Fundamental differences between dropout and weight decay for deep networks

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Abstract
We analyze dropout in deep networks with rectified linear units and the quadratic loss. Our results expose surprising differences between the behavior of dropout and more traditional regularizers like weight decay. For example, on some simple data sets dropout training produces negative weights even though the output is the sum of the inputs. This provides a counterpoint to the suggestion that dropout discourages co-adaptation of weights. We also show that the dropout penalty can grow exponentially in the depth of the network while the weight-decay penalty remains essentially linear, and that dropout is insensitive to various re-scalings of the input features, outputs, and network weights. This last insensitivity implies that there are no isolated local minima of the dropout training criterion. Our work uncovers new properties of dropout, extends our understanding of why dropout succeeds, and lays the foundation for further progress.

1 Introduction
The 2012 ImageNet Large Scale Visual Recognition challenge was won by the University of Toronto team by a surprisingly large margin. In an invited talk at NIPS, Hinton [15] credited the dropout training technique for much of their success. Dropout training is a variant of stochastic gradient descent (SGD) where, as each example is processed, the network is temporarily perturbed by randomly “dropping out” nodes of the network. The gradient calculation and weight updates are performed on the reduced network, and the dropped out nodes are then restored before the next SGD iteration. Since the ImageNet competition, dropout has been successfully applied to a variety of domains [8, 10, 9, 17, 7]; for example, it is incorporated into popular packages such as Torch [25], Caffe [6] and TensorFlow [24]. It is intriguing that crippling the network during training often leads to such dramatically improved results, and dropout has also sparked substantial research on related methods (e.g. [12, 29]).

In this work, we examine the effect of dropout on the inductive bias of the learning algorithm. A match between dropout’s inductive bias and some important applications could explain the success of dropout, and its popularity also motivates the study of its unusual properties.

Weight decay training optimizes the empirical error plus an $L_2$ regularization term, $\frac{1}{2}\|w\|_2^2$, so we call $\frac{1}{2}\|w\|_2^2$ the $L_2$ penalty of $w$ since it is the difference between training criterion evaluated at $w$ and the empirical loss of $w$. By analogy, we define the dropout penalty of $w$ to be the difference between the dropout training criterion and the empirical loss of $w$ (see Section 2). Dropout penalties measure how much dropout discriminates against weight vectors, and thus intuitively encode the inductive bias of dropout training.
Since the dropout penalty is radically non-convex even in 1-layer networks\cite{14}, conclusions drawn from (typically quadratic) approximations can be misleading. Therefore we focus on exact formal analysis of dropout in multi-layer networks to uncover important properties of the method. Of course, both dropout and deep networks are complicated beasts, and a comprehensive analysis of their combination is a truly herculean task. In this paper we further the process by exposing some of the surprising ways that dropout training differs from $L_2$ and other standard regularizers. These include:

- The dropout penalty grows exponentially in the depth of the network in cases where the $L_2$ regularizer grows linearly. This may enable dropout to penalize the complexity of the network in a way that more meaningfully reflects the richness of the network’s behaviors. (The exponential growth with $d$ of the dropout penalty is reminiscent of some regularizers for deep networks studied by Neyshabur, et al \cite{20}.)

- Unlike weight decay and other $p$-norm regularizers, dropout training is insensitive to the rescaling of input features, and largely insensitive to rescaling of the outputs; this may play a role in dropout’s practical success. Dropout is also unaffected if the weights in one layer are scaled up by a constant $c$, and the weights of another layer are scaled down by $c$; this implies that dropout training does not have isolated local minima.

- We show that dropout training can lead to negative weights even when the output is a positive multiple of the inputs. Arguably, such use of negative weights constitutes co-adaptation – this adds a counterpoint to previous analyses showing that dropout discourages co-adaptation\cite{22,14}.

- Dropout in deep networks has a variety of other behaviors different from standard regularizers. In particular: the dropout penalty for a set of weights can be negative; the dropout penalty of a set of weights depends on both the training instances and the labels; and although the dropout probability intuitively measures the strength of dropout regularization, the dropout penalties are often non-monotonic in the dropout probability. In contrast, Wager, et al \cite{28} show that when dropout is applied to Generalized Linear Models, the dropout penalty is always non-negative and does not depend on the labels.

Our analysis is for multilayer neural networks with the square loss at the output node. The hidden layers use the popular rectified linear units\cite{19} outputting $\sigma(a) = \max(0, a)$ where $a$ is the node’s activation (the weighted sum of its inputs). To keep the analysis as simple as possible, we omit explicit bias inputs. We study the minimizers of a criterion that may be viewed as the objective function when using dropout. This abstracts away sampling and optimization issues to focus on the inductive bias, as in \cite{5,30,4,18,14}. See Section 2 for a complete explanation.

Related work

A number of possible explanations have been suggested for dropout’s success. Hinton, et al \cite{16} suggest that dropout controls network complexity by restricting the ability to co-adapt weights and illustrate how it appears to learn simpler functions at the second layer. Others \cite{3,1} view dropout as an ensemble method combining the different network topologies resulting from the random deletion of nodes. Wager, et al \cite{27} observe that in 1-layer networks dropout essentially forces learning on a more challenging distribution akin to ‘altitude training’ of athletes.

Previous formal analysis of the inductive bias of dropout has concentrated on the single-layer setting, where a single neuron combines the (potentially dropped-out) inputs. Wager, et al \cite{28} considered the case that the distribution of label $y$ given feature vector $x$ is a member of the exponential family, and the log-loss is used to evaluate models. They pointed out that, in this situation, the criterion optimized by dropout can be decomposed into the original loss and a term that does not depend on the labels. They then gave approximations to this dropout regularizer and discussed its relationship with other regularizers. As we have seen, many aspects of the behavior of dropout and its relationship to other regularizers are qualitatively different when there are hidden units.

Wager, et al \cite{27} considered dropout for learning topics modeled by a Poisson generative process. They exploited the conditional independence assumptions of the generative process to show that the excess risk of
dropout training due to training set variation has a term that decays more rapidly than the straightforward empirical risk minimization, but also has a second additive term related to document length. They also discussed situations where the model learned by dropout has small bias.

Baldi and Sadowski [2] considered dropout in linear networks, and showed how dropout can be approximated by normalized geometric means of subnetworks in the nonlinear case. Gal and Ghahramani [11] describe an interpretation of dropout as an approximation to a deep Gaussian process.

The impact of dropout (and its relative dropconnect) on generalization (roughly, how much dropout restricts the search space of the learner) was studied in [29]. In the on-line learning with experts setting, [26] showed that applying dropout in the on-line trials leads to algorithms that automatically adapt to the input sequence without requiring doubling or other parameter-tuning techniques.

