Cylindrical thermal invisibility cloak based on transformation thermodynamics

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Abstract. The paper provides a theoretical proof that the methods of transformation optics can be used for thermodynamics. Thermal conductivity tensor is calculated for the development of the thermal invisibility cloak, which directs the heat flux around a certain area. The finite element method was used to simulate the cloak with parameters close to real. The simulation was performed by the finite element method for two cases: with gradient \( k \) and with copper/mica layered structure.

1. Introduction
Mathematical methods of transformational optics were proposed independently by two research groups: Ulf Leonhardt [1] and John Pendry [2, 3]. Using these methods, several exotic devices have been calculated, including the invisibility cloak, which directs the beam of a certain wavelength around the area, thereby making it invisible. Transformation optics concept was determined by Fermat’s principle. The light propagation direction depends on the material refractive index. According to this, at present, transformation optics is a unique scientific tool that allows to combine the mathematical mapping of desired distortions of space with the actual special distribution of refractive index in physical space to control light propagation.

In 2012, Sebastien Guenneau proposed to use transformation optics for thermodynamics [4], where thermal conductivity acts as analogue of the optical properties (permittivity and permeability). In this work we describe the mathematical tools to calculate the thermal conductivity tensors for creation a cloak similar to the optical invisibility cloak, which directs the heat flow around the area, and provide a simulation of the device operation.

2. Mathematical methods and calculation
2.1. Transformation optics
Transformation optics uses physical space with arbitrary spatial coordinate system \( \{x', i = 1,2,3\} \) and virtual one \( \{x', i' = 1,2,3\} \). Any optical medium distorts the geometry perceived by light (the same for medium with non-uniform thermal conductivity). This distortion is described by curved coordinate grid – physical space – in which the light follows along the coordinate lines (coordinate lines are isothermal contours for transformation thermodynamics). Physical space is transformed into a virtual space in which the effect of the medium has been fully accounted for by the shifting of the points. Virtual space demonstrates view, which is seen by observer (namely empty space). Physical and virtual spaces can be described by arbitrary coordinate system. Fig. 1 shows physical and virtual spaces of invisibility cloak in Cartesian and cylindrical coordinates.
Figure 1(a, b, c, d). (a) The virtual space of the cloak in the Cartesian coordinates; (b) The physical space of the cloak in the Cartesian coordinates; (c) The virtual space of the cloak in the cylindrical coordinate system; (d) The physical space of the cloak in the cylindrical coordinate system.

2.2. Thermal conductivity tensor
To derive the thermal conductivity tensor $k$, the two-dimensional heat equation for temperature distribution $T$ with a source $p$ is considered [4]. First, the equation is written for the virtual space in Cartesian coordinates $x' = \{x', y'\}$, where metric tensor is $g' = \text{diag}(1, 1, 1)$ (Fig. 1(a)):

$$\rho(x')c(x') \frac{\partial T}{\partial t} = \nabla \cdot (k'(x')\nabla T) + p(x', t),$$  

(1)

where $\rho(x')$ – a density, $c(x')$ – a heat capacity, $k'(x')$ – a thermal conductivity, $T$ – a temperature. To write the equation for the physical space in Cartesian coordinates $x = \{x, y\}$, where metric tensor $g$ is a metric of curved space (fig. 1(b)), some differential geometry equations are used [5]:

transformation matrix: $\Lambda_{i}' = \frac{\partial x}{\partial x'}$  

(2)

matrix equation: $g' = \Lambda^T g \Lambda$  

(3)

divergence of a vector field $\mathbf{V}$: $\nabla \cdot \mathbf{V} = \frac{1}{\sqrt{\det g}}(\sqrt{\det g} \mathbf{V}^i)_i$  

(4)

components of the gradient vector of scalar field $\psi$: $(\nabla \psi)^i = g^{ij} \psi_j$  

(5)

where a comma means partial differentiation, with the following index giving the coordinate with respect to which the derivative is taken. It is possible now writing heat equation in curved physical space using formulas (2)-(5):

$$\sqrt{\det g} \rho(x)c(x) \frac{\partial T_i}{\partial t} = (k_i(x)\sqrt{\det g} g^{ij} T_j)_i + \sqrt{\det g} p(x, t),$$  

(6)

Comparing this two equations, the equation for the thermal conductivity tensor for Cartesian coordinates can be obtained:

$$k^{ij} = \sqrt{\det g} g^{ij}.$$  

(7)

