ŁOJASIEWICZ IDEALS
IN DENJOY-CARLEMAN CLASSES

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Abstract. The classical notion of Łojasiewicz ideals of smooth functions is studied in the context of non-quasianalytic Denjoy-Carleman classes. In the case of principal ideals, we obtain a characterization of Łojasiewicz ideals in terms of properties of a generator. This characterization involves a certain type of estimates that differ from the usual Łojasiewicz inequality. We then show that basic properties of Łojasiewicz ideals in the $\mathcal{C}^\infty$ case have a Denjoy-Carleman counterpart.

1. Introduction

Let $\Omega$ be an open subset of $\mathbb{R}^n$, and let $\mathcal{C}^\infty(\Omega)$ be the Fréchet algebra of smooth functions in $\Omega$. Let $X$ be a closed subset of $\Omega$. An element $\varphi$ of $\mathcal{C}^\infty(\Omega)$ is said to satisfy the Łojasiewicz inequality with respect to $X$ if, for every compact subset $K$ of $\Omega$, there are real constants $C > 0$ and $\nu \geq 1$ such that, for any $x \in K$, we have

$$|\varphi(x)| \geq C \operatorname{dist}(x, X)^\nu.$$ (1)

For example, it is well-known that any real-analytic function satisfies the Łojasiewicz inequality with respect to its zero set.

An element of $\mathcal{C}^\infty(\Omega)$ is said to be flat on $X$ if it vanishes, together with all its derivatives, at each point of $X$. Denote by $m_X^\infty$ the ideal of functions of $\mathcal{C}^\infty(\Omega)$ that are flat on $X$. The following statement appears in [15, Section V.4] and establishes a connection between the Łojasiewicz inequality and the behavior of ideals with respect to flat functions.

Theorem 1.1. Let $\mathcal{I}$ be a finitely generated proper ideal in $\mathcal{C}^\infty(\Omega)$, and let $X$ be the zero set of $\mathcal{I}$. The following properties are equivalent:

(A) The ideal $\mathcal{I}$ contains an element $\varphi$ which satisfies the Łojasiewicz inequality with respect to $X$.
(B) $m_X^\infty \subset \mathcal{I}$.
(C) $m_X^\infty = \mathcal{I}m_X^\infty$.

2010 Mathematics Subject Classification. 26E10, 46E10.
Key words and phrases. Łojasiewicz ideals, Ultradifferentiable functions.
A finitely generated ideal $\mathcal{I}$ satisfying the equivalent properties $(A)$, $(B)$, $(C)$ is called a \textit{Łojasiewicz ideal}. A principal ideal is Łojasiewicz if and only if condition $(A)$ holds for a generator $\varphi$ of the ideal. In the general case of a finitely generated ideal with generators $\varphi_1, \ldots, \varphi_p$, one can take $\varphi = \varphi_1^2 + \cdots + \varphi_p^2$. Łojasiewicz ideals play an important role in the study of ideals of differentiable functions; see for instance \cite{8, 14, 15}. In particular, every closed ideal of finite type is Łojasiewicz, whereas the converse statement is false.

In the present paper, we study a possible approach to Łojasiewicz ideals in non-quasianalytic Denjoy-Carleman classes $\mathcal{C}_M(\Omega)$. While several papers have already been devoted to the study of closed ideals in $\mathcal{C}_M(\Omega)$ (see for example \cite{10, 11, 12}), a suitable notion of Łojasiewicz ideal is still lacking, even in the case of principal ideals. This is due to the fact that if we put $m^\infty_{X,M} = m^\infty_X \cap \mathcal{C}_M(\Omega)$ and $\mathcal{I} = \varphi \mathcal{C}_M(\Omega)$, where $\varphi$ is a given element of $\mathcal{C}_M(\Omega)$, it turns out that the usual Łojasiewicz inequality $\varphi$ is not a sufficient condition for the inclusion $\mathcal{I} \subset m^\infty_{X,M}$, let alone for the equality $m^\infty_{X,M} = \mathcal{I} m^\infty_{X,M}$. Therefore, it is natural to ask for a characterization of both of these properties in terms of the generator $\varphi$, in the spirit of the characterization given by Theorem 1.1 in the $\mathcal{C}^\infty$ case.

In the case of principal ideals, a suitable characterization will be obtained in Theorem 3.1. In the statement, the Łojasiewicz inequality $\varphi$ has to be replaced by a quite different property involving successive derivatives of $1/\varphi$, which will be shown to be equivalent to the obvious Denjoy-Carleman version of property $(C)$, that is, to the equality $m^\infty_{X,M} = \mathcal{I} m^\infty_{X,M}$. We are also able to get an equivalence with a corresponding version of property $(B)$, provided we consider the inclusion $m^\infty_{X,M} \subset \mathcal{I}$ together with a mild extra requirement on the flat points of $\varphi$.

