Estimate of convection–diffusion coefficients from modulated perturbative experiments as an inverse problem

F Sattin¹, D F Escande², Y Camenen², A T Salmi³, T Tala³ and JET EFDA Contributors

JET-EFDA, Culham Science Centre, Abingdon, OX14 3DB, UK
¹ Consorzio RFX, Associazione EURATOM-ENEA, Padova, Italy
² UMR 7345, CNRS/Aix-Marseille Université, Marseille, France
³ Association Euratom-Tekes, VTT, PO Box 1000, FI-02044 VTT, Finland

Received 25 June 2012, in final form 30 August 2012
Published 21 November 2012
Online at stacks.iop.org/PPCF/54/124025

Abstract
The estimate of coefficients of the convection–diffusion equation (CDE) from experimental measurements belongs to the category of inverse problems, which are known to come with issues of ill-conditioning or singularity. Here we concentrate on a particular class that can be reduced to a linear algebraic problem, with explicit solution. Ill-conditioning of the problem corresponds to the vanishing of one eigenvalue of the matrix to be inverted. The comparison with algorithms based on matching experimental data against numerical integration of the CDE sheds light on the accuracy of the parameter estimation procedures, and suggests a path for a more precise assessment of the profiles and of the related uncertainty. Several instances of the implementation of the algorithm to real data are presented.

(Some figures may appear in colour only in the online journal)

1. Introduction
Inverse problems are ubiquitous in applied sciences and engineering. They arise whenever one needs to extract, from measurements, some information about the object or the system measured. Their study represents almost a separate discipline, extending into sophisticated applied mathematics, with specialized scientific journals. Tomographic reconstruction represents the best known example; inverse scattering theory, both quantum and classical (e.g. in acoustics, hydrodynamics or electromagnetism), is another one. A third instance of paramount importance in magnetically confined plasmas—but relevant also in space physics (see, e.g., [1])—is the reconstruction of the equilibrium magnetic field by external magnetic field measurements via the Grad–Shafranov equation.

Transport theory is another vast field of study: it encompasses the statistical description of the dynamics of some objects while moving through a host medium. Within this field also inverse problems are common: e.g. heat conduction [2]. Analysis can also be carried out at a more formal and abstract level, partially abstracting from the physical problems, and rather focusing on the mathematical structure of the partial differential equations that describe transport [3].

Inverse problems are notoriously often ill-conditioned or even singular, i.e. small variations in the data (due, e.g., to finite instrumental precision) yield extremely large (infinite) admissible ranges for the parameters to be estimated. Qualitatively, one single effect can hardly ever be unambiguously related to a single cause; rather several possible widely different causes can be invoked. Quite often, inversion problems may reduce to linear algebraic ones. i.e. to inverting some matrices. In those cases, ill-posedness or singularity of the inverse problem amounts to the existence of small or null eigenvalues of the matrix to be inverted [4].

In this work, we consider a simple but fairly important case in transport theory, namely the extraction of transport coefficients in one-dimensional convective–diffusive transport problems when the forcing term is periodically modulated. In principle, the mathematical formalism is quite generic and can be applied...
to a large variety of conditions; however, in this work it will be biased towards magnetic confinement plasma physics, where transport analysis is fundamental, either for comparison against predictions from fundamental theories, or for using it in extrapolating current scenarios, in the absence of a satisfactory theoretical basis. The algorithm itself is rather simple, thus is likely to have been rediscovered several times. In the context of fusion plasma physics, the first explicit solution to the problem was provided in 1990 by Krieger et al [5] within the framework of impurity transport, eight years later by Takenaga et al [6] in connection with transport of the main gas, and again this year by two of us, being focused on heat transport. In this work, we review the version of the solution as developed in [7]; for our purposes, this latter formulation has two advantages: (I) it is written in a more compact, matricial form, which allows the inspection of its mathematical structure more easily, and (II) explicitly takes the source into account; we will show later that it is actually the source that makes the problem locally singular. A large part of the discussion will be then devoted to inspecting to what extent the ill-posedness of the problem affects its solution.

