On some set-valued iteration semigroups generated by interval-valued functions

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Abstract. Let $X$ be an arbitrary set. We characterize all interval-valued functions $A : X \to 2^\mathbb{R}$ for which a multifunction $F : (0, \infty) \times X \to 2^X$ of the form $F(t, x) = A^- (A(x) + \min\{t, q - \inf A(x)\})$, where $q = \sup A(X)$, is an iteration semigroup. The multifunction $F$ is the set-valued counterpart of the fundamental form of continuous iteration semigroups of single-valued functions on an interval.

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Introduction

Let $X$ be an arbitrary set. A multifunction $F : (0, \infty) \times X \to 2^X$ is said to be a set-valued iteration semigroup if

$$F(s + t, x) = F(t, F(s, x)) \quad \text{for } x \in X \quad \text{and } s, t \in (0, \infty).$$

This notion was introduced and investigated by Smajdor [11] (see also e.g. [12]), studied by J. Olko (see e.g. [9, 10]) and by Zdun [15]. In [4, 5] we introduced and studied a family of set-valued functions which now will be denoted by (A) (see Sect. 1) and we showed (see [5, Remarks 1 and 3]) that $F$ given by (A) is a set-valued counterpart of the fundamental form of iteration semigroups for single-valued functions which can be found in [2, Chap. IX, Sec. 1], [14, Theorems 5.1–8.1], [13, p. 98–99], [3, Chap. I, Sec. 1.7] (cf. also [1, Theorem 1]). In [7] we studied a lower semicontinuity of $F$ given by (A).

In [8] we gave the necessary and sufficient conditions under which $F$ given by (A) is an iteration semigroup (see [8, Theorems 2, 4 and 5]) (cf. also Sect. 1: Facts 3, 4 and 5).

In the present paper we will consider the case when the values of the generator $A$ of the multifunction $F$ are intervals. The main aim of this paper is to
show all interval-valued functions $A$ for which $F$ given by (A) is an iteration semigroup.

1. Preliminaries

Fix a set $X$ and a set-valued function $A : X \to 2^\mathbb{R}$ with non-empty values. Put

$$S := A(X) \text{ and } q := \sup S.$$ 

Throughout this paper we will always assume that $A$ satisfies the following condition:

(H) For every $s, t \in (0, \infty)$ and $x, z \in X$ with $[A(x) + s + t] \cap A(z) \neq \emptyset$ there exists $y \in X$ satisfying the conditions

$$[A(x) + s] \cap A(y) \neq \emptyset$$

and

$$[A(y) + t] \cap A(z) \neq \emptyset.$$ 

In the following remark we give some properties related to (H).

Remark 1. (i) If $S$ is an interval then (H) holds (see [5, Proposition 1(i)]).

(ii) If at least one of the values of $A$ is a singleton, then (H) holds if and only if $S$ is an interval.

(iii) Assume that all values of $A$ are intervals. If (H) holds then $(\inf S, \sup S) \subset \text{cl} S$ (see [5, Proposition 1(iii)]).

(iv) Assume that all values of $A$ are open sets. If $(\inf S, \sup S) \subset \text{cl} S$ then (H) holds (see [5, Proposition 1(ii)]).

Proof. (ii) Assume that there exist $x \in X$ and $u \in S$ such that $A(x) = \{u\}$. By (i) it is enough to show that if (H) holds then $S$ is an interval. Assume that (H) is satisfied and take $a, b \in S$, $a < b$. Let $c \in (a, b)$. We prove that $c \in S$. Take $w, z \in X$ such that $a \in A(w)$ and $b \in A(z)$. Of course if $c = u$ then $c \in S$, so assume that $c \neq u$.

If $u < c$ then there exist $s, t \in (0, \infty)$ such that

$$A(x) + s = \{c\} \text{ and } A(x) + s + t = \{b\}.$$ 

Hence $[A(x) + s + t] \cap A(z) \neq \emptyset$. By (H) there exists $y \in X$ such that $[A(x) + s] \cap A(y) \neq \emptyset$, thus $c \in A(y) \subset S$.