In order to prove these results, one has to deal with the fact that the constructive techniques used by Tougeron in the classical $\mathcal{C}^\infty$ case do not seem applicable to the $\mathcal{C}_M$ setting. Thus, the main part of our proof of Theorem 3.1 is actually based on a functional-analytic argument. Once the theorem is proven, we discuss several related properties showing that basic results of the $\mathcal{C}^\infty$ case are extended in a consistent way. For instance, we show that our $\mathcal{C}_M$ Łojasiewicz condition holds for closed principal ideals, and we also provide a non-closed example.
2. Denjoy-Carleman classes

2.1. Notation. For any multi-index $J = (j_1, \ldots, j_n)$ of $\mathbb{N}^n$, we always denote the length $j_1 + \cdots + j_n$ of $J$ by the corresponding lower case letter $j$. We put $J! = j_1! \cdots j_n!$, $D^J = \partial^{j_1} / \partial x_1^{j_1} \cdots \partial x_n^{j_n}$ and $x^J = x_1^{j_1} \cdots x_n^{j_n}$. We denote by $| \cdot |$ the euclidean norm on $\mathbb{R}^n$; balls and distances in $\mathbb{R}^n$ will always be considered with respect to that norm.

If $a$ is a point in $\mathbb{R}^n$, and if $f$ is a smooth function in a neighborhood of $a$, we denote by $T_a f$ the formal Taylor series of $f$ at $a$, that is, the element of $C[[x_1, \ldots, x_n]]$ defined by

$$T_a f = \sum_{J \in \mathbb{N}^n} \frac{1}{J!} D^J f(a) x^J.$$ 

The function $f$ is said to be flat at the point $a$ if $T_a f = 0$.

2.2. Some properties of sequences. Let $M = (M_j)_{j \geq 0}$ be a sequence of real numbers satisfying the following assumptions:

(2) the sequence $M$ is increasing, with $M_0 = 1$,

(3) the sequence $M$ is logarithmically convex.

Property (3) amounts to saying that $M_{j+1}/M_j$ is increasing. Together with (2), it implies

(4) $M_j M_k \leq M_{j+k}$ for any $(j, k) \in \mathbb{N}^2$.

We say that the moderate growth property holds if there is a constant $A > 0$ such that, conversely,

(5) $M_{j+k} \leq A^{j+k} M_j M_k$ for any $(j, k) \in \mathbb{N}^2$.

We say that $M$ satisfies the strong non-quasianalyticity condition if there is a constant $A > 0$ such that

(6) $\sum_{j \geq k} \frac{M_j}{(j+1)M_{j+1}} \leq A \frac{M_k}{M_{k+1}}$ for any $k \in \mathbb{N}$.

Notice that property (6) is indeed stronger than the classical Denjoy-Carleman non-quasianalyticity condition

(7) $\sum_{j \geq 0} \frac{M_j}{(j+1)M_{j+1}} < \infty$.

The sequence $M$ is said to be strongly regular if it satisfies (2), (3), (5) and (6).
Example 2.3. Let $\alpha$ and $\beta$ be real numbers, with $\alpha > 0$. The sequence $M$ defined by $M_j = j!^\alpha (\ln(j + e))^{\beta j}$ is strongly regular. This is the case, in particular, for Gevrey sequences $M_j = j!^\alpha$.

With every sequence $M$ satisfying (2) and (3) we also associate the function $h_M$ defined by $h_M(t) = \inf_{j \geq 0} t^j M_j$ for any real $t > 0$, and $h_M(0) = 0$. From (2) and (3), it is easy to see that the function $h_M$ is continuous, increasing, and it satisfies $h_M(t) > 0$ for $t > 0$ and $h_M(t) = 1$ for $t \geq 1/M_1$. It also fully determines the sequence $M$, since we have $M_j = \sup_{t > 0} t^{-j} h_M(t)$.

Example 2.4. Let $M$ be defined as in Example 2.3 and put $\eta(t) = \exp(- (t |\ln t|^{\beta} - 1/\alpha))$ for $t > 0$. Elementary computations show that there are constants $a > 0$, $b > 0$ such that $\eta(at) \leq h_M(t) \leq \eta(bt)$ for $t \to 0$.

A technically important consequence of the moderate growth assumption (5) is the existence of a constant $\rho \geq 1$, depending only on $M$, such that
\begin{equation}
(8) \quad h_M(t) \leq (h_M(\rho t))^2 \text{ for any } t \geq 0.
\end{equation}

We refer to [3] for a proof that (2), (3) and (5) imply (8).

2.5. Denjoy-Carleman classes. Let $\Omega$ be an open subset of $\mathbb{R}^n$, and let $M$ be a sequence of real numbers satisfying (2) and (3). We define $C_M(\Omega)$ as the space of functions $f \in C^\infty(\Omega)$ satisfying the following condition: for any compact subset $K$ of $\Omega$, one can find a real $\sigma > 0$ and a constant $C > 0$ such that
\begin{equation}
(9) \quad |D^J f(x)| \leq C\sigma^j j! M_j \text{ for any } J \in \mathbb{N}^n \text{ and } x \in K.
\end{equation}

Given a function $f$ in $C^\infty(\Omega)$, a compact subset $K$ of $\Omega$ and a real number $\sigma > 0$, put
\begin{equation}
\|f\|_{K,\sigma} = \sup_{x \in K, \ J \in \mathbb{N}^n} \frac{|D^J f(x)|}{\sigma^j j! M_j}.
\end{equation}

We see that $f$ belongs to $C_M(\Omega)$ if and only if, for any compact subset $K$ of $\Omega$, one can find a real $\sigma > 0$ such that $\|f\|_{K,\sigma}$ is finite ($\|f\|_{K,\sigma}$ then coincides with the smallest constant $C$ for which (9) holds). The function space $C_M(\Omega)$ is called the Denjoy-Carleman class of functions of class $C_M$ in the sense of Roumieu (which corresponds to $\mathcal{E}_{\{j! M_j\}}(\Omega)$ in the notation of [5]).

