Blocks with defect group $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^m}$

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Abstract In this paper, we prove that a block algebra with defect group $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^m}$, where $n \geq 2$ and $m$ is arbitrary, is Morita equivalent to its Brauer correspondent.

Keywords: Finite groups; blocks; Morita equivalences

1. Introduction

Let $p$ be a prime number and $\mathcal{O}$ a complete discrete valuation ring with an algebraically closed field $k$ of characteristic $p$. Denote by $\mathcal{K}$ a fraction field of $\mathcal{O}$. Assume that $\mathcal{K}$ contains a $|G|$-th primitive root of unity for any finite group $G$ considered below when $\mathcal{O}$ has characteristic 0.

Let $G$ be a finite group, $b$ a block of $G$ over $\mathcal{O}$ with defect group $P$, and $c$ the Brauer correspondent of $b$ in the normalizer $N_G(P)$. Assume that $P$ is abelian. Broué’s abelian defect group conjecture says that the block algebras $\mathcal{O}Gb$ and $\mathcal{O}N_G(P)c$ are derived equivalent (see [3]). Denote by $\mathbb{Z}_n$ the residue class group modulo $n$. Recently, some authors proved the conjecture for some 2-blocks with abelian defect groups, such as $\mathbb{Z}_2 \times \mathbb{Z}_2$ (see [8] and [27]), $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ (see [9]), and $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$ (see [10] for $n \geq 2$). The conjecture also has been verified in many other cases.

We are especially interested in the case $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$. In [10], the authors classified 2-blocks of quasisimple groups with abelian defect groups, and then proved that the block algebras $\mathcal{O}Gb$ and $\mathcal{O}N_G(P)c$ are Morita equivalent when $P$ is $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$ for some $n \geq 2$. By inspecting the classification, we observe that the block algebras $\mathcal{O}Gb$ and $\mathcal{O}N_G(P)c$ are Morita equivalent when $G$ is quasisimple and $P$ is $\mathbb{Z}_{2^n} \times \cdots \times \mathbb{Z}_{2^n}$ for some $n \geq 2$. So it may be interesting to ask whether such an observation holds for a general finite group $G$. In the paper, we investigate blocks with defect group $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$, where $n \geq 2$. More generally, we prove Broué’s abelian defect group conjecture for blocks with defect groups $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^m}$, where $n \geq 2$ and $m$ is arbitrary.

Theorem. Assume that $P$ is $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^m}$, where $n \geq 2$ and $m$ is arbitrary. Then the block algebras $\mathcal{O}Gb$ and $\mathcal{O}N_G(P)c$ are Morita equivalent.

Given a finite $p$-group $P$, Donovan’s conjecture says that there are finitely many Morita equivalence classes of block algebras over $k$ with defect group $P$. It is expected that Donovan’s conjecture should hold over $\mathcal{O}$ too. It is easy to see that the quotient group $N_G(P)/C_G(P)$ is trivial, or $\mathbb{Z}_3$, or $\mathbb{Z}_7$, or the Frobenius group $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$. By the structure theorem of block algebras with normal defect group, we conclude the following.

Corollary. Assume that $P$ is $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^m}$, where $n \geq 2$ and $m$ is arbitrary. Then the block algebra $\mathcal{O}Gb$ is Morita equivalent to $\mathcal{O}P$, or $\mathcal{O}(P \rtimes \mathbb{Z}_3)$, or $\mathcal{O}(P \rtimes \mathbb{Z}_7)$, or $\mathcal{O}(P \rtimes (\mathbb{Z}_7 \rtimes \mathbb{Z}_3))$. In particular, Donovan’s conjecture holds.

We remark that when $m \neq n$ the block algebra $\mathcal{O}Gb$ is Morita equivalent to $\mathcal{O}P$ or $\mathcal{O}(P \rtimes \mathbb{Z}_3)$, and that when $m = 0$, by Proposition 6.7 below, the Morita equivalence between $\mathcal{O}Gb$ and $\mathcal{O}N_G(P)c$ in Theorem is basic in the sense of [22].
The proof of Theorem heavily depends on $p$-extensions of certain perfect isometries (see Proposition 5.7 below). So the ring $\mathcal{O}$ will be always assumed to have characteristic 0 in the rest of the paper except in the proof of Theorem.

2. Preliminaries

In this section, we collect some elementary lemmas. Let $G$ and $G'$ be finite groups with common center $Z$. The central product $G \ast G'$ over $Z$ is isomorphic to the subgroup $G \otimes G'$ in the tensor product $\mathcal{O}G \otimes_{\mathcal{O}Z} \mathcal{O}G'$ and the group algebra $\mathcal{O}(G \ast G')$ is isomorphic to $\mathcal{O}G \otimes_{\mathcal{O}Z} \mathcal{O}G'$.

Lemma 2.1. Keep the notation as above. Then any block of $G \otimes G'$ is of the form $b \otimes b'$, where $b$ and $b'$ are blocks of $G$ and $G'$ respectively. If $P$ and $P'$ are defect groups of $b$ and $b'$ respectively, then $P \otimes P'$ is a defect group of $b \otimes b'$. Moreover, the order of the inertial quotient of the block $b \otimes b'$ is the product of the order of the inertial quotient of the block $b$ and that of $b'$.

Proof. The map $G \times G' \rightarrow G \otimes G', (g,g') \mapsto g \otimes g'$ is a surjective group homomorphism with kernel $\{(z,z^{-1})|z \in Z\}$ isomorphic to $Z$. Denote by $Z'$ the maximal $p'$-subgroup of $Z$. Note that the quotient group $Z/Z'$ is a $p'$-group. The homomorphism $G \times G' \rightarrow G \otimes G'$ factors through the canonical homomorphisms

$G \times G' \rightarrow (G \times G')/Z'$ and $(G \times G')/Z' \rightarrow G \otimes G'$.

Let $a \in \mathcal{O}G$ and $a' \in \mathcal{O}G'$. We denote by $a \otimes a'$ the element of the tensor product $\mathcal{O}G \otimes \mathcal{O}G'$ determined by $a$ and $a'$, in order to differentiate $a \otimes a'$ and $a \circ a'$. There is an $\mathcal{O}$-algebra isomorphism $\mathcal{O}(G \times G') \cong \mathcal{O}G \otimes \mathcal{O}G'$ sending $(x_1,x_2)$ onto $x_1 \otimes x_2$ for any $x \in G$ and any $x' \in G'$. We identify the algebras $\mathcal{O}(G \times G')$ and $\mathcal{O}G \otimes \mathcal{O}G'$. Then any block of $G \times G'$ is of form $b \otimes b'$, where $b$ and $b'$ are blocks of $G$ and $G'$. If $P$ and $P'$ are defect groups of $b$ and $b'$, then $P \otimes P'$ is a defect group of $b \otimes b'$. The inertial quotient of $b \otimes b'$ is the direct product of the inertial quotients of $b$ and $b'$.

Set $e = \frac{1}{|Z'|} \sum_{z \in Z'} z$. The homomorphism $G \times G' \rightarrow (G \times G')/Z'$ induces an algebra isomorphism $\mathcal{O}(G \times G')e \cong \mathcal{O}((G \times G')/Z')$. We identify the two algebras $\mathcal{O}(G \times G')e$ and $\mathcal{O}((G \times G')/Z')$. By the last paragraph, any block of $(G \times G')/Z'$ is of form $b \otimes b'$, where $b$ and $b'$ are blocks of $G$ and $G'$; if $P$ and $P'$ are defect groups of $b$ and $b'$, then $P \times P'$ is isomorphic to a defect group of $b \otimes b'$ (see [15, Theorem 5.7.4]); the inertial quotient of $b \otimes b'$ is isomorphic to the direct product of the inertial quotients of $b$ and $b'$ (see [19, Theorem 3.6]).

The homomorphism $(G \times G')/Z' \rightarrow G \otimes G'$ induces a surjective homomorphism $\mathcal{O}((G \times G')/Z') \rightarrow \mathcal{O}(G \otimes G')$ with kernel the central $p$-subgroup $Z/Z'$, which induces a one to one correspondence between blocks of $(G \times G')/Z'$ and $G \otimes G'$ (see [15, 5.8.11]). If $D/Z'$ is a defect group of a block of $(G \times G')/Z'$, then $D/Z$ is a defect group of the corresponding block of $G \otimes G'$ (see [15, Theorem 5.8.10]). By [20, Theorem 2.9], a block of $(G \times G')/Z'$ and the corresponding block of $G \otimes G'$ have isomorphic inertial quotients. The proof is done.

2.2. Let $\text{CF}(G, \mathcal{K})$ be the $\mathcal{K}$-vector space of all $\mathcal{K}$-valued class functions on $G$, and $\text{CF}_{p'}(G, \mathcal{K})$ the subspace of all functions in $\text{CF}(G, \mathcal{K})$ vanishing on $p$-singular elements of $G$. For any $\chi \in \text{CF}(G, \mathcal{K})$ and any central idempotent $e$ in $\mathcal{O}G$, extending $\chi$ to a function over $KG$ by the $\mathcal{K}$-linearity, we define a new class function $e \cdot \chi$ on $G$ by the equality $(e \cdot \chi)(g) = \chi(ge)$ for any $g \in G$. We set $\text{CF}(G,e,\mathcal{K}) = \{e \cdot \chi | \chi \in \text{CF}(G, \mathcal{K})\}$ and $\text{CF}_{p'}(G,e,\mathcal{K}) = \text{CF}(G,e,\mathcal{K}) \cap \text{CF}_{p'}(G, \mathcal{K})$. It is easy to see that $\text{CF}(G,e,\mathcal{K})$ and $\text{CF}_{p'}(G,e,\mathcal{K})$ are subspaces of $\text{CF}(G, \mathcal{K})$ and that we have direct sum decompositions

$$\text{CF}(G, \mathcal{K}) = \bigoplus_b \text{CF}(G, b, \mathcal{K})$$

and

$$\text{CF}_{p'}(G, \mathcal{K}) = \bigoplus_b \text{CF}_{p'}(G, b, \mathcal{K}),$$
where $b$ runs over the set of blocks of $G$. Denote by $\text{Irr}(G)$ and $\text{Irr}(G, b)$ the set of irreducible ordinary characters of the group $G$ and the set of irreducible ordinary characters in a block $b$ of $G$. It is known that $\text{Irr}(G)$ and $\text{Irr}(G, b)$ are bases of $\text{CF}(G, K)$ and $\text{CF}(G, b, K)$ and that the orthogonality relationship of characters determines the inner products on $\text{CF}(G, K)$ and $\text{CF}(G, b, K)$. We denote by $\text{CF}_{p'}(G, O)$ the $O$-submodule of all $O$-valued class functions on $G$ vanishing on all $p$-singular elements of $G$, and by $\text{CF}_{p'}(G, e, O)$ the intersection of $\text{CF}_{p'}(G, O)$ and $\text{CF}(G, e, K)$.

**2.3.** Let $b$ and $b'$ be blocks of $G$ and $G'$. Denote by $b^o$ the inverse image of $b'$ through the opposite isomorphism $O G' \cong O G$. There is an $O$-algebra isomorphism $O(\nu \otimes G') \cong O \otimes O G'$ of $G \times G'$ by the equality $\text{CF}(\nu \otimes G') = \text{CF}(\nu, G, b) \otimes \text{CF}(\nu, G', b')$ for any $\nu \in \text{Irr}(G)$, $b \in \text{blocks}$ of $G$, $b' \in \text{blocks}$ of $G'$, and $\text{Irr}(G \times G')$. Identifying the two algebras $O(\nu \otimes G')$ and $O \otimes O G'$, $b \otimes b^o$ is a block of $G \times G'$. Given a generalized character $\nu$ of $G \times G'$ in the block $b \otimes b^o$, we define a map $I_\nu : \mathbb{Z}\text{Irr}(G', b') \to \mathbb{Z}\text{Irr}(G, b)$ by the equality

$$I_\nu(\chi)(g) = \frac{1}{|G'|} \sum_{h \in G'} \nu(g, h)\alpha(h)$$

for any $\alpha \in \mathbb{Z}\text{Irr}(G', b')$ and any $g \in G$, where $\mathbb{Z}\text{Irr}(G, b)$ and $\mathbb{Z}\text{Irr}(G', b')$ are the free abelian groups generated by $\text{Irr}(G, b)$ and $\text{Irr}(G', b')$. The map $I_\nu$ can be extended to a $K$-linear map $\text{CF}(\nu \otimes G') \to \text{CF}(G, b, K)$, denoted by $I_\nu^\nu$. We remind that the definition of $I_\mu$ is slightly different from the definition of $I_\mu$ in [4, (F)]. We modify the definition of $I_\mu$ in [4, (F)], mainly to fill gaps, which are produced when we use to $\mu$ the induction and extension of generalized characters. Such a modification does not influence the use of all known theorems on perfect isometries (see [4]). If $I_\nu$ is a perfect isometry and we denote by $\chi^*$ the dual of an ordinary character $\chi$, then

$$\nu = \sum_{\chi \in \text{Irr}(G', b')} I_\nu(\chi) \times \chi^*,$$

where $I_\nu(\chi) \times \chi^*$ denotes the generalized ordinary character of $G \times G'$ determined by $\chi^*$ and $I_\nu(\chi)$.

**2.4.** Let $u$ be a $p$-element of $G$. We denote by $d_G^u$ the surjective map $\text{CF}(G, K) \to \text{CF}_{p'}(C_G(u), K)$ defined by $d_G^u(\chi)(s) = \chi(us)$ for any $\chi \in \text{CF}(G, K)$ and any $p$-regular element $s$ of $C_G(u)$, and by $e_G^u$ the map $\text{CF}_{p'}(C_G(u), K) \to \text{CF}(G, K)$ sending any $\varphi$ onto the function $e_G^u(\varphi)$ which takes $\varphi(s)$ at $us$ for any $p$-regular element $s$ of $C_G(u)$ and 0 at $g \in G$ if the $p$-part of $g$ is not conjugate to $u$. The composition $d_G^u \circ e_G^u$ is the identity map on $\text{CF}_{p'}(C_G(u), K)$. Let $(u, e)$ be a $b$-Brauer element. We denote by $d_G^{(u, e)}$ the composition of $d_G^u$ and the projection $\text{CF}_{p'}(C_G(u), K) \to \text{CF}_{p'}(C_G(u), e, K)$. By [5, Theorem A2.1] $e_G^u$ maps $\text{CF}_{p'}(C_G(u), e, K)$ into $\text{CF}(G, b, K)$ and we denote by $e_G^{(u, e)}$ the restriction of $e_G^u$ on $\text{CF}_{p'}(C_G(u), e, K)$. The composition $d_G^{(u, e)} \circ e_G^{(u, e)}$ is the identity map on $\text{CF}_{p'}(C_G(u), e, K)$.