Pass to the case $u > c$. We can find $s, t \in (0, \infty)$ such that

$$\{c\} + t = A(x) \text{ and } \{a\} + s + t = A(x).$$ 

Thus $[A(w) + s + t] \cap A(x) \neq \emptyset$ and by (H) we get

$$A(y) \cap [A(x) - t] \neq \emptyset$$

for some $y \in X$. Since $A(x) - t = \{c\}$, we obtain that $c \in A(y)$. \hfill \Box
For every $x \in X$ define
\[
\tau(x) := q - \inf A(x).
\]

Consider the following condition:

(H1) for every $x, z \in X$ and $s, t \in (0, \infty)$ with $s + t \leq \tau(x)$ if (1) and (2) hold for a $y \in X$ then $[A(x) + s + t] \cap A(z) \neq \emptyset$.

Notice that if $A$ is single-valued then (H1) holds (see also [6, Remark 1]).

Remark 2. (see [8, Remark 2]) Assume that (H1) holds and $q = \infty$. Then for every $x \in X$ either $\text{card } A(x) = 1$ or $\text{diam } A(x) = \infty$.

Prove the following easy remark.

Remark 3. Assume that the condition (H1) holds. Let $x, z \in X$ and $u < w$ for $u \in A(x)$ and $w \in A(z)$.

Then $\text{card } A(y) = 1$ for every $y \in Y$ such that
\[
\sup A(x) \leq \inf A(y) \quad \text{and} \quad \sup A(y) \leq \inf A(z).
\]

Proof. Take $y \in Y$ such that
\[
\sup A(x) \leq \inf A(y) \quad \text{and} \quad \sup A(y) \leq \inf A(z)
\]
and suppose that $\text{card } A(y) > 1$. Let $a, b \in A(y)$ and $a < b$. We can find $s \in (0, \infty)$ satisfying the conditions
\[
\sup A(x) + s < \frac{a + b}{2}
\]
and
\[
a \in A(x) + s.
\]

Similarly there exists $t \in (0, \infty)$ such that
\[
\frac{a + b}{2} < \inf A(z) - t
\]
and
\[
b \in A(z) - t.
\]

Observe that, by (3) and (5), we have
\[
s + t < \inf A(z) - \sup A(x) < q - \inf A(x) = \tau(x).
\]

Moreover, due to (4) and (6), we get
\[
[A(x) + s] \cap A(y) \neq \emptyset
\]
and
\[
A(y) \cap [A(z) - t] \neq \emptyset.
\]

Hence, according to the inequality (7) and the condition (H1),
\[
[A(x) + s] \cap [A(z) - t] \neq \emptyset.
\]
On the other hand, by (3) and (5), we obtain
\[ \sup A(x) + s < \inf A(z) - t, \]
whence
\[ [A(x) + s] \cap [A(z) - t] = \emptyset. \]
This contradiction completes the proof. \( \square \)

Define the following sets:
\[ \mathcal{L} := \{ A(x) : x \in X, \inf A(x) = -\infty \text{ and } \infty \neq q \notin A(x) \}, \]
\[ \mathcal{S} := \{ A(x) : x \in X, \text{card } A(x) = 1 \text{ and } q \notin A(x) \}, \]
\[ \mathcal{P}_{-\infty} := \{ A(x) : x \in X, \inf A(x) = -\infty \text{ and } q \in A(x) \}, \]
\[ \mathcal{P} := \{ A(x) : x \in X, \inf A(x) \in A(x) \text{ and } q \in A(x) \}, \]
\[ \mathcal{L}_{-\infty} := \{ A(x) : x \in X, \inf A(x) = -\infty \text{ and } \sup A(x) < q = \infty \}, \]
\[ \mathcal{P}_{\infty} := \{ A(x) : x \in X, \inf A(x) > -\infty \text{ and } \sup A(x) = q = \infty \}, \]
\[ \mathcal{R} := \{ A(x) : x \in X, \inf A(x) = -\infty \text{ and } \sup A(x) = q = \infty \}. \]

Assume that \( \mathcal{A} \) and \( \mathcal{B} \) are arbitrary families of subsets of \( \mathbb{R} \). We will write \( \mathcal{A} \preceq \mathcal{B} \) if
\[ \sup \mathcal{A} \leq \inf \mathcal{B} \]
for every \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \).

