Abstract

We have investigated the effect of a weak magnetic field on momentum transport in a thermal QCD medium at finite quark chemical potential using a semiclassical kinetic theory. In the presence of a magnetic field, the momentum transport coefficients acquire a tensor structure, reflected by the five shear ($\eta_0, \eta_1, \eta_2, \eta_3$, and $\eta_4$) and two bulk viscous components ($\zeta_0$ and $\zeta_1$). The weak magnetic field removes the degeneracy in the effective mass of flavours, leading to different masses for the left-handed (L) and right-handed (R) chiral modes of quarks. The coefficients, $\eta_0, \eta_1$ and $\eta_3$ decrease with magnetic field in L mode and increase in R mode, whereas, $\eta_2$ and $\eta_4$ increase in both L and R mode. The bulk viscous coefficients, $\zeta_0$ and $\zeta_1$ increase with magnetic field for both L and R mode. The shear and bulk viscous coefficients positively amplify with baryon asymmetry.

I Introduction

Ultrarelativistic heavy ion collisions at experimental facilities such as Relativistic Heavy Ion Collider (RHIC) at Brookhaven National Laboratory, Super Proton Synchrotron (SPS), Large Hadron Collider (LHC) at European Organization for Nuclear Research, aim to produce the new state of nuclear matter at extremely high temperature and/or density. The composite states, hadrons, lose their identity and dissolve into a soup of their constituents—quarks and gluons known as quark-gluon plasma (QGP). A very strong magnetic field ($eB$) is produced during the initial stage of non-central heavy-ion collisions, whose magnitude is of the order of $|eB| = 0.1 m_{\pi}^2$ ($m_{\pi} \sim 0.135$ GeV, pion mass) for SPS energy, $|eB| = m_{\pi}^2$ for RHIC and $|eB| = 15 m_{\pi}^2$ for LHC [1–3]. The possibility of existence of such a strong magnetic field has motivated the theoreticians to study the QGP under the intense field. The various novel phenomena such as chiral magnetic effect [1, 4], magnetic and inverse magnetic catalysis [5, 6], axial magnetic effect [7, 8], chiral vortical effect [9, 10], dilepton production rate [11], and various thermodynamical [12, 13] and transport coefficients [14–21] etc. have been studied in the presence of a magnetic field.
The magnetic field produced was considered to have been strong for a very short span of time, \( t \sim 0.2 \) fm/c for RHIC energies \([1, 22]\), but it was then brought out that its lifespan is prolonged by the finite electrical conductivity of the medium \([23, 24]\). The predictions of charged hadron elliptic flow from RHIC \([25]\) and their theoretical explanations using dissipative hydrodynamics \([26]\) have provided the experimental evidence of existence of the transport processes in the QGP. These transport coefficients are not directly measurable experimentally, rather serve as input parameters in the theoretical modeling of experimental observables such as directed flow, elliptic flow etc.

One of the most important transport coefficients in the hydrodynamical description is shear viscosity \((\eta)\) that governs the rate of momentum transfer in a presence of inhomogeneity of fluid velocity, while the bulk viscosity \((\zeta)\) describes the change of local pressure when the fluid element is either expanding or contracting. The relativistic hydrodynamics played one of the most important roles to extract the value of shear viscosity to entropy density ratio \((\eta/s)\) from the available experimental data \([25, 27–33]\), and the ratio is found to be below or close to the Kovtun-Son-Starinets (KSS) bound \([34]\), \(\eta/s > \frac{1}{4\pi}\). On the other hand at the classical level, the bulk viscosity vanishes for the conformal fluid and massless QGP but quantum effects break the conformal symmetry and it acquires non-zero value even for the massless QGP found in the lattice studies of SU(3) gauge theory \([35]\). In several other studies, the value of \(\eta/s\) was found to be minimum \([36, 37]\) near the phase transition whereas for bulk viscosity was found to be maximum \([38, 39]\). Taking into account all these studies, shear and bulk viscosities for a hot QCD medium have been studied using various approaches viz. kinetic theory \([40, 41]\), perturbative QCD \([42–44]\), Kubo formalism \([45]\).

The transport coefficients such as shear viscosity, bulk viscosity and electrical conductivity are taken as an input to dynamical model such as relativistic magnetohydrodynamics. Hence, it is important to calculate these transport coefficients in the presence of a magnetic field. In the present work, we have studied the shear and bulk viscosity in the presence of a weak magnetic field at finite quark chemical potential using kinetic theory. The viscous dissipative tensor gets decomposed into seven components resulting from the breaking of rotational invariance in the presence of a magnetic field. In the presence of a strong magnetic field \((q_fB >> T^2, m_f^2)\) with \(T, q_f\) and \(m_f\) being temperature of the system, absolute electric charge and current quark mass of quark of \(f\)-th flavor respectively, the motion of charged particle is restricted to the \(1 + 1\) dimensional Landau level dynamics. Hence, only the longitudinal component contributes to the viscous coefficients \([14, 17, 46, 47]\). Furthermore, explorations have been made in the presence of an arbitrary magnetic field in various models such as hadron resonance gas model \([48, 49]\), perturbative QCD \([16]\), kinetic theory \([50–54]\), linear sigma model \([55]\), in which unlike the case of strong magnetic field, transverse and Hall components also make a finite contribution to the viscous coefficients. We have explored the effect of weak magnetic field \((T^2 > q_fB > m_f^2)\) and baryon asymmetry \((\mu \neq 0)\) on the momentum transport coefficients using kinetic theory via the quasiparticle description of partons. Here, we have employed the general framework of projection tensors and written it in terms of hydrodynamical and magnetic degrees of freedom \([56]\), to calculate the viscous coefficients. In Ref. \([53]\), authors have incorporated the pure thermal mass to incorporate the interaction, while we have used the thermally generated mass with magnetic field correction. The dispersion relation of quasiparticles in weak magnetic field gives four collective modes, two from left-handed and two from right-handed modes. Various properties of dispersion relations have been discussed in Refs. \([57, 58]\). The degeneracy in left-handed (L) and right-handed (R) chiral modes of quarks is lifted up due to their different masses, which is in contrast to the strong magnetic field case, where these modes are degenerate. We have taken into account the medium-generated mass for both modes separately to estimate the momentum transport
coefficients. We have further extended the discussion to study the specific shear viscosity, specific bulk viscosity and Reynolds number.

The present work is organized as follows: In Sec. II, we have discussed the quasiparticle model of partons and hence evaluated the medium generated mass. In Sec. III, we have used this mass as an input parameter to calculate the momentum transport coefficients in the presence of a weak magnetic field using kinetic theory under relaxation time approximation (RTA) for both modes. In Sec. IV, we have discussed the result of these transport coefficients, namely, shear and bulk viscosities at different values of magnetic field and quark chemical potential. Further, we have discussed some applications of these viscosities in terms of specific shear and bulk viscosities, and Reynolds number in Sec. V. Finally, we have summarized the work in Sec. VI.

II Quasiparticle Model for Partons

The quasiparticle description of quarks and gluons encodes the interaction among partons in the form of medium-generated masses. There exists different models for quasiparticle description such as Nambu-Jona-Lasinio (NJL) model [59–62], Polyakov NJL model [63–65], Gribov-Zwanziger quantization [66, 67], linear sigma model [55, 68] etc. However, we have adopted the perturbative thermal QCD approach in which the medium generated masses for quarks and gluons are obtained from the poles of dressed propagators calculated by their respective self-energies. In pure thermal medium at finite quark chemical potential ($\mu$), the thermally generated mass for quarks and gluons are obtained as [69]

\[
m_{th}^2 = \frac{1}{8} g^2 C_F \left( T^2 + \frac{\mu^2}{\pi^2} \right),
\]

\[
m_g^2 = \frac{1}{6} g^2 T^2 \left( C_A + \frac{1}{2} N_f \right),
\]

respectively. $C_F = (N_C^2 - 1)/2N_C = \frac{4}{3}$ for $N_C = 3$, $C_A(C_A = 3)$ is the group factor, $N_f$ is the number of flavor, $g$ is the QCD coupling constant with $g^2 = 4\pi\alpha_s$, where $\alpha_s$ is the one-loop running coupling constant, which runs with temperature as [70]

\[
\alpha_s(\Lambda^2) = \frac{1}{b_1 \ln \left( \frac{\Lambda^2}{\mu^2} \right)},
\]

where $b_1 = (11N_c - 2N_f)/12\pi$ and $\Lambda_{\text{MS}} = 0.176$ GeV. The renormalization scale for quarks and gluons is chosen to be $\Lambda_{\text{q}} = 2\pi\sqrt{T^2 + \mu^2/\pi^2}$ and $\Lambda_{\text{g}} = 2\pi T$ respectively. The effect of weak magnetic field is incorporated through the medium generated masses for quarks and gluons in a weakly magnetized thermal medium, which is determined from the poles of dressed quark propagator. The gluons remains unaffected in the presence of a magnetic field, hence there will be no magnetic field contributing term in their medium dependent mass.

