On the Capacity of Waveform Channels Under Square-Law Detection of Time-Limited Signals

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Abstract

Capacity bounds for waveform channels under square-law detection of time-limited complex-valued signals are derived. The upper bound is the capacity of the channel under (complex-valued) coherent detection. The lower bound is one bit less, per dimension, than the upper bound.

I. INTRODUCTION

Square-law detection (SLD) decides based on the squared magnitude of the received complex-valued waveform, contrasting with coherent detection, in which the decision is based upon the received complex-valued waveform. The former appears in many fields, e.g., short-haul fiber-optic communication systems [1], astronomical imaging [2], X-ray crystallography [3], etc.

As the measurement in SLD depends on the magnitude of the received complex-valued signal, it is often thought that half of the degrees of freedom for data transmission are lost, when using this type of detection. Specifically, given a non-negative waveform \( s(t) \), there are many complex-valued waveforms \( y(t) \) such that \( |y(t)|^2 = s(t) \). Under some conditions on \( y(t) \), there are algorithms that retrieve the phase of \( y(t) \) from \( s(t) \). This issue is well studied in the literature on phase retrieval, e.g., see [4]–[9].

Although studying the number of bandlimited \( T \)-periodic complex functions with the same magnitude goes back more than half a century [10], its direct consequence in finding a capacity lower-bound for SLD of bandlimited signals is recent [11]. Specifically, it was shown in [11] that by using SLD, at most 1 bit per degree of freedom is lost, in comparison with complex-valued coherent detection, which suggests that noncoherent detection may remain a viable approach for emerging applications in short-haul fiber-optic communication systems.

In practice, signals are time-limited and it is the purpose of this paper to find the relative capacity of channels under SLD of time-limited signals in comparison with complex-valued coherent detection.

We adopt a similar method as in [11], except that we use a weaker condition for distinguishability of two signals. Two functions \( y_1 \) and \( y_2 \) are said to be equal almost everywhere (a.e.), written \( y_1 \overset{a.e.}\approx y_2 \), if

\[
\int_{\mathbb{R}} |y_1(t) - y_2(t)|^2 \, dt = 0;
\]

when \( y_1 \) and \( y_2 \) are not equal a.e., we write \( y_1 \overset{a.e.}\neq y_2 \). It can be shown that almost-everywhere equality is an equivalence relation. Two functions \( y_1 \) and \( y_2 \) are said to be equal up to a phase offset, written \( y_1 \sim y_2 \), if there is a \( \phi \in [-\pi, \pi) \) such that \( y_1 \overset{a.e.}\approx \exp(i\phi) y_2 \). When \( y_1 \) and \( y_2 \) are not equal up to a phase offset, then we write \( y_1 \overset{\phi}{\sim} y_2 \). Note that \( y_1 \overset{a.e.}\approx y_2 \) implies \( y_1 \sim y_2 \), but not conversely. The relation \( \sim \) is obviously reflexive and symmetric, and transitivity follows from the Cauchy-Schwarz inequality; thus \( \sim \) is an equivalence relation.

The authors of [11] assume that a coherent detector can distinguish \( y_1 \) from \( y_2 \) if and only if \( y_1 \overset{\phi}{\sim} y_2 \). Here, we assume the relaxed condition that \( y_1 \) and \( y_2 \) are distinguishable by a coherent detector if and only if \( y_1 \overset{a.e.}\neq y_2 \).

The rest of the paper is organized as follows. The problem setup is introduced in Sec. [II] In Sec. [III] some complex analysis tools are introduced, to be used in Sec. [IV] in finding capacity bounds of channels under SLD relative to coherent detection. In parallel to the 1-bit capacity gap for the bandlimited signals, which is established by [11], we derive the same gap for time-limited signals in Sec. [IV]. In Sec. [V] the paper concludes with a brief discussion of how these results can be generalized.

