TRANSITION DENSITIES OF ONE-DIMENSIONAL LÉVY PROCESSES

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ABSTRACT. In this paper, we study the existence of the transition densities of one-dimensional Lévy processes. Compared with past results, our results contain the Lévy processes whose Lévy symbols have logarithm behavior at infinity. Our results contain the Lévy symbol induced by the following Laplace exponent

\[ \psi(\xi) := \left( \ln(1 + \ln(1 + \ln(1 + \cdots \ln(1 + |\xi|)))) \right)^{\epsilon}, \quad 0 < \epsilon \leq 1, \quad 2 \leq n. \] (0.1)

We also show that \( \psi \) defined by (0.1) is a Lévy symbol with transition density.

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1. INTRODUCTION

In this paper, we study the existence of the transition densities of one-dimensional Lévy processes. Let \( \{P_t, t > 0\} \) be a distribution of the Lévy process \( X_t \) whose Fourier transform is \( \mathcal{F}(P_t)(\xi) = \int_{\mathbb{R}} e^{i\xi y} P_t(dy) = e^{-\eta(\xi)}, \) where the Lévy symbol \( \eta \) has the form

\[ \eta(\xi) = -ib\xi + \frac{1}{2}A\xi^2 - \int_{\mathbb{R}} \left( e^{i\xi y} - 1 - i\xi y I_{(-1,1)}(y) \right) \nu(dy) \]

with constant \( b, \) non-negative constant \( A \) and Lévy measure \( \nu. \) The Lévy measure \( \nu \) satisfies

\[ \int_{\mathbb{R}} \min(1, |y|^2) \nu(dy) < \infty. \]

The transition densities of the Lévy processes are important tools in theoretical probability, physics and finance. So, the existence of the transition densities and the asymptotic property of the Lévy processes, in particular \( \alpha \)-stable processes, have been studied by many mathematicians (see [2, 3, 4, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]). In those studies, transition densities have the desirable properties of continuity and boundedness, because the Lévy symbols demonstrate asymptotic, polynomial behavior at infinity.

In the present paper, we assume that the Lévy symbol \( \eta \) has logarithmic behavior at infinity (see (0.1)). To demonstrate the existence of transition density, we use Fourier-analytic methods. In general, the differentiability of \( \mathcal{F}(f) \) is related to the behavior of \( f \) at infinity and the behavior at infinity of \( \mathcal{F}(f) \) is related to the differentiability of \( f. \) Therefore, transition densities do not have good regularity (for example, continuity and boundedness), since we assume that the Lévy symbol
\( \eta \) has logarithmic behavior at infinity. Instead, to determine the behavior of transition density at infinity, we will assume that \( \eta \) lies in \( C^2(\mathbb{R} \setminus \{0\}) \). Our main results are as follows:

**Theorem 1.1.** Let \( \eta(\xi) = \eta_1(\xi) + i\eta_2(\xi) \) be a Lévy symbol of the one-dimensional Lévy process \( X_t \). Suppose that \( \eta_1 \) and \( \eta_2 \) satisfy the following assumptions: there are a \( 0 < \epsilon \leq 1 \) and \( \alpha_\epsilon > 0 \) such that, for \( \xi \in \mathbb{R} \),

\[
\eta_1(\xi) \geq \alpha_\epsilon s_n^\epsilon(\xi), \\
|\eta_2(\xi)| \leq c \begin{cases} 
|\xi|^\epsilon, & |\xi| \leq 1, \\
 s_n^\epsilon-1(\xi)r_n^{-1}(\xi), & |\xi| \geq 1,
\end{cases}
\]

where

\[
s_1(\xi) = \ln(|\xi| + 1), \\
s_n(\xi) := \ln(1 + s_n(\xi)), \quad n \geq 2, \\
r_n(\xi) := s_n(\xi)s_{n-1}(\xi) \cdots s_1(\xi)(= s_n(\xi)r_{n-1}(\xi)).
\]

Then, \( X_t \) has a transition density \( p_t \) such that for all \( 0 < t < \infty \), \( p_t \) satisfies that, if \( |x| < 1 \), then

\[
p_t(x) \leq c(t) \frac{1}{|x|^2} e^{-\alpha_\epsilon ts_n^\epsilon(\xi)} s_n^\epsilon-1(\xi)r_n^{-1}(\xi) \frac{1}{|x|^2}
\]

and, if \( |x| \geq 1 \), then

\[
p_t(x) \leq c(t) \begin{cases} 
|x|^{-1-\epsilon}, & 0 < \epsilon < 1, \\
|x|^{-2} \ln(1 + |x|), & \epsilon = 1,
\end{cases}
\]

where \( c(t) \) is positive constant dependent on \( t \) such that \( c(t) \leq ct \) for \( t \leq 1 \). Moreover, if \( |\eta''(\xi)| \) is integrable in the interval \((0, 1)\), then for \( |x| \geq 1 \),

\[
p_t(x) \leq c(t)|x|^{-2}, \quad \epsilon = 1.
\]

**Theorem 1.2.** Let \( \eta \) be a symmetric real-valued function satisfying the assumption of Theorem 1.1, such that

\[
\eta(\xi) \leq \alpha_\epsilon s_n^\epsilon(\xi), \\
-\eta''(\xi) \geq c \begin{cases} 
s_n^\epsilon-1(\xi)r_n^{-1}(\xi)(1 + \xi)^{-2}, & |\xi| \geq 1, \\
|\xi|^\epsilon-2, & |\xi| \leq 1.
\end{cases}
\]

Then, the transition density \( p_t \) of \( X_t \) satisfies that if \( |x| \leq 1 \), then

\[
p_t(x) \geq c(t) \frac{1}{|x|^2} e^{-\alpha_\epsilon ts_n^\epsilon(\xi)} s_n^\epsilon-1(\xi) \frac{1}{|x|^2}
\]
and, if $|x| \geq 1$, then

$$p_t(x) \geq c(t) \begin{cases} \frac{1}{|x|^{2-\epsilon}}, & 0 < \epsilon < 1, \\ \frac{1}{|x|^{2}} \ln(1 + |x|), & \epsilon = 1. \end{cases}$$

Moreover, if $|\eta''(\xi)|$ is integrable in the interval $(0, 1)$, then, for $|x| \geq 1$,

$$p_t(x) \geq c(t)|x|^{-2}, \quad \epsilon = 1.$$ 

The typical examples of the one-dimensional Lévy processes are subordinators. Let the Lévy process $S_t$ be a subordinator, that is,

$$S_t \geq 0 \quad \text{a.s. for each} \quad t > 0,$$

$$S_{t_1} \leq S_{t_2} \quad \text{a.s. whenever} \quad t_1 < t_2.$$ 

The Lévy symbol $\eta$ of subordinator $S_t$ has the form

$$\eta(\xi) = ib\xi - \int_{0}^{\infty} (e^{i\xi y} - 1)\lambda(dy), \quad (1.3)$$

where $b \geq 0$ and the Lévy measure $\lambda$ satisfies the additional requirements

$$\lambda(-\infty, 0) = 0, \quad \int_{0}^{\infty} \min(y, 1)\lambda(dy) < \infty. \quad (1.4)$$

Conversely, any mapping $\psi : \mathbb{R} \to \mathbb{C}$ of the form (1.3) is the Lévy symbol of a subordinator.

