Graded modules for Virasoro-like algebra

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Abstract: In this paper, we consider the classification of irreducible $\mathbb{Z}$- and $\mathbb{Z}^2$-graded modules with finite dimensional homogeneous subspaces over the Virasoro-like algebra. We first prove that such a module is a uniformly bounded module or a generalized highest weight module. Then we determine all generalized highest weight irreducible modules. As a consequence, we also determine all the modules with nonzero center. Finally, we prove that there does not exist any nontrivial $\mathbb{Z}$-graded modules of intermediate series.

Mathematics Subject Classification: 17B68, 17B65, 17B10

Keyword: graded module, generalized highest weight module, module of intermediate series.

§1 Introduction

The Virasoro algebra is playing an increasingly important role in both mathematics and physics. A book on conformal field theory by Di Francesco, Mathieu and Senechal [1] gives a great detail on the connection between the Virasoro algebra and physics. It is well-known that the Virasoro algebra acts on any highest weight module (except when the level is negative of dual coxeter number) of the affine Lie algebra through the use of famous Sugawara operators. Since 1992 when Mathieu [2] gave a classification of Harish-Chandra modules of the Virasoro algebra, it is natural for mathematicians to generalize the theory of this Lie algebra. From an algebra point of view, the Virasoro algebra can be regarded as the universal central extension of the Lie algebra of derivations (denoted by Der$A$) on the ring of Laurent polynomials $A = \mathbb{C}[t, t^{-1}]$. The Virasoro algebra has nontrivial positive-energy unitary representations only if the center is nonzero and this is one of the reasons why this algebra is more interesting than the algebra Der$A$.

So a natural generalization is the Lie algebra Der$A_\nu$ of derivations on the Laurent polynomials ring $A = \mathbb{C}[t_1^{\pm 1}, \ldots, t_\nu^{\pm 1}]$ in commuting variables. Der$A_\nu$ is also known as Lie algebra of the group of diffeomorphisms of $\nu$-dimensional torus. Several attempts have been made by physicists to give a Fock space representation of Der$A_\nu$ and its extension (see, e.g., [3]). Attempts failed to produce interesting results probably due to the lack of proper definition of “normal ordering”. The first surprising result is that Der$A_\nu$ with $\nu \geq 2$ is centrally closed ([4]). So people were

\*Supported by the National Science Foundation of China (No. 10371100, 10471096), the China Postdoctoral Science Foundation (No. 20060390693), “One Hundred Talents Program” from University of Science and Technology of China, and “New Century Talents Program” from Education Department of Fujian Province.

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forced to search other algebras (similarly to \( \text{Der} A_\nu \)) which admit nontrivial central extensions. There are two kinds of algebras being found. One is the so-called higher rank Virasoro algebras introduced in [5]. See also the papers [6–8]. Recently, Su [9] and Lu, Zhao [10] have completed the classification of Harish-Chandra modules over the higher rank Virasoro algebras. But the results of classification turn out to be disappointed as the Harish-Chandra modules are simply induced by modules of intermediate series, and thus the center of higher rank Virasoro algebras must act as zero on these modules.

The other is the Virasoro-like algebra introduced in [11]. It is the universal central extension of the skew derivation Lie algebra over the Laurent polynomials ring in two commuting variables. It can be generated by three elements and it contains many standard Heisenberg Lie subalgebras. Some relations between the Virasoro-like algebra \( L \) and the generalized Clifford algebras were given in [12]. The algebra \( L \) has many properties similar to the Virasoro and Heisenberg algebras. Many papers are devoted to the study of this algebra. The derivation Lie algebra of the centerless Virasoro-like algebra \( \bar{L} \) and the automorphism group of this derivation Lie algebra were studied in [13], while the structure of automorphism group of \( \bar{L} \) was considered in [14]. A large class of uniformly bounded \( \mathbb{Z}^2 \)-graded module over \( \bar{L} \) was constructed in [15], and a necessary condition for a nonzero level \( \mathbb{Z}^2 \)-graded irreducible module over \( L \) to be a module with finite dimensional homogeneous subspaces was given in [16]. The paper [17] presented a large class of generalized highest weight \( \mathbb{Z}^2 \)-graded irreducible modules over \( \bar{L} \), the paper [18] constructed a class of graded irreducible highest weight modules over \( L \) and classified the \( \mathbb{Z} \)-graded irreducible \( L \)-modules with nonzero center and finite dimensional homogeneous subspaces, while the paper [19] determined the structure of the Verma modules over \( L \). In the present paper, we deal with the classification of irreducible graded \( L \)-modules with finite dimensional homogeneous subspaces by using the results on the irreducible modules of Heisenberg algebra obtained in [20], and the results about the \( \mathbb{Z} \)-graded \( L \)-modules and the \( \mathbb{Z}^2 \)-graded \( L \)-modules given in [18] and [16].

The paper is arranged as follows. In section 2 we recall the concepts of the Virasoro-like algebra and its graded modules. We also collect some results about the irreducible modules of Heisenberg algebra which is crucial in the study of the classification of the graded irreducible \( L \)-modules. In section 3 we first prove that a \( \mathbb{Z} \)-graded \( L \)-module must be either a uniformly bounded, or a generalized highest weight module. Then we complete the classification of the irreducible generalized highest weight \( \mathbb{Z} \)-graded modules with finite dimensional homogeneous subspaces. In section 4 we first construct a class of irreducible generalized highest weight \( \mathbb{Z}^2 \)-graded modules with finite dimensional homogeneous subspaces by using the results obtained in the previous section. Then we prove that an irreducible generalized highest weight \( \mathbb{Z}^2 \)-graded \( L \)-modules must be isomorphic to the modules constructed in the beginning of this section. Finally, it is proved in Section 5 that there does not exist any nontrivial \( \mathbb{Z} \)-graded \( L \)-module of intermediate series.

§2 The Virasoro-like algebra and its graded modules

Throughout this paper we use \( \mathbb{C}, \mathbb{Z}, \mathbb{Z}_+, \mathbb{N} \) to denote the sets of complex numbers, integers, nonnegative integers, positive integers respectively. All spaces are over \( \mathbb{C} \). Let \( e_1 = (1, 0), e_2 = (0, 1) \) be the standard basis of \( \mathbb{C}^2 \) and let \( \Gamma = \mathbb{Z} e_1 \oplus \mathbb{Z} e_2 \), an additive group isomorphic to \( \mathbb{Z}^2 \). As usual, if \( u_1, \cdots, u_k \) are elements on some vector space, we use \( \langle u_1, \cdots, u_k \rangle \) to denote their
linear span over $C$. Let $A_2 = C[t_1^{\pm 1}, t_2^{\pm 1}]$ be the Laurent polynomial algebra with two variables. For $r = r_1e_1 + r_2e_2 \in \Gamma$, we denote

$$t^r = t_1^{r_1}t_2^{r_2} \in A_2.$$  

We use $\frac{\partial}{\partial t_i}$ to denote the partial derivative with respect to $t_i$ for $i = 1, 2$, and denote

$$d_i = t_i \frac{\partial}{\partial t_i}, \quad i = 1, 2,$$

and

$$D(r) = t^r(r_2d_1 - r_1d_2) = r_2t_1^{r_1+1}t_2^{r_2} \frac{\partial}{\partial t_1} - r_1t_1^{r_1}t_2^{r_2+1} \frac{\partial}{\partial t_2}.$$  

It is clear that the derivation algebra of $A_2$ is $\text{Der}A_2 = \langle t^r d_i | r \in \Gamma, i = 1, 2 \rangle$. One sees that the subspace

$$\mathcal{L} = \langle D(r) | r \in \Gamma \rangle$$

forms a Lie subalgebra of $\text{Der}A_2$, called the skrew derivation Lie algebra, and also called the centerless Virasoro-like algebra in [11]. For any $m = m_1e_1 + m_2e_2$, $n = n_1e_1 + n_2e_2 \in \Gamma$, let $\det(m, n) = m_1n_2 - m_2n_1$. One can easily see that

$$[D(m), D(n)] = -\det(m, n)D(m + n),$$

and that $\mathcal{L}$ is a simple Lie algebra. The universal center extension of $\mathcal{L}$ is the Lie algebra $L = \mathcal{L} \oplus \langle c_1, c_2 \rangle$, called the Virasoro-like algebra (with center), with the following Lie bracket:

$$[D(m), D(n)] = -\det(m, n)D(m + n) + \delta_{m+n, 0}h(m),$$

for $m, n \in \Gamma$, where

$$h(m) = m_1c_1 + m_2c_2.$$  

From the definition, one can directly deduces the following result.

