Darboux Transformation and solutions for an equation in 2+1 dimensions

P.G. Estévez*
Area de Física Teórica
Facultad de Física
Universidad de Salamanca
37008 Salamanca. Spain
August 8, 2018

Abstract

Painlevé analysis and the singular manifold method are the tools used in this paper to perform a complete study of an equation in 2+1 dimensions. This procedure has allowed us to obtain the Lax pair, Darboux transformation and \( \tau \) functions in such a way that a plethora of different solutions with solitonic behavior can be constructed iteratively.

PACS Numbers 02.30 and 03.40K

I. Introduction

Among the various approaches followed to study the behavior of non-linear partial differential equations (PDEs), Painlevé analysis has proved to be one of the most fruitful, providing an algorithmic procedure that affords a systematic way to deal with non-linear PDEs. Despite this, it has often been used merely as a test of integrability while other methods, as Hirota’s method or inverse scattering, have been used to obtain explicit solutions.

Our aim here is to show, for an equation in 2+1, that an approach based on Painlevé techniques, such as the singular manifold method (SMM), can be successful in identifying many of the properties of non-linear PDEs (Bäcklund and Darboux transformations, \( \tau \)-functions, etc) as well as in constructing an iterative procedure to obtain multisolitonic solutions.

*e-mail: pilar@sonia.usal.es
The subject of our study is the 2+1 PDE
\begin{align*}
0 &= V_y - (u\omega)_x \\
0 &= \lambda u_t + u_{xx} - 2uV \\
0 &= \lambda \omega_t - \omega_{xx} + 2\omega V
\end{align*} \tag{1.1}

The real version of this equation was obtained in [1] as a reduction of self-dual Yang-Mills equations while the complex version appears in [2]. The equation has the Painlevé property (PP) as it has been shown by Radha and Lakshmanan [3] (real version) and Porsezian [4] (complex version). The bilinear method was applied in [3] to obtain some soliton and dromion solutions.

For \( \lambda = i \) and \( \omega = u^* \), equation (1.1) is the expression proposed by Fokas in [5]. This case contains the non-linear Schrödinger equation when \( x = y \) [6].

Recently, [6] the author and her coworker have shown that there is a Miura transformation between (1.1) and the generalized dispersive long wave equation [7], [8].

The plan of this paper is as follows: In Section II we shall apply the singular manifold method to equation (1.1). Section III, IV and V are devoted to showing how the SMM allows us to determine algorithmically the Lax pair as well as Darboux transformations and \( \tau \)-functions. In section VI, several solutions are constructed explicitly. We close with a list of conclusions

## II. The Singular Manifold Method

The equation under study in this paper is the real version of (1.1), which reads:

\begin{align*}
0 &= m_y + u\omega \\
0 &= u_t + u_{xx} + 2um_x \\
0 &= \omega_t - \omega_{xx} - 2\omega m_x
\end{align*} \tag{2.1}

where we have set \( \lambda = 1 \) and \( V = -m_x \).

### II.1 Leading term analysis

To check the the Painlevé property [1] for (2.1), we require a generalized Laurent expansion of the fields in terms of an arbitrary singularity manifold (depending on the initial data) \( \chi(x, y, t) = 0 \). This expansion should be of the form [10]

\begin{align*}
u &= \sum_{j=0}^{\infty} u_j \chi^{j-a} \\
\omega &= \sum_{j=0}^{\infty} \omega_j \chi^{j-b} \\
m &= \sum_{j=0}^{\infty} m_j \chi^{j-c}
\end{align*} \tag{2.2}
By substituting (2.2) in (2.1), we have for the leading terms:

\[ a = b = c = 1 \quad m_0 = \chi_x, \quad u_0 \omega_0 = \chi_x \chi_y \] (2.3)

Leading analysis is able to determine the product of the dominant terms \( u_0 \omega_0 \) but not each one independently, which means that \( u \) and \( \omega \) are not good fields in which to apply the singularity analysis because their dominant behavior is not well defined. However, for the field \( m \), the leading term \( m_0 \) is well defined. This suggests that the “good field”, from the point of view of the Painlevé analysis, is \( m \). Accordingly, our first aim will be to write (2.1) as a partial differential equation only for \( m \). It is not difficult to check (see appendix) that if we identify

\[ m_t = n_x \] (2.4)

we can remove \( u \) and \( \omega \) from (2.1) to obtain the PDE:

\[ 0 = m_y^2 (n_y - m_{xy}) + m_{xy} (n_y^2 - m_{xy}^2) + 2m_y (m_{xy} m_{xxxy} - n_x n_{xy}) - 4m_y^3 m_{xx} \] (2.5)

In [6], it has been shown that there is a Miura transformation between (2.5) and the generalized long dispersive wave equation [7], [8]. This is why below we shall be referring, in the next, to (2.5) as MGLDW (modified generalized long dispersive wave equation). The study of this equation for the field \( m \) will be the subject of the rest of this paper. Furthermore, \( u \) and \( \omega \) can easily be obtained from \( m \) as:

\[ u = \sqrt{m_y} e^{\int \frac{n_y}{2m_y} dx} \] (2.6)
\[ \omega = -\sqrt{m_y} e^{-\int \frac{n_y}{2m_y} dx} \]

as we show in a detailed manner in the Appendix.

