The simplest model of jamming

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We study a well known machine learning model -the perceptron- as a simple model of jamming of hard objects. We exhibit two regimes: 1) a convex optimisation regime where jamming is hypostatic and non-critical. 2) a non convex optimisation regime where jamming is isostatic and critical. We characterise the critical jamming phase through exponents describing the distributions law of forces and gaps. Surprisingly we find that these exponents coincide with the corresponding ones recently computed in high dimensional hard spheres. In addition, modifying the perceptron to a random linear programming problem, we show that isostaticity is not a sufficient condition for singular force and gap distributions. For that, fragmentation of the space of solutions (replica symmetry breaking) appears to be a crucial ingredient. We hypothesise universality for a large class of non-convex constrained satisfaction problems with continuous variables.

PACS numbers: 63.50.Lm,45.70.-n,61.20.-p,64.70.kj

\textit{Introduction}—Jamming of hard objects is a general phenomenon that has attracted lot of interest, both experimental and theoretical, (see \cite{1,2} for recent reviews). Jamming of hard spheres, where the only interaction is excluded volume, has been widely studied: the jamming point is reached when, both under equilibrium or off-equilibrium conditions, the size of the cages where the particles can move shrinks to zero. In this limit the system is critical: the network of contacts is isostatic \cite{3,4}, and many quantities have an anomalous power behaviour with non-trivial critical exponents \cite{5\text{--7}}. While local excitations give rise to exponents that may depend on the spatial dimension \cite{8,9}, the contributions of long range excitations appear to be super-universal: numerical simulations show that upon removing the contributions of local excitation, the critical exponents have a very weak dependence on the space dimension in wide range of dimensions \cite{10,11}. Moreover, these exponents seem to be independent from the protocol used to generate jammed configurations. In the infinite dimensional limit one expects some simplifications to be present and one can study analytically jamming at equilibrium. Indeed (if we disregard crystallisation) the thermodynamics of a gas of thermal hard spheres can be solved when the dimension goes to infinity \cite{12,13}. One finds a rather unexpected scenario:

At low pressure (low density) we stay in the liquid phase. Increasing the density at a pressure \(P_g\) we enter into the glass phase where the spheres are confined into small volume cages.

Further increasing the density, at a high pressure \(P_G\) we enter into a different glass phase where the cages split into smaller cages when increasing the pressure. This process goes on up to infinite pressure.

In the infinite pressure limit one finds analytically the power laws that have been discovered in \cite{7,16,17}\(\text{: the exponents can be computed and they are compatible (within the numerical bounds) with the exponents found in dimensions from 2 upwards.}

From an abstract viewpoint the problem of packing spheres in space can be viewed as a constraint satisfaction problem where one aims to maximize the sphere’s packing fraction subject to the hard core impenetrability constraints. Statistical physics has been instrumental in the analysis of random constraint satisfaction problems, where the constraints on the variables are chosen at random from an ensemble: by changing the number or the nature of the constraints we go from a satisfiable (SAT) phase (where there is at least one configuration that satisfies all the constraints) to an unsatisfiable (UNSAT) phase (where it is impossible to satisfy all the constraints at the same time). Usually this SAT-UNSAT transition becomes sharp in the thermodynamic limit: with the exclusion of the transition point, the probability of a random problem to be satisfiable becomes 0 or 1 in the thermodynamic limit where the number of the variable of the systems (\(N\)) and the number of constraints (\(M\)) goes to infinity at fixed ratio \(\alpha = M/N\). Random constraint satisfaction problems have been widely studied by physicists in the case of discrete variables: the most celebrated case is the random K-SAT problem \cite{18}, where the physicist’s solution has been recently proved for sufficiently large \(K\) \cite{19}. Polydisperse hard spheres \cite{20} are a particular case of random constraint satisfaction problem, but they differ from the mostly studied cases by the nature of the variables which are continuous rather than discrete. The continuous nature of the variables adds a new dimension to the the problem: the SAT-UNSAT transition coincides with an equilibrium jamming transition where the volume of the space of the satisfying assignments of variables shrinks to zero and the system can become critical.

The infinite dimensional limit of hard spheres is the
first example of non-trivial continuous constraint satisfaction problems that has been analytically solved. One may wonder how much general is the critical picture of the jamming transition, and if criticality there is, if a unique or several universality classes are possible. The aim of this letter is to present a simple model of jamming, that can be solved analytically and it has the same jamming exponents of hard spheres. The model, the spherical perceptron, is very well known in machine learning and neural network theory, where it is used as linear signal classifier \[21\].

