GLOBAL WELL-POSEDNESS AND LARGE TIME BEHAVIOR OF CLASSICAL SOLUTIONS TO A GENERIC COMPRESSIBLE TWO-FLUID MODEL

GUOCHUN WU, LEI YAO, AND YINGHUI ZHANG*

Abstract. In this paper, we investigate a generic compressible two-fluid model with common pressure ($P^+ = P^-$) in $\mathbb{R}^3$. Under some smallness assumptions, Evje–Wang–Wen [Arch Rational Mech Anal 221:1285–1316, 2016] obtained the global solution and its optimal decay rate for the 3D compressible two-fluid model with unequal pressures $P^+ \neq P^-$. More precisely, the capillary pressure $f(\alpha - \rho) = P^+ - P^- \neq 0$ is taken into account, and is assumed to be a strictly decreasing function near the equilibrium. As indicated by Evje–Wang–Wen, this assumption played a key role in their analysis and appeared to have an essential stabilization effect on the model. However, global well-posedness of the 3D compressible two-fluid model with common pressure has been a challenging open problem due to the fact that the system is partially dissipative and its nonlinear structure is very terrible. In the present work, by exploiting the dissipation structure of the model and making full use of several key observations, we establish global existence and large time behavior of classical solutions to the 3D compressible two-fluid model with common pressure. One of key observations here is that to closure the higher-order energy estimates of non-dissipative variables (i.e, fraction densities $\alpha \pm \rho \pm$), we will introduce the linear combination of two velocities ($u \pm$): $v = \frac{2}{\mu + \lambda} u^+ - \frac{2}{\mu - \lambda} u^-$ and explore its good regularity, which is particularly better than ones of two velocities themselves.

1. Introduction.

1.1. Background and motivation. It is well-known that in nature, most of the flows are multi-fluid flows. Such a terminology includes the flows of non-miscible fluids such as air and water; gas, oil and water. For the flows of miscible fluids, they usually form a “new” single fluid possessing its own rheological properties. One interesting example is the stable emulsion between oil and water which is a non-Newtonian fluid, but oil and water are Newtonian ones.

One of the classic examples of multi-fluid flows is small amplitude waves propagating at the interface between air and water, which is called a separated flow. In view of modeling, each fluid obeys its own equation and couples with each other through the free surface in this case. Here, the motion of the fluid is governed by the pair of compressible Euler equations with free surface:

\begin{align}
\partial_t \rho_i + \nabla \cdot (\rho_i v_i) &= 0, \quad i = 1, 2, \quad (1.1) \\
\partial_t (\rho_i v_i) + \nabla \cdot (\rho_i v_i \otimes v_i) + \nabla p_i &= -g \rho_i e_3 \pm F_D. \quad (1.2)
\end{align}

In above equations, $\rho_i$ and $v_i$ represent the density and velocity of the upper fluid (air), and $\rho_2$ and $v_2$ denote the density and velocity of the lower fluid (water). $p_i$ denotes the pressure. $-g \rho_i e_3$ is the gravitational force with the constant $g > 0$ the acceleration of gravity and $e_3$ the vertical unit vector, and $F_D$ is the drag force. As mentioned before, the two fluids (air and water) are separated by the unknown free surface $z = \eta(x, y, t)$, which is advected with the fluids according to the kinematic relation:

$$\partial_t \eta = u_{1,z} - u_{1,x} \partial_x \eta - u_{1,y} \partial_y \eta$$

1.3}

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* Corresponding author: yinghuizhang@mailbox.gxnu.edu.cn.
on two sides of the surface $z = \eta$ and the pressure is continuous across this surface.

When the wave’s amplitude becomes large enough, wave breaking may happen. Then, in the region around the interface between air and water, small droplets of liquid appear in the gas, and bubbles of gas also appear in the liquid. These inclusions might be quite small. Due to the appearances of collapse and fragmentation, the topologies of the free surface become quite complicated and a wide range of length scales are involved. Therefore, we encounter the situation where two–fluid models become relevant if not inevitable. The classic approach to simplify the complexity of multi–phase flows and satisfy the engineer’s need of some modeling tools is the well–known volume–averaging method (see [15] [21] for details). Thus, by performing such a procedure, one can derive a model without surface: a two–fluid model. More precisely, we denote $\alpha^\pm$ by the volume fraction of the liquid (water) and gas (air), respectively. Therefore, $\alpha^+ + \alpha^- = 1$.

Applying the volume–averaging procedure to the equations (1.4) and (1.2) leads to the following generic compressible two–fluid model:

\[
\begin{align*}
\frac{\partial}{\partial t} (\alpha^\pm \rho^\pm) + \text{div} (\alpha^\pm \rho^\pm u^\pm) &= 0, \\
\frac{\partial}{\partial t} (\alpha^\pm \rho^\pm u^\pm) + \text{div} (\alpha^\pm \rho^\pm u^\pm \otimes u^\pm) + \alpha^\pm \nabla P &= -g\alpha^\pm \rho^\pm e_3 \pm F_D,
\end{align*}
\]

where the two fluids are assumed to share the common pressure $P$.

We have already discussed the case of water waves, where a separated flow can lead to a two–fluid model from the viewpoint of practical modeling. As mentioned before, two–fluid flows are very common in nature, but also in various industry applications such as nuclear power, chemical processing, oil and gas manufacturing. According to the context, the models used for simulation may be very different. However, averaged models share the same structure as (1.4). By introducing viscosity effects, one can generalize the above system (1.4) to

\[
\begin{align*}
\frac{\partial}{\partial t} (\alpha^\pm \rho^\pm) + \text{div} (\alpha^\pm \rho^\pm u^\pm) &= 0, \\
\frac{\partial}{\partial t} (\alpha^\pm \rho^\pm u^\pm) + \text{div} (\alpha^\pm \rho^\pm u^\pm \otimes u^\pm) + \alpha^\pm \nabla P &= \text{div} (\alpha^\pm \tau^\pm), \\
P &= P^\pm (\rho^\pm) = A^\pm (\rho^\pm)^{\tilde{\gamma}^\pm},
\end{align*}
\]

where $\rho^\pm(x,t) \geq 0, u^\pm(x,t)$ and $P^\pm (\rho^\pm) = A^\pm (\rho^\pm)^{\tilde{\gamma}^\pm}$ denote the densities, the velocities of each phase, and the two pressure functions, respectively. $\tilde{\gamma}^\pm \geq 1, A^\pm > 0$ are positive constants. In what follows, we set $A^+ = A^- = 1$ without loss of any generality. Moreover, $\tau^\pm$ are the viscous stress tensors

\[
\tau^\pm := \mu^\pm (\nabla u^\pm + \nabla u^\pm) + \lambda^\pm \text{div} u^\pm \text{Id},
\]

where the constants $\mu^\pm$ and $\lambda^\pm$ are shear and bulk viscosity coefficients satisfying the physical condition: $\mu^\pm > 0$ and $2\mu^\pm + 3\lambda^\pm \geq 0$, which implies that $\mu^\pm + \lambda^\pm > 0$. For more information about this model, we refer to [1] [2] [3] [4] [8] [9] [10] [11] [12] [13] [14] [22] [23] [25] [26] and references therein. However, it is well–known that as far as mathematical analysis of two–fluid model is concerned, there are many technical challenges. Some of them involve, for example:

- The two–fluid model is a partially dissipative system. More precisely, there is no dissipation on the mass conservation equations, whereas the momentum equations have viscosity dissipations;
- The corresponding linear system of the model has zero eigenvalue, which makes mathematical analysis (well–posedness and stability) of the model become quite difficult and complicated;
- Transition to single–phase regions, i.e., regions where the mass $\alpha^+ \rho^+$ or $\alpha^- \rho^-$ becomes zero, may occur when the volume fractions $\alpha^\pm$ or the densities $\rho^\pm$ become zero;
- The system is non–conservative, since the non–conservative terms $\alpha^\pm \nabla P^\pm$ are involved in the momentum equations. This brings various mathematical difficulties for us to employ methods used for single phase models to the two–fluid model.
In the excellent work \cite{13}, Evje–Wang–Wen investigated the two–fluid model (1.5) with unequal pressures. As a matter of fact, they made the following assumptions on pressures:

\[ P^+(\rho^+) - P^-(\rho^-) = (\rho^+)\gamma^+ - (\rho^-)\gamma^- = f(\alpha - \alpha^-), \]

(1.7)

where \( f \) is so–called capillary pressure which belongs to \( C^3([0, \infty)) \), and is a strictly decreasing function near the equilibrium satisfying

\[ -\frac{s^2_-(1, 1)}{\alpha^-(1, 1)} < f'(1) < \bar{\eta} - \frac{s^2_-(1, 1)}{\alpha^-(1, 1)} < 0, \]

(1.8)

where \( \bar{\eta} \) is a positive, small fixed constant, and \( s^2_\pm := \frac{dP^\pm}{d\rho^\pm}(\rho^\pm) = \gamma^\pm \frac{P^\pm(\rho^\pm)}{\rho^\pm} \) represent the sound speed of each phase respectively. Under the assumptions (1.7) and (1.8) on pressures, they obtained global existence and decay rates of the solutions when the initial perturbation is sufficiently small. However, as indicated by Evje–Wang–Wen in \cite{13}, assumptions (1.7) and (1.8) played an key role in their analysis and appeared to have an essential stabilization effect on the model in question. On the other hand, Bresch et al. in the seminal work \cite{3} considered a model similar to (1.5). More specifically, they made the following assumptions:

- inclusion of viscous terms of the form (1.1) where \( \mu^\pm(\rho) = \mu^\pm \rho^\pm \) and \( \lambda^\pm(\rho^\pm) = 0 \);
- inclusion of a third order derivative of \( \alpha^\pm \rho^\pm \), which are so–called internal capillary forces represented by the well–known Korteweg model on each phase.

They obtained the global weak solutions in the periodic domain with \( 1 < \gamma^\pm < 6 \). It is worth mentioning that as indicated by Bresch et al. in \cite{3}, their method cannot handle the case without the internal capillary forces. Later, Bresch–Huang–Li \cite{4} established the global existence of weak solutions in one space dimension without the internal capillary forces when \( \gamma^\pm > 1 \) by taking advantage of the one space dimension. However, the methods of Bresch–Huang–Li \cite{4} relied crucially on the advantage of one space dimension, and particularly cannot be applied for high dimensional problem. Recently, Cui–Wang–Yao–Zhu \cite{5} obtained the time–decay rates of classical solutions for the three–dimensional Cauchy problem by combining detailed analysis of the Green’s function to the linearized system with delicate energy estimates to the nonlinear system. It should be remarked that the internal capillary forces played an essential role in the analysis of \cite{3, 5}, which will be explained later.

In conclusion, all the works \cite{3, 4, 13} depend essentially on the internal capillary forces effects or the capillary pressure effects. Therefore, a natural and important problem is that what will happen if no internal capillary force is involved and the capillary pressure \( f = 0 \). That is to say, what about the global well–posedness and large time behavior of Cauchy problem to the two–fluid model (1.5) in high dimensions. However, to our best knowledge, so far there is no result on mathematical theory of the two–fluid model (1.5) in high dimensions. In particular, global well–posedness of the 3D compressible two–fluid model (1.5) has been a challenging open problem due to the fact that the system is partially dissipative and its nonlinear structure is very terrible. The main purpose of this work is to resolve this problem. More precisely, by exploiting the dissipation structure of the model and making full use of several key observations, we establish global existence and large time behavior of classical solutions to the 3D compressible two–fluid model (1.5). One of key observations here is that to closure the higher–order energy estimates of non–dissipative variables (i.e, fraction densities \( \alpha^\pm \rho^\pm \), we will introduce the linear combination of two velocities \( (u^\pm) : v = (2\mu^+ + \lambda^+)u^+ - (2\mu^- + \lambda^-)u^- \) and explore its good regularity, which is particularly better than ones of two velocities themselves. Particularly, our results show that even if both the capillary pressure effects and internal capillary forces effects are absence, viscosity forces still may prevent the formation of singularities for the model in question. Furthermore, the components of the solution exhibit totally distinctive behaviors: the dissipation variables have decay rates in time, but the non–dissipation variables only have uniform time–independent bounds. This phenomenon is totally new as compared to \cite{5, 13} where all the components of
1.2. New formulation of system \((1.5)\) and Main Results. In this subsection, we devote ourselves to reformulating the system \((1.5)\) and stating the main results. To begin with, noticing the relation between the pressures of \((1.5)\)

\[\frac{dP}{d\rho} = \gamma = \frac{\gamma_+ P_+ (\rho^+)}{\gamma_- P_- (\rho^-)}\]

represent the sound speed of each phase respectively. As in [3], we introduce the fraction densities

\[R_\pm = \frac{\alpha_\pm}{\rho_\pm},\]

which together with the relation:

\[\alpha_+ + \alpha_- = 1\]

gives

\[d\rho_+ = \frac{1}{\alpha_+} (dR_+ - \rho^+ d\alpha^+), \quad d\rho_- = \frac{1}{\alpha_-} (dR_- + \rho^- d\alpha^+).\]

By virtue of \((1.9)\) and \((1.10)\), we finally get

\[d\alpha^+ = \frac{\alpha_- s_+^2}{\alpha_+ \rho_+ s_+^2 + \alpha_+ \rho^- s_-^2} dR_+ - \frac{\alpha_+ s_-^2}{\alpha_- \rho_+ s_+^2 + \alpha_+ \rho^- s_-^2} dR_.\]

Substituting \((1.12)\) into \((1.11)\), we deduce the following expressions:

\[d\rho_+ = \frac{\rho^+ \rho^- r_+^2}{R_+ (\rho^+)^2 s_+^2 + R_+ (\rho^-)^2 s_-^2} (\rho^- dR_+ + \rho^+ dR_-),\]

\[d\rho_- = \frac{\rho^+ \rho^- r_+^2}{R_+ (\rho^+)^2 s_+^2 + R_+ (\rho^-)^2 s_-^2} (\rho^- dR_+ + \rho^+ dR_-),\]

which together with \((1.9)\) give the common pressure differential \(dP\)

\[dP = C (\rho^- dR_+ + \rho^+ dR_-),\]

where

\[C := \frac{s_+^2 r_+^2}{\alpha_+ \rho_+ s_+^2 + \alpha_+ \rho^- s_-^2}.\]

Next, by noting the fundamental relation: \(\alpha_+ + \alpha_- = 1\), we can get the following equality:

\[R_+ + R_- + \rho_+ = 1, \quad \text{and thus } R^- = \frac{\rho^- (\rho_+ - R_+)}{\rho_+} = \frac{P_1^{1/\gamma_-}}{P_1^{1/\gamma_+}} \left( \frac{P_1^{1/\gamma_+} - R_+}{P_1^{1/\gamma_-}} \right).\]

By virtue of \((1.5)\), \((1.10)\) and \((1.13)\), \(\alpha_\pm\) can be defined by

\[\left\{ \begin{array}{l}
\alpha_+ (P, R_+) = \frac{R_+}{P_1^{1/\gamma_+}}, \\
\alpha_- (P, R_+) = 1 - \frac{R_+}{P_1^{1/\gamma_-}}.
\end{array} \right.\]

We refer the readers to [3], P. 614 for more details.

