Pinning down $|\Delta c| = |\Delta u| = 1$ couplings with rare charm baryon decays

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We study the full angular distribution of semileptonic rare charm baryon decays in which the secondary baryon undergoes weak decay with sizable polarization parameter, $\Xi^+_c \to \Sigma^+ (\to p\pi^0)\ell^+\ell^-$, $\Xi^0_c \to \Lambda^0 (\to p\pi^-)\ell^+\ell^-$ and $\Omega^0_c \to \Xi^0 (\to \Lambda^0\pi^0)\ell^+\ell^-$. Such self-analyzing decay chains allow for seven additional observables compared to three-body decays such as $\Lambda_c \to p\ell^+\ell^-$, with different sensitivities to the $|\Delta c| = |\Delta u| = 1$ weak couplings. Opportunities to test the standard model in $c \to u$ transitions with standard model null tests and other angular observables are worked out. We show that a joint model-independent analysis of the leptonic $A^{ℓFB}_ℓ$, hadronic $A^{HFB}_H$, and combined forward-backward asymmetry $A^{HFB}_FB$ together with the fraction of longitudinally produced leptons, $F_L$, is able to pin down the dipole couplings $C_7, C_7'$ and the semileptonic (axial-) vector ones $C_{10}, C_{10}', C_{10}''$. $A^{HFB}_FB$ is also accessible with dineutrino $c \to u\nu\bar{\nu}$ modes and probes right-handed currents.

I. INTRODUCTION

Flavor changing neutral currents of charm quarks are strongly suppressed in the standard model (SM) by an efficient Glashow-Iliopoulos-Maiani (GIM) mechanism. At the same time sizable resonance contributions shadow new physics (NP) in simple observables such as branching ratios of semileptonic $c \to u\ell^+\ell^-$-induced modes [1]. This very GIM suppression, on the other hand, along with approximate symmetries of the SM gives directions for clean observables and null tests, which probe a broad range of NP phenomena. Corresponding SM tests with $|\Delta c| = |\Delta u| = 1$ transitions complement beyond standard model (BSM) searches with strange and beauty quark processes and provide novel and unique insights into flavor from the up-quark sector. Several opportunities to test the SM have been worked out for $D$-meson decays, e.g. [2–8].

Rare semileptonic decays of charm baryons have been explored as a NP probe recently [9–12]. In [12] we analyzed the NP sensitivity of rare semileptonic decays of $\Lambda_c, \Xi_c$ and $\Omega_c$-baryons, here collectively denoted by $B_0 \to B_1\ell^+\ell^-$ with the initial (daughter) baryon denoted by $B_0(B_1)$, see

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TABLE I: Self-analyzing rare charm four-body decays $B_0 \to B_1(\to B_2 \pi)\ell^+\ell^-$ and information on the branching ratios and weak decay parameters $\alpha$ of the secondary baryonic $B_1 \to B_2 \pi$ decay \cite{10}.

| $B(B_1 \to B_2 \pi)$ | $\Xi_c^+ \to \Sigma^+ (\to p\pi^0)\ell^+\ell^-$ | $\Xi_c^0 \to \Lambda^0 (\to p\pi^0)\ell^+\ell^-$ | $\Omega_c^0 \to \Xi^0 (\to \Lambda^0\pi^0)\ell^+\ell^-$ |
|----------------------|---------------------------------|---------------------------------|---------------------------------|
| $\alpha$             | $-0.98 \pm 0.01$               | $0.73 \pm 0.01$               | $-0.36 \pm 0.01$               |

\cite{10} for dineutrino modes $B_0 \to B_1\nu\bar{\nu}$. In this work we consider (quasi-) four-body decays, where the $B_1$ further decays weakly to a hyperon $B_2$ and a pion. Since kinematic observables, such as the direction of the $B_2$ momentum, provide information on the spin of the decaying $B_1$ baryon, these channels are termed ‘self-analyzing’. Advantages of such modes for NP searches are well-known in $b$-physics, notably using $\Lambda(1116) \to p\pi$ in rare decays of $\Lambda_b$-baryons, see for instance Ref. \cite{13}. Our aim is to work out null tests and to complement $\Lambda^0\pi^0$-flavor, notably using $\Lambda(1116) \to p\pi$ in rare decays of $\Lambda_b$-baryons, see for instance Ref. \cite{13}. Our aim is to work out null tests and to complement $\Lambda^0\pi^0$-flavor, notably using $\Lambda(1116) \to p\pi$ in rare decays of $\Lambda_b$-baryons, see for instance Ref. \cite{13}.

In charm, we identify the following decay channels suitable for polarization studies,

$$
\Xi_c^+ \to \Sigma^+ (\to p\pi^0)\ell^+\ell^- , \Xi_c^0 \to \Lambda^0 (\to p\pi^0)\ell^+\ell^- , \Omega_c^0 \to \Xi^0 (\to \Lambda^0\pi^0)\ell^+\ell^- ,
$$

$$
\Xi_c^+ \to \Sigma^+ (\to p\pi^0)\nu\bar{\nu} , \Xi_c^0 \to \Lambda^0 (\to p\pi^0)\nu\bar{\nu} , \Omega_c^0 \to \Xi^0 (\to \Lambda^0\pi^0)\nu\bar{\nu} ,
$$

since $\Sigma^+, \Lambda^0$ and $\Xi^0$ are self-analyzing, with sizable decay parameter $\alpha$ (we do not consider $\Xi^0 \to \Lambda\gamma$). The branching ratios and decay parameters of the secondary baryon decays are provided in Tab.\textsuperscript{I}. Our aim is to work out null tests and to complement $|\Delta c| = |\Delta u| = 1$ analyses of charmed meson decays, e.g. \cite{6,7}.

Requisite vector and tensor form factors for $\Lambda_c \to p$ transitions have been computed on the lattice \cite{9} and in quark models \cite{14}, and for $\Xi_c \to \Sigma$ from Light cone sum rules \cite{18}. As in \cite{12}, we employ $\Lambda_c \to p$ lattice form factors \cite{9} and relate them to the $\Xi_c$, $\Omega_c$ ones using $SU(3)_F$-flavor symmetries, if applicable. This procedure is improvable with better knowledge of the form factors, however, to explore NP signals in SM null tests a precise knowledge of form factors is not essential.

None of the rare charm baryon modes has been observed so far, but the upper limit on the $\Lambda_c \to p\mu^+\mu^-$ branching ratio at $\sim 10^{-7}$ by LHCb \cite{19} is close to the estimated size of the resonance contributions \cite{12}. Limits on $\Lambda_c \to pe^+e^-$ and lepton flavor violating ones $\Lambda_c \to p\bar{\nu}\nu$ are at the level of $\sim 10^{-5}$ by BaBar \cite{20}. Semileptonic rare charm baryon decays are suitable for study at high luminosity flavor facilities, such as LHCb \cite{21}, Belle II \cite{22}, BES III \cite{23}, and possible future machines \cite{24,25}.

The plan of the paper is as follows: In Sec.\textsuperscript{II} we discuss exclusive rare charm baryon decay modes within a low energy effective field theory (EFT) framework, including phenomenological resonance contributions. We also present the full angular distribution for four-body baryon decays
and review some of the simpler observables already accessible with three-body decays. We work out the impact of the new null tests and other clean NP probes in Sec. III and give an early stage strategy to disentangle NP Wilson coefficients. In Sec. IV we present further null tests, based on more advanced angular observables, with decays into dineutrinos and for decays of polarized charm baryons. We conclude in Sec. V. We present the helicity amplitudes in terms of Wilson coefficients and form factors in App. A, and provide details on the helicity amplitude description of the secondary weak decay in App. B. In App. C we give the full angular distribution for initially polarized baryon decays.

II. THEORY OF $|\Delta c| = |\Delta u| = 1$ FOUR-BODY BARYON DECAYS

We give general formulae for semileptonic rare charm baryon decays in the SM and beyond. In Sec. II A we introduce the weak Hamiltonian at the charm mass scale and discuss SM contributions. The fully differential distribution for the (quasi-)four-body decay of unpolarized charmed baryons is presented in Sec. II B.

A. An effective field theory approach to charm physics

Consider the weak effective Hamiltonian for $c \to u\ell^+\ell^-$ transitions

$$\mathcal{H}_{\text{eff}} \supset -\frac{4G_F \alpha_e}{\sqrt{2} 4\pi} \sum_{k=7,9,10} \left(C_k O_k + C'_k O'_k\right), \tag{1}$$

where $\alpha_e$ and $G_F$ denote the fine structure and Fermi’s constant, respectively. The dimension six operators are given as

$$O_7 = \frac{m_c}{e} (\overline{\pi}_L \sigma_{\mu\nu} c_R) F^{\mu\nu}, \quad O'_7 = \frac{m_c}{e} (\overline{\pi}_R \sigma_{\mu\nu} c_L) F^{\mu\nu},$$

$$O_9 = (\overline{\pi}_L \gamma_{\mu} c_L) (\overline{\ell} \gamma^\mu \ell), \quad O'_9 = (\overline{\pi}_R \gamma_{\mu} c_R) (\overline{\ell} \gamma^\mu \ell),$$

$$O_{10} = (\overline{\pi}_L \gamma_{\mu} c_L) (\overline{\ell} \gamma^{\mu\nu} \gamma_5 \ell), \quad O'_{10} = (\overline{\pi}_R \gamma_{\mu} c_R) (\overline{\ell} \gamma^{\mu\nu} \gamma_5 \ell),$$

with the electromagnetic field strength tensor $F^{\mu\nu}$, the chiral projectors $L = (1 - \gamma_5)/2$, $R = (1 + \gamma_5)/2$ and $\sigma^{\mu\nu} = \frac{i}{2} \left[\gamma^\mu, \gamma^\nu\right]$. For the mass of the charm quark we use $m_c(m_c) = 1.27$ GeV, in the $\overline{\text{MS}}$ mass scheme. SM contributions to the coefficients of the operators in Eq. (1) arise from four-quark operators at the $W$-mass scale and from intermediate resonances $M$, decaying electromagnetically to dileptons, as in the quasi four-body decay chain $B_0 \to B_1 M(\to \ell^+\ell^-) \to B_1 \ell^+\ell^- \to B_1(\to B_2 \pi) \ell^+\ell^- \to B_2 \pi \ell^+\ell^-$. Note, the lifetime of the resonances $M = \omega, \rho, \phi$ is
TABLE II: Resonance parameters $a_\omega$, $a_\phi$ defined in (3) for various rare charm baryon transitions, see text.