From now on, we will assume that the sequence $M$ is strongly regular. In particular, it satisfies (7), which implies that $C_M(\Omega)$ contains compactly supported functions. We denote by $\mathcal{D}_M(\Omega)$ the space of elements of $C_M(\Omega)$ with compact support in $\Omega$. 
For the reader’s convenience, we now recall some basic topological facts about $\mathcal{C}_M(\Omega)$ and $\mathcal{D}_M(\Omega)$, without proof (we refer to [5] for the details).

With each Whitney 1-regular compact subset $K$ of $\Omega$, and each integer $\nu \geq 1$, we associate the vector space $\mathcal{C}_{M,K,\nu}$ of all functions $f$ which are $C^\infty$-smooth on $K$ in the sense of Whitney, and such that $\|f\|_{K,\nu} < \infty$. Then $\mathcal{C}_{M,K,\nu}$ is a Banach space for the norm $\| \cdot \|_{K,\nu}$ and it can be shown that for $\nu < \nu'$, the inclusion $\mathcal{C}_{M,K',\nu} \hookrightarrow \mathcal{C}_{M,K',\nu'}$ is compact. We define the Denjoy-Carleman class $\mathcal{C}_M(\Omega)$ as the reunion of all spaces $\mathcal{C}_{M,K,\nu}$ with $\nu \geq 1$. Endowed with the inductive topology, $\mathcal{C}_M(\Omega)$ is a (DFS)-space (or Silva space). Given an exhaustion $(K_j)_{j \geq 1}$ of $\Omega$ by Whitney 1-regular compact subsets, the Denjoy-Carleman class $\mathcal{C}_M(\Omega)$ can be identified with the projective limit of all (DFS)-spaces $\mathcal{C}_M(K_j)$.

Similarly, denote by $\mathcal{D}_{M,K,\nu}$ the space of all functions $f \in C^\infty(\Omega)$ such that $\text{supp} \ f \subset K$ and $\|f\|_{K,\nu} < \infty$. Then $\mathcal{D}_{M,K,\nu}$ is a Banach space and we have the following properties: for $K \subset K'$, the space $\mathcal{D}_{M,K,\nu}$ is a closed subspace of $\mathcal{D}_{M,K',\nu}$, and for $\nu < \nu'$, the inclusion $\mathcal{D}_{M,K',\nu} \hookrightarrow \mathcal{D}_{M,K',\nu'}$ is compact. For any integer $\nu \geq 1$, put $\mathcal{D}_\nu = \mathcal{D}_{M,K_\nu,\nu}$, $\| \cdot \|_\nu = \| \cdot \|_{K_\nu,\nu}$, and notice that we have $\mathcal{D}_M(\Omega) = \bigcup_{\nu \geq 1} \mathcal{D}_\nu$ as a set. By the preceding remarks, we have a compact injection $\mathcal{D}_\nu \hookrightarrow \mathcal{D}_{\nu+1}$. Thus, the space $\mathcal{D}_M(\Omega)$ is another (DFS)-space for the corresponding inductive limit topology.

### 2.6. Some basic properties of $\mathcal{C}_M(\Omega)$

Properties (2) and (3) of the sequence $M$ ensure that $\mathcal{C}_M(\Omega)$ is an algebra containing the algebra of real-analytic functions, and that $\mathcal{C}_M$ regularity is stable under composition [9]. This implies, in particular, the following invertibility property.

**Lemma 2.7** ([9]). If the function $f$ belongs to $\mathcal{C}_M(\Omega)$ and has no zero in $\Omega$, then the function $1/f$ belongs to $\mathcal{C}_M(\Omega)$.

It is also known that the implicit function theorem holds within the framework of $\mathcal{C}_M$ regularity [6]. Thus, $\mathcal{C}_M$ manifolds and submanifolds can be defined in the usual way.

The strong regularity assumption on $M$ ensures that suitable versions of Whitney’s extension theorem and Whitney’s spectral theorem hold in $\mathcal{C}_M(\Omega)$; see [1234]. The extension result relies on a crucial construction of cutoff functions whose successive derivatives satisfy a certain type of optimal estimates. This construction is due to Bruna [2]; see also [3, Proposition 4]. Up to a rescaling in the statement of [3], the result can be written as follows.

**Lemma 2.8** ([23]). There is a constant $c > 0$ such that, for any real numbers $r > 0$ and $\sigma > 0$, one can find a function $\chi_{r,\sigma}$ belonging to $\mathcal{C}_M(\mathbb{R}^n)$,
compactly supported in the ball \( B = B(0, r) \), and such that we have \( 0 \leq \chi_{r, \sigma} \leq 1 \), \( \chi_{r, \sigma}(t) = 1 \) for \( |t| \leq r/2 \) and \( \| \chi_{r, \sigma} \|_{\mathcal{B}(c\sigma)} \leq (h_M(\sigma r))^{-1} \).

