Superconformal Covariantization Of Superdifferential Operator
On (1|1) Superspace And Classical N=2 W-superalgebras

Wen-Jui Huang

Department of Physics
National Tsing Hua University
Hsinchu, Taiwan R.O.C.

Abstract

A study of the superconformal covariantization of superdifferential operators defined on (1|1) superspace is presented. It is shown that a superdifferential operator with a particular type of constraint can be covariantized only when it is of odd order. In such a case, the action of superconformal transformation on the superdifferential operator is nothing but a hamiltonian flow defined by the corresponding supersymmetric second Gelfand-Dickey bracket. The covariant form of a superdifferential operator of odd order is given.

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1. Introduction

Since Zamolochikov introduced the W-algebras[1], W-algebras and related topics attracted a lot of attention[2-12]. Not long after Zamolochikov’s work it was realized that the classical versions of these algebras arise naturally in the context of integrable systems in 1+1 dimension[2,7,8]. Indeed, the second hamiltonian structure of the nth order KdV hierarchy provides a classical version of $W_n$-algebra. In the Lax formulation, the second hamiltonian structure is expressed elegantly by the so-called second Gelfand-Dickey bracket associated with the corresponding differential operator[13-16]. Recently, it was shown that the second Gelfand-Dickey bracket associated with a pseudodifferential operator also defines a hamiltonian structure and that the KP hierarchy is hamiltonian with respect to it[17-22]. Here, we have a different class of W-type algebras (called $W_{KP}$ algebras) from the second Gelfand-Dickey bracket. More recently, the supersymmetric version of the Gelfand-Dickey brackets have been constructed[23-25]. It was discovered that the supersymmetric second Gelfand-Dickey bracket associated with an odd-order superdifferential operator on (1|1) superspace gives (upon reduction) a superalgebra which contain the classical N=2 super Virasoro algebra as a subsuperalgebra. The analysis of the spectrum the simplest case suggests that the resulted superalgebras are N=2 W superalgebras[24,25]. However, a rigorous proof of this statement is lacking. It is the purpose of this paper to set up a formalism which could help us analyze the content of these superalgebras. More precisely, we shall study the possibility of covariantizing the superdifferential operators defined on (1|1) superspace.

To see why the covariantization of superdifferential operator is related to the spectra of the algebras resulting from the corresponding supersymmetric Gelfand-Dickey bracket, let us recall what we have learned in its bosonic counterpart. We know that the definition of the W-type algebra requires that it must contain a Virasoro subalgebra and a set of primary fields of spin higher than 2. For instance, the $W_n$ algebra has, besides a Virasoro generator, primary fields of spin up to n. On the other hand, each $W_{KP}$-
type algebra has a Virasoro generator and primary fields of spin up to $\infty$. However, the Gelfand-Dickey brackets are expressed in terms of coefficient functions of the corresponding (pseudo)differential operators, which are generally not primary fields. One therefore has to examine whether or not the required primary fields can be constructed as differential polynomials of these coefficient functions. This task has been done in refs[7-10,26]. The proofs rely on the possibility of covariantizing the corresponding (pseudo)differential operator. When a (pseudo)differential operator is properly covariantized the decompositions of the coefficient functions into primary fields then follow immediately. This suggests that the superconformal covariantization of superdifferential operator could be helpful for analyzing the spectra of the superalgebras from the supersymmetric Gelfand-Dickey brackets. Unfortunately, as we shall see later, this program does not completely solve the spectrum problem. The reason for it is the fact that we are dealing with $N = 2$ superalgebras while our differential operators are defined on (1|1) superspace and the coefficient functions are $N = 1$ superfields. As a result, there seems no natural way to identify the needed $N = 2$ supermultiplets. To be explicit, even though the flow generated by the super Virasoro generator (which is $N = 1$ superfield) allows a geometrical interpretation on (1|1) superspace the flow generated by its superpartner, the superconformal primary field of spin 1, does not. It is therefore necessary to find a different approach to handle the effect of this spin-1 flow in a systematical way.

We organize this paper as follow. In Sec. 2 we describe the supersymmetric second Gelfand-Dickey bracket briefly and derive the needed formulae. In Sec. 3 we show that an appropriate covariance condition can be imposed on a superdifferential operator of odd order and that the resulted flow is nothing but the super Virasoro flow defined by the corresponding supersymmetric Gelfand-Dickey bracket. In Sec. 4 we construct a sequence of covariant operators which are then used to decompose the coefficient functions into primary fields. In Sec. 5 we apply the result of Sec. 4 to study the simplest case in some details. Finally, we offer some concluding remarks in Sec. 6.
2. Supersymmetric Gelfand-Dickey Bracket

In this section we review briefly the supersymmetric second Gelfand-Dickey bracket for later uses. We follow the conventions used in ref.[23]. We will consider the superdifferential operators on a (1|1) superspace with coordinate \((x, \theta)\). These operators are polynomials in the supercovariant derivative \(D = \partial_\theta + \theta \partial_x\) whose coefficients are \(N = 1\) superfields; i.e.

