Infinite loop space structure(s) on the stable mapping class group

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Abstract

Tillmann introduced two infinite loop space structures on the plus construction of the classifying space of the stable mapping class group, each with different computational advantages [11, 13]. The first one uses disjoint union on a suitable cobordism category, whereas the second uses an operad which extends the pair of pants multiplication (i.e. the double loop space structure introduced by E. Y. Miller [8]). She conjectured that these two infinite loop space structures were equivalent, and managed to prove that the first delooping are the same. In this paper, we resolve the conjecture by proving that the two structures are indeed equivalent, exhibiting an explicit geometric map.

1 Introduction

Let $\Gamma_{g,n}$ be the mapping class group of a surface $F$ of genus $g$ with $n$ boundary components. The classifying space $B\Gamma_{g,n}$ has the homotopy type of the moduli space of Riemann surface of type $F$ when $n > 0$. Attaching a torus with two boundary components to the surface induces a homomorphism $\Gamma_{g,1} \to \Gamma_{g+1,1}$. Let $\Gamma_{\infty} = \lim_{g \to \infty} \Gamma_{g,1}$ denote the stable mapping class group.

The space $Z \times B\Gamma_{\infty}^+$, the group completion of $\prod_{g \geq 0} B\Gamma_{g,1}$, has a natural double loop space structure induced by the pair of pants multiplication on $\prod_{g \geq 0} B\Gamma_{g,1}$ (see also [2]). In [3] Tillmann constructed an infinite loop space operad extending the pair of pants multiplication and acting on $\prod_{g \geq 0} B\Gamma_{g,1}$, thus showing that the pair of pants multiplication actually induces an infinite loop space structure on $Z \times B\Gamma_{\infty}^+$. This multiplication plays a role in conformal field theory [10], and has also proven useful for e.g. constructing homology operations [3]. Previously [1] she had exhibited $Z \times B\Gamma_{\infty}^+$ as an infinite loop space in quite a different way, by constructing a cobordism category $\mathcal{S}$, symmetric monoidal under disjoint union of surfaces, such that $\Omega \mathcal{S} \simeq Z \times B\Gamma_{\infty}^+$. Note that the multiplication inducing the infinite loop space structure in this case is defined on $BS$, and hence on a first deloop of $Z \times B\Gamma_{\infty}^+$. This infinite loop
space structure has also proven useful. Madsen-Tillmann [5] have constructed an infinite loop map from $\mathbb{Z} \times B\Gamma_{\infty}^\infty$, with the disjoint union infinite loop space structure, to $\Omega^\infty \mathbb{C}P_{\infty}^\infty$. This map has lead recently to a proof of the Mumford conjectured (announced by I. Madsen and M. Weiss).

Tillmann conjectured [11, 3, 13] that two infinite loop space structures were equivalent and managed to prove in [11] that their first deloopings are homotopy equivalent spaces. In this paper, we resolve the conjecture of Tillmann by proving that the two structures are indeed equivalent, exhibiting an explicit geometric map. Our map sends, up to homotopy, the pair of pants multiplication to the loop on disjoint union multiplication. We show that this map preserves all higher homotopies, using the machinery of Dwyer and Kan [4], and hence produce an infinite loop map, which gives the equivalence.

To describe our result in more details, we have to introduce some notation. Let $M$ denote Tillmann’s operad and let $M$ be the associated monad. We will describe this operad in detail in Section 3. The operad has $n$th space $M(n) \simeq \bigsqcup_{g \geq 0} B\Gamma_{g,n+1}$. So $M(\ast) = M(0) \simeq \bigsqcup_{g \geq 0} B\Gamma_{g,1}$ and it is an $M$-algebra. The cobordism category $S$, described in Section 4, has objects the natural numbers. The morphism space $S(n,m)$ is the classifying space of a category with objects disjoint union of surfaces with a total of $n$ incoming and $m$ outgoing boundaries, and with morphisms the appropriate mapping class groups (see Fig. 3). The category $S$ is defined in such a way that $S(n,1) = M(n)$. So an object of $M(\ast)$ is a morphism from 0 to 1 in $S$, and thus defines a 1-simplex in $BS$. Hence there is a natural map

$$\phi : M(\ast) \times M(\ast) \rightarrow \Omega BS$$

as two elements of $S(0,1)$ define a loop in $BS$. We use Barratt and Eccles’ method to give the spectra of deloops explicitly, obtaining two sequences of simplicial spaces with space of $p$-simplices $E_p^i = G\Gamma(S^i \land M^p((M(\ast) \times M(\ast))_+))$ and $F_p^i = G\Gamma(S^{i-1} \land \Gamma^p(BS))$, where $\Gamma$ is the $E_\infty$-operad with $\Gamma(k) = E\Sigma_k$, $G$ is the group completion, and $M^p$ (resp. $\Gamma^p$) means the functor iterated $p$ times. There is a map of operads $M \rightarrow \Gamma$.

**Theorem 1.1.** The adjoint of the map $\phi : M(\ast) \times M(\ast) \rightarrow \Omega BS$ and the operad map $M \rightarrow \Gamma$ induce maps

$$f_p^i : E_p^i = G\Gamma(S^i \land M^p((M(\ast) \times M(\ast))_+)) \rightarrow F_p^i = G\Gamma(S^{i-1} \land \Gamma^p(BS))$$

for $i \geq 1$ and $p \geq 0$, which can be rectified into an equivalence of spectra

$$(f')^i : (E')^i \simeq (F')^i,$$

where $E'$ and $F'$ are spectra equivalent to $E$ and $F$ respectively.

The maps $f_p^i$ are almost simplicial maps in the sense that they satisfy all the simplicial identities except for $\delta_p f_p^i$ which is only homotopic to $f_{p-1}^i \delta_p$. The map $f'$ is a rectification of $f$ in the sense that the equivalence $E' \simeq E$ and $F' \simeq F$ is natural with respect to $f$ and $f'$. 

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In Section 2, we describe the method of rectification of diagrams which will be used in the proof. In Section 3, we give the construction of the operad $\mathcal{M}$, following [13], spelling out the details needed further on in the text and correcting a minor mistake. We also describe the spectrum of deloops of $\mathbb{Z} \times B\Gamma^+_\infty$ produced by $\mathcal{M}$. In Section 4, we give a description of the category $\mathcal{S}$, adapted to our needs, and produce an actual map inducing the equivalence $\Omega BS \simeq \mathbb{Z} \times B\Gamma^+_\infty$. In the appendix, we related this map to Tillmann’s original proof of this equivalence. In Section 5, we compare the two infinite loop space structures: we construct the map, rectify it, and show that it induces an equivalence of spectra.

We will be working most of the time in $\text{Top}_*$, the category of pointed topological spaces. Most of our spaces are realization of pointed simplicial spaces. We will use the notation $X_\bullet$ for a simplicial space and $X = |X_\bullet|$ for its realization.

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## 2 Rectification of diagrams

Suppose we have a diagram of spaces and maps (of whatever shape, possibly infinite) which commutes only up to homotopy. If there are higher homotopies, it is possible to rectify it to a strictly commutative diagram, equivalent to the one we started with (in a sense to be made precise). We give here a method which is a special case of a theory treated by Dwyer and Kan in [4]. The same construction was used by Segal in [9]. The idea is to look at a commutative diagram as a functor from a discrete category $\mathcal{D}$ to $\text{Top}_*(\ast)$, the category of (pointed) topological space, and a homotopy commutative diagram as a functor from a category $\tilde{\mathcal{D}}$ to $\text{Top}_*(\ast)$, where the spaces of maps in $\tilde{\mathcal{D}}$ are “thicker” than in $\mathcal{D}$ (see Fig. 1 for the case of a square). As long as the morphism spaces in $\tilde{\mathcal{D}}$

![Figure 1: Commutative and homotopy commutative square](image)

are homotopy equivalent to the corresponding ones in $\mathcal{D}$, a rectification can be
constructed.

We give here a precise description of the rectification in the unpointed case and prove a strong naturality statement (Proposition 2.1), as we need to know more about the rectification than what can be found in [4] or [9]. The pointed case is done similarly.

