2_{g_t}, \]

\[ + \langle \nabla_{g_t} d\dot{\Omega}_t^*, \dot{g}_t \rangle_{g_t} - 2|\nabla^\partial_{g_t} \dot{g}_t^* + \nabla_{g_t} \dot{\Omega}_t^*|_{g_t}, \]

\[ + \text{div}_{g_t} \nabla^\partial_{g_t} (\theta_t - \dot{g}_t) \]

\[ + \left( \nabla^\partial_{g_t} \nabla_{g_t} \dot{g}_t^* \right)_g - \langle \nabla_{g_t} \dot{g}_t^* \nabla_{g_t} \dot{g}_t \rangle_{g_t}, \]

\[ + 2 \langle \nabla_{g_t} \left( \nabla^\partial_{g_t} \dot{g}_t^* + \nabla_{g_t} \dot{\Omega}_t^* \right) \cdot \dot{g}_t \rangle_{g_t}, \]

\[ + \text{Tr}_R \left[ \left( \theta_t^* - \frac{d}{dt} \dot{g}_t^* \right) h_t^* - \dot{g}_t^* \left( \dot{h}_t^* - \dot{g}_t^* h_t^* \right) \right]. \]

Using the expression of \( \theta_t \), we develop the term

\[ \text{div}_{g_t} \nabla^\partial_{g_t} (\theta_t - \dot{g}_t) = \text{div}_{g_t} \nabla^\partial_{g_t} \left[ \dot{\Omega}_t^* \dot{g}_t - (\dot{g}_t^*)^2 \right]. \]

For this purpose we remind a few elementary divergence type identities. For any smooth, function \( u \), vector field \( \xi \) and endomorphism section \( A \) of \( T_X \) holds the identities

\[ \nabla^\partial_{g} (uA) = -A \cdot \nabla_{g} u + u \nabla^\partial_{g} A, \]

\[ \text{div}^\Omega (u\xi) = \langle \nabla_{g} u, \xi \rangle_g + u \text{div}^\Omega \xi, \]

\[ \nabla^\partial_{g} A^2 = -\text{Tr}_{g} (\nabla_{g} A \cdot A) + A \nabla^\partial_{g} A. \]

Furthermore if \( A \) is \( g \)-symmetric then holds also the formulas

\[ \text{div}^\Omega (A \cdot \xi) = -\langle \nabla^\partial_{g} A, \xi \rangle_g + (A, \nabla_{g} \xi)_g, \] \hfill (10)

\[ \text{div}^\Omega \text{Tr}_{g} (\nabla_{g} A \cdot A) = -\langle \nabla^\partial_{g} \nabla_{g} A, A \rangle_g + \langle \nabla_{g} A, \nabla_{g} A \rangle_g. \] \hfill (11)
Pluging this identity in the last expression of the Hessian of $H$ we obtain the equalities

$$
\text{div}^\Omega_t \nabla^\Theta^*_{g_t} \left[ \Omega_t^* g_t^* - (g_t^*)^2 \right] = \text{div}^\Omega_t \left[ -g_t^* \nabla_{g_t} \Omega_t^* + \Omega_t^* \nabla^{g_t^*} \hat{g}_t^* \right] 
+ \text{div}^\Omega_t \left[ \text{Tr}_{g_t} \left( \nabla_{g_t} \hat{g}_t^* \cdot \hat{g}_t^* \right) - \hat{g}_t^* \nabla^{g_t^*} \hat{g}_t^* \right] 
= - \left\langle \nabla_{g_t} d(\Omega_t^*; \hat{g}_t^*), g_t^* \right\rangle_{g_t} + 2 \left\langle \nabla^{g_t^*} \hat{g}_t^*, \nabla_{g_t} \Omega_t^* g_t^* \right\rangle_{g_t} 
+ \hat{\Omega}_t^* \text{div}^\Omega_t \nabla^{g_t^*} \hat{g}_t^* 
- \left\langle \nabla^{g_t^*} \hat{g}_t^* \nabla_{g_t} \hat{g}_t^*, g_t^* \right\rangle_{g_t} 
+ \left\langle \nabla_{g_t} \left( \nabla^{g_t^*} \hat{g}_t^* + \nabla_{g_t} \hat{\Omega}_t^* \right), \hat{g}_t^* \right\rangle_{g_t} 
+ \left\langle \nabla^{g_t^*} \hat{g}_t^*, \nabla^{g_t^*} \hat{g}_t^* + 2 \nabla_{g_t} \hat{\Omega}_t^* \right\rangle_{g_t} 
+ \hat{\Omega}_t^* \text{div}^\Omega_t \nabla^{g_t^*} \hat{g}_t^* 
- \left\langle \nabla^{g_t^*} \nabla_{g_t} \hat{g}_t^* \cdot \hat{g}_t^*, g_t^* \right\rangle_{g_t} 
+ \left\langle \nabla_{g_t} \hat{g}_t^* \nabla_{g_t} \hat{g}_t^* + \nabla_{g_t} \hat{g}_t^*, \hat{g}_t^* \right\rangle_{g_t}.
$$

Plugging this identity in the last expression of the Hessian of $H$ we obtain

$$
2 \nabla_D DH(g_t, \Omega_t)(\hat{g}_t, \hat{\Omega}_t; \hat{g}_t, \hat{\Omega}_t) 
= 2(\Delta^\Omega_t - \Pi) \left( \frac{d}{dt} \hat{\Omega}_t^* - \hat{\Theta}_t^* \right) + \frac{1}{4} \Delta^\Omega_t |\hat{g}_t^*|^2_{g_t} 
+ \left\langle \nabla_{g_t} d(\hat{\Omega}_t^*; \hat{g}_t^*), g_t^* \right\rangle_{g_t} - 2 \left\langle \nabla^{g_t^*} \hat{g}_t^*, \nabla_{g_t} \hat{\Omega}_t^* \right\rangle_{g_t} 
+ \left\langle \nabla_{g_t} \left( \nabla^{g_t^*} \hat{g}_t^* + \nabla_{g_t} \hat{\Omega}_t^* \right), \hat{g}_t^* \right\rangle_{g_t} 
+ \left\langle \nabla^{g_t^*} \hat{g}_t^*, \nabla^{g_t^*} \hat{g}_t^* + 2 \nabla_{g_t} \hat{\Omega}_t^* \right\rangle_{g_t} 
+ \hat{\Omega}_t^* \left( \hat{g}_t, \hat{b}_t \right)_{g_t} - \frac{1}{2} \left\langle L^\Omega_{g_t} \hat{g}_t - L^{g_t^*} \hat{g}_t^* - \nabla_{g_t} \hat{g}_t, g_t^* \right\rangle_{g_t}.
$$

