Renormalization of Position Space Amplitudes in a Massless QFT

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Abstract—Ultraviolet renormalization of position space massless Feynman amplitudes has been shown to yield associate homogeneous distributions. Their degree is determined by the degree of divergence while their order—the highest power of logarithm in the dilation anomaly—is given by the number of (sub)divergences. In the present paper we review these results and observe that (convergent) integration over internal vertices does not alter the total degree of (superficial) ultraviolet divergence. For a conformally invariant theory internal integration is also proven to preserve the order of associate homogeneity. The renormalized 4-point amplitudes in the \( \phi^4 \) theory (in four space-time dimensions) are written as (non-analytic) translation invariant functions of four complex variables with calculable conformal anomaly.

Our conclusion concerning the (off-shell) infrared finiteness of the ultraviolet renormalized massless \( \phi^4 \) theory agrees with the old result of Lowenstein and Zimmermann [23].

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1. INTRODUCTION

On-shell scattering amplitudes are simpler to study in momentum space, so perturbative renormalization in quantum field theory (QFT) was originally developed starting with an ultraviolet momentum cutoff. On the other hand, Stueckelberg and Bogolubov had realized early on (see [4] and references therein) that the principle of relativistic causality in position space may serve as a true basis of renormalization theory: not only counter-terms have to be local but causal factorization gives rise to a recursive procedure that allows to reduce ultraviolet (UV) renormalization to that of primitively divergent amplitudes. In fact, Fourier transform is an old example of what is now fashionable to call a duality transformation: it maps a large momentum problem into a small distance one. The position space approach was developed systematically by Epstein—Glaser [16] in the early seventies but only gained popularity much later. It does not use global Poincaré invariance and thus offers a way to develop perturbative QFT and operator product expansions on a curved background (see [17, 20] and references to earlier work of Brunetti, Dütsch, Fredenhagen and of Hollands et al. cited there).

It uses, following Bogolubov et al., an adiabatic procedure in which each coupling constant \( g \) is replaced by a test function \( g(x) \) that vanishes at infinity. This forces one to treat all vertices as external and allows to set up a recursive procedure for Feynman graphs of increasing order in terms of a causal factorization condition (see [NST12] and [NST]). Integrating over internal vertices—which would correspond to taking the adiabatic limit \( g(x) \to g(\neq 0) \) — would not keep track of the localization of internal vertices. Here
we demonstrate that in a conformally invariant theory, like \( \varphi^4 \) in \( D = 4 \) dimensions, such an integration actually does not pose problems. In fact, Lowenstein and Zimmermann have proven long ago [23] (in the momentum space framework) that the UV renormalized massless \( \varphi^4 \) theory is infrared finite. Various parts of the position space problem have also been addressed by a number of authors. We refer to the recent paper [19] which presents a major step in solving this problem (appearing in fact at an advanced stage of our own work on the subject) and has, in addition, the virtue of giving a careful account of earlier contributions (within a bibliography of 49 entries—see also [18] for a later survey). It provides a systematic study of all graphs up to order \( g^4 \) (and up to three loops with no tadpoles). We choose to follow the elementary readable style of this work, preferring the outline of the argument in concrete typical cases to adding to the development of the general machinery of [27]. We calculate in particular the dilation anomaly of a logarithmically divergent 4-point graph with an arbitrary 4-point subdivergence.

We start in Sect. 2 by reviewing earlier results on euclidean space renormalization of the massless \( \varphi^4 \) theory: the notion of residue and renormalization of primitively divergent graphs (Sect. 2.2), and the recursion based on the causal factorization requirement allowing to renormalize arbitrary associate homogeneous amplitudes (Sect. 2.3). We recall in Sect. 2.4 Schnetz’s vacuum completion of 4-point graphs, [29, 30]. We elaborate on his characterization of primitively divergent \( \varphi^4 \) graphs and introduce the conformal anomaly in Sect. 3.1. Sect. 3.2 surveys the associate homogeneity law for amplitudes with subdivergences. We demonstrate in Sect. 3.3 that every 4-point amplitude in the \( \varphi^4 \) theory can be presented as a translation invariant function of four complex variables. Using this representation we exhibit the dilation anomaly of a divergent 4-point graph with any primitive 4-point subdivergence (the reduced graph—in which the 4-point subgraph is substituted by a single internal vertex—is having four loops in this generic case).

