Non-linear $I(V)$ Characteristics of Luttinger Liquids and Gated Hall Bars

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Non-linear current voltage characteristics of a disordered Luttinger liquid are calculated using a perturbative formalism. One finds non-universal power law characteristics of the form $I(V) \sim V^{1/(2g-1)}$ which is valid both in the superfluid phase when $I$ is small and also in the insulator phase when $I$ is large. Mesoscopic voltage fluctuations are also calculated. One finds $\text{Var}(\Delta V) \sim I^{3g-3}$. Both the $I(V)$ characteristic and the voltage fluctuations exhibit universal power law behavior at the superfluid insulator transition where $g = \frac{3}{2}$. The possible application of these results to the non-linear transport properties of gated Hall bars is discussed.

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Recently there has been considerable interest in the transport properties of the Luttinger liquid\footnote{2}. There are many reasons for this including applications to quantum wires, quasi-one dimensional organic conductors\footnote{3} like TTF-TCNQ and possibly to high temperature superconductors\footnote{4}. Nevertheless, until the recent observation of Luttinger liquid behavior by Milliken et al.\footnote{5} clear evidence of Luttinger liquid behavior had been lacking. In their experiments, the point contact tunneling conductance between two $\nu = 1/3$ fractional quantum Hall edge channels is measured as a function of the point contact gate voltage. One observes transmission resonances whose half-widths scale with temperature as $T^{2/3}$. This is in agreement with Kane and Fisher\footnote{6}. Off resonance, the conductance scales as $T^4$ as was also predicted\footnote{7}.

In fact, the experiments of Milliken et al.\footnote{5} is only the most recent contribution to the understanding of Luttinger liquid transport. Let us begin our discussion by recounting, in a semi-historical manner, some of the major developments in the field. We will begin by recalling the work by Apel and Rice\footnote{8}, who used the Kubo formalism to show that the conductance of a clean quantum wire differed from the usual Landauer result that $G = e^2/h$\footnote{9}. Instead these authors found that $G = ge^2/h$, where $g = \pi \hbar v (\partial n/\partial \mu)$. This result may be obtained as follows: First consider the current injected from the wire into the left reservoir. This is $I_1 = \frac{1}{2} e v (\partial n/\partial \mu) \mu_1 = g(e/h) \mu_1$. Similarly the current injected into the wire from the right reservoir is $I_2 = \frac{1}{2} e v (\partial n/\partial \mu) \mu_2 = g(e/h) \mu_2$. Now the total current in the wire is $I = (I_1^2 - I_2^1)$, where $I_1^2$ is the current flowing out of the wire into the left reservoir. Eliminating $I_1^2$ in favor of $I_2^1$ gives

$$ I = g \left( \mu_1 - \mu_2 \right) - I_B $$

where $I_B = I_1^1 - I_2^1 = I^2 - I^1$ is the backscattering current. Now $I_B = 0$ for a clean wire, so eq. 1 gives $G = I/e(\mu_1 - \mu_2) = ge^2/h$.

Another major advance in transport theory was the investigation of the superfluid-insulator transition in uniformly disordered Luttinger liquids by Giamarchi and Schultz\footnote{10}. These authors found that an infinitesimal amount of disorder will localize a Luttinger liquid with $g < \frac{3}{2}$. Moreover, they find that, if $g > \frac{3}{2}$, disorder is irrelevant and the Luttinger liquid exhibits superfluid behavior.

Further progress was made by Kane and Fisher\footnote{11} who investigated a Luttinger wire with a single or double barrier. These authors found that $V_{2k_F}$, the Fourier coefficient of the potential barrier, is highly relevant if $g < 1$. Because of this, the conductance of a Luttinger liquid with a single impurity vanishes. The existence of conductance resonances may be crudely understood if one views Luttinger liquids as Wigner crystals with quasi-long range positional order\footnote{12}. According to this point of view, the coupling of a CDW to a barrier potential may be described by the pinning potential

$$ V_{\text{pin}}(\theta) = \left| V(2k_F) \cos(2k_F \theta) \right| $$

Hence, unless $V(2k_F) = 0$, $G = 0$ since the potential barrier will pin the Wigner crystal. However, when $V_{2k_F} = 0$, a conductance resonance will occur.

In this paper, we examine the non-linear $I(V)$ characteristic of a uniformly disordered Luttinger liquid. Using both renormalization group theory and perturbation theory, we will analyze the $I(V)$ characteristics several analytically tractable regimes. These regimes include the superfluid phase in the $I \to 0$ limit and the insulator phase in the $I \to \infty$ limit. We will find that, in both instances, the $I(V)$ characteristic is proportional to some non-universal power of the voltage, $V^{1/(2g-1)}$. However, at the metal-insulator transition, one obtains the universal power law dependence $I \propto V^{1/2}$. We will also briefly discuss the low voltage behavior of the $I(V)$ characteristic in the insulator phase.

