ESTIMATES FOR GENERALIZED OSCILLATORY INTEGRALS WITH POLYNOMIAL PHASE

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Abstract. In this paper we consider the problem on uniform estimates for generalized oscillatory integrals given by Mittag-Leffler functions with the homogeneous polynomial phase. We obtain a variant of Ricci-Stein Lemma and invariant estimates for corresponding integrals.

Keywords: Mittag-Leffler functions, phase function, amplitude.

1. Introduction

Many problems of harmonic analysis, analytic number theory, and mathematical physics involve trigonometric (oscillatory) integrals with polynomial phase, a common problem integration of rational polynomials instead of integrating functions [2], [3], [4], [5], [7], [8], [27]. In harmonic analysis, estimates for one dimensional oscillatory integrals can be obtained using van der Corput lemma [26]. In [12] a multidimensional version of the van der Corput lemma is considered where the decay of the oscillatory integral is established with respect to all space variables, combining the standard one-dimensional van der Corput lemma with the stationary phase method. Estimates for oscillatory integrals with polynomial phase can be found, for instance, in [5], [11].

The function $E_\alpha(z)$ is named after the great Swedish mathematician Gösta Magnus Mittag-Leffler (1846-1927) who defined it by a power series

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha \in \mathbb{C}, \text{Re}(\alpha) > 0,$$

and studied its properties in 1902-1905 in five subsequent notes [15]-[18] in connection with his summation method for divergent series.

A classic generalizations of the Mittag-Leffler function, namely the two-parametric Mittag-Leffler function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta \in \mathbb{C}, \text{Re}(\alpha) > 0,$$

which was deeply investigated independently by Humbert and Agarval in [22], [23], [25] and by Dzherbashyan in [19], [20], [21], [24].

In the current paper we replace exponential function with the Mittag-Leffler-type function and study the "generalized"oscillatory integrals. In [13] and [14] analogues of the van der Corput lemmas involving Mittag-Leffler functions for one dimensional integrals have been considered. We consider estimates for multidimensional generalization oscillatory integrals with polynomial phase. This work is analogous to [21] and application for oscillatory integrals with Mittag-Leffler functions.

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2. Preliminaries

Определение 1. (9) Если $0 < \alpha < 2$, $\beta$ is an arbitrary real number и $\mu$ is such that $\pi \alpha/2 < \mu < \min\{\pi, \pi \alpha\}$, then there is $C > 0$ such that

\[ |E_{\alpha, \beta}(z)| \leq \frac{C}{1 + |z|}, \quad z \in \mathbb{C}, \mu \leq |\arg(z)| \leq \pi. \]

Доказательство. Using Proposition 1 we obtain

Лемма 1. (9) Пусть $P(a, x) = \sum_{|\lambda| \leq d} a_{\lambda} x^\lambda$ is a polynomial in $\mathbb{R}^n$ of degree at most $d$, where

we write $x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$, $\lambda = (\lambda_1, \ldots, \lambda_n)$, with $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n$.

Определение 1. A generalization of oscillatory integral with phase $P(a, x)$ and amplitude $\psi(x)$ is an integral of the form

\[ I_{\alpha, \beta}(a) = \int_{Q^n} E_{\alpha, \beta}(iP(a, x))\psi(x)dx, \]

where $0 < \alpha < 1$, $\beta > 0$, $\psi \in C^\infty(\mathbb{R}^n)$, $Q^n := [0, 1]^n$ is $n$-dimensional cube and $P(a, x)$ polynomial. In particular if $\alpha = 1$ and $\beta = 1$ we have a classical oscillatory integral.

Теорема 1. (9) For each $d, n$ there exist a finite constant $C := C(n, d)$ such that for any multi-index $\kappa$ and any polynomial $P : \mathbb{R}^n \to \mathbb{R}$ of degree $\leq d$ satisfying $|D^\kappa P(a, x)| \geq 1$ for every $x \in Q^n := [0, 1]^n$, and for any $\mu > 0$,

\[ |\{x \in Q^n : |P(a, x)| \leq \mu}\| \leq C\mu^{1/|\kappa|}, \]

where $\kappa$ depends on $a$.