Let \( F : (0, \infty) \times X \to 2^X \) be given by the following formula
\[ F(t, x) := A^-(A(x) + \min\{t, q - \inf A(x)\}), \tag{A} \]
where \( A^-(V) := \{ x \in X : A(x) \cap V \neq \emptyset \} \) for every \( V \subset \mathbb{R} \).

In [5, Lemma 3] we proved the following fact.

**Fact 1.** (see [5, Lemma 3]) Let \( F : (0, \infty) \times X \to 2^X \) be given by (A) and let \( t \in (0, \infty) \) and \( x \in X \). If \( t < \tau(x) \) then
\[ F(t, x) = A^-(A(x) + t) \neq \emptyset \]
and if \( t \geq \tau(x) \) then
\[
F(t, x) = \begin{cases} A^-(\{q\}), & \text{if } q \in S \text{ and } \inf A(x) \in A(x); \\ \emptyset, & \text{otherwise.} \end{cases}
\]

**Fact 2.** (see [6, Theorem 3]) Let \( F : (0, \infty) \times X \to 2^X \) be given by (A). If \( A \) is single-valued then \( F \) is an iteration semigroup.

Now we present three theorems which was proved in [8]. We will use them in the proof of the main result of this paper.
**Fact 3.** (see [8, Theorem 2]) Let $F : (0, \infty) \times X \to 2^X$ be given by (A). Assume that $q = \infty$. Then $F$ is an iteration semigroup if and only if (H1) is satisfied.

**Fact 4.** (see [8, Theorem 5]) Let $F : (0, \infty) \times X \to 2^X$ be given by (A). Assume that $q \notin S$ and $q \neq \infty$. Then $F$ is an iteration semigroup if and only if (H1) is satisfied and

$$A(x) \in \mathcal{L} \cup S \quad \text{for } x \in X$$

and $\mathcal{L} \preceq S$.

**Fact 5.** (see [8, Theorem 4]) Let $F : (0, \infty) \times X \to 2^X$ be given by (A). Assume that $q \in S$. Then $F$ is an iteration semigroup if and only if the condition (H1) and all the following conditions hold:

(a) $A(x) \in \mathcal{L} \cup S \cup \mathcal{P}_{-\infty} \cup \mathcal{P}$ for every $x \in X$;
(b) $\mathcal{L} \preceq S \cup \mathcal{P}$;
(c) $S \preceq \mathcal{P}$;
(d) $S = \emptyset$ or $\mathcal{P}_{-\infty} = \emptyset$;
(e) $\mathcal{L} = \emptyset$ or $\mathcal{P}_{-\infty} = \emptyset$ or $\mathcal{P} = \emptyset$;
(f) for every $x, y \in X$ if $A(x) \in \mathcal{P}_{-\infty} \cup \mathcal{P}$, $A(y) \in \mathcal{P}$, $\inf A(x) < \inf A(y)$ and there exists $s \in (0, \inf A(y) - \inf A(x))$ satisfying (1), then for every $P \in \mathcal{P} \cup \mathcal{P}_{-\infty}$ and $t \in [\tau(y), \tau(x) - s)$ the condition

$$[A(x) + s + t] \cap P \neq \emptyset$$

holds;
(g) for every $x, y \in X$ if $A(x) \in \mathcal{L}$, $A(y) \in S \cup \mathcal{P}$ and $s \in (0, \infty)$ satisfies (1), then the condition

$$[A(x) + s + t] \cap P \neq \emptyset$$

holds for every $P \in \mathcal{P}$ and $t \geq \tau(y)$.