Let $D$ be a discrete category and let $\hat{D}$ be a category enriched over $\text{Top}$ with the same objects as $D$ and such that there is a functor (path components functor)

$$ p : \hat{D} \to D, $$

which is the identity on objects and induces a homotopy equivalence

$$ \hat{D}(x, y) \simeq D(x, y) $$

for each pair of objects $x, y$. So $\hat{D}$ has a contractible space of morphisms over each morphism in $D$ and $p$ is the projection. There is an induced functor

$$ \text{Top}^D \xrightarrow{p^*} \text{Top}^{\hat{D}}, $$

from $D$\textit{-diagrams} to $\hat{D}$\textit{-diagrams}. There is also a functor in the other direction:

$$ \text{Top}^D \xleftarrow{p_*} \text{Top}^{\hat{D}}, $$

where $p_*F$ is defined on an object $x$ of $D$ as the realization of a simplicial space whose $n$th space is

$$ (p_*F)(x)_n = \prod_{y_0, \ldots, y_n \in \text{Ob} \hat{D}} F(y_0) \times \hat{D}(y_0, y_1) \times \ldots \times \hat{D}(y_{n-1}, y_n) \times D(p(y_n), x). $$

Two functors $F$ and $G$ are said to be \textit{equivalent}, denoted $F \simeq G$, if there is a zig-zag of natural transformations $F \leftarrow F_1 \rightarrow \ldots \leftarrow F_k \rightarrow G$ which induces homotopy equivalences on objects.

\textbf{Proposition 2.1.} There is an equivalence of functors

$$ p^*p_*F \simeq F $$

for any $F$ in $\text{Top}^{\hat{D}}$, which is natural in $F$.

As $D$ and $\hat{D}$ have the same objects, this means in particular that $p_*F(x) \simeq F(x)$ for any object $x$. The functor $p_*F$ is the \textit{rectification} of $F$.

\textbf{Proof.} To prove the Proposition, we will give an explicit sequence of natural transformations giving the equivalence and show that they are moreover natural with respect to $F$.

For a functor $F : \hat{D} \to \text{Top}$, define the functor $p^*p_*F$ from $\hat{D}$ to $\text{Top}$ simplicially by

$$ (p^*p_*F)(y)_n = \prod_{y_0, \ldots, y_n \in \text{Ob} \hat{D}} F(y_0) \times \hat{D}(y_0, y_1) \times \ldots \times \hat{D}(y_{n-1}, y_n) \times \hat{D}(y_n, y). $$

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Then there are natural transformations

\[ p^*p_*F \leftarrow \overline{p^*p_*F} \rightarrow F \]

inducing equivalences \( p^*p_*F(y) \simeq \overline{p^*p_*F}(y) \simeq F(y) \), for all \( y \) in \( \mathcal{D} \). The natural transformation \( \overline{p^*p_*F} \rightarrow p^*p_*F \) is induced by the projection functor \( \mathcal{D} \rightarrow \mathcal{D} \) which is a homotopy equivalence on the space of morphisms, so it clearly induces an equivalence. Now \( \overline{p^*p_*F} \) is of the form of a two-sided bar construction \( B(F,D,DX) \). May gives an explicit simplicial homotopy for the equivalence \( B(F,D,DX) \simeq FX \) ([3], Proposition 9.9). It can be adapted to our case.

Indeed, consider the inclusion \( i: F(y) \hookrightarrow \overline{p^*p_*F}(y) \) defined by \( a \mapsto (a,id_y) \) in \( F(y) \times \mathcal{D}(y,y) \), and the obvious evaluation map \( d: \overline{p^*p_*F}(y) \rightarrow F(y) \). Clearly, \( d \circ i = id \). We are left to show that \( i \circ d \) is homotopic to the identity. The simplicial homotopy is given explicitly on \( q \)-simplices by

\[ h_i = s_i \ldots s_{i+1} \circ \eta \circ \delta_{i+1} \ldots \delta_q, \]

for \( i = 0, \ldots, q \), where \( \eta: \overline{(p^*p_*F)} \rightarrow (p^*p_*F)_{i+1} \) is defined by adding \( id_x \in \mathcal{D}(x,x) \) on the right of the simplex.

The first natural transformation is clearly natural in \( F \). The second is natural in \( F \) as the diagram

\[
\begin{array}{ccc}
F(y_0) \times \mathcal{D}(y_0,y) & \longrightarrow & F(y) \\
\downarrow & & \downarrow \\
F'(y_0) \times \mathcal{D}(y_0,y) & \longrightarrow & F'(y)
\end{array}
\]

commutes because a map of functors is itself a natural transformation. \( \square \)

Note that the inclusion \( i: F(y) \hookrightarrow \overline{p^*p_*F}(y) \) is not natural in \( y \).

3 The mapping class groups operad

In this section, we describe Tillmann’s operad \( \mathcal{M} \), correcting a minor mistake from [2] in the construction. We also give explicitly the spectrum of deloops of \( \mathbb{Z} \times B\Gamma_\infty^+ \) produced by \( \mathcal{M} \).

Let \( F_{g,n+1} \) denote an oriented surface of genus \( g \) with \( n+1 \) boundary components. One of the boundary components is marked; we call the \( n \) other components free. Each free boundary component \( \partial_i \) comes equipped with a collar, a map from \( [0,\epsilon) \times S^1 \) to a neighborhood of \( \partial_i \); for the marked boundary component, there is a map from \( (\epsilon,0] \times S^1 \) to a neighborhood of the boundary. Let \( \text{Diff}^+ (F_{g,n+1};\partial) \) be the group of orientation preserving diffeomorphisms which fix the collars, and let

\[ \Gamma_{g,n+1} = \pi_0(\text{Diff}^+ (F_{g,n+1};\partial)) \]
be its group of components, the associated *mapping class group*.

We want to construct a topological operad $\mathcal{M}$ with space of $k$-ary operations

$$\mathcal{M}(k) \simeq \coprod_{g \geq 0} B\Gamma_{g,k+1}$$

and composition maps induced by gluing surfaces. To make gluing associative, one has to replace the groups $\Gamma_{g,k+1}$ by equivalent groupoids.

### 3.1 Construction of the operad

Pick a disc $D = F_{0,1}$, a pair of pants surfaces $P = F_{0,3}$ and a torus $T = F_{1,2}$ with two boundary components, all with fixed collars of the boundary components (see Fig. 2). Define a groupoid $E_{g,n,1}$ with objects $(F,\sigma)$, where $F$ is a surface of type $F_{g,n}+1$ constructed from $D$, $P$, and $T$ by gluing the marked boundary of one surface to one of the free boundaries of another using the given parametrization, and $\sigma$ is an ordering of the $n$ free boundary components (see Fig. 2). Note that each boundary component of $F$ comes equipped with a collar. The morphisms from $(F,\sigma)$ to $(F',\sigma')$ are the homotopy classes $\Gamma(F,F') = \pi_0\text{Diff}^+(F,F';\partial)$ of orientation preserving diffeomorphisms preserving the collars and the ordering of the boundaries. The group $\Sigma_n$ acts freely on $E_{g,n,1}$ by permuting the labels and $B E_{g,n,1} \simeq B\Gamma_{g,n+1}$.

**Figure 2: Building blocks of $E_{g,k,1}$ and element of $E_{2,3,1}$**

Gluing of surfaces induces now an associative operation on the categories. Hence we have maps on the classifying spaces

$$\gamma : BE_{g,k,1} \times BE_{h_1,n_1,1} \times \ldots \times BE_{h_k,n_k,1} \to BE_{g+h_1+\ldots+h_k,n_1+\ldots+n_k,1}$$

induced by gluing the $k$ last surfaces to the first one according to the labels of its free boundaries. These maps are associative and $\Sigma$-equivariant. However, $\{\coprod_{g \geq 0} B E_{g,n,1}\}_{n \in \mathbb{N}}$ does not precisely form an operad yet as there is no unit. We will apply a quotient construction on the categories $E_{g,n,1}$ which will both provide a unit and make the product induced by the pair of pants associative and unital.

#### 3.1.1 Quotient construction

To make the multiplication induced by the pair of pants associative, we need to identify subsurfaces of the form $\gamma(P;\_\_, P)$ to subsurfaces of the form $\gamma(P;P;\_\_)$.
For the unit, we need to identify $\gamma(P; D)$ and $\gamma(P; D, \gamma)$ to a circle. This circle will also be a unit for the operad. In [13], Tillmann does a quotient construction by picking morphisms $\varphi_1 : \gamma(P; D, \gamma) \to \gamma(P; D)$ and $\varphi_2 : \gamma(P; P, P) \to \gamma(P; P, \gamma)$ and uses composition of these morphisms to identify the surfaces. She then chooses an identification of $\gamma(P; D)$ to the circle and repeats the process. This is not precisely correct as any choice of $\varphi_1, \varphi_2$ would not yield associative operad maps on the quotient categories. It will work only with the canonical choice which is the “identity”. We prove here that this canonical choice exists and that it makes the quotient construction possible. We do both quotient constructions at once.

Claim 1. For each object $F$ in $E_{g,n,1}$, there is a unique object $F'$ in $S_{g,n,1}$ obtained from $F$ by a sequence of the following operations: replacing a subsurface $\gamma(P, P)$ by $\gamma(P, P, P)$, and collapsing a subsurface of the form $\gamma(P, D)$ or $\gamma(P, D, \gamma)$ to a circle. In particular, the gluing operation on objects of $\prod S_{g,n,1}$ defined by $F \boxtimes G := F \boxtimes G$, using the gluing $\Box$ defined on $E_{g,n,1}$, is associative.

