ABELIAN DUALITY AND ABELIAN WILSON LOOPS

by

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Abstract

We consider a pure $U(1)$ quantum gauge field theory on a general Riemannian compact four manifold. We compute the partition function with Abelian Wilson loop insertions. We find its duality covariance properties and derive topological selection rules. Finally, we show that, to have manifest duality, one must assume the existence of twisted topological sectors besides the standard untwisted one.

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1. Introduction and conclusions

Electromagnetic Abelian duality is an old subject that has fascinated theoretical physicists for a long time as a means to explain the quantization of electric charge [1,2,3,4] and the apparent absence of magnetic charge [5,6,7,8,9]. Its study has also provided important clues in the analysis of analogous dualities in supersymmetric gauge theory [10,11], supergravity [12,13] and string theory [14,15,16,17]. It is also considerably interesting for the nontrivial interplay of quantum field theory, geometry and topology it shows [18,19,20,21]. The aim of this paper is to further explore these latter aspects of Abelian duality as we briefly outline next. For an updated review of these matters, see for instance refs. [22,23,24].

Consider a pure $U(1)$ gauge field theory on a general Riemannian compact four manifold $M$. The Wick rotated action is

$$S(A, \tau) = \frac{i}{2} \int_M F_A \wedge \ast F_A + \frac{q^2 \theta}{8\pi^2} \int_M F_A \wedge F_A.$$

(1.1)

Here, the charge $q$ and the angle $\theta$ are combined as the real and imaginary parts of the complex parameter

$$\tau = \frac{\theta}{2\pi} + i \frac{2\pi}{q^2}$$

(1.2)

varying in the open upper complex half plane $\mathbb{H}_+$. $A$ is the physical gauge field. Its field strength $F_A = dA$ satisfies the quantization condition

$$\frac{q}{2\pi} \int_{\Sigma} F_A \in \mathbb{Z},$$

(1.3)

for any 2–cycle $\Sigma$.

The quantization of the gauge field theory is attained as usual by summation over all topological classes of the gauge field and by functional integration of the quantum fluctuations of the gauge field about the vacuum gauge configuration of each class with the gauge group volume factored out. In this way, one can compute in principle the partition function possibly with gauge invariant insertions.

It is known that the partition function proper $Z(\tau)$ is a modular form of weights $\frac{\chi + \eta}{4}$, $\frac{\chi - \eta}{4}$ of the subgroup $\Gamma_\nu$ of the modular group generated by

$$\tau \to -1/\tau, \quad \tau \to \tau + \nu,$$

(1.4)

where $\chi$ and $\eta$ are respectively the Euler characteristic and the signature invariant of $M$ and $\nu = 1$ if $M$ is a spin manifold and $\nu = 2$ else [18]. This property of $Z(\tau)$ is what is usually
meant by Abelian duality. The natural question arises whether the partition function with simple gauge invariant insertions exhibits analogous duality covariance properties. Specifically, we shall consider the partition function with insertion of the Abelian Wilson loop associated to a 1–cycle Λ of $M$:

$$Z(\Lambda, \tau) = Z(\tau) \left< \exp \left( iq \oint_\Lambda A \right) \right>_{\tau}. \tag{1.5}$$

In due course, we shall discover the following.

a) Due to a peculiar combination of the contributions of the torsion classical topological classes and the quantum fluctuations in the field theory, the partition function $Z(\Lambda, \tau)$ vanishes unless the 1–cycle $\Lambda$ is a boundary.

b) $Z(\Lambda, \tau)$ is a member of a family of partition functions $Z_A(\Lambda, \tau)$ mixing under the transformations (1.4). $Z_A(\Lambda, \tau)$ is of the general form

$$Z_A(\Lambda, \tau) = \exp \left( -\frac{\pi \sigma(\Lambda)}{\text{Im} \tau} \right) F_A(\Lambda, \tau), \tag{1.6}$$

where $\sigma(\Lambda)$ is the renormalized selfenergy of the classical conserved current associated to the 1–cycle $\Lambda$. When the 1–boundary $\Lambda$ satisfies certain conditions, $F_A(\Lambda, \tau)$ is the $A$-th component of a vector modular form $F(\Lambda, \tau)$ of weights $\frac{\chi_+ + 2}{4}$, $\frac{\chi_- - 2}{4}$ for the subgroup $\Gamma_\nu$.

c) To have manifest duality, one must assume the existence of twisted topological sectors besides the standard untwisted one, one for each independent value of the index $A$. $Z_A(\Lambda, \tau)$ is the partition function of twisted sector $A$.

In a topologically non trivial manifold $M$, the definition of the integral $\oint_\Lambda A$ is not straightforward, as the gauge field $A$ is not a globally defined 1–form. We approach this problem using the theory of the Cheeger–Simons differential characters. This produces however a family of possible definitions of $\oint_\Lambda A$ parameterized by the choices of certain background fields. In spite of this, the result of the calculations of $Z(\Lambda, \tau)$ does not depend on the choices made as it should.

This fact is related to the $\Lambda$ selection rules mentioned above. $Z(\Lambda, \tau)$ is non zero when $\Lambda$ is a 1–boundary. When this happens, the choices entering in the definition of $\oint_\Lambda A$ turn out to be immaterial. The proof of this intriguing result involves an interesting relationship between flat Cheeger–Simons differential characters and Morgan–Sullivan torsion invariants.

The physical significance of the twisted topological sectors remains to be explored. It seems to indicate that the non perturbative structure of electrodynamics might be far richer than thought so far.
Plan of the paper

In sect. 2, we introduce the necessary topological set up. We use this to properly define the Wilson loop corresponding to a given 1–cycle. In sect. 3, we proceed to the calculation of the partition function with a Wilson loop insertion and show that it vanishes unless the associated 1–cycle is a boundary. In sect. 4, we study the duality properties of the partition function and show the existence of twisted topological sectors. Finally, in the appendix, we collect some of the technical details of the calculation of the partition function.

Conventions and notation

For a review of the mathematical formalism, see for instance [25]. For a clear exposition of its field theoretic applications, see [26].

In this paper, \( M \) denotes a compact connected oriented four manifold.

For a sheaf of Abelian groups \( \mathcal{F} \) over \( M \), \( H^p(M, \mathcal{F}) \) denotes the \( p \)–th sheaf cohomology group of \( \mathcal{F} \) and \( \text{Tor}(M, \mathcal{F}) \) its torsion subgroup. For an Abelian group \( G \), \( G \) denotes the associated constant sheaf on \( M \). For an Abelian Lie group \( G \), \( G \) denotes the sheaf of germs of smooth \( G \)–valued functions on \( M \).

\( C^p_s(M), Z^p_s(M), B^p_s(M) \) denote the groups of smooth singular \( p \)–chains, cycles and boundaries of \( M \), respectively, and \( b \) the boundary operator. \( H^p_s(M) \) denotes the \( p \)–th singular homology group and \( \text{Tor}^p_s(M) \) its torsion subgroup. For an Abelian group \( G \), \( C^p_{sG}(M), Z^p_{sG}(M), B^p_{sG}(M) \) denote the groups of smooth singular \( p \)–cochains, cocycles and coboundaries of \( M \) with coefficients in \( G \), respectively, and \( d \) the coboundary operator. \( H^p_{sG}(M) \) denotes the \( p \)–th singular cohomology group with coefficients in \( G \) and \( \text{Tor}^p_{sG}(M) \) its torsion subgroup.

\( C^p_{dR}(M), Z^p_{dR}(M), B^p_{dR}(M) \) denote the groups of general, closed and exact smooth \( p \)–forms of \( M \), respectively, and \( d \) the differential operator. \( H^p_{dR}(M) \) denotes \( p \)–th de Rham cohomology space. Further, \( Z^p_{dRZ}(M) \) denote the subgroup of closed smooth \( p \)–forms of \( M \) with integer periods and \( H^p_{dRZ}(M) \) the integer cohomology lattice in \( H^p_{dR}(M) \). \( q \) denotes the natural homomorphism of \( H^p(M, \mathbb{Z}) \) into \( H^p_{dR}(M) \). \( b_p \) denotes the \( p \)–th Betti number. When \( M \) is equipped with a metric \( g \), \( \text{Harm}^p(M) \) denotes the space of harmonic \( p \)–forms of \( M \) and \( \text{Harm}^p_{\mathbb{Z}}(M) \) the lattice \( \text{Harm}^p(M) \cap Z^p_{dRZ}(M) \). \( b^\pm_2 \) denotes the dimension of the space of (anti)selfdual harmonic 2–forms.
2. U(1) principal bundles, connections and Cheeger–Simons characters

In this section, we review well known facts about $U(1)$ principal bundles, connections and Cheeger Simons differential characters, which are relevant in the following. See ref. [27] for background material.

2.1 Smooth and flat principal bundles

The quantization of Maxwell theory involves a summation over the topological classes of the gauge field. Mathematically, these classes can be identified with the isomorphism classes of smooth $U(1)$ principal bundles, which we describe below.

The group of isomorphism classes of smooth $U(1)$ principal bundles on $M$, $\text{Princ}(M)$, can be identified with the 1–st cohomology of the sheaf $U(1)$:

$$\text{Princ}(M) = H^1(M, U(1)).$$  \hspace{1cm} (2.1.1)

There is a well known alternative more convenient characterization of $\text{Princ}(M)$ derived as follows. Consider the short exact sequence of sheaves

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \xrightarrow{i} U(1) \rightarrow 0,$$  \hspace{1cm} (2.1.2)

where $i(n) = n$ for $n \in \mathbb{Z}$ and $e(x) = \exp(2\pi ix)$ for $x \in \mathbb{R}$. The associated long exact sequence of sheaf cohomology contains the segment

$$\cdots \rightarrow H^1(M, \mathbb{R}) \xrightarrow{e_*} H^1(M, U(1)) \xrightarrow{c} H^2(M, \mathbb{Z}) \xrightarrow{i_*} H^2(M, \mathbb{R}) \rightarrow \cdots.$$  \hspace{1cm} (2.1.3)

Since $\mathbb{R}$ is a fine sheaf, $H^p(M, \mathbb{R}) = 0$ for all $p \geq 1$. Therefore, $c$ is an isomorphism $H^1(M, U(1)) \cong H^2(M, \mathbb{Z})$. It follows that

$$\text{Princ}(M) \xrightarrow{c} H^2(M, \mathbb{Z}).$$  \hspace{1cm} (2.1.4)

This isomorphism associates to any smooth $U(1)$ principal bundle $P$ its Chern class $c_P$.

Flat $U(1)$ principal bundles play an important role in determining the selection rules of the Abelian Wilson loops, as will be shown later. It is therefore necessary to understand their place within the family of smooth $U(1)$ principal bundle.

The group of isomorphism classes of flat $U(1)$ principal bundles on $M$, $\text{Flat}(M)$, can be identified with the 1–st cohomology of the constant sheaf $U(1)$:

$$\text{Flat}(M) = H^1(M, U(1)).$$  \hspace{1cm} (2.1.5)
There is an obvious natural sheaf morphism $U(1) \to U(1)$, to which there corresponds a homomorphism $H^1(M, U(1)) \to H^1(M, U(1))$ of sheaf cohomology. By (2.1.1), (2.1.5), this can be viewed as a homomorphism of Flat$(M)$ into Princ$(M)$. Its image is the subgroup of smooth isomorphism classes of flat principal bundles, Princ$_0(M)$.

On account of (2.1.4), Princ$_0(M)$ is isomorphic to a subgroup of $H^2(M, \mathbb{Z})$, which we shall identify next. Consider the short exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathbb{R} \to U(1) \to 0,$$

where $i$ and $e$ are defined as above. The associated long exact sequence of sheaf cohomology contains the segment

$$\cdots \to H^1(M, \mathbb{R}) \xrightarrow{e_*} H^1(M, U(1)) \xrightarrow{c} H^2(M, \mathbb{Z}) \xrightarrow{i_*} H^2(M, \mathbb{R}) \to \cdots.$$  

(2.1.7)

Recalling that $\text{Tor}^2(M, \mathbb{Z}) = \ker i_*|H^2(M, \mathbb{Z})$, $c$ induces an isomorphism $H^1(M, U(1))/e_*H^1(M, \mathbb{R}) \cong \text{Tor}^2(M, \mathbb{Z})$. Using the Čech realization of sheaf cohomology, it is easy to see that $H^1(M, U(1))/e_*H^1(M, \mathbb{R})$ is isomorphic to the image of $H^1(M, U(1))$ in $H^1(M, U(1))$. Therefore, we conclude that

$$\text{Princ}_0(M) \cong \text{Tor}^2(M, \mathbb{Z}).$$  

(2.1.8)

Combining (2.1.4), (2.1.8), we conclude that there is a commutative diagram

$$\begin{array}{ccc}
\text{Princ}_0(M) & \xrightarrow{c} & \text{Tor}^2(M, \mathbb{Z}) \\
\subseteq & \downarrow & \subseteq \\
\text{Princ}(M) & \xrightarrow{c} & H^2(M, \mathbb{Z}),
\end{array}$$

(2.1.9)

where the lines are isomorphisms. This describes in some detail the set of $U(1)$ principal bundles on $M$.

