SUBADDEDITIVITY OF SYZYGIES OF KOSZUL ALGEBRAS

LUCHEZAR L. AVRAMOV, ALDO CONCA, AND SRIKANTH B. IYENGAR

Abstract. Estimates are obtained for the degrees of minimal syzygies of quotient algebras of polynomial rings. For a class that includes Koszul algebras in almost all characteristics, these degrees are shown to increase by at most 2 from one syzygy module to the next one. Even slower growth is proved if, in addition, the algebra satisfies Green and Lazarsfeld’s condition $N_q$ with $q \geq 2$.

Introduction

In this paper we study homological properties of commutative graded algebras $R$, generated over a field $k$ by finitely many elements of degree one. A common approach is to choose a presentation $R \cong S/J$, where $S$ is a standard graded polynomial ring over $k$ and $J$ an ideal containing no linear forms, and use the minimal free resolution $F$ of the graded $S$-module $R$. In this context, the numbers

$$t_i^S (R) = \sup_{j \geq 0} \{ \text{Tor}_i^S (R, k)_j \neq 0 \}$$

are of significant interest, as they bound the degrees of the basis elements of $F_i$.

Explicit upper bounds exist in terms of the most accessible data: $e$, the number of variables of $S$ and $t_1^S (R)$, the maximal degrees of the generators of $J$. They are doubly exponential in $i$ and cannot be strengthened in general; see [7,4] for references.

The situation is dramatically different when $R$ is Koszul; that is, when the minimal free resolution of $k = R/R_+$ over $R$ is linear; equivalently, when $t_i^R (k) \leq i$ for all $i \geq 0$. For Koszul algebras Backelin [3] and Kempf [21] proved inequalities

(1) $t_i^S (R) \leq 2i$ for $1 \leq i \leq \text{pd}_S R$.

Cases when equality hold are described in [2].

Here we extend and sharpen these results in several directions.

First, we relax the hypothesis, assuming only $t_i^R (k) \leq i$ for $i \leq e + 1$. Such algebras are not necessarily Koszul and—unlike the latter—they are identified by a fixed finite segment of the minimal free resolution of $k$ over $R$; see Roos [31]. In Section 4 we prove that they satisfy (1) and determine when equality holds.

Second, for the algebras described above we investigate the existence of upper bounds on $t_i^S (R)$ that depend on $t_i^S (R)$ for $1 \leq a < i$. In Section 5 we prove

(2) $t_{i+1}^S (R) \leq t_i^S (R) + 2$ for $1 \leq i \leq e - \text{dim} R + 1$
when \((i+1)\) is invertible in \(k\); since then \(t^S_a(R) = 2\), this inequality refines (1) under additional hypotheses. It is itself the special case \(b = 1\) of the following inequality:

\[
t^S_{a+b}(R) \leq t^S_a(R) + t^S_b(R) \quad \text{for } a, b \geq 1 \text{ with } a + b \leq \text{pd}_S R.
\]

We conjecture that (3) holds without restrictions on the characteristic of \(k\) and review evidence gleaned from a number of sources. In particular, here we show that when \(R\) is Cohen-Macaulay and \((\binom{a+b}{a})\) is invertible in \(k\) one has

\[
t^S_{a+b}(R) \leq t^S_a(R) + t^S_b(R) + 1 \quad \text{for } a \geq 1 \text{ and } b \geq 2 \text{ with } a + b \leq \text{pd}_S R.
\]

For the algebras described above we study the impact of the Green–Lazarsfeld condition \(N_q\): The differential \(F_i \to F_{i-1}\) is given by a matrix of quadrics for \(i = 1\) and by matrices of linear forms for \(2 \leq i \leq q\); see 18. In Section 6 we show that (excluding finitely many specified characteristics) it implies a sharpening of (1):

\[
t^S_b(R) \leq 2 \begin{cases} \binom{i}{q+1} + i & \text{if } (q + 1) | i \\ 1 & \text{otherwise}. \end{cases}
\]

This inequality reveals hitherto unknown properties of the resolutions of algebras of geometric interest, even in cases such as Segre products and Veronese subalgebras of polynomial rings. For related recent progress in this context see 32, 34.

Our approach utilizes the homology of the Koszul complex \(K\) on a basis of \(R_1\).

The differential bigraded algebra \(K\) provides a bridge between the homology of the \(R\)-module \(K\) and that of the \(S\)-module \(R\); see 1. As in 2, the proof of (1) depends on the decomposable classes in \(H(K)\) contained in its subalgebra generated by \(H_1(K)\). On the other hand, the proofs of 2 and 1 involve an analysis of indecomposable homology classes; namely, of the quotient \(H_{a+b}(K)/H_a(K)\cdot H_b(K)\).

This is the main technical innovation in the paper. Its effectiveness depends on the existence (in almost all characteristics) of splitting maps for Koszul cycles, discovered in 8 and described in Section 2 from a more conceptual perspective. These maps are combined with cascades of bounds on regularities of Tor modules, obtained in Sections 1 and 3. The material in the first three sections is presented in its natural generality, which is much greater than the one needed in the sequel.

1. Complexes of flat modules

In this section we evaluate how cycles change with the introduction of coefficients, starting from a general situation and moving on to a graded one. Only basic homological considerations are involved.

1.1. Coefficients. Let \(R\) be an associative ring and \(R^e\) its center. Let

\[
F = \cdots \to F_{a+1} \xrightarrow{\partial_{a+1}} F_a \xrightarrow{\partial_a} F_{a-1} \to \cdots
\]

be a complex of right \(R\)-modules, and for every integer \(a\) set

\[
Z_a = \ker(\partial_a), \quad B_a = \text{im}(\partial_{a+1}), \quad H_a = Z_a/B_a, \quad \text{and} \quad C_a = \text{coker}(\partial_{a+1}).
\]

For each left \(R\)-module \(N\) and for \(Z_a(F \otimes_R N) = \ker(\partial_a \otimes N)\) the assignment \(z \otimes x \mapsto z \otimes x\) defines a natural in \(N\) homomorphism of abelian groups

\[
\phi_a : Z_a \otimes_R N \to Z_a(F \otimes_R N).
\]

The next lemma is a sort of “universal coefficients theorem” for cycles.
 Lemma 1.2. When $F_a$ and $F_{a-1}$ are flat there is a natural in $N$ exact sequence

$$0 \to \text{Tor}_1^R(B_{a-1}, N) \to Z_a \otimes_R N \xrightarrow{\phi_a} Z_a(F \otimes_R N) \to \text{Tor}_1^R(C_{a-1}, N) \to 0$$

of $R^e$-modules, and for every $i \geq 1$ there is a natural in $N$ isomorphism

$$\text{Tor}_{i+1}^R(B_{a-1}, N) \cong \text{Tor}_i^R(Z_a, N).$$

Proof. The maps $F_a \xrightarrow{\pi_a} B_{a-1}$ and $B_{a-1} \xrightarrow{\iota_{a-1}} F_{a-1}$ yield a commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & \text{Ker}(\pi_a \otimes_R N) \\
\downarrow & & \downarrow \\
F_a \otimes_R N & \longrightarrow & B_{a-1} \otimes_R N
\end{array}
$$

with exact rows. The Snake Lemma then produces an exact sequence

$$0 \to \text{Ker}(\pi_a \otimes_R N) \to Z_a(F \otimes_R N) \to \text{Ker}(\iota_{a-1} \otimes_R N) \to 0$$

As $F_a$ is flat, the exact sequence $0 \to Z_a \to F_a \xrightarrow{\pi_a} B_{a-1} \to 0$ induces the isomorphisms and yields an exact sequence

$$0 \to \text{Tor}_1^R(B_{a-1}, N) \to Z_a \otimes_R N \xrightarrow{\zeta_a} \text{Ker}(\pi_a \otimes_R N) \to 0$$

such that the composition of $\zeta_a$ with the inclusion $Z_a(F \otimes_R N) \subseteq F_a \otimes_R N$ is equal to $\phi_a$. As $F_{a-1}$ is flat we also have exact sequence

$$0 \to \text{Tor}_1^R(C_{a-1}, N) \to B_{a-1} \otimes_R N \xrightarrow{\iota_{a-1} \otimes_R N} F_{a-1} \otimes_R N$$

induced by $0 \to B_{a-1} \xrightarrow{\iota_{a-1}} F_{a-1} \to C_{a-1} \to 0$; it computes $\text{Ker}(\iota_{a-1} \otimes_R N)$. \qed

1.3. Graded objects. Let $k$ be a field and $V = \bigoplus_j V_j$ a graded vector space; set

$$\text{top}(V) = \sup\{j \mid V_j \neq 0\}.$$ 

thus, $\text{top}(V) = -\infty$ if and only if $V = 0$. The notation $\deg(v) = j$ means $v \in V_j$.