\[
L = D^n + U_1 D^{n-1} + U_2 D^{n-2} + \ldots + U_n
\]  

(2.1)

These operators are assumed to be homogeneous under the usual \(Z_2\) grading; that is, \(|U_i| = i (\mod 2)\). The bracket will involve functional of the form

\[
F[U] = \int_B f(U)
\]  

(2.2)

where \(f(U)\) is a homogeneous (under \(Z_2\) grading) differential polynomial of the \(U_i\)'s and \(\int_B = \int dx d\theta\) is the Berezin integral which is defined in the usual way, namely, if we write \(U_i = u_i + \theta v_i\) and \(f(U) = a(u, v) + \theta b(u, v)\) then \(\int_B f(U) = \int dx b(u, v)\). The multiplication is given by the super Leibnitz rule:

\[
D^k \Phi = \sum_{i=0}^{\infty} \left[ \begin{array}{c} k \\ i \end{array} \right] (-1)^{|\Phi|} (k-i) \Phi[i] D^{k-i},
\]

(2.3)

where \(k\) is an arbitrary integer and \(\Phi[i] = (D^i \Phi)\) and the superbinomial coefficients \(\left[ \begin{array}{c} k \\ i \end{array} \right]\) are defined by

\[
\left[ \begin{array}{c} k \\ i \end{array} \right] = \left\{ \begin{array}{cl} 0 & \text{for } i > k \text{ or } (k,i) \equiv (0,1) \pmod{2} \\ \left( \frac{k}{i} \right) & \text{otherwise} \end{array} \right. 
\]

(2.4)

where \(\left( \frac{p}{q} \right)\) is the ordinary binomial coefficient. Next, we introduce the notions of super-residue and supertrace. Given a super-pseudodifferential operator \(P = \sum p_i D^i\) we define its superresidue as

\[
sres P = p_{-1}
\]

(2.5)
and its supertrace as

$$ StrP = \int_B sresP. \quad (2.6) $$

In the usual manner it can be shown that the supertrace of a supercommutator vanishes; i.e.

$$ Str[P, Q] = 0 \quad (2.7) $$

where

$$ [P, Q] \equiv PQ - (-1)^{|P||Q|}QP. \quad (2.8) $$

Finally, for a given functional $F[U] = \int_B f(U)$ we define its gradient $dF$ by

$$ dF = \sum_{i=1}^{n} (-1)^{n+k} D^{-n+k-1} \frac{\delta f}{\delta U_k}, \quad (2.9) $$

where

$$ \frac{\delta f}{\delta U_k} = \sum_{i=0}^{\infty} (-1)^{|U_k|i+i(i+1)/2} D^i \frac{\partial f}{\partial U_k^{[i]}}. \quad (2.10) $$

Equipped with these notions we now define the supersymmetric second Gelfand-Dickey bracket as

$$ \{F, G\} = (-1)^{|F|+|G|+n} Str[L(dFL)_+dG - (LdF)_+LdG] \quad (2.11) $$

where $(\cdot)_+$ denotes the differential part of a super-pseudodifferential operator. It has been shown that (2.11) indeed defines a Hamiltonian structure: it is antisupersymmetric and satisfies the super-Jacobi identity[23].

In ref.[24] it is shown that when the constraint $U_1 = 0$ is imposed the induced bracket is well-defined only when $n$ is odd. The reason is that this constraint is second class when $n$ is odd, while becomes first class for even $n$'s. To describe these induced brackets, we need to modify at least one of $dF$ and $dG$ defined by (2.9) due to absence of $U_1$. The prescription is to add a term $D^{-n}V$ to, say, $dG$ in such a way that

$$ sres[L, D^{-n}V + dG] = 0 \quad (2.12) $$
We shall denote \( X_G = D^{-n}V + dG \) for this choice of \( V \). Replacing \( dG \) in (2.11) by \( X_G \) then gives the induced bracket. A useful operator form of the induced bracket is

\[
J(X_G) = (LX_G)_+ L - L(X_GL)_+
= \sum_{i=2}^{n} (-1)^{k|G|+1} \{U_k, G\} D^{n-k} + \left( \frac{m}{2} - 1 \right) \sum_{i=2}^{n} (-1)^{k|G|+1} \{U_k, G\} D^{n-k} (2.13)
\]

We shall also regard \( J(X_G) \) as the transformation of the superdifferential operator \( L \) under the hamiltonian flow defined by \( G \).

