Some Results on Generalized Sasakian Space Forms

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Abstract
In the present framework, we studied the semi generalized recurrent, semi generalized $\phi$-recurrent, extended generalized $\phi$-recurrent and concircularly locally $\phi$-symmetric on generalized Sasakian space forms.

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1 Introduction

The nature of a Riemannian manifold depends on the curvature tensor $R$ of the manifold. It is well known that the sectional curvatures of a manifold determine its curvature tensor completely. A Riemannian manifold with constant sectional curvature $c$ is known as a real space form and its curvature tensor is given by

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}.$$ 

A Sasakian manifold with constant $\phi$-sectional curvature is a Sasakian space form and it has a specific form of its curvature tensor. Similar notion also holds for Kenmotsu and cosymplectic space forms. In order to generalize such space forms in a common frame Alegre, Blair and Carriazo [1] introduced and studied generalized Sasakian space forms. These space forms are defined as follows:

A generalized Sasakian space form is an almost contact metric manifold $(M, \phi, \xi, \eta, g)$, whose curvature tensor is given by

$$R(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\},$$

where $f_1, f_2, f_3$ are functions on $M$.

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The Riemannian curvature tensor of a generalized Sasakian space form $M^{2n+1}(f_1, f_2, f_3)$ is simply given by

$$R = f_1R_1 + f_2R_2 + f_3R_3.$$  

where $f_1, f_2, f_3$ are differential functions on $M^{2n+1}(f_1, f_2, f_3)$ and

$$R_1(X,Y)Z = g(Y,Z)X - g(X,Z)Y,$$

$$R_2(X,Y)Z = g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z,$$

$$R_3(X,Y)Z = \eta(X)\eta(Y) - \eta(Y)\eta(X) + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi,$$

where $f_1 = \frac{c+3}{c^2}, f_2 = f_3 = \frac{1}{c - 1}$. Where $c$ denotes the constant $\phi$-sectional curvature. The properties of generalized Sasakian space form was studied by many geometers such as those mentioned in Refs. [2, 11, 12, 18, 21].

The concept of local symmetry of a Riemannian manifold has been studied by many authors in several ways to a different extent. The locally $\phi$-symmetry of Sasakian manifold was introduced by Takahashi in Ref. [26]. De et al., generalize the notion of $\phi$-symmetry and then introduced the notion of $\phi$-recurrent Sasakian manifold in Ref. [13]. Further $\phi$-recurrent condition was studied on Kenmotsu manifold [10], LP-Sasakian manifold [27] and $(LCS)_n$-manifold [22].

**Definition 1.** A Riemannian manifold $(M^{2n+1},g)$ is called a semi-generalized recurrent manifold if its curvature tensor $R$ satisfies [6, 9]

$$(\nabla_X R)(Y,Z)W = A(X)R(Y,Z)W + B(X)g(Z,W)Y,$$

where $A$ and $B$ are two 1-forms, $B$ is non-zero, $\rho_1$ and $\rho_2$ are two vector fields such that

$$g(X,\rho_1) = A(X), g(X,\rho_2) = B(X),$$

for any vector field $X,Y,Z,W$ and $\nabla$ denotes the operator of covariant differentiation with respect to the metric $g$.

**Definition 2.** A Riemannian manifold $(M^{2n+1},g)$ is semi generalized Ricci-recurrent if [6, 9]

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z) + (2n + 1)B(X)g(Y,Z),$$

where $A$ and $B$ are two 1-forms, $B$ is non-zero, $\rho_1$ and $\rho_2$ are two vector fields such that

$$g(X,\rho_1) = A(X), g(X,\rho_2) = B(X),$$

**Definition 3.** A Sasakian manifold $(M^{2n+1},\phi, \xi, \eta, g)$, $n \geq 1$, is said to be an extended generalized $\phi$-recurrent Sasakian manifold if its curvature tensor $R$ satisfies the relation

$$\phi^2(\nabla_W R)(X,Y)Z = A(W)\phi^2(R(X,Y)Z) + B(W)\phi^2(G(X,Y)Z)$$

for all vector fields $X,Y,Z,W$, where $A$ and $B$ are two non-vanishing 1-forms such that $A(X) = g(X,\rho_1), B(X) = g(X,\rho_2)$. Here $\rho_1$ and $\rho_2$ are vector fields associated with 1-forms $A$ and $B$ respectively.