## 2 Preliminaries

Throughout, we will analyze fully connected layered networks with $K$ inputs, one output, $d$ layers (counting the inputs, but not the output), and $n$ nodes in each hidden layer. We assume that $n$ is a positive multiple of $K$ to avoid floor/ceiling clutter in the analysis. We will call this the standard architecture. We use $W$ to denote a particular setting of the weights in the network and $W(x)$ to denote the network’s output on input $x$ using weights $W$. The hidden nodes are ReLUs, and the output node is linear. $W$ can be decomposed as $(W_1, ..., W_{d-1}, w)$, where each $W_j$ is the matrix of weights on connections into the $j$ hidden layer, and $w$ are the weights into the output node.

We will refer to a joint probability distribution over examples $(x, y)$ as an example distribution. We focus on square loss, so the loss of $W$ on example $(x, y)$ is $(W(x) - y)^2$. The risk is the expected loss with respect to an example distribution $P$, we denote the risk of $W$ as $R_P(W) \overset{def}{=} E_{(x,y) \sim P} ((W(x) - y)^2)$. The subscript will often be omitted when $P$ is clear from the context.

The goal of $L_2$ training is to find a weight vector minimizing the $L_2$ criterion, with regularization strength $\lambda$: $J_2(W) \overset{def}{=} R(W) + \lambda \|W\|^2$. We use $W_{L_2}$ to denote a minimizer of this criterion. The $L_2$ penalty, $\lambda \|W\|^2$, is non-negative. This is useful, for example, to bound the risk of a minimizer $W_{L_2}$ of $J_2$, since $R(W) \leq J_2(W)$.

Dropout training independently removes nodes in the network. In our analysis each non-output node is dropped out with the same probability $q$, so $p = 1 - q$ is the probability that a node is kept. (The output node is always kept; dropping it out has the effect of cancelling the training iteration.) When a node is dropped out, the node’s output is set to 0. To compensate for this reduction, the values of the kept nodes are multiplied by $1/p$. With this compensation, the dropout can be viewed as injecting zero-mean additive noise at each non-output node [28].

The dropout process is the collection of random choices, for each node in the network, of whether the node is kept or dropped out. A realization of the dropout process is a dropout pattern, which is a boolean vector indicating the kept nodes. For a network $W$, an input $x$, and dropout pattern $\hat{R}$, let $D(W, x, \hat{R})$ be the output of $W$ when nodes are dropped out or not following $\hat{R}$ (including the $1/p$ rescaling of kept nodes’ outputs). The goal of dropout training on an example distribution $P$ is to find a weight vector minimizing the dropout criterion for a given dropout probability:

$$J_D(W) \overset{def}{=} E_{\hat{R}} E_{(x,y) \sim P} ((D(W, x, \hat{R}) - y)^2).$$

This criterion is equivalent to the expected risk of the dropout-modified network, and we use $W_D$ to denote a minimizer of it. Since the selection of dropout pattern and example from $P$ are independent, the order of the two expectations can be swapped, yielding $J_D(W) = E_{(x,y) \sim P} E_{\hat{R}} ((D(W, x, \hat{R}) - y)^2)$. This motivates the study of the dropout criterion on individual examples as the dropout criterion for $P$ is just the expectation of criteria for single examples.

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[1] Some authors use a similar adjustment where the weights are scaled down at prediction time instead of inflating the kept nodes’ outputs at training time.
Consider now the example in Figure 1. The weight parameter \( \mathcal{W} \) is the all-1 vector. \( \mathcal{W}(1, -1) = 0 \) as each hidden node computes 0. Each dropout pattern indicates the subset of the four lower nodes to be kept, and when \( q = p = 1/2 \) each subset is equally likely to be kept. If \( \mathcal{R} \) is the dropout pattern where input \( x_2 \) is dropped and the other nodes are kept, then the network computes \( \mathcal{D}(\mathcal{W}, (1, -1), \mathcal{R}) = 8 \) (recall that when \( p = 1/2 \) the values of non-dropped out nodes are doubled). Only three dropout patterns produce a non-zero output, so if \( P \) is concentrated on the example \( x = (1, -1), y = 8 \) the dropout criterion is:

\[
J_D(\mathcal{W}) = \frac{1}{16}(8 - 8)^2 + \frac{2}{16}(4 - 8)^2 + \frac{13}{16}(0 - 8)^2 = 54.
\]  

As mentioned in the introduction, the dropout penalty of a weight vector for a given example distribution and dropout probability is the amount that the dropout criterion exceeds the risk, \( J_D(\mathcal{W}) - R(\mathcal{W}) \). Wager, et al [28] show that for 1-layer generalized linear models, the dropout penalty is non-negative.

Since \( \mathcal{W}(1, -1) = 0 \), we have \( R(\mathcal{W}) = 64 \), and the dropout penalty is negative in our example. In Section 6 we give a necessary condition for this.

**Definition 1** Define \( P_{(x, y)} \) as the distribution concentrating all of its weight on example \((x, y)\).

Unless we indicate otherwise, we use \( p = q = 1/2 \).

### 3 Scaling inputs, weights and outputs

#### 3.1 Dropout is scale-free

Here we prove that dropout regularizes deep networks in a manner that is independent of the scale of the input features, by freely using large weights when needed to scale up input values.

**Definition 2** For any example distribution \( P \), define the dropout aversion with \( P \) to be the maximum, over minimizers \( \mathcal{W}_D \) of the dropout risk \( J_D(\mathcal{W}_D) \), of \( R_P(\mathcal{W}_D) - \inf_{\mathcal{W}} R_P(\mathcal{W}) \).

The dropout aversion of \( P \) measures the extent to which \( P \) is incompatible with the inductive bias of dropout: when dropout is used, how much can this increase the risk?

**Definition 3** For example distribution \( P \) and square matrix \( A \), denote by \( A \circ P \) the distribution obtained by sampling \((x, y)\) from \( P \), and outputting \((Ax, y)\).

When \( A \) is diagonal and has full rank, we can think of \( A \circ P \) as changing the units of some inputs.

**Theorem 4** For any example distribution \( P \), and any diagonal full-rank \( K \times K \) matrix \( A \), the dropout aversion of \( P \) equals the dropout aversion of \( A \circ P \).