For readers convenience we show (10) and (11) in the appendix. Using the previous formulas we obtain the equalities
thanks to the variation formula of $h$ in [Pa2]. Rearranging the previous expression, we infer

$$2\nabla_G DH(g_t, \Omega_t)(\dot{g}_t, \dot{\Omega}_t; \dot{g}_t, \dot{\Omega}_t)$$

$$= 2(\Delta_{\Omega_t} - \mathbb{I}) \left( \frac{d}{dt} \dot{\Omega}_t^* - \Theta_t^* \right) + \frac{1}{4} \Delta_{\Omega_t} |\dot{g}_t|_{g_t}^2$$

$$- 2|\nabla_{g_t}^* \dot{g}_t^* + \nabla_{g_t}^* \dot{\Omega}_t^*_{g_t}|^2$$

$$+ \left\langle \nabla_{g_t} \left( 2\nabla_{g_t}^* \dot{g}_t^* + 3\nabla_{g_t} \dot{\Omega}_t^* \right) , \dot{g}_t^* \right\rangle_{g_t}$$

$$+ \left\langle \nabla_{g_t}^* \dot{g}_t^* , \nabla_{g_t}^* \dot{\Omega}_t^* \right\rangle_{g_t} + 2 \nabla_{g_t} \dot{\Omega}_t^*_{g_t}$$

$$+ \dot{\Omega}_t^* \left( \text{div}^{\Omega} \nabla_{g_t}^* \dot{g}_t^* + \langle \dot{g}_t, h_t \rangle_{g_t} \right) - \frac{1}{2} \left\langle \mathcal{L}_{g_t} \dot{g}_t^* , \dot{g}_t^* \right\rangle_{g_t}.$$  

Then the conclusion follows from the expression of $\Theta_t$. □

In [Pa2] we show that the space $G$-orthogonal to the tangent to the orbit of a point $(g, \Omega) \in \mathcal{M} \times \mathcal{V}_1$, under the action of the identity component of the diffeomorphism group, is

$$\mathbb{F}_{g,\Omega} := \{(v, V) \in T_{\mathcal{M} \times \mathcal{V}_1} \mid \nabla_g^* v_g^* + \nabla_{\Omega} V_{\Omega}^* = 0\}.$$

**Corollary 1** The Hessian form $\nabla_G DH(g, \Omega)$ of Perelman’s map $(g, \Omega) \in \mathcal{M} \times \mathcal{V}_1 \mapsto H_{g,\Omega}$, with respect to the pseudo-Riemannian structure $G$ at the point $(g, \Omega) \in \mathcal{M} \times \mathcal{V}_1$ in arbitrary directions $(v, V) \in \mathbb{F}_{g,\Omega}$, is given by the expression

$$2\nabla_G DH(g, \Omega)(v, V; v, V)$$

$$= -\frac{1}{2} \left\langle \langle \mathcal{L}_{g}^* + 2\nabla_g \nabla_{g}^* \rangle v , v \right\rangle_g$$

$$- \frac{1}{2} (\Delta_g - 2\mathbb{I}) \left[ \frac{1}{2} |v|_g^2 + (V_{\Omega}^*)^2 - \frac{1}{2} G_{g,\Omega} (v, v; v, v) \right]$$

$$+ V_{\Omega}^* \langle v, h_{g,\Omega} \rangle_g.$$  

5 Application of the weighted Bochner identity

We observe that the formal adjoint of the $\mathcal{D}_{T_X,J}$ operator with respect to the hermitian product

$$\langle \cdot , \cdot \rangle_{\omega,\Omega} := \int_X \langle \cdot , \cdot \rangle_{\omega} \Omega,$$