2. Assessing convective–diffusive transport as an inverse problem

Convection–diffusion equations (CDEs) arise in the contexts that involve the transport of one or more quantities through disordered media [8]. In the simplest version, only one passive scalar quantity $\xi$ is considered in a one-dimensional geometry, the transport equation taking the form

$$\frac{\partial \xi}{\partial t} = -\nabla \cdot (D(r) \nabla \xi) + S(r), \quad \xi = \xi(r, t) \quad \Gamma = -D \nabla \xi + V \xi.$$  \hspace{1cm} (1)

In (1), $S_E$ is the source/sink term and $r$ is the spatial coordinate. In any experiment designed to measure some kind of transport, the experimentalist uses $S_E$ as a knob to vary $\xi$, which is the output of the experiment, measured on a spatial grid and with some time resolution: $\xi(r, t) \rightarrow \xi_{\text{meas}}(r_i, t_j)$. Inferring the transport coefficients $D, V$ amounts to finding expressions for them that, once inserted into equation (1), allow us to extract solutions $\xi$ matching $\xi_{\text{meas}}(r_i, t_j)$. Here lies the nature of an inverse problem. It is trivial to show that, if we neglect time derivatives in (1), for a given measured profile $\xi_{\text{meas}}$, any couple $(D_0, V_0)$ such that $V \cdot (-D_0 \nabla \xi_{\text{meas}} + V_0 \xi_{\text{meas}}) = 0$ may be arbitrarily added to $(D, V)$. Degeneracy of the solutions can be regarded as an extreme form of ill-conditioning: even perfect knowledge of the input is not enough to fully constrain the solution. Experiments are thus designed including finite time derivatives, in the belief that this extra information is able to remove the degeneracy. As we shall show, even explicit time dependence does not warrant curing the problem. To the best of our knowledge, as far as plasma physics is concerned, the first paper that explicitly takes into account the inverse nature of the problem and addresses the issue of ill-conditioning is [9]. Here, we anticipate the claim of the next section, i.e. that the problem is ill-conditioned. We consider a one-dimensional cylindrical geometry. Equation (1), after rearrangement and a first integration over the radial coordinate, yields an expression for $\Gamma$:

$$- D \frac{\partial \xi}{\partial r} + V \xi = - \frac{1}{r} \int_0^r dz [S \xi - \xi]$$ \hspace{1cm} (2)

where we use the zero-flux boundary condition at $r = 0$. If the source is time dependent, this single equation is equivalent to infinitely many algebraic equations parametrized by time $t$, once we consider $\xi, \xi, \partial_t \xi, S_E$, as known. Obviously, the system is clearly overdetermined if $(D(r), V(r))$ are the unknowns: this is just a manifestation of ill-conditioning. Hence, adding a time dependence to problem (1) alleviates but not completely removes the issues related to its invertibility.

3. A case admitting an explicit solution for $(D, V)$

Experiments are usually performed by adding a small perturbation to an initial equilibrium condition, hence $\xi$ in equation (1) must be understood as the difference between the instantaneous total quantity measured and its equilibrium value, and the same is true for $S_E$. In any nonlinear system, the quantities in equation (1) are likely to depend on the starting equilibrium point; however, provided that the dependence is smooth and the perturbation weak enough, one can consider a linearization around the equilibrium that, hence, enters just as a parameter. We will postulate that this scenario holds; equation (1) thus becomes linear in all of the quantities appearing inside it.

As a fundamental simplification we consider those experiments where the source term is periodically modulated in time. This is not a very severe constraint, since most signals can be written as a finite sum of Fourier terms, and the algorithm is able to handle the case of either just one or a small finite number of harmonics. Furthermore, in magnetically confined plasmas, several classes of experiments, notably those of heat transport, are carried out via the periodic modulation of the source (say, radio-frequency or neutral beam injection heating).

Taking advantage of its linearity, we can Fourier-transform equation (1) in the frequency domain

$$- i \omega \xi = - \frac{1}{\mathcal{S}(r)} \frac{\partial}{\partial r} \left[ \mathcal{S}(r) \left( - D \frac{\partial \xi}{\partial r} + V \xi \right) \right] + S_E.$$ \hspace{1cm} (3)

All the quantities appearing in equation (3), except for $(D, V)$, are now understood to be complex numbers. The radial coordinate $r$ may be the true physical radius or any generalized coordinate with the dimension of length, and $\mathcal{S}(r)$ is the volume enclosed inside $r$: for a cylindrical system, $\mathcal{S}(r) \propto r$, but generically $\mathcal{S}$ can be a more complicated function of $r$ which, furthermore, may vary in time. As is convenient in these kinds of experiments, the signal $\xi$ is written in terms of an amplitude and a phase: $\xi = A e^{i \phi}$, and equation (3) is rewritten as a couple of real-valued equations.