The inverse of the dressed quark propagator using Schwinger-Dyson equation can be written as

\[
S^{-1}(P) = P^{-1}(P) - \Sigma(P)
\]

\[
= \not{P} - \Sigma(P),
\]

where $\not{P} = \gamma^\mu P^\mu$, $P$ is the momentum of external quark line, $S^{-1}(P)$ is bare inverse propagator and $\Sigma(P)$ is the quark self energy shown in Fig. (1). So, to calculate the effective quark propagator in presence
of magnetic field at finite temperature we need to evaluate the quark self energy. The quark propagator in presence of background magnetic field following the Schwinger formalism can be written in terms of Laguerre polynomial \( (L_l(2\alpha)) \) \[71\]

\[
iS(K) = \sum_{l=0}^{\infty} \frac{-id_l(\alpha)D + d_l'(\alpha)\bar{D}}{k_L^2 + 2|q_f B|} + \frac{i\gamma \cdot k}{k_L^2},
\]

where \( q_f \) is the absolute charge of \( f \)th flavor, \( l = 0, 1, 2, \ldots \) are the Landau levels, || and \( \perp \) are the parallel and perpendicular components of momentum respectively with respect to direction of magnetic field, \( \alpha = \frac{k^2_{\perp}}{|q_f B|} \), \( k_L^2 = m^2_{f0} - k^2_{||} \) and \( d_l(\alpha), d_l'(\alpha), D, \bar{D} \) are given as \[72\]

\[
d_l(\alpha) = (-1)^le^{-\alpha}C_l(2\alpha),
\]

\[
d_l'(\alpha) = \frac{\partial d_l}{\partial \alpha},
\]

\[
D = (m_{f0} + \gamma \cdot k||) + \gamma \cdot k_{\perp}\left(\frac{m^2_{f0} - k^2_{||}}{k^2_{\perp}}\right),
\]

\[
\bar{D} = \gamma_1\gamma_2(m_{f0} + \gamma \cdot k||),
\]

with \( C_l(2\alpha) = L_l(2\alpha) - L_{l-1}(2\alpha) \). In weak field limit, the quark propagator can be reorganized in power series of magnetic field \( (q_f B) \) as,

\[
iS(K) = \frac{i(K + m_{f0})}{K^2 - m^2_{f0}} - \frac{\gamma_1\gamma_2(\gamma \cdot K|| + m_{f0})}{(K^2 - m^2_{f0})^2} (q_f B),
\]

where first term in Eq.(7) is the free fermion propagator and second term is the \( O(q_f B) \) correction to it. Neglecting the current quark mass under the limit \( (m^2_{f0} < q_f B < T^2) \) in the numerator and using the following metric tensor in Eq.(7),

\[
g^{\mu\nu} = g^{\mu\nu}_{||} + g^{\mu\nu}_{\perp};
\]

\[
g^{\mu\nu}_{||} = \text{diag}(1, 0, 0, -1); \quad g^{\mu\nu}_{\perp} = \text{diag}(0, -1, -1, 0);
\]

\[
p^{\mu} = p^{\mu}_{||} + p^{\mu}_{\perp}; \quad p^{\mu}_{||} = (p^0, 0, 0, p^3);
\]

\[
p^{\mu}_{\perp} = (0, p^1, p^2, 0); \quad \slashed{p} = \gamma^\mu p_{\mu} = \slashed{p}_{||} + \slashed{p}_{\perp};
\]

\[
\slashed{p}_{||} = \gamma^0 p_0 - \gamma^3 p^3; \quad \slashed{p}_{\perp} = \gamma^1 p^1 + \gamma^2 p^2,
\]

with \[\gamma_1\gamma_2 K_{||} = -\gamma_5[(K \cdot b)\slashed{p} - (K \cdot u)\slashed{b}],\]
we obtain the quark propagator in presence of magnetic field at finite temperature as

\[ iS(K) = \frac{iK}{K^2 - m_f^2} - \frac{i\gamma_5 (K \cdot b) \hat{\gamma} - (K \cdot u) \hat{\gamma}}{(K^2 - m_f^2)^2} (q_f B), \]  

(10)

where \( u^\mu = (1, 0, 0, 0) \) denotes local rest frame of the heat bath. Introduction of a particular frame of reference breaks the Lorentz symmetry of the system. Similarly, \( b^\mu = (0, 0, 0, 1) \) denotes the preferred direction of magnetic field in our system which then breaks the rotational symmetry of the system. Using the quark propagator (10), the one-loop quark self energy up to \( \mathcal{O}(q_f B) \) in hot and weakly magnetized medium can be written as

\[ \Sigma(P) = g^2 C_F T \sum_n \int \frac{d^3k}{(2\pi)^3} \gamma_\mu \left( \frac{K}{(K^2 - m_f^2)} - \frac{\gamma_5 [(K \cdot b) \hat{\gamma} - (K \cdot u) \hat{\gamma}]}{(K^2 - m_f^2)^2} (q_f B) \right) \times \]

\[ \gamma^\nu \left( \frac{1}{P - K} \right)^2, \]  

(11)

where \( T \) is the temperature of the system and \( g^2 = 4\pi\alpha_s(A^2, |eB|) \) with \( \alpha_s (A^2, |eB|) \) is given by

\[ \alpha_s (A^2, |eB|) = \frac{\alpha_s (A^2)}{1 + b_1 \alpha_s (A^2) \ln \left( \frac{A^2}{\Lambda^2 + |eB|^2} \right)}. \]  

(12)

In Eq. (11), first term is the thermal medium contribution (\( \Sigma_0 \)) whereas second one is with magnetic field correction term (\( \Sigma_1 \)).

The general covariant structure of quark self energy at finite temperature and magnetic field can be written as \[ (\small 57) \]

\[ \Sigma(P) = - A \hat{P} - B \hat{\gamma} - C \gamma_5 \hat{\gamma} - D \gamma_5 \hat{\gamma}, \]  

(13)

where \( A, B, C, D \) are the structure functions. In the absence of a magnetic field, \( A \neq 0, B \neq 0 \) but both \( C, D \) will vanish. In the presence of a pure magnetic field without any heat bath, \( A \neq 0, B = 0 \), and the structure functions \( C \) and \( D \) will depend upon the magnetic field, as we will see later. Using Eq. (11) and (13), the general form of these structure functions are obtained as

\[ A (p_0, p, p_z) = \frac{1}{4} \frac{\text{Tr}(\Sigma(P) \hat{P}) - (P \cdot u) \text{Tr}(\Sigma(P) \hat{\gamma})}{(P \cdot u)^2 - P^2}, \]  

(14)

\[ B (p_0, p, p_z) = \frac{1}{4} \frac{(- P \cdot u) \text{Tr}(\Sigma(P) \hat{P}) + P^2 \text{Tr}(\Sigma(P) \hat{\gamma})}{(P \cdot u)^2 - P^2}, \]  

(15)

\[ C (p_0, p, p_z) = - \frac{1}{4} \text{Tr}(\gamma_5 \Sigma(P) \hat{\gamma}), \]  

(16)

\[ D (p_0, p, p_z) = \frac{1}{4} \text{Tr}(\gamma_5 \Sigma(P) \hat{\gamma}). \]  

(17)

These structure functions are found to depend upon various Lorentz scalars defined by

\[ p^0 \equiv P^\mu u_\mu = \omega, \]  

(18)

\[ p^3 \equiv P^\mu b_\mu = - p_z, \]  

(19)

\[ p_\perp \equiv [(P^\mu u_\mu)^2 - (P^\mu b_\mu)^2 - (P^\mu p_\mu)^2]^{1/2}, \]  

(20)

where \( \omega, p_\perp, p_z \) are termed as Lorentz invariant energy, transverse momentum and longitudinal momentum respectively. The detailed calculation of all these structure functions is shown in \( A \) and results are
\begin{equation}
A(p_0, |p|) = \frac{m_f^2}{|p|} Q_1 \left( \frac{p_0}{|p|} \right),
\end{equation}
\begin{equation}
B(p_0, |p|) = -\frac{n_f}{|p|} \left[ \frac{p_0}{|p|} Q_1 \left( \frac{p_0}{|p|} \right) - Q_0 \left( \frac{p_0}{|p|} \right) \right],
\end{equation}
\begin{equation}
C(p_0, |p|) = -4g^2C_F M^2 \frac{p_0}{|p|^2} Q_1 \left( \frac{p_0}{|p|} \right),
\end{equation}
\begin{equation}
D(p_0, |p|) = -4g^2C_F M^2 \frac{1}{|p|} Q_0 \left( \frac{p_0}{|p|} \right),
\end{equation}

where $Q_0$ and $Q_1$ are Legendre functions of first and second kind respectively read as
\begin{equation}
Q_0(x) = \frac{1}{2} \ln \left( \frac{x+1}{x-1} \right),
\end{equation}
\begin{equation}
Q_1(x) = \frac{x}{2} \ln \left( \frac{x+1}{x-1} \right) - 1 = xQ_0(x) - 1,
\end{equation}

with magnetic mass obtained as
\begin{equation}
M^2(T, \mu, m_f, q_f B) = \frac{|q_f B|}{16\pi^2} \left( \frac{\pi T}{2m_f} - \ln 2 + \frac{7\mu^2\zeta(3)}{8\pi^2 T^2} \right),
\end{equation}

where $\zeta$ is the Riemann zeta function. The general covariant structure of quark self energy Eq.(13) can be recast in terms of left handed ($P_L = (I - \gamma_5)/2$) and right handed ($P_R = (I + \gamma_5)/2$) chiral projection operators as
\begin{equation}
\Sigma(P) = -P_R A' P_L - P_L B' P_R,
\end{equation}

with $A'$ and $B'$ defined as
\begin{equation}
A' = A \hat{P} + (B + C) \hat{\gamma}_5 \hat{D} \hat{b},
\end{equation}
\begin{equation}
B' = A \hat{P} + (B - C) \hat{\gamma}_5 \hat{D} \hat{b}.
\end{equation}

