Observable Effects of Scalar Fields and Varying Constants

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I. INTRODUCTION

The last few years have seen a resurgence of widespread interest in the possibility that some or all of the fundamental ‘constants’ of Nature might be varying over cosmological timescales. To a large extent the revival in this field has come about because of the need to explain and understand the studies of relativistic fine structure in the absorption lines formed in dust clouds around quasars carried out by Webb et al. Using a new “many-multiplet” method which exploits the information in many wavelength separations of absorption lines with different relativistic contributions to their fine structure considerable gains in statistical significance were achieved. From a data set of 128 objects at redshifts 0.5 < z < 3, Webb et al. found their absorption spectra were consistent with a relative in the value of the fine structure constant, \( \alpha_{em}(z) \), between those redshifts and the present of \( \Delta \alpha_{em}/\alpha_{em} = \alpha_{em}(z) - \alpha_{em}(0)/\alpha_{em}(0) = -0.57 \pm 0.10 \times 10^{-5} \). In the seven years since their results were first announced, extensive analysis has yet to identify any systematic effect that could explain either its magnitude or sign. However, a small study of 23 absorption systems in one hemisphere between 0.4 ≤ z ≤ 2.3 by Chand et al., found a result consistent with no variation: \( \Delta \alpha_{em}/\alpha_{em} = -0.6 \pm 0.6 \times 10^{-6} \) but this study uses only a simplified version of the many-multiplet method in its analysis and concerns remain about calibrations and the noisiness of the data fits. Later this year, a major effort to produce a very large new data set should be reported and will clarify the status of these earlier investigations. In the meantime there have been other astronomical checks on the constancy of \( \alpha_{em} \), using a variety of different techniques; however, they have yet to reach the accuracy achieved using the many-multiplet method. All found results that are consistent with no variation but none could have seen the variation reported by Webb et al.

If \( \alpha_{em} \) can, and does, vary in time then it would seem natural to suspect that some other fundamental ‘constants’ do so also. Recently, using a study of the vibrational levels of \( \text{H}_2 \) in the absorption spectra of quasars, Reinhold et al., reported a 3.5σ indiction of a variation in the electron-proton mass ratio, \( \mu = m_e/m_p \) over the last 12 Gyrs: \( \Delta \mu/\mu = 2.0 \pm 0.6 \times 10^{-5} \). This result combines high-quality astronomical data with improvements in the measurement of crucial laboratory wavelengths to new levels of precision. It reports a variation at a level comparable to that claimed for \( \alpha_{em} \) (which partly reflects state of the art precision in spectroscopic measurements) but is theoretically much harder to understand. Variations in \( \alpha_{em} \) and \( \mu \) both lead to violations of the Weak Equivalence Principle (WEP) and these are expected to be unacceptably large if \( \Delta \mu/\mu \) varies at the \( 10^{-5} \) level. Typically, the experimental constraints on WEP violation from direct experimental studies of freefall and the study of the relative motions of the Earth-Moon system lead to upper bounds on the relative differential accelerations of different materials which are \( O(10^{-12}) \). As explained in the detailed study by Barrow and Magueijo, the Webb et al observations of varying \( \alpha_{em} \) predict violations at the \( 10^{-13} \) level but the Reinhold et al. observations of varying \( \mu \) lead us to expect violations at the \( 10^{-9} \) level.

In any study of varying constants, the data used to constrain our theories which allow variations to occur comes from a number of very different environments and scales: with densities differing by a factor of \( 10^{30} \) or more, and spanning some 12 billion years. In order to be able to use all of the information available we need to know how the results of local laboratory experiments, terrestrial or solar-system bounds from the Oklo natural reactor, and from isotope ratios in meteorites, are related to data coming from astronomical observations on extragalactic scales.
This is the ‘Local vs. Global’ problem for varying-constants. It is an important problem and yet most commentators invariably assume that the local and cosmological observations are directly comparable. This is strong assumption and is almost invariably made without any proof. A priori, it is not obvious that this assumption is true; indeed, in many other theories, not least that for gravity itself, it is not: we do not expect to be able to measure the expansion of the Universe by observing an expansion of the Earth. In this paper we describe the first rigorous proof of why, in almost all varying-constant theories, local experiments will also see any variations in ‘constants’ which occur on cosmological scales.

II. GENERAL THEORY

Before we can solve the ‘Local vs. Global’ problem, we need to introduce the general way in which a constant is promoted to become a dynamical quantity consistent with Einstein’s conception of gravity. In general, a constant, $C$, is allowed to vary by associating it with some scalar field or “dilaton”, $\phi$, i.e. $C \rightarrow C(\phi)$. We usually assume that the scalar field theory associated with $\phi$ has a canonical kinetic structure and the variations of this scalar field contribute to the spacetime curvature like all other forms of mass-energy. Variations in $\phi$ must also conserve energy and momentum and so their dynamics are constrained by a non-linear wave equation of the form

$$\Box \phi = \sum_{j,k} f_{j,\phi}(\phi) L_j(\varepsilon_k, p_k),$$

where $\phi$ is associated with the variation of one or more ‘constants’, $C_j$, via a relation $C_j = f_j(\phi)$; $f_{j,\phi}(\phi) = df_j(\phi)/d\phi$. The $L_j(\varepsilon_k, p_k)$ are some linear combinations of the density $\varepsilon_k$, and pressure, $p_k$, of the $k^{th}$ species of matter that couples to the field $\phi$. Included in this formulation are all standard theories for varying constants, like those for the variation of the Newtonian gravitation ‘constant’ $G$, $\alpha_{em}$, and the electron-proton mass ratio, as described in refs. [5, 6, 7, 9].

For local observations to be directly comparable with cosmological ones we need to know the conditions under which

$$\phi(\vec{x}, t) \approx \phi_c(t)$$

to some specified precision, with $\vec{x}$ taking values in the solar system, where the subscript $c$ labels the large-scale cosmological value of the field $\phi$. The validity of this approximate equality, the accuracy to which it holds, and the accompanying preconditions needed to support its validity are the subject of the rest of this paper.

Prior to the onset of the matter era the universe is homogeneous to a very high precision inside the horizon. Any inhomogeneities that do exist, and the evolution of $\phi$ within them, can be consistently and accurately described by linear perturbation theory and $\phi \approx \phi_c$ holds. But the study of the evolution of “constants” becomes mathematically challenging when linear theory breaks down and the inhomogeneities become non-linear. This only starts to occur during the matter era; at these epochs it is an acceptable approximation to consider the Universe to be comprised of only pressureless dust (baryonic and dark matter), density $\varepsilon$, and some cosmological constant, $\Lambda$. We usually expect that the scalar field $\phi$ will couple only to some fraction of the total dust density; for example, in varying-$\alpha_{em}$ theories it couples to the fraction that feels the electromagnetic force, and in varying $m_e$ theories it couples only to the electron density. We will assume, as is almost always the case, that the fraction of matter to which it couples is approximately constant during the epoch of interest. Under these simplifications eqn. 1 reduces to:

$$-\Box \phi = B_{\phi}(\phi)\kappa \varepsilon + V_{\phi}(\phi)$$

where $\kappa = 8\pi$, $c = G = 1$, $V(\phi)$ is the dilaton potential, and $B_{\phi}(\phi)$ is the effective dilaton-to-matter coupling. In what follows we shall assume that the varying-constant evolves according to the above conservation equation, which is certainly true of Brans-Dicke theory, BSBM and BM varying-$\mu$ theory. In what follows we will further assume that the cosmological value of $\phi$, denoted by $\phi_c$, is sufficiently far away from any extrema of the matter coupling, $B(\phi)$. We also demand that $V_{\phi}(\phi)$ is not too large. Our conditions on $B(\phi)$ and $V(\phi)$ are summarised as follows:

$$\left|\frac{B_{\phi}(\phi_c)(\phi(\vec{x}, t) - \phi_c(t))}{B_{\phi}(\phi_c)}\right| \ll 1, \quad |V_{\phi}(\phi(\vec{x}, t))| \lesssim \Lambda,$$

