Finite group symmetry breaking

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1 Introduction

It is a commonplace situation that symmetric laws of Nature give raise to physical states which are not symmetric. States related by symmetry operations are equivalent, but still Nature will select one of them.

As an example, consider a ferromagnetic system of interacting spins with no external magnetic field: the “up” and “down” states are equivalent, but one of the two will be chosen: the interaction makes states with agreeing spin orientation (and therefore macroscopic magnetization) energetically preferred, and fluctuations will decide which state is actually chosen by a given sample.

Finite group symmetry is also commonplace in Physics, in particular through crystallographic groups occurring in condensed matter physics – but also through the inversions (C,P,T and their combinations) occurring in high energy physics and field theory.

The breaking of finite groups symmetry has thus been thoroughly studied, and general approaches exist to investigate it in mathematically precise terms with physical counterparts. In particular, a widely applicable approach is provided by the Landau theory of phase transitions – whose mathematical counterpart resides in the realm of equivariant singularity and bifurcation theory. In Landau theory, the state of a system is described by a finite dimensional variable (the order parameter), and physical states correspond to minima of a potential, invariant under a group.

In this article we describe the basics of symmetry breaking analysis for systems described by a symmetric polynomial; in particular we discuss generic symmetry breakings, i.e. those determined by the symmetry properties themselves and independent on the details of the polynomial describing a concrete system. We also discuss how the plethora of invariant polynomials can be to some extent reduced by means of changes of coordinates, i.e. how one can reduce to consider certain types of polynomials with no loss of generality. Finally, we will give some indications on extension of this theory, i.e. on how one deals
with symmetry breakings for more general groups and/or more general physical systems.

2 Basic notions

Finite groups

A finite group \((G, \circ)\) is a finite set \(G\) of elements \(\{g_0, \ldots, g_N\}\) equipped with a composition law \(\circ\), and such that the following conditions hold:

\((i)\) for all \(g, h \in G\) the composition \(g \circ h\) belongs to \(G\), i.e. \(g \circ h \in G\);

\((ii)\) the composition is associative, i.e. \((g \circ h) \circ k = g \circ (h \circ k)\) for all \(g, h, k \in G\);

\((iii)\) there is an element in \(G\) – which we will denote as \(e\) – which is the identity for the action of \(\circ\) on \(G\), i.e. \(e \circ g = g = g \circ e\) for all \(g \in G\);

\((iv)\) for any \(g \in G\) there is an element \(g^{-1}\) which is the inverse of \(g\), i.e. \(g^{-1} \circ g = e = g \circ g^{-1}\).

In the following we omit the symbol \(\circ\), i.e. we write \(gh\) to mean \(g \circ h\). Similarly, we usually write simply \(G\) for the group, rather than \((G, \circ)\).

Given a subset \(H \subseteq G\), this is a subgroup of \((G, \circ)\) if \((H, \circ)\) satisfies the group axioms \((i)\)–\((iv)\) above. Note this implies that \(e \in H\) whenever \(H\) is a subgroup, and \(\{e\}\) is a subgroup. Subgroups not coinciding with the whole \(G\) and with \(\{e\}\) are said to be proper.

Given two elements \(g, h\) we say that \(ghg^{-1}\) is the conjugate of \(h\) by \(g\). The conjugate of a subgroup \(H \subseteq G\) by \(g \in G\) is the subgroup of elements conjugated to elements of \(H\), \(gHg^{-1} = \{(ghg^{-1}) : h \in H\}\).

Group action

In Physics, one is usually interested in a realization of an abstract group as a group of transformations in some set \(X\); in physical applications, this is usually a (possibly, function) space or a manifold, and we refer to elements of \(X\) as “points”. That is, there is a map \(\rho : G \mapsto \text{End}(X)\) from \(G\) to the group of endomorphisms of \(X\), such to preserve the composition law:

\[\rho(g) \cdot \rho(h) = \rho(g \circ h) \quad \forall g, h \in G.\]

