THE FACE NUMBERS OF HOMOLOGY SPHERES

KAI FONG ERNEST CHONG AND TIONG SENG TAY

Abstract. The $g$-theorem is a momentous result in combinatorics that gives a complete numerical characterization of the face numbers of simplicial convex polytopes. The $g$-conjecture asserts that the same numerical conditions given in the $g$-theorem also characterizes the face numbers of all simplicial spheres, or even more generally, all simplicial homology spheres.

In this paper, we prove the $g$-conjecture for simplicial $R$-homology spheres. A key idea in our proof is a new algebra structure for polytopal complexes. Given a polytopal $d$-complex $\Delta$, we use ideas from rigidity theory to construct a graded Artinian $R$-algebra $\Psi(\Delta, \nu)$ of stresses on a PL realization $\nu$ of $\Delta$ in $\mathbb{R}^d$, where overlapping realized $d$-faces are allowed. In particular, we prove that if $\Delta$ is a simplicial $R$-homology sphere, then for generic PL realizations $\nu$, the stress algebra $\Psi(\Delta, \nu)$ is Gorenstein and has the weak Lefschetz property.

1. Introduction and overview

The possible sequences of numbers counting the faces (for different dimensions) of a simplicial convex polytope have been completely characterized. This is known as the $g$-theorem, and it was proven in two parts by Billera–Lee [3] (sufficiency) and Stanley [50] (necessity). At first glance, characterizing these face numbers looks like a problem in combinatorics or polyhedral geometry. Indeed, Billera–Lee used an ingenious “shadow” construction on some suitable cyclic polytope to prove sufficiency. What was perhaps unexpected was Stanley’s proof of necessity: He applied the hard Lefschetz theorem (from algebraic geometry) to the intersection cohomology ring of the toric variety associated to a rational convex polytope. Subsequently, McMullen [32] (corrected and simplified in [29]) gave a different proof of necessity using convex geometry and an $R$-algebra construction [31] associated to convex polytopes. Remarkably, McMullen’s proof also gives a direct combinatorial proof of the hard Lefschetz theorem for simplicial fans.

The $g$-theorem was previously called the $g$-conjecture; this conjecture is due to McMullen [28] (1971). In his same paper [28], McMullen also remarked on extending his conjecture to all simplicial spheres. Today, the $g$-conjecture refers to the conjecture that the numerical conditions given in the $g$-theorem also characterizes the face numbers of all simplicial spheres, or even more generally, all simplicial homology spheres. For decades, this $g$-conjecture had resisted multiple attempts at a complete proof, despite much concerted effort using various methods.

In this paper, we prove the $g$-conjecture for simplicial $R$-homology spheres. Our proof requires a confluence of algebraic, combinatorial, geometric, number-theoretic, and topological ideas. For the rest of this section, we shall give a precise statement of the $g$-conjecture, discuss the prior progress made towards the $g$-conjecture, and provide an overview of our proof. For a comprehensive survey of what has been done and what strategies have been proposed (including variants and further extensions), see [53] (cf. [18]). For a rapid introduction to the subject, see Stanley’s “green book” [51].

To state the $g$-conjecture, we first need to review some definitions. Given a simplicial $d$-complex $\Delta$, its $f$-vector is $(f_0(\Delta), \ldots, f_d(\Delta))$, where each $f_i(\Delta)$ equals the number of $i$-dimensional faces of $\Delta$. The $h$-vector of $\Delta$ is $(h_0(\Delta), \ldots, h_{d+1}(\Delta))$, where

$$h_k(\Delta) = \sum_{i=0}^{k} (-1)^{k-i} \binom{d+1-i}{d+1-k} f_{i-1}(\Delta)$$

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for each $0 \leq k \leq d + 1$. (By default, $f_{-1}(\Delta) = 1$, which corresponds to the empty face $\emptyset$.) It is an easy exercise to show that the $f$-vector and the $h$-vector of $\Delta$ determine each other.

Let $k$ be a field, and let $R = \bigoplus_{i \in \mathbb{N}} R_i$ be a graded $k$-algebra generated by $R_1$. (All $k$-algebras in this paper are assumed to be unital, finitely generated, associative, and commutative.) If $R$ has Krull dimension $d$, then a fundamental result in commutative algebra says that the Hilbert series of $R$ (in terms of $t$) can be written as $\frac{h(t)}{(1-t)^d}$ for some unique polynomial $h(t) = h_0 + h_1 t + \cdots + h_d t^d \in \mathbb{Z}[t]$. The vector $(h_0, \ldots, h_d)$ is called the $h$-vector of $R$, and a sequence of integers is called an $M$-vector if it is the $h$-vector of some graded $k$-algebra generated by its degree 1 elements. A classic theorem by Macaulay [25] gives a complete numerical characterization of all possible $M$-vectors (see also [50] or [51] Sec. II.2). Thus, an assertion that some sequence of integers is an $M$-vector would be equivalent to a purely numerical condition.

**Theorem 1.1** ($g$-theorem). Let $\Delta$ be the boundary of a simplicial convex $d$-polytope. A sequence of integers $(h_0, \ldots, h_d)$ is the $h$-vector of $\Delta$ if and only if the following two conditions hold.

(i) $h_i = h_{d-i}$ for all $0 \leq i \leq d$.

(ii) $(h_0, h_1 - h_0, h_2 - h_1, \ldots, h_{\lfloor \frac{d}{2} \rfloor} - h_{\lfloor \frac{d}{2} \rfloor - 1})$ is an $M$-vector.

Condition (i) is commonly known as the Dehn–Sommerville equations [9] [12] [19]. In particular, the equation $h_0 = h_d$ is implied by the Euler–Poincaré equation, i.e. $\Delta$ has reduced Euler characteristic $(-1)^d$. The vector $(h_0, h_1 - h_0, h_2 - h_1, \ldots, h_{\lfloor \frac{d}{2} \rfloor} - h_{\lfloor \frac{d}{2} \rfloor - 1})$ in condition (ii) is also called the $g$-vector of $\Delta$, hence the name “$g$-theorem”. For convenience, we define $g_0(\Delta) = 1$ and $g_i(\Delta) := h_i(\Delta) - h_{i-1}(\Delta)$ for each $1 \leq i \leq \lfloor \frac{d}{2} \rfloor$, so that $g(\Delta) := (g_0(\Delta), \ldots, g_{\lfloor \frac{d}{2} \rfloor}(\Delta))$ is the $g$-vector of $\Delta$. The Dehn–Sommerville equations are known to hold more generally for $k$-homology spheres [20], so it is natural to extend the notion of $g$-vectors to $k$-homology spheres.

**Conjecture 1.2** ($g$-conjecture). Let $\Delta$ be a simplicial ($k$-homology) $d$-sphere. A sequence of integers $(h_0, \ldots, h_{d+1})$ is the $h$-vector of $\Delta$ if and only if the following two conditions hold.

(i) $h_i = h_{d+1-i}$ for all $0 \leq i \leq d + 1$.

(ii) $(h_0, h_1 - h_0, h_2 - h_1, \ldots, h_{\lfloor \frac{d+1}{2} \rfloor} - h_{\lfloor \frac{d+1}{2} \rfloor - 1})$ is an $M$-vector.

Notice that Billera–Lee’s result also proves sufficiency for Conjecture 1.2. Thus, the remaining (and difficult) part is to prove that the $g$-vector of a simplicial ($k$-homology) sphere is an $M$-vector.

Mani [26] proved that simplicial $d$-spheres with $\leq d + 4$ vertices are boundaries of simplicial convex $(d + 1)$-polytopes, thus the $g$-theorem implies Conjecture 1.2 in this case. The $g$-conjecture also holds for certain $k$-homology $d$-spheres $\Delta$ with small $g_2(\Delta)$. Swartz [53] (cf. [43]) proved that if $g_2(\Delta) \leq 5$, then $g(\Delta)$ is an $M$-vector for the following cases: (i) $f_0(\Delta) \leq d + 5$, (ii) $d = 5$, $g_2(\Delta) \leq 4$, (iii) $d = 6$.

If the full $g$-conjecture is true, then the $g$-vector of a simplicial ($k$-homology) sphere $\Delta$ must have non-negative entries, which we write as $g(\Delta) \geq 0$. In fact, before McMullen formulated his $g$-conjecture, Walkup [57] had already proven that $g_2(\Delta) \geq 0$ for all simplicial $d$-spheres $\Delta$ of dimension $d \leq 4$. After the $g$-theorem was proven, there was a major breakthrough due to Kalai [16], who used rigidity theory to prove that $g_2(\Delta) \geq 0$ for all simplicial $k$-homology $d$-spheres $\Delta$ satisfying $d \geq 3$. (In fact, Kalai’s result holds more generally for normal $d$-pseudo manifolds, and Nevo [42] subsequently extended Kalai’s result to 2-Cohen–Macaulay complexes, again using rigidity theory; cf. [23], [45].) Consequently, the $g$-conjecture for simplicial $k$-homology $d$-spheres is true for $d \leq 4$. Using sheaf theory, Karu [17] proved that $g(\Delta) \geq 0$ when $\Delta$ is the order complex of a Gorenstein* poset. Examples of such order complexes include the barycentric subdivisions of $k$-homology spheres. Subsequently, Kubitzke–Nevo [21] proved the $g$-conjecture for the barycentric subdivisions of $k$-homology spheres; see also [40].

PL spheres (also known as combinatorial spheres) are an important class of simplicial spheres. Strongly edge-decomposable (s.e.d.) spheres, introduced by Nevo [41], form a large subclass of PL spheres that include generalized Bier spheres [36] and Kalai’s squeezed spheres [34]. Babson–Nevo [1] proved that the $g$-conjecture holds for s.e.d. spheres. As part of their proof, they showed that a generic Artinian reduction of the Stanley–Reisner ring of a s.e.d. sphere has the strong Lefschetz property in
characteristic zero. (Murai [34] later proved this strong Lefschetz property in arbitrary characteristic; see also [48].) By studying how the strong Lefschetz property relates to bistellar moves, Swartz [53] also proved the $g$-conjecture for another subclass of PL spheres obtained from the boundary of a simplex via certain bistellar moves.

The main result of this paper is a proof of the $g$-conjecture for all $\mathbb{R}$-homology spheres:

**Theorem 1.3.** If $\Delta$ is a simplicial $\mathbb{R}$-homology $d$-sphere, then the $g$-vector of $\Delta$ is an $M$-vector.

In contrast to recent results related to the $g$-conjecture (e.g. [11, 21, 53]), we will not be using Stanley–Reisner rings. Instead, we shall look at the stresses on certain realizations, and construct what we call the “stress algebra”. One key concept we need is the notion of PL realizations. Notice that maps faces of $\Delta$ to simplices of the same dimension, since no common vertices. Consequently, a PL realization of $\Delta$ in $\mathbb{R}$, i.e.

$$\nu(F) := \nu(v_0) \wedge \cdots \wedge \nu(v_k) \in \wedge^{k+1}(\mathbb{R}^{N+1}).$$

(iii) $\pi_N(\nu(v)) \neq 0$ for all $v \in \mathcal{V}(\Delta)$, and $\nu(F) \neq 0$ for all $F \in \Delta$.

For convenience, we simply say that $(\Delta, \nu)$ is a PL realization (of $\Delta$) in $\mathbb{R}^N$. Later in Section 3 we shall define PL realizations more generally for polytopal complexes.

Given such a PL realization $\nu$, there is an associated set-valued map $\overline{\nu}: \Delta \to 2^{\mathbb{R}^N}$ (induced by $\nu$) such that every vertex $v$ is mapped to the singleton containing the point $(\overline{\nu}(\nu(v)))^{-1} \pi_N(\nu(v)) \in \mathbb{R}^N$, and more generally, every $k$-face $F = \{v_0, \ldots, v_k\} \in \Delta$ is mapped to the convex hull of $\overline{\nu}(v_0) \cup \cdots \cup \overline{\nu}(v_k)$. Notice that $\overline{\nu}$ maps faces of $\Delta$ to simplices of the same dimension, since $\nu(F) \neq 0$ for all $F \in \Delta$. Note also that we allow overlaps; $\overline{\nu}(F) \cap \overline{\nu}(F')$ could possibly be non-empty, even if $F, F'$ are faces with no common vertices. Consequently, a PL realization of $\Delta$ in $\mathbb{R}^N$ can be thought of as a realization of the vertices of $\Delta$ as points in Euclidean $N$-space, together with a choice of some fixed homogeneous coordinates for every realized vertex, such that the realized faces of $\Delta$ could possibly overlap.

Let $(\Delta, \nu)$ be a PL realization of a simplicial $d$-complex in $\mathbb{R}^d$. An $r$-stress on $(\Delta, \nu)$ is a function $a: \Delta \to \mathbb{R}$ satisfying $a(F) = 0$ whenever $\dim F \neq r - 1$, such that the equilibrium equation holds for every $(r - 2)$-face $G$ of $\Delta$, i.e.

$$\sum_{v \in \mathcal{V}(\text{Lk}_G(\Delta))} a(G \cup v)(\nu(G) \wedge \nu(v)) = 0,$$

where $\mathcal{V}(\text{Lk}_G(\Delta))$ denotes the set of vertices in the link of $G$ (in $\Delta$). Note that there are no equilibrium equations to check for 0-stresses, so a 0-stress is any scalar assignment to the empty face $0$ of $\Delta$. Denote the $\mathbb{R}$-vector space of $r$-stresses by $\Psi_{d+1-r}(\Delta, \nu)$, and define $\Psi(\Delta, \nu) := \bigoplus_{r=0}^{d+1} \Psi_r(\Delta, \nu)$. We will prove in Section 3 that $\Psi(\Delta, \nu)$ has a graded $\mathbb{R}$-algebra structure, so for this reason, we shall call $\Psi(\Delta, \nu)$ the stress algebra of $(\Delta, \nu)$.

In fact, this graded $\mathbb{R}$-algebra construction works more generally for arbitrary polytopal complexes; see Section 3 for details. In comparison, Stanley–Reisner rings are defined only for simplicial complexes, while McMullen’s polytope algebras [31, 32] (or weight algebras [29]) are defined only for convex polytopes. We also remark that when restricted to convex polytopes, the dual to the multiplication map of our stress algebra is different from McMullen’s multiplication map on weights of the corresponding dual polytopes. To prove this graded $\mathbb{R}$-algebra structure for polytopal complexes, we need to look at stresses in relation to Hodge duality, which we do so in Section 4.
In Section 8 we prove that if \( k \) is an arbitrary field, and if \( \Delta \) is an orientable simplicial \( k \)-homology \( d \)-manifold (without boundary), such that the homology group \( H_1(\Delta; \mathbb{Z}/2\mathbb{Z}) \) is trivial whenever \( d \geq 2 \), then for generic PL realizations \( \nu \) of \( \Delta \) in \( \mathbb{R}^d \), the stress algebra \( \Psi(\Delta, \nu) \) is Gorenstein and is generated by the degree 1 elements, i.e., the \( d \)-stresses on \( (\Delta, \nu) \). This genericity has a precise meaning that we cover in Section 7 and we use the term “\( \mathbb{Q} \)-generic” to refer to this precise meaning. Perhaps surprisingly, our proof of the Gorenstein property for \( \mathbb{Q} \)-generic PL realizations relies on a number theoretic result on algebraic number fields in a crucial manner.

In this same proof, we also require the homological condition that \( H_1(\Delta; \mathbb{Z}/2\mathbb{Z}) = 0 \) whenever \( d \geq 2 \). This is because we used a three-way interplay between liftings, reciprocals, and \( d \)-stresses on \( (\Delta, \nu) \) from Maxwell–Cremona theory, which is no longer true without this homological condition. (For example, there is a PL realization of a 2-torus in \( \mathbb{R}^2 \) for which this interplay does not hold.) The relevant results from Maxwell–Cremona theory, including important advances made by Rybnikov and coauthors [11, 48], and the connection to Poincaré duality, will be discussed in Section 6.

In Section 9, we introduce the notion of “pivot-compatibility”, which serves as a prelude to the much more technical Section 10. Suppose \( \Delta \) is a simplicial \( \mathbb{R} \)-homology \( d \)-sphere, and let \( \Delta' \) be a cone on \( \Delta \). In Section 10 we look at the stresses on a \( \mathbb{Q} \)-generic PL realization \( (\Delta', \nu') \) in \( \mathbb{R}^{d+1} \), and we construct a map \( \varphi : \Psi_r(\Delta', \nu') \to \Psi_{r+1}(\Delta', \nu') \). Then we use pivot-compatibility and a rather technical argument to prove that if \( \nu' \) is “sufficiently generic”, then \( \varphi \) is injective for all \( r \leq \lceil \frac{d+1}{2} \rceil \). Much of the difficulty of our proof arises from the need for “very careful bookkeeping” of various sets of parameters.

In Section 11 we use the homological interpretation of skeletal rigidity developed by Tay–Whiteley [56] as our main tool. In particular, the space of \( r \)-stresses of any simplicial complex is isomorphic to the top homology group of the \( r \)-skeletal chain complex introduced in [56]. Building on the results in Section 10, we extend \( \varphi \) to a chain map from the \( r \)-skeletal chain complex to the \((r-1)\)-skeletal chain complex. Combined with results in [56], we can then construct the following commutative diagram

\[
\begin{array}{ccc}
\Psi_r(\Delta', \nu') & \xrightarrow{\cong} & \Psi_{r-1}(\Delta, \nu) \\
\downarrow \varphi_* & & \downarrow \varphi_* \\
\Psi_{r+1}(\Delta', \nu') & \xrightarrow{\cong} & \Psi_r(\Delta, \nu)
\end{array}
\]

where the two horizontal maps in this diagram are isomorphisms, and \( \nu \) is a PL realization of \( \Delta \) in \( \mathbb{R}^d \) that is obtained from \( \nu' \) via a central projection from the conepoint of \( \Delta' \) onto a generic hyperplane. Since the first vertical map is injective for all \( r \leq \lceil \frac{d+1}{2} \rceil \) when \( \nu' \) is “sufficiently generic”, it then follows that the second vertical map is also injective for all \( r \leq \lceil \frac{d+1}{2} \rceil \) for “sufficiently generic” PL realizations \( \nu \) of \( \Delta \) in \( \mathbb{R}^d \).

We have deliberately constructed \( \varphi \), so that there is some \( \omega \in \Psi_1(\Delta, \nu) \) such that for all \( r \), the map \( \varphi_* : \Psi_{r-1}(\Delta, \nu) \to \Psi_r(\Delta, \nu) \) coincides with the multiplication map \( \omega : \Psi_{r-1}(\Delta, \nu) \to \Psi_r(\Delta, \nu) \) defined by \( x \mapsto \omega x \). Thus, combined with the Gorenstein property of \( \Psi(\Delta, \nu) \), we get the following important part of our proof.

**Theorem 1.4.** If \( (\Delta, \nu) \) is a “sufficiently generic” PL realization of a simplicial \( \mathbb{R} \)-homology \( d \)-sphere in \( \mathbb{R}^d \), then the stress algebra \( \Psi(\Delta, \nu) \) is Gorenstein and has the weak Lefschetz property.

A more precise statement of this theorem is given in Theorem 11.3 At the end of Section 11 we show how Theorem 11.3 implies our main result (Theorem 1.3).

Finally, we remark that our proof of Theorem 1.3 raises several questions. What about \( k \)-homology spheres for fields \( k \) other than \( \mathbb{R} \)? What about the strong Lefschetz property? What can we say about the face numbers of other homology manifolds? In Section 12 we conclude our paper by addressing these exciting questions with further remarks and open problems.

## 2. Basic terminology

Throughout, we use the prefix “\( k \)-” on objects (e.g., \( k \)-subspace, \( k \)-complex, \( k \)-face, etc.) to mean objects of dimension \( k \). Let \( 0 \) denote the zero vector, and let \( k \) be an arbitrary field. For any subset
A geometric polytopal complex is a finite set \( \mathcal{K} \) of polytopes in \( \mathbb{R}^N \) (for some sufficiently large \( N \)) such that the intersection of any two polytopes in \( \mathcal{K} \) is always a common (possibly empty) face of both polytopes, and such that all faces of each polytope in \( \mathcal{K} \) are also polytopes in \( \mathcal{K} \). We shall assume that the empty polytope \( \emptyset = \text{conv}(\emptyset) \subseteq \mathbb{R}^N \), which has vertex set \( \emptyset \), is always an element of \( \mathcal{K} \). An abstract polytopal complex is a set \( \Sigma \) whose elements are the vertex sets of the polytopes in some geometric polytopal complex \( \mathcal{K} \). Unless otherwise stated, a polytopal complex is assumed to be abstract. The dimension of \( F \in \Sigma \), denoted by \( \dim F \), is the dimension of the polytope in \( \mathcal{K} \) that \( F \) corresponds to. (We define \( \dim \emptyset = -1 \).) The dimension of \( \Sigma \) is \( \dim \Sigma := \max \{ \dim F \mid F \in \Sigma \} \). Elements of \( \Sigma \) are called faces, 0-faces of \( \Sigma \) are called vertices, and the inclusion-wise maximal faces of \( \Sigma \) are called facets. If all facets of \( \Sigma \) have the same dimension, then we say that \( \Sigma \) is pure. Given any \( F \in \Sigma \), we write \( F \prec G \) (or equivalently, \( G \succ F \)) to mean that \( G \) is a face of \( \Sigma \) that contains \( F \) and satisfies \( \dim G = \dim F + 1 \).

A saturated flag of faces in \( \Sigma \) is a sequence \( F_0 \prec F_1 \prec \cdots \prec F_t \) of faces in \( \Sigma \).

Given a polytopal \( d \)-complex \( \Sigma \) and any integer \( r \), let \( \mathcal{F}_r(\Sigma) \) be the set of all \( r \)-faces in \( \Sigma \). Let \( \mathcal{V}(\Sigma) \) be the set of vertices in \( \Sigma \), and for any \( F \in \Sigma \), let \( \mathcal{V}(F) \) be the set of vertices in \( F \). A subcomplex of \( \Sigma \) is a subset of \( \Sigma \) that is also a polytopal complex. The \( r \)-skeleton of \( \Sigma \) is the subcomplex consisting of all faces of \( \Sigma \) of dimension \( \leq r \). The open star of a face \( F \) in \( \Sigma \) is the set of faces \( \text{St}_\Sigma(F) := \{ G \in \Sigma \mid F \subseteq G \} \), and the antistar of \( F \) in \( \Sigma \) is the subcomplex \( \text{Ast}_\Sigma(F) := \Sigma \setminus \text{St}_\Sigma(F) \). The closed star of a face \( F \) in \( \Sigma \), denoted by \( \overline{\text{St}}_\Sigma(F) \), is the (unique) minimal subcomplex of \( \Sigma \) that contains \( \text{St}_\Sigma(F) \). The link of a face \( F \) in \( \Sigma \) is the subcomplex \( \text{Lk}_\Sigma(F) := \{ G \in \text{St}_\Sigma(F) \mid F \cap G = \emptyset \} \). A vertex \( v \in \mathcal{V}(\Sigma) \) is called a conepoint of \( \Sigma \) if \( \text{Lk}_\Sigma(v) = \text{Ast}_\Sigma(v) \). We say that \( \Sigma \) is a cone if it has at least one conepoint. Given another polytopal complex \( \Sigma' \), we say that \( \Sigma' \) is a cone on \( \Sigma \) with conepoint \( v \) if \( \text{Lk}_\Sigma(v) = \text{Ast}_\Sigma(v) = \Sigma \).