for all values of $\phi$ within the range of interest (i.e. those that can be reached from the evolution of some given initial data). The condition on the matter coupling is usually equivalent to $|B_{\phi}(\phi_c)| \ll 1$ and $B_{\phi}(\phi_c) \neq 0$. The condition on the potential must hold for $\phi = \phi_c$ to prevent the varying “constant” evolving at an unphysically fast
rate cosmologically; the assumption that holds everywhere will then be valid provided that $V_\phi(\phi)$ is suitably flat. As a result of this final assumption our results will not apply to Chameleon field theories, \cite{Chameleon}. With the major exception of theories with "Chameleonic" behavior, our model includes almost all physically viable proposals for varying-constant theories. Our results are also applicable to any scalar-field theory, not just those that describe varying-constants, provided that the scalar satisfies a conservation equation of the above form. For a general matter distribution the dilaton conservation equation is a second-order, non-linear PDE, and there is no reason to suspect that it should be easily solvable, or indeed analytically solvable at all. Even numerical calculations will generally be difficult to set-up and control. Cosmologically, we assume homogeneity and isotropy which leads us to a FRW background and to a solution for $\phi = \phi_c(t)$. Under these specifications the conservation equation for $\phi$ reduces to an ODE in time, and can be solved. The other scenario in which it reduces to an ODE is near a spherically symmetric, static body which couples to the dilaton strongly enough so that any temporal gradients of $\phi$ are negligible compared to the spatial ones. In these cases we can easily find the leading-order static mode of $\phi$, but to find the temporal evolution of $\phi$ we need to enforce the boundary condition that it match up to its cosmological value at large distances. The central technical problem is that the local, static solution was found under the assumption that the temporal derivatives were negligible, and at infinity this is no longer the case. Indeed, to get to a region of space where we know $\phi \approx \phi_c(t)$ we must certainly pass through some zone where the temporal and spatial gradients of $\phi$ are of comparable magnitude. As soon as we reach this zone, the assumptions under which the local solution was derived break down. In short: we cannot consistently apply the boundary condition at infinity to the approximate local solution since spatial infinity is far outside the range of validity of that approximation. We shall express this idea more formally below and see that it is associated with the fact that the local asymptotic approximation is not uniformly valid. To circumvent this problem, created by the presence of multiple length scales, we will use the method of matched asymptotic expansions.

III. MATCHED ASYMPTOTIC EXPANSIONS

Second-order, non-linear PDEs are difficult (and often impossible) to solve exactly. However, if one can identify some small parameter, $\delta$, in the problem then it is usually possible to find an expansion in $\delta$ which is formally asymptotic to the solution in the vicinity of some fixed point. An approximation $\sum_{n=0}^{M} f_n(x)\gamma_n(\delta)$ is asymptotic to a function $f(x, \delta)$ as $\delta \to 0$ iff

$$\frac{f(x, \delta) - \sum_{n=0}^{M} f_n(x)\gamma_n(\delta)}{f_M(x)\gamma_M(\delta)} \to 0 \text{ as } \delta \to 0,$$

for fixed $x$. If this definition holds for all $M$ then we write:

$$f(x, \delta) \sim \sum_{n=0}^{\infty} f_n(x)\gamma_n(x),$$

and $\sum_{n=0}^{\infty} f_n(x)\gamma_n(x)$ is an asymptotic expansion of $f(x, \delta)$ as $\delta \to 0$ for fixed $x$. The sum here is a formal sum, since in general it will not converge; however, as a result of the defining property of asymptotic expansions we will in general need only the first few terms of the sum to obtain a very good approximation to $f(x, \delta)$ at $x$. Asymptotic expansions are unique for each $x$, but it is also clear that an expansion that is asymptotic to $f(x, \delta)$ for some range of $x$, with $x \sim O(1)$ say, will not in general be valid in some other range of $x$, usually $x \sim O(1/\delta)$ or $x \sim O(\delta)$. In these cases the expansion is said to be not uniformly valid. If an expansion has arisen as an approximation to the solution of a PDE and is not uniformly valid, then the PDE is said to exhibit singular behaviour. Such behaviour is often associated with the presence of two or more very different length or times scales in the problem. This is precisely the case in the 'Local vs. Global' cosmological problem, where the length scale of the local inhomogeneity is very much smaller than the Hubble scale which defines the cosmological background.

We can proceed with such problems by constructing two (or more) asymptotic approximations to the solutions which are valid for different ranges of $x$, e.g. for $x \sim O(1)$ and $x/\delta = \xi \sim O(1)$, with

$$f(x, \delta) \sim \sum_{n=0}^{Q} f_n(x)\delta_n \text{ as } \delta \to 0, \quad x \text{ fixed},$$

$$f(x, \delta) \sim \sum_{n=0}^{P} g_n(\xi)\delta_n \text{ as } \delta \to 0, \quad \xi = x/\delta \text{ fixed},$$

and solving the PDE order by order in $\delta$ for both expansions w.r.t. some boundary conditions. We call expansion (2) the outer solution, and (3) the inner solution. The inner expansion is not uniformly valid in the region $\xi = O(1/\delta)$, as the outer one is not valid where $x = O(\delta)$. Because of these restrictions on the size of $x$, we will only be able to apply a subset of the boundary conditions to each expansion; in general, we will therefore be left with unknown coefficients in our asymptotic approximations. This ambiguity can be lifted if there is some intermediate region,
e.g. $x \sim \mathcal{O}(\delta^{1/2})$ where they are both valid, by appealing to the uniqueness of asymptotic expansions and matching the inner and outer solutions there. In this way we can effectively apply all boundary conditions to both solutions. This is the method of matched asymptotic expansions (MAEs). Its application to problems in general relativity was pioneered by Burke, Thorne and D'Eath \cite{17} in the 1970s. For a fuller account of MAEs we refer the reader to refs. \cite{11} and \cite{16}.

IV. GEOMETRICAL SET-UP

The experimental bounds on the permitted level of violations of the WEP due to the presence of light scalar fields demand that the dilaton field couples to matter much less strongly than gravity, so

$$|B_{,\phi}| \ll 1.$$  

As a result, the dilaton field is only weakly coupled to gravity, and so its energy density and motion create metric perturbations which have a negligible effect on the expansion of the background universe. This feature allows us to consider the dilaton evolution on a fixed background spacetime. In this work we go further than we did in ref. \cite{11} and consider not only the extent to which condition $\dot{\phi} \approx \dot{\phi}_c$ is satisfied near the surface of some spherical virialised over-density of matter, e.g. a the Earth, a star, black-hole, galaxy or galaxy cluster, but also the degree to which it is valid \textit{during} the collapse of an over-dense region. We will, however, treat the two cases separately.

In the first case, we shall refer to the virialised over-density as our ‘star’ and take it to have mass $m$ and radius $R_s$ at some time of interest $t = t_0$. Although we require that the ‘star’ itself be spherical, we do not demand that the background spacetime possess any symmetries. We do require however that, at $t = t_0$, the metric is approximately Schwarzschild, with mass $m$, inside some closed region of spacetime bounded by a surface at $r = R_s$; this region is called the \textit{interior}. The metric for $r < R_s$ is left unspecified. We allow for the possibility that $r = R_s$ is a black-hole horizon.

In the second case, we only consider the case where the spacetime is spherically-symmetric, label the mass of the collapsing region by $m$, and assume that its spatial extent is small compared to the Hubble scale. We also demand there are no black-hole horizons in the interior of the collapsing region. By using the results of the first case, however, we can, in some cases, allow for the formation of a horizon.

In both cases we demand that:

- Asymptotically, the metric must approach FRW and the whole spacetime should tend to the FRW metric in the limit $m \to 0$.
- The spacetime is approximately FRW in some open region that extends to spatial infinity, this is called the \textit{exterior}.

We are concerned with spacetimes where the matter is a pressureless dust of density $\varepsilon$, with cosmological constant, $\Lambda$. We further require that the motion of the dust particles be geodesic. In the spherically-symmetric case, all such solutions to Einstein’s equations with matter fall into the Tolman-Bondi class of metrics (for a review of these and other inhomogeneous spherically symmetric metrics see ref \cite{18}), however when condition of spherical symmetry is dropped, the general solution is not known. We can simplify our analysis greatly, however, we specify four further requirements:

1. The flow-lines of the background matter are non-rotating. This implies that the flow-lines are orthogonal to a family of spacelike hypersurfaces, $S_t$.
2. Each of the surfaces $S_t$ is conformally flat.
3. The Ricci tensor for the hypersurfaces $S_t$, $(3) R_{ab}$, has two equal eigenvalues.
4. The shear tensor, as defined for the pressureless dust background, has two equal eigenvalues.