**Lemma 2.1** If $b_1 = b_{11}e_1 + b_{12}e_2$, $b_2 = b_{21}e_1 + b_{22}e_2 \in \Gamma$ then

1. $L$ can be generated by $\{D(b_1), D(b_2), D(-b_1 - b_2)\}$ when $\{b_1, b_2\}$ is a $Z$-basis of $\Gamma$.

2. $\mathcal{H} = \langle D(kb_1), D(kb_2) | k \in Z \rangle$ is a standard Heisenberg subalgebra of $L$.

3. If $m = m_1b_1 + m_2b_2, n = n_1b_1 + n_2b_2$, then

$$[D(m), D(n)] = -\det(b_1, b_2)\det(m_1e_1 + m_2e_2, n_1e_1 + n_2e_2)D(m + n) + \delta_{m+n, 0}h(m).$$

Fix a $Z$-basis $b_1, b_2$ of $\Gamma$. One can regard $L$ as a $Z^2$-graded Lie algebra by defining the degree of the elements in $\langle D(m_1e_1 + m_2e_2) \rangle$ to be $(m_1, m_2)$ and the degree of the elements in $\langle c_1, c_2 \rangle$ to be $(0, 0)$. One can also regard $L$ as a $Z$-graded Lie algebra by defining the $j \in Z$ degree subspace of $L$ to be

$$L_j = \langle D(jb_1 + mb_2) | m \in Z \rangle \oplus \delta_{j, 0}(c_1, c_2).$$  

Next we recall the definitions of graded $L$-modules. If an $L$-module $V = \oplus_{m \in Z^2} V_m$ satisfies

$$D(m) \cdot V_n \subset V_{m+n}, \quad \forall m, n \in \Gamma,$$

then $V$ is called a $Z^2$-graded $L$-module and $V_m$ is called a homogeneous subspace of $V$ with degree $m \in Z^2$. If $L$-module $V = \oplus_{m \in Z} V_m$ satisfies

$$L_m \cdot V_n \subset V_{m+n}, \quad \forall m, n \in Z,$$

then $V$ is a $Z$-graded $L$-module and $V_m$ is called a homogeneous subspace of $V$ with degree $m \in Z$. If
then $V$ is called a $\mathbb{Z}$-graded $L$-module, and $V_m$ is called a homogeneous subspace of $V$ with degree $m \in \mathbb{Z}$.

A $\mathbb{Z}^2$- or $\mathbb{Z}$-graded module $V$ is called a quasi-finite graded module if all homogeneous subspaces are finite dimensional; a uniformly bounded module if there exists a number $n \in \mathbb{N}$ such that all dimensions of the homogeneous subspaces are $\leq n$; a module of the intermediate series if $n = 1$. Denote the sets of irreducible quasi-finite $\mathbb{Z}^2$- and $\mathbb{Z}$-graded $L$-modules by $\mathcal{O}_{\mathbb{Z}^2}$ and $\mathcal{O}_{\mathbb{Z}}$ respectively. If a $\mathbb{Z}$-graded $L$-module $V$ is generated by a vector $0 \neq v \in V$ with $L_j \cdot v = 0$, $\forall j \in \mathbb{N}$, then $V$ is called a highest weight module. Similarly, we have the notion of a lowest weight module.

Furthermore, if there is a $\mathbb{Z}$-basis $B = \{b'_1, b'_2\}$ of $\Gamma$, that is, $\mathbb{Z}b'_1 + \mathbb{Z}b'_2 = \Gamma$, and a nonzero vector $v \in V_n$ (or $v \in V_n$) such that $D(m)v = 0$, $\forall m \in \mathbb{Z}b'_1 + \mathbb{Z}b'_2$ (recall from definition that $D(m) = 0$ if $m = (0, 0)$), then $v$ is called a generalized highest weight vector corresponding to the $\mathbb{Z}$-basis $B$, and $V$ is called a generalized highest weight module with generalized highest weight vector being of degree $n$ (or $n$) corresponding to $B$.

From the definition, one can easily see that the generalized highest weight modules contain the highest weight modules and the lowest weight modules as their special cases. Since the centers $c_1$, $c_2$ of $L$ must act as scalars on any irreducible graded module, we shall use the same symbols to denote these scalars.

Now we recall the construction and some basic properties of a class of modules over the Heisenberg Lie algebra $\mathcal{H}$. Let $\psi$ be any linear function on $\mathcal{H}$ such that $\psi(h(b_2)) = 0$. We define the associative algebra homomorphism $\overline{\psi} : U(\mathcal{H}) \rightarrow \mathbb{C}[t^{\pm 1}]$ such that

$$\overline{\psi}(D(kb_2)) = \psi(D(kb_2))t^k, \quad \overline{\psi}(h(b_2)) = 0, \quad \forall k \in \mathbb{Z} : = \mathbb{Z} \setminus \{0\},$$

where $U(\mathcal{H})$ is the universal enveloping algebra of $\mathcal{H}$. Denote the image of $\overline{\psi}$ in $\mathbb{C}[t^{\pm 1}]$ by $A_{\overline{\psi}}$. Using $\overline{\psi}$, we can define a $\mathcal{H}$-module structure on the space $A_{\overline{\psi}}t^i$ for a given $i \in \mathbb{Z}$ as follows

$$D(kb_2)t^m = \overline{\psi}(D(kb_2))t^m, \quad h(b_2)t^m = 0, \quad \forall k \in \mathbb{Z}^*, \quad t^m \in A_{\overline{\psi}}t^i.$$

From the definition, we see $A_{\overline{\psi}}t^i \simeq A_{\overline{\psi}}t^j$, $\forall i, j \in \mathbb{Z}$ as $\mathcal{H}$-modules. Now, by Lemma 3.6 and Proposition 3.8 in [20], we have the following results.

**Theorem 2.2** (1) $A_{\overline{\psi}}t^i$ is an irreducible $\mathcal{H}$-module if and only if $A_{\overline{\psi}}t^i = \mathbb{C}t^i$ (denote this module by $A_{\overline{\psi}}t^i$) or there exists $s \in \mathbb{Z}^*$ such that $A_{\overline{\psi}}t^i = \mathbb{C}[t^{\pm s}]t^i$ (denote this module by $A_{\overline{\psi},i,s}$).

(2) If $V$ is an irreducible $\mathbb{Z}$-graded $\mathcal{H}$-module with zero center, then there is a linear function $\psi$ over $\mathcal{H}$ and $i \in \mathbb{Z}$ such that $V \simeq A_{\overline{\psi},i,s}$ or $V \simeq A_{0,i,0}$.

(3) If $V$ is a uniformly bounded $\mathbb{Z}$-graded $\mathcal{H}$-module, then the center $h(b_2)$ acts as zero on $V$.

The following lemma will be used in the next two sections.

**Lemma 2.3** An $L$-module $V$ is a generalized highest weight modules if there is a $\mathbb{Z}$-basis $b'_1, b'_2$ of $\Gamma$ and a homogeneous vector $v \neq 0$ such that $D(b'_1)v = D(b'_2)v = 0$.

**Proof:** By (2.1) and the assumption, we can prove $D(Nb'_1 + Nb'_2)v = 0$. Thus for the $\mathbb{Z}$-basis $m_1 = 3b'_1 + b'_2$, $m_2 = 2b'_1 + b'_2$ of $\Gamma$ we have $D(Z_{i,m_1} + Z_{i,m_2})v = 0$. Hence $V$ is a generalized highest weight module by definition. \qed
§3 The Z-graded modules over the Virasoro-like algebra $L$

**Lemma 3.1** An $L$-module $V$ is a generalized highest weight module or a uniformly bounded module if it is a $Z$-graded module.