Through this paper, \( \mathbb{N}, \mathbb{R}, \mathbb{R}^+ \) and \( \mathbb{C} \) denote the set of non-negative integers, real, non-negative real, and complex numbers, respectively. The reciprocal conjugate of \( \alpha \in \mathbb{C} \) is denoted by \( \alpha^{-*} \); hence \( \alpha^{-*} = (\alpha^*)^{-1} \). The polynomial ring over \( \mathbb{C} \) is denoted by \( \mathbb{C}[z] \), and for an integer \( n, \mathbb{C}^{\leq n}[z] \) denotes the set of polynomials in \( \mathbb{C}[z] \) of degree at most \( n \). The unit circle, i.e., \( \{ z \in \mathbb{C} : |z| = 1 \} \), is denoted by \( \mathbb{T} \). \( \mathbb{D} \) denotes the open unit disk, i.e., \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \). \( \overline{\mathbb{D}} \) denotes the closure of \( \mathbb{D} \), i.e., \( \overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{T} \), and \( \mathcal{A}(\overline{\mathbb{D}}) \) denotes the set of analytic functions on \( \overline{\mathbb{D}} \) that extend continuously to \( \overline{\mathbb{D}} \). Finally, the rectangular function is defined as

\[
\text{rect}(t) = \begin{cases} 
1, & 0 \leq t < 1; \\
0, & \text{otherwise.}
\end{cases}
\]

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A complex-valued signal, $x(t)$, whose support is a subset of $[0,1)$ is transmitted over a channel, and a complex-valued signal, $y(t)$, whose support is a subset of $[0,1)$, is received. Note that the supports of $x$ and $y$ might be different; for example, channel dispersion might broaden the support of $y$ in comparison with $x$, or the channel might compress the support. The choice of support interval does not affect the generality of the results of the paper, as is explained in Sec. V.

We assume that $x$ and $y \in L^4[0,1)$, i.e., $\int_0^1 |x(t)|^4 \, dt < \infty$, and similarly for $y$. The reason for this choice of function space will be clarified later in this section.

Two receivers are compared. The coherent receiver decides on the transmitted waveform by observing $y$, while the SLD receiver decides on the transmitted waveform by observing $s(t) \triangleq |y(t)|^2$. Since $y \in L^4[0,1)$, the waveform $s$ belongs to $L^2[0,1)$, i.e., $\int_0^1 |s(t)|^2 \, dt < \infty$. The relationships among $x$, $y$, and $s$ are shown in Fig 1.

As $y(t)$ is time-limited to $[0,1)$, we may assume that $y(t) = y_p(t)\text{rect}(t)$, where

$$y_p(t) = \sum_{k=-\infty}^{\infty} y(t-k),$$

is the periodic extension of $y(t)$ with period 1.

According to Carleson’s theorem [12], if a signal is in $L^2[0,1)$ then its periodic extension is equal a.e. to its Fourier series. Note that $L^4[0,1) \subset L^2[0,1)$ [13]; as a result,

$$y_p(t) \overset{ae}{=} \sum_{k=-\infty}^{\infty} b_k e^{i\pi k t},$$

where

$$b_k = \int_0^1 y(t) e^{-i\pi k t} \, dt.$$

We can write $y_p(t)$ as $y_p(t) \overset{ae}{=} \lim_{m \to \infty} y_{p,m}(t)$, in which

$$y_{p,m}(t) \triangleq \sum_{k=-m}^{m} b_k e^{i\pi k t},$$

is a truncated Fourier series. Writing $y(t;m) \triangleq y_{p,m}(t)\text{rect}(t)$, we then have $y(t) \overset{ae}{=} \lim_{m \to \infty} y(t;m)$. Note that there is a one-to-one correspondence between $y(t;m)$ and $y_{m+1} \overset{ae}{=} (b_{-m}, \ldots, b_m) \in \mathbb{C}^{2m+1}$. Similarly, let

$$x(t;m) \triangleq \left( \sum_{k=-m}^{m} a_k e^{i2\pi k t} \right)\text{rect}(t), \quad a_k \in \mathbb{C},$$

so that $x(t) \overset{ae}{=} \lim_{m \to \infty} x(t;m)$. Then, we can determine $x(t;m)$ uniquely from $x_{m+1} = (a_{-m}, \ldots, a_m) \in \mathbb{C}^{2m+1}$.

Square-law detection of $y(t;m)$ produces $s(t;m) \overset{ae}{=} |y(t;m)|^2$, which can be written as

$$s(t;m) = \left| \sum_{k=-m}^{m} b_k e^{i2\pi k t} \right|^2 \text{rect}(t)
= \left( \sum_{k=-m}^{m} \sum_{\ell=\max(k-m,-m)}^{\min(k+m,m)} b_{\ell} b_{\ell-k} e^{i2\pi k t} \right)\text{rect}(t)
= \left( \sum_{k=-m}^{m} c_k e^{i2\pi k t} \right)\text{rect}(t),$$

where

$$c_k = \sum_{\ell=\max(k-m,-m)}^{\min(k+m,m)} b_{\ell} b_{\ell-k}.$$
and we have used from this property that $|\text{rect}(t)|^2 = \text{rect}(t)$. Since $s(t; m)$ is a real-valued signal, we have $c_k = c_{-k}^*$. Similar to $x^{2m+1}$ and $y^{2m+1}$, there is a one-to-one correspondence between $s(t; m)$ and $s^{2m+1} = (c_0, \ldots, c_{2m}) \in \mathbb{C}^{2m+1}$.

