GRAPHS OF HYPERBOLIC GROUPS AND A LIMIT SET INTERSECTION THEOREM

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Abstract. We define the notion of limit set intersection property for a collection of subgroups of a hyperbolic group; namely, for a hyperbolic group $G$ and a collection of subgroups $\mathcal{S}$ we say that $\mathcal{S}$ satisfies the limit set intersection property if for all $H, K \in \mathcal{S}$ we have $\Lambda(H) \cap \Lambda(K) = \Lambda(H \cap K)$. Given a hyperbolic group admitting a decomposition into a finite graph of hyperbolic groups structure with QI embedded condition, we show that the set of conjugates of all the vertex and edge groups satisfy the limit set intersection property.

1. Introduction

Limit set intersection theorems first appear in the work of Susskind and Swarup ([SS92]) in the context of geometrically finite Kleinian groups. Later on Anderson ([Anda], [Andb]) undertook a detailed study of this for general Kleinian groups. In the context of (Gromov) hyperbolic groups this is true for quasiconvex subgroups ([GMRS97], Lemma 2.6). (Recently W. Yang has looked at the case of relatively quasiconvex subgroups of relatively hyperbolic groups. See [Yan12].) However, this theorem is false for general subgroups of hyperbolic groups and no characterizations other than quasi-convexity are known for a pair of subgroups $H, K$ of a hyperbolic group $G$ which guarantee that $\Lambda(H) \cap \Lambda(K) = \Lambda(H \cap K)$. This motivates us to look for subgroups other than quasiconvex subgroups which satisfy the limit set intersection property. Our starting point is the following celebrated theorem of Bestvina and Feighn.

Theorem 1.1. ([BF92]) Suppose $(\mathcal{G}, Y)$ is a graph of hyperbolic groups with QI embedded condition and the hallways flare condition. Then the fundamental group, say $G$, of this graph of groups is hyperbolic.

Graphs of groups are briefly recalled in section 3. There are many examples of hyperbolic groups admitting such a decomposition into graphs of groups where the vertex or edge groups are not quasiconvex. Nevertheless, with the terminologies of the above theorem, we have:

Theorem. The set of all conjugates of the vertex and edge groups of $G$ satisfy the limit set intersection property.

We note that a special case of our theorem was already known by results of Ilya Kapovich ([Kap01]). There the author showed that given a $k$-acylindrical graph of hyperbolic groups $(\mathcal{G}, Y)$ with quasi-isometrically embedded condition and with fundamental group $G$ (which turns out to be hyperbolic by Theorem 1.1) the
vertex groups are quasiconvex subgroups of $G$. Hence, the conjugates of all the vertex groups satisfy the limit set intersection property in this case.

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2. Boundary of Gromov hyperbolic spaces and Limit sets of subspaces

We assume that the reader is familiar with the basics of (Gromov) hyperbolic metric spaces and the coarse language. We shall however recall some basic definitions and results that will be explicitly used in the sections to follow. For details one is referred to [Gro85] or [BH99].

Notation and convention. In this section we shall assume that all the hyperbolic metric spaces are proper geodesic metric spaces. We use $\text{qi}$ to mean both quasi-isometry and quasi-isometric depending on the context. Hausdorff distance of two subsets $A, B$ of a metric space $Z$ is denoted by $Hd(A, B)$. For any subset $A$ of a metric space $Y$ and any $D \geq 0$, $N_D(A)$ will denote the $D$-neighborhood of $A$ in $Y$. We assume that all our groups are finitely generated. For any graph $Y$ we shall denote by $V(Y)$ and $E(Y)$ the vertex and the edge sets of $Y$ respectively.

Definition 2.1. 1. Suppose $G$ is a group generated by a finite set $S \subset G$ and let $\gamma \subset \Gamma(G, S)$ be a path joining two vertices $u, v \in \Gamma(G, S)$. Let $u_0 = u, u_1, u_2, ..., u_n = v$ be the consecutive vertices on $\gamma$. Let $u_{i+1} = u_i x_i, x_i \in S \cup S^{-1}$ for $0 \leq i \leq n - 1$. Then we shall say that the word $w = x_0 x_1 ... x_{n-1}$ labels the path $\gamma$.

2. Also, given $w \in F(S)$ - the free group on $S$, its image in $G$ under the natural map $F(S) \to G$ will be called the element of $G$ represented by $w$.

Definition 2.2. (See [BH99]) 1. Let $X$ be a hyperbolic metric space and $x \in X$ be a base point. Then the (Gromov) boundary $\partial X$ of $X$ is the equivalence classes of geodesic rays $\alpha$ such that $\alpha(0) = x$ where two geodesic rays $\alpha, \beta$ are said to be equivalent if $\text{Hd}(\alpha, \beta) < \infty$.

The equivalence class of a geodesic ray $\alpha$ is denoted by $\alpha(\infty)$.

2. If $\{x_n\}$ is an unbounded sequence of points in $X$, we say that $\{x_n\}$ converges to some boundary point $\xi \in \partial X$ if the following holds: Let $\alpha_n$ be any geodesic joining $x$ to $x_n$. Then any subsequence of $\{\alpha_n\}$ contains a subsequence uniformly converging on compact sets to a geodesic ray $\alpha$ such that $\alpha(\infty) = \xi$. In this case, we say that $\xi$ is the limit of $\{x_n\}$ and write $\lim_{n \to \infty} x_n = \xi$.

3. The limit set of a subset $Y$ of $X$ is the set $\{\xi \in \partial X : \exists \{y_n\} \subset Y \text{ with } \lim_{n \to \infty} y_n = \xi\}$. We denote this set by $\Lambda(Y)$.

The following lemma is a basic exercise in hyperbolic geometry and so we mention it without proof. It basically uses the thin triangle property of hyperbolic metric spaces. (See [BH99], Chapter III.H, Exercise 3.11.)