Следствие 1. (9) Let $P(a, x) = \sum_{0 < |\lambda| \leq d} a_{\lambda} x^\lambda$ is the polynomial. Then

\[ \left| \int_{Q^n} e^{iP(a, x)} dx \right| \leq C_{d, n} \left( \sum_{0 < |\lambda| \leq d} |a_{\lambda}| \right)^{-\frac{1}{2}}. \]

Moreover $C_{d, 1} \leq C d$ for an absolute constant $C$.

Лемма 1. Let $P : \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree $\leq d$ and $|a| = \max\{|a_{\lambda}|, \lambda \leq d\}$. There exists a constant $C_{d, n}$ such that if $a^0 \in S^n$ (where $N + 2$ is the dimension of space of polynomials of degree at most $d$) be a fixed point and $|D^\kappa P(a^0, x)| \geq \delta > 0$ then the following inequality holds

\[ |I_{\alpha, \beta}(\mu a^0)| \leq \frac{C_{d, n}}{|\mu|^{1/|\kappa|}}, \]

where $S^n = \{|a| = 1\}$ is the unit sphere with respect to metric $l^1$ and $\mu > 0$.

Доказательство. Using Proposition 1 we obtain

\[ |I_{\alpha, \beta}(\mu a^0)| = \left| \int_{Q^n} E_{\alpha, \beta}(iP(a^0, x))\psi(x)dx \right| \leq \int_{Q^n} \frac{\psi(x)|dx|}{1 + |P(a^0, x)|} := J(\mu a^0). \]

Now we represent $Q^n$ as union of sets

\[ Q^n = \Omega \cup A_k, \]
where
\[ \Omega = \{ x \in Q^n : |P(a^0, x)| \leq 2 \} = \left\{ x \in Q^n : \left| P\left( \frac{a^0}{\mu}, x \right) \right| \leq \frac{2}{\mu} \right\} \]

and
\[ A_k = \{ x \in Q^n : 2^k \leq |P(a^0, x)| \leq 2^{k+1} \} = \left\{ x \in Q^n : 2^k \leq \left| P\left( \frac{a^0}{\mu}, x \right) \right| \leq \frac{2^{k+1}}{\mu} \right\} , \]

where \( k = 1, 2, \ldots \).

First we represent the integral \( J(\mu a^0) \) as
\[ J(\mu a^0) = J_0 + J_k := \int_{\Omega} + \int_{A_k} . \]

Now we estimate the integral \( J_0 \) on the set \( \Omega \).
So, we obtain
\[ |J_0| \leq \int_{\Omega} \left| \frac{\psi(x)}{1 + |P(a^0, x)|} \right| dx . \]

Due to Corollary 1 we have
\[ |J_0| \leq C|\Omega| \leq \frac{C}{\mu^{2\gamma}} . \]

Then we consider the integral \( I_{\alpha, \beta} \) on the sets \( A_k \). By Theorem 1 and we have:
\[ \left| \left\{ \left| P\left( \frac{a^0}{\mu}, x \right) \right| \leq \frac{2^{k+1}}{\mu}, x \in Q^n \right\} \right| \leq C \left( \frac{2^{k+1}}{\mu} \right)^{\frac{1}{\gamma}} . \]

We write \( J(\mu a^0) \) as sum of integrals \( J_k \) and obtain
\[ J(a^0, \mu) = J_0(a^0, \mu) = \sum_{2^k \leq |P(a^0, x)| \leq 2^{k+1}} J_k = \sum_{2^k \leq |P(a^0, x)| \leq 2^{k+1}} \int_{A_k} \left| \frac{\psi(x)}{1 + |P(a^0, x)|} \right| dx \]

using Theorem 1 we find the following estimate:
\[ |J_k| = \left| \int_{A_k} \frac{\psi(x)}{1 + |P(a^0, x)|} dx \right| \leq \left( \frac{2^{k+1}}{\mu} \right)^{\frac{1}{\gamma}} \frac{C}{2^{\kappa}} . \]