A toy sphere packing problem We start from an extreme schematisation of sphere jamming problem in which we substitute the interaction between spheres with a random background and consider a single particle that should not overlap with some spherical obstacles placed in random positions in space. Both the obstacle and the single particle live on \(S_N\), the \(N\) dimensional sphere of radius \(\sqrt{N}\). Let us consider \(M = \alpha N\) point obstacles \(\xi_i^\mu\) \((i = 1, ..., N, \mu = 1, ..., M)\) in fixed random positions on the \(N\) dimensional sphere, and a particle at \(x_i\) constrained to be at a distance greater than \(\sigma\) from the obstacles: \(|\xi^\mu - x| > \sigma\). As we shall see this problem is isomorphous to the perceptron.

The perceptron – Also in this case the configuration space is the sphere \(S_N\) of normalised \(N\)-dimensional vectors \(x\) such that \(\sum_{i=1}^{N} x_i^2 = N\). We impose the following \(M = \alpha N\) constraints: we have \(M N\)-dimensional random vectors \(\xi^\mu\) (with the same normalisation) and we require that

\[
r_{\mu} = \frac{1}{\sqrt{N}} \sum_i \xi_i^\mu x_i - \kappa > 0 \quad \forall \mu = 1, ..., M. \tag{1}
\]

The quantities \(r_{\mu}\) will be called gaps in the following. In a packing perspective we see that the problem of the toy spheres coincides with the perceptron for \(\sigma^2 = 2N + 2\kappa\). The analogue of packing fraction maximisation in the problem of hard spheres is here maximisation of \(\kappa\) for fixed \(\alpha\). While in machine learning the interest is generally restricted to the positive values of \(\kappa\) (see \[22\] for an exception), for the jamming problem negative values are equally legitimate and, we will see, more interesting. In the \(\alpha - \kappa\) plane there is a SAT region where the previous inequalities have at least one solution (with probability one) and an UNSAT region where there is no solution.

The line of jamming points that separates the two regions has the shape shown in fig. 1. The value of \(\alpha_c(\kappa)\) on this line is usually called the maximum perceptron capacity in machine learning. The perceptron problem has been studied in the past with statistical physics approaches \[23, 24\] and the results of Gardner and Derrida for \(\kappa \geq 0\) are well know: defining \(D_{\sigma^2}y\) the Gaussian measure with zero average and variance \(\sigma^2\) \((Dy \equiv D_1y)\), and the error function \(H(x) = \int_x^\infty Dy\), we have

\[
\alpha_c(\kappa) = \left(\int_{-\kappa}^{\infty} Db(h - \kappa)^2\right)^{-1} \tag{2}
\]

The distribution of the gaps \(g(r)\) at jamming, that we normalize to the ratio of the number of constraints over variables \(\alpha\), is given by

\[
g(r) = \alpha(1 - H(\kappa))\delta(r) + \frac{\alpha}{\sqrt{2\pi}} e^{-(r + \kappa)^2/2\theta(r)}. \tag{3}
\]

According to equations \[23, 24\], for \(\kappa > 0\) the system is ‘hypostatic’: the weight of the delta function contribution in \([3]\) gives precisely the fraction of binding constraints in the system, for which \([4]\) is verified as an equality. This is a decreasing function of \(\kappa\), smaller than one for \(\kappa > 0\) and equal to one exactly at \(\kappa = 0\). Notice that the \(r > 0\) part of the distribution is regular and has a finite limit for \(r \to 0\). These results contrast with salient features of jamming in hard spheres: the property of isostaticity and the presence of a power law singularity in the gaps distribution at small \(r\) \([5, 6]\).

However these formulæ are valid only for positive or zero \(\kappa\) \((\alpha(0) = 2)\). In this situation, for any \(\alpha < \alpha_c(\kappa)\) the space of allowed configurations is convex on the sphere (it is the intersection of convex domains) and the situation is well under mathematical control \([22, 23]\). The case of negative \(\kappa\) is much harder, each constraint defines a non-convex allowed domain and the space of solutions can fragment in disconnected regions. In the statistical approach that we are going to describe below, this phenomenon corresponds to replica symmetry breaking, while the previous formulæ have been derived in a replica symmetric assumption. We will show that as soon as \(\kappa < 0\) replica symmetry is broken and critical universality at the jamming point emerges, the system is isostatic (i.e. the number of contacts is equal to the dimension of the space, i.e. \(N\)), \(g(r)\) displays a power law singularity \(g(r) \sim r^{-\gamma}\) at small \(r\), which is in turn associated to a pseudo-gap in the force distributions \(P(F) \sim F^{\theta}\) at small \(F\). With an argument analogous to the one used in hard spheres \([6]\), one can show (see SM) that the exponents verify the stability bound \(\gamma \geq 1/(2 + \theta)\). The arguments of \([23]\) can be used to argue that stability should be marginal and this inequality saturated. The values we find \(\gamma = 0.41269\) and \(\theta = 0.42311\) indeed saturate to numerical precision the bound and they coincide with the ones found in high dimensional hard spheres.

