The effective temperature in the quenching of coarsening systems to and to below $T_C$

Federico Corberi, Eugenio Lippiello and Marco Zannetti

Istituto Nazionale di Fisica della Materia, Unità di Salerno and Dipartimento di Fisica ‘E Caianiello’, Università di Salerno, 84081 Baronissi (Salerno), Italy
E-mail: corberi@sa.infn.it, lippiello@sa.infn.it and zannetti@na.infn.it

Received 15 November 2004
Accepted 6 December 2004
Published 20 December 2004
Online at stacks.iop.org/JSTAT/2004/P12007
doi:10.1088/1742-5468/2004/12/P12007

Abstract. We overview the general scaling behaviour of the effective temperature in quenches of simple non-disordered systems, such as ferromagnets, to and to below $T_C$. Emphasis is on the behaviour as the dimensionality is varied. Consequences for the shape of the asymptotic parametric representation are derived. In particular, this is always trivial in critical quenches with $T_C > 0$. We clarify that the quench to $T_C = 0$ at the lower critical dimensionality $d_L$, cannot be regarded as a critical quench. Implications for the behaviour of the exponent $a$ of the ageing response function in quenches to below $T_C$ are developed.

Keywords: coarsening processes (theory), slow dynamics and ageing (theory)
1. Introduction

In the study of slow relaxation phenomena [1], typically, a temperature quench is carried out at time $t = 0$ and the quantities of interest, such as the autocorrelation $C(t, t_w)$ and the linear response function $R(t, t_w)$, are monitored at subsequent times $0 < t_w < t$. The peculiar and interesting feature of these processes is that equilibrium is never reached; that is, there does not exist a finite timescale $t_{eq}$ such that for $t_w > t_{eq}$ two time quantities become time translation invariant.

Substantial progress in the study of these phenomena has been achieved since Cugliandolo and Kurchan [2] introduced the modification of the fluctuation-dissipation theorem (FDT) as a quantitative indicator of the deviation from equilibrium [3]. This can be formulated in several equivalent ways. Here, we find it convenient to use the effective temperature [4]

$$T_{eff}(t, t_w) = \frac{\partial_{t_w} C(t, t_w)}{R(t, t_w)}$$

(1)
The effective temperature in the quenching of coarsening systems to and to below $T_C$, which coincides with the temperature $T$ of the thermal bath if the FDT holds, that is in equilibrium, while it is different from $T$ off equilibrium. Because of the absence of a finite equilibration time, the characterization of the system staying out of equilibrium for an arbitrarily long time is contained in $\lim_{t_w \to \infty} T_{\text{eff}}(t, t_w)$. However, since $t_w \to \infty$ implies also $t \to \infty$, to make the limit operation meaningful one must specify how the two times are pushed to infinity. One way of doing this is to fix the value of the autocorrelation function $C(t, t_w)$ as $t_w \to \infty$. More precisely, since $C(t, t_w)$ for a given $t_w$ is a monotonically decreasing function of $t$, the time $t$ can be reparametrized in terms of $C$ obtaining $T_{\text{eff}}(t, t_w) = \hat{T}_{\text{eff}}(C, t_w)$ and, keeping $C$ fixed, the limit

$$T(C) = \lim_{t_w \to \infty} \hat{T}_{\text{eff}}(C, t_w)$$

(2)

defines the effective temperature in the time sector corresponding to the chosen value of $C$. This quantity is important because, when appropriate conditions are satisfied, it provides the connection between dynamic and static properties [7] through the relation

$$P(q) = T \frac{d}{dC} T(C)^{-1} |_{C=q}$$

(3)

where $P(q)$ is the overlap probability function of the equilibrium state. Therefore, different shapes of $T(C)$ are associated with different degrees of complexity of the equilibrium state. A first broad classification of systems has been made [8, 9], drawing the distinction between coarsening systems, structural glasses and the infinite range spin glass model on the basis of the patterns displayed by $T(C)$, or related quantities such as the fluctuation-dissipation ratio (FDR) and the zero-field cooled (ZFC) susceptibility. In this paper we focus on coarsening systems, exemplified by a non-disordered and non-frustrated system such as a ferromagnet, relaxing via domain growth after a quench to or to below the critical point [10]–[12]. We show that even within the framework of these so-called trivial systems, the spectrum of the behaviours of $T(C)$ is quite rich and interesting.

In order to illustrate the problem, in figure 1 we have drawn schematically the phase diagram, in the temperature–dimensionality plane. The critical temperature vanishes at the lower critical dimensionality $d_L$. The disordered and ordered phase are, respectively, above and below the critical line $T_C(d)$. Slow relaxation arises for quenches with the final temperature everywhere in the shaded area, including at the boundary. Usually, one considers some fixed $d > d_L$ (the dashed line in figure 1) and studies the quench to a final temperature $T \leq T_C$. The form of $T(C)$ in the critical quench with $T = T_C$ (figure 2(a)) is different from that with $T < T_C$ (figure 2(b)) and both are independent of $d$. The interesting question, then, is what happens at the special point $(d_L, T = 0)$, where critical and subcritical quenches merge. That is, will $T(C)$ display continuity with the critical or with the subcritical shape? On the basis of the evidence available from analytical solutions [13]–[16] and numerical simulations [17], the surprising answer is that it does this with neither of them. A third and non-trivial form of $T(C)$ is found (figure 2(c)) in the quench to $(d_L, T = 0)$.

In this paper we present a scaling framework which allows us to account in a unified way for this wealth of behaviour displayed by $T(C)$ upon letting $T$ and $d$ vary. This, in turn, allows us to gain insight into the challenging problem [16, 18] of the scaling of $R(t, t_w)$ in quenches to below $T_C$. Furthermore, on the basis of these ideas, predictions can be made for the interesting case of the $XY$ model. In that case, the phase diagram of
The effective temperature in the quenching of coarsening systems to and to below $T_C$

**Figure 1.** The schematic phase diagram in the $d$–$T$ plane with the critical line $T_C(d)$ and the Kosterlitz–Thouless line for the 2D $XY$ model.

**Figure 2.** Parametric plots of the effective temperature $T(C)$: (a) quenching to $T_C > 0$, (b) quenching to $T < T_C$, (c) quenching to $(d_L, T = 0)$. Panel (c) has been generated by plotting the explicit form of $T(C)$ given by equation (83) in the large $N$ model.

Figure 1 is enriched by the presence of the Kosterlitz–Thouless (KT) line of critical points at the lower critical dimensionality ($d_L = 2$), with $T \leq T_{KT}$. Therefore, slow relaxation in the $XY$ model arises also on arranging the quenches onto the KT line. Since this is a critical line, the phenomenology for $0 < T \leq T_{KT}$ is expected to be akin to that along the $T_C(d)$ line, while at $T = 0$ the switch to the third non-trivial form of $T(C)$ is expected to take place.

The paper is organized as follows. In section 2 the scaling properties of $C(t, t_w)$ and $R(t, t_w)$ are reviewed. In section 3 the general scaling behaviour of $T_{eff}(t, t_w)$ is derived, and in section 4 the parametric form $T(C)$ is obtained. The problem of the exponent $a$ of $R(t, t_w)$ below $T_C$ is presented in section 5. Sections 6 and 7 are devoted to illustration of the general concepts in the context of the large $N$ model and of the $XY$ model, respectively. Conclusions are presented in section 8.