Using Eq.(4) and (28), inverse fermion propagator can be written as
\begin{equation}
S^{*-1}(P) = \hat{P} + P_R \left[ A \hat{P} + (B + C) \hat{\gamma}_5 \hat{D} \hat{b} \right] P_L + P_L \left[ A \hat{P} + (B - C) \hat{\gamma}_5 \hat{D} \hat{b} \right] P_R,
\end{equation}

and using $P_{L,R} \gamma^\mu = \gamma^\mu P_{R,L}$ and $P_L \hat{P} P_L = P_R \hat{P} P_R = P_L P_R \hat{P} = 0$, we obtain
\begin{equation}
S^{*-1}(P) = P_R \hat{L} P_L + P_L \hat{R} P_R,
\end{equation}

where $\hat{L}$ and $\hat{R}$ are
\begin{equation}
\hat{L} = (1 + A) \hat{P} + (B + C) \hat{\gamma}_5 \hat{D} \hat{b},
\end{equation}
\begin{equation}
\hat{R} = (1 + A) \hat{P} + (B - C) \hat{\gamma}_5 \hat{D} \hat{b}.
\end{equation}

Thus, we get the effective quark propagator as
\begin{equation}
S^*(P) = \frac{1}{2} \left[ P_L \frac{\hat{L}}{L^2/2} P_R + P_R \frac{\hat{R}}{R^2/2} P_L \right],
\end{equation}
where

\[
L^2 = (1 + A)^2 P^2 + 2(1 + A)(B + C)p_0 - 2D(1 + A)p_z + (B + C)^2 - D^2, \quad (36)
\]
\[
R^2 = (1 + A)^2 P^2 + 2(1 + A)(B - C)p_0 + 2D(1 + A)p_z + (B - C)^2 - D^2. \quad (37)
\]

For pure thermal medium, \(C\) and \(D\) will vanish which leads to \(L^2 = R^2\) or \(L^\mu = R^\mu\). Therefore, the effective quark propagator becomes

\[
S^*(P) = \frac{1}{R^2} \left[ P_L \tilde{R} P_R + P_R \tilde{R} P_L \right], \quad (38)
\]
\[
= \frac{(1 + A) \hat{P} + B \gamma_5}{D}, \quad (39)
\]

where, \(D = (1 + A)^2 P^2 + 2(1 + A)B P u + B^2\). Hence, \(S^*(P)\) contains no chiral term. On the other hand, in the presence of a magnetic field at finite temperature, \(L^2 \neq R^2\) (parity violation \([58]\)) due to the non-vanishing value of \(C\) and \(D\), and hence the effective quark propagator contains the terms associated with \(\gamma_5\).

Next, we take the static limit \((p_0 = 0, |p| \to 0)\) of \(L^2/2\) and \(R^2/2\), after expanding the Legendre functions involved in structure functions in power series of \(\frac{p_0}{p_u}\). Considering upto \(O(g^2)\), we obtain

\[
\frac{L^2}{2} \big|_{p_0 = 0, |p| \to 0} = m_{th}^2 + 4g^2 C_F M^2, \quad (40)
\]
\[
\frac{R^2}{2} \big|_{p_0 = 0, |p| \to 0} = m_{th}^2 - 4g^2 C_F M^2. \quad (41)
\]

Thus, the thermal mass (squared) at finite chemical potential in presence of weak magnetic field is obtained as

\[
m_L^2 = m_{th}^2 + 4g^2 C_F M^2, \quad (42)
\]
\[
m_R^2 = m_{th}^2 - 4g^2 C_F M^2, \quad (43)
\]

where \(m_{th}^2\), Eq. \((1)\) is the thermal mass in the absence of magnetic field. In the absence of magnetic field, \(M^2 = 0\), therefore, \(m_L^2 = m_R^2 = m_{th}^2\). In the weak magnetic field limit \((T^2 > q_f B > m_f^2)\), the magnetic field contribution to the medium generated quark mass appears as a correction to the pure thermal mass, \(m_{th}\). The fact that \(L^2 \neq R^2\) in the presence of magnetic field, leads to the lifting of the degeneracy of mass of chiral modes of quarks. In the case of a strong magnetic field, however, the quasiparticle mass of the quark does not appear as a correction to \(m_{th}\), rather, the dominant contribution comes from the magnetic field. On taking the static limit \((p_0 = 0, p_z \to 0)\) of effective quark propagator in strong magnetic field limit, the medium generated mass (squared) of left and right handed chiral modes of quarks in strong magnetic field comes out as \([74]\)

\[
m_{L,R}^2 = 4g^2 C_F \frac{|q_f B|}{16\pi^2} \left( \frac{\pi T}{2m_f} - \ln 2 \right). \quad (44)
\]

Now, we will assess the momentum transport coefficients in the presence of a weak magnetic field at a finite chemical potential for both \(L\) and \(R\) modes separately.

### III Momentum Transport Coefficients

In this section, we will explore the shear and bulk viscosity for strongly interacting matter in a weakly magnetized thermal medium. The Boltzmann transport equation governs the evolution of single particle
distribution $f(x, p)$ associated with partons in our system. Since a deconfined medium of quarks and gluons is a relativistic system, therefore it allows us to use the relativistic Boltzmann transport equation (RBTE) in the partonic system. The RBTE for relativistic particle with a charge $q$ in the presence of an external electromagnetic field can be written as [75]

$$p^\mu \partial_\mu f(x, p) + q F^{\mu \nu} p_\nu \frac{\partial f(x, p)}{\partial p^\mu} = C[f],$$

(45)

where $f(x, p)$ is the slightly deviated distribution function from equilibrium distribution function $f_{eq}$ with $f = f_{eq} + \delta f$ (see $f << f_{eq}$). $F^{\mu \nu}$ is the electromagnetic field tensor which in the absence of an electric field becomes, $F^{\mu \nu} = -B b^{\mu \nu}$, where $b^{\mu \nu} = \epsilon^{\mu \nu \alpha \beta} b_\alpha u_\beta$ with unit four vector, $b^\mu = \frac{B^\mu}{T}$. The rate of change of distribution function by means of collision is described by collision integral $C[f]$, whose general form consists of absorption and emission terms in phase space volume element. This leads to the nonlinear integro-differential equation which is very cumbersome to solve. Therefore, we simplify the equation by employing RTA [76], in which the external perturbation pushes the system slightly out of equilibrium, from which it returns exponentially to equilibrium with time scale $\tau$. Eq.(45) under RTA takes the form as

$$p^\mu \partial_\mu f(x, p) + q F^{\mu \nu} p_\nu \frac{\partial f(x, p)}{\partial p^\mu} = -\frac{p^\mu u_\mu}{\tau} \delta f,$$

(46)

where first order correction is found to be

$$\delta f_1 = -\frac{\tau}{u \cdot p} \left(p^\mu \partial_\mu + q F^{\mu \nu} p_\nu \frac{\partial}{\partial p^\mu} \right) f_{eq},$$

(47)

$$= -\frac{\tau}{u \cdot p} p^\mu \partial_\mu f_{eq}; \quad q B b^{\mu \nu} p_\nu \frac{\partial f_{eq}}{\partial p^\mu} = 0.$$ Since, first order correction does not take into account the effect of magnetic field therefore, we need to take a correction term in $q F^{\mu \nu} p_\nu \frac{\partial}{\partial p^\mu}$ to see the effect of magnetic field. For small deviation from equilibrium $(\delta f << f_{eq})$, the Eq.(46) can be written as

$$p^\mu \partial_\mu f_{eq} = -\frac{p^\mu u_\mu}{\tau} \left[1 - \frac{q B r b^{\mu \nu} p_\nu}{p^\mu u_\mu} \frac{\partial}{\partial p^\mu} \right] \delta f,$$

(48)

$f_{eq}$ is the equilibrium distribution function, defined as

$$f_{eq} = \frac{1}{e^{(\sqrt{p^2 + m^2} - \mu)/T + \alpha}},$$

(49)

where, $\epsilon = \sqrt{p^2 + m^2}$ is the single particle energy and $\alpha = -1, +1, 0$ is for boson, fermion and Boltzmann gas, respectively.

The conserved net particle four-current $(N^\mu = n u^\mu + \eta^\mu)$ and energy-momentum tensor $(T^{\mu \nu} = T^{\mu \nu}(0) + \Delta T^{\mu \nu} = T^{\mu \nu}(0) - \Pi \Delta_{\mu \nu} + \pi^{\mu \nu})$, with $u^\mu$ is defined in Landau frame, can be expressed in terms of single particle phase-space distribution function as

$$N^\mu = \int dP \ p^\mu (f - \bar{f}),$$

$$T^{\mu \nu} = \int dP \ p^\mu p^\nu (f + \bar{f}).$$

(50)

Here, $dP = g \ d^3 p / (2\pi)^3 \sqrt{p^2 + m^2}$, $g$ and $m$ being the degeneracy factor and particle rest mass, $p^\mu$ is the particle four-momentum. The derivative $(\partial_\mu)$ can be split up covariantly into time and space derivative: $\partial_\mu = u_\mu D + \nabla_\mu$, where $D = u^\mu \partial_\mu$ and $\nabla_\mu = \Delta_\mu \partial^\nu = \partial_\mu - u_\mu D$, with $\Delta^{\mu \nu} = g^{\mu \nu} - u^\mu u^\nu$.
Under this decomposition, the left-hand side derivative of Eq. (48) can be expressed in terms of derivative of thermodynamic parameters as

\[
p^\mu \partial_\mu f_{eq} = \rho (u_\mu D + \nabla_\mu) f_{eq} = \left( \frac{\partial f_{eq}}{\partial T} \right) (u \cdot p) D T + \rho (\nabla_\mu) f_{eq} + \left( \frac{\partial f_{eq}}{\partial (u_\mu/T)} \right) (u \cdot p) D u^\nu + \rho \nabla_\mu (u^\nu) .
\]

Since, the thermodynamic forces do not contain the time derivative terms therefore the terms like \(DT\) and \(D(u_\mu/T)\) need to be expressed in terms of spatial derivative of thermodynamic parameters. This can be achieved by making the use of particle number conservation, energy-momentum conservation and relativistic Gibbs-Duhem relation.

