A NEW TYPE OF DARBO’S FIXED POINT THEOREM
DEFINED BY THE SEQUENCES OF FUNCTIONS

VATAN KARAKAYA, NECIP ŞİMŞEK, AND DERYA SEKMAN

Abstract. In this paper, we introduce a new type of Darbo’s fixed point theorem by using concept of function sequences with shifting distance property. Afterward, we investigate existence of fixed point under this theorem. Also we are going to give interesting example held the conditions of sequences of functions.

1. Introduction

Fixed point theory, which develops as a subbranch of operator theory, is closely related to application areas of mathematics and the various disciplines such as the geometry of Banach spaces, measure of noncompactness, game theory and economy, so on. The topological aspect of the fixed point theory, which proceeds in topological and metric terms, is based on the Brouwer’s fixed point theorem. The Brouwer’s fixed point theorem defined on finite dimensional spaces is generalized to infinite dimensional spaces under the compact operator by Schauder [1]. However, the Schauder fixed point theorem is insufficient for noncompact operators. In addition for noncompact operators, the existence of fixed point is obtained by Darbo’s fixed point theorem [8]. The definition of measure of noncompactness was introduced by Kuratowski and Hausdorff [12]. Many mathematicians have used the measure of noncompactness concept and the Darbo’s fixed point theorem to solve the integral, differential and functional equation classes and then they have achieved significant results [3, 5, 13].

The most effective and useful tools of fixed point theory are the concepts of properties in contraction mappings classes. The first of this mapping types is the Banach contraction principle [2]. This principle is used to find existence and uniqueness of solution for a class of linear and nonlinear equation systems and it has been generalized and extended by many authors under different conditions, references therein [16, 7, 10]. In 1984, Khan et al. [11] introduced the concept of altering distance functions and obtained some results on the uniqueness of fixed point in complete metric space. Rhoades [14] extended and generalized this concept to complete metric space and proved a generalized this result by Dutta and Choudhury [9]. Later, Berzig [6]
introduced the concept of shifting distance functions and established fixed point theorem which generalized Banach contraction principle. Recently, Samadi and Ghaemi [15] prove some generalizations of Darbo’s fixed point theorem associated with measure of noncompactness by using the notion of shifting distance functions and given an application of the integral equation of mixed type.

In this work, we aim to contribute to functional analysis and operator theory by making a generalization of Darbo’s fixed point theorem with the help of function sequences. We introduce a new type of Darbo’s fixed point theorem by using concept of function sequences with shifting distance property. Afterward, we investigate existence of fixed point under this theorem. Also we are going to give interesting example held the conditions of sequences of functions.

2. Preliminaries

Let $E$ be a nonempty subset of a Banach space $X$. We define $\overline{E}$ and $\text{Conv}(E)$ the closure and closed convex hull of $E$, respectively. Also, we denote by $M_X$ which is the family of all nonempty bounded subsets of $X$ and $N_X$ that is subfamily consisting of all relatively compact subsets of $X$.

**Definition 2.1** (see; [4]). A mapping $\mu : M_X \to \mathbb{R}^+$ is called a measure of noncompactness if it satisfy the following conditions

$(M_1)$ The family $\text{Ker } \mu = \{ A \in M_X : \mu(A) = 0 \}$ is nonempty and $\text{Ker } \mu \subseteq N_X$

$(M_2)$ $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$

$(M_3)$ $\mu(\bar{A}) = \mu(A)$, where $\bar{A}$ denotes the closure of $A$

$(M_4)$ $\mu(\text{conv } A) = \mu(A)$,

$(M_5)$ $\mu(\lambda A + (1 - \lambda)B) \leq \lambda \mu(A) + (1 - \lambda)\mu(B)$ for $\lambda \in [0,1]$

$(M_6)$ If $\{A_n\}$ is a sequence of closed sets in $M_X$ such that $A_{n+1} \subseteq A_n$ for $n = 1,2,\ldots$ and $\lim_{n \to \infty} \mu(A_n) = 0$, then the following intersection is nonempty.

$$A_\infty = \bigcap_{n=1}^{\infty} A_n$$

If $(M_4)$ holds, then $A_\infty \in \text{Ker } \mu$. To do this, let $\lim_{n \to \infty} \mu(A_n) = 0$. As $A_\infty \subseteq A_n$ for each $n = 0,1,2,\ldots$; by the monotonicity of $\mu$, we obtain

$$\mu(A_\infty) \leq \lim_{n \to \infty} \mu(A_n) = 0.$$ 

So, by $(M_1)$, we get that $A_\infty$ is nonempty and $A_\infty \in \text{Ker } \mu$.