Before proceeding to the next topic, the following remark is in order. The Chern class $c_P$ of a principal $U(1)$ bundle $P$ belongs by definition to the cohomology group $H^2(M, \mathbb{Z})$. Another definition identifies the Chern class of $P$ with $q(c_P)$, the natural image of $c_P$ in the integer lattice $H^2_{dR\mathbb{Z}}(M)$ of de Rham cohomology. The advantage of the first definition, adopted in this paper, is that it discriminates principal bundles differing by a flat bundle. The second, though more popular in the physics literature, does not.
2.2 The gauge group

The fixing of the gauge symmetry is an essential step of the quantization of Maxwell theory. Below, we recall the main structural properties of the gauge group.

For \( P \in \text{Princ}(M) \), the gauge group of \( P \), \( \text{Gau}(P) \), can be identified with the 0–th cohomology of the sheaf \( \underline{U}(1) \):

\[
\text{Gau}(P) = H^0(M, \underline{U}(1)).
\] (2.2.1)

Its elements are often called large gauge transformations in the physics literature.

The flat gauge group of \( P \), \( G(P) \), can similarly be identified with the 0–th cohomology of the constant sheaf \( \underline{U}(1) \):

\[
G(P) = H^0(M, U(1)).
\] (2.2.2)

Its elements are commonly called rigid gauge transformations.

The natural sheaf morphism \( \underline{U}(1) \to U(1) \) induces a homomorphism \( H^0(M, \underline{U}(1)) \to H^0(M, U(1)) \) of sheaf cohomology, which is readily seen to be an injection. Thus, \( G(P) \) is isomorphic to a subgroup \( \text{Gau}_0(P) \) of \( \text{Gau}(P) \).

Note that

\[
G(P) \cong \text{Gau}_0(P) \cong U(1).
\] (2.2.3)

\( \text{Gau}(P) \) and \( G(P) \) or \( \text{Gau}_0(P) \) do not depend on \( P \). Therefore, to emphasize this fact, we shall occasionally denote these groups by \( \text{Gau}(M) \) and \( G(M) \) or \( \text{Gau}_0(M) \), respectively.

For \( h \in H^0(M, \underline{U}(1)) \), define

\[
\alpha(h) = \frac{1}{2\pi i} h^{-1} dh.
\] (2.2.4)

It is straightforward to show that \( \alpha(h) \in Z^1_{dRZ}(M) \) and that the map \( \alpha : H^0(M, \underline{U}(1)) \to Z^1_{dRZ}(M) \) is a group homomorphism with range \( Z^1_{dRZ}(M) \) and kernel \( H^0(M, U(1)) \). Thus, on account of (2.2.1)–(2.2.3), we have the important isomorphism

\[
\text{Gau}(M)/\text{Gau}_0(M) \cong Z^1_{dRZ}(M).
\] (2.2.5)

The counterimage of \( B^1_{dR}(M) \) by \( \alpha \) is the subgroup \( \text{Gau}_c(M) \) of \( \text{Gau}(M) \) of the gauge group elements homotopic to the identity. Its elements are called small gauge transformations in the physics literature. Obviously, \( \text{Gau}_0(M) \subseteq \text{Gau}_c(M) \). Thus,

\[
\text{Gau}_c(M)/\text{Gau}_0(M) \cong B^1_{dR}(M).
\] (2.2.6)
The quotient $\text{Gau}(M)/\text{Gau}_c(M)$ is the gauge class group. By the above,

$$\text{Gau}(M)/\text{Gau}_c(M) \cong H^1_{dR}(M).$$  \tag{2.2.7}

### 2.3 Connections

After rescaling by a suitable factor $q/2\pi$, the photon gauge field of Maxwell theory can mathematically be characterized as a connection of some $U(1)$ principal bundle. Next, we recall the main properties of the set of connections of a $U(1)$ principal bundle.

For any $P \in \text{Princ}(M)$, the family of connections of $P$, $\text{Conn}(P)$, is an affine space modeled on $C^1_{dR}(M)$. For $A \in \text{Conn}(P)$,

$$F_A = dA$$  \tag{2.3.1}

is the curvature of $A$. As well known, $F_A \in Z^2_{dR}(M)$ and $q(c_P) = [F_A]_{dR}$ (cfr. eq. (2.1.4)).

If $P, P' \in \text{Princ}(M)$, $A \in \text{Conn}(P)$, $A' \in \text{Conn}(P')$, then $A + A' \in \text{Conn}(PP')$. If $P \in \text{Princ}_0(M) \subseteq \text{Princ}(M)$ is flat, then $0 \in \text{Conn}(P)$. So, if $P \in \text{Princ}(M)$, $P' \in \text{Princ}_0(M)$, then $\text{Conn}(PP') = \text{Conn}(P)$. In particular, $\text{Conn}(P') = \text{Conn}(1) = C^1_{dR}(M)$.

For $P \in \text{Princ}(M)$, $\text{Gau}(P)$ acts on $\text{Conn}(P)$ as usual, viz

$$A^h = A + \alpha(h)$$  \tag{2.3.2}

for $A \in \text{Conn}(P)$ and $h \in \text{Gau}(P)$ (cfr. eq. (2.2.4)). Note that $\text{Gau}_0(P)$ is precisely the invariance subgroup of $A$.

### 2.4 Cheeger Simons differential characters

As is well known, if $A$ is a connection of some principal $U(1)$ bundle $P$, the line integral $\int_A \Lambda$ over some closed path cannot be defined in the usual naive sense, since $A$ suffers local gauge ambiguities and, thus, is not a globally defined 1–form. Nevertheless, one can try to give a meaning to such a formal expression modulo integers using the theory of the Cheeger Simons differential characters, whose main features are described below [27,28,29,30].

A Cheeger Simons differential character is a mathematical object having the formal properties characterizing the holonomy map of a principal $U(1)$ bundle. It has however a somewhat wider scope, since it is defined for singular 1–cycles, which are objects more general than closed paths. Roughly speaking, we define the formal integral $\int_A \Lambda$ as the logarithm of a suitably chosen differential character computed at the appropriate 1–cycle $\Lambda$. 

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A Cheeger Simons differential character is a group homomorphism \( \Phi : Z^1_f(M) \to U(1) \) such that there is a 2–form \( F_\Phi \in C^2_{dR}(M) \) for which
\[
\Phi(bS) = \exp \left( 2\pi i \int_S F_\Phi \right),
\]
for \( S \in C^2_f(M) \). The Cheeger Simons differential characters form naturally a group \( CS^2(M) \).

From (2.4.1), it is simple to see that, for \( \Phi \in CS^2(M) \), \( F_\Phi \in Z^2_{dRZ}(M) \) and that the map \( F : CS^2(M) \to Z^2_{dRZ}(M) \), \( \Phi \mapsto F_\Phi \) is a group homomorphism.

To any \( \Phi \in CS^2(M) \), there is associated a class \( c_\Phi \in H^2(M,\mathbb{Z}) \) such that \( q(c_\Phi) = [F_\Phi]_{dR} \) defined as follows. Since \( U(1) \cong \mathbb{R}/\mathbb{Z} \) is a divisible group and \( Z^1_f(M) \) is a subgroup of the free group \( C^1_f(M) \), there exists a real cochain \( f \in C^1_{s\mathbb{R}}(M) \) such that \( \Phi = \exp \left( 2\pi if \right) \). Then, by (2.4.1),
\[
\varsigma(S) = f(bS) - \int_S F_\Phi, \quad S \in C^2_f(M),
\]
defines an integer cochain \( \varsigma \in C^2_{s\mathbb{R}}(M) \). It is readily checked that \( \varsigma \in Z^2_{s\mathbb{Z}}(M) \) is an integer cocycle which, viewed as a real cocycle, is cohomologically equivalent to \( F_\Phi \). The choice of \( f \) affects \( \varsigma \) at most by an integer coboundary. Hence, the class \( c_\Phi \) of \( \varsigma \) in the 2–nd integer cohomology \( H^2_{s\mathbb{Z}}(M) \) is unambiguously determined by \( \Phi \). The statement then follows from the isomorphism of integer singular and sheaf cohomology. It is simple to see that the map \( c : CS^2(M) \to H^2(M,\mathbb{Z}) \), \( \Phi \mapsto c_\Phi \) is a group homomorphism.

To any \( v \in C^1_{dR}(M) \), there is associated an element \( \chi_v \in CS^2(M) \) by
\[
\chi_v(\Lambda) = \exp \left( 2\pi i \int_{\Lambda} v \right), \quad \Lambda \in Z^1_f(M).
\]
One has \( F_{\chi_v} = dv \) and \( c_{\chi_v} = 0 \). Clearly \( \chi_v \) depends only on the class of \( v \) mod \( Z^1_{dRZ}(M) \) and the map \( \chi : C^1_{dR}(M)/Z^1_{dRZ}(M) \to CS^2(M) \), \( [v] \mapsto \chi_v \) is a group homomorphism. When \( a \in Z^1_{dRZ}(M) \subseteq C^1_{dR}(M) \), \( \chi_a \) depends only on the cohomology class of \( a \) in \( H^1_{dR}(M) \) mod \( H^1_{dRZ}(M) \) and the map \( \chi : H^1_{dR}(M)/H^1_{dRZ}(M) \to CS^2(M) \), \( [a] \mapsto \chi_a \) is again a group homomorphism.

The above properties are encoded in the short exact sequences
\[
0 \to H^1_{dR}(M)/H^1_{dRZ}(M) \xrightarrow{\chi} CS^2(M) \xrightarrow{(c,F)} A^2_{dR}(M) \to 0, \tag{2.4.4}
\]
\[
0 \to C^1_{dR}(M)/Z^1_{dRZ}(M) \xrightarrow{\chi} CS^2(M) \xrightarrow{c} H^2(M,\mathbb{Z}) \to 0. \tag{2.4.5}
\]
Here, $A^2_\mathbb{Z}(M)$ is the subset of the Cartesian product $H^2(M, \mathbb{Z}) \times Z^2_{dR\mathbb{Z}}(M)$ formed by the pairs $(e, G)$ such that $q(e) = [G]_{dR}$.

Before entering the details of the definition of the formal integral $\oint A$, with $P \in \text{Princ}(M)$, $A \in \text{Conn}(P)$ and $\Lambda \in Z^1_\mathbb{Z}(M)$, let us recall the properties which it is required to have. First, when $\Lambda$ is a boundary, so that $\Lambda = bS$ for some $S \in C^1(\mathbb{R})$, one has

$$\oint A = \int_S F_A, \quad \text{mod } \mathbb{Z},$$

(2.4.6)

where the integral in the right hand side is computed according to the ordinary differential geometric prescription. This is a formal generalization of Stokes’ theorem. Second, for $v \in C^1_{dR}(M)$, the obvious relation

$$\oint (A + v) = \oint A + \oint v, \quad \text{mod } \mathbb{Z},$$

(2.4.7)

holds, where the second integral in the right hand side is computed according to the ordinary differential geometric prescription. This property may be called semilinearity. Third, for $h \in \text{Gau}(P)$,

$$\oint A^h = \oint A \quad \text{mod } \mathbb{Z}.$$  

(2.4.8)

In this way, gauge invariance is ensured. This property, albeit important, is not independent from the others. Indeed, it follows from (2.4.7), (2.3.2) and the fact that $\alpha(h) \in Z^1_{dR\mathbb{Z}}(M)$ and, thus, $\oint \alpha(h) \in \mathbb{Z}$.

Tentatively, for $\Lambda \in Z^1_\mathbb{Z}(M)$, we define $\oint A \mod \mathbb{Z}$ as follows. We consider a character $\Phi \in CS^2(M)$ such that $c_\Phi = c_P$ and $F_\Phi = F_A$. As $q(c_P) = [F_A]_{dR}$, the condition $q(c_\Phi) = [F_\Phi]_{dR}$ is fulfilled. Then, we set

$$\Phi(\Lambda) = \exp \left(2\pi i \oint A\right).$$

(2.4.9)

The definition given is however ambiguous. Indeed, by the exact sequence (2.4.4), the character $\Phi$ with the stated properties is not unique, being defined up to a character of the form $\chi_a$ with $a \in Z^1_{dR}(M)$ defined modulo $Z^1_{dR\mathbb{Z}}(M)$. The definition is also not satisfactory, since, apparently, it yields the same result for connections differing by a closed form $a \in Z^1_{dR}(M)$.

