ON THE IDEALS OF GENERAL BINARY ORBITS:
THE LOW ORDER CASES

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ABSTRACT. Let $E$ denote a general complex binary form of order $d$ (seen as a point in $\mathbb{P}^d$), and let $\Omega_E \subseteq \mathbb{P}^d$ denote the closure of its $SL_2$-orbit.

In this note, we calculate the equivariant minimal generators of its defining ideal $I_E \subseteq \mathbb{C}[a_0, \ldots, a_d]$ for $4 \leq d \leq 10$. In order to effect the calculation, we introduce a notion called the ‘graded threshold character’ of $d$. One unexpected feature of the problem is the (rare) occurrence of the so-called ‘invisible’ generators in the ideal, and the resulting dichotomy on the set of integers $d \geq 4$.

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1. INTRODUCTION

1.1. Let

$$E = \sum_{i=0}^{d} \binom{d}{i} \alpha_i x_1^{d-i} x_2^i, \quad (\alpha_i \in \mathbb{C})$$

denote a nonzero form of order $d$ in the variables $\{x_1, x_2\}$. We will identify $E$ (distinguished up to a scalar) with the point $[\alpha_0, \ldots, \alpha_d]$ in $\mathbb{P}^d$. Define the graded polynomial ring $R = \mathbb{C}[a_0, \ldots, a_d]$ over indeterminates $a_i$, so that $\mathbb{P}^d = \text{Proj} R$.

The special linear group $SL_2 \mathbb{C}$ acts on $R$ (and hence on $\mathbb{P}^d$) as follows. Let

$$F = \sum_{i=0}^{d} \binom{d}{i} a_i x_1^{d-i} x_2^i,$$

(1)
denote the generic binary $d$-ic. Given $g = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL_2$, make substitutions

$$x_1 = p x'_1 + q x'_2, \quad x_2 = r x'_1 + s x'_2,$$

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into the right-hand side of (1), and rearrange terms to write
\[
F = \sum_{i=0}^{d} \binom{d}{i} a_i' x_1^{d-i} x_2^i.
\] (2)

Then the action of \( g \) takes \( a_i \) to \( a_i' \).

1.2. If \( d \geq 4 \), and \( E \) is a general point in \( \mathbb{P}^d \), then the closure of its \( SL_2 \)-orbit (denoted \( \Omega_E \)) is an irreducible projective variety of dimension 3. The degree of \( \Omega_E \) is 6 for \( d = 4 \), and \( d(d - 1)(d - 2) \) for \( d > 4 \) (see [11, p. 206] or [10, §8]). Its defining ideal \( I_E \) is an \( SL_2 \)-subrepresentation of \( R \), and we should like to find the equivariant minimal set of generators for \( I_E \). This is similar (but not identical) to the ‘equivalence problem for binary forms.’ (For discussions of the latter, see [3, §92] or [11, Chapter 8].)

The object of this paper is a complete determination of such generators for orders \( d \leq 10 \). The results are phrased in the language of classical invariant theory, i.e., in terms of invariants and covariants of the generic form \( F \).

The Betti numbers of \( I_E \) can be calculated by straightforward elimination (implemented here in Macaulay-2); it is rather the identification of the Betti modules \( qua \) \( SL_2 \)-representations which accounts for the bulk of the effort. In order to accomplish this, we introduce a notion called the graded threshold character of \( d \). Broadly speaking, it is designed to encode those subrepresentations of the ideal which can be detected by purely combinatorial considerations. This allows us to deduce an inequality involving the representation-theoretic character of a Betti module. It is a very surprising circumstance (to the author) that it turns out to be an equality sufficiently often for the calculation to succeed.

