BUBBLING SOLUTIONS FOR THE LIouVILLE EQUATION AROUND A QUANTIZED SINGULARITY IN SYMMETRIC DOMAINS

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(Communicated by Jaeyoung Byeon)

Abstract. We are concerned with the existence of blowing-up solutions to the following boundary value problem

\[-\Delta u = \lambda V(x)e^u - 4\pi N\delta_0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,
\]

where \(\Omega\) is a smooth and bounded domain in \(\mathbb{R}^2\) such that \(0 \in \Omega\), \(V\) is a positive smooth potential, \(N\) is a positive integer and \(\lambda > 0\) is a small parameter. Here \(\delta_0\) defines the Dirac measure with pole at \(0\). We assume that \(\Omega\) is \((N + 1)\)-symmetric and we find conditions on the potential \(V\) and the domain \(\Omega\) under which there exists a solution blowing up at \(N + 1\) points located at the vertices of a regular polygon with center \(0\).

1. Introduction. Given \(\Omega\) a smooth and bounded domain in \(\mathbb{R}^2\) containing the origin, consider the following Liouville equation with Dirac mass measure

\[
\begin{cases}
-\Delta u = \lambda V(x)e^u - 4\pi N\delta_0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega.
\end{cases}
\]

(1.1)

Here \(\lambda\) is a positive small parameter, \(\delta_0\) denotes Dirac mass supported at \(0\), \(V\) is a smooth potential satisfying \(\inf_{\Omega} V(x) > 0\) and \(N\) is a positive integer.

Singular Liouville equations appear for instance in the Abelian Maxwell-Higgs and Chern-Simons-Higgs theories of superconductivity, in the self-dual regime. For instance, (1.1) arises in the study of vortices in a planar model of Euler flows (see [14, 39]) and naturally in the construction of singular conformal metrics in \(\mathbb{R}^2\). In vortex theory the interest in constructing blowing-up solutions is related to relevant physical properties, in particular the presence of vortices with a strongly localised electromagnetic field. We refer the reader to [26, 27, 34] and references therein for recent developments in this subject and related issues.

The asymptotic behaviour of family of blowing up solutions can be referred to the papers [1, 5, 24, 25, 29, 31] for the regular problem, i.e. when \(N = 0\). An extension to the singular case \(N > 0\) is contained in [2, 3].

2020 Mathematics Subject Classification. Primary: 35J20, 35J57; Secondary: 35J61.

Key words and phrases. Singular Liouville equation, bubbling solutions, finite-dimensional reduction.

The research of T. D’Aprile is partially supported by the MIUR Excellence Department Project awarded to the Department of Mathematics, University of Rome “Tor Vergata”, CUP ES3C18000100006, and by the group GNAMPA of INdAM Istituto Nazionale di Alta Matematica.
The analysis of the blowing-up behaviour at points away from 0 actually is very similar to the asymptotic analysis arising in the regular case, which has been pursued with success and, at the present time, an accurate description of the concentration phenomenon is available. Precisely, the analysis in the above works yields that if $u_\lambda$ is an unbounded family of solutions of (1.1) for which $\lambda \int_\Omega V(x)e^{u_\lambda}$ is uniformly bounded and $u_\lambda$ is uniformly bounded in a neighborhood of 0, then, up to a subsequence, there is an integer $m \geq 1$ such that

$$\lambda \int_\Omega V(x)e^{u_\lambda}dx \to 8\pi m \quad \text{as} \quad \lambda \to 0^+. \quad (1.2)$$

Moreover there are points $\xi_1^\lambda, \ldots, \xi_m^\lambda \in \Omega$ which remain uniformly distant from the boundary $\partial \Omega$, from 0 and from one another such that

$$\lambda V(x)e^{u_\lambda} - 8\pi \sum_{j=1}^m \delta_{\xi_j^\lambda} \to 0 \quad (1.3)$$

in the measure sense. Also the location of the blowing-up points is well understood when concentration occurs away from 0. Indeed, in [29] and [31] it is established that the $m$-tuple $(\xi_1^\lambda, \ldots, \xi_m^\lambda)$ converges, up to a subsequence, to a critical point of the functional

$$\frac{1}{2} \sum_{j=1}^m H(\xi_j, \xi_j) + \frac{1}{2} \sum_{j, h=1}^m G(\xi_j, \xi_h) - \frac{N}{2} \sum_{j=1}^m G(\xi_j, 0) + \frac{1}{8\pi} \sum_{j=1}^m \log V(\xi_j). \quad (1.4)$$

Here $G(x, y)$ is the Green’s function of $-\Delta$ over $\Omega$ under Dirichlet boundary conditions and $H(x, y)$ denotes its regular part:

$$H(x, y) = G(x, y) - \frac{1}{2\pi} \log \frac{1}{|x - y|}.$$ 

The above description of blowing-up behaviour continues to work if we are in the presence of multiples singularities $\sum_i N_i \delta_{p_i}$ in (1.1), provided that we substitute the term $\sum N_i G(\xi_j, 0)$ by $\sum_i N_i \sum_j G(\xi_j, p_i)$ in (1.4).

The reciprocal issue, namely the existence of positive solutions with the property (1.3), has been addressed for the regular case $N = 0$ first in [38] in the case of a single point of concentration (i.e. $m = 1$), later generalised to the case of multiple concentration associated to any nondegenerate critical point of the functional (1.4) ([1, 9]) or, more generally, to any topologically nontrivial critical point [15, 16, 17]. In particular, still for $N = 0$, a family of solutions $u_\lambda$ concentrating at $m$-tuple of points as $\lambda \to 0^+$ has been found in some special cases: for any $m \geq 1$, provided that $\Omega$ is not simply connected ([15]), and for $m \in \{1, \ldots, h\}$ if $\Omega$ is a $h$-dumbell with thin handles ([17]). We mention that functionals similar to (1.4) occur to detect multiple-bubbling solutions in different contexts, see [4, 18, 19, 20, 36] for other related singularly perturbed problems.

The question on the existence of solutions for the problem (1.1) in the singular case $N > 0$ is far from being completely settled. Indeed only few results are available in literature. Solutions which concentrate in the measure sense at $m$ distinct points away from 0 have been built in [15] provided that $m < 1 + N$. This result has been extended in [11] to the case of multiple singular sources: in particular it is showed that, under suitable restrictions on the weights, if several sources exist then the more involved topology generates a large number of blow-up solutions. Moreover a degree formula has been obtained in [7] assuming that $N$ is not a positive integer.
A natural question for which the methods in [11, 15] fail is whether in case \( m = N + 1 \) solutions with property (1.3) exist. To our knowledge, the only paper where the above situation takes place is [14], where, for any fixed positive integer \( N \in \mathbb{N} \), it is proved the existence of a solution to (1.1), where \( \delta_0 \) is replaced by \( \delta_\lambda \) for a suitable \( \lambda \in \Omega \), with \( N + 1 \) blowing up points at the vertices of a sufficiently tiny regular polygon centered in \( p_\lambda \); moreover \( p_\lambda \) lies uniformly away from the boundary \( \partial \Omega \) but its location is determined by the geometry of the domain in a \( \lambda \)-dependent way and does not seem possible to be prescribed arbitrarily as in [11, 15].

We point out that the case \( N \in \mathbb{N} \) is more difficult to treat, and at the same time the most relevant to physical applications. Indeed, in vortex theory the number \( N \) represents vortex multiplicity, so in that context the most interesting case is precisely when it is a positive integer. The difference between the case \( N \in \mathbb{N} \) and \( N \not\in \mathbb{N} \) is also analytically essential. Indeed, as usual in problems involving small parameters and concentration phenomena like (1.1), after suitable rescaling of the blowing-up around a concentration point one sees a limiting equation. More specifically, as we will see in Section 2, we can associate to (1.1) the limiting problem of Liouville type (2.4) which will play a crucial role in the construction of solutions blowing up at \( N + 1 \) distinct points; if \( N \in \mathbb{N} \), (2.4) admits a larger class of finite mass solutions with respect to the case \( N \not\in \mathbb{N} \) since the family of all solutions extends to one carrying an extra parameter \( b \in \mathbb{R}^2 \) (see [30]). This suggests that if \( N \in \mathbb{N} \) then a blow-up scenario with \( N + 1 \) vertices allocated around the origin as asymptotic concentration set might occur.

We observe that in all the above results concentration occurs at points different from the location of the source. The problem of finding solutions with additional concentration around the source is of different nature and requires a different asymptotic analysis. In case they exist, the blowing-up at the singularity provides an additional contribution of \( 8\pi(N + 1) \) in the limit (1.2), see [2, 3, 16, 32, 33]. We refer to [6, 8, 10, 12, 16, 22, 36, 37] for construction and asymptotic estimates of solutions concentrating at 0.

In this paper we consider a symmetric setting and we provide a positive answer to the problem on the existence of a family of solutions to (1.1) consisting of exactly \( N + 1 \) blow-up points located around a fixed center. Let us pass to enumerate the hypotheses on the domain \( \Omega \) and on the potential \( V \) that will be steadily used throughout the paper: first of all

(A1) \( \Omega \) and \( V \) are \((N + 1)\)-symmetric with respect to 0, i.e.

\[
x \in \Omega \iff xe^{i \frac{2\pi i}{N + 1}} \in \Omega, \quad V(xe^{i \frac{2\pi i}{N + 1}}) = V(x).
\]

In order to state the second crucial assumption on \( \Omega \), let us consider the new domain

\[
\Omega_{N+1} := \{x^{N+1} \mid x \in \Omega\}
\]

which is smooth thanks to the \((N + 1)\)-symmetry of \( \Omega \); next, for any \( b \in \Omega_{N+1} \) let us denote by \( \beta_0, \ldots, \beta_N \in \Omega \) the \((N + 1)\)-roots of \( b \), i.e., \( \beta_i^{N+1} = b \) and \( \beta_i \neq \beta_j \) for \( i \neq j \). Then, we will assume that

(A2) the function

\[
\Lambda : b \in \Omega_{N+1} \mapsto \sum_{i,j=0}^{N} H(\beta_i, \beta_j) - N \sum_{i=0}^{N} H(\beta_i, 0) + \sum_{i=0}^{N} \log(V(\beta_i)) \frac{4\pi}{4\pi}
\]

has a compact set \( K \subset \Omega_{N+1} \setminus \{0\} \) of critical points which is \( C^1 \)-stable.
Following [23], we say that a compact set \( K \subset \Omega_{N+1} \) is \( C^1 \)-stable for \( \Lambda \) if, fixed an open neighborhood \( \mathcal{U} \supset \Lambda \), any map \( \mathcal{U} \rightarrow \mathbb{R} \) sufficiently close to \( \Lambda \) in \( C^1 \)-sense admits a critical point in \( \mathcal{U} \). It is easy to check that any compact set of local maxima or local minima, as well as any nondegenerate critical point of \( \Lambda \), is \( C^1 \) stable.

We point out that the function \( \Lambda \) is smooth in \( \Omega_{N+1} \setminus \{0\} \); moreover \( \Lambda \) can be rewritten in terms of the Robin’s function associated to the domain \( \Omega_{N+1} \), we refer to Appendix A for more details. In the case of the ball we can perform many examples of functions \( V \) satisfying assumptions (A1) – (A2) as shown in the next Remark.

**Remark 1.1.** Let \( \Omega \) be the unit ball \( B(0,1) \) and let \( V \) be radially symmetric \( V(x) = V(|x|) \); according to Remark A.1, the function \( \Lambda \) in (1.5) coincides with

\[
\Lambda(b) = \Lambda(|b|) = \frac{N+1}{4\pi} \log \left( V\left( |b| \frac{1}{2N} \right) \left( 1 - |b|^2 \right)^2 \right)
\]

Here, with a slight abuse of notation, since \( V \) continues to denote by \( V \) and let \( \Lambda \), the real functions \( r \in [0,1] \rightarrow V(r) \), \( r \in [0,1] \rightarrow \Lambda(r) \), respectively.

So, we can exhibit many examples of potentials \( V \) for which (A2) holds. In order to provide explicit examples, it is sufficient to consider any potential \( V \) admitting a nondegenerate minimum point at 0, i.e., \( V''(0) = a > 0 \); then such \( V \) satisfies (A2): to see this, observe that \( V'(r \frac{1}{N+1}) = a(1 + a(1))r \frac{1}{N+1} \) as \( r \rightarrow 0^+ \) then

\[
\Lambda'(r) \rightarrow \frac{a}{4\pi V(0)} > 0 \quad \text{as} \quad r \rightarrow 0^+ \quad \text{if} \quad N = 1, \quad \Lambda'(r) \rightarrow +\infty \quad \text{as} \quad r \rightarrow 0^+ \quad \text{if} \quad N > 1
\]

and \( \Lambda(r) \rightarrow -\infty \) as \( r \rightarrow 1^- \). Under this assumption it is immediate to prove that

\[
\max \{ \Lambda(0), \Lambda(1) \} < \sup_{0<r<1} \Lambda(r)
\]

or, equivalently,

\[
\max_{|b|=0, \, |b|=1} \Lambda(|b|) < \sup_{0<|b|<1} \Lambda(|b|)
\]

and this ensures that \( \Lambda \) admits a set of global maxima in \( B(0,1) \) which is compactly contained in \( B(0,1) \setminus \{0\} \), and this is a \( C^1 \)-stable set of critical points.

As we will see in next theorem, in the symmetric scenarios provided by assumptions (A1) – (A2) the smallness of parameter \( \lambda \) in (1.1) will yield the existence of a solution blowing up at \( N+1 \) points located at suitable vertices \( \beta_j \); roughly speaking, the solution splits into a branch of \( N+1 \) bubbles which are arranged as satellites at the vertices of a regular \( (N+1) \)-polygon with center 0. The exact formulation of the result is the following.

**Theorem 1.1.** Let \( \Omega \subset \mathbb{R}^2 \) be a smooth and bounded domain such that \( 0 \in \Omega \) and let \( V \) be a smooth potential satisfying \( \inf_{\Omega} V(x) > 0 \). Suppose, in addition that hypotheses (A1) – (A2) hold. Then, for \( \lambda \) sufficiently small the problem (1.1) has a family of solutions \( u_\lambda \) with the following property: there exist \( \mu = \mu(\lambda) > 0 \) and \( b = b(\lambda) \in \Omega_{N+1} \setminus \{0\} \) such that \( u_\lambda \) satisfies

\[
\lambda e^{\mu \lambda} - \sum_{i=0}^{N} \frac{8\mu^2}{(\mu^2 + |x - \beta_i|^2)^2} \rightarrow 0 \quad (1.6)
\]
in $L^1(\Omega)$, where

$$
\mu \sim \sqrt{\lambda}, \quad \text{dist}(b, K) \to 0.
$$

(1.7)

As a corollary, the following holds:

$$
\lambda e^{u_\lambda} - 8\pi \sum_{i=0}^{N} \delta_{b_i} \to 0
$$

(1.8)

in the measure sense.

So, taking into account of Remark 1.1, we immediately get the following corollary in the case of the ball.

**Corollary 1.2.** Let $\Omega$ be the unit ball $B(0,1)$ and let $V$ be a radially symmetric smooth potential $V(x) = V(|x|)$ satisfying $\inf_{B(0,1)} V(x) > 0$. Suppose, in addition, that $V''(0) > 0$. Then, for $\lambda$ sufficiently small the problem (1.1) has a family of solutions $u_\lambda$ with the following property: there exist $\mu = \mu(\lambda) > 0$ and $b = b(\lambda) \in B(0,1) \setminus \{0\}$ such that $u_\lambda$ satisfies (1.6)-(1.7)-(1.8) with

$$
K = \left\{ b \big| \Lambda(b) = \sup_{B(0,1)} \Lambda \right\} \subset B(0,1) \setminus \{0\}.
$$

A more precise description in terms of the $H^1_0(\Omega)$-norm of the family of solutions $u_\lambda$ is provided by Theorem 2.1. The analysis reveals that the configuration of the limiting blow-up points is determined by the balance between three crucial aspects represented by the three terms of the function $\Lambda$ in (1.5), respectively: the interplay between distinct blow-up points, the interaction with the singular source, and the effect of the potential $V$. The existence of a $C^1$-stable critical points of $\Lambda$ guarantees that an equilibrium is achieved and gives rise to a $(N+1)$-bubbling solution. In particular the shape of $\Omega$, described in terms of the regular part of the Green’s function $H(x,y)$, and the potential $V$ appear coupled at almost every point of the proof.

The proofs use singular perturbation methods which combine the variational approach with a Lyapunov-Schmidt type procedure. Roughly speaking, the first step consists in the construction of an approximate solution, which should turn out to be precise enough. In view of the expected asymptotic behavior, the shape of such approximate solution will resemble a bubble of the form (2.5) with a suitable choice of the parameter $\delta = \delta(\lambda, b)$. Then we look for a solution to (1.1) in a small neighborhood of the first approximation. As quite standard in singular perturbation theory, a crucial ingredient is nondegeneracy of the explicit family of solutions of the limiting Liouville problem (2.4), in the sense that all bounded elements in the kernel of the linearization correspond to variations along the parameters of the family, as established in [14]. This allows us to study the invertibility of the linearized operator associated to the problem (1.1) under suitable orthogonality conditions. Next we introduce an intermediate problem and a fixed point argument will provide a solution for an auxiliary equation, which turns out to be solvable for any choice of $b$. Finally we test the auxiliary equation on the elements of the kernel of the linearized operator and we find out that, in order to find an exact solution of (1.1), the location of the asymptotic peaks, which is detected by the parameter $b$, should be a critical point for the reduced finite dimensional functional. So, after this reduction process, solving (1.1) is equivalent to solving a finite-dimensional optimization problem.