In this case we say that we have a representation of the abstract group \(G\) acting in the carrier space or manifold \(X\); we also say that \(X\) is a \(G\)-space or \(G\)-manifold. We often denote by the same letter the abstract element and its representation, i.e. write simply \(g\) for \(\rho(g)\) and \(G\) for \(\rho(G)\). \(^1\)

If \(x \in X\) is a point in \(X\), the \(G\)-orbit \(G(x)\) is the set of points to which \(x\) is mapped under \(G\), i.e.

\[G(x) = \{y \in X : y = gx \, , \, g \in G\} \subseteq X.\]

\(^1\)In many physically relevant cases, but not necessarily, \(X\) has a linear structure and we consider linear endomorphisms. In this case we sometimes write \(T_g\) for the linear operator representing \(g\).
Belonging to the same orbit is obviously an equivalence relation, and partitions $X$ into equivalence classes. The orbit space for the $G$ action on $X$, also denoted as $\Omega = X/G$, is the set of these equivalence classes. It corresponds, in physical terms, to considering $X$ modulo identification of elements related by the group action.

For any point $x \in X$, the isotropy (sub)group $G_x$ is the set of elements leaving $x$ fixed,

$$G_x = \{ g \in G : gx = x \} \subseteq G .$$

Points on the same $G$-orbit have conjugated isotropy subgroups: indeed, $y = gx$ implies immediately that $G_y = gG_xg^{-1}$.

When a topology is defined on $X$, the problem arises if the $G$-action preserves it; if this is the case, we say that the $G$-action is regular. In the case of a compact Lie group (and a fortiori for a finite group) we are guaranteed the action is regular.\(^2\)

**Spontaneous symmetry breaking**

Let us now consider the case of physical systems whose state is described by a point $x$ in the $G$-space or $G$-manifold $X$, with $G$ a group acting by smooth mappings $g : X \to X$. In physical problems, $G$ is quite often acting by linear and orthogonal transformations\(^3\).

Usually $G$ represents physical equivalence of states, and $G$-orbits are collections of physically equivalent states. A point which is $G$-invariant, i.e. such that $G_x = G$, is called “symmetric” for short.

Let $\Phi$ be a scalar function (potential) defined on $X$, $\Phi : X \to \mathbb{R}$, possibly depending on some parameter $\mu$, such that the physical state corresponds to critical points – usually the (local) minima – of $\Phi$.

A concrete example is provided by the case where $\Phi$ is the Gibbs free energy; more generally, this is the framework met in the Landau theory of phase transitions (Landau 1937, Landau and Lifshits 1958).

We are interested in the case where $\Phi$ is invariant under the group action, or briefly $G$-invariant. That is, where

$$\Phi(gx) = \Phi(x) \quad \forall x \in X , \forall g \in G . \quad (1)$$

A critical point $x$ such that $G_x = G$ is a symmetrical critical point. If $G_x$ is strictly smaller than $G$, then $x$ is a symmetry breaking critical point.

If a physical system corresponds to a non symmetric critical point, we have a spontaneous symmetry breaking: albeit the physical laws (the potential function $\Phi$) are symmetric, the physical state (the critical point for $\Phi$) breaks the symmetry and chooses one of the $G$-equivalent critical points.

\(^2\)A physically relevant example of non-regular action is provided by the irrational flow on a torus. In this case $G = \mathbb{R}$, realized as the time $t$ irrational flow on the torus $X = \mathbb{T}^k$.

\(^3\)If this is not the case, the Palais-Mostow theorem guarantees that, for suitable groups (including in particular finite ones) we can reduce to this case upon embedding $X$ into a suitably larger carrier space $Y$. 

3
It follows from (1) that the gradient of $\Phi$ is covariant under $G$. If $y = g(x)$, then the differential $(Dg)$ of the map $g : X \rightarrow X$ is a linear map between the corresponding tangent spaces, $(Dg) : T_xX \rightarrow T_yX$. The covariance amounts, with $\eta$ the Riemannian metric in $X$, to $(\eta^{ij} \partial_j \Phi)(gx) = [((Dg)^i_k \eta^{km} \partial_m \Phi](x)$; this is also written compactly, with obvious notation, as

$$ (\nabla \Phi)(gx) = (Dg) \left[ (\nabla \Phi)(x) \right]. $$

As $(Dg)$ is a linear map, $(\nabla \Phi)(x) = 0$ implies the vanishing of $\nabla \Phi$ at all points on the $G$-orbit of $x$.

