Remarks on superdifferential equations
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Abstract
We show that the term superdifferential equation has been employed in the literature to refer to different types of differential equations with even and odd variables. It is justified on physical and mathematical grounds that a subclass of them, the hereafter called Grassmann-valued differential equations, cannot be effectively described through supergeometric techniques. Instead, we analyse them in terms of standard differential equations on Grassmann algebra bundles. Our considerations are illustrated through examples of physical and mathematical relevance.

Keywords: Even supervector field, Grassmann algebra, Grassmann bundle, supercoherent state, superflow, supermanifold, supermorphism, super-superposition rule.

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1 Introduction

Supergeometry have been drawing a lot of attention during the last decades (see [1, 3, 4, 9, 17, 18, 19, 24, 30, 32, 37, 44] and references therein). The main mathematical structure of this theory, the so-called supermanifold, can be roughly understood as a manifold admitting commutative (even) and anticommutative (odd) coordinates.

Mathematically, the interest in supermanifolds relies on the generalisation to a non-commutative context of the standard commutative real and complex differential geometries [2, 3, 4, 18, 24]. This is a large field of study comprehending, for instance, supersymplectic [2, 44] and super-Riemannian geometry [25]. Physically, even and odd coordinates appear as bosonic and fermionic entities, e.g. in the Faddeev–Popov’s ghosts method for the path integral formulation of quantum field theories [20]. Moreover, odd and even variables provide an adequate framework for the analysis of supermechanics [13, 28, 33], supercoherent states [15], quantization procedures [7], supergravity [41], supersymmetric models [19, 24, 26, 37], and others [16].

This work focus upon the study of the standardly denominated first-order superdifferential equations or superequations [1, 5, 6, 15, 33, 34, 35, 42, 43]. As a first contribution, we show that this term has been used in the literature with non-equivalent meanings corresponding to two main different types of such equations.
The first is the one used by Winternitz and other researchers dealing with problems from
the theory of differential equations, supercoherent states, and other geometric problems [1, 5, 6, 15, 31]. The second one emerges, for instance, in the theory of integration of
supervector fields, Lie supersymmetries, and supermechanics [23, 27, 33, 34, 35, 37]. There
still exist other types of superdifferential equations [22, 31], and several interpretations can
appear even throughout the same work (cf. [15]).

We show that the theory of supermanifolds play indeed a rôle in the superdifferential
equations appearing in supermechanics and in the integration of supervector fields [15].
These differential equations are evolution equations on supercoordinates, namely even and
odd variables, and their flows can be investigated through morphisms of supermanifolds
[34, 37]. This fact can intuitively be understood in quantum mechanical problems where
even and odd variables are related to quantum bosonic and fermionic states [14]. Then, their
evolution must be described in terms of some family of evolution operators respecting the
bosonic and fermionic character of states. In supersymmetric models a similar approach ap-
pear, but the evolution may violate the character of quantum states as fermions and bosons
become interchangeable [14, 37]. Summarising, we will call superdifferential equations on
supermanifolds the above-mentioned superdifferential equations.

There exists a second type of superdifferential equations occurring in physical and math-
ematical problems, e.g. supercoherent states, super-superposition rules, or symplectic geom-
etry [6, 11, 15], whose solutions are curves in Grassmann algebras (more generally, Grass-
mann algebra bundles over a manifold). Despite that, the ‘flow’ associated with these
superdifferential equations on the superalgebra of sections of such bundles is not related
to supermorphisms. These ideas suggest us to use purely Grassmann algebra techniques
to study the integrability of such differential equations and call them Grassmann-valued
differential equations. Actually, one can consider in a similar way other bundles of finite-
dimensional associative algebras. For instance, the matrix Riccati equations arise in this
way from matrix algebra bundles.

Relevantly, some superdifferential equations appearing in the literature can be under-
stood as both, superdifferential equations on supermanifolds or Grassmann-valued differ-
ential equations. We prove that in this case the space of solutions of the superdifferential
equation may depend on the chosen interpretation. The proper understanding must be
then selected according to the particular application of the superdifferential equation and
the expected meaning and properties of its solutions.

The structure of the paper goes as follows. Section 2 surveys the basic properties of
graded algebra bundles. Section 3 gives a brief account on supermanifolds describing the
notions and techniques to be used in this work. Section 4 is devoted to reviewing the
different types of superdifferential equations appearing in the literature. We propose and
analyse a formal definition of Grassmann-valued differential equations in Section 5. In this
section we address also the investigation on the integrability properties of such equations.
Finally, we summarize our conclusions and sketch our future work in Section 6.
2 Fundamentals on graded algebra bundles

If not otherwise stated, every field, let us say \( \mathbb{K} \), is assumed to be \( \mathbb{R} \) or \( \mathbb{C} \). Let \( \mathbb{G} \) be a commutative semigroup with composition law \(+\). A \( \mathbb{G}\)-graded vector space is a vector space \( E \) along with a decomposition \( E = \bigoplus_{i \in \mathbb{G}} E_i \) into a family of vector subspaces \( \{E_i\}_{i \in \mathbb{G}} \) of \( E \). The elements of \( E := \bigcup_{i \in \mathbb{G}} E_i \) are called homogeneous. Elements of \( E_i \) are said to be homogeneous of degree \( i \). We write \( |e| = i \) to say that \( e \in E_i \).

**Definition 2.1.** A \( \mathbb{G}\)-graded associative algebra \( \mathcal{A} \) is a unital associative \( \mathbb{K} \)-algebra \( \mathcal{A} \) together with a decomposition

\[
\mathcal{A} = \bigoplus_{i \in \mathbb{G}} \mathcal{A}_i,
\]

such that \( a_ia_j \in \mathcal{A}_{i+j} \) for every \( a_i \in \mathcal{A}_i \) and \( a_j \in \mathcal{A}_j \). Let us endow \( \mathbb{G} \) with a bi-additive mapping \( \langle \cdot | \cdot \rangle : \mathbb{G} \times \mathbb{G} \to \mathbb{Z}_2 \). If additionally \( a_ib_j = (-1)^{(i)(j)}b_ja_i \) for every \( a_i \in \mathcal{A}_i, b_j \in \mathcal{A}_j \), then \( \mathcal{A} \) is called \( \mathbb{G}\)-graded commutative or simply graded commutative if \( \mathbb{G} \) is known from context \([40, 43]\). If \( \mathcal{A} \) is \( \mathbb{Z}_2\)-graded commutative, then \( \mathcal{A} \) is called a superalgebra or, equivalently, \( \mathcal{A} \) is said to be supercommutative.

We hereupon assume \( \mathbb{G} \) to be \( \mathbb{Z} \) or \( \mathbb{Z}_2 \). In the latter case, \( E \) is called a superspace; the elements of \( E_0 \) and \( E_1 \) are said to have even and odd parity, respectively. Any \( \mathbb{Z}\)-gradation gives rise canonically to a \( \mathbb{Z}_2\)-gradation, so even and odd elements are defined also in this case. Note that a (commutative) \( \mathbb{Z}\)-graded associative algebra is in this way canonically also a superalgebra.

**Definition 2.2.** A linear mapping \( T : E \to F \) between superspaces \( E \) and \( F \) is called an even morphism, and we write \( |T| = 0 \), if \( T \) preserves parity, i.e. \( T(E_0) \subset F_0 \) and \( T(E_1) \subset F_1 \). An odd morphism of superspaces is a linear mapping \( T : E \to F \) that reverses parity, i.e. \( T(E_1) \subset F_0 \) and \( T(E_0) \subset F_1 \). Even and odd morphisms of superspaces are called homogeneous. Even morphisms of superspaces are called also supermorphisms. This denomination is due to the rôle played by supermorphisms in the theory of supermanifolds \([45]\). If not otherwise stated, we hereupon suppose that all morphisms of superspaces are supermorphisms. Similarly, in the case of \( \mathbb{Z}\)-gradations, we call morphisms linear maps preserving the degree of homogeneous elements.

Let \( \mathcal{A}, \mathcal{B} \) be superalgebras. A superalgebra morphism is a supermorphism of the form \( T : \mathcal{A} \to \mathcal{B} \) satisfying \( T(xy) = T(x)T(y) \) for every \( x, y \in \mathcal{A} \). A superalgebra automorphism is a bijective superalgebra morphism \( T : \mathcal{A} \to \mathcal{A} \). We write \( \text{Aut}(\mathcal{A}) \) for the space of superalgebra automorphisms on \( \mathcal{A} \).

**Example 2.3.** Let \( V \) be a finite-dimensional vector field and let \( \Lambda V \) be the space of multivectors on \( V \), namely the linear space of totally antisymmetric multilinear mappings on the dual space \( V^* \). The space \( \Lambda V \) is a unital associative algebra relative to the exterior product, \( \wedge \), of multivectors \([45]\). The exterior algebra, also called Grassmann algebra, associated with \( V \) is the \( \mathbb{Z}\)-graded associative algebra relative to the aforesaid unital associative algebra in \( \Lambda V \) and the decomposition \( \Lambda V = \bigoplus_{k \in \mathbb{Z}} \Lambda^kV \), where \( \Lambda^kV \) with \( k \in \mathbb{N} \) is the space
of $k$-vectors in $\Lambda V$, whereas $\Lambda^0 V := \mathbb{K}$ and $\Lambda^k V := \{0\}$ for $k < 0$. The space $\Lambda V$ becomes a superalgebra by substituting the previous decomposition with $\Lambda V = \Lambda V_0 \oplus \Lambda V_1$, where $\Lambda V_0 := \bigoplus_{k \text{ even}} \Lambda^k V$ and $\Lambda V_1 := \bigoplus_{k \text{ odd}} \Lambda^k V$.

**Example 2.4.** Let us define $A_{\mathbb{K}}^{m|q}(U) := C^\infty(U, \mathbb{K}) \otimes \Lambda \mathbb{K}^q$, where $C^\infty(U, \mathbb{K})$ is the linear space of $\mathbb{K}$-valued smooth functions on an open $U \subset \mathbb{K}^m$ and $\Lambda \mathbb{K}^q$ represents the Grassmann algebra of $\mathbb{K}^q$. The exterior product on $\Lambda \mathbb{K}^q$ extends by $C^\infty(U, \mathbb{K})$-linearity to a unital associative product on $A_{\mathbb{K}}^{m|q}(U)$. Then, the space $A_{\mathbb{K}}^{m|q}(U)$ becomes a superalgebra by considering the decomposition

$$A_{\mathbb{K}}^{m|q}(U) = (C^\infty(U, \mathbb{K}) \otimes \Lambda \mathbb{K}^q_0) \oplus (C^\infty(U, \mathbb{K}) \otimes \Lambda \mathbb{K}^q_1),$$

where $C^\infty(U, \mathbb{K}) \otimes \Lambda \mathbb{K}^q_0$ and $C^\infty(U, \mathbb{K}) \otimes \Lambda \mathbb{K}^q_1$ are the even and odd parts of $A_{\mathbb{K}}^{m|q}(U)$, respectively. \triangle

From now on all graded associative algebras will be associative and graded commutative. Hence, the terms associative and commutative will be hereafter omitted.