Claim 2. For each $F$ in $E_{g,n,1}$, one can define a morphism $\phi_F : F \to F'$ in $E_{g,n,1}$, such that all diagrams of the form

\[
\begin{array}{ccc}
F \Box G \Box H & \xrightarrow{\phi_F \Box \text{id}} & F' \Box G \Box H \\
\downarrow \phi_F \Box \varphi \downarrow & = & \downarrow \varphi \Box \varphi \\
F \Box G \Box H & \xrightarrow{\phi_F \Box \text{id}} & F' \Box G \Box H
\end{array}
\]

commute.

Proof of Claim 1. Consider first an element $(F, \sigma)$ of $E_{0,n,1}$, i.e. a surface $F$ of genus 0 together with a labeling $\sigma$ of the free boundary components ($\sigma$ is a permutation of the canonical labeling). The operations allowed do not change the number of free boundaries, nor does it permute the boundaries. In $S_{0,n,1}$ there is only one surface with labeling $\sigma$, and this surface can clearly be obtained from $F$ by a finite sequence of the prescribed moves. This surface is $F'$.  

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For a general surface $F$ with labeling $\sigma$, the moves only affect the subsurfaces of $F$ built out of $P$’s and $D$’s. Each of the maximal such subsurface has a unique image in the relevant $S_{g,n,1}$. So $\overline{F}$ is the unique surface obtained by transforming each of those subsurfaces of $F$.

Finally, gluing as defined in the Claim is associative by the uniqueness of the representative of $F \boxdot G \boxdot H$ in $S_{g,n,1}$. □

Proof of Claim 2. Fix three non-intersecting curves on the pair of pants $P$, from 0 of the marked boundary to 0 of the first free boundary, from $\pi$ of the this boundary to 0 of the second free boundary, and from $\pi$ of this boundary to $\pi$ of the marked boundary (where we think of $S^1$ as parametrized by $[0, 2\pi]$). This divides the pair of pants into two discs. Fix also a curve on the disc $D$, from 0 to $\pi$ of its boundary.

Now any surface built out of $P$’s and $D$’s comes equipped with a system of curves dividing the surface into two discs. These curves run from 0 of the marked boundary to 0 of the first (in the canonical ordering) free boundary, then from $\pi$ of that boundary to 0 of the next, and so on until one goes back to $\pi$ of the marked boundary. Choose a map $F \to \overline{F}$ which sends the curves of $F$ to the corresponding ones in $\overline{F}$. As the curves divide the surfaces into discs, by the “Alexander trick” this map is unique up to isotopy. It is, up to isotopy, the identity on the discs. Now define $\phi_F$ to be the component of this map in $\text{Diff}^+ (F, \overline{F})$.

For a general surface $F$, define $\phi_F$ to be the map defined by the above on each maximal subsurface of $F$ built out of $P$’s and $D$’s, and the identity on the tori. The diagram in the Claim commutes by the Alexander trick. □

Now one can define an operad structure on the categories $S_{g,n,1}$. The new structure maps $\gamma$ are defined on objects by taking the unique representative of the image of $\gamma$ on the surfaces:

$$\gamma(F, G_1, \ldots, G_k) = \overline{\gamma(F, G_1, \ldots, G_k)}.$$ 

On morphisms,

$$\overline{\gamma(f, g_1, \ldots, g_k)} = \phi_{H'} \gamma(f, g_1, \ldots, g_k) \phi_H^{-1},$$

where $H$ and $H'$ are the images by $\gamma$ of the sources and target surfaces of the maps $f$ and $g_i$. The associativity of $\gamma$ on objects follows from the associativity of gluing. On morphisms, it follows from the commutativity of the diagram in Claim 2.

Define the operad $\mathcal{M}$ by

$$\mathcal{M}(k) = \bigsqcup_{g \geq 0} BS_{g,k,1}$$

with structure maps induced by $\overline{\gamma}$. Note that there is a map of operads

$$\mathcal{M} \xrightarrow{\sigma} \Gamma,$$

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3.2 Infinite loop space structure

Let $G$ denote the group completion functor from the category of monoids to the category of groups and let $(M, \mu_M, \eta_M)$ denote the monad associated to the operad $M$. The pair of pants multiplication induces a monoid structure on $M$-algebras. $M(\ast)$ is an $M$-algebra and

$$GM(\ast) \cong G(\prod_{g \geq 0} B\Gamma_{g,1}) \cong \mathbb{Z} \times B\Gamma_\infty^+.$$ 

Tillmann showed that if $X$ is an $M$-algebra, $G(X)$ is weakly homotopy equivalent to an infinite loop space. In particular, $GM(\ast) \cong \mathbb{Z} \times B\Gamma_\infty^+$ is an infinite loop space. The $i$th deloop of the group completion of an $M$-algebra $X$ obtained in $G[M]$ are defined as the simplicial space with $G\mathcal{F}(S^n \wedge M^p(M(\ast) \times X))$ as space of $p$-simplices, where $G\mathcal{F}(Y)$ is the fibre of the map $GM(Y) \to GM(\ast)$. As $G\mathcal{F}(X) \cong G\Sigma(X)$, one can show that it is equivalent to the simplicial space with $p$-simplices

$$E_p^i = G\Sigma(S^i \wedge M^p((M(\ast) \times M(\ast))_+))$$

and that this equivalence induces an equivalence of spectra. We work with an added basepoint for technical reasons when constructing the map (this does not bring any new component because of the group completion).

We describe next the simplicial structure on $E^\bullet$. Let $M, \mu_M$ and $\mu_G, \eta_G$ denote the product and unit maps of the monads $M$ and $G$ and let $\theta$ denote the $G$-algebra structure map of $(M(\ast) \times M(\ast))_+$. Let $\pi : M \to G$ be the projection of operads.

For any operad $\mathcal{P}$, there is an assembly map $a : A \times P(X) \to P(A \times X)$, sending an element $(a, p, x_1, \ldots, x_k)$ to $(p, (a, x_1), \ldots, (a, x_k))$. As $\Gamma(\ast) = \{\ast\}$, the sequence of maps

$$\Gamma(A \times M(X)) \xrightarrow{a} \Gamma(M(A \times X)) \xrightarrow{\pi} \Gamma\Sigma(A \times X) \xrightarrow{\mu_G} \Gamma(A \times X)$$

induces a map on smash products

$$\lambda : \Gamma(A \wedge M(X)) \longrightarrow \Gamma(A \wedge X).$$

The simplicial structure on $E^i$, $i \geq 0$, is defined as follows:

$$\delta_0 = G(\lambda) : E_p^i \to E_p^{i+1};$$
$$\delta_i = G\Sigma(S^i \wedge M^{i-1}(\mu_M)) : E_p^i \to E_p^{i+1} \quad \text{for } 1 \leq i < p;$$
$$\delta_p = G\Sigma(S^i \wedge M^{p-1}(\theta)) : E_p^i \to E_p^{i+1};$$
$$s_i = G\Sigma(S^i \wedge M^i(\eta_M)) : E_p^i \to E_p^{i+1} \quad \text{for } 0 \leq i \leq p.$$

Let $\mathcal{F} = F_0 \leftarrow \cdots \leftarrow F_q$ and $\mathcal{G} = G_0 \leftarrow \cdots \leftarrow G_q$ denote elements of $M(\ast)$. Let $D$ denote the $0$-simplex represented by the disc.
Proposition 3.1. Let \( \phi : M(*) \rightarrow |G\Gamma((M(*) \times M(*))_+)| \) be the map which sends \( F \) to the 0-simplex \( (1,D,F) \in G\Gamma((M(*) \times M(*))_+) \). Then there is a commutative diagram

\[
\begin{array}{ccc}
M(*) & \xrightarrow{\phi} & |G\Gamma((M(*) \times M(*))_+)| \\
\downarrow & & \downarrow \cong \\
G\mathcal{M}(*) & & 
\end{array}
\]

This can be proved by studying the map of fibrations which Tillmann uses to prove the equivalence \( \mathcal{G}X \simeq \mathcal{G}\mathcal{F}(M^\bullet(M(*) \times X)) \).

4 The cobordism category

In this section, we first set up a variant of Tillmann’s cobordism category \( \mathcal{S} \). In \([12]\), the morphism spaces are categories similar to the categories \( \mathcal{E}_{g,n,1} \) defined in Section 3.1. Our version of \( \mathcal{S} \) is obtained by applying the quotient construction of Section 3.1.1 to Tillmann’s \( \mathcal{S} \). This version is more amenable to the comparison of the two infinite loop space structures. We then construct an explicit equivalence \( \mathbb{Z} \times BT^+_{\infty} \hat{\longrightarrow} \Omega BS \). We show in the appendix how this proof relates to Tillmann’s proof.