But in some cases it is convenient to use another coordinate systems, not only Cartesian. Then in equations (1) and (6) it is necessary to use also metric tensor of coordinate system $\gamma$, which is calculated by the formula (3). For curvilinear system the thermal conductivity tensor is:
\[ k_{ij} = \sqrt{\frac{\text{det} g}{\text{det} y}} g_{ij}. \]  

(8)

From matrix equation (3) it can be obtained:

\[ \sqrt{\text{det} g} = \sqrt{\frac{\text{det} g'}{\text{det} \Lambda}} \]  

(9)

Using formulas (3) and (9) equation (8) is derived in matrix form:

\[ k = \sqrt{\frac{\text{det} g'}{\text{det} y}} \Lambda (g')^{-1} \Lambda \frac{1}{\text{det} \Lambda} k' \]  

(10)

The similar equation are used for calculation permittivity and permeability in optics [5]. Here \( g' \) – a metric tensor of the virtual space, which coincides with the metric tensor of the coordinate system used, because virtual space isn’t curved, \( y \) – a metric tensor of the physical space coordinate system, \( \Lambda \) – a transformation matrix for transformation from physical to virtual space (Jacobian), \( k' \) – a thermal conductivity of the virtual space.

2.3. Thermal invisibility cloak

For the thermal cloak it is most convenient to use a cylindrical coordinates \( \{r, \phi, z\} \) (fig. 1(c, d)) with metric tensor \( y \) of physical space and metric tensor \( g' \) of virtual space:

\[ y = \text{diag}(1, \frac{1}{r^2}, 1) \]  

(11)

\[ g' = \text{diag}(1, \frac{1}{r'^2}, 1) \]  

(12)

Then the relationship between coordinates of the physical space and the virtual one is:

\[ r(r') = \frac{(R_2 - R_1)}{R_2} r' + R_1, \]

\[ \phi(\phi') = \phi' \]

\[ z(z') = z' \]  

(13)

where \( R_2 \) – an exterior cloak radius, \( R_1 \) – an interior cloak radius (fig.1). Then the thermal conductivity tensor calculated by equation (10) is:

\[ k = \text{diag}\left( \frac{r - R_1}{r}, \frac{r}{r - R_1} \right) \]  

(14)

3. Simulation

The simulation was performed by the finite element method in the COMSOL Multiphysics software. The scheme of simulation is shown in fig. 2 (a). Firstly, temperature distribution for gradient anisotropic thermal conductivity (Eq. (14)) for \( R_1 < r < R_2 \) was simulated. A silicon with \( k' = 130 \ \text{W} \cdot (\text{m} \cdot \text{K})^{-1} \) was chosen for substrate \( (r > R_2) \). The object radius is \( R_1 = 2 \ \mu\text{m} \) and the cloak radius is \( R_2 = 4 \ \mu\text{m} \).
The object inside thermal invisibility cloak keeps constant temperature 283 K in device with gradient k calculated with transformation thermodynamics mathematical methods.

It is obvious that material with anisotropic gradient thermal conductivity is difficult to make so it was suggested in the paper [6] to use layered structure with real materials (fig. 3(a)). It is material with thermal conductivity lower than substrate on the inner radius $R_1 < r < R_2$ ($k_1 < k'$) and material with thermal conductivity higher than substrate on the outer radius $R_2 < r < R_3$ ($k_2 > k'$).

In this work mica and copper are used in the inner and outer radii respectively and the silicon was used for the substrate. Mica thermal conductivity is $k_1 = 0.5$ W·(m·K)$^{-1}$ and copper $k_2 = 402$ W·(m·K)$^{-1}$. $k_1$ and $k_2$ agrees with conditions, that $k_1 < k'$ and $k_2 > k'$. The object radius is $R_1 = 2$ μm, the inner radius is $R_2 = 3.5$ μm and the outer radius is $R_3 = 4$ μm. Temperature distribution and isothermal contours demonstrated in fig. 3 (b, c). In device with layered structure (fig. 3) the object inside thermal invisibility cloak has temperature gradient of 3 K.

4. Conclusion
The mathematical methods of transformation optics were applied for thermodynamics. Thermal conductivity coefficient from thermodynamics is analogy of the optical refractive index. Thermal conductivity tensor was calculate for thermal invisibility cloak, which directs the heat flux around a certain area. The simulation was performed by the finite element method for two cases: with gradient k and with layered structure: material with $k < k'$ (mica) on the inner radius and another one with $k > k'$ on the outer radius (copper). The object in the thermal invisibility cloak keeps the constant
temperature in the first case with gradient thermal conductivity $k$ and it has temperature gradient of 3 K in the second case with the layered cloak.

References
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