FOLIATION STRUCTURE
FOR GENERALIZED HÉNON MAPPINGS

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Abstract. We consider a generalized Hénon mapping $f: \mathbb{C}^2 \to \mathbb{C}^2$ and its Green function $g^+_f: \mathbb{C}^2 \to \mathbb{R}_{\geq 0}$ (see Section 2). It is well known that each level set of the form $\{g^+_f = c\}$ for $c > 0$ is foliated by biholomorphic images of $\mathbb{C}$ and each leaf is dense. In this paper, we prove that each leaf is actually an injective Brody curve in $\mathbb{P}^2$. Namely, for any holomorphic parametrization of any leaf, the Fubini-Study metric of $\mathbb{P}^2$ of its derivative is uniformly bounded. We also study the behavior of level sets of $g^+$ near infinity.

1. Introduction

In this paper, we consider two questions about two important mathematical objects: generalized Hénon mappings and Brody curves.

Firstly, a generalized Hénon mapping is defined by

$$f(z, w) = (p(z) - aw, z)$$

as a polynomial diffeomorphisms of $\mathbb{C}^2$ where $p: \mathbb{C} \to \mathbb{C}$ is a monic polynomial and $a$ is a non-zero constant. The polynomial diffeomorphisms of this class are simple but known to have complicated dynamics such as chaotic behaviors. Indeed, in [10], Friedland and Milnor classified polynomial diffeomorphisms of $\mathbb{C}^2$ up to conjugation in the group of polynomial diffeomorphisms of $\mathbb{C}^2$ and showed that the only dynamically interesting automorphisms of $\mathbb{C}^2$ are the finite compositions of generalized Hénon mappings. So, many mathematicians have studied the generalized Hénon mappings. In particular, concerning the foliation for generalized Hénon mappings, for instance, Fornæss and Sibony ([9]), Bedford and Smillie ([3], [4]), and Bedford, Lyubich and Smillie ([2]) studied the foliation of the set $K^+$ of non-escaping points, i.e., the set of points of bounded orbit. Hubbard and Oberste-Vorth ([12]) studied the foliation of the set $U^+$ of escaping points, i.e, the complement of $K^+$.

The sets $K^+$ and $U^+$ have different dynamical properties. For instance, recurrent behavior takes place in $K^+$ but not in $U^+$. The recurrent behavior makes difference in our work. In this paper, we focus on $U^+$.

One of the useful methods to study the dynamics of $f$ is pluripotential theory. Let $g^+_f: \mathbb{C}^2 \to \mathbb{R}$ be the Green function on $\mathbb{C}^2$ for $f$ (see Section 2). By studying $g^+_f$, we can obtain information on the dynamics of $f$. The set $U^+$ can be characterized by $U^+ = \{g^+_f > 0\}$. In [12], Hubbard and Oberste-Vorth proved:

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Theorem 1.1 (See Theorem 7.2 in [12]). The set \( \mathcal{L}_c := \{ g^+ = c \} \) for \( c > 0 \) is naturally foliated by Riemann surfaces. The leaves of the natural foliation of \( \mathcal{L}_c \) are isomorphic to \( \mathbb{C} \) and each leaf is dense in \( \mathcal{L}_c \).

Then, one of the natural questions would be

“What kind of Riemann surfaces are the leaves?”

We consider this question.

Secondly, the Brody curve first appeared in Brody’s proof in [5] that every compact non-Kobayashi hyperbolic manifold contains a non-trivial holomorphic image of \( \mathbb{C} \). That non-trivial entire curve is a Brody curve.

Also, in [11], Gromov considered infinite dimensional geometry and introduced the concept of mean dimension. As a main example, he considered the space of Brody curves. This space has been further studied by Eremenko ([7], [6]), Tsukamoto ([16], [17], [18], [19]), and Matsuo and Tsukamoto ([13]). From the positivity of the mean dimension of the space of Brody curves on \( \mathbb{P}^2 \), it seems that there are many Brody curves on \( \mathbb{P}^2 \). Also, from the perspective of geometry, it might be interesting to see what Brody curves look like. However, except trivial ones such as polynomial mappings of \( \mathbb{C} \) into \( \mathbb{C}^2 \subset \mathbb{P}^2 \) and something of that sort, not many examples are known. Hence, we consider the following question in this paper:

“Are there any examples of non-trivial Brody curves?”

In general, it is difficult to find non-trivial Brody curves.

The following is the main theorem of this paper and answers the above two questions:

Theorem 1.2. Every leaf of the natural foliation of the level set \( \mathcal{L}_c \) for \( c > 0 \) is an injective Brody curve in \( \mathbb{P}^2 \) with respect to the Fubini-Study metric.

Indeed, in this paper, we prove the theorem for every finite composition of generalized Hénon mappings but for simplicity of arguments, we mainly focus on just generalized Hénon mappings and discuss the finite compositions of them at the end of the paper (see Theorem 8.1).

Here, by “non-triviality” in the second question, we mean that the closure of any leaf (an injective Brody curve) in \( \mathcal{L}_c \) is equal to \( \mathcal{L}_c \) itself in \( \mathbb{C}^2 \). Also, from the perspective of complex dynamics, this theorem implies that every leaf in \( \mathcal{L}_c \) does not fluctuate too much.

Our approach to Theorem 1.2 is modifying the Brody reparametrization lemma. With the original Brody reparametrization technique, we cannot locate limit curves in the space. This is one of the main difficulties. In order to handle this difficulty, we give a fixed point to the reparametrized family and analyse the behavior of \( \mathcal{L}_c \) near infinity. The fixed point strategy is as follows. We may assume that \( c > 0 \) is sufficiently large. We consider a specific family of analytic discs in a single leaf. We compare the Fubini-Study metric at every point on each analytic disc. These comparisons are delicate computations. Then, instead of centering some kind of the maximum derivative point as in the original Brody reparametrization lemma, we send a prescribed fixed point in the leaf to the center of every reparametrized analytic disc. In our case, the reparametrized family in this way becomes a normal family and has a fixed point. This implies that limit curves after the normal family argument passes through the fixed point. The main ingredient for this fixed
point strategy is the verticalness of leaves near a certain point at infinity and non-recurrent behavior in $U^+$. However, this does not clear up all the obstacles. Since limit curves exist in the closure of the union of the initial analytic discs, we have to handle the points at infinity. For this obstacle, we prove the following theorem in Section 3 by investigating the Green function $g^+$ near infinity (for $I_+$ and $K_c$, see Section 2):

**Theorem 1.3.** There is no non-trivial holomorphic curve in $\mathbb{P}^2$, which passes through $I_+$, and is supported in $K_c \subseteq \mathbb{P}^2$ for $c > 0$.

This theorem implies that limits curve must stay inside the level set $\mathcal{L}_c$. Then, the foliation of $\mathcal{L}_c$ tells us that limit curves must stay in a single leaf. Using a special coordinate system near a point at infinity (see Section 5), we prove that the limit map is injective. Using elementary theorems of one complex variable, we prove that actually the initial leaf itself is an injective Brody curve.

Besides the above discussions, the main theorem together with Fornæss’s result in [8], implies examples of short $C^2$ with real analytic boundary which is foliated by injective Brody curves of $\mathbb{P}^2$.

This paper is organized in the following way: In Section 2, we briefly review generalized Hénon mappings. In Section 3, Theorem 1.3 is proved. In Section 4, the concepts of the Brody curve and the injective Brody curve are explained. In Section 5, we review a special holomorphic coordinate system near a certain point at infinity and define a family of analytic discs. In Section 6, we further study the holomorphic coordinate system. In Section 7, Theorem 1.2 is proved. In Section 8, the finite compositions of generalized Hénon mappings are considered. Finally, in Section 9, short $C^2$’s are discussed.

**Notation.** We use $[z : w : t]$ for the homogeneous coordinate system of $\mathbb{P}^2$ and $(z, w)$ for the usual affine coordinate system of $\mathbb{C}^2 \subset \mathbb{P}^2$ unless stated otherwise. Let $\Delta_r$ denote the disc in $\mathbb{C}$ centered at the origin and of radius $r$, and $\Delta$ the unit disc in $\mathbb{C}$.

We denote by $\| \cdot \|$ the standard Euclidean norm and by $ds$ the Fubini-Study metric of $\mathbb{P}^2$. If necessary, we write it more precisely as $ds(p, v)$ for $p \in \mathbb{P}^2$ and $v \in T_p \mathbb{P}^2$. For simpler notation, for a holomorphic mapping $\psi : U \to \mathbb{P}^2$ with $U$ an open subset of $\mathbb{C}$, we write $\|\psi\|_{FS, \theta_0}$ to mean $ds(\psi(\theta_0), d\psi|_{\theta=\theta_0}(\frac{d}{d\theta}))$ for $\theta_0 \in U$.

For a given holomorphic endomorphism $h : \mathbb{C}^2 \to \mathbb{C}^2$, we write its $n$-th iterate as $h^n = \left(h^n_1, h^n_2\right)$. By convention, $h^0(z, w)$ means simply $(z, w)$. For a given holomorphic function $P : \mathbb{C} \to \mathbb{C}$, $P'$ denotes the derivative of $P$.

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2. Generalized Hénon mappings

We start this section by clarifying the functions that we will study in this paper. A generalized Hénon mapping is a holomorphic polynomial diffeomorphism \( f : \mathbb{C}^2 \to \mathbb{C}^2 \) defined by

\[
    f(z, w) = (p(z) - aw, z)
\]

where \( p(z) \) is a monic polynomial of one complex variable \( z \) with degree \( d \geq 2 \) and \( a \neq 0 \). Then, \( f^{-1}(z, w) = (w, (p(w) - z)/a) \).

Let \( \mathbb{P}^2 \) be the 2-dimensional complex projective space and

\[
    I_+ := [0 : 1 : 0] \quad \text{and} \quad I_- := [1 : 0 : 0]
\]

in the homogeneous coordinate system of \( \mathbb{P}^2 \). Then, \( f \) has the natural extension to \( \tilde{f} : \mathbb{P}^2 \setminus \{I_+\} \to \mathbb{P}^2 \setminus \{I_+\} \) by

\[
    \tilde{f}([z : w : t]) = \left[ t^d p(z) - awt^{d-1} : zt^{d-1} : t^d \right].
\]

Similarly, \( f^{-1} \) also has the natural extension to \( \tilde{f}^{-1} : \mathbb{P}^2 \setminus \{I_-\} \to \mathbb{P}^2 \setminus \{I_-\} \) by

\[
    \tilde{f}^{-1}([z : w : t]) = \left[ wt^{d-1} : \frac{1}{a}(t^d p(w) - zt^{d-1}) : t^d \right].
\]

We recall the following notions and properties related to the dynamics of \( f \) as in [12]. Let

\[
    K^\pm = \{ p \in \mathbb{C}^2 : \{ f^{\pm n}(p) \} \text{ is a bounded sequence of } n \},
\]

and \( J^\pm = \partial K^\pm, K = K^+ \cap K^- \) and \( U^\pm = \mathbb{C}^2 \setminus K^\pm \).

We define the Green function on \( \mathbb{C}^2 \) for \( f \) and \( f^{-1} \) by

\[
    g^+(z, w) := \lim_{n \to \infty} \frac{1}{dn} \log^+ \| g^n(z, w) \|
\]

and

\[
    g^-(z, w) := \lim_{n \to \infty} \frac{1}{dn} \log^+ \| g^{-n}(z, w) \|,
\]

respectively where \( \log^+ := \max\{0, \log\} \). Then, \( g^\pm \) are non-negative and Hölder continuous on \( \mathbb{C}^2 \), plurisubharmonic on \( \mathbb{C}^2 \), pluriharmonic on \( U^\pm \), and such that

\[
    g^+ \circ f = dg^+ \quad \text{and} \quad g^- \circ f^{-1} = dg^-.
\]

Also, it is well-known that \( K^\pm = \{ g^\pm = 0 \} \). Here, more generally, we define

\[
    K_c := \{ g^+ \leq c \}, \quad \text{for } c > 0 \quad \text{and} \quad \mathcal{L}_c := \{ g^+ = c \}.
\]

In this paper, we are focusing on the case \( c > 0 \).

**Proposition 2.1** (See [13]). \( K^\pm, U^\pm, I_\pm, \tilde{f} \) and \( \tilde{f}^{-1} \) satisfy the following properties:

1. \( I_- \) and \( I_+ \) are the super-attracting fixed points of \( \tilde{f} \) and \( \tilde{f}^{-1} \), respectively.
2. any compact subset of \( U^\pm \) uniformly converges to \( I_\pm \), respectively.
3. \( \tilde{f}(\{ t = 0 \} \setminus I_+) = I_- \) and \( \tilde{f}^{-1}(\{ t = 0 \} \setminus I_-) = I_+ \), and
4. \( K^\pm = K^+ \cup I_+ \) and \( K^- = K^- \cup I_- \).
The following filtration is also well-known. For sufficiently large $R > 0$, we have that $|z| > R$ implies either $|p(z) - aw| > |z|$ or $|z| < |w|$, or both. For such $R$ let
\[
\begin{align*}
V^+ &= \{(z, w) \in \mathbb{C}^2: R \leq |z|, |w| \leq |z|\}, \\
V^- &= \{(z, w) \in \mathbb{C}^2: R \leq |w|, |z| \leq |w|\}, \quad \text{and} \\
W &= \{(z, w) \in \mathbb{C}^2: |z|, |w| \leq R\}.
\end{align*}
\]

Then, we have

**Proposition 2.2.**

1. $f(V^+) \subseteq V^+$ and $f^{-1}(V^-) \subseteq V^-$.
2. $U^+ = \bigcup_{i=0}^\infty f^{-i}(V^+)$ and $U^- = \bigcup_{i=0}^\infty f^i(V^-)$.