II.2 Truncated expansion. Auto-Bäcklund transformations

As stated above, the singularity manifold \( \chi \) is an arbitrary function depending on the initial data. The SMM requires us to restrict ourselves to the particular cases of the singularity manifold for which the expansion (2.2) truncates at the constant level [11]. In this case the singularity manifold is not longer an arbitrary function because it is “determined” by the condition of truncation. We call it “singular manifold” and we shall use \( \phi \) to refer to it. Thus, the truncation of (2.2) is:

\[ m' = m + \frac{\phi_x}{\phi} \implies n' = n + \frac{\phi_t}{\phi} \] (2.7)

where both \( m \) and \( m' \) are solutions of (2.5). Accordingly, truncation of the Painlevé series adopts the form of an auto-Bäcklund transformation between two solutions of (2.5).
Expression of the solutions in terms of the Singular Manifold

Substitution of (2.7) in (2.5) provides a polynomial in $\phi$. The way to proceed in the SMM is to require that all the coefficients of this polynomial should be zero. The result should be:

a) The expression of $m$ in terms of $\phi$.

b) The equations to be satisfied by $\phi$.

For equation (2.5), the polynomial in $\phi$ is rather complicated. We used MAPLE to handle the calculation. This allows us to obtain the derivatives of $m$ in terms of the singular manifold. The result is:

$$4m_x = p_t - v_x - \frac{v^2 + w^2}{2} \quad (2.8)$$
$$4n_y = 2 \left( \frac{q_x + qv}{q} \right) p_y - \frac{(q_x + qv) p_{xy}}{q} + 4 \left( \frac{p_y + qp_x}{q} \right) m_y \quad (2.9)$$
$$4m_y = \frac{p^2_y - (q_x + qv)^2}{q} \quad (2.10)$$

where $p, q, w, y, v$ are defined from the singular manifold as:

$$v = \frac{\phi_{xx}}{\phi_x}$$
$$w = \frac{\phi_t}{\phi_x} = p_x \quad (2.11)$$
$$q = \frac{\phi_y}{\phi_x}$$

Singular manifold equations

The equations to be satisfied by the singular manifold $\phi$ are not difficult to obtain:

• On one hand, there are some generic equations arising from the compatibility of the definitions (2.11). These are:

$$\phi_{xxx} = \phi_{txx} \implies v_t = (w_x + vw)_x$$
$$\phi_{xxy} = \phi_{yx} \implies v_y = (q_x + qv)_x \quad (2.12)$$
$$\phi_{yt} = \phi_{ty} \implies q_t = w_y + wq_x - qw_x$$

• Also, there is an equation that is specific for (2.5) that can be determined by taking the cross derivatives in (2.8)-(2.10). This equation is:

$$p_{yt} = q_{xxx} + q(v_{xx} - vv_x) + p_x p_{xy} + \left( \frac{p^2_y - q^2_x}{q} \right)_x \quad (2.13)$$

The set (2.12)-(2.13) forms the singular manifold equations.
III. Lax pair and SMM

It is unnecessary to talk about the importance of determining the Lax pair of a non-linear PDE. Nevertheless, in most cases it is determined by inspection. We shall see here that a non trivial advantage of Painlevé analysis is that it allows us to determine the Lax pair in an algorithmic way [12], [13].

III..1 Dominant terms in singular manifold equations

Returning to the singular manifold equations (2.12)-(2.13), these can be considered to be a system of coupled non-linear PDEs. We can therefore analyze their leading terms. This requires us to set:

\[ w \sim w_0 \chi^a \]
\[ v \sim v_0 \chi^b \]
\[ q \sim q_0 \chi^c \]  

(3.1)

The balance of leading powers yields:

\[ a = b = -1 \quad c = 0 \]  

(3.2)

which means that only \( w \) and \( v \) have an expansion in negative powers of \( \chi \). Thus, the Painlevé expansion is only pertinent for them but not for \( q \). Moreover, the leading analysis provides the leading coefficients \( w_0 \) and \( v_0 \):

\[ w_0 = \pm \chi_x \]
\[ v_0 = \chi_x \]  

(3.3)

The \( \pm \) sign of \( w_0 \) means that there are two possible Painlevé expansions: The problem of systems with two Painlevé branches has been extensively discussed in [14], [15], [12] and [13]. The suggestion of the author and coworker is that, for this class of systems, it is necessary to consider both branches simultaneously by using two singular manifolds; one for each branch.