By integrating once over the radius, and imposing the natural boundary condition $\Gamma(r = 0) = 0$ at the centre, we obtain two algebraic equations with unknowns $(D(r), V(r))$. They take a particularly convenient expression when expressed in the matrix-vector form

$$M \cdot Y = \Gamma,$$ \hspace{1cm} (4)
where
\[
Y = \begin{pmatrix} D \\ V \end{pmatrix},
\]
\[
M = \begin{bmatrix} -A' \cos \varphi + A \varphi' \sin \varphi & A \cos \varphi \\ -A' \sin \varphi - A \varphi' \cos \varphi & A \sin \varphi \end{bmatrix},
\]
\[
\Gamma = \begin{pmatrix} 2' Y^{-1} \int_0^1 \overline{y}(z)(\text{Re}(S_z(z)) - \omega A(z) \sin \varphi(z)) \, dz \\ 2' Y^{-1} \int_0^1 \overline{y}(z)(\text{Im}(S_z(z)) + \omega A(z) \cos \varphi(z)) \, dz \end{pmatrix},
\]
\[
f' = \delta f.
\]
Equation (4) can be inverted at each point \( r \), to give the local values for \( Y(r) = (D(r), V(r)) \). This formally solves the problem of estimating the transport coefficients, for the specific experiment considered.

Analysis of equation (4) leads to a deep insight into the issues involved in the inversion problem. Actually, equation (4) may be inverted whenever \( \det(M) = A^2 \varphi' \neq 0 \). Since \( A \) must be nonzero in order to have a detectable signal, the inverse problem becomes ill-conditioned close to and at the singular points \( r_s : \varphi'(r_s) = 0 \). Inspection of the literature shows that this condition is often met in experiments, and actually coincides always with the location of the source [10–15]. This is easily heuristically explained by noting that the lhs of equation (3) comes as a balance between the two contributions of the rhs: the transport (\( \Gamma \)) and the source (S). If we suppose that when the source term is maximum, it dominates over the transport contribution, then we obtain \( \xi \propto S \) and \( S' = 0 \rightarrow \xi' = 0 \).

At any point \( r \) we can compute the two eigenvalues \( (\lambda_0, 1) \) and eigenvectors \( (E_0, 1) \) of the matrix \( M : M \cdot E_i = \lambda_i E_i \). By writing \( Y = \gamma_0 E_0 + \gamma_1 E_1, \Gamma = \delta_0 E_0 + \delta_1 E_1 \), equation (4) becomes an equation for the two unknowns \( \gamma_0, \gamma_1 \). The eigenvalues and eigenvectors depend on the data (A, \( \varphi \)); for instance, the explicit expression for \( \lambda \) is
\[
\lambda = \frac{1}{2} [-A' \cos \varphi + A(1 + \varphi') \sin \varphi \\
\pm \sqrt{(A' \cos \varphi - A(1 + \varphi') \sin \varphi)^2 - 4 A^2 \varphi'}].
\]

At the singular points, one of the eigenvalues vanishes: \( \lambda_0(r_s) = 0 \). Accordingly, the lhs of equation (4) aligns along the other eigenvector
\[
M \cdot Y = M \cdot (\gamma_0 E_0 + \gamma_1 E_1) = \lambda_1 \gamma_1 E_1 = \Gamma, \quad r = r_s.
\]

Physically, transport coefficients must be defined everywhere, hence equation (7) must admit a solution at \( r = r_s \). This is possible only if its rhs also aligns along \( E_1 : \Gamma(r_s) = g_1 E_1 \). Unavoidable errors present in any measurement make this case unlikely, i.e. using the data from the experiment, one expects to find points \( r_s \) where \( \lambda_0(r_s) = 0 \) but \( \Gamma(r_s) = g_0 E_0 + g_1 E_1 \), with \( g_0(r_s) \neq 0 \). More generally, any perturbation of the values \( \lambda, E \), reflects upon \( Y \) as
\[
Y = \frac{g_0}{\lambda_0} E_0 + \frac{g_1}{\lambda_1} E_1 \rightarrow \delta Y = \left( \frac{\delta g_0}{\lambda_0} - \frac{g_0}{\lambda_0^2} \delta \lambda_0 \right) E_0 \\
+ \left( \frac{\delta g_1}{\lambda_1} - \frac{g_1}{\lambda_1^2} \delta \lambda_1 \right) E_1 + \frac{g_0}{\lambda_0} \delta E_0 + \frac{g_1}{\lambda_1} \delta E_1.
\]