Now, if $S_t$ is subordinator, then, for each $t \geq 0$, the map $f(\xi) = Ee^{i\xi S_t} = e^{-t\eta(\xi)}$ can be analytically continued to the region $\{v + i\xi \mid v \in \mathbb{R}, \xi > 0\}$. Let $F(z) = e^{-t\eta(z)}$ be an analytical extension of $e^{-t\eta(\xi)}$ over $\{v + i\xi \mid v \in \mathbb{R}, \xi > 0\}$. Then, we get

$$F(i\xi) = Ee^{-\xi S_t} = e^{-t\psi(i\xi)} := e^{-t\psi(\xi)}, \quad \xi \geq 0,$$

where

$$\psi(\xi) = \eta(i\xi) = -b\xi - \int_{0}^{\infty} (e^{-\xi y} - 1)\lambda(dy) \quad (1.5)$$

for each $\xi > 0$. This is much more useful for theoretical and practical applications than is the Lévy symbol. The function $\psi$ is usually called the Laplace exponent of the subordinator.

The Laplace exponents of subordinators are characterized by the Bernstein functions. We say that the continuous function $\psi : [0, \infty) \to [0, \infty)$ is a Bernstein function if $(-1)^k \psi^{(k)}(0) \leq 0$ for all $k \in \mathbb{N}$. If $\psi$ is the Laplace exponent of the subordinator, then, from (1.5), $\psi$ is a Bernstein function.

Conversely, if $\psi$ is a Bernstein function, then there are a non-negative real number $b$ and a measure $\lambda$ defined in $\mathbb{R}$ satisfying (1.4) such that (1.3) holds. Moreover, there is a subordinator $S_t$ such that

$$\eta(\xi) = \psi(-i\xi) \quad (1.6)$$

is the Lévy symbol of $S_t$. 

This paper is organized as follows. In section 2 we introduce two lemmas to prove Theorem 1.1 and Theorem 1.2. In section 3 and section 4 we prove Theorem 1.1 and Theorem 1.2, respectively. In section 5 we introduce examples satisfying Theorem 1.1 and Theorem 1.2.

In this paper, we denote by \( c \) various generic positive constants and by \( c(\ast, \cdots, \ast) \) the constants depending only on the quantities appearing in the parenthesis.

### 2. Main Lemmas

In this section, we introduce the two lemmas for the proofs of Theorem 1.1 and Theorem 1.2. The first lemma is as follows.

**Lemma 2.1.** Suppose that, for \( \xi \in \mathbb{R} \setminus \{0\} \),

\[
\eta_1(\xi) \geq g_1(|\xi|), \quad |\eta_2(\xi)| \leq g_2(|\xi|),
\]

\[
|\eta_1'(\xi)|, \quad |\eta_2(\xi)| \leq g_3(|\xi|),
\]

\[
|\eta_1''(\xi)|, \quad |\eta_2'(\xi)| \leq g_4(|\xi|).
\]

(2.1)

Let \( f_1(\xi) = e^{-t\eta_1(\xi)} \cos t\eta_2(\xi) \), \( f_2(\xi) = e^{-t\eta_1(\xi)} \sin t\eta_2(\xi) \) and

\[
G(\xi) := e^{-t\eta_1(\xi)} \left( t^2 g_3(\xi)^2 + t g_4(\xi) \right).
\]

Suppose that \( G \) is decreasing in \( (0, \infty) \). Then, for \( x \in \mathbb{R} \) and \( t \in (0, \infty) \),

\[
|\int_{\mathbb{R}} f_1(\xi) \cos x\xi d\xi| \leq c \left( \frac{1}{|x|^2} \int_{2\pi}^{\pi} G(y) dy + \frac{1}{|x|^3} G(\frac{1}{x}) + \int_{2\pi}^{\pi} G(y) y^2 dy \right),
\]

\[
|\int_{\mathbb{R}} f_2(\xi) \sin x\xi d\xi| \leq c \left( \frac{1}{|x|^2} \int_{2\pi}^{\pi} G(y) dy + \frac{t}{|x|} \int_{0}^{\frac{\pi}{t}} e^{-\frac{t\eta_1(y)}{y}} g_2(\frac{1}{x}) dy \right).
\]

**Proof.** Note that

\[
f_1''(\xi) = e^{-t\eta_1(\xi)} \left( (-t\eta_1'')^2 \cos t\eta_2 + (-t\eta_1'') \cos t\eta_2 + (-t\eta_1'')(t\eta_2') \cos t\eta_2 + (-t\eta_1')(t\eta_2') \sin t\eta_2 - (t\eta_2')^2 \cos t\eta_2 \right),
\]

\[
f_2''(\xi) = e^{-t\eta_1(\xi)} \left( (-t\eta_1'')^2 \sin t\eta_2 + (-t\eta_1'') \sin t\eta_2 + (-t\eta_1')(t\eta_2') \cos t\eta_2 + (-t\eta_1')(t\eta_2') \cos t\eta_2 + (t\eta_2')^2 \sin t\eta_2 \right).
\]

From the assumptions (2.1) of \( \eta_1 \) and \( \eta_2 \), we get

\[
|f_1''(\xi)|, \quad |f_2''(\xi)| \leq ce^{-t\eta_1(\xi)} \left( (-t\eta_1'')^2 + |(-t\eta_1'')(t\eta_2')| + |(-t\eta_1')(t\eta_2')| + (t\eta_2')^2 \right) \leq ce^{-t\eta_1(\xi)} \left( t^2 g_3(|\xi|)^2 + t g_4(|\xi|) \right) \leq cG(|\xi|).
\]
Using the change of variables, we have

\[
\int_{\mathbb{R}} f_1(\xi) \cos(x\xi) d\xi = \frac{1}{|x|} \int_{\mathbb{R}} f_1(\frac{\xi}{x}) \cos \xi d\xi = \frac{1}{|x|} \sum_{-\infty < k < \infty} I_k^1(x),
\]

where

\[
I_k^1(x) = \int_{2\pi k}^{2\pi (k+1)} f_1(\frac{\xi}{x}) \cos \xi d\xi
\]

\[
= \int_{2\pi k}^{2\pi k + \frac{\pi}{2x}} f_1(\frac{\xi}{x}) \cos \xi d\xi + \int_{2\pi k + \frac{\pi}{2x}}^{2\pi k + \frac{\pi}{x}} f_1(\frac{\xi}{x}) \cos \xi d\xi
\]

\[
+ \int_{2\pi k + \frac{\pi}{x}}^{2\pi (k+1)} f_1(\frac{\xi}{x}) \cos \xi d\xi = \int_{0}^{\frac{\pi}{2x}} f_1\left(\frac{2\pi k + \xi}{x}\right) \cos(2\pi k + \xi) d\xi + \int_{\frac{\pi}{2x}}^{\frac{\pi}{x}} f_1\left(\frac{2\pi k + \pi - \xi}{x}\right) \cos(2\pi k + \pi - \xi) d\xi
\]

\[
+ \int_{\frac{\pi}{x}}^{\frac{\pi}{2x}} f_1\left(\frac{2\pi k + 2\pi - \xi}{x}\right) \cos(2\pi k + 2\pi - \xi) d\xi
\]

\[
= \int_{0}^{\frac{\pi}{2x}} \left( f_1\left(\frac{2\pi k + \xi}{x}\right) - f_1\left(\frac{2\pi k + \pi - \xi}{x}\right) - f_1\left(\frac{2\pi k + \pi + \xi}{x}\right) + f_1\left(\frac{2\pi k + 2\pi - \xi}{x}\right) \right) \cos \xi d\xi.
\]

