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A SPECTRAL RESULT FOR HARDY INEQUALITIES

BAPTISTE DEVYVER

Abstract. Let $P$ be a linear, elliptic second order symmetric operator, with an associated quadratic form $q$, and let $W$ be a potential such that the Hardy inequality

$$\lambda_0 \int_{\Omega} W u^2 \leq q(u)$$

holds with (non-negative) best constant $\lambda_0$. We give sufficient conditions so that the spectrum of the operator $\frac{1}{W}P$ is $[\lambda_0, \infty)$. In particular, we apply this to several well-known Hardy inequalities: (improved) Hardy inequalities on a bounded convex domain of $\mathbb{R}^n$ with potentials involving the distance to the boundary, and Hardy inequalities for minimal submanifolds of $\mathbb{R}^n$.

1. Introduction

Let $P$ be a linear elliptic, second order, symmetric, non-negative operator on a domain $\Omega$, and let $q$ be the quadratic form associated to $P$. Following Carron [12] and Tertikas [30], we will call Hardy inequality for $P$ with weight $W \geq 0$ and constant $\lambda > 0$, the following inequality:

$$\lambda \int_{\Omega} W u^2 \leq q(u), \forall u \in C^\infty_0(\Omega).$$

(1.1)

We denote by $\lambda_0 = \lambda_0(\Omega, P, W)$ the best constant $\lambda$ for which inequality (1.1) is valid. By convention, if (1.1) does not hold for any $\lambda > 0$, we will let $\lambda_0 = 0$. The inequality (1.1) aims to quantify the positivity of $P$: for instance, inequality (1.1) with $W \equiv 1$ is equivalent to the positivity of the bottom of the spectrum of (the Friedrichs extension of) $P$.

Let us give a celebrated example of Hardy inequality for $P = -\Delta$, which will be a guideline for us in this paper (see [20] for the convex case, and [7], [22] for the mean convex case):

Example 1.1. If $\Omega$ is a $C^2$, bounded, mean convex domain of $\mathbb{R}^n$, and $\delta$ is the distance to the boundary of $\Omega$, then

$$\frac{1}{4} \int_{\Omega} \frac{u^2}{\delta^2} \leq \int_{\Omega} |\nabla u|^2, \forall u \in C^\infty_0(\Omega).$$

(1.2)

We recall that $\Omega$ is called mean convex if the mean curvature of its boundary is non-negative.
Let us return to the general case.

1.1. The best constant and the existence of minimizers. A natural question is, for a given weight \( W \geq 0 \) such that the Hardy inequality (1.1) holds, to compute the best constant \( \lambda_0 \) and to discuss whether \( \lambda_0 \) is attained by a minimizer in the appropriate space or not. More precisely, define \( D^{1,2} \) to be the completion of \( C_0^\infty(\Omega) \) with respect to the norm \( \sqrt{q} \). The variationnal problem associated to the Hardy inequality (1.1) is

\[
\lambda_0 = \inf_{u \in D^{1,2}(\Omega)} \frac{q(u)}{\int_\Omega Wu^2}.
\]

(1.3)

If (1.3) is not realized by a function in \( D^{1,2} \), we will say that the Hardy inequality (1.1) with best constant \( \lambda_0 \) does not have a minimizer.

Another interesting quantity, related to the existence of minimizers, is the best constant at infinity. It is defined as follows (see [2], [20] and [15]):

**Definition 1.2.** The best constant at infinity \( \lambda_\infty = \lambda_\infty(\Omega, P, W) \) is the supremum of the set of \( \alpha \geq 0 \) such that

\[
\alpha \int_{\Omega \setminus K_\alpha} Wu^2 \leq q(u), \quad \forall u \in C_0^\infty(\Omega \setminus K_\alpha),
\]

for some \( K_\alpha \subset \subset \Omega \) compact subset of \( \Omega \).

Closely related notions have been introduced (under different names) in [30] and [17].

**Example 1.3** (Example 1.1, continued). For the Hardy inequality (1.2) with \( W = \frac{1}{\delta^2} \), the best constant and the best constant at infinity are both equal to \( \frac{1}{4} \). Furthermore, inequality (1.2) does not have a minimizer.

1.2. Improving Hardy inequalities. A very natural question is the following: for the Hardy inequality (1.1) with the best constant \( \lambda_0 \), can we improve it by adding another non-negative potential \( V \) to the left-hand side, i.e. is there a positive constant \( \mu \) and a non-negative, non-zero potential \( V \) such that

\[
\lambda_0 \int_\Omega Wu^2 + \mu \int_\Omega Vu^2 \leq q(u), \quad \forall u \in C_0^\infty(\Omega)
\]

(1.4)

Of course, such an improvement – if it exists – is not at all unique, and finding a potential \( V \) which is “as large as possible” is important.

Results by Filippas-Tertikas [17], Agmon [2], Pinchover [23], [25], Marcus-Mizel-Pinchover [20], Pinchover-Tintarev [27] among others show that the best constant at infinity, as well as the existence of minimizers, play an important role in this problem: indeed, a general result obtained by Agmon [2] (see also Pinchover [25], Lemma 4.6 for an easier and more general proof) shows that if the best constant at infinity is strictly greater than the best
constant in the inequality (1.1), then no improvement by a non-negative, non-zero potential $V$ is possible.

Also, concerning the minimizers, Pinchover-Tintarev show (Lemma 1.1 in [27]) that if $W > 0$ and if there is a minimizer for the Hardy inequality (1.1), then no improvement by a non-negative, non-zero potential $V$ is possible. In fact, if a minimizer exists then it is a ground state (in the sense of Agmon) of $P - W$.

The possibility of adding a potential $V \geq 0$ in the left-hand side of the Hardy inequality (1.1) with best constant $\lambda_0$ has to do with the criticality of the operator $P - \lambda_0 W$: there exists $V \geq 0$ such that the improved inequality (3.5) is valid (with $\mu = 1$) if and only if $P - \lambda_0 W$ is subcritical in $\Omega$. See for example [15] for details and references on this.

In the case that the best constant at infinity is strictly larger than the best constant, or that there is a minimizer, the potential $W$ “does not grow fast enough at infinity” in a certain sense. Let us illustrate this by the following example, related to Example 1.1:

**Example 1.4.** Let $P = -\Delta$ on a smooth, bounded, mean convex domain $\Omega$, and let us define, for $\alpha \in \mathbb{R}$,

$$W_\alpha = \delta^{-\alpha},$$

where we recall that $\delta$ is the distance to the boundary of $\Omega$. Then,

$$\lambda_\infty(\Omega, -\Delta, W_\alpha) = \begin{cases} +\infty, & \alpha \in (-\infty, 2) \\ \frac{1}{2}, & \alpha = 2 \\ 0, & \alpha \in (2, +\infty) \end{cases}$$

More generally, if $W$ is a small perturbation of $P$ (see [24]), then $\lambda_\infty(\Omega, P, W) = \infty$.

**Definition 1.5.** A non-negative potential $W$, satisfying the Hardy inequality (1.1), is said to belong to the class of admissible potentials $\mathcal{A}(\Omega, P)$ if

$$\lambda_0(\Omega, P, W) = \lambda_\infty(\Omega, P, W),$$

and if there is no minimizer of the associated variational problem (1.3). When the dependence with respect to $\Omega$ and $P$ will be clear, we will write $\mathcal{A}$ instead of $\mathcal{A}(\Omega, P)$.

For instance, it follows from the results of Marcus-Mizel-Pinchover [20] that if $\Omega$ is a $C^2$, bounded, mean convex domain, then the potential $\delta^{-2}$ belongs to the class $\mathcal{A}(\Omega, -\Delta)$. In this article, we will focus on Hardy inequalities (1.1) with admissible potentials. A Hardy inequality (1.1) with admissible potential can sometimes be improved, but not always. Actually, if $W$ is admissible, deciding whether the Hardy inequality (1.1) with best constant $\lambda_0$ can be improved or not, is a delicate question. We define a subclass of $\mathcal{A}$:
Definition 1.6. A non-negative potential \( W \), satisfying the Hardy inequality (1.1), is said to belong to the class of *optimal* potentials \( \mathcal{O}(\Omega, P) \) if \( W \in \mathcal{A}(\Omega, P) \) and if the Hardy inequality (1.1) with best constant \( \lambda_0 \) cannot be improved, i.e. if there is no \( V \geq 0, \mu > 0 \) such that inequality (3.5) holds. Equivalently, \( W \in \mathcal{O}(\Omega, P) \) if and only if \( W \in \mathcal{A}(\Omega, P) \) and \( P - \lambda_0 W \) is critical (see [28]). Furthermore, \( P - \lambda_0 W \) is critical if and only if it does not have a ground state in the sense of Agmon (see [28]); this of course includes an obvious case when such a ground state is a true minimizer).

Example 1.7. The potential \( \frac{1}{|x|^2} \) is an optimal potential for \( P = -\Delta \) on \( \mathbb{R}^n \) (or equivalently, on \( \mathbb{R}^n \setminus \{0\} \)) for \( n \geq 3 \) (see [15] for a short proof). Indeed, the operator \( -\Delta - \left( \frac{n-2}{2} \right)^2 \frac{1}{|x|^2} \) is critical and has ground state \( |x|^2 - \frac{n}{2} \). Recall that the potential \( \frac{1}{|x|^2} \) appears in the classical Hardy inequality in \( \mathbb{R}^n \), \( n \geq 3 \) with best constant:

\[
\left( \frac{n-2}{2} \right)^2 \int_{\mathbb{R}^n} \frac{u^2(x)}{|x|^2} \leq \int_{\mathbb{R}^n} |\nabla u|^2, \quad \forall u \in C_0^\infty(\mathbb{R}^n).
\]

However, the Hardy inequality (1.2) of Example 1.1 can be improved. The first improvement of inequality (1.2) was obtained by Brezis and Vazquez [9], for \( V \equiv 1 \), however \( \frac{1}{|x|^2} \notin \mathcal{A}(\Omega, -\Delta - \frac{1}{4|x|^2}) \) – in fact, \( V \equiv 1 \) is a small perturbation of \( -\Delta - \frac{1}{4|x|^2} \), and thus \( \lambda_\infty(\Omega, -\Delta - \frac{1}{4|x|^2}, 1) = \infty \). Later, an improvement by a potential in the class \( \mathcal{A}(\Omega, -\Delta - \frac{1}{4|x|^2}) \) was obtained by Brezis and Marcus [8]. Let us introduce the normalized logarithm function, defined by

\[
X_1(t) := (1 - \log t)^{-1}.
\]

A consequence of the work of Brezis and Marcus is the following:

Example 1.8. Let \( \Omega \) be a smooth, bounded, mean convex domain, then we have the improved Hardy inequality

\[
\frac{1}{4} \int_{\Omega} \frac{u^2}{\delta^2} X_1^2 \left( \frac{\delta}{D} \right) \leq \int_{\Omega} |\nabla u|^2 - \frac{1}{4} \int_{\Omega} \frac{u^2}{\delta^2}, \quad \forall u \in C_0^\infty(\Omega), \quad (1.5)
\]

where \( D \) is any constant such that

\[
D \geq \sup_{x \in \Omega} \delta(x).
\]

Furthermore, \( \frac{1}{4} \) is the best constant and the best constant at infinity, and there is no minimizer. In particular, \( \delta^{-2} X_1^2 \left( \frac{\delta}{D} \right) \) belongs to the class \( \mathcal{A}(\Omega, -\Delta - \frac{1}{4|x|^2}) \).

More recently, Barbatis, Filippas and Tertikas [6] have obtained a series of successive improvements of the Hardy inequality (1.2) with admissible...
potentials, generalizing the improved Hardy inequality (1.5) obtained by Brezis and Marcus. In order to present their result, let us define for $i \geq 1$,

$$X_i(t) := X_i(X_{i-1}(t)),$$

and by convention $X_0 \equiv 1$. Let us also define

$$W_i := \frac{1}{4\delta^2} \left( \sum_{k=0}^{i} X_k^2 \left( \frac{\delta}{D} \right) \right),$$

and

$$J_i := W_i - W_{i-1} = \frac{1}{4\delta^2} \left( \sum_{k=0}^{i} X_k^2 \left( \frac{\delta}{D} \right) \right).$$

The result of Barbatis, Filippas and Tertikas obtained in [6] is presented in the following example:

**Example 1.9.** Let $\Omega$ be a $C^2$, bounded, mean convex domain, then we can choose $D > \sup_{x \in \Omega} \delta(x)$ big enough (see Remark 3.4) so that for every $i \geq 1$, the following improved Hardy inequality holds

$$\int_{\Omega} J_i u^2 \leq \int_{\Omega} |\nabla u|^2 - \int_{\Omega} W_{i-1} u^2, \forall u \in C_0^\infty(\Omega).$$

(1.6)

Furthermore, 1 is the best constant and the best constant at infinity, and there is no minimizer. In particular, $J_i$ belongs to the class $A(\Omega, -\Delta - W_{i-1})$.

1.3. The spectrum of $\frac{1}{W}P$. In this subsection, we collect some results concerning the spectrum of operators of the type $\frac{1}{W}P$. For more details on this, see [2], Section 3 in [20], or Proposition 4.2 in [15].

If the potential $W$ in the Hardy inequality (1.1) is positive, then there is a spectral interpretation of the best constant $\lambda_0(\Omega, P, W)$ and of the best constant at infinity $\lambda_\infty(\Omega, P, W)$. Consider (the Friedrichs extension of) the operator $\frac{1}{W}P$, which is self-adjoint on $L^2(\Omega, Wdx)$. Then $\lambda_0(\Omega, P, W)$, the best constant in (1.1) is the infimum of the spectrum of $\frac{1}{W}P$, and $\lambda_\infty(\Omega, P, W)$, the best constant at infinity is the infimum of the essential spectrum of $\frac{1}{W}P$ (the result concerning $\lambda_\infty$ comes from the Persson’s formula, see [21], [3]).

As an immediate consequence, the following result holds:

**Lemma 1.10.** Assume that $\lambda_\infty(\Omega, P, W) = +\infty$. Then the spectrum of $\frac{1}{W}P$ is discrete.

**Example 1.11.** Let $\Omega$ be a smooth, bounded domain. Then for every $\alpha \in (2, \infty)$, the spectrum of $\delta^\alpha$ is discrete. Indeed, for every $\alpha \in (2, \infty)$, $\lambda_\infty(\Omega, -\Delta, \delta^{-\alpha}) = \infty$. 
Also, there is a minimizer of the variational problem (1.3) if and only if \( \lambda_0(\Omega, P, W) \) is an eigenvalue of \( 1_WP \). In [20], p.3246, the authors attributed the following result to Agmon:

**Claim 1.12 (Agmon).** On a smooth, bounded, mean convex domain, the spectrum of \( \delta^2(-\Delta) \) is \( \left[ \frac{1}{4}, \infty \right) \). Furthermore, without the mean convexity assumption on the domain, the essential spectrum of \( \delta^2(-\Delta) \) is \( \left[ \frac{1}{4}, \infty \right) \).