As $y \in L^4[0, 1)$, it implies that $s \in L^2[0, 1)$, which implies that $s(t) \equiv \lim_{m \to \infty} s(t; m)$. This is the reason that $y$ is considered to be in $L^4[0, 1)$, as in that case, $s$ belongs to $L^2[0, 1)$ and Carleson’s theorem guarantees equality a.e. to $s(t; m)$, in the limit as $m \to \infty$.

In summary, the system shown in Fig. 1 behaves like the system shown in Fig. 2, in the limit as $m \to \infty$.

The average mutual information between the time-limited functions $x(t; m)$ and $y(t; m)$ is defined as

$$I_m(x(t; m); y(t; m)) \triangleq \frac{I(x^{2m+1}; y^{2m+1})}{2m + 1},$$

and similarly, for $x(t; m)$ and $s(t; m)$ as

$$I_m(x(t; m); s(t; m)) \triangleq \frac{I(x^{2m+1}; s^{2m+1})}{2m + 1},$$

where $I(\cdot; \cdot)$ denotes the mutual information function. If the average mutual information per degree of freedom between $x(t)$ and $y(t)$ exists, then it is given by [14, ch. 8]

$$I(x(t); y(t)) = \lim_{m \to \infty} I_m(x(t; m); y(t; m)).$$

Similarly, if the average mutual information between $x(t)$ and $s(t)$ exists, then it can be written as

$$I(x(t); s(t)) = \lim_{m \to \infty} I_m(x(t; m); s(t; m)).$$

Note that $I(x(t); y(t))$ and $I(x(t); s(t))$ are normalized to the number of used dimensions; as a result, they are similar to the spectral efficiency under coherent detection and SLD, respectively.

In this paper, we establish bounds for $I(x(t); s(t))$, in terms of $I(x(t); y(t))$. To this aim, we establish bounds for $I_m(x(t; m); s(t; m))$, in terms of $I_m(x(t; m); y(t; m))$ and we then let $m \to \infty$.

### III. ON BLASCHKE PRODUCTS

To find a capacity lower-bound for the system shown in Fig. 2 we require some tools from complex analysis. For $\alpha \in \mathbb{D}$, the **Blaschke factor**, $B_\alpha : \mathbb{D} \to \mathbb{D}$, is defined as

$$B_\alpha(z) \triangleq \frac{\alpha - z}{1 - \alpha^* z}.$$}

Given a sequence $\alpha_1, \alpha_2, \ldots \in \mathbb{D}$, such that

$$\sum_k (1 - |\alpha_k|) < \infty,$$

and $\tau \in \mathbb{T}$, the **Blaschke product**, $B(z)$, is defined as

$$B(z) = \tau \prod_k B_{\alpha_k}(z).$$

Furthermore, if $k$ is bounded above, then $B(z)$ is called a **finite Blaschke product**. In general, a finite Blaschke product takes the form

$$B(z) = \tau z^{n_0} \prod_{k=1}^p B_{\gamma_k}^{n_k}(z),$$

for some finite $p \in \mathbb{N}$, some $\tau \in \mathbb{T}$, some distinct $\gamma_1, \ldots, \gamma_p \in \mathbb{D}\{0\}$, and some $n_0, \ldots, n_p \in \mathbb{N}$. The $z^{n_0}$ factor in (2) corresponds to the Blaschke factor $B_0(z) = -z$.

For any $\alpha \in \mathbb{D}$ and any $z \in \mathbb{T}$ we have $|B_\alpha(z)|^2 = 1$; as a result $B_\alpha(z) \in \mathbb{T}$. Consequently, any Blaschke product maps the unit circle to itself.