Equation (8) may be used to estimate visually how given errors on the raw data, translated into error bars for \( \lambda, E \), propagate onto \( Y \). It is apparent how \( \delta Y \) blows up when the singular points are approached:
\[
\delta Y \approx -\frac{g_0 \delta \lambda_0}{\lambda_0} E_0, \quad \lambda_0 \rightarrow 0.
\]

Even if \( \Gamma(r_s) \propto E_1 \) and equation (7) becomes a well-defined equation, it does not place any condition upon the full solution \( Y \) at \( r = r_s \), but only on a subspace of it: i.e. it fixes \( y_1 = g_1/\lambda_1 \) whereas the component along the subspace parallel to \( E_0 \) is undefined: any arbitrary vector aligned along the local \( E_0 \) eigenvector can be added to the solution and remains compatible with the data. This is where the degeneracy of the problem enters. An example is shown in figure 1. The clouds in the two contour plots stand for the typical distributions of the couples \( (D, V) \) that one expects after a Monte Carlo simulation taking into account errors on the data. The thick black lines

---

**Figure 1.** Contour plot for the statistical distribution of \((D,V)\) couples. Data are produced by 400 independent realizations of synthetic data with stochastic perturbations added. Black lines are parallel to the local \( E_0 \) eigenvector. On the left, a regular point; on the right a singular point.
are parallel to the local $E_0$ eigenvector. The left plot is for a regular point: independent estimates distribute roughly as a normal distribution around the average value. The right plot shows what happens when a singular point is approached: the distribution aligns along $E_0$.

If we knew exactly $Y$ at all regular points $r \neq r_*$, then $y_0(r_*)$ could easily be fixed by imposing continuity of the solution there. However, this is not the case: equation (8) tells us that the unavoidable presence of errors in the measurement of $(A, \varphi)$ implies some ignorance on $y_{0,1}$ even at the regular points. It may be extremely small from far to the singular points, but unavoidably increases without bound as soon as we approach $r_*$ (equation (9)); hence, for practical purposes we cannot have an estimate for $y_0$ not only at $r_*$ but also near it.

4. On alternative approaches to estimating $(D, V)$ and the issue of regularization

Inferring from experiment transport coefficients $(D, V)$ is an important issue in plasma physics, since the complexity of plasmas makes first-principles calculations quite hard5. Accordingly, some ingenuity has been exerted on the numerical side. Reference [16] presents a recent and extensive review of the state of the art of perturbative transport experiments, including details of numerical algorithms and related issues. The most adopted recipe (see, e.g., instances in [10–15]) makes use of transport codes6, i.e. numerical codes that, for a given geometry and for a given transport approach $Y$, output a simulated profile for $\xi$: $\xi_{\text{sim}}$. The right plot is for a given transport code: equation (1) being a second-order differential equation, for solving it one needs two boundary conditions; therefore, a priori there is no warranty that they are the most appropriate. In this specific case, the regularization imposed by the code is not harmful far from the singular points, but close to them its validity needs careful examination in order not to constrain the solution between artificially small error bars. We will discuss one such an instance in the next section.

If we wish to apply equation (4) to experiments, we have to deal with measurements $\{A(l), \varphi(l)\}$ taken at a discrete set of points $r = r(l)$. This leads to the issues of data interpolation and extrapolation. Interpolation between neighbouring points is necessary to compute the derivatives that appear in the matrix $M$ as well as to compute the integrals in $\Gamma$. Extrapolation also enters in connection with $\Gamma$, since its definition requires computing integrals from $r = 0$ onwards, whereas data are taken within some finite range, usually not including the origin. We will consider in detail these issues in section 6, in connection with the analysis of true experimental data. Here, we highlight a difference between the present approach and the transport codes: equation (1) being a second-order differential equation, for solving it one needs two boundary conditions. The first one is the already mentioned zero-flux condition at the origin; as the second condition, the value of the field $\xi$ at the outermost measurement location is commonly taken, with the consequence that $(D, V)$ can no longer be estimated there. The present approach does not need this second boundary condition, and therefore saves some (possibly valuable) information for the modelling.