The objects of \( \mathcal{S} \) are the natural numbers \( 0, 1, 2, \ldots \). The morphism space \( \mathcal{S}(n,m) = BS_{g,n,m} \), where \( S_{g,n,m} \) is a category whose objects are surfaces built out of \( P, T \) and \( D \) as in the case of the operad \( \mathcal{M} \) but allowing disjoint union of surfaces (with a component-wise quotient construction) and labeling both the inputs and the outputs (see Fig. 3). The morphisms of \( S_{g,n,m} \) are homotopy classes of diffeomorphisms preserving the orientation, the collars and the ordering of the boundary components. In particular \( \mathcal{S}(k,1) = \mathcal{M}(k) \). Also, a morphism from \( n \) to \( m \) has exactly \( m \) components. The only morphism to 0 is the identity in \( \mathcal{S}(0,0) \). Note that \( \mathcal{S}(n,n) \) contains the symmetric group, represented by disjoint copies of the circle with labels “on each side”. Composition in \( \mathcal{S} \) is induced by gluing the surfaces according to the labels, which can be done using the structure maps of the operad \( \mathcal{M} \) on each component. Disjoint union

![Figure 3: Morphism from 5 to 2 in \( \mathcal{S} \)](image-url)
of surfaces induces a symmetric monoidal structure on $S$. As $BS$ is connected, it is an infinite loop space.

We use Barratt and Eccles’ machinery \cite{1} to produce the deloops of $\Omega BS \simeq \mathbb{Z} \times B\Gamma_\infty^+$. The space of $p$-simplices of the $i$th deloops is given by

$$F^i_p = \mathcal{G}\Gamma(S^{i-1} \wedge \Gamma^p(\text{BS}))$$

for $i \geq 1$.

The simplicial structure of $F^i \cdot = \mathcal{G}\Gamma(S^{i-1} \wedge \Gamma^\cdot(\text{BS}))$ is similar to the one of $E^i \cdot$, which is given in detail in Section 3.2.

Note that $S(0,1) = M(0) = M(*)$. Recall that the pair of pants multiplication induces a monoid structure on this space and that $\mathcal{G}M(*) = \mathcal{G}(S(0,1))$ denotes its group completion, which is homotopy equivalent to $\mathbb{Z} \times B\Gamma_\infty^+$. In the following Proposition, we use the fact that a morphism in $S$ is a 1-simplex in $BS$, and hence two morphisms from 0 to 1 define a loop in $BS$.

**Proposition 4.1.** Define $\psi : S(0,1) \to \Omega BS$ by $\psi(F) = F^{(1)}_0 D$ is the loop from 0 to 1 along the morphism defined by $F$ followed by the morphism defined by the disc $D$ taken backwards. Then there is a homotopy commutative diagram

$$\begin{array}{ccc}
M(*) = S(0,1) & \xrightarrow{\psi} & \Omega BS \\
\downarrow \simeq & & \\
\mathcal{G}M(*) = \mathcal{G}S(0,1).
\end{array}$$

**Proof.** Consider the diagram

$$\begin{array}{ccc}
S(0,1) & \xrightarrow{\psi} & \Omega BS \\
\downarrow f & & \downarrow = \\
\mathcal{G}S(0,1) & \xrightarrow{\simeq} & \Omega B(S(0,1)) & \xrightarrow{\simeq} & \Omega_{1,1}(BS) & \xrightarrow{h} & \Omega BS
\end{array}$$

where $\Omega_{1,1}(BS)$ denotes the space of paths from 1 to 1 in $BS$. The map $f$ is Quillen’s group completion map, which sends an element $F$ of the monoid $S(0,1)$ to the loop it defines in its classifying space (as a monoid). To define $g$, consider the map $S(0,1) \to S(1,1)$ induced by gluing a pair of pants: $E \to E \bigwedge P$ (glue the surface to the left leg, i.e. compose $E \bigwedge S^1$ with $P$ in $S$). This induces a functor from the monoid $S(0,1)$ to the category $S$. Indeed, one can check that the pairs of pants multiplication is mapped to composition in $S$. Tillmann showed in \cite{2} (Proposition 4.1) that $g$ induces a homotopy equivalence on the classifying spaces. Finally, the map $h$ is defined by precomposing with the path from 0 to 1 (1-simplex) given by the disc and postcomposing by the same path taken backwards.

The diagram commutes up to homotopy. Indeed, starting with $F$ in $S(0,1)$, the loop obtained by following the bottom of the diagram is going along $D$ from
0 to 1, then $F \Box P$ from 1 to 1 and lastly $D$ again backwards from 1 to 0. This path is homotopy equivalent to $\psi(F)$ as $D \circ (F \Box P) = F$ in $S$, which means that there is a 2-simplex in $BS$ providing the required homotopy.

Tillmann’s proof that $\Omega BS \simeq \mathbb{Z} \times B\Gamma^\infty_\infty$ is by showing that $\mathbb{Z} \times B\Gamma^\infty_\infty$ is equivalent to a homotopy fibre which is known to be $\Omega BS$. We will show in the appendix that the map described above is a natural choice in this context to make the equivalence explicit. This leads to a more natural proof of the Proposition.

5 Comparison of the two structures

In this section, we compare the two infinite loop space structures. We first construct in 5.1 the maps $f^i_p : E^i_p \to F^i_p$ of Theorem 1.1. In 5.2, we rectify these maps to simplicial maps between simplicial spaces $(E'_i)^\bullet$ and $(F'_i)^\bullet$ equivalent to $E_i^\bullet$ and $F_i^\bullet$. In 5.3, we show that this rectification provides a map of spectra which in 5.4 is shown to be an equivalence. Theorem 5.2, Theorem 5.7 and Theorem 5.8 combine to prove the main Theorem 1.1.

5.1 Construction of a map

We want to construct a map from $E_i^p = \mathcal{G}(S^i \wedge M^p((M(*)) \times \mathcal{M}(*)))$, the space of $p$-simplices of the $i$th deloop of $\mathcal{G}M(*)$, to $F_i^p = \mathcal{G}(S^{i-1} \wedge \Gamma^p(\mathcal{B} \mathcal{S})))$, the $p$-simplices of the $(i-1)$st deloop of $\mathcal{B} \mathcal{S}$, for $i \geq 1$. By construction, an element of $\mathcal{M}(*)$ is a morphism from 0 to 1 in the category $\mathcal{S}$, and hence a 1-simplex in its classifying space $BS$. In particular, two such elements define a loop in $BS$. The map obtained using this remark is naturally bisimplicial:

![Figure 4: Map from $S^1 \wedge (M(*) \times M(*))_+$ to $BS$](image)

$$\hat{\phi}_{p,q} : S^1_p \times (M_q(*)) \times M_q(*) \to B_{p,q} \mathcal{S},$$
where $S^1$ is viewed as a simplicial space with two 0-simplices $x_0, x_1$ and two non-degenerate 1-simplices $y_1, y_2$ (see Fig. 4), and $BS$ is viewed as a bisimplicial space with the second simplicial dimension coming from the simplicial structure of its morphism spaces.

Define $\tilde{\phi}_{0,q}(x_i, F, G) = i$ for $i = 0, 1$;
$\tilde{\phi}_{1,q}(y_1, F, G) = 0 \xrightarrow{F} 1$;
$\tilde{\phi}_{1,q}(y_2, F, G) = 0 \xrightarrow{G} 1$.

This induces a map $\phi : S^1 \wedge (M(\ast) \times M(\ast))_+ \rightarrow BS$. The base point of $M(\ast) \times M(\ast)$ is $(D, D)$ and its image under $\phi$ is a contractible loop, but it is not actually the trivial loop. This is why we work with $(M(\ast) \times M(\ast))_+$ rather than $M(\ast) \times M(\ast)$.

