Understanding the Related-Key Security of Feistel Ciphers from a Provable Perspective

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Abstract—We initiate the provable related-key security treatment for models of practical Feistel ciphers. In detail, we consider Feistel networks with four whitening keys $w_i(k)$ ($i = 0, 1, 2, 3$) and round-functions of the form $f(\gamma_i(k) \oplus X)$, where $k$ is the main-key, $w_i$ and $\gamma_i$ are efficient transformations, and $f$ is a public ideal function or permutation that the adversary is allowed to query. We investigate conditions on the key-schedules that are sufficient for security against XOR-induced related-key attacks up to $2^{n/2}$ adversarial queries. When the key-schedules are nonlinear, we prove security for 4 rounds. When only affine key-schedules are used, we prove security for 6 rounds. These also imply secure tweakable Feistel ciphers in the Random Oracle model.

By shuffling the key-schedules, our model unifies both the DES-like structure (known as Feistel-2 scheme in the cryptanalytic community, a.k.a. key-alternating Feistel due to Lampe and Seurin, FSE 2014) and the Lucifer-like model (previously analyzed by Guo and Lin, TCC 2015). This allows us to derive concrete implications on these two (more common) models, and helps understanding their differences—and further understanding the related-key security of Feistel ciphers.

Index Terms—blockcipher, provable security, indistinguishability, related-key, Feistel cipher, key-alternating paradigm.

I. INTRODUCTION

Feistel-like blockciphers consist of several iterative applications of a simple Feistel permutation

$$\Phi_{G_{k_i}}(W_L || W_R) = W_R || W_L \oplus G_{k_i}(W_R)$$

for a keyed function $G : \{0, 1\}^n \times \{0, 1\}^n \rightarrow \{0, 1\}^n$ on $n$-bit strings, yielding a $2n$-bit blockcipher [1]. Such ciphers and their generalizations constitute a half proportion of modern blockciphers, including some most popular designs DES [2], Lucifer [3], GOST [4], and NSA’s SIMON family [5]. This has made it the object of a very large (and still increasing) amount of analyzes.

In information-theoretic model, the round-function $G$ would be assumed somewhat random. Without additional hardness assumption, provable security is limited to at most $2^n$ queries [6], which is much smaller than $2^{2n}$, the domain-size of the Feistel ciphers. Despite this limitation as well as the gap between the strong assumption on $G$ and the rather weak round-functions in practical ciphers, this approach excludes any possibility of generic attacks & supplies insights into the blockcipher structures. Therefore, it has found applications in both Feistel ciphers [7], [6], [8], [9], [10], [11] and their counterpart Key-Altering Ciphers (KACs) [12], [13], [14], [15].

Related-Key Attacks (RKAs) were independently introduced by Biham [16] and Knudsen [17] in early 1990s, and the attack model was later formalized by Bellare and Kohno [18]. In this setting, the adversary is allowed to query the blockcipher under multiple secret keys that satisfy adversary-chosen relations. The presence of such related-keys may be the consequence of a protocol-level key update [19], or the user key being disturbed by fault injections [20]. The goal of the adversary is to either recover the secret key(s), or to distinguish the related-key oracles from independent random permutations [18].

RKAs can be classified according to the adversary-chosen relations between the keys. Likely, the most important category is the so-called XOR-induced Related-Key Attack ($+$-RKA) [21], i.e., RKA that allows the adversary to XOR any constant of its choice to the secret user key. Such RKAs are important for at least three reasons. First, they arise naturally in a number of contexts, such as the f8 and f9 protocols of the 3GPP standard [19]. Second, from a theoretical point of view, they are the simplest kind of attacks to have the completeness property [22], namely, for any keys $k, k' \in \{0, 1\}^n$, there exists $\Delta \in \{0, 1\}^n$ such that $k \oplus \Delta = k'$.

Last—but most importantly,—$+$-RKAs are the most relevant to cryptanalytic practice. Most practical ciphers mix the keys into the state via the XOR operation. For such targets $+$-RKAs are inherent to the majority of differential-based attacks (this was a comment in [23]): because XOR key-relations leave the chance of canceling the state difference with the (chosen) round-key difference (this phenomena was named local collision [24]), thus extending differentials without decreasing their probabilities. Due to this, $+$-RKAs have been the most widely used attack model in symmetric cryptanalysis (as another example, when the powerful related-key boomerang and rectangle attacks were first introduced [25], they were in the $+$-RKA form). And they have given rise to a plenty of prominent results, including very efficient (distinguishing) attacks on many Feistel ciphers that will be mentioned in the next subsection, a practical-time attack on the 3GPP encryption algorithm KASUMI [26], and a forgery attack on 3-DES-based RMAC [27]. And their variants break full AES-192 and AES-256 [24] and 10-round AES-256 in practical-time [28].

The mentioned attack on RMAC is also a notable example of RKA weakness resulting

1KACs are blockciphers $KAC_{P_1, \ldots, P_t}$ $\forall x \in \{0, 1\}^n, k_0, k_1, \ldots, k_t, M \in \{0, 1\}^n$ $KAC_{P_1, \ldots, P_t}(M) = k_0 \oplus P_t(\ldots (k_1 \oplus P_2(k_0 \oplus M)))$, i.e., key-additions and keyless permutations are alternatively applied.

2These variants assumed XORing constants into the round-keys, and are thus called related sub-key attacks.
in more disastrous attacks on high-level symmetric primitives, showing that pursuing RKA security is not only of theoretical interest.

**Our Question.** With the above discussion, ⊕-RKAs deserve special attention on the theoretical side. Recall that such a provable security requires that with a secret key $k$, the $q$ blockcipher instances $E_{k\oplus \Delta_1}, \ldots, E_{k\oplus \Delta_q}$ queried by the attacker with distinct chosen constants $\Delta_1, \ldots, \Delta_q$ are indistinguishable from $q$ independent random permutations. Such security has been established for KACs [21], [29] and their tweakable variants [30]. It’s then natural to ask: under which conditions could Feistel ciphers be provably secure against ⊕-RKAs?

In fact, to a large extent, our motivation also stems from practice: certain structural features cause remarkable ⊕-RKA weakness in a lot of Feistel ciphers in reality. The most well-known example must be the complementation property in DES [31], i.e. $\text{DES}_X(M) = \text{DES}_{\overline{X}}(\overline{M})$ where $\overline{X}$ is the bit-by-bit complementation of $X$. This non-random behavior also exists in its variants 3-DES [32] and DESL [33]. This not only cinches efficient related-key distinguishers on DES, but also reduces its effective key-length by 1 bit in the traditional single-key attack setting. Although appearing harmless, it has been long asked how to get rid of it [34]. Other marvelous examples include ⊕-RKAs on GOST with very low complexity described in [35] and [36], and very efficient distinguisher on the SHA-3 candidate based on Lesamnta [37]. In all, it appears that the components (e.g. key-schedules) of Feistel ciphers have to be carefully designed in order to achieve RKA security. This is sharply contrast to the KAC model, for which even the simplest idea $k \oplus \text{PRF}(k \oplus \text{PRF}(k \oplus M))$ already buys some level of security (see [29]). A better understanding of Feistel ciphers in the RKA setting is thus crucial.

We have noticed two works that partially addressed our question. The first work of Barbosa and Farshim proved that the complementation property in Feistel ciphers could Feistel ciphers be provably secure against ⊕-RKAs?—including the known RKA-KAC security has been established for KACs [21], [29] and their tweakable variants [30]. Unfortunately, this model misses function [7], and have been extensively studied, with [6] using a pseudorandom function (PRF) and [8] to name a few. Hopefully, this will serve invaluable insights, and help address the challenge of designing RKA secure Feistel ciphers—and further tweakable Feistel ciphers, as RKA-secure ciphers and tweakable blockciphers [39] are strongly related [18].

### A Unified Model for Feistel Ciphers in Reality.

Practical Feistel ciphers usually employ keyless transformations for round-functions, and mix the keys into the structure via efficient group operations (usually xor). In addition, whitening keys may be used. This naturally motivates modeling the keyless round-functions as public (random) functions or permutations $f_i$, explicitly xor-ing the round-keys somewhere, and eventually adding whitening keys.

In detail, we consider Feistel networks in which the state at round $i$ is updated according to

$$W_L || W_R \mapsto W_R || W_L \oplus f_i(k_i \oplus W_R) \tag{2}$$

and four $n$-bit whitening keys $(w_0, w_1, w_2, w_3)$ are used. Among them, $w_0$ is used as the pre-whitening key, while $w_2$ is used as the post-whitening key. The special case of our model without whitening keys was named Key-Alternating Feistel (KAF) by Lampe and Seurin [9]. Due to this, we name our model Key-Alternating Feistel with Whitening keys (KAFw).

To be closer to the reality, we do not assume the components independent. Instead, we assume: (i) all the round-functions $f_1, \ldots, f_t$ are the same function $f$ and (ii) each subkey is derived from an $n$-bit master-key $k$ via an efficiently computable $n$-to-$n$-bit transformation, i.e. $k_i = \gamma_i(k)$ for $i = 1, \ldots, t$, and $w_{k_j} = w_j(k)$ for $j = 1, 2, 3, 4$. Please see Fig. 1 for the instances with 4 and 6 rounds. Denote by $(w, \gamma)$ such a key-schedule function for $t$-rounds, $w = (w_0, w_1, w_2, w_3)$, $\gamma = (\gamma_1, \ldots, \gamma_t)$; and denote by $\text{KAFw}_{f,(w,\gamma)}$ the “single-function” KAFw model with round-function $f$ and key-schedule $(w, \gamma)$.

### On Other Models.

We re-stress our model should be distinguished from the mentioned Luby-Rackoff model with pseudorandom Feistel round-functions $G_k(W_R)$. In such round-functions the keys are “embedded” in a non-obvious way, and they are thus unable to capture many structural properties in practical Feistel ciphers.

We did not notice any previous work on our KAFw model. However, by appropriately shuffling the key-schedule $(w, \gamma) = ((w_0, \ldots, w_3), (\gamma_1, \ldots, \gamma_t))$, KAFw unifies existing famous theoretical models, and captures the structures of a large range of Feistel ciphers. To see this, we first note that (as mentioned) by setting the whitening keys to 0, we recover the KAF model,

3While using $n$-bit master-keys might be uncommon in practical Feistel ciphers, as a first step it’s sufficient for serving insights (as we will see). To step further, the difficulty lies in modeling “non-trivial” key-schedules for longer master-keys: see the discussion in page 6.

4On the practical side, the cipher CLEFIA [40] recommended by the ISO/IEC standard [41] is a 4-line generalization of KAFw.
a.k.a. Feistel-2 schemes in the cryptanalytic community [42] (also see the IACR tkz library), which has been deeply understood from the cryptanalysis point of view [43], [36], [42], [44] and frequently used as instructive examples for illustrating new attacks [45], [46]. The KAF model roughly captures the structures of DES [2], GOST [4], and Camellia variant without $FL/FL^{-1}$ functions [36].

We then note that in the aforementioned Lucifer-like structure, each round-key is xored after the corresponding round-function, i.e., the state at round $i$ is updated according to

$$W_L || W_R \mapsto W_R || W_L \oplus f_i(W_R) \oplus k_i.$$  

(3)

This afterwards manner effectively eliminates the key interruption in the 1st round and in the last round and allows the analyst to analyze a shorter equivalent description [47], which is a two-round-reduced variant with the original 1st and last round-keys as whitening keys: $0||k_1$ for pre-whitening, and $k_1||0$ for post-whitening (we include a formal clarification in Appendix A). We denote by KAFv the resulted whitening key-based KAF Variant. Roughly, KAFv or the Lucifer-like model and their multi-line generalizations capture the structures of Blowfish [48], TEA [49], XTEA [50], SIMON [5], Piccolo (multi-line) [51], and RC2 [52]. Most importantly to us, each KAFv instance is also captured by a KAFw instance with a corresponding key-schedule (a formal analysis is given in section V-B). Therefore, our model KAFw seems the most general.

By the above discussion, it seems the three models KAFw, KAF, and KAFv are cryptographically equivalent modulo different key-schedules. This seems to contradict some existing understandings. For example, it was commented that the Lucifer-like structure blocks the complementation property, while in KAF the first and last rounds are more effective [47]; and that KAFw seems stronger against RKAs, which appears one of the motivations to use it [37]. And, assuming independent random round-functions and identical round-keys, the 21-round KAFv variant is indifferentiable from ideal ciphers [53], while the KAF variant is never indifferentiable [38] (even worse, such KAF would collapse to a 1-round KAC built on a keyless multi-round Feistel permutation! see page 6). As will be unveiled in this paper, this distinction stems from the fact that to achieve the same level of security, KAF and KAFv models require different properties from the involved key-schedules; and with common key-schedule designs, KAFv has a higher chance of being secure against RKAs than KAF! (For details please see below.)

**Our Contributions.** We first focus on the KAFw$^{F,(w,\gamma)}$ model and prove general results, and then derive concrete implications on the more popular KAF and KAFw models.

In detail, we analyze both the case of $(w, \gamma)$ being (highly) non-linear (with respect to $\oplus$) and the case of $(w, \gamma)$ being purely affine. In each case, (as mentioned) KAFw$^{F,(w,\gamma)}$ uses identical round-functions and sub-keys derived from an $n$-bit master-key. For the round-function $f$, we consider both $f = F$ a random $n$-to-$n$ function (denoted KAFw$^{F,(w,\gamma)}$) and $f = P$ a random $n$-bit permutation (denoted KAFw$^{P,(w,\gamma)}$). Here $w_0(k) = w_1(k) = w_2(k) = w_3(k) = 0$, i.e. no whitening keys;

$$\gamma_1(k) = M_1 \otimes k \oplus k^3, \quad \gamma_2(k) = \gamma_3(k) = 0,$$

and

$$\gamma_4(k) = M_4 \otimes k \oplus k^3,$$

where $M_1 \neq M_4$ are two non-zero constants chosen from $\{0, 1\}^n$. $\otimes$ denotes multiplications taken over the finite field $\mathbb{F}_{2^n}$. (This $M \otimes k$ should be distinguished from $M \cdot k$, the multiplication between a bit matrix $M$ and a vector $k$.)

With this key-schedule, the 4-round KAFw$^{F,(w,\gamma)}$ and KAFw$^{P,(w,\gamma)}$ are secure up to $c \cdot 2^{n/2}$ queries for a small constant $c$ (which is given in section VI).

For any cipher with an $n$-bit master-key, an RKA adversary could distinguish by detecting collisions between secret related-keys and offline guesses [18]. Such a collision can be detected with $2^{n/2}$ queries. Therefore, the birthday-type bound is tight. The round complexity 4 is also tight, since with one less round a standard (i.e., non-related-key), adaptive chosen-
Without non-linearity, using a related-key boomerang [25] distinguisher we break four rounds with any affine \((w, \gamma)\), and further using the boomerang switch trick [24] we break five rounds under one more assumption on \((w, \gamma)\). Our positive result states conditions on the key-schedule that suffice for \(2^{n/2}\) security of 6-round \(\text{KAFw}^{(w, \gamma)}\) and \(\text{KAFw}^{(w, \gamma)}\). The conditions are very simple, and (roughly) prevent self-symmetry and complementation properties. An example of good \((w, \gamma)\) is as follows (see section VI for a simpler and more surprising example):

- \(w_0(k) = w_1(k) = w_2(k) = w_3(k) = 0\), i.e. no whitening keys;
- \((\gamma_1(k), \gamma_2(k), \gamma_3(k), \gamma_4(k), \gamma_5(k), \gamma_6(k)) = (k, k, \pi(k), k, k, \pi(k))\), where \(\pi(kL || kR) = kR || kL \oplus kR\).

Note that this \(\pi\) is a linear orthomorphism. An orthomorphism is a permutation \(\pi\) of \(\{0, 1\}^n\) for which \(x \mapsto x \oplus \pi(x)\) is also a permutation. Orthomorphisms have been found helpful in establishing nice theoretical results, in particular theoretically minimal Luby-Rackoff model [55] and 2-round KACs [56].

**Implications on KAF and \(\text{KAFw}\).** From the general results on \(\text{KAFw}\) we can derive positive results on 4- and 6-round KAF and \(\text{KAFw}\), and that which conditions on the key-schedules suffice for security.

For non-linear key derivation functions (KDFs) our results indicate they could increase the \(\oplus\)-RKA security of KAF. This confirms the theoretical soundness of designs with highly non-linear key-schedules, e.g. CAST-128 [57].

For affine KDFs the situation is a bit complicated (and more interesting). Roughly speaking, for KAF (and also \(\text{KAFw}\)) ciphers, one should pay additional attention on the interaction between the KDFs at the odd rounds and even rounds respectively. On the other hand, for \(\text{KAFw}\) ciphers it (may) suffice to just focus on designing each round-KDF, without considering the interactions between different rounds. These explain the different behaviors of KAF and \(\text{KAFw}\) structures, and serve as theoretical evidence that with common ad hoc key-schedule designs, \(\text{KAFw}\) variants do have a higher chance to achieve \(\oplus\)-RKA security than KAF and \(\text{KAFw}\). This confirms the theoretical soundness of reverting to \(\text{KAFw}\) to improve RKA security, which—as mentioned—seems a folklore [47], [37], and seems the idea underlying many KAF ciphers mentioned before. For clearness, more discussion is deferred to Section V, after we present the concrete conditions on the key-schedules.

Aside from clarifying KAF and \(\text{KAFw}\) models, our results also provide new insights into designing affine key-schedules for practical Feistel ciphers, which is a long-standing open problem highlighted in e.g. [23], [12]. Note that affine key-schedules are usually preferred due to their high efficiency and compatibility with frequently rekeying. For example,

5See the end of section V-B for the use of the whitening keys \(w_0(k), w_1(k), w_2(k), w_3(k)\).

6But in practice, this should be interpreted with caution. CLEFIA also employs a highly non-linear key-schedule, but was found to suffer from (a small portion of) weak-keys [58] in the RKA setting. Weak-keys could not be covered by these theoretical analyses.

