A Dynamical System with $q$-Deformed Phase Space
Represented in Ordinary Variable Spaces

Shigefumi Naka,¹ Haruki Toyoda² and Takaoki Takanashi¹

¹Department of Physics, College of Science and Technology,
Nihon University, Tokyo 101-8308, Japan
²Junior College, Funabashi Campus, Nihon University, Funabashi 274-8501, Japan

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Dynamical systems associated with a $q$-deformed two dimensional phase space are studied as effective dynamical systems described by ordinary variables. In quantum theory, the momentum operator in such a deformed phase space becomes a difference operator instead of the differential operator. Then, using the path integral representation for such a dynamical system, we derive an effective short-time action, which contains interaction terms even for a free particle with $q$-deformed phase space. We also analyze the eigenvalue problem for a particle with $q$-deformed phase space confined in a compact space. Under some boundary conditions of the compact space, there arises fairly different structures from that of the $q = 1$ case in the energy spectrum of the particle and in the corresponding eigenspace.

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§1. Introduction

The one-dimensional dynamical system associated with the $q$-deformed phase space¹–³ $px = qp$ is known to lead to a quantum mechanical system having the canonical pairs characterized by a $q$-deformed Heisenberg algebra. The $q$-deformation, then, causes a drastic change in the dynamical structure, such as spontaneous symmetry breaking of the system. For example, in such a system, the meaning of translational invariance is changed even for a free particle;⁴ there arises a new type of discrete symmetry.⁵

As an application of the $q$-deformation in particle physics, it may be interesting to consider a particle embedded in a higher-dimensional spacetime with deformed extra dimensions. The deformation, then, breaks a higher-dimensional symmetry and yields a fairly different structure of the excitation spectrum from that of the $q = 1$ case. When we study such a deformed extra dimension, it is convenient to use a coordinate representation of the $q$-deformed Heisenberg algebra constructed out of ordinary variables. This is because compactness conditions for the extra dimensions are naturally applied not to the variables in noncommutative spaces but to those in ordinary spaces.

According to the above point of view, it is worthwhile to study the representation of the $q$-deformed Heisenberg algebra,⁶

\[ \hat{\partial}_x x - qx\hat{\partial}_x = 1, \quad (1.1) \]

coming from the $q$-deformed phase space by a combination of $x$ and its ordinary
derivative operator $\partial_x = \partial/\partial x$ in such a way that

$$\hat{\partial}_x = \partial_x \frac{q(x\partial_x) - 1}{(q - 1)(x\partial_x)}. \quad (1.2)$$

This equation defines a mapping between the space of deformed calculus and that of ordinary calculus; as a result of this mapping, the momentum operator in $q$-deformed quantum theory becomes a difference operator instead of the differential operator. Thus the purpose of this work is to study the particle dynamics in $q$-deformed phase space as an effective theory of those in ordinary phase space through the above mapping.

In the next section, we discuss a free particle in two-dimensional $q$-deformed phase space, which leads to a one-dimensional Schrödinger equation for the particle without any specific boundary conditions in quantum mechanics. Then, using a path integral representation for such a dynamical system, we derive an effective short-time action, which contains interaction terms in ordinary phase space even for a free particle in the $q$-deformed phase space.

Section 3 is the discussion on the eigenvalue problem for the $q$-deformed particle confined in a compact space; there, we confine our attention to the case of a particle in an infinite potential well, $0 < x < L$. There, the eigenvalue problem of the Hamiltonian under this situation is discussed in detail. We, then, show the fairly different structure from that of the $q = 1$ case in the energy spectrum of the particle and the corresponding eigenspace.

Section 4 is the discussion and summary. In Appendixes A and B, the mathematical background of §§2 and 3 is discussed.

§2. Free particle associated with $q$-deformed phase space

In the usual phase space dynamics, the translation from classical theories to quantum mechanical counterparts can be carried out by the substitution $p \to -i\hbar \partial_x$ in the Hamiltonian operator of a dynamical system. However, since the operator (1.2) is not an anti-hermitian operator, $-i\hbar \hat{\partial}_x$ is not a hermitian momentum operator. Then, it is necessary to modify the relation between $\hat{\partial}_x$ and the momentum operator so that

$$\hat{p}_q = -\frac{i\hbar}{2} (\hat{\partial}_x + \hat{\partial}_x^\dagger) = -i\hbar \left( \frac{q + 1}{2q} \right) D_x,$$

(2.1)

to get a hermitian momentum operator $\hat{p}_q$. Here, $D_x$ can be found to be

$$D_x = x^{-1}[\hat{N}], \quad \left[a\right] = \frac{q^a - q^{-a}}{q - q^{-1}}, \quad \hat{N} = x\partial_x$$

(2.2)

and it satisfies

$$D_x x - q^{\pm 1} x D_x = q^{\mp 1} \hat{N}.$$  

(2.3)

$^1$) To realize the algebra (1-1), we may change the roles of $\hat{\partial}_x$ and $x$. Namely, the ordinary derivative $\partial_x$ and the deformed coordinate $\hat{x} = \frac{q(x\partial_x) - 1}{(q - 1)(x\partial_x)} x$ form another set of operators satisfying (1-1). In this case, since $\hat{x}$ is not a hermitian operator, the operator $X = \frac{1}{2} (\hat{x} + \hat{x}^\dagger) = \frac{q + 1}{2q} \frac{1}{x\partial_x} x$ becomes a
The operator $D_x$ has the meaning of the $q$-difference acting on a function of $x$ as

$$D_x f(x) = \frac{f(qx) - f(q^{-1}x)}{(q - q^{-1})x},$$

(2.4)

which will reduce to the usual differential operator $\partial_x$ as $q \to 1$. From Eqs. (2.1) and (2.3), the $q$-deformed momentum operator $\hat{p}_q$ satisfies

$$\hat{p}_q x - q^{\pm 1} x \hat{p}_q = -i\hbar \left( \frac{q + 1}{2q} \right) q^{\mp 1/2} \hat{N},$$

(2.5)

which is equivalent to the following usual commutation relation:

$$[\hat{p}_q, x] = -\frac{i\hbar}{2} \left( q^{\hat{N}} + (q^{\hat{N}})^d \right).$$

(2.6)

Then, under a state $\psi(x)$ with $\langle \psi | \psi \rangle = 1$, one can obtain the uncertainty relation between $p_q$ and $x$ such as,

$$\Delta p_q \Delta x \geq \frac{\hbar}{4} \left| \int_{-\infty}^{\infty} dx \left( \psi(x)^* \psi(qx) + \psi^*(qx) \psi(x) \right) \right|.$$ 

(2.7)

The Schrödinger equation based on this deformed calculus is

$$i\hbar \frac{\partial}{\partial t} \psi(t, x) = \hat{H}_q \psi(t, x).$$

(2.8)