**Proof:** Choose a network \( \mathcal{W} = (W_1, ..., W_{d-1}, w) \). Let \( \mathcal{W}' = (W_1A^{-1}, ..., W_{d-1}, w) \). For any \( x, \mathcal{W}(x) = \mathcal{W}'(Ax) \), as \( A^{-1} \) undoes the effect of \( A \) before it gets to the rest of the network, which is unchanged. Furthermore, for any dropout pattern \( \mathcal{R} \), we have \( \mathcal{D}(\mathcal{W}, x, \mathcal{R}) = \mathcal{D}(\mathcal{W}', Ax, \mathcal{R}) \). Once again \( A^{-1} \) undoes the effects of \( A \) on kept nodes (since \( A \) is diagonal), and the rest of the network \( \mathcal{W}' \) is modified by \( \mathcal{R} \) in a manner paralleling \( \mathcal{W} \). Thus, there is bijection between networks \( \mathcal{W} \) and networks \( \mathcal{W}' \) with \( J_D(\mathcal{W}') = J_D(\mathcal{W}) \) and \( R(\mathcal{W}') = R(\mathcal{W}) \), yielding the theorem.

As we will see in Section 3.3 weight decay does not enjoy such scale-free status.
3.2 Dropout’s invariance to parameter scaling

Next, we describe an equivalence relation among parameterizations for dropout networks of depth $d \geq 2$. (A similar observation was made in a somewhat different context by Neyshabur, et al [20].)

**Theorem 5** For any input $\mathbf{x}$, dropout pattern $\mathcal{R}$, any network $\mathcal{W} = (W_1, \ldots, W_{d-1}, w)$, and any positive $c_1, \ldots, c_d$, if $\mathcal{W}' = (c_1W_1, \ldots, c_{d-1}W_{d-1}, c_d w)$, then $\mathcal{D}(\mathcal{W}', \mathbf{x}, \mathcal{R}) = (\prod_{j=1}^{d} c_j) \mathcal{D}(\mathcal{W}, \mathbf{x}, \mathcal{R})$. In particular, if $\prod_{j=1}^{d} c_j = 1$, then for any example distribution $P$, networks $\mathcal{W}$ and $\mathcal{W}'$ have the same dropout criterion, dropout penalty, and expected loss.

**Proof:** Choose an input $\mathbf{x}$ and a dropout pattern $\mathcal{R}$. Let $\mathcal{W}' = (c_1W_1, \ldots, c_{d-1}W_{d-1}, c_d w)$. For each hidden layer $j$, let $(h_{j1}, \ldots, h_{jn})$ be the $j$th hidden layer when applying $\mathcal{W}$ to $\mathbf{x}$ with $\mathcal{R}$, and let $(\hat{h}_{j1}, \ldots, \hat{h}_{jn})$ be the $j$th hidden layer when applying $\mathcal{W}'$ instead. By induction, for all $i$, $\hat{h}_{ji} = (\prod_{\ell \leq j} c_{\ell}) h_{ji}$; the key step is that the pre-rectified value used to compute $\hat{h}_{ji}$ has the same sign as for $h_{ji}$, since rescaling by $c_j$ preserves the sign. The same units are zeroed out in $\mathcal{W}$ and $\mathcal{W}'$ and $\mathcal{D}(\mathcal{W}, \mathbf{x}, \mathcal{R}) = (\prod_{j} c_j) \mathcal{D}(\mathcal{W}', \mathbf{x}, \mathcal{R})$. When $\prod_{j} c_j = 1$, this implies $\mathcal{D}(\mathcal{W}, \mathbf{x}, \mathcal{R}) = \mathcal{D}(\mathcal{W}', \mathbf{x}, \mathcal{R})$. Since this is true for all $\mathbf{x}$ and $\mathcal{R}$, we have $J_{\mathcal{D}}(\mathcal{W}) = J_{\mathcal{D}}(\mathcal{W}')$. Since, similarly, $\mathcal{W}(\mathbf{x}) = \mathcal{W}'(\mathbf{x})$ for all $\mathbf{x}$, so $R(\mathcal{W}) = R(\mathcal{W}')$, the dropout penalties for $\mathcal{W}$ and $\mathcal{W}'$ are also the same.

Theorem 5 implies that the dropout criterion never has isolated minimizers. The theorem can also be generalized to ReLU networks with biases, although the re-scaling of the biases at layer $j$ depends not only on the rescaling of the connection weights at layer $j$, but also the re-scalings at lower layers.

3.3 Output scaling with dropout

Scaling the output values of an example distribution $P$ does affect the aversion, but in a very simple way.

**Theorem 6** For any example distribution $P$, if $P'$ is obtained from $P$ by scaling the outputs of $P$ by a positive constant $c$, the dropout aversion of $P'$ is $c^2$ times the dropout aversion of $P$.

**Proof:** If a network $\mathcal{W} = (W_1, \ldots, W_{d-1}, w)$ minimizes the dropout criterion for $P$, then $\mathcal{W}' = (W_1, \ldots, W_{d-1}, cw)$ minimizes the dropout criterion for $P'$, and for any $\mathbf{x}, \mathbf{y}$, and dropout pattern $\mathcal{R}$, $(\mathcal{D}(\mathcal{W}, \mathbf{x}, \mathcal{R}) - y)^2 = (\mathcal{D}(\mathcal{W}', \mathbf{x}, \mathcal{R}) - cy)^2 / c^2$.

3.4 Scaling properties of weight decay

Weight decay does not have the same scaling properties as dropout. Define the weight-decay aversion analogously to the dropout aversion.

We analyze the $L_2$ does not hold for depth-2 networks with distributions concentrated on single examples in Appendix B resulting in the following theorem.

**Theorem 7** Fix arbitrary input vector $\mathbf{x}$ in $\mathbb{R}^K$ and positive $n$, $y$, and $\lambda$. Let weights $W_{L_2}$ be any minimizer of $J_2$ for $P_{x,y}$. If $\lambda \leq 2y||\mathbf{x}||_2$ then $R_{P_{x,y}}(W_{L_2}) = \frac{\lambda^2}{4||\mathbf{x}||_2^2}$, and if $\lambda > 2y||\mathbf{x}||_2$, then $W_{L_2}$ is the degenerate all-zero weight vector so $R_{P_{x,y}}(W_{L_2}) = y^2$. Therefore the weight-decay aversion of $P_{x,y}$ is min$(y^2, \lambda^2/4||\mathbf{x}||_2^2)$.

Theorem 7 shows that, unlike dropout, the weight decay aversion does depend on the scaling of the input features (through $||\mathbf{x}||_2^2$).

Furthermore, when $\lambda > 2y||\mathbf{x}||_2$, the weight-decay criterion has only single isolated optimum – the all-zero vector. Thus is in stark contrast to 1-layer networks, where finite $\lambda$ does not regularize away all the information from the sample.