In addition to the general discussion of non-linear transport in Luttinger liquids, we will also discuss the application of these ideas to gated Hall bars. Here we
I. MODEL

In this section, we wish to review the bosonic formulation of a uniformly disordered spinless Luttinger liquid. This well known formulation involves the two angle fields \( \theta \) and \( \phi \) which obey the commutation relations

\[
[\theta(x), \phi(x')] = -i \pi \text{ sgn}(x - x')
\]

\( \theta \) and \( \phi \) are related to the number and current densities associated with the right and left moving electrons according to

\[
\rho_{\pm}(x) = \frac{\nu}{4\pi} \left\{ \frac{\partial \theta}{\partial x} \pm \frac{2 \partial \phi}{\nu \partial x} \right\}
\]

\[
j_{\pm}(x) = -\frac{\nu}{4\pi} \left\{ \frac{\partial \theta}{\partial t} \pm \frac{2 \partial \phi}{\nu \partial t} \right\}
\]

The total charge and current densities are

\[
\rho = \frac{\nu}{2\pi} \frac{\partial \theta}{\partial x}, \quad j = -\frac{\nu}{2\pi} \frac{\partial \theta}{\partial t}
\]

which are identical to the charge and currents of a one-dimensional CDW provided that \( \theta \) is related to the displacement \( u \) according to \( \theta(x) = 2k_F(x + u(x)) \).

The parameter \( \nu \) depends on the system under consideration. For many one dimensional systems, e.g. Hubbard models, \( \nu = 1 \). However, in the context of the theory of fractional quantum Hall edge states \( \nu = 1/m \) where \( 1/m \) is the filling fraction of the (parent) fractional quantum Hall liquid [12]. Finally, we note the creation operators of right and left moving electrons may be approximately written in the form

\[
\psi_{\pm}^\dagger(x, t) = C \exp(\pm i \theta(x, t)/2) \exp(i \phi(x, t)/\nu)
\]

where \( C \approx 1/\sqrt{2\pi a} \) is a constant determined by the cut-off structure of the theory [1] (\( a \) is the microscopic lattice constant). The angle field \( \theta(x, t) \) is separated into three parts: \( \theta(x, t) = \theta_0(t) + 2k_F x + \theta(x, t) \). The first two terms describe a spatially uniform current and a time-independent charge density (cancelled by the background) through eq. [3]. The fluctuating piece \( \theta(x, t) \) then describes the quantized excitations of the Luttinger liquid.

In the absence of disorder, the Luttinger liquid Hamiltonian is taken to be

\[
\mathcal{H}_0 = \frac{\hbar v}{4\pi} \int dx \left\{ \frac{1}{\nu \partial x} \left( \frac{\partial \theta}{\partial x} \right)^2 + 2g \left( \frac{\partial \phi}{\partial x} \right)^2 - \frac{2\nu e}{\hbar v} E \right\}
\]

where \( E(x) \) is the electric field, and \( v \) is the charge velocity. However, in order to describe dirty Luttinger liquids, we must include the effect of point impurities, located at positions \( x_i \), into our Hamiltonian. This is done by including a backscattering contribution \( \mathcal{H}_B \) to our total Hamiltonian \( \mathcal{H} = \mathcal{H}_0 + \mathcal{H}_B \). \( \mathcal{H}_B \) is given by

\[
\mathcal{H}_B = -\frac{1}{2\pi a} \sum_i \left[ t_i e^{i\theta_0} e^{2ik_F x_i} e^{i\theta(x_i)} + \text{H.c.} \right]
\]

and is time-dependent through \( \theta_0(t) \). The backscattering amplitudes \( t_i \) are assumed to be uncorrelated, i.e.

\[
t_i t_j^* = |t|^2 \delta_{ij}
\]

For future reference, we introduce

\[
D \equiv n_{\text{imp}} |t|^2 a/2\pi v^2 \hbar^2,
\]

a dimensionless measure of disorder.

In the following sections, we will obtain non-linear \( I(V) \) characteristics by calculating the backscattering current which is defined by \( I_B = \int dx \, i_n(x) \), where \( \dot{\rho}_+(x) + \partial_x \rho_+(x) = -i_n(x) \). For steady states this states implies that \( I_B = I_{B+} - I_{B-} \) the definition introduced after eq. [12]. One can readily obtain an explicit expression for \( I_B \) (and \( i_n \)) by comparing the Heisenberg equations of motion with \( \dot{\rho}_+(x) + \partial_x \rho_+(x) = -i_n(x) \). The result is

\[
I_B = \frac{i}{2\pi a \hbar} \sum_i \left[ t_i e^{i\theta_0} e^{2ik_F x_i} e^{i\theta(x_i)} - \text{H.c.} \right]
\]
Since the backscattering operator \( \psi_+^\dagger(x)\psi_-(x) \), which transfers an electron from the \( - \) channel to the \( + \) channel, is proportional to \( e^{i\phi(x)} \), much of our analysis is based on the behavior of the correlation function
\[
\chi(x,t) = \frac{i}{\hbar} \langle [e^{i\phi(x,t)}, e^{-i\phi(0,0)}] \rangle \Theta(t)
\]
which has been studied by Luther and Peschel [13]. A discussion of its behavior can be found in Appendix A.