Here, $\hat{H}_q$ is the Hamiltonian operator, in which the momentum operator is given by (2.1). For a particle of mass $m$, $\hat{H}_q$ becomes

$$\hat{H}_q = \frac{1}{2m} \hat{p}_q^2 + V(x) = -\frac{\hbar^2}{2m_q} D_x^2 + V(x),$$

(2.9)

where $m_q = \left( \frac{q + 1}{2q} \right)^2 m$. It should be noted that the free Hamiltonian operator $\hat{H}_q^0 = \frac{1}{2m_q} \hat{p}_q^2$ in the deformed calculus does not represent the free particle one in the ordinary calculus, since it can be written as

$$\hat{H}_q^0 = -\frac{\hbar^2}{2m_q} \frac{1}{(q - q^{-1})^2} x^{-1} \left[ 2 \cos \left( \frac{1}{\hbar} \{ x, \hat{p} \} \log q \right) - (q + q^{-1}) \right] x^{-1},$$

(2.10)

where $\hat{p} = -i\hbar \partial_x$ is the momentum operator in the ordinary calculus. The Hamiltonian operator (2.10), however, is not a classical Hamiltonian itself, since the form physical coordinate operator, to which one can verify $\partial_x X - qX \partial_x = \frac{q+1}{2q} q^{-\theta_x}$. Substituting $-i\hbar \partial_x$ for $\hat{p}$, the $q$-commutator (2.5) with the $q^{-\theta_x}$ factor on the right-hand side is again obtained.

We further note that the operators $x^2, D^2$ and $\hat{N} + \frac{1}{2}$ form an $SU_q(2)$-like generator defined by

$$\left[ \hat{N} + \frac{1}{2}, x^2 \right] = 2x^2, \quad \left[ \hat{N} + \frac{1}{2}, D^2 \right] = -2D^2, \quad \text{and} \quad [D^2, x^2] = \left[ 2 \left( \hat{N} + \frac{1}{2} \right) \right].$$
depends on the operator ordering. Thus, as the next task we try to derive a classical 
counterpart of $\hat{H}_q^0$ in ordinary calculus by means of the path integral method.\textsuperscript{4)} For
this purpose, the role of the potential $V(x)$ is not important; so, we only consider
the $V = 0$ case in what follows.

To study the propagation kernel by the free Hamiltonian $\hat{H}_q^0$ for the finite time
interval $T = t - t'$, let us introduce the eigenstate of $\hat{p}_q$ such that

$$\hat{p}_q | k, q \rangle = \hbar k | k, q \rangle \quad \text{and} \quad \langle k, q | k', q \rangle = \delta(k - k') .$$

(2.11)

The $x$-representation of this state is given explicitly by the $q$-exponential function

$$\langle x | k, q \rangle = \frac{1}{\sqrt{2\pi a_{q}[1]}} e^{i k x} ,$$

which tends to the usual plane wave function $\langle x | k \rangle = \frac{1}{\sqrt{2\pi}} e^{i k x}$
as $q \to 1$ (Appendix A).

In terms of this eigenstate, the propagation kernel can be written as

$$\langle x | e^{-\frac{i}{\hbar} T \hat{H}_q^0} | x' \rangle = \int_{-\infty}^{\infty} dk \langle x | k, q \rangle e^{-\frac{i}{\hbar} T \frac{(k \hbar)^2}{2m_q}} \langle k, q | x' \rangle .$$

(2.12)

We note that $\langle x | k, q \rangle \langle k, q | x' \rangle$ is not a function of $x - x'$, and so, the translational
invariance of this kernel is lost owing to $q \neq 1$.

Now, according to (A.13), $\langle x | k, q \rangle \langle k, q | x' \rangle$ can be expressed as

$$\langle x | k, q \rangle \langle k, q | x' \rangle = \frac{1}{2\pi a_{q}[1]} \sum_{N=0}^{\infty} \frac{(ik)^N}{N!} (x - x')^N .$$

(2.13)

Substituting this expression into (2.12), and carrying out the Gaussian integral with
respect to $k$, we obtain a series out of $N = 2n$ terms such that

$$\langle x | e^{-\frac{i}{\hbar} T \hat{H}_q^0} | x' \rangle = \frac{1}{a_{q}[1]} \sqrt{\frac{m_q}{2\pi i \hbar}} \sum_{n=0}^{\infty} \frac{(2n)!}{n![2n]!} \left( \frac{i m_q}{\hbar} \right)^n (x - x')^{2n} T^n .$$

(2.14)

The right-hand side of this equation gives the exact form of the propagation kernel
in a finite time interval $T$, which is reduced to the ordinary free propagation kernel

$$\langle x | e^{-\frac{i}{\hbar} T \hat{H}_q^0} | x' \rangle = \sqrt{\frac{m_q}{2\pi i \hbar T}} \exp\left\{ \frac{i m_q}{\hbar} (x - x')^2 \right\}$$
in the limit $q \to 1$. However, since the $q$-binomial expansions $(x - x')^{2n}, (n = 1, 2, \cdots)$ have common zero points at $x = q^{\pm} x'$, according to (A.18), instead of $x = x'$, the $q$-propagation kernel becomes translational
invariant only for the interval $(x, q^{\pm} x)$; that is, we obtain $\langle q^{\pm} x | e^{-\frac{i}{\hbar} T \hat{H}_q^0} | x \rangle = \frac{1}{a_{q}[1]} \sqrt{\frac{m_q}{2\pi i \hbar T}}$, which should be compared with $\langle x | e^{-\frac{i}{\hbar} T \hat{H}_q^0} | x \rangle = \sqrt{\frac{m_q}{2\pi i \hbar T}}$ in the ordinary free propagation kernel.

The transition amplitude between a finite time interval $T$ by the Hamiltonian
$\hat{H}_q^0$ can be written as

$$\langle x_{b} | e^{-\frac{i}{\hbar} T \hat{H}_q^0} | x_{a} \rangle = \lim_{N \to \infty} \int \left( \prod_{i=1}^{N-1} dx_i \right) \prod_{i=1}^{N} \langle x_{i} | e^{-\frac{i}{\hbar} T \Delta t \hat{H}_q^0} | x_{i-1} \rangle ,$$

(2.15)

