Exciting Mutual Inclination in Planetary Systems with a Distant Stellar Companion: The Case of Kepler-108

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Abstract

We study the excitation of mutual inclination between planetary orbits by a novel secular-orbital resonance in multi-planet systems perturbed by binary companions, which we call “ivection.” The ivection resonance happens when the nodal precession rate of the planet matches a multiple of the orbital frequency of the binary, and its physical nature is similar to the previously studied evection resonance. Capture into an ivection resonance requires encountering the resonance with slowly increasing nodal precession rate, and it can excite the mutual inclination of the planets without affecting their eccentricities. We discuss the possible outcomes of ivection resonance capture, and we use simulations to illustrate that it is a promising mechanism for producing the mutual inclination in systems where planets have significant mutual inclination but modest eccentricity, such as Kepler-108. We also find an apparent deficit of multi-planet systems that would have a nodal precession period comparable to the binary orbital period, suggesting that ivection resonance may inhibit formation of or destabilize multi-planet systems with an external binary companion.

Unified Astronomy Thesaurus concepts: celestial mechanics (211)

1. Introduction

Thousands of exoplanets have been discovered so far, and nearly half of exoplanetary systems host multiple observed planets (Burke et al. 2014). However, among all of these systems, only three are observed to have significant (above a few degrees) measured mutual inclinations to date.3 Kepler-419 b and c have a marginally detected mutual inclination of 9°±8°, which is modest given the high eccentricities of the planets (Dawson et al. 2012). The other two systems, Kepler-108 and Upsilon Andromeda, both have significant mutual inclination and modest eccentricity. Kepler-108 b and c have a mutual inclination of 24°±11°, with eccentricities 0.135±0.11 and 0.13 ± 0.02, respectively (Mills & Fabrycky 2017). Upsilon Andromeda c and d have a mutual inclination of 30° ± 1° and eccentricities 0.245 ± 0.006 and 0.316 ± 0.006, respectively (McArthur et al. 2010).

The large mutual inclination and small eccentricities of Kepler-108 are difficult to explain, if the planets were formed in a coplanar configuration. Exciting the mutual inclination via scattering with another planet is possible, but producing such low eccentricity requires some fine-tuning since planet–planet scattering tends to produce eccentricities that are comparable to or larger than the mutual inclination (in radians; Chatterjee et al. 2008). The origin of the mutual inclination may also be due to a binary companion: Kepler-108 has a binary companion with sky-projected separation of 327 au. (The eccentricity and semimajor axis of the binary remain unknown.) However, given the large separation, the gravitational perturbation of the binary companion would be too weak to affect the evolution of the planets on a dynamical timescale. In addition, since the system hosts two relatively massive planets, the precession of the planets due to perturbation from each other completely suppresses secular inclination excitation via Lidov–Kozai oscillation (Mills & Fabrycky 2017).

Although known mechanisms are having difficulty producing the mutual inclination of Kepler-108, the similarity of the planets’ nodal precession rate and the binary’s orbital frequency suggests that the inclination may be related to a secular-orbital resonance between the planets and the binary (Mills & Fabrycky 2017). In this paper, we aim to provide a plausible explanation for the mutual inclination of Kepler-108 using a novel resonance between the nodal precession of the planets and the orbital motion of the binary. This new resonance is similar to evection resonance, a resonance between the apsidal precession of the binary and the orbital motion of the binary that can excite the eccentricities of planets in a multi-planet system (Touma & Sridhar 2015). This new resonance we identify is named “ivection” resonance, to signify that it is highly similar to evection resonance but excites the inclination instead of eccentricity of the planet.

Our discussion is organized as follows: In Section 2, we introduce the mechanism of ivection resonance and derive the Hamiltonians of different types of ivection resonance, namely first- and second-order ivection resonance (for a near-circular

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3 There are also a few circumbinary planets that are slightly inclined with respect to the binary; here we only consider planets around a single star.
4 There is one other confirmed planet (Upsilon Andromeda b) and one unconfirmed planet (Upsilon Andromeda c) in the system, but their masses are small compared to Upsilon Andromeda c and d and should not affect the evolution of Upsilon Andromeda c and d.
5 Eviction resonance is originally studied in the context of lunar evolution perturbed by the Sun (Touma & Wisdom 1998), and can also be applied to exomoons (Spalding et al. 2016) and circumbinary planets (Xu & Lai 2016).
6 Touma & Wisdom (1998) discussed a similarly named “eviction” resonance, which is an inclination-eccentricity resonance with resonant term $\pi e I$. This resonance can also excite inclination, but only when the perturbed body has finite initial eccentricity. It is different from the ivection resonance we discuss here.
binary) and eccentric ivection resonance (for a very eccentric binary). Then, in Section 3, we study ivection resonance capture during planet migration, discussing the requirements (especially on the rate of migration) and possible outcomes of resonance capture. Section 4 applies results from the previous sections to study the formation of Kepler-108, and we reproduce the observed eccentricities and mutual inclination with numerical simulation. In Section 5 we discuss the importance of ivection resonance in other exoplanetary systems. We conclude with discussions of a few topics related to this study in Section 6 and a summary of the main results in Section 7.

2. Ivection Resonances

Consider two planets with mass $m_1, m_2$ (subscript 1 denotes the inner planet) with initially coplanar orbit around their host (with mass $M_*$). The planets are perturbed by a distant binary companion with respect to the central star, and $\langle \rangle$ denotes averaging over the orbits of the two planets. In what follows, we assume that the two planets are not close to any mean-motion resonance (MMR), and that the planetary orbits remain circular. The latter is a good approximation if the initial eccentricity is small, since the secular coupling between eccentricity and inclination is at least fourth order (Murray & Dermott 1999).

The first term of Equation (1) describes the secular coupling between the two planets, which produces a nodal precession but does not directly excite mutual inclination. Let $I$ be the mutual inclination of the two planets and $\Omega$ be the longitude of ascending node of the outer planet, defined with respect to the invariant plane of the two planets (i.e., the plane aligned with their total angular momentum). Note that on long timescales the invariant plane is not fixed, but precesses around the binary angular momentum. The nodal precession rate $d\Omega/dt$ due to the planets perturbing each other is given by (see derivation in Appendix A)

$$\frac{d\Omega}{dt} = \Omega_0 \left[ 1 + \frac{1}{2} \left( \frac{f_3}{f_2} - \frac{\beta}{(1 + \beta)^2} \right) I^2 \right]$$

with

$$\Omega_0 = \frac{1}{2} n_B f_3, \quad \mu = \frac{m_1}{M_*} (1 + \beta), \quad \beta = \frac{m_2}{m_1} \left( \frac{a_2}{a_1} \right)^{1/2}.$$

Equation (2) is also of the order of unity, so that $f_3$ is of order unity, and the perturber $f_i$ is of order 0. Note that $\Omega_0$ is not a direct function of the eccentricity $e_B$.

The other two terms in Equation (1) are the result of resonances found for the limiting cases $e_B = 0$ and $e_B \neq 0$. The expression can also be used as a crude estimate for $\Omega_0$ when $a_1$ and $a_2$ are comparable (more accurate, see Bailey & Fabrycky 2020).

The replacement of “e” by “i” signifies that ivection resonances affect the inclination (instead of eccentricity) of the system. Ivection resonances happen when (assuming $a_1/a_2$ is of the order of unity, so that $f_3$ is also of the order of unity)

$$n_B \sim |\Omega| \sim n_2 \mu, \quad a_B \sim a_2 \mu^{2/3}.$$  

For planets with smaller masses or longer periods, the binary has to be farther away for ivection resonance to happen.

In the remainder of this section, we analyze the perturbation from the binary to define different types of ivection resonances and their corresponding resonant angles in Section 2.1, and construct an effective Hamiltonian for each type of ivection resonance in Section 2.2.

2.1. Resonant Perturbation from the Binary

In this subsection we study the secular perturbation from the binary. For simplicity, we only consider two limiting cases: when the binary has zero eccentricity ($e_B = 0$), and when the binary has very high eccentricity ($1 - e_B \ll 1$). The results for the limiting cases can be interpreted as the lowest-order term of the resonant perturbation potential expanded in powers of $e_B$ and $(1 - e_B)$, respectively. For intermediate $e_B$ (where expanding in $e_B$ or $(1 - e_B)$ is a bad approximation), it is likely that the resonances found for the limiting cases remain qualitatively the same, except that their strengths (amplitudes of the resonant potentials) will be affected by $e_B$.

2.1.1. Circular Binary

First consider a circular binary. To quadrupole order in $a_i/a_B$ and lowest order in $m_i/M_*$, the coupling between a planet and the binary is given by

$$\langle \Phi_{IB} \rangle = -Gm_i m_B \left( \frac{1}{|r_B - r_i|} - \frac{r_B \cdot r_i}{r_B^3} \right)$$

$$= \frac{3}{4} \frac{G m_i a_i^2}{a_B^3} (\hat{r}_B \cdot \hat{n}_i)^2 + \text{constant.}$$

Here $\hat{r}_B$ is the direction of the location of the binary, and $\hat{n}_i$ is the direction of angular momentum of planet $i$. Let $I_i$ and $\Omega_i$ be the inclination of longitude of ascending node of each planet. Expanding $\langle \Phi_{IB} \rangle$ up to second order in $\sin I_i$ and removing approximately given by

$$\frac{2\pi}{|\Omega_0|} \approx 1300 \text{ yr} \times \left( \frac{P_2}{1 \text{ yr}} \right) \left( \frac{P_1}{P_2} \right)^{-4/3} \left( \frac{\mu}{10^{-3}} \right)^{-1}.$$  

Here $P_1$ and $P_2$ are the orbital periods of the planets. This expression can also be used as a crude estimate for $\Omega_0$ when $a_1$ and $a_2$ are comparable (for more accuracy, see Bailey & Fabrycky 2020).

2.1.2. Eccentric Binary

To leading order in $(1 - e_B)$, the coupling between a planet and the binary is given by

$$\langle \Phi_{IB} \rangle = -Gm_i m_B \left( \frac{1}{|r_B - r_i|} - \frac{r_B \cdot r_i}{r_B^3} \right)$$

$$= \frac{3}{4} \frac{G m_i a_i^2}{a_B^3} (\hat{r}_B \cdot \hat{n}_i)^2 + \text{constant.}$$

Here $\hat{r}_B$ is the direction of the location of the binary, and $\hat{n}_i$ is the direction of angular momentum of planet $i$. Let $I_i$ and $\Omega_i$ be the inclination of longitude of ascending node of each planet. Expanding $\langle \Phi_{IB} \rangle$ up to second order in $\sin I_i$ and removing approximately given by

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Here $P_1$ and $P_2$ are the orbital periods of the planets. This expression can also be used as a crude estimate for $\Omega_0$ when $a_1$ and $a_2$ are comparable (for more accuracy, see Bailey & Fabrycky 2020).