2. EUCLIDEAN SPACE RENORMALIZATION OF A MASSLESS QFT

2.1. Terminology and Conventions

We call a graph \( n \)-point if it has \( n \) external (half-) lines. Thus, each of the three graphs on Fig. 1 corresponds to a 4-point vertex function (i.e. a Feynman amplitude without propagators attached to the external lines). Any subgraph of a graph of \( \Gamma \) is obtained by eliminating a non-empty subset of the vertices of \( \Gamma \) together with the adjacent half-edges. (We use the terms “line” and “edge” interchangeably.) The 2-point vertex graph \( \Gamma_1 \) of Fig. 2 is a subgraph of the corresponding 2-point amplitude \( G_1 \) but the 4-point graph \( \Gamma_2 \) of Fig. 1 is not a subgraph of \( \Gamma_1 \) (since the two graphs have the same set of vertices). There are only graphs with an even number of external lines (2n-point graphs) in the \( \varphi^4 \) theory.

The (euclidean) position space Feynman rules for a massless theory can be summarized as follows. To each vertex \( i (=1, ..., V(\Gamma)) \) of the graph \( \Gamma \) we associate an euclidean 4-vector \( x_i = x_i^a, \alpha = 1, 2, 3, 4 \). To an internal line with end points \( i, j \) corresponds a (massless) propagator

\[
G_{ij}(x) = \frac{P_{ij}(x)}{x^{2m_{ij}}}, \quad x = x_j - x_i, \\
x^2 = \sum_{\alpha=1}^{4} (x^\alpha)^2, \quad m_{ij} \in \mathbb{N},
\]

where \( P_{ij}(x) \) are homogeneous polynomials in the components \( x^\alpha \) of \( x \). (In a scalar QFT

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{Four point graphs with one (\( \Gamma_a \)) , two (\( \Gamma_b \)) and three (\( \Gamma_c \)) independent external 4-vectors.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig2.png}
\caption{Self-energy graph without (\( \Gamma_1 \)) and with (\( G_1 \)) external propagators.}
\end{figure}
$P_g = \text{const}, m_{ij} = 1$.) Each internal vertex $x$ is integrated with a measure $\frac{d^4 x}{\pi^2}$ (a convention that goes back to old work of Broadhurst and helps displaying the number theoretic content of renormalization—see the discussion after Proposition 2.1 below). Thus, the Feynman amplitude $G_4$ corresponding to the 4-loop graph $\Gamma_4$ on Fig. 3 is given by

$$G_4(x_1, \ldots, x_4) = \frac{1}{x_1^2 x_2^2 x_3^2 x_4^2} \prod_{i=1}^{4} \frac{1}{(x_i - x)^2} \frac{d^4 x}{\pi^2}. \quad (2.2)$$

Note that the integral $G_4$ is absolutely convergent for non-coinciding arguments, $x_i \neq x_j$ for $i \neq j$.

Each $G_4$ is a locally integrable homogeneous function for non-coinciding arguments and defines a Schwartz distribution off the large diagonal. It may diverge—i.e., not admit a homogeneous extension as a distribution on the entire space $\mathbb{R}^{4(V(\Gamma) - 1)}$—if its product with the volume form (of order $4(V(\Gamma) - 1)$) has a non-positive degree of homogeneity, i.e. if

$$\kappa := 2L(\Gamma) - 4(V(\Gamma) - 1) \geq 0, \quad L(\Gamma) \text{ being the number of internal lines of } \Gamma. \quad (2.3)$$

A divergent graph $\Gamma$ is primitive if each of its proper connected subgraphs $\gamma$ is convergent—i.e., if $4(V(\gamma) - 1) > 2L(\gamma)$.

\section*{2.2. Renormalization of Primitively Divergent Graphs}

In a massless QFT a Feynman amplitude $G(\bar{x})$ is a homogeneous function of $\bar{x} \in \mathbb{R}^N$; if $G$ corresponds to a connected graph with $V$ vertices then $N = 4(V - 1)$. It is superficially divergent if $G$ defines a homogeneous density in $\mathbb{R}^N \setminus \mathbb{D}$ of non-positive degree, where $\mathbb{D}$ is a subvariety (diagonal) of lower dimension:

$$G(\lambda \bar{x}) d^N \lambda \bar{x} = \lambda^{-\kappa} G(\bar{x}) d^N \bar{x}, \quad (2.3)$$

where summation is assumed (from 1 to $N$) over the repeated indices $i_1, \ldots, i_N$.

\kappa is called (superficial) degree of divergence. In a scalar QFT with propagators $\frac{1}{x_i^2}$ a connected graph with a set $\mathcal{L}$ of internal lines gives rise to a Feynman amplitude that is a multiple of the product

$$G(\bar{x}) = \prod_{(i,j) \in \mathcal{L}} \frac{1}{x_{ij}^2}. \quad (2.4)$$

If $G$ is superficially divergent (i.e. if $\kappa = 2L - N \geq 0$ where $L$ is the number of lines in $L$) then it is divergent—that is, it does not admit a homogeneous extension as a distribution on $\mathbb{R}^N$. (For more general spin-tensor fields whose propagator has a polynomial numerator a superficially divergent amplitude may, in fact, turn out to be convergent—see Sect. 5.2 of [27].) For a primitively divergent amplitude the following proposition (Theorem 2.3 of [26]) serves as a definition of both a residue $\text{Res} G$ and of a renormalized amplitude $G^0(\bar{x})$.

