On Lagrangian and Hamiltonian systems with homogeneous trajectories

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Abstract
Motivated by various results on homogeneous geodesics of Riemannian spaces, we study homogeneous trajectories, i.e. trajectories which are orbits of a one-parameter symmetry group, of Lagrangian and Hamiltonian systems. We present criteria under which an orbit of a one-parameter subgroup of a symmetry group $G$ is a solution of the Euler–Lagrange or Hamiltonian equations. In particular, we generalize the ‘geodesic lemma’ known in Riemannian geometry to Lagrangian and Hamiltonian systems. We present results on the existence of homogeneous trajectories of Lagrangian systems. We study Hamiltonian and Lagrangian geodesic orbit (g.o.) spaces, i.e. homogeneous spaces $G/H$ with $G$-invariant Lagrangian or Hamiltonian functions on which every solution of the equations of motion is homogeneous. We show that the Hamiltonian g.o. spaces are related to the functions that are invariant under the coadjoint action of $G$. Riemannian g.o. spaces thus correspond to special $\text{Ad}^*(G)$-invariant functions. An $\text{Ad}^*(G)$-invariant function that is related to a g.o. space also serves as a potential for the mapping called the ‘geodesic graph’. As an illustration we discuss the Riemannian g.o. metrics on $\text{SU}(3)/\text{SU}(2)$.

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1. Introduction

Let $M$ be a Riemannian manifold. A geodesic in $M$ is called homogeneous if it is the orbit of a one-parameter group of isometries of $M$. A homogeneous Riemannian manifold $M = G/K$, where $G$ is a connected Lie group and $K$ is a closed subgroup, is a geodesic orbit (g.o.) space with respect to $G$, if every geodesic in it is the orbit of a one-parameter subgroup of $G$.

The homogeneous space $M = G/K$ is called a reductive space if there exists a direct sum decomposition (called reductive decomposition) $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$ of the Lie algebra of $G$, where $\mathfrak{m}$ is an $\text{ad}(K)$-invariant linear subspace of $\mathfrak{g}$ and $\mathfrak{k}$ is the Lie algebra of $K$. It is known that...
all Riemannian homogeneous spaces are reductive. If $M = G/K$ is Riemannian and there exists a reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$ such that each geodesic in $M$ starting at the origin $o \in M$ is an orbit of a one-parameter subgroup of $G$ generated by some element of $\mathfrak{m}$, then $M$ is called a naturally reductive space with respect to $G$, and $\mathfrak{m}$ is called a natural complement. The origin $o$ is the image of $K$ by the canonical projection $G \rightarrow G/K$.

Obviously, every naturally reductive space is a g.o. space as well. It was believed some decades ago that the converse is also true, i.e. every g.o. space is isometric to some naturally reductive space. A counterexample, however, was found by Kaplan [1], initiating the extensive study of g.o. spaces [3–21]. Pseudo-Riemannian g.o. spaces were also investigated recently [22–24]. Before Kaplan’s example appeared, Szenthe discovered a geometrical background for the situation when a g.o. space is not naturally reductive [2], not knowing whether such a situation can be realized or not. This result had considerable influence on the later studies.

In general, it is possible that a homogeneous Riemannian space $M = G/K$ is not naturally reductive with respect to $G$, but one can take other groups $G'$ and $K'$ so that $M = G'/K'$ and $M$ is naturally reductive with respect to $G'$. The same situation can occur for g.o. spaces as well. It is also possible in some cases that a g.o. space can be made naturally reductive by taking a different symmetry group $G'$, but there also exist g.o. spaces for which this is not possible, i.e. which are in no way naturally reductive. Kaplan’s example is of the latter type.

Since Riemannian (and pseudo-Riemannian) manifolds can be viewed as a special class of the manifolds with a Lagrangian or Hamiltonian function, it is interesting to consider the generalization of the g.o. property to homogeneous spaces with invariant Lagrangian and Hamiltonian functions and to ask whether the known results for the Riemannian spaces can be generalized, and whether the techniques of Lagrangian or Hamiltonian dynamics can be used for the study of Riemannian g.o. spaces. In this paper we present the results that we obtained in relation to these questions.

A subject closely related to the study of g.o. spaces is the characterization of the homogeneous geodesics in Riemannian manifolds. Homogeneous geodesics are of interest also in Finsler geometry, pseudo-Riemannian geometry and in dynamics. We refer the reader to [25–41] and further references therein. The present paper is also concerned with the characterization of homogeneous trajectories in Lagrangian and Hamiltonian dynamical systems, partly because this is necessary for the study of dynamical systems that have the g.o. property. In the physics literature the homogeneous geodesics are usually called relative equilibria; therefore, we shall also use this term, along with the term homogeneous trajectory. We mention that another name for homogeneous geodesics that appears in the literature is stationary geodesic. At times we shall use the terms Lagrangian space and Hamiltonian space for Lagrangian and Hamiltonian dynamical systems, in analogy with the term Riemannian space.

The paper is organized as follows. In section 2 we discuss the case of Lagrangian systems. We describe criteria for an orbit of a one-parameter subgroup to be a solution of the Euler–Lagrange equations, including the Lagrangian version of the ‘geodesic lemma’. We also present results concerning the existence of relative equilibria.

In section 3 we discuss the case of Hamiltonian systems. We describe criteria for an orbit of a one-parameter subgroup to be a solution of the Hamiltonian equations, including the Hamiltonian version of the geodesic lemma. Then we turn to the characterization of Hamiltonian g.o. spaces. In particular, we show that the Hamiltonian g.o. spaces are closely related to the functions which are invariant under the coadjoint action of $G$. Riemannian g.o. spaces correspond, of course, to special $Ad^*(G)$-invariant functions. Naturally reductive metrics, in particular, are known to correspond to quadratic $Ad^*(G)$-invariant polynomials [42, 43]. An $Ad^*(G)$-invariant function that is related to a g.o. space also serves as a potential
for the mapping called the geodesic graph, which was introduced originally by Szenthe [2] and which has proved to be useful for the description of Riemannian g.o. spaces. We present certain results on geodesic graphs, and then we describe a criterion based on the relation between g.o. spaces and $Ad^*(G)$-invariant functions that can be used to find g.o. Hamiltonians or metrics. We also describe a generalization of the notion of Hamiltonian g.o. space.

In section 4 we discuss the two-parameter family of Riemannian g.o. metrics on $SU(3)/SU(2)$ for the illustration of the results of section 3. We calculate the geodesic graph in a new way, utilizing the relation between g.o. spaces and $Ad^*(G)$-invariant functions.

2. Lagrangian systems with homogeneous trajectories

Let $M$ be a connected manifold with a Lagrangian function $L : TM \to \mathbb{R}$ on it. The Euler–Lagrange equation for a curve $\gamma : I \to M$, where $I$ is an interval, is

$$\frac{\partial L}{\partial x^j}(\gamma(t), \dot{\gamma}(t)) = \frac{d}{dt} \left( \frac{\partial L}{\partial v^j}(\gamma, \dot{\gamma}) \right)(t) \quad \forall t \in I,$$

or, expanding the right-hand side,

$$\frac{\partial L}{\partial x^j}(\gamma(t), \dot{\gamma}(t)) = \frac{\partial^2 L}{\partial x^j \partial v^j}(\gamma(t), \dot{\gamma}(t))\dot{\gamma}^j(t) + \frac{\partial^2 L}{\partial v^j \partial v^j}(\gamma(t), \dot{\gamma}(t))\ddot{\gamma}^j(t).$$

Here and throughout the paper we use the Einstein summation convention for indices of coordinates related to $M$. In the special case when $L$ is the quadratic form corresponding to a Riemannian or pseudo-Riemannian metric, a solution $\gamma : I \to M$ of the Euler–Lagrange equations is a geodesic with affine parametrization.

The Lagrangian is regular if the bilinear form $\frac{\partial^2 L}{\partial v^j \partial v^j}(x, v)$ is nondegenerate for any $(x, v) \in TM$. The regularity of a Lagrangian implies that the solution of the Euler–Lagrange equations is unique for given initial data $(x, v) \in TM$. If a Lagrangian corresponds to a metric, then it is regular.

In the following we assume that $L$ is invariant under the action of a connected Lie group $G$ on $TM$ induced by an action of $G$ on $M$. We denote the Lie derivative with respect to a vector field $Z$ as $L_Z$. We use the notation $\circ$ for the composition of two functions, i.e. if $f$ and $g$ are two functions, then $f \circ g$ is the function for which $(f \circ g)(x) = f(g(x))$.

In the derivation of the results of this section the Euler–Lagrange equation, an equation expressing the invariance of $L$ and equations characterizing the velocity and acceleration of orbits have important roles.