These conditions are automatic if spherical symmetry is required, and in general they specify the Szekeres-Szafron class of solutions, \cite{14,15}, of which the Tolman-Bondi solutions, \cite{20,21}, are the spherically symmetric limit. We require the inhomogeneity to be of finite spatial extent, this limits us to consider only the quasi-spherical Szekeres solutions, which are described by the metric:

$$d s^2 = dt^2 - \frac{(1 + \nu_R R)^2 R^2 d\tau^2}{1 - k(r)} - R^2 e^{2\nu} \left( d\chi^2 + dy^2 \right), \quad (4)$$
where $\nu_R := \nu_r/R$ and

$$e^{-\nu} = A(r)(x^2 + y^2) + 2B_1(r)x + 2B_2(r)y + C(r),$$

$$AC - B_1^2 - B_2^2 = \frac{1}{\tau},$$

and:

$$R_{r}^2 = -k(r) + 2M(r)/R + \frac{1}{3}\Lambda R^2.$$ 

In this quasi-spherically symmetric subcase of the Szekeres-Szaf ron spacetimes the surfaces of constant curvature, $(t, r) = \text{const}$, are 2-spheres [22]; however, they are not necessarily concentric. These 2-spheres have surface area $4\pi R^2$, and so we deem $R$ to be the physical radial coordinate. In the limit $\nu_r \to 0$, the $(t, r) = \text{const}$ spheres becomes concentric. We can make one further coordinate transformation so that the metric on the surfaces of constant curvature, $(t, r) = \text{const}$, is the canonical metric on $S^2$ i.e. $d\theta^2 + \sin^2 \theta d\phi^2$:

$$X \rightarrow X = 2(A(r)x + B_1(r)), \quad Y \rightarrow Y = 2(A(r)y + B_2(r)),$$

where $X + iY = e^{i\phi} \cot \theta/2$. This yields

$$-\nu_r |_{x,y} = \frac{\lambda_z(X^2 + Y^2 - 1) + 2\lambda_x X + 2\lambda_y Y}{X^2 + Y^2 + 1} = \lambda_z(r) \cos \theta + \lambda_x(r) \sin \theta \cos \varphi + \lambda_y(r) \sin \theta \sin \varphi,$$

where we have defined:

$$\lambda_z(r) := \frac{A'}{A}, \quad \lambda_x(r) := \left(\frac{2B_1}{A}\right)' A, \quad \lambda_y(r) := \left(\frac{2B_2}{A}\right)' A.$$

With this choice of coordinates, the local energy density of the dust separates uniquely into a spherical symmetric part, $\varepsilon_s$, and and a non-spherical part, $\varepsilon_{ns}$:

$$\varepsilon = \varepsilon_s(t, R) + \varepsilon_{ns}(t, R, \theta, \phi),$$

where:

$$\kappa\varepsilon_s = \frac{2M_r}{R^2}, \quad (5)$$

$$\kappa\varepsilon_{ns} = -\frac{\nu_r R}{1 + \nu_r R} \left(\frac{2M}{R^2}\right)_R. \quad (6)$$

We define $M_r = M_r/R$. We use the remaining freedom to choose $r$ to demand that $r = R$ at $t = t_0$.

In the virialised case, we follow the conventions of our earlier papers and write $M(r) := m + Z(r)$, where $m$ is the gravitational mass of our ‘star’. In the spherically-symmetric case $\kappa\varepsilon_{ns} = 0$ and the metric is of Tolman-Bondi form:

$$ds^2 = dt^2 - \frac{R_x^2}{1 - k(r)} - R^2 \left\{d\theta^2 + \sin^2 \theta d\phi^2\right\}.$$ 

V. VIRIALISED CASE

In ref. [11] we considered whether $\dot{\phi}(\vec{x}, t) \approx \dot{\phi}_c(t)$ near the surface of some virialised over-density of matter, which might be a planet, a black-hole, star, or cluster of galaxies. In general, to specify initial data for the Szekeres-Szaf ron solution we must give both the energy density on some initial hypersurface, $\kappa\varepsilon$, and the spatial curvature of that hypersurface (given by $k(r)$). In [11] we considered the two sub-cases of the full Szekeres-Szaf ron metric, compatible with our geometric set-up, where the solution is completely specified by giving the energy density, $\kappa\varepsilon$. :
• The ‘Gautreau’ case: the hypersurfaces $t = const$ are spatially flat, $k(r) = 0$. In this case the big-bang is not simultaneous along the past world lines of all geodesic observers; i.e. it does not occur everywhere for a single value of $t$, at $t = 0$ say. In these cases the flow lines of matter move out of our ‘star’ - and the mass of ‘star’ decreases of time.

• The simultaneous big-bang case: the big-bang singularity occurs at $t = 0$ for all geodesic observers. In these cases the flow lines of matter move into the ‘star’ - and its mass increases with time.

The first of these cases is simpler to analyse but the second is more physically reasonable, since we expect the gravity to pull matter onto our star rather than expel it. For this reason we will only explicitly consider the simultaneous big-bang case in this paper. The results for the Gautreau case are very similar and the simultaneity of the big bang is not a significant factor for the late-time evolutionary problem that we are considering.

The ‘interior region’, which is immediately outside the surface of the ‘star’, is approximately Schwarzschild, and so an intrinsic interior length scale, $L_I$, of a sphere centred on the Schwarzschild mass with surface area $4\pi R_s^2$ is defined by the Riemann invariant:

$$L_I = \left( \frac{1}{12} R_{abcd} R^{abcd} \right)^{-1/4} = \frac{R_s^{3/2}}{(2m)^{1/2}}.$$  

(7)

In the asymptotically FRW, or exterior, region, the intrinsic length scale is proportional to the inverse root of the local energy density: $1/\sqrt{\kappa \varepsilon_c + \Lambda}$, where $\varepsilon_c$ is the total cosmological energy density of matter. We shall assume that the FRW region is approximately flat ($k = 0$), and we define a length scale appropriate for this exterior region at epoch at $t = t_0$ equal to the Hubble radius, which is defined by the inverse Hubble parameter at that time:

$$L_E = 1/H_0.$$  

For realistic models $L_E \gg L_I$ and so we define $\delta$ to be a small parameter given by

$$\delta = L_I/L_E.$$  

We assume that the whole spacetime metric is Szekeres-Szafran and define dimensionless coordinates appropriate to both the interior and exterior near some epoch of interest at time $t = t_0$. In the interior:

$$T = L_I^{-1}(t - t_0), \quad \xi = R_s^{-1}R,$$

where $R$ is the physical radial coordinate; $T$ and $\xi$ are $O(1)$ in the interior and we take the ratio $2m/R_s$ to be fixed. It is also helpful to define

$$\eta = \left( \xi^{3/2} - 3T/2 \right)^{2/3}; \quad R_s \eta = r + O(\delta^9, \delta^{2/3}).$$

In the exterior we define:

$$\tau = H_0t, \quad \rho = H_0r,$$

where $r$ is the unphysical radial labelling coordinate used in the metric. We define the interior limit by $\delta \to 0$ for fixed $T$ and $\xi$, and the exterior limit by $\delta \to 0$ with $\tau$ and $\rho$ held fixed.

A. The Exterior Limit

According to our prescription that the metric be FRW to zeroth order in $\delta$, we write

$$H_0 Z(\rho) \sim \frac{1}{2} \Omega_m \rho^3 + \delta^p z_1(r) + o(\delta^p),$$

and

$$H_0^{-1} \lambda_i \sim \delta^s l_i(\rho) + o(\delta^s),$$

where $z_1(r)$ and $l_i(\rho)$ are functions of the FLRW metric.
where $s$ and $p$ are positive numbers which depend on the particular form of the initial matter distribution. The exterior expansion of $k(r)$ can be found using the exact solutions for the Szekeres metrics with cosmological constant $\Omega$.

Since $H_0^{-2} \left( \frac{2M}{R^2} \right)_R \sim O(\delta^p, \delta)$, we have that: $H_0^{-2} \kappa \varepsilon \phi_{ns} \sim O(\delta^{p+s}, \delta^{1+s})$ whereas $H_0^{-2} \kappa \varepsilon \sim O(\delta^p, \delta)$. Thus, the non-spherical perturbation to the energy density is always of sub-leading order compared to the first order in spherical perturbation. The first-order, non-spherical, metric perturbation appears at $O(\delta^s)$; however, this is equivalent to a coordinate transform on $(r, \theta, \phi)$ and does not source a non-spherically symmetric physical perturbation to the dilaton evolution at this order. For the dilaton field, $\phi$, then, the first non-spherical perturbation is always sourced at subleading order compared to the first spherically symmetric one.

In the exterior we can apply the boundary condition that $\phi \to \phi_c(t)$ as $r \to \infty$. In addition to this we also have the stronger condition that, as $\delta \to 0$, the inhomogeneity should disappear and $\phi \to \phi_c(t)$. Thus to zeroth order in the exterior $\phi \sim \phi_c(t) + O(\delta^p, \delta)$. Since we are only really interested in the behaviour of $\phi$ in the interior we do not need to calculate the higher-order terms in the exterior limit explicitly, we only need to know enough about their behaviour to be able to perform the matching in some intermediate scaling region. We will consider that behaviour later.