\textbf{Proposition 2.1.} If $G(\bar{x})$ (2.4) is primitively divergent then for any smooth norm $p(\bar{x})$ on $\mathbb{R}^N$ one has

$$|p(\bar{x})|^4 G(\bar{x}) - \frac{1}{\epsilon} (\text{Res} G)(\bar{x}) = G^0(\bar{x}) + O(\epsilon). \quad (2.5)$$

Here $\text{Res} G$ is a distribution with support at the origin. Its calculation is reduced to the case $\kappa = 0$ of a logarithmically divergent graph by using the identity

$$(\text{Res} G)(\bar{x}) = (-1)^{\kappa} \frac{1}{\kappa!} \partial_{i_1} \cdots \partial_{i_{\kappa}} \text{Res} (x^{i_1} \cdots x^{i_\kappa} G)(\bar{x}). \quad (2.6)$$

where summation is assumed (from 1 to $N$) over the repeated indices $i_1, \ldots, i_{\kappa}$. If $G$ is homogeneous of degree—$N$ then

$$(\text{Res} G)(\bar{x}) = \text{res} G(\hat{\bar{x}}) \quad (\text{for } \hat{\partial}_{i} (x^{i} G) = 0). \quad (2.7)$$

Here the numerical residue $\text{res} G$ is given by an integral over the hypersurface $\Sigma_p = \{ \bar{x} | p(\bar{x}) = 1 \}$:

$$\text{res} G = \frac{1}{\pi^{N/2}} \int_{\Sigma_p} G(\bar{x}) \sum_{i=1}^{N} (-1)^{i-1} x^{i} d^{N-1} x = \prod_{i=1}^{N} (-1)^{i-1} x^{i} d^{N-1} x. \quad (2.8)$$

(a hat over an argument meaning, as usual, that this argument is omitted). The residue $\text{res} G$ is independent of the (transverse to the dilation) surface $\Sigma_p$ since the form in the integrant is closed in the projective space $P^{N-1}$.

We note that $N$ is even, in fact, divisible by 4, so that the $P^{N-1}$ is orientable.

\textbf{Remark 2.1.} The use of a (homogeneous) norm as a regulator appears to be more flexible than dimen-
sional regularization and should also be applicable in the presence of an axial (or chiral) anomaly.

The functional \( \text{res} G \) is a period according to the definition of [22, 25]. Such residues are sometimes called Feynman or quantum periods in the present context (see e.g. [29]). We shall use in what follows “residue” and “period” interchangeably.

The convention of accompanying the 4D volume \( d^4 x \) by a \( \pi^{-2} \) (\( 2\pi^2 \) being the volume of the unit sphere \( S^3 \) in four dimensions), reflected in the prefactor, goes back at least to Broadhurst [5, 7] and is adopted in [12, 29]; it yields rational residues for the graph \( \Gamma_a \) of Fig. 1 and for that of Fig. 2. For graphs with three or higher number of loops \( \ell (= h_1(\Gamma)) \), the first Betti number of the graph) one encounters, in general, multiple zeta values of overall weight not exceeding \( 2\ell - 3 \) (cf. [7, 29, 30]). If we write as above \( L = L(\Gamma) \) and \( V = V(\Gamma) \) for the numbers of internal lines and vertices of a connected graph \( \Gamma \) then \( \ell = L - V + 1 \) (\( = V - 1 \) for a connected 4-point graph in the \( \phi^4 \) theory). Examples of graphs with three and four loops are \( \Gamma_a \) on Fig. 1 and \( \Gamma_1 \) on Fig. 2. With the above choice of the 4D volume form the only residues at three, four and five loops (in the \( \phi^4 \) theory) are integer multiples of \( \zeta(3), \zeta(5) \) and \( \zeta(7) \), respectively. The first double zeta value, \( \zeta(5,3) \), appears at six loops (with a rational coefficient) (see the census in [29]). All known residues were (up to 2013) rational linear combinations of multiple zeta values [7, 29]. A seven loop graph was recently demonstrated [6, 28] to involve multiple Deligne values—i.e., values of hyperlogarithms at sixth roots of unity.