The dual graph of a pure polytopal complex \( \Sigma \) is the graph whose vertices are the facets of \( \Sigma \), such that two vertices in this graph are adjacent if and only if the corresponding facets share a common codimension 1 face in \( \Sigma \). We say \( \Sigma \) is strongly connected if its dual graph is connected. A pseudomanifold is a strongly connected pure polytopal complex for which every codimension 1 face is contained in exactly two facets. A \( k \)-homology \( d \)-manifold is a \( d \)-pseudomanifold \( \Sigma \) such that \( \text{Lk}_\Sigma(F) \) has the same homology groups (over \( \mathbb{Z} \)) as a \((d - \dim F - 1)\)-sphere for all non-empty faces \( F \in \Sigma \). A \( k \)-homology \( d \)-sphere is a \( k \)-homology \( d \)-manifold with the same homology groups (over \( \mathbb{Z} \)) as a \( d \)-sphere.

A polytopal complex \( \Delta \) such that \( \dim F = |F| - 1 \) for all \( F \in \Delta \) is called simplicial or a simplicial complex. Subcomplexes of simplicial complexes are simplicial. Given another simplicial complex \( \Delta' \) such that \( \mathcal{V}(\Delta) \cap \mathcal{V}(\Delta') = \emptyset \), the join of \( \Delta \) and \( \Delta' \) is \( \Delta \ast \Delta' := \{ F \cup G \mid F \in \Delta, G \in \Delta' \} \). In particular, a cone on \( \Delta \) with conepoint \( a \) is the simplicial complex \( \Delta \ast \{a\} \).

3. Stresses on PL realizations of polytopal complexes

In this section, we extend the definitions of PL realizations and \( r \)-stresses (as given in Section 1) to allow for the consideration of polytopal complexes. We shall also discuss some basic properties of stresses that will be used throughout this paper.

Let \( 1 \leq d \leq N \) be integers. First, we construct the map \( \tau : 2^{\mathbb{R}^{N+1}} \to \Lambda(\mathbb{R}^{N+1}) \) as follows. Define \( \tau(0) = 1 \in \Lambda^0(\mathbb{R}^{N+1}) \), and for each \( u \in \mathbb{R}^{N+1} \), define \( \tau\{u\} = u \in \Lambda^1(\mathbb{R}^{N+1}) \). For every non-empty ordered \((k+1)\)-set \( U = \{u_0,...,u_k\} \subseteq \mathbb{R}^{N+1} \) such that \( \text{aff}(U) \) is an affine \( k \)-subspace of \( \mathbb{R}^{N+1} \), define

\[
\tau(U) := \tau(u_0) \wedge \cdots \wedge \tau(u_k) \in \Lambda^{k+1}(\mathbb{R}^{N+1}).
\]

More generally, for every non-empty finite ordered set \( U' \subseteq \mathbb{R}^N \) such that \( \text{aff}(U') \) is an affine \( k \)-subspace of \( \mathbb{R}^{N+1} \), let \( \mathcal{T} \) be any oriented triangulation of the polytope \( \text{conv}(U') \) whose orientation is consistent with the given linear order on \( U' \) (e.g. \( \mathcal{T} \) could be a barycentric subdivision), and define \( \tau(U') := \sum_{F \in \mathcal{T}} \tau(F) \), where the sum is over all ordered \( k \)-simplices \( F \) in \( \mathcal{T} \) (treated as ordered \((k+1)\)-sets).
Note that $\tau(U')$ does not depend on the choice of $T$, and note that $||\tau(U')|| = (k!) \text{vol}(\text{conv}(U')) > 0$. In fact, for any ordered $(k + 1)$-subset $T$ of $U'$ that forms an affine basis for $\text{aff}(U')$, we can always write $\tau(U')$ as a scalar multiple of $\tau(T)$, so in particular, $\tau(U')$ is decomposable, i.e. the wedge product of $k + 1$ elements of $\mathbb{R}^{N+1}$.

**Definition 3.1.** Let $\Sigma$ be a polytopal $d$-complex, and fix a linear order on $\mathcal{V}(\Sigma)$. A PL realization of $\Sigma$ in $\mathbb{R}^N$ is a map $\nu: \Sigma \to \bigwedge (\mathbb{R}^{N+1})$ satisfying the following conditions.

(i) $\nu(\emptyset) = 1 \in \bigwedge^0 (\mathbb{R}^{N+1})$, and $\nu(F) \in \bigwedge^\dim F + 1 (\mathbb{R}^{N+1})$ for all $F \in \Sigma$.

(ii) If $F \in \mathcal{F}_k(\Sigma)$, and if $\nu(F) = \{v_0, \ldots, v_1\}$ is treated as an ordered set whose order is consistent with the given linear order on $\mathcal{V}(\Sigma)$, then

$$\nu(F) := \tau(\{\nu(v_0), \ldots, \nu(v_1)\}) \in \bigwedge^{k+1}(\mathbb{R}^{N+1}).$$

(iii) $\hat{\pi}_N(\nu(v)) \neq 0$ for all $v \in \mathcal{V}(\Sigma)$, and $\nu(F) \neq 0$ for all $F \in \Sigma$.

For convenience, we simply say that $(\Sigma, \nu)$ is a PL realization (of $\Sigma$) in $\mathbb{R}^N$.

For the rest of this section, let $\Sigma$ be a polytopal $d$-complex. A PL set-valued map on $\Sigma$ is a map $\psi: \Sigma \to 2^{\mathbb{R}^N}$ satisfying $\psi(F) = \text{conv}(\bigcup_{v \in \mathcal{V}(F)} \psi(v))$ for all $F \in \Sigma$, such that each $\psi(F)$ is a polytope in $\mathbb{R}^N$ of dimension $\dim F$. In particular, $\psi(\emptyset) = \emptyset$, and $\psi(v)$ is a singleton for all $v \in \mathcal{V}(\Sigma)$. Given any PL realization $\nu$ of $\Sigma$ in $\mathbb{R}^N$, the associated set-valued map induced by $\nu$ is the map $\Psi: \Sigma \to 2^{\mathbb{R}^N}$ such that every vertex $v$ of $\Sigma$ is mapped to the singleton containing the point $(\hat{\pi}_N(\nu(v)))^{-1} \pi_N(\nu(v)) \in \mathbb{R}^N$, and more generally, every $k$-face $F = \{v_0, \ldots, v_l\} \subset \Sigma$ is mapped to $\text{conv}(\hat{\pi}(v_0) \cup \cdots \cup \hat{\pi}(v_l))$. By default, let $\hat{\pi}(\emptyset) = \emptyset$. (Whenever $\nu$ is a PL realization, we always reserve the overline in $\hat{\pi}$ to mean this associated set-valued map induced by $\nu$.) Since $\nu(F) \neq 0$ for all $F \in \Sigma$, it follows that $\Psi$ maps faces of $\Sigma$ to polytopes of the same dimension, thus $\Psi$ is a PL set-valued map. Similar to the case of simplicial complexes, $\hat{\pi}(F)$ and $\hat{\pi}(F')$ could possibly overlap for any $F, F' \subset \Sigma$, even if they share no common vertices.

**Remark 3.2.** If $\psi: \Sigma \to 2^{\mathbb{R}^N}$ is a PL set-valued map on $\Sigma$, then for any function $\lambda: \mathcal{V}(\Sigma) \to \mathbb{R}\backslash\{0\}$, consider the map on $\mathcal{V}(\Sigma)$ given by $v \mapsto (\lambda(v)v, \lambda_v) \in \mathbb{R}^{N+1}$, where $v$ denotes the unique $v \in \mathbb{R}^N$ contained in the singleton $\psi(v)$. This map on $\mathcal{V}(\Sigma)$ extends (uniquely) to a PL realization of $\Sigma$ in $\mathbb{R}^N$, and we shall call it the PL realization induced by $(\psi, \lambda)$. More generally, we say that a PL realization of $\Sigma$ is induced by $\psi$ if it is the PL realization induced by $(\psi, \lambda)$ for some $\lambda: \mathcal{V}(\Sigma) \to \mathbb{R}\backslash\{0\}$.

**Definition 3.3.** Let $(\Sigma, \nu)$ be a PL realization of a polytopal $d$-complex in $\mathbb{R}^N$. An $r$-stress on $(\Sigma, \nu)$ is a function $a: \Sigma \to \mathbb{R}$ satisfying $a(F) = 0$ whenever $\dim F \neq r - 1$, such that the equilibrium equation holds for every $G \in \mathcal{F}_{r-2}(\Sigma)$, i.e.

$$\sum_{v \in \mathcal{V}(\text{link}_G(\Sigma))} a(G \cup v)(\nu(G) \wedge \nu(v)) = 0. \tag{2}$$

Let $\Psi_r(\Sigma, \nu)$ be the $\mathbb{R}$-vector space of $(d + 1 - r)$-stresses on $(\Sigma, \nu)$, and let $\Psi(\Sigma, \nu) := \bigoplus_{r=0}^{d+1} \Psi_r(\Sigma, \nu)$. By default, $\Psi_r(\Delta, \nu) = 0$ when $r < 0$ or $r > d + 1$.

An $r$-stress $a$ on $(\Sigma, \nu)$ is called trivial if $a(F) = 0$ for all $F \in \Sigma$, and called non-trivial otherwise. A 0-stress on $(\Sigma, \nu)$ is any scalar assignment to the empty face $\emptyset$ of $\Sigma$, hence $\Psi_{d+1}(\Sigma, \nu) \cong \mathbb{R}$. Note that the 1-stresses on $(\Sigma, \nu)$ encode the linear relations on $\{\nu(v) | v \in \mathcal{V}(\Sigma)\}$, or equivalently, the affine relations on $\bigcup_{v \in \mathcal{V}(\Sigma)} \hat{\pi}(v)$. For $r \geq 2$, we have the following geometric interpretation of $r$-stresses.

**Theorem 3.4** ([13; cf. [24]]). Let $2 \leq r \leq d + 1$, and let $a: \Sigma \to \mathbb{R}$ be a function satisfying $a(F) = 0$ whenever $\dim F \neq r - 1$. Then $a$ is an $r$-stress on $(\Sigma, \nu)$ if and only if for all $G \in \mathcal{F}_{r-2}(\Sigma)$,

$$\sum_{F: F \supset G} a(F) \text{vol}(\hat{\pi}(F)) n_{\hat{\pi}(G), \hat{\pi}(F)} = 0, \tag{3}$$

where $n_{\hat{\pi}(G), \hat{\pi}(F)}$ denotes the outer unit normal vector to $\hat{\pi}(F)$ at the codimension 1 face $\hat{\pi}(G)$. 
Remark 3.5. Theorem 3.4 implies that the definition of an \( r \)-stress on \( (\Sigma, \nu) \) is independent of the choice of a linear order on \( V(\Sigma) \).

Usually, we consider PL realizations in \( \mathbb{R}^d \). A simplicial \( d \)-complex obviously admits PL realizations in \( \mathbb{R}^d \); simply consider any PL set-valued map that maps the vertices to points in general position; such a PL set-valued map induces a PL realization in \( \mathbb{R}^d \) (see Remark 3.2). For an arbitrary polytopal \( d \)-complex, begin with an embedding in \( \mathbb{R}^{2d+1} \), then project onto a generic affine \( d \)-subspace of \( \mathbb{R}^{2d+1} \) to get a PL set-valued map that induces a PL realization in \( \mathbb{R}^d \).

Suppose \( (\Sigma, \nu) \) is a PL realization in \( \mathbb{R}^d \). If \( \Sigma \) has \( n \) vertices, then \( \Psi_d(\Sigma, \nu) \cong \mathbb{R}^{n-d-1} \). To see why, consider any \( a \in \Psi_d(\Sigma, \nu) \), choose a linear basis \( \{\nu(v_1), \ldots, \nu(v_{d+1})\} \) of \( \mathbb{R}^{d+1} \) from among the vectors in \( \{\nu(v)|v \in \mathcal{V}(\Sigma)\} \), and notice that \( a(v) \) can be assigned arbitrarily for those vertices \( v \) not equal to \( v_1, \ldots, v_{d+1} \). Also, if \( \Sigma \) is a \( d \)-pseudo-manifold, then \( \Psi_0(\Sigma, \nu) \cong \mathbb{R} \) if \( \Sigma \) is orientable, and \( \Psi_0(\Sigma, \nu) = 0 \) otherwise. Indeed, for any \( a \in \Psi_0(\Sigma, \nu) \), if \( G \) is a \((d-1)\)-face of \( \Sigma \) contained in \( d \)-faces \( F_1 \) and \( F_2 \), then (3) says that \( \text{vol}(\pi(F_1))a(F_1) = \varepsilon \text{vol}(\pi(F_2))a(F_2) \) for some sign \( \varepsilon \in \{\pm 1\} \) completely determined by the PL realization \( \nu \). Since \( \Sigma \) is strongly connected, \( \text{vol}(\pi(F))a(F) \) must have a common value (up to sign) on all \( d \)-faces \( F \) of \( \Sigma \). This common value can be arbitrarily chosen when \( \Sigma \) is orientable, and must be 0 when \( \Sigma \) is non-orientable. For an arbitrary polytopal \( d \)-complex \( \Sigma \), the vector space \( \Psi_0(\Sigma, \nu) \) has dimension \( \beta_d \), where \( \beta_d \) denotes the \( d \)-th Betti number of \( \Sigma \); see [54 Thm. 4.1].

A local \( r \)-stress of \( (\Sigma, \nu) \) on a face \( H \) in \( \Sigma \) is a function \( a : \text{St}_r(\Sigma, \nu) \to \mathbb{R} \) such that the equilibrium equation (2) holds for every \( G \in \mathcal{F}_{r-2}(\text{St}_r(\Sigma, \nu)) \). Note that if \( a' \) is an \( r \)-stress on \( (\Sigma, \nu) \), then the map \( \text{St}_r(\Sigma, \nu) \) given by \( F \mapsto a'(F) \) is a local \( r \)-stress of \( (\Sigma, \nu) \) on \( H \). In particular, a local \( r \)-stress of \( (\Sigma, \nu) \) on the empty face \( \emptyset \) is precisely an \( r \)-stress on \( (\Sigma, \nu) \). Similar to the case of the (usual) \( r \)-stresses, we shall denote the \( \mathbb{R} \)-vector space of local \( r \)-stresses on \( H \) by \( \Psi_{d+1-r}(\text{St}_r(\Sigma, \nu)|_{\text{St}_r(\Sigma, \nu)}) \), and we define \( \Psi(\text{St}_r(\Sigma, \nu)|_{\text{St}_r(\Sigma, \nu)}) := \bigoplus_{r=0}^{d+1} \Psi_r(\text{St}_r(\Sigma, \nu)|_{\text{St}_r(\Sigma, \nu)}) \).

We end this section with comments on the terminology used by other authors. Stresses first appeared in Maxwell’s study of planar frameworks in classical mechanics [27]; they coincide with our notion of \( a \)-stresses on 1-dimensional simplicial complexes. A generalized notion of \( a \)-stresses in [24]. In fact, Lee defined two types of \( a \)-stresses: affine and linear. Given a simplicial \( d \)-complex \( \Delta \) and any integer \( N \geq d \), the space of Lee’s linear \( r \)-stresses on a PL realization of \( \Delta \) in \( \mathbb{R}^{N+1} \) is isomorphic to our space of \( r \)-stresses on a PL realization of \( \Delta \) in \( \mathbb{R}^N \). Another common equivalent definition for the space of (affine) \( r \)-stresses is to define it as the cokernel of some \( r \)-rational matrix. (There are several “kinds” of rigidity matrices.) The next section gives another useful equivalent definition of stresses. For more details and other equivalent definitions of (affine) \( r \)-stresses, see [48, 54, 55].

4. STRESSES AND HODGE DUALITY

In this section, we give another (equivalent) definition for stresses that is related to Hodge duality. An expert familiar with rigidity theory would notice that it corresponds to the cokernel of the minimal rigidity matrix. We shall use this definition to show that stresses “remain” as stresses on links of faces in a natural way (Theorem 3.2). Later in Section 5 we use this definition in a crucial way to construct a (well-defined) multiplication map on stresses.

Let \( N \geq d \geq 0 \) be integers. Assume that \( \mathbb{R}^{N+1} \) is equipped with the usual inner product \( \langle \cdot, \cdot \rangle \), and fix the standard orthonormal basis \( e_1, \ldots, e_{N+1} \) of \( \mathbb{R}^{N+1} \). Clearly, \( \langle \cdot, \cdot \rangle \) extends to an inner product on the exterior algebra \( \bigwedge(\mathbb{R}^{N+1}) \). Let \( \xi := e_1 \wedge \cdots \wedge e_{N+1} \in \bigwedge^{N+1}(\mathbb{R}^{N+1}) \). Elements of \( \bigwedge^r(\mathbb{R}^{N+1}) \) are called \( r \)-vectors. The \textit{Hodge star operator} \( * \) on \( \bigwedge(\mathbb{R}^{N+1}) \) is a linear operator that is completely determined by the relation \( \alpha \wedge (* \beta) = \langle \alpha, \beta \rangle \xi \) for all \( r \)-vectors \( \alpha, \beta \), where \( 0 \leq r \leq N + 1 \). Notice that \( * \) maps \( r \)-vectors to \((N + 1 - r)\)-vectors for all \( 0 \leq r \leq N + 1 \). \textit{Hodge duality} refers to the fact that \( *\langle \alpha \rangle = (-1)^r(N+1-r)\alpha \) for all \( r \)-vectors \( \alpha \), hence we can define an inverse \( *^{-1} : \bigwedge(\mathbb{R}^{N+1}) \to \bigwedge(\mathbb{R}^{N+1}) \) by \( \alpha \mapsto (-1)^r(N+1-r)*\alpha \) for all \( r \)-vectors \( \alpha \). The \textit{Grassmann–Cayley operator} on \( \bigwedge(\mathbb{R}^{N+1}) \), which we denote by \( \tilde{\bigwedge} \), is defined by \( \alpha \tilde{\bigwedge} \beta := *^{-1}((* \alpha) \wedge (* \beta)) \) for arbitrary \( \alpha, \beta \in \bigwedge(\mathbb{R}^{N+1}) \). An exterior algebra
equipped with the Grassmann–Cayley operator is also known as the Grassmann–Cayley algebra. For such algebras, we caution the reader that some authors (e.g. in [10]) instead use ∨ and ∧ to denote wedge product and the Grassmann–Cayley operator respectively.

Given an ordered set $U$, and any ordered subset $U'$ of $U$, let $\hat{U}'$ be the complement of $U'$ relative to $U$ (i.e. $U' \cup \hat{U}' = U$ and $U' \cap \hat{U}' = \emptyset$), and order the elements of $\hat{U}'$ so that they are consistent with the fixed linear order on $U$. Thus, by treating ordered sets as sequences, the concatenated sequence $(U', \hat{U}')$ is a permutation of $U$. We shall denote the sign of this permutation by $\text{Sgn}[U', U]$.

Let $(\Sigma, \nu)$ be a PL realization of a polytopal $d$-complex in $\mathbb{R}^N$, and fix a linear order on $\hat{\nu}(\Sigma)$. Given any faces $F' \subseteq F$ in $\Sigma$, let $\text{Sgn}[F', F]$ denote the sign $\text{Sgn}[\nu(F'), \nu(F)]$, where $\nu(F')$ and $\nu(F)$ are treated as ordered sets, whose orders are consistent with the given linear order on $\nu(\Sigma)$. Also, define $\mathbf{m}^\nu_{F', F} := (\nu(F))\overline{\nu}(\nu(F')))$, which is a non-zero $(\dim F - \dim F')$-vector in $\Lambda(\mathbb{R}^{N+1})$.

**Proposition 4.1** ([55]). Let $0 \leq r \leq d + 1$, and let $a : \Sigma \to \mathbb{R}$ be a function satisfying $a(F) = 0$ whenever $\dim F \neq r - 1$. Then $a$ is an $r$-stress if and only if for every $F' \in \mathcal{F}_{r-2}(\Delta)$,

$$\sum_{F:F\triangleright F'} a(F) \text{Sgn}[F', F] \mathbf{m}^\nu_{F', F} = 0.$$  

For the rest of this section, let $(\Delta, \nu)$ be a PL realization of a simplicial $d$-complex in $\mathbb{R}^N$. For any saturated flag $F_0 \prec F_1 \prec \cdots \prec F_t$ of faces in $\Delta$, let $v_j := F_j \setminus F_{j-1}$ for each $1 \leq j \leq t$. The following identities are straightforward consequences of the definition of the Grassmann–Cayley operator:

$$\mathbf{m}^\nu_{F_0, F_1} \wedge \cdots \wedge \mathbf{m}^\nu_{F_{t-1}, F_t} = \mathbf{m}^\nu_{F_0, F_0 \cup v_1} \wedge \cdots \wedge \mathbf{m}^\nu_{F_0, F_0 \cup v_t} = \prod_{1 \leq i \leq t} \text{Sgn}[F_{i-1}, F_i] \mathbf{m}^\nu_{F_0, F_t}.$$  

Given any $F \in \mathcal{F}_k(\Delta)$, write $(\nu(F'))$ as a decomposable $(N - k)$-vector $x_1 \wedge \cdots \wedge x_{N-k}$ (which is possible since $\nu(F)$ is decomposable), and define $\hat{\nu}_F := \text{lin}\{x_1, \ldots, x_{N-k}\} \subseteq \mathbb{R}^{N+1}$. Note that $\mathbf{m}^\nu_{F', G} \in \Lambda(\hat{\nu}_F)$ for all $G', G \in \Delta$ satisfying $F \subseteq G' \subseteq G$. Given any ordered basis $\mathcal{B} = \{e^F_1, \ldots, e^F_{N-k}\}$ for $\hat{\nu}_F$, define the linear map $T_{\mathcal{B}} : \hat{\nu}_F \to \mathbb{R}^{N-k}$ by

$$\begin{align*}
\alpha_1 e^F_1 + \cdots + \alpha_{N-k} e^F_{N-k} \mapsto (\alpha_1, \ldots, \alpha_{N-k}).
\end{align*}$$

We say that $\mathcal{B}$ is distinguished if $\hat{\pi}_{N-k-1}(T_{\mathcal{B}}(\mathbf{m}^\nu_{F', G})) \neq 0$ for all faces $F \subseteq G' \prec G$ in $\Delta$. Notice that the distinguished ordered bases for $\hat{\nu}_F$ are dense among arbitrary ordered bases for $\hat{\nu}_F$.

Given a distinguished ordered basis $\mathcal{B}$ for $\hat{\nu}_F$, let $\nu_{\mathcal{B}}$ be the PL realization of $\text{Lk}_\Delta(F)$ in $\mathbb{R}^{N-k-1}$ that is uniquely determined by $\nu_{\mathcal{B}}(v) = T_{\mathcal{B}}(\mathbf{m}^\nu_{F', F_0 \cup v})$ for all $v \in \nu(\text{Lk}_\Delta(F))$. Note in particular that $\nu(G) \neq 0$ for all $G \in \Delta$ implies $\nu_{\mathcal{B}}(G) \neq 0$ for all $G \in \text{Lk}_\Delta(F)$.