B. Interior Limit

To lowest order in the interior region, we write $Z \sim \delta^q R_s \mu_1$, and $\lambda_i := \delta^q R_s^{-1} b_i$, where $i = \{x, y, z\}$: $q$ and $q'$ are determined by specify a particular matter distribution. The condition that $\kappa \varepsilon > 0$ everywhere requires $q' \geq q$. From the exact solutions we find:

$$k(r) \sim \delta^{2/3} k_0 (1 + \delta^q \mu_1 (\eta) + O(\delta^q)) + O\left(\delta^{5/3}\right),$$

where $k_0(\delta T) = (2m/R_s) (\pi/(H_0 t_0 + \delta T))^{2/3}$. In refs. [11] we sought to remove the effect of the $O(\delta^{2/3})$ in $k(r)$ by a transformation of the time coordinate. However, it is not clear that this new time coordinate is well-defined near a black-hole horizon; we now believe this procedure to have been technically incorrect (although it did not effect the results). We correct it in this work by implementing the $O(\delta^{2/3})$ correction differently. If $q' > q$ then to the next-to-leading order, we need only consider the spherically-symmetric modes to find interior expansion of $\phi$. We could also include a non-spherical vacuum component for $\phi$ at next-to-leading order; however, this will be entirely determined by a boundary condition on $R = R_s$ and the need that it should vanish for large $R$. To find the leading-order behaviour of the $\phi_T$ we need to know $\phi$ at next-to-leading order. Hence, the only case where we must explicitly consider non-spherically symmetric effects is when $q' = q$, i.e. $\kappa \varepsilon \phi_{ns} = O(\kappa \varepsilon)$. In what follows it is natural to consider the spherically symmetric and non-spherically symmetric modes of $\phi$ separately.

Before we can solve the $\phi$ equations in the interior limit, we need a boundary condition at $R = R_s$. At leading order we take this to be:

$$R_s^2 \left( 1 - \frac{2m}{R_s} \right) \partial_R \phi_0 \big|_{R \to R_s} = 2m F(\bar{\phi}_0) = \int_0^{R_s} dR' R'^2 B_{\phi}(\phi_0(R', t)) \kappa \varepsilon (R'),$$

The no-hair theorem for black holes implies that $F(\bar{\phi}_0) = 0$, however for bodies where $2m/R_s \ll 1$ we expect $F(\bar{\phi}_0) \approx B_{\phi}(\phi_c, \phi_s)$. At higher orders we find the flux $F$ by perturbing the above expression as explained in [11]. The zeroth-order mode of $\phi$ in the interior is then found to be:

$$\phi^{(0)} = \phi^{(0)}(T, \xi) := \phi_c(\delta T) + F(\bar{\phi}_0) \ln \left( 1 - \frac{2m}{R_s \xi} \right).$$

The matching procedure gives $\phi_c(\delta T) = \phi_c(t)$.

1. Spherically Symmetric Perturbations

In [11] we considered the spherically symmetric perturbations of $\phi$ that occur at order $\delta^q$ and at order $\delta$. Here, we will also consider the perturbations at order $\delta^{2/3}$. The perturbation at order $\delta$ is sourced by the $\square_0 \phi_c(\delta T)$ term,
where $\Box_0$ is the d’Alembertian of the Schwarzschild metric. This effect of perturbation acts like a drag term, and is equivalent to the change $\phi_c(\delta T) \to \phi_c(\delta T')$ with:

$$T' = T - 2 \left( \frac{2m}{R_s} \right)^{3/2} \left( \sqrt{\frac{R_s}{2m} - \ln \left| 1 + \sqrt{\frac{2m}{R_s}} \right|} \right) + C,$$

where $C$ is some constant which is determined by the matching process. When the matching is performed one finds:

$$T' \sim T + \left( \frac{2m}{R_s} \right) \int_{\xi}^{\ell} d\xi' \xi' \left( \xi T(\xi', T) \xi'^2 - \xi T(1, T) \xi'^2(1 - k(\xi', T) - (2m/R_s)^{1/2} \xi'_{T}(\xi', T)) \right)$$

where $\xi_T(\xi, T) = R_t(R_t = R_s \xi, t = t_0 + L_t T)$. When the ‘star’ is actually a black hole, the above expression reproduces, to leading order in $\delta$, the result found by Jacobson [24]. As for the $O(\delta^{5/3})$ correction to the metric coming from $k(t)$, we find that it only affects $\phi$ at order $O(\delta^{5/3}, \delta^{3+2/3})$; as such, they can be ignored since they are always smaller than the effects that we have included.

In [11] we calculated the order $\delta'$ correction to $\phi$, $\phi^{(q)}(T, \xi)$. In cases where the surface of our ‘star’ is far outside its Schwarzschild horizon $(2m/R_s)$, $\phi^{(q)}$ is given by:

$$\phi^{(q)}_I \sim \frac{2m}{R_s} B_{\phi}(\phi_c) \left( \int_{\eta}^{\eta'} d\eta' \frac{\mu_1(\eta')}{\xi(\eta', T)} - \frac{\mu_1(\eta)}{\xi} + \left( 1 - \frac{F(\tilde{\phi}_0)}{B(\phi_c)} - \frac{\mu_1(\eta(\xi = 1, T)}{\xi} + D(T) \right) + O \left( \frac{2m}{R_s} \right)^2 \right).$$

In the cases where $2m/R_s \approx 1$ it was not possible to find a closed analytical expression for $\phi^{(q)}_I$; however, from the its equation of motion, it can be easily seen that $\phi^{(q)}_I$ will be of the same order of magnitude as the above expression. The function $D(T)$ is a constant of integration, and it must be found via the matching procedure. Before we perform this matching, we will consider the non-spherically symmetric modes.

2. Non-Spherically Symmetric Perturbations

Non-spherically symmetric modes in the interior approximation to $\phi$ will be sourced at order $\delta'$, where $q' \geq q$. We studied these modes and found that, to order $\delta'$, they only have a dipole moment. We write the order $\delta'$ non-spherically symmetric, modes as $\delta' \phi^{(q')}_I$ where:

$$\phi^{(q')}_I := \phi^{(q')}_I(\xi, T) \cos \theta + \phi^{(q')}_{Ix}(\xi, T) \sin \theta \cos \varphi + \phi^{(q')}_{Iy}(\xi, T) \sin \theta \sin \varphi.$$

As before, when $2m/R_s \ll 1$, we can find analytic expressions for these modes:

$$\phi^{(q')}_I \sim -\frac{2m}{R_s} B_{\phi}(\phi_c) \frac{\mu_1(\eta)}{\xi} \int_{\eta}^{\eta'} d\eta' \frac{b_1(\eta')}{\xi'^2} + \frac{2m}{R_s} B_{\phi}(\phi_0^\prime) \frac{1}{\xi^2} \int_{\eta}^{\eta'} d\eta' b_1(\eta')$$

$$- \frac{2m}{R_s} F(\tilde{\phi}_0) \frac{1}{\xi^2} \int_{\eta}^{\eta'} d\eta' b_1(\eta') \xi + \frac{C_i(T)}{\xi^2} + D_i(T) \xi + O(2m/R_s^2),$$

where $i = x, y, z$. When $2m/R_s \approx 1$ the above expression can be seen as an order of magnitude estimate for the $\phi^{(q')}_I$. The $D_i(T)$ and $C_i(T)$ are constants of integration, with $D_i(T)$ determined by the matching procedure. The value of $C_i(T)$ should be set by a boundary condition on $R = R_s$. We cannot specify $C_i(T)$ exactly without further information about the interior of our ‘star’ in $R < R_s$. If we assume that the prescription for the sub-leading order boundary condition given above is correct then we find:

$$\partial_\xi \phi^{(q')}_I \big|_{\xi = 1} \sim -\frac{2m}{R_s} \frac{b_1(\eta)}{\eta'^2} \bigg|_{\xi = 1} F(\tilde{\phi}_0) + O((2m/R_s)^2)$$

$$\Rightarrow C_i = -\frac{m}{R_s} B_{\phi}(\phi_0^\prime) \int_{\eta}^{\eta'} d\eta' \frac{b_1(\eta')}{\xi'^2} + \frac{1}{2} D_i$$