Let $Z_a : M \to TM$ and $\hat{Z}_a : TM \to TTM$, where $a \in g$, be the infinitesimal generator vector fields for the action of $G$ on $M$ and $TM$, respectively. Their coordinate form is

$$Z_a(x) = \frac{\partial \phi^i}{\partial \tau}(0, x) \frac{\partial}{\partial x^i}, \quad x \in M$$

and

$$\hat{Z}_a(x, v) = \frac{\partial \phi^i}{\partial \tau}(0, x) \frac{\partial}{\partial x^i} + \frac{\partial^2 \phi^i}{\partial \tau \partial x^j}(0, x) v^j \frac{\partial}{\partial v^i}, \quad (x, v) \in TM,$$

where $\phi_a : \mathbb{R} \times M \to M$ is the action of the one-parameter subgroup generated by $a \in g$ and $\tau$ denotes the first variable of $\phi_a$.

The invariance of $L$ under the action of $G$ implies the following symmetry condition:

$$L_{\hat{Z}_a}L(x, v) = \frac{\partial L}{\partial x^j}(x, v) \frac{\partial \phi^i}{\partial \tau}(0, x) + \frac{\partial L}{\partial v^j}(x, v) \frac{\partial^2 \phi^i}{\partial \tau \partial x^j}(0, x) v^j = 0,$$

where $a \in g$. This equation holds for all $(x, v) \in TM$. 3
The orbit of the one-parameter subgroup generated by \( a \in g \) in \( M \) with the initial point \( x \) is the curve \( \gamma : I \to M, t \mapsto \phi_a(t, x) \). For the velocity

\[
\dot{\gamma}(t) = \frac{\partial \phi_a}{\partial \tau}(t, x)
\]

of this orbit the equation

\[
\dot{\gamma}^i(t) = \frac{\partial \phi_a^i}{\partial x^j}(t, x) \dot{\gamma}^j(0) = \frac{\partial \phi_a^i}{\partial \tau}(0, x)
\]

holds because of the group property. For the acceleration we have

\[
\ddot{\gamma}^i(t) = \frac{\partial^2 \phi_a^i}{\partial \tau \partial x^j}(t, x) \frac{\partial \phi_a^j}{\partial \tau}(0, x) = \frac{\partial^2 \phi_a^i}{\partial \tau^2}(t, x).
\]

**Theorem 2.1.** The orbit of a one-parameter subgroup of \( G \) starting at \( x \in M \) is a solution of the Euler–Lagrange equations corresponding to the (not necessarily regular) Lagrangian \( L \) if and only if \( x \) is a critical point of the function \( L \circ Z_a \), i.e.

\[
d(L \circ Z_a)(x) = 0,
\]

where \( Z_a \) is the generator vector field of the subgroup.

**Proof.** Because of the invariance of the Lagrangian, an orbit of a one-parameter symmetry group is a solution of the Euler–Lagrange equations if and only if it satisfies the Euler–Lagrange equations at the initial point. First, let us assume that the orbit is a solution of the Euler–Lagrange equations. Differentiating the symmetry condition (5) with respect to \( v^j \) yields

\[
0 = \frac{\partial}{\partial v^j} L \cdot Z_a(x, v) = \frac{\partial^2 L}{\partial x^i \partial v^j}(x, v) \frac{\partial \phi_a^i}{\partial \tau}(0, x) + \frac{\partial^2 L}{\partial v^i \partial v^j}(x, v) \frac{\partial^2 \phi_a^i}{\partial \tau \partial x^j}(0, x).
\]

Substituting the right-hand side of (8) for \( \ddot{\gamma} \) in the Euler–Lagrange equation (2) at \( t = 0 \) gives

\[
0 = \frac{\partial^2 L}{\partial x^i \partial v^j}(x, v) \frac{\partial \phi_a^i}{\partial \tau}(0, x) + \frac{\partial^2 L}{\partial v^i \partial v^j}(x, v) \frac{\partial^2 \phi_a^i}{\partial \tau \partial x^j}(0, x).
\]

where \( v = \dot{\gamma}(0) \). Setting \( v = \dot{\gamma}(0) \) also in (10) and subtracting from (11) gives

\[
\frac{\partial L}{\partial x^i}(x, v) + \frac{\partial L}{\partial v^j}(x, v) \frac{\partial^2 \phi_a^i}{\partial \tau \partial x^j}(0, x) = 0,
\]

where \( v = \dot{\gamma}(0) \), which is just the coordinate form of (9). Considering the reverse direction of the statement, it is clear now that if (12) and (10) hold, then (11) follows. \( \square \)

A similar theorem is stated in [31] (see also [41]). However, our proof is different from those given in [31] and [41]. The function \( L \circ Z_a \) is called augmented Lagrangian in [31] and locked Lagrangian in [41].

**Definition 2.1.** An element \( a \) of \( g \) is called a relative equilibrium vector at \( x \in M \) if the orbit of the one-parameter subgroup of \( G \) generated by \( a \) and starting at \( x \) is a solution of the Euler–Lagrange equations.

In Riemannian geometry the interesting relative equilibrium vectors are, of course, those which generate orbits that are not single points in \( M \). We note that in Riemannian geometry the relative equilibrium vectors are usually called geodesic vectors.
The set of relative equilibrium vectors at \( x \) is invariant under \( G_x \), the stabilizer of \( x \). If \( g x = y \) for some \( x, y \in M \) and \( g \in G \), then the set of relative equilibrium vectors at \( y \) can be obtained from that at \( x \) by the adjoint action of \( g \).

As regards the existence of relative equilibria, the following corollary of theorem 2.1 can be stated.

**Theorem 2.2.** Let \( M, G, L \) be as in theorem 2.1 and let \( M \) be compact. For any \( a \in g \) there exists at least one solution of the Euler–Lagrange equations which is the orbit of the one-parameter subgroup generated by \( a \). If there exists an \( a \in g \) such that \( Z_a(x) \neq 0 \) \( \forall x \in M \), then there exists at least one solution of the Euler–Lagrange equations which is the orbit of the one-parameter subgroup generated by \( a \) and is not a single point in \( M \). If, in addition, \( M \) is also homogeneous with respect to the action of \( G \), then there exists at least one nonzero relative equilibrium vector at every point in \( M \), which generates an orbit that is not a single point.

This result can be found e.g. in [27] (proposition 5.2) for the special case of Lagrangians that describe geodesic motion in Riemannian manifolds.

In the rest of this section we consider the case when \( M \) is a homogeneous space. For a homogeneous space \( M = G/K \) there is a linear map \( f_x : g \rightarrow T_xM, a \mapsto Z_a(x) \) for each point \( x \in M \). We use the notation \( f \) for \( f_0 \) (i.e. we omit the subscript \( 0 \) denoting the origin in \( G/K \)).

The dual of a vector space \( V \) will be denoted by \( V^* \). The contraction (or natural pairing) between \( V \) and \( V^* \) will be denoted in the following way: \( (w|v) \), where \( w \in V^* \) and \( v \in V \). The transpose of a linear map \( A : V \rightarrow W \) will be denoted by \( A^* \) (it is defined as \( A^* : W^* \rightarrow V^*, w \mapsto w \circ A \)).

The following lemma, which concerns homogeneous manifolds with invariant Lagrangians and is the generalization of the known ‘geodesic lemma’ for the Riemannian case [6] (see also, for example, [7, 9, 33]), gives a condition for an element of \( g \) to be a relative equilibrium vector at \( o \). This is a local condition in the sense that it is given in terms of \( L \) restricted to \( T_oM \), the elements of \( g \) and the values of the infinitesimal generator vector fields at \( o \). In Riemannian geometry the geodesic lemma has proved to be very useful in the study of homogeneous geodesics.

**Lemma 2.1** (Geodesic lemma). Let \( M = G/K \) be a homogeneous space with a \( G \)-invariant Lagrangian \( L : TM \rightarrow \mathbb{R} \). An element \( a \in g \) is a relative equilibrium vector at \( o \) if and only if

\[
(dL_o(f(a)))[f([a, b])] = 0 \quad \forall b \in g. \tag{13}
\]

where \( L_o \) is \( L \) restricted to \( T_oM \). In particular, if \( L \) corresponds to a Riemannian metric, then (13) takes the form

\[
(f([a, b]), f(a)) = 0 \quad \forall b \in g. \tag{14}
\]

or, equivalently,

\[
([a, b]_m, a_m) = 0 \quad \forall b \in g. \tag{15}
\]

where the index \( m \) denotes the \( m \)-component related to a reductive decomposition \( g = k \oplus m \), and \( m \) is assumed to be identified with \( T_oM \) by \( f \).

