Numerical analysis of nonlinear forced vibrations of a cylindrical shell with combinational internal resonance in a fractional viscoelastic medium

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Abstract. Nonlinear force driven vibrations of thin cylindrical shells in a viscoelastic medium, the damping properties of which are described by the fractional derivative Kelvin–Voigt model, are studied. A comparative analysis has been carried out for numerical results of both free and forced shell’s oscillations using two different numerical methods for the case of the combination internal resonance.

1. Introduction
The analysis of forced nonlinear vibrations of thin cylindrical shell is the important area of applied mechanics, since shells are widely used as structural elements in many fields of industry and technology [1]. It will suffice to mention the state-of-the-art articles [2-3] and the monograph [4], involving the extensive review of literature in the field of internal resonances in different mechanical systems. Various types of the internal resonance: one-to-one, two-to-one, three-to-one, as well as a variety of combinational resonances, when three and more natural modes interact, have been discussed. The enumerated internal resonances were investigated in various mechanical systems with multiple degree-of-freedom, as well as in strings, beams, plates, and shells.

Free damped nonlinear vibrations of cylindrical shells in a fractional derivative medium have been studied in [5], wherein the procedure resulting in decoupling linear parts of nonlinear equations has been proposed with the further utilization of the method of multiple scales for solving nonlinear governing equations of motion, in so doing the amplitude functions are expanded into power series in terms of the small parameter and depend on different time scales.

It has been shown that the phenomenon of the internal resonance between vibrational subsystems of the cylindrical shell under consideration could be very critical, since in the circular cylindrical shell of such a type the two-to-one [5], one-to-one, three-to-one [6] internal resonances, as well as combinational internal resonances [7] could occur, which are governed by the order of smallness of viscosity. All possible cases of the internal resonance have been recently revealed in [7], which belong to the resonances of the constructive type, since all of them depend on the geometrical dimensions of the shell under consideration and its mechanical characteristics, that is why such resonances could not be ignored and eliminated for a particularly designed shell. It has been shown that the energy exchange could occur between two or three subsystems at a time: normal vibrations of the shell, its torsional vibrations and shear vibrations along the shell axis. Such an energy exchange, if it takes place for a rather long time, could result in crack formation in the shell, and finally to its failure. The
energy exchange has been illustrated pictorially by the phase portraits, wherein the phase trajectories of the phase fluid motion are depicted [5-7].

In the present paper, the approach proposed in [5] has been extended to the force driven vibrations of a thin cylindrical shell under the combinational internal resonance with the force frequency approximately equal to one of natural frequencies.

2. Problem formulation

Let us consider the dynamic behavior of a simply-supported thin cylindrical shell with geometrical nonlinearities, vibrations of which in a viscoelastic fractional derivative medium are described in the general coordinates by the following three differential equations written in the dimensionless form (free damped nonlinear vibrations described by the partial nonlinear differential equations presented in [5] are supplemented herein by the orthogonal harmonic force applied to the thin cylindrical shell middle surface at the point with coordinates x₀,φ₀,r):

\[
\begin{align*}
\frac{\partial^2 u_{xx}}{\partial t^2} + \frac{1-v}{2} \beta^1_1 u_{yy} + \frac{1+v}{2} \beta^1_1 v_{yy} - v^2 \beta^1_1 w_r + w_z \left( \frac{\partial^2 w_{xx}}{\partial t^2} + \frac{1-v}{2} \beta^1_1 \right) \\
\frac{1+v}{2} \beta^1_1 \beta^2_1 w_{yy} \phi_{yy} + \frac{1-v}{2} \beta^1_1 \phi_{yy} + \frac{1+v}{2} \beta^1_1 \phi_{yy} + \frac{1-v}{2} \beta^1_1 \phi_{yy} + \frac{1+v}{2} \beta^1_1 \phi_{yy}
\end{align*}
\]

subjected to the initial
\[
u \mid_{t=0} = v \mid_{t=0} = w \mid_{t=0} = 0,
\quad \phi \mid_{t=0} = \phi \mid_{t=0} = \phi \mid_{t=0} = 0
\]
as well as the boundary conditions
\[
w \mid_{r=0} = w \mid_{r=\pi} = 0,
\quad v \mid_{r=0} = v \mid_{r=\pi} = 0,
\quad u_x \mid_{x=0} = u_x \mid_{x=\pi} = 0,
\quad w_x \mid_{x=0} = w_x \mid_{x=\pi} = 0,
\]
where the x-axis is directed along the axis of the cylinder, \( r \) and \( \phi \) is the polar radius and angle, respectively, \( u = u(x,\phi,t) \), \( v = v(x,\phi,t) \), and \( w = w(x,\phi,t) \) are the displacements of points located in the shell’s middle surface in three mutually orthogonal directions \( x,\phi,r \), \( \beta_1 = 1/R \), \( \beta_2 = h/l \) are parameters defining the dimensions of the shell, \( h \) is the thickness, \( l \) is the length, \( R \) is the radius, \( t \) is time, an over dot denotes the time-derivative, lower indices label the derivatives with respect to the corresponding coordinates, \( F_0 \cos(\Omega \pi t) \) is the harmonic force with the amplitude \( F_0 = \text{const} \) and frequency \( \Omega \), \( \delta \) is the Dirac delta function, \( \alpha \) are damping coefficients, \( \left( \frac{\partial}{\partial t} \right)^{\beta} \) is the fractional order
of the operator of differentiation which is equivalent to the Riemann-Liouville fractional derivative of the \( \gamma \)-order \([8]\)