**Theorem 4.2.** Let $(\Delta, \nu)$ be a PL realization of a simplicial $d$-complex in $\mathbb{R}^N$. Let $a \in \Psi_r(\Delta, \nu)$, let $F \in \mathcal{F}_k(\Delta)$, and define the function $a' : \text{Lk}_\Delta(F) \to \mathbb{R}$ by $a'(G) := a(F \cup G)$. If $\mathcal{B}$ is a distinguished ordered basis for $\hat{\nu}_F$, then $a' \in \Psi_r(\text{Lk}_\Delta(F), \nu_{\mathcal{B}})$.

**Proof.** Consider an arbitrary $G' \in \mathcal{F}_{d-k-r-2}(\text{Lk}_\Delta(F))$. Since

$$\nu(F \cup G') \wedge \nu(G', G') = \text{Sgn}[F, F \cup G'] \text{Sgn}[G', G'] \nu(F) \wedge \nu(G)$$

for all $G \in \text{Lk}_\Delta(F)$ satisfying $G > G'$, it follows from the definition of a $(d + 1 - r)$-stress that

$$\sum_{G' \in \text{Lk}_\Delta(F), G' > G'} a(F \cup G) \text{Sgn}[F, F \cup G'] \text{Sgn}[G', G'] \nu(F) \wedge \nu(G) = 0,$$

so the identity $\nu(F') \wedge \nu(G') = \nu(F) \wedge \mathbf{m}^\nu_{F', F \cup G}$ yields

$$\nu(F) \wedge \sum_{G' \in \text{Lk}_\Delta(F), G' > G'} a(F \cup G) \text{Sgn}[G', G] \mathbf{m}^\nu_{F', F \cup G} = 0,$$
which implies
\[ \sum_{G \in \text{Lk}_\Delta(F)} a'(G) \text{Sgn}[G', G] \mathbf{m}_{F,F \cup G}' = 0. \]

Now, use (5) and apply the linear map \( T_R : \hat{\nabla}_F \to \mathbb{R}^{N-k} \) to get
\[ \sum_{v \in \mathcal{V}(\text{Lk}_\Delta(F \cup G'))} a'(G' \cup v)(\nu_B(G') \land \nu_B(v)) = 0. \]

Therefore \( a' \) is a \((d-k-r)\)-stress on \((\text{Lk}_\Delta(F), \nu_B)\), i.e. \( a' \in \Psi_r(\text{Lk}_\Delta(F), \nu_B) \).

\[ \square \]

5. STRESS ALGEBRA

Throughout this section, let \((\Sigma, \nu)\) be a PL realization of a polytopal \(d\)-complex in \(\mathbb{R}^d\). The main aim of this section is to define multiplication on stresses of different dimensions, and prove that \(\Psi(\Sigma, \nu)\) is an associative commutative graded \(\mathbb{R}\)-algebra.

For every \(F \in \Sigma\), let \(\mathcal{P}_F\) be the set of all pairs \((G, H) \in \Sigma \times \Sigma\) satisfying \(\text{aff}(\overline{\nu}(G) \cup \overline{\nu}(H)) = \mathbb{R}^d\) and \(G \cap H = F\). Note that if \(\Sigma\) is pure, then \(\mathcal{P}_F\) is non-empty for all \(F \in \Sigma\). For every \(P' = (G', H') \in \Sigma \times \Sigma\), let \(\mathcal{A}_{P'}\) be the set of all pairs \((G'', H'') \in \Sigma \times \Sigma\) satisfying either \(G'' = G', H'' \succ H'\), or \(G'' \succ G', H'' = H'\), such that \(G'' \cap H'' \in \mathcal{V}(\Sigma)\) whenever \(P' \in \mathcal{P}_\emptyset\).

**Lemma 5.1.** If \(F' \in \Sigma\), then
\[ \bigcup_{F:F' \succ F'} \mathcal{P}_F = \bigcup_{P' \in \mathcal{P}_{F'}} \mathcal{A}_{P'} \]
as sets of pairs in \(\Sigma \times \Sigma\).

**Proof.** The assertion is true by definition if \(F' = \emptyset\), so assume henceforth that \(F' \neq \emptyset\).

Suppose \((G, H) \in \mathcal{A}_{P'}\) for some \(P' = (G', H') \in \mathcal{P}_{F'}\). If \(G = G', H \succ H'\) (resp. \(G \succ G', H = H'\)), then since \(\text{aff}(\overline{\nu}(G') \cup \overline{\nu}(H')) = \mathbb{R}^d\) and \(G' \cap H' = F'\), it follows that \(\mathbf{m}_{\overline{\nu}(G'), \overline{\nu}(H')} \in \text{lin}(\overline{\nu}(G'))\) (resp. \(\mathbf{m}_{\overline{\nu}(G'), \overline{\nu}(G')} \in \text{lin}(\overline{\nu}(H'))\), thus \(G \cap H \succ F'\), and therefore \((G, H) \in \mathcal{P}_{G \cap H}\).

Conversely, suppose instead that \((G, H) \in \mathcal{P}_{F'}\) for some face \(F' \succ F'\) of \(\Sigma\). This means there is some vertex \(v \in \mathcal{V}(F')\) that is not a vertex of \(F'\). Let \(\hat{G}\) be an inclusion-wise maximal face of \(G\) that does not contain \(v\). Note that \(\hat{G} \prec G, \hat{G} \cap H = F'\), and \(\text{aff}(\overline{\nu}(G) \cup \overline{\nu}(H)) = \mathbb{R}^d\), thus by definition, \((\hat{G}, H) \in \mathcal{P}_{F'},\) and \((G, H) \in \mathcal{A}_{(\hat{G}, H)}\).

Given stresses \(a \in \Psi_r(\Sigma, \nu)\) and \(b \in \Psi_s(\Sigma, \nu)\), define their multiplication \(ab : \Sigma \to \mathbb{R}\) by
\[ (ab)(F) := \sum_{(G,H) \in \mathcal{P}_F} a(G)b(H). \]

By default, \((ab)(F) = 0\) if \(\mathcal{P}_F = \emptyset\).

**Theorem 5.2.** If \(a \in \Psi_r(\Sigma, \nu)\) and \(b \in \Psi_s(\Sigma, \nu)\), then \(ab \in \Psi_{r+s}(\Sigma, \nu)\).

**Proof.** Consider an arbitrary \(F' \in \mathcal{F}_{d-r-s-1}(\Sigma)\), and let \(P' = (G', H') \in \mathcal{P}_{F'}\). By Proposition 1.1
\[ \sum_{(G', H') \in \mathcal{A}_{P'}} \text{Sgn}[G', G]a(G)b(H')(\mathbf{m}_{F', G} \otimes 1) = 0 \in (\bigwedge (\mathbb{R}^{d+1})) \otimes \mathbb{R}(\bigwedge (\mathbb{R}^{d+1})), \]
and similarly,
\[ \sum_{(H', H) \in \mathcal{A}_{P'}} \text{Sgn}[H', H]a(G)b(H)(1 \otimes \mathbf{m}_{H', H}) = 0 \in (\bigwedge (\mathbb{R}^{d+1})) \otimes \mathbb{R}(\bigwedge (\mathbb{R}^{d+1})), \]
thus
\[ (6) \sum_{(G,H) \in \mathcal{A}_{P'}} \text{Sgn}[G', G] \text{Sgn}[H', H]a(G)b(H)(\mathbf{m}_{G', G} \otimes \mathbf{m}_{H', H}) = 0. \]
Next, let \( \phi_P : (\bigwedge (\mathbb{R}^{d+1})) \otimes \mathbb{R} \to (\bigwedge (\mathbb{R}^{d+1})) \) be the map defined by
\[
x \otimes y \mapsto (\nu(F') \wedge x \wedge y) \wedge (\nu(F')).
\]
Given any \((G, H) \in A_P\), note that
\[
m^\nu_{(G', G)} \otimes m^\nu_{(H', H)} = \begin{cases} m^\nu_{G', G} \otimes 1 & \text{if } G' \supseteq G', H = H'; \\ 1 \otimes m^\nu_{H', H} & \text{if } G = G', H \supset H'; \end{cases}
\]
so since \(\text{aff}(\mathcal{P}(G')) \cup \mathcal{P}(H')) = \mathbb{R}^d\) and \(G' \cap H' = F'\), it follows that
\[
\phi_P (\text{Sgn}[G', G] m^\nu_{(G', G)} \otimes \text{Sgn}[H', H] m^\nu_{(H', H)}) = \text{Sgn}[F', G \cap H] m^\nu_{F', G \cap H}
\]
for all \((G, H) \in A_P\).

Consequently, by applying \(\phi_P\) to (6) and summing over all \(P' \in P_F\), we get
\[
\sum_{P' \in P_F} \sum_{(G, H) \in A_P} \text{Sgn}[F', G \cap H] a(G) b(H) m^\nu_{F', G \cap H} = 0.
\]
It then follows from Lemma 5.1 that
\[
\sum_{F: F' \supset F} \sum_{(G, H) \in P_F} \text{Sgn}[F', F] a(G) b(H) m^\nu_{F', F} = 0,
\]
or equivalently,
\[
\sum_{F: F' \supset F} \text{Sgn}[F', F] ab(F) m^\nu_{F', F} = 0.
\]
Finally, for any \(F \in \Sigma\), the definition of \(P_F\) implies that \(a(G) b(H) = 0\) whenever \((G, H) \in P_F\) satisfies \(\dim F \neq d - r - s\), therefore Proposition 4.1 implies \(ab \in \Psi_{r+s}^t(\Sigma, \nu)\). \(\square\)

In the rest of this paper, we shall call \(\Psi(\Sigma, \nu)\) the stress algebra of \((\Sigma, \nu)\). The following theorem justifies our terminology.

**Theorem 5.3.** If \((\Sigma, \nu)\) is a PL realization of a polytopal d-complex in \(\mathbb{R}^d\), then \(\Psi(\Sigma, \nu)\) is an associative commutative graded \(\mathbb{R}\)-algebra.

**Proof.** Theorem 5.2 gives us \(\Psi_r(\Sigma, \nu) \cdot \Psi_s(\Sigma, \nu) \subseteq \Psi_{r+s}(\Sigma, \nu)\) for all \(r, s \in \mathbb{Z}\), thus \(\Psi(\Sigma, \nu)\) is a graded \(\mathbb{R}\)-algebra. (Recall that \(\Psi_k(\Sigma, \nu) = 0\) if \(k < 0\) or \(k > d + 1\).) The commutativity of \(\Psi(\Sigma, \nu)\) follows from the symmetry in the definition of \(P_F\), while the associativity of \(\Psi(\Sigma, \nu)\) is obvious. \(\square\)

**Remark 5.4.** Since \(P_F \subseteq \text{St}_{\Sigma}(F) \times \text{St}_{\Sigma}(F)\) by definition, the proofs of the results in this section hold verbatim when \(r\)-stresses are replaced by local \(r\)-stresses. The multiplication of local stresses is defined in exactly the same manner and yields another local stress. In particular, \(\Psi(\text{St}_{\Sigma}(H), \nu|_{\text{St}_{\Sigma}(H)})\) is an associative commutative graded \(\mathbb{R}\)-algebra for any \(H \in \Sigma\).

For readers familiar with McMullen’s proof of the \(g\)-theorem [29], recall that the multiplication of two weights on a polytope \(P \subseteq \mathbb{R}^{d+1}\) requires the notion of tight coherent subdivisions of \(P\) in its definition. McMullen proved that this multiplication map is well-defined (independent of the choice of the tight coherent subdivision) by using properties of fiber polytopes [4]. As proven by Lee [24], the \(r\)-weights of \(P\) coincide with the \((d + 1 - r)\)-stresses on the boundary complex of the polar dual \(P^*\) of \(P\) realized in \(\mathbb{R}^d\), hence a multiplication map on stresses analogous to McMullen’s multiplication of weights would require a dual notion to tight coherent subdivisions (as well as a dual notion to fiber polytopes). It is possible to define such duals to tight coherent subdivisions for faces \(F\) of \(P\); they are certain sets \(\mathcal{C}\) of pairs of polytopes \((G, H)\) satisfying \(G \cap H = F\). To define multiplication on stresses in terms of \(\mathcal{C}\), so that it agrees with McMullen’s multiplication on weights of the dual polytope, it would actually take considerable effort to show that the definition is independent of the choice of \(\mathcal{C}\).

Instead, we have defined multiplication on stresses in terms of \(P_F\), which corresponds to the union of all possible \(\mathcal{C}\) (for a fixed \(F\)). Dually, we get a different multiplication map on weights of the dual polytope \(P^*\) that is defined in terms of the union of all possible tight coherent subdivisions of \((\text{faces of}) \ P^*\), which would simplify McMullen’s proof of the \(g\)-theorem without the need for fiber polytopes.
6. Liftings, Reciprocals, and d-Stresses on Orientable Homology Manifolds

Liftings and reciprocals are fundamental concepts in Maxwell–Cremona theory \([7,8]\) that deal with the realizations of pure polytopal complexes. Intuitively, liftings (into Euclidean spaces of one dimension higher) are the inverses of vertical projections onto hyperplanes, while reciprocals are rectilinear realizations of dual graphs whose edge directions are fixed in a specific manner. In this section, we build upon the work by Rybnikov and coauthors \([11,48]\) on the close connections between d-stresses, liftings, and reciprocals. We will relate \(\Psi_1(\Sigma, \nu)\) to the liftings and reciprocals of \((\Sigma, \nu)\) in the case when \(\Sigma\) is an orientable (polytopal) \(k\)-homology manifold satisfying a certain homological condition, and in particular, we will consider the special subcase when \(\Sigma\) is simplicial.

Let \((\Sigma, \nu)\) be a PL realization of a pure polytopal \(d\)-complex in \(\mathbb{R}^d\). For brevity, denote the projection maps \(\pi_d\) and \(\widehat{\pi}_d\) simply as \(\pi\) and \(\widehat{\pi}\) respectively. Given \(\mathbb{R}\)-vector spaces \(U, V\), let the space of \(V\)-valued affine functions on \(U\) be denoted by \(\text{Aff}(U, V)\). A lifting of \((\Sigma, \nu)\) is a map \(\mu : \mathcal{F}_d(\Sigma) \to \text{Aff}(\mathbb{R}^d, \mathbb{R}^{d+1})\) such that \(\pi((\mu(F))(\widehat{\pi}(F))) = \pi(F)\) for all \(F \in \mathcal{F}_d(\Sigma)\), and \(\mu(F)|_{\mathbb{R}^{d-1}\cap \mathbb{R}^{d'}} = \mu(F')|_{\mathbb{R}^{d-1}\cap \mathbb{R}^{d'}}\) for all \(F, F' \in \mathcal{F}_d(\Sigma)\). Note that adjacent \(d\)-faces of \(\Sigma\) are not required to be mapped to affine maps with distinct \(\pi\)-stresses, \(\nu\)-stresses, and \(d\)-stresses.

By definition, every lifting \(\mu\) is completely determined by the map \(\widehat{\pi}(\mu) : \mathcal{F}_d(\Sigma) \to \text{Aff}(\mathbb{R}^d, \mathbb{R}^{d+1})\) defined by \(F \mapsto \pi \circ \mu(F)\). Thus, we can add two liftings \(\mu, \mu'\) of \((\Sigma, \nu)\) by specifying that \(\pi(\mu + \mu') = \pi(\mu) + \pi(\mu')\).

Similarly, scalar multiplication is defined by \(\pi(\alpha \mu) = \alpha(\pi(\mu))\) for all \(\alpha \in \mathbb{R}\).

Theorem 6.1 \([48]\). Let \((\Sigma, \nu)\) be a PL realization of an orientable (polytopal) \(k\)-homology \(d\)-manifold in \(\mathbb{R}^d\) with trivial first homology group over \(\mathbb{Z}/2\mathbb{Z}\) whenever \(d \geq 2\). Then for any \(Q \in \mathcal{F}_d(\Sigma)\), there is an isomorphism between \(\Psi_1(\Sigma, \nu)\) and \(\text{Lift}(\Sigma, \nu, Q)\) as \(\mathbb{R}\)-vector spaces.

Remark 6.2. For any \(k\)-homology manifold \(\Sigma\), the condition \(H_1(\Sigma; \mathbb{Z}/2\mathbb{Z}) = 0\) necessarily implies that \(\Sigma\) is orientable; see \([14]\) Cor. 3.28. This homological condition cannot be omitted from Theorem 6.1.

For example, Rybnikov gave a PL realization of a 2-torus that has non-trivial 2-stresses, but does not admit any non-trivial liftings; see \([14]\) Fig. 3.

Theorem 6.3. Let \((\Delta, \nu)\) be a PL realization of an orientable simplicial \(k\)-homology \(d\)-manifold in \(\mathbb{R}^d\), such that \(H_1(\Delta; \mathbb{Z}/2\mathbb{Z}) = 0\) whenever \(d \geq 2\). Then \(\Psi_1(\Delta, \nu) \cong \mathbb{R}^{n-d-1}\), where \(n\) is the number of vertices in \(\Delta\).

Proof. Let \(Q\) be an arbitrary \(d\)-face of \(\Delta\). Theorem 5.1 yields the isomorphism \(\Psi_1(\Delta, \nu) \cong \text{Lift}(\Delta, \nu, Q)\). Every lifting \(\mu \in \text{Lift}(\Delta, \nu, Q)\) is completely determined once we know the values of \(\widehat{\pi}(\mu(v))\) for all vertices \(v \in \mathcal{V}(\Delta)\). Since \(\Delta\) is simplicial, all the vertices not in \(Q\) can take on arbitrary real values, therefore \(\Psi_1(\Delta, \nu) \cong \mathbb{R}^{n-|\mathcal{V}(\Delta)|} = \mathbb{R}^{n-d-1}\). \(\square\)

Remark 6.4. In general, for non-simplicial \(\Sigma\), the dimension of \(\Psi_1(\Sigma, \nu)\) would depend on the combinatorial structure of \(\Sigma\). Without the explicit knowledge of this combinatorial structure, we can only conclude that \(\text{dim}(\Psi_1(\Sigma, \nu)) \leq n - d'\), where \(d'\) is the maximum number of vertices in any \(d\)-face of \(\Sigma\). See \([48]\) for an algorithmic approach to finding the exact value of the dimension of the space of liftings for certain classes of non-simplicial \(k\)-homology manifolds.

For the rest of this section, let \((\Sigma, \nu)\) be a PL realization of a (polytopal) \(k\)-homology \(d\)-manifold in \(\mathbb{R}^d\). A reciprocal of \((\Sigma, \nu)\) is a map \(R : \mathcal{F}_d(\Sigma) \to \mathbb{R}^d\) such that \(R(F') - R(F)\) is parallel to \(n_{\widehat{\pi}(F)\cap \widehat{\pi}(F')}\) for all adjacent \(d\)-faces \(F, F'\) of \(\Sigma\). We say \(R\) is non-degenerate if \(R(F) \neq R(F')\) for all adjacent \(d\)-faces \(F, F'\). Typically, we consider reciprocals when \(\Sigma\) is orientable. A PL orientation of \((\Sigma, \nu)\) is a map \(\rho : \mathcal{F}_d(\Sigma) \to \{\pm 1\}\) such that adjacent \(d\)-faces \(F, F'\) of \(\Sigma\) satisfy \(\rho(F) = \rho(F')\) if and only if the outer unit normal vectors to \(\widehat{\pi}(F)\) and \(\widehat{\pi}(F')\) at their common codimension 1 face \(\widehat{\pi}(F \cap F')\) have opposite directions. It can be shown that \(\Sigma\) is orientable in the usual sense (e.g. as defined in \([14]\)) if and only if
(Σ, ν) admits a PL orientation [48]. In particular, every orientable Σ has two possible PL orientations: Starting with any d-face F of Σ, the value of ρ(F), together with ν, would uniquely determine the values of ρ(F′) for all remaining faces F′ ∈ F_d(Σ).

Suppose ρ is a PL orientation of (Σ, ν), and let R be a reciprocal of (Σ, ν). It follows from definition that for every pair of adjacent d-faces F, F′ of Σ, there exists a unique scalar ℓ_{F′→F} ∈ ℜ such that

\[ R(F′) - R(F) = ℓ_{F′→F}ρ(F)n_{π(F′→F),π(F)}. \]

The map L_R : F_{d-1}(Σ) → ℜ defined by G ↦ ℓ_G is called the edge-length map of R, and the map \( \hat{L}_R : F_{d-1}(Σ) \rightarrow \hat{R} \) defined by G ↦ \( \frac{1}{\text{vol}(π(F))]ℓ_G \) is called the normalized edge-length map of R. Notice that L_R uniquely determines R up to translation. Thus, if Rec(Σ, ν, Q) denotes the set of all reciprocals R of (Σ, ν) satisfying R(0) = 0 for some given d-face Q of Σ, then every reciprocal R ∈ Rec(Σ, ν, Q) can be identified with its edge-length map L_R. We check that for any reciprocals R, R′ ∈ Rec(Σ, ν, Q), the sum of their edge-length maps is always the edge-length map of a uniquely determined reciprocal in Rec(Σ, ν, Q). Thus, we can define addition on Rec(Σ, ν, Q) by specifying that L_{R+R′} = L_R + L_{R′}. Scalar multiplication on Rec(Σ, ν, Q) is obvious, therefore Rec(Σ, ν, Q) is an ℜ-vector space.

**Theorem 6.5 ([48]).** Let (Σ, ν) be a PL realization of an orientable (polytopal) k-homology d-manifold in \( ℜ^d \). Then for any Q ∈ F_d(Σ), there is an isomorphism of ℜ-vector spaces between Lift(Σ, ν, Q) and Rec(Σ, ν, Q).

**Remark 6.6.** Note that Theorem 6.5 does not require the homological condition \( H_1(Σ; ℤ/2ℓZ) = 0 \) (for \( d ≥ 2 \)). However, we do require that Σ is orientable so that the addition of reciprocals is well-defined.

Given a lifting µ of (Σ, ν) and any F ∈ F_d(Σ), the map \( (\pi ∘ µ(F)): ℝ^d \rightarrow ℝ \) is by definition an affine map, so there is a unique vector \( m_µ^F \in ℝ^d \), and a unique scalar \( c_µ^F \in ℜ \), such that \( \pi ∘ µ(F) = \pi ∘ (m_µ^F, x) + c_µ^F \). The following two theorems refine Theorem 6.1 and Theorem 6.5 respectively.

**Theorem 6.7 ([48]).** Let (Σ, ν) be a PL realization of an orientable (polytopal) k-homology d-manifold in \( ℜ^d \). Then for every Q ∈ F_d(Σ), there is an injective map \( \varphi : \text{Lift}(Σ, ν, Q) \rightarrow Ψ_1(Σ, ν) \) (of ℜ-vector spaces) given by

\[ \mu \mapsto \left((F \cap F′) \mapsto \frac{ρ(F)}{\text{vol}(π(F \cap F′))}(m_µ^F - m_µ^{F′}, n_{π(F′→F),π(F)})\right), \]

where \( F \cap F′ \) is a (d−1)-face of Σ contained in the two (uniquely determined) d-faces F, F′ of Σ, and ρ is a PL orientation of (Σ, ν). Furthermore, if \( H_1(Σ; ℤ/2ℓZ) = 0 \) or \( d ≤ 1 \), then \( \varphi \) is an isomorphism.