\textsuperscript{4)} There is another approach to the path integral representation to the propagator\textsuperscript{7)} that is based on the $q$-deformed phase space. Since the configuration space is different from ordinary space, the result is different from ours.
where \( x_N = x_b, \) \( x_0 = x_a, \) and \( \Delta t = T/N. \) The classical action associated with \( \hat{H}_q^0 \) then appears as the phase factor of the transition kernel between two neighboring points \((x_i, x_{i-1})\) with short time interval \( \Delta t. \) Within the first order approximation of \( \Delta t, \) we can evaluate the kernel as

\[
\langle x_i | e^{-\frac{i}{\hbar} \Delta t \hat{H}_q^0} | x_{i-1} \rangle \simeq \langle x_i | x_{i-1} \rangle + \frac{i}{\hbar} \Delta t \frac{\hbar^2}{2m_q} D_i^2 \langle x_i | x_{i-1} \rangle ,
\]

which is consistent with (2.12) within the approximation up to the first order of \( T, \) since \( \langle x|^k \rangle q^2 \langle k, q | x'\rangle = -D_i^2 \langle x|^k \rangle q(k, q | x') \) and \( \int_{-\infty}^{\infty} dk \langle x|^k \rangle q(k, q | x') = \langle x | x'\rangle. \)

Using the momentum expansion of \( \langle x_i | x_{i-1} \rangle = \int_{-\infty}^{\infty} \frac{dp}{2\pi \hbar} e^{ip(x_i-x_{i-1})}, \) the second derivative of \( \langle x_i | x_{i-1} \rangle \) by \( D_i \) on the right-hand side can be evaluated as

\[
D_i^2 \langle x_i | x_{i-1} \rangle = \int \frac{dp}{2\pi \hbar} e^{ip(x_i-x_{i-1})/\hbar} q + q^{-1} \left( \frac{q - q^{-1}}{\hbar} \right)^2 \left\{ \cos \left( \frac{q - q^{-1}}{\hbar} \left( \frac{x_i + x_{i-1}}{2} \right) \right) - 1 \right\} .
\]

Substituting (2.17) for (2.16), the short-time-propagation kernel can be written as

\[
\langle x_i | e^{-\frac{i}{\hbar} \Delta t \hat{H}_q^0} | x_{i-1} \rangle \simeq \int \frac{dp}{2\pi \hbar} \exp \left\{ \frac{i}{\hbar} \Delta t (L(p, x, x_{i-1}) \right\} \text{. Here, } L(p, x, x_{i-1}) \text{ is the phase space Lagrangian of the system in the interval } (x_i, x_{i-1}), \text{ which formally tends to }
\]

\[
L(p, x, \dot{x}) = p \dot{x} - \frac{\hbar^2}{2m_q} \frac{q + q^{-1}}{(q - q^{-1})^2 x^2} \left\{ 1 - \cos \left( \frac{2p}{q + q^{-1}} \frac{q - q^{-1}}{\hbar} \right) \right\} \quad \text{(2.18)}
\]

on assuming \( x_i - x_{i-1} \sim 0(\Delta t) \) as \( \Delta t \to 0. \) The variable \( p \) has the meaning of a momentum conjugate to \( x \) because of \( p = \frac{\partial L}{\partial \dot{x}}, \) and so, the Hamiltonian of the system \( H(x, p) = p \dot{x} - L \) becomes

\[
H(x, p) = \frac{\hbar^2}{2m_q} \frac{q + q^{-1}}{(q - q^{-1})^2 x^2} \sin^2 \left( \frac{q - q^{-1}}{\hbar} \frac{q - q^{-1}}{q + q^{-1}} \right) , \quad \text{(2.19)}
\]

to which one can verify \( H(x, p) \to \frac{1}{2m_q} \dot{x}^2, \) \( (q \to 1). \) If we put \( H = E(= \text{const}), \) we obtain trajectories in the \((x, p)\) phase space, on which the total energy of the system is fixed at \( E. \) The trajectory is a straight line with \( p_0 = \sqrt{(q + q^{-1})m_q E} \) near \( x = 0; \) then, the line will arrive at a turning point \( x_{\text{max}} \) after stretching to some length (Fig. 1), since \( |x| \) is bounded by \( |x| \leq \frac{\hbar}{|q - q^{-1}|} \sqrt{\frac{q + q^{-1}}{m_q E}} \) for a given \( E. \)

\[\]
Furthermore, the constraint $\frac{\partial L}{\partial \dot{p}} = 0$ allows us to solve $p$ as a function of $\dot{x}$, and we obtain

$$p_n = \frac{\hbar}{2x} \frac{q + q^{-1}}{q - q^{-1}} \left[ \sin^{-1} \left\{ \frac{m_q}{\hbar} (q - q^{-1}) \dot{x} \right\} + 2\pi n \right]. \quad (n = 0, \pm 1, \pm 2, \cdots) \quad (2.20)$$

In the above equation, $n$ implies the number that distinguishes branches of $\sin^{-1} \theta$ for $-1 < \theta < 1$. Substituting (2.20) for (2.18), the Lagrangian as a function of $x$ and $\dot{x}$ in the $n$-th branch is given by

$$L_n(x, \dot{x}) = \frac{\hbar}{2x} \frac{q + q^{-1}}{q - q^{-1}} \left[ \sin^{-1} \left\{ \frac{m_q}{\hbar} (q - q^{-1}) \dot{x} \right\} + 2\pi n \right] + \frac{\hbar^2}{2m_q} \frac{q + q^{-1}}{(q - q^{-1})^2 x^2} \left[ \sqrt{1 - \left\{ \frac{m_q}{\hbar} (q - q^{-1}) \dot{x} \right\}^2} - 1 \right]. \quad (2.21)$$

From this Lagrangian, by taking into account that the $p \dot{x}$ term does not contribute to the Lagrange equation, a little calculation leads to the equation of motion of $x$, for example, for $n = 0$, such that

$$m_q \ddot{x} = -m_q \frac{\dot{x}^2}{x} - \frac{2\hbar^2}{m_q} \sqrt{1 - \left\{ \frac{m_q}{\hbar} (q - q^{-1}) \dot{x} \right\}^2} \left[ \sqrt{1 - \left\{ \frac{m_q}{\hbar} (q - q^{-1}) \dot{x} \right\}^2} - 1 \right], \quad (2.22)$$

to which one can verify $\ddot{x} \to 0$, $(q \to 1)$. It is interesting that the customary function for a harmonic oscillator $x(t) = A \sin\{\omega(t - t_0)\}$ becomes a solution of the above equation of motion provided that $\frac{m_q \omega^2 A^2}{2} = \frac{\hbar}{|q - q^{-1}|}$. This indicates that the free

\[ \text{constraint in the main text, the \"sin\" function becomes a special solution of Eq. (2.22).} \]
particle in $q$-deformed phase space allows bounded motions in ordinary phase space. The result, however, depends on the validity of the Lagrangian (2.21) beyond a short time interval.