To the author’s knowledge, Agmon never published this result, but proved a closely related result in [1]. The validity of Agmon’s claim 1.12 implies at once that if \( \Omega \) is a smooth, bounded, mean convex domain, the best constant and the best constant at infinity for the Hardy inequality (1.1) are both equal to \( \frac{1}{4} \). Agmon’s claim has to be compared to Example 1.11.

### 1.4. Optimal Hardy inequalities and the supersolution construction

In the article [15], starting from a general subcritical operator \( P \) on a punctured domain \( \Omega^* = \Omega \setminus \{0\} \), we have constructed an optimal potential \( W \) satisfying the Hardy inequality (1.1) with best constant 1. This is actually a generalization of Example 1.7, since for the case \( P = -\Delta \) with \( \Omega = \mathbb{R}^n \), \( n \geq 3 \), the constructed potential \( W \) is equal to \( \left(\frac{n-2}{2}\right)^2 \frac{1}{|x|^2} \). The most remarkable property of these constructed potentials \( W \) is the criticality of \( P - \lambda_0W \): as we have mentioned above, if \( W \in A(\Omega, P) \), the criticality of \( P - \lambda_0W \) is a delicate property.

These optimal potentials are obtained through a construction that we have called in [15] the *supersolution construction*, and that we recall now:

**Proposition 1.13 (Supersolution construction, see [15], Lemma 5.1 and Corollary 5.2).** Assume that \( P \) is subcritical in \( \Omega \), and that there exist \( u_0 \) and \( u_1 \) two linearly independent, positive supersolutions of \( Pu = 0 \), in \( \Omega \setminus \{0\} \), and if moreover

\[
\lim_{x \to \infty} \frac{u_0(x)}{u_1(x)} = 0, \quad \lim_{x \to 0} \frac{u_0(x)}{u_1(x)} = \infty,
\]

then \( W(u_0, u_1) \) is an optimal potential (having a singularity at 0). Actually, \( u_0 \) as above is necessarily the minimal, positive Green function of \( P \) with pole 0. These optimal potentials obtained via the supersolution construction have another interesting property: if \( W(u_0, u_1) \) is positive, then the spectrum of the operator \( \frac{1}{W(u_0, u_1)}P \) is \( [1, \infty) \). The proof of this last fact relies on the existence of generalized eigenfunctions of exponential type for
However, the optimal potentials \( W(u_0, u_1) \) have two drawbacks: first, the existence of two linearly independent positive solutions \( u_0, u_1 \) of \\

\[
Pu_i = 0
\]

does not always hold: one generally needs to remove one point in \( \Omega \) in order to guarantee this condition, and the potential obtained will have a singularity at that point. Secondly, finding the asymptotic of \( W(u_0, u_1) \) at infinity, or even a lower bound of \( W(u_0, u_1) \) in order to get a more explicit Hardy inequality, is a difficult problem. Actually, it is an open problem to give sufficient conditions guaranteeing that \( W(u_0, u_1) \) is positive in a neighborhood of infinity. This is an interesting question, since the interpretation of the best constant and the best constant at infinity in term of the spectrum of the operator \( \frac{1}{W(u_0, u_1)} P \) holds only if \( W(u_0, u_1) \) is positive.

It is sometimes more natural to work with non-optimal (but good enough) potentials given by an explicit formula. In the supersolution construction, instead of two positive solutions \( u_i \) of \( Pu = 0 \), one can take two (well chosen) positive supersolutions. For example, the potential \( \frac{1}{\delta^2} \) of the Hardy inequality of Example 1.1 is obtained by applying the supersolution construction with \( u_0 = 1 \) and \( u_1 = \delta \) (which is a supersolution, but not a solution of \( P = -\Delta \)). Furthermore, the potentials of the Examples 1.1, 1.8 and 1.9 are not optimal, because the corresponding Hardy inequality can be improved, yet they are admissible potentials. We will also see (in Proposition 3.2) that each improvement of the Hardy inequality of Example 1.9 can also be obtained using the supersolution construction, with explicit supersolutions \( u_0 \) and \( u_1 \) which are functions of \( \delta \). It is thus tempting to ask whether the property that the spectrum of \( \frac{1}{W(u_0, u_1)} P \) is \([1, \infty)\), valid in the case of the optimal potentials obtained in [15], remains true for only admissible potentials (that is, potentials in the class \( \mathcal{A}(\Omega, P) \). In particular, it is interesting to ask the following question:

**Question 1.14.** For the Hardy inequalities of Examples 1.1 and 1.9 obtained respectively by Brezis-Marcus and Barbati-Filippas-Tertikas, does it hold that the spectrum of \( \frac{1}{W(u_0, u_1)} P \) is \([1, \infty)\)?

1.5. **Our results and organization of the article.** In this article, we generalize the results of [15] concerning the spectrum of \( \frac{1}{W(u_0, u_1)} P \) for optimal potentials \( W(u_0, u_1) \), to non-optimal potentials \( W(u_0, u_1) \) obtained by the supersolution construction, provided that the function \( u_0 \) and \( u_1 \) are optimal approximate solutions of \( P \) at infinity. We will define later what optimal approximate solutions of \( P \) at infinity precisely means (see Definition
of This Article can then be roughly stated as follows (for a precise formulation, see Theorem 2.17):

**Theorem 1.15.** Let \((u_0,u_1)\) be a pair of optimal approximate solutions at infinity for \(P\). Recall that \(W(u_0,u_1) := \frac{1}{4} \left| \nabla \log \frac{u_0}{u_1} \right|^2\). Then the essential spectrum of \(\frac{1}{W(u_0,u_1)} P\) is equal to \([1, \infty)\), and its spectrum below 1 consists at most of a finite number of eigenvalues with finite multiplicity.

Roughly speaking, the proof relies on the fact that there are approximate generalized eigenfunctions of exponential type for \(\frac{1}{W(u_0,u_1)} P\). As a main corollary of Theorem 1.15, we will be able to answer Question 1.14 (see Theorem 3.5):

**Theorem 1.16.** Let \(\Omega\) be a \(C^2\), bounded, mean convex domain of \(\mathbb{R}^n\), and assume that \(D\) is chosen as in Example 1.9. Then for every \(i \geq 1\), the spectrum of \(\frac{1}{\eta^2} (-\Delta - W_{i-1})\) is \([1, \infty)\). Furthermore, without the mean convexity assumption on \(\Omega\), the essential spectrum of \(\frac{1}{\eta^2} (-\Delta - W_{i-1})\) is \([1, \infty)\), and the intersection of the spectrum of \(\frac{1}{\eta^2} (-\Delta - W_{i-1})\) with \((-\infty, 1)\) consists of (at most) a finite number of eigenvalues with finite multiplicity.

In particular, this proves Agmon’s claim 1.12 about the spectrum of \(\frac{1}{\eta^2} (-\Delta)\). As an immediate corollary, we recover the value of the best constant and of the best constant at infinity in the improved Hardy inequalities (1.6), a result already proved in [6].

Another result that we obtain as a consequence of Theorem 1.15 concerns Hardy inequalities on minimal submanifolds \(M^n \hookrightarrow \mathbb{R}^N\). Let us fix \(x_0 \in \mathbb{R}^N\), and denote by \(r := d_{\mathbb{R}^N}(x_0, \cdot)\), where \(d_{\mathbb{R}^N}\) is the Euclidean distance. Carron obtained in [12] the following Hardy inequality

\[
\left( \frac{n-2}{2} \right)^2 \int_M \frac{u^2}{r^2} \leq \int_M |\nabla u|^2, \forall u \in C^\infty_0(M).
\]

Let us denote by \(\Pi\) the second fundamental form of the isometric immersion \(M^n \hookrightarrow \mathbb{R}^N\), and denote by \(\Delta_M\) the (negative) Laplacian on \(M\). As a consequence of Theorem 1.15, we get the following result (see Theorem 4.1):

**Theorem 1.17.** Assume that the total curvature of the minimal isometric immersion \(M^n \hookrightarrow \mathbb{R}^N\) is finite, i.e. that

\[
\int_M |\Pi|^2 < \infty.
\]

Then the spectrum of \(r^2 (-\Delta_M)\) is \([\left( \frac{n-2}{2} \right)^2, \infty)\).

To conclude this introduction, we propose the following open problem:

**Question 1.18.** Are there embedded eigenvalues for the operators \(\frac{1}{\eta^2} (-\Delta - W_{i-1})\) and \(r^2 (-\Delta_M)\), appearing respectively in Theorem 1.16 and Theorem 1.17?
The structure of the article is as follows: in Section 2, we establish the general result (Theorem 1.15) that will be the key to studying the spectrum of our operators. In Section 3, we apply this result to the Hardy inequality (1.6), and we prove Theorem 1.16. In Section 4, we consider the case of the Hardy inequality on minimal submanifolds of $\mathbb{R}^n$, and we prove Theorem 1.17. In Section 5, we study an exponential volume growth property, which shows up naturally for Hardy inequalities with admissible potentials.

2. General theory

2.1. Preliminaries. Throughout the paper, the potentials appearing in the various Hardy inequalities will always be non-negative. As usual, $C$ will denote a generic constant, whose value can change from line to line.

**Notation:** For two positive functions $f$ and $g$, we will write $f \asymp g$ if there is a positive constant $C$ such that

$$C^{-1}f \leq g \leq Cf.$$  

Let $\Omega$, $n \geq 2$ be a smooth domain in $\mathbb{R}^n$ (or more generally, a smooth, connected manifold of dimension $n$). The infinity of $\Omega$ is the ideal point in the one-point compactification of $\Omega$. From this, we derive the notions of neighborhood of infinity in $\Omega$, of convergence at infinity for a real function defined in $\Omega$, etc... Let $\nu$ be a positive measure on $\Omega$. Consider a symmetric second-order elliptic operator $L$ on $\Omega$ with real coefficients in divergence form of the type

$$Pu = -\text{div}(A\nabla u) + cu,$$  

(2.1)

Here, $-\text{div}$ is the formal adjoint of the gradient with respect to the measure $\nu$. We assume that for every $x \in \Omega$ the matrix $A(x) = (a^{ij}(x))_{i,j}$ is symmetric and that the real quadratic form

$$\langle \xi, A(x)\xi \rangle := \sum_{i,j=1}^{n} \xi_i a^{ij}(x)\xi_j \quad \xi \in \mathbb{R}^n$$  

(2.2)

is positive definite. We will denote the norm associated to this quadratic form by $|\cdot|_A$, that is

$$|\xi|_A := \langle \xi, A(x)\xi \rangle.$$  

Moreover, it is assumed that $P$ is locally uniformly elliptic, and that $A$ and $c$ are locally bounded in $\Omega$.

We denote by $q$ the quadratic form associated to $P$, defined by

$$q(u) = \int_{\Omega} \left( \langle A\nabla u, \nabla u \rangle + cu^2 \right) d\nu, \ \forall u \in C_0^\infty(\Omega).$$
We will assume that $q$ is nonnegative, and consider the Friedrichs extension of $P$, that we will also denote $P$. It is a self-adjoint operator on $L^2(\Omega, d\nu)$. If $u \in W^{1,2}_{\text{loc}}(\Omega)$ and $f \in L^\infty_{\text{loc}}(\Omega)$, we will say that the equation

$$Pu = f$$

holds in $\Omega$ in the weak sense if for every $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega \left( (A\nabla u, \nabla \varphi) + cu \varphi \right) d\nu = \int_\Omega f \varphi d\nu.$$

In a similar way, we define the notion of weak supersolutions/subsolutions for $P$.

We will make use of the following two constructions. The first one is called the $h$-transform and is defined as follows: if $h > 0$ is in $C^{1,\alpha}_{\text{loc}}(\Omega)$ and $Ph \in L^\infty_{\text{loc}}(\Omega)$, we define the operator

$$P_h := h^{-1} Ph,$$

which is self-adjoint on $L^2(\Omega, h^2 d\nu)$. Also, the following formula holds in the weak sense:

$$P_h u = -\text{div}_h (A\nabla u) + \frac{Ph}{h} u,$$

where

$$\text{div}_h (X) = \text{div}(X) + 2 \langle h^{-1} \nabla h, X \rangle$$

is the divergence with respect to the measure $h^2 d\nu$.

We call the second construction the change of measure. Let $W > 0$, then we can consider the operator $\frac{1}{W}P$. It is a self-adjoint operator on $L^2(\Omega, W d\nu)$, and its quadratic form is the same as $P$. If $\text{div}_W$ is the divergence with respect to the measure $W d\nu$, then

$$\frac{1}{W}Pu = -\text{div}_W \left( \frac{A}{W} \nabla u \right) + \frac{c}{W} u,$$

which follows from the formula

$$\text{div}_W (X) = \frac{1}{W} \text{div}(WX).$$

2.2. A spectral result. We now present a general spectral result concerning operators of the form $\frac{1}{W}P$. It is probable that this result, maybe in a weaker form, is already known to experts, even if we have been unable to find a reference in the literature that covers such a general case.
Theorem 2.1. Assume that $W_1$ and $W_2$ are two positive (in a neighborhood of infinity) potentials in $L^\infty_{\text{loc}}(\Omega)$, such that $W_1(x) \sim W_2(x)$ when $x \to \infty$. Then $\sigma_{\text{ess}}\left(\frac{1}{W_1}P\right) = \sigma_{\text{ess}}\left(\frac{1}{W_2}P\right)$.