As already stated, the model \((1.5)\) is a partially dissipative system, which brings various difficulties for our studies on its mathematical properties. Therefore, to tackle with this difficulty, we need to explore new dissipative variable by making full use of the structure of the model \((1.5)\). One key observation in this paper is that the common pressure \(P\) is a dissipative variable. With this crucial observation, the system \((1.5)\) can be rewritten in terms of the variables
employ Duhamel principle and uniform time–independent energy estimates on the nonlinear terms to prove Remark 1.4. and thus omit the details for the sake of simplicity. (1.18)–(1.20) show that the components of the solution exhibit totally distinctive behaviors. More precisely, the dissipation variables \( \lambda^+ \), \( \lambda^- \), \( \mu^+ \), \( \mu^- \) have decay rates in time, but the non–dissipation variables \( R^+, R^- \) only have uniform time–independent bound. This phenomenon is totally new as compared to \([5, 13]\) where all the components of the solution show the same behaviors (i.e, have decay rates in time), and is the most important difference between partially dissipative system and dissipative system.

Now, we are in a position to state our main result.

**Theorem 1.1.** Assume that \( R_0^+ - 1 \in H^3(\mathbb{R}^3), P_0 - \bar{P}, u_0^+, u_0^- \in H^3(\mathbb{R}^3) \cap L^1(\mathbb{R}^3), \) then there exists a constant \( \delta_0 \) such that if

\[
K_0 := \left\| (R_0^+ - 1) \right\|_{H^3} + \left\| (P_0 - \bar{P}, u_0^+, u_0^-) \right\|_{H^3 \cap L^1} \leq \delta_0,
\]

then the Cauchy problem (1.15)–(1.16) admits a unique solution \( (R^+, P, u^+, u^-) \) globally in time, satisfying

\[
R^+ - 1, P - \bar{P} \in C^0([0, \infty); H^3(\mathbb{R}^3)) \cap C^1([0, \infty); H^2(\mathbb{R}^3)),
\]

\[
u^+, u^- \in C^0([0, \infty); H^3(\mathbb{R}^3)) \cap C^1([0, \infty); H^1(\mathbb{R}^3)).
\]

Moreover, for any \( t \geq 0 \), there exists a positive constant \( C_0 \) independent of \( t \) such that the solution \( (R^+, P, u^+, u^-) \) satisfies the following estimates:

\[
\left\| (P - \bar{P}, u^+, u^-) (t) \right\|_{L^2} \leq C_0 K_0 (1 + t)^{-\frac{3}{4}},
\]

\[
\left\| \nabla P (t) \right\|_{H^1} + \left\| \nabla (u^+, u^-) (t) \right\|_{H^2} \leq C_0 K_0 (1 + t)^{-\frac{3}{4}},
\]

and

\[
\left\| \nabla^3 P (t) \right\|_{L^2} + \left\| (R^+ - 1, R^- - 1) (t) \right\|_{H^3} \leq C_0 K_0.
\]

**Remark 1.2.** (1.18)–(1.20) show that the components of the solution exhibit totally distinctive behaviors. More precisely, the dissipation variables \( (P, u^+, u^-) \) have decay rates in time, but the non–dissipation variables \( (R^+, R^-) \) only have uniform time–independent bound. This phenomenon is totally new as compared to \([5, 13]\) where all the components of the solution show the same behaviors (i.e, have decay rates in time), and is the most important difference between partially dissipative system and dissipative system.

**Remark 1.3.** Noticing the lower bound on linear \( L^2 \) decay rates in Proposition 2.6 we can employ Duhamel principle and uniform time–independent energy estimates on the nonlinear terms to prove

\[
\left\| (P - \bar{P}, u^+, u^-) (t) \right\|_{L^2} \geq C(1 + t)^{-\frac{3}{4}},
\]

where the rate is the same as the upper decay rate in (1.18). Therefore, our \( L^2 \) decay rate on \( (P - \bar{P}, u^+, u^-) \) is optimal in this sense. However, this is not our main concern. We will focus our attention on global well–posedness of the solutions and thus omit the details for the sake of simplicity.

**Remark 1.4.** Our methods can be applied to study bounded domain problem to the 3D compressible two–fluid model (1.5). This will be reported in our forthcoming work [24].

Now, let us illustrate the main difficulties encountered in proving Theorem 1.1 and explain our strategies to overcome them. Different from the models in [13] [5], where either the capillary pressure effects or the internal capillary forces effects are involved, we consider the two–fluid model (1.5) with common pressure \( P^+ = P^- \) and no capillary forces effects being involved.
Therefore, the model in the present paper is a partially dissipative system which brings many essential difficulties. We will develop new ideas to overcome these difficulties as explained below.

To begin with, we give a heuristic description of the significant difference between model (1.5) and those in [13, 5]. By taking $n^+ = R^\pm - 1$, one can write the corresponding linear system of the model (1.5) in terms of the variables $(n^+, u^+, n^-, u^-)$:

$$
\begin{align*}
\partial_t n^+ + \text{div} u^+ &= 0, \\
\partial_t u^+ + \beta_1 \nabla n^+ + \beta_2 \nabla n^- - \nu_1^+ \Delta u^+ - \nu_2^+ \nabla \text{div} u^+ = 0, \\
\partial_t n^- + \text{div} u^- &= 0, \\
\partial_t u^- + \beta_3 \nabla n^+ + \beta_4 \nabla n^- - \nu_1^- \Delta u^- - \nu_2^- \nabla \text{div} u^- &= 0,
\end{align*}
$$

(1.21)

where $\beta_1 = \frac{c(1.1)\rho(-1.1)}{\rho(1.1)}$, $\beta_2 = \beta_3 = C(1, 1)$, $\beta_4 = \frac{c(1.1)\rho_1(1.1)}{\rho_1(-1.1)}$, $\nu_1^\pm = \frac{\mu^\pm}{\rho^\pm(1, 1)}$, and $\nu_2^\pm = \frac{\mu^\pm + \lambda^\pm}{\rho(1, 1)} > 0$. Multiplying (1.21) and (1.22) by $\frac{3}{2} u^+, \frac{1}{\beta_2} u^+, \frac{3}{2} n^-,$ and $\frac{1}{\beta_2} u^-$, one can easily get the nature energy equation of the linear system (1.22):

$$
\begin{align*}
\partial_t E_0(t) + D_0(t) := \partial_t \int_{\mathbb{R}^3} \left( \frac{3}{2} \beta_2 |u^+|^2 + \frac{1}{2} \beta_3 |n^-|^2 + \frac{1}{2} \beta_3 |n^+|^2 + \frac{1}{2} \beta_3 |u^-|^2 \right) dx \\
+ \int_{\mathbb{R}^3} \left( \nu_1^+ |\nabla u^+|^2 + \nu_2^+ |\text{div} u^+|^2 \right) dx + \frac{1}{\beta_3} \left( \nu_1^- |\nabla u^-|^2 + \nu_2^- |\text{div} u^-|^2 \right) dx = 0,
\end{align*}
$$

(1.22)

where $E_0(t)$ and $D_0(t)$ denote the nature energy and dissipation, respectively. Noticing the fact that $\beta_1 \beta_4 = \beta_2 \beta_3 = \beta_3^2$, it is clear that

$$
E_0(t) = \frac{1}{2} \int_{\mathbb{R}^3} \left( \sqrt{\frac{\beta_1}{\beta_2}} n^+ + \sqrt{\frac{\beta_3}{\beta_4}} n^- \right)^2 dx.
$$

This together with the energy equation (1.22) makes it impossible for us to get the uniform energy estimates of the non–dissipative variables $n^+$ and $n^-$ simultaneously, even though in the linear level, but possibly the uniform energy estimate of their linear combination $\sqrt{\frac{\beta_1}{\beta_2}} n^+ + \sqrt{\frac{\beta_3}{\beta_4}} n^-$. On the other hand, Evje–Wang–Wen [13] considered the capillary pressure $f = P^+ - P^-$ which is a strictly decreasing function near the equilibrium and satisfies assumption (1.8). Similarly, we can get the same nature energy equation (1.22) except replacing the expressions of $\beta_2$ and $\beta_4$ by $\beta_2 = C(1, 1) + \frac{c(1.1)\rho_1(-1.1)}{\rho_1(1.1)} f(1)$ and $\beta_4 = C(1, 1) - \frac{c(1.1)\rho_1(-1.1)}{\rho_1(1.1)} f(1)$. Then, we can rewrite the energy $E_0(t)$ into:

$$
E_0(t) = \frac{1}{2} \int_{\mathbb{R}^3} \left( \left( \sqrt{\frac{\beta_1}{\beta_2}} n^+ + \sqrt{\frac{\beta_3}{\beta_4}} n^- \right)^2 + \frac{1}{\beta_2} |u^+|^2 + \frac{1}{\beta_3} |u^-|^2 \right) dx
$$

$$
= \frac{1}{2} \int_{\mathbb{R}^3} \left( \left( \sqrt{\frac{\beta_1}{\beta_2}} n^+ + \sqrt{\frac{\beta_3}{\beta_4}} n^- \right)^2 - \frac{C^2(1, 1) f(1)}{\beta_1 \beta_2 \rho_1(1)} |n^-|^2 \right) dx + \frac{1}{\beta_2} |u^+|^2 + \frac{1}{\beta_3} |u^-|^2 dx,
$$

which together with the key assumption (1.8) implies that Evje–Wang–Wen [13] can get the uniform energy estimates of $n^+$ and $n^-$ simultaneously, at least in the linear level. As for Cui–Wang–Yao–Zhu [5], where the internal capillary effects in terms of a third–order derivative of fraction density $(a^3 \rho^2)$ are involved in the momentum equations, one can get the similar nature energy equation (1.22) except adding the term $\int_{\mathbb{R}^3} \frac{\sigma^+}{\beta_2} |\nabla n^+|^2 + \frac{\sigma^-}{\beta_3} |\nabla n^-|^2 dx$ into the energy $E_0(t)$ with the capillary coefficients $\sigma^\pm > 0$. This again enables Cui–Wang–Yao–Zhu [5] to obtain the uniform energy estimates of $n^+$ and $n^-$, at least in the linear level. Therefore, the methods...
of \[13\] relying heavily on the capillary pressure effects or the internal capillary forces effects are invalid in our problem. As already stated, for our studies on the partially dissipative system \[13, 5\], it is essential to explore some new potential dissipative variable by fully using the specific structure of the system, and divide the analysis into two parts, one for the dissipative variables and another one for the non–dissipative variables. As mentioned above, the linear combination:

\[
\sqrt{\beta_1 n^+} + \sqrt{\beta_2 n^-} = \frac{1}{\sqrt{\rho^-(1,1)\rho^+(1,1)}} (\rho^-(1,1)n^+ + \rho^+(1,1)n^-)
\]

may be dissipative. On the other hand, by virtue of Mean Value Theorem, we have \(P - \bar{P} \sim C(1,1) (\rho^-(1,1)n^+ + \rho^+(1,1)n^-)\).

In the spirit of these heuristic observations, it is nature to choose the common pressure \(P\) as a new variable, and thus, in terms of the variables \((R^+, P, u^+, u^-)\), we can reformulate the Cauchy problem of the model \((2.3)\) into the Cauchy problem \((1.5) - (1.10)\). Then, we first do energy estimates on the dissipative variables. Next, we deduce energy estimates on the non–dissipative variables to close the a priori assumption \((3.1)\) by making full use of the obtained energy estimates on the dissipative variables. Roughly speaking, our proofs mainly involves the following three steps.

First, we make spectral analysis and linear \(L^2\) estimates on dissipative components \((\theta, u^+, u^-)\) of the solution to the linear system of \((2.1) - (2.2)\). To derive time–decay estimates of the linear system \((2.11)\), it requires us to make a detailed analysis on the properties of the semigroup. To simplify the analysis of the Green function which is a \(7 \times 7\) system, we will employ the Hodge decomposition technique firstly introduced by Danchin [6] to split the linear system into three systems. One has three distinct eigenvalues and the other two are classic heat equations. By making careful pointwise estimates on the Fourier transform of Green’s function to the linear equations, we can obtain the desired linear \(L^2\) estimates, and refer to the proof of Proposition \(2.6\) for details.

Second, we deduce decay estimates of \((\theta, u^+, u^-)\). Under a priori assumption: \(\|n^+, \theta, u^\pm\|_{H^3} \leq \delta \ll 1\), we first deduce \(H^2\)–energy estimates of \((\theta, u^+, u^-)\). Here, it is worth mentioning that at this stage, we cannot derive \(H^4\)–energy estimates as the classic results for the dissipation system. Indeed, since the common pressure \(P\) has been chosen as a new variable in our analysis, the cost is that the strongly coupling terms like \(C \rho^u \cdot \nabla n^+\) and \(C \rho^u \cdot \nabla n^-\) are involved in the right–hand side of equation \((2.1)\). Therefore, in the derivation of the a priori energy estimate for \(\nabla^3 \theta\), we encounter the trouble term \(\int_{\mathbb{R}^3} C \rho^u \cdot \nabla \nabla^3 n^+ \nabla^3 \theta \, dx\), which however is out of control in the \(H^3\) setting. By making careful analysis and interpolation tricks, we can get

\[
\frac{d}{dt} E_1(t) + C \left( \|\nabla^2 u^\pm\|^2_{L^2} + \|\nabla \text{div} u^\pm\|^2_{L^2} \right) \lesssim \delta \left( \|\nabla \theta\|^2_{L^2} + \|\nabla u^\pm\|^2_{L^2} \right),
\]

where \(E_1(t)\) is equivalent to \(\|\nabla(\theta, u^+, u^-)\|^2_{H^1}\). Next, by making energy estimates on the time derivatives of \((\theta, u^+, u^-)\) and fully using parabolic properties of the momentum equations in \((2.1)\), we deduce that

\[
\frac{d}{dt} E_2(t) + C (E_2(t) + \|u^\pm\|^2_{L^2}) \lesssim \delta \|\nabla(\theta^l, u^{l, \pm})\|^2_{L^2},
\]

where \(E_2(t) = E_1(t) + \|(\theta^l, u^l)^+, \nabla u^l_+\|^2_{L^2}\). Therefore, to close the energy estimate of \(E_2(t)\), it suffices to show that \(\|\nabla(\theta^l, u^{l, \pm})\|^2_{L^2}\) decays sufficiently quickly. Applying Duhamel principle, linear estimates obtained in Step 1, Plancherel theorem, Hölder inequality and Hausdorff–Young inequality, one has

