ON ASYMPTOTIC DIMENSION OF COUNTABLE ABELIAN GROUPS

J. Smith

Abstract. We compute the asymptotic dimension of the rationals given with an invariant proper metric. Also we show that a countable torsion abelian group taken with an invariant proper metric has asymptotic dimension zero.

§1 Introduction

Gromov introduced the notion of asymptotic dimension as an invariant of finitely generated discrete groups [Gr]. This invariant was studied in a sequel of papers [B-D1], [B-D2], [B-D3], [B-D-K], [Ji] and others. The notion of asymptotic dimension can be extended to the class of all countable groups and most of the results for finitely generated groups are valid for countable groups [D-S]. To define asymptotic dimension for a general countable group one should consider a left invariant proper metric on it. It turns out that such metrics alway exists and any two such metrics on a group \( \Gamma \) are coarsely equivalent, i.e., they lead to the same number \( \text{asdim} \Gamma \).

Even for very familiar infinitely generated countable groups, like the group of rationals \( \mathbb{Q} \), an invariant proper metric turns them into a quite complicated geometrical object. To give an idea, we notice that an invariant metric on \( \mathbb{Z}_{\left\lceil \frac{1}{2} \right\rceil} \subset \mathbb{Q} \) can be defined as induced from the metric graph obtained by gluing together infinitely many isosceles triangles with sides \( 2^n - 2^{n-1} \), \( n \in \mathbb{N} \), by the following rule. First we glue triangles with sides 2-2-1 to all intervals \( [n, n+1] \subset \mathbb{R} \), \( n \in \mathbb{Z} \) and mark their free vertices by the averages of the endpoints, \( \frac{2n+1}{2} \) (see Figure 1).
Then to every edge of length 2 we glue a triangle with sides 4-4-2 and mark its free vertex by the average of the base and so on. Then $\mathbb{Z}[\frac{1}{2}]$ is identified with the set of vertices of this graph.

The corresponding picture for $\mathbb{Q}$ is more complicated. Nevertheless, in §3 we compute that $\operatorname{asdim} \mathbb{Q} = 1$. In particular, $\operatorname{asdim} \mathbb{Z}[\frac{1}{2}] = 1$.

In the case of finitely generated groups, if $\operatorname{asdim} \Gamma = 0$, then the group $\Gamma$ is finite. This is not true for countable groups. In §2 of the paper we give a criterion for a group to have asymptotic dimension 0. As a corollary we obtain that all torsion countable abelian groups have asymptotic dimension 0.

In §4 we show that the asymptotic dimension of the rationals taken with $p$-adic norm is zero. Since $p$-adic norm is not proper on $\mathbb{Q}$ this cannot be done by the criterion of §2.

**Preliminaries.**

The asymptotic dimension is defined for metric spaces.

**Definition [Gr].** We say that a metric space $X$ has asymptotic dimension $\leq n$ if, for every $d > 0$, there is an $R$ and $n + 1$ $d$–disjoint, $R$–bounded families $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_n$ of subsets of $X$ such that $\bigcup \mathcal{U}_i$ is a cover of $X$.

We say that a family $\mathcal{U}$ of subsets of $X$ is $R$–bounded if $\sup \{\operatorname{diam} U | U \in \mathcal{U} \} \leq R$. Also, $\mathcal{U}$ is said to be $d$–disjoint if $d(x, y) > d$ whenever $x \in U$, $y \in V$, $U \in \mathcal{U}$, $V \in \mathcal{U}$, and $U \neq V$.

The notion $\operatorname{asdim}$ is a coarse invariant (see [Roe]).

Let $f : X \to Y$ be a map between metric spaces. If, for each $R > 0$, there is an $S > 0$ such that $d(f(x), f(y)) < S$ whenever $d(x, y) < R$, then we say that $f$ is bornologous. If the preimage of each bounded subset of $Y$ is a bounded subset of $X$, then we say that $f$ is metrically proper. A map is said to be coarse if it is both metrically proper and bornologous. Also, given two maps $f, f' : X \to Y$, where $X$ is a set and $Y$ is a metric
space, then we say that \( f \) and \( f' \) are close if \( \sup_{x \in X} d(f(x), f'(x)) < \infty \). We say that a metric space \((X, d)\) is proper if closed, bounded sets are compact.