Proof. Let $\lambda \in \sigma_{\text{ess}}\left(\frac{1}{W_1}P\right)$. Let $(\Omega_n)_{n \in \mathbb{N}}$ be an exhaustion of $\Omega$. Then, using a ground state transform, it follows essentially from the decomposition principle for the essential spectrum (see [16] or [18]) that for every compact set $K \subseteq \Omega$, there is a Weyl sequence $(u_n)_{n \in \mathbb{N}}$ associated to $\lambda$, orthonogonal in $L^2(W_1d\nu)$, such that for every $n \in \mathbb{N}$, the support of $u_n$ is included in $\Omega \setminus \Omega_n$. By definition of a Weyl sequence, if we denote by $|| \cdot ||$ the norm in $L^2(W_1d\nu)$, there holds

$$\lim_{n \to \infty} \frac{||(\frac{1}{W_1}P - \lambda)u_n||}{||u_n||} = 0.$$ 

Since $W_1(x) \sim_{x \to \infty} W_2(x)$ and given the hypothesis on the support of $u_n$, one has, when $n \to \infty$,

$$||u_n|| \sim ||u_n||_{L^2(W_2d\nu)}.$$ 

Also, using the hypothesis on the support of $u_n$, and the fact that $W_1$ and $W_2$ are equivalent at infinity, one has for $n$ big enough,

$$||(\frac{1}{W_2}P - \lambda)u_n||_{L^2(W_2d\nu)} = \left(\int_{\Omega} |(P - \lambda W_2)u_n|^2 d\nu\right)^{1/2} \leq \left(\int_{\Omega} |(P - \lambda W_1)u_n|^2 d\nu\right)^{1/2} + |\lambda| \left(\int_{\Omega} |(W_1 - W_2)u_n|^2 d\nu\right)^{1/2} \leq 2||((\frac{1}{W_1}P - \lambda)u_n|| + |\lambda| \left(\int_{\Omega} \left|\frac{W_1 - W_2}{W_2}\right|^2 u_n^2 W_2 d\nu\right)^{1/2} \leq o(||u_n||).$$

Consequently, $u_n$ is also a Weyl sequence for $\frac{1}{W_2}P$, associated to $\lambda$. The hypothesis on the support of $u_n$ now implies that $\lambda$ is in the essential spectrum of $\frac{1}{W_2}P$. 

$\square$
Theorem 2.1 actually allows us to get a first quick proof of Agmon’s claim 1.12, relying on the results of [15]. An alternative proof, which works more generally for improved Hardy inequalities (1.6), will be given in Section 3.

Corollary 2.2. If \( \Omega \) is a bounded, \( C^2 \) domain in \( \mathbb{R}^n \), then the essential spectrum of \( \delta^{-2} \Delta \) is \([ \frac{1}{4}, \infty ) \).

Proof. Let \( W = \frac{1}{4} \left| \nabla G \right|^2 \) be an optimal weight in the sense of [15], where \( G \) is the Green function of \( \Delta \) with pole at some fixed point \( x_0 \in \Omega \). It has been proved in [15, Example 13.2] that as \( x \to \partial \Omega \),

\[
W \sim \frac{1}{4 \delta^2}.
\]

Furthermore, according to [15, Theorem 2.2], the (essential) spectrum of \( \frac{1}{W} \Delta \) is \([ 1, \infty ) \). By Theorem 2.1, the essential spectrum of \( \delta^{-2} \Delta \) is equal to the essential spectrum of \( 4 \frac{1}{W} \Delta \), and thus is equal to \([ \frac{1}{4}, \infty ) \). \( \square \)

Another direct application of Theorem 2.1 and of the results of [15] is to multipolar Hardy inequalities. Let \( x_1, \cdots, x_N, \ N \geq 2 \) be distinct points in \( \mathbb{R}^n \), and consider the positive weight

\[
W = \left( \sum_{1 \leq i < j \leq N} \frac{\left| x_i - x_j \right|^2}{|x - x_i|^2 |x - x_j|^2} \right).
\]

In [13], the following multipolar Hardy inequality was shown:

\[
\int_{\mathbb{R}^n} |\nabla u|^2 \geq \left( \frac{n - 2}{N} \right)^2 \int_{\mathbb{R}^n} Wu^2, \quad \forall u \in C_0^\infty(\mathbb{R}^n). \tag{2.5}
\]

It was proved in [15, Remark B2 and B.3] that \( \Delta - \left( \frac{n-2}{N} \right)^2 W \) is critical, but the weight \( W \) is not optimal, i.e. does not belong to \( O(\mathbb{R}^n \setminus \{x_1, \cdots, x_N\}) \) (one reason is that the constant \( \left( \frac{n-2}{N} \right)^2 \) in (2.5) is not optimal for test functions supported outside a ball \( B(0, R) \), for \( R \) large enough). From Theorem 2.1, one can deduce the following result:

Corollary 2.3. (Spectrum for multipolar Hardy inequalities). Denote \( C(N) = \frac{N^2}{4(N-1)} \). If \( N = 2 \), then the (essential) spectrum of \( \frac{1}{W} \Delta \) is \( \left( \left( \frac{n-2}{2} \right)^2, \infty \right) \). If \( N > 2 \), then the essential spectrum of \( \frac{1}{W} \Delta \) is \( C(N) \left( \frac{n-2}{N} \right)^2, \infty \), the bottom of the spectrum of \( \frac{1}{W} \Delta \) is \( \left( \frac{n-2}{N} \right)^2 \), and the positive function \( v = \prod_{i=1}^{N} |x - x_i|^{\frac{2-n}{N}} \) is eigenfunction for \( \frac{1}{W} \Delta \), associated to the eigenvalue \( \left( \frac{n-2}{N} \right)^2 \).

Remark 2.4. One can actually show, using arguments similar to the one appearing in the proofs of Lemma 2.13 and Proposition 2.14, that if \( N > 2 \), \( \frac{1}{W} \Delta \) has a finite number of eigenvalues, each with finite multiplicity, belonging to \( \left( \left( \frac{n-2}{N} \right)^2, C(N) \left( \frac{n-2}{N} \right)^2 \right) \). This in turn implies (see [14]) that the eigenspace associated to \( C(N) \left( \frac{n-2}{N} \right)^2 \) is finite dimensional.
Proof. Denote $\mathcal{L} := \frac{1}{W}\Delta$. Let $\varepsilon > 0$ be small enough such that the balls $B(x_j, \varepsilon)$ are disjoint. Let $K_1$ be a regular compact set containing the balls $B(x_j, \varepsilon)$, $j = 1, \cdots, N$, and denote $K_2$ the complement in $K_1$ of $\bigcup_{j=1}^N B(x_j, \varepsilon)$. The decomposition principle for the essential spectrum (see [16] or [18]) implies that the essential spectrum of $\mathcal{L}$ is equal to the essential spectrum of $\mathcal{L}$ on $L^2(\mathbb{R}^n \setminus K_1, Wdx)$, with Neumann boundary conditions on $\partial K_1$. Thus, it consists of the union of the essential spectrum of $\mathcal{L}$ on $B(x_j, \varepsilon)$, $j = 1, \cdots, N$ with Neumann boundary conditions, and of the essential spectrum of $\mathcal{L}$ on $\mathbb{R}^n \setminus K_1$, with Neumann boundary conditions on $\partial K_1$. When $x \to x_j$, one has (cf [15, Remark B.3])

$$\left(\frac{n-2}{N}\right)^2 W \sim C(N)^{-1} \frac{C_H}{|x-x_j|^2} = C(N)^{-1} W_{opt,j},$$

where $C_H := \left(\frac{n-2}{N}\right)^2$. Since $W_{opt,j}$ is an optimal weight, by [15, Theorem 2.2], the essential spectrum of $W_{opt,j}^{-1}\Delta$ is $[1, \infty)$. More precisely, for every $\lambda \in [1, \infty)$, one can find a Weyl sequence for $W_{opt,i}^{-1}\Delta$ supported in $B(x_j, \frac{\varepsilon}{2})$. This implies that the essential spectrum of $W_{opt,i}^{-1}\Delta$ on $B(x_j, \varepsilon)$, with Neumann boundary conditions, is equal to $[1, \infty)$. Applying Theorem 2.1, one obtains that the essential spectrum of $\mathcal{L}$ on $B(x_j, \varepsilon)$, $j = 1, \cdots, N$ with Neumann boundary conditions is equal to $[C(N)\left(\frac{n-2}{N}\right)^2, \infty)$. Moreover, when $|x| \to \infty$, it holds that

$$W \sim \frac{1}{|x|^4},$$

and therefore

$$\lambda_\infty(\mathbb{R}^n \setminus K_1, \Delta, W) = \infty,$$

which implies, according to Persson’s formula, that $\mathcal{L}$ on $\mathbb{R}^n \setminus K_1$, with Neumann boundary conditions on $\partial K_1$, has no essential spectrum. Therefore, the essential spectrum of $\mathcal{L}$ on $\mathbb{R}^n$ is $[C(N)\left(\frac{n-2}{N}\right)^2, \infty)$. The statement that if $N > 2$, the bottom of the spectrum of $\mathcal{L}$ is $\left(\frac{n-2}{N}\right)^2$ and that $v$ is eigenfunction follows from [15, Remark B.2].

However, it is more delicate to use Theorem 2.1 and [15, Theorem 2.2] in order to prove Theorems 1.16 and 1.17 in a similar way. More precisely, in order to prove Theorem 1.17, one would have to show that as $x \to \infty$ in $M$,

$$\frac{1}{4} \left| \frac{\nabla G(x)}{G(x)} \right|^2 \sim \left(\frac{n-2}{2}\right)^2 \frac{1}{r^2(x)},$$

where $G$ is the Green function of the Laplacian on $M$, with pole at some fixed point of $M$. In the case of Theorem 1.16, one would have to show that, as $x \to \partial \Omega$, 

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\[
\frac{1}{4} \frac{\nabla G_{i-1}(x)}{G_{i-1}(x)}^2 \sim \mathcal{I}(x), \quad (2.7)
\]

where \( G_{i-1} \) is the Green function of \( \Delta - W_{i-1} \), with pole at some fixed point of \( \Omega \). We do not know if the two estimates (2.6) and (2.7) hold. In Section 3 and 4, we will prove respectively Theorem 1.16 and 1.17, \textit{without having to prove gradient estimates for Green functions}. This is an important point, since in general finding the asymptotic at infinity of \( |\nabla \log G| \) for a Green function \( G \) is a difficult task, that can be achieved only in some particular cases. Instead, in our general approach that will lead to the proof of Theorems 1.16 and 1.17, we will not have to estimate the gradient of a Green function. This general approach is based on the results of the next subsection.

2.3. Another spectral result. We first introduce some definitions and notations. Denote by \( \mathcal{C} \) the set of positive measures on \([0, \infty)\) which are absolutely continuous with respect to the Lebesgue measure, and have infinite mass. For \( \chi \in \mathcal{C} \) and \( r \geq 0 \), denote

\[
V(r) := \int_0^r d\chi
\]

and

\[
S(r) := \int_r^{r+1} d\chi.
\]

Also, define \( \sigma \), the \textit{exponential rate of volume growth} of \( \chi \) by

\[
\sigma(\chi) := \lim_{r \to \infty} \sup \frac{1}{r} \log V(r).
\]

We have the following elementary technical Lemma:

\textbf{Lemma 2.5.} If \( \sigma(\chi) = 0 \), then for every \( a > 0, \varepsilon > 0 \) and \( d > 0 \), there exists \( b > a \) such that \( |b - a| \geq d \) and

\[
\frac{S(b)}{V(b) - V(a)} < \varepsilon.
\]

Conversely, if for every \( \varepsilon > 0 \), there exists \( a > 0 \) and \( d > 0 \) such that for every \( b > a + d \), the inequality

\[
\frac{S(b)}{V(b) - V(a)} < \varepsilon
\]

holds, then \( \sigma = 0 \).

\textit{Proof.} The proof is elementary, but we provide it for the sake of completeness. Assume that \( \sigma = 0 \). We proceed by contradiction: define \( f(r) := V(r) - V(a) \), and assume that there is \( \varepsilon > 0 \) and \( R \) big enough such that for every \( r \geq R \),
\[ \frac{f(r+1) - f(r)}{f(r)} > \varepsilon. \]

Define \( g(r) := e^{\nu r} \), with \( \nu > 0 \) chosen so that \( e^{\nu} - 1 < \varepsilon \), then

\[ \frac{g(r+1) - g(r)}{g(r)} = e^{\nu} - 1 < \varepsilon < \frac{f(r+1) - f(r)}{f(r)}. \]

From this, we deduce at once that there is a constant \( C \) such that for every \( r > 0 \),

\[ f(r) \geq C g(r). \]

But this implies that

\[ \sigma \geq \nu, \]

which is impossible.

Conversely, if \( \varepsilon > 0 \) is fixed and for every \( r > R \),

\[ \frac{f(r+1) - f(r)}{f(r)} < \varepsilon, \]

then proceeding as above, by comparison with \( g(r) := e^{\nu r} \) with \( \nu \) chosen so that \( e^{\nu} - 1 > \varepsilon \), we find that

\[ f(r) \leq C e^{\nu r}, \]

which yields that \( \sigma \leq \nu \). Letting \( \varepsilon \to 0 \), we can let \( \nu \to 0 \) and conclude that \( \sigma = 0. \)

\[ \square \]

**Definition 2.6.** A positive measure \( \chi \in \mathcal{C} \) is said to have *subexponential volume growth* if it satisfies \( \sigma(\chi) = 0 \). In other words, \( \chi \) has subexponential volume growth iff \( V(r) = e^{o(r)} \).

We will consider push-forward measures, of which we recall the definition:

**Definition 2.7.** If \( X \) and \( Y \) are measured spaces, \( f: X \to Y \) is measurable and \( \mu \) is a measure on \( X \), then the *push-forward* of \( \mu \) by \( f \) is the measure \( f_* \mu \), defined so that for every measurable set \( B \) of \( Y \),

\[ (f_* \mu)(B) = \mu(f^{-1}(B)), \]

where by definition \( f^{-1}(B) \) is the subset of \( X \) defined by

\[ f^{-1}(B) = \{ x \in X : f(x) \in B \}. \]

Then, for every \( g: Y \to \mathbb{R} \) measurable, we have the *change of variable formula*

\[ \int_Y g \, d(f_* \mu) = \int_X g \circ f \, d\mu. \quad (2.8) \]
After these preliminaries, let us now turn to our first general spectral result. Let $L$ be a symmetric, second-order elliptic operator of the form (2.1). We assume that $L1 = 0$ (that is, $c = 0$), then by Allegretto-Piepenbrink theory (see [4], or Lemma 3.10 in [26]), $L$ extends to a self-adjoint operator on $L^2(\Omega, d\nu)$, whose spectrum is included in $[0, \infty)$. In the following Proposition, we give conditions guaranteeing that the spectrum of $L$ is the whole $[0, \infty)$.