III..2 Eigenfunctions and the singular manifold

With this idea in mind, for the dominant terms of \( w \) and \( v \) we can write:

\[ v = \frac{\psi^+}{\psi^+} + \frac{\psi^-}{\psi^-} \]
\[ w = \frac{\psi^+}{\psi^+} - \frac{\psi^-}{\psi^-} \]  

(3.4)

where we have used \( \psi^+ \) for the singular manifold of the positive branch and \( \psi^- \) for the negative one. As we will seen later on, \( \psi^+ \) and \( \psi^- \) will be the eigenfunctions of the Lax pair and hereafter we will designate them as eigenfunctions.
Taking the derivatives of (3.4) with respect to \(t\) and \(y\) and using (2.12) to integrate them in \(x\), we can write:

\[
w_x + wv = \frac{\psi^+_t}{\psi^+} + \frac{\psi^-_t}{\psi^-}
\]

\[
p_t = \frac{\psi^+_t}{\psi^+} - \frac{\psi^-_t}{\psi^-}
\]

and

\[
q_x + qv = \frac{\psi^+_y}{\psi^+} + \frac{\psi^-_y}{\psi^-}
\]

\[
p_y = \frac{\psi^+_y}{\psi^+} - \frac{\psi^-_y}{\psi^-}
\]

Expressions (3.4)-(3.6) allow us to write the logarithmic derivatives of the eigenfunctions \(\psi^+\) and \(\psi^-\) in terms of the singular manifold as:

\[
\alpha^+ = \frac{\psi^+_x}{\psi^+} = \frac{v + w}{2} \quad \alpha^- = \frac{\psi^-_x}{\psi^-} = \frac{v - w}{2}
\]

\[
\beta^+ = \frac{\psi^+_y}{\psi^+} = \frac{q_x + qv + p_y}{2} \quad \beta^- = \frac{\psi^-_y}{\psi^-} = \frac{q_x + qv - p_y}{2}
\]

\[
\gamma^+ = \frac{\psi^+_t}{\psi^+} = \frac{w_x + wv + p_t}{2} \quad \gamma^- = \frac{\psi^-_t}{\psi^-} = \frac{w_x + wv - p_t}{2}
\]

\(\alpha, \beta, \gamma\) have been introduced with the single purpose of simplifying later calculations.

Conversely, the determination of \(\phi\) from \(\psi^+\) and \(\psi^-\) is not difficult taking into account (2.11), which allows us integrate (3.4) with respect to \(x\), which yields:

\[
\phi_x = \psi^+ \psi^-
\]

where the integration constant has been set at zero with no loss of generality (because the singular manifold is defined except for a multiplicative constant). The \(t\) derivative of \(\phi\) can be obtained by combining (2.11), (3.4) and (3.10) to obtain:

\[
\phi_t = \psi^- \psi_x^+ - \psi^+ \psi_x^-
\]

and, similarly, \(\phi_y\) arises from (2.10), (2.11) and (3.6) as:

\[
\phi_y = -\frac{\psi^+_y \psi^-}{m_y}
\]

Equations (3.10), (3.11) and (3.12) allow us to construct \(\phi\) from \(\psi^+\) and \(\psi^-\). Accordingly, the total correspondence between singular manifolds and eigenfunctions is explicitly constructed.
III..3 Linearization of the singular manifold equations: The Lax pair

We return to equations (2.8)-(2.10). These equations are the expression of the seminal solution $m$ in terms of the singular manifold. At the same time, (3.7)-(3.12) relate the singular manifold to the eigenfunctions. The question is now: How can we express $m$ in terms of $\psi^+$ and $\psi^-$?

- As a previous step, it is easy to see that (2.10) can be combined with (3.8), yielding:

$$ m_y = -\frac{\beta^+\beta^-}{q} \Rightarrow \frac{\psi_y^+\psi_y^-}{\psi^+\psi^-} = -qm_y $$

(3.13)

which shows the coupling between $\psi^+$ and $\psi^-$. 

- Let us return to (2.8). To write this in terms of the eigenfunctions, we need to substitute $v$ and $p_t$ from (3.7) and (3.9)

$$ 4m_x = 2\gamma^+ - 2\alpha^+_x - 2(\alpha^+)^2, \quad \text{or} \quad 4m_x = -2\gamma^- - 2\alpha^-_x - 2(\alpha^-)^2 $$

Now, by substituting $\alpha$ and $\gamma$:

$$ 0 = \psi_t^+ - \psi_{xx}^+ - 2m_x\psi^+, \quad \text{or} \quad 0 = \psi_t^- + \psi_{xx}^- + 2m_x\psi^- $$

(3.14)

and this can be considered as the temporal part of the Lax pair.

- Finally, by combining it with (3.7)-(3.8) (2.9) can be written as:

$$ qn_y = \left[\beta^+\beta^- - \beta^-\beta^+_x\right] + m_y \left[(\beta^+ - \beta^-) + q(\alpha^+ - \alpha^-)\right] $$

(3.15)

If we use $\beta^+\beta^- = -qm_y$ and $\alpha^+ + \alpha^- = v$ to remove from (3.15) $(\beta^-, \alpha^-)$ or $(\beta^+, \alpha^+)$. 

$$ n_y = -m_{xy} + 2m_y\left(\frac{\beta^+_x}{\beta^+} + \alpha^+ + \frac{m_y}{\beta^+}\right), \quad n_y = m_{xy} - 2m_y\left(\frac{\beta^-_x}{\beta^-} + \alpha^- + \frac{m_y}{\beta^-}\right) $$

or

$$ (n_y + m_{xy})\psi_y^+ = 2m_y(\psi_{xy}^+ + m_y\psi^+), \quad (-n_y + m_{xy})\psi_y^- = 2m_y(\psi_{xy}^- + m_y\psi^-) $$

(3.16)

and this can be considered the spatial part of the Lax pair. 