Lemma 2.3. Suppose $\{x_n\}, \{y_n\}$ are two sequences in a hyperbolic metric space $X$ both converging to some points of $\partial X$. If $\{d(x_n, y_n)\}$ is bounded then $\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n$. 


Lemma 2.4. 1. There is a natural topology on the boundary $\partial X$ of any proper hyperbolic metric space $X$ with respect to which $\partial X$ becomes a compact space.

2. If $f : X \to Y$ is a quasi-isometric embedding of proper hyperbolic metric spaces then $f$ induces a topological embedding $\partial f : \partial Y \to \partial X$.

If $f$ is a quasi-isometry then $\partial f$ is a homeomorphism.

We refer the reader to Proposition 3.7 and Theorem 3.9 in Chapter III.H of [BH99] for a proof of Lemma 2.4.

Definition 2.5. 1. A map $f : Y \to X$ between two metric spaces is said to be a proper embedding if for all $M > 0$ there is $N > 0$ such that $d_X(f(x), f(y)) \leq M$ implies $d_Y(x, y) \leq N$ for all $x, y \in Y$.

A family of proper embeddings between metric spaces $f_i : X_i \to Y$, $i \in I$- where $I$ is an indexing set, is said to be uniformly proper if for all $M > 0$ there is an $N > 0$ such that for all $i \in I$ and $x, y \in X_i$, $d_Y(f_i(x), f_i(y)) \leq N$ implies that $d_X(x, y) \leq M$.

2. If $f : Y \to X$ is a proper embeddings of proper hyperbolic metric spaces then we say that Cannon-Thurston (CT) map exists for $f$ if $f$ gives rise to a continuous map $\partial f : \partial Y \to \partial X$.

This means that given a sequence of points $\{y_n\}$ in $Y$ converging to $\xi \in \partial Y$, the sequence $\{f(y_n)\}$ converges to a point of $\partial X$ and the resulting map is continuous.

Note that our terminology is slightly different from Mitra ([Mit98]). The following lemma is immediate.

Lemma 2.6. Suppose $X, Y$ are hyperbolic metric spaces and $f : Y \to X$ is a proper embedding. If the CT map exists for $f$ then we have $\Lambda(f(Y)) = \partial f(\partial Y)$.

We mention the following lemma with brief remarks about proofs since it states some standard facts from hyperbolic geometry.

Lemma 2.7. Suppose $Z$ is proper $\delta$-hyperbolic metric space and $\{x_n\}$ and $\{y_n\}$ are two sequences in $Z$ such that $\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = \xi \in \partial X$. For each $n$ let $\alpha_n, \beta_n$ be two geodesics in $X$ joining $x_1, x_n$ and $y_1, y_n$ respectively.

1. Then there is subsequences $\{n_k\}$ of natural numbers such that the sequences of geodesics $\{\alpha_{n_k}\}$ and $\{\beta_{n_k}\}$ converge uniformly on compact sets to two geodesics $\alpha, \beta$ joining $x_1, y_1$ respectively to $\xi$.

2. Moreover, there is a constant $D$ depending only on $\delta$ and $d(x_1, y_1)$ and sequences of points $p_{n_k} \in \alpha_{n_k}$ and $q_{n_k} \in \beta_{n_k}$ such that $d(p_{n_k}, q_{n_k}) \leq D$, and $\lim_{k \to \infty} p_{n_k} = \lim_{k \to \infty} q_{n_k} = \xi$.

3. The conclusion (2) remains valid if we replace $\alpha_n, \beta_n$ by $K$-quasi-geodesics for some $K \geq 1$; in other words if $x_n, y_n$ are joined to $x_1, y_1$ by $K$-quasi-geodesics $\alpha_n, \beta_n$ respectively then there is a constant $D$ depending on $\delta$, $d(x_1, y_1)$ and $K$, a subsequence $\{n_k\}$ of natural numbers and sequences of points $p_{n_k} \in \alpha_{n_k}$, $q_{n_k} \in \beta_{n_k}$ such that $d(p_{n_k}, q_{n_k}) \leq D$, and $\lim_{k \to \infty} p_{n_k} = \lim_{k \to \infty} q_{n_k} = \xi$.

For a proof of (1), (2) see Lemma 3.3, and Lemma 3.13 and for (3) Theorem 1.7(stability of quasi-geodesics) in Chapter III.H of [BH99]. More precisely, for proving (3) we may choose geodesic segments $\alpha_n’s$, $\beta_n’s$ connecting the endpoints of the quasi-geodesics $\alpha_n’s$ and $\beta_n’s$ respectively and then apply (1) for these geodesics to extract subsequences $\{\alpha_{n_k}\}$ and $\{\beta_{n_k}\}$ of $\{\alpha_n\}$ and $\{\beta_n\}$ respectively both converging uniformly on compact sets. Then we can find two sequences of points
We shall denote by \( Y \) a graph of groups where \( Y \) consists of the following data:

- \( \alpha_n \), \( \beta_n \) \( n \in \mathbb{N} \) satisfying (2). Finally, by stability of quasi-geodesics for all \( k \) there are \( p_n \), \( q_n \) \( n \in \mathbb{N} \) such that \( d(p_n, q_n) \) are uniformly small. That will prove (3).

**Definition 3.2.** Suppose \( G \) is a Gromov hyperbolic group. Let \( S \) be any collection of subgroups of \( G \). We say that \( S \) has the limit set intersection property if for all \( H, K \in S \) we have \( \Lambda(H) \cap \Lambda(K) = \Lambda(H \cap K) \).

We state two elementary results on limit sets for future use.

**Lemma 2.9.** Suppose \( G \) is a hyperbolic group and \( H \) is any subset of \( G \). Then for all \( x \in G \) we have

1. \( \Lambda(xH) = \Lambda(xHx^{-1}) \).
2. \( \Lambda(xH) = x\Lambda(H) \).