**Proposition 2.8.** Assume that (outside a compact set) there exists a function $v \in C^{1,\alpha}_{loc}$, $\alpha \in (0, 1)$ such that, for some positive constant $C$,

$$\lim_{x \to \infty} |\nabla v(x)|_A \to C,$$

and such that

$$\lim_{x \to \infty} v(x) = +\infty$$ (2.9)

and

$$\lim_{x \to \infty} Lv(x) = 0 \text{ (pointwise).}$$ (2.10)

Assume also that the push-forward measure $\chi := v_\ast \nu$ is in $C$ and has subexponential volume growth. Then the spectrum of $L$ is $[0, \infty)$: more precisely, for every $\eta \geq 0$, we can construct a Weyl sequence whose support goes to infinity, i.e. a sequence $(\varphi_n)_{n \in \mathbb{N}}$ of smooth, compactly supported functions such that

$$\lim_{n \to \infty} \frac{||[(L - \eta)\varphi_n]\|2}{||\varphi_n||2} = 0$$

and for every compact $K \subset \Omega$, there is an $N$ such that the support of $\varphi_n$ is in $\Omega \setminus K$ for every $n \geq N$.

**Remark 2.9.** (1) The infinity is the ideal point in the one-point compactification of $\Omega$.

(2) The condition that $v_\ast \nu$ be in $C$ is quite weak: for example, if $\nu$ is the Lebesgue measure on $\mathbb{R}^n$, then by the co-area formula, $v_\ast \nu \in C$ if and only if the function $f(t) := \text{Vol}_{H^{n-1}}(\{v = t\})$ is locally $L^1$ and $\int_0^\infty f(t)dt = \infty$. Here, $\text{Vol}_{H^{n-1}}$ is the volume with respect to $H^{n-1}$, the $(n - 1)$-dimensional Hausdorff measure.

(3) As follows from the proof, Proposition 2.8 holds under the weaker condition on $\chi$ that for every $a > 0$, $\varepsilon > 0$ and $d > 0$, there exists $b > a$ such that $|b - a| \geq d$ and

$$\frac{S(b)}{V(b) - V(a)} < \varepsilon.$$ 

This is indeed (slightly) weaker than subexponential volume growth, by Lemma 2.5.
The idea behind Proposition 2.8 is that \( L \), restricted to the set of “radial” functions, that is functions of the form \( f(v) \), has spectrum \([0, \infty)\). Thus it can be thought of in a way as a result about a “one-dimensional operator”. In fact, for every \( \eta \geq 0 \), the function \( e^{i\sqrt{\eta}} \) is an approximate solution at infinity of \( (L - \eta)u = 0 \), and the Weil sequence will be constructed as a sequence of functions that approximate \( e^{i\sqrt{\eta}} \). Let us turn to the detailed proof of Proposition 2.8:

Proof. Without loss of generality, we can assume that \( |\nabla v(x)|_A \to 1 \) when \( x \to \infty \). We will use the fact that since \( L1 = 0, L \) satisfies the two formulae (in the weak sense):

\[
L(f(u)) = f'(u)Lu - f''(u)|\nabla u|_A^2 \quad (2.11)
\]

and if \( g \) and \( h \) are \( C^1_{loc}(\Omega) \),

\[
L(gh) = gL(h) + hL(g) - 2A\nabla g \cdot \nabla h. \quad (2.12)
\]

Fix \( \eta \geq 0 \), and let \( \mu := \sqrt{\eta} \). Define \( \varphi = e^{i\mu v} \). Then, using formula (2.11), we get that in the weak sense,

\[
L\varphi = i\mu(Lv)\varphi + \mu^2|\nabla v|_A^2\varphi,
\]

that is

\[
(L - \eta)\varphi = i\mu(Lv)\varphi + \eta(|\nabla v|_A^2 - 1)\varphi.
\]

We want to define \( \varphi_n := \psi_n(v)\varphi \), where \( \psi_n(v) \) is going to play the role of a “radial” cut-off function. We first compute (using formulae (2.11) and (2.12)) that in the weak sense,

\[
(L - \eta)\varphi_n = \psi_n(v)[(L - \eta)\varphi] + \varphi(L\psi_n(v)) - 2\langle A\nabla \varphi, \nabla \psi_n(v) \rangle
\]

\[
= i\mu\varphi_n(Lv) + \eta(|\nabla v|_A^2 - 1)\varphi_n
\]

\[
+ \varphi \left( \psi_n'(v)(Lv) - \psi_n''(v)|\nabla v|_A^2 \right) - 2i\mu\psi_n'(v)\varphi|\nabla v|_A^2 \quad (2.13)
\]

We now define the real function \( \psi_n \): we take \( \psi_n(t) \) equal to 1 if \( t \in [a_n + 1, b_n - 1] \), 0 if \( t \notin [a_n, b_n] - a_n \) and \( b_n \) to be chosen later, and such that there is a constant \( C \) independent of \( n \) satisfying

\[
|\psi_n'| + |\psi_n''| \leq C.
\]

We are now ready to estimate each of the terms appearing in the computation of \( (L - \eta)\varphi_n \): we first have, using the property of change of variable formula (2.8) of the push-forward measure,

\[
\|\varphi_n\|_2^2 = \|\psi_n(v)\varphi\|_2^2 = \int_{a_n + 1}^{b_n - 1} \psi_n^2(t)d\chi(t) \geq \int_{a_n + 1}^{b_n - 1} d\chi(t).
\]
Moreover, using that $Lv$ and $|\nabla v|_A$ are bounded, and the fact that $\psi'_n$ and $\psi''_n$ are supported in the union of the intervals $[a_n, a_n + 1] \cup [b_n - 1, b_n]$, we get (using again the change of variable formula (2.8))

$$
||\varphi \left( \psi'_n(v)(Lv) - \psi''_n(v)|\nabla v|_A^2 \right) - 2i\mu \psi'_n(v)|\nabla v|_A^2 \varphi||_2 \leq C \int_{[a_n, a_n + 1] \cup [b_n - 1, b_n]} d\chi(t).
$$

Furthermore, since $|\nabla v|_A^2 \to 1$ at infinity, we have

$$
||\eta(|\nabla v|_A^2 - 1)\psi_n(v)\varphi||_2^2 = o \left( \int \psi_n^2(t) d\chi(t) \right) \text{ when } n \to \infty.
$$

We now choose the sequences $(a_n, b_n)_{n \in \mathbb{N}}$ inductively: suppose that $(a_{n-1}, b_{n-1})$ is defined, then take $a_n > \min(b_{n-1}, n)$ such that

$$
|Lv| \leq \frac{1}{n}
$$
on the set $\{v \geq a_n\}$ (here we use hypotheses (2.9) and (2.10)), and take $b_n$ big enough such that

$$
\frac{\int_{[a_n, a_n + 1] \cup [b_n - 1, b_n]} d\chi(t)}{\int_{a_n + 1}^{b_n} d\chi(t)} \leq \frac{1}{n}.
$$

This is possible by Lemma 2.5, since $\chi$ has subexponential volume growth. Collecting all the estimates, we get that

$$
\lim_{n \to \infty} \frac{||(L - \eta)\varphi_n||_2}{||\varphi_n||_2} = 0,
$$

which concludes the proof. 

\[\square\]

2.4. Spectral result for Hardy inequalities. We now apply Proposition 2.8 to study Hardy inequalities. We consider a symmetric operator $P$ on $L^2(\Omega, d\nu)$ of the form (2.1). We take $u_0, u_1$ positive functions on $\Omega$ which, for some $\alpha \in (0, 1)$ and $K \subset \subset \Omega$ compact subset of $\Omega$, are $C^{1,\alpha}_{loc}(\Omega \setminus K)$. Denote

$$
V_i = \frac{Pu_i}{u_i}.
$$

Recall that $X_1$ is defined by

$$
X_1(t) := (1 - \log(t))^{-1},
$$

and consider the non-negative weight

$$
W(u_0, u_1) := \frac{1}{4} \left| \nabla \log \left( \frac{u_0}{u_1} \right) \right|^2_A = \frac{1}{4} \left| \nabla X_1^{-1} \left( \frac{u_0}{u_1} \right) \right|^2_A.
$$
We emphasize that here and everywhere in the paper, $X_1^{-1}$ is a notation for $\frac{1}{X_1}$ and not for the inverse of $X_1$. We know from the supersolution construction of [15] that in the case where $u_0$, $u_1$ are solutions of $P$, then $u_{1/2} := \sqrt{u_0 u_1}$ and $u_{1/2} X_1^{-1} \left( \frac{u_0}{u_1} \right)$ are solutions of $P - W(u_0, u_1)$. Furthermore, if $u_0$ and $u_1$ are only supersolutions of $P$, then $u_{1/2}$ is supersolution of $P - W(u_0, u_1)$. In this subsection, we will be interested in the case where $u_0$ and $u_1$ are approximate solutions of $P$. In the rest of this subsection, we will denote $W(u_0, u_1)$ by $W$. We will need the following general computational lemma (see [15] for the proof of the first equality):

**Lemma 2.10.** The following equalities hold, in the weak sense:

\[
\left( P - \frac{1}{2} (V_0 + V_1) - W \right) u_{1/2} = 0,
\]

and

\[
\left( P - \frac{1}{2} (V_0 + V_1) + (V_0 - V_1) X_1 \left( \frac{u_0}{u_1} \right) - W \right) u_{1/2} X_1^{-1} \left( \frac{u_0}{u_1} \right) = 0.
\]

For the rest of this section, we assume that $W$ is positive in a neighborhood of infinity in $\Omega$: more precisely, we will assume that $W > 0$ on $\Omega \setminus K$. From the assumption that $u_0$ and $u_1$ belong to $C^{1,\alpha}(\Omega \setminus K)$, it follows that $W$ and $\frac{1}{W}$ are continuous, and in particular locally bounded, on $\Omega \setminus K$. In order to study the spectral properties of $\frac{1}{W} P$, we perform simultaneously a $h$-transform and a change of measure: we consider the operator

\[
L := u_{1/2}^{-1} (W^{-1} P - 1) u_{1/2},
\]

which is symmetric on $L^2(\Omega, u_{1/2}^2 W d\nu)$ and unitarily equivalent to $(W^{-1} P - 1)$. We compute from the formulae (2.3) and (2.4) that

\[
L = -\text{div} \left( \frac{A}{W} \nabla \cdot \right) + \frac{(W^{-1} P - 1) u_{1/2}}{u_{1/2}^2},
\]

where the divergence is for the measure $u_{1/2}^2 W d\nu$. Let us denote by $V$ the potential

\[
V := \frac{(W^{-1} P - 1) u_{1/2}}{u_{1/2}} = \frac{1}{2W} (V_0 + V_1),
\]

(we have used here the first equality in Lemma 2.10), and by $\tilde{L}$ the symmetric operator

\[
\tilde{L} := -\text{div} \left( \frac{A}{W} \nabla \cdot \right)
\]

acting on $L^2(\Omega, u_{1/2}^2 W d\nu)$, so that
$$L = \tilde{L} + V.$$ 

We have then the following consequence of Proposition 2.8:

**Proposition 2.11.** Assume that the following conditions are satisfied:

1. \( \lim_{x \to \infty} \frac{u_0(x)}{u_1(x)} = 0 \),

2. \( \lim_{x \to \infty} \frac{1}{2W} (V_0 + V_1) X_1^{-1} \left( \frac{u_0(x)}{u_1(x)} \right) = 0 \),

3. \( \lim_{x \to \infty} \frac{V_0 - V_1}{W} = 0 \),

4. The push-forward measure \( (X_1^{-1} \left( \frac{u_0}{u_1} \right))_{\ast} (u_0 u_1 W \, d\nu) \) is in \( \mathcal{C} \) and has subexponential volume growth.

Then the essential spectrum of \( W^{-1} P \) on \( L^2(\Omega, W \, d\nu) \) is \( [1, +\infty) \).

**Remark 2.12.**

1. It will be clear from the proof that if \( \tilde{W} = W \) in a neighborhood of infinity of \( \Omega \), then the essential spectrum of \( \tilde{W}^{-1} P \) is also \( [1, \infty) \).

2. Concerning the hypotheses made in Proposition 2.11: condition (1) expresses the fact that \( u_0 \) has “minimal growth” at infinity; conditions (2) and (3) express in a quantitative way that \( u_0 \) and \( u_1 \) are “approximate solutions” of \( P \) at infinity; condition (4) is satisfied for optimal potentials obtained by the supersolution construction, i.e. if \( u_0 \) and \( u_1 \) are solutions of \( P \); indeed, in this case, \( (X_1^{-1} \left( \frac{u_0}{u_1} \right))_{\ast} (u_0 u_1 W \, d\nu) \) has linear volume growth (see [15]).

Therefore, the hypotheses (1)–(4) can be considered to express in a quantitative way that \( W \) is “optimal at infinity”. We will discuss with greater details the relevance of condition (4) in Section 5.

**Proof.** Since \( W^{-1} P - 1 \) and \( L \) are unitarily equivalent, it is enough to show that the essential spectrum of \( L \) is \( [0, \infty) \). By Lemma 2.10, we have

\[
L X_1^{-1} \left( \frac{u_0}{u_1} \right) = \frac{1}{2W} (V_0 + V_1) X_1^{-1} \left( \frac{u_0}{u_1} \right) + \frac{V_0 - V_1}{W}.
\]

Define \( v := X_1^{-1} \left( \frac{u_0}{u_1} \right) \), which is \( C_{1,\text{loc}}^{1,\alpha}(\Omega \setminus K) \) for some \( K \) compact subset of \( \Omega \). The hypotheses made imply that

\[
\lim_{x \to \infty} v(x) = +\infty,
\]

\[
\lim_{x \to \infty} V(x) v(x) = 0,
\]
in particular,
\[ \lim_{x \to \infty} V(x) = 0, \]
and finally,
\[ \lim_{x \to \infty} \tilde{L}v(x) = 0. \]
Let us remark that by definition of \( W \),
\[ |\nabla v|_A/W = W^{-1}|\nabla v|_A = 4. \]
We can apply Proposition 2.8 to \( \tilde{L} \) (notice that the matrix \( \frac{A}{W} \) appearing in the definition of \( \tilde{L} \) is locally bounded in \( \Omega \setminus K \)): for every \( \eta \geq 0 \), there is a Weyl sequence \((\varphi_n)_{n \in \mathbb{N}}\) for \( \tilde{L} - \eta \), with the support of \( \varphi_n \) going to infinity.