**Example 2.5.** Let us endow the ring $\mathbb{K}$ with its natural unital algebra structure and the decomposition $\mathbb{K} = \mathbb{K}_0 \oplus \mathbb{K}_1$, with $\mathbb{K}_0 := \mathbb{K}$ and $\mathbb{K}_1 := \{0\}$. This turns $\mathbb{K}$ into a superalgebra. Given a $\mathbb{Z}_2$-graded algebra $\Lambda V$ and its ideal, $\Lambda^{k\geq 1} V$, of $k$-vectors with $k \geq 1$, one has a superalgebra isomorphism $\Lambda V/\Lambda^{k\geq 1} V \simeq \mathbb{K}$. The quotient map $\pi : \Lambda V \mapsto \Lambda V/\Lambda^{k\geq 1} V \simeq \mathbb{K}$, the so-called augmentation map, is a superalgebra morphism. \triangle

**Definition 2.6.** A $\mathbb{Z}$-graded algebra bundle is a vector bundle $\pi : F \to M$ whose fibres are isomorphic to a $\mathbb{Z}$-graded algebra $A$ and there exists a covering $\{U_\alpha\}_{\alpha \in I}$ of trivialising open subsets of $\pi$ and a cocycle $\{g_{\alpha\beta}\}_{\alpha, \beta \in I}$ of functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \operatorname{Aut}(A)$.

**Example 2.7.** If $\pi : F \to M$ is a $\mathbb{Z}$-graded algebra bundle, then the $\mathbb{Z}$-graded structure of the fibres, namely $\pi^{-1}(x) = \bigoplus_{g \in \mathbb{Z}} (F_g)_x$ for every $x \in M$, allows us to define a family of vector subbundles $\pi_g : F_g \to M$, with $F_g := \bigcup_{x \in M} (F_g)_x$ for every $g \in \mathbb{Z}$. Then, the space of sections of $\pi : F \to M$, let us say $\Gamma(M, F)$, is a $\mathbb{Z}$-graded algebra with respect to the decomposition $\Gamma(M, F) = \bigoplus_{g \in \mathbb{Z}} \Gamma(M, F_g)$ and the natural unital product on $\Gamma(M, F)$ of the form

$$\forall \gamma_1, \gamma_2 \in \Gamma(M, F), \quad (\gamma_1\gamma_2)(x) := \gamma_1(x)\gamma_2(x), \quad \forall x \in M.$$ \triangle

**Example 2.8.** Let $\pi : E \to M$ be a vector bundle and let us define $E_x := \pi^{-1}(x)$, with $x \in M$, and $\Lambda E := \bigsqcup_{x \in M} \Lambda E_x$. We can define then a new vector bundle $\Lambda \pi : \Lambda E \to M$ with fibres of the form $(\Lambda E)_x := \Lambda E_x$ for every $x \in M$. The natural Grassmann algebra structure of the fibres of $\Lambda \pi : \Lambda E \to M$ allows us to understand it as a $\mathbb{Z}$-graded algebra bundle, the so-called exterior bundle of $\pi : E \to M$. Conversely, it is immediate to see that a Grassmann algebra bundle can be considered as the exterior bundle of a vector bundle. \triangle
Consider the Grassmann bundle $\Lambda \pi : \Lambda E \to M$ associated with the vector bundle $\pi : E \to M$. Let $U \subset M$ be an open coordinated by $\{x_1, \ldots, x_m\}$ and let $\Psi : U \times \Lambda \mathbb{K}^q \to \Lambda \pi^{-1}(U)$ be a local trivialization of $\Lambda \pi : \Lambda E \to M$. Consider a set of odd generators $\theta_1, \ldots, \theta_q$ of $\Lambda \mathbb{K}^q$. The space $\Lambda \mathbb{K}^q$ has dimension $2^q$. Indeed, every $z \in \Lambda \mathbb{K}^q$ can be brought into the form $z = \sum_j z_j \theta^j$, where $z_j \in \mathbb{K}$, $\theta^0 = 1$ and $\theta^j := \theta_{i_1} \wedge \ldots \wedge \theta_{i_s}$ with $j := (i_1, \ldots, i_s)$ for $i_1 < \ldots < i_s \in \{1, \ldots, q\}$ and $s = 1, \ldots, q$. This enables us to define a coordinate system on $\Lambda \mathbb{K}^q$ given by $2^q$ functions $u_j(z) := z_j$. Then, $\{x_1, \ldots, x_m, u_j\}$ for $\Lambda \pi^{-1}(U)$ becomes, through the local trivialization $\Psi$, a local coordinate system for $\Lambda E$.

### 3 Fundamentals on supermanifolds

We hereafter assume structures to be smooth and well-defined globally. This enables us to avoid minor technical problems and to focus on the key parts of our presentation. We refer to [37, 45] for further details.

**Definition 3.1.** A supermanifold $\mathcal{M}$ of dimension $(m|q)$ is a pair $\mathcal{M} := (M, \mathcal{A}_M)$, where $M$ is an $m$-dimensional second-countable Hausdorff manifold, called the reduced manifold or body of $\mathcal{M}$, and $\mathcal{A}_M$, the structure sheaf of $\mathcal{M}$, is a sheaf of superalgebras which is locally isomorphic to the sheaf $\mathcal{A}_K^{m|q}$ attaching every open $U \subset M$ to $\mathcal{A}_K^{m|q}(U)$. The elements of each $\mathcal{A}_K^{m|q}(U)$ are called superfunctions.

**Example 3.2.** Every manifold $M$ gives rise to an $(m|0)$-dimensional supermanifold $(M, C^\infty)$, where $C^\infty$ stands for the sheaf attaching every open $U \subset M$ onto the space $C^\infty(U, \mathbb{K})$ of $\mathbb{K}$-valued smooth functions on $M$.

**Example 3.3.** Every pair of the form $(\mathbb{K}^m, \mathcal{A}_K^{m|q})$ gives rise to an $(m|q)$-dimensional supermanifold denoted by $\mathbb{K}^{m|q}$.

**Example 3.4.** Let $\pi : E \to M$ be a vector bundle of rank $k$ and let $\Lambda \pi : \Lambda E \to M$ be its associated exterior bundle. We write $\mathcal{A}_{A\pi}$ for the sheaf of sections of $\Lambda \pi$. Then, the pair $\prod E := (M, \mathcal{A}_M)$ is a supermanifold of dimension $(m|2^k)$. The superfunctions are sections of $\Lambda E$, i.e. sections of a Grassmann bundle. The relevance of $\prod E$ is due to the fact that in many cases the study of a given supermanifold can be reduced to analysing a supermanifold of this type (see [4, 24]).

To explain how the study of every supermanifold can be reduced to studying a supermanifold of the form $\prod E$, we introduce the notion of a supermanifold morphism. This also entails recalling that every sheaf $\mathcal{A}_M$ is endowed with the restriction morphisms [37], namely the family of superalgebra mappings $\tau_{U,V}^M : \mathcal{A}_M(U) \to \mathcal{A}_M(V)$ for every pair of open subsets $V \subset U \subset M$.

**Definition 3.5.** A supermanifold morphism between the supermanifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ is a morphism between their sheaves $\mathcal{A}_{\mathcal{M}_1}$ and $\mathcal{A}_{\mathcal{M}_2}$, i.e. a pair $(\mathcal{S}, \mathcal{S}^*)$, where $\mathcal{S} : M_1 \to M_2$ and $\mathcal{S}^*$ is a correspondence attaching every open $U \subset M_2$ to a superalgebra morphism $\mathcal{S}^*(U) : \mathcal{A}_{\mathcal{M}_2}(U) \to \mathcal{A}_{\mathcal{M}_1}(\mathcal{S}^{-1}(U))$ commuting with the restriction morphisms, namely $\mathcal{S}^*(V) \circ \tau_{U,V}^{M_1} = \tau_{\mathcal{S}^{-1}(U), \mathcal{S}^{-1}(V)}^{M_1} \circ \mathcal{S}^*(U)$ for all opens $V \subset U \subset M$. 

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For simplicity, the correspondence $S^*$ will be hereafter denoted by $S^* : \mathcal{A}_{M_2} \to \mathcal{A}_{M_1} \circ \tilde{S}^{-1}$. Also to simplify our notation, we will denote $S^*(M_2)$ by $S^*$ when it is clear what we mean.

If $U$ is an open coordinate neighbourhood of $M$, then $U := (U, \mathcal{A}^{m|q}_{K}|_{U})$, where $\mathcal{A}^{m|q}_{K}|_{U}$ is the restriction of the sheaf of $\mathcal{A}^{m|q}_{K}$ to opens within $U$, is called a superdomain or a splitting neighbourhood. Every supermanifold is by definition locally isomorphic to a superdomain. Assume that $U$ is an open coordinate subset of $M$. Let $\{x^1, \ldots, x^m\}$ and $\{\theta^1, \ldots, \theta^q\}$ be even and odd elements of $\mathcal{A}_{M}$, respectively. If $\mathcal{A}_{M}(U)$ and $C^\infty(x^1, \ldots, x^m) \otimes \Lambda(\theta^1, \ldots, \theta^q)$ are isomorphic superalgebras, then we say that $(x^i, \theta^\alpha)$ is a supercoordinate system for $\mathcal{M}$ on $U$.

**Example 3.6.** For every supermanifold $(M, \mathcal{A}_{M})$ and every open $U \subset M$, the projection $\epsilon^*(U) : \mathcal{A}_{M}(U) \to \mathcal{A}_{M}(U)/\mathcal{N}(U)$, where $\mathcal{N}(U)$ is the ideal of nilpotent elements of $\mathcal{A}_{M}(U)$, is a superalgebra morphism. Since $\mathcal{A}_{M}$ is locally isomorphic to $\mathcal{A}^{m|q}_{K}$, one can prove that $\mathcal{A}_{M}(U)/\mathcal{N}(U)$ and $C^\infty(U)$ are isomorphic superalgebras and the morphisms $\epsilon^*(U)$ commute with restriction morphisms (cf. [4, 37]). Then, the pair $(\text{Id}_M, \epsilon^*)$ becomes a supermorphism $\epsilon : (M, C^\infty) \to M$. We call $\epsilon$ the augmentation morphism of $M$. If $f \in \mathcal{A}_{M}(U)$, then we write $\tilde{f} := \epsilon^*(f)$.