**Proof:** (1) follows from Lemma 2.3. For (2) one notes that \( G \) acts naturally on a Cayley graph \( X \) of \( G \) by isometries and thus by homeomorphisms on \( \partial X = \partial G \) by Lemma 2.4. \( \square \)

3. **Graphs of groups**

We presume that the reader is familiar with the Bass-Serre theory. However, we briefly recall all the concepts that we shall need. For details one is referred to section 5.3 of J.P. Serre’s book *Trees* ([Ser00]). Although we always work with nonoriented metric graphs like Cayley graphs, we need oriented graphs possibly with multiple edges between adjacent vertices and loops to describe graphs of groups. Hence the following definition is quoted from [Ser00].

**Definition 3.1.** A graph \( Y \) is a pair \((V, E)\) together with two maps

\[ E \to V \times V, \quad e \mapsto (o(e), t(e)) \]

\[ E \to E, \quad e \mapsto \bar{e} \]

such that \( o(\bar{e}) = t(e), t(\bar{e}) = o(e) \) and \( \bar{e} = e \) for all \( e \in E \).

For an edge \( e \) we refer to \( o(e) \) as the origin and \( t(e) \) as the terminus of \( e \); the edge \( \bar{e} \) is the same edge \( e \) with opposite orientation. We write \( V(Y) \) for \( V \) and \( E(Y) \) for \( E \). We refer to \( V(Y) \) as the set of vertices of \( Y \) and \( E(Y) \) as the set of edges of \( Y \). We shall denote by \(|e|\) the edge \( e \) without any orientation.

**Definition 3.2.** A graph of groups \((G, Y)\) consists of the following data:

1. A (finite) graph \( Y \) as defined above,
2. For all \( v \in V(Y) \) (and edge \( e \in E(Y) \)) there is group \( G_v \) (respectively \( G_e \)) together with two injective homomorphisms \( \phi_{e, o(e)} : G_e \to G_{o(e)} \) and \( \phi_{e, t(e)} : G_e \to G_{t(e)} \) for all \( e \in E(Y) \) such that the following conditions hold:

   i. \( G_o = G_{\bar{e}} \),
   ii. \( \phi_{e, o(e)} = \phi_{\bar{e}, t(e)} \) and \( \phi_{e, t(e)} = \phi_{\bar{e}, o(e)} \).

We shall refer to the maps \( \phi_{e, v} \)'s as the canonical maps of the graph of groups. We shall refer to the groups \( G_v \) and \( G_e \), \( v \in V(Y) \) and \( e \in E(Y) \) as vertex groups and edge groups respectively. For topological motivations of graph of groups and the following definition of the fundamental group of a graph of groups one is referred to [SW79] or [Hat01].

**Definition 3.3.** Fundamental group of a graph of groups Suppose \((G, Y)\) is a graph of groups where \( Y \) is a (finite) connected oriented graph. Let \( T \subset Y \) be a
maximal tree. Then the fundamental group $G = \pi_1(\mathcal{G}, Y, T)$ of $(\mathcal{G}, Y)$ is defined in terms of generators and relations as follows:

The generators of $G$ are the elements of the disjoint union of the generating sets of the vertex groups $G_v$, $v \in V(Y)$ and the set $E(Y)$ of oriented edges of $Y$.

The relations are of four types: (1) Those coming from the vertex groups; (2) $\tilde{e} = e^{-1}$ for all edge $e$, (3) $e = 1$ for $|e| \in T$ and (4) $e\phi_{e, t(e)}(a)e^{-1} = \phi_{e, o(e)}(a)$ for all oriented edge $e$ and $a \in G_e$.

**Bass-Serre tree of a graph of groups**

Suppose $(\mathcal{G}, Y)$ is a graph of groups and let $T$ be a maximal tree in $Y$ as in the above definition. Let $G = \pi_1(\mathcal{G}, Y, T)$ be the fundamental group of the graph of groups. The Bass-Serre tree, say $T$, is the tree with vertex set $\bigcup_{v \in V(Y)} G/G_v$ and edge set $\bigcup_{e \in E(Y)} G/G^e_e$ where $G^e_e = \phi_{e, t(e)}(G_e) < G_{t(e)}$. The edge relations are given by

$$t(gG^e_e) = geG_{t(e)}, \quad o(gG^e_e) = gG_{o(e)}$$

Note that when $|e| \in T$ then we have $e = 1$ in $G$.

**Tree of metric spaces from a graph of groups**

Given a graph of groups $(\mathcal{G}, Y)$ and a maximal tree $T \subset Y$ one can form in a natural way a graph, say $X$, on which the fundamental group $G = \pi_1(\mathcal{G}, Y, T)$ acts by isometries properly and cocompactly and which admits a simplicial Lipschitz $G$-equivariant map $X \to T$. The construction of $X$ can be described as follows.

We assume that $Y$ is a finite connected graph and all the vertex groups and the edge groups are finitely generated. We fix a finite generating set $S_e$ for each one of the vertex groups $G_v$; similarly for each edge group $G_e$ we fix a finite generating set $S_e$ and assume that $\phi_{e, t(e)}(S_e) \subset S_{t(e)}$ for all $e \in E(Y)$. Let $S = \bigcup_{v \in V(Y)} S_v \cup (E(Y) \setminus E(T))$ be a generating set of $G$ where in $E(Y) \setminus E(T)$ we shall include only nonoriented edges of $Y$ not in $T$. We define $X$ from the disjoint union of the following graphs by introducing some extra edges as follows:

1. **Vertex spaces**: For all $v = gG_v \in V(T)$, where $v \in Y$ and $g \in G$ we let $X_v$ denote the subgraph of $\Gamma(G, S)$ with vertex set the coset $gG_v$; two vertices $gx, gy \in X_v$ are connected by an edge iff $x^{-1}y \in S_v$. We shall refer to these subspaces of $X$ as vertex spaces.