\[
D^\gamma F = \frac{d}{dt} \int_0^t F(t-t') (1-\gamma) dt'.
\] (6)

To solve the partial differential equations (1)-(3) subjected to the initial (4) and boundary (5) conditions, we will use the approach suggested in [5], which involves the decoupling of linear parts of equations (1)-(3) with further utilization of the generalized method of multiple time scales [9] to expand the amplitude functions into power series in terms of the small parameter. Assuming thereafter that the vibrational process occurs in such a way that only three natural modes corresponding to the generalized displacements \( X_{i\alpha} \) \((i=1, 2, 3)\) predominate and are coupled by the combinational resonance, resulting in the energy transfer between the coupled modes. Thus, equations (1)-(3) are reduced to the following set of equations:

\[
\begin{align*}
\dot{X}_{1\alpha} + \alpha_1 D^\gamma X_{1\alpha} + \Omega_{1\alpha}^2 X_{1\alpha} &= -\frac{3}{3} F_{1\alpha} l_{1\alpha}^I - \tilde{F} \cos(\Omega_{1\alpha} t) l_{1\alpha}^I, \\
\dot{X}_{2\alpha} + \alpha_2 D^\gamma X_{2\alpha} + \Omega_{2\alpha}^2 X_{2\alpha} &= -\frac{3}{3} F_{2\alpha} l_{2\alpha}^I - \tilde{F} \cos(\Omega_{2\alpha} t) l_{2\alpha}^I, \\
\dot{X}_{3\alpha} + \alpha_3 D^\gamma X_{3\alpha} + \Omega_{3\alpha}^2 X_{3\alpha} &= -\frac{3}{3} F_{3\alpha} l_{3\alpha}^I - \tilde{F} \cos(\Omega_{3\alpha} t) l_{3\alpha}^I,
\end{align*}
\] (7)-(9)

where \( m \) and \( n \) are integers, \( \tilde{F} = F_0 \cos(\Omega_{2\alpha} t) \sin(\pi m T_0) \sin(\pi n \phi_0) \), \( \Omega_{1\alpha} \), \( \Omega_{2\alpha} \) and \( \Omega_{3\alpha} \) are the eigenvalues of the matrix \( S^\alpha_{ij} \) and \( l_{1\alpha}^I \), \( l_{2\alpha}^I \) and \( l_{3\alpha}^I \) are eigenvectors of the same matrix \( S^\alpha_{ij} \) with the elements

\[
S^\alpha_{ij} = \begin{bmatrix}
S^{\alpha}_{11} & S^{\alpha}_{12} & S^{\alpha}_{13} \\
S^{\alpha}_{21} & S^{\alpha}_{22} & S^{\alpha}_{23} \\
S^{\alpha}_{31} & S^{\alpha}_{32} & S^{\alpha}_{33}
\end{bmatrix} = \begin{bmatrix}
\left(\pi m^2 + \frac{1-v}{2} \beta_1^2 n^2\right) & \frac{1+v}{2} \beta_1 \pi mn & v\beta_1 \pi m \\
\frac{1+v}{2} \beta_1 \pi mn & \left(\frac{1-v}{2} \pi^2 m^2 + \beta_1^2 n^2\right) & \beta_1 n \\
v\beta_1 \pi m & \beta_1 n & \frac{\beta_1^2}{12} \left(\pi^2 m^2 + \beta_1^2 n^2\right)^2 + \beta_1
\end{bmatrix}
\] (10)

3. Methods of solution

Utilizing the procedure described in detail in [7] and assuming that \( \tilde{F} = \varepsilon^2 f \), we will solve equations (7)-(9) using two different methods for the case of the combinational internal resonance \( \Omega_2 = \Omega_1 + \Omega_3 \) accompanied by the primary resonance, i.e. when the frequency of the external force is close to the natural frequency of the vertical mode \( \Omega_1 = \Omega_1 + \varepsilon \sigma \), where \( \sigma \) is the detuning parameter.