From now on, we set $C_i = 0$, for simplicity. Even when this is not exactly satisfied, we do not expect the magnitude of $C_i$ or $C_i, T$ to be larger than any of the other terms in $\phi^{(1)}_I$ or $\phi^{(1)}_{I, T}$, respectively.
C. Validity of Matching Procedure

Before we can apply the matching procedure, we must ensure that it is applicable to our problem i.e. that there exists some intermediate region where both the interior and exterior approximations are simultaneously valid. We considered these conditions in ref. [11]. We define coefficients \( a > 0 \) and \( d_i > 0 \) by \( \mu_1(\eta) \sim \eta^a \) and \( b_i(\eta) \sim \eta^{b_i} \) as \( \eta \to \infty \) respectively. Writing \( H_0^{-1} \chi_i \sim \delta^i l_i(\rho) \), we also define coefficients \( m \) and \( f_i \) by \( z_1(\rho) \sim \rho^m \) and \( l_i(\rho) \sim \rho^{-f_i} \), as \( \rho \to 0 \). For both the exterior and interior to be simultaneously valid in some intermediate scaling region where \( \eta, \chi, \xi \sim \delta^{-\alpha} \) with \( 0 < \alpha < 1 \), we need there to exist some \( \alpha \) such that:

\[
\max \left( 0, 1 - \frac{p}{1 - m} \right) < \alpha < \frac{\eta}{n}
\]
\[
\max_i p_i' + (1 - \alpha)(f_i + m_i) > -p \text{ if } p \leq 1,
\]
\[
\max_i p_i' + (1 - \alpha)f_i > -1 \text{ if } p \geq 1,
\]
\[
\alpha - \max_i q_i/d_i > 0
\]

These conditions can, in almost all cases, be rephrased as:

\[
\lim_{\delta \to 0} R^2 \kappa \Delta \varepsilon = o(1)
\]
\[
\lim_{\delta \to 0} 2(m + Z)/R = o(1)
\]
as \( \delta \to 0 \), with \( L_1^q L_E^{1-\alpha}(t - t_0), L_2^q L_E^{1-\alpha} \) held fixed, for all \( \alpha \in (0, 1) \).

D. Matching and Results

We are interested in time derivatives of the \( \phi \) field. From the expression for \( \phi \) in the interior, we see that \( t' = L_t T' \) seems to play the role of a natural time coordinate. At radii where \( 2m/R \ll 1 \), the interior metric is close to diagonal when written in \( (t', R) \) coordinates; it is in this sense a natural time coordinate for an observer at fixed \( R \). In this region, \( t' \) coincides with the standard Schwarzschild time coordinate. As \( R \to \infty \), \( t' \to t \). We therefore consider \( \phi, t' \). Whenever the required conditions of the previous section hold, the matching procedure is valid, and we find

\[
\phi_{t', \rho}(r, t) \approx \phi_{t', \rho}^{(0)} + \delta^q \phi_{t', \rho}^{(0)} + \delta^q \phi_{t', \rho}^{(0)} + o(\delta^q, \delta^q)
\]
in the interior, where the \( \approx \) sign means that whilst this is not a formal asymptotic series (since there may be excluded terms that are bigger than some of the included ones) this is a good numerical estimate since at least one of the included terms will be bigger than all the excluded ones. The terms in this expression are given by:

\[
\phi_{t', t}^{(0)} \sim \dot{\phi}(t) + \Delta t(t, t) \dot{\phi}(t),
\]
\[
\Delta t(t, t) = \int_{t_0}^{t} \frac{\Delta(R_e)R^2 - R^2 \Delta(R_e)}{R^2(1 - k(r') - R^2)}
\]

where \( \Delta R_e = R_e - HR \),

\[
\delta^q \phi_{t', \rho}^{(0)} \sim -B_{ph}(\phi) \left( \int_{t_0}^{t} dr' \Delta R_{e} R_{e}^4 \kappa \varepsilon_n(r', t) + \left( 1 - \frac{F(\delta_0)}{B_{ph}(\phi)} \right) \frac{R_{e}^2 \kappa \varepsilon_n(r', t)}{R} \right) + O \left( \left( \frac{2m}{R_e} \right)^2 \right)
\]

and

\[
\delta^q \phi_{t', \rho}^{(0)} \sim -\frac{2}{3} B_{ph}(\phi) \left( \int_{t_0}^{t} dr' R_{e} R_{e}^4 \kappa \varepsilon_n(r', t) - \frac{1}{3} B_{ph}(\phi) \frac{R_{e}^2}{R^2} \int_{R_e}^{R} dr' R_{e} R_{e}^4 \kappa \varepsilon_n(r', t) \right) + \frac{1}{3} F(\delta_0) \frac{R_{e}^2}{R^2} \int_{R_e}^{R} dr' R_{e} R_{e}^4 \kappa \varepsilon_n(r', t) + O \left( \left( \frac{2m}{R_e} \right)^2 \right).
\]

Equations 13 and 14 are derived for the case \( 2m/R_e \ll 1 \). For \( 2m/R_e \approx 1 \), they are accurate when \( R \gg 2m \) and otherwise provide order-of-magnitude estimates for \( \delta^q \phi_{t', \rho}^{(0)} \) and \( \delta^q \phi_{t', \rho}^{(0)} \) respectively.
We can now evaluate these terms to find out when $\dot{\phi}(\vec{x}, t) \approx \dot{\phi}_c(t)$ and also state the precision to which this approximate equality holds. In many cases, however, a lot of the terms in the above expression are negligible or cancel, and so we can find a more succinct necessary and sufficient condition for $\dot{\phi}(\vec{x}, t) \approx \dot{\phi}_c(t)$ to hold. When $2m/R_s \gg 1$ we expect $\mathcal{F}(\dot{\phi}_0) \approx B_{\phi} (\dot{\phi}_0^2) \left(1 + \mathcal{O}(2m/R_s)\right)$, and so:

$$\phi_{I,'} - \phi_{c,t} \approx -B_{\phi} (\phi_c) \int_\infty^\infty dr' R_{r}R_{ct} \kappa \Delta \epsilon_s (r', t) - \frac{2}{3} B_{\phi} (\phi_c) R \int_\infty^r dr' R_{r}R_{ct} \kappa \epsilon ns (r', t) \frac{\epsilon ns}{R}$$

(15)

$$- \frac{1}{3} B_{\phi} (\phi_c) RR_{r} \kappa \epsilon ns (r, t) + \Delta t(r, t) \dot{\phi}_c(t).$$

(16)

We will refer to this last term as the drag term, and it is responsible for the local value of $\dot{\phi}$ lagging slightly behind the cosmological one - this effect was first observed by Jacobson in the study of gravitational memory, [24]. Whenever the cosmological $\phi$ is not potential dominated, and our ‘star’ resides in a local overdensity of matter, the drag term will be negligible compared to the other terms in this expression. If the potential term dominates the cosmological evolution then it is possible for the drag term to give the dominant effect, even if we have an local over-density of matter. But whenever this happens we always have $|\Delta t \phi_c|/\phi_c \ll 1$ and so we have $\dot{\phi}(\vec{x}, t) \approx \dot{\phi}_c(t)$.

Independent of the nature of over-density, we saw in our previous papers that potential domination of the cosmological $\phi$ evolution acts only to strengthen the degree to which $\phi(\vec{x}, t) \approx \phi_c(t)$.

Even if we ignore the drag term, the above expression for $\phi_{I,'} - \phi_{c,t}$ is still rather unwieldy. In almost all cases, integrating over the non-spherically symmetric modes of $\kappa \epsilon$ in the same way as we did for the spherically symmetric ones only acts to increase $|\phi_{I,'} - \phi_{c,t}|$. Therefore, we define the quantity $\mathcal{I}$ by:

$$\mathcal{I} := B_{\phi} (\phi_c) \int_\infty^\infty dr \frac{\max\theta (\sin \Delta (v \epsilon))}{\phi_c}$$

(17)

where $v$ is the radial velocity of the matter particles (i.e. $v = R_{r}$). When the background spacetime is Szekeres-Szafron we have that:

$$\left| \frac{\phi_{I,'} - \phi_{c,t}}{\phi_{c,t}} \right| \lesssim \mathcal{I}$$

with equality in the spherically symmetric case. The strong inequality $\mathcal{I} \ll 1$ is therefore a sufficient condition for $\phi(\vec{x}, t) \approx \phi_c(t)$, and the value of $\mathcal{I}$ gives a measure of the amount by which $\phi_{I,'}$ and $\phi_{c,t}$ differ. In addition to having shown this for cases where the background is Szekeres-Szafron, and that matching conditions hold, we also conjecture that, even if the matching conditions formally fail, for other classes of spacetime, that $\mathcal{I} \ll 1$ is a sufficient condition for $\dot{\phi}(\vec{x}, t) \approx \dot{\phi}_c(t)$.