**Tweakable Feistel Ciphers.** By the general result of Bellare and Kohno [18], given a \(\oplus\) RKA secure blockcipher \(E_k(M)\) with \(n\)-bit \(k\), XORing the tweak \(t\) into the key, i.e. \(E_{k\oplus t}(M)\), gives rise to a tweakable blockcipher (TBC) with \(n\)-bit tweaks and keys and provable security against \(2^{n/2}\) queries. Therefore, efficient tweakable Feistel ciphers with birthday security could be obtained from our results. We stress that tweakable Feistel ciphers obtained via our approach are in the Random Oracle Model, i.e. with the round-functions public and random, which significantly deviates from the tweakable Luby-Rackoff ciphers [59] built upon secret random functions.

**Modes for Permutations.** Alternatively, the variants \(\text{KAFw}^{P,(w, \gamma)}\), \(\text{KAFp}_{\gamma}\), and \(\text{KAFp}_{\gamma^*}\) can be viewed as modes for cryptographic permutations. With the appearance of reliable permutations such as the permutations underlying SHA-3 [54] and the Simpria family [60], our results allow creating highly modular wide-block ciphers with some level of provable \(\oplus\)-RKA security support. As mentioned, by further XORing the tweak into the key, wide-block tweakable Feistel ciphers are obtained. These may find application in various settings, for example, instantiating provably secure Onion-AE [61] and disk encryption [62]. In addition, given a wide-block TBC, using the nonce and associated data as the tweak and performing an encode-then-encrypt process give rise to a robust authenticated encryption scheme [63], [60], which could be a quite attractive application.

For comparison, the KAC results [29], [21], [30], [64] also offered such permutation modes. But our \(\text{KAFw}^{P,(w, \gamma)}\) approach achieves domain extension at the same time, i.e. it offers a provable way to build such a TBC from “smaller” cryptographic permutations. This may reduce implementation cost & increase security confidence.

The recent SPN approach also achieves domain extension [65]. A more detailed comparison will appear later.

Finally, we remind the reader that all of our results and implications are derived in the Random Oracle Model. Once instantiated, arguments and security insurance turn heuristic [66].

**Related Work and Comparison.** As mentioned, Barbosa and Farshim (BF) have studied provable RKA-security of Feistel ciphers (more precisely, of Luby-Rackoff models) [10]. Here we make a comprehensive comparison. In detail, BF proved the following 4-round Luby-Rackoff variant (see Eq. (1) for the function \(\Phi_{G_{k_1}}(x)\))

\[
\text{LR}_{k_1, k_2}(M) = \Phi_{G_{k_2}}(\Phi_{G_{k_1}}(\Phi_{G_{k_2}}(\Phi_{G_{k_1}}(M))))
\]

is CCA secure against RKAs, if \(G\) is an RKA-secure PRF.

BF’s work has two advantages:

(i) Their results covered a much wider range of Related-Key Derivation (RKD) function set. Informally, this means

\[7\]We are not trying to make a comparison to [62], since our main goal is to deepen the understanding of blockcipher structures.

DES, GOST, SIMON all used affine schedules.
LR_{k_1,k_2} is secure even if the attacker queries LR_{ψ(k_1,k_2)} for ψ more complicated than (k_1||k_2) ⊕ Δ.

(ii) Their round-functions are more “generic”, and could be instantiated under the complexity assumptions.

For the first point, as we argued, we aim at bridging theory and reality. The most widely-used attack model is ⊕-RKA, and it’s not clear whether the complicated RKD functions are indeed possible to appear in reality. Moreover, for KAFw, RKA security against larger RKD sets isn’t “for-free”: since the sufficient conditions on the key-schedules heavily depend on the concrete RKD function (e.g. see Definition 1), more complicated key-schedules are likely required. Random oracle KDFs should be sufficient for all the “interesting” RKD sets, but they fall short of providing insights into practical designs. In all, it seems questionable to spend a lot of complexity on complicated key-schedules to buy security against somewhat artificial RKD functions. They shed lights on SPNs, while we on Feistel. In all, the two works are complementary.

For the second point, we argue switching from the Luby-Rackoff model to KAFw variants represents a significant step in cryptography along two axes.

First, viewing Feistel networks as abstract models of real-world blockciphers, we already argued the (seemingly more generic) Luby-Rackoff model LR_{k_1,k_2}(M) misses many important structural features in the RKA setting, and is arguably too far from cryptographic reality (see page 2). Even from a theoretical point of view there remains disadvantages: the Luby-Rackoff model falls short of showing how to concretely design keyed primitives (BCs or keyed round-functions) from (conceptually) simpler keyless primitives; it just “defers” the task to designing keyed round-function G_{k_i}. In the RKA setting, this requires an RKA-secure PRF G_{k_i} from keyless primitives, which is even harder.

In contrast, results on KAFw variants demonstrate how to construct secure blockciphers from keyless permutations or functions, which was a hot topic (see the KAC papers [12]), and has been recently re-emphasized by Diffie (in Leiden, March, 2018). This nicely fills in the gap left by Luby-Rackoff results.

Second, viewing Feistel networks as modes, this represents switching from modes for PRFs/blockciphers to modes for cryptographic keyless permutations. Permutation-based modes not only provides more choices, but also reduce the burden of designers (they could focus on designing one permutation without considering RKA issues). Therefore, it has been a long trend, with prominent examples include permutation-based variant of the IAPM authenticated encryption mode [67], the popular multi-purpose sponge functions [68] and many sponge-based CAESAR submissions [69], permutation-based hash functions [70] and random functions [71].

In summary, BF’s work is more foundational, and shows how to build RKA secure PRPs from RKA secure PRFs, while our work tries to shed more light on the practical side. BF’s Luby-Rackoff approach also gives rise to RKA-secure ciphers and TBCs, but it requires an RKA secure PRF (as the round-function), for which it may not be easy to find an efficient and reliable candidate (especially when a large block-size is desired).

A concurrent work of Cogliati and Lee shows how to construct wide-block TBC from SPNs [65]. They focus on beyond-birthday security, which is better than our birthday-type bounds. On the other hand, we proved ⊕-RKA security, which is not known to be implied by strong tweakable pseudorandomness; and we allow using non-invertible random functions. They shed lights on SPNs, while we on Feistel. In all, the two works are complementary.

Concentrating on Feistel ciphers in the ideal model, previous works only considered KAF and KAFv. In the provable setting, KAF has been analyzed by Lampe and Seurin [9]. Yet, they assumed completely independent round-functions and independent round-keys, and they only considered the single-key security (on the other hand, they proved stronger security against 2^{t+1} queries for 6t rounds). These results were recently improved to birthday 2^{n/2} multi-user security at 4 rounds and beyond-birthday 2^{n/3} multi-user security at 6 rounds, with correlated round-keys [72]. The 4-round “minimal” KAF scheme given in [72] is very similar to ours: but we additionally considered round permutation case (giving rise to modes for permutations) and our results indicate that stronger assumptions (i.e., non-linearity) on the key schedule functions buy ⊕-RKA security. Another (aforementioned) work is the indifferentiability of KAFv analyzed by Guo and Lin [38], the security bound of which was however too weak.

Initiated by Lucks [73], a series of papers established efficient generic approach to obtain RKA secure blockciphers from classical PRPs [10], [74], which are complementary to our “concrete” results. Generic transformations however fall short of deepening the understanding of widely-deployed structures.

Finally, in the ideal model, conditions on the key-schedules that suffice for some level of security have been characterized for single-key security of Luby-Rackoff [75], KAC [56], and SPNs [76], for ⊕-RKA security of KAC [21], and for indifferentiability of Luby-Rackoff [11] and KACs, see [15] and the reference therein. These results are complementary to ours. Since we identified concrete conditions on the key-schedules, our work is closer to the series [56], [21], [76].

Possible Future Works include many directions. For example, investigating RKA security of KAFw with respect to larger RKD function sets (as discussed in page 5), posing beyond-birthday secure tweakable KAFw variants, generalizing the results to multi-line generalized Feistel networks (this might be more helpful for designing), or studying which type of key-schedules are sufficient for chosen-key security [77]. The most attractive direction seems to prove beyond-birthday security for KAFw models with $\geq 2n$-bit master-keys. This is much closer to reality. This requires modeling the combinatorial properties of “non-trivial” key-schedules for longer master-keys, which seems quite hard. For RKA security, some level of dependence has
to be assumed between the round-keys [12]. The dependence should be both close to reality and enough for proofs. So which type of dependence is satisfying? Following [78], a possible choice is to consider an alternating form of round-keys \((\gamma_1(k), \gamma_2(k'), \gamma_3(k), \gamma_4(k'), \ldots)\), where \(k\) and \(k'\) are the two halves of a 2\(n\)-bit master-key. But this model seems too artificial.

**Organization.** Section II presents notations, definitions, and tools. In Sections III and IV, we analyze the \(\oplus\)-RKA security of \(\text{KAF}w^{f,(w,\gamma)}\) with non-linear \((w, \gamma)\) and affine \((w, \gamma)\) respectively. Then, from the KAFw results we derive results on KAF and KAFv in Section V, and make discussion on theoretically best possible results in Section VI. The complementing attacks are given in Appendix B to help understanding our proofs.

## II. Preliminaries

**General Notation.** For integers \(1 \leq b \leq a\), we write \((a)_b = a(a-1) \cdots (a-b+1)\) and \((a)_0 = 1\) by convention. In all the following, we fix an integer \(n \geq 1\) and denote \(N = 2^n\). Further denote by \(\mathcal{F}(n)\) the set of all functions from \(\{0,1\}^n\) to \(\{0,1\}^n\), by \(\mathcal{P}(n)\) the set of all permutations on \(\{0,1\}^n\), and by \(\mathcal{B}(n, 2n)\) the set of all blockciphers with \(2n\)-bit block size and \(n\)-bit keys. For a finite set \(\mathcal{X}\), \(X \xleftarrow{\$} \mathcal{X}\) means that an element \(X\) is selected from \(\mathcal{X}\) uniformly at random. Finally, for \(X, Y \in \{0,1\}^n\), \(X\parallel Y\) or simply \(XY\) denotes their concatenation.

**Non-linear and Affine Functions.** A function \(\gamma : \{0,1\}^n \to \{0,1\}^n\), its non-linearity could be measured by
\[
\delta = \max_{\alpha, \beta \in \{0,1\}^n, \alpha \neq 0} \left| \left\{ k \in \{0,1\}^n : \gamma(k \oplus \alpha) \oplus \gamma(k) = \beta \right\} \right|. 
\]
(4)

On the other hand, viewing the \(n\)-bit input \(k\) as an \(n\)-dimensional vector over \(\mathbb{F}_2\), an \(n\)-bit affine function \(\gamma\) can be defined as
\[
\gamma(k) = M \cdot k + C
\]
for a fixed \(n \times n\) matrix over \(\mathbb{F}_2\) and a fixed \(n\)-dimensional vector \(C\) over \(\mathbb{F}_2\). By these, a \(t\)-round affine key-schedule \((w, \gamma) = ((w_0, w_1, w_2, w_3), (\gamma_1, \ldots, \gamma_t))\) (as mentioned in the Introduction) would be specified by \(t + 4\) fixed matrices
\[
M_0^{(w)}, M_1^{(w)}, M_2^{(w)}, M_3^{(w)}, M_1, \ldots, M_t,
\]
and \(t + 4\) fixed vectors/\(n\)-bit constants
\[
C_0^{(w)}, C_1^{(w)}, C_2^{(w)}, C_3^{(w)}, C_1, \ldots, C_t,
\]
as
\[
w_i(k) = M_i^{(w)} \cdot k \oplus C_i^{(w)}, \quad i = 1, 2, 3, 4,
\]
and
\[
\gamma_i(k) = M_j \cdot k \oplus C_j, \quad j = 1, \ldots, t.
\]

**KAFw Ciphers.** As mentioned in the Introduction, we focus on \(\text{KAF}w^{f,(w,\gamma)}\), the KAFw variants with the following two features:

(i) the same function \(f : \{0,1\}^n \to \{0,1\}^n\) is used at each round, and
(ii) the key-schedule is \((w, \gamma) = ((w_0, w_1, w_2, w_3), (\gamma_1, \ldots, \gamma_t)), i.e., the \(i\)-th whitening key \(w_k_i\) is derived from the \(n\)-bit master-key \(k\) via \(w_k_i = w_i(k)\), and the \(i\)-th round-key \(k_i\) is \(k_i = \gamma_i(k)\).

For such variants, the \(i\)-th round transformation is defined as
\[
\Psi^{f}_{\gamma_i(k)}(W_L||W_R) = W_R||W_L \oplus f(\gamma_i(k) \oplus W_R), \tag{7}
\]
where \(W_L\) and \(W_R\) are respectively the left and right \(n\)-bit halves of the input. Then the \(t\)-round \(\text{KAF}w^{f,(w,\gamma)}\) variant is defined as (cf. Fig. 1)
\[
\text{KAF}w^{f,(w,\gamma)}(k)(W) = (w_2(k)||w_3(k)) \oplus \Psi^{f}_{\gamma_1(k)} \circ \ldots \circ \Psi^{f}_{\gamma_t(k)}((w_0(k)||w_1(k)) \oplus W). 
\]

To make it more precise and introduce our notations (which belong to a classical notation system used for Luby-Rackoff [6]), we give formal descriptions for the 4- and 6-round \(\text{KAF}w^{f,(w,\gamma)}\) that will be studied later. For the 4-round \(\text{KAF}w^{f,(w,\gamma)}\), on the \(2n\)-bit input \(W\) which is parsed into \(L||R\), the computation proceeds in 4 steps:

(i) \(x_1 \leftarrow \gamma_1(k) \oplus w_1(k) \oplus R, y_1 \leftarrow f(x_1), X = w_0(k) \oplus L \oplus y_1;\)
(ii) \(x_2 \leftarrow \gamma_2(k) \oplus X, y_2 \leftarrow f(x_2), Y \leftarrow w_1(k) \oplus R \oplus y_2;\)
(iii) \(x_3 \leftarrow \gamma_3(k) \oplus Y, y_3 \leftarrow f(x_3), S \leftarrow X \oplus y_3 \oplus w_2(k);\)
(iv) \(x_4 \leftarrow \gamma_4(k) \oplus w_2(k) \oplus S, y_4 \leftarrow f(x_4), T \leftarrow Y \oplus y_4 \oplus w_3(k).\)

One could see Fig. 1 (left) for illustration. For the 6-round \(\text{KAF}w^{f,(w,\gamma)}\), on input \(W = L||R\), the computation proceeds in 6 steps (as in Fig. 1 (right)):

(i) \(x_1 \leftarrow \gamma_1(k) \oplus w_1(k) \oplus R, y_1 \leftarrow f(x_1), X = w_0(k) \oplus L \oplus y_1;\)
(ii) \(x_2 \leftarrow \gamma_2(k) \oplus X, y_2 \leftarrow f(x_2), Y \leftarrow w_1(k) \oplus R \oplus y_2;\)
(iii) \(x_3 \leftarrow \gamma_3(k) \oplus Y, y_3 \leftarrow f(x_3), Z \leftarrow X \oplus y_3;\)
(iv) \(x_4 \leftarrow \gamma_4(k) \oplus Z, y_4 \leftarrow f(x_4), A \leftarrow Y \oplus y_4;\)
(v) \(x_5 \leftarrow \gamma_5(k) \oplus A, y_5 \leftarrow f(x_5), S \leftarrow Z \oplus y_5 \oplus w_2(k);\)
(vi) \(x_6 \leftarrow \gamma_6(k) \oplus w_2(k) \oplus S, y_6 \leftarrow f(x_6), T \leftarrow A \oplus y_6 \oplus w_3(k).\)

As noted in [79], a KAFw cipher (even with independent round-functions) with an even number of rounds can be seen as a special case of a KAC. In detail, the \(i\)-th and \((i+1)\)-th rounds with round-functions \(f_i\) and \(f_{i+1}\) and round-keys \(k_i\) and \(k_{i+1}\) can be rewritten as
\[
\Psi^{f_{i+1}}_{k_{i+1}} \circ \Psi^{f_i}_{k_i}(W) = (k_{i+1}||k_i) \oplus \Psi^{f_{i+1}}_0 \circ \Psi^{f_i}_0((k_{i+1}||k_i) \oplus W),
\]
where \(\Psi^{f_i}_0 \circ \Psi^{f_i}_0\) is a keyless 2-round Feistel permutation. However, provable results on KAFw cannot be derived by black-box composition of existing results on KACs and keyless Feistel, since no provable results can be seen on \(\Psi^{f_{i+1}}_0 \circ \Psi^{f_i}_0\) (let alone the even weaker \(\Psi^{f_i}_0 \circ \Psi^{f_i}_0\)).

As a side remark, for a 2t-round KAFw cipher, if the 2t round-keys are identical \(k' = \gamma_1(k) = \ldots = \gamma_{2t}(k)\), then it can be seen it’s essentially a 1-round KAC, i.e., \((w_2(k) \oplus k'||w_3(k) \oplus k'||\ldots \pi((w_0(k) \oplus k'||w_1(k) \oplus k') \oplus W), where \(\pi = \Psi^{f_{2t}}_0 \circ \ldots \circ \Psi^{f_1}_0\) is a keyless permutation. This is
**⊕-RKA Security.** We follow Cogliati and Seurin [21] to formalize ⊕-RKA security in the ideal model. In detail, let $E$ be a $(n, 2n)$-blockcipher, and fix a key $k \in \{0, 1\}^n$. We define the ⊕-restricted related-key oracle $\text{RK}_E(k)$, which takes as input an “offset” $\Delta \in \{0, 1\}^n$ and a plaintext $LR \in \{0, 1\}^{2n}$, and returns $\text{RK}_E(k)(\Delta, LR) := E_{k⊕\Delta}(LR)$. The oracle can be queried backward, which we denote $\text{RK}_E(k)^{-1}(\Delta, ST) := E_{k⊕\Delta}(ST)$. Then, we consider a ⊕-restricted related-key adversary $D$ which has access to a function oracle $f$ as well as a related-key oracle, and must distinguish between the following two worlds:

- the “real” world, where it interacts with $(\text{RK}_E(k), f)$, and $k$ is randomly drawn;
- the “ideal” world where it interacts with $(\text{RK} | \text{IC}_E, f)$, where $\text{IC}$ is an ideal cipher independent from $f$, and $k$ is randomly drawn.