**Proof.** Let us first assume that \( a \) is a relative equilibrium vector. Equation (9) in theorem 2.1 is equivalent to \( \mathcal{L}_{Z_a}(L \circ Z_a)(o) = 0 \forall b \in g \). In the coordinate form
\( \mathcal{L}_{Z_0}(L \circ Z_0)(o) = \frac{\partial \phi^i}{\partial \tau}(0, o) \frac{\partial L}{\partial x^i}(o, \frac{\partial \phi_a}{\partial \tau}(0, o)) + \frac{\partial L}{\partial v^i}(0, o) \frac{\partial \phi^j}{\partial \tau}(0, o) \frac{\partial^2 \phi^i}{\partial \tau \partial x^j}(0, o) = 0. \)

Taking the symmetry condition (5) at the point \((o, \frac{\partial \phi_a}{\partial \tau}(0, o))\), we get

\( \mathcal{L}_{Z_0} \left( o, \frac{\partial \phi_a}{\partial \tau}(0, o) \right) = \frac{\partial \phi^i}{\partial \tau}(0, o) \frac{\partial L}{\partial x^i}(o, \frac{\partial \phi_a}{\partial \tau}(0, o)) + \frac{\partial L}{\partial v^i}(o, \frac{\partial \phi_a}{\partial \tau}(0, o)) \frac{\partial^2 \phi^i}{\partial \tau \partial x^j}(0, o) \frac{\partial \phi_j}{\partial \tau}(0, o) = 0. \)

Subtracting these two equations gives

\[
\frac{\partial L}{\partial v^i} \left( o, \frac{\partial \phi_a}{\partial \tau}(0, o) \right) \left[ \frac{\partial \phi^i}{\partial \tau}(0, o) \frac{\partial^2 \phi^j}{\partial \tau \partial x^i}(0, o) - \frac{\partial \phi^j}{\partial \tau}(0, o) \frac{\partial^2 \phi^i}{\partial \tau \partial x^j}(0, o) \right] = 0,
\]

which is the coordinate expression for (13). Conversely, assuming that (18) holds and using (17) one obtains (16). The second part of the lemma concerning the Riemannian case follows obviously from the first part.

\[ \square \]

Formula (15) for Riemannian spaces is well known and is also a generalization of Arnold’s result about homogeneous geodesics of left-invariant metrics on Lie groups [26].

Let \( r : \mathbb{R} \to \mathfrak{g} \) be the adjoint orbit starting at \( a \) and generated by \( b \). \( f(\{a, b\}) \) is the tangent vector of the curve \( f \circ r \) at the point \( f(a) \). Equation (13) means that the derivative of \( L_o \) at \( f(a) \) along this tangent vector is 0.

The following theorems 2.3 and 2.4 are about the existence of relative equilibria.

**Theorem 2.3.** Let \( M = G/K \) be a homogeneous space with a \( G \)-invariant Lagrangian \( L : TM \to \mathbb{R} \). If \( G \) is compact, then each adjoint orbit of \( G \) contains at least one relative equilibrium vector at \( o \), and each adjoint orbit of \( G \) that is not contained entirely by \( \mathfrak{k} \) contains at least one relative equilibrium vector at \( o \) which generates an orbit that is not a single point.

**Proof.** Any adjoint orbit \( O \) of \( G \) is compact. \( f(O) \) is also compact and \( L_o \) is continuous on it; thus, there exists at least one \( \bar{v} \in f(O) \) so that \( L_o|_{f(O)} \) is minimal or maximal at \( \bar{v} \). Because of this extremality the derivative of \( L_o \) is zero at \( \bar{v} \) along any curve that lies in \( f(O) \) and passes through \( \bar{v} \). It is clear from the remark after the proof of the geodesic lemma that any element of \( f^{-1}(\bar{v}) \cap O \) is a relative equilibrium vector at \( o \).

If an adjoint orbit \( O \) is not contained entirely by \( \mathfrak{k} \), then \( f(O) \neq \{0\} \); thus, there exists at least one \( \bar{v} \in f(O) \) so that \( \bar{v} \neq 0 \) and \( L_o|_{f(O)} \) is minimal or maximal at \( \bar{v} \). Any element of \( f^{-1}(\bar{v}) \cap O \) is a relative equilibrium vector at \( o \) that generates an orbit that is not a single point.

\[ \square \]

**Theorem 2.4.** Let \( M = G/K \) be a homogeneous space with a \( G \)-invariant Lagrangian \( L : TM \to \mathbb{R} \). If \( G \) is solvable and the image space of \( dL_o|_{T_M(0)} \) contains vectors of arbitrary direction, then there exists at least one relative equilibrium vector at \( o \), which generates an orbit that is not a single point.

**Proof.** Consider the derived series of \( \mathfrak{g} \), i.e. the sequence

\[
\mathfrak{g}^{(0)} \supset \mathfrak{g}^{(1)} \supset \cdots \supset \mathfrak{g}^{(i)} \supset \cdots,
\]

where \( \mathfrak{g}^{(0)} = \mathfrak{g} \) and \( \mathfrak{g}^{(i)} = [\mathfrak{g}^{(i-1)}, \mathfrak{g}^{(i-1)}] \) for \( i = 1, 2, \ldots \). Because of the solvability of \( G \), the derived series strictly decreases and ends in the null space. Consequently, there exists an index
Proposition 2.1. Let \( r \geq 0 \) such that \( f(g^{(r)}) = T_0 M \), but \( f(g^{(r+1)}) \) is a proper subspace of \( T_0 M \). The connected subgroup \( G^{(r)} \) corresponding to \( g^{(r)} \) still acts transitively on \( M \); therefore, it is necessary and sufficient for a vector to be a relative equilibrium vector that (13) holds for all \( b \in g^{(r)} \). The condition imposed on \( dL_0 \) in the theorem ensures that there exists an \( \tilde{v} \in T_0 M \setminus \{0\} \) such that \( (dL_0(\tilde{v})/f([g^{(r)}, g^{(r)}])) = 0 \), implying that any element of \( f^{-1}(\tilde{v}) \cap g^{(r)} \) is a relative equilibrium vector.

This theorem is similar to some parts of proposition 3 of [33]. It is clear from the proof that the solvability of \( G \) is not necessary, it can be replaced by the weaker condition that there exists an element \( g^{(r+1)} \) of the derived series of \( g \) such that \( f(g^{(r+1)}) \) is a proper subspace of \( T_0 M \).

The condition of regularity has not been imposed on the Lagrangians so far. It is assumed, however, in proposition 2.1 and in theorem 2.5, which characterize Lagrangian g.o. spaces.

Definition 2.2. Let \( M = G/K \) be a homogeneous space and let \( L : M \to \mathbb{R} \) be a \( G \)-invariant Lagrangian function. \((M, L)\) is a called a Lagrangian geodesic orbit (g.o.) space with respect to \( G \), if every solution of the Euler–Lagrange equations corresponding to \( L \) is an orbit of a one-parameter subgroup of \( G \).

In other words, a Lagrangian g.o. space is defined by the property that every solution of the Euler–Lagrange equations is a relative equilibrium. In the general Lagrangian mechanical context one could introduce a new name instead of ‘geodesic orbit space’, since the latter bears a reference to Riemannian geometry. In this paper, however, we shall not introduce such a name. This applies also to the ‘geodesic lemma’ and to the ‘geodesic graph’ defined below.

Proposition 2.1. Let \( M = G/K \) and \( L \) be as in definition 2.2 and assume that \( L \) is regular. The Lagrangian dynamical system \((M, L)\) has the g.o. property with respect to \( G \) if and only if for all \( v \in T_0 M \) there exists an \( a \in g \) such that \( f(a) = v \) and \( a \) is a relative equilibrium vector.

Definition 2.3. Let \( (M = G/K, L) \) be a Lagrangian system that has the g.o. property with respect to \( G \). A mapping \( \xi : T_0 M \to g \) with the properties that \( f(\xi(v)) = v \) and \( \xi(v) \) is a relative equilibrium vector at \( o \) for all \( v \in T_0 M \) is called a geodesic graph. Obviously, there exists at least one geodesic graph for every Lagrangian system that has the g.o. property. \( f(\xi(v)) = v \) means that the velocity of the orbit generated by \( \xi(v) \) is \( v \) at \( o \).