To solve these problems, we proceed as follows. With some natural criterion, we fix a reference connection $A_P \in \text{Conn}(P)$ and a fiducial character $\Phi_P \in CS^2(M)$ such that $c_{\Phi_P} = c_P$ and $F_{\Phi_P} = F_{A_P}$ and declare $\oint A_P$ to be given mod $\mathbb{Z}$ by the above procedure:

$$\Phi_P(\Lambda) = \exp \left(2\pi i \oint A_P\right).$$

(2.4.10)
Next, for a generic connection \( A \in \text{Conn}(P) \), we define a form \( v_A \in C^1_{dR}(M) \) depending on \( A \) by the relation
\[
A = A_P + v_A. \tag{2.4.11}
\]
Then, we set
\[
\oint_A A = \oint_A A_P + \oint_A v_A \mod Z. \tag{2.4.12}
\]
It is easy to check that this definition of \( \oint_A A \) has the required properties (2.4.6)–(2.4.8).

Note that \( \oint_A A \) depends on \( P \) via its Chern class \( c_P \) and not simply via \( q(c_P) = [F_A]_{dR} \).
It is therefore sensitive to torsion. By the isomorphism (2.1.8), the torsion part of \( c_P \) reflects the flat factors of \( P \). Thus, \( \oint_A A \) depends explicitly on these latter.

Needless to say, what we have done here is to provide a family of definitions of \( \oint_A A \) parameterized by the choices of \( A_P \) and \( \Phi_P \). In the next subsection, we shall devise a way of restricting the amount of arbitrariness involved.

2.5 Background connection and character assignments

We consider below the group isomorphism that associates to any \( c \in H^2(M, \mathbb{Z}) \) the unique (up to smooth equivalence) \( U(1) \) principal bundle \( P_c \) such that \( c_{P_c} = c \). This map is the inverse of the isomorphism (2.1.4).

A background connection assignment is a map that associates to any \( c \in H^2(M, \mathbb{Z}) \) a connection \( A_c \in \text{Conn}(P_c) \) in such a way that
\[
A_{c+c'} = A_c + A_{c'}, \quad c, c' \in H^2(M, \mathbb{Z}), \tag{2.5.1}
\]
\[
A_t = 0, \quad t \in \text{Tor}^2(M, \mathbb{Z}). \tag{2.5.2}
\]
We set \( F_c = F_{A_c} \).

A background character assignment compatible with a background connection assignment \( c \mapsto A_c \) is a map that associates to any \( c \in H^2(M, \mathbb{Z}) \) a character \( \Phi_c \in CS^2(M) \) such that \( c_{\Phi_c} = c \) and \( F_{\Phi_c} = F_c \) and that
\[
\Phi_{c+c'} = \Phi_c \cdot \Phi_{c'}, \quad c, c' \in H^2(M, \mathbb{Z}). \tag{2.5.3}
\]

A background connection assignment \( c \mapsto A_c \) and a compatible background character assignment \( c \mapsto \Phi_c \) can be constructed as follows. Let \( f_r, r = 1, \ldots, b_2 \) and \( t_\rho, \rho = 1, \ldots, t_2 \) a set of independent generators of \( H^2(M, \mathbb{Z}) \), where the \( f_r \) are free and the \( t_\rho \) are torsion of order \( \kappa_\rho \). Every \( c \in H^2(M, \mathbb{Z}) \) can be written uniquely as
\[
c = \sum_r n^r(c) f_r + \sum_\rho k^\rho(c) t_\rho, \tag{2.5.4}
\]
for certain \( n^r(c) \in \mathbb{Z} \) depending linearly on \( c \) and \( k^\rho(c) = 1, \ldots, \kappa_\rho - 1 \) depending linearly on \( c \) modulo \( \kappa_\rho \). Next, choose \( A_r \in \text{Conn}(P_{f_r}) \) with curvature \( F_{A_r} = F_r \). Then, set

\[
A_c = \sum_r n^r(c) A_r. \tag{2.5.5}
\]

Similarly, choose \( \Phi_r \in CS^2(M) \) with \( c_{\Phi_r} = f_r \) and \( F_{\Phi_r} = F_r \) and \( \Phi_\rho \in CS^2(M) \) with \( c_{\Phi_\rho} = t_\rho \) and \( F_{\Phi_\rho} = 0 \). As \( \kappa_\rho t_\rho = 0 \), \( \Phi_\rho^{\kappa_\rho} = \chi_a \) for some \( a \in Z^1_{dR}(M) \), by the exact sequence (2.4.4). Redefining \( \Phi_\rho \) into \( \Phi_\rho \chi_a/\kappa_\rho \), one can impose

\[
\Phi_\rho^{\kappa_\rho} = 1. \tag{2.5.6}
\]

Then, set

\[
\Phi_c = \prod_r \Phi_r^{n^r(c)} \cdot \prod_\rho \Phi_\rho^{k^\rho(c)}. \tag{2.5.7}
\]

Then, the maps \( c \mapsto A_c \) and \( c \mapsto \Phi_c \) are respectively a connection and a compatible character assignment.

Let a background connection assignment \( c \mapsto A_c \) and a compatible background character assignment \( c \mapsto \Phi_c \) be given. For \( \Lambda \in Z^s_1(M) \) and \( A \in \text{Conn}(P_c) \), we define \( \mathcal{f}_\Lambda A \) by the procedure expounded in the previous subsection by taking \( A_{P_c} = A_c \) and \( \Phi_{P_c} = \Phi_c \), for \( c \in H^2(M, \mathbb{Z}) \). In this way, (2.4.10)–(2.4.12) hold with \( A_\rho \) and \( \Phi_\rho \) replaced by \( A_c \) and \( \Phi_c \). It is convenient, though not necessary, to choose \( A_c, \Phi_c \) of the form (2.5.5), (2.5.7).

In this way, the arbitrariness inherent in the definition of \( \mathcal{f}_\Lambda A \), discussed at the end of the previous subsection, is reduced to that associated with the choice of \( A_r, \Phi_r, \Phi_\rho \).

2.6 Example, the 4–torus

Since the formalism expounded above is rather abstract, we illustrate it with a simple example. We consider the case where \( M \) is the 4–torus \( T^4 \). As coordinates of \( T^4 \), we use angles \( \theta^i \in [0, 2\pi], 1 \leq i \leq 4 \).

The 4-torus \( T^4 \) has the nice property that torsion vanishes both in homology and in cohomology. Thus, we have the isomorphisms \( H^*_p(T^4) \cong H^p_{dRZ}(T^4) \cong H^p(T^4, \mathbb{Z}) \cong \mathbb{Z}^{C^4_p} \), where \( C^4_p = b_p \) is a binomial coefficient. A standard basis of \( H^*_p(T^4) \) consists of the homology classes of the singular \( p \)-cycles \( \Lambda_{a_1 \cdots a_p} \in Z^*_p(T^4), 1 \leq a_1 < \cdots < a_p \leq 4 \), defined by

\[
\theta^i(t_1, \cdots, t_p) = 2\pi \sum_{s=1}^p \delta_{a_s}^i t_s, \quad 0 \leq t_1, \cdots, t_p < 1. \tag{2.6.1}
\]
A standard basis of $H^p_{dR\mathbb{Z}}(T^4)$ consists of the cohomology classes of the integer period $p$–forms $ω^{a_1⋯a_p} ∈ Z^p_{dR\mathbb{Z}}(T^4)$, $1 ≤ a_1 < ⋯ < a_p ≤ 4$, defined by

$$ω^{a_1⋯a_p} = \frac{1}{(2\pi)^p}dθ^{a_1} ∧ ⋯ ∧ dθ^{a_p}.$$  \hfill (2.6.2)

For a given $p$, the homology and cohomology basis are reciprocally dual.

Since $H^2(T^4, \mathbb{Z}) \cong H^2_{dR\mathbb{Z}}(T^4)$, a principal $U(1)$ bundle on $T^4$ is determined up to equivalence by the de Rham cohomology class of the curvature of any connection. We consider the principal $U(1)$ bundle $P^{ab} ∈ \text{Princ}(T^4)$ defined by the de Rham cohomology class of the 2–form

$$F^{ab} = ω^{ab} ∈ Z^2_{dR\mathbb{Z}}(T^4),$$  \hfill (2.6.3)

with $1 ≤ a < b ≤ 4$. $P^{ab}$ is described concretely by the monodromy of a section of the associated line bundle around the 1–cycles $Λ_c$

$$T^{ab}c = \exp(iδ^a c θ^b - iδ^b c θ^a)$$  \hfill (2.6.4)

Any $P ∈ \text{Princ}(T^4)$ is expressible as a product of $P^{ab}$’s and their inverses. A connection $A^{ab} ∈ \text{Conn}(P^{ab})$ with curvature $F^{ab}$ is

$$A^{ab} = \frac{1}{2(2\pi)^2}(θ^a dθ^b - θ^b dθ^a).$$  \hfill (2.6.5)

$[F^{ab}]_{dR} ∈ H^2_{dR\mathbb{Z}}(T^4)$ determines unambiguously a class $c^{ab} ∈ H^2(T^4, \mathbb{Z})$. There is a unique Cheeger Simons character $Φ^{ab} ∈ CS^2(T^4)$ such that $F_{Φ^{ab}} = F^{ab}$, $c_{Φ^{ab}} = c^{ab}$ and that

$$Φ^{ab}(Λ_c) = 1, \quad 1 ≤ c ≤ 4.$$  \hfill (2.6.6)

Indeed, (2.6.6) selects unambiguously a unique character among those such that $F_{Φ^{ab}} = F^{ab}$, $c_{Φ^{ab}} = c^{ab}$ (cfr. the exact sequence (2.4.5)). By (2.4.1), (2.6.6)

$$Φ^{ab}(Λ) = \exp \left(2πi \int_Σ F^{ab} \right),$$  \hfill (2.6.7)

for $Λ = \sum_{a=1}^4 n_a Λ_a + bS ∈ H^1_s(T^4)$ with $n_a ∈ \mathbb{Z}$ and $S ∈ C^2_s(T^4)$ a 2–chain.

A background connection assignment and a compatible background character assignment are given by

$$A_c = \sum_{1 ≤ a < b ≤ 4} n_{ab}(c)A^{ab},$$  \hfill (2.6.8)
\[ \Phi_c = \prod_{1 \leq a < b \leq 4} (\Phi^{ab})^{n_{ab}(c)}, \]  

for \( c = \sum_{1 \leq a < b \leq 4} n_{ab}(c)c^{ab} \in H^2(T^4, \mathbb{Z}). \)

### 3. The gauge partition function

The physical motivation of the following construction has been given in the introduction.

To begin with, to properly define the kinetic term of the photon action and to carry out the gauge fixing and quantization program, we endow \( M \) with a fixed background Riemannian metric \( g \).

#### 3.1 The photon action

For any \( P \in \text{Princ}(M) \) and any \( A \in \text{Conn}(P) \), the Wick rotated photon action \( S(A, \tau) \) is given by \(^1\)

\[ S(A, \tau) = \pi \int_M F_A \wedge \hat{\tau} F_A. \]  

(3.1.1)

Here, \( \tau \) varies in the open upper complex half plane \( \mathbb{H}^+ \),

\[ \tau = \tau_1 + i\tau_2, \quad \tau_1 \in \mathbb{R}, \quad \tau_2 \in \mathbb{R}^+ \]  

(3.1.2)

and \( \hat{\tau} \) is the operator

\[ \hat{\tau} = \tau_1 + i \ast \tau_2. \]  

(3.1.3)

The action \( S(A, \tau) \) takes the form (1.1) upon expressing \( \tau \) as in (1.2) and rescaling \( A \) into \( (q/2\pi)A \). The integrality of the de Rham cohomology class of \( F_A \) translates in the flux quantization condition (1.3) after the rescaling.

The action \( S(A, \tau) \) has the obvious symmetry

\[ A \rightarrow A + a \]  

(3.1.4)

where \( a \in \mathbb{Z}_{dR}^1(M) \). Unless \( H^1(M, \mathbb{R}) = 0 \), this symmetry is larger than gauge symmetry, which corresponds to \( a \in \mathbb{Z}_{dR}^1(M) \) (cfr. subsect. 2.2).

