Unbounded number of channel uses are required to see quantum capacity

Toby Cubitt, 1 David Elkouss, 2 William Matthews, 1,3 Maris Ozols, 1 David Pérez-García, 2 and Sergii Strelchuk 1

1 Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 0WA, U.K.
2 Departamento de Análisis Matemático and Instituto de Matemática Interdisciplinar, Universidad Complutense de Madrid, 28040 Madrid, Spain
3 Statistical Laboratory, University of Cambridge, Wilberforce Road, Cambridge CB3 0WB, U.K.

(Dated: August 22, 2014)

Transmitting data reliably over noisy communication channels is one of the most important applications of information theory, and well understood when the channel is accurately modelled by classical physics. However, when quantum effects are involved, we do not know how to compute channel capacities. The capacity to transmit quantum information is essential to quantum cryptography and computing, but the formula involves maximising the coherent information over arbitrarily many channel uses [1–3]. This is because entanglement across channel uses can increase the coherent information [7], even from zero to non-zero [8]! However, in all known examples, at least to detect whether the capacity is non-zero, two channel uses already suffice [8, 24]. Maybe a finite number of channel uses is always sufficient? Here, we show this is emphatically not the case: for any n, there are channels for which the coherent information is zero for n uses, but which nonetheless have capacity. This may be a first indication that the quantum capacity is uncomputable.

In the classical case, not only can we exactly characterise the maximum rate of communication over any channel – its capacity – we also have practical error-correcting codes that attain this theoretical limit. It is instructive to review why the capacity of classical channels is much richer. Indeed, we do not even have a complete characterisation of which channels have zero quantum capacity.

To date, we know of only two kinds of channels with zero quantum capacity: antidegradable channels [10, 11] and entanglement-binding channels [12]. The former has the property that the environment can reproduce the output, thus Q = 0 by the no-cloning theorem [13]. The latter can only distribute PPT entanglement, which cannot be distilled by local operations and classical communication [14], which again implies Q = 0.

This has dramatic consequences. It is possible to take two quantum channels above, N1 antidegradable and N2 entanglement-binding, which individually have no capacity whatsoever, yet when used together can transmit quantum information reliably (Q(N1 ⊗ N2) > 0). This superactivation phenomenon was discovered recently by Smith and Yard [8]. They also used their examples to construct a single channel N exhibiting an extreme form of superadditivity of the coherent information, where 0 = Q(1)(N) < Q(2)(N). (In their construction, having two uses of N effectively enables one use of N1 and one of N2.) Even stronger superactivation phenomena have
been shown in the context of zero-error communication over quantum channels [15–19].

On the one hand, additivity violation means regularisation is required in formulas for computing capacities. On the other hand, it also means that entanglement can protect information from noise (the coherent information is additive for unentangled input states).

But just how bad can this additivity violation be? One might hope that, at least in determining whether the quantum capacity is non-zero, one need only consider a finite number of uses of a channel. Indeed, since the Smith and Yard construction relies on combining the only two known types of zero-capacity channels, one might dare to hope that even two uses suffice. (Similarly, for the classical capacity of quantum channels the only known method for constructing examples of additivity violation [4, 5] cannot give a violation for more than two uses of a channel, and there is some evidence that this may be more than just a limitation of the proof techniques [6].) Was this indeed the case, additivity violation would be reduced to something relatively benign: entangling the inputs across more than two uses of the channel would give no advantage. And one would be able to compute the quantum capacity by optimising the coherent information over two uses of the channel, which is not substantially more difficult than the optimisation over a single channel use.

In this paper, we show for the first time that this is not the case: additivity violation is as bad as it could possibly be. We prove that, for any $n$, one can construct a channel $N$ for which the coherent information of $n$ uses is zero $\langle Q^{(n)}(N) \rangle = 0$, yet for a larger number of uses the coherent information is strictly positive, implying that the channel has non-zero quantum capacity $\langle Q(N) \rangle > 0$. This is also the first proof that there can be a gap between $Q^{(n)}(N)$ and the quantum capacity for an arbitrarily large number $n$ of uses of the channel. Our result implies that, in general, one must consider an arbitrarily large number of uses of the channel just to decide whether the channel has any quantum capacity at all!

Perhaps the earliest indication that determining the quantum capacity may be a difficult problem comes from the work of Watrous [20], who showed that an arbitrarily large number of copies of a bipartite quantum state can be required for entanglement distillation assisted by two-way classical communication. Our result can be regarded as the counterpart of [20] for the quantum capacity (which is mathematically equivalent to entanglement distillation assisted by one-way communication). However, the proof ideas and techniques of [20] require two-way communication, thus they do not apply to the usual capacity setting. Our result is instead based on the ideas of Smith and Yard, in particular the intuition provided by Oppenheim’s commentary thereon [25].

This intuition comes from a class of bipartite quantum states called phbits (private bits) [21]:

$$\rho_{\text{phbit}} = \frac{1}{2} (|\phi^{+}\rangle \langle\phi^{+}|_{ab} \otimes \sigma_{\text{AB}} + |\phi^{-}\rangle \langle\phi^{-}|_{ab} \otimes \sigma_{\text{AB}}),$$

together with the standard equivalences between quantum capacity (sending entanglement over a channel) and distilling entanglement from the Choi-Jamiolkowsky state associated with the channel. Here, $|\phi^{\pm}\rangle$ are Bell states, and $\sigma^{\pm}$ are hiding states [22]. The latter are orthogonal (globally perfectly distinguishable), but cannot be distinguished using local operations and classical communication (LOCC).

If $\rho_{\text{phbit}}$ is shared between Alice (who holds $aA$) and Bob (who holds $bB$), then they share at least one ebit of entanglement due to the Bell states. But this entanglement is inaccessible to them unless they can determine which of the two Bell states they share. This they could do if only they could determine which hiding state they have. But $\sigma^{\pm}$ cannot be distinguished by LOCC, preventing them from extracting the entanglement from $\rho_{\text{phbit}}$. The $ab$ part of the system is usually called the “key”, and $\text{AB}$ the “shield” (as it decouples the systems $ab$ from any external system).