Later in Section 5, we will choose sufficiently large $R$ with more conditions for our purpose.

### 3. Behavior of level sets $\mathcal{L}_c := \{g^+ = c\}$ near infinity

In this section, we prove Theorem 1.3. We start by recalling the following definitions and properties related to $\bar{f}$ over $\mathbb{P}^2$. For details, see [14].

For the study of $K_\bar{c}$ near infinity, we rather consider $\bar{f}$ on $\mathbb{P}^2$. Correspondingly, we consider the Green function $G$ on $\mathbb{C}^3$ of $\bar{f}$ as a rational map of $\mathbb{P}^2$. Let $\pi : \mathbb{C}^3 \setminus \{0\} \to \mathbb{P}^2$ denote the natural projection and $\bar{F}$ a lifting of $\bar{f}$ to $\mathbb{C}^3 \setminus \{0\}$ such that $\sup((\|\zeta, \omega, t\| = 1) \|\bar{F}(\zeta, \omega, t)\| = 1$. Since $\bar{F}$ has the holomorphic extension to the whole $\mathbb{C}^3$, we can think of $\bar{F}$ as being defined over $\mathbb{C}^3$.

The Green function $G$ is defined by
\[
G(\zeta, \omega, t) := \lim_{n \to \infty} \frac{1}{d^n} \log \|\bar{F}^n(\zeta, \omega, t)\|
\]
over $\mathbb{C}^3$. For the convergence of this function, see [14]. Then, $G$ satisfies
\[
\left\{ \begin{array}{l}
G(\lambda \zeta, \lambda \omega, \lambda t) = |\lambda| + G(\zeta, \omega, t) \\
G(\bar{F}(\zeta, \omega, t)) = d \cdot G(\zeta, \omega, t),
\end{array} \right.
\]
where $(\zeta, \omega, t) \in \mathbb{C}^3$ and $\lambda \in \mathbb{C} \setminus \{0\}$ is a constant. By Theorem 1.6.5 in [14], $G$ is continuous over $\mathbb{C}^3 \setminus (\{0\} \cup \pi^{-1}(\{I_+\}))$.

Let $(\zeta, \omega, t) \in \mathbb{C}^3 \setminus (\{0\} \cup \pi^{-1}(\{I_+\}))$. Then, for $t \neq 0$, by the homogeneity and regularity of $\bar{F}$, we have
\[
\begin{align*}
G(\zeta, \omega, t) &= \lim_{n \to \infty} \frac{1}{d^n} \log \|\bar{F}^n(\zeta, \omega, t)\| \\
&= \lim_{n \to \infty} \frac{1}{d^n} \left( \log |t^n| + \log \left\|\bar{F}^n \left( \frac{\zeta}{t}, \frac{\omega}{t}, 1 \right) \right\| \right) \\
&= \log |t| + G \left( \frac{\zeta}{t}, \frac{\omega}{t}, 1 \right) \\
&= \log |t| + g^+ \left( \frac{\zeta}{t}, \frac{\omega}{t} \right).
\end{align*}
\]
(3.1)

The last line is due to $\lim_{m \to \infty} [\log(a_m + 1)]/m = \lim_{m \to \infty} [\log^+ a_m]/m$ for $a_m \geq 0$. 
The following lemma describes the behavior of the Green function $g^+$ near $I_-; g^+$ diverges to $\infty$ near $I_-.$

**Lemma 3.1.** For any $M > 0,$ there exist sufficiently small $\epsilon_1, \epsilon_2 > 0$ such that $g^+ > M$ on $U_M \cap \mathbb{C}^2$ for $U_M = \{1 : w : t\} \in \mathbb{P}^2 : |w| < \epsilon_1, |t| < \epsilon_2\}.$

**Proof.** We fix the affine coordinate chart $\{(1, w, t) : w, t \in \mathbb{C}\} = \{1\} \times \mathbb{C}^2 \subseteq \mathbb{C}^3 \setminus \{0\}$ centered at $I_-.$ Since $G$ is continuous over $\mathbb{C}^3 \setminus \{0\},$ $G$ restricted to $\{1\} \times \mathbb{C}^2$ is continuous. Take a neighborhood $U$ of $I_-$ with compact closure in the coordinate chart $\{(1, w, t) : w, t \in \mathbb{C}\}.$ Then, $G$ is bounded over $U.$ Let $m$ be the bound. We take $\epsilon_2$ small enough so that $-\log |\epsilon_2| - m > M$ and $\epsilon_1$ small enough to satisfy $\{(1, w, t) : |w| < \epsilon_1 \text{ and } |t| < \epsilon_2\} \subseteq U.$ Let $U_M = \{1 : w : t\} : |w| < \epsilon_1 \text{ and } |t| < \epsilon_2\}$ in $\mathbb{P}^2.$ Then, by (3.1) and the choice of $\epsilon_1, \epsilon_2 > 0,$ we have $g^+(1/t, w/t) > -\log |t| - m > M$ over $U_M \setminus \{t = 0\}.$

The following proposition implies that the level set $\mathcal{L}_c$ with $c > 0$ can only accumulate at $I_+$ near $\{t = 0\}$ in $\mathbb{P}^2.$

**Proposition 3.2.**

$$K_c = K_c \cup \{I_+\}.$$ 

**Proof.** By Proposition 2.1 $K_c \cup \{I_+\} \subseteq K_c.$ It suffices to show that any sequence $\{x_n\} \subseteq K_c$ with $\|x_n\| \to \infty$ as $n \to \infty$ converges to $I_+$ in $\mathbb{P}^2.$ Consider the inverse images $\{f^{-1}(x_n)\}$ of $\{x_n\}$ under $f$ in $\mathbb{C}^2 \subset \mathbb{P}^2.$ Then, since $\mathbb{P}^2$ is compact, by passing to a convergent subsequence, we may assume that $f^{-1}(x_n) \to L$ for some $L \in \mathbb{P}^2.$ If $L \in \mathbb{C}^2,$ then $x_n \to f(L) \in \mathbb{C}^2$ and this contradicts $\|x_n\| \to \infty.$ Therefore, $L \in \{t = 0\}.$ From Proposition 2.1 we have $f(\{t = 0\}) \subseteq \mathbb{C}.$ Since $\tilde{f}|_{\mathbb{C}} = f,$ we conclude that $\{x_n\}$ should converge either to $I_+$ or $I_-.$ However, by Lemma 3.1 $g^+$ is not bounded near $I_-,$ so $I_-$ cannot be a limit point of $K_c.$ This proves the statement.

**Proof of Theorem 1.3.** For the proof, we use the blow-up of $\mathbb{P}^2$ at $I_+$ as in Section 1.2 of [14]. Let $\mathbb{P}^2_{I_+}$ be the blow-up of $\mathbb{P}^2$ at $I_+$ and $\pi_{I_+} : \mathbb{P}^2_{I_+} \to \mathbb{P}^2$ be the corresponding birational map. By abuse of notation, we keep writing $\tilde{f} : \mathbb{P}^2_{I_+} \to \mathbb{P}^2$ for the lift of $\tilde{f} : \mathbb{P}^2 \to \mathbb{P}^2.$

We prove this theorem by contradiction. Suppose to the contrary that there exists a non-trivial holomorphic curve $\mathcal{C}$ as in the statement. Let $\mathcal{C}_{I_+}$ be a lift of the curve $\mathcal{C}$ in $\mathbb{P}^2_{I_+}.$ Note that we are not assuming any regularity nor connectedness of the curve on the exceptional set $\pi_{I_+}^{-1}(I_+).$ Then, we consider the image of $\mathcal{C}_{I_+}$ under $\tilde{f}$ in $\mathbb{P}^2.$ Observe that the restriction to $\mathbb{C}^2$ of $\tilde{f}(\mathcal{C}_{I_+})$ is simply $f(\mathcal{C} \setminus \{I_+\}).$ If we have $\{I_+\} \neq \mathcal{C}_{I_+} \cap \pi_{I_+}^{-1}(I_+),$ then $f(\mathcal{C} \setminus \{I_+\})$ has a limit point other than $I_+$ in $\{t = 0\} \subset \mathbb{P}^2.$ This is a contradiction to Proposition 3.2 since $f(K_c) = K_{\text{deg}}.$ Hence, it suffices to prove that $\{I_+\} \notin \text{the limit points of } f(\mathcal{C} \setminus \{I_+\}).$

Let $\phi : \Delta \to \mathbb{P}^2$ be a parametrization of $\mathcal{C}$ and we write $\phi(\theta) = [z(\theta) : 1 : t(\theta)]$ where $z, t$ are holomorphic functions of $\theta$ with $z(0) = t(0) = 0.$ The case where either $z$ or $t$ is identically 0 is obvious. So, assume that both $z$ and $t$ have discrete zeros. By reparametrizing and restricting, we may further assume that $z$ and $t$
vanish only at $\theta = 0$. Then, for $\theta \neq 0$, we have
\[
\bar{f}(z(\theta) : 1 : t(\theta)) = \left[ t(\theta) \frac{\partial}{\partial \theta} \left( \frac{z(\theta)}{t(\theta)} \right) - \frac{a}{z(\theta)} : t(\theta) - \frac{a}{z(\theta)} : 1 : t(\theta)^d \right].
\]

In order to have $\{I_+\} \in \text{the limit points of } f(C \setminus \{I_+\})$, $t(\theta)/z(\theta)p(z(\theta)/t(\theta)) - a/z(\theta)$ and $t(\theta)/z(\theta)$ both shrink to 0 simultaneously as $\theta \to 0$. This is impossible since $(z(\theta)/t(\theta))^{d-1}$ becomes the dominating term in $t(\theta)/z(\theta)p(z(\theta)/t(\theta)) - a/z(\theta)$ and blows off to infinity as $t(\theta)/z(\theta)$ shrinks to 0. This completes the proof.

\section{Brody Curves}

In this section, we briefly introduce the concepts of the \textit{Brody curve} and the \textit{injective Brody curve}.

\textbf{Definition 4.1} (Brody Curve). Let $M$ be a compact complex manifold with a smooth metric $ds_M$. Let $\psi : \mathbb{C} \to M$ be a non-constant holomorphic map.

The map $\psi$ is said to be Brody if $\sup_{\theta \in \mathbb{C}} ds_M(\psi(\theta), d\psi(\theta/\psi)) < C_M$ for some constant $C_M > 0$. We call the image $\psi(\mathbb{C})$ a Brody curve in $M$. The curve $\psi(\mathbb{C})$ is said to be \textit{injective Brody} if the parametrization $\psi$ is injective.

In the rest of the paper, we only consider the Brody curves in $\mathbb{P}^2$ with respect to the Fubini-Study metric of $\mathbb{P}^2$.

Below, we consider some trivial examples. The proofs are all straightforward computations and so, we omit them.

\textbf{Proposition 4.2.} Let $a$ be a complex constant and $p, q$ polynomials of one complex variable $z$. Then, all curves of the form $[z : p(z) : 1]$ and of the form $[p(z) \exp(z) : q(z) \exp(\alpha z) : 1]$ are Brody in $\mathbb{P}^2$.

However, not all holomorphic curves from $\mathbb{C}$ to $\mathbb{P}^2$ are Brody. The mapping $z \to [e^z : e^{iz^2} : 1]$ is not Brody. For the verification, simply take $z = bi$ for real $b$ and let $b \to \infty$. Even if we require them to be injective, not all injective curves from $\mathbb{C}$ to $\mathbb{P}^2$ are Brody. The following gives us some examples of injective but non-Brody holomorphic maps of $\mathbb{C}$ to $\mathbb{P}^2$.

\textbf{Proposition 4.3.} The map $f_n : z \to (z, \exp(z^n))$ is not Brody in $\mathbb{C}^2 \subset \mathbb{P}^2$ for $n \geq 3$. In particular, not all biholomorphic images of $\mathbb{C}$ in $\mathbb{P}^2$ are Brody.

We close this section by pointing out a property of injective Brody curves.

\textbf{Proposition 4.4.} For an injective Brody curve $C$ in $\mathbb{P}^2$, every parametrization of $C$ has uniformly bounded Fubini-Study metrics. In short, the injective Brody property does not depend on the choice of the parametrization.