**Theorem 2.2** (see; [1]). *Let $E$ be a closed and convex subset of a Banach space $X$. Then every compact, continuous map $T : E \to E$ has at least one fixed point.*

**Theorem 2.3** (see; [4]). *Let $E$ be a nonempty, bounded, closed and convex subset of a Banach space $X$ and let $T : E \to E$ be a continuous mapping.*
Suppose that there exists a constant \( k \in [0, 1) \) such that
\[
\mu(T(A)) \leq k \mu(A)
\]
for any subset \( A \) of \( E \), then \( T \) has a fixed point.

**Definition 2.4** (see: [6]). Let \( \psi, \phi : [0, \infty) \to \mathbb{R} \) be two functions. The pair of functions \((\psi, \phi)\) is said to be a pair of shifting distance function, if the following conditions hold

(i) for \( u, v \in [0, \infty) \) if \( \psi(u) \leq \phi(v) \), then \( u \leq v \),

(ii) for \( \{u_k\}, \{v_k\} \subset [0, \infty) \) with \( \lim_{k \to \infty} u_k = \lim_{k \to \infty} v_k = w \), if \( \psi(u_k) \leq \phi(v_k) \) for all \( k \in \mathbb{N} \), then \( w = 0 \).

3. **Main Result**

Now we will give the definition of a pair of sequences of functions with shifting distance property and by using this function class, we introduce a new type of Darbo’s fixed point theorem. Afterward, we investigate fixed point of mapping according to generalized new Darbo’s fixed point theorem. Also we are going to give one interesting example.

**Definition 3.1.** Let \( \psi_n, \phi_n : [0, \infty) \to \mathbb{R} \) be two sequences of functions. The pair of sequences of functions \((\psi_n, \phi_n)\) is said to be a pair of sequences of functions with shifting distance property which satisfy the following conditions

(i) for \( u, v \in [0, \infty) \) if \( \psi_n(u) \leq \phi_n(v) \) \( \Rightarrow \psi(u) \leq \phi(v) \) uniformly in \( n \), then \( u \leq v \),

(ii) for \( \{u_k\}, \{v_k\} \subset [0, \infty) \) with \( \lim_{k \to \infty} u_k = \lim_{k \to \infty} v_k = w \), if \( \psi_n(u_k) \leq \phi_n(v_k) \) \( \Rightarrow \psi(u_k) \leq \phi(v_k) \) for all \( k \in \mathbb{N} \), uniformly in \( n \) , then \( w = 0 \).

**Definition 3.2.** The pair \((\psi_n, \phi_n)\) is said to be having shifting distance property if \((\psi_n, \phi_n) \to (\psi, \phi)\) uniformly in \( n \) and the pair \((\psi, \phi)\) is shifting distance function.

**Lemma 3.3.** Let \( \psi_n, \phi_n : [0, \infty) \to \mathbb{R} \) be two sequences of functions. Assume that the sequences of functions hold following conditions

(i) if \((\psi_n)\) upper semi-continuous sequence of functions and \( \psi_n \leq \psi_{n+1} \), then \( \psi_n \to \psi \) convergent (uniformly in \( n \)),

(ii) if \((\phi_n)\) lower semi-continuous sequence of functions and \( \phi_n \geq \phi_{n+1} \), then \( \phi_n \to \phi \) convergent (uniformly in \( n \)).

Then, \((\psi_n, \phi_n)\) is called the pair of sequences of functions having shifting distance property.