**Proof:** Let $V = \oplus_{m \in Z} V_m$. If $V$ is not a generalized highest weight module, then for any $m \in Z$, by considering the following maps

$$D(-mb_1 + b_2) : V_m \to V_0, \quad D((1-m)b_1 + b_2) : V_m \to V_1,$$

we have

$$\text{ker}D(-mb_1 + b_2) \cap \text{ker}D((1-m)b_1 + b_2) = 0,$$

by Lemma 2.3. Therefore $\dim V_m \leq \dim V_0 + \dim V_1$. So $V$ is a uniformly bounded module. \hfill $\Box$

**Lemma 3.2** If $V$ is an irreducible nontrivial generalized highest weight $Z$-graded $L$-module corresponding to the $Z$-basis $B = \{b'_1, b'_2\}$ then

1. For any $v \in V$ there is some $p \in \mathbb{N}$ such that $D(m_1b'_1 + m_2b'_2)v = 0$ for all $m_1, m_2 \geq p$.
2. For any $0 \neq v \in V$ and $m_1, m_2 > 0$, we have $D(-m_1b'_1 - m_2b'_2)v \neq 0$.

**Proof:** We may assume that $v_0$ is a generalized highest weight vector corresponding to a $Z$-basis $B = \{b'_1, b'_2\}$.

1. By the irreducibility of $V$ and the PBW Theorem, there exists $u \in U(L)$ such that $v = u \cdot v_0$, where $u$ is a linear combination of elements of the form

$$u_n = D(i_1 b'_1 + j_1 b'_2) \cdots D(i_n b'_1 + j_n b'_2).$$

Thus without loss of generality, we may assume $u = u_n$. Take

$$p_1 = -\sum_{i_s < 0} i_s + 1, \quad p_2 = -\sum_{j_s < 0} j_s + 1.$$

By induction on $n$ one gets that $D(ib'_1 + jb'_2) \cdot v = 0$ for $i \geq p_1$ and $j \geq p_2$, which gives the result with $p = \max\{p_1, p_2\}$.

2. If there are $0 \neq v \in V$ and $m_1, m_2 > 0$ with $D(-m_1b'_1 - m_2b'_2)v = 0$, let $p$ be as in the proof of (1), then we see

$$D(-m_1b'_1 - m_2b'_2), \quad D(b'_1 + p(m_1b'_1 + m_2b'_2)), \quad D(b'_2 + p(m_1b'_1 + m_2b'_2)),$$

act trivially on $v$. By (2.1), the above elements generate the Lie algebra $L$. So $V$ is a trivial module, a contradiction with the assumption. \hfill $\Box$

**Lemma 3.3** An $L$-module $V \in \mathcal{O}_Z$ is a highest weight module or a lowest weight module if it is a generalized highest weight module corresponding to a $Z$-basis $B = \{b'_1, b'_2\}$.

**Proof:** For convenience, we suppose $V$ is a generalized highest weight module with generalized highest weight vector being of degree 0, corresponding to the $Z$-basis $b'_1 = b'_{11}b_1 + b'_{12}b_2$, $b'_2 = b'_{21}b_1 + b'_{22}b_2$. Let $a = b'_{11} + b'_{21}$ and denote

$$\varphi(V) = \{m \in Z \mid V_m \neq 0\}.$$ 

If necessary, by replacing $b'_1, b'_2$ by $b''_1 = 3b'_1 + b'_2$, $b''_2 = 2b'_1 + b'_2$, we can suppose $a \neq 0$. 


First we prove that if \( a > 0 \) then \( V \) is a highest weight module. Let

\[
A_i = \{ j \in \mathbb{Z} \mid i + aj \in \varphi(V) \}, \quad \forall 0 \leq i < a,
\]

then there is \( m_i \in \mathbb{Z} \) such that \( A_i = \{ j \in \mathbb{Z} \mid j \leq m_i \} \) or \( A_i = \mathbb{Z} \) by Lemma 3.2(2).

Set \( b = b_1' + b_2' \). Now we prove \( A_i \neq \mathbb{Z} \) for all \( 0 \leq i < a \). Otherwise, we may assume \( A_0 = \mathbb{Z} \). Thus we can choose \( 0 \neq v_j \in V_{a_j} \) for any \( j \in \mathbb{Z} \). By Lemma 3.2(1), we know that there is \( p_{v_j} > 0 \) with

\[
D(s_1b_1' + s_2b_2') \cdot v_j = 0, \quad \forall s_1, s_2 > p_{v_j},
\]

Choose \( \{ k_j \in \mathbb{N} \mid j \in \mathbb{N} \} \) and \( v_{k_j} \in V_{ak_j} \) such that

\[
k_{j+1} > k_j + p_{v_{k_j}} + 2.
\]

We prove that \( \{ D(-k_jb) \cdot v_{k_j} \mid j \in \mathbb{N} \} \subset V_0 \) is a set of linearly independent vectors, from this we will get a contradiction which gives the result as required. Indeed, for any \( r \in \mathbb{N} \), there exists \( a_r \in \mathbb{N} \) such that

\[
D(xb + b_1')v_{k_r} = 0, \quad \forall x \geq a_r \text{ by Lemma 3.2(1)}.\]

On the other hand, we have

\[
D(xb + b_1') \cdot v_{k_r} \neq 0 \text{ for any } x < -1 \text{ by Lemma 3.2(2)}.\]

Thus we can choose \( s_r \geq -2 \) such that

\[
D(s_rb + b_1') \cdot v_{k_r} \neq 0, \quad D(xb + b_1') \cdot v_{k_r} = 0, \quad \forall x > s_r.
\]

By (3.2) we have \( k_r + s_r - k_j > p_{v_{k_j}} \) for all \( 1 \leq j < r \). Hence by (3.1) we know that for all \( 1 \leq j < r \),

\[
D((k_r + s_r)b + b_1') \cdot D(-k_jb)v_{k_j} = [D((k_r + s_r)b + b_1'), D(-k_jb)]v_{k_j} = k_j \det(b_1', b_2')D((k_r + s_r - k_j)b + b_1') \cdot v_{k_j} = 0.
\]

Now by (3.2) and (3.3), one gets

\[
D((k_r + s_r)b + b_1') \cdot D(-k_r b)v_{k_r} = [D((k_r + s_r)b + b_1'), D(-k_r b)]v_{k_r} = k_r \det(b_1', b_2')D(s_r b + b_1') \cdot v_{k_r} \neq 0.
\]

Hence if \( \sum_{j=1}^{r} \lambda_j D(-k_jb) \cdot v_{k_j} = 0 \) then \( \lambda_r = \lambda_{r-1} = \cdots = \lambda_1 = 0 \) by the arbitrariness of \( r \). So we see \( \{ D(-k_jb) \cdot v_{k_j} \mid j \in \mathbb{N} \} \subset V_0 \) is a set of linearly independent vectors which contradicts the fact that \( V \in \mathcal{O}_\mathbb{Z} \). Therefore, for any \( 0 \leq i < a \), there is \( m_i \in \mathbb{Z} \) such that \( A_i = \{ j \in \mathbb{Z} \mid j \leq m_i \} \) which implies that \( V \) is a highest weight module since \( \varphi(V) = \bigcup_{i=0}^{a-1} A_i \).