For a polynomial $f \in \mathbb{C}[z]$ let $Z_f = \{\alpha \in \mathbb{C} : f(\alpha) = 0\}$ be the zero set of $f$. Let

$$Z'_f = \{\alpha \in \mathbb{D}\{0\} : f(\alpha) f(\alpha^*) = 0\},$$
be the set of points, \( \alpha \in \mathbb{D} \setminus \{0\} \), such that either \( f(\alpha) = 0 \) or \( f(\alpha^{-1}) = 0 \). Finally, let \( Z_f'' = Z_f \cap \mathbb{T} \) be the set of zeros of \( f \) that are on the unit circle. We extend the usual notion of root-multiplicity to the entire complex plane as follows. For an arbitrary \( \alpha \in \mathbb{C} \), let \( d_f(\alpha) \) be the multiplicity of \( \alpha \) as a root of \( f \); if \( \alpha \notin Z_f \), then let \( d_f(\alpha) = 0 \).

The following theorem plays an important role in proving the subsequent theorems.

**Theorem 1. (Fatou)** If \( f(z) \in \mathcal{A}(\mathbb{D}) \) and \( f(\mathbb{T}) \subseteq \mathbb{T} \), then \( f \) is a finite Blaschke product.

**Proof.** See [15, Theorem 3.5.2] and [16]. \( \square \)

The next theorem gives a necessary and sufficient condition for two nonzero complex polynomials to have a constant magnitude-ratio on the unit circle.

**Theorem 2.** Let \( f \) and \( g \in \mathbb{C}[z] \) be two nonzero polynomials. Then \( |f(z)| = \kappa |g(z)| \) for all \( z \in \mathbb{T} \) and for some \( \kappa \in \mathbb{R}^+ \) if and only if for all \( z \in \mathbb{C} \setminus \{0\} \),

\[
d_f(z) + d_f(z^{-1}) = d_g(z) + d_g(z^{-1}).
\]

**Proof.** Suppose that \( |f(z)| = \kappa |g(z)| \) for a \( \kappa \in \mathbb{R}^+ \) and for any \( z \in \mathbb{T} \). Let

\[
B(z) = \prod_{\alpha \in Z_f \cap \mathbb{D}} B^f_{\alpha}(z)
\]

be the Blaschke product produced by the zeros of \( f \) that are inside the unit disk. Furthermore, let

\[
H(z) \triangleq \begin{cases} \frac{\kappa g(z)B(z)}{f(z)}, & z \in \mathbb{D}; \\ \lim_{w \to z} \frac{\kappa g(w)B(w)}{f(w)}, & z \in \mathbb{T}. \end{cases}
\]

As \( B(z) \) is a Blaschke product, it maps the unit circle to itself. Furthermore, \( |f(z)| = \kappa |g(z)| \) for all \( z \in \mathbb{T} \); consequently, \( H(\mathbb{T}) \subseteq \mathbb{T} \). In addition to that, the zeros of \( f \) that are in \( \mathbb{D} \) are cancelled by \( B(z) \); as a result, \( H(z) \in \mathcal{A}(\mathbb{D}) \). By Theorem 1 it follows that \( H(z) \) can be written as a finite Blaschke product, i.e.,

\[
H(z) = \tau z^n \prod_{k=1}^{p} B^n_{\gamma_k}(z),
\]

for some finite \( p \in \mathbb{N} \), some \( \tau \in \mathbb{T} \), some distinct \( \gamma_1, \ldots, \gamma_p \in \mathbb{D} \setminus \{0\} \), and some \( n_0, \ldots, n_p \in \mathbb{N} \). As \( \kappa g(z)B(z) = f(z)H(z) \), by substituting \( B(z) \) from (4) and \( H(z) \) from (5) and then multiplying by the denominator polynomials of \( B(z) \) and \( H(z) \), we have

\[
\kappa g(z) \prod_{\alpha \in Z_f \cap \mathbb{D}} (\alpha - z)^{d_f(\alpha)} \prod_{k=1}^{p} (1 - \gamma_k^* z)^{n_k} = \tau f(z)z^{n_0} \prod_{\alpha \in Z_f \cap \mathbb{D}} (1 - \alpha^* z)^{d_f(\alpha)} \prod_{k=1}^{p} (\gamma_k - z)^{n_k}.
\]

Let

\[
g'(z) \triangleq \prod_{\alpha \in Z_f \cap \mathbb{D}} (\alpha - z)^{d_f(\alpha)}, \quad g''(z) \triangleq \prod_{k=1}^{p} (1 - \gamma_k z)^{n_k},
\]

\[
f'(z) \triangleq \prod_{\alpha \in Z_f \cap \mathbb{D}} (1 - \alpha z)^{d_f(\alpha)}, \quad f''(z) \triangleq \prod_{k=1}^{p} (\gamma_k - z)^{n_k};
\]

then, we can write (6) as

\[
\kappa g(z)g'(z)g''(z) = \tau z^n f(z)f'(z)f''(z).
\]

