ON A Topological $\Gamma$ - Rings

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Abstract

In this study we introduce the concepts of topological $\Gamma$-ring as generalization of Topological ring. We also present and study the concepts of topological $\Gamma$-ring, Homomorphism of topological $\Gamma$-ring, and compact of topological $\Gamma$-ring. The results have confirmed that: if $(G, +, \cdot, J)$ is a compact topological $\Gamma$-ring and $f : (G, +, \cdot, J) \rightarrow (G', +', \cdot', J')$ epimorphism topological $\Gamma$-ring Then $G'$ is compact.

Keywords: topological $\Gamma$-ring, norm, homomorphism and compact.

1. INTRODUCTION

The concepts of topological ring is one of the most important topics in topological algebra. The objective of the concept of topological $\Gamma$-ring is to generalize the definition of topological ring.

The concept of $\Gamma$-ring was presented by Nobusawa 1964 in [1], and it was generalized by Barnes 1966 [2] as below:

Let $M$ and $\Gamma$ be two additive abelian groups. Suppose that there is a mapping from $M \times \Gamma \times M \rightarrow M$, the image of $(a, \alpha, b)$ is denoted by $a\alpha b$, $a, b \in M$ and $\alpha \in \Gamma$, that satisfies the following for all $a, b, c \in M$, $\alpha, \beta \in \Gamma$:

1. $a + (b\alpha c) = a\alpha (b + c)$
2. $a(\alpha + \beta)c = a\alpha c + a\beta c$
3. $a(a + b) = a\alpha b + a\beta c$
4. $(a\alpha b)\beta c = a\alpha (b\beta c)$

Then $M$ is called a $\Gamma$-ring., where every ring is a $\Gamma$-ring. $M$ is said to be 2-torsion free if $2a = 0$ implies $a = 0$ for all $a \in M$. Besides, $M$ is called a prime $\Gamma$-ring if for all $a, b \in M$, $a\Gamma Mb = (0)$ implies either $a = 0$ or $b = 0$, and $M$ is called a semiprime if $a\Gamma Ma = (0)$ with $a \in M$ implies $a = 0$.

Note that every prime $\Gamma$-ring is obviously a semiprime [3], [4], [5-10].

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In this paper we present the concept of topological $\Gamma$-ring as generalization of topological ring where we get every topological ring is topological $\Gamma$-ring. In general the converse is not true.

In this paper we introduce and study the concepts of topological $\Gamma$-ring, norm, homomorphism of topological $\Gamma$-ring some type of homomorphism kernel of homomorphism, and natural mapping as well as we present the compact of topological $\Gamma$-rings. We also prove that: If $f: (G, +, \cdot, T) \rightarrow (G_1, +, \cdot, T^*)$ is homomorphism of topological $\Gamma$-ring $G$ onto topological $\Gamma$-ring $G_1$ then

1) If $I$ is an ideal of topological $\Gamma$-ring of $G$ then $f(I)$ is an ideal of topological $\Gamma$-ring $G_1$.

2) If $J$ is an ideal of topological $\Gamma$-ring of $G_1$ then $f^{-1}(J)$ is an ideal of topological $\Gamma$-ring $G$.

The results have confirmed that: If $f: (G, +, \cdot, J) \rightarrow (G^*, +, \cdot, \cdot, T^*)$ is isomorphism topological $\Gamma$-ring from $G$ into compact topological $\Gamma$-ring $G^*$ then $G$ is compact topological $\Gamma$-ring.

2. Topological $\Gamma$-Rings:

Basic concepts related to the talked topic are given as follows with few important properties has been presented in the next section.