A graded $k$-algebra is a graded vector space $R = \bigoplus_{i \in \mathbb{Z}} R_i$ with $R_0 = k$, $R_i = 0$ for $i < 0$, and associative $k$-bilinear products $R_i \times R_j \to R_{i+j}$. A left graded $R$-module is a graded vector space $N = \bigoplus_{j \in \mathbb{Z}} N_j$ with $N_j = 0$ for $j < 0$ and associative $k$-bilinear products $R_i \times N_j \to N_{i+j}$. Graded right $R$-modules are defined similarly.

Homomorphisms of graded $R$-modules are homogeneous of degree 0. Thus, when $L$ is a right $R$-module and $N$ a left one, $\text{Tor}_i^R(L, N)$ is a graded $R'$-module.

As usual, given an graded $R$-module $N$ and integer $n$, we write $N(n)$ for the graded $R$-module with $N(n)_i = N_{n+i}$ for all $i \in \mathbb{Z}$.

By abuse of notation, $k$ also stands for the module $R/\bigoplus_{i \geq 1} R_i$. The number

$$t_i^R(N) = \text{top}(\text{Tor}_i^R(k, N))$$

then equals the largest degree of a minimal $i$-syzygy of the $R$-module $N$.

Lemma 1.4. When $L$ is a right graded $R$-module and $N$ a left one, one has

$$\text{top}(\text{Tor}_i^R(L, N)) \leq \text{top}(L) + t_i^R(N)$$

for all $i \in \mathbb{Z}$. 


Proof. Set \( t = \text{top}(L) \). We may assume \( t \) is finite, and then \( s = t - \min \{ j \mid L_j \neq 0 \} \) is a non-negative integer. If \( s = 0 \), then \( \text{Tor}_i^R(L, N) \cong L \otimes_k \text{Tor}_i^R(k, N) \), so equality holds in (1.4.1). The exact sequence \( 0 \to L(-t) \to L \to L/L(-t) \to 0 \) of graded \( R \)-modules induces an exact sequence of graded Tor vector spaces, from which the inequality (1.4.1) follows by induction on \( s \).

Lemma 1.5. Let \( R \) be a graded algebra and \( F \) a complex of flat graded \( R \)-modules. For all integers \( i \geq 1 \) and \( a \geq 0 \) the following inequalities hold:

\[
\begin{align*}
(1.5.1) \quad & \text{top}(\text{Tor}_i^R(B_a, N)) \leq \max_{0 \leq j < a} \{ \text{top}(H_j) + t_{a+i+1-j}^R(N) \} \\
(1.5.2) \quad & \text{top}(\text{Tor}_i^R(Z_a, N)) \leq \max_{0 \leq j < a} \{ \text{top}(H_j) + t_{a+i+1-j}^R(N) \} \\
(1.5.3) \quad & \text{top}(\text{Tor}_i^R(C_{a-1}, N)) \leq \max_{0 \leq j < a} \{ \text{top}(H_j) + t_{a-1+i+1-j}^R(N) \}
\end{align*}
\]

Proof. Fix some integer \( i \geq 1 \). In view of (1.2.1), we obtain an exact sequence

\[
\text{Tor}_{i+1}^R(H_a, N) \to \text{Tor}_i^R(B_a, N) \to \text{Tor}_{i+1}^R(B_{a-1}, N)
\]

from the one induced by \( 0 \to B_a \to Z_a \to H_a \to 0 \). Now (1.5.4) and (1.4.1) yield

\[
\text{top}(\text{Tor}_i^R(B_a, N)) \leq \max \{ \text{top}(H_a) + t_{i+1}^R(N), \text{top}(\text{Tor}_{i+1}^R(B_{a-1}, N)) \}
\]

The proof of (1.5.1) is completed by iterating this procedure.

The isomorphisms (1.2.1) show that formula (1.5.2) follows from (1.5.1).

The exact sequence \( 0 \to H_{a-1} \to C_{a-1} \to B_{a-2} \to 0 \) induces an exact sequence

\[
\text{Tor}_i^R(H_{a-1}, N) \to \text{Tor}_i^R(C_{a-1}, N) \to \text{Tor}_i^R(B_{a-2}, N)
\]

from which the inequality (1.5.3) follows, due to formulas (1.4.1) and (1.5.4).

2. Indecomposable Koszul homology

In this section \( R \) denotes a commutative ring and \( M \) an \( R \)-module.

2.1. Koszul complexes. Let \( E = \bigwedge_R E_1 \) be the exterior algebra on a free \( R \)-module \( E_1 \) sitting in homological degree 1, and \( \mu : E \otimes_R E \to E \) the product map.

The algebra \( E \) has following universal property: Each \( R \)-linear map \( E_1 \to R \) extends uniquely to an \( R \)-linear map \( \partial : E \to E \) of homological degree \(-1\), satisfying

\[
\partial \mu = \mu(\partial \otimes E + E \otimes \partial).
\]

As \( \partial^2(E_1) = 0 \) holds for degree reasons, this implies \( \partial^2 = 0 \). The complex \( K \) with underlying graded module \( E \) and differential \( \partial \) is the Koszul complex of \( E_1 \to R \).

Set \( K^M = K \otimes_R M \). In view of (2.1.1), \( K \) is a DG algebra and \( K^M \) is a DG \( K \)-module. Thus, \( H(K) \) is a graded algebra and \( H(K^M) \) is a graded \( H(K) \)-module.

If \( R \) is a graded \( k \)-algebra and \( E_1 \to R \) a homomorphism of graded \( R \)-modules, see 1.3 then the differential algebra \( K \) and its differential module \( K^M \) are bigraded: their differentials decrease homological degrees by 1 and preserve internal degrees. Thus, \( H(R) \) is naturally a bigraded algebra and \( H(M) \) a bigraded module over it.

Theorem 2.2. For each pair \((a, b) \in \mathbb{Z}^2\) there exists a natural \( R \)-linear map

\[
(2.2.1) \quad \gamma_{a,b} : \text{Tor}_1^R(C_{a-1}(K), Z_b(K^M)) \to \frac{H_{a+b}(K^M)}{H_a(K)H_b(K^M)}
\]

described in (2.1.10) which is surjective when \((a+b) \) is invertible in \( R \).

When \( R \) and \( M \) are graded \( \gamma_{a,b} \) is a homomorphism of graded modules.
The theorem is proved at the end of the section.

**Corollary 2.3.** Assume \( H_a(K) = 0 \) for \( a > 1 \) (for instance, \( E_1 \) has a basis \( e_1, \ldots, e_r \) such that the sequence \( \delta(e_1), \ldots, \delta(e_r) \) is \( R \)-regular) and set \( I = \delta(E_1) \).

If \( \binom{a+b}{a} \) is invertible in \( R \), then there is a surjective \( R \)-linear map

\[
\gamma'_{a,b} : \text{Tor}_a^R(R/I, Z_b(K^M)) \to \text{Tor}_a^R(R/I, M)
\]

**Proof.** The hypothesis yields \( H_{a+b}(K^M) \cong \text{Tor}_a^R(R/I, M) \) and an exact sequence

\[
0 \to C_{a-1}(K) \to K_{a-1} \to \cdots \to K_0 \to R/I \to 0
\]

of \( R \)-modules. Since \( K_i \) is flat for \( 0 \leq i \leq a - 1 \), the iterated connecting map \( \text{Tor}_a^R(R/I, Z_b(K^M)) \to \text{Tor}_1^R(C_{a-1}(K), Z_b(K^M)) \) is bijective. \( \Box \)

As a special case, we obtain an (unexpected) inequality between Betti numbers:

**Corollary 2.4.** Let \( R \) be a regular local ring, \( K \) the Koszul complex on a minimal generating set of the maximal ideal of \( R \), and \( M \) a finitely generated \( R \)-module.