§3. Particle in a box

The particle in a box, a square well potential with perfectly rigid walls, extending from 0 to $L$, is another interesting solvable problem,\textsuperscript{8) to which the results suffer fairly large modification from those in the $q = 1$ case. As usual, the eigenvalue problem of $\hat{H}_0$ is set as

$$\hat{H}_0 \psi(x) = -\hbar^2 \frac{D_x^2 \psi(x)}{2m_q} = E_q \psi(x), \quad (0 < x < L)$$  \hspace{1cm} (3.1)

with the boundary conditions

$$\psi(0) = \psi(L) = 0. \hspace{1cm} (3.2)$$

If we exclude singular solutions, the functional space of eigenstates is constructed of a pair of $q$-deformed sin functions defined by

$$\sin_q(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n + 1]!} x^{2n+1}, \hspace{1cm} (3.3)$$

$$\bar{\sin}_q(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n + 1]!} q^{-\frac{1}{2} n(n-1)} x^{2n+1}. \hspace{1cm} (3.4)$$

In what follows, we simply write $S_n(x) = N_n \sin_q(k_n x)$ and $\bar{S}_n(x) = N_n \bar{\sin}_q(k_n x)$, where $N_n$ is a normalization constant. By definition, these functions obviously satisfy $S_n(0) = \bar{S}_n(0) = 0$. Furthermore, one can show the following (Appendix B).

1) The $\{S_n(x)\}$ are eigenfunctions of $D_x^2$ characterized by $D_x^2 S_n(x) = -k_n^2 S_n(x)$ and the orthogonality $\langle S_n S_m \rangle = \delta_{n,m}$. Here, $\langle \cdots \rangle$ is the $q$-integral defined by (B.24) with $a = 0$ and $b = q^{-1}L$. Contrary to the case of $q = 1$, however, the boundary condition $\psi(L) = 0$ is not satisfied by a $S_n(x)$ alone.

2) The independent functions chosen on the basis of the boundary condition $\psi(L) = 0$ are $\{S_n(x)\}$, for which $k_n$ is given by

$$k_n = \frac{\pi_q(n)}{L}, \quad (n = 1, 2, \cdots) \hspace{1cm} (3.5)$$

where $\pi_q(n)$ is the solution of $\pi n = \sum_{k=0}^{\infty} \tan^{-1}\{(1 - q^{-2})q^{-2k}x\}$ with respect to $x$. Although $\pi_q(x)$ is not an elementary function of $x$, it obviously tends to $\pi n$ in the limit $q \to 1$. The $\pi_q(n)$ can be expanded in an odd power series of $\pi n$, and a few of its terms are given by (B.15)–(B.18)

$$\pi_q(n) = (\pi n) + \frac{1}{3} \frac{(1 - q^{-2})^3}{1 - q^{-6}} (\pi n)^3$$
Table I. Several approximate values of $\pi_q(n)$ for $q = 1.5.$

| $n$ | 1    | 2    | 3    | 4    |
|-----|------|------|------|------|
| $\pi_q(n)$ | 5.28 | 3.00×10 | 1.55×10² | 7.87×10² |

$$+ \left[ -\frac{1}{5} \frac{(1-q^{-2})^5}{1-q^{-10}} + \frac{1}{3} \left\{ \frac{(1-q^{-2})^3}{1-q^{-6}} \right\}^2 \right] (\pi n)^5 + \cdots.$$ (3.6)

We note that no $\bar{S}_n(x)$ is an eigenfunction of $D_x^2$ in the usual sense, since they satisfy $D_x^2 \bar{S}_n(x) = -q^{-1}k^2_n \bar{S}_n(q^{-2}x);$ however, it can be verified that $D_x^2 \bar{S}_n(x) = -k^2_n \bar{S}_n(x)$ for $D_x \equiv x^{-1}[\hat{N}, q^2].$

3) Each $S_n(x)$ can be expanded as a linear combination of $\{\bar{S}_n(x)\}$ and vice versa; in this sense, the boundary condition at $x = L$ is also satisfied by $\{S_n(x)\}$. Therefore, the energy eigenvalues for a particle in the box under consideration can be written as

$$E_n = \frac{\hbar^2}{2m_l} \left( \frac{\pi_q(n)}{L} \right)^2 \quad (n = 1, 2, \cdots)$$ (3.7)

It should then be stressed that for a large deformation such as $q \geq 1.2$, the $\pi_q(n)$ increases rapidly in response to an increase in $n$, as can be seen from Table I and Fig. 2. The result shows that $\log \pi_q(n)$ is almost linear in this interval of $n$.

In this section, we have studied the eigenstates characterized by (3.1) and (3.2) within the framework of regular functions. If we allow a singular structure of wave functions, however, we may multiply $\{S_n(x)\}$ by the singular phase functions

$$Q_m(x) = \exp \left\{ i \frac{2\pi m}{\log q} \log \left| \frac{x}{L} \right| \right\} \quad (m = \pm 1, \pm 2, \cdots)$$ (3.8)

The $S_n(x)$ and $Q_m(x)S_n(x)$ satisfy the same eigenvalue equation; the singular structure disappears from the probability amplitude $|Q_m(x)S_n(x)|^2 = |S_n(x)|^2$. Thus, $Q_m(x)S_n(x)$ also satisfies the same boundary condition as $S_n(x)$. This phase ambiguity is due to the invariance of the Hamiltonian $\hat{H}_0$, such as

$$\hat{H}_0(x, \partial_x) = \hat{H}_0 \left(x, \partial_x + i \frac{2\pi m}{\log q} \right) \quad (0 < x < L)$$ (3.9)

In other words, we can say that the gauge-potential-like term $\frac{2\pi m}{\log q} \frac{1}{x}$ exerts no physical effect on the $q$-deformed particle under consideration. This is a peculiar property of such a particle, which appears only when $q \neq 1$.

§4. Summary and discussion

We have studied the particle associated with the $q$-deformed phase space, to which the momentum operator in quantized theory is given by $\hat{p}_q = -\frac{i\hbar}{2}(\partial_x + \partial_x)$, where $\partial_x$ is the operator characterized by the $q$-deformed Heisenberg algebra $\partial_x x -$
Our approach to this $q$-deformed phase space is to represent the deformed operator $\hat{\partial}_x$ as being in ordinary space by means of mapping from operators $(x, \partial_x)$; in other words, we tried to represent the $q$-deformed dynamics as an effective theory in ordinary variable spaces.

In §2, we discussed the free particle in the $q$-deformed phase space without a boundary. We first derived an effective action for such a particle represented in the ordinary variable spaces by the path integral method. The form of the action is evaluated from the short time propagator, though it is limited in applications. This is because the $q$-exponential function giving the plane wave in the deformed space does not satisfy the associative law in the usual sense. Rather, if we apply this action to a long-time motion, however, we can derive an interesting result wherein the trajectories of the particle are drawn as if they are bounded in configuration space. Furthermore, we evaluated the exact propagator, and found that the propagation from $x$ to $q^\pm x$ in the $q$-deformed phase space corresponds to one from $x$ to $x$ in ordinary phase space. We note that for the short-time action, the introduction of external potential is simply an additional effect.