II. SUPERFLUID-INSULATOR TRANSITION

A renormalization group treatment of the above model has been given by Giamarchi and Schultz [3]. They find that when the microscopic cutoff length \( a \rightarrow a e^l \), the coupling constants flow to \( \tilde{g}(l) \), \( D(l) \), \( v(l) \) and \( E(l) \) which obey the flow equations
\[
\begin{align*}
\frac{d\tilde{g}(l)}{dl} &= -\tilde{g}(l)^2 D(l) \\
\frac{dD(l)}{dl} &= -2(\tilde{g}(l) - \frac{1}{2})D(l) \\
\frac{dv(l)}{dl} &= -\tilde{g}(l) v(l) D(l) \\
\frac{dE(l)}{dl} &= 0
\end{align*}
\]
which are illustrated in Fig. 2. For \( E = 0 \), there exists a superfluid phase which is the domain of attraction of a line of fixed points and an insulator phase. The phase boundary is at \( \Delta = 0 \) where \( \Delta^2 = \frac{2}{\nu} D - (\tilde{g} - \frac{1}{2})^2 \). This is a Kosterlitz-Thouless transition since, near the \( (D, \tilde{g}) = (0, \frac{1}{2}) \) fixed point, the RG flow equations are equivalent to the Kosterlitz Thouless RG equations. Because of this, a variety of well known results apply to this model. For instance, in the insulating state, near the MI transition, the localization length is simply \( \xi_L = \exp(\pi/\Delta) = \exp(\pi/\sqrt{D-D_c}) \). This has the same form as the correlation length of the XY model. Now invariance under the renormalization group implies that \( I(E; X, a) = I(E; X, l(a)e^l) \). A simple rescaling of \( (x, t) \rightarrow (x e^{-l}, t e^{-l}) \) will, according to dimensional analysis, give
\[
I(E; X, a) = e^{-l} I(e^{2l} E; X(l), a)
\]
This general result will be used in the next section to check perturbative calculation of \( I(V) \).

III. THE NON-LINEAR \( I(V) \) CHARACTERISTIC OF DIRTY LUTTINGER LIQUIDS

In this section, we wish to perturbatively calculate the non-linear \( I(V) \) characteristic of a Luttinger liquid. The result gives non-universal power law \( I(V) \) characteristics. Using eq. [13] we will argue that the perturbative \( I(V) \) characteristics are valid both in the superfluid phase when \( I \rightarrow 0 \) and in the insulator phase when \( I \rightarrow \infty \).

Our approach to obtaining \( I(V) \) begins with a calculation of \( I_B(V) \) to order \( O(|t|^2) \) in perturbation theory. To do this, we assume that a steady-state current \( I \) flows along the wire. This is described by taking \( \theta_0(t) = \Omega t \), with \( \Omega = 2\pi I / \nu e \). Now according to linear response theory the backscattering current to leading order in \( t_i \) is
\[
I_B = -\frac{iI}{\hbar} |C|^4 \sum_{i,j} \langle t_i \bar{t}_j e^{ig(x_i-x_j)} \rangle \chi(x_i-x_j, \Omega) - c.c.
\]
where \( \chi(x,t) \) is the response function discussed in Appendix A, and where \( g = 2k_B + d/L^2 \). If we now average over the impurity backscattering amplitudes as in eqs. [10]-[11] we find
\[
\langle I_B \rangle = \frac{2e}{\hbar} |C|^4 n_{imp}|t|^2 L \chi''(x = 0, \Omega)
\]
\[
= \frac{D}{\Gamma(2\tilde{g})} \left( \frac{L}{a} \right) \left( \frac{\nu e}{a} \right) \left( \frac{2\pi a |I|}{\nu e v} \right)^{2\tilde{g}-1} \text{sgn}(I) \quad (T = 0)
\]
\[
\simeq C(\tilde{g}) \cdot 2\pi D \left( \frac{L}{a} \right) \left( \frac{\tilde{g} k_B T a}{\hbar v} \right)^{2\tilde{g}-2} I \quad (T \gg \frac{\hbar I}{\nu e k_B})
\]
where \( \langle I_B \rangle \) denotes an average with respect to the impurity positions, and \( C(\tilde{g}) \) is a numerical factor (see Appendix A).

To determine when this is valid, we calculated the higher order terms in perturbation theory. We obtain, at \( T = 0 \),
\[
\frac{\langle I_B \rangle}{I} = \sum_{n=1}^{\infty} a_n D^n I^{(2\tilde{g}-1)+(2\tilde{g}-3)(n-1)}
\]
where \( a_n \) are numerical coefficients. The above result implies that higher order corrections to eq. [17] are negligible if \( |D(Ia/ev)^{2\tilde{g}-1}| < < 1 \). This means that eq. [13] is valid in the superfluid phase for small \( I \) and is valid in insulator phase for large \( I \). These conclusions may also be reached from renormalization group arguments. However, these arguments indicate that the observed exponent depends not on the bare \( \tilde{g} \) but rather on the \( g \), the governing fixed point. Depending on whether \( I \rightarrow 0 \) or \( I \rightarrow \infty \) limit is taken, the fixed point may be an infrared or ultraviolet fixed point.
Now using eq. [14] and eq. [1] one can solve for $I(\mu_1 - \mu_2)$. In the limit where $L \to \infty$ but $I$ is held fixed, one obtains a non-linear $I(V)$ characteristic of the form

$$I \propto e^{\frac{\nu}{a}} \left( \frac{e E a^2}{\hbar v D} \right)^{1/2} \approx 1 \text{ (19)}$$

There are several comments to be made about the above result. The first is that the linear response regime was not considered since it automatically breaks down in the $L \to \infty$ limit. The next comment is that, when $D << \Delta$, the above result is consistent with the scaling identity derived in eq. [4]. The verification is straightforward, one simply demonstrates that $e^{2iE/LD(l)}^{(1/2}\delta_{\nu - 1})$ is $l$-independent. For example, when $\tilde{g} > \frac{1}{2}$, one uses $D(l) \approx 4\Delta^2/(9\sinh^2(\Delta l + b_0)) \sim e^{-2\Delta l}$ where $\Delta \approx (\frac{1}{2} - \tilde{g})$ to verify the $l$-independence. $\tilde{g} < \frac{1}{2}$ may similarly be demonstrated provided that $l$ is not so large that $D(l) << \Delta$ is violated. The last comment regarding eq. [22] is that the insulating phase can display a negative differential resistance depending on whether $\tilde{g}_s < 1/2$ or $\tilde{g}_s > 1/2$, eq. [22].