DEFINITION [Roe]. Suppose \( f : X \to Y \) is a coarse map between metric spaces. \( f \) is a coarse equivalence if there is a coarse map \( g : Y \to X \) such that \( f \circ g \) is close to the identity function on \( Y \) and \( g \circ f \) is close to the identity function on \( X \).

Let \( G \) be a group. A map \( \| \cdot \| : G \to [0, \infty) \) is said to be a norm on \( G \) if \( \| x \| = 0 \) if and only if \( x = 1_G \), \( \| x^{-1} \| = \| x \| \) for all \( x \in G \), and \( \| xy \| \leq \| x \| + \| y \| \) for all \( x, y \in G \).

Given a norm \( \| \cdot \| \) on \( G \), define \( d : G \times G \to [0, \infty) \) by \( d(x, y) = \| x^{-1} y \| \), where \( x, y \in G \). It is easy to verify that \( d \) is a left invariant metric. We say that a norm on \( G \) is proper if it has the property that for each \( R > 0 \), there are only finitely many \( x \in G \) such that \( \| x \| \leq R \). Then \( d \) will induce the discrete topology on \( \Gamma \) and \( d \) will be a proper metric.

Let \( G \) be a finitely generated group with finite generating set \( S \). We define \( \| x \| = \inf \{ n | x = \gamma_1 \gamma_2 \cdots \gamma_n, \gamma_i \in S \cup S^{-1} \} \). This can be shown to be a norm on the group \( G \). The metric induced by this norm is known as the word metric on \( G \) associated with \( S \).

Since the word metrics associated with any two finite generating sets of a finitely generated group are coarsely equivalent (even quasi-isometric), the asymptotic dimension is a group invariant. Below we show that every countable group admits a left invariant proper metric. In view of Proposition 1, one can extend the invariant asdim to all countable groups (not necessarily finitely generated).

For countable groups, note that a left invariant, proper metric induces the discrete topology. To see this, observe that for a proper metric, the group with this metric is complete. To get a contradiction, suppose that there are no isolated points; then, as a consequence of the Baire category theorem, the group is not countable, a contradiction. Thus, the group contains an isolated point. As the metric is left invariant, left multiplication by a fixed element is an isometry and hence a homeomorphism. This implies that every point in the group is isolated, so that the metric induces the discrete topology. Thus, we have that a left invariant metric on a countable group is proper if and only if each bounded set is finite.

**Proposition 1.** For a countable group, any two left invariant, proper metrics are coarsely equivalent.

**Proof.** Let \( G \) be a group with left invariant, proper metrics \( d \) and \( d' \). First, we show \( id : (X, d) \to (X, d') \) is bornologous. Let \( R > 0 \) be given. Let \( B(R, d) = \{ g \in G | d(g, 1) \leq R \} \). Since \( d \) is proper, \( B(R, d) \) is finite. Thus, there is an \( S > 0 \) such that \( B(R, d) \subset B(S, d') \). So if \( d(x, y) \leq R \), then \( d(1, x^{-1} y) \leq R \) since \( d \) is left invariant. Thus, \( x^{-1} y \in B(R, d) \), and so \( x^{-1} y \in B(S, d') \), or \( d'(1, x^{-1} y) \leq S \). Hence \( d'(x, y) \leq S \). So \( id \) is bornologous. By a similar argument, \( id^{-1} \) is bornologous. This shows that \( id \) and \( id^{-1} \) are proper. So
id is a coarse equivalence. □

DEFINITION. Let Γ be a countable group. Let S be a generating set (possibly infinite) for Γ. A weight function \( w : S \to [0, \infty) \) on S is a function such that the following properties hold:

1. if \( w(s) = 0 \), then \( 1_\Gamma \in S \) and \( s = 1_\Gamma \),
2. \( w(s) = w(s^{-1}) \) whenever \( s, s^{-1} \in S \), and
3. for each \( N \in \mathbb{N} \), \( w^{-1}[0, N] \) is a finite set.