(12)
is the operator
\[
\overline{\partial}^{g,\Omega}_{T_X,J} := e^f \overline{\partial}^{g}_{T_X,J} (e^{-f} \bullet).
\]
With this notation, we define the anti-holomorphic Ω-Hodge-Witten Laplacian operator acting on \(T_X\)-valued \(q\)-forms as
\[
\Delta^{\Omega,-J}_{T_X,g} := \frac{1}{q} \overline{\partial}^{\Omega}_{T_X,J} \overline{\partial}^{g,\Omega}_{T_X,J} T_X,J \big( e^{-f} \bullet \big)
\]
with the usual convention \(\infty \cdot 0 = 0\), and the functorial convention on the scalar product in \([\text{Pal1}]\). We will omit the symbol Ω in the Hodge-Witten Laplacian operator, when Ω = Cst \(dV_g\). We define the vector space
\[
H^0_{\Omega,1} g, \Omega (T_X,J) := \text{Ker} \Delta^{\Omega,-J}_{T_X,g} \cap C^\infty (X, T^*_X, -J \otimes T_X,J).
\]
It has been showed in \([\text{Pal2}]\), that for any smooth \(J\)-anti-linear endomorphism section \(A\) of the tangent bundle holds the fundamental Bochner type formula
\[
\mathcal{L}^0_g A = 2 \Delta^{J}_{T_X,g} A + [\text{Ric}^*(g), A] + \nabla_g f - \nabla_g A. \tag{13}
\]
We observe that for bidegree reasons holds the equalities
\[
\overline{\partial}^{g,\Omega}_{T_X,J} A = \nabla^*_g A
\]
\[
= \nabla^*_g A + A \nabla_g f
\]
\[
= \overline{\partial}^{g}_{T_X,J} A + A \nabla_g f.
\]
Using the last equality we obtain the expression
\[
\overline{\partial}^{g,\Omega}_{T_X,J} A = \overline{\partial}^{g}_{T_X,J} \nabla^*_g A + \overline{\partial}^{0,1}_{g,J} A \nabla_g f + A \overline{\partial}^{g}_{T_X,J} \nabla_g f.
\]
We observe indeed
\[
2 \overline{\partial}^{g}_{T_X,J} (A \nabla_g f) = \nabla_g (A \nabla_g f) + J \nabla_g, J. (A \nabla_g f)
\]
\[
= \nabla_g A \nabla_g f + A \nabla_g^2 f + J \nabla_g, J. A \nabla_g f + J A \nabla_g, J \nabla_g f
\]
\[
= 2 \nabla_{g,J}^0 A \nabla_g f + A (\nabla_g^2 f - J \nabla_g, J \nabla_g f)
\]
\[
= 2 \nabla_{g,J}^0 A \nabla_g f + 2 A \overline{\partial}^{g}_{T_X,J} \nabla_g f.
\]
For bidegree reasons holds also the identities
\[
\frac{1}{2} \overline{\partial}^{g,\Omega}_{T_X,J} \overline{\partial}^{g}_{T_X,J} A = \frac{1}{2} \nabla^*_g \overline{\partial}^{g}_{T_X,J} A
\]
\[
= \nabla^*_g \overline{\partial}^{g}_{T_X,J} A
\]
\[
= \nabla^*_g \overline{\partial}^{g}_{T_X,J} A + \nabla_g f - \overline{\partial}^{g}_{T_X,J} A.
\]
Thus
\[
\frac{1}{2} \partial_{T X,J}^* \partial_{T X,J} A = \frac{1}{2} \nabla^*_{T X} \partial_{T X,J} A + \nabla_g f - \partial_{T X,J} A
\]
Combining the identities obtained so far we deduce the expression
\[
\Delta_{T X,g} J T X, \Omega = \Delta_{T X,g} A + \nabla_g f - \nabla^0_{g,J} A - \nabla^0_{g,J} \nabla_g f.
\]
(14)
Plugging this in the fundamental identity (13) we obtain the equalities
\[
\mathcal{L}_g^\Omega A = 2 \Delta_{T X,g} J T X, \Omega + [\text{Ric}^*(g), A] - 2 A \partial^g_{T X,J} \nabla_g f
\]
Thus, if \( A \in \mathcal{H}^{0,1}_{g,J} (T X, J) \), then holds the stability identity
\[
\langle \mathcal{L}_g^\Omega A, A \rangle_g = -2 \langle \nabla^2_g f, A^2 \rangle_g + \langle J \nabla_g f - \nabla_g A, J A \rangle_g.
\]
(15)
6 Variations of \( \omega \)-compatible complex structures
Let \((X, J, g, \omega)\) be a Fano manifold such that \( \omega = \text{Ric}_J (\Omega) \), with \( \Omega \in \mathcal{V}_1 \) and let \((J_t)_t \subset J_\omega\) be a smooth curve such that \( J_0 = J \). We differentiate the definition \( g_t := -\omega J_t \). We obtain \( \dot{g}_t = -J_t \dot{J}_t \) and \( \ddot{g}_t = -J_t \ddot{J}_t \). On the other hand, deriving twice the condition \( J_t^2 = -\mathbb{I} \), we obtain \(-\langle J_t, \dot{J}_t \rangle^{1,0}_t = J_t^2 \) and thus \( (\ddot{g}_t)^{1,0}_t = (\ddot{g}_t)^2 \). The latter gives
\[
(\ddot{g}_t)^{1,0}_t = \ddot{g}_t - (\dot{g}_t)^2 = \frac{d}{dt} \ddot{g}_t.
\]
For any endomorphism \( A \) of the tangent bundle and for any bilinear form \( B \) over it we define the contraction operation \( A \cdot B := \text{Alt} (B \circ A) \), where \( \text{Alt} \) is the alternating operator and the composition operator \( \circ \) act on the first entry of \( B \). Let \( N_J \) be the Nijenhuis tensor of an arbitrary almost complex structure \( J \). Then the general formula
\[
\frac{d}{dt} N_{J_t} = \ddot{J}_t - N_{J_t} - \dot{J}_t N_{J_t} + \partial_{T X,J_t} \dot{J}_t.
\]
(see a computation in [Pa1]), implies \( \partial_{TX,J_t} \hat{J}_t \equiv 0 \), in our case. Thus time deriving the identity

\[
\frac{d}{dt} \partial_{TX,J_t} \hat{g}_t^* = 0,
\]

we obtain the property

\[
\partial_{TX,J_t} \frac{d}{dt} \hat{g}_t^* = \hat{g}_t^* - \nabla_{g_t^*,J_t} \hat{g}_t^*.
\]

(16)

Indeed we prove the variation formula

\[
\left( \frac{d}{dt} \partial_{TX,J_t} \right) \hat{g}_t^* = -\hat{g}_t^* - \nabla_{g_t^*,J_t} \hat{g}_t^*.
\]

For this purpose we expand the derivative of \( \partial_{TX,J_t} \) acting on a smooth endomorphism section \( A \) of \( TX \). We obtain

\[
2 \left[ \left( \frac{d}{dt} \partial_{TX,J_t} \right) A \right] (\xi,\eta) = 2 \frac{d}{dt} \text{Alt} \left[ \nabla_{g_t,J_t} A (\xi,\eta) \right] \\
= \text{Alt} \frac{d}{dt} [\nabla_{g_t} A (\xi,\eta) + J_t \nabla_{g_t} A (J_t \xi,\eta)] \\
= \text{Alt} [\nabla_{g_t} A (\xi,\eta) + J_t \nabla_{g_t} A (J_t \xi,\eta)] \\
+ \text{Alt} [J_t \nabla_{g_t} A (J_t \xi,\eta) + J_t \nabla_{g_t} A \left( J_t^* \xi,\eta \right)].
\]