The theorem also indicates that the weight decay aversion either is more stable to rescaling the output labels (when $\lambda \leq 2y||\mathbf{x}||_2$) or changes at the same $c^2$ rate as dropout when the weights are scaled up by a factor of $c$. 
The “vertical” flexibility to rescale weights between layers enjoyed by dropout (Theorem 5) does not hold for $L_2$: one can always drive the $L_2$ penalty to infinity by scaling one layer up by a large enough positive $c$, even while scaling another down by $c$. On the other hand, the proof of Theorem 7 shows that the $L_2$ criterion has an interesting “horizontal” flexibility involving the rescaling of weights across nodes on the hidden layer. Lemma 18 shows that at the optimizers each hidden node’s contributions to the output are a constant (depending on $x$) times their contribution to the the $L_2$ penalty. The magnitude of these contributions can be transferred between hidden nodes leading to alternative weights that compute the same value and have the same weight decay penalty.

4 Growth of the dropout penalty as a function of $d$

In this section, we show that the dropout penalty can grow exponentially in the depth $d$ even when the size of individual weights remains constant.

**Theorem 8** If $x = (1, 1, ..., 1)$ and $0 \leq y \leq Kn^{d-1}$, for $P_{x,y}$ there are weights $W$ for the standard architecture with $R(W) = 0$ such that (a) every weight has magnitude at most one, but (b) $J_D(W) \geq \frac{y^2}{K+1}$, whereas (c) $J_2(W) \leq \frac{\lambda c^2}{d} (Kn + n^2(d-2) + n)$.

**Proof:** Let $W$ be the network whose weights are all $c = \frac{y^{1/d}}{K^{1/d} n^{d-1/d}}$, so the $L_2$ penalty is the number of weights times $\lambda c^2/2$. It is a simple induction to show that, for these weights and input $(1, 1, ..., 1)$, the value computed at each hidden node on level $j$ is $c^j Kn^{j-1}$, so the network outputs $c^d Kn^{d-1}$, and has zero square loss (since $W(x) = c^d Kn^{d-1} = y$).

Consider now dropout on this network. This is equivalent to changing all of the weights from $c$ to $2c$ and, independently with probability $1/2$, replacing the value of each node with 0. For a fixed dropout pattern, each node on a given layer has the same weights, and receives the same (kept) inputs. Thus, the value computed at every node on the same layer is the same. For each $j$, let $H_j$ be the value computed by the units in the $j$th hidden layer.

If $k_0$ is the number of input nodes kept under dropout, and, for each $j \in \{1, ..., d-1\}$, $k_j$ is the number of hidden nodes kept in layer $j$, a straightforward induction shows that, for all $\ell$, we have $H_\ell = (2c)^\ell \prod_{j=0}^{\ell-1} k_j$, so that the output $\hat{y}$ of the network is $(2c)^d \prod_{j=0}^{d-1} k_j$.

Using a bias-variance decomposition, $E((\hat{y} - y)^2) = E[\hat{y} - y]^2 + \text{Var}(\hat{y})$. Since each $k_j$ is binomially distributed, and $k_0, ..., k_{d-1}$ are independent, we have $E(\hat{y}) = (2c)^d(K/2)(n/2)^{d-1} = c^d Kn^{d-1} = y$, so $E((\hat{y} - y)^2) = \text{Var}(\hat{y})$. Since $E(\hat{y}^2) = (2c)^{2d}(K(K+1)/4)(n(n+1)/4)^{d-1} = y^2(1 + 1/K)(1 + 1/n)^{d-1}$, we have $\text{Var}(\hat{y}) = E(\hat{y}^2) - E(\hat{y})^2 = y^2(1 + 1/K)(1 + 1/n)^{d-1} - 1 \geq y^2/K$, completing the proof.

If $y = \exp(\Theta(d))$, the dropout penalty grows exponentially in $d$, whereas the $L_2$ penalty grows polynomially.

5 Dropout uses negative weights

In this section we exploit the variance of the dropout process to show the following theorem.

**Theorem 9** For the standard architecture, if $n > 2K^2$, $K \geq d$, $K \geq 6$, $x$ is all one’s, and $y$ is positive then every optimizer of the dropout criterion for $P_{(x,y)}$ uses negative weights.

To prove Theorem 9 first, we prove an upper bound on $J_D(W_{neg})$ for a network $W_{neg}$ using negative weights, and then we prove a lower bound that holds for all networks using only non-negative weights.

A key building block in the definition of $W_{neg}$ is a block of hidden units that we call the first-one gadget. Each such block has $K$ hidden nodes, and takes its input from the $K$ input nodes. The $i$th hidden node in the block takes the value 1 if the $K$th input node is 1, and all inputs $x_{i'}$ for $i' < i$ are 0. This can be
accomplished with a weight vector $w$ with $w_i = -1$ for $i' < i$, with $w_i = 1$, and with $w_{i'} = 0$ for $i' > i$. The first hidden layer of $\mathcal{W}_{\text{neg}}$ comprises $n/K$ copies of the first-one gadget.

Informally, this construction removes most of the variance in the number of 1’s in the input, as recorded in the following lemma.

**Lemma 10** On any input $x \in \{0, 1\}^n$ except $(0, 0, \ldots, 0)$, the sum of the values on the first hidden layer of $\mathcal{W}_{\text{neg}}$ is exactly $n/K$.

The weights into the remaining hidden layers of $\mathcal{W}_{\text{neg}}$ are all 1, and all the weights into the output layer take a value $c = \frac{K^2}{2n^{d-1}(1+\frac{K}{n})(1+\frac{1}{n})^{d-2}}$. The following lemma analyzes $\mathcal{W}_{\text{neg}}$.

**Lemma 11** $J_D(\mathcal{W}_{\text{neg}}) = y^2 \left(1 - \frac{(1-2^{-K})}{(1+\frac{K}{n})(1+\frac{1}{n})^{d-2}}\right)$.

**Proof:** Consider a random computation of $\mathcal{W}(1,1,\ldots,1)$ under dropout. Let $k_0$ be the number of input nodes kept, and, for each $j \geq 2$, let $k_j$ be the number of nodes in the $j$th hidden layer kept. Call the node in each first-one gadget that computes 1 a key node, and if no node in the gadget computes 1 because the input is all dropped, arbitrarily make the gadget’s first hidden node the key node. This ensures there is exactly one key node per gadget, and every non-key node computes 0. Let $k_1$ be the number of kept key nodes on the first hidden layer. If $k_0 = 0$, the output $\hat{y}$ of the network is 0. Otherwise, $\hat{y} = c^2 \prod_{j=1}^{d-1} k_j$.