Using the mean-value theorem, there are \( \epsilon_1 \in (\xi, \pi - \xi), \epsilon_2 \in (\pi + \xi, 2\pi - \xi) \) such that

\[
f_1\left(\frac{2\pi k + \xi}{x}\right) - f_1\left(\frac{2\pi k + \pi - \xi}{x}\right) = -\frac{1}{x} f_1'\left(\frac{2\pi k + \epsilon_1}{x}\right) (\pi - 2\xi),
\]

\[
f_1\left(\frac{2\pi k + \pi + \xi}{x}\right) - f_1\left(\frac{2\pi k + 2\pi - \xi}{x}\right) = -\frac{1}{x} f_1'\left(\frac{2\pi k + \epsilon_2}{x}\right) (\pi - 2\xi).
\]

We use the mean-value theorem again such that

\[-\frac{1}{x} f_1'\left(\frac{2\pi k + \epsilon_1}{x}\right) (\pi - 2\xi) + \frac{1}{x} f_1'\left(\frac{2\pi k + \epsilon_2}{x}\right) (\pi - 2\xi) = \frac{1}{x^2} (\pi - 2\xi) (\epsilon_2 - \epsilon_1) f_1''\left(\frac{2k\pi + \epsilon_3}{x}\right),
\]

where \( \epsilon_3 \) lies between \( \epsilon_1 \) and \( \epsilon_2 \). Hence, we obtain

\[
I_k^1(x) = \frac{1}{|x|^2} \int_{0}^{\frac{\pi}{2x}} (\pi - 2\xi) (\epsilon_2 - \epsilon_1) f_1''\left(\frac{2k\pi + \epsilon_3}{x}\right) \cos \xi d\xi.
\]

Since \( f_1'' \) is dominated by the positive and decreasing function \( G \), we have

\[
|I_k^1(x)| \leq c \frac{1}{|x|^2} \int_{0}^{\frac{\pi}{2x}} G\left(\frac{2\pi k + \epsilon_3}{|x|}\right) d\xi
\]

\[
\leq c \begin{cases} 
\frac{1}{|x|^2} G\left(\frac{2\pi k}{|x|}\right), & k \geq 1, \\
\frac{1}{|x|^2} G\left(\frac{2\pi (k+1)}{|x|}\right), & k \leq -2.
\end{cases}
\]
Since $G : (0, \infty) \to (0, \infty)$ is a decreasing function, we have

\[
\sum_{2 \leq |k|} |I_k(x)| \leq c \frac{1}{x^2} \left( \sum_{2 \leq k} G\left(\frac{2\pi|k|}{|x|}\right) + \sum_{k \leq -2} G\left(\frac{2\pi|k+1|}{|x|}\right) \right)
\]

\[
\leq c \frac{1}{x^2} \sum_{2 \leq k < \infty} G\left(\frac{2\pi k}{|x|}\right)
\]

\[
\leq c \frac{1}{x^2} \int_1^{\infty} G\left(\frac{2\pi y}{|x|}\right) dy
\]

\[
= c \frac{1}{2\pi|x|} \int_{\frac{2\pi}{|x|}}^{\infty} G(y) dy.
\]

(2.6)

In the case $k = 0$, from (2.3), we have

\[
\frac{1}{x} \int_0^{2\pi} f_1\left(\frac{\xi}{x}\right) \cos \xi d\xi = \frac{1}{x} \int_0^{\frac{2\pi}{x}} \left( f_1\left(\frac{\xi}{x}\right) - f_1\left(\frac{\pi - \xi}{x}\right) - f_1\left(\frac{\pi + \xi}{x}\right) + f_1\left(\frac{2\pi - \xi}{x}\right) \right) \cos \xi d\xi
\]

\[
= \frac{1}{x} \int_0^{\frac{2\pi}{x}} \left( - \int_{\frac{\pi - \xi}{x}}^{\frac{\pi + \xi}{x}} f_1'(y) dy + \int_{\frac{2\pi - \xi}{x}}^{\frac{2\pi}{x}} f_1'(y) dy \right) \cos \xi d\xi
\]

\[
= \frac{1}{x} \int_0^{\frac{2\pi}{x}} \int_{\frac{\pi - \xi}{x}}^{\frac{\pi + \xi}{x}} (- f_1'(y) + f_1'(\frac{\pi}{x} + y)) dy \cos \xi d\xi
\]

\[
= \frac{1}{x} \int_0^{\frac{2\pi}{x}} \int_{\frac{\pi - \xi}{x}}^{\frac{\pi + \xi}{x}} f_1''(z) dz dy \cos \xi d\xi.
\]

Hence, using Fubini’s theorem, for $x > 0$, we have

\[
\frac{1}{x} \int_0^{2\pi} f_1\left(\frac{\xi}{x}\right) \cos \xi d\xi = \frac{1}{x} \int_0^{\frac{2\pi}{x}} \int_{\frac{\pi - \xi}{x}}^{\frac{\pi + \xi}{x}} G(z) dz dy \cos \xi d\xi
\]

\[
= \frac{1}{x} \int_0^{\frac{2\pi}{x}} \int_{\frac{\pi - \xi}{x}}^{\frac{\pi + \xi}{x}} G(z) dz dy \cos \xi d\xi
\]

\[
+ \frac{1}{x} \int_0^{\frac{2\pi}{x}} \int_{\frac{2\pi - \xi}{x}}^{\frac{2\pi}{x}} G(z) dz dy \cos \xi d\xi
\]

\[
= \frac{1}{x} \int_0^{\frac{2\pi}{x}} \int_{\frac{\pi - \xi}{x}}^{\frac{\pi + \xi}{x}} G(z)(z - \frac{\xi}{x}) dz dy \cos \xi d\xi
\]

\[
+ \frac{1}{x} \int_0^{\frac{2\pi}{x}} \int_{\frac{\pi - \xi}{x}}^{\frac{\pi + \xi}{x}} G(z)(2\pi - \xi)(\frac{2\pi - \xi}{x} - z) dz dy \cos \xi d\xi
\]

\[
:= I_{0,1}(x) + I_{0,2}(x) + I_{0,3}(x).
\]
Using Fubini’s theorem, we get
\[ I_{0,1}(x) = \frac{1}{x} \int_0^{\pi x} G(z) \int_0^{x \pi} \left( z - \frac{\xi}{x} \right) d\xi dz + \frac{1}{x} \int_0^{\pi x} G(z) \int_0^{\pi - x \pi} \left( z - \frac{\xi}{x} \right) d\xi dz \]
\[ = \frac{1}{2} \int_0^{\pi x} G(z) z^2 dz + \frac{1}{2} \int_0^{\pi x} G(z)(z^2 - (\frac{\pi}{x})^2) dz \]
\[ \leq \frac{1}{2} \int_0^{\pi x} G(z) z^2 dz + \frac{1}{2} \frac{G(\pi)}{2x} \int_0^{\pi x} (z^2 - (\frac{\pi}{x})^2) dz \]
\[ \leq c \left( \int_0^{\pi x} G(z) z^2 dz + \frac{1}{x} G(\frac{1}{x}) \right). \] (2.7)