Recall from [1] that for the monad $\Gamma$ there is an assembly map $a : A \wedge \Gamma(X) \rightarrow \Gamma(A \wedge X)$, defined by $a(y, \sigma, x_1, \ldots, x_n) = (\sigma, [y, x_1], \ldots, [y, x_n])$. This also induces a map $\mathcal{A} \wedge \Gamma(X) \rightarrow \mathcal{G}(A \wedge X)$. Combining $a$, $\phi$ and the operad map $\pi : \mathcal{M} \rightarrow \Gamma$, we get a map

$$
\begin{align*}
\mathcal{G}(S^i \wedge \Gamma^p((M(\ast) \times M(\ast))_+)) & \xrightarrow{f_p^i} \mathcal{G}(S^{i-1} \wedge \Gamma^p(BS)) \\
\mathcal{G}(S^i \wedge \pi^p) & \downarrow \\
\mathcal{G}(S^i \wedge \Gamma^p((M(\ast) \times M(\ast))_+)) & \xrightarrow{\mathcal{G}(S^{i-1} \wedge \Gamma^p(\phi))} \\
\mathcal{G}(S^{i-1} \wedge \Gamma^p(S^1 \wedge (M(\ast) \times M(\ast))_+)).
\end{align*}
$$

**Proposition 5.1.** Let $\delta_j$ and $s_j$ denote the boundary and degeneracy maps of both the simplicial spaces $E^i_\ast$ and $F^i_\ast$. Then the maps $f^i_p : E^i_p \rightarrow F^i_p$ for $i \geq 1$ and $p \geq 0$ satisfy

$$
\delta_j f^i_p = f^i_{p-1} \delta_j \quad \text{for} \quad 0 \leq j < p
$$

and

$$
\delta_j f^i_p = f^i_{p+1} \delta_j \quad \text{for} \quad 0 \leq j < p.
$$

**Proof.** The boundary maps $\delta_j$, for $0 \leq j < p$, and the degeneracies for $E^i_\ast$ and $F^i_\ast$ are defined in terms of the operad structure maps of $\mathcal{M}$ and $\Gamma$. The map $f^i_\ast$ commutes with all of those maps because $f$ maps $\mathcal{M}$ to $\Gamma$ via an operad map.\[\Box\]

The last commutation relation necessary to have a simplicial map, $\delta_p f_p = f_{p-1} \delta_p$, is satisfied only up to homotopy. We show in the next section that it preserves all the higher homotopies and hence that it can be rectified into a simplicial map.
5.2 Rectification of the map

Theorem 5.2. There exist simplicial spaces \((E')^i\) and \((F')^i\), equivalent to the simplicial spaces \(E^i\) and \(F^i\), and a simplicial map \((f')^i: (E')^i \to (F')^i\) such that the following diagram commutes:

\[
\begin{array}{ccc}
E^i_p & \xrightarrow{f_p^i} & F^i_p \\
\downarrow & & \downarrow \\
(E')^i_p & \xrightarrow{(f')^i_p} & (F')^i_p
\end{array}
\]

To prove this theorem, we will use the method described in Section 2. We first need to construct the categories \(D\) and \(\tilde{D}\) relevant to our situation.

Let \(\Delta^{op}\) denote the simplicial category: the objects of \(\Delta^{op}\) are the natural numbers and there are maps \(\delta_i: p \to p - 1\) and \(s_i: p \to p + 1\) for each \(i = 0, \ldots, p\), satisfying the simplicial identities. So a \(\Delta^{op}\)-diagram is a simplicial space. Any morphism in \(\Delta^{op}(p, q)\) can be expressed uniquely as a sequence \(s_{j_t} \cdots s_{j_1} \delta_{i_t} \cdots \delta_{i_1}\) with \(0 \leq i_s < \cdots < i_1 \leq p\) and \(0 \leq j_1 < \cdots < j_t \leq q\) and \(q - t + s = p\).

Let \(D\) be the category whose \(D\)-diagrams are precisely a couple of simplicial spaces with a simplicial map between them (see Fig. 5). So \(D\) has two copies of the natural numbers as set of objects, denoted \(E_p\) and \(F_p\) for \(p \in \mathbb{N}\). The full subcategory of \(D\) containing all the \(E_p\)'s is isomorphic to \(\Delta^{op}\). So \(D(E_p, E_q) = \Delta^{op}(p, q)\). Similarly \(D(F_p, F_q) = \Delta^{op}(p, q)\). Finally, there is a unique map \(f_p \in D(E_p, F_p)\) and it satisfies the simplicial identities \(\delta_i f_p = f_{p-1}\delta_i\) and \(s_i f_p = f_{p+1}s_i\) for \(i = 0, \ldots, p\). So any morphism in \(D\) from \(E_p\) to \(F_q\) can be written uniquely as a sequence \(s_{j_t} \cdots s_{j_1} \delta_{i_t} \cdots \delta_{i_1} f_p\) where the indices are as above.
We now define the category \( \tilde{D} \) in such a way that the data given in \( \Box \) will induce a functor from \( \tilde{D} \) to \( \text{Top}_* \). \( \tilde{D} \) has the same objects as \( D \) and we will again denote them by \( E_p \) and \( F_p \) for \( p \in \mathbb{N} \). Also, \( D(E_p, E_q) = D(E_p, E_q) \) and \( \tilde{D}(E_p, F_q) = \tilde{D}(F_p, F_q) \). To describe \( \tilde{D}(E_p, F_q) \), we first need to define the \textit{degeneracy degree} \( d(g) \) of a morphism \( g \in D(E_p, F_q) \). If \( g = s_{j_1} \ldots s_{j_k} \delta_{i_k} \ldots \delta_{i_1} f_p \) in the above notation, then \( d(g) \) is the biggest \( k \) such that \( i_k = p - k + 1 \), and \( d(g) = 0 \) if no such \( k \) exists. In other words, \( d \) counts the number of “bad” maps, i.e. last boundary maps, occurring in \( g \). Define

\[
\tilde{D}(E_p, F_q) = \coprod_{g \in D(E_p, F_q)} \Delta_{d(g)},
\]

where \( \Delta_d = \{(t_0, \ldots, t_d) \in \mathbb{R}^{d+1} | t_i \geq 0, \Sigma t_i = 1 \} \) is the standard \( d \)-simplex. Note that for each \( g \in D(E_p, F_q) \), there is an inclusion

\[
\tilde{g} : \Delta_{d(g)} \hookrightarrow \tilde{D}(E_p, F_q)
\]

whose image is the space of morphisms “sitting over \( g \”).

Recall that all the simplicial identities between the \( \delta_i \)'s and \( s_j \)'s are satisfied in \( \tilde{D} \). Also, note that there is a unique map \((0\text{-simplex})\) between \( E_p \) and \( F_p \) sitting over the simplicial map \( f_p \). We denote this map again by \( f_p \) and we set the relation \( s_i f_p = f_{p+1} s_i \) for \( 0 \leq i \leq p \) and \( \delta_i f_p = f_{p-1} \delta_i \) for \( 0 \leq i < p \). Because the last relation, when \( i = p \), does not hold in \( \tilde{D} \), there are exactly \( d(g) + 1 \) maps formed of compositions of \( \delta_i \)'s, \( s_j \)'s and an \( f_k \) projecting down to \( g \) in \( D \). We define those compositions to be the vertices of \( \Delta_{d(g)} \). More precisely, for \( 0 \leq k \leq d(g) \), define

\[
g_k := s_{j_k} \ldots s_{j_1} \delta_{i_k} \ldots \delta_{i_1} = \tilde{g}(0, \ldots, 1, \ldots, 0),
\]

where 1 is in the \( k \)th position counting backwards.

Now composition in \( \tilde{D} \) is determined by the vertices of the simplices. A map in \( \tilde{D}(E_p, F_q) \) can only be pre-composed by a map in \( \tilde{D}(E_r, E_p) \) or post-composed by a map in \( \tilde{D}(F_q, F_s) \). Using the identities given above, we know how those compositions are defined on the vertices of the simplices of \( \tilde{D}(E_p, F_q) \). We then extend the composition simplicially.

**Theorem 5.3.** Let \( \tilde{D} \) be the category defined above. For each \( i \geq 1 \), there is a functor

\[
L_i : \tilde{D} \rightarrow \text{Top}_*
\]

such that \( L_i(E_*) = E_i^*, \ L_i(F_*) = F_i^* \), where \( E_* \) and \( F_* \) denote the two subcategories of \( D \) isomorphic to \( \Delta^{op} \), and \( L_i(f_p) = f_p^i \).

**Proof.** As \( E_i^* \) and \( F_i^* \) are simplicial spaces, the restriction of \( L_i \) to each copy of \( \Delta^{op} \) in \( \tilde{D} \) is a well-defined functor. As the map \( f_p^i \) satisfies the identities satisfied by \( f_p \) with the boundary and degeneracy maps, \( L_i \) is also well defined on the
vertices of the simplices of the morphism spaces $\hat{D}(E_p, E_q)$. We have to show that we can extend the definition of $L_i$ to the whole simplices.