Therefore, Eq. (51) takes the form as

\[
p^\mu \partial_\mu f_{eq} = \left( \frac{1 - \alpha f_{eq}}{T} \right) X \nabla \mu u^\nu - \rho p^\nu \nabla \mu u^\nu - \frac{1}{3} \Delta_{\mu\nu} \nabla \sigma u^\sigma + \left( \frac{1 - u \cdot p}{h} \right) T p^\nu \nabla \mu \left( \frac{\mu}{T} \right) ,
\]

where \(X = (u \cdot p)^2 \left( \frac{4}{3} - \lambda' \right) - \frac{4}{3} m^2 + (u \cdot p) [(\lambda'' - 1) h - \lambda''' T] \) and \(h\) is the enthalpy per particle. The details of the calculation along with expressions of \(\lambda', \lambda''\) and \(\lambda'''\) are given in B.

The term \(\delta f\) in Eq. (48) characterizes the deviation of the distribution function from equilibrium distribution function. \(\delta f\) can be constituted as linear combination of the thermodynamic forces \((Y_{\mu\nu})\) times appropriate tensorial coefficients, resulting in a Lorentz scalar, \(\delta f\) as,

\[
\delta f = A Y + B^\mu Y_\mu + C^{\mu\nu} Y_{\mu\nu} .
\]

By substituting the above form of \(\delta f\) in Boltzmann transport equation and comparing the coefficients of thermodynamic forces, we obtain the unknown coefficients \(A, B^\mu\) and \(C^{\mu\nu}\) in the expression of \(\delta f\); and then by incorporating the \(\delta f\) in the thermodynamic flows, we can obtain the transport coefficients.

We are interested in the computation of momentum transport coefficients, specifically, shear and bulk viscosity, as discussed in the following subsection.

### III.A Shear Viscosity

The general form of \(\delta f\) for shear viscosity in the presence of a magnetic field can be expressed in terms of fourth-rank projection tensors as [48, 49, 75]

\[
\delta f = \sum_{r=0}^{4} a_r A^{(r)}_{\mu\nu\alpha\beta} p^\mu p^\nu V^{\alpha\beta} = a_0 P^{(0)}_{<\mu\nu>,\alpha\beta} + a_1 \left( P^{(1)}_{<\mu\nu>,\alpha\beta} + P^{(-1)}_{<\mu\nu>,\alpha\beta} \right) + i a_2 \left( P^{(1)}_{<\mu\nu>,\alpha\beta} - P^{(-1)}_{<\mu\nu>,\alpha\beta} \right) + a_3 \left( P^{(2)}_{<\mu\nu>,\alpha\beta} + P^{(-2)}_{<\mu\nu>,\alpha\beta} \right) + i a_4 \left( P^{(2)}_{<\mu\nu>,\alpha\beta} - P^{(-2)}_{<\mu\nu>,\alpha\beta} \right) p^\mu p^\nu V^{\alpha\beta} .
\]
where $V^{\alpha\beta} = \frac{1}{2} \left( \frac{\partial u^\alpha}{\partial x^\nu} + \frac{\partial u^\beta}{\partial x^\nu} \right)$ and $P^{(c)}_{<\mu\nu>\alpha\beta} = P^{(c)}_{\mu\alpha\beta} + P^{(c)}_{\nu\alpha\beta}$. In general, the fourth rank projection tensor is defined in terms of second rank projection tensor as [56],

$$P^{(m)}_{\mu\nu, \mu'\nu'} \equiv \sum_{m_1=-1}^{1} \sum_{m_2=-1}^{1} P^{(m_1)}_{\mu\mu'} P^{(m_2)}_{\nu\nu'} \delta (m, m_1 + m_2),$$

and second rank projection tensor is defined as,

$$P^{0}_{\mu\nu} = b_{\mu} b_{\nu},$$

$$P^{1}_{\mu\nu} = \frac{1}{2} (\Delta_{\mu\nu} - b_{\mu} b_{\nu} + i b_{\mu\nu}),$$

$$P^{-1}_{\mu\nu} = \frac{1}{2} (\Delta_{\mu\nu} - b_{\mu} b_{\nu} - i b_{\mu\nu}).$$

The properties of second rank projection tensor are as follows

$$P^{(m)}_{\mu\nu} P^{(m')}_{\nu\nu'} = \delta_{m m'} P^{(m)}_{\mu\nu},$$

$$\left( p^{(m)}_{\mu\nu} \right)^{\dagger} = p^{(-m)}_{\mu\nu} = p^{(m)}_{\nu\mu},$$

$$\sum_{m=-1}^{1} p^{(m)}_{\mu\nu} = \delta_{\mu\nu}, \quad p^{(m)}_{\mu\mu} = 1.$$

The left-hand side of Eq.(48) can be expressed in terms of 4-rank projection tensors $P^{(r)}_{\mu\nu, \alpha\beta}$ as

$$-\frac{p^{\mu} p^{\nu}}{2T} f_{eq}(1 - \alpha f_{eq}) V^{\alpha\beta} \left( P^{(0)}_{\mu\nu, \alpha\beta} + P^{(1)}_{\mu\nu, \alpha\beta} + P^{(-1)}_{\mu\nu, \alpha\beta} + P^{(2)}_{\mu\nu, \alpha\beta} + P^{(-2)}_{\mu\nu, \alpha\beta} \right).$$

Substituting $\delta f$ on right hand side of Eq.(48),

$$-\left( \frac{u \cdot p}{\tau} \right) \left[ 1 - \frac{q B T b^{\mu\nu} p_{\nu}}{u \cdot p} \frac{\partial}{\partial p_{\mu}} \right] \sum_{r=0}^{4} \alpha_r A^{(r)}_{\mu\nu, \rho\sigma} p^{\rho} p^{\sigma} V^{\alpha\beta}$$

and equating the coefficients of $P^{(r)}_{\mu\nu, \alpha\beta}$ in Eq.(58) and (59), we get the coefficients for a system quarks of multiple charge species as [48, 49, 77]

$$a_0 = \sum_{f} \frac{1}{2T} f_{eq, f} (1 - f_{eq, f}) \frac{\tau_f}{(u \cdot p)},$$

$$a_1 = \sum_{f} \frac{(u \cdot p) f_{eq, f} (1 - f_{eq, f}) \tau_f}{2T \left[ (u \cdot p)^2 + (q f B \tau_f)^2 \right]},$$

$$a_2 = \sum_{f} \frac{(q f B) f_{eq, f} (1 - f_{eq, f}) \tau_f^2}{2T \left[ (u \cdot p)^2 + (q f B \tau_f)^2 \right]},$$

$$a_3 = \sum_{f} \frac{(u \cdot p) f_{eq, f} (1 - f_{eq, f}) \tau_f}{2T \left[ (u \cdot p)^2 + (2 q f B \tau_f)^2 \right]},$$

$$a_4 = \sum_{f} \frac{(q f B) f_{eq, f} (1 - f_{eq, f}) \tau_f^2}{2T \left[ (u \cdot p)^2 + (2 q f B \tau_f)^2 \right]}.$$

where, $f$ stands for flavor. Here we have used $f = u$ (up) and down (d). Similarly, the above coefficients for antiquarks can be obtained using, $f_{eq} = \frac{1}{e^{(\sqrt{p^2 + \mu^2} - m_f) / \tau + 1}}, \quad q_f \rightarrow q_f, \quad \tau_f \rightarrow \tau_f$ and $\delta f = \sum_{r=0}^{4} \bar{a}_r A^{(r)}_{\mu\nu, \alpha\beta} p^{\mu} p^{\nu} V^{\alpha\beta}$. 

10
\( \tau_{f(\bar{f})} \) is the relaxation time for quarks (antiquarks) expressed as follows [78]

\[
\tau_{f(\bar{f})} = \frac{1}{5.1T \alpha_s^2 (\Lambda^2, |eB|) \log \left( \frac{1}{\alpha_s(\Lambda^2, |eB|)} \right) (1 + 0.12 (2N_f + 1))}
\]

(65)

It was argued in [79] that finite parton mass has little effect on scattering cross section and hence on relaxation time. This leads to the qualitatively same result for massless and massive partons. The current light quark \((m_{u,d})\) masses are chosen to be 0.1 times the strange quark mass \((m_s)\) which is in compliance with chiral perturbation theory [80, 81]. The parameters were adjusted to get the best fitted lattice data with \(m_s = 80\) MeV [82].