In Riemannian geometry the geodesic graph is very useful for studying g.o. spaces. Important results about its properties were obtained in [2, 9]. It follows directly from the definition of naturally reductive metrics in the introduction and from definition 2.3 that the naturally reductive spaces are precisely those Riemannian g.o. spaces that admit a \( K \)-equivariant linear geodesic graph. In order to see this in detail, assume first that \( M = G/K \) is a naturally reductive space with the natural reductive decomposition \( g = \mathfrak{k} \oplus m \). Then \( f|m \) is a linear bijection between \( m \) and \( T_0 M \), and its inverse \( \xi = (f|m)^{-1} \) obviously has the property \( f(\xi(v)) = v \). \( \xi \) is also \( K \)-equivariant, since \( m \) is an \( Ad(K) \)-invariant subspace of \( g \). The natural reductivity of \( M \) implies that for any \( v \in T_0 M \) there is an \( a \in m \) so that the orbit generated by \( a \) and starting at \( o \) coincides with the geodesic with initial velocity \( v \). However, the initial velocity of the orbit generated by \( a \) is \( f(a) \); therefore, \( a = \xi(v) \). This shows that \( \xi \) is a \( K \)-equivariant linear geodesic graph. Conversely, if \( M \) is a Riemannian g.o. space and \( \xi \) is a \( K \)-equivariant linear geodesic graph, then \( \xi(T_0 M) \) is an \( Ad(K) \)-invariant linear subspace of \( g \), due to the linearity and \( K \)-equivariance of \( \xi \). \( \xi(T_0 M) \) is complementary to \( \mathfrak{k} \), because \( f(\mathfrak{k}) = 0 \) and \( f(\xi(T_0 M)) = T_0 M \). \( g = \mathfrak{k} \oplus \xi(T_0 M) \) is thus a reductive decomposition. By definition 2.3, for an arbitrary geodesic \( \gamma \) starting at \( o \) the Lie
algebra element \( \xi(v) \in \xi(T_{\gamma} M) \), where \( v \) is the initial velocity of \( \gamma \), generates an orbit \( \tilde{\gamma} \) that is also a geodesic with initial velocity \( v \). Since geodesics are uniquely determined by their initial data, \( \gamma \) and \( \tilde{\gamma} \) coincide. This shows that the reductive decomposition \( g = \mathfrak{k} \oplus \xi(T_{\gamma} M) \) is also natural.

We note that there is a minor difference between our definition of the geodesic graph and the usual definition; in the usual definition one has a direct sum decomposition \( g = m \oplus k \), and one takes the \( k \)-component of \( \xi(v) \) as the value of the geodesic graph at \( v \), since the \( m \)-component is uniquely determined by the property \( f(\xi(v)) = v \). In fact, in the literature \( m \) is often identified with \( T_{\mathcal{O}} M \) by \( f \). It is also usual in the literature to include in the definition of the geodesic graph the requirement that it should be \( K \)-equivariant.

The following consequence of proposition 2.1 and of the geodesic lemma, in particular of (13), applying to the special case \( M = G \), is well known [28].

**Theorem 2.5.** If \( M = G \), i.e. \( L \) is a regular left-invariant Lagrangian on \( G \), then \( (M, L) \) is a g.o. space with respect to \( G \) if and only if \( L_e = L|_{T_e G} \) (where \( e \) is the unit element of \( G \)) is invariant under the adjoint action of \( G \). Any function on \( T_e G \) can be extended uniquely to a left-invariant function on \( G \); therefore, the Lagrangians on \( G \) that have the g.o. property with respect to \( G \) are in one-to-one correspondence with the regular \( \text{Ad} \)-invariant functions on \( g \).

We note that in the case \( M = G \) the equation (13) expresses the \( \text{Ad}(G) \)-invariance of \( L_e \).

In the next section we turn to the Hamiltonian formalism, which is better suited to the characterization of g.o. spaces than the Lagrangian formalism.

### 3. Hamiltonian systems with homogeneous trajectories

Let \( M \) be a manifold with a Hamiltonian function \( H : T^* M \to \mathbb{R} \). We denote the Hamiltonian vector field generated by \( H \) on the symplectic manifold \( T^* M \) by \( X_H \). In the coordinates, \( X_H \) is given by

\[
X_H (x, p) = \left( \frac{\partial H}{\partial p_i}(x, p), -\frac{\partial H}{\partial x^i}(x, p) \right).
\]

The Hamiltonian equations for a curve \( \gamma : I \to T^* M \) are the following:

\[
X_H (\gamma(t)) = \dot{\gamma}(t) \quad \forall t \in I, \tag{19}
\]

or equivalently

\[
\frac{\partial H}{\partial p_i}(x, p) = \dot{x}^i \tag{20}
\]

\[
-\frac{\partial H}{\partial x^i}(x, p) = \dot{p}_i. \tag{21}
\]

The projection of a solution \( \gamma : I \to T^* M \) on \( M \) is a geodesic with affine parametrization in the special case when \( H \) is the quadratic form corresponding to a Riemannian or pseudo-Riemannian metric.

In the following we assume that \( H \) is invariant under the action of a connected Lie group \( G \) on \( T^* M \) induced by an action of \( G \) on \( M \). Let \( \tilde{Z}_a : T^* M \to TT^* M, a \in g \), be the infinitesimal generator vector fields for the action of \( G \) on \( T^* M \). Their coordinate form is

\[
\tilde{Z}_a (x, p) = \frac{\partial \phi^j}{\partial \tau}(0, x) \frac{\partial}{\partial x^j} + \frac{\partial^2 \phi^j}{\partial \tau \partial x^j}(0, x) p_j \frac{\partial}{\partial p_i}, \tag{22}
\]

where \( \phi_a \) is the same object as in section 2.

The invariance of \( H \) implies the following symmetry condition:

\[
L_{\tilde{Z}_a} H(x, p) = \frac{\partial H}{\partial x^j}(x, p) \frac{\partial \phi^j}{\partial \tau}(0, x) - \frac{\partial H}{\partial p_i}(x, p) \frac{\partial^2 \phi^j}{\partial \tau \partial x^j}(0, x) p_j = 0, \tag{23}
\]
where \( b \in g \). This equation holds for all \((x, p) \in T^*M\).

We recall that the momentum map for the action of \( G \) on \( T^*M \) is \( P : T^*M \rightarrow g^* \), \((x, p) \mapsto f^*_x(p)\), where \( f_x \) is the linear mapping introduced in section 2 after theorem 2.2. Clearly \( P \) is linear on each cotangent space \( T^*_xM \), \( x \in M \), and it is also equivariant. \( P \) restricted to the cotangent space \( T^*_xM \) at \( x \in M \) is the transpose of \( f_x \). \( P \) has the property that

\[
X(P|_a) = \hat{Z}^*_a \quad \forall a \in g.
\]

This property implies

\[
[X(P|_a), X(P|_b)] = X(P|[a, b]),
\]

where \( [\cdot, \cdot] \) on the left-hand side denotes the Lie bracket of vector fields. The functions \( P|_a \), \( a \in g \), are conserved quantities, i.e. the function \( P \) (and thus \( P|_a \) for all \( a \in g \)) is constant along the solutions of the Hamiltonian equations.

**Definition 3.1.** An element \( a \) of \( g \) is called a relative equilibrium vector at \((x, p) \in T^*M\) if the orbit of the corresponding one-parameter subgroup starting at \((x, p)\) is a solution of the Hamiltonian equations.

Since the momentum map is constant along the solutions of the Hamiltonian equations, if \( a \in g \) is a relative equilibrium vector at \((x, p) \in T^*M\), then \( a \) is an element of the stabilizer subgroup of \( P(x, p) \) with respect to the coadjoint action of \( G \).

**Lemma 3.1.** Let \( H : T^*M \rightarrow \mathbb{R} \) be a Hamiltonian function that is invariant under the action of a connected Lie group \( G \). \( a \in g \) is a relative equilibrium vector at \((x, p) \in T^*M\) if and only if

\[
X_H(x, p) = \hat{Z}^*_a(x, p),
\]

or, equivalently,

\[
d(H - (P|_a))(x, p) = 0,
\]

where \( P \) is the momentum mapping for the action of \( G \) on \( T^*M \).

The proof of this lemma can be found in [28] (proposition 4.3.7), for instance.

The following generalization of the geodesic lemma can be stated for homogeneous spaces with invariant Hamiltonians.

**Lemma 3.2 (Geodesic lemma).** Let \( M = G/K \) be a homogeneous space and \( H : T^*M \rightarrow \mathbb{R} \) a \( G \)-invariant Hamiltonian function. An element \( a \in g \) is a relative equilibrium vector at \((o, p)\), where \( o \) denotes the origin, if and only if

\[
dH_o(p) = f(a)
\]

and

\[
(f^*(p)|[a, b]) = 0 \quad \forall b \in g
\]

hold, where \( H_o \) is \( H \) restricted to \( T^*_oM \). Equation (28) is equivalent to the condition that the one-parameter subgroup generated by \( a \) is contained by the stabilizer subgroup of \( f^*(p) \in g^* \) with respect to the coadjoint action of \( G \).

