BICONVEX POLYTOPES AND TROPICAL LINEARITY

JAEHO SHIN

Dedicated to Bernd Sturmfels on the occasion of his 60th birthday

Abstract. A biconvex polytope is a convex tropical polytope. For a biconvex polytope with the maximum number of vertices, we assign to each vertex of it a cycle-free bigraph and construct a matroid base polytope from the graph so that the collection of the matroid base polytopes thus obtained is a matroid subdivision of a hypersimplex dual to the biconvex polytope, and thereby prove a biconvex polytope arises as a cell of a tropical linear space. We also show that there is an injection from the proper faces of a \((k - 1)\)-dimensional biconvex polytope into the monomials of degree \(\leq k - 1\) in \(k\) indeterminates.

Contents

1. Introduction 1
2. Preliminaries 3
3. Cycle-Free Bigraphs and Matroids 5
4. Tropical Setting 12
5. Cycle-Free Bigraphs and Biconvex Polytopes 14
6. Biconvex Polytopes and Tropical Linearity 18
References 24

1. Introduction

Tropical geometry is geometry over min-plus or max-plus algebra. Many notions in classical geometry can be tropicalized, but tropicalized notions are delicate and often do not conform to our usual geometric intuition. In this paper, we deepen our understanding of tropical convexity and tropical linearity by studying convex polyhedra that are also tropically convex, which we call biconvex polytopes, and investigating their relationship to tropical linear spaces.

For a tropical linear space, there corresponds a regular matroid subdivision of a matroid base polytope that is dual to it, and we expect that matroid subdivisions serve as a bridge between those two tropicalized notions. It has been known that every bounded cell of a tropical linear space is a biconvex polytope. If we prove that for an arbitrary biconvex polytope there is a matroid subdivision dual to it, then it will follow that a biconvex polytope arises as a cell of a tropical linear space.

2020 Mathematics Subject Classification. Primary 14T15; Secondary 05B35, 05C30, 52B40.
Key words and phrases. biconvex polytope, tropical linearity, cycle-free bipartite graph, matroid subdivision, combinatorial log map.

Some authors call them polytropes, but the "r" in polytrope is apt to be blown past, and we call them biconvex polytopes instead.
To do so, we first construct a matroid base polytope from a cycle-free bigraph, and show that to each vertex of a biconvex polytope with the maximum number of vertices there corresponds a cycle-free bigraph where our bigraphs need not be connected and vertices are vertices of a classical polyhedron in a quotient space \( \mathbb{R}^k / \mathbb{R} \mathbb{I} \) for some integer \( k \geq 2 \). And then, we show that the collection of matroid base polytopes thus obtained is a matroid subdivision of a hypersimplex that is dual to the biconvex polytope.

We also show there is an injection from the proper faces of a \((k-1)\)-dimensional biconvex polytope into the monomials of degree \(< k\) in \(k\) indeterminates.

We assume familiarity with matroid theory and refer readers who are unfamiliar with it to [Oxl11, Sch03] for the references. We refer readers to [MS15] for standard tropical theory and terminology. In particular, we assume that our tropical semiring is min-plus algebra throughout the paper.

**Conventions and notations.** For a finite set \( S \), we denote by \( \mathbb{R}^S \) the Cartesian product of \( |S| \) copies of \( \mathbb{R} \) that are labeled by the elements of \( S \).

The rank-\( k \) uniform matroid on \( S \) is denoted by \( U^k_S \). The matroid base polytope of \( U^k_S \) is denoted by \( \Delta^k_S \) which is called a hypersimplex. If \( S = [n] := \{1, \ldots, n\} \) for a positive integer \( n \), we simply write \( U^k_n \) and \( \Delta^k_n \) for \( U^k_{[n]} \) and \( \Delta^k_{[n]} \), respectively.

For all \( i \in S \) we understand \( x_i \) as coordinate functions of \( \mathbb{R}^S \) or indeterminates for the coordinates. For a vector \( v \in \mathbb{R}^S \) and \( i \in S \) we denote by \( x_i(v) \) the \( i \)-th coordinate of \( v \). For a nonempty subset \( A \) of \( S \), we denote

\[
x(A) = \sum_{i \in A} x_i.
\]

Let \( Q \) be a polyhedron in \( \mathbb{R}^S \) and \( Q \) a set of describing equations and inequalities of it. We often write \( Q \) for \( Q \) (even though there can be many different such sets). For instance, \( \{x(S) = k\} \) denotes the polyhedron in an ambient space determined by the equation \( x(S) = k \), and we write \( \Delta^k_S = [0,1]^S \cap \{x(S) = k\} \).

We will use the following standard notation of graph theory.

For a graph \( G \), we denote by \( V_G \) the set of its nodes, by \( E_G \) the set of its edges, and by \( \deg_G(j) \) the number of edges of \( G \) adjacent to \( j \in V_G \). We also denote

\[
N_G(j) = \{ \text{the nodes of } G \text{ adjacent to } j \}
\]

\[
N_G[j] = N_G(j) \cup \{j\}
\]

where if \( G \) is a simple graph, we have \( |N_G(j)| = \deg_G(j) \) and \( |N_G[j]| = \deg_G(j) + 1 \).

**Acknowledgements.** The author learned of the possible relationship between tropical linear spaces and biconvex polytopes from Bernd Sturmfels and is truly grateful to him. He is also grateful to Günter Ziegler for his interest and advice. Special thanks to Thomas Zaslavsky for his comments and helpful conversations. The author would like to acknowledge conversations over email with Michael Joswig, Benjamin Schröter and David Speyer.

This research was partially supported by the grant (NRF-2019R1A2C3010487) of the National Research Foundation funded by the Korean government at the final stage, and he would like to thank JongHae Keum for the support.
2. Preliminaries

In this section, we provide a quick primer on matroid tilings. Let $S$ be a finite set and $M$ be a matroid on $S$ with rank function $r$ throughout the section.

A pair $\{A, B\}$ of subsets of $S$ is called a modular pair of $M$ if equality holds in the submodular inequality $r(A) + r(B) \geq r(A \cup B) + r(A \cap B)$, that is,

$$r(A) + r(B) = r(A \cup B) + r(A \cap B).$$

A subset $A \subseteq S$ is called a separator of $M$ if $\{A, S - A\}$ is a modular pair. Both $S$ and $\emptyset$ are separators which are called trivial separators. Then, $M$ and its dual matroid $M^*$ have the same set of separators.

A matroid $M$ on $S$ is called connected if it has no nontrivial separators, and disconnected otherwise. A subset $A \subseteq S$ is called connected if the restriction matroid $M|_A$ is, and disconnected otherwise.

We denote $\kappa(M)$ : the number of all nonempty inclusionwise minimal separators of $M$ where the inclusion is set inclusion. Let $S_1, \ldots, S_{\kappa(M)}$ be all nonempty inclusionwise minimal separators of $M$, then $M$ is written as

$$M = M|_{S_1} \oplus \cdots \oplus M|_{S_{\kappa(M)}}.$$  

Here, $M|_{S_i}$ are called the connected components of $M$.

For a subset $A \subseteq S$, we denote

$$M(A) := M|_A \oplus M/A.$$  

Then, $A$ is called a non-degenerate\(^2\) subset of $M$ if

$$\kappa(M(A)) = \kappa(M) + 1$$

and a degenerate subset otherwise. Every separator is degenerate. If $M$ is a disconnected matroid, there can be different non-degenerate subsets $A_1, \ldots, A_m$ such that $M(A_1) = \cdots = M(A_m)$, but there exists a unique inclusionwise minimal such. For subsets $S_1, \ldots, S_m \subseteq S$, we write

$$M(S_1)(S_2) \cdots (S_m) = (\cdots ((M(S_1))(S_2)) \cdots )(S_m).$$

If $A$ is a non-degenerate subset of $M$, then $S - A$ is a non-degenerate subset of $M^*$.

The indicator vector of $A \subseteq S$ is the vector $v \in \mathbb{R}^S$ such that $x_i(v) = 1$ if $i \in A$ and $0$ otherwise, which we denote by $1^A$:

$$x_i(1^A) = \begin{cases} 1 & \text{if } i \in A, \\ 0 & \text{if } i \not\in A. \end{cases}$$

The set of bases of $M$ is denoted by $B(M)$. The matroid base polytope or simply the matroid polytope of $M$ is the convex hull of all indicator vectors $1^B$ of $B \in B(M)$, which we denote by $\text{BP}_M$ whose dimension is

$$\dim \text{BP}_M = |S| - \kappa(M).$$

Using inequalities, $\text{BP}_M$ is written as

$$\text{BP}_M = \{x_i \geq 0 : i \in S\} \cap \{x(A) \leq r(A) : A \in 2^S\}.$$  

\(^2\)The definition of non-degenerate subsets was originally given in [GS87] for connected matroids, and generalized to the current form in [Shi19].
The correspondence between matroids and matroid polytopes is one-to-one. Note that $BP_M$ is full-dimensional if and only if $M$ is connected.

A face matroid of $M$ is the matroid of a face of $BP_M$.

We say that two polytopes are face-fitting if their intersection is a common face of both.

A $(k, S)$-tiling or simply a tiling is a finite collection of polytopes in $\Delta^k$ that are pairwise face-fitting. The support $|\Sigma|$ of a tiling $\Sigma$ is the union of its members. The dimension of $\Sigma$ is the dimension of $|\Sigma|$. Throughout the paper, a tiling is assumed equidimensional, i.e. all of its members have the same dimension.

When mentioning cells of $\Sigma$, we identify $\Sigma$ with the polyhedral complex that its polytopes generate with intersections.

A matroid tiling is a tiling whose members are matroid polytopes, which is well defined because every face of a matroid polytope is again a matroid polytope.

A matroid subdivision is a matroid tiling whose support is a matroid polytope.

The base intersection of two matroids $M_1$ and $M_2$ is the intersection of the base collections of $M_1$ and $M_2$, which we denote by $M_1 \cap M_2$.

When $M_1 \cap M_2$ is the base collection of a matroid, by abuse of notation, we denote the matroid by $M_1 \cap M_2$. For instance, if $M_1$ and $M_2$ are face matroids of the same matroid, then $BP_{M_1} \cap BP_{M_2}$ is the maximum common face of $BP_{M_1}$ and $BP_{M_2}$, whose matroid is denoted by $M_1 \cap M_2$.

