COMPARISON RESULTS FOR SOLUTIONS TO THE ANISOTROPIC LAPLACIAN WITH ROBIN BOUNDARY CONDITIONS

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ABSTRACT. In this paper we consider PDE’s problems involving the anisotropic Laplacian operator, with Robin boundary conditions. By means of Talenti techniques, widely used in the last decades, we prove a comparison result between the solutions of the above-mentioned problems and the solutions of the symmetrized ones. As a consequence of these results, a Bossel-Daners type inequality can be shown in dimension 2.

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1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, with Lipschitz boundary. Let us consider the following anisotropic problem with Robin boundary conditions

$$
\begin{aligned}
-\text{div}(H(\nabla u)H_\xi(\nabla u)) &= f & \text{in } \Omega \\
H(\nabla u)H_\xi(\nabla u) \cdot \nu + \beta H(\nu)u &= 0 & \text{on } \partial \Omega,
\end{aligned}
$$

where $f \geq 0$ (not identically zero) belongs to $L^2(\Omega)$, $H$ is a sufficiently smooth norm in $\mathbb{R}^n$, $\nu$ is the outer unit normal to $\partial \Omega$ and $\beta > 0$ is a positive real parameter.

A weak solution to problem (1.1) is a function $u \in H^1(\Omega)$ that satisfies

$$
\int_\Omega H(\nabla u)H_\xi(\nabla u) \cdot \nabla \varphi \, dx + \beta \int_{\partial \Omega} H(\nu)u \varphi \, d\mathcal{H}^{n-1} = \int_\Omega f \varphi \quad \forall \varphi \in H^1(\Omega).
$$

The aim of the paper is to establish a comparison result with the solution to the following symmetrized problem

$$
\begin{aligned}
-\text{div}(H(\nabla v)H_\xi(\nabla v)) &= f^* & \text{in } \Omega^* \\
H(\nabla v)H_\xi(\nabla v) \cdot \nu + \beta H(\nu)v &= 0 & \text{on } \partial \Omega^*,
\end{aligned}
$$

where $f^*$ is the convex symmetrization of $f$ and $\Omega^*$ is a set homothetic to the Wulff Shape $\mathcal{W}$ such that $|\Omega^*| = |\Omega|$ (for the exact definitions, see section 2).

It is well known that, when $H$ is the euclidean norm in $\mathbb{R}^n$, then the convex
symmetrization coincides with the Schwarz symmetrization (to know more about
Schwarz symmetrization see [K]). Moreover if the Robin boundary condition is
replaced by the Dirichlet boundary condition, problem (1.1) becomes
\[
\begin{cases}
-\Delta u = f & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
with \( f \in L^2(\Omega) \) (non-negative and not identically zero) and \( \Omega \) an open subset of \( \mathbb{R}^n \). This problem has been widely studied, with Talenti being the pioneer in this
direction. Indeed the author (see [T]) proved, via rearrangements arguments, that
the Schwarz symmetrization of the solution to problem (1.4) is pointwise bounded
by the solution to the following symmetrized problem
\[
\begin{cases}
-\Delta v = f^\sharp & \text{in } \Omega^\sharp \\
v = 0 & \text{on } \partial \Omega^\sharp,
\end{cases}
\]
with \( f^\sharp \) being the Schwarz decreasing rearrangement of \( f \) and \( \Omega^\sharp \) the ball centered
at the origin having the same measure as \( \Omega \). Talenti, with his techniques, gave
birth to a series of generalizations and results that still now take his name. For
instance see [AFLT, ALT, T1, T2] for generalizations to other kind of operators.
When we have Robin boundary conditions with positive parameter, problem (1.4)
becomes
\[
\begin{cases}
-\Delta u = f & \text{in } \Omega \\
\frac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
To our knowledge, in literature, there are few comparison results à la Talenti for
this kind of problem. A result of this type has been proved only recently by [ANT],
where they have highlighted the importance of the dependence on the dimension
of the space. The authors, in fact, managed to compare the Lorentz norm (see
[L]) of the solution to problem (1.6) with that of the solution to the symmetrized
problem
\[
\begin{cases}
-\Delta v = f^\sharp & \text{in } \Omega^\sharp \\
\frac{\partial v}{\partial \nu} + \beta u = 0 & \text{on } \partial \Omega^\sharp,
\end{cases}
\]
where the exponents of these norms depend on the dimension of the space. In
particular they proved, for \( n \geq 2 \), that
\[
\|u\|_{L^{p,1}(\Omega)} \leq \|v\|_{L^{p,1}(\Omega^\sharp)} \quad \text{for all } 0 < p \leq \frac{n}{2n - 2}
\]
and
\[
\|u\|_{L^{2p,2}(\Omega)} \leq \|v\|_{L^{2p,2}(\Omega^\sharp)} \quad \text{for all } 0 < p \leq \frac{n}{3n - 4},
\]
with \( u \) solution to (1.6) and \( v \) to (1.7). Moreover, when \( f \equiv 1 \) in \( \Omega \) and \( n=2 \), they
showed that
\[
u^\sharp(x) \leq v(x) \quad x \in \Omega^\sharp,
\]
and, for $n \geq 3$, that
\[
\|u\|_{L^p,1(\Omega)} \leq \|v\|_{L^p,1(\Omega^\flat)}
\]
\[
\|u\|_{L^{2p/2}(\Omega)} \leq \|v\|_{L^{2p/2}(\Omega^\flat)}
\]
for all $0 < p \leq \frac{n}{n-2}$.

With a different approach [BG] proved that
\[
\|u\|_{L^1(\Omega)} \leq \|v\|_{L^1(\Omega^\flat)}.
\]

In [S] one can find the computation of the shape derivative of the $L^\infty$ and $L^p$ norms of the solution to the problem (1.6) and the proof of the stationarity of the ball, when fixing the volume.

The main theorems of this paper show that the proves proposed by [ANT] can be adapted even when considering the anisotropic Laplacian operator. Let us now state these results.

**Theorem 1.1.** Let be $n \geq 2$. If $u$ and $v$ are the solutions to problems (1.1) and (1.3) respectively, then
\[
\|u\|_{L^p,1(\Omega)} \leq \|v\|_{L^p,1(\Omega^\flat)} \quad \text{for all} \quad 0 < p \leq \frac{n}{2n-2} \quad (1.9)
\]
and
\[
\|u\|_{L^{2p/2}(\Omega)} \leq \|v\|_{L^{2p/2}(\Omega^\flat)} \quad \text{for all} \quad 0 < p \leq \frac{n}{3n-4}. \quad (1.10)
\]

**Theorem 1.2.** Let $n = 2$, $f \equiv 1$ in $\Omega$. If $u$ and $v$ are the solutions to problems (1.1) and (1.3) respectively. Then
\[
u^*(x) \leq v(x) \quad x \in \Omega^*, \quad (1.11)
\]
where $u^*$ is the convex symmetrization of $u$.

**Theorem 1.3.** Let $n \geq 3$ and $f \equiv 1$. If $u$ and $v$ are the solutions to problems (1.1) and (1.3) respectively, then
\[
\|u\|_{L^p,1(\Omega)} \leq \|v\|_{L^p,1(\Omega^\flat)} \quad (1.12)
\]
and
\[
\|u\|_{L^{2p/2}(\Omega)} \leq \|v\|_{L^{2p/2}(\Omega^\flat)}, \quad (1.13)
\]
for all $0 < p \leq \frac{n}{n-2}$.

