Non-Fermi-liquid behavior from critical electromagnetic vacuum fluctuations

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We study two-dimensional materials where electrons are coupled to the vacuum electromagnetic field of a cavity. We show that, at the onset of the superradiant phase transition towards a macroscopic photon occupation of the cavity, the critical electromagnetic fluctuations, consisting of photons strongly overdamped by their interaction with electrons, can in turn lead to the absence of electronic quasiparticles. Since transverse photons couple to the electronic current, the appearance of non-Fermi-Liquid behavior strongly depends on the lattice. In particular, we find that in a square lattice the phase space for electron-photon scattering is reduced in such a way to preserve the quasiparticles, while in a honeycomb lattice the latter are removed due to a non-analytical frequency dependence of the damping \( \propto |\omega|^{2/3} \). Standard cavity probes could allow to measure the characteristic frequency spectrum of the overdamped critical electromagnetic modes responsible for the non-Fermi-liquid behavior.

**Introduction.** — Certain strongly correlated metals do not behave according to Landau’s Fermi-liquid theory (see [1] for a recent classification). In most of the cases this happens in correspondence to a quantum critical point separating a normal metallic phase and a symmetry-broken phase [2, 3]. Within this scenario which has special relevance in two-dimensional materials, the strong coupling between the Fermi surface and critical order-parameter fluctuations leads to the loss of Landau’s quasiparticles and thus to non-Fermi-Liquid behavior [4–17]. A direct signature is a non-analytical frequency dependence of the quasiparticle damping \( \sim |\omega|^\alpha \), with \( \alpha < 1 \), as opposed to the usual Fermi-liquid damping \( \sim \omega^2 \) which becomes instead increasingly irrelevant towards the Fermi surface \( \omega \to 0 \). In order to improve our understanding of the emergence of non-Fermi-liquid behavior and its experimental relevance, it would be highly desirable to determine the microscopic origin of the bosonic degree of freedom whose critical fluctuations are responsible for removing the electronic quasiparticles.

The recent experimental possibility to strongly couple electrons in two-dimensional materials with photons confined within cavities [18, 19] has opened new avenues for controlling collective electronic phenomena and explore them in novel scenarios. These include cavity induced superconductivity [20–28], magnetism [29–33], ferroelectricity [34, 35], as well as topological phenomena [36–43], that could be realized by engineering the quantum electrodynamical (QED) coupling between matter and light. These scenarios can also be investigated with synthetic matter made of ultracold atoms [44].

In this work, we show that cavity QED within two-dimensional materials offers ideal conditions to implement and observe non-Fermi-liquid behavior. The fluctuations of the emergent bosonic degree of freedom which induce non-Fermi-liquid behavior in the standard scenario are here substituted by the fluctuations of the vacuum electromagnetic field, i.e., a microscopic degree of freedom whose dynamics and coupling with electrons can be controlled by cavity engineering. Moreover, two-dimensional materials within layered structures [19] (and even more so synthetic ultracold-atomic systems [44]) offer an enhanced tuneability of electronic properties, including Coulomb interactions as well as the role of impurities and phonons. This potentially allows to realize a situation where the QED coupling with cavity photons is dominant. Vacuum electromagnetic fluctuations can be made critical by reaching the transition point towards superradiance [45–51], beyond which selected modes are macroscopically occupied by photons. Since the QED coupling depends on the electron momentum, we find that non-Fermi-liquid behavior can be controlled via the choice of the underlying lattice. We consider here a square and a honeycomb lattice away from unit filling. While in both cases polarization effects lead to overdamped electromagnetic modes at the critical point, the phase space for electron-photon scattering is such that non-Fermi-liquid behaviour is absent for the square lattice, where the electron quasiparticle damping is \( \propto \omega^2 \log |\omega| \), but present for the honeycomb lattice, where the damping is \( \propto |\omega|^{2/3} \). We also argue that measurements of the cavity spectrum could show, instead of a well-defined resonance, non-analytical power-law tails revealing the presence of the critical bosonic fluctuations responsible for the non-Fermi-liquid behavior.

**Model.** — We assume the cavity to consist of two parallel perfect conducting mirrors and choose the Coulomb gauge: \( \text{div} \mathbf{A} = 0 \) for the electromagnetic field. The two-dimensional electronic system is placed in the middle of the two plates as illustrated in Fig. 1. Due to the boundary condition at the mirrors, the photon dispersion acquires a finite ‘mass’ \( \omega_0 \) of the order of the inverse distance between two plates [52]:

\[
\omega_k^2 = k^2 + \omega_0^2, \tag{1}
\]

where \( \mathbf{k} \) is the 2D photon momentum in the plane of the
Setting:

Fermi-surface:
Polarization:
QP decay:

FIG. 1: Summary of the main results for the two cases considered: cavity photons coupled to electrons on a two-dimensional square lattice (left column) and honeycomb lattice (right column, red and blue atoms belong to the A, B sublattices).

lattice. The free particle Hamiltonian is:

\[ H_0 = -t \sum_{i,j,\alpha=\pm 1} \epsilon_{\alpha,i} c_{\alpha,i} c_{\alpha,j} + \sum_k \omega_k \left( b_k^\dagger b_k + \frac{1}{2} \right). \]  

(2)

Here \( i,j \) are lattice sites and \( \alpha \) is the spin index. The first term in (2) is the tight-binding electron Hamiltonian, and the second term describes the transverse photon modes. The free transverse photon Green’s function has the form [52, 53]:

\[ D_{0,ij}(k) = -\frac{2}{\omega^2 - \omega_k^2} e_i(k) e_j^\dagger(k), \]  

(3)

where \( e(k) \) is the transverse polarization vector: \( k . e(k) = 0 \). Electron-photon coupling is introduced via the Peierls substitution: \( c_i \rightarrow c_i \exp \int_0^{r_i} A . dr \) in \( H_0 \) [54]. For our purpose it is sufficient to keep the paramagnetic part linear in \( A \) [51]:

\[ V = -j . A, \quad H = H_0 + V. \]  

(4)

The structure of the current operator \( j \) depends on the geometry of the underlying lattice. Note that in principle, higher-order terms are required for guaranteeing gauge invariance of the theory. In particular, this means that superradiance cannot happen at vanishing photon momenta which corresponds to a pure gauge. In this case the paramagnetic contribution is cancelled exactly by the diamagnetic part.