From here we find the sum \( J_k \) and, by estimating the integral \( I_{\alpha, \beta}(a^0, \mu) \). So
\[ |I_{\alpha, \beta}(a^0, \mu)| = |J_0| + \sum_{k=1}^{\infty} J_k \leq \frac{C}{\mu^{2\gamma}} + \sum_{k=1}^{\infty} \left( \frac{2^{k+1}}{\mu} \right)^{\frac{1}{\gamma}} \frac{C}{2^{\kappa}} \leq \frac{C}{\mu^{2\gamma}} + \frac{C}{\mu^{\gamma}} \sum_{k=1}^{\infty} 2^{\frac{k+1}{\gamma} - k} . \]

As the last series is convergence since \( |\kappa| \geq 2 \) then we obtain proof of Lemma 1. □

3. Relation to Mittag-Leffler functions

Theorem 2. Let \( 0 < \alpha < 1, \beta > 0 \). There exists positive number \( C_d \) such that for the integral \( I_{\alpha, \beta}(a, x) \) following inequality holds
\[ (5) \quad |I_{\alpha, \beta}| \leq \frac{C_d}{\|a\|^2} \]

where \( \|a\| = \sum_{|\lambda| \leq d} |a_{\lambda}| . \)
Remark. Theorem 2 is an analog of the Ricci-Stein lemma [7].

Proof of Theorem 2. We use inequality (3) for the integral (4) and obtain

\[ |I_{\alpha,\beta}| = \left| \int_{Q^n} E_{\alpha,\beta}(iP(a,x))\psi(x) dx \right| \leq \int_{Q^n} |E_{\alpha,\beta}(iP(a,x))| |\psi(x)| dx \]

(6)

\[ \leq C \int_{Q^n} \frac{|\psi(x)| dx}{1 + |P(a,x)|}. \]

We represent cube as union of following subspaces

\[ Q^n = \Delta_1 \cup \Delta_2 := \{ x \in Q^n : |P(a,x)| < 2 \} \cup \{ x \in Q^n : |P(a,x)| \geq 2 \}. \]

First we estimate the last integral (6) over the \( \Delta_1 \)

\[ |I_{\alpha,\beta}| \leq \int_{\Delta_1} \frac{|\psi(x)| dx}{1 + |P(a,x)|}. \]

Using the results sublevel set estimates Theorem 1 and we have

\[ \left| \int_{\Delta_1} \frac{|\psi(x)| dx}{1 + |P(a,x)|} \right| \leq \frac{c \max_{x \in Q^n} |\psi(x)|}{|a|^{\frac{3}{\alpha}}} \leq \frac{c}{|a|^{\frac{3}{\alpha}}}. \]

Now we estimate the integral (6) over the \( \Delta_2 \)

\[ |I_{\alpha,\beta}| \leq \int_{\Delta_2} \frac{|\psi(x)| dx}{1 + |P(a,x)|}. \]

We use Lemma 1 and we get

\[ |I_{\alpha,\beta}| \leq \frac{c}{|a|^{\frac{3}{\alpha}}}. \]

4. THE CASE \( d = 3 \)

Now we consider the following example for \( d = 3 \).

Let \( 0 < \delta < 1 \). We consider the integral:

(7)

\[ J_{\alpha,\beta} = \int_{|x| \leq 1} E_{\alpha,\beta}(i(x^3 + px + q)) dx, \]

where \( 0 < \alpha < 1, \beta > 0 \).

Теорема 3. 1) If \( \frac{1}{3} \leq \delta \leq \frac{1}{2} \) then for the integral (7) following estimate holds

(8)

\[ |J_{\alpha,\beta}| \leq \frac{c_5}{\left( \frac{|p|^3}{2^\gamma} + \frac{q^2}{4} \right)^{\frac{2}{3} - \frac{\alpha}{3}}}. \]

2) If \( \frac{1}{2} < \delta < 1 \) then for the integral (7) following estimate holds

(9)

\[ |J_{\alpha,\beta}| \leq \frac{c_5}{|D|^{\delta - \frac{1}{2}} \left( \frac{|p|^3}{2^\gamma} + \frac{q^2}{4} \right)^{\frac{2}{3} - \frac{\alpha}{3}}}, \]

where \( D = \frac{|p|^3}{2^\gamma} + \frac{q^2}{4} \) is discriminant of polynomial \( x^3 + px + q \) and \( c_5 \) any positive only depending to \( \delta \).
Remark. If \( \delta \leq \frac{1}{3} \), then according to Theorem 2 the integral (7) bounded.  

