A non-abelian quasi-particle model for gluon plasma

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We propose a quasi-particle model which takes into account the non-abelian characteristics of the gluonic field in a twofold way. Firstly, in counting the gluonic density of states we assume that the microstates of the gluon field are non-linear plane waves in a finite volume. Secondly, inspired by the fact that the gluon mass is a dynamically generated quantity, we assume microstates with different masses for the gluon field. To describe the associated mass distribution we use a quasi-Gaussian ansatz with mean value and standard deviation depending on the temperature. The temperature dependence of the mean mass in the critical region is determined via recent results of lattice SU(3) calculations of the gluon propagator at finite temperature. The temperature dependence of the standard deviation is used to fit the lattice results for the equation of state of pure SU(3) as reported in latest lattice results. Thus, our model being also free from thermodynamic inconsistencies, is in very good agreement with two lattice studies on different observables.

I. INTRODUCTION

Strongly interacting non-abelian gauge theories are described by SU(N) algebras. These theories possess a strong coupling in the low temperature regime prohibiting perturbative treatment. In addition, the associated degrees of freedom occur exclusively in a confined phase. In higher temperatures, near and above the critical point of SU(3) gauge theories, it is expected that the interaction of fermions and gauge fields, namely quarks and gluons in the case of SU(3) color (QCD), is significantly weaker, leading to the deconfined phase known as quark gluon plasma (QGP) [1]. The thermodynamical treatment of QGP has led to the introduction of quasi-particle models (QPMs) [2, 3], primarily aiming to explain the QGP equation of state as obtained from lattice gauge theory simulations of QCD at finite temperature. In these models and at temperatures higher than the critical value it was assumed that the thermodynamics of a system of interacting massless gluons may be approached by an ideal gas of massive noninteracting gluons. However, the above models [4] failed to explain the most recent lattice results [5].

Due to asymptotic freedom, the interaction of quarks and gluons is expected to be very weak at extremely high temperatures. Thus, in these thermodynamic conditions the picture of an ideal gas of (almost) non-interacting particles for the quark-gluon system should provide a good approximation. Nevertheless, as supported by the results of the experiments at the Relativistic Heavy Ion Collider (RHIC) [6], in the neighbourhood of the critical temperature $T_c$, associated with the transition from hadronic matter to QGP, the interaction is strong and the quark-gluon system is far from the ideal gas scenario sharing features of a perfect fluid [8]. This holds in particular also for the gluon field alone where accurate lattice results [9] demonstrate that the gluon system remains far from the ideal behaviour even for temperatures 5 times larger than $T_c$.

A useful and common strategy is to restrict the analysis to the gluonic sector considering the emergence of a (non-ideal) gluon plasma above the associated critical temperature. To capture this non-ideal behaviour the QPMs [10–21] introduce temperature dependent parameters which are suitably adjusted in order to fit the existing lattice results. A basic assumption of these models is the presence of a temperature dependent mass for the gluons, a property which may lead to thermodynamic inconsistencies [21] within the Landau statistical approach [22]. This is due to the fact that the temperature dependent mass becomes a thermodynamic quantity affecting the usual relations connecting pressure with energy density. This inconsistency may be healed by introducing an appropriate constraint in models involving vacuum energy $B(T)$ [21]. However, a much more natural way to overcome this issue is to use the Pathria [23] approach starting the calculations of thermodynamic quantities and equation of state from the energy density instead of the pressure [24]. The pressure is obtained via the integration of a fundamental thermodynamic relation, taking into account the temperature dependence of the gluon mass [24]. The latter is determined by fitting the lattice results for the equation of state of pure SU(3) at finite temperature [5]. With the suitable temperature dependence for the gluon mass, the description of lattice data with such an improved QPM turns out to be quite satisfactory. However, despite of being free from thermodynamic inconsistencies QPMs still include controversial issues from the physical point of view. The use of massive gluons and their treatment as free particles is not fully justified. To resolve this issue, a recent work [25] assumed that the gluon mass emerges through the propagation of gluons in a plasma environment as a collective effect and is furthermore related to the associated plasma frequency. Lattice results for the equation of state are
sufficiently reproduced with the use of a single temperature dependent parameter. Regarding the origin of gluon mass, a consistent interpretation is possible in terms of classical solutions of the corresponding gauge theory equations of motion (e.g. see [26] and references therein). In particular, [27] derives a class of nonlinear plane wave SU(2) solutions with the gauge field acquiring a mass dynamically due to its non-linear self-interaction. However, such a scenario has not up to now been associated with the gluon mass in QPMs.