**Definition 4.** A generalized Sasakian space form is said to be locally $\phi$-symmetric if

$$\phi^2(\nabla_W R)(X,Y)Z = 0$$

for all vector fields $X,Y,Z$ orthogonal to $\xi$. This notion was introduced by T. Takahashi for Sasakian manifolds [26].
In 1940, Yano introduce the concircular curvature tensor. A \((2n+1)\) dimensional concircular curvature tensor \(C\) is given by [30, 31]

\[
C(X,Y)Z = R(X,Y)Z - \frac{r}{2n(2n+1)} \{ g(Y,Z)X - g(X,Z)Y \},
\]

where \(R\) and \(r\) are the Riemannian curvature tensor and scalar curvature tensor, respectively.

Author in Ref. [5] studies the symmetric conditons of generalized Sasakian space forms with concircular curvature tensor such as \(C(\xi,X) \cdot C = 0\), \(C(\xi,X) \cdot R = 0\), \(C(\xi,X) \cdot S = 0\) and \(C(\xi,X) \cdot P = 0\). Recently, researcher in Ref. [28] investigate some symmetric condition on generalized Sasakian space forms with \(W_2\)-curvature tensor, such as pseudosymmetric, locally symmetric, locally \(\phi\)-symmetric and \(\phi\)-recurrent. Moreover many geometer’s studied the generalized Sasakian space forms with different conditions such as those mentioned in Refs. [11–13, 15, 16].

2 Generalized Sasakian space-forms

A \((2n+1)\)-dimensional Riemannian manifold is called an almost contact metric manifold if the following result holds [6], [7]:

\[
\phi^2 X = -X + \eta(X)\xi, \tag{6}
\]

\[
\eta(\xi) = 1, \quad \phi \xi = 0, \quad \eta(\phi X) = 0, \quad g(X,\xi) = \eta(X), \tag{7}
\]

\[
g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{8}
\]

\[
g(\phi X,Y) = -g(X,\phi Y), \quad g(\phi X,X) = 0 \tag{9}
\]

\[
(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y) \tag{10}
\]

for all vector field \(X\) and \(Y\). On a generalized Sasakian space form \(M^{2n+1}(f_1, f_2, f_3)\), we have ( [1, 15])

\[
(\nabla_X \phi)Y = (f_1 - f_3)(g(X,Y)\xi - \eta(Y)X), \tag{11}
\]

\[
\nabla_X \xi = -(f_1 - f_3)\phi X. \tag{12}
\]

Again, we know that from Ref. [1], \((2n+1)\)-dimensional generalized Sasakian space forms holds the following relations:

\[
S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y), \tag{13}
\]

\[
R(X,Y)\xi = (f_1 - f_3)\{ \eta(Y)X - \eta(X)Y \}, \tag{14}
\]

\[
R(\xi,X)Y = (f_1 - f_3)\{ g(X,Y)\xi - \eta(Y)X \}, \tag{15}
\]

\[
\eta(R(X,Y)Z) = (f_1 - f_3)\{ g(Y,Z)\eta(X) - g(X,Z)\eta(Y) \}, \tag{16}
\]

\[
S(X,\xi) = 2nf_1 f_2 - f_3 \eta(X). \tag{17}
\]

3 Semi generalized recurrent generalized Sasakian space forms

**Definition 5.** A generalized Sasakian space form \((M^{2n+1}, g)\) is semi-generalized recurrent manifold if

\[
(\nabla_X R)(Y,Z)W = A(X)R(Y,Z)W + B(X)g(Z,W)Y, \tag{18}
\]

here \(A\) and \(B\) are two 1-forms, \(B\) is non-zero, \(\rho_1\) and \(\rho_2\) are two vector fields such that

\[
A(X) = g(X,\rho_1) \quad \text{and} \quad B(X) = g(X,\rho_2)
\]
Definition 6. A generalized Sasakian space forms \((M^{2n+1}, g)\) is semi generalized Ricci-recurrent if
\[
(\nabla_X S)(Y, Z) = A(X)S(Y, Z) + (2n + 1)B(X)g(Y, Z).
\]
(19)