The self-energy graph \( \Gamma_1 \) on Fig. 2 is viewed as primitively (quadratically) divergent in the configuration space—while it is treated as a diagram with overlapping divergences in the traditional momentum space picture. Its renormalized contribution can be obtained from that of the logarithmically divergent 4-point graph at \( \Gamma_a \) of Fig. 1 using the identity (\( \Box \) stands for the 4-dimensional Laplace \( \delta \)-function and \( \ell \) is a free parameter.

As noted in the beginning—and illustrated in the above example—the extension of a homogeneous primitively divergent amplitude is no longer homogeneous. It satisfies instead an associate homogeneity condition which fixes the dilation anomaly; in particular,

\[
\lambda^N \Gamma_1(\lambda x) = \Gamma_1(x) + \frac{1}{4} \delta(x) \ln \lambda. \quad (2.10)
\]

The renormalized Feynman amplitude \( G(x_1, \ldots, x_4) \) of an arbitrary primitively divergent 4-point graph (with a single external half-line at each external vertex) is an associate homogeneous distribution (of order one):

\[
\lambda^{12} G(\lambda x_1, \ldots, \lambda x_4) = G(x_1, \ldots, x_4) + res(G) \delta(x_{12}) \delta(x_{23}) \delta(x_{34}) \ln \lambda. \quad (2.11)
\]

For graphs with subdivergences one first renormalizes the contributions of all primitively divergent subgraphs, then a similar procedure is applied to the resulting associate homogeneous amplitude (see [27], Sect. 4 and Appendix D.2). Remarkably, at each step one just solves a 1-dimensional problem. For a renormalized 4-point function with \( n \) (sub) divergences one then has an order \( n \) associate homogeneity law:

\[
\lambda^N G(\lambda \vec{x}) = G(\vec{x}) + \sum_{j=1}^{n} R_j(G)(\vec{x}) (\ln \lambda)^j / j!, \quad (2.12)
\]

where the distributions \( R_j(G) \) can be viewed as generalized residues:

\[
R_j(G) = Res \left( (\bar{\mathcal{E}} + N)^{-1} G(\vec{x}) \right). \quad (2.13)
\]

One proves that only the coefficient to the highest power of the logarithm,

\[
R_p(G) = res \left( (\bar{\mathcal{E}} + N)^{p-1} G(\vec{x}) \right) \bar{\mathcal{E}}, \quad (2.14)
\]

is independent of the ambiguity of renormalization (i.e., independent, in our case, of the scale parameters—like \( \ell \) in Eq. (2.9)). The standard normalization condition consists in fixing a zero of the Fourier transform of Feynman amplitudes. For instance, the Fourier transform of \( \Gamma_1(x) \) (2.9),

\[
\hat{\Gamma}_1(p) = \frac{p^2}{8} \ln \left( \frac{p^2}{\mu^2} \right) \quad (2.15)
\]

vanishes for \( p^2 = \mu^2 \) (while \( \Gamma_1(x) \) only vanishes for \( x \to \infty \)).
2.3. Renormalization of Associate Homogeneous Distributions; Causal Factorization

In order to treat the general case of a graph with subdivergences we shall define ultraviolet renormalization by induction with respect to the number of vertices. Assume that all contributions of diagrams with less than \( n \) points are renormalized. If then \( \Gamma \) is an arbitrary connected \( n \)-point graph its renormalized contribution should satisfy the following inductive causal factorization requirement.

Let the index set \( I = I(n) = \{1, \ldots, n\} \) of \( \Gamma \) be split into any two non-empty non-intersecting subsets \( I = I_1 \cup I_2(I_1 \neq \emptyset, I_2 \neq \emptyset), \quad I_1 \cap I_2 = \emptyset. \)

Let \( \mathcal{G}_{I_1,I_2} = \{(x_i) \in \mathbb{R}^4n = (\mathbb{R}^4)^n; \quad x_{j_1} \neq x_{j_2} \quad \text{for} \quad j_1 \in I_1, j_2 \in I_2 \} = \mathcal{G}_{I_1,I_2}^R \). Let further \( G_{I_1}^R \) and \( G_{I_2}^R \) be the renormalized distributions associated with the subgraphs whose vertices belong to the subsets \( I_1 \) and \( I_2 \), respectively. We demand that for each such splitting our euclidean distribution \( \mathcal{G}_{I_1,I_2}^R \), defined on all partial diagonals, exhibits the factorization property:

\[
G_{I}^R = G_{I_1}^R \prod_{i \in I_2} G_{ij}^R G_{I_2}^R \quad \text{on} \quad \mathcal{G}_{I_1,I_2}^R, \quad (2.16)
\]

where \( G_{ij} \) are factors (of type (2.1)) in the rational function \( G_{I} \), which are understood as multiplicators on \( \mathcal{G}_{I_1,I_2}^R \).