The general form of shear viscous tensor \((\pi^{\mu\nu})\) in terms of 4-rank projection tensor can be written as

\[
\pi^{\mu\nu} = \eta^{\mu\nu\alpha\beta} \sigma_{\alpha\beta},
\]

(66)

\[
= \sum_{r=0}^{4} \eta^{(r)} A^{\mu\nu\alpha\beta} \sigma_{\alpha\beta},
\]

where \(\sigma^{\alpha\beta} = V^{\alpha\beta} - \theta \Delta^{\alpha\beta} / 3\), \(\theta = \nabla_\mu u^\mu\) is the scalar expansion. Employing the integral form of shear viscous tensor for quark and antiquark

\[
\pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta} \int dP \ p^\alpha p^\beta (\delta f + \delta \bar{f}),
\]

(67)

and using Eq. (54), we obtained the five shear viscosity components as follows:

\[
\eta_0 = \sum_f \beta \tau_f J_{42}^{(1)+},
\]

(68)

\[
\eta_1 = \sum_f \beta \tau_f Y_{42}^{(1)+},
\]

(69)

\[
\eta_2 = \sum_f \beta q_f B \tau_f^2 Y_{42}^{(2)-},
\]

(70)

\[
\eta_3 = \sum_f \beta \tau_f Z_{42}^{(1)+},
\]

(71)

\[
\eta_4 = \sum_f 2 \beta q_f B \tau_f^2 Z_{42}^{(2)-}.
\]

(72)

Here, \(J_{42}^{(1)+}, Y_{42}^{(1)+}, Y_{42}^{(2)-}, Z_{42}^{(1)+}, Z_{42}^{(2)-}\) are thermodynamic integrals (defined in C). One may compare the above coefficients qualitatively with Ref. [53, 83]. The difference arises due to the incorporation of quasiparticle mass in weak magnetic field limit. Depending upon the direction of magnetic field and current, we can have longitudinal, transverse and Hall effect. \(\eta_0\) is the longitudinal component, \(\eta_1, \eta_3\) are the transverse components and \(\eta_2, \eta_4\) are the Hall components of shear viscosity. In the limit of vanishing magnetic field, \(qB \to 0\), \(\eta_2\) and \(\eta_4\) reduces to zero while \(\eta_0, \eta_1\) and \(\eta_3\) becomes equal as expected. We shall take into account the quasiparticle mass that was shown to differ for L and R chiral modes of quarks.

Now, we will calculate the gluonic contribution to the shear viscosity. Since, gluons do not interact with electromagnetic field therefore there will be no magnetic field dependent term and shear viscosity for gluons turns out to be

\[
\eta_g = 2 \beta \tau_g J_{42}^{(1)},
\]

(73)

where, \(\tau_g\) is the relaxation time for gluons [78],

\[
\tau_g = \frac{1}{22.5T \alpha_s^2 (\Lambda^2) \log \left( \frac{1}{\alpha_s(\Lambda^2)} \right) (1 + 0.06N_f)}.
\]

(74)
III.B Bulk Viscosity

Now, we will discuss the computation of bulk viscosity in a weakly magnetized thermal medium. There are three components of bulk viscosity in the presence of a magnetic field. The general form of $\delta f$ associated to them is

$$\delta f = \sum_{r=1}^{3} c_r A_r^{(r)} \partial^\mu u^\nu,$$

where $A_r^{(r)}$ are the magnetic field components. Using the properties of second rank projection tensor, the right-hand side of Eq.(48) becomes,

$$- \frac{u \cdot p}{\tau} \{ c_1 (b_\mu b_\nu) + c_2 (\Delta_{\mu\nu} - b_\mu b_\nu) + ic_3 b_\mu b_\nu \} (\partial^\mu u^\nu),$$

and equating the coefficients of $\Delta_{\mu\nu} \partial^\mu u^\nu$, $b_\mu b_\nu \partial^\mu u^\nu$ and $b_\mu \partial^\mu u^\nu$ from Eq.(52) and (76), we get

$$c_1 = c_2 = - \sum_f \frac{\tau_f X f, f (1 - f_{eq,f})}{T (u \cdot p)},$$

$$c_3 = 0.$$

Similarly, the coefficients $\tilde{c}_1$, $\tilde{c}_2$ and $\tilde{c}_3$ associated to antiquarks can be evaluated. The general form of bulk viscous pressure is given as

$$\Pi = \zeta^{\mu\nu} \partial_\mu u_\nu,$$

where the tensor coefficient $\zeta^{\mu\nu}$ is known as bulk viscosity coefficient. $\zeta^{\mu\nu}$ can be expressed in terms of projection basis as

$$\zeta^{\mu\nu} = \zeta_1 A^{(1)}_{\mu\nu} + \zeta_2 A^{(2)}_{\mu\nu} + \zeta_3 A^{(3)}_{\mu\nu},$$

where $\zeta_1$, $\zeta_2$ and $\zeta_3$ is the longitudinal, transverse and Hall component of bulk viscosity respectively. Using the integral form of bulk viscous pressure

$$\Pi = \frac{\Delta^{\mu\nu}}{3} \int dP p^\mu p^\nu (\delta f + \delta \bar{f}),$$

we obtained $\zeta_1$, $\zeta_2$ and $\zeta_3$ as

$$\zeta_1 = \zeta_2 = \frac{\Delta^{\mu\nu}}{3} \int dP p_\mu p_\nu c_1,$$

and

$$\zeta_3 = 0.$$

Since, the dissipative part of energy-momentum tensor, $\Delta T^{\mu\nu}$ and the particle current, $n^\mu$ cannot be determined uniquely by the second law of thermodynamics, one usually introduces some constraints to fix them, they are known as matching conditions. The matching (or fitting) conditions are also introduced because of necessity of matching the energy density and baryonic charge density in non-equilibrium state to the corresponding equilibrium densities, or equivalently,

$$u_\mu u_\nu \Delta T^{\mu\nu} = 0; \quad u_\mu \Delta N^\mu = 0.$$

With the help of above matching conditions and using the definition of $X$, Eq.(82) can be written as

$$\zeta_1 = \zeta_2 = \sum_f \frac{1}{3T} \int dP \frac{X_f^2}{\epsilon_f} \tau_f \left[ f_{eq,f} (1 - f_{eq,f}) + \bar{f}_{eq,f} (1 - \bar{f}_{eq,f}) \right].$$
The longitudinal (\(\zeta_1\)) and transverse (\(\zeta_2\)) component of bulk viscosity are same and the effect of magnetic field comes through the thermal mass (squared) of quarks and antiquarks with magnetic field correction. The above form of bulk viscosity is qualitatively similar to the Ref. [48, 53], where difference arises due to the mass. We have so far determined how quarks and antiquarks contribute to bulk viscosity. We shall now determine the gluonic contribution to bulk viscosity. Since, gluons do not interact with electromagnetic field, the Boltzmann transport equation has the form as follows

\[
p^\mu \partial_\mu f_{eq,g} = -\frac{u \cdot p}{\tau} \delta f_g.
\]

Using the Eq.(135) and taking gluon chemical potential to be zero, the left-hand side of Eq.(86) can be expressed as follows,

\[
p^\mu \partial_\mu f_{eq,g} = \frac{1}{f_{eq,g}} (1 + f_{eq,g}) \left[ X'\nabla_\mu u^\mu - p^\mu p^\nu \left( \nabla_\mu u^\nu - \frac{1}{3} \Delta_{\mu\nu} \nabla_\sigma u^\sigma \right) \right],
\]

where \(X' = (u \cdot p)^2 \left( \frac{4}{3} - \lambda' \right) - \frac{4}{3} m^2\). On employing the Eq.(75) to the right-hand side of Eq.(86), it can be written in terms of the basis of projection tensor as

\[
-\frac{u \cdot p}{\tau} \delta f_g = -\frac{u \cdot p}{\tau} \left[ c_1 (b_\mu b_\nu) + c_2 (\Delta_{\mu\nu} - b_\mu b_\nu) + i c_3 b_\mu b_\nu \right] (\partial^\mu u^\nu).
\]

On equating the coefficients of \(\Delta_{\mu\nu} \partial^\mu u^\nu, b^\mu b^\nu \partial_\mu u_\nu, b^\mu \partial_\mu u_\nu\) on both sides of Eq.(88) and (89), we obtain

\[
c_2 = \frac{\tau}{(u \cdot p) T} \left[ (u \cdot p)^2 (1 - \lambda') \right] f_{eq,g} (1 + f_{eq,g}).
\]

Hence, the bulk viscosity coefficient for gluons is obtained to be as

\[
\zeta_g = \frac{\tau}{T} \int dP \frac{X'^2}{\epsilon g^2} f_{eq,g} (1 + f_{eq,g}).
\]

This gluonic contribution of bulk viscosity (\(\zeta_g\)) will be added to the fermionic contribution of bulk viscosity (\(\zeta_1, \zeta_2\)).

**IV Results and Discussions**

We have summarized the results for the anisotropic components of shear and bulk viscosity coefficients in RTA for non-zero magnetic field and quark chemical potential.

**IV.A Shear Viscosity**

The interaction among partons has been incorporated via the quasiparticle mass. Thus, in order to study the results on momentum transport, we need to understand the effects of \(m_{L/R}^2\) on the quark distribution function which is shown in Fig. (2). \(f_{B \neq 0}\) for L and R mode is defined using the mass \(m_L^2\) and \(m_R^2\) whereas \(f_{B=0}\) is defined using \(m_{th}^2\). As can be seen from the figures, this ratio is found to be smaller than unity for the L mode and larger than unity for the R mode, which suggests that the presence of a magnetic field causes a decrease in the L mode distribution function, and an increase in the R mode.
Figure 2: Variation of $f_{B \neq 0}/f_{B = 0}$ for L mode (a) and R mode (b) with momentum at low and high temperature.

Figure 3: Variation of $\eta_0$ with temperature at different fixed values of magnetic field (a) and quark chemical potential (b) for both L and R modes.
distribution function, compared to the distribution function at $B = 0$. This effect of mass will also be reflected in the magnitude of shear and bulk viscosity.