For a subcollection $A$ of the power set $2^S$ of $S$, let $P_A$ be the convex hull of the indicator vectors $1^A$ for all $A \subseteq A$. Then, [Sch03, Corollary 41.12d] says

$$BP_{M_1} \cap BP_{M_2} = P_{M_1 \cap M_2}.$$

**Lemma 2.1 ([Shi19]).** Let $M$ be a rank-$k$ matroid on $S$ with rank function $r$.

1. If there is a subset $A \subseteq S$ of size $k + 1$ such that $M|_A = U^k_A$, then $M \setminus \emptyset_M$ is a connected matroid where $\emptyset_M$ denotes the set of loops of $M$.

2. Suppose $M = M|_{S_1} \oplus \cdots \oplus M|_{S_{a(M)}}$ is loopless. Then, $BP_M$ is determined by $\kappa(M)$ equations $x(S_i) = r(S_i)$ and the following system of inequalities:

$$\begin{align*}
 &x(i) \geq 0 \quad \text{for all } i \in S, \\
 &x(F) \leq r(F) \quad \text{for all the minimal non-degenerate flats } F \text{ of } M.
\end{align*}$$

3. For two subsets $F$ and $L$ of $S$, we have $M(F) \cap M(L) \neq \emptyset$ if and only if \( \{F, L\} \) is a modular pair.

4. Suppose $M$ is a connected matroid with $k \geq 3$. Let \( \{F, L\} \) be a modular pair of distinct non-degenerate flats such that $M(F) \cap M(L)$ is loopless and $BP_{M(F)} \cap BP_{M(L)}$ has codimension 2 in $BP_M$. Then, precisely one of the following 4 cases happens.

| $F \cap L = \emptyset$ | $M(F) \cap M(L) = M(F \cup L)$ with $M|_{F \cup L} = M|F \oplus M|L$ |
| $F \cup L = S$ | $M(F) \cap M(L) = M(F \cap L)$ with $M|(F \cap L) = M|F \oplus M|L$ |
| $F \supset L$ | $M(F) \cap M(L) = M|F \oplus M|L/F \oplus M|L$ |
| $F \subset L$ | $M(F) \cap M(L) = M|L \oplus M|L/F \oplus M|F$ |
3. Cycle-Free Bigraphs and Matroids

A bipartite graph or a bigraph for short is a graph that does not contain any odd cycle. In this section, we extend the definition of bigraph and construct matroids from cycle-free bigraphs.

3.1. Bigraphs of our interest. We extend the notion of bigraph at the cost of connectedness of graph. If \( G \) is a graph whose nodes are divided into two disjoint sets \( I \) and \( I^c \) such that there is at most one edge between two nodes and in both sets no two nodes are adjacent, then we will say that \( G \) is a bipartite graph or a bigraph for short, with (ordered) parts \((I, I^c)\). Thus, whether or not a graph is connected, if it has no odd length cycles, it has a bipartite structure and vice versa. Note that one of \( I \) and \( I^c \) can be empty (in this case \( G \) is a set of isolated nodes), but not both of them can.

Let \( G \) be a cycle-free bigraph with \(|V_G| = k\) nodes \((k \geq 1)\). Then, \(|E_G| \leq k - 1\) where equality holds if and only if \( G \) is connected. For all \( j \in V_G \), denote

\[
V^j_G = \{ j \} \cup (N_G(j) \cap I) \quad \text{and} \quad V^j_G = \{ j \} \cup (N_G(j) \cap I^c).
\]

In other words,

\[
V^j_G = \begin{cases} 
N_G[j] & \text{if } j \in I^c \\
\{j\} & \text{if } j \in I
\end{cases} \quad \text{and} \quad V^j_G = \begin{cases} 
\{j\} & \text{if } j \in I^c \\
N_G[j] & \text{if } j \in I
\end{cases}.
\]

If \( i \in I \) and \( c \in I^c \) are adjacent, denote by \((i, c)\) the unique edge that connects \( i \) to \( c \), and denote by \( G(i, c) \) the graph obtained from \( G \) by removing \((i, c)\) from it. Then, \( G(i, c) \) is a bigraph with induced bipartite structure.

Let \( G^+(i, c) \) and \( G^-(i, c) \) be the connected components of \( G(i, c) \) containing \( i \) and \( c \), respectively, both of which are nonempty and have induced bipartite structure. Note that \( G^+(i, c) \) and \( G^-(i, c) \) are connected whether or not \( G \) is connected, and that two node sets \( V_{G^+(i,c)} \) and \( V_{G^-(i,c)} \) are disjoint, which partition the node set of the connected component of \( G \) that contains \((i, c)\).

3.2. Construction of matroids from cycle-free bigraphs. Let \( G \) be a cycle-free bigraph with parts \((I, I^c)\) and \( k = |V_G| \geq 1 \), and \( G_1, \ldots, G_m \) be its connected components. Given a cycle-free bigraph, we choose a partition \( \{S_j : j \in V_G\} \) of a finite set \( S \) satisfying that \(|S_j| \geq 2\) for all \( j \in V_G \), which we will call an underlying partition of the cycle-free bigraph \( G \). For any nonempty subset \( J \subseteq V_G \), denote

\[
S_J = \bigcup_{j \in J} S_j.
\]

For any subgraph \( G' \) of \( G \), we will write \( S_{G'} \) instead of \( S_{V_{G'}} \) for simplicity.

**Theorem 3.1.** Let \( \mathcal{B} \) be the set of all those \( k \)-element subsets \( X \) of \( S \) satisfying that for all edges \((i, c) \in E_G \),

\[
|X \cap S_{G^+(i,c)}| \leq |V_{G^+(i,c)}|.
\]

Then, \( \mathcal{B} \) is nonempty and is the base collection of a rank-\( k \) loopless matroid \( M \) on \( S \), where the number of connected components of \( M \) equals the number of connected components of \( G \). In particular, \( M \) is connected if and only if \( G \) is connected.
Proof. Because (3.1) is independently applied to each connected component of \( G \), we may assume \( G \) is connected, and also (3.1) is equivalent to
\[
|X \cap S_{G^- (i, c)}| \geq |V_{G^- (i, c)}|.
\]
Let \( X \) be a \( k \)-element subset of \( S \) satisfying (3.1). For a fixed \( i \in I \), the node sets \( V_{G^- (i, c)} \) for all \( c \in N_G (i) \) partition \( V_G - \{ i \} \), and
\[
|X \cap S_i| = |X| - \sum_{c \in N_G (i)} |X \cap S_{G^- (i, c)}| \\
\leq k - \sum_{c \in N_G (i)} |V_{G^- (i, c)}| \\
= 1.
\]
If \( |X \cap S_i| = 1 \), then \( |X \cap S_{G^- (i, c)}| = |V_{G^- (i, c)}| \) for all \( c \in N_G (i) \). If \( |X \cap S_i| = 0 \), there is one and only one \( c \in N_G (i) \) with \( |X \cap S_{G^- (i, c)}| > |V_{G^- (i, c)}| \), more precisely, with \( |X \cap S_{G^- (i, c)}| = |V_{G^- (i, c)}| + 1 \). We define a map \( \nu_X : I \rightarrow V_G \) such that
\[
\nu_X (i) = \begin{cases} 
  i & \text{if } |X \cap S_{G^- (i, c)}| = |V_{G^- (i, c)}| + 1, \\
  c & \text{otherwise}. 
\end{cases}
\]
Then, for all edges \((i, c) \in E_G, \)
\[
|X \cap S_{G^+ (i, c)}| = \begin{cases} 
  |V_{G^+ (i, c)}| - 1 & \text{if } i \in v_X^{-1} (c), \\
  |V_{G^+ (i, c)}| & \text{otherwise}.
\end{cases}
\]
For a fixed \( c \in I^c \), the node sets \( V_{G^+ (i, c)} \) for all \( i \in N_G (c) \) partition \( V_G - \{ c \} \), and
\[
|X \cap S_c| = |X| - \sum_{i \in N_G (c)} |X \cap S_{G^+ (i, c)}| \\
= k - \sum_{i \in N_G (c)} |V_{G^+ (i, c)}| + |v_X^{-1} (c)| \\
= |v_X^{-1} (c)| + 1.
\]
Therefore, there is an injection \( f : V_G \rightarrow S \) whose image is \( X \), satisfying that
\[
(3.2) \quad f(j) \in S_{V_G^+} \quad \text{for all } j \in V_G.
\]
More specifically, let \( \tilde{\nu}_X : V_G \rightarrow V_G \) be an extension of \( v_X \) fixing \( I^c \), i.e. \( \nu_X |I^c = v_X \) and \( \tilde{\nu}_X |I = id \), then
\[
f(j) \in S_{\tilde{\nu}_X (j)} \quad \text{for all } j \in V_G.
\]
Conversely, if \( f : V_G \rightarrow S \) is an injection satisfying (3.2), then there is a unique map \( \nu : I \rightarrow V_G \) with
\[
\nu (i) \in V_G^ I \quad \text{for all } i \in I
\]
such that the extension \( \tilde{\nu} : V_G \rightarrow V_G \) of \( \nu \) fixing \( I^c \) satisfies
\[
f(j) \in S_{\tilde{\nu} (j)} \quad \text{for all } j \in V_G
\]
and the image \( X = \text{im} f \) is a \( k \)-element subset of \( S \) with (3.1) and a member of \( B \).

We call \( \nu \) a lift map and \( f \) an injection associated to \( \nu \). There always exists an injection associated to a lift map. If \( f \) and \( f' \) are two injections associated to \( \nu \), then there is a permutation \( \pi \) on \( V_G \) with \( f' = f \circ \pi \) satisfying that \( \pi |_{I^c - \nu^{-1} (I^c)} = id \) and \( \pi |_{\nu^{-1} (I^c)} = \pi |_{\nu^{-1} (I^c)} \) for each \( c \in I^c \) is a permutation.
Now, we show that $B$ is a nonempty collection with the base exchange property. Let $f$ be an injection associated to a lift map $id : I \to I \subset V_G$, then $\text{im} f \in B \neq \emptyset$. To show $B$ has the base exchange property, take two members $B$ and $B'$ from $B$, and let $f$ and $f'$ be injections associated to them, respectively. Pick any $b \in B - B' \neq \emptyset$, then there is $j \in V_G$ with $b = f(j) \in S_{\text{im} f(j)}$.