Thus, the SMM allows us to define two eigenfunctions, $\psi^+$ and $\psi^-$, such that the expression of the truncated solutions in terms of these eigenfunctions is precisely the Lax pair (3.15)-(3.16).
IV. Darboux Transformations

This section will be devoted to determining an algorithmic procedure for constructing solutions.

1. We shall summarize the results obtained in the previous section: Let $m$ be a solution of (2.5), and $\phi_1$ a singular manifold for it. This singular manifold can be constructed by means of two eigenfunctions $\psi_1^+$ and $\psi_1^-$ through:

$$
\begin{align*}
\phi_{1x} &= \psi_1^+ \psi_1^- \\
m_y \phi_{1y} &= -\psi_{1y}^+ \psi_{1y}^- \\
\phi_{1t} &= \psi_1^- \psi_{1x}^+ - \psi_1^+ \psi_{1x}^-
\end{align*}
$$

where $\psi_1^+$ and $\psi_1^-$ satisfy the Lax pairs:

$$
\begin{align*}
0 &= \psi_{1t}^+ - \psi_{1xx}^+ - 2m_x \psi_1^+ & 0 &= \psi_{1t}^- + \psi_{1xx}^- + 2m_x \psi_1^- \\
0 &= 2m_y \psi_{1xy}^+ - (m_{xy} + n_y) \psi_{1y}^+ + 2m_y^2 \psi_1^+ & 0 &= 2m_y \psi_{1xy}^- - (m_{xy} - n_y) \psi_{1y}^- + 2m_y^2 \psi_1^-
\end{align*}
$$

2. According to (2.7), the singular manifold $\phi_1$ allows us to define a new solution $m'$

$$
m' = m + \frac{\phi_{1x}}{\phi_1} \implies n' = n + \frac{\phi_{1t}}{\phi_1}
$$

whose Lax pairs will be

$$
\begin{align*}
0 &= \psi_{t}^{'+} - \psi_{x}^{'+} - 2m_x' \psi_1^{'+} & 0 &= \psi_{t}^{'-} + \psi_{x}^{'-} + 2m_x' \psi_1^{'-} \\
0 &= 2m_y' \psi_{xy}^{'+} - (m_{xy}^' + n_y^') \psi_{y}^{'+} + 2m_y'^2 \psi_1^{'+} & 0 &= 2m_y' \psi_{xy}^{'-} - (m_{xy}^' - n_y^') \psi_{y}^{'-} + 2m_y'^2 \psi_1^{'-}
\end{align*}
$$

and $\psi^{'+}$ and $\psi^{'-}$ can be used to construct, for $m'$, a singular manifold $\phi'$ defined as:

$$
\begin{align*}
\phi'_x &= \psi'^+ \psi'^- \\
m_y' \phi'_y &= -\psi_y'^+ \psi_y'^- \\
\phi'_t &= \psi'^- \psi_{x}^{'+} - \psi'^+ \psi_{x}^{'-}
\end{align*}
$$

Truncated expansion in the Lax pair

A Lax pair such as (4.4) is usually considered to be a linear system for $\psi'$, where $m'$ is the potential and hence the inverse scattering method can be applied.

A different interpretation \[16\], \[12\] of (4.4) is to consider it as a coupled “non linear” system of PDEs for the fields $m', n', \psi^{'+}, \psi^{'-}$. In this case, the singular manifold method can be applied to the Lax pair itself and the truncated expansion (4.3) for $m$ and $n$ should
Darboux Transformation and solutions for an equation in 2+1 dimensions

be combined with a similar expansion for $\psi'^{+}$ and $\psi'^{-}$. In fact, this expansion could be written as:

$$
\psi'^{+} = \psi'^+_2 + \frac{\psi'^+_0}{\phi_1} \quad \psi'^{-} = \psi'^-_2 + \frac{\psi'^-_0}{\phi_1}
$$

where $\psi'^+_0, \psi'^-_0$ are the dominant terms. It is useful. For later calculations, it is useful to set $\psi'^+_0 = -\psi'^+_1 \Omega^+$, and $\psi'^-_0 = -\psi'^-_1 \Omega^-$. Therefore:

$$
\psi'^{+} = \psi'^+_2 - \psi'^+_1 \frac{\Omega^+}{\phi_1} \quad \psi'^{-} = \psi'^-_2 - \psi'^-_1 \frac{\Omega^-}{\phi_1}
$$

(4.6)

Substitution of the truncated expansions (4.3) and (4.6) in the Lax pairs (4.4) provides the following results (we used MAPLE for the calculations):