At \( \omega = 0 \), the longitudinal cavity mode decouples from electrons as a result of charge conservation \( k . j = 0 \).

\[ \Pi(k) = \begin{pmatrix} \Lambda(p,k) & c^\dagger_p k \end{pmatrix} \begin{pmatrix} c^\dagger_p k & \Lambda(p,k) \end{pmatrix}. \]  

(5)

\[ \Sigma(p) = \int \exp(i\omega_\tau - k . r) d\tau d^2 x; \quad \omega_n = 2\pi n T. \]

Therefore, at small frequency, only the transverse cavity mode couples strongly to the Fermi-surface. In what follows we shall only consider coupling between the transverse mode and electrons.

Square Lattice. — Let us start with the square lattice below unit filling. We shall use the method of Matsubara Green’s functions to obtain results at finite temperature \( T \) and then take limit \( T \rightarrow 0 \). Our goal is to compute the polarization correction to the transverse photon Green’s function (3) to one-loop order, as illustrated diagrammatically in Fig. 2(a):

\[ \Pi(\omega_n, k) = c^\dagger_i(k) c(k) \int \langle j_i(\tau, r) j_k(0, 0) \rangle \times \exp(i\omega_n \tau - k . r) d\tau d^2 x; \quad \omega_n = 2\pi n T. \]

The electron Green’s function is given by:

\[ G(\nu_n, p) = \frac{1}{i\nu_n - \epsilon(p) + \mu}; \quad \nu_n = (2n + 1) \pi T, \]  

and \( \epsilon(p) = -2t(\cos p_x a + \cos p_y a) \) is the electron band dispersion corresponding to (2). The current operator in the tight-binding approximation is given by:

\[ j(k) = \sum_p \Lambda(p,k) c^\dagger_p k c_p, \]  

(6)

\[ \Lambda(p,k) = -ael \sum_{i} \sin \left( \frac{\left( p + \frac{k}{2} \right) \cdot a_i}{a} \right) a_i. \]  

(7)

Here \( a_i \) is the \( i \)-th lattice vector and \( a \) is the lattice constant. There are points on the Fermi surface which are most strongly affected by the quantum electromagnetic fluctuations. These hot spots are four pairs at opposite sites of the Fermi surface indicated in Fig. 1. There, the nesting momentum \( Q \) is shown for only one pair. At the hot-spots the electron dispersion has the form (8):

\[ \xi_{\pm}(k) = \pm v_F k_\perp + \frac{k_\parallel^2}{2m}, \]  

(8)

where \( k_\perp \) and \( k_\parallel \) are components of \( k \) parallel and orthogonal to \( Q \), and \( v_F, m \) are parameters of the Fermi-surface. We have assumed that we are sufficiently away
from unit filling that the Fermi velocity $v_F$, which vanishes at the corners of the square Fermi-surface at unit filling, can be regarded as constant here. In all cases the current-photon vertex that connect pairs of hot-spots in Eq. (7) has the same form:

$$\Lambda \cdot e \approx -a^2 e t \left( p_\parallel + \frac{k_\parallel}{2} \right).$$  \hspace{1cm} (9)

Here $p$ is the quasimomentum measured from one of the hot-spots. The momentum-dependent vertex in (9) affects the one-loop polarization function for the transverse cavity photons, which takes the form:

$$\Pi(\omega_n, Q + k) = 2a^4(\epsilon t)^2 T \sum_m \int \left( p_\parallel + \frac{k_\parallel}{2} \right)^2$$

$$\times G(\nu_m, p) G(\omega_n + \nu_m, p + Q + k) \frac{d^2 p}{(2\pi)^2}. \hspace{1cm} (10)$$

The integral is computed in the Supplementary Materials [53]. The result consists of an non-analytical contribution from near the Fermi-surface:

$$- \frac{a^4(\epsilon t)^2 m^2}{6\pi v_F} \left[ \left( E(k) + \omega + i0 \right)^\frac{3}{2} + \left( E(k) - \omega - i0 \right)^\frac{3}{2} \right], \hspace{1cm} (11)$$

where $E(k) = v_F k_\perp + k_\parallel^2/(4m)$. Subsequent terms in $\Pi(k) - \Pi(0, Q)$ receive contributions far from the Fermi-surface and can be written in powers of $E(k)$. The retarded photon Green’s function is:

$$\left[ D^R(\omega, k) \right]^{-1} = \left[ D^R_0(\omega, k) \right]^{-1} + \Pi^R(\omega, k), \hspace{1cm} (12)$$

see Fig. 2(b). Assuming $\Pi(0, Q)$ cancels the ‘photon mass’ term $\omega_k^2$ in $(D^R_0)^{-1}$, the retarded transverse photon Green’s function is then proportional to:

$$D^R(\omega, Q + k) = \left[ \left( E(k) + \omega + i0 \right)^\frac{3}{2} + \left( E(k) - \omega - i0 \right)^\frac{3}{2} + b E(k) + c E(k)^2 \right]^{-1} \hspace{1cm} (13)$$

where we have included up to quadratic powers of $E(k)$ in the expansion. As will be shown below, for the calculation of quasi-particle decay, $\xi_+(k)$ in the expansion can be neglected. The constants $b$ and $c$ are given by contributions in (10) far from the Fermi-surface. The non-analytic terms induce an imaginary part in Eq. (13) at large negative $E(k)$ corresponding to strong Landau damping of the photon:

$$\left| -E(k) \right| \pm (\omega + i0)^\frac{3}{2} \to \pm i\left| E(k) \right| \mp (\omega + i0)^\frac{3}{2}. \hspace{1cm} (14)$$