**Theorem 6.8 ([48]).** Let (Σ, ν) be a PL realization of an orientable (polytopal) k-homology d-manifold in \( ℜ^d \). Then for every Q ∈ F_d(Σ), there is an isomorphism \( \varphi′ : \text{Lift}(Σ, ν, Q) \rightarrow \text{Rec}(Σ, ν, Q) \) (of ℜ-vector spaces) given by \( μ \mapsto (F \mapsto m_µ^F) \).

**Remark 6.9.** The proof of Theorem 6.8 (as given in [48]) requires the existence of dual polyhedral decompositions, which are used in proving Poincaré duality for k-homology manifolds.

**Remark 6.10.** Similar to local d-stresses, we can define local liftings and local reciprocals by replacing every instance of (Σ, ν) with (St_{Σ}(H), ν|_{St_{Σ}(H)}) for some \( H ∈ Σ \); cf. Remark 6.4. The local versions of Theorem 6.7 and Theorem 6.8 still hold true. (Their proofs in [48] hold verbatim.) Thus, for any \( H ∈ Σ \) and any Q ∈ F_d(St_{Σ}(H)), we have an isomorphism of ℜ-vector spaces:

\[ Ψ_1(St_{Σ}(H), ν|_{St_{Σ}(H)}) \cong \text{Lift}(St_{Σ}(H), ν|_{St_{Σ}(H)}, Q) \cong \text{Rec}(St_{Σ}(H), ν|_{St_{Σ}(H)}, Q). \]

Recall that a collection of vectors in \( ℜ^d \) is said to be in general position if every subcollection of at most \( d+1 \) vectors is affinely independent. Given any PL realization (Σ, ν), we say that the vertices of Σ are realized in general position if \( \bigcup_{v ∈ Ψ(Σ)} D(v) \) is a collection of vectors in general position. Notice that such a polytopal complex Σ is necessarily simplicial. By combining Theorem 6.7 and Theorem 6.8 with our definition of the multiplication of stresses of different dimensions (see Theorem 5.3), we get the following useful corollary.
Corollary 6.11. Let $(\Delta, \nu)$ be a PL realization of an orientable simplicial $k$-homology $d$-manifold in $\mathbb{R}^d$ such that $H_1(\Delta; \mathbb{Z}/2\mathbb{Z}) = 0$ whenever $d \geq 2$. Let $a \in \Psi_{1}(\Delta, \nu)$ and $b \in \Psi_{d}(\Delta, \nu)$. Fix some $d$-face $Q$ of $\Delta$, and let $\hat{G}_a : F_{d-1}(\Delta) \to \mathbb{R}$ be the normalized edge-length map of the reciprocal corresponding to the $d$-stress $a$ under the isomorphism $\Psi_{1}(\Delta, \nu) \cong \text{Rec}(\Delta, \nu, Q)$. If the vertices of $\Delta$ are realized in general position, then $ab$ is the 0-stress given by

$$(ab)(\emptyset) = \sum_{v \in V(\Delta)} \left( \sum_{G \in G_v} \hat{G}(G) \right) b(v),$$

where $G_v := \{ G \in F_{d-1}(\Delta) | v \notin V(G) \}$.

7. Generic PL realizations and multiplication by $d$-stresses

Throughout this section, let $(\Delta, \nu)$ be a PL realization of an oriented simplicial $k$-homology $d$-manifold in $\mathbb{R}^d$ satisfying $H_1(\Delta; \mathbb{Z}/2\mathbb{Z}) = 0$ whenever $d \geq 2$, and let $\rho$ be the PL orientation of $(\Delta, \nu)$. The purpose of this section is twofold: First, we introduce a specific notion of genericity, which we call $\mathbb{Q}$-genericity. Next, we prove that if $\nu$ is $\mathbb{Q}$-generic, then for every non-trivial 1-stress $b \in \Psi_{d}(\Delta, \nu)$, we can always find some $d$-stress $a \in \Psi_{1}(\Delta, \nu)$ such that $ab$ is non-trivial. The existence of such a $d$-stress would serve as the base case for our induction argument in Section 8, where we prove that the stress algebra $\Psi(\Delta, \nu)$ is Gorenstein for $\mathbb{Q}$-generic PL realizations $\nu$.

Intuitively, choosing a $\mathbb{Q}$-generic PL realization $\nu$ amounts to choosing rational coordinates for a certain collection of points (arising from certain “basis reciprocals” and completely determined by $\nu$) such that the “normalized” distance between any pair of distinct points in this collection must be the square root of a squarefree rational number. (A rational number $z$ is called squarefree if $\sqrt{|z|} \not\in \mathbb{Q}$.) We now state a number theoretic result on square roots that will be useful later.

Lemma 7.1 (33). Let $F$ be a real algebraic number field, let $\zeta_1, \ldots, \zeta_r \in \mathbb{R}$ be square roots of elements in $F$, and suppose that the product of the elements in every non-empty subcollection of $\{\zeta_1, \ldots, \zeta_r\}$ is not contained in $F$. If $P(x_1, \ldots, x_r)$ is a polynomial in $r$ variables with coefficients in $F$, such that $P$ is linear in each variable $x_i$, then $P(\zeta_1, \ldots, \zeta_r) = 0$ if and only if all the coefficients of $P$ are 0.

Fix a linear order $v_1, \ldots, v_n$ on the vertices of $\Delta$. For each set $S = \{v_1, \ldots, v_r\}$ of vertices such that $t_1 < \cdots < t_r$, let $W'_S$ be the $r$-by-$(d+1)$ matrix whose rows are $\nu(v_{t_1}), \ldots, \nu(v_{t_r}) \in \mathbb{R}^{d+1}$ (as row vectors), and let $W_S$ be the matrix obtained from $W'_S$ as follows: For every $1 \leq i \leq r$, divide all the entries in the $i$-th row of $W'_S$ by the scalar $\pi_d(\nu(v_{t_i}))$. Notice that $W_S$ would have a column of ones as its rightmost column. For convenience, we write $W_{\Psi(F)}$ simply as $W_F$ for each face $F$ of $\Delta$. Recall that the Gram matrix of a collection $\{u_1, \ldots, u_k\}$ of vectors in $\mathbb{R}^{d+1}$ is the $k \times k$ matrix whose $(i, j)$-th entry equals the inner product $\langle u_i, u_j \rangle$. Given any matrix $W$, let Gram($W$) be the Gram matrix of the collection of the rows of $W$ (treated as vectors).

Following the notation in Section 6 we recall that any lifting $\mu$ of $(\Delta, \nu)$ is completely determined by the collection of real scalars $\{ \pi(\mu(v)) | v \in V(\Delta) \}$. For each $1 \leq i \leq n$, let $\mu_i$ be the lifting of $(\Delta, \nu)$ given by $\pi(\mu_i(v_i)) = 1$ and all remaining $v \neq v_i$. Given any $d$-face $F$ of $\Delta$, it follows from definition that $m^\mu_F$ is 0 if $v_i$ is not a vertex of $F$, and $m^\mu_F = \pi_d(W_F^{-1}e_i) \in \mathbb{R}^d$ if $v_i$ is the $k$-th vertex of $F$ induced by the linear order $v_1, \ldots, v_n$, where $e_i \in \mathbb{R}^{d+1}$ is the standard basis column vector whose $k$-th entry equals 1. In this latter case, Cramer’s rule says that the $j$-th entry of $m^\mu_F$ equals $\frac{\text{det}(W'_F)}{\text{det}(W_F)}$, where $W'_F$ is the matrix obtained from replacing the $j$-th column of $W_F$ by $e_j$. Consequently, if $\nu(v) \in \mathbb{Q}^{d+1}$ for all $v \in V(\Delta)$ (in which case we say that $\nu$ is rational), then $m^\mu_F \in \mathbb{Q}^d$ for all $1 \leq i \leq n$ and all $F \in F_d(\Delta)$.

Consider an arbitrary $G \in F_{d-1}(\Delta)$. Since $\Delta$ is a $k$-homology manifold, there are two uniquely determined $d$-faces $G', G''$ of $\Delta$ that contain $G$. For each $v_1 \in V(\Delta)$, define the non-negative real scalar

$$\zeta^\nu_{(G, v_1)} := \frac{\| m^\mu_{G'} - m^\mu_{G''} \|}{\text{vol}(\Gamma(G))},$$

where $\text{vol}(\Gamma(G))$ is the volume of the paraboloid determined by $G$. This completes the proof of Theorem 1.1.
where \( \| \cdot \| \) denotes the usual Euclidean norm, and define the real scalar
\[
\zeta^\nu_{(G,v_i)} := \frac{\rho(G)}{\text{vol}(\mathcal{P}(G))} (m^\mu_{G'} - m^\mu_{G''}, m_{\mathcal{P}(G),G''-}).
\]

Notice that Theorem 6.7 and Theorem 6.8 together imply that \( \zeta^\nu_{(G,v_i)} \) is the absolute value of \( c^\nu_{(G,v_i)} \).

**Definition 7.2.** A PL realization \( \nu \) of \( \Delta \) in \( \mathbb{R}^d \) is called \( \mathbb{Q} \)-generic if \( \nu \) is rational, the vertices of \( \Delta \) are realized in general position, the collection of real scalars
\[
\mathcal{S}_\nu := \left\{ \zeta^\nu_{(G,v_i)} \mid 1 \leq i \leq n, G \in \mathcal{F}_{d-1}(\Delta), v_i \in \mathcal{V}(\mathcal{S}_\Delta(G)) \right\}
\]
is linearly independent over \( \mathbb{Q} \) (i.e. the scalars in \( \mathcal{S}_\nu \) are distinct, non-zero, and do not satisfy any non-trivial linear equations with coefficients in \( \mathbb{Q} \)), and the product of the elements in every non-empty subcollection of \( \mathcal{S}_\nu \) is irrational.

Geometrically, the last two conditions in this definition mean that the normalized edge-length maps of the \( n \) reciprocals \( (F \mapsto m^\mu_i) \) \( (i = 1, \ldots, n) \) map each \((d-1)\)-face of \( \Delta \) to either 0 or an irrational number (cf. Theorem 6.8), such that the irrational “normalized edge-lengths” (over all these \( n \) reciprocals) do not satisfy any non-trivial monomial or linear equations over \( \mathbb{Q} \).

Notice that every PL realization \( \nu \) of \( \Delta \) is completely determined by \( \{\nu(v_i) \in \mathbb{R}^{d+1} \mid 1 \leq i \leq n\} \) and hence can be identified with a vector in \( \mathbb{R}^{n(d+1)} \). Under this identification, the following proposition justifies our terminology “\( \mathbb{Q} \)-generic”.

**Proposition 7.3.** The set of all \( \mathbb{Q} \)-generic PL realizations of \( \Delta \) is a dense subset of all (not necessarily rational) PL realizations of \( \Delta \) (with respect to the usual Euclidean metric).

**Proof.** First of all, rational PL realizations and PL realizations with vertices realized in general position, are each dense among arbitrary PL realizations. Hence, if \( \mathcal{Y} \) denotes the set of rational PL realizations of \( \Delta \) with vertices realized in general position, then it suffices to show that the set of \( \mathbb{Q} \)-generic PL realizations of \( \Delta \) is a dense subset of \( \mathcal{Y} \).

Suppose \( \nu \in \mathcal{Y} \). Let \( G \in \mathcal{F}_{d-1}(\Delta) \), let \( G', G'' \) be the two uniquely determined \( d \)-faces of \( \Delta \) that contain \( G \), and let \( v_i \in \mathcal{V}(\mathcal{S}_\Delta(G)) \). Note that \( \frac{1}{d+1} |\det(W_{G'})| = \text{vol}(\mathcal{P}(G')) \), while for each \( 1 \leq j \leq d \), the scalar \( \frac{1}{d+1} |\det(\tilde{W}_{G,j})| \) equals the \((d-1)\)-volume of the orthogonal projection of \( \mathcal{P}(G) \) onto the coordinate hyperplane containing all points whose \( j \)-th coordinate is zero. (Similar statements hold when \( G' \) is replaced by \( G'' \).) By definition, \( |\text{vol}(\mathcal{P}(G))|^2 = |\det(W_{G})| \). Note also that
\[
\|m^\mu_i_{G'} - m^\mu_i_{G''}\|^2 = \sum_{j=1}^{d} \left[ \frac{|\det(\tilde{W}_{G,j})|}{|\det(W_{G})|} - \frac{|\det(\tilde{W}_{G'',j})|}{|\det(W_{G''})|} \right]^2.
\]

Consequently, \( \|m^\mu_i_{G'} - m^\mu_i_{G''}\|^2 \) is a rational function with rational coefficients in terms of the \((d+1)(d+2)\) coordinates of the \( d + 2 \) vectors in \( \{\nu(u)u \in \mathcal{V}(G') \cup \mathcal{V}(G'')\} \) as its variables. We check that this rational function is not the square of any rational function (with rational coefficients) over the same variables.

Now, the set of squarefree rational numbers is dense in \( \mathbb{Q} \), thus if some scalar in \( \mathcal{S}_\nu \) is rational, then we can always perturb \( \nu \) (as a vector in \( \mathbb{Q}^{n(d+1)} \)) so that the scalar becomes the square root of a squarefree rational number, i.e. the scalar becomes irrational. Consequently, there is a dense subset \( \mathcal{Y}' \) of \( \mathcal{Y} \) such that for every PL realization \( \nu \) in \( \mathcal{Y}' \), the scalars in \( \mathcal{S}_\nu \) are distinct and irrational.

Finally, a routine argument shows that there is a dense subset \( \mathcal{Y}'' \) of \( \mathcal{Y}' \) such that for any \( \nu \in \mathcal{Y}'' \), the product of scalars in every non-empty subset of \( \mathcal{S}_\nu \) is irrational. Therefore, it follows from Lemma 7.1 that \( \mathcal{S}_\nu \) is linearly independent over \( \mathbb{Q} \). By definition, this dense subset \( \mathcal{Y}'' \) is identically the set of \( \mathbb{Q} \)-generic PL realizations of \( \Delta \). \( \square \)

Before we prove the main result (Theorem 7.7) of this section, we need to introduce more notation and definitions. Given a real algebraic number field \( \mathbb{F} \), and any set \( S \) of real numbers, let \( \mathbb{F}[S] \) denote
the smallest subfield of \( \mathbb{R} \) that contains \( F \) and \( S \). Given any matrix \( X \), let \( X^T \) be its transpose, and let \( \text{Row}(X) \) be its row space. For any pure simplicial \( d \)-complex \( \Gamma \) with \( n \) linearly ordered vertices and \( m \) linearly ordered \((d - 1)\)-faces, the vertex-ridge incidence matrix of \( \Gamma \) is an \( n \times m \) real matrix whose \((i, j)\)-entry equals 1 if the \( i \)-th vertex is contained in the \( j \)-th \((d - 1)\)-face, and equals 0 otherwise.

**Lemma 7.4** ([5 Cor. 3]). If \( \Gamma \) is a pure strongly connected simplicial complex, then its vertex-ridge incidence matrix has full rank.

For the rest of this section, fix a linear order \( G_1, \ldots, G_m \) on the \((d - 1)\)-faces of \( \Delta \). Let \( A \) be an \( n \times m \) matrix whose \((i, j)\)-th entry equals \( \zeta_{i,j}^{\rho}(G_{i,j}) \). Let \( B \) be an \( n \times m \) real matrix whose \((i, j)\)-th entry equals 1 if \( v_i \not\in V(G_j) \), and equals 0 otherwise. Also, let \( \overline{B} \) be the vertex-ridge incidence matrix of \( \Delta \). Note that \( B, \overline{B} \) have constant column sums \( n - d, d \) respectively, so \( \text{Row}(B) = \text{Row}(\overline{B}) \), and by Lemma 7.4 both \( B \) and \( \overline{B} \) have full rank. An elementary fact from linear algebra says that if \( X, Y \) are \( n \times m \) matrices (note that \( m \geq n \)), then \( XY^T \) is invertible if and only if \( Y \) has full rank and \( \text{Row}(X)^T \cap \text{Row}(Y) = \{0\} \), thus \( AB^T \) is invertible if and only if \( A\overline{B}^T \) is invertible.

**Lemma 7.5.** Let \( t_1, \ldots, t_n \) be positive integers. For each \( 1 \leq i \leq n \), let \( Z_i := \{\zeta_{1,i}, \ldots, \zeta_{t_i,i}\} \) be a set of \( t_i \) distinct square roots of squarefree positive rational numbers, and assume that the product of the elements in every non-empty subcollection of \( Z := Z_1 \cup \cdots \cup Z_n \) is irrational. Suppose \( X = (X_{i,j})_{1 \leq i,j \leq n} \) is an \( n \times n \) matrix that satisfies the following conditions.

(i) Every row or column of \( X \) has at least one non-zero entry.

(ii) For each \( 1 \leq i \leq n \), the entries in the \( i \)-th row of \( X \) are elements of the number field \( \mathbb{Q}[Z_i] \), and the non-zero entries among them are linearly independent over \( \mathbb{Q} \).

Then \( X \) must be an invertible matrix.

**Proof.** Let \( X_1 := \{X_{1,1}, \ldots, X_{1,r_1}\} \subseteq \mathbb{R} \) be the set of all non-zero entries in the first row of \( X \), and define \( \widetilde{Z}_1 := \mathbb{Z} \setminus \widetilde{Z}_i \). We shall first show that \( X_1 \) is linearly independent over \( \mathbb{Q}[\widetilde{Z}_1] \). Suppose not, then there exist \( \alpha_1, \ldots, \alpha_s \in \mathbb{Q}[\widetilde{Z}_1] \), not all zero, such that \( \alpha_1 X_{1,r_1} + \cdots + \alpha_s X_{1,r_s} = 0 \). Let \( x = (x_1, \ldots, x_{t_1}) \) be a vector of \( t_1 \) variables, and let \( \tilde{x} = (x_{i,j}) \) be a vector of doubly-indexed variables, where the indices satisfy \( 2 \leq i \leq n \) and \( 1 \leq j \leq t_i \). For every \( 1 \leq k \leq s \), since \( X_{1,r_k} \in \mathbb{Q}[\widetilde{Z}_1] \) (resp. \( \alpha_k \in \mathbb{Q}[\widetilde{Z}_1] \)), there exists a polynomial \( P_k(x) \) (resp. \( \tilde{P}_k(\tilde{x}) \)) in terms of variables \( x \) (resp. \( \tilde{x} \)), such that \( X_{1,r_k} \) is the value of \( P_k \) evaluated at \( x_i = \zeta_{1,i} \) for \( 1 \leq i \leq t_1 \) (resp. \( \alpha_k = \text{value of } \tilde{P}_k \text{ evaluated at } x_{i,j} = \zeta_{i,j} \) for \( 2 \leq i \leq n \), \( 1 \leq j \leq t_i \)).

Since \( \alpha_1, \ldots, \alpha_s \) are not all zero, at least one of \( P_1, \ldots, \tilde{P}_s \) is not the zero polynomial. By assumption, \( X_{1,r_1}, \ldots, X_{1,r_s} \) are non-zero, so none of \( P_1, \ldots, P_s \) is the zero polynomial. Choose some \( \bar{y} \in \mathbb{Q}[z_2, \ldots, z_n] \) such that \( \tilde{P}_k(\bar{y}) \neq 0 \) whenever \( P_k \neq 0 \). Define the polynomial \( P(x, \bar{x}) := \sum_{k=1}^s P_k(x) \tilde{P}_k(\bar{x}) \), and note that \( P \) evaluated at \( x = (\zeta_{1,1}, \ldots, \zeta_{1,t_1}) \) and \( \bar{x} = \bar{y} \) gives a non-trivial \( \mathbb{Q} \)-linear combination of \( X_1 \). This \( \mathbb{Q} \)-linear combination must be non-zero by condition [ii] thus \( P \) is not the zero polynomial. Now, \( \alpha_1 X_{1,r_1} + \cdots + \alpha_s X_{1,r_s} \) is the value of \( P \) evaluated at \( x_i = \zeta_{1,i} \) for \( 1 \leq i \leq t_1 \) and \( x_{i,j} = \zeta_{i,j} \) for \( 2 \leq i \leq n \), \( 1 \leq j \leq t_i \), hence Lemma 7.4 forces the contradiction that \( \alpha_1 X_{1,r_1} + \cdots + \alpha_s X_{1,r_s} \neq 0 \). Consequently, \( X_1 \) is linearly independent over \( \mathbb{Q}[\widetilde{Z}_1] \) as claimed.

Next, we prove the lemma by induction on \( n \). Using Laplace’s expansion along the first row, we have \( \det(X) = \sum_{j=1}^m X_{1,r_j}C_{1,r_j} \), where each \( C_{1,r_j} \) is the \((1, r_j)\)-cofactor of \( X \). This implies that \( \det(X) \) is a \( \mathbb{Q}[\widetilde{Z}_1] \)-linear combination of the elements of \( X_1 \). Now, by condition [i] we know that there is a permutation \( \varphi \) on \( \{1, \ldots, n\} \) such that \( X_{i,\varphi(i)} \neq 0 \) for all \( 1 \leq i \leq n \), thus \( C_{1,\varphi(1)} \neq 0 \) by induction hypothesis. Therefore, it follows from the linear independence of \( X_1 \) over \( \mathbb{Q}[\widetilde{Z}_1] \) that \( \det(X) \neq 0 \). \( \square \)

**Proposition 7.6.** If \( \nu \) is \( \mathbb{Q} \)-generic, then \( A\overline{B}^T \) is invertible.

**Proof.** We shall prove the equivalent statement that \( A\overline{B}^T \) is invertible. Let \( S_\nu \) be the collection of non-zero scalars as defined in [9]. For each \( 1 \leq i \leq n \), let \( R_i \) be the reciprocal of \( (\Delta, \nu) \) given by \( (F \mapsto m_F^{\mu_i}) \), and recall that \( \hat{L}_{R_i} \) denotes the normalized edge-length map of \( R_i \). By definition, the
(i, j)-th entry of $A$ equals $\zeta^p_{a_{ij}} = \tilde{\mathcal{L}}_{R_i}(G_j)$. Thus, every non-zero entry of $A$ is an element of $\mathcal{S}_\nu$ up to some sign. Since $\nu$ is $\mathbb{Q}$-generic, the absolute values of these non-zero entries are distinct square roots of squarefree positive rational numbers, and the product of any non-empty subset of these absolute values is irrational.

For each $1 \leq i \leq n$, let $Z_i$ be the set of the absolute values of all non-zero entries in the $i$-th row of $A$, and note that the entries in the $i$-th row of $AB^T$ are elements of the number field $\mathbb{Q}[Z_i]$. By definition, the $(i, i)$-th entry of $AB^T$ equals $\sum G \tilde{\mathcal{L}}_{R_i}(G)$, where the sum is over all $G \in \mathcal{F}_{d-1}(St_\Delta(v_i))$. This sum is in particular a non-trivial $\mathbb{Q}$-linear combination of the elements in $\mathcal{S}_\nu$ (where the coefficients are in $\{-1, 0, 1\}$), so it must be non-zero by the $\mathbb{Q}$-genericity of $\nu$.