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1 We use the notation $\sim$ to denote quantities which in the limit $\lambda \to 0^+$ are of the same order.
The question on the existence of solutions having more than \(N+1\) blow-up points remains open: in this case the techniques used in [15] and in this paper to catch a critical point for the reduced functional stop to work because of the lack of a crucial compactness condition which prevents the collapsing of parts of the blow-up points onto the singularity. On the other hand, according to the computation by Chen and Lin in [7] the Leray-Schauder degree does not change when the energy (1.2) is bigger than \(8\pi(N+1)\), and this suggests that blowing-up solutions having more than \(N+1\) blow-up points might not occur for (1.1), in general.

The rest of the paper is organized as follows. Section 2 is devoted to some preliminary results, notation, and the definition of the approximating solution. Moreover, a more general version of Theorem 1.1 is stated there (see Theorem 2.1). In Section 3 we prove the solvability of the linearized problem. The error up to which the approximating solution solves problem (1.1) is estimated in Section 4. Section 5 considers the solvability of an auxiliary problem by a contraction argument. In Section 6 we reduce the problem to finite dimension by the Liapunov-Schmidt reduction method and we compute the reduced energy. Finally in Section 7 we complete the proof of Theorem 1.1. In Appendix A, B we collect some results, most of them well-known, which are usually referred to throughout the paper.

Notation: In our estimates throughout the paper, we will frequently denote by \(C > 0, c > 0\) fixed constants, that may change from line to line, but are always independent of the variables under consideration. We also use the notations \(O(1), o(1), O(\lambda), o(\lambda)\) to describe the asymptotic behaviors of quantities in a standard way.

2. Preliminaries and statement of the main result. We are going to provide an equivalent formulation of problem (1.1) and Theorem 1.1. Indeed, let us set

\[
\alpha := N + 1 \geq 2
\]

and let us observe that, setting \(v\) the regular part of \(u\), namely

\[
v = u + 4\pi(\alpha - 1)G(x,0), \quad \alpha = N + 1,
\]

problem (1.1) is then equivalent to solving the following (regular) boundary value problem

\[
\begin{cases}
-\Delta v = \lambda \tilde{V}(x)|x|^{2(\alpha-1)}e^v & \text{in } \Omega \\
v = 0 & \text{on } \partial \Omega
\end{cases},
\]

where \(\tilde{V}(x)\) is the function

\[
\tilde{V}(x) = V(x)e^{-4\pi(\alpha-1)H(x,0)}.
\]

Here \(G\) and \(H\) are the Green’s function and its regular part as defined in the introduction. Theorem 1.1 will be a consequence of a more general result concerning Liouville-type problem (2.2). In order to provide such a result for (2.2), we now give a construction of a suitable approximate solution for (2.2). In what follows, we identify \(x = (x_1, x_2) \in \mathbb{R}^2\) with \(x_1 + ix_2 \in \mathbb{C}\). Moreover, \((x_1, x_2)\) stands for the inner product between the vectors \(x_1, x_2 \in \mathbb{R}^2\), whereas we denote by \(x_1x_2\) the multiplication of the complex numbers \(x_1, x_2\) and, analogously, by \(x^\alpha\) the power of the complex number \(x\). Clearly \((x_1, x_2) = \text{Re}(x_1\overline{x_2})\).

For any \(\alpha \in \mathbb{N}\), we can associate to (2.2) a limiting problem of Liouville type which will play a crucial role in the construction of the \((N+1)\)-bubbling solutions
as $\lambda \to 0^+$:

$$-
\Delta w = |x|^{2(\alpha - 1)} e^w \quad \text{in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} |x|^{2(\alpha - 1)} e^w dx < +\infty. \quad (2.4)$$

All solutions of this problem are given, in complex notation, by the three-parameter family of functions

$$w_{\delta,b}^{\alpha}(x) := \log \frac{8\alpha^2 \delta^{2\alpha}}{(\delta^{2\alpha} + |x^\alpha - b|^2)^2} \quad \delta > 0, \ b \in \mathbb{C}. \quad (2.5)$$

The following quantization property holds:

$$\int_{\mathbb{R}^2} |x|^{2(\alpha - 1)} e^{w_{\delta,b}^{\alpha}(x)} dx = 8\pi \alpha. \quad (2.6)$$

In the following we agree that

$$W_\lambda = w_{\delta,b}^{\alpha}(x),$$

where the value $\delta = \delta(\lambda, b)$ is defined as:

$$\delta^{2\alpha} := \frac{\lambda}{8\alpha^2} V(\beta_0) e^{8\pi \mathcal{H}_\alpha(b,b) - 4\pi \frac{\alpha - 1}{2} \mathcal{H}_\alpha(b,0)}, \quad \beta_0^\alpha = b, \quad (2.7)$$

and the function $\mathcal{H}_\alpha$ has been introduced in Appendix A.

To obtain a better first approximation, we need to modify the functions $W_\lambda$ in order to satisfy the zero boundary condition. Precisely, we consider the projections $PW_\lambda$ onto the space $H_0^1(\Omega)$, where the projection $P : H^1(\mathbb{R}^N) \to H_0^1(\Omega)$ is defined as the unique solution of the problem

$$\Delta P v = \Delta v \quad \text{in } \Omega, \quad P v = 0 \quad \text{on } \partial \Omega.$$

We recall by Appendix A that, setting $\Omega_\alpha = \{ x^\alpha \mid x \in \Omega \}$, for any $b \in \Omega_\alpha$ the function $\mathcal{H}_\alpha(x^\alpha, b)$ is harmonic in $\Omega$ and satisfies $\mathcal{H}_\alpha(x^\alpha, b) = \frac{1}{2\pi} \log |x^\alpha - b|$ on $\partial \Omega$.

A straightforward computation gives that for any $x \in \partial \Omega$

$$|PW_\lambda - W_\lambda + \log \left( 8\alpha^2 \delta^{2\alpha} \right) - 8\pi \mathcal{H}_\alpha(x^\alpha, b)| = |W_\lambda - \log \left( 8\alpha^2 \delta^{2\alpha} \right) + 4\log |x^\alpha - b| | \leq C \delta^{2\alpha}.$$

Since the expressions considered inside the absolute values are harmonic in $\Omega$, then the maximum principle applies and implies the following asymptotic expansion

$$PW_\lambda = W_\lambda - \log \left( 8\alpha^2 \delta^{2\alpha} \right) + 8\pi \mathcal{H}_\alpha(x^\alpha, b) + O(\delta^{2\alpha})$$

$$= -2 \log \left( \delta^{2\alpha} + |x^\alpha - b|^2 \right) + 8\pi \mathcal{H}_\alpha(x^\alpha, b) + O(\delta^{2\alpha}) \quad (2.8)$$

uniformly for $x \in \Omega$ and $b$ on a compact subset of $\Omega_\alpha$.

We point out that, in order to simplify the notation, in our estimates throughout the paper we will describe the asymptotic behaviors of quantities under considerations in terms of $\delta = \delta(\lambda, b)$ instead of the parameter $\lambda$ of the equation. Clearly according to (2.7) $\delta$ has the same rate as $\lambda^{\frac{1}{2\alpha}}$, so at each step we can easily pass to the analogous asymptotic in terms of $\lambda$: for instance, in (2.8) the error term “$O(\delta^{2\alpha})$” can be equivalently replaced by “$O(\lambda)$”.

We shall look for a solution to (1.1) in a small neighborhood of the first approximation, namely a solution of the form

$$u_\lambda = PW_\lambda + \phi_\lambda,$$

where the rest term $\phi_\lambda$ is small in $H^1(\Omega)$-norm.
In order to state the main theorem of the paper, let us reformulate the two assumptions \((A1) - (A2)\) in an equivalent way according to the new framework in terms of \(\alpha\) instead of \(N\) and the function \(H_\alpha\) in the place of \(H\) (see \((A.1)\)):

\[(A1)^* \Omega \text{ and } V \text{ are } \alpha\text{-symmetric with respect to the origin, i.e.}
\]

\[x \in \Omega \iff xe^{i\frac{\pi}{\alpha}} \in \Omega, \quad V(xe^{i\frac{\pi}{\alpha}}) = V(x);
\]

\[(A2)^* \text{ the function}
\]

\[
\Lambda : b \in \Omega_\alpha \longrightarrow \alpha H_\alpha(b, b) - (\alpha - 1)H_\alpha(b, 0) + \frac{\alpha}{4\pi} \sum_{i=0}^{\alpha-1} \log(V(\beta_i))
\]

has a compact set \(K \subset \Omega_\alpha \setminus \{0\}\) of critical points which is \(C^1\)-stable.

**Theorem 2.1.** Let \(\Omega \subset \mathbb{R}^2\) be a smooth and bounded domain such that \(0 \in \Omega\) and let \(V\) be a smooth potential satisfying \(\inf_{\Omega} V(x) > 0\). Suppose, in addition, that \(\Omega\) and \(V\) satisfy assumptions \((A1)^* - (A2)^*\). Then, for \(\lambda\) sufficiently small the problem \((2.2)\) has a family of solutions \(v_\lambda\) satisfying

\[v_\lambda = -2\log (\delta^{2\alpha} + |x^\alpha - b_\lambda|^2) + 8\pi H_\alpha(x^\alpha, b_\lambda) + o(1)
\]

in \(H^1\)-sense, where

\[\text{dist}(b_\lambda, K) \to 0 \text{ as } \lambda \to 0^+.
\]

In the remaining part of this paper we will prove Theorem 2.1 and at the end of Section 7 we shall see how Theorem 1.1 follows quite directly as a corollary according to the change of variable \((2.1)\).

We end up this section by setting notation and basic well-known facts which will be of use in the rest of the paper. We denote by \(\|\cdot\|\) according to the change of variable \((2.1)\).

In next lemma we recall the well-known Moser-Trudinger inequality ([28, 35]).

**Lemma 2.2.** There exists \(C > 0\) such that for any bounded domain \(\Omega\) in \(\mathbb{R}^2\)

\[
\int_\Omega e^{4\pi u^2} \, dx \leq C|\Omega| \quad \forall u \in H^1_0(\Omega),
\]

where \(|\Omega|\) stands for the measure of the domain \(\Omega\). In particular, for any \(q \geq 1\)

\[
\|e^u\|_q \leq C \frac{1}{q} |\Omega|^{\frac{1}{q}} e^{\frac{\pi}{2}q^2 \|u\|^2} \quad \forall u \in H^1_0(\Omega).
\]

For any \(\alpha \geq 1\) we will make use of the Hilbert spaces

\[
L_\alpha(\mathbb{R}^2) := \left\{ u \in L^2_{\text{loc}}(\mathbb{R}^2) : \| \frac{|y|^{\alpha-1}}{1 + |y|^{2\alpha}} u \|_{L^2(\mathbb{R}^2)} < +\infty \right\}
\]

and

\[
H_\alpha(\mathbb{R}^2) := \left\{ u \in W^{1,2}_{\text{loc}}(\mathbb{R}^2) : \| \nabla u \|_{L^2(\mathbb{R}^2)} + \left\| \frac{|y|^{\alpha-1}}{1 + |y|^{2\alpha}} u \right\|_{L^2(\mathbb{R}^2)} < +\infty \right\},
\]

endowed with the norms

\[
\|u\|_{L_\alpha} := \left\| \frac{|y|^{\alpha-1}}{1 + |y|^{2\alpha}} u \right\|_{L^2(\mathbb{R}^2)}, \quad \|u\|_{H_\alpha} := \left( \| \nabla u \|_{L^2(\mathbb{R}^2)}^2 + \left\| \frac{|y|^{\alpha-1}}{1 + |y|^{2\alpha}} u \right\|_{L^2(\mathbb{R}^2)}^2 \right)^{1/2}.
\]

**Proposition 2.3.** For any \(\alpha \geq 1\) the embedding \(H_\alpha(\mathbb{R}^2) \hookrightarrow L_\alpha(\mathbb{R}^2)\) is compact.
Proof. See [21, Proposition 6.1].

As commented in the introduction, our proof uses the singular perturbation methods. For that, the nondegeneracy of the functions that we use to build our approximating solution is essential. Next proposition is devoted to the nondegeneracy of the finite mass solutions of the Liouville equation (regular and singular).

Proposition 2.4. Assume that $\xi \in \mathbb{R}^2$ and $\phi : \mathbb{R}^2 \to \mathbb{R}$ solves the problem

$$
- \Delta \phi = 8 \alpha^2 \frac{|y|^{2(\alpha - 1)}}{(1 + |y^\alpha - \xi|^2)^2} \phi \quad \text{in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} |\nabla \phi(y)|^2 dy < +\infty. \quad (2.9)
$$

Then there exists $c_0, c_1, c_2 \in \mathbb{R}$ such that

$$
\phi(y) = c_0 Z_0 + c_1 Z_1 + c_2 Z_2.
$$

Proof. In [21, Theorem 6.1] it was proved that any solution $\phi$ of (2.9) is actually a bounded solution. Therefore we can apply the result in [13] to conclude that $\phi = c_0 Z_0 + c_1 Z_1 + c_2 Z_2$ for some $c_0, c_1, c_2 \in \mathbb{R}$.

3. Analysis of the linearized operator. According to Proposition 2.4, by the change of variable $x = \delta y$, we immediately get that all solutions $\psi \in H_\delta(\mathbb{R}^2)$ of

$$
- \Delta \psi = 8 \alpha^2 \frac{\delta^{2\alpha}|x|^{2(\alpha - 1)}}{(\delta^{2\alpha} + |x^\alpha - b|^2)^2} \psi = |x|^{2(\alpha - 1)} e^{W_\lambda} \psi \quad \text{in } \mathbb{R}^2
$$

are linear combinations of the functions

$$
Z^0_{\delta, b}(x) = \frac{\delta^{2\alpha} - |x^\alpha - b|^2}{\delta^{2\alpha} + |x^\alpha - b|^2}, \quad Z^1_{\delta, b}(x) = \frac{\delta^\alpha \Re (x^\alpha - b)}{\delta^{2\alpha} + |x^\alpha - b|^2}, \quad Z^2_{\delta, b}(x) = \frac{\delta^\alpha \Im (x^\alpha - b)}{\delta^{2\alpha} + |x^\alpha - b|^2}.
$$

We introduce the projections $PZ^j_{\delta, b}$ onto $H^1_\delta(\Omega)$. It is immediate that

$$
PZ^0_{\delta, b}(x) = Z^0_{\delta, b}(x) + 1 + O(\delta^{2\alpha}) = \frac{2\delta^{2\alpha}}{\delta^{2\alpha} + |x^\alpha - b|^2} + O(\delta^{2\alpha}) \quad (3.1)
$$

and

$$
PZ^j_{\delta, b}(x) = Z^j_{\delta, b}(x) + O(\delta^{\alpha}) \text{ for } j = 1, 2 \quad (3.2)
$$

uniformly with respect to $x \in \Omega$ and $b$ on compact subsets of $\Omega_\lambda$.

We agree that $Z^j_{\lambda} := Z^j_{\delta, b}$ for any $j = 0, 1, 2$, where $\delta$ is defined in terms of $\lambda$ and $b$ according to (2.7). Motivated by the symmetry of the domain in assumption (A1)*, we consider the subspaces $H^1_{\delta, \lambda}(\Omega)$, $L^p_{\lambda}(\Omega)$ made up of $\alpha$-symmetric functions:

$$
H^1_{\delta, \lambda}(\Omega) = \{ u \in H^1_\delta(\Omega) \mid u(x^\frac{2\pi}{\delta}) = u(x) \}, \quad L^p_{\lambda}(\Omega) = \{ u \in L^p(\Omega) \mid u(x^\frac{2\pi}{\delta}) = u(x) \}.
$$

Clearly $PW_{\lambda}, PZ^j_{\lambda} \in H^1_{\delta, \lambda}(\Omega)$. Let us consider the following linear problem: given $h \in H^1_{\delta, \lambda}(\Omega)$, find a function $\phi \in H^1_{\delta, \lambda}(\Omega)$ satisfying

$$
\begin{cases}
- \Delta \phi - \lambda \tilde{W}(x)|x|^{2(\alpha - 1)} e^{PW_{\lambda}} \phi = \Delta h \\
\int_{\Omega} \nabla \phi \nabla PZ^j_{\lambda} = 0 \quad j = 1, 2
\end{cases} \quad (3.3)
$$
Before going on, we recall the following identities which follow by straightforward computations: for every \( \xi \in \mathbb{R}^2 \)
\[
\int_{\mathbb{R}^2} \log(1 + |y|^2) \frac{1 - |y|^2}{(1 + |y|^2)^3} dy = -\frac{\pi}{2}, \tag{3.4}
\]
\[
\int_{\mathbb{R}^2} \frac{1 - |y|^2}{(1 + |y|^2)} dy = 0, \tag{3.5}
\]
\[
\int_{\mathbb{R}^2} \frac{y^2}{(1 + |y|^2)} dy = \int_{\mathbb{R}^2} \frac{y^2}{(1 + |y|^2)^2} dy = \frac{1}{2} \int_{\mathbb{R}^2} \frac{|y|^2}{(1 + |y|^2)^2} dy = \frac{\pi}{12}. \tag{3.6}
\]

**Proposition 3.1.** There exist \( \lambda_0 > 0 \) and \( C > 0 \) such that for any \( \lambda \in (0, \lambda_0) \), any \( b \in \mathbb{R}^2 \) in a compact subset of \( \Omega_\alpha \setminus \{0\} \) and any \( h \in H^1_{0,\alpha}(\Omega) \), if \( \phi \in H^1_{0,\alpha}(\Omega) \) solves (3.3), then the following holds
\[
\|\phi\| \leq C|\log \delta\|\|h\|. \]

**Proof.** We argue by contradiction. Assume that there exist sequences \( \lambda_n \to 0 \), \( h_n \in H^1_{0,\alpha}(\Omega) \), \( b_n \) in a compact subset of \( \Omega_\alpha \setminus \{0\} \) and \( \phi_n \in H^1_{0,\alpha}(\Omega) \) which solves (3.3) and
\[
\|\phi_n\| = 1, \quad |\log \delta_n\|\|h_n\| \to 0. \tag{3.7}
\]

We define \( \tilde{\Omega}_n := \frac{\Omega}{\delta_n} \) and
\[
\tilde{\phi}_n(y) := \begin{cases} 
\phi_n(\delta_n y) & \text{if } y \in \tilde{\Omega}_n \\
0 & \text{if } y \in \mathbb{R}^2 \setminus \tilde{\Omega}_n.
\end{cases}
\]

We split the remaining argument into five steps. In what follows at many steps of the reasoning we will pass to a subsequence, without further notice.