**2.5.** Let $(P, f)$ be a maximal $b$-Brauer pair. For any subgroup $Q$ of $P$, there is a unique $b$-Brauer pair $(Q, f_Q)$ contained in $(P, f)$. We denote by $\text{Br}(P, f)$ the Brauer category (see [29]), whose objects are all $b$-Brauer pairs $(Q, f_Q)$ contained in $(P, f)$ and whose morphisms from $(Q, f_Q)$ to $(R, f_R)$ are group homomorphisms $Q \to R$ induced by elements $g \in G$ such that $(Q, f_Q)^g \leq (R, f_R)$. We assume that the blocks $b$ and $b'$ have a common defect group $P$ and that the inclusions $P \subset G$ and $P \subset G'$ induce an isomorphism between the categories $\text{Br}(P, f)$ and $\text{Br}(P, f')$, where $(P, f')$ is a suitable maximal $b'$-Brauer pair. The map $I_{\nu}^b : \text{CF}(G', b', K) \to \text{CF}(G, b, K)$ is said to be compatible with a local system $\{I_T : \text{CF}_{p'}(C_G(T), f_T, K) \to \text{CF}_{p'}(C_G(T), f_T, K)\}_{\{T_{(cyclic)}\subset P\}}$ (see [4, Definition 4.3]), if for any cyclic subgroup $T$ of $P$ and any generator $x$ of $T$, we have $d_G^{(x, f_T)} \circ I_{\nu}^b = I_T \circ d_G^{(x, f_T)}$. By the isomorphism theorem of groups, the local system is uniquely determined by $I_{\nu}^b$ if it exists.
Lemma 2.6. Keep the notation and the assumption of 2.5 and assume that \( P \) is abelian and that the map \( I^\nu : \text{CF}(G', b', \mathcal{K}) \to \text{CF}(G, b, \mathcal{K}) \) is compatible with a local system \( \{ I_T \}_{T \text{(cyclic)} \subset P} \). Then we have \( I^\nu = \sum_{x \in J} e_G^{(x, f(x))} \circ I(x) \circ d_{G'}^{(x, f(x))} \), where \( (x) \) is the cyclic subgroup generated by \( x \) and \( J \) is a set of representatives of orbits of the conjugation action of \( N_G(P, f) \) on \( P \).

Proof. Obviously \( \{(x, f(x)) | x \in J\} \) is a set of representatives of orbits of the action of \( N_G(P, f) \) on the set of \( b' \)-Brauer elements contained in \( (P, f) \), and since \( P \) is abelian, \( \{(x, f(x)) | x \in J\} \) is a set of representatives of \( G \)-conjugacy classes of \( b' \)-Brauer elements. The family \( \{d_{G'}^{(x, f(x))} | x \in J\} \) induces a \( \mathcal{K} \)-linear isomorphism \( \text{CF}(G, b, \mathcal{K}) \cong \bigoplus_{x \in J} \text{CF}_{G'}(C_G(x), f(x), \mathcal{K}) \) (see the last paragraph of [4, 4A]). Since \( d_{G}^{(x', f(x'))} \circ e_{G}^{(x, f(x))} = 0 \) for different \( x, x' \in J \), we have

\[
d_{G}^{(x, f(x))} \circ I^\nu = d_{G}^{(x, f(x))} \circ \left( \sum_{y \in J} e_{G}^{(y, f(y))} \circ I(y) \circ d_{G'}^{(y, f(y))} \right).
\]

So \( I^\nu = \sum_{y \in J} e_{G}^{(y, f(y))} \circ I(y) \circ d_{G'}^{(y, f(y))} \).

The main idea of the following proposition is already included in the first sentence of [26, 1.6].

Proposition 2.7. Keep the notation and the assumption of 2.5 and assume that \( P \) is abelian. Then the map \( I^\nu : \text{CF}(G', b', \mathcal{K}) \to \text{CF}(G, b, \mathcal{K}) \) is compatible with a local system \( \{ I_T \}_{T \text{(cyclic)} \subset P} \) if and only if \( I^\nu(\lambda \ast \chi) = \lambda \ast I^\nu(\chi) \) for any \( \chi \in \text{CF}(G', b', \mathcal{K}) \) and any \( N_G(P, f) \)-stable character \( \lambda \) of \( P \), where \( \ast \) denotes the \( \ast \)-construction of characters due to Broué and Puig.

Before the proof, we remark that any ordinary character \( \lambda \) of \( P \) is \( N_G(P, f) \)-stable if and only if it is \( N_G(P, f') \)-stable, since the Brauer categories \( \text{Br}(P, f) \) and \( \text{Br}(P, f') \) are isomorphic.

Proof. We assume that the map \( I^\nu \) is compatible with a local system \( \{ I_T \}_{T \text{(cyclic)} \subset P} \). For any \( b' \)-Brauer element \((u, f(u))\) and any \( \chi \in \text{CF}(G, b, \mathcal{K}) \), set \( \chi(u, f(u)) = (e^{(u, f(u))} \circ d^{(u, f(u))}(\chi)) \). Let \( J \) be a set of representatives of orbits of the conjugation action of \( N_G(P, f) \) on \( P \). Since \( P \) is abelian, \( \{(x, f(x)) | x \in J\} \) is a set of representatives of \( G \)-conjugacy classes of \( b' \)-Brauer elements. Since the categories \( \text{Br}(P, f) \) and \( \text{Br}(P, f') \) are isomorphic, \( \{(x, f(x)) | x \in J\} \) is a set of representatives of orbits of the action of \( N_G(P, f') \) on the set of \( b' \)-Brauer elements contained in \( (P, f') \) and \( \{(x, f(x)) | x \in J\} \) is also a set of representatives of the \( G' \)-conjugacy classes of \( b' \)-Brauer elements.

For any \( \chi \in \text{CF}(G', b', \mathcal{K}) \), we have \( \chi = \sum_{x \in J} \chi(x, f(x)) \). Since \( I^\nu(\lambda \ast \chi)(x, f(x)) = \lambda(x)e^{(x, f(x))} \circ I(x) \circ d_{G'}^{(x', f(x'))}(\chi(x', f(x'))) = (\lambda \ast I^\nu(\chi))(x, f(x)) \) (see Lemma 2.6), we have

\[
I^\nu(\lambda \ast \chi) = \sum_{x \in J} I^\nu(\lambda \ast \chi(x, f(x))) = \sum_{x \in J} (\lambda \ast I^\nu(\chi))(x, f(x)) = \lambda \ast I^\nu(\chi).
\]

Conversely, we assume that \( I^\nu(\lambda \ast \chi) = \lambda \ast I^\nu(\chi) \) for any \( \chi \in \text{CF}(G', b', \mathcal{K}) \) and any \( N_G(P, f) \)-stable character \( \lambda \) of \( P \). Obviously \( N_G(P, f) \) acts on \( \text{Irr}(P) \) and by Brauer’s permutation Lemma, the number of orbits of the action of \( N_G(P, f) \) on \( \text{Irr}(P) \) is equal to the cardinality of \( J \). We list the orbit sums of the action of \( N_G(P, f) \) on \( \text{Irr}(P) \) as \( \lambda_j \), where \( 1 \leq j \leq n \) and \( n = |J| \). For any \( \lambda_i \) and any \( \chi \in \text{Irr}(G', b') \), we have \( \lambda_i \ast \chi = \sum_{u \in J} \lambda_i(u)\chi^{(u, f(u))} \). The matrix \( (\lambda_i(u)) \) is an invertible matrix, since \( \text{Irr}(P) \) is a basis of \( \text{CF}(P, \mathcal{K}) \). Denoting by \( (a_{iu}) \) the inverse of \( (\lambda_i(u)) \), we have

\[
\chi^{(u, f(u))} = \sum_i a_{iu} \lambda_i \ast \chi.
\]
Since $I^K_G(\lambda \ast \chi) = \lambda \ast I^K_G(\chi) = \sum_{u \in J} \lambda_i(u)(I^K_G(\chi))^{(u, f(u))}$, similarly we have

$$\tag{2.7.2} (I^K_G(\chi))^{(u, f(u))} = \sum_i a_{iu} I^K_G(\lambda_i \ast \chi).$$

Combining the equalities 2.7.1 and 2.7.2, we have

$$\tag{2.7.3} (I^K_G(\chi))^{(u, f(u))} = I^K_G(\chi^{(u, f(u))}).$$

For any $u \in J$, we denote by $V^{(u, f(u))}$ the image of the homomorphism $e^{(u, f(u))}$. Obviously the set $\{\chi^{(u, f(u))}| \chi \in \text{Irr}(G, b)\}$ is a generator set of the space $V^{(u, f(u))}$, the space $\text{CF}(G, b, K)$ is equal to the sum $\bigoplus_{u \in J} V^{(u, f(u))}$ since the family $\{d_G^{(u, f(u))}| u \in J\}$ induces a $K$-linear isomorphism $\text{CF}(G, b, K) \cong \bigoplus_{u \in J} \text{CF}_p(C_G(u, f(u)), K)$, and the kernel of $d_G^{(u, f(u))}$ is equal to the sum $\bigoplus_{v \in J - \{u\}} V^{(v, f(v))}$. By the equality 2.7.3, $I^K_G$ maps $V^{(u, f(u))}$ onto $V^{(u, f(u))}$ and thus it maps the kernel of $d_G^{(u, f(u))}$ onto the kernel of $d_G^{(u, f(u))}$. Therefore $I^K_G$ induces a $K$-linear homomorphism

$$I_{(x)}: \text{CF}_p(C_G(\langle x \rangle, f'_x), b(x), \mathcal{O}) \to \text{CF}_p(C_G(\langle x \rangle, f'_x), \mathcal{O})$$

such that $d_G^{(u, f(u))} \circ I^K_G = I_{(x)} \circ d_G^{(u, f(u))}$.

**Lemma 2.8.** Keep the notation and the assumption of 2.5 and assume that $P$ is abelian. Assume that the map $I^K_G: \text{CF}(G', b', K) \to \text{CF}(G, b, K)$ is compatible with a local system $\{I_T\}_{T(\text{cyclic}) \subset P}$ and that $I^K_G$ maps $\text{CF}(G', b', \mathcal{O})$ into $\text{CF}(G, b, \mathcal{O})$. Then for any cyclic subgroup $T$ of $P$, $I_T$ maps $\text{CF}_p(C_G(T, f_T), b_T, \mathcal{O})$ into $\text{CF}_p(C_G(T, f_T), \mathcal{O})$.

**Proof.** Take $x \in P$. Since $e_G^{(x, f(x))} \circ I_{(x)} \circ d_G^{(x, f(x))} = e_G^{(x, f(x))} \circ d_G^{(x, f(x))} \circ I^K_G = I^K_G \circ e_G^{(x, f(x))} \circ d_G^{(x, f(x))}$ (the second equality is obtained by the equality 2.7.3), we have $e_G^{(x, f(x))} \circ I_{(x)} = I^K_G \circ e_G^{(x, f(x))}$. Since the homomorphisms $e_G^{(x, f(x))}$ maps $\text{CF}_p(C_G(\langle x \rangle, f'_x), b(x), \mathcal{O})$ into $\text{CF}(G, b, \mathcal{O})$ and $I^K_G$ maps $\text{CF}(G', b', \mathcal{O})$ onto $\text{CF}(G, b, \mathcal{O})$, the image of $\text{CF}_p(C_G(\langle x \rangle, f'_x), \mathcal{O})$ through $e_G^{(x, f(x))} \circ I_{(x)}$ is contained in $\text{CF}(G, b, \mathcal{O})$. Since

$$d_G^{(x, f(x))}(\text{CF}(G, b, \mathcal{O})) \subset \text{CF}_p(C_G(\langle x \rangle, f'_x), \mathcal{O}),$$

$I_{(x)}$ maps $\text{CF}_p(C_G(\langle x \rangle, f'_x), b(x), \mathcal{O})$ into $\text{CF}_p(C_G(\langle x \rangle, b(x), \mathcal{O})$.

**2.9.** Now we take an $(OG, O'G')$-bimodule $M$ inducing a Morita equivalence between $OGb$ and $O'G'b'$. By [1, Theorem 2.1 and Exercise 4.5], the Morita equivalence induced by $M$ induces an algebra isomorphism

$$\rho: Z(OG'b') \cong Z(OGb)$$

such that for any $a \in Z(OGb)$ and any $a' \in Z(O'G'b')$, $a$ and $a'$ correspond to each other if and only if $am = ma'$ for any $m \in M$, where $Z(OGb)$ and $Z(O'G'b')$ are the centers of the algebras $OGb$ and $O'G'b'$. In order to consider the character of the bimodule $M$ and basic Morita equivalences in Section 3, we also regard the module $M$ as an $O(G \times G')$-bimodule by the equality $(g, g'^{-1})m = gmg'$ for any $g \in G$, any $g' \in G'$ and any $m \in M$. Let $\mu$ be the character of $M$. The
map \( I_\mu : Z\text{Irr}(G', b') \rightarrow Z\text{Irr}(G, b) \) is a perfect isometry. By [4, Theorem 1.5], the perfect isometry \( I_\mu \) induces an algebra isomorphism
\[
\rho' : Z(OG'b') \cong Z(OGb).
\]

**Lemma 2.10.** Keep the notation above. The isomorphisms \( \rho \) and \( \rho' \) coincide.

**Proof.** Consider the extensions of the isomorphisms \( \rho \) and \( \rho' \) by the \( K \)-linearity
\[
\rho^K : Z(KG'b') \cong Z(KGb) \quad \text{and} \quad \rho'^K : Z(KG'b') \cong Z(KGb).
\]
For any \( \chi \in \text{Irr}(G) \), we denote by \( e_\chi \) the central primitive idempotent \( \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g)g^{-1} \) in \( KG \).

Set \( e_\chi = \rho^K(e_\chi') \) for any \( \chi' \in \text{Irr}(G', b') \). We have \( e_\chi m = me_{\chi^*} \) for any \( m \in M \), and thus \( \mu(e_\chi, 1) = \mu(1, e_{\chi^*}) \neq 0 \). Since \( \mu = \sum_{\chi' \in \text{Irr}(G', b')} (I_\mu(\chi') \times \chi'^*) \), \( I_\mu(\chi')(e_\chi) \) is nonzero and \( I_\mu(\chi') \) is equal to \( \chi \). On the other hand, by the proof of [4, Theorem 1.5] the isomorphism \( \rho^K \) also sends \( e_{\chi^*} \) onto \( e_\chi \). So \( \rho'^K = \rho^K \).