| $\Lambda_c \to p$ | $\Xi_c^+ \to \Sigma^+$ | $\Xi^0_c \to \Lambda^0$ | $\Omega^0_c \to \Sigma^0$ |
|------------------|-------------------------|------------------------|--------------------------|
| $a_\omega$       | $0.062 \pm 0.009$       | $\sim 0.06$           | $\sim 0.05$              |
| $a_\phi$         | $0.110 \pm 0.008$       | $\sim 0.1$            | $\sim 0.09$              |

much shorter than the one of the daughter baryons $B_1 = \Sigma^+, \Lambda, \Xi^0$, which decay weakly after the dileptons have been produced. The resonance contributions are taken into account with a phenomenological ansatz, as

$$C_R(q^2) = a_\omega e^{i \delta_\omega} \left( \frac{1}{q^2 - m_\omega^2 + i m_\omega \Gamma_\omega} - \frac{3}{q^2 - m_\rho^2 + i m_\rho \Gamma_\rho} \right) + a_\phi e^{i \delta_\phi},$$

implying a contribution to $O_9$. Here, $m_M$ and $\Gamma_M$ denote the mass and total width of the meson $M$.

The strong phases $\delta_M$ are unknown and provide a significant amount of theoretical uncertainty. We neglect effects from intermediate $\eta, \eta'$ mesons as they are strongly localized and have a negligible effect on the (differential) branching ratio [12]. We further use isospin to relate the $\rho$ and $\omega$ contributions [26], as no data on any of the $B_0 \to B_1 \rho$ branching ratios is available. Experimental input on the parameters $a_M$ is provided in Tab. II. Note that due to Belle’s recent measurement of $B(\Lambda_c^+ \to p \omega)$ [27] the corresponding entry slightly differs from the one in [12].

For the form factors we use the same helicity-based definition as in [9, 12]. Form factors from lattice computations for $\Lambda_c \to p$ transitions are obtained in Ref. [9]. We obtain the form factors for the baryon transitions studied in this work via flavor symmetries, see Refs. [12, 28] for details. Consequently, we find for any of the ten form factors commonly denoted here as $f_{B_0 \to B_1}$

$$f_{\Lambda_c \to p} = f_{\Xi_c^+ \to \Sigma^+} = \sqrt{6} f_{\Xi^0_c \to \Lambda^0} \simeq f_{\Omega^0_c \to \Sigma^0}.$$

We emphasize that all but the last relation follow from $SU(3)_F$ symmetry. The connection to the $\Omega_c$ is broken as it sits in a different multiplet. In absence of form factor determinations for the latter at the same level as those for the $\Lambda_c \to p$ we use this simple relation to be able to make progress. We stress that this ansatz does not affect the null test features discussed in this work.

The relations (4) have also been used for $B_0 \to B_1 (\phi, \omega)$ to obtain the $a_M$ factors for the decays other than $\Lambda_c \to p (\phi, \omega)$ presented in Tab. II. Specifically, the $\Lambda_c^+ \to p$ parameters serve as an input to all other modes, as branching ratio data for the latter are not available. An exception is $B(\Xi^0_c \to \Lambda^0 \phi) = (4.9 \pm 1.5) \times 10^{-4}$ [29], which gives $a_\phi = 0.080 \pm 0.013$, consistent with the value in Tab. II.

Due to the severe GIM cancellation in rare charm decays, the perturbative SM contributions are overwhelmed by the effects from intermediate resonances: Perturbatively, $C^{\text{eff}}_7(q^2) \sim 10^{-3}$,
\( C^\text{eff}(q^2) \sim 10^{-2} \), whereas the \( \rho, \omega, \phi \) resonances yield \( C^R_q(q^2) \sim \mathcal{O}(10) \) on resonance peaks and \( \sim \mathcal{O}(1) \) off peak, see [12], based on results in [5, 30, 31]. The primed Wilson coefficients of (2) are suppressed by \( m_u/m_c \) and are negligible in the SM. The Wilson coefficient \( C_{10} \) vanishes in the SM, and therefore leptonic axial vector currents do, too, providing a prime opportunity for null test searches in charm. Electromagnetic loop contributions to the matrix element of 4-quark operators, or mixing, induce contributions not exceeding permille level [6].

B. Fully differential distribution for \( m_\ell \neq 0 \)

We present the full differential decay distribution for four-body decays \( B_0 \rightarrow B_1(\rightarrow B_2\pi)\ell^+\ell^- \). A brief discussion of \( B_0 \rightarrow B_1(\rightarrow B_2\pi)\nu\bar{\nu} \) decays is deferred to Sec. IVB. We compute the distribution using the helicity formalism [32, 33] for unpolarized charmed baryons and keeping finite lepton masses, \( m_\ell \neq 0 \). Details on the helicity amplitudes are given in App. A. To be specific, expressions are given for the decay \( \Xi_+^c \rightarrow \Sigma^+(\rightarrow p\pi^0)\ell^+\ell^- \), however, with replacements of masses, form factors and \( B_1 \rightarrow B_2\pi \) branching ratios, the same holds for any of the other modes in Tab. I. The fully differential distribution can be parameterized in terms of the ten \( q^2 \)-dependent angular observables

\[
K_i = K_i(q^2) = \frac{d^4\Gamma}{dq^2d\cos\theta_\ell d\cos\theta_\pi d\phi}
\]

as

\[
d^4\Gamma = \frac{3}{8\pi} \left[ K_{1ss} \sin^2\theta_\ell + K_{1cc} \cos^2\theta_\ell + K_{1c} \cos\theta_\ell \\
+ (K_{2ss} \sin^2\theta_\ell + K_{2cc} \cos^2\theta_\ell + K_{2c} \cos\theta_\ell) \cos\theta_\pi \\
+ (K_{3sc} \sin\theta_\ell \cos\theta_\ell + K_{3s} \sin\theta_\ell) \sin\theta_\pi \sin\phi \\
+ (K_{4sc} \sin\theta_\ell \cos\theta_\ell + K_{4s} \sin\theta_\ell) \sin\theta_\pi \cos\phi \right].
\]

Here, \( \theta_\ell \) is the angle of the \( \ell^+ \) with respect to the negative direction of flight of the charmed baryon \( (\Xi^+_c) \) in the dilepton rest frame. Similarly, \( \theta_\pi \) is the angle between the momentum of the final state \( (B_2) \) baryon \( (p) \) and the negative direction of flight of the \( B_1 \) baryon \( (\Sigma^+) \) in the proton-pion center-of-mass frame. The azimuthal angle \( \phi \) describes the angle between the dilepton and the \( p\pi^0 \) decay planes. The allowed regions for the angles \( \theta_\ell, \theta_\pi, \phi \) are \(-1 \leq \cos\theta_\ell \leq +1, -1 < \cos\theta_\pi < 1 \) and \( 0 < \phi < 2\pi \).
The $q^2$-dependent coefficients $K_i$ are given as\footnote{We adapt the notation of helicity expressions $I_{i}^{P\mu\nu}$, $i = 1, 2, 3, 4$ from\cite{14}, however use them to formulate angular observables in a notation similar to\cite{15}. Note that we dropped the subscript $P$ from $I_{i}^{mm'}$, $I_{3}^{mm'}$ since these two interference terms are parity-even.}

\begin{align*}
\frac{K_{1ss}}{\mathcal{B}(\Sigma^+ \to p\pi^0)} &= q^2 v^2 \left( \frac{1}{2} U^{11+22} + L^{11+22} \right) + 4m_\ell^2 (U^{11} + L^{11} + S^{22}) , \\
\frac{K_{1cc}}{\mathcal{B}(\Sigma^+ \to p\pi^0)} &= q^2 v^2 U^{11+22} + 4m_\ell^2 (U^{11} + L^{11} + S^{22}) , \\
\frac{K_{1c}}{\mathcal{B}(\Sigma^+ \to p\pi^0)} &= -2q^2 v P^{12} , \\
\frac{K_{2ss}}{\mathcal{B}(\Sigma^+ \to p\pi^0)} &= q^2 v^2 \left( \frac{1}{2} P^{11+22} + L^{11+22} \right) + 4m_\ell^2 (P^{11} + L^{11} + S^{22}) , \\
\frac{K_{2cc}}{\mathcal{B}(\Sigma^+ \to p\pi^0)} &= q^2 v^2 P^{11+22} + 4m_\ell^2 (P^{11} + L^{11} + S^{22}) , \\
\frac{K_{2c}}{\mathcal{B}(\Sigma^+ \to p\pi^0)} &= -2q^2 v U^{12} , \\
\frac{K_{3sc}}{\mathcal{B}(\Sigma^+ \to p\pi^0)} &= -2\sqrt{2}q^2 v I_{2}^{11+22} , \\
\frac{K_{3s}}{\mathcal{B}(\Sigma^+ \to p\pi^0)} &= 4\sqrt{2}q^2 v I_{4}^{12} , \\
\frac{K_{4sc}}{\mathcal{B}(\Sigma^+ \to p\pi^0)} &= 2\sqrt{2}q^2 v I_{2}^{11+22} , \\
\frac{K_{4s}}{\mathcal{B}(\Sigma^+ \to p\pi^0)} &= -4\sqrt{2}q^2 v I_{3}^{12} ,
\end{align*}

in agreement with our own computation and\cite{34}. Here, $v = \sqrt{1 - \frac{4m_\ell^2}{q^2}}$, $U^{11+22} = U^{11} + U^{22}$ and likewise for $L, P, I_{1P}, I_{2}$. The $q^2$-dependent terms $U, L, S, P, L_P, S_P$ denote quadratic expressions of helicity amplitudes and correspond to unpolarized transverse, longitudinal, scalar, transverse parity-odd, longitudinal parity-odd and scalar parity-odd contributions, respectively. The coefficients $I_{1P}, I_{4P}$ and $I_{2}, I_{3}$ correspond to longitudinal-transverse interference terms, where the subscript $P$ refers to the parity-odd ones. We refer to App.\cite{A} for expressions in terms of Wilson coefficients and hadronic form factors, $f_i(q^2), g_i(q^2), i = +, - , 0$ and $h_j(q^2), \tilde{h}_j(q^2), j = +, - , \perp$, which are defined in Ref.\cite{9,12}.