If \( \binom{a+b}{a} \) is invertible in \( R \), then the \( b \)-cycles of \( K^M = K \otimes_R M \) satisfy

\[
\beta_a^R(Z_b(K^M)) \geq \beta_{a+b}^R(M).
\]

In the graded setup such inequalities hold for graded Betti numbers. \( \Box \)

**2.5. Diagonal maps.** Equipping \( E \otimes_R E \) with the product

\[
(x_1 \otimes x_2) \cdot (y_1 \otimes y_2) = (-1)^{|x_2||y_1|}(x_1y_1 \otimes x_2y_2).
\]

turns it into an \( R \)-algebra that is strictly graded-commutative for the homological degree. The universal property of exterior algebras yields a unique homomorphism \( \Delta : E \to E \otimes_R E \) of graded \( R \)-algebras, such that \( \Delta(x) = x \otimes 1 + 1 \otimes x \) for \( x \in E_1 \).

Set \( \delta' = \partial \otimes E \) and \( \delta'' = E \otimes \partial \). These maps satisfy the equalities

\[
\begin{align*}
\delta' \Delta &= \Delta \delta = \delta'' \Delta \\
\delta' \delta'' + \delta'' \delta' &= 0
\end{align*}
\]

Indeed, the maps in (2.5.1) and (2.5.2) are graded derivations of the \( R \)-algebras \( E \) and \( E \otimes_R E \), respectively. They are generated in homological degree 1, so it suffices to verify agreement on \( E_1 \) and \( (E \otimes_R E)_1 \), respectively. This is straightforward.

**2.6. Partial cycles.** As \( E \) is a graded free \( R \)-module, we have

\[
E \otimes_R Z(K^M) = E \otimes_R \ker(\delta \otimes M) = \ker(\delta'' \otimes M)
\]

From (2.6.1) we obtain \( (\delta' \otimes M)(\ker(\delta'' \otimes M)) \subseteq \ker(\delta'' \otimes M) \), so \( \delta' \otimes M \) turns \( \ker(\delta'' \otimes M) \) into a complex. It is equal to \( K \otimes_R Z(K^M) \), so we get

\[
Z(K \otimes_R Z(K^M)) = \ker(\delta' \otimes M) \cap \ker(\delta'' \otimes M)
\]

as graded submodules of \( E \otimes_R E \otimes_R M \). For every integer \( n \) there is an equality

\[
Z_n(K \otimes_R Z(K^M)) = \bigoplus_{i+j=n} Z_i(K \otimes_R Z_j(K^M))
\]

Let \( \iota_{i,j} \) and \( \pi_{i,j} \) denote the canonical maps from and to \( Z_i(K \otimes_R Z_j(K^M)) \), respectively, induced by the decomposition (2.6.2).
2.7. The maps $\alpha$. The equalities (2.1.1) and (2.5.2) imply
\[(\mu \otimes M)(\text{Ker}(\partial \otimes M + \partial' \otimes M)) \subseteq \text{Ker}(\partial \otimes M) = Z(K^M).\]
In view of (2.6.1), $\mu \otimes M$ restricts to a homomorphism of graded $R$-modules
\[\alpha: Z(K \otimes R Z(K^M)) \to Z(K^M).\]
Setting $n = a + b$, we define $\alpha_{a,b}$ to be the composed map
\[Z_a(K \otimes R Z_b(K^M)) \xrightarrow{\alpha_{a,b}} Z_n(K \otimes R Z(K^M)) \xrightarrow{\alpha_n} Z_n(K^M)\]

2.8. The maps $\beta$. The equalities (2.6.1) imply
\[(\Delta \otimes R M)(\text{Ker}(\partial \otimes M)) \subseteq \text{Ker}(\partial' \otimes M) \cap \text{Ker}(\partial'' \otimes M).\]
In view of (2.6.1), $\Delta \otimes R M$ restricts to a homomorphism of graded $R$-modules
\[\beta: Z(K \otimes M) \to Z(K \otimes R Z(K^M)).\]
Setting $n = a + b$, we define $\beta_{a,b}$ to be the composed map
\[Z_n(K^M) \xrightarrow{\beta_n} Z_n(K \otimes R Z(K^M)) \xrightarrow{\alpha_{n,b}} Z_n(K \otimes R Z_b(K^M))\]

The next result is proved by Bruns, Conca, and Römer in [8, 2.4] through direct calculations. Our proof mines the graded Hopf algebra structure of $E$.

Lemma 2.9. For each pair $(a, b)$ of non-negative integers there is an equality
\[\alpha_{a,b} \circ \beta_{a,b} = \left(\begin{array}{c} a + b \\ a \end{array}\right) \text{id}_{Z_{a+b}(K^M)}.\]

Proof. Set $n = a + b$ and note that $\alpha_{a,b} \circ \beta_{a,b}$ is induced by the composed map
\[\delta_{a,b}: E_n \xrightarrow{\Delta} (E \otimes R E)_n \xrightarrow{\rho_{a,b}} (E \otimes R E)_n \xrightarrow{\mu_n} E_n\]
where $\rho_{a,b}$ projects $(E \otimes R E)_n = \bigoplus_{i+j=n} E_i \otimes R E_j$ onto its summand $E_a \otimes E_b$.

Let $n$ be a positive integer and let $\mathbf{n}$ denote the sequence $(1, \ldots, n)$. For each sequence $\mathbf{i} = (u_1, \ldots, u_i)$ of integers satisfying $1 < u_1 < \cdots < u_i < n$, write $n \setminus \mathbf{i} = (v_1, \ldots, v_{n-i})$ with $v_1 < \cdots < v_{n-i}$ and let $\text{sgn}(\mathbf{i}, n \setminus \mathbf{i})$ denote the sign of the permutation $(v_1, \ldots, u_i, v_1, \ldots, v_{n-i})$ of $\mathbf{n}$. Given a subset $x = \{x_1, \ldots, x_n\}$ of $E_1$, setting $x_i = x_{u_i} \cdots x_{u_1} \in E_1$ we obtain the following equalities:
\[\delta_{a,b}(x_n) = \mu_n \circ \rho_{a,b} \left( \sum_{\mathbf{i} \in \mathbf{n}} \text{sgn}(\mathbf{i}, n \setminus \mathbf{i}) x_i \otimes x_{n \setminus \mathbf{i}} \right) = \mu_n \left( \sum_{\text{card } \mathbf{i} = a} \text{sgn}(\mathbf{i}, n \setminus \mathbf{i}) x_i \otimes x_{n \setminus \mathbf{i}} \right) = \left(\begin{array}{c} n \\ a \end{array}\right) x_n\]

It remains to remark that $E_n$ is additively generated by elements of the form $x_n$. \(\square\)

2.10. The maps $\gamma$. Each pair $(a, b)$ of integers defines a diagram of $R$-linear maps
\[
\begin{aligned}
Z_a(K) \otimes Z_b(K^M) &\xrightarrow{\chi_a \otimes \chi_b} H_a(K) \otimes H_b(K^M) \\
Z_a(K \otimes R Z_b(K^M)) &\xrightarrow{\alpha_{a,b}} Z_{a+b}(K^M) &\xrightarrow{\chi_{a+b}} H_{a+b}(K^M)
\end{aligned}
\]
where the maps $\chi$ are canonical, $\mu^S_{a,b}$ and $\mu^R_{a,b}$ are induced by $\mu$ in view of (2.1.1), $\alpha_{a,b}$ comes from (2.2.1), and $\phi_a$ is given by (1.1.1) with $F = K$ and $N = Z_0(K^M)$. The definitions show that diagram (2.10.1) commutes, so it yields natural $R$-linear maps (2.10.2)  
\[ \text{Coker}(\phi_a) \to \text{Coker}(\mu^S_{a,b}) \to \text{Coker}(\mu^R_{a,b}) \]

The isomorphism $\text{Tor}_1^R(C_{a-1}(K), Z_0(K^M)) \cong \text{Coker}(\phi_a)$ from Lemma 1.2 composed with the maps in (2.10.2), defines the map $\gamma_{a,b}$ in (2.2.1).

**Proof of Theorem 2.2.** Since $\chi_{a+b}$ is surjective, so is the second map in (2.10.2).

If $(a+b)$ is invertible in $R$, then $\alpha_{a,b}$ is surjective by Lemma 2.9, hence so is the first map in (2.2.1). It follows that $\gamma_{a,b}$ is surjective, as desired.