Section 3 concerned the case of compact space bounded by perfectly rigid walls placed in $x = 0$ and $x = L$; that is, we discussed the particle in the box $0 < x < L$. Then, we can solve the eigenvalue problem using a pair of $q$-sin functions such that one of those are independent functions governed by the boundary conditions $\hat{S}_n(0) = \hat{S}(L)_n = 0$, and the others are characterized by the eigenvalue equation $\hat{D}_x^2 \hat{S}_n = -k_n^2 \hat{S}_n$. These two sin functions tend to the standard sin function as $q \to 1$. By virtue of those sin functions, the energy eigenvalues can be determined as $E_n = \frac{\hbar^2}{2m_q} \left( \frac{\pi_n}{L} \right)^2$, $(n = 1, 2, \cdots)$, where $\pi_q(n) (\to \pi n, \quad q \to 1)$ is a $q$-dependent function of $n$. The $q$ dependence of $\pi_q(n)$ is fairly large; and, it becomes a rapid increase function of $n$ for a large $q$.

The boundary condition is not limited to the above case; and, we can consider the cases of $\psi(0) = 1$ and $\psi(\frac{L}{2}) = 0$. In these cases, we need a pair of $q$-cos functions $\cos_q(x)$ and $\bar{\cos}_q(x)$ to construct a space of wave functions belonging to the eigenspace of $\hat{D}_x^2$ followed by the boundary conditions. By definition (B·5), $\cos_q(x)$ has the product form

$$\cos_q(x) = \frac{1}{2} \lim_{N \to \infty} \left[ \left( 1 + \frac{ix}{[N]} \right)^N_q + \left( 1 - \frac{ix}{[N]} \right)^N_q \right] \propto \prod_{k=0}^{N-1} e^{i\theta_k^{(N)}(x)} + 1, \quad (4.1)$$

where $\theta_k^{(N)}$ is given by (B·9). Thus, for zero $\cos_q(x)$, $\theta_k^{(N)}(x)$ satisfies $\sum_{k=0}^{N-1} \theta_k^{(N)}(x) = (2n+1)\pi$ with an integral $n$. From these conditions, one can find the zeros of $\cos_q(x)$ similarly to the case of $\sin(x)$ at $x = \pi_q(n + \frac{1}{2})$, $(n = 0, \pm 1, \pm 2, \cdots)$. Thus, under the normalization $\cos_q(0) = 1$, we can obtain the expression

$$\cos_q(x) = \prod_{n=1}^{\infty} \left[ 1 - \left( \frac{x}{\pi_q(n + \frac{1}{2})} \right)^2 \right]. \quad (4.2)$$

The $\cos_q(x)$ is clearly an even function; however, its odd property between $x = 0$ and $x = L$ is lost because of $\cos_q(\pi) \neq -1$. 

$qx \hat{\partial}_x = 1$. The $q$-deformed phase space is to represent the deformed operator $\hat{\partial}_x$ as being in ordinary space by means of mapping from operators $(x, \partial_x)$; in other words, we tried to represent the $q$-deformed dynamics as an effective theory in ordinary variable spaces.
The results obtained in §3 gives us insight concerning the mass spectrum of a particle embedded in a higher-dimensional spacetime with this type of compact space as the extra dimensions. For example, the Klein-Gordon equation in five-dimensional spacetime with $q$-deformed fifth dimension,

$$ (\partial_\mu \partial^\mu - D_5^2 + m^2) \psi(x^\mu, x^5) = 0 \quad (4.3) $$

leads to a mass spectrum such that only a few states correspond to light-mass particles upon adjusting the parameter $q$ suitably. These problems will be discussed elsewhere.

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**Appendix A**

--- On the Eigenstates of $\hat{p}_q$ in a Free Boundary space ---

The key to studying the eigenstates of $\hat{p}_q = -i\hbar (\frac{q+1}{2q}) D_x$ is the equation

$$ D_x x^n = [n] x^{n-1}. \quad (n = 0, 1, 2, \cdots) \quad (A.1) $$

From this equation, one can verify that the $q$-exponential function defined by

$$ e_q(x) = e^x_q = \sum_{n=0}^{\infty} \frac{1}{[n]!} x^n, \quad \left( [0]! \equiv 1 \text{ and } [n]! = \prod_{k=1}^{n} [k] \text{ for } n \geq 1 \right) \quad (A.2) $$

satisfies

$$ D_x e_q(ax) = ae_q(ax) \quad (a = \text{const}) \quad (A.3) $$

It is obvious, by definition, that $e_q(x) \to e^x, \ (q \to 1)$ and $e_q(x)e_q(y) \neq e_q(x+y)$ for $q \neq 1$. Since $[n] \geq n$, the series of $e_q(x)$ is more rapidly convergent than the usual exponential function. In particular, $e_q(x) \to 0$ as $x \to -\infty$. Using this $q$-exponential function we can introduce the $q$-analog of $e^{ix} = \cos(x) + i \sin(x)$ so that

$$ e_q(ix) = \cos_q(x) + i \sin_q(x) \quad (A.4) $$

from which we can explicitly write

$$ \cos_q(x) = \frac{1}{2} (e_q(ix) + e_q(-ix)) = \sum_{k=0}^{\infty} \frac{(-1)^k}{[2k]!} x^{2k}, \quad (A.5) $$

$$ \sin_q(x) = \frac{1}{2i} (e_q(ix) - e_q(-ix)) = \sum_{k=0}^{\infty} \frac{(-1)^k}{[2k+1]!} x^{2k+1}. \quad (A.6) $$
Now the function
\[ \langle x|k, q \rangle = N_q e_q(ikx) \], \quad (N_q = \text{const}) \tag{A.7} \]
is an eigenstate of \( \hat{p}_q \) belonging to the eigenvalue \( \hbar k \), though it is not a simple plane-wave function in the ordinary \( x \) space. Here, the wave number \( k \) may be an ordinary continuous real number running from \(-\infty\) to \( \infty \).