Let $d = d(g)$ and $g_k$ be the $k$th vertex of the $d$-simplex over $g$ as described above. To simplify notations, let

$$X = (M(\ast) \times M(\ast))_+.$$ 

Because only the first $d$ last boundaries appear in $g$, one can factorize the map $g_k$ for any $0 \leq k \leq d$ as

$$g_k : E^i_p = \mathcal{G}(S^i \wedge M^p(X)) \xrightarrow{\alpha} \mathcal{G}(S^{i-1} \wedge \Gamma^{p-d}(S^1 \wedge M^d(X)))$$

where $\alpha$ is a composition of $\pi : M \to \Gamma$ and the assembly map, $\beta$ is a composition of boundaries and degeneracies and $A$ is the functor $\mathcal{G}(S^{i-1} \wedge \Gamma^{p-d}(-))$. Both $\alpha$ and $\beta$ are independent of $k$.

We first construct a $d$-simplex of maps $S^1 \wedge M^d(X) \to BS$ having the maps $h_k$ as vertices. For $0 \leq l < k \leq d$, $h_k$ and $h_l$ are given by the two sides of the following diagram:

$$S^1 \wedge M^d(X) \xrightarrow{\delta_{i_k} \ldots \delta_{i_1}} S^1 \wedge M^{d-k} M^{k-l}(X) \xrightarrow{\Gamma^{d-k} \Gamma^{k-l}(BS)}$$

$$M^{d-k}(S^1 \wedge X) \xrightarrow{\delta_{i_k} \ldots \delta_{i_{l+1}}} \Gamma^{d-k}(BS)$$

We think of an element of $M^d(X)$ as being divided into $d + 1$ levels $d, \ldots, 0$: level $d$ is the surface (with elements of the mapping class group) coming from the $M$ the most on the left, level $d - 1$ is composed of $m_{d-1}$ surfaces coming from the second $M$, and so on up to level 0 which is composed of $m_0$ elements of $X = (M(\ast) \times M(\ast))_+$. In the above diagram, we start by gluing the levels $l, \ldots, 1$ to level 0. The maps $\delta_{i_k} \ldots \delta_{i_{l+1}}$ going down glue similarly the levels $k, \ldots, l + 1$, and the image of $h_k$ is the disjoint union of the surfaces obtained, which is a couple of morphisms $F_{k,0}, G_{k,0}$ from 0 to $m_{k}$ in $S$. The map $h_l$ is obtained by following the top of the diagram. Once the levels $l, \ldots, 1$ glued to level 0, $h_l$ takes the disjoint union, producing two morphisms $F_{l,0}, G_{l,0}$ from 0 to $m_l$ in $S$. Now the surface $H_{k,l}$ obtained by gluing together the levels $k, \ldots, l + 1$ gives a morphism from $m_l$ to $m_k$ in $S$ which is such that $H_{k,l} \circ F_{l,0} = F_{k,0}$ and $H_{k,l} \circ G_{l,0} = G_{k,0}$. This produces two 2-simplices in $BS$ and hence a homotopy equivalence (see Fig. 3).

More precisely, consider the two $d + 1$-simplices of $BS$

$$m_d \xrightarrow{F_i} \ldots \xrightarrow{F_i} m_0 \xrightarrow{F_i} 0 \quad \text{and} \quad m_d \xrightarrow{F_i} \ldots \xrightarrow{F_i} m_0 \xrightarrow{G_i} 0,$$

where $F_i$ is the disjoint union of the surfaces of level $i$ of an element $F$ of $M^d(X)$ for $i \geq 1$ and $(F_0, G_0)$ is the disjoint union, component-wise, of level 0. Then $h_k(S^1, F)$ is the loop in $BS$ from 0 to $m_k$ along $F_{k,0} = F_k \circ \cdots \circ F_0$ and back to 0 along $G_{k,0} = F_{k,0} \circ \cdots \circ F_{k,0}$. The morphism $H_{k,l} = F_{k,0} \circ \cdots \circ F_{l+1}$ induces
a homotopy between $h_k$ and $h_l$ as explained above. Note that the homotopies $H_{k,l}$ form the edges of a $d$-simplex.

Define

$$
\tilde{h} : \Delta_d \times S^1 \land M^d(X) \longrightarrow BS
$$

for $F = (E_1, \ldots, E_d, (E_0, G_0))$ by

$$
\begin{align*}
\tilde{h}(\Delta_d, x_0, \mathbf{F}) &= 0 \\
\tilde{h}(\Delta_d, x_1, \mathbf{F}) &= m_d \overleftarrow{E_d} \ldots \overleftarrow{E_1} m_0 \\
\tilde{h}(\Delta_d, y_1, \mathbf{F}) &= m_d \overleftarrow{E_d} \ldots \overleftarrow{E_1} m_0 \overleftarrow{E_0} 0 \\
\tilde{h}(\Delta_d, y_2, \mathbf{F}) &= m_d \overleftarrow{E_d} \ldots \overleftarrow{E_1} m_0 \overleftarrow{G_0} 0
\end{align*}
$$

(so $\tilde{h}$ maps $\Delta_d \times I$ to $\Delta_{d+1}$ by collapsing $\Delta_d \times \{0\}$). This induces a map

$$
\tilde{g} : \Delta_d \times E^i_p \longrightarrow F^i_q
$$

by the continuity of the functor $A = \mathcal{G}(S^i \land \Gamma^{d-1}(\mathcal{W}))$, which extends the definition already given on the vertices of $\Delta_d$.

**Proof of Theorem 5.2** The projection $p : \tilde{D} \to D$ induces homotopy equivalences $\tilde{D}(A,B) \cong D(A,B)$. Hence the functor $L_i$ defined in Theorem 5.3 has a rectification $L'_i = p_*(L_i) : D \to \text{Top}_*$. Denote by $(E'_i)_p$, $(F'_i)_p$ and $(f'_i)_p$ the images of $E_p$, $F_p$ and $f'_i$ via $L'_i$. By definition of the category $\mathcal{D}$, $(E'_i)_*$ and $(F'_i)_*$ are simplicial spaces and $(f'_i)_* : (E'_i)_* \to (F'_i)_*$ is a simplicial map.
Lemma 5.5. For this, we need two lemmas. 

Proof. $\Sigma L_i$ and $L_{i+1}$ are functors from $\tilde{\mathcal{D}}$ to $\text{Top}_*$. We already know that the $\tau^i_p$'s form a couple of simplicial maps and commute with the maps $f_p$ (Proposition 5.4). So we only need to check that the $\tau^i_p$ commute with all the homotopies. This follows from the fact that the map $\tau^i_p$ is induced by an assembly map $\Sigma \Gamma X \to \Gamma \Sigma X$, which is natural in $X$.

Lemma 5.6. For any functor $F : \mathcal{D} \to \text{Top}_*$, there is a natural transformation $\beta : \Sigma (p^*p_*) F \to p^*p_*(\Sigma F)$.
such that the following diagram is commutative:

\[
\begin{array}{ccc}
\Sigma(p^*p_\ast F) & \xrightarrow{\Sigma(\approx)} & \Sigma F \\
\downarrow{\beta} & & \downarrow{\approx} \\
p^*p_\ast(\Sigma F) & \xleftarrow{\approx} & \Sigma F
\end{array}
\]

In particular, \( \beta \) is an equivalence.

**Proof.** There is a natural map \( \beta_n : (\Sigma(p_\ast F)(x)) \to (p_\ast(\Sigma F)(x)) \) on each simplicial level as the second space is a quotient of the first. The resulting map \( \beta \) collapses a contractible subspace of the first simplicial space. One then checks that the diagram commutes. \( \square \)

**Theorem 5.7.** The spaces \( (E')^i \) and \( (F')^i \) for \( i \geq 1 \) form two spectra which are equivalent to the spectra \( E^i \) and \( F^i \), and the maps \( (f')^i : (E')^i \to (F')^i \) form a map of spectra.

**Proof.** Recall that \( p_\ast \) is a functor \( \text{Top}^\Delta_\sim \to \text{Top}^\Delta \). By Lemma 5.5, we have a natural transformation \( \Sigma L_i \xrightarrow{\tau} L_{i+1} \). Denote by \( (\Sigma L_i)' \xrightarrow{\tau'} L_{i+1}' \) its image under \( p_\ast \). We need to construct maps \( \lambda^i : (E')^i \xrightarrow{\approx} \Omega(E')^{i+1} \) and \( \lambda^i : (F')^i \xrightarrow{\approx} \Omega(F')^{i+1} \). We define their adjoint \( \lambda \) simplicially in the following diagram:

\[
\begin{array}{ccc}
\Sigma(E')^i_p & \xrightarrow{\Sigma(\approx)} & \Sigma(F')^i_p \\
\downarrow{\beta_p} & & \downarrow{\beta_p} \\
\Sigma(E^i_p) & \xrightarrow{\Sigma(\approx)} & \Sigma(F^i_p) \\
\downarrow{\tau_p} & & \downarrow{\tau_p} \\
E^i_{p+1} & \xrightarrow{f^i_{p+1}} & F^i_{p+1} \\
\downarrow{\approx} & & \downarrow{\approx} \\
(E')^i_{p+1} & \xleftarrow{\approx} & (F')^i_{p+1}
\end{array}
\]

This diagram commutes by Proposition 5.4 for the commutation of the square in the center, Proposition 2.1 (naturality in \( F \) of the equivalence \( p^*p_\ast F \approx F \)) for the commutation of the left and right squares, because the equivalence is a natural transformation of functors for the top and bottom squares, and by Lemma 5.6 for the two triangles.

