Abstract

Possible ways of constructing extended fermionic strings with $N = 4$ world-sheet supersymmetry are reviewed. String theory constraints form, in general, a non-linear quasi(super)conformal algebra, and can have conformal dimensions $\geq 1$. When $N = 4$, the most general $N = 4$ quasi-superconformal algebra to consider for string theory building is $\hat{D}(1,2;\alpha)$, whose linearisation is the so-called ‘large’ $N = 4$ superconformal algebra. The $\hat{D}(1,2;\alpha)$ algebra has $\widehat{su}(2)_{k^+} \oplus \widehat{su}(2)_{k^-} \oplus \widehat{u}(1)$ Kac-Moody component, and $\alpha = k^-/k^+$. We check the Jacobi identities and construct a BRST charge for the $\hat{D}(1,2;\alpha)$ algebra. The quantum BRST operator can be made nilpotent only when $k^+ = k^- = -2$. The $\hat{D}(1,2;1)$ algebra is actually isomorphic to the $SO(4)$-based Bershadsky-Knizhnik non-linear quasi-superconformal algebra. We argue about the existence of a string theory associated with the latter, and propose the (non-covariant) hamiltonian action for this new $N = 4$ string theory. Our results imply the existence of two different $N = 4$ fermionic string theories: the old one based on the ‘small’ linear $N = 4$ superconformal algebra and having the total ghost central charge $c_{gh} = +12$, and the new one with non-linearly realised $N = 4$ supersymmetry, based on the $SO(4)$ quasi-superconformal algebra and having $c_{gh} = +6$. Both critical string theories have negative ‘critical dimensions’ and do not admit unitary matter representations.

1Supported in part by the ‘Deutsche Forschungsgemeinschaft’ and the NATO grant CRG 930789
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1 Introduction

Any known critical \(N\)-extended fermionic string theory with \(N \leq 4\) world-sheet supersymmetries is based on a two-dimensional (2d) linear \(N\)-extended superconformal algebra (SCA) which is gauged \([1]\). The string world-sheet fields usually form a linear \(N\)-extended superconformal multiplet coupled to the \(N\)-extended conformal supergravity fields which are the gauge fields of the \(N\)-extended SCA. This pattern worked well for the \(N = 1\) and \(N = 2\) strings, and it was always expected to be also true for \(N = 4\) strings. In the past, only one \(N = 4\) string theory had been actually constructed \([2]\) by gauging the ‘small’ linear \(N = 4\) SCA with \(SU(2)\) internal symmetry. The 2d covariant action of this \(N = 4\) string can be found in ref. \([3]\). An apparent drawback of this \(N = 4\) string theory is the ‘negative’ value of its (quaternionic) critical dimension, \(D_c = -2\), which always prevented this theory from having physical applications. This explains, in particular, why nobody succeeded in constructing \(N = 4\) string scattering amplitudes.

Still, it is of interest to know how many different \(N = 4\) string theories can be constructed at all. Any \(N = 4\) string constraints are going to be very strong in two dimensions, so that their explicit realisation should always imply very non-trivial interplay between geometry, conformal invariance and extended supersymmetry. The \(N = 4\) strings are also going to be relevant in the search for the ‘universal string theory’ \([4]\). In addition, strings with \(N = 4\) supersymmetry are expected to have deep connections with integrable models \([5, 6]\), so that we believe they are worthy to be studied.

It has been known for some time that there are actually two different linear \(N = 4\) SCAs which are (affine versions of) finitely-generated Lie superalgebras: the so-called ‘small’ linear \(N = 4\) SCA with \(SU(2)\) internal symmetry \([1, 2]\), and the so-called ‘large’ linear \(N = 4\) SCA with \(SU(2) \times SU(2) \times U(1)\) internal symmetry \([7, 8]\). Unlike the ‘small’ \(N = 4\) SCA mentioned above, the ‘large’ \(N = 4\) SCA has subcanonical charges, or currents of conformal dimension 1/2. This observation already implies that no supergravity or string theory based on the ‘large’ \(N = 4\) SCA exists, because there are no 2d gauge fields which would correspond to the fermionic charges of dimension 1/2. \(^3\)

When a number of world-sheet supersymmetries exceeds two, there are, in fact, more opportunities to build up new string theories, by using 2d non-linear quasi-superconformal algebras which are known to exist for an arbitrary \(N > 2\). By an

\(^3\)In conformal field theory, ‘currents’ of dimension 1/2 are just free fermions \([3]\).
\section*{Extended Quasi-Superconformal Algebra (QSCA)}

We mean a graded associative algebra whose contents is restricted to canonical charges of dimension 2, 3/2 and 1, which (i) contains the Virasoro subalgebra, and (ii) $N$ real supercurrents of conformal dimension 3/2, whose \textit{operator product expansion} (OPE) has a stress tensor of dimension 2, (iii) satisfies the Jacobi identity, and (iv) has the usual spin-statistics relation.\footnote{By definition, a QSCA is an ‘almost’ usual SCA, except it may not be a Lie superalgebra but its OPEs have to be closed on quadratic composites of the fundamental set of canonical generators. The QSCAs can, therefore, be considered on equal footing with the $W$ algebras \cite{10} without, however, having currents of spin higher than two. Though QSCAs do not belong, in general, to ordinary (finitely-generated) affine Lie superalgebras, but, so to say, to infinitely-generated Lie superalgebras, they are still closely related with finite Lie superalgebras \cite{11}.}

$N$-extended \textit{quasi-superconformal algebra} (QSCA) we mean a graded associative algebra whose contents is restricted to canonical charges of dimension 2, 3/2 and 1, which (i) contains the Virasoro subalgebra, and (ii) $N$ real supercurrents of conformal dimension 3/2, whose \textit{operator product expansion} (OPE) has a stress tensor of dimension 2, (iii) satisfies the Jacobi identity, and (iv) has the usual spin-statistics relation.\footnote{By definition, a QSCA is an ‘almost’ usual SCA, except it may not be a Lie superalgebra but its OPEs have to be closed on quadratic composites of the fundamental set of canonical generators. The QSCAs can, therefore, be considered on equal footing with the $W$ algebras \cite{10} without, however, having currents of spin higher than two. Though QSCAs do not belong, in general, to ordinary (finitely-generated) affine Lie superalgebras, but, so to say, to infinitely-generated Lie superalgebras, they are still closely related with finite Lie superalgebras \cite{11}.}

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The full classification of QSCAs has been done by Fradkin and Linetsky \cite{11}.\footnote{See also refs. \cite{12,13} for earlier works on classification of linear SCAs.}

The classification \cite{11} is based on the classical results of Kač \cite{14} on classification of finite simple Lie superalgebras, while their relation to QSCAs is very similar to the one existing between ordinary Lie algebras $sl(n)$ and $W_n$ algebras via a Drinfeld-Sokolov-type reduction \cite{10}. There are three classical families of QSCAs, namely

$$osp(N|2; R), \quad su(1, 1|2M), \quad osp(4^*|2M),$$

one continuous family $\hat{D}(2, 1; \alpha)$ parametrised by a real parameter $\alpha$,\footnote{The algebras $\hat{D}(2, 1; \alpha)$ and $\hat{D}(2, 1; \alpha^{-1})$ are isomorphic.} and the two exceptional QSCAs with $N = 7$ and $N = 8$ supersymmetry $\cite{11,15}$.

When $N = 4$, the only different QSCAs are just $su(1, 1|2)$ and $\hat{D}(2, 1; \alpha)$, since $osp(4^*|2)$ is isomorphic to $osp(4|2; R)$, whereas the latter can actually be considered as a particular member of the family $\hat{D}(2, 1; \alpha)$, as we are going to demonstrate in this paper. The $su(1, 1|2)$ QSCA is, in fact, a linear $N = 4$ SCA which is isomorphic to the ‘small’ $SU(2)$-based $N = 4$ SCA.

The $osp(N|2; R)$ and $su(1, 1|N)$ series of QSCAs with the $SO(N)$ and $U(N)$ Kač-Moody (KM) symmetries, respectively, were discovered by Knizhnik \cite{16} and Bershadsky \cite{17}, whereas the non-linear $\hat{D}(2, 1; \alpha)$ QSCA was originally extracted by Goddard and Schwimmer \cite{18} from the ‘large’ linear $N = 4$ SCA of ref. \cite{7,8} by factoring out free fermions and boson. In our paper, we give a straightforward

\footnote{We exclude from our analysis all kinds of twisted (Q)SCAs with unusual relations between spin and statistics (they are, however, relevant for topological field theory and topological strings \cite{19}). We also ignore all kinds of reducible combinations of SCAs and QSCAs.}
construction of this QSCA, and emphasize on some peculiarities of the $N = 4$ case in the orthogonal (Bershadsky-Knizhnik) series of the QSCAs.

Since the $N = 4$ supersymmetric $\hat{D}(2, 1; \alpha)$ QSCA includes only canonical charges, despite of its apparent non-linearity it seems possible to construct the associated 2d, $N = 4$ conformal supergravity and, hence, a new $N = 4$ string theory by coupling this supergravity to an appropriate 2d matter, completely along the lines of constructing the $W$ gravities and $W$ strings.

The paper is organised as follows. In sect. 2 we review the Bershadsky-Knizhnik orthogonal series of non-linear $N$-extended QSCAs in two dimensions, paying special attention to a peculiar nature of the $N = 4$ case. In sect. 3, the $N = 4$ extended nonlinear $\hat{D}(1, 2; \alpha)$ QSCA is introduced, and its relation to the ‘large’ linear $N = 4$ SCA is explained. In sect. 4 we discuss a construction of classical superconformal field theories and supergravities, in which quasi(super)conformal algebras appear as symmetry algebras, since we are interested in local realisations of the corresponding symmetries, needed for supergravity and string theory building. The hamiltonian (2d non-covariant) form of invariant action is also presented in sect. 4. The BRST quantisation and the construction of quantum BRST charges for both the $SO(N)$-based Bershadsky-Knizhnik QSCA and the $\hat{D}(1, 2; \alpha)$ QSCA are given in sect. 5. Sect. 6 comprises our conclusion. In Appendix, the relevant facts, needed for a construction of the classical BRST charge in a gauge theory with first-class constraints satisfying a non-linear algebra, are summarized.