2. Preliminaries

The ansatz used in this paper is similar to the one in [2, §1], and the reader will find there detailed explanations of many of the notions used below. We refer to [7, Lecture 11] and [15, Chapter 4] for the basic representation theory of \( SL_2 \). Classical accounts of the invariant theory of binary forms may be found in [8, 12], and more modern expositions in [4, 11, 14]. For the necessary facts from commutative algebra, reference [5] is more than adequate.
2.1. The base field will be $\mathbb{C}$. Let $S_q$ denote the $(q + 1)$-dimensional vector space of binary forms of order $q$ in $\{x_1, x_2\}$. Then $\{S_q : q \geq 0\}$ is the totality of all finite dimensional irreducible $SL_2$-representations. Since $SL_2$ is a linearly reductive group, each finite dimensional representation decomposes as a direct sum of irreducibles. We will need two specific decomposition formulae: the Clebsch-Gordan formula

$$S_p \otimes S_q \simeq \bigoplus_{r=0}^{\min(p,q)} S_{p+q-2r}, \tag{3}$$

and the Cayley-Sylvester formula

$$\text{Sym}^p(S_q) \simeq \bigoplus_{r=0}^{\left\lfloor \frac{pq}{2} \right\rfloor} (S_{pq-2r})^{\pi(r,p,q) - \pi(r-1,p,q)}.$$ \tag{4}

Here $\pi(a, b, c)$ denotes the number of partitions of $a$ into $b$ parts such that no part exceeds $c$.

2.2. Given forms $A \in S_p$ and $B \in S_q$, the image of $A \otimes B$ via the projection map $S_p \otimes S_q \longrightarrow S_{p+q-2r}$ is called their $r$-th transvectant, denoted by $(A, B)_r$. It is given by the formula

$$(A, B)_r = \frac{(p-r)! (q-r)!}{p! q!} \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{\partial^r A}{\partial x_1^{r-i} \partial x_2^i} \frac{\partial^r B}{\partial x_1^r \partial x_2^{-i}}.$$

Two forms $A, B \in S_p$ are said to be apolar if $(A, B)_p = 0$. The pairing $S_p \otimes S_p \longrightarrow S_0$ is nondegenerate, hence there is a $p$-dimensional space of forms apolar to any specific nonzero form $A \in S_p$ (see [8, Chapter XI]).

2.3. Let $\Gamma$ denote the representation ring of $SL_2$, i.e., it is a free abelian group on generators $s_0, s_1, s_2, \ldots$ etc., with multiplication corresponding to the tensor product of representations. Given a finite-dimensional $SL_2$-representation $U$, let $[U] \in \Gamma$ denote its character. For instance,

$$[S_5 \otimes S_3] = s_5 \cdot s_3 = s_8 + s_6 + s_4 + s_2,$$

by formula (3). We will write $s_p \circ s_q$ for $[\text{Sym}^p(S_q)]$, e.g.,

$$s_4 \circ s_7 = s_{28} + s_{24} + s_{22} + 2 s_{20} + s_{18} + 3 s_{16} + 2 s_{14} + 3 s_{12} + 2 s_{10} + 3 s_8 + s_6 + 3 s_4 + s_0,$$

by formula (4).
Given elements \( a = \sum \alpha_i s_i \) and \( b = \sum \beta_i s_i \) in \( \Gamma \), write \( a \geq b \) if \( \alpha_i \geq \beta_i \) for all \( i \). Define
\[
\text{sup}(a,b) = \sum_i \max(\alpha_i, \beta_i) s_i.
\]

2.4. There is an isomorphism of \( SL_2 \)-representations
\[
j : R_1 \sim \rightarrow S_d, \quad a_i \rightarrow (-1)^i x_1^i x_2^{d-i},
\]
which allows us to make the identification
\[
R = \bigoplus_{m \geq 0} R_m = \bigoplus_{m \geq 0} \text{Sym}^m(S_d).
\]
Let \( W_{m,q} \subseteq R_m \) denote the span of the images of all \( SL_2 \)-equivariant maps \( S_q \rightarrow R_m \). Then there is a decomposition of representations
\[
R_m = \bigoplus_q W_{m,q}.
\]
Similarly \( (I_E)_m = \bigoplus_q (I_E)_{m,q} \), where \( (I_E)_{m,q} \) is an \( SL_2 \)-invariant subspace of \( W_{m,q} \).