Since \( \lim_{x \to \infty} V(x) = 0 \), we conclude that \((\varphi_n)_{n \in \mathbb{N}}\) is also a Weyl sequence for \( L - \eta \). Hence the essential spectrum of \( L \) contains \([0, \infty)\). For the inverse inclusion, it is enough to prove that 0 is the infimum of the essential spectrum of \( L \). By Persson’s formula (see [21] or [3]), \( \lambda_\infty(L) \), the infimum of the essential spectrum of \( L \), is given by
\[ \lambda_\infty(L) = \sup\{\lambda : \exists K \subset \subset \Omega, \exists u_\lambda > 0, \text{ s.t. } (L - \lambda)u_\lambda = 0 \text{ on } \Omega \setminus K\}. \tag{2.14} \]
Let \( \varepsilon > 0 \). Since \( V \) tends to 0, we can find a compact set \( K_0 \) containing \( K \) such that
\[ |V| \leq \varepsilon \text{ on } \Omega \setminus K_0. \]
Since \( \tilde{L}1 = 0 \), by Allegretto-Piepenbrink \( \tilde{L} \) is nonnegative. Therefore, again by Allegretto-Piepenbrink one can find a positive function \( u \), solution of
\[ (L + \varepsilon)u = (\tilde{L} + V + \varepsilon)u = 0 \text{ on } \Omega \setminus K_0. \]
Persson’s formula (2.14) now implies that \( \lambda_\infty(L) \geq -\varepsilon \). Letting \( \varepsilon \to 0 \), we conclude that
\[ \lambda_\infty(L) \geq 0. \]

We will be also interested in the part of the essential spectrum below the essential spectrum. For this, we will use the following Lemma:

**Lemma 2.13.** Let \( \mathcal{L} \) be an operator of the form (2.1). Assume that in a neighborhood of infinity there is a positive \( C^{1,\alpha}_{\text{loc}} \) function \( v \) such that when \( x \to \infty \), there holds
\[ |\nabla v(x)|_A \geq C > 0 \]
and
\[
\left| \frac{L_v}{v} + \mathcal{L}1 \right| = o(v^{-2}).
\]

Then the negative spectrum of \( \mathcal{L} \) consists of (at most) a finite number of eigenvalues, each with finite multiplicity.

**Proof.** We apply Lemma 2.10 to get, in a neighborhood of infinity,

\[
\mathcal{L}^{1/2} = \left( \frac{1}{2} \left( \frac{L_v}{v} + \mathcal{L}1 \right) + \frac{1}{4} \frac{|\nabla v|^2}{v^2} \right)^{1/2}.
\]

Using the hypotheses, we see that in a neighborhood of infinity,

\[
\mathcal{L}^{1/2} \geq 0.
\]

According to [14], the existence of a positive supersolution of \( L \) outside a compact set of \( \Omega \) is equivalent to the fact that the negative spectrum of \( L \) consists of (at most) a finite number of eigenvalues, with finite multiplicity, hence the conclusion.

\[\square\]

We apply this in the framework of Proposition 2.11:

**Proposition 2.14.** Let \( P, W, u_0, u_1, V_0, V_1 \) be as defined above Proposition 2.11. Assume that when \( x \to \infty \),

\[
(1) \quad \lim_{x \to \infty} \frac{1}{2W}(V_0 + V_1) = o \left( X_1^2 \left( \frac{u_0(x)}{u_1(x)} \right) \right),
\]

\[
(2) \quad \lim_{x \to \infty} \frac{V_0 - V_1}{W} = o \left( X_1 \left( \frac{u_0(x)}{u_1(x)} \right) \right),
\]

Then the spectrum of \( W^{-1}P \) strictly below 1 consists (at most) of a finite number of eigenvalues of finite multiplicity.

**Remark 2.15.** It is clear from the proof that the same conclusion also holds for \( W^{-1}P \), if \( W = W \) in a neighborhood of infinity of \( \Omega \).

**Proof.** It is a direct consequence of Lemma 2.13, applied with \( \mathcal{L} = L = u_{1/2}^{-1}(\frac{1}{W}P - 1)u_{1/2} \) and \( v = X_1^{-1} \left( \frac{u_0(x)}{u_1(x)} \right) \).

\[\square\]

The results of this section lead us naturally to the concept of “approximate solutions”, as announced in the introduction. We introduce the following definition:

**Definition 2.16.** Recall that \( W(u_0, u_1) := \frac{1}{4} \left| \nabla \log \frac{u_0}{u_1} \right|_{A}^2 \), and assume that \( W(u_0, u_1) \) is positive in a neighborhood of infinity. We say that \((u_0, u_1)\) is a pair of optimal approximate solutions at infinity for \( P \) if the following conditions are satisfied:
(1) \[ \lim_{x \to \infty} \frac{u_0(x)}{u_1(x)} = 0. \]

(2) When \( x \to \infty \),
\[ \frac{1}{W(u_0, u_1)} \left( \frac{Pu_0}{u_0} + \frac{Pu_1}{u_1} \right) = o \left( X_1^2 \left( \frac{u_0(x)}{u_1(x)} \right) \right). \]

(3) When \( x \to \infty \),
\[ \frac{1}{W(u_0, u_1)} \left( \frac{Pu_0}{u_0} - \frac{Pu_1}{u_1} \right) = o \left( X_1 \left( \frac{u_0(x)}{u_1(x)} \right) \right). \]

(4) The push-forward measure \( \left( X_1^{-1} \left( \frac{u_0}{u_1} \right) \right)^* \) is in \( C \) and has subexponential volume growth.

Summarizing the results of this section, we obtain one of the main results of this paper (as a consequence of Propositions 2.11 and 2.14):

**Theorem 2.17.** Let \( P \) be a symmetric, second-order elliptic operator of the form (2.1). Let \((u_0, u_1)\) be a pair of optimal approximate solutions at infinity for \( P \). Recall that \( W(u_0, u_1) = \frac{1}{4} \left| \nabla \log \frac{u_0}{u_1} \right|^2 \). Then the essential spectrum of the operator \( \frac{1}{W(u_0, u_1)}P \) on \( L^2(\Omega, W(u_0, u_1)\nu) \) is equal to \([1, \infty)\), and furthermore the spectrum below 1 consists at most of a finite number of eigenvalues with finite multiplicity.

### 3. Bounded domains of \( \mathbb{R}^n \)

In this section, \( \Omega \) will be a general (not necessarily mean convex, otherwise stated) \( C^2 \) bounded domain of \( \mathbb{R}^n \), and \( \delta \) is the distance to the boundary of \( \Omega \).

In this section, we study the spectrum of the operator \( J_{i-1}^{-1}(\Delta - \mathcal{W}_{i-1}) \), associated with the Hardy inequality (1.6) obtained in [6]. We will make use of the following properties of the function \( \delta \) (see [20] and [22], and references therein):

**Lemma 3.1.** Let \( \Omega \subset \mathbb{R}^n \) be an open set. Then \( \delta \) is Lipschitz in \( \Omega \), and
\[ |\nabla \delta| = 1 \text{ a.e. on } \Omega. \]

If \( \Omega \) is \( C^2 \), then there exists \( U \) neighborhood of \( \partial \Omega \) such that \( \delta \) is \( C^2 \) in \( U \) (in particular, \( |\nabla \delta| = 1 \) in \( U \)) and \( \Delta \delta \) is bounded in \( U \). If \( \Omega \) is \( C^2 \), then \( -\Delta \delta \geq 0 \) in the distribution sense in \( \Omega \) if and only if \( \Omega \) is mean-convex.

The equivalence between mean-convexity and non-negativity of \( -\Delta \delta \geq 0 \) goes back to Gromov, and was established for the first time in [22].

Let us recall the definition of the functions \( X_i \): we set by convention \( X_0 \equiv 1 \).
and for all $i \geq 1$,

$$X_{i+1}(t) = X_1(X_i(t)).$$

Then for every $i \geq 1$,

$$X_i((0, 1]) = (0, 1],$$

and

$$X_i(1) = 1, \quad \lim_{t \to 0} X_i(t) = 0.$$  

We will use the following formula which gives, for $i \geq 1$, the derivative of $X_i$, and which is extracted from [6]:

$$X_i'(t) = \frac{1}{t} X_1(t) \cdots X_{i-1}(t) X_i^2(t).$$

Now, we have the following computation, which is a direct consequence of Lemma 2.10, and which implies the Hardy inequality (1.6) obtained in [6] (related computations have been performed in [17]).

**Proposition 3.2.** Let $i \geq 0$, and define

$$U_{0,i} := \left( \frac{\delta}{D} X_0^{-1} \frac{\delta}{D} \cdots X_i^{-1} \frac{\delta}{D} \right)^{1/2},$$

$$U_{1,i} := U_{0,i} X_{i+1}^{-1} \frac{\delta}{D},$$

$$R_i := \sum_{k=1}^{i} X_1 \frac{\delta}{D} \cdots X_k \frac{\delta}{D} (R_0 = 0 \text{ by convention}),$$

and

$$H_i := \frac{1}{4} \sum_{k=1}^{i+1} \left| \nabla X_k^{-1} \frac{\delta}{D} \right|^2.$$

Then in $\Omega$,

$$\left( -\Delta - \frac{\Delta \delta}{2\delta} (1 - R_i) - H_i \right) U_{0,i} = 0,$$

(3.1)

and

$$\left( -\Delta - \frac{\Delta \delta}{2\delta} (1 - R_i) + \frac{\Delta \delta}{\delta} X_1 \frac{\delta}{D} \cdots X_{i+1} \frac{\delta}{D} - H_i \right) U_{1,i} = 0,$$

(3.2)

Moreover, if we denote
\[ W_i := \frac{1}{4\delta^2} \left( \sum_{k=0}^{i} X_0^2 \left( \frac{\delta}{D} \right) \cdots X_k^2 \left( \frac{\delta}{D} \right) \right), \]

then

\[ H_i = W_i \text{ a.e. on } \Omega, \]

and if \( U \) is the neighborhood of \( \partial \Omega \) given by Lemma 3.1, then

\[ H_i = W_i \text{ on } U. \]

**Corollary 3.3.** If \( \Omega \) is mean-convex, and if \( D \) is chosen such that in \( \Omega \), one has

\[ R_i = \sum_{k=1}^{i} X_1 \left( \frac{\delta}{D} \right) \cdots X_k \left( \frac{\delta}{D} \right) \leq 1, \]

then the Hardy inequality (1.6) takes place on \( \Omega \).

**Remark 3.4.** Actually, \( D \) in Corollary 3.3 can be chosen independently of \( i \): it is a consequence of the fact that the series

\[ \sum_{k=0}^{\infty} X_1(t) \cdots X_k(t) \]

converges for every \( t \in [0, 1) \). For a proof of this fact (kindly provided to us by A. Tertikas), see the Appendix.

**Proof of Corollary 3.3.** The hypothesis on \( D \) gives that \( R_i \leq 1 \), so that by Proposition 3.2 and the fact that \( -\Delta \delta \geq 0 \), we have (in the weak sense)

\[ (-\Delta - H_i) U_{0,i} \geq 0. \]

Given that \( U_{i,0} > 0 \), this implies by Allegretto-Piepenbrink theory that

\[ -\Delta - H_i \geq 0, \]

which is equivalent to saying that the following Hardy inequality holds:

\[ \int_{\Omega} H_i u^2 \leq \int_{\Omega} |\nabla u|^2, \quad \forall u \in C_0^\infty(\Omega). \]

Since \( \delta \) is Lipschitz on \( \Omega \), it is not hard to see that \( W_i \) and \( H_i \) are in \( L^\infty(\Omega) \). By Proposition 3.2, there is a set \( A \subset \Omega \) of zero measure such that \( H_i = W_i \) on \( \Omega \setminus A \). Let us fix \( u \in C_0^\infty(\Omega) \); then

\[ \int_{\Omega \setminus A} W_i u^2 + \int_A H_i u^2 \leq \int_{\Omega} |\nabla u|^2. \]

Since \( \delta \) is Lipschitz on \( \Omega \), it is not hard to see that \( H_i \) and \( W_i \) are in \( L^\infty(\Omega) \), and thus \( \int_A H_i u^2 = \int_A W_i = 0 \). Therefore,
\[
\int_{\Omega} \mathcal{W}_i u^2 \leq \int_{\Omega} |\nabla u|^2.
\]
Consequently, the Hardy inequality (1.6) holds on \( \Omega \).

\[\square\]

**Proof of Proposition 3.2.** The proof of the formulae (3.1) and (3.2) is by induction on \( i \), using the construction described in Lemma 2.10. Indeed, to pass from (3.1) and (3.2) of index \( i \) to (3.1) and (3.2) of index \( i + 1 \), just apply Lemma 2.10 with \( P = -\Delta + H_i \), \( u_0 = U_{0,i} \) and \( u_1 = U_{i,1} \). In order to initialize the induction, we apply Lemma 2.10 with \( P = -\Delta \), \( u_0 = \frac{\delta}{D} \), \( V_0 = \frac{-\Delta \delta}{\delta} \) and \( u_1 = 1 \), \( V_1 = 0 \), and we get exactly the formulae (3.1) and (3.2) for \( i = 0 \). Now assume that the formulae (3.1) and (3.2) are true for the index \( i \), and apply Lemma 2.10 with \( P = -\Delta + H_i \), \( u_0 = U_{0,i} \) and \( u_1 = U_{i,1} \). By the induction hypothesis,

\[ V_0 = \frac{-\Delta \delta}{2\delta} (1 - R_i), \]

and

\[ V_1 = \frac{-\Delta \delta}{2\delta} (1 - R_i) - \frac{-\Delta \delta}{\delta} X_1 \left( \frac{\delta}{D} \right) \cdots X_{i+1} \left( \frac{\delta}{D} \right). \]

The formulae of Lemma 2.10 gives (3.1) and (3.2) for the index \( i + 1 \), upon noticing that

\[ X_1 \left( \frac{u_0}{u_1} \right) = X_1 \left( X_i \left( \frac{\delta}{D} \right) \right) = X_{i+1} \left( \frac{\delta}{D} \right). \]

The fact that \( \mathcal{W}_i = H_i \) on \( U \) follows from the following computation:

\[ |\nabla X_{k+1}^{-1} \left( \frac{\delta}{D} \right)| = \frac{1}{D} X_{k+1} \left( \frac{\delta}{D} \right) X_{k+1}^{-2} \left( \frac{\delta}{D} \right) \]

\[ = \frac{1}{D} \left[ \frac{\partial}{\partial} X_1 \left( \frac{\delta}{D} \right) \cdots X_k \left( \frac{\delta}{D} \right) X_{k+1}^2 \left( \frac{\delta}{D} \right) \right] X_{k+1}^{-2} \left( \frac{\delta}{D} \right) \]

\[ = \frac{1}{D} X_1 \left( \frac{\delta}{D} \right) \cdots X_k \left( \frac{\delta}{D} \right), \]

and therefore \( \mathcal{W}_i = H_i \) on \( U \).

\[ \square\]

We now prove the announced spectral result for the Hardy inequality (1.6), as a consequence of Proposition 2.11. Recall that for \( i \geq 1 \),

\[ \mathcal{J}_i = \mathcal{W}_i - \mathcal{W}_{i-1}, \]

and by convention

\[ \mathcal{J}_0 = \frac{1}{4\delta^2}. \]
Theorem 3.5. For every $i \geq 0$, the essential spectrum of the operator $L_i := J_i^{-1}(-\Delta - W_{i-1})$ in $L^2(\Omega, J_i\,dx)$ is $[1, \infty)$. Furthermore, the spectrum of $L_i$ strictly below 1 consists (at most) of a finite number of eigenvalues with finite multiplicity.