**Proof.** Let \((\psi_n)\) be increasing and bounded by \( \psi(u) \), also let \((\phi_n)\) be decreasing and bounded by \( \phi(v) \). By using condition (i) of Definition 2.4 we get
\[
\psi_1(u) < \psi_2(u) < \cdots < \psi_n(u) < \psi(u) \leq \phi(v) < \phi_n(v) < \cdots < \phi_2(v) < \phi_1(v).
\]
Hence we can write for all $n \in \mathbb{N}$

$$
\psi_n(u) \leq \phi_n(v).
$$

Taking limit on both sides of (3.1),

$$
\lim_{n \to \infty} \psi_n(u) \leq \lim_{n \to \infty} \phi_n(v)
$$

By using condition $(ii)$ of Definition 2.4, for \{u_k\}, \{v_k\} ⊂ [0, ∞) with

$$
\lim_{k \to \infty} u_k = \lim_{k \to \infty} v_k = w \quad \text{if} \quad \psi_n(u_k) \leq \phi_n(v_k) \quad \text{for all} \quad n, k \in \mathbb{N},
$$

taking limit and we get

$$
\lim_{n \to \infty} \psi_n(u_k) \leq \lim_{n \to \infty} \phi_n(v_k)
$$

That is $(\psi_n, \phi_n)$ is the pair of sequences of functions with shifting distance property.

**Theorem 3.4.** Let $E$ be a nonempty, bounded, closed and convex subset of the Banach space $X$. Suppose that $T : E \to E$ is a continuous mapping such that

$$
\psi_n(\mu(TA)) \leq \phi_n(\mu(A))
$$

for any nonempty subset $A \subset E$, where $\mu$ is an arbitrary measure of non-compactness and $\psi_n, \phi_n : [0, \infty) \to \mathbb{R}$ be the pair of sequences of functions with shifting distance property. Then, $T$ has a fixed point in $E$.

**Proof.** We define a sequence \{A_k\} such that $A_0 = A$ and $A_k = \text{Conv} (TA_{k-1})$ for all $k \geq 1$. Then we get

$$
TA_0 = TA \subseteq A = A_0
$$

$$
A_1 = \text{Conv} (TA_0) \subseteq A = A_0.
$$

By repeating process mentioned above, we have

$$
A_0 \supseteq A_1 \supseteq E_2 \supseteq \cdots \supseteq A_k \supseteq \cdots
$$

If there exists an integer $k \geq 0$ such that $\mu(A_k) = 0$, then $A_k$ is relatively compact and since

$$
TA_k \subseteq \text{Conv} (TA_k) = A_{k+1} \subseteq A_k,
$$

Theorem 2.2 implies that $T$ has a fixed point in Schauder’s sense on the set $A_k$ for all $k \geq 0$. Now we assume that $\mu(A_k) > 0$ for all $k \geq 0$. By using (3.2) we have

$$
\psi_n(\mu(A_{k+1})) = \psi_n(\mu(\text{Conv} (TA_k)))
$$

$$
= \psi_n(\mu(TA_k))
$$

$$
\leq \phi_n(\mu(A_k)).
$$
Let us consider (3.2). Then we obtain that \( \{\mu(A_k)\} \) is a decreasing sequence of positive real numbers and there exists \( p \geq 0 \) such that \( \mu(A_k) \to p \) as \( k \to \infty \). By using (3.3) and Lemma 3.3, we have
\[
\psi_n(\mu(A_{k+1})) \to \psi(\mu(A_{k+1})), \quad \text{uniformly in } n
\]
and
\[
(3.4) \quad \psi(\mu(A_{k+1})) = \psi(\mu(TA_k)).
\]
Also, if \( \mu(A_k) \to p \) as \( k \to \infty \), then \( \mu(A_{k+1}) \to p \) as \( k \to \infty \). Hence we have
\[
\lim_{k \to \infty} \psi(\mu(A_{k+1})) = \lim_{k \to \infty} \psi(\mu(TA_k)) \leq \lim_{k \to \infty} \phi(\mu(A_k))
\]
\[
\psi(p) \leq \phi(p).
\]
By condition (ii) of Definition 3.1, we get \( p = 0 \). So we have \( \mu(A_k) \to 0 \) as \( k \to \infty \). On the other hand, since \( A_{k+1} \subseteq A_k \), \( TA_k \subseteq A_k \) and \( \mu(A_k) \to 0 \) as \( k \to \infty \). Using (M6) of Definition 2.1, \( A_\infty = \cap_{k=1}^\infty A_k \) is nonempty, closed, convex, and invariant under \( T \). Hence the mapping \( T \) belong to \( \text{Ker} \mu \). Therefore, Schauder’s fixed point theorem implies that has a fixed point in \( A_\infty \subset A \). \( \square 