2. **Edge spaces**: Similarly for any edge $\tilde{e} = gG^e_e \in T$, let $X_{\tilde{e}}$ denote the subgraph of $\Gamma(G, S)$ with vertex set $gG^e_e$ where where two vertices $gxe, gye$ are connected by an edge iff $x^{-1}y \in \phi_{e, t(e)}(S_e)$. We shall refer to these subspaces of $X$ as edge spaces.

3. The extra edges connect the edge spaces with the vertex spaces as follows:

For all edge $\tilde{e} = gG^e_e$ of $T$ connecting the vertices $\tilde{u} = gG_{o(e)}$ and $\tilde{v} = gG_{t(e)}$ of $T$, and $x \in G^e_e$ join $gxe \in X_{\tilde{u}} = gG^e_e$ to $gxe \in X_{\tilde{v}} = gG_{t(e)}$ and $gxe^{-1} \in X_{\tilde{u}} = gG_{o(e)}$ by edges of length $1/2$ each. We define $f_{e, \tilde{u}} : X_{\tilde{u}} \to X_{\tilde{v}}$ and $f_{e, \tilde{v}} : X_{\tilde{v}} \to X_{\tilde{u}}$ by setting $f_{e, \tilde{u}}(gxe) = yxe$ and $f_{e, \tilde{u}}(gxe) = gxe^{-1}$.

We have a natural simplicial map $\pi : X \to T$ (more precisely to the first barycentric subdivision of $T$). This map is the coarse analog of the tree of metric spaces introduced by [BF92] (see also [Mit98]). By abuse of terminology we shall refer to this also as a tree of metric spaces or graphs. We recall some notations and definitions from [Mit98] and collect some basic properties.

1. We note that $X_u = \pi^{-1}(u)$ and $X_{\tilde{e}} = \pi^{-1}(e)$ for all $u \in V(T)$ and $e \in E(T)$. For all $u \in V(T)$ the intrinsic path metric of $X_u$ will be denoted by $d_u$. Similarly,
we use $d_e$ for the intrinsic path metric on $X_e$. It follows that with these intrinsic
metrics the metric spaces $X_e, X_u$ are isometric to the Cayley graphs $\Gamma(G_e, S_e)$ and
$\Gamma(G_u, S_u)$ respectively. Therefore, if all the vertex and edge groups are Gromov
hyperbolic then the vertex and edge spaces of $X$ are uniformly hyperbolic metric
spaces.

(2) **Quasi-isometric lifts of geodesics:** Suppose $u, v \in T$ and let $[u, v]$ denote
the geodesic in $T$ joining them. A $K$-QI section of $\pi$ over $[u, v]$ or a $K$-QI lift of
$[u, v]$ (in $X$) is a set theoretic section $s : [u, v] \to X$ of $\pi$ which is also a $K$-QI
embedding. In general, we are only interested in defining these sections over the
vertices in $[u, v]$.

(3) **Hallways flare condition:** We will say that $\pi : X \to T$ satisfies the
hallways flare condition if for all $K \geq 1$ there are numbers $\lambda_K > 1, M_K \geq 1, n_K \geq 1$
such that given a geodesic $\alpha : [-n_K, n_K] \to T$ and two $K$-QI lifts $\alpha_1, \alpha_2$ of $\alpha$, if
distance between the two lifts $\alpha_1, \alpha_2$ is at least $M_K$. Let $d_{\alpha(0)}(\alpha_1(0), \alpha_2(0))$.

(4) **Graphs of groups with QI embedded conditions.** Suppose $(G, Y)$ is
a graph of groups such that each vertex and edge group is finitely generated. We
say that it satisfies the QI embedded condition if all the inclusion maps of the edge
groups into the vertex groups are quasi-isometric embeddings with respect any choice
of finite generating sets for the vertex and edge groups.

It is clear that if $(G, Y)$ is a graph of groups with QI embedded condition then
all the maps $f_{e,u} : X_e \to X_u$ are uniform QI embeddings.

**Lemma 3.4.** There is a naturally defined proper and cocompact action of $G$ on $X$
such that the map $\pi : X \to T$ is $G$-equivariant.

**Proof:** We note that $X$ is obtained from the disjoint union of the cosets of the
vertex and edge groups of $(G, Y)$. The group $G$ has a natural action on this disjoint
union. It is also easy to check that under this action adjacent vertices of $X$ go to
adjacent vertices. Thus we have a simplicial $G$-action on $X$. Clearly the natural
map $\pi : X \to T$ is $G$-equivariant. To show that the action is proper it is enough to
show that the vertex stabilizers are uniformly finite. However, if a point $x \in G_v$
is fixed by an element $h \in G$ then $h$ fixes $gG_v \in V(T)$. However, stabilizers of $gG_v$
is simply $gG_v g^{-1}$ and the action of $gG_v g^{-1}$ on $gG_v \subset X$ is fixed point free. Hence,
the $G$-action on $X$ is fixed point free.

That the $G$-action is cocompact on $X$ follows from the fact that the $G$-actions
on $V(T)$ and $E(T)$ are cofinite. \Box

Fix a vertex $v_0 \in Y$ and the vertex $G_{v_0} \in V(T)$. Look at the corresponding
vertex space $G_{v_0} \subset X$ and let $x_0$ denote $1 \in G_{v_0}$. Let $\Theta : G \to X$ denote the orbit
map $g \mapsto gx_0$. By Milnor-Schwarz lemma this orbit map is a quasi-isometry since
the $G$-action is proper and cocompact by the above lemma.

**Lemma 3.5.** There is a constant $D_0$ such that for all vertex space $gG_v \subset X$ we
have $Hd(\Theta(gG_v), gG_v) \leq D_0$.

It follows that for any $g, x \in gG_v \subset X$ we have $g, x \in \Theta^{-1}(B(gx, D_0))$.