2.5. Let \( A_{m,q} \) denote the space of covariants of \( F \) in degree \( m \) and order \( q \). Each element \( \Phi \in A_{m,q} \) may be written as
\[
\sum_{i=0}^q \varphi_i(a_0, \ldots, a_d) x_1^{q-i} x_2^i,
\]
where \( \varphi_i \) are homogeneous forms of degree \( m \) in \( a_0, \ldots, a_d \). Now \( \Phi \) defines an equivariant morphism
\[
S_q \rightarrow R_m, \quad A(x_1, x_2) \rightarrow (\Phi, A)_q,
\]
whose image is \( \text{Span} \{ \varphi_0, \ldots, \varphi_q \} \subseteq W_{m,q} \). Every such morphism comes from a covariant, i.e., we have an isomorphism
\[
A_{m,q} \simeq \text{Hom}_{SL_2}(S_q, W_{m,q}).
\]
This induces a bijection between subspaces of \( A_{m,q} \) and \( SL_2 \)-invariant subspaces of \( W_{m,q} \). It associates to a subspace \( U \subseteq A_{m,q} \), the span of all the coefficients of all the elements in \( U \) (to be denoted by \( U^\circ \)).

It is a standard fact (see \cite{8} §86)) that \( A_{m,q} \) admits a basis each of whose elements is a compound transvectant in \( F \). E.g., for \( d = 7 \), the space \( A_{3,9} \) is 2-dimensional with a basis \( \{ F(F, F)_6, (F, (F, F))_4 \} \). By formula (4),
\[
\zeta_{m,q} = \dim A_{m,q} = \pi \left( \frac{md-q}{2}, m, d \right) - \pi \left( \frac{md-q-2}{2}, m, d \right).
\]
2.6. Given a specific form $E \in S_d$, there is an evaluation map
\[ \theta_E : \mathcal{A}_{m,q} \rightarrow S_q \]
which substitutes the coefficients of $E$ for the indeterminates $a_i$. Write $(K_E)_{m,q} = \ker \theta_E$. Henceforth we may omit $E$ from the notation if no confusion is likely; it is understood that $K, I, J$ etc depend upon the choice of $E$.

**Lemma 2.1.** We have an equality $K_{m,q}^o = I_{m,q}$.

**Proof.** Indeed, each element in $K_{m,q}^o$ vanishes on $E$, and hence by equivariance on $\Omega_E$. Alternately, let $e \in I_{m,q}$ denote a nonzero element. Then $e$ belongs to a unique smallest $SL_2$-invariant subspace $V \subseteq I_{m,q}$. Let $\Psi \in \mathcal{A}_{m,q}$ denote the covariant (unique up to a constant) whose coefficients give a basis of $V$. It immediately follows that $\Psi \in K_{m,q}$, hence $e \in K_{m,q}^o$. \( \square \)

Since $\dim K_{m,q}$ is no smaller than $\max(0, \zeta_{m,q} - q - 1)$, we will define the **threshold character (of $d$) in degree $m$** to be the element
\[ T_m = \sum_{q \geq 0} \max(0, \zeta_{m,q} - q - 1) s_q \in \Gamma. \]

For instance, let $d = 5$. Then $\zeta_{14,10} = 17$, hence the coefficient of $s_{10}$ in $T_{14}$ is $17 - 11 = 6$. In fact, the full expression is
\[ T_{14} = s_{22} + 4 s_{18} + 2 s_{16} + 6 s_{14} + 3 s_{12} + 6 s_{10} + 2 s_{8} + 5 s_{6} + s_{4} + 3 s_{2}. \] (6)

2.7. The minimal resolution of $I$ will be written as
\[ 0 \leftarrow I \leftarrow E_0 \leftarrow E_1 \leftarrow \cdots \]
where $E_r = \bigoplus_{j \geq 0} B(r, j) \otimes R(-j)$, are graded free $R$-modules of finite rank. Thus $B(0, -)$ are the minimal generators of $I_E$, and $B(r, -)$ are the $r$-th syzygy modules. Each Betti module $B(r, j)$ is an $SL_2$-representation. For each $m$, let $J_m \subseteq I_m$ denote the subspace generated by $\bigoplus_{r < m} I_r$ (the ideal elements in earlier degrees). Write
\[ I_m = [I_m], \quad J_m = [J_m], \quad B_m = [B(0, m)] \]
for the corresponding elements in $\Gamma$. By construction, $I_m \geq T_m$ (which justifies the term ‘threshold’), and hence
\[ B_m = I_m - J_m \geq \sup(\overline{J_m, T_m}) - J_m. \]
Henceforth \( E \) will be assumed to be sufficiently general, which ensures that \( I, J, \mathcal{B} \) etc are independent of \( E \).