Using the variation formula

\[
\nabla_{g_t} A (\xi,\eta) = \nabla_{g_t} (\xi,A\eta) - A \nabla_{g_t} (\xi,\eta),
\]

and the fact that the bilinear form \( \nabla_{g_t} \) is symmetric we deduce the formula

\[
2 \left[ \left( \frac{d}{dt} \partial_{TX,J_t} \right) A \right] (\xi,\eta) \\
= \text{Alt} [\nabla_{g_t} (\xi,A\eta) + J_t \nabla_{g_t} A (J_t \xi,\eta)] \\
+ \text{Alt} [J_t \nabla_{g_t} (J_t \xi,\eta) - J_t A \nabla_{g_t} (J_t \xi,\eta) + J_t \nabla_{g_t} A \left( J_t^* \xi,\eta \right)].
\]

We remind now (see [Pa1]), that time deriving the Kähler condition \( \nabla_{g_t} J_t \equiv 0 \), we obtain the identity

\[
\nabla_{g_t}(\eta,\xi) + J_t \nabla_{g_t} (J_t \eta,\xi) + J_t \nabla_{g_t} J_t (\xi,\eta) = 0,
\]

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Using this in the previous formula with \( A = \dot{g}_t^* = -J_t \dot{J}_t \) we obtain
\[
2 \left[ \left( \frac{d}{dt} \mathcal{J}_{\gamma_t,J_t} \right) \dot{g}_t^* \right] (\xi, \eta)
\]

\[
= \text{Alt} \left[ -J_t \nabla_{g_t} \dot{J}_t (\dot{g}_t^* \eta, \xi) - \dot{g}_t^* J_t \nabla_{g_t} \dot{J}_t (\eta, \xi) \right]
\]

\[
+ \text{Alt} \left[ \dot{J}_t \nabla_{g_t} \dot{g}_t^* (J_t \xi, \eta) + J_t \nabla_{g_t} \dot{g}_t^* \left( \dot{J}_t \xi, \eta \right) \right]
\]

\[
= \text{Alt} \left[ \nabla_{g_t} \dot{g}_t^* (\dot{g}_t^* \eta, \xi) + \dot{J}_t \nabla_{g_t} \dot{g}_t^* (J_t \xi, \eta) + J_t \nabla_{g_t} \dot{g}_t^* \left( \dot{J}_t \xi, \eta \right) \right]
\]

\[= \dot{g}_t^* \partial_{\mathcal{J}_{\gamma_t,J_t}} \dot{g}_t^* (\xi, \eta),\]
and thus the required formula. The latter can also be obtained deriving the Maurer-Cartan equation, which writes in the Kähler case (see the appendix) as
\[
\mathcal{J}_{\gamma_t,J_t} \mu_t + \mu_t \nabla_{g_t}^1 \mu_t = 0,
\]
with \( \mu_t \) the Cayley transform of \( J_t \) with respect to \( J \).

We remind now (see [Pal2]), that for any smooth family \((g_t, \Omega_t)_t \subset \mathcal{S}_\omega\), holds the identity
\[
\Delta_{\mathcal{J}_{\gamma_t,J_t}} \dot{g}_t^* = \left( \Delta_{\mathcal{J}_{\gamma_t,J_t}} \dot{g}_t^* \right)^T_{g_t},
\]
with \( J_t := -\omega^{-1} g_t \). The latter rewrites as
\[
\partial_{\mathcal{J}_{\gamma_t,J_t}} \nabla_{g_t}^* \dot{g}_t^* = \left( \partial_{\mathcal{J}_{\gamma_t,J_t}} \nabla_{g_t}^* \dot{g}_t^* \right)^T_{g_t}. \tag{17}
\]

\textbf{Lemma 4}  
For any smooth family \((g_t, \Omega_t)_t \subset \mathcal{S}_\omega\), with \((g, \Omega) = (g_0, \Omega_0)\) and \((\dot{g}_0, \dot{\Omega}_0) \in \mathbb{F}_g^J,\) holds the symmetry property
\[
\partial_{\mathcal{J}_{\gamma_t,J_t}} \nabla_{g_t}^* \frac{d}{dt} \dot{g}_t^* = \left( \partial_{\mathcal{J}_{\gamma_t,J_t}} \nabla_{g_t}^* \frac{d}{dt} \dot{g}_t^* \right)^T_{g_t} + \left[ \partial_{\mathcal{J}_{\gamma_t,J_t}} \nabla_{g_t}^* \dot{g}_t^* \right]. \tag{18}
\]

\textbf{Proof}  
Let \( A \) be a smooth \( g \)-symmetric endomorphism section of \( T_X \). Differentiating in the variables \((g, \Omega)\) the trivial identity \( \nabla^* \; A = g^{-1} \nabla^* (g A) \), we obtain
\[
\left( D_{g,\Omega} \nabla^* \right)(v, V) A = -v^* \nabla^* A + g^{-1} \left( D_{g,\Omega} \nabla^* \right)(v, V) (g A)
\]
\[+ \nabla^* \left( v^* A \right).\]
We observe now the identities

\[
M_g(v, v) = 2g Tr_g (\nabla_g v^*_g \cdot v^*_g) + \frac{1}{2} d|v^*_g|^2
\]

\[
= 2v^*_g \nabla_g^* v^*_g - 2g \nabla_g^* (v^*_g)^2 + \frac{1}{2} d|v^*_g|^2.
\]

Then using the variation formula (5) we infer the fundamental identity

\[
2 \left[ (D_{g, t}) \nabla^* (v, V) \right] v^*_g = \frac{1}{2} \nabla_g|v|^2 - 2v^*_g \cdot (\nabla_g^* v^*_g + \nabla V^*_g).
\] (19)

The variation formula for the \(\partial T X, J_t\)-operator acting on vector fields in lemma 1 of [Pal3] writes as

\[
2 \frac{d}{dt} \left( \partial T X, J_t \xi \right) = \xi - \nabla_g \dot{\xi} - \left[ \partial_{T X, J_t} \xi, \dot{g}_t^* \right] + \left[ \partial T X, J_t \xi, \dot{g}_t^* \right].
\]