## 2. Scalings of $C(t, t_w)$ and $R(t, t_w)$

In the following we consider quenches from an initial disordered state above the critical line to any one of the states in the shaded area of figure 1. For definiteness, we shall...
The effective temperature in the quenching of coarsening systems to and to below \( T_C \) refer to purely relaxational dynamics without conservation of the order parameter, but the results are of general validity. The common feature of all these processes is that, after a short transient, dynamical scaling sets in with a characteristic length growing with the power law \( L(t) \sim t^{1/z} \). [10]. The scaling properties with \( T = T_C \) are different from those with \( T < T_C \) [11]. In the renormalization group language this means that for a given \( d \) there are two fixed points, one unstable at \( T_C \) and the other stable and attractive at \( T = 0 \) (thermal fluctuations are irrelevant below \( T_C \)) [19]. The dynamical exponent \( z \) on the \( T_C(d) \) line coincides with the dynamical critical exponent \( z_c \) and depends on the dimensionality, while below criticality it takes the dimensionality independent value \( z = 2 \). These values become identical at \( (d_L, T = 0) \), since \( \lim_{d \to d_L} z_c = 2 \) [20].

Let us, next, review the scaling properties of \( C(t, t_w) \) and \( R(t, t_w) \). The fixed point structure enters in two ways: in the values of the exponents involved and, less obviously, in the form of the functions \( C(t, t_w) \) and \( R(t, t_w) \). In analysing the asymptotic behaviour of these quantities, it is necessary to distinguish between the short time and the large time behaviours obtained by letting the waiting time \( t_w \) become large, while keeping, respectively, either the time difference \( \tau = t - t_w \) or the time ratio \( x = t/t_w > 1 \) fixed. Notice that from \( x = 1 - \tau/t_w \) it follows that the short time regime, when using the \( x \) variable, gets compressed to \( x = 1 \).

An important general property of two time quantities in slow relaxation phenomena is that in the short time regime the system appears equilibrated and the FDT is satisfied, while the off-equilibrium character of the dynamics, or ageing, shows up in the large time regime [1,3,12]. The mechanism underlying this property, however, is quite different for quenches to and to below \( T_C \).

### 2.1. Critical quenching: multiplicative structure

In the critical quenches, for \( t_w \) sufficiently large, \( C(t, t_w, t_0) \) and \( R(t, t_w, t_0) \) take the product forms [11,12]

\[
C(t, t_w, t_0) = (\tau + t_0)^{-b} g_C(x, y) \tag{4}
\]

\[
R(t, t_w, t_0) = (\tau + t_0)^{-(1+a)} g_R(x, y) \tag{5}
\]

where \( g_{C,R}(x, y) \) are smooth functions of \( x \) with a weak dependence on \( y = t_0/t_w \). In addition to the observation times, \( t \) and \( t_w \), we have explicitly included also the dependence on a microscopic time \( t_0 \), which is needed to regularize these functions at equal times. In particular, we shall take \( C(t, t, t_0) = 1 \) throughout. In the short time regime,

\[
C(t, t_w, t_0) = (\tau + t_0)^{-b} g_C(1, 0) \tag{6}
\]

\[
R(t, t_w, t_0) = (\tau + t_0)^{-(1+a)} g_R(1, 0) \tag{7}
\]

which are the autocorrelation and response function of the stationary critical dynamics satisfying, therefore, the equilibrium FDT, i.e. \( T_C R(\tau) = -\partial_\tau C(\tau) \), which implies

\[
T_C g_R(1, 0) = b g_C(1, 0) \tag{8}
\]

and

\[
a = b. \tag{9}
\]
We emphasize that, in the critical quench, as a consequence of the multiplicative structure and of the FDT, the exponents $a$ and $b$ are not independent and coincide. Furthermore, their common value is given by \[ a = b = (d - 2 + \eta)/z_c = 2\beta/\nu z_c \] (10)

where $\eta$, $\beta$ and $\nu$ are the usual static exponents. Using the geometrical interpretation of the critical properties, equation (10) can be rewritten as $a = b = 2(d - D)/z_c$, where $D$ is the fractal dimensionality of the correlated critical clusters [21]. These become compact as $T_C \to 0$ yielding

\[
\lim_{d \to d_L} a = b = 0.
\] (11)

On the KT line, where $d = 2$, equation (10) is replaced by $a = b = \eta(T)/z$ with $\eta(T)$ vanishing as $T \to 0$ [22]. Equations (4) and (5) can be recast in the simple ageing form

\[
C(t, t_w, t_0) = t_w^{-b} f_C(x, y)
\] (12)

\[
R(t, t_w, t_0) = t_w^{-(1+a)} f_R(x, y)
\] (13)

where

\[
f_C(x, y) = (x - 1 + y)^{-b} g_C(x, y)
\] (14)

\[
f_R(x, y) = (x - 1 + y)^{-(1+a)} g_R(x, y).\] (15)

These functions, for large $x$ and $t_w$, decay with the same power law

\[
f_{C,R}(x, 0) = A_{C,R} x^{-\lambda_c/z_c}
\] (16)

where $\lambda_c$ is the autocorrelation exponent [11, 12].

### 2.2. Quenching to below the critical line: additive structure

Below $T_C$, the picture is different because the stationary and ageing components enter additively into the autocorrelation function [23, 1, 3]

\[
C(t, t_w, t_0) = C_{\text{st}}(\tau) + C_{\text{ag}}(t, t_w, t_0)
\] (17)

where $C_{\text{st}}(\tau)$ is the equilibrium autocorrelation function in the broken symmetry pure state at the temperature $T$. From the equal time properties

\[
C(t, t, t_0) = 1 \quad C_{\text{st}}(\tau = 0) = 1 - M^2
\] (18)

which imply

\[
C_{\text{ag}}(t, t, t_0) = M^2
\] (19)

where $M$ is the spontaneous magnetization, it follows that in equation (17) the stationary component corresponds to the quasi-equilibrium decay to the plateau at the Edwards–Anderson order parameter $q_{\text{EA}} = M^2$, while the ageing component $C_{\text{ag}}$ describes the off-equilibrium decay from the plateau at much larger times. In the short time regime, then, we have

\[
C(\tau + t_w, t_w, t_0) = C_{\text{st}}(\tau) + M^2
\] (20)
The effective temperature in the quenching of coarsening systems to and to below $T_C$

while, taking into account that $C_{st}(\tau)$ vanishes as $\tau \to \infty$, in the large time regime

$$C(t, t_w, t_0) = C_{ag}(t, t_w, t_0).$$

(21)

From this it is easy to verify that, contrary to what happens with the multiplicative structure of equation (4), the limits $t_w \to \infty$ and $t \to \infty$ do not commute:

$$\lim_{t \to \infty} \lim_{t_w \to \infty} C(t, t_w, t_0) = M^2$$

(22)

$$\lim_{t_w \to \infty} \lim_{t \to \infty} C(t, t_w, t_0) = 0$$

(23)

yielding weak ergodicity breaking [1], characteristic of quenches to below $T_C$.