Similarly, one can prove that if \( a < 0 \) then \( V \) is a lowest weight module. \( \square \)

In order to complete the classification of the highest and lowest weight \( \mathbb{Z} \)-graded \( L \)-module, we first give a triangular decomposition of \( L \) and construct a class of highest (lowest) weight \( \mathbb{Z} \)-graded irreducible modules which are similar to that described in [18]. Let

\[
L_+ = \bigoplus_{j>0} L_j, \quad L_- = \bigoplus_{j<0} L_j,
\]

where \( L_j = \langle D(jb_1 + mb_2) \mid m \in \mathbb{Z} \rangle \oplus \delta_{j,0}\langle c_1, c_2 \rangle \), then \( L \) has the following triangular decomposition

\[
L = L_+ \oplus L_0 \oplus L_-.
\]
For any linear function $\psi$ over $L_0$ with $\psi(h(b_2)) = 0$, we define a one dimensional $(L_0 + L_+)$-module $Cv_0$ as follows

$$L_jv_0 = 0, \quad xv_0 = \psi(x)v_0, \quad \forall \; j > 0, \; x \in L_0.$$ 

Then we get an induced $L$-module

$$\nabla^+(\psi) = \text{Ind}_{L_0 + L_+}^L Cv_0 = U(L) \otimes_{U(L_0 + L_+)} Cv_0,$$

where $U(L)$ is the universal enveloping algebra of $L$. Set the degree of $v_0$ to be 0 then $\nabla^+(\psi)$ becomes a $\mathbb{Z}$-graded module. It is obvious that $\nabla^+(\psi) \cong U(L_-)$ as vector spaces and $\nabla^+(\psi)$ has an unique maximal proper submodule $J$. Then we obtain an irreducible $\mathbb{Z}$-graded highest weight $L$-module

$$\hat{\nabla}^+(\psi) = \nabla^+(\psi)/J.$$

Similarly, we can define an irreducible lowest weight $\mathbb{Z}$-graded $L$-module $\hat{\nabla}^-(\psi)$ for any linear function $\psi$ over $L_0$ with $\psi(h(b_2)) = 0$.

For convenience, we introduce the following definition, which is similar to a definition in [17].

**Definition 3.4** If $\psi$ is a nonzero linear function over $L_0$ with $\psi(h(b_2)) = 0$ and there exist $b_{10}, b_{11}, \ldots, b_{1s_1}, \ldots, b_{r0}, \ldots, b_{rs_r} \in C, \; \alpha_1, \ldots, \alpha_r \in C \setminus \{0\}$ such that

$$\psi(D(kb_2)) = \frac{(b_{10} + b_{11}k + \cdots + b_{1s_1}k^{s_1})\alpha_k^1 + \cdots + (b_{r0} + b_{r1}k + \cdots + b_{rs_r}k^{s_r})\alpha_k^r}{k}, \quad \forall \; k \in \mathbb{Z}^*, \quad \alpha_0 = 1;$$

$$\psi(h(b_1)) = -\det(b_1, b_2)(b_{10} + b_{20} + \cdots + b_{r0}),$$

then $\psi$ is called an exp-polynomial function over $L_0$.

**Remark 3.5** Set $f_k = \psi(kD(kb_2))$, \; $f_0 = -\det(b_1, b_2)\psi(h(b_1))$. Then by a well-known combinatorial formula we see that $\psi \neq 0$ is an exp-polynomial function over $L_0$ if and only if there are $a_0, \ldots, a_n \in C$, with $a_0a_n \neq 0$ such that

$$\sum_{i=0}^n a_if_{k+i} = 0, \; \psi(h(b_2)) = 0, \; \forall \; k \in \mathbb{Z}.$$

By using a technique in [18], we prove the following theorem.

**Theorem 3.6** The nontrivial $\mathbb{Z}$-graded $L$-module $\hat{\nabla}^+(\psi)$ is in $\mathcal{O}_\mathbb{Z}$ (or $\hat{\nabla}^-(\psi)$ is in $\mathcal{O}_\mathbb{Z}$) if and only if $\psi$ is an exp-polynomial function over $L_0$.

**Proof:** For convenience, we introduce a linear map $\Psi : C[t_1^{\pm 1}, t_2^{\pm 1}] \to L$ by defining

$$\Psi(t_1^{m_1}t_2^{m_2}) = D(m_1b_1 + m_2mb_2), \quad \forall \; m_1, m_2 \in \mathbb{Z}.$$ 

By Remark 3.5, it is sufficient for us to prove the following claim.

**Claim 1.** $\hat{\nabla}^+(\psi) \in \mathcal{O}_\mathbb{Z}$ if and only if there exists a nonzero polynomial $P(t_2) = \sum_{i=0}^n a_it_2^i \in C[t_2]$ with $a_0a_n \neq 0$ such that

$$\psi(\Psi(d_2(t_2^kP(t_2)))) - a_k\det(b_1, b_2)h(b_1)) = 0, \quad \forall k \in \mathbb{Z},$$

where $a_k = 0$ if $k \not\in \{0, 1, \ldots, n\}$. 

7
“⇒” Since \( \dim V_1 < \infty \) and \( \Psi(t_1^{-1} t_2^k) \cdot v_0 \in V_1 \) for all \( i \in \mathbb{Z} \), there exists a \( k \in \mathbb{Z} \) and a nonzero polynomial \( P(t_2) = \sum_{i=0}^{n} a_i t_2^i \in \mathbb{C}[t_2] \) with \( a_0a_n \neq 0 \) such that
\[
\Psi(t_1^{-1} t_2^k P(t_2)) \cdot v_0 = 0.
\]
Applying \( \Psi(t_1 t_2^s) \) for any \( s \in \mathbb{Z} \) to the above equality, we get
\[
0 = \Psi(t_1 t_2^s) \cdot \Psi(t_1^{-1} t_2^k P(t_2)) \cdot v_0 = [D(b_1 + s b_2), \sum_{i=0}^{n} a_i D(-b_1 + (k + i)b_2)] \cdot v_0 \\
= \left( -\sum_{i=0}^{n} a_i \det(b_1, b_2)(s + i + k)D((s + i + k)b_2) + a_{-s-k} h(b_1) \right) \cdot v_0 \\
= \left( \Psi(-\det(b_1, b_2)d_2(t_2^{s+k} P(t_2))) + a_{-s-k} h(b_1) \right) \cdot v_0,
\]
which deduces the result as we hope.

“⇐” Since \( L_+ \) is generated by \( L_{-1} \), and \( L_+ \) is generated by \( L_1 \), one sees that
\[
L_{-1} \cdot V_{-i} = V_{-i-1}, \quad \forall i \in \mathbb{Z}_+,
\]
and \( v = 0 \) if \( L_1 \cdot v = 0 \).

Next, we will show the following claim by induction on \( l \).

**Claim 2.** For any \( l \in \mathbb{Z}_+ \), there exists a nonzero polynomial \( P_l(t_2) = \sum_{i \in \mathbb{Z}} a_i^{(l)} t_2^i \in \mathbb{C}[t_2] \) such that
\[
\left( \Psi(d_2(t_2^k P_l(t_2))) - a_{-k} \det(b_1, b_2) h(b_1) \right) \cdot V_{-l} = 0,
\]
\[
\Psi(t_1^{-1} t_2^k P_l(t_2)) \cdot V_{-l} = 0, \quad \forall k \in \mathbb{Z}.
\]
By the assumption, (3.4) holds for \( l = 0 \) with \( P_0(t_2) = P(t_2) \). On the other hand, we have
\[
\Psi(t_1 t_2^s) \cdot \Psi(t_1^{-1} t_2^k P(t_2)) \cdot V_0 = \left( \Psi(-\det(b_1, b_2)d_2(t_2^{s+k} P(t_2))) + a_{-s-k} h(b_1) \right) \cdot V_0 = 0, \quad \forall s, k \in \mathbb{Z},
\]
which deduces that (3.5) also holds for \( l = 0 \) with \( P_0(t_2) = P(t_2) \). Thus the claim holds for \( s = 0 \).

Suppose the claim holds for \( l \). From (3.4) and (3.5), we have that
\[
\left( \Psi(d_2(Q(t_2))) - a_Q \det(b_1, b_2) h(b_1) \right) \cdot V_{-l} = 0,
\]
\[
\Psi(t_1^{-1} Q(t_2)) \cdot V_{-l} = 0,
\]
for any \( Q(t_2) \in \mathbb{C}[t^{\pm 1}] \) with \( P_l(t_2) | Q(t_2) \), where \( a_Q \) is the constant in \( Q(t_2) \).