The distinguisher is adaptive, and can make two-sided queries to the related-key oracle. Note that in the ideal world, the oracle $\text{RK} | \text{IC}_E$ essentially implements an independent random permutation for each offset $\Delta \in \{0, 1\}^n$. Formally, when $f = F$ is a random function, $D$’s distinguishing advantage on $\text{KAF}_F(x, w, γ)$ is defined as

$$\text{Adv}_{\text{KAF}_F(x, w, γ)}(D) = |\Pr[D_{\text{RK} | \text{IC}_E}, f = 1] - \Pr[D_{\text{RK} | \text{KAF}_F(x, w, γ)}, f = 1]|,$$

where the former probability is taken over the random draw of $\text{IC} \leftarrow \text{BC}(n, 2n)$, $k \leftarrow \{0, 1\}^n$, $F \leftarrow \text{F}(n)$, and the latter probability is taken over $k \leftarrow \{0, 1\}^n$, $F \leftarrow \text{F}(n)$.

For $\text{KAF}_F(x, w, γ)$, $f = P$ is randomly picked from the set $\mathcal{P}(n)$, i.e. $P \leftarrow \mathcal{P}(n)$.

Furthermore, we consider computationally unbounded distinguishers, and we assume without loss of generality (wlog) that the distinguisher is deterministic and never makes redundant queries. For non-negative integers $q_f$, $q_e$, we define the insecurity of the $\text{KAF}_F(x, w, γ)$ cipher against ⊕-restricted related-key attacks as

$$\text{Adv}_{\text{KAF}_F(x, w, γ)}(q_f, q_e) = \max_{D} \text{Adv}_{\text{KAF}_F(x, w, γ)}(D),$$

where the maximum is taken over all distinguishers $D$ making exactly $q_f$ queries to the function oracle and in total $q_e$ queries to the related-key oracle (termed as $(q_f, q_e)$-distinguishers).

**The H-Coefficients Technique.** We employ the H-coefficient technique [80], and follow the paradigm of Chen and Steinberger [13]. To this end, we summarize the information gathered by the distinguisher in tuples $Q_E$ and $Q_f$. The tuple

$$Q_E = ((\Delta_1, L_1R_1, S_1T_1), \ldots, (\Delta_{q_e}, L_{q_e}R_{q_e}, S_{q_e}T_{q_e}))$$

summarizes the queries to the related-key oracle, and means that the $j$-th query was either a forward query $(\Delta_j, L_jR_j)$ with answer $S_jT_j$, or a backward query $(\Delta_j, S_jT_j)$ with answer $L_jR_j$. Throughout the remaining, we’ll use the bold letter $t$ as a simplified notation for a tuple $(\Delta, LR, ST)$ in $Q_E$.

Similarly to $Q_E$, the tuple

$$Q_f = ((x_1, y_1), \ldots, (x_{q_f}, y_{q_f}))$$

summarizes the queries to the round-function $f$, and

- when $f = P$ is an invertible permutation, it means the $j$-th query was either a forward query $x_j$ with answer $y_j$ or a backward query $y_j$ with answer $x_j$;
- when $f = F$ is a non-invertible function, it means $F$ was queried on $x_1, \ldots, x_{q_f}$ and answered $y_1, \ldots, y_{q_f}$ correspondingly.

To simplify the arguments (in particular, the definition of “bad transcripts”), we reveal to the distinguisher the key $k$ at the end of the interaction. This is wlog since $D$ is free to ignore this additional information to compute its output bit. Formally, we append $k$ to $(Q_E, Q_f)$ and obtain what we call the transcript $\tau = (Q_E, Q_f, k)$ of the attack. With respect to some fixed distinguisher $D$, a transcript $\tau$ is said attainable if there exists oracles $(\text{IC}, f)$ such that the interaction of $D$ with the ideal world $(\text{RK} | \text{IC}_E, f)$ yields $\tau$. We denote $\mathcal{T}$ the set of attainable transcripts. In all the following, we denote $T_{re}$, resp. $T_{id}$, the probability distribution of the transcript $\tau$ induced by the real world, resp. the ideal world (note that these two probability distributions depend on the distinguisher). By extension, we use the same notation to denote a random variable distributed according to each distribution. To further simplify notations, we let $\Pr_{T_{re}}(\tau) = \Pr[T_{re} = \tau]$ and $\Pr_{T_{id}}(\tau) = \Pr[T_{id} = \tau]$.

Given a tuple $Q_f$ of function queries and a function $f$, we say that $f$ extends $Q_f$, denoted $f \triangleright Q_f$, if $f(x) = y$ for all $(x, y) \in Q_f$. Similarly, given a related-key oracle transcript $Q_E$, a blockcipher $E$, and a key $k \in \{0, 1\}^n$, we say the related-key oracle $\text{RK}_E(k)$ extends $Q_E$, denoted $\text{RK}_E(k) \triangleright Q_E$, if $E_{k⊕\Delta}(LR) = ST$ for all $(\Delta, LR, ST) \in Q_E$. It is easy to see that for any attainable transcript $\tau = (Q_E, Q_f, k)$, the interaction of the distinguisher with oracles $(\text{RK} | \text{IC}_E, f)$ produces $\tau$ if and only if $\text{RK}_E(k) \triangleright Q_E$ and $f \triangleright Q_f$.

With all the above definitions, the main lemma of H-coefficients technique is as follows.

**Lemma 1 (Lemma 1 in [56])** Fix a distinguisher $D$. Let $\mathcal{T} = \mathcal{T}_{\text{good}} \cup \mathcal{T}_{\text{bad}}$ be a partition of the set of attainable transcripts $\mathcal{T}$. Assume that there exists $\varepsilon_1$ such that for any $\tau \in \mathcal{T}_{\text{good}}$, one has

$$\Pr_{T_{re}}(\tau) \geq 1 - \varepsilon_1,$$

and that there exists $\varepsilon_2$ such that $\Pr[T_{id} \in T_{\text{bad}}] \leq \varepsilon_2$. Then

$$\text{Adv}(D) \leq \varepsilon_1 + \varepsilon_2.$$

A proof could be found in [56].

Finally, it’s not hard to see

$$\Pr[T_{id}(\tau) = \Pr[f \triangleright Q_f] \cdot \text{Pr}[\text{RK} | \text{IC}_E \triangleright Q_E] \leq \Pr[f \triangleright Q_f] \cdot \left(\frac{1}{N^2 - q_e}\right)^{q_e}. $$
III. KAFw with Non-linear Key-Schedules

It is well-known that 3-round Feistel networks are not CCA secure even in the single-key setting. So we consider 4-round KAFw. First, in section III-A, we present conditions on the key-schedules that are sufficient for the \( \oplus \)-RKA security of the 4-round KAFw\(_{F,(w,\gamma)}\) (which also turn out sufficient for 4-round KAFw\(_{P,(w,\gamma)}\)). Then, since the security proof for KAFw\(_{F,(w,\gamma)}\) can be easily obtained by “dropping” some modules from the proof for KAFw\(_{P,(w,\gamma)}\), we start from KAFw\(_{P,(w,\gamma)}\), analyze it in section III-B, and then discuss how to adapt the proof for the 4-round KAFw\(_{F,(w,\gamma)}\) variant in section III-C.

A. Conditions on the Key-Schedules

4-round key-schedules defined as follows would suffice.

**Definition 1 (Good Key-Schedule for 4-Round KAFw\(_{F,(w,\gamma)}\))**

Consider a 4-round key-schedule \((w,\gamma)\), where \(w = (w_0,w_1,w_2,w_3)\) for \(w_i : \{0,1\}^n \to \{0,1\}^n\), and \(\gamma = (\gamma_1,\gamma_2,\gamma_3,\gamma_4)\) for \(\gamma_i : \{0,1\}^n \to \{0,1\}^n\). Then \((w,\gamma)\) is good, if \(\phi_1(k) = w_1(k) \oplus \gamma_1(k)\) and \(\phi_4(k) = w_2(k) \oplus \gamma_4(k)\) posses four properties as follows:

(i) \(\delta_1\)-uniformness: for \(i = 1,4\), for any image \(k_i \in \{0,1\}^n\), the number of master-keys \(k\) such that \(\phi_i(k) = k_i\) is at most \(\delta_1 = \text{poly}(n)\);

(ii) \(\delta_2\)-non-linearity: the non-linearity measure (Eq. (4)) of \(\phi_1\) and \(\phi_4\) is \(\delta_2 = \text{poly}(n)\);

(iii) \(\delta_3\)-“mutual-uniformness”: for any \(k^*\), the number of master-keys \(k\) such that \(\phi_1(k) \oplus \phi_4(k) = k^*\) is at most \(\delta_3 = \text{poly}(n)\);

(iv) \(\delta_4\)-“mutual-non-linearity”: for any pair \((\nabla,\Delta) \in (\{0,1\}^n \setminus \{0\}) \times \{0,1\}^n\), the number of master-keys \(k\) such that \(\phi_1(k) \oplus \phi_4(k \oplus \nabla) = \Delta\) is at most \(\delta_4 = \text{poly}(n)\).

Note that \(\phi_2(k)\) and \(\phi_3(k)\) effectively mask (and protect) the inputs to the 1st and last round-functions respectively. This protection would be ineffective if the \(\delta_1\)-uniformness property is seriously compromised. An extreme example is \(\phi_1(k) = 0\), for which an adversary could freely compute the 2nd-round intermediate value as \(R[|L\oplus F(R)]\). \(\delta_2\)-non-linearity is intended to reduce the probability of 1-round related-key differentials with non-zero master-key differences; see the argument for condition (B-2) in page 9.

Finally, \(\delta_3\)-“mutual-uniformness” and \(\delta_4\)-“mutual-linearity” are intended to prevent the derived round-keys from “palindrome-like” properties in the RKA setting, which have been found harmful [75]. For example, consider a key-schedule \((w,\gamma)\) such that \(w(k) = (0,0,0,0)\) and one could easily derive a value \(\Delta\) for which it holds \(\gamma(k) = (k^*,0,0,0)\) and \(\gamma(k \oplus \Delta) = (k',0,0,k''')\) for any master-key \(k\). In other words, when \(\Delta \neq 0\), “mutual-non-linearity” is lacked in \((w,\gamma)\), and “mutual-uniformness” is lacked otherwise. Then it can be distinguished by querying \(\text{RK}[E_k(0,LR) \to ST, \text{RK}[E_k(\Delta,TS) \to R'L']\), and checking if \(R = R'\).

Actually, it might be possible to prove security without the 3rd and 4th properties. But this requires \(\gamma_2\) and \(\gamma_3\) to fulfill more involved conditions. Therefore, our Definition 1, with no requirement on \(\gamma_2\) and \(\gamma_3\) at the expense of slightly more requirements on \(\varphi_1\) and \(\varphi_4\), captures a “minimal” group of conditions to some extent.

As a side remark, Barbosa and Fastshim conjectured that for a Luby-Rackoff cipher, if the derived round-keys are palindrome-free in the RKA setting, then the cipher is RKA-secure [10]. This means (informally) it’s unlikely to find two related-key derivation functions for which the derived round-key vectors \((k_1,\ldots,k_t)\) and \((k'_1,\ldots,k'_t)\) are such that \(k_i = k'_{i+i-1}\) for \(i = 1,\ldots,t\). Clearly, the two round-key vectors \((k'',0,0,k')\) and \((k',0,0,k'')\) appeared in our instructive example are not palindrome. Yet, their occurrence still ruins the RKA security of KAFw. This comparison again highlights the gap between Luby-Rackoff and KAFw models (and likely, the reality).

B. Security for 4 Rounds with Good Key-Schedules and \(f=P\)

Instantiated with a good key-schedule, 4-round KAFw\(_{P,(w,\gamma)}\) is secure against \(\oplus\)-RKAs.

**Theorem 1** When \(q_f + 2q_e \leq N/2\), for the 4-round, random permutation-based KAFw\(_{P,(w,\gamma)}\) cipher with a good key-schedule \((w,\gamma)\) as specified in Definition 1, it holds

\[
\text{Adv}_{\oplus-\text{RKA}}(\text{KAFw}_{P,(w,\gamma)})(q_f,q_e) \leq \frac{2(\delta_1 + 4)q_f + (\delta_2 + \delta_3 + 27)q_e^2 + 4q_e}{N}.
\]

**Proof.** Following Lemma 1, with respect to a fixed \((q_f,q_e)\)-distinguisher \(D\), we first identify bad transcripts and bound their probability of occurring in the ideal world, then lower bound the ratio \(\text{Pr}_{\text{rel}}(\tau)/\text{Pr}_{\text{rel}}(\tau)\) for good \(\tau\). In detail, let \(\tau = (Q_E, Q_P, k)\) be an attainable transcript, with \(|Q_E| = q_e\) and \(|Q_P| = q_f\). For convenience, for the involved transcript \(Q_P = \{(x_1,y_1), (x_2,y_2), \ldots, (x_{q_f}, y_{q_f})\}\), we define two sets

\[
X(\tau) \overset{\text{def}}{=} \{x_1, x_2, \ldots, x_{q_f}\}, \quad \text{and} \quad Y(\tau) \overset{\text{def}}{=} \{y_1, y_2, \ldots, y_{q_f}\}.
\]

We introduce a series of notations that will be used in the subsequent analysis: for any tuple \(t = (\Delta,LR,ST)\) in \(Q_E\) and any function \(f = P\) or \(F\); the former is the focus of this subsection), define 10 functions

\[
x_1(t) = \phi_1(k \oplus \Delta) \oplus R, \\
y_1(t,f) = f(x_1(t)), \\
X(t,f) = L \oplus w_0(k \oplus \Delta) \oplus y_1(t,f), \\
x_2(t,f) = \gamma_2(k \oplus \Delta) \oplus X(t,f), \\
y_2(t,f) = R \oplus w_1(k \oplus \Delta) \oplus Y(t,f), \\
Y(t,f) = T \oplus w_3(k \oplus \Delta) \oplus y_4(t,f), \\
x_3(t,f) = \gamma_3(k \oplus \Delta) \oplus Y(t,f), \\
y_3(t,f) = S \oplus w_2(k \oplus \Delta) \oplus X(t,f), \\
x_4(t) = \phi_4(k \oplus \Delta) \oplus S, \\
y_4(t,f) = f(x_4(t)).
\]

The suffix \(f\) is intended to emphasize the functions depends on the function values of \(f\). Note that these values are derived in an “LR \rightarrow X, Y ← ST” manner, rather than the “LR \rightarrow X \rightarrow Y \rightarrow ST” manner. Moreover, the two function
values \( x_1(t) \) and \( x_4(t) \) only depend on \( \tau \).

**Bad Transcripts** are now defined as follows.

**Definition 2 (Bad Transcripts for 4-Round KAFW)\(^{P,(w,\gamma)}\)** 
An attainable transcript \( \tau = (Q_E, Q_P, k) \) is bad, if at least one of the following conditions is fulfilled:

- **(B-1)** \( \exists t \in Q_E : x_1(t) \in X(\tau) \) or \( x_4(t) \in X(\tau) \);
- **(B-2)** \( \exists t = (\Delta, LR, ST) \) and \( t' = (\Delta', LR', ST') \) in \( Q_E \) such that \( \Delta \neq \Delta' \), whereas \( x_1(t) = x_1(t') \) or \( x_4(t) = x_4(t') \);
- **(B-3)** \( \exists t, t' \in Q_E : x_1(t) = x_4(t') \) (it could be \( t = t' \));
- **(B-4)** there exists two distinct queries \((\Delta, LR, ST)\) and \((\Delta', LR', ST')\) in \( Q_E \) such that \( \Delta = \Delta' \), whereas
  - \( L \odot L' = S \odot S' \), or \( R \odot R' = T \odot T' \);
- **(B-5)** there exists \((\Delta, LR, ST) \in Q_E\) such that
  - \( L \odot w_0(k + \Delta) = S \odot w_2(k + \Delta) \), or \( R \odot w_1(k + \Delta) = T \odot w_3(k + \Delta) \).

Otherwise we say \( \tau \) is good. Denote by \( \mathcal{T}_{bad} \) the set of bad transcripts.

We analyze the conditions in turn, with (B-1) the first. For any \( t = (\Delta, LR, ST) \) in \( Q_E \) and any \( x \), by the \( \delta_1 \)-uniformity of \( \varphi_1 \) (cf. Definition 1), the number of \( \kappa_\Delta \) such that \( \varphi_1(k_\Delta) = R \odot x \) is at most \( \delta_1 \). Therefore,

\[
\Pr[x_1(t) \in X(\tau)] = \Pr[\exists x \in X(\tau) : \varphi_1(k_\Delta) = R \odot x] \leq \delta_1 q_1 \frac{q_1}{N}.
\]

Similarly, \( \Pr[x_4(t) \in X(\tau)] \leq \delta_1 q_1 \frac{q_1}{N} \). Since we have \( q_e \) choices for \( t \), it holds

\[
\Pr[(B-1)] \leq \frac{2 \delta_1 q_1 q_1}{N}.
\]

For (B-2), we utilize the \( \delta_2 \)-non-linearity: since the number of \( \kappa_\Delta \) such that \( \varphi_1(k_\Delta) \odot \varphi_1(k_\Delta \odot (\Delta + \Delta')) \) is at most \( \delta_2 \) for \( i = 1, 4 \), for each pair \((t, t')\) with \( t = (\Delta, LR, ST) \) and \( t' = (\Delta', LR', ST', T') \),

\[
\Pr[x_1(t) = x_1(t') \text{ or } x_4(t) = x_4(t')] = \Pr[\varphi_1(k_\Delta) \odot R = \varphi_1(k_\Delta \odot (\Delta + \Delta')) \odot R']
\]

or \( \varphi_4(k_\Delta) \odot S = \varphi_4(k_\Delta \odot (\Delta + \Delta')) \odot S' \) \( \leq \frac{2 \delta_2}{N} \).