One rather surprising result appears when one looks how errors upon measurements7 ($\{A(l) \pm \delta A(l), \varphi(l) \pm \delta \varphi(l)\}$) propagate onto $Y$. Let $W = M^{-1}$. Writing (4) using Einstein’s convention on indices yields:

$$Y_i(r(l)) = W_{ij} \Gamma_j$$

$$\rightarrow \delta Y_i(r(l)) = \left( \frac{\partial W_{ij}}{\partial A(l)} + \frac{\partial \Gamma_j}{\partial A(l)} \right) \delta A(l)$$

$$+ \left( \frac{\partial W_{ij}}{\partial \varphi(l)} + \frac{\partial \Gamma_j}{\partial \varphi(l)} \right) \delta \varphi(l)$$

$$+ \sum_{k<l} \left( \frac{\partial \Gamma_j}{\partial A(l)} \delta A(k) + \frac{\partial \Gamma_j}{\partial \varphi(l)} \delta \varphi(k) \right).$$

(10)

The total error on $Y_i$ is due to a local contribution (depending on $r(l)$ alone) that includes the first two terms, and another

5 In view of the difficulty in carefully assessing $(D, V)$, sometimes simplified approaches are used, where it is postulated that the whole transport may be taken into account by just one mechanism, the diffusion, thus requiring just $D$ to be measured. In earlier works [17, 18] we showed that it is not always justified.

6 Of course, this same procedure may hold for other fields, not just plasma physics.

7 In actual experiments, the source $S$, too, is only known to within some finite precision. However, including errors on it in the analysis is straightforward.
one from the data taken at points \( r(k) < r(l) \), which appear because of their presence within the integrals. Allowing for independently distributed errors, the sum grows in absolute value roughly like the square root of the number of summands (i.e. roughly as \( r(l)^{1/2} \)). Hence, equation (10) hints to a generic propensity of modelling to become less and less accurate as the radius grows, although the precise behaviour can be easily reversed in particular cases due to the specific dependence of \( W \) and \( \Gamma \) on \( \{A(l), \phi(l)\} \).

5. Multiple experiments and extended transport models

The algorithm in section 3 refers to a very specific case, where a single Fourier component from one experiment is taken. However, nothing prevents considering several experiments or Fourier modes simultaneously. The only unavoidable constraint is that the plasma conditions remain the same, so that one can reasonably claim that transport (i.e. \( D, V \)) is also the same. Each set of data yields an instance of equation (4):

\[
M_i \cdot Y = \Gamma_i.
\]

This leads to an overdetermined problem, which allows recovering a common estimate for \( Y \) via, e.g., some weighted average. We consider the simplest case of two data sets. At any point \( r \) we obtain two estimates, each with its error bar, \( Y_1 \pm \Delta Y_1, Y_2 \pm \Delta Y_2 \). If the two values are consistent, within the error bars, we can extract a common estimate endowed with an error bar which is given by the overlap between the two independent estimates, hence is smaller than both \( \Delta Y_1 \) and \( \Delta Y_2 \). This is extremely valuable when \( r \) is a singular point for, say, \( Y_1 \), since we can estimate the local \( Y \) using \( Y_2 \). A different case is when, at all points, \( Y_1 \approx Y_2, \Delta Y_1 \approx \Delta Y_2 \). This means that we are practically dealing with two repetitions of the same experiment, hence no new information can be gained this way. The opposite and interesting case is when, even taking into account the error bars, the two estimates cannot agree: \( Y_1 \neq Y_2 \). This conflicting evidence is the proof that something is wrong either at the level of the data (i.e. at least one set of measurements is flawed) or at the level of modelling (that is, equation (1) is not the appropriate framework for modelling this kind of transport, or the transport coefficients are not the same between the two experiments).