VI. COLLAPSING CASE

When a spacetime undergoes gravitational collapse it is possible for a black-hole to form inside the collapsing region. In the Tolman-Bondi model a black-hole horizon appears when $2M(r)/R = 1$. If we have $\kappa \Delta \epsilon R^2 = (\kappa \epsilon - \kappa \epsilon_c) R^2 \ll 1$ outside the horizon then we can apply the results of the previous section, taking the surface of our ‘star’ to be the black-hole horizon. This is also true for any virialised region in the interior of the collapsing region, not just for black holes.

The results of the previous section can also be extended to the case where the collapsing interior region has no central black-hole or virialised region. For simplicity we consider only spherically-symmetric, dust-plus-$\Lambda$ cosmologies i.e. Tolman-Bondi models. We do not require the big bang to be simultaneous for all observers. This extension requires that curvature of the interior spacetime be in some sense weak so that the metric is Minkowski to zeroth order. We require:

$$R^2 \kappa \Delta \epsilon \ll 1, \quad 2 \Delta M/R \ll 1$$

everywhere; $\Delta \epsilon = \epsilon - \epsilon_c$ and $\Delta M = M - M_{\kappa \epsilon_c}$, in the interior. In the exterior we assume, as before, that the spacetime is FRW to zeroth order. It is clear that in this model the following parameters, $\delta_1$ and $\delta_2$ are everywhere small:

$$\delta_1 (R, t) = R^2 (\kappa \epsilon - \kappa \epsilon_c) = \frac{2M}{R^3} - 3 \Omega_m H^2 R^2,$$

$$\delta_2 (R, t) = \frac{2M}{R} - \Omega_m H^2 R^2.$$
In addition, the following parameter, $\delta_3$, is small in interior region but $O(1)$ in the exterior:

$$\delta_3(R, t) = H^2R^2.$$  

In the interior $\delta_3 \ll \delta_1, \delta_2$. For the purposes of our asymptotic expansions we treat and $\delta_1$ and $\delta_2$ as being of the same order. For the interior to be collapsing we need $k(r) > 0$. The condition that $R^2_\perp = -k(r) + \delta_2 + (\Omega_m + \Omega_\Lambda)\delta_3 > 0$ implies that $k(r) \leq O(\delta_3)$, and $R^2_\perp \sim O(\delta_2)$ in the interior.

We will perform the matching in an intermediate region where $\delta_1 \sim \delta_2 \sim \delta_3 \ll 1$. It is clear that with these definitions that such an intermediate region must always exist.

A. The Interior

In the interior we write the metric as:

$$ds^2 = \left(1 - \delta_2 + \delta_3(\Omega_m + \Omega_\Lambda)\right)dt^2 + \frac{2R_\perp dR dt}{1 - k(r)} - \frac{dR^2}{1 - k(r)} - R^2 \{d\theta^2 + \sin^2 \theta d\varphi^2\},$$

which is flat spacetime to lowest order in the $\delta_i$. The dilaton, $\phi$, obeys:

$$-R^2 \Box \phi = B_\phi(\phi)(\delta_1 + 3\Omega_m \delta_3) + V_\phi(\phi)R^2,$$

and, in line with our previous assumptions, we have $V_\phi(\phi)R^2 \sim O(\Omega_\Lambda \delta_3)$. We note that $\partial_\phi \delta_1 \sim O(\delta_1^{1/2} \delta_2) = o(\delta_1^{1/2})$ and so $\delta_1$ is quasi-static; as such we expect $\phi$ to also be quasi-static in the interior. We can solve the equations for $\phi$ order-by-order in the interior, requiring (as a boundary condition) that $\phi$ is regular at $R = 0$:

$$\phi \approx \phi_c(t) + B_\phi(\phi_c) \int_C \frac{dR'}{R'} \delta_2(R', t) + \phi_c(t) \int_D dR' (R_\perp - HR') + \frac{1}{6} \left[R^2 V_\phi(\phi_c) + 3B_\phi \Omega_m \delta_3 + (\phi_c(t) + 3H \phi_c(t))R^2\right] + O\left(\delta_1^{1/2} \delta_2^2, (\delta_3 R_{\phi_c})^3, \delta_3(R_{\phi_c})\right),$$

where $\phi_c(t)$ is $O(1)$ but quasi-static i.e. $R_{\phi_c} = o(\delta_1^{1/2}, \delta_2^{1/2})$; the third term is $O(\delta_1^{1/2} R_{\phi_c})$. Since the above expression is not a formal asymptotic expansion as such we cannot be sure that the neglected terms are smaller than all of the included terms; indeed we shall see that the matching ensures the vanishing of the term in $[..]$ This is because we do not know precisely how the sizes of $\delta_3$ and $R_{\phi_c}$ relate to those of $\delta_1$ and $\delta_2$. What we do know is that, in the interior, the excluded terms are smaller than at least one of the included terms. The limits $C$ and $D$ as well as $\phi_c(t)$ must be found matching the interior expansion to the exterior one.

B. The Exterior

In the exterior we define a coordinate $\varrho = R/a(t) \sim r$ where $a(t)$ is the FRW scalar factor. In $(t, \varrho)$ coordinates the metric reads:

$$ds^2 = dt^2(1 + O(\delta_2^2, (\Delta k)^2) + \frac{2(R_\perp - HR) \varrho dt}{1 - k(r)} - \frac{a^2 d\varrho^2}{1 - k(r)} - a^2 \varrho^2 \{d\theta^2 + \sin^2 \theta d\varphi^2\}$$

where $\Delta k = k(r) - k_0 \varrho^2$, and $k_0 = \lim_{r \to \infty} k(r)/r^2$, $(1 - \Omega_m - \Omega_\Lambda)H^2 = -k_0^2/a^2$ and $2(R_\perp - HR) \sim (\Delta k + \delta_2)/HR$. As $R_\perp, \varrho \to \infty$ and the inhomogeneity is removed (i.e. $\Delta k, \delta_1, \delta_2 \to 0$) we require that $\phi \to \phi_v(t)$. As with the virialised case our only interest in the subleading order behaviour of $\phi$ in the exterior is so as to match it to the interior approximation. It is only necessary therefore to consider how the exterior approximation to $\phi$ behaves in the intermediate region where all the $\delta_i$ are small. In the intermediate region the exterior approximation is:

$$\phi \approx \phi_c(t) + \phi_v(\varrho, t) + B_\phi(\phi_c) \int_\infty^R \frac{dR'}{R'} \delta_2(R', t) + \phi_c(t) \int_\infty^R dR' (R_\perp - HR') + O(\delta_1^{1/2}, \delta_2^2, \delta_3^3),$$

where $\phi_v(\varrho, t) = o(1)$ is some vacuum mode i.e. $\Box_{\text{FRW}} \phi_v = 0$.  

C. Matching and Results

Now we match the interior and exterior expansions and find that $C = D = \infty$, $\phi_c(t) = \phi_v(t)$ and $\phi_s = 0$. The requirement that $\phi_c(t) = \phi_v(t)$, combined with the cosmological evolution equation for $\phi_v(t)$ ensure that the term in [..] in eqn. (13) vanishes. We have found that the matching interior approximation is therefore given by:

$$
\phi \sim \phi_c(t) + B_\phi(\phi_c) \int_{\infty}^{R} \frac{dR'}{R'} \delta_2(R', t) + \phi_v(t) \int_{\infty}^{R} dR' (R, t - H R') + \mathcal{O}(\delta_1^2, \delta_2^2, \delta_3^2)
$$

$$
\sim \phi_c(t' = t + \Delta t) + B_\phi(\phi_c) \int_{\infty}^{R} \frac{dR'}{R'} \delta_2(R', t) + \mathcal{O}(\delta_1^2, \delta_2^2, \delta_3^2).
$$

where the lag in the time coordinate, $\Delta t$, is given by:

$$
\Delta t = \int_{\infty}^{R} dR' (R, t - H R').
$$

This coincides (to leading order) with the expression for the virialised case, eqn. (12), for $R_s = 0$. It seems natural, in the interior, to consider the time derivative of $\phi$ w.r.t. $t'$. As noted in the previous section, $t'$ will look like the Schwarzschild time coordinate near the surface of a massive body (that is far outside its Schwarzschild radius), and $t' \rightarrow t$ as $R \rightarrow \infty$. We find that:

$$
\phi_{t'}(r, t') - \phi_{c,t} \sim -B_\phi(\phi_c) \int_{\infty}^{R} dR' \Delta(R, t_c \kappa \varepsilon)(R', t) - \phi_v(t) \Delta t + \mathcal{O}(\delta_1^2, \delta_2^2, \delta_3^2)
$$

where $\Delta(R, t_c \kappa \varepsilon)(R, t) = R, t_c \kappa \varepsilon(R, t) - H R \kappa \varepsilon_c$. We note that this is the same as the spherically-symmetric limit of the result found in eqn. (13) for the virialised case.