A novel aspect in the framework of QPMs emerged from recent Lattice SU(3) calculations of the temperature dependence of the dynamically generated gluon mass by estimating the inverse gluon propagator in the infra-red limit [29]. These calculations obtained the gluon mass for a temperature regime just above the critical point, providing a further constraint in the phenomenology of QPMs. The stiff increase of quasiparticle masses as the critical region is approached from above is a general characteristic not in accordance with the Lattice [29]. A solution to this controversy has been proposed in the framework of QPMs through a Polyakov loop coupling to the quasiparticles [30].

In the present work we develop a QPM which takes into account the Lattice gluon mass data [29] and at the same time is in consistency with the older lattice calculations of the equation of state [5]. The proposed model remains intact with the dynamical origin of the gluon mass suggesting that the associated microstates are in fact non-linear plane wave solutions of the equations of motion of the non-abelian gluon field. We develop such a scenario using a simplified SU(2) description for the gluon and assuming that the basic characteristics of its non-abelian character are captured by our model. We demonstrate that the non-abelian character of the gluons introduces significant changes in their thermodynamical treatment which are taken into account in the proposed QPM. Furthermore, in order to be consistent with the thermodynamic nature of the mass, we introduce fluctuations in its statistical description promoting the mass itself to a random variable with determined probability density function. For simplicity we choose a truncated Gaussian distribution where the associated parameters (mean, variance) are temperature dependent and are constrained to fit the Lattice results. We show that the proposed model provides a very good description of two different Lattice results for the gluon field theory: the gluon equation of state presented in [5] and the recent gluon transverse mass calculation in [29].

The paper is organized as follows. In Section II we present the non-linear plane wave solutions of the SU(2) Yang-Mills theory and we reveal their properties which are relevant for the subsequent formulation of the non-abelian QPM. In section III we introduce the non-abelian QPM for the gluon plasma and the corresponding statistical treatment. In section IV we discuss our results concerning the dependence of the parameters of the proposed model on temperature as well as their compatibility with existing Lattice results. Finally, in section V we give our concluding remarks.

II. NON LINEAR PLANE WAVES IN SU(2) YANG-MILLS THEORY

A simplified description of the gluon field, capturing the basic phenomenological characteristics, can be obtained by the classical SU(2) Yang-Mills theory. Since SU(2) is a subgroup of SU(3), we will restrict our analysis to this case which is easier to handle and at the end of this section we will give some arguments supporting that our treatment can be transferred to the more realistic description with colored SU(3) Yang-Mills. Neglecting fermionic (matter) degrees of freedom the Lagrangian density of this model is written as:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

(1)

where $F_{\mu\nu}$ is the antisymmetric field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - g \epsilon_{abc} A_\mu^a A_\nu^b$ of the gauge field $A_\mu^a$. The corresponding classical equations of motion for the gauge field are:

$$\partial_\mu F^{\mu a} + g \epsilon_{abc} A_\mu^b F^{\nu c} = 0$$

(2)

The equations (2) permit plane wave solutions of the form:

$$A_\mu^a = \frac{1}{g} \epsilon_{\mu a} h (\omega \cdot t - \vec{k} \cdot \vec{x})$$