Definition (2-1): A topology $\tau$ on a $\Gamma$-ring $G$ is a topological $\Gamma$-ring, which is denoted by $(G, +, \cdot, \tau)$, if and only if:

i) $(m, n) \rightarrow m + n$ is continuous from $G \times G$ into $G$

ii) $(m, y)_a \rightarrow m \cdot a$ is continuous from $G \times G$ into $G$

iii) $m \rightarrow m$ is continuous from $G \times G$ into $G$

where $G$ is given topology $\tau$ and $G \times G$ the Cartesian product topology determined by $\tau$.

Example (2-2): Let $G$ be any $\Gamma$-ring and be discrete topology on $G$ then $(G, +, \cdot, \tau)$ is topology $\Gamma$-ring, which is called discrete topological $\Gamma$-ring.

Example (2-3): Let $G$ be any $\Gamma$-ring and be indiscrete topology on $G$ then $(G, +, \cdot, \tau)$ is topology $\Gamma$-ring, which is called indiscrete topological $\Gamma$-ring.

Example (2-4): Let $R$ be the set of all real numbers and $Q$ be the set of all rational numbers then $R$ is $Q$-ring if $\tau$ is usual topology on $R$, and then $(R, +, \cdot, \tau)$ is topological $Q$-ring, which is called usual topological $\Gamma$-ring. By Definition (2.1), $G = R$ and $\Gamma = Q$, also the operations $+$, $\cdot$ and $-$ are continuous from $R \times R$ into $R$.

Since every ring is $\Gamma$-ring, then we can get the following lemma.

Lemma (2-5): Every topological ring is topological $\Gamma$-ring.

The next example shows that the converse of Lemma (2-5) is not true in general.

Example (2-6): Let $G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d, s, t \in \mathbb{Z} \right\}$ and $\Gamma = \left\{ \begin{pmatrix} x & y & z \\ w & u & v \end{pmatrix} : x, y, z, w, u, v \in \mathbb{Z} \right\}$ then $G$ is $\Gamma$-ring, if we define the discrete topology $\tau$ on $G$ then $(G, +, \cdot, \tau)$ is topological $\Gamma$-ring. However $(G, +, \cdot, \tau)$ is not topological ring.

Definition (2-7): A function $N$ from a $\Gamma$-ring $G$ into $R^+ \cup \{0\}$ is a norm if the following conditions hold for all $p, h \in G$ and $\alpha \in \Gamma$:

1) $N(0) = 0$

2) $N(p + h) \leq N(p) + N(h)$

3) $N(p) = N(-p)$

4) $N(p\alpha) \leq N(p)N(\alpha)$

5) $N(p) = 0$ if and only if $p = 0$.

Remark (2-8): If $N$ is a norm on a $\Gamma$-ring $G$, then $d$ defined by $d(p, h) = N(p - h)$ for all $p, h \in G$ and $\alpha \in \Gamma$, is a metric. Indeed (1) and (5) imply that $d(p, h) = 0$ iff $p = h$, (3) implies that $d(p, h) = d(h, p)$, and (2) yields the triangle inequality.

Theorem (2.8): Let $N$ be a norm on a $\Gamma$-ring. The topology is given by the metric $d$ which is defined by $N$ is a $\Gamma$-ring topology.

Proof: Let $a, b, p, h \in G$ and $\alpha \in \Gamma$. 

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\[d(p+h, a+b) = N((p+h)-(a+b)) = N((p-a)+(h-b)) \leq N(p-a) + N(h-b) = d(p,a) + d(h-b)\]

Hence the definition (2-1) (1) holds, for all \(a, b \in G\)

\[d(-p-a) = N(-p+a) = N(p-a) = d(p,a)\]

By (3) of definition 2-6 hence the definition 2-1 (2) holds. Finally for all \(p, a \in G\)

\[d(pah, aab) = N((p-a) \alpha (h-b) + a\alpha(h-b) + (p-a)(ab)) \leq N(a)N(h-b) + N(a)\alpha N(h-b) + N(p-a)\alpha N(b)\]

Hence the definition 2-1 (3) holds, and the proof is finished

Let \(f\) be a partial function from the carrier of \(S\) to \(R\), we recall that \(f\) is said to be uniformly continuous on \(X\) if and only if the following conditions are satisfied (i) \(X \subseteq \text{dom } f\), and (ii) for every \(r > 0\) there exists \(s > 0\) such that \(|f(x_1) - f(x_2)| < r\) from \(f\) to \(G\) into \(R^+ \cup \{0\}\).