\[
\|\nabla(\theta^l, u^{l, \pm})\|^2_{L^2} \lesssim K_0 (1 + t)^{-\frac{2}{5}} + \int_0^t \|\nabla e^{(t-\tau)} F(\tau)\|^2_{L^2} d\tau,
\]

where \(F = (F^1, F^2, F^3)^t\) denote the nonlinear source terms. On the other hand, since \(n^\pm\) are non–dissipative, the strongly coupling terms \(C \rho^u \cdot \nabla n^+\) and \(C \rho^u \cdot \nabla n^-\) in \((2.3)\) denote the slowest time–decay rates into the second term on the right–hand side of \((1.23)\). Therefore, this
together with (1.28) implies that one can only get the following estimate:

\[
\| \nabla (\theta, u^{\pm}) (t) \|_{L^2} \lesssim K_0 (1 + t)^{-\frac{5}{4} + \delta_0} \int_0^t \left( \| e^{(t-\tau)B} F^l (\tau) \|_{L^2} \right) d\tau + \delta_0 \int_{\frac{t}{2}}^t \left( \| e^{(t-\tau)B} F^l (\tau) \|_{L^2} \right) d\tau
\]

\[
\lesssim K_0 (1 + t)^{-\frac{5}{4} + \delta_0} \int_0^t (1 + t - \tau)^{-\frac{5}{4}} \| F^l (\tau) \|_{L^1} d\tau + \delta_0 \int_{\frac{t}{2}}^t (1 + t - \tau)^{-\frac{5}{4}} \| F^l (\tau) \|_{L^2} d\tau
\]

\[
\lesssim K_0 (1 + t)^{-\frac{5}{4} + \delta_0} \int_0^t (1 + t - \tau)^{-\frac{5}{4}} (1 + \tau)^{-\frac{5}{4}} d\tau + \delta_0 \int_{\frac{t}{2}}^t (1 + t - \tau)^{-\frac{5}{4}} (1 + \tau)^{-\frac{5}{4}} d\tau
\]

\[
\lesssim K_0 (1 + t)^{-\frac{5}{4} + \delta_0} (1 + t)^{-1},
\]

which however is not quickly enough for us to close energy estimates of the non-dissipative variables \( n^\pm \) (see the proof of Lemma 3.5 for details). To overcome this difficulty, it is essential to develop new ideas to deal with the trouble terms \( C \rho^- u^+ \cdot \nabla n^+ + C \rho^+ u^- \cdot \nabla n^- \). The key idea here is that we consider the two trouble terms as one: \( C (\rho^- u^+ - \rho^+ u^-) \cdot \nabla n - \rho^- s^2_2 - \rho^+ s^2_2 \neq 0 \) and fully using the subtle relation between the variables, we surprisingly find that

\[
C \rho^- u^+ \cdot \nabla n^+ + C \rho^+ u^- \cdot \nabla n^-
\]

\[
= \text{div} \left[ \frac{s^2_2}{\rho^+} \rho^- (u^+ - u^-) \frac{\rho^+}{\rho^-} \text{ln} \left( \frac{(\rho^+ s^2_2 + (\rho^+ s^2_2)}{(\rho^+ s^2_2 + (\rho^+ s^2_2)} \right) \right] + \text{good terms}
\]

\[
\begin{align*}
\text{div} R_1 \text{ + good terms.}
\end{align*}
\]

With this crucial observation, one can shift the derivative of \( R_1 \) onto the solution semigroup to derive the desired decay estimates of \( \| \nabla (\theta, u^{\pm}) \|_{L^2} \), and refer to the proof of (3.32) for details. Consequently, by noticing the definition of \( E_2 (t) \) and using the parabolicity of the momentum equations in (2.1), we can obtain the key uniform estimate on \( \| \nabla \theta \|_{H^1} \) and \( \| \nabla u^{\pm} \|_{H^2} \). Finally, by using this crucial uniform estimate and employing similar arguments, we can get the desired uniform estimate of \( \| (\theta, u^+, u^-) \|_{L^2} \). 

In the last step, we close energy estimates of \( n^\pm \). Compared to [5, 13], we need to develop new ingredients in the proof to handle with the difficulties arising from the partially dissipative system, which requires some new thoughts. Indeed, as in [5, 13], one may consider the corresponding linearized system of the model (1.3):

\[
\begin{aligned}
\partial_t n^+ + \text{div} u^+ &= S^+_1, \\
\partial_t u^+ + \beta_1 \nabla n^+ + \beta_2 \nabla n^- - \nu_1^+ \Delta u^+ - \nu_2^+ \text{div} u^+ &= S^+_2, \\
\partial_t n^- + \text{div} u^- &= S^-_1, \\
\partial_t u^- + \beta_3 \nabla n^+ + \beta_4 \nabla n^- - \nu_1^- \Delta u^- - \nu_2^- \text{div} u^- &= S^-_2,
\end{aligned}
\]

where \( S^+_1 = -\text{div} (n^+ u^+) \), and

\[
S^+_2 = - \left( \frac{C^2 \rho^+ (n^+ + 1, \theta + \bar{P}) - C^2 \rho^+ (1, \bar{P})}{\rho^+ (n^+ + 1, \theta + \bar{P})} - C^2 \rho^+ (1, \bar{P}) \right) \nabla n^+ - \left( C^2 (n^+ + 1, \theta + \bar{P}) - C^2 (1, \bar{P}) \right) \nabla n^+ + \text{good terms.}
\]

Then, by virtue of the first and third equations in (1.25), it is easily to see that

\[
\frac{d}{dt} \| n^\pm \|_{H^k} \lesssim \| n^\pm \|_{H^k}, \text{ for } k = 2, 3,
\]

which together with the key uniform decay estimates of \( \| \nabla u^{\pm} \|_{H^2} \) in (3.22) implies the uniform boundedness of \( \| n^\pm \|_{H^2} \) directly. However, since we don’t know whether the \( L^1 (0, t) \)–norm of \( \| \nabla^4 u^\pm \|_{L^2} \) is uniformly bounded or not, it seems impossible to derive the uniform estimates on \( \| \nabla^3 n^\pm \|_{L^2} \) from (1.27). Therefore, we must pursue another route by resorting to the momentum.
By virtue of (1.25), By multiplying $\nabla^3 (1.26)_2$ and $\nabla^3 (1.26)_4$ by $\nabla^3 u^+$ and $\nabla^3 u^+$, and summing up, one has from (1.26)
\[
\frac{d}{dt} \int_{\mathbb{R}^3} \left| \nabla^3 u^+ \right|^2 + \beta_1 \left| \nabla^3 n^+ \right|^2 + \beta_3 \left| \nabla^3 n^- \right|^2 \, dx
+ \int_{\mathbb{R}^3} \nu_1^+ \left| \nabla^4 u^+ \right|^2 + \nu_2^+ \left| \nabla^3 \text{div} u^+ \right|^2 \, dx
\approx \int_{\mathbb{R}^3} \nabla^3 u^+. \nabla^3 (n^+ \nabla n^+) \, dx + \text{other terms.}
\] (1.28)

Noticing that $n^\pm$ has no decay rate in time, the term $\mathcal{R}_2$ in the right-hand of (1.28) is out of control since we don’t know whether the $L^1(0, t)$-norm of $\|\nabla^4 u^\pm\|_{L^2}$ is uniformly bounded or not. To tackle with this difficulty, our idea here is to introduce a new semi-linearized system of the model (1.25) (refer to (3.47)), where the term $\mathcal{R}_2$ exactly disappears. However, we encounter an alternative trouble term coming from strong couplings between the two fluids (refer to (5.49):
\[
\mathcal{R}_3 = \int_{\mathbb{R}^3} \nabla^3 \left[ C \left( \rho^- \nabla n^+ + \rho^+ \nabla n^- \right) \right] \cdot \nabla^3 u^\pm \, dx.
\]
By virtue of (1.25)_1 and (1.25)_3, we can rewrite $\mathcal{R}_3$ as follows:
\[
\mathcal{R}_3 = -\int_{\mathbb{R}^3} C \rho^- \nabla^3 n^+ \nabla^3 \text{div} u^+ \, dx - \int_{\mathbb{R}^3} C \rho^+ \nabla^3 n^- \nabla^3 \text{div} u^+ \, dx + \text{other terms.}
\]

For $\mathcal{R}_{3,1}$, we can employ the equation (1.25)_1 to deal with. However, to bound $\mathcal{R}_{3,2}$, because of the strongly coupling of the system and non-dissipation of $n^-$, it requires us to develop new thoughts. The key idea here is to fully use the special structure of the equation (3.27) and the fact that the pressures of the two fluids are equal. To see this, we first introduce the linear combination of two velocities $u^\pm$ as follows:
\[
v = (2\mu^+ + \lambda^+) u^+ - (2\mu^- + \lambda^-) u^-, \quad (1.29)
\]
which satisfies the following elliptic equation
\[
-\Delta \text{div} v = -\text{div}(\rho^+ u_t^+ - \rho^- u_t^-) + \text{div}(G_1 - G_2),
\] (1.30)
where $G_1$ and $G_2$ are defined in (3.37). Then by employing the classic elliptic estimate for (1.30), we have (refer to (5.42)):
\[
\|\nabla^3 \text{div} v\|_{L^2} \lesssim \|\nabla u_t^+\|_{H^1} + \|\nabla u^\pm\|_{H^2} \\
\lesssim \|(u_{tt}^+, u_t^+, \nabla u_t^+)\|_{L^2} + \|\nabla u^\pm\|_{H^2},
\]
which together with the uniform decay estimate on $\|u_t^\pm\|_{H^1}$, $\|\nabla u^\pm\|_{H^2}$ and time-weighted estimate on $\|u_{tt}^\pm\|_{L^2}$ in Step 2 (see also (3.22) and (3.23)) leads to the following key estimate:
\[
\int_0^t \|\nabla^3 \text{div} \tau\|_{L^2} \, d\tau \leq CK_0.
\] (1.31)
This is very remarkable, since the forth order spatial derivatives of $u^\pm$ have no such good estimate. We think that this phenomenon should owe to the special structure of the system. It should be mentioned that this key observation plays an essential role in our analysis. Next, we rewrite the term $\mathcal{R}_{3,2}$ as follows:
\[
\mathcal{R}_{3,2} = \int_{\mathbb{R}^3} C \rho^+ \nabla^3 n^- \nabla^3 \left( \frac{(2\mu^+ + \lambda^+) \text{div} u^\pm \text{div} v}{2\mu^\pm + \lambda^\pm} \right) \, dx.
\] (1.32)
Finally, by virtue of (1.31) and (1.32), we can control the trouble term $R_{3,2}$ appropriately. We refer to the proof of (3.4) for details.

1.3. Notations and conventions. Throughout this paper, we use $H^k(\mathbb{R}^3)$ to denote the usual Sobolev space with norm $\| \cdot \|_{H^k}$ and $L^p$, $1 \leq p \leq \infty$ to denote the usual $L^p(\mathbb{R}^3)$ space with norm $\| \cdot \|_{L^p}$. For the sake of conciseness, we do not precise in functional space names when they are concerned with scalar–valued or vector–valued functions. $(f, g)$ denotes $\| f \|_X + \| g \|_X$. We will employ the notation $a \preceq b$ to mean that $a \leq Cb$ for a universal constant $C > 0$ that only depends on the parameters coming from the problem. We denote $\nabla = \partial_x = (\partial_1, \partial_2, \partial_3)$, where $\partial_i = \partial_i$, $\nabla_i = \partial_i$, and put $\partial_x^k f = \nabla^k f = \nabla(\nabla^{k-1} f)$. Let $A^s$ be the pseudo differential operator defined by

$$A^s f = \mathfrak{g}^{-1}(|\xi|^s \hat{f}),$$

where $\hat{f}$ and $\mathfrak{g}(f)$ are the Fourier transform of $f$. The homogenous Sobolev space $H^s(\mathbb{R}^3)$ with norm given by $\| f \|_{H^s} \triangleq \| A^s f \|_{L^2}$. For a radial function $\phi \in C_0^\infty(\mathbb{R}^3)$ such that $\phi(\xi) = 1$ when $|\xi| \leq \frac{\eta}{\xi}$ and $\phi(\xi) = 0$ when $|\xi| \geq \eta$, where $\eta$ is defined in Lemma 2.1 we define the low–frequency part of $f$ by

$$f^l = \mathfrak{g}^{-1}[\phi(\xi)\hat{f}]$$

and the high–frequency part of $f$ by

$$f^h = \mathfrak{g}^{-1}[(1 - \phi(\xi))\hat{f}].$$

It is direct to check that $f = f^l + f^h$ if the Fourier transform of $f$ exists.

2. Spectral analysis and linear $L^2$ estimates.

2.1. Reformulation. In this subsection, we first reformulate the system. Setting

$$n^\pm = R^\pm - 1 \quad \text{and} \quad \theta = P - \tilde{P},$$

the Cauchy problem (1.15) and (1.16) can be reformulated as

$$\begin{cases}
\partial_t n^+ + \text{div} u^+ = -\text{div}(n^+ u^+), \\
\partial_t \theta + \beta_1 \text{div} u^+ + \beta_2 \text{div} u^- = F_1, \\
\partial_t u^+ + \beta_3 \nabla \theta - \nu_1^+ \Delta u^+ - \nu_2^+ \nabla \nabla u^+ = F_2, \\
\partial_t u^- + \beta_4 \nabla \theta - \nu_1^- \Delta u^- - \nu_2^- \nabla \nabla u^- = F_3,
\end{cases}$$

(2.1)

with initial data

$$(n^+, \theta, u^+, u^-)(x, 0) = (n_0^+, \theta_0, u_0^+, u_0^-)(x) \to (0, 0, 0, 0, \tilde{\theta}), \quad \text{as} \ |x| \to +\infty.$$  \hspace{1cm} (2.2)

Here $\beta_1 = C(1, \tilde{P}) \rho^-(\tilde{P})$, $\beta_2 = C(1, \tilde{P}) \rho^+(\tilde{P})$, $\beta_3 = \frac{1}{\rho^+(\tilde{P})}$, $\beta_4 = \frac{1}{\rho^-(\tilde{P})}$, $\nu_1^\pm = \frac{\mu^\pm + \lambda^\pm}{\rho^\pm(\tilde{P})} > 0$ and the nonlinear terms are given by

$$\begin{align*}
F_1 &= -g_1^+(n^+, \theta) \text{div} u^+ - g_1^-(n^+, \theta) \text{div} u^- - C\rho^- u^+ \cdot \nabla n^+ - C\rho^+ u^- \cdot \nabla n^-, \\
F_2 &= -u^+. \nabla u^+ - g_2^+ \nabla \theta + g_3^+ \Delta u^+ + g_4^+ \nabla \nabla u^+ + \frac{\mu^+ (\nabla u^+ + \nabla^l u^+)}{R^+} + \nu^+ \text{div} u^+ \nabla\alpha^+, \\
F_3 &= -u^- \cdot \nabla u^- - g_2^- \nabla \theta + g_3^- \Delta u^- + g_4^- \nabla \nabla u^- + \frac{\mu^- (\nabla u^- + \nabla^l u^-)}{R^-} + \nu^- \text{div} u^- \nabla\alpha^-,
\end{align*}$$