The field equations can be written as

\[ d\hat{\tau} F_A = 0. \]  

(3.1.5)

\(^1\) The Wick rotated action \( S \) is related to the Euclidean action \( S_E \) as \( S = iS_E \).
They are equivalent to the vacuum Maxwell equations and the Bianchi identity

\[ dF_A = 0, \quad d * F_A = 0. \]  
(3.1.6)

3.2 The Wilson loop action

The insertion of a Wilson loop along a cycle \( \Lambda \in Z^1(M) \) is equivalent to add to the photon action a coupling of the gauge field \( A \) to a one dimensional defect represented by \( \Lambda \). For any \( A \in \text{Conn}(P) \), the interaction term of \( A \) and \( \Lambda \) is then

\[ W(A, \Lambda) = 2\pi \oint_{\Lambda} A \mod 2\pi \mathbb{Z}, \]  
(3.2.1)

where the right hand side is defined in the way expounded in subsect. 2.4. The fact that \( \Lambda \) is a 1–cycle is equivalent to the conservation of the associated current. (See subsect. 3.5 below.)

As explained in subsect. 2.4, the definition of \( \oint_{\Lambda} A \) involves choices and, thus, is not unique. It will be necessary to check at the end that the result of our calculations does not depend on the conventions used.

3.3 The partition function

The partition function with a Wilson loop insertion is given by

\[ Z(\Lambda, \tau) = \sum_{P \in \text{Princ}(M)} \int_{A \in \text{Conn}(P)} \frac{DA}{\text{vol}(\text{Gau}(P))} \exp(iS(A, \tau) + iW(A, \Lambda)) \]  
(3.3.1)

[18, 19, 20, 21]. The right hand side of this expression is the formal mathematical statement of the physical quantization prescription consisting in a summation over all topological classes of the gauge field and a functional integration of the quantum fluctuations of the gauge field about the vacuum gauge configuration of each class with the gauge group volume divided out.

To compute the above formal expression, we exploit heavily the results of subsect. 2.5. We first set \( P = P_c \) with \( c \in H^2(M, \mathbb{Z}) \) and transform the summation over \( P \) into one over \( c \). Next, we choose a background connection assignment \( c \mapsto A_c \) and write a generic \( A \in \text{Conn}(P_c) \) as

\[ A = A_c + v, \]  
(3.3.2)

where \( v \in C^1_{dR}(M) \) is a fluctuation, and transform the integration over \( A \) into one over \( v \). To evaluate the Wilson loop action, we further pick a background character assignment \( c \mapsto \Phi_c \) compatible with the connection assignment \( c \mapsto A_c \).
It is possible and convenient to impose that the connections $A_c$ of the connection assignment satisfy the Maxwell equation

$$d \ast F_c = 0. \quad (3.3.3)$$

To keep the arbitrariness involved in the various choices as controlled as possible, we assume further that the background connection and character assignments $c \mapsto A_c$, $c \mapsto \Phi_c$ are of the form (2.5.5), (2.5.7), respectively.

Proceeding in this way, we find that the partition function factorizes in a classical background and a quantum fluctuation factor,

$$Z(\Lambda, \tau) = Z_{\text{cl}}(\Lambda, \tau) \cdot Z_{\text{qu}}(\Lambda, \tau), \quad (3.3.4)$$

where

$$Z_{\text{cl}}(\Lambda, \tau) = \sum_{c \in H^2(M, \mathbb{Z})} \exp \left( i\pi \int_M F_c \wedge \hat{\tau} F_c + 2\pi i \oint_{\Lambda} A_c \right), \quad (3.3.5)$$

$$Z_{\text{qu}}(\Lambda, \tau) = \int_{v \in C^1_{\text{dr}}(M)} \frac{\varrho Dv}{\text{vol}(Z^1_dRZ(M))} \exp \left( -\pi \tau_2 \int_M dv \wedge *dv + 2\pi i \oint_{\Lambda} v \right). \quad (3.3.6)$$

$\varrho$ is a universal Jacobian relating the formal volumes $\text{vol}(\text{Gau}(M))$ and $\text{vol}(Z^1_dRZ(M))$ (cfr. subsect. 2.2).

3.4 Evaluation of the classical partition function

In order (3.3.3) to hold, the curvatures $F_r$ of the connections $A_r$ appearing in (2.5.5) all satisfy (3.3.3). Hence, the $F_r$ form a basis of the lattice $\text{Harm}^2 \subseteq \text{Harm}^2(M)$. The inverse intersection matrix $Q$ is defined by

$$Q_{rs} = \int_M F_r \wedge F_s. \quad (3.4.1)$$

As well known, $Q$ is a unimodular symmetric integer $b_2 \times b_2$ matrix characterizing the topology of $M$ and $Q$ is even or odd according to whether $M$ is spin or not. As $\ast \text{Harm}^2(M) \subseteq \text{Harm}^2(M)$ and $\ast^2 = 1$ on $\text{Harm}^2(M)$, one has

$$\ast F_r = \sum_s H^s \ast F_s, \quad (3.4.2)$$

where $H$ is a non singular real $b_2 \times b_2$ matrix such that $H^2 = 1$. As $\int_M F \wedge \ast F$ is a norm on $\text{Harm}^2(M)$, $QH$ is a positive definite symmetric $b_2 \times b_2$ matrix.
From (2.5.5), one has immediately that

\[ F_c = \sum_r n^r(c) F_r. \]  \tag{3.4.3}  

Recalling from subsect. 2.5 that \( \exp \left( 2\pi i \oint_{\Lambda} A_c \right) = \Phi_c(\Lambda) \) and using (2.5.7), we find

\[ \exp \left( 2\pi i \oint_{\Lambda} A_c \right) = \exp \left( 2\pi i \sum_r n^r(c) \oint_{\Lambda} A_r \right) \prod_{\rho} \Phi_{\rho}(\Lambda)^{k^\rho(c)} \]  \tag{3.4.4}  

Using (3.3.5), (3.1.3), (3.4.1)–(3.4.4), we obtain

\[ Z_{cl}(\Lambda, \tau) = \sum_{c \in H^2(M, \mathbb{Z})} \exp \left( i \pi n(c)Q(\tau_1 1 + i\tau_2 H) n + 2\pi i n^t(\gamma(\Lambda)) \right) \prod_{\rho} \Phi_{\rho}(\Lambda)^{k^\rho(c)}, \]  \tag{3.4.5}  

where

\[ \gamma_r(\Lambda) = \oint_{\Lambda} A_r. \]  \tag{3.4.6}  

From (2.5.4), by setting \( n^r = n^r(c) \) and \( k^\rho = k^\rho(c) \), we can transform the summation over \( c \in H^2(M, \mathbb{Z}) \) in a summation over \( n^r \in \mathbb{Z} \) and \( k^\rho = 0, 1, \ldots, \kappa_{\rho} - 1 \). Using (2.5.6), it is easy to see that

\[ \sum_{k^\rho=0,\ldots,\kappa_{\rho}-1} \prod_{\beta} \Phi_{\beta}(\Lambda)^{k^\beta} = \prod_{\rho} \kappa_{\rho} \varsigma(\Lambda), \]  \tag{3.4.7}  

where the characteristic map \( \varsigma \) is defined by

\[ \varsigma(\Lambda) = 1 \text{ if } \Phi_{\rho}(\Lambda) = 1 \text{ for all } \rho, \quad \varsigma(\Lambda) = 0 \text{ else}. \]  \tag{3.4.8}  

Thus,

\[ Z_{cl}(\Lambda, \tau) = \sum_{n \in \mathbb{Z}^{k_2}} \exp \left( i \pi n^t Q(\tau_1 1 + i\tau_2 H) n + 2\pi i n^t(\gamma(\Lambda)) \right) \prod_{\rho} \kappa_{\rho} \varsigma(\Lambda), \]  \tag{3.4.9}  

which is our final expression of the classical partition function.

The origin of the strange looking factor \( \varsigma(\Lambda) \) is not difficult to interpret intuitively. Comparing (3.4.3), (3.4.4), we notice that, while the gauge curvature \( F_c \) is not sensitive to the torsion part of \( c \) (cfr. eq. (2.5.4)), the Abelian Wilson loop \( \exp \left( 2\pi i \oint_{\Lambda} A_c \right) \) is. When we sum over all classes \( c \in H^2(M, \mathbb{Z}) \) in (3.3.5), a finite subsum over all torsion classes \( t \in \text{Tor}^2(M, \mathbb{Z}) \) is involved. By (3.4.3), (3.4.4), the terms of the subsum differ only by phases, which, on account of (2.5.6), are rational. The superposition of these phases leads to either constructive or destructive interference and yields the factor \( \varsigma(\Lambda) \). As explained
in subsect. 2.4, the dependence of the Abelian Wilson loop $\exp(2\pi i \oint_{\Lambda} A_c)$ on the torsion part of $c$ can be traced to its dependence on the flat factors of the underlying principal bundle $P_c$. Thus, the factor $\zeta(\Lambda)$ can ultimately be attributed to an interference effect of the flat topological classes in (3.3.1).

3.5 Evaluation of the quantum partition function

The computation of the quantum partition function proceeds through two basic steps [26]. Firstly, one endows the relevant field spaces with suitable Hilbert structures in order to define the corresponding functional measures. Secondly, one determines the appropriate field kinetic operators required by the definition of the perturbative expansion. In our case, the problem is simplified by the fact that the field theory we are dealing with is free. There are however complications related to gauge invariance and the consequent need for gauge fixing.

In our model, the relevant field spaces are $C^p_{dR}(M)$ with $p = 0, 1$, corresponding to the Faddeev–Popov ghost field and photon field. The Hilbert structure of $C^p_{dR}(M)$ is defined as usual by

$\langle u, v \rangle = \int_M u \wedge * v, \quad u, v \in C^p_{dR}(M). \quad (3.5.1)$

The relevant kinetic operators are the standard form Laplacians $\Delta_p$ acting on $C^p_{dR}(M)$

$\Delta_p = (d^\dagger d + dd^\dagger)_p. \quad (3.5.2)$

which are order 2 elliptic non negative self adjoint operators.

Since we are using a Hilbert space formalism, it is convenient to express the argument of the exponential in (3.3.6) in terms of the Hilbert structure (3.5.1). To this end, for a cycle $\Lambda \in Z^s_1(M)$, we define a distribution $j_\Lambda$ on $C^1_{dR}(M)$ by

$\langle j_\Lambda, \omega \rangle = \oint_{\Lambda} \omega, \quad \omega \in C^1_{dR}(M). \quad (3.5.3)$

As a consequence of the relation $b\Lambda = 0$, one has

$d^\dagger j_\Lambda = 0. \quad (3.5.4)$

Intuitively, $j_\Lambda$ is the current associated to the 1–cycle $\Lambda$ and (3.5.4) is the statement that $j_\Lambda$ is conserved.

As one is computing the partition function of a field theory on a generally topologically non trivial manifold, particular care must be taken for a proper treatment of the zero modes.
of the kinetic operators. The $p = 0$ ghost zero modes form the 1–dimensional vector space of constant functions on $M$, Harm$^0(M)$. As a basis of this, we choose the constant scalar 1.

The $p = 1$ photon zero modes form the $b_1$–dimensional vector space of harmonic 1–forms of $M$, Harm$^1(M)$. As a basis of this, we choose a basis $\{\omega_m\}, m = 1, \ldots, b_1$, of the lattice Harm$^1_Z(M)$ for convenience.

We fix the gauge by imposing the customary Lorentz fixing gauge condition. By using standard Faddeev–Popov type manipulations to perform the gauge fixing, we find

$$Z_{\text{qu}}(\Lambda, \tau) = \left(\frac{\det G_1}{\text{vol } M}\right)^{\frac{3}{2}} \left(\frac{1}{(2\pi)^{b_1-\frac{1}{2}}}ight) \prod_n \delta(\langle j_\Lambda, \omega_n \rangle, 0) \times \left[\det'(\Delta_0) \det'(2\pi\tau_2\Delta_0)\right]^{\frac{1}{2}} \exp \left(-\pi^2 \langle j_\Lambda, (\pi\tau_2\Delta_1)^{-1}'j_\Lambda \rangle\right).$$  \hspace{1cm} (3.5.5)

Here, $\det'(\Delta)$ and $\Delta^{-1'}$ denote the determinant and the inverse of the restriction of $\Delta$ to the orthogonal complement of its kernel, respectively, and

$$G_{1mn} = \langle \omega_m, \omega_n \rangle. \hspace{1cm} (3.5.6)$$

We collect in the appendix the details of the derivation of (3.5.5). Without going through all that, we can intuitively understand the origin of the various factors appearing in (3.5.5). $[\det'(2\pi\tau_2\Delta_1)]^{-\frac{1}{2}}$ is the photon determinant. Roughly speaking, the combination $[\det'(\Delta_0) \det'(2\pi\tau_2\Delta_0)]^{\frac{1}{2}}$ is the ghost determinant, since the second determinant equals the first up to a $\tau_2$ dependent constant. The factor $\prod_n \delta(\langle j_\Lambda, \omega_n \rangle, 0)$ is yielded by the integration over the photon zero modes that satisfy the Lorentz gauge fixing condition with the volume of the residual gauge symmetry divided out. The zero modes live in the torus Harm$^1(M)/\text{Harm}^1_Z(M)$. Only the integral $\exp \left(2\pi i \oint \Lambda v\right)$ in (3.3.6) depends on them. Integration of this phase on the zero modes torus produces the above combination of Kronecker delta functions. Finally, the exponential factor $\exp \left(-\pi^2 \langle j_\Lambda, (\pi\tau_2\Delta_1)^{-1'}j_\Lambda \rangle\right)$ is the result of the Gaussian integration in (3.3.6) and represents the selfenergy of the current $j_\Lambda$. The remaining factors are just normalization constants.