**Step 1.** Using the polar coordinates \((\rho, \theta)\) let us set, according to (A.5),
\[
\tilde{\Phi}_n(\rho e^{i\theta}) = \tilde{\phi}_n(\rho^{-\alpha} e^{i\frac{\theta}{\alpha}}) \quad \rho \geq 0, \quad \theta \in [-\pi, \pi).
\]

We will show that
\[
\tilde{\Phi}_n \in H^1(\mathbb{R}^2) \quad \text{and} \quad \tilde{\Phi}_n(\cdot + \delta_n^{-\alpha} b_n) \text{ is bounded in } H^1(\mathbb{R}^2).
\]

It is immediate to check that
\[
\int_{\mathbb{R}^2} |\nabla \tilde{\phi}_n|^2 dy = \int_{\Omega} |\nabla \phi_n|^2 dx = 1. \tag{3.8}
\]

Next, we multiply the equation in (3.3) by \( \phi_n \); then we integrate over \( \Omega \) to obtain
\[
\lambda_n \int_{\Omega} \tilde{V}(x) |x|^{2(\alpha - 1)} e^{P_{\lambda_n} \phi_n^2} dx = \int_{\Omega} |\nabla \phi_n|^2 dx + \int_{\Omega} \nabla h_n \nabla \phi_n dx \leq C
\]
by (3.7). So Proposition 4.2 (taking \( p \) close enough to 1) gives \( \int_{\Omega} |x|^{2(\alpha - 1)} e^{W_{\lambda_n} \phi_n^2} \leq C \) or, equivalently,
\[
\int_{\mathbb{R}^2} \frac{|y|^{2(\alpha - 1)}}{(1 + |y|^{\alpha - \delta_n^{-\alpha} |b_n|^2})^2} \phi_n^2 dy \leq C. \tag{3.9}
\]

We deduce that \( \tilde{\phi}_n \) belongs to \( H^1(\mathbb{R}^2) \) and satisfies (3.8)-(3.9). Thanks to Lemma A.2 we get \( \tilde{\Phi}_n \in H^1(\mathbb{R}^2) \) and
\[
\int_{\mathbb{R}^2} |\nabla \tilde{\Phi}_n|^2 dy = \frac{1}{\alpha} \int_{\mathbb{R}^2} |\nabla \tilde{\phi}_n|^2 dy = \frac{1}{\alpha}.
\]
and
\[
\int_{\mathbb{R}^2} \frac{1}{1 + |y|^2} |\tilde{\Phi}_n(y + \delta_n^{-\alpha} b_n)|^2 dy = \int_{\mathbb{R}^2} \frac{1}{1 + |y - \delta_n^{-\alpha} b_n|^2} |\tilde{\Phi}_n|^2 dy = \alpha \int_{\mathbb{R}^2} \frac{|y|^{2(\alpha - 1)}}{1 + |y - \delta_n^{-\alpha} b_n|^2} |\tilde{\phi}_n|^2 dy \leq C.
\]

We have thus proved that the sequence \( \tilde{\Phi}_n(y + \delta_n^{-\alpha} b_n) \) \((n \in \mathbb{N})\) is bounded in \( H_1(\mathbb{R}^2) \).

**Step 2. We will show that, for some \( \gamma_0 \in \mathbb{R} \):
\[
\tilde{\Phi}_n(y + \delta_n^{-\alpha} b_n) \rightarrow \gamma_0 \frac{1 - |y|^2}{1 + |y|^2} \quad \text{weakly in } H_1(\mathbb{R}^2) \quad \text{and strongly in } L_1(\mathbb{R}^2).
\]

Step 1 and Proposition 2.3 give
\[
\tilde{\Phi}_n(y + \delta_n^{-\alpha} b_n) \rightarrow f \quad \text{weakly in } H_1(\mathbb{R}^2) \quad \text{and strongly in } L_1(\mathbb{R}^2)
\]
for some \( f \in H_1(\mathbb{R}^2) \). Let \( \psi \in C_c(\mathbb{R}^2) \) and set
\[
\psi_n(x) = \psi \left( \frac{x^\alpha - b_n}{\delta_n^\alpha} \right) \quad \tilde{\psi}_n(y) = \psi_n(\delta_n y).
\]

Setting, according to (A.5), in polar coordinates
\[
\tilde{\psi}_n(\rho e^{i\theta}) = \tilde{\psi}_n(\rho \frac{1}{\delta_n} e^{i\frac{\theta}{\delta_n}}) \quad \rho \geq 0, \quad \theta \in [-\pi, \pi),
\]
we immediately get
\[
\tilde{\psi}_n(y) = \psi(y - \delta_n^{-\alpha} b_n).
\]

We have \( \psi_n \in C_c(\Omega) \), for large \( n \). Then by applying Corollary A.3 we get
\[
\int_{\Omega} \nabla \phi_n \cdot \nabla \psi_n dx = \int_{\mathbb{R}^2} \nabla \tilde{\phi}_n \cdot \nabla \tilde{\psi}_n dy = \alpha \int_{\mathbb{R}^2} \nabla \tilde{\Phi}_n \cdot \nabla \tilde{\psi}_n dy
= \alpha \int_{\mathbb{R}^2} \nabla f \cdot \nabla \psi dy + o(1). \tag{3.11}
\]

Similarly we compute
\[
\int_{\Omega} |x|^{2(\alpha - 1)} e^{\lambda_n x_n} \phi_n \psi_n dx = 8\alpha^2 \int_{\Omega} \frac{\delta_{2\alpha}^2 |x|^{2(\alpha - 1)} \phi_n \psi_n dx}{\delta_n^{2(\alpha - 1)} + |x^\alpha - b_n|^2} \phi_n \psi_n dx
= 8\alpha^2 \int_{\mathbb{R}^2} \frac{|y|^{2(\alpha - 1)} \tilde{\phi}_n(y) \tilde{\psi}_n(y) dy}{(1 + |y - \delta_n^{-\alpha} b_n|^2)^2} \tilde{\Phi}_n \tilde{\psi}_n dy
= 8\alpha \int_{\mathbb{R}^2} \frac{1}{(1 + |y - \delta_n^{-\alpha} b_n|^2)^2} \tilde{\Phi}_n \tilde{\psi}_n dy
= 8\alpha \int_{\mathbb{R}^2} \frac{1}{(1 + |y|^2)^2} \tilde{\Phi}_n(y + \delta_n^{-\alpha} b_n) \psi dy
= 8\alpha \int_{\mathbb{R}^2} \frac{1}{(1 + |y|^2)^2} f \psi dy + o(1),
\]
by which, using Proposition 4.2 we deduce
\[
\lambda_n \int_{\Omega} |x|^{2(\alpha - 1)} \tilde{V}(x) e^{\lambda_n x_n} \phi_n \psi_n dx = \int_{\Omega} |x|^{2(\alpha - 1)} e^{\lambda_n x_n} \phi_n \psi_n dx + o(1)
\]
Finally we estimate
\[
\int_{\Omega} \nabla h_n \nabla \psi_n \, dx = O(\|h_n\|) = o(1). \tag{3.13}
\]
We multiply the equation in (3.3) by \(\psi_n\), we integrate over \(\Omega\) and pass to the limit as \(n \to +\infty\); combining (3.11)-(3.12)-(3.13) we obtain
\[
\int_{\mathbb{R}^2} \nabla f \nabla \psi \, dy - \int_{\mathbb{R}^2} \frac{8}{(1 + |y|^2)^2} f \psi \, dy = 0.
\]
Thus, we deduce that the function \(f \in H_1(\mathbb{R}^2)\) is a solution of the equation
\[
-\Delta f = \frac{8}{(1 + |y|^2)^2} f \quad \text{in } \mathbb{R}^2.
\]
Proposition 2.4 gives
\[
f = \gamma_0 \frac{1 - |y|^2}{1 + |y|^2} + \sum_{j=1,2} \gamma_j \frac{y_j}{1 + |y|^2}
\]
for suitable constants \(\gamma_0, \gamma_1, \gamma_2 \in \mathbb{R}\). Now we use the orthogonality \(\int_{\Omega} \nabla \phi_n \nabla PZ^1_{\lambda_n} = 0\) in (3.3) to obtain
\[
0 = \int_{\Omega} \nabla \phi_n \nabla PZ^1_{\lambda_n} \, dx = \int_{\Omega} |x|^{2(\alpha - 1)} e^{W_{\lambda_n} \phi_n} Z^1_{\lambda_n} \, dx
\]
\[
= 8\alpha \int_{\mathbb{R}^2} \frac{|y|^{2(\alpha - 1)}}{(1 + |y|^{2\alpha - 2})^2} \tilde{\phi}_n(y) \frac{\Re(y^{\alpha - \delta_n \alpha b_n})}{1 + |y|^{\alpha - \delta_n \alpha b_n}} \, dy
\]
\[
= 8\alpha \int_{\mathbb{R}^2} \frac{\Re(y - \delta_n \alpha b_n)}{1 + |y|^{\alpha - \delta_n \alpha b_n}} \hat{\phi}_n(y) \, dy
\]
\[
= 8\alpha \int_{\mathbb{R}^2} \frac{y_1}{(1 + |y|^2)^3} \tilde{\phi}_n(y + \delta_n \alpha b_n) \, dy. \tag{3.14}
\]
Then we pass to the limit when \(n \to +\infty\) and we obtain
\[
0 = \int_{\mathbb{R}^2} \frac{y_1}{(1 + |y|^2)^3} f \, dy = \gamma_1 \int_{\mathbb{R}^2} \frac{y_1^2}{(1 + |y|^2)^3} \, dy = \frac{\pi}{12} \gamma_1
\]
by (3.6). So \(\gamma_1 = 0\) and, similarly, \(\gamma_2 = 0\).

Step 3. We will show that
\[
\int_{\Omega} |x|^{2(\alpha - 1)} e^{W_{\lambda_n} \phi_n} \phi_n \, dx = o\left(\frac{1}{|\log \delta_n|}\right).
\]
We multiply the equation in (3.3) by \(PZ^0_{\lambda_n}\), we integrate over \(\Omega\) and we get
\[
\int_{\Omega} \nabla \phi_n \nabla PZ^0_{\lambda_n} \, dx - \lambda_n \int_{\Omega} \tilde{V}(x) |x|^{2(\alpha - 1)} e^{P_{\lambda_n} \phi_n} PZ^0_{\lambda_n} \, dx = - \int_{\Omega} \nabla h_n \nabla PZ^0_{\lambda_n} \, dx. \tag{3.15}
\]
We are now concerned with the estimates of each term of the above expression. First, we compute
\[
\int_{\Omega} \nabla \phi_n \nabla PZ^0_{\lambda_n} \, dx = \int_{\Omega} |x|^{2(\alpha - 1)} e^{W_{\lambda_n} \phi_n} Z^0_{\lambda_n} \, dx. \tag{3.16}
\]
Using Proposition 4.2 and (3.1), we obtain
\[
\lambda_n \int_\Omega \tilde{V}(x) |x|^{2(\alpha - 1)} e^{PW_\lambda} \phi_n PW_\lambda dx
\]
\[
= \int_\Omega |x|^{2(\alpha - 1)} e^{W_\lambda} \phi_n (Z_\lambda^0 + 1) dx + o\left( \frac{1}{|\log \delta_n|} \right)
\]
\[
= \int_\Omega |x|^{2(\alpha - 1)} e^{W_\lambda} \phi_n Z_\lambda^0 dx + \int_{\mathbb{R}^2} |x|^{2(\alpha - 1)} e^{W_\lambda} \phi_n dx + o\left( \frac{1}{|\log \delta_n|} \right). \tag{3.17}
\]

Finally, since \( PZ_\lambda^0 = O(1) \), we have \( \int_\Omega |\nabla PZ_\lambda^0|^2 = \int_\Omega |x|^{2(\alpha - 1)} e^{W_\lambda} PZ_\lambda^0 = O(1) \), by which, owing to (3.7),
\[
\int_\Omega |\nabla h_n| |\nabla PZ_\lambda^0| dx \leq \|h_n\| \|PZ_\lambda^0\| = o\left( \frac{1}{|\log \delta_n|} \right). \tag{3.18}
\]

We now multiply (3.15) by \( \log \delta_n \) and pass to the limit: inserting (3.16), (3.17), (3.18), we obtain the thesis of the step.

Step 4. We will show that \( \gamma_0 = 0 \).

We multiply the equation in (3.3) by \( PW_\lambda \), we integrate over \( \Omega \) and we get
\[
\int_\Omega \nabla \phi_n \nabla PW_\lambda dx - \lambda_n \int_\Omega \tilde{V}(x) |x|^{2(\alpha - 1)} e^{PW_\lambda} \phi_n PW_\lambda dx
\]
\[
= - \int_\Omega \nabla h_n \nabla PW_\lambda dx. \tag{3.19}
\]

Let us estimate each of the terms above. Let us begin with:
\[
\int_\Omega \nabla \phi_n \nabla PW_\lambda dx = \int_\Omega \phi_n |x|^{2(\alpha - 1)} e^{W_\lambda} dx = o(1) \tag{3.20}
\]
by step 3. By Proposition 4.2 and (3.7), using that \( |PW_\lambda| = O(|\log \delta|) \), we get
\[
\lambda_n \int_\Omega \tilde{V}(x) |x|^{2(\alpha - 1)} e^{PW_\lambda} \phi_n PW_\lambda dx
\]
\[
= \int_\Omega |x|^{2(\alpha - 1)} e^{W_\lambda} \phi_n PW_\lambda dx + o(1). \tag{3.21}
\]

Observe that by (2.8) we have
\[
PW_\lambda(x)
\]
\[
= - 2 \log(\delta^{2\alpha} + |x^\alpha - b|^2) + 8\pi \mathcal{H}_\alpha(b, b) + O(|x^\alpha - b|) + O(\delta^{2\alpha})
\]
\[
= - 4\alpha \log \delta - 2 \log(1 + \delta^{-2\alpha}|x^\alpha - b|^2) + 8\pi \mathcal{H}_\alpha(b, b) + O(|x^\alpha - b|) + O(\delta^{2\alpha}).
\]

Recalling Step 3 and Lemma 4.1 we obtain
\[
\int_\Omega |x|^{2(\alpha - 1)} e^{W_\lambda} \phi_n PW_\lambda dx
\]
\[
= - 2 \int_\Omega |x|^{2(\alpha - 1)} e^{W_\lambda} \phi_n \log(1 + \delta^{-2\alpha}|x^\alpha - b_n|^2) dx
\]
\[
+ 8\pi \mathcal{H}_\alpha(b_n, b_n) \int_\Omega |x|^{2(\alpha - 1)} e^{W_\lambda} \phi_n dx + o(1)
\]
\[
= - 16\alpha^2 \int_{\mathbb{R}^2} \frac{|y|^{2(\alpha - 1)}}{(1 + |y^\alpha - \delta_n^{-\alpha} b_n|^2)^\alpha} \tilde{\phi}_n \log(1 + |y^\alpha - \delta_n^{-\alpha} b_n|^2) dy
\]
\[
+ 64\pi^2 \mathcal{H}_\alpha(b_n, b_n) \int_{\mathbb{R}^2} \frac{|y|^{2(\alpha - 1)}}{(1 + |y^\alpha - \delta_n^{-\alpha} b_n|^2)^\alpha} \tilde{\phi}_n dy + o(1)
\]
and, using Corollary A.3,

\[
\int_{\Omega} |x|^{2(\alpha-1)} e^{W_{\lambda_n}} \phi_n PW_{\lambda_n} \, dx \\
= -16\alpha \int_{\mathbb{R}^2} \frac{1}{(1 + |y - \delta_n^{-\alpha}b_n|^2)^2} \tilde{\Phi}_n \log(1 + |y - \delta_n^{-\alpha}b_n|^2) dy \\
+ 64\pi\alpha \mathcal{H}_\alpha(b_n, b_n) \int_{\mathbb{R}^2} \frac{1}{(1 + |y - \delta_n^{-\alpha}b_n|^2)^2} \tilde{\Phi}_n dy + o(1).
\]

Now, by Step 2,

\[
\int_{\Omega} |x|^{2(\alpha-1)} e^{W_{\lambda_n}} \phi_n PW_{\lambda_n} \, dx \\
= -16\alpha \gamma_0 \int_{\mathbb{R}^2} \frac{1}{(1 + |y|^2)^2} \frac{1 - |y|^2}{1 + |y|^2} \log(1 + |y|^2) dy \\
+ 64\pi\alpha \gamma_0 \mathcal{H}_\alpha(b_n, b_n) \int_{\mathbb{R}^2} \frac{1}{(1 + |y|^2)^2} \frac{1 - |y|^2}{1 + |y|^2} dy + o(1) \\
= 8\alpha \gamma_0 \pi + o(1) \quad (3.22)
\]

by (3.4)-(3.5). Finally, since \( PW_{\lambda} = O(|\log \delta|) \), then we have \( \int_{\Omega} |\nabla PW_{\lambda}|^2 = \int_{\Omega} |x|^{2(\alpha-1)} e^{W_{\lambda}} PW_{\lambda} = O(|\log \delta|) \), by which, owing to (3.7),

\[
\int_{\Omega} |\nabla h_n| |\nabla PW_{\lambda_n}| \, dx \leq \|h_n\| \|PW_{\lambda_n}\| = o(1). \quad (3.23)
\]

By inserting (3.20), (3.22) and (3.23) into (3.19) and passing to the limit we deduce \( \gamma_0 = 0 \).