In order to study the model it is convenient to introduce an Hamiltonian \(H(x)\) that is non zero only if all the constraints are violated. A choice analogous to the soft sphere Hamiltonian is

\[
H(x) = \frac{1}{2} \sum_{\mu=1}^{M} r_{\mu}^2 \theta(r_{\mu}), \tag{4}
\]
where $\theta$ is the standard Heaviside function. In the following we will concentrate mainly on analytic computation of the Gardner volume $\mathcal{V}$ of the satisfying assignments

$$
\mathcal{V}(\alpha, \kappa) = e^{NS(\alpha, \kappa)} = \int_{S_N} dx \prod_{\mu=1}^{\alpha N} \theta(r_\mu)
$$

The equilibrium jamming transition line is the locus of points where this volume shrinks to zero, and we will approach it from the SAT phase. To test our theoretical findings, and show that also in the perceptron the critical properties of jamming are independent of the preparation of the jammed configurations, we will also present numerical simulations, where we generate non-equilibrium jammed configurations through local minimisation algorithms. These last can be defined as isolated points of minimum of $\mathcal{H}$ with $\mathcal{H} = 0$.

**The Gardner Volume** The quenched average of the entropy $S(\alpha, \kappa)$ over the random vectors $\xi^\mu$ can be performed with replicas. Alternatively the more transparent but cumbersome cavity method could be used [27]. Following standard computations, see e.g. [28], one finds [24] that the entropy can be expressed as a saddle point over the overlap matrix between solutions in different replicas $Q_{ab} = N^{-1} \sum_{i} (x^a_i x^b_i)$ where $a, b = 1, \ldots, n$ with $n \to 0$ at the end [29]:

$$
nS(Q) = 1/2 \text{tr} \log Q + \alpha \log \left( \sum_{ab} Q_{ab} \prod_a \theta(h_a - \kappa) \right)_{h_a = 0}
$$

Assuming that the replica symmetry is broken in the usual hierarchical ultrametric way [28], one gets an explicit form of the entropy that we report for reader convenience. First of all, using the parameterization the matrix $Q$ in terms the function $x(q)$ which varies in an interval $[q_0, q_1]$ [28], we have

$$
\frac{1}{n} \text{tr} \log Q = \log(1 - q_1) + \frac{q_0}{\lambda(q_0)} + \int_{q_0}^{q_1} dq \frac{1}{\lambda(q)}
$$

$$
\lambda(q) = 1 - q_1 + \int_{q}^{q_1} dq' x(q').
$$

Secondly, the remaining term in the entropy, can be written as $-na \int D_h(h - \kappa) f(q_0, h)$, where, indicating with dots $q$-derivatives and with primes $h$-derivatives, the function $f(q, h)$ verifies the partial differential equation [30]:

$$
f' = -\frac{1}{2}(f'' + x f'^2)
$$

with boundary condition

$$
f(q_1, h) = -\log H \left( \frac{\kappa - h}{\sqrt{1 - q_1}} \right).
$$

As usual, in order to write the variational equations for $x(q)$ it is useful to define the distribution of the local gaps at level $q$: $P(q, h)$ which verifies [31, 32]

$$
\dot{P} = \frac{1}{2} (P'' - 2x (m P'))
$$

$$
P(q_0, h) = D_{q_0}(h)/dh.,
$$

where we introduced $m(q, h) = f'(q, h)$ that verifies

$$
m = -\frac{1}{2} (m'' + 2x mm').
$$

The variational equations with respect to $x(q)$ read

$$
\frac{1}{2} \left( \frac{q_0}{\lambda(q_0)^2} + \int_{q_0}^{q_1} dq' \frac{1}{\lambda(q')^2} \right) - \frac{\alpha}{2} \int dh P m'^2 = 0.
$$