The redundancy provided by multiple experiments can also be employed for assigning values to further parameters, within the framework of models that extend the physics beyond the convective-diffusive picture of equation (1). This is the case, e.g., for the transport of toroidal angular momentum in magnetically confined plasmas. As explained in detail in [19], an accurate modelling of toroidal rotation \( \Omega \) might require the explicit consideration of additive contributions to the flux, which do not directly depend on the rotation or on the external torque. At this stage, we do not afford a full-fledged investigation attempting to identify separate ingredients to these contributions: in principle, they could be neoclassical toroidal viscosity, residual stress terms, some ignorance about the NBI modulation, or other unidentified torques. In this work, we limit to the simple question: is it possible from the data and using the above method to infer anything about the possible existence of additional flux terms in equation (3), which do not directly depend on the measured quantity (toroidal momentum in our case)? An additional contribution to the flux writes, for the equilibrium and modulation:

\[
\Gamma = -D \nabla \Omega + V \Omega \rightarrow -D \nabla \Omega + V \Omega + R
\]

and, for perturbed quantities:

\[
\Gamma \rightarrow \Gamma + \delta \Gamma, \quad \delta \Gamma = -D \nabla \delta \Omega + V \delta \Omega + \delta R.
\]

After temporal Fourier transform, we may thus write the like of equations (4)–(6):

\[
M \cdot Y + R = \Gamma,
\]

where \( R \) is an array containing the real and imaginary part of \( R \). At this level of description, it must be treated as an unknown, on the same footing as \( Y \). Hence, equation (13) alone is not sufficient for fixing all the variables. However, using two independent experiments, we can write

\[
\begin{align*}
M_1 \cdot Y + R &= \Gamma_1, \\
M_2 \cdot Y + R &= \Gamma_2.
\end{align*}
\]

It is now straightforward to solve equations (14) for \( (Y, R) \).

To conclude this section, we recall that an important and straightforward instance of multiple-experiment analysis is by considering, together with modulation data, also the steady-state profiles. This information is often used in transport codes as a useful constraint.

6. Simulations of experimental data

We present two instances of the use of the present algorithm, all of them based on JET data. We summarize in a brief recipe how calculations are performed: (a) the fundamental bricks are the data \( \{A(l), \phi(l)\} \) along with the source \( S \). Prior to the analysis they are interpolated using smooth curves (cubic splines proved to work fine). However, if the data appear noisy, this procedure alone is not sufficient. The reason is that random fluctuations between neighbouring points, particularly if the average slope is small, may yield interpolating curves whose derivatives vanish at several points. These singular points are actually artefacts, but in view of the analysis of section 3, they produce wild fluctuations in the local estimate of transport coefficients. As a remedy, in the case of noisy data, we pre-processed them with a Gaussian filter. (b) The functions \( A(r), \phi(r), S(r) \) are used to compute the entries of the matrix \( M \) as well as of \( \Gamma \). The remaining part of the integrals, from \( r = 0 \) to \( r_1 \), the innermost measurement point, have been approximated using the trapezoidal rule:

\[
\int_0^{r_1} \frac{f(z)}{\Gamma} \, dz \approx \frac{(r_1)^2}{2} f(r_1)/2.
\]

(c) The couple \( (D, V) \) at each point is recovered from inversion of equation (4) (or, the triple \( (D, V, R) \) from system (14)).

(d) The sensitivity study is carried out via Monte Carlo techniques: each datum is at a first stage independently varied \( (A(l), \phi(l)) \rightarrow (A(l) \pm \delta A(l), \phi(l) \pm \delta \phi(l)) \), where \( \delta A(l), \delta \phi(l) \) are randomly picked up from a normal distribution with zero mean and standard deviation given by experimental errors. The steps (a–c) are then repeated with the new set of data, and new
coefficients \((D, V)\) computed. Repeating the whole procedure over a large number of independent runs yields a statistical distribution for \((D, V)\), from which confidence intervals can be drawn.