3. Morita equivalences and the \( * \)-structure

In this section, we will prove that if two block algebras with abelian defect groups are basically Morita equivalent, then there is a Morita equivalence between the two corresponding block algebras inducing a perfect isometry compatible with a local system (see Proposition 3.6). In order to do that, we firstly recall the computation of generalized decomposition numbers in [25].

3.1. Throughout the section, all \( O \)-algebras and \( O \)-modules are \( O \)-free of finite \( O \)-rank. For any ring \( R \), denote by \( R^* \) the multiplicative group of \( R \). Let \( G \) be a finite group. An \( O \)-algebra \( A \) is an interior \( G \)-algebra if there is a group homomorphism \( G \rightarrow A^* \). The conjugation of \( G \) in \( A \) induces a group homomorphism \( G \rightarrow \text{Aut}(A) \) so that \( A \) becomes a \( G \)-algebra, where \( \text{Aut}(A) \) denotes the automorphism group of the \( O \)-algebra \( A \). For any subgroup \( H \) of \( G \), we denote by \( A^H \) the subalgebra of all \( H \)-fixed elements in the \( G \)-algebra \( A \). A pointed group on \( A \) is a pair \( (H, \alpha) \), where \( H \) is a subgroup of \( G \) and \( \alpha \) is an \( (A^H)^* \)-conjugacy class of primitive idempotents in \( A^H \). We often write \( (H, \alpha) \) as \( H_\alpha \), and say that \( \alpha \) is a point of \( H \) on \( A \). The pointed group \( H_\alpha \) is contained in another pointed group \( K_\beta \), denoted by \( H_\alpha \leq K_\beta \), if \( H \) is a subgroup of \( K \) and there are \( i \in \alpha \) and \( j \in \beta \) such that \( ji = ij = i \).

3.2. Let \( H \) and \( K \) be two subgroups of \( G \) such that \( K \leq H \). We denote by \( \text{Tr}^H_K : A^K \rightarrow A^H \) the relative trace map and by \( A^K_H \) the image of \( \text{Tr}^H_K \). We consider the quotient
\[
A(H) = A^H / \left( J(O)A^H + \sum K A^K_H \right)
\]
where \( K \) runs over all proper subgroups of \( H \), and denote by \( \text{Br}^A_H \) the canonical surjective algebra homomorphism from \( A^H \) to \( A(H) \). If \( A(H) \) is not zero, then \( H \) has to be a \( p \)-subgroup. Clearly the inclusion \( G \subset OG \) endows \( OG \) an interior \( G \)-algebra structure. Let \( Q \) be a \( p \)-subgroup of \( G \). Then the inclusion \( OG_Q(Q) \subset (OG)^Q \) induces an algebra isomorphism \( (OG)^Q(Q) \cong kC_G(Q) \). We identify \( (OG)^Q(Q) \) and \( kC_G(Q) \) in the sequel.

3.3. A pointed group \( H_\alpha \) on \( A \) is local if \( H \) is a \( p \)-subgroup of \( G \) and the image \( \text{Br}^A_H(\alpha) \neq \{0\} \). In this case, \( \alpha \) is called a local point of \( H \) on \( A \), and since \( \text{Br}^A_H \) is a surjective algebra homomorphism,
Br^A_G(\alpha) is a set of primitive idempotents in A(H) and by the lifting theorem of idempotents, the correspondence \alpha \mapsto Br^A_G(\alpha) gives a bijection between the local points of H on A and the points of H on A(H). Let x be an element of G. We call x_\alpha a local pointed element if \langle x \rangle_\alpha is a local pointed group on A. The local pointed element x_\alpha is contained in a pointed group K_\beta if \langle x \rangle_\alpha is contained in K_\beta.

3.4. Now we consider the group algebra OG. Let b be a block of G. Then \{b\} is a point of G on OG. Let x_\alpha be a local pointed element contained in the pointed group G_{\{b\}} on OG, and take j \in \alpha. Let \chi \in \Irr(G,b), afforded by a KG-module M. The product jM of j and M is an \O(u)-module, whose character is denoted by \chi^\alpha. Denote by P_{OG}(x) the set of all \alpha such that x_\alpha is a local pointed element on OG. By [25, Corollary 4.4], for any \gamma'-element s of C_G(x), we have

\[ \chi(xs) = \sum_{\alpha \in P_{OG}(x)} \chi^\alpha(x) \varphi_\alpha(s). \]

If x_\alpha is not contained in G_{\{b\}}, then \chi^\alpha(x) is equal to 0.

3.5. By [25, Theorem 1.2], all maximal local pointed groups contained in G_{\{b\}} are G-conjugate to each other. Let P_\gamma be a maximal local pointed group contained in G_{\{b\}}, choose i \in \gamma and set A_\gamma = iAi. Obviously A_\gamma is an OG-algebra and admits a group homomorphism \psi : P \to A_\gamma^*, \gamma \mapsto u_i. So A_\gamma is an endopermutation algebra. Such an endopermutation algebra A_\gamma is called a source algebra of the block algebra OGb. By [25, Corollary 3.5], A_\gamma and OGb are Morita equivalent. In particular, given a simple KG-module M in the block b, iM is a simple K \otimes_O A_\gamma-module, and the correspondence

\[ M \mapsto iM \]

gives a bijection between the set of isomorphism classes of simple KG-modules in the block b and the set of isomorphism classes of simple K \otimes_O A_\gamma-modules.

We say that the block algebras OGb and OGb' are basically Morita equivalent if there is an OG(G \times G')-module with endopermutation source, inducing a Morita equivalence between the block algebras OGb and OGb'. In this case, such a Morita equivalence is called a basic Morita equivalence between OGb and OGb' (see [22]).

**Proposition 3.6.** Let G and G' be finite groups and b and b' blocks of G and G'. Assume that the block algebras OGb and OGb' are basically Morita equivalent and that defect groups of b and b' are abelian. Then there is a Morita equivalence between the block algebras OGb and OGb' inducing a perfect isometry I_\nu : \Z\Irr(G',b') \to \Z\Irr(G,b) such that I_\nu^C is compatible with a local system \{I_T\}_{T(cyclic) \subset P}.

**Proof.** Since OGb and OGb' are basically Morita equivalent, by [22, Corollary 7.4] they have isomorphic defect groups. For convenience, we assume without loss of generality that b and b' have a common defect group P. By [22, 6.9.3 and Corollary 7.4], there are maximal local pointed groups P_\gamma on OGb and P_\gamma' on OGb' and an endopermutation OP-module N with vertex P, such that setting (OG)_\gamma = i(OG)i and (OG')_\gamma' = i'(OG')i' for some i \in \gamma and some i' \in \gamma', we have an interior P-algebra embedding

\[ f : (OG)_\gamma \to \End_O(N) \otimes_O (OG')_\gamma'. \]
Explicitly, \( f \) is an injective \( \mathcal{O} \)-algebra homomorphism, the image of \( f \) is equal to \( f(i)(\text{End}_{\mathcal{O}}(N) \otimes_{\mathcal{O}} (\mathcal{O}G')_{\gamma'} f(i) \), and \( f \) preserves interior \( P \)-algebra structures (which is an interior \( P \)-algebra homomorphism in the sense of [25, Definition 3.1]); the interior \( P \)-algebra structure on \( \text{End}_{\mathcal{O}}(N) \otimes_{\mathcal{O}} (\mathcal{O}G')_{\gamma'} \) is determined diagonally by the interior \( P \)-algebra structures on \( \text{End}_{\mathcal{O}}(N) \) and \( (\mathcal{O}G')_{\gamma'} \). We identify \( (\mathcal{O}G)_{\gamma} \) as its image through \( f \), so that

\[
3.6.1 \quad (\mathcal{O}G)_{\gamma} = i\left(\text{End}_{\mathcal{O}}(N) \otimes_{\mathcal{O}} (\mathcal{O}G')_{\gamma'}\right)i.
\]

Since \( \mathcal{O}Gb \) and \( \mathcal{O}G'b' \) are basically Morita equivalent, \( (\mathcal{O}G)_{\gamma} \) and \( (\mathcal{O}G')_{\gamma'} \) have the same numbers of isomorphism classes of simple \( (\mathcal{O}G)_{\gamma} \)- and \( (\mathcal{O}G')_{\gamma'} \)-modules. Clearly \( \text{End}_{\mathcal{O}}(N) \otimes_{\mathcal{O}} (\mathcal{O}G')_{\gamma'} \) and \( (\mathcal{O}G')_{\gamma'} \) have the same numbers of isomorphism classes of simple \( (\text{End}_{\mathcal{O}}(N) \otimes_{\mathcal{O}} (\mathcal{O}G')_{\gamma'}) \)- and \( (\mathcal{O}G')_{\gamma'} \)-modules. So \( (\mathcal{O}G)_{\gamma} \) and \( \text{End}_{\mathcal{O}}(N) \otimes_{\mathcal{O}} (\mathcal{O}G')_{\gamma'} \) have the same numbers of isomorphism classes of simple \( (\mathcal{O}G)_{\gamma} \)- and \( (\text{End}_{\mathcal{O}}(N) \otimes_{\mathcal{O}} (\mathcal{O}G')_{\gamma'}) \)-modules. By [29, Theorem 9.9], the correspondence

\[
3.6.2 \quad W \to i(N \otimes_{\mathcal{O}} W)
\]

gives a Morita equivalence between the categories of \( (\mathcal{O}G')_{\gamma'} \)- and \( (\mathcal{O}G)_{\gamma} \)-modules. By composing the Morita equivalences between \( \mathcal{O}G'b' \) and \( (\mathcal{O}G')_{\gamma'} \) (see 3.5), between \( (\mathcal{O}G')_{\gamma'} \) and \( (\mathcal{O}G)_{\gamma} \) and between \( (\mathcal{O}G)_{\gamma} \) and \( \mathcal{O}Gb \) (see 3.5), we get a Morita equivalence \( \Phi \) between \( \mathcal{O}G'b' \) and \( \mathcal{O}Gb \).

Note that \( \Phi \) may be different from the Morita equivalence \( \Psi \) between \( \mathcal{O}G'b' \) and \( \mathcal{O}Gb \) in the assumption. For a \( K \)-algebra \( A \), we denote by \( \mathcal{M}(A) \) the set of isomorphism classes of simple \( A \)-modules. The purpose of constructing \( \Phi \) is to get the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{M}(KGb) & \xrightarrow{\Phi} & \mathcal{M}((\mathcal{O}G'b')_{\gamma'}) \\
\downarrow & & \downarrow \\
\mathcal{M}(K \otimes_{\mathcal{O}} (\mathcal{O}G')_{\gamma'}) & \xrightarrow{\Phi} & \mathcal{M}(\mathcal{O}G)
\end{array}
\]

where the vertical arrows are obtained by the correspondence 3.5.1 applied to \( \mathcal{O}Gb \) and \( \mathcal{O}G'b' \), the top arrow \( \phi \) is induced by \( \Phi \) and the bottom arrow is determined by the correspondence 3.6.2. The diagram is a consequence of the Morita equivalence \( \Phi \) in the last paragraph. The bijection \( \mathcal{M}(KGb) \to \mathcal{M}(\mathcal{O}G) \) induced by \( \Psi \) may not make the above diagram commutative.

We say that a local pointed group \( Q_{\delta} \) on \( \mathcal{O}G \) is associated to a Brauer pair \((Q, g)\), if

\[\text{Br}_{Q}(f)\text{Br}_{Q}(\delta) = \text{Br}_{Q}(\delta)\]

and that a local pointed element \( u_\alpha \) is associated to a Brauer element \((u, g)\) if the local pointed group \((u_\delta)\) is associated to the Brauer pair \((u, g)\). Let \((P, f)\) and \((P', f')\) be Brauer pairs, to which \( P_{\gamma} \) and \( P_{\gamma'} \) are associated. Let \( M \) be a simple \( KGb \)-module and \( \chi \) its ordinary character. Let \((u, h)\) be a Brauer element contained in \((P, f)\). Since \( P \) is abelian, any local pointed element \( u_\alpha \) associated to the Brauer element \((u, h)\) is contained in \( P_{\gamma} \) and conversely any local pointed element \( u_\alpha \) contained in \( P_{\gamma} \) is associated to \((u, h)\). By 3.4.1, we have

\[
3.6.3 \quad d_{G}^{(u, h)}(\chi) = \sum_{u_\alpha \in \mathcal{P}(u, h)} \chi^\alpha(u)\varphi_\alpha,
\]

where \( \mathcal{P}(u, h) \) denotes the set of all local pointed elements associated to the Brauer element \((u, h)\). Let \( M' \) be a simple \( KGb' \)-module corresponding to \( M \) under the bijection \( \phi \), and \( \chi' \) its ordinary
character. Let \((u, h')\) be the Brauer pair contained in \((P, f')\). Similarly, we have

\[
d^\alpha_{G'}(u, h') = \sum_{\alpha' \in \mathcal{P}(u, h')}(\chi')^{\alpha'}(u)\varphi_{\alpha'}.
\]

Given \(u_\alpha \in \mathcal{P}(u, h)\), by [29, Proposition 15.1], the intersection \(\alpha \cap (OG)_\gamma\) is still a local point of \(\langle u \rangle\) on \((OG)_\gamma\) and the correspondence \(\alpha \mapsto \alpha \cap (OG)_\gamma\) gives a bijection between \(\mathcal{P}(u, h)\) and the set of local points of \(\langle u \rangle\) on \((OG)_\gamma\). Through the bijection, we identify \(\mathcal{P}(u, h)\) as a set of local pointed elements on \((OG)_\gamma\). By [22, 7.6.2] applied to the equality 3.6.1, we get a bijection

\[
\mathcal{P}(u, h) \to \mathcal{P}(u, h'), \; u_\alpha \mapsto u_{\alpha'}
\]

where \(u_\alpha\) and \(u_{\alpha'}\) correspond to each other if and only if there are \(j \in \alpha\) and \(j' \in \alpha'\) such that

\[
j(\ell \otimes j') = j = (\ell \otimes j')j,
\]

where \(\ell\) is a primitive idempotent in the unique local point of \(\langle u \rangle\) on \(\text{End}_O(\nu)\). We are going to prove that

\[
\chi^\alpha(u) = \omega(u)(\chi')^{\alpha'}(u)
\]

under the bijection, where \(\omega\) denotes the character of the \((OG)\)-module \(\ell(\nu)\).