In (3.5.5), both the determinants and the argument of the exponential suffer ultraviolet divergences which have to be regularized and renormalized.

We regularize the determinants using Schwinger’s proper time method, which now we briefly review [26]. Let $\Delta$ be an elliptic non negative self adjoin operator in some Hilbert space of fields on a manifold $X$. Its proper time regularized determinant is given by

$$\det'(\epsilon)(\Delta) = \exp \left(-\int_\epsilon^\infty \frac{dt}{t} \left(\text{tr } \exp(-t\Delta) - \dim \ker \Delta\right)\right), \hspace{1cm} (3.5.7)$$
where $\epsilon > 0$ is a small ultraviolet cut off of mass dimension exponent $-2$. According to the heat kernel expansion

$$\text{tr} \exp(-t\Delta) \sim \sum_{k=0}^{\infty} t^{k-\dim X \ord \Delta} \int_X a_k(\Delta), \quad t \to 0^+,$$

where $a_k(\Delta)$ is a dim $X$–form depending locally on the background geometry. Using (3.5.7), (3.5.8), it is easy to show that

$$\det'_\epsilon(\Delta) = \epsilon^{-\dim \ker \Delta} \exp \left\{ - \sum_{l=1}^{\dim X} \frac{\epsilon^{-l/\ord \Delta}}{l/\ord \Delta} \int_X a_{\dim X-l}(\Delta) \right\} \det'_\epsilon \ms(\Delta).$$

(3.5.9)

Here, $\det'_\ms(\Delta)$ is the finite minimally subtracted renormalized determinant.

We note that, for any $\kappa > 0$, one has

$$\det'_\epsilon(\kappa \Delta) = \det'_{\kappa \epsilon}(\Delta),$$

(3.5.10)

a simple property that will be useful in the calculations below.

We replace the formally divergent determinants appearing in (3.5.5) with their proper time regularized counterparts and use the expansion (3.5.9). The expressions of the heat kernel forms $a_k(\Delta_p)$ are well known in the literature [31]. In this way, one finds

$$\left[ \frac{\det'_\epsilon(\Delta_0)}{\det'_\epsilon(2\pi \tau_2 \Delta_1)} \right]^{\frac{1}{2}} = \epsilon^{\frac{b_1+1}{2}} \exp \left\{ \frac{1}{(8\pi)^2} \left( \frac{3}{(2\pi \tau_2)^2} - 1 \right) \frac{1}{\epsilon^2} \int_M d^4x g^{\frac{3}{2}} ight. + \frac{1}{(2\pi \tau_2^2) + \frac{1}{3}} \frac{1}{\epsilon} \int_M d^4x g^{\frac{1}{2}} R + \frac{1}{(8\pi)^2} \frac{1}{90} \ln \epsilon \int_M d^4x g^{\frac{1}{2}} \left( 25R^2 - 88R_{ij}R_{ij} \right. \\
+ 13R_{ijkl}R_{ijkl} \right) + \frac{1}{(8\pi)^2} \frac{\ln(2\pi \tau_2)}{60} \int_M d^4x g^{\frac{1}{2}} \left( 15R^2 - 58R_{ij}R_{ij} + 8R_{ijkl}R_{ijkl} \right) + O(\epsilon) \right\}$$

$$\times (2\pi \tau_2)^{\frac{b_1-1}{2}} \frac{\det'_\ms(\Delta_0)}{\det'_\ms(\Delta_1)^{\frac{1}{2}}}.$$

(3.5.11)

The prefactor $\epsilon^{\frac{b_1+1}{2}}$ can be absorbed into an appropriate $\epsilon$ dependent normalization of the zero mode part of the partition function measure. The local divergences appearing in the
exponential can be removed by adding to the action \( S(A, \tau) \) (cfr. eq. (3.1.1)) local counterterms with suitable \( \epsilon \) dependent coefficients. The general form of these counterterms, predicted also by standard power counting considerations, is

\[
\Delta S_\epsilon(\tau) = \frac{i}{(8\pi)^2} \int_M d^4x g^{1/2} \left( c_4(\epsilon, \tau) + c_2(\epsilon, \tau)R 
+ c_0(\epsilon, \tau)R^2 + c_0'(\epsilon, \tau)R_{ij} + c_0''(\epsilon, \tau)R^{ijkl}R_{ijkl} \right),
\]

where the suffix of the numerical coefficients denotes the exponent of their mass dimension. If one adopts the minimal subtraction renormalization scheme, one obtains

\[
\left[ \frac{\det'(\Delta_0)}{\det'(2\pi\tau_2\Delta_0)} \right]^{1/2}_{\text{ms}} = \exp \left\{ \frac{1}{(8\pi)^2} \int_M d^4x g^{1/2} \left( 15R^2 - 58R_{ij}R_{ij} 
+ 8R^{ijkl}R_{ijkl} \right) \right\} \times \left( 2\pi\tau_2 \right)^{b_1-1} \frac{\det'(\Delta_0)}{\det'(\Delta_1)^{1/2}}.
\]

As it turns out, the \( \tau_2 \) dependence of the resulting renormalized product of determinants has bad duality covariance properties due to the exponential factor. It is possible to remove the latter by adjusting the finite part of the local counterterms. This amounts to adopting another duality covariant renormalization scheme for which

\[
\left[ \frac{\det'(\Delta_0)}{\det'(2\pi\tau_2\Delta_0)} \right]^{1/2}_{\text{dc}} = (2\pi\tau_2)^{b_1-1} \frac{\det'(\Delta_0)}{\det'(\Delta_1)^{1/2}}.
\]

It is Witten’s choice [18] and also ours.

Next, we regularize the Green function by using again Schwinger’s proper time method, as described below [26]. Let \( \Delta \) be an elliptic non negative self adjoint operator in some Hilbert space of fields on a manifold \( X \) as before. Its proper time regularized Green function is

\[
\Delta^{-1/\epsilon} = \int_{\epsilon}^\infty dt \left( \exp(-t\Delta) - P(\ker \Delta) \right),
\]

where \( P(\ker \Delta) \) is the orthogonal projector on \( \ker \Delta \) and \( \epsilon > 0 \) is a small ultraviolet cut off of mass dimension exponent \(-2\). Indeed, carrying out the integration explicitly, one has

\[
\Delta^{-1/\epsilon} = \Delta^{-1/\epsilon} \exp(-\epsilon\Delta).
\]

We note that, for any \( \kappa > 0 \), one has

\[
(\kappa \Delta)^{-1/\epsilon} = \kappa^{-1} \Delta^{-1/\epsilon},
\]

where the suffix of the numerical coefficients denotes the exponent of their mass dimension. If one adopts the minimal subtraction renormalization scheme, one obtains

\[
\left[ \frac{\det'(\Delta_0)}{\det'(2\pi\tau_2\Delta_0)} \right]^{1/2}_{\text{ms}} = \exp \left\{ \frac{1}{(8\pi)^2} \int_M d^4x g^{1/2} \left( 15R^2 - 58R_{ij}R_{ij} 
+ 8R^{ijkl}R_{ijkl} \right) \right\} \times \left( 2\pi\tau_2 \right)^{b_1-1} \frac{\det'(\Delta_0)}{\det'(\Delta_1)^{1/2}}.
\]

As it turns out, the \( \tau_2 \) dependence of the resulting renormalized product of determinants has bad duality covariance properties due to the exponential factor. It is possible to remove the latter by adjusting the finite part of the local counterterms. This amounts to adopting another duality covariant renormalization scheme for which

\[
\left[ \frac{\det'(\Delta_0)}{\det'(2\pi\tau_2\Delta_0)} \right]^{1/2}_{\text{dc}} = (2\pi\tau_2)^{b_1-1} \frac{\det'(\Delta_0)}{\det'(\Delta_1)^{1/2}}.
\]

It is Witten’s choice [18] and also ours.
∂′ appearing in (3.5.21) are easily interpreted. \(\det\) which is our final expression of the renormalized quantum partition function. The factors with a suitably adjusted \(\epsilon\) placing \((\pi\tau)\) was explained below (3.5.6).

From (3.4.8), (3.4.9), it follows that

\[
Z = \exp(-W) \langle \sigma \rangle + O(\epsilon^2),
\]

The heat kernel \(\exp(-t\Delta)(x,x')\), with \(x, x' \in M\), is a bitensor with the small \(t\) expansion

\[
\exp(-t\Delta)(x,x') \sim \frac{1}{(4\pi t)^{\dim X/2}} \exp \left( -\frac{\sigma(x,x')}{2t} \right) \sum_{l=0}^{\infty} t^l f_l(x,x'), \quad t \to 0 + .
\]

Here, \(\sigma(x,x')\) is half the square geodesic distance of \(x, x'\). The \(f_l(x,x')\) are certain bitensors of the same type as \(\exp(-t\Delta)(x,x')\) [31].

We regularize the formal expression \(\langle j_\Lambda, (\pi\tau_2\Delta_1)^{-1'} j_\Lambda \rangle\) appearing in (3.5.5) by replacing \((\pi\tau_2\Delta_1)^{-1'}\) with \((\pi\tau_2\Delta_1)^{-1}\). The only thing one needs to know about the small \(t\) expansion of the heat kernel \(\exp(-t\Delta)_{ij}(x,x')\) is that \(f_{0ij'}(x,x')|_{x'=x} = g_{ij}(x)\) and \(\partial_{k'} f_{0ij'}(x,x')|_{x'=x} = g_{kl}\Gamma_{ij}^{l}(x)\). In this way, one finds

\[
\langle j_\Lambda, (\pi\tau_2\Delta_1)^{-1'} j_\Lambda \rangle = \frac{2}{(4\pi^2 \tau_2)^{3/2}} \frac{1}{\epsilon^2} \int_0^1 dt (\Lambda^* g_{tt})^{1/2} + \frac{1}{\pi \tau_2} \sigma(\Lambda) + O(\epsilon^{1/2}),
\]

where \(\sigma(\Lambda)\) is a finite constant depending on \(\Lambda\). In the first term, the 1–cycle \(\Lambda\) is viewed as a parameterized path \(\Lambda : [0,1] \to M\) and the value of the integral is just the length of the path as measured by the metric \(g\). The divergent part can be removed by adding to the interaction action \(W(A, \Lambda)\) (cfr. eq. (3.2.1)) a local counterterm of the form

\[
\Delta W_\epsilon(\Lambda, \tau) = i c_1(\epsilon, \tau) \int_0^1 dt (\Lambda^* g_{tt})^{1/2}
\]

with a suitably adjusted \(\epsilon\) dependent coefficient of mass dimension exponent 1.

One finds in this way

\[
Z_{\text{qu ren}}(\Lambda, \tau) = \left( \frac{\det G_1}{\text{vol} M} \right)^{1/2} \frac{\det'_{\text{ms}}(\Delta_0)}{\det'_{\text{ms}}(\Delta_1)^{1/2}} \prod_n \delta_{(j_\Lambda, \omega_n), 0} \tau_2^{\frac{\delta_{i-1}}{2}} \exp \left( -\frac{\pi \sigma(\Lambda)}{\tau_2} \right),
\]

which is our final expression of the renormalized quantum partition function. The factors appearing in (3.5.21) are easily interpreted. \(\det'_{\text{ms}}(\Delta_0), \det'_{\text{ms}}(\Delta_1)^{-1/2}\) are the renormalized ghost and photon determinants, respectively. \(\tau_2^{\frac{\delta_{i-1}}{2}}\) is the explicit \(\tau_2\) dependence of the renormalized determinants. \(\sigma(\Lambda)\) is the conventionally normalized renormalized selfenergy of the conserved current \(j_\Lambda\) associated with \(\Lambda\). The origin of the combination \(\prod_n \delta_{(j_\Lambda, \omega_n), 0}\) was explained below (3.5.6).

### 3.6 Selection rules

Let us examine the implications of the above calculation. Consider a cycle \(\Lambda \in Z_\Lambda^1(M)\).