**Remark 2.2.** In the Lorentzian signature case one demands that the points indexed by the set \( I_1 \) precede those of \( I_2 \) and uses Wightman functions instead of \( G_{ij} \) in the counterpart of (2.16)—thus justifying the term causal (see Sect. 2.2 of [27]).

We shall add to this basic physical requirement two more mathematical conventions (MC) which will substantially restrict the notion of renormalization used in this paper.

(MC1) Renormalization maps rational homogeneous functions onto associate homogeneous distributions of the same degree of homogeneity; it extends associate homogeneous distributions defined off the small diagonal to associate homogeneous distributions of the same degree (but possibly of higher order) defined everywhere on \( \mathbb{R}^N \).

(MC2) The renormalization map commutes with multiplication by polynomials. If we extend the class of our distributions by allowing multiplication with smooth functions of no more than polynomial growth (in the domain of definition of the corresponding functionals), then this requirement will imply commutativity of the renormalization map with such multipliers.

The induction is based on the following.

**Proposition 2.2.** The complement \( C(\Delta_n) \) of the small diagonal is the union of all \( \mathcal{G}_{I_1,I_2} \) for all pairs of disjoint \( I_1, I_2 \) with \( I_1 \cup I_2 = \{1, \ldots, n\} \), i.e.,

\[
C(\Delta_n) = \bigcup_{I_1 \cup I_2 = \{1, \ldots, n\}} \mathcal{G}_{I_1,I_2}.
\]

**Proof.** Let \( (x_1, \ldots, x_n) \in C(\Delta_n) \). Then there are at least two different points \( x_i \neq x_j \). We define \( I_1 \) as the set of all indices \( i \) of \( I = I(n) \) for which \( x_i \neq x_j \) and \( I_2 := I \setminus I_1 \). Hence, \( C(\Delta_n) \) is included in the union of all such pairs. Each \( \mathcal{G}_{I_1,I_2} \), on the other hand, is defined to belong to \( C(\Delta_n) \). This completes the proof of our statement.

The first step in implementing the above inductive procedure consists in the renormalization of primitively divergent graphs surveyed in Sect. 2.2. Schnetz’s notion of completion of 4-point graphs, reviewed in the next subsection, offers a general picture of primitively divergent graphs in the \( \phi^4 \)-theory.

2.4. Vacuum Completion of Four-Point Graphs

Following Schnetz [29, 30] we associate to each 4-point graph \( \Gamma \) of the \( \phi^4 \) theory a completed vacuum graph \( \bar{\Gamma} \), obtained from \( \Gamma \) by joining all four external lines in a new vertex “at infinity”. An \( n \)-vertex 4-regular vacuum graph—having four edges incident with each vertex and no tadpole loops—gives rise to \( n \) 4-point graphs (with \( n \) vertices each) corresponding to the \( n \) possible choices of the vertex at infinity. The introduction of such completed graphs is justified by the following result (see Sect. 2.3, Proposition 2.6 and Sect. 2.4, Theorem 2.7 of [29] as well as Sect. 3.1 below).

**Proposition 2.3.** A 4-regular vacuum graph \( \bar{\Gamma} \) with at least three vertices is said to be completed primitive if the only way to split it by a four edge cut is by splitting off one vertex. A 4-point Feynman amplitude corresponding to a connected 4-regular graph \( \Gamma \) is primitively divergent iff its completion \( \bar{\Gamma} \) is completed primitive. All 4-point graphs with the same primitive completion have the same residue.

There are infinitely many primitive 4-point graphs while there is a single primitive 2-point graph: the self-energy graph \( \Gamma_1 \) of Fig. 2 (Proposition 3.1 below). The only primitive 4-point graph with a rational period is the one loop graph \( \Gamma_5 \) of Fig. 1 (with residue 1). The n loop zig-zag graph [12] has a residue that is a rational multiple of \( \zeta(2n-3), n = 3, 4, \ldots \). The first two zig-zag diagrams are the graphs \( \Gamma_5 \) of Fig. 1 and \( \Gamma_4 \) on Fig. 3. Their residues are \( \sum_{n=3}^{4} \zeta(2n-3), n = 3, 4 \) (see [31] for an elementary derivation and further references).
3. Renormalized Position Space Amplitudes

3.1. Primitively Divergent $\varphi^4$ Graphs. Conformal Anomaly

We first note that the primitively divergent vacuum graphs of the $\varphi^4$ theory have either two or four external legs. The only 4-regular vacuum graph with three vertices is the completion $\Gamma_{a}$ of $\Gamma_a$ (Fig. 1). Calling a vacuum graph simple if it contains at most one edge joining any two of its vertices, one can prove that $\Gamma_{a}$ is the only non-simple completed primitive graph.