The other two terms in Equation (1) describe the perturbation from the binary. Such perturbation is typically small, but it is nontrivial when the precession rate of the planets becomes commensurate with some integer multiple of the orbital frequency of the binary. We call this secular-orbital resonance the ivection resonance. The name “ivection” is derived from evocation resonance, which is the resonance between the apsidal precession and the orbital frequency of some distant perturber (Touma & Wisdom 1998; Touma & Sridhar 2015).
nonresonant terms gives

\[ \langle \Phi_{ib} \rangle = \frac{3}{4} \frac{GM_m a_i^2}{a_b^2} \times \left[ -\sin I_B \left( 1 - \cos I_B \right) \sin I \cos(-2\lambda_B + 3\Omega_B - \Omega_i) + \left( 1 - \cos I_B \right)^2 \sin^2 I \sin^2(-\lambda_B + 2\Omega_B - \Omega_i) \right]. \]  

(7)

Here \( I_B, \Omega_B, \) and \( \lambda_B \) are the inclination, longitude of ascending node, and mean longitude of the binary, respectively. \( I_B \) and \( \Omega_B \) are approximately constant. We ignore the nonresonant terms on the grounds that such terms only change \( \Omega_i \) by \( \sim n_B a_i^2/a_b^2 \), which is negligible when the system is not too far from ivection resonance. If one wishes to be more exact, the leading order secular planet–binary interaction can be absorbed into terms with the same \( I_i \) dependence in the secular planet–planet interaction potential. The form of the resulting Hamiltonian should remain the same.

The first term in \( \langle \Phi_{ib} \rangle \) gives rise to a resonance with resonant angle \((-2\lambda_B + 3\Omega_B - \Omega_i))\), which we call a first-order ivection resonance (because the resonant term is \( \propto \sin I \)). The second term gives rise to another resonance with resonant angle \((-\lambda_B + 2\Omega_B - \Omega_i))\), which we call a second-order ivection resonance (because the resonant term is \( \propto \sin^2 I \)). Note that \( \Omega_i < 0, \lambda_B > 0 \) and \( \Omega_B \approx 0 \).

Equation (7) shows that the resonant perturbation for the first-order ivection resonance vanishes when the binary is aligned or anti-aligned with the planets \( (I_B = 0 \) or \( \pi) \), and the resonant perturbation for second-order ivection resonance vanishes when the binary is aligned with the planets \( (I_B = 0) \). In general, the resonant perturbation remains qualitatively the same for a prograde and retrograde binary; \( I_B \) only affects the strength of the resonant perturbation. We therefore do not discuss prograde and retrograde binaries separately.

2.1.2. Very Eccentric Binary

The other limit is when the binary is very eccentric, with \( (1-e_B) \ll 1 \). In this case, the perturbation from the binary produces discrete kicks in planet eccentricity and inclination when the binary passes periastron. Each periastron passage changes inclination by (Kobayashi & Ida 2001)\footnote{Kobayashi & Ida (2001) derived this relation for planetesimals perturbed by the single passage of a binary with parabolic or hyperbolic orbit. The result should also be applicable to elliptic orbits when \( e_B \to 1 \). We also assume that the change in the vector \((I, \sin(\Omega - \Omega_B), I \sin(\Omega - \Omega_B))\) due to one periastron passage is independent of the initial inclination, which is a relatively good approximation when the initial inclination is small.}

\[
\Delta [I \cos(\Omega - \Omega_B)] \approx 0, \\
\Delta [I \sin(\Omega - \Omega_B)] \approx \frac{3}{8\sqrt{2}} \frac{q}{\sqrt{1 + q}} \left( \frac{a_i}{D} \right)^{3/2} \sin 2I_B. 
\]  

(8)

Here \( q \equiv M_B/M_\star \) is the binary mass ratio, and \( D = a_B(1-e_B) \) is the periastron distance. The eccentricity of the planet is also perturbed during periastron passage, but the change in eccentricity is smaller than the change in inclination by a factor of \( a_i/D \) (Kobayashi & Ida 2001). One can safely ignore the eccentricity kicks as long as the apsidal precession frequency is not commensurate with the binary orbital frequency (i.e., the system is far from an “eccentric ivection resonance”).

The inclination kicks are resonant (i.e., the kicks are in the same direction in the frame co-precessing with the planet) if the orbital period of the binary is close to an integer multiple of the period of nodal precession. We call this resonance eccentric ivection resonance (or eccentric first-order ivection resonance), to distinguish it from the ivection resonances for circular binaries that we discussed previously.

To derive the resonant term in the secular binary perturbation potential, we Fourier expand \( \langle \Phi_{ib} \rangle \) with respect to the binary mean anomaly and compute the Fourier amplitudes in the limit of small \( (1-e_B) \). When the binary orbital period is \( \approx j \) times the planet precession period (with \( j \) being a positive integer), the resonant term \( \langle \Phi_{ib} \rangle \) is given by

\[
\langle \Phi_{ib} \rangle_{\text{res}} \approx -\frac{3}{16\sqrt{2}} \frac{GM_m a_i^2}{a_b^3(1-e_B)^{3/2}} \sin(2I_B) \\
\times \sin I \cos[ -j(\lambda_B - \omega_B) - (\Omega_i - \Omega_B)].
\]  

(9)

The above expression is a good approximation when \( j \ll (1-e_B)^{-3/2} \). The full derivation of this resonant potential is given in Appendix B. The same result can also be obtained intuitively by approximating the the discrete kicks in inclination with a continuous perturbation corresponding to a potential \( \propto \cos[ -j(\lambda_B - \omega_B) - (\Omega_i - \Omega_B)] \). The resonant potential of an eccentric ivection resonance is similar to that of a (circular) first-order ivection resonance.

2.2. Hamiltonian of the System

For each type of ivection resonance discussed above, we can construct an effective Hamiltonian to describe the evolution of mutual inclination \( I \). In order to do this, we first find the equations of motion for each planet’s inclination \( I \) and resonant angle \( \theta \) using the secular Hamiltonian \( H_{sec} \). Then we relate \( I \) and \( \theta \) to the mutual inclination \( I \) and resonant angle \( \theta \) (which is \( \theta \) with \( \Omega_i \) replaced by \( \Omega \)), and find the equations of motion for a pair of variables \((X, Y) \approx (I \cos \theta, I \sin \theta)\) (see their exact definition in Appendix A). Given \( dX/dt, dY/dt \), we can build an effective Hamiltonian \( H_{eff}(X, Y) \) that gives the correct equations of motion (i.e., \( dX/dt = -\partial H_{eff}/\partial Y, dY/dt = \partial H_{eff}/\partial X \)). Finally, we scale the Hamiltonian \( H_{eff} \) to a dimensionless form \( \mathcal{H} \).

For brevity, the full derivation of the effective Hamiltonian is delegated to Appendix A, and below we only summarize the main results.

2.2.1. First-order Ivection Resonance

For a first-order ivection resonance, the Hamiltonian of the system can be written in the following dimensionless form:

\[
\mathcal{H} = \eta(x^2 + y^2) - (x^2 + y^2) - x. 
\]  

(10)

\((x, y)\) are a pair of conjugate variables for \( \mathcal{H} \) given by (for small mutual inclination)

\[
x \approx (I/I_0) \cos \theta, \quad y \approx (I/I_0) \sin \theta.
\]  

(11)

The resonant angle \( \theta \) and the characteristic inclination \( I_0 \) are

\[
\theta = -2\lambda_B + 3\Omega_B - \Omega, \\
I_0 = \Omega_B.
\]  

(12)
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\[ I_0 = \frac{1}{4} \left( \frac{f_b}{f_5} + \frac{\beta}{(1 + \beta)^2} \right)^{1/3} \times \left[ \frac{3}{4} \frac{q}{1 + q} \frac{1 - \cos I_B}{2} \sin I_B \frac{(n_1 - n_2)n_B}{n_1 n_2} \right]^{1/3}. \] (13)

The time unit for this dimensionless Hamiltonian \( T_0 \) is given by

\[ T_0 = n_B^{-1} \left[ \frac{1}{4} \left( \frac{f_b}{f_5} + \frac{\beta}{(1 + \beta)^2} \right)^{1/3} \times \left[ \frac{3}{4} \frac{q}{1 + q} \frac{1 - \cos I_B}{2} \sin I_B \frac{(n_1 - n_2)n_B}{n_1 n_2} \right]^{-2/3} \right]. \] (14)

\( \eta \) is a constant characterizing how far the system is from the resonance, and is defined as

\[ \eta = \frac{1}{2} \left( \frac{\partial \theta}{\partial \lambda_B} n_B - \Omega_0 \right) T_0. \] (15)

For this resonance, \( \partial \theta / \partial \lambda_B = -2. \)

Physically, the first two terms of \( \mathcal{H} \) represent the secular coupling between the planets, and the last term represents the resonant perturbation of the binary. For binary mass ratio \( q \approx 1 \) and planets with similar orbital period (at some timescale \( P \)), \( I_0 \) and \( T_0 \) scale as

\[ I_0 \sim \mu^{1/3}, \ T_0 \sim \mu^{-5/3} P \sim \mu^{-2/3} P_B. \] (16)

Here \( P_B \) is the binary orbital period.

2.2.2. Second-order Ivection Resonance

For a second-order ivection resonance, the effective Hamiltonian of the system can be written as

\[ \mathcal{H} = \eta (x^2 + y^2) - (x^2 + y^2)^2 + y^2, \] (17)

with all definitions identical to the previous case except that

\[ \theta = -\lambda_B + 2Q_B - \Omega, \] (18)

\[ I_0 = \left[ \frac{1}{6} q \left( \frac{f_b}{f_5} + \frac{\beta}{(1 + \beta)^2} \right) \right]^{1/2} \times \left[ \frac{1 - \cos I_B}{2} \right]^{-2/3} \left( \frac{1 + \beta}{1 + \beta n_2 n_B} \right)^{1/2} \sim \mu^{1/2}, \] (19)

\[ T_0 = \frac{4}{3} q \left( \frac{1 - \cos I_B}{2} \right) ^{-2/3} \left( \frac{1 + \beta}{1 + \beta n_2 n_B} \right)^{1/2} \sim \mu^{-2} P. \] (20)

Compared to first-order ivection resonance, the characteristic inclination \( I_0 \) (which is comparable to the maximum inclination excitation of a non-dissipative system with zero initial inclination) is smaller, and the unit time \( T_0 \) (which characterizes the timescale of libration) is longer, suggesting that second-order ivection resonance is weaker than first-order ivection resonance.