When $R$ is a graded ring and $M$ is a graded $R$-module the constructions of $\alpha_{a,b}$, $\beta_{a,b}$, and $\gamma_{a,b}$ show that they are homomorphisms of graded $R$-modules.

### 3. Regularity

In this section $k$ is a field and $R$ a graded commutative $k$-algebra with $R_0 = k$, $R = k[R_1]$, and rank$_k R_1$ finite. We write $S$ for the symmetric $k$-algebra $S$ on $R_1$ and let $S \to R$ be the canonical surjective homomorphism of graded $k$-algebras.

#### 3.1. Regularities.

The $n$th partial regularity of an $R$-module $M$ is the number

\[ \text{reg}_n^R(M) = \max \{ \text{reg}_i^R(M) - i \} \]

Thus, for every $n \in \mathbb{Z}$ there is an inequality (3.1.2)

\[ \text{reg}_n(R) \leq \text{reg}_{n+1}(R) \]

and $\sup \{ \text{reg}_n^R(M) \}_{n \in \mathbb{Z}}$ is the Castelnuovo-Mumford regularity $\text{reg}^R(M)$.

The goal of this section is to prove the following result.

**Theorem 3.2.** Let $a$, $b$ be non-negative integers satisfying $a + b \leq \text{pd}_S R$.

If $(a+b)$ is invertible in $k$, then for every $R$-module $M$ one has

\[ t^S_{a+b}(M) \leq \max \left\{ t^S_a(R) + t^S_b(M) + \text{reg}_a^S(R) + \text{reg}_{a+b}^R(M) + a + b \right\} \]

We single out a special case of the theorem:

**Corollary 3.3.** If $\text{reg}_{a+b+1}^R(k) = 0 = \text{reg}_{a+b}^R(M)$, then there is an inequality (3.3.1)

\[ t^S_{a+b}(M) \leq \max \left\{ t^S_a(R) + t^S_b(M) + \text{reg}_{a-1}^S(R) + \text{reg}_{b-1}^S(M) + a + b + 1 \right\} \]

The proof of Theorem 3.2 utilizes Koszul complexes.

#### 3.4. Koszul homology.

The inclusion $R_1(-1) \subseteq R$ defines an $R$-linear map $R \otimes_k R_1(-1) \to R$, and hence a Koszul complex $K$; see 2.1. Set $K^M = K \otimes_R M$.

For each $i \in \mathbb{N}$ we have an isomorphism (3.4.1) $H_i(K^M) \cong \text{Tor}^S_i(k, M)$

of graded $R$-modules; it gives an equality (3.4.2) $t^S_i(M) = \text{top}(H_i(K^M))$
As \((R_+)H(K^M) = 0\), for all non-negative integers \(i, j\) the preceding formula yields

\[(3.4.3) \quad t_j^R(H_i(K^M)) = t_i^S(M) + t_j^R(k)\]

By applying Theorem 2.2, then formula \((1.5.3)\) with \(F = K\) and \(N = Z_b(K^M)\), and finally formula \((3.4.2)\) with \(M = R\) we obtain:

**Lemma 3.5.** If \(\binom{a+b}{b}\) is invertible in \(R\), then there is an inequality

\[(3.5.1) \quad \text{top} \left( \frac{H_{a+b}(K^M)}{H_a(K)H_b(K^M)} \right) \leq \max_{0 \leq j < a} \{ t_j^S(R) + t_{a-j}^R(Z_b(K^M)) \}. \]

Next we estimate the partial regularities of the cycles appearing in \((3.5.1)\).

**Lemma 3.6.** For each \(R\)-module \(M\) the following inequality holds:

\[(3.6.1) \quad t_a^R(Z_b(K^M)) \leq \max_{1 \leq j \leq b} \left\{ \frac{t_a^R(M) + b}{t_{a+j}^R(M) + b - j}, \frac{t_{b-j}^R(M) + t_{a+j+1}^R(k)}{} \right\} \]

**Proof.** From the exact sequences of \(R\)-modules

\[
\begin{align*}
0 & \longrightarrow Z_b(K^M) \longrightarrow K_b^M \longrightarrow B_{b-1}(K^M) \longrightarrow 0 \\
0 & \longrightarrow B_{b-1}(K^M) \longrightarrow Z_{b-1}(K^M) \longrightarrow H_{b-1}(K^M) \longrightarrow 0
\end{align*}
\]

we obtain the following inequalities:

\[
\begin{align*}
t_a^R(Z_b(K^M)) & \leq \max \{ t_{a+1}^R(B_{b-1}(K^M)), t_a^R(M) + b \} \\
t_{a+1}^R(B_{b-1}(K^M)) & \leq \max \{ t_{a+2}^R(H_{b-1}(K^M)), t_{a+1}^R(Z_{b-1}(K^M)) \}.
\end{align*}
\]

By concatenating these relations and invoking \((3.4.3)\) we get

\[
t_a^R(Z_b(K^M)) \leq \max \left\{ \frac{t_a^R(M) + b}{t_{a+1}^R(Z_{b-1}(K^M))}, \frac{t_{b-1}^S(M) + t_{a+2}^R(k)}{} \right\}
\]

The desired result follows from the preceding formula by induction on \(b\). \(\square\)

**Proof of Theorem 3.2** Products in homology yield an exact sequence

\[
H_a(K) \otimes H_b(K^M) \rightarrow H_{a+b}(K^M) \rightarrow \frac{H_{a+b}(K^M)}{H_a(K)H_b(K^M)} \rightarrow 0
\]

of graded \(k\)-vector spaces. It accounts for the first inequality in the following display, where the equality comes from \((3.4.2)\), the second inequality from \((3.5.1)\), and the
last inequality from (3.6.1):
\[ t^S_{a+b}(M) = \text{top}(H_{a+b}(K^M)) \]
\[ \leq \max \left\{ \text{top}(H_a(K)) + \text{top}(H_b(K^M)), \frac{H_{a+b}(K^M)}{H_a(K)H_b(K^M)} \right\} \]
\[ \leq \max_{0 \leq i < a} \{ t^S_i(R) + t^S_b(M), t^S_i(R) + t^R_{a-i}(Z_b(K^M)) \} \]
\[ \leq \max_{0 \leq i < a, 1 \leq j \leq b} \left\{ \begin{array}{l}
t^S_i(R) + t^R_{a-i}(M) + b \\
t^S_i(R) + t^R_{a-i+j}(M) + b - j \\
t^S_i(R) + t^S_{b-j}(M) + t^R_{a-i+j+1}(k) \end{array} \right\} \]

For each pair \((i, j)\) with \(0 \leq i < a\) and \(0 \leq j \leq b\), in view of (3.1.1) we have
\[ t^S_i(R) + t^R_{a-i+j}(M) + b - j \]
\[ \leq [\text{reg}^S_{a-i-1}(R) + i] + [\text{reg}^R_{a+b}(M) + a - i + j] + b - j \]
\[ = \text{reg}^S_{a-1}(R) + \text{reg}^R_{a+b}(M) + a + b \]

Similarly, for each pair \((i, j)\) with \(0 \leq i < a\) and \(1 \leq j \leq b\), we obtain
\[ t^S_i(R) + t^S_{b-j}(M) + t^R_{a-i+j+1}(k) \]
\[ \leq [\text{reg}^S_{a-1}(R) + i] + [\text{reg}^S_{a-1}(M) + b - j] + [\text{reg}^R_{a+b+1}(k) + a - i + j + 1] \]
\[ = \text{reg}^S_{a-1}(R) + \text{reg}^S_{b-1}(M) + \text{reg}^R_{a+b+1}(k) + a + b + 1 \]

It remains to assemble the inequalities in the last three displays.

4. Decomposable Koszul homology

We keep the hypotheses on \(R\) and \(S\) from Section 3 and let \(J\) denote the kernel of the surjection \(S \to R\) of graded \(k\)-algebras.

Theorem 4.1. Let \(K\) be the Koszul complex described in 3.3.

If \(\text{reg}^R_{a+b+1}(k) = 0\) for some \(n \geq 0\), then for \(0 \leq i \leq n\) there are equalities
\[ H_i(K)_j = 0 \quad \text{when} \quad j > 2i \ . \]
\[ H_i(K)_{2i} = (H_1(K)_{2i})^i . \]

Formula (3.4.2) gives a numerical translation of the equality (4.1.1).