The RS solution is recovered from the above formulation in the limit $q_1 = q_0$. If there is a continuous part in $x(q)$ it is useful to consider the derivatives of the [33] w.r.t. $q$

$$
\frac{1}{2 \lambda(q)^2} - \frac{\alpha}{2} \int dh P m'^2 = 0.
$$

which, as it is well known, signals that continuous RSB solutions are marginally stable with a divergent spin glass susceptibility. In any discontinuous solution, stability requires positivity of the l.h.s. of (14). For each value of $\kappa$ at sufficiently low values of $\alpha$ the system is in the replica symmetric “liquid” phase: the space of SAT assignments is simply connected and one can go with continuity from one solution to the others. At higher values of $\alpha$ replica symmetry breaks down and the space of solution becomes disconnected. The line of transition for $\kappa < 0$, along with the jamming line estimated from the RS solution, are presented in fig. 1. The RSB transition occurs in the SAT phase for $\kappa < 0$, and as announced, the jamming line lies in the glassy RSB phase. Generically, the RSB solution space fragmentation can occur either through a second order transition to a continuous RSB phase or through a discontinuous Random First Order Transition [28]. Close to $\kappa = 0$, and down to $\kappa = \kappa_{RSB} \approx -2.05$ one finds RSB to a continuous solution occurring via a de Almeida-Thouless [33] instability of the RS solution. Below the value $\kappa_{RSB}$ one finds a transition to a discontinuous “one step” solution. It is important to remark however, that upon increasing $\kappa$ at fixed $\alpha$ so to approach jamming, a second transition to a continuous solution, the so called Gardner transition should be expected, so that for all $\kappa < 0$ jamming is described by a continuous solution.

**Jamming** The jamming transition is the point where the space of solutions shrinks to a point, the entropy goes to $-\infty$, and the solution’s self-overlap $q_1 \to 1$. In this conditions, the boundary condition [10] for $f$ in $q_1$ reduces to

$$
f(q_1, h) \approx \frac{-h^2 \theta(-h)}{2(1 - q_1)} q_1 \to 1.
$$

As for jamming in spheres or for the low temperature limit of the SK model, we can expect a scaling regime
as estimated from the RS solution. The jamming line is exact for $\kappa > 0$ while corrections are to be expected for $\kappa < 0$, where the RS solution just provides an upper bound to the true value.

FIG. 1. Phase diagram of the model. The de Almeida-Thouless (AT) line of instability of the RS solution in the $\alpha - \kappa$ plane (red line), together with the jamming line (blue) as estimated from the RS solution. The jamming line is ex-

A simple variant: random linear programming. Our results are quite surprising, the jamming transitions in the perceptron and in hard spheres, which are rather different problems are in the same universality class. Both in the high dimensional spheres and in the present case, isostaticity as well as the singular behaviour of the gap and force distribution at small argument appear as non-trivial consequences of the full-RSB solution to the problem. It is therefore natural to ask if RSB is a necessary ingredient, if isostaticity and power laws are always associated or there are models where isostaticity hold while the system remains replica symmetric, and if in this case the singular behaviour in $g$ and $P$ is found. To address these questions we slightly modify the problem [1]. On the one hand we relax the spherical constraint on $x$, which then becomes a vector of $N$ real variables, on the other we modify the constraints to:

$$\frac{1}{\sqrt{N}} \sum_i \xi^\mu_i x_i + s^\mu - \kappa > 0 \quad \forall \mu = i, M.$$ (17)

where, as before the $\xi^\mu$ are vectors on the sphere and we introduced new constants $s^\mu$ that we take as Gaussian numbers of zero average and variance $\sigma$. The maximisation of $\kappa$, and the simultaneous determination of the maximiser vector $x$ is a linear programming problem which is convex for all values of $\alpha$. For $\alpha < 2$ and all $\kappa$ the constraints [17] define an open region of space and no finite maximum for $\kappa$ exists. We concentrate then on $\alpha \geq 2$ where the constraints define a closed region of space and a maximum $\kappa$ exists. The analysis of the Gardner volume for this model is very similar to the one of the perceptron (see SM) with the important difference that here replica symmetry always holds. Remarkably, the RS solution is marginally stable and isostaticity holds.

Simulations — In order to check the soundness of these predictions we have performed the following numerical experiment. We have started from a random configuration at $\alpha = 3$ and we have found a minimum of the Hamiltonian [1] in the UNSAT region $\kappa > \kappa_c(\alpha)$, and we have then approached the jamming point decreasing the value of $\kappa$ until the point where the energy is about $10^{-12}$. We obtain in this way a jammed configuration with $\kappa = \kappa^* \approx -0.431$ (the value we obtain slightly depends on the search procedure). Notice we did not try to equilibrate the system: the jammed configuration that we reach can be expected to be different from the one computed analytically studying the Gardner volume and $\kappa^* < \kappa_c(3)$. We can however study the probability distribution of the gaps and of the forces in the configurations found in this way. We have done this analysis for moderate values of $N$ (in the range $50-400$), and found that for both quantities the distributions at small values are compatible with the thermodynamical predictions. This is same qualitative coincidence of exponents in equilibrium

The difference between the two cases comes in the distributions of gaps and forces at large values that are non-universal.