The additive structure of the response function

$$R(t, t_w, t_0) = R_{st}(\tau) + R_{ag}(t, t_w, t_0)$$

(24)

is obtained in the following way [3]: in the short time regime stationarity holds:

$$R(t, t_w, t_0) = R_{st}(\tau),$$

and $R_{st}(\tau)$ is defined from the FDT

$$R_{st}(\tau) = -\frac{1}{T} \frac{\partial}{\partial \tau} C_{st}(\tau)$$

(25)

whilst $R_{ag}(t, t_w, t_0)$ remains defined by equation (24). This implies that $R_{ag}(t, t_w, t_0)$ vanishes in the short time regime and since, conversely, $R_{st}(\tau)$ vanishes in the large time regime, we may write

$$R(t, t_w, t_0) = \begin{cases} R_{st}(\tau) & \text{for } x = 1 \\ R_{ag}(t, t_w, t_0) & \text{for } x > 1. \end{cases}$$

(26)

The components $C_{ag}(t, t_w, t_0)$ and $R_{ag}(t, t_w, t_0)$ obey simple ageing forms such as (12) and (13) with scaling functions $f_{C,R}(x, y)$ decaying, for large $x$, with a power law of the form (16). It is understood that all the exponents must be replaced by their values below $T_C$, which are different from those at $T_C$. In particular, from phase ordering theory it is well known [10] that $b = 0$ independently from dimensionality. This can be readily understood from $b = 2(d - D)/z$, which holds in general, and from the compact nature of the coarsening domains, which gives $D = d$. Quite different is the case of the exponent $a$. Since the FDT relates only the stationary components $C_{st}(\tau)$ and $R_{st}(\tau)$, due to the additive structure of equations (17) and (24) we no longer have the constraint $a = b$ as in the critical quench. That is, in the quench to below $T_C$ the value of $a$ is decoupled from that of $b$. As a matter of fact, the determination of $a$ below $T_C$ is a difficult and challenging problem [17,18], which will be discussed in section 5.

2.3. Quenching to $(d_L, T = 0)$: additive structure

As we have seen above, the structures of $C(t, t_w, t_0)$ and $R(t, t_w, t_0)$ follow different patterns in the quenches to and to below $T_C$. This raises the question of which one applies when the quench is carried out at $(d_L, T = 0)$. In other words, should we consider the state $(d_L, T = 0)$ as a critical point with $T_C = 0$, as is often done in the literature, or should we consider it as the continuation to $d_L$ of the $T = 0$ states below criticality? The correct answer is the second one, since at $d_L$ the system orders, as in all the states below the critical line, while on the $T_C(d)$ line there is no ordering. In particular, this implies that weak
ergodicity breaking takes place also in the quench to \((d_L, T = 0)\). From equations (17) and (18) it then follows that \(C_{st}(\tau) \equiv 0\) and
\[
C(t, t_w, t_0) = f_C(x, y)
\]
with \(f_C(1, 0) = 1\), since \(M^2 = 1\) at \(T = 0\).

The fact that the quench to \((d_L, T = 0)\) belongs to the additive scheme becomes important when considering the response function, because if \(C_{st}(\tau)\) vanishes in the ground state, the same is not true for \(R_{st}(\tau)\). Therefore, equation (24) holds with \(R_{st}(\tau) \neq 0\) and it is important to subtract this contribution from \(R(t, t_w, t_0)\) if one wants to study the scaling properties of \(R_{ag}(t, t_w, t_0)\). We shall come back on this point in section 6, when treating the explicit example of the large \(N\) model. It should be mentioned, here, that the linear response function at \(T = 0\) is well defined only for systems with soft spins. In the case of hard spins there does not exist a linear response regime when \(T = 0\). For a discussion of this problem and how to bypass it in Ising systems, we refer the reader to [13,16].

2.4. Comparison with theory

A natural question is that of how the picture outlined above compares with theoretical approaches. In the case of the quench to \(T_C\) renormalization group methods are available [24,12], which account for the multiplicative structure of the autocorrelation and response function and give explicit expressions for the quantities of interest up to two-loop order. The development of systematic expansion methods for the quench to below \(T_C\) is much more difficult [25]. Recently, Henkel and collaborators [26,27] have developed a method based on the requirement of local scale invariance. According to this approach the response function obeys the multiplicative structure (5) both in the quench to \(T_C\) and in that to below \(T_C\). Problems, then, arise in the latter case. The main one is of conceptual nature, since, as explained above, the weak ergodicity breaking scenario requires the autocorrelation function to obey the additive structure (17). Then, in the short time regime, where equation (20) holds, from the FDT one has \(\partial_{t_w} C_{st}(\tau) \sim (\tau + t_0)^{-(1+a)}\), which implies \(C_{st}(\tau) \sim (\tau + t_0)^{-a}\). In other words, if the local scale invariance prediction for the response function is valid, then the stationary correlation function must decay with a power law. This is the case, as we have seen above, in the critical quenches and, as we shall see in section 6, in the large \(N\) model. Conversely, in all other cases where the equilibrium correlation function below \(T_C\) decays exponentially, the response function cannot be of the form (5). Nonetheless, Henkel et al criticize [27] the additive structure below \(T_C\).

The second problem is that existing results, exact [13]–[15] and approximated [16,28,25] as well as numerical [18,29], lead to an ageing part of the response function which, for systems with scalar order parameter\(^1\), is of the form
\[
R_{ag}(t, t_w, t_0) = t_w^{-1/z}(\tau + t_0)^{-(a-1/z+1)}g_R(x, y).
\]
This is qualitatively different from the multiplicative form (5), both in the structure and in the exponents. As mentioned above, an exception is provided by the large \(N\) model,

\(^1\) For systems with vector order parameter there is not, as of yet, enough information, analytical or numerical, to conjecture a general form of the response function.
The effective temperature in the quenching of coarsening systems to and to below $T_C$

where $R(t, t_w, t_0)$ is of the form (5) also below $T_C$. Yet, as we show in section 6, the additive structure applies to the large $N$ model as in all cases of quenching to below $T_C$.

3. Effective temperature

The next step is to see how these different behaviours of $C(t, t_w, t_0)$ and $R(t, t_w, t_0)$ affect the behaviour of $T_{\text{eff}}(t, t_w, t_0)$.

3.1. Critical quenching

From equations (12), (13) and (9), using the definition (1), we get

$$T_{\text{eff}}(t, t_w, t_0) = F(x, y)$$

with

$$F(x, y) = -f_{OC}(x, y)/f_R(x, y)$$

and

$$f_{OC}(x, y) = b f_C(x, y) + \left[ x \frac{\partial}{\partial x} f_C(x, y) + y \frac{\partial}{\partial y} f_C(x, y) \right].$$

In the short time regime, equations (6)–(8) lead to the boundary condition

$$F(1, y) = T_C$$

while, in the ageing regime, from equation (16) it follows that

$$\lim_{x \to \infty} F(x, 0) = T_\infty = \left( \frac{\lambda_c}{z_c} - b \right) \frac{A_C}{A_R} = \frac{T_C}{X_\infty}$$

where $X_\infty$ is the limit FDR of Godrèche and Luck [11].