• 1) $\psi'^+_2$ and $\psi'^-_2$ are eigenfunctions for $m$. Consequently, they satisfy Lax pairs such as:

$$
\begin{align*}
0 &= \psi'^+_2 - \psi'^+_2 xx - 2 m_x \psi'^+_2 \\
0 &= 2 m_y \psi'^+_2 xy - (m_{xy} + n_y) \psi'^+_2 + 2 m_y^2 \psi'^+_2 \\
0 &= 2 m_y \psi'^+_2 xy - (m_{xy} - n_y) \psi'^-_2 + 2 m_y^2 \psi'^-_2
\end{align*}
$$

(4.7)

• 2) $\Omega^+$ and $\Omega^-$ are related to the eigenfunctions in the following way:

$$
\begin{align*}
\Omega^+_x &= \psi'^+_1 \psi'^-_2 \\
m_y \Omega^+_y &= -\psi'^+_1 \psi'^-_2 \\
\Omega^+_t &= \psi'^+_1 \psi'^-_2 - \psi'^+_1 \psi'^-_2
\end{align*}
$$

(4.8)

$$
\begin{align*}
\Omega^+_x &= \psi'^+_2 \psi'^-_1 \\
m_y \Omega^+_y &= -\psi'^+_2 \psi'^-_1 \\
\Omega^+_t &= \psi'^+_2 \psi'^-_1 - \psi'^+_2 \psi'^-_1
\end{align*}
$$

(4.9)

• To summarize: Two pairs of eigenfunctions $(\psi'^+_1, \psi'^-_1), (\psi'^+_2, \psi'^-_2)$ for a solution $(m, n)$ are sufficient to construct the following transformation:

$$
\begin{align*}
m' &= m + \frac{\phi_{1x}}{\phi_1} \\
n' &= n + \frac{\phi_{1t}}{\phi_1} \\
\psi'^{+} &= \psi'^+_2 - \psi'^+_1 \frac{\Omega^+}{\phi_1} \\
\psi'^{-} &= \psi'^-_2 - \psi'^-_1 \frac{\Omega^-}{\phi_1}
\end{align*}
$$

(4.10)

where $\phi_1, \Omega^+$ and $\Omega^-$ are related to the eigenfunction through (4.1), (4.8) and (4.9).

Equation (4.10) is a transformation of potentials and eigenfunctions that leaves invariant the Lax pairs. It should therefore be considered a Darboux transformation [17].
V. Iteration of the singular manifold: \( \tau \)-functions

A well known method for obtaining multisolitonic solutions of PDEs is the bilinear Hirota method. Indeed, some solutions of (2.1) have been identified with this method [3]. Let us address ourselves to the task of establishing, by explicit construction, the relationship between the singular manifold and the \( \tau \)-functions of Hirota’s method.

- Equation (4.5) could be considered as a non linear system among \( \phi' \), \( \psi'^+ \) and \( \psi'^- \). For this system we can use the same criterion used in the previous section. It requires that the expansion

\[
\begin{align*}
\psi'^+ &= \psi^+_2 - \psi^+_1 \frac{\Omega^+}{\phi_1} \\
\psi'^- &= \psi^-_2 - \psi^-_1 \frac{\Omega^-}{\phi_1}
\end{align*}
\]  

(5.1)

for \( \psi'^+ \) and \( \psi'^- \) should be combined with a truncated expansion for \( \phi' \).

\[
\phi' = \phi_2 + \frac{\Delta}{\phi_1}
\]  

(5.2)

It is not difficult to prove that the substitution of (5.1)-(5.2) in (4.5) gives:

\[
\Delta = -\Omega^+\Omega^-
\]  

(5.3)

while \( \phi_2 \) is the singular manifold for \( m \) related to \( \psi^+_2 \) and \( \psi^-_2 \), which means:

\[
\begin{align*}
\phi_{2x} &= \psi^+_2 \psi^-_2 \\
m_y \phi_{2y} &= -\psi^+_2 \psi^-_2 \\
\phi_{2t} &= \psi^-_2 \psi^+_2 - \psi^+_2 \psi^-_2
\end{align*}
\]  

(5.4)

- As far as (5.2) defines a singular manifold for \( m' \), it can be used to obtain a new solution:

\[
\begin{align*}
m'' &= m' + \frac{\phi'_{x}}{\phi'} \\
n'' &= n' + \frac{\phi'_{t}}{\phi'}
\end{align*}
\]  

(5.5)

which, combined with (4.3), is:

\[
\begin{align*}
m'' &= m + \frac{\tau_{x}}{\tau} \\
n'' &= n + \frac{\tau_{t}}{\tau}
\end{align*}
\]  

(5.6)

where

\[
\tau = \phi' \phi_1 = \phi_1 \phi_2 - \Omega^+\Omega^-
\]  

(5.7)

In the previous section we have shown that \( \phi_1, \phi_2, \Omega^+, \Omega^- \) are obtained from the eigenfunctions \( (\psi^+_1, \psi^-_1), (\psi^+_2, \psi^-_2) \). Therefore Equation (5.7) affords the relationship between \( \tau \)-functions, on one hand, and singular manifolds, on the other hand.
VI. Solutions

From the previous results we can derive an iterative procedure to construct solutions. It can be summarized as follows:

1) Starting with a seminal solution \( m \), solve the Lax pairs (4.2) and (4.8) to obtain \( \psi_1^+, \psi_1^- \), \( \psi_2^+, \psi_2^- \).