Now imagine they have access to a quantum erasure channel $E_\phi$, which with probability $1/2$ transmits its input perfectly, and with probability $1/2$ completely erases it. It is well known that such a channel cannot be used to transmit any entanglement. However, if they also share $\rho_{\text{phbit}}$, Alice can use the erasure channel to send her part $A$ of the shield to Bob. If the erasure channel transmits, Bob now holds the entire $\text{AB}$ system and can now distinguish $\sigma^{\pm}$. Thus, with probability $1/2$, Alice and Bob can now extract the entanglement from $\rho_{\text{phbit}}$.

Instead of supplying Alice and Bob with the state $\rho_{\text{phbit}}$ and an erasure channel, we instead supply them with a switched channel. This has an auxiliary classical input that controls whether the channel acts as $E_\phi$ or $\Gamma$, where $\Gamma$ is the channel with Choi-Jamiolkowsky state $\rho_{\text{phbit}}$. The above argument then implies that no quantum information can be sent over a single use of the channel, but it can be sent using two uses, by switching one to $E_\phi$ and the other to $\Gamma$.

This is the intuition behind the Smith and Yard construction [25]. However, because it is constructed out of two very particular types of quantum channels, this idea does not seem to extend to larger numbers of uses. Nonetheless, the intuition behind our result is based on a refinement of these ideas, which we now sketch.

We want to achieve two seemingly contradictory goals: (1) To prevent Alice from sending any quantum information to Bob over $n$ uses of the channel. (2) To permit this when Alice has access to some larger number of uses $N > n$. We can achieve (1) by increasing the erasure probability of the erasure channel to something much closer to 1, and also adding noise to the $\Gamma$ channel; the noise then swamps any entanglement. The problem is that this seems to also render (2) impossible. If the channel is so noisy that it destroys all entanglement sent through it, then (by definition) no amount of coding over multiple uses of the channel can succeed in transmitting quantum information.

However, note that the information that Alice needs to send to Bob in order to extract entanglement from the phbit $\rho_{\text{phbit}}$ is essentially classical. Bob just needs to know
one classical bit of information to distinguish the two hiding states. This suggests that classical error correction might help Alice send this information to Bob, even when the channel is very noisy. The intuition behind our proof is that a simple classical repetition code suffices. Instead of the pbit $\rho_{aB}$, we use a pbit $\frac{1}{2}(|\phi^+\rangle\langle\phi^+|_{ab} \otimes \sigma^+_{A_1B_1} \otimes \cdots \otimes \sigma^+_{A_NB_N} + |\phi^-\rangle\langle\phi^-|_{ab} \otimes \sigma^-_{A_1B_1} \otimes \cdots \otimes \sigma^-_{A_NB_N})$ that contains $N$ copies of the shield. For Bob to distinguish the hiding states, it suffices for one copy to make it through the erasure channel. Alice now tries to send all of the copies of the shield through many uses of the erasure channel. However high the erasure probability, the probability that at least one will get through can be made arbitrarily high for sufficiently many attempts.

We now give a more precise description of our construction. The erasure channel with erasure probability $p$ is $E_p := (1-p)|0\rangle\langle 0| \otimes I^A \otimes B + p|1\rangle\langle 1| \otimes I^A \otimes B / \text{dim}(B)$, where $I^A \otimes B$ is the identity channel from $A$ to $B$, and $F$ is the erasure flag. The channel $\Gamma_{A \rightarrow B}$ belongs to the class of PPT entanglement-binding channels whose Choi state is an approximate pbit [21]. We show that $\Gamma$ can be constructed with $A := A_1 \cdots A_N$ and $B := B_1 \cdots B_N$ consisting of $N$ parts, such that even if Bob only receives part $A_i$ of Alice’s shield for any $i$, they obtain approximately one bit of one-way distillable entanglement. Let $\tilde{\Gamma}_\kappa := \sigma^B \rightarrow F \circ \Gamma_{A \rightarrow B}$ be a noisy version of the channel $\Gamma$. Our construction uses channels of the form

$$M_{SA \rightarrow SFB} := \mathcal{P}_0^{S \rightarrow S} \otimes \tilde{\Gamma}_\kappa \rightarrow FB + \mathcal{P}_1^{S \rightarrow S} \otimes \tilde{\Gamma}_\kappa \rightarrow FB.$$ (1)

Here $\mathcal{P}_i^{S \rightarrow S}$ projects onto the $i$-th computational basis vector of the qubit system $S$ which thereby acts as a classical switch allowing Alice to choose whether the channel acts as $E_p$ or $\tilde{\Gamma}_\kappa$ on the main input $A$. $S$ is retained in the output which lets Bob learn which choice was made.

Making the above intuition rigorous for this channel is non-trivial: First, we must prove that the coherent information of $n$ uses of the channel is strictly zero, for any input to the channel (not just the input states from the above intuition). To this end, we cannot just directly use a pbit with $N$-copy shield of the form given above, as it would have distillable entanglement. Fortunately, we find that an approximate pbit construction from [21] can be adapted for the role. But then we must take this approximation into account in the proof that the channel does have capacity. This requires a careful analysis of the various parameters of our channel to show that both of the desired properties can hold simultaneously, which requires a somewhat delicate argument. The technical arguments are described in the Methods section.

One natural question (which we leave open) is whether we can obtain a stronger form of our result with a constant upper bound on the channel dimension. It would also be interesting to see if one can obtain a result analogous to ours for the private capacity of quantum channels. Finally, our result gives a first indication that the quantum capacity of a channel might well be an uncomputable quantity; uncomputability of the quantum capacity would necessarily imply the behaviour we have shown here.