If $j \in I^c$ and $v^{-1}_B(j) = \emptyset$, let $b' = f'(j)$ then $(B - \{b\}) \cup \{b'\} \in B$.

If $j \in I^c$ and $v^{-1}_B(j) \neq \emptyset$, by switching $j$ and any element of $v^{-1}_B(j) \neq \emptyset$ we may assume that $j \in I$.

If $j \in I$, let $i_0 := j$. We recursively construct $i_{l+1}$ for $l = 0, 1, \ldots$ by the following process which we stop if we find $b' \in B' - B$ with $(B - \{b\}) \cup \{b'\} \in B$.

1. If $v_{B'}(i_l) = i_l$, let $b' := f'(i_l)$, then $(B - \{b\}) \cup \{b'\} \in B$.
2. If $v_{B'}(i_l) \neq i_l$ and there is $b' \in (B' - B) \cap S_{v_{B'}(i_l)}$, then $(B - \{b\}) \cup \{b'\} \in B$.
3. If $v_{B'}(i_l) \neq i_l$ and $(B' - B) \cap S_{v_{B'}(i_l)} = \emptyset$, let $c_l := v_{B'}(i_l)$, then

$$|v^{-1}_B(c_l) - \{i_l\}| < |v^{-1}_B(c_l) - \{i_l\}|$$

and there is $i_{l+1} \in v^{-1}_B(c_l) - v^{-1}_B(c_l) - \{i_l\}$ with $v_{B'}(i_{l+1}) \neq c_l$.

This process terminates because all $i_l$ and $c_l$ are distinct while $V_G$ is finite.

Thus, the base exchange property of $B$ is proved, and $B$ is the base collection of a rank-$k$ matroid $M$ on $S$ which has no loops.

To prove the remaining statement, it suffices to assume $G$ is connected and prove that $M$ is connected because once proved, if $G$ is disconnected, $M$ is the direct sum of those matroids constructed from the connected components of $G$ along (3.1), all of which are connected matroids. Hence, the number of connected components of $M$ equals the number of connected components of $G$, and it immediately follows that $M$ is connected if and only if $G$ is connected.

So, suppose $G$ is connected. If $|I^c| = 0$, then $|I| = 1$ and $k = 1$. Also, if $|I| = 0$, then $|I^c| = 1$ and $k = 1$. In both cases, $M$ is the rank-$1$ uniform matroid on $S = S_1$, which is a connected matroid.

If $|I^c| \geq 1$ and $|I| \geq 1$, fix $c_0 \in I^c$. For every $j \in V_G$ there is a unique simple path from $c_0$ to $j$ with length $d(j)$. Let $\nu : I \to V_G$ be a lift map with $d(\nu(i)) \geq d(i)$ such that equality holds only when $\deg_G(i) = 1$, and let $f$ be an injection associated to it. Pick an element $s_0 \in S_{c_0} - \text{im} f \neq \emptyset$, and let $A = \text{im} f \cup \{s_0\}$. Then, any $k$-element subset of $A$ satisfies (3.1) and is a base of $M$. Thus, $M|_A = U^k_A$ and $M$ is a connected matroid by Lemma 2.1. The proof is complete. \hfill $\Box$

Notation 3.2. We denote by $M[G]$ the matroid constructed from $G$ in Theorem 3.1.

Henceforth, we assume that our bigraph is cycle-free and has $k$ nodes. We also assume that a bigraph $G$ has parts $(I, I^c)$ and is equipped with an underlying partition $\{S_j : j \in V_G\}$ whose union is $S$. We will simply write $V^G_G$ for $V^G_G$.

Corollary 3.3. Let $G$ be a bigraph with connected components $G_1, \ldots, G_m$. Then, $M[G_1], \ldots, M[G_m]$ are connected matroids and

$$M[G] = M[G_1] \oplus \cdots \oplus M[G_m].$$

Remark 3.4. Regardless of its bipartite structure, the matroid of an isolated node is a rank-$1$ uniform matroid. So, irrespective of the bipartite structures, we often assume that sets of the same number of isolated nodes are the same bigraph.
Example 3.5. For a bigraph $G$, let $G''$ be the set of isolated nodes of $G$, and $G'$ be the union of the other connected components of $G$. Then, 

$$M[G] = M[G'] \oplus \bigoplus_{j \in V_{G''}} U_{S_j}^j.$$ 

From the inequality (3.1), we see that $S_{G^++(i,c)}$ for all edges $(i,c) \in E_G$ are flats of $M[G]$ whose ranks are $|V_{G^+(i,c)}|$. We can say more:

Corollary 3.6. Let $G$ be a bigraph. If $J$ is a subset of $V_G$, then $\bigcup_{j \in J} S_{V_j^G}$ is a flat of $M[G]$ of rank $|\bigcup_{j \in J} V_j^G|$.

Proof. By Corollary 3.3, we may assume $G$ is a connected bigraph with $|V_G| \geq 2$. Then, $|J| \geq 2$ and $|I^c| \geq 2$. Let $G'$ be the maximum subgraph of $G$ with $V_{G'} = \bigcup_{j \in J} V_j^G$, and we may also assume $G'$ is connected. Note that $S_{G'} = \bigcup_{j \in J} S_{V_j^G}$.

If $V_{G'} = V_G$, then $S_{G'} = S$ is a flat of $M[G]$ of rank $|V_{G'}|$.

If $V_{G'} \neq V_G$, let $J'$ be the collection of those vertices $i$ of $G'$ contained in $J$ with $\deg_{G'}(i) = 1$, then $J' \neq \emptyset$. For each $i \in J'$, denote by $(i,c_i)$ with $c_i \in I^c$ the unique edge of $G'$ connected to $i$. Then,

$$V_{G'} = \bigcap_{i \in J'} \bigcap_{l \in N_G(i) - \{c_i\}} V_{G^+(i,l)}$$

which proves that $S_{G'}$ is a flat of $M[G]$ of rank $|V_{G'}|$.

\hfill \Box

3.3. Non-degenerate flats of $M[G]$ and the matroid $M[G(i,c)]$. Let $G$ be a bigraph with connected components $G_1, \ldots, G_m$. We classify the non-degenerate flats of $M[G]$. By Corollary 3.3, any flat of $M[G]$ is a union of flats of $M[G_i]$, $i \in [m]$, and we may assume that $G$ is connected, or equivalently, $M[G]$ is connected.

Let $(i,c)$ be an edge of $G$, then $S_{G^+(i,c)}$ is a flat of $M[G]$ of rank $|V_{G^+(i,c)}|$, and

$$M[G^+(i,c)] = M[G]|_{S_{G^+(i,c)}}$$

which is connected since $G^+(i,c)$ is connected. Also, $M[G^-(i,c)]$ is connected since $G^-(i,c)$ is connected. By (3.1), a base of

$$M[G(i,c)] = M[G^+(i,c)] \oplus M[G^-(i,c)]$$

is a base of $M[G]$ whose intersection with $S_{G^+(i,c)}$ is a base of $M[G]|_{S_{G^+(i,c)}}$ and vice versa, which is precisely a base of

$$M[G](S_{G^+(i,c)}) = M[G]|_{S_{G^+(i,c)}} \oplus M[G]/S_{G^+(i,c)}.$$ 

Therefore,

$$(3.3) \quad M[G(i,c)] = M[G](S_{G^+(i,c)})$$

and in particular,

$$M[G^-(i,c)] = M[G]/S_{G^+(i,c)}.$$ 

Because $\kappa(M[G](S_{G^+(i,c)})) = 2$, the set $S_{G^+(i,c)}$ is a non-degenerate flat of $M[G]$ whose size is $\geq 2$. Then, since $G$ is a tree with $k$ nodes and $k - 1$ edges, $M[G]$ has at least $k - 1$ non-degenerate flats of size $\geq 2$ which are $S_{G^+(i,c)}$ for all $k - 1$ edges $(i,c)$ of $G$ whose ranks are $|V_{G^+(i,c)}|$. The next lemma shows that there are no more non-degenerate flats of size $\geq 2$ and further classifies all the non-degenerate flats of $M[G]$. 

Lemma 3.7. Let $G$ be a connected bigraph with $k = |V_G| \geq 2$. The matroid $M[G]$ has precisely $k - 1$ non-degenerate flats of size $\geq 2$ which are $S_{G^+(i,c)}$ for all $k - 1$ edges $(i,c)$ of $G$ whose ranks are $|V_{G^+(i,c)}|$. The size-1 non-degenerate flats are all the singletons contained in $S_F$.

Proof. For a subset $F$ of $S$, let $V_F^+$ be the set of vertices $j$ of $G$ with $F \cap S_j \neq \emptyset$, and $G_F^+$ the maximum subgraph of $G$ with $V_{G_F^+} = V_F^+$. Let $V_F^-$ be the set of vertices $j$ of $G$ with $F \cap S_j = \emptyset$, and $G_F^-$ the maximum subgraph of $G$ with $V_{G_F^-} = V_F^-$. If $F$ is a non-degenerate flat of $M[G]$, since $M[G]|_F$ is connected, $G_F^+$ is connected by Theorem 3.1. Also, $G_F^-$ is connected since $M[G]/F$ is.

Fix $c \in F$ and let $F \subseteq V_G^c$ be a flat of $M[G]$. Then, $S_i \subseteq S$ is a rank-$|I|$ flat by Corollary 3.6, and $F \cap S_j$ is a flat whose rank is the number of those $i$ with $F \cap S_i \neq \emptyset$ where $S_i \subseteq F$ if $F \cap S_i \neq \emptyset$.

If $|F - S_i| + r(F \cap S_i) = 1$, then either $F = \{s_0\}$ for some $s_0 \in S_c$ or $F = S_i$ for some $i \in I$. In the latter case, $F$ is a connected flat of size 1. In the same way as done in the proof of Theorem 3.1, we can construct a $(k + 1)$-element subset $A \subseteq S$ containing $s_0$ such that $M[G]|_A = U_k^2$, then $M[G]/\{s_0\} \cong U_k^{-1}$ and $M[G]/\{s_0\}$ is a connected matroid by Lemma 2.1(1). Hence, $F = \{s_0\}$ is a non-degenerate flat.