Permutating equation (3) twice with respect to \(X, Y, Z\), adding the three equations and using Bianchi second identity, we have
\[
A(X)R(Y, Z)W + B(X)g(Z, W) + A(Y)R(Z, X)W
+ B(Y)g(X, W)Z + A(Z)R(X, Y)W + B(Z)g(Y, W) = 0.
\]
(20)

Contracting (20) with respect to \(Y\), we get
\[
A(X)S(Z, W) + B(X)g(Z, W) - g(R(Z, X)\rho, W)
B(Z)g(X, W) - A(Z)S(X, W) + B(Z)g(X, W) = 0.
\]
(21)

Setting \(S(Y, Z) = g(QY, Z)\) in (21) and factoring off \(W\), we get
\[
A(X)QZ + (2n + 1)B(X)Z - R(Z, X)\rho + 2B(Z)X - A(Z)QX = 0.
\]
(22)

Again contracting with respect to \(Z\) and then substitute \(X = \xi\) in (22), one can get
\[
r = -\frac{1}{\eta(\rho_1)} \{((2n + 1)^2 - 2\eta(\rho_2) - 2n(f_1 - f_3)[\eta(\rho_2) + \eta(\rho_1)]\}.
\]
(23)

Now, we can state the following statement

Theorem 1. The scalar curvature \(r\) of a semi-generalized recurrent generalized Sasakian space forms is related in terms of contact forms \(\eta(\rho_1)\) and \(\eta(\rho_2)\) is given in (23).

Next, we prove the semi generalized Ricci-recurrent generalized Sasakian space form, inserting \(Z = \xi\) in (19), we have
\[
2n(f_1 - f_3)^2g(W, \phi Y) + (f_1 - f_3)S(Y, \phi W) = A(X)2n(f_1 - f_3)\eta(Y) + (2n + 1)B(X)\eta(Y).
\]
(24)

Again setting \(Y = \xi\) in (24), we get
\[
A(X)2n(f_1 - f_3) + (2n + 1)B(X) = 0.
\]
(25)

Now, we can state the following theorem

Theorem 2. A semi-generalized Ricci-recurrent generalized Sasakian space forms, the 1-form \(A\) and \(B\) holds (25)

4 Semi generalized \(\phi\)-recurrent generalized Sasakian space forms

Definition 7. A generalized Sasakian space form \((M^{2n+1}, g)\) is called semi-generalized \(\phi\)-recurrent if its curvature tensor \(R\) satisfies the condition
\[
\phi^2(\nabla_W R)(X, Y)Z = A(W)R(X, Y)Z + B(W)g(Y, Z)X
\]
(26)

where \(A\) and \(B\) are two 1-forms, \(B\) is non-zero and these are defined by
\[
A(W) = (W, \rho_1), \quad B(W) = (W, \rho_2)
\]
and \(\rho_1\) and \(\rho_2\) are vector fields associated with 1-forms \(A\) and \(B\) respectively.
Let us consider a semi-generalized $\phi$-recurrent generalized Sasakian space forms. Then by virtue of (6) and (26), we have
\[
- (\nabla_W R)(X,Y)Z + \eta((\nabla_W R)(X,Y)Z)\xi = A(W)R(X,Y)Z + B(W)g(Y,Z)X. \tag{27}
\]
it follows that
\[
- g((\nabla_W R)(X,Y)Z,U) + \eta((\nabla_W R)(X,Y)Z)\eta(U) = A(W)g(R(X,Y)Z,U) + B(W)g(Y,Z)g(X,U). \tag{28}
\]
Let $e_i, i = 1,2,..n$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (28) and taking summation over $i, 1 \leq i \leq (2n+1)$, we get
\[
- (\nabla_W S)(Y,Z) + \sum_{i=1}^{2n} \eta((\nabla_W R)(e_i,Y)Z)\eta(e_i) = A(W)2n(f_1 - f_3)S(Y,Z) + B(W)(2n+1)g(Y,Z). \tag{29}
\]
The second term of left hand side of (29) by putting $Z = \xi$ takes the form $((\nabla_W R)(e_i,Y)Z,\xi) = 0$. So, by replacing $Z$ by $\xi$ in (29) and with the help of (7) and (12), we get
\[
- 2n(f_1 - f_3)^2g(W,\phi Y) + (f_1 - f_3)S(Y,\phi W) = A(W)2n(f_1 - f_3)\eta(Y) + B(W)(2n+1)g(Y,Z). \tag{30}
\]
Inserting $Y = \xi$ in (30) and using (7), we have
\[
- 2n(f_1 - f_3)A(W) = (2n+1)B(W). \tag{31}
\]
In view of (31) and replace $Y$ by $\phi Y$, (30) yields
\[
S(Y,W) = 2n(f_1 - f_3)g(Y,W). \tag{32}
\]