The third property says that \( w \) is a proper map, where \( \Gamma \) has the discrete topology. Also, this property can essentially be viewed as the requirement that \( \lim w = \infty \).

It is not hard to see that for any countable group \( \Gamma \), there is a weight function. In fact, for any generating set \( S \), there is a weight function with domain \( S \). Also, a weight function \( w : S \to [0, \infty) \) can be extended to a weight function on \( S \cup S^{-1} \) (or \( S \cup S^{-1} \cup 1_\Gamma \)).

**Theorem 1.** A weight function on the countable group \( \Gamma \) induces a proper norm \( \| \cdot \| \), and so a weight function induces a left invariant, proper metric \( d \).

**Proof.** Given a weight function \( w : S \to [0, \infty) \), where \( S \) is a generating set for the countable group \( \Gamma \), define \( \| x \| = \inf \{ \sum_{i=1}^{n} w(s_i) | x = s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_n^{\epsilon_n}, s_i \in S, \epsilon_i \in \{ \pm 1 \} \} \). Note that if we view \( 1_\Gamma \) as an empty product, \( \| 1_\Gamma \| = 0 \). The proof that \( \| \cdot \| \) is a norm is left to the reader.

Let \( R > 0 \) be given. Let \( r \) be a nonzero value that the weight function assumes (otherwise, the weight function is always zero, in which case \( \Gamma \) is trivial, and so the theorem holds). So \( \{ s \in S | 0 < w(s) \leq r \} \) is nonempty and finite by definition. Thus, there is a \( t \in S \) such that \( w(t) = \min \{ w(s) | s \in S, 0 < w(s) \leq r \} \). It is immediate that \( 0 < w(t) \leq w(s) \) for all \( s \in S \setminus 1_\Gamma \). Now, suppose \( x \) is such that \( \| x \| \leq R \) and \( x \neq 1_\Gamma \). Then \( \| x \| < R + 1 \). So there are \( s_1, s_2, \ldots, s_n \in S \), and \( \epsilon_1, \epsilon_2, \ldots, \epsilon_n \in \{ \pm 1 \} \) such that \( x = s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_n^{\epsilon_n} \) and \( \sum_{i=1}^{n} w(s_i) < R + 1 \). Further, we may assume that \( s_i \neq 1_\Gamma \) for each \( i \). Thus, \( s_i \in \{ s \in S | w(s) \leq R + 1 \} \) for all \( i \). Also, \( R + 1 > \sum_{i=1}^{n} w(s_i) \geq nw(t) \), so that \( n < (R + 1)/w(t) \). Thus, \( x \) is an element of \( \{ t_1^{\delta_1} t_2^{\delta_2} \cdots t_m^{\delta_m} | t_i \in S \setminus 1_\Gamma, w(t_i) \leq R + 1, \delta_i \in \{ \pm 1 \}, m < (R + 1)/w(t) \} \), a finite set. This shows that \( \{ x \ | \ \| x \| \leq R \} \) is finite. □

Note that the infimum in the definition of \( \| x \| \) is actually a minimum. To see this, simply modify the argument in the last paragraph to show that the set of elements of \( \{ \sum_{i=1}^{n} w(s_i) | x = s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_n^{\epsilon_n}, s_i \in S, \epsilon_i \in \{ \pm 1 \} \} \) less than \( \| x \| + 1 \) is a finite set.

§2 Groups with asymptotic dimension 0

The following theorem gives a necessary and sufficient for a countable group to have
asymptotic dimension zero. This condition relies only on the algebraic structure of the group.

**Theorem 2.** Let $G$ be a countable group. Then $\text{asdim} G = 0$ if and only if every finitely generated subgroup of $G$ is finite.

**Proof.** Let $w : G \to [0, \infty)$ be a weight function on the generating set $G$. Let $\| \cdot \|$ and $d$ be the induced norm and metric, respectively.