As we have at most \( \left( \frac{q_e}{2} \right)^2 \) choices for \((t, t')\) it holds

\[
\Pr[(B-2)] \leq \frac{2 \delta_2 q_1^2}{N}.
\]

For (B-3), we utilize the so-called \( \delta_3 \)-“mutual-uniformity” and \( \delta_4 \)-“mutual-non-linearity” defined on \( \varphi_1 \) and \( \varphi_4 \). In detail, for each pair \((t, t')\), consider two cases:

- **Case 1:** \( \Delta = \Delta' \). Then \( x_1(t) = x_4(t') \) implies \( \varphi_1(k_\Delta) \odot R = \varphi_4(k_\Delta \odot \Delta + \Delta') \odot S' \) and further
  \[
  \varphi_1(k_\Delta \odot \Delta + \Delta') = R \odot S' \quad (8)
  \]

  By Definition 1, the number of \( k_\Delta \) satisfying Eq. (8) is at most \( \delta_3 \). Thus the bound \( \frac{1}{N} \) for this case;

- **Case 2:** \( \Delta \neq \Delta' \). Then \( x_1(t) = x_4(t') \) implies \( \varphi_1(k_\Delta) \odot R = \varphi_4(k_\Delta \odot \Delta + \Delta') \odot S' \) and further
  \[
  \varphi_1(k_\Delta) \odot \varphi_4(k_\Delta \odot \Delta + \Delta') = R \odot S' \quad (9)
  \]

  for \( k_\Delta = k_\Delta \). By Definition 1, the number of \( k_\Delta \) satisfying Eq. (9) is at most \( \delta_4 \). Thus the bound \( \frac{1}{N} \).

In all, for each \((t, t')\) the probability is \( \leq (\delta_3 + \delta_4)/N \), thus \( \Pr[(B-3)] \leq \frac{(\delta_3 + \delta_4) q_e}{2} \).

For (B-4), consider a pair \((t, t')\). Wlog assume that \( t' \) comes after \( t \). If \( t' \) was forward \( \mathcal{R}_{KAFW}^P(\Delta, LR', ST') \), then the obtained \( S' \) can be seen as uniform in a set of size at least \( N - q_e \), and thus

\[
\Pr[S' = L \odot L' + S] \leq \frac{1}{N - q_e}.
\]

Similarly, \( \Pr[S' = R \odot R' + T] \leq \frac{1}{N - q_e} \). If \( t' \) was backward \( \mathcal{R}_{KAFW}^P(\Delta, LR', ST') \), then similarly

\[
\Pr[S' = L \odot S' + S] \leq \frac{1}{N - q_e}, \quad \Pr[S' = R \odot T' + T] \leq \frac{1}{N - q_e}.
\]

Therefore, for each of the \( \left( \frac{q_e}{2} \right)^2 \) pairs \((t, t')\), (B-4) is fulfilled with probability at most \( 2/\left( N - q_e \right) \). Thus \( \Pr[(B-4)] \leq \frac{2 q_e}{N - q_e} \).

Finally consider (B-5). Fix a query \( t = (\Delta, LR, ST) \). For \( k \in \{0, 1\}^n \), denote by \( R_k \) the set of possible values of \( w_0(k + \Delta) \odot w_2(k + \Delta) \), and by \( R_3 \) the set of values of \( w_1(k + \Delta) \odot w_3(k + \Delta) \). If \( t \) was forward, then the obtained \( S \) and \( T \) are uniform in \( N - q_e \) values, and \( \Pr[L \odot S = v] \leq \frac{1}{N - q_e} \) for any fixed value \( v \in R_3 \). Then \( \Pr_{\mathcal{T}_{bad}} \leq \frac{2 q_e}{N - q_e} \).

Summing over the above yields

\[
\Pr[T_{id} \in \mathcal{T}_{bad}] \leq \frac{2 \delta_1 q_1 q_1 + (\delta_2 + \delta_3 + \delta_4) q_e^2 + q_1^2 + 2 q_e}{N}.
\]

**Ratio** \( \Pr_{\mathcal{T}_{id}}(\tau)/\Pr_{\mathcal{T}_{id}}(\tau) \) for Good \( \tau \). Fix a good transcript \( \tau \). Following the “predicate” approach in [76], we define a “bad” predicate \( B(P) \) on \( P \) such that:

- \( \Pr_{P}[B(P) \mid P \vdash Q_P] \) can be bounded to negligible, and for every \( t = (\Delta, LR, ST) \),

\[
\Pr_{P}[-B(P) \mid P \vdash Q_P \land \neg B(P)] + \Pr_{P}[-B(P) \mid P \vdash Q_P \land B(P)] \geq \Pr_{P}[P(x_1(t, P)) = y_1(t, P) \land P(x_3(t, P)) = y_3(t, P) \land P \vdash Q_P \land \neg B(P)] \geq \frac{1}{N}. \quad (12)
\]

- For every \( t = (\Delta, LR, ST) \),

\[
\Pr_{P}[-B(P) \mid P \vdash Q_P \land \neg B(P)] + \Pr_{P}[-B(P) \mid P \vdash Q_P \land B(P)] \geq \Pr_{P}[P(x_1(t, P)) = y_1(t, P) \land P(x_3(t, P)) = y_3(t, P) \land P \vdash Q_P \land \neg B(P)] \geq \frac{1}{N}. \quad (13)
\]
By these, we have
\[
\Pr_{P}(\tau) = \Pr_{P}[\text{RK}|\text{AKFw} \wedge (u, \gamma) \vdash Q_E \land F \vdash Q_F] \\
\geq \Pr_{P}[\text{RK}|\text{AKFw} \wedge (u, \gamma) \vdash Q_E \land F \vdash Q_F \land \lnot B(F)] \\
\geq p \left(1 - \Pr_{P}[B(F) | F \vdash Q_F]\right) \cdot \Pr_{P}[F \vdash Q_F] \\
\geq \frac{\Pr_{P}[F \vdash Q_F]}{N^{2q_e}} \left(1 - \Pr_{P}[B(F) | F \vdash Q_F]\right). \tag{14}
\]

The correctness of Eq. (11) can be seen from the definitions of $x_2(t, P)$, etc in page 8. We proceed to define $B(F)$ in the next paragraph, from which Eq. (12) will become clear. Last, to show Eq. (13), we need to show
\[
\{P(x_2(t, P)) = y_2(t, P), P(x_3(t, P)) = y_3(t, P) \mid t \in Q_E\} \tag{15}
\]
are $2q_e$ distinct equations; this will be shown in the next-next paragraph.

**The Bad Predicate $B(F)$**. For any $P \vdash Q_P$, the predicate $B(F)$ holds, if any of the following conditions is fulfilled:

- (C-1) $\exists t', t'' \in Q_E : x_1(t) \neq x_1(t'')$, yet either $x_2(t, P) = x_2(t', P)$ or $y_3(t, P) = y_3(t', P)$.
- (C-2) $\exists t' \in Q_E$ (could be $t = t'$):
  - $x_2(t, P) \in X(\tau)$ or $y_3(t, P) \in Y(\tau)$, or
  - $x_2(t, P) = x_2(t', P)$ or $y_3(t, P) = y_3(t', P)$, or
  - $y_3(t, P) = y_3(t', P)$.
- (C-3) $\exists t' \in Q_E : x_1(t) \neq x_1(t')$, yet either $x_3(t', P) = x_3(t, P)$ or $y_2(t', P) = y_2(t, P)$.
- (C-4) $\exists t' \in Q_E$ (could be $t = t'$):
  - $x_3(t, P) \in X(\tau)$ or $y_2(t, P) \in Y(\tau)$, or
  - $x_3(t, P) \in \{x_2(t', P), y_3(t', P), x_3(t', P)\}$, or
  - $y_2(t, P) = \{y_1(t', P), y_2(t', P), y_4(t', P)\}.

Let $t = (\Delta, L.R, ST)$. Consider (C-1) first. Since $x_1(t) \neq x_1(t')$, the two function values $P(x_1(t))$ and $P(x_1(t'))$ are independent. Furthermore, by $\sim$(B-1), it holds $x_1(t) \notin X(\tau)$, so $P(x_1(t))$ remains uniform conditioned on $P \vdash Q_P$. Conditioned on the $\leq 2q_e$ function values $\{P(x_1(t')) \mid t' \in Q_E, i = 1, 4, x_1(t') \neq x_1(t)\}$ and further $P \vdash Q_P$, $P(x_1(t))$ and thereby $X(t, P) = L \oplus w_0(k + \Delta) \oplus P(x_1(t))$ is uniform in a set of size at least $N - q_f - 2q_e$. Therefore, for each of the $\binom{q_e}{2} \leq \frac{q_e^2}{2}$ pairs $(t', t'')$, it holds
\[
\Pr[x_2(t, P) = x_2(t', P)] = \Pr[P(x_1(t)) \in X(t, P)] \leq \frac{q_e}{N - q_f - 2q_e}, \quad \text{and}
\]
\[
\Pr[y_3(t, P) = y_3(t', P)] = \Pr[X(t', P) \oplus w_2(k + \Delta) \oplus S = y_3(t', P)] \leq \frac{1}{N - q_f - 2q_e}.
\]

Therefore, $\Pr[(C-1)] \leq \frac{q_e^2}{N - q_f - 2q_e}$. For (C-3) it’s symmetrical, yielding $\Pr[(C-3)] \leq \frac{q_e^2}{N - q_f - 2q_e} \leq \frac{q_e^2}{N - q_f - 2q_e}$.

We next consider (C-2). As argued, for any $t, X(t, P)$ is uniform in $\geq N - q_f - 2q_e$ possibilities. On the other hand, all the values in $X(\tau)$ are fixed by $\tau$ and thus independent from the function values of $P$. Therefore,
\[
\Pr[x_2(t, P) \in X(\tau)] = \Pr[\gamma_2(k + \Delta) \oplus X(t, P) \in X(\tau)] \leq \frac{q_f}{N - q_f - 2q_e}. \tag{16}
\]

Similarly,
\[
\Pr[\exists t' : x_2(t, P) = x_2(t') \text{ or } x_2(t, P) = x_2(t')] \leq \frac{2q_e}{N - q_f - 2q_e}, \tag{17}
\]
\[
\Pr[y_3(t, P) \in Y(\tau)] \leq \frac{q_f}{N - q_f - 2q_e}. \tag{18}
\]

Now consider $\Pr[\exists t' : y_3(t, P) = y_3(t')]$. If this event happens, then
\[
L \oplus w_0(k + \Delta) \oplus P(x_1(t)) \oplus w_2(k + \Delta) \oplus S = P(x_1(t')).
\]

By $\sim$(B-3) we have $x_1(t) \neq x_1(t')$, so $P(x_2(t'))$ is independent from the left hand side. Therefore,
\[
\Pr[\exists t' : y_3(t, P) = y_3(t')] \leq \frac{q_e}{N - q_f - 2q_e}. \tag{19}
\]

Finally consider $\Pr[\exists t' : y_3(t, P) = y_3(t')]$. If this event happens, then for $t$ there exists $t' \in Q_E$ such that
\[
L \oplus w_0(k + \Delta) \oplus P(x_1(t)) \oplus w_2(k + \Delta) \oplus S = P(x_1(t')).
\]

We distinguish two cases:

(i) Case 1: $x_1(t) \neq x_1(t')$. Then $P(x_1(t))$ and $P(x_1(t'))$ are independent, and $\Pr[\text{Eq. (20)}] \leq \frac{1}{N - q_f - 2q_e}$.

(ii) Case 2: $x_1(t) = x_1(t')$. Then for this tuple we have $L \oplus w_0(k + \Delta) = w_2(k + \Delta) \oplus S$, which contradicts $\sim$(B-5) (Definition 2).

As we have $q_e$ choices for $t'$ we obtain
\[
\Pr[\exists t' : y_3(t, P) = y_3(t')] \leq \frac{q_e}{N - q_f - 2q_e}. \tag{21}
\]

Summing over (16), (17), (18), (19), and (21), and taking union bound on the $q_e$ choices of $t$, we obtain
\[
\Pr[(C-2)] \leq \frac{q_e(q_f + 2q_e + q_f + q_e + q_e)}{N - q_f - 2q_e} \leq 2q_e(q_f + 2q_e). \tag{22}
\]

The analysis for (C-4) is similar by symmetry; for each $t = (\Delta, L.R, ST) \in Q_E$, $P(x_2(t))$ and further $Y(t, F) = T \oplus w_3(k + \Delta) \oplus P(x_2(t), x_3(t), P)$, and $y_2(t, P)$ are uniform. By this, for $t$,
\[
\Pr[x_3(t, P) \in X(\tau) \text{ or } y_2(t, P) \in Y(\tau)] \leq \frac{2q_f}{N - q_f - 2q_e}. \tag{23}
\]
\[
\Pr[\exists t' : x_3(t, P) = x_3(t')] \text{ or } x_3(t, P) = x_3(t') \leq \frac{2q_e}{N - q_f - 2q_e}. \tag{24}
\]

Now consider $\Pr[\exists t' : x_3(t, P) = x_3(t')]$. If this event happens, then for $t$ there exists $t' = (\Delta', L'R', ST')$ such that
\[
\gamma_3(k + \Delta) \oplus T \oplus w_3(k + \Delta) \oplus P(x_3(t)) \oplus P(x_3(t')).
\]

By $\sim$(B-3) we have $x_3(t) \neq x_3(t')$, so the right hand side of (25) is independent from $P(x_3(t))$. Thus we have $\Pr[\text{Eq. (25)}] \leq \frac{1}{N - q_f - 2q_e}$, and further
\[
\Pr[\exists t' : x_3(t, P) = x_3(t')] \leq \frac{q_e}{N - q_f - 2q_e}. \tag{26}
\]
By similar arguments, it can be shown
\[
\Pr[\exists t': y_2(t, P) = y_1(t') \text{ or } y_2(t, P) = y_3(t', P)] \leq \frac{2q_e}{N - q_f - 2q_e}.
\] (27)

Finally consider \(\Pr[\exists t' : y_2(t, P) = y_4(t', P)]\). If it happens then for \(t\) there exists \(t' = (\Delta', L'R', S'T')\) such that
\[R \oplus w_1(k \oplus \Delta) \oplus T \oplus w_3(k \oplus \Delta) \oplus P(x_4(t)) = P(x_4(t')).\] (28)
If \(x_4(t) \neq x_4(t')\) then the right hand side of (28) is independent from \(P(x_4(t))\) and thus \(\Pr[\text{Eq. (28)}] \leq \frac{1}{N - q_f - 2q_e}\); otherwise i.e. \(x_4(t) = x_4(t')\), then it implies \(R \oplus w_1(k \oplus \Delta) = T \oplus w_3(k \oplus \Delta)\), contradicting \(\neg(B-5)\). So
\[
\Pr[\exists t' : y_2(t, P) = y_4(t', P)] \leq \frac{q_e}{N - q_f - 2q_e}.
\] (29)

Summing over (23), (24), (26), (27), and (29), and taking union over \(q_e\) yield
\[
\Pr([C-4]) \leq \frac{q_e(2q_f + 2q_e + q_e + 2q_e + q_e)}{N - q_f - 2q_e} \\
\leq \frac{2q_e(q_f + 3q_e)}{N - q_f - 2q_e}.
\]

Finally, summing over the four conditions yields
\[
\Pr[P = \mathcal{B}(n) : \mathcal{B}(P) \mid P \Rightarrow Q_P] \leq \frac{4q_eq_f + 12q_e^2}{N - q_f - 2q_e}. \tag{30}
\]

Eq. (30) is now clear. In detail, conditioned on \(P \Rightarrow Q_P\) and \(\neg\mathcal{B}(P)\), the following function values must satisfy some conditions, and cannot be seen as uniform:

- \(\{P(x), P^{-1}(y) \mid (x, y) \in Q_P\}\), and
- \(\{P(x_1(t), P^{-1}(y_1(t, P)), P(x_4(t)), P^{-1}(y_4(t, P)) \mid t \in Q_E\}\).

Yet, from the definition \(\mathcal{B}(P)\) it can be seen for any \(t\) it holds
- \(x_2(t, P), x_3(t, P) \notin (\mathcal{A}(\tau) \cup \{x_1(t), x_2(t) \mid t \in Q_E\})\),
- \(y_2(t, P), y_3(t, P) \notin (\mathcal{A}(\tau) \cup \{y_1(t, P), y_4(t, P) \mid t \in Q_E\})\).

This shows Eq. (12).

2q_e. Equations for Good P. As discussed, the duty of this paragraph is to argue the set in (15) contains \(2q_e\)-distinct equations. For this we show \(\{x_2(t, P), x_3(t, P) \mid t \in Q_E\}\) and \(\{y_2(t, P), y_3(t, P) \mid t \in Q_E\}\) are \(2q_e\) each. First, by \(\neg\mathcal{B}(P)\) (more precisely, \(\neg(C-4)\)), for any pair \((t, t')\), it holds \(x_2(t, P) \neq x_3(t', P)\) and \(y_2(t, P) \neq y_3(t', P)\) by \(\neg(C-4)\). It remains to show
- \(x_2(t, P) \neq x_2(t', P), y_2(t, P) \neq y_2(t', P)\), and
- \(x_3(t, P) \neq x_3(t', P), y_3(t, P) \neq y_3(t', P)\).

Consider \((x_2(t, P), x_2(t', P))\) and \((y_3(t, P), y_3(t', P))\) first: their proof flows are similar. In detail, let \(t = (\Delta, LR, ST)\) and \(t' = (\Delta', L'R', S'T')\), then we exclude possibility of \(x_2(t, P) = x_2(t', P)\) or \(y_3(t, P) = y_3(t', P)\) for each case as follows:

(i) Case 1: \(\Delta \neq \Delta'\). Then \(x_1(t, P) \neq x_1(t', P)\) by \(\neg(B-2)\) (see Definition 2), and further \(x_2(t, P) \neq x_2(t', P)\) and \(y_3(t, P) \neq y_3(t', P)\) by \(\neg(C-1)\);

(ii) Case 2: \(\Delta = \Delta'\), yet \(R \neq R'\). Then still \(x_1(t, P) \neq x_1(t', P)\), thus further \(x_2(t, P) \neq x_2(t', P)\) and \(y_3(t, P) \neq y_3(t', P)\);

(iii) Case 3: \(\Delta = \Delta'\) and \(R = R'\). Then it has to be \(L \neq L'\) since \(t' \neq t\). Now:

- On one hand, \(L \neq L'\) immediately implies \(x_2(t, P) = L \oplus w_0(k \oplus \Delta) \oplus y_1(t, P) \oplus y_2(t, P)\) and \(y_3(t, P) = L \oplus w_0(k \oplus \Delta) \oplus y_1(t, P) \oplus y_2(t, P)\) are distinct;

- On the other hand, \(\Delta = \Delta'\) and \(R = R'\) imply \(X(t, P) \oplus X(t', P) = L \oplus L'\). By this, \(y_3(t, P) = X(t, P) \oplus w_3(k \oplus \Delta) \oplus y_1(t, P) \oplus y_2(t, P) \oplus w_3(k \oplus \Delta) \oplus S = y_3(t, P) = X(t', P) \oplus w_3(k \oplus \Delta) \oplus S\) would imply \(L \oplus L' = S \oplus S\), contradicting \(\neg(B-4)\).