The augmentation morphism shows that, given a supermorphism $(\tilde{S}, S^*)$, the mapping $\tilde{S}$ can be obtained from $S^*$. In fact, since the mappings $S^*(U)$ commute with restriction morphisms, $S^*$ induces a morphism of sheaves from $C^\infty_{M_2}$ to $C^\infty_{M_1}$. Dualising, one gets a mapping from $M_1$ to $M_2$ which coincides with $\tilde{S}$ (see [3]).

**Theorem 3.7. (The Batchelor–Gawędzki Theorem [4, 24])** For every supermanifold $\mathcal{M}$, there exists a vector bundle $\pi : E_{\mathcal{M}} \to M$ such that $\mathcal{M}$ is isomorphic to $\coprod E_{\mathcal{M}}$.

The supermanifold morphism between $\mathcal{M}$ and $\coprod E_{\mathcal{M}}$ is non-canonical but the vector bundle $\pi : E_{\mathcal{M}} \to M$ is unique up to a vector bundle isomorphism [24]. This justifies to call $\pi : E_{\mathcal{M}} \to M$ the structural bundle of $\mathcal{M}$ [37]. The non-canonicity is very important as it implies that a supermanifold cannot be considered as a privileged type of vector bundle and there are morphisms in the category of supermanifolds which are not induced by vector bundle morphisms. Hence, supermanifolds represent a separate field of study on their own [36].

**Definition 3.8.** A homogeneous supervector field $X$ on $\mathcal{M}$ is a homogeneous superderivation of the sheaf of superalgebras $\mathcal{A}_{M}$, i.e. $X$ attaches every open $U \subset M$ to a homogeneous linear morphism $X|_{U} : \mathcal{A}_{M}(U) \to \mathcal{A}_{M}(U)$ satisfying the graded Leibniz rule, namely

$$X|_{U}(fg) = X|_{U}(f)g + (-1)^{|f||X|}fX|_{U}(g), \quad \forall f, g \in \mathcal{A}_{M}(U),$$

where $|X|$ stands for the parity of $X$ as a morphism of superspaces. A supervector field on $\mathcal{M}$ is any sum of homogeneous supervector fields on $\mathcal{M}$.
We write \( \mathfrak{X}_0(\mathcal{M}) \) for the sheaf of even superderivations on \( \mathcal{M} \) and \( \mathfrak{X}_1(\mathcal{M}) \) for the sheaf of odd ones. Hence, a supervector field on \( \mathcal{M} \) is an element of \( \mathfrak{X}(\mathcal{M}) := \mathfrak{X}_0(\mathcal{M}) \oplus \mathfrak{X}_1(\mathcal{M}) \). Thus, every supervector field \( X \) on \( \mathcal{M} \) can be written in a unique way as \( X = X_0 + X_1 \), where \( X_0 \) and \( X_1 \) are uniquely defined even and odd homogeneous supervector fields on \( \mathcal{M} \), respectively.

If \( \mathcal{M} \) is a supermanifold of dimension \((m|q)\), then \( \mathfrak{X}(\mathcal{M}) \) is a sheaf of \( \mathcal{A}_\mathcal{M} \)-modules of dimension \((m|q)\) on \( \mathcal{M} \) [37]. If \( \pi : E_M \to M \) is the structural bundle of \( \mathcal{M} \), then we can define a supertangent manifold of the form \( T\mathcal{M} := (TM, TA_M) \) of dimension \((2m|2q)\) whose structural bundle is \( T\pi : TE_M \to TM \) and a supermanifold isomorphism between \( T\mathcal{M} \) and \( (TM, A_{AT\mathcal{M}}) \) (see [13] for details). As a consequence of the above construction, every local supercoordinate system \((x^i, \theta^\alpha)\) on \( \mathcal{M} \) induces a local supercoordinate system \((x^i, \dot{x}^i, \theta^\alpha, \dot{\theta}^\alpha)\) on \( T\mathcal{M} \).

We write \((\{\ast\}, \mathbb{K})\), where \( \{\ast\} \) is a point, for the terminal object in the category of supermanifolds over \( \mathbb{K} \) [39]. The terminal morphism \( C \), namely the supermanifold morphism \( C : \mathcal{M} \to (\{\ast\}, \mathcal{A}_\mathbb{K}^{\{\ast\}}) \), with \( \tilde{C}(x) := \{\ast\} \), and \( (C^\ast \lambda)(x) := \lambda(x) \) for every \( x \in M \) and \( \lambda \in \mathbb{K} \), plays a fundamental rôle. The supermanifold \((M, C^\infty)\) can be ‘embedded’ naturally into any supermanifold \((\mathcal{M}, \mathcal{A}_\mathcal{M})\) through the augmentation morphism \( \epsilon : (M, C^\infty) \to (\mathcal{M}, \mathcal{A}_\mathcal{M}) \). The map \( \epsilon \) enables us to relate each supervector field \( X \) on \( \mathcal{M} \) to a unique vector field \( \tilde{X} \) on \( M \) given by \( \tilde{X}f := \tilde{X}_0f \) for every \( f \in \mathcal{A}_\mathcal{M} \) (cf. [34]).

4 On different types of superdifferential equations

We recall that this work focuses on first-order superdifferential equations. More particularly, this section has two aims. First, it highlights that there exist two main different types of superdifferential equations to be studied in Sections 4.1 and 4.2. Only the type described in Section 4.2 can be fully described in terms of supermanifold techniques. Consequently, the term superdifferential equation on a supermanifold will be reserved for them, while those superdifferential equations appearing in Section 4.2 will be called Grassmann-valued differential equations for reasons to be stated soon.

Another goal of this section is to show that some superdifferential equations appearing in the physics literature can potentially be interpreted simultaneously in the two aforesaid different ways. Depending on the interpretation, their solutions are defined in a different manner, which may lead to different descriptions of the same physical phenomena. This imposes physical restrictions on how a superdifferential equation must be understood. This will be the main topic of Section 4.3.

4.1 Differential equations on Grassmann bundles

This subsection presents a special class of superdifferential equations appearing in the theory of fluids, the study of super-superposition rules, supercoherent states, and others [5, 6, 15, 22, 31]. This illustrates their interest and justifies that they should not be called superdifferential equations or superequations, as done in the literature, because their evolu-
tion cannot be described through the structures attached to a supermanifold. Instead, we propose they should be studied as differential equations on Grassmann bundles.

Let us analyse a simple instance of a superdifferential equation occurring in the theory of supercoherent states (see [15] for details). The propagator associated with supercoherent states for the super orthosymplectic Lie group $\text{Osp}(1|2, \mathbb{R})$, where the associated Lie group is two-dimensional and the Grassmann algebra has one odd generator, can be obtained by solving the two systems of complex superdifferential equations [15]

\[
\frac{d\alpha}{dt}(t) = \frac{i}{\hbar} \left( f^*(t) + \frac{A(t)}{2} \alpha(t) \mp f(t)\alpha^2(t) \pm \frac{\bar{\Theta}(t)}{\sqrt{2}} \Theta(t) \mp \frac{\theta(t)}{\sqrt{2}} \alpha(t)\Theta(t) \right),
\]

\[
\frac{d\Theta}{dt}(t) = \frac{i}{\hbar} \left( \frac{\bar{\Theta}(t)}{\sqrt{2}} - \frac{\theta(t)}{\sqrt{2}} \alpha(t) + \frac{A(t)}{2} \Theta(t) \mp f(t)\alpha(t)\Theta(t) \right),
\]

where $\hbar$ is the Planck constant, $\Theta(t), \bar{\Theta}(t), \theta(t)$ are odd elements of a complex Grassmann algebra $\Lambda V$ for every $t \in \mathbb{R}$, the elements $\alpha(t)$ are even, and $A(t), f(t)$ are complex numbers. The first system in (4.1) is obtained by choosing the upper signs in $\mp$ and $\pm$, and the second by choosing the lower ones. In any case, a particular solution to (4.1) is given by a curve $(\alpha(t), \Theta(t))$ in $\Lambda V$ satisfying (4.1). Moreover, it is always possible to reduce (4.1) to a standard system of differential equations by choosing a basis of the Grassmann algebra $\Lambda V$ (cf. [5, 15]).

Contrary to a quite general assumption for different types of superdifferential equations [39], the evolution of system (4.1) does not generally give rise to a one parameter group of supermorphisms on the Grassmann algebra $\Lambda V$ (which can be understood as the sheaf of the supermanifold $\mathbb{R}^{0|\dim V}$). Instead, the evolution will only give rise to a family of curves in $\Lambda V$.

To illustrate our previous claims, we will study the case $V = \mathbb{C}$. Let $\vartheta$ be an odd generator of the Grassmann algebra $\Lambda \mathbb{C}$. Then, $\alpha(t) = \alpha_{(0)}(t)1$, $\Theta(t) = \xi_{(1)}(t)\vartheta$ for certain $t$-dependent complex functions $\alpha_{(0)}(t), \xi_{(1)}(t)$. Let us consider also for simplicity $A(t) = 0$, $\theta(t) = 0$, $f(t) = 1$, and the minus option in $\mp$. Then, system (4.1) reduces to a first-order system of differential equations (cf. [15, p. 3390])

\[
\frac{d\alpha_{(0)}}{dt}(t) = \frac{i}{\hbar} \left( 1 - \alpha_{(0)}^2(t) \right), \quad \frac{d\xi_{(1)}}{dt}(t) = -\frac{i}{\hbar} \alpha_{(0)}(t)\xi_{(1)}(t).
\]

This illustrates that Grassmann–differential equations (4.1) can be reduced to systems of standard differential equations. The general solution, $z(t) = \alpha_{(0)}(t) + \xi_{(1)}(t)\vartheta$, to (4.2) with the initial condition $z(0) = \alpha_0 + \xi_0\vartheta$ reads

\[
z(t) = \frac{\alpha_0 \cos(t/\hbar) + i \sin(t/\hbar) + \xi_0 \vartheta}{\frac{i}{\alpha_0} \sin(t/\hbar) + \cos(t/\hbar)}.
\]

\footnote{The system (4.1) corrects two typos in (4.18a) of [15, p. 3389]. It is also worth noting that the work [15] contains an obvious typo in the initial conditions of system (4.15) and a less obvious one in (3.8b), which should take the form $\tilde{\chi} = -u^*\vartheta \pm v\vartheta$. Previous minor corrections are in order to fully understand the form of system (4.1) and its applications.}
Let us analyse the flow of system (4.2) so as to understand geometrically its properties. Let \( z_0(t) \) be the particular solution to (4.2) with initial condition \( z_0 \in \Lambda C \). The evolution of system (4.2) is described by a one-parametric family of mappings \( G^*_t : z_0 \in \Lambda C \mapsto z_0(t) \in \Lambda C \) with \( t \in \mathbb{R} \). Although the particular solution \( z_1(t) \) of (4.2) with initial condition \( z_1(0) = 1 \) is, by virtue of (4.3), given by \( z_1(t) = 1 \) and, in consequence, \( G^*_1(1) = 1 \), we will now show that the mappings \( G^*_t \) are not superalgebra morphisms.