Now let us consider the claim for \( l + 1 \). Let \( P_{l+1}(t_2) = P_l(t_2)^3 = \sum_{i \in \mathbb{Z}} a_i^{(l+1)} t_2^i \). Then \( P_l(t_2) | d_2(t_2^k P_{l+1}(t_2)) \) and \( P_l(t_2) | d_2 \cdot d_2(t_2^k P_{l+1}(t_2)) \) for any \( k \in \mathbb{Z} \). By using (3.6) and (3.7), for any \( s, k \in \mathbb{Z} \), we have that
\[
\left( \Psi(d_2(t_2^k P_{l+1}(t_2))) - a_{-k}^{(l+1)} \det(b_1, b_2) h(b_1) \right) \cdot \Psi(t_1^{-1} t_2^s) \cdot V_{-l} \\
= \left[ \Psi(d_2(t_2^k P_{l+1}(t_2))), \Psi(t_1^{-1} t_2^s) \right] \cdot V_{-l} = -\det(b_1, b_2) \Psi(t_1^{-1} t_2^s (d_2 \cdot d_2 (t_2^k P_{l+1}(t_2)))) \cdot V_{-l} = 0.
\]
Thus (3.4) holds for $l+1$. From the above equality and (3.7), we deduce that, for any $m, k, s \in \mathbb{Z}$,
\[
\Psi(t_{11}^{m}) \Psi(t_{1}^{-1} t_{2}^{P_{l+1}}(t_{2})) \Psi(t_{2}^{-1} t_{2}^{P_{l+1}}(t_{2})) \Psi(t_{3}^{-1} t_{2}^{P_{l+1}}(t_{2})).
\]
\[
= [\Psi(t_{11}^{m}), \Psi(t_{1}^{-1} t_{2}^{P_{l+1}}(t_{2})), \Psi(t_{2}^{-1} t_{2}^{P_{l+1}}(t_{2})), \Psi(t_{3}^{-1} t_{2}^{P_{l+1}}(t_{2})).] \Psi(t_{11}^{m}) \Psi(t_{1}^{-1} t_{2}^{P_{l+1}}(t_{2})).
\]
\[
= (\Psi(-\det(b_{1}, b_{2}) d_{2}(t_{2}^{m+k} P_{l+1}(t_{2}))) + a_{m-k}^{l+1} h(b_{1})) \Psi(t_{11}^{m}) \Psi(t_{1}^{-1} t_{2}^{P_{l+1}}(t_{2})).
\]
\[
= 0.
\]

So we have (3.4) also holds for $l+1$. This completes the proof of Claim 2.

From Claim 2, we deduce that
\[
\dim V_{l-1} \leq \deg P_{l+1}(t_{2}) \cdot \dim V_{l-1}, \quad \forall l \in \mathbb{Z}_{+}.
\]

Thus $\hat{V}^{+}(\psi) \in \mathcal{O}_{\mathbb{Z}}$. This completes the proof of our theorem. \hfill \Box

**Theorem 3.7** If a $\mathbb{Z}$-graded $L$-module $V$ is in $\mathcal{O}_{\mathbb{Z}}$ then one and only one of the following cases holds.

1. $V$ is a uniformly bounded module.
2. There exists an exp-polynomial function $\psi$ over $L_{0}$ such that $V \simeq \hat{V}^{+}(\psi)$.
3. There exists an exp-polynomial function $\psi$ over $L_{0}$ such that $V \simeq \hat{V}^{-}(\psi)$.

**Proof:** If $V$ is not a uniformly bounded module then by Lemmas 3.1 and 3.3 we obtain that $V$ is a highest weight module or a lowest weight module. By the property of a Verma module we know that there is a nonzero linear function $\psi$ over $L_{0}$ such that $V \simeq \hat{V}^{+}(\psi)$ or $V \simeq \hat{V}^{-}(\psi)$. Now by Theorem 3.6, case (2) or case (3) must hold.

Now we prove that only one of the above three cases holds. It is obvious that only one of case (2) and case (3) holds. If case (2) holds then there is $k \in \mathbb{Z}^{*}$ with $\psi(D(kb_{2})) \neq 0$ which implies that $D(kb_{2}) v_{0} \neq 0$. Similar to the proof of part (2) in Lemma 3.2, one can easily deduce that $D(-b_{1}) v_{k} \neq 0$ for any nonzero homogeneous vector $v_{k} \in V_{k}$ of $V$. (In fact, if $D(-b_{1}) v_{k} = 0$ then $D((-k+1)b_{1} + b_{2}), D((-k+1)b_{1} - b_{2})$ and $D(-b_{1})$ act trivially on $v_{k}$ by the construction of $\hat{V}^{+}(\psi)$. Thus $V$ is a trivial module by the irreducibility of $V$ since from (2.1) one can easily check that the Lie algebra $L$ is generated by the above three elements. Hence we reach a contradiction.) Meanwhile
\[
\mathcal{B} = \{ D(-b_{1})^{j} D((-n + j)b_{1} + kb_{2}) v_{0} \mid 0 \leq j < n \} \subset \hat{V}^{+}(\psi)_{-n}, \forall n \in \mathbb{N}.
\]

Next we prove that $\mathcal{B}$ is a set of linear independent vectors in $\hat{V}^{+}(\psi)_{-n}$. Indeed, if
\[
\sum_{j=0}^{n-1} \lambda_{j} D(-b_{1})^{j} D((-n + j)b_{1} + kb_{2}) v_{0} = 0,
\]
then for any $0 \leq i < n - 1$ we have
\[
0 = D((-n - i)b_{1}) \sum_{j=0}^{n-1} \lambda_{j} D(-b_{1})^{j} D((-n + j)b_{1} + kb_{2}) v_{0}
\]
\[
= - \sum_{j=0}^{i} \lambda_{j} k(n - i) \det(b_{1}, b_{2}) D(-b_{1})^{j} D((-i)b_{1} + kb_{2}) v_{0},
\]
which implies $\lambda_{0} = \cdots = \lambda_{n-1} = 0$. Hence $\mathcal{B}$ is a set of linear independent vectors in $\hat{V}^{+}(\psi)_{-n}$ and thus
\[
\dim(\hat{V}^{+}(\psi)_{-n}) \geq n.
\]
Therefore $\hat{V}^+(\psi)$ is not a uniformly bounded module by the arbitrariness of $n$. So only one of case (2) and (1) holds.

Similarly we can prove that only one of case (1) and case (3) holds. \qed

\textbf{Remark 3.8} In Section 5, we will prove that there does not exist a nontrivial $\mathbb{Z}$-graded $L$-module of the intermediate series. We conjecture that there does not exist any nontrivial uniformly bounded $\mathbb{Z}$-graded irreducible $L$-module.

From Theorem 3.7, we can recover Theorem 3.2 in [18] below.

\textbf{Corollary 3.9} If a center of $L$ acts as a nonzero scalar on the $\mathbb{Z}$-graded module $V$, then $V \in \mathcal{O}_{\mathbb{Z}}$ if and only if there is an exp-polynomial function $\psi$ over $L_0$ such that $V \simeq \hat{V}^+(\psi)$ or $V \simeq \hat{V}^-(\psi)$.

\textbf{Proof:} Set $V = \oplus_{n \in \mathbb{Z}} V_n$. We have $h(b_2) = 0$ by considering $V_0$ as a finite dimensional $L_0$-module. Thus $h(b_1) \neq 0$ since a center of $L$ acts as a nonzero scalar on $V$ and $\{b_1, b_2\}$ is a $\mathbb{Z}$-basis of $\Gamma$. Therefore, by Theorem 2.2(3), the module $V$ is not a uniformly bounded module since the Lie subalgebra $\langle D(mb_1), h(b_1) \mid m \in \mathbb{Z} \rangle$ of $L$ is isomorphic to $\mathcal{H}$ and its center $h(b_1)$ acts as a nonzero scalar on $V$. Thus we obtain the result as we wish by Theorem 3.7. \qed