Using this, the variation formula (19) and the assumption on the initial speed of the curve \((g_t, \Omega_t)\), we infer

\[
2 \frac{d}{dt} \left| t=0 \right| \left( \partial T X, J_t \nabla^*_g \Omega \right) \dot{g}_0^* = \nabla^*_g \Omega \dot{g}_0^* - \left[ \partial^g_{T X, J_t} \nabla^*_g \dot{g}_0^*, \dot{g}_t^* \right] + \left[ \partial^g_{T X, J_t} \nabla^*_g \dot{g}_0^*, \dot{g}_t^* \right].
\]

Using this equality, the elementary identity

\[
\frac{d}{dt} A^T_{g_0} = [A^T_{g_0}, \dot{g}_t^*],
\]

for arbitrary endomorphism section \(A\) of \(T X\) and time deriving the identity (17), we obtain the required conclusion. (Notice that the endomorphism section \(\partial^g_{T X, J_t} \nabla^*_g \dot{g}_0^*\) is \(g\)-symmetric thanks to the assumption \(\nabla^*_g \dot{g}_0^* = -\nabla_g \dot{\Omega}_0^*\).)

**Corollary 2** Let \((J_t)_t \subset J_\omega\) be a smooth curve such that \(\dot{J}_0 \in H_{g, \Omega}^{0, 1} (T X, \omega)\) then

\[
\nabla^*_g \left( \dot{J}_0 - \nabla^1_0, J_0 \right) = \left[ \nabla^*_g \left( \dot{J}_0 - \nabla^1_0, J_0 \right) \right]^T.
\]

**Proof** The identity (16) implies

\[
\partial T X, J_t \nabla^*_g \frac{d}{dt} \bigg|_{t=0} \dot{g}_t^* = \Delta_{T X, J_t}^g \frac{d}{dt} \bigg|_{t=0} \dot{g}_t^* - \nabla^*_g \left( g_0^* - \nabla^1_0 J_0 \right).
\]

Plugging this in the equality (18) and using the fact that the Laplace term is \(g\)-symmetric (see [Pal2]), we infer the required conclusion.
Lemma 5 Let \((X,J,g,\omega)\) be a Fano manifold such that \(\omega = \text{Ric}_J(\Omega)\), with \(\Omega \in \mathcal{V}_1\) and let \((\gamma_t)_t \subset \mathcal{J}_\omega\) be a smooth curve such that \(\gamma_0 = J\) and \(\nabla^*_{\omega}J_0 = 0\). Then there exists unique \((\psi, A_1) \in \Lambda_{g,J}^{\nabla,1} \oplus \mathcal{H}_{g,J}(T_X,J)\) such that

\[
\frac{d}{dt}\bigg|_{t=0} \psi_t^* + \nabla^* g \left( \Delta_{T_X,g}^{\Omega,-J} \right)^{-1} \left( J_0 - \nabla^*_{g,J} J_0 \right) = \nabla_{T_X,J} \nabla_{g,J} \psi + A_1.
\]

Proof The identity (16) implies

\[
\nabla_{T_X,J} \left[ \frac{d}{dt}\bigg|_{t=0} \psi_t^* + \nabla^* g \left( \Delta_{T_X,g}^{\Omega,-J} \right)^{-1} \left( J_0 - \nabla^*_{g,J} J_0 \right) \right] = 0.
\]

Moreover the endomorphism

\[
\frac{d}{dt}\bigg|_{t=0} \psi_t^* + \nabla^* g \left( \Delta_{T_X,g}^{\Omega,-J} \right)^{-1} \left( J_0 - \nabla^*_{g,J} J_0 \right)
= \frac{d}{dt}\bigg|_{t=0} \psi_t^* + \left( \Delta_{T_X,g}^{\Omega,-J} \right)^{-1} \nabla^* g \left( J_0 - \nabla^*_{g,J} J_0 \right),
\]

is \(g\)-symmetric thanks to corollary \([\text{Pal2}]\) lemma \(13\) in \([\text{Pal2}]\) and identity (14.7) in \([\text{Pal2}]\). By corollary \(3\) in \([\text{Pal2}]\), we infer the required conclusion. Notice that \((\psi, A_1)\) is uniquely determined by \(J_0\) and \(\tilde{J}_0\).

\[\square\]

7 Proof of theorem \([1]\)

For any smooth family \((g_t,\Omega_t)_{t \in \mathcal{S}_\omega}\), with \((g_0,\Omega_0) = (g,\Omega)\), we consider the smooth curve \(t \mapsto \gamma_t := H_{g_t,\Omega_t}/\Omega_t \in C^\infty(X,\mathbb{R})_0\). Then \((g_t,\Omega_t) \equiv (J_t,\omega)_t\) is a family of Kähler-Ricci solitons if and only if \(\gamma_t \equiv 0\). We assume this identity and we notice that \(0 = \gamma_0 = D_{g,\Omega} H(g_0,\Omega_0)\). We write

\[
\dot{g}_t^* = -J \dot{J}_0 = \nabla_{T_X,J} \nabla_{g,J} \theta + 2A,
\]

with \((\theta, A) \in \Lambda_{g,J}^{\nabla,1} \oplus \mathcal{H}_{g,J}(T_X,J)\). The properties of the first variation of \(H\) imply \(\theta \in \mathcal{O}_{J,\Omega}\). According to the isomorphism \(\tau\) in remark \([1]\) we pick the unique \(u \in \Lambda_{g,J}^{\nabla,1}\) such that \(\theta - iu \in \Lambda_{g,J}^{\nabla,1}\) and we consider the one parameter subgroup of \(\omega\)-symplectomorphisms \((\Psi_t)_t\), \(\Psi_0 = \text{id}_X\), given by \(2\Psi_t = -(\omega^{-1} du) \circ \Psi_t\). Then \((\Psi^*_t J_t,\omega)_t\) is still a family of Kähler-Ricci solitons and

\[
\frac{d}{dt}\bigg|_{t=0} \Psi_t^* J_t = \dot{J}_0 - \frac{1}{2} L_{\omega^{-1} du} J
= \dot{\nabla}_{T_X,J} \nabla_{g,J} (\theta - iu) + 2JA
= 2JA.
\]

