A Global Uniqueness on Spherically Stratified Dielectric Medium in Time-Harmonic Maxwell Equation with Interior Transmission Eigenvalues

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Abstract

A set of regularly distributed transmission eigenvalues generates a density function. We use such a density function to inversely determine the form of the indicator function. Using the entire function theory, we reduce an uniqueness problem with interior transmission eigenvalues induced by time-harmonic Maxwell equation to an uniqueness problem in entire function theory. In such an inverse problem, the definite integral of the square root of refraction index is the main parameter.

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1 Introduction and Preliminaries

In this paper, we consider the time-harmonic Maxwell equation with non-absorbing refraction index in the following setting:

\[ n(x) = n(r) > 0, \quad r = |x|, \text{ when } r \in [0, 1]; \quad 3n = 0; \quad n(r) = 1, \text{ when } r \geq 1; \quad n \in C^2[0, \infty); \] (1.1)

such that

\[ \begin{align*}
\nabla \times E_1 - ikH_1 &= 0, \quad \nabla \times H_1 + ikn(r)E_1 = 0, \quad \text{in } B; \\
\nabla \times E_0 - ikH_0 &= 0, \quad \nabla \times H_0 + ikE_0 = 0, \quad \text{in } B;
\end{align*} \]

(1.2)

with boundary condition

\[ \nu \times (E_1 - E_0), \nu \times (H_1 - H_0) = 0, \quad \text{on } \partial B, \]

(1.3)

where \( E_0, H_0 \) is an electromagnetic Herglotz pair, \( B \) is an open ball of radius 1 in \( \mathbb{R}^3 \) with exterior unit normal vector \( \nu \). We will look for a non-trivial solution to this homogeneous electromagnetic interior transmission problem (1.2) and (1.3). For each \( k \in \mathbb{C} \) such that (1.2) and (1.3) has a set of non-trivial solution is called an interior transmission eigenvalue. We reduce such an electromagnetic interior transmission problem to the acoustic interior transmission problem:

\[ \begin{align*}
\Delta w + k^2n(r)w &= 0, \quad \text{in } B; \\
\Delta v + k^2v &= 0 \quad \text{in } B; \\
w = v, \quad \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad \text{on } \partial B,
\end{align*} \]

(1.4)

where \( w, v \in C^3(B) \). To see that, we consider the following quantity

\[ \begin{align*}
E_1(x) &= \nabla \times \{ xw(x) \}; \\
H_1(x) &= \frac{1}{ik} \nabla \times \{ E_1(x) \}; \\
E_0(x) &= \nabla \times \{ xv(x) \}; \\
H_0(x) &= \frac{1}{ik} \nabla \times \{ E_0(x) \},
\end{align*} \]

(1.5)
from which one can obtain a set of solution to the electromagnetic interior transmission problem \((1.2)\) and \((1.3)\). We refer the induction to the Colton and Kress [4].

We need to consider the solutions \(w, v\) to \((1.4)\) that are not spherically symmetric. Therefore, we look for non-trivial solutions \(w, v\) in the following form:

\[
v(r, \theta) = a_l j_l(kr)P_l(\cos \theta);
\]

\[
w(r, \theta) = b_l \frac{y_l(r)}{r} P_l(\cos \theta),
\]

where \(P_l\) is Legendre's polynomial, \(j_l\) is the spherical Bessel function of degree \(l\), \(a_l\) and \(b_l\) depend on \(k\) and the function \(y_l\) is a solution of

\[
y_l'' + \left(k^2 n(r) - \frac{l(l+1)}{r^2}\right)y_l = 0, 0 < r < 1,
\]

\[
\lim_{r \to 0}\left\{\frac{y_l(r)}{r} - j_l(kr)\right\} = 0,
\]

where \(y_l\) is continuous for \(r \geq 0\). Moreover, as demonstrated in [4], we consider the non-spherically symmetric \(w, v\). In such a magnetic problem, we are asked to consider \(l \geq 1\). Furthermore, we see from \((1.9)\) that

\[
y_l(0) = 0; \quad y_l'(0) = 0.
\]

We will show there exist a set of \(k \in \mathbb{C}\) with its maximal density and constants \(a_l = a_l(k), b_l = b_l(k)\), such that \((1.6)\) and \((1.7)\) is a set of non-trivial solution to the interior transmission problem \((1.4)\). Considering \((1.9)\), we see that, for any such value of \(k\), the set of the electric far field patterns is not complete in certain functional space. See the discussion in [4].