Step 5. End of the proof.

We will show that a contradiction arises. According to Step 2 and Step 4 we have

\( \tilde{\Phi}_n(y + \delta_n^{-\alpha}b_n) \to 0 \) weakly in \( H_1(\mathbb{R}^2) \) and strongly in \( L_1(\mathbb{R}^2) \).

By Proposition 4.2 and (3.7)

\[
\lambda_n \int_{\Omega} \tilde{V}(x)|x|^{2(\alpha-1)} e^{PW_{\lambda_n}} \phi_n^2 \, dx = \int_{\Omega} |x|^{2(\alpha-1)} e^{W_{\lambda_n}} \phi_n^2 \, dx + o(1).
\]

Now, using Lemma A.2,

\[
\lambda_n \int_{\Omega} |x|^{2(\alpha-1)} e^{W_{\lambda_n}} \phi_n^2 \, dx = 8\alpha^2 \int_{\mathbb{R}^2} \frac{|y|^{2(\alpha-1)}}{(1 + |y - \delta_n^{-\alpha}b_n|^2)^2} \tilde{\phi}_n^2 dy \\
= 8\alpha \int_{\mathbb{R}^2} \frac{1}{(1 + |y - \delta_n^{-\alpha}b_n|^2)^2} \tilde{\phi}_n^2 dy \\
= 8\alpha \|\tilde{\Phi}_n(\cdot + \delta_n^{-\alpha}b_n)\|_{L^1}^2 = o(1).
\]

Moreover, by (3.7),

\[
\int_{\Omega} \nabla h_n \nabla \phi_n \, dx = o(1).
\]

We multiply the equation in (3.3) by \( \phi_n \), we integrate over \( \Omega \) and we obtain

\[
\int_{\Omega} |\nabla \phi_n|^2 \, dx = \lambda_n \int_{\Omega} \tilde{V}(x)|x|^{2(\alpha-1)} e^{PW_{\lambda_n}} \phi_n^2 \, dx - \int_{\Omega} \nabla h_n \nabla \phi_n \, dx = o(1),
\]

in contradiction with (3.7). This concludes the proof of the proposition.
In addition to (3.3), let us consider the following linear problem: given \( h \in H^1_{0,\alpha}(\Omega) \), find a function \( \phi \in H^1_{0,\star}(\Omega) \) and constants \( c_1, c_2 \in \mathbb{R} \) satisfying

\[
\begin{aligned}
-\Delta \phi - \lambda \tilde{V}(x)|x|^{2(\alpha-1)}e^{PW_\lambda}\phi &= \Delta h + \sum_{j=1,2} c_j \int_{\Omega}|x|^{2(\alpha-1)}e^{W_\lambda}\int_{\Omega} \nabla \phi \nabla Z^1_{\lambda} = 0, \quad j = 1, 2 \\
\int_{\Omega} \phi \nabla \nabla Z^1_{\lambda} &= 0.
\end{aligned}
\] (3.24)

In order to solve problem (3.24), we need to establish an a priori estimate analogous to that of Proposition 3.1.

**Proposition 3.2.** There exist \( \lambda_0 > 0 \) and \( C > 0 \) such that for any \( \lambda \in (0, \lambda_0) \), any \( b \in \mathbb{R}^2 \) in a compact subset of \( \Omega_\alpha \setminus \{0\} \) and any \( h \in H^1_{0,\star}(\Omega) \), if \( (\phi, c_1, c_2) \in H^1_{0,\lambda}(\Omega) \times \mathbb{R}^2 \) solves (3.24), then the following holds

\[ \|\phi\| \leq C|\log \delta|\|h\|. \]

**Proof.** First observe that by (3.2)

\[
\int_{\Omega} \nabla PZ^1_\lambda \nabla PZ^2_\lambda dx = \int_{\mathbb{R}^2} |x|^{2(\alpha-1)}e^{W_\lambda} Z^1_\lambda PZ^2_\lambda dx = \int_{\mathbb{R}^2} |x|^{2(\alpha-1)}e^{W_\lambda} Z^1_\lambda Z^2_\lambda dx + o(1)
\]

Similarly

\[ \|PZ^1_\lambda\|^2 = \int_{\Omega} |x|^{2(\alpha-1)}e^{W_\lambda} Z^1_\lambda PZ^1_\lambda dx = \int_{\Omega} |x|^{2(\alpha-1)}e^{W_\lambda}\phi PZ^1_\lambda dx + o(1) \]

\[ = 8\alpha^2 \int_{\mathbb{R}^2} \frac{|y|^{2(\alpha-1)}|\text{Re}(y^\alpha - \delta^{-\alpha}b)|^2}{(1 + |y^\alpha - \delta^{-\alpha}b|^2)^4} dy + o(1) \]

\[ = 8\alpha \int_{\mathbb{R}^2} \frac{y^2}{1 + |y|^2} dy = \frac{2}{3} \pi + o(1) \] (3.26)

where we have used Lemma A.1 and (3.6). Analogously \( \|PZ^2_\lambda\|^2 = \frac{2}{3} \pi + o(1) \).

Then, taking into account that \( -\Delta PZ^j_\lambda = |x|^{2(\alpha-1)}e^{W_\lambda} Z^j_\lambda \), according to Proposition 3.1 we have

\[ \|\phi\| \leq C|\log \delta|\|h\| \] (3.27)

Hence it suffices to estimate the values of the constants \( c_j \). We multiply the equation in (3.24) by \( PZ^1_\lambda \) and we find

\[
\begin{aligned}
\int_{\Omega} \phi |x|^{2(\alpha-1)}e^{W_\lambda} Z^1_\lambda dx - \lambda \int_{\Omega} \tilde{V}(x)|x|^{2(\alpha-1)}e^{PW_\lambda} \phi PZ^1_\lambda dx &= 2\pi \alpha c_1 + o(c_1) + o(c_2) + O(\|h\|).
\end{aligned}
\] (3.28)

Let us fix \( p \in (1, +\infty) \) sufficiently close to 1. Then, by (3.2) and (4.1) we may estimate

\[
\int_{\Omega} |x|^{2(\alpha-1)}e^{W_\lambda} |\phi||PZ^1_\lambda - Z^1_\lambda| dx \leq C\delta^{\alpha} \int_{\Omega} |x|^{2(\alpha-1)}e^{W_\lambda} |\phi| dx
\]

\[
\leq C\delta^{\alpha} \|\phi\| \|x|^{2(\alpha-1)}e^{W_\lambda}\|_p \leq C\delta^{\alpha-2\alpha\frac{p-1}{p}} \|\phi\| \leq \delta^{\frac{\alpha}{p}} \|\phi\|
\]

and, since \( PZ^1_\lambda = O(1) \), using Proposition 4.2,

\[
\int_{\Omega} |x|^{2(\alpha-1)}e^{W_\lambda} - \lambda \tilde{V}(x)|x|^{2(\alpha-1)}e^{PW_\lambda} |\phi||PZ^1_\lambda| dx
\]
\[ \leq C \int_{\Omega} |R_\lambda||\phi|dx \leq C\delta^\alpha - 2\alpha \frac{p-1}{p} \|\phi\| \leq \delta^\frac{2}{p} \|\phi\|. \]

By inserting the above estimate into (3.28) we obtain
\[ |c_1| + o(c_2) \leq C\|h\| + C\delta^\frac{2}{p} \|\phi\|. \]

We multiply the equation in (3.24) by \( PZ^2_\lambda \) and, by a similar argument as above, we find
\[ |c_2| + o(c_1) \leq C\|h\| + C\delta^\frac{2}{p} \|\phi\|, \]
and so
\[ |c_1| + |c_2| \leq C\|h\| + C\delta^\frac{2}{p} \|\phi\|. \]

Combining this with (3.27) we obtain the thesis. \( \square \)

4. **Estimate of the error term.** The goal of this section is to provide an estimate of the error up to which the function \( W_\lambda \) solves problem (1.1). First of all, we perform the following estimates.

**Lemma 4.1.** Let \( \gamma = 0, 1, 2 \) and \( p > 1 \) be fixed. The following holds:
\[ \| |x|^{2(\alpha-1)}|x^\alpha - b|^\gamma e^{W_\lambda} \|_p \leq C\delta^\alpha \delta^{-2\alpha \frac{p-1}{p}}, \] uniformly for \( b \in \Omega_\alpha \). Moreover (4.1) holds also for \( \gamma = 0, 1 \) and \( p = 1 \).

**Proof.** By Lemma A.1 we compute
\[ \| |x|^{2(\alpha-1)}|x^\alpha - b|^\gamma e^{W_\lambda} \|_p = (8\alpha^2)p\delta^{2\alpha p} \int_{\Omega} |x|^{2(\alpha-1)p}|x^\alpha - b|^\gamma p \frac{dx}{(\delta^{2\alpha} + |x^\alpha - b|^2)^p} \]
\[ \leq C(8\alpha^2)p\delta^{2\alpha p} \int_{\Omega} |x|^{2(\alpha-1)p}|x^\alpha - b|^\gamma p \frac{dx}{(\delta^{2\alpha} + |x^\alpha - b|^2)^p} \]
\[ = C(8\alpha^2)p\delta^{\alpha \gamma p - 2\alpha (p-1)} \int_{\mathbb{R}^2} \frac{|y|^{2(\alpha-1)}|y^\alpha - \delta^{-\alpha}b|^\gamma p}{(1 + |y^\alpha - \delta^{-\alpha}b|^2)^p} dy \]
\[ = C(8\alpha^2)p\delta^{\alpha \gamma p - 2\alpha (p-1)} \int_{\mathbb{R}^2} \frac{|y|^{\gamma p}}{(1 + |y|^2)^p} dy. \]

Taking into account that the last integral is finite for \( \gamma = 0, 1, 2 \) and \( p > 1 \) (and also for \( \gamma = 0, 1 \) and \( p = 1 \)) we deduce (4.1). \( \square \)

**Proposition 4.2.** Define
\[ R_\lambda := -\Delta PW_\lambda - \Lambda \tilde{V}(x)|x|^{2(\alpha-1)}e^{PW_\lambda} = |x|^{2(\alpha-1)}e^{PW_\lambda} - \Lambda \tilde{V}(x)|x|^{2(\alpha-1)}e^{PW_\lambda}. \] Then
\[ R_\lambda = -\frac{4\pi}{\alpha} \langle \nabla \Lambda(b), x^\alpha - b \rangle |x|^{2(\alpha-1)}e^{PW_\lambda} + \left(O(|x^\alpha - b|^2) + O(\delta^{2\alpha})\right) |x|^{2(\alpha-1)}e^{PW_\lambda} \]
uniformly for \( b \) on compact subsets of \( \Omega_\alpha \setminus \{0\} \), where \( \Lambda \) is the function defined in (1.5). Moreover, for any fixed \( p \geq 1 \) the following holds
\[ \| R_\lambda \|_p \leq C\delta^{\alpha - 2\alpha \frac{p-1}{p}}, \quad \left\| \frac{\partial R_\lambda}{\partial b_j} \right\|_p \leq C\delta^{-2\alpha \frac{p-1}{p}} \quad j = 1, 2 \]
uniformly for \( b \) on compact subsets of \( \Omega_\alpha \setminus \{0\} \). Consequently, by (4.1), if \( p \geq 1 \)
\[ \| \Lambda \tilde{V}(x)|x|^{2(\alpha-1)}e^{PW_\lambda} \|_p = \| |x|^{2(\alpha-1)}e^{W_\lambda} \|_p + O(\delta^{\alpha - 2\alpha \frac{p-1}{p}}) = O(\delta^{-2\alpha \frac{p-1}{p}}) \]
uniformly for \( b \) on compact subsets of \( \Omega_\alpha \setminus \{0\} \).
Proof. By (2.8) and the choice of $\delta$ in (2.7) we derive
\[\lambda \tilde{V}(x)|x|^{2(\alpha-1)} e^{PW_\lambda}\]
\[= \frac{\lambda}{8\alpha^2 \delta^2 \alpha} \tilde{V}(x)|x|^{2(\alpha-1)} e^{PW_\lambda} + 8\pi \mathcal{H}_\alpha(x^n,b) + O(\delta^{2\alpha})\]
\[= |x|^{2(\alpha-1)} e^{PW_\lambda} \frac{V(x)}{V(\beta)} e^{-4\pi (\alpha-1)(H(x,0) - \mathcal{H}_\alpha(b,0))} + 8\pi (\mathcal{H}_\alpha(x^n,b) - \mathcal{H}_\alpha(b,0)) + O(\delta^{2\alpha})\]
\[= |x|^{2(\alpha-1)} e^{PW_\lambda} \frac{V(x)}{V(\beta)} e^{-4\pi \frac{\alpha}{4\pi} (\mathcal{H}_\alpha(x^n,0) - \mathcal{H}_\alpha(b,0))} + 8\pi (\mathcal{H}_\alpha(x^n,b) - \mathcal{H}_\alpha(b,0)) + O(\delta^{2\alpha})\] (4.4)

By (A.2)
\[e^{-4\pi \frac{\alpha}{4\pi} (\mathcal{H}_\alpha(x^n,0) - \mathcal{H}_\alpha(b,0))} + 8\pi (\mathcal{H}_\alpha(x^n,b) - \mathcal{H}_\alpha(b,0)) + O(\delta^{2\alpha})\]
\[= e^{\frac{2\pi}{\alpha} (\nabla_b (\alpha \mathcal{H}_\alpha(b)) - (\alpha - 1) \mathcal{H}_\alpha(b,0))} + O(|x^n - b|^2) + O(\delta^{2\alpha})\] (4.5)

uniformly for $b$ on compact subsets of $\Omega_\alpha$. Now, using assumption (A1)* let us set
\[V_\alpha(b) = V(\beta) = V(\beta) \quad \forall b \in \Omega_\alpha;\]

observe that $V_\alpha$ is smooth in $\Omega_\alpha \setminus \{0\}$. We have
\[V(x) = V_\alpha(x^n) = V_\alpha(b) + \langle \nabla V_\alpha(b), x^n - b \rangle + O(|x^n - b|^2)\]
\[= 1 + \langle \nabla_b V_\alpha(b), x^n - b \rangle + O(|x^n - b|^2)\]
\[= 1 + \langle \nabla_b (\log V_\alpha(b)), x^n - b \rangle |x|^{2(\alpha-1)} e^{PW_\lambda} + O(|x^n - b|^2)\] (4.6)

uniformly for $b$ on compact subsets of $\Omega_\alpha \setminus \{0\}$. So combining (4.5) with (4.6), (4.4) reduces to
\[\lambda \tilde{V}(x)|x|^{2(\alpha-1)} e^{PW_\lambda}\]
\[= |x|^{2(\alpha-1)} e^{PW_\lambda} + \frac{4\pi}{\alpha} \langle \nabla_b (\alpha \mathcal{H}_\alpha(b) - (\alpha - 1) \mathcal{H}_\alpha(b,0)), x^n - b \rangle\]
\[\times |x|^{2(\alpha-1)} e^{PW_\lambda} + \left( O(|x^n - b|^2) + O(\delta^{2\alpha}) \right)^{|x|^{2(\alpha-1)} e^{PW_\lambda}}\]
\[= |x|^{2(\alpha-1)} e^{PW_\lambda} + \frac{4\pi}{\alpha} \langle \nabla_b A(x), x^n - b \rangle + \left( O(|x^n - b|^2) + O(\delta^{2\alpha}) \right)|x|^{2(\alpha-1)} e^{PW_\lambda}\]

uniformly for $b$ on compact subsets of $\Omega_\alpha \setminus \{0\}$ and the first part of the thesis follows.

In particular
\[\lambda \tilde{V}(x)|x|^{2(\alpha-1)} e^{PW_\lambda} = |x|^{2(\alpha-1)} e^{PW_\lambda} + \left( O(|x^n - b|) + O(\delta^{2\alpha}) \right)|x|^{2(\alpha-1)} e^{PW_\lambda}\] (4.8)

uniformly for $b$ on compact subsets of $\Omega_\alpha \setminus \{0\}$, and the first part of (4.2) follows by Lemma 4.1. Next, in order to prove the second part of (4.2), observe that
\[\frac{\partial W_\lambda}{\partial b_j}, \frac{\partial PW_\lambda}{\partial b_j} = 4\delta^{-\alpha} PZ_\lambda^j + O(1) \quad j = 1, 2\] (4.9)

$C^1$-uniformly for $b$ on compact subsets of $\Omega_\alpha$, therefore by (4.8)
\[\frac{\partial R_\lambda}{\partial b_j} = |x|^{2(\alpha-1)} e^{PW_\lambda} \left( 4\delta^{-\alpha} PZ_\lambda^j + O(1) \right) - \lambda \tilde{V}(x)|x|^{2(\alpha-1)} e^{PW_\lambda} \left( 4\delta^{-\alpha} PZ_\lambda^j + O(1) \right)\]
\[= \left( O(\delta^{-\alpha} |x^n - b|) + O(1) \right)|x|^{2(\alpha-1)} e^{PW_\lambda}\]
5. The nonlinear problem: a contraction argument. In order to solve (1.1), let us consider the following intermediate problem:

\[
\begin{align*}
- \Delta (PW_\lambda + \phi) - \lambda \tilde{V}(x)|x|^{2(\alpha-1)}e^{PW_\lambda + \phi} &= \sum_{j=1,2} c_j Z_j^1 |x|^{2(\alpha-1)}e^{W_\lambda} + \tilde{\nu}_j (x)^2 (\alpha-1) e^{PW_\lambda} + \nu_j \\
\phi \in H^1_0,(\Omega) , \quad \int_\Omega \nabla \phi \nabla \tilde{Z}_j dx &= 0, j = 1,2.
\end{align*}
\]

(5.1)

Then it is convenient to solve as a first step the problem for \( \phi \) as a function of \( b \).