2.2.3. Eccentric Ivection Resonance

For an eccentric ivection resonance, the form of the effective Hamiltonian is identical to that of a first-order ivection resonance for a circular binary (Equation (10)) except that the last term is \( \mp x \) for prograde \( (I_B < \pi/2) \) and retrograde \( (I_B > \pi/2) \) binaries, respectively. The resonant angle \( \theta \) and normalizations \( I_0 \) and \( T_0 \) are now

\[ \theta = -j(\lambda_B - \omega_B) - (\Omega - \Omega_B) \] (21)

\[ I_0 = \left[ \frac{f_b}{8} \left( \frac{f_b}{f_5} + \frac{\beta}{(1 + \beta)^2} \right) \right]^{1/3} \times \left[ \frac{3}{16 \sqrt{2}} q \frac{n_B(n_1 - n_2)}{n_1 n_2} (1 - e_B)^{-3/2} |\sin 2I_B| \right]^{1/3} \sim \mu^{1/3} (1 - e_B)^{-1/2}, \] (22)

\[ T_0 = n_B^{-1} \left[ \frac{f_b}{8} \left( \frac{f_b}{f_5} + \frac{\beta}{(1 + \beta)^2} \right) \right]^{1/3} \times \left[ \frac{3}{16 \sqrt{2}} q \frac{n_B(n_1 - n_2)}{n_1 n_2} (1 - e_B)^{-3/2} |\sin 2I_B| \right]^{-2/3} \sim \mu^{-5/3} (1 - e_B) P. \] (23)

The large binary eccentricity increases \( I_0 \) and decreases \( T_0 \). Note that since \( n_B \sim n_2 \mu, \ I_0 \) is also approximately given by \( I_0 \sim (a_2/D)^{1/2} \) where \( D = a_1(1 - e_B) \) is theperiastron distance. The stability of the system requires relatively large \( D/a_2 \), so \( I_0 \) will never be too large.

3. Conditions and Outcomes of Ivection Resonance Capture

In the absence of dissipative mechanisms, the maximum mutual inclination that initially coplanar planets can reach is \( \sim I_0 \), which is usually a few degrees or less, given its dependence on \( \mu \) (see Equations (16), (19), and (22)). Moreover, the width of the resonance (i.e., the region in parameter space where the inclination of an initially coplanar system can be nontrivially excited) is small, making the probability of a system forming near an ivection resonance low.

However, planets often undergo migration after their formation, which leads to a smooth variation of \( \Omega \) that increases the likelihood for a system to encounter an ivection resonance during its migration. As we will show below, if the migration is in the desired direction and is sufficiently slow, the inclination of the system can be excited to \( \gg I_0 \) via resonance capture.

In this section, we first study the process of iveryon resonance capture and find the conditions of capture. This is mainly done with the help of the Hamiltonian model, which relates the iveryon resonances to other previously studied resonances. Then we discuss the possible outcomes of iveryon resonance, with a focus on the disruption of iveryon resonance due to the encounter with another resonance, which cannot be captured by the Hamiltonian model.

3.1. Ivection Resonance Capture

First we study the outcomes when the system encounters an iveryon resonance during migration. The dimensionless Hamiltonians (Equations (10) and (17)) have forms identical to first- and second-order MMRs up to some sign changes, so many of the results for MMR capture can be directly applied. The similarity can be easily seen from the contour of the Hamiltonians shown in Figure 1. Due to this similarity, here we only summarize the possible outcomes of a resonance encounter. (For MMR capture and its outcomes, see Peale 1976;
Near the resonance, \( \eta \) is no longer constant since migration (or change in binary separation) changes \( n_B \) or \( \omega_0 \). The speed and direction of migration can be characterized by the parameter \( d\eta/d\tau \), where \( \tau = t/T_0 \) is the dimensionless time associated with the Hamiltonian. When the system encounters the resonance with \( d\eta/d\tau < 0 \), it cannot be captured into the resonance. The inclination may increase by at most \( \sim I_0 \) as the system crosses the resonance.

When \( d\eta/d\tau > 0 \), resonance capture becomes possible. Whether the system can be captured into the resonance depends on the migration rate and the initial mutual inclination. When migration is slow \( (d\eta/d\tau \lesssim 1) \) and initial inclination small \( (I/I_0 \lesssim 1) \), resonance capture is guaranteed. When migration is fast \( (d\eta/d\tau \gg 1) \) or initial inclination is large \( (I/I_0 \gg 1) \), resonance capture is impossible. Between these two limits, capture is in general probabilistic. We numerically compute the probability of capture as a function of \( d\eta/d\tau \) and initial \( I/I_0 \), and the result is shown in Figure 2.

An example of capturing into a first-order icketion resonance is shown in Figure 3. For this example, the primary and the
binary are both 1$M_\odot$ stars, on a circular orbit with $a_B \approx 200$ au and $i_B = 120^\circ$ initially. The two planets have masses $m_1 = 10M_E$ and $m_2 = 10M_{Jup}$ and periods $P_1 = 1$ yr and $P_2 = 3.7$ yr, respectively, with initially circular and coplanar orbits. The binary migrates outward at a timescale of 40 Myr, increasing initially circular and coplanar orbits. The system is captured into a first-order ionection resonance at $\sim 10$ Myr. Once captured, the resonant angle librates around $\pi$, and the mutual inclination keeps increasing; meanwhile, the mutual inclination and the inclination evolution after resonance capture agree well with the numerical result up to $I \sim 30^\circ$.

Hamiltonian located at (for $\eta \gtrsim 1$)

$$I/I_0 \approx \sqrt{x^2 + y^2} \approx \sqrt{\eta/2}.$$  

(24)

As $\eta$ continues to increase, $\theta$ increases approximately linearly in time. In reality, the growth of inclination stops when the migration rate changes such that $\eta$ no longer increases, or when the system is knocked out of resonance when passing another resonance (such as an MMR; see more in Section 3.3). In either case, the final inclination is not directly limited by $I_0$, which is usually small for small $\mu$.

### 3.2. Conditions of Ivection Resonance Capture

To capture the system into resonance, there are two main requirements: $\eta$ needs to be increasing when the system encounters resonance, and the migration rate needs to be sufficiently slow ($dI/d\tau \lesssim 1$).\(^8\) Here we discuss the physical meaning of these requirements.

#### 3.2.1. Direction of Migration

The requirement that $dI/d\tau > 0$ physically means that the ratio between the precession rate and the binary frequency $|\dot{\Omega}_0|/\eta_B$ needs to be increasing. If we assume that $\eta_B$ remains fixed, this requires the planets to migrate convergently (with increasing $a_1/a_2$) or inwardly.\(^9\)

It is worth noting that the direction of migration required for evocation resonance capture is usually the opposite. Evocation resonance capture requires the apsidal precession rate $|\dot{\omega}|$ to be decreasing, which requires the planets to migrate outward or divergently.

#### 3.2.2. Critical Migration Timescale

Ivection resonance capture also requires the migration to be sufficiently slow. The physical timescale of migration can be characterized by the timescale of the evolution of $|\dot{\Omega}_0|$, 

$$T_\Omega \equiv \left| \frac{d \ln |\dot{\Omega}_0|}{dt} \right|^{-1}.$$  

(25)

Normally, this timescale is comparable to the minimum of the planets’ migration timescale (defined as $T_{\text{m},i} \equiv |a_i/a_\text{m}|^{-1}$), 

$$T_\Omega$$ is related to the dimensionless parameter $|dI/d\tau|$ via (assuming fixed $n_B$)

$$\left| \frac{dI}{d\tau} \right| = \frac{1}{2} T_\Omega^2 |\dot{\Omega}_0| T_{\text{c},i}^{-1}.$$  

(26)

Therefore, the requirement $|dI/d\tau| \lesssim 1$ corresponds to

$$\min(T_{\text{m},1}, T_{\text{m},2}) \sim T_\Omega \gtrsim T_{\text{c},i},$$  

(27)

where the critical migration timescale $T_{\text{c},i}$ is defined as

$$T_{\text{c},i} \equiv \frac{1}{2} T_\Omega^2 |\dot{\Omega}_0|.$$  

(28)

---

\(^8\) There is a third requirement that the initial mutual inclination should be $\lesssim I_0$. This condition is easily satisfied for typical giants with $\mu \gtrsim 10^{-3}$, since the initial inclination should be no more than a few degrees if the planets form in a mostly planar disk.

\(^9\) Having convergently or inwardly migrating planets is a necessary but insufficient condition for having an increasing $|\dot{\Omega}_0|$. 

---

**Figure 3.** An example of capturing into a first-order ionection resonance. The primary and the binary are both 1$M_\odot$ stars, on a circular orbit with $a_B \approx 200$ au and $i_B = 120^\circ$. The two planets have masses $m_1 = 10M_E$ and $m_2 = 10M_{Jup}$ and periods $P_1 = 1$ yr and $P_2 = 3.7$ yr, respectively, with initially circular and coplanar orbits. The binary migrates outward at a timescale of 40 Myr, increasing $|\dot{\Omega}_0|/\eta_B$. The system is captured into a first-order ionection resonance at $\sim 10$ Myr. Once captured, the resonant angle librates around $\pi$, and the mutual inclination keeps increasing; meanwhile, the eccentricities of both planets remain small. The orange dashed curve in the top panel shows an analytic prediction of mutual inclination evolution based on the Hamiltonian in Equation (10). Both the location of the resonance and the inclination evolution after resonance capture agree well with the numerical result up to $I \sim 30^\circ$.
Note that \( \min(T_{in,1}, T_{in,2}) \gtrsim T_{crit} \) means that the time it takes for the system to migrate across the width of the resonance (i.e., changing \( \eta \) by a few) is longer than the timescale of libration.

Equation (27) is a relatively coarse approximation of the condition of ivection resonance capture, mainly because the maximum \( dw/dt \) for which capture probability is nontrivial can vary by a couple orders of magnitude depending on the initial inclination of the system. For instance, the probability of capture is still \( \sim 10\% \) when \( dw/dt \sim 10 \) for first-order ivection resonance and eccentric ivection resonance and \( \sim 100 \) for second-order ivection resonance, if the initial inclination is optimal (see Figure 2).