The behaviours of $p_1(u)$ at large positive and negative argument are related respectively to the small gap and the small force distributions and behaves as

$$p_1(u) \sim \begin{cases} u^{-\gamma} & u \to \infty^+ \\ |u|^\theta & u \to \infty^- \end{cases}$$ (16)

with $\gamma = 2a/k = 0.41269$, $\theta = \frac{k-1-a}{1-k/2} = 0.42311$. The relation $\gamma = 1/(2 + \theta)$ is verified within numerical errors and isostaticity holds.

A different approach: random linear programming. Our results are quite surprising, the jamming transitions in the perceptron and in hard spheres, which are rather different problems are in the same universality class. Both in the high dimensional spheres and in the present case, isostaticity as well as the singular behaviour of the gap and force distribution at small argument appear as non-trivial consequences of the full-RSB solution to the problem. It is therefore natural to ask if RSB is a necessary ingredient, if isostaticity and power laws are always associated or there are models where isostaticity hold while the system remains replica symmetric, and if in this case the singular behaviour in $g$ and $P$ is found. To address these questions we slightly modify the problem [1]. On the one hand we relax the spherical constraint on $x$, which then becomes a vector of $N$ real variables, on the other we modify the constraints to:

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and off-equilibrium that is observed in the field distribution of SK model at zero temperature and in hard spheres. The data of best quality are these for the force distribution and are presented in figure [2]. We also simulated in a similar way the random linear programming model, where thanks to convexity the limit of thermodynamic jamming can be readily reached, and we find force and gap distributions in perfect agreement with the RS predictions with no singularity at small argument.

**Conclusions** — The most important results of this letter could be summarised as follow: jamming in the perceptron is hypostatic and stable in the region $\kappa > 0$ where it defines a convex optimisation problem, conversely in the non convex region $\kappa < 0$ it is isostatic and marginally stable. Remarkably jamming is in the same universality class as hard spheres. The salient feature of jamming at jamming for $\alpha = 4$ and $N = 50, 100, 200, 400$ in a log-log scale. The straight line is a one parameter fit of the kind $P(f) = a f^\theta$ with the predicted value of $\theta = 0.42311$. tends to systems with constraints of a different nature, e.g. where the number of non-zero $x_i$ in each constraint is finite. It has been argued in [24] that isostaticity gives rise to infinite correlations, and that for this reason at jamming hard spheres behave under stress as a system with long range forces. Our hypothesis suggests that this could be the case for a large class of systems.

The perceptron is an extreme limit of the hard sphere problem in infinite dimensions where all the particles except one are pinned in random positions. It is known that in low dimensional hard sphere systems the exponents are independent of the fraction of pinned particles [30]. One of the interests of the perceptron is that it is a much simpler model than hard spheres -even in the infinite dimensional limit-. The derivation of the replica effective action is much more direct and the study with the cavity method would be straightforward. Though the connection of this model with jamming had never been underlined before, a lot is known on the model even at the rigorous level. Many questions that one could ask about jamming could be answered more easily in this context than in the hard spheres. Our present interests include the computations of spectrum of vibrational normal modes at small temperature and the low temperature specific heat in a quantum version of the model. This is technically feasible: more surprises are to be waited.

**Acknowledgments** — We would like to thank G. Biroli, P. Charbonneau, M. Lenz, M. Müller, M. Wyart for very useful suggestions and P. Urbani and F. Zamponi for extremely valuable discussions and for careful reading of the manuscript. The European Research Council has provided financial support through ERC grant agreement no. 247328. SF acknowledges the hospitality of the Physics Department of the “Sapienza” University of Rome.

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[1] A. J. Liu, S. R. Nagel, W. Van Saarloos, and M. Wyart, Dynamical heterogeneities in glasses, colloids, and granular media Editors: L. Berthier, G. Biroli, J-P Bouchaud, L. Cipelletti and W. van Saarloos Oxford University Press 2010 (2010).

[2] S. Torquato and F. H. Stillinger, Reviews of modern physics 82, 2633 (2010).

[3] C. F. Moukarzel, Physical review letters 81, 1634 (1998).

[4] A. V. Tkachenko and T. A. Witten, Physical Review E 60, 687 (1999).

[5] C. S. O’Hern, S. A. Langer, A. J. Liu, and S. R. Nagel, Physical Review Letters 88, 075507 (2002).

[6] C. S. O’Hern, L. E. Silbert, A. J. Liu, and S. R. Nagel, Physical Review E 68, 011306 (2003).