3.2. Quenching to below the critical line

Below $T_C$, where $b = 0$, from the definition (1), the additive structure and equation (25) we get

$$T_{\text{eff}}(t, t_w, t_0) = \frac{T R_{\text{st}}(\tau)}{R_{\text{st}}(\tau) + R_{\text{ag}}(t, t_w, t_0)} + \frac{\partial_{t_w} C_{\text{ag}}(t, t_w, t_0)}{R_{\text{st}}(\tau) + R_{\text{ag}}(t, t_w, t_0)}$$

which can be rewritten as

$$T_{\text{eff}}(t, t_w, t_0) = T \left[ \frac{R_{\text{st}}(\tau)}{R_{\text{st}}(\tau) + R_{\text{ag}}(t, t_w, t_0)} \right] + t_w^a F(x, y) \left[ \frac{R_{\text{ag}}(t, t_w, t_0)}{R_{\text{st}}(\tau) + R_{\text{ag}}(t, t_w, t_0)} \right]$$

where $F(x, y)$ is still defined by equations (30), (31) with $f_C(x, y)$, $f_R(x, y)$ the scaling functions of $C_{\text{ag}}(t, t_w, t_0)$ and $R_{\text{ag}}(t, t_w, t_0)$.

Using equation (26) we may approximate equation (35) with

$$T_{\text{eff}}(t, t_w, t_0) = T H(x) + t_w^a F(x, y)[1 - H(x)]$$
The effective temperature in the quenching of coarsening systems to and to below $T_C$

where

\[ H(x) = \begin{cases} 
1 & \text{for } x = 1 \\
0 & \text{for } x > 1.
\end{cases} \]  

(37)

Therefore, for large $t_w$ the general formula containing both critical and subcritical quenches is given by\(^2\)

\[ T_{\text{eff}}(t, t_w, t_0) = \begin{cases} 
T & \text{for } x = 1 \\
t_w^{-b} F(x, y) & \text{for } x > 1.
\end{cases} \]  

(38)

Of course, this also includes the quench to $(d_L, T = 0)$.

4. Parametric representation

In order to derive the parametric representation, as explained in the introduction, we must express the time dependence of $T_{\text{eff}}(t, t_w, t_0)$ through $C(t, t_w, t_0)$ and then let $t_w \to \infty$, yielding $T(C)$. From equations (4) and (17), for large $t_w$, the general form of the autocorrelation function can be written as

\[ C(t, t_w, t_0) = \begin{cases} 
1 & \text{for } x = 1 \\
t_w^{-b} f_C(x, y) & \text{for } x > 1.
\end{cases} \]  

(39)

The task is to eliminate $x$ between equations (38) and (39), which we now do separately for quenches into the different regions of the phase diagram.

4.1. Critical quenching

In this case $a = b > 0$. From equations (12) and (14), letting $t_w \to \infty$ with $x$ fixed, we obtain the singular limit

\[ C(x) = \begin{cases} 
1 & \text{for } x = 1 \\
0 & \text{for } x > 1
\end{cases} \]  

(40)

whose inverse is readily obtained exchanging the horizontal with the vertical axis:

\[ x(C) = \begin{cases} 
\infty & \text{for } C = 0 \\
1 & \text{for } 0 < C \leq 1.
\end{cases} \]  

(41)

Inserting into equation (38) we obtain (figure 2(a))

\[ T(C) = \begin{cases} 
T_{\infty} > T_C & \text{for } C = 0 \\
T_C & \text{for } 0 < C \leq 1
\end{cases} \]  

(42)

where we have used equations (32) and (33). This is a universal result, since all the non-universal features of $F(x, 0)$ for the intermediate values of $x$ have been eliminated in the limit process. Therefore, we have that for all quenches to $T_C > 0$, except for the value $T_{\infty}$ at $C = 0$, the parametric plot of $T(C)$ is trivial in the sense that the effective temperature

\[ ^2 \text{The equivalent formula for the FDR was derived in [11].} \]
The effective temperature in the quenching of coarsening systems to and to below $T_C$ coincides with the temperature $T_C$ of the thermal bath. Notice that this implies that for the ZFC susceptibility

$$\chi(t, t_w) = \int_{t_w}^{t} ds R(t, s)$$

(43)

whose parametric form is related to $T(C)$ by

$$\chi(C) = \int_{C}^{1} \frac{dC'}{T(C')}$$

(44)

a linear plot is obtained:

$$\chi(C) = (1 - C)/T_C$$

(45)

as for equilibrated systems [3].

4.2. Quenching to below the critical line

In this case $b = 0$ and from equation (39) it follows that

$$C(x) = \begin{cases} 1 & \text{for } x = 1 \\ f_C(x, 0) & \text{for } x > 1 \end{cases}$$

(46)

where, recalling equations (18) and (21), $f_C(x, 0)$ is the smooth function describing the fall below the plateau at $M^2$. Therefore, the inverse function is given by

$$x(C) = \begin{cases} f_C^{-1}(C) > 1 & \text{for } C < M^2 \\ 1 & \text{for } M^2 \leq C \leq 1 \end{cases}$$

(47)

and inserting into equation (38) we find

$$\hat{T}_{\text{eff}}(C, t_w) = \begin{cases} t_w F(f_C^{-1}(C), 0) & \text{for } C < M^2 \\ T & \text{for } M^2 \leq C \leq 1 \end{cases}$$

(48)

Although the actual value of $a$, as will be explained in section 5, is to some extent a debated issue, there is consensus that $a > 0$ for $d > d_L$. Therefore, taking the $t_w \to \infty$ limit we recover the well known result [1, 3] (figure 2(b))

$$T(C) = \begin{cases} \infty & \text{for } C < M^2 \\ T & \text{for } M^2 \leq C \leq 1 \end{cases}$$

(49)

Again, all the details of $F(x, 0)$ having disappeared, the function $T(C)$ is universal.

Let us remark here that, as a further manifestation of the difference between quenches to and to below $T_C$, the behaviour (42) of $T(C)$ cannot be recovered by letting $T \to T_C^-$ and $M \to 0$ in (49). In fact, in the latter case we find

$$\lim_{T \to T_C^-} T(C) = \begin{cases} \infty & \text{for } C = 0 \\ T_C & \text{for } 0 < C \leq 1 \end{cases}$$

(50)

which differs from equation (42), where $T_\infty$ is a finite number.
The effective temperature in the quenching of coarsening systems to and to below $T_C$

4.3. Quenching to $(d_L, T = 0)$

If we take the limit $d \to d_L$ in equations (42) and (49) we obtain, respectively,

$$T(C) = \begin{cases} 
T_\infty & \text{for } C = 0 \\
0 & \text{for } 0 < C \leq 1 
\end{cases} \quad (51)$$

and

$$T(C) = \begin{cases} 
\infty & \text{for } C < 1 \\
0 & \text{for } C = 1 
\end{cases} \quad (52)$$

which are very different from one another. So, there is the problem of which is the form of $T(C)$ in the quench to $(d_L, T = 0)$. A statement in this regard can be made on the basis of the substantial amount of information by now accumulated from sources as diverse as the exact solutions of the 1D Ising model [13, 14] and of the 2D large $N$ model [15], the Ohta–Jasnow–Kawasaki type of approximation with $d = 1$ [16], in addition to numerical simulations [17] of systems at $d_L$ with scalar and vector order parameter, with and without conserved dynamics. In all of these cases we have obtained the parametric plot (44) of the ZFC susceptibility, which gives for $T(C)$ at $(d_L, T = 0)$ a non-trivial and non-universal finite function of the type depicted in figure 2(c). By contrast, there does not exist, up to now, any evidence for behaviours of $T(C)$ of the type (51) or (52).