The organization of the paper is the following. In Section 2 the reader can find notations and preliminaries concerning the anisotropic norm, its properties and the convex symmetrization of a function. Section 3 contains results about existence, uniqueness and other properties of the solutions to problem (1.1) and (1.3). In section 4 the main theorems are proved. Section 5 is dedicated to the application of these results to prove, in dimension 2, the Bossel-Daners inequality for the anisotropic operator. Last section will include some counterexamples and problems that still are open.
2. Notations and Preliminaries

2.1. Anisotropy. What follows can be found in [AFLT]. Let $H : \mathbb{R}^n \rightarrow [0, +\infty]$, $n \geq 2$, be a $C^2(\mathbb{R}^n \setminus \{0\})$ convex functions that satisfies the following homogeneity property

$$H(t\xi) = |t|H(\xi) \quad \forall \xi \in \mathbb{R}^n, \forall t \in \mathbb{R},$$

(2.1)

and such that

$$\gamma |\xi| \leq H(\xi) \leq \delta |\xi|,$$

(2.2)

for some positive constants $\gamma \leq \delta$. Because of (2.1), we can assume that the set

$$K = \{\xi \in \mathbb{R}^n : H(\xi) \leq 1\}$$

is such that $|K|$ is equal to the measure $\omega_n$ of the unit sphere in $\mathbb{R}^n$. We can define the support function of $K$ as

$$H^\circ(x) = \sup_{\xi \in K} \langle x, \xi \rangle,$$

(2.3)

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^n$. $H^\circ : \mathbb{R}^n \rightarrow [0, +\infty]$ is a convex, homogeneous function. Moreover $H$ and $H^\circ$ are polar to each other, in the sense that

$$H^\circ(\xi) = \sup_{\xi \neq 0} \frac{\langle x, \xi \rangle}{H(\xi)},$$

and

$$H(x) = \sup_{\xi \neq 0} \frac{\langle x, \xi \rangle}{H^\circ(\xi)}.$$

$H^\circ$ is the support function of the set

$$K^\circ = \{x \in \mathbb{R}^n : H^\circ(x) \leq 1\}.$$ 

The set $\mathcal{W} = \{x \in \mathbb{R}^n : H^\circ(x) < 1\}$ is the so-called Wulff shape centered at the origin. We set $k_n = |\mathcal{W}|$. More generally we will denote by $\mathcal{W}_R(x_0)$ the Wulff shape centered in $x_0 \in \mathbb{R}^n$ with measure $k_nR^n$ the set $R\mathcal{W} + x_0$, and $\mathcal{W}_R(0) = \mathcal{W}_R$.

$H$ and $H^\circ$ satisfy the following properties:

$$H_\xi(\xi) \cdot H^\circ_\xi(\xi) = H(\xi), \quad H^\circ(\xi) \cdot H^\circ_\xi(\xi) = H^\circ(\xi),$$

(2.4)

$$H(H^\circ_\xi(\xi)) = H^\circ(H_\xi(\xi)) = 1 \quad \forall \xi \in \mathbb{R}^n \setminus \{0\},$$

(2.5)

$$H^\circ(\xi)H_\xi(H^\circ_\xi(\xi)) = H(\xi)H^\circ_\xi(H_\xi(\xi)) = \xi \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$ 

(2.6)

If $\Omega \subset \mathbb{R}^n$ is an open bounded set with Lipschitz boundary and $E$ is an open subset of $\mathbb{R}^n$, we can give a generalized definition of perimeter of $E$ with respect to the anisotropic norm as follows

$$P_H(E, \Omega) = \int_{\partial E \cap \Omega} H(\nu_E) d\mathcal{H}^{n-1},$$

where $\nu_E$ is the exterior unit normal to $E$.
where $\partial^* E$ is the reduced boundary of $E$ and $\nu^E$ is its outer normal. Clearly, if $E$ is open, bounded and Lipschitz, then the outer unit normal exists almost everywhere and
\[ P_H(E, \mathbb{R}^n) := P_H(E) = \int_{\partial E} H(\nu^E) \, d\mathcal{H}^{n-1}. \] (2.7)

By (2.2) we have that
\[ \gamma P(E) \leq P_H(E) \leq \delta P(E). \]

In [AB, AFLT] it is shown that if $u \in W^{1,1}(\Omega)$, then for a.e. $t > 0$
\[ - \frac{d}{dt} \int_{\{u > t\}} H(\nabla u) \, dx = P_H(\{u > t\}, \Omega) = \int_{\partial^* \{u > t\} \cap \Omega} \frac{H(\nabla u)}{|\nabla u|} \, d\mathcal{H}^{n-1}. \] (2.8)

In particular an isoperimetric inequality for the anisotropic perimeter holds (for instance see [AFLT, Bu, DP, FM])
\[ P_H(E) \geq nk_1 \frac{1}{n} |E|^{1-\frac{1}{n}}. \] (2.9)

2.2. **Convex symmetrization.** Let $f : \Omega \rightarrow [0, +\infty]$ be a measurable function.

The decreasing rearrangement $f^*$ of $f$ is defined as follows
\[ f^*(s) = \inf \{t \geq 0 : |\{x \in \Omega : |f(x)| > t\}| < s\} \quad s \in [0, |\Omega|], \]
which is the generalized inverse function of the distribution function of $f$. We define the convex symmetrization $f^\ast$ of $f$ as
\[ f^\ast(x) = f^*(k_n H^\circ(x)^n) \quad x \in \Omega^\ast. \]

In particular it is well known that the functions $f$, $f^*$ and $f^\ast$ are equimisurable, i.e.
\[ |\{f > t\}| = |\{f^* > t\}| = |\{f^\ast > t\}| \quad t \geq 0. \]

As a consequence, if $f \in L^p(\Omega)$, $p \geq 1$, then $f^* \in L^p([0,|\Omega|])$, $f^\ast \in L^p(\Omega^\ast)$ and
\[ \|f\|_{L^p(\Omega)} = \|f^*\|_{L^p([0,|\Omega|])} = \|f^\ast\|_{L^p(\Omega^\ast)}. \]

Moreover the Hardy-Littlewood inequality holds (see [K])
\[ \int_{\Omega} |f(x)g(x)| \, dx \leq \int_0^{\mu(\Omega)} f^*(s)g^*(s) \, ds. \] (2.10)

So, if we consider $g$ as the characteristic function of the set $\{x \in \Omega : u(x) > t\}$, for some measurable function $u : \Omega \rightarrow \mathbb{R}$ and $t \geq 0$, then we get
\[ \int_{\{u > t\}} f(x) \, dx \leq \int_0^{\mu(t)} f^*(s) \, ds, \] (2.11)

where, again, $\mu(t)$ is the distribution function of $u$.

3. **Existence, uniqueness and properties of the solution.**

In this section we want to prove that the solution to problem (1.1) exists, it is unique and non-negative.
3.1. **Existence.** Let us consider the following energy functional
\[
E[w] = \frac{1}{2} \int_{\Omega} H^2(\nabla w) \, dx + \frac{\beta}{2} \int_{\partial\Omega} H(\nu) \, w^2 \, dH^{n-1} - \int_{\Omega} f(w) \, dx, \quad w \in H^1(\Omega). \tag{3.1}
\]
We notice that the Euler-Lagrange equations of this functional is (1.1), hence if \( E[\cdot] \) has minima, there will exist a solution to the considered problem. Let us proceed with the classical Calculus of Variation method to prove the existence of a minimum.