We conclude that critical points of a $G$-invariant potential come in $G$-orbits: if $x$ is a critical point for $\Phi$, then all the $y \in G(x)$ are also critical points for $\Phi$. We speak therefore of critical orbits for $\Phi$.

It is thus possible (thanks to the regularity of the $G$-action), and actually convenient, to study spontaneous symmetry breaking in the orbit space $\Omega = X/G$ rather than in the carrier manifold $X$ (Michel 1971).

If $G$ describes physical equivalence, physical states whose symmetries are $G$-conjugated should be seen as physically equivalent. An equivalence class of isotropy types under conjugation will be said to be a symmetry type. We are thus interested, given a $G$-invariant polynomial $\Phi$, to know the symmetry types of its critical points. We denote symmetry types as $[H] = \{gHg^{-1}\}$, and say that $[H] < [K]$ if a group conjugated to $H$ is strictly contained in a group conjugated to $K$.

As we have seen, points on the same $G$-orbit have the same symmetry type. On the other hand, points on different $G$-orbits can have the same isotropy type (e.g., for the standard action of $O(n)$ in $\mathbb{R}^n$, all collinear nonzero points will have the same isotropy subgroup but will lie on distinct group orbits).

### 3 G-invariant polynomials

Consider a finite group $G$ acting in $X$.\(^5\) We look at the ring of $G$-invariant scalar polynomials in $x_1, ..., x_n$.

By the *Hilbert basis theorem*, there is a set $\{J_1(x), ..., J_k(x)\}$ of $G$-invariant homogeneous polynomials of degrees $\{d_1, ..., d_k\}$ such that any $G$-invariant polynomial $\Phi(x)$ can be written as a polynomial in the $\{J_1, ..., J_k\}$, i.e.

$$ \Phi(x) = \Psi [J_1(x), ..., J_k(x)] $$

with $\Psi$ a polynomial. (A similar theorem holds for smooth functions.)

The algebra of $G$-invariant polynomials is finitely generated, i.e. we can choose $k$ finite. When the $J_a$ are chosen so that none of them can be writ-

\(^4\)In the case of euclidean spaces ($\eta = \delta$) and linear actions described by matrices $T_g$, the covariance condition reduces to $(\nabla \Phi)^i(T_g x) = (T_g)^i_j [\nabla \Phi]^j(x)$.

\(^5\)Many of the notions and results mentioned in this section have a much wider range of applicability.
ten as a polynomial of the others⁶ and \( r \) has the smallest possible value (this value depends on \( G \)), we say that they are a \textit{minimal integrity basis (MIB)}. In this case we say that the \( \{ J_a \} \) are a set of \textit{basic invariants} for \( G \). There is obviously some arbitrariness in the choice of the \( J_a \) in a MIB, but the degrees \( \{ d_1, \ldots, d_k \} \) of \( \{ J_1, \ldots, J_k \} \) are fixed by \( G \). (In mathematical terms, they are determined through the Poincaré series of the graded algebra \( P_G \) of \( G \)-invariant polynomials.)

We will from now on assume we have chosen a MIB, with elements \( \{ J_1, \ldots, J_k \} \) of degrees \( \{ d_1, \ldots, d_k \} \) in \( x \), say with \( d_1 \leq d_2 \leq \ldots \leq d_k \).

When the elements of a MIB for \( G \) are algebraically independent, we say that the MIB is \textit{regular}; if \( G \) admits a regular MIB we say that \( G \) is \textit{coregular}.

An algebraic relation between elements \( J_\alpha \) of the MIB is said to be a relation of the first kind. The algebraic relations among the \( J \) are a set of polynomials in the \( \{ J_1, \ldots, J_r \} \) which are identically zero when seen as polynomials in \( x \). If there are algebraic relations among these, they are called relations of the second kind, and so on. A theorem by Hilbert guarantees that the chain of \( r \) relations has finite maximal length. (This is the homological dimension of the graded algebra \( P_G \) mentioned above.)