(2.3)

(2.4)

(2.5)

where the nonlinear functions $g_i^\pm (1 \leq i \leq 4)$ are defined by

$$\begin{cases}
g_i^+(n^+, \theta) = C(n^+ + 1, \theta + \tilde{P})\rho^- R^+ - \beta_1, \\
g_i^-(n^+, \theta) = C(n^+ + 1, \theta + \tilde{P})\rho^+ R^- - \beta_2,
\end{cases}$$

(2.6)
The linearized system corresponding to the Cauchy problem \((2.1)–(2.2)\) reads

\[
\begin{align*}
\partial_t n^+ + \text{div} \, u^+ & = 0, \\
\partial_t \theta + \beta_1 \text{div} \, u^+ + \beta_2 \text{div} \, u^- & = 0, \\
\partial_t u^+ + \beta_3 \nabla \theta - \nu_1^+ \Delta u^+ - \nu_2^+ \nabla \text{div} \, u^+ & = 0, \\
\partial_t u^- + \beta_3 \nabla \theta - \nu_1^- \Delta u^- - \nu_2^- \nabla \text{div} \, u^- & = 0, \\
(n^+, \theta, u^+, u^-)(x, 0) & = (n_0^+, \theta_0, u_0^+, u_0^-)(x).
\end{align*}
\]  

(2.10)

It is easy to check that there is a zero eigenvalue for the linearized problem \((2.10)\), which makes the problem become much more difficult and complicated. Noticing that the equations \((2.10)\), \((2.11)\), and \((2.12)\) are decoupled with \(n^+\), thus we consider the following Cauchy problem for \((\theta, u^+, u^-)\), which enables us to exclude the case of zero eigenvalue

\[
\begin{align*}
\partial_t \theta + \beta_1 \text{div} \, u^+ + \beta_2 \text{div} \, u^- & = 0, \\
\partial_t u^+ + \beta_3 \nabla \theta - \nu_1^+ \Delta u^+ - \nu_2^+ \nabla \text{div} \, u^+ & = 0, \\
\partial_t u^- + \beta_3 \nabla \theta - \nu_1^- \Delta u^- - \nu_2^- \nabla \text{div} \, u^- & = 0, \\
(\theta, u^+, u^-)(x, 0) & = (\theta_0, u_0^+, u_0^-)(x).
\end{align*}
\]  

(2.11)

In terms of the semigroup theory for evolutionary equation, the solution \((\theta, u^+, u^-)\) of linear Cauchy problem \((2.11)\) can be expressed via the Cauchy problem for \(U = (\theta, u^+, u^-)\) as

\[
\begin{align*}
U_t & = BU, \\
U|_{t=0} & = U_0,
\end{align*}
\]  

(2.12)

where the operator \(B\) is given by

\[
B = \begin{pmatrix}
0 & -\nu_1^+ \Delta + \nu_2^+ \nabla \text{div} \\
-\beta_3 \nabla & -\beta_1 \text{div} \\
-\beta_3 \nabla & 0
\end{pmatrix}.
\]

Applying Fourier transform to the system \((2.12)\), one has

\[
\begin{align*}
\tilde{U}_t & = A(\xi)\tilde{U}, \\
\tilde{U}|_{t=0} & = \tilde{U}_0,
\end{align*}
\]  

(2.13)

where \(\tilde{U}(\xi, t) = \mathcal{F}(U(x, t))\), \(\xi = (\xi^1, \xi^2, \xi^3)^t\) and \(A(\xi)\) is defined by

\[
A(\xi) = \begin{pmatrix}
0 & -i\beta_1 \xi^1 & -i\beta_2 \xi^1 \\
-i\beta_3 \xi & -\nu_1^+ |\xi|^2_{I_{3 \times 3}} - \nu_2^+ \xi \otimes \xi & 0 \\
-i\beta_3 \xi & 0 & -\nu_1^- |\xi|^2_{I_{3 \times 3}} - \nu_2^- \xi \otimes \xi
\end{pmatrix}.
\]

To derive the linear time–decay estimates, by using a real method as in \([17, 18, 19]\), one need to make a detailed analysis on the properties of the semigroup. To simplify the analysis of the Green function which is a \(7 \times 7\) system, we will employ the Hodge decomposition technique firstly introduced by Danchin \([6]\) to split the linear system into three systems. One only has three equations and its characteristic polynomial possesses three distinct roots, the other two systems are the heat equation. This key observation allows us to get the optimal linear convergence rates.
Let \( \varphi^\pm = \Lambda^{-1} \text{div} u^\pm \) be the “compressible part” of the velocities \( u^\pm \), and denote \( \phi^\pm = \Lambda^{-1} \text{curl} u^\pm \) (with \( \text{curl} z_i^j = \partial_x z^j_i - \partial_x z^i_j \)) by the “incompressible part” of the velocities \( u^\pm \). Then, we can rewrite the system (2.11) as follows:

\[
\begin{align*}
\begin{cases} 
\partial_t \theta + \beta_1 \Lambda \varphi^+ + \beta_2 \Lambda \varphi^- = 0, \\
\partial_t \varphi^+ - \beta_3 \Lambda \theta + \nu^+ \Lambda^2 \varphi^+ = 0, \\
\partial_t \varphi^- - \beta_4 \Lambda \theta + \nu^- \Lambda^2 \varphi^- = 0, \\
(\theta, \varphi^+, \varphi^-)|_{t=0} = (\theta_0, \Lambda^{-1} \text{div} u_0^+, \Lambda^{-1} \text{div} u_0^-)(x),
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
\begin{cases} 
\partial_t \phi^\pm + \nu^\pm \Lambda^2 \phi^\pm = 0, \\
\phi^\pm|_{t=0} = \Lambda^{-1} \text{curl} u_0^+(x),
\end{cases}
\end{align*}
\]

where \( \nu^\pm = \nu_1^\pm + \nu_2^\pm \).

2.2. Spectral analysis for IVP (2.14). In terms of the semigroup theory, we may represent the IVP (2.14) for \( \mathbf{U} = (\theta, \varphi^+, \varphi^-)^t \) as

\[
\begin{align*}
\begin{cases} 
\dot{\mathbf{U}}_t = B_1 \mathbf{U}, \\
\mathbf{U}|_{t=0} = \mathbf{U}_0,
\end{cases}
\end{align*}
\]

where the operator \( B_1 \) is defined by

\[
B_1 = \begin{pmatrix} 0 & -\beta_1 \Lambda & -\beta_2 \Lambda \\ \beta_3 \Lambda & -\nu^+ \Lambda^2 & 0 \\ \beta_4 \Lambda & 0 & -\nu^- \Lambda^2 \end{pmatrix}.
\]

Taking the Fourier transform to the system (2.16), we obtain

\[
\begin{align*}
\begin{cases} 
\hat{\mathbf{U}}_t = A_1(\xi) \hat{\mathbf{U}}, \\
\hat{\mathbf{U}}|_{t=0} = \hat{\mathbf{U}}_0,
\end{cases}
\end{align*}
\]

where \( \hat{\mathbf{U}}(\xi, t) = \mathcal{F}(\mathbf{U}(x, t)) \) and \( A_1(\xi) \) is given by

\[
A_1(\xi) = \begin{pmatrix} 0 & -\beta_1 |\xi| & -\beta_2 |\xi| \\ \beta_3 |\xi| & -\nu^+ |\xi|^2 & 0 \\ \beta_4 |\xi| & 0 & -\nu^- |\xi|^2 \end{pmatrix}.
\]

We compute the eigenvalues of matrix \( A_1(\xi) \) from the determinant

\[
\det(\lambda I - A_1(\xi)) = \lambda^3 + (\nu^+ + \nu^-)|\xi|^2 \lambda^2 + (\beta_1 \beta_3 + \beta_2 \beta_4 + \nu^+ \nu^- |\xi|^2) \lambda^2 + (\beta_1 \beta_3 \nu^- + \beta_2 \beta_4 \nu^+)|\xi|^4,
\]

which implies that matrix \( A_1(\xi) \) possesses three different eigenvalues:

\[\lambda_1 = \lambda_1(\xi), \quad \lambda_2 = \lambda_2(\xi), \quad \lambda_3 = \lambda_3(\xi).\]

Consequently, the semigroup \( e^{t A_1} \) can be decomposed into

\[
e^{t A_1(\xi)} = \sum_{k=1}^3 e^{\lambda_k t} P_k(\xi),
\]

where the projector \( P_k(\xi) \) is defined by

\[
P_k(\xi) = \prod_{j \neq k} \frac{A_1(\xi) - \lambda_j I}{\lambda_k - \lambda_j}, \quad k, j = 1, 2, 3.
\]

Thus, the solution of IVP (2.17) can be expressed as

\[
\hat{\mathbf{U}}(\xi, t) = e^{t A_1(\xi)} \hat{\mathbf{U}}_0(\xi) = \left( \sum_{k=1}^3 e^{\lambda_k t} P_k(\xi) \right) \hat{\mathbf{U}}_0(\xi).
\]
To derive long time properties of the semigroup $e^{tA_1}$ in $L^2$ framework, one need to analyze the asymptotical expansions of $\lambda_k$, $P_k$ ($k = 1, 2, 3$) and $e^{tA_1(t)}$ in the low frequency part. Employing the similar argument of Taylor series expansion as in [17] [18] [19], we have from tedious calculations that

**Lemma 2.1.** There exists a positive constants $\eta \ll 1$ such that, for $|\xi| \leq \eta$, the spectral has the following Taylor series expansion

$$
\begin{align*}
\lambda_1 &= -\frac{\beta_1\beta_3\nu^+ + \beta_2\beta_4\nu^-}{2(\beta_1\beta_3 + \beta_2\beta_4)}|\xi|^2 + \sqrt{\beta_1\beta_3 + \beta_2\beta_4}|\xi| + O(|\xi|^3), \\
\lambda_2 &= -\frac{\beta_1\beta_3\nu^+ + \beta_2\beta_4\nu^-}{2(\beta_1\beta_3 + \beta_2\beta_4)}|\xi|^2 - \sqrt{\beta_1\beta_3 + \beta_2\beta_4}|\xi| + O(|\xi|^3), \\
\lambda_3 &= -\frac{\beta_1\beta_3\nu^- + \beta_2\beta_4\nu^+}{\beta_1\beta_3 + \beta_2\beta_4}|\xi|^2 + O(|\xi|^3).
\end{align*}
$$

(2.22)

By virtue of (2.20) and (2.22), we can establish the following estimates for the low-frequency part of the solutions $U(t, \xi)$ to the IVP (2.17):

**Lemma 2.2.** Let $\bar{\nu} = \min \left\{ \frac{\beta_1\beta_3\nu^+ + \beta_2\beta_4\nu^-}{2(\beta_1\beta_3 + \beta_2\beta_4)} : \frac{\nu^-}{\beta_1\beta_3 + \beta_2\beta_4} \right\} > 0$, we have

$$
|\tilde{\theta}|, |\tilde{\varphi}^+|, |\tilde{\varphi}^-| \lesssim e^{-\bar{\nu} |\xi|^2 t} (|\tilde{\theta}| + |\tilde{\varphi}^+| + |\tilde{\varphi}^-|),
$$

(2.23)

for any $|\xi| \leq \eta$.

**Proof.** By virtue of formula (2.20) and Taylor series expansion of $\lambda_k$ ($k = 1, 2, 3$) in (2.22), we can represent $P_k$ ($k = 1, 2, 3$) as follows:

$$
P_1(\xi) = \begin{pmatrix}
\frac{1}{2} & \frac{\beta_1}{\beta_1\beta_3 + \beta_2\beta_4} & \frac{\beta_2}{\beta_1\beta_3 + \beta_2\beta_4} \\
-\frac{\beta_1}{\beta_1\beta_3 + \beta_2\beta_4} & \frac{2\sqrt{\beta_1\beta_3 + \beta_2\beta_4}}{2(\beta_1\beta_3 + \beta_2\beta_4)} & \frac{2\sqrt{\beta_1\beta_3 + \beta_2\beta_4}}{2(\beta_1\beta_3 + \beta_2\beta_4)} \\
-\frac{\beta_2}{\beta_1\beta_3 + \beta_2\beta_4} & \frac{2\sqrt{\beta_1\beta_3 + \beta_2\beta_4}}{2(\beta_1\beta_3 + \beta_2\beta_4)} & \frac{2\sqrt{\beta_1\beta_3 + \beta_2\beta_4}}{2(\beta_1\beta_3 + \beta_2\beta_4)}
\end{pmatrix} + O(|\xi|),
$$

(2.24)

and

$$
P_2(\xi) = \begin{pmatrix}
\frac{1}{2} & \frac{-\beta_1}{\beta_1\beta_3 + \beta_2\beta_4} & \frac{-\beta_2}{\beta_1\beta_3 + \beta_2\beta_4} \\
\frac{\beta_1}{\beta_1\beta_3 + \beta_2\beta_4} & \frac{2\sqrt{\beta_1\beta_3 + \beta_2\beta_4}}{2(\beta_1\beta_3 + \beta_2\beta_4)} & \frac{2\sqrt{\beta_1\beta_3 + \beta_2\beta_4}}{2(\beta_1\beta_3 + \beta_2\beta_4)} \\
\frac{\beta_2}{\beta_1\beta_3 + \beta_2\beta_4} & \frac{2\sqrt{\beta_1\beta_3 + \beta_2\beta_4}}{2(\beta_1\beta_3 + \beta_2\beta_4)} & \frac{2\sqrt{\beta_1\beta_3 + \beta_2\beta_4}}{2(\beta_1\beta_3 + \beta_2\beta_4)}
\end{pmatrix} + O(|\xi|),
$$

(2.25)

and

$$
P_3(\xi) = \begin{pmatrix}
0 & \frac{\beta_3}{\beta_1\beta_3 + \beta_2\beta_4} & \frac{\beta_4}{\beta_1\beta_3 + \beta_2\beta_4} \\
0 & \frac{\beta_1\beta_3 + \beta_2\beta_4}{\beta_1\beta_3 + \beta_2\beta_4} & \frac{\beta_1\beta_3 + \beta_2\beta_4}{\beta_1\beta_3 + \beta_2\beta_4} \\
0 & \frac{\beta_1\beta_3 + \beta_2\beta_4}{\beta_1\beta_3 + \beta_2\beta_4} & \frac{\beta_1\beta_3 + \beta_2\beta_4}{\beta_1\beta_3 + \beta_2\beta_4}
\end{pmatrix} + O(|\xi|),
$$

(2.26)

for any $|\xi| \leq \eta$. Therefore, (2.23) follows from (2.21)–(2.22) and (2.24)–(2.26) immediately.