In the latter case, if $\deg_G(i) = 1$, then $G_F^+ = G^+(i,c)$ and $G_F^- = G^-(i,c)$, and $F = S_i = S_{G^+(i,c)}$ is a non-degenerate flat of size $\geq 2$. If $\deg_G(i) > 1$, then $G_F^-$ is disconnected, and $F$ is degenerate.

This proves all the singletons contained in $S_F$ are the size-1 non-degenerate flats.

If $1 < |F - S_i| + r(F \cap S_i) < |V_G^c|$, then $F$ is disconnected.

If $|F - S_i| + r(F \cap S_i) \geq |V_G^c|$, then $F = V_G^c$ which is connected.

Thus, if $X$ is a non-degenerate flat, $G_X^+$ and $G_X^-$ are disjoint connected graphs and $X$ is a union of $V_G^c$ for $j \in V_G$. So, there is an edge $(i,c)$ of $G$ with $G_X^+ = G^+(i,c)$ and $G_X^- = G^-(i,c)$. The proof is complete. \(\square\)

3.4. Faces of $BP_{M[G]}$ not contained in the boundary $\partial \Delta^k_S$ of $\Delta^k_S$. Let $G$ be a connected bigraph with $V_G = [k]$. If $G'$ is a subgraph of $G$ obtained by deleting edges from $G$, then $BP_{M[G']} = \Phi_{j \in V_G} U_{S_j}^1$ is a face of $BP_{M[G]}$ because a face of a face is a face, which is not contained in the boundary $\partial \Delta^k_S$ of $\Delta^k_S$. In particular, $V_G$ is a subgraph of $G$ obtained by deleting all edges from $G$, with $M[V_G] = \Phi_{j \in V_G} U_{S_j}^1$, and

$$BP_{M[V_G]} = BP_{\Phi_{j \in V_G} U_{S_j}^1}$$

is a codimension-$(k - 1)$ face of $BP_{M[G]}$ which is not contained in $\partial \Delta^k_S$.

Let $E_G = \{e_1, \ldots, e_{k-1}\}$. A face of $BP_{M[G]}$ that is not contained in $\partial \Delta^k_S$ is an intersection of facets $BP_{M[G(e_1)]}, \ldots, BP_{M[G(e_{k-1})]}$ by Lemma 3.7 and is written as $\bigcap_{l \in \Lambda} BP_{M[G(e_l)]}$ for a subset $\Lambda$ of $[k]$ whose size is its codimension. Let $G'$ be the maximum common subgraph of $G(e_l)$ for all $l \in \Lambda$, which is obtained by deleting all those edges $e_l$ from $G$, then we have

$$\bigcap_{l \in \Lambda} BP_{M[G(e_l)]} = BP_{M[G']}$$

which contains $BP_{M[V_G]}$. Thus, $BP_{M[V_G]}$ is a unique codimension-$(k - 1)$ face of $BP_{M[G]}$ that is not contained in $\partial \Delta^k_S$. 
3.5. Collections of bigraphs and matroid tilings in $\Delta^k_S$. Let $G_1$ and $G_2$ be bigraphs with the same node set $V_{G_1} = V_{G_2} = [k]$ and the same underlying partition. Then, $BP_{M[V_{G_1}]} = BP_{M[V_{G_2}]}$ is a common face of both $BP_{M[G_1]}$ and $BP_{M[G_2]}$, which is not contained in $\partial \Delta^k_S$. Suppose that $G_1$ is not a subgraph of $G_2$ and vice versa. Then, $BP_{M[G_1]} \not\subseteq BP_{M[G_2]}$ and $BP_{M[G_1]} \nsubseteq BP_{M[G_2]}$.

If $BP_{M[G_1]}$ and $BP_{M[G_2]}$ are face-fitting, then $G_1$ and $G_2$ have subgraphs $G'_1$ and $G'_2$ with edges $e_1 = (i_1, c_1)$ and $e_2 = (i_2, c_2)$, respectively, such that

$$\begin{align*}
G'_1(e_1) &= G'_2(e_2) \\
(G'_1)^+(e_1) &= (G'_2)^-(e_2) \\
(G'_1)^-(e_1) &= (G'_2)^+(e_2)
\end{align*}$$

(3.4)

and $S_{(G_i)^+(e_i)}$ for $l = 1, 2$ are non-degenerate flats of $M[G'_i]$ so that $BP_{M[G'_1(e_1) = BP_{M[G'_2(e_2)]}}$ is the maximum common face of both $BP_{M[G_1]}$ and $BP_{M[G_2]}$, cf. Lemma 2.1(2) and (3.3).

Conversely, if $G_1$ and $G_2$ have subgraphs $G'_1$ and $G'_2$ with edges $e_1$ and $e_2$, respectively, satisfying (3.4), then $BP_{M[G_1]}$ and $BP_{M[G_2]}$ are face-fitting.

**Definition 3.8.** Two bigraphs $G_1$ and $G_2$ with the same node set and the same underlying partition are said to be face-fitting if one is a subgraph of the other, or they have subgraphs $G'_1$ and $G'_2$ with edges $e_1$ and $e_2$, respectively, satisfying (3.4). We say multiple bigraphs are face-fitting if they are pairwise face-fitting.

**Example 3.9.** The bigraphs $G_1$ and $G_2$ of Figure 3.1 have parts $\{(1), \{2, 3, 4\}\}$ and $\{(1, 4), \{2, 3\}\}$ respectively, and are face-fitting since $G_1(1, 4) = G_2(3, 4)$ with $G'_1(1, 4) = G'_2(3, 4)$ and $G'_1(1, 4) = G'_2(3, 4)$. Fix an underlying partition, then

$$BP_{M[G_1]} \cap BP_{M[G_2]} = BP_{M[G_1(1, 4)]} = BP_{M[G_2(3, 4)]}$$

is a codimension-1 common face of $BP_{M[G_1]}$ and $BP_{M[G_2]}$, and $\{BP_{M[G_1]}, BP_{M[G_2]}\}$ is a matroid tiling that is connected in codimension 1.

**Example 3.10.** The 6 graphs $G_1, \ldots, G_6$ of Figure 3.2 are face-fitting bigraphs, and $\Sigma = \{BP_{M[G_i]} : l \in \{6\}\}$ is a matroid tiling connected in codimension 1 with 5 connecting facets, i.e. 5 common facets of two polytopes of $\Sigma$, whose matroids are

- $M(G_2(4, 2)) = M(G_1(1, 4))$,
- $M(G_2(1, 3)) = M(G_5(3, 2))$,
- $M(G_3(1, 3)) = M(G_4(4, 1))$,
- $M(G_3(4, 2)) = M(G_6(2, 3))$,
- $M(G_2(1, 2)) = M(G_3(4, 3))$.

**Example 3.11.** Replace $G_2$ and $G_3$ of Figure 3.2 with $G'_2$ and $G'_3$ of Figure 3.3, then $G_1, G_4, G_5, G_6, G'_2, G'_3$ are face-fitting bigraphs and the corresponding matroid tilting is connected in codimension 1 with 5 connecting facets whose matroids are

- $M(G'_2(4, 3)) = M(G_1(1, 4))$,
- $M(G'_2(1, 2)) = M(G_6(2, 3))$,
- $M(G'_3(1, 2)) = M(G_4(4, 1))$,
- $M(G'_3(4, 3)) = M(G_5(3, 2))$,
- $M(G'_2(1, 3)) = M(G'_3(4, 2))$.

\(^{3}\text{We draw a bigraph such that its ordered parts are } (I \text{ = lower vertices}, I^c = \text{upper vertices}).\)
**Remark 3.12.** There are no other pairs of bigraphs than \( \{G_2, G_3\} \) and \( \{G_2', G_3'\} \) that can be added to \( \{G_1, G_4, G_5, G_6\} \) to extend \( \{\text{BP}_{M[G_1]}, \text{BP}_{M[G_4]}, \text{BP}_{M[G_5]}, \text{BP}_{M[G_6]}\} \) to a matroid tiling connected in codimension 1.
4. Tropical Setting

For $a, b \in \mathbb{R}$, the tropical sum of $a$ and $b$ is $a \boxplus b = \min \{a, b\}$ and the tropical product of $a$ and $b$ is $a \odot b = a + b$. Then, $(\mathbb{R} \cup \{\infty\}, \boxplus, \odot)$ is a semiring which is called min-plus algebra. By replacing minimum with maximum, we obtain another semiring $(\mathbb{R} \cup \{-\infty\}, \max, +)$ which is called max-plus algebra and is isomorphic to min-plus algebra.

4.1. Tropical polytopes and biconvex polytopes. Let $k$ be a positive integer. The vector $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^k$ is called the all-one vector. For $a \in \mathbb{R}$ and $x \in \mathbb{R}^k$, the tropical scalar multiplication $a \odot x$ is

$$a \odot x = a \mathbf{1} + x.$$  

For vectors $x_1, \ldots, x_n \in \mathbb{R}^k$, their tropical sum $x_1 \oplus \cdots \oplus x_n$ is entrywise defined, that is, the $j$-th entry of $x_1 \oplus \cdots \oplus x_n$ is $x_{1j} \oplus \cdots \oplus x_{nj}$ where $x_{ij}$ is the $j$-th entry of $x_i$. A tropical linear sum or a tropical linear combination of $x_1, \ldots, x_n$ is

$$a_1 \odot x_1 \oplus \cdots \oplus a_n \odot x_n$$

for some $a_1, \ldots, a_n \in \mathbb{R}$.

A subset of $\mathbb{R}^k$ is called tropically convex if it is closed under the operation of tropical linear sum.

The tropical convex hull of a subset $V \subseteq \mathbb{R}^k$ is the smallest tropically convex subset that contains $V$, denoted by $\text{tconv}(V)$. We say $V$ generates $\text{tconv}(V)$.

Remark 4.1. Tropical convex hull is well-defined over $\mathbb{R}^k/\mathbb{R}\mathbf{1}$ although tropical linear sum is not.