In the following we will consider a matrix built with the gradients of basic invariants, the \( \mathcal{P} \)-matrix (Sartori). This is defined as

\[
\mathcal{P}_{ab}(x) := \langle \nabla J_a(x), \nabla J_b(x) \rangle
\]

with \( \langle \cdot, \cdot \rangle \) the scalar product in \( T_x X \).

The gradient of an invariant is necessarily a covariant quantity; the scalar product of two covariant quantities is an invariant one, and thus can be expressed again in terms of the basic invariants. Thus, the \( \mathcal{P} \)-matrix can always be written in terms of the basic invariants themselves.

4 Geometry of group action

The use of a MIB allows to introduce a map \( \mathbf{J} : x \rightarrow \{ J_1(x), \ldots, J_k(x) \} \) from \( X \) to a subset \( P \) of \( \mathbb{R}^k \). If the MIB is regular, \( P = \mathbb{R}^k \), while if the \( J_i \) satisfy some relation then \( P \subset \mathbb{R}^k \) is the submanifold satisfying the corresponding relations. The manifold \( P \) is isomorphic to the orbit space \( \Omega = X/G \) (the isomorphism being realized by the \( \mathbf{J} \) map) and provides a more convenient framework to study \( \Omega \).

As mentioned above, on physical terms we are mainly interested in the orbit space up to equivalence of symmetry type. The set of points in \( X \) (of orbits in \( \Omega \)) with the same symmetry type will be called a \textit{G-stratum} in \( X \) (a \( G \)-stratum)

⁶Note that some of the \( J_a \) could be written as non-polynomial functions of the others, and the \( J_a \) could verify polynomial relations. E.g., consider the group \( \mathbb{Z}_2 \) acting in \( \mathbb{R}^2 \) via \( g : (x, y) \rightarrow (-x, -y) \); a MIB is made of \( J_1(x, y) = x^2 \), \( J_2(x, y) = y^2 \), and \( J_3(x, y) = xy \). None of these can be written as a polynomial function of the others, but \( J_1 J_2 = J_3^2 \).
in \( \Omega \); the \( G \)-stratum of the point \( x \) will be denoted as \( \sigma(x) \subset X \) (the \( G \)-stratum of the orbit \( \omega \) as \( \Sigma(\omega) \subset \Omega \)).

It results that the \( G \)-stratification is compatible with the topological stratification. Indeed \( P \) is a semialgebraic (i.e., is defined by algebraic equalities and inequalities) stratified manifold in \( \mathbb{R}^k \); the image of any \( G \)-stratum in \( \Omega \) belongs to a single topological stratum in \( P \), and topological strata in \( P \) are the union of images of \( G \)-strata in \( \Omega \).

Moreover, the subgroup relations correspond to bordering relations between \( G \)-strata: if \([G_x] < [G_y]\), then \( \sigma(y) \in \partial \sigma(x) \) and (with \( \omega_x \) the orbit of \( x \)) \( \Sigma(\omega_y) \in \partial \Sigma(\omega_x) \).

There is a stratum, called the principal stratum \( \sigma_0 \), which corresponds to minimal isotropy, open and dense in \( X \); similarly, the principal stratum \( \Sigma_0 \) is open and dense in \( \Omega \).

## 5 Landau polynomial

In the Landau theory of phase transitions (Landau 1937) the state of the system under study is described by a \( G \)-invariant polynomial \( \Phi : X \to \mathbb{R} \) having a critical point in the origin, with at least some of its coefficients – in particular those controlling stability of the zero critical point – depending on external control parameters (usually, \( X = \mathbb{R}^n \) and \( G \subseteq O(n) \); in particular, in solid state physics \( G \) is a crystallographic group). This should be chosen as the most general \( G \)-invariant polynomial of the lowest degree \( \ell \) sufficient to ensure thermodynamic stability; in mathematical terms, this amounts to the requirement that there is some open set \( \mathcal{B} \) containing the origin and such that – for all values of the control parameters – \( -\nabla \Phi \) points inwards at all points of \( \partial \mathcal{B} \) (i.e. \( \mathcal{B} \) is invariant under the gradient flow of \( \Phi \)). If the polynomials in the MIB are of degree \( d_1 \leq d_2 \ldots \leq d_r \), then usually \( \ell = 2d_r \).

