A response to an article of Xu-Jia Wang

YanYan Li * and Luc Nguyen †

Abstract

This is a response to the article [arXiv:1212.3130v1] by Xu-Jia Wang, where he attempted to address a mathematical question we raised. We point out that, and explain why, the article is far from answering our objections. Moreover, we have more recently found more serious trouble in the paper under discussion based on the false assertion there that minimal radial functions of superharmonic functions (with respect to a conformal Laplacian of a Riemannian metric) are superharmonic.

1

This is a response to the article [W] by Xu-Jia Wang. We point out that the article is far from answering our objections.

Firstly, we disagree with the claim of Wang, made in the last paragraph of Section 3 of [W], that Section 2 of [W] answers our objections to the proof of the main theorem Theorem 1.3 in [TW]. The reasons are given in the next section.

Secondly, we don’t understand why the equicontinuity of $h(r)(\cdot) := h(r\cdot)$, for $0 < r < 1$, when $h$ is locally bounded, is clear to Wang, as claimed in Section 4 of [W] (line 4 on page 4). Here $h$ is the function in Lemma 3.4 of [W]. We have repeatedly asked Wang, starting from Nov. 16, to provide a proof of this assertion made in his email on Nov. 14, but he has never given one. Pages of detailed arguments in our paper [LN] can be used to prove this.

Thirdly, having studied [TW] in more detail, we have found more serious trouble based on the false assertion on line -9 of page 2445 of the paper that minimal radial functions of superharmonic functions (with respect to a conformal Laplacian of a Riemannian metric) are superharmonic. Paper [TW] has made essential use

*Department of Mathematics, Rutgers University
†Department of Mathematics, Princeton University
of the false assertion. In our paper [LN] we have used at one point \( \min_{\partial B_r} v \), what Wang calls a minimal radial function of \( v \), but we do not suppose that \( \min_{\partial B_r} v \) is superharmonic.

So our objections have not been answered, and new objections have arisen.

2

In the last paragraph of Section 3 of [W], Wang claimed that Section 2 of [W] answers our objections to the proof of the main theorem Theorem 1.3 in [TW]. We disagree on that, and explain the reasons below.

Let us start by recalling Theorem 1.3 in [TW]:

**Theorem 1.3.** ([TW]) Assume that \( \sigma \) satisfies \( C_1 - C_4 \), \( \varphi \in C^0(\mathcal{M}) \), \( \varphi \geq c_0 > 0 \), and \( \Gamma \) is a convex cone satisfying \( G_1 \) and \( G_2 \). Let \( g_j = v_j^{\frac{2}{n-2}} g_0 \) be a sequence of solutions to (1.6). Then \( v_j/\inf_{\mathcal{M}} v_j \) converges in \( W^{1,p} \) (for any \( 1 < p < \frac{n}{n-1} \)) to an admissible function \( v \). Moreover, if \( x_0 \) is a singular point of \( v \), then near \( x_0 \),

\[ v(x) = \frac{C_0 + o(1)}{d(x, x_0)^{n-2}}, \tag{1.22} \]

where \( C_0 \) is a positive constant, \( d(x, x_0) \) denotes the geodesic distance from \( x \) to \( x_0 \) in the metric \( g_0 \). Furthermore, each singular point is isolated.

As defined in the first two lines of page 2443 of [TW], \( x_0 \) is a singular point of \( v \) if there is a sequence of points \( x_j \in \mathcal{M} \) such that \( v_j(x_j)/\inf_{\mathcal{M}} v_j \to \infty \) and \( x_j \to x_0 \).

In (1.22), the \( o(1) \) term is to be understood in the usual sense, i.e.

\[ \lim_{x_j \to x_0} (d(x_j, x_0)^{n-2}v(x_j) - C_0) = 0. \]

This is agreed by Wang in the second to the last paragraph of Section 3 of [W]. Furthermore this usual sense of convergence is needed to obtain Theorem 1.1 and 1.2 in [TW]. However we do not see this usual sense of convergence in Theorem 1.3 is being established in both [TW] and Sections 2-3 of [W], as explained below.

The proof of Theorem 1.3 on page 2 of [W] consists of three paragraphs. We will call them paragraph 1, paragraph 2, and paragraph 3 respectively.

We now phrase our question as we follow these three paragraphs. We raise our question in the simplest situation in order to more easily convey the ideas.

Paragraph 1 says: “Let \( x_j \) be the absolute maximum point of \( v_j \). As above we may assume \( x_j \) is a fixed point and \( x_j = 0 \). By Lemma 3.4, the function \( w_j \) given
in (3.11) converges in $W^{1,p}$ to the function $w$ in (3.29). We need to show that 0 is an isolated singular point of $w$.

Let us look at a simplified situation that 0 is the only singular point of $w$. More precisely, $v_j/\inf_M v_j$ is locally bounded in $\mathcal{M} \setminus \{0\}$ and $v_j(0)/\inf_M v_j \to \infty$.

In this case, paragraph 2 is not needed, since that is used to prove that 0 is an isolated singular point of $w$. The part “Therefore the absolute maximum point is an isolated singular point...... The above arguments also leads to a contradiction.” in paragraph 3 is also not needed. There is only one sentence left in paragraph 3 which asserts that the proof of Theorem 1.3 is completed. We do not see why this is the case — estimate (1.22) has not been established.

To make our point more precise, note that

$$w = -\frac{2}{n-2} \log v, \quad h(x) := w(x) - 2 \log |x|,$$

so (1.22) is equivalent to

$$h(x) = o(1) \quad \text{in the usual sense as} \quad x \to 0. \quad (1)$$

However what has been proved in Lemma 3.4 in the form stated in [W] is (*) in [W], i.e.

$$\lim_{r \to 0} \int_{r < |x| < 2r} |h(x)|dx = 0,$$

which is much weaker than (1).

[LN] Yanyan Li and Luc Nguyen, *A compactness theorem for a fully nonlinear Yamabe problem under a lower Ricci curvature bound*, arXiv:1212.0460v1 [math.AP] 3 Dec 2012.

[TW] Neil Trudinger and Xu-Jia Wang, *The intermediate case of the Yamabe problem for higher order curvatures*, International Mathematics Research Notices, Vol 2010, no. 13, pp. 2437-2458.

[W] Xu-Jia Wang, *Response to a question of Yanyan Li and Luc Nguyen in their paper “A compactness theorem for a fully nonlinear Yamabe problem under a lower Ricci curvature bound”*, arXiv:1212.0460v1 arXiv:1212.3130v1 [math.AP].