2.8. The Betti numbers (in the free resolution of) \( I \) can be calculated as follows. For illustration, let \( d = 6 \). Choose a ‘general’ form \( E(x_1, x_2) \) in \( S_6 \), and make simultaneous substitutions

\[
x_1 \rightarrow p x_1 + q x_2, \quad x_2 \rightarrow r x_1 + s x_2,
\]

into \( E \) to construct a new form

\[
\sum_{i=0}^{6} \binom{6}{i} \psi_i(p, q, r, s) x_1^{6-i} x_2^i.
\]

This defines a ring morphism

\[
\Psi_E : C[a_0, \ldots, a_6] \rightarrow C[p, q, r, s], \quad a_i \mapsto \psi_i(p, q, r, s).
\]

Then \( I = \ker \Psi_E \). The actual calculation shows that the Betti numbers of \( I \) are as in the following table:

|   | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 4 | 1 |
| 6 | 1 |
| 9 | 1 |
| 10 | 1 |
| 12 | 97 | 222 | 114 | 7 | 1 |
| 13 | 27 | 235 | 609 | 587 | 233 | 30 |
| 15 | | |

The entry in the row labelled \( i \) and column labelled \( j \) gives the dimension of \( B(j, i+j) \), e.g., \( \dim B(1, 14) = 235 \). In practice, for each \( d \), I have repeated the calculation for several random choices of \( E \) to eliminate any likelihood of error. Our task is to identify the \( B(0, m) \) qua \( SL_2 \)-representations, and secondly to identify the corresponding ideal generators.

2.9. It is a paradoxical feature of the subsequent calculations that the higher syzygies do not enter into them.\[\text{[Def.]}\]

Define

\[\overline{J}_m = [(\mathcal{E}_0)_m] - \mathcal{B}_m = \sum_{j<m} [B(0, j) \otimes R_{m-j}] = \sum_{j<m} \mathcal{B}_j \cdot (s_{m-j} \circ s_d).\]

This should be thought of as an approximation to \( J_m \), but with all higher syzygies ignored. Clearly \( \overline{J}_m \geq J_m \). Define \( \overline{Q}_m = \sup(\overline{J}_m, T_m) - \overline{J}_m. \)

\[\text{[With one small exception, noted later.]}\]
Lemma 2.2. We have an inequality $\mathcal{Q}_m \geq \tilde{\mathcal{Q}}_m$.

PROOF. Fix an integer $q$, and let $a, b, c$ denote the coefficients of $s_q$ in $\tilde{J}_m, J_m$ and $T_m$ respectively. Since $a \geq b$, the result follows from the obvious inequality $\max(b, c) - b \geq \max(a, c) - a$. □

As a consequence, we have the crucial inequality

$$\mathbb{B}_m \geq \tilde{\mathcal{Q}}_m$$

which will serve as our workhorse throughout the next section.

3. Computations

In this section we will describe the solution for each $d$. The calculations for order $d$ are to be found in §3.d. Of course, the results are valid only for $E$ belonging to a dense open subset of $\mathbb{P}^d$. E.g., if $E = x_1^d$, then $\Omega_E$ is the rational normal curve whose ideal is generated by quadrics.

Henceforth we will write $\beta_m$ for $\dim B(0, m)$, to be called the generator dimensions of $I$. As mentioned earlier, they were all calculated using MACAULAY-2. Formulae (3), (4) as well as the rest of the calculations in the representation ring $\Gamma$ were programmed in MAPLE by the author.