Let us prove that, for instance, the mappings \( G^*_t \) are not even morphisms. If \( z_\vartheta(t) \) is the particular solution of (4.2) with \( z_\vartheta(0) = \vartheta \), then the explicit expression of \( z_\vartheta(t) \) can be obtained by choosing \( \alpha_0 = 0, \xi_0 = 1 \) in (4.3), which gives

\[
     z_\vartheta(t) = i \tan(t/\hbar) + \sec(t/\hbar) \vartheta.
\]

In consequence, \( z_\vartheta(0) \wedge z_\vartheta(0) = 0 \) whereas

\[
     z_\vartheta(t) \wedge z_\vartheta(t) = - \tan^2(t/\hbar) + 2i\vartheta \sec(t/\hbar) \tan(t/\hbar)
\]

is different from zero if and only if \( t \notin \hbar \pi \mathbb{Z} \). Then,

\[
     G^*_t(z_\vartheta(0)) \wedge G^*_t(z_\vartheta(0)) = z_\vartheta(t) \wedge z_\vartheta(t) \neq z_\vartheta(0) \wedge z_\vartheta(0) = 0 \iff t \notin \hbar \pi \mathbb{Z}.
\]

Therefore, the mappings \( G^*_t \) do not preserve parity: although \( z_\vartheta(0) \) is odd, one has that \( z_\vartheta(t) \) is not odd for \( t \notin \hbar \pi \mathbb{Z} \).

Moreover, the mappings \( G^*_t \) are not even linear. This can be proved by using the particular solutions \( z_1(t), z_\vartheta(t), \) and \( z_{1+\vartheta}(t) \). In view of (4.3), the particular solution to (4.2) with initial condition \( 1 + \vartheta \) takes the form

\[
     z_{1+\vartheta}(t) = 1 + \frac{\vartheta}{i \sin(t/\hbar) + \cos(t/\hbar)}
\]

Then,

\[
     1 + \frac{\vartheta}{i \sin(t/\hbar) + \cos(t/\hbar)} = G^*_t(1 + \vartheta) = G^*_t(z_1(0) + z_\vartheta(0)),
\]

whereas

\[
     G^*_t(z_1(0)) + G^*_t(z_\vartheta(0)) = 1 + i \tan(t/\hbar) + \sec(t/\hbar) \vartheta.
\]

Hence, \( G^*_t(z_1(0) + z_\vartheta(0)) \neq G^*_t(z_1(0)) + G^*_t(z_\vartheta(0)) \) for \( t \notin \hbar \pi \mathbb{Z} \).

Despite the above, the \( \{G^*_t\}_{t \in \mathbb{R}} \) generate a one-parametric family of diffeomorphisms of \( \Lambda C \). Indeed, one can only ensure that (4.2) behaves as a standard system of differential equations whose dependent variables take values in the Grassmann algebra \( \Lambda C \). It is immediate to observe that the same applies to (4.1). Moreover, other superdifferential equations in the literature of supercoherent states (see [21, 22]) share the same properties of (4.1).

One can tackle a different approach to study (4.2) by studying it as a superdifferential equation on the supermanifold \( \mathbb{C}^{1|1} \). This will be done in Section 4.2. Interestingly, this new interpretation will lead to define solutions of (4.2) in a different way, which in turn will give rise to a different family of solutions. It is therefore important to highlight that the interpretation of (4.2) given in [15] is the one given in this section. The approach to (4.2)
given in Section 4.2 will lead to other solutions that will not have an appropriate physical meaning within the theory described in [15].

Superdifferential equations similar to (4.1) appear in other research areas different from the field of supercoherent states. For instance, let $\Lambda V$ be an arbitrary Grassmann algebra and consider the superdifferential equation [5, p. 116]

\[
\begin{align*}
\frac{d\eta}{dt}(t) &= \rho(t) + \sigma(t)w(t) - a(t)\eta(t) - b(t)w(t)\eta(t), \\
\frac{dw}{dt}(t) &= c(t) - 2a(t)w(t) - \rho(t)\eta(t) - \sigma(t)w(t)\eta(t) - b(t)w^2(t),
\end{align*}
\]

where $w(t), a(t), b(t), c(t) \in \Lambda V_0$, $\eta(t), \rho(t), \sigma(t) \in \Lambda V_1$. Then, (4.4) becomes a superdifferential equation on the unknown variable $\Omega = \eta + w \in \Lambda V$. Its solutions are curves in $\Lambda V$. As in the case of (4.1), the system (4.4) can be reduced to a system of standard differential equations by choosing a particular basis of $\Lambda V$ (cf. [5, 15]).

Restricting ourselves to the case of $a(t), b(t), c(t), \rho(t), \sigma(t)$ being constants, we can easily repeat the same analysis of the previous example (4.2) giving rise to analogous results. Consequently, although (4.4) can be understood as a differential equation on $\mathbb{R}^{0|\dim V}$, this is the Grassmann algebra structure in fibres of $\Lambda V$ which is necessary to define the right-hand side of (4.4). Meanwhile, the solutions of (4.4) cannot be obtained through supermanifold morphisms or other related structures. Hence, (4.4) should be treated as a standard differential equation on $\Lambda V$.

All previous superdifferential equations could be reduced to systems of ordinary differential equations. Let us comment in a more general case, the so-called *super Riccati differential equations*

\[
\frac{dz}{dt}(t, x) = a^0_t(x) + a^1_t(x)z(t, x) + a^2_t(x)z^2(t, x), \quad a^0_t(\cdot), a^1_t(\cdot), a^2_t(\cdot), z(t, \cdot) \in \mathcal{A}_{\mathbb{R}}^{m|q},
\]  

which will require a slightly more general approach. The study of particular types of such super Riccati differential equations were proposed by Winternitz and collaborators within the framework of the theory of super-superposition rules in [6].

If we consider $r = q = 1$, $a_0 = a_2 = 1$, $a_1 = 0$, and supercoordinates $(x, \vartheta)$ in $\mathbb{R}^{1|1}$, then the general solution to (4.5) reads

\[
z(t, x) = \tan(t + \lambda_1(x)) + \vartheta \lambda_2(x) \sec^2(t + \lambda_1(x)),
\]

for arbitrary $x$-dependent real functions $\lambda_1(x), \lambda_2(x)$. The induced flow is not related to supermanifold morphisms $G^*_t : \mathcal{A}_{\mathbb{R}}^{1|1}(\mathbb{R}) \to \mathcal{A}_{\mathbb{R}}^{1|1}(\mathbb{R})$, e.g. the particular solution with initial condition $z(t, x) = 1$ is $z(t, x) = \tan(t + \pi/4)$ and therefore $G^*_t(1) \neq 1$. It is also worth noting that (4.5) does not give rise to any differential equation in the body of the supermanifold $\mathbb{R}^{m|q}$.

**Remark 4.1.** Note that, as Grassmann-valued differential equations can be reduced to a system of ordinary differential equations, the fact that we deal with a superalgebra plays in fact no rôle. Actually, we can replace the Grassmann algebra with any finite-dimensional (graded) associative algebra.
Consider now the projection $\tau : T^*M \to M$, the associated Grassmann bundle $\Lambda \tau : \Lambda T^*M \to M$, a one-form $\omega_1$ on $U$, and the superdifferential equation on $M$ given by
\[
\frac{\partial \omega}{\partial t} = \omega_1, \quad \omega \in \Gamma(M, \Lambda T^*M),
\]
appearing, for instance, in the proof of the Moser’s theorem in symplectic geometry [11].

Let us check whether it makes sense to study (4.6) as a superdifferential equation on a supermanifold $(M, \Gamma_{\Lambda\tau})$.

The general solution to (4.6) is given by $\omega(t) = \omega_0 + t\omega_1 =: G_t^*\omega_0$, where $\omega_0$ is any arbitrary element of $\Gamma(M, \Lambda T^*M)$. This gives rise to a $t$-parametric family $\{G_t^*\}_{t \in \mathbb{R}}$ of mappings $G_t^* : \Gamma(M, \Lambda T^*M) \to \Gamma(M, \Lambda T^*M)$. Such mappings are not generally superalgebra morphisms, e.g. $G_t^*(1) = 1 + t\omega_1 \neq 1$ for $\omega_1 \neq 0$ and $t \neq 0$. Moreover, the $\{G_t^*\}_{t \in \mathbb{R}}$ are not even linear morphisms since $G_t^*(0)$ is generally different from zero. It is worth noting also that the superdifferential equation (4.6) does not lead to a differential equation on the reduced manifold $M$.

Above-mentioned examples can be considered as particular cases of a type of differential equation on a Grassmann bundle $\Lambda\pi : \Lambda E \to M$. More specifically, all previous models can be understood as differential equations whose particular solutions are curves $z_t$ within the space of sections $\Gamma(M, \Lambda E)$ satisfying that
\[
\frac{dz_t(x)}{dt} = f_x(t, z_t(x)),
\]
for a certain family of functions $f_x : \mathbb{R} \times \Lambda_x E \to \Lambda_x E$ with $x \in M$. A more precise geometric description of (4.7) will be accomplished in Section 5. For the time being, we merely propose to hereafter call the superdifferential equations (4.1), (4.4), (4.5), (4.6), and all similar ones in works [5, 6] Grassmann-valued differential equations, as this last term reflects better their properties.

### 4.2 ‘True’ superdifferential equations

A second type of superdifferential equations is related to the integration of supervector fields [27, 34, 39] and a generalisation of supermechanics [10, 12, 14, 28, 33]. We now briefly survey the main properties of the theory of integration of even supervector fields (see [37] for details). This theory will be a key to study the properties of the superdifferential equations in this section, which prompts us to call them superdifferential equations on supermanifolds or simply superdifferential equations.