In the limit of small \( a_1/a_2 \), \( T_{crit} \) is given by

\[
T_{crit} \approx \begin{cases}
146 \text{ Myr} \times \left( \frac{P_2}{1 \text{ yr}} \right) \left( \frac{P_1}{P_2} \right)^{28/9} \left( \frac{\mu}{10^{-3}} \right)^{-7/3} \left[ \frac{q}{1 + q} (1 - \cos I_B) \sin I_B \right]^{-4/3} & \text{1st order} \\
5.4 \text{ Gyr} \times \left( \frac{P_2}{1 \text{ yr}} \right) \left( \frac{P_1}{P_2} \right)^{-6} \left( \frac{\mu}{10^{-3}} \right)^{-3} \left[ \frac{q}{1 + q} (1 - \cos I_B)^{-4} \right] & \text{2nd order} \\
0.23 \text{ Myr} \times \left( \frac{P_2}{1 \text{ yr}} \right) \left( \frac{P_1}{P_2} \right)^{28/9} \left( \frac{\mu}{10^{-3}} \right)^{-7/3} \left[ 1 - e_B \right]^{2/3} \left( \frac{1 + q}{q} \right) \sin 2I_B \sin I_B \right]^{-4/3} & \text{eccentric.}
\end{cases}
\]

This can still be used as a coarse estimation of \( T_{crit} \) when \( a_1 \) and \( a_2 \) are comparable. For eccentric ivection resonance, the integer \( j \) is the ratio between planet precession rate and binary orbital frequency (\( \Omega_0 \approx \mu n_B \)). From this estimation, we can see that \( T_{crit} \) tends to be relatively large for ivection resonance with a circular binary. In particular, the large \( T_{crit} \) makes capturing into a second-order ivection resonance very unlikely, unless the planets are very massive (e.g., \( \mu \gtrsim 10^{-2} \)). Meanwhile, if the binary is highly eccentric (e.g., with \( e_B > 0.9 \) or 0.95), \( T_{crit} \) can easily becomes smaller than the typical planet migration timescale. Equation (29) also shows that \( T_{crit} \) has a strong dependence on the planetary mass (the \( \mu \) parameter). Therefore, among systems that have binary companions with suitable periods, ivection resonance is significantly more likely to affect those with more massive planets.

### 3.3. Possible Outcomes of Ivection Resonance Capture

So far we have mainly discussed the process of ivection resonance capture, which can be well described by the Hamiltonian model. The final outcome of ivection resonance capture, i.e., the final state of the system that we observe today, depends heavily on other factors, such as the encounter with a different resonance. In this section, we broadly discuss the possible outcomes of ivection resonance capture.

#### 3.3.1. Maintaining Ivection Resonance

If the planets have not migrated very far after they are captured into the ivection resonance, so that they never encounter a second resonance or reach a very large mutual inclination, they will likely remain inside the ivection resonance. This usually leads to a modestly large mutual inclination, small eccentricities (unless eccentricities have been excited before encountering the ivection resonance), and the resonant angle should librate around \( \pi \) for first-order ivection resonance or eccentric ivection resonance with prograde binary, 0 for eccentric ivection resonance with retrograde binary, and \( \pm \pi/2 \) for second-order ivection resonance. (or 0 if the resonance is an eccentric ivection resonance and the binary is retrograde). The mutual inclination today should be of the order of

\[
I \approx I_0 \sqrt{\eta} \sim \left( 1 - \frac{\partial \theta}{\partial n_B} [\Omega_0] \right). \tag{30}
\]

This gives \( I = \mathcal{O}(1) \) rad if migration changes \( n_B/\Omega_0 \) by \( \mathcal{O}(1) \) after the resonance capture.

#### 3.3.2. Disruption of Ivection Resonance

In reality, a migrating system is very likely to encounter another (secular or mean-motion) resonance before migration stops. In most (if not all) scenarios, encountering another resonance will disrupt the ivection resonance.

For example, if we integrate the system in Figure 3 for longer time, the system will eventually encounter an eviction resonance, which is another secular-orbital resonance with resonant interaction \( \propto \varepsilon_I^2 I \) and resonant angle \( 2\lambda_B - 2\Omega_B + \Omega - 2\omega_I \) (Touma & Wisdom 1998). The system’s evolution is shown in Figure 4. When the system gets close to the eviction resonance, it is affected by both ivection and eviction resonance, making the dynamics of the system chaotic while the two resonances overlap. This chaotic behavior knocks the system out of the ivection resonance’s resonant zone (i.e., the region in the phase space in which the resonant angle librates). The inclination ceases to increase, and the planets end up being out of resonance, with nearly constant inclination and small eccentricity (excited during the transient chaos). This is the typical outcome of encountering a relatively weak resonance whose width is narrower and libration timescale is longer than those of ivection resonance (but still short compared to the migration timescale; otherwise the effect of that resonance will be totally negligible). In principle, it is possible for the system to remain inside ivection resonance after encountering a relatively weak resonance, but it should be very rare (because it requires the system to luckily return to the resonant zone after the transient chaos), and we never observed it in our simulations.

Another example is when the system encounters a relatively strong MMR after being captured into an ivection resonance. This is relevant for real systems where one or both planets migrate and their semimajor axis ratio changes. For instance, consider a system with the same parameters as that shown in Figure 4, except that the binary has fixed orbit at 215 au and the outer planet migrates inward with a timescale of 80 Myr. The
This happens before the system actually reaches 1:3 MMR, because MMR encounters a 1:3 MMR after being captured into an 1:3 resonance. As the system approaches 1:3 MMR, the eccentricities of both planets are excited. Eventually, the system is forced out of the 1:3 resonance, and the inner planet becomes unstable and is ejected soon after that. This is the typical outcome of encountering a resonance stronger than the 1:3 resonance: the resonance will be disrupted, and if the inclination is sufficiently large when the system reaches the second resonance, the smaller of the two planets may be ejected.

It is worth noting that the relative strength of the encountered resonance can depend on the inclination at which the encounter happens. For instance, for the inclination mode of a second-order MMR, the resonant force in inclination is \( \propto I \), making it stronger than 1:3 resonance if encountered at sufficiently large \( I \) but weaker if encountered at very small \( I \).

### 3.3.3. Other Possibilities

There are a few other possible outcomes that we have not discussed above; these scenarios are either less common or involve physics beyond the scope of this paper. One possibility is that the system manages to migrate a long way inside 1:3 resonance (without being disrupted) and reaches large inclination. In this scenario, the evolution has to be studied with numerical simulations since the Hamiltonian model no longer applies. We do not consider this possibility in this paper because, in reality, the system will most likely encounter another resonance—either another secular resonance or an MMR—before this happens.

Another important mechanism that we have not discussed is the damping of inclination. When strong enough, this can prevent the mutual inclination from reaching large values or significantly damp the mutual inclination after the system leaves 1:3 resonance. Moreover, inclination damping may change the stability of resonant libration. Without inclination damping, libration is stable; but including damping may make libration overstable (i.e., the amplitude of \( \theta \) libration increases for every cycle), which eventually drives the system out of resonance. This mechanism is essentially the same as the resonance escape mechanism for MMR discussed in Goldreich & Schlichting (2014). The effect of inclination damping to the outcome of 1:3 resonance capture should be studied in more detail in future studies.

### 4. Formation of Kepler-108

In this section we apply the results from previous sections to discuss a possible formation scenario of Kepler-108 (Mills & Fabrycky 2017). The system hosts two planets with eccentricity \( e_1 \sim 0.1 \) and mutual inclination \( I \approx 24^\circ \), and is perturbed by a binary companion with a sky-projected separation of \( \sim 300 \text{ au} \). The orbital period of the binary is comparable to the timescale of nodal precession (which is \( \approx 5700 \text{ yr} \)), suggesting that 1:3 resonance may have played an important role in the formation of the system.\(^{11}\)

To model the migration of the planets, we assume that the planets start far from the resonance and migrate with constant \( T_{\Delta i} = |a_i/a_j|^{-1} \) for a given amount of time, then the migration stops and we wait until the system reaches a steady state. This model is oversimplified in that it does not capture the time dependence of the migration rate, the precession of the planets caused by the disk, and possible eccentricity and inclination damping. Still, we expect that these oversimplifications do not qualitatively affect the results.

\(^{11}\) Mutual inclination can also be excited when another planet is scattered out of the system, or by Lidov–Kozai oscillation. However, exciting the mutual inclination of Kepler-108 through these mechanisms is unlikely, since exciting such significant mutual inclination by planet–planet scattering tends to produce larger eccentricities than the values observed in Kepler-108, and the Lidov–Kozai mechanism is completely suppressed by the fast precession of the planets (see Section 6 of Mills & Fabrycky 2017).
We attempt to reproduce the currently observed orbital configuration using an eccentric ivection resonance, because the resonant perturbation will be much weaker, and resonance capture will require unrealistically slow migration if the binary has small eccentricity instead. We fix the binary eccentricity at 0.95 and initial inclination \( I_B = 45^\circ \). These parameters only affect the strength of the resonance; for a smaller binary eccentricity, we can get similar results by correspondingly decreasing the migration rates. We assume that both planets migrate inwards, and fix \( T_{m,2} \) at 8 Myr. The initial semimajor axes and the time to stop migration are chosen to reproduce the observed planet semimajor axes, and we tune \( T_{m,1} \) and the binary period \( P_B \) (which together determine when eccentric ivection resonance and MMRs are encountered) to reproduce the observed eccentricity and inclination.

Figure 5 shows an example that manages to reproduce the currently observed orbital configuration of the system. The planet and stellar masses are set to the observed values. For this simulation, \( T_{m,1} = 6.2 \) Myr and \( P_B = 7000 \) yr. In this case the two planets migrate divergently (so they cannot be captured into any MMR), but the nodal precession rate \( |\Omega| \) increases. We start the simulation at \( t = 7 \) Myr, with a planet period ratio < 2. After passing a 2:1 MMR at \( t \sim 5.5 \) Myr, the planets gain some finite eccentricities \( e_1 \sim 0.03, \ e_2 \sim 0.06 \). (The evolution for \( t < 0 \) is not shown in Figure 6 since we want to focus on the evolution of the system after it gets close to the ivection resonance.) Having some finite eccentricity before the system encounters ivection resonance is necessary in order to produce eccentricities consistent with observation (because ivection resonance does not excite eccentricity), but such eccentricity need not come from crossing a 1:2 MMR.

At \( t \sim 1 \) Myr, the system approaches and gets captured into an eccentric ivection resonance (with \( |\Omega| \approx \eta_B \)) and the resonant angle begins to librate. The finite “initial” eccentricity does not significantly affect capturing into the ivection resonance, because there is no low-order coupling between eccentricity and inclination. Once the system is inside the ivection resonance, inclination begins to increase as migration drives the system deeper into resonance. Shortly after the capture, the system crosses 1:3 MMR at \( t \sim 2 \) Myr, which further increases the planet eccentricities. Crossing this 1:3 MMR does not disrupt the ivection resonance, because the mutual inclination and planet eccentricities are all still small, and the perturbation due to a 1:3 MMR—which is \( \propto I \) for the inclination mode and \( \propto e_i \) for the eccentricity mode—is weak.

The ivection resonance is disrupted at \( T \sim 4.5 \) Myr, and the resonant angle ceases to librate. This is likely due to the encounter of a 2:7 (1:3:5) MMR, which allows the coupling between eccentricity, inclination, and semimajor axis. Note that since the mutual inclination of the system is large, the \( I'e \) mode of this resonance has strength comparable to a first-order MMR. Once the system is no longer inside ivection resonance, the mutual inclination stops increasing. Secular coupling between the planets causes the eccentricities to oscillate with relatively large amplitude.