The the interior transmission problem \((1.4)\) and \((1.5)\) admits a set of non-trivial solution \(v, w\) if there exists a set of non-trivial solutions \(a_l, b_l\) to the following homogenous system

\[
b_l y_l(1) - a_l j_l(1) = 0; \quad (1.11)
\]

\[
b_l \frac{d}{dr} \left(\frac{y_l(r)}{r}\right)_{r=1} - a_l k j_l'(k) = 0. \quad (1.12)
\]

Such a system admits a set of non-trivial solutions \(a_l = a_l(k), b_l = b_l(k)\) if and only if the determinant

\[
d_l(k) := \det \begin{pmatrix} y_l(1) & -j_l(k) \\ \frac{d}{dr} \left(\frac{y_l(r)}{r}\right)_{r=1} & -k j_l'(k) \end{pmatrix} = 0. \quad (1.13)
\]

In this paper, we apply the following setting.

\[
l = 1; \quad (1.14)
\]

\[
b_1(k) = b(k), a_1(k) = a(k); \quad (1.15)
\]

\[
da_1(k) := D(k), k \in \mathbb{C}; \quad (1.16)
\]

\[
y_1(r; k) := y(r; k), k \in \mathbb{C}. \quad (1.17)
\]

Hence,

\[
D(k) = (-k)y(1; k) j_1'(k) + y'(1; k) j_1(k) - y(1; k) j_1(k), \quad (1.18)
\]

where

\[
j_1(t) = \frac{\sin t}{t^2} - \frac{\cos t}{t}. \quad (1.19)
\]

We consider in the spherical coordinate \((r, \theta, \phi)\) that

\[
\Phi(r; k) := b(k)y(r; k); \quad (1.20)
\]

\[
\Phi_0(r; k) := a(k)r j_1(kr). \quad (1.21)
\]

Hence, using \((1.10)\),

\[
\Phi(0; k) = 0; \quad (1.22)
\]

\[
\partial_r \Phi(0; k) = b(k)\partial_r y(r; k)\big|_{(r=0)} = 0; \quad (1.23)
\]
using (1.11) and (1.12),
\[ \Phi(1; k) = \Phi_0(1; k); \]  
\[ \partial_r \Phi(1; k) = b(k)\partial_r y(r; k)|_{r=1} = \partial_r \Phi_0(1; k). \]
Therefore, it is obviously that
\[ \begin{cases} 
\partial_r \Phi + (k^2 n(r) - \frac{2}{r^2})\Phi = 0, & 0 < r < 1; \\
\partial_r \Phi_0 + (k^2 - \frac{2}{r^2})\Phi_0 = 0, & 0 < r < 1; \\
\Phi(0) = \Phi_0(0) = 0, & \Phi(1) = \Phi(1), \partial_r \Phi(1) = \partial_r \Phi_0(1). 
\end{cases} \]  
(1.26)

We also observe that initial value problem (1.22), (1.23) and the first equation in (1.25) imply
\[ \Phi(r; k) = y(r; k). \]  
(1.27)

This is due to the uniqueness of the initial value problem (1.8) and (1.9) with \( l = 1 \). Therefore, \( \Phi(r; k) \) is entire function of the same type and order as \( y(r; k) \).