The GIM mechanism is responsible for the absence of leptonic axial-vector currents in rare charm decays. Therefore, neglecting higher order electromagnetic contributions to $C_{10}$\cite{6},

$$K_{1c}^{SM} = K_{2c}^{SM} = K_{3s}^{SM} = K_{4s}^{SM} = 0 .$$

At the same time, these angular observables serve as clean null tests of the SM. The first one, $K_{1c}$, has already been studied in $\Lambda_c \rightarrow p\mu^+\mu^-$\cite{9} and three-body $1/2 \rightarrow 1/2\ell^+\ell^-$ decays of $\Lambda_c, \Xi_c$ and
$\Omega_c$’s [12]. The other three null tests, $K_{2c}, K_{3s}$ and $K_{4s}$ are a new result of this work. They become accessible in four-body decays, and vanish for $\alpha = 0$. We analyze the NP sensitivity in Section III.

Let us recap basic features of the distribution Eq. (5). If both $\theta_\pi$ and $\phi$ are not measured, only the first line survives and one recovers the double differential distribution for three-body decays:

$$\frac{d^2\Gamma}{dq^2 d \cos \theta_\ell} = \int_{-1}^{1} \int_{0}^{2\pi} \frac{d^4 \Gamma}{dq^2 d \cos \theta_\ell d \cos \theta_\pi d \phi} d \phi \cos \theta_\pi \frac{3}{2} \left( K_{1ss} \sin^2 \theta_\ell + K_{1cc} \cos^2 \theta_\ell + K_{1c} \cos \theta_\ell \right).$$

(8)

From here follows the $q^2$-differential decay rate

$$\frac{d\Gamma}{dq^2} = \int_{-1}^{1} \frac{d^2 \Gamma}{dq^2 d \cos \theta_\ell} d \cos \theta_\ell = 2 K_{1ss} + K_{1cc},$$

(9)

the longitudinal fraction of the dilepton system, $F_L$,

$$F_L = \frac{2 K_{1ss} - K_{1cc}}{2 K_{1ss} + K_{1cc}},$$

(10)

and the forward-backward asymmetry of the leptonic scattering angle, $A_{FB}$:

$$A_{FB}^L \equiv \frac{1}{d\Gamma/dq^2} \left[ \int_{0}^{1} - \int_{-1}^{0} \right] \frac{d^2 \Gamma}{dq^2 d \cos \theta_\ell} d \cos \theta_\ell = \frac{3}{2} \frac{K_{1c}}{2 K_{1ss} + K_{1cc}},$$

(11)

see [12] for a detailed discussion of the phenomenology in and beyond the SM.

Kinematic endpoints are $q_{\min}^2 = 4m_\ell^2$, corresponding to maximum hadronic recoil, and $q_{\max}^2 = (m_{B_0} - m_{B_1})^2$, corresponding to zero hadronic recoil. The latter is subject to symmetry relations, enforcing $K_{1ss} = K_{1cc}$, hence $F_L = 1/3$ and similarly $K_{2ss} = K_{2cc}$ model-independently at this point [33]. These relations hold also at the other end of the spectrum, at $q_{\min}^2$, because here the four-momenta of the leptons coincide which leads also to a reduction of Lorentz structures [12].

The integrated decay rate is obtained as

$$\Gamma = \int_{q_{\min}^2}^{q_{\max}^2} (2 K_{1ss} + K_{1cc}) dq^2,$$

(12)

where phase space cuts may be applied. Integrating the full $q^2$ region with $\pm 40\text{MeV}$ cuts around the $\omega$ and the $\phi$ resonances, we find

$$B(\Xi_c^+ \rightarrow \Sigma^+ (\rightarrow p\pi^0) \mu^+ \mu^-) \sim 1.8 \times 10^{-8},$$

$$B(\Xi_c^0 \rightarrow \Lambda^0 (\rightarrow p\pi^-) \mu^+ \mu^-) \sim 2.4 \times 10^{-9},$$

(13)

$$B(\Omega_c^0 \rightarrow \Xi^0 (\rightarrow \Lambda^0 \pi^0) \mu^+ \mu^-) \sim 2.5 \times 10^{-8},$$

in agreement with results in Ref. [12] multiplied with $B(B_1 \rightarrow B_2 \pi)$ given in Tab. I.
III. PROBING NP WITH ANGULAR OBSERVABLES

In this section we discuss angular observables in rare, semileptonic charm baryon decays that can cleanly signal NP, and work out sensitivities to specific Wilson coefficients. The full angular distribution features four GIM-based null tests, $K_{1c}, K_{2c}, K_{3s}$ and $K_{4s}$, which vanish in the SM. In addition, $F_L$ is also a sensitive probe of NP. In Sec. IIIA we work out BSM signatures in $F_L$ and in a similarly simple and sensitive observable, the hadronic forward-backward asymmetry, $A_{FB}$. $K_{1c}$ and $K_{2c}$ correspond to the leptonic and combined leptonic-hadronic forward-backward asymmetries, respectively. They are discussed in Sec. IIIB. Null tests in the longitudinal-transverse interference terms $K_{4s} \sim I_{12}^L$ and $K_{3s} \sim I_{12}^H$ are analyzed in the next Sec. IV. We summarize a strategy to disentangle Wilson coefficients based on three asymmetries and $F_L$ in Sec. IIIC.

A. $A_{FB}^H$ and $F_L$

The hadronic forward-backward asymmetry $A_{FB}^H$ is defined similar to the leptonic one, $A_{FB}^L$, Eq. (11), as

$$A_{FB}^H = \frac{1}{d\Gamma/dq^2} \left[ \int_0^1 - \int_{-1}^0 \right] \int_0^{2\pi} \frac{d^4\Gamma}{dq^2 d\cos \theta_e d\cos \theta_\pi d\phi} d\phi d\cos \theta_e d\cos \theta_\pi$$

$$= \frac{1}{2} \frac{K_{2ss} + K_{2cc}}{2K_{1ss} + K_{1cc}}.$$  

Unlike $A_{FB}^L$, $A_{FB}^H$ is not a null test of the SM, however, it turns out to be a highly sensitive probe of right-handed quark currents as illustrated in the left plot of Fig. 1. Here, the orange curve displays the SM expectation, and several NP benchmarks are shown in red, green and blue. The brackets in the subscripts are understood as or, for instance, $C_{9,(10)} = 0.5$ is short for $C_{9} = 0.5$ or $C_{10} = 0.5$. We learn that $A_{FB}^H$ is sensitive to $C_{7}^r, C_{9}^r$ and $C_{10}$. $A_{FB}^H$ shares features with $F_L$, shown in the right plot of Fig. 1. Cancellation of hadronic uncertainties in the SM (orange), strong sensitivity to NP contributions in some Wilson coefficients and large uncertainties in NP scenarios due to unknown strong phases, observed previously for $F_L$ in 12. The main differences between these two angular observables are the following:

- $F_L = 1/3$ at both kinematic endpoints of maximum and zero recoil, whereas $A_{FB}^H$ is unconstrained at low $q^2$ and vanishes at maximum $q^2$.
- $F_L$ is mostly sensitive to radiative dipole couplings $C_7$ and $C_7^r$, see the blue and green bands in the right plot of Fig. 1. $A_{FB}^H$ is similar (equal) to the SM in scenarios involving $C_7$ or $C_9$. 

FIG. 1: The hadronic forward-backward asymmetry $A_{FB}^H$ (left plot) for $\Xi_c^+ \rightarrow \Sigma^+(\rightarrow p\pi^0)\mu^+\mu^-$ decays in the SM (orange) and in NP scenarios with $C_7$ or $C_9 = 0.3$, $C_9'$ or $C_{10}' = 0.5$ and $C_7' = 0.3$ in blue, red and green, respectively. A NP scenario with $C_{10}$ only is not shown, as it is indistinguishable from the SM. The right plot shows the fraction of longitudinally polarized dimuons $F_L$ in the SM (orange) and NP scenarios $C_7 = 0.3$, $C_9 = 0.5$ and $C_7' = 0.3$ in blue, red and green, respectively. Scenarios with $C_9$ and $C_{10}'$ can not be distinguished from the SM with $F_L$ and are not shown. The width of the bands stem predominantly from unknown strong phases.