From (3.4.8), (3.4.9), it follows that \(Z_{cl}(\Lambda, \tau) = 0\) unless \(\Phi_\rho(\Lambda) = 1\) for all \(\rho\), that is \(\Lambda\) is
contained in the kernel of all characters \( \Phi \in CS^2(M) \) such that \( c_\Phi \in \text{Tor}^2(M, \mathbb{Z}) \). This is the classical selection rule. From (3.5.21), recalling that \( \langle j_\Lambda, \omega_k \rangle = \int_\Lambda \omega_k \) by (3.5.3), it follows that \( Z_{qu}(\Lambda, \tau) = 0 \) unless \( \int_\Lambda \omega_k = 0 \) for all \( k \), that is \( \Lambda \) is a torsion cycle, i. e. \( [\Lambda]^s \in \text{Tor}_1^s(M) \). This is the quantum selection rule. From (3.3.4) and the above, we conclude that \( Z(\Lambda, \tau) = 0 \) identically unless \( \Lambda \in Z_1^s(M) \) satisfies

\[
[\Lambda]^s \in \text{Tor}_1^s(M), \tag{3.6.1}
\]

\[
\Phi(\Lambda) = 1, \quad \text{for all } \Phi \in CS^2(M) \text{ with } c_\Phi = 0. \tag{3.6.2}
\]

### 3.7 Flat characters and the Morgan–Sullivan torsion invariant

Let \( \Phi \in CS^2(M) \) be a flat character, i. e. such that \( F_\Phi = 0 \). Then, \( c_\Phi \in \text{Tor}^2(M, \mathbb{Z}) \cong \text{Tor}_{s\mathbb{Z}}^2(M) \). Therefore, there exist a minimal integer \( \nu_\Phi \in \mathbb{N} \), an integer cocycle \( \rho \in Z_{s\mathbb{Z}}^2(M) \) and an integer cochain \( s \in C_{s\mathbb{Z}}^1(M) \) such that \( c_\Phi = [\rho]_{s\mathbb{Z}} \) and \( \nu_\Phi \rho = ds \). On the other hand, as explained in subsect. 2.4, there is a real cochain \( f \in C_{s\mathbb{R}}^1(M) \) such that \( \Phi = \exp \left( 2\pi i [Z_1^s(M)] \right) \), \( df \in Z_{s\mathbb{Z}}^2(M) \) and \( c_\Phi = [df]_{s\mathbb{Z}} \). We thus have, \( df = \rho + dt \) for some integer cochain \( t \in C_{s\mathbb{Z}}^1(M) \).

Let \( \Lambda \in Z_1^s(M) \) such that \( [\Lambda]^s \in \text{Tor}_1^s(M) \). Then, there are a minimal \( \nu_\Lambda \in \mathbb{N} \) and \( S \in C_2^s(M) \) such that \( \nu_\Lambda \Lambda = bS \).

Using the above relations, one easily shows that \( \nu_\Lambda f(\Lambda) = \rho(S) + \nu_\Lambda t(\Lambda) \) and \( \nu_\Phi \rho(S) = \nu_\Lambda s(\Lambda) \). Thus

\[
f(\Lambda) = \rho(S)/\nu_\Lambda = s(\Lambda)/\nu_\Phi \mod \mathbb{Z}. \tag{3.7.1}
\]

Now, using (3.7.1), it is easy to check that \( f(\Lambda) \) depends only on the cohomology class \( c_\Phi \) of \( \rho \) and the homology class \( [\Lambda]^s \) of \( \Lambda \mod \mathbb{Z} \). Hence, the object defined by

\[
\langle [\Lambda]^s, c_\Phi \rangle = f(\Lambda) \mod \mathbb{Z} \tag{3.7.2}
\]

is a topological invariant. It is called Morgan–Sullivan torsion invariant pairing \([32,33]\). It is \( Z \) linear in both arguments and non singular.

From the above, we conclude that, for a character \( \Phi \in CS^2(M) \) such that \( F_\Phi = 0 \),

\[
\Phi(\Lambda) = \exp \left( 2\pi i \langle [\Lambda]^s, c_\Phi \rangle \right), \tag{3.7.3}
\]

for all \( \Lambda \in Z_1^s(M) \) such that \([\Lambda]^s \in \text{Tor}_1^s(M)\).

### 3.8 The final form of the selection rules
Using the results of the previous subsection, we can restate the selection rules (3.6.1), (3.6.2) as follows:

\[ [\Lambda]^s \in \text{Tor}_1^s(M), \]  
\[ \langle [\Lambda]^s, c \rangle = 0 \mod \mathbb{Z}, \quad c \in \text{Tor}_2^s(M). \]

As the Morgan–Sullivan pairing is non singular, these are equivalent to

\[ \Lambda \in B_1^s(M). \]

Thus, the partition function \( Z(\Lambda, \tau) \) vanishes unless \( \Lambda \) is a 1–boundary. This is the final form of the selection rules of the Abelian Wilson loops. Note that they originate from a non trivial combination of the classical and quantum selection rules.

Remarkably, in spite of the ambiguity inherent in the definition of the integral \( \int_{\Lambda} A \), the partition function \( Z(\Lambda, \tau) \) is unambiguously defined. Indeed, as explained in subsect. 2.4, the indetermination of \( \int_{\Lambda} A \) is of the form \( \int_{\Lambda} a \mod \mathbb{Z} \) with \( a \in Z_1^{dR}(M) \) and this object vanishes when \( \Lambda \) is a boundary. When, conversely, \( \Lambda \) is not a boundary, \( Z(\Lambda, \tau) \) vanishes identically, regardless the way the ambiguity of \( \int_{\Lambda} A \) is fixed.

This selection rule found is rather surprising when compared to the result for Abelian Chern Simons theory [34], where non trivial Abelian Wilson loops are found for non trivial knots. This calls for an explanation. As a gauge theory on a topologically non trivial manifold \( M \), Chern Simons theory is rather trivial, since the underlying principal bundle is trivial. For non trivial bundles, the Chern Simons Lagrangian would not be globally defined on \( M \) in general and thus could not be integrated to yield an action. Further, it is implicitly assumed that there are no photon zero modes. This restricts the manifold \( M \) to be such that \( H^1(M, \mathbb{R}) = 0 \). Thus, unlike for Maxwell theory, the quantization of Chern Simons theory involves no sum over the topological classes of the gauge field, since only the trivial class is involved. For this reason, the basic interference mechanism involving flat bundles which is partly responsible for the selection rule of Abelian Wilson loops of Maxwell theory is not working in Abelian Chern Simons theory. Further, all 1–cycles \( \Lambda \) one is dealing with are torsion from the start. Finally, in the Abelian Chern Simons model the relevant invariants of a knot are given in terms of the selfenergy of the current associated to the knot, which is of a topological nature. In Maxwell theory, the self energy of a 1–cycle is obviously not topological.

3.9 Example, the 4–torus
We illustrate the above results with an example. We consider again the case where
$M$ is 4–torus $T^4$, which was already discussed in subsect. 2.6.

It is not difficult to compute the $\tau$ dependent factor of the partition function $Z(\Lambda, \tau)$. $Z(\Lambda, \tau)$ is given by (3.3.4) with $Z_{cl}(\Lambda, \tau), Z_{qu}(\Lambda, \tau)$ given respectively by (3.4.9), (3.5.21)
(after renormalization). Since $\text{Tor}^2(T^4, \mathbb{Z}) = 0$, the factor $\prod_\rho \kappa_\rho \varsigma(\Lambda)$ appearing in (3.4.9)
is identically 1. The Betti numbers $b_1, b_2$ of the 4–torus $T^4$ are 4, 6, respectively. It follows
that, for a 1–boundary $\Lambda \in B_1^2(T^4)$,

$$Z(\Lambda, \tau) = Z_0 \tau_2^{\frac{3}{2}} \exp \left( -\frac{\pi \sigma(\Lambda)}{\tau_2} \right) \Psi(\gamma(\Lambda), \tau),$$

(3.9.1)

where $Z_0$ is a constant independent from $\Lambda, \tau, \gamma(\Lambda)$ is defined in (3.4.6) and $\Psi(\gamma, \tau)$ is a
certain function of $\gamma \in \mathbb{C}^6, \tau \in \mathbb{H}^+$, given by (3.4.9) with $\gamma(\Lambda)$ replaced by $\gamma$ and $\prod_\rho \kappa_\rho \varsigma(\Lambda)$
set to 1.

It is not difficult to compute $\Psi(\gamma, \tau)$ when $T^4$ is endowed with the standard flat metric

$$g = \delta_{ij} d\theta^i \otimes d\theta^j.$$  

(3.9.2)

The 2–forms $\omega^{ab}, 1 \leq a < b \leq 4$, defined in (2.6.2), belong to $\text{Harm}_2(T^4)$ and form a
basis of this latter. A simple calculations shows that $Q^{ab,cd} = \int_{T^4} \omega^{ab} \wedge \omega^{cd} = \epsilon^{abcd}$ and
$QH^{ab,cd} = \int_{T^4} \omega^{ab} \wedge *\omega^{cd} = \delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc}$. If we use the index $r = 1, 2, 3, 4, 5, 6$ for the pairs $(ab) = (12), (34), (13), (24), (14), (23)$, $Q$ and $QH$ are representable as the $6 \times 6$ matrices

$$Q = \sigma_1 \oplus -\sigma_1 \oplus \sigma_1, \quad QH = l_2 \oplus l_2 \oplus l_2,$$  

(3.9.3)

where $l_2$ is the $2 \times 2$ unit matrix and $\sigma_1$ is a Pauli matrix. Using (3.9.3), it is straightforward
to show that

$$\Psi(\gamma, \tau) = \psi(\gamma^{(1)}, \tau)\psi(\gamma^{(2)}, -\bar{\tau})\psi(\gamma^{(3)}, \tau), \quad \gamma = \gamma^{(1)} \oplus \gamma^{(2)} \oplus \gamma^{(3)}$$

(3.9.4)

where $\gamma^{(h)} \in \mathbb{C}^2$ and, for $\tau \in \mathbb{H}^+, g \in \mathbb{C}^2$,

$$\psi(g, \tau) = \vartheta_2(g_1 + g_2|2\tau)\vartheta_2(\bar{g}_1 - \bar{g}_2|2\tau) + \vartheta_3(g_1 + g_2|2\tau)\vartheta_3(\bar{g}_1 - \bar{g}_2|2\tau),$$

(3.9.5)

$\vartheta_2, \vartheta_3$ being standard Jacobi theta functions.

4. Analysis of Abelian duality

We now come to the analysis of the duality covariance properties of the partition
function with Wilson loop insertion $Z(\Lambda, \tau)$, which is the main subject of he paper.
4.1 Study of the $\tau$ dependence and duality

We next study the $\tau$ dependence of the partition function $Z(\Lambda, \tau)$. This resides essentially in a $\vartheta$ function of the appropriate characteristics. It is therefore necessary to review first some of the basics of the theory of $\vartheta$ functions. See for instance [35] for background.