To find the estimates on the solution \( y(r; k) \). We use the Liouville transform.
\[ z(\xi; k) := [n(r)]^{\frac{1}{2}} y(r; k), \text{ where } \xi := \int_0^r [n(\rho)]^{\frac{1}{2}} d\rho. \]  
(1.28)

Moreover, if we set
\[ B := \int_0^1 [n(\rho)]^{\frac{1}{2}} d\rho, \]  
(1.29)

then
\[ \begin{cases} 
z'' + [k^2 - p(\xi)]z = 0; \\
z(0) = 0; (-k)z(B; k)j_1'(k) + z'(B; k)j_1(k) - z(B; k)j_1(k) = 0, \end{cases} \]  
(1.30)

where
\[ p(\xi) := \frac{n''(r)}{4[n(r)]^2} - \frac{5}{16} \frac{[n'(r)]^2}{[n(r)]^3} + \frac{2}{r^2 n(r)} \cdot \]  
(1.31)

We rephrase the system above again.
\[ \begin{cases} 
z'' + [k^2 - q(\xi) - \frac{2}{r^2}]z = 0; \\
z(0) = 0; D(k) = (-k)z(B; k)j_1'(k) + z'(B; k)j_1(k) - z(B; k)j_1(k) = 0, \end{cases} \]  
(1.32)

where
\[ q(\xi) := \frac{n''(r)}{4[n(r)]^2} - \frac{5}{16} \frac{[n'(r)]^2}{[n(r)]^3} + \frac{2}{r^2 n(r)} - \frac{2}{\xi}. \]  
(1.33)

The fundamental estimates of its solution is found in Somasundaram [10] Lemma 5.5 which is based on the methods in [11]. The (5.25) in [10] needs to be considered subject to its (5.27) on page 45. In particular, we need the following estimates to (1.32). For \( |k| > 1 \),
\[ z(\xi; k) = \frac{3 \sin(k\xi)}{k^2 \xi} - \frac{3 \cos(k\xi)}{k^2} + O\left(\exp\{8\|q\| \sqrt{\xi}\} \exp\{|3k|\xi\}\right); \]  
(1.34)
\[ z'(\xi; k) = \frac{3 \sin(k\xi)}{k^2} \left(\frac{1}{k^2 \xi^2} + k\right) + \frac{3 \cos(k\xi)}{k^2 \xi} + O\left(\exp\{8\|q\| \sqrt{\xi}\} \exp\{|3k|\xi\}\right). \]  
(1.35)

They are bounded over \( 0i + \mathbb{R} \). We will apply Cartwright’s theory to such entire functions.

May we ask that if the set of the interior transmission eigenvalues of (1.2), in particular, the set of interior transmission eigenvalues of the acoustic system (1.4) or zeros of \( D(k) \), can uniquely determine the refraction index \( n^i(\gamma) \)?

Following the local uniqueness results in [7, 8], we state the uniqueness result in this paper.

**Theorem 1.1** Let the functional determinant \( D^i(z) \), \( i = 1, 2 \), be defined by refraction index \( n^i(\gamma) \) as in (1.15) and (1.10). If the zeros of \( D^i(z) \) inside any of the angular wedges
\[ \Sigma_1 := \{ k \in \mathbb{C} | -\epsilon < \arg k \leq \epsilon \}, \]  
(1.36)
\[ \Sigma_2 := \{ k \in \mathbb{C} | \pi - \epsilon \leq \arg k \leq \pi + \epsilon \}, \forall \epsilon > 0, \]  
(1.37)

coincide, then \( n^1(\gamma) \equiv n^2(\gamma) \).
2 Counting the Zeros: Cartwright’s Theorem

From (1.13), we compute the $D(k)$ as follows.

$$D(k) = (-k)z(B;k)j'_1(k) + z'(B;k)j_1(k) - z(B;k)j_1(k),$$

where

$$j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z},$$

which is an entire function of order 1. Moreover, using the asymptotics (1.24) and (1.35), we compute the following asymptotics.

$$D(k) = (-k)z(B;k)\left(\frac{2 \cos(k)}{k^2} + \left(\frac{1}{k} \frac{2}{k^3}\right) \sin k + z'(B;k)\left(\frac{\sin k}{k^2} - \frac{\cos k}{k}\right) - z(B;k)\left(\frac{\sin k}{k^2} - \frac{\cos k}{k}\right)\right)$$

$$= (-k)z(B;k)\left(\frac{3 \cos(kB)}{k^2} + \left(\frac{1}{k} \frac{1}{k^3}\right) \sin k + z'(B;k)\left(\frac{\sin k}{k^2} - \frac{\cos k}{k}\right)\right)$$

$$+ 3 \sin(kB)\left(\frac{\sin k}{k^2}\right) - \frac{\cos k}{k}[1 + O(\frac{1}{k})], \forall k \notin 0i + \mathbb{R},$$

where we have used the fact that $\tan z$ and $\cot z$ are bounded outside $0i + \mathbb{R}$.