$(C_{10})$, but strongly altered in scenarios involving right-handed currents $C_7'$, $C_9'$, $C_{10}'$, see the green and red bands in the left plot of Fig. 1

The different impact of left-handed and right-handed NP contributions to $A_{FB}^H$ can be attributed to the parity behavior of the angular observables. While $K_{1ss}$ and $K_{1cc}$ are P-even observables, $K_{2ss}$ and $K_{2cc}$ are P-odd. This leads to cancellations between numerator and denominator only in the case of left-handed contributions. To illustrate this consider $A_{FB}^H$ for $m_\ell = 0$ in scenarios with $C_9$ and $C_{10}'$ and all other NP coefficients switched off. It can be written as

$$A_{FB}^H = -\alpha \cdot \frac{\left(|C_9|^2 + |C_{10}|^2 - |C_{10}'|^2\right) A(q^2)\sqrt{s_+s_-}}{\left(|C_9|^2 + |C_{10} - C_{10}'|^2\right) B(q^2)s_+ + \left(|C_9|^2 + |C_{10} + C_{10}'|^2\right) C(q^2)s_-},$$

where $A(q^2)$, $B(q^2)$ and $C(q^2)$ contain form factors and kinematics and are given in App. A and $s_\pm = (m_{B_0} \pm m_{B_1})^2 - q^2$. For $C_{10}' = 0$ the coefficient $C_9$ cancels as in $F_L$, leading to the thin SM (orange) band. For $C_{10} \neq 0$ the same effect happens and $|C_9|^2 + |C_{10}|^2$ drops out. On the other hand, for $C_{10}' \neq 0$ the numerator is proportional to $|C_9|^2 - |C_{10}'|^2$, which does not cancel against the denominator and leads to NP deviations with $q^2$-shape driven by $C_9^R(q^2)$. The same arguments holds for dipole couplings $C_7$ and $C_7'$: $A_{FB}^H$ is strongly sensitive to the latter, but not the former.
Note, interference terms between $C_9$ and $C_7$ softly break the exact cancellation (blue band around the SM). Again we stress that the requisite additional minus sign in front of the primed Wilson coefficients arises because $K_{2ss}$ and $K_{2cc}$ are P-odd.

B. $A_{FB}^H$ and $A_{FB}^L$

A third forward-backward asymmetry arises from combining leptonic and hadronic ones, $A_{FB}^H$:

$$A_{FB}^H = \frac{1}{d\Gamma/dq^2} \left[ \int_0^1 - \int_{-1}^0 \right] \left[ \int_0^1 - \int_{-1}^0 \right] \int_0^{2\pi} \frac{d^4\Gamma}{dq^2 d\theta d\theta'} d\phi d\cos \theta d\cos \theta'$$

$$= \frac{3}{4} \frac{K_{2c}}{2K_{1ss} + K_{1cc}}.$$  \hspace{1cm} (16)

It is yet another charming null test of the SM, because $C_{10}$ or $C'_{10}$ are required to observe a non-vanishing signal.

**FIG. 2:** The left plot shows the forward-backward asymmetry in both hadronic and leptonic scattering angles, $A_{FB}^H$\(\text{[16]}\) for $\Xi^+_c \rightarrow \Sigma^+_c (\rightarrow p\pi^0)\mu^+\mu^-$ decays in NP scenarios with $C_{10}$ or $C'_{10} = 0.3$, $C_{10} = -C'_{10} = 0.3$ and $C_{10} = C'_{10} = 0.3$ in red, green and blue, respectively. The right plot shows $A_{FB}^L$\(\text{[11]}\) only in a $C_{10} = 0.3$ scenario. The width of the bands stem predominantly from unknown strong phases. Both $A_{FB}^H$ and $A_{FB}^L$ are null tests of the SM\(\text{[7]}\).

In Fig. 2 $A_{FB}^H$ is shown (left plot) for three different NP scenarios in red, green and blue for $C_{10}$ or $C'_{10} = 0.3$, $C_{10} = -C'_{10} = 0.3$ and $C_{10} = C'_{10} = 0.3$, respectively. These benchmarks are chosen to illustrate the following: Firstly, $C_{10}$ and $C'_{10}$ contributions are indistinguishable within the large uncertainties induced by unknown strong phases entering in Eq. (3) and varied in the plot. Secondly, a scenario with $C_{10} = C'_{10}$ leads to a partial cancellation of contributions leading to a decreased signal with respect to $C_{10} = -C'_{10}$ scenarios. We also show $A_{FB}^L$\(\text{[11]}\) in Fig. 2 (right plot)
for $C_{10} = 0.3$. We recall that $A_{FB}^\ell$ is a charm specific null test with sensitivity to the axial-vector coupling $C_{10}$ down to the percent level. $A_{FB}^\ell$ vanishes at both, the low and the high $q^2$ endpoints.

The main benefit in studying $A_{FB}^{H}$ in addition to $A_{FB}^\ell$ is complementarity. As pointed out in Ref. [12], $A_{FB}^\ell$ has sensitivity to $C_{10}$, but not necessarily $C_{10}'$, as this would require also NP contributions in $C_9'$. In $A_{FB}^{H}$ interference terms of type $C_9 C_{10}'$ exist, which are needed to observe a NP signal in a $C_{10}'$-only scenario. In addition, $A_{FB}^{H}$ does not necessarily vanish at the high $q^2$ endpoint, and rather assumes a model-dependent value [35].

### C. Model-independent analysis

To outline the strategy for disentangling NP contributions in charm baryon decays, we first summarize the sensitivities to single Wilson coefficients. The observables $A_{FB}^\ell$ and $F_L$ appear in three-body decays, while $A_{FB}^H$ and $A_{FB}^{H}$ are arise in self-analyzing four-body decays discussed in this work.

1. $A_{FB}^\ell$ is a null test that probes $C_{10}$
2. $A_{FB}^{H}$ is a null test that probes $C_{10}$ and $C_{10}'$
3. $F_L$ probes $C_7$ and $C_7'$
4. $A_{FB}^{H}$ probes $C_7', C_9'$ and $C_{10}'$

If there is no signal observed for the null tests $A_{FB}^\ell$ and $A_{FB}^{H}$, one concludes that both $C_{10}$ and $C_{10}'$ are well below the percent level. In a next step $F_L$ can be used to probe dipole contributions $C_7$ or $C_7'$. To differentiate between the left-handed and right-handed dipole operators, $A_{FB}^{H}$ can be employed. Similarly $C_9'$ can be extracted from $A_{FB}^{H}$, if $F_L$ is SM-like. In a scenario with non-vanishing null tests, NP contributions to $C_{10}'(0)$ are evident. Here, again $A_{FB}^{H}$ differentiates $C_{10}$ and $C_{10}'$. In addition, $A_{FB}^\ell$ and $A_{FB}^{H}$ reveal information on $C_{10}'$ contributions. The only coefficient which can not be probed efficiently is $C_9$, as it is dominated by the resonances $C_{10}^{R}$. As anticipated already in [12], a future simultaneous fit of Wilson coefficients and resonance parameters is then needed.

### IV. FURTHER NULL TESTS

In this section we discuss further null test opportunities for rare charm baryon decays. We begin with the angular observables $K_{3s}$ and $K_{4s}$ (7) in Sec. [IVA] discuss dineutrino modes in Sec. [IVB]
and present null tests that become available if the initial charm baryon is polarized (Sec. IVC). The study of charged lepton flavor violating modes offers even more clean null tests but is beyond the scope of this work.

A. $K_{3s}$ and $K_{4s}$

The angular observables $K_{3s}$ and $K_{4s}$ vanish in the SM (7) or any SM extension with vanishing $C_{10}$ and $C'_{10}$. Both $K_{3s}$ and $K_{4s}$ contain terms of the form $C_9C_{10}$ and $C_9C'_{10}$, just like $K_{2c} \propto A^{\text{FB}}_F$. The latter requires additional NP coefficients to be sensitive to $C'_{10}$. This way, $K_{3s}$ and $K_{4s}$ are structurally similar to $A^{\text{FB}}_F$, discussed in Sec. IIIB. All three of them probe $C_{10}$ and $C'_{10}$, although with different combinations of form factors and NP coefficients.

At zero recoil, $K_{3s} = K_{1c} = 0$, and $K_{4s}(d\Gamma/dq^2)^{-1} = -K_{2c}(d\Gamma/dq^2)^{-1}$ and in general finite, with the value dependent on the model (5). In Fig. 3 we show $K_{3s}$ (right panel) and $K_{4s}$ (left panel), normalized to the differential decay rate, for the same benchmarks with NP in $C_{10}$ as for $A^{\text{FB}}_F$ in Fig. 2. While the different Wilson coefficient and form factor combinations of $K_{3s}$ and $K_{4s}$ only offer little qualitative complementarity compared to $A^{\text{FB}}_F$, they do increase the statistics and enhance the sensitivity in a global analysis.

![Fig. 3: The angular observables $K_{4s}$ (left) and $K_{3s}$ (right) normalized to the differential decay rate for $\Xi_c^+ \rightarrow \Sigma^+(\rightarrow p\pi^0)\mu^+\mu^-$ decays in different NP scenarios for $C_{10}$ and $C'_{10}$. The width of the bands stem predominantly from unknown strong phases. Both observables are clean SM null tests (7).](image-url)
B. Baryonic dineutrino modes

Dineutrino modes induced by $c \rightarrow u\nu\bar{\nu}$ transitions are severely GIM suppressed and negligible in the SM \cite{1}. Any observation hence signals NP, making them prime candidates for searches \cite{10}. The effective Hamiltonian reads

$$H_{\text{eff}} = -\frac{4G_F}{\sqrt{2}} \sum_{ij} \left( C_L^{ij} Q_L^{ij} + C_R^{ij} Q_R^{ij} \right),$$

with $C_{L,R}$ negligible in the SM and

$$Q_L^{ij} = (\pi L \gamma_\mu c_L)(\bar{\nu}_{Lj} \gamma^\mu \nu_{Li}), \quad Q_R^{ij} = (\pi R \gamma_\mu c_R)(\bar{\nu}_{Lj} \gamma^\mu \nu_{Li}).$$

Assuming the absence of light right-handed neutrinos, only these two operators exist for each combination of neutrino flavors $i,j$.