(3)

where $h$ is a scalar function of the plane wave phase

$$\xi = \omega \cdot t - \vec{k} \cdot \vec{x}$$

(4)

and $\epsilon_{\mu a}$ (for $a = 1, 2, 3$ and $\mu = 0, 1, 2, 3$) is given by

$$\epsilon_{\mu a} = \begin{pmatrix}
-k_1 m + k_1^2 + k_2^2 + k_3^2 + \omega^2 & -k_2 \omega + m k_2 & -k_3 \omega + m k_3 \\
-k_2 \omega + m k_2 & k_2^2 m + \omega^2 & -k_3 \omega + m k_3 \\
-k_3 \omega + m k_3 & -k_2 \omega + m k_2 & k_3^2 m + \omega^2 \\
\omega & \omega & \omega
\end{pmatrix}

(5)

The momentum four-vector $k^\mu = (\omega, \vec{k})$ satisfies the dispersion relation $\omega^2 = k^2 + m^2$ for an arbitrary mass parameter $m$. Note that the three columns ($\alpha = 1, 2, 3$) of the above matrix (5) are non-other than the three orthonormal space-like vectors orthogonal also to the timelike four-momentum vector $k^\mu$. As such, the solution obeys automatically the Lorentz gauge condition $\partial_\mu A^{\mu a} = 0$ since for plane waves it becomes equivalent to the transversality condition $k_\mu A^{\mu a} = 0$.

The solution (3) is most conveniently derived by the authors of [27] on the proper time frame $\nu^\mu = (m, \vec{0})$ with the gauge fixing condition $A_0^a = 0$. On that frame, $A_1^1 = A_2^2 = A_3^3 = (m/g) h(\xi)$ with all other components equal to zero and $h(\xi)$ satisfying the equation:

$$h''(\xi) + 2 h(\xi)^3 = 0.$$

Note that $\epsilon_{\mu a} \epsilon_{\nu b} = -m^2 \delta^{ab}$ holds.
The above equation possesses solutions of the form:

\[ h(ξ) = sn[ξ; -1] = cn[√2ξ; \frac{1}{2}] \]  

(7)

where \( sn[ξ; k^2](cn[ξ; k^2]) \) is the Jacobi elliptic sine (cosine) function with elliptic modulus \( k \) \[28\]. Thus, the solutions \( (7) \) describe periodic (anharmonic) plane waves with period \( P = 4K(k)/√2 \) where \( K(k) \) is the complete elliptic integral of the first kind. For \( k^2 = 1/2 \), the period numerically becomes \( P = 5.244 \). A Lorentz boost with parameters \( γ = ω/m, β = k/ω \) leads immediately to the general form \( 3 \). The mass of the gluon field, \( m \), is dynamically generated due to the gluon self-interaction whose origin is traced in the non-linear terms in the Lagrangian \( 1 \). Furthermore, it is free to take any positive value, reflecting the scale invariance of \( 1 \). Thus, although scale-free, the Yang-Mills classical solutions depend on an arbitrary scale \( m \) via the nonlinearity of the theory.

The solution \( (7) \) is the non-generalization of the usual linear plane waves describing the microstates of an ideal gas in a box. This is a key observation for the formulation of the non-abelian QPM in the next section. Solutions of the form \( (7) \) are also found in the \( φ^4 \) scalar field theory forming a complete (non-orthogonal) basis \( 31 \) for this system. In \[32\] the existence of an (approximate) mapping of the Yang-Mills \( SU(N) \) theory to the scalar \( φ^4 \) theory is shown. Thus one may argue that the solutions \( (7) \) provide also a non-orthogonal basis for \( SU(3) \) Yang-Mills gauge theory. The question is if and how the non-linear plane waves \( (7) \) are related to the thermodynamics of the gluonic system. Undoubtedly they will influence the counting of gluon microstates contained in a finite volume \( V \). The period of the non-linear plane waves is not \( 2π \) but \( P = 5.244 \), thus the number of such stationary states fitting in a fixed volume \( V \) is greater than the corresponding number for linear plane waves by a factor \( (\frac{2π}{P})^3 \). A comment is here in order. In lattice calculations \[29\] the gluon transverse mass is calculated only for a restricted region above the critical temperature \( T_c \), (namely in the range \([T_c, 1.9T_c]\)) and therefore it has to be treated as a free parameter in the region \( T ≥ 1.9T_c \) where no lattice data are available. Actually, we follow exactly this strategy which in turn suggests that in the temperature range \( T ≥ 1.9T_c \) the fitting parameters are two. However, in this temperature region, as we will see in the next section, the mean mass is approximately analogous to the variance. Therefore, practically in this regime only a single parameter (the variance) has to be fitted. Thus the proposed non-abelian quasi-particle model is in fact an one-parameter model.