\textbf{Proof.} Let $\phi_1, \phi_2 : \mathbb{C} \to C$ be two biholomorphic parametrizations of $C$. The composition $\phi_2^{-1} \circ \phi_1 : \mathbb{C} \to C$ is a biholomorphism of $C$ on $C$. From a theorem of one complex variable, $\phi_2^{-1} \circ \phi_1(z) = az + b$ for constants $a, b \in \mathbb{C}$ with $a \neq 0$.

\section{Family of analytic discs}

In this section, we construct a family of analytic discs, to which we will apply our modified Brody reparametrization technique.
5.1. Choices of large numbers. We list some technical conditions on $R$ in the filtration for $f$ and $c$ in the level set $L_c$ that will be used later.

**Constant $R$ in the filtration for $f$.** Write $p(z) = \sum_{i=0}^{d} a_i z^i$ with $a_d = 1$ and $q(z) = p(z) - z^d = \sum_{i=0}^{d'} a_i z^i$ with $a_{d'} \neq 0$ and $d' \leq d-1$. Here, $q(z)$ may be 0 if no such $a_{d'}$ exists. Given a polynomial $H(z) = \sum_{i=0}^{d} h_i z^i$, we define a real polynomial $|H|(x) := \sum_{i=0}^{d} |h_i| x^i$.

We can choose $R > 2$ with the properties listed below:

1. $R$ is $\geq$ the largest absolute value of the roots of the real polynomial equations $5x^d/4 - |p(x) - (|a| + 2)x = 0$ and $5dx^{d-1}/4 - |p'(x)| - 1 = 0$. Indeed, this condition implies that for any $z$ with $|z| > R$,

\[
\frac{3}{4}|z|^d \leq |p(z)| - (|a| + 2)|z| \leq |p(z)| \leq |p(z)| + (|a| + 2)|z| \leq \frac{5}{4}|z|^d
\]

and

\[
\frac{3}{4}d|z|^{d-1} \leq |p'(z)| - 1 \leq |p'(z)| \leq |p'(z)| + 1 \leq \frac{5}{4}d|z|^{d-1}.
\]

Note that this condition implies that $R$ satisfies the filtration property.

2. If $d' > 1$, $R$ is $\geq$ the largest absolute value of the roots of the real polynomial equations $5|a_{d'}|x^d/4 - |q(x)| - |a| x = 0$ and $5d'|a_{d'}|x^{d'-1}/4 - |q'(x)| - |a| = 0$. Indeed, this condition implies that

\[
\frac{3}{4}|a_{d'}||z|^d \leq |p(z) - z^d| - |az| \leq |p(z) - z^d| \leq |p(z) - z^d| + |az| \leq \frac{5}{4}|a_{d'}||z|^d
\]

and

\[
\frac{3}{4}d'|a_{d'}||z|^{d-1} \leq |p'(z) - dz^{d-1}| - |a| \leq |p'(z) - dz^{d-1}| + |a| \leq \frac{5}{4}d'|a_{d'}||z|^{d-1}.
\]

If $d' \leq 1$, we disregard this condition on $R$.

3. For $z \in \mathbb{C}$ with $|z| > R$, \[|p'(z)z - p(z)| - d' \leq \frac{1}{d'}.
\]

4. \[\frac{|a_{d'}| + |a|}{2R} < \frac{1}{100^d}.
\]

5. $R$ is greater than both $1/r_1$ and $1/r_2$ where $r_1$ and $r_2$ are as in the following lemma:

**Lemma 5.1** (See Lemma 6.3 of [12]). Let $F$ be the space of analytic functions $h : \Delta_{1/R_0} \to \Delta$ such that $h(0) = 0$ and $h'(0) = 1$, where $R_0$ is as in Condition 1. Then, $F$ is compact with respect to the compact-open topology. In particular, there exists $r_1, r_2 > 0$ independent of $h \in F$ such that every $h \in F$ is injective in $\Delta_{1/r_1}$ and satisfies $\Delta_{1/r_2} \subseteq h(\Delta_{1/r_2})$.

Indeed, this condition is for the HO-coordinate system in the next subsection.

**Constant $c$ in $L_c$.** For notational convenience we will use $r$ such that $r := \exp(c) > 1$ instead of $c$ in the following list of conditions.

1. $r > \exp(\max_{z,w} \in W \ g^+(z, w))$. This condition implies that $f^n(L_c) \cap W = \emptyset$ for any $n = 0, 1, \cdots$, where the set $W$ is from the definition of the filtration.

2. $r > 5R$. 

In the rest of the paper, we assume that $R$ and $c$ satisfy all the above conditions unless stated otherwise.

5.2. Coordinate system near $I_-$. We recall a theorem of Hubbard and Oberste-Vorth.

Proposition 5.2 (See Proposition 5.2 in [12]). There exist analytic functions $\varphi_\pm : V_\pm \to \mathbb{C} \setminus \Delta$ such that

$$
\varphi_+(f(z, w)) = (\varphi_+(z, w))^d \text{ and } \varphi_-(f^{-1}(z, w)) = (\varphi_-(z, w))^d,
$$

$$
\lim_{\|(z, w)\| \to \infty} \frac{\varphi_+(z, w)}{z} = 1 \text{ in } V^+ \text{ and } \lim_{\|(z, w)\| \to \infty} \frac{\varphi_-(z, w)}{Aw} = 1 \text{ in } V^-,
$$

where $A$ is a non-zero constant only depending on $a$ in $f$.

We recall the proof below in order to use some estimate in the proof later.

Proof. We only prove the statements for $f$ and $\varphi_+$. Those for $f^{-1}$ and $\varphi_-$ are analogous. For simpler notation, we write $z_n = f_1^n(z, w)$ and $w_n = f_2^n(z, w)$. As in [12], we define $\varphi_+$ to be the following telescoping-looking infinite product:

$$
\varphi_+(z, w) = z \cdot \left(\frac{z_1}{z_n^d}\right)^{1/d} \cdots \left(\frac{z_{n+1}}{z_n^d}\right)^{1/d^{n+1}} \cdots.
$$

We claim that the limit function exists and is analytic on $V^+$. We first check the well-definedness of the $d^{n+1}$-st root in each factor $(z_{n+1}/z_n^d)^{1/d^{n+1}}$. We have

$$
\left(\frac{z_{n+1}}{z_n^d}\right)^{1/d^{n+1}} = \left(p(z_n) - aw_n\right)^{1/d^{n+1}} = \left(1 + \sum_{i=0}^{d'} a_i z_n^i - aw_n\right)^{1/d^{n+1}}.
$$

By Proposition 2.2 we have that $(z_{n+1}, w_{n+1}) \in V^+$ whenever $(z_n, w_n) \in V^+$. By the triangle inequality, Conditions 2 and 4 on $R$, and the relationship $|z_n| \geq |w_n|$, we have $|\left(\sum_{i=0}^{d'} a_i z_n^i - aw_n\right)/z_n^d| \leq 1/4$ for all $(z, w) \in V^+$. So, the angle of $z_{n+1}/z_n^d \in (-\arctan 1/d, \arctan 1/d) \subseteq (-\pi, \pi)$ for all $n > 0$ and all $(z, w) \in V^+$. Thus, we can take the principle branch of the $d^{n+1}$-st root in each factor so that each factor is well-defined and holomorphic in $V^+$.

Now we check the convergence of (5.1) on $V^+$. We estimate $|z_{n+1}/z_n^d|^{1/d^{n+1}}$. Assume that $d' \geq 2$. By the triangle inequality, Condition 2 on $R$, and the relationships $R < |z| < |z_n|$ and $|w_n| \leq |z_n|$ on $V^+$, we have

$$
1 + \frac{5|a_{d'}|}{4|z|^d d'} \geq 1 + \frac{\sum_{i=0}^{d'} a_i z_n^i - aw_n}{z_n^d} \geq \frac{z_{n+1}}{z_n^d} \geq 1 - \frac{\sum_{i=0}^{d'} a_i z_n^i - aw_n}{z_n^d} \geq 1 - \frac{5|a_{d'}|}{4|z|^d d'}.
$$

In particular, from Condition 4 on $R$, we have

$$
1 - \frac{1}{4} \leq \frac{z_{n+1}}{z_n^d} \leq 1 + \frac{1}{4}
$$

and therefore,

$$
\frac{z_{n+1}}{z_n^d}^{1/d^{n+1}} - 1 = \frac{\sum_{i=0}^{d^{n+1}-1} |z_{n+1}/z_n^d|^{1/d^{n+1}}}{\sum_{i=0}^{d^{n+1}-1} |z_{n+1}/z_n^d|^{1/d^{n+1}}} \leq \frac{1}{d^{n+1} \cdot 4} = \frac{1}{3d^{n+1}}.
$$
This shows the uniform convergence of $\sum_{i=0}^{\infty} \|z_{n+1}/z_{n}^{d+1/d-1} - 1\|$. Hence, an elementary theorem of one complex variable about infinite products proves the existence and the analyticity of $\varphi_+$. The property $\varphi_+(f(z, w)) = (\varphi_+(z, w))^d$ is clear from the definition.

We prove the property about the asymptotic behavior of $\varphi_+$ near infinity. From (5.2), we can also estimate the difference between $\varphi_+(z, w)$ and $z$ for $|z| > R$:

$$\left(1 - \frac{5|a_0|}{4|z|^{d-d'}}\right)^{1/(d-1)} \leq \left|\varphi_+(z, w)/z\right| \leq \left(1 + \frac{5|a_0|}{4|z|^{d-d'}}\right)^{1/(d-1)}.$$

Since $|w| < |z|$ in $V^+$, we have just proved $|\varphi_+(z, w)/z| \to 1$ as $\|(z, w)\| \to \infty$. We need to show the angle of $\varphi_+(z, w)/z$ converges to 0 as $\|(z, w)\| \to \infty$.

The angle of $(z_{n+1}/z_{n})^{d+1/d-1}$ lies between $-(\sin^{-1} 5|a_d|/(4|z|^{d-d'}))/d^{m+1}$ and $(\sin^{-1} 5|a_d|/(4|z|^{d-d'}))/d^{m+1}$ where we take the branch of $\sin^{-1}$ to be $[-\pi/2, \pi/2]$. Since

$$\sum_{i=0}^{\infty} \frac{1}{d^{n+1}} \sin^{-1} \left(\frac{5|a_d|}{4|z|^{d-d'}}\right) = \frac{1}{d-1} \sin^{-1} \left(\frac{5|a_d|}{4|z|^{d-d'}}\right) \approx \frac{1}{d-1} \cdot \frac{5|a_d|}{4|z|^{d-d'}}.$$ 

the angle of $\varphi_+(z, w)/z$ is between $-5|a_d|/(4(d - 1)|z|^{d-d'})$ and $5|a_d|/(4(d - 1)|z|^{d-d'})$. Thus, again, since $|w| < |z|$, as $\|(z, w)\| \to \infty$, the angle of $\varphi_+(z, w)/z$ also shrinks to 0.

These two approximations prove the asymptotic behavior of $\varphi_+$ near $I_+$. From (5.3) with Condition 4 on $R$, we know that $\varphi_+(V^+) \subseteq C \setminus \Delta$.

In the case of $d' \leq 1$, we can apply the same arguments with better approximations.

We use the work of Hubbard and Oberste-Vorth in [12] to find a local coordinate chart near $I_-$. For details, see [12]. For any $(z, w) \in V^+$,

$$\varphi_{HO} : (z, w) \to (\varphi_+(z, w), w/z).$$

This defines a local biholomorphism between $(c_0, \infty) \times \Delta$ and $V^+ \cap \{ g > c_0 \}$ due to Condition 5 on $R$ where $c_0 > 0$ is a constant satisfying the conditions in the second part of Subsection 5.1 and $c_0 < c$. For notational convenience, in the rest of the paper, we write $(z, w)$ for the usual Euclidean coordinate system of $C^2$, and $(x, y)$ for the local coordinate system near $I_-$ defined just above and call it the HO-coordinate system.

We observe that by use of this local holomorphic coordinate system, the Green function of $f$ can be written simply as $g(z, w) = \log |\varphi_+(z, w)|$.

Hence, each level set $L_c$ inside $V^+$ is a vertical set $\{ |x| = \exp(c), |y| < 1 \}$ and each leaf inside $V^+$ is a vertical complex line $\{ x = s \}$ for $s \in C$ with $|s| = \exp(c)$ in the HO-coordinate system.

5.3. Family of analytic discs. Let $i_\alpha : \Delta \to \{ \alpha \} \times \Delta$ be an obvious biholomorphic map for $\alpha \in C$.