The polynomials on both sides of (7) must have the same roots with the same multiplicities. As a result, for all \( z \in \mathbb{C} \setminus \{0\} \) we have

\[
d_g(z) + d_{g'}(z) + d_{g''}(z) = d_f(z) + d_f'(z) + d_f''(z),
\]

and consequently,

\[
d_g(z) + d_{g'}(z) + d_{g''}(z) = d_f(z) + d_f'(z) + d_f''(z).
\]

Note that \( d_{g'}(z) = d_f'(z^{-1}) \) and \( d_{g''}(z) = d_f''(z^{-1}) \), for all \( z \in \mathbb{C} \setminus \{0\} \). As a result, (8) simplifies to (3).

Conversely, assume that (3) holds for all \( z \in \mathbb{C} \setminus \{0\} \). If \( \alpha \in Z_f \setminus \{0\} \), then \( d_f(\alpha) + d_f(\alpha^{-1}) > 0 \), which by (3) implies that \( d_g(\alpha) + d_g(\alpha^{-1}) > 0 \). As a result, either \( \alpha \) or \( \alpha^{-1} \) belongs to \( Z_g \), which implies that \( Z_f' = Z_g' \). Furthermore, if \( \alpha \in Z_f'' \), then \( \alpha^{-1} = \alpha \), which implies that \( \alpha \) is a zero of \( g \) with the same multiplicity as of \( f \). As a result, \( Z_f'' = Z_g'' \).
For some \(a_f\) and \(a_g\) \(\in \mathbb{C}\setminus \{0\}\) and some \(n_f\) and \(n_g\) \(\in \mathbb{N}\) we have
\[
f(z) = a_f z^{n_f} \prod_{\alpha \in Z_f'} (z - \alpha)^{d_f(\alpha)} (z - \alpha^{-*})^{d_f(\alpha^{-*})} \prod_{\alpha \in Z_f''} (z - \alpha)^{d_f(\alpha)},
\]
and
\[
g(z) = a_g z^{n_g} \prod_{\alpha \in Z_g'} (z - \alpha)^{d_g(\alpha)} (z - \alpha^{-*})^{d_g(\alpha^{-*})} \prod_{\alpha \in Z_g''} (z - \alpha)^{d_g(\alpha)}.
\]
Let \(K(z) \triangleq \frac{f(z)}{g(z)}\), then
\[
K(z) = \frac{a_f}{a_g} z^{n_f - n_g} \prod_{\alpha \in Z_f'} \frac{(z - \alpha)^{d_f(\alpha) - d_g(\alpha)}}{(z - \alpha^{-*})^{d_g(\alpha^{-*})}} \prod_{\alpha \in Z_f''} (z - \alpha)^{d_f(\alpha)},
\]
which, as \(d_f(\alpha) = d_g(\alpha)\) for \(\alpha \in Z_f''\), can be simplified as
\[
K(z) = \frac{a_f}{a_g} z^{n_f - n_g} \prod_{\alpha \in Z_f'} B^{d_f(\alpha) - d_g(\alpha)}_\alpha (z)(\alpha^*)^{d_f(\alpha) - d_g(\alpha)}.
\]
As a result, for all \(z \in \mathbb{T}\),
\[
|K(z)| = \left| \frac{a_f}{a_g} \prod_{\alpha \in Z_f'} |\alpha|^{d_f(\alpha) - d_g(\alpha)} \right|,
\]
which is a constant number, independent of \(z\). Due to the definition of \(K(z)\), we then have \(|f(z)| = \kappa |g(z)|\), where \(\kappa = |K(z)|\) is given in (10).

Corollary 1. Let \(f\) and \(g\) \(\in \mathbb{C}[z]\) be nonzero polynomials such that \(|f(z)| = \kappa |g(z)|\) for some \(\kappa \in \mathbb{R}^+\) and for all \(z \in \mathbb{T}\). Then \(\deg(f) = \deg(g)\) if and only if \(f(0) = g(0)\).