**Proof:** Proving the lemma let $\gamma_v$ be a geodesic in $X$ joining $x_0$ to the identity
element of $G_v$. Then for all $x \in G_v, g_x \gamma_v$ is a path joining $gx_0$ and $gx \in gG_v$.
Hence one can choose $D_0$ to be the maximum of the lengths of $\gamma_v$'s, $v \in V(Y)$.$\Box$

The following corollary is an immediate consequence of the above two lemmas.
Corollary 3.6. The vertex spaces and edge spaces of $X$ are uniformly properly embedded in $X$.

Notation: We shall use $i_w : X_w \to X$ denote the canonical inclusion of the vertex and edge spaces of $X$ into $X$. Let $\tilde{v} = gG_v \in V(T)$. It follows from the above corollary that $\Theta$ induces a coarsely well-defined quasi-isometry from $gG_v \subset G$ to $X_{\tilde{v}}$. Namely, we can send any $x \in gG_v$ to a point $y$ of $X_{\tilde{v}}$ such that $d_X(\Theta(x), y) \leq D_0$, where $D_0$ is as in the above corollary. We shall denote this by $\Theta_{g,v} : gG_v \to X_{\tilde{v}}$.

4. The main theorem

For the rest of the paper we shall assume that $G$ is a hyperbolic group which admits a graph of groups decomposition $(\mathcal{G}, Y)$ with the QI embedded condition where all the vertex and edge groups are hyperbolic. Let $T$ be the Bass-Serre tree of this graph of groups.

We aim to show that in $G$ the family of subgroups $\{G_v : v \in V(T)\}$ satisfies the limit set intersection property:

Theorem 4.1. Suppose a hyperbolic group $G$ admits a decomposition into a graph of hyperbolic groups $(\mathcal{G}, Y)$ with quasi-isometrically embedded condition and suppose $T$ is the corresponding Bass-Serre tree. Then for all $w_1, w_2 \in V(T)$ we have $\Lambda(G_{w_1}) \cap \Lambda(G_{w_2}) = \Lambda(G_{w_1} \cap G_{w_2})$.

The idea of the proof is to pass to the tree of space $\pi : X \to T$ using the orbit map $\Theta : G \to X$ defined in the previous section and then use the techniques of [Mit98]. The following theorem is an important ingredient of the proof.

Theorem 4.2. (Mit98) The inclusion maps $i_w : X_w \to X$ admit CT maps $\partial i_w : \partial X_w \to \partial X$ for all $w \in V(T)$.

Recall that if $u, v \in V(T)$ are connected by an edge $e$ there are natural maps $f_{e,u} : X_e \to X_u$ and $f_{e,v} : X_e \to X_v$. We know that these maps are uniform QI embeddings. We assume that they are all $K$-QI embeddings for some $K > 1$. They induce embeddings $\partial f_{e,u} : \partial X_e \to \partial X_u$ and $\partial f_{e,v} : \partial X_e \to \partial X_v$ by Lemma 2.4. Therefore, we get partially defined maps from $\partial X_u$ to $\partial X_v$ with domain $\Lambda(\partial f_{e,u})$. Let us denote this by $\psi_{u,v} : \partial X_u \to \partial X_v$. By definition for all $x \in X_e$ we have $\psi_{u,v}(f_{e,u}(x)) = f_{e,v}(x)$.

Definition 4.3. (1) If $\xi \in \partial X_u$ is in the domain of $\psi_{u,v}(\xi) = \eta$ then we say that $\eta$ is a flow of $\xi$ and that $\xi$ can be flowed to $\partial X_v$.

(2) Suppose $w_0 \neq w_n \in V(T)$ and $w_0, w_1, \ldots, w_n$ are consecutive vertices of the geodesic $[w_0, w_n] \subset T$. We say that a point $\xi \in \partial X_{w_0}$ can be flowed to $\partial X_{w_n}$ if there are $\xi_i \in \partial X_{w_i}, 0 \leq i \leq n$ where $\xi_0 = \xi$ such that $\xi_{i+1} = \psi_{w_i, w_{i+1}}(\xi_i), 0 \leq i \leq n - 1$. In this case, $\xi_n$ is called the flow of $\xi_0$ in $X_{w_n}$.

Since the maps $\psi_{u,v}$ are injective on the their domains for all $u \neq v \in V(T)$ and $\xi \in \partial X_u$ the flow of $\xi$ in $\partial X_v$ is unique if it exists.

Lemma 4.4. Suppose $w_1, w_2 \in V(T)$ and $\xi_i \in \partial X_{w_i}, i = 1, 2$ such that $\xi_2$ is a flow of $\xi_1$. Let $\alpha_i$ be a geodesic in the vertex space $X_{w_i}$ such that $\alpha_i(\infty) = \xi_i, i = 1, 2$. Then $\text{Hd}(\alpha_1, \alpha_2) < \infty$. 
Proof: It is enough to check it when \( w_1, w_2 \) are adjacent vertices. Suppose \( e \) is the edge connecting \( w_1, w_2 \). The lemma follows from the stability of quasi-geodesics in the hyperbolic space \( X_w \)'s and the fact that every point of \( X_e \) is at distance \( 1/2 \) from \( X_{w_i}, i = 1, 2 \).

Corollary 4.5. Under the CT maps \( \partial v_{w_j} : \partial X_{w_j} \to \partial X \), \( j = 1, 2 \) the points \( \xi_1, \xi_2 \) go the same point of \( \partial X \), i.e. \( \partial v_{w_1}(\xi_1) = \partial v_{w_2}(\xi_2) \).

Lemma 4.6. Let \( w_1, w_2 \in T \) and suppose they are joined by an edge \( e \). Suppose \( \xi_1 \in \partial X_{w_1} \) cannot be flowed to \( \partial X_{w_2} \). Let \( \alpha \subset X_{w_1} \) be a geodesic ray such that \( \alpha(\infty) = \xi_1 \). Then for all \( D > 0 \) the set \( N_D(\alpha) \cap f_{e,w_1}(X_e) \) is bounded.