Note that $k_0$ is zero with probability $2^{-K}$. Whenever $k_0 \geq 1$, $k_1$ is distributed as $B(n/K,1/2)$. Each other $k_j$ is distributed as $B(n,1/2)$, and $k_1,k_2,\ldots,k_{d-1}$ are independent of one another.

$$E(\hat{y}) = \Pr(k_0 \geq 1)c^2 \prod_{j=2}^{d-1} E[k_j] = (1-2^{-K})c^2 \left(\frac{n}{2K}\right)\left(\frac{n}{2}\right)^{d-2} = \frac{2c}{K}(1-2^{-K})n^{d-1}.$$ 

Using the value of the second moment of the binomial, we get

$$E(\hat{y}^2) = E \left[ \left( \prod_{j=1}^{d-1} k_j \right)^2 \right] = 4c^2(1-2^{-K})\left(\frac{n}{K}\right)\left(\frac{n}{n} + 1\right)^{d-2}$$

$$= \frac{4c^2(1-2^{-K})}{K^2}n^{2(d-1)}\left(1 + \frac{K}{n}\right)\left(1 + \frac{1}{n}\right)^{d-2}.$$ 

Thus,

$$R_D(\mathcal{W}_{\text{neg}}) = y^2 - 2yE(\hat{y}) + E(\hat{y}^2)$$

$$= y^2 - \frac{4c}{K}(1-2^{-K})n^{d-1} + \frac{4c^2(1-2^{-K})}{K^2}n^{2(d-1)}\left(1 + \frac{K}{n}\right)\left(1 + \frac{1}{n}\right)^{d-2}$$

$$= y^2 \left(1 - \frac{(1-2^{-K})}{(1+\frac{K}{n})(1+\frac{1}{n})^{d-2}}\right),$$

since $c = \frac{K^2}{2n^{d-1}(1+\frac{K}{n})(1+\frac{1}{n})^{d-2}}$, completing the proof.

Next we prove a lower bound for networks with nonnegative weights.

**Theorem 12** For any $W$ with all nonnegative weights and the standard architecture, $J_D(W) \geq \frac{y^2}{K+1}$.
When all the weights and inputs in a network of rectified linear units are positive, then the rectified linear weighted sum of its inputs. Therefore the network with weights \( W \) behaves linearly, and this linear behavior is preserved under dropout patterns. In particular, if \( x \) and \( x' \) have no negative components then for all dropout patterns \( R: D(W_D, x, R) + D(W_D, x', R) = D(W_D, x + x', R) \) (see [2] for additional properties of dropout in linear networks).

Let \( r_0 \in \{0, 1\}^K \) be the dropout pattern concerning the input layer, and let \( R' \) be the portion of the dropout pattern regarding the hidden layers. We have

\[
J_D(W) = E_{r_0} [E_{R'} [D(W, 1^K, (r_0, R')) - y]^2] \geq E_{r_0} [E_{R'} [D(W, 1^K, (r_0, R'))] - y]^2,
\]

by Jensen’s inequality. However, for any \( r_0, D(W, 1^K, (r_0, R')) = D(W, r_0, (1^K, R')) \), and, since the network obtained from \( W \) by dropping out according to \( R' \) is linear for non-negative inputs, this implies

\[
J_D(W) \geq E_{r_0} [E_{R'} [D(W, r_0, (1^K, R'))] - y]^2 \geq E_{r_0} [(\sum_i r_0_i E_{R'} [D(W, e_i, (1^K, R'))] - y)^2],
\]

where \( e_i \) is the \( i \)th standard basis vector. If we denote \( E_{R'} [D(W, e_i, (1^K, R'))] \) by \( \mu_i \), we get \( J_D(W) \geq E_{r_0} [(\sum_i r_0_i \mu_i - y)^2] \). The RHS is convex, and symmetric with respect to permutations on \( \mu_1, ..., \mu_K \). Thus, replacing each \( \mu_i \) with \( \mu = \frac{1}{K} \sum_{i=1}^K \mu_i \) cannot increase it. Thus,

\[
J_D(W) \geq E_{r_0} \left[ \mu \sum_i r_0_i - y \right]^2 = \mu^2 \frac{K(K+1)}{4} - 2\mu y \frac{K}{2} + y^2.
\]

This is minimized when \( \mu = \frac{2y}{K+1} \), where it takes a value \( \frac{y^2}{K+1} \), completing the proof.

Now we are ready to put everything together.

**Proof of Theorem 9** By Theorem 12, \( R_D(W_{\text{neg}}) < \frac{y^2}{K+1} \) suffices to prove Theorem 9. By Lemma 11, it suffices for \( \left( 1 - \frac{(1-2^{-K})}{(1+\frac{K}{n})(1+\frac{K}{n})^{\frac{n-2}{2}}} \right) < \frac{1}{K+1} \). The LHS is decreasing in \( n \), so, if \( n \geq 2K^2 \), then

\[
\left( 1 - \frac{(1-2^{-K})}{(1+\frac{K}{n})(1+\frac{K}{n})^{\frac{n-2}{2}}} \right) < \frac{1}{K+1} \]

suffices. Finally, if \( d \leq K \), then

\[
\left( 1 - \frac{(1-2^{-K})}{(1+\frac{K}{d})(1+\frac{K}{d})^\frac{d-2}{2}} \right) < \frac{1}{K+1} \]

suffices, and this holds for \( K \geq 6 \).

6 **A necessary condition for negative dropout penalty**

Section 2 contains an example where the dropout penalty is negative. The following theorem includes a necessary condition.

**Theorem 13** The dropout penalty can be negative. For all example distributions, a necessary condition for this in rectified linear networks is that some weights (or inputs) are negative.

**Proof:** [2] show that for networks of linear units (as opposed to the non-linear rectified linear units we focus on) the network’s output without dropout equals the expected output over dropout patterns, so in our notation: \( W(x) \) equals \( E_R(D(W, x, R)) \). Assume for the moment that the network consists of linear units and the example distribution is concentrated on the single example \( x, y \). Using the bias-variance decomposition for square loss and this property of linear networks,

\[
J_D(W) = E_R \left( (D(W, x, R) - y)^2 \right) = E_R(D(W, x, R) - y)^2 + \text{Var}_R(D(W, x, R)) \geq (W(x) - y)^2
\]

and the dropout penalty is again non-negative. Since the same calculations go through when averaging over multiple examples, we see that the dropout penalty is always non-negative for networks of linear nodes. When all the weights and inputs in a network of rectified linear units are positive, then the rectified linear units behave as linear units, so the dropout penalty will again be non-negative.
7 Multi-layer dropout penalty does depend on labels

In contrast with its behavior on a variety of linear models including logistic regression, the dropout penalty can depend on the value of the response variable in deep networks with ReLUs and the quadratic loss.