The \( G \)-invariance of \( \Phi \) and the results recalled above mean that we can always write it in terms of the polynomials in a MIB for \( G \) as in (3), \( \Phi(x) = \Psi[J(x)] \).

The discussion of previous sections show that we can study symmetry breakings for \( \Phi : X \to \mathbb{R} \) by studying critical points of \( \Psi : P \to \mathbb{R} \); in other words, \textit{Landau theory can be worked out in the \( G \)-orbit space} \( \Omega := M/G \). The polynomial \( \Psi \) – providing a representation of the Landau polynomial in the orbit space – will also be called \textit{Landau-Michel polynomial}.

In this way the evaluation of the map \( \Phi : X \to \mathbb{R} \) is in principles substituted by evaluation of two maps, \( J : X \to P \) and \( \Psi : P \to \mathbb{R} \). However, if – as in Landau theory – we have to consider the most general \( G \)-invariant polynomial on \( X \), we can just consider the most general polynomial on \( P \).

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7. The notion of stratum was introduced by Whitney in topology; a \textit{stratified manifold} is a set which can be decomposed as the disjoint union of smooth manifolds of different dimensions, the topological (or Whitney) strata: \( M = \sqcup M^k \), with \( M^k \subset \partial M^j \) for all \( k < j \).

8. Louis Michel (1923-1999) pioneered the use of orbit space techniques in Physics and Nonlinear Dynamics, originally motivated by the study of hadronic interactions.
6 Critical points of the Landau polynomial and geometry of orbit space

The $G$-invariance has consequences on the critical points of $\Phi$. We have already seen one such consequence: critical points come in $G$-orbits. This, however, is not all. Indeed, $G$-invariance enforces the presence of a certain set $\chi(G) \in X$ of critical points, and conversely if we look for points which are critical under any $G$-invariant potential, these are precisely the points in $\chi(G)$; the critical points on $\chi(G)$ correspond to critical orbits which we call principal critical orbits.

The set $\chi(G)$ can be determined on the basis of the geometry of the $G$-action. Indeed (Michel 1971): An orbit $\omega$ is a principal critical orbit if and only if it is isolated in its stratum.

For the linear orthogonal group actions in $\mathbb{R}^n$ often occurring in Physics, no nonzero point or orbit can be isolated in its stratum. However, we can quotient out the radial degeneracy and work on $X = S^{n-1} \subset \mathbb{R}^n$. In this case, a $G$-orbit $\omega_1$ in $S^{n-1}$ which is isolated in its stratum corresponds to a one-dimensional family $\{\omega_r\}$ of $G$-orbits in $\mathbb{R}^n$, call $X_0$ the corresponding submanifold in $X$; the gradient of $\Phi$ at $x \in X_0$ points along $T_xX_0$. We can thus reduce to consider the restriction $\Phi_0$ of the potential $\Phi$ to $X_0$. (See also the reduction lemma of Golubitsky and Stewart in this context.)

Correspondingly, if $P_0 \subset P$ is the submanifold in $P$ image of $X_0$, i.e. $P_0 = J(X_0)$, we can reduce to consider the restriction $\Psi_0$ of $\Psi$ to $P_0$.

As these become one-dimensional problems, general results are available. In particular, one can provide general conditions ensuring the existence of one-dimensional branches of symmetry-breaking solutions bifurcating from zero along any such $X_0$ or $P_0$; this is also known as the equivariant branching lemma of Cicogna and Vanderbauwhede.

7 Reduction of the Landau potential

In realistic problems, $\Phi$ becomes quickly extremely complicated, i.e. includes a high number of terms and therefore of coefficients. A thorough study of different symmetry breaking patterns, i.e. of the symmetry type of minima of $\Phi$ for different values of these coefficients and of the external control parameter, is in this case a prohibitive task. It is possible to reduce the generality of the Landau polynomial with no loss of generality for the corresponding physical problem. Indeed, a change of coordinates in the $X$ space will produce a formally different – but obviously equivalent – Landau polynomial; it is convenient to use coordinates in which the Landau polynomial is simpler.