It is known that if we define

\[
T = U_3 - \frac{1}{2} U_2' \quad J = U_2
\]

where \( V' = (DV), V'' = (D^2V), \ldots \) etc, then \( T \) and \( J \) obey[24]

\[
\{T(X), T(Y)\} = \left[ \frac{1}{4} m(m+1)D^5 + \frac{3}{2} T(X)D^2 + \frac{1}{2} T'(X)D + T''(X) \right] \delta(X - Y),
\]

\[
\{T(X), J(Y)\} = \left[ -J(X)D^2 + \frac{1}{2} J'(X)D - \frac{1}{2} J''(X) \right] \delta(X - Y),
\]

\[
\{J(X), T(Y)\} = \left[ J(X)D^2 - \frac{1}{2} J'(X)D + J''(X) \right] \delta(X - Y),
\]

\[
\{J(X), J(Y)\} = -\left[ m(m+1)D^3 + 2T(X) \right] \delta(X - Y),
\]

where we have written \( n = 2m + 1 \) and \( \delta(X - Y) = \delta(x - y)(\theta - w) \). (2.15) is the classical \( N = 2 \) super Virasoro algebra. It is conjectured that each remaining field \( U_j \) for \( j \) even gives rise to an \( N = 2 \) superconformal primary field \( W_j \) obtained by deforming \( U_j \) via the addition of differential polynomials in the \( U_{i < j} \) and that the remaining \( U_j \) with \( j \) odd give rise to their partners. This conjecture naturally leads us to consider the hamiltonian flows defined by the two linear functionals:

\[
G = \int_B T\xi = \int_B (U_3\xi + \frac{1}{2} U_2\xi')
\]

\[
H = \int_B J\zeta = \int_B U_2\zeta
\]

where \( |\xi(x, \theta)| = |\zeta(x, \theta)| = 0 \). Putting (2.16) into (2.13) we obtain

\[
J(X_G) = [\xi D^2 + \frac{1}{2} \xi' D + \frac{(m+1)}{2} \xi'']L - L[\xi D^2 + \frac{1}{2} \xi' D - \frac{m}{2} \xi'']
\]

\[
J(X_H) = [-\zeta D - (m+1)\zeta']L - L[-\zeta D + m\zeta']
\]

(2.17)
Since $T$ is the super Virasoro generator, $J(X_G)$ is called the super Virasoro flow. We shall prove in the next section that $J(X_G)$ in (2.17) arises quite naturally once we impose on $L$ a covariance condition which amounts to requiring $L$ to satisfy a particular transformation law under the superconformal transformation on $(1|1)$ superspace.

3. Superconformal Covariance And Super Virasoro Flow

Let us consider the $(1|1)$ superspace with coordinate $X = (x, \theta)$. The most general superdiffeomorphism has the form

$$
\begin{align*}
\tilde{x} &= g(x) + \theta \kappa(x) \\
\tilde{\theta} &= \chi(x) + \theta B(x)
\end{align*}
$$

(3.1)

where $|g| = |B| = 0$ and $|\kappa| = |\chi| = 1$. Under the superdiffeomorphism (3.1) the superderivative transforms as follows:

$$
D = (D\tilde{\theta})\tilde{D} + [(D\tilde{x}) - \tilde{\theta}(D\tilde{\theta})](\tilde{D})^2
$$

(3.2)

We call the superdiffeomorphism (3.1) a superconformal transformation if

$$
D = (D\tilde{\theta})\tilde{D}
$$

(3.3)

or, equivalently,

$$
D\tilde{x} = \tilde{\theta}(D\tilde{\theta})
$$

(3.4)

A function $f(X)$ is called a superconformal primary field of spin $h$ if, under superconformal transformation, it transforms as

$$
f(\tilde{X}) = (D\tilde{\theta})^{-2h}f(X)
$$

(3.5)

We shall denote by $F_h$ the space of all superconformal primary fields of spin $h$. As usual, a superdifferential operator $\Delta$ is called a covariant operator if it maps $F_h$ to $F_l$ for some $h$ and $l$. 

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We are ready to study the covariant property of the superdifferential operator

\[ L = D^n + U_2D^{n-2} + U_3D^{n-3} + \ldots + U_n \]  

(3.6)

where we have set \( U_1 \) to be zero. As in the bosonic case, we like to impose the covariance
condition:

\[ L : F_h \rightarrow F_l \]  

(3.7)

or, equivalently,

\[ L(X') = (D\tilde{\theta})^{-2l}L(X)(D\tilde{\theta})^{2h} \]  

(3.8)

for some \( h \) and \( l \). In other words, we like to see if there exists a transformation of the
functions \( U_2, \ldots, U_n \) such that the operator \( L \) is a covariant operator. We expect, as in the
bosonic case, the constraint \( U_1 = 0 \) determines both the values \( h \) and \( l \). To this purpose,
we rewrite (3.8) as

\[ L(X')(D\tilde{\theta})^{-2h} = (D\tilde{\theta})^{-2l}L(X) \]  

(3.9)

By using (3.3) the first term on the left hand side of (3.9) can be expanded as

\[ (\tilde{D})^n(D\tilde{\theta})^{-2h} = (D\tilde{\theta})^{-2h-n}(D^n + A_{n-1}D^{n-1} + A_{n-2}D^{n-2} + \ldots) \]  

(3.10)