By the above, it does hold \(x_2(t, P) \neq x_2(t', P)\) and \(y_3(t, P) \neq y_3(t', P)\) for any \(t' \neq t\). A symmetrical argument could establish \(x_3(t, P) \neq x_3(t', P)\) and \(y_2(t, P) \neq y_2(t', P)\) for any \(t' \neq t\).

The Final Counting. By the above discussion and (14) and (30), when \(q_f + 2q_e \leq N/2\), for any \(\tau \in T_{good}\) we have

\[
\frac{Pr[v \neq \mathcal{B}(P)]}{Pr[\neg \mathcal{B}(P)]} = \frac{Pr[P \Rightarrow Q_P]}{Pr[P \Rightarrow Q_P]} \frac{\left(1 - Pr[\mathcal{B}(P)]\right)}{\left(1 - Pr[\neg \mathcal{B}(P)]\right)} \frac{Pr[P \Rightarrow Q_P]}{Pr[P \Rightarrow Q_P]} \leq \frac{4q_eq_f + 12q_e^2}{N - q_f - 2q_e} \leq \frac{1}{N^2 - q_e^2} \leq \frac{1}{N^2 - q_e^2}.
\]

Gathering this, (10), and Lemma 1 yields Theorem 1 (when \(q_f + 2q_e \leq N/2\), we also have \(q_e^2 + 2q_e \leq N^2 - 2q_e^2\).)

C. When f=F is a Random Function

With a good key-schedule specified in Definition 1, the \(+\)RKA security claim still holds when we use a random function \(F\) for \(f\). For the proof, we make some moderate modifications to the previous proof for KAFW\(_F^{\ell,(\omega,\gamma)}\). First, (of course) the helper functions \(y_1(t, P), X(t, P), \ldots\) here are defined on \(F\) instead of \(P\), i.e. \(y_1(t, F), X(t, F), \ldots\)

Then, note that since \(F\) is a random function, for the to-be-derived \(2q_e\) equalities
\(\{F(x_2(t, F)) = y_2(t, F), F(x_3(t, F)) = y_3(t, F) \mid t \in Q_E\}\), collisions within the image set \(\{y_2(t, F), y_3(t, F) \mid t \in Q_E\}\) would not be troublesome. Therefore, the main task is to drop definitions and arguments concerning these image values.

In detail, we recall that in the definition of bad transcripts (Definition 2),

- the condition \(B-4\) is only used for proving \(\{y_2(t, F) \mid t \in Q_E\} = \{y_3(t, F) \mid t \in Q_E\}\) in the subsequent analysis, cf. the Case 3 in page 11, and
• (B-5) is only used for bounding \( \Pr[\exists t, t' : y_2(t, F) = y_4(t', F)] \) and \( \Pr[\exists t, t' : y_3(t, F) = y_1(t', F)] \), cf. Eq. (20) and (28) in page 10.

So both (B-4) and (B-5) could be dropped. On the other hand, (B-1), (B-2), and (B-3) and their probabilities remain unchanged. Subtracting the corresponding terms from (10) yields the following bound for 4-round KAFw\(^F,(w,γ)\)

\[
\Pr[T_{id} \in T_{bad}] \leq \frac{2δ_1 q_2 q_f + (δ_2 + δ_3 + δ_4)q_e^2}{N},
\]

(31)

We then modify the definition of \( B(F) \) into \( B(F) \). We remark that for any value \( x \) such that \( F(x) \) remains unknown, the function value \( F(x) \) is uniform in \( \{0,1\}^n \), which slightly deviates from the permutation case. Then, following the idea as before, we make the following modifications:

(i) Dropping \( y_3(t, F) = y_3(t', F) \) in (C-1). This decreases \( \Pr[[C-1]] \) to \( \frac{q_e^2}{N^2} \) (with the above remark in mind);

(ii) Dropping the condition(s) \( \exists t', t' : y_3(t, F) \in Y' \lor y_3(t, F) = y_1(t', F) \lor y_3(t, F) = y_3(t', F) \) in (C-2).

This decreases \( \Pr[[C-2]] \) to \( \frac{q_e^2}{Nq_e^2} \);

(iii) Dropping \( y_3(t, F) = y_2(t', F) \) in (C-3). This decreases \( \Pr[[C-3]] \) to \( \frac{q_e^2}{N^2} \);

(iv) Dropping the condition(s) \( \exists t, t' : y_2(t, F) \in Y' \lor y_2(t, F) = y_1(t', F) \lor y_2(t, F) = y_3(t', F) \lor y_2(t, F) = y_4(t', F) \) in (C-4). This decreases \( \Pr[[C-2]] \) to \( \frac{q_e^2}{N^2} \).

In total we have

\[
\Pr[F \not\subseteq F(n) : B(F) \lor \{F \} = \Pr[F \not\subseteq F(n) : B(F)] \leq \frac{2q_e q_f + 6q_e^2}{N}.
\]

Finally,

\[
\Pr[F \not\subseteq F(n) : B(F) \lor \{F \} = \Pr[F \not\subseteq F(n) : B(F)] \leq \Pr[F \not\subseteq F(n) : B(F)] = \frac{1}{N^{2q_e}}.
\]

Therefore,

\[
\Pr_{re}(τ) \geq 1 - \frac{2q_e q_f + 6q_e^2}{N} = \frac{q_e^2}{N^2}.
\]

Therefore, gathering (31) and (32) yields

**Theorem 2** For the 4-round, random function-based KAFw\(^F,(w,γ)\) cipher with a good key-schedule \((w, γ)\) as specified in Definition 1, it holds

\[
\text{Adv}_{\text{rkf}} \text{KAFw}_{k,(w, γ)}(q_f, q_e) \leq \frac{2(δ_1 + 1)q_e q_f + (δ_2 + δ_3 + δ_4 + 7)q_e^2}{N}.
\]

IV. KAFW WITH AFFINE KEY-SCHEDULES

This section provides a comprehensive analysis of KAFW with affine key-schedules. First, in section IV-A, we describe an attack against 4-round KAFW with any affine key-schedules. Then, in section IV-B, we describe an attack against 5-round KAFW with affine key-schedules that satisfy some additional assumptions. These attacks can be easily adapted to KAF, which seems to be of more general interest for attacks. After these, we give positive results on 6-round KAFw\(^F,(w,γ)\) and KAFw\(^F,(w,γ)\) in sections IV-C and IV-D respectively.

A. Insecurity for 4 Rounds with Any Affine Key-Schedules

From a cryptanalytic point of view, note that for KAFw with affine key-schedules, we have 2-round related-key differential characteristics with probability 1: see Eq. (33) and (34) below. Concatenating them would yield a 4-round related-key boomerang distinguisher with probability 1 [25] and cinch distinguishing using only four related-key oracle queries. Formally, we have

**Theorem 3** Let \((w, γ)\) be a 4-round affine key-schedule, where \(w\) and \(γ\) are as defined in Eq. (5) and (6), and \( \overline{F} = (f_1, f_2, f_3, f_4) \) be any 4 functions. Then the 4-round KAFw cipher

\[
\text{KAFw}_{k,(w, γ)}(W) = (w_2(k) || w_3(k)) \oplus \Psi_{f_{γ_2}(k)} \oplus \Psi_{f_{γ_1}(k)} \oplus \Psi_{f_2}(k) \oplus \Psi_{f_1}(k)((w_0(k) || w_1(k)) \oplus W),
\]

is insecure against \( \oplus\)-RKA:

\[
\text{Adv}_{\oplus\text{RKA}} \text{KAFw}_{k,(w, γ)}(0, 4) \geq 1 - \frac{1}{N^2}.
\]

This insecurity claim does not require identical round-functions. The same for the attack in the next subsection.

**Proof:** We denote generically \( (RK[E_k], \overline{F}) \) the oracles to which the adversary has access, where \( E \) is either KAFw\(^F,(w,γ)\) or an ideal cipher IC. Consider the following distinguisher:

1. Choose arbitrary values \( L, R, Δ ∈ \{0,1\}^n \), \( Δ ≠ 0 \), let \( \nabla_1 = (M_0^{(w)} + M_2) \cdot Δ, \nabla_2 = (M_1^{(w)} + M_1) \cdot Δ, \nabla_3 = (M_4 + M_2^{(w)}) \cdot Δ, \) and \( \nabla_4 = (M_3 + M_3^{(w)}) \cdot Δ \). Make two queries \( RK[E_k](0, L) || R \rightarrow S \oplus T \) and \( RK[E_k](Δ, L) \oplus \nabla_1 \oplus \nabla_2 \rightarrow S' \oplus T' \);
2. Make two decryption queries \( RK[E_k]^{-1}(Δ, S' || T') \rightarrow L'' || R'' \) and \( RK[E_k]^{-1}(0, S' || T') \rightarrow L''' || R''' \), for \( S'' || T'' = S \oplus \nabla_3 \oplus T \oplus \nabla_4 \) and \( S''' || T''' = S' \oplus \nabla_3 \oplus T' \oplus \nabla_4 \).
3. If \( L'' || R'' \oplus L''' || R''' \oplus S' || T' \oplus \nabla_4 \) then output 1 to indicate \( E \) is KAFw\(^F,(w,γ)\), and otherwise 0: \( E \) is IC.

We show the output is always 1 when \( E \) is KAFw\(^F,(w,γ)\). To make it cleaner let \( k_Δ = k \oplus Δ \). It’s not hard to see for any \( i \) and any \( V, Δ ∈ \{0,1\}^n \), it holds

\[
γ_i(k) \oplus V = γ_i(k_Δ) \oplus V \oplus M_i \cdot Δ.
\]

By this, for any \( Δ \), it holds

\[
\Pr_{k, L, R, Δ}[\Psi_{f_1}(V || W) \oplus \Psi_{f_1}(V || M_{i+1} \cdot Δ || W \oplus M_i \cdot Δ)] = M_i \cdot Δ || M_{i+1} \cdot Δ] = 1.
\]

This is essentially a 1-round related-key differential with probability 1. For the ease of exposition, we follow the notation in [23] and denote this phenomena by

\[
\Pr(M_{i+1} \cdot Δ || M_i \cdot Δ \frac{Ψ_{f_1}(W)}{Δ} \rightarrow M_i \cdot Δ || M_{i+1} \cdot Δ) = 1.
\]
Concatenating two such differentials gives rise to two 2-round related-key differentials with probability 1 as follows
\[
\Pr\left(\nabla_2 \Theta_{2(k)} \circ \Theta_{1(k)} \circ \Theta_{w(k)} \circ \Theta_{w(k)} \mid M_4 \cdot \Delta \mid M_1 \cdot \Delta\right) = 1,
\]
(33)
\[
\Pr\left(M_4 \cdot \Delta \mid M_3 \cdot \Delta \circ \Theta_{w(k)} \circ \Theta_{w(k)} \circ \Theta_{w(k)} \circ \Theta_{w(k)} \mid \nabla_3 \mid \nabla_4\right) = 1,
\]
(34)
where \(\Theta_{w(k)} \circ \Theta_{w(k)}(W) = w_1(k) \parallel w_2(k) + W\).

Therefore, for the two forward queries, if we assume
\[
(\Psi_{f_2(k)} \circ \Psi_{f_1(k)})(\langle w_0(k) \parallel w_1(k) \rangle \parallel (L \parallel R)) = X \parallel Y,
\]
then by (33) it holds
\[
(\Psi_{f_2(k)} \circ \Psi_{f_1(k)})(\langle w_0(k) \parallel w_1(k) \rangle \parallel (L \parallel \nabla_1 \parallel R \parallel \nabla_2)) = X \parallel M_2 \cdot \Delta \parallel Y \parallel M_1 \cdot \Delta,
\]
(36)
Eq. (35) and (36) also mean
\[
\left(\Psi_{f_3(k)} \circ \Psi_{f_2(k)} \circ \Psi_{f_1(k)} \right)^{-1}(w_2(k) \parallel w_3(k)) = (S' \parallel T') = X \parallel Y,
\]
\[
(\Psi_{f_3(k)} \circ \Psi_{f_2(k)} \circ \Psi_{f_1(k)})(w_2(k) \parallel w_3(k)) = (S' \parallel T').
\]
By (34) we have
\[
X'' \parallel Y'' = X \parallel M_4 \cdot \Delta \parallel Y \parallel M_3 \cdot \Delta,
\]
\[
X''' \parallel Y''' = X \parallel M_2 \cdot \Delta \parallel M_4 \cdot \Delta \parallel Y \parallel M_1 \cdot \Delta \parallel M_3 \cdot \Delta,
\]
thus \((X'' \parallel Y'') \parallel (X''' \parallel Y''') = M_2 \cdot \Delta \parallel M_1 \cdot \Delta\). By this and (33) it can be seen \((L'' \parallel R'') \parallel L''' \parallel R''' = \nabla_3 \parallel \nabla_2\) always holds.

On the other hand, when interacting with \(\text{RK}[\mathcal{E}](k)\), the last response \(L''' \parallel R'''\) is uniform in \(\{0,1\}^{2n} \setminus \{LR\}\). So \(\Pr_{\mathcal{E}}[L'' \parallel R'' \parallel L''' \parallel R''' = \nabla_3 \parallel \nabla_2] = \frac{1}{2^{2n-1}},\) which is also the probability that the distinguisher outputs 1 in the ideal world. Thus, the claimed assumption holds.

\(\Box\)

B. (In)security for 5 Rounds

We first exhibit an attack. The attack makes only one additional assumption on the key-schedule: it's easy to derive \(\Delta \neq 0\) such that \(M_1 \cdot \Delta = M_5 \cdot \Delta\). This is possible: for example, if \(\gamma_1\) and \(\gamma_5\) are two bit-permutations, then for \(\Delta = 0\), \(M_5 \cdot \Delta = 0\), \(\Delta = 0\), \(\Delta = 0\). From a cryptographic viewpoint of the core trick is: in the boomerang attack setting, under some conditions, Feistel schemes allow a boomerang switch trick called "Feistel switch" [24], which enables the boomerang to penetrate one more round. Applying this trick to the 4-round related-key boomerang mentioned before yields a 5-round related-key boomerang distinguisher. Formally,

**Theorem 4** Let \((w, \gamma)\) be a 5-round affine key-schedule, where \(w\) and \(\gamma\) are as defined in Eq. (5) and (6) and satisfy that it's easy to derive \(\Delta \neq 0\) such that \(M_1 \cdot \Delta = M_5 \cdot \Delta,\) and \(\nabla_5 \parallel \nabla_5 = (f_1, f_2, f_3, f_4, f_5)\) be any 5 functions. Then the 5-round KAFW cipher
\[
\text{KAFW}_{k}^{(w, \gamma)}(X) = (w_2(k) \parallel w_3(k)) \circ \Psi_{f_5(\gamma)}(w_2(k) \parallel \Psi_{f_4(\gamma)}(w_2(k) \parallel \Psi_{f_3(\gamma)}(w_2(k) \parallel \Psi_{f_2(\gamma)}(w_2(k) \parallel \Psi_{f_1(\gamma)}(w_2(k) \parallel W),
\]
is insecure against \(\oplus\)-RKA:
\[
\text{Adv}^{\oplus\text{-rka}}_{\text{KAFW}_{k}^{(w, \gamma)}}(0, 4, \mathcal{E}) \geq 1 - \frac{1}{N^2}.
\]

**Proof:** In a similar vein to the proof of Theorem 3, consider the following distinguishing attack:

(1) derive a non-zero difference \(\Delta\) such that \(M_1 \cdot \Delta = M_5 \cdot \Delta;\)

(2) choose two arbitrary values \(L, R \in \{0, 1\}^{n}\), and let \(\nabla_1 = (M_0(w) \parallel M_2(w)) \cdot \Delta, \nabla_2 = (M_1(w) \parallel M_1(w)) \cdot \Delta, \nabla_3 = (M_2(w) \parallel M_2(w)) \cdot \Delta,\) and \(\nabla_4 = (M_3(w) \parallel M_3(w)) \cdot \Delta.\)

Make two queries \(\text{RK}[\mathcal{E}](k)\) \(L \parallel R \) \(\rightarrow S[T.R];\)

(3) query \(\text{RK}[\mathcal{E}](k)\) \(M \parallel \nabla_3 \parallel T \parallel \nabla_4 \) \(\rightarrow L'' \parallel R'';\)

(4) if \(L'' \parallel R'' = L''' \parallel R'''\) \(\rightarrow \nabla_1 \parallel \nabla_2\) then output 1 to indicate \(E\) is KAFW \((w, \gamma), k\), and otherwise 0: \(E\) is IC.