**Proposition 3.1.** The only primitively divergent 2-point Feynman amplitude corresponds to the graph $\Gamma_1$ of Fig. 2.

**Proof.** Cutting off an external vertex of a given 2-point graph $\Gamma$ we obtain a 4-point graph that is the trivial single vertex graph for $\Gamma = \Gamma_1$. The Proposition then follows from the following simple fact about 4-point graphs.

**Lemma 3.2.** Each non-trivial connected 4-point graph of the $\varphi^4$ theory is either primitive logarithmically divergent or contains a subdivergence.

The Lemma follows from the fact that for a connected 4-point 4-regular graph the number of internal lines is $L = 2(V - 1)$ and hence the superficial degree of divergence is $\kappa = 2L - 4V + 1 = 0$.

**Proposition 3.3.** The period of a completed primitive graph $\Gamma$ is equal to the residue of each 4-point graph $\Gamma = \Gamma_{a}$ (obtained from $\Gamma$ by cutting off an arbitrary vertex $v$). The resulting common period can be evaluated from $\Gamma_{a}$ by choosing arbitrarily three vertices $\{0, e, s.t. e^2 = 1, \infty\}$, setting all propagators corresponding to edges of the type $(x, \infty)$ equal to 1 and integrating over the remaining $n - 2$ vertices of $\Gamma$ ($n = V(\Gamma)$):

\[
\text{Per}(\Gamma) \equiv \text{res}(\Gamma) \int \Gamma_{a}(e, x_2, \ldots, x_{n-1}, 0) \prod_{i=2}^{n-1} \frac{d^4 x_i}{\pi^2}. \tag{3.1}
\]

**Sketch of proof.** For a given choice of the vertex at infinity (3.1) follows from (2.7). The independence of the choice of the point at infinity follows from conformal invariance; the conformal inversion $I_r : x_i \rightarrow \frac{x_i}{x_i^2}$, $i = 2, \ldots, n$, exchanges the (arbitrarily chosen) $x_1 = 0$ and $\infty$ while the integral remains invariant since

\[
I_r : \frac{1}{x_i^2} \rightarrow \frac{x_i^2}{x_i^2}, \quad d^4 x \rightarrow d^4 x \left(\frac{x^2}{x_i^2}\right)^4. \tag{3.2}
\]

The conformal invariance is broken in a controllable way by renormalization. For a special conformal transformation

\[
g(x) = \frac{x + cx^2}{\omega(c, x)}, \quad \Delta g(x) = \frac{d^2 x^2}{\omega(c, x)^2}, \quad \omega(c, x) = 1 + 2cx + c^2 x^2 \tag{3.3}
\]

one obtains the conformal anomaly by substituting $\lambda$ in (2.11) by $\frac{1}{\omega(c, x)}$ for any $i \in (1, 2, 3, 4)$. The $\delta$-function ensures that the result is independent of the choice of $i$. The cocycle condition that implements the group law is satisfied because of the identity

\[
\omega(c_1 + c_2, x) = \omega(c_1, x)\omega(c_2, g_{c_1}, x). \tag{3.4}
\]

3.2. Associate Homogeneity Law for Amplitudes with Subdivergences

The study of graphs with a 2-point subdivergence requires the computation of the dressed propagator $G_i$ of Fig. 2:

\[
G_i(x_{12}) = \int \frac{d^4 x_1}{\pi^2} \frac{d^4 y}{\pi^2} \frac{\Gamma_i(x - y)}{(x_1 - x)^2(y - x_2)^2}. \tag{3.5}
\]

An intelligent way to compute $G_i$ consists in using (2.12) to first derive its dilation law:

\[
\lambda^2 G_i(\lambda x_{12}) = G_i(x_{12}) - \frac{\ln \lambda}{\pi^2 x_{12}}. \tag{3.6}
\]

where we have used the fact that the integrand in (3.5) involves the Green function of the 4D Laplacian:

\[
\Box \frac{1}{4\pi^2 (x_1 - x)^2} = \delta(x_1 - x). \tag{3.7}
\]

The general form of $G_i(x)$ satisfying (3.6) is:

\[
G_i(x) = \frac{\ln \left(\frac{x^2}{x_{12}^2}\right)}{2\pi^2 x^2}. \tag{3.8}
\]

We observe that (convergent) integration over internal vertices preserves the order of associate homogeneity of the integrand. The power of the logarithm only increases if one encounters another ultraviolet divergence (typically in an UV divergent graph with a subdivergence) as illustrated by the amplitude $St(x)$ corresponding to the “stye graph” displayed on Fig. 4:

\[
St(x) = \frac{G_i(x)}{x^2} \text{ for } x \neq 0. \tag{3.9}
\]