2) Perform the integration of (4.1), (4.8), (4.9) and (5.4) to get \( \phi_1, \Omega^+, \Omega^- \) and \( \phi_2 \): Use (5.7) to construct \( \tau \).

3) Use (4.3) to obtain the solution \( m' \) for the first iteration and (5.5) for the second one \( m'' \).

The easiest way to obtain explicit solutions is to apply the above explained procedure, starting with a trivial seminal solution. We shall use as seminal solutions \( m = m_0 y \) and \( m = 0 \). From the dependence on \( y \) of (4.1), (4.9), (4.10) and (5.4) it is that the behavior is totally different, depending on whether \( m_y \) is zero or not and giving rise to line-soliton or dromion behavior, respectively.

VI.1 Line solitons \( m = \omega_0 y \)

- The easiest solutions of (4.2) and (4.7) are:

\[
\begin{align*}
\psi_1^+ &= \exp\left[a_1^+ x - \frac{\omega_0}{a_1} y + a_1^+ t\right] \\
\psi_2^+ &= \exp\left[a_2^+ x - \frac{\omega_0}{a_2} y + a_2^+ t\right] \\
\psi_1^- &= \exp\left[a_1^- x - \frac{\omega_0}{a_1} y - a_1^- t\right] \\
\psi_2^- &= \exp\left[a_2^- x - \frac{\omega_0}{a_2} y - a_2^- t\right]
\end{align*}
\]

where \( a_1^+, a_1^-, a_2^+, a_2^- \) are arbitrary constants.

- Integration of (4.1), (4.8), (4.9) and (5.4) affords:

\[
\begin{align*}
\phi_1 &= \frac{1}{a_1 + a_1}(b_1 + \psi_1^+ \psi_1^-) \\
\Omega^+ &= \frac{1}{a_2^+ + a_1}(c^+ + \psi_1^+ \psi_2^-) \\
\phi_2 &= \frac{1}{a_2 + a_2}(b_2 + \psi_2^+ \psi_2^-) \\
\Omega^- &= \frac{1}{a_1^+ + a_2}(c^- + \psi_2^+ \psi_1^-)
\end{align*}
\]

where \( b_1, b_2, c^+, c^- \) are arbitrary constants.

- The first iteration provides the solution (figure 1):

\[
m'_y = \omega_0 + \partial_{xy} \left[ \ln \phi_1 \right]
\]

and the second (figure 2):

\[
m''_y = \omega_0 + \partial_{xy} \left[ \ln \tau \right]
\]

where

\[
\phi_1 = \frac{b_1}{a_1 + a_1} (1 + F_1)
\]
Darboux Transformation and solutions for an equation in 2+1 dimensions

\[ \tau = \phi_1 \phi_2 - \Omega^+ \Omega^- = \frac{b_1 b_2}{(a_1^+ + a_1^-)(a_2^+ + a_2^-)}[1 + F_1 + F_2 + AF_1 F_2] \]  

(6.6)

and

\[ F_i(x, y, t) = \exp \left[ (a_i^+ + a_i^-) \left\{ x - \frac{\omega_0}{a_i^+ a_i^-} y + (a_i^+ - a_i^-) t \right\} + \varphi_i \right] \]

(6.7)

\[ A = \frac{(a_2^+ - a_1^+)(a_2^- - a_1^-)}{(a_2^+ + a_1^-)(a_2^- + a_1^+)} \]  

(6.8)

and \( b_i \) has been redefined as: \( b_i = e^{-\varphi_i} \)

**Particular case:** When \( a_2^+ = a_1^+ \), or \( a_2^- = a_1^- \), \( A = 0 \) and this is said to be resonant state \[18\]

### VI..2 Dromions \( m = 0 \)

In this case (4.1), (4.8), (4.9) and (5.4) require that

\[ \psi_1^+ \psi_1^- = \psi_2^+ \psi_2^- = \psi_1^+ \psi_2^- = \psi_2^+ \psi_1^- = 0 \]

Therefore it is compulsory that \( \psi_1^y = \psi_2^y = 0 \), or \( \psi_1^y = \psi_2^y = 0 \)

- If we choose the possibility \( \psi_1^y = \psi_2^y = 0 \), then simple solutions of (4.2) and (4.7) are:

  \[ \psi_1^- = e^{a_1^-} x - a_1^-^2 t \quad \psi_1^+ = \left( e^{a_1^+} x + a_1^+^2 t \right) E_1(y) = Q_1^+(x, t) E_1(y) \]

  \[ \psi_2^- = e^{a_2^-} x - a_2^-^2 t \quad \psi_2^+ = \left( e^{a_2^+} x + a_2^+^2 t \right) E_2(y) = Q_2^+(x, t) E_2(y) \]