2 Bershadsky-Knizhnik orthogonal QSCA series

The current contents of the 2d $N$-extended Bershadsky-Knizhnik QSCA \cite{16, 17} is given by the holomorphic fields $T(z)$, $G^i(z)$ and $J^a(z)$, all having the standard mode expansions

$$T(z) = \sum_n L_n z^{-n-2},$$

$$G^i(z) = \sum_r G^i_r z^{-r-3/2},$$

$$J^a(z) = \sum_n J^a_n z^{-n-1},$$

and (conformal) dimensions 2, 3/2 and 1, respectively. The real supersymmetry generators $G^i$, $i = 1, \ldots, N$, are defined in the fundamental (vector) representation of the internal symmetry group $SO(N)$ generated by the zero modes of the currents $J^a$, $a = 1, \ldots, \frac{1}{2}N(N - 1)$, in the adjoint representation.
Most of the OPEs defining the $SO(N)$-based Bershadsky-Knizhnik QSCA take the standard linear form, viz.

\[
T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w},
\]

\[
T(z)G^i(w) \sim \frac{2G^i(w)}{(z-w)^2} + \frac{\partial G^i(w)}{z-w},
\]

\[
T(z)J^a(w) \sim \frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w},
\]

\[
J^a(z)G^i(w) \sim \frac{(t^a)^{ij}G^j(w)}{z-w},
\]

\[
J^a(z)J^b(w) \sim \frac{f^{abc}J^c(w)}{z-w} + \frac{-k\delta^{ab}}{(z-w)^2},
\]

where $k$ is an arbitrary ‘level’ of the KM subalgebra, $f^{abc}$ are $SO(N)$ structure constants, and $(t^a)^{ij}$ are generators of $SO(N)$ in the fundamental (vector) representation, \n
\[
[t^a, t^b] = f^{abc}t^c, \quad f^{abc}f^{ab} = 2(N - 2)\delta^{cd},
\]

\[
\mathrm{tr}(t^a t^b) = -2\delta^{ab}, \quad (t^a)^{ij}(t^a)^{kl} = \delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk}.
\]

On symmetry and dimensional reasons, the only non-trivial OPE defining the non-linear supersymmetry component of QSCA can be of the form

\[
G^i(z)G^j(w) \sim a_1 \frac{\delta^{ij}}{(z-w)^3} + a_2 \frac{(t^a)^{ij}J^a(w)}{(z-w)^2} + \frac{1}{z-w} \left[ 2\delta^{ij}T(w) + \frac{1}{2}a_2(t^a)^{ij}\partial J^a(w) \right]
\]

\[
+ \frac{1}{z-w} \left[ a_3 \left( t^{(a \rho b)} \right)^{ij} + a_4\delta^{ab}\delta^{ij} \right] : J^a J^b : (w),
\]

where $a_1, a_2, a_3$ and $a_4$ are parameters to be determined by solving the Jacobi identity, and the normal ordering is defined by

\[
: J^a J^b : (w) = \lim_{z \to w} \left[ J^{(a)}(z)J^{(b)}(w) + \frac{k\delta^{ab}}{(z-w)^2} \right].
\]

Indices in brackets mean symmetrization with unit weight, e.g. $t^{(a \rho b)} \equiv \frac{1}{2}(t^a t^b + t^b t^a)$. Eq. (2.4) can be considered as the general ansatz for supersymmetry algebra.

Demanding consistency of the whole algebra determines the parameters \[10, 17\]:

\[
a_1 = \frac{k(N + 2k - 4)}{N + k - 3}, \quad a_2 = \frac{N + 2k - 4}{N + k - 3},
\]

\[
a_3 = a_4 = \frac{1}{N + k - 3},
\]

5
while the Virasoro central charge of this QSCA is also quantized [16, 17]:
\[ c = \frac{k(N^2 + 6k - 10)}{2(N + k - 3)} \] (2.7)
The KM parameter \( k \) remains arbitrary in this construction.

In the case of \( N = 2 \) QSCA, the non-linearity actually disappears and the algebra becomes the \( N = 2 \) linear SCA, since the total coefficient in front of the sum of two last terms in the second line of eq. (2.4) vanishes \( (i^2 + 1 = 0) \) after substituting \( U(1) \cong SO(2) \) and the last eq. (2.6). Therefore, the non-linear structure of Bershadsky-Knizhnik QSCAs is only relevant when \( N \geq 3 \).

The regular series of SCAs [1, 2] with \( N > 4 \) seem to be irrelevant for constructing \( N \)-extended string theories since they do not admit central extensions and always have subcanonical charges [1, 12, 13]. In addition, there is no obvious \( N \)-extended supersymmetric matter to represent world-sheet string fields in the case of (Q)SCAs with \( N > 4 \). In what follows, we are going to consider the \( N = 4 \) QSCAs in more details.

Eq. (2.4) at \( N = 4 \) takes the form
\[
g^i(z)g^j(w) \sim \frac{2k^2}{(k+1)} \frac{\delta^{ij}}{(z-w)^2} + \frac{2k}{(k+1)} \frac{(t^a)^{ij}J^a(w)}{(z-w)^2} ,
\]
\[
+ \frac{1}{z-w} \left[ 2\delta^{ij}T(w) + \frac{k}{k+1}(t^a)^{ij}\partial J^a(w) \right]
\]
\[
+ \frac{1}{(k+1)(z-w)} \left[ (t^{(ab)})^{ij} + \delta^{ab}\delta^{ij} \right] : J^aJ^b : (w) ,
\] (2.8)
whereas the Virasoro central charge of the \( SO(4) \)-based QSCA is simply
\[ c = 3k \] (2.9)

The \( N = 3 \) Bershadsky-Knizhnik QSCA with \( SO(3) \) KM symmetry can be easily obtained by truncation of the \( N = 4 \) algebra.

An analogue of the Sugawara-Sommerfeld relation for the \( SO(4) \) Bershadsky-Knizhnik QSCA takes the weaker form [19]
\[
\partial T(z) = \frac{1}{4} : g^i g^i : (z) - \frac{1}{4(k+1)} \partial(: J^aJ^a:) (z) ,
\] (2.10)
which is consistent with all the commutation relations.

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7The *unitary* series of non-linear Bershadsky-Knizhnik QSCAs with \( N > 2 \) are also irrelevant in the search for new string theories because of another reason, see sect. 6.

8Still, one may imagine that *all* string world-sheet fields could be in one (presumably, non-linearly realised) representation of \( N > 4 \) QSCA. This would mean, however, very radical changes in the philosophy underlying string theory nowadays. 2d supergravities based on the QSCAs with \( N > 4 \) may also exist.
3 $D(1,2;\alpha)$ QSCA and ‘large’ $N = 4$ SCA

The construction of the previous section in the case of $N = 4$ QSCA is, however, incomplete. Because of the isomorphism $SO(4) \cong SU(2) \otimes SU(2)$, the supersymmetry generators transform in a reducible $(2,2)$ representation of $SU(2) \otimes SU(2)$, while each of the $su(2)$ KM components can have its own ‘level’, thus opening the way for generalisations. Therefore, the derivation of the $SO(4)$-based QSCA in sect. 2 needs to be reconsidered, partly because of some additional identities hold for $SO(4)$.

Let $J^a_\pm(z)$ be the internal symmetry currents, where $a, b, \ldots$ are now the adjoint indices of $SU(2)$, and $\pm$ distinguishes between the two $SU(2)$ factors. We still label the four-dimensional fundamental (vector) representation space of $SO(4)$ by indices $i, j, \ldots$, as before. The self-dual components of the KM currents, $J^\pm_i(z)$ can be unified into an antisymmetric tensor $J^{ij}(z)$ in the adjoint of $SO(4)$,

$$J^{ij}(z) = (t^{a-})^{ij} J^a_-(z) + (t^{a+})^{ij} J^a_+(z),$$

(3.1)

where the antisymmetric $4 \times 4$ matrices $t^{a\pm}$ satisfy the relations

$$[t^{a\pm}, t^{b\pm}] = -2\varepsilon^{abc} t^{c\pm}, \quad [t^{a+}, t^{a-}] = 0, \quad \{t^{a\pm}, t^{b\pm}\} = -2\delta^{ab}.$$

(3.2)

These matrices can be explicitly represented as

$$(t^{a\pm})^{ij} = \varepsilon^{aij} \pm (\delta^i_a \delta^j_4 - \delta^j_a \delta^i_4),$$

(3.3)

and satisfy the identity

$$\sum_a (t^{a\pm})^{ij} (t^{a\pm})^{kl} = \delta^{ik} \delta^{jl} - \delta^{il} \delta^{jk} \pm \varepsilon^{ijkl}.$$

(3.4)

The OPEs describing the action of $J^a_\pm(z)$ read

$$J^a_\pm(z) J^{b\pm}(w) \sim \frac{\varepsilon^{abc} J^c_\pm(w)}{z-w} - \frac{k^\pm \delta^{ab}}{2(z-w)^2},$$

$$J^a_\pm(z) G^i(w) \sim \frac{1}{2} (t^{a\pm})^{ij} G^j(w),$$

(3.5)

where two arbitrary ‘levels’ $k^\pm$ for both independent $su(2)$ KM components have been introduced.

The general ansatz for the OPE of two fermionic supercurrents can be written as

$$G^i(z) G^j(w) \sim b_1 \frac{\delta^{ij}}{(z-w)^3} + \frac{2T(w) \delta^{ij}}{z-w} + \frac{1}{2} (b_2 + b_3) \left[ \frac{J^{ij}(w)}{(z-w)^2} + \frac{1}{2} \partial J^{ij}(w) \right].$$

7
+ \frac{1}{7}(b_2 - b_3) \varepsilon^{ijkl} \left[ \frac{J^{kl}(w)}{(z - w)^2} + \frac{1}{2} \partial J^{kl}(w) \right] + \frac{1}{4} b_4 \varepsilon^{iklm} \varepsilon^{jlpq} : \frac{J^{lm} J^{pq}}{z - w} : (w) , \quad (3.6a)

in the vector notation, or, equivalently,

\begin{align*}
G^i(z)G^j(w) & \sim \frac{b_1 \delta^{ij}}{(z - w)^3} + \frac{1}{(z - w)^2} \left[ b_2 (t^a + )^{ij} J^a + b_3 (t^a - )^{ij} J^a \right] \\
& \quad - \frac{1}{z - w} \left[ 2 T(w) \delta^{ij} + \frac{1}{2} \partial \left\{ b_2 (t^a + )^{ij} J^a + b_3 (t^a - )^{ij} J^a \right\} \right] \\
& \quad + \frac{b_4}{z - w} : (t^a + )^{ij} (t^a - )^{kj} \left( t^b + j^b - t^b - j^b \right)^{il} : (w) ,
\end{align*}

in our new notation, where we have used the fact that

\begin{equation}
\frac{1}{2} \varepsilon^{ijkl} T^{kl}(z) = (t^a + )^{ij} J^a(z) - (t^a - )^{ij} J^a(z) , \quad (3.7)
\end{equation}

as a consequence of eq. (3.1). Compared to eq. (2.4), note the presence of extra terms in eq. (3.6a) due to the \(\varepsilon\)-symbol, and in eq. (3.6b) due to the \((\pm)\)-distinction.