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Now the adjoint of $\lambda$ are equivalences as

\[
\begin{array}{ccc}
\Sigma(E')^i & \xrightarrow{\Omega(\sim)} & \Sigma E^i \\
\bigg\downarrow\Sigma & & \downarrow e^i \\
(E')^{i+1} & \xrightarrow{\sim} & E^{i+1}
\end{array}
\implies
\begin{array}{ccc}
(E')^i & \xrightarrow{\sim} & E^i \\
\bigg\downarrow\Omega(\sim) & & \downarrow e^i \\
\Omega(E')^{i+1} & \xrightarrow{\sim} & \Omega E^{i+1}
\end{array}
\]

the commutation of the left diagram implies the commutation of the right one, and the maps $e^i$ are equivalences. Moreover, this last diagram shows that the equivalences $E^i \simeq (E')^i$ and $F^i \simeq (F')^i$ are equivalences of spectra.

Finally, the commutation of the larger square in the big diagram implies that the maps $(f')^i$ form a map of spectra.

### 5.4 Equivalence

So far, we have defined the spectra $(E')^i$ and $(F')^i$ only for $i \geq 1$, i.e. starting with the first deloop of $\mathbb{Z} \times B\Gamma_{\infty}^+$. Define $(E')_0^1 := \Omega(E')^1$, $(F')_0^1 := \Omega(F')^1$ and

\[
(f')_0^1 := \Omega(f')^1 : (E')^0 \longrightarrow (F')^0.
\]

Note that $\Omega(E')^1 \simeq \mathbb{Z} \times B\Gamma_{\infty}^+ \simeq \Omega(F')^1$.

**Theorem 5.8.** The map of spectra $\{(f')^i\}_{i \geq 0} : \{(E')^i\} \longrightarrow \{(F')^i\}$ is an equivalence.

**Lemma 5.9.** There are maps $\phi_p : M(*) \rightarrow \mathcal{G}\Gamma(M^p((M(* \times M(*))_+))$ and $\psi : M(*) \rightarrow \Omega BS$ such that the following diagram commutes.

$$
\begin{array}{ccc}
(E')^1_p & \xrightarrow{(f')^1_p} & (F')^1_p \\
\downarrow \cong & & \downarrow \cong \\
E^1_p = \mathcal{G}\Gamma(S^1 \wedge M^p((M(* \times M(*))_+)) & \xrightarrow{f^1_p} & \mathcal{G}\Gamma(\Gamma^p(BS)) = F^1_p \\
\downarrow & & \downarrow \\
\Sigma\mathcal{G}\Gamma(M^p((M(* \times M(*))_+)) & \xrightarrow{(\Sigma\phi_p)} & BS \\
\downarrow \Sigma M(*) & & \downarrow \Sigma \psi \\
\end{array}
$$

**Proof.** Let $F$ be an element of $M(*)$. Define $\phi_p(F) := (1, \ldots, 1,(F,D))$, where $D$ is the disc, and let $\psi$ be the map defined in Proposition 4.1 which sends $F$ to the loop in $BS$ going from 0 to 1 along $F$ and back to 0 along $D$. The bottom part of the diagram is easily seen to commute. The top part commutes by naturality of the equivalence. \qed
Proof of Theorem 5.8. The spectra $(E^i)'$ and $(F^i)'$, for $i \geq 0$ are connective. Indeed, they are equivalent to the spectra $E^i$ and $F^i$. As the functor $\Gamma$ preserves connectedness, $E^i$ and $F^{i+1}$ are connected for $i \geq 1$. Moreover, $F^1 \simeq B\mathcal{S}$ is also connected. Hence it is enough to show that $(f')^0$ is an equivalence.

Thinking of $M(\ast)$ as a constant simplicial space, Lemma 5.9 yields a commutative diagram of simplicial spaces (with no map from $E^1_\ast$ to $F^1_\ast$). Taking adjoints and using Propositions 3.1 and 4.1 we get a homotopy commutative diagram

\[
\begin{array}{cccccc}
(E')^0 & \rightarrow & (F')^0 \\
\phi & & \psi \\
\downarrow \simeq & & \downarrow \simeq \\
\varphi & & \psi \Omega \mathcal{B}S \\
\downarrow \simeq & & \downarrow \simeq \\
GM(*') & \rightarrow & \Omega \mathcal{B}S \\
\end{array}
\]

(the left triangle commutes only up to homotopy). Hence $(f')^0$ is a homotopy equivalence. □

Appendix

The proof that $\Omega \mathcal{B}S \simeq \mathbb{Z} \times \mathcal{B} \Gamma_\infty^+$ which appeared in [11, 12] relies on a generalized group completion theorem. Tillmann constructs a homology fibration with fibre $\mathbb{Z} \times \mathcal{B} \Gamma_\infty^+$ and homotopy fibre of the homotopy type of $\Omega \mathcal{B}S$. The canonical map from the fibre to the canonical homotopy fibre is thus a homology equivalence. We use an explicit identification of the homotopy fibre with $\Omega \mathcal{B}S$ (before stabilization) to show that the map given in Proposition 4.1 induces the homology equivalence.

Consider the simplicial space with space of $n$-simplices

\[(E_S S)_n = \bigtimes_{m_0, \ldots, m_n \in \text{Ob}_S} \mathcal{S}(m_0, m_1) \times \cdots \times \mathcal{S}(m_{n-1}, m_n) \times \mathcal{S}(m_n, 1)\]

and boundary maps induced by composition in $\mathcal{S}$. Consider also the telescope

\[S_\infty(n) = \text{Tel}(\mathcal{S}(n, 1) \xrightarrow{T} \mathcal{S}(n, 1) \xrightarrow{T} \cdots)\]

where $\mathcal{S}(n, 1) \xrightarrow{T} \mathcal{S}(n, 1)$ is induced by gluing the torus $T$. Note that $S_\infty^0(0) \simeq \mathbb{Z} \times B \Gamma_\infty^+$. As composition induces maps $\mathcal{S}(m, n) \times S_\infty^0(n) \rightarrow S_\infty^0(n)$, we can also define a simplicial space $E_S S_\infty = \text{Tel}(E_S S_1 \xrightarrow{T} \cdots)$. The map

\[\pi : E_S S_\infty \rightarrow BS,\]

induced by collapsing $S_\infty(n)$ to $\{n\}$, is a homology fibration. Let $hF_\infty := PBS \times BS$ $E_S S_\infty$ denote the homotopy fibre. As $E_S S_\infty$ is contractible, $hF_\infty$ is of the homotopy type of $\Omega BS$. Hence, we have

\[\mathbb{Z} \times B \Gamma_\infty^+ \simeq S_\infty^0(0) \xrightarrow{\mu^S} hF_\infty \simeq \Omega BS.\]
**Theorem 5.10.** The map $\psi : S(0,1) \to \Omega BS$ defined by $\psi(F) = F(1)$ induces the homology equivalence $Z \times B\Gamma_\infty \simeq H_* \Omega BS$, i.e. there is a commutative diagram

\[
\begin{array}{ccc}
S(0,1) & \xrightarrow{\psi} & \Omega BS \\
\downarrow & & \downarrow \simeq_{H_*} \\
S_\infty(0) & & 
\end{array}
\]

To prove the Theorem, we first study the non-stable case. Consider the map $\pi_1 : E_S S_1 \to BS$, and let $hF_1 := PBS \times_{BS} E_S S_1$ denote its homotopy fiber. As $E_S S_1$ is also contractible, $hF_1$ is homotopy equivalent to $\Omega BS$ (but not equivalent to the fiber in this case). We want to construct an explicit homotopy equivalence $\rho : hF_1 \longrightarrow \Omega_0 BS$

where $\Omega_0 BS$ is the space of loops in $BS$ starting at 0 and ending at 1. To a $q$-simplex $\sigma = (F_1, \ldots, F_q, \xi) \in S(n_0, n_1) \times \ldots \times S(n_{q-1}, n_q) \times S(n_q, 1)$ of $E_S S_1$ corresponds a $q+1$-simplex of $BS$

\[
\sigma = n_0 \xrightarrow{F_1} n_1 \longrightarrow \ldots \longrightarrow n_q \xrightarrow{\xi} 1
\]

having 1 as last vertex. The face opposite to 1 in $\sigma$ is $\pi_1(F_1, \ldots, F_q, \xi)$. For $e \in E_S S_1$, define the path $\delta_e$ from $\pi_1(e)$ to 1, to be the straight line in $\sigma$ between $\pi_1(e)$ and 1 (see figure). Now for $(p, e) \in hF_1$ ($e \in E_S S_1$ and $p$ is a path in $BS$ from 0 to $\pi(e)$), define $\rho(p, e)$ to be the product of paths $p.\delta_e$.