We shall also need a basic result on flat functions. Given a closed subset \( Z \) of \( \Omega \), recall that \( m_{\infty,Z,M} \) denotes the ideal of functions of \( \mathcal{C}_M(\Omega) \) which are flat at each point of \( Z \).

**Lemma 2.9.** Let \( f \) be an element of \( m_{\infty,Z,M} \). For any compact subset \( K \) of \( \Omega \), there are positive constants \( c_1 \) and \( c_2 \) such that, for any multi-index \( I \) in \( \mathbb{N}^n \) and any \( x \) in \( K \), we have

\[
|D^I f(x)| \leq c_1 c_2^I! M_I h_M(c_2 \text{dist}(x, Z)).
\]

**Proof.** For any real \( r > 0 \), put \( K_r = \{ y \in \Omega : \text{dist}(y, K) \leq r \} \). If \( r \) is chosen small enough, \( K_r \) is a compact subset of \( \Omega \). Thus, there is a constant \( \sigma > 0 \) such that, for any \( y \in K_r \), \( I \in \mathbb{N}^n \) and \( J \in \mathbb{N}^n \), we have \( |D^{I+J} f(y)| \leq \|f\|_{K_r, \sigma} \sigma^I (i + j)! M_{i+j} \). Using (11) and the elementary estimate \((i + j)! \leq 2^{i+j} j!\), we get

\[
|D^{I+J} f(y)| \leq c_1 c_2^I! M_I c_2^J! M_j.
\]

with \( c_1 = \|f\|_{K_r, \sigma} \) and \( c_2 = 2A\sigma \). Now let \( x \) be a point in \( K \), and let \( z \) be a point in \( Z \) such that

\[
|x - z| = \text{dist}(x, Z).
\]

If \( \text{dist}(x, Z) \leq r \), then the segment \([x, z]\) is contained in \( K_r \). Let \( j \) be an integer. Since \( D^I f \) is flat at \( z \), the Taylor formula easily yields \( |D^I f(x)| \leq n! \sup_{|I|=j, y \in K_r} |D^{I+J} f(y)||x - z|^j / j! \). Using (11) and (12), and taking the infimum with respect to \( j \), we obtain (10) up to the replacement of \( c_2 \) by \( nc_2 \). If \( \text{dist}(x, Z) > r \), the estimate is a simple consequence of the definition of \( \mathcal{C}_M(\Omega) \), up to another modification of \( c_1 \) and \( c_2 \). \( \square \)

### 3. Łojasiewicz ideals

The following notion will serve as a replacement for the standard Łojasiewicz inequality.

**Definition 3.1.** Let \( \varphi \) be a non-zero element of \( \mathcal{C}_M(\Omega) \) and let \( X \) be the zero set of \( \varphi \). We say that \( \varphi \) satisfies the \( \mathcal{C}_M \) Łojasiewicz condition if, for any compact subset \( K \) of \( \Omega \) and any real \( \lambda > 0 \), one can find positive constants \( C \) and \( \sigma \) (depending on \( K \) and \( \lambda \)) such that, for any multi-index \( J \in \mathbb{N}^n \) and any \( x \in K \setminus X \), we have

\[
|D^J (1/\varphi)(x)| \leq \frac{C \sigma^j M_j}{h_M(\lambda \text{dist}(x, X))}.
\]
Remark 3.2. From the basic properties of $h_M$ in Section 2.2 we see that, on a given open subset $\{x \in \Omega : \text{dist}(x, X) > \delta\}$ with $\delta > 0$, the $C_M$ Łojasiewicz condition amounts to nothing more than the conclusion of Lemma 2.7. It is relevant only as a bound on the explosion of $1/\varphi$ and its derivatives in a neighborhood of the zeros of $\varphi$.

In Section 4 we will provide examples of functions for which the $C_M$ Łojasiewicz condition holds. Lemma 3.3 below shows that such functions cannot have “too many flat points” on the boundary of their zero set.

Lemma 3.3. Let $\varphi$ be a non-zero element of $C_M(\Omega)$ and let $X$ be its zero set. Assume that $\varphi$ satisfies the $C_M$ Łojasiewicz condition, and let $X_\infty = \{a \in X : T_a \varphi = 0\}$ be the set of points of flatness of $\varphi$. Then $X \setminus X_\infty$ is dense in the boundary $\partial X$ of $X$.