**Theorem (2-9):** Let \(N\) be a norm on a \(\Gamma\)-ring \(G\). For all \(p, h \in G\), then \(|N(p) - N(h)| \leq N(p-h)\), and hence \(N\) is a uniformly continuous function.

**Proof:** \(N(p) = N((p-h)+h) \leq N(p-h)+N(h)\)

Therefore \(|N(p) - N(h)| \leq N(p-h)\)

**Definition (2-10):** Let \((G, +, \cdot)\) be a \(\Gamma\)-ring. A basis of \(\tau\) is a collection \(\mathcal{B}\) of \(\tau\) such that \(\mathcal{B} = \bigcup\{B : B \in \mathcal{B}_1\}\).

**Theorem (2-11):** The Cartesian product of \(\Gamma\)-rings is a \(\Gamma\)-ring.

**Proof:** Let \((G, +, \cdot)\) and \((G_1, +, \cdot)\) are two \(\Gamma\)-rings and \(\mathcal{B}\) be a basis of \(\tau\) and \(\mathcal{C}\) be a basis for \(\tau_1\) then

\[\mathcal{H} = \{B \times C : B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}\]

is a basis for the product topology \((G \times G_1, +, \cdot, \mathcal{W})\).

Let \((x,y)\) be any point of \(G \times G_1\) and \(N\) be a neighborhood of \((x,y)\)

Since \(N = \{X \times Y : x \in \tau \text{ and } Y \in \tau_1\}\) is a basis for \(\mathcal{W}\) there exists a member \(X \times Y\) of \(\tau\) such that \((x,y)\) is in \(X \times Y\) (1)

Since \(X\) is \(\tau\)-open and \(B\) is a basis for \(\tau\) there exists some \(B \in \mathcal{B}\) such that \(x \in B \subseteq X\). Similarly there exists some \(C \in \mathcal{C}\) such that \(y \in C \subseteq Y\). It follows that

\[(x,y) \in B \times C \subseteq X \times Y\] (2)

Hence from (1) and (2), we get \((x,y) \in B \times C \subseteq N\). This implies that \(\mathcal{H}\) is basis for \(G \times G_1\).

### 3. HOMOMORPHISM OF TOPOLOGICAL \(\Gamma\)-RINGS:

We introduce and study in this section the following concepts: subtopological \(\Gamma\)-ring, ideal of topological \(\Gamma\)-ring, and homomorphism of topological \(\Gamma\)-ring as well as some properties of them are given.

**Definition (3-1):** Let \((G, +, \cdot, T)\) be topological \(\Gamma\)-ring a subring \(S\) of \(G\)-ring is called a subtopological \(\Gamma\)-subring (or called subtopological \(\Gamma\)-ring) denoted by \((S, +, \cdot, T_S)\) where \(A_S \in T_S\) \(A_S = A \cap S\), for all \(A \in T\).

Every topological \(\Gamma\)-ring \((G, +, \cdot, T)\) have two topologies \(\Gamma\)-subrings \((0)\) and \(G\) itself which is called trivial topological \(\Gamma\)-subrings. Any topological \(\Gamma\)-subring except trivial is called proper topological \(\Gamma\)-subring.

**Definition (3-2):** A topological \(\Gamma\)-subring \((I, +, \cdot, T)\) of topological \(\Gamma\)-ring \((G, +, \cdot, T)\) is called an ideal of \((G, +, \cdot, T)\) if \(I \subseteq I\) and \(G I \subseteq I\).