With simple algebras we find

\[ A_{n-1} = \begin{cases} m & n = 2m \\ -2h - m & n = 2m + 1 \end{cases} \]  

(3.11)

Thus, for even \( n \) the constraint \( U_1 = 0 \) cannot be preserved under superconformal trans-
formation. But when

\[ n = 2m + 1 \]  

(3.12)

the constraint is preserved if one chooses

\[ h = -\frac{1}{4}(n - 1) = -\frac{1}{2}m \]  

(3.13)
As a result of (3.9), (3.10) and (3.13), the only choice of $l$ is then
\[ l = \frac{1}{4}(n + 1) = \frac{1}{2}(m + 1) \quad (3.14) \]

With these choices the covariance condition (3.8) reads
\[ L(\tilde{X}) = (D\tilde{\theta})^{-m} \]
\[ L(X)(D\tilde{\theta})^{-m} \quad (3.15) \]

which then determines how the functions $U_k$’s transform under superconformal transformation. For example, simple computations yield the transformation laws of $U_2$ and $U_3$:
\[ U_2(X) = U_2(\tilde{X})(D\tilde{\theta})^2 \]
\[ U_3(X) = U_3(\tilde{X})(D\tilde{\theta})^3 + U_2(\tilde{X})(D^2\tilde{\theta}) + \frac{1}{2}m(m + 1)S(\tilde{X}, X) \quad (3.16) \]

where $S(\tilde{X}, X)$ is the superschwarzian defined by
\[ S(\tilde{X}, X) = \frac{D^4\tilde{\theta}}{D\theta} - 2\left(\frac{D^3\tilde{\theta}}{D\theta}\right)\left(\frac{D^2\tilde{\theta}}{D\theta}\right) \quad (3.17) \]

One recognizes at once that $U_2$ is a superconformal primary field of spin 1. Moreover, using (3.16) we find that $T$ defined by (2.14) transforms as
\[ T(X) = T(\tilde{X})(D\tilde{\theta})^3 + \frac{1}{2}m(m + 1)S(\tilde{X}, X) \quad (3.18) \]

We therefore see that $T$ has the same transformation law as the energy-momentum tensor in the superconformal theory. It is not hard to verify that the infinitesimal forms of (3.16) and (3.18) is the same as the corresponding transformation laws from $J(X_G)$ of (2.17).

As a matter of fact, we can prove that, with a suitable identification of parameter $\xi$, the infinitesimal form of (3.15) is precisely equal to $J(X_G)$. To prove this statement, we first write down the most general infinitesimal form of superconformal transformation:
\[ \tilde{x} = x - \epsilon(x) - \theta\eta(x) \]
\[ \tilde{\theta} = \theta - \frac{1}{2}\partial_x\epsilon(x)\theta - \eta(x) \quad (3.19) \]
where $|\epsilon| = 0$ and $|\eta| = 1$. From now on we shall keep terms up to linear in $\epsilon$ and $\eta$ in all computations. Define $\xi(x, \theta) = \frac{1}{2}\epsilon(x) + \theta\eta(x)$ we find

$$\tilde{D}\tilde{\theta} = 1 - \xi''$$

$$\tilde{D} = D + \xi''D = D + D[D, \xi]D + [D, \xi]D^2$$

(3.20)

By induction, we derive from (3.20) the formula:

$$(\tilde{D})^k = D^k + D[D^k, \xi]D + [D^k, \xi]D^2$$

(3.21)

which has a more useful equivalent form

$$(\tilde{D})^k = D^k + D^k(\xi' D) - (\xi' D)D^k + 2[D^k, \xi]D^2$$

(3.22)

Secondly, we note

$$U_k(\tilde{X}) = U_k(x - \epsilon - \theta\eta, \theta - \frac{1}{2}\partial_x \epsilon \theta - \eta) + \delta\xi U_k$$

(3.23)

$$= U_k(x) - 2\xi \partial_x U_k - \xi'(DU_k) + \delta\xi U_k$$

Now (3.20), (3.22) and (3.23) together yield

$$(D\tilde{\theta})^{-m-1}L(X)(D\tilde{\theta})^{-m} = (1 - \xi'')^{-m-1}L(X)(1 - \xi'')^{-m}$$

$$= L(X) + (m + 1)\xi''L + mL\xi''$$

(3.24)

and

$$L(\tilde{X}) = L(X) - 2\xi[\partial_x, L] - \xi' \sum_{k=2}^{2m+1} (DU_k)D^{2m+1-k} + \delta\xi L + L(\xi'D)$$

$$- \sum_{k=2}^{2m+1} U_k\xi'DD^{2m+1-k} + 2[L, \xi]\partial_x$$

(3.25)

$$= L(X) - [2\xi D^2 + \xi' D]L + L[\xi' D + 2\xi D^2] + \delta\xi L$$

Equating (3.24) with (3.25) we obtain the infinitesimal form of (3.15):

$$\delta\xi L = [2\xi D^2 + \xi' D + (m + 1)\xi'']L - L[2\xi D^2 + \xi' D - m\xi]$$

(3.26)
which is equal to \( J(X_G) \) given by (2.17), provided that the trivial redefinition \( \xi \rightarrow \frac{1}{2} \xi \) is taken. We therefore have shown that the infinitesimal form of covariance condition is nothing but the super Virasoro flow.