Next, we estimate \( I_{0,2} \). Using Fubini’s theorem, we have
\[ I_{0,2} \leq c \frac{1}{x^2} \int_0^{\pi x} \int_0^{\pi x} G(z) dz d\xi \]
\[ = c \frac{1}{x^2} \int_0^{\pi x} G(z) dz \]
\[ \leq c \frac{1}{x^3} G(\frac{1}{x}). \] (2.8)

Next, we estimate \( I_{0,3} \). Using Fubini’s theorem, we get
\[ I_{0,3} = \frac{1}{x} \int_0^{\pi x} G(z) \int_0^{x \pi - \pi} \left( \frac{2\pi - \xi}{x} - z \right) d\xi dz + \frac{1}{x} \int_0^{\pi x} G(z) \int_0^{\pi x - 2\pi} \left( \frac{2\pi - \xi}{x} - z \right) d\xi dz \]
\[ = \frac{1}{x} \int_0^{\pi x} G(z)(xz - \pi)(\frac{5\pi}{2x} - \frac{3z}{2}) dz + \frac{1}{x} \int_0^{\pi x} G(z)(-xz + 2\pi)(\frac{\pi}{x} - \frac{1}{2}z) dz \]
\[ \leq c \frac{1}{x^3} G(\frac{1}{x}). \] (2.9)

Hence, from (2.7), (2.8) and (2.9), we have that, for \( 0 < x \),
\[ I_0 \leq \frac{1}{x} \int_0^{2\pi} f_1(\frac{\xi}{x}) \cos \xi d\xi \leq c \left( \frac{1}{|x|^3} G(\frac{1}{x}) + \int_0^{\pi x} G(z) z^2 dz \right). \] (2.10)

By similar calculation, (2.10) holds for \( x < 0 \). Using the same argument, we obtain
\[ I_1 \leq \frac{1}{x} \int_{-2\pi}^0 f_1(\frac{\xi}{x}) \cos \xi d\xi \leq c \left( \frac{1}{|x|^3} G(\frac{1}{x}) + \int_0^{\pi x} G(z) z^2 dz \right). \] (2.11)

Hence, by (2.2), (2.3), (2.10) and (2.11), we obtain the first inequality of Theorem 2.1.

Next, we have
\[ \int_R f_2(\xi) \sin(x\xi) d\xi = \frac{1}{|x|} \int_R f_2(\frac{\xi}{x}) \sin \xi d\xi \]
\[ = \frac{1}{|x|} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f_2(\frac{\xi}{x}) \sin \xi d\xi + \frac{1}{|x|} \sum_{0 \leq k < \infty} (I_{k,1}^{2.1} + I_{k,2}^{2.2}), \]
where for \( k > 0 \),
\[ I_{k,1}^{2.1}(x) = \int_{2\pi(k+1)\frac{\xi}{x}}^{2\pi(k+1)\frac{\xi}{x} + \frac{\pi}{x}} f_2(\frac{\xi}{x}) \sin \xi d\xi, \quad I_{k,2}^{2.2}(x) = \int_{2\pi(k-1)\frac{\xi}{x} - \frac{\pi}{x}}^{2\pi(k-1)\frac{\xi}{x} - \frac{\pi}{x} + \frac{\pi}{x}} f_2(\frac{\xi}{x}) \sin \xi d\xi. \]
Using two applications of mean-value theorem, there are \( \epsilon_1 \in (\frac{1}{2}\pi + \xi, \frac{3}{2}\pi - \xi) \), \( \epsilon_2 \in (\frac{3}{2}\pi + \xi, \frac{5}{2}\pi - \xi) \) and \( \epsilon_3 \in (\epsilon_1, \epsilon_2) \) such that

\[
I_{2,k}^1(x) = \int_{2\pi k + \frac{1}{2}\pi}^{2\pi (k+1) + \frac{1}{2}\pi} f_2(\frac{\xi}{x}) \sin \xi \, d\xi
\]

\[
= \int_0^{\frac{\pi}{2}} \left( f_2\left(\frac{2\pi k + \frac{3}{2}\pi + \xi}{x}\right) - f_2\left(\frac{2\pi k + \frac{3}{2}\pi - \xi}{x}\right)ight) \sin \xi \, d\xi
- f_2\left(\frac{2\pi k + \frac{3}{2}\pi + \xi}{x}\right) + f_2\left(\frac{2\pi k + \frac{3}{2}\pi - \xi}{x}\right) \sin \xi \, d\xi
\]

\[
= \frac{1}{\pi} \int_0^{\frac{\pi}{2}} 2\xi (\epsilon_2 - \epsilon_1) f_2''\left(\frac{2\pi k + \epsilon_3}{x}\right) \sin \xi \, d\xi.
\]

Hence, we have

\[
|I_{2,k}^1(x)| \leq C \frac{1}{|x|^2} \int_0^{\frac{\pi}{2}} G\left(\frac{2\pi k + \epsilon_3}{|x|}\right) \, d\xi
\]

\[
\leq C \frac{1}{|x|^2} \int_0^{\frac{\pi}{2}} G\left(\frac{2\pi k + \frac{1}{2}\pi}{|x|}\right) \, d\xi
\]

\[
\leq C \frac{1}{|x|^2} G\left(\frac{2\pi k + \frac{1}{2}\pi}{|x|}\right).
\] (2.12)

Using similar calculation to \( I_{2,k}^1(x) \), we have

\[
|I_{2,k}^2(x)| \leq C \frac{1}{|x|^2} G\left(\frac{2\pi |k| + \frac{1}{2}\pi}{|x|}\right).
\] (2.13)

Hence, we have

\[
\sum_{0 \leq k < \infty} (|I_{2,k}^1(x)| + |I_{2,k}^2(x)|) \leq C \frac{1}{|x|^2} \sum_{1 \leq k < \infty} G\left(\frac{2\pi k + \frac{1}{2}\pi}{|x|}\right)
\]

\[
\approx C \frac{1}{|x|^2} \int_0^{\infty} G\left(\frac{2\pi y + \frac{1}{2}\pi}{|x|}\right) \, dy
\]

\[
= C \frac{1}{|x|^2} \int_{\frac{\pi}{2\pi}}^{\infty} G(y) \, dy.
\]

Note that

\[
|\frac{1}{x} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f_2\left(\frac{\xi}{x}\right) \sin \xi \, d\xi| \leq C \frac{1}{|x|^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-tg_1(\frac{\xi}{x})} \, d\xi
\]

\[
\leq C \frac{t}{|x|} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-tg_1(\frac{\xi}{x})} \, d\xi.
\]

Hence, we obtain the second inequality of Lemma 2.1. We complete the proof of Lemma 2.1.

\(\square\)

The second lemma is as follow.