\section{The $\mathbb{Z}^2$-graded modules over the Virasoro-like algebra $L$}

We first construct a class of irreducible generalized highest weight $\mathbb{Z}^2$-graded $L$-modules by using the $\mathbb{Z}$-graded $L$-module $\hat{V}^+(\psi)$. For any linear function $\psi$ over $L_0$ with $\psi(h(b_2)) = 0$, we set $V(\psi) = \hat{V}^+(\psi) \otimes \mathbb{C}[t^{\pm 1}]$, and define the actions of the elements of $L$ on $V(\psi)$ as follows

$$D(id_1 + jb_2)(v \otimes t^k) = (D(id_1 + jb_2)v) \otimes t^{k+j}, \quad (4.1)$$

where $v \in \hat{V}^+(\psi)$, $i,j,k \in \mathbb{Z}$. For any homogeneous element $v$ with degree $m$ in $\mathbb{Z}$-graded module $\hat{V}^+(\psi)$, define the degree of $v \otimes t^k$ to be $mb_1 + kb_2$. Then one can easily see that $V(\psi)$ becomes a $\mathbb{Z}^2$-graded $L$-module. Denote $U(\mathcal{H})(v_0 \otimes t^t)$ by $W_i$, then $W_i \simeq A_{\psi,i,s}$ as $\mathcal{H}$-modules. If $W_i$ is an irreducible $\mathcal{H}$-module then $W_i \simeq A_{\psi,i,s}$ or $W_i \simeq A_{0,i,0}$ by Theorem 2.2. Thus by the construction of $L$-module $V(\psi)$ we know that there exists a unique maximal proper submodule either $W_{\psi,i,s}$ or $W_{0,i,0}$ which intersects trivially with $W_i$. Then we have the irreducible $\mathbb{Z}^2$-graded $L$-module either

$$\nabla(\psi, i, s) = V(\psi)/W_{\psi,i,s}, \quad \text{or} \quad \nabla(0, i, 0) = V(\psi)/W_{0,i,0}. \quad (4.2)$$

One can easily check that there exists $s \in \mathbb{N}$ such that

$$W_i = v_0 \otimes \mathbb{C}[t^{\pm s}] \cdot t^t \quad \text{if} \ \psi \text{ is an exp-polynomial function over } L_0.$$ 

Hence by Theorem 2.2 we know $W_i$ is an irreducible $\mathcal{H}$-module in this instance. Now by Theorem 3.6 and the construction of $\nabla(\psi, i, s)$ we obtain the following result.

\textbf{Lemma 4.1} If $\psi$ is an exp-polynomial function over $L_0$ then

(1) $\nabla(\psi, i, s) \in \mathcal{O}_{\mathbb{Z}^2}$; \quad (2) $V(\psi) \simeq \nabla(\psi, i, s)$ if and only if $s = 1$.
Remark 4.2 \( V(\psi) \) is not an irreducible \( L \)-module in general. For example, we can define a linear function \( \psi \) over \( L_0 \) such that
\[
\psi(D(kb_2)) = \frac{(-1)^k + 1}{k}, \quad \forall k \in \mathbb{Z}^*, \quad \psi(h(b_1)) = -2 \det(b_1, b_2), \quad \psi(h(b_2)) = 0.
\]
Thus one can easily see \( \psi \) is an exp-polynomial function over \( L_0 \). On the other hand, by the PBW Theorem and the construction of \( L \)-module \( V(\psi) \), we can deduce
\[
W_0 = v_0 \otimes \mathbb{C}[t^{\pm 2}], \quad (U(L).(v_0 \otimes t)) \cap W_0 = 0.
\]
Thus \( U(L) \cdot (v_0 \otimes t) \) is a nonzero proper submodule of \( V(\psi) \) which implies that \( V(\psi) \) is not irreducible.

Lemma 4.3 A \( \mathbb{Z}^2 \)-graded \( L \)-module is a generalized highest weight module or a uniformly bounded module.

Proof: Suppose \( V = \bigoplus_{m \in \mathbb{Z}^2} V_m \). If \( V \) is not a generalized highest weight module, then for any \((m_1, m_2) \in \mathbb{Z}^2\), by considering the linear maps
\[
D(-m_1 e_1 + e_2) : V_{(m_1, m_2)} \rightarrow V_{(0, m_2 + 1)},
\]
\[
D((1 - m_1)e_1 + e_2) : V_{(m_1, m_2)} \rightarrow V_{(1, m_2 + 1)},
\]
we obtain by Lemma 2.3
\[
\ker D(-m_1 e_1 + e_2) \cap \ker D((1 - m_1)e_1 + e_2) = 0.
\]
Thus
\[
\dim V_{(m_1, m_2)} \leq \dim V_{(0, m_2 + 1)} + \dim V_{(1, m_2 + 1)}.
\]

Now consider the following linear maps
\[
D(-e_1 - m_2 e_2) : V_{(0, m_2 + 1)} \rightarrow V_{(-1, 1)},
\]
\[
D(-e_1 + (1 - m_2)e_2) : V_{(0, m_2 + 1)} \rightarrow V_{(-1, 2)}.
\]

By Lemma 2.3 we have
\[
\ker D(-e_1 - m_2 e_2) \cap \ker D(-e_1 + (1 - m_2)e_2) = 0.
\]
Thus \( \dim V_{(0, m_2 + 1)} \leq \dim V_{(-1, 1)} + \dim V_{(-1, 2)} \).

Similarly we can deduce
\[
\dim V_{(1, m_2 + 1)} \leq \dim V_{(0, 1)} + \dim V_{(0, 2)}.
\]

Therefore \( V \) is a uniformly bounded module. \( \square \)

Now we recall the following proposition from [16].

Theorem 4.4 ([16] Proposition 3.9) If \( V \) is a generalized highest weight \( \mathbb{Z}^2 \)-graded \( L \)-module and \( V \in O_{\mathbb{Z}^2} \), then there exists a \( \mathbb{Z} \)-basis \( B = \{b_1, b_2\} \) of \( \Gamma \) and a linear function \( \psi \) over \( L_0 = \langle mb_2 \rangle | m \in \mathbb{Z} \) \( \oplus \langle c_1, c_2 \rangle \) with \( \psi(h(b_2)) = 0 \) such that \( V \) is isomorphic to \( V(\psi, i, s) \) or \( V(0, i, 0) \), where \( i \in \mathbb{Z}, s \in \mathbb{N} \).
Lemma 4.5  

(1) If $\psi(h(b_1)) = 0$ then $\nabla(0, i, 0)$ is a trivial module.

(2) If $\psi(h(b_1)) \neq 0$ then $\nabla(0, i, 0) \not\in \mathcal{O}_{\mathbb{Z}^2}$.

Proof:  
(1) By the construction of $\nabla(0, i, 0)$, one can easily deduce the result.

(2) By definition we have

$$\psi(D(mb_2)) = \psi(h(b_2)) = 0, \forall m \in \mathbb{Z},$$

and

$$\mathcal{B}_n = \{D(-b_1 + mb_2)D(-b_1 - mb_2) \cdot (v_0 \otimes t^i) \mid 1 \leq m \leq n \} \subset \nabla_{-2b_1 + ib_2}, \forall n \in \mathbb{N}.$$ 

Since

$$D(b_1 + mb_2)D(-b_1 - mb_2) \cdot (v_0 \otimes t^i) = h(b_1 + mb_2) \cdot (v_0 \otimes t^i) \neq 0, \forall m \in \mathbb{Z},$$

we have

$$D(-b_1 - mb_2) \cdot (v_0 \otimes t^i) \neq 0.$$

Now we prove that $\mathcal{B}_n$ is a set of linear independent elements in $\nabla_{-2b_1 + ib_2}$. If there are $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ such that

$$\sum_{j=1}^{n} \lambda_j D(-b_1 + j b_2)D(-b_1 - j b_2) \cdot (v_0 \otimes t^i) = 0,$$

then for any $1 \leq k \leq n$ we have

$$0 = D(b_1 - kb_2) \sum_{j=1}^{n} \lambda_j D(-b_1 + j b_2)D(-b_1 - j b_2) \cdot (v_0 \otimes t^i)$$

$$= \lambda_k \psi(h(b_1))D(-b_1 - kb_2) \cdot (v_0 \otimes t^i) + \sum_{j=1}^{n} \lambda_j (j-k)^2(\det(b_1, b_2))D(-b_1 - kb_2) \cdot (v_0 \otimes t^i)$$

$$= \lambda_k \psi(h(b_1))D(-b_1 - kb_2) \cdot (v_0 \otimes t^i) + \sum_{j=1}^{n} \lambda_j (j-k)^2 D(-b_1 - kb_2) \cdot (v_0 \otimes t^i).$$