4. Superconformal Covariantization of \( L \)

In this section we shall covariantize the superdifferential operator (3.6). The construction will be parallel to that for the bosonic case. First, we define

\[
B(\tilde{X}, X) = \frac{D^2 \tilde{\theta}}{D\theta} \tag{4.1}
\]

We can show easily that \( B(\tilde{X}, X) \) has the following transformation law:

\[
B(\tilde{\tilde{X}}, \tilde{X}) = (D\tilde{\theta})B(\tilde{X}, \tilde{X}) + B(\tilde{X}, X) \tag{4.2}
\]

and that the superschwarzian can be represented as

\[
S(\tilde{X}, X) = D^2 B(\tilde{X}, X) - (DB(\tilde{X}, X))B(\tilde{X}, X) \tag{4.3}
\]

Using (4.2) we can verify that the superschwarzian satisfies

\[
S(\tilde{\tilde{X}}, \tilde{X}) = (D\tilde{\theta})^3 S(\tilde{X}, \tilde{X}) + S(\tilde{X}, X) \tag{4.4}
\]

Now we choose a particular coordinate \( Z = (z, \vartheta) \) and demand

\[
T(X) = \frac{m(m+1)}{2} S(Z, X) \tag{4.5}
\]

The transformation law (4.4) then guarantees \( T(X) \) transforms as (3.18). Obviously, this choice of coordinate is to make \( T \) vanish identically; i.e. \( T(Z) = 0 \). We are not going to concern with the problem of existence of such a coordinate, which is beyond the scope of this paper, but simply insist the identification (4.5). For the rest of this section we shall use the notation:

\[
B(X) \equiv B(Z, X) \tag{4.6}
\]
and the representation of $T$:

$$T(X) = \frac{m(m + 1)}{2}[D^2B(X) - (DB(X))B(X)]$$

(4.7)

One should note that different $B(X)$’s may define the same $T(X)$. Indeed, if we replace $B$ by $B + \delta B$ and demand $\delta B$ satisfy

$$D^2(\delta B) - [D(\delta B)]B - (DB)\delta B = 0$$

(4.8)

then $T$ is not changed.

The definition of $B(X)$ enables us to introduce a covariant superderivative defined by

$$\hat{D}_{2k} \equiv D - 2kB(X)$$

(4.9)

One can verify easily that $\hat{D}_{2k}$ maps from $F_k$ to $F_{k+\frac{3}{2}}$. Hence the operator

$$\hat{D}_{2k}^l \equiv \hat{D}_{2k+l-1} \hat{D}_{2k+l-2} \ldots \hat{D}_{2k} \quad (l > 0)$$

(4.10)

$$= [D - (2k + l - 1)B][D - (2k + l - 2)B] \ldots [D - 2kB]$$

maps from $F_k$ to $F_{k+\frac{3}{2}}$, that is, it transforms, under superconformal transformation, as

$$\hat{D}_{2k}^l(\tilde{X}) = (D\tilde{\theta})^{-2k-l} \hat{D}_{2k}^l(D\tilde{\theta})^{2k}$$

(4.11)

We list here two useful relations following from the definition (4.9) of covariant superderivative. The first one is

$$\hat{D}_{2k}\delta B = -\delta B \hat{D}_{2k-1} + \triangle B$$

(4.12)

where $\delta B$ is an arbitrary variation and

$$\triangle B \equiv D(\delta B) - B\delta B$$

(4.13)

The other one is an equivalent form of (4.8):

$$\hat{D}_{2k+1} \hat{D}_{2k} \delta B = \delta B \hat{D}_{2k} \hat{D}_{2k-1}$$

(4.14)
By using (4.13) and (4.14) we can easily derive the variation of \( \hat{D}_{2k}^l \) due to \( \delta B \) subjected to (4.8). The results are

\[
\delta_B \hat{D}_{2k}^{2m} = -\delta B (m \hat{D}_{2k}^{2m-1}) - \Delta B [m(2k + m - 1) \hat{D}_{2k}^{2m-2}] \quad (4.15)
\]

and

\[
\delta_B \hat{D}_{2k}^{2m+1} = -\delta B [(2k + m) \hat{D}_{2k}^{2m}] - \Delta B [m(2k + m) \hat{D}_{2k}^{2m-1}] \quad (4.16)
\]