Since the \(KG\)-module \(M\) and the \(KG'\)-module \(M'\) correspond to each other under the bijection \(\phi\), by the construction of \(\Phi\) and 3.6.2, we have

\[
iM \cong i(N \otimes_O i'M)
\]

as \(K \otimes_O (OG)\)-modules. On the other hand, since \(u_\alpha\) is contained in \(P_\gamma\), there is \(j'' \in \alpha \cap (OG)_\gamma\) such that \(ij'' = j'' = j''i\). So

\[
j''M \cong j''(N \otimes_O i'M)
\]

as \((K \otimes_O (OG)_\alpha)\)-modules, where \((OG)_\alpha = j''(OG)j''\). We note that both \(j\) and \(j''\) are inside \(\alpha \cap (OG)_\gamma\) and thus that the idempotents \(j''\) and \(j\) are conjugate in \((\text{End}_O(\nu) \otimes (OG')_\gamma)(\nu)\). In particular, we have

\[
j''(N \otimes_O i'M) \cong j(N \otimes_O i'M)
\]

as \(O(\nu)\)-modules. Now, summarizing the above three isomorphisms, we conclude that the value of \(\chi^\alpha\) at \(u\) is equal to the value at \(u\) of the character of the \(O(\nu)\)-module \(j(N \otimes_O i'M)\).

Since \(j(\ell \otimes j') = j = (\ell \otimes j')j\), we have \(j(N \otimes_O i'M) = j(N \otimes_O j'M)\). By [24, 5.6.3], \(\text{Br}_{\mathcal{F}(\nu)}(\text{End}_O(\nu) \otimes (OG')_\gamma)(\ell \otimes j' - j) = 0\). So the \(O(\nu)\)-module \((\ell \otimes j' - j)(N \otimes_O i'M)\) is the direct sum of indecomposable \(O(\nu)\)-modules with vertex properly contained in \(\langle u \rangle\), its character has the value 0 at \(u\), and the value at \(u\) of the character of the \(O(\nu)\)-module \(j(N \otimes_O i'M)\) is equal to the value at \(u\) of the character of the \(O(\nu)\)-module \(\ell(N) \otimes_O i'M\), which is equal to \(\omega(u)(\chi')^{\alpha'}(u)\). Until now, the equality 3.6.5 is proved.

The bijection \(\text{Irr}(G', b') \to \text{Irr}(G, b), \; \chi' \mapsto \chi\) induced by \(\Phi\) can be extended to a perfect isometry \(I_\nu : \text{ZIrr}(G', b') \to \text{ZIrr}(G, b)\) by the \(Z\)-linearity. We define a \(K\)-linear isomorphism

\[
I_u : \text{CF}(C_G'(u), h', K) \to \text{CF}(C_G(u), h, K),
\]

which sends \(\varphi_{\alpha'}\) onto \(\omega(u)\varphi_{\alpha}\) if local pointed elements \(u_\alpha \in \mathcal{P}(u, h)\) and \(u_{\alpha'} \in \mathcal{P}(u, h')\) correspond under the bijection \(\mathcal{P}(u, h) \to \mathcal{P}(u, h')\). By the equalities 3.6.3, 3.6.4 and 3.6.5 we have \(d^\alpha(u, h') \circ I_u^\nu = I_u \circ d^\alpha(u, h')\). In particular, \(I_u^\nu\) is compatible with a local system \(\{I_u|u \in P\}\).
4. Reduction

In this section, we give several lemmas for the reduction of the proof of Theorem.

4.1. Let $G$ be a finite group and $b$ a block of $G$ with maximal $b$-Brauer pair $(P, f)$. Obviously $N_G(P, f)$ acts on the simple algebra $k \otimes_{OZ(P)} OC_G(P)f$ by the conjugation, where $Z(P)$ is the center of $P$. We denote by $\hat{N}_G(P, f)$ the set of all pairs $(x, a_x)$, where $x \in N_G(P, f)$ and $a_x$ is an invertible element of $k \otimes_{OZ(P)} OC_G(P)f$ such that $d^{a_x} = d$ for all $d \in k \otimes_{OZ(P)} OC_G(P)f$. Obviously $\hat{N}_G(P, f)$ is a subgroup of the direct product $\hat{N}_G(P, f) \times (k \otimes_{OZ(P)} OC_G(P)f)^*$. The maps $k^* \to \hat{N}_G(P, f), \lambda \mapsto (1, \lambda)$ and $C_G(P) \to \hat{N}_G(P, f), x \mapsto (x, x(1 \otimes f))$ are injective group homomorphisms.

The image of $k^*$ is central in $\hat{N}_G(P, f)$, the image of $C_G(P)$ is normal in $\hat{N}_G(P, f)$ and they intersect trivially. We identify $k^*$ and $C_G(P)$ with their images through the two homomorphisms. The quotient of $\hat{N}_G(P, f)$ by $k^*$ is isomorphic to $N_G(P, f)$ and $\hat{N}_G(P, f)$ is a central extension of $N_G(P, f)$ by $k^*$. Set $E_G(P, f) = N_G(P, f)/C_G(P)$ and $\hat{E}_G(P, f) = \hat{N}_G(P, f)/C_G(P)$. The quotient group $E_G(P, f)$ is the inertial quotient of the block $b$ if $P$ is abelian. The homomorphism $k^* \to N_G(P, f), \lambda \mapsto (1, \lambda)$ induces a new injective group homomorphism $k^* \to \hat{E}_G(P, f)$, and the quotient of $\hat{E}_G(P, f)$ by the image of $k^*$ is isomorphic to $E_G(P, f)$. So $\hat{E}_G(P, f)$ is a central extension of $E_G(P, f)$ by $k^*$.

4.2. In this section, we always assume that $P$ is abelian and that the central extension $\hat{E}_G(P, f)$ splits. The $N_G(P, f)$-conjugation induces actions of $E_G(P, f)$ on $P$. Set $K = P \rtimes E_G(P, f)$. We say that a perfect isometry $I_\nu : Z(G) \to Z(G, b)$ is compatible with the $*$-structure if $I_\nu (\lambda \ast \chi) = \lambda \ast I_\nu (\chi)$ for any $\chi \in \text{Irr}(K)$ and any $N_G(P, f)$-stable character $\lambda$ of $P$, and that a Morita equivalence between the algebras $OGb$ and $OK$ is a Morita equivalence compatible with the $*$-structure if the Morita equivalence induces a perfect isometry compatible with the $*$-structure.

4.3. Let $N$ be a normal subgroup of $G$ and $h$ a block of $N$ covered by the block $b$. Let $H$ be the stabilizer of $h$ under the $G$-conjugation action. Then there is a suitable block $e$ of $H$ such that $b = \sum_{x} xex^{-1}$, where $x$ runs over a set of representatives of left cosets of $H$ in $G$, and such that $ee^* = 0$ for any $x \in G - H$. The blocks $b$ and $e$ have common defect groups and the block algebra $OGb$ is isomorphic to the induction $\text{Ind}^G_N(OHe)$ of the block algebra $OHe$ (see [29, §16]). In particular, there is a Morita equivalence between the block algebras $OGb$ and $OHe$ induced by an indecomposable $O(G \times H)$-module with trivial sources. We adjust the choice of $P$, so that $P$ is also a defect group of the block $e$. By [22, 7.6.5], $\hat{E}_G(P, f)$ is isomorphic to $\hat{E}_H(P, f')$ as central extensions for some maximal $e$-Brauer pair $(P, f')$. In particular, $E_G(P, f)$ is isomorphic to $E_H(P, f')$. We identify $E_G(P, f)$ and $E_H(P, f')$. Since the central extension $\hat{E}_G(P, f)$ is assumed to split, $\hat{E}_H(P, f')$ splits.

Lemma 4.4. Keep the notation and the assumptions of 4.2 and 4.3. Assume that there is a Morita equivalence between the algebras $OHe$ and $OK$ compatible with the $*$-structure. There is a Morita equivalence between the block algebras $OGb$ and $OK$ compatible with the $*$-structure.

Proof. Since there is a Morita equivalence between the block algebras $OGb$ and $OHe$ induced by an indecomposable $O(G \times H)$-module with trivial sources, by Propositions 2.7 and 3.6, there is a Morita equivalence inducing a perfect isometry $I_\nu : Z(G, e) \to Z(G, b)$ such that $I_\nu (\lambda \ast \chi) = \lambda \ast I_\nu (\chi)$ for any $\chi \in \text{Irr}(H, e)$ and any $N_G(P, f)$-stable character of $P$. The composition of $I_\nu$ and the perfect isometry induced by the Morita equivalence between $OHe$ and $OK$ is a perfect isometry compatible with the $*$-structure, which is induced by the composition of the latter Morita equivalence between $OGb$ and $OHe$ and the Morita equivalence between $OHe$ and $OK$. 

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Lemma 4.5.  Keep the notation and the assumption of 4.2 and 4.3. Set $Q = P \cap N$. Then $Q$ is a defect group of the block $h$. Assume that $H = G$ and that the block $h$ is nilpotent. Then there is a finite group $L$ such that

4.5.1. $Q$ is normal in $L$ and $P$ is a Sylow $p$-subgroup of $L$,
4.5.2. the quotient groups $G/N$ and $L/Q$ are isomorphic, and
4.5.3. there are a $p'$-central extension $\hat{L}$ of $L$ and a block $\ell$ of $\hat{L}$ with defect group $P$, such that the block algebras $OGb$ and $O\hat{L}\ell$ are basically Morita equivalent.

Proof. This lemma follows from [13, Theorem 1.12].

By [22, 7.6.5], $\hat{E}_G(P,f)$ is isomorphic to $\hat{E}_L(P,f')$ as central extensions for some maximal $\ell$-Brauer pair $(P,f')$. In particular, $E_G(P,f)$ is isomorphic to $E_L(P,f')$. We identify $E_G(P,f)$ and $E_L(P,f')$. Since the central extension $\hat{E}_G(P,f)$ is assumed to split, $\hat{E}_L(P,f')$ splits.

Lemma 4.6. Keep the notation and the assumptions in Lemma 4.5. Assume that there is a Morita equivalence between the algebras $OGb$ and $OK$ compatible with the $\ast$-structure. Then there is a Morita equivalence between the algebras $OGb$ and $OK$ compatible with the $\ast$-structure.

Proof. One can prove the lemma, using the same proof method for Lemma 4.4.

5. Extensions of Morita equivalences

In this section, we mainly give an extension of Morita equivalences compatible with the $\ast$-structure (see Proposition 5.9 below).

5.1. Let $G$ be a finite group, $H$ a normal subgroup of $G$ with $p$-power index, and $b$ a $G$-stable block of $H$. Then $b$ is also a block of $G$. We always assume that $b$ as a block of $G$ has abelian defect group $P$. Set $Q = P \cap H$. By [13, Proposition 5.3], $Q$ is a defect group of the block $b$ of $H$ and $G$ is equal to $PH$. Let $(P,f)$ be a maximal Brauer pair of the block $b$ of $G$. Denote by $T$ the subgroup of all $E_G(P,f)$-fixed elements in $P$ and by $R$ the commutator $[P,N_G(P,f)]$. We have $P = T \times R$, $R = [P,N_H(P,f)] \subset H$, and $R$ is contained in $Q$. In particular, we have $G = TH$. Set $E = E_G(P,f)$, $K = P \times E$ and $L = Q \times E$. Then $K = LT$.

Lemma 5.2. Keep the notation and the assumption of 5.1. Any simple module in the block $b$ of $H$ is extendible to $G$ and any irreducible ordinary character in the block $b$ of $H$ is extendible to $G$.

Proof. See [12, Theorem].

We label characters in $\text{Irr}(K)$ as $\zeta_j$, $j \in J$, and choose a subset $I \subset J$, such that all these restrictions $(\zeta_i)_L$, $i \in I$, are exactly all characters in $\text{Irr}(L)$. For any $i \in I$, set $\phi_i = (\zeta_i)_L$. Then $\text{Irr}(L) = \{ \phi_i \mid i \in I \}$.

Lemma 5.3. Keep the notation and the assumption of 5.1. Assume that $I_{\nu} : Z\text{Irr}(K) \rightarrow Z\text{Irr}(G,b)$ is a perfect isometry compatible with the $\ast$-structure. Then there is a perfect isometry $I_{\nu} : Z\text{Irr}(L) \rightarrow Z\text{Irr}(H,b)$ compatible the $\ast$-structure such that the diagram

\[
\begin{array}{ccc}
Z\text{Irr}(K) & \xrightarrow{I_{\nu}} & Z\text{Irr}(G,b) \\
\downarrow & & \downarrow \\
Z\text{Irr}(L) & \xrightarrow{I_{\nu}} & Z\text{Irr}(H,b)
\end{array}
\]
is commutative, where the two down arrows are determined by the restriction of characters. Moreover, \( \nu' \) has a suitable extension \( \tilde{\nu'} \) to \((H \times L)\Delta(P)\) such that \( \nu = \text{Ind}_{(H \times L)\Delta(P)}^{G \times K}(\tilde{\nu'}) \).

**Proof.** Since \( I_{\nu} : \text{ZIrr}(K) \rightarrow \text{ZIrr}(G, b) \) is a perfect isometry compatible with the \(*\)-structure,

\[
I_{\nu}(\lambda * \chi) = \lambda * I_{\nu}(\chi)
\]

for any \( \chi \in \text{ZIrr}(K) \) and any \( N_{G}(P, f) \)-stable character \( \lambda \) of \( P \).

For any \( i, j \in I \), there exist \( \epsilon_{i}, \epsilon_{j} \in \{ \pm 1 \} \) and \( \chi_{i}, \chi_{j} \in \text{Irr}(G, b) \), such that \( \epsilon_{i}\chi_{i} = I_{\nu}(\zeta_{i}) \) and \( \epsilon_{j}\chi_{j} = I_{\nu}(\zeta_{j}) \). By Lemma 5.2, the restrictions \( \psi_{i} \) and \( \psi_{j} \) of \( \chi_{i} \) and \( \chi_{j} \) to \( H \) are irreducible. Suppose \( \psi_{i} = \psi_{j} \) for different \( i \) and \( j \). Set \( S = T \cap H \). The quotient group \( G/H \) is naturally isomorphic to \( T/S \). Through the isomorphism, any linear character \( \lambda \) of \( T \) with kernel containing \( S \) can be regarded as a character of \( G \) with kernel containing \( H \). Since \( \psi_{i} = \psi_{j} \), there exists a linear character \( \lambda \) of \( T \) with kernel containing \( S \), such that \( \chi_{i} = \chi_{j} \). Obviously \( \lambda \) is \( E_{G}(P, f) \)-stable, and by [6, Corollary] \( \chi_{j}\lambda \) is equal to \( \lambda * \chi_{j} \). By 5.3.2, \( \lambda\epsilon_{j}\zeta_{j} = \epsilon_{i}\zeta_{i}, \epsilon_{i} = \epsilon_{j} \) and \( \zeta_{i} = \zeta_{j} \). Thus we have \( \phi_{i} = \phi_{j} \). That contradicts the characters \( \phi_{i} \) and \( \phi_{j} \) being different for different \( i \) and \( j \). On the other hand, denoting by \( \text{Irr}(T/S) \) the set of irreducible ordinary characters of \( T \) with kernel containing \( S \), by 5.3.2, we have \( \text{Irr}(G, b) = \{ \lambda\chi_{i} \mid i \in I; \lambda \in \text{Irr}(T/S) \} \). Therefore \( \text{Irr}(H, b) = \{ \psi_{i} \mid i \in I \} \).