(6.9)

where \( a_1^+, a_1^-, a_2^+, a_2^- \) are arbitrary constants while \( E_i \) are arbitrary functions of \( y \)

- We can now perform now the integration of (4.1), (4.8), (4.9) and (5.4) to obtain:

  \[ \phi_1 = \frac{1}{a_1^+ + a_1^-} \left( E_1 Q_1^+ \psi_1^- + M_1(y) \right) \quad \Omega^+ = \frac{1}{a_2^+ + a_1^-} \left( E_2 \psi_2^- Q_2^+ + N^+(y) \right) \]

  \[ \phi_2 = \frac{1}{a_2^+ + a_2^-} \left( E_2 \psi_2^- Q_2^+ + M_2(y) \right) \quad \Omega^- = \frac{1}{a_1^+ + a_2^-} \left( E_1 \psi_1^- Q_1^+ + N^-(y) \right) \]

(6.10)

\( N^+, N^- \) and \( M_i^+ \) are arbitrary functions of \( y \). The arbitrariness of the six functions \( E_i \), \( M_i \), \( N^j \) and the four constants \( a_i^+, a_i^- \) implies that there are many particular cases. We list some of them:
VI..2.1 1+1 dromions:
These can be obtained by choosing
\[
E_i(y) = 1 + b_i e^{c_i y} \quad M_i(y) = 1 + e^{c_i y} \quad N^* = N^- = 0 \quad (6.11)
\]
where \( b_i \) and \( c_i \) are arbitrary constants.

- The first and second iteration yield:
  \[
  m'_y = \partial_{xy}[\ln \phi_1] \quad (6.12)
  \]
  \[
  m''_y = \partial_{xy}[\ln \tau] \quad (6.13)
  \]
where
\[
\phi_1 = \frac{1}{a_1^+ + a_1^-} (M_1 + E_1 F_1) \quad (6.14)
\]
\[
\tau = \phi_1 \phi_2 - \Omega^+ \Omega^- = \frac{1}{(a_1^+ + a_1^-)(a_2^+ + a_2^-)} [M_1 M_2 + M_1 E_2 F_2 + M_2 E_1 F_1 + AE_1 E_2 F_1 F_2] \quad (6.15)
\]
and
\[
F_i(x, t) = \exp \left[ (a_i^+ + a_i^-) \left\{ x + (a_i^+ - a_i^-) t \right\} \right] \quad (6.16)
\]
\[
A = \frac{(a_2^+ - a_1^+)(a_2^- - a_1^-)}{(a_2^+ + a_1^-)(a_2^- + a_1^+)} \quad (6.17)
\]
The behavior of (6.12) and (6.13) are represented in Figure 3 and Figure 4 respectively.

VI..2.2 1+n dromion:
Dromions with several jumps in the \( y \) direction can be obtained by choosing
\[
E_i = 1 + \sum_{j=1}^{n} b_{ij} e^{c_{ij} y} \quad M_i = 1 + \sum_{j=1}^{n} e^{c_{ij} y}
\]
The first iteration
\[
m'_y = \partial_{xy}(\ln \phi_1)
\]
describes a structure with \( n \) jumps located along the \( y \)-direction, moving in the \( x \)-direction with velocity \( a_1^+ - a_1^- \)

Figure 5 represents one of these structures with \( n = 2 \) and \( c_{11} > 0, c_{12} < 0 \)

Figure 6 corresponds to \( n = 3 \) and \( c_{11} > 0, c_{12} > 0, c_{13} > 0 \).

The solution that we have obtained in this section generalizes the solutions found in [13] by means of the bilinear method.
Conclusions

- A system of non-linear PDEs proposed by different authors as one of the simplest equations in 2+1 dimensions is studied from the point of view of the Painlevé property. The dominant behavior indicates the best field to use Painlevé analysis. On this basis, we rewrite the system as a PDE (2.5) with only one field. This equation can be considered as the modified version of the generalized long dispersive wave equation. This is why we have call it MGLDW (modified generalized long dispersive wave equation).

- The singular manifold method was applied to MGLDW in section II. The singular manifold equations, as well as the expression of the seminal field in terms of the singular manifold were obtained.

- In section III, we linearized the singular manifold equations to obtain the Lax pair. The relation between the singular manifold and the eigenfunctions of the Lax pair is constructed explicitly.

- In section IV the Lax pair was considered as a system of non linear coupled PDE. We applied the singular manifold method to the Lax pair itself. The bonus is the construction of Darboux transformations for MGLDW. Its transformations allow us to determine an iterative method for obtaining solutions. The relation between this method and the Hirota \( \tau \)-functions is shown in section V.

- Finally section VI is devoted to the construction of solitonic solutions of MGLDW. A rich collection of solutions with different solitonic behavior appear depending on the seminal solution that we have chosen.