Assuming that the generic behaviour of $T(C)$ is of the type shown in figure 2(c), as the amount of evidence quoted above strongly suggests, this is compatible with equation (38) only if $a = 0$. Then, from equations (46) to (48) with $M^2 = 1$, we may write

$$T(C) = F(f_C^{-1}(C), 0) \quad (53)$$

which represents a smooth, finite, non-trivial function decreasing smoothly from $T_\infty = F(\infty, 0)$ at $C = 0$ toward zero at $C = 1$, preserving all the non-universal features of $f_C(x, 0)$ and $F(x, 0)$ for the intermediate values of $C$.

4.4. The physical temperature and the connection with statics

We conclude this section with comments on the interpretation of $T(C)$ as a true temperature and on its connection with the equilibrium properties of the system.

The relation of the effective temperature with a physically measurable temperature is a very interesting issue, which has been thoroughly investigated in [4]–[6]. A necessary condition for the identification of the effective temperature with a true temperature is its uniqueness, that is the independence from the observable used in the study of the deviation from FDT. As regards the quenches to $T_C$, in [6] it is shown that the multiplicative structure (4) and (5) is observable independent. Therefore, the derivation of equation (42) is also observable independent, except as regards the actual value of $T_\infty$, which does depend on the observable [6]. In this case, the criterion of uniqueness does indeed lead to the identification of the effective temperature with a physical temperature, since $T(C)$, for $C > 0$, coincides with the temperature of the thermal bath $T_C$. Conversely $T_\infty$, due to the non-uniqueness of its value, cannot be considered as a physical temperature. Although results of comparable generality are lacking for quenches to below $T_C$, we also expect equation (49) to be observable independent, due to the universality of $T(C)$ (for observables with $a > 0$). Finally, the effective temperature (53), in the quench
The effective temperature in the quenching of coarsening systems to and to below $T_C$ to $(d_L, T = 0)$, is certainly not physical, since, as explained above, it does depend on the details of the scaling function, which makes it observable dependent. To this case belong the results for the $d = 1$ Ising model [5], which do indeed show observable dependence of the effective temperature.

The connection between static and dynamic properties is given by equation (3). We can now assess the status of this relation, in the light of the results derived above. In all cases, the overlap probability function is given by $P(q) = \delta(q - M^2)$. We must, then, establish whether the rhs of equation (3) actually is a $\delta$ function. From equations (42), (49), (53) it follows that $T(\frac{d}{dC})T(C)^{-1}|_{C=q}$ vanishes everywhere, except at $C = M^2$. Therefore, in order to establish whether this is a $\delta$ function, we must look at the integral $\int_0^1 dC(\frac{d}{dC})(T/T(C))$. This gives finite values in the quenches to $T_C$ and to below $T_C$. The validity of equation (3), however, cannot be established in the quench to $(d_L, T = 0)$.

In that case the value of the integral is not determined, since the contribution at the upper limit of integration is given by the ratio of two vanishing quantities.

5. The response function exponent

With the results for $T(C)$ illustrated in the previous section we get to grips with the problem of the exponent $a$. The vanishing of $a$, required by the behaviour of $T(C)$ in the quench to $(d_L, T = 0)$, cannot be accounted for by equation (11) because, as explained in section 2.3, the quench to $(d_L, T = 0)$ is not a critical quench. Hence, the vanishing of $a$ must be framed within the behaviour of $a$ in the quenches to below the critical line.

According to a popular conjecture [30], $a$ for $T < T_C$ should coincide with the exponent $n/z$ entering in the time dependence of the density of defects, which goes like $t^{-n/z}$ with $n = 1$ or 2 for a scalar or vector order parameter, respectively [10]. We recall that for $T < T_C$ the dynamical exponent $z = 2$ does not depend on $d$. Thus, in the scalar case one ought to have $a = 1/2$ and in the vector case $a = 1$, independently from $d$ and, therefore, also at $d_L$. This is obviously incompatible with the large body of evidence for $a = 0$ at $d_L$, quoted in the previous section.

This difficulty is circumvented in the alternative proposal which we have put forward [16,18] for the behaviour of $a$ as $d$ is varied, on the basis of the exact solution of the large $N$ model [15] for arbitrary $d$ and the Ohta–Jasnow–Kawasaki type of approximation [16,28,25], also for arbitrary $d$. In these two cases one finds that $a$ does depend on $d$ according to

$$a = \frac{n}{z} \left( \frac{d - d_L}{d_U - d_L} \right)$$

where $d_U$ is a parameter dependent on the universality class. We have proposed promoting this result to a phenomenological formula, whose general validity we have tested by undertaking a systematic numerical investigation of systems in different classes of universality and at different dimensionalities [16]–[18]. The results obtained are in quite good agreement with equation (54), taking $d_U = 3$ and $d_U = 4$, respectively, for scalar$^3$ and vector order parameters. We emphasize that equation (54) gives $\lim_{d \to d_L} a = 0$ when

$^3$ In the Ohta–Jasnow–Kawasaki approximation, $d_U = 2$. A possible origin of the discrepancy with $d_U = 3$ found in the simulations of scalar systems is discussed in [16].
The effective temperature in the quenching of coarsening systems to and below $T_C$ can be defined as below $T_C$, and this should not be confused with equation (11) holding for the critical quenches.

The final remark is that, up to now, equation (54) has remained a phenomenological formula. The challenge is to construct a theory for it. A preliminary attempt to relate the dimensionality dependence of $a$ to the roughening of interfaces in the scalar case has been made in [17]. However, the explanation of equation (54) as a result of general validity seems to require an understanding of the response function much deeper than the one we currently have.

6. The large $N$ model

In this section all the concepts introduced above are explicitly illustrated through the exact solution of the large $N$ model. This model, or the equivalent spherical model, has been solved analytically in a number of papers [31]–[33]. Here, we follow our own solution in [15], where we showed that for quenches to $T \leq T_C$ the order parameter can be split into the sum of two stochastic processes, $\phi(\vec{x}, t) = \sigma(\vec{x}, t) + \psi(\vec{x}, t)$, which, for $t$ sufficiently large, are independent and mimic the separation between the interface and bulk fluctuations taking place in domain forming systems [34]. Correspondingly, the autocorrelation function decomposes into the sum of the autocorrelations of $\sigma$ and $\psi$:

$$C(t, t_w, t_0) = C_\sigma(t, t_w, t_0) + C_\psi(t, t_w, t_0)$$

which obey the scaling forms

$$C_\sigma(t, t_w, t_0) = t_w^{-b_\sigma} f_\sigma(x, y)$$

and

$$C_\psi(t, t_w, t_0) = t_w^{-b_\psi} f_\psi(x, y).$$

For the purposes of the present paper it is sufficient to limit the analysis to $d < 4$, where the exponents and the scaling functions are given by