We will determine the $\mathbb{B}_m$ successively for increasing $m$. If the characters $\mathbb{B}_r$ for $r < m$ are known, then the calculation of $\tilde{\mathcal{Q}}_m$ is a purely mechanical task. Now our governing principle is simple: if the dimensions of $\mathbb{B}_m$ and $\tilde{\mathcal{Q}}_m$ coincide, then we must have equality in (♯). At first blush, this seems optimistic beyond reason. However, it is an intriguing but pleasing circumstance that (♯) is an equality in all the cases below, with only two exceptions. Moreover, each of the exceptions is ‘thematic’ in a sense which will be readily understood once it is encountered.

We will say that all the ideal generators in degree $m$ are visible if (♯) is an equality; if not, the ideal $I$ is said to have invisible generators in degree $m$. These phrases are to be understood atomically; it is meaningless to speak of any specific element in the ideal as being visible or otherwise.

3.4. Quartics. The variety $\Omega_E$ is a hypersurface of degree 6. Since $\zeta_{6,0} = 2$, the space $K_{6,0}$ is one-dimensional, and its generator gives the defining equation for $\Omega_E$. Said differently, define invariants

$$g_2 = (\mathbb{F}, \mathbb{F})_4, \quad g_3 = (\mathbb{F}, (\mathbb{F}, \mathbb{F})_2)_4,$$

‡The dimension of an element in $\Gamma$ is understood in the obvious sense.
in degrees 2 and 3 respectively. Then \( \{g_2^3, g_3^2\} \) is a basis of \( A_{6,0} \), and hence
\[
\theta_E(g_3)^2 g_2^3 - \theta_E(g_2)^3 g_3^2
\]
is the generator of \( I \).

3.5. Quintics. The generator dimensions are
\[
\beta_8 = 1, \quad \beta_{12} = 1, \quad \beta_{14} = 60.
\]
Evidently, \( B_8 = B_{12} = s_0 \). Define covariants (cf. [8] p. 131))
\[
H = (F, F)_2, \quad \iota = (F, F)_4, \\
A = (\iota, \iota)_2, \quad B = (\iota^3, H)_6, \quad C = (\iota^5, F^2)_10.
\]
Now \( \zeta_{8,0} = 2, \zeta_{12,0} = 3 \), and
\[
\{A^2, B\}, \quad \{A^3, AB, C\}
\]
are respectively bases of \( A_{8,0} \) and \( A_{12,0} \). As in the previous section, the degree 8 generator of \( I \) can be taken to be \( Z_8 = \theta_E(B) A^2 - \theta_E(A)^2 B \). Then the new generator in degree 12 can be chosen to be any element in \( K_{12,0} \) which is not a multiple of \( A Z_8 \), e.g., \( Z_{12} = \theta_E(A B) C - \theta_E(C) A B \).

Now \( \tilde{\eta}_{14} = s_6 \circ s_5 + s_2 \circ s_5 \), which evaluates to
\[
s_{30} + s_{26} + s_{24} + 2 s_{22} + 2 s_{20} + 3 s_{18} + 2 s_{16} + 4 s_{14} + 3 s_{12} + \\
5 s_{10} + 2 s_8 + 5 s_6 + s_4 + 3 s_2,
\]
by formula (4). Using the expression for \( T_{14} \) from (6), one arrives at
\[
\tilde{Q}_{14} = s_{18} + 2 s_{14} + s_{10},
\]
which has dimension 19 + 2 \cdot 15 + 11 = 60. Hence \( \mathbb{B}_{14} = \tilde{Q}_{14} \) in (5).

We introduce some notation in order to describe the generators succinctly. There is a 2-dimensional subspace \( V \subseteq K_{14,14} \), such that \( V^\circ \) accounts for the new generators in degree 14. Now \( V \) is not uniquely determined, but all choices satisfying the condition \( V^\circ \cap J_{14} = \{0\} \) are valid. Henceforth we will write \( G^\circ(2, K_{14,14}) \) for such a \( V^\circ \). In general, \( G^\circ(r, K) \) will stand for the span of coefficients of an \( r \)-dimensional subspace \( V \subseteq K \), where \( V \) is chosen to lie outside a (tacitly specified) proper subvariety in the Grassmannian of \( r \)-subspaces of \( K \).