**Definition (3-3):** Let \((G, +, \cdot, T)\) and \((G_1, +, \cdot, T)\) be two topological \(\Gamma\)-rings.

A map \(f: (G, +, \cdot, T) \rightarrow (G_1, +, \cdot, T)\) is called homomorphism of topological \(\Gamma\)-ring, if
\( f: (G, T) \rightarrow (G_1, T^\ast) \) is continuous and \( f \) satisfies:

\[ f(p + h) = f(p) + f(h), \text{and } f(p \cdot h) = f(p) \cdot f(h), \text{ for all } p, h \in G \text{ and } \alpha \in \Gamma. \]

An epimorphism topological \( \Gamma \)-ring if \( f \) is continuous map and epimorphism \( \Gamma \)-ring, a monomorphism topological \( \Gamma \)-ring is a continuous mapping and monomorphism \( \Gamma \)-ring. An isomorphism topological \( \Gamma \)-ring is a homeomorphism topological map (A mapping \( f: (X, T_X) \rightarrow (Y, T_Y) \) is called homeomorphism if \( f \) is 1-1 and onto also \( f \) and \( f^{-1} \) are continuous) and isomorphic \( \Gamma \)-rings, an automorphism topological \( \Gamma \)-ring is a continuous mapping and automorphism \( \Gamma \)-ring.

**Lemma (3-4):** Let \( f: (G, +, \cdot, T) \rightarrow (G_1, +', \cdot', T^\ast) \) be isomorphism of topological \( \Gamma \)-ring then:

1) If \( S \) is a subtopological \( \Gamma \)-ring of \( G \), then \( f(S) \) is a subtopological \( \Gamma \)-ring of \( G_1 \).
2) If \( E \) is a subtopological \( \Gamma \)-ring of \( G_1 \), then \( f^{-1}(E) \) is a subtopological \( \Gamma \)-ring of \( G \).

**Proof:** We prove (2) and (1) by using the same technical

2) By assumption we get

\((E, +, \cdot, T^\ast_E)\) is subtopological \( \Gamma \)-ring of \((G_1, +', \cdot', T^\ast)\)

Since \( f \) is homomorphism

Therefore for all \( E \in T^\ast_E \) then \( f^{-1}(E) \in T^\ast_{f^{-1}(E)} \)

Now, let \( p, h \in f^{-1}(E) \) then \( f(p), 22222a \) \( f(h) \in E \) and \( \alpha \in \Gamma \).

i) \( f(p) - f(h) = f(p - h) \)

Since \( f(p), f(h) \in E \) and \( E \) is subtopological \( \Gamma \)-ring then \( f(p) - f(h) \in E \), and hence

\[ p \in f^{-1}(E) \]

ii) \( f(p \cdot h) = f(p) f(h) \)

Since \( f(p), f(h) \in E \) and \( E \) is subtopological \( \Gamma \)-ring then \( f(p \cdot h) \in E \) hence \( p \cdot h \in f^{-1}(E) \).

Therefore \( f^{-1}(E) \) is subtopological \( \Gamma \)-ring.

**Lemma (3-5):** If \( f: (G, +, \cdot, T) \rightarrow (G_1, +', \cdot', T^\ast) \) is homomorphism of topological \( \Gamma \)-ring \( G \) onto topological \( \Gamma \)-ring \( G_1 \), then

1) If \( I \) is an ideal of topological \( \Gamma \)-ring \( G \) then \( f(I) \) is an ideal of topological \( \Gamma \)-ring \( G_1 \).
2) If \( J \) is an ideal of topological \( \Gamma \)-ring of \( G_1 \) then \( f^{-1}(J) \) is an ideal of topological \( \Gamma \)-ring \( G \).

**Proof:**

1) \( i) \) By theorem (3-4) \( f(I) \) is a subtopological \( \Gamma \)-ring

(\( i.e. \) \( f(I) \neq \emptyset \) and \( a - b \in f(I), \forall a, b \in f(I) \))

\( ii) \) Let \( \cdot \in G_1 \) and for all \( a \in f(I) \), and \( \alpha \in \Gamma \).