With the help of (2.23), we can get the following Proposition which is concerned with the optimal $L^2$ convergence rate on the low-frequency part of the solution.

**Proposition 2.3 (L²–theory).** Let $k \geq 0$ and $1 \leq p \leq 2$, it holds that

$$
\|\nabla^k e^{tB_1} U(0)\|_{L^2} \lesssim (1 + t)^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}} \|U(0)\|_{L^p},
$$

(2.27)

for any $t \geq 0$. 
Proposition 2.5. That the solution \( \phi \) for the IVP \( (2.15) \) satisfies the following decay estimate.

\[
\| \nabla^k e^{tA} \phi (0) \|_{L^2} \leq (1 + t)^{-\frac{3}{2} + \frac{k}{4}} \| \phi (0) \|_{L^p},
\]

for any \( t \geq 0 \).

Proof. Due to (2.23) and Plancherel theorem, we have

\[
\| \nabla^k e^{tA} \phi (0) \|_{L^2}^2 = \| \xi^k e^{tA} \hat{\phi} (0) \|_{L^2}^2 \leq \int_{|\xi| \leq \eta} e^{-2q|\xi|^2 t} |\xi|^{2k} |\hat{\phi} (0)|^2 d\xi \leq (1 + t)^{-3(\frac{k}{4} - \frac{1}{2}) - k} \| \hat{\phi} (0) \|_{L^q}^2 \leq (1 + t)^{-3(\frac{k}{4} - \frac{1}{2}) - k} \| \phi (0) \|_{L^p}^2,
\]

where \( q \) satisfies \( \frac{1}{p} + \frac{1}{q} = 1 \). This implies (2.27) immediately, and thus the proof of Proposition 2.5 is completed. \( \square \)

It should be mentioned that the \( L^2 \)-convergence rates derived above are optimal. Indeed, we have the lower-bound on the convergence rates which is stated in the following Proposition 2.6.

Proposition 2.4. Assume that \( \left( \hat{\theta}, \hat{\varphi}^+, \hat{\varphi}^- \right) \in L^1 \) satisfies

\[
\hat{\varphi}^+ (\xi) = \hat{\varphi}^- (\xi) = 0 \quad \text{and} \quad |\hat{\theta} (\xi)| \geq c_0,
\]

for any \( |\xi| \leq \eta \). Then it holds that the global solution \( (\theta, \varphi^+, \varphi^-) \) of the IVP (2.16) satisfies

\[
\min \left\{ \| \theta \|_{L^2}, \| \varphi^+ \|_{L^2}, \| \varphi^- \|_{L^2} \right\} \geq c_0 (1 + t)^{-\frac{3}{4} - \frac{3}{8}},
\]

for large enough \( t \).

Proof. Let \( \hat{\nu} = \frac{\beta_1 \beta_2 \nu + \beta_2 \beta_3 \nu^-}{2 (\beta_1 \beta_3 + \beta_2 \beta_4)} > 0 \). Due to (2.28), it follows from (2.21), (2.22) and (2.23) that

\[
\hat{\theta} \sim e^{-\hat{\nu} |\xi|^2 t} \cos \left( \sqrt{\beta_1 \beta_3 + \beta_2 \beta_4} |\xi| t + O(|\xi|^3 t) \right) \hat{\theta}_0
\]

This together with Plancherel theorem and the double angle formula gives that

\[
\| \theta \|_{L^2}^2 = \| \theta \|_{L^2}^2 \geq \frac{c_0^2}{2} \int_{|\xi| \leq \eta} e^{-2\hat{\nu} |\xi|^2 t} \cos^2 \left( \sqrt{\beta_1 \beta_3 + \beta_2 \beta_4} |\xi| t + O(|\xi|^3 t) \right) d\xi
\]

\[
= \frac{c_0^2}{4} \int_{|\xi| \leq \eta} e^{-2\hat{\nu} |\xi|^2 t} d\xi + \frac{c_0^2}{4} \int_{|\xi| \leq \eta} e^{-2\hat{\nu} |\xi|^2 t} \cos \left( 2 \sqrt{\beta_1 \beta_3 + \beta_2 \beta_4} |\xi| t + O(|\xi|^3 t) \right) d\xi
\]

\[
\geq \frac{c_0^2}{4} (1 + t)^{-\frac{3}{2}} - C c_0^2 (1 + t)^{-\frac{3}{8}} (1 + t)^{-\frac{3}{4}}
\]

\[
\geq \frac{c_0^2}{4} (1 + t)^{-\frac{3}{2}},
\]

if \( t \) is large enough. Using a similar procedure as in (2.28) to handle \( \| (\varphi^+, \varphi^-) \|_{L^2} \), one has (2.29). Therefore, we have completed the proof of Proposition 2.4. \( \square \)

2.3. Spectral analysis for IVP (2.15). From the classic theory of the heat equation, it is clear that the solution \( \phi^\pm \) to the IVP (2.15) satisfies the following decay estimate.

Proposition 2.5 \( (L^2\text{-theory}) \). Let \( k \geq 0 \) and \( 1 \leq p \leq 2 \), it holds that

\[
\| \nabla^k e^{-\hat{\nu}^2 \lambda_2 t} \phi^\pm (0) \|_{L^2} \leq (1 + t)^{-\frac{3}{4} + \frac{k}{4} - \frac{3}{8}} \| \phi^\pm (0) \|_{L^p},
\]

for any \( t \geq 0 \).
2.4. \(L^2\) decay estimates for IVP \((2.12)\). By virtue of the definition of \(\varphi^\pm\) and \(\phi^\pm\), and the fact that the relations

\[ u^\pm = -\lambda^{-1} \nabla \varphi^\pm - \lambda^{-1} \text{div} \phi^\pm \]

involves pseudo–differential operators of degree zero, the estimates in space \(H^k(\mathbb{R}^3)\) for the original function \(u^\pm\) will be the same as for \((\varphi^\pm, \phi^\pm)\). Combining Propositions 2.3, 2.4 and 2.5, we have the following result concerning long time properties for the solution semigroup \(e^{-tA}\).

**Proposition 2.6.** Let \(k \geq 0\) and \(1 \leq p \leq 2\). Assume that the initial data \(U_0 \in L^p(\mathbb{R}^3)\), then for any \(t \geq 0\), the global solution \(U = (\theta, u^+, u^-)^t\) of the IVP \((2.11)\) satisfies

\[ \|\nabla^k e^{-tB} U(t)\|_{L^2} \leq C(1 + t)^{-\frac{k}{2}(\frac{1}{p} - \frac{1}{2})} \|U(0)\|_{L^p}. \]  

(2.32)

If additionally the initial data satisfies \((2.28)\), we also have the following lower–bound on convergence rate

\[ \min \left\{ \|\theta(t)\|_{L^2}, \|u^+(t)\|_{L^2}, \|u^-(t)\|_{L^2} \right\} \geq C_1 c_0 (1 + t)^{-\frac{3}{2}} \]

(2.33)

if \(t\) large enough.

3. Proof of Theorem 1.1

By a classic argument, the global existence of solutions will be obtained by combining the local existence result with a priori estimates. Since the local classical solutions can be proved by a standard argument of the Lax–Milgram theorem and Schauder–Tychonoff fixed–point theorem as in \([18, 19]\) whose details are omitted, global existence of classical solutions will follow in a standard continuity argument after we have established a priori estimates. Therefore, we assume a priori that

\[ \| (n^+, \theta, u^+, n^-, u^-) \|_{H^3} \leq \delta \ll 1, \]

(3.1)

here \(\delta \sim \delta_0\) is small enough. This together with Sobolev inequality in \([7, 27]\) particularly implies that

\[ \| (n^+, \theta, u^+, n^-, u^-) \|_{W^{1, \infty}} \lesssim \delta. \]

(3.2)

In what follows, a series of lemmas on the energy estimates are given. First, the zero–order energy estimate of \((\theta, u^+, u^-)\) is obtained in the following lemma.

**Lemma 3.1.** Assume that the notations and hypotheses of Theorem 1.1 and 3.1 are in force, then

\[
\frac{1}{2} \frac{d}{dt} \left\{ \|\theta\|_{L^2}^2 + \frac{\beta_1}{\beta_3} \|u^+\|_{L^2}^2 + \frac{\beta_2}{\beta_4} \|u^-\|_{L^2}^2 \right\} + \frac{3\beta_1}{4\beta_3} \left( \nu_1^- \|\nabla u^-\|_{L^2} + \nu_2^- \|\text{div} u^-\|_{L^2} \right) + \frac{3\beta_2}{4\beta_4} \left( \nu_1^+ \|\nabla u^+\|_{L^2} + \nu_2^+ \|\text{div} u^+\|_{L^2} \right) \leq C \delta \|\nabla \theta\|_{L^2}^2. 
\]

(3.3)

**Proof.** Multiplying \((2.1)\), \((2.2)\), \((2.3)\) by \(\theta, \frac{\beta_1}{\beta_3} u^+, \frac{\beta_2}{\beta_4} u^-\), respectively, summing up and then integrating the resultant equation over \(\mathbb{R}^3\), we have

\[
\frac{1}{2} \frac{d}{dt} \left\{ \|\theta\|_{L^2}^2 + \frac{\beta_1}{\beta_3} \|u^+\|_{L^2}^2 + \frac{\beta_2}{\beta_4} \|u^-\|_{L^2}^2 \right\} \\
+ \frac{\beta_1}{\beta_3} \left( \nu_1^- \|\nabla u^-\|_{L^2} + \nu_2^- \|\text{div} u^-\|_{L^2} \right) + \frac{\beta_2}{\beta_4} \left( \nu_1^+ \|\nabla u^+\|_{L^2} + \nu_2^+ \|\text{div} u^+\|_{L^2} \right) \\
= \int_{\mathbb{R}^3} \left( F_1 \theta + \frac{\beta_1}{\beta_3} F_2 u^+ + \frac{\beta_2}{\beta_4} F_3 u^- \right) \, dx \\
=: I_1 + I_2 + I_3.
\]

(3.4)
Using (3.1), integration by parts, Lemma 3.1 Hölder inequality, Sobolev inequality and Young inequality, we have

\[
|I_1| \lesssim \int_{\mathbb{R}^3} |(n^+, \theta)| \cdot |\nabla(u^+, u^-)| \cdot |\theta| dx \\
+ \int_{\mathbb{R}^3} |C \rho^T||n^+||u^+||\nabla\theta| + |C \rho^T||n^+||\text{div}u^+||\theta| + |\nabla(C \rho^T)||n^+||u^+||\theta| dx \\
\lesssim \|(n^+, \theta)\|_{L^3} \|\nabla(u^+, u^-)\|_{L^2} \|\theta\|_{L^6} + \|\nabla\theta\|_{L^2} \|(u^+, u^-)\|_{L^6} \|(n^+, n^-)\|_{L^3}^3 \\
+ \|\theta\|_{L^6} \|(\text{div}u^+, \text{div}u^-)\|_{L^2} \|(n^+, n^-)\|_{L^3}^2 \\
+ \|\theta\|_{L^6} \|(u^+, u^-)\|_{L^6} \|(n^+, n^-)\|_{L^3} \|\nabla(n^+, \theta)\|_{L^3} \\
\lesssim \delta \|\nabla(\theta, u^+, u^-)\|_{L^2}^2.
\]

(3.5)

For the term $I_2$, by employing integration by parts, Hölder inequality, Sobolev inequality and Young inequality, we obtain

\[
|I_2| \lesssim \int_{\mathbb{R}^3} \left((u^+||\nabla u^+| + |\theta||\nabla\theta| + |n^+||\nabla u^+| + |\nabla\theta||\nabla u^+|) |u^+| + |\theta||\nabla u^+|^2 dx \\
\lesssim \|u^+\|_{L^2} \|\nabla u^+\|_{L^2} \|u^+\|_{L^3} \|\theta\|_{L^6} \|\nabla\theta\|_{L^2} \|u^+\|_{L^3} \|\nabla n^+\|_{L^3} \|\nabla u^+\|_{L^2} \|u^+\|_{L^6} \\
+ \|\nabla\theta\|_{L^2} \|\nabla u^+\|_{L^2} \|u^+\|_{L^n} \|\theta\|_{L^n} \|\nabla u^+\|_{L^n} \\
\lesssim \delta \|\nabla(\theta, u^+)^2\|_{L^2}^2.
\]

(3.6)

Similarly, we have

\[
|I_3| \lesssim \delta \|\nabla(\theta, u^-)\|_{L^2}^2.
\]

(3.7)

Substituting (3.5)–(3.7) into (3.3) and using the smallness of $\delta$ yield (3.3) immediately. Therefore, we complete the proof of Lemma 3.1.

Next, we deduce energy estimates on the first and second order spatial derivatives of $(\theta, u^+, u^-)$, which are stated in the following lemma.