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Thus we can assume, without loss of generality in the statement of the theorem, that the family of Kähler-Ricci solitons \((J_t, \omega)_t\) satisfies \(J_0 \in \mathcal{H}^{0,1}_{g, \Omega}(T_{X,J})\). Using this assumption, we explicit the second variation of the map \((g, \Omega) \mapsto H_{g, \Omega}\). The fact that \(\dot{g}_0 = 2A\), implies \(\dot{\Omega}_0 = 0\), thanks to the equations defining the space \(T_{g, \Omega}^{J}\). Thus

\[
2 \frac{d^2}{dt^2} \Big|_{t=0} H_{g, t, \Omega} = 2 \nabla G D H (g, \Omega)(\dot{g}_0, 0; \dot{g}_0, 0) + 2 D_{g, \Omega} H (\xi, \Xi),
\]

with

\[
\begin{align*}
\xi_g^* &:= \frac{d}{dt} \big|_{t=0} \dot{g}_t^*, \\
\Xi_{\Omega}^* &:= \frac{d}{dt} \big|_{t=0} \dot{\Omega}_t^* + \frac{1}{4} |\dot{g}_0|_g^2 - \frac{1}{4} G_{g, \Omega}(\dot{g}_0, 0; \dot{g}_0, 0).
\end{align*}
\]

Using the fact that \((g, \Omega)\) is a soliton and the first and second variation formulas for Perelman’s functions \(H\) (in [Pal2] and corollary [1]), and \(W\) (in [Pal2]), we infer

\[
2 \frac{d^2}{dt^2} \big|_{t=0} H_{g, t, \Omega} = \nabla G D (2H - W) (g, \Omega)(\dot{g}_0, 0; \dot{g}_0, 0) + 2 D_{g, \Omega} H (\xi, \Xi)
\]

\[
= -2 \left( \mathcal{L}_{g}^{\Omega} A, A \right)_g - \left( \Delta_{g}^{\Omega} - 2 \| \right) |A|_g^2 - 2 \int_X |A|_g^2 \Omega
\]

\[
+ 2 \int_X |A|_g^2 F \Omega + 2(\Delta_{g}^{\Omega} - \| \Xi_{\Omega}^* - \text{div} \nabla^{* u} \xi_g^* \nabla g, A) + \Delta_{g}^{\Omega} |A|_g^2
\]

\[
+ 2(\Delta_{g}^{\Omega} - \| \frac{d}{dt} \big|_{t=0} \dot{\Omega}_t^* - \text{div} \nabla^{* u} \frac{d}{dt} \big|_{t=0} \dot{g}_t^*).
\]

Using lemma [3] and the weighted complex Bochner formula (13.9) in [Pal2], we obtain

\[
\nabla^{* u} \frac{d}{dt} \big|_{t=0} \dot{g}_t = \overline{\partial}_{T_{X,J}} \overline{\partial}_{T_{X,J}} \nabla_{g, J} \psi = \frac{1}{2} \nabla_{g, J} (\Delta_{g, J}^{\Omega} - 2 \| \psi),
\]

and thus

\[
- \text{div} \nabla^{* u} \frac{d}{dt} \big|_{t=0} \dot{g}_t = \frac{1}{2} \Delta_{g}^{\Omega} R_{\psi} + \frac{1}{2} B_{g, J}^{\Omega} I_{\psi},
\]

\[
R_{\psi} := \text{Re} \left[ (\Delta_{g, J}^{\Omega} - 2 \| \psi) \right],
\]

\[
I_{\psi} := \text{Im} \left[ (\Delta_{g, J}^{\Omega} - 2 \| \psi) \right].
\]
(Here we use the notation $z = \text{Re } z + i \text{Im } z$, for any $z \in \mathbb{C}$). Differentiating the tangential identity $2dd^c J^* \dot{\Omega}^*_t = -d \left[ \nabla_{g_t}^{\ast \Omega_t} \dot{g}_t^* - \omega \right]$, we obtain,

$$2dd^c \left. \frac{d}{dt} \dot{\Omega}^*_t \right|_{t=0} = -d \left[ \left. \frac{d}{dt} \nabla_{g_t}^{\ast \Omega_t} \dot{g}_t^* \right|_{t=0} - \omega \right].$$

Using the variation formula (19), and the identity (20) we obtain

$$\left. \frac{d}{dt} \right|_{t=0} \nabla_{g_t} \left( g_t \dot{g}_t^* \right) = \nabla_g |A|^2_g + \frac{1}{2} \nabla_g, J (\Delta_{g,J}^0 - 2\Pi) \psi,$$

and thus

$$\left. \frac{d}{dt} \dot{\Omega}^*_t \right|_{t=0} = -\frac{1}{2} R_{\psi} - |A|^2_g + \int_X |A|^2_g \Omega.$$

We obtain in conclusion the variation formula

$$2 \frac{d^2}{dt^2} \left|_{t=0} \frac{H_{g_t,\Omega_t}}{2} \right. = 2 \int_X |A|^2_g F \Omega - 2 \left( \mathcal{L}_{g}^0 A, A \right)_g$$

$$- (\Delta_{g}^0 - 2\Pi) |A|^2_g - 2 \int_X |A|^2 \Omega$$

$$- \frac{1}{2} (\Delta_{g}^0 - 2\Pi) R_{\psi} + \frac{1}{2} B_{g,J}^0 \psi$$

$$= -2 \left( J \nabla_g f - \nabla_g A, J A \right)_g + 4 \left( \nabla_g^2 f, A^2 \right)_g + 2 \int_X |A|^2_g F \Omega$$

$$- (\Delta_{g}^0 - 2\Pi) |A|^2_g - 2 \int_X |A|^2 \Omega - \frac{1}{2} \mathcal{P}^0_{g,J} \text{ Re } \psi,$$

thanks to identity (15) and a computation in the proof of lemma 25 in [Pal2].