To determine the constant \( N_q \), let us evaluate the inner product \( \langle k, q|k', q \rangle \) by assuming \(|k| < |k'|\) without loss of generality. Then, introducing a factor \( \epsilon(\rightarrow +0) \) to define a convergent inner product, we have
\[
\langle k, q|k', q \rangle = |N_q|^2 \int_{-\infty}^{\infty} dx e_q \left( -ikx - \frac{i}{2} \epsilon |x| \right) e_q \left( ik'x - \frac{i}{2} \epsilon |x| \right) = |N_q|^2 \sum_{n=0}^{\infty} \left\{ -i \left( k - \frac{i \epsilon}{2} \right) \right\}^n \frac{1}{[n]!} \int_0^{\infty} dx^n e_q \left( i(k' + \frac{\epsilon}{2})x \right) + \text{c.c.} \tag{A.8} \\
= |N_q|^2 \sum_{n=0}^{\infty} \frac{i}{k' + \frac{i \epsilon}{2}} \left( \frac{k - \frac{i \epsilon}{2}}{k' + \frac{i \epsilon}{2}} \right)^n \frac{1}{[n]!} \int_0^{\infty} dt e_q^{-t}t^n + \text{c.c.} \tag{A.8} ,
\]

where the analytic continuation \( i(k' + \frac{i \epsilon}{2})x \rightarrow -t \) has been performed. Furthermore, due to the scale invariance of \( dt / \{ (q - q^{-1})t \} \), it holds that \( \int_0^{\infty} dt \{ D_t f(t) \} g(t) = -\int_0^{\infty} dt f(t) \{ D_t g(t) \} \) for a convergent integral. Thus, taking (A.1) and (A.3) into account, we obtain
\[
\int_0^{\infty} dt e_q^{-t}t^n = -\int_0^{\infty} dt \{ D_t e_q^{-t} \} t^n = [n] \int_0^{\infty} dt e_q^{-t}t^{n-1} = \cdots = [n]! \Gamma_q[1] \tag{A.9}
\]
with the definition of the \( q \)-gamma function
\[
\Gamma_q[n] = \int_0^{\infty} dt e_q^{-t}t^{n-1}, \quad (n = 1, 2, \cdots) \tag{A.10}
\]
for which one can see that \( \Gamma_q[1] \neq 1 \). Therefore, substituting (A.9) for (A.8), we arrive at the expression \( ^{13} \)
\[
\langle k, q|k', q \rangle = |N_q|^2 \left( \frac{i}{k' - k + \frac{i \epsilon}{2}} + \text{c.c.} \right) \times \Gamma_q[1] = 2\pi |N_q|^2 \times \frac{1}{\pi} \frac{\epsilon}{(k' - k)^2 + \epsilon^2} \times \Gamma_q[1] = \delta(k' - k) \tag{A.11} \\
\]
provided that
\[ N_q = \frac{1}{\sqrt{2\pi \Gamma_q[1]}} \tag{A.12} \]

To apply \( \langle x|k, q \rangle \) to the propagation kernel, it is worthwhile to study, in more detail, the product \( e_q(x)e_q(y) \), which can be written as
\[ e_q(x)e_q(y) = \sum_{N=0}^{\infty} \frac{1}{[N]!} (x + y)^N_q \tag{A.13} \]
where \((x + y)^N_q\) is the \(q\)-binomial expansion defined by

\[
(x + y)^N_q = \sum_{n+m=N} \frac{[N]!}{[n]![m]!} x^n y^m .
\] (A.14)

We have used the symbol “\(\dagger\)” to stress that \((x + y)^N_q\) is not a function of \(x + y\). Now, the \(q\)-binomial expansion satisfies the following recursion formula:

\[
(x + y)^{N+1}_q = (x + q^N y)(x + q^{-1}y)^N_q .
\] (A.15)

Indeed, it is not difficult to verify that

\[
x(x + q^{-1}y)^N_q = \sum_{n=0}^{N} \frac{[N]!}{[n]![N-n]!} x^{n+1} y^{N-n} q^{-N+n} , \quad (k = n + 1)
\]

\[
= \sum_{k=1}^{N+1} \frac{[N + 1]!}{[k]![N + 1 - k]!} x^k y^{(N+1)-k} \times \frac{[k]q^{-(N+1)+k}}{[N + 1]} \quad \text{(A.16)}
\]

and

\[
q^N y(x + q^{-1}y)^N_q = \sum_{n=0}^{N} \frac{[N]!}{[n]![N-n]!} x^n y^{(N+1)-n} q^n , \quad (k = n)
\]

\[
= \sum_{k=1}^{N+1} \frac{[N + 1]!}{[k]![N + 1 - k]!} x^k y^{(N+1)-k} \times \frac{[(N + 1) - k]q^k}{[N + 1]} . \quad \text{(A.17)}
\]

Here, \(\frac{[k]q^{-(N+1)+k}}{[N + 1]}\) \(\big|_{k=N+1} = 1\), \(\frac{[(N + 1) - k]q^k}{[N + 1]}\) \(\big|_{k=0} = 1\) and \(\frac{[(N + 1) - k]q^k}{[N + 1]} + \frac{[(N + 1) - k]q^k}{[N + 1]} = 1\) for \(1 \leq k \leq N\); thus, the sum of Eqs. (A.16) and (A.17) becomes the left-hand side of Eq. (A.15). Using Eq. (A.15) repeatedly, we can also obtain a factorized form to the \(q\)-binomial expansion:

\[
(x + y)_q^N = (x + q^{-N+1}y)(x + q^{-N+3}y) \cdots (x + q^{-N-1}y)
\]

\[
= \prod_{k=0}^{N-1} (x + q^{N-1-2k}y) . \quad \text{(A.18)}
\]

If necessary, \(e_q^x e_q^{-y} \), \((x - y)_q^N\) and its factorized form are obtained by substituting \(-y\) for \(y\) in Eqs. (A.13), (A.14) and (A.18), respectively. Then, it can be found that \((x - y)_q^N\) has simple zeros at \(x = q^{N-1-2k}y\), \((k = 0, 1, \cdots N-1)\), in contrast to \((x - y)_q^N\) having a zero of order \(N\) at \(x = y\). In particular, since \((x - y)^{2k}(k = 1, 2, \cdots)\) have zeros at \(y = q^{\pm x}\), we have

\[
\text{Re}\{e_q(ix)e_q(-i\cdot q^{\pm 1}x)\} = \cos_q(x) \cos_q(q^{\pm 1}x) + \sin_q(x) \sin_q(q^{\pm 1}x)
\]

\[
= \sum_{k=0}^{\infty} \frac{(-1)^k}{[2k]!} (x \cdot q^{\pm 1}x)^{2k} = 1 . \quad \text{(A.19)}
\]
Appendix B

--- Eigenvalue Problem of \( \hat{p}_q^2 \) for a Particle in a Box ---

The eigenvalue problem for a particle in the box \( 0 \leq x \leq L \) shows different characteristics from those of the \( q = 1 \) case. To study the problem, we introduce another type of \( q \)-exponential function defined by\(^{10),11}\)

\[
\bar{e}_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]!} q^{-\frac{1}{2}n(n-1)} = \sum_{n=0}^{\infty} \frac{x^n}{[n, q^2]!},
\]

where

\[
[n, q^2] = \frac{q^{2n} - 1}{q^2 - 1}.
\]