A systematic and algorithmic reduction procedure – based on perturbative expansion near the origin – is well known in dynamical systems theory (Poincaré-
Birkhoff normal forms), and can be adapted to the reduction of Landau polynomials.\(^{10}\)

We work near the origin, so that we can assume \(X = \mathbb{R}^n\) (with metric \(\eta\)), and for simplicity we also take the case where \(G\) acts via a linear representation \(T_g\). We consider changes of coordinates of the (Poincaré) form

\[
x^i = y^i + h^i(y),
\]

generated by a \(G\)-invariant function \(H\): \(h^i(y) = \eta^{ij}(\partial H(y)/\partial y^j)\); this guarantees that (5) preserves the \(G\)-invariance of \(\Phi\). The action of (5) on \(\Phi\) can be read from its action on the basic invariants \(J_a\). It results

\[
J_a(x) = J_a(y) + (\delta J_a)(y); \quad \delta J_a := \mathcal{P}_{ab}(\partial H/\partial J_b). \quad (6)
\]

Let us now consider the reduction of an invariant polynomial \(\Phi(x) = \Psi(J)\). We write

\[
\Psi(J) \rightarrow \Psi(J + \delta J) = \Psi(J) + \sum_{a=1}^r \frac{\partial \Psi(J)}{\partial J_a} \delta J_a + \text{h.o.t.}.
\]

Disregarding higher order terms and using (6) and (4), we get

\[
\delta \Psi = \frac{\partial \Psi}{\partial J_a} \mathcal{P}_{a\beta} \frac{\partial H}{\partial J_\beta} = (D_\alpha \Psi) \mathcal{P}_{a\beta} (D_\beta H). \quad (7)
\]

We expand \(\Phi\) as a sum of homogeneous polynomials, and write \(\Phi(x) = \sum_{k=0}^l \Phi_k(x)\), where \(\Phi_k(ax) = a^{k+1} \Phi_k(x)\). Also, write \(\Psi = \sum_k \Psi_k\), where \(\Phi_k(x) := \Psi_k[J(x)]\).

It results that under a change of coordinates (5) generated by \(H = H_m\) homogeneous of degree \(m+1\), the terms \(\Psi_k\) with \(k \leq m\) are not changed, while the terms \(\Psi_{m+p}\) change according to

\[
\Psi_{m+p} \rightarrow \hat{\Psi}_{m+p} = \Psi_{m+p} + (D_\alpha \Psi_p)\mathcal{P}_{a\beta}(D_\beta H_m) + \text{h.o.t.}. \quad (8)
\]

We can then operate sequentially with \(H_m\) of degree 3, 4, ...; at each stage (generator \(H_m\)) we are not affecting the terms \(\Psi_k\) with \(k \leq m\). Moreover, we can just consider (8), as higher order terms are generic and will be taken care of in subsequent steps. (This procedure requires to determine suitable generating functions \(H_m\); these are obtained as solutions to homological equations.)

In the above we disregarded dependence on the control parameters, such as temperature, pression, magnetic field, etc. i.e. we implicitly considered fixed values for these. However, they have to change for a phase transition to take place. If we consider a full range of values – including in particular critical ones – for the control parameters, say \(\lambda \in \Lambda\), we should take care that the concerned quantities and operators are nonsingular uniformly in \(\Lambda\).