Let $c$ be a sufficiently large positive real number as in the second part of Subsection 5.1. We define a family of analytic discs for $s \in C$ with $|s| = \exp(c)$. For
each \( n \in \mathbb{N} \), we define an analytic disc \( \phi_{s,n} : \Delta \to \mathcal{L}_c \) by
\[
\phi_{s,n}(\theta) = f^{-n}(\varphi_H^{-1}(i_{s,n}(\theta))), \quad \text{for each } \theta \in \Delta
\]
and write \( \phi_{s,n} = ([\phi_{s,n}], [\phi_{s,n}]_2) \). Also, we denote \( \Phi_{s,n} := \phi_{s,n}(\Delta) \). It is not difficult to check that \( \Phi_{s,n} \subset \Phi_{s,n+1} \) from direct computations. Indeed, the leaves of \( \mathcal{L}_c \) in Theorem 1.1 are of the form \( \cup_{n=0}^{\infty} \Phi_{s,n} \). In the rest of the paper, we denote the leaf corresponding to the parameter \( s \in \mathbb{C} \) with \( |s| = \exp(c) \) for \( c \) sufficiently large as in the second part of Subsection 5.1 by \( C_s := \cup_{n=0}^{\infty} \Phi_{s,n} \). It is also not difficult to see that \( C_s \) is a smooth complex manifold. For the proofs in detail, see [12] and also [1] in the second part of Subsection 5.1.

6. Structure of the level sets \( \{g^+ = c\} \) in \( V^+ \)

In this section, we study the derivatives of the function \( \varphi_+ \) on \( V^+ \). As a main lemma in this section, Lemma 6.7 states that when \( c \) is sufficiently large as in the second part of Subsection 5.1, the leaves of \( \{g^+ = c\} \) in \( V^+ \) can be understood as almost vertical curves. This verticalness is one of the main ingredients for our modified reparametrization.

In the following series of lemmas, we study the derivatives of \( f_1^n \) and \( f_2^n \) in order to study the derivatives of \( \varphi_+ \). We first recall the following chain rules.
\[
\begin{pmatrix}
\partial f_1^{n+1}/\partial z & \partial f_1^{n+1}/\partial w \\
\partial f_2^{n+1}/\partial z & \partial f_2^{n+1}/\partial w
\end{pmatrix}
= \begin{pmatrix} p'(f_1^n) & -a \\ 1 & 0 \end{pmatrix}
\begin{pmatrix} \partial f_1^n/\partial z & \partial f_1^n/\partial w \\ \partial f_2^n/\partial z & \partial f_2^n/\partial w \end{pmatrix}
\]
and also
\[
\begin{pmatrix}
\partial f_1^1/\partial z & \partial f_1^1/\partial w \\
\partial f_2^1/\partial z & \partial f_2^1/\partial w
\end{pmatrix}
= \begin{pmatrix} p'(z) & -a \\ 1 & 0 \end{pmatrix}.
\]

We start with the following proposition as a preliminary step.

**Proposition 6.1.** For \((z, w) \in V^+ \) and all \( n \geq 0 \), \( |f_1^n(z, w)| \geq |z|^n \).

**Proof.** The proof is by induction. The cases \( n = 0 \) and \( n = 1 \) are obvious from Condition 1 on \( R \). Similarly, when \( n = 2 \), we have
\[
|f_1^2(z, w)| \geq \frac{3}{4} |f_1^1(z, w)|^d \geq \frac{3}{4} |f_1^2(z, w)|^2 \geq \frac{3}{4} \left( \frac{3}{4} |z|^d \right)^2 \geq \frac{27}{64} |z|^4 \geq |z|^2.
\]
Suppose that the inequality is true for \( n = k \geq 2 \). Then, from the relationship \( |f_1^k(z, w)| > |z| > R \) and Condition 1 on \( R \), we have
\[
|f_1^{k+1}(z, w)| \geq |p(f_1^k(z, w)) - af_1^k(z, w)| \geq \frac{3}{4} |f_1^k(z, w)|^d \geq \frac{3}{4} |z|^{dk} \geq |z|^{k+1}.
\]
Hence, it is proved.

**Lemma 6.2.** For \((z, w) \in V^+ \) and all \( n \geq 0 \),
\[
\frac{\partial f_{1, n+1}^n}{\partial z} \neq 0 \quad \text{and} \quad \left| \frac{\partial f_{2, n+1}^n/\partial z}{\partial f_{1, n+1}^n/\partial z} \right| \leq \frac{2}{d|f_{1, n}^1|}.
\]
Proof. The proof is by induction. We consider the initial case \( n = 0 \) first. If \( n = 0 \), then by Condition 1 on \( R > 0 \),
\[
\frac{\partial f_1^1}{\partial z} = |p'(z)| > 0 \quad \text{and} \quad \frac{\partial f_2^1}{\partial z} = \left| \frac{1}{p'(z)} \right| < \frac{4}{3d|z|}.
\]
Suppose that the statements are true for \( n = k - 1 \). Then, from Conditions 1 and 4 on \( R \), Proposition 6.4 and the induction hypotheses, it is straightforward that \( |\partial f_{k+1}^1/\partial z| > 0 \) and also, we have
\[
\left| \frac{\partial f_{k+1}^1}{\partial f_{k}^1} \right| = \left| \frac{\partial f_1^1}{\partial z} \frac{\partial f_2^1}{\partial z} - a \frac{\partial f_2^1}{\partial f_1^1} \right| \leq 1 \left( \frac{|p'(f_1^1)| - |a| \left| \frac{\partial f_2^1}{\partial z} \right|}{d|f_{k-1}^1|} \right) \leq \frac{2}{d|f_1^1|}.
\]
Hence, the lemma is proved. \( \square \)

Lemma 6.3. For \((z, w) \in V^+ \) and all \( n \geq 0 \),
\[
\frac{\partial f_{n+1}^1}{\partial w} \neq 0 \quad \text{and} \quad \left| \frac{\partial f_{n+1}^1}{\partial w} \right| \leq \frac{2}{d|f_1^1|}.
\]
Proof. The proof is the same as for Lemma 6.2. \( \square \)

Lemma 6.4.
\[
\left| \frac{\partial f_n^w}{\partial f_1^w} \right| \leq \left[ \prod_{i=1}^{n-1} \left( 1 + \frac{8|a|}{d|z|} \right) \right] \left| \frac{a}{|p'(z)|} \right| \text{ for } (z, w) \in V^+ \text{ and all } n > 0.
\]
Proof.
\[
\left| \frac{\partial f_{n+1}^1}{\partial f_{n}^1} \right| = \left| \frac{p'(f_1^n)\partial f_1^n/\partial w - a\partial f_2^n/\partial w}{p'(f_1^n)\partial f_1^n/\partial w - a\partial f_2^n/\partial w} \right| \leq \frac{|p'(f_1^n)\partial f_1^n/\partial w + |a|\partial f_2^n/\partial w|}{|p'(f_1^n)\partial f_1^n/\partial w - |a|\partial f_2^n/\partial w|} \leq 1 + \frac{|a| \partial f_2^n/\partial w}{|p'(f_1^n)\partial f_1^n/\partial w|} \leq 1 + \frac{|a| \partial f_2^n/\partial w}{|p'(f_1^n)\partial f_1^n/\partial w|} \leq 1 + \frac{8|a|}{d|p'(f_1^n)||f_1^n|} \left| \frac{\partial f_n^w}{\partial f_1^w} \right| \leq \left( 1 + \frac{8|a|}{d|p'(f_1^n)||f_1^n|} \right) \left| \frac{\partial f_n^w}{\partial f_1^w} \right| \leq \left( 1 + \frac{8|a|}{d|f_1^n|} \right) \left| \frac{\partial f_n^w}{\partial f_1^w} \right|.
\]
In the second last line, Lemma 6.2 and Lemma 6.3 are used. In the second last and last inequalities, Conditions 1 and 4 on \( R \) and Proposition 6.4 are used. The initial value \(|(\partial f_1^1/\partial w)/(\partial f_1^1/\partial z)|\) is easy to compute; we differentiate \( p(z) - aw \) with respect to \( z \) and \( w \) and take the ratio. Proposition 6.5 completes the proof of the lemma. \( \square \)

We estimate the following infinite product for the main lemma in this section.

Proposition 6.5. For some \( r_0 > 0 \) and sufficiently large \( r > 1 \) such that \( r_0/r < 1 \),
\[
\sum_{n=1}^{\infty} \log \left( 1 + \frac{r_0}{r^n} \right) \leq \frac{r_0}{r - 1}.
\]
Proof. We consider the function $h(x) = \log |1 + x|$ of real numbers over a closed interval $[-r_0/r, r_0/r]$. Elementary calculus tells us that for all $n > 0$, $\log(1 + r_0/r^n) \leq r_0/r^n$. This inequality proves the statement. \hfill \Box

Now, we are ready to study the derivatives of $\varphi_+$.

Lemma 6.6.

$$\frac{\partial \varphi_+}{\partial z} \neq 0 \text{ over } V^+.$$

Proof. Recall \textbf{[5.1]}. We define a sequence $\{\varphi_+\}_n$ of partial products of $\varphi_+$ on $V^+$ by

$$(\varphi_+)_n(z, w) = z \prod_{i=0}^{n-1} \left( \frac{f_{i+1}(z, w)}{(f_i(z, w))^d} \right)^{1/d+1},$$

where $(\varphi_+)_0(z, w) = (z, w)$. We know that $\varphi_+$ is analytic on $V^+$ and $(\varphi_+)_n \to \varphi_+$ locally uniformly on $V^+$. Observe that $[(\varphi_+)_n]^{d^n} = f_{1^n}$ and $(\varphi_+)_n \neq 0$ on $V^+$. Taking a partial derivative with respect to $z$, we have

$$d^n[(\varphi_+)_n]^{d^n-1} \frac{\partial (\varphi_+)_n}{\partial z} = \frac{\partial f_{1^n}}{\partial z}.$$ 

Here, all the denominators and all the numerators of the fractions are non-zero. Then, observe that

$$\frac{\partial \varphi_+}{\partial z} = \lim_{N \to \infty} \prod_{n=0}^{N} \frac{\partial (\varphi_+)_n+1/\partial z}{\partial (\varphi_+)_n/\partial z}.$$ 

Due to Proposition \textbf{[5.2]} it suffices to consider

$$\lim_{N \to \infty} \prod_{n=0}^{N} \frac{1}{d} \cdot \frac{f_{1^n}}{f_{1^{n+1}}} \cdot \frac{\partial f_{1^n+1}}{\partial z}.$$
If \( d' > 1 \), by Condition 2 on \( R > 0 \) in Subsection 5.1 Lemma 6.1 and Lemma 6.2 we have

\[
1 - (3d'|a_{d'}|/4 + |a|\mu)|f^n_1|^{-(d-d')/d} \
1 + 5|a_{d'}||f^n_1|^{-(d-d')/4}
\]

\[= \frac{1}{d} \cdot \frac{|f^n_1|}{|f^n_1|^d + 5|a_{d'}||f^n_1|^d/4} \cdot \left( |f^n_1|^d - 3d'|a_{d'}||f^n_1|^d - 1/4 - |a|\mu \right)\]

\[\leq \frac{1}{d} \cdot \frac{|f^n_1|}{|f^n_1|^d + 3|a_{d'}||f^n_1|^d/4} \cdot \left( |f^n_1|^d + 5d'|a_{d'}||f^n_1|^d - 1/4 + |a|\mu \right)\]

\[= \frac{1}{d} \cdot \frac{|f^n_1|}{|f^n_1|^d + 3|a_{d'}||f^n_1|^d/4} \cdot \left( |f^n_1|^d + 5d'|a_{d'}||f^n_1|^d - 1/4 + |a|\mu \right)\]

\[= \frac{1}{d} \cdot \frac{1}{1 - 3|a_{d'}||f^n_1|^{-(d-d')/d}}.
\]

where \( \mu \) is an upper bound in Lemma 6.2 and 6.3 for instance, say \( 1/R \). As in Proposition 5.2 from this approximation together with an elementary theorem of one complex variable about the convergence of infinite products, it is not difficult to see that \( \partial \varphi^+ / \partial z \) converges to a non-zero value on \( V^+ \).

If \( d' \leq 1 \), we can apply the same argument with better approximations. Hence, the proof is completed. \( \Box \)

The following lemma together with the previous lemma implies that the analytic discs look vertical on \( V^+ \). We prove the following:

**Lemma 6.7.**

\[
\left| \frac{\partial \varphi^+/\partial w}{\partial \varphi^+/\partial z} \right| \leq 2 \left| \frac{\alpha}{p'(z)} \right| \text{ for } (z, w) \in V^+.
\]

**Proof.** Recall the definition of the sequence \( \{(\varphi^n_+)\} \) in the proof of Lemma 6.6. Taking a partial derivative to \( [(\varphi^n_+)]^{d'n} = f^n_1 \) with respect to \( z \), we have \( d'n[(\varphi^n_+)]^{d'-1}\partial(\varphi^n_+)/\partial z = \partial f^n_1 / \partial z \). Do the same with respect to \( w \). Note that \( (\varphi^n_+) \neq 0 \), \( \partial \varphi^+/\partial z \neq 0 \) and \( \partial f^n_1 / \partial z \neq 0 \) on \( V^+ \). Thus, we have

\[
\frac{\partial \varphi^+/\partial w}{\partial \varphi^+/\partial z} = \lim_{n \to \infty} \frac{\partial(\varphi^n_+)/\partial w}{\partial(\varphi^n_+)/\partial z} = \lim_{n \to \infty} \frac{\partial f^n_1 / \partial w}{\partial f^n_1 / \partial z}.
\]
The limit is bounded as follows:
\[
\left| \frac{\partial \varphi_+}{\partial x} \right| \leq \lim_{n \to \infty} \left[ \prod_{i=1}^{n} \left( 1 + \frac{8|a|}{d|z|^{2n}} \right) \right] \left| \frac{a}{p'(z)} \right|
\]
\[
\leq \lim_{n \to \infty} \left[ \prod_{i=1}^{n} \left( 1 + \frac{z^2}{10} \cdot \frac{1}{|z|^{2n}} \right) \right] \left| \frac{a}{p'(z)} \right|
\]
\[
\leq \exp(0.25) \cdot \left| \frac{a}{p'(z)} \right| \leq 2 \left| \frac{a}{p'(z)} \right| \quad \text{for} \ (z, w) \in V^+.
\]

The first inequality is from Lemma 6.4, the second one from Condition 4 on \( R \), and the third one from Proposition 6.5. We have just proved the lemma.

\[\square\]

7. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. Recall the family of analytic discs in Subsection 5.3.