[7] M. Wyart, Physical review letters 109, 125502 (2012).

[8] E. Lerner, G. Düring, and M. Wyart, EPL (Europhysics Letters) 99, 58003 (2012).

[9] E. Lerner, G. Düring, and M. Wyart, Soft Matter 9, 8252 (2013).
q is solution of the saddle point equation, b) the l.h.s. of eq. (14), which should be positive in a stable solution, vanishes on the line, c) the RS solution coincides with a degenerate RSB solution with $q_0 = q_1 = q$ and the “breaking point” $x_c = \lim_{q_1 \to q} x(q_1)$ lies in the interval $[0,1]$. The breaking point $x_c$ can be computed from (30). Consistency requires $x_c < 1$; if $x_c$ so identified is $x_c > 1$, this is a signal that a 1RSB transition occurs before the RS solution becomes unstable. In fig. 1 we show that full RSB transition occurs in the range $0 < \kappa < \kappa_{1RSB} \approx -2.05$ where $x_c < 1$, for $\kappa < \kappa_{1RSB}$ the found value is $x_c > 1$ and a 1RSB - or random first order transition- can be expected.

**Asymptotic solution close to jamming** In order to characterise the scaling regime close to jamming, where $1 - q_1 \to 0$, let study first the solution to the equations for large values of $|h|$ and consider the equation for $m$ first. For large negative $h$ one can assume a form $m(q,h) = -\frac{b}{r(q)}$, which inserted into (12) gives

$$\dot{r}(q) = -x(q).$$

The solution that respects the condition in $q_1$ is $r(q) = \lambda(q)$. On the other extreme, for $h \to +\infty$ one trivially has $m(q,h) = 0$. Let us now study the equation for $P(q,h)$. For large positive $h$, one just have a diffusion equation, and since for $q \to q_0$ $P(q_0,h) = D_{q_0} h/dh$, then $P(q_0,h) = D_q h/dh$, which tends to a Gaussian with unit variance for $q \to 1$. For large negative $h$ on the other hand, it is simple to show that $P(q,h)$ must have the form $P(q,h) = B(q)e^{-A(q)\frac{x^2}{\lambda}}$. The factors $A$ and $B$ should verify:

$$\frac{1}{2} \frac{A}{A} - \frac{\dot{B}}{B} = -\frac{A}{2} \frac{x}{\lambda}.$$
In order to solve these equations to the leading order, we should know something more about the behaviour of the function \( x(q) \). We are interested to the scaling regime when \( 1 - q_1 \ll 1 \). Since \( x(q) \) is the weight of the overlap lower than \( q \) one can expect it to go to zero in the jamming limit, we therefore write

\[
 x(q) = x_1 \left( \frac{1 - q_1}{\Delta} \right)^{\frac{1}{2}} 
\]

with \( 1 - q_1 \ll x_1 \ll 1 \), where

\[
 \lambda(q) \approx \int_q^1 x(q) \approx \frac{k}{k - 1} x(q) \Delta 
\]

and we denoted \( \Delta = 1 - q \). In this regime, supposing self-consistently \( 2/k > 1 \), we can solve the equations \[19\] supposing that \( A \ll x/\lambda \approx 1 \). In this case, we find \( A(q) \sim \Delta^{-a/2}k^2 \) and \( B(q) \sim \Delta^{-1+1/k} \). We conclude that \( P \) and \( m \) behave respectively as \( P(q, h) \approx p_0(h^\Delta) \Delta^{-a/2}k^2 / (a < b) \) and \( m(q, h) \approx -h/\lambda(q) \Delta^{-a/2}k^2 / (a > b) \). These regimes should be matched by functions \( \Delta^{-a/2}k_1(h\Delta^{-b/2}) \) with \( a < b \) and \( \Delta^{-a/2}k_1(h\Delta^{-b/2}) \). Notice that the in the jamming limit \( q \to 1 \),

\[
 P(q, h) = A\delta(h) + p_2(h) 
\]

with \( A = \int du p_0(u) \). Apart for the \( \delta \)-function term, \( p_2(h) \) is the physical distribution of the gaps at jamming and should scale as \( p_2(h) \sim h^{-\gamma} \) at small argument. We notice at this point that for \( q \to 1 \), eq. \[14\] reduces to

\[
 1 = \alpha \int_{-\infty}^0 dh \ p_0(h) 
\]