By 5.3.2, the perfect isometry \( I_{\nu} \) sends \( \lambda\zeta_{i} \) onto \( \epsilon_{i}\lambda\chi_{i} \) for any \( i \in I \) and any \( \lambda \in \text{Irr}(T/S) \).

So \( \nu = \sum_{i \in I} \sum_{\lambda \in \text{Irr}(T/S)} \epsilon_{i}(\lambda\chi_{i}) \times (\lambda\zeta_{i})^{*} \). Set \( \nu' = \sum_{i \in I} \epsilon_{i}(\chi_{i} \times \zeta_{i}^{*}) \). Then \( \nu = \text{Ind}_{(H \times L)\Delta(P)}^{G \times K}(\nu') \).

Considering \( \nu' \) as a generalized character on \( H \times L \) by restriction, the map \( I_{\nu'} : \text{ZIrr}(L) \rightarrow \text{ZIrr}(H, b) \) sends \( \phi_{i} \) onto \( \epsilon_{i}\psi_{i} \) for any \( i \in I \) and it is a bijective isometry. Moreover, it is trivial to see that \( I_{\nu} \) and \( I_{\nu'} \) satisfies the diagram 5.3.1. By the following lemma, the proof is done.

**Lemma 5.4.** Keep the notation and the assumptions in Lemma 5.3 and its proof. Then the map \( I_{\nu'} \) is a perfect isometry compatible with the \(*\)-structure.

**Proof.** Let \((x, f_{(x)})\) be a \( b \)-Brauer element contained in \((P, f)\). The pair \((P, f)\) is also a maximal \( b_{(x)} \)-Brauer pair and the quotient group \( N_{C_{G}(x)}(P, f)/C_{G}(P) \) is isomorphic to \( C_{E}(x) \). Set \( K_{(x)} = P \times C_{E}(x) \). By Proposition 2.7, there is a \( K \)-linear isomorphism \( I_{(x)} : C_{F_{\nu}'}(K_{(x)}, K) \rightarrow C_{F_{\nu}'}(C_{G}(x), b_{(x)}, K) \), such that

\[
I_{(x)} \circ d_{K}^{(x, b_{(x)})} = d_{G}^{(x, b_{(x)})} \circ I_{\nu}^{K}.
\]

Denote by \( \text{Res}_{H}^{G} \) the restriction map from \( C_{F}(G, b, K) \rightarrow C_{F}(H, b, K) \) and by \( \text{Res}_{H, p'}^{G} \) the restriction map from \( C_{F_{\nu}'}(G, b, K) \rightarrow C_{F_{\nu}'}(H, b, K) \). By Lemma 5.2, \( \text{Res}_{H}^{G} \) is surjective and \( \text{Res}_{H, p'}^{G} \) is a \( K \)-linear isomorphism.

Assume that \( x \) lies in \( Q \). Note that \( b_{(x)} \) is a block of \( C_{H}(x) \) with defect group \( Q \). We set

\[
L_{(x)} = Q \times C_{E}(x) \quad \text{and} \quad I'_{(x)} = \text{Res}_{C_{H}(x), p'}^{C_{G}(x)} \circ I_{(x)} \circ (\text{Res}_{L_{(x)}, p'}^{K})^{-1}.
\]

We claim that \( d_{H}^{(x, b_{(x)})} \circ I'_{\nu} = I'_{(x)} \circ d_{L}^{(x, b_{(x)})} \). Indeed, it is easy to prove that the equalities

\[
\text{Res}_{C_{H}(x), p'}^{C_{G}(x)} d_{G}^{(x, b_{(x)})} = d_{H}^{(x, b_{(x)})} \circ \text{Res}_{H}^{G} \quad \text{and} \quad \text{Res}_{L_{(x)}, p'}^{K} d_{L}^{(x, b_{(x)})} = d_{L}^{(x, b_{(x)})} \circ \text{Res}_{L}^{K}.
\]
hold. Since
\[
d^{z,b(x)}_H \circ I'_{\nu} \circ \text{Res}_L^K = \left(d^{z,b(x)}_H \circ \text{Res}_H^G \circ I'_{\nu}\right) = \left(\text{Res}_{C_G(x), \nu'} \circ d_G^{z,b(x)} \circ I'_{\nu}\right) = \left(\text{Res}_{C_G(x), \nu'} \circ I(x) \circ d_K^x\right) = I'(x) \circ \text{Res}_{L(x), \nu'} \circ d_K^x = I'(x) \circ d_L^x \circ \text{Res}_L^K
\]
where the first equality is obtained by 5.3.1, the second by 5.4.3, the third by 5.4.1, the fourth by 5.4.2 and the fifth by 5.4.3, the claim is done. Therefore $I^K_{\nu}$ is compatible with a local system $\{I'_{\nu}(x) \mid x \in P\}$ in the sense of [4, Definition 4.3]. By Proposition 2.7, $I^K_{\nu}$ is compatible with the $\ast$-structure.

In order to prove that $I_{\nu}$ is a perfect isometry, by [4, Lemma 4.5], it remains to prove that $I'_{\nu}(x)$ maps $\text{CF}_{\nu'}(L(x), O)$ onto $\text{CF}_{\nu'}(C_H(x), b(x), O)$. Since $I'(x) = \text{Res}_{C_H(x), \psi} \circ I_{\nu} \circ (\text{Res}_{L(x), \nu'})^{-1}$, it suffices to show that $I'_{\nu}$ maps $\text{CF}_{\nu'}(K(x), O)$ onto $\text{CF}_{\nu'}(C_G(x), b(x), O)$. This follows from Lemma 2.8.

Set $L' = R \times E$. Then $L$ is equal to the direct product $L' \times T'$, where $T'$ is the subgroup of all $E$-fixed elements in $Q$. Let $I_{\psi} : \text{ZIrr}(L') \rightarrow \text{ZIrr}(L)$ be a perfect isometry compatible with the $\ast$-structure. Since any irreducible ordinary character of $L$ is of the form $\phi \times \lambda$, where $\phi \in \text{Irr}(L')$ and $\lambda \in \text{Irr}(T')$, we extend $I_{\psi}$ to a $Z$-linear map $I_{\psi} : \text{ZIrr}(L) \rightarrow \text{ZIrr}(L)$ sending $\phi \times \lambda$ onto $I_{\psi}((\phi) \times \lambda)$ for any $\phi \in \text{Irr}(L')$ and any $\lambda \in \text{Irr}(T')$. Set $\psi' = \psi' \times 1_{\Delta(T')}$, since $(L' \times L') \Delta(T') = (L' \times L') \times \Delta(T')$.

**Lemma 5.5.** Keep the notation and the assumption as above. Then $I_{\psi}$ is a perfect isometry compatible with the $\ast$-structure, $\psi = \text{Ind}_{(L' \times L') \ast(T')}^{L \times L}(\tilde{\psi})'$, and the diagram

\[
\begin{array}{ccc}
\text{ZIrr}(L) & \longrightarrow & \text{ZIrr}(L) \\
\downarrow & & \downarrow \\
\text{ZIrr}(L') & \longrightarrow & \text{ZIrr}(L')
\end{array}
\]

is commutative, where the downarrows are induced by the restriction of characters.

**Proof.** The last statement of the lemma is obvious. Since
\[
\psi = \sum_{\phi \in \text{Irr}(L')} \sum_{\lambda \in \text{Irr}(T')} (I_{\psi'}((\phi) \times \lambda) \times (\phi^* \times \lambda^*)) = \psi' \times \left(\sum_{\lambda \in \text{Irr}(T')} \lambda \times \lambda^*\right),
\]
it is easy to see $\psi = \text{Ind}_{(L' \times L') \ast(T')}^{L \times L}(\tilde{\psi})'$. It remains to show that $I_{\psi}$ is a perfect isometry compatible with the $\ast$-structure.

Since any irreducible ordinary character of $L$ is of the form $\phi \times \lambda$, where $\phi \in \text{Irr}(L')$ and $\lambda \in \text{Irr}(T')$, there is an obvious group isomorphism
\[
\text{ZIrr}(L) \cong \text{ZIrr}(L') \bigotimes_{Z} \text{ZIrr}(T').
\]
We identify $\text{ZIrr}(L)$ and $\text{ZIrr}(L') \bigotimes_{Z} \text{ZIrr}(T')$ through the isomorphism. Since $Q = R \times T'$, we also have a $\mathcal{K}$-linear isomorphism
\[
\text{CF}_{\nu'}(Q, \mathcal{K}) \cong \text{CF}_{\nu'}(R, \mathcal{K}) \bigotimes_{\mathcal{K}} \text{CF}_{\nu'}(T', \mathcal{K}), 1^Q_R \mapsto 1^R_R \bigotimes 1^T_T.
\]
where $1^\circ_Q$ denotes the unique irreducible Brauer character on $Q$. We identify $\text{CF}_{p'}(Q, \mathcal{K})$ and $\text{CF}_{p'}(R, \mathcal{K}) \otimes_{\mathcal{K}} \text{CF}_{p'}(T', \mathcal{K})$. With the identifications as above, we have

5.5.1

$$I_{\psi} = I_{\psi} \otimes \text{Id}_{\text{Irr}(T')}$$

where $\text{Id}_{\text{Irr}(T')}$ is the identity map on $\text{Irr}(T')$, and

5.5.2

$$d^{xu}_L = d^x_L \otimes d^u_{T'}$$

for any $x \in R$ and any $u \in T'$. On the other hand, since $I_{\psi'}$ is a perfect isometry compatible with the $*$-structure, by Proposition 2.7, for any $x \in R$, there is a $K$-linear map $I_x : \text{CF}_{p'}(C_L'(x), \mathcal{K}) \to \text{CF}_{p'}(C_L'(x), \mathcal{K})$ such that

5.5.3

$$d^x_L \circ I^K_{\psi'} = d^x_L \circ I_x.$$

Now set $I_{xu} = I_x \otimes_K \text{Id}_{\text{CF}_{p'}(T', \mathcal{K})}$, where $\text{Id}_{\text{CF}_{p'}(T', \mathcal{K})}$ is the identity map on $\text{CF}_{p'}(T', \mathcal{K})$. By 5.5.1, 5.5.2 and 5.5.3, we have $I_{xu} \circ d^{xu}_L = d^{xu}_L \circ I^K_{\psi'}$. By Proposition 2.7, $I_{\psi}$ is compatible with the $*$-structure. Since $I_{\psi'}$ is a perfect isometry, by Lemma 2.8, $I_x$ maps $\text{CF}_{p'}(C_L'(x), \mathcal{O})$ onto $\text{CF}_{p'}(C_L'(x), \mathcal{O})$. Thus for any $x \in R$ and $u \in T'$, $I_{xu}$ maps $\text{CF}_{p'}(C_L(xu), \mathcal{O})$ onto $\text{CF}_{p'}(C_L(xu), \mathcal{O})$. By [4, Lemma 4.5], $I_{\psi}$ is a perfect isometry.

**Lemma 5.6.** Keep the notation and the assumption of 5.1. Assume that $E$ is cyclic and acts freely on $R - \{1\}$ and that the map $I_{\psi} : \text{Irr}(L) \to \text{Irr}(L)$ is a perfect isometry compatible with the $*$-structure. Then $\psi$ has a suitable extension $\tilde{\psi}$ to $(L \times L)\Delta(K)$, such that setting $\tilde{\psi} = \text{Ind}_{(L \times L)\Delta(K)}^{K \times K}(\tilde{\psi})$, the map $I_{\tilde{\psi}} : \text{Irr}(K) \to \text{Irr}(K)$ is a perfect isometry compatible with the $*$-structure and makes the following diagram commutative

$$\begin{array}{ccc}
\text{Irr}(K) & \longrightarrow & \text{Irr}(K) \\
\downarrow & & \downarrow \\
\text{Irr}(L) & \longrightarrow & \text{Irr}(L)
\end{array}$$

where the downarrows are induced by the restriction of characters.

**Proof.** By Lemma 5.3, there is a generalized character $\psi'$ of $(L' \times L')\Delta(T')$ such that $\psi = \text{Ind}_{(L' \times L')\Delta(T')}^{L \times L}(\psi')$ and such that, restricting $\psi'$ to $L' \times L'$, the map $I_{\psi'} : \text{Irr}(L') \to \text{Irr}(L')$ is a perfect isometry compatible with the $*$-structure and makes the following diagram commutative

$$\begin{array}{ccc}
\text{Irr}(L) & \longrightarrow & \text{Irr}(L) \\
\downarrow & & \downarrow \\
\text{Irr}(L') & \longrightarrow & \text{Irr}(L')
\end{array}$$

where the two down arrows are determined by the restriction of characters.

Set $\tilde{\psi}' = \psi' \times 1_{\Delta(T')}$ and $\tilde{\psi}'' = \text{Ind}_{(L' \times L')\Delta(T')}^{L \times L}(\tilde{\psi}')$. By Lemma 5.5 applied to $I_{\psi''}$ and the group $L$, the map $I_{\psi''} : \text{Irr}(L) \to \text{Irr}(L)$ is a perfect isometry compatible with the $*$-structure and the diagram

$$\begin{array}{ccc}
\text{Irr}(L) & \longrightarrow & \text{Irr}(L) \\
\downarrow & & \downarrow \\
\text{Irr}(L') & \longrightarrow & \text{Irr}(L')
\end{array}$$
is commutative where the downarrows are induced by the restriction of characters. Note \((L' \times L')\Delta(K) = (L' \times L') \times \Delta(T)\). Set \(\tilde{\psi}' = \psi' \times 1_{\Delta(T)}\) and \(\psi''' = \Ind_{(L' \times L')\Delta(T)}^{K \times K} (\tilde{\psi}')\). Similarly, the map 
\(I_{\psi''' : Z\text{Irr}(K) \to Z\text{Irr}(K)}\) is a perfect isometry compatible with the \(*\)-structure and the diagram

\[
\begin{array}{ccc}
Z\text{Irr}(K) & \longrightarrow & Z\text{Irr}(K) \\
\downarrow & & \downarrow \\
Z\text{Irr}(L') & \longrightarrow & Z\text{Irr}(L')
\end{array}
\]

is commutative, where the downarrows are induced by the restriction of characters.