**METHODS**

We state and outline the proof of our main result – for any number of uses we can show that there exists a channel with positive capacity but zero coherent information. Formally, we prove the following:

**Theorem.** Let $\mathcal{M}$ be the channel defined in Eq. (1). For any positive integer $n$, if $\kappa \in (0, 1/2)$ and $p \in [(1 + \kappa^n)^{-1/n}, 1]$ then we can choose $N$ and $\Gamma$ such that:

1. $Q(n)(\mathcal{M}) = 0$ and
2. $Q(n+1)(\mathcal{M}) > 0$, and therefore $Q(\mathcal{M}) > 0$.

The proof is divided in two parts. We first prove that, given $n$ and $\kappa$, for any $\Gamma$ with zero capacity there is a range of $p$ that makes the coherent information of $\mathcal{M}^\otimes n$ zero. In the second part we prove that there exists $\Gamma$ with zero capacity such that $\mathcal{M}$ has positive capacity.

For the first part we can simplify the analysis of $\mathcal{M}^\otimes n$ by showing that it is optimal to make a definite choice (i.e. a computational basis state input) for each of the $n$ switch registers. For each possible setting of the $n$ switches, the coherent information is a convex combination of the coherent information for three cases, weighted by their probabilities: (a) every channel erases, (b) all of the $E_p$ erase but not all $\Gamma$ erase, (c) at least one of the $E_p$ does not erase (and therefore acts as the identity channel). The coherent information for cases (b) and (c) can be upper bounded respectively by zero and $H(R)$, where $R$ is a system that purifies the input. For (a) it is bounded above by $-H(R)$. Weighting by the probabilities, we find that the total coherent information is upper-bounded by $1 - (1 + \kappa^n)p^nH(R)$. This allows us to conclude that for any $n$ and $\kappa$ we can find $p$ such that the coherent information of $n$ uses of the channel is zero.

To prove the second part, we show that for fixed $\kappa, p$ we can find a $\Gamma$ with an $N$-copy shield such that the coherent information of $N + 1$ uses of the channel $\mathcal{M}$ is positive for some $N + 1 > n$. We number the channel uses $0, \ldots, N$ and label the systems involved in the $i$-th use of the channel with superscript $i$. Consider the following input. The switch registers are set to choose $\Gamma_\kappa$ for use 0 and $E_p$ for the remaining uses 1, $\ldots, N$. We maximally entangle subsystem $A_0^t$ of $A^t$ (which is acted on by $\tilde{\Gamma}_\kappa$) with subsystem $A_1^t$ of $A^t$ (acted on by an erasure channel). We also maximally entangle subsystem $a_0^t$ of $A^t$ with a purifying reference system $a$ which is retained by Alice. The remaining input subsystems are set to an arbitrary pure state. The resulting coherent information is a convex combination of cases where (a) $\tilde{\Gamma}_\kappa$ erases, (b) $\tilde{\Gamma}_\kappa$ does not erase but all the $E_p$ erase, and (c) $\tilde{\Gamma}_\kappa$ and at least one $E_p$ do not erase. Case (a) contributes coherent information $-1$ weighted by its probability $\kappa$. Case (b) contributes approximately zero coherent information (due to a standard property of pbits).
In case (c), after channel use 0, Alice and Bob share the Choi state of $\Gamma$ on systems $A_0^b B_0^a \ldots A_N^b B_N^a$, and after the $N$ uses of $E_p$ at least one of $A_1^a_1 \ldots A_N^a$ reaches Bob unerased. They then share a state with approximately one ebit of one-way distillable entanglement (coherent information +1). This contribution is weighted by the probability $(1 - \kappa)(1 - p^N)$. We show that for $p \in (0, 1)$, $\kappa \in (0, 1/2)$, we can find a $\Gamma$ with large enough $N$ for which the overall coherent information is positive, proving that $Q(M) > 0$. Further mathematical details are given in the Supplementary Information.

**Acknowledgements:** DE and DP acknowledge financial support from the European CHIST-ERA project CQC (funded partially by MINECO grant PRI-PIMCHI-2011-1071). TSC is supported by the Royal Society. MO acknowledges financial support from European Union under project QALGO (Grant Agreement No. 600700).

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For any switched channel, To see this, note that any purification to produce a state of the “main output” as a classical switch allowing the sender to choose between different channels channel is a channel of the form \( \mathcal{N}^A \rightarrow \mathcal{B} \) which follows easily from \( \mathcal{I}^A \rightarrow \mathcal{B} \) denote the identity channel between them, and \( F \) be a binary erasure flag. The total erasure channel \( \mathcal{E}^A \rightarrow \mathcal{FB} \) maps any input state to \( |1\rangle \langle 1| \otimes \mathcal{I}^B \), while \( \mathcal{E}^A \rightarrow \mathcal{FB} := (1 - p)|0\rangle \langle 0| \otimes \mathcal{I}^A \rightarrow \mathcal{B} + \mathcal{P}^A \rightarrow \mathcal{FB} \) denotes the erasure channel with erasure probability \( p \). For any number of uses of \( \mathcal{E}_1 \) and any input state \( \rho \) we have

\[
I_{\text{coh}}(\mathcal{E}_1^{\otimes n}, \rho) = -S(\rho).
\] (S1)

For any register \( F \), a flagged channel is of the form \( \mathcal{N}^A \rightarrow \mathcal{FB} = \sum_{i=0}^{\dim(F)-1} p_i |i\rangle \langle i| \otimes \mathcal{N}_i^A \rightarrow \mathcal{B} \). An example is \( \mathcal{E}^A \rightarrow \mathcal{FB} \).