**Lemma 2.2.** Let \( 0 < \epsilon \leq 1 \) and \( n \geq 2 \).
Proof. To prove (1) of Lemma 2.2, it is sufficient to show

\[ \lim_{a \to \infty} \frac{\int_1^a z^\alpha e^{-ta_\alpha s'_n(z)} s^\epsilon_{n-1}(z)(z)dz}{a^{\alpha+1}e^{-\alpha s'_n(a)}s^\epsilon_{n-1}(a)r_{n-1}^{-1}(a)} = (\alpha + 1)^{-1}. \]

Using L’Hospital’s theorem, we have

\[ \lim_{a \to \infty} \frac{\int_1^a z^\alpha e^{-ta_\alpha s'_n(z)} s^\epsilon_{n-1}(z)(z)dz}{a^{\alpha+1}e^{-\alpha s'_n(a)}s^\epsilon_{n-1}(a)r_{n-1}^{-1}(a)} = \lim_{a \to \infty} \frac{a^{\alpha+1}e^{-\alpha s'_n(a)}s^\epsilon_{n-1}(a)r_{n-1}^{-1}(a)}{T(a)}, \]

where

\[ T(a) = \frac{d}{da}(a^{\alpha+1}e^{-\alpha s'_n(a)}s^\epsilon_{n-1}(a)r_{n-1}^{-1}(a)) \]

\[ = (\alpha + 1)a^{\alpha+1}e^{-\alpha s'_n(a)}s^\epsilon_{n-1}(a)r_{n-1}^{-1}(a) - \epsilon c_\alpha a^{\alpha+1}e^{-\alpha s'_n(a)}s^{2-2}(a)r_{n-1}^{-1}(a)(s_n(a))' \]

\[ + (\epsilon - 1)a^{\alpha+1}e^{-\alpha s'_n(a)}s^{1-2}(a)r_{n-1}^{-1}(a)(s_n(a))' - a^{\alpha+1}e^{-\alpha s'_n(a)}s^\epsilon_{n-1}(a)r_{n-1}^{-1}(a)r_{n-1}^{-1}(a). \]

Note that

\[ s'_n(a) = \prod_{k=1}^{k=n-1} (1 + a)^{-1}(a)(1 + a)^{-1}, \]

\[ r'_n(a) = \sum_{1 \leq k \leq n} t_k(a) \prod_{l=1}^{l=k-1} (1 + s_l)^{-1}(a)(1 + a)^{-1}, \]

where \( t_k(a) = \frac{r_n(k)}{s_k}\). Hence, we get

\[ \lim_{a \to \infty} \frac{\int_1^a z^\alpha e^{-ta_\alpha s'_n(z)} s^\epsilon_{n-1}(z)(z)dz}{a^{\alpha+1}e^{-\alpha s'_n(a)}s^\epsilon_{n-1}(a)r_{n-1}^{-1}(a)} \]

\[ = \lim_{a \to \infty} (\alpha + 1) + \frac{a}{1+a} (\epsilon - \alpha s^\epsilon_{n-1}(a) + (\epsilon - 1)s^\epsilon_{n-1}(a)) \prod_{k=1}^{k=n-1} (1 + s_k(a))^{-1} - \frac{a}{1+a} r_{n-1}^{-1}(a) \sum_{1 \leq k \leq n} t_k(a)r_{n-1}^{-1}(a) \]

\[ = (\alpha + 1)^{-1}. \]

This completes the proof of (1) of Lemma 2.2.
Using the change of variables and integration by parts sequentially, we get

\[
\int_{a_0}^a (z + 1)^\alpha e^{-\tau z} s_n^\alpha(z) s_{n-1}^{r-1}(z)dz = \int_{\ln(1+a_0)}^{\ln(1+a)} e^{(1+\alpha)z} z^{-1} e^{-\tau z} s_{n-1}^{\alpha-1}(z) s_{n-1}^{r-1}(z) r_{n-2}^{-1}(z)dz
\]

\[
= (1 + \alpha)^{-1} e^{(1+\alpha)z} z^{-1} e^{-\tau z} s_{n-1}^{\alpha-1}(z) s_{n-1}^{r-1}(z) r_{n-2}^{-1}(z) \int_{\ln(1+a_0)}^{\ln(1+a)} dz
\]

\[
- (1 + \alpha)^{-1} \int_{\ln(1+a_0)}^{\ln(1+a)} e^{(1+\alpha)z} \frac{dz}{dz} \left( z^{-1} e^{-\tau z} s_{n-1}^{\alpha-1}(z) s_{n-1}^{r-1}(z) r_{n-2}^{-1}(z) \right)dz.
\]

Note that

\[
- \frac{d}{dz} \left( z^{-1} e^{-\tau z} s_{n-1}^{\alpha-1}(z) s_{n-1}^{r-1}(z) r_{n-2}^{1}(z) \right)
\]

\[
= z^{-1} e^{-\tau z} s_{n-1}^{\alpha-1}(z) s_{n-1}^{r-1}(z) r_{n-2}^{1}(z) \left( z^{-1} + \epsilon \tau s_{n-1}^{\alpha-1}(z) \prod_{k=1}^{k=n-2} (1 + s_k(z))^{-1} (1 + z)^{-1}
\]

\[
- (\epsilon - 1) s_{n-1}^{\alpha-1}(z) \prod_{k=1}^{k=n-2} (1 + s_k(z))^{-1} (1 + z)^{-1} + r_{n-2}^{1}(z) \sum_{1 \leq k \leq n-2} t_k(a) \prod_{l=1}^{l=k-1} (1 + s_l(a))^{-1} (1 + a)^{-1}
\]

\[
= z^{-1} e^{-\tau z} s_{n-1}^{\alpha-1}(z) s_{n-1}^{r-1}(z) r_{n-2}^{1}(z) R(z).
\]

Since \( R(z) \) is a decreasing function, for \( (a_0 + 1) \leq z \), we get

\[
R(z) \leq \ln(a_0 + 1)^{-1} + \epsilon \tau s_{n-1}^{\alpha-1}(z) \ln(a_0 + 1)^{-1} - (\epsilon - 1) \ln(a_0 + 1)^{-1} + (n - 2) \ln(a_0 + 1)^{-1}
\]

\[
\leq (n - \epsilon + \epsilon \tau s_{n-1}^{\alpha-1}(z)) \ln(a_0 + 1)^{-1}.
\]

Since \( (1 + \alpha)^{-1} (n - \epsilon + \epsilon \tau s_{n-1}^{\alpha-1}(z)) \ln(1 + a_0)^{-1} = \frac{1}{2} \), we have

\[
-(1 + \alpha)^{-1} \frac{d}{dz} \left( z^{-1} e^{-\tau z} s_{n-1}^{\alpha-1}(z) s_{n-1}^{r-1}(z) r_{n-2}^{1}(z) \right) \leq \frac{1}{2} z^{-1} e^{-\tau z} s_{n-1}^{\alpha-1}(z) s_{n-1}^{r-1}(z) r_{n-2}^{1}(z), \quad z \geq \ln(1 + a_0).
\]

Since \( s_{n-1}(\ln(1 + a)) = s_n(a) \) and \( \ln(1 + a) r_{n-2}(\ln(1 + a)) = r_{n-1}(a) \), we have

\[
\int_{a_0}^a (z + 1)^\alpha e^{-\tau z} s_n^\alpha(z) s_{n-1}^{r-1}(z)dz \leq 2(1 + \alpha)^{-1} e^{(1+\alpha)z} z^{-1} e^{-\tau z} s_{n-1}^{\alpha-1}(z) s_{n-1}^{r-1}(z) r_{n-2}^{1}(z) \int_{\ln(1+a)}^{\ln(1+a)} dz
\]

\[
\leq c_\alpha d^{1+\alpha} e^{-\tau z} s_n^{\alpha-1}(a) s_{n-1}^{r-1}(a) r_{n-1}^{1}(a).
\]

Hence, we complete the proof of (2) of the Lemma 2.22.