Notice that the restriction of \(\tilde{\psi}'\) to \((L' \times L')\Delta(T')\) is equal to \(\tilde{\psi}'\). Set \(\tilde{\psi}'' = \Ind_{(L' \times L')\Delta(T)}^{K \times K} (\tilde{\psi}')\).

The restriction to \(L \times L\) of \(\tilde{\psi}''\) is equal to \(\psi''\) and \(\Ind_{(L \times L)\Delta(T)}^{K \times K} (\tilde{\psi}'')\) is equal to \(\psi'''\). Moreover, \(I_{\psi''}\) and \(I_{\psi'''}\) are perfect isometries compatible with the \(*\)-structure.

In order to prove the claim, we take \(\phi \in Z\text{Irr}(L')\) and set \(\tilde{\psi}' = \Ind_{(L' \times L')\Delta(T)}^{K \times K} (\tilde{\psi}')\) to verify that \(R\) acts freely on \(\tilde{\psi}'\) and fix an extension \(\tilde{\psi}\) of \(\tilde{\psi}'\). For any \(v \in T'\), \(\tilde{\psi}\) is a perfect isometry compatible with the \(*\)-structure, there is a \(K\)-linear isomorphism \(I_{(uv)} : CF_{p'}(Q, K) \to CF_{p'}(Q, K)\) such that

\[
I_{(uv)} \circ d_{L'}^{K'p'} = d_{L'}^{K'p'} \circ I_{\tilde{\psi}''}.
\]

Applying both sides of the equality to \(\phi \times 1_{T'}\) and \(\phi' \times 1_{T'}\), we get \(I_{(uv)}(\phi(u)1_{Q}) = \phi(v)\lambda_{d}(v)1_{Q}\), \(I_{(uv)}(\phi'(u)1_{Q}) = \phi'(u)\lambda_{d'}(v)1_{Q}\), and thus \(\lambda_{d}(v) = \lambda_{d'}(v)\) for any \(v \in T'\). The claim is done.

We set \(\pi = \lambda_{1_{T'}}\) and an extension \(\pi\) of \(\pi\) to \(T\). We extend \(I_{\tilde{\psi}}\) to a \(Z\)-linear map \(I_{\tilde{\psi}} : Z\text{Irr}(K) \to Z\text{Irr}(K)\) sending \(\phi \times \lambda\) onto \(\phi \times \pi \lambda\) for any \(\phi \in Z\text{Irr}(L')\) and any \(\lambda \in Z\text{Irr}(T)\). It is trivial to verify that \(I_{\tilde{\psi}}\) is a perfect isometry compatible with the \(*\)-structure, that \(\tilde{\psi} = \Ind_{(L' \times L')\Delta(T)}^{K \times K} (\tilde{\psi}')\) for some suitable extension \(\tilde{\psi}\) of \(\psi\) to \((L \times L)\Delta(T)\), and that \(I_{\tilde{\psi}}\) and \(I_{\tilde{\psi}}\) satisfy the diagram 5.6.1.

**Proposition 5.7.** Keep the notation and the assumption of 5.1 and assume that \(E\) is cyclic and acts freely on \(R - \{1\}\) and that \(I_{\nu : Z\text{Irr}(L) \to Z\text{Irr}(H, b)}\) is a perfect isometry compatible with the \(*\)-structure. Then there is a suitable extension \(\nu'\) of \(\nu\) to \((H \times L)\Delta(P)\) such that setting \(\psi = \Ind_{(H \times L)\Delta(P)}^{K \times K} (\nu)\), the map \(I_{\nu : Z\text{Irr}(K) \to Z\text{Irr}(G, b)}\) is a perfect isometry compatible with the \(*\)-structure and the diagram is commutative:

\[
\begin{array}{ccc}
Z\text{Irr}(K) & \longrightarrow & Z\text{Irr}(G, b) \\
\downarrow & & \downarrow \\
Z\text{Irr}(L) & \longrightarrow & Z\text{Irr}(H, b)
\end{array}
\]
where the down arrows are induced by the restriction of characters.

**Proof.** Denote by $\ell_G(b)$ the number of isomorphism classes of simple modules in the block $b$ of $G$. Since $\ell_H(b)$ is equal to $|E_G(P, f)|$ and $\ell_G(b)$ is equal to $\ell_H(b)$ (see Lemma 5.2), $\ell_G(b)$ is equal to $E_G(P, f)$. By [31, Theorem 1], there is a perfect isometry $\text{ZIrr}(K) \to \text{ZIrr}(G, b)$ compatible with the $*$-structure. By Lemma 5.3, there is a perfect isometry $I_{\mu'} : \text{ZIrr}(L) \to \text{ZIrr}(H, b)$ compatible with the $*$-structure, $\mu = \text{Ind}^{G \times K}_{(H \times L)\Delta(P)}(\hat{\mu'})$ for some extension $\hat{\mu'}$ of $\mu'$ to $(H \times L)\Delta(P)$, and the diagram

\[
\begin{array}{ccc}
\text{ZIrr}(K) & \xrightarrow{I_{\mu}} & \text{ZIrr}(G, b) \\
\downarrow & & \downarrow \\
\text{ZIrr}(L) & \xrightarrow{I_{\mu'}} & \text{ZIrr}(H, b)
\end{array}
\]

is commutative, where the two down arrows are determined by the restriction of characters.

Obviously $I_{\mu'}^{-1} \circ I_{\nu'}$ is a perfect isometry compatible with the $*$-structure. Suppose $I_{\nu'} = I_{\mu'}^{-1} \circ I_{\nu'}$. By Lemma 5.6, $\eta'$ has an extension $\tilde{\eta}'$ to $(L \times L)\Delta(K)$, such that setting $\eta = \text{Ind}^{K \times K}_{(L \times L)\Delta(K)}(\tilde{\eta}')$, the map $I_{\eta} : \text{ZIrr}(K) \to \text{ZIrr}(K)$ is a perfect isometry compatible with the $*$-structure and makes the following diagram commutative

\[
\begin{array}{ccc}
\text{ZIrr}(K) & \xrightarrow{I_{\eta}} & \text{ZIrr}(K) \\
\downarrow & & \downarrow \\
\text{ZIrr}(L) & \xrightarrow{I_{\nu'}} & \text{ZIrr}(L)
\end{array}
\]

where the downarrows are induced by the restriction of characters. Now we compose $I_{\eta}$ and $I_{\eta'}$ and suppose $I_{\eta} = I_{\mu} \circ I_{\eta'}$. Then $I_{\eta}$ is a perfect isometry compatible with the $*$-structure, and it makes commutative the diagram in the proposition. Notice that $I_{\eta}$ uniquely determines $I_{\nu'}$. By Lemma 5.3, $\nu'$ has an extension $\tilde{\nu}'$ to $(H \times L)\Delta(P)$ and $\nu = \text{Ind}^{G \times K}_{(H \times L)\Delta(P)}(\tilde{\nu}')$.

**5.8.** Assume that there are perfect isometries

$I_{\nu} : \text{ZIrr}(K) \to \text{ZIrr}(G, b)$ and $I_{\nu'} : \text{ZIrr}(L) \to \text{ZIrr}(H, b)$,

such that $\nu = \text{Ind}^{G \times K}_{(H \times L)\Delta(P)}(\tilde{\nu}')$ for a suitable extension $\tilde{\nu}'$ of $\nu'$ to $(H \times L)\Delta(P)$, and such that $I_{\nu}$ and $I_{\nu'}$ make the following diagram commutative

\[
\begin{array}{ccc}
\text{ZIrr}(K) & \rightarrow & \text{ZIrr}(G, b) \\
\downarrow & & \downarrow \\
\text{ZIrr}(L) & \rightarrow & \text{ZIrr}(H, b)
\end{array}
\]

where the downarrows are induced by the restrictions of characters. By [4, Theorem 1.5], the perfect isometries $I_{\nu}$ and $I_{\nu'}$ induce algebra isomorphisms

$$\pi : Z(OL) \cong Z(OHb) \quad \text{and} \quad \tau : Z(OK) \cong Z(OGb).$$

Obviously $Z(OGb)$ and $Z(OK)$ are $P/Q$-graded algebras with the $\bar{u}$-components $Z(OGb) \cap OHu$ and $Z(OK) \cap OLu$, where $\bar{u} \in P/Q$ and $u$ is a representative of $\bar{u}$ in $P$.

**Lemma 5.9.** Keep the notation and the assumption of 5.8. Then $\tau$ is a $P/Q$-graded algebra isomorphism and $\pi$ is the restriction of $\tau$ to the 1-components of $Z(OGb)$ and $Z(OK)$.

**Proof.** For any $r \in Z(KGb)$ and any $s \in Z(KK)$, denote by $r(g)$ the class function on $G$ such that $r = \sum_{g \in G} r(g^{-1})g$, and by $s(t)$ the class function on $K$ such that $s = \sum_{t \in K} s(t^{-1})t$. We
define a $\mathcal{K}$-linear map $R^*_\nu : Z(\mathcal{K}G) \to Z(\mathcal{K}K)$, $r \mapsto \sum_{t \in K} \left( \frac{1}{|T|} \sum_{g \in G} \nu(g, t)r(g) \right) t^{-1}$, and a $\mathcal{K}$-linear map $I^*_\nu : Z(\mathcal{K}K) \to Z(\mathcal{K}G)$, $s \mapsto \sum_{g \in G} \left( \frac{1}{|T|} \sum_{t \in K} \nu(g, t)s(t) \right) g^{-1}$. By [4, Theorem 1.5], the isomorphism $\tau$ maps $r$ onto $I^*_\nu(rR^*_\nu(b))$ for any $r \in Z(\mathcal{O}K)$. Since $\nu = \text{Ind}_{(H \times L)\Delta(P)}^{G \times K}(\nu')$, it is trivial to prove that $I^*_\nu$ and $R^*_\nu$ are $P/Q$-graded linear maps and thus that $\tau$ is a $P/Q$-graded algebra isomorphism.

We extend the isomorphisms $\tau$ and $\pi$ by the $\mathcal{K}$-linearity, and get $\mathcal{K}$-algebra isomorphisms $\tau^K : Z(\mathcal{K}K) \cong Z(\mathcal{K}G)$ and $\pi^K : Z(\mathcal{K}L) \cong Z(\mathcal{K}Hb)$. Note that $Z(\mathcal{K}L)$ is the 1-component of $Z(\mathcal{K}K)$. Since $\tau^K(Z(\mathcal{K}L)) = Z(\mathcal{K}G)$ and $Z(\mathcal{K}Hb)$, and the algebras $Z(\mathcal{K}L)$ and $Z(\mathcal{K}Hb)$ have the same $\mathcal{K}$-dimension, we have $\tau^K(Z(\mathcal{K}L)) = Z(\mathcal{K}b)$.

Since $P$ is abelian, by Lemma 5.2, any character $\chi \in \text{Irr}(H, b)$ is extendible to $G$. Consequently, for any $\chi \in \text{Irr}(H, b)$ and any $\hat{\chi} \in \text{Irr}(G, b)$, $\hat{\chi}$ is an extension of $\chi$ if and only if $e_{\chi}e_{\hat{\chi}} \neq 0$.

Fix $\chi \in \text{Irr}(L)$ and $\chi \in \text{Irr}(K)$ such that $\hat{\chi}$ is an extension of $\chi$. Since $e_{\chi}$ is primitive in $Z(\mathcal{K}L)$, $\tau^K(e_{\chi}) = e_\varphi$ for a character $\varphi \in \text{Irr}(H, b)$. Similarly, $\tau^K(e_{\hat{\chi}}) = e_{\hat{\varphi}}$ for some $\hat{\varphi} \in \text{Irr}(G, b)$. By the proof of [4, Theorem 1.5], $I^*_\nu(\hat{\chi}) = \pm \hat{\varphi}$. Then we use the diagram 5.8.1 and obtain $I^*_\nu(\chi) = \pm \varphi$. Again, by the proof of [4, Theorem 1.5], we have $\tau^K(e_{\chi}) = e_\varphi$. The proof is done.

**Proposition 5.10.** Keep the notation and the assumption of 5.1. Assume that $E$ is cyclic and acts freely on $R - \{1\}$ and that there is a Morita equivalence between $\mathcal{O}b$ and $\mathcal{O}L$ compatible with the $*$-structure. Then there is a Morita equivalence between $\mathcal{O}Gb$ and $\mathcal{O}K$ compatible with the $*$-structure.

**Proof.** We suppose that the $\mathcal{O}(H \times L)$-module $M$ induces the Morita equivalence between $\mathcal{O}b$ and $\mathcal{O}L$, and denote by $\mu'$ the character of $M$. Then the map $I^*_\mu : \mathcal{Z}_\text{Irr}(L) \to \mathcal{Z}_\text{Irr}(H, b)$ is a perfect isometry compatible with the $*$-structure, which induces an algebra isomorphism

$$
\sigma : Z(\mathcal{O}L) \cong Z(\mathcal{O}b).
$$

By Proposition 5.7, the character $\mu'$ has a suitable extension $\tilde{\mu}'$ to $(H \times L)\Delta(P)$ such that setting $\mu = \text{Ind}_{(H \times L)\Delta(P)}^{G \times K}(\tilde{\mu}')$, the map $I^*_\mu : \mathcal{Z}_\text{Irr}(K) \to \mathcal{Z}_\text{Irr}(G, b)$ is a perfect isometry compatible with the $*$-structure and the diagram

$$
\mathcal{Z}_\text{Irr}(K) \xrightarrow{I^*_\mu} \mathcal{Z}_\text{Irr}(G, b) \\
\downarrow \quad \downarrow \\
\mathcal{Z}_\text{Irr}(L) \xrightarrow{I^*_\nu} \mathcal{Z}_\text{Irr}(H, b).
$$

is commutative, where the two down arrows are induced by the restrictions of characters. Since $I^*_\mu$ maps ordinary characters of $L$ onto ordinary characters of $H$, by the diagram, $I^*_\mu$ maps ordinary characters of $K$ onto ordinary characters of $G$. In particular, $\mu$ is an ordinary character of $G \times K$.