This contributes to the imaginary part in the electron self-energy given by Fig. 2(c):

$$\text{Im} \Sigma^R \left( \omega, \frac{Q}{2} \right) \propto \int k_\parallel^2 \text{Im} D^R[\xi_+(k) - \omega, k - Q] $$

$$\frac{d^2 k}{(2\pi)^2} \hspace{1cm} (15)$$

Here $0 < \xi_+(k) < \omega$. As discussed in [13] for the case of incommensurate density-wave quantum critical points and their non-Fermi-liquid behavior, the terms linear in $E(k)$ need in general to be kept. Here, contrary to that case, the free photon dispersion (1) gives in principle also powers of $\xi_+(k)$ in the denominator of $D(k)$. However, as will be shown below, at sufficiently small $\omega$, the dominant contribution in (14) comes from the region in the phase space where $|E(k)| \gg \xi_+(k) \sim \omega > 0$. This will allow us to neglect $\xi_+(k)$. We substitute Eq. (13) into Eq. (14), and make the change of variables [13]:

$$k' = -E(k), \omega' = \xi_+(k). \hspace{1cm} (16)$$

This then gives:

$$\text{Im} \Sigma^R \left( \omega, \frac{Q}{2} \right) \sim \omega^2 \log |\omega|. \hspace{1cm} (17)$$

The electron quasiparticles are thus well-defined in this case. This can be attributed to the suppression of Landau damping near the Fermi-surface due to the additional powers of momenta in the current operator; compare with Eq. (25) on the honeycomb lattice below.

Honeycomb Lattice.— We now turn to the honeycomb lattice. For simplicity we shall focus on the $K'$-point phase momentum space. The single-particle Hamiltonian $H_0$ then has the form:

$$H_0(p) = v_F \sigma \cdot p. \hspace{1cm} (18)$$

Here $\sigma$ are Pauli matrices in pseudo-spin space and $p$ is a 2D quasimomentum measured from $K'$. Eq. (17) gives two particle-hole symmetric bands with energies $\varepsilon_\lambda(p)$ and eigenstates $u_\lambda(p)$:

$$\varepsilon_\lambda(p) = \lambda v_F p, \hspace{1cm} u_\lambda(p) = \frac{1}{v_F} \left( \begin{array}{c} 1 \\ \lambda \exp(i\varphi_p) \end{array} \right), \hspace{1cm} (19)$$

where $\lambda = \pm$ and $\varphi_p$ is the polar angle of $p$. At finite positive chemical potential $\mu$, the Fermi-surface is a circle on the electron band. Contrary to the square lattice case, the current operator has no momentum dependence:

$$j = \frac{\delta}{\delta A} H_0(p - eA) = -e v_F \sigma. \hspace{1cm} (20)$$
The free electron Green’s function is a matrix in spin and pseudo-spin space:

$$G_{\alpha\beta}(\nu_n, p) = G(\nu_n, p)\delta_{\alpha\beta},$$

$$G(\omega_n, p) = \sum_{\lambda} G_{\lambda}(\omega_n, p)u_{\lambda}(p)u_{\lambda}^*(p),$$

(20)

$$G_{\lambda}(\omega_n, p) = \frac{1}{\omega_n - \epsilon_{\lambda}(p) + \mu}.$$  

$$G_{\lambda}(\nu_n, p)$$ are the Green’s functions of the electron and hole bands for $$\lambda = \pm$$ respectively. We rewrite Eq. (20) by substituting (18) to give:

$$G(\nu_n, p) = \frac{1}{2} \left( [G_+(\nu_n, p) + G_-(\nu_n, p)] + \sigma_n [G_+(\nu_n, p) - G_-(\nu_n, p)] \right).$$

(21)

Here $$n$$ is the directional vector of $$p$$.

At $$Q = 2p_F$$, the photons couple diametrically opposite points of the Fermi-surface. Near each of these hot-spots the dispersion has the form (8) with $$m = p_F/v_F$$. At low-enough energies, we will assume each of the infinitely many hot-spot pairs to contribute independently [10] and therefore consider only one such pair. To leading order, the polarization tensor $$\Pi_{\mu\nu}(\omega_n, Q + k)$$ in Fig. 2(a) near $$Q$$ is proportional to:

$$T \sum_m \int \text{Tr} [\sigma_\mu G(\nu_m, p)\sigma_\nu G(\omega_n + \nu_m, p + Q + k)] \frac{d^2 p}{(2\pi)^2}.$$  

(22)

We substitute (21) into (22) and project to the transverse mode. Assuming again that $$\Pi(0, Q)$$ cancels the $$\omega^2_k = Q$$ part in $$(D^R)^{-1}$$, at small $$\omega$$ and $$k$$ the non-analytical terms come from contributions near the Fermi-surface, i.e. from products of the form $$G_+ G_+$$; the other terms give regular powers of $$\omega$$ and $$k$$. Retaining only these terms in (22) gives:

$$\frac{e^2 v_F \sqrt{m}}{2\pi} \left( \left[ E(k) + i\omega \right]^{3/2} + \left[ E(k) - i\omega \right]^{3/2} \right).$$

(23)

where we have neglected terms of order $$k^2/p_F^2$$. The details of computing (22) are given in [53].