For (3) of Lemma 2.22 using the change of variables and integration by parts sequentially, we get

\[
\int_{a_0}^\infty (z + 1)^\alpha e^{-\tau z} s_n^\alpha(z) s_{n-1}^{r-1}(z)dz = \int_{\ln(1+a)}^{\ln(1+a)} e^{(1+\alpha)z} z^{-1} e^{-\tau z} s_{n-1}^{\alpha-1}(z) s_{n-1}^{r-1}(z) r_{n-2}^{1}(z)dz
\]

\[
= (1 + \alpha)^{-1} e^{(1+\alpha)z} z^{-1} e^{-\tau z} s_{n-1}^{\alpha-1}(z) s_{n-1}^{r-1}(z) r_{n-2}^{1}(z) \int_{\ln(1+a)}^{\ln(1+a)} dz
\]

\[
- (1 + \alpha)^{-1} \int_{\ln(1+a)}^{\ln(1+a)} e^{(1+\alpha)z} \frac{dz}{dz} \left( z^{-1} e^{-\tau z} s_{n-1}^{\alpha-1}(z) s_{n-1}^{r-1}(z) r_{n-2}^{1}(z) \right)dz.
\]
Hence, we complete the proof of (3) of Lemma 2.2.

Since $z^{-1}e^{-t_{2\alpha,s_{n-1}}(z)s_{n-1}(z)r_{n-2}(z)}$ is a decreasing function, \( \frac{d}{dz}(z^{-1}e^{-t_{2\alpha,s_{n-1}}(z)s_{n-1}(z)r_{n-2}(z)}) \) is a non-positive function. Since $\alpha < -1$, we get

\[
\int_{a}^{\infty} (z + 1)^{\alpha} e^{-t_{2\alpha,s_{n}}(z)s_{n}(z)r_{n-1}(z)} \leq (1 + \alpha)^{1+(n-1)}e^{-t_{2\alpha,s_{n-1}}(z)s_{n-1}(z)r_{n-2}(z)} \left[ \ln(1+n) \right]^{\infty}_{a} = -(1 + \alpha)^{1+(n-1)}e^{-t_{2\alpha,s_{n}}(z)s_{n}(z)r_{n-1}(z)} \leq -(1 + \alpha)^{1+(n-1)}e^{-t_{2\alpha,s_{n}}(z)s_{n}(z)r_{n-1}(z)}(a).
\]

Hence, we complete the proof of (3) of Lemma 2.2.

\( \square \)

3. PROOF OF THEOREM 1.1

In this section, we prove Theorem 1.1.

Let

\[
g_{1}(\xi) := \alpha_{s_{n}}(\xi),
g_{2}(\xi) := c \begin{cases} |\xi|^{r}, & |\xi| \leq 1, \\ s_{n}^{-1}(\xi)\frac{1}{1+r_{n-1}(\xi)}, & |\xi| \geq 1, \end{cases}
g_{3}(\xi) := c \begin{cases} |\xi|^{r-1}, & |\xi| \leq 1, \\ s_{n}^{-1}(\xi)r_{n-1}(\xi)(1 + |\xi|)^{-1}, & |\xi| \geq 1, \end{cases}
g_{4}(\xi) := c \begin{cases} |\xi|^{r-2}, & |\xi| \leq 1, \\ s_{n}^{-1}(\xi)r_{n-1}(\xi)(1 + |\xi|)^{-2}, & |\xi| \geq 1. \end{cases}
\] (3.1)

From the Lemma 2.1, it is sufficient to estimate the following:

\[
\frac{1}{|x|^{2}} \int_{0}^{\infty} G(\xi) d\xi, \quad \frac{t}{|x|} \int_{0}^{\frac{2\pi}{|x|}} e^{-t_{g_{1}}(\xi)} g_{2}(\frac{\xi}{x}) \xi, \quad \frac{1}{|x|^{2}} G(\frac{1}{x}), \quad \int_{0}^{\frac{2\pi}{|x|}} G(\xi) \xi^{2} d\xi
\]

with $G(\xi) := e^{-t_{g_{1}}(\xi)}(t^{2}g_{3}(\xi)^{2} + t g_{4}(\xi))$. Note that, by the assumption (1.1) of $\eta_{1}$, we have

\[
e^{-t_{\eta_{1}}(\xi)} \leq e^{-t_{g_{1}}(\xi)} = e^{-\alpha_{s_{n}}(\xi)}
\]

and

\[
G(\xi) \leq c \begin{cases} e^{-\alpha_{s_{n}}(\xi)}(t^{2}|\xi|^{2r-2} + t|\xi|^{r-2}), & |\xi| \leq 1, \\ t(1+e^{-t_{\alpha,s_{n}}(\xi)})(1 + |\xi|)^{-2}, & |\xi| \geq 1. \end{cases}
\]
(1) In the case of $|x| \geq 1$.

By direct calculation, we have

$$
\frac{1}{|x|^2} \int_{\mathbb{R}} G(\xi) d\xi \leq \frac{1}{|x|^2} \int_{\mathbb{R}} t(t + 1) e^{-\alpha ts_n(\xi)} s_{n-1}^\epsilon(\xi) r_{n-1}(\xi) (1 + |\xi|)^{-2} d\xi
$$

$$
+ \frac{1}{|x|^2} \int_{\mathbb{R}} e^{-\alpha ts_n(\xi)} (t^2 |\xi|^{2\epsilon - 2} + t |\xi|^{\epsilon - 2}) d\xi
$$

$$
\leq ct(t + 1) \left\{ \begin{array}{ll}
\frac{1}{|x|^{1+\epsilon}}, & 0 < \epsilon < 1, \\
\frac{\ln(1+|x|)}{|x|^2}, & \epsilon = 1.
\end{array} \right.
$$

Using the change of variables, we get

$$
\frac{t}{|x|} \int_{0}^{\pi} e^{-t g_1(\xi)} g_2(\xi) d\xi \leq \frac{ct}{|x|} \int_{0}^{\pi} e^{-t g_1(\xi)} g_2(\xi) d\xi
$$

$$
\leq \frac{ct}{|x|} \int_{0}^{\pi} e^{-\alpha ts_n(\xi)} |\xi|^\epsilon d\xi
$$

$$
\leq \frac{ct}{|x|^{1+\epsilon}}.
$$

(3.2)

Since $\frac{1}{|x|} \leq 1$, we have

$$
\frac{1}{|x|^2} G(\frac{t}{x}) \leq \frac{1}{|x|^2} e^{-\alpha ts_n(\xi)} (t^2 |x|^{2\epsilon - 2} + t |x|^{\epsilon - 2})
$$

$$
\leq ct(t + 1) \frac{1}{|x|^2} |\xi|^{\epsilon - 2}
$$

$$
\leq ct(t + 1) \frac{1}{|x|^{1+\epsilon}}
$$

(3.3)

and

$$
\int_{0}^{\pi} G(\xi) \xi^2 d\xi \leq \int_{0}^{\pi} \xi^2 d\xi = ct(t + 1) \frac{1}{|x|^{1+\epsilon}}.
$$

(3.5)

Hence, by (3.2), (3.3), (3.4), (3.5) and Lemma 2.1 Theorem 1.1 holds for $|x| \geq 1$.

(1) In the case of $|x| \leq 1$.