which gives \( A = 1/\alpha \) i.e. the condition that the system is isostatic. Generalising the thermodynamic calculation to compute the ground state energy slightly above jamming, for \( \kappa > \kappa_c(\alpha) \), one sees that, apart from a multiplicative constant, the negative part of the distribution, \( p_0(-f) \) can be identified with the force distribution \[14\]. We can therefore assume a power scaling \( p_0(u) \sim |u|^q \) at small argument, to be matched with the behaviour of \( p_1(z) \) at large negative \( z \), analogously, the behaviour \( p_2(h) \sim |h|^{-\gamma} \) at small positive argument should be matched with the behaviour of \( p_1 \) at large positive \( z \). Consistency requires that \( \theta = \frac{b \gamma}{b - 1} \), \( c = k - 1 \), \( \gamma = a/b \) and \( b/k = 1/2 \). We find at this point the remarkable result that the equations and boundary conditions that determine \( p_1 \) and \( m_1 \) coincide with the corresponding ones in the case of jamming of hard spheres. Writing them explicitly we have:

\[
 \frac{a}{k} p_1 + \frac{1}{2} z p'_1 = \frac{1}{2} \left( p''_1 + \frac{c}{k} (m_1 p'_1) \right) 
\]

\[
 \frac{c}{k} m_1 + \frac{1}{2} m'_1 = -\frac{1}{2} \left( m''_1 + \frac{c}{k} m_1 m'_1 \right) 
\]

with boundary conditions

\[
 p_1(z) = \begin{cases} \end{cases} 
\]

\[
 m_1(z) = \begin{cases} \end{cases} 
\]

In addition the solution should verifies

\[
 \int du \ p_1(u) \left[ ((k - 1) m'_1(u)^2 (1 + m''(u)) - km''_1(u)^2) \right] = 0 
\]

that follows from eq. \[14\] and its derivative w.r.t. \( q \) in the scaling domain. We can just then quote from \[14\] the values of \( \gamma = 0.41269 \) and \( \theta = 0.42311 \). We should note at this point that while the exponents and the scaling functions \( p_1 \) and \( m_1 \) appear to be universal, the functions \( p_0 \) and \( p_2 \), apart for their small argument part, are system specific and non universal.

**Random Linear Programming** The replicated entropy of the RLP model is very similar to the one of the percolation, indeed formula \(7\) is substituted by:

\[
 nS[Q] = \frac{1}{2} \text{tr} \log Q + \left( 29 \right) 
\]

\[
 \alpha \log \left( e^{\frac{r}{2} \sum_{a, b} \left[ Q_{a,b} \sigma \theta_{a,b} \prod_{a} \theta(h_{a} - \kappa) \right]} \right) 
\]

with the important difference that here since there is no spherical constraint the diagonal value of \( Q_{a,b} \), \( Q_{a,a} = \tilde{q} = \frac{1}{N} \sum_{i=1}^{N} x_{i}^{2} \) is a variational parameter to be determined by the saddle point equations. Since the problem is convex replica symmetry always holds, and

\[
 S = \frac{1}{2} \left[ \log(\tilde{q} - q) + \frac{q}{\tilde{q} - q} \right] + \alpha \int D_{\sigma + \theta} \log H \left( \frac{\kappa - y \sqrt{q - q}}{\sqrt{q - q}} \right) 
\]

in the jamming limit in which \( \Delta = \tilde{q} - q \to 0 \), \( S \) should not diverge faster than log \( \Delta \). Expanding \( S \) in \( \Delta \) we have

\[
 S \approx \frac{1}{2 \Delta} \left[ q - \alpha \int D_{\sigma + \theta} (y - \kappa)^{2} \theta(\kappa - y) \right] 
\]

both this term and its \( q \)-derivative should vanish; we get

\[
 0 = q - \alpha \int D_{\sigma + \theta} (y - \kappa)^{2} \theta(\kappa - y) \quad (32) 
\]

\[
 0 = 1 - \alpha \int D_{\sigma + \theta} \theta(\kappa - y), \quad (33) 
\]

which have a finite \( q \) and \( \kappa \) solution if \( \alpha > 2 \). Eq. \[33\] is the RS version of the condition of marginal stability \[14\] and implies isostaticity. From an explicit computation one sees that the field distribution is given by

\[
 g(r) = \delta(r) + \frac{\alpha}{\sqrt{2\pi} (\sigma + q)} e^{-(r + \kappa)^{2}/2(\sigma + q)} \theta(r) \quad (34) 
\]
We see that the solution is marginally stable and isostatic, but \( g(r) \) is regular in the origin. The singular behaviour of \( g(r) \) appears critically associated to replica symmetry breaking. This requires an interpretation which is at present lacking.