Proof: If \( N_D(\alpha) \cap f_{e,w_1}(X_e) \) is not bounded for some \( D > 0 \) then \( \xi_1 \) is in the limit set of \( f_{e,w_1}(X_e) \) and so \( \xi_1 \) can be flowed to \( \partial X_{w_2} \) by Lemma 2.6. This contradiction proves the lemma.

Now we briefly recall the ladder construction of Mitra which was crucial for the proof of the main theorem of [Mitra98]. We shall need it for the proof of Theorem 4.7.

Mitra’s Ladder \( B(\lambda) \).
Fix \( D_0, D_1 > 0 \). Let \( v \in V(T) \) and \( \lambda \) be a finite geodesic segment of \( X_v \). We shall define the set \( B(\lambda) \) to be a union of vertex space geodesics \( \lambda_w \subset X_w \) where \( w \) is in a subtree \( T_1 \) of \( T \) containing \( v \). The construction is inductive. Inductively one constructs the \( n \)-sphere \( S_{T_1}(v, n) \) of \( T_1 \) centred at \( v \) and the corresponding \( \lambda_w \)'s, \( w \in S(v, n) \).

\( S_{T_1}(v, 1) \): There are only finitely many edges \( e \) incident on \( v \) such that \( N_{D_0}(\lambda) \cap f_{e,v}(X_e) \neq \emptyset \). Then \( S(v, 1) \) is the set of terminal points of all the edges \( e \) that start at \( v \) such that the diameter of \( N_{D_0}(\lambda) \cap f_{e,v}(X_e) \) is at least \( D_1 \). In this case, for each edge \( e \) connecting \( v \) to say \( v_1 \in S(v, 1) \), we choose two points, say \( x, y \in N_{D_0}(\lambda) \cap f_{e,v}(X_e) \) such that \( d_v(x, y) \) is maximum. Then we choose \( x_1, y_1 \in X_{w_1} \) such that \( d(x, x_1) = 1 \) and \( d(y, y_1) = 1 \) and define \( \lambda_{w_1} \) to be a geodesic in \( X_{w_1} \) joining \( x_1, y_1 \).

\( S_{T_1}(v, n + 1) \) from \( S_{T_1}(v, n) \): Suppose \( w_1 \in S(v, n) \). Then a vertex \( w_2 \) adjacent to \( w_1 \) with \( d_T(v, w_2) = n + 1 \) belongs to \( S(v, n + 1) \) if the diameter of \( N_{D_0}(\lambda_{w_1}) \cap f_{e,w_1}(X_e) \) is at least \( D_1 \), where \( e \) is the edge connecting \( w_1, w_2 \), has diameter at least \( D_1 \) in \( X_{w_1} \). To define \( \lambda_{w_2} \) one chooses two points \( x, y \in N_{D_0}(\lambda_{w_1}) \cap f_{e,w_1}(X_e) \) such that \( d_{w_1}(x, y) \) is maximum, then let \( x_1, y_1 \in X_{w_2} \) be such that \( d(x, x_1) = 1 \) and \( d(y, y_1) = 1 \) and define \( \lambda_{w_2} \) to be a geodesic in \( X_{w_2} \) joining \( x_1, y_1 \).

Theorem 4.7. (Mitra [Mitra98]) There are constants \( D_0 > 0, D_1 > 0 \) and \( C > 0 \) depending on the defining parameters of the tree of metric spaces \( \pi : X \to T \) such that the following holds:

For any \( v \in V(T) \) and a geodesic segment \( \lambda \subset X_v \), the corresponding ladder \( B(\lambda) \) is a \( C \)-quasi-convex subset of \( X \).

To prove this theorem, Mitra defines a coarse Lipschitz retraction map \( P : X \to B(\lambda) \) which we now recall. For the proof of how this works one is referred to [Mitra98]. However, we shall subsequently assume that appropriate choices of \( D_0, D_1 \) are made in our context so that all the ladders are uniformly quasi-convex subsets of \( X \).

Coarsely Lipschitz retraction on the ladders
Suppose \( \lambda \subset X_v \) is a geodesic. Let \( T_1 = \pi(B(\lambda)) \). For each \( w \in T_1 \), \( \lambda_w = X_w \cap B(\lambda) \) is a geodesic in \( X_w \). We know that there is a coarsely well defined
nearest point projection $P_w : X_w \to \lambda_w$. (See Proposition 3.11 in Chapter III.1 of BH99.) Now for each $x \in X_w$, $w \in T_1$ define $P(x) = P_w(x)$. If $x \in X_w$ and $w \not\in T_1$ then connect $w$ to $T_1$ by a geodesic in $T$. Since $T$ is a tree there is a unique such geodesic. Let $w_1 \in T_1$ be the end point of this geodesic and let $e$ be the edge on this geodesic incident on $w_1$ going out of $T_1$. Mitra proved that in this case the projection of $f_{x,w_1}(x)$ on $\lambda_{w_1}$ is uniformly small. It follows by careful choice of $D_0,D_1$. (See Lemma 3.1 in BH98.) Choose a point $x_{w_1}$ on this projection. Define $P(x) = x_{w_1}$.

**Theorem 4.8.** (BH98) The map $P : X \to B(\lambda)$ is a coarsely Lipschitz retraction.

In other words, it is a retraction and there are constants $A,B$ such that $d(P(x), P(y)) \leq A d(x,y) + B$ for all $x,y \in X$.

Using the above theorems of Mitra now we shall prove the converse of Corollary 4.5. This is the last ingredient for the proof of Theorem 4.1.

**Proposition 4.9.** Suppose $v \neq w \in T$ and there are points $\xi_v \in \partial X_v$ and $\xi_w \in \partial X_w$ which map to the same point $\xi \in \partial X$ under the CT maps $\partial X_v \to \partial X$ and $\partial X_w \to \partial X$ respectively. Then $\xi_v$ can be flowed to $\partial X_w$.