1) **Lower bound.** Let us prove that the functional (3.1) is lower bounded. From (2.2) and the generalized Young’s inequality we have
\[
E[u] \geq \frac{\gamma^2}{2} \|\nabla u\|^2_{L^2(\Omega)} + \frac{\beta\gamma}{2} \|u\|^2_{L^2(\partial\Omega)} - \frac{\epsilon}{2} \|u\|^2_{L^2(\Omega)} - \frac{1}{2\epsilon} \|f\|^2_{L^2(\Omega)}
\geq C_1 \left( \|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\partial\Omega)} \right) - \frac{\epsilon}{2} \|u\|^2_{L^2(\Omega)} - \frac{1}{2\epsilon} \|f\|^2_{L^2(\Omega)}
\geq \left( C_2 - \frac{\epsilon}{2} \right) \|u\|^2_{L^2(\Omega)} - \frac{1}{2\epsilon} \|f\|^2_{L^2(\Omega)}.
\]
In the last inequality we have used a Poincaré inequality with trace term (see for instance \[BGT\]). Here \( C_2 = C_2(\beta, \gamma, \Omega) \). If we choose \( \epsilon \) small enough, then
\[
E[u] \geq -\frac{1}{2\epsilon} \|f\|^2_{L^2(\Omega)} > -\infty.
\]
We have proved in this way that the functional is bounded from below. Let be
\[
m := \inf_{w \in H^1(\Omega)} E[w]. \tag{3.2}
\]
and let \( \{u_k\} \subset H^1(\Omega) \) be a minimizing sequence, i.e.
\[
\lim_{k \to \infty} E[u_k] = m.
\]
We can suppose that \( E[u_k] \leq m + 1 \) for all \( k \in \mathbb{N} \).

2) **Compactness and lower semicontinuity.** Using again the generalized Young’s inequality and the Poincaré inequality with trace term, we have
\[
m + 1 \geq \frac{\gamma^2}{4} \|\nabla u_k\|^2_{L^2(\Omega)} - \frac{\epsilon}{2} \|u_k\|^2_{L^2(\Omega)}
+ \frac{\gamma^2}{4} \|\nabla u_k\|^2_{L^2(\Omega)} + \frac{\beta\gamma}{2} \|u_k\|^2_{L^2(\partial\Omega)} - \frac{1}{2\epsilon} \|f\|^2_{L^2(\Omega)}
\geq \frac{\gamma^2}{4} \|\nabla u_k\|^2_{L^2(\Omega)} - \frac{\epsilon}{2} \|u_k\|^2_{L^2(\Omega)} + C_3 \|u_k\|^2_{L^2(\Omega)} - \frac{1}{2\epsilon} \|f\|^2_{L^2(\Omega)},
\]
where \( C_3 = C_3(\beta, \gamma, \Omega) \). Choosing \( \epsilon \) small enough and calling \( C_4 = \min(\frac{\gamma^2}{4}, C_3 - \frac{\epsilon}{2}) \) then
\[
\|u_k\|_{H^1(\Omega)} \leq \frac{m + 1}{C_4} + \frac{1}{2\epsilon C_4} \|f\|^2_{L^2(\Omega)} < \infty.
\]
Hence \( \{ u_k \} \) is bounded in \( H^1(\Omega) \), so it exists a subsequence \( \{ u_{k_j} \} \subset \{ u_k \} \) that converges weakly in \( H^1(\Omega) \) and strongly in \( L^2(\Omega) \) to a function \( u \in H^1(\Omega) \). To simplify the notation let us continue to call the subsequence as \( \{ u_k \} \).

By the strict convexity of the functions \( t \rightarrow t^2 \) and \( H^2 \), we have the following inequalities

\[
\begin{align*}
    u_k^2 &\geq u^2 + 2u(u_k - u) \quad (3.3) \\
    H^2(\nabla u_k) &\geq H^2(\nabla u) + 2H(\nabla u)H_\xi(\nabla u) \cdot (\nabla u_k - \nabla u). \quad (3.4)
\end{align*}
\]

By (3.3) and (3.4), we have

\[
E[u_k] \geq \frac{1}{2} \int_\Omega H^2(\nabla u) \, dx + \frac{\beta}{2} \int_{\partial \Omega} u^2 \, d\mathcal{H}^{n-1} - \int_\Omega fu_k \, dx + \int_\Omega H(\nabla u)H_\xi(\nabla u) \cdot (\nabla u_k - \nabla u) \, dx + \int_{\partial \Omega} u(u_k - u) \, d\mathcal{H}^{n-1}.
\]

By the weak convergence of \( u_k \) in \( H^1(\Omega) \) and being \( H^1(\Omega) \) compactly embedded in \( L^2(\partial \Omega) \), passing to the limit for \( k \to \infty \) we have that

\[
E[u] \leq m.
\]

Eventually \( E[u] = m \) and \( u \) is a minimum.

### 3.2. Uniqueness and non-negativeness

Let us prove now that the minimum of (3.1) is unique. If \( u, v \in H^1(\Omega) \), by the strict convexity of the function \( H^2 \) we know that if \( t \in [0, 1] \), then

\[
H^2(t\nabla u + (1 - t)\nabla v) \leq tH^2(\nabla u) + (1 - t)H^2(\nabla v). \quad (3.5)
\]

The equality occurs if and only if \( t = 0 \) or \( t = 1 \). Analogously

\[
[tu + (1 - t)v]^2 \leq tu^2 + (1 - t)v^2, \quad t \in [0, 1]. \quad (3.6)
\]

Let \( u \in H^1(\Omega) \) be a minimizer of (3.1) and let us suppose that there exists another minimizer \( v \in H^1(\Omega) \), such that \( u \neq v \). Hence \( E[u] = E[v] = m \). Let us denote by \( w = u + v \) and choose \( t = \frac{1}{2} \), then by (3.5) and (3.6)

\[
E[w] < \frac{E[u]}{2} + \frac{E[v]}{2} = m.
\]

This fact contradicts the minimality of \( u \) and so the minimum must be unique.

Let us now show the non-negativeness of the solution. Let \( u \) be the unique minimum of (3.1), namely \( E[u] = m \). If we consider \( |u| \), by (2.1), we get

\[
H^2(\nabla |u|) = H^2\left( \frac{u}{|u|} \nabla u \right) = H^2(\nabla u).
\]
Hence

\[ E[|u|] = \frac{1}{2} \int_{\Omega} H^2(\nabla |u|) \, dx + \frac{\beta}{2} \int_{\partial\Omega} H(\nu)|u|^2 \, dH^{n-1} - \int_{\Omega} f |u| \, dx \]

\[ = m + \int_{\Omega} f(u - |u|) \, dx = m + 2 \int_{\{u \leq 0\}} f u \leq m. \]

By the uniqueness of the minimizer it must be \( u = |u| \) in \( \Omega \). Eventually \( u \geq 0 \) in \( \Omega \).

3.3. The anisotropic radial case. Let us consider problem (1.3). Because of uniqueness of the solution and the symmetry of the problem we look for a solution of the type \( v(x) = v(H^s(x)) \). This function solves the following ODE

\[
\begin{cases}
\frac{1}{r^{n-1}}v''(r) = f^r(k_n r^n) & \text{for } r \in (0, R) \\
v'(0) = 0 \\
v'(R) + \beta v(R) = 0,
\end{cases}
\]

(3.7)

where \( r = H^s(x) \) and \( R \) is such that \( \Omega^* = \mathcal{W}_R \) and \( |\Omega| = |\Omega^*| \).