Proof of Theorem 3. If \( p = q = 0 \), then the required estimate is trivially satisfied. We use inequality (3) for the integral (7) and we obtain

\[
|J_{a, b}| \leq \int_{|x| \leq 1} \frac{dx}{1 + |x^3 + px + q|} \leq \int_{|x| \leq 1} \frac{dx}{|x^3 + px + q|^\delta} := J(p, q).
\]

Let \( \frac{1}{3} < \delta < \frac{1}{2} \) and \( (p, q) \neq (0, 0) \). Now we consider the following cases separately.

\[
\max \left\{ \frac{|p|^3}{27}, \frac{q^2}{4} \right\} = \frac{|p|^3}{27}
\]

and

\[
\max \left\{ \frac{|p|^3}{27}, \frac{q^2}{4} \right\} = \frac{q^2}{4}.
\]

Suppose the condition (11) holds, then we use change the variables \( x = |p|^\frac{1}{2} y \) and obtain:

\[
|J(p, q)| = \int_{-|p|^{-\frac{1}{2}}}^{|p|^{-\frac{1}{2}}} \frac{|p|^\frac{1}{2} dy}{|y^3 + \frac{|p|^2}{|p|} p\frac{1}{2} y + q|^\delta} = \frac{1}{|p|^{\frac{1}{2} + \frac{1}{2}}} \int_{-|p|^{-\frac{1}{2}}}^{|p|^{-\frac{1}{2}}} \frac{dy}{|y^3 + \frac{\text{sgn}(p)y + q|}|^\delta |p|^{-\frac{3}{2}}}.\]

We consider the integral

\[
|J_1(p, q)| = \int_{-|p|^{-\frac{1}{2}}}^{|p|^{-\frac{1}{2}}} \frac{dy}{|y^3 + \text{sgn}(p)y + B|^\delta},
\]

where \( |B| = \left| \frac{q}{|p|^{\frac{1}{2}}} \right| \leq \frac{2}{3\sqrt{3}} \). We show that this integral is bounded, when \( \frac{1}{3} < \delta < \frac{1}{2} \).

If \( |y| \geq 2 \), then \( |y|^3 \left| 1 + \frac{1}{y^2} + \frac{B}{y} \right| \geq |y|^3 \left| 1 - \frac{1}{12\sqrt{3}} \right| \geq \frac{|y|^3}{4} \). Consequently,

\[
|J(p, q)| = \int_{-2}^{2} \frac{dy}{|y^3 + \text{sgn}(p)y + B|^\delta} + R(B),
\]

where \( R(B) \) is a bounded function of \( B \in \left[ -\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}} \right] \).

Let

\[
|J_{10}| := \int_{-2}^{2} \frac{dy}{|y^3 + \text{sgn}(p)y + B|^\delta}.
\]

Suppose \( \text{sgn}(p) = 1 \). Then polynomial \( y^3 + y + B \) has no multiple roots. In fact, for any \( B \) it has one simple root. Thus,

\[
|J_0| = \int_{-2}^{2} \frac{dy}{|y^3 + y + B|^\delta} \leq c,
\]

for \( \delta < \frac{1}{2} < 1 \).

Suppose \( \text{sgn}(p) = -1 \). Then, since \( B \) below in a compact set, it suffices to obtain the corresponding "local" estimate. Let \( \varepsilon \) is sufficiently small positive number. For a number \( B \), consider two sets

\[
A_1 = \left\{ \left| B^2 - \frac{4}{27} \right| \geq \varepsilon \right\}, \ A_2 = \left\{ \left| B^2 - \frac{4}{27} \right| \leq \varepsilon \right\}.
\]
First, consider the estimate for the integral when $B$ lies in the set $A_1$. In this case equation $y^3 - y + B = 0$ has three different roots $y_1(B), y_2(B), y_3(B)$, since $|y_j(B) - y_k(B)| \geq \Delta(\varepsilon) > 0$ when $j \neq k$. Since $\delta < 1$, then the following integral convergences

\begin{align}
J_{10} &:= \int_{-2}^{2} \frac{dy}{(y - y_1(B))(y - y_2(B))(y - y_3(B))}.
\end{align}