**Proof.** Assume first that \( a \) is a relative equilibrium vector. Equation (27) is just the first of the two Hamiltonian equations at the initial point and in the coordinate form it reads as follows:

\[
\frac{\partial \phi^a_{\tau}}{\partial \tau}(x, 0) = \frac{\partial H}{\partial p_i}(x, p).
\]
The second Hamiltonian equation at the initial point is
\[
\frac{\partial^2 \phi^j_i}{\partial \tau \partial x^j}(0, x)p_j = \frac{\partial H}{\partial x^i}(x, p).
\] (30)

Substituting the left-hand sides of (29) and (30) for the right-hand sides of (29) and (30) in (23) gives
\[
p_j \left[ \frac{\partial \phi^i_j}{\partial \tau}(0, x) \frac{\partial^2 \phi^j_i}{\partial \tau \partial x^j}(0, x) - \frac{\partial \phi^i_j}{\partial \tau}(0, x) \frac{\partial^2 \phi^j_i}{\partial \tau \partial x^j}(0, x) \right] = 0 \quad \forall b \in \mathfrak{g},
\] (31)
which is just the coordinate form of the equation
\[
(p|\Za,Zb)(o)) = 0 \quad \forall b \in \mathfrak{g}.
\] (32)

This is equivalent to (28), because $[\Za,Zb](o) = f([a, b])$ and $(p| f([a, b])) = (f^*(p)||a, b|).$ Considering the reverse direction, it is clear that (30) can be obtained from (31), (29) and (23).

**Proposition 3.1.** The set of relative equilibrium vectors at any point $(o, p)$ is an affine subspace of $\mathfrak{g}.$

**Proof.** For any fixed $p$ equations (27) and (28) constitute an inhomogeneous linear system of equations for $a;$ thus, the solutions constitute an affine subspace in $\mathfrak{g}.$

A similar result holds for Lagrangian systems as well; in this case the statement is that the set of relative equilibrium vectors $a$ at $o$ for which $f(a)$ (which is the initial velocity of the orbit generated by $a$) is fixed is an affine subspace of $\mathfrak{g}.$ This follows from the fact that the equations $f(a) = v$ and (13), where $v$ is fixed, constitute an inhomogeneous linear system for $a.$

**Definition 3.2.** Let $M = G/K$ be a homogeneous space and $H : T^*M \rightarrow \mathbb{R}$ a $G$-invariant Hamiltonian function. $(M, H)$ is called a Hamiltonian geodesic orbit (g.o.) space with respect to $G,$ if every solution of the Hamiltonian equations is an orbit of a one-parameter subgroup of $G.$

In the following propositions 3.2 and 3.3 elementary conditions are given under which a homogeneous space with an invariant Hamiltonian has the g.o. property. They are the direct consequences of lemma 3.1 and lemma 3.2.

**Proposition 3.2.** Let $M = G/K$ be a homogeneous space and $H : T^*M \rightarrow \mathbb{R}$ a $G$-invariant Hamiltonian function. This dynamical system has the g.o. property with respect to $G$ if and only if
\[
\frac{dH(o, p)}{d\tau}(o, p) \in \{d(P|b)(o, p) : b \in \mathfrak{g}\} \quad \forall (o, p) \in T^*_oM
\] (33)
or, equivalently,
\[
X_H(o, p) \in \{\Za(o, p) : b \in \mathfrak{g}\} \quad \forall (o, p) \in T^*_oM.
\] (34)

**Proposition 3.3.** Let $M$ and $H$ be the same as in the previous proposition. $(M, H)$ is a g.o. space with respect to $G$ if and only if for all $p \in T^*_oM$ there exists an $a \in \mathfrak{g}$ such that
\[
\frac{dH_a(p)}{d\tau} = f(a)
\] (35)

and
\[
(f^*(p)||a, b|) = 0 \quad \forall b \in \mathfrak{g}
\] (36)
**Definition 3.3.** Let $M = G/K$ be a Hamiltonian g.o. space with respect to $G$. A mapping $\xi : T^*_o M \rightarrow g$ with the property that $\xi(p)$ is a relative equilibrium vector at $(o, p)$ for all $p \in T^*_o M$ is called a geodesic graph. Obviously, there exists at least one geodesic graph for every Hamiltonian g.o. space.

The naturally reductive spaces are precisely those Riemannian g.o. spaces which admit a linear $K$-equivariant geodesic graph. If $M = G/K$ is naturally reductive and $g = \mathfrak{t} \oplus \mathfrak{m}$ is a natural reductive decomposition, then the mapping $\xi$ defined as $\xi(p) = (f|_m)^{-1}(dH_o(p))$ is a linear $K$-equivariant geodesic graph. The mapping $p \mapsto dH_o(p)$ is a linear bijection between $T_o^* M$ and $T_o M$ in this case, since $H_o$ is quadratic and nondegenerate. If $M$ is a Riemannian g.o. space and $\xi$ is a linear $K$-equivariant geodesic graph, then $g = \mathfrak{t} \oplus \xi(T_o^* M)$ is a natural reductive decomposition. (See also the remarks after definition 2.3.)

In the last part of this section we describe the relation between g.o. spaces and $Ad^*(G)$-invariant functions, and we describe how an $Ad^*(G)$-invariant function that corresponds to a g.o. space can be used to obtain a geodesic graph. We present certain results on geodesic graphs and we discuss Riemannian g.o. spaces and naturally reductive spaces. We also describe a criterion that can be used to find Hamiltonians or metrics that have the g.o. property. Finally, we briefly discuss a generalization of the notion of Hamiltonian g.o. space.

**Lemma 3.3.** Let $M = G/K$ be a homogeneous space and $H : T^* M \rightarrow \mathbb{R}$ a $G$-invariant Hamiltonian function that has the g.o. property with respect to $G$. If $P$ is constant along a smooth curve $\gamma : I \rightarrow T^* M$, then $H$ is also constant along this curve.

**Proof.** The derivative $\frac{dH(\gamma)}{dt}$ of $H$ along $\gamma$ at $t \in I$ equals $(dH(\gamma(t)))\dot{\gamma}(t))$. It is sufficient to show that this number is zero for any $t \in I$. Let $t$ be a fixed element of $I$. It follows from proposition 3.2. that $(dH(\gamma(t)))\dot{\gamma}(t)) = (d(P|b)(\gamma(t)))\dot{\gamma}(t))$ for some $b \in g$. Since $P$ is constant along $\gamma$, the derivative of $P$ along $\gamma$ is zero; therefore, the derivative of $(P|b)$ is also zero, and thus $(d(P|b)(\gamma(t)))\dot{\gamma}(t)) = 0$. \qed

The following theorem is a direct consequence of lemma 3.3.

**Theorem 3.1.** Let $M = G/K$ be a homogeneous space and $H : T^* M \rightarrow \mathbb{R}$ a $G$-invariant Hamiltonian function that has the g.o. property with respect to $G$. If the connected components of the level sets of the momentum mapping $P$ have the property that any two points in them can be connected by a piecewise smooth curve, then $H$ is constant on the connected components of the level sets of $P$. If, in addition, $H$ takes the same value on all connected components of any level set of $P$, then $H$ takes the form

$$H = h \circ P,$$

where $h : g^* \rightarrow \mathbb{R}$ is an $Ad^*(G)$-invariant function.

$P$ is an analytic function, and therefore its rank is maximal on an open dense subset $N$ of $T^* M$, which is $G$-invariant. It follows that in $N$ the level sets of $P$ are submanifolds; therefore, the condition of theorem 3.1 is satisfied and thus $H$ is constant on the connected components of the level sets of $P|_N$.

The formula $H = h \circ P$ always holds locally in $N$; if $(o, p)$ is in $N$, then there exists a suitable open neighborhood $O$ of $(o, p)$ in $N$ so that in this neighborhood $H$ takes the form $H = h \circ P$, where $h$ is a (locally) $Ad^*(G)$-invariant smooth function on $P(O)$. Furthermore, it follows from the proof of theorem 3.2 that if $\dim P(O) = \dim G$, then $\xi : p' \mapsto dh(P(o, p'))$ is a smooth (locally) $K$-equivariant geodesic graph in an open neighborhood of $p$ in $T^*_o M$. If $\dim P(O) < \dim G$, then $h$ can be extended to an open neighborhood of $P(O)$, and this
The following theorem is a converse of theorem 3.1. The summation over the index $n$ is implied in the formulas (39) and (41).

**Theorem 3.2.** Let $h : \mathfrak{g}^* \rightarrow \mathbb{R}$ be an $\text{Ad}^*(G)$-invariant function with the properties that $h \circ f^*$ is smooth and $h$ is differentiable at the points of the image space of $f^*$ (which is $\mathfrak{f}^*(\mathfrak{T}_o^* M)$). The Hamiltonian function defined as

$$H = h \circ P$$

is $G$-invariant and has the g.o. property. The vector

$$dh(P(o, p)) = \frac{\partial h}{\partial g_n}(P(o, p)) dg_n,$$  \hspace{1cm} (39)

where $g_n$ are some linear coordinates on $\mathfrak{g}^*$, is a relative equilibrium vector at $(o, p) \in T^* M$; thus, the mapping

$$\xi : P(o, p) \mapsto dh(P(o, p)) \equiv (dh \circ f^*)(p)$$ \hspace{1cm} (40)

is a $K$-equivariant geodesic graph.