To this aim, first let us rewrite problem (5.1) in a more convenient way. For any \( p > 1 \), let

\[
i^*_p : L^p(\Omega) \to H^1_0,(\Omega)
\]

be the adjoint operator of the embedding \( i_p : H^1_0,(\Omega) \hookrightarrow L^{\frac{p}{p-1}}(\Omega) \), i.e. \( u = i^*_p(v) \) if and only if \( -\Delta u = v \) in \( \Omega \), \( u = 0 \) on \( \partial \Omega \). We point out that \( i^*_p \) is a continuous mapping, namely

\[
\|i^*_p(v)\| \leq c_p \|v\|_p, \text{ for any } v \in L^p(\Omega),
\]

(5.3)

for some constant \( c_p \) which depends on \( \Omega \) and \( p \).

Next let us set

\[
K := \text{span} \{ PZ_1^1, PZ_2^1 \}
\]

and

\[
K^\perp := \left\{ \phi \in H^1_0,(\Omega) : \int_\Omega \nabla \phi \nabla PZ_1^1 dx = \int_\Omega \nabla \phi \nabla PZ_2^1 dx = 0 \right\}
\]

and denote by

\[
\Pi : H^1_0,(\Omega) \to K, \quad \Pi^\perp : H^1_0,(\Omega) \to K^\perp
\]

the corresponding projections. Let \( L : K^\perp \to K^\perp \) be the linear operator defined by

\[
L(\phi) := \Pi^\perp \left( i^*_p (\lambda \tilde{V}(x)|x|^{2(\alpha-1)}e^{PW_\lambda} \phi) \right) - \phi.
\]

(5.4)

Notice that problem (3.24) reduces to

\[
L(\phi) = \Pi^\perp h, \quad \phi \in K^\perp.
\]

As a consequence of Proposition 3.2 we derive the invertibility of \( L \).

**Proposition 5.1.** For any \( p > 1 \) there exist \( \lambda_0 > 0 \) and \( C > 0 \) such that for any \( \lambda \in (0, \lambda_0) \), any \( b \in \mathbb{R}^2 \) in a compact subset of \( \Omega_\alpha \setminus \{0\} \) and any \( h \in K^\perp \) there is a unique solution \( \phi \in K^\perp \) to the problem

\[
L(\phi) = h.
\]

In particular, \( L \) is invertible; moreover,

\[
\|L^{-1}\| \leq C |\log \delta|.
\]

**Proof.** Observe that the operator \( \phi \mapsto \Pi^\perp \left( i^*_p (\lambda V(x)|x|^{2(\alpha-1)}e^{PW_\lambda} \phi) \right) \) is a compact operator in \( K^\perp \). Let us consider the case \( h = 0 \), and take \( \phi \in K^\perp \) with \( L(\phi) = 0 \). In other words, \( \phi \) solves the system (3.24) with \( h = 0 \) for some \( c_1, c_2 \in \mathbb{R} \). Proposition 3.2 implies \( \phi \equiv 0 \). Then, Fredholm’s alternative implies the existence and uniqueness result.

Once we have existence, the norm estimate follows directly from Proposition 3.2.
Now we come back to our goal of finding a solution to problem (5.1). In what follows we denote by $N : H^1_{0,*} \to K^\perp$ the nonlinear operator

$$N(\phi) = \Pi^\perp \left( i_p^* (\lambda \tilde{V}(x)|x|^{2(\alpha-1)}e^{PW_\lambda}(e^\phi - 1 - \phi)) \right).$$

Therefore problem (5.1) turns out to be equivalent to the problem

$$L(\phi) + N(\phi) = \tilde{R}, \quad \phi \in K^\perp$$

where, recalling Lemma 4.1,

$$\tilde{R} = \Pi^\perp \left( i_p^*(R_\lambda) \right) = \Pi^\perp \left( PW_\lambda - \lambda \sum_{p} \alpha \frac{e^{\phi_1} + e^{\phi_2}}{1 - e^{\phi_1} + e^{\phi_2}} \right).$$

We need the following two auxiliary lemmas.

**Lemma 5.2.** For any $p > 1$ and any $\phi_1, \phi_2 \in H^1_{0,*}(\Omega)$ with $\|\phi_1\|, \|\phi_2\| < 1$ the following holds

$$\|e^{\phi_1} - e^{\phi_2} + \phi_2\|_p \leq C(\|\phi_1\| + \|\phi_2\|)\|\phi_1 - \phi_2\|,$$  \hspace{1cm} (5.6)

$$\|N(\phi_1) - N(\phi_2)\| \leq C(\|\phi_1\| + \|\phi_2\|)\|\phi_1 - \phi_2\|,$$  \hspace{1cm} (5.7)

uniformly for $b$ in a compact subset of $\Omega_n \setminus \{0\}$.

**Proof.** A straightforward computation gives that the inequality $|e^a - e^b + b| \leq e^{|a|+|b|}(|a| + |b|)|a - b|$ holds for all $a, b \in \mathbb{R}$. Then, by applying Hölder's inequality with $\frac{1}{q} + \frac{1}{r} + \frac{1}{t} = 1$, we derive

$$\|e^{\phi_1} - e^{\phi_2} + \phi_2\|_p \leq C\|e^{\phi_1} + \phi_2\|_p(\|\phi_1\| + \|\phi_2\|)\|\phi_1 - \phi_2\|_p$$

and (5.6) follows by using Lemma 2.2 and the continuity of the embeddings $H^1_{0}(\Omega) \subset L^{pr}(\Omega)$ and $H^1_{0}(\Omega) \subset L^{pt}(\Omega)$. Let us prove (5.7). According to (5.3) we get

$$\|N(\phi_1) - N(\phi_2)\| \leq C\|\tilde{V}(x)|x|^{2(\alpha-1)}e^{PW_\lambda}(e^{\phi_1} - e^{\phi_2} + \phi_2)\|_p,$$

and by Hölder's inequality with $\frac{1}{r} + \frac{1}{q} = 1$ we derive

$$\|N(\phi_1) - N(\phi_2)\| \leq C\|\lambda x|^{2(\alpha-1)}\tilde{V}(x)e^{PW_\lambda}\|_p(\|\phi_1 - \phi_2\| + \|\phi_2\|)\|\phi_1 - \phi_2\|_p,$$

by (5.6), and the conclusion follows recalling (4.3). \hfill \Box

**Corollary 5.3.** For any $p > 1$ and any $\phi \in H^1_{0,*}(\Omega)$ with $\|\phi\| < 1$ the following holds

$$\|e^{\phi} - 1 - \phi\|_p \leq C\|\phi\|, \quad \|e^{\phi} - 1\|_p \leq C\|\phi\|.$$  \hspace{1cm} (5.8)

**Proof.** It is sufficient to apply (5.6) by taking $\phi_2 = 0$. \hfill \Box

**Lemma 5.4.** For any $\phi \in H^1_{0,*}(\Omega)$ the following holds:

$$\left\| \frac{\partial \Pi}{\partial b_i}(\phi) \right\| = O(\delta^{-\alpha})\|\phi\|, \quad \left\| \frac{\partial \Pi}{\partial b_i}(\phi) \right\| = O(\delta^{-\alpha})\|\phi\|$$

uniformly for $b$ on compact subsets of $\Omega_n$. \hspace{1cm} (5.8)
Proof. Given any \( \phi \in H^1_{0,*}(\Omega) \), we write \( \Pi \phi \) in coordinates

\[
\Pi \phi = \sum_{j=1,2} a_j PZ^j_{\lambda}.
\]

Here the coefficients \( a_j = a_j(b) \) solve

\[
\langle \phi, PZ^k_{\lambda} \rangle = \sum_{j=1,2} a_j \langle PZ^j_{\lambda}, PZ^k_{\lambda} \rangle, \quad k = 1, 2,
\]

where the scalar product \( \langle \cdot, \cdot \rangle \) is taken in \( H^1_{0}(\Omega) \). In other words, the vector \( a = (a_1, a_2) \) solves the linear system:

\[
A \cdot a = \xi,
\]

with \( \xi = \xi(b) = (\langle \phi, PZ^j_{\lambda} \rangle)_{j=1,2} \) and

\[
A = A(b) = ((PZ^j_{\lambda}, PZ^k_{\lambda}))_{2 \times 2}.
\]

Estimates (3.25)-(3.26) imply that \( A \) is invertible and \( \|A\| = O(1), \|A^{-1}\| = O(1) \). So

\[
a = A^{-1} : \xi \Rightarrow \|a\| \leq C \|
\phi\|.
\]

(5.9)

Computing now the derivative with respect to \( b_i \), we obtain:

\[
\frac{\partial a}{\partial b_i} = A^{-1} \cdot \frac{\partial \xi}{\partial b_i} - A^{-1} \cdot \frac{\partial A}{\partial b_i} \cdot A^{-1} \cdot \xi.
\]

A straightforward computation gives \( \|\frac{\partial PZ^j_{\lambda}}{\partial b_i}\| = O(\delta^{-\alpha}) \) uniformly for \( b \) on compact subsets of \( \Omega_{\alpha} \), by which \( \frac{d\phi}{db_i} = O(\delta^{-\alpha})\|\phi\|, \frac{dA}{db_i} = O(\delta^{-\alpha}) \). Therefore,

\[
\left| \frac{\partial a}{\partial b_i} \right| \leq C\delta^{-\alpha} \|\phi\|.
\]

(5.10)

Finally, observe that

\[
\frac{\partial \Pi}{\partial b_i}(\phi) = \sum_{j=1,2} a_j \frac{\partial Z^j_{\lambda}}{\partial b_i} + \frac{\partial a_j}{\partial b_i} \cdot PZ^j_{\lambda}.
\]

Taking into account of (5.9)-(5.10), we obtain the first estimate of (5.8). The second follows from \( \Pi^\perp \phi = \phi - \Pi \phi \).

Problem (5.1) or, equivalently, problem (5.5) turns out to be solvable for any choice of point \( b \) in a compact subset of \( \Omega_{\alpha} \setminus \{0\} \), provided that \( \lambda \) is sufficiently small. Indeed we have the following result.

**Proposition 5.5.** Let \( \varepsilon > 0 \) be a fixed small number. Then there exists \( \lambda_0 > 0 \) such that for any \( \lambda \in (0, \lambda_0) \) and any \( b \in \mathbb{R}^2 \) in a compact subset of \( \Omega_{\alpha} \setminus \{0\} \) there is a unique \( \phi_{\lambda} = \phi_{\lambda,b} \in K^\perp \) satisfying (5.1) for some \( c_1, c_2 \in \mathbb{R} \) and

\[
\|\phi_{\lambda}\| \leq \delta^{\alpha - \varepsilon}.
\]

Moreover the map \( b \mapsto \phi_{\lambda,b} \in H^1_{0,*}(\Omega) \) is \( C^1 \) and

\[
\left\| \frac{\partial \phi_{\lambda,b}}{\partial b_i} \right\| \leq \delta^{-3\varepsilon}.
\]

(5.11)
Proof. Since problem (5.5) is equivalent to problem (5.1), we will show that problem (5.5) can be solved via a contraction mapping argument. Indeed, in virtue of Proposition 5.1, let us introduce the map

\[ T := L^{-1}(\tilde{R} - N(\phi)), \quad \phi \in K^{\perp}. \]

Let us fix

\[ 0 < \eta < \varepsilon \]

and \( p > 1 \) sufficiently close to 1. According to (5.3) and Proposition 4.2 we have

\[ \| \tilde{R} \| \leq C\delta^{\alpha - \eta}. \quad (5.12) \]

Similarly, by (5.7),

\[ \| N(\phi_1) - N(\phi_2) \| \leq C\delta^{-\eta}(\| \phi_1 \| + \| \phi_2 \|)\| \phi_1 - \phi_2 \| \quad \forall \phi_1, \phi_2 \in H_{0,*}^{1}(\Omega), \| \phi_1 \|, \| \phi_2 \| < 1. \quad (5.13) \]

In particular, by taking \( \phi_2 = 0 \),

\[ \| N(\phi) \| \leq C\delta^{-\eta}\| \phi \|^{2} \quad \forall \phi \in H_{0,*}^{1}(\Omega), \| \phi \| < 1. \quad (5.14) \]

We claim that \( T \) is a contraction map over the ball

\[ \mathcal{B} := \left\{ \phi \in K^{\perp} \mid \| \phi \| \leq \delta^{\alpha - \varepsilon} \right\} \]

provided that \( \lambda \) is small enough. Indeed, combining Proposition 5.1, (5.12), (5.13), (5.14) with the choice of \( \eta (0 < \eta < \varepsilon) \), for any \( \phi, \phi_1, \phi_2 \in \mathcal{B} \) we have

\[ \| T(\phi) \| \leq C| \log \delta(\| \tilde{R} \| + \| N(\phi) \|) \|

\[ \leq C| \log \delta \delta^{-\eta}(\delta^{\alpha} + \| \phi \|^{2}) < \delta^{\alpha - \varepsilon}, \| T(\phi_1) - T(\phi_2) \| \leq C| \log \delta\| N(\phi_1) - N(\phi_2) \|

\[ \leq C\delta^{-\eta}| \log \delta(\| \phi_1 \| + \| \phi_2 \|)\| \phi_1 - \phi_2 \| \leq \frac{1}{2}\| \phi_1 - \phi_2 \|. \]

We now consider the dependence of \( \phi_{\lambda,b} \) on \( b \). In order to prove that the map \( b \to \phi_{\lambda,b} \) is \( C^{1} \), we apply the Implicit Function Theorem to the function

\[ \Phi(b, \phi) = \phi + \Pi^{\perp} \left( PW_{\lambda} - i_{p}^{*}(\lambda \tilde{V}(x)|x|^{2(\alpha - 1)}e^{PW_{\lambda} + \Pi^{\perp} \phi}) \right), \quad \phi \in H_{0,*}^{1}(\Omega). \]

Indeed \( \Phi(b, \phi_{\lambda,b}) = 0 \) and the linear operator: \( \frac{\partial \Phi}{\partial \phi}(b, \phi_{\lambda,b}) : H_{0,*}^{1}(\Omega) \to H_{0,*}^{1}(\Omega) \) is given by

\[ \frac{\partial \Phi}{\partial \phi}(b, \phi_{\lambda,b})(\psi) = \psi - \Pi^{\perp} \left( i_{p}^{*}(\lambda \tilde{V}(x)|x|^{2(\alpha - 1)}e^{PW_{\lambda} + \phi_{\lambda,b} + \Pi^{\perp} \psi}) \right). \]