Since $B$ belongs to the compact set $A_1$, the integral (16) is uniformly bounded for $\delta < 1$. Therefore, we have

$$|J| = \frac{1}{|p|^{\Delta - 1}} \int_{-|p|^{\Delta - 1}/2}^{|p|^{\Delta - 1}/2} \frac{dy}{|y^3 - y + B|^\delta} \leq \frac{c_3}{|p|^{\Delta - 1/2}},$$

as $B \in A_1$. Now we consider the integral in the case when $B \in A_2$. For the sake of definiteness, we can assume that $|B - \frac{2}{3\sqrt{3}}| < \varepsilon$, where $0 < \varepsilon < \frac{1}{6\sqrt{3}}$ a fixed positive number. Note that $y_{1,2} = \pm \frac{1}{\sqrt{3}}$ are critical points of the function $F(y) = y^3 - y + B$. Thus, the number $B = B_0 = \frac{2}{3\sqrt{3}}$ for the function $F(y, B_0)$ is a simple critical value. Let us study the behavior of the integral as $|B - B_0| < \Delta$, where $\Delta$ is sufficiently small fixed positive number. Since $F\left(\frac{1}{\sqrt{3}}\right) = 0$, then

$$F\left(y + \frac{1}{\sqrt{3}}, B\right) = y^3 + \sqrt{3}y^2 - \frac{2}{3\sqrt{3}} + B = y^2\left(y + \sqrt{3}\right) + H,$$

where $H = B - \frac{2}{3\sqrt{3}}$.

We consider the following integral

\begin{align}
\int_{-\sigma}^{\sigma} \frac{dy}{y^2\left(y + \sqrt{3}\right) + H}.
\end{align}

where $\sigma > 0$ sufficiently small fixed positive number. If $\frac{1}{3} < \delta < \frac{1}{2}$, then according to the (7), the integral (17) is uniformly bounded with respect to $H$. This completes the proof of the first part of Theorem 3.

Next, suppose that $\frac{1}{2} < \delta < 1$. We use change of variables as $z = y\sqrt{y + \sqrt{3}}$ and and denoting the inverse function by $y(y(z))$, we have:

$$\int_{-\sigma}^{\sigma} \frac{dy}{y^2 \left(y + \sqrt{3}\right) + H} = \int_{-\sigma_1}^{\sigma_2} y'(z)dz = \int_{-\sigma_1}^{\sigma_2} \frac{dz}{|z^2 + H|^\delta} + c \int_{-\sigma_1}^{\sigma_2} \frac{z\varphi(z)dz}{|z^2 + H|^\delta} = J_1 + J_2,$$

where $\varphi$ is any smooth function, $\sigma_1$ and $\sigma_2$ determined from the conditions $y(\sigma_1) = -\sigma$, $y(\sigma_2) = \sigma$. In the integrals $J_1$ and $J_2$ we make a linear change $z = |H|^{1/2}t$. Then

\begin{align}
J_1' &= \frac{1}{\sqrt{3}} \int_{-\sigma_1}^{\sigma_2} |H|^{-1/2} \frac{dt}{|t^2 + 1|^\delta},
\end{align}

$$J_2' = \frac{1}{\sqrt{3}} \int_{-\sigma_1}^{\sigma_2} |H|^{-1/2} \frac{dt}{|t^2 + 1|^\delta}. $$
It is easy to show that the integral \([15]\) is uniformly bounded with respect to \(H\), for \(\frac{1}{2} < \delta < 1\). From here we get:

\[
|J_1'| \leq \frac{c}{H^{\delta - 1/2}}.
\]

The integral \(J_2\) is estimated as follows:

\[
J_2' = c \int_{-\sigma_1}^{\sigma_2} \frac{z\varphi(z)dz}{|z^2 + H|^{\delta}} = c \int_{-\sigma_1/H^{1/2}}^{\sigma_2/H^{1/2}} \frac{Ht\varphi(|H|^{1/2} t)dt}{|Ht^2 + H|^{\delta}} \leq \frac{c}{H^{\delta - 1}} \int_{-\sigma_1/H^{1/2}}^{\sigma_2/H^{1/2}} \frac{t\varphi(|H|^{1/2} t)dt}{|t^2 \pm 1|^{\delta}} \leq \frac{c_1}{H^{\delta - 1}} \int_{1/2}^{1/2} t^{1-2\delta}dt + \frac{c_2}{H^{\delta - 1}} = c_1 \frac{H^{1-\delta}}{2 - 2\delta} t^{2-2\delta} |t|^{1/2} + c_2 \frac{H^{1-\delta}}{2 - 2\delta} [H^{\delta - 1} - 2^{2-2\delta}] + c_2 \frac{c_1 H^{1-\delta}}{H^{\delta - 1}} \leq C,
\]