On the other hand, for any $k > n$, we have

$$0 = D(b_1 - kb_2) \sum_{j=1}^{n} \lambda_j D(-b_1 + j b_2)D(-b_1 - j b_2) \cdot (v_0 \otimes t^i)$$

$$= \sum_{j=1}^{n} \lambda_j (j-k)^2 D(-b_1 - kb_2) \cdot (v_0 \otimes t^i).$$

Thus

$$\lambda_k \psi(h(b_1)) + \sum_{j=1}^{n} \lambda_j (j-k)^2 = 0, \forall 1 \leq k \leq n; \quad (4.3)$$

$$\sum_{j=1}^{n} \lambda_j (j-k)^2 = 0, \forall k > n. \quad (4.4)$$

Hence (4.4) implies that the polynomial $\sum_{j=1}^{n} \lambda_j (j - x)^2 = 0$ has infinite many roots, and therefore $\sum_{j=1}^{n} \lambda_j (j-x)^2$ is a zero polynomial. This, together with (4.3), gives $\lambda_k = 0, \forall 1 \leq k \leq n$. Hence $\mathcal{B}_n$ is a set of linear independent vectors in $\nabla_{-2b_1 + ib_2}$ which implies $\dim \nabla_{-2b_1 + ib_1} \geq n, \forall n \in \mathbb{N}$. Thus $\nabla(0, i, 0) \not\in \mathcal{O}_{\mathbb{Z}^2}$.  

\[ \square \]
Theorem 4.6  If $V$ is a nontrivial generalized highest weight $\mathbb{Z}^2$-graded $L$-module, then $V \in \mathcal{O}_{\mathbb{Z}^2}$ if and only if there exists a $\mathbb{Z}$-basis $B = \{b_1, b_2\}$ of $\Gamma$ and an exp-polynomial function $\psi$ over $L_0 = \langle mb_2 \rangle$ such that $V$ is isomorphism to $\overline{V}(\psi, i, s)$, where $i \in \mathbb{Z}$, $s \in \mathbb{N}$.

**Proof:** The sufficiency follows directly from part (1) of Lemma 4.1. Now we prove the necessity. By Theorem 4.4 and Lemma 4.5 we know there exist a $\mathbb{Z}$-basis $B = \{b_1, b_2\}$ of $\Gamma$ and a linear function $\psi$ over $L_0$ with $\psi(h(b_2)) = 0$ such that $V$ is isomorphism to $\overline{V}(\psi, i, s)$, where $i \in \mathbb{Z}$, $s \in \mathbb{N}$.

If $s = 1$ then by Lemma 4.1(2) we have $\overline{V}(\psi, i, s) \cong V(\psi)$. Thus by the construction of $V(\psi)$ we see that the dimensions of the homogeneous subspaces $\overline{V}(\psi, i, s)m_{b_1+b_2}$ of $\overline{V}(\psi, i, s)$ equal to those of the corresponding homogeneous subspaces $\overline{V}^+(\psi)_m$ of $\overline{V}^+(\psi)$. Then the result for the case $s = 1$ follows directly from Theorem 3.6.

If $s \neq 1$ then $\sum_{i=1}^s \overline{V}(\psi, i, s)$ is a quasi-finite $\mathbb{Z}^2$-graded module. Moreover, since

$$v_j = D(-b_1 + jb_2) \cdot (v_0 \otimes t^{-j}) \in \left(\sum_{i=1}^s \overline{V}(\psi, i, s)\right)_{-b_1}, \quad \forall j \in \mathbb{Z},$$

there exist an integer $k \in \mathbb{Z}$, and numbers $a_0, \cdots, a_n \in \mathbb{C}$ with $a_0a_n \neq 0$ such that $\sum_{j=0}^n a_jv_{k+j} = 0$. For convenience, we set $a_i = 0$ for $i < 0$ and $i > n$. Then for any $s \in \mathbb{Z}$ we have

$$0 = D(b_1 - sb_2) \sum_{j=0}^n a_jv_{k+j}$$

$$= \sum_{j=0}^n a_j(s-(k+j))\text{det}(b_1, b_2)\psi(D((k+j-s)b_2))(v_0 \otimes t^{-s}) + a_{s-k}\psi(h(b_1))(v_0 \otimes t^{-s}).$$

Thus

$$\sum_{j=0}^n a_j\psi((k+j)D((k+j)b_2)) - a_{s-k}\text{det}(b_1, b_2)\psi(h(b_1)) = 0, \quad \forall k \in \mathbb{Z}.$$

Denote

$$f_k = \psi(kD(b_2)), \quad \forall 0 \neq k \in \mathbb{Z}, \quad f_0 = -\text{det}(b_1, b_2)\psi(h(b_1)),$$

then

$$\sum_{j=0}^n a_jf_{k+j} = 0, \quad \forall k \in \mathbb{Z}.$$

Therefore by Remark 3.5 we see $\psi$ is an exp-polynomial function over $L_0$. \hfill $\Box$

By similar arguments as in the proof of Theorem 3.7 and Corollary 3.9, one can prove the following two results by applying Lemma 4.3 and Theorems 4.6 and 2.2.

**Theorem 4.7**  If $\mathbb{Z}^2$-graded $L$-module $V \in \mathcal{O}_{\mathbb{Z}^2}$, then one and only one of the following cases holds.

1. $V$ is a uniformly bounded module.
2. There exist a $\mathbb{Z}$-basis $B = \{b_1, b_2\}$ of $\Gamma$ and an exp-polynomial function $\psi$ over $L_0 = \langle mb_2 \rangle$ such that $V$ is isomorphism to $\overline{V}(\psi, i, s)$, where $i \in \mathbb{Z}$, $s \in \mathbb{N}$.

**Corollary 4.8**  If $V$ is a $\mathbb{Z}^2$-graded $L$-module with nonzero centers, then $V \in \mathcal{O}_{\mathbb{Z}^2}$ if and only if there exist a $\mathbb{Z}$-basis $B = \{b_1, b_2\}$ of $\Gamma$ and an exp-polynomial function $\psi$ over $L_0 = \langle mb_2 \rangle$ such that $V$ is isomorphism to $\overline{V}(\psi, i, s)$, where $i \in \mathbb{Z}$, $s \in \mathbb{N}$.

We would like to conclude this section by recalling some results from [15] which show that there are many uniformly bounded irreducible $\mathbb{Z}^2$-graded $L$-modules. A question one may ask
is: whether or not the modules constructed in the following exhaust all the uniformly bounded irreducible $\mathbb{Z}^2$-graded $L$-modules.

Let $x_+, x_-, h$ be the Chevalley basis of the simple Lie algebra $sl_2$, that is

$$[h, x_+] = 2x_+, \quad [h, x_-] = -2x_-, \quad [x_+, x_-] = h.$$ 

For any irreducible $sl_2$-module $V$ and $\alpha_1, \alpha_2 \in \mathbb{C}$, let $V(A) = V \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}]$, we define the action of $L$ on $V(A)$ as follows

$$D(m_1 e_1 + m_2 b_2) \cdot (v \otimes t_1^{n_1} t_2^{n_2}) = (m_2 (\alpha_1 + n_1) - m_1 (\alpha_2 + n_2)) v \otimes t_1^{n_1 + m_1} t_2^{n_2 + m_2}$$

$$+ ((m_2^2 x_- - m_1^2 x_+ m_2 h) \cdot v) \otimes t_1^{n_1 + m_1} t_2^{n_2 + m_2},$$

$$c_1(v \otimes t_1^{n_1} t_2^{n_2}) = c_2(v \otimes t_1^{n_1} t_2^{n_2}) = 0, \quad \forall v \in V.$$ 

Then one can easily see that $V(A)$ becomes a $\mathbb{Z}^2$-graded $L$-module. And all the dimensions of the homogeneous subspaces of $V(A)$ are dim $V$. The following result was obtained in [15].

**Theorem 4.9** ([15])

1. If $\dim V \geq 3$ then $V(A)$ is an irreducible $L$-module.
2. If $\dim V = 2$ then $V(A)$ has a unique proper submodule when $(\alpha_1, \alpha_2) \notin \mathbb{Z}^2$, otherwise, $V(A)$ has two proper submodules.
3. If $\dim V = 1$ then $V(A)$ is an irreducible $L$-module when $(\alpha_1, \alpha_2) \notin \mathbb{Z}^2$, otherwise, $V(A)$ can be decomposed into the direct sum of two irreducible $L$-module.