We observe that \( \frac{\partial \Phi}{\partial \phi}(b, \phi_{\lambda,b}) \) is a Fredholm’s operator. By comparing \( \frac{\partial \Phi}{\partial \phi}(b, \phi_{\lambda,b}) \) with the definition of \( L \) in (5.4), we have that

\[ \frac{\partial \Phi}{\partial \phi}(b, \phi_{\lambda,b})(\psi) = \Pi^{\perp}(\psi) - L(\Pi^{\perp} \psi) - \Pi^{\perp} \left( i_{p}^{*}(\lambda \tilde{V}(x)|x|^{2(\alpha - 1)}e^{PW_{\lambda}(e^{\phi_{\lambda,b}} - 1)\Pi^{\perp} \psi}) \right) \]

by which, using Lemma 5.1,

\[ \left\| \frac{\partial \Phi}{\partial \phi}(b, \phi_{\lambda,b})(\psi) \right\| \geq \sqrt{\| \Pi^{\perp}(\psi) \|^{2} + \| L(\Pi^{\perp} \psi) \|^{2}} \]

\[ - \left\| \Pi^{\perp} \left( i_{p}^{*}(\lambda \tilde{V}(x)|x|^{2(\alpha - 1)}e^{PW_{\lambda}(e^{\phi_{\lambda,b}} - 1)\Pi^{\perp} \psi}) \right) \right\| \]

\[ \geq \frac{c}{| \log \delta |} \| \psi \| - c\| \lambda \tilde{V}(x)|x|^{2(\alpha - 1)}e^{PW_{\lambda}(e^{\phi_{\lambda,b}} - 1)\Pi^{\perp} \psi} \|_{\tilde{P}}. \quad (5.15) \]
Now, using Hölder’s inequality with \( \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 \) we compute
\[
\|\lambda \widetilde{V}(x)|x|^{2(\alpha-1)}e^{PW_{\lambda}(\phi_{\lambda,b}) - 1}\|_p
\leq \|\lambda \widetilde{V}(x)|x|^{2(\alpha-1)}e^{PW_{\lambda}}\|_p\|e^{\phi_{\lambda,b}} - 1\|_{pr}
\leq C\delta^{\alpha-\varepsilon-2\alpha\frac{\varepsilon^2-1}{r^2}}\|\psi\|
\]
by (4.3) and Corollary 5.3. Inserting this into (5.15), we conclude that
\[
\left\| \frac{\partial \Phi}{\partial \phi}(b, \phi_{\lambda,b})(\psi) \right\| \geq \frac{c}{|\log \delta|}\|\psi\|
\]
which guarantees the invertibility of the operator \( \frac{\partial \Phi}{\partial \phi}(b, \phi_{\lambda,b}) \), and moreover
\[
\left\| \left( \frac{\partial \Phi}{\partial b}(b, \phi_{\lambda,b}) \right)^{-1} \right\| \leq C|\log \delta|.
\]
By the implicit function theorem, the map \( b \mapsto \phi_{\lambda,b} \) is \( C^1 \) and
\[
\frac{\partial \phi_{\lambda,b}}{\partial b_i} = -\left( \frac{\partial \Phi}{\partial \phi}(b, \phi_{\lambda,b}) \right)^{-1} \left( \frac{\partial \Phi}{\partial b}(b, \phi_{\lambda,b}) \right)
\]
So we need to estimate \( \frac{\partial \Phi}{\partial b_i}(b, \phi_{\lambda,b}) \in H^1_{b_i}(\Omega) \). Taking into account that \( PW_{\lambda} = i_p^*(|x|^{2(\alpha-1)}e^{W_{\lambda}}) \), we can write \( \Phi(b, \phi) \) as
\[
\Phi(b, \phi) = \phi + \Pi_\perp(i_p^*(F(b, \phi))) \quad F(b, \phi) := R_{\lambda} - \lambda \widetilde{V}(x)|x|^{2(\alpha-1)}e^{PW_{\lambda}(\phi_{\lambda,b} - 1)}
\]
by which we compute
\[
\frac{\partial \Phi}{\partial \phi}(b, \phi) = \partial \left( \Pi_\perp(i_p^*(F(b, \phi))) \right) = \Pi_\perp\left( i_p^*(F(b, \phi) \right) \left( i_p^*(\frac{\partial F}{\partial b}(b, \phi)) \right)
\]
Now, using Hölder’s inequality with \( \frac{1}{p} + \frac{1}{s} = 1 \), and recalling (4.3) and Corollary 5.3, we compute
\[
\|\lambda \widetilde{V}(x)|x|^{2(\alpha-1)}e^{PW_{\lambda}(\phi_{\lambda,b} - 1)}\|_p
\leq \|\lambda \widetilde{V}(x)|x|^{2(\alpha-1)}e^{PW_{\lambda}}\|_p\|e^{\phi_{\lambda,b} - 1}\|_{ps}
\leq C\delta^{\alpha-\varepsilon-2\alpha\frac{\varepsilon^2-1}{r^2}}
\]
and, taking into account of (4.2), this implies
\[
\|F(b, \phi_{\lambda,b})\|_p \leq C\delta^{\alpha-\varepsilon-2\alpha\frac{\varepsilon^2-1}{r^2}}.
\]
Moreover, using again (4.2) and the estimate \( \frac{\partial PW_{\lambda}}{\partial b_i} = O(\delta^{-\alpha}) \) by (4.9),
\[
\left\| \frac{\partial F}{\partial b}(b, \phi_{\lambda,b}) \right\|_p = \left\| \frac{\partial R_{\lambda}}{\partial b_i} + \lambda \widetilde{V}(x)|x|^{2(\alpha-1)}e^{PW_{\lambda}} \frac{\partial PW_{\lambda}}{\partial b_i}(\phi_{\lambda,b} - 1) \right\|_p
\leq C\delta^{\varepsilon-2\alpha\frac{\varepsilon^2-1}{r^2}}.
\]
By inserting (5.17)-(5.18) into (5.16), and recalling Lemma 5.4, we get
\[
\left\| \frac{\partial \Phi}{\partial b}(b, \phi_{\lambda,b}) \right\| \leq \left\| \Pi_\perp(i_p^*(F(b, \phi_{\lambda,b}))) \right\| \left\| i_p^*(\frac{\partial F}{\partial b}(b, \phi_{\lambda,b})) \right\| \leq C\delta^{-\alpha}\left\| i_p^*(F(b, \phi_{\lambda,b})) \right\| \left\| i_p^*(\frac{\partial F}{\partial b}(b, \phi_{\lambda,b})) \right\| \leq C\delta^{-\alpha}\|F(b, \phi_{\lambda,b})\|_p \left\| \frac{\partial F}{\partial b}(b, \phi_{\lambda,b}) \right\|_p \leq C\delta^{-\varepsilon-4\alpha\frac{\varepsilon^2-1}{r^2}}
\]
and (5.11) follows by taking \( p \) sufficiently close to 1. \( \square \)

6. The finite dimensional reduction. After problem (5.1) has been solved according to Proposition 5.5, then we find a solution to the original problem (2.2) if \( b \in \Omega_\alpha \setminus \{0\} \) is such that
\[
c_j = 0 \text{ for } j = 1, 2.
\]

Let us find the condition on \( b \) in order to get the \( c_j \)'s equal to zero. This problem is actually variational; more precisely, it is equivalent to find a critical point of a function of \( b \).

Indeed, let us consider the following energy functional associated with (2.2):
\[
I_\lambda(v) = \frac{1}{2} \int_\Omega |\nabla v|^2 \, dx - \lambda \int_\Omega \tilde{V}(x) |x|^{2(\alpha-1)} e^p \, dx, \quad v \in H^1_0(\Omega).
\]

Solutions of (2.2) correspond to critical points of \( I_\lambda \). Now we introduce the new functional
\[
J_\lambda(b) = I_\lambda(PW_\lambda + \phi_\lambda)
\]
defined on compact subsets of \( \Omega_\alpha \setminus \{0\} \), where \( \phi_\lambda = \phi_{\lambda,b} \) has been constructed in Proposition 5.5. The next proposition reduces the problem (2.2) to the one of finding critical points of the functional \( J_\lambda \).

**Proposition 6.1.** If \( b \in \Omega_\alpha \setminus \{0\} \) is a critical point of \( J_\lambda \), then the corresponding function \( v_\lambda = PW_\lambda + \phi_\lambda \) is a solution of (2.2).

**Proof.** Let \( b \) be a critical point of \( J_\lambda \):
\[
\frac{\partial J_\lambda}{\partial b_1}(b) = \frac{\partial J_\lambda}{\partial b_2}(b) = 0.
\]

Using Proposition 5.5 we can differentiate directly \( I_\lambda(PW_\lambda + \phi_\lambda) \) under the integral sign, so that for \( i = 1, 2 \)
\[
\int_\Omega \nabla(PW_\lambda + \phi_\lambda) \nabla \frac{\partial (PW_\lambda + \phi_\lambda)}{\partial b_i} \, dx = \lambda \int_\Omega \tilde{V}(x) |x|^{2(\alpha-1)} e^{pW_\lambda + \phi_\lambda} \frac{\partial (PW_\lambda + \phi_\lambda)}{\partial b_i} = 0.
\]

Taking into account that \( \phi_\lambda \) solves problem (5.1), this is equivalent to
\[
\sum_{j=1,2} c_j \int_\Omega Z_j^i |x|^{2(\alpha-1)} e^{W_\lambda} \frac{\partial (PW_\lambda + \phi_\lambda)}{\partial b_i} \, dx = 0, \quad i = 1, 2.
\]

Now, let \( p > 1 \) be sufficiently close to 1 and let \( 1 < q < +\infty \) be such that \( \frac{1}{p} + \frac{1}{q} = 1 \); then Lemma 4.1 and (5.11) give
\[
\int_\Omega Z_j^i |x|^{2(\alpha-1)} e^{W_\lambda} \frac{\partial \phi_\lambda}{\partial b_i} \, dx = O \left( \|x|^{2(\alpha-1)} e^{W_\lambda} \|_p \left\| \frac{\partial \phi_\lambda}{\partial b_i} \right\|_q \right) = O(\delta^{-2\alpha + 1} - \varepsilon) = o(\delta^{-\alpha})
\]
provided that \( p \) is chosen sufficiently close to 1 and \( \varepsilon \) sufficiently close to 0. Using (4.9) we deduce that the system (6.4) can be rewritten as
\[
\sum_{j=1,2} c_j \int_\Omega Z_j^i |x|^{2(\alpha-1)} e^{W_\lambda} PZ_j \, dx + o(c_j) = 0, \quad i = 1, 2.
\]

By (3.25)-(3.26) we obtain that the system (6.5) is diagonal dominant and then we achieve \( c_1 = c_2 = 0 \). \( \square \)

Next purpose of this section is to provide an asymptotic expansion of the energy \( I_\lambda(PW_\lambda) \), where \( I_\lambda \) is the energy functional in (6.1).
Proposition 6.2. The following asymptotic expansion holds:

\[ I_\lambda(PW_\lambda) = -8\alpha \pi \left(2 + \log \lambda - \log(8\lambda^2)\right) - 32\pi^2 \Lambda(b) + O(\delta^\alpha) \]

\(C^1\)-uniformly with respect to \(b\) in compact subsets of \(\Omega_\alpha \setminus \{0\}\).

Proof. First by Lemma A.1 we compute

\[
\int_{\Omega} |x|^{2(\alpha - 1)} e^{W_\lambda} \log(\delta^{2\alpha} + |x^\alpha - b|^2) dx
\]

\[
= 8\alpha^2 \int_{\mathbb{R}^2} \frac{|y|^{2(\alpha - 1)}}{1 + |y^\alpha - \delta^{-\alpha} b|^2} \left(\log(\delta^{2\alpha}) + \log(1 + |y^\alpha - \delta^{-\alpha} b|^2)\right) dy
\]

\[
= 8\alpha \int_{\mathbb{R}^2} \frac{\log(\delta^{2\alpha}) + \log(1 + |y^\alpha - \delta^{-\alpha} b|^2)}{(1 + |y - \delta^{-\alpha} b|^2)^2} dy
\]

\[
= 8\alpha \int_{\mathbb{R}^2} \frac{\log(\delta^{2\alpha}) + \log(1 + |y|^2)}{(1 + |y|^2)^2} dy + O(\delta^{2\alpha})\log(\delta)\]

\[
= 8\pi \alpha (1 + 2\alpha \log \delta) + O(\delta^{2\alpha})\log(\delta)
\]

since \(\int_{\mathbb{R}^2} \frac{1}{1 + |y|^2} = \int_{\mathbb{R}^2} \frac{\log(1 + |y|^2)}{(1 + |y|^2)^2} = \pi\). On the other hand, since \(\mathcal{H}_\alpha(x^\alpha, b) = \mathcal{H}_\alpha(b, b) + O(|x^\alpha - b|)\) uniformly for \(x \in \Omega_\alpha\) and \(b\) on compact subsets of \(\Omega\), recalling (2.6) and Lemma 4.1, we get

\[
\int_{\Omega} |x|^{2(\alpha - 1)} e^{W_\lambda} \mathcal{H}(x^\alpha, b) dx
\]

\[
= \mathcal{H}_\alpha(b, b) \int_{\Omega} |x|^{2(\alpha - 1)} e^{W_\lambda} dx + \int_{\Omega} |x|^{2(\alpha - 1)} e^{W_\lambda} O(|x^\alpha - b|) = 8\pi \alpha \mathcal{H}_\alpha(b, b) + O(\delta^{\alpha}).
\]

So by (2.8), combining the above computations, we can write:

\[
\frac{1}{2} \int_{\Omega} |\nabla PW_\lambda|^2 dx
\]

\[
= \frac{1}{2} \int_{\Omega} |x|^{2(\alpha - 1)} e^{W_\lambda} P W_\lambda dx
\]

\[
= - \int_{\Omega} |x|^{2(\alpha - 1)} e^{W_\lambda} \log(\delta^{2\alpha} + |x^\alpha - b|^2) dx + 4\pi \int_{\Omega} |x|^{2(\alpha - 1)} e^{W_\lambda} \mathcal{H}_\alpha(x^\alpha, b) dx + O(\delta^{2\alpha})
\]

\[
= -8\pi \alpha(1 + \log \delta^{2\alpha}) + 32\pi^2 \alpha \mathcal{H}_\alpha(b, b) + O(\delta^{\alpha})
\]

\[
= -8\pi \alpha \left(1 + \log \lambda - \log(8\lambda^2) + \log(V(\beta_0))\right)
\]

\[
+ 32\pi(\alpha - 1) \mathcal{H}_\alpha(b, 0) - 32\pi^2 \alpha \mathcal{H}_\alpha(b, b) + O(\delta^{\alpha}). \tag{6.6}
\]

Now, the expansion in (4.7) gives

\[
\lambda \int_{\Omega} \bar{V}(x) |x|^{2(\alpha - 1)} e^{PW_\lambda} dx \tag{6.7}
\]

\[
= \int_{\Omega} |x|^{2(\alpha - 1)} e^{W_\lambda} dx + \int_{\Omega} O(|x^\alpha - b|) |x|^{2(\alpha - 1)} e^{W_\lambda} dx + O(\delta^{2\alpha}) = 8\pi \alpha + O(\delta^{\alpha})
\]

where we have used Lemma 4.1. Combing (6.6)-(6.7) we get

\[
I_\lambda(PW_\lambda) = \frac{1}{2} \int_{\Omega} |\nabla PW_\lambda|^2 dx - \lambda \int_{\Omega} |x|^{2(\alpha - 1)} \bar{V}(x) e^{PW_\lambda} dx
\]

\[
= -8\pi \alpha \left(2 + \log \lambda - \log(8\lambda^2) + \log(V(\beta_0))\right) + 32\pi(\alpha - 1) \mathcal{H}_\alpha(b, 0)
\]
Proposition 6.2. The following expansion holds:

\[-32\pi^2 \alpha \mathcal{H}_\alpha(b, b) + O(\delta^\alpha)\]

uniformly for \(b\) on compact subsets of \(\Omega_\alpha \setminus \{0\}\), and the first part of the thesis follows.

In order to prove the \(C^1\)-expansion, observe that using (4.9) and Proposition 4.2 we compute

\[
\frac{\partial}{\partial b_j} I_\lambda(W_\lambda) = 7 \int_\Omega \left( - \Delta PW_\lambda - \lambda \bar{V}(x)|x|^{2(\alpha - 1)}e^{W_\lambda} \right) \frac{\partial PW_\lambda}{\partial b_j} dx
\]

\[
= 4\delta^{-\alpha} \int_\Omega R_\lambda Z_\lambda^j dx + O\left( \int_\Omega |R_\lambda| dx \right)
\]

\[
= -\frac{16\pi}{\alpha} \delta^{-\alpha} \int_\Omega (\nabla_b \Lambda, x^\alpha - b)|x|^{2(\alpha - 1)}e^{W_\lambda} Z_\lambda^j dx
\]

\[
+ \int_\Omega O(|x^\alpha - b|^2 + O(\delta^{2\alpha}))(\nabla_b \Lambda, x^\alpha - b)|x|^{2(\alpha - 1)}e^{W_\lambda} dx + O(\delta^\alpha)
\]

by Lemma 4.1. Now we use Lemma A.1

\[
\delta^{-\alpha} \int_\Omega (\nabla_b \Lambda, x^\alpha - b)|x|^{2(\alpha - 1)}e^{W_\lambda} Z_\lambda^j dx
\]

\[
= 8\alpha^2 \int_{\mathbb{R}^2} (\nabla_b \Lambda, y^\alpha - \delta^{-\alpha} b)|y|^{2(\alpha - 1)}(1 + |y^\alpha - \delta^{-\alpha} b|^2)^3 \text{Re}(y^\alpha - \delta^{-\alpha} b) dy
\]

\[
= 8\alpha^2 \frac{\partial \Lambda}{\partial b_1} \int_{\mathbb{R}^2} |y|^2 (1 + |y|^2)^3 dy + O(\delta^\alpha) = 2\pi \alpha \frac{\partial \Lambda}{\partial b_1} + O(\delta^\alpha).
\]

(6.9)

since \(\int_{\mathbb{R}^2} \frac{y^2}{(1 + |y|^2)^3} dy = \frac{\pi}{4}\). Similarly

\[
\delta^{-\alpha} \int_\Omega (\nabla_b \Lambda, x^\alpha - b)|x|^{2(\alpha - 1)}e^{W_\lambda} Z_\lambda^j dx = 2\pi \alpha \frac{\partial \Lambda}{\partial b_2} + O(\delta^\alpha).
\]

(6.10)

Combining (6.9)-(6.10) with (6.8) we obtain

\[
\frac{\partial}{\partial b_j} I_\lambda(W_\lambda) = -32\pi^2 \frac{\partial \Lambda}{\partial b_j} + O(\delta^\alpha), \quad j = 1, 2
\]

uniformly for \(b\) in compact subsets of \(\Omega_\alpha \setminus \{0\}\), and the thesis follows. \(\square\)

Finally we describe an expansion for the functional \(J_\lambda\) defined in (6.2), a key step is its expected closeness to the functional \(I_\lambda(W_\lambda)\) analyzed in the previous proposition.