The extension of the distribution $St$ to the entire $\mathbb{R}^4$ is again reduced to an 1-dimensional problem by
integrating the corresponding density with respect to the angles:

\[ r^3 \int_{\mathbb{S}^3} S(t(\omega) \frac{d^3 \omega}{\pi^2} = \frac{2}{r} \ln \frac{r}{r_0} dr. \]

Its general associate homogeneous extension, the renormalized stye 1-form \( \mathcal{R}(S) \) is:

\[ \mathcal{R}(S) := r^3 \int_{\mathbb{S}^3} S(t(\omega) \frac{d^3 \omega}{\pi^2} \bigg| \ln \frac{r}{r_0} - \ln \frac{r}{r_0} \bigg) \bigg| \Theta(r), \]

where \( \ell' > 0 \) is another scale and \( \Theta(r) \) is the Heaviside step function. (For \( \ell' = \ell \) we recover the extension given by Proposition A.1 of [27].) The associate homogeneity law for \( \mathcal{R}(S) \) reads:

\[ \mathcal{R}(S)(\lambda r) = \mathcal{R}(S)(r) - 2d \left( \ln \frac{r}{\ell'} \right) \ln \lambda - \Theta(r)(\ln \lambda)^2. \]

We see that the term with \( (\ln \lambda)^2 \) is indeed independent of the ambiguity \( (\ell) \), while the coefficient to \( \ln \lambda \) is \( 2d \left( \ln \frac{r}{\ell'} \right) \) and thus depends on \( \ell \).

We shall consider the case of 4-point subdivergences within our treatment of 4-point functions in the \( \phi^4 \) theory in the next subsection.

### 3.3. Four-Point \( \phi^4 \) Amplitudes

#### as Conformal Invariant Functions

of Four Complex Variables

Every four points, \( x_1, \ldots, x_4 \), can be confined by a conformal transformation to a 2-plane (for instance by sending a point to infinity and using translation invariance). Then we can represent each point \( x_i \) by a complex number \( z_i \) such that:

\[ x_i^2 = [z_i] = (z_i - \bar{z}_i)(\bar{z}_i - z_i). \]

To make the correspondence between 4-vectors \( x \) and complex numbers \( z \) explicit we fix a unit vector \( e \) and let \( n \) be a variable unit vector parametrizing a 2-sphere orthogonal to \( e \). Then any euclidean 4-vector \( x \) can be written (in spherical coordinates) in the form:

\[ x = r(\cos \rho e + \sin \rho n), \quad e^2 = 1 = n^2, \quad 0 \geq \rho \geq 0, \quad 0 \leq \rho \leq \pi. \]

The 4D volume element is written in these coordinates as

\[ d^4x = r^3 dr \sin^2 \rho d\rho d^2n, \quad \int d^2n = 4\pi. \]

We associate with the vector \( x \) (3.13) a complex number \( z \) such that:

\[ z = re^{i\theta} \rightarrow x^2(= r^2) = z\bar{z}, \quad (x - e)^2 = |z - \bar{z}|^2 = (1 - z)(1 - \bar{z}) \]

\[ \int_{\mathbb{S}^2} d^4x = z - \bar{z}^2 \frac{d^2z}{\pi}. \]

As massless Feynman integrands give rise to well defined functions for non-coinciding arguments the \( \phi^4 \) primitively divergent 4-point amplitudes are conformally covariant for such arguments. For a graph with four distinct external vertices such an amplitude (integrated over the internal vertices) has scale dimension 12 (in mass or inverse length units) and can be written in the form:

\[ G(x_1, \ldots, x_4) = g(u, v) = \prod_{i<j} \frac{F(z)}{|z_i - z_j|^2} \]

where the indices run in the range \( 1 \leq i < j \leq 4 \), the (positive real) variables \( u, v \), and (the complex) \( z \) are conformally invariant crossratios:

\[ u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = z\bar{z}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = |z - \bar{z}|^2, \]

\[ z = \frac{z_{12} z_{34}}{z_{13} z_{24}}. \]

We shall outline how one can compute the amplitude \( G_4 \) corresponding to the graph \( \Gamma_4 \) of Fig. 3 in terms of the variables \( z \) (see [30, 31]). We shall then use the integral in (2.2),

\[ I(x_1, \ldots, x_4) := \frac{1}{x_{12}^2 x_{34}^2 x_{23}^2 x_{14}^2} G_4(x_1, \ldots, x_4) \]

\[ = \int_{\mathbb{S}^2} \frac{d^4x}{(x_r - x_i)^2} \]

to calculate the dilation anomaly of a 4-point graph with a primitive 4-point subdivergence. According to (3.17) and (3.18) this amounts to evaluating the conformal invariant amplitude \( F(z) = x_{13} x_{24} I(x_1, \ldots, x_4) \).