We assume that the constant \( c > 0 \) is fixed sufficiently large as in Subsection 5.3 and another constant \( s \in \mathbb{C} \) is fixed such that \( |s| = \exp(c) \) until the proof of the main theorem. For notational simplicity, we write \( \varphi_n \) for \( \varphi_{s,n} \). In the same way, we omit the parameter \( s \) below unless stated otherwise.

We define the following set:
\[ \Theta_{n,i} := \{ \theta \in \Delta : f^j(\varphi_n(\theta)) \in V^+ \ \text{for all} \ j \ \text{with} \ i \leq j \leq n \}. \]

Then, \( \Theta_{n,i} \subset \Theta_{n,i+1} \) is clear. Intuitively speaking, \( \Theta_{n,i} \) is the set of \( \theta \)'s in \( \Delta \) corresponding to the points in \( \varphi_{1}^{-1}(\{ \varphi_s \} \times \Delta) \) whose images under \( f \), for \( k = 0, \cdots, n-i \), remain in \( V^+ \). In particular, \( \Theta_{n,0} \) is the set of \( \theta \)'s with \( \varphi_n(\theta) \in V^+ \) due to Proposition 2.2 and our choice of \( R \) (Condition 1 on \( R \)).

We first consider the influence of the mapping \( f^{-1} \) on the Fubini-Study metric of the analytic disc \( f(\varphi_n(\Delta)) \). Due to the filtration for \( f \), Proposition 2.2 Condition 1 on \( R \) and Condition 1 on \( c \), we know that \( f^j(\varphi_n(\Delta)) \) lies in \( V^+ \cup V^- \) and it suffices to consider \( f^{-1} \) in three cases:

**Case i** \( f^{-1} : f(\varphi_n(\Theta_{n,i})) \to f^{-1}(\varphi_n(\Delta)) \cap V^+ \) maps points of \( f^i(\varphi_n(\Delta)) \) from \( V^+ \) to \( V^+ \).

**Case ii** \( f^{-1} : f(\varphi_n(\Theta_{n,i} \setminus \Theta_{n,i-1})) \to f^{-1}(\varphi_n(\Delta)) \cap V^- \) maps points of \( f^i(\varphi_n(\Delta)) \) from \( V^+ \) to \( V^- \), and

**Case iii** \( f^{-1} : f(\varphi_n(\Delta \setminus \Theta_{n,i})) \to f^{-1}(\varphi_n(\Delta)) \cap V^- \) maps points of \( f^i(\varphi_n(\Delta)) \) from \( V^- \) to \( V^- \).

We study Case i. We start by proving the following lemma.

**Lemma 7.1.** Let \( i, n \in \mathbb{Z} \) be given such that \( 0 \leq i \leq n \). Let \( (z, w) = f^i \circ \varphi_{s,n} \) and \( (z', w') = [d(f^i \circ \varphi_{s,n})][\frac{d}{d\theta}] \). Then, for any \( \theta \in \Theta_{n,i} \), we have \( |z'| \leq \mu |w'| \) and

\[
|w'|^2 \leq \frac{|z'|^2 + |w'|^2 + |z'w - zw'|^2}{4(1 + |z|^2 + |w|^2)} \leq \frac{3|w'|^2}{1 + |z|^2 + |w|^2}
\]

where \( \mu \) is a small positive real number only depending on \( R \) in the filtration for \( f \).
Proof. Note that \((z, w) \in V^+, |w| \leq |z|\). From Lemma 6.7 Condition 1 on \(R\) and Proposition 6.1 the first assertion is straightforward. Then,

\[
\frac{|z'|^2 + |w'|^2 + |z'w - zw'|^2}{(1 + |z|^2 + |w|^2)^2} \geq \frac{|w'|^2 + |zw'|^2/2}{(1 + |z|^2 + |w|^2)^2} \geq \frac{|w|^2}{4(1 + |z|^2 + |w|^2)^2}
\]

On the other hand,

\[
\frac{|z'|^2 + |w'|^2 + |z'w - zw'|^2}{(1 + |z|^2 + |w|^2)^2} \leq \frac{|z'|^2 + |w'|^2 + 2(|z'|^2 + |zw'|^2)}{(1 + |z|^2 + |w|^2)^2} \leq 2\frac{|z'|^2 + |w'|^2}{1 + |z|^2 + |w|^2} \leq \frac{3|w'|^2}{1 + |z|^2 + |w|^2}.
\]

\(\square\)

The following proposition implies that in Case i, \(f^{-1}\) increases the Fubini-Study metric at least by a fixed ratio.

Lemma 7.2. Let \(1 \leq i \leq n\).

\[
1 < \frac{d^2}{256 \cdot 15^2 |a|^2} \left| s^{d^4} \right|^{4 - 4/d} \leq \inf_{\theta \in \Theta_{n, i - 1}} \left\| f^{-1} \circ \phi_n \right\|_{FS, \theta}^2 \leq \left\| f \circ \phi_n \right\|_{FS, \theta}^2.
\]

Proof. Let \(\theta \in \Theta_{n, i - 1}\). For simpler computations, we use \((z, w) = f^i \circ \phi_n, (z', w') = \{d(f^i \circ \phi_n)\}(\frac{z}{a})\), and \((\phi_n, \omega_{n}) = f^{-i} \circ \phi_n\). Then, \((z, w, \omega) \in \{\varphi_+ = s^{d^4}\} \cap V^+\) and so \(|z'| \leq 4|w'|/100^d\) from Lemma 6.7 and Conditions 1 and 4 on \(R\). Also, \((\phi_n, \omega_{n}) \in \{\varphi_+ = s^{d^4}\} \cap V^+\).

Since \((\phi_n, \omega_{n}) \in V^+, |w| > R\) and \(|w| \geq |p(w) - z|/|a|\). Then, from (6.3) with Condition 4 on \(R\) and Condition 2 on \(c\), we have an approximation of \(w\) in terms of \(s\) as follows:

\[
\frac{3}{2} \left| s^{d^4} \right| \geq |z| \geq |p(w)| - |aw| \geq \frac{3}{4}|w|^d \quad \text{and} \quad \frac{5}{4}|w|^d \geq |p(w)| + |aw| \geq \frac{2}{3}|s^{d^4}|.
\]

and so \(2s^{d^4/1/d} \geq |w| \geq 8s^{d^4/15} \).

With this range of \(w\) and \(|z'| \leq 4|w'|/100^d\), we estimate the increasing rate of the Fubini-Study metrics.

\[
\left\| f \circ \phi_n \right\|_{FS, \theta}^2 = \frac{|z|^2 + |w|^2 + |z'w - zw'|^2}{(1 + |z|^2 + |w|^2)^2} \leq \frac{|z'|^2 + |w'|^2 + 2(|z'|^2 + |zw'|^2)}{(1 + |z|^2 + |w|^2)^2} \leq 2\frac{|z'|^2 + |w'|^2}{1 + |z|^2 + |w|^2} \leq \frac{3|w'|^2}{|z|^2}.
\]

By Lemma 7.1 with \((w, (p(w) - z)/a) \in V^+\), we have

\[
\left\| f^{-1} \circ \phi_n \right\|_{FS, \theta}^2 = \frac{|w'|^2 + | - z'/a + p'(w)w'/a|^2 + |(p(w) - z)w'/a - w(-z'/a + p'(w)w'/a)|^2}{(1 + |w|^2 + |(p(w) - z)/a|^2)^2} \geq \frac{| - z'/a + p'(w)w'/a|^2}{4(1 + |w|^2 + |(p(w) - z)/a|^2)^2} \geq \frac{|p'(w)w'|^2}{24|a|^2|w|^2}.
\]
Lemma 7.3. Let \( |z'| \leq 4|w'|/100^d \) with Condition 1 on \( R \). Hence,
\[
\frac{|p'(w)|^2|z|^2}{72 |a|^2 |w|^2} \leq \inf_{\theta \in \Theta_{n,1}} \|f^{i-1} \circ \phi_n\|_{F, \theta}^2.
\]
Again, by (5.3), the range of \( w \) bounded above by the lower bound of Lemma 7.2.

\[
\text{The last inequality is from Condition 4 on } R \text{ and Condition 2 on } c.
\]

We consider Case iii.

Lemma 7.3. Let \( 1 \leq i \leq n \).
\[
\sup_{\theta \in \Delta \setminus \Theta_{n,i}} \|f^{i-1} \circ \phi_n\|_{F, \theta} \leq 2|a|^2 \left( \frac{20}{9} \right)^2 (1 + d + 1/d^2 (1 + 2R^2)^2).
\]
Here, this upper bound is bounded above by the lower bound in Lemma 7.2.

Proof. Let \( \theta \in \Delta \setminus \Theta_{n,i} \). For simpler computations, we use \((z, w) = f^i \circ \phi_n, (z', w') = d(f^i \circ \phi_n)(\frac{z'}{a})\). Then, \((z, w) \in \{ \varphi_+ = s^d \} \cap V^-, \) and so \( |z| \leq |w| \) and \( R \leq |w| \).

We estimate the ratio \( \|f^{i-1} \circ \phi_n\|_{F, \theta}^2/\|f^i \circ \phi_n\|_{F, \theta}^2 \). We have
\[
\|f^i \circ \phi_n\|_{F, \theta}^2 = \frac{|z'|^2 + |w'|^2 + |z'-w'|^2}{(1 + |z|^2 + |w|^2)^2} \geq \frac{|z'|^2 + |w'|^2}{(1 + 2|w|^2)^2}.
\]

On the other hand, we have
\[
\|f^{i-1} \circ \phi_n\|_{F, \theta}^2 = \frac{|w|^2 + |z'/a - p'(w)w'/a|^2 + |w(-z'/a + p'(w)w'/a) - (p(w) - z)w'/a|^2}{(1 + |w|^2 + |p(w)|^2)^2} \leq \frac{|a|^2 |aw|^2 + |z'|^2 + |p'(w)w'|^2 + |wz'|^2 + |p'(w)w| + |p(w)| + |w| |w'|^2}{(1 + |w|^2 + |p(w)|^2)^2} \leq \frac{|a|^2 |aw|^2 + 2|p'(w)|^2 + |p(w)| + |p'(w)| + |w| |w'|^2}{(3 |w|^d/4)^2} \max\{|z'|, |w'|\}^2 \leq \frac{2|a|^2 (1 + d + 1/d)^2 (5 |w|^d/4)^2}{(3 |w|^d/4)^2} \max\{|z'|, |w'|\}^2.
\]

The third inequality is from Condition 1. The second last inequality is due to Conditions 1, 3 and 4 on \( R \). The last inequality is due to the Condition 1 on \( R \).

Thus, the ratio is bounded by
\[
2|a|^2 \left( \frac{20}{9} \right)^2 (1 + d + 1/d)^2 (1 + 2R^2)^2 \frac{R^2d}{R^2d}.
\]

By Condition 4 on \( R \) and Condition 2 on \( c \), the upper bound of Lemma 7.3 is bounded above by the lower bound of Lemma 7.2.
We finally consider Case ii. Here, we show that the increasing rate of the Fubini-Story metric in this case is dominated by the increasing rate of the Fubini-Story metric in Case i up to a uniform constant.