This completes the extension of our analysis to the case of spherically symmetric collapsing spacetimes. It is clear that the quantity $I$, defined in the analysis of the virialised case, will be also be a good measure of $|\phi - \phi_c|/\phi_c$ in the collapsing case. The validity of the matching procedure is this case is assured by the condition that $\delta_1, \delta_2 \ll 1$ holds everywhere.

VII. RESULTS AND CONSEQUENCES

We now consider the astronomical consequences of our results for observations here on Earth, and answer the basic question of whether local experiments will detect cosmologically varying constants. We can evaluate the quantity $I$ explicitly for an Earth-based experiment assuming the varying constant to be the Newtonian gravitation “constant” $G$ governed by Brans-Dicke theory (since in this case the cosmological evolution of $\phi$ is easy to solve). We expect similar values for BSBM, BM and other non-potential-dominated theories for varying $\alpha$ and $\mu$.

We will consider a star (and associated planetary system) inside a galaxy that is itself embedded in a large galactic cluster. The cluster is assumed to have virialised and be of size $R_{\text{clust}}$. Close to the edge of the cluster we allow for some dust to be unvirialised and still undergoing collapse. There are three main contributions to $I$ coming from the star, the galaxy, and the galaxy cluster, respectively, and of these the galaxy cluster contribution is by far the biggest. This can be understood by noting that the galaxy cluster is the deepest gravitational potential well, and the galaxy and star are only small perturbations to it. The contribution to $I$ from the galaxy cluster is found to be:

$$
I_{\text{clust}} \lesssim \frac{3}{2} H_0 (s - 1/2)^{-1} \sqrt{2 M_{\text{clust}} R_{\text{clust}}} \frac{\Omega_{\text{clust}}}{\varepsilon_c}
$$

$$
= \frac{10[3/(2s - 1)] v_{\text{clust}}^2 (1 + z_{\text{vir}})^{3/2} \Delta_{\text{vir}}^{1/2}}{3 M_{\text{vir}}^{1/2} \Omega_m^{1/2} \Omega_m^{1/2}} = 1.01 \times 10^{-5} \left( \frac{[3/(2s - 1)] (1 + z_{\text{vir}})^{5/2} \Delta_{\text{vir}}^{5/6}}{\Omega_m^{1/6}} \right) \left( \frac{h M_{\text{clust}}}{10^{15} M_\odot} \right)^{2/3},
$$

$$
= 4.95 \times 10^{-5} \left( \frac{v_{\text{clust}}}{10^3 \text{km s}^{-1}} (1 + z_{\text{vir}})^{3/2} \right)^2,
$$

$$
\approx 1.61 \times 10^{-3} \left( \frac{v_{\text{clust}}}{10^3 \text{km s}^{-1}} (1 + z_{\text{vir}})^{3/2} \right)^2 \ll 1
$$

where we have used $3 M_{\text{clust}} / 5 R_{\text{clust}} = v_{\text{clust}}^2 = 3 \sigma_v^2$ and $\kappa_{\varepsilon_{\text{clust}}} = 6 M_{\text{clust}} / R_{\text{clust}}^3$; $\sigma_v$ is the 1-D velocity dispersion and $\Delta_{\text{vir}} \approx 178$ is the density contrast between the cluster and the background at virialisation. In the final line of the approximation we have used the representative value $\sigma_v = 1040 \text{km s}^{-1} \Rightarrow v_{\text{vir}} = 1800 \text{km s}^{-1}$ appropriate for a rich
cluster like Coma. Taking a cosmological density parameter equal to $\Omega_m = 0.27$, in accordance with WMAP, we expect that for a typical cluster which virialised at a redshift $z_{vir} \ll 1$, we would have $I_{clust} \approx 0.31[3/(2s - 1)] \times 10^{-2}$. The term in $[.]$ is unity when $s = 2$, i.e. $2GM/R \rightarrow const$; such a matter distribution is characteristic of dark matter halos. Different choices of $s > 1/2$ only change this estimate by a factor that is $O(1)$. We note that, since $2G\Delta M/R$ (where $\Delta M = M - \frac{1}{3}c, R^3$) is required to be small as $R \rightarrow \infty$ by the matching conditions, the model used here is only valid for $s \leq 2$ and hence the singularity in $I_{clust}$ at $s = 1/2$ is fictitious. If we were to have $I_{clust} \gtrsim 1$ then we would require a large virial velocity: $v_{clust} \gtrsim 32, 400[3/(2s - 1)]^{-1/2}(1 + z_{vir})^{-3/4} \text{km s}^{-1}$.

It is clear that in theories like Brans-Dicke, which have their cosmological evolution dominated by the matter-to-dilaton coupling, $B_{\phi}$, the local time variation of $\phi$ and the associated constant differs from its cosmological value by at most about 1%. In theories where the potential dominates the cosmological evolution this result becomes even stronger and we expect any deviations to occur only at the $0.4|B_{\phi}/V_{\phi}|\%$ level, where $|B_{\phi}/V_{\phi}| \ll 1$.

We have also seen that $I$ is also good measure of $|\phi - \phi_c|/\phi_c$ inside spherically-symmetric regions that are still undergoing gravitational collapse. Let us now evaluate $I$ for the case of a collapsing cluster in Brans-Dicke theory in the matter era. We assume that the cluster is approximately homogeneous. We further assume that when the cluster eventually virialises, at time $t_{vir}$, it has a virialisation velocity $v_{vir}$. We use the spherical approximation detailed in Chapter 5 of [13] to model the collapse of the the cluster. We define $\theta$ by $t = t_{vir}(\theta - \sin \theta)/2\pi$, and find:

$$I(t) = \frac{\Delta v_{vir}^2}{2} \left( \frac{\Delta v_{vir}^2 f(\theta)g(\theta)}{2\sqrt{2\pi}} \right) + \frac{\sqrt{2\pi}g(\theta)}{\Delta v_{vir}^2(\theta - \sin \theta)} = \left( \frac{\Delta v_{vir}^2}{4} \left( -\frac{3f(\theta)g(\theta)}{3(\theta - \sin \theta)} + \frac{2g(\theta)}{3(\theta - \sin \theta)} \right) \right),$$

where $\Delta v_{vir} = 18\pi^2 \approx 178$ is the density contrast at virialisation when $\theta = 2\pi$. When $\theta \leq 3\pi/2$, $f(\theta) = \sin(1 - \cos \theta)^{-1}$ and $g(\theta) = \sin \theta$; for $\theta > 3\pi/2$, $f(\theta) = g(\theta) = -1$. In this evaluation we have included the effect of the ‘drag term’; this is important up to turnaround but it becomes negligible soon afterwards. Turnaround occurs at $\theta = \pi$, $t = t_{turn}$.

![FIG. 1: Plot of $\phi_{c,t} - \phi_{c,t}$ vs. time for Brans-Dicke theory at the centre of a collapsing cluster with $v_{vir} = 1800\text{km s}^{-1}$.](image)

In deriving the above expression we have used $3M_{clust}/5R_{vir} = v_{vir}^2$, where $R_{vir}$ is the radius of the cluster after virialisation and $M_{clust}$ its mass. The conditions required for the matching procedure to be valid are equivalent to $10v_{vir}^2/(1 - \cos \theta) \ll 1$, and it is clear that this will not be satisfied all the way down to $\theta = 0$. For $v_{vir} = 1800\text{km s}^{-1}$, our method will be valid for $\theta > 0.027$ and for the matching conditions to hold from turnaround to virialisation we require $v_{vir} \ll 95000\text{km s}^{-1}$. Assuming that the cluster virialises at an epoch that is close to the present day, this bound on $v_{vir}$ translates to requiring $R_{vir} \ll 432h^{-1}(1 + z_{vir})^{-3/2}\text{Mpc}$, where $H_0 = 100h\text{km s}^{-1}\text{Mpc}^{-1}$. We observe that $I$ is small up until turnaround and then grows quickly until virialisation. At turnaround $I = 0$, and at virialisation we find

$$I(t_{vir}) = \left( \frac{15\pi}{2} - \frac{5}{12\pi} \right) v_{vir}^2 = 2.61 \times 10^{-4} \left( \frac{v_{vir}}{10^3\text{km s}^{-1}} \right)^2 \approx 0.85 \times 10^{-3}. $$
For the final evaluation we have taken $v_{\text{vir}} = 1800\text{km}\text{s}^{-1}$ (as appropriate for the Coma cluster). The vanishing of $I$ at turnaround is specific to Brans-Dicke theory, more generally: $I(t_{\text{turn}}) = 400v_{\text{vir}}^2 [\dot{\phi}_c - B_{\phi}(\phi_c)\kappa\epsilon_c/H\dot{\phi}_c]/27\pi^2$. In theories where the matter coupling is strongly dominant cosmologically, $|B_{\phi}(\phi_c)\kappa\epsilon_c| \gg |V_c(\phi_c)|$, we find $I(t_{\text{turn}}) \approx 160v_{\text{vir}}^2 |B_{\phi}(\phi_c)|/27\pi^2 \ll 1$.