First suppose that $\text{asdim} G = 0$. Let $T \subset G$ be a finite set. Take $d > \max_{g \in T} \|g\|$. As $\text{asdim} G = 0$, there is a $d-$disjoint, uniformly bounded cover $U$ of $G$. Choose $U \subset U$ with $1 \in U$. We will show that $\langle T \rangle \subset U$. To do this, we will show by induction that every product of $k$ ($k \geq 0$) elements of $T \cup T^{-1}$ lies in $U$. This is true for $k = 0$, as $1 \in U$. Now suppose it is true for $k - 1$, $k \geq 1$. Consider $x = t_1^{e_1} t_2^{e_2} \cdots t_k^{e_k}$, where $t_i \in T$ and $e_i \in \{ \pm 1 \}$. Set $y = t_1^{e_1} t_2^{e_2} \cdots t_{k-1}^{e_{k-1}}$. By the induction assumption, $y \in U$. Since $d(y, x) = \| y^{-1} x \| = \| t_k^{e_k} \| = \| t_k \| < d$, and because $U$ is a $d-$disjoint cover, we must have $x \notin U$. Thus, each product of $k$ elements of $T \cup T^{-1}$ lies in $U$. Therefore, $\langle T \rangle \subset U$. As $U$ is uniformly bounded, $U$ is bounded, and so $U$ and $\langle T \rangle$ are finite.

Conversely, suppose every finitely generated subgroup of $G$ is finite. Let $d > 0$ be given. Define $T = \{ s \in G | w(s) < d \}$ and $H = \langle T \rangle$. By definition of weight function, $T$ is finite. By our assumption, $H$ is finite as well. Let $U = \{ gH | g \in G \}$ be the collection of left cosets. So $U$ is a uniformly bounded cover, as left multiplication by a fixed element is an isometry of $G$. Further, suppose $gH \neq hH$. Let $x \in gH$ and $y \in hH$. So $xH = gH$ and $yH = hH$. As $gH \neq hH$, $xH \neq yH$, and so $y^{-1}x \notin H$. Hence $y^{-1}x$ cannot be written as a product of elements of $T \cup T^{-1}$. So if we take $s_i \in G$ such that $y^{-1}x = s_1 s_2 \cdots s_n$ and $\| y^{-1}x \| = \sum w(s_i)$, then there is a $j$ such that $s_j \notin T$. Hence $w(s_j) \geq d$, and so $d(y, x) = \| y^{-1}x \| \geq d$. Therefore $U$ is a $d-$disjoint, uniformly bounded cover. Since $d > 0$ was arbitrary, $\text{asdim} G = 0$. This completes the proof. □

The following corollaries are immediate consequences.

**Corollary 1.** Let $G$ be a finitely generated group. Then $\text{asdim} G = 0$ iff $G$ is a finite group.

**Corollary 2.** Let $G$ be a countable abelian group. Then $\text{asdim} G = 0$ if and only if $G$ is a torsion group.

**REMARK.** The last corollary shows that $\oplus_i \mathbb{Z}_{m_i}$, $\mathbb{Q}/\mathbb{Z}$, and $\mathbb{Z}_{p^\infty} = \lim_{\to} \mathbb{Z}_{p^k}$ all have asymptotic dimension 0.

The next theorem states that the epimorphic image of a zero dimensional countable group is zero dimensional. This is not true for one dimensional groups. Moreover, every countable group is an epimorphic image of a free group which is one dimensional.
Theorem 3. Let \( \phi : G \to H \) be an epimorphism of countable groups. If \( \text{asdim} \, G = 0 \), then \( \text{asdim} \, H = 0 \).

Proof. We will show that every finitely generated subgroup of \( H \) is finite. Let \( T \) be a finite subset of \( H \). Since \( \phi \) is an epimorphism, for each \( t \in T \) we can find a \( g_t \in G \) for which \( \phi(g_t) = t \). Let \( S = \{g_t | t \in T\} \). As \( S \) is finite, and since \( \text{asdim} \, G = 0 \), we have that \( \langle S \rangle \) is finite by the theorem. Thus, \( \langle T \rangle = \langle \phi(S) \rangle = \phi(\langle S \rangle) \) is a finite set. By the theorem, \( \text{asdim} \, H = 0 \). \( \square \)

§3 Asymptotic dimension of the rationals

We will now show that \( \text{asdim} \, \mathbb{Q} = 1 \). Once more we note that we are not computing the dimension of \( \mathbb{Q} \) with the Euclidean metric here, but rather with a proper, left-invariant metric.