We have arrived at the following result:

**Theorem 3.1.** For a general quintic \( E \), the ideal \( I \) is minimally generated by the following subspaces:
\[
K_{8,0}, \quad G^\circ(1, K_{12,0}), \quad G^\circ(1, K_{14,18}), \quad G^\circ(2, K_{14,14}), \quad G^\circ(1, K_{14,10}).
\]
3.6. **Sextics.** The generator dimensions are
\[ \beta_4 = 1, \quad \beta_6 = 1, \quad \beta_{10} = 1, \quad \beta_{12} = 97, \quad \beta_{13} = 27. \]

Hence \( B_4 = B_6 = B_{10} = s_0 \). A preliminary manoeuvre is necessary before proceeding to degree 12. Notice that the generators in degrees 4, 6 must give rise to a first syzygy in degree 10. Its contribution to \( I_m \) can be cancelled against that of the degree 10 generator. Thus, for the purposes of calculating \( \tilde{Q}_{12} \) and \( \tilde{Q}_{13} \), we will henceforth ignore \( B_{10} \). Then one gets
\[ \tilde{Q}_{12} = s_{24} + 2s_{20} + s_{16} + s_{12}, \]
which is 97-dimensional; hence \( B_{12} = \tilde{Q}_{12} \). Similarly,
\[ B_{13} = \tilde{Q}_{13} = s_{26}, \]
which completes the calculation. We have proved the following result.

**Theorem 3.2.** For a general sextic \( E \), the ideal \( I \) is minimally generated by the following subspaces:
\[
K_{4,0}, \quad G^o(1, K_{6,0}), \quad G^o(1, K_{10,0}), \quad G^o(1, K_{12,24}),
\]
\[
G^o(2, K_{12,20}), \quad G^o(1, K_{12,16}), \quad G^o(1, K_{12,12}), \quad G^o(1, K_{13,26}).
\]

We will no longer state such theorems explicitly, since they can be written down ritually once the \( B_m \) are known.

3.7. **Septimics.** The generator dimensions are
\[ \beta_6 = 10, \quad \beta_8 = 40, \quad \beta_9 = 106, \quad \beta_{10} = 89. \]

A calculation shows that \( T_6 \) (and hence \( \tilde{Q}_6 \)) equals \( s_2 \). It follows that (\( \sharp \)) must be a strict inequality, i.e., there are invisible generators in degree 6. The explanation lies behind the following algebraic peculiarity of the ring of covariants for binary septimics.

The spaces \( A_{4,6} \) and \( A_{6,6} \) are respectively of dimensions 1 and 7. Let \( \Delta \) denote a generator of the former. Septimics have no invariant in degree 10, i.e., \( A_{10,0} = 0 \). It follows that for any \( \Phi \in A_{6,6} \), we must have \( (\Phi, \Delta)_6 = 0 \).

But then \( (\theta_6(\Phi), \theta_6(\Delta))_6 = 0 \), i.e., the image of the evaluation map
\[
\theta_6 : A_{6,6} \longrightarrow S_6
\]
is contained in the 6-dimensional subspace of sextics which are apolar to \( \theta_6(\Delta) \). Hence \( K_{6,6} \neq 0 \). It follows that \( s_6 \) must be a summand in \( B_6 \), and hence on dimensional grounds \( B_6 = s_6 + s_2 \).

\(^5\)One can choose \( \Delta = ((F, F)_4, (F, F)_6)_1 \), but the precise expression is not relevant to the argument.
The rest of the generators are all visible, hence the calculation is straightforward. The Betti modules are
\[
\begin{align*}
B_8 &= s_{16} + s_{12} + s_8 + s_0, \\
B_9 &= s_{23} + 2s_{21} + s_{19} + s_{17}, \\
B_{10} &= 2s_{30} + s_{26}.
\end{align*}
\]