Shortly speaking, the integration of an even supervector field is the determination of a superflow, namely a generalisation to the supermanifold context of the flow for standard vector fields [37]. More specifically, let $X$ be an even supervector field on an $(m|q)$-dimensional supermanifold $\mathcal{M}$. On a superdomain $(x^i, \theta^\alpha)$ of $\mathcal{M}$, one has that $X$ can locally be written as
\[
X = \sum_{i=1}^m f^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q g^\alpha \frac{\partial}{\partial \theta^\alpha},
\]

for a certain family of functions $f^i, g^\alpha : \mathbb{R} \times \Lambda_x E \to \Lambda_x E$. A more precise geometric description of (4.7) will be accomplished in Section 5. For the time being, we merely propose to hereafter call the superdifferential equations (4.1), (4.4), (4.5), (4.6), and all similar ones in works [5, 6] Grassmann-valued differential equations, as this last term reflects better their properties.
where the functions $f^i$ and $g^\alpha$ are even and odd, respectively.

To define the integration of $X$, it is convenient to recall that a supermanifold $M$ gives rise to a new supermanifold $\mathbb{R} \times M$ whose reduced space is $\mathbb{R} \times M$ and whose sheaf, $\mathcal{A}_{\mathbb{R} \times M}$, is such that for every open $U \subset M$ of a superdomain of $M$ one has that $\mathcal{A}_{\mathbb{R} \times M}(\mathbb{R} \times U) = C^\infty(\mathbb{R} \times U) \otimes \Lambda K^q(U)$. Then, the integration of $X$ is the process of finding a superflow, namely a supermorphism $\mathcal{G} : \mathbb{R} \times M \to M$ such that $\tilde{\mathcal{G}} : \mathbb{R} \times M \to M$ is the flow of the standard vector field $\tilde{X}$, while $\mathcal{G}^* : \mathcal{A}_M \to \mathcal{A}_{\mathbb{R} \times M} \circ \tilde{G}^{-1}$ satisfies that

$$\partial_t \mathcal{G}^* f = \mathcal{G}^*(X f), \quad (\mathcal{G}^* f)(0, \cdot) = f(\cdot), \quad \forall f \in \mathcal{A}_M(M). \tag{4.9}$$

It can be proved that the conditions (4.9) ensure the local existence and uniqueness of $\mathcal{G}$ [34, 37].

If we define $\tilde{G}_t : x \in M \mapsto \tilde{G}(t, x) \in M$ and $G_t^* (U) : f(\cdot) \in \mathcal{A}_M(U) \mapsto (G^*(U)f)(t, \cdot) \in \mathcal{A}_M(G_t^{-1}(U))$ for every $t \in \mathbb{R}$, then each pair $(\tilde{G}_t, G_t^*)$ is a supermorphism of $M$ to $M$. Assuming that the integral curves of $X$ are globally defined, the curves in $\mathcal{A}_M(M)$ of the form $z(t) = G_t^* z_0$, with $z_0 \in \mathcal{A}_M(M)$, are understood as the integral curves of $X$ while $z_0$ is therefore the initial condition of $z(t)$. The space of integral curves of $X$ becomes then a Grassmann algebra via the identification of every particular solution with its initial condition. Moreover, if the initial condition $z(0)$ of an integral curve $z(t)$ of $X$ is even or odd, then every $z(t)$ remains so at any $t$.

The system of superdifferential equations (4.9) can be reduced to solving (4.9) for a family of supercoordinates $(x^i, \theta^\alpha)$. In other words, defining $x^i(t) = G_t^* x^i$ and $\theta^\alpha(t) = G_t^* \theta^\alpha$, we obtain that (4.9) reduces to

$$\frac{dx^i(t)}{dt} = x^i(t)^* f^i, \quad \frac{d\theta^\alpha(t)}{dt} = \theta^\alpha(t)^* g^\alpha. \tag{4.10}$$

In short, the systems (4.10) are denoted in physical works by (cf. [13, 14, 28, 33])

$$\frac{dx^i}{dt} = f^i, \quad \frac{d\theta^\alpha}{dt} = g^\alpha. \tag{4.11}$$

Each of the separate differential equations of (4.11) describes the evolution under the superflow of $X$ of a particular solution to (4.9) with a different initial condition given by a super-coordinate. Then, (4.11) describes only the evolution of $m + q$ different particular solutions of (4.9). Such particular solutions allow us to determine $G_t^*$ by using that each $G_t^*$ is a superalgebra morphism. The integral curves $z(t) = G_t^* z_0$ of $X$ are then called particular solutions to (4.11). Nevertheless, it is important to recall that, strictly speaking, $z(t)$ is not a solution to equations (4.11), which describe only $m + q$ particular solutions of (4.9).

The above shows that, although the differential equations of the Section 4.1, e.g. (4.2), and (4.4) may have a similar notation, their meanings are absolutely different.

For the sake of completeness, let us analyse an example coming from supermechanics [13]. Consider the supermanifold $T\mathbb{R}^{1|1}$. It admits a supercoordinate system $(x, \dot{x}, \theta, \dot{\theta})$, which gives rise to a supermanifold isomorphism $T\mathbb{R}^{1|1} \simeq \mathbb{R}^{2|2}$. Consider the set of superdifferential equations

$$\frac{dx}{dt} = \dot{x}, \quad \frac{d\dot{x}}{dt} = -x, \quad \frac{d\theta}{dt} = \dot{\theta}, \quad \frac{d\dot{\theta}}{dt} = -\theta \tag{4.12}$$
describing the so-called super-harmonic oscillator [33, eq. (1.93)]. Particular solutions 
\( z(t) \in \mathcal{A}_R^{2|2} \) of the above superdifferential equation are understood as integral curves of the even supervector field on \( \mathbb{R}^{2|2} \) given by

\[
X = \dot{x} \frac{\partial}{\partial x} - x \frac{\partial}{\partial \dot{x}} + \dot{\theta} \frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial \dot{\theta}}.
\]

On the one hand, \( X \) gives rise to a vector field \( \tilde{X} \) on \( \mathbb{R}^2 \) given by

\[
\tilde{X} = \dot{x} \frac{\partial}{\partial x} - x \frac{\partial}{\partial \dot{x}},
\]

whose integration gives rise to a standard one-parametric group of diffeomorphisms \( \{ \tilde{G}_t : \mathbb{R}^2 \to \mathbb{R}^2 \}_{t \in \mathbb{R}} \), with \( t \in \mathbb{R} \).

On the other hand, particular solutions to (4.12) take the form

\[
z(t) = \mathcal{G}_t^* z(0), \quad \forall t \in \mathbb{R},
\]

for a certain supermorphism \( (\tilde{G} : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2, \mathcal{G}^* : \mathcal{A}_R^{2|2} \to \mathcal{A}_R^{3|2} \circ \tilde{G}^{-1}) \) satisfying

\[
\frac{\partial \mathcal{G}_t^* z_0}{\partial t} = \mathcal{G}_t^* (X z_0), \quad \mathcal{G}_t^* z_0(0, \cdot) = z_0(\cdot), \quad \forall z_0 \in \mathcal{A}_R^{2|2}(\mathbb{R}^2). \tag{4.13}
\]

In consequence, assuming \( z_0 \) to take the values \( x, \dot{x}, \theta, \dot{\theta} \), we obtain that the differential equation (4.13) reduces to

\[
\frac{\partial}{\partial t} \mathcal{G}_t^* x = \mathcal{G}_t^* \dot{x}, \quad \frac{\partial}{\partial t} \mathcal{G}_t^* \dot{x} = -\mathcal{G}_t^* x, \quad \frac{\partial}{\partial t} \mathcal{G}_t^* \theta = \mathcal{G}_t^* \dot{\theta}, \quad \frac{\partial}{\partial t} \mathcal{G}_t^* \dot{\theta} = -\mathcal{G}_t^* \theta. \tag{4.14}
\]

Since \( \mathcal{G}^* \) is an even morphism of sheaves, one has that

\[
\mathcal{G}_t^* x = \mathcal{G}_t^x(t, x, \dot{x}) + \mathcal{G}^x_{\theta\theta}(t, x, \dot{x}) \theta \dot{\theta}, \quad \mathcal{G}_t^* \dot{x} = \mathcal{G}_t^{\dot{x}}(t, x, \dot{x}) + \mathcal{G}^{\dot{x}}_{\theta\theta}(t, x, \dot{x}) \theta \dot{\theta},
\]

\[
\mathcal{G}_t^* \theta = \mathcal{G}_t^{\theta}(t, x, \dot{x}) \theta + \mathcal{G}^{\theta}_{\theta\theta}(t, x, \dot{x}) \theta \dot{\theta}, \quad \mathcal{G}_t^* \dot{\theta} = \mathcal{G}_t^{\dot{\theta}}(t, x, \dot{x}) \theta + \mathcal{G}^{\dot{\theta}}_{\theta\theta}(t, x, \dot{x}) \theta \dot{\theta}, \tag{4.15}
\]

where \( \mathcal{G}^A_B(t, x, \dot{x}) \in C^\infty(\mathbb{R}^3) \) for arbitrary indexes \( A, B \in \{ x, \dot{x}, \theta, \dot{\theta} \} \). Moreover, the equalities (4.15) allow us to stablsh the value of \( \mathcal{G}_t^* \) on any element of \( \mathcal{A}_R^{2|2} \):

\[
\mathcal{G}_t^* \left( \sum_{J, K} \mathcal{A}_{JK}(x, \dot{x}) \theta^J \dot{\theta}^K \right) = \sum_{J, K} \mathcal{A}_{JK}(\mathcal{G}_t^* x, \mathcal{G}_t^* \dot{x}) \mathcal{G}_t^* \theta^J \mathcal{G}_t^* \dot{\theta}^K.
\]

In view of the relations (4.15), the differential equations (4.14) read

\[
\frac{\partial}{\partial t} \mathcal{G}_t^* x = \mathcal{G}_t^* (X x) \Rightarrow \frac{\partial}{\partial t} \mathcal{G}^x + \frac{\partial}{\partial t} \mathcal{G}^{\dot{x}}_{\theta\theta} \theta \dot{\theta} = \mathcal{G}_t^{\ddot{x}} + \mathcal{G}_t^{\dot{x}} \theta \dot{\theta} = \mathcal{G}_t^{\ddot{x}} = \mathcal{G}_t^{\dot{x}}, \quad \frac{\partial}{\partial t} \mathcal{G}_t^x = \mathcal{G}_t^{\ddot{x}} = \mathcal{G}_t^{\dot{x}}, \quad \frac{\partial}{\partial t} \mathcal{G}_t^{\dot{x}} = -\mathcal{G}_t^x, \quad \frac{\partial}{\partial t} \mathcal{G}_t^\theta = -\mathcal{G}_t^{\dot{\theta}}.
\]
Moreover,
\[
\frac{\partial}{\partial t} G^x \phi = G^x_t, \quad \frac{\partial}{\partial t} G^\theta = \frac{\partial}{\partial t} G^\theta_t \Rightarrow \frac{\partial}{\partial t} G^\theta_t = G^\theta_t - G^\theta_t = -G^\theta_t - G^\theta_t = -G^\theta_t.
\]