Next, we prove that the non-zero entries in each row of $AB^T$ are linearly independent over $\mathbb{Q}$. For each $1 \leq i \leq n$, let $\tilde{A}_i$ be the row vector consisting of all non-zero entries in the $i$-th row of $A$, and let $\tilde{B}_i$ be the vertex-ridge incidence matrix of $\tilde{S}_\Delta(v_i)$. Note that the entries in $\tilde{A}_i$ correspond to the $(d - 1)$-faces in $\tilde{S}_\Delta(v_i)$. By definition, the non-zero entries in the $i$-th row of $AB^T$, when written as a row vector, equals the product $\tilde{A}_i(\tilde{B}_i)^T$. Since $\tilde{S}_\Delta(v_i)$ is a pure strongly connected simplicial complex, Lemma 7.3 says that $\tilde{B}_i$ has full rank. Now, since the $\mathbb{Q}$-genericity of $\nu$ implies that the entries in $\tilde{A}_i$ are linearly independent over $\mathbb{Q}$, it then follows that the entries in $\tilde{A}_i(\tilde{B}_i)^T$ are linearly independent over $\mathbb{Q}$. Therefore, we can apply Lemma 7.5 and conclude that $AB^T$ is invertible.

**Theorem 7.7.** If $(\Delta, \nu)$ is a $\mathbb{Q}$-generic PL realization of an oriented simplicial $k$-homology manifold in $\mathbb{R}^d$, such that $H_1(\Delta; \mathbb{Z}/2\mathbb{Z}) = 0$ whenever $d \geq 2$, then for every non-trivial $b \in \Psi_d(\Delta, \nu)$, there exists some $a \in \Psi_1(\Delta, \nu)$ such that $ab$ is non-trivial.

**Proof.** By relabeling the vertices of $\Delta$ if necessary, assume that the last $d + 1$ vertices $v_{n-d}, \ldots, v_n$ form a $d$-face $Q$ of $\Delta$. Choose arbitrary $a \in \Psi_1(\Delta, \nu)$ and $b \in \Psi_d(\Delta, \nu)$, and let $\varphi : \text{Lift}(\Delta, \nu, Q) \rightarrow \Psi_1(\Delta, \nu)$ be the isomorphism of $\mathbb{R}$-vector spaces given in Theorem 6.7. By Theorem 6.8, $\{\varphi(\mu_1), \ldots, \varphi(\mu_{n-d-1})\}$ is a basis for $\Psi_1(\Delta, \nu)$, thus we can write $a = \alpha_1 \varphi(\mu_1) + \cdots + \alpha_{n-d-1} \varphi(\mu_{n-d-1})$ for some uniquely determined $\alpha_1, \ldots, \alpha_{n-d-1} \in \mathbb{R}$. For convenience, let $a$ denote the row vector $(\alpha_1, \ldots, \alpha_{n-d-1})$, and let $b$ denote the column vector $(b(v_1), \ldots, b(v_n))$. Notice that Corollary 6.11 yields $(ab)(\emptyset) = a \tilde{A} B^T b$, where $\tilde{A}$ is the $(n - d + 1) \times m$ submatrix of $A$ corresponding to the first $n - d - 1$ rows.

By the definition of a 1-stress, $b(v_{n-d-1+i})$ is a $\mathbb{Q}$-linear combination of $\{b(v_1), \ldots, b(v_{n-d-1})\}$ for all $1 \leq i \leq d + 1$. (The coefficients are rational since $\nu$ is rational.) Explicitly, if we define the matrix $W_Q := W_{\tilde{V}_i(\Delta) \setminus \{v_i\}}$, and let $w_{i,j}$ denote the $(i, j)$-th entry of the rational matrix $(W_Q^{-1})^T$, then $b(v_{n-d-1+i}) = -\left(\sum_{j=1}^{n-d-1} w_{i,j} b(v_j)\right)$ for all $1 \leq i \leq d + 1$.

Let $C'$ (resp. $\overline{C}$) be the matrix obtained by applying the following sequence of column operations to $AB^T$ (resp. $AB^T$): For each $1 \leq i \leq d + 1$, $1 \leq j \leq n - d - 1$, subtract $w_{i,j}$ times the $(n - d - 1 + i)$-th column from the $j$-th column. Let $C$ (resp. $\overline{C}$) be the principal submatrix of $C'$ (resp. $\overline{C}$) containing the first $(n - d - 1)$ rows and columns of $C'$ (resp. $\overline{C}$). Similar to how we showed that $AB^T$ is invertible if and only if $AB^T$ is invertible, we can also show that $C$ is invertible if and only if $\overline{C}$ is invertible.

In the proof of Proposition 7.6, we showed that $AB^T$ satisfies the conditions of Lemma 7.5. Since the non-zero entries in each row of $AB^T$ are linearly independent over $\mathbb{Q}$, and since every $w_{i,j}$ is rational, we infer that $\overline{C}$ also satisfies the conditions of Lemma 7.5. In particular, the diagonal entries of $\overline{C}$ are non-zero. Consequently, $\overline{C}$ also satisfies the conditions of Lemma 7.5, which implies $C$ is invertible.

Observe that $a \tilde{A} B^T b = a C b'$, where $b' = (b(v_1), \ldots, b(v_{n-d-1})) \in \mathbb{R}^{n-d-1}$ is a column vector. By assumption, the vertices of $\Delta$ are realized in general position, so $\{\nu(v_{n-d}), \ldots, \nu(v_n)\}$ is a linear basis of $\mathbb{R}^{d+1}$, and $\Psi_d(\Delta, \nu) \cong \mathbb{R}^{n-d-1}$, whereby each $b' \in \mathbb{R}^{n-d-1}$ uniquely determines $b$. Therefore, if $b$ is non-trivial, then $b' \neq 0$, and it follows from the invertibility of $C$ that we can always choose some $a \in \mathbb{R}^{n-d-1}$ (which corresponds uniquely to some $d$-stress $a$) such that $(ab)(\emptyset) = a C b' \neq 0$. \qed
8. Gorenstein property of the stress algebra

In this section, we shall establish a symmetry in the h-vector of the stress algebra. Specifically, if \((\Delta, \nu)\) is a \(\mathbb{Q}\)-generic PL realization of an orientable simplicial \(k\)-homology \(d\)-manifold in \(\mathbb{R}^d\), such that \(H_1(\Delta; \mathbb{Z}/2\mathbb{Z}) = 0\) whenever \(d \geq 2\), then we show that \(\Psi(\Delta, \nu)\) is an Artinian Gorenstein \(\mathbb{R}\)-algebra generated by the degree 1 elements, so in particular, the \(h\)-vector \((h_0, \ldots, h_{d+1})\) of \(\Psi(\Delta, \nu)\) satisfies \(h_k = h_{d+1-k}\) for all \(0 \leq k \leq d + 1\).

**Theorem 8.1.** Let \((\Delta, \nu)\) be a \(\mathbb{Q}\)-generic PL realization of an orientable simplicial \(k\)-homology \(d\)-manifold in \(\mathbb{R}^d\), such that \(H_1(\Delta; \mathbb{Z}/2\mathbb{Z}) = 0\) whenever \(d \geq 2\). Then for every \(1 \leq r \leq d + 1\) and every non-trivial \(b \in \Psi_{d+1-r}(\Delta, \nu)\), there exists some \(a \in \Psi_1(\Delta, \nu)\) such that \(ab\) is non-trivial.

**Proof.** The case \(r = d + 1\) is trivial (since \(\Psi_0(\Delta, \nu) \cong \mathbb{R}\)), while the case \(r = 1\) is proven in Theorem 7.7 so assume that \(2 \leq r \leq d\). Let \(b \in \Psi_{d+1-r}(\Delta, \nu)\) be non-trivial, and suppose \(b(F) \neq 0\) for some \(F \in \mathcal{F}_{r-1}(\Delta)\). Let \(G\) be any \((r-2)\)-face of \(\Delta\) contained in \(F\), and define the vertex \(v := F \cap G\). Note that \(\text{Lk}_\Delta(G)\) is a \(k\)-homology \((d+1-r)\)-sphere. Choose a distinguished ordered basis \(\mathcal{B}\) for \(\tilde{V}_G\) such that the PL realization \(\nu \mathcal{B}\) is \(\mathbb{Q}\)-generic; the existence of such a \(\mathcal{B}\) is implied by Proposition 7.3. Define the map \(b' : \text{Lk}_\Delta(G) \to \mathbb{R}\) by \(F' \mapsto b(G \cup F')\) for all \(F' \in \text{Lk}_\Delta(G)\). Theorem 4.2 implies that \(b' \in \Psi_{d+1-r}(\text{Lk}_\Delta(G), \nu_G)\) is a 1-stress. Since \(b'(v) = b(F) \neq 0\), it follows from Theorem 7.7 that there exists some \((d+1-r)\)-stress \(a' \in \Psi_1(\text{Lk}_\Delta(G), \nu_G)\) such that \(a'b'\) is non-trivial, i.e. \((a'b')(\emptyset) \neq 0\).

Define \(\tilde{a} : \text{St}_\Delta(G) \to \mathbb{R}\) by \(G \cup H \mapsto a'(H)\) for all \(H \in \text{Lk}_\Delta(G)\). Using Theorem 4.2 we check that \(\tilde{a}\) is a local \(d\)-stress of \((\Delta, \nu)\) on \(G\), and we get

\[
\sum_{(H, H') \in \mathcal{P}_G} \tilde{a}(H)b(H') = (a'b')(\emptyset) \neq 0.
\]

Fix some \(Q \in \mathcal{F}_d(\text{St}_\Delta(G))\). By Theorem 6.7 and Remark 6.10 there is an isomorphism

\[
\varphi_G : \text{Lift}(\text{St}_\Delta(G), \nu|_{\text{St}_\Delta(G)}, Q) \to \Psi_1(\text{St}_\Delta(G), \nu|_{\text{St}_\Delta(G)})
\]

of \(\mathbb{R}\)-vector spaces that extends to the isomorphism \(\varphi : \text{Lift}(\Delta, \nu, Q) \to \Psi_1(\Delta, \nu)\) given by (7).

Let \(\mu_G := \varphi^{-1}_G(\tilde{a})\). Following the notation in Section 6 note that the values of the scalars \(\pi(\mu_G(v)) \in \mathbb{R}\) for all \(v \in \mathcal{V}(\text{St}_\Delta(G))\). Now, choose an arbitrary assignment of real scalars to the remaining vertices in \(\mathcal{V}(\Delta) \setminus \mathcal{V}(\text{St}_\Delta(G))\), so that we get a lifting \(\mu \in \text{Lift}(\Delta, \nu, Q)\) satisfying \(\mu(H) = \mu_G(H)\) for all \(H \in \mathcal{F}_d(\text{St}_\Delta(G))\). Finally, define \(a = \varphi^{-1}(\mu) \in \Psi_1(\Delta, \nu)\). Note that \(a(H) = \tilde{a}(H)\) for all \(H \in \text{St}_\Delta(G)\), therefore (10) implies that \((ab)(\emptyset) \neq 0\).

For each \(1 \leq k \leq d + 1\), let \((\Psi_1(\Delta, \nu))^k \) denote the \(k\)-fold products of elements in \(\Psi_1(\Delta, \nu)\). Also, define \((\Psi_1(\Delta, \nu))^0 = \Psi_0(\Delta, \nu) \cong \mathbb{R}\).

**Corollary 8.2.** Let \((\Delta, \nu)\) be a \(\mathbb{Q}\)-generic PL realization of an orientable simplicial \(k\)-homology \(d\)-manifold in \(\mathbb{R}^d\), such that \(H_1(\Delta; \mathbb{Z}/2\mathbb{Z}) = 0\) whenever \(d \geq 2\). Then for every \(0 \leq r \leq d + 1\) and every non-trivial \(b \in \Psi_r(\Delta, \nu)\), there exists some \(a \in (\Psi_1(\Delta, \nu))^r \subseteq \Psi_{d+1-r}(\Delta, \nu)\) such that \(ab\) is non-trivial.

**Theorem 8.3.** If \((\Delta, \nu)\) is a \(\mathbb{Q}\)-generic PL realization of an orientable simplicial \(k\)-homology \(d\)-manifold in \(\mathbb{R}^d\), such that \(H_1(\Delta; \mathbb{Z}/2\mathbb{Z}) = 0\) whenever \(d \geq 2\), then \(\dim_\mathbb{R}(\Psi_r(\Delta, \nu)) = \dim_\mathbb{R}(\Psi_{d+1-r}(\Delta, \nu))\) and \(\Psi_r(\Delta, \nu) = (\Psi_1(\Delta, \nu))^r\) for all \(0 \leq r \leq d + 1\). In particular, the stress algebra \(\Psi(\Delta, \nu)\) is generated as an \(\mathbb{R}\)-algebra by \(\Psi_1(\Delta, \nu)\).

**Proof.** For any subspaces \(U \subseteq \Psi_k(\Delta, \nu)\), \(U' \subseteq \Psi_{d+1-k}(\Delta, \nu)\), we say that \(U\) separates \(U'\) if for every non-trivial \(b \in U\), there exists some \(a \in U\) such that \(ab\) is non-trivial. Note that if \(U\) separates \(U'\), then \(\dim_\mathbb{R}U' \leq \dim_\mathbb{R}U\). Given any \(0 \leq r \leq d + 1\), Corollary 8.2 says in particular that \(\Psi_{d+1-r}(\Delta, \nu)\) separates \(\Psi_r(\Delta, \nu)\), and that \((\Psi_1(\Delta, \nu))^r\) separates \(\Psi_{d+1-r}(\Delta, \nu)\). Now, since \((\Psi_1(\Delta, \nu))^r \subseteq \Psi_r(\Delta, \nu)\), a dimension count yields

\[
\dim_\mathbb{R}(\Psi_r(\Delta, \nu)) \leq \dim_\mathbb{R}(\Psi_{d+1-r}(\Delta, \nu)) \leq \dim_\mathbb{R}(\Psi_1(\Delta, \nu))^r \leq \dim_\mathbb{R}(\Psi_r(\Delta, \nu)).
\]

Thus, all dimensions must be equal, and we get \(\Psi_r(\Delta, \nu) = (\Psi_1(\Delta, \nu))^r\).
Given an Artinian graded \( \mathbb{R} \)-algebra \( A = \bigoplus_{i \geq 0} A_i \), that is generated (as an \( \mathbb{R} \)-algebra) by \( A_1 \), the socle of \( A \) (as an \( A \)-module) is \( \text{Soc} A := \{ a \in A | xa = 0 \ \forall x \in A_1 \} \). We say that \( A \) is \textit{Gorenstein} if \( \text{dim}_\mathbb{R}(\text{Soc} A) = 1 \). Note that \( \text{dim}_\mathbb{R}(\Psi_{d+1}(\Delta, \nu)) = 1 \), and that \( x\Psi_{d+1}(\Delta, \nu) = 0 \) for all \( x \in \Psi_1(\Delta, \nu) \), thus we have the following immediate consequence of Theorem 8.1 and Theorem 8.3.

\[ \text{Corollary 8.4. If } (\Delta, \nu) \text{ is a } \mathbb{Q}\text{-generic PL realization of an orientable simplicial } k\text{-homology d-manifold in } \mathbb{R}^d, \text{ such that } H_1(\Delta; \mathbb{Z}/2\mathbb{Z}) = 0 \text{ whenever } d \geq 2, \text{ then } \Psi(\Delta, \nu) \text{ is Gorenstein.} \]

9. Pivot-compatibility and the rigidity matrix

In this section, we shall introduce the notion of “pivot-compatibility” for simplicial complexes whose vertices are realized in general position. We begin with a brief overview of what it entails, and why it is important. Recall that the space of \((k + 1)\)-stresses on a PL realization of any simplicial \( d \)-complex \( \Delta \) in \( \mathbb{R}^d \) is isomorphic to the cokernel of some \((k + 1)\)-rigidity matrix. The rows of this rigidity matrix correspond bijectively to the \( k \)-faces of \( \Delta \), so if we reduce the transpose of this matrix to reduced row-echelon form, then the \( k \)-faces that correspond to the pivot columns of the resulting matrix in reduced row-echelon form would depend on some choice of a linear order on \( \mathcal{F}_k(\Delta) \). Roughly speaking, we want to choose a suitable linear order on \( \mathcal{F}_k(\Delta) \) so that the non-pivot columns correspond to a subset \( \mathcal{H} \subseteq \mathcal{F}_k(\Delta) \) that satisfies a certain “nice” property. We say that \( \mathcal{H} \) is “pivot-compatible” if such a suitable linear order on \( \mathcal{F}_k(\Delta) \) exists. The existence of a pivot-compatible \( \mathcal{H} \) will later be crucial in Section 10, where we use it to prove a technical result that our proof of the \( g \)-conjecture will subsequently rely on.

For the rest of this section, let \( 0 \leq k \leq d \), let \( \Delta \) be an arbitrary simplicial \( d \)-complex, and let \((\Delta, \nu)\) be a PL realization in \( \mathbb{R}^d \), such that the vertices of \( \Delta \) are realized in general position. Let \((f_0, \ldots, f_d)\) be the \( f \)-vector of \( \Delta \), and let \( N := \text{dim}_\mathbb{R}(\Psi_{d-k}(\Delta, \nu)) \).

Let \( \{x_H\}_{H \in \mathcal{F}_k(\Delta)} \) be a set of variables indexed by the \( k \)-faces of \( \Delta \), and consider the following system of vector equations

\[ (11) \sum_{v \in \mathcal{V}(\text{Lk}_\Delta(F))} \nu(v)x_{F \cup v} = 0, \]

where \( F \) ranges over all \((k - 1)\)-faces of \( \Delta \). Since every \( \nu(v) \) is in \( \mathbb{R}^{d+1} \), these vector equations are equivalent to a system of \((d + 1)f_{k-1}\) linear equations in the \( f_k \) variables \( \{x_H\}_{H \in \mathcal{F}_k(\Delta)} \). Given some choice of linear orders on \( \mathcal{F}_{k-1}(\Delta) \) and \( \mathcal{F}_k(\Delta) \) respectively, let \( \mathbf{R}_v^k \) be the coefficient matrix of this linear system (that is consistent with the given linear orders). Notice that \( \mathbf{R}_v^k \) is the transpose of the truncated face-ring rigidity matrix of \((\Delta, \nu)\) as defined in [54, Sec. 6]. In particular, [54, Prop. 6.1] says that the nullspace of \( \mathbf{R}_v^k \) is isomorphic to \( \Psi_{d-k}(\Delta, \nu) \) as \( \mathbb{R} \)-vector spaces.

Let \( \mathbf{R}_w^k \) be the reduction of \( \mathbf{R}_v^k \) to reduced row-echelon form. By assumption, the nullspace of \( \mathbf{R}_w^k \) has dimension \( N \). A \textit{linear order} on \( \mathcal{F}_k(\Delta) \) is a bijective map \( \phi : \mathcal{F}_k(\Delta) \rightarrow \{1, \ldots, f_k\} \). Given a linear order \( \phi \) on \( \mathcal{F}_k(\Delta) \) such that \( \phi^{-1}(i) = H_i \) for all \( 1 \leq i \leq f_k \), we say that \( \phi \) is \textit{pivotal} if the first \( f_k - N \) variables \( x_{H_1}, \ldots, x_{H_{f_k-N}} \) of the above linear system correspond to pivot columns of \( \mathbf{R}_w^k \), or equivalently, if \( \mathbf{R}_w^k \) has the matrix structure

\[ \mathbf{R}_w^k = \begin{bmatrix} \mathbf{I}_{f_k-N} & \tilde{\mathbf{R}}_w^k \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \]

where \( \mathbf{I}_{f_k-N} \) denotes the \((f_k-N)\)-by-\((f_k-N)\) identity matrix, \( \mathbf{0} \) denotes a (possibly empty) zero matrix of an appropriate size, and \( \tilde{\mathbf{R}}_w^k \) is a \((f_k-N)\)-by-\(N\) submatrix of \( \mathbf{R}_w^k \). For convenience, let \( \mathcal{H}[\phi] \) denote the set \( \{H_1, \ldots, H_{f_k-N}\} \) consisting of the first \((f_k-N)\) \( k \)-faces of \( \Delta \) (relative to \( \phi \)), and let \( \mathcal{H}[\phi] \) denote the set \( \{H_{f_k-N+1}, \ldots, H_{f_k}\} \) consisting of the last \( N \) \( k \)-faces of \( \Delta \) (relative to \( \phi \)). For any pivotal linear order \( \phi \) on \( \mathcal{F}_k(\Delta) \), we call the elements of \( \mathcal{H}[\phi] \) (resp. \( \mathcal{H}[\phi] \)) the \textit{pivot} \( k \)-\textit{faces} (resp. \textit{non-pivot} \( k \)-\textit{faces}) of \( \Delta \) (relative to \( \phi \)). For each \( i \in \{1, \ldots, f_k-N\} \) (resp. \( i \in \{1, \ldots, N\} \)), let \( \mathcal{H}_\phi(i) \) (resp. \( \mathcal{H}_\phi(i) \)) denote the \( i \)-th pivot (resp. non-pivot) \( k \)-face of \( \Delta \) relative to \( \phi \).
Let \( \phi \) be a pivotal linear order on \( \mathcal{F}_k(\Delta) \), and let \( x \in \Psi_{d-k}(\Delta, \nu) \). By definition, knowing the values of \( x(H) \) for all non-pivot \( k \)-faces \( H \in \mathcal{H}[\phi] \) would uniquely determine the values of \( x(\hat{H}) \) for all pivot \( k \)-faces \( \hat{H} \in \hat{\mathcal{H}}[\phi] \). Furthermore, the values of \( x(H) \) for non-pivot \( k \)-faces \( H \in \mathcal{H}[\phi] \) can be arbitrarily chosen. Since the vertices of \( \Delta \) are realized in general position, it follows from the vector equation (11) that for each \( F \in \mathcal{F}_{k-1}(\Delta) \), knowing the values of \( x(F \cup v) \) on \( |V(\text{Lk}_d(F))| - d - 1 \) vertices in \( V(\text{Lk}_d(F)) \) would uniquely determine the values of \( x(F \cup v) \) for all \( d + 1 \) remaining vertices in \( V(\text{Lk}_d(F)) \).

Given any \( H \in \mathcal{F}_k(\Delta) \), the pivotal weight of \( H \) relative to \( \mathcal{H}[\phi] \), which we denote by \( \text{wt}_{\mathcal{H}[\phi]}(H) \), is defined recursively to be the smallest integer \( t \in \{1, \ldots, N\} \) such that the value of \( x(H) \) can be uniquely determined once we know the values of \( x(H') \) for \( H' = \mathcal{H}_\phi(t) \) and for all \( H' \in \mathcal{F}_k(\Delta) \) satisfying \( 1 \leq \text{wt}_{\mathcal{H}[\phi]}(H') < t \). Equivalently, \( \text{wt}_{\mathcal{H}[\phi]}(H) \) is the smallest integer \( t \in \{1, \ldots, N\} \) such that knowing the values of \( x \) on the first \( t \) non-pivot \( k \)-faces (relative to \( \phi \)) would uniquely determine the value of \( x(H) \). Note in particular that \( \text{wt}_{\mathcal{H}[\phi]}(\mathcal{H}_\phi(t)) = t \) for all \( 1 \leq t \leq N \).