For any flagged channel we have

\[
I_{\text{coh}}(\mathcal{N}^F_{\text{BA}}, \rho^A) = \sum_i p_i I_{\text{coh}}(\mathcal{N}_i^A \rightarrow \mathcal{B}, \rho^A).
\] (S2)

which follows easily from

\[
I(R)B = \sum_i p_i \rho^{i\otimes |i\rangle \langle i|} = \sum_i p_i I(R)B_{\rho^i}.
\] (S3)

For any \( i \in \{0, \ldots, \dim(S) - 1\} \), let \( \mathcal{P}_i^{S \rightarrow S} \) denote the completely positive map \( X^S \rightarrow |i\rangle \langle i|^S X^S |i\rangle \langle i|^S \). A switched channel is a channel of the form \( \sum_{i=0}^{\dim(S)-1} \mathcal{P}_i^{S \rightarrow S} \otimes \mathcal{N}_i^A \rightarrow \mathcal{B} \) where each \( \mathcal{N}_i \) is a quantum channel. The register \( S \) acts as a classical switch allowing the sender to choose between different channels \( \mathcal{N}_i \) to be applied on the “main input” \( A \) to produce a state of the “main output” \( B \). We will need the following simple lemma regarding switched channels:

**Lemma 1.** For any switched channel,

\[
\max_{\rho^{SA}} I_{\text{coh}}(\mathcal{N}^S_{\text{A} \rightarrow \text{SB}}, \rho^{SA}) = \max_i \max_{\rho^A} I_{\text{coh}}(\mathcal{N}_i^A \rightarrow \mathcal{B}, \rho^A)
\] (S4)

where \( 0 \leq i \leq \dim(S) - 1 \).

**Proof.** To see this, note that any purification \( \rho^{\text{SAR}} \) of \( \rho^{\text{SA}} \) can be written in the form

\[
|\rho^{\text{SAR}}\rangle = \sum_i \sqrt{p_i} |i\rangle^S \otimes |\rho_i\rangle^\text{AR}.
\] (S5)

Here \( p_i \) is the probability that the switch is set to \( i \), and \( |\rho_i\rangle^\text{AR} \) is a purification of the channel input state \( \rho_i^A \) conditioned on that setting. Conversely, given probabilities \( p_i \) and states \( \rho_i^A \) for each switch value, we can always find \( |\rho^{\text{SAR}}\rangle \) satisfying (S5). Given this, we see that

\[
\mathcal{N}^{S \rightarrow \text{SB}}(\rho^{\text{SAR}}) = \sum_i p_i |i\rangle^S \otimes \mathcal{N}_i^A \rightarrow \mathcal{B}(\rho_i^\text{AR})
\] (S6)

where \( \rho_i^\text{AR} := |\rho_i\rangle \langle \rho_i|^\text{AR} \). From (S2) it follows that

\[
I_{\text{coh}}(\mathcal{N}^{S \rightarrow \text{SB}}, \rho^{SA}) = \sum_i p_i I_{\text{coh}}(\mathcal{N}_i^A \rightarrow \mathcal{B}, \rho_i^A)
\] (S7)

\[
\leq \sum_i p_i \max_{\rho_i^A} I_{\text{coh}}(\mathcal{N}_i^A \rightarrow \mathcal{B}, \rho_i^A)
\] (S8)

\[
\leq \max_{\rho^A} \max_i I_{\text{coh}}(\mathcal{N}_i^A \rightarrow \mathcal{B}, \rho_i^A)
\] (S9)

which completes the proof. □
We will also require some basic facts about pbits (“private bits”) [21], which we gather here. Given a bipartite system ab with dim a = dim b = 2 and a bipartite system AB with dim A = dim B, a perfect pbit with key ab and shield AB is a state γ_{abAB} of the form
\[ γ_{abAB} := U_{abAB}(φ_{ab} ⊗ σ_{AB})(U_{1})_{abAB}, \] (S10)
where φ_{ab} is the projector onto |φ⟩_{ab} := \frac{1}{\sqrt{2}}(|00⟩ + |11⟩)^{ab}, σ_{AB} is some mixed state, and
\[ U_{abAB} := \sum_{i,j=0}^{1} |i⟩⟨i|_{a} ⊗ |j⟩⟨j|_{b} ⊗ U_{ij}^{AB} \] (S11)
is a twisting unitary controlled by the key ab and acting on the shield AB as some unitary U^{AB}_{ij}. Note that due to the form of φ_{ab} and U_{abAB}, we have
\[ γ_{abAB} = \frac{1}{2} \sum_{k,l=0}^{1} |k,k⟩^{ab} ⟨l,l|^{ab} ⊗ U_{kk}^{AB} σ_{AB}^{1} (U_{1})^{AB}. \] (S12)
Let us define \( U^{abAB} := \sum_{i,j=0}^{1} |i⟩⟨j|^{b} ⊗ U^{AB}_{ij} \). If Bob has access to \( b \) and the whole shield AB then he can apply the unitary operation \( (U_{1})^{bAB} \) to these systems, yielding a 2-qubit maximally entangled state on ab. Therefore,
\[ I(\alpha|b)γ_{ab} ≥ 0 \text{ and thus} \]
\[ I(\alpha|b)γ_{ab1} ⊗ μ_{A1} ≥ 0. \] (S14)

### Channel construction

We will now describe the input and output systems of our channel \( \mathcal{M} \). Let a and b be two-dimensional systems (qubits). We call ab the “key”. Let \( A_{i,k} \) and \( B_{i,k} \) be \( d \)-dimensional systems for all \( i ∈ [N], j ∈ [r], k ∈ [m] \) where \( |n| := \{1, \ldots, n\} \). We define composite systems \( A_{i} := \{A_{i,j,k} : j ∈ [r], k ∈ [m]\} \) and \( A := \{A_{i} : i ∈ [N]\} \) for Alice, and similar systems \( B_{i} \) and \( B \) for Bob. We call AB the “shield” and call \( A_{i} \) “Alice’s i-th share of the shield”. Let F be a qubit called “the erasure flag”. Let \( F := aA, \) and \( B := bB. \)

Our construction is a switched channel
\[ \mathcal{M}^{A→SF} := P_{0}^{S→S} ⊗ Γ^{A→F}|_{\kappa} + P_{1}^{S→S} ⊗ C_{p}^{A→F}. \] (S15)
It depends on parameters \( N, r, m ∈ N \) and \( p, κ, q ∈ [0, 1] \), where \( q \) is an implicit parameter of \( Γ_{\kappa}. \) We define \( Γ^{A→F}|_{\kappa} \) to be the composite channel
\[ \hat{Γ}_{\kappa}^{A→F} := C_{p}^{B→F} \circ Γ^{A→B}. \] (S16)
A useful fact regarding compositions is that
\[ I_{coh}(N_{1}, ρ) ≥ I_{coh}(N_{2} ⊗ N_{1}, ρ), \] (S17)
which is just the quantum data processing inequality for coherent information [25].