At this point, we wish to discuss briefly the issue of mesoscopic fluctuations. This is a very interesting topic which will be discussed at length elsewhere [4]. However, here we simply wish to understand when $\text{Var}(\Delta \mu) << |\Delta \mu|$ which the condition which must be met in order that the predicted $I(V)$ characteristic will not be obscured by mesoscopic voltage fluctuations. To do this, we first calculate $\text{Var}(I_B)$ at $T = 0$ using eq. [12]. One finds that

$$\text{Var}(I_B) \approx \left( \frac{\hbar e^2 D}{\pi a^3} \right)^2 \int \frac{dg}{2\pi} \left[ \chi''(2k_F, \Omega) \right]^2 \text{ (20)}$$

where $\chi(g, \omega)$ is the Fourier transform of $\chi(x, t)$. Then, using $\Delta \mu \equiv \mu_1 - \mu_2 = \hbar I_B/\nu^2 e\tilde{g}$, and evaluating eq. [20], we find

$$\text{Var}(\Delta \mu) \approx 16\pi \frac{\Gamma(2\tilde{g} - 1)^2}{\tilde{g}^2 \Gamma(\tilde{g})^2 \Gamma(4\tilde{g} - 2)} \left( \frac{D\hbar}{\nu^2 a} \right)^2 \left( \frac{\nu}{e v} \right)^4 \tilde{g}^{-3} \text{ (21)}$$

This implies

$$\frac{\text{Var}(\Delta \mu)}{|\Delta \mu|^2} \approx \frac{\pi \Gamma(2\tilde{g} - 1)^2}{\Gamma(\tilde{g})^2 \Gamma(4\tilde{g} - 2)} \frac{\nu v}{L} \text{ (22)}$$

where $\xi_L \equiv \nu v/2\pi I$ is a length scale associated with the finite current. Mesoscopic voltage fluctuations will be small if $L \gg \xi_L$.

**IV. DEVICES**

The experimental work of Milliken et al. [3], has raised the hope of using gated semiconductor heterostructure devices as convenient laboratory models of the Luttinger liquid. It is therefore of interest to consider devices which might be used to experimentally investigate dirty Luttinger liquids. Of course, the main requirement of such a device is that it allows backscattering to occur at random points, uniformly distributed along a finite length of a pair of oppositely directed but parallel edge channels. This geometry may be achieved in several devices. One device is a mesoscopic Hall bar of width $w$ of order of a few magnetic lengths. This sort of device has been studied by Simmons et al. [12] in the context of measuring the quasi-particle charge.

Another interesting device is the gated Hall bar (see Fig. 1). In this second device [4], one has a long but very narrow gate (i.e. gate width $d = \mathcal{O}(\ell)$ where $\ell = \sqrt{\hbar/eB}$ is the magnetic length). We will suppose that one has an incompressible $\nu = 1/m$ fractional quantum Hall liquid and that the gate is straddled by only a single pair of edge channels. The two pair of edge channels will contribute the left and right movers of a chiral Luttinger liquid model equivalent to that discussed in section 1. To see this we follow Wen [17,18] and write down a simple edge Hamiltonian

$$H_0 = \frac{1}{2} \int dx \left\{ U_{RR} |\rho_R^2 + \rho_L^2| + 2U_{RL} |\rho_R \rho_L| \right\} \text{ (23)}$$

where $|\rho_R(x), \rho_L(x')| = (\nu/2\pi) \delta_{\varphi_0} \delta(x - x')$. If we identify $\rho_R + \rho_L = -\nu \partial_x \psi/2\pi$ and $\rho_R - \rho_L = -\partial_x \phi/\pi$, then $H_0$ reduces to the model defined in eqn. [8] and the commutators reduce to that given in eqn. [8] where

$$g = \nu \left( \frac{U_{RR} - U_{RL}}{U_{RR} + U_{RL}} \right)^{1/2} = \nu^2 \tilde{g} \text{ (24)}$$

Next we consider interchannel tunneling, the analogue of backscattering. Provided that one neglects inter-Landau level mixing associated with the gate potential, then it is well known that a tunneling event involves a change of momentum equal to $\Delta k = d/\ell^2$. Hence, tunneling is suppressed between a pair of infinite rectilinear edges. More realistically, however, the gate potential will not be uniform in systems with such extremely narrow gates. Instead it will exhibit random spatial irregularities and as a result the edges follow the irregular equipotential contours of Fig. 3. In this case, interchannel tunneling will occur particularly near the points of close contact. The points of close contact in Fig. 3 are denoted with dots. Now we observe that, since the path of each chiral edge is not straight, the distance traveled between a pair of adjacent impurities is different for the two edges. One must therefore write the physical tunneling operator as $t_i \psi_+^i(x_i - \frac{1}{2} s_i) \psi_-^i(x_i + \frac{1}{2} s_i) + \text{H.c.}$ where $x_i - x_{i-1} \equiv (s_i - s_{i-1})$ is the path length between the $i^{th}$ and $(i-1)^{th}$ impurities on the $\pm$ edge channels.