**Lemma 3.2.** Assume that the notations and hypotheses of Theorem 1.1 and (3.1) are in force, then

\[
\frac{1}{2} \frac{d}{dt} \left( \|\nabla\theta\|^2_{H^1} + \frac{\beta_3}{\beta_4} \|\nabla u^+\|^2_{H^1} + \frac{\beta_3}{\beta_4} \|\nabla u^-\|^2_{H^1} \right) + \frac{3\beta_3}{4\beta_4} \left( \nu_1 \|\nabla^2 u^+\|^2_{H^1} + \nu_2 \|\nabla \text{div} u^+\|^2_{H^1} \right) \\
+ \frac{3\beta_2}{4\beta_4} \left( \nu_1 \|\nabla^2 u^-\|^2_{H^1} + \nu_2 \|\nabla \text{div} u^-\|^2_{H^1} \right) \leq C \delta \left( \|\nabla\theta\|^2_{H^1} + \|\nabla(\theta, u^+, u^-)\|^2_{H^2} \right).
\]

(3.8)

**Proof.** For $k = 1, 2$, multiplying $\nabla^k (2.1)_2$, $\nabla^k (2.1)_3$, $\nabla^k (2.1)_4$ by $\nabla^k \theta$, $\frac{\beta_1}{\beta_3} \nabla^k u^+$, $\frac{\beta_2}{\beta_3} \nabla^k u^-$, respectively, summing up and then integrating the resultant equation over $\mathbb{R}^3$, we have

\[
\frac{1}{2} \frac{d}{dt} \left( \|\nabla^k \theta\|^2_{L^2} + \frac{\beta_1}{\beta_3} \|\nabla^k u^+\|^2_{L^2} + \frac{\beta_2}{\beta_3} \|\nabla^k u^-\|^2_{L^2} \right) \\
+ \frac{\beta_1}{\beta_3} \left( \nu_1 \|\nabla^{k+1} u^+\|^2_{L^2} + \nu_2 \|\nabla \text{div} u^+\|^2_{L^2} \right) + \frac{\beta_2}{\beta_3} \left( \nu_1 \|\nabla^{k+1} u^-\|^2_{L^2} + \nu_2 \|\nabla \text{div} u^-\|^2_{L^2} \right) \\
= \int_{\mathbb{R}^3} \left( \nabla^k F_1 \cdot \nabla^k \theta + \frac{\beta_1}{\beta_3} \nabla^k F_2 \cdot \nabla^k u^+ + \frac{\beta_2}{\beta_3} \nabla^k F_3 \cdot \nabla^k u^- \right) dx \\
=: J_1^k + J_2^k + J_3^k.
\]

(3.9)
By using integration by parts, Lemmas A.1–A.2 Hölder inequality, Sobolev inequality and Young inequality, we have

\[
\begin{align*}
|J_1^k| &\leq \int_{\mathbb{R}^3} \left[ |\nabla^k \theta| |\nabla^k (g_1^{-} (n^+, \theta) \text{div} u^+)| + |\nabla^k \theta| |\nabla^k (g_1^{+} (n^+, \theta) \text{div} u^-)| \\
&\quad + |\nabla^k \theta| |\nabla^k (C \rho^+ u^+ \cdot \nabla n^+)| + |\nabla^k \theta| |\nabla^k (C \rho^- u^- \cdot \nabla n^-)| \right] \, dx \\
&\lesssim \left( \left( \|n^+, \theta\|_{L^\infty} \|\nabla (u^+, u^-)\|_{H^k} + \|\nabla (u^+, u^-)\|_{L^\infty} \|\|n^+, \theta\|_{H^k} \\
&\quad + \|C \rho^+ \cdot C \rho^-\|_{L^\infty} \|\|u^+, u^-\|_{L^\infty} \|\nabla (n^+, n^-)\|_{H^{k-1}} \\
&\quad + \|\nabla (u^+, u^-)\|_{L^\infty} \|\nabla (n^+, n^-)\|_{L^\infty} \right) \||\nabla^k \theta|_{L^2} \\
&\lesssim \delta \left( \||\nabla \theta|^2_{H^1} + \||\nabla (u^+, u^-)|^2_{H^2} \right). \quad (3.10)
\end{align*}
\]

For the terms \( J_2^k \), by employing integration by parts, Lemmas A.1–A.2 Hölder inequality, Sobolev inequality and Young inequality, we obtain

\[
\begin{align*}
|J_2^k| &= -\frac{\beta_1}{\beta_3} \int_{\mathbb{R}^3} \nabla^{k-1} F_2 \cdot \nabla^{k+1} u^+ \, dx \\
&\lesssim \|u^+\|_{L^\infty} \|\nabla u^+\|_{L^2} \|\nabla^2 u^+\|_{L^2} + \|\theta\|_{L^\infty} \|\nabla \theta\|_{L^2} \|\nabla^2 u^+\|_{L^2} \\
&\quad + \|\nabla u^+\|_{L^\infty} \|\nabla \theta\|_{L^2} \|\nabla^3 u^+\|_{L^2} + \|\nabla^3 u^+\|_{L^2} \|\nabla (n^+, \theta)\|_{L^\infty} \|\nabla^3 u^+\|_{L^2} \\
&\quad + \|\nabla \theta\|_{L^\infty} \|\nabla^3 u^+\|_{L^2} \|\nabla^3 u^+\|_{L^2} + \|\theta\|_{L^\infty} \|\nabla \theta\|_{L^2} \|\nabla^3 u^+\|_{L^2} \\
&\quad + \|\nabla \theta\|_{L^\infty} \|\nabla^3 u^+\|_{L^2} \|\nabla^3 u^+\|_{L^2} + \|\nabla (n^+, \theta)\|_{L^\infty} \|\nabla^3 u^+\|_{L^2} \|\nabla^3 u^+\|_{L^2} \\
&\lesssim \delta \left( \||\nabla \theta|^2_{H^1} + \||\nabla u^+|^2_{H^2} \right). \quad (3.11)
\end{align*}
\]

Similarly, we have

\[
|J_3^k| \lesssim \delta \left( \||\nabla \theta|^2_{H^1} + \||\nabla u^-|^2_{H^2} \right). \quad (3.12)
\]

Substituting (3.10)–(3.12) into (3.9), then summing \( k \) from 1 to 2, we obtain (3.8) if \( \delta \) is small enough. The proof of Lemma 3.2 is completed. \( \square \)

In the following lemma, we give the energy estimate of the time derivative for \( (\theta, u^+, u^-) \).

**Lemma 3.3.** Assume that the notations and hypotheses of Theorem 1.1 and 3.1 are in force, then

\[
\frac{1}{2} \frac{d}{dt} \left\{ \frac{\beta_3}{\beta_1} \|\theta_t\|^2_{L^2} + \|u^+\|^2_{L^2} \right\} + \left( \nu_1^+ \|\nabla u^+_t\|^2_{L^2} + \nu_2^+ \|\text{div} u^+_t\|^2_{L^2} \right) \leq C \delta \left( \|\nabla u^+(t)\|^2_{H^1} + \|\nabla u^+(t)\|^2_{L^2} \right), \quad (3.13)
\]

and

\[
\frac{d}{dt} \left\{ \mu^+ \|\nabla u^+_t\|^2_{L^2} + (\mu^+ + \lambda^+) \|\text{div} u^+_t\|^2_{L^2} \right\} + \|\sqrt{\rho^+} u^+_t\|^2_{L^2} \leq C \left( \|\nabla u^+\|^2_{H^1} + \delta \|\nabla u^+_t\|^2_{L^2} \right). \quad (3.14)
\]

**Proof.** Differentiating (2.1) with respect to \( t \), one has

\[
\begin{align*}
\begin{cases}
\partial_t \theta + \beta_1 \text{div} u^+_t + \beta_2 \text{div} u^-_t &= (F_1)_t, \\
\partial_t u^+ + \beta_5 \nabla \theta_t - \nu_1^+ \Delta u^+_t - \nu_2^+ \nabla \text{div} u^+_t &= (F_2)_t, \\
\partial_t u^- + \beta_3 \nabla \theta_t - \nu_1^- \Delta u^-_t - \nu_2^- \nabla \text{div} u^-_t &= (F_3)_t.
\end{cases}
\end{align*}
\]
From integration by parts, we have

\[ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \beta_3 \frac{d}{dt} \theta_1^2 + |u_1^+|^2 \, dx + \int_{\mathbb{R}^3} \nu_1^+ \| \nabla u_1^+ (t) \|^2 \, dx + \int_{\mathbb{R}^3} | \text{div} u_1^+ (t) |^2 \, dx \]

\[ = \int_{\mathbb{R}^3} \frac{\beta_3}{\beta_1} (F_1)_t \theta_1 + (F_2)_t u_1^+ + (F_3)_t u_1^- \, dx. \]

Using (1.15), (2.1), we finally have

\[ (3.16) \]

Multiplying (3.15), by \( \beta_3 \theta_1, u_1^+, u_1^- \) respectively, summing up and then integrating the resulting equality over \( \mathbb{R}^3 \), one has

\[ (3.16) \]

Substituting (3.17) and (3.19) into (3.16), we obtain

\[ (3.17) \]

Next, we turn to the proof of (3.14). To do this, we first rewrite (3.15) and the similar argument in (3.10), we obtain

\[ (3.17) \]

Performing the similar procedure as (3.17), we finally have

\[ (3.18) \]

Substituting (3.17) and (3.19) into (3.16), we obtain (3.14).

Next, we turn to the proof of (3.14). To do this, we first rewrite (3.15) and (3.15) in the following form:

\[ (3.20) \]
Similarly, we also have

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \mu^+ |\nabla u_t^+|^2 + (\mu^+ + \lambda^+) \text{div} u_t^+ |\nabla|^2 + \int_{\mathbb{R}^3} \rho^+ |u_t^+(t)|^2 dx
\]

\[
= \int_{\mathbb{R}^3} \nabla \theta u_t^+ dx - \int_{\mathbb{R}^3} \rho_t^+ u_t^+ dx
\]

\[
- \int_{\mathbb{R}^3} \left[ \rho^+ u_t^+ \cdot \nabla u_t^+ + \frac{\mu^+}{\alpha^+} (\nabla u_t^+ + \nabla^t u_t^+) \right] \nabla \alpha^+ + \frac{\lambda^+}{\alpha^+} \text{div} u_t^+ \nabla \alpha^+ \bigg] u_t^+ dx
\]

\[
\leq C \left( \|\nabla \theta_t\| \|u_t^+\| \|L^2 \| + \|\rho_t^+\| \|u_t^+\| \|L^2 \| + \|u_t^+\| \|L^\infty \| \|\nabla u_t^+\| \|L^2 \| + \|\nabla u_t^+\| \|L^\infty \| \|\nabla \alpha_t^+\| \|L^2 \| + \|\nabla u_t^+\| \|L^\infty \| \|\nabla \alpha_t^+\| \|L^2 \| \right)
\]

\[
\leq C \left( \|\nabla u_t^+\|^2_{H^1} + \delta \|\nabla u_t^+\|^2_{L^2} + \frac{1}{2} \int_{\mathbb{R}^3} \rho^+ |u_t^+(t)|^2 dx, \right)
\]

which shows (3.14). The proof of Lemma 3.3 is completed.

Now, we are in a position to derive the time decay rates for \((\theta, u^+, u^-)\).

**Theorem 3.4.** Under the conditions of Theorem 1.1 and (3.3), there exists a positive constant \(C\) such that

\[
\|(\theta, u^+)(t)\| \|L^2 \| \leq C K_0 (1 + t)^{-\frac{\delta}{2}}, \quad (3.21)
\]

\[
\|\nabla \theta(t)\| \|H^1 \| + \|\nabla u^t(t)\| \|L^2 \| + \|(\theta_t, u_t^+, \nabla u_t^+)^2\| \|L^2 \| \leq C K_0 (1 + t)^{-\frac{\delta}{2}} \quad (3.22)
\]

and

\[
\int_0^t \left( 1 + \tau \right)^{\frac{\delta}{2}} \|u_t^+(\tau)\|^2_{L^2} d\tau \leq C K_0. \quad (3.23)
\]

**Proof.** From (2.1), we see that

\[
\|u_t^+\| \|L^2 \| \leq C \left( \|(\nabla^2 u^t, \nabla \theta)\| \|L^2 \| + \delta \|\nabla u^+\| \|H^1 \| + \delta \|\nabla^2 \theta\| \|L^2 \|, \right) \quad (3.24)
\]

\[
\|\theta_t\| \|L^2 \| \leq C \|\nabla u^+\| \|L^2 \|, \quad (3.25)
\]

and

\[
\|\nabla^2 \theta\| \|L^2 \| \preceq \frac{C}{\delta} \left( \|\nabla u_t^+ \|, \|\nabla u^+ \|, \|\nabla \theta \| \right) \|L^2 \| \leq C \left( \|(\nabla u_t^+, \nabla^3 u^+)^2\| \|L^2 \| + \delta \|\nabla^2 u^+\| \|L^2 \| + \delta \|\nabla^2 \theta\| \|L^2 \| \right). \quad (3.26)
\]

Therefore, we have

\[
\|\nabla^2 \theta\| \|L^2 \| \leq C \left( \|(\nabla u_t^+, \nabla^3 u^+)^2\| \|L^2 \| + \delta \|\nabla^2 u^+\| \|L^2 \| \right). \quad (3.26)
\]

Similarly, we also have

\[
\|\nabla^3 u^+\| \|L^2 \| \leq C \left( \|(\nabla u_t^+, \nabla^2 \theta)\| \|L^2 \| + \delta \|\nabla^2 u^+\| \|L^2 \| \right). \quad (3.27)
\]

Let \(D > 0\) be a large but fixed constant. Using (3.21) – (3.27) and summing up \(D \times(3.8) + (3.13) + (3.14)\), then there exists an energy functional \(H(\theta, u^+)\) which is equivalent to \(\|\nabla \theta\| \|H^1 \| + \|\nabla u^+\| \|H^2 \| + \|(\theta_t, u_t^+, \nabla u_t^+)^2\| \|L^2 \| \) such that

\[
\frac{d}{dt} H(t) + C_1 \left( H(t) + \|u_t^+\| \|L^2 \| \right) \leq C \|\nabla (\theta_t, u_t^+)^2\| \|L^2 \|, \quad (3.28)
\]

for some positive constant \(C_1\). We also define the time–weighted energy functional

\[
\mathcal{E}(t) = \sup_{\theta \leq \tau \leq t} \left\{ (1 + \tau)^{\frac{\delta}{2}} H(\tau) \right\}. \quad (3.29)
\]
It is clear that \( E(t) \) is non-decreasing. Denoting

\[ \mathcal{F} = (F^1, F^2, F^3)^t, \]

we have from Duhamel principle that

\[
U^t = e^{tB}U^t(0) + \int_0^t e^{(t-\tau)B} \mathcal{F}^t(\tau) d\tau,
\]

which together with Proposition\( 2.6 \) Plancherel theorem, Hölder inequality and Hausdorff–Young inequality leads to

\[
\|\nabla(\theta^t, u^\pm, l)(t)\|_{L^2} \lesssim (1 + t)^{-\frac{n}{2}} \|\nabla((\theta, u^\pm)(0))\|_{L^1} + \int_0^t \|\nabla(e^{(t-\tau)B} \mathcal{F}^t(\tau))\|_{L^2} d\tau.
\] (3.30)