$$b_\sigma = \begin{cases} d - 2 & \text{for } T = T_C \\ 0 & \text{for } T < T_C \end{cases}$$

$$b_\psi = (d - 2)/2$$

$$f_\sigma(x, y) = A_\sigma x^{-\omega/2} [\frac{1}{2}(x + 1 + y)]^{-d/2}$$

$$f_\psi(x, y) = \frac{2T}{(4\pi)^{d/2} x^{-\omega/2}} \int_0^1 dz \, z^{\omega} (x + 1 - 2z + y)^{-d/2}$$

$$\omega = \begin{cases} d/2 - 2 & \text{for } T = T_C \\ -d/2 & \text{for } T < T_C \end{cases}$$

and $A_\sigma$ is a constant, which coincides with $M^2$ in the quenches to below $T_C$. The scaling function $f_\psi(x, y)$ can also be rewritten as

$$f_\psi(x, y) = (x - 1 + y)^{-b_\psi} g_\psi(x, y)$$
The effective temperature in the quenching of coarsening systems to and to below $T_C$ with

$$g_\psi(x, y) = \frac{T}{(4\pi)^{d/2}} x^{-\omega/2} \int_0^{2/(x-1+y)} dz \left(1 + \frac{d/2}{2(x - 1 + y)z}\right)^\omega$$

(64)

which allows us to recast $C_\psi(t, t_w, t_0)$ in the multiplicative form

$$C_\psi(t, t_w, t_0) = (\tau + t_0)^{-\omega} g_\psi(x, y).$$

(65)

The response function is given by

$$R(t, t_w, t_0) = t_w^{-(1+a)} f_R(x, y)$$

(66)

with

$$f_R(x, y) = (4\pi)^{-d/2}(x - 1 + y)^{-(1+a)x^{-\omega/2}}$$

(67)

and

$$a = b_\psi = (d - 2)/2.$$  

(68)

Finally, in the phase diagram of figure 1 the critical line is a straight line:

$$T_C = (4\pi)^{d/2} \Lambda^{2-d} (d - d_L)/2$$

(69)

where $\Lambda$ is the momentum cut-off and $d_L = 2$.

### 6.1. Scalings of $C(t, t_w, t_0)$ and $R(t, t_w, t_0)$

Let us now extract from the above results the scaling properties of $C(t, t_w, t_0)$ and $R(t, t_w, t_0)$ in the different regions of the phase diagram.

(1) **Critical quenching.** The first observation is that the autocorrelation and response function have the multiplicative structures (4) and (5). For $R(t, t_w, t_0)$ this is immediately evident from equation (67), where $f_R(x, y)$ is of the form (15) with $g_R(x, y) = (4\pi)^{-d/2} x^{1-d/4}$.

For $C(t, t_w, t_0)$, from $b_\psi = 2b_\sigma$ it follows that for large $t_w$ the first term in equation (55) is negligible with respect to the second one, yielding

$$C(t, t_w, t_0) = C_\psi(t, t_w, t_0)$$  

(70)

and this is of the form (4) and (12) with the identifications $b = b_\psi$, $f_C(x, y) = f_\psi(x, y)$ and $g_C(x, y) = g_\psi(x, y)$. Furthermore, from equations (59) and (68) it follows that

$$a = b = (d - 2)/2$$

(71)

in agreement with equation (10), since in the large $N$ model $\eta = 0$ and $z_c = 2$.

Finally, from equation (64) we have $g_C(1, 0) = (4\pi)^{-d/2}2T_C/(d - 2)$ and, using $g_R(1, 0) = (4\pi)^{-d/2}$, it is easy to check that equation (8) is verified.

(2) **Quenching to below the critical line.** For large $t_w$, the additive structure of equation (17) is found ready made in equation (55), with the identifications

$$C_{\text{st}}(\tau) = C_\psi(\tau) = (\tau + t_0)^{-\omega} g_\psi(1, 0)$$

(72)

where $g_\psi(1, 0) = (4\pi)^{-d/2}2T/(d - 2)$ and

$$C_{\text{ag}}(t, t_w, t_0) = C_\sigma(t, t_w, t_0)$$

(73)
The effective temperature in the quenching of coarsening systems to and to below $T_C$

which gives $b = b_\sigma = 0$ and $f_C(x, y) = f_\sigma(x, y)$. The power law decay (72) of the stationary component $C_{st}(\tau)$ is a peculiarity of the large $N$ model, since below $T_C$ to lowest order in $1/N$ only the Goldstone modes contribute to thermal fluctuations [35]. Carrying out the prescription outlined in section 2.2 for the construction of the corresponding components of the response function, from equation (25) we have

$$R_{st}(\tau) = (4\pi)^{-d/2}(\tau + t_0)^{-(1 + b_\psi)}$$

(74)

and using the identity $b_\psi = a$ we find

$$R_{ag}(t, t_w, t_0) = R(t, t_w, t_0) - R_{st}(\tau)$$

$$= t_w^{-(1+a)}f_R(x, y)$$

(75)

with

$$f_R(x, y) = (4\pi)^{-d/2} \frac{x^{d/4} - 1}{(x - 1 + y)^{1+a}}$$

(76)

6.2. The effective temperature $T(C)$

(1) Critical quenching. From the explicit expressions for $f_C(x, y)$ and $f_R(x, y)$ in the critical quenches it follows that

$$T_{eff}(t, t_w, t_0) = F(x, y)$$

$$= dT_C \left\{ (x + y) \int_0^1 dz \, z^{d/2-2}(x + 1 - 2z + y)^{-1-d/2} - \frac{1}{2} \int_0^1 dz \, z^{d/2-2}(x + 1 - 2z + y)^{-d/2} \right\} (x - 1 + y)^{d/2}.$$  

(78)

Evaluating numerically the right-hand side, as $y \to 0$ the curve $F(x, y)$ approaches (figure 3) the limit function $F(x, 0)$ rising from $F(1, 0) = T_C$ to $T_\infty = \lim_{x \to \infty} F(x, 0) = T_C/X_\infty$, where $X_\infty = (d - 2)/d$ [32]. Similarly, carrying out
The effective temperature in the quenching of coarsening systems to and to below $T_C$.

Figure 3. The approach to the limiting curve $F(x, 0)$ (thick curve) for $t_w = 1, 2, 10, 100$ (top to bottom) in the critical quench of the 3D large $N$ model.

Figure 4. The approach to the limiting function $x(C)$ of equation (41) for $t_w = 10, 10^2, 10^3, 10^4$ (top to bottom) in the critical quench of the 3D large $N$ model.

numerically the inversion of equation (70), the function $x = f_C^{-1}(t_w C, y)$ shows (figure 4) the approach to singular behaviour (41) in the limit $y \to 0$. The parametric plot in figure 5 displays the approach of $\hat{T}(C, t_w)$ toward $T(C)$ of the form of equation (42) as $t_w \to \infty$, which can be very slow if $b$ is small. With the purpose of comparing later with data for the $XY$ model, in figure 6 we have also shown the parametric plot of the ZFC susceptibility. This has been obtained by plotting $\chi(t, t_w)$ against $C(t, t_w)$ for fixed $t_w$ in two different cases: in panel (a), with $d = 2.5$ corresponding to $b = 0.25$, the approach to the asymptotic linear plot (45) is evident, while in panel (b), with $d = 2.06$ corresponding to the much smaller value $b = 0.03$, there is a much longer preasymptotic regime preceding the onset of the linear behaviour.
The effective temperature in the quenching of coarsening systems to and to below $T_C$

**Figure 5.** The approach of $\hat{T}_{\text{eff}}(C,t_w)$ toward $T(C)$ of the form of equation (42) for $t_w = 10, 10^2, 10^3, 10^4$ (top to bottom) in the critical quench of the 3D large $N$ model.