We denote respectively by $\pi_1$ and $\pi_2$ the projection to the first and second factor of the decomposition (4). Then the identity

$$0 = \pi_2 \gamma_0 = \pi_2 \frac{d^2}{dt^2} |_{t=0} \frac{H_{g_t,\Omega_t}}{2},$$

is equivalent to the identity

$$\int_X \mu_1 \left[ 4 \left( \nabla_g^2 f, A^2 \right)_g - 2 \left( J \nabla_g f - \nabla_g A, J A \right)_g - (\Delta_{g}^0 - 2\Pi) |A|^2_g \right] \Omega = 0,$$  \hspace{1cm} (21)
for any $u = u_1 + iu_2 \in \Lambda^\Omega_{g,J}$, with $u_1$, $u_2$, real valued. We observe now the equalities
\[
\int_X u_1 (\Delta^\Omega_g - 2\mathbb{II}) |A|^2 \Omega = - \int_X B^\Omega_{g,j} u_2 |A|^2 \Omega
\]
\[
= \int_X u_2 B^\Omega_{g,j} |A|^2 \Omega
\]
\[
= \int_X u_2 (J \nabla_g f) \cdot |A|^2 \Omega
\]
\[
= 2 \int_X u_2 \langle J \nabla_g f - \nabla_g A, A \rangle_{g,\Omega}.
\]
We conclude that the identity (21) is equivalent to
\[
2 \int_X u_1 \langle \nabla^2_g f, A^2 \rangle_{g,\Omega} = \int_X \langle J \nabla_g f - \nabla_g A, \nabla \times_j A \rangle_{g,\Omega},
\]
which shows the required conclusion.

8 Appendix

8.1 Proof of the identities (10) and (11)

By definition of the $\Omega$-divergence operator and using the symmetry of $A$ we infer
\[
\text{div}^\Omega (A \cdot \xi) = g(\nabla_{g,e_k} (A \cdot \xi), e_k) - g(A \cdot \xi, \nabla_g f)
\]
\[
= g(\nabla_{g,e_k} A \cdot \xi + A \cdot \nabla_{g,e_k} \xi, e_k) - g(\xi, A \cdot \nabla_g f)
\]
\[
= g(\xi, \nabla_{g,e_k} A \cdot e_k - A \cdot \nabla_g f) + g(\nabla_{g,e_k} \xi, Ae_k),
\]
and thus the identity (10). We expand now the term
\[
\text{div}^\Omega \text{Tr}_g (\nabla_g A \cdot A) = \text{div}^\Omega (\nabla_{g,e_k} A \cdot Ae_k)
\]
\[
= g(\nabla_{g,e_l} (\nabla_{g,e_k} A \cdot Ae_k), e_l) - g(\nabla_{g,e_k} A \cdot Ae_k, \nabla_g f)
\]
\[
= g(\nabla_{g,e_l} \nabla_{g,e_k} A \cdot Ae_k + \nabla_{g,e_k} A \cdot \nabla_{g,e_l} A \cdot e_k, e_l)
\]
\[
- g(Ae_k, \nabla_{g,e_k} A \cdot \nabla_g f).
\]
Expanding further we infer
\[
\text{div}^\Omega \text{Tr}_g (\nabla_g A \cdot A) = g(Ae_k, \nabla_{g,ei} \nabla_{g,ek} A \cdot e_l) + g(\nabla_{g,ei} A \cdot e_k, \nabla_{g,ek} A \cdot e_l) \\
- g(Ae_k, \nabla_g A \cdot \nabla f) \\
= g(Ae_k, \nabla_{g,ei} \nabla_{g,ek} A(e_l, e_k) - \nabla_{g}A(\nabla g, e_l)) \\
+ \left\langle \nabla_g A, \nabla_{g}A \right\rangle_g,
\]
and thus the identity (11).

### 8.2 The Maurer-Cartan equation in the Kähler case

We observe that for any vector spaces \(V\) and \(E\), we can define a contraction operation

\[
\neg : (\Lambda^p V^* \otimes V) \times (\Lambda^q V^* \otimes E) \rightarrow \Lambda^{p+q-1} V^* \otimes E
\]

by the expression

\[
(\alpha \neg \beta)(\xi) := \sum_{|I|=\deg \alpha} \varepsilon_I \beta(\alpha(\xi_I), \xi_{\bar{I}}).
\]

This map restricts to

\[
\neg : \mathcal{E}^{0,p}(T^{1,0}_X) \times \mathcal{E}^{r,q} \rightarrow \mathcal{E}^{r-1,p+q}.
\]

We notice indeed the identity \(\alpha \neg \beta = \bar{\zeta}_I^* \wedge (\alpha_I \neg \beta)\), where \(\alpha = \alpha_I \otimes \bar{\zeta}_I^*\), with \((\zeta_k)_k \subset C^\infty(U, T^{1,0}_X)\) a local frame. (We use from now on the Einstein convention for sums). Obviously, the contraction operation \(\neg\), generalises the one used in the previous sections.