Notation 4.2. Equality “$\equiv$” in $\mathbb{R}^k/\mathbb{R}\mathbf{1}$ is equality modulo $\mathbb{R}\mathbf{1}$ in $\mathbb{R}^k$: for $a, b \in \mathbb{R}^k$, $a \equiv b$ if and only if $a - b = t \cdot \mathbf{1}$ for some real number $t$.\footnote{We use $\boxplus$ to denote tropical sum, reserving $\oplus$ for direct sum notation.}
A tropically convex subset $X$ in $\mathbb{R}^k$ is an unbounded polyhedron which is closed under tropical scalar multiplication, and we usually identify it with its image in the quotient space $\mathbb{R}^k/\mathbb{R}1$, which is bounded.

**Definition 4.3.** A **tropical polytope** is the tropical convex hull of a finite subset in $\mathbb{R}^k/\mathbb{R}1$ for a positive integer $k$.

A tropical polytope $P = \text{tconv}(v_1, \ldots, v_m) \subset \mathbb{R}^k/\mathbb{R}1$ is not (classically) convex, but is a union of convex polytopes.

**Definition 4.4.** A **biconvex polytope** is a convex tropical polytope.

Let $tconv(v_1, \ldots, v_k) \subset \mathbb{R}^k/\mathbb{R}1$ be full-dimensional, then it contains a unique full-dimensional cell $P$, and henceforth we may assume that

$$P = \text{tconv}(v_1, \ldots, v_k).$$

Then, $\{v_1, \ldots, v_k\}$ is a unique inclusionwise minimal set of points in $\mathbb{R}^k/\mathbb{R}1$ that generates $P$ where the inclusion is set inclusion, [DS04, Proposition 21]. Denote

$$\text{Vert}(P) = \{\text{the vertices of } P\}.$$

The cardinality of $\text{Vert}(P)$ is at least $k$ and at most $(2k-2)/k-1$, [DS04, Proposition 19], and it makes sense to introduce the following notation

$$\text{Vert}^0(P) := \text{Vert}(P) - \{v_1, \ldots, v_k\}.$$

**Definition 4.5.** A **maximal** biconvex polytope is a full-dimensional one with the maximum number of vertices.

Unless otherwise stated we assume our biconvex polytope is maximal because a biconvex polytope of lower dimension or with fewer number of vertices is obtained as a tropical degeneration of a maximal biconvex polytope $tconv(v_1, \ldots, v_k)$ for some integer $k$ as varying the $k$ points $v_1, \ldots, v_k$.

4.2. **Tropical hyperplanes.** The **tropical hyperplane** at $a = (a_1, \ldots, a_k) \in \mathbb{R}^k$ is defined as the set of points $(x_1, \ldots, x_k)$ such that the minimum

$$a_i \odot x_j$$

occurs at least twice, that is, the minimum equals $a_i \odot x_i$ and $a_j \odot x_j$ for some $i$ and $j$ with $i \neq j$, which is the following union of subsets of $\mathbb{R}^k$:

$$H^a := \bigcup_{i,j \in [k], i \neq j} \bigcap_{t \in [k] - \{i,j\}} \{a_t + x_t \geq a_i + x_i = a_j + x_j\}.$$

Tropical hyperplanes are sometimes called **min-plus hyperplanes**. We call each subset $\bigcap_{t \in [k] - \{i,j\}} \{a_t + x_t \geq a_i + x_i = a_j + x_j\}$ with $i \neq j$ in the above formula the **$\{i,j\}$-min-branch** of the tropical hyperplane.
The **i-th min-sector** for \(i \in [k]\) by the tropical hyperplane at \(a\) is defined as

\[
E_i^a := \bigcap_{j \in [k] - \{i\}} \{a_j + x_j \geq a_i + x_i\}.
\]

**Remark 4.6.** Tropical hyperplanes, min-branches and min-sectors are tropically convex, cf. [DS04, Proposition 6 and Corollary 7].

In the same manner, the **max-plus hyperplane** at \(a = (a_1, \ldots, a_k)\) is defined as the set of points \((x_1, \ldots, x_k)\) such that the maximum

\[
\max(a_1 + x_1, \ldots, a_k + x_k)
\]

occurs at least twice, which is

\[
\hat{H}^a := \bigcup_{i, j \in [k], i \neq j} \bigcap_{i \in [k] - \{i, j\}} \{a_i + x_i \leq a_j + x_j\}
\]

where we will call each \(\bigcap_{i \in [k] - \{i, j\}} \{a_i + x_i \leq a_j + x_j\}\) with \(i \neq j\) in the above formula the **\(i, j\)-max-branch** of the max-plus hyperplane.

The **i-th max-sector** for \(i \in [k]\) by the max-plus hyperplane at \(a\) is defined as

\[
E_i^a := \bigcap_{j \in [k] - \{i\}} \{a_j + x_j \leq a_i + x_i\}.
\]

where max-sectors and min-sectors are related as follows: for \(i \in [k]\) and \(a \in \mathbb{R}^k\),

\[
E_i^a = -E_i^{(-a)}.
\]

**Remark 4.7.** Max-plus hyperplanes are closed under tropical scalar multiplication and so are max-branches and max-sectors. Thus, all of tropical hyperplanes, max-plus hyperplanes, branches and sectors are well-defined over \(\mathbb{R}^k/\mathbb{R}\).

A branch has codimension 1, and the boundary of a sector is a union of branches. The \(\{i, j\}\)-branch of a sector will be called the **\(i, j\)-facet** of the sector.

5. **Cycle-Free Bigraphs and Biconvex Polytopes**

5.1. **Vertices of biconvex polytopes.** Let \(P = \text{tconv}(v_1, \ldots, v_k) \subset \mathbb{R}^k/\mathbb{R}\) be a maximal biconvex polytope, then \(\{v_1, \ldots, v_k\} \subset \text{Vert}(P)\) generates \(P\). At each vertex \(v_i\) of \(P\) there is a unique max-sector that contains \(P\), say \(E_{\pi(i)}^v\) for some permutation \(\pi : [k] \to [k]\), and we have

\[
P = \bigcap_{i \in [k]} \bar{E}_i.
\]

From now on, by rearranging indices if necessary, we may assume \(\pi = \text{id}\) and write

\[
E_i = E_i^v = E_{\pi(i)}^v.
\]

Take any vertex \(w \in \text{Vert}(P)\) and define

\[
I^w := \{i \in [k] : w \in \text{(a facet of } \bar{E}_i)\}.
\]

Then, \(I^w \neq \emptyset\). For each \(i \in I^w\), define

\[
C_i^w := \{l \in [k] - \{i\} : w \in \text{(the } \{i, l\}-\text{facet of } \bar{E}_i)\},
\]

\[
D_i^w := [k] - \{i\} - C_i^w.
\]
Then, $i \notin C_i^w$ and $i \notin D_i^w$. In particular, for $w = \{v_1, \ldots, v_k\}$, say for $w = v_j$,
\[
I^{v_j} = \{j\}
\]
\[
C_j^{v_j} = [k] - \{j\}
\]
\[
D_j^{v_j} = \emptyset
\]

We write $I$, $C_i$ and $D_i$ without the superscript $w$ unless confusion could arise.

Let $e_1, \ldots, e_k$ be the standard basis vectors of $\mathbb{R}^k$. For two $i, j \in I$, there are nonnegative real numbers $a_l$ for $l \in D_i$ and $b_m$ for $m \in D_j$ such that
\[
w \equiv v_i + \sum_{l \in D_i} a_l(-e_l) \equiv v_j + \sum_{m \in D_j} b_m(-e_m).
\]
Because the maximum of coordinates of $w - v_i$ is its $i$-th coordinate, $i \in D_j$ with $b_i \neq 0$, and similarly $j \in D_i$ with $a_j \neq 0$. Thus, $I \subseteq D_i \cup \{i\}$ for any $i \in I$, and
\[
I \cap C_i = \emptyset.
\]

The vertex $w$ is an intersection of $k - 1$ branches, each of which contains one and only one vertex in $\{v_i : i \in I\}$ by maximality of $P$. More specifically,
\[
w = \bigcap_{i \in I} \bigcap_{l \in C_i} \text{(the } \{i, l\} \text{-facet of } E_i)\]
where $\bigcap_{l \in C_i} \text{(the } \{i, l\} \text{-facet of } E_i)$ for $i \in I$ has codimension $|C_i|$ and
\[
\sum_{i \in I} |C_i| = k - 1.
\]
A computation shows that
\[
\bigcup_{i \in I} C_i = [k] - I \text{ and } \bigcap_{i \in I} D_i = \emptyset.
\]
If $w \in \text{Vert}^0(P)$, then $|I^w| \geq 2$ and vice versa, and we have
\[
0 \leq \bigcap_{i \in I} C_i \leq 1 \text{ and } k - 1 \leq \bigcup_{i \in I} D_i \leq k.
\]
Now, for any vertex $w \in \text{Vert}(P)$ and for all $i \in [k] - I$ define
\[
C_i := \emptyset \text{ and } D_i := \emptyset.
\]
Hence, $C_i$ and $D_i$ are defined for all $i \in [k]$, and $C_i \neq \emptyset$ if and only if $i \in I$.

**Notation 5.1.** For a vertex $w \in \text{Vert}(P)$, we introduce the following notation
\[
w = v_{1}^{C_1} \cdots v_{k}^{C_k} = \prod_{i \in [k]} v_i^{C_i}.
\]
Practically, we remove all $v_i^{C_i}$ with $C_i = \emptyset$ from the notation and write
\[
w = \prod_{i \in I} v_i^{C_i}.
\]
In particular, $v_i = v_i^{[k] - \{i\}}$ for a vertex $v_i$. 
5.2. Faces of biconvex polytopes. Let \( Q \) be a face of \( P \). For \( i \in [k] \), denote
\[
\text{Vert}_i(Q) := \{ w \in \text{Vert}(Q) : C^w_i \neq \emptyset \}
\]
and define
\[
I^Q := \{ i \in [k] : \text{Vert}_i(Q) \neq \emptyset \},
\]
\[
C^Q_i := \bigcap_{w \in \text{Vert}_i(Q)} C^w_i,
\]
\[
D^Q_i := [k] - \{ i \} - C^Q_i.
\]
Then,
\[
Q = \bigcap_{i \in I^Q} \bigcap_{l \in C^Q_i} \text{the } \{i, l\}-\text{facet of } \overline{E}_i \cap P
\]
whose codimension is
\[
\text{codim}(Q) = \sum_{i \in I^Q} |C^Q_i|
\]
which are consistent with (5.2) and (5.3). Note that \( I^Q \neq \emptyset \) and \( C^Q_i \neq \emptyset \) for all \( i \in I^Q \). For all \( i \in [k] - I^Q \) define
\[
C^Q_i := \emptyset \quad \text{and} \quad D^Q_i := \emptyset.
\]
Hence, \( C^Q_i \) and \( D^Q_i \) are defined for all \( i \in [k] \), and \( C^Q_i \neq \emptyset \) if and only if \( i \in I^Q \). Thus, we generalize the notions for vertices. We also generalize Notation 5.1.