**Proof.** We note that $P(o, p) = f^*(p)$, by definition. $H$ is obviously $G$-invariant. The property that $h \circ f^*$ is smooth implies the smoothness of $H$. We have

$$dH = \frac{\partial h}{\partial g_n} \frac{\partial P_n}{\partial x^j} dx^j + \frac{\partial h}{\partial g_n} \frac{\partial P_n}{\partial p_i} dp_i,$$ \hspace{1cm} (41)

where $P_n$ are the components of $P$ with respect to the coordinates $g_n$. This shows that at $(o, p) \in T^* M$ the vector $b \in \mathfrak{g}$ that has the components $\frac{\partial h}{\partial g_n}(P(o, p))$ has the property $dH(o, p) = d(P|b)(o, p)$; thus, the condition of proposition 3.2 is fulfilled. Clearly $\frac{\partial h}{\partial g_n}(P(o, p))$ are just the components of $dh(P(o, p))$ with respect to the coordinates $g_n$. □

It is also clear from the proof of theorem 3.2 that

**Proposition 3.4.** If $h : \mathfrak{g}^* \rightarrow \mathbb{R}$ is an $\text{Ad}^*(G)$-invariant function, $H = h \circ P$ is a smooth Hamiltonian function and $h$ is differentiable at $P(o, p)$ for some $p \in T^*_o M$; then, $dh(P(o, p))$ is a relative equilibrium vector at $(o, p)$.

The condition imposed on $h$ in theorem 3.2 could probably be weakened; in particular, we do not expect that the differentiability of $h$ in every point of $\mathfrak{f}^*(\mathfrak{T}_o^* M)$ is necessary for $h \circ P$ to be a g.o. Hamiltonian.

Propositions 3.5–3.8, theorem 3.4 and partly theorem 3.3 are about Riemannian and pseudo-Riemannian spaces.

**Proposition 3.5.** If $h$ is a quadratic $\text{Ad}^*(G)$-invariant polynomial on $\mathfrak{g}^*$ and the polynomial $h \circ f^*$ is homogeneous, quadratic and nondegenerate, then $h$ gives rise to a Riemannian or pseudo-Riemannian g.o. metric on $M = G/K$. On $T^*_o M$ the quadratic polynomial that corresponds to the metric is $h \circ f^*$. The geodesic graph $\xi : p \mapsto dh(P(o, p))$ is linear in this case. If $h \circ f^*$ is positive definite, then the metric is naturally reductive.

**Proof.** The Hamiltonian $H = h \circ P$ restricted to $T^*_o M$ is $h \circ f^*$, and the latter is a nondegenerate homogeneous quadratic polynomial; therefore, $H$ corresponds to a Riemannian or pseudo-Riemannian metric. The quadraticity of $h$ implies that $dh$ is linear. $P(x, p)$ is also linear in the second variable; therefore, $\xi : p \mapsto dh(P(o, p))$ is a linear map. If $h \circ f^*$ is positive definite, then the corresponding metric on $M$ is Riemannian. The linearity (and the
Proposition 3.6. If $h$ is a smooth $Ad^*(G)$-invariant function on $\mathfrak{g}^*$ and $h \circ f^*$ is a homogeneous positive definite quadratic polynomial, then $h$ defines a naturally reductive space.

Proof. $h$ gives rise to a Riemannian metric, since $h \circ f^*$ is a homogeneous positive definite quadratic polynomial. $h$ is smooth; therefore, we can take its quadratic part $h^{(2)}$ at $0 \in \mathfrak{g}^*$. $h^{(2)}$ is defined as $h^{(2)}(a) = \frac{1}{4} \sum_{a, m} \partial_a^2 h(0) a_a a_m$, where $g_n$ are linear coordinates on $\mathfrak{g}^*$, $a \in \mathfrak{g}^*$, and $a_n$ are the components of $a$ with respect to the coordinates $g_n$. $h$ is $Ad^*(G)$-invariant and the action of $Ad^*(G)$ is linear; therefore, $\partial_a^2 h(0)$, as an element of $\mathfrak{g} \otimes \mathfrak{g}$, is a $G$-invariant tensor, and thus $h^{(2)}$ is also $Ad^*(G)$-invariant. Moreover, $h \circ f^* = h^{(2)} \circ f^*$, since $f^*$ is linear and injective, thus $h^{(2)}$ gives rise to the same metric as $h$. As a consequence, $\xi : p \mapsto dh^{(2)}(P(o, p))$ is a $K$-equivariant linear geodesic graph, implying that the metric defined by $h$ is naturally reductive.

The positive definiteness of $h \circ f^*$ is not essential in the proof of this proposition; it is needed only to ensure that the metric which $h$ gives rise to is positive definite. The condition that $h$ is smooth can also be relaxed to the condition that $h$ is twice differentiable at 0.

From a theorem of Kostant [42] generalized by D’Atri and Ziller [43] it also follows that all naturally reductive metrics can be obtained from $h$ functions that are nondegenerate (not necessarily positive definite) quadratic polynomials. More specifically, let $M = G/K$ be a naturally reductive Riemannian space with respect to $G$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ a natural reductive decomposition, and assume that $G$ acts almost effectively on $M$ (i.e. the subgroup of elements that act as the identity transformation is discrete). Then there exists an analytic subgroup $\tilde{G}$ of $G$ and an analytic subgroup $\tilde{K}$ of $K$ so that $M = \tilde{G}/\tilde{K}$ and the metric is naturally reductive with respect to $\tilde{G}$ and it arises from an $Ad^*(\tilde{G})$-invariant function that is a nondegenerate quadratic polynomial. The subgroup $\tilde{G}$ is generated by $\tilde{g} = \mathfrak{m} + [\mathfrak{m}, \mathfrak{m}]$, which is an ideal in $\mathfrak{g}$, and the Lie algebra of $\tilde{K}$ is $\tilde{\mathfrak{k}} = \mathfrak{g} \cap \mathfrak{t}$. The articles [42, 43] also contain the result that if $h$ is a nondegenerate quadratic $Ad^*(G)$-invariant polynomial on $\mathfrak{g}^*$ and $h \circ f^*$ is positive definite, then the metric defined by $h$ is naturally reductive.

Proposition 3.7. Let $M = G/K$ be a Riemannian or pseudo-Riemannian homogeneous space. If $a \in \mathfrak{g}$ is a relative equilibrium vector at $(o, p)$, then $\lambda a$ is also a relative equilibrium vector at $(o, \lambda p)$ for any $\lambda \in \mathbb{R}$.

Proof. This result follows easily from lemma 3.2.

Proposition 3.8. Let $M = G/K$ be a Riemannian or pseudo-Riemannian g.o. space. If there exists a geodesic graph $\xi$ so that $\xi(0) = 0$ and $\xi$ is differentiable at 0, then there also exists a corresponding geodesic graph that is linear. If, in addition, $\xi$ is $K$-equivariant, then the corresponding linear geodesic graph is also $K$-equivariant.

Proof. The differentiability of $\xi$ and $\xi(0) = 0$ imply that $\xi$ can be written as $\xi = \xi^{(1)} + \tilde{\xi}$, where $\xi^{(1)}$ is linear and $\tilde{\xi}$ has the property that $\lim_{\lambda \to 0} \tilde{\xi}(\lambda p)/\lambda = 0$. $\xi^{(1)}$ is uniquely determined by $\xi$. It follows from proposition 3.7 that $\xi(\lambda p) = \xi(p)$ for any $\lambda > 0$. We have $\lim_{\lambda \to 0} \xi_\lambda(p) = \xi^{(1)}(p)$; thus, $\xi^{(1)}$ is also a geodesic graph. If $\xi$ is $K$-equivariant, then obviously $\xi^{(1)}$ is also $K$-equivariant.

Theorem 3.3. Let $M = G/K$ be a Hamiltonian g.o. space. If $K$ is compact, then there exists a $K$-equivariant geodesic graph. If $K$ is compact and the space is Riemannian, then there
exists a $K$-equivariant geodesic graph $\xi$ with the property that $\xi(\lambda p) = \lambda \xi(p)$ for all $\lambda \in \mathbb{R}$ (i.e. $\xi$ is first order homogeneous).

**Proof.** Due to the compactness of $K$ there exists a positive definite $Ad(K)$-invariant scalar product $Q$ on $\mathfrak{g}$. For any $(o, p) \in T^*_o M$, consider the set of all relative equilibrium vectors at $(o, p)$, which is an affine subspace of $\mathfrak{g}$ according to proposition 3.1. Let the value of the geodesic graph at $p$ be that unique element of this affine subspace which has the smallest norm with respect to $Q$. Since $Q$ is $Ad(K)$-invariant, the geodesic graph defined in this way is obviously $K$-equivariant. Taking into consideration proposition 3.7, it is also obvious that this geodesic graph has the property $\xi(\lambda p) = \lambda \xi(p)$ for all $\lambda \in \mathbb{R}$ if the Hamiltonian defines a Riemannian metric. □

It is easy to see that a similar theorem with a similar proof holds for Lagrangian g.o. spaces as well. An $Ad(K)$-invariant scalar product on $\mathfrak{g}$ also exists if $M = G/K$ is a Riemannian g.o. space and the action of $G$ on $M$ is effective (i.e. the compactness of $K$ is not necessary). See e.g. [33], proposition I for a proof. The proof of proposition 3.8 also shows that if a geodesic graph $\xi$ has the properties that $\xi(\lambda p) = \lambda \xi(p)$ for any $\lambda \in \mathbb{R}$ and it isdifferentiable at 0, then $\xi$ is linear. Consequently, we can state the following theorem.