**Proof:** Using Lemma 4.3 we can assume that the point $v$ is such that $\xi_v$ can not further be flowed along $vw$ and similarly $\xi_w$ can not be flowed in the direction of $vw$ where $v \neq w$. Let $\alpha : [0,\infty) \to X_v$ and $\beta : [0,\infty) \to X_w$ be geodesic rays in $X_v,X_w$ respectively such that $\alpha(\infty) = \xi_v$ and $\beta(\infty) = \xi_w$. Let $e_v,e_w$ be the first edges from the points $v,w$ along the direction of $vw$ and $vw$ respectively. Then $\partial D_\ell (\alpha) \cap f_{e_v,w}(X_{e_v}) \subset X_v$ and $\partial D_\ell (\beta) \cap f_{e_w,w}(X_{e_w}) \subset X_w$ are both bounded sets by Lemma 4.9 where $D_\ell$ is as in Theorem 4.7.

For all $n \in \mathbb{N}$ let $\alpha_n := \alpha|_{[0,n]}$ and $\beta_n := \beta|_{[0,n]}$ respectively. The ladders $B(\alpha_n), B(\beta_n)$ are uniformly quasi-convex subsets of $X$ by Theorem 4.8. Hence there are uniform ambient quasi-geodesics of $X$ in these ladders joining $\alpha(0),\alpha(n)$ and $\beta(0),\beta(n)$ respectively. Choose one such for each one of them and let us call them $\gamma_n$ and $\nu_n$ respectively. Now, since $\alpha$ and $\beta$ limit on the same point $\xi \in \partial X$, by Lemma 2.7(3) there is a uniform constant $D$ such that for a subsequence $\{n_k\}$ of natural numbers there are points $x_{n_k} \in \gamma_{n_k}$ and $y_{n_k} \in \nu_{n_k}$ such that $d(x_{n_k}, y_{n_k}) \leq D$ and $\lim_{k \to \infty} x_{n_k} = \lim_{k \to \infty} y_{n_k} = \xi \in \partial X$.

Let $v_1,w_1$ be the vertices on the geodesic $[v,w] \subset T$ adjacent to $v,w$ respectively. Let $A_{v_k} := B(\alpha_{n_k}) \cap X_{v_1}$ and $A_{w_k} := B(\beta_{n_k}) \cap X_{w_1}$ respectively.

If we remove the edge space $X_{e_v}$ from $X$ then the remaining space has two components one containing $X_v$ and the other containing $X_w$. Call them $Y_1,Y_2$ respectively. We note that since the diameter of $A_{v_k}$ is uniformly bounded, if at all nonempty, the portion of $\gamma_{n_k}$ contained in $Y_2$, if at all it travels into $Y_2$, is uniformly bounded. This implies that the portion of $\gamma_{n_k}$ joining $\alpha(0)$ and $A_{v_k}$ is uniformly small if $A_{v_k} \neq \emptyset$.

Hence, there are infinitely many $k \in \mathbb{N}$ such that $x_{n_k} \in Y_1$ and $y_{n_k} \in Y_2$. Since we are dealing with a tree of spaces and $d(x_{n_k}, y_{n_k}) \leq D$ for all $k \in \mathbb{N}$, this implies there are points $z_k \in f_{e_v}(X_{e_v})$ such that $d(x_{n_k}, z_k) \leq D$ for all $k \in \mathbb{N}$. Thus $\xi_v$ can be flowed to $\partial X_{v_1}$ by Lemma 2.6. This contradiction proves the proposition. □

**Proof of Theorem 4.1**

Suppose $w_i = g_i G_{v_i}$, $i = 1,2$, for some $g_1,g_2 \in G$ and $v_1,v_2 \in V(Y)$. This implies $G_{w_1} = g_1 G_{v_1} g_1^{-1}$, $i = 1,2$. Also, $\Lambda(g_1 G_{v_1} g_1^{-1}) = \Lambda(g_2 G_{v_2})$ by Lemma 2.9(1).
Hence we need to show that $\Lambda(g_1 G_v) \cap \Lambda(g_2 G_v) = \Lambda(g_1 G_v, g_1^{-1} \cap g_2 G_v, g_2^{-1})$. Using Lemma 2.9(2), therefore, it is enough to show that $\Lambda(G_{v_1}) \cap \Lambda(g G_{v_2}) = \Lambda(G_{v_1} \cap g G_{v_2} g^{-1})$ for all $v_1, v_2 \in V(Y), g \in G$.

Clearly, $\Lambda(G_{v_1} \cap g G_{v_2} g^{-1}) \subseteq \Lambda(G_{v_1}) \cap \Lambda(g G_{v_2})$. Thus we need to show that $\Lambda(G_{v_1}) \cap \Lambda(g G_{v_2}) \subseteq \Lambda(G_{v_1} \cap g G_{v_2} g^{-1})$.

Given an element $\xi \in \Lambda(G_{v_1}) \cap \Lambda(g G_{v_2})$ there are $\xi_1 \in \partial G_{v_1}$ and $\xi_2 \in \partial(g G_{v_2})$ both of which map to $\xi$ under the CT maps $\partial G_{v_1} \to \partial G$ and $\partial(g G_{v_2}) \to \partial G$ by Lemma 2.6. Now, we have a quasi-isometry $\Theta : \Gamma(G, S) \to X$. By Lemma 5.3 each coset of any vertex group in $G$ is mapped uniformly Hausdorff close to the same coset in $X$. Hence $\Theta$ induces uniform quasi-isometries $\Theta_{v_1, v}$ from $g G_{v_1} \subseteq \Gamma(G, S)$ to $g G_{v} \subseteq X$ for all $g \in G, v \in Y$. For avoiding confusion let us denote the subset $G_{v_1} \subset X$ by $X_{v_1}$ and $g G_{v_2} \subset X$ by $X_{v_2}$. It follows that $\partial \Theta_{1, v_1}(\xi_1) \in \partial X_{v_1}$ and $\partial \Theta_{v_1, v}(\xi_2) \in \partial X_{v_2}$ are mapped to the same element of $\partial X$ under the CT maps $\partial X_{v_1} \to \partial X, i = 1, 2$. Hence by Proposition 4.4, $\partial \Theta_{1, v_1}(\xi_1)$ can be flowed to, say $\xi'_2 \in \partial X_{w_2}$. By Lemma 4.4, the image of $\xi'$ and $\Theta_{v_1, v}(\xi_2)$ under the CT map $\partial X_{w_2} \to \partial X$ are the same. Hence, we can replace $\xi_2$ by $\Theta_{v_1, v}(\xi_2)^{-1}(\xi_2)$ and assume that $\Theta_{1, v_1}(\xi_1)$ flows to $\Theta_{v_2, v}(\xi_2)$.