Notation 5.2. For a face \( Q \) of \( P \), we denote
\[
Q = v_1^{C^Q_1} \cdots v_k^{C^Q_k} = \prod_{i \in [k]} v_i^{C^Q_i}.
\]

Remark 5.3. Notation 5.2 describes how to obtain a face \( Q \) of \( P \) from the unique generating set of vertices \( \{v_1, \ldots, v_k\} \). It further generalizes to a biconvex polytope that is not maximal and the uniqueness of expression still holds.

5.3. Bigraphs and faces of biconvex polytopes. To each vertex \( w \) of \( P \), assign the graph \( G_w \) with the node set \( [k] \) satisfying that for \( i, c \in [k] \),
\[
(i, c) \text{ is an edge of } G_w \quad \text{if and only if} \quad i \in I^w \text{ and } c \in C^w_i.
\]
Then, \( G_w \) is a bigraph with parts \( (I^w, [k] - I^w) \) by (5.1) and is a connected cycle-free bigraph since it is a tree by (5.3).

We assume all bigraphs \( G_w \) have the same underlying partition \( S = \bigcup_{i \in [k]} S_i \).

Let \( Q \) be a nonempty proper face of \( P \). Define a graph \( G_Q \) with the node set \( [k] \) such that
\[
(i, c) \text{ is an edge of } G_Q \quad \text{if and only if} \quad i \in I^Q \text{ and } c \in C^Q_i.
\]
Then, \( G_Q \) is another notation for a face \( Q \) of \( P \) which is equivalent to Notation 5.2. Given a vertex \( w \) of \( Q \), the graph \( G_Q \) is obtained from \( G_w \) by deleting edges \( (i, c) \) with \( i \in I^w \) and \( c \in C^w_i - C^Q_i \), that is,
\[
Q = \prod_{i \in [k]} v_i^{C^w_i - (C^w_i - C^Q_i)}.
\]
Hence, $G_Q$ is a cycle-free bigraph. Note that its bipartite structure is induced from that of $G_w$, but does not depend on the choice of the vertex. The number of its connected components minus the number of its isolated nodes is

\[
\dim(Q) = \sum_{i \in I^w} \left| C_i^w - C_i^Q \right|.
\]

5.4. **Edges of biconvex polytopes.** Let $Q$ be an edge of $P$ with vertices $\{w_1, w_2\}$, then there are edges $(i_1, c_1)$ and $(i_2, c_2)$ of $G_{w_1}$ and $G_{w_2}$, respectively, such that

\[
Q = \left( \prod_{i \in I^w - \{i_1\}} v_i^{C_{i1}^w} \right) v_{i1}^{C_{i1}^w - \{c_1\}} = \left( \prod_{i \in I^w - \{i_2\}} v_i^{C_{i2}^w} \right) v_{i2}^{C_{i2}^w - \{c_2\}}
\]

where $i_1 \in I^w$ and $c_l \in C_{i_l}^w$ for $l = 1, 2$. In other words,

\[
G_Q = G_{w_1}(i_1, c_1) = G_{w_2}(i_2, c_2)
\]

with

\[
G_1' := G_{w_1}^+(i_1, c_1) = G_{w_2}^-(i_2, c_2),
G_2' := G_{w_1}^-(i_1, c_1) = G_{w_2}^+(i_2, c_2).
\]

Let $I_l = I^w \cap V_{G_l}$ and $C_l = V_{G_l} - I_l$ for $l = 1, 2$. Then, $G_1'$ and $G_2'$ are the two connected components of $G_Q$ containing $\{i_1, c_2\}$ and $\{i_2, c_1\}$, respectively, which are connected cycle-free bigraphs with parts $(I_1, C^1)$ and $(I_2, C^2)$, respectively, and

\[
G_Q = G_1' \cup G_2'
\]

is a cycle-free bigraph with parts $(I_1 \cup I_2, C^1 \cup C^2)$.

Let $Q_1$ and $Q_2$ be the faces of $P$ corresponding to two bigraphs $G_1' \cup V_{G_2}'$ and $G_2' \cup V_{G_1}'$, respectively, which are biconvex polytopes, then

\[
Q = Q_1 \cap Q_2
\]

where the sets of isolated nodes $V_{G_2}' = [k] - V_{G_1}'$ and $V_{G_1}' = [k] - V_{G_2}'$ are dummy.

Any point of $Q_1$ is contained in the $\{l, c\}$-max-facet at $v_l$ for all $l \in I_1$ and $c \in C_l^Q$, and for each $i \in V_{G_1}'$ its $i$-th coordinate is a fixed real number, say $a_i$. Also, for any point of $Q_2$, its $j$-th coordinate for each $j \in V_{G_2}'$ is a fixed real number, say $b_j$. So, given a point $u$ of $Q$ there are real numbers $x_i$ for $i \in V_{G_1}'$ and $y_j$ for $j \in V_{G_2}'$ such that

\[
u \equiv \sum_{i \in V_{G_1}'} a_i e_i + \sum_{j \in V_{G_2}'} y_j e_j = \sum_{i \in V_{G_1}'} x_i e_i + \sum_{j \in V_{G_2}'} b_j e_j.
\]

Then, because $i_1 \in V_{G_1}'$ and $i_2 \in V_{G_2}'$ there are positive real numbers $t_1$ and $t_2$ with

\[
w_1 = \sum_{i \in V_{G_1}'} a_i e_i + \sum_{j \in V_{G_2}'} b_j e_j - t_1 1_{V_{G_1}'},
\]

\[
w_2 = \sum_{i \in V_{G_1}'} a_i e_i + \sum_{j \in V_{G_2}'} b_j e_j - t_2 1_{V_{G_1}'}.
\]

Therefore, since $1_{V_{G_2}'} = -1_{V_{G_1}'}$, we have

\[
w_2 - w_1 = -t_2 1_{V_{G_1}'} + t_1 1_{V_{G_2}'} = -(t_1 + t_2) 1_{V_{G_1}'}.
\]
Definition 5.4. Let $P = \text{tconv}(v_1, \ldots, v_k) \subset \mathbb{R}^k/\mathbb{R}\mathbb{I}$ be a maximal biconvex polytope. For each vertex $w$ of $P$, there are exactly $k - 1$ edges $Q$ of $P$. For each edge $Q$ with $\text{Vert}(Q) = \{w, v\}$, there is a unique subset $W$ of $[k]$ such that
\[ v \equiv w - t \cdot 1^W \]
for a positive real number $t \in \mathbb{R}$ which is a unique such. We define $L^w$ such that
\[ L^w : \{k - 1 \text{ edges } Q \text{ of } P \text{ containing } w\} \rightarrow 2^{[k]} \]
\[ Q \mapsto W \]
which we call the combinatorial log map for $P$, see [GKZ94, Chapter 6.1.B] for the usual log map. In particular, $\emptyset \neq L^w(Q) \neq [k]$ and $L^w(Q) = [k] - L^w(Q)$.

Remark 5.5. Let $P'$ be a tropical degeneration of $P$. Because the direction vectors of edges of $P'$ are direction vectors of edges of $P$, the combinatorial log map is defined for all biconvex polytopes.

6. Biconvex Polytopes and Tropical Linearity

6.1. Tropical linear spaces. Let $M$ be a rank-$k$ connected matroid on $S$ with $n = |S|$, and $B$ its base collection. The Dressian $D_M$ of $M$ is the intersection of the tropical hypersurfaces in $\mathbb{R}^B/\mathbb{R}\mathbb{I}$ defined by the tropicalized Plücker relations for $M$, where a tropicalized Plücker relation for $M$ is a tropical polynomial
\[ \bigoplus_{j \in \tau'} y_{\sigma \cup \{j\}} \odot y_{\tau - \{j\}} \]
for a coordinate vector $y = (y_B)_{B \in B} \in \mathbb{R}^B/\mathbb{R}\mathbb{I}$ that satisfies the following:
- $\sigma$ is an independent set of $M$ of size $k - 1$,
- $\tau$ is a rank-$k$ subset of $S$ of size $k + 1$ with $\sigma \not\subseteq \tau$,
- $\tau'$ is the set of $j$ in $\tau$ such that both $\sigma \cup \{j\}$ and $\tau - \{j\}$ are bases of $M$.

Fix any point $y$ in $D_M \subset \mathbb{R}^B/\mathbb{R}\mathbb{I}$. For any rank-$k$ subset $\tau$ of $S$ of size $k + 1$ with $\tau' = \{j \in \tau : \tau - \{j\} \in B\}$, we denote by $L_{\tau}(y)$ the tropical hyperplane in $\mathbb{R}^S/\mathbb{R}\mathbb{I}$ that is defined by a tropical polynomial
\[ \bigoplus_{j \in \tau'} y_{\tau - \{j\}} \odot x_j \]
for a coordinate vector $x = (x_j)_{j \in S} \in \mathbb{R}^S/\mathbb{R}\mathbb{I}$. Now, we define
\[ L_y := \bigcap_{\tau} L_{\tau}(y) \]
which is a $(k - 1)$-dimensional balanced contractible polyhedral complex in $\mathbb{R}^S/\mathbb{R}\mathbb{I}$, and called a tropical linear space. Then, the Dressian $D_M$ is a moduli space whose fibers are $(k - 1)$-dimensional tropical linear spaces in the $(n - 1)$-dimensional tropical projective space.