T.P. \( \alpha' \cdot \alpha' \cdot r' \in f(I) \) and \( \alpha' \cdot \alpha' \cdot a \in f(I) \)

\( \therefore \) \( f \) is onto, then \( \exists r \in G \) s.t., \( r = f(r) \) \( \cdots \cdots \ast \)

and \( \exists x \in I \) s.t. \( a = f(x) \) \( \cdots \cdots \ast \)

since, \( r \in R \), \( x \in I \) and since \( I \) is an ideal of a ring

\[ \Rightarrow x \cdot a \cdot r \in I \text{ and } r \cdot a \cdot x \in I \]

\[ \Rightarrow f(x) \cdot a \cdot r \in f(I) \text{ and } f(r) \cdot a \cdot x \in f(I) \]

\[ \therefore f(x) \cdot a \cdot r \in f(I) \text{ and } f(r) \cdot a \cdot x \in f(I) \]

\[ \therefore a' \cdot \alpha' \cdot r \in f(I) \text{ and } \alpha' \cdot \alpha' \cdot a \in f(I) \]

Therefore, \( f(I) \) is an ideal of topological \( \Gamma \)-ring \( G_1 \).

2) By using the same technic of (1) we get the require result.

**Theorem (3-6):** If \( f: (M, +, \cdot, T) \rightarrow (M_1, +', \cdot', T_1) \) is homomorphism of topological \( \Gamma \)-ring \( M \) onto topological \( \Gamma \)-ring \( M_1 \) then every homomorphism image of

1) If \( M \) is commutative topological \( \Gamma \)-ring then \( M_1 \) is a commutative \( \Gamma \)-ring.
2) If \( M \) is topological \( \Gamma \)-ring with identity \( 1 \) then \( M_1 \) is a topological \( \Gamma \)-ring with identity \( 1' \).

Proof: Since \( f \) is a homomorphism from a topological \( \Gamma \)-ring \((M, +, \cdot, T)\) into a topological \( \Gamma \)-ring \((M_1, +', \cdot', T_1)\).

1) Suppose that \((M, +, \cdot, T)\) be a commutative topological \( \Gamma \)-ring

Let \( a, b \in M_1 \), \( \alpha \in \Gamma \) and since \( f \) is onto ,
Then, \( x, y \in M \) s.t. \( a = f(x) \) and \( b = f(y) \)
\[ a \alpha b = f(x)af(y) = f(x \alpha y) \quad \text{(since } f \text{ is homo.)} \]
\[ = f(y \alpha x) \quad \text{(since } M \text{ is a comm. topological } \Gamma \text{-ring)} \]
\[ = f(y) \cdot f(x) \quad \text{(since } f \text{ is homo.)} \]
\[ = b \alpha a \]

Hence, \( M_1 \) is a commutative topological \( \Gamma \)-ring

2) Suppose that \( M \) be a topological \( \Gamma \)-ring with identity
Let \( a \in M_1 \) since \( f \) is onto therefore \( \exists x \in M \) s.t. \( a = f(x) \)
\[ x \alpha 1 = 1 \alpha x = x, \forall x \in M, a \in \Gamma. \]
\[ \Rightarrow f(x \alpha 1) = f(1 \alpha x) = f(x) \]
\[ \Rightarrow f(x) = f(1) \alpha f(x) \quad \text{[} f \text{ is homomorphism]} \]
\[ \Rightarrow a \alpha 1' = 1' \alpha a = a, \text{ for all } a \in M_1. \]
Thus, \( M_1 \) is a topological \( \Gamma \)-ring with identity \( 1' \).