Proof. Notice that $\varphi$ is necessarily flat at each interior point of $X$, hence the inclusion $X \setminus X_\infty \subset \partial X$. We prove the density property by contradiction. If the property is not true, there are a point $a$ in $\partial X$ and an open neighborhood $\omega$ of $a$ in $\Omega$, such that $\varphi$ is flat on $\omega \cap \partial X$. Put $K = B(a, r)$ with $r = \frac{1}{2} \text{dist}(a, X \setminus \omega)$. Then $K$ is a compact subset of $\omega$ and we have

$$\text{dist}(x, \omega \cap \partial X) = \text{dist}(x, \partial X) = \text{dist}(x, X) \text{ for any } x \in K. \tag{14}$$

Using Lemma 2.9 on the open set $\omega$, with $f = \varphi|\omega$, $Z = \omega \cap \partial X$ and $I = 0$, we see that there are constants $c_1$ and $c_2$ such that we have $|\varphi(x)| \leq c_1 h_M(c_2 \text{dist}(x, \omega \cap \partial X))$ for any $x \in K$. Taking property (8) into account, we obtain, for any $x \in K$,

$$|\varphi(x)| \leq c_1 h_M(c_3 \text{dist}(x, \omega \cap \partial X))^2 \tag{15}$$

with $c_3 = \rho c_2$. On the other hand, using the $C_M$ Łojasiewicz condition with $\lambda = c_3$ and $J = 0$, we obtain a constant $c_4 > 0$ such that, for any $x \in K \setminus X$,

$$|\varphi(x)| \geq c_4 h_M(c_3 \text{dist}(x, X)). \tag{16}$$

Gathering (14), (15) and (16), we obtain $h_M(c_3d(x, X)) \geq c_4/c_1$ for any $x \in K \setminus X$, which is impossible since $K \setminus X$ has at least an accumulation point on $X$, namely the point $a$. \hfill \Box

We are now able to state the main result.

Theorem 3.4. Let $\varphi$ be a non-zero element of $C_M(\Omega)$, let $X$ be its zero set, and let $X_\infty$ be its set of points of flatness. Put $\mathcal{I} = \varphi C_M(\Omega)$. The following properties are equivalent:

$(A')$ The function $\varphi$ satisfies the $C_M$ Łojasiewicz condition.
\((B')\) \(m_{X,M}^\infty \subset I\) and \(X \setminus X_\infty\) is dense in \(\partial X\).

\((C')\) \(m_{X,M}^\infty = m_{X,M}^\infty\).

**Proof.** We prove the implication \((C') \Rightarrow (A')\) first. We use the (DFS)-space \(D_M(\Omega) = \lim_n D_\nu\) defined in Section 2.2. The intersection \(D_M(\Omega) \cap m_{X,M}^\infty\) is obviously closed in \(D_M(\Omega)\), hence it is also a (DFS)-space with step spaces \(E_\nu = D_\nu \cap m_{X,M}^\infty\).

It is easy to see that the map \(\Lambda : D_M(\Omega) \cap m_{X,M}^\infty \to D_M(\Omega) \cap m_{X,M}^\infty\) defined by \(\Lambda(f) = \varphi f\) is continuous. Moreover, given an element \(g\) of \(D_M(\Omega) \cap m_{X,M}^\infty\), the assumption implies that it can be written \(\varphi h\) for some \(h \in m_{X,M}^\infty\). If \(\chi\) is an element of \(D_M(\Omega)\) such that \(\chi = 1\) on \(\text{supp} g\), we then have \(g = \chi g = \varphi f\) with \(f = \chi h \in D_M(\Omega) \cap m_{X,M}^\infty\). Thus, \(\Lambda\) is also surjective.

We can therefore apply the De Wilde open mapping theorem ([7, Chapter 24]), which yields the following property: for any \(\nu \geq 1\), there exist an integer \(\mu_\nu \geq 1\) and a real constant \(C_\nu > 0\) such that, for any \(g \in E_\nu\), one can find an element \(f\) of \(E_{\mu_\nu}\) such that

\[\varphi f = g \quad \text{and} \quad \|f\|_{\mu_\nu} \leq C_\nu\|g\|_{\nu}.\]

Now, let \(x\) be a point in \(K \setminus X\), let \(d_K\) be a real number such that \(0 < d_K < \text{dist}(K, \mathbb{R}^n \setminus \Omega)\), and put \(r_x = \min(\text{dist}(x, X), d_K)\). Given \(\lambda > 0\), we apply Lemma 2.8 with \(r = 2r_x / 3\) and \(\sigma = 3\lambda / 2\). We set \(g_x(y) = \chi_{r,\sigma}(y - x)\). Then \(g_x\) belongs to \(C_M(\Omega)\) and is compactly supported in the ball \(B_x = B(x, 2r_x / 3)\). Obviously \(B_x\) is contained in \(K' = \{y \in \Omega : \text{dist}(y, K) \leq 2d_K / 3\}\), which is a compact subset of \(\Omega\). For a sufficiently large integer \(\nu\), depending only on \(K\) and \(\lambda\), we have \(\nu \geq c\sigma\) and \(K' \subset \hat{K}_\nu\), so that \(g_x\) belongs to \(E_\nu\) and

\[\|g_x\|_\nu = \|g_x\|_{\nu, r} \leq \|g_x\|_{\nu, \sigma} \leq (h_M(\lambda r_x))^{-1}.\]

Since \(h_M(\lambda r_x)\) equals either \(h_M(\lambda \text{dist}(x, X))\) or \(h_M(\lambda d_K)\), and since we have \(h_M(t) \leq 1\) for every \(t > 0\), we see that

\[h_M(\lambda r_x) \geq h_M(\lambda d_K) h_M(\lambda \text{dist}(x, X)).\]