3.8. Octavics. The generator dimensions are
\[
\begin{align*}
\beta_4 &= 1, & \beta_5 &= 1, & \beta_6 &= 7, & \beta_7 &= 106, \\
\beta_8 &= 264, & \beta_9 &= 97, & \beta_{10} &= 82.
\end{align*}
\]
All the generators are visible, and the Betti modules are
\[
\begin{align*}
B_4 &= s_0, \\
B_5 &= s_0, \\
B_6 &= s_4 + 2s_0, \\
B_7 &= 2s_{16} + s_{14} + 2s_{12} + s_{10} + s_8 + 2s_4 + s_0, \\
B_8 &= 4s_{24} + 2s_{22} + 4s_{20} + 2s_{16}, \\
B_9 &= 2s_{32} + s_{30}, \\
B_{10} &= 2s_{40}.
\end{align*}
\]

3.9. Nonics. The generator dimensions are
\[
\begin{align*}
\beta_4 &= 1, & \beta_6 &= 71, & \beta_7 &= 508, & \beta_8 &= 324, \\
\beta_9 &= 86, & \beta_{10} &= 51.
\end{align*}
\]
Once again, all the generators are visible. The Betti modules are
\[
\begin{align*}
B_4 &= s_0, \\
B_6 &= s_{14} + 2s_{10} + 4s_6 + 2s_2, \\
B_7 &= 3s_{23} + 5s_{21} + 5s_{19} + 6s_{17} + 4s_{15} + 3s_{13} + s_{11}, \\
B_8 &= s_{34} + 6s_{32} + 2s_{30} + s_{28}, \\
B_9 &= s_{43} + s_{41}, \\
B_{10} &= s_{50}.
\end{align*}
\]

3.10. Decimics. The generator dimensions are
\[
\begin{align*}
\beta_4 &= 1, & \beta_5 &= 3, & \beta_6 &= 367, & \beta_7 &= 679, \\
\beta_8 &= 324, & \beta_9 &= 151, & \beta_{10} &= 61.
\end{align*}
\]
The generators in degrees 4, 5 are visible, which gives \( B_4 = s_0 \) and \( B_5 = s_2 \). In degree 6, one gets

\[
\tilde{Q}_6 = 2s_{20} + 5s_{16} + 2s_{14} + 7s_{12} + s_{10} + 6s_8 + 2s_6 + 5s_4 + 4s_0,
\]

which is 356-dimensional, hence there exist invisible generators. The explanation is similar to the case of septimics.

The space \( A_{6,10} \) is 13-dimensional, whereas \( A_{7,0} = 0 \). Thus every element in \( A_{6,10} \) is apolar to \( \mathbb{F} \). It follows that the map \( A_{6,10} \xrightarrow{\theta_\mathbb{F}} S_{10} \) is not surjective, and hence its kernel is at least 3-dimensional. The coefficient of \( s_{10} \) in \( \tilde{Q}_6 \) is 1, hence \( B_6 \) must contain at least two copies of \( s_{10} \). This forces \( B_6 = \tilde{Q}_6 + s_{10} \), since the additional term precisely compensates for the missing dimensions (367 = 356 + 11).

From degree 7 onwards, all the generators are visible and the modules are

\[
\begin{align*}
B_7 &= 4s_{30} + 6s_{28} + 7s_{26} + 4s_{24} + 4s_{22}, \\
B_8 &= 6s_{40} + 2s_{38}, \\
B_9 &= 2s_{50} + s_{48}, \\
B_{10} &= s_{60}.
\end{align*}
\]

I know of no general method for identifying the characters corresponding to invisible generators. In either of the cases above, it is only by educated guesswork that we have succeeded in doing so.

4. MISCELLANEOUS REMARKS

4.1. Let us say (for the present purposes) that an integer \( d \geq 4 \) is ‘prosaic’ if \((\#)\) is an equality for all \( m \), and ‘erratic’ otherwise. Our calculations show that \( d = 4, 5, 8, 9 \) are prosaic, whereas \( d = 7, 10 \) are erratic.