Demanding associativity of the combinations \(TGG\), \(JGG\) and \(GGG\) determines the parameters \(b_1\), \(b_2\), \(b_3\), \(b_4\), and, hence, all of the QSCA 3- and 4-point ‘structure constants’, viz.

\begin{align*}
b_1 & = \frac{4k^+ k^-}{k^+ + k^- + 2}, & b_4 & = \frac{-2}{k^+ + k^- + 2}, \\
b_2 & = \frac{-4k^-}{k^+ + k^- + 2}, & b_3 & = \frac{-4k^+}{k^+ + k^- + 2},
\end{align*}

as well as the central charge,

\begin{equation}
c = \frac{6(k^+ + 1)k^- + 1}{k^+ + k^- + 2} - 3 , \quad (3.9)
\end{equation}

in agreement with refs. [15, 18, 23].

We define \(\alpha\)-parameter of this \(\hat{D}(1,2;\alpha)\) QSCA as a ratio of its two KM ‘levels’,

\begin{equation}
\alpha \equiv \frac{k^-}{k^+} , \quad (3.10)
\end{equation}

which measures the relative asymmetry between the two \(\hat{su}(2)\) KM algebras in the whole algebra. When \(\alpha = 1\), i.e. \(k^- = k^+ \equiv k\), the \(\hat{D}(1,2;1)\) QSCA is just the \(SO(4)\) Bershadsky-Knizhnik QSCA considered in the previous section, with the central charge \(c = 3k\), as in eq. (2.9).

In the vector notation, the \(\hat{D}(1,2;\alpha)\) QSCA non-trivial OPEs take the form

\begin{equation}
T^{ij}(z)G^k(w) \sim \frac{1}{z - w} \left[ \delta^{ik} G^j(w) - \delta^{jk} G^i(w) \right] , \quad (3.11a)
\end{equation}
\[ J^{ij}(z)J^{kl}(w) \sim \frac{1}{z-w} \left[ \delta^{ik}J^{jl}(w) - \delta^{ik}J^{jl}(w) + \delta^{jl}J^{ik}(w) - \delta^{jl}J^{ik}(w) \right] \\
- \frac{1}{2}(k^+ + k^-) \delta^{ij} \frac{1}{(z-w)^2} - \frac{1}{2}(k^+ - k^-) \frac{\varepsilon^{ijkl}}{(z-w)^2}, \]

When using mode decompositions, like in eq. (2.1), we find instead

\[ G^i(z)G^j(w) \sim \frac{4k^+k^-}{(k^+ + k^- + 2)(z-w)^2} + \frac{2T(w)\delta^{ij}}{z-w} - \frac{k^+ + k^-}{k^+ + k^- + 2} \left[ \frac{2J^{ij}(w)}{z-w} + \partial J^{ij}(w) \right] \]

\[ + \frac{k^+ - k^-}{k^+ + k^- + 2} \varepsilon^{ijkl} \left[ \frac{2J^{kl}(w)}{(z-w)^2} + \partial J^{kl}(w) \right] - \frac{\varepsilon^{iklm}\varepsilon^{jkl}}{2(k^+ + k^- + 2)(z-w)} . \]

Though \( \hat{D}(1;2;\alpha) \) is a non-linear QSCA, it can be turned into a linear SCA by adding some ‘auxiliary’ fields, namely, four free fermions \( \psi^i(z) \) of dimension 1/2, and a free Fubini-Veneziano bosonic current \( U(z) \) of dimension 1, defining a \( \widehat{U}(1) \) KM algebra [12]. The new fields have canonical OPEs,

\[ \psi^i(z)\psi^j(w) \sim \frac{-\delta^{ij}}{z-w} , \]

\[ U(z)U(w) \sim \frac{-1}{(z-w)^2} . \]

The fermionic fields \( \psi^i(z) \) transform in a \((2,2)\) representation of \( SU(2) \otimes SU(2) \),

\[ J^{a\pm}(z)\psi^j(w) \sim \frac{1}{2}(t^{a\pm}ij\psi^j(w)) \left. \right|_{z-w} , \]

whereas the singlet \( U(1) \)-current \( U(z) \) can be thought of as derivative of a free scalar boson, \( U(z) = i\partial\phi(z) \).
Let us now define the new currents \[ T_{\text{tot}} = T - \frac{1}{2} : U^2 : - \frac{1}{2} : \partial \psi^i \psi^i : \]
\[ G^i_{\text{tot}} = G^i - U \psi^i + \frac{1}{3 \sqrt{2(k^+ + k^- + 2)}} \varepsilon^{ijkl} \psi^k \psi^l \]
\[ - \sqrt{\frac{2}{k^+ + k^- + 2}} \psi^j \left[ (t^{a+})^{ji} J^{a+} - (t^{a-})^{ji} J^{a-} \right], \]
\[ J^a_{\text{tot}} = J^a + \frac{1}{4} (t^{a+})^{ji} \psi^i \psi^j, \]
in terms of the initial $\hat{D}(1,2;\alpha)$ QSCA currents $T$, $G^i$ and $J^a_{\pm}$. Then the following set of affine generators

\[
\{ T_{\text{tot}} , \ G^i_{\text{tot}} , \ J^a_{\text{tot}} , \ \psi^i , \ U \} \]

has closed OPEs among themselves, defining a linear ‘large’ $N = 4$ SCA with the $su(2) \oplus su(2) \oplus u(1)$ KM component! Explicitly, the non-trivial OPEs of this ‘large’ $N = 4$ SCA are given by (cf refs. \[ \text{[8, 23]} \])

| Expression                                                                 | Equation                                                                                           |
|---------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------|
| $T_{\text{tot}}(z)T_{\text{tot}}(w)$                                    | $\frac{1}{4}(c + 3) \frac{2 T_{\text{tot}}(w)}{(z - w)^4} + \frac{2 T_{\text{tot}}(w)}{(z - w)^2} + \frac{\partial T_{\text{tot}}(w)}{z - w}$ |
| $T_{\text{tot}}(z)\mathcal{O}(w)$                                       | $\frac{h \mathcal{O}(w)}{(z - w)^2} + \frac{\partial \mathcal{O}(w)}{z - w}$                     |
| $J^a_{\text{tot}}(z)J^a_{\text{tot}}(w)$                                | $\frac{\varepsilon^{abc} J^c_{\text{tot}}(w)}{z - w} - (k^+ + 1)\delta^{ab} \frac{2}{(z - w)^2}$ |
| $J^a_{\text{tot}}(z)G^i_{\text{tot}}(w)$                                | $\frac{1}{4} (t^{a+})^{ij} G^j_{\text{tot}}(w) \mp \frac{k^+ + 1}{\sqrt{2(k^+ + k^- + 2)} (z - w)^2} \text{OPE}$ |
| $G^i_{\text{tot}}(z)G^j_{\text{tot}}(w)$                                | $\frac{2}{3}(c + 3)\delta^{ij} \frac{2 T_{\text{tot}}(w)\delta^{ij}}{(z - w)^3} - \frac{2}{k^+ + k^- + 2} \left[ \frac{2}{(z - w)^2} + \frac{1}{z - w} \partial_w \right]$ |
|                                                                           | $\times \left[ (k^- + 1)(t^{a+})^{ij} J^{a+}_{\text{tot}}(w) + (k^+ + 1)(t^{a-})^{ij} J^{a-}_{\text{tot}}(w) \right]$ |
| $\psi^i(z)G^j_{\text{tot}}(w)$                                          | $\frac{1}{z - w} \sqrt{\frac{2}{k^+ + k^- + 2}} \left[ (t^{a+})^{ij} J^{a+}_{\text{tot}}(w) - (t^{a-})^{ij} J^{a-}_{\text{tot}}(w) \right] + \frac{U(w)\delta^{ij}}{z - w}$ |
| $U(z)G^i_{\text{tot}}(w)$                                               | $\frac{\psi^i(w)}{(z - w)^2}$                                                                     |

where $\mathcal{O}$ stands for the generators $G_{\text{tot}}$, $J_{\text{tot}}$ and $\psi$ of dimension $3/2$, 1 and $1/2$, respectively, and the $\hat{D}(1,2;\alpha)$ QSCA central charge $c$ is given by eq. (3.9). Unlike refs. \[ \text{[8, 23]} \], we always put forward the underlying QSCA structure in our notation.
Having restricted ourselves to the (Neveu-Schwarz–type, for definiteness) modes $(L_{\text{tot}})_{\pm 1,0}$, $(G^{\text{tot}}_{\text{ini}})_{\pm 1/2}$, and $(J^{\text{tot}}_{\text{ini}})_{0}$, we get a finite-dimensional Lie superalgebra which is isomorphic to the simple Lie superalgebra $D(1,2;\alpha)$ from the Kač list \[14\]. This explains the reason why we use almost the same (with hat) notation for our affine (infinite-dimensional) QSCA $\hat{D}(1,2;\alpha)$ defined by eqs. (3.11) or (3.12). Note that the finite Lie superalgebra of the ‘large’ $N = 4$ SCA in eq. (3.17), defining a ‘linearised’ version of the $\hat{D}(1,2;\alpha)$ QSCA in eq. (3.11), is not simple, but contains a $U(1)$ piece, in addition to the finite-dimensinal $D(1,2;\alpha)$ subalgebra. The finite-dimensional simple Lie superalgebras $D(2,1;\alpha)$ at various $\alpha$ values are not, in general, isomorphic to each other (except of the isomorphism under $\alpha \to \alpha^{-1}$, interchanging the two $\widehat{su}(2)$ factors) \[14\]. This is enough to argue about the non-equivalence (for different $\alpha$) of the $\hat{D}(1,2;\alpha)$ QSCAs, which are their affine generalisations.

It is also worthy to notice that the KM ‘levels’ and the central charge of the ‘large’ $N = 4$ SCA and those of the underlying $\hat{D}(1,2;\alpha)$ QSCA are different according to eq. (3.16), namely

\[ k_{\text{large}}^\pm = k^\pm + 1, \quad c_{\text{large}} = c + 3, \tag{3.18} \]

which is quite obvious because of the new fields introduced. The exceptional ‘small’ $N = 4$ SCA with the $\widehat{su}(2)$ KM component \[1\] follows from eq. (3.17) in the limit $\alpha \to \infty$ or $\alpha \to 0$, where either $k^- \to \infty$ or $k^+ \to \infty$, respectively, and the $\widehat{su}(2) \oplus u(1)$ KM component decouples from the rest of the algebra. Taking the limit results in the central charge

\[ c_{\text{small}} = 6k, \tag{3.19} \]

where $k$ is an arbitrary ‘level’ of the remaining $\widehat{su}(2)$ KM component. For an arbitrary $\alpha$, the ‘large’ $N = 4$ SCA contains two ‘small’ $N = 4$ SCAs \[8\].