Theorem 14 There are joint distributions $P$ and $Q$, and weights $W$ such that, for all dropout probabilities $q \in (0, 1)$, (a) the marginals of $P$ and $Q$ on the input variables are equal, but (b) the dropout penalties of $W$ with respect to $P$ and $Q$ are different.

We will prove Theorem 14 by describing a general, somewhat technical, condition that implies that $P$ and $Q$ are witnesses to Theorem 14.

For each input $x$ and dropout pattern $R$, let $H(W, x, R)$ be the values presented to the output node with dropout. As before, let $w \in \mathbb{R}^n$ be those weights of $W$ on connections directly into the output node. Let $r \in \{0, 1\}^n$ be the indicator variables for whether the various nodes connecting to the output node are kept.

Proof (of Theorem 14): Suppose that $P$ is concentrated on a single $(x, y)$ pair. We will then get $Q$ by modifying $y$.

Let $h$ be the values coming into the output node in the non-dropped out network. Therefore the output of the non-dropout network is $w \cdot h$ while the output of the network with dropout is $w \cdot H(W, x, R)$. We now examine the dropout penalty, which is the expected dropout loss minus the non-dropout loss. We will use $\delta$ as a shorthand for $w \cdot (H(W, x, R) - h)$.

\[
\text{dropout penalty} = \mathbb{E} \left( (w \cdot H(W, x, R) - y)^2 \right) - (w \cdot h - y)^2
\]
\[
= \mathbb{E} \left( (w \cdot H(W, x, R) - w \cdot h + w \cdot h - y)^2 \right) - (w \cdot h - y)^2
\]
\[
= \mathbb{E} (\delta^2) + 2(w \cdot h - y)\mathbb{E}(\delta)
\]

which depends on the label $y$ unless $\mathbb{E}(\delta) = 0$.

Typically $\mathbb{E}(\delta) \neq 0$. To prove the theorem, consider the case where $x = (1, -2)$ and there are two hidden nodes with weights 1 on their connections to the output node. The value at the hidden nodes without dropout is 0, but with dropout the hidden nodes never negative and computes positive values when only the negative input is dropped, so the expectation of $\delta$ is positive.

\[\blacksquare\]

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### A Table of Notation

| Notation | Meaning |
|----------|---------|
| \( \mathbb{1}_{\text{set}} \) | indicator function for “set” |
| \((x, y)\) | an example with feature vector \( x \) and label \( y \) |
| \( \sigma(\cdot) \) | the rectified linear unit computing \( \max(0, \cdot) \) |
| \( \mathcal{W} \) | an arbitrary weight setting for the network |
| \( w, v \) | specific weights, often subscripted |
| \( W(x) \) | the output value produced by weight setting \( W \) on input \( x \) |
| \( P \) | an arbitrary source distribution over \((x, y)\) pairs |
| \( P_{x,y} \) | the source distribution concentrated on the single example \((x, y)\) |
| \( R_P(W) \) | the risk (expected square loss) of \( W \) under source \( P \) |
| \( q, p \) | probabilities that a node is dropped out \((q)\) or kept \((p)\) by the dropout process |
| \( \mathcal{R} \) | a dropout pattern, indicates the kept nodes |
| \( r, s \) | dropout patterns on subsets of the nodes |
| \( D(W, x, \mathcal{R}) \) | Output of dropout with network weights \( W \), input \( x \), and dropout pattern \( \mathcal{R} \) |
| \( J_D(W) \) | the dropout criterion |
| \( J_2(W) \) | the \( L_2 \) criterion |
| \( \lambda \) | the \( L_2 \) regularization strength parameter |
| \( \mathcal{W}_{L_2} \) | an optimizer of the dropout criterion |
| \( \mathcal{W}_{L_2} \) | an optimizer of the \( L_2 \) criterion |
| \( n, d \) | the network width and depth |
| \( K \) | the number of input nodes |

### B Proof of Theorem 7

We will focus on the standard architecture with depth \( d = 2 \) and distributions of the form \( P_{x,y} \).

Here we prove Theorem 7 showing that the weight-decay aversion depends on the values of the inputs and the number of input nodes \( K \), but, perhaps surprisingly, not on the number of hidden nodes \( n \). Furthermore, unlike the single-layer case, the \( L_2 \) regularization strength has a threshold where the minimizer of the \( L_2 \) criterion degenerates to the all-zero network.

Recall that Theorem 7 says for any minimizer \( \mathcal{W}_{L_2} \) of the weight decay criterion, if \( \lambda \leq 2y||x||_2 \) then \( R_{P_{x,y}}(\mathcal{W}_{L_2}) = \frac{\lambda^2}{4||x||_2^2} \), and if \( \lambda > 2y||x||_2 \), then \( \mathcal{W}_{L_2} \) is the degenerate all-zero weight vector so \( R_{P_{x,y}}(\mathcal{W}_{L_2}) = y^2 \).

The proof of Theorem 7 involves a series of lemmas. We first explore essential properties of the \( L_2 \) criterion minimizers and then use these properties to bound any hidden node’s effect on the output by the regularization penalty on the weights in and out of that node. This allows us to treat optimizing the \( L_2 \) criterion as a one-dimensional problem, whose solution yields the theorem.

With respect to a minimizer \( \mathcal{W}_{L_2} \), let \( v^*_j \) denote the vector of weights into hidden node \( j \) and \( w^*_j \) denote the weight from \( j \) to the output node.

Our first lemma collects together some properties of those \( \mathcal{W}_{L_2} \) minimizing the \( L_2 \) criterion.

**Lemma 15.** Any optimal \( \mathcal{W}_{L_2} = (V^*, W^*) \) satisfies the following conditions:

1. For each hidden node \( j \in \{1, \ldots, n\} \), vector \( v^*_j \) is a non-negative multiple of \( x \);
2. \( \mathcal{W}_{L_2}(\epsilon) \leq y \);
3. \( \mathcal{W}_{L_2} \) maximizes \( \mathcal{W}(\epsilon) \) among those weight vectors with \( ||W||_2 = ||W_{L_2}||_2 \);

**Proof:** Most parts are proven by contradicting the optimality of \( \mathcal{W}_{L_2} \).

(a) If \( v^*_j \) is not a non-negative multiple of \( x \) then replacing \( v^*_j \) with the zero vector (when \( v^*_j \cdot x \leq 0 \)) or the projection of \( v^*_j \) onto \( x \) (when \( v^*_j \cdot x > 0 \)) leads to a network computing the same value but with a lower \( L_2 \) penalty.