Now expanding the bosonized tunneling operator in $s_i$ and dropping gradient terms in $\theta$ gives
\[ \mathcal{H}_t \approx -\int dx \left[ \hat{t}(x) e^{ix/\ell^2} e^{i\theta(x)} + H.c. \right] \]

\[ \hat{t}(x) = \frac{1}{2\pi a} \sum_i \hat{t}_i \delta(x-x_i) \]

We observe that whether or not the phase of \( \hat{t}_i \) is random, the phase of \( \hat{t}_i = t_i e^{i\phi_i}/\ell^2 \) will be random provided that \( \text{Var}(\phi_i) \gg (4\pi \ell^2/d)^2 \). If this is the case, then our model reduces to that discussed in section 1. We note that the model does not include quasiparticle tunneling which is suppressed by the \( \nu = 0 \) depletion zone separating the edge channels.

Consider now the behavior of \((\tilde{g}, D)\) as a function of the gate voltage in this device. When the gate has a large negative bias, the two edges will be well separated, backscattering will be suppressed, and \((\tilde{g}, D) \rightarrow (1/\nu, 0)\). This puts the system on the superfluid side of the phase diagram. The edge channels straddling the gate can then readily transport charge across the Hall bar. Now suppose that the \( |V_G| \) decreases. As this occurs, the edges get closer together. This increases \( D \) and decreases \( \tilde{g} \).

The antiwire Hamiltonian, therefore, moves across the phase diagram on the dotted line as shown schematically in Fig. 2. At some gate voltage \( V^* \), a superfluid-insulator transition occurs. For smaller \( |V_G| \) the antiwire is localized.

What is the manifestation of the superfluid-insulator transition in Hall bar? To understand this, we need to know how \( I(V) \) characteristics of the two terminal antiwire are related to the four terminal characteristics of the gated Hall bar. Qualitatively, the longitudinal voltage drop \( \mu_6 - \mu_5 \) vs. source-drain current \( I_{14} \) is similar to the \( I(V) \) characteristic of the two terminal antiwire. The reasons for this are twofold. First, the source-drain current \( I_{14} \) gives rise to a voltage drop across the antiwire. Secondly the antiwire current \( I_A \) produces a longitudinal voltage \( \mu_6 - \mu_5 \) in the Hall bar.

Below we will derive the actual quantitative relation between Hall bar and antiwire characteristics. We will find that

\[ \mu_6 - \mu_5 = \frac{h}{\nu e} \left[ (\mu_6 - \mu_5)/e + (h/\nu e^2)I_{14} \right] \]

\[ \mu_2 - \mu_6 = \frac{h}{\nu e} I_{14} \]

(25)

In the above equations, \( I_A = F(V_A) \) is the non-linear characteristic of a segmented wire which we will construct from the edge channels straddling the gate and from the edges which connect the antiwire to terminals \#2, 3, 5, and 6. These connecting channels are paired up, as shown in Fig. 4, in order to give two non-chiral Luttinger liquids connected to the two ends of the antiwire.

The three pieces of the segmented wire, labeled 1, 2, and 3, are characterized by different \( g \)'s. In particular, \( g_1 = g_3 = \nu \). However, the renormalized \( g_2 \equiv \nu^2 \tilde{g} \) of the disordered central segment depends on the interaction strengths \( U_{AB}, U_{AC} \), and \( \nu \), and the disorder \( D \). Now because \( g \) at the end segments is not equal to that in the interior segment, eq. (4) is modified to

\[ I_A = F(V_A) = \frac{\nu^2}{\hbar} V_A - I_B(I_A, \tilde{g}) \]

(26)

where \( I_B(I_A, \tilde{g}) \) is the backscattering current which occurs only in segment B. One remarkable feature of this result is that, in the absence of backscattering, the conductance of the segmented antiwire \( G_A \equiv I_A/V_A \) is independent of \( g_2 \).

At this point, we need to introduce the notation \( I_{+}^{[a]} \) (where \( a = 1, \ldots, 6 \)) which denotes the chiral edge current entering (−) or leaving (+) reservoir \( a \). We should note that the chiral currents are excess currents defined relative to a reference state in which the source-drain current \( I_{14} \) vanishes. The sign convention for these currents is as follows: \( I_{+}^{[a]} \) is taken to be positive if the \( \textit{excess current} \) has the same sense as the propulsion direction of the capillary waves i.e. it is in the direction of the arrows decorating the edge channels in Fig. 1.

We can now derive eqn. (23). First we must enumerate the current conservation conditions: The first of these are the four conditions of the form \( I_{+}^{[1]} = I^{[2]}, I_{+}^{[3]} = I^{[5]}, I_{+}^{[4]} = I^{[6]} \). In addition, there are 6 additional equations associated with current conservation at the terminals. These are \( I^{[1]} - I^{[3]} = I_{14}, I^{[4]} - I^{[4]} = I_{14} \), and \( I_{+}^{[a]} = I^{[a]} \) for \( a = 2, 3, 5, 6 \). Finally, there is a single constraint associated with current conservation across the antiwire. This is \( I_{+}^{[5]} - I^{[5]} = I^{[5]} - I_{+}^{[4]} \).

Now in order to obtain 12 equations in the 12 unknown \( I_{+}^{[a]} \), we need to augment the current conservation equations with the antiwire characteristic:

\[ I_{+}^{[2]} - I^{[3]} = F(\nu^2/h)(I^{[2]} - I^{[3]}) \]

(27)

The form of the antiwire characteristic follows from the fact that the voltage across the antiwire is \( \Delta V_A = (\mu_5 - \mu_2) = (h/\nu e)(I^{[2]} - I^{[3]}) \), whereas the antiwire current is \( I_A = I_{+}^{[2]} - I^{[3]} \).