As regards the representation theory, it is also advantageous to express a given algebra in terms of the smaller number of fundamental charges, whenever it is possible, since it makes more evident the structure of its representations. The representation theory of the $SO(N)$-based Bershadsky-Knizhnik QSCAs was developed in ref. \[19\]. Some of these representations can be naturally constructed by using the Kazama-Suzuki method on quaternionic (Wolf) spaces \[20\]. A construction of unitary highest weight (positive energy) representations of the ‘small’ and ‘large’ $N = 4$ SCAs can be found in ref. \[21\] and refs. \[22, 23\], respectively. Since both the $\hat{D}(1,2;\alpha)$ QSCA and the ‘large’ $N = 4$ SCA contain two $SU(2)$ KM subalgebras, unitary requires both levels $k^\pm$ to be non-negative integers, in particular. The Kazama-Suzuki method on

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\[^9\] See eq. (2.1) for their definition.
the quaternionic spaces $\text{Wolf} \otimes SU(2) \otimes U(1)$ can also be used to get unitary highest weight representations of the ‘large’ $N = 4$ SCA for $3 \leq c_{\text{large}} < 6$ \cite{22}.

The ‘basic’ unitary (non-linear) representation \cite{7} of the ‘large’ $N = 4$ SCA in terms of four free fermions and one boson ($\psi^i, \phi$) can be immediately obtained from eq. (3.15) by taking $T(z) = G^i(z) = J^{a\pm}(z) = 0$. This representation has the lowest possible central charge value, $c = 3$, among all the unitary representations of the ‘large’ $N = 4$ algebra, having the central charge (2.9) for non-negative integers $k$. This representation is actually realised in a model with defected Ising chains \cite{24}. More unitary representations can be constructed by taking tensor products of the ‘basic’ representation \cite{7, 8}.

4 Gauging a Classical QSCA

2d quantum conformal field theory requires the closure of its OPEs \cite{3}. There is nothing wrong there if the closure actually takes place in the $W$-sense, i.e. in terms of (regularised) local products of the fundamental fields. When building a $W$-type (super)gravity or a $W$-type (super)string, we need to represent the underlying non-linear algebra as a symmetry algebra, with the symmetries to be locally realised.

All affine algebras considered in the previous sections are the quantum (Q)SCAs. Their classical versions can be recovered in the limit $\hbar \to 0$, after making the substitutions $\mathcal{O} \to \tilde{\mathcal{O}} \equiv \hbar \mathcal{O}$ for all the (Q)SCA generators $\mathcal{O}$, and using the standard correspondence between the classical (graded) Poisson brackets and the (anti)commutators,

$$\{\tilde{\mathcal{O}}_1, \tilde{\mathcal{O}}_2\}_{P.B.} = \lim_{\hbar \to 0} \frac{1}{\hbar} [\tilde{\mathcal{O}}_1, \tilde{\mathcal{O}}_2] . \quad (4.1)$$

It should be stressed, however, that, in order to get a correct classical result, one should take into account an additional factor of $\hbar$ in front of the current bilinear in a quantum QSCA.

Therefore, when starting from a quantum QSCA, we first need to identify its classical analogue and its non-anomalous component. Both are of crucial importance for the gauging procedure. The main goal of this section is to outline a construction of the new ($W$-type) $N = 4$ extended conformal supergravity based on the classical version of the non-linear quantum algebra $\hat{D}(1, 2; \alpha)$ introduced in the previous section. We are going to compare this new $N = 4$ conformal supergravity with the standard $SU(2)$-based $N = 4$ conformal supergravity in two dimensions. More details about the new $N = 4$ supergravity are going to be reported in a separate publication \cite{23}.
As to the regular (linear) $N$-extended SCAs constructed by Ademollo et al. \[\textit{[4]}\], the corresponding classical symmetry algebras with vanishing central terms can be easily realised in terms of the restricted (= superconformal) superdiffeomorphisms in the $N$-extended light-cone superspace \[\textit{[4]}\], where the vanishing $N$-extended schwartzian derivative just means no anomaly. The last condition is known to be solved by the superprojective transformations forming the ‘little superconformal group’ $OSP(N/2)$ \[\textit{[23]}\]. The regular SCAs do not admit central extensions for $N \geq 5$ \[\textit{[27]}\], and, therefore, they are of no interest for superconformal quantum field theory where anomalous extensions, it cannot be gauged in full because of the presence of matter (\[\psi^i, \phi\]) among its ‘currents’. The only case of linear quantum $N = 4$ SCA which can be locally realised is the exceptional ‘small’ $N = 4$ SCA equivalent to the $SU(2)$-based Bershadsky-Knizhnik (Q)SCA \[\textit{[16, 17]}\]. The corresponding 2d, $N = 4$ conformal supergravity was constructed by gauging this ‘small’ $N = 4$ SCA in ref. \[\textit{[29]}\]. The $N = 4$ locally supersymmetric string action was known even before \[\textit{[3]}\].

As far as the non-linear $\tilde{D}(1, 2; \alpha)$ QSCA is concerned, its classical limit ( $\equiv \tilde{D}_4$) with vanishing central terms takes the form

$$
\begin{align*}
\{T(\zeta^+), T(\xi^+)\}_{\text{P.B.}} & = -\delta'(\zeta^+ - \xi^+) \left[ T(\zeta^+) + T(\xi^+) \right], \\
\{T(\zeta^+), G^i(\xi^+)\}_{\text{P.B.}} & = -\delta'(\zeta^+ - \xi^+) \left[ G^i(\zeta^+) + \frac{1}{2} G^i(\xi^+) \right], \\
\{T(\zeta^+), J^{a\pm}(\xi^+)\}_{\text{P.B.}} & = -\delta'(\zeta^+ - \xi^+) \left[ J^{a\pm}(\zeta^+) - \frac{1}{2} J^{a\pm}(\xi^+) \right], \\
\{J^{a\pm}(\zeta^+), G^i(\xi^+)\}_{\text{P.B.}} & = \delta(\zeta^+ - \xi^+) \frac{1}{2} (\xi^\pm)^{ij} G^j(\xi^+), \\
\{J^{a\pm}(\zeta^+), J^{b\pm}(\xi^+)\}_{\text{P.B.}} & = \delta(\zeta^+ - \xi^+) \epsilon^{abc} J^c(\xi^+), \\
\{G^i(\zeta^+), G^j(\xi^+)\}_{\text{P.B.}} & = \delta(\zeta^+ - \xi^+) \left[ 2 \delta^{ij} T(\xi^+) - \Lambda^{ij}(\xi^+) \right],
\end{align*}
$$

(4.2)

where the new composite generator ($A = +, -$)

$$
\Lambda^{(i)j} = \Lambda_{aAB}^{ij} J^a(\zeta^+), J^b(\xi^+)
$$

\[\begin{equation}
\equiv (t^a, J^a(\zeta^+) - t^a, J^a(-\zeta^+))^{(i)}_k (t^b, J^b(\zeta^+) - t^b, J^b(-\zeta^+))^{j)}_k,
\end{equation}\]

(4.3)

and the 2d space-time light-cone coordinates $\zeta^\pm$, related to the usual 2d Cartesian coordinates $(t, \sigma)$ by $\zeta^\pm = \frac{1}{\sqrt{2}} (t \pm \sigma)$, have been introduced. The holomorphic coordinate $z$, appearing in the OPEs, is related to them via the Wick rotation, $\tau = it$, and the exponential map, $z = e^{\tau + i\sigma}$, as is usual in conformal field theory \[\textit{[3]}\]. We also have to assume here that all currents in eq. (4.2) arise from some classical conformal field theory, so that the bracket on the l.h.s. of that equation is the graded Poisson
bracket in a canonical formalism with $\zeta$ as the time variable. Note that, in classical theory, there is no problem in defining the product of currents in eq. (4.3) at coincidence limit, unlike the situation in quantum theory where one uses normal ordering and divergence subtraction. The classical algebra $\tilde{D}_4$ defined by eqs. (4.2) and (4.3) can be considered as a particular supersymmetric version of the Gel’fand-Dikii-type algebras known in the theory of integrable models [30].

When following the lines of construction of the conventional extended conformal supergravity theories in two dimensions [29], it would be natural to pick up

$$L_{\pm 1,0}, \quad G_{\pm 1/2}^{i}, \quad J_{0}^{a \pm },$$

as the non-anomalous generators. However, according to eq. (3.12), they do not form a closed subalgebra in the case of $\hat{D}(1,2;\alpha)$, because of the non-linear term present in the anticommutator of the supersymmetry charges,

$$\{G_{r}^{i}, G_{s}^{j}\} = 2\delta^{ij}L_{r+s} + \frac{2}{k^{+} + k^{-} + 2}(s - r) \left[k^{-}(t^{a+})^{ij}J_{r+s}^{a+} + k^{+}(t^{a-})^{ij}J_{r+s}^{a-}\right] - \frac{2}{k^{+} + k^{-} + 2} \left(t^{a+}J_{r+s}^{a+} - t^{a-}J_{r+s}^{a-}\right)^{(ij}_{k} \left(t^{b+}J_{r+s}^{b+} - t^{b-}J_{r+s}^{b-}\right)^{j)}_{k}, (4.5)$$

where $r, s = \pm 1/2, \quad r + s = -1, 0, 1$. Though the second term on the r.h.s. of this equation never contributes when $s = r = \pm \frac{1}{2}$ and, hence, it does not actually depend on $J_{r+s}^{a \pm }$, the last term vanishes only in the limit $k^{+} + k^{-} \equiv 2k \rightarrow \infty$ where the finite set (4.4) formally constitutes a linear algebra. [30]

The ‘small’ (non-chiral) $SU(2)$-based $N = 4$ conformal supergravity [3] is known to be obtained by gauging the linear $ssu(1,1|2) \oplus ssu(1,1|2)$ Lie superalgebra, whereas its chiral $(4,0)$ supersymmetric version is based on a factor $ssu(1,1|2)$ [26]. The superalgebra $ssu(1,1|2)$ appears, in particular, in the Kač list of finite-dimensional simple Lie superalgebras [14], and its internal symmetry generators are just the self-dual or anti-self-dual ones, $J_{a}^{a+}$ or $J_{a}^{a-}$. When comparing its contents and (anti)commutation relations with what we have in eq. (4.2), it is clear that conformal supergravity based on gauging the classical $W$-type non-linear algebra $\tilde{D}_4$ is going to be different from the ‘small’ $N = 4$ conformal supergravity. Indeed, the former has six (instead of three) spin-1 gauge fields in the vector representation of $SO(4)$, and its underlying algebra is non-linear.