Now, if \(f_x\) denotes the element of \(E_{\mu_\nu}\) associated with \(g_x\) by property \((17)\), we therefore have \(\varphi f_x = g_x\) and, thanks to \((18)\) and \((19)\),

\[\|f_x\|_{\mu_\nu} \leq C'_\nu (h_M(\lambda \text{dist}(x, X)))^{-1}\]

with \(C'_\nu = C_\nu / h_M(\lambda d_K)\). For any \(y\) in \(B'_x = B(x, r_x / 3)\), we have \(g_x(y) = 1\), hence

\[f_x(y) = 1 / \varphi(y).\]
In particular, we have \( f_x(y) \neq 0 \). Thus, we derive \( B'_x \subset \text{supp} \ f_x \subset K_{\mu \nu} \), which implies, for any \( y \in B'_x \) and any multi-index \( J \),

\[
|D^J f_x(y)| \leq \| f_x \|_{\mu \nu} |J|! M_j.
\]

Combining (20), (21) and (22), we get the desired estimate (13) with suitable constants \( A = C'_\nu \) and \( B = \mu \nu \) depending only on \( \nu \), hence only on \( K \) and \( \lambda \).

We now prove the implication \((A') \Rightarrow (B')\). By Lemma 3.3 the assumption implies that \( X \setminus X_\infty \) is dense in \( \partial X \). The proof of the inclusion \( m_{X,M}^\infty \subset I \) is a variant of the proof of [10, Theorem 2.3]; we give some details for the reader’s convenience. Let \( f \) be an element of \( m_{X,M}^\infty \). For any \( x \in \Omega \setminus X \) and any multi-index \( P \in \mathbb{N}^n \), the Leibniz formula yields

\[
D^P(f/\varphi)(x) = \sum_{I+J = P} \frac{P!}{I!J!} D^I f(x) D^J (1/\varphi)(x).
\]

Let \( K \) be a compact subset of \( \Omega \). For \( x \in K \setminus X \), we combine the \( C_M \) Łojasiewicz condition with Lemma 2.9 in order to obtain an estimate for all the terms \( D^I f(x) D^J (1/\varphi)(x) \) that appear in (23). Lemma 2.9, together with (3), yields \( |D^I f(x)| \leq c_1 c_2^i M_i (h_M (c_3 \text{dist}(x,X)))^2 \) with \( c_3 = \rho c_2 \).

Applying the \( C_M \) Łojasiewicz condition with \( \lambda = c_3 \), we therefore get \( |D^I f(x) D^J (1/\varphi)(x)| \leq c_2 C c_2^j \sigma^j |J|! M_i M_j h_M (c_3 \text{dist}(x,X)) \). Since \( i + j = p \), we have \( |J| \leq \rho \), as well as \( M_i M_j \leq M_p \) by (4). Inserting these estimates in (23), we obtain, for every multi-index \( P \) and every \( x \in K \setminus X \),

\[
|D^P(f/\varphi)(x)| \leq c_5 c_3^p M_p h_M (c_3 \text{dist}(x,X))
\]

with \( c_5 = c_2 C \) and \( c_4 = c_2 + \sigma \). Using (24) and the Hestenes lemma, we see that the function \( g \) defined by \( g(x) = f(x)/\varphi(x) \) for \( x \in \Omega \setminus X \) and \( g(x) = 0 \) for \( x \in X \) belongs to \( C_M(\Omega) \). Obviously, we have \( f = \varphi g \), hence \( f \in I \).

Finally, we prove the implication \((B') \Rightarrow (C')\). Let \( f \) be an element of \( m_{X,M}^\infty \). By assumption, there is \( g \in C_M(\Omega) \) such that \( f = \varphi g \). Let \( a \) be a point of \( X \setminus X_\infty \). In the ring of formal power series, we have \( 0 = T_a f = (T_a \varphi)(T_a g) \) with \( T_a \varphi \neq 0 \), which implies \( T_a g = 0 \). Thus, \( g \) is flat on \( X \setminus X_\infty \), hence on \( \partial X \) since it is assumed that \( X \setminus X_\infty \) is dense in \( \partial X \). Put \( \tilde{g}(x) = g(x) \) for \( x \in \Omega \setminus X \) and \( \tilde{g}(x) = 0 \) for \( x \in X \). By the Hestenes lemma, it is then readily seen that \( \tilde{g} \in m_{X,M}^\infty \). Moreover, we have \( f = \varphi \tilde{g} \), hence \( f \in I m_{X,M}^\infty \), and the proof is complete.

**Remark 3.5.** We do not know whether the implication \((B') \Rightarrow (C')\) still holds without the additional assumption on \( X \setminus X_\infty \) in \((B')\). This is true
when $X$ is a real-analytic submanifold of $\Omega$: indeed, according to [13, Theorem 4.2.4], we then have $m_{X,M}^\infty = m_{X,M}^\infty m_{X,M}^\infty$. Thus, in this case, the inclusion $m_{X,M}^\infty \subset I$ easily implies $(C')$.

**Remark 3.6.** Using the equivalence $(A') \Leftrightarrow (C')$, we see that if $\varphi$ satisfies the $C_M$ Łojasiewicz condition and if $h$ is an invertible element of the algebra $C_M(\Omega)$, so that $\varphi$ and $h\varphi$ generate the same ideal $I$, then $h\varphi$ also satisfies the $C_M$ Łojasiewicz condition. This can also be checked by a direct computation with the Leibniz formula.