We recall that the standard $\vartheta$ function with characteristics is defined by

$$\vartheta_b \left[ \begin{array}{c} x \\ y \end{array} \right] (K) = \sum_{n \in \mathbb{Z}^b + x} \exp \left( i\pi n^t Kn + 2i\pi n^t y \right),$$

(4.1.1)

where $b \in \mathbb{N}$, $x, y \in \mathbb{R}^b$ and $K \in \mathbb{C}(b)$ such that $K = K^t$ and $\text{Im} K > 0$. The main properties of $\vartheta_b \left[ \begin{array}{c} x \\ y \end{array} \right] (K)$ used below are the following. Using the Poisson resummation formula, one can show that the $\vartheta$ function satisfies the relation

$$\vartheta_b \left[ \begin{array}{c} x \\ y \end{array} \right] (K) = \det (-iK)^{-\frac{1}{2}} \exp \left( 2\pi i x^t y \right) \vartheta_b \left[ \begin{array}{c} y \\ -x \end{array} \right] (-K^{-1}),$$

(4.1.2)

where the branch of the square root used is that for which $u^{\frac{1}{2}} > 0$ for $u > 0$. If $L \in \mathbb{R}(b)$ induces an automorphism of the lattice $\mathbb{Z}^b$, one has

$$\vartheta_b \left[ \begin{array}{c} x \\ y \end{array} \right] (K) = \vartheta_b \left[ \begin{array}{c} L^{-1}x \\ L^t y \end{array} \right] (L^t KL).$$

(4.1.3)

An element $Z \in \mathbb{Z}(b)$ with $Z = Z^t$ is said even if $n^t Z n \in 2\mathbb{Z}$ for any $n \in \mathbb{Z}^b$ and odd else. We set $\nu_Z = 1$ if $Z$ is even and $\nu_Z = 2$ if $Z$ is odd. Then, one has

$$\vartheta_b \left[ \begin{array}{c} x \\ y \end{array} \right] (K) = \exp \left( \nu_Z \pi i x^t Z x \right) \vartheta_b \left[ \begin{array}{c} x \\ y - \nu_Z Z x \end{array} \right] (K + \nu_Z Z)$$

(4.1.4)

From (3.3.4), (3.4.9), (3.5.21), the $\tau$ dependent factor of the partition function $Z(\Lambda, \tau)$ can be written as

$$Z(\Lambda, \tau) = \exp \left( -\frac{\pi \sigma(\Lambda)}{\tau_2} \right) F(\Lambda, \tau),$$

(4.1.5)

where

$$F(\Lambda, \tau) = \tau_2 \frac{b_1 - 1}{\pi} \vartheta_{b_2} \left[ \begin{array}{c} 0 \\ \gamma(\Lambda) \end{array} \right] (K(\tau)).$$

(4.1.6)

Here, $\tau = \tau_1 + i\tau_2$ varies in the open upper complex half plane $\mathbb{H}_+$. On account of the selection rules derived in subsect. 3.8, we can assume that $\Lambda \in B_1^b(M)$ is a boundary. $K(\tau)$ is given by

$$K(\tau) = Q(\tau_1 + i\tau_2 H),$$

(4.1.7)
where $Q$ and $H$ are defined by (3.4.1), (3.4.2), respectively. Since $Q, H \in \mathbb{R}(b_2)$, $Q = Q^t$, $QH = (QH)^t$ and $QH > 0$, $K(\tau) \in \mathbb{C}(b_2)$, $K(\tau) = K(\tau)^t$ and $\text{Im}K(\tau) > 0$, as required. The vector $\gamma(\Lambda) \in \mathbb{R}^{b_2}$ is given by (3.4.6). $\gamma(\Lambda)$ is defined modulo $\mathbb{Z}^{b_2}$. Since $\Lambda$ is a boundary and the curvatures $F_r$ of the connections $A_r$ satisfy the Maxwell equations (3.3.3), $\gamma(\Lambda)$ does not depend on the choice of the $A_r$ modulo $\mathbb{Z}^{b_2}$. For convenience, we have extracted the exponential factor $\exp(-\pi \sigma(\Lambda)/\tau^2)$, whose $\tau$ dependence is anyway quite simple.

The analysis of duality reduces essentially to the study of the covariance properties of the function $Z(\Lambda, \tau)$ under a suitable subgroup of the modular group [18,19], whose main properties we now briefly review [36].

The modular group $\tilde{\Gamma}[1]$ consists of all transformations of the open upper complex half plane $\mathbb{H}_+$ of the form

$$u(\tau) = \frac{a\tau + b}{c\tau + d}, \quad \text{with } a, b, c, d \in \mathbb{Z}, \ ad - bc = 1. \quad (4.1.8)$$

As is well known, $\tilde{\Gamma}[1]$ is generated by two elements $s, t$ defined by

$$s(\tau) = -1/\tau, \quad t(\tau) = \tau + 1. \quad (4.1.9)$$

These satisfy the relations

$$s^2 = \text{id}, \quad (st)^3 = \text{id}. \quad (4.1.10)$$

The modular group $\tilde{\Gamma}[1]$ is isomorphic to the group $\text{PSL}(2,\mathbb{Z}) \cong \text{SL}(2,\mathbb{Z})/\{-1,1\}$, the isomorphism being defined by

$$A(u) = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (4.1.11)$$

with $u \in \tilde{\Gamma}[1]$ given by (4.1.18). In particular,

$$A(s) = \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad A(t) = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \quad (4.1.12)$$

To efficiently study the duality covariance of $F(\Lambda, \tau)$, it is necessary to introduce a class of functions of $\tau \in \mathbb{H}_+$ defined as follows. Recall that $Q \in \mathbb{Z}(b_2)$ and $Q = Q^t$ and, so, $Q$ can be even or odd (according to whether $M$ is spin or not). For $k, l \in \mathbb{Z}$ with $kl \in \nu_Q\mathbb{Z}$, we set

$$F_{(k,l)}(\Lambda, \tau) = \tau^n_2 \frac{b_1}{\tau} \exp\left(-i\pi kl\gamma(\Lambda)^tQ^{-1}\gamma(\Lambda)\right) \vartheta_{b_2}\left[ \begin{array}{c} kQ^{-1}\gamma(\Lambda) \\ l\gamma(\Lambda) \end{array} \right](K(\tau)). \quad (4.1.13)$$
It is readily checked that this expression is defined unambiguously in spite of the \( \mathbb{Z}^{b_2} \) indeterminacy of \( \gamma(\Lambda) \). Our function \( \mathcal{F}(\Lambda, \tau) \) is actually a member of this function class, since indeed

\[
\mathcal{F}(\Lambda, \tau) = \mathcal{F}_{(0,1)}(\Lambda, \tau). \tag{4.1.14}
\]

A simple analysis shows that

\[
\mathcal{F}_{(k,l)}(\Lambda, \tau) = e^{\frac{i\pi}{4} \eta \tau - \frac{\chi + \eta}{4} \tau - \frac{\chi - \eta}{4}} \mathcal{F}_{(l,-k)}(\Lambda, -1/\tau). \tag{4.1.15}
\]

Here, \( \chi \) and \( \eta \) are respectively the Euler and signature invariant of \( M \) and are given by

\[
\chi = 2(1 - b_1) + b_2, \tag{4.1.16}
\]
\[
\eta = b_2^+ - b_2^- . \tag{4.1.17}
\]

To prove (4.1.15), one uses (4.1.2), (4.1.3) with \( L = Q \), and the relations \( b_2 = b_2^+ + b_2^- \) and

\[
det (-iK(\tau))^{\frac{1}{2}} = e^{-\frac{i\pi}{4} \eta \tau b_2^+ / 2 - \eta b_2^- / 2}, \tag{4.1.18}
\]
\[
- K(\tau)^{-1} = Q^{-1} K(-1/\tau) Q^{-1}. \tag{4.1.19}
\]

Using (4.1.4), one shows similarly that

\[
\mathcal{F}_{(k,l)}(\Lambda, \tau) = \mathcal{F}_{(k,l-\nu_Q k)}(\Lambda, \tau + \nu Q). \tag{4.1.20}
\]

Let \( G_{\nu Q} \) be the subgroup of \( \tilde{\Gamma}[1] \) generated by \( s \) and \( t^{\nu Q} \). Specifically, \( G_1 = \tilde{\Gamma}[1] \) and \( G_2 = \tilde{\Gamma}_{\theta} \), the so called Hecke subgroup of \( \tilde{\Gamma}[1] \). In [18,19], it was shown that \( G_{\nu Q} \) is the duality group, the subgroup of \( \tilde{\Gamma}[1] \) under which the partition function without insertions behaves as a modular form of weights \( \chi + \eta / 4 \), \( \chi - \eta / 4 \). Now, (4.1.14) and (4.1.20) can be written as

\[
\mathcal{F}_{(k,l)}(\Lambda, \tau) = e^{\frac{i\pi}{4} \eta \tau - \frac{\chi + \eta}{4} \tau - \frac{\chi - \eta}{4}} \mathcal{F}_{(k,l) A(s)^{-1}}(\Lambda, s(\tau))
\]
\[= \mathcal{F}_{(k,l) A(t^{\nu Q})^{-1}}(\Lambda, t^{\nu Q}(\tau)). \tag{4.1.21}
\]

Since \( \mathcal{F}_{(k,l)}(\Lambda, \tau) = \mathcal{F}_{(-k,-l)}(\Lambda, \tau) \), as is easy to show from (4.1.13) using (4.1.1), the above expressions are unambiguously defined in spite of the sign indeterminacy of \( A(s) \) and \( A(t^{\nu Q}) \). (4.1.21) states that \( \mathcal{F}_{(k,l)}(\Lambda, \tau) \) is a generalized modular form of \( G_{\nu Q} \) of weights \( \chi + \eta / 4 \), \( \chi - \eta / 4 \). In this sense, \( G_{\nu Q} \) continues to be the duality group also for the partition function with Wilson loop insertions.

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We denote by $E_{\nu Q}(\Lambda)$ the subspace of $\text{Fun}(\mathbb{H}_+)$ spanned by the functions $\mathcal{F}_{(k,l)}(\Lambda, \tau)$. We note that, when $\gamma(\Lambda)$ satisfies certain restrictions, the functions $\mathcal{F}_{(k,l)}(\Lambda, \tau)$ are not all independent. For instance, if $\gamma(\Lambda) = 0 \mod \mathbb{Z}^{b_2}$, $\mathcal{F}_{(k,l)}(\Lambda, \tau)$ is actually independent from $k, l$. So, $E_{\nu Q}(\Lambda)$ may in some instance be finite dimensional. To see how this can come about in greater detail, suppose that $\gamma(\Lambda) \in \mathbb{Q}^{b_2}$. Then, there is a minimal $p \in \mathbb{N}$ such that $p\gamma(\Lambda) \in \mathbb{Z}^{b_2}$. Let $k, l \in \mathbb{Z}$ such that $k\ell \in \nu Q\mathbb{Z}$. Let further $m, n \in \mathbb{Z}$ such that $(kn + lm + mnp)p \in \nu Q\mathbb{Z}$. Then, $(k + mp)(l + np) \in \nu Q\mathbb{Z}$ and, as is easy to show from (4.1.13), one has

\[
\mathcal{F}_{(k+mp, l+np)}(\Lambda, \tau) = \exp(2\pi i (nk - ml - mnp)w(\Lambda)/\nu Qp)\mathcal{F}_{(k,l)}(\Lambda, \tau),
\]  

where $w(\Lambda) \in \mathbb{Z}$ is given by

\[
w(\Lambda) = \frac{1}{2}\nu Qp^2\gamma(\Lambda)^t Q^{-1} \gamma(\Lambda).
\]

The phase factor is a $\nu Qp$–th root of unity independent from $\tau$. Therefore, when $\gamma(\Lambda)$ satisfies the above condition, $E_{\nu Q}(\Lambda)$ is finite dimensional. A standard basis of $E_{\nu Q}(\Lambda)$ consists of the $F_{(k,l)}(\Lambda, \tau)$ such that $0 \leq k, l \leq p - 1$. The dimension of $E_{\nu Q}(\Lambda)$ is therefore

\[
n_p = p^2 - [p/2]^2(\nu Q - 1).
\]

Denote by $\mathcal{F}_A(\Lambda, \tau)$ the standard basis of $E_{\nu Q}(\Lambda)$. Combining (4.1.15), (4.1.20) and (4.1.22), it is simple to show that there are invertible $n_p \times n_p$ complex matrices $S_{AB}(\Lambda)$ and $T_{\nu Q AB}(\Lambda)$ such that

\[
\mathcal{F}_A(\Lambda, \tau) = e^{\frac{\pi i}{4} \eta \tau - \frac{\chi_{+n}}{4} \bar{\tau} - \frac{\chi_{-n}}{4}} \sum_B S_{AB}(\Lambda)\mathcal{F}_B(\Lambda, -1/\tau),
\]

\[
\mathcal{F}_A(\Lambda, \tau) = \sum_B T_{\nu Q AB}(\Lambda)\mathcal{F}_B(\Lambda, \tau + \nu Q).
\]

This means that $\mathcal{F}_A(\Lambda, \tau)$ is the $A$-th component of a vector modular form $\mathcal{F}(\Lambda, \tau)$ of $G_{\nu Q}$ of weights $\frac{\chi_{+n}}{4}, \frac{\chi_{-n}}{4}$.

The matrices $S_{AB}(\Lambda)$ and $T_{\nu Q AB}(\Lambda)$ have the property that only one matrix element in each row and column is non zero. For instance, if $p = 2$ and $\nu Q = 1$, one has $n_p = 4$, $A = (0, 0), (0, 1), (1, 0), (1, 1)$ and

\[
S(\Lambda) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & \varepsilon_\Lambda
\end{pmatrix},
\]

\[
T(\Lambda) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & \varepsilon_\Lambda \\
0 & 0 & 1 & 0
\end{pmatrix},
\]

\[
\varepsilon_\Lambda = \exp(-i\pi w(\Lambda)).
\]
For $p = 2$, $\nu_Q = 2$, one has $n_p = 3$, $A = (0,0)$, $(0,1)$, $(1,0)$ and

$$S(\Lambda) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T^2(\Lambda) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon_\Lambda \end{pmatrix}, \quad \varepsilon_\Lambda = \exp(-i\pi w(\Lambda)/2). \quad (4.1.28)$$

### 4.2 Duality and Twisted sectors

The question arises whether the formal considerations expounded in the previous subsection have a physical interpretation. Here, we propose one.