The full 2d algebra to be gauged is the direct sum of two light-cone parts, each one being isomorphic to $\tilde{D}_4$. Gauging only one factor should correspond to a ‘chiral’

\[\text{10}^\text{th} \text{This is to be compared with the known fact that, e.g. in the Zamolodchikov’s } W_3 \text{ algebra, the coefficient in front of the only non-linear term in the commutator of two spin-3 generators vanishes in the limit } c \rightarrow \infty, \text{ where } c \text{ is the quantum } W_3 \text{ central charge [28]}.\]
\( \tilde{D}_4 \) supergravity. It seems to be quite appropriate to identify it as the classical \((4,0)\) supersymmetric ‘heterotic’ (or chiral) \( \tilde{D}_4 \) conformal supergravity, whereas the full (non-chiral) 2d theory based on gauging the \( \tilde{D}_4 \oplus \tilde{D}_4 \) classical algebra should be called the \((4,4)\) supersymmetric (non-chiral) \( \tilde{D}_4 \) conformal supergravity. Because of the classical isomorphisms \( su(1,1) \sim so(2,1) \sim sl(2) \) and \( so(2,2) \sim sl(2) \oplus sl(2) \), the full algebra \( \tilde{D}_4 \oplus \tilde{D}_4 \) obviously contains the 2d ‘little’ (finite-dimensional) conformal algebra \( so(2,2) \), as it should \[31\]. In addition, it has four supersymmetry charges of each chirality. Hence, we are dealing with an \( N = 4 \) extended conformal 2d supergravity indeed.

Consider now any 2d classical superconformal field theory with the \( \tilde{D}_4 \) or \( \tilde{D}_4 \oplus \tilde{D}_4 \) symmetry. It could be, e.g., a \((4,0)\) or \((4,4)\) supersymmetric WZNW model on a quaternionic (Wolf) space. \[11\] Let \( S_0 \) be its classical action, and \( W_{\pm M} \) the \( \tilde{D}_4 \oplus \tilde{D}_4 \) currents labelled by some (generalised) index \( M \) and satisfying the constraints \( \partial_\pm W_{\pm M} = 0 \) where the signs are correlated. Then the gauging procedure is very similar to the one known for the classical \( W \) algebras. \[12\] One introduces gauge fields \( h^{\pm M} \) for each current and adds the Noether (minimal) coupling to \( S_0 \). To lowest order, the action is given by \[35\]

\[
S = S_0 + \sum_M \int d^2 \zeta \left[ h^{+M} W_{+M} + h^{-M} W_{-M} \right] + O(h^2) ,
\]

where the higher order corrections could, in principle, be calculated using the Noether (trial and error) method. Transformations laws of the gauge fields get fixed by imposing the (anti)commutation relations of the symmetry algebra \( \tilde{D}_4 \oplus \tilde{D}_4 \). As is usual for the classical \( W \) algebras, the Noether coupling alone gives the full gauge-invariant action for the chiral gauge theory of \( \tilde{D}_4 \), where only the gauge fields \( h^{+M} \) are present \[35\].

In the non-chiral case, the full gauge-invariant action is non-polynomial in the gauge fields, and, in a non-covariant form, can be constructed within the canonical approach as follows \[36\]. \[38\] After rewriting the action \( S_0 \) to the hamiltonian (first-order) form and replacing time derivatives of fields in the \( \tilde{D}_4 \) currents by the corresponding momenta, the gauge-invariant action is obtained by simply adding the Noether coupling of these currents to Lagrange multipier gauge fields. The full action reads

\[
S = \int dt \left[ p_A \partial_t q^A - h^M(W_M(p,q)) \right] ,
\]

\[4.7\]

\[11\] The Wolf spaces are also selected for the couplings of \( N = 2 \) scalar multiplets to \( N = 2 \) extended supergravity in four dimensions \[32\].

\[12\] See, e.g., refs. \[33, 34\] for a review.

\[13\] The related approach based on introducing auxiliary fields \[37\] can also be used for this purpose.
in terms of the generalised coordinates $q^A(t)$ and the momenta $p_A(t)$. The gauge fields impose the first-class constraints $W_M \sim 0$, whose Poisson bracket algebra obviously closes in the weak sense (on the constraints),

$$\{W_M, W_N\}_{\text{P.B.}} = f_{MN}^L(p, q)W_L,$$

(4.8)

where some of the structure ‘constants’ $f_{MN}^L$ in the case of the non-linear algebra $\tilde{D}_4 \oplus \tilde{D}_4$ are dependent on phase space variables. The action (4.8) is invariant under the following local symmetries with parameters $\varepsilon^M(t)$:

$$\delta p_A = \varepsilon^M\{W_M, p_A\}_{\text{P.B.}},$$

$$\delta q^A = \varepsilon^M\{W_M, q^A\}_{\text{P.B.}},$$

$$\delta h^M = \partial_t\varepsilon^M - f_{NL}^M h^N\varepsilon^L.$$  

(4.9)

Elimination of momenta is, however, non-trivial in the action (4.7), but this is a technical problem.

In summary, the gauge field contents of the 2d, $\tilde{D}_4$ conformal supergravity is

$$h_{\mu\nu}, \quad \chi^i_\mu, \quad A^a_{\mu} \pm ,$$

(4.10)

where $h_{\mu\nu}$ is spin-2 graviton, $\chi^i_\mu$ are four spin-3/2 Majorana gravitinos, and $A^a_{\mu} \pm$ are six spin-1 gauge fields. The gauge field contents is balanced by the gauge symmetries as usual, which implies no off-shell degrees of freedom for this new $SO(4)$-extended 2d conformal supergravity. In other words, the classical 2d, $\tilde{D}_4$ conformal supergravity is trivial (up to moduli). However, in quantum theory, some of the gauge symmetries may become anomalous and thereby some of the gauge degrees of freedom may become physical. The ghost contributions to the central charge of the BRST quantised $N = 4$ conformal supergravities are collected in Table I (see sect. 5 for a derivation and more results).

Table I. The ghost contributions to the $N = 4$ central charge.

| dimension | $c_j$ | ‘small’ SCA | $\hat{D}(1, 2; \alpha)$ |
|-----------|------|-------------|------------------------|
| 3/2       | +11  | +44         | +44                    |
| 1         | -2   | -6          | -12                    |
| total     | +12  | +6          |                        |

\[^{14}\text{We use here the condensed notation in which the generalised indices } A, B, \ldots \text{ represent both the discrete indices and the continuous variable } \sigma. \text{ The same convention applies to summations.}\]
5 BRST Charge

In this section, the nilpotent quantum BRST charges for the non-linear QSCAs introduced in sects. 2 and 3 are constructed. Despite of the apparent non-linearity of these algebras, their quantum BRST charges should be in correspondence with their classical BRST charges, up to renormalisation. A classical BRST charge having the vanishing Poisson bracket with itself can, in fact, be constructed for an arbitrary algebra of first-class constraints \[40\]. \[^{15}\] This procedure was already applied to obtain the quantum BRST charge for the non-linear quantum \(W_3\) algebra \[38\], and later generalised to any quadratically non-linear \(W\)-type algebra in ref. \[39\]. In particular, the quantum BRST charges for the orthogonal and unitary series of Berschadsky-Knizhnik QSCAs were also calculated in ref. \[39\], but the analysis of the BRST charge nilpotency conditions given there was, however, incomplete. The nilpotency conditions always require the total (matter + ghosts) central charge to vanish, but also lead to some more constraints on the QSCA parameters, whose consistency is not guaranteed. This is because the ‘new’ constraints imposed by the BRST charge nilpotency condition may be in conflict with the ‘old’ constraints dictated by the QSCA Jacobi identities. Our calculations of the quantum BRST charge for the Berschadsky-Knizhnik \(SO(N)\)-based QSCAs confirm ‘almost’ all of the results of ref. \[39\], but one. Namely, we find that the only nilpotent solution exists at \(N = 4\). We then make a similar calculation for the new case of the \(N = 4\) supersymmetric \(\hat{D}(1, 2; \alpha)\) QSCA, obtain the corresponding quantum BRST charge and its nilpotency conditions. Again, we find only one solution, namely, just in the case when the \(\hat{D}(1, 2; \alpha)\) QSCA reduces to the \(SO(4)\) Berschadsky-Knizhnik QSCA. Some consequences of our results for string theory are discussed in Conclusions, sect. 6.

We begin with the Berschadsky-Knizhnik \(SO(N)\)-based QSCA introduced in sect. 2. The BRST ghosts appropriate for this case are:

- the conformal ghosts \((b, c)\), an anticommuting pair of world-sheet free fermions of conformal dimensions \((2, -1)\), respectively;
- the \(N\)-extended superconformal ghosts \((\beta^i, \gamma^i)\) of conformal dimensions \((\frac{3}{2}, -\frac{1}{2})\), respectively, in the fundamental (vector) representation of \(SO(N)\);
- the \(SO(N)\) internal symmetry ghosts \((\tilde{b}^a, \tilde{c}^a)\) of conformal dimensions \((1, 0)\), respectively, in the adjoint representation of \(SO(N)\).

\[^{15}\]See Appendix for details.
The reparametrisation ghosts

\[
  b(z) = \sum_{n \in \mathbb{Z}} b_n z^{-n-2}, \quad c(z) = \sum_{n \in \mathbb{Z}} c_n z^{-n+1},
\]

have the following OPE and anticommutation relations:

\[
  b(z) c(w) \sim \frac{1}{z-w}, \quad \{c_m, b_n\} = \delta_{m+n,0}.
\]

The superconformal ghosts

\[
  \beta^i(z) = \sum_{r \in \mathbb{Z}(-1/2)} \beta^i_r z^{-r-3/2}, \quad \gamma^i(z) = \sum_{r \in \mathbb{Z}(1/2)} \gamma^i_r z^{-r+1/2},
\]

satisfy the OPE

\[
  \beta^i(z) \gamma^j(w) \sim \frac{-\delta^{ij}}{z-w},
\]

which implies the only non-vanishing commutation relations

\[
  [\gamma^i_r, \beta^j_s] = \delta_{r+s,0}.
\]

An integer or half-integer moding of these generators corresponds to the usual distinction between the Ramond- and Neveu-Schwarz-type sectors.