Then by Lemma 4.4, for any geodesic rays $\alpha_i \subset X_{w_i}$ with $\alpha_i(\infty) = \partial \Theta_{x_i, v_1}(\xi_i)$ for $i = 1, 2$ where $x_1 = 1$ and $x_2 = g$ we have $Hd(\alpha_1, \alpha_2) < \infty$. Pulling back these geodesics by $\Theta_{1, v_1}$ and $\Theta_{v_1, v}$ we get uniform quasi-geodesic rays, say $\beta_1 \subset G_{v_1}$ and $\beta_2 \subset g G_{v_2}$ such that $\beta_i(\infty) = \xi_i, i = 1, 2$ and $Hd(\beta_1, \beta_2) < \infty$.

Now let $p_i = \beta_1(i)$ and $q_i = \beta_2(i), i \in \mathbb{N}$ be such that $d(p_i, q_i) \leq D$ where $Hd(\beta_1, \beta_2) = D$. Join $p_i$ to $q_i$ by a geodesic in $\Gamma(G, S)$. Suppose $w_i$ is the word labeling this geodesic. Since there are only finitely many possibilities for such words, there is a constant subsequence $\{w_{n_k}\}$ of $\{w_n\}$. Let $h_k = p_n^{-1} p_{n_k}$ and $h'_k = q_n^{-1} q_{n_k}$. Let $x$ be the group element represented by $w_{n_k}$. Then we have $p_i, h_k x = p_{n_k} x h'_k$ or $h'_k x = x h_k x^{-1}$. Since $h'_k$ connects two elements of $g G_{v_1}$, it is in $g G_{v_2}$. Hence $h_k \subseteq G_{v_1} \cap x G_{v_2} x^{-1}$. Thus $p_i, h_k p_{n_k}^{-1} \subseteq p_{n_k} G_{v_1} p_{n_k}^{-1} \cap p_{n_k} x G_{v_2} (p_{n_k} x)^{-1} = G_{v_1} \cap g G_{v_2} g^{-1}$. Finally, since $d(p_i, h_k p_{n_k}^{-1}, p_{n_k}) = d(p_i, h_k p_{n_k}^{-1}, p_{n_k}) = d(1, p_{n_k})$ for all $k \in \mathbb{N}$, $\lim_{n \to \infty} p_i, h_k p_{n_k}^{-1} = \lim_{n \to \infty} p_{n_k} = \xi_1$. This completes the proof. \[\]

The following corollary has been pointed out by Mahan Mj. We use the same notations as in the main theorem.

**Corollary 4.10.** If $H_i \subset G_{w_i}, i = 1, 2$ are two quasiconvex subgroups then $\Lambda(H_1) \cap \Lambda(H_2) = \Lambda(H_1 \cap H_2)$.

**Proof**: Assume that $w_i = g_i G_{v_i}, i = 1, 2$. Then $G_{w_i} = g_i G_{v_i} g_i^{-1}$. Let $K_i = g_i^{-1} H_i g_i \subset G_{v_i}$. We may construct a new finite graph starting from $Y$ by adding two vertices $u_1, u_2$ where $u_i$ is connected to $v_i$ by an edge $e_i, i = 1, 2$. Let us call this graph $Y_1$. Define a new graph of groups $(G_1, Y_1)$ by keeping the definition same on $Y$ and setting $G_{w_i} = G_{v_i} = K_i, i = 1, 2$ and defining $\phi_{e_i, u_i} = 1_{K_i}$ and $\phi_{e_i, v_i}$ to be the inclusion map $K_i \subset G_{v_i}$. This produces a new graph of groups with QI embedded condition and with fundamental group isomorphic to $G$. Suppose the Bass-Serre tree of the new graph of groups is $T_1$.

Now we can apply Theorem 4.4 to $G_{w_i}, i = 1, 2$ where $w_i = g_i K_i \in T_1, i = 1, 2$ to finish the proof. \[\]

**Example 4.11.** We now give an example where intersection of limit sets is not equal to the limit set of the intersection. Suppose $G$ is a hyperbolic group with an
infinite normal subgroup $H$ such that $G/H$ is not torsion. Let $g \in G$ be such that its image in $G/H$ is an element of infinite order. Let $K = \langle g \rangle$. Then $H \cap K = \{1\}$ whence $\Lambda(H \cap K) = \emptyset$. However, $H$ being an infinite normal subgroup of $G$ we have $\Lambda(H) = \partial G$. Thus $\Lambda(H) \cap \Lambda(K) = \Lambda(K) \neq \emptyset$.

We end with a question.

**Question 4.12.** If a hyperbolic group $G$ admits a decomposition into a graph of hyperbolic groups with qi embedded condition and $G_v$ is a vertex group how to describe $\Lambda(G_v) \subset \partial G$?

It has been pointed out to me by Prof. Ilya Kapovich that the first interesting case where this question should be considered is a hyperbolic strictly ascending HNN extension of a finitely generated nonabelian free group $F$

$$G = \langle F, t | t^{-1} \phi(w)t = \phi(w), \forall w \in F \rangle$$

where $\phi : F \to F$ is an injective but not surjective endomorphism of $F$. One would also like to describe $\partial G$ in this case.

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