- We believe that the equation analyzed in depth in this paper is a good example of how to obtain maximum information about the equation using Painlevé analysis and the singular manifold method as the only tools.

ACKNOWLEDGEMENTS This research has been supported in part by DGICYT under contract PB95-0947. I would like to thank Professor J. M. Cerveró for encouragement and a careful reading of the manuscript.

Appendix

We first attempt to write:

\[ 0 = u_t + u_{xx} + 2um_x \quad (A.1) \]
\[ 0 = \omega_t - \omega_{xx} - 2\omega m_x \quad (A.2) \]
as an equation for only one field

Taking \( u \) out of (A.3) and substituting it in (A.1), we obtain:

\[
0 = \frac{m_y + m_{xy}}{\omega} - 2m_{xy}\frac{\omega_x}{\omega^2} + m_y \left( \frac{2m_x}{\omega} - \frac{\omega_t}{\omega^2} - \frac{\omega_{xx}}{\omega^2} - \frac{2\omega_x^2}{\omega^3} \right) \tag{A.4}
\]

Using (A.2) in (A.4), we also obtain:

\[
0 = m_{xy} + m_y - \left( \frac{2m_y}{\omega} \right)_x \tag{A.5}
\]

which can easily be integrated by setting \( m_t = n_x \), which yields:

\[
\frac{\omega_x}{\omega} = \frac{m_{xy} + n_y}{2m_y} \tag{A.6}
\]

Substituting (A.6) in (A.2), we obtain:

\[
\frac{\omega_t}{\omega} = 2m_x + \frac{m_{xy} + n_{xy}}{2m_y} - \frac{m_{xy}^2 - n_y^2}{4m_y^2} \tag{A.7}
\]

Next, we calculate the identity \( \left( \frac{\omega_t}{\omega} \right)_x = \left( \frac{\omega_x}{\omega} \right)_t \) using (A.6) and (A.7), and finally we obtain for \( m \) the equation:

\[
0 = m_t - n_x \tag{A.8}
\]

\[
0 = m_y^2(n_y - m_{xy}) + m_{xy}(n_y^2 - m_{xy}) + 2m_y (m_{xy}m_{xy} - n_y n_{xy}) - 4m_y^3 m_{xx} \tag{A.8}
\]

The integration of (A.6) is:

\[
u = \sqrt{m_y} e^{\int \frac{n_y}{2m_y} \, dx} \tag{A.9}
\]

And from (3.4) we finally obtain:

\[
\omega = -\sqrt{m_y} e^{\int -\frac{n_y}{2m_y} \, dx} \tag{A.10}
\]

References

[1] Chakravarty S., Kent S.L and Newman E.T: J. Math. Phys 36, 763-772 (1995).
[2] Maccari A. J. Math. Physics 37, 6207-6212, (1996).
[3] Radha R. and Lakshmanan, J. Math. Physics 38, 292-299 (1997).
[4] Porsezian K.,J. Math. Physics 38, 4675-4679 (1997).
Darboux Transformation and solutions for an equation in 2+1 dimensions

[5] Fokas A., *Inverse Problems* 10, L19-L22 (1994).

[6] Cerveró J. M. and Estévez P. G. *J. Math. Phys.*, 39, 2800-2807, (1998).

[7] Boiti M., Leon J. and Pempinelli F, *Inv. Problems* 3, 37-45 (1987).

[8] Ablowitz M. J and Clarkson P., “Solitons, Nonlinear Evolution Equations and Inverse Scattering”. London Mathematical Society. Lecture Note Series 149, Cambridge University Press. (1991).

[9] P. Painlevé, *Acta Mathematica*, Paris, (1900).

[10] Weiss J., Tabor M. and Carnevale G., *J. Math. Physics* 24, 522-526 (1983).

[11] Weiss J., *J. Math. Phys.* 24, 1405-14013 (1983).

[12] Estévez P.G. and Gordoa P.R., *Inverse Problems* 13, 939-957 (1997).

[13] Estévez P.G., Conde-Calvo E. and Gordoa P.R., *Jour. Nonlinear. Math. Phys* 5, 82-114, (1998).

[14] Conte R. Musette M. and Pickering A., *J. Phys A: Math. Gen.* 27, 2831-2836 (1994).

[15] Estévez P.G., Gordoa P.R., Martinez-Alonso L. and Medina-Reus E., *J. Phys. A* 26, 1915-1925 (1993).

[16] Konopelchenko B.G. and Stramp W., *J.Math. Phys.*, 24, 40-49, (1991).

[17] Matveev V.B.and Salle M.A., “Darboux transformations and solitons”, Springer Series in Nonlinear Dynamics, Springer-Verlag (1991).

[18] Estévez P.G. and Leble S. B., *Inverse Problems* 11, 925-937, (1995).
Figure 1: line soliton
Figure 2: Interaction of two line solitons
Figure 3: One dromion
Figure 4b: Interaction of two dromions, t=0
Figure 5: 2+1 dromion
Figure 4a: Interaction of two dromions, $t<0$
Figure 4c: Interaction of two dromions, $t>0$