Note that for $|x| \leq 1$, taking $a = \frac{\pi}{|x|}$ and $\alpha = -2$ in (3) of Lemma 2.2 we have

$$
\frac{1}{|x|^2} \int_{\mathbb{R}} G(\xi) d\xi \leq \frac{ct}{|x|} \int_{\mathbb{R}} t(t + 1) e^{-\alpha ts_n(\xi)} s_{n-1}^\epsilon(\xi) r_{n-1}(\xi) (1 + |\xi|)^{-2} d\xi
$$

$$
\leq \frac{ct}{|x|} \int_{\mathbb{R}} t(t + 1) e^{-\alpha ts_n(\xi)} s_{n-1}^\epsilon(\xi) r_{n-1}(\xi) (1 + |\xi|)^{-1}
$$

$$
\leq \frac{ct}{|x|} t(t + 1) e^{-\alpha ts_n(\xi)} s_{n-1}^\epsilon(\xi) r_{n-1}(\xi).
$$

(3.6)

with $c(t) \leq ct$ for $t \leq 1$. By (1) of Lemma 2.2 we have

$$
\int_{0}^{\pi} G(\xi) \xi^2 d\xi \leq ct(t + 1) \int_{0}^{\pi} |\xi|^{-2} d\xi + ct(t + 1) \int_{1}^{\pi} e^{-\alpha ts_n(\xi)} s_{n-1}^\epsilon(\xi) r_{n-1}(\xi) d\xi
$$

$$
\leq ct(t + 1) e^{-\alpha ts_n(\xi)} s_{n-1}^\epsilon(\xi) r_{n-1}(\xi).
$$

(3.7)
Moreover, if \( t \leq 1 \), then by (1) of Lemma 2.2, we have
\[
\int_0^{\pi} G(\xi)^2 d\xi \leq c(t+1) \int_0^{a_0} |\xi|^{r} d\xi + c(t+1) \int_0^{\pi} e^{-\alpha t \xi} (\xi) s_n^{r-1}(\xi) d\xi
\]
\[
\leq c(t) \frac{1}{|x|} e^{-t\alpha s_n^r(\xi)} s_n^{r-1}(\frac{\xi}{x}) r_n^{-1}(\frac{\xi}{x}),
\]
where \( a_0 \) is a constant defined in Lemma 2.2. By direct calculation, we get
\[
\frac{1}{|x|^3} G(\frac{1}{x}) \leq c(t+1) \frac{1}{|x|} e^{-\alpha t \xi} (\xi) s_n^{r-1}(\frac{\xi}{x}) r_n^{-1}(\frac{\xi}{x}).
\]
By (2) of Lemma 2.2, we have
\[
\frac{t}{|x|} \int_0^{\frac{\pi}{2}} e^{-t \xi^2} g_2(x) d\xi \leq c(t+1) \frac{1}{|x|} \int_0^{\pi} |\xi|^{r} d\xi + c(t+1) \int_1^{\pi} e^{-\alpha t \xi} (\xi) s_n^{r-1}(\xi) d\xi
\]
\[
\leq c(t) \frac{1}{|x|} t(t+1) e^{-\alpha t \xi} (\xi) s_n^{r-1}(\frac{\xi}{x}) r_n^{-1}(\frac{\xi}{x}).
\]
Hence, from (3.6) to (3.10), and Lemma 2.1 Theorem 1.1 holds for \(|x| \leq 1\). \( \square \)

**Remark 3.1.** Note that \( p_1 \in L^1(\mathbb{R}) \). In fact, using the change of variables \((x = y^{-1})\), we get
\[
\int_{|x| < 1} \frac{1}{|x|} e^{-\alpha t \xi} (\xi) s_n^{r-1}(\frac{\xi}{x}) r_n^{-1}(\frac{\xi}{x}) dx = 2 \int_0^1 \frac{1}{x} e^{-\alpha t x} (\xi) s_n^{r-1}(\frac{\xi}{x}) r_n^{-1}(\frac{\xi}{x}) dx
\]
\[
= 2 \int_0^\infty \frac{1}{x} e^{-\alpha t x} (x) s_n^{r-1}(x) r_n^{-1}(x) dx.
\]
Use the change of variables \((\ln(x+1) = y)\) again, we get
\[
\leq c \int_{\ln 2}^\infty \frac{1}{x} e^{-\alpha t (\ln x+1)^r} (\ln (x+1))^{-1} dx
\]
\[
\leq c \int_{s_n^{-1}(1)}^\infty e^{-\alpha t x^r} x^{-1} dx
\]
\[
< \infty.
\]

4. **Proof of Theorem 1.2**

Since \( \eta = \eta_1 \) and \( \eta \) is symmetric, \( \int e^{-t\eta} \sin(x\xi)d\xi = 0 \). Hence, the inverse Fourier transform of \( f := e^{-t\eta} \) is real and
\[
\int e^{-t\eta} e^{-ix\xi} d\xi = \int e^{-t\eta} \cos(x\xi) d\xi
\]
\[
= \frac{1}{|x|} \int e^{-t\eta} \cos \xi d\xi
\]
\[
= \sum_{-\infty<k<\infty} \frac{1}{|x|} \int_{2k\pi}^{2(k+1)\pi} f\left(\frac{\xi}{x}\right) \cos \xi d\xi.
\]
Hence, we have
\[ \epsilon^2 \alpha \left| \sum_{n=1}^{k-1} (1 + |\xi|)^{-2} \right| \geq 1, \]
where \( \epsilon \) is a positive constant such that \( g \) is decreasing. Then, for \( k \in \mathbb{Z} \), as the proof of Lemma 2.1 there are \( \epsilon_1 \in (\xi, \pi - \xi) \), \( \epsilon_2 \in (\pi + \xi, 2\pi - \xi) \) and \( \epsilon_3 \in (\epsilon_1, \epsilon_2) \) such that
\[
\int_{\mathbb{R}} e^{-t\eta(|\xi|)} e^{-ix\xi} d\xi = \frac{1}{|x|^2} \int_{2k\pi}^{2(k+1)\pi} f\left(\frac{\xi}{x}\right) \cos \xi d\xi.
\]
Hence, we have
\[
\int_{\mathbb{R}} e^{-t\eta(|\xi|)} e^{-ix\xi} d\xi \geq c \frac{1}{|x|^2} \int_{\pi \pi} g(y) dy.
\]
Hence, for \( |x| \geq 1 \), we have
\[
\int_{\mathbb{R}} e^{-t\eta(|\xi|)} e^{-ix\xi} d\xi \geq c \frac{t}{|x|^2} \int_{1}^{\infty} e^{-t\alpha \frac{s_n}{s_n} (\xi)} s_n^{-1} \left( \frac{\eta}{\alpha} \right)^{-2} d\xi + c \frac{t}{|x|^2} \int_{1}^{\infty} e^{-\epsilon t \frac{a}{a} \ln(1 + |x|)}, \quad \epsilon = 1.
\]
Applying (3) of Lemma 2.2 for \( \alpha = -2 \) and \( a = \frac{1}{\pi \pi} \) for \( |x| \leq 1 \), we have
\[
\int_{\mathbb{R}} e^{-t\eta(|\xi|)} e^{-ix\xi} d\xi \geq c(t) \frac{1}{|x|^2} e^{-t\alpha \frac{s_n}{s_n} (\xi)} s_n^{-1} \left( \frac{\eta}{\alpha} \right)^{-2} d\xi.
\]
5. **Examples**

In this section, we show that the Lévy symbol induced from the Laplacian exponent \( \psi^{e,n}(\xi) = (s_n(\xi))^e \), \( 0 < e \leq 1 \), \( 1 \leq n \) satisfies assumption (1.1). We also show that \( \psi^{e,n} \) satisfies assumption (1.2).
1. We show that the Lévy symbol induced from the Laplacian exponent $\psi^{\epsilon,n}$ satisfies assumption (1.1).