We define \( \hat{Γ}^{A→B} \) by giving its Choi state, which depends on the parameters \( N, r, m, \) and \( q. \) Defining the composite systems \( C_{i,j,k} := A_{i,j,k} B_{i,j,k} \) and \( C_{k} := \{C_{i,j,k} : i ∈ [N], j ∈ [r]\} \), the Choi state of \( \hat{Γ}^{A→F}|_{\kappa} \) is proportional to
\[ \zeta_{abAB} = \langle 00|⟨00⟩^{ab} ⊗ \bigotimes_{k=1}^{m} \frac{q}{2}(σ + q)^{C_{k}} + \langle 01|⟨01⟩^{ab} ⊗ \bigotimes_{k=1}^{m} \frac{q}{2}(σ - q)^{C_{k}} + \langle 10|⟨10⟩^{ab} ⊗ \bigotimes_{k=1}^{m} (1 - q)^{C_{k}}, \] (S18)
where \( \omega^C_k := \bigotimes_{i=1}^N \bigotimes_{j=1}^r \frac{1}{2}(\mu^C_{i,j,k} + \mu^C_{i,j,k}) \), and \( \sigma^C_k := \bigotimes_{i=1}^N \bigotimes_{j=1}^r \mu^C_{i,j,k} \) are the Eggeling-Werner data hiding states [22]. Here \( \mu^C_{i,j,k} \) and \( \mu^C_{i,j,k} \) are the states proportional to the projectors onto the symmetric and anti-symmetric subspaces respectively.

In Eq. (139) of [21], a state \( \rho_{\text{rec}}^{(p,d,k)} \) is defined. Apart from \( p, d \) and \( k \), it also implicitly depends on a parameter \( m \), so we will denote it by \( \rho_{\text{rec}}^{(p,d,k;m)} \). Our \( \gamma^{abAB} \) is precisely \( \rho_{\text{rec}}^{(p,d,r;N,m)} \). From Sections X-A (in particular Lemma 5) and X-B of [21] we see that \( \rho_{\text{rec}}^{(q,d,r;N,m)} \) is PPT if

\[
0 < q \leq 1/3 \quad \text{and} \quad \frac{1 - q}{q} \geq \left( \frac{d}{d-1} \right)^r N.
\]

(S19)

Since a channel is PPT-binding iff its Choi matrix is PPT, the same conditions suffice for \( \Gamma \) to be PPT-binding. This condition is key to our subsequent analysis.

We will now derive from [21] another important fact about \( \gamma^{abAB} \): Defining

\[
\zeta^{abA_1B_1} := \text{Tr}_A_{2}B_{2} \cdots A_{N}B_{N} \gamma^{abAB},
\]

(S20)

for an appropriate choices of parameters, \( \rho^{abA_1B_1} \) can be made arbitrarily close to a perfect pbit \( \gamma^{abA_1B_1} \) with key \( ab \) and shield \( A_1B_1 \). In particular, we will use

**Lemma 2.** Let \( q := 1/3 \) and \( r := 2m + \lceil \log_2 m \rceil \). Then \( \tau := \| \rho^{abA_1B_1} - \gamma^{abA_1B_1} \|_1 \leq 16m^{1/2}2^{-m/4} \) for some perfect pbit \( \gamma^{abA_1B_1} \), where \( \| \cdot \|_1 \) denotes the trace norm.

**Proof.** First note that the \( \rho^{abA_1B_1} \) is simply \( \rho_{\text{rec}}^{(q,d,r;N,m)} \). Adopting the notation of [21], let \( \| A_{0011} \|_1 \) be the norm of the upper right block of the matrix \( \rho_{\text{rec}}^{(q,d,r;N,m)} \) expanded in the computational basis of the key system \( ab \). In Proposition 4 of [21], it is shown that if \( 1/2 - \| A_{0011} \|_1 < \epsilon < 1/8 \) then \( \tau \leq \delta(\epsilon) \) for some function \( \delta(\epsilon) \). The function \( \delta \) is given in Eq. (70) of [21] as

\[
\delta(\epsilon) := 2\left( 8\sqrt{2} + h(2\sqrt{2}\epsilon) \right)^{1/2} + 2\sqrt{2}\epsilon
\]

(S21)

where \( h(\epsilon) := -\epsilon \log_2 \epsilon - (1 - \epsilon) \log_2 (1 - \epsilon) \) is the binary entropy function. Provided \( 0 \leq \epsilon \leq 1/2 \), \( h(\epsilon) \) is an increasing function of \( \epsilon \) and

\[
h(\epsilon) \leq \epsilon \log_2 \left( \frac{1}{2\epsilon} \right).
\]

(S22)

In particular, if we assume that \( 0 < 2\sqrt{2}\epsilon < 1/2 \), i.e.,

\[
0 < \epsilon < 1/32,
\]

(S23)

then \( h(2\sqrt{2}\epsilon) \leq \sqrt{8\epsilon} \log_2 \frac{1}{8\epsilon} \) and thus

\[
\delta(\epsilon) \leq 2\left( 4\sqrt{8\epsilon} + \sqrt{8\epsilon} \log_2 \frac{1}{8\epsilon} \right)^{1/2} + \sqrt{8\epsilon}.
\]

(S24)

From Eq. (S23) we also get \( \log_2 \frac{1}{8\epsilon} > 1 \). By inserting this extra factor next to \( 4\sqrt{8\epsilon} \) in Eq. (S24) we get

\[
\delta(\epsilon) \leq 2\left( 5\sqrt{8\epsilon} \log_2 \frac{1}{8\epsilon} \right)^{1/2} + \sqrt{8\epsilon}.
\]