Also, given any \( F \in \mathcal{F}_{k-1}(\Delta) \), the pivotal subweight of \( F \) relative to \( \mathcal{H}[\phi] \), denoted by \( \text{subwt}_{\mathcal{H}[\phi]}(F) \), is the smallest integer \( t \in \{1, \ldots, N\} \) such that knowing the values of \( x \) on the first \( t \) non-pivot \( k \)-faces (relative to \( \phi \)) would uniquely determine the value of \( x(H) \) for every \( H \in \mathcal{F}_k(\Delta) \) containing \( F \). The following lemma is a direct consequence of the definitions of pivotal weights and pivotal subweights.

**Lemma 9.1.** For any pivotal linear order \( \phi \) on \( \mathcal{F}_k(\Delta) \), the following statements hold.

(i) If \( H \in \mathcal{H}[\phi] \), then \( \text{wt}_{\mathcal{H}[\phi]}(H) = \min\{\text{subwt}_{\mathcal{H}[\phi]}(F')|F' \in \mathcal{F}_{k-1}(\Delta), F' \subseteq H\} \).

(ii) If \( F \in \mathcal{F}_{k-1}(\Delta) \), then \( \text{subwt}_{\mathcal{H}[\phi]}(F) = \max\{\text{wt}_{\mathcal{H}[\phi]}(H')|H' \in \mathcal{F}_k(\Delta), F \subseteq H'\} \).

Given any permutations \( \tilde{\sigma} \) and \( \sigma \) on \( \{1, \ldots, f_k - N\} \) and \( \{1, \ldots, N\} \) respectively, let \( [\tilde{\sigma}, \sigma] : \mathcal{F}_k(\Delta) \rightarrow \{1, \ldots, f_k\} \) given by

\[
([\tilde{\sigma}, \sigma] \cdot \phi)^{-1}(i) = \begin{cases} 
\phi^{-1}(\tilde{\sigma}(i)), & \text{if } 1 \leq i \leq f_k - N; \\
\phi^{-1}(f_k - N + \sigma(i - f_k + N)), & \text{if } f_k - N < i \leq f_k.
\end{cases}
\]

Notice that \( [\tilde{\sigma}, \sigma] \cdot \phi \) is always a pivotal linear order on \( \mathcal{F}_k(\Delta) \) for all possible permutations \( \tilde{\sigma} \) and \( \sigma \). (A much more general theory on permutations of rows and columns of matrices in relation to pivot columns is known as “pivoting” in numerical linear algebra.)

Given any linear order \( \phi' \) on \( \mathcal{F}_k(\Delta) \), we say that \( \phi' \) is a pivotal reordering of \( \phi \) if \( \phi' = [\tilde{\sigma}, \sigma] \cdot \phi \) for some permutations \( \tilde{\sigma} \) and \( \sigma \) on \( \{1, \ldots, f_k - N\} \) and \( \{1, \ldots, N\} \) respectively.

For any \( \hat{H} \in \hat{\mathcal{H}}[\phi] \) satisfying \( \text{wt}_{\mathcal{H}[\phi]}(\hat{H}) = N \), define the bijective map \( \phi(\hat{H}) : \mathcal{F}_k(\Delta) \rightarrow \{1, \ldots, f_k\} \)
so that

\[
\phi(\hat{H})(H) := \begin{cases} 
\phi(\mathcal{H}_\phi(N)), & \text{if } H = \hat{H}; \\
\phi(\hat{H}), & \text{if } H = \mathcal{H}_\phi(N); \\
\hat{H}, & \text{otherwise}.
\end{cases}
\]

By the definition of a pivotal weight, the \( \phi(\hat{H}) \)-th column of \( \mathbb{R}^{f_k}_k \) is not contained in the subspace spanned by the first \( N - 1 \) columns of \( \mathbb{R}^{f_k}_k \), thus \( \phi(\hat{H}) \) must also be a pivotal linear order on \( \mathcal{F}_k(\Delta) \).

**Definition 9.2.** Given any set \( \hat{\mathcal{H}} \subseteq \mathcal{F}_k(\Delta) \), we say that \( \hat{\mathcal{H}} \) is pivot-compatible if \( \hat{\mathcal{H}} \subseteq \hat{\mathcal{H}}[\phi] \) for some pivotal linear order \( \phi \) on \( \mathcal{F}_k(\Delta) \), and for every \( \hat{H} \in \hat{\mathcal{H}} \), there exists some \( (k - 1) \)-face \( F \) contained in \( \hat{H} \) that is not contained in any other \( k \)-face in \( \hat{\mathcal{H}} \), i.e. if \( F \subseteq \hat{H} \) for some \( \hat{H}' \in \hat{\mathcal{H}} \), then \( \hat{H}' = \hat{H} \).

**Definition 9.3.** A subset of vertices \( \mathcal{A} \subseteq \mathcal{V}(\Delta) \) is called \( k \)-autonomous if for every \( k \)-face \( F \) of \( \Delta \) that does not contain any of the vertices in \( \mathcal{A} \), there exists some \( v \in \mathcal{A} \) such that \( v \) is a vertex of \( \text{Lk}_d(F) \).

By default, we define \( \mathcal{A} \) to be \((-1)\)-autonomous if and only if \( \mathcal{A} \) is non-empty.

**Theorem 9.4.** Let \( \mathcal{A} \subseteq \mathcal{V}(\Delta) \) be a \((k - 1)\)-autonomous set of vertices. Let \( F_1, \ldots, F_m \) be all the distinct \((k - 1)\)-faces of \( \Delta \) that do not contain any of the vertices in \( \mathcal{A} \), and for each \( 1 \leq i \leq m \), let \( v_i \) be any vertex in \( \mathcal{A} \) such that \( F_i \cup v_i \in \mathcal{F}_k(\Delta) \). (Such a vertex \( v_i \) exists since \( \mathcal{A} \) is \((k - 1)\)-autonomous.) Then the set \( \hat{\mathcal{H}} := \{F_i \cup v_1, \ldots, F_m \cup v_m\} \subseteq \mathcal{F}_k(\Delta) \) is pivot-compatible.
Proof. First of all, for each $1 \leq i \leq m$, note that $F_i \cup v_i$ is the only $k$-face in $\hat{H}$ that contains $F_i$. Let $\phi$ be a pivotal linear order on $\mathcal{F}_k(\Delta)$ such that the value $|\hat{H} \cap \mathcal{H}[\phi]|$ is minimized over all possible pivotal linear orders on $\mathcal{F}_k(\Delta)$. Suppose $|\hat{H} \cap \mathcal{H}[\phi]| \geq 1$, and let $H \in \hat{H} \cap \mathcal{H}[\phi]$. Without loss of generality, assume that $H = F_i \cup v_i$. Since $\mathcal{H}[\phi'] = \mathcal{H}[\phi]$ for all pivotal reorderings $\phi'$ of $\phi$, we can assume that $\text{wt}_{\mathcal{H}[\phi]}(H) = N$. By Lemma 9.4.11, $\text{subwt}_{\mathcal{H}[\phi]}(F_i) = N$, thus there exist at least $d$ $k$-faces of $\Delta$ distinct from $H$ that contain $F_i$ and have pivotal weight $N$ relative to $\phi$.

Let $\hat{H}$ be one such $k$-face. Since $\text{wt}_{\mathcal{H}[\phi]}(\hat{H}) = N$ and $\hat{H} \not\in H$, we get $\hat{H} \in \hat{H}[\phi]$, so we can obtain a new pivotal linear order $\phi(\hat{H})$ from $\phi$. Now, $H$ is the only $k$-face in $\hat{H}$ that contains $F_i$, thus $\hat{H} \not\in \hat{H}$, and we have $|\hat{H} \cap \mathcal{H}[\phi(\hat{H})]| < |\hat{H} \cap \mathcal{H}[\phi]|$, which contradicts the minimality of $|\hat{H} \cap \mathcal{H}[\phi]|$. Therefore, $|\hat{H} \cap \mathcal{H}[\phi]| = 0$, i.e. $\hat{H} \subseteq \hat{H}[\phi]$, and so $\hat{H}$ is pivot-compatible. \hfill $\square$

10. STRESSES ON CONES OF HOMOLOGY SPHERES

Our strategy for proving the $g$-conjecture for $\mathbb{R}$-homology spheres is to look at the stresses on cones of $\mathbb{R}$-homology spheres instead, which (fortunately for us) is easier to work with. In this section, we shall construct a map on the $k$-stresses on the cone of an $\mathbb{R}$-homology $d$-sphere, and prove that if our PL realization is "sufficiently generic", then this map is injective for all $k > \lceil \frac{d+1}{2} \rceil$ (see Theorem 10.5). Our proof on the injectivity of this map is rather technical and will later (in Section 11) play a crucial role in establishing that the stress algebra of an $\mathbb{R}$-homology sphere has the weak Lefschetz property.

Homology spheres and cones on them are examples of Cohen–Macaulay complexes. Recall that a simplicial $d$-complex is called $k$-Cohen–Macaulay if $H_i(\text{Lk}_d(\Delta); k) = 0$ for all $F \in \Delta$ and all $0 \leq i < \dim(\text{Lk}_d(\Delta))$. Here, $H_i(\text{Lk}_d(\Delta); k)$ denotes the $i$-th reduced homology group of $\text{Lk}_d(\Delta)$ with coefficients in $k$.

**Theorem 10.1** ([24] 56). Let $(\Delta, \nu)$ be a PL realization of a simplicial $d$-complex in $\mathbb{R}^d$, such that the vertices of $\Delta$ are realized in general position. If $\Delta$ is $\mathbb{R}$-Cohen–Macaulay, and if $(h_0, \ldots, h_{d+1})$ is the $h$-vector of $\Delta$, then $h_i = \dim \mathbb{R}(\psi_{d+1-i}(\Delta, \nu))$ for all $0 \leq i \leq d+1$.

For the rest of this section, fix some integers $1 \leq k \leq d$, let $\Delta$ be a simplicial $\mathbb{R}$-homology $d$-sphere, and fix some $d$-face $Q$ of $\Delta$. Also, let $\Delta'$ be a cone on $\Delta$ with conepoint $a$, and note that $\dim(\Delta') = d+1$. For convenience, let $\mathcal{Q}_{\text{PL}}^{\text{gen}}(\Delta)$ (resp. $\mathcal{Q}_{\text{PL}}^{\text{gen}}(\Delta')$) denote the set of all $\mathcal{Q}$-generic PL realizations of $\Delta$ in $\mathbb{R}^d$ (resp. $\Delta'$ in $\mathbb{R}^{d+1}$).

Consider arbitrary PL realizations $\nu \in \mathcal{Q}_{\text{PL}}^{\text{gen}}(\Delta)$ and $\nu' \in \mathcal{Q}_{\text{PL}}^{\text{gen}}(\Delta')$, and let $\phi$ be a pivotal linear order on $\mathcal{F}_k(\Delta')$ that corresponds to $\nu'$. Recall from Section 9 that $\hat{R}_{\nu'}^k$ is the transpose of the truncated face-ring rigidity matrix of $(\Delta', \nu')$, the null-space of $\hat{R}_{\nu'}^k$ is isomorphic to $\mathcal{Q}_{d+1-k}(\Delta', \nu')$ as $\mathbb{R}$-vector spaces, and $\hat{R}_{\nu'}^k$ is the reduction of $R_{\nu'}^k$ to reduced row-echelon form, i.e. $\hat{R}_{\nu'}^k$ has the matrix structure

$$
\hat{R}_{\nu'}^k = \begin{bmatrix} I & \tilde{R}_{\nu'}^k \\ 0 & 0 \end{bmatrix}.
$$

Since $\Delta'$ is $\mathbb{R}$-Cohen–Macaulay, it follows from Theorem 10.1 that the nullspace of $\tilde{R}_{\nu'}^k$ has dimension $h_{k+1}(\Delta')$, hence $\hat{R}_{\nu'}^k$ is a $(f_k(\Delta') - h_{k+1}(\Delta'))$-by-$h_{k+1}(\Delta')$ submatrix of $R_{\nu'}^k$. Note that $\hat{R}_{\nu'}^k$ has rational entries, since $\nu'$ is rational. Conversely, every $(f_k(\Delta') - h_{k+1}(\Delta'))$-by-$h_{k+1}(\Delta')$ matrix with rational entries equals $\hat{R}_{\nu'}^k$, for some rational PL realization $\nu''$ on $\Delta'$. Thus by Proposition 7.3, the set $\mathcal{Y} := \{\hat{R}_{\nu'}^k | \nu' \in \mathcal{Q}_{\text{PL}}^{\text{gen}}(\Delta')\}$ is dense among arbitrary $(f_k(\Delta') - h_{k+1}(\Delta'))$-by-$h_{k+1}(\Delta')$ matrices with rational entries.

Recall that $\hat{H}[\phi]$ and $\mathcal{H}[\phi]$ are the sets of pivot $k$-faces and non-pivot $k$-faces of $\Delta'$ respectively (relative to $\phi$), and that $\mathcal{H}[\phi](t)$ denotes the $t$-th non-pivot $k$-face of $\Delta'$ relative to $\phi$. For each $(\hat{H}, H) \in \hat{H}[\phi] \times \mathcal{H}[\phi]$ such that $H = \mathcal{H}[\phi](t)$, let $\nu''_{\hat{H}, H}$ be the $(\phi(\hat{H}), t)$-th entry of $\hat{R}_{\nu'}^k$. Let $\mathcal{E}$ be the $\mathbb{Q}$-vector space isomorphic to $Q_{f_0(\Delta')-d-1}$ that contains all $\mathbb{Q}$-valued maps on $V(\Delta) \setminus V(Q)$, and let $\Xi_0$ be the
dense subset
\[ \Xi_0 := \left\{ w \in \Xi \mid w(v) \neq 0 \text{ for all } v \in V(\Delta) \setminus V(Q) \right\} \subseteq \Xi. \]
Since \( Q \) is a fixed \( d \)-face of \( \Delta \), each choice of \( \nu \in Q_{PL}^{\text{gen}}(\Delta) \) would uniquely determine a PL orientation \( \rho_\nu \) on \( \Delta \) that satisfies \( \rho_\nu(Q) = 1 \). For each \( w \in \Xi \) and \( G \in \mathcal{F}_{d-1}(\Delta) \), define the scalar
\[ \zeta_\nu^w(G) := \sum_{v \in V(S\Delta(G))} w(v) \zeta(\nu,v), \]
where we note that \( \zeta_\nu^w(G) \) was previously already defined in \([8]\).

Next, for every pair \((F,H) \in \mathcal{F}_{k-1}(\Delta') \times \mathcal{H}[\phi]\), and every \( w \in \Xi \), define
\[ \xi_{w,F,H} := \sum_{\nu \in Q_{PL}(\Delta')} \left( \sum_{H \subseteq F} \sum_{\phi \in \mathcal{Q}^{-\Delta}} -r_{\nu,F}^H \zeta_\nu^w(G) \right) \]
if \( F \neq H \), and define
\[ \xi_{(F,H)} := \left( \sum_{G \in \mathcal{F}_{d-1}(\Delta)} \zeta_\nu^w(G) \right) + \left( \sum_{H \in \mathcal{H}[\phi]} \sum_{G \subseteq F} \sum_{\nu \in Q_{PL}(\Delta')} -r_{\nu,H}^F \zeta_\nu^w(G) \right) \]
if \( F < H \). Notice that if \( F \in \mathcal{F}_{k-1}(\Delta') \setminus \mathcal{F}_{k-1}(\Delta) \), then \( \xi_{w,F,H} = 0 \). Our motivation for defining \( \xi_{w,F,H} \) will later become apparent in the proof of Theorem \([11,3]\).

Given any \( \nu \in Q_{PL}^{\text{gen}}(\Delta) \), \( \nu' \in Q_{PL}^{\text{gen}}(\Delta') \), \( w \in \Xi \), any pivotal linear order \( \phi \) on \( \mathcal{F}_{k}(\Delta') \) that corresponds to \( \nu' \), and any non-empty subset \( \mathcal{H} \subseteq \mathcal{H}[\phi] \), let \( M_{\nu,H}^{\nu',\mathcal{W}} \) denote the \( f_{k-1}(\Delta') \)-by-\( |\mathcal{H}| \) matrix whose entries are indexed by pairs \((F,H) \in \mathcal{F}_{k-1}(\Delta') \times \mathcal{H} \), such that the \((F,H)\)-th entry of \( M_{\nu,H}^{\nu',\mathcal{W}} \) equals \( \zeta_{\nu,H}^{F,H} \).

Notice that every entry of \( M_{\nu,H}^{\nu',\mathcal{W}} \) is a \( Q \)-linear combination of the irrational scalars in \( S_{\nu} \). Given any non-empty subset \( \mathcal{F} \subseteq \mathcal{F}_{k-1}(\Delta') \), let \( M_{\nu,\mathcal{F}}^{\nu',\mathcal{W}} \) denote the \(|\mathcal{F}|\)-by-\( |\mathcal{H}| \) submatrix of \( M_{\nu,H}^{\nu',\mathcal{W}} \) induced by the rows indexed by \( \mathcal{F} \).

We shall prove a technical lemma that says \( M_{\nu,H}^{\nu',\mathcal{W}} \) has full rank when \( w \in \Xi_0 \) and \( \nu' \) is “sufficiently generic” (Lemma \([11,4]\), where “sufficiently generic” has a precise meaning that we shall determine. To state this technical lemma, we need the following definitions. (By default, we define the empty product to be equal to 1 in \( Q \).)

**Definition 10.2.** Let \( \nu \in Q_{PL}^{\text{gen}}(\Delta) \), and define \( G_{\nu,Q}(\Delta) := \{(G,v) \mid G \in \mathcal{F}_{d-1}(\Delta), v \in V(S\Delta(G)) \setminus V(Q)\} \). Given any real number \( \xi \in Q[S_{\nu}] \), we say that the \( \nu \)-support of \( \xi \), denoted by \( \text{supp}_\nu(\xi) \), is the set \( T \) of (possibly empty) subsets of \( G_{\nu,Q}(\Delta) \), such that \( \xi \) can be written as
\[ \xi = \sum_{A \subseteq T} r_A \prod_{(G,v) \in A} \zeta_{\nu,H}^{F,H} \]
for some collection of non-zero rational scalars \( \{r_A \mid A \in T\} \). (By Lemma \([7,2]\) the scalars \( r_A \) are uniquely determined once \( \nu \) and \( \xi \) are fixed.) For each \( A \in \text{supp}_\nu(\xi) \), we say that \( r_A \) is the rational coefficient of \( A \) in \( \xi \) relative to \( \nu \), while for each subset \( A' \) of \( G_{\nu,Q}(\Delta) \) not in \( \text{supp}_\nu(\xi) \), we say that \( 0 \) is the rational coefficient of \( A' \) in \( \xi \) relative to \( \nu \). Observe that \( \text{supp}_\nu(\xi) = \{\emptyset\} \) if and only if \( \xi \) is a non-zero rational number, while \( \text{supp}_\nu(\xi) = \emptyset \) if and only if \( \xi = 0 \).

Let \( X = \{x_{\tilde{H},H} \mid (\tilde{H},H) \in \tilde{H}[\phi] \times \mathcal{H}[\phi]\} \) be a set of doubly indexed variables, and let \( Q[X] \) be the ring of polynomials on the variables in \( X \) with coefficients in \( Q \). For any polynomial \( q = q(X) \in Q[X] \), we write \( q(x_{\tilde{H},H} = r_{\nu,H}^{\nu'} \) to denote the evaluation of \( q \) on the values \( x_{\tilde{H},H} = r_{\nu,H}^{\nu'} \) for all \((\tilde{H},H)\) in \( \tilde{H}[\phi] \times \mathcal{H}[\phi]\). Given non-empty subsets \( \mathcal{F} \subseteq \mathcal{F}_{k-1}(\Delta) \), \( \mathcal{H} \subseteq \mathcal{H}[\phi] \) satisfying \(|\mathcal{F}| = |\mathcal{H}|\), any map \( w \in \Xi \),
and any subset $\mathcal{A} \subseteq \mathcal{G}_{\mathcal{V}}(\Delta)$, let $q^{A,w}_{F,H} = q^{A,w}_{F,H}(X)$ denote the uniquely determined polynomial in $\mathbb{Q}[X]$, such that for every $\nu \in \mathbb{Q}^{\operatorname{gen}}_{\mathcal{V}}(\Delta)$ and $\nu' \in \mathbb{Q}^{\operatorname{gen}}_{\mathcal{V}}(\Delta')$, the evaluation $q^{A,w}_{F,H}(x_{\hat{H},H} = r^{\nu'}_{\hat{H},H})$ equals the rational coefficient of $\mathcal{A}$ in $\det(M^{\nu,\nu',w}_{F,H})$ relative to $\nu$.

**Lemma 10.3.** Let $m \geq 1$ be an integer, let $\nu \in \mathbb{Q}^{\operatorname{gen}}_{\mathcal{V}}(\Delta)$, $\nu' \in \mathbb{Q}^{\operatorname{gen}}_{\mathcal{V}}(\Delta')$, $w \in \mathbb{Z}$, and let $\phi$ be a pivoting linear order on $\mathcal{F}_k(\Delta')$ that corresponds to $\nu'$. Also, let $F = \{F_1, \ldots, F_m\} \subseteq \mathcal{F}_{k-1}(\Delta)$, let $\mathcal{H} = \{H_1, \ldots, H_m\} \subseteq \mathcal{H}[\phi]$, and let $\hat{H}_1, \ldots, \hat{H}_m$ be a sequence of (not necessarily distinct) pivot $k$-faces in $\hat{H}[\phi]$. If $A = \{(G_1, v_1), \ldots, (G_m, v_m)\} \subseteq \mathcal{G}_{\mathcal{V}}(\Delta)$ such that the coefficient of the monomial $q^{A,w}_{F,H}(X)$ is non-zero, then there exists some permutation $\sigma$ on $\{1, \ldots, m\}$ such that $F_i \subseteq G_{\sigma(i)}$ for all $1 \leq i \leq m$. 

**Proof.** This follows from the Leibniz formula for determinants applied to $M^{\nu,\nu',w}_{F,H}$, as well as from the definitions of $\xi_{(F,H)}^{\nu,\nu',w}$ and $q^{A,w}_{F,H}(X)$. \hfill \Box

**Lemma 10.4.** Let $m \geq 1$, let $\nu \in \mathbb{Q}^{\operatorname{gen}}_{\mathcal{V}}(\Delta)$, let $w \in \mathbb{Z}$, and let $F = \{F_1, \ldots, F_n\} \subseteq \mathcal{F}_{k-1}(\Delta)$. Also, let $\phi$ be a pivoting linear order on $\mathcal{F}_k(\Delta')$ such that $\{F_1 \cup a, \ldots, F_n \cup a\} \subseteq \hat{H}[\phi]$. (The existence of $\phi$ is implied by Theorem 9.4, since $\{a\}$ is $(k-1)$-autonomous.) Suppose that $\mathcal{H} := \{H_1, \ldots, H_m\} \subseteq \mathcal{H}[\phi]$, and that $A = \{(G_1, v_1), \ldots, (G_m, v_m)\} \subseteq \mathcal{G}_{\mathcal{V}}(\Delta)$ satisfies $F_i \subseteq G_{\sigma(i)}$ for each $1 \leq i \leq m$. (In particular, $r^{w}_{A}$ is a non-zero rational scalar, and $p_A$ is a positive integer.)