In Fig. (3), we have shown the variation of longitudinal shear viscosity ($\eta_0$) with temperature at different fixed values of magnetic field and quark chemical potential for both L and R modes. For L mode, $\eta_0$ decreases with magnetic field, whereas it increases for the R mode. This opposite behaviour is a result of the different mass of the L/R mode, which increases/decreases with magnetic field. The appearance of this mass in the denominator of $\eta_0$, results in the decreasing/increasing behaviour of $\eta_0$ with magnetic field. Further, the increasing behaviour of $\eta_0$ for both modes with temperature is attributed to the Boltzmann factor $\exp(-\varepsilon/(p)/T)$ in the distribution function. $\eta_0$ will have nonvanishing contribution for zero magnetic field and quark chemical potential. Fig. (4) shows the variation of $\eta_1$ (transverse component of shear viscosity) with temperature at different fixed values of magnetic field and quark chemical potential respectively. Similar to the $\eta_0$, $\eta_1$ decreases for L mode and increases for R mode with magnetic field and increases with quark chemical potential for both modes. In the absence of magnetic field, transverse component becomes equal to the longitudinal component. Also it possesses the finite contribution at $\mu = 0$ due to the equal contribution from quarks and antiquarks in same direction.

Figs. (5) and (6) shows the variation of $\eta_2$ (Hall component of shear viscosity) with temperature at different fixed values of magnetic field and quark chemical potential respectively. Since, $\eta_2$ is proportional to $qB$, therefore it increases with magnetic field for both modes. Further, its magnitude amplifies with quark chemical potential. This Hall-type shear viscosity vanishes in the absence of magnetic field. At zero chemical potential, the number of particles and antiparticles are same and they will give equal and opposite contribution to the shear viscosity component, $\eta_2$. Hence, it vanishes for zero quark chemical potential even in the ambience of nonvanishing magnetic field.

Figure 4: Variation of $\eta_1$ with temperature at different fixed values of magnetic field (a) and quark chemical potential (b) for both L and R modes.
Figure 5: Variation of $\eta_2$ with temperature at different fixed values of magnetic field for both L and R modes.

Figure 6: Variation of $\eta_2$ for L mode (a) and R mode (b) with temperature at different fixed values of quark chemical potential.
The variation of another transverse component of shear viscosity ($\eta_3$) and Hall component ($\eta_4$) is shown in Fig. (7). Similar to the $\eta_1$, $\eta_3$ decreases/increases with magnetic field for L/R mode of quarks. In the absence of magnetic field, $\eta_1$ and $\eta_3$ becomes equal to the longitudinal component, i.e. $\eta_0 = \eta_1 = \eta_3$. The same as $\eta_0$ and $\eta_1$, $\eta_3$ will have a finite contribution at the vanishing quark chemical potential. The another Hall component of shear viscosity ($\eta_4$) shows the same behaviour as $\eta_2$ with temperature and magnetic field. It also vanishes for zero magnetic field and for symmetric quark chemical potential at finite magnetic field. The increasing behaviour of all the coefficients with temperature is due to the Boltzmann factor $\exp(-\varepsilon(p)/T)$. They also exhibit an increase with quark chemical potential, similar to the other shear viscous coefficients.

**IV.B Bulk Viscosity**

Figure 8: Variation of $\zeta_0, \zeta_4$ for L mode (a) and R mode (b) with temperature at different fixed values of magnetic field.
The variation of longitudinal and transverse bulk viscosity ($\zeta_0$ and $\zeta_1$) with temperature at different fixed values of magnetic field and quark chemical potential is shown in Fig. (8). Both L and R mode bulk viscosity have higher magnitude at $eB = 0.2m_\pi^2$ than at $eB = 0.1m_\pi^2$. This behaviour is attributed due to the factor, $X = (u \cdot p)^2 (\frac{4}{3} - \lambda') - \frac{1}{3}m^2 + (u \cdot p) [(\lambda'' - 1) h - \lambda'''T]$. Further, the Boltzmann factor $f_{eq}(1 - f_{eq})$ leads to the increasing behaviour of bulk viscosity with temperature. The increment of bulk viscosity with quark chemical potential for both modes is due to the increased contribution from quarks than antiquarks. $\zeta_0$ and $\zeta_1$ will have finite contribution at zero quark chemical potential even in the absence of magnetic field.

V  Applications

In this part, we will examine the influence of a weak magnetic field on the specific shear and bulk viscosity, and Reynolds number.

V.A  Specific shear and bulk viscosities

The specific shear ($\eta/s$) and bulk viscosity ($\zeta/s$) are the viscosity to entropy density ratio. A small value for $\eta/s$ was estimated from the analysis of the elliptic flow data [25] and was found to be close to the conjectured lower bound $\eta/s|_{KSS}$ from AdS/CFT [84]. This led to claim that the QGP formed at RHIC was the most perfect fluid ever observed. We have plotted the five specific shear viscous and two specific bulk viscous coefficients.

Figure 9: Variation of $\eta_0/s$ (a) and $\eta_1/s$ (b) with temperature for both L and R mode at different fixed values of magnetic field.

![Figure 9: Variation of $\eta_0/s$ (a) and $\eta_1/s$ (b) with temperature for both L and R mode at different fixed values of magnetic field.](image-url)
The variation of $\eta_0/s$ and $\eta_1/s$ with temperature at different fixed values of magnetic field is shown in Fig. (9). The entropy density for L mode decreases and increases for R mode with magnetic field as shown in Fig. (10). This behaviour of entropy density with $B$ is attributable to the Boltzmann factor $f_{eq}(1 - f_{eq})$. The rate of change of magnitude for shear viscosity with magnetic field is higher than the entropy density, hence $\eta_0/s$ and $\eta_1/s$ shows the same behaviour as $\eta_0$ and $\eta_1$ with magnetic field.

Figure 11: Variation of $\eta_2/s$ (a) and $\eta_3/s$ (b) with temperature at different fixed values of magnetic field for both L and R modes.
The behaviour of $\eta_2/s$ and $\eta_3/s$ with temperature is shown in Fig. (11). The behaviour of $\eta_2/s$ for L mode is same as $\eta_2$ but opposite for R mode. The rate of change of magnitude of entropy density with magnetic field is higher than the $\eta_2$, hence entropy density determines the trend of $\eta_2/s$. Whereas the rate of change of magnitude of $\eta_3$ with magnetic field is higher than the entropy density, hence $\eta_3/s$ shows the same behaviour as $\eta_3$. The variation of $\eta_4/s$ with temperature is shown in Fig. (12) behaving in the same manner $\eta_2/s$. We observed that the shear viscosity to entropy density ratio for longitudinal and transverse components in weak field limit is greater than $\frac{1}{4\pi}$ as expected because in absence of magnetic field, this ratio is higher than KSS lower bound. The Hall component of shear viscosity vanishes in the absence of a magnetic field and its magnitude is very small as compared to the longitudinal and transverse components. The corresponding ratio for Hall component is found to be less than $\frac{1}{4\pi}$.

The variation of bulk viscosity to entropy density ratio, $\zeta_0/s$ and $\zeta_1/s$, with temperature at different fixed values of magnetic field is shown in Fig. (13). Similar to the $\zeta_0$ and $\zeta_1$, the specific bulk viscosities have greater magnitude at $eB = 0.2m_\pi^2$ compared to $eB = 0.1m_\pi^2$. The $\zeta_0/s$ and $\zeta_1/s$ were found to be
less than shear viscosity to entropy density ratio. Both $\zeta_0/s$ and $\zeta_1/s$ for L and R mode exhibits the non-monotonic behaviour with temperature. The ratio for L mode shows a minimum at $T=0.205$ GeV for $eB = 0.1m_\pi^2$ and at $T = 0.195$ GeV for $eB = 0.2m_\pi^2$. The R mode ratio for $eB = 0.1m_\pi^2$ has a minimum at $T=0.235$ GeV and maximum at $T=0.265$ GeV for $eB = 0.2m_\pi^2$. This non-monotonic behaviour has also been observed in the estimations based on NJL model [85], linear sigma model at large N limit [86]. The non-zero value of bulk viscosity suggests the departure from conformality.

V.B Reynolds number

Reynolds number (RI) is one of the most important dimensionless quantities in microfluidics. RI is significant for characterizing a fluid’s transport characteristics and for figuring out the type of flow pattern a fluid exhibits. It is defined in terms of characteristic length ($L$), relative velocity of the fluid ($v$) and kinematic viscosity ($\eta/\rho$) as

$$\text{RI} = \frac{Lv}{\eta/\rho}.$$  \hspace{1cm} (92)

The small value of RI suggests the laminar flow, which characterizes the more viscous fluid, while its large value, typically in thousands, indicates the turbulent flow.

Figure 14: Variation of Reynolds number associated with longitudinal, transverse (a) and Hall (b) component of shear viscosity, with temperature.