The self-analyzing four-body decays offer further opportunities for rare charm decays into dineutrinos. Specifically, this concerns the decays $\Xi_c^+ \rightarrow \Sigma^+ (\rightarrow p\pi^0)\nu\bar{\nu}$, $\Xi_c^0 (\rightarrow p\pi^-)\nu\bar{\nu}$, and $\Omega_c^0 (\rightarrow \Lambda^0\pi^0)\nu\bar{\nu}$. The angular distribution is then given by

$$\frac{d^2\Gamma}{dq^2 d\cos\theta_\pi} = \int_{-1}^{1} \int_{0}^{2\pi} \frac{d^4\Gamma}{dq^2 d\cos\theta_\pi d\cos\theta_\ell d\phi} d\phi d\cos\theta_\ell = 2K_{1ss} + K_{1cc} + (2K_{2ss} + K_{2cc}) \cos\theta_\pi,$$

and accessible without reconstructing the neutrinos \cite{35}. Note, here, in $K_{1ss}$, $K_{1cc}$, $K_{2ss}$ and $K_{2cc}$ one has to replace $C_9 = -C_{10} = \frac{4\pi}{\alpha e} C_L^{ij}/2$ and $C'_9 = -C'_{10} = \frac{4\pi}{\alpha e} C_R^{ij}/2$, skip the $m_\ell^2$ terms and incoherently sum the neutrino flavors $ij$. This gives rise to two independent observables, the differential decay rate and the hadronic forward-backward asymmetry $A_{\text{FB}}^H$ \cite{14}.

The hadronic forward-backward asymmetry $A_{\text{FB}}^H$ is obtained by integrating over the leptons phase space, and therefore can be measured in dineutrino modes. It reads

$$A_{\text{FB}}^H(B_0 \rightarrow B_1 (\rightarrow B_2\pi)\nu\bar{\nu}) = -\alpha \cdot \frac{(|C_L|^2 - |C_R|^2) A(q^2) \sqrt{s_+ s_-}}{|C_L - C_R|^2 B(q^2) s_+ + |C_L + C_R|^2 C(q^2) s_-}$$

and probes $C_R/C_L$. To ease notation here we omit the flavor indices. In the limit $C_R = 0$, the asymmetry becomes free of Wilson coefficients,

$$A_{\text{FB}}^H(B_0 \rightarrow B_1 (\rightarrow B_2\pi)\nu\bar{\nu}) = -\alpha \cdot \frac{A(q^2) \sqrt{s_+ s_-}}{B(q^2)s_+ + C(q^2)s_-}. $$

The functions $A(q^2), B(q^2)$ and $C(q^2)$ contain form factors and kinematics and can be seen in App. A.
Upper limits on the branching ratios can be derived from a global EFT-analysis \cite{10}

\begin{align}
B(\Xi_c^+ \to \Sigma^+ (\to pn^0)\nu\bar{\nu}) &\lesssim 3.9 \times 10^{-5}, \\
B(\Xi_c^0 \to \Lambda^0 (\to p\pi^-)\nu\bar{\nu}) &\lesssim 3.6 \times 10^{-6}, \\
B(\Omega_c^0 \to \Xi^0 (\to \Lambda^0\pi^0)\nu\bar{\nu}) &\lesssim 7.1 \times 10^{-5},
\end{align}  
(22)

in agreement with results in Ref. \cite{12}. The upper limits are stronger when assumptions on the lepton flavor are made \cite{10}. These are given in the next equation, with the first entry corresponding to charged lepton flavor conservation, and the even stronger one in parentheses assuming lepton universality:

\begin{align}
B(\Xi_c^+ \to \Sigma^+ (\to pn^0)\nu\bar{\nu}) &\lesssim 1.1 \times 10^{-5}, \ (1.9 \times 10^{-6}), \\
B(\Xi_c^0 \to \Lambda^0 (\to p\pi^-)\nu\bar{\nu}) &\lesssim 1.0 \times 10^{-6}, \ (1.7 \times 10^{-7}), \\
B(\Omega_c^0 \to \Xi^0 (\to \Lambda^0\pi^0)\nu\bar{\nu}) &\lesssim 1.9 \times 10^{-5}, \ (3.4 \times 10^{-6}).
\end{align}  
(23)

C. Polarized Charm baryons

Let us point out that studying decays of polarized charmed baryons introduces further null test observables on top of those in (7). We identify in total eight additional angular null tests probing leptonic axial vector currents, \textit{i.e.}, \textit{C}^{(t)}_{10} \textit{E}, which are proportional to the initial \textit{B}_0\textit{-polarization} \textit{P} \textit{B}_0, \\

\begin{align}
K_{13}, K_{16}, K_{18}, K_{20}, K_{22}, K_{24}, K_{26}, K_{28}|_{\text{SM}} &\simeq 0, 
\end{align}  
(24)

with the differential distribution $\text{[34]}$ given in App. C. Among these, \textit{K} \textit{13}, \textit{K} \textit{22} and \textit{K} \textit{24} do not vanish for $\alpha = 0$ and hence can be studied in the simpler three-body decays, including $\Lambda_c \to p\mu^+\mu^-$. We note that $\text{K}_{13} = -P_{B_0}K_{2c}$, anticipating that a study with polarized baryons can access some of the null tests, here $\text{A}^{(t)}_{FB}$, that otherwise require a self-analyzing four-body decay. If in the future high luminosity sources of polarized $\Lambda_c$’s or other charmed baryons can be used, see [36] for LHC and $e^+e^-$-collider possibilities, we would like to come back to explore these observables and their NP reach in charm in more detail.

V. CONCLUSIONS

We perform a full angular analysis of baryonic $|\Delta c| = |\Delta u| = 1$ four-body decays with self-analyzing secondary baryon to explore the BSM reach. We identify several such modes:
\( \Xi^+_c \rightarrow \Sigma^+ (\rightarrow p\pi^0)\ell^+\ell^- \), \( \Xi^+_c \rightarrow \Lambda^0 (\rightarrow p\pi^-)\ell^+\ell^- \) and \( \Omega^0_c \rightarrow \Xi^0 (\rightarrow \Lambda^0\pi^0)\ell^+\ell^- \), with sizable decay parameter, see Table I. The full differential distribution of an unpolarized initial charm baryon features ten angular observables, seven more, and with different NP sensitivities, than the ones available with three-body decays such as \( \Lambda_c \rightarrow p\ell^+\ell^- \). We point out three new, clean null tests of the SM, \( K_2c \), \( K_3s \) and \( K_4s \). Just like the leptonic forward-backward asymmetry \( A_{FB}^\ell \propto K_1c \), their SM contribution can be safely neglected because the GIM-mechanism switches off axial-vector couplings of the leptons, a feature previously exploited also for the \( D \rightarrow \pi\pi\ell^+\ell^- \) angular distribution \([6]\). We also find that the hadronic forward-backward asymmetry \( A_{FB}^H \) \([14]\), although not a null test, can cleanly signal BSM physics in right-handed currents, \( C^\prime_{7,9,10} \), illustrated in Fig. I.

The angular observables in four-body decays enable highly diagnostic tests of BSM couplings. Concrete analysis of the NP sensitivity, see Section III, shows that already four observables, the longitudinal polarization fraction \( F_L \) \([10]\), together with the forward-backward asymmetries \( A_{FB}^\ell \) and \( A_{FB}^H \) \([16]\) allow to pin down BSM Wilson coefficients in one go:

- If there is sizable NP only in \( F_L \), it is \( C_7 \).
- If there is sizable NP only in \( F_L \) and \( A_{FB}^H \), it is \( C^\prime_7 \).
- If there is sizable NP only in \( A_{FB}^H \), it is \( C^\prime_9 \).
- If there is sizable NP only in \( A_{FB}^H \) and \( A_{FB}^H \), it is \( C^\prime_{10} \).
- If there is sizable NP only in \( A_{FB}^\ell \) and \( A_{FB}^H \), it is \( C_{10} \).

The price to pay for the self-analyzing (quasi) four-body decay is the limitation to specific charm baryon modes; the suppression from the secondary baryon decays is modest since branching ratios are at least 50\%. Experimental analysis is suitable for (advanced stages of) high luminosity flavor factories LHCb \([21]\), Belle II \([22]\), BES III \([23]\), and possible future machines \([24, 25]\).

Decays of polarized charmed baryon offer further GIM-based null tests \([24]\), some of which persist in the simpler three-body decays. Their exploration should be pursued further if in the future high luminosity sources of polarized \( \Lambda_c \)’s or other charmed baryons become available.

We note in passing that the hadronic forward-backward asymmetry is obtained by integrating over the leptons phase space, and therefore can be measured in dineutrino modes \( \Xi^+_c \rightarrow \Sigma^+ (\rightarrow p\pi^0)\bar{\nu}\nu \), \( \Xi^+_c \rightarrow \Lambda^0 (\rightarrow p\pi^-)\bar{\nu}\nu \), and \( \Omega^0_c \rightarrow \Xi^0 (\rightarrow \Lambda^0\pi^0)\bar{\nu}\nu \), briefly discussed in Sec. IVB too.