\(^{10}\)An alternative and more general – but also much more demanding – approach is provided by the spectral sequence approach, also originating in normal forms theory.
This leads to reduction criteria for the Landau and Landau-Michel polynomials (Gufan). Define, for \( i = 1, ..., k \) and with \( (., .) \) the scalar product in \( X = \mathbb{R}^n \), the quantities

\[
U_i(J_1, ..., J_k) := (\partial F/\partial J_i) \mathcal{P}_{si}.
\]

**Reduction criterion.** For \( \Phi(x) = \Psi(J_1, ..., J_k) : \mathbb{R}^n \to \mathbb{R} \) a \( G \)-invariant potential depending on physical parameters \( \lambda \in \Lambda \), there is a sequence of Poincaré changes of coordinates such that \( \Phi \) is expressed in the new coordinates \( y \) as \( \hat{\Phi}(y) = \hat{\Psi}(J) \), where terms which can be written (up to higher order terms) uniformly in \( \Lambda \) as \( \sum_{\alpha=1}^k Q_\alpha(J_1, ..., J_k) U_\alpha(J_1, ..., J_k) \), with \( Q_\alpha \) polynomials in \( J_1, ..., J_k \) satisfying the compatibility condition

\[
(\partial Q_\beta/\partial J_\alpha) = (\partial Q_\alpha/\partial J_\beta),
\]

are not present in \( \hat{\Psi} \).

### 8 Non-stationary and non-variational problems

So far we have considered stationary physical states. In some cases, one is not satisfied with such a description, and wants to study time evolution. A model framework for this is provided by the Ginzburg-Landau equation

\[
\dot{x} = f(x)
\]

(9)

where \( f = \eta(\nabla \Phi) : X \to TX \) (see above for notation). In this case, \( G \)-invariance of \( \Phi \) implies equivariance of (9). More generally, we can consider (9) for an equivariant smooth \( f \) (not necessarily a gradient), i.e. \( f^i(gx) = (Dg)^i_j f^j(x) \).

In this case one shows that

\[
f(x) \in T_x \sigma(x),
\]

(10)

so that closure of \( G \)-strata are dynamically invariant, and the dynamics can be reduced to them. This is of special interest for the “most singular” strata, i.e. those of lower dimension. The reduction lemma and the equivariant branching lemma mentioned above also hold (and were originally formulated) in this context.

The relation (10) also implies that one can project the dynamics (9) in \( X \) to a smooth dynamics \( \dot{p} = F(p) \) in the orbit space: this satisfies \( F[J(x)] = (DJ)[f(x)] \). In the gradient case this (together with initial conditions) embodies the full dynamics in \( X \), while in the generic case one looses all information about motions along group orbits (note these correspond to phonon modes).

An orbit \( \omega \) isolated in its stratum is still an orbit of fixed points for any \( G \)-equivariant dynamics in \( X \) in the gradient case, while in the generic case it corresponds to a fixed point for \( F \) and to relative equilibria (dynamical orbits which belong to a single group orbit) in \( X \). In this case, time averages of physical quantities can be \( G \)-invariant for nontrivial relative equilibria.

### 9 Extensions and physical applications

We have discussed finite groups symmetry breaking and focused on polynomial potentials (which can be thought of as Taylor expansions around critical points).
For non finite groups, and in particular non compact ones, the situation can be considerably more complicated.

(1) An extension of the theory sketched here is provided by Palais’ theory, and in particular by his symmetric criticality principle which applies in Hilbert or Banach spaces of sections of a fiber bundle satisfying certain conditions. This is specially relevant in connection with field theory and gauge groups.

(2) We focused on the situation discussed in Classical Physics. Finite groups symmetry breaking is of course also relevant in Quantum Mechanics; this is discussed e.g. in the classical books by Weyl and Wigner, and in the review by Michel, Kim, Zak and Zhilinskii.

(3) One speaks of “explicit symmetry breaking” when a non-symmetric perturbation is introduced in a symmetric problem. In the Hamiltonian case (or in the Lagrangian one for Noether symmetries), hamiltonian symmetries correspond to conserved quantities, and non-symmetric perturbations make these become approximate constants of motion.

(4) The symmetry of differential equations – as well as symmetric and symmetry-breaking solutions for symmetric equations – can be studied in general mathematical terms (see e.g. Olver).

(5) Physical applications of the theory discussed here abound in the literature, in particular through the Landau theory of phase transitions. A number of these, together with a deeper discussion of the underlying theory, is given in the monumental review paper by Michel, Kim, Zak and Zhilinskii (see “further reading”).

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