Theorem 4. \( \text{asdim} \, \mathbb{Q} = 1 \).

Proof. First, we will show that \( \text{asdim} \, \mathbb{Q} \cap [0,1) = 0 \). We will then use this to prove the result.

On \( \mathbb{Q} \), define \( \|m/n\|_\mathbb{Q} = |m/n| + \ln(n) \) when \( m \) and \( n \) are relatively prime integers and \( n \) is positive (when \( m \) and \( n \) have these properties, we will say that \( m/n \) is in standard form). Here \(| \cdot |\) denotes the usual absolute value. It is easy to show that \( \| \cdot \|_\mathbb{Q} \) is a proper norm on \( \mathbb{Q} \). This norm induces a metric \( d_\mathbb{Q} \) in the usual way. Also, it is not hard to show that, on \( \mathbb{Q} \cap (-1,1) \), \( \ln n \leq \|m/n\|_\mathbb{Q} \leq 3 \ln n \) whenever \( m/n \) is in standard form.

Let \( \Gamma = \mathbb{Q}/\mathbb{Z} \). Let \( p : \mathbb{Q} \to \Gamma \) be the usual projection map. \( p \) is a surjective homomorphism and \( \ker p = \mathbb{Z} \). As the topology induced by \( \| \cdot \|_\mathbb{Q} \) is discrete, \( \mathbb{Z} \) is closed in \( \mathbb{Q} \). Thus, this norm induces a norm on \( \mathbb{Q}/\mathbb{Z} \), given by

\[
\|\overline{x}\| = \inf \{\|x + m'\|_\mathbb{Q} \mid m' \in \mathbb{Z}\}.
\]

This is a proper norm and hence its associated metric \( d \) will be a left-invariant, proper metric on \( \mathbb{Q}/\mathbb{Z} \).

Define \( i : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q} \cap [0,1) \) as follows: For \( r \in \mathbb{Q} \), there is a unique \( r' \in \mathbb{Q} \cap [0,1) \) such that \( \overline{r} = \overline{r'} \); set \( i(\overline{r}) = r' \). It is easy to see that \( i = p|_{\mathbb{Q} \cap [0,1)} \).

We will show that \( i \) is a coarse equivalence. First, \( \|p(r)\| = \|\overline{r}\| \leq \|r\|_\mathbb{Q} \), so \( p \) and hence \( p|_{\mathbb{Q} \cap [0,1)} \) is bornologous. Further, suppose \( \frac{m}{n} \in \mathbb{Q} \cap (-1,1) \), \( (m,n) = 1 \), \( m, n \in \mathbb{Z} \), and \( n > 0 \). For \( m' \in \mathbb{Z} \), we know that \( (m + nm', n) = 1 \). Thus,

\[
\left\| \frac{m}{n} + m' \right\|_\mathbb{Q} = \left\| \frac{m + nm'}{n} \right\|_\mathbb{Q} = \left| \frac{m}{n} + m' \right| + \ln(n) \geq \ln(n).
\]
Also, since $-1 < \frac{m}{n} < 1$, we have that $\ln(n) \geq \frac{1}{3}\|\frac{m}{n}\|_Q$. Hence

$$\|\frac{m}{n}\| = \inf\{\|\frac{m}{n} + m'\|_Q \mid m' \in \mathbb{Z}\} \geq \ln(n) \geq \frac{1}{3}\|\frac{m}{n}\|_Q.$$ 

Since each $r \in Q \cap (-1,1)$ can be expressed in standard form, $\|r\|_Q \leq 3\|\tau\|$. So for $r, s \in Q \cap [0,1)$, we have $s - r \in Q \cap (-1,1)$ and so

$$d_Q(i(\tau), i(\bar{\tau})) = \|i(\tau) + i(\bar{\tau})\|_Q = \|s - r\|_Q \leq 3\|\tau - \bar{\tau}\| = 3\|\tau - \bar{\tau}\| = 3d(\tau, \bar{\tau}).$$

Since each $r \in Q$, there is a $r' \in Q \cap [0,1)$ such that $\tau = r'\bar{\tau}$, we have that $d_Q(i(\tau), i(\bar{\tau})) \leq d(\tau, \bar{\tau})$ for $r, s \in Q$. Thus, $i$ is bornologous. As $p|_{Q \cap [0,1)}$ and $i$ are inverses, each is proper and $i$ is a coarse equivalence of $Q/\mathbb{Z}$ and $Q \cap [0,1)$. By Corollary 2, $\text{asdim} Q \cap [0,1) = \text{asdim} Q/\mathbb{Z} = 0$.