Fig. (14) shows the variation of Reynolds number associated with $\eta_0, \eta_1$ and $\eta_2$, with temperature at $eB = 0.1m_\pi^2$ and $\mu = 30$ MeV. The mass density for L mode is higher than R mode whereas the magnitude of shear viscosity for L mode is lower than R mode. Therefore, RI associated with $\eta_0, \eta_1$ and $\eta_2$ for L mode has higher magnitude than for R mode. Further, RI associated with $\eta_0$ and $\eta_1$ has the same magnitude in their respective modes. Csernai et al. estimated the value of RI in the range between 3 to 10 for initial QGP with $\eta/s = 0.1$ using a (3+1)-dimensional fluid dynamical model [87]. Moreover, an upper bound on Reynolds number has been investigated from holographic point of view for nearly central collisions at given temperature and also at different values of magnetic field [88].
VI Conclusions

In this work, we aimed to investigate the impact of weak magnetic field and baryon asymmetry on shear and bulk viscosities in a hot QCD medium, using kinetic theory under RTA. In a weak magnetic field, we have found that the L and R chiral modes of quarks get separated due to difference in their mass and become non-degenerate contrary to the strong magnetic field case. Moreover, the introduction of magnetic field breaks the isotropy of the medium, resulting in a decomposition of shear and bulk viscosity into distinct components. The shear viscosity is decomposed into five components, including the longitudinal component ($\eta_0$), transverse components ($\eta_1$ and $\eta_3$) and Hall components ($\eta_2$ and $\eta_4$). In the absence of magnetic field, the transverse components assume the same form as longitudinal component, while the Hall components vanish for both modes. The decrease or increase of $\eta_0$, $\eta_1$ and $\eta_3$ with the magnetic field for L or R mode is attributed to the different values of effective quark mass for both modes. On the other hand, the increase of $\eta_2$ and $\eta_4$ with magnetic field is due to the direct dependence of magnetic field on Hall components. The longitudinal, transverse and Hall components positively amplify with the quark chemical potential for both L and R modes. In the absence of quark chemical potential, the Hall component for the shear viscosity vanishes for both modes, even in the presence of a magnetic field. The bulk viscosity is decomposed into two components, including the longitudinal ($\zeta_0$) and transverse ($\zeta_1$) component. Both $\zeta_0$ and $\zeta_1$ have higher magnitude at $eB = 0.2m_\pi^2$ than at $eB = 0.1m_\pi^2$ in both L and R mode. Similar to the shear viscosity, bulk viscosity also increases with quark chemical potential. The shear viscosity to entropy density ratio for longitudinal ($\eta_0/s$) and transverse components ($\eta_1/s$, $\eta_3/s$) exhibit the decrease for L mode and increase for R mode as the magnetic field intensity increases. Furthermore, the Hall component of specific shear viscosity increases with magnetic field for L mode and decreases for R mode. Moreover, the ratios of shear viscosity to entropy density are found to be greater than $1/(4\pi)$ for longitudinal and transverse components, whereas the ratio is less than $1/(4\pi)$ for Hall component. The bulk viscosity to entropy density ratios ($\zeta_0/s$ and $\zeta_1/s$) show an increase with magnetic field for both L and R modes, and these ratios exhibit non-monotonic behaviour with temperature, showing minima or maxima around $T \sim 200$ MeV. Additionally, the RI for L mode is higher than the R mode due to the different mass densities. The small value of RI suggests the more viscous fluid, hence describing the laminar flow.

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A Calculation of structure functions

Here, we will show the computation of structure functions from Eq. (14) to (17) in one-loop order for hot and weakly magnetized medium under HTL approximation. Since, trace of odd number of gamma matrices is zero, the Eq. (14) can be written as

$$\mathcal{A} = \frac{1}{4} \frac{\text{Tr}(\Sigma_0 \tilde{P}) - (P \cdot u) \text{Tr}(\Sigma_0 \tilde{\eta})}{(P \cdot u)^2 - P^2},$$

(93)
where,
\[
\Sigma_0 = g^2 C_F T \sum_n \int \frac{d^3k}{(2\pi)^3} \frac{k}{K^2 - m_f^2} \gamma^\mu \frac{1}{(P - K)^2}.
\]  

Using the following two traces:
\[
\begin{align*}
\text{Tr} \left[ \gamma_\mu K^\nu \slashed{P} \right] &= -8K \cdot P, \\
\text{Tr} \left[ \gamma_\mu K^\nu \slashed{k} \right] &= -8K \cdot u,
\end{align*}
\]

we obtain,
\[
\mathcal{A}(P) = \frac{1}{4|\mathbf{p}|} g^2 C_F \left[ I_1(P) + I_2(P) \right],
\]

where \((P \cdot u)^2 - P^2 = |\mathbf{p}|^2\). We will use the frequency sum to evaluate \(I_1(P)\) and \(I_2(P)\) with \(k_0 = i\omega_n\), \(p_0 = i\omega,\ E_1 = \sqrt{k^2 + m_f^2}\) and \(E_2 = \sqrt{(p - k)^2}\). The frequency sum for fermion-boson case is [69]
\[
T \sum_n \tilde{\Delta}_{s_1}(i\omega_n, E_1) \Delta_{s_2}(i(\omega - \omega_n), E_2) = \sum_{s_1,s_2 = \pm 1} -\frac{s_1s_2}{4E_1E_2} \frac{1 - f(s_1E_1) + f(s_2E_2)}{i\omega - s_1E_1 - s_2E_2}.
\]

The leading \(T^2\) behaviour will come from \(s_1 = -s_2 = 1\) with \(E_1 \approx k\) and \(E_2 = |\mathbf{p} - \mathbf{k}|\). Defining light-like four-vector \(\tilde{K} = (-i, \mathbf{\tilde{k}})\) and \(\tilde{K}' = (-i, -\mathbf{\tilde{k}})\), we have,
\[
\begin{align*}
i\omega + E_1 - E_2 &\approx i\omega + \mathbf{p} \cdot \mathbf{\tilde{k}} = P \cdot \tilde{K}, \\
i\omega - E_1 + E_2 &\approx i\omega - \mathbf{p} \cdot \mathbf{\tilde{k}} = P \cdot \tilde{K}',
\end{align*}
\]

and using the angular integration under HTL approximation,
\[
\int \frac{d\Omega \cdot \tilde{K} \cdot u}{4\pi P \cdot \tilde{K}} = \frac{1}{|\mathbf{p}|} Q_0 \left( \frac{p_0}{|\mathbf{p}|} \right),
\]

we get,
\[
\mathcal{A}(p_0, |\mathbf{p}|) = \frac{m_f^2}{|\mathbf{p}|^2} Q_1 \left( \frac{p_0}{|\mathbf{p}|} \right).
\]

Similarly, structure function \(B\) can be evaluated as
\[
B(p_0, |\mathbf{p}|) = -\frac{m_f^2}{|\mathbf{p}|^2} \left[ \frac{p_0}{|\mathbf{p}|} Q_1 \left( \frac{p_0}{|\mathbf{p}|} \right) - Q_0 \left( \frac{p_0}{|\mathbf{p}|} \right) \right].
\]

Using Eq.(11) in (16) and (17), where the contribution from \(\Sigma_0\) vanishes due to the trace of odd no. of gamma matrices and we get the non-vanishing contribution form \(\Sigma_1\) only and hence we get,
\[
\begin{align*}
\mathcal{C}(p_0, |\mathbf{p}|) &= \frac{1}{4} \text{Tr}(\gamma_5 \Sigma_1 \gamma_\mu) , \\
\mathcal{D}(p_0, |\mathbf{p}|) &= \frac{1}{4} \text{Tr}(\gamma_5 \Sigma_1 \gamma_\mu).
\end{align*}
\]

Using the following two traces
\[
\begin{align*}
\text{Tr} \left[ \gamma_5 \gamma_\mu \gamma_5 \left[ (K \cdot b) \gamma_\mu - (K \cdot u) \gamma_\mu \right] \right] &= 8(K \cdot b), \\
\text{Tr} \left[ \gamma_5 \gamma_\mu \gamma_5 \left[ (K \cdot b) \gamma_\mu - (K \cdot u) \gamma_\mu \right] \right] &= 8(K \cdot u),
\end{align*}
\]
we obtain,
\[
C = \frac{g^2 C_F q_f B}{4} T \sum_n \int \frac{d^3 k}{(2\pi)^3} \frac{8(K \cdot b)}{(K^2 - m^2_{f_0})^2(P - K)^2},
\]
(109)
\[
D = -\frac{g^2 C_F q_f B}{4} T \sum_n \int \frac{d^3 k}{(2\pi)^3} \frac{8(K \cdot u)}{(K^2 - m^2_{f_0})^2(P - K)^2},
\]
(110)
which in turn requires the calculation of frequency sum [70]
\[
Y = T \sum_n \Delta_F(K) \Delta_B(P - K),
\]
(111)
\[
= \left( -\frac{\partial}{\partial m^2_{f_0}} \right) T \sum_n \Delta_F(K) \Delta_B(P - K),
\]
where,
\[
T \sum_n \Delta_F(K) \Delta_B(P - K) = \sum_{s_1, s_2 = \pm 1} -s_1 s_2 \frac{(1 - f(s_1 E_1) + f(s_2 E_2))}{4 E_1 E_2 i \omega - s_1 E_1 - s_2 E_2}.
\]
(112)
For \(s_1 = -s_2 = 1\), we get,
\[
C = \frac{4 g^2 C_F q_f B}{16\pi^2} \left( \frac{\pi T}{2 m_{f_0}} - \ln 2 + \frac{7 \mu^2 \zeta(3)}{8 \pi^2 T^2} \right) \left[ -\frac{p_z}{|p|^2} Q_1 \left( \frac{p_0}{|p|} \right) \right],
\]
(113)
\[
D = -\frac{4 g^2 C_F q_f B}{16\pi^2} \left( \frac{\pi T}{2 m_{f_0}} - \ln 2 + \frac{7 \mu^2 \zeta(3)}{8 \pi^2 T^2} \right) \left[ \frac{1}{|p|^2} Q_0 \left( \frac{p_0}{|p|} \right) \right].
\]
(114)
(115)

B Derivation of Eq. (52)