6.2. Biconvex polytopes and dual matroid subdivisions. Every $y \in \mathbb{R}^B/\mathbb{R}\mathbb{I}$ induces a regular subdivision of the matroid polytope $BP_M$. Furthermore,

Proposition 6.1. A point $y$ of $\mathbb{R}^B/\mathbb{R}\mathbb{I}$ is contained in $D_M$ if and only if the regular subdivision of $BP_M$ that $y$ induces is a matroid subdivision.
Let \( \Sigma \) be the regular matroid subdivision of \( \text{BP}_M \subseteq \Delta^k_S \) that \( y \) induces. For any matroid polytope \( \text{BP}_{M_i} \subseteq \Delta^k_S \) of \( \Sigma \), its involution \( \mathbb{1} - \text{BP}_{M_i} \subseteq \Delta^{n-k}_S \) is the matroid polytope \( \text{BP}_{M_i}^* \) of the dual matroid \( M_i^* \), and
\[
\Sigma^* := \{ \mathbb{1} - P \subseteq \Delta^{n-k}_S : P \in \Sigma \}
\]
is a matroid subdivision of \( \text{BP}_{M_i}^* \subseteq \Delta^{n-k}_S \) where \( (\Sigma^*)^* \) = \( \Sigma \).

There is a bijection between \( \Sigma^* \) and the set of vertices of the tropical linear space \( L_\Sigma \), which satisfies the following two conditions, and we say that \( \Sigma^* \) is dual to \( L_\Sigma \).

(D1) For each line segment \( w_1w_2 \) of \( L_\Sigma \) with two vertices \( w_1 \) and \( w_2 \), there is a positive real number \( t \) such that for the vector \( w_1w_2 \),
\[
\overrightarrow{w_1w_2} \equiv -t \cdot 1^w(w_1w_2).
\]
I.e. the two corresponding matroid polytopes \( \text{BP}_{w_1} \) and \( \text{BP}_{w_2} \) of \( \Sigma^* \) have a common facet through which they are face-fitting, and the vector \( w_1w_2 \) is out normal to \( \text{BP}_{w_1} \).

(D2) Each ray at a vertex of \( L_\Sigma \) is perpendicular to a facet of the corresponding matroid polytope that is not contained in a coordinate hyperplane of \( \mathbb{R}^n \), and conversely for every such facet there is a perpendicular ray of \( L_\Sigma \).

**Definition 6.2.** Let \( P \) be a biconvex polytope. If \( \Sigma^* \) is a matroid tiling satisfying (D1) with \( L_\Sigma \) replaced by \( P \), we say that \( \Sigma^* \) is dual to \( P \).

If \( \Sigma^* \) is a matroid subdivision dual to a biconvex polytope, it is automatically regular. Now, we prove our main theorem.

**Theorem 6.3.** Let \( P = \text{tconv} (v_1, \ldots, v_k) \subseteq \mathbb{R}^k/\mathbb{Z} \) be a biconvex polytope. Then, there exists a matroid subdivision of a hypersimplex that is dual to \( P \). Thus, every biconvex polytope arises as a bounded cell of a tropical linear space.

**Proof.** We may assume \( P \) is a maximal biconvex polytope. For all vertices \( w \) of \( P \), the corresponding bigraphs \( G_w \) have the same node set \([k]\) and let \( S = \bigcup_{j \in [k]} S_j \) with \( n = |S| \) be a common underlying partition. Then,
\[
\text{BP}_{M[\text{G}_w]} = \text{BP}_{\oplus_{i \in [k]} U_i^k,
\]
is a codimension-\((k-1)\) matroid polytope that is not contained in the boundary of the hypersimplex \( \Delta^k_S \), which is a common face of all matroid polytopes \( \text{BP}_{M[\text{G}_w]} \) with \( w \in \text{Vert}(P) \). Let
\[
\Sigma = \{ \text{BP}_{M[\text{G}_w]} \subseteq \Delta^k_S : w \in \text{Vert}(P) \}.
\]
For each vertex \( w \) of \( P \), there are precisely \( k-1 \) edges \( \overrightarrow{wv} \) of \( P \) with \( v \in \text{Vert}(P) \) so that two bigraphs \( G_w \) and \( G_v \) are face-fitting and there is a unique positive real number \( t \) with
\[
\overrightarrow{wv} \equiv -t \cdot 1^w(\overrightarrow{wv}).
\]
Two subsets \( S_{\text{Lw}(\overrightarrow{wv})} \) and \( S_{\text{Lv}(\overrightarrow{wv})} \) of \( S \) are non-degenerate flats of the two matroids \( M[\text{G}_w] \) and \( M[\text{G}_v] \), respectively, and
\[
\text{BP}_{M[\text{G}_w]}|_{S_{\text{Lw}(\overrightarrow{wv})}} = \text{BP}_{M[\text{G}_v]}|_{S_{\text{Lv}(\overrightarrow{wv})}}
\]
is a common facet of both \( \text{BP}_{M[\text{G}_w]} \) and \( \text{BP}_{M[\text{G}_v]} \) through which these two matroid polytopes are face-fitting. In particular, the two vectors \( \overrightarrow{wv} \) and \( \overrightarrow{vw} \) are out normal to \( \text{BP}_{M[\text{G}_w]} \) and \( \text{BP}_{M[\text{G}_v]} \) in due order.
Because this is true for all vertices $w$ of $P$, the collection $\Sigma$ is a matroid tiling. Moreover, all codimension-1 cells of $\Sigma$ that are not shared by two matroid polytopes of $\Sigma$ are contained in the boundary of $\Delta_S^k$ by Lemmas 3.7 and 2.1(2). Therefore, $\Sigma$ is a matroid subdivision of $\Delta_S^k$. Now,

$$\Sigma^* = \{1 - BP_{M(G_w)} \subseteq \Delta_S^{n-k} : w \in \text{Vert}(P)\}$$

is a matroid subdivision of $\Delta_S^{n-k}$, and $\Sigma = (\Sigma^*)^*$ is dual to $P$. Thus, $P$ is a bounded cell of a tropical linear space. \hfill \qed

6.3. Faces of biconvex polytopes revisited.

Proposition 6.4. For a $(k-1)$-dimensional biconvex polytope $P$, let $\Sigma$ be a matroid subdivision of $\Delta_S^k$ dual to it with an underlying partition $S = \bigcup_{i \in [k]} S_i$. Then, there is a one-to-one correspondence between the set of the nonempty proper faces of $P$ and the set of the cells of $\Sigma$ that strictly contain $BP_{\bigoplus_{i \in [k]} U_i}$.

Proof. We may assume that $P = \text{tconv}(v_1, \ldots, v_k) \subseteq \mathbb{R}^k/\mathbb{R}1$ is a maximal biconvex polytope. Let $Q$ be a nonempty proper face of $P$ and $G_Q$ the corresponding bigraph. Let $w$ be a vertex of $Q$, then $BP_{M(G_Q)}$ is a face of $BP_{M(G_w)}$ and is a cell of $\Sigma$. Further, $G_Q$ has an edge by (5.5), and $BP_{M(G_Q)}$ strictly contains $BP_{M[\text{Vert}(Q)]} = BP_{\bigoplus_{i \in [k]} U_i}$. Conversely, any cell of $\Sigma$ that strictly contains $BP_{\bigoplus_{i \in [k]} U_i}$ is a face of $BP_{M(G_w)}$ for a vertex $w$ of $P$ that is expressed as $BP_{M[G]}$ for a bigraph $G$ that is obtained by deleting edges from $G_w$. Hence, there is a unique (nonempty proper) face $Q$ of $P$ with $G = G_Q$. The proof is done. \hfill \qed

Definition 6.5. For any full-dimensional biconvex polytope $P = \text{tconv}(v_1, \ldots, v_k)$ in $\mathbb{R}^k/\mathbb{R}1$ for some $k \geq 2$, we define a map $\mu$ such that

$$\mu : \{\text{the proper faces of } P\} \rightarrow \{\text{the monomials in } x_1, \ldots, x_k \text{ of degree } \leq k-1\}$$

$$Q = v_1^{C_Q} \cdots v_k^{C_Q} \quad \Rightarrow \quad x_1^{C_Q} \cdot \cdots \cdot x_k^{C_Q}$$

Proposition 6.6. The map $\mu$ is injective.

Proof. We prove by induction. For all 1-dimensional biconvex polytopes the map $\mu$ is injective, and the base case holds. Suppose that $\mu$ is injective for all biconvex polytopes of dimension $< k$. We may assume that $P$ is a maximal biconvex polytope of dimension $k$. Let $Q_1$ and $Q_2$ be nonempty proper faces of $P$ with $\mu(Q_1) = \mu(Q_2)$, then $I^{Q_1} = I^{Q_2}$.

We first show that $G_{Q_1}$ and $G_{Q_2}$ have a common edge. If $G_{Q_1}$ and $G_{Q_2}$ have no common edges, then their maximum common subgraph is $V_{G_{Q_1}} = V_{G_{Q_2}}$ while they are face-fitting bigraphs by Theorem 6.3. Since none of them is a subgraph of the other, they have subgraphs $G'_1$ and $G'_2$ with edges $e_1 = (i_1, c_1)$ and $e_2 = (i_2, c_2)$, respectively, satisfying (3.4). Then, for $l = 1, 2$ we have $G'_l = \{e_l\} \cup V_{G_{Q_l}}$ with

$$(G'_1)^+(e_l) = \{i_l\} \quad \text{and} \quad (G'_1)^-(e_l) = \{c_l\}.$$ 

However, by (3.4),

$$\{i_1\} = (G'_1)^+(e_1) = (G'_2)^-(e_2) = \{c_2\}$$

and therefore

$$c_2 \in I^{Q_1} \cap C_{i_2}^{Q_2} = I^{Q_2} \cap C_{i_2}^{Q_2} = \emptyset$$

which is a contradiction. Thus, $G_{Q_1}$ and $G_{Q_2}$ have a common edge, say $(i, c)$. 
Then, $Q_1$ and $Q_2$ are contained in the $\{i, c\}$-facet of $E_i$ whose intersection with $P$ is a biconvex polytope of dimension $< k$, which has a unique inclusionwise minimal generating set, cf. [DS04, Proposition 21], and inherits its geometry from $P$. Thus, $Q_1 = Q_2$ by the induction hypothesis, which proves that $\mu$ is injective. \qed

**Remark 6.7.** If $P$ is a maximal biconvex polytope, the map

$$
\text{Vert}(P) \xrightarrow{\mu} \{\text{the degree-}(k-1)\text{ monomials in } x_1, \ldots, x_k\}
$$

is a bijection because the maximum number of vertices of $P$ equals the number of degree-$(k-1)$ monomials in $k$ indeterminates, which is $\binom{2k}{k-1}$.