Hence,
\[
\frac{\partial^2}{\partial t^2} G^x = -G^x, \quad \frac{\partial^2}{\partial t^2} G^\theta = -G^\theta, \quad \frac{\partial^2}{\partial t^2} G^\theta_t = -G^\theta_t.
\]

The previous system of differential equations can straightforwardly be solved and the solutions take the form
\[
G^x = A(x, \dot{x}) \cos(t) + \dot{A}(x, \dot{x}) \sin(t), \quad G^x = -A(x, \dot{x}) \sin(t) + \dot{A}(x, \dot{x}) \cos(t),
\]
\[
G^\theta = B(x, \dot{x}) \cos(t) + \dot{B}(x, \dot{x}) \sin(t), \quad G^\theta = -B(x, \dot{x}) \sin(t) + \dot{B}(x, \dot{x}) \cos(t),
\]
\[
G^\theta = C(x, \dot{x}) \cos(t) + \dot{C}(x, \dot{x}) \sin(t), \quad G^\theta = -C(x, \dot{x}) \sin(t) - \dot{C}(x, \dot{x}) \cos(t),
\]
\[
G^\theta_t = D(x, \dot{x}) \cos(t) + \dot{D}(x, \dot{x}) \sin(t), \quad G^\theta_t = -D(x, \dot{x}) \sin(t) + \dot{D}(x, \dot{x}) \cos(t),
\]
for certain functions \(A(x, \dot{x}), B(x, \dot{x}), C(x, \dot{x}), D(x, \dot{x})\). Using the condition in (4.13) for \(t = 0\), we obtain that
\[
G^x x(0, x, \dot{x}) = A(x, \dot{x}) + \dot{A}(x, \dot{x}) \theta = x, \quad G^x \dot{x}(0, x, \dot{x}) = \dot{A}(x, \dot{x}) + \dot{B}(x, \dot{x}) \theta = \dot{x},
\]
\[
G^\theta(0, x, \dot{x}) = B(x, \dot{x}) \theta + \dot{B}(x, \dot{x}) \theta = \theta, \quad G^\theta(0, x, \dot{x}) = C(x, \dot{x}) \theta + \dot{C}(x, \dot{x}) \theta = \dot{\theta},
\]
then
\[
A(x, \dot{x}) = x, \quad \dot{A}(x, \dot{x}) = \dot{x}, \quad B(x, \dot{x}) = 1, \quad \dot{B}(x, \dot{x}) = 0,
\]
\[
C(x, \dot{x}) = 1, \quad \dot{C}(x, \dot{x}) = 0, \quad D(x, \dot{x}) = 0, \quad \dot{D}(x, \dot{x}) = 0
\]
and finally
\[
G^x x = x \cos(t) + \dot{x} \sin(t), \quad G^x \dot{x} = -x \sin(t) + \dot{x} \cos(t),
\]
\[
G^\theta = \cos(t) \theta + \sin(t) \dot{\theta}, \quad G^\theta = \cos(t) \dot{\theta} - \sin(t) \dot{\theta}.
\]

The above can be summarised by writing
\[
G^x = \exp(tB) \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}, \quad B := \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.
\]

**Remark 4.2.** Superflows can be defined also for odd supervector fields (see for example [43]). The difference is that the ‘time’ is odd, so we deal with a supermorphism \(G : \mathbb{R}^{0|1} \times \mathcal{M} \rightarrow \mathcal{M}\).
4.3 A case with a double interpretation

This section illustrates that there exist superdifferential equations with physical applications that can be understood simultaneously in the different ways described in the two previous subsections. As a consequence, their particular solutions will become different, which may have physical implications.

For instance, let us understand the superdifferential equation (4.2) as a superdifferential equation [34, 38]. This is accomplished by considering (4.2) as the superdifferential equation for the integration of an even supervector field [38] on $\mathbb{C}^{1|1}$ of the form

$$X = \frac{i}{\hbar} \left[ (1 - \alpha^2) \frac{\partial}{\partial \alpha} - \alpha \theta \frac{\partial}{\partial \vartheta} \right].$$

the superflow of $X$ gives rise to a supermanifold mapping $\mathcal{G} : \mathbb{R} \times \mathbb{C}^{1|1} \rightarrow \mathbb{C}^{1|1}$ associated with a one-parametric family of diffeomorphisms $\{\mathcal{G}_t : \mathbb{C} \rightarrow \mathbb{C}\}_{t \in \mathbb{R}}$ and a mapping $\mathcal{G}^* : \mathcal{A}_C^{1|1} \rightarrow \mathcal{A}_C^{2|1} \circ \mathcal{G}^{-1}$ such that $\mathcal{G}_t^* z_0 = z(t)$ for every $z_0 \in \mathcal{A}_C^{1|1}(\mathbb{C})$ is a particular solution to

$$\frac{\partial}{\partial t} \mathcal{G}^* z = \mathcal{G}^*(X z), \quad \mathcal{G}^* z(0, \cdot) = z(\cdot), \quad \forall z \in \mathcal{A}_C^{1|1}(\mathbb{C}). \quad (4.16)$$

In this approach, we demand $\mathcal{G}_t^* : z_0 \in \mathcal{A}_C^{1|1}(\mathbb{C}) \mapsto \mathcal{G}_t^* z_0 \in \mathcal{A}_C^{1|1}(\mathbb{C})_{t \in \mathbb{R}}$ to give rise to a one-parametric family of superalgebra morphisms. Since this is always possible due to the fact that $X$ is even (see [36, 39]), the particular solutions, $z(t)$, obtained in this interpretation must necessarily differ from the ones given in Section 4.1, which did not give rise to such Grassmann algebra morphisms. Moreover, note that solutions to (4.16) are curves in $\mathcal{A}_C^{1|1}(\mathbb{C})$, while interpreting (4.2) as a Grassmann-valued differential equation gave rise to solutions which are curves in $\mathcal{A}_C^{0|1}(\mathbb{C})$. In any case, let us obtain explicitly the supermorphisms $\mathcal{G}_t^*$ to dissipate any doubt about our latter claims.

Since each $\mathcal{G}_t^*$ should be a morphism of superalgebras, one can write $\mathcal{G}^* \alpha = \mathcal{G}_\alpha(t, \alpha)$ and $\mathcal{G}^* \vartheta = \mathcal{G}_\vartheta(t, \alpha) \vartheta$ for certain complex valued functions $\mathcal{G}_\alpha, \mathcal{G}_\vartheta$ depending on $t$ and $\alpha$. The differential equations determining $\mathcal{G}_\vartheta$ and $\mathcal{G}_\alpha$ are obtained by assuming $z = \alpha$ and $z = \vartheta$ in (4.16), respectively. This gives us

$$\frac{\partial \mathcal{G}_\alpha}{\partial t} = \frac{\partial \mathcal{G}_t \alpha}{\partial t} = \mathcal{G}_t^* (X \alpha) = \frac{i}{\hbar} \mathcal{G}_t^* (1 - \alpha^2) = \frac{i}{\hbar} (1 - \mathcal{G}_\alpha^2), \quad \mathcal{G}_\alpha(0, \alpha) = \alpha$$

$$\frac{\partial \mathcal{G}_\vartheta}{\partial t} = \frac{\partial \mathcal{G}_t \vartheta}{\partial t} = \mathcal{G}_t^*(X \vartheta) = -\frac{i}{\hbar} \mathcal{G}_t^*(\alpha \vartheta) = -\frac{i}{\hbar} \mathcal{G}_\alpha \mathcal{G}_\vartheta \vartheta, \quad \mathcal{G}_\vartheta(0, \alpha) = 1.$$ 

Solving the above system of PDEs, one obtains

$$\mathcal{G}_\alpha = \frac{\alpha \cos(t/\hbar) + i \sin(t/\hbar)}{i \alpha \sin(t/\hbar) + \cos(t/\hbar)}, \quad \mathcal{G}_\vartheta = \frac{1}{\cos(t/\hbar) + \alpha i \sin(t/\hbar)}.$$ 

Recall that the general solution to system (4.2), as a Grassmann-valued differential equation, takes the form of a curve $z(t) = \mathcal{G}_\alpha(t, \alpha) + \xi_0 \mathcal{G}_\vartheta(t, \alpha) \vartheta$ in the Grassmann algebra $\Lambda \mathbb{C}$, where
α + ξ_0θ is the initial condition of each particular solution. Meanwhile, the solutions of (4.2) as a superdifferential equation are curves in \( A_C^{11}(C) \) giving rise to a superflow. For instance, given a function \( f_1(α) + f_2(α)θ \) of \( A_C^{11}(C) \), the solution with this initial condition is

\[
z(t) = f_1(G_α(t, α)) + f_2(G_α(t, α))G_θ(α, t)θ,
\]

which is a curve in \( A_C^{11}(C) \). Hence, one sees that the nature of the solutions and the problem are absolutely different. The way of understanding superdifferential equations must therefore depend on the nature of the problem under study. It is worth noting that the way of understanding (4.2) in [15] is the one given in Section 4.1.

5 Integration of Grassmann-valued differential equations

Based on the examples of Section 4.1, we propose now a geometric description for Grassmann-valued differential equations. Hereafter, \( π_F : F → M \) stands for a Grassmann bundle. The tangent vector bundle \( τ_F : TF → F \) admits a vector subbundle \( τ_F|VF : VF → F \) where \( VF = \ker Tπ_F \). In turn, we can still construct a new vector bundle \( Vπ_F : VF → M \), where

\[
Vπ_F = π_F ◦ τ_F|VF.
\]

Observe that every vertical bundle morphism \( F : F → F \), namely \( π_F ◦ F = π_F \), gives rise to a section of the vertical bundle \( τ_F|TV : VF → F \) via the natural isomorphism \( VF ≃ F ⊕_M F \) and vice versa. That is why we will identify vertical bundle morphisms \( F \) and sections of \( τ_F|TV \) hereafter.