The first method resides in the discretization of all derivatives in equations (7)-(10). For this purpose we will follow the procedures suggested by Diethelm in [10,11].

Let us introduce the following notation:

\[
\begin{align*}
Y_i &= X_i, \quad Y_2 = D^\gamma X_2 = D^\gamma Y_2, \quad Y_3 = DX_3 = DY_3, \\
\dot{X}_1 &= DDX_1 = DY_1 = -\sum_{i=1}^3 F_{i\alpha} l_{i\alpha}^I - \alpha_1 Y_1 - \Omega_{1\alpha} Y_1 - \tilde{F} \cos(\Omega_{1\alpha} t) l_{1\alpha}^I.
\end{align*}
\] (11)

The first time-derivative could be discretized using the trapezoidal rule

\[
DY_1 = Y_3 \to Y_{1,i} = Y_{1,i-1} + \frac{1}{2} h \left( Y_{1,i} + Y_{1,i-1} \right),
\] (12)

therefore

\[
Y_{1,i} = \frac{1}{2} h Y_{1,i} = Y_{1,i-1} + \frac{1}{2} h \left( Y_{3,i-1} \right) = s_{1,i-1}.
\] (13)

The fractional derivative could be discretized using the Diethelm’s method [10,11] according to the definition
\[ D' Y_1 = \frac{1}{\alpha} \chi_i \left( \sum_{k=0}^{i} \gamma_{0,k} Y_{i-k} + \frac{Y_{i0}}{\gamma} \right) = Y_2, \]  
whence it follows
\[ \gamma \chi_i Y_2 - \gamma \alpha_{0,i} Y_{i+1} = \left( \sum_{k=0}^{i} \gamma_{0,k} Y_{i-k} + \frac{Y_{i0}}{\gamma} \right). \]

Assuming that \( s_{1,i} = \left( \sum_{k=0}^{i} \gamma_{0,k} Y_{i-k} + \frac{Y_{i0}}{\gamma} \right) \) and applying the described above procedure for other two equations, equations (7)-(9) could be represented in the following matrix form:

\[
\begin{bmatrix}
-\gamma \omega_{0,i} & \gamma \chi_i & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -\frac{h}{2} & 0 & 0 & 0 & 0 & 0 \\
\frac{h}{2} \Omega_1^2 & \frac{h}{2} \alpha_1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\gamma \omega_{0,i} & \gamma \chi_i & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -\frac{h}{2} & 0 & 0 \\
0 & 0 & \frac{h}{2} \Omega_2^2 & \frac{h}{2} \alpha_2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\gamma \omega_{0,i} & \gamma \chi_i & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -\frac{h}{2} \\
0 & 0 & 0 & 0 & 0 & \frac{h}{2} \Omega_3^2 & \frac{h}{2} \alpha_3 & 1
\end{bmatrix}
\begin{bmatrix}
Y_{1,i} \\
Y_{2,i} \\
Y_{3,i} \\
Y_{4,i} \\
Y_{5,i} \\
Y_{6,i} \\
Y_{7,i} \\
Y_{8,i} \\
Y_{9,i}
\end{bmatrix}
= \begin{bmatrix}
s_{1,i-1} \\
s_{2,i-1} \\
s_{3,i-1} \\
s_{4,i-1} \\
s_{5,i-1} \\
s_{6,i-1} \\
s_{7,i-1} \\
s_{8,i-1} \\
s_{9,i-1}
\end{bmatrix}
\]  
(16)

It is quite straightforward to solve (16) for \( Y \) column by finding the inverse of the left matrix and multiplying it by \( s \) (the right column which represents the already known values or initial conditions from the previous iteration step).

The second method is based on the generalized multiple time scales approach proposed by Rossikhin and Shitikova [9], resulting in a set of six nonlinear differential equations in terms of amplitudes \( a_i \) and phases \( \phi_i \):

\[
\begin{align*}
(a_i^1) + s_i a_i^2 &= \Omega_1 a_i a_i a_i a_i \sin \delta, \\
(a_i^2) + s_i a_i^3 &= -\Omega_2 a_i a_i a_i a_i \sin \delta, \\
(a_i^3) + s_i a_i^4 &= -\Omega_3 a_i a_i a_i a_i \sin \delta, \\
\phi_i - \frac{1}{2} \sigma_i - \frac{1}{2} \frac{\partial a_i^1}{\partial a_i} \cos \delta &= 0, \\
\phi_i - \frac{1}{2} \sigma_i - \frac{1}{2} \frac{\partial a_i^2}{\partial a_i} \cos \delta &= 0, \\
\phi_i - \frac{1}{2} \sigma_i - \frac{1}{2} \frac{\partial a_i^3}{\partial a_i} \cos \delta &= 0,
\end{align*}
\]  
(17)

where \( \delta = \phi_i + \phi_i - \phi_i \) is the phase difference, and all coefficients are given in [17].