The map $I^*_\mu$ induces an algebra isomorphism

$$
\zeta : Z(\mathcal{O}K) \cong Z(\mathcal{O}b).
$$

By Lemma 5.9, the isomorphism $\zeta$ is a $P/Q$-graded algebra isomorphism and the restriction of $\zeta$ to the 1-component of $Z(\mathcal{O}b)$ coincides with $\sigma$. Since $T$ is contained in $Z(\mathcal{O}K)$, for any $t \in T$, $\zeta(t)$ is contained in the $\tilde{t}$-component of $Z(\mathcal{O}b)$ and $t\zeta(t^{-1})$ is inside $\mathcal{O}b$. The map $\rho : T \to (\mathcal{O}b)^*, t \mapsto t \zeta(t^{-1})$ is a group homomorphism. Now we extend the $\mathcal{O}(H \times L)$-module $M$ to $(H \times L)\Delta(P)$ by the equality $(t, t) \cdot m = \rho(t)m$ for any $t \in T$ and any $m \in M$. We claim
that such an extension is well defined. Since \( O(H \times L) \)-module \( M \) induces the Morita equivalence between \( OUb \) and \( OL \), by Lemma 2.10, we have \( \sigma(a)m = ma \) for any \( a \in Z(OL) \) and any \( m \in M \). So for any \( t \in T \cap H \) and any \( m \in M \), we have

\[
(t, t)m = tm = tσ(t^{-1})m = t\zeta(t^{-1})m = (t, t) \cdot m.
\]

The claim is done. Now by [14, Theorem 3.4 (a)], the \( O(G \times K) \)-module \( \text{Ind}_{(H \times L)^{\Delta(P)}}^{G \times K} (M) \) induces a Morita equivalence between \( OGb \) and \( OK \). Now it remains to show that the Morita equivalence is compatible with the \( \ast \)-structure.

Set \( \tilde{M} = \text{Ind}_{(H \times L)^{\Delta(P)}}^{G \times K} (M) \). The Morita equivalence induced by \( \tilde{M} \) induces an algebra isomorphism (see Paragraph 2.9)

\[
ρ : Z(OK) \cong Z(OGb)
\]

such that for any \( a \in Z(OGb) \) and any \( a' \in Z(OK) \), \( a \) and \( a' \) correspond to each other if and only if \( am = ma' \) for any \( m \in \tilde{M} \). We claim that \( ρ \) and \( ς \) are the same. For any \( a \in Z(OL) \) and any \( m \in M \), we have

\[
(σ(a) \otimes 1)m = (1 \otimes 1)σ(a)m = (1 \otimes 1)ma = (1 \otimes a^o) \otimes m,
\]

where \( a^o \) is the image of \( a \) through the opposite ring isomorphism \( OL \to OL \) sending \( x \) onto \( x^{-1} \) for any \( x \in L \). So \( ρ(a) = σ(a) = ς(a) \). On the other hand, given \( t \in T \) and \( m \in M \), we have

\[
(ς(t) \otimes t)m = ((t \otimes t)(t^{-1}ς(t) \otimes 1)) \otimes m = (t \otimes t) \otimes t^{-1}ς(t)m = (1 \otimes 1) \otimes m
\]

and thus \( (ς(t) \otimes 1)m = (1 \otimes t^{-1})m \). So \( ρ \) maps \( t \) onto \( ς(t) \). Since \( Z(OK) = ∑_{t ∈ T} Z(OL)t \), the claim is done.

Finally, we claim that the character of \( \text{Ind}_{(H \times L)^{\Delta(P)}}^{G \times K} (M) \) is equal to \( ρ \). In particular, the Morita equivalence between \( OGb \) and \( OK \) induced by \( \text{Ind}_{(H \times L)^{\Delta(P)}}^{G \times K} (M) \) induces a perfect isometry compatible with the \( \ast \)-structure. Indeed, since the isomorphisms \( ρ \) and \( ς \) coincide with each other, setting \( ρ(ε_χ) = ε_χ' \) for any \( χ \in \text{Irr}(K) \), by the proof of [4, Theorem 1.5], both \( ρ \) and the character of \( \tilde{M} \) are equal to the sum \( ∑_{χ ∈ \text{Irr}(K)} χ' \times χ^\ast \). The claim is done.

6. Proof of the Theorem

In this section, we prove Theorem in the introduction. We borrow the notation and the assumption in Theorem. Note that \( p \) is 2 and that the defect group \( P \) is abelian. However, some results, such as Lemmas 6.2, 6.5 and 6.6, hold in general. Let \( (P, f) \) be a maximal Brauer pair associated with the block \( b \). The inertial quotient \( E_G(P, f) \) may be of order 1, 3, 7 or 21.

**Lemma 6.1.** Let \( n \) be a nilpotent block of a normal subgroup \( N \) of the group \( G \) covered by the block \( b \), and set \( N' = NZ(G) \). Then there is a nilpotent block \( n' \) of \( N' \) covered by the block \( b \).

**Proof.** We consider the group homomorphism \( N \times Z(G) \to N', (x, y) \mapsto xy \) and the induced algebra homomorphism \( O(N \times Z(G)) \to ON' \). Since the homomorphism \( O(N \times Z(G)) \to ON' \) maps \( n \otimes 1 \) onto \( n \), by the same proof method as that of Lemma 2.1, we prove that any block of \( N' \) covering \( n \) is the image of the block \( n \otimes w \) of \( N \times Z(G) \) for some suitable block \( w \) of \( Z(G) \), that its inertial quotient has order 1, and that it is nilpotent.
Lemma 6.2. Assume that there is a finite group $\tilde{G}$, such that $G$ is normal in $\tilde{G}$ and $\tilde{G}$ has a nilpotent block $\tilde{b}$ covering $b$. Then $O\tilde{b}$ and $O_{\tilde{G}}(P)c$ are basically Morita equivalent.

Proof. By the main theorem of [17], the block $b$ is inertial. That is to say, $O\tilde{b}$ and $O_{\tilde{G}}(P)c$ are basically Morita equivalent.

Lemma 6.3. Let $G$ be a finite group and $N$ a normal subgroup of $G$ with index an odd prime. Let $b$ be a block of $G$ with defect group $P$, and $n$ a $G$-stable block of $N$ covered by $b$. Assume that $P$ is $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$, that the inertial quotient of the block $b$ is of order 7 or 21 and that the block $n$ is not nilpotent. Then the inertial quotient of the block $n$ is of order 7 or 21.

Proof. Since $N$ is a normal subgroup of $G$ with index an odd prime and $b$ covers $n$, $P$ is also a defect group of $n$; moreover, we can choose a maximal $n$-Brauer pair $(P, h)$ so that $f$ covers $h$.

Suppose that $N$ contains $C_G(P)$. Then $f = h$, $N_N(P, h)/C_N(P)$ is a normal subgroup of $N_G(P, f)/C_G(P)$, and $N_N(P, h)/C_N(P)$ has order 7 or 21 since the block $n$ is not nilpotent.

Suppose that $N$ does not contain $C_G(P)$ and that $C_G(P) = C_G(P, h)$, where $C_G(P, h)$ is the intersection of $C_G(P)$ and the stabilizer $N_G(P, h)$ of $(P, h)$ under the $G$-conjugation. Then $h$ is a central idempotent in $\mathcal{O}_{C_G(P)}(P)$ and we have $h^f = h$. For any $x \in N_G(P, f)$, we have $xhx^{-1} = f$ and $xhx^{-1} = h$ and $x \in N_G(P, h)$. So $N_G(P, f) \leq N_G(P, h)$. Since the index of $N$ in $G$ is an odd prime and $N$ does not contain $C_G(P)$, $G = NC_G(P)$ and $N_G(P, h) = N_N(P, h)cG(P)$. The inclusion $N_G(P, f) \subset N_G(P, h)$ induces an injective group homomorphism

$$N_G(P, f)/C_G(P) \rightarrow N_G(P, h)/C_G(P) \cong N_N(P, h)/C_N(P).$$

This implies that the inertial quotient of the block $n$ has order 7 or 21.

Suppose that $N$ does not contain $C_G(P)$ and that $C_G(P) \neq C_G(P, h)$. Since the index of $N$ in $G$ is an odd prime $q$, the index of $C_N(P)$ in $C_G(P)$ is $q$ and $C_N(P) = C_G(P, h)$. That is to say, the stabilizer of $h$ under the $C_G(P)$-conjugation is $C_N(P)$. Since $f$ covers $h$, we have $f = \sum_{x \in I} xhx^{-1}$, where $I$ is a complete set of representatives of $C_N(P)$ in $C_G(P)$ and $xhx^{-1}h = 0$ for any $x \in I$ outside $C_N(P)$. Obviously we have $N_G(P, h) \subset N_G(P, f)$. For any $y \in N_G(P, f)$, there is $z \in C_G(P)$ such that $z^{-1}y$ centralizes $h$. So $N_G(P, f) \subset N_G(P, h)cG(P)$ and then $N_G(P, f) = N_G(P, h)cG(P)$. Now it is easily seen that the inclusion $N_N(P, h) \subset N_G(P, f)$ induces an injective group homomorphism

$$N_N(P, h)/C_N(P) \rightarrow N_G(P, f)/C_G(P),$$

whose image is normal. So the inertial quotient of the block $n$ has order 7 or 21.

Proposition 6.4. Assume that $P$ is $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$ and that $E_G(P, f)$ is of order 7 or 21. If $n \geq 2$, then $O\tilde{b}$ and $O_{\tilde{G}}(P)c$ are basically Morita equivalent.

Proof. Suppose that $b$ is a block of $G$ with defect group $P$ and the inertial quotient of order 7 or 21, such that $([G : Z(G)], |G|)$ is minimal in the lexicographic ordering and that $O\tilde{b}$ and $O_{\tilde{G}}(P)c$ are not basically Morita equivalent. Then the block $b$ has to be quasiprimitive.

Suppose that $O_2(G)$ is not trivial. Set $H = C_G(O_2(G))$. Let $h$ be a block of $H$ covered by the block $b$. Then $P$ is a defect group of the block $h$. Since $C_G(P) \subset H$, we may adjust the choice of $h$ so that $(P, f)$ is a maximal Brauer pair of the block $h$. By [18, Proposition 15.10], $h$ is equal to $b$ and the quotient group $G/H$ is an odd group. The inertial quotient $E_H(P, f)$ is a normal subgroup of $E_G(P, f)$. Since $E_G(P, f)$ is of order 7, $E_H(P, f)$ is of order 1, or 7, or 21. Suppose that $E_H(P, f)$ is of order 1. Then the block $b$ of $H$ is nilpotent and by [32, Theorem], $O\tilde{b}$ and $O_{\tilde{G}}(P)c$ are basically Morita equivalent. That contradicts the choice of the block $b$. Suppose that
$E_H(P,h)$ is of order 7 or 21. Then the commutator subgroup of $P$ and $E_H(P,h)$ is $P$. But since $O_2(G)$ is in the center of $H$, that is impossible. So $O_2(G)$ has to be trivial and $Z(G)$ is a subgroup of odd order.

Suppose that the block $b$ covers a nilpotent block $n$ of a normal subgroup $N$ of $G$. Set $N' = NZ(G)$. By Lemma 6.1, there is a nilpotent block $n'$ covered by the block $b$. By Lemma 4.5 applied to the normal subgroup $N'$ and its block $n'$ and by the minimality of $(|G : Z(G)|,|G|)$, $N'$ has to be equal to $Z(G)$ and thus $N$ is contained in $Z(G)$.

Denote by $E(G)$ the layer of the group $G$ and by $F(G)$ the Fitting subgroup of $G$. By the choice of the block $b$, $F(G)$ is equal to $O_2(G)$. Set $F^*(G) = E(G)F(G)$, the generalized Fitting subgroup of $G$. Suppose that $E(G)$ is the central product $L_1 * L_2 * \cdots * L_t$, where $L_1, L_2, \cdots, L_t$ are components of $G$. Since $C_G(F^*(G)) \leq F^*(G)$, by the choice of the block $b$, $t \geq 1$.

Let $e$ be the unique block of $E(G)$ covered by the block $b$ and $e_i$ a block of $L_i$ covered by the block $e$. Suppose $t > 1$. Since the block $b$ has defect group $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$, by Lemma 2.1 some block $e_i$ has to be nilpotent and then the block $b$ covers a nilpotent block of a normal subgroup generated by all $G$-conjugates of $L_i$. So $L_i \leq Z(G)$ and this contradicts $L_i$ being a component of $G$. So $t = 1$ and then $F^*(G) = L_1O_2(G)$.

By the Schreier conjecture, $G/L_1$ is solvable. We claim that $G/L_1$ is trivial.

Suppose that $G$ has a normal subgroup $H$ of index 2. By [13, Proposition 5.3] $G = PH$. Since $P = [P,E_G(P,f)]$, by [21, Proposition 4.2] $P$ is contained in $H$ and so $G = H$. That contradicts the choice of $H$. So $G$ has no normal subgroup $H$ of index 2.

Suppose that $G$ has a normal subgroup $H$ with an odd prime index containing $L_i$. Let $h$ be the unique block of $H$ covered by $b$. Note that $P$ is a defect group of the block $h$. By [32, Theorem], the block algebra $OHHh$ is not basically Morita equivalent to its Brauer correspondent in $N_H(P)$. By Lemma 6.3, the inertial quotient of the block $h$ has order 7 or 21. This is against the minimality of $(|G : Z(G)|,|G|)$.

So $G$ is equal to $L_1$. Since $P$ is $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$ for $n \geq 2$, only the case (iii) among the four cases of [10, Theorem 6.1] happens for the block $b$ of $G$. In this case, there is a finite group $\tilde{G}$, such that $G$ is normal in $\tilde{G}$ and $\tilde{G}$ has a nilpotent block $\tilde{b}$ covering $b$. By Lemma 6.2, $O\tilde{G}b$ and $O\tilde{N}/G(P)c$ are basically Morita equivalent. That contradicts the choice of the block $b$.

**Lemma 6.5.** Let $H$ and $H'$ be finite groups and $h$ and $h'$ blocks of $H$ and $H'$. Assume that $M$ is an indecomposable $O(H \times H')$-module inducing a Morita stable equivalence between $OHHh$ and $OHH'h'$. Let $\tilde{R}$ be a vertex of $\tilde{M}$ and $R'$ the images of $\tilde{R}$ through the projections $H \times H' \to H$ and $H \times H' \to H'$. Then $R$ and $R'$ are defect groups of $h$ and $h'$.