**Proof.** For any $\nu' \in \mathbb{Q}^{\operatorname{gen}}_{\mathcal{V}}(\Delta')$, note that $q^{A,w}_{F,H}(x_{\hat{H},H} = r^{\nu'}_{\hat{H},H})$ is by definition the rational coefficient of $\mathcal{A}$ in $\det(M^{\nu,\nu',w}_{F,H})$ relative to $\nu$. If statement [ii] holds, then the coefficient of $\prod_{i=1}^{m} x_{F_i \cup a, H_i}$ in the polynomial $q^{A,w}_{F,H}(X)$ is non-zero, so in particular, $q^{A,w}_{F,H}(X)$ is not the zero polynomial. This implies that the subset $\mathcal{X} \subseteq \mathcal{X}$, defined by the condition that the polynomial equation $q^{A,w}_{F,H}(x_{\hat{H},H} = r^{\nu'}_{\hat{H},H}) \neq 0$ holds for all $\nu' \in \mathbb{Q}^{\operatorname{gen}}_{\mathcal{V}}(\Delta')$ satisfying $\hat{R}_{\nu'} \in \mathcal{X}$, is a Zariski dense subset of $\mathcal{X}$. Consequently, statement [ii] implies statement [i] by induction on $m$. The base case $m = 1$ trivially follows from the definitions of $\xi_{(F,H)}^{\nu,\nu',w}$ and $\Xi_0$, so assume that $m > 1$. Let $\mathcal{H}' := \{H_1, \ldots, H_{m-1}\}$, and for each $1 \leq j \leq m$, define $\mathcal{A}_j := \mathcal{A}\setminus\{(G_j, v_j)\}$. By the cofactor expansion along the $m$-th column,

$$\det(M^{\nu,\nu',w}_{F,H}) = \sum_{i=1}^{m} (-1)^{m+i} \xi_{(F_i, H_m)}^{\nu,\nu',w} \det(M^{\nu,\nu',w}_{F_i, H_m}),$$

thus the rational coefficient of $\mathcal{A}$ in $\det(M^{\nu,\nu',w}_{F,H})$ relative to $\nu$ equals

$$\sum_{i=1}^{m} (-1)^{m+i} \sum_{j=1}^{m} q^{\nu,\nu',w}_{\{F_j, \{H_m\}\}}(x_{\hat{H},H} = r^{\nu'}_{\hat{H},H}) q^{A,w}_{F_j, H_m, H'}(x_{\hat{H},H} = r^{\nu'}_{\hat{H},H}).$$

For each $1 \leq i \leq m$, let $\hat{H}_i := F_i \cup a$, and let

$$\hat{q}_i(X) := \sum_{j=1}^{m} q^{\nu,\nu',w}_{\{F_j, \{H_m\}\}}(X) q^{A,w}_{F_j, H_m, H'}(X).$$

Next, let $j = j_1, \ldots, j_s$ be all the indices in $\{1, \ldots, m\}$ that satisfy all of the following conditions.
(i) $F_m \subseteq G_j$.
(ii) There exists a bijective map $\sigma_j : \{1, \ldots, m-1\} \to \{1, \ldots, m\}\setminus\{j\}$ such that $F_i \subseteq G_{\sigma_j(i)}$ for all $1 \leq i \leq m-1$.

In particular, $m$ is among these indices $j_1, \ldots, j_s$, so $s \geq 1$. By Lemma 10.3, the coefficient of $x_{\tilde{H}_m, H_m}$ in the polynomial $q_{\{(G_j, v_j)\}; (F_m, H_m)}(X)$ is non-zero only if condition (3) holds, while the coefficient of $m_{i=1}^{m-1} x_{\tilde{H}_i, H_i}$ in the polynomial $q_{\{(G_j, v_j)\}; (F_m, H_m')}(X)$ is non-zero only if condition (2) holds, thus it follows from (13) that the coefficient of $m_{i=1}^{m} x_{\tilde{H}_i, H_i}$ in $\tilde{q}_m(X)$ is identical to the coefficient of $m_{i=1}^{m} x_{\tilde{H}_i, H_i}$ in the following polynomial:

\[(14) \sum_{t=1}^{s} \left[q_{\{(G_j, v_j)\}; (F_m, H_m)}(X)\right] \left[q_{\mathcal{F}\setminus\{(F_m, H_m')}(X)\right].\]

For every $1 \leq t \leq s$, notice that
\[q_{\{(G_j, v_j)\}; (F_m, H_m)}(x_{\tilde{H}_t, H_t}) = r_t^{m'},\]

is the rational coefficient of $\{(G_j, v_j)\}$ in $\xi_{\mathcal{F}\setminus\{(F_m, H_m')}$ relative to $v_t$, thus the definition of $\xi_{\mathcal{F}\setminus\{(F_m, H_m')}$ implies that the coefficient of $\tilde{x}_{\tilde{H}_m, H_m}$ in the linear polynomial $q_{\{(G_j, v_j)\}; (F_m, H_m)}(X)$ is the non-zero rational scalar $-w(v_j)$. Let $X_{H_m} := \{x_{\tilde{H}_t, H_t} \in X | H = H_m\}$, $\tilde{X}_{H_m} := \{x_{\tilde{H}_t, H_t} \in X | H \neq H_m\}$ be subsets of $X$, and note that $X = X_{H_m} \cup \tilde{X}_{H_m}$. Also, let $-1 \in \mathbb{Z}$ be the map such that $-1(v) = -1$ for all $v \in \mathcal{V}(\Delta) \setminus \mathcal{V}(Q)$. Treat $\tilde{q}_m(X)$ as a polynomial in the variables in $X_{H_m}$ with coefficients in $\mathbb{Q}[\tilde{H}_{H_m}]$. (Here, $\mathbb{Q}[X_{H_m}]$ denotes the polynomial ring on the set of variables $X_{H_m}$ with coefficients in $\mathbb{Q}$.) Then from (13), the coefficient (contained in $\mathbb{Q}[\tilde{X}_{H_m}]$) of $\tilde{x}_{\tilde{H}_m, H_m}$ in $\tilde{q}_m(X)$ is the polynomial

\[(15) (-1)^{m-r_{A'}} \left(\sum_{t=1}^{s} q_{\mathcal{F}\setminus\{(F_m, H_m')(X)\}\}} \in \mathbb{Q}[\tilde{X}_{H_m}]\]

of degree $m-1$.

Suppose $1 \leq t \leq s$. By induction hypothesis, the coefficient of $m_{i=1}^{m-1} x_{\tilde{H}_i, H_i}$ in $q_{\mathcal{F}\setminus\{(F_m, H_m')(X)$ is a positive integer that equals the number of bijective maps $\sigma : \{1, \ldots, m-1\} \to \{1, \ldots, m\}\setminus\{j_i\}$ satisfying $F_i \subseteq G_{\sigma(j_i)}$ for all $1 \leq i \leq m-1$. This coefficient is positive, since by assumption, there is a bijective map $\sigma_i : \{1, \ldots, m-1\} \to \{1, \ldots, m\}\setminus\{j_i\}$ such that $F_i \subseteq G_{\sigma_i(j_i)}$ for all $1 \leq i \leq m-1$. Thus, it follows from (15) that the coefficient of the monomial $m_{i=1}^{m} x_{\tilde{H}_i, H_i}$ in $\tilde{q}_m(X)$ (as a polynomial in variables in $X$ with rational coefficients) equals $(-1)^{m-r_{A'}}$. For every $1 \leq t' \leq m-1$, note that $F_{t'} \neq F_{m-t} = \tilde{H}_m$. Observe that if we treat $\tilde{q}_{t'}(X)$ as a polynomial in the variables in $X_{H_m}$ with coefficients in $\mathbb{Q}[\tilde{X}_{H_m}]$, then any variable $x_{\tilde{H}_m, H_m}$ with a non-zero coefficient in $\tilde{q}_{t'}(X)$ must necessarily satisfy $F_{t'} \notin \tilde{H}_m$. Consequently, the coefficient of the monomial $m_{i=1}^{m} x_{\tilde{H}_i, H_i}$ in $\tilde{q}_{t'}(X)$ (as a polynomial in variables $X$ with rational coefficients) equals 0. Finally, from (12), we conclude that the coefficient of $m_{i=1}^{m} x_{\tilde{H}_i, H_i}$ in $q_{\mathcal{F}\setminus\{(F_m, H_m')(X)} equals $(-1)^{m+m} (-1)^{m-r_{A'}} = (-1)^{m-r_{A'}}$. □

**Theorem 10.5.** Let $w \in \Xi_0$ and $\nu \in \mathbb{Q}_{PL}^{\text{gen}}(\Delta')$. There exists a dense subset $\tilde{\Psi}$ of $\mathbb{Q}_{PL}^{\text{gen}}(\Delta')$ such that for every PL realization $(\Delta', \nu') \in \mathbb{R}^{d+1}$ satisfying $\nu' \in \tilde{\Psi}$, the map $\varphi'_{w, \nu} : \Psi(\Delta', \nu') \to \text{Hom}(\Delta', \mathbb{R})$ defined by

\[x \mapsto \begin{cases} \sum_{G \in \mathcal{F}_{d+1}(\Delta')} \zeta_w(G)x(H), & \text{if } F \in \mathcal{F}_{d-r}(\Delta'); \\ 0, & \text{if } F \notin \mathcal{F}_{d-r}(\Delta'); \end{cases} \]

is injective for all $1 \leq r \leq \left\lfloor \frac{d+1}{2} \right\rfloor$. 

Proof. For each $1 \leq k \leq d$, note that $\{a\} \subseteq \mathcal{V}((\Delta'))$ is $(k-1)$-autonomous, thus by Theorem 9.4 the set $\hat{\mathcal{H}}_k := \{F \cup a | F \in \mathcal{F}_{k-1}(\Delta)\} \subseteq \mathcal{F}_k(\Delta')$ is pivot-compatible. In particular, there exists a pivotal linear order $\phi_k$ on $\mathcal{F}_k(\Delta')$ such that $\hat{\mathcal{H}}_k \subseteq \hat{\mathcal{H}}[\phi_k]$. Observe that an equivalent reformulation of (16) yields

\[
|\hat{\mathcal{H}}_k| = f_{k-1}(\Delta) = \sum_{i=0}^{k} \frac{d + 1 - i}{d + 1 - k} h_i(\Delta).
\]

Note also that $\Delta$ is $\mathbb{R}$-Cohen–Macaulay, which implies $h_i(\Delta) \geq 0$ for all $0 \leq i \leq d + 1$ (see [51 Thm. II.3.3]). If $\left\lceil \frac{d+1}{2} \right\rceil \leq k \leq d$, then $0 \leq d - k \leq k$, thus it follows from [16], Theorem 8.3 and Theorem 10.1 that

\[
|\hat{\mathcal{H}}_k| \geq h_{d-k}(\Delta) = h_{k+1}(\Delta) = h_{k+1}(\Delta').
\]

Consider an arbitrary integer $k$ satisfying $\left\lceil \frac{d+1}{2} \right\rceil \leq k \leq d$. Let $F_1, \ldots, F_{h_k(\Delta')}$ be an enumeration of all $(k-1)$-faces in $\Delta$, and let $\mathcal{F}_k := \{F_1, \ldots, F_{h_k(\Delta')}\} \subseteq \mathcal{F}_{k-1}(\Delta)$ be the ordered set consisting of the first $h_k(\Delta')$ $(k-1)$-faces in $\Delta$. Then by Lemma 10.2(1), there exists a dense subset $\Upsilon'_k$ of $\{\hat{R}_\nu | \nu' \in Q_{PL}^\text{gen}(\Delta')\}$ such that for every $\nu' \in Q_{PL}^\text{gen}(\Delta')$ satisfying $\hat{R}_\nu' \in \Upsilon'_k$, and every $\mathcal{A}_k := \{(G_1, v_1), \ldots, (G_{h_{k+1}(\Delta')}, v_{h_{k+1}(\Delta')})\} \subseteq \mathcal{G}_Q(\Delta)$ satisfying $F_i \subseteq G_i$ for all $1 \leq i \leq h_k(\Delta')$, the $\nu'$-support of $\det(M_{\hat{F}_k, \mathcal{H}[\phi_k]})$ contains $\mathcal{A}_k$. Since $\nu$ is $\mathbb{Q}$-generic (cf. Lemma 7.1), we would then get that $\det(M_{\hat{F}_k, \mathcal{H}[\phi_k]}) \neq 0$, hence $M_{\hat{F}_k, \mathcal{H}[\phi_k]}$ would be a matrix of rank $h_{k+1}(\Delta')$ (i.e. of full rank). Now, define

\[
\tilde{\Upsilon} := \{\nu' \in Q_{PL}^\text{gen}(\Delta') | \hat{R}_\nu' \in \Upsilon'_k \text{ for all } \left\lceil \frac{d+1}{2} \right\rceil \leq k \leq d\}.
\]

The intersection of finitely many dense subsets is dense, thus $\tilde{\Upsilon}$ is a dense subset of $Q_{PL}^\text{gen}(\Delta')$. Henceforth, we shall fix some $\nu' \in \tilde{\Upsilon}$.

Let $1 \leq r \leq \left\lfloor \frac{d+1}{2} \right\rfloor$, and let $x \in \Psi_r(\Delta', \nu')$. Given any $F \in \mathcal{F}_{d-r}(\Delta')$, it follows from definition that

\[
(\varphi^{\nu',w}(x))(F) = \sum_{\begin{array}{c} G \in \mathcal{F}_{d-1}(\Delta) \\ H \in \mathcal{F}_{d-r+1}(\Delta') \\ G \cap H = F \end{array}} \zeta_{\nu}(G)x(H).
\]

By the definition of the pivotal linear order $\phi_{d-r+1}$ on $\mathcal{F}_{d-r+1}(\Delta')$, the value of $x(\hat{H})$ on every pivot $(d-r+1)$-face $\hat{H} \in \hat{\mathcal{H}}[\phi_{d-r+1}]$ is completely determined once the values of $x(H)$ on all non-pivot $(d-r+1)$-faces $H \in \mathcal{H}[\phi_{d-r+1}]$ are known. In particular,

\[
x(\hat{H}) = \sum_{H \in \mathcal{H}[\phi_{d-r+1}]} -\nu_{\hat{H},H}',x(H)
\]

for every $\hat{H} \in \hat{\mathcal{H}}[\phi_{d-r+1}]$. Thus, (17) and (18) together imply that

\[
(\varphi^{\nu',w}(x))(F) = \sum_{H \in \mathcal{H}[\phi_{d-r+1}]} \zeta_{\nu}(F,H)x(H)
\]

for all $F \in \mathcal{F}_{d-r}(\Delta')$.

Since $1 \leq r \leq \left\lfloor \frac{d+1}{2} \right\rfloor$ and only if $\left\lceil \frac{d+1}{2} \right\rceil \leq d - r + 1 \leq d$, it then follows from the definition of $\tilde{\Upsilon}$ that the matrix $M^{\nu',w}_{\mathcal{H}[\phi_{d-r+1}]}$ has rank $h_{d-r+2}(\Delta')$ (i.e. full rank). Consequently, it follows from (19) that $(\varphi^{\nu',w}(x))(F) = 0$ for all $F \in \mathcal{F}_{d-r}(\Delta')$ if and only if $x(H) = 0$ for all $H \in \mathcal{H}[\phi_{d-r+1}]$. Therefore, $\varphi^{\nu',w}$ is injective. \hfill \square
11. The weak Lefschetz property for homology spheres

An Artinian graded $\mathbb{R}$-algebra $A = \bigoplus_{i \geq 0} A_i$ is said to have the weak Lefschetz property if there exists some $\omega \in A_1$ such that the multiplication map $\omega : A_i \to A_{i+1}$ (given by $x \mapsto \omega x$) is either injective or surjective for all $i \geq 0$. This element $\omega$ is called a weak Lefschetz element of $A$. In this section, we shall prove that if $(\Delta, \nu)$ is a “sufficiently generic” PL realization of an $\mathbb{R}$-homology $d$-sphere in $\mathbb{R}^d$, then the stress algebra $\Psi(\Delta, \nu)$ has the weak Lefschetz property, where “sufficiently generic” here has a precise meaning; see Theorem [1.3]. We end this section by completing the proof of Theorem [1.3] (i.e. the $g$-conjecture for $\mathbb{R}$-homology spheres) using this weak Lefschetz property.

A key tool used in this section is the homological interpretation of skeletal rigidity developed by Tay–Whiteley [56], which requires some preparation. We shall build on the notation introduced in Section 4 and Section 10. Let $(\Delta, \nu)$ be a PL realization of a simplicial $d$-complex in $\mathbb{R}^d$. For each $F \in \Delta$ and $0 \leq k \leq d+1$, note that $\nu(F)$ induces an equivalence relation $\sim$ on $\bigwedge^k(\mathbb{R}^{d+1})$ given by $x \sim y \iff x \land \nu(F) = y \land \nu(F)$. Let $\mathbb{W}_F^{(k)}$ be the quotient space $\bigwedge^k(\mathbb{R}^{d+1})/ \sim$. Given any chain complex $C$, let $H_i(C)$ denote the $i$-th homology group of $C$.

Let $0 \leq r \leq d+1$. The $r$-skeletal chain complex of $(\Delta, \nu)$, which we denote by $R_r(\Delta, \nu)$, is

$$0 \longrightarrow \bigoplus_{F \in F_r(\Delta)} \mathbb{W}_F^{(r)} \longrightarrow \cdots \longrightarrow \bigoplus_{F \in F_{r-1}(\Delta)} \mathbb{W}_F^{(r-1)} \longrightarrow \cdots \longrightarrow \bigoplus_{F \in F_1(\Delta)} \mathbb{W}_F^{(0)} \longrightarrow 0,$$

where the boundary map $\partial_i$ is defined by

$$\partial_i(\alpha \cdot [F]) = \sum_{v \in \mathcal{V}(F)} (\alpha \land \nu(v)) \cdot [F \setminus v]$$

for all $F \in F_i(\Delta)$ and all $\alpha \in \mathbb{W}_F^{(r-1-i)}$. From [56] Thm. 4.1(i), we have

$$\Psi_{d+1-r}(\Delta, \nu) \cong H_{r-1}(R_r(\Delta, \nu))$$

as $\mathbb{R}$-vector spaces. For convenience, let

$$R_r(\Delta, \nu)_i := \bigoplus_{F \in F_i(\Delta)} \mathbb{W}_F^{(r-1-i)}$$

for each $-1 \leq i \leq r-1$.

Let $\Delta' := \Delta \ast \{a\}$ be the cone on $\Delta$ with conepoint $a$, and let $(\Delta', \nu')$ be a PL realization in $\mathbb{R}^{d+1}$. Fix a codimension 1 subspace $U$ of $\mathbb{R}^{d+2}$ that does not contain $\nu'(a)$, and let $\pi_a : \mathbb{R}^{d+2} \to U$ be the central projection from $\nu'(a)$ to $U$. Identify $U$ with $\mathbb{R}^{d+1}$, and let $\pi_a, \nu' : \Delta \to \bigwedge(\mathbb{R}^{d+1})$ denote the PL realization of $\Delta$ in $\mathbb{R}^d$ determined by $(\pi_a, \nu')(v) = \pi_a(\nu'(v)) \in \mathbb{R}^{d+1}$ for all $v \in \mathcal{V}(\Delta)$. Next, extend $\pi_a$ linearly so that it becomes the map $\pi_a : \bigwedge(\mathbb{R}^{d+2}) \to \bigwedge(\mathbb{R}^{d+1})$.

Let $\Pi = \Pi_\bullet$ be a sequence of homomorphisms $\Pi_i : R_r(\Delta', \nu')_i \to R_r(\Delta, \pi_a \nu')_i$ determined by

$$\Pi_i(\alpha \cdot [F]) = \pi_a(\alpha) \cdot [F]$$

on elementary chains $\alpha \cdot [F] \in R_r(\Delta', \nu')_i$. As proven in [56] Thm. 8.1, $\Pi : R_r(\Delta', \nu') \to R_r(\Delta, \pi_a \nu')$ is a (well-defined) surjective chain map. This chain map $\Pi$ descends to a map on homology

$$(\Pi_\bullet)_* : H_* (R_r(\Delta', \nu')) \to H_* (R_r(\Delta, \pi_a \nu')),$$

and it was further proven in [56] Thm. 8.2(iv)] that

$$(\Pi_{r-1})_* : H_{r-1} (R_r(\Delta', \nu')) \to H_{r-1} (R_r(\Delta, \pi_a \nu'))$$

is an isomorphism.

Suppose $(\Gamma, \nu')$ is an arbitrary PL realization of a simplicial $d'$-complex in $\mathbb{R}^{d'}$. Given $w \in \Xi_0$, $\nu \in \mathbb{Q}_{\text{PL}}(\Delta)$ and any $0 \leq r \leq d' + 1$, let $\varphi_{\nu', w} = \varphi_{\nu', \nu}$ be a sequence of homomorphisms $\varphi_{\nu', w} :
\( \mathcal{R}_r(\Gamma, \nu'')_i \rightarrow \mathcal{R}_{r-1}(\Gamma, \nu'')_{i-1} \) determined by

\[
\varphi^\nu_w(\alpha \cdot [H]) = \sum_{v \in V(H)} \sum_{G \in \mathcal{F}_{d-1}(\Delta) \atop H \setminus v \subset G} \zeta^\nu_w(G) \alpha \cdot [H \setminus v]
\]
on elementary chains \( \alpha \cdot [H] \in \mathcal{R}_r(\Gamma, \nu'')_i \).

**Lemma 11.1.**

(i) \( \varphi^\nu_w : \mathcal{R}_r(\Gamma, \nu'') \rightarrow \mathcal{R}_{r-1}(\Gamma, \nu'') \) is a chain map.

(ii) The chain maps \( \varphi^\nu_w \) and \( \Pi \) commute, i.e. if \( \Delta' = \Delta \ast \{a\} \) is a cone on \( \Delta \) with conepoint \( a \) as above, then the following diagram

\[
\begin{array}{ccc}
\mathcal{R}_r(\Delta', \nu')_i & \xrightarrow{\Pi_i} & \mathcal{R}_r(\Delta, \pi_a \nu')_i \\
\downarrow_{\varphi^\nu_w} & & \downarrow_{\varphi^\nu_w} \\
\mathcal{R}_{r-1}(\Delta', \nu')_{i-1} & \xrightarrow{\Pi_{i-1}} & \mathcal{R}_{r-1}(\Delta, \pi_a \nu')_{i-1}
\end{array}
\]

commutes for all \( 0 \leq i \leq r \).