**Theorem 3.** A semi generalized $\phi$-recurrent generalized Sasakian space forms $(M^{2n+1}, g)$ is an Einstein manifold and moreover: the 1-forms $A$ and $B$ are related as $-2n(f_1 - f_3)A(W) = (2n+1)B(W)$.

## 5 Extended generalized $\phi$-recurrent generalized Sasakian space forms

According to the definition of extended generalized $\phi$-recurrent Sasakian manifolds, we will define the Extended generalized $\phi$-recurrent generalized Sasakian space forms

**Definition 8.** A generalized Sasakian space forms $(M^{2n+1}, \phi, \xi, \eta, g)$, $n \geq 1$, is said to be an extended generalized $\phi$-recurrent generalized Sasakian space forms if its curvature tenor $R$ satisfies the relation
\[
\phi^2(\nabla_W R)(X,Y)Z = A(W)\phi^2(R(X,Y)Z) + B(W)\phi^2(G(X,Y)Z) \tag{32}
\]
for all vector fields $X, Y, Z, W$, where $A$ and $B$ are two non-vanishing 1-forms such that $A(X) = g(X, \rho_1), \ B(X) = g(X, \rho_2)$. Here $\rho_1$ and $\rho_2$ are vector fields associated with 1-forms $A$ and $B$ respectively.

Let us consider an extended generalized $\phi$-recurrent generalized Sasakian space forms. Then by virtue of (6), we have
\[
- (\nabla_W R)(X,Y)Z + \eta((\nabla_W R)(X,Y)Z)\xi = A(W)[-R(X,Y)Z + \eta(R(X,Y)Z)]
+ B(W)[-G(X,Y)Z + \eta(G(X,Y)Z)]. \tag{33}
\]
From which it follows that

\[-g((\nabla_w R)(X,Y)Z,U) + \eta((\nabla_w R)(X,Y)Z)\eta(U)\]

\[= A(W)\{-g(R(X,Y)Z,U) + \eta(R(X,Y)Z)\eta(U)\} + B(W)\{-g(G(X,Y)Z,U) + \eta(G(X,Y)Z)\eta(U)\}. \tag{34}\]

Let \(e_i, \ i = 1, 2, \ldots, n\) be an orthonormal basis of the tangent space at any point of the manifold. Then putting \(X = U = e_i\) in (34) and taking summation over \(i, \ 1 \leq i \leq (2n + 1)\), and the relation \(g((\nabla_w R)(X,Y)Z,U) = -g((\nabla_w R)(X,Y)U,Z)\), we get

\[-(\nabla_w S)(Y,Z) = A(W)\{-S(Y,Z) + \eta(R(\xi,Y)Z)\} + B(W)\{-(2n-1)g(Y,Z) - \eta(Y)\eta(Z)\}. \tag{35}\]

It follows that,

\[(\nabla_w S)(Y,Z) = A \otimes S(Y,Z) + Kg(Y,Z) + \mu \eta(Y)(Z). \tag{36}\]

where \(K = [(2n-1)B(W) - A(W)(f_1 - f_3)]\) and \(\mu = [(f_1 - f_3)A(W) + B(W)]\).