Finally, the fermionic \(SO(N)\) ghosts

\[
  \tilde{b}^a(z) = \sum_{n \in \mathbb{Z}} \tilde{b}_n^a z^{-n-1}, \quad \tilde{c}^a(z) = \sum_{n \in \mathbb{Z}} \tilde{c}_n^a z^{-n},
\]

have

\[
  \tilde{b}^a(z) \tilde{c}^a(w) \sim \frac{\delta^{ab}}{z-w}, \quad \{\tilde{c}_m^a, \tilde{b}_n^b\} = \delta^{ab} \delta_{m+n,0}.
\]

The construction of a classical BRST charge for any (non-linear) algebra of first-class constraints is reviewed in Appendix. This construction provides us with the reasonable ansatz for the quantum BRST charge\cite{39} associated with the quantum \(SO(N)\)-based Bershadsky-Knizhnik QSCA in the form (cf ref.\cite{39})

\[
  Q_{\text{BRST}} = c_{-n} L_n + \gamma^i_r G^i_r + \tilde{c}^a_{-n} J^a_n - \frac{1}{2} (m - n) c_{-m} c_{-n} b_{m+n} + nc_{-m} \tilde{b}^a_{-n} \tilde{b}_m^a
  + \left( \frac{m}{2} - r \right) c_{-m} \gamma^i_r \gamma^j_r - b_{r+s} \gamma^i_r \gamma^j_r - \tilde{c}^a_{-m} \beta^i_{m+r} (t^a)^{ij} \gamma^j_r
  + \eta a_2 (r-s) b^a_{r+s} \gamma^i_r \gamma^j_r - \frac{1}{2} f^{abc} c_{-m} \tilde{c}^b_{-n} \tilde{b}_m^c
  - \frac{1}{2} a_4 \left[ (t^a t^b)^{ij} + \delta^{ab} \delta^{ij} \right] J^a_{r+s+m} \tilde{b}_m^i \gamma^j_r - \frac{1}{24} a_4^2 \left[ (t^a t^b)^{ij} + \delta^{ab} \delta^{ij} \right]
  \times \left[ (t^c t^d)^{kl} + \delta^{cd} \delta^{kl} \right] f^{ace} \delta_{m+n+p,q+r+s+t+u} \tilde{b}_m^b \tilde{b}_n^c \tilde{b}_p^d \gamma^j_r \gamma^j_r \gamma^j_r \gamma^j_u,
\]

\footnote{The normal ordering is implicit below.}
where a quantum renormalisation parameter \( \eta \) has been introduced. Its value is going to be fixed by the BRST charge nilpotency conditions. The coefficients \( a_2 \) and \( a_4 \) have already been fixed by eq. (2.6).

We find always useful to represent a quantum BRST charge as

\[
Q_{\text{BRST}} = \oint \frac{dz}{2\pi i} j_{\text{BRST}}(z),
\]

where the BRST current \( j_{\text{BRST}}(z) \) is defined \textit{modulo} total derivative. In particular, the BRST current \( j_{\text{BRST}}(z) \) corresponding to the BRST charge of eq. (5.8) is given by

\[
j_{\text{BRST}}(z) = cT + \gamma^i G^i + \bar{c}^a J^a + bc \partial c - \bar{c}^a \partial \bar{c}^a - \frac{1}{2} c^i \partial \beta^i - \frac{3}{2} c^i \partial \gamma^i - b \gamma^i \gamma^i - \eta^a \bar{b}^a \bar{\eta} \bar{c}^a \bar{c}^a \beta^i \gamma^i - \frac{1}{2} f^{abc} \bar{c}^a \bar{c}^b \bar{c}^c - bc \partial c - \frac{1}{2} f^{abc} \bar{c}^a \bar{c}^b \bar{c}^c - \frac{1}{2} f^{abc} \bar{c}^a \bar{c}^b \bar{c}^c - \frac{1}{2} a_4 \left[ (t^a t^b)_{ij} \beta^i \gamma^j + \delta^{ab} \delta^{ij} \right] J^a \bar{b}^b \gamma^j \gamma^j - \frac{1}{24} a_4^2 \left[ (t^a t^b)_{ij} \beta^i \gamma^j + \delta^{ab} \delta^{ij} \right] J^a \bar{b}^b \gamma^j \gamma^j - \frac{1}{24} a_4^2 \left[ (t^a t^b)_{ij} \beta^i \gamma^j + \delta^{ab} \delta^{ij} \right] J^a \bar{b}^b \gamma^j \gamma^j - \frac{1}{24} a_4^2 \left[ (t^a t^b)_{ij} \beta^i \gamma^j + \delta^{ab} \delta^{ij} \right] J^a \bar{b}^b \gamma^j \gamma^j.
\]

The quantum BRST charge for any (quasi)SCA can always be written in the standard form,

\[
Q_{\text{BRST}} = \oint \frac{dz}{2\pi i} \left[ c(T + \frac{1}{2} T_{gh}) + \gamma^i (G^i + \frac{1}{2} G^i_{gh}) + \bar{c}^a (J^a + \frac{1}{2} J^a_{gh}) \right],
\]

where the ghost contributions which can be read off from an explicit formula for \( Q_{\text{BRST}} \). There are, however, some important differences between the linear and nonlinear (Q)SCAs because, as is clear, e.g., from eq. (5.10), the QSCA ghost contributions are \textit{dependent} on the matter currents and include terms of higher order in the (anti)ghosts. In particular, the ghost supercurrent \( G^i_{gh} \) in eqs. (5.10) and (5.11) involves the spin-1 matter currents \( J^a \), which is intuitively reasonable since one can view the non-linear bilinear on the r.h.s. of the supersymmetry algebra in eq. (2.8) as being like a linear algebra but with \( J \)-dependent structure ‘constants’. These structure constants then appear in the ghost currents. The ghost currents alone need not satisfy the QSCA, and they actually do not. In addition, central extensions (anomalies) of the ghost-extended QSCA need not form a linear supermultiplet, and they actually do not also. Therefore, the vanishing of any anomaly alone does \textit{not} automatically imply the vanishing of the others, unlike in the linear case.

The most tedious part of calculational handwork in computing \( Q_{\text{BRST}}^2 \) can be avoided when using either the Mathematica Package for computing OPEs \cite{41} or

\footnote{The total derivative can be fixed by requiring the \( j_{\text{BRST}}(z) \) to transform as a primary field.}
some of the general results in ref. [39]. In particular, as was shown in ref. [39], quantum renormalisation of the 3-point structure constants in the quantum BRST charge should be *multiplicative*, whereas the non-linearity 4-point ‘structure constants’ should *not* be renormalised at all — the facts already used in the BRST charge ansatz above. Most importantly, among the contributions to the $Q^2_{\text{BRST}}$, only the terms *quadratic* in the ghosts are relevant. Their vanishing imposes the constraints on the central extension coefficients of the QSCA and simultaneously determines the renormalisation parameter $\eta$. The details can be found in the appendices of ref. [39]. The same conclusion comes as a result of straightforward calculation on computer. Therefore, finding out the nilpotency conditions amounts to calculating only a few terms ‘by hands’, namely, those which are quadratic in the ghosts. This makes the whole calculation as simple as that in ordinary string theories based on linear SCAs [3].

The 2-ghost terms in the $Q^2_{\text{BRST}}$ arise from single contractions of the first three linear (in the ghosts) terms of $Q_{\text{BRST}}$ with themselves and with the next cubic terms of eq. (5.10), and from double contractions of the latter among themselves. They result in the pole contributions to $j_{\text{BRST}}(z)j_{\text{BRST}}(w)$, proportional to $(z-w)^{-n}$ with $n = 1, 2, 3, 4$. All the residues have to vanish modulo total derivative. We find

\[
j_{\text{BRST}}(z)j_{\text{BRST}}(w) \sim \frac{c(z)c(w)}{2(z-w)^4} \left[ c - N^2 + 12N - 26 \right]
+ \frac{\gamma^i(z)\gamma^i(w)}{(z-w)^3} \left[ a_1 - \frac{ka_4}{2}(N-1)(N-2) - 4\eta a_2(N-1) + 2 \right]
+ \frac{\bar{c}^a(z)\bar{c}^a(w)}{(z-w)^2} \left[ -k - 2(N-2) + 2 \right]
+ \frac{J^a(w)\mu^i j^i(w)\partial\gamma^j(w)}{z-w} \left[ -4\eta a_2 - 4a_4 \left( 1 - \frac{N}{2} \right) \right] + \ldots ,
\]

where the dots stand for the other terms of higher order in the (anti)ghosts, while the coefficients $a_1, a_2, a_4$ and $c$ are given by eqs. (2.6) and (2.7), respectively.
Eq. (5.12) immediately yields the BRST charge nilpotency conditions:

\[
\begin{align*}
    c_{\text{tot}} &\equiv c + c_{\text{gh}} = \frac{k(N^2 + 6k - 10)}{2(N + k - 3)} - N^2 + 12N - 26 = 0 , \\
    s_{\text{tot}} &\equiv a_1 + (a_1)_{\text{gh}} = \frac{k(N + 2k - 4)}{N + k - 3} - \frac{k(N - 1)(N - 2)}{2(N + k - 3)} \\
    &\quad - \frac{4\eta(N - 1)}{N + k - 3} + 2 = 0 , \\
    k_{\text{tot}} &\equiv k + k_{\text{gh}} = k + 2N - 6 = 0 , \\
    \eta(N + 2k - 4) &\quad \frac{N - 2}{2(N + k - 3)} = 0 .
\end{align*}
\] (5.13)

The first line of eq. (5.13) just means the vanishing total central charge, where the value of \( c_{\text{gh}} \) is dictated by the standard formula of conformal field theory \[6\]
\[
    c_{\text{gh}} = 2 \sum_\lambda n_\lambda (-1)^{2\lambda+1} \left( 6\lambda^2 - 6\lambda + 1 \right)
\]
\[
    = 1 \times (-26) + N \times (+11) + \frac{1}{2}N(N - 1) \times (-2) = -26 + 12N - N^2 ,
\] (5.14)

\( \lambda \) is conformal dimension and \( n_\lambda \) is a number of the conjugated ghost pairs: \( \lambda = 2, 3/2, 1 \) and \( n_\lambda = 1, N, \frac{1}{2}N(N - 1), \) respectively. The zero central charge condition alone has two solutions, 

\[
    k = 6 - 2N , \quad 6k = N^2 - 12N + 26 ,
\] (5.15)

but only the first of them is compatible with the third equation (5.13). The fourth equation (5.13) just determines the renormalisation parameter \( \eta \). Finally, the second equation (5.13) can be interpreted as the vanishing total supersymmetric anomaly. Since the supersymmetry is non-linearly realised, this anomaly does not have to vanish as a consequence of the other equations (5.13), but restricts \( N \) as the only remaining parameter. Substituting 

\[
    k = 6 - 2N , \quad \text{and} \quad \eta = \frac{N - 2}{2(8 - 3N)} ,
\] (5.16)

into the second equation (5.13), we find 

\[
    6(N - 3) + \frac{(N - 1)(N - 2)(N - 5)}{N - 3} = 0 ,
\] (5.17)

which has only one solution, \( N = 4 \). Therefore, though the system of four equations (5.13) for only three parameters \( \eta, k \) and \( N \) is clearly overdetermined (while \( N \) is a positive integer!), there is still the only solution, namely 

\[
    N = 4 , \quad k = -2 , \quad \eta = -\frac{1}{4} .
\] (5.18)
The nilpotent quantum BRST charge for the Bershadsky-Knizhnik $SO(N)$-extended QSCAs exists, therefore, only when $N = 4$. The corresponding BRST current reads

$$j_{\text{BRST}}(z) = cT + \gamma^i G_i + c^a J^a + bc\partial c - \tilde{b}^a \partial \tilde{c}^a - \frac{1}{2} c\gamma^i \partial \beta^i - \frac{3}{2} c\beta^i \partial \gamma^i$$

$$- b\gamma^i \gamma^i + \tilde{b}^a (t^a)^{ij} \left( \gamma^j \partial \gamma^i - \gamma^i \partial \gamma^j \right) - c^a (t^a)^{ij} \beta^i \gamma^j$$

$$- \frac{1}{2} f^{abc} \tilde{c}^a \tilde{b}^b \tilde{c}^c - \left[ (t^a t^b + t^b t^a)^{ij} + 2 \delta^{ab} \delta^{ij} \right] J^a \tilde{b}^b \gamma^i \gamma^j$$

$$- \frac{1}{6} \left[ (t^a t^b + t^b t^a)^{ij} + 2 \delta^{ab} \delta^{ij} \right] \left[ (t^c t^d + t^d t^c)^{kl} + 2 \delta^{cd} \delta^{kl} \right] \right] \times$$

$$\times f^{ace} \tilde{b}^b \tilde{b}^d \tilde{c}^e \gamma^i \gamma^j \gamma^k \lambda \lambda'.$$