Moreover, we define

$$D_k := \frac{k^4 D(k)}{3}$$

$$\cos(Bk) \cos(k[|k| + (k^2 - 1) \tan k][1 + O(\frac{1}{k})])$$

$$+ \tan(Bk)\left|k\tan k - k^2\right|[1 + O(\frac{1}{k})]), \forall k \notin 0i + \mathbb{R}.$$ (2.5)

(2.6)

For such a representation form of an entire function, we consider one type of the theorems concerning the distribution of the zeros of certain class of entire functions. We apply the Cartwright’s theory. We refer the Cartwright’s theory to the Levin’s book [5, 6] and [2, 3]. Let us review the following verbatim.

**Definition 2.1.** Let $f(z)$ be an entire function. Let $M_f(r) := \max_{|z| = r} |f(z)|$. An entire function of $f(z)$ is said to be a function of finite order if there exists a positive constant $k$ such that the inequality

$$M_f(r) < e^{rk}$$

is valid for all sufficiently large values of $r$. The greatest lower bound of such numbers $k$ is called the order of the entire function $f(z)$. By the type $\sigma$ of an entire function $f(z)$ of order $\rho$, we mean the greatest lower bound of positive number $\lambda$ for which asymptotically we have

$$M_f(r) < e^{A\rho^\sigma}.$$ (2.8)

That is

$$\sigma := \limsup_{r \to \infty} \frac{\ln M_f(r)}{r^\rho}.\) (2.9)

If $0 < \sigma < \infty$, then we say $f(z)$ is of normal type or mean type.

We also see that

$$e^{(\sigma - \epsilon) \rho^\sigma} \leq M_f(r) \leq e^{(\sigma + \epsilon) \rho^\sigma},$$ (2.10)

where the first inequality holds for some sequence going to infinity and the second one holds asymptotically.

**Definition 2.2.** If an entire function $f(z)$ is of order one and of normal type, then we say it is an entire function of exponential type $\sigma$. 


Definition 2.3 Let \( \rho \in \mathbb{R} \) and \( \rho(r) : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \). We say \( \rho(r) \) is a proximate order to \( \rho \) if
\[
\lim_{r \to \infty} \rho(r) = \rho \geq 0; \lim_{r \to \infty} r \rho'(r) \ln r = 0.
\] (2.11)

Definition 2.4 Let \( f(z) \) be an integral function of finite order in the angle \([\theta_1, \theta_2]\). We call the following quantity as the generalized indicator of the function \( f(z) \).
\[
h_f(\theta) := \limsup_{r \to \infty} \frac{\ln |f(re^{i\theta})|}{r \rho(r)}, \quad \theta_1 \leq \theta \leq \theta_2,
\] (2.12)
where \( \rho(r) \) is some proximate order.

The order and the type of an integral function in an angle can be defined similarly. The connection between the indicator \( h_f(\theta) \) and its type \( \sigma_f \) is specified by the following theorem.

Lemma 2.5 (Levin [5], p.72) The maximum value of the indicator \( h_f(\theta) \) of the function \( f(z) \) on the interval \( \alpha \leq \theta \leq \beta \) is equal to the type \( \sigma_f \) of this function inside the angle \( \alpha \leq \arg z \leq \beta \).

Lemma 2.6 Let \( a, b \) be real constants.
\[
h_{\sin(az+b)}(\theta) = |a \sin \theta|;
\] (2.13)
if \( p(z) \) is a polynomial with bounded holomorphic coefficients, then
\[
h_{p(z)}(\theta) = 0.
\] (2.14)

Proof We apply definition (2.12), we prove the lemma. \( \square \)

We mention two more inequalities for indicator functions.