Inserting \(Z = \xi\) (35) and using (12), (17) and (7), we get

\[2n(f_1 - f_3)^2g(W,\phi Y) + (f_1 - f_3)S(Y,\phi W) = \{2n(f_1 - f_3)A(W) + 2nB(W)\} \eta(Y). \tag{37}\]

Again inserting \(Y = \xi\) and using (7), (37) yields

\[2n(f_1 - f_3)A(W) + 2nB(W) = 0. \tag{38}\]

By taking the account of (38) in (37) and then replace \(Y\) by \(\phi Y\), we get

\[S(Y,W) = 2n(f_1 - f_3)g(Y,W). \]

Thus we have the following assertion

**Theorem 4.** An extended generalized \(\phi\)-recurrent generalized Sasakian space forms is an Einstein manifold and moreover the associated 1-forms \(A\) and \(B\) are related by \((f_1 - f_3)A + B = 0\).

It is known that a generalized Sasakian space form is Ricci-semisymmetric if and only if it is an Einstein manifold. In fact, by **Theorem 4**, we have the following:

**Corollary 5.** An extended generalized \(\phi\)-recurrent generalized Sasakian space forms is Ricci-semisymmetric.

### 6 Concircularly locally \(\phi\)-symmetric generalized Sasakian space forms

**Definition 9.** A \((2n + 1)\) dimensional \((n > 1)\) generalized Sasakian space form is called concircularly locally \(\phi\)-symmetric if it satisfies [12].

\[\phi^2(\nabla_w C)(X,Y)Z = 0.\]

for all vector fields \(X, Y, Z\) are orthogonal to \(\xi\) and an arbitrary vector field \(W\).

Differentiate covariantly with respect \(W\), we have

\[(\nabla_w C)(X,Y)Z = (\nabla_w R)(X,Y)Z - \frac{dr(W)}{2n(2n+1)}\{g(Y,Z)X - g(X,Z)Y\}. \tag{39}\]
Operate $\phi^2$ on both side, we have

$$\phi^2(\nabla_W C)(X,Y)Z) = \phi^2((\nabla_W R)(X,Y)Z) - \frac{dr(W)}{2n(2n+1)} \{g(Y,Z)\phi^2 X - g(X,Z)\phi^2 Y\}. \tag{40}$$

In view of (6), and taking the help of relation (1) with $X,Y,Z$ are orthogonal vector field, one can get

$$\phi^2(\nabla_W C)(X,Y)Z) = df_1(W)\{g(Y,Z)X - g(X,Z)Y\}$$
$$+ df_2(W)\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\}$$
$$+ f_2\{g(X,\phi Z)(\nabla_W \phi)Y + g(X,(\nabla_W \phi)Z)\phi Y$$
$$- g(Y,\phi Z)(\nabla_W \phi)X - g(Y,(\nabla_W \phi)Z)\phi X$$
$$+ 2g(X,\phi Y)(\nabla_W \phi)Z + 2g(X,(\nabla_W \phi)Y)\phi Z\}$$
$$+ \frac{dr(W)}{2n(2n+1)} \{g(Y,Z)X - g(X,Z)Y\}. \tag{41}$$

If the manifold is conformally flat then $f_2 = 0$. Therefore, (41) yields

$$\phi^2(\nabla_W C)(X,Y)Z) = \left\{df_1(W) + \frac{dr(W)}{2n(2n+1)}\right\} \{g(Y,Z)X - g(X,Z)Y\}.$$

Hence we can state the following theorem

**Theorem 6.** A generalized Sasakian space forms is concircularly locally $\phi$-symmetric if and only if $f_1$ and the scalar curvature are constant

**Note 7.** In [18], U. K. Kim studied generalized Sasakian space forms and proved that if a generalized Sasakian space forms $M^{2n+1}(f_1,f_2,f_3)$ of dimension greater than three is conformally flat and $\xi$ is Killing, then it is locally symmetric. Moreover, if $M^{2n+1}(f_1,f_2,f_3)$ is locally symmetric, then $f_1 - f_3$ is constant. In the above theorem it is shown that a conformally flat generalized Sasakian space form of dimension greater than 3 is locally $\phi$-symmetric if and only if $f_1$ and scalar curvature is constant. Thus, we observe the difference between locally symmetric generalized Sasakian space forms and conformally locally $\phi$-symmetric generalized Sasakian space forms.

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