Integrating the first equation in (3.7), calling \( \tilde{t} = k_n t^n \), we get

\[ v'(r) = -\frac{1}{r^{n-1}} \int_0^r t^{n-1} f^r(k_n t^n) \, dt + C_1 = -\frac{1}{nk_n r^{n-1}} \int_0^{k_n r^n} f^r(\tilde{t}) \, d\tilde{t} + C_1. \]

Since \( v'(0) = 0 \) then \( C_1 = 0 \). By denoting \( \tilde{s} = k_n s^n \), another integration gives

\[ v(r) = -\int_0^r \frac{1}{nk_n s^{n-1}} \int_0^{k_n s^n} f^r(\tilde{t}) \, d\tilde{t} \, ds + C_2 = \]

\[ -\int_0^{k_n r^n} \frac{1}{nk_n s^{n-1}} \int_0^{k_n s^n} f^r(\tilde{t}) \, d\tilde{t} \, ds + C_2. \]

From \( v'(R) + \beta v(R) = 0 \) we compute \( C_2 \), hence

\[ v(r) = -\int_0^{k_n r^n} \frac{1}{nk_n s^{n-1}} \int_0^{k_n s^n} f^r(\tilde{t}) \, d\tilde{t} \, ds + \]

\[ \int_0^{k_n R^n} \frac{1}{nk_n s^{n-1}} \int_0^{k_n s^n} f^r(\tilde{t}) \, d\tilde{t} \, ds + \frac{1}{\beta nk_n R^{n-1}} \int_0^{k_n R^n} f^r(\tilde{t}) \, d\tilde{t}. \]

Therefore

\[ v(r) = \int_{k_n r^n}^{k_n R^n} \frac{1}{nk_n s^{n-1}} \int_0^{k_n s^n} f^r(\tilde{t}) \, d\tilde{t} \, ds + \frac{1}{\beta nk_n R^{n-1}} \int_0^{k_n R^n} f^r(\tilde{t}) \, d\tilde{t}. \]

(3.8)

In this way we have shown that the unique solution to problem (1.3) is radially symmetric with respect to anisotropic norm and its value on the boundary is given
by
\[
v(R) = \frac{1}{\beta n k_n R^{n-1}} \int_0^{k_n R^2} f^*(\tilde{t}) \, d\tilde{t} \geq 0.
\]

**Remark 3.1.** We stress that if \( f \equiv 1 \) in \( \Omega \), then by (3.8) the solution to problem (1.3) can be written explicitly as follows
\[
v(x) = v(H^o(x)) = \frac{R}{\beta n} + \frac{1}{2n} (R^2 - H^o(x)^2).
\]
The solution is a paraboloid with respect to the anisotropic norm. Moreover if \( H \) is the euclidean norm in \( \mathbb{R}^n \), we step back to the classical torsion problem (or Saint Venant problem) with robin boundary conditions, whose radial solution is a concave paraboloid.

### 3.4. Level sets and distribution functions.
If \( u \) is a solution to problem (1.1), we will denote by
\[
U_t = \{ x \in \Omega : u(x) > t \}
\]
for a non-negative real number \( t \geq 0 \). It is clear that if \( t \leq u_{\text{min}} \), then \( U_t = \Omega \) and that if \( t > u_{\text{max}} \), then \( U_t = \emptyset \). With \( u_{\text{min}} \) and \( u_{\text{max}} \) we have denoted the minimum and the maximum of \( u \) in \( \Omega \). We will denote by
\[
\partial U_t^\text{int} = \Omega \cap \partial U_t, \quad \partial U_t^\text{ext} = \partial \Omega \cap \partial U_t
\]
the internal and external boundary of \( U_t \) with respect to \( \Omega \), and by
\[
\mu(t) = |U_t|
\]
the distribution function of \( u \).

If \( v \) is solution to problem (1.3), for \( t \geq 0 \), we will indicate with
\[
V_t = \{ x \in \Omega^* : v(x) > t \}, \quad \phi(t) = |V_t|
\]
the superlevel sets and the distribution function of \( v \), respectively. Furthermore, for \( 0 \leq t \leq v_{\text{min}} \), \( V_t = \Omega \), while for \( v_{\text{min}} < t < v_{\text{max}} \), the superlevel sets \( V_t \) are Wulff shapes homothetic to \( \Omega^* \) and strictly contained in it. Again, \( v_{\text{min}} \) and \( v_{\text{max}} \) are the minimum and the maximum of \( v \) in \( \Omega^* \).

### 4. Main results
Before proving the main results let us state Gronwall Lemma and prove two others lemmata that will have a central importance for what will follow.

**Lemma 4.1.** (Gronwall) Let \( \xi(t) \) be a continuously differentiable function satisfying for some non-negative constant \( C \), the following differential inequality
\[
\tau \xi'(\tau) \leq \xi(\tau) + C,
\]
for all \( \tau \geq \tau_0 > 0 \). Then we have
\[ \xi(\tau) \leq \tau \frac{\xi(\tau_0) + C}{\tau_0} + C \]

and

\[ \xi'(\tau) \leq \frac{\xi(\tau_0) + C}{\tau_0}, \]

for all \( \tau \geq \tau_0 \).

**Lemma 4.2.** Let \( u \) and \( v \) be the solutions to problems (1.1) and (1.3) respectively. Then for a.e. \( t > 0 \) we have

\[
\begin{align*}
\frac{n^2 k_n^2}{\beta} f^*(s) &\geq (-\phi' (t) + \frac{1}{\beta} \int_{\partial U^\text{ext}_t} \frac{H(\nu)}{u} dH^{n-1}) \int_0^{\mu(t)} f^*(s) \, ds \quad (4.1) \\
\frac{n^2 k_n^2}{\beta} f^*(s) &\geq (-\mu' (t) + \frac{1}{\beta} \int_{\partial U^\text{ext}_t} \frac{H(\nu)}{u} dH^{n-1}) \int_0^{\mu(t)} f^*(s) \, ds. \quad (4.2)
\end{align*}
\]

**Proof.** Let \( t, h > 0 \) and let us consider the following test function in \( H^1(\Omega) \)

\[
\varphi_h(x) = \begin{cases} 
0 & u \leq t \\
\frac{u - t}{h} & t < u \leq t + h \\
h & u \geq t + h.
\end{cases}
\]

Substituting this in (1.2) we have

\[
\begin{align*}
\int_{U_t \setminus U_{t+h}} H(\nabla u) H(\nabla u) \cdot \nabla u \, dx + \beta \int_{\partial U^\text{ext}_t \setminus \partial U^\text{ext}_{t+h}} H(\nu)(u - t) u \, dH^{n-1} \\
+ \beta h \int_{\partial U^\text{ext}_{t+h}} H(\nu) u \, dH^{n-1} &= \int_{U_t \setminus U_{t+h}} f(u - t) \, dx + h \int_{U_{t+h}} f \, dx.
\end{align*}
\]

Applying (2.4) in the first integral, dividing by \( h \) and applying Coarea Formula, we have for a.e. \( t > 0 \)

\[
\begin{align*}
\frac{1}{h} \int_{t}^{t+h} \int_{\partial U^\text{int}_t} \frac{H^2(\nabla u)}{|\nabla u|} \, dH^{n-1} \, d\tau &+ \beta \frac{1}{h} \int_{\partial U^\text{ext}_t \setminus \partial U^\text{ext}_{t+h}} H(\nu)(u - t) u \, dH^{n-1} \\
+ \beta \int_{\partial U^\text{ext}_{t+h}} H(\nu) u \, dH^{n-1} &= \frac{1}{h} \int_{U_t \setminus U_{t+h}} f(u - t) \, dx + \int_{U_{t+h}} f \, dx.
\end{align*}
\]

