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The deformed Hermitian-Yang-Mills equation, the Positivstellensatz, and the solvability. (English)
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Summary: Let \((M, \omega)\) be a compact connected Kähler manifold of complex dimension four and let \(\chi \in H^{1,1}(M; \mathbb{R})\). We confirm the conjecture by Collins-Jacob-Yau \([8]\) of the solvability of the deformed Hermitian-Yang-Mills equation, which is given by the following nonlinear elliptic equation
\[
\sum_i \arctan(\lambda_i) = \hat{\theta},
\]
where \(\lambda_i\) are the eigenvalues of \(\chi\) with respect to \(\omega\) and \(\hat{\theta}\) is a topological constant. This conjecture was stated in \([8]\), wherein they proved that the existence of a supercritical \(C\)-subsolution or the existence of a \(C\)-subsolution when \(\hat{\theta} \in [(n - 2)/2, n\pi/2)\) will give the solvability of the deformed Hermitian-Yang-Mills equation. Collins-Jacob-Yau conjectured that their existence theorem can be improved to \(\hat{\theta} > (n - 2)\pi/2\), where \(n\) is the complex dimension of the manifold. In this paper, we confirm their conjecture that when the complex dimension equals four and \(\hat{\theta}\) is close to the supercritical phase \(\pi\) from the right, then the existence of a \(C\)-subsolution implies the solvability of the deformed Hermitian-Yang-Mills equation.

MSC:
32Q15 Kähler manifolds
32W50 Other partial differential equations of complex analysis in several variables
53C55 Global differential geometry of Hermitian and Kählerian manifolds

Keywords:
differential geometry; geometric analysis; deformed Hermitian-Yang-Mills equation

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