In the case where $\Omega$ is mean-convex and if $D$ is chosen as in Corollary 3.3, then the spectrum of $L_i$ in $L^2(\Omega, J_i\,dx)$ is $[1, \infty)$, and in particular 1 is the best constant at infinity in the Hardy inequality (1.6).

Proof. By simplicity, we will assume that

$$H_i = W_{i-1}$$

on all $\Omega$ (and not only in a neighborhood of infinity). If it is not the case, one has to use the Remarks 2.12 and 2.15. The modifications are left to the reader.

For $k \geq 1$, define $Y_k := X_k^{-1}\left(\frac{\delta}{D}\right)$. Fix $i \geq 0$. We have seen in the proof of Proposition 3.2 that the improved Hardy inequality (1.6) at step $i$ is obtained by applying the construction of Lemma 2.10 with $P = -\Delta - W_{i-1}$, $u_0 = U_{0,i-1}$, $u_1 = U_{0,i-1}Y_i$ (for $i = 0$, we have to take $P = -\Delta$, $u_0 = \frac{\delta}{D}$ and $u_1 = 1$). For this choice of $u_0$ and $u_1$, we have

$$V_0 := \frac{Pu_0}{u_0} = O(\delta^{-1})$$

and

$$V_1 := \frac{Pu_1}{u_1} = O(\delta^{-1}).$$

We want to apply Proposition 2.11 to $P, u_0, u_1, W := J_i$. We have to check the three corresponding conditions. Notice that

$$\frac{u_0}{u_1} = X_i\left(\frac{\delta}{D}\right) \to 0 \text{ when } \delta \to 0.$$

Next, remark that

$$X_i^{-1}\left(\frac{u_0}{u_1}\right) = X_i^{-1}\left(X_i\left(\frac{\delta}{D}\right)\right) = Y_{i+1}.$$

Therefore,

$$\frac{1}{2J_i}(V_0 + V_1)X_i^{-1}\left(\frac{u_0}{u_1}\right) = \frac{1}{2J_i}Y_{i+1}O(\delta^{-1}) = O(\delta^{1-\varepsilon}).$$

Also,

$$\frac{1}{2J_i}(V_0 - V_1) = O(\delta^{1-\varepsilon}),$$
so the second and the third conditions of Proposition 2.11 are satisfied. It remains to verify the condition on the measure. Noticing that

\[ u_0 u_1 = U_{0,i}^2, \]

we see that the measure is \( d\nu = \mathcal{J} U_{0,i}^2 dx \). We compute

\[
\mathcal{J} U_{0,i}^2 = \frac{1}{4D} X_1^2 \cdots X_i^2 \times \frac{\delta}{D} X_{1}^{-1} \cdots X_{i}^{-1} \\
= \frac{1}{4D} (\frac{\delta}{D} X_1 \cdots X_i) \\
= \frac{1}{4D} |\nabla Y_{i+1}| 
\]

Therefore, by the coarea formula,

\[
\int_{\{a \leq Y_{i+1} \leq b\}} \mathcal{J} U_{0,i}^2 dx = \frac{1}{4D} \int_{\{a \leq Y_{i+1} \leq b\}} |\nabla Y_{i+1}| dx \\
= \frac{1}{4D} \int_a^b \left( \int_{\{Y_{i+1} = t\}} \frac{d\sigma}{dt} \right) dt 
\]

But \( Y_{i+1} = t \) if and only if \( \delta = \varphi(t) \), for a decreasing function \( \varphi \) such that

\[
\lim_{t \to \infty} \varphi(t) = 0. 
\]

When \( \varepsilon \to 0 \), the surface measure of the level set \( \{ \delta = \varepsilon \} \) is equivalent to \( |\partial \Omega| \), and therefore we obtain that when \( a \) and \( b \) go to \( +\infty \),

\[
\int_{\{a \leq Y_{i+1} \leq b\}} \mathcal{J} U_{0,i}^2 dx \sim (b - a). 
\]

Applying now Proposition 2.8 gives that the essential spectrum of \( L_i \) is \([1, \infty)\). Concerning the finiteness of the spectrum below 1, it is immediate to see that the previous computations imply that the hypotheses of Proposition 2.14 are satisfied, which gives the result.

\[ \square \]

The analogue of Proposition 3.2 and Theorem 3.5 holds for the improved Hardy inequalities considered in [17], where \( \delta \) is replaced by \( \varphi(x) := |x|^{2-n} \), which is \textit{harmonic} (and not only superharmonic). Let us explain this: for \( n \geq 3 \), we consider \( \Omega \) a bounded domain of \( \mathbb{R}^n \) containing 0, and we define

\[
Z_i := \left( \varphi(x) X_0^{-1} \left( \frac{|x|}{D} \right) \cdots X_i^{-1} \left( \frac{|x|}{D} \right) \right)^{1/2}, 
\]

we have for \( D \geq \sup_{\Omega} |x| \),

\[
\left( -\Delta - \frac{1}{4} \sum_{k=1}^{i+1} \nabla X_k^{-1} \left( \frac{|x|}{D} \right)^2 \right) Z_i = 0, \tag{3.3} 
\]

and
\[-\Delta - \frac{1}{4} \sum_{k=1}^{i+1} \left| \nabla X_k \right|^{-1} \left( \frac{|x|}{D} \right)^{2} \right) Z_i X_{i+1}^{-1} \left( \frac{|x|}{D} \right) = 0. \quad (3.4)\]

Equation (3.3) has already been obtained in [17]. We then have the following result, which is proved exactly like Theorem 3.5, considering what is happening around zero rather than at the boundary of $\Omega$:

**Theorem 3.6.** For $n \geq 3$, let $\Omega$ is a bounded domain of $\mathbb{R}^n$ containing 0. Define

$$H_i := \frac{1}{4} \sum_{k=1}^{i+1} \left| \nabla X_k \right|^{-1} \left( \frac{|x|}{D} \right)^{2}$$

and

$$R_i := H_i - H_{i-1}.$$  

Then the (essential) spectrum of the operator $R_i^{-1}(-\Delta - W_{i-1})$ in $L^2(\Omega, R_i \, dx)$ is $[1, \infty)$. In particular, 1 is the best constant around zero in the improved Hardy inequality

$$\int \Omega R_i u^2 \leq \int \Omega |\nabla u|^2 - \int \Omega H_{i-1} u^2, \forall u \in C_0^\infty(\Omega). \quad (4.1)$$

**Remark 3.7.** The fact that 1 is the best constant around zero was already shown in [17].

### 4. Minimal immersions of the Euclidean space

We consider a minimal isometric immersion $M^n \hookrightarrow \mathbb{R}^N$, for $n \geq 3$. We will denote by $\Pi$ the second fundamental form of this immersion. Let $x_0$ be any point of $\mathbb{R}^N$, and let $r = d_{\mathbb{R}^N}(x_0, \cdot)$ be the Euclidean distance. G. Carron [12] has shown the following Hardy inequality:

$$\left( \frac{n-2}{2} \right)^2 \int_M \frac{u^2}{r^2} \leq \int_M |\nabla u|^2, \forall u \in C_0^\infty(M). \quad (4.1)$$

Of course, if $x_0$ is a point of $M$, then an easy argument using test functions localized close to $x_0$ shows that $\left( \frac{n-2}{2} \right)^2$ is the best constant in the Hardy inequality (4.1). An interesting question is to what extent the weight $W := \left( \frac{n-2}{2} \right)^2 \frac{1}{r^2}$ has the “best behavior” at infinity in $M$. We show that

**Theorem 4.1.** Let us assume that the total curvature $\int_M |\Pi|^{n/2}$ is finite. Then the operator $r^2(-\Delta)$ has spectrum $\left[ \left( \frac{n-2}{2} \right)^2, \infty \right)$ (here, the Laplacian is on $M$). In particular, $\left( \frac{n-2}{2} \right)^2$ is the best constant at infinity in the Hardy inequality (4.1).
This result is not so surprising, since by results of Anderson [5], we know that the condition on the second fundamental form implies that $M$ is asymptotically Euclidean. In fact, we will use some estimates obtained in [5] in order to establish Theorem 4.1.

**Proof.** The proof is once again an application of Proposition 2.11. First, let us recall Carron’s computation that leads to (4.1). We have

$$\Delta r^2 = n,$$

which implies that

$$\left( -\Delta - \frac{1}{r^2} \left( \left( \frac{n-2}{2} \right)^2 + \frac{(2-n)(n+2)}{4} \left( 1 - |\nabla r|^2 \right) \right) \right) r^{\frac{2-n}{2}} = 0. \quad (4.2)$$

Here, $\nabla$ is the gradient on $M$ (and not on $\mathbb{R}^n$). Notice that $|\nabla r| \leq 1$, which implies by the above equation that

$$\left( -\Delta - \left( \frac{n-2}{2} \right)^2 \frac{1}{r^2} \right) r^{\frac{2-n}{2}} \geq 0.$$

By Allegretto-Piepenbrink theory, this yields the Hardy inequality (4.1). We will use the following Lemma, consequence of the work of Anderson [5] and Shen-Zhu [29]:

**Lemma 4.2.** The volume of $M$ is Euclidean at infinity, i.e. if $B_r(x_0)$ is the Euclidean ball centered in $x_0$ and $V(r)$ the volume of $M \cap B_R(x_0)$, then as $r \to \infty$, $V(r)$ is comparable to $r^n$. Also, $M$ has a finite number of ends, and at infinity in each end, $M$ tends to a linear subspace of dimension of $n$; moreover, the second fundamental form satisfies the estimate

$$|\Pi| = O(r^{-n/2}),$$

and as $r \to \infty$,

$$|\nabla r|^2 - 1 = O(r^{-1}).$$

For the estimate of $|\nabla r|^2 - 1$, see in particular the proof of Lemma 2.4 in [5].

Define

$$V_0 := \frac{(2-n)(n+2)}{4} \left( 1 - |\nabla r|^2 \right),$$

then

$$(-\Delta - V_0 - W)r^{\frac{2-n}{2}} = 0,$$

and by Lemma 2.10,
\[-\Delta - \frac{n(n-2)(1-|\nabla r|^2)}{r^2} X_1(r^{2-n}) - V_0 - W \right) r^{2-n} X_1^{-1}(r^{2-n}) = 0.\]

Furthermore,
\[
\frac{1}{W} |\nabla X_1^{-1}(r^{2-n})|^2 \to 1 \text{ when } r \to \infty.
\]

The proof of Proposition 2.11, which relies on Proposition 2.8, shows that it is enough to prove the following three properties:

1. \[
\lim_{r \to \infty} \frac{V_0}{W} X_1^{-1}(r^{2-n}) = 0,
\]
2. \[
\lim_{r \to \infty} \frac{1}{W} \frac{n(n-2)(1-|\nabla r|^2)}{r^2} = 0,
\]
3. The measure \(r^{2-n}Wdx\) on \(M\) satisfies the hypothesis (4) of Proposition 2.8.

The first two claims are consequences of the estimate of \(1-|\nabla r|^2\) given by Lemma 4.2. For the last one, since \(X_1^{-1}(r^{2-n}) \sim (n-2) \log r\) when \(r \to \infty\), we see that it is enough to prove the estimate on the measure with \(\nu = \log r\).

By the co-area formula,
\[
\int_{a \leq \log r \leq b} r^{2-n} W = \int_{a \leq \log r \leq b} r^{2-n} r^{-2} \approx \int_{a \leq \log r \leq b} r^{-n} |\nabla r| \approx \int_{e^a}^{e^b} t^{-n} dV(t),
\]
so that the corresponding measure on \(\mathbb{R}_+\) is
\[
d\chi(t) = t^{-n} dV(t).
\]

We have to check that for any \(a\) big enough,
\[
\lim_{b \to \infty} \frac{\int_a^b t^{-n} dV(t)}{\int_a^b \frac{r^{2-n} W}{r^2}} = 0.
\]

Integrating by part gives the formula
\[
\int_c^d t^{-n} dV(t) = d^{-n} V(d) - c^{-n} V(c) + n \int_c^d t^{-n-1} V(t) dt.
\]

Using the hypothesis that the volume growth in \(M\) is Euclidean at infinity (Lemma 4.2), we see that
\[
\frac{\int_{b-1}^{b} t^{-n} dV(t)}{\int_{a}^{b} t^{-n} dV(t)} \lesssim \frac{1}{\log(b)}
\]
hence goes to 0 when \( b \to \infty \). This gives that \( \chi \) is in the class \( \mathcal{C} \) of measure with subexponential volume growth.

\[\square\]

5. Spectrum and Agmon metric

In this section, we study the relationship between good Hardy inequalities and some weak “hyperbolicity” properties. The motivation comes from the following example: in the case of the Euclidean unit ball \( B \), consider the Hardy inequality

\[
\frac{1}{4} \int_{B} \frac{u^2}{\delta^2} \leq \int_{B} |\nabla u|^2, \; \forall u \in C_0^\infty(B).
\]

Define the following metric (the associated Agmon metric, see the paragraph below):

\[
ds^2 = \frac{1}{\delta^2} dx^2,
\]
then \( ds^2 \) is the hyperbolic metric on the ball \( B \). We ask the following question:

**Question 5.1.** For a general Hardy inequality with an admissible potential, does the corresponding Agmon metric retain some (weak) hyperbolicity properties, i.e. properties similar to the one of the hyperbolic metric \( \frac{1}{\delta^2} dx^2 \) on \( B \)?

We will show in this section that in the case of a general good Hardy inequalities, the Agmon metric satisfies a property of exponential growth of volume, similar to the exponential volume growth of the hyperbolic space. We will then apply this to the example of improved Hardy inequalities (1.9) on a mean convex domain of the Euclidean space. In passing, we will show a connection between this property of exponential growth of volume, and the condition (4) (the condition of subexponential volume growth of the measure) in Proposition 2.11.