We conclude that rare decays of charm baryons contribute extensively to our endeavor to search for NP. Dedicated computations of form factors for \( \Xi^+_c \rightarrow \Sigma^+ \), \( \Xi^+_c \rightarrow \Lambda^0 \) and \( \Omega^0_c \rightarrow \Xi^0 \) transitions are desirable. As anticipated in \([12]\), a simultaneous fit of \( |\Delta c| = |\Delta u| = 1 \) Wilson coefficients and resonance parameters is called for. The sensitivity of such a fit is enriched by the new presented observables in four-body baryon decays.
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Appendix A: Helicity Amplitudes

Following [14] we introduce the contributions to the angular observables [6]

\[ S_{\text{mm}} = N^2 \cdot \text{Re} \left[ \mathcal{H}_{1/2,1}^{m} \mathcal{H}_{1/2,1}^{m'} + \mathcal{H}_{3/2,0}^{m} \mathcal{H}_{3/2,0}^{m'} \right], \]

\[ S_{P \text{mm}} = N^2 \cdot \text{Re} \left[ \mathcal{H}_{1/2,1}^{m} \mathcal{H}_{1/2,1}^{m'} - \mathcal{H}_{3/2,0}^{m} \mathcal{H}_{3/2,0}^{m'} \right], \]

\[ U_{\text{mm}} = N^2 \cdot \text{Re} \left[ \mathcal{H}_{1/2,1}^{m} \mathcal{H}_{1/2,1}^{m'} + \mathcal{H}_{1/2,1}^{m} \mathcal{H}_{1/2,1}^{m'} \right], \]

\[ P_{\text{mm}} = N^2 \cdot \text{Re} \left[ \mathcal{H}_{1/2,1}^{m} \mathcal{H}_{1/2,1}^{m'} - \mathcal{H}_{1/2,1}^{m} \mathcal{H}_{1/2,1}^{m'} \right], \]

\[ L_{\text{mm}} = N^2 \cdot \text{Re} \left[ \mathcal{H}_{1/2,0}^{m} \mathcal{H}_{1/2,0}^{m'} + \mathcal{H}_{1/2,0}^{m} \mathcal{H}_{1/2,0}^{m'} \right], \]

\[ L_{P \text{mm}} = N^2 \cdot \text{Re} \left[ \mathcal{H}_{1/2,0}^{m} \mathcal{H}_{1/2,0}^{m'} - \mathcal{H}_{1/2,0}^{m} \mathcal{H}_{1/2,0}^{m'} \right], \]

\[ I_{1P \text{m}} = \frac{N^2}{4} \text{Im} \left[ \mathcal{H}_{1/2,1}^{m} \mathcal{H}_{1/2,1}^{m'} - \mathcal{H}_{3/2,0}^{m} \mathcal{H}_{3/2,0}^{m'} + \mathcal{H}_{1/2,1}^{m} \mathcal{H}_{1/2,1}^{m'} - \mathcal{H}_{1/2,1}^{m} \mathcal{H}_{1/2,1}^{m'} \right], \]

\[ I_{2 \text{mm}} = \frac{N^2}{4} \text{Im} \left[ \mathcal{H}_{1/2,1}^{m} \mathcal{H}_{1/2,1}^{m'} - \mathcal{H}_{3/2,0}^{m} \mathcal{H}_{3/2,0}^{m'} - \mathcal{H}_{3/2,0}^{m} \mathcal{H}_{3/2,0}^{m'} + \mathcal{H}_{1/2,1}^{m} \mathcal{H}_{1/2,1}^{m'} + \mathcal{H}_{1/2,1}^{m} \mathcal{H}_{1/2,1}^{m'} \right], \]

\[ I_{3 \text{mm}} = \frac{N^2}{4} \text{Im} \left[ \mathcal{H}_{1/2,1}^{m} \mathcal{H}_{1/2,1}^{m'} + \mathcal{H}_{3/2,0}^{m} \mathcal{H}_{3/2,0}^{m'} + \mathcal{H}_{3/2,0}^{m} \mathcal{H}_{3/2,0}^{m'} + \mathcal{H}_{1/2,1}^{m} \mathcal{H}_{1/2,1}^{m'} - \mathcal{H}_{1/2,1}^{m} \mathcal{H}_{1/2,1}^{m'} \right], \]

in terms of helicity amplitudes \( \mathcal{H}_{\lambda_{\Sigma},\lambda_{\gamma}}^{m} \), where \( \lambda_{\Sigma} \) and \( \lambda_{\gamma} \) denote the helicities of the \( \Sigma \) baryon and the effective current \( \gamma^* (\to \ell \ell) \), respectively. \( \lambda_{\Sigma} \) can therefore assume values of \( \pm \frac{1}{2} \), whereas \( \lambda_{\gamma} \) takes values of 0, \( \pm 1 \), and we further distinguish \( \lambda_{\gamma} = t \) in the \( J_{\gamma} = 0 \) case and \( \lambda_{\gamma} = 0 \) in the \( J_{\gamma} = 1 \) case. The superscript \( m(i) \) distinguishes between leptonic vector \( (m(i) = 1) \) and axial-vector \( (m(i) = 2) \) contributions. While the former receive contributions from \( C_{7}^{(i)} \) and \( C_{9}^{(i)} \), the latter are induced by \( C_{10}^{(i)} \), hence vanish in the SM. The helicity amplitudes \( \mathcal{H}_{\lambda_{\Sigma},\lambda_{\gamma}}^{m} \) are obtained by summing the contributions from individual hadronic matrix elements, \( \mathcal{H}_{\lambda_{\Sigma},\lambda_{\gamma}}^{a,m} \), hence \( \mathcal{H}_{\lambda_{\Sigma},\lambda_{\gamma}}^{m} = \sum_{a} \mathcal{H}_{\lambda_{\Sigma},\lambda_{\gamma}}^{a,m} \). Here, \( a = 1, 2 \) correspond to dipole contributions from \( C_{7}^{(i)} \) and \( a = 3, 4 \) to those from 4-fermion operators \( C_{9}^{(i)} \) and \( C_{10}^{(i)} \). Contributions \( a = 1 \) and \( a = 3 \) are induced by quark-level vector currents.
hence proportional to $C + C'$, whereas contributions $a = 2$ and $a = 4$ are induced by quark-level axial-vector currents, hence $\propto C - C'$. Due to parity, flipping the helicities results in a minus sign for the amplitudes $a = 2$ and $a = 4$. The amplitudes $H_{\lambda_\Sigma,\lambda_\gamma}^{1,2}$ are then decomposed as

\begin{align}
H_{\lambda_\Sigma,\lambda_\gamma}^{1,1} &= H_{\lambda_\Sigma,\lambda_\gamma}^{1,1} + H_{\lambda_\Sigma,\lambda_\gamma}^{2,1} + H_{\lambda_\Sigma,\lambda_\gamma}^{3,1} + H_{\lambda_\Sigma,\lambda_\gamma}^{4,1}, \\
H_{-\lambda_\Sigma,\lambda_\gamma}^{1,1} &= H_{\lambda_\Sigma,\lambda_\gamma}^{1,1} - H_{\lambda_\Sigma,\lambda_\gamma}^{2,1} + H_{\lambda_\Sigma,\lambda_\gamma}^{3,1} - H_{\lambda_\Sigma,\lambda_\gamma}^{4,1}, \\
H_{\lambda_\Sigma,\lambda_\gamma}^{2,1} &= H_{\lambda_\Sigma,\lambda_\gamma}^{3,2} + H_{\lambda_\Sigma,\lambda_\gamma}^{4,2}, \\
H_{-\lambda_\Sigma,\lambda_\gamma}^{2,1} &= H_{\lambda_\Sigma,\lambda_\gamma}^{3,2} - H_{\lambda_\Sigma,\lambda_\gamma}^{4,2}. 
\end{align}

(A2)

For convenience we give in the following a list of single contributions but stress that except for the different decay modes these equations are equivalent to Eqs. (C3)-(C5) of Ref. [12]. The helicity of the initial, charm baryon satisfies $\lambda_{\Xi_c} = -\lambda_\Sigma + \lambda_\gamma$.

$\lambda_{\Xi_c} = \frac{1}{2}$, $\lambda_\gamma = t$:

\begin{align}
H_{-\frac{1}{2},t}^{1,1} &= 0, \\
H_{-\frac{1}{2},t}^{2,1} &= 0, \\
H_{-\frac{1}{2},t}^{3,1(2)} &= (C_{9(10)} + C_{9(10)}') \sqrt{\frac{s_+}{q^2}} f_0(q^2)(m_{\Xi_c} - m_{\Sigma}), \\
H_{-\frac{1}{2},t}^{4,1(2)} &= (C_{9(10)} - C_{9(10)}') \sqrt{\frac{s_-}{q^2}} g_0(q^2)(m_{\Xi_c} + m_{\Sigma}), 
\end{align}

(A3)

$\lambda_{\Xi_c} = -\frac{1}{2}$, $\lambda_\gamma = 0$:

\begin{align}
H_{\frac{1}{2},0}^{1,1} &= (C_7 + C_7') \frac{2m_c}{q^2} \sqrt{s_- h_+(q^2)}, \\
H_{\frac{1}{2},0}^{2,1} &= -(C_7 - C_7') \frac{2m_c}{q^2} \sqrt{s_+ h_+(q^2)}, \\
H_{\frac{1}{2},0}^{3,1(2)} &= (C_{9(10)} + C_{9(10)}') \frac{1}{q^2} \sqrt{s_- f_+(q^2)(m_{\Xi_c} + m_{\Sigma})}, \\
H_{\frac{1}{2},0}^{4,1(2)} &= -(C_{9(10)} - C_{9(10)}') \frac{1}{q^2} \sqrt{s_+ g_+(q^2)(m_{\Xi_c} - m_{\Sigma})}, 
\end{align}

(A4)