The twelve equations are readily solved to give \( I_{+}^{[a]} \) in terms of the source-drain current \( I_{14} \). From this, one then calculates the voltage drop across an arbitrary pair of terminals using \( I_{+}^{[a]} = (\nu e/h)\mu_a \). In this manner, eq. (23) is obtained.

Notice that eq. (25) says that the antiwire current is

\[ I_A = \frac{\nu e}{h} (\mu_6 - \mu_5) \]

This result, together with eq. (26) gives us the \( I(V) \) characteristic
\[ I_{14} = I_B(I_\lambda) = I_B \left( \frac{\mu_6 - \mu_5}{\epsilon V_0} \right)^{2g-1}, \]

which, through eq. [14], may be written in a form appropriate for the gated Hall bar:

\[ I_{14} = I_0 \left( \frac{L}{L_D} \right) \left( \frac{\mu_6 - \mu_5}{\epsilon V_0} \right)^{2g-1}, \]

where \( V_0 \equiv \hbar v/ea, \) \( I_0 \equiv \nu e \nu /2\pi a, \) and \( L_D \equiv \nu \Gamma(2\tilde{g}) a /2\pi D. \) The above non-universal power law is valid in the superfluid phase when \( \mu_6 - \mu_5 \) is small, and in the insulator phase when \( \mu_6 - \mu_5 \) is large. At the superfluid-insulator transition, the above result gives a universal power law behavior of the form \( I_{14} / L \propto (\mu_6 - \mu_5)^{3}. \)

In the superfluid phase, the above \( I(V) \) characteristic should be observable throughout a window \( I_{\text{min}} \ll I_{14} \ll I_{\text{max}} \) whose upper limit is set by the applicability of perturbation theory, and whose lower limit is set by the condition that the results not be obscured by mesoscopic fluctuations. From Appendix B, we see that higher order terms in perturbation theory come in powers of \( D(I/I_0)^{2g-3}, \) hence a rough estimate for \( I_{\text{max}} \) is given by

\[ I_{\text{max}} \approx I_0 D^{-1/(2g-3)}. \]

From eq. [22], the condition \( \left[ I_{14} / I_0 \right] \approx 1 \) gives us \( I_{\text{min}} \approx I_0 (L / L_D) (A_0 a / L)^{2g-1}, \) where \( A_0 \) is a known numerical factor. We assume that the magnetic length \( \ell \) provides a rough lower bound to the ultraviolet cutoff \( a; \) at \( B = 10^4 \) T, then, \( a \approx \ell = 81 \) A. We also assume \( \nu = \frac{\ell}{a}, \) so that \( \tilde{g} = 3 \) in the limit of widely separated edge channels (a wide barrier), and a capillarity wave velocity of \( v = 10^6 \) m/s. Now \( D \propto |t|^2 \) is a Gaussian function of the distance between the two edge channels and can therefore be arbitrarily small; at the superfluid-insulator transition, \( D \sim O(1). \) Thus \( I_0 \approx 1 \mu A, V_0 \approx 80 \) nV, and \( L_D \approx (500 / D) \) A. For an antwire of length \( L = 20 \mu m, \) then, we estimate \( I_{\text{min}} \approx (8D) nA \) and \( I_{\text{max}} \approx D^{-1/3} \mu A. \) The window \( I_{\text{min}} \ll I \ll I_{\text{max}} \) is larger for wider barriers.

Although at the time of writing, there have been several experiments involving gated hall bars [10,11], no experiments studying either the superfluid insulator transition or the non-linear transport properties of a Luttinger liquid have been attempted. Clearly an experimental investigation into either of these aspects of Luttinger liquid physics would be most welcome.

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**APPENDIX A: CORRELATIONS AT FINITE TEMPERATURE**

The correlation function

\[ \chi(x,t) = \frac{i}{\hbar} \left[ \left| e^{i\theta(x,t)} e^{0,0} \right| \right] \Theta(t) \]

has been calculated by Luther and Peschel [8]. In real space, one finds

\[ \chi(x,t) = \frac{2i}{\hbar} \Theta(t) \text{Im} \left[ J_\tilde{g}(vt-x) J_\tilde{g}(vt+x) \right] \]

where

\[ J_\gamma(z) = \left[ \frac{a - iz - \hbar v}{\pi z k_n T} \sinh \left( \frac{\pi z k_n T}{\hbar v} \right) \right]^{-\gamma}. \]

Here \( T \) is the temperature and \( a \) is a short distance cutoff. The Fourier transform \( \chi(q,\omega) \) is then

\[ \chi(q,\omega) = \frac{1}{4\pi \hbar v} \int_{-\infty}^{\infty} ds \int_{\epsilon > 0} \left[ K_\tilde{g} (s + \omega - vq) K_\tilde{g} (s + \omega + vq) - K_\tilde{g} (-s - \omega + vq) K_\tilde{g} (-s - \omega - vq) \right] \]

\[ K_\gamma(\eta) = \int_{-\infty}^{\infty} du e^{i\eta u} \left( \frac{a + i u}{a + i u} \right)^\gamma \left( \frac{\pi k_n T a}{\hbar v} \cosh \left( \frac{\pi k_n T a}{\hbar v} \right) \right)^\gamma \]

\[ \approx \Gamma(1 - \gamma) \text{Im} \left( -i \pi \gamma k_n T / 2 \hbar v \right)^{\gamma - 1}, \]