Corollary 4.2. For \(0 \leq i \leq n\) the following inequalities hold:
\[ t^S_i(R) \leq 2i . \]

When \(R\) is Koszul, that is to say, when \(\text{reg}^R(k) = 0\), these results hold for all integers \(i \geq 0\). This is known: (4.2.1) is proved by Kempf \[21\] Lemma 4] and is a special case of a result of Backelin \[3\] Corollary]; see also \[21\] 4.1(a) and 1.4]. On the other hand, (4.1.2) is a special case of \[21\] 4.1(a) and (3.8.3)].

The proof of Theorem 4.1 does not use the material developed in previous sections; rather, it adapts ideas from our proof of \[2\] 3.1]. We start by presenting the relevant background material.
4.3. Minimal models. Given a set $X = \bigsqcup_{i,j \geq 1} X_{i,j}$, let $k[X]$ denote the $k$-algebra $\bigotimes_{i=1}^{\infty} k[X_i]$, where $X_i = \bigsqcup_{j} X_{i,j}$ and $k[X_i]$ is the exterior algebra on $X_i$ when $i$ is odd, respectively, the symmetric algebra on $X_i$ when $i$ is even.

For $x \in X_{i,j}$ set $|x| = i$ and $\deg(x) = j$. A $k$-basis of $k[X]$ is given by the element $1 \in k$ and the monomials $x_1^{d_1} \cdots x_u^{d_u}$ with $x_1 \in X$ and $d_i = 1$ when $|x_i|$ is odd, respectively, with $d_i \geq 1$ when $|x_i|$ is even. We bigrade $k[X]$ by setting

$$|x_1^{d_1} \cdots x_u^{d_u}| = \sum_{i=1}^{u} d_i |x_i| \quad \text{and} \quad \deg(x_1^{d_1} \cdots x_u^{d_u}) = \sum_{i=1}^{u} d_i \deg(x_i).$$

We also bigrade $S$ by $|y| = 0$ and $\deg(y) = i$ for $y \in S_i$.

A model of $R$ is a differential bigraded $S$-algebra $S[X]$ with underlying algebra $S \otimes_k k[X]$ and differential satisfying $|\partial(y)| = |y| - 1$, $\deg(\partial(y)) = \deg(y)$, and

$$\partial(s \otimes y y') = s \otimes \partial(y y') + (-1)^{|s|} s \otimes y \partial(y'),$$

together with an $S$-linear map $\varepsilon$ that makes the following sequence exact:

$$(4.3.1) \quad \cdots \to S[X]_{1,*} \xrightarrow{\partial} S[X]_{0,*} \to \cdots \to S[X]_{0,0} \xrightarrow{\varepsilon} R \to 0$$

A model $S[X]$ is said to be minimal if it satisfies $\partial(S[X]) \subseteq (S_+ + X^2)S[X]$. Such a model always exists, and any two are isomorphic as differential bigraded $S$-algebras; see [1, 7.2.4]. In every minimal model $\partial(X_i)$ is a minimal set of generators of $J$ and $S[X_1]$ is the Koszul complex of the restriction $S X_1 \to S$ of $\partial_1$.

Let $X_{0,0}$ be a $k$-basis of $S_1$, and define a set $X' = \bigsqcup_{i,j \geq 0} X'_{i,j}$ by the formula

$$(4.3.2) \quad X'_{i,j} = \{ x' \in X_{i-1,j} \}.$$

Set $k\langle X' \rangle = \bigotimes_{i=1}^{\infty} k\langle X'_i \rangle$, where $X'_i = \bigsqcup_{j} X'_{i,j}$ and $k\langle X'_i \rangle$ denotes the exterior algebra on $X'_i$ when $i$ is odd, respectively, the divided powers algebra on that space when $i$ is even.

A $k$-basis of $k\langle X' \rangle$ is given by $1 \in k$ and $x_1^{(d_1)} \cdots x_u^{(d_u)}$ with $x_1 \in X'$ and $d_i = 1$ if $|x_i'|$ is odd, respectively, with $d_i \geq 1$ if $|x_i'|$ is even. It is bigraded by setting

$$|x_1^{(d_1)} \cdots x_u^{(d_u)}| = \sum_{i=1}^{u} d_i |x_i'| \quad \text{and} \quad \deg(x_1^{(d_1)} \cdots x_u^{(d_u)}) = \sum_{i=1}^{u} d_i \deg(x'_i).$$

By [1, 7.2.6], there exists an isomorphism of bigraded $k$-vector spaces

$$(4.3.3) \quad \operatorname{Tor}^R(k, k) \cong \bigotimes_{i=1}^{\infty} k\langle X'_i \rangle$$

**Proof of Theorem 4.1.** Let $S[X]$ be a minimal model of $R$. Set $k[X] = k \otimes_S S[X]$ and note that (4.4.1) and (4.3.1) yield isomorphisms of bigraded $k$-algebras

$$(4.4.1) \quad H(k[X]) \cong \operatorname{Tor}^S(k, R) \cong H(K).$$

This induces an isomorphism $kX_{1,2} \cong H_1(K)$, by the minimality of $S[X]$.

Let $x'$ be the element of $X'_{i,j}$ corresponding to the element $x$ of $X_{i-1,j}$; see (4.3.2). For $i + 1 \leq n + 1$, from $\operatorname{res}^R_{n+1}(k) = 0$ and (4.3.3) we get $j = i + 1$. Thus, $|x| \leq n$ implies $\deg(x) = |x| + 1$. Since $|x| \geq 1$ holds for $i \leq n$, for any monomial $x_1^{d_1} \cdots x_u^{d_u}$
with $|x^{d_1}_1 \cdots x^{d_u}_u| \leq n$ we obtain the relations
\[
\deg (x^{d_1}_1 \cdots x^{d_u}_u) = \sum_{i=1}^{u} d_i \deg (x_i) = \sum_{i=1}^{u} d_i(|x_i| + 1) = |x^{d_1}_1 \cdots x^{d_u}_u| + \sum_{i=1}^{u} d_i
\]
\[
\leq 2|x^{d_1}_1 \cdots x^{d_u}_u|
\]

For $j > 2i$ these relations yield $(k[X])_{i,j} = 0$. As a consequence, $H_i(k[X])_j = 0$ holds for $j > 2i$ and $H_i(k[X])_{2i}$ is a quotient of $(k[X])_{i,2i}$. In view of (4.4.1), now (4.4.1) is proved, and for (4.4.2) it remains to show $(k[X])_{1,2i} = (k[X_{1,2}])^i$.

When $\deg (x^{d_1}_1 \cdots x^{d_u}_u) = 2|x^{d_1}_1 \cdots x^{d_u}_u|$ the inequality in the last display becomes an equality. All $d_i$ and $|x_i|$ being positive integers, for $1 \leq t \leq u$ we get first $|x_i| = 1$, then $|x_i| = \deg(x_i) - |x_i|$: that is, $\deg(x_i) = 2$. As a consequence, we get $k[X]_{1,2i} = k[X]_{1,2i} = (k[X_{1,2}])^i$, as desired. $\square$

5. Subadditivity of syzygies

We keep the hypotheses on $R$ and $S$ from Section 3. In this section we pick up a theme from the introduction—linearity at the start of the resolution of $k$ over $R$ implies subadditivity of the degree of the syzygies of $R$ over $S$. We set

$$t_i(R) = t^S_i(R), \quad \text{reg}(R) = \text{reg}^S(R), \quad \text{and} \quad \text{reg}_i(R) = \text{reg}^S_i(R)$$

for $i \in \mathbb{Z}$. Some of our results depend on the number

$$m(R) = \min\{i \in \mathbb{Z} \mid t_i(R) \geq t_{i+1}(R)\}.$$

Lemma 5.1. The following inequalities hold:

(5.1.1) $\text{rank}_k R_1 - \dim R \leq m(R) \leq \text{pd}_S R$.

Proof. We set $m = m(R)$ and let $F$ be a minimal free resolution of $R$ over $S$.

For $h = \text{pd}_S R$ we have $t_h(R) > 0$ and $t_{h+1}(R) = -\infty$, hence $m \leq h$.