We are now in a position to consider a construction of the quantum BRST charge for the $\hat{D}(1,2;\alpha)$ QSCA generalising the $SO(4)$-based Bershadsky-Knizhnik QSCA at $N = 4$. The (anti)commutation relations of the $\hat{D}(1,2;\alpha)$ QSCA were given in sect. 3. The ghost/antighost fields for the internal $su(2) \oplus su(2)$ KM component of the $\hat{D}(1,2;\alpha)$ QSCA are now denoted by $\tilde{c}^A(z), \tilde{b}^A(z)$, where $A = +, -, -.$

The natural ansatz for the quantum BRST charge of the $\hat{D}(1,2;\alpha)$ QSCA is given by (see Appendix)

$$Q_{\text{BRST}} = c_n L_n + \gamma^i G^i + c^a J^a + c^a L^a = - \frac{1}{2} (m - n) c_m c_n b_{m+n} + n c_m c^A n b^A_{m+n}$$

$$+ \left( \frac{m}{2} - r \right) c_m \beta^i m_r \gamma^i + b_r \gamma^i \gamma^i - \frac{1}{2} c_{m} c^A \left( t^A \right)^{ij} \beta^i m_r \gamma^j$$

$$+ \tilde{b}^a (r - s) \tilde{b}^a (t^a)^{ij} \gamma^i \gamma^j + \tilde{b}^a (r - s) \tilde{b}^a (t^a)^{ij} \gamma^i \gamma^j$$

$$- \frac{1}{2} \gamma e \gamma^i m_n \gamma^i - \frac{1}{2} \gamma e \gamma^i m_n \gamma^i - \frac{1}{2} b^a A^{ij} A^{ij} + \frac{1}{2} b^a A^{ij} A^{ij}$$

$$- \frac{1}{24} b^a A^{ij} A^{ij} A^{kl} A^{kl} = c e \delta^i m_n \gamma^i \gamma^i$$

or, equivalently,

$$j_{\text{BRST}}(z) = cT + \gamma^i G^i + c^a J^a + bc\partial c - \tilde{b}^a \partial \tilde{c}^a - \frac{1}{2} c\gamma^i \partial \beta^i - \frac{3}{2} c\beta^i \partial \gamma^i$$

$$- b\gamma^i \gamma^i + \tilde{b}^a (t^a)^{ij} \left( \gamma^j \partial \gamma^i - \gamma^i \partial \gamma^j \right) - c^a (t^a)^{ij} \beta^i \gamma^j$$

$$- \frac{1}{2} f^{abc} \tilde{c}^a \tilde{b}^b \tilde{c}^c - \left[ (t^a t^b + t^b t^a)^{ij} + 2 \delta^{ab} \delta^{ij} \right] J^a \tilde{b}^b \gamma^i \gamma^j$$

$$- \frac{1}{6} \left[ (t^a t^b + t^b t^a)^{ij} + 2 \delta^{ab} \delta^{ij} \right] \left[ (t^c t^d + t^d t^c)^{kl} + 2 \delta^{cd} \delta^{kl} \right] \times$$

$$\times f^{ace} \tilde{b}^b \tilde{b}^d \tilde{c}^e \gamma^i \gamma^j \gamma^k \lambda \lambda'.$$

where two quantum renormalisation parameters $\eta_2$ and $\eta_3$ have been introduced, and $\Lambda_{A^{ij}}$ denote the $\hat{D}(1,2;\alpha)$ QSCA 4-point ‘structure constants’, see eq. (4.3).

Requiring the quantum BRST charge (5.20) to be nilpotent, yields the following equations:
\begin{itemize}
  \item from the terms $c(z)c(w)/(z - w)^4$:
  \begin{equation}
  c_{\text{tot}} \equiv c + c_{gh} = \left[\frac{6(k^+ + 1)(k^- + 1)}{k^+ + k^- + 2} - 3\right] + 6 = 0 ,
  \end{equation}
  where the central charge $c$ is now given by eq. (3.9) and $c_{gh} = +6$ according to eq. (5.14);
  \item from the terms $\gamma^i(z)\gamma^i(w)/(z - w)^3$:
  \begin{equation}
  s_{\text{tot}} \equiv b_1 + (b_1)_{gh} = b_1 + \frac{3}{2} b_4 (k^+ + k^-) - 6(\eta_2 b_2 + \eta_3 b_3) + 2 = 0 ,
  \end{equation}
  where the parameters $b_1$, $b_2$ and $b_4$ are given by eq. (3.8);
  \item from the terms $c^{a\pm}(z)c^{a\pm}(w)/(z - w)^2$:
  \begin{equation}
  k_{\text{tot}}^{\pm} \equiv k^{\pm} + 2 = 0 ,
  \end{equation}
  \item from the terms $J^{a\pm}(t^{a\pm})^{ij}\gamma^i\partial\gamma^j/(z - w)$:
  \begin{equation}
  -2\eta_2 b_2 - 2b_4 = -2\eta_3 b_3 - 2b_4 = 0 .
  \end{equation}
\end{itemize}

Note that eqs. (5.13) and (5.21) are consistent with each other in the case of $N = 4$ and
\begin{equation}
  k = k^+ = k^- = -2 ,
\end{equation}
as they should. It provides a good check of our calculations. Moreover, eq. (5.22) is, in fact, the only consistent solution to eq. (5.21). This means that the BRST quantisation of the non-linear $\hat{D}(1, 2; \alpha)$ QSCA can only be consistent if both its $su(2)$ KM components enter symmetrically, i.e. when this quantum non-linear algebra is actually the $SO(4)$-based Bershadsky-Knizhnik QSCA, with $k = -2$ and $c = 3k = -6$. This is to be compared with the known fact \cite{12} that the quantum BRST charge for the ‘small’ $N = 4$ SCA, whose all central terms are related and proportional to central charge, is only nilpotent when $c = -12$ (see also Table I).

A connection between the non-linear $SO(4)$-based Bershadsky-Knizhnik QSCA and the ‘small’ linear $SU(2)$-based SCA exists via the linearisation of the former into the ‘large’ linear $SU(2) \otimes SU(2) \otimes U(1)$-based SCA and taking the limit either $k^+ \to 0$ or $k^- \to 0$ (see sect. 3). Since (i) there is no nilpotent QSCA BRST charge for the case of $k^+ \neq k^-$, and (ii) it does not make sense to gauge and BRST quantise all generators of the ‘large’ $N = 4$ linear SCA, there seems to be no direct connection between the nilpotent BRST operators for these two $N = 4$ (Q)SCAs.
6 Conclusion

In our paper we constructed the quantum BRST charges for the orthogonal (i.e. with the $\hat{so}(N)$ KM component) series of Bershadsky-Knizhnik non-linear QSCAs and for the particular quantum $\hat{D}(2,1;\alpha)$ QSCA generalising them at $N = 4$. We found only one nilpotent solution, namely, when $N = 4$, $k = -2$ and $\alpha = 1$. Eqs. (5.18) and (5.22) constitute our main results. They are apparently in line with the analogous fact [39] that the BRST quantisation for all unitary (i.e. with the $\hat{u}(N)$ KM component) Bershadsky-Knizhnik QSCAs breaks down for $N \geq 3$, since their BRST charge nilpotency conditions are always in conflict with the Jacobi identities. [18]

The existence of the nilpotent quantum BRST operator for the non-linear $SO(4)$-based Bershadsky-Knizhnik QSCA, and the existence of the corresponding $W$-type $N = 4$ conformal supergravity in two dimensions, imply the existence of a new $N = 4$ supersymmetric $W$-type string theory. The 2d supergravity field equations of motion, which follow from the action (4.6) or (4.7), impose the vanishing of the corresponding currents. At the quantum level, these conditions can be interpreted, as in ordinary string theories, as operator constraints on physical states. By interpreting zero modes of scalar fields in matter QSCA realisations as spacetime coordinates, one arrives at a first-quantised description of string oscillations. The structure of a matter part of the new string theory action will be discussed elsewhere [25].

Gauging the local symmetries of the $SO(4)$-based Bershadsky-Knizhnik QSCA results in the positive total ghost central charge contribution, $c_{gh} = 6$. When adding the matter $(\psi^i, \phi)$ to linearise this algebra (see sect. 3), one adds $+3$ to the total central charge. In addition, the anomaly-free solution requires $k = -2 < 0$. Therefore, there is no way to build an anomaly-free string theory by using only unitary representations. For example, in the ‘basic’ $c = 3$ unitary non-linear representation (sect. 3), the matter represented by $(\psi^i, \phi)$ is unified with the currents $(T_{tot}, G_{tot}^i, J_{tot}^{a+\pm})$ into one $N = 4$ linear supermultiplet with $8_B \oplus 8_F$ components. As was suggested in ref. [7], one may try to interpret the only bosonic coordinate $\phi$ there as a string world-sheet coordinate, with the non-linearly realised world-sheet supersymmetry. However, it would then be impossible to match the anomaly-free condition.

When choosing, instead, a non-unitary representation of the $SO(4)$-based QSCA with $k = -2$, one can get the desired anomaly-free matter contribution, $c_m = -6$, but then a space-time interpretation and a physical significance of the construction, if any,

\footnote{The unitary series of Bershadsky-Knizhnik QSCAs do not admit unitary representations for $N \geq 3$ also [26].}
become obscure. Despite of all this, we believe that it is worthy to know how many string models, consistent from the mathematical point of view, can be constructed at all. Relying on the argument based on the existence of a gauge-invariant action and a nilpotent BRST operator, our answer reads: at $N = 4$ there are only two different critical string theories within the requirements mentioned in the Introduction.