The first dataset refers to JET pulses 73701-2, 73704, 73707-9. They are part of a set of discharges explicitly designed for measuring momentum transport, as documented in [20]; hence, represent an excellent workbench for the present method. Here we do not provide information about the experimental setup, referring the reader to [20]. Only the relevant output is shown in figure 2: the amplitude and phase of the measured signal (the toroidal velocity of rotation), both taken at the fundamental frequency \((v_1 = 8.33 \text{ Hz})\) and the second harmonic \((v_2 = 2 \times v_1)\), together with the radial profile of the torque (obtained by modulating the NBI). Figure 3 presents the corresponding \((D, V)\) couples computed for both harmonics using equation (4). Data have been pre-filtered, with a kernel’s width equal to 20% of the minor radius\(^8\). Error bars have been built by running 20 independent realization of the data, each point being perturbed with a Gaussian noise of amplitude 20% (in the amplitude \(A\)) or 7.5/15° (in the phase \(\phi\), depending on the harmonic). Our results should be compared against, e.g., figures 9(a) and (b) of [20]. There is qualitative agreement, since both studies yield increasing trends for \(D\) and \(|V|\) in the outward direction, with a ratio of about (2–3) between the edge and the core values, whereas quantitatively our results somewhat exceed those of [20]. If we compare

\(^8\) The need for filtering small scales is apparent from figure 2: the fluctuations in the phase make several spurious singular points \((\phi' = 0)\) appear, which make reconstruction of transport coefficient there impossible. However, it can also be justified on the basis of the way the data in figure 2 were produced: they come from the mapping on plasma coordinates of measurements taken at 12 locations on the equatorial plane. This means that roughly any structure spanning less than about \(1/10\) of the radius is not likely to have physical meaning.
Figure 3. Transport coefficients computed from the data of figure 2: Diffusivity \( D \) (m\(^2\) s\(^{-1}\)) and (- pinch V) (m s\(^{-1}\)) for both the fundamental frequency of modulation \( \nu_1 \) and first overtone \( \nu_2 \). The colour code is the same as in figure 2. Different sets of data have been slightly horizontally shifted with respect to each other. Reconstructed coefficients span the range \( \rho < 0.55 \) since in the region \( 0.6 < \rho < 0.8 \) the phase of the signal is almost constant (figure 2, bottom row), making the reconstruction unfeasible: this is made apparent by the large error bars at the largest radii. For the same reason, some points are missing below \( \rho = 0.2 \).

the results obtained by analysing the data relative to the two frequencies \( \nu_1, \nu_2 \), they are fully compatible, although the fundamental harmonic yields more accurate estimates, having the larger amplitude.

There is some shot-to-shot difference in the \((D, V)\) profiles. A more complete investigation is obviously desirable attempting to assess what is the cause. It is beyond the scope and the length limits of this work, and will be postponed to future studies. However, for the moment we note that the discharges are not identical: they differ slightly in the equilibria (density scale length, \( q \) profile and equilibrium toroidal rotation), hence it is quite likely that at least part of the differences is for this reason.

As a second dataset we refer to the shots extensively reported in [15]: a set of JET L-mode-confinement discharges designed to study the transport of toroidal momentum in the presence of different torques and of magnetic ripple. The presence of finite ripple is expected to exert some supplementary torque, hence these discharges appear promising candidates for detecting further contributions to the flux, \( R \). We considered in detail shots 77090, 77091: in these shots, the imposed ripple was quite similar, except for a small difference in the equilibrium rotation, hence, we could speculate that the additional contribution is almost the same and use the formalism of equation (14). Figure 4 shows the \((A, \varphi, S)\) profiles (only the fundamental harmonic is considered in this analysis), and figure 5 the results \((D, V, R)\); 20 independent statistical realizations were used for obtaining the error bars.

Notice that, from figure 5, the three contributions carry approximately equal fractions of the total flux: \( D \partial_r \xi \approx V \xi \approx R \).

In a second simulation (figure 6), we compute the couples \((D, V)\) separately from each shot without accounting for the additional term \( R \) appearing in equations (11)–(14): \( R \rightarrow 0 \). There is good agreement between the two shots for radii smaller than about 0.5. This suggests that, inside this region, the presence or lack of \( R \) is not a discriminant. Conversely, in the outermost half-radius, the two \( D \) estimates in figure 6 are clearly incompatible. This implies that either the two shots do not share the same set of transport coefficients or that some contribution from \( R \) is needed. Since \( D \) in shot 77090 (red triangles) takes vanishingly small values—whose physical meaning is dubious—it is plausible to argue that a correct book-keeping can be restored only with the second option.