Definition 5.1. Given a curve \( z : t ∈ \mathbb{R} → z_t ∈ Γ(M, F) \) in the space of sections of the Grassmann bundle \( π_F : F → M \), we call tangent vector of \( z \) at \( t \) the section \( Vz_t ∈ Γ(M, VF) \) of the form

\[
Vz_t(x) := \frac{dz_t(x)}{dt} ∈ (VF)_{z_t}, \quad ∀x ∈ M, \quad ∀t ∈ \mathbb{R}.
\]

Definition 5.2. A differential equation on a Grassmann bundle \( π_F : F → M \) is a differential equation

\[
\frac{dz_t(x)}{dt} = F_t(z_t), \quad z_t ∈ Γ(M, F), \quad ∀t ∈ \mathbb{R}, \tag{5.1}
\]

where each \( F_t \) is a section of the bundle \( Vπ_M : VF → F \) and \( z_t \) is a curve in \( Γ(M, F) \). The curves in \( Γ(M, F) \) satisfying (5.1) are called particular solutions of (5.1).

A particular solution to a first-order ordinary differential equation on a manifold can be interpreted as a curve in the manifold. Analogously, a particular solution to a differential equation on a Grassmann bundle is a curve in the space of sections of the associated bundle.

Example 5.3. Let us study the Grassmann Riccati differential equations on \( \mathbb{R}^{m|q} \), namely

\[
\frac{dz}{dt} = a_t^0z + a_t^1z + a_t^2z^2, \quad a_t^0, a_t^1, a_t^2, z ∈ A_{\mathbb{R}}^{m|q}(\mathbb{R}^m), ∀t ∈ \mathbb{R}, \tag{5.2}
\]
by means of Grassmann-valued differential equations. Classically, each particular solution to (5.2) is considered as a curve $z : t \in \mathbb{R} \mapsto z(t) \in A^m_{\mathbb{R}}(\mathbb{R}^m)$ whose derivative $dz/dt$ at $t$ coincides with the value of the right-hand side of (5.2) for $z = z(t)$.

The Bachelor–Gawędzki theorem ensures that there exists a natural non-canonical isomorphism of sheaves of superalgebras $A^m_{\mathbb{R}}|_q \simeq A_{A\pi}$, where $A_{A\pi}$ is the sheaf of sections of the exterior bundle of the vector bundle $\pi : (w_1, w_2) \in E := \mathbb{R}^m \times \mathbb{R}^q \mapsto w_1 \in \mathbb{R}^m$. The right-hand side of (5.2) can now be interpreted as a $t$-dependent family of sections $F^R_t : \Lambda \rightarrow V(\Lambda E) \simeq \Lambda E \oplus M \Lambda E$ of the form

$$F^R_t(w, \vartheta) = (w, \vartheta, a_0(t, w) + a_1(t, w)\vartheta + a_2(t, w)\vartheta^2), \quad w \in M, \vartheta \in \Lambda \mathbb{R}^q, \forall t \in \mathbb{R},$$

where the multiplication on the right-hand side is the one naturally induced by the Grassmann structure in $\Lambda \mathbb{R}^q$. Therefore, (5.2) can be recast as

$$\frac{dz_t}{dt}(x) = F^R_t(z_t(x)), \quad z_t \in \Gamma(\mathbb{R}^m, \Lambda E), \quad \forall t \in \mathbb{R}, \forall x \in \mathbb{R}^m.$$ 

**Example 5.4.** The superdifferential equations (4.1), for $t$-independent coefficients, can also be interpreted as Grassmann-valued differential equations. This is achieved by realizing that $\Lambda V$ is the space of sections of the Grassmann bundle $\Lambda \pi : \Lambda V \rightarrow \{\ast\}$ obtained from $\pi : V \rightarrow \{\ast\}$. Hence, (4.1) can be written as a differential equation on the Grassmann bundle defined by $\pi_V$.

The above examples can be extended to a more general family of evolution differential equations on supermanifolds.

**Proposition 5.5.** The superdifferential equation on $\mathcal{M} := (M, A_M)$ of the form

$$\frac{dz}{dt} = F_t(z), \quad z \in A_M(M),$$

where $F_t(z)$ is, for every $t \in \mathbb{R}$, a polynomial function in $z$ and $dz/dt$ is understood as an element of $A_M(M)$, can be understood as a differential equation on a Grassmann bundle.

**Proof.** The Bachelor–Gawędzki Theorem ensures the existence of an isomorphism of sheaves of superalgebras $\Psi_{BG} : A_M \simeq \Gamma_{A\pi}$ for the exterior bundle $\Lambda \pi : \Lambda E_M \rightarrow M$ associated with the structural bundle $\pi : E_M \rightarrow M$ of $\mathcal{M}$. Since $\Psi_{BG}$ is, in particular, a linear morphism of sheaves, we have that $z^* := \Psi_{BG}(z)$ satisfies

$$\frac{dz^*}{dt} = \Psi_{BG} \circ F_t \circ \Psi^{-1}_{BG}(z^*), \quad z \in \Gamma_{A\pi}(M).$$

Since $F_t(z) = \sum_J a_J(t)z^J$ is a polynomial in $z$ and $\Psi_{BG}$ is a morphism of sheaves of superalgebras, we can write

$$\Psi_{BG} \circ F_t \circ \Psi^{-1}_{BG}(z^*) = \sum_J \tilde{a}_J(t)(z^*)^J = \frac{dz^*}{dt} \quad (5.4)$$

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for certain coefficients $\bar{a}_J(t)$. Taking into account the natural isomorphism $V(\Lambda E_M) \simeq \Lambda E_M \oplus_M \Lambda E_M$ where the vertical bundle is considered relative to the projection $\Lambda E_M \to M$, we can interpret the latter expression as a differential equation on the Grassmann bundle $\Lambda \pi : \Lambda E_M \to M$. Hence, every solution of (5.3) gives rise to a particular solution to the differential equation on a Grassmann bundle (5.4). The converse is trivial from the above, which concludes the proof.

By using local trivializations, every superdifferential equation (5.1) can be transformed into a normal system of first-order ordinary differential equations in the coefficients of the decomposition of $z$ in a local base of sections of $\Gamma(M, F)$. The theorem of existence and uniqueness of solutions for first-order ordinary differential equations allow us to prove the following proposition.

**Proposition 5.6.** Given a differential equation (5.1) on a Grassmann bundle $\pi : F \to M$, there always exists locally a particular solution with initial condition $\gamma_0 \in \Gamma(M, F)$.

This section depicts several integration properties of a certain class of differential equations on Grassmann bundle steaming from their associated Grassmann algebra structures.

**Theorem 5.7.** The integration of every Grassmann-bundle differential equation on $\pi : F \to M$ of the form

$$\frac{dz}{dt} = F_t(z), \quad z \in \Gamma(M, F)$$

where $F_t(z)$ is, for every $t \in \mathbb{R}$, a polynomial in $z$ relative to the product in $\Gamma(M, F)$, can be reduced to the integration of a hierarchy of nested standard differential equations. The first member of the hierarchy is given by an Abel differential equation.

**Proof.** Using a local trivialisation of $F$, we can assume locally that $F$ is of the form $M \times \Lambda K^q$. Consider a set $\{\theta_1, \ldots, \theta_q\}$ of odd generators of $\Lambda K^q$. We write $I_k$ for the ideal of $\Lambda K^q$ generated by the elements $\{\theta_k, \ldots, \theta_q\}$ for $k = 1, \ldots, q$. Then, $\Lambda K^q/I_k \simeq \Lambda K^{q-k}$ as Grassmann algebras and we have a commutative diagram of superalgebra morphisms

$$
\begin{array}{ccccccc}
\Lambda K^q & \xrightarrow{\epsilon_q} & \Lambda K^q/I_q & \xrightarrow{\epsilon_{q-1}} & \Lambda K^q/I_{q-1} & \xrightarrow{\epsilon_{q-2}} & \cdots & \Lambda K^q/I_1 \\
\end{array}
$$

where the mappings $\epsilon_k^*$, with $k = 1, \ldots, q$, are the corresponding quotient maps. Observe that $\epsilon_1^* := \epsilon^*$ is the augmentation morphism and all $\epsilon_k^*$ are superalgebra morphisms. If $\Delta_k$ is the ideal of $\Lambda K^q$ generated by $\theta_k$, then one has a Grassmann algebra isomorphism

$$(\Lambda K^q/I_k)/\Delta_{k-1} \simeq \Lambda K^q/I_{k-1},$$

which implies that each $\epsilon_{k-1}^* : \Lambda K^q/I_k \to \Lambda K^q/I_{k-1}$ is a Grassmann algebra morphism.

We write $z_i = \epsilon_i^*(z)$ for $i = 1, \ldots, q$. Due to the linearity of the morphisms $\epsilon_i^*$, we have

$$\frac{dz_i}{dt} = \epsilon_i^* \left( \frac{dz}{dt} \right) = \epsilon_i^*(F_t(z)), \quad i = 1, \ldots, q,$$
where, if \( \mathcal{F}_t(z) := \sum_{\alpha} c_{\alpha}(t)z^\alpha \), then \( \epsilon^*_t(\mathcal{F}_t(z)) = \sum_{\alpha} \epsilon^*_t(c_{\alpha}(t))z^\alpha \). In particular, we have the generalised Abel equation

\[
\frac{dz_1}{dt} = \sum_{\alpha} \epsilon^*_1(c_{\alpha}(t))z_1^\alpha.
\]

Once the general solution to this equation is known, the differential equation for \( z_2 \) can be determined by a differential equation

\[
\frac{dz_2}{dt} = \sum_{\alpha} \epsilon^*_2(c_{\alpha}(t))z_2^\alpha,
\]

whose has a right-hand side that is polynomial in \( z_2 \). If a particular solution for the equation for \( z_1 \) is known, then we can assume that \( \epsilon^*_2 z_2 = z_1 \), which gives a restriction to the possible values of \( z_2 \). Assuming a particular solution for \( z_2 \), we can iteratively apply the same procedure to obtain \( z_3 \) and the remaining projections of \( z \) up to \( z \). Then, the general solution to (5.5) can be obtained inductively through a hierarchy of standard differential equations.