To calculate the thermodynamic properties of the gluon system within the non-abelian QPM we follow the procedure described in \[29\] to avoid thermodynamic inconsistencies. To be self-contained we present here the basic formulas of the model. The energy density is given by (\( β = 1/T, k_B = 1 \)):

\[ ε = g_f \int dm \, N(μ, σ) \int \frac{d^3 k}{P^3} \frac{ω(\vec{k}, m)}{e^{\kappa ω(\vec{k}, m)} - 1} \]

\[ = N \cdot \frac{4π}{P^3} \cdot g_f \cdot T^4 \cdot \int_0^∞ \int_0^∞ dx \cdot x^2 \cdot \frac{\sqrt{x^2 + m^2/T^2}}{e^{x^2 + m^2/T^2} - 1} \cdot e^{-\frac{(m-μ)^2}{2σ^2}} \]

(8)

where in the counting of microstates we have included the prefactor \( (\frac{2π}{P})^3 \) and \( N \) is a normalization factor for the truncated normal distribution describing the gluon mass fluctuations. The number of degrees of freedom \( g_f \) is 3·8 = 24 since
that we have located the optimal values fast to the optimal values. This is crucial since the numerical calculations of the integrals in eqs. (9-10) are performed with high accuracy consuming CPU time. The results of our numerical analysis concerning the equation of state of the gluonic system are presented in Fig. 1.

\[ \epsilon = 24N \cdot \frac{4\pi}{P^3} \cdot T^4 \cdot \int_0^{\infty} \frac{e^{(m/a^2)^2}}{l} \, dl \]

\[ \sum_{l=1}^{\infty} \frac{1}{(k_T)^3} \cdot \left[ (\frac{m}{T})^3 \cdot K_1\left(\frac{m}{T}\cdot l\right) + 3(\frac{m}{T} \cdot l)^2 \cdot K_2\left(\frac{m}{T} \cdot l\right) \right] \]

with \( K_0 \) the modified Bessel functions. The mass integral is performed numerically. Having calculated the energy density, the pressure is obtained \([25]\) via integrating the thermodynamically consistent relation \( \epsilon = T^2 \partial P/\partial T - P \):

\[ \frac{P}{T} = \frac{P_0}{T_0} + \int_{T_0}^{T} T \, \frac{\epsilon}{T^2} \, dT \]  

The integral in (10) is also performed numerically. The final result for the energy density as well as the pressure, depends of course on the parameters \( \mu(T) \) and \( \sigma(T) \).

**IV. NUMERICAL RESULTS**

Using equations (9-10) we calculate the energy density and pressure for the gluon system at finite temperature. The dependence of the gluon mass on the temperature is obtained by a linear fit to the transverse mass lattice data presented in [29], leading to:

\[ \mu(T) = 1.18T + 0.48T_c \]  

with \( T_c \approx 270 \text{ MeV} \). We use eq. (11) to determine the mean gluon mass in the temperature range \([T_c, 1.9T_c]\) while the variance \( \sigma \) is determined by a Monte-Carlo search such that the lattice data for the energy density and pressure [5] are fitted within 10^{-4} accuracy. For \( T > 1.9T_c \) the Monte-Carlo search is performed in the two-dimensional space \((\mu, \sigma)\). In order to guide the search to profiles \((\mu(T), \sigma(T))\) which approximate a continuous function we use the following strategy: Assume that we want to determine the parameters of our model for an (ordered) sequence of temperatures \( T_i, i = 1, 2, \ldots, N \) with \( T_{i+1} > T_i \). Furthermore, assume that we have located the optimal values \((\mu(T_i), \sigma(T_i))\) for the temperature \( T_i \). To estimate the values \((\mu(T_{i+1}), \sigma(T_{i+1}))\) at the subsequent temperature \( T_{i+1} \) we explore a region centered at \((\mu(T_i), \sigma(T_i))\) and extending up to 50% around it in each direction. This procedure turns out to converge surprisingly