Due to non-dissipation property of the variable \( n^\pm \), it requires us to develop new thoughts to deal with the last two terms of \( F_1 \) on the right-hand side of (2.12). The main idea here is that we consider the two trouble terms as one: \(-C\rho^- u^+ \cdot \nabla n^+ - C\rho^+ u^- \cdot \nabla n^-\), and then rewrite it in a clever way. To see this, by noticing that \( \rho^- s_2^2 - \rho^+ s_3^2 \neq 0 \), and fully using the subtle relation between the variables, we surprisingly find that the trouble term \(-C\rho^- u^+ \cdot \nabla n^+ - C\rho^+ u^- \cdot \nabla n^-\) can be rewritten as

\[
\begin{align*}
- C\rho^- u^+ \cdot \nabla n^+ - C\rho^+ u^- \cdot \nabla n^- \\
= - C\rho^- u^+ \cdot \nabla n^+ - C\rho^+ u^- \cdot \left( \frac{\nabla \theta}{C\rho^+} - \frac{\rho^- \nabla n^+}{\rho^+} \right) \\
= - u^- \cdot \nabla \theta + C\rho^- (u^+ - u^-) \cdot \nabla n^+ \\
= - u^- \cdot \nabla \theta + \frac{s_2^2 s_3^2 \rho^+ \rho^- (u^+ - u^-)}{(\rho^+ s_3^2)^2 + (\rho^- s_2^2 - \rho^+ s_3^2)R^+} \cdot \nabla n^+ \\
= - u^- \cdot \nabla \theta + \frac{s_2^2 s_3^2 \rho^+ \rho^- (u^+ - u^-)}{\rho^- s_2^2 - \rho^+ s_3^2} \cdot \nabla \ln \left( (\rho^+ s_3^2)^2 + (\rho^- s_2^2 - \rho^+ s_3^2)R^+ \right) \\
+ \frac{s_2^2 s_3^2 \rho^+ \rho^- (u^+ - u^-)}{\rho^- s_2^2 - \rho^+ s_3^2} \cdot \left( \nabla (\rho^+ s_3^2)^2 + R^+ \nabla (\rho^- s_2^2 - \rho^+ s_3^2) \right) \\
= - u^- \cdot \nabla \theta + \ln \left( \frac{(\rho^+ s_3^2)^2 + (\rho^- s_2^2 - \rho^+ s_3^2)R^+}{(\rho^+ s_3^2)^2 + (\rho^- s_2^2 - \rho^+ s_3^2)R^+} \right) \\
+ \ln \left( \frac{(\rho^+ s_3^2)^2 + (\rho^- s_2^2 - \rho^+ s_3^2)R^+}{(\rho^+ s_3^2)^2 + (\rho^- s_2^2 - \rho^+ s_3^2)R^+} \right) \\
+ \frac{s_2^2 s_3^2 \rho^+ \rho^- (u^+ - u^-)}{\rho^- s_2^2 - \rho^+ s_3^2} \cdot \left( \nabla (\rho^+ s_3^2)^2 + R^+ \nabla (\rho^- s_2^2 - \rho^+ s_3^2) \right) \\
= - u^- \cdot \nabla \theta + \left( \frac{(\rho^+ s_3^2)^2 + (\rho^- s_2^2 - \rho^+ s_3^2)R^+}{(\rho^+ s_3^2)^2 + (\rho^- s_2^2 - \rho^+ s_3^2)R^+} \right) \\
+ \frac{s_2^2 s_3^2 \rho^+ \rho^- (u^+ - u^-)}{\rho^- s_2^2 - \rho^+ s_3^2} \cdot \left( \nabla (\rho^+ s_3^2)^2 + R^+ \nabla (\rho^- s_2^2 - \rho^+ s_3^2) \right) + \text{div} F_{11}.
\end{align*}
\] (3.31)
which shows (3.22) if Applying Gronwall inequality to (3.33), we have that is,
\[
\|\nabla(\theta^t, u^\pm_t)(t)\|_{L^2} \\
\lesssim (1 + t)^{-\frac{\delta}{4}}\|(\theta, u^\pm)(0)\|_{L^1} + \int_0^t (1 + t - \tau)^{-\frac{\delta}{4}}\|\mathcal{F}_2(\tau)\|_{L^1}d\tau \\
+ \int_0^t (1 + t - \tau)^{-\frac{\delta}{4}}\|F_{11}\|_{L^2}d\tau \\
\lesssim (1 + t)^{-\frac{\delta}{4}}K_0 + \int_0^t (1 + t - \tau)^{-\frac{\delta}{4}}\|(n^\pm, \theta, u^\pm, \nabla n^\pm, \nabla \theta)(\tau)\|_{L^2}\|\nabla(\nabla \theta, \nabla u^\pm, \nabla^2 u^\pm)(\tau)\|_{L^2}d\tau \\
+ \int_0^t (1 + t - \tau)^{-\frac{\delta}{4}}\|(n^\pm, \theta)(\tau)\|_{L^2}\|u^\pm(\tau)\|_{L^6}d\tau \\
\lesssim (1 + t)^{-\frac{\delta}{4}}K_0 + \delta \int_0^t (1 + t - \tau)^{-\frac{\delta}{4}}(1 + \tau)^{-\frac{\delta}{4}}\sqrt{\mathcal{E}(\tau)}d\tau \\
\lesssim (1 + t)^{-\frac{\delta}{4}}K_0 + \delta(1 + t)^{-\frac{\delta}{4}}\sqrt{\mathcal{E}(t)}. 
\]
(3.32)

Substituting (3.32) into (3.28), we have
\[
\frac{d}{dt}H(t) + C_1 \left( H(t) + \|u_{tt1}\|_{L^2}^2 \right) \leq C(1 + t)^{-\frac{\delta}{4}}K_0^2 + C\delta^2(1 + t)^{-\frac{\delta}{4}}\mathcal{E}(t). 
\]
(3.33)

Applying Gronwall inequality to (3.33), we have
\[
H(t) \leq C(1 + t)^{-\frac{\delta}{4}}K_0^2 + C\delta^2(1 + t)^{-\frac{\delta}{4}}\mathcal{E}(t), 
\]
that is,
\[
\mathcal{E}(t) \leq CK_0^2 + C\delta^2\mathcal{E}(t), 
\]
which shows (3.22) if \( \delta \) is small enough.

Employing the similar argument as in (3.32), we have
\[
\|\nabla(\theta^t, u_{t}^\pm)(t)\|_{L^2} \\
\lesssim (1 + t)^{-\frac{\delta}{4}}\|(\theta, u^\pm)(0)\|_{L^1} + \int_0^t (1 + t - \tau)^{-\frac{\delta}{4}}\|\mathcal{F}_2(\tau)\|_{L^1}d\tau \\
+ \int_0^t (1 + t - \tau)^{-\frac{\delta}{4}}\|F_{11}\|_{L^2}d\tau \\
\lesssim (1 + t)^{-\frac{\delta}{4}}K_0 + \int_0^t (1 + t - \tau)^{-\frac{\delta}{4}}\|(n^\pm, u^\pm, \nabla n^\pm)(\tau)\|_{L^2}\|\nabla(\nabla \theta, \nabla u^\pm, \nabla^2 u^\pm)(\tau)\|_{L^2}d\tau \\
+ \int_0^t (1 + t - \tau)^{-\frac{\delta}{4}}\|(n^\pm, \theta)(\tau)\|_{L^2}\|u^\pm(\tau)\|_{L^6}d\tau \\
\lesssim (1 + t)^{-\frac{\delta}{4}}K_0 + \delta^2 \int_0^t (1 + t - \tau)^{-\frac{\delta}{4}}(1 + \tau)^{-\frac{\delta}{4}}d\tau \\
\lesssim (1 + t)^{-\frac{\delta}{4}}K_0 + \delta^2(1 + t)^{-\frac{\delta}{4}}. 
\]
(3.34)
Substituting (3.34) into (3.3), there exists a positive constant $C_2$ such that
\[
\frac{d}{dt} \left\{ ||\theta||_{L^2}^2 + \frac{\beta_1}{\beta_3} ||u^+||_{L^2}^2 + \frac{\beta_2}{\beta_4} ||u^-||_{L^2}^2 \right\} 
+ C_2 \left\{ ||\theta||_{L^2}^2 + \frac{\beta_1}{\beta_3} ||u^+||_{L^2}^2 + \frac{\beta_2}{\beta_4} ||u^-||_{L^2}^2 \right\} 
\lesssim ||(\theta^t, u^{\pm t})||_{L^2}^2 + ||\nabla \theta||_{L^2}^2 
\lesssim (1 + t)^{-\frac{3}{2} K_0^2}.
\]

Applying Gronwall inequality to the above inequality, we obtain (3.21).

Multiplying (3.33) by $(1 + t)^{\frac{5}{2} K_0^2}$, we have
\[
\frac{d}{dt} \left\{ (1 + t)^{\frac{5}{2} K_0^2} H(t) + C_1 (1 + t)^{\frac{5}{2} K_0^2} (H(t) + ||u^t||_{L^2}^2) \right\} 
\leq C \left\{ (1 + t)^{\frac{5}{2} K_0^2} (1 + t)^{-\frac{3}{2} K_0^2} + (1 + t)^{\frac{5}{2} K_0^2} H(t) \right\} 
\leq C \left\{ (1 + t)^{\frac{5}{2} K_0^2} (1 + t)^{-\frac{3}{2} K_0^2} (1 + t)^{\frac{5}{2} K_0^2} (1 + t)^{-\frac{3}{2} K_0^2} \right\} 
\leq C (1 + t)^{-\frac{3}{2} K_0^2}.
\]

Integrating (3.35) from 0 to $t$, we obtain (3.28). The proof of Theorem 3.4 is completed.

Now, we are in a position to establish the a priori estimate for $(n^+, n^-)$.

**Lemma 3.5.** Assume that the notations and hypotheses of Theorem 1.1 and (3.1) are in force, then
\[
||(n^+, n^-)||_{H^3} \leq CK_0.
\]

**Proof.** As already stated, we cannot work on the linearized system (1.25). Instead, we introduce the following new semi-linearized system of model (1.3):
\[
\begin{align*}
\partial_t n^+ + (1 + n^+) \text{div} u^+ + u^+ \cdot \nabla n^+ &= 0, \\
\rho^+ \partial_t u^+ + C \left( \rho^+ \nabla n^+ + \rho^+ \nabla n^- \right) - \mu^+ \Delta u^+ - (\mu^+ + \lambda^+) \nabla \text{div} u^+ &= G_1, \\
\partial_t n^- + (1 + n^-) \text{div} u^- + u^- \cdot \nabla n^- &= 0, \\
\rho^- \partial_t u^- + C \left( \rho^- \nabla n^+ + \rho^- \nabla n^- \right) - \mu^- \Delta u^- - (\mu^- + \lambda^-) \nabla \text{div} u^- &= G_2,
\end{align*}
\]
where
\[
\begin{align*}
G_1 &= -\rho^+ u^+ \cdot \nabla u^+ + \frac{\mu^+ (\nabla u^+ + \nabla u^+ \nabla \alpha^+)}{\alpha^+} + \frac{\lambda^+ \text{div} u^+ \nabla \alpha^+}{\alpha^+}, \\
G_2 &= -\rho^- u^- \cdot \nabla u^- + \frac{\mu^- (\nabla u^- + \nabla u^- \nabla \alpha^-)}{\alpha^-} + \frac{\lambda^- \text{div} u^- \nabla \alpha^-}{\alpha^-}.
\end{align*}
\]

For $0 \leq \ell \leq 2$, multiplying $\nabla^\ell (3.37)_1$ and $\nabla^\ell (3.37)_3$ by $\nabla^\ell n^+$ and $\nabla^\ell n^-$, and then integrating over $\mathbb{R}^3$, we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla^\ell n^+|^2 dx = -\int_{\mathbb{R}^3} \nabla^\ell \left( \text{div} u^+ + n^+ \text{div} u^+ + u^+ \cdot \nabla n^+ \right) \nabla^\ell n^+ dx.
\]

Integrating by parts, we see that
\[
\int_{\mathbb{R}^3} u^+ \cdot \nabla \nabla^\ell n^+ \nabla^\ell n^+ dx = \frac{1}{2} \int_{\mathbb{R}^3} \text{div} u^+ |\nabla^\ell n^+|^2 dx.
\]

Then using Lemmas A.1 A.2 and Hölder inequality, we can deal with (3.38) as follows
\[
\frac{d}{dt} \int_{\mathbb{R}^3} |\nabla^\ell n^+|^2 dx \leq C ||\nabla u^+||_{H^2} ||n^+||_{H^2}.
\]
Therefore, we have
\[ \frac{d}{dt} \| n^\pm \|_{H^2} \leq C \| \nabla u^\pm \|_{H^2}. \] (3.39)

In virtue of Theorem 3.4, we see that by applying the classic \( t^L \) estimate of elliptic system to (3.41), we can obtain the following key estimate of \( (u^\pm, \nabla u^\pm) \) in \( L^\infty \). To begin with, by applying the classic \( L^p \)–estimate of elliptic system to (3.40), we have
\[ \| \nabla^2 u^\pm_i \|_{L^2} \lesssim \| (u^\pm_{t_i}, \nabla \theta_i) \|_{L^2} + \| (\rho^\pm_i, \nabla u^\pm_i) \|_{L^\infty} \| (u^\pm_{t_i}, \nabla n^\pm_i) \|_{L^2} + \| (u^\pm, \nabla n^\pm) \|_{L^\infty} \| \nabla u^\pm_i \|_{L^2} \]
\[ + \| \rho^\pm_i \|_{L^\infty} \| \nabla u^\pm \|_{L^2} + \| (n^\pm_i, \theta_i) \|_{L^2} \| \nabla u^\pm \|_{L^\infty} \| (\nabla n^\pm, \nabla \theta) \|_{L^\infty} \]
\[ \lesssim \| (u^\pm_{t_i}, u^\pm_i, \nabla u^\pm_i) \|_{L^2} + \| \nabla u^\pm \|_{H^1}. \] (3.40)

Summing up div\( 3.37 \)_2 and div\( 3.37 \)_4, we obtain
\[ \text{div}(\rho^+ u^+_i - \rho^- u^-_i) - \Delta [(2\mu^+ + \lambda^+) \text{div} u^+ - (2\mu^- + \lambda^-) \text{div} u^-] = \text{div}(G_1 - G_2). \] (3.41)

Denote the linear combination of the two velocities by \( v = (2\mu^+ + \lambda^+) u^+ - (2\mu^- + \lambda^-) u^- \). Applying standard \( L^p \)–estimate of elliptic system to (3.44), we can obtain the following key estimate of \( v \)
\[ \| \nabla^2 \text{div} v \|_{L^2} \lesssim \| \nabla^2 (\rho^+ u^+_i, G_1 - G_2) \|_{L^2} \]
\[ \lesssim \| \rho^\pm \|_{W^{1,\infty}} \| \nabla u^\pm_\|_{H^1} + \| \nabla^2 \rho^\pm \|_{L^2} \| u^\pm_\|_{L^6} + \| \nabla^2 (\nabla u^\pm \nabla \theta) \|_{L^2} \]
\[ + \| \nabla u^\pm \|_{L^\infty} \| (\nabla u^\pm, \nabla n^\pm) \|_{H^2} + \| (u^\pm, \nabla n^\pm) \|_{L^\infty} \| \nabla u^\pm \|_{H^2} \]
\[ \lesssim \| \nabla u^\pm_\|_{H^1} + \| \nabla u^\pm \|_{H^2}. \] (3.42)