An important consequence of (4.15) and (4.16) is that the covariant operator \( \hat{D}_{2k}^l \) depends explicitly on \( B \) except when \( l = 2m+1 \) and \( k = -\frac{m}{2} \). In these exceptional cases, it depends on \( B \) only through \( T \). This result, of course, can be expected from (3.12) and (3.13). Now we are ready to construct covariant operators involving superconformal primary fields. Let us consider

\[
\Delta^{(2m+1)}_{2p}(W_{2p}, T) = \sum_{i=0}^{2m+1-2p} \alpha_{2p,i} (\hat{D}_{2p}^i W_{2p}) \hat{D}_{-m}^{2m+1-2p-i} \quad \alpha_{2p,0} = 1 \quad (4.17)
\]

where \( W_{2p} \) is a superconformal primary field of spin \( p \). We like to choose \( \alpha_{2p,i} \)'s in such a way that the right hand side of (4.16) depends on \( B \) only through \( T \). To this end, we have to compute the variation with respect to \( B \) with \( \delta B \) constrained by (4.14). For integer \( p \) we find

\[
\delta_B \Delta^{(2m+1)}_{2p} \equiv \delta B \left( \frac{\delta \Delta^{(2m+1)}_{2p}}{\delta B} \right) + \Delta B \left( \frac{\delta \Delta^{(2m+1)}_{2p}}{\delta B} \right) \quad (4.18)
\]

where

\[
\frac{\delta \Delta^{(2m+1)}_{2p}}{\delta B} = - \sum_{i=1}^{m-p} [i \alpha_{2p,2i} - (m - p - i + 1) \alpha_{2p,2i-1}] (\hat{D}_{2p}^{2i-1} W_{2p}) \hat{D}_{-m}^{2(m-p-i)+1} - \sum_{i=1}^{m-p+1} [(p + i - 1) \alpha_{2p,2i-2} - (2p + i - 1) \alpha_{2p,2i-1}] (\hat{D}_{2p}^{2i-2} W_{2p}) \hat{D}_{-m}^{2(m-p-i)+1} \quad (4.19)
\]
and

\[
\frac{\delta \Delta^{(2m+1)}_{2p}}{\Delta B} = \sum_{i=0}^{m-p-1} [(p+i)(m-p-i)\alpha_{2p,2i} - (i+1)(2p+i)\alpha_{2p,2i+2}] (\hat{D}^{2i}_{2p} W_{2p}) \hat{D}^{2(m-p-i)-1}_{-m} \\
- \sum_{i=1}^{m-p} [i(2p+i)\alpha_{2p,2i+1} - (m-p-i+1)(p+i)\alpha_{2p,2i-1}] (\hat{D}^{2i-1}_{2p} W_{2p}) \hat{D}^{2(m-p-i)}_{-m} 
\]

Demanding \( \frac{\delta \Delta^{(2m+1)}_{2p}}{\delta B} = 0 \) and \( \frac{\delta \Delta^{(2m+1)}_{2p}}{\Delta B} = 0 \) gives, respectively,

\[
\alpha_{2p,2i+1} = \frac{p+i}{2p+i} \alpha_{2p,2i} \\
\alpha_{2p,2i} = \frac{(m-p-i+1)}{i} \alpha_{2p,2i-1}
\]

and

\[
\alpha_{2p,2i+2} = \frac{(p+i)(m-p-i)}{(i+1)(2p+i)} \alpha_{2p,2i} \\
\alpha_{2p,2i+1} = \frac{(p+i)(m-p-i+1)}{i(2p+i)} \alpha_{2p,2i-1}
\]

Remarkably, (4.21) implies (4.22). Therefore, \( \alpha_{2p,i} \)'s are determined unambiguously. For a half integer \( p = q + \frac{1}{2} \) the calculation is much the same. We simply write down the resulted recursion relations:

\[
\alpha_{2q+1,2i} = \frac{q+i}{i} \alpha_{2q+1,2i-1} \\
\alpha_{2q+1,2i+1} = \frac{m-q-i}{2q+i+1} \alpha_{2q+1,2i}
\]

Solving (4.21) and (4.23) then yields

\[
\alpha_{2p,2l} = (-1)^l \left( \begin{array}{c} l+p-m-1 \\ l \end{array} \right) \left( \begin{array}{c} p+l-1 \\ l \end{array} \right) \\
\alpha_{2p,2l+1} = \frac{(-1)^l}{2} \left( \begin{array}{c} p+l-m-1 \\ l \end{array} \right) \left( \begin{array}{c} p+l \\ l \end{array} \right)
\]

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\[ \alpha_{2q+1,2l} = (-1)^l \frac{\binom{q+l-m-1}{l} \binom{q+l}{l}}{\binom{2q+l}{l}} \]

\[ \alpha_{2q+1,2l+1} = (-1)^l \frac{m-q}{2q+1} \frac{\binom{q+l-m}{l} \binom{q+l}{l}}{\binom{2q+l+1}{l}} \] (4.25)