Our results differ greatly from those that were found using the spherical collapse model used in [25]; where $I(t_{\text{vir}}) \approx 200$. In that model, the spatial derivatives of $\phi$ were assumed to be negligible and are neglected. However, this is can only be a realistic approximation when the collapsing region is as large as the cosmological horizon; for a cluster virialising today that would require $R_{\text{vir}} \gtrsim 5\text{Gpc}$. Since our method will fail for $R_{\text{vir}} \lesssim 432h^{-1}(1 + z_{\text{vir}})^{-3/2}\text{Mpc}$, there must be some region of intermediate behavior, $500\text{Mpc} \lesssim R_{\text{vir}} \lesssim 5\text{Gpc}$, that is not described by either the spherical collapse model or our present analysis. We derived these results for Brans-Dicke theory, where $\phi \propto G^{-1}$, however we should expect similar numbers for all varying-constant theories where the cosmological dilaton evolution is dominated by its matter coupling, $B_{\phi}\kappa\epsilon_c$. In potential-dominated theories, the above numbers will be reduced by a factor of $|B_{\phi}(\phi_c)\kappa\epsilon_c/V_c(\phi_c)| \ll 1$. As in the post-virialisation case, potential domination of the cosmological evolution strengthens the amount to which local experiments will see cosmologically varying constants.

In conclusion: we have used the method of matched asymptotic expansions to find a sufficient condition for the time-variation of a scalar field, and any related varying physical ‘constants’ whose variation is driven by such a field, to track its cosmological evolution. We have extended our previous analyses by allowing ‘local’ also to include being time-variation of a scalar field, and any related varying physical ‘constants’ whose variation is driven by such a field, and then used that condition to sufficiently constrain the cosmological time variation of many supposed ‘constants’ of Nature.

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[1] J.K. Webb et al, Phys. Rev. Lett. 82, 884 (1999); M. T. Murphy et al, Mon. Not. Roy. Astron. Soc. 327, 1208 (2001); J.K. Webb et al, Phys. Rev. Lett. 87, 091301 (2001); M.T. Murphy, J.K. Webb and V.V. Flambaum, Mon. Not R. astron. Soc. 345, 609 (2003).
[2] H. Chand et al., Astron. Astrophys. 417, 853 (2004); R. Srianand et al., Phys. Rev. Lett. 92, 121302 (2004).
[3] J.D. Barrow, Phil. Trans. Roy. Soc. 363, 2139 (2005).
[4] E. Reinhold, et al. Phys. Rev. Lett. 96, 151101 (2006).
[5] C. Brans and R.H. Dicke, Phys. Rev. 124, 925 (1961).
[6] J.D. Bekenstein, Phys. Rev. D 25, 1527 (1982).
[7] H. Sandvik, J.D. Barrow and J. Magueijo, Phys. Rev. Lett. 88, 031302 (2002); J. D. Barrow, H. B. Sandvik and J. Magueijo, Phys. Rev. D 65, 063504 (2002); J. D. Barrow, H. B. Sandvik and J. Magueijo, Phys. Rev. D 65, 123501 (2002); J. D. Barrow, J. Magueijo and H. B. Sandvik, Phys. Rev. D 66, 043515 (2002); J. Magueijo, J. D. Barrow and H. B. Sandvik, Phys. Lett. B 541, 201 (2002); H. Sandvik, J.D. Barrow and J. Magueijo, Phys. Lett. B 549, 284 (2002); J.D. Barrow, D. Kimberly and J. Magueijo, Class. Quantum Grav. 21, 4289 (2004).
[8] D. Kimberly and J. Magueijo, Phys. Lett. B 584, 8 (2004); D. Shaw and J.D. Barrow, Phys. Rev. D 71, 063525 (2005); D. Shaw, Phys. Lett. B 632, 105 (2006).
[9] J.D. Barrow and J. Magueijo, Phys. Rev. D 72, 043521 (2005); J.D. Barrow, Phys. Rev. D 71, 083520 (2005).
[10] J.P. Uzan, Rev. Mod. Phys. J.P. Uzan, Rev. Mod. Phys. 75, 403 (2003): J.-P. Uzan, astro-ph/0409424 K.A. Olive and Y-Z. Qian, Physics Today, pp. 40-5 (Oct. 2004); J.D. Barrow, The Constants of Nature: from alpha to omega, (Vintage, London, 2002);[11] D. Shaw and J.D. Barrow, gr-qc/0512022 and gr-qc/0601056. D. Shaw and J.D. Barrow, gr-qc/0512117[12] J. Khoury and A. Weltman, astro-ph/0309300; Phys. Rev. D 69, 044026 (2004); P. Brax, C. van de Bruck, A.-C. Davis, J. Khoury and A. Weltman, astro-ph/0408415[13] T. Padmanabhan, Theoretical Astrophysics, Volume III: Galaxies and Cosmology, (Cambridge University Press, Cambridge, 2002).
[14] P. Szekeres, Commun. Math. Phys. 41, 55 (1975)
[15] D.A. Szafron, J. Math. Phys. 18, 1673 (1977); D.A. Szafron and J. Wainwright, J. Math. Phys. 18, 1668 (1977).
[16] E. J. Hinch, Perturbation methods, (Cambridge UP, Cambridge, 1991); J. D. Cole, Perturbation methods in applied mathematics, (Blaisdell, Waltham, Mass., 1968).
[17] W. L. Burke, J. Math. Phys. **12**, 401 (1971); P. D. D'Eath, Phys. Rev. D. **11**, 1387 (1975); P. D. D'Eath, Phys. Rev. D. **12**, 2183 (1975); W. L. Burke and K. Thorne, in *Relativity*, edited by M. Carmeli, S. Fickler, and L. Witten (Plenum Press, New York and London, 1970), pp. 209-228.

[18] A. Krasinski, *Inhomogeneous Cosmological Models*, (Cambridge UP, Cambridge, 1996).

[19] J. D. Barrow and J. Stein-Schabes, Phys. Lett. **103A**, 315 (1984); U. Debnath, S. Chakraborty, and J.D. Barrow, Gen. Rel. Gravitation **36**, 231 (2004).

[20] G. Lemaitre, C. R. Acad. Sci. Paris **196**, 903 & 1085 (1933)

[21] R.C. Tolman, Proc. Nat. Acad. Sci. USA 20, 169 (1934); H. Bondi, Mon. Not. R. astron. Soc. **107**, 410 (1947)

[22] P. Szekeres, Phys. Rev. D **12**, 2941 (1975).

[23] R. Gautreau, Phys. Rev. D. **29**, 198 (1984).

[24] T. Jacobson, Phys. Rev. Lett. **83**, 2699 (1999).

[25] D. Mota and J.D. Barrow, Mon. Not. Roy. Astron. Soc. **349**, 281 (2004); J.D. Barrow and D. Mota, Phys. Lett. B **581**, 141 (2004); T. Clifton, D. Mota and J.D. Barrow, Mon. Not. R. Astron. Soc. **358**, 601 (2005).

[26] P.J.E. Peebles, Astron. J. **75**, 13 (1970).

[27] J.D. Barrow, Phys. Rev. D **46**, R3227 (1992); J.D. Barrow, Gen. Rel. Gravitation **26**, 1 (1992); J.D. Barrow and B.J. Carr, Phys. Rev. D **54**, 3920 (1996); H. Saida and J. Soda, Class. Quantum Gravity **17**, 4967 (2000); J.D. Barrow and N. Sakai, Class. Quantum Gravity **18**, 4717 (2001); T. Harada, C. Goymer and B. J. Carr, Phys. Rev. D. **66**, 104023 (2002).