To anticipate, to each boundary $\Lambda \in B_1^s(M)$, there is associated a family $\mathcal{T}_\Lambda$ of twisted sectors of the quantum field theory. $\mathcal{T}_\Lambda$ is characterized by a point of the cohomology torus $H^2_{dR}(M)/H^2_{dRZ}(M)$ and is parameterized by a pair of integers $k$, $l \in \mathbb{Z}$ such that $kl \in \nu_Q \mathbb{Z}$ and satisfying further restrictions when $\gamma(\Lambda) \in Q^{p_2}$, as explained earlier. In turn, each sector is a collection of topological vacua in one–to–one correspondence with $\text{Princ}(M)$, as usual. The $\tau$ dependent factor of the partition function with a Wilson loop insertion associated to $\Lambda$ of the sector $k,l$ is

$$Z_{(k,l)}(\Lambda, \tau) = \exp \left( -\frac{\pi \sigma(\Lambda)}{\tau_2} \right) F_{(k,l)}(\Lambda, \tau) \quad (4.2.1)$$

(cfr. eq. (4.1.5)). In the rest of the subsection, we shall try to justify the claims made.

For $\Lambda \in B_1^s(M)$, we define first

$$B_\Lambda = \sum_{rs} Q^{-1rs} \left( \oint_\Lambda A_r \right) A_s. \quad (4.2.2)$$

$$G_\Lambda = dB_\Lambda = \sum_{rs} Q^{-1rs} \left( \oint_\Lambda A_r \right) F_s. \quad (4.2.3)$$

As is easy to see from (3.4.1),

$$\oint_\Lambda B_\Lambda = \int_M G_\Lambda \wedge G_\Lambda. \quad (4.2.4)$$

Next, for $k$, $l \in \mathbb{Z}$ with $kl \in \nu_Q \mathbb{Z}$, we define the action

$$S_{(k,l)}(A, \Lambda, \tau) = \pi \int_M (F_A + kG_\Lambda) \wedge \hat{\tau}(F_A + kG_\Lambda) + 2\pi l \oint_\Lambda (A + kB_\Lambda)$$

$$- \pi kl \int_M G_\Lambda \wedge G_\Lambda, \quad (4.2.5)$$
where \( A \in \text{Conn}(P) \) with \( P \in \text{Princ}(M) \) (cfr. eqs. (3.1.1)-(3.1.3) and (3.2.1)). We shall consider now the quantum field theory defined by \( S_{(k,l)}(A, \Lambda, \tau) \). But first a few remarks are in order.

Since \( \int \Lambda A \not\in \mathbb{Z} \) is defined up to an arbitrary integer \( m_r \), \( B_\Lambda \) is defined up to a shift of the form \( B_m = \sum_{rs} Q^{-1} m_r A_s \). Correspondingly, \( G_\Lambda \) is defined up to a shift of the form \( G_m = \sum_{rs} Q^{-1} m_s F_s \). Note that \( B_m \) is a connection of a \( U(1) \) principal bundle \( Q_m \) such that \( n^r(c_{Q_m}) = \sum_{rs} Q^{-1} m_s \) (cfr. eqs. (2.5.4), (2.5.5)) and that \( G_m \) is its curvature.

If we make the replacements \( B_\Lambda \rightarrow B_\Lambda + B_m \) and \( G_\Lambda \rightarrow G_\Lambda + G_m \), one has

\[
S_{(k,l)}(A, \Lambda, \tau) \rightarrow S_{(k,l)}(A + kB_m, \Lambda, \tau) + \pi kl \int_M G_m \wedge G_m. \tag{4.2.6}
\]

Note that \( A + kB_m \in \text{Conn}(PQ_m^k) \). Further, \( kl \int_M G_m \wedge G_m \in 2\mathbb{Z} \).

Next, we come to the quantum field theory defined by the action \( S_{(k,l)}(A, \Lambda, \tau) \). Its partition function is computed summing over all topological vacua of \( \text{Princ}(M) \) and factoring the classical and quantum fluctuation contributions, as usual. As is easy to see, the ambiguity (4.2.6) is absorbed by exponentiation and topological vacua summation.

A calculation completely analogous to that expounded in sect. 3 for the partition function \( Z(\Lambda, \tau) \) shows that the \( \tau \) dependent factor of the partition function is precisely \( Z_{(k,l)}(\Lambda, \tau) \), eq (4.2.1).

The class of \( G_\Lambda \) in \( Z^2_{dR}(M) \) modulo \( Z^2_{dR\mathbb{Z}}(M) \) is the point of \( H^2_{dR}(M)/H^2_{dR\mathbb{Z}}(M) \) characterizing \( \mathcal{T}_\Lambda \) mentioned at the beginning of the subsection.

The conclusion of the analysis is that, to preserve Abelian duality in the presence of Wilson loops, it is necessary to assume the existence of twisted sectors.

**Appendix**

In this appendix, we provide briefly the details of the derivation of the formal expression (3.5.5) of the quantum partition function \( Z_{\text{qu}}(\Lambda, \tau) \). The starting expression of \( Z_{\text{qu}}(\Lambda, \tau) \), given in (3.3.6), is a formal functional integral which requires a careful treatment.

We normalize conventionally the functional measure \( D\varphi \) on a Hilbert space \( \mathcal{F} \) of fields \( \varphi \) so that

\[
\int_{\varphi \in \mathcal{F}} D\varphi \exp \left( -\frac{1}{2} \langle \varphi, \varphi \rangle \right) = 1. \tag{A.1}
\]

In our case, the relevant field Hilbert spaces are certain subspaces of \( C^p_{dR}(M) \) with \( p = 0, 1 \) equipped with the Hilbert space structure defined by (3.5.2). The corresponding functional measures are characterized by (A.1).
The invariant measure on the gauge group \( \text{Gau}(M) \) is defined by the translation of that on its Lie algebra \( \text{LieGau}(M) \) once the normalization of the exponential map is chosen. Recall that \( \text{LieGau}(M) \cong C^0_{dR}(M) \). We fix the normalization by writing \( h \in \text{Gau}(M) \) near 1 as \( h = \exp(2\pi if) \) with \( f \in C^0_{dR}(M) \) and choose \( Df \) as the measure on \( \text{LieGau}(M) \).

Let us go back to (3.3.6). We fix the gauge by imposing a generalized Lorentz condition

\[
d_1^\dagger v = a, \quad v \in C^1_{dR}(M),
\]

where \( a \in \text{ran}d_1^\dagger t \). We then employ a slight variant of the Faddeev–Popov trick.

We define a functional \( B(v, a) \) of the fields \( v \in C^1_{dR}(M) \), \( a \in \text{ran}d_1^\dagger \) through the identity

\[
1 = B(v, a) \int_{x \in \text{ran}d_0} Dx \delta_{\text{ran}d_1^\dagger}(d_1^\dagger(v + x) - a).
\]

It is easy to show that

\[
B(v - x, a) = B(v, a), \quad x \in \text{ran}d_0.
\]

Further, when \( v \) satisfies the gauge fixing condition (A.2),

\[
B(v, a) = B_0,
\]

where \( B_0 \) is a constant. We now insert these relations in the functional integral (3.3.6) and, after some straightforward manipulations, we obtain

\[
Z_{\text{qu}}(\Lambda, \tau) = \frac{\varrho B_0}{\text{vol}(\text{Harm}^1_Z(M))} \int_{v \in C^1_{dR}(M)} Dv \delta_{\text{ran}d_1^\dagger}(d_1^\dagger v - a)
\]

\[
\times \exp \left( -\langle v, (\pi \tau_2 d^\dagger d)_{1} v \rangle + 2\pi i \langle j_\Lambda, v \rangle \right),
\]

where \( j_\Lambda \) is defined in (3.5.3). Here, we have used the identity \( \text{ran}d_0 = B^1_{dR}(M) \) and the formal relation

\[
\text{vol}(Z^1_{dRZ}(M))/\text{vol}(B^1_{dR}(M)) = \text{vol}(\text{Harm}^1_Z(M)).
\]

Next, we define a function \( \Gamma(\xi) \) of the parameter \( \xi > 0 \) by the formal identity

\[
1 = \Gamma(\xi) \int_{a \in \text{ran}d_1^\dagger} Da \exp(-\xi \langle a, a \rangle).
\]

Introducing the above relation in the functional integral (A.6), we eliminate the \( \delta \) function, obtaining

\[
Z_{\text{qu}}(\Lambda, \tau) = \frac{\varrho B_0 \Gamma(\xi)}{\text{vol}(\text{Harm}^1_Z(M))} \int_{v \in C^1_{dR}(M)} Dv \exp \left( -\langle v, (\pi \tau_2 d^\dagger d + \xi dd^\dagger)_{1} v \rangle + 2\pi i \langle j_\Lambda, v \rangle \right).
\]

(4.9)
We compute first the Jacobian $\varrho$. Recalling the facts about the structure of the gauge group $\text{Gau}(M)$ expounded in subsect. 2.2, we find the formal relation

$$\varrho = \frac{\text{vol}(Z_{\text{der}}(M))}{\text{vol}(\text{Gau}(M))} = \frac{\text{vol}(B_{\text{der}}(M))}{\text{vol}(\text{Gau}_c(M))}.$$  \hfill (A.10)

The tangent map of the isomorphism $\alpha : \text{Gau}_c(M)/\text{Gau}_0(M) \to B_{\text{der}}(M)$ at the identity is just $d_0|_{\ker d_0^\perp}$. From here, we have $\varrho = \frac{\det' (\langle dd_1^\dagger d_0^\dagger \rangle)^{1/2}}{(\text{vol}(G(M))^{1/2})}$. One easily computes $\text{vol}(G(M)) = (\text{vol}(M)/2\pi)^{1/2}$. Thus,

$$\varrho = \left[\frac{2\pi \det' (\langle dd_1^\dagger d_0^\dagger \rangle)}{\text{vol}(M)}\right]^{1/2}.$$  \hfill (A.11)

The constant $B_0$ is easily computed from (A.3), taking (A.2) into account and writing $x = df$ with $f \in \ker d_0^\perp$. The result is

$$B_0 = \det' (\langle dd_1^\dagger d_0^\dagger \rangle)^{1/2}.$$  \hfill (A.12)

Similarly, $\Gamma(\xi)$ is easily computed from (A.8), writing $a = d_1^\dagger x$ with $x \in \ker d_1^{\dagger \perp}$:

$$\Gamma(\xi) = \left[\frac{\det' (2\xi \langle dd_1^\dagger \rangle)^{1/2}}{\det' (\langle dd_1^\dagger \rangle)}\right]^{1/2}.$$  \hfill (A.13)

The functional integrand (A.9) is invariant under the shifts $v \to v + \hat{v}_0$, where $\hat{v}_0 \in \text{Harm}_1^1(M)$, as is easy to see. Thus, we can factorize the functional integration as follows

$$\frac{1}{\text{vol}(\text{Harm}_1^1(M))} \int_{v \in C_{\text{der}}^1(M)} Dv = \int_{v_0 \in \text{Harm}^1(M)/\text{Harm}_1^1(M)} Dv_0 \int_{v' \in \text{Harm}^1(M)^{\dagger}} Dv'.$$  \hfill (A.14)

Proceeding in this way, we carry out the Gaussian integration straightforwardly and obtain

$$\int_{v \in C_{\text{der}}^1(M)} Dv \exp \left( -\langle v, (\pi \tau_2 d_1^\dagger d + \xi dd_1^\dagger) v \rangle + 2\pi i \langle j_\Lambda, v \rangle \right) = \left[\frac{\det G_1}{(2\pi)^{n_1}}\right]^{1/2} \prod_k \delta_{\langle j_\Lambda, \omega_k \rangle, 0} \times \det' \left( (2\pi \tau_2 d_1^\dagger d + 2\xi dd_1^\dagger)^{-1/2} \exp \left(-\pi^2 \langle j_\Lambda, (\tau_2 d_1^\dagger d + \xi dd_1^\dagger)^{-1} j_\Lambda \rangle \right) \right),$$  \hfill (A.15)

where $G_1$ is the matrix given by (3.5.6).

Next, we substitute (A.11), (A.12), (A.13) and (A.15) into (A.9). The resulting expression can be simplified noting that the operators $(d_1^\dagger d_0^\dagger), (dd_1^\dagger)$ have the same non zero spectrum counting also multiplicity and, thus, equal determinants and that

$$\det' (pd_1^\dagger d + qdd_1^\dagger) = \det' (p(d_1^\dagger d)) \det' (q(dd_1^\dagger)),$$  \hfill (A.16)
with \( p, q > 0 \). Proceeding in this way, the \( \xi \) gauge independence of \( Z_{\text{qu}}(\Lambda, \tau) \) becomes manifest and one straightforwardly obtains (3.5.5).

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