We will now complete the proof that $\text{asdim} Q = 1$. Let $d > 0$ be given. Since $\{x \in Q \mid \|x\|_Q \leq d\}$ is a finite set, there is an $R \in \mathbb{Z}_+$ such that $\|x\|_Q \leq d$ implies $|x| < R$. For $n \in \mathbb{Z}$, define $A_n = Q \cap [nR, (n+1)R)$. Notice $\text{asdim} A_0 = 0$ by the finite union theorem of [B-D1]. So there is a $d$–disjoint, $S$–bounded covering $\{A_{0,k}|k = 1, 2, \ldots\}$ of $A_0$. Since the map $x \mapsto x + nR$ (n fixed) is an isometry $[0, R] \to [nR, (n+1)R]$, the covering $\{A_{n,k}|k = 1, 2, \ldots\}$ of $A_n$, where $A_{n,k} = nR + A_{0,k}$, is $d$–disjoint and $S$–bounded.

Let

$$U_0 = \{A_{n,k}|n \text{ even, } k = 1, 2, \ldots\} \quad \text{and} \quad U_1 = \{A_{n,k}|n \text{ odd, } k = 1, 2, \ldots\}.$$ 

Note that $U_0 \cup U_1$ is a cover of $Q$, and $U_0$ and $U_1$ are both $S$–bounded. We will now show that $U_0$ is $d$–disjoint. Consider $A_{n,k}$ and $A_{n',k'}$, where $(n, k) \neq (n', k')$ and $n, n'$ are even. Suppose first that $n \neq n'$. Without loss of generality, take $n < n'$. Let $x \in A_{n,k}$ and $y \in A_{n',k'}$. Then since $A_{n,k} \subset A_n$ and $A_{n',k'} \subset A_{n'}$, we have $nR \leq x < (n+1)R$ and $n'R \leq y < (n' + 1)R$, and so $y - x \geq (n' - n - 1)R \geq 2R$. Hence $|y - x| \geq R$. But by our choice of $R$, this implies $\|y - x\|_Q \geq d$. Now we will consider the case when $n = n'$. This forces $k \neq k'$. By our construction of the $A_{n,k}$, $A_{n,k}$ and $A_{n',k'} = A_{n,k'}$ are $d$–disjoint.

Similarly, $U_1$ is a $d$–disjoint family. Since $d > 0$ was arbitrary, $\text{asdim} Q \leq 1$. Finally, since $d_Q$ restricts to the Euclidean metric on $\mathbb{Z}$, we have $\text{asdim} Q \geq 1$. Therefore, $\text{asdim} Q = 1$. □

§4 Asymptotic dimension of the rationals with $p$-adic norm

We will now consider $Q$ with the $p$-adic norm $\|\cdot\|_p$. Namely, if $m = p^am'$ and $n = p^bn'$, where $p$ divides neither $m'$ nor $n'$, then

$$\|\frac{m}{n}\|_p = p^{b-a} = \frac{1}{p^{a-b}}.$$
Let \( d_p \) denote the metric obtained from this norm. Unlike the previous examples, \( Q \) with the metric \( d_p \) is not proper. To differentiate the dimension with respect to this metric from the one in Theorem 4, we will always write \( \text{asdim}(Q, \| \cdot \|_p) \).