We now substitute (23) into the imaginary part of the electron self-energy given by Fig. 2(c):

$$\text{Im} \Sigma(\omega_n, p_F) \sim \text{Im} T \sum_{m} \int D(\Omega_m, k)(e, \sigma) \times G(\Omega_n + \omega_n, p + k)(e, \sigma) d^2 k.$$  

(24)

As is the case for the square lattice, powers of $$\xi_+$$ do not need to be included in the self-energy integral. After analytical continuation $$\omega_n \to \omega + i0$$, formally this integral is the same as setting $$b = 0$$ in Eq. (15) instead of $$c = 0$$, and receives the dominant contribution from $$k' \sim \Lambda_0(\omega/\Lambda_0)^{2/3} \gg \omega$$. Therefore, apart from the additional pseudospin structure, we expect in this case:

$$\text{Im} \Sigma^R(\omega, p_F) \sim |\omega|^{2/3}.$$  

(25)

Therefore, quasiparticle decay exhibits non-FL scaling contrary to the square lattice case, despite the presence of strong Landau damping for both lattices. This is because, as we have seen, on the honeycomb lattice the dominant contribution to the non-FL term in the self-energy comes from $$k' \sim \Lambda_0(\omega/\Lambda_0)^{2/3} \gg \omega$$, whereas on the square lattice this region is suppressed by the additional powers of momenta from the current operator (7).

We note that, the non-analytical form of the quasiparticle damping in (25) is the same as predicted for non-commensurate charge-density-wave [4, 13], as well as Fulde-Ferrel-Larkin-Ovchinnikov [14, 15] quantum critical points.

Implementation and observability. — The light-matter coupling (4) is relevant for electrons in solid state, which couple to photons via the current [19]. For neutral atomic gases instead, the dispersive coupling between photons and the atomic center-of-mass is independent of the momentum of the latter [44]. The same situation can be engineered also in solid state using two-photon transitions exploiting the diamagnetic coupling [55] or auxiliary electronic bands [32]. In all these cases, the non-Fermi-liquid behavior of Eq. (25) will be present regardless of the lattice, as long as unit filling is avoided.

With the purpose of probing the non-Fermi-liquid behavior, layered structures allow to precisely measure the electronic spectral function of an embedded 2D material using techniques like momentum-and-energy-resolved tunneling spectroscopy (MERTS) [56] and thus potentially to directly probe the non-analytic behavior of the quasiparticle damping. Such measurements are standardly available also for neutral atomic gases through radio-frequency spectroscopy [57].

Since in our scenario the bosonic order parameter fluctuations affecting the electrons are in fact photons, cavity probes can provide direct access to their characteristic overdamped behavior responsible in turn for the electronic quasiparticle damping. At sufficiently low frequencies $$|\omega| < -E(k)$$, the usual resonant peaks at $$\omega_k$$ in the cavity spectral function $$A(\omega, Q + k) = \text{Im} D^R(\omega, Q + k)/\pi$$ are substituted by a continuum of over-damped modes:

$$A(\omega, Q + k) \sim \frac{[\omega + |E(k)|]^\alpha - |E(k)|^\alpha}{[\omega + |E(k)|]^\alpha - (|E(k)| - \omega)^\alpha} + b^2 E(k)^2,$$

(26)

where $$\alpha = 3/2$$ for the square lattice and $$\alpha = 1/2$$ for the honeycomb. The distinct power-law dependence in (26) at small $$\omega$$ can be an experimental signature of the critical electromagnetic fluctuations affecting the electrons. While such frequency- and momentum-resolved cavity
probes are available in state-of-the-art experiments with atomic gases at the relevant frequencies and momenta [58], they seem rather challenging in the solid-state case, since the momentum scale \( Q \sim p_F \) is much larger than the characteristic photon wave vectors.

Conclusions.— We have shown that cavity QED with 2D materials allows to implement and probe non-Fermi-liquid behavior under pristine conditions: 1) the underlying critical bosonic fluctuations which destroy the electronic quasiparticles are provided by the electromagnetic vacuum field and are thus controllable via cavity engineering; 2) 2D materials (or the synthetic versions based on ultracold gases) allow for enhanced tuneability of electronic properties and control over unwanted effects.

Our work introduces a new (experimentally relevant) model for the emergence of non-Fermi-liquid behavior, thus offering a new playground for controlled experimental investigations as well as for theoretical approaches.

Future studies shall provide improved theories for the superradiant criticality and the associated non-Fermi-liquid behavior (also including the unit-filling case on the square lattice where logarithmic divergences appear in the polarization), possibly identifying deviations from the universality class predicted for non-commensurate charge-density wave [4, 13] or Fulde-Ferrell-Larkin-Ovchinnikov [14, 15] quantum critical points.

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Supplemental Materials: Non-Fermi-liquid behavior from critical electromagnetic vacuum fluctuations

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QUANTIZATION OF ELECTROMAGNETIC FIELD IN A蔡

In this Appendix, we review the quantization of electromagnetic field in a cavity and derive the Green’s function for cavity photons Eq. (3) in the main text [1]. We begin by considering arbitrary cavity geometry.

We choose the Coulomb gauge for the four-potential operator $A^\mu = (\phi, A)$:

$$\text{div } A = 0. \quad (S1)$$

Under this gauge, the scalar potential $\phi$ mediates the direct Coulomb interaction between electrons. The electric and magnetic fields are then given by:

$$E = -\frac{\partial A}{\partial t} - \nabla \phi, \quad B = \nabla \times A.$$ 

The vector potential operator has the form:

$$A = \sum_{a,n,k} \left( b^a_n A^a e^{-i\omega_n t} + b^a_\dagger_n A^{a*} e^{i\omega_n t} \right) \quad (S2)$$

Here $a$ and $n$ are indices for polarization and photon energy levels $\omega_n$, and $b^a_n$ is the corresponding annihilation operator for the photons. A similar expression exists for $\phi$. Under the gauge (S1), the equation of motion for the eigenfunctions $A^a_n$ is:

$$\omega^2_n A^a_n + \Delta A^a_n = 0, \quad (S3)$$

where $\Delta$ is the Laplacian. The scalar potential $\phi$ satisfies $\Delta \phi = 0$.