Passing to the limit for \( h \to 0^+ \) we have

\[
\int_{\partial U^\text{int}_t} \frac{H^2(\nabla u)}{|\nabla u|} \, dH^{n-1} + \beta \int_{\partial U^\text{ext}_t} H(\nu) u \, dH^{n-1} = \int_{U_t} f \, dx.
\]
Let us set
\[
g(x) = \begin{cases} \frac{H(\nabla u)}{\beta u} & \text{if } \partial U^\text{int}_t, \\ \frac{\partial U^\text{ext}_t}{\nabla u} & \text{if } \partial U^\text{ext}_t. \end{cases} \tag{4.3}
\]

We want to end the proof using the anisotropic version of the isoperimetric inequality and to this aim it is necessary to write properly the anisotropic perimeter of \(U_t\). Because of the regularity of \(\partial \Omega\) we know that \(\partial U^\text{ext}_t\) is sufficiently regular and a normal vector can be defined. Being \(u \in H^1(\Omega)\) and \(f \in L^2(\Omega)\), then \(\partial U^\text{int}_t\) can't have any good regularity property. By (2.8) we can write for a.e. \(t > 0\)
\[
P_H(U_t) = \int_{\partial U^\text{int}_t} \frac{H(\nabla u)}{|\nabla u|} d\mathcal{H}^{n-1} + \int_{\partial U^\text{ext}_t} H(\nu) d\mathcal{H}^{n-1},
\]
where \(\nu\) is the outer unit normal to \(\Omega\). If we set
\[
h(x) = \begin{cases} \frac{H(\nabla u)}{|\nabla u|} & \text{if } \partial U^\text{int}_t, \\ H(\nu) & \text{if } \partial U^\text{ext}_t, \end{cases} \tag{4.4}
\]
then
\[
P_H(U_t) = \int_{\partial U_t} h(x) d\mathcal{H}^{n-1}.
\]
Furthermore we note that
\[
\int_{\partial U_t} h(x)g(x) d\mathcal{H}^{n-1} = \int_{U_t} f dx. \tag{4.5}
\]
Therefore by Schwarz inequality and (4.5), we have for a.e. \(t > 0\)
\[
P_H^2(U_t) = \left( \int_{\partial U_t} h(x) d\mathcal{H}^{n-1} \right)^2 \leq \left( \int_{\partial U_t} \sqrt{h(x)g(x)} \sqrt{h(x)g(x)} d\mathcal{H}^{n-1} \right)^2 \leq \int_{\partial U_t} h(x)g(x) d\mathcal{H}^{n-1} \left( \int_{\partial U^\text{int}_t} \frac{1}{\nabla u} d\mathcal{H}^{n-1} + \frac{1}{\beta} \int_{\partial U^\text{ext}_t} \frac{H(\nu)}{u} d\mathcal{H}^{n-1} \right) \leq \left( -\mu'(t) + \frac{1}{\beta} \int_{\partial U^\text{ext}_t} \frac{H(\nu)}{u} d\mathcal{H}^{n-1} \right) \int_0^\mu(t) f^*(s) ds.
\]
Hence, by (2.9)
\[
n^2 k_n^2 \mu(t) \frac{2n-2}{n} \leq \left( -\mu'(t) + \frac{1}{\beta} \int_{\partial U^\text{ext}_t} \frac{H(\nu)}{u} d\mathcal{H}^{n-1} \right) \int_0^\mu(t) f^*(s) ds.
\]
If we do the same computations, replacing \(v\) with \(u\), all the previous inequalities become equalities and we have (4.1).
In particular, if \(f \equiv 1\) in \(\Omega\), we have
\[
n^2 k_n^2 \mu(t) \frac{2n-2}{n} \leq -\mu'(t) + \frac{1}{\beta} \int_{\partial U^\text{ext}_t} \frac{H(\nu)}{u} d\mathcal{H}^{n-1}, \tag{4.6}
\]
Lemma 4.4. For all \( t \) and if \( t < v_{\min} \), then, using (1.1), (1.3) and the isoperimetric inequality

\[ \int_{\partial U_{\text{ext}}} H(\nu) u(x) d\mathcal{H}^{n-1} = \frac{1}{\beta} \int_{\partial \Omega^*} H(\nabla \nu) H(\nabla v) \cdot \nu d\mathcal{H}^{n-1} \]

\( \therefore \]

Remark 4.3. Let us notice that \( u_{\text{min}} \leq v_{\text{min}} \). Indeed, being the level sets of \( v \) homothetic to \( \Omega^* \), then, using (1.1), (1.3) and the isoperimetric inequality

\[ v_{\text{min}}P_H(\Omega^*) = \int_{\partial \Omega^*} H(\nu) v(x) d\mathcal{H}^{n-1} = -\frac{1}{\beta} \int_{\partial \Omega^*} H(\nabla v) H(\nabla v) \cdot \nu d\mathcal{H}^{n-1} \]

\[ = \frac{1}{\beta} \int_{\partial \Omega^*} H(\nabla u) H(\nabla u) \cdot \nu d\mathcal{H}^{n-1} \]

\[ \geq u_{\text{min}}P_H(\Omega) \geq u_{\text{min}}P_H(\Omega^*). \]

As a consequence for all \( 0 < t < v_{\text{min}} \) we have that

\[ \mu(t) \leq \phi(t) = |\Omega|. \] (4.9)

Lemma 4.4. For all \( t \geq v_{\text{min}} \) we have

\[ \int_{\partial \Omega^*} H(\nu) v(x) d\mathcal{H}^{n-1} = -\frac{1}{\beta} \int_{\partial \Omega^*} H(\nabla v) H(\nabla v) \cdot \nu d\mathcal{H}^{n-1} \]

\[ \therefore \]

Proof. By Fubini’s Theorem and using (1.1), we have that

\[ \int_{\partial U_{\text{ext}}} H(\nu) u(x) d\mathcal{H}^{n-1} = \int_{\partial \Omega} \left( \int_{\partial U_{\text{ext}}} \frac{H(\nu)}{u(x)} d\mathcal{H}^{n-1} \right) d\tau = \int_{\partial \Omega} \left( \int_{\partial \Omega} \frac{H(\nu)}{u(x)} d\tau \right) d\mathcal{H}^{n-1} \]

\[ = \int_{\partial \Omega} \frac{H(\nu) u(x)}{2} d\mathcal{H}^{n-1} = \frac{1}{2 \beta} \int_{\partial \Omega^*} f^*(s) ds. \] (4.10)

Analogously,

\[ \int_{\partial \Omega^*} H(\nu) v(x) d\mathcal{H}^{n-1} = \frac{1}{2 \beta} \int_{\partial \Omega^*} f^*(s) ds. \] (4.11)

By monotonicity of the integral we have that for \( t \geq 0 \)

\[ \int_{\partial \Omega} \left( \int_{\partial U_{\text{ext}}} \frac{H(\nu)}{u(x)} d\mathcal{H}^{n-1} \right) d\tau \leq \int_{\partial \Omega} \left( \int_{\partial U_{\text{ext}}} \frac{H(\nu)}{u(x)} d\mathcal{H}^{n-1} \right) d\tau \]

and if \( t \geq v_{\text{min}} \), then \( \partial V_t \cap \partial \Omega^* = \emptyset \). Hence

\[ \int_{\partial \Omega} \left( \int_{\partial U_{\text{ext}}} \frac{H(\nu)}{u(x)} d\mathcal{H}^{n-1} \right) d\tau = \int_{\partial \Omega} \left( \int_{\partial U_{\text{ext}}} \frac{H(\nu)}{u(x)} d\mathcal{H}^{n-1} \right) d\tau, \]
and we have (4.11), (4.10).