§5  $\mathbb{Z}$-graded modules of the intermediate series over the Virasoro-like algebra

The main result in this section is the following.

**Theorem 5.1.** There does not exist any nontrivial $\mathbb{Z}$-graded irreducible $L$-module of the intermediate series.

**Proof:** If $V$ is a nontrivial $\mathbb{Z}$-graded irreducible module of the intermediate series, then $c_1 = c_2 = 0$ and the action of $D(b_1)$ is nondegenerate. In fact, if it is degenerate then there exists a vector $0 \neq v_i \in V_i$ such that $D(b_1) \cdot v_i = 0$. Hence we have

$$0 = D(b_1) \cdot D(b_2) \cdot v_i = -\det(b_1, b_2) D(b_1 + b_2) \cdot v_i,$$

by the definition of a $\mathbb{Z}$-graded $L$-module of the intermediate series, which implies that $V$ is a generalize highest weight module. Thus $V$ is not a uniformly bounded $\mathbb{Z}$-graded module by Theorem 3.7 which is absurd.

Similarly, one can prove that the action of $D(-b_1)$ is nondegenerate. Therefore $\dim V_i = 1$ for all $i \in \mathbb{Z}$. Thus we can choose a basis $\{v_i \in V_i \mid i \in \mathbb{Z}\}$ of $V$ such that $D(b_1) \cdot v_i = av_{i+1}$ and $D(-b_1) \cdot v_i = av_{i-1}$ for all $i \in \mathbb{Z}$, where $a \neq 0$. Denote $D((l b_1 + k b_2) \cdot v_i = f(l, k, i) v_{i+l}$. We can deduce $af(l, 0, i) = af(0, i, i)$ since $D(l b_1) D(\pm b_1) \cdot v_i = D(\pm b_1) D(l b_1) \cdot v_i$. Hence $f(l, 0, i)$ are independent of $i$ for all $l \neq 0$. Set $\varepsilon = \det(b_1, b_2) \overline{e} \in \{ \pm 1 \}$ since $b_1, b_2$ is a $\mathbb{Z}$-basis of $\Gamma$.

For any $l, i, k \in \mathbb{Z}$ with $k \neq 0$, we have

$$\varepsilon k f(l, k, i) v_{i+j} = \varepsilon k D(l b_1 + k b_2) v_i$$

$$= D((l-1) b_1 + k b_2) D(b_1) v_i - D(b_1) D((l-1) b_1 + k b_2) v_i$$

$$= af(l-1, k, i+1) - f(l-1, k, i)) v_{i+1},$$

14
Thus, by (5.5) and (5.4), we have
\[ \varepsilon f(l, k, i) = \frac{a}{k}(f(l - 1, k, i + 1) - f(l - 1, k, i)). \]  
(5.1)

Similarly, we have
\[ \varepsilon k D((l - 1)b_1 + kb_2)v_i = D(-b_1)D(lb_1 + kb_2)v_i - D(lb_1 + kb_2)D(-b_1)v_i, \]
which implies
\[ \varepsilon f(l - 1, k, i) = \frac{a}{k}(f(l, k, i) - f(l, k, i - 1)). \]  
(5.2)

Substituting (5.2) into (5.1), we have
\[ a^2 f(l, k, i + 1) - (2a^2 + k^2)f(l, k, i) + a^2 f(l, k, i - 1) = 0. \]  
(5.3)

Set \( x_k = \frac{2a^2+k^2+(4a^2k^2+k^4)^{\frac{1}{2}}}{2a^2}. \) We can choose \( k \in \mathbb{N} \) such that \( x_k \neq x_k^{-1} \) and \( |x_k| > 1 \) since \( \lim_{k \to +\infty} x_k = \infty. \) Therefore, the equation \( a^2T^2 - (2a^2 + k^2)T + a^2 = 0 \) has different roots \( x_k \) and \( x_k^{-1} \), so we have
\[ f(l, k, i) = a(l, k)x_k^i + b(l, k)x_k^{-i}, \quad \forall i \in \mathbb{Z}, \]  
(5.4)

for some \( a(l, k), b(l, k) \in \mathbb{C} \) by (5.3). Since
\[ \varepsilon kl D(lb_1)v_i = D(lb_1 + kb_2)D(-kb_2)v_i - D(-kb_2)D(lb_1 + kb_2)v_i, \]
we obtain
\[ \varepsilon kl f(l, 0, i) = f(l, k, i)(f(0, -k, i) - f(0, -k, l + i)). \]

Thus, by (5.4) and the fact \( x_k = x_{-k} \), we have
\[
\varepsilon kl f(l, 0, i) = a(l, k)a(0, -k)(1 - x_k^i)x_k^{2i} + b(l, k)b(0, -k)(1 - x_k^{-i})x_k^{-2i}
+ a(l, k)b(0, -k)(1 - x_k^{-i}) + b(l, k)a(0, -k)(1 - x_k^i).
\]  
(5.5)

Since \( |x_k| > 1 \), we see that if \( a(l, k)a(0, -k) \neq 0 \) then
\[
\lim_{i \to +\infty} (a(l, k)a(0, -k)(1 - x_k^i)x_k^{2i} + b(l, k)b(0, -k)(1 - x_k^{-i})x_k^{-2i}
+ a(l, k)b(0, -k)(1 - x_k^{-i}) + b(l, k)a(0, -k)(1 - x_k^i)) = \infty,
\]
which contradicts (5.5) and the fact that \( f(l, 0, i) \) is independent of \( i \). Hence \( a(l, k)a(0, -k) = 0. \)

Similarly, we can deduce \( b(l, k)b(0, -k) = 0 \). If \( a(0, -k) = b(0, -k) = 0 \) or \( a(l, k) = b(l, k) = 0 \) then \( \varepsilon kl f(l, 0, i) \equiv 0 \) by (5.5) which is a contradiction. Hence without loss of generality, we may assume that
\[ a(0, -k) = b(l, k) = 0. \]  
(5.6)

Thus, by (5.5) and (5.4), we have
\[ f(l, 0, i) = \frac{a(l, k)b(0, -k)(1 - x_k^{-i})}{\varepsilon lk}, \quad f(0, -k, i) = b(0, -k)x_k^{-i}, \quad f(l, k, i) = a(l, k)x_k^i. \]  
(5.7)

Now we can substitute (5.7) into (5.1) to get
\[ \frac{a(l, k)}{a(l - 1, k)} = a\varepsilon^{-1}k^{-1}(x_k - 1). \]
Thus $a(l, k) = a(0, k)(ae^{-1}k^{-1}(x_k - 1))^l$. From this equation and (5.7), we can deduce that

$$f(l, k, i) = a(0, k)(ae^{-1}k^{-1}(x_k - 1))^lx_k, \quad \forall l \in \mathbb{Z}, \quad (5.8)$$

and

$$f(l, 0, i) = \frac{a(0, k)b(0, -k)(1 - x_k^{-1})(ae^{-1}k^{-1}(x_k - 1))^l}{\varepsilon lk}, \quad \forall l \neq 0. \quad (5.9)$$

Since $\varepsilon lkD(lb_1 + kb_2)v_i = D(kb_2)D(lb_1)v_i - D(lb_1)D(kb_2)v_i$, we have

$$\varepsilon lk f(l, k, i) = a(0, k)x_k^l(x_k^l - 1)f(l, 0, i),$$

by (5.8). Thus

$$\frac{f(l, k, i)}{f(l, 0, i)} = \frac{a(0, k)x_k^l(x_k^l - 1)}{\varepsilon lk}, \quad \forall 0 \neq l \in \mathbb{Z}. \quad (5.10)$$

Now by (5.8), (5.9) and (5.10), we have

$$1 = a(0, k)b(0, -k)l^{-2}k^{-2}(x_k^l - 1)(1 - x_k^{-l}), \quad \forall 0 \neq l \in \mathbb{Z}.$$

But $\lim_{l \to \infty} (l^{-2}(x_k^l - 1)(1 - x_k^{-l})) = \infty$ since $|x_k| > 1$, this is a contradiction with the above equation. Thus there does not exist a nontrivial $\mathbb{Z}$-graded irreducible $L$-module of the intermediate series.

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