**Theorem 3.4.** Let $M = G/K$ be a Riemannian g.o. space and assume that the action of $G$ on $M$ is effective. Then there exists at least one $K$-equivariant geodesic graph $\xi$ with the property that $\xi(\lambda p) = \lambda \xi(p)$ for any $\lambda \in \mathbb{R}$. If $\xi$ is differentiable at 0, then $\xi$ is linear and thus $M$ is a naturally reductive space with respect to $G$.

A geodesic graph that have the stated properties can be constructed in the same way as in the proof of theorem 3.3. The main result in Szente’s paper [2], which he obtained for affine g.o. manifolds with torsion-free affine connection and for compact $K$, is similar to theorem 3.4. Our construction of the $K$-equivariant geodesic graph is simpler than that given in [2] (constructions similar to that in [2] can also be found in [6, 9]). For further results on the geodesic graphs of Riemannian g.o. spaces we refer the reader to [9, 18].

In section 4 we discuss an example where $h$ is a complicated function; nevertheless, $h \circ f^*$ is a homogeneous quadratic polynomial and it is also positive definite; thus, $h$ still gives rise to a Riemannian metric on $G/K$. This metric has the g.o. property, but the geodesic graph, which is unique in this example on an open dense set, is not linear and is not differentiable at $p = 0$, and the metric is not naturally reductive with respect to $G$, in accordance with theorems 3.3 and 3.4. In addition to the nondifferentiability at 0, the geodesic graph is also discontinuous along a one-dimensional subspace (from which 0 is excluded).

As the example shows, in the Riemannian case the function $h$ is not necessarily simple even though $H|_{T^*_o M} \equiv H_o = h \circ f^*$, and thus $h|_{m^*}$, where $m^*$ is defined as $m^* = f^*(T^*_o M)$, is also a quadratic polynomial. However, $h|_{m^*}$ is sufficient for determining $H_o$ (since $H_o = h|_{m^*} \circ f^*$), and thus $H$. Therefore, in order to specify a Riemannian g.o. space it is sufficient to specify the polynomial $h|_{m^*}$, for which we introduce the notation $h_o = h|_{m^*}$. The g.o. property implies that there is an open dense subset $N_o$ of $m^*$ such that at any point $b \in N_o$ the derivative of $h_o$ has to be zero in any direction $ad_a^*(b)$, where $a \in \mathfrak{g}$ is such that $ad_a^*(b) \in m^*$. That is to say, at any point $b \in N_o$ the equation

$$
(dh_o(b) | ad_a^*(b)) = 0
$$

has to hold for all $a \in \mathfrak{g}$ for which $ad_a^*(b) \in m^*$. This equation can be used in practice for finding suitable $h_o$ functions, i.e. for finding g.o. metrics or g.o. Hamiltonians, or to test whether a given metric or Hamiltonian function has the g.o. property. In terms of $H_o$, $h_o$ is given as $h_o = H_o \circ (f^*)^{-1}$, of course.
The $\text{Ad}^*(K)$-invariance of $h_0$ is necessary and sufficient for the $G$-invariance of the Hamiltonian function defined by $h_0$. If $a \in \mathfrak{t}$ and $b \in N_{\mathfrak{g}}$, then $\text{ad}^*_G(b) \in \mathfrak{m}^*$; thus, (42) has to be satisfied. However, if $h_0$ is $\text{Ad}^*(K)$-invariant, then (42) obviously holds if $a \in \mathfrak{t}$. Condition (42) is therefore interesting mainly for those elements $a$ of $\mathfrak{g}$ which are not in $\mathfrak{t}$.

The construction of g.o. Hamiltonian functions as $H = h \circ P$ can be generalized in the following way.

**Theorem 3.5.** Let $M$ be a manifold and $P$ a mapping $T^*M \to \mathfrak{g}^*$, where $\mathfrak{g}$ is a Lie algebra of a Lie group $G$, with the property $[X_{(P)a}, X_{(P)b}] = X_{(P)[a,b]}$ for all $a, b \in \mathfrak{g}$. Let $h$ be a smooth $\text{Ad}^*(G)$-invariant function. The Hamiltonian function $H = h \circ P$ is $G$-invariant with respect to $G$ in the sense that $H$ is constant along the integral curves of $X_{(P)a}$ for all $a \in \mathfrak{g}$. Any integral curve of $X_H$ coincides with an integral curve of $X_{(P)a}$ for some $a \in \mathfrak{g}$. In particular, the integral curve of $X_H$ starting at the point $(x, p) \in T^*M$ coincides with the integral curve of $X_{(P)a}$, where $a = \text{dh}(P(x, p))$, starting at $(x, p)$.

In a more general form of the theorem the condition that $h$ should be smooth could be relaxed. Certain notable dynamical systems, for example the system of two pointlike bodies which interact by the Newtonian gravitational force (the Kepler problem) and the harmonic oscillator, admit a formulation in this framework with noncommutative groups $G$. Completely integrable systems can also be formulated in the framework of theorem 3.5 with commutative symmetry groups.

**4. Example**

In this section, we discuss the example when $G = SU(3)$ and $K = SU(2)$ in order to give an illustration to the second part of section 3. The $SU(3)$-invariant metrics on $SU(3)/SU(2)$, which is diffeomorphic to the sphere $S^7$, constitute a two-parameter family. These metrics were described e.g. in [44], where a complete description of the homogeneous metrics on the spheres was given. In [6] it was found that all the $SU(3)$-invariant metrics on $SU(3)/SU(2)$ have the g.o. property, but only a one-parameter subfamily is naturally reductive with respect to $SU(3)$. Further results, in particular concerning the geodesic graph, were obtained in [9].

We note that these metrics belong to the type of g.o. metrics which are naturally reductive with respect to a suitable larger symmetry group [9]. This larger group is $U(3)$ in the present case, and the stability subgroup of the origin is $U(2)$.

The Lie algebras of $SU(3)$ and $SU(2)$ are as follows:

$$
\begin{align*}
\mathfrak{su}(3) &= \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m} \\
\mathfrak{su}(2) &= \mathfrak{t} = \text{span}(A, B, C) \\
\mathfrak{m} &= \text{span}(E_1, E_2, E_3, E_4, Z)
\end{align*}
$$

where the matrices $A, B, C, E_1, E_2, E_3, E_4, Z$ are as follows:

$$
\begin{align*}
[A, B] &= 2C & [A, Z] &= 0 & [A, E_1] &= -E_2 & [B, E_1] &= E_3 & [C, E_1] &= E_4 \\
[B, C] &= 2A & [B, Z] &= 0 & [A, E_2] &= E_1 & [B, E_2] &= E_4 & [C, E_2] &= -E_3 \\
[C, A] &= 2B & [C, Z] &= 0 & [A, E_3] &= E_4 & [B, E_3] &= -E_1 & [C, E_3] &= E_2 \\
[\mathfrak{z}, E_1] &= E_2 & [E_1, E_2] &= Z - \frac{1}{2}A & [E_2, E_4] &= \frac{1}{2}B \\
[\mathfrak{z}, E_2] &= -E_1 & [E_1, E_3] &= \frac{3}{2}B & [E_3, E_4] &= Z + \frac{1}{2}A \\
[\mathfrak{z}, E_3] &= E_4 & [E_1, E_4] &= \frac{3}{2}C \\
[\mathfrak{z}, E_4] &= -E_3 & [E_2, E_3] &= -\frac{1}{3}C.
\end{align*}
$$
There exists one (up to multiplication by a constant) quadratic homogeneous invariant polynomial on $su(3)$:
\[ Y_1 = a'^2 + b'^2 + c'^2 + e_1^2 + e_2^2 + e_3^2 + e_4^2 + z^2, \tag{43} \]
where $a'$, $b'$, $c'$, $e_1$, $e_2$, $e_3$, $e_4$, $z$ denote the coordinates corresponding to the basis vectors $A' = \frac{A}{\sqrt{3}}$, $B' = \frac{B}{\sqrt{3}}$, $C' = \frac{C}{\sqrt{3}}$, $E_1$, $E_2$, $E_3$, $E_4$, $Z$ of $su(3)$. $Y_1$ defines a positive definite $Ad$-invariant quadratic form on $su(3)$, allowing the identification of $su(3)$ and $su(3)^*$ and implying the equivalence of the coadjoint and adjoint actions of $SU(3)$. The bases $A'$, $B'$, $C'$, $E_1$, $E_2$, $E_3$, $E_4$, $Z$ are orthonormal with respect to the quadratic form defined by $Y_1$. We use the same notation for the corresponding orthonormal basis in $su(3)^*$. $Y_1$ can now be taken as an invariant polynomial on $su(3)^*$ as well. $f$ can be used to identify $T_o M$ with $m$, and then the momentum mapping restricted to $T_o^* M$, i.e. $f^*$, is the trivial embedding $m \rightarrow m \oplus \mathfrak{f}$. The polynomial $Y_1$ composed with $f^*$ thus takes the form
\[ y_1 = Y_1 \circ f^* = e_1^2 + e_2^2 + e_3^2 + e_4^2 + z^2, \tag{44} \]
where we have introduced the notation $y_1$ for $Y_1 \circ f^*$. The metric on $SU(3)/SU(2)$ corresponding to $y_1$ is naturally reductive. In [6] it was found that the complete family of Riemannian g.o. metrics on $SU(3)/SU(2)$ is given on $T_o^* M \equiv m$ by
\[ \alpha(e_1^2 + e_2^2 + e_3^2 + e_4^2) + \beta z^2, \quad \alpha > 0, \quad \beta > 0, \tag{45} \]
where $\alpha$ and $\beta$ are real numbers. The metric (45) is naturally reductive if and only if $\alpha = \beta$ [6], which corresponds to $h = \alpha Y$. The family of polynomials (45) coincides with the complete family of positive definite $Ad^*(K)$-invariant quadratic homogeneous polynomials on $m$. It is not difficult to verify that the metrics (45) also satisfy condition (42).