We stop the migration at \( t = 7 \) Myr, when the semimajor axes of the planets reach the observed values. The system ends up in a quasi-steady state where the inclination is nearly constant and the eccentricities oscillate at constant amplitude. The final eccentricities are slightly smaller than the observed value, but the difference is within 1\( \sigma \) of observational uncertainty.

This example does require a certain amount of fine-tuning: we tuned the relative locations of the ivection resonance and MMRs so that the 2:7 MMR can disrupt the ivection resonance at the desired inclination, and the 1:3 MMR is encountered while \( I \) is still small (or before encountering the ivection resonance), avoiding disruption of the ivection resonance. Still, the qualitative behavior that the system ends up in a nonresonant state with significant \( I \) and smaller \( e_i \) is fairly generic, and can be observed for a wide range of parameter choices (as long as ivection resonance capture is allowed). The main limiting factor that may prevent such behavior is that capturing into ivection resonance requires large binary eccentricity or very slow migration. As we discuss in the next section, this is also why ivection resonance capture may be relatively uncommon. Overall, we do not intend to claim that this example, with an oversimplified migration model,
represents the actual formation scenario of Kepler-108. (For example, the actual eccentricity evolution is likely different.) Instead, it illustrates that ivection resonance is a promising mechanism for exciting inclinations in systems like Kepler-108 that have significant mutual inclination and modest eccentricities today.

5. Application to Other Exoplanetary Systems

The most direct consequence of ivection resonance capture is the excitation of mutual inclination. However, even if exciting mutual inclination by ivection resonance is common, finding another system like Kepler-108 will still be very difficult since we seldom manage to observe the mutual inclination of planets (except when they are nearly coplanar). In this section, we determine whether ivection resonance can be common among exoplanetary systems with external binary companions using indirect evidence such as the overall statistics of the precession period and critical migration timescale.

5.1. Can Precession Period Match Binary Period?

Ivection resonance happens when the period of nodal precession ($P_{\text{prec}}$) is commensurate with the binary period ($P_B$). Therefore, we can infer the likelihood of passing ivection resonance during migration from the distribution of $P_{\text{prec}}$ versus $P_B$.

This distribution of $P_{\text{prec}}$ and $P_B$ is shown in Figure 7. The precession period is estimated as follows: for single-planet systems, the precession rate is estimated by setting $\mu$ to the planet–star mass ratio and evaluating $f_3(\alpha)$ at $\alpha = 2.57$, which

![Figure 6.](image)

**Figure 6.** A simulation that reproduces the orbital configuration of Kepler-108 from initially coplanar planets. The dashed lines in the top-left, middle-left, middle-right, and bottom-right panels show observed system parameters, with the 1\(\sigma\) observational uncertainty of eccentricity (middle left) and inclination (top left) marked by shades. The mutual inclination is excited due to capturing into an eccentric ivection resonance with an $e_B = 0.95$, $I_B = 45^\circ$, and $P_B = 7000$ yr binary. The system leaves the resonance (at $t \sim 4.5$ Myr) probably due to encountering a 2:7 MMR. The initial eccentricities are due to a passage through a 1:2 MMR at $t < 0$. The eccentricities are further excited when passing a 1:3 MMR at $\approx 1.8$ Myr and after the ivection resonance is disrupted by the 2.7 MMR. The final eccentricities and mutual inclination are consistent with observation. See Section 4 for more discussion of this simulation.

![Figure 7.](image)

**Figure 7.** Estimates for the nodal precession timescale $P_{\text{prec}}$ and binary period $P_B$ of systems with known external binary companion. Color marks whether the system hosts multiple observed planets, and marker shape corresponds to detection method. The gray dashed line marks $P_{\text{prec}} = P_B$. 

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is the median $\alpha$ for adjacent planets in multi-planet systems detected by radial velocity (RV).\textsuperscript{12} Physically, this gives the precession rate if the system hosts, or once hosted, another planet with similar or smaller mass. For multiple-planet systems, the precession period of each pair of planets is evaluated. This overestimates the precession period, if there are more than two planets.

In Figure 7, there is no significant correlation between $P_B$ and $P_{\text{prec}}$, and many systems have $P_B \sim P_{\text{prec}}$. More quantitatively, $\sim$10\% of the systems in Figure 7 can pass an iveauion resonance if $P_{\text{prec}}$ changes by one order of magnitude during migration. Note that the actual distribution may be different from Figure 7 since we do not account for observational bias and only include systems in which the mass (or $M \sin i$) of at least one planet is known in this figure.

\subsection*{5.2. Is Migration Slow Enough?}

Iveauion resonance capture is also limited by the migration rate of the planets, with resonance capture possible only when the migration timescale is no shorter than $T_{\text{crit}}$. The critical migration timescale $T_{\text{crit}}$ corresponding to the systems in Figure 7 is shown in Figures 8. We estimate $T_{\text{crit}}$ for first- and second-order iveauion resonance at $e_B = 0$ and for eccentric iveauion resonance at $e_B = 0.95$ using Equation (29), assuming optimal $I_B$. When estimating $T_{\text{crit}}$, we scale the semimajor axes of the planets (while maintaining the semimajor axis ratio) so that the nodal precession period is commensurate with the binary period; this partially accounts for the migration after the system passes the resonance. For systems with a single observed planet, we use the approximation (29) with $P_1/P_2 = 1$.

The first two panels of Figure 8 show that when the binary eccentricity is small, $T_{\text{crit}}$ tends to be large. Very few systems have relatively small $T_{\text{crit}}$ (i.e., $\lesssim$10 Myr) for first-order iveauion resonance, and $T_{\text{crit}}$ for second-order iveauion resonance is at least $\sim$1 Gyr. Meanwhile, as shown in the last panel of Figure 8, when the binary eccentricity is large, it becomes much easier to have low $T_{\text{crit}}$. Therefore, we expect iveauion resonance to be important only when binary eccentricity is large.

\subsection*{5.3. An Observational Signature?}

Although the critical migration timescale $T_{\text{crit}}$ tends to be very large, iveauion (and eveauion) resonance may still play an important role in the evolution of exoplanetary systems with binary companions. This can be noticed by inspecting the distribution of $P_B/P_{\text{prec}}$ (Figure 9), which appears to show a gap at estimated $P_B/P_{\text{prec}}$ between 0 and 0.5 dex for systems

\textsuperscript{12} Here we choose an $\alpha$ value representative of systems discovered by RV (instead of all systems, which would give median $\alpha = 1.67$), because planets prone to iveauion and eveauion resonances are mainly detected by RV due to their relatively large mass and semimajor axis.
with a single observed planet. Meanwhile, such a gap does not show up for systems with multiple observed planets, possibly because ivection (and evection) resonances are generally weaker in this sample (see the systematic difference in \( T_{\text{crit}} \) between the samples with single and multiple observed planets in Figure 8). A deficiency of systems close to ivection resonance is also visible in Figure 7.

Given the relatively small sample size, one might suspect that the gap in Figure 9 is just a coincidence or the result of a particular binning. We show in Appendix C that the gap is not an artifact due to binning, and the probability of observing a similar gap from a gap-less distribution is \( \sim 6\% \). We also comment that our estimates of \( P_B \) and \( P_{\text{prec}} \) both have relatively large uncertainties (from estimating \( a_B \) from sky-projected distance, using \( M \sin i \) for planet mass, etc.), especially for systems with one observed planet (where the location of another hidden/ejected planet also comes from a coarse estimate). These uncertainties could smear out features in the distribution or slightly shift the location of features (via systematic errors in \( P_B \) and \( P_{\text{prec}} \) estimates). However, they could not create a gap from a gap-less distribution as long as the errors are not strongly correlated with \( P_B/P_{\text{prec}} \). Therefore, the existence of a gap in the estimated distribution of \( P_B/P_{\text{prec}} \) should suggest that the true distribution of \( P_B/P_{\text{prec}} \) likely contains a more prominent gap.

Given that the distribution of \( P_B \) and \( P_{\text{prec}} \) are both wide and gap-less, the most reasonable explanation for the gap, if it is not a coincidence, is that it is associated with some mechanism that only operates at \( P_B/P_{\text{prec}} \sim 1 \), which is most likely ivection or evection resonance. One possibility is that the gap is due to scattering of planets when a resonant system becomes dynamically unstable (e.g., due to encountering another strong resonance, as discussed in Section 3.3). This could either push the semimajor axes away from the resonant values, or cause ejection of the less-massive planet. If such mechanism has a strong effect, there should be a deficiency of planets in regions of the parameter space where it is more likely to be in resonance (i.e., when estimated \( P_B/P_{\text{prec}} \sim 1 \)). If this is the case, the gap should exist in the distribution of both single- and multi-planet systems, although it may be less observable for multi-planet systems due to the much smaller sample size.

However, such dynamical instability requires the planets to be captured into an ivection (or evection) resonance in the first place. As shown in Figure 8, for most systems, the critical migration timescale for ivection resonance is extremely long, making resonance capture unlikely. Evection resonance suffer from the same problem, since \( T_{\text{crit}} \) for evection resonance should be comparable to that of second-order ivection resonance.

Another possible explanation is that the deficiency of the planets near \( P_B/P_{\text{prec}} \sim 1 \) is due to a suppression of planet formation, rather than removal of planets captured into ivection or evection resonance; i.e., it is possible that formation of planet (at a certain semimajor axis) is suppressed when the local precession period of planetesimals (due to, for instance, perturbation from the surrounding disk) is commensurate with \( P_B \). This explanation is physically reasonable, since ivection resonance can drive planetesimals away from the midplane, thereby reducing their collision rate and suppressing planet formation.

Overall, more data (and further physical and statistical analysis) are required to verify whether the gap in \( P_B/P_{\text{prec}} \) distribution is physical, and, if the gap is indeed physical, how (and whether) ivection/evection resonance creates this gap.

### 6. Discussion

#### 6.1. Comparison with Evection Resonance

Ivection resonance is very similar to evection resonance in that they both originate from the commensurability between the precession of the planets and the orbital motion of the binary. However, there are a few important differences between them.

Table 1 summarizes the key properties and scaling relations for ivection and evection resonances; properties for MMR are also given as reference. As shown in Table 1, evection resonance is a second-order resonance (i.e., with resonant term \( \pi \epsilon^2 \)), with scalings identical to second-order ivection resonance. To quadrupole order of the binary perturbation potential, this is the only type of evection resonance; there is no evection counterpart of first-order ivection resonance or eccentric ivection resonance.

The direction of migration required for resonance capture is also different for ivection and evection resonance. For ivection resonance, resonance capture requires \( \varpi B > 0 \), which requires planets to migrate convergently or inward. Meanwhile, evection resonance capture requires \( \varpi B < 0 \) (note that the apsidal precession rate \( \dot{\varpi} \sim 0 \)), which requires planets to migrate divergently or outward (see the examples in Touma & Sridhar 2015).