If \( f \equiv 1 \) in \( \Omega \), then
\[
\int_0^t \tau \left( \int_{\partial U_{ext} t} \frac{H(\nu)}{u(x)} dH^{n-1} \right) d\tau \leq \frac{|\Omega|}{2\beta} \tag{4.12}
\]
and
\[
\int_0^t \tau \left( \int_{\partial V_{ext} \cap \partial \Omega^*} \frac{H(\nu)}{v(x)} dH^{n-1} \right) d\tau = \frac{|\Omega|}{2\beta}. \tag{4.13}
\]
\[\square\]

Proof of Theorem 1.1. Let \( 0 < p \leq \frac{n}{2n-2} \) and let us denote \( K_n = n^2 k_n^2 \). Let us multiply (4.2) by \( t\mu(t)^p \), where \( \eta = \frac{1}{p} - \frac{2n-2}{n} \geq 0 \), and integrate from 0 to \( \tau \geq v_{\min} \)
\[
\int_0^\tau K_n t \mu(t)^p \, dt \leq \int_0^\tau -t \mu'(t) t \mu(t)^p \left( \int_0^{\mu(t)} f^*(s) \, ds \right) \, dt + \frac{1}{\beta} \int_0^\tau t \mu(t)^p \left( \int_{\partial U_{ext} t} \frac{H(\nu)}{u(x)} dH^{n-1} \int_0^{\mu(t)} f^*(s) \, ds \right) \, dt
\]
\[
\leq \int_0^\tau -t \mu(t)^p \left( \int_0^{\mu(t)} f^*(s) \, ds \right) \, d\mu(t) + \frac{|\Omega|^\eta}{2\beta^2} \left( \int_0^{[\Omega]} f^*(s) \, ds \right)^2,
\]
where we applied Lemma 4.4 and the fact that \( \mu(t) \) is a monotone non-increasing function.

By setting \( F(l) = \int_0^l w^n \int_0^{w^p} f^*(s) \, ds \, dw \) and integrating by parts the first and last members in this chain of inequalities we have
\[
\tau F(\mu(\tau)) + \tau \int_0^\tau K_n \mu(t)^{\frac{p}{p}} \, dt \leq \int_0^\tau F(\mu(t)) \, dt + \int_0^\tau \int_0^t K_n \mu(t)^{\frac{p}{p}} \, dr \, dt + \frac{|\Omega|^\eta}{2\beta^2} \left( \int_0^{[\Omega]} f^*(s) \, ds \right)^2
\]
By applying Lemma 4.1 with
\[
\xi(\tau) = \int_0^\tau F(\mu(t)) \, dt + \int_0^\tau \int_0^t K_n \mu(t)^{\frac{p}{p}} \, dr \, dt,
\]
\[
C = \frac{|\Omega|^\eta}{2\beta^2} \left( \int_0^{[\Omega]} f^*(s) \, ds \right)^2 \text{ and } \tau_0 = v_{\min},
\]
we have that
\[
F(\mu(\tau)) + \int_0^\tau K_n \mu(t)^{\frac{p}{p}} \, dt \leq \frac{1}{v_{\min}} \left( \int_0^{v_{\min}} F(\mu(t)) \, dt \right) + \int_0^{v_{\min}} \int_0^t K_n \mu(r)^{\frac{p}{p}} \, dr \, dt + \frac{|\Omega|^\eta}{2\beta^2} \left( \int_0^{[\Omega]} f^*(s) \, ds \right)^2. \tag{4.14}
\]
Analogously
\[ F(\phi(\tau)) + \int_0^\tau K_n \phi(t)^{\frac{1}{p}} \, dt = \frac{1}{v_{\text{min}}} \left( \int_0^{v_{\text{min}}} F(t) \, dt \right) + \int_0^{v_{\text{min}}} \int_0^t K_n \phi(r)^{\frac{1}{p}} \, dr \, dt + \frac{|\Omega|^{\eta}}{2\beta^2} \left( \int_0^{\Omega} f^*(s) \, ds \right)^2. \] (4.15)

By (4.8) and (1.9), then we can compare directly the right-hand sides of (4.14) and (4.15). So
\[ F(\mu(\tau)) + \int_0^\tau K_n \mu(t)^{\frac{1}{p}} \, dt \leq F(\phi(\tau)) + \int_0^\tau K_n \phi(t)^{\frac{1}{p}} \, dt \]
For \( \tau \to +\infty \) we have
\[ \int_0^\infty \mu(t)^{\frac{1}{p}} \, dt \leq \int_0^\infty \phi(t)^{\frac{1}{p}} \, dt, \]
which is (1.9).
Now we want to prove (1.10). In order to obtain this result let us pass to the limit as \( \tau \to \infty \) the following inequality:
\[ \int_0^\tau K_n t \mu(t)^{\frac{1}{p}} \, dt \leq \int_0^\tau -t \mu(t)^{\eta} \left( \int_0^{\Omega} f^*(s) \, ds \right) \, d\mu(t) + \frac{|\Omega|^{\eta}}{2\beta^2} \left( \int_0^{\Omega} f^*(s) \, ds \right)^2. \]
After an integration by parts we get
\[ \int_0^\infty K_n t \mu(t)^{\frac{1}{p}} \, dt \leq \int_0^\infty F(\mu(t)) \, dt + \frac{|\Omega|^{\eta}}{2\beta^2} \left( \int_0^{\Omega} f^*(s) \, ds \right)^2. \]
On the other hand
\[ \int_0^\infty K_n t \phi(t)^{\frac{1}{p}} \, dt = \int_0^\infty F(\phi(t)) \, dt + \frac{|\Omega|^{\eta}}{2\beta^2} \left( \int_0^{\Omega} f^*(s) \, ds \right)^2. \]
So we need just to show that
\[ \int_0^\infty F(\mu(t)) \, dt \leq \int_0^\infty F(\phi(t)) \, dt. \] (4.16)
To this aim we multiply (4.12) by \( tF(\mu(t)) \mu(t)^{-\frac{2n-2}{2}} \). Since \( F(l)l^{-\frac{2n-2}{2}} \) is a non-decreasing function in \( l \), when \( 0 < p \leq \frac{n}{9n-4} \), we can integrate from 0 to \( \tau \geq v_{\text{min}} \).
This inequality holds as an equality when we have $\phi$ in place of $\mu$, so as before

$$\int_0^\tau K_n F(\mu(t)) dt + J(\mu(t)) \leq \int_0^\tau K_n F(\phi(t)) dt + J(\phi(t)).$$

For $\tau \to \infty$ we have (1.10), which concludes the proof. \hfill \Box
Proof of Theorem 1.2. Multiplying by \( t \geq 0 \) inequality (4.6) and integrating from 0 to \( \tau \geq v_{\min} \), we have that
\[
2k_2\tau^2 \leq \int_0^\tau -\mu'(t)t \, dt + \frac{|\Omega|}{2\beta^2}.
\]
Here we applied Lemma 4.4. Analogously for (4.7)
\[
2k_2\tau^2 = \int_0^\tau -\phi'(t)t \, dt + \frac{|\Omega|}{2\beta^2}.
\]
Then
\[
\int_0^\tau t (-d\mu(t)) \geq \int_0^\tau t (-d\phi(t)),
\]
for every \( \tau \geq v_{\min} \). Integrating by parts
\[
\mu(\tau) \leq \phi(\tau) \quad \tau \geq v_{\min}.
\]  
(4.17)
Since \( u_{\min} \leq v_{\min} \), inequality (4.17) holds for \( t \geq 0 \) and the claim is proved. \( \square \)