where the last line is an approximation given by Luther and Peschel. For our purposes, we need \( \chi''(q,\omega), \) which at \( T = 0 \) is \[ \chi''(q,\omega) = \frac{\pi^2 q^{2g}}{2} \left( \frac{\omega^2 - v^2 q^2}{4v^2} \right)^{\tilde{g} - 1} \Theta(\omega^2 - v^2 q^2) \text{sgn}(\omega) \]

and at high temperatures given by

\[ \chi''(q,\omega) \approx C(\tilde{g}) \frac{2g}{\hbar v} \left( \frac{\omega}{2v} \right)^{2g-3} \Theta(\pi \tilde{g} k_n T - \hbar v |q|) \]

where \( C(\tilde{g}) \) is independent of \( \omega, q, \) and \( T. \) We also need the \( \chi(x = 0, \omega), \) the integral of \( \chi(q,\omega) \) over all \( q. \) This is given by

\[ \chi''(x = 0, \omega) = \frac{\pi e^{2g}}{2} \left( \frac{\omega}{v} \right)^{2g-3} \text{sgn}(\omega) \]

at low temperatures and
\[ \chi''(x = 0, \omega) \simeq C(\hat{g}) \frac{\alpha^2 g}{hv} \left( \frac{\omega}{\nu} \right) \left( \frac{\pi \hat{g} \kappa T}{hv} \right)^{2\hat{g} - 2} \]

at high temperatures.

**APPENDIX B: DIMENSIONAL ANALYSIS OF PERTURBATION SERIES**

Here we discuss the derivation of Eq. (28), in which it was claimed that the perturbative result for \( I_B \) is

\[ I_B = \sum_{n=1}^{\infty} a_n D^n \int (2\hat{g} - 3)n + 2. \]  

(28)

This says that the higher order terms in the series are small for large \( I \) if \( g < \frac{3}{2} \), and for small \( I \) if \( g > \frac{3}{2} \).

To derive this result, we begin with the Euclidean action \( S = S_0 + S_1 \),

\[ S_0 = \frac{1}{8\pi v g} \int dx d\tau \left[ \left( \frac{\partial \tilde{\theta}}{\partial \tau} \right)^2 + v^2 \left( \frac{\partial \tilde{\theta}}{\partial x} \right)^2 \right] \]

\[ S_1 = \int dx d\tau \left[ e^{-i\Omega \tau} \tilde{i}(x) e^{-i\theta(x, \tau)} + e^{i\Omega \tau} \tilde{i}^*(x) e^{i\theta(x, \tau)} \right] \]

\[ Z = Z[\tilde{i}(x, \tau), \tilde{i}^*(x, \tau)] \equiv \int \mathcal{D}\tilde{\theta} e^{-S_0(S_0+S_1)} , \]

where \( \tilde{i}(x, \tau) \equiv \tilde{i}(x)e^{-i\Omega \tau} \), and consider the Green’s function

\[ \langle \exp \left[ \sum_{k=1}^{n} \left( \tilde{\theta}(x_k^+, \tau_k^+) - \tilde{\theta}(x_k^-, \tau_k^-) \right) \right] \rangle_0 = e^{-nG(0)} \]

\[ \times \prod_{i,j} \left| \tilde{z}_i^+ - \tilde{z}_j^- \right|^{-2\hat{g}} \prod_{i<j} \left| \tilde{z}_i^+ - \tilde{z}_j^+ \right|^{-2\hat{g}} \left| \tilde{z}_i^- - \tilde{z}_j^- \right|^{-2\hat{g}} \]  

(29)

where \( \tilde{z}_j^\pm \equiv x_j^+ + i\tilde{\tau}_j^\pm \) are the complexified space-time coordinates. Here, \( G(x, \tau) \simeq 2\hat{g} \ln(R/|z|) \) is proportional to the Green’s function for the Laplacian in a two-dimensional disk of size \( R \). The limit \( G(0) \) is rendered finite by a suitable ultraviolet cutoff. Now the numerator of eq. (23) is homogeneous and of degree \(-2n^2\hat{g}\), while the denominator is homogeneous and of degree \(-2n(n-1)\hat{g}\).

A general term of order \(|\tilde{i}|^{2n} \) in the perturbation expansion for \( I_B(\tau) \) will involve an integral over \( 2n \) space coordinates \( \{x^+_1, \ldots, x^+_n \} \) and \( 2n - 1 \) time coordinates \( \{\tau^+_1, \tau^+_2, \ldots, \tau^+_n \} \). Clearly the integral is invariant under an overall spatial translation \( x^+_k \rightarrow x^+_k + \Delta x \), so the \( 4n - 1 \) integrations produce an overall factor of the length \( L \) and result in a degree of homogeneity of \( 4n - 2 \). Note that since \( \tau \equiv \tau^+_1 \) is not integrated, there is no overall time translational invariance. Finally, the divergence average pairs each \( x^+_j \) with some \( x^-_k \), resulting in a loss of \( n \) spatial integration variables. The overall degree of homogeneity is then

\[ \deg(n) = (4n - 2) - n - 2n^2\hat{g} + 2n(n-1)\hat{g} = (3 - 2\hat{g})n - 2. \]

Thus, assuming that the ultraviolet divergences are cancelled (as in the linear response case), we conclude that the \( n^{\text{th}} \) order term in the perturbation expansion behaves as

\[ I_B^{(n)} \propto \Omega^{2+(2\hat{g} - 3)n} \propto \int \Omega^{2+(2\hat{g} - 3)(n-1)} \]  

(30)

which is what we set out to show.