Proposition 6.3. The following expansion holds:

\[
J_\lambda(b) = I_\lambda(PW_\lambda) + o(1)
\]

\(C^1\)-uniformly with respect to \(b\) on compact subsets of \(\Omega_\alpha \setminus \{0\}\). As a corollary, by Proposition 6.2,

\[
J_\lambda(b) = -8\alpha \pi \left( 2 + \log \lambda - \log(8\alpha^2) \right) - 32\pi^2 \Lambda(b) + o(1)
\]

\(C^1\)-uniformly with respect to \(b\) on compact subsets of \(\Omega_\alpha \setminus \{0\}\).
Proof. We compute:

\[ J(\lambda) = \frac{1}{2} \int \nabla(\nabla \phi(x) + \phi(x))^2 - \lambda \int \nabla \phi(x)|x|^{2(\alpha-1)} e^{PW\phi} \, dx \]

\[ = I(\lambda) + \frac{1}{2} \int \nabla \phi(x))^2 dx + \int \nabla \phi(x) \nabla PW \phi \, dx \]

\[ - \lambda \int \nabla \phi(x)|x|^{2(\alpha-1)} e^{PW\phi} \, dx \]

where \( R \) is the error term defined in Proposition 4.2. Let us fix \( \varepsilon > 0 \) sufficiently small and let \( p = 1 + \frac{\varepsilon}{\delta} \). Next let \( 1 < q < \infty \) be such that \( \frac{1}{p} + \frac{1}{q} = 1 \). Then by Proposition 4.2 and Proposition 5.5 for \( \lambda \) sufficiently small we have

\[ \| \phi \| \leq C \delta^{\alpha-\varepsilon}, \quad \| \frac{\partial \phi}{\partial \theta_i} \| \leq C \delta^{-\varepsilon}, \quad \| R \| \leq C \delta^{\alpha-\varepsilon}. \]

Then

\[ \int \nabla \phi(x)^2 \, dx = \| \phi \|^2 \leq C \delta^{2(\alpha-\varepsilon)}. \]

\[ \int |R \phi(x)| \, dx \leq \| R \| \| \phi \| \leq C \| R \| \| \phi \| \leq C \delta^{2(\alpha-\varepsilon)}. \]

Next, (4.3) and Corollary (5.3) imply

\[ \lambda \int \nabla \phi(x)|x|^{2(\alpha-1)} e^{PW\phi} (e^{\phi} - 1) \, dx \]

\[ = O(\| \nabla \phi(x)|x|^{2(\alpha-1)} e^{PW\phi} \| \| e^{\phi} - 1 \| ) \]

\[ = O(\delta^{-2\alpha+3\varepsilon}). \]

The thesis follows by inserting (6.12), (6.13) and (6.14) into (6.11). In order to prove the \( C^1 \)-closeness, we compute

\[ \frac{\partial}{\partial \theta_i} J(\lambda) \]

\[ = I(\lambda) (PW \phi) \left( \frac{\partial PW \phi}{\partial \theta_i} + \frac{\partial \phi}{\partial \theta_i} \right) \]

\[ = \int \nabla (PW \phi) \nabla \left( \frac{\partial PW \phi}{\partial \theta_i} + \frac{\partial \phi}{\partial \theta_i} \right) - \lambda \int \nabla \phi(x)|x|^{2(\alpha-1)} e^{PW\phi} \left( \frac{\partial PW \phi}{\partial \theta_i} + \frac{\partial \phi}{\partial \theta_i} \right) \, dx \]

\[ = \frac{\partial}{\partial \theta_i} I(\lambda) + \int \nabla \phi \nabla \left( \frac{\partial \phi}{\partial \theta_i} \right) \, dx + \int \nabla \phi \left( \frac{\partial PW \phi}{\partial \theta_i} \right) \, dx \]

\[ - \lambda \int \nabla \phi(x)|x|^{2(\alpha-1)} e^{PW\phi} \left( \frac{\partial PW \phi}{\partial \theta_i} + \frac{\partial \phi}{\partial \theta_i} \right) \, dx \]

We easily estimate

\[ \int \nabla \phi \nabla \left( \frac{\partial \phi}{\partial \theta_i} \right) \, dx = O(\| \phi \| \| \frac{\partial \phi}{\partial \theta_i} \|) = O(\delta^{-\alpha+2\varepsilon}), \]

\[ \int \nabla \phi \left( \frac{\partial PW \phi}{\partial \theta_i} \right) \, dx = O(\| \phi \| \| \frac{\partial PW \phi}{\partial \theta_i} \|) = O(\delta^{-\alpha+2\varepsilon}) \]
and, by (4.9), recalling that \( \int_{\Omega} \nabla \phi_{\lambda} \nabla P Z_{\lambda}^{i} = 0 \),
\[
\int_{\Omega} \nabla \phi_{\lambda} \nabla \left( \frac{\partial P W_{\lambda}}{\partial b_{i}} \right) \, dx = \int_{\Omega} \nabla \phi_{\lambda} \left( \nabla 4^{-40} P Z_{\lambda}^{i} + O(1) \right) \, dx = O\left( \int_{\Omega} |\nabla \phi_{\lambda}| \, dx \right) = O(\delta^{s-\varepsilon}).
\] (6.18)

Let us consider \( 1 < s, t < \infty \) be such that \( \frac{1}{p} + \frac{1}{t} + \frac{1}{\ell} = 1 \). Then, by Hölder’s inequality, using (4.3) and Corollary 5.3
\[
\lambda \int_{\Omega} \tilde{V}(x)|x|^{2(\alpha-1)} e^{PW_{\lambda}} (e^{\phi_{\lambda}} - 1) \frac{\partial \phi_{\lambda}}{\partial b_{i}} \, dx = O\left( \lambda \|\tilde{V}(x)|x|^{2(\alpha-1)} e^{PW_{\lambda}} \|_{p} \|e^{\phi_{\lambda}} - 1\|_{s} \|\frac{\partial \phi_{\lambda}}{\partial b_{i}}\|_{t} \right) = O(\delta^{s-3\varepsilon}).
\] (6.19)

Now we use again (4.9) and we recall that \( \phi_{\lambda} \) satisfies \( \int_{\Omega} |x|^{2(\alpha-1)} e^{W_{\lambda}} \phi_{\lambda} Z_{\lambda}^{i} = \int_{\Omega} \nabla \phi_{\lambda} \nabla P Z_{\lambda}^{i} = 0 \); so
\[
\int_{\Omega} \left| x \right|^{2(\alpha-1)} e^{W_{\lambda}} (e^{\phi_{\lambda}} - 1) \frac{\partial P W_{\lambda}}{\partial b_{i}} \, dx = \int_{\Omega} \left| x \right|^{2(\alpha-1)} e^{W_{\lambda}} (e^{\phi_{\lambda}} - 1) (4\delta^{-\alpha} Z_{\lambda}^{i} + O(1)) \, dx = 4\delta^{-\alpha} \int_{\Omega} \left| x \right|^{2(\alpha-1)} e^{W_{\lambda}} (e^{\phi_{\lambda}} - 1 - \phi) Z_{\lambda}^{i} + O\left( \int_{\Omega} \left| x \right|^{2(\alpha-1)} e^{W_{\lambda}} (e^{\phi_{\lambda}} - 1) \, dx \right) = 4\delta^{-\alpha} O\left( \|\|x\|^{2(\alpha-1)} e^{W_{\lambda}}\|_{p} \|e^{\phi_{\lambda}} - 1 - \phi\|_{q} \right) + O(\|\|x\|^{2(\alpha-1)} e^{W_{\lambda}}\|_{p} \|e^{\phi_{\lambda}} - 1\|_{q}) = O(\delta^{s-3\varepsilon})
\] (6.20)
where last identity follows by (4.1) and Corollary 5.3. Moreover Proposition 4.2 and Corollary 5.3 give
\[
\int_{\Omega} R_{\lambda}(\phi_{\lambda} - 1) \frac{\partial P W_{\lambda}}{\partial b_{i}} \, dx = \delta^{-\alpha} O\left( \int_{\Omega} R_{\lambda} \|\phi_{\lambda} - 1\| \, dx \right) = \delta^{-\alpha} O(\|R_{\lambda}\|_{p} \|\phi_{\lambda} - 1\|) = O(\delta^{s-2\varepsilon}).
\] (6.21)

Finally we combine (6.20) with (6.21) to arrive at
\[
\lambda \int_{\Omega} \tilde{V}(x)|x|^{2(\alpha-1)} e^{PW_{\lambda}} (e^{\phi_{\lambda}} - 1) \frac{\partial P W_{\lambda}}{\partial b_{i}} \, dx = \int_{\Omega} \left| x \right|^{2(\alpha-1)} e^{W_{\lambda}} (e^{\phi_{\lambda}} - 1) \frac{\partial P W_{\lambda}}{\partial b_{i}} \, dx - \int_{\Omega} R_{\lambda}(\phi_{\lambda} - 1) \frac{\partial P W_{\lambda}}{\partial b_{i}} \, dx = O(\delta^{s-3\varepsilon}).
\] (6.22)

By inserting (6.16), (6.17), (6.18), (6.19) and (6.22) into (6.15) we get the thesis for \( \varepsilon \) sufficiently small.

\[\square\]

7. Proof of Theorem 2.1 and Theorem 1.1.

Proof of Theorem 2.1. According to assumption (A2)*, let \( \mathcal{K} \subset \Omega_{a} \setminus \{0\} \) be a \( C^{1} \)-stable compact set of critical points for \( \Lambda \). Then, Proposition 6.3 implies that for \( \lambda > 0 \) sufficiently small there exists \( b_{\lambda} \) a critical point of \( J_{\lambda} \) such that
\[
\text{dist}(b_{\lambda}, \mathcal{K}) \to 0.
\]
In view of Proposition 6.1, we find a solution to the original problem (2.2) of the form \(v_\lambda = PW_\lambda + \phi_\lambda\) associated to such \(b_\lambda\). Theorems 2.1 is thus completely proved.

**Proof of Theorem 1.1.** Let \(v_\lambda\) be the solution to (2.2) provided by Theorem 2.1. Clearly, by (2.1),

\[
  u_\lambda = v_\lambda - 4\pi(\alpha - 1)G(x, 0)
\]
solves equation (1.1). Moreover, combining (4.3), Corollary 5.3 and Proposition 5.5, by Hölder’s inequality with \(\frac{1}{p} + \frac{1}{q} = 1\) we get

\[
  \lambda \|	ilde{V}(x)|x|^{2(\alpha - 1)}(e^{\phi_\lambda} - e^{PW_\lambda})\|_1 = \lambda \|	ilde{V}(x)|x|^{2(\alpha - 1)}e^{PW_\lambda}(e^{\phi_\lambda} - 1)\|_1
\]

\[
  \leq \lambda \|	ilde{V}(x)|x|^{2(\alpha - 1)}e^{PW_\lambda}\|_p \|e^{\phi_\lambda} - 1\|_q
\]

\[
  = O(\delta^{-2\alpha \frac{p - 1}{p-\alpha}}) = o(1),
\]

if \(p\) is chosen sufficiently close to 1 and \(\varepsilon\) sufficiently close to 0. Similarly, by Proposition 4.2,

\[
  \|\lambda \tilde{V}(x)|x|^{2(\alpha - 1)}e^{PW_\lambda} - |x|^{2(\alpha - 1)}e^{W_\lambda}\|_1 = \|R_\lambda\|_1 \leq C\|R_\lambda\|_p = O(\delta^{-2\alpha \frac{p - 1}{p-\alpha}}) = o(1).
\]

Therefore

\[
  \|\lambda e^{\alpha \lambda} - |x|^{2(\alpha - 1)}e^{W_\lambda}\|_1 = \|\lambda|x|^{2(\alpha - 1)}\tilde{V}(x)e^{\alpha \lambda} - |x|^{2(\alpha - 1)}e^{W_\lambda}\|_1 = o(1). \tag{7.1}
\]

In order to complete the proof of Theorem 1.1, let us split \(\Omega\) as follows:

\[
  \Omega = D_0 \cup \ldots \cup D_{\alpha - 1} \cup (\Omega \setminus (D_0 \cup \ldots \cup D_{\alpha - 1})
\]

where

\[
  D_i := \{ x \in \Omega \mid |x - \beta_i \lambda| \leq \delta \log \delta \}.
\]

Let us fix \(i = 0, 1, \ldots, \alpha - 1\) and \(x \in D_i\): we have \(|x - \beta_i \lambda| \leq \delta \log \delta = o(1)\), by which for \(j \neq i\) we deduce

\[
  |x - \beta_j \lambda| = |\beta_i \lambda - \beta_j \lambda| + O(|x - \beta_i \lambda|) = |\beta_i \lambda - \beta_j \lambda| + o(1) = |\beta_i \lambda - \beta_j \lambda|(1 + o(1)).
\]

Therefore

\[
  |x| = |\beta_i \lambda|(1 + o(1)) \text{ unif. in } D_i,
\]

\[
  |x^\alpha - b_\lambda| = |x - \beta_i \lambda| \prod_{j \neq i} |\beta_i \lambda - \beta_j \lambda|(1 + o(1))
\]

\[
  = |x - \beta_i \lambda| |\beta_i \lambda|^{\alpha - 1} \prod_{k=1}^{\alpha - 1} (1 - e^{2\pi i k}) (1 + o(1))
\]

\[
  = o|\beta_i \lambda|^{\alpha - 1}|x - \beta_i \lambda|(1 + o(1)) \text{ unif. in } D_i
\]

where we have used Lemma 7.1. We have thus proved that

\[
  |x|^{2(\alpha - 1)}e^{W_\lambda} = \frac{8\alpha^2 \delta^{2\alpha} |x|^{2(\alpha - 1)}}{2\delta^{2\alpha} + |x^\alpha - b_\lambda|^2} = \frac{8\alpha^2 \delta^{2\alpha} |\beta_i \lambda|^{2\alpha - 1}}{2\delta^{2\alpha} + \alpha^2 |\beta_i \lambda|^{2\alpha - 1} |x - \beta_i \lambda|^2} (1 + o(1)) \tag{7.2}
\]

uniformly in \(D_i\). On the other hand in \(\Omega \setminus (D_0 \cup \ldots \cup D_{\alpha - 1})\) we have \(|x^\alpha - b_\lambda| \geq \delta^\alpha |\log \delta|^{\alpha}\), which implies, using Lemma A.1,

\[
  \int_{\Omega \setminus (D_0 \cup \ldots \cup D_{\alpha - 1})} |x|^{2(\alpha - 1)}e^{W_\lambda} dx \leq 8\alpha^2 \delta^{2\alpha} \int_{|x^\alpha - b_\lambda| \geq \delta^\alpha |\log \delta|^{\alpha}} \frac{|x|^{2(\alpha - 1)} dx}{|x^\alpha - b_\lambda|^4}
\]

\[
  = 8\alpha^2 \delta^{2\alpha} \int_{|y - b_\lambda| \geq \delta^\alpha |\log \delta|^{\alpha}} \frac{1}{|y - b_\lambda|^4} dy
\]
\[ \leq C \left| \log \delta \right|^{2\alpha} = o(1) \]  

(7.3)

and, similarly,

\[ \int_{\Omega \setminus D} \frac{8\alpha^2 \delta^{2\alpha} |b_{\lambda}|^{2\frac{\alpha - 1}{\alpha}}}{(\delta^{2\alpha} + \alpha^2 |b_{\lambda}|^{2\frac{\alpha - 1}{\alpha}} |x - \beta_{i\lambda}|^2)^2} \leq \frac{8\delta^{2\alpha}}{\alpha^2 |b_{\lambda}|^{2\frac{\alpha - 1}{\alpha}}} \int_{|x - \beta_{i\lambda}| \geq \delta |\log \delta|} \frac{1}{|x - \beta_{i\lambda}|^2} dx \leq C \frac{\delta^{2(\alpha - 1)}}{\left| \log \delta \right|^2} = o(1). \]  

(7.4)

By (7.2)-(7.3)-(7.4) we deduce

\[ |x|^{2(\alpha - 1)} e^{W_{\lambda}} - \sum_{i=0}^{\alpha - 1} \frac{8\alpha^2 \delta^{2\alpha} |b_{\lambda}|^{2\frac{\alpha - 1}{\alpha}}}{(\delta^{2\alpha} + \alpha^2 |b_{\lambda}|^{2\frac{\alpha - 1}{\alpha}} |x - \beta_{i\lambda}|^2)^2} = |x|^{2(\alpha - 1)} e^{W_{\lambda}} - \frac{8\mu_{\lambda}^2}{(\mu_{\lambda}^2 + |x - \beta_{i\lambda}|^2)^2} \to 0 \]

in \( L^1(\Omega) \), where

\[ \mu = \mu_{\lambda} = \frac{\delta^\alpha}{\alpha |b_{\lambda}|^{\frac{\alpha - 1}{\alpha}}}. \]

Combining this with (7.1), and recalling (2.7), we get (1.6)-(1.7). Finally, since

\[ \frac{8\mu_{\lambda}^2}{(\mu_{\lambda}^2 + |x - \beta_{i\lambda}|^2)^2} - 8\pi \delta_{\beta_{i\lambda}} \to 0 \]

in the measure sense, we get (1.8) and this concludes the proof of Theorem 1.1. \( \square \)

We conclude with the following geometric lemma concerning the roots of unity.