5.1. **General case.** In this section, we consider a general Hardy inequality. Let \( W \) be a positive potential, and \( P \) of the form (2.1), such that the following Hardy inequality takes place for some \( \lambda > 0 \):

\[
\lambda \int_{\Omega} W u^2 d\nu \leq q(u), \; \forall u \in C_0^\infty(\Omega),
\]

where \( q \) is the quadratic form of \( P \). Our first result is a direct generalization Brooks’ results [10], [11], who proved an estimate for the bottom of
the essential spectrum of the Laplace-Beltrami operator on a complete Riemannian manifold, in term of the exponential growth of the volume of the geodesic balls. Brooks’ results are consequence of Agmon’s work on exponential decay of solutions of second-order elliptic equations [3]. Here we will show that Brooks’ results can be formulated in the more general context of Hardy inequalities: our result is an estimate of the best constant at infinity in the Hardy inequality (5.1), in term of the exponential volume growth of some measure. Before presenting our result, we need to introduce some definitions and notations. Let \( \phi \) be a positive solution of

\[ P\phi = 0, \]

and let us perform a \( h \)-transform with respect to \( \phi \): define

\[ \tilde{P} := \varphi^{-1}P\varphi, \]

which is self-adjoint on \( L^2(\Omega, \varphi^2d\nu) \). By formula (2.3), \( \tilde{P} \) is given by

\[ \tilde{P}u = -\text{div}_\varphi(A\nabla u). \]

Now perform a change of measure: introduce the measure

\[ d\mu := \varphi^2Wd\nu, \]

and define

\[ L := \frac{1}{W}\tilde{P}, \]

which is self-adjoint on \( L^2(\Omega, \varphi^2Wd\nu) \), so that the Hardy inequality is equivalent to

\[ \lambda \int_\Omega u^2 d\mu \leq \int_\Omega |\nabla u|^2_A d\mu, \forall u \in C_0^\infty(\Omega), \]  

(5.2)

where \( \langle \xi, \xi \rangle_A := \langle A^{-1}\xi, \xi \rangle \). The term on the right-hand side is the quadratic form associated to \( L \). Furthermore, the best constant \( \lambda_0 \) (resp. best constant at infinity \( \lambda_\infty \)) in (5.1) is equal to the best constant (resp. best constant at infinity) in (5.2). The Hardy inequality (5.2) expresses that \( L \) has a spectral gap, indeed as we have already indicated in the introduction, the bottom of the spectrum (resp. essential spectrum) of \( L \) is \( \lambda_0 \) (resp. \( \lambda_\infty \)). Let us define the Agmon metric

\[ |\xi|_{Ag}^2 := W\langle A^{-1}\xi, \xi \rangle. \]  

(5.3)

Denote by \( \rho \) the distance function in this Agmon metric. If \( \mu(\Omega) = \infty \), then for a fixed \( x_0 \in \Omega \), denote by \( V(r) \) the volume for the measure \( \mu \) of the ball \( B_{Ag}(x_0, r) \) of center \( x_0 \) and of radius \( r \) (with respect to the distance \( \rho \)). If \( \mu(\Omega) < \infty \), then define \( V(r)^{-1} \) to be the volume of \( \Omega \setminus B_{Ag}(x_0, r) \) for the measure \( \mu \) instead. Finally, define \( \sigma \) by
\[ \sigma := \lim_{r \to \infty} \sup \frac{1}{r} \log V(r). \]  

If \( \mu(\Omega) = \infty \), then \( \sigma \) is the exponential rate of volume growth for the measure \( \mu \), in the Agmon metric (5.3). If \( \mu(\Omega) < \infty \), then \( \sigma \) is the exponential rate of convergence to \( \mu(\Omega) \) of the volume of balls in the Agmon metric. This definition does not depend on the choice of \( x_0 \). Our result is

**Theorem 5.2.** Assume that the Agmon metric is complete. Then the following inequality takes place:

\[ \lambda_\infty \leq \frac{\sigma^2}{4}. \]

**Proof.** The proof follows closely Brook’s proofs in [10], [11], once the reduction to inequality (5.2) has been made. The need to use the Agmon metric (5.3) instead of the Riemannian metric on \( \Omega \) comes from the following fact, that we extract from [3], Theorem 1.4:

**Lemma 5.3.** The distance \( \rho \) for the Agmon metric (5.3) satisfies

\[ |\nabla \rho(x_0, \cdot)|^2 \leq 1. \]

With this at hand, the adaptation of Brook’s proof is quite straightforward, replacing the Riemannian metric by the Agmon metric and the Riemannian volume form by the measure \( \mu \). We leave the details to the reader.

\[ \square \]

**Remark 5.4.** If \( P = -\text{div}(A \nabla \cdot) \) and \( W = |\nabla h|^2_A \) for some real-valued Lipschitz function \( h \), then we have the following formula for the distance in the Agmon metric (5.3) (for a proof, see Lemma 10.5 in [15]):

\[ \rho(x, y) = |h(x) - h(y)|. \]  

(5.5)

The completeness of \( \Omega \) for the Agmon metric is then equivalent to

\[ \lim_{x \to \infty} |h(x)| = \infty \]

(see Lemma A1.2 in [3]). In the case of the optimal potentials obtained by the supersolution construction in [15], the Agmon metric is complete (Lemma 10.5 in [15]) and the inequality of Theorem 5.2 is an equality (see Lemma 7.2 in [15]).

In fact, following the argument of Li and Wang [19, Theorem 1.3], more can be said:

**Theorem 5.5.** For every \( \varepsilon > 0 \), the following holds:
(1) if $\mu(\Omega) < \infty$, then there is a constant $C$ such that

$$V(\infty) - V(r) \leq C \exp \left( -(2 - \varepsilon)\sqrt{\lambda_{\infty}} \right),$$

where $V(\infty) := \mu(\Omega)$.

(2) if $\mu(\Omega) = \infty$, then there is a constant $C$ such that

$$V(r) \geq C \exp \left( (2 - \varepsilon)\sqrt{\lambda_{\infty}} \right).$$

Furthermore, if the spectrum of $L$ below $\lambda_{\infty}$ consists of a finite number of eigenvalues, each with finite multiplicity, then we can take $\varepsilon = 0$ in the above inequalities.

To conclude this general subsection, we present a result about the discrete spectrum case, related to Lemma 1.10. Let $P$ be of the form (2.1) with quadratic form $q$, and let $W$ be a positive potential. Define $D^{1,2}(\Omega, P, W)$ to be the completion of $C_0^\infty(\Omega)$ for the norm

$$\left( q(u)^2 + \int_{\Omega} |u|^2 W d\nu \right)^{1/2}.$$

**Theorem 5.6.** Assume that for some $\lambda > 0$, the Hardy inequality (1.1) holds. Let $V$ be a positive potential such that

$$\lim_{x \to \infty} V(x) W(x) = 0.$$ 

Then $\lambda_{\infty}(\Omega, P, V) = +\infty$. In particular, the spectrum of $V^{-1}P$ consists of an increasing sequence of eigenvalues, tending to $+\infty$, and if $\lambda \in \mathbb{R}$ does not belong to the spectrum of $\frac{1}{W}P$, then the resolvent $(\frac{1}{W}P - \lambda)^{-1}$ is compact.

If moreover the Agmon metric

$$|\xi|_A^2 := W(A^{-1}\xi, \xi)$$

is complete and $V \in L^\infty_{\text{loc}}(\Omega)$, then $D^{1,2}(\Omega, P, V)$ injects compactly into $L^2(\Omega, V d\nu)$.

**Proof.** Let us prove the first part of the theorem. Let $\varepsilon > 0$. Then there is a compact set $K$ such that for every $x \in \Omega \setminus K$,

$$\frac{V(x)}{W(x)} \leq \varepsilon.$$ 

Therefore, using the fact that the Hardy inequality (1.1) is satisfied by assumption, we obtain that for every $u \in C_0^\infty(\Omega \setminus K)$,

$$\lambda \varepsilon^{-1} \int_{\Omega \setminus K} Vu^2 d\nu \leq \lambda \int_{\Omega \setminus K} Wu^2 d\nu \leq q(u).$$

Therefore,
\[ \lambda_\infty(\Omega, P, W) \geq \lambda \varepsilon^{-1}. \]

Letting \( \varepsilon \to 0 \), we get that

\[ \lambda_\infty(\Omega, P, W) = +\infty. \]

This concludes the proof of the first part of the theorem.

Let us now prove the second part. First, using a \( h \)-transform, we can assume that \( P\mathbf{1} = 0 \). We denote in all the proof \( \mathcal{D}^{1,2}(\Omega, P, W) = \mathcal{D}^{1,2} \). Fix a point \( x_0 \in \Omega \), and let \( \varepsilon > 0 \). By hypothesis, there exists a compact set \( K_0 \) such that for every \( x \in \Omega \setminus K_0 \),

\[ \frac{V(x)}{W(x)} \leq \varepsilon. \]

Let \( K \) compact set of \( \Omega \), containing \( K_0 \) in its interior. As a consequence of the completeness of the Agmon metric, there is a constant \( C \) (independant of \( \varepsilon \)) such that for every \( R > 0 \) big enough, there exists a \( C^\infty_0 \) cut-off function \( \varphi_R \) equal to 1 on \( B_{Ag}(x_0, R) \setminus K \), zero outside \( B_{Ag}(x_0, R + 1) \), and zero in \( K_0 \), such that

\[ \|\varphi_R\|_\infty + |\nabla \varphi_R|_W \leq C. \]

Let \( (w_n)_{n \in \mathbb{N}} \) be a bounded sequence in \( \mathcal{D}^{1,2}(\Omega, P, W) \). Up to a subsequence, one can assume that \( (w_n)_{n \in \mathbb{N}} \) converges weakly in \( \mathcal{D}^{1,2}(\Omega, P, W) \), and by substracting the limit, one can assume that \( (w_n)_{n \in \mathbb{N}} \) converges weakly to zero in \( \mathcal{D}^{1,2}(\Omega, P, W) \). Let us compute
\[ \int_\Omega (A \nabla (\varphi_R w_n) \cdot \nabla (\varphi_R w_n)) = \int_\Omega (A \nabla w_n \cdot \nabla w_n) \varphi_R^2 \]

\[ + 2 \int_\Omega (A \nabla w_n \cdot \nabla \varphi_R) w_n \varphi_R + \int_\Omega (A \nabla \varphi_R \cdot \nabla \varphi_R) w_n^2 \]

\[ \leq C \int_\Omega (A \nabla w_n \cdot \nabla w_n) + \int_\Omega (A \nabla \varphi_R \cdot \nabla \varphi_R) w_n^2 \]

\[ + 2C \left( \int_\Omega (A \nabla w_n \cdot \nabla w_n) \right)^{1/2} \left( \int_\Omega W w_n^2 \right)^{1/2} \]

\[ \leq C \int_\Omega (A \nabla w_n \cdot \nabla w_n) + \int_\Omega W w_n^2 \]

\[ + 2C \left( \int_\Omega (A \nabla w_n \cdot \nabla w_n) \right)^{1/2} \left( \int_\Omega W w_n^2 \right)^{1/2} \]

\[ \leq C, \]

where \( C \) is independent of \( R \). Here, we have successively used that \( (A \nabla \varphi_R, \nabla \varphi_R) \leq CW \), and the Hardy inequality (1.1), satisfied by hypothesis. Also, using the hypothesis on \( V \), the definition of \( \varphi_R \) and the Hardy inequality (1.1),

\[ \int_{\Omega \setminus K_0} V(\varphi_R w_n)^2 \leq \varepsilon \int_{\Omega \setminus K_0} W(\varphi_R w_n)^2 \]

\[ \leq \varepsilon q(\varphi_R w_n) = \varepsilon \int_\Omega (A \nabla (\varphi_R w_n) \cdot \nabla (\varphi_R w_n)) \]

(recall that we have assumed that \( P1 = 0 \)). Therefore,

\[ \int_{\Omega \setminus K_0} V(\varphi_R w_n)^2 \leq \varepsilon C. \]

Since \( C \) is independent of \( R \), we get, letting \( R \to \infty \):

\[ \int_{\Omega \setminus K} V w_n^2 \leq \varepsilon C. \]

By local uniform ellipticity of \( P \) and the Rellich theorem, \((w_n)_{n \in \mathbb{N}}\) converges to zero in \( L^2_{loc}(\Omega) \). Since \( V \in L^\infty_{loc}(\Omega) \),

\[ \lim_{n \to \infty} \int_K V w_n^2 = 0. \]

Finally, we obtain that

\[ \limsup_{n \to \infty} \int \Omega V w_n^2 \leq \varepsilon C. \]
Letting $\varepsilon \to 0$, we conclude that
\[
\lim_{n \to \infty} \int_{\Omega} V w_n^2 = 0,
\]
and the proof of the second part of the theorem is complete. \hfill \Box

5.2. The case of approximate solutions and the role of subexponential growth. In this subsection, we investigate what happens for a Hardy inequality with a weight obtained by the supersolution construction of [15]. We let $u_0$ and $u_1$ be positive supersolutions of $P$, and we recall the notation $W(u_0, u_1) := \frac{1}{4} |\nabla X^{-1} \left( \frac{u_0}{u_1} \right) |_A^2$. By Lemma 5.1 in [15], the following Hardy inequality takes place for $\lambda = 1$:
\[
\lambda \int_{\Omega} W u^2 d\nu \leq q(u), \quad \forall u \in C_0^\infty(\Omega),
\]
where $q$ is the quadratic form of $P$, and define $\lambda_0$ and $\lambda_\infty$ to be respectively the best constant and the best constant at infinity in (5.6). Define also the measures $\mu_i$ for $i = 0, 1$ by
\[
\mu_i := u_i^2 W(u_0, u_1) \nu,
\]
and the measure
\[
\mu := u_0 u_1 W(u_0, u_1) \nu,
\]
where we recall that $\nu$ is the underlying measure. Define the Agmon metric
\[
|\xi|^2_{\text{Ag}} := W(u_0, u_1) \langle A^{-1} \xi, \xi \rangle,
\]
and let $\rho$ be the distance for the Agmon metric. By Remark 5.4, $\rho$ is given by formula (5.5) with $h = \frac{1}{4} X^{-1} \left( \frac{u_0}{u_1} \right)$. Define also $\sigma_i$ for $i = 1, 2$, being the exponential rate of volume growth of $\sigma_i$, as in Definition 5.4 with $\mu$ replaced by $\mu_i$. Let us first consider an example, which introduces the results that we want to present in this subsection:

Example 5.7. Let $\Omega$ be a bounded, smooth domain, with $P = -\Delta$, $\nu = dx$, $W = \frac{1}{4|x|^2} = W(u_0, u_1)$ for $u_0 = \delta$, $u_1 = 1$. Then $\mu_0 = \frac{dx}{4\delta^2}$, $\mu_1 = \frac{dx}{4\delta^2}$ and $\mu = \frac{dx}{4\delta^2}$. Also, the distance in the Agmon metric $\frac{dx}{4\delta^2}$ is
\[
\rho(x, y) = \frac{1}{2} \log \left( \frac{\delta(x)}{\delta(y)} \right).
\]
Moreover, elementary computations show that:

1. $\mu_0$ has finite volume, $\mu_1$ has infinite volume.
2. $1 = \lambda_\infty(\Omega, -\Delta, \frac{1}{4|x|^2}) = \sigma_0^2 = \sigma_1^2 = \frac{\sigma_i^2}{4}$. 