**Lemma 5.11.** The map $\rho : hF_1 \longrightarrow \Omega_0 BS$ is a homotopy equivalence.

**Proof.** Define the map $\xi : \Omega_0 BS \longrightarrow hF_1$ by $\xi(\lambda) = (\lambda, Id_1)$, where $Id_1 \in S(1,1)$ is the identity at 1. Then $\rho \circ \xi$ is the identity on $\Omega_0 BS$. On the other hand, $\xi \circ \rho$ is homotopic to the identity on $hF_1$. Indeed, for $e$ in the $q$-simplex $(F_1, \ldots, F_q, \xi)$, we have $(\xi \circ \rho)(p, e) = (p.\delta_e, Id_1)$. Consider the $q+1$-simplex of $E_S S_1$ defined by $(F_1, \ldots, F_q, \xi, Id_1)$. As in the case of $\delta_e$, we can define a straight line $\gamma_e$ in the $q+1$-simplex from $e$ to $Id_1$ ($e$ lies in the face opposite to $Id_1$). Now $\pi_1(\gamma_e) = \delta_e$.

This induces the required homotopy in $hF_1$ by truncating the path $p.\delta_e$ at $\pi_1(\gamma_e(t))$ (explicitly $H(t, p, e) = (p.\delta_{|\pi_1(\gamma_e(t))}, \gamma_e(t))$). \qed
Proof of Theorem 5.10. Consider the diagram

\[
\begin{array}{ccccccc}
S(0,1) & \xrightarrow{T} & S(0,1) & \xrightarrow{T} & S(0,1) & \xrightarrow{} & \cdots \\
\rho \circ j & \downarrow & \rho \circ j & \downarrow & \rho \circ j & \downarrow & \\
\Omega_0 BS & \xrightarrow{\lambda_T} & \Omega_0 BS & \xrightarrow{\lambda_T} & \Omega_0 BS & \xrightarrow{} & \cdots \\
\end{array}
\]

where \( j : S(0,1) \to hF_1 \) is the canonical map from the fiber to the homotopy fiber, \( \rho \) and \( T \) are defined above and \( \lambda_T : \Omega_0 BS \to \Omega_0 BS \) is the multiplication with the loop from 1 to 1 defined by the torus \( T \). As the squares commute only up to homotopy, we need to rectify this diagram to get a map of telescopes.

Let \( D \) be the discrete category with two copies of \( \mathbb{N} \) as set of objects and morphisms as shown in the following diagram:

\[
\begin{array}{ccccccc}
A_0 & \to & A_1 & \to & A_2 & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
B_0 & \to & B_1 & \to & B_2 & \to & \cdots \\
\end{array}
\]

where the squares commute strictly. Now consider the standard \( k \)-simplex \( \Delta_k \subset \mathbb{R}^{k+1} \) and consider the path space

\[ P_k = \{ \delta : I \to \Delta_k | \delta(0) = (1,0,\ldots,0) \text{ and } \delta(1) = (0,\ldots,0,1) \}. \]

So \( P_k \simeq \Omega \Delta_k \) is a contractible space. Let \( \tilde{D} \) be the category enriched over \( \text{Top} \) having the same objects as \( D \) and morphism spaces defined as follows: for \( n \in \mathbb{N} \) and \( k \geq 0 \),

\[
\begin{align*}
\tilde{D}(A_n, A_{n+k}) & = D(A_n, A_{n+k}) = \{ * \} \\
\tilde{D}(B_n, B_{n+k}) & = D(B_n, B_{n+k}) = \{ * \} \\
\tilde{D}(A_n, B_{n+k}) & = P_{k+1},
\end{align*}
\]

the other morphism spaces being empty. Labeling the \( k + 2 \) vertices of \( \Delta_{k+1} \) with \( A_n, B_n, \ldots, B_{n+k} \) induces a face inclusion \( i : \Delta_{k+1} \hookrightarrow \Delta_{n+k+i+1} \). The composition of a morphism \( f : A_n \to B_{n+k} \) with the unique morphism \( g : B_{n+k} \to B_{n+k+i} \) is defined to be the product of paths \((i \circ f)p \) in \( \Delta_{n+k+i+1} \), where \( p \) is the path from \( B_{n+k} \) to \( B_{n+k+i} \) following the edges \( B_{n+k} \sim B_{n+k+1} \sim \cdots \sim B_{n+k+i} \). To compose the morphism \( f : A_n \to A_{n+k} \) with a morphism \( g : A_{n+k} \to B_{n+k+i} \), one uses the inclusion \( j : \Delta_{i+1} \hookrightarrow \Delta_{k+i+1} \) sending \( \Delta_{i+1} \) to the face having vertices labeled \( A_n, B_{n+k}, B_{n+k+1}, \ldots, B_{n+k+i} \). Define \( g \circ f \) to be \( j \circ g \).

The categories \( D \) and \( \tilde{D} \) satisfy the hypothesis of Section 4. We need to show that our data gives a functor \( J : \tilde{D} \to \text{Top} \). Define \( J \) on object by \( J(A_n) := S(0,1) \) and \( J(B_n) := \Omega_0 BS \). The diagram given at the beginning of the proof defines \( J \) on the morphisms of the type \( A_n \to A_{n+k}, B_n \to B_{n+k} \) and \( A_n \to B_n \) as well as their composition. Such compositions between \( A_n \) and \( B_{n+k} \) are paths following the edges in the relevant simplex. The figure shows in the case of \( A_n \to B_{n+2} \) that the images are actually paths following the edges in a simplex of \( BS \). For any morphism \( \delta : A_n \to B_{n+k} \), define \( J(\delta) : S(0,1) \to \Omega_0 BS \) by setting \( J(\delta)(\mathcal{F}) \) to be the corresponding path in the \( k + 1 \)-simplex of \( BS \)

\[
\begin{array}{ccccccc}
0 & \xrightarrow{\mathcal{F}} & 1 & \xrightarrow{T} & 1 & \xrightarrow{T} & \cdots & \xrightarrow{T} & 1.
\end{array}
\]
$J$ is functorial essentially because we defined composition in $\hat{D}$ precisely to make it functorial.

Let $J' = p_* J : D \to \text{Top}$ be the rectification of $J$. The rectification produces two telescopes $J'(A_\ast)$ and $J'(B_\ast)$ which are equivalent to the one we started with by naturality of the equivalence $p^* p_* J \simeq J$. Moreover, we now have a map of telescopes $f : \text{Tel}(J'(A_\ast)) \to \text{Tel}(J'(B_\ast))$. From [11], we know that the map $S_\infty(0) \to hF_\infty = PBS \times_{BS} ES \times S_\infty$ is a homology equivalence. Putting all this information together, we have

$$
\begin{array}{c}
\text{Tel}(J'(A_\ast)) & \xrightarrow{\simeq} & S_\infty(0) & \xrightarrow{\simeq_{h_*}} & hF_\infty \\
\downarrow & & \downarrow \\
\text{Tel}(J'(B_\ast)) & \xrightarrow{\simeq} & \text{Tel}(\Omega_{01}BS)
\end{array}
$$

Now on each “level” of the telescope, by naturality of the equivalence and by Lemma 5.11 we have a commutative diagram

$$
\begin{array}{c}
J'(A_\ast) & \xrightarrow{\simeq} & S(0,1) & \xrightarrow{j} & hF_1 \\
\downarrow^{\rho \circ j} & & \downarrow_{\rho} & & \downarrow_{\rho} \\
J'(B_\ast) & \xrightarrow{\simeq} & \Omega_{01}BS
\end{array}
$$

It follows that the map $f : \text{Tel}(J'(A_\ast)) \to \text{Tel}(J'(B_\ast))$ is a homology equivalence. Hence we have a commutative diagram

$$
\begin{array}{c}
S(0,1) & \xrightarrow{\psi} & S_\infty(0) \\
\Omega BS & \xrightarrow{\rho \circ j} & \Omega_{01}BS & \xrightarrow{\simeq_{h_*}} & \text{Tel}(\Omega_{01}BS)
\end{array}
$$

where the map $\Omega_{01}BS \to \Omega BS$ is the multiplication with the path from 0 to 1 along the disc (taken backwards), and $\text{Tel}(\Omega_{01}BS) \simeq \Omega_{01}BS$ as the telescope structure map has a homotopy inverse. \qed
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