**Lemma 7.1.** For any \( \alpha \geq 2 \) the following identity holds:

\[ \prod_{k=1}^{\alpha - 1} \left( 1 - e^{i\frac{2\pi k}{\alpha}} \right) = \alpha. \]

**Proof.** For any \( z \in \mathbb{C} \) we have

\[ z^\alpha - 1 = (z - 1)(z^{\alpha - 1} + z^{\alpha - 2} + \ldots + z + 1). \]

On the other hand

\[ z^\alpha - 1 = \prod_{k=0}^{\alpha - 1} (z - e^{i\frac{2\pi k}{\alpha}}) \]

by which

\[ \prod_{k=0}^{\alpha - 1} (z - e^{i\frac{2\pi k}{\alpha}}) = (z - 1)(z^{\alpha - 1} + z^{\alpha - 2} + \ldots + z + 1). \]

Let us divide both sides by \( z - 1 \):

\[ \prod_{k=1}^{\alpha - 1} (z - e^{i\frac{2\pi k}{\alpha}}) = z^{\alpha - 1} + z^{\alpha - 2} + \ldots + z + 1; \]

secondly, evaluating at \( z = 1 \) we get the thesis. \( \square \)
Appendix A. Appendix. In this appendix we carry out some asymptotic expansions involving the regular part \( H(x, y) \) of the Green’s function in the case of symmetric domains. According to hypothesis (A1)*, we assume that \( \Omega \) is \( \alpha \)-symmetric:

\[
x \in \Omega \iff xe^{i\frac{2\pi}{\alpha}} \in \Omega,
\]

and this implies that the new domain

\[
\Omega_{\alpha} := \{x^{\alpha} \mid x \in \Omega\}
\]

is smooth. Let us denote by \( \mathcal{H}_{\alpha}(z, b) \) the regular part of the Green’s function of \(-\Delta\) in \( \Omega_{\alpha}\); so, for any fixed \( b \in \Omega_{\alpha}\) the function \( \mathcal{H}_{\alpha}(\cdot, b) \) satisfies

\[
\Delta_{\alpha}\mathcal{H}_{\alpha}(z, b) = 0 \text{ in } \Omega_{\alpha}, \quad \mathcal{H}_{\alpha}(z, b) = 2\pi \log |z - b| \text{ on } \partial\Omega_{\alpha}.
\]

Now, for any fixed \( b \in \Omega_{\alpha}\) we have that the function \( x \in \Omega \mapsto \mathcal{H}_{\alpha}(x^\alpha, b) \) is harmonic in \( \Omega \) and satisfies \( \mathcal{H}_{\alpha}(x^\alpha, y) = 2\pi \log |x^\alpha - b| = 2\pi \sum_{i=0}^{\alpha-1} \log |x - \beta_i| \) on \( \partial\Omega \), which implies

\[
\sum_{i=0}^{\alpha-1} H(x, \beta_i) = \mathcal{H}_{\alpha}(x^\alpha, b) \text{ in } \Omega.
\]

In particular, for any \( b \in \Omega_{\alpha}\)

\[
\sum_{i,j=0}^{\alpha-1} H(\beta_i, \beta_j) - (\alpha - 1) \sum_{i=0}^{\alpha-1} H(\beta_i, 0) = \alpha \mathcal{H}_{\alpha}(b, b) - (\alpha - 1) \mathcal{H}_{\alpha}(b, 0) \quad (A.1)
\]

Thanks to the symmetry of \( \mathcal{H}_{\alpha}(z, b) \), we get\(^2\)

\[
\nabla_b \left( \alpha \mathcal{H}_{\alpha}(b, b) - (\alpha - 1) \mathcal{H}_{\alpha}(b, 0) \right) = 2\alpha \nabla_{\alpha} \mathcal{H}(b, b) - (\alpha - 1) \nabla_{\alpha} \mathcal{H}_{\alpha}(b, 0).
\]

Moreover, by Taylor expansion,

\[
2\alpha \left( \mathcal{H}_{\alpha}(x^\alpha, b) - \mathcal{H}_{\alpha}(b, b) \right) - (\alpha - 1) \left( \mathcal{H}_{\alpha}(x^\alpha, 0) - \mathcal{H}_{\alpha}(b, 0) \right)
\]

\[
= 2\alpha \left( \nabla_{\alpha} \mathcal{H}_{\alpha}(b, b), x^\alpha - b \right) - (\alpha - 1) \left( \nabla_{\alpha} \mathcal{H}_{\alpha}(b, 0), x^\alpha - b \right) + O(|x^\alpha - b|^2)
\]

\[
= \left( \nabla_b \left( \alpha \mathcal{H}_{\alpha}(b, b) - (\alpha - 1) \mathcal{H}_{\alpha}(b, 0) \right), x^\alpha - b \right) + O(|x^\alpha - b|^2) \quad (A.2)
\]

uniformly for \( x \in \Omega \) and \( b \) on compact subsets of \( \Omega_{\alpha} \).

Remark A.1. If \( \Omega \) is the unit ball \( \Omega = B(0, 1) \), then \( \Omega_{\alpha} = \Omega = B(0, 1) \) and so

\[
\mathcal{H}_{\alpha}(z, b) = H(z, b) = \frac{1}{2\pi} \log \left( |b| \left| z - \frac{b}{|b|^2} \right| \right).
\]

Consequently, by (A.1), for any \( b \in B(0, 1) \)

\[
\sum_{i,j=0}^{\alpha-1} H(\beta_i, \beta_j) - (\alpha - 1) \sum_{i=0}^{\alpha-1} H(\beta_i, 0) = \alpha H_{\alpha}(b, b) - (\alpha - 1) \mathcal{H}_{\alpha}(b, 0)
\]

\[
= \alpha H(b, b) = \frac{\alpha}{2\pi} \log(1 - |b|^2). \quad (A.3)
\]

This appendix is devoted to deducing some integral identities associated to the change of variable \( x \mapsto x^\alpha \) which appears frequently when dealing with functions in spaces \( H_{\alpha}(\mathbb{R}^2) \), \( L_{\alpha}(\mathbb{R}^2) \) introduced in Section 2.

\(^2\)Here \( \nabla_{\alpha} \mathcal{H}_{\alpha} \) denotes the gradient of the function \( \mathcal{H}_{\alpha}(\cdot, \cdot) \) with respect to the first variable.
Lemma A.1. Let $\xi \in \mathbb{R}^2$. For any $f \in L_1(\mathbb{R}^2)$ we have that $f(y^\alpha) \in L_\alpha(\mathbb{R}^2)$ and
\[
\int_{\mathbb{R}^2} \frac{|y|^{2(\alpha-1)}}{(1 + |y^\alpha - \xi|^2)^2} |f(y^\alpha)|^2 dy = \frac{1}{\alpha} \int_{\mathbb{R}^2} \frac{1}{(1 + |y|^2)^2} |f(y)|^2 dy. \tag{A.4}
\]
Moreover, if $f \in H_1(\mathbb{R}^2)$, then $f(y^\alpha) \in H_\alpha(\mathbb{R}^2)$
\[
\int_{\mathbb{R}^2} |\nabla (f(y^\alpha))|^2 dy = \alpha \int_{\mathbb{R}^2} |\nabla f|^2 dy.
\]

Proof. It is sufficient to prove the thesis for a smooth function $f$. Using the polar coordinates $(\rho, \theta)$ and then applying the change of variables $(\rho', \theta') = (\rho^\alpha, \alpha \theta)$
\[
\int_{\mathbb{R}^2} \frac{|y|^{2(\alpha-1)}}{(1 + |y^\alpha - \xi|^2)^2} |f(y^\alpha)|^2 dy
\]
\[
= \frac{\alpha^2}{\int_{\mathbb{R}^2} \frac{1}{(1 + |\rho^\alpha e^{i\alpha \theta} - \xi|^2)^2} |f(\rho^\alpha e^{i\alpha \theta})|^2 d\theta
\]
\[
= \alpha \int_{\mathbb{R}^2} \frac{1}{(1 + |\rho e^{i \theta} - \xi|^2)^2} |f(\rho e^{i \theta})|^2 d\theta
\]
\[
= \alpha \int_{\mathbb{R}^2} |\nabla (f(y^\alpha))|^2 dy.
\]

Similarly, we get
\[
\int_{\mathbb{R}^2} |\nabla (f(y^\alpha))|^2 dy
\]
\[
= \int_{\mathbb{R}^2} |\nabla f|^2 dy
\]
\[
= \frac{\alpha^2}{\int_{\mathbb{R}^2} \frac{1}{(1 + |\rho e^{i \theta} - \xi|^2)^2} |f(\rho e^{i \theta})|^2 d\theta}
\]
\[
= \alpha \int_{\mathbb{R}^2} |\nabla f|^2 dy.
\]

Now we are going to obtain a sort of counterpart of Lemma A.1 which converts a $\alpha$-symmetric function in $L_\alpha(\mathbb{R}^2)$ (in $H_\alpha(\mathbb{R}^2)$ respectively) into a function in $L_1(\mathbb{R}^2)$ (in $H_1(\mathbb{R}^2)$ respectively) by a suitable change of variables.

Lemma A.2. Let $\xi \in \mathbb{R}^2$ and let $f \in L_\alpha(\mathbb{R}^2)$ be $\alpha$-symmetric, i.e.
\[
f(xe^{i\frac{\theta}{\alpha}}) = f(x) \quad \forall x \in \mathbb{R}^2
\]
and set
\[
F : \mathbb{R}^2 \to \mathbb{R}, \quad F(\rho e^{i \theta}) = f(\rho^{\frac{\theta}{\alpha}} e^{i \frac{\theta}{\alpha}}) \quad \rho \geq 0, \quad \theta \in [-\pi, \pi). \tag{A.5}
\]

Then $F \in L_1(\mathbb{R}^2)$ and
\[
\int_{\mathbb{R}^2} \frac{1}{(1 + |y - \xi|^2)^2} |F(y)|^2 dy = \alpha \int_{\mathbb{R}^2} \frac{|y|^{2(\alpha-1)}}{(1 + |y^\alpha - \xi|^2)^2} |f(y)|^2 dy. \tag{A.6}
\]
Moreover, if \( f \in H_\alpha(\mathbb{R}^2) \), then \( F \in H_1(\mathbb{R}^2) \) and
\[
\int_{\mathbb{R}^2} |\nabla F|^2 \, dy = \frac{1}{\alpha} \int_{\mathbb{R}^2} |\nabla f|^2 \, dy. \tag{A.7}
\]

Proof. Taking into account that a single point has capacity 0 in \( \mathbb{R}^2 \), it is sufficient to prove the thesis for a smooth function \( f \) such that \( f = 0 \) in a neighborhood of 0. Since by definition
\[ f(y) = F(y^\alpha) \quad \text{if} \quad y \in \mathbb{R}^2, \]
then the thesis follows by applying Lemma A.1.

An analogous identity holds for the scalar product associated to (A.6)-(A.7) as stated in the following corollary.

Corollary A.3. Let \( \xi \in \mathbb{R}^2 \).

- For any \( f, g \in L_\alpha(\mathbb{R}^2) \) we have that
  \[
  \int_{\mathbb{R}^2} \frac{1}{(1 + |y - \xi|^2)^2} F G \, dy = \frac{\alpha}{\alpha - 1} \int_{\mathbb{R}^2} \frac{|y|^{2(\alpha - 1)}}{(1 + |y|^2 - \xi^2)^2} f g \, dy;
  \]
- For any \( f, g \in H_\alpha(\mathbb{R}^2) \) we have that
  \[
  \int_{\mathbb{R}^2} \nabla F \nabla G \, dy = \frac{1}{\alpha} \int_{\mathbb{R}^2} \nabla f \nabla g \, dy.
  \]

where \( F, G \) are the functions defined according to (A.5) starting from \( f, g \), respectively.

REFERENCES

[1] S. Baraket and F. Pacard, Construction of singular limits for a semilinear elliptic equation in dimension 2, Calc. Var. Partial Differ. Equ., 6 (1998), 1–38.
[2] D. Bartolucci, C. C. Chen, C. S. Lin and G. Tarantello, Profile of blow-up solutions to mean field equations with singular data, Commun. Partial Differ. Equ., 29 (2004), 1241–1265.
[3] D. Bartolucci and G. Tarantello, Liouville type equations with singular data and their application to periodic multivortices for the electroweak theory, Commun. Math. Phys., 229 (2002), 3–47.
[4] T. Bartsch, A. Pistoia and T. Weth, N-vortex equilibria for ideal fluids in bounded planar domains and new nodal solutions of the sinh-Poisson and the Lane-Emden-Fowler equations, Commun. Math. Phys., 297 (2010), 653–686.
[5] H. Brezis and F. Merle, Uniform estimates and blow-up behavior for solutions of \(-\Delta u = V(x) e^u\) in two dimensions, Commun. Partial Differ. Equ., 16 (1991), 1223–1253.
[6] X. Chen, Remarks on the existence of branch bubbles on the blowup analysis of equation \(-\Delta u = e^{2u}\) in dimension two, Commun. Anal. Geom., 7 (1999), 295–302.
[7] C. C. Chen and C. C. Lin, Mean field equation of Liouville type with singular data: topological degree, Commun. Pure Appl. Math., 68 (2015), 887–947.
[8] C. C. Chen and C. C. Lin, Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces, Commun. Pure Appl. Math., 55 (2002), 728–771.
[9] W. Chen and C. Li, Classification of solutions of some nonlinear elliptic equations, Duke Math. J., 63 (1991), 615–623.
[10] T. D’Aprile, Blow-up phenomena for the Liouville equation with a singular source of integer multiplicity, J. Differ. Equ., 266 (2019), 7379–7415.
[11] T. D’Aprile, Multiple blow-up solutions for the Liouville equation with singular data, Commun. Partial Differ. Equ., 38 (2013), 1409–1436.
[12] T. D’Aprile and J. Wei, Bubbling solutions for the Liouville equation with singular sources: non-simple blow-up, J. Funct. Anal., 279 (2020), Art. 108605.
[13] M. Del Pino, P. Esposito and M. Musso, Nondegeneracy of entire solutions of a singular Liouville equation, Proc. Am. Math. Soc., 140 (2012), 581–588.
[14] M. Del Pino, P. Esposito and M. Musso, Two dimensional Euler flows with concentrated vorticities, Trans. Am. Math. Soc., 362 (2010), 6381–6395.
[15] M. Del Pino, M. Kowalczyk and M. Musso, Singular limits in Liouville-type equation, Calc. Var. Partial Differ. Equ., 24 (2005), 47–81.
[16] P. Esposito, Blow up solutions for a Liouville equation with singular data, SIAM J. Math. Anal., 36 (2005), 1310–1345.
[17] P. Esposito, M. Grossi and A. Pistoia, On the existence of blowing-up solutions for a mean field equation, Ann. Inst. H. Poincaré Anal. Non Linéaire, 22 (2005), 227–257.
[18] P. Esposito, M. Musso and A. Pistoia, On the existence and profile of nodal solutions for a two-dimensional elliptic problem with large exponent in nonlinearity, Proc. Lond. Math. Soc., 94 (2007), 497–519.
[19] P. Esposito, M. Musso and A. Pistoia, Concentrating solutions for a planar elliptic problem involving nonlinearities with large exponent, J. Differ. Equ., 227 (2006), 29–68.
[20] P. Esposito, A. Pistoia and J. Wei, Concentrating solutions for the Hénon equation in $\mathbb{R}^2$, J. Anal. Math., 100 (2006), 249–280.
[21] M. Grossi and A. Pistoia, Multiple blow-up phenomena for the sinh-Poisson equation, Arch. Ration. Mech. Anal., 209 (2013), 287–320.
[22] T. J. Kuo and C. S. Lin, Estimates of the mean field equations with integer singular sources: non-simple blowup, J. Differ. Geom., 103 (2016), 377–424.
[23] Y. Y. Li, On a singularly perturbed elliptic equation, Adv. Differ. Equ., 2 (1997), 955–980.
[24] Y. Y. Li and I. Shafrir, Blow-up analysis for solutions of $-\Delta u = Ve^u$ in dimension two, Indiana Univ. Math. J., 43 (1994), 1255–1270.
[25] L. Ma and J. Wei, Convergence for a Liouville equation, Comment. Math. Helv., 76 (2001), 506–514.
[26] C. S. Lin and C. L. Wang, Elliptic functions, Green functions and the mean field equation on tori, Ann. of Math., 172 (2010), 911–954.
[27] C. S. Lin and S. Yan, Bubbling solutions for relativistic abelian Chern-Simons model on a torus, Commun. Math. Phys., 297 (2010), 733–758.
[28] J. Moser, A sharp form of an inequality by N. Trudinger, Indiana Univ. Math. J., 20 (1970/71), 1077–1092.
[29] K. Nagasaki and T. Suzuki, Asymptotic analysis for two-dimensional elliptic eigenvalue problems with exponentially dominated nonlinearities, Asymptotic Anal., 3 (1990), 173–188.
[30] J. Prajapat and G. Tarantello, On a class of elliptic problem in $\mathbb{R}^2$: Symmetry and uniqueness results, Proc. Roy. Soc. Edinburgh Sect. A, 131 (2001), 967–985.
[31] T. Suzuki, Two-dimensional Emden-Fowler equation with exponential nonlinearity, Nonlinear diffusion equations and their equilibrium states, 3 (Gregynog, 1989), 493–512. Progr. Nonlinear Differential Equations Appl., 7, Birkhäuser Boston, Boston, MA, 1992.
[32] G. Tarantello, Analytical Aspects of Liouville-Type Equations with Singular Sources, North-Holland, Amsterdam, 2004.
[33] G. Tarantello, A quantization property for blow up solutions of singular Liouville-type equations, J. Funct. Anal., 219 (2005), 368–399.
[34] G. Tarantello, Progress in Nonlinear Differential Equations and their Applications, Birkhäuser Boston Inc., Boston, MA, 2008.
[35] N. S. Trudinger, On imbeddings into Orlicz spaces and some applications, J. Math. Mech., 17 (1967), 473–483.
[36] J. Wei, D. Ye and F. Zhou, Bubbling solutions for an anisotropic Emden-Fowler equation, Calc. Var. Partial Differ. Equ., 28 (2007), 217–247.
[37] J. Wei and L. Zhang, Estimates for Liouville equation with quantized singularities, arXiv:1905.04123.
[38] V. H. Weston, On the asymptotic solution of a partial differential equation with an exponential nonlinearity, SIAM J. Math. Anal., 9 (1978), 1030–1053.
[39] Y. Yang, Solutions in Field Theory and Nonlinear Analysis, Springer-Verlag, New York, 2001.

Received June 2020; revised August 2020.

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