\[ \frac{\partial^2}{\partial \sigma^2} \]

\[ \text{II. RESULTS AND DISCUSSION} \]

We observe an excellent agreement between the lattice results via the calculations using the non-abelian QPM introduced in the present work. In Fig. 2(a,b) we show the dependence of the parameters \( \mu \) and \( \sigma \) on the temperature \( T \) as obtained by the Monte-Carlo optimization procedure described above. In the region \([T_c, 1.9T_c]\) the mass values \( \mu(T) \) lie on the line (11) (see Fig. 2b). For \( T > 1.9T_c \) the search is performed in a two-dimensional space. However, as can be seen in Fig. 2a, in this temperature region the ratio \( \frac{\sigma}{\mu} \) is approximately constant and therefore practically only one relevant parameter remains in the model. Thus the presented quasiparticle model has in fact only one free parameter.

A final comment is in order here. As seen in Fig. 2a, when approaching the critical point a rapid increase of the variance \( \sigma \) is observed while the behaviour of the transverse mass is smooth. This is in accordance with the lattice findings in [29]. There, the critical behaviour is traced in the abrupt dependence of the longitudinal gluon mass on the temperature while in our model criticality is reflected in the temperature dependence of \( \sigma \) which defines naturally a measure for the mass fluctuations. This allows for an interpretation of the gluon deconfining transition as a change in the spectral properties of the dynamically generated mass.

\[ \frac{\partial^2}{\partial \sigma^2} \]


V. CONCLUDING REMARKS

We have introduced a quasi-particle model for the thermodynamical description of the \( SU(3) \) gauge field (gluon) which takes the non-abelian characteristics into account. The microstates building up the grand canonical ensemble for the gluon are assumed to consist of transverse non-linear plane waves in a box of finite volume \( V \). The non-abelian character is taken into account in a twofold way: (i) counting the number of single gluon microstates contained in the box based on the periodicity of the non-linear waves and (ii) assuming a massive gluon similarly to other QPMs. Regarding (ii), we attempt to bridge the gap with the lattice treatment —where the gluon mass is a quantity estimated after averaging over many gluon field configurations— via considering the gluon mass as a quasi-Gaussian random (restricted only on positive values) variable with well defined, temperature dependent, mean and variance. Close to the critical temperature the only relevant parameter of the model (this associated with the mean mass) is determined by recent lattice calculations of the gluon inverse propagator \cite{29}. Tuning in turn appropriately the free parameter \( \sigma \) we reproduce with high accuracy the results of the lattice calculations for the gluon field equation of state. Within our treatment the traces of the transition to the gluon plasma phase are imprinted on the rapid decrease of the transverse gluon mass fluctuations just above the corresponding critical point. It would be interesting to look for a similar behaviour in the phase diagram when matter degrees of freedom are included within our approach. However, this is left for future investigations.

\begin{figure}
\centering
\begin{subfigure}{0.4\textwidth}
\includegraphics[width=\textwidth]{fig2a.png}
\caption{(Color online) (a) The ratio \( \mu(T)/T \) as a function of the temperature \( T \).}
\end{subfigure}\hspace{0.5cm}
\begin{subfigure}{0.4\textwidth}
\includegraphics[width=\textwidth]{fig2b.png}
\caption{(b) The temperature dependence of the transverse gluon mass \( T \).}
\end{subfigure}
\caption{(Color online) (a) The ratio \( \mu(T)/T \) as a function of the temperature \( T \). With stars we show the lattice data taken from \cite{29}.}
\end{figure}

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