4. **Additional properties and examples**

4.1. **On the zero set.** We have a Denjoy-Carleman counterpart of [15, Proposition V.4.6].

**Proposition 4.2.** Let $\varphi$ be an element of $C_M(\Omega)$ that satisfies the $C_M$ Łojasiewicz condition, and let $X$ be its zero set. Then there is a $C_M$-smooth submanifold $Y$ of $\Omega$ such that $X = \overline{Y}$.

**Proof.** We notice first that the conclusion of Lemma 3.3 only requires a weaker property than the $C_M$ Łojasiewicz condition: more precisely, the proof remains valid as soon as, for any compact subset $K$ of $\Omega$ and any real $\lambda > 0$, one can find a constant $C > 0$ such that the inequality $|\varphi(x)| \geq Ch_M(\lambda \text{dist}(x,X))$ holds for any $x \in K$. It is then fairly easy to check that the proof by induction given in [15] for the usual Łojasiewicz inequality on $C^\infty$ functions remains valid in the $C_M$ case, up to minor modifications. □

4.3. **Connection with closedness.** In this section, we show that the $C_M$ Łojasiewicz condition behaves as expected with respect to closedness properties of ideals.

**Proposition 4.4.** Let $\varphi$ be a non-zero element of $C_M(\Omega)$ that generates a closed ideal in $C_M(\Omega)$. Then $\varphi$ satisfies the $C_M$ Łojasiewicz condition. Moreover, both properties are equivalent when the zeros of $\varphi$ are isolated.

**Proof.** We use the same notation as in the proof of the implication $(C') \Rightarrow (A')$ of Theorem 3.4. Put $I = \varphi C_M(\Omega)$ and assume that $I$ is closed in $C_M(\Omega)$. Since the inclusion $D_M(\Omega) \hookrightarrow C_M(\Omega)$ is continuous, $I \cap D_M(\Omega)$ is closed in $D_M(\Omega)$. Using cutoff functions, it is also easy to see that $I \cap D_M(\Omega) = \varphi D_M(\Omega)$. It is then possible to duplicate the proof of the implication $(C') \Rightarrow (A')$.

---

1The result in [15] is actually a local version of that statement, but it can be globalized, using partitions of unity.
Łojasiewicz condition: for instance, given an integer \( k \), include any homogeneous polynomial with an isolated real critical point at \( \psi \) with \( \Psi^{k} = 0 \). In particular, the Łojasiewicz condition does not imply closedness in general.

The converse in the case of isolated zeros is based on a variant of the argument leading to \cite{10} Proposition 4.1 (which deals with a singleton). Assume that \( \varphi \) satisfies the \( C_{M} \) Łojasiewicz condition and that its zero set \( X \) consists of isolated points, so that \( X \) is a countable subset \( \{a_{j} : j \geq 1\} \) of \( \Omega \). Put \( \mathcal{I} = \varphi C_{M}(\Omega) \) and let \( f \) be an element of the closure \( \overline{\mathcal{I}} \). By the \( C_{M} \) version of Whitney’s spectral theorem \cite{4}, for every \( j \geq 1 \) there is a function \( g_{j} \) of \( C_{M}(\Omega) \) such that \( f - \varphi g_{j} \) is flat at \( a_{j} \). Let \( (\chi_{j})_{j \geq 1} \) be a sequence of compactly supported elements of \( C_{M}(\Omega) \) such that \( \chi_{j} = 1 \) in a neighborhood of \( a_{j} \) and \( \text{supp} \chi_{j} \cap \text{supp} \chi_{k} = \emptyset \) for \( k \neq j \). Then the (locally finite) series \( g = \sum_{j \geq 1} \chi_{j} g_{j} \) defines an element of \( C_{M}(\Omega) \) and we have \( f - \varphi g \in \overline{\mathcal{I}} \). Since \( (B') \) holds, this yields \( f \in \mathcal{I} \), hence the result.

\( \square \)

**Example 4.5.** According to Proposition 4.4 and the results in \cite{10, 12}, examples of functions \( \varphi \) which satisfy the \( C_{M} \) Łojasiewicz condition will include any homogeneous polynomial with an isolated real critical point at 0, as well as real analytic functions whose germs of complex zeros intersect \( \mathbb{R}^{n} \) at isolated points with Łojasiewicz exponent 1 for the regular separation property. On the other hand, some analytic functions do not satisfy the \( C_{M} \) Łojasiewicz condition: for instance, given an integer \( k \geq 2 \), the polynomial \( \psi(x) = x_{1}^{2} + x_{2}^{2k} \) does not satisfy the \( C_{M} \) Łojasiewicz condition in \( \mathbb{R}^{2} \), as can be seen from the results in \cite{10} (property \( B' \) fails).

We now give an example showing that the converse to Proposition 4.4 is false without the assumption of isolated zeros. In particular, the \( C_{M} \) Łojasiewicz condition does not imply closedness in general.