(b) If \( \mathcal{W}_{L_2}(x) > y \) then scaling down the output weights would reduce both parts of the \( L_2 \) criterion.
(c) Let \( y_{\text{max}} \) be the maximum of \( W(x) \), from among networks with \( ||W||_2 = ||W_{L_2}||_2 \). If \( y_{\text{max}} > y \), then we could improve on \( J_2(W_{L_2}) \) by scaling down the weights in the output layer of \( W \) until the resulting weights \( \tilde{W} \) had \( \tilde{W}(x) = y \). This \( \tilde{W} \) would have zero error, and a smaller norm than \( W_{L_2} \).

If \( y_{\text{max}} \leq y \), then any \( W \) with \( ||W||_2 = ||W_{L_2}||_2 \) and \( W_{L_2}(x) < W(x) \leq y \), would have smaller error but the same penalty, and therefore \( J_2(W) < J_2(W_{L_2}) \).

Informally, parts (b) and (c) of Lemma 15 engender a view of the learner straining against the yolk of the \( L_2 \) penalty to produce a large enough output on \( \epsilon \). This motivates us to ask how large \( W(\epsilon) \) can be, for a given value of \( ||W||_2 \). We can restrict our attention to networks satisfying part (a) of Lemma 15, and the following definitions capture how much each hidden node contributes to each of these quantities.

**Definition 16** For each hidden node \( j \), let \( \alpha_j \) be the constant such \( v_j = \alpha_j x \), and \( \underline{c} = x \cdot x \), so the value computed at hidden node \( j \) is \( c \alpha_j \).

**Definition 17** The contribution to the activation at the output due to hidden node \( j \) is \( c \alpha_j w_j \), and the contribution to the \( L_2 \) penalty from these weights is \( \frac{1}{2} (w_j^2 + c \alpha_j^2) \).

We now bound the contribution to the activation in terms of the contribution to the \( L_2 \) penalty. Note that as the \( L_2 \) “budget” increases, so does the maximum possible contribution to the output node’s activation.

**Lemma 18** If hidden node \( j \)’s weight-decay contribution, \( w_j^2 + c \alpha_j^2 = B \) then hidden node \( k \)’s contribution to the output node’s activation is maximized when \( w_j = \sqrt{B/2} \) and \( \alpha_j = \sqrt{B/2c} \), where it achieves the value \( B \sqrt{c}/2 \)

**Proof:** Since \( c \alpha_j^2 + w_j^2 = B \), we have \( w_j = \sqrt{B - c \alpha_j^2} \), so the contribution to the activation can be re-written as \( c \alpha_j \sqrt{B - c \alpha_j^2} \). Taking the derivative with respect to \( \alpha_j \), and solving, we get \( \alpha_j = \pm \sqrt{B/2c} \) and only the positive solution is feasible (otherwise the node outputs 0). When \( \alpha_j = \sqrt{B/2c} \) we have \( w_j = \sqrt{B/2} \) and thus the maximum contribution of the hidden node is

\[
\frac{c}{2} \sqrt{B/2c} \left( \sqrt{B/2} \right) = \frac{B \sqrt{c}}{2}.
\]

**Lemma 18** immediately implies the following.

**Lemma 19** The maximum of \( W(\epsilon) \), subject to \( ||W||_2^2 \leq A \), is \( A \sqrt{c}/2 \).

**Proof:** When maximized, the contribution of each hidden node to the activation at the output is \( \sqrt{c}/2 \) times the hidden node’s contribution to the sum of squared-weights. Since each weight in \( W \) is used in exactly one hidden node’s contribution to the output node’s activation, this completes the proof.

Note that this bound is independent of \( n \), the number of hidden units, but does depend on the input \( x \) through \( c = x \cdot x \).

**Proof (of Theorem 7):** Combining Lemma 15 and Lemma 19, \( W_{L_2} \) minimizes

\[
\left( \frac{\sqrt{c} ||W||_2^2}{2} - y \right)^2 + \frac{\lambda}{2} ||W||_2^2.
\]

Its derivative with respect to \( ||W||_2^2 \) is

\[
\sqrt{c} \left( \frac{\sqrt{c} ||W||_2^2}{2} - y \right) + \frac{\lambda}{2}.
\]
If $\sqrt{c} \left( \frac{\sqrt{c}||W||^2}{2} - y \right) + \lambda/2 > 0$ whenever $||W||^2 \geq 0$ (i.e. $\lambda > 2y\sqrt{c}$), then the $L_2$ criterion is minimized only when $||W||^2 = 0$. In this case $W_{L_2}$ is the all-zero vector and the $L_2$ criterion is $y^2$.

Otherwise, setting this derivative to 0 and solving for $||W||^2$ gives us that the minimum of the criterion occurs when
\[ ||W||^2 = \frac{2y}{\sqrt{c}} - \frac{\lambda}{c}. \]

Evaluating the risk using Lemma 19 we see for any $W_{L_2}$ minimizing the criterion in this case,
\[ R(W_{L_2}) = \left( \frac{\sqrt{c}}{2} \left( \frac{2y}{\sqrt{c}} - \frac{\lambda}{c} \right) - y \right)^2 = \left( \frac{\lambda}{2\sqrt{c}} \right)^2 = \frac{\lambda^2}{4c}. \]

Thus, overall the risk of the minimizer of the $L_2$ criterion is $\min \left\{ y^2, \frac{\lambda^2}{4c} \right\}$. \[\blacksquare\]
This is a staging document for holding latex that proves that weight decay is sensitive to scaling of the inputs.

The $L_2$ regularization parameter is fixed at an arbitrary positive value $\lambda$. We need to find $P$ and $Q$ such that the weight-decay aversion with $P$ and $Q$ are different.

If, instead, we can define a distribution parameterized by a parameter $\epsilon$ such that the aversion changes with $\epsilon$, this may be simpler and better.

The source $P_2$ in this section is concentrated on the single example $x = (\epsilon, \epsilon, ..., \epsilon)$ and $y = 1$.

We will denote a minimizer by $W_{L_2} = (W_1^*, ..., W_{d-1}^*, w^*)$.

Our first lemma collects together some properties of optimal network.

**Lemma 1** Any optimal $W_{L_2} = (W_1^*, ..., W_{d-1}^*, w^*)$ for $P_2$ satisfies the following conditions:

(a) for each hidden layer $j$, all hidden nodes compute the same value

(b) for each hidden layer $j$, all weights into the layer are

(c) $W_{L_2}$ maximizes $\mathcal{W}(\epsilon, \epsilon)$ among those weight vectors with $||W||_2 = ||W_{L_2}||_2$;

(d) all weights of $W_{L_2}$ are non-negative.