Since in this paper we are interested in multifunctions $F$ given by (A) which are generated by interval-valued functions $A$, below we prove two properties of the multifunction $A$ under this assumption.

**Remark 4.** Assume that the values of $A$ are intervals and the condition (H1) holds. Let $x, y, z \in X$.

(i) If

$$\sup A(x) \leq \inf A(y) \leq \inf A(z),$$

then $\text{card } A(y) = 1$ or $\inf A(y) = \inf A(z)$.

(ii) If $\sup A(x) \leq \sup A(y) \leq \inf A(z)$, then $\text{card } A(y) = 1$ or $\sup A(x) = \sup A(y)$.

**Proof.** (i) Assume (8) and suppose that $\text{card } A(y) > 1$ and

$$\inf A(y) < \inf A(z).$$

(9)
Then, by Remark 3, we have
\[ \inf A(z) < \sup A(y). \] (10)

Let
\[ t := \frac{\inf A(z) - \inf A(y)}{4}, \]
and
\[ s := \inf A(y) - \sup A(x) + t. \]

Of course \( s, t \in (0, \infty) \) and
\[ \inf A(y) < \inf A(y) + t = \sup A(x) + s. \]

Due to (9) we obtain
\[ \inf A(z) - t = \inf A(z) - 2t + t = \frac{\inf A(z) + \inf A(y)}{2} + t \]
\[ > \inf A(y) + t = \sup A(x) + s. \] (11)

Hence, by (10),
\[ \inf A(y) < \sup A(x) + s < \inf A(z) - t < \sup A(y). \]

Therefore, since the values of \( A \) are intervals, we get
\[ [A(x) + s] \cap A(y) \neq \emptyset \text{ and } [A(z) - t] \cap A(y) \neq \emptyset. \]

On the other hand, by (11)
\[ [A(x) + s] \cap [A(z) - t] = \emptyset, \]
which, by the following inequality
\[ s + t = \frac{\inf A(y) + \inf A(z) - 2 \sup A(x)}{2} < \inf A(z) - \sup A(x) < \]
\[ < q - \inf A(x) = \tau(x), \]
contradicts condition (H1) and completes the proof of (i).

The proof of (ii) is similar. It is enough to take
\[ s := \frac{\sup A(y) - \sup A(x)}{4} \text{ and } t := \inf A(z) - \sup A(y) + s. \]
Notice that if $A$ is interval-valued then for every $x \in X$ we get

- $A(x) \in \mathcal{L}$ iff $A(x) = (-\infty, a_x)$ for some $a_x \leq q < \infty$ or $A(x) = (-\infty, a_x]$ for some $a_x < q < \infty$,
- $A(x) \in \mathcal{S}$ iff $A(x) = \{a_x\}$ for some $a_x \neq q$,
- $A(x) \in \mathcal{P}_{-\infty}$ iff $A(x) = (-\infty, q]$,
- $A(x) \in \mathcal{P}$ iff $A(x) = [a_x, q]$ for some $a_x \leq q$,
- $A(x) \in \mathcal{L}_{-\infty}$ iff $A(x) = (-\infty, a_x)$ for some $a_x < q = \infty$ or $A(x) = (-\infty, a_x]$ for some $a_x < q < \infty$,
- $A(x) \in \mathcal{S} \iff A(x) = \{a_x\}$ for some $a_x \neq q$,
- $A(x) \in \mathcal{P}_{-\infty}$ iff $A(x) = (-\infty, a_x]$ for some $a_x > q = \infty$ or $A(x) = (-\infty, a_x]$, for some $a_x > q = \infty$,
- $A(x) \in \mathcal{S}$ iff $A(x) = \{a_x\}$ for some $a_x \neq q$,
- $A(x) \in \mathcal{P}_{\infty}$ iff $A(x) = (a_x, \infty)$ for some $a_x \in \mathbb{R}$ or $A(x) = [a_x, \infty)$ for some $a_x \in \mathbb{R}$,
- $A(x) \in \mathcal{R}$ iff $A(x) = \mathbb{R}$.