We show the output is always 1 when \(E\) is KAFW \((w, \gamma), k\). Let \(\hat{k} = k \oplus \Delta\). Assume that
\[
(\Psi_{f_2(k)} \circ \Psi_{f_1(k)})(\langle w_0(k) \parallel w_1(k) \rangle \parallel (L \parallel R)) = X \parallel Y,
\]
then by (33) we have
\[
(\Psi_{f_2(k)} \circ \Psi_{f_1(k)})(\langle w_0(k) \parallel w_1(k) \rangle \parallel (L \parallel \nabla_1 \parallel R \parallel \nabla_2)) = X \parallel M_2 \cdot \Delta \parallel Y \parallel M_1 \cdot \Delta.
\]
Computing one more round, we obtain
\[
(\Psi_{f_3(k)} \circ \Psi_{f_2(k)} \circ \Psi_{f_1(k)})(\langle w_0(k) \parallel w_1(k) \rangle \parallel (L \parallel R)) = Y \parallel X \parallel f_3(\gamma_3(k) \parallel Y),
\]
and
\[
(\Psi_{f_3(k)} \circ \Psi_{f_2(k)} \circ \Psi_{f_1(k)})(\langle w_0(k) \parallel w_1(k) \rangle \parallel (L \parallel \nabla_1 \parallel R \parallel \nabla_2)) = Y \parallel X \parallel f_3(\gamma_3(k) \parallel M_2 \cdot \Delta \parallel \Delta \parallel Y \parallel M_1 \cdot \Delta).
\]
The differential Eq. (34) should be adapted to the 5-round case:
\[
\Pr\left(M_5 \cdot \Delta \mid M_4 \cdot \Delta \circ \Theta_{w(k)} \circ \Theta_{w(k)} \circ \Theta_{w(k)} \circ \Theta_{w(k)} \circ \Theta_{w(k)} \parallel \nabla_3 \parallel \nabla_4\right) = 1
\]
(39)
By this and Eq. (37) and (38), if we assume
\[
Y'' \parallel Z'' = (\Psi_{f_3(\gamma_3(k))} \circ \Psi_{f_2(\gamma_3(k))} \circ \Psi_{f_1(\gamma_3(k))})(\langle w_2(k) \parallel w_3(k) \rangle \parallel (S \parallel \nabla_3 \parallel T \parallel \nabla_4)),
\]
\[
X'' \parallel Y'' = (\Psi_{f_2(\gamma_3(k))} \circ \Psi_{f_1(\gamma_3(k))})(\langle w_2(k) \parallel w_3(k) \rangle \parallel (S \parallel \nabla_3 \parallel T \parallel \nabla_4),
\]
\[
X'' \parallel Y'' = (\Psi_{f_3(\gamma_3(k))} \circ \Psi_{f_2(\gamma_3(k))} \circ \Psi_{f_1(\gamma_3(k))})(\langle w_2(k) \parallel w_3(k) \rangle \parallel (S \parallel \nabla_3 \parallel T \parallel \nabla_4),
\]
then it necessarily holds
\[ Y''' = Y' + M_3 \cdot \Delta | X + f_3(\gamma_3(k) + Y) \oplus M_4 \cdot \Delta, \]
\[ X'' = X + f_3(\gamma_3(k) + Y) \oplus M_4 \cdot \Delta \]
\[ Y''' = Y + M_1 \cdot \Delta + M_5 \cdot \Delta | X + M_2 \cdot \Delta \]
\[ = \gamma, \text{ since } M_1 \cdot \Delta = M_5 \cdot \Delta \]
\[ \oplus f_3(\gamma_3(k) \oplus M_3 \cdot \Delta + Y + M_1 \cdot \Delta) + M_4 \cdot \Delta, \]
\[ X'' = X + M_2 \cdot \Delta + f_3(\gamma_3(k) \oplus M_3 \cdot \Delta + Y + M_1 \cdot \Delta) \]
\[ \oplus M_4 \cdot \Delta + f_3(\gamma_3(k) \oplus Y). \]

Now since \( M_5 \cdot \Delta = M_1 \cdot \Delta \), it can be seen
\[ (X''') (Y''') \oplus (X''')(Y''') = M_2 \cdot \Delta | M_5 \cdot \Delta = M_2 \cdot \Delta | M_1 \cdot \Delta, \]
which further indicates \((L''') (R''') \oplus (L''')(R''') = \Delta \parallel \Delta \) by Eq. (33).

We have proved that the probability of outputting 1 in the ideal world is \( 1/(N^2 - 1) \) in the proof of Theorem 3. Thus the claim.

Note that we did not assume \( \exists \Delta \not= 0 \) such that \( M_1 \cdot \Delta \not= M_3 \cdot \Delta \) or \( M_3 \cdot \Delta \not= M_5 \cdot \Delta \). In this case, the scheme suffers from simpler complementation-based attacks, see Appendix B. On the other hand, one may notice that if there is no \( \Delta \not= 0 \) such that \( M_1 \cdot \Delta = M_5 \cdot \Delta \), then the above attack is not effective any more. In fact, we conjecture security in this (latter) case. However, we expect that the proof would be much more complicated than those given in this paper, and have to follow a significantly different framework. Moreover, it’s inferior in the sense that it requires additional assumptions on the key-schedule (i.e., \( \forall \Delta \not= 0, M_1 \cdot \Delta \not= M_5 \cdot \Delta \)). We thereby leave it for future, and revert to 6 rounds.

C. Security for 6 Rounds when \( f=P \)

We first present the conditions on the key-schedule \((w, \gamma)\) that are sufficient for security proof for 6-round KAFW\(_k^{\ell,(w,\gamma)}\).

Definition 3 (Good Key-Schedule for 6-Round KAFW\(_k^{\ell,(w,\gamma)}\))

We say that a 6-round key-schedule \((w, \gamma)\), for which \( w = (w_0, w_1, w_2, w_3) \), \( w_i(k) = M_{i}^{(w)} \cdot k \oplus C_{i}^{(w)} \), \( \gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6) \), and \( \gamma_{i}(k) = M_{i} \cdot k \oplus C_{i} \), is good, if it satisfies the following conditions:

1. \( \varphi_1, \varphi_6, \text{ and } \varphi_1 \otimes \varphi_6 \) are bijective maps of \( \{0,1\}(8) \), where \( \varphi_1(k) = w_1(k) \oplus \gamma_1(k) \), \( \varphi_6(k) = w_2(k) \oplus \gamma_6(k) \);

2. for any \( \Delta \neq 0 \), \( M_1 \cdot \Delta \neq M_3 \cdot \Delta \), \( M_1 \cdot \Delta \neq M_5 \cdot \Delta \).

The 1st condition resembles those appeared in Definition 1. On the other hand, the 2nd condition is intended to prevent the complementing attacks. One could see Appendix B to gain further insights.

Theorem 5 When \( q_f + 4q_c \leq N/2 \), for the 6-round, random permutation-based KAFW\(_k^{\ell,(w,\gamma)}\) cipher with a good key-schedule \((w, \gamma)\) as specified in Definition 3, it holds
\[ \text{Adv}_{k}^{\text{Adv}}(\text{KAFW}_{k}^{\ell,(w,\gamma)})(q_f, q_c) \leq \frac{14q_f q_f + 57q_c^2 + 4q_c}{N}. \]

Proof. Similarly to section III, for any function transcript \( Q_f = ((x_1, y_1), \ldots, (x_{q_f}, y_{q_f})) \), we define \( X(\tau) \) and \( Y(\tau) \) as the sets \( \{x_1, \ldots, x_{q_f}\} \) and \( \{y_1, \ldots, y_{q_f}\} \). We also define 16 functions for any tuple \( t = (\Delta, LR, ST) \) in \( Q_f \) and any function \( f = P \in P(n) \) in this subsection:

\- \( x_1(t) = \varphi_1(k \oplus \Delta) \oplus R, \)
\- \( y_1(t, f) = f(x_1(t)), \)
\- \( X(t, f) = L \oplus w_0(k \oplus \Delta) \oplus y_1(t, f), \)
\- \( y_2(t, f) = f(x_2(t, f)), \)
\- \( Y(t, f) = R \oplus w_1(k \oplus \Delta) \oplus y_2(t, f), \)
\- \( x_3(t, f) = \gamma_3(k \oplus \Delta) \oplus Y(t, f), \)
\- \( y_3(t, f) = X(t, f) \oplus Z(t, f), \)
\- \( Z(t, f) = S \oplus w_2(k \oplus \Delta) \oplus y_3(t, f), \)
\- \( x_4(t, f) = \gamma_4(k \oplus \Delta) \oplus Z(t, f), \)
\- \( y_4(t, f) = Y(t, f) \oplus A(t, f), \)
\- \( A(t, f) = T \oplus w_3(k \oplus \Delta) \oplus y_4(t, f), \)
\- \( x_5(t, f) = \gamma_5(k \oplus \Delta) \oplus A(t, f), \)
\- \( y_5(t, f) = f(x_5(t, f)), \)
\- \( x_6(t) = \varphi_6(k \oplus \Delta) \oplus S, \)
\- \( y_6(t, f) = f(x_6(t)). \)

Bad Transcripts are then defined as follows.

Definition 4 (Bad Transcripts for 6-Round KAFW\(_k^{\ell,(w,\gamma)}\))

An attainable transcript \( \tau = (Q_E, Q_P, k) \) is bad, if at least one of the following conditions is fulfilled:

\- (B-1) \( \exists t \in Q_E : x_1(t) \in X(\tau) \) or \( x_2(t) \in X(\tau) \);
\- (B-2) \( \exists t, t' \in Q_E : x_1(t) = x_2(t'), \)
\- (B-3) there exists two queries \( t = (\Delta, LR, ST) \) and \( t' = (\Delta', LR', ST') \) in \( Q_E \) such that \( \Delta \neq \Delta' \), and \( R \oplus R' = (M_1^{(w)} \oplus M_1) \cdot (\Delta \oplus \Delta') \) and \( S \oplus S' = (M_6 \oplus M_2^{(w)}) \cdot (\Delta \oplus \Delta') \).

Otherwise we say \( \tau \) is good. Denote by \( T_{\text{bad}} \) the set of bad transcripts.

Recall that
\[ \varphi_1(k) = w_1(k) \oplus \gamma_1(k) = M_1^{(w)} \cdot k \oplus C_1^{(w)} \oplus M_1 \cdot k \oplus C_1, \]
\[ \varphi_6(k) = w_2(k) \oplus \gamma_6(k) = M_2^{(w)} \cdot k \oplus C_2^{(w)} \oplus M_6 \cdot k \oplus C_6. \]
Since both \( \varphi_1 \) and \( \varphi_6 \) are bijective maps of \( \mathbb{F}_2^n \), \( \text{Pr}((B-1)) \leq \frac{q_f q_f}{2q_f q_f} \) is obvious. On the other hand, since \( \varphi_1 \otimes \varphi_6 \) is also bijective, for each choice of \( t = (\Delta, LR, ST) \) and \( t' = (\Delta', LR', ST') \) it holds
\[ \text{Pr}[x_1(t) = x_6(t')] = \text{Pr}[(M_1^{(w)} \oplus M_1) \cdot (k \oplus \Delta) \oplus C_1^{(w)} \oplus C_1 \oplus R] = \text{Pr}[(M_2^{(w)} \oplus M_6) \cdot (k \oplus \Delta) \oplus (M_2^{(w)} \oplus M_6) \cdot (\Delta \oplus \Delta')] + \text{Pr}(C_2^{(w)} \oplus C_6 \oplus S') = \frac{1}{N}. \]
Therefore, \( \text{Pr}((B-2)) \leq \frac{q_c^2}{N} \). Ultimately, for (B-3), for any such two queries \( (\Delta, LR, ST) \) and \( (\Delta', LR', ST') \), following an analysis similar to (B-4) in Definition 2, the probability that \( R \oplus R' = (M_1^{(w)} \oplus M_1) \cdot (\Delta \oplus \Delta') \) and \( S \oplus S' = (M_6 \oplus \Delta \oplus \Delta') \) and \( S \oplus S' = (M_6 \oplus \Delta \oplus \Delta') \) and \( S \oplus S' = (M_6 \oplus \Delta \oplus \Delta') \) holds.
\[ M_2^{(w)} \cdot (\Delta \oplus \Delta') \text{ are both fulfilled is at most } 1/(N - q_e). \text{ Thus } \Pr[(B-3)] \leq \frac{q_e^2}{2N - q_e}. \text{ In all, when } q_e \leq N/2, \text{ we have} \]
\[ \Pr[T_{id} \in \bar{T}_{bad}] \leq 2q_eq_f + \frac{q_e^2}{2(N - q_e)} \leq 2q_eq_f + 2q_e^2 \frac{e_e}{N}. \] (40)

**Ratio** \( \Pr_{rec}(\tau)/\Pr_{rec}(\tau) \) for Good \( \tau \). We define two bad predicates on \( P \) in turn. We finally argue that if neither of the two predicates holds, then the event \( \text{KAFW}_n^{(w, \gamma)} \subset Q_E \) is equivalent to \( P \) satisfying \( 2q_e \) fresh and distinct equations, and thus the bound.

**First Bad Predicate.** For any \( P \vdash Q_P \), the predicate \( B1(P) \) holds, if any of the following conditions are fulfilled:

- (C-11) there exists \( t = (\Delta, LR, ST) \) and \( t' = (\Delta', LR', ST') \) in \( Q_E \) such that \( x_1(t) \neq x_1(t') \), yet \( x_2(t, P) = x_2(t', P) \) or \( X(t) \oplus X(t', P) = M_6 \cdot (\Delta \oplus \Delta') \);
- (C-12) \( \exists t, t' \in Q_E \) (could be \( t = t' \)): \( x_2(t, P) \in X(\tau), \) or \( x_2(t, P) = x_2(t') \); \( x_2(t, P) = x_2(t'); \)
- (C-13) there exists \( \exists t = (\Delta, LR, ST) \) and \( t' = (\Delta', LR', ST') \) in \( Q_E \) such that \( x_2(t) \neq x_2(t') \), yet \( x_2(t, P) = x_2(t', P) \) or \( A(t) \oplus A(t', P) = M_1 \cdot (\Delta \oplus \Delta') \);
- (C-14) \( \exists t, t' \in Q_E \) (could be \( t = t' \)): \( x_5(t, P) \in X(\tau) \), or \( x_5(t, P) = x_5(t', P) \);
- (C-15) there exists a query \( t = (\Delta, LR, ST) \) in \( Q_E \) such that

\[ L \oplus w_0(k \oplus \Delta) \oplus S \oplus w_2(k \oplus \Delta) = P(x_1(t)) \]
\[ \text{or} \]
\[ R \oplus w_1(k \oplus \Delta) \oplus T \oplus w_3(k \oplus \Delta) = P(x_1(t)). \]

For (C-11), for each pair \((t, t')\) with \( t = (\Delta, LR, ST) \) and \( t' = (\Delta', LR', ST') \), the event \( x_2(t, P) = x_2(t', P) \) is equivalent to \( X(t) \oplus X(t', P) = M_2 \cdot (\Delta \oplus \Delta') \), which is further equivalent to

\[ L \oplus w_0(k \oplus \Delta) \oplus P(x_1(t)) \]
\[ = (\Delta', LR', ST') \oplus P(x_1(t')) \oplus M_2 \cdot (\Delta \oplus \Delta'). \] (41)

Since \( x_1(t) \neq x_1(t') \), \( P(x_1(t)) \) and \( P(x_1(t')) \) are independent. So the right-hand side of Eq. (41) is independent from the left-hand side. Furthermore, since \( \tau \) is good, it holds \( x_1(t) \notin X(\tau) \). Conditioned on \( P \vdash Q_P \) and the \( 2q_e \) function values \( \{P(x_1(t'))\} \mid t' \in Q_E, i = 1, 6, x_2(t') \neq x_1(t') \}, \) \( P(x_1(t)) \) is uniform in at least \( N - q_f - 2q_e \) possibilities. Therefore, for each pair \((t, t')\), \( \Pr[x_2(t, P) = x_2(t', P)] = \Pr_{rec}(\text{Eq. (41)}) \leq \frac{1}{N - q_f - 2q_e}. \) For the same reason \( \Pr[X(t) \oplus X(t', P) = M_6 \cdot (\Delta \oplus \Delta')] \leq \frac{1}{N - q_f - 2q_e}. \) Thus

\[ \Pr[(C-11)] \leq \frac{q_e}{2} \cdot \frac{2}{N - q_f - 2q_e} \leq \frac{q_e^2}{N - q_f - 2q_e}. \]

Then, the value \( x_2(t, P) \) relies on \( P(x_1(t)) \), and is thus uniform. Since the values in \( X(\tau) \) and the values of the form \( x_1(t') \) and \( x_6(t') \) are all independent from \( P(x_1(t)) \), it holds

\[ \Pr[(C-12)] \leq \frac{q_eq_f}{N - q_f - 2q_e} + \frac{q_e^2}{N - q_f - 2q_e} = \frac{q_e(q_f + 2q_e)}{N - q_f - 2q_e}. \]

For (C-13) the analysis is similar to (C-11) by symmetry, yielding the same bound

\[ \Pr[(C-13)] \leq \frac{q_e}{2} \cdot \frac{2}{N - q_f - 2q_e} \leq \frac{q_e^2}{N - q_f - 2q_e}. \]

Similarly, the main claim in (C-14) can be bounded:

\[ \Pr[(C-14)] \leq \frac{q_eq_f + 2q_e^2}{N - q_f - 2q_e}. \]

The remaining subevent of (C-14), i.e. \( \exists t, t' : x_5(t, P) = x_2(t', P) \), is equivalent to

\[ \gamma_5(k \oplus \Delta) \oplus T \oplus u_3(k \oplus \Delta) \oplus P(x_6(t)) \]
\[ = \gamma_2(k \oplus \Delta') \oplus L' \oplus w_0(k \oplus \Delta') \oplus P(x_1(t')). \] (42)

By \(-\text{(B-2)}, x_1(t') \neq x_6(t) \), thus \( P(x_1(t')) \)—as well as the entire right hand side—is independent from \( P(x_6(t)) \). Thus \( \Pr[(C-14)] \leq q_e(q_f + 3q_e) \frac{1}{N - q_f - 2q_e} \).

Finally, since both \( P(x_1(t)) \) and \( P(x_6(t)) \) are uniform for each \( t \), we immediately obtain \( \Pr[(C-15)] \leq \frac{2q_e}{N - q_f - 2q_e} \).