**Proof:** Most parts are proven by contradicting the optimality of $W_{L_2}$.

(a) If $v_{k1} \neq v_{k2}$, then replacing each of them with their average results in a set of weights computing the same values at all nodes, but with a smaller $L_2$ penalty.

(b) If $W_{L_2}(\epsilon, \epsilon) > 1$ then scaling down the output weights would reduce both parts of the $L_2$ criterion.

(c) Let $y_{\text{max}}$ be the maximum of $\mathcal{W}(\epsilon, \epsilon)$, from among networks with $||W||_2 = ||W_{L_2}||_2$. If $y_{\text{max}} > 1$, then we could improve on $J_2(W_{L_2})$ by scaling down the weights in the output layer of $W$ until the resulting weights $\tilde{W}$ had $\tilde{W}(\epsilon, \epsilon) = 1$. This $\tilde{W}$ would have zero error, and a smaller norm that $W_{L_2}$.

If $y_{\text{max}} \leq 1$, then any $W$ with $||W||_2 = ||W_{L_2}||_2$ and $W_{L_2}(\epsilon, \epsilon) < \mathcal{W}(\epsilon, \epsilon) \leq 1$, would have smaller error but the same penalty, and therefore $J_2(W) < J_2(W_{L_2})$.

(d) If any hidden node $k$ has a negative $v_{k1}$ or $v_{k2}$ then both are negative by (a). Replacing both weights by 0 reduces the $L_2$ penalty without changing the network’s output on $P_2$.

If any hidden node $k$ has a negative $w_{k}$ then setting $w_{k}$ to 0 reduces the $L_2$ penalty without decreasing the network’s output. Scaling up the output weights so that the modified weights $\tilde{W}$ have $||\tilde{W}||_2 = ||W_{L_2}||_2$ will then increase the network’s output, contradicting (c).

This allows us to restrict our attention to network weights $W$ where for each $k$ we have $v_{k1} = v_{k2}$. We use $v_{k1}$ to denote the shared value of these weights.

Informally, parts (b) and (c) of Lemma 1 engender a view of the learner straining against the yolk of the $L_2$ penalty to produce a large enough output on $(\epsilon, \epsilon)$. This motivates us to ask how large $\mathcal{W}(\epsilon, \epsilon)$ can be, for a given value of $||W||_2^2$. The following definition captures how much each hidden node contributes to each of these quantities.

**Definition 1** The contribution of hidden node $k$ to the activation at the output node is $2\epsilon w_{k} v_{k1}$, and the contribution to the $L_2$ penalty from these weights is $\frac{\lambda}{2}(2v_{k1}^2 + w_{k}^2)$. 


We now bound the contribution to the activation in terms of the contribution to the $L_2$ penalty. Note that as the $L_2$ “budget” increases, so does the maximum possible contribution to the output node’s activation.

**Lemma 2** If $w_k^2 + v_{k1}^2 + v_{k2}^2 = B$ then hidden node $k$’s contribution to the output node’s activation is maximized when $w_k = \sqrt{B/2}$ and $v_{k1} = v_{k2} = \sqrt{B/2}$, where it achieves the value $\frac{\epsilon B}{\sqrt{2}}$.

**Proof:** Since $2v_{k1}^2 + w_k^2 = B$, we have $v_{k1} = \sqrt{B - 2v_{k1}^2}$, so the contribution to the activation can be re-written as $2\epsilon v_{k1}\sqrt{B - 2v_{k1}^2}$. Taking the derivative with respect to $v_{k1}$, and solving, we get $v_{k1} = \pm \sqrt{B/2}$ and the maximum is at the positive solution (when $v_{k1}$ is positive). When $v_{k1} = \sqrt{B/2}$ we have $w_k = \sqrt{B/2}$ and thus the maximum contribution of the hidden node is $2\epsilon \sqrt{B/2} \left( \frac{\sqrt{B/2}}{\sqrt{2}} \right) = \frac{\epsilon B}{\sqrt{2}}$.

Lemma 2 immediately implies the following.

**Lemma 3** The maximum of $\mathcal{W}(\epsilon, \epsilon)$, subject to $||\mathcal{W}||_2^2 \leq A$, is $\frac{\epsilon A}{\sqrt{2}}$.

**Proof:** When maximized, the contribution of each hidden node to the activation at the output is $\epsilon/\sqrt{2}$ times the hidden node’s contribution to the sum of squared-weights. Since each weight in $\mathcal{W}$ is used in exactly one hidden node’s contribution to the output node’s activation, this completes the proof.

Note that this bound is independent of $K$, the number of hidden units.

**Proof (of Theorem ??):** Combining Lemma 1 and Lemma 3, $\mathcal{W}_{L_2}$ minimizes

$$\left( \frac{\epsilon ||\mathcal{W}||_2^2}{\sqrt{2}} - 1 \right)^2 + \frac{\lambda}{2} ||\mathcal{W}||_2^2.$$ 

Its derivative with respect to $||\mathcal{W}||_2^2$ is

$$\sqrt{2} \epsilon \left( \frac{\epsilon ||\mathcal{W}||_2^2}{\sqrt{2}} - 1 \right) + \frac{\lambda}{2}.$$

If $\sqrt{2} \epsilon \left( \frac{\epsilon ||\mathcal{W}||_2^2}{\sqrt{2}} - 1 \right) + \frac{\lambda}{2} > 0$ whenever $||\mathcal{W}||_2^2 \geq 1$ (i.e. $\lambda > \sqrt{8} \epsilon$), then the $L_2$ criterion is minimized when $||\mathcal{W}||_2^2 = 0$, where it takes the value 1.

Otherwise, setting this derivative to 0 and solving for $||\mathcal{W}||_2^2$ gives us that the minimum of the criterion occurs when

$$||\mathcal{W}||_2^2 = \frac{\sqrt{2} \epsilon}{\epsilon} - \frac{\lambda}{2\epsilon^2}.$$ 

Evaluating the risk using Lemma 3,

$$R(\mathcal{W}_{L_2}) = \left( \frac{\epsilon}{\sqrt{2}} \left( \frac{\sqrt{2}}{\epsilon} - \frac{\lambda}{2\epsilon^2} \right) - 1 \right)^2 = \left( \frac{\lambda}{2\sqrt{2} \epsilon} \right)^2 = \frac{\lambda^2}{8\epsilon^2}.$$
for the weights minimizing the $L_2$ criterion in this case.

Thus, overall the risk of the minimizer of the $L_2$ criterion is $\min\left\{1, \frac{\lambda^2}{8\sigma^2}\right\}$. ■