(S25)

We can upper bound the last term as \( \sqrt{8\epsilon} < (\sqrt{8\epsilon})^{1/2} < (\sqrt{8\epsilon} \log_2 \frac{1}{8\epsilon})^{1/2} \) and the whole expression as

\[
\delta(\epsilon) \leq 2^{5/2} \left( \sqrt{8\epsilon} \log_2 \frac{1}{8\epsilon} \right)^{1/2}.
\]

(S26)

Thus, we get:

\[
\tau \leq 2^{5/2} \left( \sqrt{8\epsilon} \log_2 \frac{1}{8\epsilon} \right)^{1/2}.
\]

(S27)
Rearranging Eq. (142) in the proof of Theorem 6 of [21] we find \(1/2 - \|A_{0011}\|_1 = \frac{1}{2} \left( 1 - \frac{(1-2^{-r})^m}{1 + (\frac{1}{2q} - 1)^m} \right)\). By omitting the factor 1/2 we get:

\[
\frac{1}{2} - \|A_{0011}\|_1 = \frac{1}{2} \left( 1 - \frac{(1-2^{-r})^m}{1 + (\frac{1}{2q} - 1)^m} \right)
\]

\(\leq \frac{1 + \left( \frac{1}{2q} - 1 \right)^m - (1-2^{-r})^m}{1 + (\frac{1}{2q} - 1)^m}.\) (S28)

Setting \(q = 1/3\) and using

\[(1 - x)^m \geq 1 - mx\]

for all \(m \in \mathbb{N}\) and \(x \in (0, 1)\), we have

\[
\frac{1}{2} - \|A_{0011}\|_1 < \frac{1 + 2^{-m} - (1 - 2^{-r})^m}{1 + 2^{-m}} \leq \frac{1 + 2^{-m} - (1 - m2^{-r})}{1 + 2^{-m}} = \frac{2^{-m} + m2^{-r}}{1 + 2^{-m}}\]

which is a decreasing function of \(r\). Setting \(r = 2m + \lceil \log_2 m \rceil \) we get

\[
\frac{1}{2} - \|A_{0011}\|_1 < \frac{2^{-m} + m2^{-(2m+\log_2 m)}}{1 + 2^{-m}} = \frac{2^{-m} + 2^{-2m}}{1 + 2^{-m}} = 2^{-m}.\] (S33)

Therefore, for any \(m > 5\), and substituting \(\epsilon = 2^{-m}\) into (S27) we obtain

\[
\tau \leq 2^{5/2} \left( \sqrt{8\epsilon \log_2 2} \frac{1}{8\epsilon} \right)^{1/2} \leq \frac{2^{5/2}(\sqrt{2^{-m}(m - 3)})^{1/2}}{16 \times 2^{-m/4} m^{1/2} \leq 2^{5/2} \left( \sqrt{8\epsilon \log_2 2} \frac{1}{8\epsilon} \right)^{1/2}}\]

as desired. \(\square\)

**Main result**

The proof of our main result is based on the following two key lemmas. The first proves the coherent information is zero up to \(n\) uses of the channel. The second proves that it is non-zero for some larger number of uses, hence the quantum capacity is positive.

**Lemma 3.** If \(\Gamma\) is PPT-binding, then for \(\kappa \in [0, 1]\), \(p \in [(1 + \kappa^n)^{-1/n}, 1]\), the coherent information of \(n\) uses of the channel \(\mathcal{M}\) is zero: \(Q^{(n)}(\mathcal{M}) = 0\).

**Proof.** Using (S4) and the general fact that

\[
\max_{\rho} I_{\text{coh}}(\mathcal{N} \otimes \mathcal{M}, \rho) = \max_{\rho} I_{\text{coh}}(\mathcal{M} \otimes \mathcal{N}, \rho)\]

we have \(Q^{(n)}(\mathcal{M}) = \frac{1}{n} \max_{0 \leq t \leq n} I_t\), where

\[
I_t := I_{\text{coh}}(\Gamma_t \otimes \mathcal{E}_{p} \otimes (n-t), \rho_t^{\hat{A}^t \cdots \hat{A}^n})\]

and \(l\) is the number of switches set to use \(\Gamma_t\). Here \(\rho_t^{\hat{A}^t \cdots \hat{A}^n}\) is an input state for \(n\) uses of the channel that maximises the RHS of (S40), where \(\hat{A}^i := a^i A^i_1 \cdots A^i_N\) is the main input system for the \(i\)-th use of the channel.
From the definition (S16) and Eq. (S2), we see that $I_I$ can be written as a sum of $2^n$ terms, each corresponding to a possible setting of the $n$ erasure flags. We get

$$I_I \leq \kappa^i p^{n-i}(-S(\rho_i)) + (1 - \kappa^i)p^{n-i}I_{coh}(\Gamma^\otimes l \otimes \mathcal{E}_{1}^{\otimes n-i}, \rho_i) + (1 - p^{n-i})S(\rho_i). \quad (S41)$$

The first term in this bound is the case where all $n$ channel uses erase and it follows from (S1). The second term upper bounds the cases where all of the $\mathcal{E}_p$ uses erase but not all of the $\Gamma_\kappa$ channels do, obtained via (S17). The final term upper bounds the contribution from the remaining cases using the trivial bound.

Using (S17) and the fact that $\Gamma$ is PPT-binding, we obtain $I_{coh}(\Gamma^\otimes l \otimes \mathcal{E}_{1}^{\otimes n-i}, \rho_i) \leq I_{coh}(\Gamma^\otimes n, \rho_i) \leq 0$ and thus we can drop the second term in (S41):

$$I_I \leq (-\kappa^i p^{n-i} + 1 - p^{n-i})S(\rho_i) \leq (1 - (1 + \kappa^i)p^n)S(\rho_i), \quad (S42)$$

where the second inequality follows from $p, \kappa \in [0, 1]$. We find that $I_I \leq 0$ provided that

$$p \geq (1 + \kappa^n)^{-1/n}. \quad (S43)$$

On the other hand, $I_I \geq 0$ since we can always choose $\rho_i$ to be pure. This implies $I_I = 0$ and thus $Q^{(n)}(\mathcal{M}) = 0$, which completes the proof.