**Proof.** It is straightforward to check that both \( \partial_{i-1}(\varphi^\nu_w(\alpha \cdot [H])) \) and \( \varphi^\nu_w(\partial_i(\alpha \cdot [H])) \) equal

\[
\alpha \land \sum_{u, v \in V(H) \atop u \neq v} \left( \sum_{G \in \mathcal{F}_{d-1}(\Delta) \atop H \setminus w \cup v \subset G} \zeta^\nu_w(G) \nu''(u) + \sum_{G \in \mathcal{F}_{d-1}(\Delta) \atop H \setminus w \cup v \subset G} \zeta^\nu_w(G) \nu''(v) \right) \cdot [H \setminus (u \cup v)]
\]

for all elementary chains \( \alpha \cdot [H] \in \mathcal{R}_r(\Gamma, \nu'')_i \), thus \( \partial \circ \varphi^\nu_w = \varphi^\nu_w \circ \partial \), i.e. \( \varphi^\nu_w \) is a chain map.

Also, we check that both \( \varphi^\nu_w(\Pi_i(\alpha \cdot [H])) \) and \( \Pi_{i-1}(\varphi^\nu_w(\alpha \cdot [H])) \) equal

\[
\sum_{v \in V(H)} \sum_{G \in \mathcal{F}_{d-1}(\Delta) \atop H \setminus v \subset G} \zeta^\nu_w(G) \pi_a(\alpha) \cdot [H \setminus v]
\]

for all elementary chains \( \alpha \cdot [H] \in \mathcal{R}_r(\Delta', \nu')_i \), therefore \( \varphi^\nu_w \) and \( \Pi \) commute. \( \square \)

**Lemma 11.2.** Let \( \Delta \) be a simplicial \( \mathbb{R} \)-homology d-sphere, and let \( \Delta' = \Delta \ast \{a\} \) be a cone on \( \Delta \) with conepoint \( a \). Let \( \mathbf{w} \in \Xi_0 \) and \( \nu \in \mathbb{Q}_{\text{gen}}(\Delta) \). Then there exists a dense subset \( \hat{\mathbf{Y}} \) of \( \mathbb{Q}_{\text{gen}}(\Delta') \) such that for every PL realization \( (\Delta', \nu') \) in \( \mathbb{R}^{d+1} \) satisfying \( \nu' \in \hat{\mathbf{Y}} \), the chain map \( \tilde{\varphi}^\nu_w \) induces an injective map

\[
(\tilde{\varphi}^\nu_w)_{d+1-r} : H_{d+1-r}(\mathcal{R}_{d+2-r}(\Delta', \nu')) \rightarrow H_{d-r}(\mathcal{R}_{d+1-r}(\Delta', \nu'))
\]

for all \( 1 \leq r \leq \left\lceil \frac{d+1}{2} \right\rceil \).

**Proof.** This is a straightforward translation of Theorem 10.5. In particular, we will have to use the isomorphism \( \Psi_{d+2-r}(\Delta', \nu') \cong H_{r-1}(\mathcal{R}_r(\Delta', \nu')) \) given in (20). \( \square \)

**Theorem 11.3.** Let \( \Delta \) be a simplicial \( \mathbb{R} \)-homology d-sphere. Then there exists a dense subset \( \hat{\mathbf{Y}} \) of \( \mathbb{Q}_{\text{gen}}(\Delta) \) such that for every PL realization \( (\Delta, \nu) \) in \( \mathbb{R}^d \) satisfying \( \nu \in \hat{\mathbf{Y}} \), the stress algebra \( \Psi(\Delta, \nu) \) has the weak Lefschetz property. In particular, for every \( \nu \in \hat{\mathbf{Y}} \), there exists a weak Lefschetz element \( \omega \) such that the multiplication map \( \cdot \omega : \Psi_{r-1}(\Delta, \nu) \rightarrow \Psi_r(\Delta, \nu) \) is injective for \( r \leq \left\lceil \frac{d+1}{2} \right\rceil \) and surjective for \( r > \left\lceil \frac{d+1}{2} \right\rceil \).

**Proof.** Let \( \Delta' = \Delta \ast \{a\} \) be a cone on \( \Delta \) with conepoint \( a \), and let \( (\Delta', \nu') \) be a PL realization in \( \mathbb{R}^{d+1} \). Recall that \( \Pi : \mathcal{R}_r(\Delta, \nu') \rightarrow \mathcal{R}_r(\Delta, \pi_a \nu') \) is a surjective chain map, so we have the following short exact sequence of chain complexes

\[
0 \rightarrow \text{Ker} \, \Pi \xrightarrow{i} \mathcal{R}_r(\Delta, \nu') \xrightarrow{\Pi} \mathcal{R}_r(\Delta, \pi_a \nu') \rightarrow 0,
\]
where \( \ell = \ell \ast \) is the chain map induced by the natural inclusions \( \text{Ker} \Pi_i \hookrightarrow \mathcal{R}_r(\Delta') \). Thus by the snake lemma, we have the following induced long exact sequence:

\[
\cdots \rightarrow H_{r-1}(\text{Ker} \Pi) \xrightarrow{\ell_\ast} H_{r-1}(\mathcal{R}_r(\Delta', \nu')) \xrightarrow{\Pi_\ast} H_{r-1}(\mathcal{R}_r(\Delta, \pi_a \nu')) \xrightarrow{\delta} H_{r-2}(\text{Ker} \Pi) \rightarrow \cdots,
\]

where \( \delta \) denotes the connecting homomorphism.

Let \( w \in \Xi_0 \) and \( \nu \in \mathcal{Q}_{\text{PL}}^{\text{gen}}(\Delta) \). Lemma [11.1] says that the chain maps \( \varphi_{r,w}^{\nu} \) and \( \Pi \) commute, thus for all \( 1 \leq r \leq d + 1 \), the following diagram commutes.

\[
\begin{array}{ccc}
H_{r-1}(\text{Ker} \Pi) & \xrightarrow{\ell_\ast} & H_{r-1}(\mathcal{R}_r(\Delta', \nu')) \\
\downarrow{\varphi_{r,w}^{\nu}} & & \downarrow{\varphi_{r,w}^{\nu}} \\
H_{r-2}(\text{Ker} \Pi) & \xrightarrow{\ell_\ast} & H_{r-2}(\mathcal{R}_r(\Delta', \nu')) \\
\end{array}
\]

Thus it follows from (20) that for all \( 1 \leq r \leq d + 1 \), the following diagram commutes.

\[
\begin{array}{ccc}
0 & \xrightarrow{} & \Psi_r(\Delta', \nu') \\
\downarrow & & \downarrow & & \downarrow \\
0 & \xrightarrow{} & \Psi_{r+1}(\Delta', \nu') \\
\end{array}
\]

(22)

Now, by Lemma [11.2], there exists a dense subset \( \tilde{\Upsilon} \) of \( \mathcal{Q}_{\text{PL}}^{\text{gen}}(\Delta) \) such that for every PL realization \( (\Delta', \nu') \) in \( \mathbb{R}^{d+1} \) satisfying \( \nu' \in \tilde{\Upsilon} \), the second vertical map \( (\varphi_{d+1-r,w}^{\nu,w})_* : \Psi_r(\Delta', \nu') \rightarrow \Psi_{r+1}(\Delta', \nu') \) in the commutative diagram (22) is injective for all \( 1 \leq r \leq \left\lfloor \frac{d+1}{2} \right\rfloor \). Define

\[
\tilde{\Upsilon} := \{ \nu \in \mathcal{Q}_{\text{PL}}^{\text{gen}}(\Delta) | \nu = \pi_a \nu' \text{ for some } \nu' \in \tilde{\Upsilon} \},
\]

and note that \( \tilde{\Upsilon} \) is a dense subset of \( \mathcal{Q}_{\text{PL}}^{\text{gen}}(\Delta) \). Henceforth, fix some \( \nu' \in \tilde{\Upsilon} \) such that \( \nu := \pi_a \nu' \in \tilde{\Upsilon} \).

Let \( 1 \leq r \leq \left\lfloor \frac{d+1}{2} \right\rfloor \). By the four lemma, the third vertical map \( (\varphi_{d+1-r,w}^{\nu,w})_* : \Psi_{r-1}(\Delta, \nu) \rightarrow \Psi_r(\Delta, \nu) \) in the commutative diagram (22) is injective. By definition,

\[
\left( (\varphi_{d+1-r,w}^{\nu,w})_*(x) \right)(F) = \sum_{G \in \mathcal{F}_{d-1}(\Delta)} \sum_{H \in \mathcal{F}_{d+1-r}(\Delta)} \zeta_{w}^\nu(G) x(H)
\]

for all \( x \in \Psi_{r-1}(\Delta, \nu) \) and all \( F \in \mathcal{F}_{d-r}(\Delta) \).

Recall that in the definition of \( w \in \Xi_0 \), we have implicitly fixed some \( d \)-face \( Q \) of \( \Delta \). For each \( v \in \mathcal{V}(\Delta) \), let \( \mu_v \) be the lifting of \( (\Delta, \nu) \) defined by \( \tilde{\pi}_d(\mu_v(v)) = 1 \), and \( \tilde{\pi}_d(\mu_v(v')) = 0 \) for all remaining vertices \( v' \neq v \). (Recall that \( \tilde{\pi}_d : \mathbb{R}^{d+1} \rightarrow \mathbb{R} \) denotes the projection map \( (x_1, \ldots, x_{d+1}) \mapsto x_{d+1} \).) Let \( \varphi : \text{Lift}(\Delta, \nu, Q) \rightarrow \Psi_1(\Delta, \nu) \) be the isomorphism given by Theorem [6.7], and note that Theorem [6.3] says \( \{ \varphi(\mu_v) | v \in \mathcal{V}(\Delta) \} \) is a basis for \( \Psi_1(\Delta, \nu) \). Now, define

\[
\omega = \sum_{v \in \mathcal{V}(\Delta)} \sum_{v \notin \mathcal{V}(Q)} \omega(v) \varphi(\mu_v) \in \Psi_1(\Delta, \nu),
\]

and observe that \( \zeta_{w}^\nu(G) = \omega(G) \) by definition. Since the vertices of \( \Delta \) are realized in general position, it thus follows from the definition of the multiplication of stresses that

\[
\left( (\varphi_{d+1-r,w}^{\nu,w})_*(x) \right)(F) = (\omega x)(F)
\]

for all \( F \in \Delta \), which implies that the multiplication map \( \omega : \Psi_{r-1}(\Delta, \nu) \rightarrow \Psi_r(\Delta, \nu) \) is injective for all \( 1 \leq r \leq \left\lfloor \frac{d+1}{2} \right\rfloor \). It then follows that the dual map \( \omega : \text{Hom}_\mathbb{R}(\Psi_r(\Delta, \nu), \mathbb{R}) \rightarrow \text{Hom}_\mathbb{R}(\Psi_{r-1}(\Delta, \nu), \mathbb{R}) \) is surjective for all \( 1 \leq r \leq \left\lfloor \frac{d+1}{2} \right\rfloor \).
Finally, since $\Psi(\Delta, \nu)$ is Gorenstein (by Corollary 8.3), it follows from [51] Thm. I.12.5 and [51], Thm. I.12.10 that $\text{Hom}_R(\Psi_r(\Delta, \nu), R) \cong \Psi_{d+1-r}(\Delta, \nu)$, hence $\omega : \Psi_{r-\nu-1}(\Delta, \nu) \rightarrow \Psi_r(\Delta, \nu)$ is surjective for all $\lfloor \frac{d+1}{2} \rfloor + 1 \leq r \leq d + 1$, and therefore $\Psi(\Delta, \nu)$ has the weak Lefschetz property. 

We now complete the proof of the $g$-conjecture for $\mathbb{R}$-homology spheres.

**Proof of Theorem 11.3.** By Theorem 11.3 we can choose some $\mathbb{Q}$-generic PL realization $\nu$ of $\Delta$ in $\mathbb{R}^d$, such that the stress algebra $\Psi(\Delta, \nu)$ has a weak Lefschetz element $\omega \in \Psi(\Delta, \nu)$ for which the multiplication map $\omega : \Psi_{r-\nu-1}(\Delta, \nu) \rightarrow \Psi_r(\Delta, \nu)$ is injective for $r \leq \lfloor \frac{d+1}{2} \rfloor$, and surjective for $r > \lfloor \frac{d+1}{2} \rfloor$. Let $R = \bigoplus_{i \geq 0} R_i$ be the graded quotient ring $\Psi(\Delta, \nu)/(\omega)$. For each $0 \leq i \leq \lfloor \frac{d+1}{2} \rfloor$, consider the following short exact sequence:

$$0 \longrightarrow \Psi_{i-\nu-1}(\Delta, \nu) \longrightarrow \Psi_i(\Delta, \nu) \longrightarrow R_i \longrightarrow 0.$$ 

By Theorem 10.1 and Theorem 8.3, the $h$-vector of $\Delta$ is the $h$-vector of $\Psi(\Delta, \nu)$. Consequently, by the additivity of dimensions (of $\mathbb{R}$-vector spaces) on exact sequences, we get $\dim(R_i) = h_i(\Delta) - h_{i-1}(\Delta)$ (where $h_{-1}(\Delta) = 0$), thus the $h$-vector of the truncated ring $\bigoplus_{i = 0}^{\lfloor \frac{d+1}{2} \rfloor} R_i$ is the $g$-vector of $\Delta$, and therefore the $g$-vector of $\Delta$ is an $M$-vector. 

12. Further Remarks

12.1. *Homology spheres over other fields.* Our proof of Theorem 11.3 uses the assumption that $\Delta$ is a homology sphere over the reals in several instances. To show both the Gorenstein property and the weak Lefschetz property in Theorem 11.4, we used Mordell’s result on algebraic number fields (Lemma 7.1), as well as the fact that $\mathbb{R}$ is a field extension of $\mathbb{Q}$ containing infinitely many quadratic fields. (Although we work with $\mathbb{R}$-homology spheres, our proof actually works for homology spheres over any subfield of $\mathbb{R}$ that contains the square roots of all squarefree positive rational numbers.) To show that $\mathbb{Q}$-generic PL realizations are dense among arbitrary PL realizations (Proposition 7.3), we used the fact that $\mathbb{R}$ is an infinite metric space.

To relate rigidity theory to $f$-vector theory, we used Theorem 10.1, which relates the $h$-vector of an $\mathbb{R}$-Cohen–Macaulay simplicial $d$-complex $\Delta$ to the $h$-vector of the stress algebra on a generic PL realization of $\Delta$ in $\mathbb{R}^d$. Tay–Whiteley [56] gave a homological proof of Theorem 10.1, so an analogous statement holds for generic PL realizations of $k$-Cohen–Macaulay simplicial $d$-complexes in $k^d$. (The definitions of PL realizations and stresses extend in the obvious way.) However, we do not require PL realizations in $\mathbb{R}^d$ to apply Rybnikov’s results in Section 6. Specifically, we used the fact that $\mathbb{R}^d$ is an inner product space over an ordered field, so that the notion of outer unit normal vectors makes sense. Thus, without an analogous of Maxwell–Cremona theory (and in particular, an analogous three-way interplay between liftings, reciprocals, and $d$-stresses) over non-ordered fields, we do not know how to extend our proof of Theorem 11.3 to homology spheres over fields of non-zero characteristic.

12.2. *Lefschetz properties.* Given an Artinian Gorenstein graded $k$-algebra $A = \bigoplus_{i = 0}^{d+1} A_i$ that is generated (as a $k$-algebra) by $A_1$, and whose socle is contained in $A_{d+1}$, we say that $A$ has the strong Lefschetz property if there exists some $\omega \in A_1$ such that $\omega^{d+1-2r}A_r = A_{d+1-r}$ for all $0 \leq r \leq \lceil \frac{d+1}{2} \rceil$. It is easy to see that the strong Lefschetz property implies the weak Lefschetz property. A less obvious fact is that this implication is strict: There are Artinian Gorenstein graded algebras that have the weak Lefschetz property but not the strong Lefschetz property [13] (cf. [11]).

**Problem 12.1.** Let $(\Delta, \nu)$ be a $\mathbb{Q}$-generic PL realization of a simplicial $\mathbb{R}$-homology $d$-sphere in $\mathbb{R}^d$. Is it possible for $\Psi(\Delta, \nu)$ to have the weak Lefschetz property, but not the strong Lefschetz property?

The strong Lefschetz property for generic Artinian reductions of Stanley–Reisner rings is preserved under joins, connected sums, stellar subdivisions, and certain bistellar moves [11, 53]. Does an analogous statement hold for stress algebras? More fundamentally, how are Stanley–Reisner rings related to stress algebras?
Problem 12.2. Given a PL realization \((\Delta, \nu)\) of an \(\mathbb{R}\)-Cohen–Macaulay simplicial \(d\)-complex in \(\mathbb{R}^d\), is it possible to find an Artinian reduction (in terms of \(\nu\)) of the Stanley–Reisner ring of \(\Delta\) (over \(\mathbb{R}\)) that is isomorphic to \(\Psi(\Delta, \nu)\) as graded \(\mathbb{R}\)-algebras?

The generalized lower bound conjecture (GLBC), posed by McMullen–Walkup [30] (cf. [12, Sec. 10]), characterizes “stacked” simplicial convex polytopes in terms of their \(d\)-vectors. A \(k\)-homology \(d\)-ball \(\Sigma\) is called \(r\)-stacked if every face of dimension \(\leq d-r-1\) intersects the boundary \(\partial \Sigma\) non-trivially. A \(k\)-homology \(d\)-sphere is called \(r\)-stacked if it is the boundary of an \(r\)-stacked \(k\)-homology \((d+1)\)-ball.

Conjecture 12.3 (McMullen–Walkup (1971)). Let \(\Delta\) be the boundary of a simplicial convex \(d\)-polytope. For every \(1 \leq r \leq \lfloor \frac{d+1}{2} \rfloor\), we have \(g_r(\Delta) \geq 0\), with equality holding if and only if \(\Delta\) is \((r-1)\)-stacked.

The \(g\)-theorem implies that \(g(\Delta) \geq 0\), while McMullen–Walkup [30] had already proven (when they proposed the conjecture) that if \(\Delta\) is \((r-1)\)-stacked, then \(g_r(\Delta) = 0\). Recently, Murai–Nevo [57] (cf. [38]) proved the remaining (difficult) part of the conjecture. In fact, they proved the following more general result.

Theorem 12.4 ([57, Thm. 1.3]). Let \(\Delta\) be a simplicial \(k\)-homology \(d\)-sphere. If \(k\) has characteristic 0, and if there exists an Artinian reduction of the Stanley–Reisner ring of \(\Delta\) (over \(k\)) that has the weak Lefschetz property, then for every \(1 \leq r \leq \lfloor \frac{d+1}{2} \rfloor\), the equality \(g_r(\Delta) = 0\) implies \(\Delta\) is \((r-1)\)-stacked.

Is there an analog of Theorem 12.4 in terms of stress algebras? If so, then Theorem 12.4 would yield a proof of an extension of the GLBC to \(\mathbb{R}\)-homology spheres.

More recently, Klee–Novik [19] introduced the notion of “\(\mathcal{F}\)-vector” for balanced simplicial complexes, and they proposed a balanced analog of the GLBC in terms of \(\mathcal{F}\)-vectors. (A simplicial \((d-1)\)-complex is called balanced if its 1-skeleton, treated as a graph, admits a \(d\)-coloring.) Soon after, Kubitzke–Murai [15] proved the first part of the balanced GLBC, i.e. if \(\Delta\) is the boundary of a balanced simplicial convex polytope, then the \(\mathcal{F}\)-vector of \(\Delta\) has non-negative entries. In their proof, they showed a certain “weaker Lefschetz property” for a particular Artinian reduction of the Stanley–Reisner ring of \(\Delta\). Can we use stress algebras to show the remaining part of the balanced GLBC, as well as prove an extension of the balanced GLBC for balanced \(\mathbb{R}\)-homology spheres?

12.3. Extending the \(g\)-conjecture to homology manifolds. Kalai’s manifold \(g\)-conjecture [41] (see also [18, 53]) is a far-reaching generalization of the \(g\)-conjecture to orientable \(k\)-homology manifolds without boundary. To state this conjecture, we need the notion of \(h''\)-vectors introduced by Kalai.

Given a simplicial \(d\)-complex \(\Delta\), define \(\beta_i(\Delta) := \dim_k(\overline{H}_i(\Delta; k))\) for each integer \(i\). Let \(h'_0(\Delta) = 1\), and for every \(1 \leq j \leq d + 1\), define

\[
h'_j(\Delta) := h_j(\Delta) + \binom{d+1}{j} \sum_{i=0}^{j-1} (-1)^{j-i-1} \beta_{i-1}(\Delta).
\]

Next, define \(h''(\Delta) := h'_j(\Delta) - \binom{d+1}{j} \beta_{j-1} \) for each \(0 \leq j \leq d\), and define \(h''(\Delta) := h''(\Delta) + h''(\Delta)\) is the \(h''\)-vector of \(\Delta\).

Conjecture 12.5 (Kalai’s manifold \(g\)-conjecture). Let \(\Delta\) be an orientable simplicial \(k\)-homology \(d\)-manifold without boundary. A sequence of integers \(h''_0, \ldots, h''_{d+1}\) is the \(h''\)-vector of \(\Delta\) if and only if the following two conditions hold.

\((i)\) \(h''_i = h''_{d+1-i}\) for all \(0 \leq i \leq d + 1\).

\((ii)\) \(\langle h''_0, h''_1, h''_2, \ldots, h'', \frac{d+1}{2}, \frac{d+1}{2} \rangle\) is an \(M\)-vector.

Condition \((i)\) was proven combinatorially by Novik [44], and subsequently proven algebraically by Novik–Swartz [46]. See [32, 47] for a general algebraic treatment of the face numbers of \(k\)-Buchsbaum complexes. (\(k\)-homology manifolds are examples of \(k\)-Buchsbaum complexes.) Murai [35] showed that condition \((i)\) is true when \(\Delta\) is the barycentric subdivision of an orientable simplicial \(k\)-homology \(d\)-manifold. Swartz [44, 53] (cf. [52]) proved that condition \((i)\) holds if the generic Artinian reductions of
the Stanley–Reisner rings of the links (in $\Delta$) of at least $f_0(\Delta) - d - 1$ vertices have the weak Lefschetz property. We believe that stress algebra analogs of the results by Swartz and Novik–Swartz hold.

To tackle Kalai’s manifold $g$-conjecture, we pose the following two conjectures, which if true would prove Conjecture [12.5] for a large class of orientable $\mathbb{R}$-homology manifolds (without boundary).

**Conjecture 12.6.** Let $\Delta$ be an orientable simplicial $\mathbb{R}$-homology $d$-manifold (without boundary), such that $H_i(\Delta; \mathbb{Z}/2\mathbb{Z}) = 0$ whenever $d \geq 2$. There exists a dense subset $Y$ of all $\mathbb{Q}$-generic PL realizations of $\Delta$ in $\mathbb{R}^d$, such that for every $\nu \in Y$, the stress algebra $\Psi(\Delta, \nu)$ has the weak Lefschetz property.

**Conjecture 12.7.** Let $(\Delta, \nu)$ be a PL realization of an orientable simplicial $\mathbb{R}$-homology $d$-manifold (without boundary) in $\mathbb{R}^d$. If the vertices are realized in general position, then $\text{dim}_\mathbb{R}(\Psi_{d+1-i}(\Delta, \nu)) = h_i^d(\Delta)$ for all $0 \leq i \leq d + 1$.

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