the validity of the last inequality follows from the condition \(\delta < 1\). Thus, the integral \(J_2'\) is uniformly bounded. Summing up the obtained estimates, we have:

\[
\int_{-\sigma}^{\sigma} \frac{dy}{|y^2 (y + \sqrt{3}) + H|^\delta} \leq \frac{c}{|H|^{\delta - 1/2}}.
\]

Thus, under the conditions \([11]\) and \(\frac{1}{2} < \delta < 1\), we get the estimate:

\[
|J(p, q)| = \int_{-1}^{1} \frac{dx}{|x^3 + px + q|^\delta} \leq \frac{1}{|p|^{\frac{3\delta - 1}{2}}} \frac{c}{|B - \frac{2}{3\sqrt{3}}|^\delta} \frac{1}{\frac{13 + 2}{2\sqrt{3}}} \times \frac{c}{\left|\frac{q}{(-p)^{3/2}} - \frac{2}{3\sqrt{3}}\right|^{\delta - 1/2}} \leq \frac{c}{D^{\delta - 1/2}} \left(\frac{|p|^{3/2} + \frac{2}{3\sqrt{3}}}{4}\right)^{\frac{2-\delta}{4}}.
\]

Now suppose that the condition \([12]\) holds true. In this case, we use the change of variables \(x = |q|^{1/3} z\) for the integral \([10]\). Then

\[
|J(p, q)| = \frac{1}{|q|^{\frac{3\delta - 1}{2}}} \int_{-|q)^{1/3}|}^{\frac{3}{2\sqrt{3}}} \frac{dy}{|z^3 + Az + sgn(q)|^\delta},
\]

where \(|A| = \left|\frac{p}{q}\right| \leq \frac{3}{\sqrt{3}}\).

If \(A \in \left[-\frac{3}{\sqrt{3}} + \varepsilon, \frac{3}{\sqrt{3}} - \varepsilon\right]\) (where \(\varepsilon\) sufficiently small positive fixed number), then again the integral is uniformly bounded. Case \(|A^2 - \frac{9}{\sqrt{16}}| < \varepsilon\) is considered similarly to the case \(|B^2 - \frac{4}{27}| < \varepsilon\).

The case when \(\delta = 1\) is trivially holds and we can obtain an analogical estimate for the case when \(\delta = \frac{1}{2}\). Considering separately the cases \(|x^3 + px + q| > 1\) and \(|x^3 + px + q| \leq 1\), Theorem \([3]\) is proved.
5. INVARIANT ESTIMATES FOR HOMOGENEOUS POLYNOMIAL PHASE

In this section, we consider generalized oscillatory integrals with phase function which is a homogeneous polynomial of degree three in two variables
\[ P_3(a, x) = a_0 x_1^3 + 3a_1 x_1^2 x_2 + 3a_2 x_1 x_2^2 + a_3 x_2^3. \]
We consider the following integral
\[ I_{\alpha, \beta} = \int_{|x| \leq 1} E_{\alpha, \beta} (i P_3(a, x)) \psi(x) dx_1 dx_2, \]
where $|x| \leq 1$ is the unite circle. We consider behavior of the integral (20) in the case when the coefficients of the polynomial tend to infinity. Let us obtain estimates for integral (21) in terms of the invariants of the group of motions of the Euclidean plane. Note that, the discriminant of a polynomial denoted by $D$ is defined by the formula:
\[ D = 3a_1^2 a_2^2 + 6a_0 a_1 a_2 a_3 - 4a_0 a_2^3 - 4a_1^3 a_3 - a_0^2 a_3^2 \]

\textbf{Theorem 4.} For the integral (19) with phase (19) following inequality
\[ |I_{\alpha, \beta}| \leq c \|\psi\|_{L^\infty} |D|^r, \]
holds, where $c$ is a constant and $D$ is discriminant of the polynomial $P_3$.