The importance of this function is that with the help of \( \frac{[N]!}{[r]![N-r]!} \sim \frac{[N]^r}{[r]^r} q^{-\frac{1}{2}r(r-1)} \) for large \( N \), one can obtain the expression

\[
\bar{e}_q(x) = \lim_{N \to \infty} \left(1 + \frac{x}{[N]}\right)^N.
\]  

(B-3)

The \( q \)-sin/cos functions in this case can be defined, as usual, by

\[
\bar{\sin}_q(x) = \frac{1}{2i} (\bar{e}_q(ix) - \bar{e}_q(-ix)),
\]

(B-4)

\[
\bar{\cos}_q(x) = \frac{1}{2} (\bar{e}_q(ix) + \bar{e}_q(-ix)).
\]

(B-5)

It should be noted that \( e_q(x) \) and \( \bar{e}_q(x) \) are eigenfunctions of the difference operators \( D_x \equiv x^{-1}[\hat{N}] \) and \( \bar{D}_x \equiv x^{-1}[\hat{N}, q^2] \), respectively; that is, we have

\[
D_x e_q(ax) = a e_q(ax), \quad \bar{D}_x \bar{e}_q(ax) = a \bar{e}_q(ax).
\]

(B-6)

However, we can also verify that

\[
D_x \bar{e}_q(ax) = a \bar{e}_q(a^{-1}x), \quad \bar{D}_x e_q(ax) = a e_q(ax).
\]

(B-7)

Now, the characteristics of \( \bar{e}_q(x) \) is that \( \bar{\sin}(x) = \text{Im} \bar{e}_q(ix) \) has an infinite product representation, an analogue of \( \sin(x) = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{(\pi n)^2}\right) \), for \( q = 1 \), with the aid of (B-3). To derive the formula, we apply the \( q \)-binomial expansion (A.18) to (B-3). Then, we obtain

\[
\bar{\sin}_q(x) = \frac{1}{2i} \lim_{N \to \infty} \left[ \left(1 + \frac{ix}{[N]}\right)_q^N - \left(1 - \frac{ix}{[N]}\right)_q^N \right]
= \frac{1}{2i} \lim_{N \to \infty} \prod_{k=0}^{N-1} \left(1 - q^{N-1-2k} \frac{ix}{[N]}\right) \times \prod_{k=0}^{N-1} \left(\frac{1 + q^{N-1-2k} \frac{ix}{[N]}}{1 - q^{N-1-2k} \frac{ix}{[N]}}\right) - 1.
\]  

(B-8)
It is obvious that the zeros of $\overline{\sin}_q(x)$ exist in the factor $\prod_{k=0}^{N-1} e^{i\theta_k^{(N)}(x)}$ on the right-hand side of (B.19). Here, we have put

$$e^{i\theta_k^{(N)}(x)} = \frac{1 + q^{N-1-2k} \frac{ix}{[N]}}{1 - q^{N-1-2k} \frac{ix}{[N]}}. \quad (B.9)$$

This implies that if $x$ is a zero of $\overline{\sin}_q(x)$, then $\theta_k^{(N)}(x)$ satisfies

$$\sum_{k=0}^{N-1} \theta_k^{(N)}(x) = 2\pi n. \quad (n = 0, \pm 1, \pm 2, \ldots) \quad (B.10)$$

Furthermore, since $\theta_k^{(N)}(x)$ in Eq. (B.9) can be solved with respect to $x$ as

$$x = q^{-N+1+2k[N]} \tan \frac{\theta_k^{(N)}(x)}{2}, \quad (B.11)$$

we can get the expression

$$\Theta^{(N)}(x) = \frac{1}{2} \sum_{k=0}^{N-1} \theta_k^{(N)}(x) = \sum_{k=0}^{N-1} \left[ \tan^{-1} \left\{ q^{N-1} \frac{1}{[N]} q^{-2k} x \right\} + \delta \right]. \quad (B.12)$$

Here, $\delta = \pi \times \text{(integer)}$ is an undetermined phase originating from the periodicity of $\tan(x)$, which can be absorbed by the right-hand side of Eq. (B.12); thus, we hereafter set $\delta = 0$.

This equation tells us the structure of zeros of $\overline{\sin}_q(x)$. First, it is obvious that $\Theta^{(N)}(x)$ tends to $x$ as $q \to 1$ because $q^{N-1}/[N] \sim N^{-1}$ for $q \sim 1$. Namely, the zeros of $\overline{\sin}_q(x)$ become $\pi n (n = 0, \pm 1, \ldots)$, as expected in this limit.

Secondly, for $q \neq 1$, we obtain the asymptotic behavior $\Theta^{(N)}(x) \sim \frac{N\pi}{2} - \frac{[N]^2}{x}$ for a large $x$ by taking $\tan^{-1}(ax) \sim \frac{x}{2} - \frac{1}{ax}$ into account. Then, $\Theta^{(N)}(x)$ tends to the upper bound $\frac{\pi N}{2}$ as $x \to \infty$ for fixed $N$; therefore, the number of zeros determined by $\Theta^{(N)}(x) = \pi n$ becomes finite. In other words, $\Theta^{(N)}(x)$ does not have an inverse function. However, since $\Theta^{(N)}(x)$ is an increase function of $N$, $y = \Theta(x) \equiv \lim_{N \to \infty} \Theta^{(N)}(x)$ can be solved with respect to $x$. To do so, we write

$$\Theta(x) = \sum_{k=1}^{\infty} a_k x^k, \quad (B.13)$$

where $a_2 = a_4 = \cdots = 0$ and

$$a_{2k-1} = \frac{(-1)^{k-1} (1 - q^{-2})^{2k-1}}{2k-1 \ 1 - q^{-2(2k-1)}}, \quad (k = 1, 2, \ldots) \quad (B.14)$$

Since $\Theta(x)$ is a single-valued analytic function of $x$, $y = \Theta(x)$ can be inverted by the
Lagrange expansion\(^*) x = \sum_{k=1}^{\infty} b_k y^k \) with \( b_2 = b_4 = \cdots = 0 \) and

\[
b_1 = \frac{1}{a_1} = 1,
\]

\[
b_3 = -\frac{a_3}{a_1^3} = \frac{1}{3} \frac{(1 - q^{-2})^3}{1 - q^{-6}},
\]

\[
b_5 = -\frac{a_5}{a_1^5} + \frac{3a_3^2}{a_1^7} = -\frac{1}{5} \frac{(1 - q^{-2})^5}{1 - q^{-10}} + \frac{1}{3} \left\{ \frac{(1 - q^{-2})^3}{1 - q^{-6}} \right\}^2,
\]

\[
b_7 = -\frac{a_7}{a_1^7} + 4 \frac{a_3 a_5}{a_1^9} - 12 \frac{a_3^3}{a_1^{10}} = \frac{1}{7} \frac{(1 - q^{-2})^7}{1 - q^{-14}} - \frac{8}{15} \frac{(1 - q^{-2})^8}{(1 - q^{-6})(1 - q^{-10})} + \frac{4}{9} (1 - q^{-2})^9,
\]

and so on. Therefore, writing \( \pi_q(n) = \sum_{k=1}^{\infty} b_k (\pi n)^k \) and taking into account that \( \Theta(x) \) is an odd function of \( x \), we obtain the expression

\[
\sin_q(x) = x \prod_{n=0}^{\infty} \left[ 1 - \left( \frac{x}{\pi_q(n)} \right)^2 \right]
\]

under the normalization \( \sin_q(x)/x \to 1, (x \to 0) \). Since \( \pi_q(n) \) obviously tends to \( \pi n \) as \( q \to 1 \), the right-hand side of (B.19) reduces to the standard product formula of \( \sin(x) \) in this limit.