7. Conclusions

To summarize, we have discussed a computationally very light approach (hereafter, the matricial approach—MA) to the inversion of the one-dimensional convective–diffusive equation (1) under periodic forcing. Its major advantage is that it is direct, providing an explicit solution. This avoids, to a large extent, all the issues typical of iterative methods, which include the choice of the parametrization of the class of trial solution functions—and more generally their regularization—and the
Figure 4. Left, amplitude; centre, phase; right, torque. The data are the same as in [15] and refer to the fundamental modulation frequency $\nu = 6.25$ Hz. Red curves with triangles refer to shot 77090; blue curves with circles, to shot 77091.

Figure 5. Left, diffusivity $D$; centre, pinch $V$; additional flux, $R$, obtained from solving equation (14) with the data of figure 4. The results refer to the fundamental harmonic, $\nu = 6.25$ Hz.

Figure 6. Left, diffusivity $D$; right, pinch $V$ for the shots 77090 (red symbols), and 77091 (blue symbols). Transport coefficients are computed using equation (4), i.e. neglecting $R$.

numerically heavy minimization procedure. Furthermore, it provides a clear geometrical foundation to the nature and size of uncertainties in profile reconstruction. The radius-by-radius reconstruction enables us to make the local uncertainty $\delta Y(r)$ explicit. The algebraic inversion yields a high precision in the reconstruction of transport profiles: indeed, this method is not restricted by the $a$ priori guess of the trial profiles, but by that of these derivatives of $(A, \varphi)$, which is generally much more reliable and controllable. In the presence of singular points, the related huge uncertainty makes the estimate of $Y(r)$ useless for practical purposes: in this case, some regularization is still useful. This is achieved by overlapping the solutions’ uncertainty intervals from various experiments, where the same transport is assumed to hold: it provides either a way to improve the precision of the reconstruction (the case of a nonvanishing overlap) or to prove the set of initial assumptions in the reconstruction to be wrong (the case of a vanishing overlap). The MA can help in designing $a$ priori the combination of perturbation measurements susceptible of improving the precision of the reconstruction of transport profiles. It can also be easily extended to include parameters other than $D$ and $V$.

Acknowledgments

This work was supported by EURATOM and carried out within the framework of the European Fusion Development Agreement. The views and opinions expressed herein do not necessarily reflect those of the European Commission. The authors thank R Paccagnella for careful reading of the manuscript. The authors wish to thank an anonymous referee for pointing [5] to their attention and for other useful suggestions.

© Euratom 2012.

References

[1] Sonnerup B U Õ, Hasegawa H. The W-L and Hau L-N 2006 J. Geophys. Res. 111 A09204
[2] Necat Ozisik M and Orlande H R B 2000 Inverse Heat Transfer (London: Taylor and Francis)
[3] Isakov V 2005 Inverse Problems for Partial Differential Equations (Berlin: Springer)
[4] Neumaier A 1998 SIAM Rev. 40 636
[5] Krieger K et al 1990 Nucl. Fusion 30 2392
[6] Takenaga H et al 1998 Plasma Phys. Control. Fusion 40 183
[7] Escande D F and Sattin F 2012 Phys. Rev. Lett. 108 125007
[8] Risken H 1996 The Fokker–Planck Equation (Berlin: Springer)
[9] Andreev V F and Kasyanova N V 2005 Plasma Phys. Rep. 31 709
[10] Lopes Cardozo N J 1995 Plasma Phys. Control. Fusion 37 799
[11] Ryter F et al 2000 Nucl. Fusion 40 1917
[12] Mantica P et al 2005 Phys. Rev. Lett. 95 185002
[13] Mantica P et al 2006 Plasma Phys. Control. Fusion 48 385
[14] Mantica P et al 2008 Fusion Sci. Technol. 53 1152
[15] Salmi A T et al 2011 Plasma Phys. Control. Fusion 53 085005
[16] Ryter F, Dux R, Mantica P and Tala T 2010 Plasma Phys. Control. Fusion 52 124043
[17] Escande D F and Sattin F 2007 Phys. Rev. Lett. 99 185005
[18] Escande D F and Sattin F 2008 Plasma Phys. Control. Fusion 50 124023
[19] Peeters A G et al 2011 Nucl. Fusion 51 094027
[20] Tala T et al 2011 Nucl. Fusion 51 123002