**Proof.** This lemma follows from [22, Theorem 6.9].

**Lemma 6.6.** Let $H$, $H'$ and $H''$ be finite groups, and $h$, $h'$ and $h''$ blocks of $H$, $H'$ and $H''$. Assume that $R$ is a common $p$-subgroup of $H$, $H'$ and $H''$, that $M$ is an indecomposable $O(H \times H')$-module with vertex $\Delta_\sigma(R) = \{(u,\sigma(u))|u \in R\}$ for some group automorphism $\sigma$ on $R$, and that $M'$ is an indecomposable $O(H' \times H'')$-module with vertex $\Delta_{\sigma'}(R) = \{(u,\sigma'(u))|u \in R\}$ for some group automorphism $\sigma'$ on $R$. Then the order of vertex of any indecomposable direct summand of $M \otimes_{OH'} M'$ is at most $|R|$.  

**Proof.** Let $\mathcal{O}\Delta_\sigma(R)$-module $S$ be a source of $M$ and $\mathcal{O}\Delta_{\sigma'}(R)$-module $S'$ a source of $M'$. Then $M$ is a direct summand of $\text{Ind}^{H \times H'}_{\Delta_\sigma(R)}(x)$ and $M'$ is a direct summand of $\text{Ind}^{H' \times H''}_{\Delta_{\sigma'}(R)}(S')$. We have an $O(H \times H'')$-module isomorphism

$$\text{Ind}^{H \times H'}_{\Delta_\sigma(R)}(O(H' \otimes_{\mathcal{O}} \text{Res}_T(S')) \otimes_{\mathcal{O}} \text{Res}_T(S'')) \cong \text{Ind}^{H \times H'}_{\Delta_\sigma(R)}(x) \otimes_{OH'} \text{Ind}^{H' \times H''}_{\Delta_{\sigma'}(R)}(S').$$
mapping \((x \otimes y) \otimes (z \otimes s \otimes s')\) onto \(((x \otimes z) \otimes s') \otimes ((1 \otimes y) \otimes s')\) for any \(x \in O\), any \(y \in O\), any \(z \in OH\), any \(s \in S\) and \(s' \in S'\), where \(\tau\) is the homomorphism \(R \times R \rightarrow \Delta_\sigma(R), (u, v) \mapsto (u, \sigma(u))\), \(\tau'\) is the homomorphism \(R \times R \rightarrow \Delta_\sigma'(R), (u, v) \mapsto (\sigma'(v), v)\) and \(OH\) is an \(O(R \times R)\)-module defined by the equality
\[
(u, v)w = \sigma(u)w\sigma(v)^{-1}
\]
for any \(u, v \in R\) and any \(w \in OH\). Since \(OH' \cong \oplus_{x \in R/H'/R} \text{Ind}^{R \times R}_{R'}(O)\), where \(R/H'/R\) denotes a set of representatives of the double cosets of \(R\), then there is a Morita equivalence between \(O(R \times R)\)-module sources and vertex \(\Delta\). By [10, Theorem 1.1], there is a Morita equivalence between \(O(R)\)-module inducing a Morita stable self-equivalence on \(\Delta\).

**Proposition 6.7.** Let \(h\) be a block of \(H\) with defect group \(R = \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}\), where \(n \geq 2\). Then the block algebra \(O\) and its Brauer correspondent \(O\) are basically Morita equivalent.

**Proof.** Let \((R, e)\) be a maximal \(h\)-Brauer pair and set \(K = R \times E_H(R, e)\). By the structure theorem of blocks with normal defect groups, there is a Morita equivalence between \(ON_H(R)d\) and \(OK\) induced by a \(p\)-permutation bimodule.

By [10, Theorem 1.1], \(O\) and \(OK\) are Morita equivalent. Suppose that an \(O(H \times K)\)-module \(M\) induces the Morita equivalence. By [23, Remarque 6.8] there is a stable equivalence between \(O\) and \(OK\) induced by an indecomposable \(O(H \times K)\)-module \(M\) with endopermutation module structure and vertex \(\Delta(R)\). Denote by \(\hat{M}\) the dual of the \(O(H \times K)\)-module \(M\). Then \(\hat{M} \otimes_{OG} M\) is an indecomposable \(O(K \times K)\)-module inducing a Morita stable self-equivalence on \(OK\). By [7, Corollary 3.3], there is an integer \(r\) such that \(\Omega^r(\hat{M} \otimes_{OG} M)\) induces a Morita self-equivalence on \(OK\), where \(\Omega\) denotes the Heller translate. By [34, Theorem 2] and [22, Corollary 7.4], \(\Omega^r(\hat{M} \otimes_{OG} M)\) has vertex \(\Delta_\sigma(R) = \{(u, \sigma(u)) | u \in R\}\) for some group automorphism \(\sigma\) on \(R\), and so does \(\hat{M} \otimes_{OG} M\). Since \(M\) is a direct summand of \(\hat{M} \otimes_{OK} \hat{M} \otimes_{OG} M\) and the order of vertex of any indecomposable direct summand of \(\hat{M} \otimes_{OK} \hat{M} \otimes_{OG} M\) is at most \(|R|\), by Lemma 6.5 the order of vertex of \(M\) has to be equal to \(|R|\). Then by [22, Corollary 7.4], \(M\) induces a basic Morita equivalence between \(O\) and \(OK\).

**Lemma 6.8.** Let \(G\) be a finite group and \(N\) a normal subgroup of \(G\) with index an odd prime. Let \(b\) be a block of \(G\) with defect group \(P\), and \(n\) a \(G\)-stable block of \(N\) covered by \(b\). Assume that \(P = \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}\), that the inertial quotient of the block \(b\) is of order 3 and that the block \(n\) is not nilpotent. Then the inertial quotient of the block \(n\) has order 3 or 21.

**Proof.** One uses the proof of Lemma 6.3 to prove the lemma.

**Lemma 6.9.** Let \(G\) be a finite group and \(b\) a block of \(G\) with defect group \(P\). Assume that \(P = \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}\) and that the inertial quotient of the block \(b\) is of order 3. Then \(\ell_G(b) = 3\).

**Proof.** See [11, Theorem 1.1 (i)] or alternatively use [30, Theorem 1] to prove the lemma.

**Proposition 6.10.** Assume that \(P = \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}\) and that \(E_G(P, f)\) is of order 3. If \(n \geq 2\), then there is a Morita equivalence between the block algebras \(OG\) and \(ON_G(P)c\) compatible with the *-structure, where \(c\) denotes the Brauer correspondent of \(b\) in \(N_G(P)\).

**Proof.** Set \(K = P \times E_G(P, f)\). By the structure theorem of blocks with normal defect groups, there is a Morita equivalence between \(ON_G(P)c\) and \(OK\) induced by a \(p\)-permutation bimodule. Therefore, in order to prove the lemma, by Propositions 2.7 and 3.6, it suffices to show that there is a Morita equivalence between \(OG\) and \(OK\) compatible with the *-structure.
When \( m = 0 \), by Propositions 6.7, 2.7 and 3.6 there is a Morita equivalence between \( OGb \) and \( OK \) compatible with the \(*\)-structure. We go by induction on \( m \). Suppose that \( b \) is a block of \( G \) with defect group \( P \) and the inertial quotient of order 3 such that \( (|G : Z(G)|, |G|) \) is minimal in the lexicographical order and that there is not a Morita equivalence between \( OGb \) and \( OK \) compatible with the \(*\)-structure. By Lemma 3.4, the block \( b \) has to be quasiprimitive.

Set \( H = C_G(O_2(G)) \). Let \( h \) be the block of \( H \) covered by the block \( b \). Then \( P \) is a defectgroup of the block \( h \). Since \( C_G(P) \subseteq H \), we may assume without loss that \((P, f)\) is a maximal Brauer pair of the block \( h \). By [18, Proposition 15.10], \( h \) is equal to \( b \) and the quotient group \( G/H \) is an odd group. The inertial quotient \( E_H(P, f) \) is a normal subgroup of \( E_G(P, f) \). Since \( E_G(P, f) \) is of order 3, \( E_H(P, f) \) is of order 1 or 3. If \( E_H(P, f) \) is of order 1, then the block \( b \) of \( H \) is nilpotent and by [32], \( OGb \) and \( OK \) are basically Morita equivalent. By Proposition 3.6, there is a Morita equivalence between \( OGb \) and \( OK \) compatible with the \(*\)-structure. That contradicts the choice of the block \( b \). So \( E_H(P, f) \) is of order 3 and \( N_G(P, f) = N_H(P, f)C_G(P) \). By a Frattini argument, we have \( G = N_G(P, f)H \). So \( G = H \) and \( O_2(G) \) is central.

Suppose that the block \( b \) covers a nilpotent block of a normal subgroup \( N \) of \( G \). Set \( N' = NZ(G) \). By Lemma 6.1, there is a nilpotent block \( n' \) covered by the block \( b \). By Lemma 3.5 applied to the normal subgroup \( N' \) and its block \( n' \) and the minimality of \((|G : Z(G)|, |G|)\), \( N' \) has to be equal to \( Z(G) \) and thus \( N \) is contained in \( Z(G) \).

Denote by \( E(G) \) the layer of the group \( G \) and by \( F(G) \) the Fitting subgroup of \( G \). Then \( F(G) \) is equal to \( Z(G) \). Set \( F^*(G) = E(G)F(G) \), the generalized Fitting subgroup of \( G \). Suppose that \( E(G) \) is the central product \( L_1 \ast L_2 \ast \cdots \ast L_t \), where \( L_1, L_2, \cdots, L_t \) are the components of \( G \). Since \( C_G(F^*(G)) \leq F^*(G) \), by the choice of the block \( b, t \geq 1 \).

Let \( e \) be the unique block of \( E(G) \) covered by the block \( b \) and \( e_i \) a block of \( L_i \) covered by the block \( e \). By the choice of the block \( b \), the order of the inertial quotient of the block \( e \) may be 3, 7, or 21. Suppose \( t > 1 \). By Lemma 2.1, some block \( e_i \) has to be nilpotent and then the block \( b \) covers a nilpotent block of a normal subgroup generated by all \( G \)-conjugates of \( L_i \). So \( L_i \) is contained in \( Z(G) \). This contradicts \( L_i \) being a component of \( G \). So \( t = 1 \) and \( F^*(G) = L_1Z(G) \).

By the Schreier conjecture, \( G/L_1 \) is solvable. We claim that \( G/L_1 \) is trivial.

Suppose that \( G \) has a normal subgroup \( H \) of index 2. Since \( E_G(P, f) \) is of order 3, we may assume without loss of generality that the commutator \([P, E_G(P, f)]\) is equal to \( \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \) and that the subgroup of \( E_G(P, f) \)-fixed elements of \( P \) is \( \mathbb{Z}_{2m} \). Let \( h \) be the unique block of \( H \) covered by the block \( b \). Then the intersection \( Q = P \cap H \) is a defect group of the block \( h \). By [21, Proposition 4.2] \( \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \leq Q \) and so \( Q \) is \( \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times \mathbb{Z}_{2n-1} \) since \( G = PH \) (see [13, Proposition 5.3]). By [33, Lemma 3.6] the block \( h \) has the inertial quotient of order 3. By induction on \( m \), there is a Morita equivalence between \( OHHh \) and \( OL \) compatible with the \(*\)-structure, where \( L = Q \times E_G(P, f) \). By Proposition 5.10 there is a Morita equivalence between \( OGb \) and \( OK \) compatible with the \(*\)-structure. This contradicts the choice of the block \( b \). So \( G \) has no normal subgroup \( H \) of index 2.

Suppose that \( G \) has a normal subgroup \( H \) of an odd prime index containing \( L_1 \). Let \( h \) be the unique block of \( H \) covered by the block \( b \). By Lemma 6.8, the inertial quotient of the block \( h \) has order 3 or 21. By Proposition 6.4 and [32, Theorem], we may exclude the case that the inertial quotient of the block \( h \) has order 21. Now by Lemma 6.9, \( \ell_H(b) = \ell_H(h) = 3 \). Since \( H \) has odd prime index in \( G \), by the last paragraph of the proof of [8, Theorem 1.1], that is impossible.

So \( G = L_1 \). Then as in the last paragraph of the proof of Proposition 6.4, we prove that the block algebra \( OGb \) and \( OK \) are basically Morita equivalent. Then by Propositions 2.7 and 3.6, there is a Morita equivalence between \( OGb \) and \( OK \) compatible with the \(*\)-structure. That contradicts the choice of the block \( b \).
6.11. Proof of Theorem

Note that $O$ unnecessarily has characteristic 0 in Theorem. We divide the proof of Theorem into the characteristic zero case and characteristic nonzero case.

Firstly, we assume that $O$ has characteristic 0. When $E_G(P, f)$ is 1, the block $b$ of $G$ is nilpotent and Theorem is true (see [24]). When $E_G(P, f)$ is 7 or 21, $n = m$ and Theorem follows from Proposition 6.4. When $E_G(P, f)$ is 3, Theorem follows from Proposition 6.10.

Finally, we assume that $O$ has characteristic $p$. The blocks $b$ and $c$ determine blocks $\bar{b}$ and $\bar{c}$ of $kG$ and $kN_G(P)$ with defect group $P$. By [28, Chapter II, Theorem 3], there is a complete discrete valuation ring $\tilde{O}$ with characteristic 0 and with residue field $k$. The blocks $\bar{b}$ and $\bar{c}$ can be lifted to blocks $\tilde{b}$ and $\tilde{c}$ of $\tilde{O}G$ and $\tilde{O}N_G(P)$ with defect group $P$. By [28, Chapter II, Proposition 3], without loss of generality, we may assume that $\tilde{O}$ contains a $|G|$-th primitive root of unity. By the last paragraph, the block algebras $\tilde{O}\tilde{b}$ and $\tilde{O}\tilde{c}$ are Morita equivalent. So the block algebras $k\tilde{b}$ and $k\tilde{N}_G(P)\tilde{c}$ are Morita equivalent. Since $O$ and $k$ have the same characteristic $p$, by [28, Chapter II, Proposition 8], $k$ can be identified with a subring of $O$. Moreover, it is easy to see that $O\tilde{b}$ and $O\tilde{N}_G(P)c$ are equal to $O \otimes_k k\tilde{b}$ and $O \otimes_k k\tilde{N}_G(P)c$ respectively. Therefore the block algebras $O\tilde{b}$ and $O\tilde{N}_G(P)c$ are Morita equivalent.

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