**Definition (3-7):** Let \( f: (M, +, \cdot, T) \to (M_1, +', \cdot', T_1) \) is homomorphism of topological \( \Gamma \)-ring s.th. then the kernel of \( f \) is denoted by \( \ker. (f) \), and it is defined by \( \ker. (f) = \{ meM: f(m) = 0_{M_1} \} \)

**Lemma (3-8):** If \( f: (G, +, \cdot, T) \to (G^*, +', \cdot', T^*) \) is homomorphism of topological \( \Gamma \)-ring \( G \) into topological \( \Gamma \)-ring \( G^* \), then the \( \ker. (f) \) is an ideal of topological \( \Gamma \)-ring \( G \).

**Proof:** 1) Let \( p, h \in \ker. (f) \) and \( f \) is homomorphism of topological \( \Gamma \)-ring ..
we get \( f(p - h) = f(p) - f(h) = 0_G \)
then \( p - h \in \ker. (f) \)
2) Let \( p, h \in \ker. (f), a \in \Gamma \) and \( f \) is homomorphism of topological \( \Gamma \)-ring
we get \( f(pah) = f(p)af(h) = 0_G \)
then \( pah \in \ker. (f) \)
Hence \( \ker. (f) \) is an ideal of topological \( \Gamma \)-ring \( (G, +, \cdot, T) \).

**Theorem (3-9):** If \( f \) is homomorphism of topological \( \Gamma \)-ring \( (G, +, \cdot, T) \) into topological \( \Gamma \)-ring \( (G^*, +', \cdot', T^*) \) then \( f \) is one-one if \( \ker. (f) = \{0\} \).

**Proof:** Suppose that \( \ker. (f) = \{0\} \)
Let \( r, e \in G \) s.t. \( f(r) = f(e) \) \( \Rightarrow f(r - e) = 0_G \)
\( f(r - e) = 0_G \quad \text{[} f \text{ is homo.]} \)
\( r - e \in \ker. f \)
But, \( \ker. f = \{0\} \) \( \Rightarrow r - e = 0 \Rightarrow r = e \)
Suppose that \( f \) is one to one.
Let \( r \in \ker. f \) \( \Rightarrow f(r) = 0 \) and since \( f(0) = 0 \), therefore \( f(r) = f(0) \)
Since, \( f \) is one to one \( \Rightarrow r = 0 \Rightarrow r \in \{0\} \Rightarrow \ker. (f) \subseteq \{0\} \).
Since \( f(0) = 0 \)
\( \Rightarrow 0 \in \ker. f \Rightarrow \{0\} \subseteq \ker. (f) \)
\( \ker. (f) = \{0\} \).

**Theorem (3-10):** If \( (I, +, \cdot) \) is an ideal of the topological \( \Gamma \)-ring \( (M, +, \cdot, T) \) then the natural map is a homomorphism from a topological \( \Gamma \)-ring \( M \) to the topological quotient \( \Gamma \)-ring \( (M/I, +, \cdot, T) \) with kernel equal to \( I \).

**Proof:** Let \( m, n \in M, \alpha \in \Gamma \)
1- \( nat_1(m + n) = (m + n) + 1 = (m + 1) + (E + I) = nat_1(m) + nat_1(n) \)
2- \( nat_1(m \alpha n) = (m \alpha n) + 1 = (m + 1) \alpha (E + I) = nat_1(m \alpha n) \)
\( nat_1: (M, T) \to (M/I, T) \) is continuous map since every open set \( V \) in \( M/I \) then \( nat_1^{-1}(V) \) is open in \( M \).

Hence \( nat_1 \) is homomorphism
Now, to prove \( nat_1 \) is onto
\( \forall m + I \in R/I \Rightarrow \exists m \in R \) s.t. \( nat_1(m) = m + I \)
\[ \text{nat}_I \text{ is onto.} \]

\[ \text{Ker.}(\text{nat}_I) = \{ m \in R : \text{nat}_I(m) = 0 + I \} = \{ m \in R : m + I = 0 + I \} = \{ a + I : a \in I \} = I \]

**Theorem (3-11):** If \( f \) is homomorphism of topological \( \Gamma \)-ring \((G,+, . , T)\) into topological \( \Gamma \)-ring \((G^*,+, . , T^*)\) then \( M/\ker.((f)) \) is isomorphic to topological \( \Gamma \)-ring \( G^* \).