**Lemma 6 (Expression of the exterior Lie product).** Let \((X, J, \omega)\) be a Kähler manifold and let \(\alpha, \beta \in C^\infty(X, \Lambda^0 J^* T_X \otimes \omega T^{1,0}_X)\). Then holds the identity

\[
[\alpha, \beta] = \alpha \neg \partial^\omega_{T^{1,0}_X} \beta - (-1)^{||\alpha||} \beta \neg \partial^\omega_{T^{1,0}_X} \alpha.
\]

**Proof** In the case \(|\alpha| = |\beta| = 0\), the identity follows from an elementary computation in geodesic holomorphic coordinates. In order to show the general
case, let \((\zeta_k)_k \subset \mathcal{O}(U,T^{1,0}_{X,J})\) be a local frame. We consider the local expressions 
\[ \alpha = \alpha_K \otimes \bar{\zeta}_K, \quad \beta = \beta_L \otimes \bar{\zeta}_L. \]
Then 
\[ [\alpha, \beta] = [\alpha_K, \beta_L] \otimes (\bar{\zeta}_K \wedge \bar{\zeta}_L). \]

The identity \(\partial_{T^{1,0}_{X,J}} \zeta_k = 0\) implies \(\partial_j \bar{\zeta}_K = 0\). We infer 
\[ \partial_{T^{1,0}_{X,J}} \alpha = \partial_{T^{1,0}_{X,J}} \alpha_K \wedge \bar{\zeta}_K, \]
and a similar local expression for \(\beta\). Thus using the identity 
\[ \alpha - \gamma = \bar{\zeta}_K \wedge (\alpha_K - \gamma), \]
with \(\gamma\) arbitrary, we deduce 
\[ \alpha - \partial_{T^{1,0}_{X,J}} \beta = (\alpha_K - \partial_{T^{1,0}_{X,J}} \beta_L) \otimes (\bar{\zeta}_K \wedge \bar{\zeta}_L), \]
\[ \beta - \partial_{T^{1,0}_{X,J}} \alpha = (\beta_L - \partial_{T^{1,0}_{X,J}} \alpha_K) \otimes (\bar{\zeta}_L \wedge \bar{\zeta}_K), \]
and thus the required conclusion. \(\square\)

We deduce that over a Kähler manifold the Maurer-Cartan equation 
\[ \partial_{T^{1,0}_{X,J}} \theta + \frac{1}{2} [\theta, \theta] = 0, \]
writes as 
\[ \partial_{T^{1,0}_{X,J}} \theta + \partial_{T^{1,0}_{X,J}}^{\omega} \theta = 0. \]
(22)

We show below that we can rewrite the Maurer-Cartan equation in equivalent real terms as 
\[ \partial_{T^{1,0}_{X,J}} \mu + \mu - \nabla_{g,J}^{1,0} \mu = 0, \]
(23)
or in more explicit terms 
\[ (\| + \mu) J \nabla_{g,J} \mu = (\| + \mu) J \nabla_{g,J} \mu. \]

In order to show (23) we expand, for any \(u,v \in T_X\), the term 
\[ \left( \theta - \partial_{T^{1,0}_{X,J}} \theta \right)(u,v) = \partial_{T^{1,0}_{X,J}} \theta (\theta u, v) + \partial_{T^{1,0}_{X,J}} \theta (u, \theta v) \]
\[ = \nabla_{g,J}^{1,0} \theta (\theta u, v) - \nabla_{g,J}^{1,0} \theta (v, \theta u) \]
\[ + \nabla_{g,J}^{1,0} \theta (u, \theta v) - \nabla_{g,J}^{1,0} \theta (v, \theta u). \]
Expanding further we obtain
\[
2 \left( \theta - \partial_{T_{X,J}^{1,0}} \theta \right)(u,v) = \nabla_g \theta (\theta u, v) - i \nabla_g \theta (Ju, v) \\
- \nabla_g \theta (v, \theta u) + i \nabla_g \theta (Jv, \theta u) \\
+ \nabla_g \theta (u, \theta v) - i \nabla_g \theta (Ju, \theta v) \\
- \nabla_g \theta (\theta v, u) + i \nabla_g \theta (J\theta v, u).
\]

Using the fact that \( \theta \) takes values in \( T_{X,J}^{1,0} \) we obtain
\[
2 \left( \theta - \partial_{T_{X,J}^{1,0}} \theta \right)(u,v) = 2 \nabla_g \theta (\theta u, v) - \nabla_g \theta (v, \theta u) + i \nabla_g \theta (Jv, \theta u) \\
- 2 \nabla_g \theta (\theta v, u) + \nabla_g \theta (u, \theta v) - i \nabla_g \theta (Ju, \theta v).
\]

Replacing on the right hand side of this equality the identity \( 2 \theta = \mu - iJ\mu \) and adding the conjugate of both sides we infer
\[
8 \left( \theta - \partial_{T_{X,J}^{1,0}} \theta \right)(u,v) + 8 \left( \theta - \partial_{T_{X,J}^{1,0}} \theta \right)(u,v) = 4 \nabla_g \mu (\mu u, v) - 4J \nabla_g \mu (J\mu u, v) \\
- 2 \nabla_g \mu (v, \mu u) + 2J \nabla_g \mu (v, J\mu u) \\
+ 2 \nabla_g \mu (Jv, J\mu u) + 2J \nabla_g \mu (Jv, \mu u) \\
+ 2 \nabla_g \mu (u, \mu v) - 2J \nabla_g \mu (u, J\mu u) \\
- 2 \nabla_g \mu (Ju, J\mu v) - 2J \nabla_g \mu (Ju, \mu v) \\
- 4 \nabla_g \mu (\mu v, u) + 4J \nabla_g \mu (J\mu v, u).
\]
Using the anti $J$-linearity of $\nabla_{g,J} \mu$ we deduce

$$
8 \left( \theta - \partial_{T_{X,0}} \nabla_{g,g} \theta \right) (u, v) + 8 \left( \theta - \partial_{T_{X,0}} \nabla_{g,g} \theta \right) (u, v)
$$

$$
= 4 \nabla_{g,J} (\mu u, v) - 4 J \nabla_{g,J} (J \mu u, v)
$$

$$
- 4 \nabla_{g,J} (\mu v, u) + 4 J \nabla_{g,J} (J \mu v, u)
$$

$$
= 8 \nabla_{g,J}^{1,0} (\mu u, v) - 8 \nabla_{g,J}^{1,0} (\mu v, u)
$$

$$
= 8 \left( \mu - \nabla_{g,J}^{1,0} \right) (u, v).
$$

The latter combined with

$$
\bar{\mathcal{J}}_{T;X,J} \theta (u, v) + \bar{\mathcal{J}}_{T;X,J} \theta (u, v) = \mathcal{J}_{T;X,J} \mu (u, v),
$$

and (22) implies the required identity (23).

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