**Theorem 5.** \( \text{asdim}(Q, \| \cdot \|_p) = 0. \)

**Proof.** Set \( L = \{ \frac{m}{p^a} | a \in \mathbb{N}, m \in \mathbb{Z} \} \) and \( X = L \cap [0, 1). \)

We will now show that \( N_1(X) = Q \), where \( N_1(X) = \{ y \in Q | d_p(y, X) \leq 1 \} \). Suppose \( r \in Q \). If \( r = 0 \), then \( r \in X \subseteq N_1(X) \). Now suppose \( r \neq 0 \). Then there are \( m', n' \in \mathbb{Z} \setminus 0 \) such that \( r = \frac{m'}{n'} \). So \( m' = p^a m \) and \( n' = p^b n \) for some \( a, b \in \mathbb{N} \) and \( m, n \in \mathbb{Z} \setminus 0 \) such that \( p \) divides neither \( m \) nor \( n \). Thus, \( \frac{m'}{n'} = p^{a-b} \frac{m}{n} \). First consider the case when \( a \geq b \). Then \( \| \frac{m'}{n'} \|_p = \frac{1}{p^{r-\ell}} \leq 1 \). So \( r = \frac{m'}{n'} \in N_1(0) \subseteq N_1(X) \). Now suppose \( a < b \). Set \( c = b - a \), so \( \frac{m'}{n'} = \frac{m}{p^c n} \). Since \( (n, p^c) = 1 \), \( p \in \mathbb{Z}_{p^c} \) is a generator. Hence there exists an \( \ell \) such that \( 0 \leq \ell < p^c \) and \( m = \ell m' \). Therefore, \( p^c | m - \ell n \). Now take \( d \geq 0 \) such that \( p^d | m - \ell n \) but \( p^{d+1} \) does not divide \( m - \ell n \). So \( d \geq c \). Thus,

\[
\| \frac{m'}{n'} - \ell \|_p = \| \frac{m}{p^c n} - \ell \|_p = \frac{1}{p^{d-c}} \leq 1.
\]

As \( \ell \in X \), \( r = \frac{m'}{n'} \in N_1(X) \). Therefore, \( Q = N_1(X) \).

This means that the inclusion \( X \hookrightarrow Q \) is a coarse equivalence (see [Roe]), where \( X \) has the restricted metric \( d_p \). Thus, \( \text{asdim}(X, d_p) = \text{asdim}(Q, \| \cdot \|_p) \). Let \( \| \cdot \|_Q \) be the norm from theorem 4, and we consider its restriction to \( X' = L \cap (-1, 1) \). Let \( r \in X' \setminus 0 \). So \( r = \frac{m}{p^a} \) for some \( a \geq 0 \) and \( m \in \mathbb{Z} \setminus 0 \) such that \( (m, p^a) = 1 \). Since we also have \(-1 < \frac{m}{p^a} < 1 \), it follows that \( p \) does not divide \( m \). Then, since \( \frac{m}{p^a} \) is in standard form and \( \frac{m}{p^a} \in (-1, 1) \), we have

\[
\ln(p^a) \leq \| \frac{m}{p^a} \|_Q \leq 3 \ln(p^a).
\]

But \( p^a = \| \frac{m}{p^a} \|_p \) since \( m \neq 0 \) and \( p \) does not divide \( m \). So

\[
\ln(\| \frac{m}{p^a} \|_p) \leq \| \frac{m}{p^a} \|_Q \leq 3 \ln(\| \frac{m}{p^a} \|_p).
\]

Thus, \( \ln(\| r \|_p) \leq \| r \|_Q \leq 3 \ln(\| r \|_p) \). For \( x, y \in X \) such that \( x \neq y \), we have \( y - x \in X' \setminus 0 \), and so

\[
\ln(d_p(x, y)) \leq d_Q(x, y) \leq 3 \ln(d_p(x, y)).
\]

From this it is immediate that \( \text{id} : (X, d_Q) \rightarrow (X, d_p) \) and its inverse are bornologous, and so they are coarse equivalences as well. From the results of Theorem 4, \( \text{asdim}((X, d_p) = \text{asdim}(X, d_Q) \leq \text{asdim}(Q \cap [0, 1), d_Q) = 0 \). Thus, \( \text{asdim}(Q, \| \cdot \|_p) = 0. \) \( \Box \)
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University of Florida, Department of Mathematics, P.O. Box 118105, 358 Little Hall, Gainesville, FL 32611-8105, USA
E-mail address: justins@math.ufl.edu