**Proof:** We define a map \( g \) from topological \( \Gamma \)-ring \((M/\ker.((f)),+, . , T)\) into topological \( \Gamma \)-ring \((G^*,+, . , T^*)\) by \( g(E+\ker.(f)) = f(E) \), for all \( E \in G \).

It's clear that \( g \) is well defined and one to one.

Now, to show \( g \) is homomorphism.

Let \( p \in \ker.((f)) \) and \( h \in \ker.((f)) \)\)

\[ g(p+h) = g(p) + g(h) = f(p) + f(h) = f(p+h) \]

To prove \( g \) is onto.

Let \( E \in \ker.((f)) \) where \( m \in G \), since \( f \) is onto then there exist \( f(E) \in G^* \) such that \( g(E+\ker.((f))) = f(E) \) hence \( g \) is onto.

Hence \( g \) is isomorphism \( \Gamma \)-ring.

Now, since \( f \) is continuous therefore \( g \) is continuous.

Let \( U \) be an open set in \( G/\ker(f) \)

Since \( g(E+\ker(f)) = f(E) \) and \( f \) is continuous, so it \( g^{-1} \) is continuous.

\[ g^{-1} : G^* \rightarrow G/\ker.((f)) \]

Hence the topological \( \Gamma \)-ring \( G/\ker.((f)) \) is isomorphic topological \( \Gamma \)-ring to \( G^* \).

**Theorem (3-12):** If \( M \) is a topological \( \Gamma \)-ring let \( \Sigma \) be a subring, and \( I \) be an ideal of \( M \).

Then:

1. \( S + I = \{ s + a : s \in S, a \in I \} \) is a topological subring of \( M \).
2. \( S \cap I \) is an ideal of \( S \).
3. \( S + I/ I \) is isomorphic topological \( \Gamma \)-ring to \( S/ I \cap I \).

**Proof:** It is clear so that the proof is omitted.

**Theorem (3-13):** Let \((M,+, . , T)\) be a topological \( \Gamma \)-ring and \( J \subset I \) be ideals of \( M \). Then \( I/J \) is an ideal of \( M/J \) and

\[ \frac{M/I}{J} \cong \frac{M}{J} \]

**Proof:** Since \( I \) and \( J \) are ideals, they are non-empty and so that \( I/J = \{ a + J : a \in I \} \) is also non-empty. Let \( a_1, a_2 \in I \); \( m \in M \), and \( \alpha \in \Gamma \). By definition of addition, and multiplication of the cosets, we have \((a_1 + J) + (a_2 + J) = a_1 + a_2 + J \), \((m + J)\alpha(a_1 + J) = m\alpha a_1 + J \), and \((a_1 + J)\alpha(m + J) = a_1\alpha m + J \). Since \( I \) is an ideal, \( a_1 + a_2 \), \( m\alpha a_1 \), and \( a_1\alpha m \) are contained in \( I \) so that \( I/J \) is an ideal of \( M/J \). Let \( \varphi : M/J \rightarrow M/I \) that sends \( m + J \) to \( m + I \). It is clear that this is a well-defined surjective homomorphism with kernel equal to \( I/J \), and \( \varphi \) is also open map. Then \((M/J)/(I/J)\) is isomorphism topological \( \Gamma \)-ring to \( M/I \) by the first isomorphism theorem.