Equations (17) could be solved numerically using the Runge-Kutta fourth-order algorithm via MatLab.

4. Numerical results

Reference to figure 1 shows the relation between \( \beta_2 \) and \( \beta_1 \) representing shell’s parameters under the combinational internal resonance \( \Omega_2 = \Omega_3 + \Omega_1 \). Figure 2 represent the interdependency between the shell’s parameters and the natural frequencies satisfying the condition of the combinational resonance under investigation. From Figure 2(A) it is seen that the shell parameters \( \beta_2 \) and \( \beta_1 \) under the combinational internal resonance are exponentially interrelated, while from Figures 2(B) and 2(C) it is evident that magnitudes of \( \beta_2 \) are linearly related to every frequency of combinational internal resonance in the case of \( \Omega_4 = \Omega_3 + \Omega_2 \), and \( \beta_1 \) are exponentially related to them.
Figure 1 Shell’s parameters under the condition of the combinational internal resonance ($\Omega_1=\Omega_2+\Omega_3$)

Figure 2 Interdependency between parameters $\beta_1$, $\beta_2$ and frequencies $\Omega_1$, $\Omega_2$, $\Omega_3$: (A) $\beta_1$ - $\beta_2$ dependence at $m_1=n_1=m_2=n_2=4$, $n_3=2$, (B) $\beta_2$-dependence of frequencies - $\cdot$ $\Omega_1$, - $\circ$ $\Omega_2$, - $\square$ $\Omega_3$, (C) $\beta_1$-dependence of frequencies

The numerical solution of (16) using the multi-step method has been carried out for the dimensionless parameters given in figure 1, and the results are shown in figure 3 which represents the time-dependence estimation of the generalized displacements $X_1$, $X_2$, and $X_3$ during forced vibrations at the fractional parameter $\gamma=0.25$. 
The solution of equations (17) has been carried out numerically using the Runge-Kutta forth-order algorithm, and the results are shown for forced and free vibrations in Figures 4 and 5, respectively. The magnitudes of the fractional parameter $\gamma$ are shown by figures near the corresponding curves. Figures 4 and 5 show the extensive energy exchange between the coupled modes of vibrations, in so doing the external force increases the amplitudes of vibrations. With the increase of the fractional parameter from zero (undamped vibrations) to unit (conventional Kelvin-Voigt model) attenuation of vibrations increases.

**Figure 3** Time-dependence of the generalized displacements $X_1$, $X_2$, and $X_3$ calculated via the Diethelm’s multi-term method

**Figure 4**: Amplitudes vs $T_1$: forced vibrations analyzed by the Rossikhin-Shitikova generalized method of multiple-time scales
Figure 5 Amplitudes vs $T_1$: free vibrations analyzed by calculated
by the Rossikhin-Shitikova generalized method of multiple-time
scales

Conclusion
In the present paper, nonlinear force driven vibrations of thin cylindrical shell in a viscoelastic
medium have been studied, when the motion of the cylindrical shell is described by a set of three
coupled nonlinear differential equations subjected to the conditions of the combinational internal
resonance, resulting in the interaction three orthogonal modes, corresponding to the mutually
orthogonal displacements, using two different numerical methods. To describe the nonlinear damped
vibrations of the thin shell under consideration, the fractional derivative Kelvin-Voigt model is used,
because its prediction is in a good compliance with experimental data. The nonlinear set of resolving
equations has been obtained in terms of the generalized displacements and in terms of the amplitudes
and phases. The two systems have been solved numerically by two different methods.

Within the first method, the generalized displacements of a coupled set of nonlinear ordinary
differential equations are estimated using numerical solution of nonlinear multi-term fractional
differential equations by the procedure based on the reduction of the problem to a system of fractional
differential equations. According to the second method, the amplitudes and phases of nonlinear
vibrations are estimated from the governing nonlinear differential equations describing amplitude-and-
phase modulations for the case of the combinational internal resonance. A good agreement between
the results obtained by the two methods has been found.

Relationships (16) and (17) allow one to trace the external force influence on vibrational process,
resulting in the energy exchange between the coupled modes. The interrelation between the parameters
$\beta_2$ and $\beta_1$, which define the geometrical parameters of the shell, and the shell’s natural frequencies $\Omega_1$, $\Omega_2$, and $\Omega_3$, which could fall within the condition of the internal combinational resonance $\Omega_1 = \Omega_2 + \Omega_3$, has been revealed in the form of linear and exponential relations. It has been shown that with the increase of the fractional parameter from zero (undamped vibrations) to unit (conventional Kelvin-
Voigt model) attenuation of vibrations increases, and the external force influence becomes less
effective as the fractional parameter $\gamma$ approaches to 1.

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