With the coefficients \( \alpha_{2p,l} \)'s given by (4.24) and (4.25) we now write

\[ L = \tilde{D}^{2m+1} + U_2 D^{2m-1} + \ldots + U_{2m+1} \]

\[ = \hat{D}^{2m+1}_m + \Delta^{(2m+1)}_2(U_2, T) + \sum_{k=4}^{2m+1} \Delta^{(2m+1)}_k(W_k, T) \] (4.26)

which is the desired covariant form. If one works out explicitly the right hand side of (4.26), one would obtain decomposition of the form

\[ U_k = W_k + G_k(W_{k-1}, \ldots, W_4, T, U_2) \quad (k \geq 4) \] (4.27)

where \( G_k \) is a differential polynomial in \( W_{k-1}, \ldots, W_4, T, U_2 \). Inverting (4.27) gives the definitions of superconformal primary fields in terms of coefficient functions:

\[ W_k = U_k + H_k(U_{k-1}, \ldots, U_4, T, U_2) \quad (k \geq 4) \] (4.28)

where \( H_k \) is again a differential polynomial. This completes the covariantization of \( L \).

Before ending this section we like to remark that so far we have only taken care of super Virasoro flow. In other words, what we have done is to decompose the coefficient functions into superconformal primary fields which satisfy

\[ \delta_\xi W_k = -\{W_k, G\} = \frac{k}{2} W_k \xi'' + \frac{(-1)^{k+1}}{2} W_k' \xi' + W_k'' \xi \] (4.29)

We do not know yet how \( W_k \)'s transform under \( J(X_H) \), the flow generated by the spin-1 current \( J \). As a result, we can not expect that \( W_{2k} \) and \( W_{2k+1} \) do form a \( N = 2 \) supermultiplet. To identify the supermultiplets some redefinitions of primary fields should
be expected. For example, to obtain the first two supermultiplets the following redefinitions should be considered

\[
\begin{align*}
\hat{W}_4 &= W_4 + aJ^2 \\
\hat{W}_5 &= W_5 \\
\hat{W}_6 &= W_6 + bJW_4 + cJ^3 \\
\hat{W}_7 &= W_7 + eJW_5
\end{align*}
\]

where \(a, b, c\) and \(e\) are constants. Indeed, we shall see in the next section, where the simplest nontrivial case is studied, that redefinitions of this sort must be done in order to get the desired supermultiplets.

5. An Explicit Example

In this section we study the simplest nontrivial case:

\[
L = D^5 + U_2D^3 + U_3D^2 + U_4D + U_5
\]

Even though this case has been studied in the literature\[25\] we like to use it to illustrate the usefulness of the results in the previous section. By (4.26) we have

\[
L = \hat{D}_2^{-5} + \Delta_2^{(5)}(J, T) + \Delta_4^{(5)}(W_4, T) + \Delta_5^{(5)}(W_5, T)
\]

By using (4.24) and (4.25) we find

\[
\begin{align*}
\hat{D}_2^{-5} &= (D - 2B)(D - B)D(D + B)(D + 2B) \\
&= D^5 + TD^2 + \frac{1}{3}T'D + \frac{2}{3}T''
\end{align*}
\]

\[
\Delta_2^{(5)}(J, T) = J\hat{D}_2^{-3} + \frac{1}{2}(\hat{D}_2J)\hat{D}_2^{-2} + \frac{1}{2}(\hat{D}_2^2J)\hat{D}_2^{-1} + \frac{1}{3}(\hat{D}_2^3J)
\]

\[
= JD^3 + \frac{1}{2}J'D^2 + \frac{1}{2}J''D + \frac{1}{3}J''' + \frac{4}{9}JT
\]

\[
\begin{align*}
\Delta_4^{(5)}(W_4, T) &= W_4\hat{D}_2^{-2} + \frac{1}{2}(\hat{D}_4W_4) \\
&= W_4D + \frac{1}{2}W'_4
\end{align*}
\]

\[
\Delta_5^{(5)}(W_5, T) = W_5
\]

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Thus, we have the following decompositions:

\[ U_2 = J \]
\[ U_3 = T + \frac{1}{2} J' \]
\[ U_4 = W_4 + \frac{1}{3} T'' + \frac{1}{2} J'' \]
\[ U_5 = W_5 + \frac{1}{2} W'_4 + \frac{2}{3} T''' + \frac{1}{3} J''' + \frac{4}{9} JT \]

(5.4)

Inverting (5.4) then gives the definitions primary fields in terms of coefficient functions:

\[ J = U_2 \]
\[ T = U_3 - \frac{1}{2} U'_2 \]
\[ W_4 = U_4 - \frac{1}{3} U'_3 - \frac{1}{3} U''_2 \]
\[ W_5 = U_5 - \frac{1}{2} U'_4 - \frac{1}{2} U'''_3 + \frac{1}{6} U'''_2 - \frac{4}{9} T U_2 \]

(5.5)

As we explained at the end of section 4 that \( W_4 \) and \( W_5 \) may not form a \( N = 2 \) supermultiplet. To check this point we have to compute \( J(X_H) \) given by (2.17) explicitly.