**Example 5.8.** Let us describe a Grassmann-valued equation of the type (5.5) as a differential equations on a Grassmann bundle. Consider for instance the Grassmann-valued Riccati equation on \( \mathbb{R}^{1|2} \) of the form

\[
\frac{dz}{dt} = \mathcal{F}_R(z) := 1 + z^2, \quad z \in \mathcal{A}^{2|1}_R(\mathbb{R}^2).
\]

(5.6)

Let \((x, \theta_1, \theta_2)\) be a global supercoordinate system on \( \mathcal{A}^{1|2}_R \) with \( x \) being a coordinate on \( \mathbb{R} \). This global coordinate system induces a Grassmann algebra isomorphism \( \mathcal{A}^{1|2}_R(\mathbb{R}) \cong \Gamma(\mathbb{R}, \Lambda E) \), where the structural bundle is \( E := \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R} \). Hence, \( z \) can be identified with an element of \( \Gamma(\mathbb{R}, \Lambda E) \), the variable \( x \) is by assumption a coordinate for \( \mathbb{R} \) and \( \theta_1, \theta_2 \) can be understood as odd generators of \( \Lambda \mathbb{R}^2 \). This allows us to consider (5.6) as a differential equation on a Grassmann bundle where \( 1 + z^2 \) is to be understood as an element of \( V_z \Lambda \mathbb{R}^2 \) via the natural linear isomorphism \( V_z \Lambda \mathbb{R}^2 \cong \Lambda \mathbb{R}^2 \).

A section \( z \in \Gamma(\mathbb{R}, \Lambda E) \) can be written in a unique way as

\[
z(x) = z_{(0,0)}(x) + z_{(1,0)}(x)\theta_1 + z_{(0,1)}(x)\theta_2 + z_{(1,1)}(x)\theta_1\theta_2,
\]

for functions \( z_{(0,0)}, z_{(1,0)}, z_{(0,1)}, z_{(1,1)} \in C^\infty(\mathbb{R}) \), which are even elements of \( \Gamma(M, \Lambda E) \). Let us apply the method given in Theorem 5.7. Construct the ideals \( I_1 := \langle \theta_1, \theta_2 \rangle, I_2 := \langle \theta_2 \rangle \). Obviously, \( \Lambda \mathbb{R}^2 / I_1 \cong \mathbb{R} \). Applying the augmentation morphism to (5.6) we obtain the differential equation

\[
\frac{\partial z_{(0,0)}}{\partial t} = 1 + z_{(0,0)}^2.
\]

(5.7)

It is immediate that its solution for each \( x \) reduces to solving a Riccati differential equation with constant coefficients, whose solutions are known and can be considered as a type of
Abel differential equation \([29]\). Next, we consider the projection \(\varepsilon_2^* : \Lambda K^2 \rightarrow \Lambda K^2/I_2\) of the equation (5.6), which reads

\[
\frac{\partial z_{(0,0)}}{\partial t} = 1 + z^2_{(0,0)}, \quad \frac{\partial z_{(1,0)}}{\partial t} = z_{(0,0)}z_{(1,0)}.
\]

(5.8)

For each particular solution of the equation (5.7) for \(z_{(0,0)}\), we can determine the some particular values of \(z_{(1,0)}\). Again, particular solutions for \(z_{(0,0)}\) and \(z_{(1,0)}\) allow us to obtain the solutions to \(z_{(0,1)}\) and \(z_{(1,1)}\) by solving

\[
\frac{\partial z_{(0,0)}}{\partial t} = 1 + z^2_{(0,0)}, \quad \frac{\partial z_{(1,0)}}{\partial t} = z_{(0,0)}z_{(1,0)}, \quad \frac{\partial z_{(0,1)}}{\partial t} = z_{(0,0)}z_{(0,1)}, \quad \frac{\partial z_{(1,1)}}{\partial t} = 0.
\]

This enables us to determine the solutions to (5.6) by solving a hierarchy of standard differential equations.

The last step of the previous example demands to solve simultaneously the differential equations on the unknowns \(z_{(0,1)}\) and \(z_{(1,1)}\) in terms of particular known solutions concerning the remaining variables. In differential equations of the type (5.5) where the odd dimension will be larger, this situation can be even more complicated because of having to solve many different variables simultaneously. It is therefore interesting to provide a method to determine the form of a unique \(z_{(i,j)}\) in terms of previous solved coordinates of \(z\) at each stage. This can be more easily afforded through the following result.

**Proposition 5.9.** Take a basis of odd generators \(\{\theta_1, \ldots, \theta_q\}\) of \(\Lambda K^q\) and define \(I_k := \langle \theta_k, \ldots, \theta_q \rangle\), for \(k = 1, \ldots, q\). We have the following commutative diagram of algebra morphisms:

![Diagram](image)

where \(\delta^p_r : \Lambda K^q_{\Lambda > r + 1K^q + I_r} \rightarrow \Lambda K^q_{\Lambda > rK^q + I_{r-1}}\) and \(\epsilon^*_r : \Lambda K^q_{\Lambda > rK^q + I_{r-1}} \rightarrow \Lambda K^q_{\Lambda > r+1K^q + I_r}\) are quotient maps.
Proof. It is an immediate consequence of the fact that \( I_k \), with \( k = 1, \ldots, q \), and \( \Lambda^{>r}\mathbb{K}^{q} \), for \( r = 0, \ldots, q - 1 \), are ideals of \( \Lambda\mathbb{K}^{q} \) and the commutativity of the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\pi_{1}^{0}} & A \\
\downarrow{\pi_{1}^{0}} & & \downarrow{\pi_{1}^{0}} \\
A & \xrightarrow{\pi_{2}^{0}} & A \\
\downarrow{\pi_{2}^{0}} & & \downarrow{\pi_{2}^{0}} \\
\end{array}
\]

where \( \pi^{B} \) is the projection \( \pi^{B} : A/B \to A/C \) for arbitrary ideals \( B, C \) in \( A \), e.g. \( I_{1}, I_{2} \).

Example 5.10. Proposition 5.9 allows us to simplify the integration of Grassmann-valued differential equations of the form (5.5) by means of a hierarchy of nested differential equations. Let us apply it to the case of the Grassmann-valued differential equation on \( \pi : \mathbb{K} \times \Lambda\mathbb{K}^{2} \to \mathbb{K} \) of the form

\[
\frac{dz}{dt} = \theta_{1} + \theta_{2} z^{2},
\]

where \( \theta_{1}, \theta_{2} \) are a basis of odd variables in \( \Lambda\mathbb{K}^{2} \). Considering the spaces \( I_{1} = \langle \theta_{1}, \theta_{2} \rangle \) and \( I_{2} = \langle \theta_{2} \rangle \), we have the diagrams

\[
\begin{array}{ccc}
\Lambda\mathbb{K}^{2} & \xrightarrow{\delta_{1}} & \Lambda\mathbb{K}^{2}/\Lambda\mathbb{K}^{2} \delta_{1}^{2} \\
\downarrow{\epsilon_{2}^{2}} & & \downarrow{\epsilon_{2}^{2}} \\
\Lambda\mathbb{K}^{2} & \xrightarrow{\delta_{1}} & \Lambda\mathbb{K}^{2}/\Lambda\mathbb{K}^{2} \delta_{1}^{2} \\
\downarrow{\epsilon_{1}^{2}} & & \downarrow{\epsilon_{1}^{2}} \\
\mathbb{K} & \xrightarrow{\text{Id}} & \mathbb{K} \\
\end{array}
\]

\[
\begin{array}{ccc}
\langle 1, \theta_{1}, \theta_{2}, \theta_{1} \wedge \theta_{2} \rangle & \xrightarrow{\delta_{2}} & \langle 1, \theta_{1}, \theta_{2} \rangle \\
\downarrow{\epsilon_{2}^{2}} & & \downarrow{\epsilon_{2}^{2}} \\
\langle 1, \theta_{1} \rangle & \xrightarrow{\delta_{1}} & \langle 1, \theta_{1} \rangle \\
\downarrow{\epsilon_{1}^{2}} & & \downarrow{\epsilon_{1}^{2}} \\
\langle 1 \rangle & \xrightarrow{\text{Id}} & \langle 1 \rangle \\
\end{array}
\]

where every element of the right-hand side diagram stands for the equivalence class containing that element.

Now, the differential equation (5.6) projects onto each space of the above diagram giving rise to a hierarchy of differential equations which can be solved by integrating a partial differential equation involving only a new variable at each time, e.g. (I), (II), (III) and (IV), consecutively.

\[
\frac{dz}{dt} = \theta_{2} + \theta_{1} z^{2} \quad (IV) \quad \frac{\partial z(0,0)}{\partial t} = 0, \quad \frac{\partial z(1,0)}{\partial t} = z_{2}^{2}(0,0), \quad \frac{\partial z(0,1)}{\partial t} = 1 \quad (III) \quad \frac{\partial z(0,0)}{\partial t} = 0
\]

\[
\frac{\partial z(0,0)}{\partial t} = 0, \quad \frac{\partial z(0,1)}{\partial t} = 1 \quad (II) \quad \frac{\partial z(0,0)}{\partial t} = 0, \quad \frac{\partial z(0,1)}{\partial t} = 1 \quad \frac{\partial z(0,0)}{\partial t} = 0
\]

\[
\frac{\partial z(0,0)}{\partial t} = 0 \quad (I) \quad \frac{\partial z(0,0)}{\partial t} = 0 \quad \frac{\partial z(0,0)}{\partial t} = 0
\]
6 Conclusions and Outlook

It is strongly believed that the so-called superequations or superdifferential equations are firmly related to the properties of supermanifolds. The very same term “superequation” seems to support this idea. Despite that, we have studied several types of superdifferential equations and we have shown that the properties of a certain subclass, e.g. their evolution, cannot be described through supermanifold morphisms. This suggested us to develop a theory to analyse these superequations, the here called Grassmann-valued differential equations. These Grassmann-valued differential equations appeared in the theory of supercoherent states, super-Riccati differential equations, and super-superposition rules. Meanwhile, the term superdifferential equation on supermanifolds was reserved for superdifferential equations that can be fully described through supermanifolds, e.g. the ones appearing in supermechanics and in the integration of supervector fields [13, 33].

The mathematical properties of above-mentioned classes of superdifferential equations, e.g. their solutions, were proved to have a different nature. This has relevant applications, as certain superdifferential equations in the literature may be understood in both mentioned ways. In these cases, the proper interpretation has to be determined according to the expected properties of their solutions.

In the future, we will apply our theory to study Grassmann-valued differential equations of physical and mathematical interest, e.g. Grassmann Riccati equations. It seems that the study of these differential equations can lead to the generalisation of the theory of Lie systems [46] to Grassmann-valued differential equations. There exists also a big deal of Grassmann-valued partial differential equations appearing as supersymmetric generalisations of standard differential equations [1, 26]. It is still an open question to provide a geometric understanding of such partial differential equations.

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