(3) \( \mu \) has linear volume growth:

\[
\mu(B_{A_0}(x_0, R)) \asymp R, \quad \forall R \geq 1,
\]

where we recall that \( B_{A_0}(x_0, R) \) is the geodesic ball of center \( x_0 \) and radius \( R \) in the Agmon metric \( \frac{dx^2}{4s^2} \).

In the rest of this subsection, we will show that properties (1), (2) of Example 5.7 hold in more general situations. Let us begin to show an analogue of Theorem 5.2:

**Proposition 5.8.** Assume that \( W(u_0, u_1) \) is positive, that

\[
\lim_{x \to \infty} \frac{u_0(x)}{u_1(x)} = 0,
\]

and that for \( i = 0, 1 \),

\[
\lim_{x \to \infty} \frac{1}{W(u_0, u_1)} Pu_i = 0.
\]

Then for \( i = 1, 2 \), the inequality

\[
\lambda_\infty \leq \frac{\sigma_i^2}{4}
\]

holds.

**Proof.** The completeness of the Agmon metric follow from Remark 5.4 and the hypothesis that

\[
\lim_{x \to \infty} \frac{u_0(x)}{u_1(x)} = 0.
\]

In the rest of the proof, we will denote \( W(u_0, u_1) \) by \( W \), and we let \( V_i := \frac{Pu_i}{u_i} \).

As in section 5.1, we successively perform an \( h \)-transform and a change of measure. But this time, the \( h \)-transform is performed with respect to an approximate solution of \( P - W \), and not to a solution: we define two operators

\[
L_i := \frac{1}{W} u_i^{-1} Pu_i,
\]

where \( i \in \{0; 1\} \), where \( L_i \) is self-adjoint on \( L^2(\Omega, u_i^2 W \, d\nu) \). By formulae (2.3) and (2.4), we have

\[
L_i = -\text{div}_i \left( \frac{A}{W} \right) + \frac{V_i}{W},
\]

where the divergence \( \text{div}_i \) is for the measure \( \mu_i \). Denote by \( \tilde{L}_i \) the operator

\[
\tilde{L}_i := -\text{div}_i \left( \frac{A}{W} \right).
\]
Under the assumptions on $V_i$ made, an argument involving Persson’s formula (see the proof of Proposition 2.11) shows that for $i = 1, 2$, the bottom of the essential spectrum of $L_i$ is equal to the bottom of the essential spectrum of $\tilde{L}_i$. But since $L_i$ is unitarily equivalent to $\frac{1}{W}P$, its bottom of the spectrum is $\lambda_\infty$, and therefore the bottom of the spectrum of $\tilde{L}_i$ is $\lambda_\infty$. We can now apply Theorem 5.2 to $\tilde{L}_i$ to get the result.

We now turn to the reverse inequalities for $\sigma_0$, $\sigma_1$. This, as we shall see, requires conditions on the growth of the measure $\mu = u_0 u_1 W(u_0, u_1) \nu$. These conditions generalize Property (3) (linear growth of $\mu$) of Example 5.7, and are linked with the condition (4) appearing in Proposition 2.11. Denote by $\chi$ the push-forward of the metric $\mu$ by $X^{-1}_1$:

$$\chi := (X^{-1}_1 \left( \begin{array}{c} u_0 \\ u_1 \end{array} \right) ) \ast \mu.$$  

The measure $\chi$ is the measure which appears in the condition (4) of Proposition 2.11.

**Definition 5.9.** We will say that $\mu$ has at most $\varepsilon$-exponential growth and decay if there exists a constant $C > 0$ such that for every $t > 0$,

$$C^{-1} e^{-\varepsilon t} \leq \chi(t) \leq Ce^{\varepsilon t}.$$  

**Remark 5.10.** The inequality $\chi(t) \leq Ce^{\varepsilon t}$ is related – but not equivalent – to subexponential growth for $\chi$ (that is, is related to the condition (4) in Proposition 2.11). For example, $\mu = \frac{d}{dx} \delta_0$ of Example 5.7, has $\varepsilon$–exponential growth and decay, for every $\varepsilon > 0$.

With this definition, we show the following result:

**Proposition 5.11.** Assume that $\frac{u_0}{u_1}$ is bounded from above, that

$$\lim_{x \to \infty} \frac{u_0(x)}{u_1(x)} = 0,$$

and that for some $0 < \varepsilon < 1$, $\mu$ has at most $\varepsilon$–exponential growth and decay. Then $\mu_0(\Omega) < \infty$, $\mu_1(\Omega) = \infty$, and for $i = 0, 1$, the reverse inequality

$$\frac{\sigma_i^2}{4} \leq 1$$

holds.

As a direct consequence of Proposition 5.8 and Proposition 5.11, we get the following corollary:

**Corollary 5.12.** Assume that $\frac{u_0}{u_1}$ is bounded from above, that assumptions of Proposition 2.11 are satisfied, that there is $0 < \varepsilon < 1$ and some constant
C such that \( \mu \) has at most \( \varepsilon \)-exponential growth and decay, and moreover that for \( i = 0, 1 \),

\[
\lim_{x \to \infty} \frac{1}{W(u_0, u_1)} \frac{P u_i}{u_i} = 0.
\]

Then \( \mu_0(\Omega) < \infty, \mu_1(\Omega) = \infty \), the Agmon metric (5.7) is complete and for \( i = 0, 1 \),

\[
\frac{\sigma_i^2}{4} = \lambda_\infty = 1.
\]

**Proof.** By normalization of \( u_0 \) and \( u_1 \), we will assume without loss of generality that

\[
\frac{u_0}{u_1} \leq 1.
\]

We will denote \( W := W(u_0, u_1) \) and \( V_i := \frac{P u_i}{u_i} \). Let us also denote by \( \psi \) the inverse function of \( X_1^{-1} \). Since

\[
X_1^{-1}(t) \sim -\log(t) \text{ as } t \to 0,
\]

we have

\[
\psi(t) \sim e^{-t} \text{ as } t \to 0.
\]

Now,

\[
\psi \left( X_1^{-1} \left( \frac{u_0}{u_1} \right) \right) \mu = u_0^2 W \nu = \mu_0,
\]

and

\[
\left( \frac{1}{\psi} \right) \left( X_1^{-1} \left( \frac{u_0}{u_1} \right) \right) \mu = u_1^2 W \nu = \mu_1.
\]

Thus, using the change of variable formula (2.8), we see that

\[
\mu_0 \left( a \leq X_1^{-1} \left( \frac{u_0}{u_1} \right) \leq b \right) = \int_a^b \psi(t) d\chi(t),
\]

and

\[
\mu_1 \left( a \leq X_1^{-1} \left( \frac{u_0}{u_1} \right) \leq b \right) = \int_a^b \frac{d\chi(t)}{\psi(t)}
\]

By the hypothesis that

\[
\lim_{x \to \infty} \frac{u_0}{u_1} = 0,
\]

we have
\[
\mu_0(\Omega) = \mu_0 \left( 1 \leq X_1^{-1} \left( \frac{u_0}{u_1} \right) < \infty \right) = \int_1^\infty \psi(t) d\chi(t),
\]
and
\[
\mu_1(\Omega) = \mu_1 \left( 1 \leq X_1^{-1} \left( \frac{u_0}{u_1} \right) < \infty \right) = \int_1^\infty \frac{d\chi(t)}{\psi(t)}.
\]

Since \( \psi(t) \sim e^{-t} \) as \( t \to \infty \), \( \mu_0(\Omega) < \infty \) (resp. \( \mu_1(\Omega) = \infty \)) is equivalent to \( \int_1^\infty e^{-t} d\chi(t) < \infty \) (resp. \( \int_1^\infty e^t d\chi(t) = \infty \). But by the integration by part formula, valid for Stieltjes measures,

\[
\int_1^\infty e^{-t} d\chi(t) = \left[ e^{-t} \chi(t) \right]_1^\infty + \int_1^\infty e^{-t} \chi(t) dt,
\]
and given the hypothesis on \( \chi \),

\[
\lim_{t \to \infty} e^{-t} \chi(t) = 0,
\]
and

\[
\int_1^\infty e^{-t} \chi(t) dt < \infty.
\]

This proves that \( \mu_0(\Omega) < \infty \). For \( \mu_1(\Omega) \), we have again by integration by parts

\[
\int_1^\infty e^t d\chi(t) = \left[ e^t \chi(t) \right]_1^\infty + \int_1^\infty e^t \chi(t) dt,
\]
and given the hypothesis on \( \chi \),

\[
\int_1^\infty e^t \chi(t) dt = \infty,
\]
which yields \( \mu_1(\Omega) = \infty \). Now we turn to the estimates on \( \sigma_0 \) and \( \sigma_1 \). Since the Agmon metric is given by formula (5.5) with \( h = X_1^{-1} \left( \frac{u_0}{u_1} \right) \), we see that in the definition of \( \sigma_i \) we can replace the ball \( B(x_0, r) \) in the Agmon metric by the set \( \{ 2 \leq X_1^{-1} \left( \frac{u_0}{u_1} \right) \leq 2r \} \). Thus, using the change of variable formula (2.8), and the fact that \( \psi(t) \sim e^{-t} \) when \( t \to \infty \), we see that

\[
\sigma_0 = \lim_{r \to \infty} \sup \frac{1}{r} \log \int_{2r}^{\infty} e^{-t} d\chi(t),
\]
and

\[
\sigma_1 = \lim_{r \to \infty} \sup \frac{1}{r} \log \int_1^{2r} e^t d\chi(t).
\]
But using as above the integration by parts formula for Stieltjes measures and the the hypothesis on \( \chi \), we see that there is a constant \( c > 0 \) such that for \( r > 0 \) big enough,
\[ \int_1^{2r} e^t \, d\chi(t) \leq c e^{(2-\varepsilon)r} \]

and

\[ \int_{2r}^{\infty} e^{-t} \, d\chi(t) \geq c^{-1} e^{-(2-\varepsilon)r}. \]

This implies at once that \( \sigma_i \leq 2 \), which concludes the proof. \( \square \)

5.3. **Volume growth for the improved Hardy inequalities on a convex set.** In this subsection, we show how the general theory developed in subsection 5.2 applies to the particular example of the improved Hardy inequalities on a bounded domain \( \Omega \) of \( \mathbb{R}^n \). Fix \( i \geq 0 \), and define

\[
P := -\Delta - W_{i-1}
\]

(\( W_{-1} = 0 \) by convention), where \( W_i \) is the weight

\[
W_i := \frac{1}{4\delta^2} \left( \sum_{k=0}^i X_0^2 \left( \frac{\delta}{D} \right) \cdots X_k^2 \left( \frac{\delta}{D} \right) \right),
\]

(\( W_{-1} = 0 \) by convention). Recall also the definition of

\[
J_i := W_i - W_{i-1} = \frac{1}{4\delta^2} X_0^2 \left( \frac{\delta}{D} \right) \cdots X_i^2 \left( \frac{\delta}{D} \right).
\]

From Section 3, recall the definition of \( U_{0,j} \) and \( U_{1,j} \), and define

\[
u_0 := U_{0,i-1},
\]

\[
u_1 := U_{1,i-1}.
\]

Define as in subsection 5.2, for \( k = 0, 1 \)

\[
\mu_{i,k} := \nu_k^2 J_i \, dx.
\]

Define also the associated volume growth rate \( \sigma_{i,k} \), for \( k = 0, 1 \). Then as a consequence of Corollary 5.12, we have

**Theorem 5.13.** For every \( i \geq 0 \), the measure \( \mu_{i,0} \) (resp. \( \mu_{i,1} \)) has finite (resp. infinite) mass, and the convergence of volumes is exponential: for \( k = 0, 1 \),

\[
\frac{\sigma_{i,k}^2}{4} = 1.
\]
Proof. Denote $d\chi$ the push-forward measure of $u_0 u_1 \mathcal{J} dx$ by $X_1^{-1} \left( \frac{u_0}{u_1} \right)$, then the computations done in the proof of Theorem 3.5 show that $\chi$ has linear growth. Also, for $k = 0, 1$, denote 

$$V_k := \frac{P u_k}{u_k}.$$ 

Then by Proposition 3.2, in a neighborhood of $\partial \Omega$ we have 

$$V_0 = -\frac{\Delta \delta}{2 \delta} (1 - R_{i-1}),$$ 

and 

$$V_1 = -\frac{\Delta \delta}{2 \delta} (1 - R_{i-1}) - \frac{\Delta \delta}{\delta} X_1 \left( \frac{\delta}{D} \right) \cdots X_{i+1} \left( \frac{\delta}{D} \right).$$ 

It is immediate to check that for $k = 0, 1$, 

$$\lim_{\delta \to 0} \frac{V_k}{W} = 0.$$ 

Thus we can apply Corollary 5.12 to $\mu_{i,k}$, which gives the result.

\[ \square \]

6. Appendix

Here, we give a proof of the fact that the series 

$$\sum_{k \geq 1} X_1(t) \cdots X_k(t)$$ 

converges for every $t \in [0, 1)$. We thank A. Tertikas for having provided us with the proof. For every $\varphi$ defined on the unit ball $B_1$, we have the following Hardy inequality (as a simple consequence of Allegretto-Piepenbrink theory, or by direct integration by parts) 

$$\int_{B_1} |\nabla u|^2 \, dx \geq \int_{B_1} \frac{-\Delta \varphi}{\varphi} u^2 \, dx, \quad u \in C_0^\infty(B_1)$$

We make the choice 

$$\varphi = X_1^{-1/2}(|x|) \cdots X_k^{-1/2}(|x|),$$

and compute (see Lemma 6.3 in [17]) 

$$\frac{-\Delta \varphi}{\varphi} = \frac{n - 2}{2|x|^2} \sum_{i=1}^k X_1(|x|) \cdots X_i(|x|) + \frac{1}{4|x|^2} \sum_{i=1}^k X_1^2(|x|) \cdots X_i^2(|x|)$$

$$\geq \frac{n - 2}{2|x|^2} \sum_{i=1}^k X_1(|x|) \cdots X_i(|x|)$$

Applying it for $n \geq 3$, we conclude the the convergence of the required series for $t \in (0, 1)$.

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