For $t = t_m(R)$ we have $t_{m-1}(R) < t \geq t_{m+1}(R)$. The functor $?^* = \text{Hom}_S(?, S)$ gives homogeneous $R$-linear maps of graded $R$-modules, and hence $k$-linear maps

$$(F_{m-1})^*_{-t} \xrightarrow{(\tilde{\epsilon}_{m-1})^*_{-t}} (F_m)^*_{-t} \xrightarrow{(\tilde{\epsilon}_{m+1})^*_{-t}} (F_{m+1})^*_{-t}$$

The inequalities for $t$ imply $(F_{m-1})^*_{-t} = 0$ and $(F_{m+1})_j^* = 0$ for $j < -t$, hence

$$\text{Im}((\tilde{\epsilon}_{m+1})^*_{-t}) \subseteq (S_i(F_{m+1})^*)_{-t} = \sum_{i \geq 1} S_i(F_{m+1})^*_{-t-i} = 0$$

Thus, we get $\text{Ext}^n_S(R,S)^i = (F_m)^*_{-t} \neq 0$, and hence $\text{grade}_S R \leq m$. This is the desired lower bound, as $\text{grade}_S R = \text{rank}_k R_1 - \dim R$ because $S$ is regular. $\square$

Theorem 5.2. Let $a, b$ be non-negative integers such that $\binom{a+b}{a}$ is invertible in $k$.

When $\text{reg}_{a+b+1}^R(k) = 0$ and $\max\{a, b\} \leq m(R)$ hold there is an inequality

(5.2.1) $t_{a+b}(R) \leq \max \left\{ t_a(R) + t_b(R), t_{a-1}(R) + t_{b-1}(R) + 3 \right\}$.

In particular, there are inequalities

(5.2.2) $t_{a+1}(R) \leq t_a(R) + 2 \quad \text{when} \quad b = 1$.

(5.2.3) $t_{a+b}(R) \leq t_a(R) + t_b(R) + 1 \quad \text{for} \quad b \geq 2$. 
Proof. Formula \((3.3.1)\) and the hypothesis on \(a, b\) yield
\[
 t_{a+b}(R) \leq \max \left\{ t_a(R) + t_b(R), \quad t_{a-1}(R) - (a-1) + t_{b-1}(R) - (b-1) + a + b + 1 \right\}
\]
This simplifies to \((5.2.1)\). The other inequalities follow, since \(t_0(R) = 0\), \(t_1(R) = 2\), and \(t_{i-1} \leq t_i - 1\) for \(i \leq m(R)\).

Unless \(R\) is a polynomial ring, \(\text{reg}^R(k) = 0\) implies \(t_1(R) = 2\). Thus, \((5.2.1)\) gives an upper bound on \(t_a(R)\) by a function depending only on \(a\) and \(t_1(R)\). Such a property does not hold in general:

**Example 5.3.** For \(R = k[x_1, \ldots, x_e]/J\) and \(J\) generated by \(e+1\) general quadrics
\[
 t_1(R) = 2 \quad \text{and} \quad t_2(R) = \left\lfloor \frac{e}{2} \right\rfloor + 2.
\]
This is seen from the free resolution given by Migliore and Miró-Roig in \([29, 5.4]\).

On the other hand, we have:

### 5.4. General upper bounds

For every algebra \(R\) the number \(t_a(R)\) is bounded above by a function of the numbers \(a, t_1(R)\), and \(e = \text{rank}_k R_1\); specifically, Bayer and Mumford \([5, 3.7]\) and Caviglia and Sbarra \([10, 2.7]\) give the upper bound
\[
 t_a(R) \leq (2t_1(R))^{2^{e-2}} + a - 1 \quad \text{for every} \quad a \geq 1.
\]
It cannot be tightened substantially, since variations of the Mayr-Meyer ideals \([26]\) yield algebras with the same type of doubly exponential syzygy growth, albeit with different coefficients; see Bayer and Stillman \([6, 2.6]\) and Koh \([22\text{ Theorem}, p. 233]\).

The results in this section suggest the following

**Conjecture 5.5.** If \(\text{reg}^R_{a+b+1}(k) = 0\), then the following inequality holds:
\[(5.5.1) \quad t_{a+b}(R) \leq t_a(R) + t_b(R) \quad \text{whenever} \quad a + b \leq \text{pd}_R R.
\]
Additional supporting evidence is reviewed below. The algebra \(R = S/J\) has **monomial relations** if \(J\) can be generated by products of elements in some \(k\)-basis of \(R_1\); such an algebra is Koszul if and only if \(J\) is generated by quadratic; see \([15]\).

**Examples 5.6.** Set \(e = \text{rank}_k R_1\) and \(h = \text{pd}_R R\).

The inequality \((5.5.1)\) holds in the following cases:

1. \(\dim R \leq 1\), depth \(R = 0\), and \(a + b = e\): Eisenbud, Huneke, and Ulrich \([14, 4.1]\).
2. \(b = 1\) when \(R\) has monomial relations: Fernández-Ramos and Gimenez \([13, 1.9]\) in the quadratic case, Herzog and Srinivasan \([19]\) in general.
3. \(e \leq 7\) when \(R\) has quadratic monomial relations and \(\text{char}(k) = 0\): Verified by McCullough by using MACAULAY2 \([17]\) and Nauty \([28]\).

In addition, the following weaker inequalities are known to hold:

4. \(t_{a+b}(R) \leq t_a(R) + t_b(R) + 1\) when \(R\) is Cohen-Macaulay, \(\text{reg}^R_{a+b+1}(k) = 0\), and \(\binom{a+b}{a}\) is invertible in \(k\): Formulas \((5.1.1)\) and \((5.2.3)\).
5. \(t_{a+b}(R) \leq \max_{a+b \geq 1}(t_a(R) + t_b(R))\) when \(a + b = e\): McCullough \([27, 4.4]\).

When \(p\) is a prime number and \(n\) a non-negative integer, we say that \(p\) is **good for** \(n\) if \(\binom{n}{i} \equiv 0 \pmod{p}\) holds for all \(i \in \mathbb{Z}\) with \(0 \leq i \leq n\).
Lemma 5.7. A prime number $p$ is good for an integer $n \geq 0$ if and only if one of the following holds: (a) $p = 0$, or (b) $p > n$, or (c) $n + 1 = p^s u$ for some integers $s \geq 1$ and $2 \leq u \leq p$; furthermore, at most one pair $(p, n)$ satisfies (c).

Proof. We may assume $0 < p \leq n$. Let $\sum_{j=0}^{s-1} n_j p^j$ and $\sum_{j=0}^{s-1} i_j p^j$ be the $p$-adic expansions of $n$ and $i$, respectively; thus, $s \geq 1$ and $1 \leq n_s < p$. Lucas’ congruence

$$\binom{n}{i} \equiv \prod_{j=0}^{s-1} \binom{n_j}{i_j} \pmod{p}$$

shows that $\binom{n}{i}$ is invertible mod $p$ if and only if so is $\binom{n_j}{i_j}$ for each $j$. One has $0 \leq i_j, n_j < p$ by definition, so $\binom{n_j}{i_j}$ is invertible mod $p$ if and only if $n_j \geq i_j$. In conclusion, $p$ is good for $n$ if and only if $n_j \geq i_j$ holds for $j = 0, \ldots, s$ and $n \geq i \geq 0$.

An inequality $n \geq i \geq 0$ means $s \geq r$ and there is an integer $l$ with $0 \geq l \geq s$, satisfying $n_j = i_j$ for $j > l$ and $n_l > i_l$. This holds for all $i$ with $n \geq i \geq 0$ if and only if $n_s \geq i_s$ and $n_j = p - 1$ for $s > j \geq 0$; that is, if and only if

$$n = n_sp^s + \sum_{j=0}^{s-1} (p - 1)p^j = n_sp^s + (p - 1)\frac{p^s - 1}{p - 1} = (n_s + 1)p^s - 1.$$

Assume $up^s = up^{s'}$ holds for some prime number $p'$ and integers $u'$, $s'$ satisfying $s' \geq 1$ and $2 \leq u' \leq p$. If $p' > p$, then $p'|u$, and hence $u \geq p' > p$, which is impossible. By symmetry, so is $p > p'$, so we get $p' = p$. Now $s \neq s'$ implies $p|u$ or $p|u'$, which again is impossible. Thus, we must have $s = s'$, and hence $u = u'$.