**Example 4.6.** We put \( n = 2, \Omega = \mathbb{R}^{2} \), and \( \varphi(x) = x_{1} \psi(x) \) where \( \psi \) is the polynomial mentioned in Example 4.5. We then have \( X = \{x \in \mathbb{R}^{2} : x_{1} = 0\} \) and \( \text{dist}(x, X) = |x_{1}| \). Let \( x \) be a point in \( \mathbb{R}^{2} \setminus X \). For any \( v = (v_{1}, v_{2}) \in \mathbb{C}^{2} \), we have

\[
|\psi(x + v) - \psi(x)| \leq 2|x_{1}||v_{1}| + |v_{1}|^{2} + \sum_{p=1}^{2k} \binom{2k}{p} |x_{2}|^{2k-p} |v_{2}|^{p}.
\]

We also have the obvious inequalities \( |x_{1}| \leq (\psi(x))^{1/2} \) and \( |x_{2}| \leq (\psi(x))^{1/2k} \). Thus, if we assume \( |v_{1}| \leq \delta (\psi(x))^{1/2} \) and \( |v_{2}| \leq \delta (\psi(x))^{1/2k} \) for some real number \( \delta \) with \( 0 < \delta < 1 \), we get

\[
|\psi(x + v) - \psi(x)| \leq \left( 2\delta + \delta^{2} + \sum_{p=1}^{2k} \binom{2k}{p} \delta^{p} \right) \psi(x) \leq (2^{2k} + 2)\delta \psi(x).
\]
Setting $\delta = (2^{2k+1} + 4)^{-1}$, we obtain $|\psi(\zeta)| \geq \frac{1}{2} \psi(x)$ for every point $\zeta$ in the bidisc $\{\zeta \in \mathbb{C}^2 : |\zeta_1 - x_1| \leq \delta(\psi(x))^{1/2}, |\zeta_2 - x_2| \leq \delta(\psi(x))^{1/2k}\}$. The Cauchy formula then yields, for every $(i, j) \in \mathbb{N}^2$,
\[
\frac{\partial^{i+j}}{\partial x_1^i x_2^j} \left( \frac{1}{\psi(x)} \right) \leq 2\delta^{-(i+j)i!j!}(\psi(x))^{-(\frac{i}{2} + \frac{j}{k} + 1)},
\]
which easily implies
\[
(25) \quad \frac{\partial^{i+j}}{\partial x_1^i x_2^j} \left( \frac{1}{\psi(x)} \right) \leq 2\delta^{-(i+j)i!j!}|x_1|^{-(i+j+2)}
\]
provided we assume $|x_1| < 1$. Using (25), the definition of $\varphi$, and the Leibniz formula, we then get
\[
\frac{\partial^{i+j}}{\partial x_1^i x_2^j} \left( \frac{1}{\varphi(x)} \right) \leq B^{i+j+1} i!j!|x_1|^{-(i+j+2)}
\]
for some suitable constant $B > 0$. Let $\lambda$ be a given positive real number. We write $|x_1|^{-(i+j+2)} = \frac{\lambda^{i+j+2} \lambda^{i+j+2}}{(\lambda|x_1|)^{i+j+2}}$. The definition of $h_M$ implies $(\lambda|x_1|)^{i+j+2} M_{i+j+2} \geq h_M(\lambda|x_1|) = h_M(\lambda \text{dist}(x, X))$, whereas (5) yields $M_{i+j+2} \leq A^{i+j+2} M_{i+j}$. Gathering these inequalities, we eventually obtain $|x_1|^{-(i+j+2)} \leq (A\lambda)^{i+j+2} (h_M(\lambda \text{dist}(x, X)))^{-1}$ and
\[
\frac{\partial^{i+j}}{\partial x_1^i x_2^j} \left( \frac{1}{\varphi(x)} \right) \leq C\sigma^{i+j}(i+j!M_{i+j})\frac{h_M(\lambda \text{dist}(x, X))}{h_M(\lambda \text{dist}(x, X))}
\]
with $C = A^2 B^2$ and $\sigma = A B \lambda$. Thus, we have established the desired estimate for $|x_1| = \text{dist}(x, X) < 1$, which suffices to conclude that $\varphi$ satisfies the $C_M$ Łojasiewicz condition (see Remark 3.2). However, the ideal $I = \varphi C_M(\mathbb{R}^2)$ is not closed for $k \geq 2$. Indeed, in this case, it has been shown in [10] that the ideal $J = \psi C_M(\mathbb{R}^2)$ is not closed. Since $J$ is the preimage of $I$ under the continuous mapping $\Pi : C_M(\mathbb{R}^2) \to C_M(\mathbb{R}^2)$ defined by $\Pi(f)(x) = x_1 f(x)$, we see that $I$ is not closed either.

We conclude with a natural question.

**Problem.** Is it possible to extend the above results to the general case of finitely generated ideals? A first idea is to mimic the definition of Łojasiewicz ideals in the $C^\infty$ case, and say that a finitely generated ideal of $C_M(\Omega)$ is Łojasiewicz if it contains an element $\varphi$ which satisfies the $C_M$ Łojasiewicz condition. However, this definition doesn’t seem to allow an immediate extension of the crucial implication $(C') \Rightarrow (A')$, whose proof is quite different from the $C^\infty$ case and doesn’t seem easily adaptable to the case of several generators.
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