Eq. (S3) must be supplemented by certain boundary conditions on the surface of the cavity. We shall assume that the cavity mirrors are perfectly conducting and there is no energy loss from the cavity: the surface impedance $\zeta = (\mu/\varepsilon) = 0$. [2] The boundary conditions are that the tangent components of $E$ and the normal component of $H$ to the conducting surface vanish there [3]:

$$E_{||} = 0, \quad B_{\perp} = 0. \quad (S4)$$

However, Eq. (S4) is formulated in physical gauge-invariant quantities $E$ and $B$. The boundary conditions in terms of $\phi, A$ are not unique. We shall choose the following boundary conditions which separate $\phi$ that mediates direct Coulomb interactions between electrons, and the vacuum cavity mode given by $A$:

$$A_{||} = 0, \quad \phi = \text{const.}, \quad (\nabla \times A)_{\perp} = 0. \quad (S5)$$

This is because in the absence of matter, one can impose the further condition $\phi = 0$. Then the boundary condition (S4) for the vacuum cavity modes becomes (S5). In this paper we shall neglect effects from $\phi$ and are mainly concerned with interactions mediated by $A$. 

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quantization of electromagnetic field in a cavity

in this appendix, we review the quantization of electromagnetic field in a cavity and derive the green’s function for cavity photons eq. (3) in the main text [1]. we begin by considering arbitrary cavity geometry.

we choose the coulomb gauge for the four-potential operator $a^\mu = (\phi, a)$:

$$\text{div } a = 0. \quad (s1)$$

under this gauge, the scalar potential $\phi$ mediates the direct coulomb interaction between electrons. the electric and magnetic fields are then given by:

$$e = -\frac{\partial a}{\partial t} - \nabla \phi, \quad b = \nabla \times a.$$ 

the vector potential operator has the form:

$$a = \sum_{a,n,k} \left( b^a_n a^a e^{-i\omega_n t} + b^a_\dagger_n a^{a*} e^{i\omega_n t} \right) \quad (s2)$$

here $a$ and $n$ are indices for polarization and photon energy levels $\omega_n$, and $b^a_n$ is the corresponding annihilation operator for the photons. a similar expression exists for $\phi$. under the gauge (s1), the equation of motion for the eigenfunctions $a^a_n$ is:

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$$a_{||} = 0, \quad \phi = \text{const.}, \quad (\nabla \times a)_{\perp} = 0. \quad (s5)$$

this is because in the absence of matter, one can impose the further condition $\phi = 0$. then the boundary condition (s4) for the vacuum cavity modes becomes (s5). in this paper we shall neglect effects from $\phi$ and are mainly concerned with interactions mediated by $a$. 
Additionally, \( A_a^n \) satisfies the following normalization condition:

\[
\int \overline{A_a^n} A_b^m dV = \frac{1}{2\omega_n} \delta_{nm} \delta_{ab}. \tag{S6}
\]

Eq. (S6) is necessary for correctly quantizing the photon Hamiltonian:

\[
H = \int \frac{1}{2} (E^2 + B^2) dV \tag{S7}
\]

where the volume integral is taken over the entire cavity. Using the gauge condition (S1) and the vector identity:

\[
(\nabla \times A)^2 = (\nabla \times A) \cdot H = \nabla \cdot (A \times H) + A \cdot (\nabla \times H); \\
A \cdot (\nabla \times H) = A \cdot [\nabla \times (\nabla \times A)] = -A \cdot \Delta A,
\]

Eq. (S7) is transformed as:

\[
H = \int \frac{1}{2} \left[ \left( \frac{\partial A}{\partial t} \right)^2 + (\nabla \phi)^2 - A \cdot \Delta A \right] dV + \\
+ \int \left[ A \times (\nabla \times A) + \phi \frac{\partial A}{\partial t} \right] dS. \tag{S8}
\]

The second integral is taken over the cavity surface and corresponds to energy loss to the cavity. With the perfect conductor boundary condition (S5), this term vanishes as it should. Substituting Eqs. (S2) and (S3), and the normalization condition (S6) into (S8) then gives:

\[
H = \sum_{n,a} \left( b_n^a b_n^a + \frac{1}{2} \right) \omega_n. \tag{S9}
\]

Thus the normalization condition (S6) gives the quantized photon Hamiltonian. Note that with finite energy loss, the surface integral in (S8) introduces an imaginary, dissipative term to (S9), as coherent cavity modes become ill-defined.

Eqs. (S3), (S5) and (S6) allow one to find the vacuum photon modes for any cavity geometry. In this paper, the cavity is taken to be two perfectly conducting planes perpendicular to the \( z \)-axis at \( z = 0, L \). Let us now find the corresponding eigenfunctions \( A_a^n \). Since the \((x,y)\)-plane is infinite, \( A_a^n \) depends on the two-dimensional position vector \( r = (x,y) \) via \( \exp(ik \cdot r) \). The photons are bounded along the \( z \)-direction due to Eq. (S5), which makes the corresponding contribution to the energy levels discrete. Denoting from now on \( n \) this discrete index:

\[
A_a = \sum_{a,n,k} \frac{1}{\sqrt{2\omega_n k}} \left( e_n(a) b_n^{ak} e^{-ikx} + e_n^{(*)} b_n^{ak\dagger} e^{ikx} \right). \tag{S10}
\]

Here \( n \) is the index due to the finite boundary condition along the \( z \)-axis, \( kx = \omega t - k \cdot r \). The polarization vector \( e_n^{(a)} \) depends in general on \( z, k \) and \( n \).