**Lemma 4.** For $p \in (0, 1)$, $\kappa \in (0, 1/2)$, we can choose the parameters $q, N, r, m, d$ such that the PPT condition (S19) holds and $Q^{(N+1)}(\mathcal{M}) > 0$.

**Proof.** Our proof has two parts. In part (i) we prove a lower bound on $Q^{(N+1)}(\mathcal{M})$ by analysing a particular input to the channel. In part (ii) we show that the channel parameters can be chosen to make this lower bound strictly positive while, at the same time, satisfying (S19).

(i) We number the $N + 1$ channel uses by $\{0, 1, \ldots, N\}$, and label the systems involved in the $i$-th channel use with superscript $i$. The switch systems are set so that the first use of the channel acts as $\Gamma_\kappa$ on its main input, and the remaining $N$ uses act as $\mathcal{E}_p$.

If $X$ and $Y$ are two systems of equal dimensions, we use $\phi^{XY} := |\phi\rangle\langle\phi|_{XY}$ to denote the maximally entangled state on $XY$ where $|\phi^{XY}\rangle := \sum_{i=0}^{\dim(X)}|i\rangle_Y|i\rangle_Y/\sqrt{\dim(X)}$. Alice prepares maximally entangled states on subsystems $a^i\alpha$ and on $A_i^0A_i^1$ for all $i \in [N]$. The purification of the overall input to the $N + 1$ uses of the channel $\mathcal{M}$ is

$$|\nu\rangle := |0\rangle^S|\phi\rangle^{ab^0} \bigotimes_{i=1}^{N} (|1\rangle^{S_i}|\alpha\rangle^a_i|\phi\rangle^{A_i^0A_i^1} \bigotimes_{j=2}^{N} |\beta\rangle^{A_i^j} \big) \quad (S44)$$

shown in Fig. 1, where $|\alpha\rangle$ and $|\beta\rangle$ are arbitrary pure states, $\alpha$ is a reference system, and $S_i$ and $\hat{A}_i$ are the switch and main input systems for the $i$-th use of $\mathcal{M}$, respectively.

The switch settings cause the first use of $\mathcal{M}$ to act as $\Gamma_\kappa$ on $\hat{A}_0^0 = a^0\alpha_0^0 \cdots A_N^0$ (see (S16)). With probability $\kappa, \Gamma_\kappa$ erases, yielding $|1\rangle|0\rangle^F \otimes \mu^B$. With probability $1 - \kappa$, it sets the erasure flag to $|0\rangle|0\rangle^F$ and acts as $\Gamma_\kappa$ on $\hat{A}_0^0$, producing $\hat{B}_0^0$. At this point the state of $a^0\hat{A}_0^0 \cdots A_N^0$ is just the Choi state $\zeta^{aB_0B}$ defined in (S18) with its systems relabeled as follows: $\hat{B}_0 \rightarrow \hat{B}_0^0$ and $A_j \rightarrow A_j^0$ for all $j \in [N]$. The switches are set so that the remaining $N$ uses of $\mathcal{M}$ apply $\mathcal{E}_p$ to the each of the systems $\hat{A}_j^0$ for each $j \in [N]$. Bob now applies a simple post-processing operation to the output systems of all $N + 1$ channel uses to obtain a state of a system $bA_1^0B_1^G^0$: He first measures the erasure flags $F^1 \cdots F^N$. With probability $1 - p^N$, at least one of these flags, say $F^j$, will be in the state $|0\rangle|0\rangle^F$, and the state of $A_1^0$ has been perfectly transferred to his system $B_1^0$. Otherwise, with probability $p^N$, Bob picks an arbitrary $j \in [N]$. In this case the state of $F^j$ is $|1\rangle|1\rangle^F$ and the state of $B_1^j$ is maximally mixed and uncorrelated with any other system. Now, as depicted in Fig. 1, Bob transfers the state of $F^j$ to a system $G$, the state of $B_1^0$ to $A_1^0$, the state of $B_0^0$ to $B_1^0$, and $b^0$ to $b$. Bob then discards all of his systems except for $bA_1^0B_1^G^0$, which are now in the state

$$\rho^{bA_1^0B_1^G^0} := \kappa \mu_2^a \otimes \sigma^{bA_1^0B_1^G} \otimes |1\rangle|1\rangle^F$$

$$+ (1 - \kappa)p^N\zeta^{aB_1^0} \mu_{A_1^0} \otimes |1\rangle|1\rangle^G \otimes |0\rangle|0\rangle^F$$

$$+ (1 - \kappa)(1 - p^N)\zeta^{aB_1^0} \mu_{A_1^0} \otimes |0\rangle|0\rangle^G \otimes |0\rangle|0\rangle^F. \quad (S45)$$

Here $\zeta^{aB_1^0} := \mathcal{I}_{A_1^0 \rightarrow A_1^0}(\zeta^{aB_1^0})$, where $\zeta^{aB_1^0}$ is the state (S20) from Lemma 2. The details of $\sigma^{bA_1^0B_1^G}$ are unimportant. The first term in (S45) corresponds to the case where the first channel use erases. When the first use doesn’t
erase, the case where all other uses erase yields the second term, and the case where at least one of the other uses does not erase gives the third term.