Lemma 2.7 Let \( f, g \) be two entire functions. Then, the following two inequalities hold.
\[
h_{fg}(\theta) = h_f(\theta) + h_g(\theta), \text{ if one limit exists;}
\] (2.15)
\[
h_{f+g}(\theta) \leq \max_{\theta} \{h_f(\theta), h_g(\theta)\},
\] (2.16)
where if the indicator of the two summands are not equal at some \( \theta_0 \), then the equality holds in (2.16).

Proof We can find these in [5, p.51]. \( \square \)

Definition 2.8 The following quantity is called the width of the indicator diagram of entire function \( f \):
\[
\tilde{d} = h_f\left(\frac{\pi}{2}\right) + h_f\left(-\frac{\pi}{2}\right).
\] (2.17)

Definition 2.9 Let \( f(z) \) be an entire function of order \( \rho(r) \). We use \( N(f, [\alpha, \beta], r) \) to denote the number of the zeros of \( f(z) \) inside the angle \([\alpha, \beta]\) and \( |z| \leq r \); we define the density function
\[
\Delta_f(\alpha, \beta) := \limsup_{r \to \infty} \frac{N(f, [\alpha, \beta], r)}{r \rho(r)},
\] (2.18)
and
\[
\Delta(\beta) := \Delta(\alpha_0, \beta),
\] (2.19)
with fixed \( \alpha_0 \notin E \) with \( E \) as an at most countable set.

The distribution on the zeros of an entire function is described precisely by the following Cartwright’s theorem [2, 3, 5, 6]. The following statements are from Levin [5, ch.5, sec.4].
**Theorem 2.10 (Cartwright)** If an entire function of exponential type satisfies one of the following conditions:

\[ \int_0^\infty \ln|f(x)| \frac{dx}{1 + x^2} \text{ exists, and } h_f(0) = h_f(\pi) = 0, \quad (2.20) \]

then

1. \( f(z) \) is of class A and is of completely regular growth and its indicator diagram is an interval on the imaginary axis. In particular, for some constant \( \kappa \), we have
   \[ h_f(\theta) = \kappa \sin \theta; \quad (2.25) \]

2. all of the zeros of the function \( f(z) \), except possibly those of a set of zero density, lie inside arbitrarily small angles \( |\arg z| < \epsilon \) and \( |\arg z - \pi| < \epsilon \), where the density
   \[ \Delta_i = \lim_{r \to \infty} \frac{n_i(r)}{r}, \quad i = 1, 2, \quad (2.26) \]

of the set of zeros within each of these angles is equal to \( \frac{\delta}{\pi} \), where \( \delta \) is the width of the indicator diagram in \( (2.17) \). Moreover, \( n_i(r), \quad i = 1, 2 \), here is understood as the number of the zeros that fall in the wedge \( |\arg z| < \epsilon \) and \( |\arg z - \pi| < \epsilon \) respectively. Furthermore, the limit \( \delta = \lim_{r \to \infty} \delta(r) \) exists, where
   \[ \delta(r) := \sum_{|a_k| < r} \frac{1}{|a_k|}; \quad (2.27) \]

3. moreover,
   \[ \Delta_f(\epsilon, \pi - \epsilon) = \Delta_f(\pi + \epsilon, -\epsilon) = 0, \quad (2.28) \]

4. the function \( f(z) \) can be represented in the form
   \[ f(z) = cz^m e^{iCz} \lim_{r \to \infty} \prod_{|a_k| < r} (1 - \frac{z}{a_k}), \quad (2.29) \]

where \( c, m, B \) are constants and \( C \) is real.

Therefore, we apply this theorem to make the following conclusion.