**Lemma 1.** Assume that the values of $A$ are intervals, (H1) holds and $q = \infty$. Then all the following conditions are satisfied

(i) $A(x) \in \mathcal{L}_{-\infty} \cup \mathcal{S} \cup \mathcal{P}_{\infty} \cup \mathcal{R}$ for every $x \in X$;
(ii) $\mathcal{L}_{-\infty} \preceq \mathcal{S} \cup \mathcal{P}_{\infty}$;
(iii) $\mathcal{S} \preceq \mathcal{P}_{\infty}$;
(iv) $\mathcal{S} = \emptyset$ or $\mathcal{R} = \emptyset$;
(v) $\mathcal{L}_{-\infty} = \emptyset$ or $\mathcal{R} = \emptyset$ or $\mathcal{P}_{\infty} = \emptyset$.

**Proof.** Notice that $\tau(x) = \infty$ for every $x \in X$. The condition (i) follows immediately from Remark 2.

Pass to the proof of (ii). Suppose that there exist $x, y, z \in X$ such that $A(x) \in \mathcal{S} \cup \mathcal{P}_{\infty}$, $A(y) \in \mathcal{L}_{-\infty}$ and $\sup A(y) > \inf A(x)$. Take

$$s := \frac{\sup A(y) - \inf A(x)}{2},$$

and $t \in (0, \infty)$, $t > s$. Of course $s, t \in (0, \infty)$ and $s + t < \tau(x)$. Since the values of $A$ are intervals, notice that

$$[A(x) + s] \cap A(y) \neq \emptyset \quad \text{and} \quad [A(y) - t] \cap A(y) \neq \emptyset,$$

but $[A(x) + s + t] \cap A(y) = \emptyset$, which contradicts (H1).

Now we prove (iii) and (iv). Suppose that there exist $x, y \in X$ such that

- $A(x) \in \mathcal{S}$ and $A(y) \in \mathcal{P}_{\infty}$ and $\inf A(y) < \sup A(x)$

or

- $A(x) \in \mathcal{S}$ and $A(y) \in \mathcal{R}$.

Let $s \in (0, \infty)$ and $t \in (0, \sup A(x) - \inf A(y))$. By our assumptions

$$[A(x) + s] \cap A(y) \neq \emptyset \quad \text{and} \quad [A(x) - t] \cap A(y) \neq \emptyset.$$
On the other hand
\[ [A(x) + s] \cap [A(x) - t] = \emptyset, \]
which contradicts (H1).

To prove (v) suppose that there exist \( x, y, z \in X \) such that \( A(z) \in \mathcal{L}_{-\infty} \), \( A(y) \in \mathcal{R} \) and \( A(x) \in \mathcal{P}_\infty \). By (ii) we get
\[
\sup A(z) \leq \inf A(x). \tag{12}
\]
For every \( s, t \in (0, \infty) \) we obtain
\[ [A(x) + s] \cap A(y) \neq \emptyset \quad \text{and} \quad [A(z) - t] \cap A(y) \neq \emptyset. \]
Due to (12)
\[ [A(x) + s] \cap [A(z) - t] = \emptyset, \]
which contradicts (H1) and completes the proof of (v).

\( \square \)

2. Main result

Let \( \mathcal{A}, \mathcal{B} \subset 2^R \). We will say that \( A \) has values of type \( \mathcal{A} \), if \( A(x) \in \mathcal{A} \) for every \( x \in X \).

We will say that \( A \) has values of type \( \mathcal{A}\mathcal{B} \), if \( A(x) \in \mathcal{A} \cup \mathcal{B} \) for every \( x \in X \) and \( \mathcal{A} \neq \emptyset \) and \( \mathcal{B} \neq \emptyset \). Similarly for three classes of sets.