\textbf{Proof.} For the sake of defined we suppose that $|a_0| = \max\{|a_i|, i = 0, 3\}$, otherwise, by rotating the coordinate axes, we can reduce the general case to the case under consideration. Since $D$ is the invariant of the group $SL(2, \mathbb{C})$, the integral and estimate do not depend on the choice of such a change of variables. Moreover, the norm of the amplitude is uniformly bounded because the rotation group is compact. Let’s represent the polynomial in the form
\[ P_3 = a_0 \left( x^3 + \frac{3a_1}{a_0} x^2 y + \frac{3a_2}{a_0} xy^2 + \frac{a_3}{a_0} y^3 \right). \]

Let we make a change of variables as follows $x_1 = x + \frac{a_1}{a_0} y$, $y_1 = y$. Since $|\frac{a_1}{a_0}| \leq 1$, then the Jacobian and the transformation norm are uniformly bounded, which is important in what follows. As a result, we get
\[ P_3 = a_0 \left( x_1^3 + p x_1 y_1^2 + q y_1^3 \right) = a_0 \Phi, \]
where $p = \frac{3a_0 a_2 - 3a_1^2}{a_0^2}$ and $q = \frac{a_2^3 + 2a_1^3 - 3a_0 a_1 a_2}{a_0^3}$. Note that $|p| \leq 6, |q| \leq 6$.

Hence the integral has the form
\[ J_{\alpha, \beta} := \int E_{\alpha, \beta} (i a_0 \Phi) \psi(x_1, y_1) dx_1 dy_1. \]

Applying the polar coordinate system $x_1 = r \cos \theta, \ y_1 = r \sin \theta$, we get
\[ J_{\alpha, \beta} \leq \int_0^{2\pi} \int_0^\infty E_{\alpha, \beta} (i a_0 r^3 \phi(\cos \theta, \sin \theta)) r \psi(r \cos \theta, r \sin \theta) dr d\theta. \]
For the inner integral (22), i.e. for
\[ J_{in} := \int_0^\infty E_{\alpha, \beta} (i a_0 r^3 \phi(\cos \theta, \sin \theta)) r \psi(r \cos \theta, r \sin \theta) dr \]
Then where \( J \)

Using Proposition 1, we get

\[
|J_{in}| \leq \int_0^\infty \frac{r|\psi(r \cos \theta, r \sin \theta)|dr}{1 + |a_0 r^3 \phi(\cos \theta, \sin \theta)|}. 
\]

We make change the variable as \( \rho = r^3 |a_0 \phi(\cos \theta, \sin \theta)| \). So, we have

\[
|J_{in}| \leq \frac{c \|\psi\|_{L^\infty}}{|a_0 \phi(\cos \theta, \sin \theta)|} \int_0^\infty \frac{d\rho}{\rho^2 (1 + \rho)}.
\]

As the last integral convergence, we obtain

\[
|J_{in}| \leq \frac{c \|\psi\|_{L^\infty}}{|a_0 \phi(\cos \theta, \sin \theta)|}.
\]

Thus, the integral \( J \) has the estimate

\[
|J_{\alpha, \beta}| \leq \frac{c \|\psi\|_{L^\infty}}{|a_0|^{\alpha/2}} \int_0^{2\pi} \frac{d\theta}{|\phi(\cos \theta, \sin \theta)|^{\beta/2}},
\]

where \( \phi = \cos^3 \theta + p \cos \theta \sin^2 \theta + q \sin^3 \theta \).

We introduce the following integral

\[
J_2 := \int_0^{2\pi} \frac{d\theta}{|\phi(\cos \theta, \sin \theta)|^{\alpha}},
\]

where \( \phi = \cos^3 \theta + p \cos \theta \sin^2 \theta + q \sin^3 \theta \).

**Lemma 2.** For the integral \( J_2 \), following estimate holds true

\[
|J_2| = \left| \int_0^{2\pi} \frac{d\theta}{|\cos^3 \theta + p \cos \theta \sin^2 \theta + q \sin^3 \theta|^{\alpha}} \right| \leq \frac{c}{|D(\phi)|^{\alpha/2}},
\]

where \( D(\phi) = \frac{\rho^3}{2\pi} + \frac{q^2}{\pi} \).