Example 3.5. The under the conditions of Definition 3.1, we consider the following sequence of functions
\[
\psi_n(u) = \frac{2n(1+u) + 2u + 1}{n+1}, \quad \phi_n(v) = \frac{n(2 + v) + 1}{n}.
\]
It is clear that \( \psi_n(u) \leq \phi_n(v) \), for all \( n \in \mathbb{N} \) and \( u, v \in [0, \infty) \). It is easy to see that the pairs \( (\psi_n, \phi_n) \to (\psi, \phi) \) are shifting distance function. To see this, we have
\[
\lim_{n \to \infty} \frac{2n(1+u) + 2u + 1}{n+1} = 2 + 2u \leq 2 + v = \lim_{n \to \infty} \frac{n(2 + v) + 1}{n}.
\]
Therefore \((\psi, \phi)\) is shifting distance functions.
Now we suppose that \( u = \mu(TA) \) and \( v = \mu(A) \). Since
\[
\frac{2n(1+\mu(TA)) + 2\mu(TA) + 1}{n+1} \leq \frac{n(2 + \mu(A)) + 1}{n},
\]
we have
\[
(3.5) \quad 2\mu(TA) - \mu(A) \leq \frac{2n + 1}{n(n+1)}.
\]
If limit goes to infinity in (3.5), we obtain
\[
2\mu(TA) - \mu(A) \leq 0.
\]
As a result,
\[
2\mu(TA) \leq \mu(A)
\]
\[
\mu(TA) \leq \frac{1}{2} \mu(A).
\]
Therefore, according to condition of Darbo’s fixed point theorem, \( T \) has a fixed point under continuous sequences of functions. Hence this completes the proof.

If we take \( \psi_n = I_n \) such that \( \lim_{n \to \infty} I_n = I \) uniformly convergence for all \( n \in \mathbb{N} \) in Theorem 3.4, we obtain the following result.

**Corollary 3.6.** Let \( E \) be a nonempty, bounded, closed and convex subset of the Banach space \( X \). Suppose that \( T : E \to E \) is a continuous function such that

\[
I_n (\mu(TA)) \leq \phi_n (\mu(A))
\]

for any nonempty subset of \( A \subset E \), where \( \mu \) is an arbitrary measure of noncompactness and \( \phi_n : [0, \infty) \to \mathbb{R} \) be a sequence of function such that

(a) for \( u, v \in [0, \infty) \) if \( I_n (u) \leq \phi_n (v) \), then \( u \leq v \),

(b) for \( \{u_k\}, \{v_k\} \subset [0, \infty) \) with \( \lim_{k \to \infty} u_k = \lim_{k \to \infty} v_k = w \), if \( I_n (u_k) \leq \phi_n (v_k) \)

for all \( n, k \in \mathbb{N} \), then \( w = 0 \).

Then, \( T \) has a fixed point in \( E \).

**Corollary 3.7.** Let \( E \) be a nonempty, bounded, closed and convex subset of the Banach space \( X \). Suppose that \( T : E \to E \) is a continuous mapping such that

\[
\psi_n (\mu(TA)) \leq \psi_n (\mu(A)) - \phi_n (\mu(A))
\]

for any nonempty subset of \( A \subset E \), where \( \mu \) is an arbitrary measure of noncompactness and \( \psi_n, \phi_n : [0, \infty) \to \mathbb{R}^+ \) be a pair having shifting distance property. Also the pair \( (\psi, \phi) \) is two nondecreasing and continuous functions satisfying \( \psi(t) = \phi(t) \) if and only if \( t = 0 \). Then, \( T \) has a fixed point in \( E \).

**Proof.** Assume that (3.6) holds. If by taking limit on (3.6), we get

\[
\psi (\mu(TA)) \leq \psi (\mu(A)) - \phi (\mu(A)).
\]

Besides, by using hypothesis in statement, we suppose that \( \psi (\mu(A)) = \phi (\mu(A)) \). Then we get \( \mu(A) = 0 \). Under the conditions of Theorem 3.4, \( A \) is relatively compact and then Theorem 2.2 implies that \( T \) has a fixed point in \( E \). Conversely, we suppose that \( \mu(A) = 0 \). Then in (3.7) \( \psi (\mu(A)) = \phi (\mu(A)) \). Since \( \mu(A) = 0 \), it is clear that \( A \) is relatively compact. Hence using Theorem 2.2 again, we say that \( T \) has a fixed point in \( E \). Also since \( (\psi, \phi) \in \mathbb{R}^+ \), \( \mu(TA) = 0 \). So by repeating the conditions of Theorem 3.4 we obtain that \( T \) belong to Ker \( \mu \). As a result, mapping \( T \) has a fixed point in \( A_{\infty} \subset A \).