There are two ways to do it: one may either expand the original \( x \)-space integrand in Gegenbauer polynomials [13] (after sending \( x_i \) to infinity) or use the theory of single-valued multiple polylogarithms [9, 30] (for a
pedagogical derivation and more references—see [31]). The result is:

\[
F(z) = \frac{2L_i(z) - 2L_i(z_\infty)}{z - z_\infty} + \ln z_\infty \ln \frac{1 - z}{1 - z_\infty} = \frac{4iD(z)}{z - z_\infty},
\]

(3.21)

where \(D(z)\) is the Bloch–Wigner single-valued dilogarithm [1, 33]. The real valued function

\[
\tilde{D}(z_1, z_2, z_3, z_4) = D\left(\frac{z_{1234}}{z_{1324}}\right)
\]

(3.22)

is known since Lobachevsky to give the volume of the (oriented) ideal tetrahedron with vertices \(z_{1}, \ldots, z_{4}\) on the absolute (horosphere) of the 3-dimensional hyperbolic space [24]. It has the symmetry of a dimensionalless fermionic 4-point function (in a logarithmic conformal field theory). It is symmetric under even permutations and changes sign under odd permutations of the arguments \(z_{1}, \ldots, z_{4}\). The complete 4-point integral \(I\) can be written in the form

\[
I = \tilde{J}(z_1, \ldots, z_4) = \int d^2z \frac{z - z_1}{z - z_i} \prod_{i=1}^{4} |z_i - z|^2
\]

\[
= \frac{4iD(z)}{z - z_1} \prod_{i=1}^{4} |z_i - z|^2
\]

(3.23)

(3.24)

implies that the dilation anomaly of \(G\) for non-coinciding arguments is

\[
\lambda^{12}G_S(\lambda x_1, \ldots, \lambda x_4) - G_S(x_1, \ldots, x_4) = G_4(x_1, \ldots, x_4)\text{res}(S)\text{ln } \lambda,
\]

(3.25)

where \(G_4\) is given by (2.2). It follows that the coefficient \(\text{res}(G_4)\) to \((\text{ln } \lambda)^2\) which is independent of the renormalization ambiguity is given by the product of residues:

\[
\text{res}_2(G_4) = \text{res}(G_4)\text{res}(S) (\text{res}(G_4) = 20\zeta(5)).
\]

(3.26)

The symmetry of the ratio (3.22) allows to determine its behaviour for various pairs of coinciding arguments by just considering the limit in which one of them, say \(z_{1234}\), is small:

\[
\tilde{J}(z_1, \ldots, z_4) \sim |z_{1324}|^{-2} \partial_z (P_{10} - P_{01}) \mathcal{I}_{00},
\]

(3.27)

Thus \(J\) only has logarithmic singularities; it follows that any finite power of the function (3.22) is locally integrable and hence defines a distribution in the whole space \(\mathbb{C}^3\).

Let \(S(y_1, \ldots, y_4)\) be a renormalized primatively divergent 4-point amplitude that appears as a sub-divergence in

\[
G_S(x_1, \ldots, x_4) = \int S(y_1, \ldots, y_4) \prod_{i=1}^{4} \frac{d^4y_i}{\pi} (x_i - y_i)^2.
\]

4. OUTLOOK

Quantum field theory which once signaled, according to Freeman Dyson [15], a divorce between mathematics and physics, now seems to be the best common playground of the two sciences. Not only did renormalization theory, which was viewed as a liability, become respectable in the Epstein–Glaser approach, but the key role, which the notion of residue and the applications of the (Hopf) algebra of hyperlogarithms play in it, relates it to current work in algebraic geometry and number theory (see e.g. [2, 3, 8, 11]).
Hörmander [21]). Here we complete this study in the case of the conformally invariant (at least at the classical level) euclidean $\phi^4$ theory by including integration over internal vertices.

There seems to exist a parallel between the study of massless QFT and neglecting friction by the founding fathers of classical mechanics starting with Galileo. Such an idealization made it easier to find the simple basic laws of mechanics. Taking subsequently the corrections due to friction into account just added minor technical details to the general picture. We feel that at least as far as UV renormalization is concerned, the role of residues (that also appear in the renormalization group beta function) and their relation to modern study of periods in number theory, taking masses into account will not change substantially the overall picture and can be advantageously postponed to a later stage (hadronic masses appearing, without having been put in, as a result of the strong interaction—cf. [32]).

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