Lemma 2 was proved in [10]. For the convenience of readers we give a proof of Lemma 2.

**Proof of Lemma 2.** Note that,

\[
J_2 = 2 \int_0^{\pi/2} \frac{d\theta}{|\cos^3 \theta + p \cos \theta \sin^2 \theta + q \sin^3 \theta|^{\alpha}} = 2(J_{20} + J_{21}),
\]

where \( J_{20} = \int_0^{\pi/2} \frac{d\theta}{|\cos^3 \theta + p \cos \theta \sin^2 \theta + q \sin^3 \theta|^{\alpha}}, \quad J_{21} = \int_0^{\pi/2} \frac{d\theta}{|\cos^3 \theta + p \cos \theta \sin^2 \theta + q \sin^3 \theta|^{\alpha}} \).

First, we consider

\[
J_{20} = \int_0^{\pi/2} \frac{d\theta}{|\cos^3 \theta + p \cos \theta \sin^2 \theta + q \sin^3 \theta|^{\alpha}} = \int_0^{\pi/2} \frac{d\theta}{|\cos^3 \theta + p \cos \theta \sin^2 \theta + q \sin^3 \theta|^{\alpha}} + \int_0^{\pi/2} \frac{d\theta}{|\cos^3 \theta + p \cos \theta \sin^2 \theta + q \sin^3 \theta|^{\alpha}}.
\]

Then

\[
J_{20}' = \int_0^{\pi/2} \frac{d\theta}{|\cos^3 \theta + p \cos \theta \sin^2 \theta + q \sin^3 \theta|^{\alpha}}
\]

making the change of variables \( t \sin \theta = t \), we get

\[
J_{20}' = \int_0^1 \frac{dt}{|qt^3 + pt^2 + 1|^{\alpha/2}}.
\]
Let \(|t| \leq \frac{1}{4}\) as \(|p| \leq 6, |q| \leq 6\), then \(|qt^3| \leq \frac{6}{64}, |pt^2| \leq \frac{6}{16}\). Hence,
\[|t^3 + pt^2| \leq \frac{30}{64} \quad \text{and} \quad 1 + qt^3 + pt^2 \geq 1 - \frac{30}{64} = \frac{34}{64}.
\]
Then
\[
\begin{align*}
|J_{20}| & = \int_0^1 \frac{dt}{|qt^3 + pt^2 + 1|^{3/4}} = \int_0^{1} \frac{dt}{|qt^3 + pt^2 + 1|^{3/4}} \\
+ \int_{\frac{1}{4}}^{1} \frac{dt}{|qt^3 + pt^2 + 1|^{3/4}} & \leq 1 + \int_{\frac{1}{4}}^{1} \frac{dt}{|qt^3 + pt^2 + 1|^{3/4}}.
\end{align*}
\]
By changing the variables \(x = \frac{1}{t}\) the last integral is reduced to the form:
\[
\begin{align*}
\int_{\frac{1}{4}}^{1} \frac{dt}{|qt^3 + pt^2 + 1|^{3/4}} & = \int_{1}^{4} \frac{dx}{|x^3 + px + q|^{3/4}}.
\end{align*}
\]
It is obvious that the equality
\[
\int_{\frac{1}{4}}^{1} \frac{d\theta}{|\cos^3 \theta + p \cos \theta \sin^2 \theta + q \sin^3 \theta|^{3/4}} = \int_{0}^{1} \frac{dx}{|x^3 + px + q|^{3/4}}
\]
Thus, the problem reduces to estimating an integral of the form
\[
\int_{N_1}^{N_2} \frac{dx}{|x^3 + px + q|^{3/4}},
\]
where \(N_1, N_2\) are fixed numbers. Finally, the desired estimate for the integral \(J_{21}\) follows easily from Theorem 3. The estimate for the integral \(J_{21}\) is similarly performed. Which completes the proof of Lemma 2.

Applying Lemma 2 for the integral (23), we obtain the required estimate. Theorem 5.1 is proved.

**Declaration of competing interest.**
This work does not have any conflicts of interest.

**Data availability.** My manuscript has no associated data.

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