**Corollary 3.8.** Let \( E \) be a nonempty, bounded, closed and convex subset of a Banach space \( X \) and let \( T : E \to E \) be a continuous mapping. Suppose that there exists a constant \( k \in [0, 1) \) such that

\[
\mu(TA) \leq k \mu(A),
\]

for any subset \( A \subset E \), then \( T \) has a fixed point.
Proof. Taking $\psi_n(t) = I_n$ and $\phi_n(t) = kI_n$ such that $I_n \to I$ uniformly $n$ in Theorem 3.4 we get Darbo’s fixed point theorem, where $k \in [0,1)$. □

Acknowledgement. This work was supported by the Ahi Evran University Scientific Research Projects Coordination Unit. Project Number: RKT. A3.17.001.

References

[1] R. Agarwal, M. Meehan, D. O’Regan, Fixed point theory and applications, Cambridge University, Cambridge, 2004.
[2] S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations integrales, Publie dans Fund. Math.3, p.133-181, 1922.
[3] J. Banas, Measures of noncompactness in the study of solutions of nonlinear differential and integral equations, J. Cent. Eur. J. Math., 2012. doi: 10.2478/s11533-012-0120-9.
[4] J. Banas, K. Goebel, Measure of noncompactness in Banach spaces, Lecture Notes in Pure and Applied Mathematics, 60 (1980), Dekker, New York.
[5] J. Banas, T. Zajac, Solvability of a functional integral equation of fractional order in the class of functions having Limits at infinity, Nonlinear Anal., 71(11): 5491–5500, 2009.
[6] M. Berzig, Generalization of the Banach Contraction Principle, 2013. arXiv:1310.0995 [math.CA]
[7] N.E.H. Bouzara, V. Karakaya, On different type of fixed point theorem for multivalued mappings via measure of noncompactness, Adv. Oper. Theory 3 (2018), no. 2, 326–336.
[8] G., Darbo, Punti uniti in trasformazioni a codominio non compatto, Rend. Sem. Mat.Univ. Padova, 24, 84-92, 1955.
[9] P.N. Dutta, B.S. Choudhury, A generalization of contraction principle in metric spaces, Fixed Point Theory Appl. 2008, 1–8, 2008.
[10] V. Karakaya, Y. Atalan, K. Doğan, N.E.H. Bouzara, Some fixed point results for a new three steps iteration process in Banach spaces, Fixed Point Theory and Applications, 18(2): 625-640, 2017.
[11] M.S. Khan, M. Swaleh, S. Sessa, Fixed point theorems by altering distances between the points, Bull. Aust. Math. Soc. 30, 1–9, 1984.
[12] K., Kuratowski, Sur les espaces complets, Fund. Math.15, 301-309, 1930.
[13] M. Mursaleen, S.A. Mohiuddine, Applications of measures of noncompactness to the infinite system of differential equations in lp spaces, Nonlinear Anal. 75(4): 2111–2115, 2012.
[14] B.E. Rhoades, Some theorems on weakly contractive maps, Nonlinear Anal. 47, 2683–2693, 2001.
[15] A. Samadi and M. B. Ghaemi, An extension of Darbo’s theorem and its application, Abstract and Applied Analysis, Vol. 2014, Article ID 852324, 11 pages.
[16] D. Sekman, N.E.H. Bouzara, V. Karakaya, n-tuplet fixed points of multivalued mappings via measure of noncompactness, Communications in Optimization Theory, Vol. 2017, Article ID 24, pp. 1-13, 2017.
(V. Karakaya) Department of Mathematical Engineering, Yıldız Technical University, Davutpasa Campus, Esenler, 34210 Istanbul, Turkey
E-mail address: vkkaya@yahoo.com

(N. Şimşek) Department of Mathematics, Istanbul Commerce University, Sütlüce Campus, Beyoğlu, 34445 Istanbul, Turkey
E-mail address: nesimsin@yahoo.com

(D. Sekman) Department of Mathematics, Ahi Evran University, Bağbaşı Campus, 40100 Kirşehir, Turkey
E-mail address: deryasekman@gmail.com