The equilibrium number density \(n = N^\mu u_\mu\) for system of quarks for \(f\)th flavor is given by
\[
n = \sum_f g_f \int \frac{d^3 p}{(2\pi)^3} \frac{p^\mu u_\mu}{p^0} J_{eq,f},
\]
(116)
where \(g_f = 3 \times 2\) is the color and spin degeneracy factor. For ease of the calculation, we introduce the two dimensionless quantities as,
\[
y = \frac{m}{T}, \quad \xi = \frac{p^\mu u_\mu}{p^0} = \frac{1}{T} \left( p^2 + m^2 \right)^{1/2}.
\]
(117)
In local rest frame, \(\frac{d^3 p}{p^0}\) can be expressed as follows using the abovementioned two quantities
\[
\frac{d^3 p}{p^0} = T^2 (\xi^2 - y^2)^{1/2} d\xi d\Omega,
\]
(118)
where \(d\Omega = d(\cos \theta)d\phi\) is the differential solid angle. Using the expansion identity \(\frac{1}{
{\frac{y}{\sqrt{1 - y^2}}} = \sum_{k=1}^\infty (ye^{-x})^k\),
the integral Eq. (116) can be expressed to a sum of the integral as
\[
n = \sum_f \frac{g_f}{2\pi^2} y^2 T^3 \sum_{k=1}^\infty e^{\xi k T} k^{-1} K_2(ky)
\]
(119)
\[
= \sum_f \frac{g_f}{2\pi^2} y^2 T^3 S_2(y),
\]
(120)
where $S^a_j(y) = (\pi)^{k-1} \sum_{n=1}^{\infty} e^{k \mu/T} k^{-\alpha} K_j(ky)$ for fermionic (bosonic), $K_j(y)$ is the modified Bessel function of second kind of order $j$. Similarly, we can obtain the equilibrium formulas for energy density $(en = u_\mu T^{\mu\nu} u_\nu)$, pressure $(P = -\Delta_{\mu\nu} T^{\mu\nu}/3)$ and enthalpy per particle $(h)$ as

$$en = \sum_f g_f \int \frac{d^3p}{(2\pi)^3} \frac{(p^\mu u_\mu)^2}{\rho^2} f_{eq,f} = \sum_f g_f \frac{y^2 T^4}{2\pi^2} [y S^1_j(y) - S^2_j(y)],$$

$$P = \sum_f g_f \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{3\rho^2} f_{eq,f} = \sum_f g_f \frac{y^2 T^4 S^2_j(y)},$$

$$h = e + \frac{P}{n} = yT \frac{S^1_j(y)}{S^2_j(y)}.$$  \hspace{1cm} (121, 122, 123)

The particle number conservation ($\partial_\mu N^\mu = 0$) and contraction of energy-momentum conservation with $u_\mu$ and $\Delta_{\mu\nu}$ ($u_\mu \partial_\nu T^{\mu\nu} = 0$, $\Delta_{\mu} \partial_\nu T^{\mu\nu} = 0$) leads to the continuity equation, equation of energy and equation of motion respectively as follows

$$Dn = -n \partial_\mu u_\mu,$$

$$De = -\frac{P}{n} \partial_\mu u_\mu,$$

$$Du_\mu = \frac{1}{nh} \nabla_\mu P.$$  \hspace{1cm} (124, 125, 126)

In the hydrodynamic regime slightly away from equilibrium, the relativistic Gibbs-Duhem relation is given as

$$\partial_\nu P = nT \partial_\nu \left( \frac{\mu}{T} \right) + nh T^{-1} (\partial_\nu T),$$

which on contraction with $u_\mu$ can be rewritten as

$$Dh = TD \left( \frac{\mu}{T} \right) + h T^{-1} (DT).$$  \hspace{1cm} (127, 128)

Eq. (126) and (128) can be expanded in terms of derivative of temperature and chemical potential over temperature as

$$\left( \frac{\partial e}{\partial T} \right)_{\mu/T} DT + \left( \frac{\partial e}{\partial (\mu/T)} \right)_{T} D \left( \frac{\mu}{T} \right) = -\frac{P}{n} \nabla_\mu u_\mu,$$

$$\left[ \frac{\partial h}{\partial T} \right]_{\mu/T} - h T^{-1} \left[ D + \left( \frac{\partial h}{\partial (\mu/T)} \right)_{T} - T \right] D \left( \frac{\mu}{T} \right) = 0,$$

and on further using Eq. (122) and (123), we can derive the following quantities

$$\frac{\partial h}{\partial T} \left( \frac{\mu}{T} \right) = y \left[ \frac{S^1_1}{S^2_2} + \frac{S^0_1}{S^2_2} - y \frac{S^1_1 S^0_1}{(S^2_2)^2} \right],$$

$$\frac{\partial e}{\partial T} \left( \frac{\mu}{T} \right) = 4y \frac{S^1_1}{S^2_2} + \frac{S^2_2 S^0_1}{(S^2_2)^2} - \frac{S^2_2}{S^2_2} + y^2 \left[ \frac{S^0_1}{S^2_2} - \frac{S^1_1 S^0_1}{(S^2_2)^2} \right],$$

$$\frac{\partial h}{\partial (\mu/T)} \left( \frac{\mu}{T} \right) = Ty \left[ \frac{S^0_1}{S^2_2} - \frac{S^1_1}{S^2_2} \right],$$

$$\frac{\partial e}{\partial (\mu/T)} \left( \frac{\mu}{T} \right) = -T \left[ 1 - \frac{S^2_2 S^0_1}{(S^2_2)^2} \right] + Ty \left[ \frac{S^0_1}{S^2_2} - \frac{S^1_1 S^0_1}{(S^2_2)^2} \right].$$  \hspace{1cm} (131, 132, 133, 134)
We can solve for $DT$ and $D \left( \frac{\mu}{T} \right)$ using the above equations and we get

\[
\frac{1}{T} DT = (1 - \lambda') \nabla \mu u^\mu, \tag{135}
\]

\[
TD \left( \frac{\mu}{T} \right) = [(\lambda'' - 1) h - \lambda''' T] \nabla \mu u^\mu, \tag{136}
\]

where,

\[
\lambda' = \frac{(S_0^2/S_1^2)^2 - (S_0^2/S_1^2)^2 + 4y^{-1}S_0^2S_1^2/(S_1^2)^2 + y^{-1}(S_0^2/S_1^2)}{(S_0^2/S_1^2)^2 - (S_0^2/S_1^2)^2 + 3y^{-1}S_0^2S_1^2/(S_1^2)^2 + 2y^{-1}(S_0^2/S_1^2) - y^{-2}}, \tag{137}
\]

\[
\lambda'' = 1 + \frac{y^{-2}}{(S_0^2/S_1^2)^2 - (S_0^2/S_1^2)^2 + 3y^{-1}S_0^2S_1^2/(S_1^2)^2 + 2y^{-1}(S_0^2/S_1^2) - y^{-2}}, \tag{138}
\]

\[
\lambda''' = \frac{(S_0^2/S_1^2)^2 - (S_0^2/S_1^2)^2 + 5y^{-1}(S_0^2S_1^2/(S_1^2)^2 + 2y^{-1}(S_0^2/S_1^2) - y^{-2})}{(S_0^2/S_1^2)^2 - (S_0^2/S_1^2)^2 + 3y^{-1}S_0^2S_1^2/(S_1^2)^2 + 2y^{-1}(S_0^2/S_1^2) - y^{-2}}. \tag{139}
\]

Now, replacing the time derivative using Eq.(135) and (136) in Eq.(51), we get

\[
p^\mu \partial_\mu f_{eq} = \frac{1}{T} f_{eq}(1 - f_{eq}) \left[ (u \cdot p)^2 (1 - \lambda') \nabla \mu u^\mu + \left( \frac{u \cdot p}{T} \right) p^\mu \nabla \mu T + (u \cdot p) \{(\lambda'' - 1) h - \lambda''' T\} \nabla \mu u^\mu + \left( \frac{\mu}{T} \right) \right]
\]

E\[p^\mu \partial_\mu f_{eq} = \frac{1}{T} f_{eq}(1 - f_{eq}) \left[ (u \cdot p)^2 \left( \frac{4}{3} - \lambda' \right) - \frac{1}{3} m^2 + (u \cdot p) \{(\lambda'' - 1) h - \lambda''' T\} \nabla \mu u^\mu - p^\mu p^\nu \left( \nabla \mu u^\nu - \frac{1}{3} \Delta_{\mu \nu} \nabla u^\sigma \right) + \left( 1 - \frac{u \cdot p}{h} \right) T p^\mu \nabla \mu \left( \frac{\mu}{T} \right) \right].
\]

**C Thermodynamic integrals**

We define the thermodynamic functions

\[
J_{nq}^{(m)\pm} = \sum_f \frac{1}{(2q + 1)!} \int dP(u \cdot p)^{n-2q-m} (\Delta_{\alpha \beta} p^\alpha p^\beta)^q \left( f_{eq,f} \tilde{f}_{eq,f} \pm f_{eq,f} \tilde{f}_{eq,f} \right), \tag{142}
\]

where $dP = g_f d^3p / [2(2\pi)^3 \varepsilon_f]$, $\tilde{f}_{eq} = 1 - \alpha f_{eq}$. Similarly, we define

\[
Y_{nq}^{(m)\pm} = \sum_f \frac{1}{(2q + 1)!} \int dP \left[ \frac{u \cdot p)^{n-2q-m} \left( \Delta_{\alpha \beta} p^\alpha p^\beta \right)^q \left( f_{eq,f} \tilde{f}_{eq,f} \pm f_{eq,f} \tilde{f}_{eq,f} \right) \right], \tag{143}
\]

and

\[
Z_{nq}^{(m)\pm} = \sum_f \frac{1}{(2q + 1)!} \int dP \left[ \frac{(u \cdot p)^{n-2q-m} \left( \Delta_{\alpha \beta} p^\alpha p^\beta \right)^q \left( f_{eq,f} \tilde{f}_{eq,f} \pm f_{eq,f} \tilde{f}_{eq,f} \right) \right]. \tag{144}
\]
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