Now, with the aid of the product formula (B.19), one can find that the function

\[
\tilde{S}_n(x) \equiv N_n \sin_q(k_n x), \quad (N_n = \text{const}, \ k_n = \pi_q(n)/L)
\]

is a solution of the eigenvalue problem

\[
\tilde{D}^2 \tilde{S}_n(x) = -k_n^2 \tilde{S}_n(x), \quad (n = 1, 2, \cdots)
\]

under the boundary conditions \( \tilde{S}_n(0) = \tilde{S}_n(L) = 0 \).

Turning back to the eigenvalue problem of \( D^2 \), let us define

\[
S_n(x) \equiv N_n \sin_q(x) = q^{\frac{1}{2}} \tilde{N}(\tilde{N}-1) \tilde{S}_n(x).
\]

Then, taking \( D^2 = q^{\frac{1}{2}} \tilde{N}(\tilde{N}-1) \tilde{D}^2 q^{-\frac{1}{2}} \tilde{N}(\tilde{N}-1) \) into account, we obtain

\[
(D^2 + k_n^2)S_n(x) = q^{\frac{1}{2}} \tilde{N}(\tilde{N}-1) \left\{ (D^2 + k_n^2)S_n(x) \right\} = 0.
\]

\(^*)\) The \( b_n \) can be calculated by\(^{12)} \) \( b_n = \frac{1}{a_1^n} \left[ \left( \frac{d}{dx} \right)^n \left( \frac{\Theta(z)}{z} \right)^n \right]_{x=0}, \ (n = 1, 2, \cdots) \). Here, writing \( \Theta^\infty(x)/x = a_1(1 + B(x)) \) using \( B(x) = \tilde{a}_2 x^2 + \tilde{a}_4 x^4 + \tilde{a}_6 x^6 + \cdots \) with \( \tilde{a}_k = a_{k+1}/a_1 \), we obtain the expression

\[
b_n = \frac{1}{a_1^n} \left[ \left( \frac{d}{dx} \right)^n \sum_{k=0}^{n-1} \left( \begin{array}{c} -n \\ k \end{array} \right) B(x)^k \right]_{x=0}, \quad \text{which leads to, for } n = 2m + 1,
\]

\[
b_{2m+1} = \frac{1}{(2m+1)a_1^{2m+1}} \sum_{k=1}^{m} \frac{(-2m+1)}{k} \sum_{i_1+\cdots+i_k=2m} \tilde{a}_{i_1} \cdots \tilde{a}_{i_k}. \quad (m = 1, 2, \cdots)
\]
Thus, the function $S_n(x)$ is also an eigenstate of $D^2$ belonging to the eigenvalue $-k_n^2$, which is determined by the zero point of the function $\tilde{S}_n(x)$ instead of $S_n(x)$.

The normalization of $\{S_n(x)\}$ can be done by introducing the $q$-integral defined by the inverse $D^{-1} = [\hat{N}]^{-1}x = x[\hat{N} + 1]^{-1}$ in such a way that

$$\int^b_a \Delta_q f(x) \equiv D^{-1}f(x)|^b_a = \sum_{k=0}^{\infty} \Delta_q x q^{-2(2k+1)} f(q^{-2(2k+1)}x)|^b_a, \quad (q > 1) \quad \text{(B.24)}$$

where $\Delta_q x = (q - q^{-1})x$. The right-hand side of (B.24) reduces to the standard Riemann integral of $f(x)$ as $q \to 1$; one can also verify that

$$\int^b_a \Delta_q x \{Df(x)\} = f(x_b) - f(x_a), \quad \text{(B.25)}$$

provided that the infinite series of (B.24) converges. Then, integrating both sides of the equation

$$D[S_n(qx)DS_m(x)] = DS_n(qx)DS_m(qx) + S_n(x)D^2S_m(x) \quad \text{(B.26)}$$

from 0 to $x_0$, we obtain

$$[S_n(qx)DS_m(x) - S_m(qx)DS_n(x)]^x_0 = -(k^2_m - k^2_n) \langle S_nS_m \rangle, \quad \text{(B.27)}$$

where $\langle \cdots \rangle = \int^x_0 \Delta_q x \langle \cdots \rangle$ and $x_0$ is a point characterized by $S_n(qx_0) = 0$, $(n = 1, 2, \cdots)$. Thus, $S_n$ satisfies the orthogonality $\langle S_nS_m \rangle = \delta_{n,m}$ upon adjusting the normalization constants $N_n$ appropriately. We here assume that $\{S_n(x)\}$ forms a complete basis of the $\{\tilde{S}_n(x)\}$ space. Then, we have

$$\tilde{S}_n(x) = q^{-\frac{1}{2}}\hat{N}(\hat{N}-1)S_n(x) = \sum_{m=1}^{\infty} S_m(x)\langle S_m q^{-\frac{1}{2}}\hat{N}(\hat{N}-1)S_n \rangle, \quad \text{(B.28)}$$
from which follows
\[
S_n(x) = q^{1/2} \hat{N}(\hat{N} - 1) \bar{S}_n(x) = q^{1/2} \hat{N}(\hat{N} - 1) \sum_{m=1}^{\infty} S_m(x) \langle S_m q^{-1/2} \hat{N}(\hat{N} - 1) S_n \rangle \\
= \sum_{m=1}^{\infty} \bar{S}_m(x) \langle S_m q^{-1/2} \hat{N}(\hat{N} - 1) S_n \rangle. \tag{B.29}
\]

The expansion (B.29) means that \( S_n(x) \) satisfies the boundary condition \( S_n(L) = 0 \) in addition to \( S_n(0) = 0 \). On the other hand, the expansion (B.28) requires \( \bar{S}_n(qx_0) = 0 \) to ensure the orthogonality of \( \{ S_n(x) \} \). Therefore, by choosing \( x_0 = q^{-1}L \), both expansions (B.28) and (B.29) become consistent.

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