**Stability Bound** It is possible to obtain the relation between the force and the gap exponents \( \gamma \geq 1/(2 + \theta) \) in the perceptron, generalising the argument of stability used by Wyart in hard spheres [7]. To this aim let us note that for fixed \( \alpha \) the maximisation of \( \kappa \) on the sphere with fixed \( \mathbf{x}^2 \) is equivalent to the dual optimisation problem of finding an the maximum (resp. minimum) value in free space \( R^N \) of \( \mathbf{x}^2 \) at fixed value of \( \kappa \) negative (resp. positive). Let us consider the most interesting case \( \kappa < 0 \).

The core of the argument is that in isostatic configurations, in absence of small enough gaps one could increase the value of \( \mathbf{x}^2 \) just by following the ‘floppy mode’ that ensue from the opening of a contact; stability then requires that small forces be in correspondence with small enough gaps.

The maximisation of \( \mathbf{x}^2 \) under the perceptron constraints can be performed introducing Karush-Kuhn-Tucker multipliers \( F^\mu \) (forces) and the objective function:

\[
L(\mathbf{x}) = \mathbf{x}^2 + \sum_{\mu=1}^{\alpha N} F^\mu [\frac{1}{\sqrt{N}} \mathbf{x} \cdot \xi^\mu - \kappa].
\] (35)

The first order maximisation conditions read:

\[
\frac{\partial L}{\partial x_i} = 2x_i + \frac{1}{\sqrt{N}} \sum_{\mu} F^\mu \xi_i^\mu = 0.
\] (36)

In an isostatic maximum, there are \( N \) positive forces \( F^\mu \) in correspondence with the binding constraints while all the others are zero. Let us suppose without loss of generality to renumber the constraints so that the \( 0 < F^1 < F^2 < \ldots < F^N \) while \( F^\mu = 0 \) for \( \mu > N \), and study the effect of unbinding the first contact by an extent \( s \). We change then \( \mathbf{x} \) into \( \mathbf{x} + \delta \mathbf{x} \) in such a way

\[
\frac{1}{\sqrt{N}} (\mathbf{x} + \delta \mathbf{x}) \cdot \xi^1 = \kappa + s,
\]

while keeping the remaining contacts binding:

\[
\frac{\delta \mathbf{x} \cdot \xi^\mu}{\sqrt{N}} = s\delta_{\mu,1} \text{ for } \mu = 1, \ldots, N
\] (37)

Eq. (37) has a unique solution and implies that each of the \( \delta x_i \) for \( i = 1, \ldots, N \) is of order \( s \). Using (36,37), the variation of \( \mathbf{x}^2 \), \( \Delta \mathbf{x}^2 = 2 \mathbf{x} \cdot \delta \mathbf{x} + (\delta \mathbf{x})^2 \) can be then written as

\[
\Delta \mathbf{x}^2 = -F^1 s + ANs^2
\] (38)

where \( A \) is a constant of order 1. Notice that \( \Delta \mathbf{x}^2 < 0 \) for small \( s \), while it changes sign for \( s = s^* \equiv F^1/(AN) \). The maximum is stable if before reaching that point, a new contacts forms, that prevents further maximisation of \( \mathbf{x}^2 \). At this point the argument proceeds verbatim as in [7], we reproduce it here just for completeness. The less restrictive assumption one can make on the force distribution is the presence of a power singularity \( P(F) \sim F^{-\gamma} \), which implies \( F^1 = F_{\text{min}} \sim N^{-1-\gamma/\theta} \). Opening the weakest contact would therefore imply a growth in \( \mathbf{x}^2 \) if \( s > s^* \sim N^{-1-\gamma/\theta} \). However, a new contact is formed and blocks the floppy mode at a value of \( s \) of the order of \( r_{\text{min}} \) the minimum gap in the system. Stability requires therefore \( r_{\text{min}} \geq s^* \). Consequently, the distribution \( g(r) \) should be power law in the origin \( g(r) \sim r^{-\gamma} \) in such a way that \( r_{\text{min}} \sim N^{-1-\gamma/\theta} \). The inequality \( \gamma \geq 1/(2 + \theta) \) follows readily. Two facts should be noted:

1. If \( \kappa > 0 \) maximisation of \( \kappa \) at fixed \( \mathbf{x}^2 \) is dual (equivalent) to minimisation of \( \mathbf{x}^2 \) at fixed \( \kappa \).

In this case, the forces are negative, and both terms in (38) lead to an increase of \( \mathbf{x}^2 \).

2. For the RLP \( \kappa \) maximisation is by no means equivalent to \( \mathbf{x}^2 \) maximisation.

In both cases, small forces do not need to be compensated by the existence of small gaps and there is no necessarily a relation between the two distributions.