[1] J.M. Luttinger, *J. Math. Phys.* 4, 1154 (1963); E.H. Lieb, D.C. Mattis, *J. Math. Phys.* 6, 304 (1965); F.D.M. Haldane, *J. Phys. C* 14, 2585 (1981); F.D.M. Haldane, *Phys. Rev. Lett.* 47, 1840 (1981). For a review, see V. J. Emery in *Highly Conducting One-Dimensional Solids*, J. T. Devreese et al. eds. (Plenum, New York, 1979).

[2] D. Jérome, F. Creuzet, and C. Bourbonnais, *Physica Scripta* T27, 130 (1988).

[3] S. Chakravarty, A. Sudbø, P. W. Anderson, and S. Strong, *Science* 261, 337 (1993).

[4] F.P. Milikken, C.P. Umbach, R.A. Webb, submitted to *Phys. Rev. Lett.*; B. G. Levi, *Physics Today* 47, 21 (1994).

[5] C.L. Kane and M.P.A. Fisher, *Phys. Rev. B* 46, 15233 (1992).

[6] W. Apeil and T.M. Rice, *Phys. Rev. B* 26, 7063 (1982).

[7] R. Landauer, *Philos. Mag.* 21, 863 (1970); M. Buettiker, *Phys. Rev. Lett.* 57, 1761 (1986).

[8] T. Giamarchi and H.J. Schultz, *Phys. Rev. B* 37, 325 (1988); *Europhys. Lett.* 3, 1287 (1987).

[9] This viewpoint has been advocated by a number of authors. See L.I. Glazman, I.M. Ruzin, and B.I. Shklovskii, *Phys. Rev. B* 45, 8454 (1992); H.J. Schultz, *Phys. Rev. Lett.* 71, 1864 (1993); H. Maurey and T. Giamarchi, cond-mat 9408094 (preprint, 1994); and X.G. Wen, *Phys. Rev. B* 44, 5708 (1991).

[10] R. Hanh, A.H. MacDonald, P. Streda, K. von Klitzing, *Phys. Rev. Lett.* 61, 2797 (1988).

[11] S. Washburn, A.B. Fowler, H. Schmidt, D. Kern, *Phys. Rev. Lett.* 61, 2801 (1988).

[12] We consider only simple quantum Hall edge states, arising from so-called “principal” bulk QHE fluids, and not the composite edges which occur, for example, at filling fraction \( \nu = 2/3 \).

[13] A. Luther and I. Peschel, *Phys. Rev. B* 9, 2911 (1974); A. Luther and V. J. Emery, *Phys. Rev. Lett.* 33, 589 (1974).

[14] D.P. Arovas and S.R. Renn, work in progress.

[15] A. Siddons, W.P. Wei, L.W. Engle, D.C. Tsui, and M. Shayegan, *Phys. Rev. Lett.* 63, 1731 (1989); J.A. Siddons, S.W. Hwang, D.C. Tsui, H.P. Wei, L.W. Engle
and M. Shayegan, Phys. Rev. B 44, 12933 (1991); S.W. Hwang, J.A. Simmons, D.C. Tsui, and M. Shayegan, Surface Science 263, 72 (1992); J.K. Jain and S.A. Kivelson, Phys. Rev. Lett. 60, 1542 (1988); S.A. Kivelson and V.L. Pokrovsky, Phys. Rev. B 40, 1373 (1989).

[16] This is essentially identical with devices studied by Haug et al.

[17] X.G. Wen, Phys. Rev. B 44, 5708 (1991).

[18] X.G. Wen, Int. J. Mod. Phys. 6, 1711 (1992).

[19] For a derivation of $I_+ = ν e h μ$, see, for instance, p. 207-208 of review by C.W.J. Beenakker and H. van Houten in Solid State Physics, vol. 44 (Academic Press, San Diego, 1991).

FIGURE CAPTIONS

FIG. 1 A Hall bar with a long but narrow gate. (The gate width $w \sim O(ℓ)$). Terminals #1 and #4 are the source and drain terminals, respectively. The Hall bar is decorated with the edge state geometry discussed in the text. The two parallel edge segments straddling the gate are referred to as the antiwire in the text. As discussed in the text, the antiwire provides an experimental realization of a dirty Luttinger liquid.

FIG. 2 The renormalization group flows of a dirty spinless Luttinger liquid. The superfluid and insulator phases are separated by the line $D^{1/2} = 2g/3 - 1$. The dashed line indicated the trajectory of an antiwire system as a function of gate voltage.

FIG. 3 A schematic drawing of a pair of edge channels under an ultranarrow gate. Because of the irregular nature of the equipotentials, the depletion gap between the edge channels exhibits spatially random fluctuations. Electron tunneling across the $ν = 0$ depletion gap is assumed to occur at points of close contact which are indicated in the figure with dots.

FIG. 4 The tri-segmented antiwire which is constructed from the pair of edge channels straddling the gate plus the four edge channels which connect the gate to terminals #2, #3, #5, and #6. The connecting edge channels to terminals #2 and #3 are paired to give segment 1. The edge channels straddling the gate form segment 2, and the connecting edge channels connecting the antiwire to terminals #5 and #6 form segment 3. Note that $g_1 = g_3 = ν$, but because of disorder and interaction effects, $g_2 ≠ ν$. 

8