Our results could still be generalised. In addition to the critical $N = 2$ extended fermionic strings, based on complex numbers and the linear $N = 2$ SCA, and the critical $N = 4$ extended fermionic strings, based on quaternionic numbers and on either the linear ‘small’ $N = 4$ SCA or the non-linear $SO(4)$-based Bershadsky-Knizhnik QSCA, new consistent string theories, based on octonionic numbers and having $N = 7$ or $N = 8$ world-sheet supersymmetry, may exist. The exceptional $N = 7$ and $N = 8$ QSCAs are known to arise via Drinfeld-Sokolov-type reduction from affine versions of the exceptional Lie superalgebras $G(3)$ and $F(4)$ \cite{12,14}. Their central charges are

$$c_7 = \frac{k(9k + 31)}{2(k + 3)}, \quad c_8 = \frac{2k(2k + 11)}{k + 4},$$

where $k$ is an arbitrary ‘level’ of the corresponding KM subalgebra. Those exceptional QSCAs are indeed related with octonions \cite{12,14}, and may even be identified with the octonionic generalisations of the $SO(3)$- and $SO(4)$-based Bershadsky-Knizhnik QSCAs, respectively.

Like the $W$ algebras and unlike the linear SCAs, the non-linear QSCAs do not yet have a natural geometrical interpretation, which is yet another obstruction for their applications in physics.

**Acknowledgements**

I acknowledge discussions with S. J. Gates Jr., J. W. van Holten and O. Lechtenfeld.
In this Appendix we describe the procedure to obtain a nilpotent quantum BRST operator $Q_{\text{BRST}}$ directly from a quadratically non-linear operator algebra. We first show how to construct a classical BRST charge $Q$, satisfying the classical ‘master equation’ $\{Q,Q\}_{\text{P.B.}} = 0$, for an arbitrary classical (quadratically) non-linear algebra and, then, how to ‘renormalise’ the naively quantised operator $Q$ to a nilpotent quantum BRST charge $Q_{\text{BRST}}^2 = 0$.

In classical theory, an algorithm for the construction of a BRST operator $Q$ is known due to Fradkin and Fradkina [40]. Consider a set of bosonic generators $B_i$ and fermionic generators $F_\alpha$, which satisfy a graded non-linear algebra of the form

\begin{align}
\{B_i, B_j\}_{\text{P.B.}} &= f_{ij}^k B_k, \\
\{B_i, F_\alpha\}_{\text{P.B.}} &= f_{i\alpha}^\beta F_\beta, \\
\{F_\alpha, F_\beta\}_{\text{P.B.}} &= f_{\alpha\beta}^i B_i + \Lambda_{\alpha\beta}^{ij} B_i B_j,
\end{align}

(A.1)

in terms of the graded Poisson (or Dirac) brackets, with some 3-point and 4-point ‘structure constants’, $f_{ij}^k$, $f_{i\alpha}^\beta$, $f_{\alpha\beta}^i$ and $\Lambda_{\alpha\beta}^{ij}$, respectively, which have to be ordinary numbers. The symmetry properties of these constants with respect to exchanging their indices obviously follow from their definition by eq. (A.1), and they are going to be implicitly assumed below. When using the unified index notation, $A \equiv (i, \alpha), \ldots$, the Jacobi identities for the classical graded algebra of eq. (A.1) take the form

\begin{align}
f_{[AB]D} f_{CDE} &= 0, \\
\Lambda_{[AB]D} f_{CDE} + \Lambda_{[AB]DF} f_{CDE} + f_{[AB]D} \Lambda_{CDE} &= 0.
\end{align}

(A.2)

As is clear from eq. (A.2), $f_{AB}^C$ are to be the structure constants of a graded Lie algebra.\footnote{We assume that all symmetry operations with unified indices also have to be understood in the graded sense. In particular, a graded ‘antisymmetrisation’ of indices with unit weight (denoted by mixed brackets $\{ \}$ here) actually means the antisymmetrisation for bosonic-bosonic or bosonic-fermionic index pairs, but the symmetrisation for indices which are both fermionic.}

According to the classical BRST procedure,\footnote{See, e.g., ref. [10] for a review.} one introduces an anticommuting ghost-antighost pair $(c^m, b_m)$ for each of the bosonic generators $B_m$, and the commuting ghost-antighost pair $(\gamma^\alpha, \beta_\alpha)$ for each of the fermionic generators $F_\alpha$. The ghosts satisfy (graded) bracket relations

\begin{align}
\{c^m, c^n\}_{\text{P.B.}} &= \{b_m, b_n\}_{\text{P.B.}} = 0, \quad \{c^m, b_n\}_{\text{P.B.}} = \delta^m_n, \\
\{\gamma^\alpha, \gamma^\beta\}_{\text{P.B.}} &= \{\beta_\alpha, \beta_\beta\}_{\text{P.B.}} = 0, \quad \{\gamma^\alpha, \beta_\beta\}_{\text{P.B.}} = \delta^\alpha_\beta.
\end{align}

(A.3)
Additional ghosts for the composite generators $B_i B_j$ are not needed since invariance of the classical theory under $B_i$ already implies invariance under $B_i B_j$.\[40\]

The classical BRST charge $Q$ is given by

$$Q = c^n B_n + \gamma^\alpha F_\alpha + \frac{1}{2} f_{ij} k^i c^j c^i + f_{i\alpha} \beta_\beta \gamma^\alpha c^i - \frac{1}{2} f_{\alpha\beta} n^\gamma \gamma^\beta \gamma^\alpha$$

\[A.4\]

Compared to the standard expression for the linear algebras ($\Lambda = 0$), the BRST charge in eq. (A.4) has the additional 3-(anti)ghost terms, dependent on the initial bosonic generators $B_i$, and the 7-(anti)ghost terms as well. It is easy to check that the classical ‘master equation’

$$\{Q, Q\}_{P.B.} = 0\quad (A.5)$$

follows from eq. (A.2) and the related identity

$$\Lambda_{[\alpha \beta} \Lambda_{\gamma \delta]}^{kl} f_{ik}^m = \Lambda_{[\alpha \beta} \Lambda_{\gamma \delta]}^{kl} f_{ik}^m$$

\[A.6\]

The graded classical algebra (A.1) can be extended to include (classical) central extensions. However, the condition (A.5) forces them to vanish \[39\]. This explains why we have not introduced central extensions for the classical $\tilde{D}_4$ algebra in sect. 4.

The classical BRST charge (A.4) may serve as the starting point in a construction of nilpotent quantum BRST charge $Q_{\text{BRST}}$ associated with the corresponding graded non-linear quantum algebra \[39\]. Since we are actually interested in quantizing QSCAs, we can assume that all operators are just currents, with a holomorphic dependence on $z$. In other words, let generators of the non-linear algebra under consideration carry an additional (affine) index (see eq. (2.1), for example). In particular, in eq. (A.3) one should simply replace the (graded) Poisson brackets by (anti)commutators,

$$[c^m, c^n] = [b_m, b_n] = 0, \quad [c^m, b_n] = \delta^m_n,$$

\[A.7\]

$$\{\gamma^\alpha, \gamma^\beta\} = \{\beta_\alpha, \beta_\beta\} = 0, \quad \{\gamma^\alpha, \beta_\beta\} = \delta^\alpha_\beta,$$

assuming that the indices $n$ and $\alpha$ are, in fact, multi-indices $nn'$ and $\alpha r'$, where $n'$ and $r'$ are affine indices (of Fourier modes) while $n$ and $\alpha$ count different bosonic and fermionic currents, respectively.

In addition, in quantum theory, one must take into account central extensions and the normal ordering needed for defining products of bosonic generators. This results
in the quantum (anti)commutation relations

\[ [B_i, B_j] = f_{ij}^k B_k + h_{ij} Z , \]
\[ [B_i, F_\alpha] = f_{\alpha \beta} B_\beta , \]
\[ \{ F_\alpha, F_\beta \} = h_{\alpha \beta} Z + f_{\alpha \beta}^i B_i + \Lambda_{\alpha \beta}^{ij} : B_i B_j : , \]

where the central charge generator \( Z \) commutes with all the other generators, and the new constants \( h_{ij} \) and \( h_{\alpha \beta} \) are supposed to be restricted by the Jacobi identities (see sects. 2 and 3). The ghost/anti-ghosts become operators, and they also require normal ordering for their products. Although no general procedure seems to exist, which would explain how to fully ‘renormalise’ the naively quantised (i.e. only normally-ordered) charge \( Q \) to a nilpotent quantum-mechanical operator \( Q_{BRST} \), the answer is known for a particular class of quantum algebras of the \( W \)-type due to refs. [38, 39]. The quantum algebra \( \hat{D}(1, 2; \alpha) \) falls into this class, so that we may expect that, similarly to the quantum \( W \) algebras considered in refs. [38, 39], the only non-trivial modification of eq. (A.4) in quantum theory amounts to a multiplicative renormalisation of the structure constants \( f_{\alpha \beta}^i \),

\[ f_{\alpha \beta}^i \rightarrow (f_{\text{ren}})_{\alpha \beta}^i \equiv \eta f_{\alpha \beta}^i , \]

after the formal replacement of the graded Poisson brackets by (anti)commutators and the normal ordering, namely

\[ Q_{BRST} = c^n B_n + \gamma^\alpha F_\alpha + \frac{1}{2} f_{ij}^k B_k c^i c^j + f_{\alpha \beta}^i : \beta_\beta \gamma^\alpha : c^i - \frac{1}{2} \eta f_{\alpha \beta}^m b_\alpha \gamma^\beta \gamma^\beta \]
\[ - \frac{1}{2} \Lambda_{\alpha \beta}^{ij} B_i B_j \gamma^\alpha \gamma^\beta - \frac{1}{24} \Lambda_{\alpha \beta}^{ij} \Lambda_{\gamma \delta}^{kl} f_{ik}^m b_j b_m \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta . \]

This ansatz for the quantum BRST operator introduces only a few (normally, only one) additional renormalization parameters \( \eta \) to be determined from the nilpotency condition. Since the central extension parameters of the quantum non-linear algebra are severely restricted by the Jacobi identities, whereas the quantum BRST charge nilpotency conditions normally lead to some more restrictions on their values, the procedure could make the parameters to be overdetermined, in general. Therefore, an existence of the quantum BRST charge is not guaranteed, and it is important to check consistency in each particular case.

Eq. (A.10) was used in the text (sect. 5), in order to write down the ansätze (5.8) and (5.20) for a quantum BRST operator \( Q_{BRST} \) in the cases of Bershadsky-Knizhnik orthogonal series of non-linear algebras and \( \hat{D}(2, 1; \alpha) \) QSCA, respectively. Since there was \( \dot{a} \) priori no guarantee that the quantum BRST charges constructed this way are going to be nilpotent, it is not very surprising that the whole construction turns out to be consistent only when \( N = 4 \) and \( \alpha = 1 \), i.e. for the orthogonal \( SO(4) \)-based Bershadsky-Knizhnik quantum quasi-superconformal algebra alone (sect. 5).
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