We have treated the case \( d = 6, m = 10 \) as anomalous. Following the definition literally, one gets \( \tilde{Q}_{10} = 0 \), i.e., we have strict inequality in \((\#)\). Nevertheless, (as we have seen) it is easy to restore equality by cancelling \( B_{10} \) against a first syzygy. This suggests that our definitions of ‘prosaic’ and ‘erratic’ are not in their final shape, and a more refined understanding of the problem will modify them. However, even in their present formulation they do seem to capture a valuable distinction.
It would be an interesting (but immensely ambitious) undertaking to arrive
at such a classification for all $d$. The problem implicitly involves the struc-
ture of the ring of covariants $\bigoplus_{m,q} \mathcal{A}_{m,q}$. Such rings are in general very com-
plicated, and it is not obvious how to proceed in the general case.

4.2. The process we have used to calculate the ideal generators is analo-
gous to the minimal resolution conjecture ($\text{MRC}$) for general points in $\mathbb{P}^n$
(see [9]). To see the parallel, consider the following example: let $X$ denote
a set of 8 general points in $\mathbb{P}^2$, and we are to find the generator degrees of its
defining ideal $I_X \subseteq \mathbb{R} = \mathbb{C}[z_0, z_1, z_2]$. The heuristic reasoning goes as fol-
lows. Since $\dim \mathbb{R}_3 = 10$, the evaluation map $e_X : \mathbb{R}_3 \longrightarrow \mathbb{C}^8$ has kernel
dimension $\geq 2$. Since the points are general, we may assume equality, i.e.,
$\dim (I_X)_3 = 2$. By the same reasoning, $\dim (I_X)_4 = 15 - 8 = 7$. Now one
assumes that the rank of the map $(I_X)_3 \otimes \mathbb{R}_1 \longrightarrow (I_X)_4$ is the maximum
possible, which is $2 \times 3 = 6$. Hence there should be one new generator in
degree 4. The process detects no further generators in degree 5, hence we
have an expected presentation

$$0 \leftarrow \mathbb{R}/I_X \leftarrow \mathbb{R} \leftarrow \mathbb{R}(-3)^2 \oplus \mathbb{R}(-4) \leftarrow \ldots$$

The argument can be continued to obtain the module of first syzygies of
$I_X$ (which would be $\mathbb{R}(-5)^2$ in this case), but I have not succeeded in the
analogous calculation for $I_E$. Although MRC is false in general (see [6]),
it is known to be true in many cases (in particular for $\mathbb{P}^2$). Thus, broadly
speaking, the dichotomy between prosaic and erratic integers corresponds
to the one between true and false instances of MRC.

4.3. There is an evidently analogous problem of calculating $I_E$ for the
action of $SL_n$ on the space of $n$-ary $d$-ics. To the best of my knowledge,
the answer is known only in the case $d = n = 3$. Ternary cubics have two
invariants $G_4, G_6$ in degrees 4, 6 respectively (cf. [13, §198], where they are
labelled $S$ and $T$). For a general cubic curve $E$, the hypersurface $\Omega_E \subseteq \mathbb{P}^9$
is of degree 12, with defining equation

$$\theta_E(G_6)^2 G_4 - \theta_E(G_4)^3 G_6 = 0.$$

Much to my chagrin, I have found that at present even the case of ternary
quartics seems too large for computational experimentation.
4.4. If one considers the same problem (in the binary case) over a field of characteristic $p > 0$, then preliminary calculations show that the generator dimensions of $I_E$ depend on $p$. Here are some data for $d = 5$.

| characteristic | $\beta_8 = \beta_{12} = 1$, $\beta_{13} = 12$, $\beta_{16} = 18$ |
|---------------|---------------------------------------------------------------|
| 2             | $\beta_8 = \beta_{12} = 1$, $\beta_{13} = 6$, $\beta_{14} = 32$, $\beta_{15} = 6$ |
| 3             | same as characteristic zero                               |
| 5             | $\beta_8 = \beta_{12} = 1$, $\beta_{13} = 2$, $\beta_{14} = 48$ |
| 7             | same as characteristic zero                               |
| 11            | same as characteristic zero                               |

Since $SL_2$ is no longer linearly reductive, many of the techniques used here are no longer applicable.

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