4. **COMPACT TOPOLOGICAL \( \Gamma \)-RINGS:**

**Definition (4-1):** Let \((G,+, . , J)\) be a topological \( \Gamma \)-ring, the family\{ \( G_i \in \{ (G_n,+, . , J) \} \) is a proper subrings of \( G \), for all \( i \in \Lambda \) is a cover topological \( \Gamma \)-ring of \((G,+, . , J)\) if \( G = \bigcup_{i \in \Lambda} G_i \).

**Definition (4-2):** Let \((G,+, . , J)\) be topological \( \Gamma \)-ring then \((G,+, . , J)\) is compact topological \( \Gamma \)-ring if for every cover topological rings of \((G,+, . , J)\) there is a finite sub cover topological rings.

**Example (4-3):** Every finite topological \( \Gamma \)-ring is compact topological \( \Gamma \)-ring.

**Theorem (4-4):** If \( f : (G,+, . , J) \rightarrow (G^*,+, . , T^*) \) is isomorphism topological \( \Gamma \)-ring from \( G \) ito compact topological \( \Gamma \)-ring \( G^* \) then \( G \) is compact topological \( \Gamma \)-ring. **Proof:** Suppose that \( S = \{
G_i \in \mathcal{J}: (G_i + , , ) is a proper subring of G, for all \( i \in \Lambda \) is a proper subring of G. That means \( G = \bigcup_{i \in \Lambda} G_i \). Therefore \( f(G) = G^* = \bigcup_{i \in \Lambda} f(G_i) \). Since \( G^* \) is compact, it follows that there is a finite subcover \( \mathcal{H} \subseteq S \) such that \( G^* \subseteq \mathcal{H} \).

Since \( f \) is isomorphism we get \( G = f^{-1}(\mathcal{H}) \). where \( f^{-1}(\mathcal{H}) \) is a finite subcover of G.

**Theorem (4-5):** Every closed subset of a compact topological \( \Gamma \)-ring space is compact.

**Proof:** Let \( B \) be a closed subset of the compact topological \( \Gamma \)-ring \((G, +, ., \mathcal{J})\). Let \( D^* = \{ G_i \in \mathcal{J}: (G_i + , , ) is a proper subring of G, for all \( i \in \Lambda \) \} be a cover topological \( \Gamma \)-ring of \( D^* \). Since \( B \) is closed then \( G-B \) is open, and \( C = D^* \cup (G-B) \) is an open cover of G. Since G is compact topological ring, it has a finite subcover, containing only finitely many members \( G_i, \ldots, G_n \) of \( D^* \) and may contain \( G-B \). Since \( G = (G-B) \bigcup_{i=1}^{n} G_i \) it follows that \( B \subseteq \bigcup_{i=1}^{n} G_i \) and \( D^* \) has a finite subcover.

**Theorem (4-6):** If \((G, +, , , \mathcal{J})\) is a compact topological \( \Gamma \)-ring, and \( f : (G, +, ., \mathcal{J}) \rightarrow (G^*, +, , \mathcal{J}^* ) \) epimorphism topological \( \Gamma \)-ring, then \( G^* \) is compact.

**Proof:** Assume that \( S^* = \{ G^*_i \in \mathcal{T}^*: (G^*_i +, , ) is a proper subring of \( G^* \), for all \( i \in \Lambda \) \} is a cover topological \( \Gamma \)-ring of \( S^* \) that is \( G^* = \bigcup_{i \in \Lambda} G^*_i \). Since \( f \) is epimorphism topological \( \Gamma \)-ring this implies that \( G = \bigcup_{i \in \Lambda} f^{-1}(G^*_i) \), where \( f^{-1}(G^*_i) \in \mathcal{J} \). Since G is compact topological \( \Gamma \)-ring, this gives \( G = \bigcup_{i=1}^{n} f^{-1}(G^*_i) \). Hence \( f \) is epimorphism topological \( \Gamma \)-ring \( G^* = \bigcup_{i=1}^{n} G^*_i \). Thus \( G^* \) is compact topological \( \Gamma \)-ring.

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