Proof. If \(\deg(f) \neq \deg(g)\) then, according to Theorem 2, the difference between the degrees can only be due to the \(z\) factor. The converse proof is similar. \(\square\)

In Sec. 1 we introduced the equivalence relations \(\cong \) and \(\overset{\phi}{\cong}\) for functions taking real arguments. In parallel to that, we define similar relations for functions that have complex arguments. Two functions \(f\) and \(g\) \(\in \mathbb{C}^\mathbb{C}\) are said to be equal almost everywhere, written \(f \cong g\), if and only if
\[
\int_{\mathbb{C}} |f(z) - g(z)|^2 \, dz = 0.
\]
Similarly, two functions \(f\) and \(g\) \(\in \mathbb{C}^\mathbb{C}\) are said to be equal up to a phase offset, written \(f \overset{\phi}{\cong} g\), if and only if there is a \(\phi \in (-\pi, \pi)\) such that \(f \cong e^{i\phi} g\). If \(f\) and \(g\) are not equal up to a phase offset, we write \(f \ncong g\). The relations \(\cong\) and \(\overset{\phi}{\cong}\) for functions in \(\mathbb{C}^\mathbb{C}\) are equivalence relations.

If two polynomials \(f\) and \(g\) \(\in \mathbb{C}[z]\) are equal a.e., then they are identical, i.e., \(f \cong g\) implies \(f = g\).

The next theorem plays a key role in computing a lower bound for the capacity of the channel that outputs \(s(t; m)\) (see Fig. 4).

Theorem 3. For every \(n \in \mathbb{N}\), given \(f \in \mathbb{C}^\mathbb{C}[z]\) and \(\kappa \in \mathbb{R}\), let \(S\) be any set of complex polynomials of degree at most \(n\) for which \(h \overset{\phi}{\cong} g\) for all \(h \in S\) and \(g \in S\), and \(|f(z)| = \kappa |g(z)|\) for all \(z \in \mathbb{T}\). Then \(|S| \leq 2^{n+1}\).

Proof. For a \(g \in S\), let \(K(z) = \frac{f(z)}{g(z)}\), then \(K(z)\) can be written as in (9). Note that by fixing \(n_g\) and \(d_g(\alpha)\) for all \(\alpha \in Z_f'\), \(|a_g|\) is determined uniquely by \(\kappa\) from (10). As \(0 \leq n_g \leq n - \deg(f) + d_f(0)\) and \(0 \leq d_g(\alpha) \leq d_f(\alpha) + d_f(\alpha^{-*})\) for all \(\alpha \in Z_f'\), then
\[
|S| \leq (n + 1 - \deg(f) + d_f(0)) \prod_{\alpha \in Z_f'} (d_f(\alpha) + d_f(\alpha^{-*}) + 1)
\]
\[
\leq (n + 1 - \deg(f) + d_f(0)) \prod_{\alpha \in Z_f'} (d_f(\alpha) + d_f(\alpha^{-*}) + 1) \prod_{\alpha \in Z_f''} (d_f(\alpha) + 1).
\]
By using the arithmetic-geometric-mean inequality we have
\[
|S| \leq \left( \frac{n + |Z_f'| + |Z_f''| + 1}{|Z_f'| + |Z_f''| + 1} \right)^{|Z_f'| + |Z_f''| + 1} \leq \left( \frac{n + |Z_f'| + 1}{|Z_f'| + 1} \right)^{|Z_f'| + 1},
\]
Let $f(t) \in V_m$ be a non-zero function, and let $S$ be any subset of $V_m$ such that $h \approx g$ for all $h$ and $g \in S$, and $|g(t)| = |f(t)|$. Then $|S| \leq 2^{2m+1}$.

**Proof.** The proof is similar to the proofs given in [10], [11]. Specifically, let $S' = \{ P_g(z) : g(t) \in S \}$; clearly $|S'| = |S|$. Note that for all $g$ and $h \in S$, $g \approx h$ implies $P_g \approx P_h$. As a result, by Theorem 3, $|S'| \leq 2^{2m+1}$.

Let $Q_m : [-\pi, \pi) \to \{ 0, \pm \frac{2\pi}{m}, \pm \frac{4\pi}{m}, \ldots, \pm \frac{2\pi}{m} \}$ be a phase-quantizer, which maps $\theta \in [-\pi, \pi)$ to the nearest point in its range, breaking ties by rotating counterclockwise. Specifically,

$$Q_m(\theta) = \left\lfloor \frac{\theta + \frac{\pi}{m}}{\frac{2\pi}{m}} \right\rfloor \frac{2\pi}{m},$$

in which $\lfloor \cdot \rfloor$ denotes the floor function. Furthermore, for any $z \in \mathbb{C}$, let $\Theta_m : \mathbb{C} \to \left[ \frac{-\pi}{m}, \frac{\pi}{m} \right]$ be defined as

$$\Theta_m(z) \triangleq Q_m(\arg(z)) - \arg(z).$$

In another words, $\Theta_m(z)$ denotes the rotation angle which maps $z$ to the point $|z| \exp(iQ_m(\arg(z)))$.