**Figure 6.** Parametric plots of the ZFC susceptibility in the critical quench of the large $N$ model: (a) $d = 2.5$, $t_w = 10^2, 10^3, 10^4$ (bottom to top), (b) $d = 2.06$, $t_w = 10^2, 3 \times 10^2, 10^3, 3 \times 10^3, 10^4, 10^6, 10^9, 10^{12}$ (bottom to top).

(2) **Quenching to below the critical line.** Using equations (74) and (75) we may rewrite equation (35) as

$$
T_{\text{eff}}(t, t_w, t_0) = T x^{-d/4} + t_w^{d/2-1} F(x, y)(1 - x^{-d/4})
$$

where

$$
F(x, y) = M^2 (8\pi)^{d/2} d \frac{x^{d/4}}{4 (x^{d/4} - 1)} \left( \frac{x - 1 + y}{x + 1 + y} \right)^{1+d/2}.
$$

In figure 7 we have plotted $T_{\text{eff}}(t, t_w, t_0)$ against the exact $C(t, t_w, t_0)$ obtaining $\hat{T}(C, t_w)$, which shows the approach toward the form (49) of $T(C)$ as $t_w$ grows.
The effective temperature in the quenching of coarsening systems to and below $T_C$

![Image](image.png)

**Figure 7.** The approach of $\tilde{T}_{\text{eff}}(C, t_w)$ toward $T(C)$ of the form of equation (49) for $t_w = 10^2, 10^3, 10^4, 10^5$ (bottom to top) in the quench to $T = T_c/2$ of the 3D large $N$ model.

(3) **Quenching to $T = 0$ with $d = d_L$.** At $d = 2$ and $T = 0$, the second term in equation (55) vanishes and $b_\nu = 0$, yielding

$$C(x, y) = 2x^{1/2}(x + 1 + y)^{-1}$$

while from equations (79) and (80)

$$T_{\text{eff}}(t, t_w, t_0) = (4\pi) \left( \frac{x - 1 + y}{x + 1 + y} \right)^2.$$  

Letting $y \to 0$ and eliminating $x$ between the above two equations we obtain the explicit non-trivial expression

$$T(C) = (4\pi) \left( \frac{1 - C^2 + \sqrt{1 - C^2}}{1 + \sqrt{1 - C^2}} \right)^2.$$  

plotted in figure 2(c). We recall that the specific form of the function $T(C)$ in the quench to $(d_L, T = 0)$ is non-universal.

7. **The XY model**

In the XY model the phase diagram of figure 1 includes the KT line and, as anticipated in the introduction, in the $d = 2$ case the behaviour of $T(C)$ for $0 < T \leq T_{KT}$ is expected to be given by equation (42). This is hard to detect from existing data, since the small values of $b = \eta(T)/z$ at the temperatures used in [36, 37] would require reaching huge values of $t_w$ in order to observe the approach to (42). In [36], data are presented for the ZFC susceptibility. According to equation (44), asymptotically one should see the approach to the linear parametric plot (45), as illustrated in figure 6(a) for the large $N$ model. The data in figure 10 of [36] seem to be far from this behaviour and, indeed, in [36] these data have been regarded as reminiscent of the non-trivial parametric plot in
the $d = 3$ Edwards–Anderson model. However, we believe that the non-trivial pattern displayed by these data must be attributed to the large preasymptotic effect due to the small value of $b = \eta(T)/z = 0.03$. Indeed, the pattern displayed by $\chi(C, t_w)$ in figure 10 of [36] is strikingly similar to the one in figure 6(b) for the critical quench in the large $N$ model, despite the fact that in the large $N$ model there is no KT transition. The data in figure 6(b) have been generated in order to reproduce the same value of $b$ as in the quench of the $XY$ model considered in [36].

It would be interesting to have data for the $XY$ model quenched at $d_L$ and $T = 0$, where, according to the general argument of section 3, $\chi(C)$ is expected to display the non-trivial behaviour discussed in section 4.

In the $d = 3$ case in [38], the parametric plots of the FDR

$$X(C) = \lim_{t_w \to \infty} \frac{T}{T_{\text{eff}}(C, t_w)} = \frac{T}{T(C)}$$

are presented in the quenches to and to below $T_C$. The data display the approach to the asymptotic parametric forms

$$X(C) = \begin{cases} X_\infty = T_C/T_\infty & \text{for } C = 0 \\ 1 & \text{for } 0 < C \leq 1 \end{cases}$$

in the quench to $T_C$ and

$$X(C) = \begin{cases} 0 & \text{for } C < M^2 \\ 1 & \text{for } M^2 \leq C \leq 1 \end{cases}$$

in the quench to below $T_C$, which correspond to equations (42) and (49) for $T(C)$.

Furthermore, in the $d = 3$ model, of particular interest is the measurement of the exponent $a$ below $T_C$. The value $a = 1/2$ has been obtained both from the ZFC susceptibility [17] and from the direct measurement of $R(t, t_w, t_0)$ [38]. This value gives additional and clear-cut evidence in favour of equation (54), which, in fact, predicts $a = 1/2$ when $d = 3$ and the order parameter is non-conserved ($z = 2$) and vectorial ($n = 2, d_U = 4$). Conversely, the conjecture [30] relating $a$ to the density of defects would predict $a = 1$, since in the $XY$ model, with a vectorial order parameter, the density of defects goes like $t^{-1}$ [10, 39].

8. Conclusions

In this paper we have overviewed the behaviour of the effective temperature in the slow relaxation processes arising when systems, such as a ferromagnet, with a simple pattern of ergodicity breaking in the low temperature state are quenched from high temperature to or to below $T_C$. On the basis of very general assumptions on the scaling properties of the autocorrelation and response functions, we have derived the scaling behaviour of the effective temperature, which allows us to account in a unified way for the different patterns displayed by $T(C)$ in the different regions of the phase diagram. The primary results are as follows. (i) In the critical quenches with $T_C > 0$, as a consequence of $a = b > 0$, $T(C)$ always displays the form (42), which is trivial in the sense that $T(C) = T_C$ except for the limiting value $T_\infty$ of Godrèche and Luck at $C = 0$. In particular, this implies that in the $d = 2$ $XY$ model quenched to $0 < T \leq T_{KT}$ $T(C)$, and the equivalent parametric
representations of the FDR or the ZFC susceptibility, must be trivial. The non-trivial behaviour reported in [36], then, must be regarded as preasymptotic. (ii) The non-trivial form of $T(C)$ in the quenches to $(d_1, T = 0)$ requires $a = 0$. Since, due to weak ergodicity breaking, this process cannot be regarded as the continuation to $T_c = 0$ of the critical quenches, $a = 0$ cannot be explained on the basis of equation (11). Rather, the exponent $a$ in the quenches to below $T_c$ must be expected to have a dimensionality dependence such that $\lim_{d \to d_c} a = 0$. The phenomenological formula (54) fits quite well within this framework, although a deeper theoretical understanding of the problem is needed in order to put it on firmer grounds.