**Definition 6.8.** Let $P = \text{tconv}(v_1, \ldots, v_k) \subset \mathbb{R}^k/\mathbb{Z}$ be a maximal biconvex polytope. Along the formula (5.4) we define the type of a vertex $w \in \text{Vert}^0(P)$:

$$
\text{type}(w) = \begin{cases} 0 & \text{if } \bigcap_{i \in I^w} C^w_i = \emptyset, \\ 1 & \text{otherwise}. \end{cases}
$$

**Example 6.9.** If $k \leq 4$, every vertex $w \in \text{Vert}^0(P)$ has type 1.

**Example 6.10.** The edge structure at a type-1 vertex $w \in \text{Vert}^0(P)$ is particularly nice because if $\bigcap_{i \in I^w} C^w_i = 1$, all subsets $C^w_j - \bigcap_{i \in I^w} C^w_i$ of $[k]$ with $j \in I^w$ are mutually disjoint, and the bigraph $G_w$ has a nice structure, see Figure 6.1. It is easy to find all $k-1$ edges $Q$ of $P$ containing $w$, whose images under $L^w$ are

$$
L^w(Q) = \begin{cases} C^w_j - \bigcap_{i \in I^w} C^w_i & \text{for } j \in I^w, \\ [k] - \{j\} & \text{for } j \in [k] - \bigcap_{i \in I^w} C^w_i - I^w. \end{cases}
$$

**Figure 6.1.** The Bigraph $G_w$ of a Type-1 Vertex $w$

### 6.4. Cutting hypersimplices into matroid polytopes.

**Lemma 6.11 ([Shi19]).** Let $S$ be a finite set and $k$ an integer with $1 \leq k < n = |S|$. Let $F \subset S$ be a proper subset of size $\geq 2$ and $\rho$ a positive integer satisfying that

$$
k - n + |F| < \rho < \min \{k, |F|\}.
$$

Then, $P_1 = \Delta^k_S \cap \{x(F) \leq \rho\}$ and $P_2 = \Delta^k_S \cap \{x(S-F) \leq k-\rho\}$ are face-fitting full-dimensional matroid polytopes which form a matroid subdivision of $\Delta^k_S$. Let $P_1 = \text{BP}_{M_1}$ and $P_2 = \text{BP}_{M_2}$, then $F$ and $S - F$ are the only non-degenerate flats of size $\geq 2$ of $M_1$ and $M_2$, respectively, whose ranks are $\rho$ and $k-\rho$.

**Lemma 6.12 ([Shi19]).** Let $S$ be a finite set and $k$ an integer with $1 < k < |S|$. For a partition $S = \bigcup_{j \in [k]} S_j$, cutting $\Delta^k_S$ with all $k$ hyperplanes of the form $\{x(S_j) = 1\}$ produces a matroid subdivision.
Example 6.13. Let $k = 4$. By cutting $\Delta_4^{k}$ with all 4 hyperplanes $\{x(S_i) = 1\}$ we get the matroid subdivision of Figure 6.2 in which polytopes are quotient polytopes, modulo $\text{Aff}_0(Q)$, where

$$Q = \text{BP} \bigoplus_{i \in [4]} U_{S_i}$$

is their maximum common face which is represented as a big dot in the barycenter, and $\text{Aff}_0(Q)$ is the linear span of $Q - q$ for a point $q \in Q$. See also Figure 6.3.

Figure 6.2. The matroid subdivision of Lemma 6.12 for $k = 4$, represented as a quotient subdivision, modulo $\text{Aff}_0(Q)$.

Figure 6.3. Three matroid polytopes of the matroid subdivision of Example 6.13 with the corresponding bigraphs (if applicable) which are from Figures 3.2 and 3.3.
6.5. **A rank-4 example of our main theorem.** The 3-dimensional maximal biconvex polytopes are computed in [JK10]. In this subsection, we study matroid subdivisions of the hypersimplex $\Delta^4_S$ whose maximum common cell is a matroid polytope of codimension 3 that is not contained in the boundary of $\Delta^4_S$, where regular ones among them are dual to 3-dimensional biconvex polytopes.

Observe that in Example 6.13, by cutting $\Delta^4_S$ with 4 hyperplanes $\{x(S - S_i) = 3\}$ instead, we obtain the same matroid subdivision because

$$\{x(S - S_i) = 3\} = \{x(S_i) = 1\}.$$ 

Observe also that the 4 polytopes

$$\bigcap_{j \in [4] - \{i\}} \{x(S_j) \geq 1\}$$

whose quotients are positioned at the 4 corners of the tetrahedron of Figure 6.2 cannot be further cut into full-dimensional matroid polytopes and neither can the 4 polytopes

$$\bigcap_{j \in [4] - \{i\}} \{x(S_j) \leq 1\}$$

whose quotients are positioned at the centers of the 4 facets of the tetrahedron.

Therefore if we want to obtain a matroid subdivision of $\Delta^4_S$ with $Q = BP_{\bigoplus_{i \in [4]} U^1_{S_i}}$ being a common cell that is finer than the matroid subdivision of Example 6.13, we must cut each of its 6 polytopes other than the 8 polytopes whose quotients are parallelepipeds or tetrahedra, with hyperplanes of the form

$$\{x(A) = 2\}$$

for a subset $A \subset S$ because it contains $Q$, cf. Lemma 2.1(2). Then, $A$ is a rank-2 flat of $\bigoplus_{i \in [4]} U^1_{S_i}$, and is a union of two of $S_1, \ldots, S_4$. Thus, the hyperplanes we cut with have the form

$$\{x(S_J) = 2\} \quad \text{for a size-2 subset } J \subset [4].$$

Moreover, by the following lemma, the number of such cutting hyperplanes cannot exceed 1, and hence is 1.

**Lemma 6.14.** Let $M$ be a rank-4 connected matroid with a rank-2 non-degenerate flat $F$. If $L$ is a non-degenerate flat such that $BP_{M(F)} \cap BP_{M(L)}$ is a codimension-2 face of $BP_M$ that is not contained in a coordinate hyperplane, then $r(L) \neq 2$.

**Proof.** To prove by contrapositive, suppose $r(L) = 2$. Then, $r(L \cap F) < 2$ since $L \neq F$, and $\{F, L\}$ is a modular pair by Lemma 2.1(3) since $BP_{M(F)} \cap BP_{M(L)}$ is nonempty. So, we have either $r(F \cap L) = 0$ and $r(F \cup L) = 4$, or $r(F \cap L) = 1$ and $r(F \cup L) = 3$.

In the former case, $F \cap L = \emptyset$ which implies that $F \cup L$ is a non-flat of rank 4 by Lemma 2.1(4), but then $M(F) \cap M(L) = M(F \cup L)$ has a loop, a contradiction. In the latter case, $r(F) = 2$ and moreover $F \subset L$ or $L \subset F$ by Lemma 2.1(4) since $F \cap L \neq \emptyset$ and $F \cup L \neq S$, but this contradicts that $r(F) = r(L) = 2$. Thus, we conclude that $r(L) \neq 2$. \hfill \box

Lemmas 6.12 and 6.14 tell that for a matroid subdivision of $\Delta^4_S$, if $Q = BP_{\bigoplus_{i \in [4]} U^1_{S_i}}$ is a codimension-3 common cell of its members, then the number of its members does not exceed $4 + 4 + 2 \cdot 6 = 20$ which is the maximum number of vertices of a
3-dimensional biconvex polytope. Note that this maximum is attained by Theorem 6.3. It also tells that every finest such matroid subdivision is obtained from the matroid subdivision of Example 6.13 by cutting its members with some hyperplanes of the form (6.1). Then, the following question arises.

**Question.** Do we always obtain a matroid subdivision by cutting each member of the matroid subdivision of Example 6.13, with a hyperplane of the form (6.1)?

The answer is YES because there are precisely 2 candidates for those subdivisions of $P$ of Figure 6.3 with $Q$ being a common cell of codimension 3, and our bigraph argument proves that those candidates are indeed matroid subdivisions, see Figures 6.4 and 6.5, which applies to the other 5 matroid polytopes in the same manner.

Another way to see that the answer is yes is to observe that the above mentioned 2 candidates are subtilings of larger polyhedral subdivisions of a matroid polytope whose matroid has the simplification that is a rank-3 uniform matroid, and prove that the polyhedral subdivision is actually a matroid subdivision, for which one can use Lemma 6.11, see Figure 6.6.

**References**

[DS04] M. Develin and B. Sturmfels, *Tropical convexity*, Doc. Math. 9 (2004), 1–27. 13, 14, 21

[GKZ94] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, Resultants, and Multidimensional Determinants*, Mathematics: Theory & Applications, Birkhauser Boston Inc., Boston, MA, 1994. 18

[GS87] I. M. Gelfand and V. V. Serganova, *Combinatorial geometries and the strata of a torus on homogeneous compact manifolds*, Uspekhi Mat. Nauk 42 (1987), no. 2(254), 107–134. 3

[JK10] M. Joswig and K. Kulas, *Tropical and ordinary convexity combined*, Adv. Geom. 10 (2010), no. 2, 333–352. 23

[MS15] D. Maclagan and B. Sturmfels, *Introduction to Tropical Geometry*, Grad. Stud. Math. vol. 161, Amer. Math. Soc., Providence, RI, 2015. 2

[Oxl11] J. Oxley, *Matroid Theory*, second edition, Oxf. Grad. Texts Math., vol. 21, Oxford Univ. Press, New York, 2011. 2

[Sch03] A. Schrijver, *Combinatorial Optimization: Polyhedra and Efficiency*, Springer-Verlag, Berlin, 2003. 2, 4

[Shi19] J. Shin, *Geometry of matroids and hyperplane arrangements*, arXiv:1912.12449. 3, 4, 21

Korea Institute for Advanced Study, 85 Hoegiro, Seoul 02455, South Korea

Email address: shin@kias.re.kr
Figure 6.4. A matroid subdivision of P of Figure 6.3 and the corresponding bigraphs I, cf. Figure 3.2.
Figure 6.5. A matroid subdivision of $P$ of Figure 6.3 and the corresponding bigraphs II, cf. Figure 3.3.
Figure 6.6. Two matroid subdivisions of $P$ as subtilings of larger matroid subdivisions.