In Theorem 4, the elements of $S$ are not equal up to a phase offset. In order to weaken this condition to have waveforms that are equal up to a phase offset but not everywhere, an auxiliary channel is introduced whose input is $y(t; m)$ and whose output is $z(t; m) = \exp(i\Theta_m(b_0)) y(t; m)$. As a result,

$$z(t; m) = \sum_{k=-m}^{m} d_k e^{i2\pi kt},$$

in which $d_k = \exp(i\Theta_m(b_0)) b_k$. Fig. 3 shows the system, including the auxiliary channel. Let $z(t) \triangleq \lim_{m \to \infty} z(t; m)$.
Then the system from $x(t)$ to $z(t)$ behaves like the coherent channel, i.e., the system that observes $y(t)$. To see this, for $z^{2m+1} \triangleq (d_{-m} \ldots d_{m}) \in \mathbb{C}^{2m+1}$, let

$$I(x(t); z(t;m)) = \frac{I(x^{2m+1}; z^{2m+1})}{2m+1},$$

and

$$I(x(t); z(t)) = \lim_{m \to \infty} I(x(t); z(t;m)).$$

Due to the chain rule for mutual information, we have

$$I(x^{2m+1}; y^{2m+1}, z^{2m+1}) = I(x^{2m+1}; y^{2m+1}) + I(x^{2m+1}; z^{2m+1} | y^{2m+1})$$

$$= I(x^{2m+1}; z^{2m+1}) + I(x^{2m+1}; y^{2m+1} | z^{2m+1}).$$

As $x^{2m+1} - y^{2m+1} - z^{2m+1}$ is a Markov chain, we have $I(x^{2m+1}; z^{2m+1} | y^{2m+1}) = 0$, and as a result,

$$I(x^{2m+1}; y^{2m+1}) = I(x^{2m+1}; z^{2m+1}) + I(x^{2m+1}; y^{2m+1} | z^{2m+1})$$

$$= I(x^{2m+1}; z^{2m+1}) + I(x^{2m+1}; \Theta_m(b_0) | z^{2m+1}).$$

By taking the limit as $m \to \infty$ we have

$$I(x(t); y(t)) = I(x(t); z(t)) + \lim_{m \to \infty} \frac{I(x^{2m+1}; \Theta_m(b_0) | z^{2m+1})}{2m+1}. $$

Note that as $m \to \infty$, the interval which $\Theta_m(b_0)$ takes values in, i.e., $[\frac{x}{m}, \frac{z}{m}]$, shrinks to zero, which means that $\Theta_m(b_0)$ will take a deterministic value as $m \to \infty$. It implies that

$$\lim_{m \to \infty} \frac{I(x^{2m+1}; \Theta_m(b_0) | z^{2m+1})}{2m+1} = 0.$$ 

As the channel from $x(t;m)$ to $z(t;m)$ behaves like coherent channel when $m \to \infty$, instead of finding bounds for $I(x^{2m+1}; y^{2m+1}, z^{2m+1})$ in terms of $I(x^{2m+1}; y^{2m+1})$, we find bounds in terms of $I(x^{2m+1}; z^{2m+1})$.

By using the chain rule for the mutual information we have

$$I(x^{2m+1}; z^{2m+1}, s^{2m+1}) = I(x^{2m+1}; z^{2m+1}) + I(x^{2m+1}; s^{2m+1} | z^{2m+1})$$

$$= I(x^{2m+1}; s^{2m+1}) + I(x^{2m+1}; z^{2m+1} | s^{2m+1}).$$

Note that $|z(t;m)| = |y(t;m)|$ and, as a result, $s(t;m) = |z(t;m)|^2$. Consequently, $x^{2m+1} - z^{2m+1} - s^{2m+1}$ form a Markov chain. This implies that

$$I(x^{2m+1}; s^{2m+1} | z^{2m+1}) = 0,$$

and as a result,

$$I(x^{2m+1}; s^{2m+1}) = I(x^{2m+1}; z^{2m+1}) - I(x^{2m+1}; z^{2m+1} | s^{2m+1}).$$

(11)