Let us call Bob's post-processing operation \( \mathcal{P} \). Using the state \( \nu := |\nu \rangle \langle \nu | \) from (S44), we can write
\[
(N + 1)Q^{(N+1)}(\mathcal{M}) \geq I_{\text{coh}}(\mathcal{M}^\otimes N, \nu) \geq I_{\text{coh}}(\mathcal{P} \circ \mathcal{M}^\otimes N, \nu) = I(a)_{\mathcal{A}1}B1_{\mathcal{B}}G^0_{\mathcal{B}} \otimes \nu_{\mathcal{B}1}, \omega^o,
\]
where the composition property (S17) was used. Given the “flagged” structure of (S45), we can use (S3):
\[
(N + 1)Q^{(N+1)}(\mathcal{M}) \geq kI(a)_{\mathcal{A}1}B1_{\mathcal{B}}G_{\mu^s \otimes \omega^s_{\mathcal{B}1}} + (1 - \kappa)p^N I(a)_{\mathcal{A}1}B1_{\mathcal{B}} \gamma_{ab}^{\mathcal{B}1} + (1 - \kappa)(1 - p^N)I(a)_{\mathcal{A}1}B1_{\mathcal{B}} \gamma_{ab}^{\mathcal{B}1}. \tag{S47}
\]
The first term is \(-kS(\mu^s) = -k\). If \(\tau = \|\gamma_{ab}^{\mathcal{B}1} - \gamma_{ab}^{\mathcal{B}1}\|_1\) for some perfect pbib \(\gamma_{ab}^{\mathcal{B}1}\), then by the monotonicity of the trace distance under CPTP maps
\[
\tau \geq \|\gamma_{ab}^{\mathcal{B}1} \otimes \mu_{\mathcal{A}1} - \gamma_{ab}^{\mathcal{B}1} \otimes \mu_{\mathcal{A}1}\|_1. \tag{S48}
\]
In what follows, we will use the Alicki-Fannes inequality [26]. This states that for \(\rho^{RB}\) and \(\sigma^{RB}\) such that \(\tau := \|\rho^{RB} - \sigma^{RB}\|_1 < 1\) we get
\[
|I(R)_{\rho_{\mathcal{B}}} - I(R)_{\sigma_{\mathcal{B}}}| \leq 4\log_2 \text{dim}(R) + 2h(\tau). \tag{S49}
\]
Using (S48) and properties (S13), (S14), \(\dim(a) = 2\) together with the Alicki-Fannes inequality we have
\[
I(a)_{\mathcal{A}1}B1_{\mathcal{B}} \gamma_{ab}^{\mathcal{B}1} \geq 1 - \Delta, \quad I(a)_{\mathcal{A}1}B1_{\mathcal{B}} \gamma_{ab}^{\mathcal{B}1} \geq -\Delta, \tag{S50, S51}
\]
where
\[
\Delta := 4\tau + 2h(\tau). \tag{S52}
\]
Therefore, \((N + 1)Q^{(N+1)}(\mathcal{M}) \geq (1 - \kappa)(1 - p^N - \Delta) - \kappa\) which is strictly positive if
\[
\Delta < 1 - p^N - \frac{\kappa}{1 - \kappa}. \tag{S53}
\]

(ii) We will now show how the parameters must be chosen. First, to ensure that (S19) is satisfied, we specify that \(d := 2N\tau\) and \(q := 1/3\). Now, if \(\kappa \in (0, 1/2)\) then \(\kappa/(1 - \kappa) \in (0, 1)\), so for any \(p \in (0, 1)\) we can always choose \(N\) large enough to make the RHS of (S53) positive. Fixing this value of \(N\), we then must choose \(m\) and \(\tau\) to make \(\Delta\) small enough to satisfy (S53). Lemma 2 tells us that with \(q = 1/3\) and \(\tau = 2m + \log_2 m\), we have \(\tau \leq 16m^{1/2}2^{-m/4}\). Recall that \(\Delta = 4\tau + 2h(\tau)\), into which we are substituting \(\tau \leq 16m^{1/2}2^{-m/4}\). Provided \(0 \leq x \leq 1/2\), \(h(x)\) is an increasing function of \(x\), and \(h(x) \leq 2x \log_2 \frac{1}{x}, \) so \(h(\tau) \leq h(16m^{1/2}2^{-m/4}) \leq 4m^{3/2}2^{-m/4}\) (provided \(16m^{1/2}2^{-m/4} \leq 1/2\)). We get
\[
\Delta \leq 64m^{1/2}2^{-m/4} + 8m^{3/2}2^{-m/4} \leq 72 \times 2^{-m/4}m^{3/2}. \tag{S54}
\]
One can choose \(m\) to make this as small as required.

We can now prove our main result.
Theorem. Let $\mathcal{M}$ be the channel defined in Eq. (S15). For any positive integer $n$, if $\kappa \in (0, 1/2)$ and $p \in [(1 + \kappa^n)^{-1/n}, 1]$ then we can choose $q, d, r, N, m$ such that:

1. $Q^{(n)}(\mathcal{M}) = 0$ and
2. $Q^{(N+1)}(\mathcal{M}) > 0$, and therefore $Q(\mathcal{M}) > 0$.

Proof. In Lemma 3 we show that if $\Gamma$ is PPT-binding and $\kappa, p$ satisfy $\kappa \in [0, 1]$ and $p \in [(1 + \kappa^n)^{-1/n}, 1]$, then the first statement holds. In Lemma 4 we show that for any $\kappa \in (0, 1/2)$ and $p \in (0, 1)$, we can choose the parameters $q, d, r, N, m$ so that the second statement holds and $\Gamma$ is PPT-binding. Therefore, for $(\kappa, p)$ in the intersection of the two regions, the channel $\mathcal{M}$ satisfies both statements.

To be concrete, we can choose $\kappa = 1/4$ (so that $\kappa/(1 - \kappa) = 1/3$) and choose $p = (1 + \kappa^n)^{-1/n}$. We can then choose $N$ so that $1 - p^N \geq 2/3$: we require $(1 + \kappa^n)^{-N/n} < 1/3$. Taking logs of both sides, rearranging and using $x/\ln(2) \geq \log_2(1 + x)$, we have $N > (\log_2 3)(\ln 2)n^4$, so let us take $N = 2n^4$. We must now choose $m$ large enough ($m \geq 68$) that $\Delta < 1/3$ in (S54).

SUPPLEMENTARY REFERENCES

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