Multiplying \( \nabla^2 \)\( 3.37 \)_2 and \( \nabla^2 \)\( 3.37 \)_4 by \( \nabla^3 u^+ \) and \( \nabla^3 u^- \), and then integrating over \( \mathbb{R}^3 \), we obtain
\[ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \rho^\pm \| \nabla^3 u^\pm \|^2 \, dx + \mu^\pm \int_{\mathbb{R}^3} |\nabla^4 u^\pm|^2 \, dx + (\mu^\pm + \nu^\pm) \int_{\mathbb{R}^3} |\nabla^3 \text{div} u^\pm|^2 \, dx \]
\[ = \frac{1}{2} \int_{\mathbb{R}^3} \rho^\pm_i |\nabla^3 u^\pm|^2 \, dx - \sum_{m=0}^2 \int_{\mathbb{R}^3} \nabla^{3-m} \rho^\pm \nabla^m u^\pm_i \nabla^3 u^\pm \, dx \]
\[ - \int_{\mathbb{R}^3} \nabla^3 [C (\rho^- \nabla n^- + \rho^+ \nabla n^+)] \nabla^3 u^+ \, dx \]
\[ - \int_{\mathbb{R}^3} \nabla^3 [C (\rho^- \nabla n^- + \rho^+ \nabla n^+)] \nabla^3 u^- \, dx + \int_{\mathbb{R}^3} \nabla^3 G_1 \nabla^3 u^+ \, dx + \int_{\mathbb{R}^3} \nabla^3 G_2 \nabla^3 u^- \, dx \]
\[ = \sum_{i=1}^6 I_i. \]

From (2.14), we have
\[ |I_1| \lesssim \| \rho^\pm \|_{L^\infty} \| \nabla^3 u^\pm \|_{L^2} \lesssim \| u^\pm \|_{W^{1,\infty}} \| \nabla^3 u^\pm \|_{L^2} \lesssim \delta \| \nabla^3 u^\pm \|_{L^2} \] (3.44)

and from (3.40), we also have
\[ |I_2| \lesssim \left( \| \nabla \rho^\pm \|_{L^\infty} \| \nabla^2 u^\pm_i \|_{L^2} + \| \nabla^2 \rho^\pm \|_{L^3} \| \nabla u^\pm_i \|_{L^6} + \| \nabla^3 \rho^\pm \|_{L^2} \| u^\pm_i \|_{L^\infty} \right) \| \nabla^3 u^\pm \|_{L^2} \]
\[ \lesssim \delta \left( \| (u^\pm_i, u^\pm_i, \nabla u^\pm_i) \|_{L^2} + \| \nabla u^\pm \|_{H^2} \right) \] (3.45)
By virtue of (3.37), (3.37), (3.40) and (3.42), we get

\[
I_3 = - \int_{\mathbb{R}^3} \nabla^3 \left[ C \left( \rho^{-} \nabla n^+ + \rho^{+} \nabla n^- \right) \right] \nabla^3 u^+ dx
\]

\[
= - \int_{\mathbb{R}^3} \left[ C \left( \rho^{-} \nabla \nabla^3 n^+ + \rho^{+} \nabla \nabla^3 n^- \right) \right] \nabla^3 u^+ dx - \sum_{m=0}^{2} \int_{\mathbb{R}^3} \nabla^{3-m} (C \rho^{-}) \nabla \nabla^3 n^+ \nabla^3 u^+ dx
\]

\[
- \sum_{m=0}^{2} \int_{\mathbb{R}^3} \nabla^{3-m} (C \rho^{+}) \nabla \nabla^3 n^- \nabla^3 u^+ dx
\]

\[
= \int_{\mathbb{R}^3} C \rho^{-} \nabla^3 n^+ \nabla^3 u^+ dx + \int_{\mathbb{R}^3} C \rho^{+} \nabla^3 n^- \nabla^3 u^+ dx
\]

\[
+ \int_{\mathbb{R}^3} \left[ \nabla (C \rho^{-}) \nabla^3 n^+ + \nabla (C \rho^{+}) \nabla^3 n^- \right] \nabla^3 u^+ dx
\]

\[
- \sum_{m=0}^{2} \int_{\mathbb{R}^3} t^2 \nabla^{3-m} (C \rho^{-}) \nabla \nabla^3 n^+ \nabla^3 u^+ dx - \sum_{m=0}^{2} \int_{\mathbb{R}^3} \nabla^{3-m} (C \rho^{+}) \nabla \nabla^3 n^- \nabla^3 u^+ dx
\]

\[
= - \int_{\mathbb{R}^3} C \rho^{-} \nabla^3 n^+ \nabla^3 \left( \frac{n_t^+ + u^+ \cdot \nabla n^+}{1 + n^+} \right) dx
\]

\[
+ \int_{\mathbb{R}^3} C \rho^{+} \nabla^3 n^- \nabla^3 \left( \frac{\text{div} + (2 \mu^- + \lambda^-) \text{div} u^-}{2 \mu^+ + \lambda^+} \right) dx
\]

\[
+ \int_{\mathbb{R}^3} \left[ \nabla (C \rho^{-}) \nabla^3 n^+ + \nabla (C \rho^{+}) \nabla^3 n^- \right] \nabla^3 u^+ dx - \sum_{m=0}^{2} \int_{\mathbb{R}^3} \nabla^{3-m} (C \rho^{-}) \nabla \nabla^3 n^+ \nabla^3 u^+ dx
\]

\[
- \sum_{m=0}^{2} \int_{\mathbb{R}^3} \nabla^{3-m} (C \rho^{+}) \nabla \nabla^3 n^- \nabla^3 u^+ dx
\]

\[
= - \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} C \rho^{-} \left| \nabla^3 n^+ \right|^2 \frac{1}{1 + n^+} dx + \frac{1}{2} \int_{\mathbb{R}^3} \left| \nabla^3 n^+ \right|^2 \left( \frac{C \rho^{-}}{1 + n^+} \right)_t dx
\]

\[
- \sum_{m=0}^{2} \int_{\mathbb{R}^3} C \rho^{-} \nabla^3 n^+ \nabla^m n_t^+ \nabla^3-m \left( \frac{1}{1 + n^+} \right)_t dx
\]

\[
+ \frac{1}{2} \int_{\mathbb{R}^3} \left| \nabla^3 n^+ \right|^2 \text{div} \left( \frac{C \rho^{-} u^+}{1 + n^+} \right) dx - \sum_{m=0}^{2} \int_{\mathbb{R}^3} C \rho^{-} \nabla^3 n^+ \nabla^3-m \left( \frac{u^+}{1 + n^+} \right) \cdot \nabla \nabla^3 n^+ dx
\]

\[
+ \frac{2 \mu^- + \lambda^-}{2 \mu^+ + \lambda^+} \left( \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} C \rho^{+} \left| \nabla^3 n^- \right|^2 \frac{1}{1 + n^-} dx + \frac{1}{2} \int_{\mathbb{R}^3} \left| \nabla^3 n^- \right|^2 \left( \frac{C \rho^{+}}{1 + n^-} \right)_t dx
\]

\[
- \sum_{m=0}^{2} \int_{\mathbb{R}^3} C \rho^{+} \nabla^3 n^- \nabla^m n_t^- \nabla^3-m \left( \frac{1}{1 + n^-} \right)_t dx + \frac{1}{2} \int_{\mathbb{R}^3} \left| \nabla^3 n^- \right|^2 \text{div} \left( \frac{C \rho^{+} u^-}{1 + n^-} \right) dx
\]

\[
- \sum_{m=0}^{2} \int_{\mathbb{R}^3} C \rho^{+} \nabla^3 n^- \nabla^3-m \left( \frac{u^-}{1 + n^-} \right) \cdot \nabla \nabla^3 n^- dx + \frac{1}{2 \mu^+ + \lambda^+} \int_{\mathbb{R}^3} C \rho^{+} \nabla^3 n^- \nabla^3 \text{div} dx
\]

\[
+ \int_{\mathbb{R}^3} \left[ \nabla (C \rho^{-}) \nabla^3 n^+ + \nabla (C \rho^{+}) \nabla^3 n^- \right] \nabla^3 u^+ dx - \sum_{m=0}^{2} \int_{\mathbb{R}^3} \nabla^{3-m} (C \rho^{-}) \nabla \nabla^3 n^+ \nabla^3 u^+ dx
\]

\[
- \sum_{m=0}^{2} \int_{\mathbb{R}^3} \nabla^{3-m} (C \rho^{+}) \nabla \nabla^3 n^- \nabla^3 u^+ dx
\]

\[
\leq - \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} C \rho^{-} \left| \nabla^3 n^+ \right|^2 \frac{1}{1 + n^+} dx - \frac{2 \mu^- + \lambda^-}{2 (2 \mu^+ + \lambda^+)} \frac{d}{dt} \int_{\mathbb{R}^3} C \rho^{+} \left| \nabla^3 n^- \right|^2 \frac{1}{1 + n^-} dx
\]
Lemma A.1. Let \( 0 \leq i, j \leq k \), then we have
\[
\|\nabla^i f\|_{L^p} \lesssim \|\nabla^j f\|_{L^q}^{1-a} \|\nabla^k f\|_{L^r}^a
\]
where \( a \) belongs to \( \left[ \frac{i}{k}, 1 \right] \) and satisfies
\[
\frac{i}{3} - \frac{1}{p} = \left( \frac{j}{3} - \frac{1}{q} \right) (1-a) + \left( \frac{k}{3} - \frac{1}{r} \right) a.
\]

Employing the similar argument as in (3.46), we obtain
\[
I_4 \leq -\frac{1}{2t} \int_{\mathbb{R}^3} C\rho^+ |\nabla^3 n^-|^2 dx - \frac{2\mu^+ + \lambda^+}{2(2\mu^+ + \lambda^+)} \int_{\mathbb{R}^3} C\rho^- |\nabla^3 n^+|^2 dx
+ C\delta \left( \| (u_{t_i}^\pm, u_t^\pm, \nabla u_{t_i}^\pm) \|_{L^2} + \| \nabla u^\pm \|_{H^2} \right) (3.47)
\]

By applying integration by parts, we have
\[
I_5 = -\int_{\mathbb{R}^3} \nabla^2 G_1 \nabla^4 u^+ dx
\leq C \| \nabla^2 G_1 \|_{L^2} \| \nabla^4 u^+ \|_{L^2}
\leq C \left( \| (u^\pm, \nabla n^\pm) \|_{H^2} \| \nabla u^\pm \|_{L^\infty} + \| (u^\pm, \nabla n^\pm) \|_{L^\infty} \| \nabla u^\pm \|_{H^2} \right) \| \nabla^4 u^+ \|_{L^2}
\leq C\delta \left( \| \nabla u^\pm \|_{H^2}^2 + \| \nabla^4 u^+ \|_{L^2}^2 \right). (3.48)
\]

Similarly, for \( I_6 \), we have
\[
I_6 \leq C\delta \left( \| \nabla u^\pm \|_{H^2}^2 + \| \nabla^4 u^+ \|_{L^2}^2 \right). (3.49)
\]

Substituting (3.44) - (3.49) into (3.43) yields
\[
\frac{d}{dt} \int_{\mathbb{R}^3} \rho^+ |\nabla^3 u^+|^2 + \frac{C\rho^- |\nabla^3 n^+|^2}{1 + n^+} + \frac{2\mu^- + \lambda^-}{1 + n^-} \frac{C\rho^+ |\nabla^3 n^-|^2}{1 + n^-}
+ \frac{C\rho^+ |\nabla^3 n^-|^2}{1 + n^-} + \frac{2\mu^+ + \lambda^+}{2\mu^+ + \lambda^+} \frac{C\rho^- |\nabla^3 n^+|^2}{1 + n^+} dx
+ \mu^+ \int_{\mathbb{R}^3} |\nabla^4 u^+|^2 dx + (\mu^+ + \nu^+) \int_{\mathbb{R}^3} |\nabla^3 \nabla u^+|^2 dx
\leq C\delta \left( \| (u_{t_i}^\pm, u_t^\pm, \nabla u_{t_i}^\pm) \|_{L^2} + \| \nabla u^\pm \|_{H^2} \right). (3.50)
\]

In virtue of (3.22) and (3.23), for any \( t \geq 0 \), it holds that
\[
\int_0^t \left( \| (u_{t_i}^\pm, u_t^\pm, \nabla u_{t_i}^\pm) \|_{L^2} + \| \nabla u^\pm (\tau) \|_{H^2} \right) d\tau \leq CK_0.
\]

Consequently, integrating (3.50) from 0 to \( t \), we finally deduce that
\[
\| \nabla^3 n^\pm \|_{L^2} \leq CK_0. (3.51)
\]

Therefore, we complete the proof of Lemma 3.5. \( \square \)

Proof of Theorem 1.1. Using Theorem 3.4 and Lemma 3.5 we can obtain Theorem 1.1 immediately.

Appendix A. Analytic tools

We recall the Sobolev interpolation of the Gagliardo–Nirenberg inequality.

Lemma A.1. Let \( 0 \leq i, j \leq k \), then we have
\[
\|\nabla^i f\|_{L^p} \lesssim \|\nabla^j f\|_{L^q}^{1-a} \|\nabla^k f\|_{L^r}^a
\]
where \( a \) belongs to \( \left[ \frac{i}{k}, 1 \right] \) and satisfies
\[
\frac{i}{3} - \frac{1}{p} = \left( \frac{j}{3} - \frac{1}{q} \right) (1-a) + \left( \frac{k}{3} - \frac{1}{r} \right) a.
\]
Particularly, when \( p = q = r = 2 \), we have
\[
\left\| \nabla^i f \right\|_{L^2} \lesssim \left\| \nabla^j f \right\|_{L^2}^{\frac{k}{L^2}} \left\| \nabla^k f \right\|_{L^2}^{\frac{1}{L^2}}.
\]

Proof. This is a special case of [20, pp. 125, THEOREM]. □

Lemma A.2 ([16]). For any integer \( k \geq 1 \), we have
\[
\left\| \nabla^k (fg) \right\|_{L^p} \lesssim \left\| f \right\|_{L^{p_1}} \left\| \nabla^k g \right\|_{L^{p_2}} + \left\| \nabla^k f \right\|_{L^{p_3}} \left\| g \right\|_{L^{p_4}},
\]
and
\[
\left\| \nabla^k (fg) - f \nabla^k g \right\|_{L^p} \lesssim \left\| \nabla^k f \right\|_{L^{p_1}} \left\| \nabla^{k-1} g \right\|_{L^{p_2}} + \left\| \nabla^k f \right\|_{L^{p_3}} \left\| g \right\|_{L^{p_4}},
\]
where \( p, p_1, p_2, p_3, p_4 \in [1, \infty] \) and
\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.
\]

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GUOCHUN WU
FUJIAN PROVINCE UNIVERSITY KEY LABORATORY OF COMPUTATIONAL SCIENCE, SCHOOL OF MATHEMATICAL SCIENCES, HUAQIAO UNIVERSITY, QUANZHOU 362021, P.R. CHINA.
Email address: guochunwu@126.com

LEI YAO
SCHOOL OF MATHEMATICS AND CENTER FOR NONLINEAR STUDIES, NORTHWEST UNIVERSITY, XI’AN 710127, P.R. CHINA.
Email address: leiyao@nwu.edu.cn

YINGHUI ZHANG
SCHOOL OF MATHEMATICS AND STATISTICS, GUANGXI NORMAL UNIVERSITY, GUILIN, GUANGXI 541004, P.R. CHINA
Email address: yinghuizhang@mailbox.gznu.edu.cn