Lemma 7.4. For all \( i, n \) such that \( 1 \leq i \leq n \), there exists a constant \( C_s > 0 \) independent of \( i, n \) such that

\[
\sup_{\theta \in \Theta_{n,i} \setminus \Theta_{n,-1}} \| f^{i-1} \circ \phi_n \|_{F,\theta}^2 \leq C_s \inf_{\theta \in \Theta_{n,i-1}} \| f^{i-1} \circ \phi_n \|_{F,\theta}^2.
\]

Proof. Let \( \theta \in \Theta_{n,i} \setminus \Theta_{n,-1} \). Then, \( f^i(\phi_n(\theta)) \in V^+ \) but \( f^{i-1}(\phi_n(\theta)) \in V^- \). For simpler computations, we use \((z, w) = f^i \circ \phi_n, (z', w') = |d(f^i \circ \phi_n)(\frac{d}{d\theta})|, (z*, w*) = f^{i-1} \circ \phi_n \) and \((z'_*, w'_*) = |d(f^{i-1} \circ \phi_n)(\frac{d}{d\theta})| \). Then, \((z, w) \in \{ \varphi_+ = s^{d^d} \} \cap V^+\) and so \( |z'| \leq 4|w'|/100d \) from Lemma [6.7] and Conditions 1 and 4 on \( R \). Also, \((z*, w*) \in \{ \varphi_* = s^{d^d+1} \} \cap V^-\) and \(|w| \leq |(p(z) - w)/a|\).

Recall that

\[
(z'_*, w'_*) = \begin{pmatrix} 0 & 1 \\ -1/a & p'(w)/a \end{pmatrix} \begin{pmatrix} z' \\ w' \end{pmatrix}.
\]

We have

\[
\| f^{i-1} \circ \phi_n \|_{F,\theta}^2 = \frac{|w'|^2 + |\varphi'(w)w'|^2 + |(p(w) - z)w'/a - w(\varphi'(w)w'/a)|^2}{(1 + |w|^2 + |(p(w) - z)/a|^2)^2} \leq \frac{|w'|^2 + |\varphi'(w)w'|^2 + |(p(w) - z)w'/a - w(\varphi'(w)w'/a)|^2}{(1 + |w|^2 + |(p(w) - z)/a|^2)^2} \leq 2 \frac{|w'|^2 + |\varphi'(w)w'|^2 + |(p(w) - z)w'/a - w(\varphi'(w)w'/a)|^2}{(1 + |w|^2 + |(p(w) - z)/a|^2)^2} \leq 2 \frac{2|p'(w)/a|^2 + |(p(w) - z)/a|^2}{(1 + |w|^2 + |(p(w) - z)/a|^2)^2} \leq 2 \frac{2\varphi'(|w|) + 2|p'(w)/a|^2}{(1 + |w|^2 + |(p(w) - z)/a|^2)^2} = \frac{4|w'|^2}{1 + |w|^2 + |(p(w) - z)/a|^2} + 4 \frac{|w|^2 + |p'(w)|^2 + \varphi'(w)|^2}{(1 + |w|^2 + |(p(w) - z)/a|^2)^2} |w'|^2.
\]

The fourth inequality is from Condition 1 on \( R \).

From Lemma [7.1], we have

\[
\| f^i \circ \phi_n \|_{F,\theta}^2 \geq \frac{|w'|^2}{4(1 + |z|^2 + |w|^2)^2} \geq \frac{|w'|^2}{12|z|^2}.
\]

Thus, the ratio \( \| f^{i-1} \circ \phi_n \|_{F,\theta}^2/\| f^i \circ \phi_n \|_{F,\theta}^2 \) is bounded above by

\[
\frac{48|z|^2}{1 + |w|^2 + |(p(w) - z)/a|^2} + 48 \frac{|p'(w)/a|^2 + \varphi'(w)|^2|z|^2}{(1 + |w|^2 + |(p(w) - z)/a|^2)^2}.
\]

We estimate upper bounds of (7.2) in 4 cases:

Case 1 \(|w| < R\),
Case 2 \( R \leq |w| < \left| \frac{s^{d'}}{5} \right|^{1/d} \),

Case 3 \( \left| \frac{s^{d'}}{5} \right|^{1/d} \leq |w| < 3 \left| s^{d'} \right|^{1/d} \) and

Case 4 \( 3 \left| s^{d'} \right|^{1/d} \leq |w| \).

Here, observe that from Condition 2 on \( c \), we have for \( i \geq 1 \),

\[
R < \frac{|s|}{5} < \left| \frac{s^{d'}}{5} \right|^{1/d}.
\]

Let \( M'_R := \sup_{|z| \leq R} |p'(z)| \).

Case 1. This case is easy to see that

\[
\text{(7.2)} \leq 48 \left| 2s^{d'} \right|^2 + 48 \frac{(M'_R)^2 R^2 + 1}{|a|^2} \left| 2s^{d'} \right|^2
\]

from (5.3), Condition 2 on \( c \) and Condition 4 on \( R \).

Case 2. In this case, we have

\[
\frac{1}{4} \left| s^{d'} \right| \leq \frac{1}{2} \left| s^{d'} \right| - \frac{5}{4} |w|^d \leq |z| - |p(w)| \leq |p(w) - z|
\]

from (5.3), Condition 2 on \( c \) and Conditions 1 and 4 on \( R \). Then, again, from (5.3), Condition 2 on \( c \) and Condition 1 on \( R \), (7.2) is bounded as follows:

\[
\text{(7.2)} \leq \frac{48|z|^2}{((|z| - |p(w)|)/|a|^2)^2} + \frac{48|z|^2(\left| p'(w)/a^2 \right| + |wp'(w)/a^2|)}{((|z| - |p(w)|)/|a|^4)^2}
\]

\[
\leq \frac{48|a|^2 |z|^2}{((|z| - |p(w)|)^2 / |a|^2)} + \frac{48|a|^2 |z|^2(5d|w|^{d-1}/4)^2 + (M'_R)^2((|z|^2 + 1)}{((|z| - |p(w)|)^4}
\]

\[
\leq \frac{48|a|^2 \left| 2s^{d'} \right|^2}{s^{d'/4}^2}
\]

\[
\text{(7.2)} \leq 4^2 \cdot 2^2 \cdot 48|a|^2 + 4^4 \cdot 48 \cdot 4 \cdot 3|a^2 \left( (\frac{d}{4})^2 + \frac{(M'_R)^2}{5} \right).}
\]
Case 3. Similarly to Case 2, from (5.3), Condition 2 on c and Condition 1 on R, we have
\[
\begin{align*}
\frac{|z|^2}{|w|^2} & \leq \frac{48|z|^2}{|w|^4} + \frac{48|z|^2 (|p'(w)/a|^2 + |wp'(w)/a|^2)}{|w|^4} \\
& \leq \frac{48|z|^2}{|w|^4} + \frac{48|z|^2 ((5d|w|^{d-1}/4)^2 + (M_R')^2)(|w|^2 + 1)}{|a|^2|w|^4} \\
& \leq \frac{48}{|s^d/5|^{2/d}} \\
& \leq 5 \cdot 2^2 \cdot 48 \left| s^d \right|^{2-2/d} + \frac{5^2 \cdot 48 \cdot 2^2 \cdot 2}{|a|^2} \left( \frac{15}{4} \right)^2 d^2 + 3(M_R')^2 \left| s^d \right|^{4-4/d}.
\end{align*}
\]

Case 4. In this case, we have |w| > R and
\[
\frac{1}{12} |w|^{d-2} \left| s^d \right| \leq |p(w)| - |z| \leq |p(w) - z| \leq \frac{3}{4} |w|^{d-2} \left| s^d \right|
\]
from (5.3), Condition 2 on c and Conditions 1 and 4 on R. As above, again, from (5.3), Condition 2 on c and Condition 1 on R, we have
\[
\begin{align*}
\frac{|z|^2}{[(p(w)) - |z|]/a]^2} & \leq \frac{48|z|^2}{|w|^2d/12a^2} + \frac{48|z|^2 (|p'(w)/a|^2 + |wp'(w)/a|^2)}{|w|^2d/12a^2} \\
& \leq \frac{48|z|^2}{|w|^2d/12a^2} + \frac{48|z|^2 ((5d|w|^{d-1}/4)^2 + (M_R')^2)(|w|^2 + 1)}{|w|^4d/12a^4} \\
& \leq \frac{48}{|s^d/5|^{2/d}} \\
& \leq 5 \cdot 2^2 \cdot 48 \left| s^d \right|^{2-2/d} + \frac{5^2 \cdot 48 \cdot 2^2 \cdot 2}{|s^d|^{4/d}} \left( \frac{15}{4} \right)^2 d^2 + 3(M_R')^2 \left| s^d \right|^{4-4/d}.
\end{align*}
\]

Now, we compare the bounds of (7.2) obtained from the 4 cases to the lower bound in Lemma 7.2. Since the upper bounds of (7.2) have the same or less order of \(|s^d|\) than the lower bound in Lemma 7.2, thus, we can find the maximum ratio of the bounds of (7.2) to the lower bound in Lemma 7.2. This maximum ratio is
the desired constant for $C_s$. So, the statement is proved. Independence is clear from the proof. \qed

So far, by Lemma 7.2, Lemma 7.3 and Lemma 7.4 together with $\Theta_{n,n} = \Delta$, we have proved that the maximum of the Fubini-Study metric on $\Delta \setminus \Theta_{n,0}$ is dominated by the infimum of the Fubini-Study metric on $\Theta_{n,0}$ when $n \in \mathbb{N}$. Hence, we are going to compare the values of the Fubini-Study metric over $\Theta_{n,0}$. We consider a more general statement.

**Proposition 7.5.** \(\sup_{\theta \in \Theta_{n,i}} \|f^i \circ \phi_n\|_{F_{\theta}}^2_{\theta} \) is uniformly bounded for \(0 \leq i \leq n\).

We write \(f^i \circ \phi_n = ([f^i \circ \phi_n]_1, [f^i \circ \phi_n]_2)\). We first prove that \(\sup_{\theta \in \Theta_{n,i}} |\partial [f^i \circ \phi_n]_2/\partial \theta|/\inf_{\theta \in \Theta_{n,i}} |\partial [f^i \circ \phi_n]_2/\partial \theta|\) is uniformly bounded for \(0 \leq i \leq n\) and prove Proposition 7.5 using (5.3) and Lemma 7.3.

**Proposition 7.6.** For all \(n = 0, 1, 2, \ldots\),

\[
\frac{\sup_{\theta \in \Delta} |\partial [\varphi_{HO}^{-1} \circ \iota_{s^2}]_2/\partial \theta|}{\inf_{\theta \in \Delta} |\partial [\varphi_{HO}^{-1} \circ \iota_{s^2}]_2/\partial \theta|} = \frac{\sup_{\theta \in \Theta_{n,i}} |\partial [f^n \circ \phi_n]_2/\partial \theta|}{\inf_{\theta \in \Theta_{n,i}} |\partial [f^n \circ \phi_n]_2/\partial \theta|} \leq \left(\frac{1 + c_1/|s|^2}{1 - c_1/|s|^2}\right)^{1/(d-1)} \frac{100^d + 4}{100^d - 4},
\]

where \(\varphi_{HO}^{-1} \circ \iota_{s^2} = ([\varphi_{HO}^{-1} \circ \iota_{s^2}]_1, [\varphi_{HO}^{-1} \circ \iota_{s^2}]_2)\) and \(c_1, c_2\) are small constants depending only on \(f\).

**Proof.** The first equality is just from definition. Note that $\Theta_{n,n} = \Delta$ and so \(f^n(\phi_n(\theta)) \in \{\varphi_s = s^{d^2}\} \cap V^+\) for $\theta \in \Delta$.

Let $\theta \in \Delta$. For simplicity, we use $(z, w) = f^n \circ \phi_n$, $(z', w') = [d(f^n \circ \phi_n)](\partial \omega)$. We have

\[
\begin{cases}
\varphi+(z, w) = s^{d^2} \\
w = z\theta
\end{cases}
\]

Differentiating with respect to $\theta$ and solving this system for $w'$, we have $w' = z^2/(z - \mu w)$, where $\mu$ is a small constant satisfying Lemma 6.7 with Conditions 1 and 4 on $R$, for example, $|\mu| < 4/100^d$. Then, we have

\[
\frac{|z|}{1 + |\mu|} = \frac{|z|^2}{|z| + |\mu z|} \leq |w'| \leq \frac{|z|^2}{|z| - |\mu z|} = \frac{|z|}{1 - |\mu|}.
\]

From (5.3), we have

\[
\frac{\sup_{\theta \in \Delta} |w'|}{\inf_{\theta \in \Delta} |w'|} \leq \left(\frac{1 + c_1/|s|^{d^2}}{1 - c_1/|s|^{d^2}}\right)^{1/(d-1)} \frac{1 + |\mu|}{1 - |\mu|} \leq \left(\frac{1 + c_1/|s|^{d^2}}{1 - c_1/|s|^{d^2}}\right)^{1/(d-1)} \frac{100^d + 4}{100^d - 4},
\]

where $c_1, c_2$ are small positive constants depending only on $f$ and therefore it is bounded independently of $n$. \qed

In order to estimate \(\sup_{\theta \in \Theta_{n,i}} |\partial [f^i \circ \phi_n]_2/\partial \theta|/\inf_{\theta \in \Theta_{n,i}} |\partial [f^i \circ \phi_n]_2/\partial \theta|\) inductively from Lemma 7.3, we prepare the following lemma:
Lemma 7.7. For all \( \theta \in \Theta_{n,i} \) and \( i \leq j \leq n - 1 \),
\[
\frac{|p'(f^j \circ \phi_n)_1| - 1}{|a|} \leq \frac{\partial [f^{j+1} \circ \phi_n]_2}{\partial \theta} \leq \frac{|p'(f^j \circ \phi_n)_1| + 1}{|a|} \frac{\partial [f^{j+1} \circ \phi_n]_2}{\partial \theta}.
\]

Proof. The lemma is a direct result from Condition 1 on \( R, \{ \partial [f^{j+1} \circ \phi_n]_2/\partial \theta \} \) for \( \theta \in \Theta_{n,i} \) by Lemma 6.7 and
\[
df^{-1} f^{j+1}(\phi_n(\theta)) = \left( \frac{p'(f^j \circ \phi_n)_1}{1}, -a \right)^{-1} \left( -\frac{1}{a} 1 \right) \left( -\frac{1}{a} p'(f^j \circ \phi_n)_1/a \right),
\]
where \( \mu \) is a small constant satisfying Lemma 6.7 with Conditions 1 and 4 on \( R \), for example, \( |\mu| < 4/100^d \).