**Proposition 2.11** The indicator function of \( D(z) \) and \( D(z) \) are equal and

\[ h_D(\theta) = |1 + B||\sin \theta|. \quad (2.30) \]

**Proof** We examine \( (2.5) \). When \( \theta \neq 0, \pi \), we see that \( \tan z \) and \( \cot z \) are bounded functions. Hence, we can use the product to sum formula to see that

\[ D(k) = \left\{ \frac{1}{2} \cos((B - 1)k) + \frac{1}{2} \cos((B + 1)k) \right\} \]
\[ \times \{|k + (k^2 - 1) \tan k||1 + O(\frac{1}{k})| + \tan(Bk)|k \tan k - k^2|(1 + O(\frac{1}{k}))\}. \quad (2.31) \]

Hence, using Lemma 2.6 and Lemma 2.7

\[ h_D(\theta) = \max\{|1 - B||\sin \theta|, (1 + B)|\sin \theta|\} = (1 + B)|\sin \theta|, \quad \theta \neq 0, \pi. \quad (2.32) \]

Given \( D(z) \) is entire, \( h(\theta) \) is a continuous function of \( \theta \). We refer this to Levin [5, p.54]. Hence, the statement is proved for \( D(z) \) for all \( \theta \in [0, 2\pi] \). Surely, \( D(z) \) and \( D(z) \) have the same indicator function.
Moreover, we prove the following density theorem.

**Theorem 2.12** The length of the indicator diagram of $D(z)$ is $2(1 + B)$. The density in each of the two small angles along real axis is

$$\Delta_D(-\epsilon, \epsilon) = \Delta_D(\pi - \epsilon, \pi + \epsilon) = (1 + B)/\pi.$$  \hfill (2.33)

**Proof** This follows from Cartwright’s theorem as (2.26) and definition (2.17). □

## 3 Proof of Theorem 1.1

Let $D^i(z)$ be the functional determinant corresponding to refraction index $n^i(r)$, $i = 1, 2$. If the zeros of $D^i$ in either wedge coincide, then Theorem 2.12 tells us that

$$B^1 = B^2,$$ \hfill (3.1)

where $B^i := \int_0^1 \sqrt{n^i(\rho)}d\rho$. Let us consider

$$F(k) := y^1(1;k) - y^2(1;k).$$ \hfill (3.2)

Let $k_j$ be a common zero of $D^1(k)$ and $D^2(k)$, then, using the boundary condition in the third equation in system (1.26),

$$F(k_j) = y^1(1;k_j) - y^2(1;k_j) = 0.$$ \hfill (3.3)

Moreover, we use (1.34) and (2.13) to obtain

$$h_F(\theta) \leq B^1|\sin \theta|.$$ \hfill (3.4)

Therefore, its indicator diagram is

$$h_F(-\frac{\pi}{2}) + h_F(\frac{\pi}{2}) \leq 2B^1.$$ \hfill (3.5)

To consider an uniqueness problem in entire function theory, we apply a generalized Carlson’s theorem from Levin [5, p.190].

**Theorem 3.1** Let $F(k)$ be holomorphic and at most of normal type with respect to the proximate order $\rho(r)$ in the angle $\alpha \leq \arg k \leq \alpha + \pi/\rho$ and vanish on a set $N := \{a_k\}$ in this angle, with angular density $\Delta_N(\psi)$. Let

$$H_N(\theta) := \pi \int_\alpha^{\alpha + \pi/\rho} \sin |\psi - \theta|d\Delta_N(\psi),$$

when $\rho$ is integral. Then, if $F(k)$ is not identically zero,

$$h_F(\alpha) + h_F(\alpha + \pi/\rho) \geq H_N(\alpha) + H_N(\alpha + \pi/\rho).$$ \hfill (3.6)

In this paper, we consider $\rho \equiv 1$, $\alpha = -\frac{\pi}{2}$. We set the collection of interior transmission eigenvalues as

$$N := \{k_1, k_2, \cdots\}.$$ \hfill (3.7)

From Theorem 2.12 and (2.28),

$$\Delta_D(-\epsilon, \epsilon) = \Delta_D(\pi - \epsilon, \pi + \epsilon) = (1 + B^1)/\pi;$$ \hfill (3.8)

$$\Delta_D(\epsilon, \pi - \epsilon) = \Delta_D(\pi + \epsilon, -\epsilon) = 0.$$ \hfill (3.9)

Therefore,

$$H_N(\theta) = \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin |\psi - \theta|d\Delta_K(\psi) = (1 + B^1)|\sin \theta|, \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}].$$ \hfill (3.10)