**Acknowledgments**

We wish to thank Pasquale Calabrese, Silvio Franz and Mauro Sellitto for very useful discussions. This work was partially supported by MURST through PRIN-2002.

**References**

[1] For a recent review see Cugliandolo L F, *Dynamics of glassy systems*, 2002 Preprint cond-mat/0210312

[2] Cugliandolo L F and Kurchan J, *Analytical solution of the off-equilibrium dynamics of a long-range spin-glass model*, 1993 Phys. Rev. Lett. 71 173 [cond-mat/9303030]

Cugliandolo L F and Kurchan J, *On the out-of-equilibrium relaxation of the Sherrington-Kirkpatrick model*, 1994 J. Phys. A: Math. Gen. 27 5749 [cond-mat/9311016]

[3] Crisanti A and Ritort F, *Violation of the fluctuation-dissipation theorem in glassy systems: basic notions and the numerical evidence*, 2003 J. Phys. A: Math. Gen. 36 R181 [cond-mat/0212190]

[4] Cugliandolo L F, Kurchan J and Peliti L, *Energy flow, partial equilibration, and effective temperatures in systems with slow dynamics*, 1997 Phys. Rev. E 55 3898 [cond-mat/9611044]

[5] Sollich P, Fielding S and Mayer P, *Fluctuation-dissipation relations and effective temperatures in simple non-mean field systems*, 2002 J. Phys.: Condens. Matter 14 1683 [cond-mat/0111241]

[6] Calabrese P and Gambassi A, *On the definition of a unique effective temperature for non-equilibrium critical systems*, 2004 Preprint cond-mat/0406289

[7] Franz S, Mézard M, Parisi G and Peliti L, *Measuring equilibrium properties in aging systems*, 1998 Phys. Rev. Lett. 81 1758 [cond-mat/9803108]

Franz S, Mézard M, Parisi G and Peliti L, *The response of glassy systems to random perturbations: a bridge between equilibrium and off-equilibrium*, 1999 J. Stat. Phys. 97 459 [cond-mat/9903370]

[8] Cugliandolo L F, *Effective temperatures out of equilibrium*, 1999 Preprint cond-mat/9903250

[9] Parisi G, Ricci-Tersenghi F and Ruiz-Lorenzo J J, *Generalized off-equilibrium fluctuation-dissipation relations in random Ising systems*, 1999 Eur. Phys. J. B 11 317 [cond-mat/9811374]

[10] Bray A J, *Theory of phase-ordering kinetics*, 1994 Adv. Phys. 43 357

[11] Godrèche C and Luck J M, *Nonequilibrium critical dynamics of ferromagnetic spin systems*, 2002 J. Phys.: Condens. Matter 14 1589 [cond-mat/0109212]

[12] Calabrese P and Gambassi A, *Ageing properties of critical systems*, 2004 Preprint cond-mat/0410357

[13] Lippiello E and Zannetti M, *Fluctuation dissipation ratio in the one dimensional kinetic Ising model*, 2000 Phys. Rev. E 61 3360 [cond-mat/0001103]

[14] Godrèche C and Luck J M, *Response of non-equilibrium systems at criticality: exact results for the Glauber–Ising chain*, 2000 J. Phys. A: Math. Gen. 33 1151 [cond-mat/9911348]

[15] Corberi F, Lippiello E and Zannetti M, *Slow relaxation in the large $N$ model for phase ordering*, 2002 Phys. Rev. E 66 046136 [cond-mat/0202091]

[16] Corberi F, Lippiello E and Zannetti M, *On the connection between off-equilibrium response and statics in non-disordered coarsening systems*, 2001 Eur. Phys. J. B 24 359 [cond-mat/0110651]

Corberi F, Lippiello E and Zannetti M, *Interface fluctuations, bulk fluctuations, and dimensionality in the off-equilibrium response of coarsening systems*, 2001 Phys. Rev. E 63 001506 [cond-mat/0007021]

[17] Corberi F, Castellano C, Lippiello E and Zannetti M, *Generic features of the fluctuation dissipation relation in coarsening systems*, 2004 Phys. Rev. E 70 017103 [cond-mat/0311046]

[18] Corberi F, Lippiello E and Zannetti M, *Comment on “Aging, phase ordering and conformal invariance”,* 2003 Phys. Rev. Lett. 90 099601 [cond-mat/0211600]
Corberi F, Lippiello E and Zannetti M, *Scaling of the linear response function from zero-field-cooled and thermoremanent magnetization in phase-ordering kinetics*, 2003 Phys. Rev. E 68 046131 [cond-mat/0307542]

[19] Mazenko G F, Valls O T and Zhang F C, *Kinetics of first-order phase transitions: Monte Carlo simulations, renormalization-group methods, and scaling for critical quenches*, 1985 Phys. Rev. B 31 4453

Bray A J, *Renormalization-group approach to domain-growth scaling*, 1990 Phys. Rev. B 41 6724

[20] Hohenberg P C and Halperin B I, *Theory of dynamic critical phenomena*, 1977 Rev. Mod. Phys. 49 435

[21] Mazenko G F, Valles O T and Zhang F C, *Kinetics of first-order phase transitions: Monte Carlo simulations, renormalization-group methods, and scaling for critical quenches*, 1985 Phys. Rev. B 31 4453

[22] Bray A J, *Renormalization-group approach to domain-growth scaling*, 1990 Phys. Rev. B 41 6724

[23] Hohenberg P C and Halperin B I, *Theory of dynamic critical phenomena*, 1977 Rev. Mod. Phys. 49 435

[24] Mazenko G F, *Response functions in phase-ordering kinetics*, 2004 Phys. Rev. E 69 016114 [cond-mat/0308169]

[25] Henkel M, *Phenomenology of local scale invariance: from conformal invariance to dynamical scaling*, 2002 Nucl. Phys. B 641 405 [hep-th/0205256]

Henkel M, Pleimling M, Godrèche C and Luck J M, *Aging, phase ordering and conformal invariance*, 2001 Phys. Rev. Lett. 87 265701 [hep-th/0107122]

[26] Henkel M, Paessens M and Pleimling M, *Scaling of the linear response in simple aging systems without disorder*, 2004 Phys. Rev. E 69 056109 [cond-mat/0310761]

[27] Berthier L, Barrat J L and Kurchan J, *Response function of coarsening systems*, 1999 Eur. Phys. J. B 11 635 [cond-mat/9907438]

[28] Abriet S and Karevski D, *Off-equilibrium dynamics in the 3d-XY system*, 2004 Preprint cond-mat/0405598

[29] Blundell R E and Bray A J, *Phase-ordering dynamics of the O(n) model: exact predictions and numerical results*, 1994 Phys. Rev. E 49 4925 [cond-mat/9310075]