Subadditivity is also reflected in the behavior of partial regularity. Recall from (3.1.2) that one has always an inequality $\text{reg}_a(R) \leq \text{reg}_{a+1}(R)$.

Proposition 5.8. Let $a, b$ be non-negative integers.

If $\text{reg}_{a+b+1}(k) = 0$ and $p$ is good for $(a + b)$, then the following inequalities hold:

\begin{align*}
(5.8.1) & \quad \text{reg}_{a+1}(R) \leq \text{reg}_a(R) + 1 & \text{when } b = 1, \\
(5.8.2) & \quad \text{reg}_{a+b}(R) \leq \max \left\{ \text{reg}_a(R) + \text{reg}_b(R), \text{reg}_{a-1}(R) + \text{reg}_{b-1}(R) + 1 \right\} & \text{for } b \geq 2.
\end{align*}

Proof. Pick any integer $i$ with $0 \leq i \leq a + b$ and write it as $i = a' + b'$ with integer $a'$ and $b'$ satisfying $0 \leq a' \leq a$ and $0 \leq b' \leq b$. The number $(a'+b')$ is invertible in $k$ due to the hypothesis on char$(k)$, so for $b \geq 1$ formulas (3.3.1) and (3.1.2) yield:

$$t_i(R) - i \leq \max \left\{ \frac{(t_{a'}(R) - a') + (t_{b'}(R) - b')}{\text{reg}_{a'-1}(R) + \text{reg}_{b'-1}(R) + 1} \right\}$$

$$\leq \max \left\{ \frac{\text{reg}_{a'}(R) + \text{reg}_{b'}(R)}{\text{reg}_{a'-1}(R) + \text{reg}_{b'-1}(R) + 1} \right\}$$

$$\leq \max \left\{ \frac{\text{reg}_a(R) + \text{reg}_b(R)}{\text{reg}_{a-1}(R) + \text{reg}_{b-1}(R) + 1} \right\}$$

This implies (5.8.2), and also (5.8.1) because $\text{reg}_j(R) = j$ holds for $j = 0, 1$. 

6. The Green-Lazarsfeld property \( N_q \)

We keep the hypotheses on \( R \) and \( S \) from Section 3, and set
\[
p = \text{char}(k), \quad h = \text{pd}_R S, \quad e = \text{rank}_k R_1.
\]

Green and Lazarsfeld [18, p. 85] say that \( R \) has property \( N_q \) for a positive integer \( q \) if \( t_i(R) = i + 1 \) holds for \( 1 \leq i \leq q \). This is equivalent to \( \text{reg}_q^S(R) = 1 \), see 3.1.

**Theorem 6.1.** Let \( R \) be a graded algebra satisfying \( \text{reg}_n^R \leq 1 \) for some \( n \geq 2 \).

When \( R \) satisfies \( N_q \) with \( q \geq 2 \) the following inequalities hold:
\[
t_i(R) \leq 2i - 1 \quad \text{for} \quad 2 \leq i \leq n.
\]

If, furthermore, \( p \) is good for \( i \), then also the following inequalities hold:
\[
t_i(R) \leq 2 \left\lceil \frac{i}{q + 1} \right\rceil + i + \begin{cases} 0 & \text{if} \quad (q + 1)|i \\ 1 & \text{otherwise.} \end{cases}
\]

\[
\text{reg}_n(R) \leq 2 \left\lceil \frac{n}{q + 1} \right\rceil + 1
\]

**Remark.** Both (6.1.1) and (6.1.2) sharpen (4.2.1). The condition \( q \geq 2 \) is necessary, since a complete intersection of quadrics \( R \) has \( N_1 \) and \( t_a(R) = 2a \) for \( a \leq h \).

**Proof.** The isomorphisms (3.4.2) and (4.1.2) give for each \( i \geq 2 \) equalities
\[
\text{Tor}_1^S(R,k)_2i = (\text{Tor}_1^S(R,k)_2)^2(\text{Tor}_1^S(R,k)_2)^{i-2} = \text{Tor}_2^S(R,k)_4(\text{Tor}_1^S(R,k)_2)^{i-2}
\]
We have \( \text{Tor}_2^S(R,k)_4 = 0 \) because \( R \) satisfies \( N_2 \), and hence \( \text{Tor}_1^S(R,k)_2i = 0 \). Since (4.4.1) gives \( t_{2i} \leq 2i \) for \( i \geq 1 \), we conclude that \( t_{2i} \leq 2i - 1 \) holds for \( i \geq 2 \).

If \( i \leq q \), then (6.1.2) holds by hypothesis. When \( i \geq q + 1 \) we may assume by induction on \( i \) that for \( 0 \leq j < i \) the following inequalities hold:
\[
\text{reg}_j(R) \leq 2 \left\lceil \frac{j}{q + 1} \right\rceil + \begin{cases} 0 & \text{if} \quad (q + 1)|j \\ 1 & \text{otherwise.} \end{cases}
\]

Euclidean division yields integers \( 0 \leq u \leq q \) and \( v \geq 1 \), such that
\[
i = (q + 1)v + u.
\]

In case \( u = 0 \) by applying first formula (3.3.1) with \( a = i - 1 \) and \( b = 1 \), then formula (6.1.4) with \( j = (q + 1)v - 1 \), \( j = (q + 1)v - 2 \), and \( j = 0 \) we obtain
\[
t_i(R) \leq \max \left\{ t_{(q+1)v-1}(R) + t_1(R) \right\} \leq \max \left\{ 2(v - 1) + (i - 1) + 1 + 2 \right\} = 2v + i
\]

In case 1 \( u \leq q \) by applying formula (3.3.1) with \( a = (q+1)v \) and \( b = u \), then formula (6.1.4) with \( j = (q+1)v, j = u, j = (q+1)v - 1 \), and \( j = u-1 \) we get

\[
t_i(R) \leq \max \left\{ \frac{t_{(q+1)v}(R)}{\text{reg}_{(q+1)v-1}(R)} + \frac{t_u(R)}{\text{reg}_{u-1}(R)} + i + 1 \right\}
\]

\[
\leq \max \left\{ \frac{2v + (q+1)v + u + 1}{2(v - 1) + 1 + i + 1} \right\}
\]

\[
= 2v + i + 1.
\]

This concludes the proof of (6.1.2). Formula (6.1.3) is a direct consequence. □

Conjecture 5.5 predicts that the bounds in Theorem 6.1 can be strengthened:

**Lemma 6.2.** Assume \( R \) has property \( N_q \) for some \( q \geq 2 \).

If \( t_{i+q}(R) \leq t_i(R) + t_q(R) \) holds for \( i \leq h - q \), then there are inequalities

\[
t_i(R) \leq \left\lfloor \frac{i}{q} \right\rfloor + i \quad \text{for} \quad i \geq 1.
\]

(6.2.1)

\[
\text{reg} R \leq \left\lfloor \frac{h}{q} \right\rfloor.
\]

(6.2.2)

**Proof.** For \( 1 \leq i \leq q \) the inequality (6.2.1) is just condition \( N_q \). Thus, by induction on \( i \) we may assume that the inequality holds for some \( i \geq q \). The calculation

\[
t_{i+1}(R) \leq t_{i+1-q}(R) + t_q(R)
\]

\[
\leq \left\lfloor \frac{i + 1 - q}{q} \right\rfloor + i + 1 - q + q + 1
\]

\[
= \left\lfloor \frac{i + 1}{q} \right\rfloor + i + 1
\]

completes the induction step for formula (6.2.1). The latter implies (6.2.2). □

Next we describe graphical displays that help visualize such inequalities.

**6.3. Betti templates.** Recall that the *Betti table* of \( R \) is the matrix that has \( \beta_{i,i+j}(R) \) as the entry in its \( i \)th column and \( j \)th row. In order to present information on the vanishing of Betti numbers, for each integer \( q \geq 1 \) we form a *Betti template* by replacing \( \beta_{i,i+j}(R) \) with one of the symbols from the following list:

- *non-zero due to condition \( N_q \).*
- *zero due to condition \( N_q \).*
- *zero by (6.1.1).*
- *zero by (6.1.2) when \( p \) is good for \( i \), predicted by (6.2.1) in general.
- *zero predicted by (6.2.1).*
- *no information available.*

Thus, columns 0 through \( i \) of the template contain the available and the conjectural information on the vanishing of \( \beta_{i,i+j}(R) \) when \( \text{reg}^R_{i+1}(k) = 0 \) and \( R \) satisfies \( N_q \).