According to Theorem 4, for a particular $y_{p,m}(t)$ and up to a constant phase ambiguity, there are at most $2^{2m+1}$ functions in $V_m$ that have the same magnitude as $y_{p,m}(t)$. We have $y(t;m) = y_{p,m}(t)\text{rect}(t)$, so up to a multiplication by some $r \in \mathbb{T}$, there are at most $2^{2m+1}$ waveforms of the form

$$\left( \sum_{k=-m}^{m} g_k e^{i2\pi k t} \right) \text{rect}(t), \quad g_k \in \mathbb{C},$$

which have the same magnitude as $y(t;m)$, hence as $z(t;m)$. As a result, for the system shown in Fig. 3 for a given $s(t;m)$, there are at most $m2^{2m+1}$ possibilities for $z(t;m)$, where the $m$ factor multiplying $2^{2m+1}$ is due to the $m$ possibilities for $\arg(d_0)$. Consequently,

$$I(x^{2m+1}; z^{2m+1} | s^{2m+1}) \leq H(z^{2m+1} | s^{2m+1}) \leq 2m + 1 + \log(m),$$

in which $H$ denotes the entropy function. As a result, from (11) and by using the data-processing inequality, we have

$$I(x^{2m+1}; z^{2m+1}) - (2m + 1 + \log(m)) \leq I(x^{2m+1}; s^{2m+1}) \leq I(x^{2m+1}; y^{2m+1}),$$

thus,

$$I(x(t;m); z(t;m)) - 1 - \frac{\log(m)}{2m+1} \leq I(x(t;m); s(t;m)) \leq I(x(t;m); z(t;m)).$$

(12)

By taking the limit as $m \to \infty$, (12) reduces to

$$I(x(t); z(t)) - 1 \leq I(x(t); s(t)) \leq I(x(t); z(t)), $$
and as a result
\[ I(x(t); y(t)) - 1 \leq I(x(t); s(t)) \leq I(x(t); y(t)). \tag{13} \]

Let \( p(x(t)) \) denote the probability density function of \( x(t) \), and define
\[ p_1 \triangleq \arg \max_{p(x(t))} I(x(t); y(t)), \]
and
\[ p_2 \triangleq \arg \max_{p(x(t))} I(x(t); s(t)). \]

Correspondingly, let \( I_1(\cdot; \cdot) \) and \( I_2(\cdot; \cdot) \) denote the mutual information, computed by \( p_1 \) and \( p_2 \), respectively. Then, by \( \tag{13} \) and the definitions of \( p_1 \) and \( p_2 \), we have
\[ I_1(x(t); y(t)) - 1 \leq I_1(x(t); s(t)) \leq I_2(x(t); s(t)) \leq I_2(x(t); y(t)) \leq I_1(x(t); y(t)). \tag{14} \]

The channel capacity under coherent detection is \( C_{\text{coh}} \triangleq I_1(x(t); y(t)) \), and under SLD it is \( C_{\text{std}} \triangleq I_2(x(t); s(t)) \), so we have
\[ C_{\text{coh}} - 1 \leq C_{\text{std}} \leq C_{\text{coh}}. \tag{15} \]

V. Discussion

In Sec. \[ \text{II} \] we made the assumption that the support of \( x(t) \) and \( y(t) \) is limited to \([0, 1)\). Restricting the time interval to \([0, 1)\) does not affect \( \tag{15} \), as in the general case, we may assume that their support is \([t_1, t_2)\), for \( t_1 < t_2 \). Then we can write \( y(t) \) as
\[ y(t) = y_\text{p}(t) \text{rect} \left( \frac{t - t_1}{t_2 - t_1} \right), \]
in which
\[ y_\text{p} = \sum_{k=-\infty}^{\infty} y(t - k(t_2 - t_1)) \]
is the periodic extension of \( y(t) \) with period \( t_2 - t_1 \). Note that the Fourier series of \( y_\text{p} \) is expressed in terms of \( \exp \left( i \frac{2\pi k}{t_2 - t_1} t \right) \), instead of \( \exp (i 2\pi k t) \). Then the computations are similar to the ones done for the support \([0, 1)\).

Although \( \tag{15} \) is derived for square-law detection, the capacity bounds are true for any invertible function of \( |y(t)| \), as well. An example in which we may measure some other functions of \( x(t) \) than \( s(t) \) is the direct detection of optical waveform, using a photo-diode. Generally, diodes have a non-linear input-output relationship, in which, in certain operating regimes, it might be approximated by some simple functions, e.g., quadratic function. While this approximation works in those specific regimes, it might fail in some other. However, as long as the measurement is an invertible function of the magnitude waveform, the discussed concepts are still true.

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