We take the direction of \( k \) to be along the \( x \)-axis so that \( A \) does not depend on \( y \). To find the two physical polarizations, we first substitute the gauge condition (S1), which gives:

\[
k \cdot e^{(a)} - i \frac{\partial e^{(a)}}{\partial z} = 0. \tag{S11}
\]

The first solution is obvious: \( e^{(1)} \) is along the \( y \)-axis. The boundary conditions (S5) and (S6) then gives:

\[
e_n^{(1)} = \sqrt{2} e_y \sin \left( \frac{n\pi z}{L} \right), n = 1, 2...
\]

For the magnetic field:

\[
B_y = 0, B_z = \frac{\partial A_y}{\partial x}, B_x = -\frac{\partial A_y}{\partial z}.
\]
Then \( B_z = 0 \) is also satisfied at the boundary. Substituting our solution into Eq. (S3) gives for the cavity photon dispersion:

\[
\omega_{nk}^2 = k^2 + \omega_0^2, \quad \omega_0 = \frac{n\pi z}{L}.
\]  

(S12)

Since \( n \neq 0 \), the photon dispersion is always gapped. The second polarisation lies in the \((x, z)\)-plane and \( A_y = 0 \). This gives for the magnetic field:

\[
B_x = 0, \quad B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \quad B_z = 0.
\]

Then \( B_z = 0 \) automatically. The boundary conditions for \( A \) in (S5) gives:

\[
A_x \propto \sin \left( \frac{n\pi z}{L} \right).
\]

We then have from the gauge condition (S11):

\[
kA_x - i \frac{\partial A_z}{\partial z} \rightarrow A_z \propto \cos \left( \frac{n\pi z}{L} \right),
\]

and the cavity photon dispersion is also given by (S12). The polarisation vector here satisfies:

\[
e_n^{(2)} = a \sin \left( \frac{n\pi z}{L} \right) e_x + b \cos \left( \frac{n\pi z}{L} \right) e_z,
\]

\[
|a|^2 + |b|^2 = 2, \quad b = -\frac{ikL}{n\pi} a.
\]

The second and third equalities are obtained by substituting (S6) and (S1). This gives (without loss of generality \( a \) can be taken real):

\[
a = \left[ \frac{2}{1 + (kL/n\pi)^2} \right]^{1/2}.
\]

The electrons are placed at \( z = L/2 \). Note that in this case \( n = 0 \) can be taken and the photon mode is gapless. But then \( A_x = A_y = 0 \) and the system does not couple to two-dimensional currents. Therefore, we only consider \( n = 1 \) below and omit this index. In the limit \( k \ll \pi/L \), both polarisation modes are in plane and orthogonal and:

\[
e^{(2)} \approx \sqrt{2}e_x.
\]

We are now in a position to calculate the photon Green’s function:

\[
D_{ik}(t, r) = i \langle 0 | T A_i(t, r) A_k(0, 0) | 0 \rangle,
\]  

(S13)

where \( T \) is time-ordering. Substituting (S2) and using the identities:

\[
\langle 0 | b^\dagger_{k'} b^\dagger_k | 0 \rangle = \delta_{\alpha\beta} \delta(\mathbf{k} - \mathbf{k'}), \quad \sum_a e^{(a)}_i e^{(a)*}_k = 2 \delta_{ik},
\]

(the second equality can be shown by choosing \( e^{1,2} \) as along the \( x-, y\)-axis, so only diagonal components are non-zero), we obtain:

\[
D_{ik}(t, r) = 2 \delta_{ik} D(t, r),
\]

\[
D(t, r) = -\sum_k \frac{1}{2\omega_k} \begin{cases} 
\exp(-i\omega_k t + \mathbf{k}.\mathbf{r}), \ t > 0; \\
\exp(i\omega_k t - i\mathbf{k}.\mathbf{r}), \ t < 0,
\end{cases}
\]

where \( \omega_k = k^2 + \omega_0^2, \omega_0 = \pi/L \). Taking the Fourier transform of \( D(t, r) \) and regularizing the integral over \( t \) by displacing \( \omega_k \to \omega_k + i0 \) at \( t \to \pm\infty \) then gives:

\[
D_{ik}(k) = -\frac{2\delta_{ik}}{\omega^2 - \omega_k^2 + i0}.
\]  

(S14)

Projecting to the transverse mode then gives Eq. (3) in the main text. Remember that the \( \delta_{ik} \) tensor structure in (S14) is only true for small quasi-momenta \( k \ll \pi/L \), for which the longitudinal polarization \( e^{(2)} \) is approximately in the \((x, y)\)-plane. The transverse mode on the other hand, is always inside the plane and does not depend on \( n \).
\[ \Pi(k) = \includegraphics{Diag1} \]

\[ D(k) = D_0(k) + D_0(k)\Pi(k)D(k) \]

\[ \Sigma(p) = \includegraphics{Diag2} \]

**FIG. 1:** One loop diagrammatic representation for: (a) the Polarization function \( \Pi(k) \); (b) the Dyson equation for photon propagator \( D(k) \); and (c) electron self-energy \( \Sigma(p) \).

**COMPUTATION OF PHOTON POLARIZATION FUNCTION ON THE SQUARE LATTICE**

In this section we compute Eq. (11) in the main text. The integral is given by the diagram in Fig. 1(a):

\[ \Pi(\omega_n, Q + k) = 2a^4(e^\gamma)^2 T \sum_m \int \left( \frac{p^\parallel + k^\parallel}{2} \right)^2 G(\nu_m, p)G(\omega_n + \nu_m, p + Q + k) \frac{d^2\rho}{(2\pi)^2}. \tag{S15} \]

In Eq. (S15) the sum over Matsubara frequency is first carried out. This gives:

\[ -\frac{2a^4(e^\gamma)^2}{(2\pi)^2} \int \left[ p^\parallel + \frac{k^\parallel}{2} \right]^2 \int \left( n_+(p + k) - n_-(p) \right) \frac{d^2\rho}{(2\pi)^2} \]

\[ \omega_n - \left[ \xi_+(p + k) - \xi_-(p) \right] d^2\rho, \tag{S16} \]

where \( \xi_{\pm}(k) = \frac{k^2}{2m} \pm v_F k^\perp \) is the dispersion near the ‘hot-spots’ and \( n_{\pm}(p) \) is the Fermi distribution for energy \( \xi_{\pm}(p) \). For the first term in the bracket, we perform the substitution \( p + k \rightarrow p \) and take \( T \rightarrow 0 \):

\[ -\frac{2a^4(e^\gamma)^2}{(2\pi)^2} \int d\rho^\parallel \int_{-\Lambda_0}^{\Lambda_0} \left( p^\parallel - \frac{k^\parallel}{2} \right)^2 \log \left[ i\omega_n + v_F k^\perp + \frac{k^2}{4m} + \frac{(p^\parallel - k^\parallel/2)^2}{m} \right] d\rho^\parallel. \tag{S17} \]