$\lambda_{\Xi_c} = \frac{1}{2}$, $\lambda_\gamma = 1$:

\begin{align}
H_{\frac{1}{2},1}^{1,1} &= \sqrt{2}(C_7 + C_7') \frac{2m_c}{q^2} \sqrt{s_- h_\perp(q^2)(m_{\Xi_c} + m_{\Sigma})}, \\
H_{\frac{1}{2},1}^{2,1} &= -\sqrt{2}(C_7 - C_7') \frac{2m_c}{q^2} \sqrt{s_+ h_\perp(q^2)(m_{\Xi_c} - m_{\Sigma})}, \\
H_{\frac{1}{2},1}^{3,1(2)} &= \sqrt{2}(C_{9(10)} + C_{9(10)}') \sqrt{s_- f_\perp(q^2)}, \\
H_{\frac{1}{2},1}^{4,1(2)} &= -\sqrt{2}(C_{9(10)} - C_{9(10)}') \sqrt{s_+ g_\perp(q^2)}. 
\end{align}

(A5)
Using the helicity amplitudes we obtain for the contributions in Eq. (A1)

\[ U^{11} = 4N^2 \cdot \left[ (C_7 + C_7') \frac{2mc}{q^2} (m_{\Xi^-} + m_{\Sigma^+}) h_\perp + (C_9 + C_9') f_\perp \right]^2 \cdot s_- \\
+ \left[ (C_7 - C_7') \frac{2mc}{q^2} (m_{\Xi^-} - m_{\Sigma^+}) \tilde{h}_\perp + (C_9 - C_9') g_\perp \right]^2 \cdot s_+ , \]

\[ L^{11} = \frac{2N^2}{q^2} \cdot \left[ (C_7 + C_7') 2mc h_+ + (C_9 + C_9') (m_{\Xi^-} + m_{\Sigma^+}) f_+ \right]^2 \cdot s_- \\
+ \left[ (C_7 - C_7') 2mc \tilde{h}_+ + (C_9 - C_9') (m_{\Xi^-} - m_{\Sigma^+}) g_+ \right]^2 \cdot s_+ , \]

\[ U^{22} = 4N^2 \cdot \left[ (C_{10} + C_{10}') f_\perp \right]^2 \cdot s_- + \left[ (C_{10} - C_{10}') g_\perp \right]^2 \cdot s_+ , \]

\[ L^{22} = \frac{2N^2}{q^2} \cdot \left[ (C_{10} + C_{10}') (m_{\Xi^-} + m_{\Sigma^+}) f_+ \right]^2 \cdot s_- + \left[ (C_{10} - C_{10}') (m_{\Xi^-} - m_{\Sigma^+}) g_+ \right]^2 \cdot s_+ , \]

\[ S^{22} = \frac{2N^2}{q^2} \cdot \left[ (C_{10} + C_{10}') (m_{\Xi^-} - m_{\Sigma^+}) f_0 \right]^2 \cdot s_+ + \left[ (C_{10} - C_{10}') (m_{\Xi^-} + m_{\Sigma^+}) g_0 \right]^2 \cdot s_- , \]

\[ P^{12} = -8N^2 \cdot \left[ \text{Re}((C_7 - C_7') (C_{10}^* + C_{10}')) \frac{mc}{q^2} (m_{\Xi^-} - m_{\Sigma^+}) f_\perp \tilde{h}_\perp \right. \\
+ \left. \text{Re}((C_7 + C_7') (C_{10}^* - C_{10}')) \frac{mc}{q^2} (m_{\Xi^-} + m_{\Sigma^+}) g_\perp h_\perp \right] \cdot \sqrt{s_+ s_-} . \]

(A6)

Here, \( N^2 = \frac{G_F^2 N_c^2 \sqrt{\lambda(a, b, c)}}{3 - 2\lambda^2 + \lambda^2} \) with the Källén function \( \lambda(a, b, c) = a^2 + b^2 + c^2 - 2(ab + ac + bc) \) and \( s_+ = (m_{\Xi^-} \pm m_{\Sigma^+})^2 - q^2 \). The contributions in Eq. (A6) are those relevant to three-body decays and have already been given in Ref. [12]. The additional contributions that arise in the full, four-
body angular distribution [5] read

\[
L_{11}^P = - \frac{4N^2}{q^2} \cdot \text{Re} \left[ \left( C_7 + C_9' \right) 2m_e h_+ + \left( C_9 + C_9' \right) (m_{\Xi^+} + m_{\Sigma^+}) f_+ \right]
\]

\[
\cdot \left( C_7^* - C_7'' \right) 2m_e \tilde{h}_+ + \left( C_9^* - C_9'' \right) (m_{\Xi^+} - m_{\Sigma^+}) g_+ \right] \cdot \sqrt{s_+ s_-},
\]

\[
P^{11} = -8N^2 \cdot \text{Re} \left[ \left( C_7 + C_9' \right) \frac{2m_e}{q^2} h_\perp (m_{\Xi^+} + m_{\Sigma^+}) + \left( C_9 + C_9' \right) f_\perp \right]
\]

\[
\cdot \left( C_7^* - C_7'' \right) \frac{2m_e}{q^2} \tilde{h}_\perp (m_{\Xi^+} - m_{\Sigma^+}) + \left( C_9^* - C_9'' \right) g_\perp \right] \cdot \sqrt{s_+ s_-},
\]

\[
L_{P}^{22} = - \frac{4N^2}{q^2} \left[ \left| C_{10} \right|^2 - \left| C_{10}' \right|^2 \right] f_+ g_+ (m_{\Xi^+}^2 - m_{\Sigma^+}^2) \right] \cdot \sqrt{s_+ s_-},
\]

\[
P^{22} = -8N^2 \left[ \left| C_{10} \right|^2 - \left| C_{10}' \right|^2 \right] f_\perp g_\perp \right] \cdot \sqrt{s_+ s_-},
\]

\[
U^{12} = 4N^2 \left[ \left( \text{Re}((C_7 + C_9')(C_{10} + C_{10}')) f_+ h_+ \frac{2m_e}{q^2} (m_{\Xi^+} + m_{\Sigma^+}) \right.ight.
\]

\[
+ \left( \text{Re}((C_9 + C_9')(C_{10} + C_{10}')) f_\perp \right) \cdot s_-
\]

\[
+ \left( \text{Re}((C_7 - C_9')(C_{10} - C_{10}')) g_+ \frac{2m_e}{q^2} (m_{\Xi^+} - m_{\Sigma^+}) \right.
\]

\[
+ \left( \text{Re}((C_9 - C_9')(C_{10} - C_{10}')) g_\perp \right) \cdot s_+ \right]
\]

\[
S_{P}^{22} = - \frac{4N^2}{q^2} \left[ \left| C_{10} \right|^2 - \left| C_{10}' \right|^2 \right] f_0 g_0 (m_{\Xi^+}^2 - m_{\Sigma^+}^2) \right] \cdot \sqrt{s_+ s_-}.
\]
The interference terms are given by

\[
I_{1P}^{11} = N^2 \frac{2}{q^2} \left[ \Re((C_7 - C'_7)(C'^*_7 + C''_7)) \frac{4m^2_c}{q^2} \left( \tilde{h}_+ \tilde{h}_\perp (m_{\bar{2}^+_c} + m_{\Sigma^+}) - h_+ \tilde{h}_\perp (m_{\bar{2}^+_c} - m_{\Sigma^+}) \right) \\
+ \Re((C_9 - C'_9)(C'^*_9 + C''_9)) \cdot \left( g_+ f_\perp (m_{\bar{2}^+_c} - m_{\Sigma^+}) - f_+ g_\perp (m_{\bar{2}^+_c} + m_{\Sigma^+}) \right) \\
+ \Re((C_9 - C'_9)(C'^*_9 + C''_9)) 2m_c \cdot \left( g_\perp \tilde{h}_\perp - f_\perp \tilde{h}_\perp \frac{m_{\bar{2}^+_c}^2 - m_{\Sigma^+}^2}{q^2} \right) \\
+ \Re((C_7 - C'_7)(C'^*_7 + C''_7)) 2m_c \cdot \left( g_\perp \tilde{f}_\perp - f_\perp \tilde{h}_\perp \frac{m_{\bar{2}^+_c}^2 - m_{\Sigma^+}^2}{q^2} \right) \right] \cdot \sqrt{s_+s_-},
\]

\[
I_{1P}^{22} = N^2 \frac{2}{q^2} \left[ (|C_{10}|^2 - |C'_{10}|^2) \cdot \left( f_\perp g_+(m_{\bar{2}^+_c} - m_{\Sigma^+}) - f_+ g_\perp (m_{\bar{2}^+_c} + m_{\Sigma^+}) \right) \right] \cdot \sqrt{s_+s_-},
\]

\[
I_{2}^{11} = 2m_c N^2 \frac{2}{q^2} \left[ \Im((C_9 + C'_9)(C'^*_9 + C''_9)) \cdot \left( f_\perp h_+ - f_\perp h_\perp \frac{(m_{\bar{2}^+_c} + m_{\Sigma^+})^2}{q^2} \right) \cdot s_- \\
- \Im((C_9 - C'_9)(C'_7 - C''_7)) \cdot \left( g_\perp \tilde{h}_\perp - g_\perp \tilde{h}_\perp \frac{(m_{\bar{2}^+_c} - m_{\Sigma^+})^2}{q^2} \right) \cdot s_+ \right],
\]

\[
I_{2}^{22} = 0,
\]

\[
I_{3}^{12} = N^2 \frac{2}{q^2} \left[ \Re((C_7 + C'_7)(C'^*_7 + C''_7)) \frac{4m^2_c}{q^2} \left( h_+ f_\perp + h_\perp f_\perp \frac{(m_{\bar{2}^+_c} + m_{\Sigma^+})^2}{q^2} \right) \cdot s_- \\
- \Re((C_7 - C'_7)(C'^*_7 + C''_7)) \frac{4m^2_c}{q^2} \left( h_+ f_\perp + h_\perp f_\perp \frac{(m_{\bar{2}^+_c} - m_{\Sigma^+})^2}{q^2} \right) \cdot s_+ \\
+ \Re((C_9 + C'_9)(C'^*_9 + C''_9)) \cdot \left( f_\perp f_\perp (m_{\bar{2}^+_c} + m_{\Sigma^+}) \right) \cdot s_- \\
- \Re((C_9 - C'_9)(C'^*_9 + C''_9)) \cdot \left( g_\perp g_\perp (m_{\bar{2}^+_c} - m_{\Sigma^+}) \right) \cdot s_+ \right],
\]