By solving the partial differential equations that express the $Ad^*(G)$-invariance of a function we find that the $Ad^*(G)$-invariant functions are of the form $G(Y_1, Y_2)$, where $G$ is an arbitrary function of two variables and $Y_2$ is the homogeneous third-order polynomial
\[ Y_2 = \sqrt{3}\sigma_3 + z(\sigma_2 - 2\sigma_1) + \frac{1}{2}z^3, \tag{46} \]
where $\sigma_1$, $\sigma_2$ and $\sigma_3$ are the following $Ad^*(K)$-invariant polynomials:
\[ \sigma_1 = a'^2 + b'^2 + c'^2 \tag{47} \]
\[ \sigma_2 = e_1^2 + e_2^2 + e_3^2 + e_4^2 \tag{48} \]
\[ \sigma_3 = a'(e_1^2 + e_2^2 - e_3^2 - e_4^2) + 2b'(e_1e_4 - e_2e_3) - 2c'(e_1e_3 + e_2e_4). \tag{49} \]
We have
\[ y_2 = Y_2 \circ f^* = z(e_1^2 + e_2^2 + e_3^2 + e_4^2) + \frac{1}{2}z^3, \tag{50} \]
where the notation $y_2$ is introduced for $Y_2 \circ f^*$. In order to get the $G$ function for which $G(Y_1, Y_2) \circ f^*$ equals (45), one has to solve equations (44) and (50) for $e_1^2 + e_2^2 + e_3^2 + e_4^2$ and $z$. This involves the solution of a third-order algebraic equation; therefore, the result is a complicated formula that we do not write here. This example shows that the function $h$ (which is $G(Y_1, Y_2)$ in the present case) can be complicated even though $h \circ f^*$ is a quadratic polynomial.

The geodesic graph can be calculated directly by solving the equations in lemma 3.2 or in lemma 2.1, as is done in [9] (it is equation (15) that is actually used); it is not necessary for this to know $h$. The result, which can be found written explicitly below in equation (63) and in [9], has a relatively simple form. The geodesic graph can also be calculated from the formula
\[ \xi = dh \circ f^*, \]
where the necessary derivatives of $h$ can be determined from (42). As a third
approach, one can utilize the knowledge of the invariant polynomials $Y_1$ and $Y_2$ to calculate $d\theta \circ f^\ast$. Here we calculate the geodesic graph in this way, using (44), (50) and (45). We have

$$d(G(Y_1, Y_2)) = \frac{\partial G}{\partial Y_1} dY_1 + \frac{\partial G}{\partial Y_2} dY_2;$$

(51)

thus we have to calculate the partial derivatives of $G$. Equations (44), (45) and (50) can be written as

$$G(y_1, y_2) = \alpha r^2 + \beta z^2 \quad (52)$$

$$y_1 = z^2 + r^2 \quad (53)$$

$$y_2 = \frac{z^2}{r^2} + z r^2. \quad (54)$$

where

$$r^2 = e_1^2 + e_2^2 + e_3^2 + e_4^2. \quad (55)$$

We have

$$\frac{\partial G}{\partial y_1} = \frac{\partial G}{\partial r} \frac{\partial r}{\partial y_1} + \frac{\partial G}{\partial z} \frac{\partial z}{\partial y_1} \quad (56)$$

$$\frac{\partial G}{\partial y_2} = \frac{\partial G}{\partial r} \frac{\partial r}{\partial y_2} + \frac{\partial G}{\partial z} \frac{\partial z}{\partial y_2} \quad (57)$$

For $\frac{\partial G}{\partial y_1}$ and $\frac{\partial G}{\partial y_2}$ we obtain

$$\frac{\partial G}{\partial y_1} = \frac{\partial G}{\partial r} \frac{\partial r}{\partial y_1} + \frac{\partial G}{\partial z} \frac{\partial z}{\partial y_1} = 2\alpha r + 2\beta z \quad (58)$$

from (52). The partial derivatives $\frac{\partial r}{\partial y_1}, \frac{\partial r}{\partial y_2}, \frac{\partial z}{\partial y_1}$ and $\frac{\partial z}{\partial y_2}$ can be calculated by taking partial derivatives of equations (53) and (54) with respect to $y_1$ and $y_2$, and then solving the obtained four equations for $\frac{\partial r}{\partial y_1}, \frac{\partial r}{\partial y_2}, \frac{\partial z}{\partial y_1}$ and $\frac{\partial z}{\partial y_2}$. The result is

$$\frac{\partial r}{\partial y_1} = \frac{z^2}{r^3} + \frac{1}{2r} \quad (59)$$

$$\frac{\partial r}{\partial y_2} = \frac{z}{r^3} \quad (59)$$

$$\frac{\partial z}{\partial y_1} = -\frac{z}{r^2} \quad (60)$$

$$\frac{\partial z}{\partial y_2} = -\frac{1}{r^2} \quad (60)$$

Taking into consideration (56) and (57) and using the results (58), (59) and (60) we obtain for $\frac{\partial G}{\partial y_1}$ and $\frac{\partial G}{\partial y_2}$ that

$$\frac{\partial G}{\partial y_1} = \alpha + (\alpha - \beta) \frac{2z^2}{r^2} \quad (61)$$

$$\frac{\partial G}{\partial y_2} = -(\alpha - \beta) \frac{2z^2}{r^2} \quad (62)$$

$dY_1$ and $dY_2$ are straightforward to calculate, and the result for the geodesic graph is

$$[dG(Y_1, Y_2) \circ f^\ast](e_1 E_1 + e_2 E_2 + e_3 E_3 + e_4 E_4 + z Z)$$

$$= 2\alpha(e_1 E_1 + e_2 E_2 + e_3 E_3 + e_4 E_4) + 2\beta z Z$$

$$+ (\beta - \alpha) \frac{2\sqrt{3}z}{r^2} \left[ (e_1^2 + e_2^2 - e_3^2) A' + 2(e_1 e_4 - e_2 e_3) B' - 2(e_1 e_3 + e_2 e_4) C' \right]. \quad (63)$$
which agrees with the result obtained in [9], if we take into consideration the differences between the definitions in this paper and in [9]. One difference that is worth noting is that in [9] the geodesic graph is defined in such a way that only the $\mathbf{e}_4$-component is kept, i.e. the obvious $2\alpha(e_1E_1 + e_2E_2 + e_3E_3 + e_4E_4) + 2\beta zZ$ part is subtracted.

Equation (63) is well defined on an open dense subset of $T^*_\o M$, but it does not have well-defined values at $r = 0$ if $\alpha \neq \beta$. It can be verified using (27) and (28) that at $zZ$ (i.e. when $r = 0$) all vectors $2\beta zZ + aA' + bB' + cC'$, $a, b, c \in \mathbb{R}$, are relative equilibrium vectors. The limit of (63) in the points characterized by $r = 0$ and $z \neq 0$ depends on the path (assumed to lie in the domain where $r \neq 0$) along which the limit is taken; therefore, the geodesic graph is necessarily discontinuous at these points.

Several other examples of Riemannian g.o. spaces can be found in the literature (see e.g. [6, 7, 9]), which would also be interesting to discuss in a similar way.

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