We will say that \( A \) has property \( \mathcal{W} \) and write \( \mathcal{W}(\mathcal{A}) \), if
\[ \text{int } \mathcal{P} = \text{int } \mathcal{R}, \quad \text{for every } \mathcal{P}, \mathcal{R} \in \mathcal{A}, \]
(where \( \text{int } \mathcal{P} \) denotes the interior of the set \( \mathcal{P} \)).

Now we present the main result of this paper.

**Theorem.** Assume that the values of \( A \) are intervals and \( F : (0, \infty) \times X \to 2^X \) is given by (A). Then \( F \) is an iteration semigroup if and only if one of the following conditions holds:

(i) \( A \) has values of type \( \mathcal{P} \);
(ii) \( A \) has values of type \( \mathcal{P}_{-\infty} \);
(iii) \( A \) has values of type \( \mathcal{P}\mathcal{P}_{-\infty} \);
(iv) \( A \) has values of type \( \mathcal{S}\mathcal{P} \) and
\[ \mathcal{S} \preceq \mathcal{P} \quad \text{and} \quad \text{card } \mathcal{P} = 1; \]
(v) \( A \) has values of type \( \mathcal{L}\mathcal{P} \) and
\[ \mathcal{L} \preceq \mathcal{P} \quad \text{and} \quad \mathcal{W}(\mathcal{L}) \quad \text{and} \quad \text{card } \mathcal{P} = 1; \]
(vi) \( A \) has values of type \( \mathcal{L}\mathcal{P}_{-\infty} \);
(vii) \( A \) has values of type \( \mathcal{L}\mathcal{S}\mathcal{P} \) and
\[ \mathcal{L} \preceq \mathcal{S} \preceq \mathcal{P} \quad \text{and} \quad \mathcal{W}(\mathcal{L}) \quad \text{and} \quad \text{card } \mathcal{P} = 1; \]
(viii) \( A \) has values of type \( \mathcal{S} \);
(ix) $A$ has values of type $L$;
(x) $A$ has values of type $LS$ and
\[
L \preceq S \quad \text{and} \quad W(L);
\]
(xi) $A$ has values of type $P_\infty$;
(xii) $A$ has values of type $L_{-\infty}$;
(xiii) $A$ has values of type $R$;
(xiv) $A$ has values of type $L_{-\infty}P_\infty$ and
\[
L_{-\infty} \preceq P_\infty \quad \text{and} \quad W(L_{-\infty}) \quad \text{and} \quad W(P_\infty);
\]
(xv) $A$ has values of type $SP_\infty$ and
\[
S \preceq P_\infty \quad \text{and} \quad W(P_\infty);
\]
(xvi) $A$ has values of type $L_{-\infty}R$;
(xvii) $A$ has values of type $RP_\infty$;
(xviii) $A$ has values of type $L_{-\infty}S$ and
\[
L_{-\infty} \preceq S \quad \text{and} \quad W(L_{-\infty});
\]
(xix) $A$ has values of type $L_{-\infty}SP_\infty$ and
\[
L_{-\infty} \preceq S \preceq P_\infty \quad \text{and} \quad W(L_{-\infty}) \quad \text{and} \quad W(P_\infty).
\]

Proof. At first assume that $F$ is an iteration semigroup. Of course, by Facts 3, 4 and 5, condition (H1) is satisfied. We show that one of the conditions (i)–(xix) holds.

First we prove that the multifunction $A$ can have only the values mentioned in conditions (i)–(xix).

Consider the case $q \in S$. Then, by Fact 5, $F$ has the values in the set $L \cup S \cup P_{-\infty} \cup P$. Notice that if all of the values of $A$ belong to the same class, then it can be $P_{-\infty}$ [cf. (ii)] or $P$ [cf. (i)] (because $q \in S$). Assume that $A$ has values in exactly two classes of sets. Then we have 6 possibilities. Since $q \in S$ it cannot be $LS$, and, by the condition (d) of Fact 5, it cannot be $SP_{-\infty}$. Thus we obtain the types from (iii)–(vi). Notice that if $A$ has values in exactly three classes of sets, then, by Fact 5(d) and (e), it has to be the type $LSP$ [cf. (vii)]. Due to the condition (e) of Fact 5, there does not exist $x \in X$ such that $A(x) \in LSP_{-\infty}P$.