Clearly, $\psi^{\epsilon,n}$ is a Bernstein function and so $\psi^{\epsilon,n}$ is a Laplace exponent of some subordinators. First, we consider in the case of $\epsilon = 1$ and $n = 1$. Let $\psi_{1,1} := \psi^1$. By (1.6), we have

$$
\eta_1^1(\xi) = \psi^1(-i\xi) = \ln(1 - i\xi) = \frac{1}{2}\ln(1 + \xi^2) - iTan^{-1}\xi.
$$

Hence, we have

$$
\eta_1^1(\xi) = \frac{1}{2}\ln(1 + \xi^2),
$$

$$
\eta_2^1(\xi) = Tan^{-1}\xi.
$$

It is easy to show that $\eta_1^1$ and $\eta_2^1$ satisfy the assumption in Theorem 1.1 for $n = 1$. Using the mathematical induction. Suppose that $s_n(\xi)$ satisfies the assumption of Theorem 1.1. Note that

$$
\eta_{n+1}^1(\xi) = \psi_{n+1}^1(-i\xi) = \ln(1 + s_n(-i\xi)) = \ln(1 + \eta_1^n(\xi) - i\eta_2^n(\xi)) = \frac{1}{2}\ln((1 + \eta_1^n(\xi))^2 + (\eta_2^n(\xi))^2) - iTan^{-1}\frac{\eta_2^n(\xi)}{1 + \eta_1^n(\xi)}.
$$

Hence, we get

$$
\eta_{n+1}^1(\xi) = \frac{1}{2}\ln\left((1 + \eta_1^n(\xi))^2 + (\eta_2^n(\xi))^2\right),
$$

$$
\eta_{n+1}^2(\xi) = Tan^{-1}\frac{\eta_2^n(\xi)}{1 + \eta_1^n(\xi)}.
$$

Under the assumption that $\eta_1^n$ and $\eta_2^n$ satisfies (1.1), it is easy to show that $\eta_{n+1}^1$ and $\eta_{n+1}^2$ satisfy (1.1). Hence, by mathematical induction, $s_n$ satisfies (1.1) for all $n \geq 1$.

Next, let $0 < \epsilon < 1$. By (1.6), we have

$$
\eta_{\epsilon}^{n+1}(\xi) = (s_n(-i\xi))^\epsilon = \ln(1 + s_{n-1}(-i\xi))^\epsilon = \ln(1 + \eta_1^n(\xi) - i\eta_2^n(\xi))^\epsilon = \frac{1}{2}\ln\left((1 + \eta_1^n(\xi))^2 + (\eta_2^n(\xi))^2\right) - iTan^{-1}\frac{\eta_2^n(\xi)}{1 + \eta_1^n(\xi)})^\epsilon = e^{\frac{1}{2}\ln\left((1 + \eta_1^n(\xi))^2 + (\eta_2^n(\xi))^2\right) - iTan^{-1}\frac{\eta_2^n(\xi)}{1 + \eta_1^n(\xi))}}^\epsilon = e\eta_{\epsilon}^{n+1}(\xi)\left(\frac{Tan^{-1}\frac{\eta_2^n(\xi)}{1 + \eta_1^n(\xi))}}{\frac{Tan^{-1}\frac{\eta_2^n(\xi)}{1 + \eta_1^n(\xi))}}\right)\left(Tan^{-1}\frac{\eta_2^n(\xi)}{1 + \eta_1^n(\xi)}\right)^\epsilon.
$$
Hence, we get

\[
\eta_{n1}^{n+1}(\xi) = \left(\frac{1}{2} \ln \left( (1 + \eta_1^n(\xi))^2 + (\eta_2^n(\xi))^2 \right) \right)^{\frac{1}{2}} \tan^{-1} \left( \frac{\eta_2^n(\xi)}{(1 + \eta_1^n(\xi))} \right) \right)^{\frac{1}{2}} \cos \tan^{-1} \left( \frac{\eta_2^n(\xi)}{(1 + \eta_1^n(\xi))} \right),
\]

\[
\eta_{n2}^{n+1}(\xi) = \left(\frac{1}{2} \ln \left( (1 + \eta_1^n(\xi))^2 + (\eta_2^n(\xi))^2 \right) \right)^{\frac{1}{2}} \tan^{-1} \left( \frac{\eta_2^n(\xi)}{(1 + \eta_1^n(\xi))} \right) \right)^{\frac{1}{2}} \sin \tan^{-1} \left( \frac{\eta_2^n(\xi)}{(1 + \eta_1^n(\xi))} \right).
\]

Since \( \eta_1^n(\xi) \) and \( \eta_2^n(\xi) \) satisfy (1.1), it is easy to show that \( \eta_{n1}^n(\xi) \) and \( \eta_{n2}^n(\xi) \) satisfy (1.1).

(2) Second, we show that \( \psi^{\varepsilon,n}(\xi) = (s_n(\xi))^\varepsilon \) satisfies assumption (1.2).

First, we consider the case of \( \varepsilon = 1 \). By direct calculus, we have

\[
(\psi^{1,n}(\xi))' = s_n'(\xi) = A_{n-1}(\xi), \quad (\psi^{1,n}(\xi))'' = s_n''(\xi) = - \sum_{1 \leq k \leq n-1} B_k(\xi),
\]

where for \( 1 \leq k \leq n-1 \),

\[
A_{n-1}(\xi) : = (1 + s_{n-1}(\xi))^{-1}(1 + s_{n-2}(\xi))^{-1}\cdots(1 + s_1(\xi))^{-1}(1 + \xi)^{-1},
\]

\[
B_k(\xi) : = A_{n-1}(\xi)A_k(\xi), \quad 1 \leq k \leq n-1,
\]

\[
B_0(\xi) : = A_{n-1}(\xi)(1 + \xi)^{-1}.
\]

Since \( B_k(\xi) > 0 \) for all \( 0 \leq k \leq n-1 \), we have

\[
-(\psi^{1,n}(\xi))'' = \sum_{1 \leq k \leq n-1} B_k(\xi) > B_0(\xi) \geq c\varepsilon_{n-1}(\xi)(1 + \xi)^{-2}.
\]

Next, we consider the case of \( 0 < \varepsilon < 1 \). By direct calculus, we have

\[
-(\psi^{\varepsilon,n}(\xi))'' = -((\varepsilon - 1)s_{n-2}^{\varepsilon-2}(\xi)A_{n-1}^2(\xi) + s_{n-1}^{\varepsilon-1}(\xi) \sum_{1 \leq k \leq n-1} B_k(\xi)
\]

\[
\geq s_{n-1}^{\varepsilon-1}(\xi)B_0(\xi)
\]

\[
\geq c\varepsilon_{n-1}(\xi)r_{n-1}(\xi)(1 + \xi)^{-2}.
\]

\[\square\]

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