An additional row at the bottom of the template displays—if it exists—the only good characteristic \( p \) for \( i \) other that \( p = 0 \) or \( p > i \); see Lemma 5.7.

A natural question is how tightly these templates can be filled in. Precise answers would be interesting both in their own right and for applications to coordinate rings of classical algebraic varieties. We present estimates by drawing on existing
examples, and also record new information obtained from our results. It is worth noting that the algebras in these examples are Koszul and many are Gorenstein.

Recall that the Hilbert series $H_R(t) = \sum_{i \geq 0} \text{rank}_k R_i t^i$ is a rational function with numerator in $\mathbb{Z}[t]$ and denominator equal to $(1 - t)^{\dim R}$.

### 6.4. Minimally elliptic singularities.

Let $k = \mathbb{C}$ and let $R$ be the associated graded ring of a rationally elliptic surface singularity; see Laufer [24]. Wahl [35, 2.8] proved that $R$ is a Gorenstein ring, $\text{pd}_S(R) = \frac{e - 2}{2}$, $\beta_{i,j}(R) = 0$ for $j \neq i + 1$ and $1 \leq i \leq e - 3$, $\beta_{e-2,j}(R) = 0$ for $j \neq e$, and $H_R(t) = (1 + et + t^2)/(1 - t)^2$.

When $k$ is infinite Wahl’s arguments apply to any Gorenstein $k$-algebra that has $H_R(t) = (1 + et + t^2)/(1 - t)^{\dim R}$. Furthermore, such an $R$ is Koszul: By factoring out a maximal $R$-regular sequence of linear forms it suffices to treat the case of dimension zero, where the desired assertion follows from [25, Theorem 3(1)].

In particular, for every $q \geq 2$ there is a Koszul $k$-algebra $R$ that is a Gorenstein ring, and for which equality holds in (6.2.1) with $i = q + 1$.

**Example 6.5.** The first 13 columns and 8 rows of the Betti template for $N_2$:

|       | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|-------|---|---|---|---|---|---|---|---|---|---|----|----|----|----|
| 0     | # | - | - | - | - | - | - | - | - | - | -  | -  | -  | -  |
| 1     |   | * | * | * | - | * | * | * | - | * | -  | -  | -  | -  |
| 2     |   |   |   |   |   |   |   |   |   |   |   10| 16 | 10 | -  |
| 3     |   |   |   |   |   |   |   |   |   |   |    | -  | -  | -  |
| 4     |   |   |   |   |   |   |   |   |   |   |    |    | -  | -  |
| 5     |   |   |   |   |   |   |   |   |   |   |    |    |    | -  |
| 6     |   |   |   |   |   |   |   |   |   |   |    |    |    |   1|
| 7     |   |   |   |   |   |   |   |   |   |   |    |    |    |    |
| 8     |   |   |   |   |   |   |   |   |   |   |    |    |    |    |

Let $R$ be a general Gorenstein algebra with Hilbert series $1 + 5z + 5z^2 + z^3$. It is Koszul by [11, 6.3]. When $p \neq 2$, by specializing Schreyer’s result [33, p. 107] to the Artinian case one sees that the Betti table of $R$ is given by

|       | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|-------|---|---|---|---|---|---|---|
| 0     | 1 | - | - | - | - | - | - |
| 1     |   | 10| 16| 10| - | - | - |
| 2     |   |   |   |   |   |   | - |
| 3     |   |   |   |   |   |   | - |
| 4     |   |   |   |   |   |   | - |

Thus, when $q = 2$ there is a Koszul $k$-algebra $R$ that is a Gorenstein ring, and for which equalities hold in (6.2.1) with $i = 3$ and $i = 5$.

Property $N_3$ appears prominently in recent studies of a basic construction:

### 6.6. Segre products.

Let $(e_1, \ldots, e_s)$ be an $s$-tuple of positive integers with $s \geq 2$.

Let $R^{(e_1, \ldots, e_s)}$ be the coordinate ring of the Segre embedding $\mathbb{P}^{e_1}_k \times \cdots \times \mathbb{P}^{e_s}_k \to \mathbb{P}^{e_s}_k$, where $e = \prod_{s=1}^{e_s} (e_i + 1) - 1$. It is Koszul; see Bărcănescu and Manolache [41, 2.1].

When $k = \mathbb{C}$ Rubel [52, Theorem 10] proved that $R^{(e_1, \ldots, e_s)}$ satisfies $N_3$, but not $N_4$ unless $s = 2$ and $e_1 = 1$ or $e_2 = 1$. Snowden developed a functorial method for building its syzygies: “One consequence of this result is that almost any question
about \( p \)-syzygies—such as, is the module of 13-syzygies of every Segre embedding generated in degrees at most 20?—can be resolved by a finite computation (in theory);\(^7\) see [34, p. 226]. Column 13 in the next example gives a positive answer.

**Example 6.7.** The first 16 columns and 8 rows of the Betti template for \( N_3 \):

|   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|
| 0 | * | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 1 | - | * | * | * | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 2 | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 3 | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 4 | - | - | - | - | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | - | - | - | - | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6 | - | - | - | - | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | - | - | - | - | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | - | - | - | - | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

The algebra \( R^{(1,1,1)} \) has \( N_3 \) by \[6.6\], it is Gorenstein with \( H_R = (1+4t+t^2)/(1-t)^4 \) by [1] 3.2(7)], and so it is Koszul by \[6.3\].

Equality can hold in (6.2.2) for any \( q \geq h \geq 1 \) by \[6.4\] and with \( p = 2q \) by Example \[5.5\]. However, \[6.2.2\] may be far from optimal when \( p \) is large. In fact, we know of no family of Koszul algebras satisfying \( N_q \) and with projective dimensions bounded above by some linear function of their regularities. On the other hand, families with quadratic bounds do exist:

6.8. Grassmannians. Let \( R^{[n]} \) be the coordinate ring of the Grassmannian of 2-dimensional subspaces of the affine space \( \mathbb{A}^n_k \). It is classically known that the \( k \)-algebra \( R^{[n]} \) is minimally generated by \( \binom{n}{2} \) elements, and Hochster [20, 3.2] proved that it is a Gorenstein ring of Krull dimension \( 2n - 3 \). It is also Koszul, see [21], and when \( p = 0 \) or \( p > \frac{1}{2}(n - 4) \) has property \( N_q \) by Kurano [23, 5.3]. The following hold: \( \text{reg} \left( R^{[n]} \right) = n - 3 \) by [9, 1.4] but \( \text{pd} \left( R^{[n]} \right) = \binom{n - 2}{2} \).

In a different direction, Dao, Huneke, and Schweig [12, 4.8] obtained an upper bound on the regularity of Koszul algebras with monomial relations satisfying \( N_q \) for some \( q \geq 2 \) in terms of the number \( e \) of algebra generators: They exhibit an explicit polynomial \( f_q(x) \in \mathbb{R}[x] \) and a real number \( \alpha_q \in \mathbb{R} \) with \( \alpha_q > 1 \), such that

\[
\text{reg } R \leq \text{log}_{\alpha_q}(f_q(e)).
\]

Our last example shows that no bound of similar type can exist for the class of all Koszul algebras satisfying \( N_q \), since it contains the following family:

6.9. Veronese subalgebras. Let \( R^{q,n} \) be the \( k \)-subalgebra generated by all the monomials of degree \( q \) in a polynomial ring in \( qn \) variables. It is Koszul by [4, 2.1], has \( N_q \) by [7, 4.2], and needs \( \binom{q^2 + q - 1}{q - 1} \) generators, and its regularity is equal to \( (q - 1)n \); the last expression is well known: It follows easily from the results of Goto and Watanabe in [16, Ch. 3] and is given explicitly by Nitsche in [30, 4.2].

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Department of Mathematics, University of Nebraska, Lincoln, NE 68588, U.S.A.

E-mail address: avramov@math.unl.edu

Dipartimento di Matematica, Università di Genova, Via Dodecaneso 35, I-16146 Genova, Italy

E-mail address: conca@dima.unige.it

Department of Mathematics, University of Nebraska, Lincoln, NE 68588, U.S.A.

E-mail address: s.b.iyengar@unl.edu