#### 6.2. The Effect of Finite Disk Mass

In our analysis, we have only considered the case when the precession of the planet is driven by another planet. However, the gravitational perturbation from the disk can also drive the planet to precess. Consider a single planet migrating in a disk; the gravitational perturbation of the disk mass drives the planet to precess with

\[
\frac{2\pi}{|\Omega|} \sim \mu_B^{-1} P, \quad \text{with} \quad \mu_B = \frac{\Sigma_D a^2}{M_*}.
\]

Here, \( \Sigma_D \) is the disk surface density evaluated near the planet. (The secular planet–disk interaction can be calculated more...
precisely with the method given in Heppenheimer 1980; for our purpose, a coarse scaling estimate is enough since we do not know the detailed disk profile in the first place.) In other words, the disk acts like another planet with mass $\mu m_B$, and all scaling relations in Table 1 should still hold (with $\mu$ replaced by $\mu m_B$).

Migration of the planet and evolution of the disk can change the precession rate, providing another mechanism of encountering (and capturing into) 1vection resonances.

As we discussed in Section 3.2.2, one major factor that limits the possibility of 1vection (and ejection) resonance capture is that unless $(1 - e_B)$ is small, $\mu$ has to be large ($\gtrsim$ a few $10^{-2}$) for $T_{\text{crit}}$ to be sufficiently small. Since $\mu m_B$ can be comparable to the disk-to-star mass ratio (which is often $\gtrsim 10^{-2}$), having $T_{\text{crit}}$ below the actual migration timescale of the planet is much easier.

One caveat is that the dispersal of disk tends to decrease the precession rate, while 1vection resonance capture requires increasing precession rate. Capture into 1vection resonance becomes possible only when the migration of the planet tends to increase the precession rate, and the effect of migration overshadows the effect of disk dispersal. Meanwhile, the dispersal of the disk makes the system more likely to cross 1vection resonance in the right direction for capture.

### 6.3. A Note on Lidov–Kozai Mechanism

The Lidov–Kozai mechanism can also excite eccentricity and inclination of the planet. However, when 1vection or ejection resonance is important, the nodal precession timescale of the planets ($\sim P_B$) is generally short enough to completely suppress mutual inclination perturbation due to the Lidov–Kozai mechanism, which occurs at a timescale of $\sim (P_B^2/P)(1 - e_B^2)^{3/2}$. The ratio between the two timescales is $\sim (P_B^2/P)(1 - e_B^2)^{3/2} \sim \mu^{-1}(1 - e_B^2)^{3/2}$, and the Lidov–Kozai mechanism can be important only when the binary eccentricity is extremely high ($e_B > 0.99$) for Jupiter-mass planets.

On the other hand, the Lidov–Kozai mechanism may affect the eccentricity evolution of the planets if at least one of them has a relatively low apsidal precession rate, as discussed in Takeda et al. (2008). This mechanism is irrelevant for systems like Kepler-108 where both planets are massive and undergo fast apsidal precession. However, it can become relevant for systems with more extreme mass ratios, where the apsidal precession of the more massive planet is slow.

### 7. Summary

We identify a new type of secular-orbital resonance, 1vection resonance, that can excite the mutual inclination of planets in a multi-planet system (or a single-planet system with a relatively massive disk). Ivection resonance happens when the nodal precession rate of the planet (due to perturbation from another planet or the protoplanetary disk) is commensurate with the orbital frequency of an external binary perturber. We study several types of 1vection resonances (Section 2), including first- and second-order 1vection resonance (for a circular binary perturber) and eccentric 1vection resonance (for a very eccentric binary). They share a similar physical nature, but their strengths vary. Their properties are summarized in Table 1.

Capturing into an 1vection resonance happens when the system encounters an 1vection resonance with slowly increasing $|\dot{\Omega}|/n_B$. More precisely, the planet migration timescale needs to be shorter than a critical migration timescale $T_{\text{crit}}$ (see Section 3.2.2, which is usually very large unless the planets are relatively massive or the binary is highly eccentric.

After capturing into an 1vection resonance, the mutual inclination increases as the system continues to migrate, while the eccentricities remain unaffected by 1vection resonance. If migration stops before the system encounters another resonance (such as an MMR), the system can stay resonant. Otherwise, encountering another resonance often disrupts 1vection resonance. Depending on the strength of this second resonance, the system is either left at a mutually inclined nonresonant configuration or becomes dynamically unstable. In the latter case, the smaller planet may be ejected.

We propose 1vection resonance as a promising mechanism for producing the orbital configuration of Kepler-108, a system hosting two planets with mild eccentricities but significant mutual inclination. We use simulations to reproduce the observed configuration of the system in Section 4. The importance of 1vection resonance in other exoplanetary systems with relatively close binary companions is also investigated, with some indirect evidence (Section 5.3) suggesting that 1vection resonance may significantly affect the formation and/or evolution of such systems. However, given the large $T_{\text{crit}}$ of most systems, our current theory of 1vection resonance capture cannot account for such a significant effect, and how 1vection resonance affects the formation and evolution of planets perturbed by an external binary companion should be investigated in future studies.
Appendix A
Derivation of the Effective Hamiltonian

A.1. Frame of Reference and the Evolution of Mutual Inclination

We start by finding a frame of reference for orbital elements in which the evolution of the two planets’ mutual inclination is easily related to the evolution of their individual inclinations \( \iota_i \) and longitudes of ascending node \( \Omega_i \).

We can define a (generally time-dependent) “standard” frame of reference, where the plane of reference is the invariable plane if one only takes into account the two planets (i.e., the plane normal to their total angular momentum), and the reference direction \( \hat{x}(t) \) in this plane is defined such that \( d\hat{x}/dt \) is perpendicular to the plane of reference. A “mutual” longitude of ascending node \( \Omega \) can be defined as the outer planet’s longitude of ascending node in this standard reference frame. This standard reference frame rotates, mainly due to the precession of the invariable plane around the binary angular momentum. For the systems we are interested in, the planet–planet coupling is much stronger than the planet–binary coupling, and the rotation of this frame is much slower than the nodal precession rate of the planets.

We then define a fixed frame of reference that coincides with our standard frame of reference at a certain time \( t = t_0 \). If we define \( I_1 \) and \( \Omega_1 \) with respect to this fixed frame of reference, \( I \) and \( \Omega \) at \( t_0 \) are simply given by \( I = I_1 + I_2 \) and \( \Omega = \Omega_1 + \pi = \Omega_2 \). Moreover, since \( d\hat{x}/dt \) is perpendicular to the plane of reference, it is easy to prove that at \( t = t_0 \),

\[
\frac{d}{dt} (F(I) \cos \Omega) = \frac{d}{dt} (\sin I_2 \cos \Omega_2 - \sin I_1 \cos \Omega_1),
\]

\[
\frac{d}{dt} (F(I) \sin \Omega) = \frac{d}{dt} (\sin I_2 \sin \Omega_2 - \sin I_1 \sin \Omega_1). \tag{A1}
\]

Here \( F(I) \) is a function of \( I \), defined as \( F(I) = \sin I_1 + \sin I_2 \) where \( I_1 \) is the inclination of planet \( i \) evaluated in the standard reference frame. Note that the \( I_1 \)’s at \( t = t_0 \) and \( F(I) = I + O(I^3) \) for small \( I \).

Now consider the resonant angles, \( \varphi_i \), which are of the form \( \varphi_i = -\Omega_1 + \varphi(t) \) and \( \varphi = -\Omega + \varphi(t) \). Here \( \varphi(t) \) is some linear combination of the binary’s orbital elements \( \Omega_B, \omega_B, \lambda_B \), evaluated with respect to the fixed frame defined above. \( \varphi(t) \) is the same as \( \varphi(t) \), except that it is evaluated with respect to the standard reference frame defined above. Since the rotation of the standard reference frame is slow and nonresonant (i.e., at small \( I \), the rate of rotation is insensitive to \( \varphi \)), in this paper we ignore this rotation when evaluating the time derivatives of \( \varphi(t) \) and \( \varphi(t) \) and assume

\[
\frac{d}{dt} \varphi(t) \approx \frac{d}{dt} \varphi(t) \approx \frac{\partial \varphi}{\partial \lambda_B} n_B. \tag{A2}
\]

In the remainder of this Appendix, we do not distinguish between \( \varphi(t) \) and \( \varphi(t) \). Under this approximation, Equation (A1) suggests that at \( t = t_0 \),

\[
\frac{d}{dt} (F(I) \cos \theta) = \frac{d}{dt} (\sin I_2 \cos \theta_2 - \sin I_1 \cos \theta_1),
\]

\[
\frac{d}{dt} (F(I) \sin \theta) = \frac{d}{dt} (\sin I_2 \sin \theta_2 - \sin I_1 \sin \theta_1). \tag{A3}
\]

Note that here \( I \) and \( \theta \) are defined with respect to the time-dependent standard reference frame, and \( I_1 \) and \( \theta_1 \) are defined with respect to the fixed reference frame. The above result holds only at \( t = t_0 \). At a different time, we need to evaluate the RHS with respect to a different frame.

In the calculations below, we will first find the equations of motion for \( I_1 \) and \( \theta_1 \), then use them to get the equations of motion for variables related to \( I \) and \( \theta \).

A.2. Equations of Motion for Inclinations of Individual Planets

The secular (averaged over planet orbits) Hamiltonian of the system can be written as

\[
H_{sec} = \langle \Phi_{1B} \rangle + \langle \Phi_{1B, res} \rangle + \langle \Phi_{2B} \rangle. \tag{A4}
\]

\( \langle \Phi_{1B} \rangle \) is the secular interaction between the two planets, and to fourth order in inclination (Murray & Dermott 1999)

\[
\langle \Phi_{12} \rangle = \frac{-Gm_1m_2}{a_2^2} \left[ (s_1^2 + s_2^2) f_3 + (s_1^4 + s_2^4) f_8 + s_1^2 s_2^2 f_0 \right.
\]

\[
+ s_1 s_2 f_4 \cos(\Omega_2 - \Omega_1)
\]

\[
+ s_1 s_2 (s_1^2 + s_2^2) f_6 \cos(2\Omega_2 - 2\Omega_1)
\]

\[
+ s_1^2 s_2^2 f_{26} \cos(2\Omega_2 - 2\Omega_1) \right]. \tag{A5}
\]

Here \( s_i = \sin(I_i/2) \) and \( f_i \) are functions of \( \alpha = a_1/a_2 \) given in Appendix B of Murray & Dermott (1999) evaluated at \( j = 0 \). For simplicity, we assume that the planet orbits remain circular. \( \langle \Phi_{1B, res} \rangle \) includes only the resonant terms in \( \langle \Phi_{1B} \rangle \). We neglect all nonresonant terms in \( \langle \Phi_{1B} \rangle \), since the planets are tightly coupled and \( \langle \Phi_{12} \rangle \gg \langle \Phi_{1B} \rangle \). \( \langle \Phi_{2B, res} \rangle \) for different types of ivection resonances are given in Section 2.1 and Appendix B.