The integral over \( p^\parallel \) is formally divergent. Thus we need to extract from the following integral the contribution close to the Fermi-surface:

\[ \int x^2 \log(x^2 + a) dx = \frac{x^3}{3} \log(x^2 + a) - \frac{2}{9} x^3 + \frac{2}{3} a x - \frac{2}{3} \int \frac{x^2}{x^2 + a} dx; \quad a = i\omega_n + v_F k^\perp + \frac{k^2}{4m}. \]

Only the last term converges near the Fermi surface to give \(-2\pi a^{3/2}/3\). This gives:

\[ -\frac{(ae^\gamma)^2}{6\pi v_F} m^{3/2} \left( v_F k^\perp + \frac{k^2}{4m} + i\omega_n \right)^{3/2} \]

For the second term in Eq. (S16) we get:

\[ -\frac{(ae^\gamma)^2}{(2\pi)^2} \int d\rho^\parallel \int_{-\Lambda_0}^{\Lambda_0} \left( p^\parallel + \frac{k^\parallel}{2} \right)^2 \frac{d\rho^\parallel}{(2\pi)^2} \]

\[ \frac{1}{2(2\pi)^2 v_F} \int_{-\Lambda_0}^{\Lambda_0} \left( p^\parallel + \frac{k^\parallel}{2} \right)^2 \log \left( v_F k^\perp + \frac{k^2}{4m} - i\omega_n + \frac{(p^\parallel + k^\parallel/2)^2}{m} \right) d\rho^\parallel. \tag{S18} \]
And we get:
\[-\frac{(a_{et})^2}{6\pi v_F}m^{3/2} \left(v_F k_\perp + \frac{k_\parallel^2}{4m} - i\omega_n\right)^{3/2}.\]

Summing the two parts and performing analytical continuation \(i\omega_n \to \omega + i0\) we return to Eq. (11) in the main text. Alternatively, one can use the zero temperature diagram technique, and integrate successively over \(p_\perp\) then \(p_0\).

**COMPUTATION OF ELECTRON SELF-ENERGY ON THE SQUARE LATTICE**

Eq. (14) in the main text [given by Fig. 1(c)] is computed as follows. Under the change of variables to \(k', \omega'\):

\[-k' = e(k), \omega' = \varepsilon(k) = \frac{k_\parallel^2}{2m} + k_\perp, J = 2m/k_\parallel = \sqrt{m/(\omega' + k')}, k_\parallel^2 = 4m(\omega' + k'),\]

where \(J\) is the Jacobian of the integral, Eq. (14) becomes:

\[\text{Im } \Sigma \left(\omega, \frac{Q}{2}\right) \propto \int_0^\omega d\omega' \int_0^{\Lambda_0} dk' \frac{\sqrt{\omega' + k'}}{i(\omega' + \omega)^{3/2} - i(k' + \omega' - \omega)^{3/2} + bk' + ck'^2}.\] (S19)

The dominant contribution comes from \(k' \gg \omega - \omega' > 0\), at which the non-analytical term in the photon Green’s function Eq. (11) gives a large imaginary part that is responsible for Landau damping [4]:

\[-k' \pm (\omega' - \omega + i0)^{3/2} \to \pm i|k'| \mp (\omega' - \omega + i0)^{3/2};\]

\[i(k' - \omega' + \omega)^{3/2} - i(k' + \omega' - \omega)^{3/2} \approx 3i(\omega - \omega')\sqrt{k'^2}.\]

This gives for Eq. (S19):

\[\text{Im } \Sigma \left(\omega, \frac{Q}{2}\right) \propto \int_0^\omega d\omega' \int_0^{\Lambda_0} dk' \frac{\omega - \omega'}{(\omega - \omega')^2 + (b + ck')^2k'^2}.\] (S20)

Here we have defined new constants \(b, c \to b/3, c/3\). The upper limit is at \(\Lambda_0\), since the expressions for polarization Eq. (11) and self-energy (14) in the main text no longer hold at large \(k' \sim \Lambda_0\). Another change into dimensionless variables:

\[\omega' = \omega_1, k' = \omega k_1,\]

gives for the self-energy:

\[\omega \int_0^1 d\omega_1 \int_0^{\Lambda_0/\omega} dk' \frac{1 - \omega_1}{(1 - \omega_1)^2 + (b/\sqrt{\omega} + ck_1 \sqrt{\omega})^2k_1}.\] (S21)

The lower limit for \(k'\) is strictly speaking not zero but a term of the magnitude \(\omega - \omega'.\) This substitution then results in an error of \(\omega^2\) which is not important here. In the second bracket in the denominator of (S21), the first term is dominant at \(k_1 \ll b/(\omega c)\). But since \(b\) and \(c\) comes from integration far from the Fermi-surface, the only momentum scale is \(\Lambda_0\). Hence by dimensional arguments \(b/(\omega c) \sim \Lambda_0/\omega\), which is just the upper-limit in (S21). Therefore we can set \(c = 0\) in (S21) and obtain, up to logarithmic accuracy:

\[\text{Im } \Sigma \left(\omega, \frac{Q}{2}\right) \sim \omega^2 \log |\omega|.\] (S22)

This is Eq. (16) in the main text.