\[
I_{4}^{12} = N^2 \frac{2}{q^2} \left[ \Im((C_7 + C'_7)(C'^*_7 + C''_7)) \frac{4m^2_c}{q^2} \left( h_+ g_\perp + h_\perp g_\perp \frac{m_{\bar{2}^+_c}^2 - m_{\Sigma^+}^2}{q^2} \right) \\
+ \Im((C_7 - C'_7)(C'^*_7 + C''_7)) \frac{4m^2_c}{q^2} \left( h_+ g_\perp + h_\perp g_\perp \frac{m_{\bar{2}^+_c}^2 - m_{\Sigma^+}^2}{q^2} \right) \\
+ \Im((C_9 + C'_9)(C'^*_9 + C''_9)) \cdot \left( f_\perp g_+(m_{\bar{2}^+_c} - m_{\Sigma^+}) + f_+ g_\perp (m_{\bar{2}^+_c} + m_{\Sigma^+}) \right) \\
+ \Im((C_9 - C'_9)(C'^*_9 + C''_9)) \cdot \left( g_\perp f_+(m_{\bar{2}^+_c} + m_{\Sigma^+}) + g_\perp f_+(m_{\bar{2}^+_c} - m_{\Sigma^+}) \right) \right] \cdot \sqrt{s_+s_-}.
\]

All additional contributions except \(U^{12}, I_{2}^{11}\) and \(I_{3}^{12}\) are P-odd, that is, change sign for \(C_i \leftrightarrow C'_i\), and vanish for \(C_i = C'_i\).
In Eqs. (15), (20) and (21) the following $q^2$-dependent functions appear

\[
A(q^2) = 2 f_\perp g_\perp + f_+ g_+ \frac{m_{B_0}^2 - m_{B_1}^2}{q^2},
\]
\[
B(q^2) = 2 g_\perp^2 + g_+^2 \frac{(m_{B_0} - m_{B_1})^2}{q^2},
\]
\[
C(q^2) = 2 f_\perp^2 + f_+^2 \frac{(m_{B_0} + m_{B_1})^2}{q^2}.
\]

(A9)

Appendix B: Helicity amplitude description of $B_1 \to B_2\pi$

The secondary baryonic decay $B_1 \to B_2\pi$, here discussed for $\Sigma^+ \to p\pi^0$, can be parameterized by the sum, $\alpha_+$, and the difference, $\alpha_-$, of the helicity amplitudes $h^{\Sigma}_{\lambda}(\lambda_{\Sigma})$ squared

\[
\alpha_\pm = \left| h^{\Sigma}_{\pm} \left( \lambda_{\Sigma} = \frac{1}{2} \right) \right|^2 \pm \left| h^{\Sigma}_{\mp} \left( \lambda_{\Sigma} = -\frac{1}{2} \right) \right|^2.
\]

(B1)

The helicity amplitudes of non-leptonic baryon decays involving a spin-0 meson, here taken to be a pion, can be parametrized as

\[
h^{\Sigma}_{\lambda}(\lambda_{\Sigma}) = G_F m_\pi^2 u_p(\lambda_p)(A - B\gamma_5) u_{\Sigma}(\lambda_{\Sigma}),
\]

(B2)

where $A$ and $B$ are complex constants [37] and $m_\pi$ the pion mass. We compute the amplitude in the rest frame of the $\Sigma^+$ with the $z$-axis pointing in the direction of the proton momentum. The spinors then take the form

\[
u_{\Sigma}(p, \lambda_{\Sigma} = \pm \frac{1}{2}) = \sqrt{2m_{\Sigma}} \begin{pmatrix} \chi_\pm \\ 0 \end{pmatrix},
\]
\[
\bar{u}_p(k, \lambda_p = \pm \frac{1}{2}) = \sqrt{E_p + m_p} \begin{pmatrix} \chi_\pm \frac{\mp|\vec{k}|}{E_p + m_p} \chi_\pm \end{pmatrix},
\]

(B3)

where $p$ ($k$) denotes the four-momentum of the $\Sigma^+$ (proton), with $p^0 = m_{\Sigma}$, $|\vec{p}| = 0$ and $E_p = \sqrt{|\vec{k}|^2 + m_p^2}$ is the energy of the proton, hence $k = (E_p, 0, 0, |\vec{k}|)^T$, and $\chi_+ = (1, 0)^T$, $\chi_- = (0, 1)^T$. Plugging the spinors into (B2) and simplifying, we arrive at

\[
h^{\Sigma}_{\frac{1}{2}} \left( \lambda_{\Sigma} = \frac{1}{2} \right) = \sqrt{2m_{\Sigma}} G_F m_\pi^2 (\sqrt{r_+} A + \sqrt{r_-} B),
\]
\[
h^{\Sigma}_{\frac{-1}{2}} \left( \lambda_{\Sigma} = -\frac{1}{2} \right) = \sqrt{2m_{\Sigma}} G_F m_\pi^2 (\sqrt{r_+} A - \sqrt{r_-} B),
\]

(B4)

with $r_\pm = \sqrt{E_p \pm m_p}$. Using these helicity amplitudes we can express $\alpha_\pm$ as

\[
\alpha_+ = 4G_F^2 m_\pi^4 m_{\Sigma} (r_+ |A|^2 + r_- |B|^2),
\]
\[
\alpha_- = 8G_F^2 m_\pi^4 m_{\Sigma} \sqrt{r_+ r_-} \text{Re}(AB^*),
\]

(B5)
and obtain for their ratio

$$\frac{\alpha_0}{\alpha_+} = \frac{2\sqrt{\frac{r_1}{r_2}} \text{Re}(AB^*)}{|A|^2 + \sqrt{\frac{r_1}{r_2}} |B|^2} = \alpha,$$

(B6)

which corresponds to the decay parameter $\alpha$ in [16]. We can therefore factorize $\alpha_+$ from the angular distribution and use $\alpha_+ = \mathcal{B}(\Sigma^+ \to p\pi)$ and $\frac{\alpha}{\alpha_+} = \alpha$ to arrive at the expressions given in Sec. II B.

Appendix C: Angular distribution for polarized initial baryons

Taking into account initial state polarization, the differential decay distribution depends on five angles and $q^2$ and reads

$$\frac{d^6\Gamma}{dq^2 d\Omega} = \frac{3}{32\pi^2} \left( (K_{1ss} \sin^2 \theta_\ell + K_{1cc} \cos^2 \theta_\ell + K_{1c} \cos \theta_\ell) + (K_{2ss} \sin^2 \theta_\ell + K_{2cc} \cos^2 \theta_\ell + K_{2c} \cos \theta_\ell) \cos \theta_\pi + (K_{3sc} \sin \theta_\ell \cos \theta_\ell + K_{3s} \sin \theta_\ell) \sin \theta_\pi \sin (\phi_c + \phi_\ell) + (K_{4sc} \sin \theta_\ell \cos \theta_\ell + K_{4s} \sin \theta_\ell) \sin \theta_\pi \cos (\phi_c + \phi_\ell) + (K_{11} \sin^2 \theta_\ell + K_{12} \cos^2 \theta_\ell + K_{13} \cos \theta_\ell) \cos \theta_c + (K_{14} \sin^2 \theta_\ell + K_{15} \cos^2 \theta_\ell + K_{16} \cos \theta_\ell) \cos \theta_\pi \cos \theta_c + (K_{17} \sin \theta_\ell \cos \theta_\ell + K_{18} \sin \theta_\ell) \sin \theta_\pi \cos (\phi_c + \phi_\ell) \cos \theta_c + (K_{19} \sin \theta_\ell \cos \theta_\ell + K_{20} \sin \theta_\ell) \sin \theta_\pi \sin (\phi_c + \phi_\ell) \cos \theta_c + (K_{21} \cos \theta_\ell \sin \phi_\ell \sin \theta_c + K_{22} \sin \theta_\ell \sin \phi_\ell \sin \theta_c + K_{23} \cos \theta_\ell \sin \phi_\ell \sin \theta_c + K_{24} \sin \theta_\ell \sin \phi_\ell \sin \theta_c + K_{25} \cos \theta_\ell \sin \phi_\ell \sin \theta_\pi \sin \theta_c + K_{27} \cos \theta_\ell \sin \phi_\ell \cos \theta_\pi \sin \theta_c + K_{29} \cos^2 \theta_\ell + K_{30} \sin^2 \theta_\ell) \sin \theta_\pi \sin \phi_c \sin \theta_c + K_{31} \cos^2 \theta_\ell + K_{32} \sin^2 \theta_\ell \sin \theta_\pi \cos \phi_c \sin \theta_c + (K_{33} \sin^2 \theta_\ell) \sin \theta_\pi \cos (2\phi_\ell + \phi_c) \sin \theta_c + (K_{34} \sin^2 \theta_\ell) \sin \theta_\pi \sin (2\phi_\ell + \phi_c) \sin \theta_c \right).$$

(C1)

$K_{11}, K_{12}, K_{13}, K_{21}, K_{22}, K_{23}$ and $K_{24}$ survive in the limit $\alpha = 0$ of which $K_{13}, K_{22}$ and $K_{24}$ are null tests. The first four lines are identical to Eq. (5) with $\phi_c + \phi_\ell = \phi$. $\phi_c$ and $\theta_c$ are new.
angles related to the initial state polarization, see Ref. [34] for details.

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