Now pass to the case when $q \notin S$ and $q \neq \infty$. Therefore, according to Fact 4, the values of $A$ can be of the types: $L$, $S$ or $LS$ (cf. (viii)–(x)).

Assume that $q = \infty$. Then, by Lemma 1, for every $x \in X$
\[
A(x) \in L_{-\infty} \cup S \cup P_\infty \cup R.
\]

If $A$ has only one kind of values then each of the types: $L_{-\infty}$, $S$, $P_\infty$, $R$ is possible [cf. (viii), (xi)–(xiii)]. If $A$ has values in exactly two classes of sets then by Lemma 1(iv), it cannot be $SR$. Thus the conditions (xiv)–(xviii) describe all the types of the values in this case. Observe that if $A$ has values in exactly three
classes of sets, then, by Lemma 1[(iv) and (v)], it has to be the type $L_{-\infty}SP_{\infty}$ [cf. (xix)] and according to the condition (v) of Lemma 1, the multifunction $A$ cannot have values of the type $L_{-\infty}SP_{\infty}R$.

The relations “$\preceq$” between the classes of sets follow immediately from Fact 5 [in the cases (iv), (v) and (vii)], from Fact 4 [in the case (x)] and from Lemma 1 [in the cases (xiv)–(xv) and (xviii)–(xix)].

The conditions relating to the cardinality of sets and property $W$ follow immediately from Remark 4.

Now pass to the proof of the contrary implication. At first notice that if (viii) holds then $A$ is single-valued and, by Fact 2, $F$ is an iteration semigroup. In the cases (i)–(iii), (vi), (ix), (xi)–(xiii) and (xvi)–(xvii) the condition

$$[A(x) + s] \cap A(z) \neq \emptyset$$

is satisfied for all $x, z \in X$ and $s \in (0, \tau(x)] \cap \mathbb{R}$. Hence (H1) holds. Moreover in this cases we have $F \equiv X$, so the multifunction $A$ generates an iteration semigroup $F$. It easy to observe that also in other cases condition (H1) is satisfied. Of course each of the conditions (xiv)–(xv) and (xviii)–(xix) defines the multifunction $A$ for which $q = \infty$. Therefore by Fact 3 in all these cases $F$ is an iteration semigroup. In the cases (iv), (v) and (vii) we obtain that $q \in S$ and the conditions (a)–(g) of Fact 5 hold, thus $F$ is an iteration semigroup. Now assume (x). Then $q \notin S$ and $q \neq \infty$, so due to Fact 4 we obtain that $F$ is an iteration semigroup.

□

As follows from the above proof if $A$ satisfies one of conditions: (i)–(iii), (vi), (ix), (xi)–(xiii), (xvi) or (xvii) of the Theorem, then $F \equiv X$.

Since in our paper we always assume that $A$ satisfies condition (H), we can ask what it means if one of conditions (i)–(xix) of the Theorem holds. Notice that for an arbitrary interval-valued function $A$ we obtain:

- if $A$ satisfies one of the conditions: (i)–(iii), (vi), (ix), (xi)–(xiii), (xvi), (xvii), then $S$ is an interval, thus by Remark 1 condition (H) holds;
- if at least one of the values of $A$ is a singleton [see (iv), (vii), (viii), (x), (xv), (xviii), (xix)] then by Remark 1 we get: (H) holds if and only if $S$ is an interval;
- if $A$ satisfies (v) then it is easy to see that: (H) holds if and only if $S$ is an interval;
- if $A$ satisfies (xiv) then we can notice that: if at least one of values of $A$ is a closed set, then (H) holds if and only if $S$ is an interval, i.e. $S = \mathbb{R}$; otherwise: (H) holds if and only if card $(\mathbb{R} \setminus S) \leq 1$. 
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