Straightforward calculations yield

\[ J(X_H) = [-\zeta D - 3\zeta']L - L[-\zeta D + 2\zeta'] \]
\[ \equiv (\delta \zeta U_2)D^3 + (\delta \zeta U_3)D^2 + (\delta \zeta U_4)D + (\delta \zeta U_5) \]

(5.6)

where

\[ \delta \zeta U_2 = [6D^3 + (2U_3 - U'_2)]\zeta \]
\[ \delta \zeta U_3 = [-3D^4 - U_2 D^2 + U_3 D - U'_3]\zeta \]
\[ \delta \zeta U_4 = [3D^5 + 3U_2 D^3 + U_3 D^2 + (2U_5 - U'_4)]\zeta \]
\[ \delta \zeta U_5 = [-2D^6 - 2U_2 D^4 - 2U_3 D^3 - 2U_4 D^2 + U_5 D - U'_5]\zeta \]

(5.7)

Using (5.4) and (5.5) we further find

\[ \delta \zeta J = [6D^3 + 2T]\zeta \]
\[ \delta \zeta T = [-JD^2 + \frac{1}{2} J'D - \frac{1}{2} J'']\zeta \]
\[ \delta \zeta W_4 = \left[ \frac{8}{3} JD^3 + \frac{8}{9} JT + 2W_5 \right] \zeta \]
\[ \delta \zeta W_5 = [-2(W_4 - \frac{2}{9} J^2)D^2 + \frac{1}{2}(W_4 - \frac{2}{9} J^2)'D - \frac{1}{2}(W_4 - \frac{2}{9} J^2)'')] \zeta \]

(5.8)
The first two equations of (5.8) give rise to, with the help of (2.13), Poisson brackets as expected from (2.15). On the other hand, since $J$ and $T$ show up in $\delta \xi W_4$ and $\delta \xi W_5$, $W_4$ and $W_5$ do not form a $N = 2$ supermultiplet. Hence, a redefinition of $W_4$ of the form of (4.30) is necessary. Indeed, the last of (5.8) does suggest the following redefinition:

$$W_4 = W_4 - \frac{2}{9} J^2$$

$$= U_4 - \frac{1}{3} U_3 - \frac{1}{3} U_2' - \frac{2}{9} U_2'' (5.9)$$

With (5.9) we then obtain

$$\bar{W}_4 = 2W_5 \xi$$

$$W_5 = [-2\bar{W}_4 D^2 + \frac{1}{2} \bar{W}_4' D - \frac{1}{2} \bar{W}_4''] \xi (5.10)$$

The corresponding Poisson brackets can be easily read off:

$$\{\bar{W}_4(X), J(Y)\} = -2W_5 \delta(X - Y)$$

$$\{W_5(X), J(Y)\} = [-2\bar{W}_4 D^2 + \frac{1}{2} \bar{W}_4' D - \frac{1}{2} \bar{W}_4''] \delta(X - Y) (5.11)$$

We therefore conclude that $\bar{W}_4$ and $W_5$ form a $N = 2$ supermultiplet.

6. Conclusions

In this paper we have carried out the study of superconformal covariantization of superdifferential operators. We have shown that when the constraint $U_1 = 0$ is imposed only those of odd order can be consistently covariantized. The covariance condition is then shown to be equivalent to the superVirasoro flow. As a result, the covariant form of a superdifferential operator immediately leads to the decompositions of coefficient functions into differential polynomials of spin-1 supercurrent, superVirasoro generator and superconformal primary fields of spin higher than $\frac{3}{2}$. However, to prove the corresponding superalgebra to be a $N = 2$ W-superalgebra this is only half the way. The essential point is that the superdifferential operators are defined on the $(1|1)$ superspace and hence there is no natural way to interpret the flow generated by the spin-1 supercurrent in a geometrical
manner. As illustrated by the simplest nontrivial example, explicit calculations and further redefinitions of superconformal primary fields are required to identify the desired \( N = 2 \) supermultiplets. The problem of systemmatical identifications of \( N = 2 \) supermultiplets for superdifferential operators of high orders therefore remains open.

Finally we like to remark that there exists an interesting link between the covariant differential operators and a class of singular vectors in Virasoro modules in the classical limit\[27\]. As known, this link is manifest when the Drinfeld and Sokolovs’ matrix representation of differential operators\[28\] is exploited. Presumably, a similar link between the superconformally covariant superdifferential operators and a certain class of singular vectors in super-Virasoro modules in the classical limit should also exist. A systematical investigation of this link would be a very interesting task.

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Note added in proof

After submitting this work the author became aware of the works by Gieres and Theisen \[29-31\]. In ref.\[30\] the results of sections 4 and 5 of this paper had been derived in the same spirit and a matrix representation of covariant superdifferential operators was also obtained. The author also likes to recommend refs. \[29,30\] to those readers who are interested in general aspects of covariant operators and the relation between classical W-superalgebras and Lie superalgebras.
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