We estimate \( \sup_{\theta \in \Theta_{n,i}} |\partial [f^i \circ \phi_n]_2/\partial \theta|/\inf_{\theta \in \Theta_{n,i}} |\partial [f^i \circ \phi_n]_2/\partial \theta| \).

Lemma 7.8. Let \( i, n \in \mathbb{Z} \) be such that \( 0 \leq i \leq n - 1 \). There exists a constant \( C > 0 \) independent of \( i, n \) such that
\[
\sup_{\theta \in \Theta_{n,i}} |\partial [f^i \circ \phi_n]_2/\partial \theta| \leq C \inf_{\theta \in \Theta_{n,i}} |\partial [f^{n} \circ \phi_n]_2/\partial \theta|.
\]

Proof. Let \( \theta \in \Delta \). For the simplicity of the proof, we use \( (z, w) = \phi_n, (z', w') = d\phi_n((d f^i \circ \phi_n), (z_j, w_j) = f^i \circ \phi_n, \text{ and } (z_j', w_j') = d(f^i \circ \phi_n)((d f^i \circ \phi_n)) \). Then \( (z, w) \in \{ \phi_+ = s \} \) and \( (z_j, w_j) \in \{ \phi_+ = s^{d^j} \} \). Note that \( |p'(z_j)| \gg 1 \) on \( \Theta_{n,j} \). From Lemma 7.7, we have
\[
\frac{\sup_{\theta \in \Theta_{n,i}} |w_j'|}{\inf_{\theta \in \Theta_{n,i}} |w_j'|} \leq \frac{\sup_{\theta \in \Theta_{n,j}} |p'(z_j)| + 1}{\inf_{\theta \in \Theta_{n,j}} |p'(z_j)| - 1} \frac{\sup_{\theta \in \Theta_{n,i+1}} |w_{j+1}'|}{\inf_{\theta \in \Theta_{n,i+1}} |w_{j+1}'|}.
\]
Multiple use of this inequality gives us
\[
(7.3)
\sup_{\theta \in \Theta_{n,i}} |w_j'| \leq \sup_{\theta \in \Theta_{n,i}} |w_j'| \prod_{j=1}^{n-1} \sup_{\theta \in \Theta_{n,j}} |p'(z_j)| + 1 \sup_{\theta \in \Theta_{n,i+1}} |w_{j+1}'|.
\]
So, we estimate \( \prod_{j=1}^{n-1} (\sup_{\theta \in \Theta_{n,j}} |p'(z_j)| + 1)/\inf_{\theta \in \Theta_{n,j}} |p'(z_j)| - 1) \). Let \( \theta_a, \theta_b \in \Theta_{n,i} \),
\[
z_a = f^i(\phi_n(\theta_a)) \in \{ \phi_+ = s^{d^j} \} \text{ and } z_b = f^i(\phi_+(\theta_b)) \in \{ \phi_+ = s^{d^j} \}.
\]
Then, by (5,3), we have
\[
1 - c_1/|s^d|^{c_2} \leq \frac{(1 - c_1/|s^d|^{c_2})^{1/(d-1)}}{1 + c_1/|s^d|^{c_2}} \leq \frac{|z_a|}{|z_b|} \leq \frac{1 + c_1/|s^d|^{c_2})^{1/(d-1)}}{1 - c_1/|s^d|^{c_2}} \leq \frac{1 + c_1/|s^d|^{c_2})^{1/(d-1)}}{1 - c_1/|s^d|^{c_2}}
\]
(7.4)
where \( c_1, c_2 \) are small positive constants only depending on \( f \). Also, we have
\[
|z_a - z_b| = |z_a|^2 + |z_b|^2 - 2z_a \cdot z_b
\leq |z_a|^2 + |z_b|^2 - 2|z_a||z_b| \sqrt{1 - \frac{2c_1}{(d-1)|s^d|^{c_2}}}^2
\leq (|z_a| - |z_b|)^2 + 2|z_a||z_b| \left( \frac{2c_1}{(d-1)|s^d|^{c_2}} \right)^2.
\]
Note that in the computation of $z_a \cdot z_b$, we used the bound of the angle in Proposition 5.2 and an elementary calculus fact that $\sin t \leq t$ for small positive $t$.

Thus, together with (7.4), we have

$$
\left| \frac{z_a - z_b}{z_b} \right|^2 \leq \left( \left| \frac{z_a}{z_b} \right| - 1 \right)^2 + 8 \left( \frac{c_1}{d-1} \right)^2 \left| \frac{z_a}{z_b} \right| \\
\leq \left( \frac{1 + c_1}{d-1} \right)^2 + 16 \left( \frac{c_1}{d-1} \right)^2 \\
\leq \frac{C}{|s|^{d-1}},
$$

where $C \geq 0$ is a constant independent of $z_a, z_b, i, n$.

Suppose that $d' > 1$. We now consider each term in the product to be estimated.

$$
\frac{|p'(z_a)| + 1}{|p'(z_b)| - 1} = \frac{(p'(z_a) - p'(z_b)) + p'(z_b)| + 1}{|p'(z_b)| - 1} \leq \frac{|p'(z_a) - p'(z_b)| + 2}{|p'(z_b)| - 1} + 1.
$$

Write $p'(z) = dz^{d-1} + q'(z)$. By Condition 1 on $R$ and $R < |z_b|$, we have $|p'(z_b)| - 1 \geq 3d|z_b|^{d-1}/4$ and by Condition 2 on $R$ and $R < |z_a|, |z_b|$, we have $|q'(z)| \leq 5d'a_{d'}|z_a|^{d-1}/4$ and $|q'(z)| \leq 5d'a_{d'}|z_a|^{d-1}/4$. So, together with the estimates in the above, we have

$$
\frac{|p'(z_a) - p'(z_b)| + 2}{|p'(z_b)| - 1} + 1 \leq \frac{4d(z_a^{d-1} - z_b^{d-1}) + (q'(z_a) - q'(z_b))| + 2}{d|z_b|^{d-1}} + 1 \\
\leq \frac{4}{3} \left( \frac{z_a}{z_b} \right)^{d-1} - 1 \\
+ \frac{4}{3} \left[ 5d'a_{d'}|z_a|^{d-1}/4 \right] + \frac{5d'a_{d'}|z_a|^{d-1}/4}{d|z_b|^{d-1}} + 1 \\
\leq \frac{4}{3} \left( \frac{z_a}{z_b} \right)^{d-1} - 1 \\
+ \frac{4}{3} \left[ 5d'a_{d'}|z_a|^{d-1}/4 \right] + \frac{5d'a_{d'}|z_a|^{d-1}/4}{d|z_b|^{d-1}} + 1 \\
\leq \frac{4}{3} \left( (\sqrt{r} + 1)^{d-1} - 1 \right) + \frac{8}{3} \left( \frac{5d'a_{d'}|z_a|^{d-1}/4}{d|z_b|^{d-1}} \right) + 1 \\
\leq \frac{C}{|s|^{d-1} \max\{c_2/2, 1\}} + 1,
$$

where $C$ is a constant independent of $z_a, z_b, i, n$ and $r := C/|s|^{d-1}$. From the convergence of $\sum_{i=1}^{\infty} 1/(s^{d-1}) \max\{c_2/2, 1\}$, we have the convergence of (7.8) to a finite number by an elementary calculus theorem about the convergence of infinite products.

In the case of $d' \leq 1$, we can apply the same arguments with better approximations. This completes the proof of Lemma 7.8. \qed
Proof of Proposition 7.9. So far, in Lemmas 7.6 and 7.8 we have proved that 
\[\sup_{\theta \in \Theta_{n,i}} |\partial [f^i \circ \phi_n]|_{2/\partial \theta} / \inf_{\theta \in \Theta_{n,i}} |\partial [f^i \circ \phi_n]|_{2/\partial \theta}\] is uniformly bounded for \(0 \leq i \leq n\).

Since \(f^i(\phi_n(\theta)) \in V^+\) on \(\Theta_{n,i}\), by Lemma 7.1 we have
\[
\sup_{\theta \in \Theta_{n,i}} \|f^i \circ \phi_n\|_{FS,\theta} \leq 36 \cdot \sup_{\theta \in \Theta_{n,i}} |[f^i \circ \phi_n]|_1 \cdot \inf_{\theta \in \Theta_{n,i}} |[f^i \circ \phi_n]|_1 \cdot \sup_{\theta \in \Theta_{n,i}} |\partial [f^i \circ \phi_n]|_{2/\partial \theta} / \inf_{\theta \in \Theta_{n,i}} |\partial [f^i \circ \phi_n]|_{2/\partial \theta}.
\]

Finally, (5.3) with Condition 4 on \(R\) implies the desired boundedness. □

Let \(\theta_n \in \Delta\) be such that \(\phi_n(\theta_n) = (z_K, 0)\) for a fixed point \((z_K, 0) \in C_a \cap V^+\).

Summarizing the results from Lemma 7.2, Lemma 7.4, Lemma 7.5 and Lemma 7.10, we obtain the following crucial lemma in the proof of Theorem 1.2.

Lemma 7.9. The sequence \(\left\{\sup_{\delta \in \Delta} \|\phi_n\|_{FS,\theta} / \|\phi_n\|_{FS,\theta_n}\right\}\) as a sequence of \(n\) is bounded above. We call this bound \(M_n\).

We consider a numerical sequence \(\{\|\phi_n\|_{FS,\theta_n}\}_{n=1}^{\infty}\) for the reparametrization in the proof of the main theorem.

Lemma 7.10. We have \(\|\phi_n\|_{FS,\theta_n} \to \infty\) as \(n \to \infty\).

Proof. By Lemma 7.1 it suffices to show that \(|\partial [\phi_n]|_{2/\partial \theta}\to \infty\) as \(n \to \infty\).

Write \((z_i, w_i) = f^i(\phi_n(\theta_n))\) and \((z'_i, w'_i) = \overline{d}\{f^i(\phi_n)\}_{\theta=\theta_n}\) for \(0 \leq i \leq n\).

From the definition, \(f^n \circ \phi_n = \varphi_\theta^{1/2} \circ \phi_{\partial \theta}^{1/2}\). Note that for any \(n = 0, 1, \cdots\), we have \(f^n(z_K, 0) \in V^+\). Hence, as in Proposition 7.6 from Lemma 7.1, we have
\[
\frac{|z_n|}{1 + |\mu|} = \frac{|z_n|^2}{|z_n| + |\mu z_n|} \leq |w'_n| \leq \frac{|z_n|^2}{|z_n| - |\mu z_n|} = \frac{|z_n|}{1 - |\mu|}.
\]

where \(\mu\) is a small constant satisfying Lemma 6.7 with Conditions 1 and 4 on \(R\), for example, \(|\mu| < 4/100^4\). From (5.3) and Condition 4 on \(R\), we have
\[
\frac{|s^{d^n}|}{2(1 + |\mu|)} \leq |w'_n|.
\]

Next, we consider the relationship
\[
\begin{pmatrix} z'_i \\ w'_i \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1/a & \rho'(z_i)/a \end{pmatrix} \begin{pmatrix} z'_{i+1} \\ w'_{i+1} \end{pmatrix}.
\]

We have
\[
|w'_{i+1}| < \left(\frac{|\rho'(z_i)|}{a} - \frac{2}{|\rho'(z_{i+1})|}\right) |w'_i| \leq |\rho'(z_i)| |w'_i| - \frac{|z'_{i+1}|}{a} \leq |w'_i|
\]

from Conditions 1 and 4 on \(R\), and Lemma 6.7. With these two inequalities, inductive arguments show that
\[
\frac{|s^{d^n}|}{2(1 + |\mu|)} \leq |w'_0| = |\partial [\phi_n]|_{2/\partial \theta}^{\infty}
\]
and this shows the desired divergence as \(n \to \infty\). □

Remark 7.11. We can use the arguments in the proof of Lemma 7.10 to analytically show that the Kobayashi pseudometric of \(C_a \equiv 0\).
We show some kind of rigidity phenomena that a holomorphic image of $\mathbb{C}$ into $\mathcal{L}_c$ must sit inside a single leaf of the foliation of $\mathcal{L}_c$. This is due to the pluriharmonicity of $g^+$ in $U^+$.