Consider the canonically conjugate variables \( \Theta \equiv m_i \sqrt{GM} a_i (2s_i^2) \) and \( \Theta_i \equiv -\theta_i \). We want to make a canonical transform so that one of the conjugate variables is the resonant angle

\[
\Theta_1 \equiv \begin{cases} -2\lambda_B + 3\Omega_B - \Omega_1 & (1st order ivection) \\ -\lambda_B + 2\Omega_B - 2\Omega_1 & (2nd order ivection) \\ -j(\lambda_B - \omega_B) - (\Omega_1 - \Omega_B) & (eccentric ivection). \end{cases} \tag{A6}
\]

In general, we can write \( \Theta = -\Omega + \varphi(t) \) [and \( \Theta = -\Omega + \varphi(t) \)], with \( \varphi(t) \) being a linear combination of \( \Omega_B, \omega_B, \lambda_B \), and \( d\varphi(t)/dt \approx n_B \partial \varphi / \partial \lambda_B \). Using a canonical transform with type-3 generating function \( G_3(\Theta_i, \Theta) = \sum_{i=1,2} [\theta_i - \varphi(t)] \Theta_i \), we get a new set of canonical variables \( (\Theta_i, \theta_i) \) with \( \Theta_i \equiv \Theta_i \), and the corresponding Hamiltonian is

\[
\hat{H}_{sec}(\Theta_i, \theta_i) = H_{sec} + \frac{\partial G_3}{\partial \Theta} = H_{sec} + \frac{\partial \varphi}{\partial \lambda_B} n_B \sum_{i=1,2} \Theta_i. \tag{A7}
\]
\((\Theta_2, \theta_2)\) can then be transformed to another pair of conjugate variables, \((\sqrt{2} \Theta_2 \cos \theta_2, \sqrt{2} \Theta_2 \sin \theta_2)\). For convenience, we define a pair of dimensionless variables \((X_i, Y_i)\) as

\[
(X_i, Y_i) = \frac{1}{m_i \sqrt{GM_a a_i}} (\sqrt{2} \Theta_i \cos \Theta_i, \sqrt{2} \Theta_i \sin \Theta_i)
\]

\[
= (2s_i \cos \theta_i, 2s_i \sin \theta_i).
\]

(A8)

The equations of motion for \((X_i, Y_i)\) are then

\[
\frac{dX_i}{dt} = -\frac{1}{m_i \sqrt{GM_a a_i}} \frac{\partial \dot{H}_{sec}}{\partial Y_i}, \quad \frac{dY_i}{dt} = \frac{1}{m_i \sqrt{GM_a a_i}} \frac{\partial \dot{H}_{sec}}{\partial X_i}.
\]

(A9)

This gives the evolution of the inclinations of individual planets. Note that \((X_i, Y_i)\) can be defined with respect to any fixed reference frame.

A.3. Equations of Motion for Mutual Inclination

Now let us consider the equations of motion for

\[
X \equiv F(I) \cos \theta, \quad Y \equiv F(I) \sin \theta.
\]

(A10)

\(F(I)\) is defined in A.1 and \(F(I) = I + O(I^3)\) for small \(I\). Using the results from A.1,

\[
\frac{d}{dt} X = \frac{d}{dt} (\sin l_2 \cos \theta_2 - \sin l_1 \cos \theta_1),
\]

\[
\frac{d}{dt} Y = \frac{d}{dt} (\sin l_1 \sin \theta_2 - \sin l_1 \sin \theta_1).
\]

(A11)

Since \(\sin l_1 = 2s_1(1 - \frac{1}{8} I_1^2) + O(I_1^4)\), the RHS can be written in terms of \(X_i, Y_i\) as (up to third order in inclination)

\[
\frac{d}{dt} X = \Sigma_{i=1,2} (-1)^i \left[ \frac{1}{8} \left( X_i^2 + Y_i^2 \right) \right] \frac{dX_i}{dt}
\]

\[
- \frac{1}{4} \left[ X_i \frac{dX_i}{dt} + X_i Y_i \frac{dY_i}{dt} \right],
\]

\[
\frac{d}{dt} X = \Sigma_{i=1,2} (-1)^i \left[ \frac{1}{8} \left( X_i^2 + Y_i^2 \right) \right] \frac{dY_i}{dt}
\]

\[
- \frac{1}{4} \left[ Y_i \frac{dY_i}{dt} + X_i Y_i \frac{dX_i}{dt} \right].
\]

(A12)

Here \(d(X_i, Y_i)/dt\) are given by Equation (A9). Then we can rewrite the RHS in terms of \(X, Y\) using the relation

\[
(X_i, Y_i) = -\frac{\beta}{1 + \beta}
\]

\[
\times \left[ 1 + \frac{1}{8} \frac{\beta^2}{(1 + \beta)^2} (X^2 + Y^2) + O(I^4) \right] (X, Y),
\]

\[
(X_2, Y_2) = \frac{1}{1 + \beta}
\]

\[
\times \left[ 1 + \frac{1}{8} \frac{1}{(1 + \beta)^2} (X^2 + Y^2) + O(I^4) \right] (X, Y).
\]

(A13)

Here \(\beta = (m_2/m_1)(a_2/a_1)^{1/2}\) is the ratio between the angular momenta of the planets.

Below we summarize the explicit expressions for \(d(X, Y)/dt\) in terms of \((X, Y)\). We can split \(d(X, Y)/dt\) into nonresonant terms (those associated with \(\Phi_{I2} + \partial G_3/\partial t\)) and resonant terms (those associated with \(\Phi_{I2, res}\)). We keep nonresonant terms up to third order in inclination, and only keep the lowest-order resonant term because \(|\Phi_{I2}| \ll |\Phi_{I2, res}|\).

The nonresonant terms give (to third order in inclination)

\[
\frac{dX}{dt}_{pp} = \left\{ - \frac{\partial \theta}{\partial \lambda_B} n_B + \Omega_\theta \left[ 1 + \frac{1}{2} \left( \frac{f_s}{f_3} \right) \left( X^2 + Y^2 \right) \right] Y, \right. \\
\frac{dY}{dt}_{pp} = \left\{ \frac{\partial \theta}{\partial \lambda_B} n_B - \Omega_\theta \left[ 1 + \frac{1}{2} \left( \frac{f_s}{f_3} \right) \left( X^2 + Y^2 \right) \right] X. \right. \]

(A14)

The subscript “pp” denotes \(d(X, Y)/dt\) due to the nonresonant planet–planet interaction, and

\[
\dot{\Omega}_0 = \frac{1}{2} f_1 n_2 m_1 (1 + \beta) \frac{M_B}{M_e}.
\]

(A15)

To obtain the above results, we have used the following relations between \(f_i\): \(f_{14} = -2 f_3\), \(f_{16} = f_3 - 4 f_8\) and \(f_9 + f_{26} = 6 f_8 - 2 f_3\). These relations can be easily proven using the expressions for \(f_i\) given in Murray & Dermott (1999), or using the physical argument that rotating the plane of reference should leave \(\Phi_{I2}\) invariant. The mutual precession rate due to planet–planet interactions can also be obtained from the above result. To second order in inclination, the mutual precession rate is

\[
\frac{d\Omega}{dt}_{pp} = \Omega_\theta \left[ 1 + \frac{1}{2} \left( \frac{f_s}{f_3} \right) \right] \left( X^2 + Y^2 \right) \frac{\partial \theta}{\partial \lambda_B} n_B - \Omega_\theta \left[ 1 + \frac{1}{2} \left( \frac{f_s}{f_3} \right) \right] \left( X^2 + Y^2 \right) \frac{\partial \theta}{\partial \lambda_B} n_B.
\]

(A16)

For the resonant terms, using \(\Phi_{I2, res}\) from Section 2.1 and Appendix B, we find that (to lowest order in inclination),

\[
\frac{dX}{dt}_{res} = 0, \quad \frac{dY}{dt}_{res} = -\frac{3}{4} \sin l_B \left( \frac{1 - \cos l_B}{2} \right) \frac{q n_2^2 (n_1 - n_2)}{1 + q n_2 n_1}
\]

\[
\times \quad \text{for first order inclination resonance;}
\]

\[
\frac{dY}{dt}_{res} = 0, \quad \frac{dX}{dt}_{res} = -\frac{3}{2} \left( \frac{1 - \cos l_B}{2} \right)^2 \frac{q n_2^2 (1 + \beta)}{1 + q n_2 n_1} Y
\]

\[
\times \quad \text{for second order inclination resonance;}
\]

\[
\frac{dX}{dt}_{res} = 0, \quad \frac{dY}{dt}_{res} = -\frac{3}{16 \sqrt{2}} \sin (2l_B) \frac{q n_2^2 (n_1 - n_2)}{1 + q n_2 n_1}
\]

\[
\times \quad \text{for eccentric inclination resonance.}
\]

(A17)

Here \(q = MB/M_e\) is the binary mass ratio. Adding Equations (A17) and (A18), or Equations (A19)–(A14) then gives the equations of motion for \((X, Y)\).
A.4. Constructing the Effective Hamiltonian

At small inclination, the equations of motion for \((X, Y)\) can also be described by an effective Hamiltonian given by

\[
H_{\text{eff}} = \frac{1}{2} \left( \frac{\partial \theta}{\partial \beta} n_B - \Omega_0 \right) (X^2 + Y^2) - \frac{1}{8} \Omega_0 \times \left( \frac{f_8}{f_5} - \frac{\beta}{(1 - \beta)^2} \right) (X^2 + Y^2)^2 + \Phi_{\text{eff, res}}, \quad (A20)
\]

where

\[
\Phi_{\text{eff, res}} = \begin{cases} 
- \frac{3}{4} \sin l_B \left( 1 - \cos l_B \right) & \text{for first order inclination resonance,} \\
\frac{3}{4} \left( 1 - \cos l_B \right)^2 & \text{for second order inclination resonance,} \\
\frac{3}{16 \sqrt{2}} (1 - e_B)^3 & \text{for eccentric inclination resonance.}
\end{cases}
\]

It is easy to see that \(H_{\text{eff}}\) reproduces the correct equations of motion for \((X, Y)\), up to \(O(I^3)\) for the nonresonant (planet-planet) terms and to lowest nontrivial order for the resonant (planet-binary) term. This effective Hamiltonian makes it easier to study the evolution of mutual inclination as the system passes the resonance.