Proof of Theorem 1.3. Let \( 0 < p \leq \frac{n}{n-2} \). Let us multiply (4.6) by \( t\mu(t)^\eta \), where \( \eta = \frac{1}{p} - \frac{n-2}{n} \geq 0 \), and integrate from 0 to \( \tau \geq v_{\min} \)
\[
\int_0^\tau K_n t\mu(t)^\frac{1}{p} \, dt \leq \int_0^\tau -\mu'(t)t\mu(t)^\eta \, dt + \frac{1}{\beta} \int_0^\tau t\mu(t)^\eta \int_{\partial U_{\text{ext}}} \frac{H(\nu)}{u(x)} \, d\mathcal{H}^{n-1} \]
\[
\leq \int_0^\tau -\mu'(t)t\mu(t)^\eta \, dt + \frac{\eta}{\beta} \int_0^\tau t \int_{\partial U_{\text{ext}}} \frac{H(\nu)}{u(x)} \, d\mathcal{H}^{n-1} \]
\[
\leq \int_0^\tau -\mu'(t)t\mu(t)^\eta \, dt + \frac{\eta}{\beta} \int_0^\tau t \, d\mu(t) \leq \frac{|\Omega|\eta+1}{2\beta^2}
\]
where, again, \( K_n = n^2k_n^\frac{2}{n} \), in the third inequality we applied Lemma 4.4 and in the last the fact that \( \mu(t) \) is a monotone non increasing function.

By setting \( G(t) = \int_0^t w^\eta = \frac{t^{\eta+1}}{\eta+1} \) and integrating by parts the first and last members in this chain of inequalities we have
\[
\tau G(\mu(\tau)) + \tau \int_0^\tau K_n t\mu(t)^\frac{1}{p} \, dt \leq \int_0^\tau G(\mu(t)) \, dt + \int_0^\tau \int_0^t K_n \mu(t)^\frac{1}{p} \, dr \, dt + \frac{|\Omega|\eta+1}{2\beta^2}
\]
By applying Lemma 4.1 with
\[
\xi(t) = \int_0^t G(\mu(t)) \, dt + \int_0^\tau \int_0^t K_n \mu(t)^\frac{1}{p} \, dr \, dt,
\]
$C = \frac{\|\Omega\|^{\eta+1}}{2\beta^2}$ and $\tau_0 = v_{\min}$, we have that

$$G(\mu(\tau)) + \int_0^\tau K_n \mu(t)^{\frac{1}{p}} \, dt \leq \frac{1}{v_{\min}} \left( \int_0^{v_{\min}} G(\mu(t)) \, dt \right)$$

$$+ \int_0^{v_{\min}} \int_0^t K_n \mu(r)^{\frac{1}{p}} \, dr \, dt + \frac{\|\Omega\|^{\eta+1}}{2\beta^2}. \tag{4.18}$$

Analogously

$$G(\phi(\tau)) + \int_0^\tau K_n \phi(t)^{\frac{1}{p}} \, dt = \frac{1}{v_{\min}} \left( \int_0^{v_{\min}} G(\phi(t)) \, dt \right)$$

$$+ \int_0^{v_{\min}} \int_0^t K_n \phi(r)^{\frac{1}{p}} \, dr \, dt + \frac{\|\Omega\|^{\eta+1}}{2\beta^2}. \tag{4.19}$$

By (4.18) and (4.19), we compare directly the righthand sides of (4.18) and (4.19). So

$$G(\mu(\tau)) + \int_0^\tau K_n \mu(t)^{\frac{1}{p}} \, dt \leq G(\phi(\tau)) + \int_0^\tau K_n \phi(t)^{\frac{1}{p}} \, dt$$

For $\tau \to +\infty$ we have

$$\int_0^\infty \mu(t)^{\frac{1}{p}} \, dt \leq \int_0^\infty \phi(t)^{\frac{1}{p}} \, dt,$$

which is (1.12). Now we want to prove (1.13). In order to obtain this result let us pass to the limit as $\tau \to \infty$ the following inequality:

$$\int_0^\tau K_n t \mu(t)^{\frac{1}{p}} \, dt \leq \int_0^\tau -t \mu(t)^{\eta} \, d\mu(t) + \frac{\|\Omega\|^{\eta+1}}{2\beta^2}.$$

After an integration by parts we get

$$\int_0^\infty K_n t \mu(t)^{\frac{1}{p}} \, dt \leq \int_0^\infty G(\mu(t)) \, dt + \frac{\|\Omega\|^{\eta+1}}{2\beta^2}.$$

On the other hand

$$\int_0^\infty K_n t \phi(t)^{\frac{1}{p}} \, dt = \int_0^\infty G(\phi(t)) \, dt + \frac{\|\Omega\|^{\eta+1}}{2\beta^2}.$$

So we need just to show that

$$\int_0^\infty G(\mu(t)) \, dt \leq \int_0^\infty G(\phi(t)) \, dt. \tag{4.20}$$
To this aim we multiply (4.6) by \( tG(\mu(t))\mu(t)^{-\frac{n-2}{2}} \). Since \( G(l)l^{-\frac{n-2}{2}} \) is a non-decreasing function in \( l \), we can integrate from 0 to \( \tau \geq \tau_{\min} \), to obtain
\[
\int_0^\tau K_n tG(\mu(t)) dt \leq \int_0^\tau -t\mu(t)^{-\frac{n-2}{2}} G(\mu(t)) d\mu(t) + \frac{1}{\beta} \int_0^\tau tG(\mu(t))\mu(t)^{-\frac{n-2}{2}} \int_{\partial U_{\text{ext}}} \frac{H(\nu)}{u} dH^{n-1} dt
\]
\[
\leq \int_0^\tau -t\mu(t)^{-\frac{n-2}{2}} G(\mu(t)) d\mu(t) + \frac{1}{\beta} G(|\Omega||\Omega|^{-\frac{n-2}{2}}) \int_0^\tau t \int_{\partial U_{\text{ext}}} \frac{H(\nu)}{u} dH^{n-1} dt
\]
\[
\leq \int_0^\tau -t\mu(t)^{-\frac{n-2}{2}} G(\mu(t)) d\mu(t) + G(|\Omega|) \frac{|\Omega|^2}{2\beta^2},
\]
where, again, we applied 4.4. Now, if we call \( C = G(|\Omega|) \frac{|\Omega|^2}{2\beta^2} \) and set \( J(l) = \int_l^1 w^{-\frac{n-2}{2}} G(w) dw \), integrating by parts the first and last member of the previous chain of inequalities, we have
\[
\tau \int_0^\tau K_n G(\mu(t)) dt + \tau J(\mu(\tau)) \leq \int_0^\tau \int_0^\tau K_n G(\mu(z)) dz dr + \int_0^\tau J(\mu(t)) dt + C.
\]
Setting
\[
\xi(t) = \int_0^\tau \int_0^\tau K_n G(\mu(z)) dz dr + \int_0^\tau J(\mu(t)) dt,
\]
and applying 4.1 with \( \tau_0 = \tau_{\min} \) we deduce that
\[
\int_0^\tau K_n G(\mu(t)) dt + J(\mu(\tau)) \leq \frac{1}{\tau_{\min}} \left( \int_0^\tau \int_0^\tau K_n G(\mu(z)) dz dr + \int_0^\tau J(\mu(t)) dt + C \right).
\]
This inequality holds as an equality when we have \( \phi \) in place of \( \mu \), so as before
\[
\int_0^\tau K_n G(\mu(t)) dt + J(\mu(t)) \leq \int_0^\tau K_n G(\phi(t)) dt + J(\phi(t)) .
\]
For \( \tau \rightarrow \infty \) we have (4.20), which concludes the proof. \( \square \)