Summing over \( \Pr[(C-11)] \) to \( \Pr[(C-15)] \), we reach

\[ \Pr[P \triangleleft P(n) : B1(P) \mid P \vdash Q_P] \leq 2q_eq_f + 7q_e^2 + 2q_e \frac{e}{N - q_f - 2q_e}. \] (43)

**Second Bad Predicate.** We then consider a random permutation \( P \) such that \( P \vdash Q_P \) and \( \neg B1(P) \). For \( P \), the predicate \( B2(P) \) holds if any of the following conditions is fulfilled:

- (C-21) \( \exists t, t' \in Q_E : x_2(t, P) \neq x_2(t', P), \) yet either \( x_3(t, P) = x_3(t', P) \) or \( y_1(t, P) = y_1(t', P); \)
- (C-22) \( \exists t, t' \in Q_E \) (could be \( t = t' \)): \( x_3(t, P) \in X(\tau) \) or \( y_3(t, P) \in Y(\tau) \), or \( x_3(t, P) \in \{x_1(t'), x_2(t', P), x_5(t', P), x_6(t'), P\}, \) or \( y_3(t, P) \in \{y_1(t'), y_2(t', P), y_3(t', P), y_6(t'), P\}, \)
- (C-23) \( \exists t, t' \in Q_E : x_5(t, P) \neq x_5(t', P), \) yet either \( x_3(t, P) = x_3(t', P) \) or \( y_3(t, P) = y_3(t', P); \)
- (C-24) \( \exists t, t' \in Q_E \) (could be \( t = t' \)): \( x_3(t, P) \in X(\tau) \) or \( y_3(t, P) \in Y(\tau) \), or \( x_3(t, P) \in \{x_1(t'), x_2(t', P), x_5(t', P), x_6(t'), P\}, \) or \( y_3(t, P) \in \{y_1(t'), y_2(t', P), y_3(t', P), y_6(t'), P\}, \)

First, for each \( t = (\Delta, LR, ST) \), conditioned on \( P \vdash Q_P \) and the \( \leq 4q_e \) values

\[ \{P(x_i(t')) \mid t' \in Q_E, i = 1, 6, j = 2, 5, \}
\[ x_1(t') \neq x_2(t, P) \}\],

the value \( y_2(t, P) = P(x_2(t, P)) \) remains uniform in at least \( N - q_f - 4q_e \) possibilities. So \( Y(t, P), x_3(t, P), \) and \( y_1(t, P) \) derived from \( y_2(t, P) \) are all uniform. These show:
\[ \Pr[\text{C-21}] \leq \left( \frac{q_e}{2} \right) \cdot \frac{2}{N - q_f - 4q_e} \leq \frac{q_e^2}{N - q_f - 4q_e}, \]
\[ \Pr[\exists t : x_3(t, P) \in \mathcal{X}(\tau)] \leq \frac{q_e \cdot q_f}{N - q_f - 4q_e}, \quad \text{(44)} \]
\[ \Pr[\exists t, t' : x_3(t, P) \in \{ x_1(t'), x_6(t') \}] \leq \frac{q_e \cdot 2q_f}{N - q_f - 4q_e}, \quad \text{(45)} \]
\[ \Pr[\exists t : y_4(t, P) \in \mathcal{Y}(\tau)] \leq \frac{q_e \cdot q_f}{N - q_f - 4q_e}. \quad \text{(46)} \]

Second, for cleanliness let \( k_\Delta = k + \Delta \) and \( k_{\Delta'} = k + \Delta' \), then
\[
\Pr[\exists t, t' : x_3(t, P) = x_2(t', P)]
= \Pr[\exists t, t' : \gamma_3(k_\Delta) \otimes R \oplus u_1(k_\Delta) \otimes P(x_2(t, P))]
= \gamma_2(k_{\Delta'}) \oplus L' \oplus u_0(k_{\Delta'}) \oplus P(x_3(t')).
\]
By \( \neg(\text{C-12}) \), \( x_3(t, P) \neq x_1(t') \), \( x_2(t, P) \neq x_6(t') \), \( x_2(t, P) \neq x_6(t) \), so for the involved equality the right hand side is independent from the left hand side. Therefore,
\[
\Pr[\exists t, t' : x_3(t, P) = x_2(t', P)] \leq \frac{q_e^2}{N - q_f - 4q_e}. \quad \text{(47)}
\]
For similar reasons,
\[
\Pr[\exists t, t' : x_3(t, P) = x_2(t', P)]
= \Pr[\exists t, t' : CON_1 \oplus P(x_2(t, P))]
= \gamma_3(k_\Delta) \oplus T' \oplus u_3(k_{\Delta'}) \oplus P(x_3(t'))
\leq \frac{q_e^2}{N - q_f - 4q_e}.
\]
\[
\Pr[\exists t, t' : y_4(t, P) = y_1(t', P)]
= \Pr[\exists t, t' : (R \oplus u_1(k_\Delta) \otimes P(x_2(t, P)))
\oplus (T \oplus u_3(k_{\Delta'}) \oplus P(x_3(t'))) = P(x_1(t'))]
= \Pr[\exists t, t' : P(x_2(t, P)) \oplus P(x_3(t'))]
\leq \frac{q_e^2}{N - q_f - 4q_e}.
\]
\[
\Pr[\exists t, t' : y_4(t, P) = y_6(t', P)]
= \Pr[\exists t, t' : P(x_2(t, P)) \oplus P(x_3(t'))]
\leq \frac{q_e^2}{N - q_f - 4q_e}. \quad \text{(50)}
\]
Furthermore, by \( \neg(\text{C-14}) \), \( \forall t, t', x_3(t, P) \neq x_5(t', P) \). So
\[
\Pr[\exists t, t' : y_4(t, P) = y_5(t', P)]
= \Pr[\exists t, t' : R \oplus u_1(k_\Delta) \otimes P(x_2(t, P)) \oplus T \oplus u_3(k_\Delta) \oplus P(x_0(t))]
\leq \frac{q_e^2}{N - q_f - 4q_e}.
\]
\[
\text{Finally, for a pair } (t, t'), y_4(t, P) = y_2(t', P) \text{ would imply}
R \oplus u_1(k_\Delta) \otimes P(x_2(t, P)) \oplus T \oplus u_3(k_\Delta) \oplus P(x_6(t))
= P(x_2(t', P)). \quad \text{(52)}
\]

Then,
1. If \( x_2(t, P) = x_2(t', P) \), then for \( t = (\Delta, LR, ST) \) it holds
\[
R \oplus u_1(k_\Delta) \oplus T \oplus u_3(k_\Delta) = P(x_6(t)),
\]
contradicting \( \neg(\text{C-15}) \);
2. Otherwise, \( P(x_2(t', P)) \) is independent from the left hand side of \( (52) \), thus \( \Pr[\text{Eq. } (52)] \leq \frac{1}{N - q_f - 4q_e} \).

As the number of pairs \( (t, t') \) is at most \( q_e^2 \),
\[
\Pr[\exists t, t' : y_4(t, P) = y_2(t', P)] \leq \frac{q_e^2}{N - q_f - 4q_e}. \quad \text{(53)}
\]
Summing over \( (44)-(53) \), we obtain
\[
\Pr[\text{C-22}] \leq \frac{2q_e(q_f + 4q_e)}{N - q_f - 4q_e}.
\]

Third, symmetrically, for each \( t = (\Delta, LR, ST) \in \mathcal{Q}_E \), the value \( y_5(t, P) = P(x_5(t, P)) \) remains random. So \( Z(t, P), x_4(t, P), \) and \( y_3(t, P) \) are all uniform. Therefore, \( \Pr[\text{C-23}] \leq \frac{q_e^2}{N - q_f - 4q_e} \). In addition, in a similar vein to the analysis of \( (C-22) \), we have
\[
\Pr[\exists t, t' : x_4(t, P) \in \mathcal{X}(\tau) \text{ or } y_3(t, P) \in \mathcal{Y}(\tau)] \leq \frac{N - q_f - 4q_e}{N - q_f - 4q_e}, \quad \text{(54)}
\]
\[
\Pr[\exists t, t' : x_4(t, P) = x_1(t') \text{ or } x_4(t, P) = x_6(t')] \leq \frac{N - q_f - 4q_e}{N - q_f - 4q_e}. \quad \text{(55)}
\]
By \( \neg(\text{C-14}) \), \( \forall t, t', x_5(t, P) \neq x_1(t') \). So \( (k_\Delta = k + \Delta, k_{\Delta'} = k + \Delta') \)
\[
\Pr[\exists t, t' : x_4(t, P) = x_2(t', P)]
= \Pr[\gamma(x_4(k_\Delta) + S + u_2(k_\Delta) \oplus P(x_5(t, P))]
\quad \text{CON}_2, \quad \text{will be used below}
= \gamma_2(k_{\Delta'}) \oplus L' \oplus u_0(k_{\Delta'}) \oplus P(x_1(t'))
\leq \frac{q_e^2}{N - q_f - 4q_e}, \quad \text{(56)}
\]
and
\[
\Pr[\exists t, t' : y_3(t, P) = y_1(t', P)]
= \Pr[(L \oplus u_0(k_\Delta) \oplus P(x_1(t))) \oplus (S \oplus u_2(k_\Delta) \oplus P(x_5(t, P))]
= \Pr[P(x_5(t, P) = L \oplus u_0(k_\Delta) \oplus S \oplus u_2(k_\Delta)]
\quad \text{CON}_3, \quad \text{will be used later}
\leq \frac{q_e^2}{N - q_f - 4q_e}.
\]
By \( \neg(\text{C-14}) \), \( \forall t, t', x_5(t, P) \neq x_2(t', P) \). So
\[
\Pr[\exists t, t' : x_4(t, P) = x_3(t', P)]
= \Pr[\text{CON}_2 \oplus P(x_5(t, P))]
\quad \text{CON}_2, \quad \text{will be used below}
= \gamma_3(k_{\Delta'}) \oplus R' \oplus u_1(k_{\Delta'}) \oplus P(x_2(t', P))]
\leq \frac{q_e^2}{N - q_f - 4q_e}, \quad \text{(57)}
\]
and
\[
\Pr[\exists t, t' : y_3(t, P) = y_2(t', P)]
= \Pr[P(x_5(t, P) = \text{CON}_3 \oplus P(x_1(t)) \oplus P(x_2(t', P))]
\leq \frac{q_e^2}{N - q_f - 4q_e}.
\]
By \( (C-14) \), \( \forall t, t', x_5(t, P) \neq x_6(t') \). So
\[
\Pr[\exists t, t' : x_4(t, P) = x_5(t', P)] = \Pr[CON_2 \oplus P(x_5(t, P)) = \gamma_5(k_{\Delta}) \oplus T' \oplus w_3(k_{\Delta'}) \oplus P(x_6(t'))] \\
\leq \frac{q_e^2}{N - q_f - 4q_e},
\]
and
\[
\Pr[\exists t, t' : y_3(t, P) = y_6(t')] = \Pr[P(x_5(t, P)) = CON_3 \oplus P(x_1(t)) \oplus P(x_6(t'))] \\
\leq \frac{q_e^2}{N - q_f - 4q_e}.
\]
By \( (C-14) \), \( \forall t, t', x_5(t, P) \neq x_2(t', P) \) and \( x_5(t, P) \neq x_6(t') \). So
\[
\Pr[\exists t, t' : y_3(t, P) = y_5(t', P)] = \Pr[P(x_5(t, F)) = CON_3 \oplus P(x_1(t)) \oplus \{R' \oplus w_3(k_{\Delta'}) \oplus P(x_2(t', P)) \} \oplus (T' \oplus w_3(k_{\Delta'}) \oplus P(x_6(t'))) \\
\leq \frac{q_e^2}{N - q_f - 4q_e}.
\]
Finally consider \( \Pr[\exists t, t' : y_3(t, P) = y_5(t', P)] \). If it happens then we have
\[
L \oplus w_0(k_{\Delta}) \oplus P(x_1(t)) \oplus S \oplus w_2(k_{\Delta}) \oplus P(x_5(t, P)) \\
= P(x_5(t', P)). \tag{54}
\]
If \( x_5(t, P) \neq x_2(t', P) \) then the right hand side of (54) is independent from \( P(x_5(t, P)) \) and \( \Pr[\text{Eq. (54)}] \leq \frac{1}{N - q_f - 2q_e} \); otherwise we reach \( L \oplus w_0(k_{\Delta}) \oplus S \oplus w_2(k_{\Delta}) = P(x_1(t)) \), contradicting \( (C-15) \). So
\[
\Pr[\exists t, t' : y_3(t, P) = y_5(t', P)] \leq \frac{q_e^2}{N - q_f - 2q_e}. \tag{55}
\]
In all, we have
\[
\Pr[(C-24)] \leq \frac{2q_e(q_f + 5q_e)}{N - q_f - 4q_e},
\]
and further
\[
\Pr[P \not\equiv P(n) : B2(P) \mid P \not\equiv Q_P \land \neg B1(P)] \leq \frac{4q_e q_f + 20q_e^2}{N - q_f - 4q_e}. \tag{56}
\]
Define \( B(P) = B1(P) \lor B2(P) \). Then Eq. (43) and (56) yield
\[
\Pr[P \not\equiv P(n) : B(P) \mid P \not\equiv Q_P] \leq \frac{6q_e q_f + 27q_e^2 + 2q_e}{N - q_f - 4q_e}. \tag{57}
\]
\( 2q_e \) Equations. Conditioned on \( P \not\equiv Q_P \land \neg B(P) \), we show
\[
\Pr[P \not\equiv Q_P \land \neg B(P) \mid Q_E] = \Pr[P \not\equiv Q_E : (x_3(t, P) = y_3(t, P) \land P(x_4(t, P)) = y_4(t, P))] = \frac{1}{N - 2q_e}.
\]
Similarly to the 4-round case, \( \neg B(P) \) indicates
\[
\forall t \in Q_E : i = 3, 4, x_i(t, P) \not\equiv \lambda(t), y_i(t, P) \not\equiv \lambda(t), 
\]
and
\[
\{x_i(t, P) \mid i = 3, 4, t \in Q_E \} \cap \{y_j(t, P) \mid j = 1, 2, 5, 6, t \in Q_E \} = \emptyset,
\]
and
\[
\{y_i(t, P) \mid i = 3, 4, t \in Q_E \} \cap \{y_j(t, P) \mid j = 1, 2, 5, 6, t \in Q_E \} = \emptyset,
\]
and
\[
\forall t, t' \in Q_E : x_4(t, P) \neq x_4(t', P), y_3(t, P) \neq y_4(t', P).
\]
It remains to show
\[
\{x_i(t, P) \mid t \in Q_E \} \mid \{y_i(t, P) \mid t \in Q_E \} = q_e
\]
for \( i = 3, 4 \). For this, we argue \( t \neq t' \Rightarrow x_3(t, P) \neq x_3(t', P) \) and \( y_4(t, P) \neq y_4(t', P) \) for any \( t = (\Delta, LR, ST) \) and \( t' = (\Delta', LR', ST') \):

**Case 1:** \( t \) and \( t' \) are such that \( R \oplus R' = (M_1^{(w)} \oplus M_1^{(w)}) \cdot (\Delta \oplus \Delta') \) and \( L \oplus L' = (M_2^{(w)} \oplus M_2) \cdot (\Delta \oplus \Delta') \). By the definition of \( \varphi_0, \varphi_1, \gamma_1 \), and \( \gamma_2 \), the former implies \( \varphi_1(k \oplus \Delta) = \varphi_1(k \oplus \Delta') \oplus R, \) i.e. \( x_1(t) = x_1(t') \); and the latter further implies
\[
\gamma_2(k \oplus \Delta) \oplus L \oplus w_0(k \oplus \Delta) \oplus P(x_1(t)) = \gamma_2(k \oplus \Delta') \oplus L' \oplus w_0(k \oplus \Delta') \oplus P(x_1(t)),
\]
i.e. \( x_2(t, P) = x_2(t', P) \). And it necessarily be \( \Delta \neq \Delta' \), otherwise \( \Delta \oplus \Delta' = 0 \) and thus \( R = R' \) and \( L = L' \) and \( t = t' \), a contradiction. Then:

- \( x_3(t, P) \neq x_3(t', P) \), otherwise it implies
\[
\gamma_3(k \oplus \Delta) \oplus R \oplus w_1(k \oplus \Delta) \oplus P(x_2(t, P)) = \gamma_3(k \oplus \Delta') \oplus R' \oplus w_1(k \oplus \Delta') \oplus P(x_2(t', P)),
\]

Then we have
\[
\gamma_3(k \oplus \Delta) \oplus \gamma_3(k \oplus \Delta') = R \oplus R' \oplus M_1^{(w)} \cdot (\Delta \oplus \Delta') = M_3 \oplus \Delta \oplus \Delta',
\]

thus \( M_3 \cdot (\Delta \oplus \Delta') = M_1 \cdot (\Delta \oplus \Delta'), \) contradicting condition (2) (good key-schedule for 6 rounds);

- \( y_4(t, P) \neq y_4(t', P) \). Because the assumption on \( R \oplus R' \) implies \( S \oplus S' \neq (M_6 \oplus M_2^{(w)}) \cdot (\Delta \oplus \Delta') \) by \( (B-3) \). By \( (C-13) \) we further have \( A(t, P) \oplus A(t', P) \neq M_1 \cdot (\Delta \oplus \Delta') \). However, in this case, it necessarily be
\[
Y(t, P) \oplus Y(t', P) = R \oplus R' \oplus w_1(k \oplus \Delta) \oplus w_1(k \oplus \Delta') = M_1 \cdot (\Delta \oplus \Delta'),
\]

Therefore, we must have \( Y(t, P) \oplus Y(t', P) \neq A(t, P) \oplus A(t', P) \), i.e. \( y_4(t, P) \neq y_4(t', P) \);

**Case 2:** for \( (t', t) \), \( x_1(t) = x_1(t') \), yet \( x_2(t, P) \neq x_2(t', P) \). Then by \( (C-21) \) we immediately have \( x_3(t, P) \neq x_3(t', P) \) and \( y_4(t, P) \neq y_4(t', P) \);

**Case 3:** for \( (t', t) \), \( x_1(t) \neq x_1(t') \). This implies \( x_2(t, P) \neq x_2(t', P) \) by \( (C-11) \), and further \( x_3(t, P) \neq x_3(t', P) \) and \( y_4(t, P) \neq y_4(t', P) \) by \( (C-21) \).

So \( \{x_3(t, P) \mid t \in Q_E \} = \{y_4(t, P) \mid t \in Q_E \} = q_e \).