We can scale this effective Hamiltonian into a dimensionless form, in order to compare it with other resonances (e.g., MMR and evocation resonance) with similar Hamiltonians. We make the following transforms:

\[
\mathcal{H} = I_0^{-2} H_{\text{eff}}, \quad (x, y) = (X, Y)/I_0, \quad \tau = t/T_0. \quad (A22)
\]

After the transform, the equations of motion become

\[
\frac{dx}{d\tau} = -\frac{\partial \mathcal{H}}{\partial y}, \quad \frac{dy}{d\tau} = \frac{\partial \mathcal{H}}{\partial x}. \quad (A23)
\]

We choose the scaling coefficients \(I_0\) and \(T_0\) such that \(\mathcal{H}\) is of the form \(\eta(x^2 + y^2) - (x^2 + y^2)^2 \pm x \) for first-order and eccentric evocation resonance and of the form \(\eta(x^2 + y^2) - (x^2 + y^2)^2 \) for second-order evocation resonance. The signs of the resonant terms are chosen such that \(I_0\) and \(T_0\) are positive, \(I_0\) and \(T_0\) for different types of resonances have been summarized in Section 2.2.

Appendix B

Resonant Perturbation Potential for Eccentric Ivection Resonance

In this Appendix, we calculate the resonant perturbation potential \(\Phi_{\text{res}}\) for eccentric Ivection resonance. The secular perturbation potential due to the binary is

\[
\langle \Phi_{\text{rb}} \rangle = -GM_B m_i \left( \frac{1}{|r_B - r_i|} - \frac{r_B \cdot r_i}{r_B^3} \right). \quad (B1)
\]

In the calculation below, we assume that the planet orbits remain circular and \(m_i/M_\ast \rightarrow 0\). The second term in Equation (B1) therefore vanishes after averaging over the planet orbit. We then expand the potential in \(a_i/r_B\) up to quadrupole order and expand in \(\sin I\) up to first order. This gives

\[
\langle \Phi_{\text{rb}} \rangle = \frac{3}{2} \frac{GM_B m_i a_i^2}{r_B^5} \sin I (\sin \Omega_i x_B z_B - \cos \Omega_i y_B z_B) + \text{(terms independent of } I \text{ and } \Omega_i) + \text{(high - order terms).} \quad (B2)
\]

Here \((x_B, y_B, z_B)\) is the location of the binary, defined in a coordinate system where \((\hat{x}, \hat{y})\) spans the plane of reference, and \(\hat{x}\) is aligned with the reference direction.

Now we Fourier expand this potential in terms of the mean anomaly \(M = \lambda_B - \omega_B\) (not to be confused with the stellar masses \(M_\ast\) and \(M_B\)):

\[
\frac{3}{2} \frac{GM_B m_i a_i^2}{r_B^5} \sin I (\sin \Omega_i x_B z_B - \cos \Omega_i y_B z_B) = \sum_{k=-\infty}^{\infty} A_k e^{ikM}. \quad (B3)
\]

Here the Fourier amplitude \(A_k\) is independent of \(M\) (or \(\lambda_B\)).

In order to find \(A_k\), we first compute the Fourier amplitudes of \((x_B')^2/r_B^5, (y_B')^2/r_B^5\) and \((x_B' y_B')/r_B^7\), where \(x_B' \equiv r_B \cos M\) and \(y_B' \equiv r_B \sin M\). For example, the Fourier amplitude of \((x_B')^2/r_B^5\) will be

\[
\left( \frac{(x_B')^2}{r_B^5} \right)_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} dM e^{-ikM} (x_B)^2. \quad (B4)
\]

\(x_B' y_B', r_B, M\) and \(M\) can be related to the eccentric anomaly \(E\) by

\[
\begin{align*}
    r_B &= a_B (1 - e_B \cos E), \quad x_B' = a_B (\cos E - e_B), \\
    y_B' &= a_B \sin E \sqrt{1 - e_B^2}, \quad M = E - e_B \sin E.
\end{align*} \quad (B5)
\]

For small \((1 - e_B)\), the integral for Fourier amplitude in Equation (B4) is dominated by contribution from small \(E\). To simplify the integral, we define a scaled variable \(\zeta \equiv E/\sqrt{1 - e_B}\), and push the limits of the integral from \(\zeta = \pm \pi / \sqrt{1 - e_B}\) to \(\zeta = \pm \infty\). We also expand \(x_B', y_B', r_B, \exp(-ikM)\) and \(dM\) in terms of \(\zeta\) and keep only the first two terms.\(^{14}\) This allows the Fourier amplitudes to be written as some coefficient (involving \(a_B\) and \(1 - e_B\)) times an integral

\[^{14}\text{Here keeping only one term is not enough, because the two lowest-order terms can often have similar coefficients. For example, } r_B \approx a_B (1 - e_B) (1 - \zeta^2/2). \text{ The coefficients for higher-order terms, on the other hand, are smaller at least by a factor of } (1 - e_B).\]
that depends only on $\zeta$. Physically, this approximation corresponds to expanding the Fourier amplitude in terms of $(1 - e_B)$ and keeping only the lowest-order term. After evaluating the integrals, we get

$$
\left( \frac{x_k^2}{r_B^3} \right) \approx \left( \frac{y_k^2}{r_B^3} \right) \approx \frac{1}{4} \frac{a_B^3}{1 - e_B}^{3/2},
$$

$$
\left( x_B y_B \right) \approx \mathcal{O}(a_B^3(1 - e_B)^{-1}) \approx 0.
$$

Note that the above result is valid only for relatively small $k$: our expansion of the integrand is valid only when $|kM| \ll 1$ at $\zeta \sim 1$, and this requires $k \ll (1 - e_B)^{-3/2}$. We can then obtain $A_k$ by relating $(x_B, y_B, z_B)$ to $(x_k, y_k)$ via a rotation. To lowest order in $(1 - e_B)$,

$$
A_k = -\frac{3}{16\sqrt{2}} \frac{GMm_i}{a_B^3(1 - e_B)^{3/2}} \sin I_i \sin(2I_B) \cos(\Omega_i - \Omega_B).
$$

Therefore, for an eccentric ivection resonance with $\Omega \approx -m_B$ ($j$ is a positive integer, since $\Omega < 0$), the resonant term in $\langle \Phi_{jB} \rangle$ is given by

$$
\langle \Phi_{jB} \rangle_{\text{res}} = -\frac{3}{16\sqrt{2}} \frac{GMm_i}{a_B^3(1 - e_B)^{3/2}} \sin I_i \sin(2I_B) \cos \theta_i,
$$

where $\theta_i = -(\Omega_i - \Omega_B) - j(\lambda_B - \omega_B)$ is the resonant angle.

\section*{Appendix C}

\textbf{Testing the Gap in $P_B/P_{\text{prec}}$ Distribution}

In Figure 9 the distribution of estimated $P_B/P_{\text{prec}}$ for systems with a single observed planet appears to have a gap near $P_B/P_{\text{prec}} = 1$, while the same distribution for a system with multiple observed planets does not. Here we discuss the statistical significance of this gap, and whether it is an artifact due to binning.

Consider the probability of observing a system with certain $P_B/P_{\text{prec}}$. If this probability distribution does not contain any gaps associated with ivection/evection resonance, it should be approximately given by “smearing out” the observed distribution; i.e., we can approximate the actual distribution by replacing each observed system with a normal distribution $[\log_{10}(P_B/P_{\text{crit}})]$ centered at the observed value with a certain standard deviation $\sigma$. Here we choose $\sigma = 1$ dex, which is about the smallest $\sigma$ for which the smeared distribution is no longer bimodal. This smeared distribution, after a normalization, gives our estimated probability distribution of observing a system with certain $P_B/P_{\text{prec}}$ if there is no gap.

We then consider whether the observed data is consistent with this estimated gap-less distribution. For each bin in the histogram, we calculate $p_{\text{c}}$, the probability that a sample drawn from the estimated distribution with the same sample size as observation contains $\leq N_{\text{obs}}$ system in this bin, where $N_{\text{obs}}$ is the number of observed systems in this bin.

We perform this analysis for $P_B/P_{\text{prec}}$ distribution of systems with a single or multiple observed planets, and the results are shown in Figure 10. For systems with a single observed planet, $p_{\text{c}} \approx 0.015$ for the bin between 0 and 0.5 dex. We choose our null hypothesis to be that none of the four bins covering $P_B/P_{\text{prec}}$ between $-1$ and 1 dex show a gap and apply a Bonferroni correction. This gives a significance level of 0.06, which is not (although close to) statistically significant. For systems with multiple observed planets, there is neither a visible nor statistically significant gap, probably due to the smaller sample size.

We also perform the same analysis for $P_B$ and $P_{\text{prec}}$. We find that $p_{\text{c}}$ is always large in their distributions, suggesting that the gap in the left panel of Figure 10, if not a coincidence, should be due to an effect(s) directly associated with $P_B/P_{\text{prec}}$ such as ivection/evection resonance.

Additionally, we address the concern of whether the gap in the left panel of Figure 10 is due to a particular binning choice, by varying the number of bins. This changes both bin size and bin location at the gap. We find that for a wide range of bin numbers (15–25 bins from −5 dex to 4 dex, while Figure 10 has 18), the minimum $p_{\text{c}}$ is $\lesssim 0.015$ more than half of the time (since the gap width may be comparable to the bin size, it is natural for the gap to appear less significant when it is between two bins), confirming that this gap is not just due to the binning.
References

Bailey, N., & Fabrycky, D. 2020, AJ, 159, 217
Borderies, N., & Goldreich, P. 1984, CeMec, 32, 127
Burke, C. J., Bryson, S. T., Mullally, F., et al. 2014, ApJS, 210, 19
Chambers, J. E. 2012, Mercury: A Software Package for Orbital Dynamics, Astrophyysics Source Code Library, ascl:1201.008
Chatterjee, S., Ford, E. B., Matsumura, S., & Rasio, F. A. 2008, ApJ, 686, 580
Dawson, R. I., Johnson, J. A., Morton, T. D., et al. 2012, ApJ, 761, 163
Goldreich, P., & Schlichting, H. E. 2014, AJ, 147, 32
Heppenheimer, T. A. 1980, Icar, 41, 76
Kobayashi, H., & Ida, S. 2001, Icar, 153, 416
McArthur, B. E., Benedict, G. F., Barnes, R., et al. 2010, ApJ, 715, 1203
Mills, S. M., & Fabrycky, D. C. 2017, AJ, 153, 45
Murray, C. D., & Dermott, S. F. 1999, Solar System Dynamics (Cambridge: Cambridge Univ. Press)
Mustill, A. J., & Wyatt, M. C. 2011, MNRAS, 413, 554
Peale, S. J. 1976, ARA&A, 14, 121
Spalding, C., Batygin, K., & Adams, F. C. 2016, ApJ, 817, 18
Takeda, G., Kita, R., & Rasio, F. A. 2008, ApJ, 683, 1063
Touma, J., & Wisdom, J. 1653, AJ, 115, 1653
Touma, J. R., & Sridhar, S. 2015, Natur, 524, 439
Xu, W., & Lai, D. 2016, MNRAS, 459, 2925
Xu, W., & Lai, D. 2017, MNRAS, 468, 3223