Note that the self-energy for the electrons on a honeycomb lattice corresponds to formally setting \(b = 0\) in Eq. (S21). The integral is then convergent and gives:

\[\text{Im } \Sigma \left(\omega, \frac{Q}{2}\right) \sim \omega \int_0^1 d\omega_1 \int_0^{\infty} \frac{1 - \omega_1}{(1 - \omega_1)^2 + c^2k_1^2} d\omega_1 dk_1 = \left(\frac{\omega}{c}\right)^{2/3} \int_0^1 (1 - \omega_1)^{-1/3} d\omega_1 \int_0^{\infty} \frac{dx}{1 + x^3} = \frac{\pi}{\sqrt{3}} \left(\frac{\omega}{c}\right)^{2/3}.\] (S23)

Here the dominant contribution comes from \(k' \sim \Lambda_0(\omega/\Lambda_0)^{2/3} \gg \omega\).
COMPUTATION OF POLARIZATION FUNCTION ON THE HONEYCOMB LATTICE

To obtain Eq. (23) for the polarization function on the honeycomb lattice in the main text, we substitute Eq. (21) into (22). The result is simplified using the Pauli matrix identities (terms with the anti-symmetric tensor \( e_{\mu \nu \delta} \) vanish since \( \mu, \nu = 1, 2 \):)

\[
\text{Tr}(\sigma_a \sigma_b) = 2\delta_{ab}, \quad \text{Tr}(\sigma_a \sigma_b \sigma_c \sigma_d) = 2(\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}),
\]

then contracted with:

\[
e_\mu(Q + k)e_\nu(Q + k) \approx e_\mu(Q)e_\nu(Q).
\]

After some algebra we obtain for the \( G_+G_+ \) term in \( \Pi(\omega_n, Q + k) \) up to terms of order \( k^2/p_F^2 \):

\[
2e^2v_F^2T \sum_m \int G_+(\nu_m + \omega_n, p + Q + k)G_+^*(\nu_m, p) \frac{d^2p}{(2\pi)^2} \tag{S24}
\]

The integral has the form:

\[
\sum_m \int G_+(\nu_m + \omega_n, p + Q + k)G_+^*(\nu_m, p) \frac{d^2p}{(2\pi)^2} = \sum_m \int \frac{d^2p}{(2\pi)^2} \left( i\nu_m + v_F p_\perp - \frac{p_\parallel}{2m} \right)^{-1} \left[ i(\nu_m + \omega_n) - v_F(p_\perp + k_\perp) - \frac{(p_\parallel + k_\parallel)^2}{2m} \right]^{-1}. \tag{S25}
\]

This is calculated analogously to Eq. (S15) by summing over Matsubara frequency \( \nu_m \) then taking \( T \to 0 \). This gives:

\[
- \frac{1}{(2\pi)^2} \int \left( n_+(p + k) - n_-(p) \right) i\omega_n - [\xi_+(p + k) - \xi_-(p)] \frac{d^2p}{(2\pi)^2} = - \frac{1}{(2\pi)^2} \int \left( \frac{n_+(p)}{i\omega_n - [\xi_+(p + k) - \xi_-(p)]} - \frac{n_-(p)}{i\omega_n - [\xi_+(p + k) - \xi_-(p)]} \right) d^2p. \tag{S26}
\]

At \( T = 0 \), we consider the first term in the bracket in (S26) and integrate over \( p_\perp \):

\[
- \frac{1}{(2\pi)^2} \int \left( \frac{n_+(p)}{i\omega_n + v_F k_\perp + k_\parallel^2/(2m)} - \frac{n_-(p)}{i\omega_n - [\xi_+(p + k) - \xi_-(p)]} \right) d^2p = \frac{1}{2(2\pi)^2v_F} \int_{-\Lambda_0}^{\Lambda_0} \log \left( i\omega_n + v_F k_\perp + k_\parallel^2/4m + \left( \frac{p_\parallel - k_\parallel/2}{m} \right)^2 \right) dp_\parallel. \tag{S27}
\]

Here \( \Lambda_0 \) is a cut-off of the order of the bandwidth. In calculating the integral over the logarithm we extract the contribution from close to the Fermi-surface in the same way as in the square lattice case:

\[
\int \log(x^2 + a)dx = 2\Lambda_0 \log(\Lambda_0^2 + a) - 4\Lambda_0 + 2a \int \frac{1}{x^2 + a} dx; \quad a = i\omega_n + v_F k_\perp + \frac{k_\parallel^2}{4m}.
\]

The logarithmic terms can be expanded in powers of \( a \) which would give far from Fermi surface corrections. The last term is convergent and we have:

\[
\left( v_F k_\perp + \frac{k_\parallel^2}{4m} + i\omega_n \right)^{1/2}. \tag{S28}
\]

The second term in (S26) gives:

\[
\left( v_F k_\perp + \frac{k_\parallel^2}{4m} - i\omega_n \right)^{1/2}. \tag{S29}
\]

Adding them together gives Eq. (23) in the main text.
[1] K. Kakazu and Y. S. Kim, “Quantization of electromagnetic fields in cavities and spontaneous emission,” Phys. Rev. A 50, 1830–1839 (1994).

[2] In principle, $\zeta(\omega)$ is a function of frequency. Here the approximation is that $\zeta(\omega) \approx 0$ in the frequency range of interest.

[3] L.D. Landau, J.S. Bell, M.J. Kearsley, L.P. Pitaevskii, E.M. Lifshitz, and J.B. Sykes, Electrodynamics of Continuous Media, COURSE OF THEORETICAL PHYSICS (Elsevier Science, 2013).

[4] Here the convention for circumventing the branch-cuts in complex plane of $k'$ is: for $(\omega' - \omega - k' + i0)^{3/2}$ as $k'$ increases it goes from under the branch-cut to the real axis, whereas for $(\omega - \omega' - k' - i0)^{3/2}$ it goes from above.