5. APPLICATION TO PDE’S: BOSSLER-DANERS INEQUALITY

Let \( \Omega \) be a bounded and smooth open set in \( \mathbb{R}^n \). Let us denote by \( \nu \) the outer unit normal to \( \partial \Omega \) and let \( \beta > 0 \) be a positive real number. It is well known that for the following Laplacian eigenvalue problem with Robin boundary conditions
\[
\begin{cases}
-\Delta u = \lambda(\Omega) u & \text{in } \Omega \\
\frac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
a Faber-Krahn type inequality for the first eigenvalue holds. It is famous under the name of Bossel-Daners inequality and it can be read as follows

\[ \lambda_{1,\beta}(\Omega) \geq \lambda_{1,\beta}(\Omega^\sharp), \]

where \( \Omega^\sharp \) is the ball centered at the origin with the same measure as \( \Omega \). Equality holds if and only if \( \Omega \) is a ball.

Let us consider now the anisotropic case. If \( f = \lambda(\Omega)u \), then (1.1) can be written in this way

\[
\begin{aligned}
-\text{div}(H(\nabla u)H_\xi(\nabla u)) &= \lambda(\Omega)u \quad \text{in} \ \Omega \\
H(\nabla u)H_\xi(\nabla u) \cdot \nu + \beta H(\nu)u &= 0 \quad \text{on} \ \partial \Omega.
\end{aligned}
\] (5.1)

The variational characterization for the first eigenvalue is

\[ \lambda_1(\Omega) = \min_{w \in H^1(\Omega) \setminus \{0\}} J[w], \] (5.2)

where

\[ J[w] = \frac{\int_{\Omega} H^2(\nabla u) \, dx + \beta \int_{\partial \Omega} u^2 H(\nu) \, dH^{n-1}}{\int_{\Omega} u^2 \, dx}. \] (5.3)

In [DG] the authors proved a Bossel-Daners type inequality for the anisotropic \( p \)-Laplacian problem. Indeed, they proved that

\[ \lambda_{1,\beta}(\Omega) \geq \lambda_{1,\beta}(\Omega^\star), \] (5.4)

where \( \Omega^\star \) is the set homothetic to the Wulff Shape having the same measure as \( \Omega \). In particular the equality case holds if and only if \( \Omega \) is a set homothetic to the Wulff Shape.

In this section we want to give an alternative proof of (5.4) in the planar case, using the results found in the previous section.

**Corollary 5.1.** Let \( n = 2 \) and \( u \) be the first eigenfunction associated to \( \lambda_{1,\beta}(\Omega) \), solution to problem (5.1). If \( z \) is the solution to the symmetrized problem

\[
\begin{aligned}
-\text{div}(H(\nabla z)H_\xi(\nabla z)) &= \lambda_{1,\beta}(\Omega^\star)u^* \quad \text{in} \ \Omega^\star \\
H(\nabla z)H_\xi(\nabla z) \cdot \nu + \beta H(\nu)z &= 0 \quad \text{on} \ \partial \Omega^\star.
\end{aligned}
\] (5.5)

then we have that

\[ \lambda_{1,\beta}(\Omega) \geq \lambda_{1,\beta}(\Omega^\star). \] (5.6)

Equality occurs if and only if \( \Omega \) is homothetic to the Wulff Shape with same measure as \( \Omega \).
Proof. By theorem 1.2 we know that
\[ \int_{\Omega} u^2 \, dx = \int_{\Omega^\star} (u^\star)^2 \, dx \leq \int_{\Omega^\star} z^2 \, dx. \]
So, by Cauchy-Schwarz inequality we have
\[ \int_{\Omega^\star} u^\star z \, dx \leq \int_{\Omega^\star} z^2 \, dx, \]
and this ends the corollary, indeed
\[ \lambda_{1,\beta}(\Omega) = \frac{\int_{\Omega} H^2(\nabla z) \, dx + \beta \int_{\partial \Omega} z^2 H(\nu) \, d\mathcal{H}^{n-1}}{\int_{\Omega^\star} z^2 \, dx} \geq \frac{\int_{\Omega} H^2(\nabla z) \, dx + \beta \int_{\partial \Omega} z^2 H(\nu) \, d\mathcal{H}^{n-1}}{\int_{\Omega} z^2 \, dx} = \lambda_{1,\beta}(\Omega^\star). \]
\[ \square \]

6. Conclusions and open problems

As in the euclidean case, we have proved that these comparison results depend on the dimension of the space. In particular if we are in the hypothesis of theorem 1.1 when \( n = 2 \), then
\[ \|u\|_{L^1(\Omega)} \leq \|v\|_{L^1(\Omega^\star)}, \]
and
\[ \|u\|_{L^2(\Omega)} \leq \|v\|_{L^2(\Omega^\star)}. \]
Therefore a question arises spontaneously. Is it true that
\[ \|u\|_{L^p(\Omega)} \leq \|v\|_{L^p(\Omega^\star)} \quad (6.1) \]
for all values of \( p \)? In dimension 2 the answer is negative for large values of \( p \). Next example will show that (6.1) is untrue when \( p = \infty \) and \( n = 2 \).

Example 1. Let \( \Omega \) be the union of two disjoint bidimensional Wulff shapes \( \mathcal{W} \) and \( \mathcal{W}_r \), with radii 1 and \( r \) respectively. If we choose \( \beta = \frac{1}{2} \) and \( f \) such that it is constantly 1 in \( \mathcal{W} \) and constantly zero in \( \mathcal{W}_r \), then the solutions to problem (1.1) and (1.3) can be explicitly computed. In particular it is possible to prove that there exists a positive constant \( c \) such that
\[ \|u\|_{L^\infty(\Omega)} - \|v\|_{L^\infty(\Omega^\star)} = cr^2 + o(r^2). \]
Now someone could ask if (6.1) can be true when \( n \geq 3 \). Next counterexample will show its untruthfulness when \( n = 3 \) and \( p = 2 \).
Example 2. If we consider $\Omega$, $\beta$ and $f$ as in the example 1 in the corresponding three-dimensional case, then, in the hypothesis of theorem 1.1, the solutions to problem (1.1) and (1.3) can be explicitly computed. It is possible to prove that there exists a positive constant $d$ such that
\[ \|u\|_{L^2(\Omega)} - \|v\|_{L^2(\Omega^*)} = dr^3 + o(r^3). \]

A problem that is still open is the following

**Open Problem 1.** In the hypothesis of theorem 1.1, (6.1) is true for $p = 1$ and $n \geq 3$?

If we now consider the theorem 1.2, we have proved that when $n = 2$ and $f \equiv 1$ in $\Omega$, then
\[ u^*(x) \leq v(x) \quad x \in \Omega^*. \]
In doing so, another question arises:

**Open Problem 2.** In the hypothesis of theorem 1.2, (6) is true even when $n \geq 3$?

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