**Lemma 7.12.** Let $\xi : \mathbb{C} \to \mathcal{L}_c$ be a holomorphic mapping. Then, the image $\xi(\mathbb{C})$ should be inside a single leaf of the foliation of $\mathcal{L}_c$.

**Proof.** Suppose, to the contrary, that the image does not lie inside a single leaf. Then, we can find an open subset $U \subset \mathbb{C}$ with compact closure such that $\xi(U)$ is not contained in a single leaf. Since $\xi(U)$ has compact closure and $\xi(U) \subset \mathcal{L}_c \subset U^+$, we can find a sufficiently large number $n$ such that $f^n(\xi(U)) \subset V^+$. Note that $f^n(\xi(U))$ does not still lie inside a single leaf. We consider a holomorphic function $\varphi_+ \circ f^n \circ \xi : U \to \mathbb{C}$. Since $f^n(\xi(U))$ does not belong to a single leaf, $\varphi_+ \circ f^n \circ \xi : U \to \mathbb{C}$ is a non-constant holomorphic function over the open subset $U$. So, the open mapping theorem of one complex variable implies that $\varphi_+ \circ f^n \circ \xi(U)$ should be open in $\mathbb{C}$. However, $\xi(U) \subset \mathcal{L}_c$ means that $\log |\varphi_+ \circ f^n \circ \xi| \equiv d^n c$ on $U$, and therefore, $\varphi_+ \circ f^n \circ \xi(U)$ is a subset of a circle $|z| = \exp(d^n c)$. This is a contradiction. So, the lemma is proved. \qed

Now, we are ready to prove the main theorem of this paper.

**Proof of Theorem 1.2.** At the moment, we continue to assume sufficiently large $c > 0$ and $s \in \mathbb{C}$ such that $c$ satisfies the conditions in the second part of Subsection 5.1 and $|s| = \exp(c)$ as done previously in the above discussions and to use the same notation. The general case will be considered later.

We first prove that there exists a Brody curve $B_s$ in $\mathcal{C}_s$. We modify the Brody reparametrization lemma.

Recall the family $\{\phi_n : \Delta \to \mathcal{C}_s\}_{n=0}^{\infty}$ of analytic discs. By rescaling $\theta \to (1 - \epsilon)\theta$ for a very small $\epsilon > 0$, we may assume that our parametrizations are holomorphic over $\Delta$.

Let $H_s : \Delta \to [0, \infty)$ be defined by $H_s(\theta) = \|\phi_n\|_{FS, \theta}(1 - |\theta|^2)$. Let $\theta_n \in \Delta$ be a point such that $\phi_n(\theta_n) = (z_K, 0)$ for a prescribed fixed point $(z_K, 0) \in \mathcal{C}_c \cap V^+$. We define a sequence of Möbius transformations $\mu_n(\omega) = (\omega + \theta_n)/(1 + \overline{\theta_n}\omega)$. Consider a sequence $\{\omega_n\}$ of mappings $g_n : \Delta \to \mathcal{C}_s$ defined by $g_n = \phi_n \circ \mu_n$. Then,

$$\|g_n\|_{FS, \omega}(1 - |\omega|^2) = \|\phi_n\|_{FS, \theta}|\mu_n'(\omega)|(1 - |\omega|^2) = \|\phi_n\|_{FS, \theta}(1 - |\theta|^2).$$

By Lemma 7.9 we have

$$\|g_n\|_{FS, \omega} \leq \frac{M_s}{1 - |\omega|^2},$$

where $M_s$ denotes a bound for the sequence in Lemma 7.9.

Let $R_n = \|g_n\|_{FS, 0}$. By the definition of our parametrization $\phi_n$ in Section 5.3 we have $\theta_n = f^n_+(z^+, 0)/f^n_+(z^+, 0)$. Since $f^n(z^+, 0)$ converges to $L^-$, $\theta_n$ converges to 0. Then, $|\mu_n'(0)|$ converges to 1 and so, Lemma 7.10 implies $R_n \to \infty$. We define $k_n(\theta) := g_n(\theta/R_n)$.

Then, over $\Delta_{R_n/2}$, we have

$$\|k_n\|_{FS, \theta} = \frac{\|g_n\|_{FS, \theta/R_n}}{R_n} \leq \frac{M_s\|g_n\|_{FS, 0}}{R_n} \frac{1}{1 - |\theta/R_n|^2} \leq 2M_s.$$

Thus, the set $\{k_n'(\theta)\}$ of derivatives is uniformly bounded and $k_n(\theta)$ has a fixed point, that is, $k_n(0) = (z_K, 0)$ for every $n$ and therefore, $\{k_n(\theta)\}$ is a normal family.
Moreover, \( \|k_n\|_{F_{S,0}} = 1 \). Thus, the limit maps of \( \{k_n(\theta)\} \) are non-constant Brody maps.

We denote a limit map by \( \kappa_s \) and its image by \( B_s \). Notice that \( B_s \) is a holomorphic curve and passes through the fixed point \((z_K, 0)\). We have \( B_s \subseteq \overline{C_s} \subseteq \mathcal{L}_c \cup I^+ \) by Proposition 6.2. By Theorem 1.3, we have \( B_s \subseteq \mathcal{L}_c \). So, we can apply Lemma 7.12. Hence, the limit curve \( B_s \) should sit in one single leaf of the natural foliation of \( \mathcal{L}_c \). However, the fixed point for the family \( \{k_n\} \) implies that the limit map also satisfies \( \kappa_s(0) = (z^+, 0) \) and that its image \( B_s \) should sit inside \( \mathcal{C}_s \).

Next, we prove the injectivity of the limit map \( \kappa_s : \mathbb{C} \to \mathcal{C}_s \). Without loss of generality, by passing to a convergent subsequence, we may assume that \( \{k_n\} \) is a convergent to \( \kappa_s \). It suffices to prove that for \( \theta_a, \theta_b \in \mathbb{C}, \kappa_s(\theta_a) \neq \kappa_s(\theta_b) \).

Let \( r > 0 \) be such that \( \theta_a, \theta_b \in \Delta_r \subset \mathbb{C} \). Let \( U_{\Delta_r} \) denote an open subset of \( U^+ \) with compact closure with respect to the standard Euclidean topology of \( \mathbb{C}^2 \) such that \( \kappa_s(\Delta_r) \subset U_{\Delta_r} \). Since \( f \) is a super-attracting point, there exist a large number \( N \in \mathbb{N} \) such that \( f^N(U_{\Delta_r}) \subset V^+ \) by Proposition 7.4. Since \( k_n \to \kappa_s \) locally uniformly from the normal family argument, we can find another large number \( N' \in \mathbb{N} \) such that for all \( n \geq N' \), \( k_n \to \kappa_s \) is uniform over \( \Delta_r \), and \( \kappa_s(\Delta_r) \subset U_{\Delta_r} \).

Note that \( f^N(\kappa_s(\Delta_r)) \subset V^+ \cap \Phi_{s,dN} \) and \( f^N(k_n(\Delta_r)) \subset V^+ \cap \Phi_{s,dN} \) from Subsection 5.3. Recall the biholomorphism \( \varphi_{HO} \). Since \( f^N(\kappa_s(\Delta_r)) \) lives in \( V^+ \cap \{\varphi_+ = s^{dN}\} \), it suffices to show the injectivity of \( f^N_2 \circ \kappa_s / f^N_1 \circ \kappa_s \). From the locally uniform convergence of \( \{k_n\} \) to \( \kappa_s \), we have that

\[
\frac{f^N_2 \circ k_n}{f^N_1 \circ k_n} \to \frac{f^N_2 \circ \kappa_s}{f^N_1 \circ \kappa_s}
\]

is uniform for \( n \geq N' \). The functions \( \{f^N_2 \circ k_n / f^N_1 \circ k_n\} \) are a priori injective over \( \Delta_r \) from our definitions of \( \{\phi_n\} \). Since the limit map is not constant, by Hurwitz’s theorem, we have just proved that \( f^N_2 \circ \kappa_s / f^N_1 \circ \kappa_s \) is injective over \( \Delta_r \). This proves the injectivity of \( \kappa_s \).

Finally, we show that \( B_s = \mathcal{C}_s \). So far, we have proved that \( B_s \) is a biholomorphic image of \( \mathbb{C} \) in \( \mathcal{C}_s \). Suppose to the contrary that \( B_s \neq \mathcal{C}_s \). Then, we can find a biholomorphic mapping from \( \mathbb{C} \) onto a proper subset of \( \mathbb{C} \). This is a contradiction since there is no such mapping for one complex variable. Hence, the injective Brody curve \( B_s \) should be the whole leaf \( \mathcal{C}_s \) itself. The theorem is proved.

So far, we have proved the theorem for sufficiently large \( c \) in \( \mathcal{L}_c \). The general case is obtained by making \( c \) large enough by applying \( f \) sufficiently many times so that we can apply the above argument. Then, together with the properties of the filtration, the same argument as in Lemma 7.8 implies that the image of an injective Brody curve under \( f^{-N} \) is still an injective Brody curve for any finite \( N \in \mathbb{N} \). This proves the general case.

8. Finite compositions of generalized Hénon mappings

We can apply our method to finite compositions of generalized Hénon mappings as well. There are three essential parts: filtration, the choice of the constant \( c \) and the function \( \varphi_+ \). Once we have them, then the remaining is quite straightforward. So, we discuss about them.
Let \( f = f_\eta \circ \cdots \circ f_1 \) where each \( f_i \) is a generalized Hénon mapping with the degree \( d_i \geq 2 \) for \( 1 \leq i \leq \eta \). Then, for each \( f_i \), take \( R_i \) as in Subsection 5.1. Simply \( R = \max_{1 \leq i \leq \eta} R_i \) works for the filtration of \( f \). Concerning the constant \( c \), we take the maximum over the \( c \)'s for all \( f_i \)'s with \( 1 \leq i \leq \eta \). For \( \varphi^+ \), we replace the definition \( \varphi^+ \) in the proof of Proposition 5.2 by the below:

\[
\begin{align*}
    z & \cdot \left( \frac{z_1}{z_0} \right)^{1/d_1} \cdot \left( \frac{z_2}{z_1} \right)^{1/(d_1d_2)} \cdots \left( \frac{z_{n\eta+j}}{z_{n\eta+j-1}} \right)^{1/(d_1d_2\cdots d_j)} \\
    & \cdots 
\end{align*}
\]

where \( z_{n\eta+j} = [f_j \circ \cdots \circ f_1 \circ f^n]_1 \) and \( d = \prod_{1 \leq i \leq \eta} d_i \). Then, we obtain

**Theorem 8.1.** Let \( f \) be a finite composition of generalized Hénon mappings and \( g^+ \) its Green function. Then, the set \( \{ g^+ = c \} \) for \( c > 0 \) is naturally foliated by Riemann surfaces, each leaf is dense in \( \{ g^+ = c \} \) and every leaf is an injective Brody curve in \( \mathbb{P}^2 \) with respect to the Fubini-Study metric.

9. **Short \( \mathbb{C}^2 \)**

A domain in \( \mathbb{C}^2 \) which is not the entire \( \mathbb{C}^2 \) but is biholomorphic to \( \mathbb{C}^2 \) is called a Fatou-Bieberbach domain. The existence of the Fatou-Bieberbach domain in \( \mathbb{C}^2 \) shows the crucial difference between the theory of one complex variable and that of several complex variables. Usually, such domains are constructed from dynamical methods. In particular, such a domain is given as a basin of attraction. In general, the boundary of a basin of attraction does not have good smoothness. In [15], Stensønes used a sequence of shear maps to construct a Fatou-Bieberbach domain with smooth boundary. Also, the boundary is foliated by Riemann surfaces.

In Stensønes’s construction (and also the construction using Hénon mappings), the resulting Fatou-Bieberbach domains are unions of holomorphic balls. It was an interesting question what an increasing union of holomorphic balls in \( \mathbb{C}^k \) with Kobayashi metric identically 0 would be. In [8], it turned out that it may not be biholomorphic to \( \mathbb{C}^k \). Such domains are called short \( \mathbb{C}^k \).

It might be interesting to think of the same boundary regularity question for short \( \mathbb{C}^k \)'s as for Fatou-Bieberbach domains. We consider the boundary behavior of short \( \mathbb{C}^2 \) domains. According to Theorem 1.12 in [8], the set \( K_c = \{ g < c \} \) for \( c > 0 \) is a short \( \mathbb{C}^2 \). Since \( g^+ \) is pluri-harmonic in \( U^+ \), the boundary \( \partial K_c = \mathcal{L}_c \) is real analytic. Moreover, due to our main result, we know that the boundary is foliated by injective Brody curves. Hence, we have the following:

**Corollary 9.1.** The set \( K_c \) with \( c > 0 \) is a short \( \mathbb{C}^2 \) with real analytic boundary foliated by injective Brody curves.

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