The argument for \( x_4(t, P) \) and \( y_4(t, P) \) is similar by symmetry (utilizing the property \( M_4 \cdot \Delta \neq M_6 \cdot \Delta \) for \( \Delta \neq 0 \) given
Theorem 6 For the 6-round, random function-based $\text{KAF}^{f,w,\gamma}$ cipher with a good key-schedule $(w, \gamma)$ as specified in Definition 3, it holds

$$\text{Adv}_{\text{KAF}^{f,w,\gamma}}(q_f, q_e) \leq \frac{6q_f + 18q_e^2}{N}.$$  

V. DERIVING RESULTS ON KAF AND KAFV CIPHERS

A. Results on KAF

Since KAF ciphers are $\text{KAF}^w$ ciphers with no whitening keys, results on the latter can be immediately transposed to the former. In detail, denote by $\gamma = (\gamma_1, \ldots, \gamma_t)$ a t-round key-schedule of KAF, then with the function $\Psi_k^f$ defined by Eq. (7) in section II, the t-round $\text{KAF}^{f,\gamma}$ cipher is defined as

$$\text{KAF}^{f,\gamma}_k(W) = \Psi_k^f(\gamma_1) \circ \cdots \circ \Psi_k^f(\gamma_t)(W).$$

Setting $\phi_i = \gamma_i$ for $i = 1, 4, 6$ in Definitions 1 and 3 (good key-schedules for $\text{KAF}^{f,w,\gamma}$) yields the corollary.

Corollary 1 A 4-round non-linear key-schedule $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ is good for $\text{KAF}^{f,\gamma}$, if $\gamma_1 = \gamma_2$ and $\gamma_3$ possesses the $\delta_1$-uniformness, $\delta_2$-non-linearity, $\delta_2$-“mutual-uniformness”, and $\delta_4$-“mutual-non-linearity” properties defined in Definition 1.

A 6-round affine key-schedule $\gamma = (\gamma_1, \ldots, \gamma_6)$, where $\gamma_i(k) = M_i \cdot k + C_i$, is good for $\text{KAF}^{f,\gamma}$, if:

1. in (C-13) only used $\gamma_4(k)$ to fulfill some conditions, i.e. $\gamma_4$ need not be beneficial for RKA security (since the “inner” KDFs remain “bad”).

For affine schedules, both Corollary 1 and Definition 3 require the “inner” KDFs $\gamma_3$ and $\gamma_4$ to fulfill some conditions, i.e. $M_1 \cdot \Delta \neq M_3 \cdot \Delta$ and $M_4 \cdot \Delta \neq M_6 \cdot \Delta$. This means for KAF instances that suffer from $\oplus$-RKA, adding whitening keys derived by affine KDFs would probably not be beneficial for RKA security (since the “inner” KDFs remain “bad”). For example, consider the attempt to prevent DES from the complementation property via using a DESX-like [81] structure $\text{DESX}^L_k(M) = k \oplus \text{DES}_K(k \oplus M)$, where the 56-bit DES key $k'$ are 56 bits chosen from the 64-bit main-key $k$. It can be seen while $\text{DESX}^L_k(M) = \text{DESX}^L_k(M)$ does not necessarily hold, the $\text{DESX}^L_k(M)$ still suffers from a (less trivial) complementation-based property $\text{DESX}^L_k(M) = \text{DESX}^L_k(M)$.

B. Results on KAFV

For the KAFV model the transition is a bit more complicated. Formally, KAFV relies on the following round transformation

$$\Psi_k^f(W_L \| W_R) = W_R \| W_L \oplus f(W_R) \oplus k.$$  

(60)

With this, a t-round KAFV needs $t + 2$ sub-keys. To make a clear distinction from the notations for KAF, we denote by $\gamma^* = (\gamma_0^*, \gamma_1^*, \ldots, \gamma_{t+1}^*)$ a t-round key-schedule for KAFV: $\gamma_1^*, \ldots, \gamma_t^*$ for the t-round keys, while $\gamma_0^*$ and $\gamma_{t+1}^*$ for the two whitening keys. Then the entire $\text{KAFV}^{f,\gamma^*}$ variant is

$$\text{KAFV}^{f,\gamma^*}_k(W) = (\gamma_{t+1}^*(k)\|0) \circ \tilde{\Psi}_{\gamma_t^*(k)} \circ \cdots \circ \tilde{\Psi}_{\gamma_1^*(k)}(0\|\gamma_0^*(k)) \oplus W.$$  

For $\text{KAFV}^{f,\gamma^*}$ we have

...
Corollary 2 A 4-round non-linear key-schedule $\gamma^* = (\gamma_0^*, \ldots, \gamma_5^*)$ is good for $\operatorname{KAF}_P^{P, \gamma}$, if $\varphi_1 = \gamma_0^*$ and $\varphi_4 = \gamma_5^*$ possesses the four properties defined in Definition 1.

A 6-round affine key-schedule $\gamma^* = (\gamma_0^*, \ldots, \gamma_5^*)$, where $\gamma_3^* = \gamma_3^*$, is good for $\operatorname{KAF}_P^{P, \gamma}$, if:

1. $\gamma_0^*, \gamma_1^*, \gamma_2^*$ and $\gamma_5^*$ are bijective maps of $\{0, 1\}^n$,
2. for any $\Delta \neq 0$, $M_3 \cdot \Delta = 0$, $M_2 \cdot \Delta = 0, \Delta = 0$.

With such good key-schedules, 4- and 6-round idealized $\operatorname{KAF}_P^{P, \gamma}$ and $\operatorname{KAF}_P^{P, \gamma}$ ensure the same security bounds as described in Theorems 1, 2, 5, and 6.

Proof: For a $t$-round $\operatorname{KAF}_P$ key-schedule $\gamma^*$, define a $t$-round $\operatorname{KAF}_P$ schedule $(w, \gamma)$ as follows:

- $\gamma_{2t+1} = \bigoplus_{i=0}^t \gamma_{2i}$, where $\ell = 0, \ldots, \lfloor \frac{t-1}{2} \rfloor$, and
- $\gamma_{2t+2} = \bigoplus_{i=0}^t \gamma_{2i+1}$, where $\ell = 0, \ldots, \lfloor \frac{t-2}{2} \rfloor$, and
- $w_0 = \gamma_0 \oplus \gamma_{t+1}, w_3 = \gamma_{t+2} \oplus \gamma_{t+4}$, while $w_0(k) = w_1(k) = 0$.

Then it can be seen a t-round $\operatorname{KAF}_P$ with the key-schedule $\gamma^*$ is a $\operatorname{KAF}_P$ instance with $(w, \gamma)$, i.e.

$$\operatorname{KAF}_P^{f, \gamma^*}(W) = \operatorname{KAF}_P^{f, (w, \gamma^*}(W). \quad (61)$$

Concretely, the 4-round $\operatorname{KAF}_P^{f, \gamma^*}(\gamma_0^*, \ldots, \gamma_5^*)$ corresponds to the 4-round $\operatorname{KAF}_P^{f, (w, \gamma^*}(w, \gamma)$ with $w_0(k) = w_1(k) = 0, \gamma_1 = \gamma_0^* (\text{thus } \varphi_1 = w_1 \oplus \gamma_1 = \gamma_0^*)$, and $\varphi_3 = \gamma_3 \oplus w_2 = \gamma_5^*$. The first half of the corollary thus follows from Definition 1.

On the other hand, the 6-round affine key-schedule $\gamma^* = (\gamma_0^*, \ldots, \gamma_5^*)$ corresponds to the 6-affine affine schedule $(w, \gamma)$, in which $\varphi_1 = w_1 \oplus \gamma_1 = \gamma_0^*, \varphi_6 = \gamma_6 \oplus w_2 = \gamma_5^*$, and:

1. $\gamma_1 \oplus \gamma_3 = \gamma_2^*, \text{ and thus } M_2^* = M_1 \oplus M_3$;
2. $\gamma_4 \oplus \gamma_6 = \gamma_5^*$, and thus $M_5^* = M_4 \oplus M_6$.

Therefore, the second half follows from Definition 3.

We believe the requirements on schedules of $\operatorname{KAF}_P^{f, \gamma}$ are more relaxed than that required by $\operatorname{KAF}_P^{f, \gamma}$ (Corollary 1), since its condition (2) only requires to carefully design $\gamma_2$ and $\gamma_5$, without considering the more complicated interactions between different round-KDFs (as required by the second condition in Definition 3). In particular, when designing affine key-schedules in practice, one tends to choose invertible matrices for $M_0, \ldots, M_t+1$ in order to ensure the largest possible amount of entropy in the round-keys, e.g. the bit-permutation-based key-schedules (which was used in DES). In this case, condition (2) is naturally satisfied, yet the second condition in Definition 3 may not be satisfied! (And when $M_1 \cdot k$ and $M_3 \cdot k$ define two bit-permutations, the latter condition is indeed not satisfied since $M_1 \cdot \Delta = M_3 \cdot \Delta = \Delta$ for $\Delta = 0x\text{FF} \ldots \text{FF}$. This matches the fact that DES is vulnerable to complementing attacks.)

Finally, we remark that whitening keys play a crucial role in the transformation Eq. (61). This means $\operatorname{KAF}_P$—as well as the Lucifer-like model—cannot be precisely captured by $\operatorname{KAF}_P$, the variant of $\operatorname{KAF}_P$ without whitening keys.

VI. TOWARDS MINIMALISM

For better efficiency when used as permutation modes, we make discussion on the theoretically “minimal” constructions derived from our general results. We focus on $\operatorname{KAF}_P^{f, \gamma}$ since it’s of the most general interest, and it’s also wlog, since minimal $\operatorname{KAF}_P^{f, (w, \gamma)}$ and $\operatorname{KAF}_P^{f, \gamma}$ schemes can be easily derived in a similar vein.

First, for the 4-round $\operatorname{KAF}_P^{f, \gamma}$, $\gamma_1(k) = M_1 \oplus k + k^3, \gamma_2(k) = \gamma_3(k) = 0$, and $\gamma_4(k) = M_4 \oplus k + k^3$ is a group of good choices, where $M_1 \neq M_3$ are two non-zero constants in $\{0, 1\}^n$, and $\otimes$ denotes multiplications taken over the finite field $\mathbb{F}_{2^q}$. With this choice, it can be seen the four parameters defined in Definition 1 are such that $\delta_1 \leq 3, \delta_2 \leq 2, \delta_3 = 1$, and $\delta_4 \leq 2$, and the concrete advantage bound is a classical birthday one $\frac{1}{\sqrt{q_1 - q_2^3} - k_2}$.

Our choice of $\gamma_1$ and $\gamma_4$ is motivated by $\oplus$-RKA secure almost xor-universal hash functions introduced by Wang et al. [82]. On the other hand, since no requirement is placed on $\gamma_2$ and $\gamma_3$ (see Corollary 1 or Definition 1), they are completely absent: this matches the existing result that the two middle round-functions of 4-round Feistel do not need be secret/"protected" by round-keys [83]. This $\operatorname{KAF}_P^{f, \gamma}$ variant seems “minimal” in the sense that removing any component harms security: reducing rounds ruins CCA security, choosing $M_1 = M_3$ introduces the weakness $\operatorname{KAF}_P^{f, \gamma}(LR) = ST \iff \operatorname{KAF}_P^{f, \gamma}(TS) = RL$ and allows trivially distinguishing, while reducing the non-linearity of KDFs would introduce related-key differentials with higher probability and compromise the concrete security.

Second, for 6-round $\operatorname{KAF}_P^{f, \gamma}$, it can be seen the key-schedule $k \mapsto (k, 0, 0, 0, 0, \pi(k))$ is sufficient, where $\pi$ is a linear oromorphism. It may be quite hard to believe many carefully designed sophisticated key-schedules (e.g. DES) are insufficient to prevent complementing attacks, while such an exotic design should be good. The reason is that the absence of the 3rd and 4th round-keys incidently prevents the occurrence of complementation properties.

We stress that the key-schedule instances mentioned in this discussion, which have a lot of “blanks”, are only for theoretically minimalism, not for general purpose Feistel ciphers.

For the latter purpose, one could (actually, should) “fill in the blanks”. For example, if $\pi$ is the linear oromorphism $\pi(k_L, k_R) = k_R \oplus k_L$ mentioned in the Introduction, then it can be seen $k \mapsto (k, k, \pi(k), k, k, \pi(k))$ is a good key-schedule for 6-round $\operatorname{KAF}_P^{f, \gamma}$

VII. CONCLUSION

We’ve studied provable security of key-alternating Feistel/Feistel-2 variants against $\oplus$-induced related-key attacks, which better model the reality of Feistel blockciphers. Assuming key-schedules being non-linear or purely affine, we identify (different) conditions on the key-schedules that are sufficient for a birthday-type security up to $2^{n/2}$ queries. The results and implications make a step towards understanding the behaviors of existing different Feistel cipher structures, and offer new insights.
Appendix A
Lucifer-like Model and $\text{KAF}_V$

The Lucifer-like model $\text{Luc}$ also relies on the round transformation $\Psi_k$ in Eq. (60). With this, a $t$-round $\text{Luc}$ model built upon $t$ round-functions $f_1, \ldots, f_t$ uses $t$ round-keys $k_1, \ldots, k_t$, and is

$$\text{Luc}_{k_1, \ldots, k_t}(W) = \tilde{\Psi}_t \circ \cdots \circ \tilde{\Psi}_1(W).$$  \hfill (62)

From section V-B we know a $(t-2)$-round $\text{KAF}_V$ uses $t-2$ round-functions $f_1, \ldots, f_{t-2}$ and $t$ sub-keys $k_1, \ldots, k_t$:

$$\text{KAF}_{V_{k_1, \ldots, k_t}}(W) = (k_t||0) \oplus \tilde{\Psi}_{k_2-1} \circ \cdots \circ \tilde{\Psi}_1 (0||k_1) \oplus W.$$

By these, it's not hard to see when $t \geq 2$,

$$\text{Luc}_{f_1, \ldots, f_t}(W) = \tilde{\Psi}_t \circ \text{KAF}_{f_2, \ldots, f_{t-1}} \circ \tilde{\Psi}_0(W),$$

where $\tilde{\Psi}_0$ and $\tilde{\Psi}_t$ are two keyless permutations that can be freely evaluated by the adversary. It can be seen within a large range, any ACA attack $A$ on $(t-2)$-round $\text{KAF}_V$ can be turned into a ACA attack $\mathcal{A}'$ on $t$-round $\text{Luc}$: whenever $A$ queries $\text{RK}[\text{KAF}_{V_{k_1, \ldots, k_t}}](\Delta, LR)$, $\mathcal{A}'$ queries $\text{RK}[\text{Luc}_{f_1, \ldots, f_t}](\Delta, \tilde{\Psi}_0^{-1}(LR))$; whenever $A$ queries $\text{RK}[\text{KAF}_{V_{k_1, \ldots, k_t}}](\Delta, ST)$, $\mathcal{A}'$ queries $\text{RK}[\text{Luc}_{f_1, \ldots, f_t}](\Delta, \tilde{\Psi}_0^{-1}(ST))$. The formal characterization is out of the scope of this paper.

Appendix B
Complementing Attacks

We don't claim novelty for these attacks, see [36]. We just include them to help understanding our provable results. We focus on $\text{KAF}_w$ variants with key-schedules that do not satisfy condition (2) in Definition 3. We first briefly describe how to break more than 4 rounds, then describe the attack against any number of rounds for "bad enough" key-schedules.

On 5 Rounds. Consider a 5-round key-schedule $(w, \gamma)$, where $w = (w_0, w_1, w_2, w_3)$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5)$, and $\gamma$ is such that $M_0 \cdot \Delta = M_3 \cdot \Delta$ for a non-zero value $\Delta$. Then there exists a 1-round related-key differential for the 3rd round, i.e.

$$\Pr \left( M_2 \cdot \Delta \| M_1 \cdot \Delta \xrightarrow{\varphi_{f_3} / \Delta} M_1 \cdot \Delta \| M_2 \cdot \Delta \right) = 1.$$  \hfill (1)

Concatenating this differential with the mentioned 2-round related-key differential Eq. (33) gives a 3-round differential:

$$\Pr \left( \varphi_{f_3} / \Delta \circ \varphi_{f_2} / \Delta \circ \varphi_{f_1} / \Delta \circ \text{XOR}_{w_0} \circ w_1 / \Delta \xrightarrow{\Delta} M_1 \cdot \Delta \| M_2 \cdot \Delta \right) = 1,$$

where $\varphi_{f_i} = (M_0^{(w_i)} \oplus M_2) \cdot \Delta$, $\varphi_{f_i} = (M_0^{(w_i)} \oplus M_1) \cdot \Delta$. Further concatenating this differential with the 2-round related-key differential Eq. (39) yields a 5-round related-key boomerang distinguisher, which allows distinguishing 5 rounds with 2 queries.

On Any Rounds. Consider a $2t$-round schedule $(w, \gamma)$ with $w = (w_0, w_1, w_2, w_3)$ and $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_{2t})$, and $\gamma$ satisfies the following condition: it's easy to derive $\Delta \neq 0$ such that

- $\Delta_1 = M_1 \cdot \Delta = M_3 \cdot \Delta = M_5 \cdot \Delta = \cdots = M_{2t-1} \cdot \Delta$;
- $\Delta_2 = M_2 \cdot \Delta = M_4 \cdot \Delta = M_6 \cdot \Delta = \cdots = M_{2t} \cdot \Delta$.

Then it can be seen there exists related-key differentials with any number of rounds:

$$\Pr \left( \varphi_{f_1} / \Delta \circ \text{XOR}_{w_0} \circ w_1 / \Delta \xrightarrow{\Delta} \varphi_{f_2} / \Delta \circ \text{XOR}_{w_2} \circ w_3 / \Delta \circ \varphi_{f_1} / \Delta \circ \text{XOR}_{w_0} \circ w_1 / \Delta \xrightarrow{\Delta} \varphi_{f_2} / \Delta \circ \text{XOR}_{w_2} \circ w_3 / \Delta \circ \varphi_{f_1} / \Delta \circ \text{XOR}_{w_0} \circ w_1 / \Delta \xrightarrow{\Delta} \cdots \right) = 1,$$

where $\varphi_{f_i} = (M_0^{(w_i)} \oplus M_2) \cdot \Delta$, $\varphi_{f_i} = (M_0^{(w_i)} \oplus M_1) \cdot \Delta$, the output difference $\delta = \Delta_2 \oplus \Delta_1 \cdot \Delta_2 \oplus \Delta_3$, $\Delta$ when $t$ is even, and $\delta = \Delta_1 \oplus M_3^{(w_3)} \oplus \Delta_2 \oplus M_2^{(w_2)} \cdot \Delta$ otherwise. This allows distinguishing any $t$ rounds with 2 queries. To save space we omit detailed descriptions of these two (innovative) variants of complementing attacks.

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