We refer to [5, p.91; Ch 2. Sec 3.] for a complete introduction on this set indicator function. Hence,

$$H_N(-\frac{\pi}{2}) + H_N(\frac{\pi}{2}) = 2(1 + B^1).$$ \hfill (3.11)
Now (3.10), (3.11) and Theorem 3.1 imply that
\[ y^1(1; k) \equiv y^2(1; k). \]  
(3.12)

Similar argument can show that
\[ (y^1)'(1; k) \equiv (y^2)'(1; k). \]  
(3.13)

We seek to identify \( n(r) \) by inverse Sturm-Liouville theorem as in [1, 7, 8, 9]. In particular, we apply the methods reviewed in [8].

**Theorem 3.2 (McLaughlin)** We consider the following Sturm-Liouville problem
\[
\begin{align*}
  z'' + (k^2 - q)z &= 0, \quad 0 < x < 1; \\
  z(0) &= z(1) = 0,
\end{align*}
\]  
(3.14)

where \( q \in L^2(0,1) \). For another boundary condition,
\[
  z(0) = z'(1) = 0. \]  
(3.16)

Suppose \( q_1, q_2 \in L^2(0,1) \) and, \( \lambda_n(q_1) = \lambda_n(q_2) \), the eigenvalues to (3.14) and (3.15), \( \mu_n(q_1) = \mu_n(q_2) \), the eigenvalues to (3.13) and (3.16), \( \forall n \in \mathbb{N} \). Then, \( q_1 \equiv q_2 \), a.e.

Under the Liouville transform (1.28), the zeros of \( z(B; k) \) exactly corresponds to the eigenvalues of the Sturm-Liouville problem
\[
\begin{align*}
  z'' + [k^2 - p(\xi)]z &= 0, \quad 0 < \xi < B; \\
  z(0) &= z(B) = 0.
\end{align*}
\]  
(3.17)

Similarly, the zeros of \( z'(B; k) \) exactly corresponds to the eigenvalues of the Sturm-Liouville problem
\[
\begin{align*}
  z'' + [k^2 - p(\xi)]z' &= 0, \quad 0 < \xi < B; \\
  z(0) &= z'(B) = 0.
\end{align*}
\]  
(3.18)

Now \( z^i(B; k) \) and \( (z^i)'(B; k) \) corresponding to refraction index \( n^i(r) \) have common zeros by (3.12) and (3.13). Hence, the Sturm-Liouville problems
\[
\begin{align*}
  (z^i)'' + [k^2 - p^i(\xi)]z^i &= 0, \quad 0 < \xi < B; \\
  z^i(0) &= z^i(B) = 0,
\end{align*}
\]  
(3.19)

have the same eigenvalues for both \( i = 1, 2 \). Similarly,
\[
\begin{align*}
  (z^i)''' + [k^2 - p^i(\xi)]z^i &= 0, \quad 0 < \xi < B; \\
  z^i(0) &= (z^i)'(B) = 0.
\end{align*}
\]  
(3.20)

have the same eigenvalues for both \( i = 1, 2 \). Given \( p^i \in L^2, \ i = 1, 2 \), Theorem 3.2 says these imply \( p^1 \equiv p^2 \) almost everywhere. However, (1.31) and (1.32) implies \( p^1 \equiv p^2 \). The solution is unique, so we have
\[
z^1(\xi; k) \equiv z^2(\xi; k), \forall \xi, k.
\]  
(3.21)

This says that
\[
[n^1(r)]^\ddagger \equiv [n^2(r)]^\ddagger, \forall r, k. 
\]  
(3.22)

Since \( n^i \) never vanishes, the solutions \( y^1(r; k), y^2(r; k) \) have common zero set in \( \mathbb{C} \) of its maximal density as described by Cartwright’s theory for any fixed \( r \). From (1.31) and (2.13), their common density is \( \int_0^{\infty} \sqrt{n^1(\rho)} d\rho \). Hence,
\[
\int_0^r [n^1(\rho)]^\ddagger d\rho = \int_0^r [n^2(\rho)]^\ddagger d\rho, \forall r. 
\]  
(3.23)

This implies that \( n^1 \equiv n^2 \). □
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