Bilevel Aggregator-Prosumers’ Optimization Problem in Real-Time: A Convex Optimization Approach

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ABSTRACT
This paper proposes a Real-Time Market (RTM) platform for an aggregator and its corresponding prosumers to participate in the electricity wholesale market. The proposed energy market platform is modeled as a bilevel optimization problem where the aggregator and the prosumers are considered as self-interest agents. The current state-of-the-art Mathematical Programming with Equilibrium Constraints (MPEC) and Mixed-Integer Programming (MIP) based approaches to solve bilevel optimization problems are not satisfactory for real-time applications. The computation time for such approaches grows exponentially as the number of prosumers and decision variables increase. This paper presents a convex optimization problem which can capture a subset of the set of global optima of the bilevel problem as its optimal solution.

Keywords Bilevel optimization · Convex optimization · Real-time electricity market · Computational efficiency

1 Introduction

Power systems are experiencing a fundamental transition. Previously, the energy was generated in the bulk power plants and it was flowing through transmission and distribution networks to the consumers. The massive installation of Renewable Energy Sources (RESs) at the household level has challenged this structure. Therefore, new schemes and models are needed to efficiently cope with this transition [Bollen and Hassan, 2011].

The emergence of the energy producing consumers, i.e., prosumers and recent Information and Communications Technology (ICT) developments in the paradigm of smart grid [Gungor et al., 2011] have opened up new horizons for less grid-dependent households. Since output generation of RESs is volatile due to their intrinsic environmental dependency, researchers have proposed different approaches to address demand and supply matching for a group of prosumers. Utilization of storage devices [Roberts and Sandberg, 2011], bilateral energy transactions between prosumers [Bedoya et al., 2019], and bilateral energy transaction between prosumers and the wholesale market [Zugno et al., 2013] are among the most prominent of those approaches.
Bilevel Aggregator-Prosumers’ Optimization Problem: A Convex Approach

Here in this paper we focus on a real-time grid-prosumers energy transaction through an aggregator as the mechanism to address demand and supply matching. The aggregator’s role is to gather and manage a group of prosumers in order to participate in the real-time wholesale market.

Many types of aggregator with dissimilar goals have been studied in different financial and market structures in the area of electricity markets [Martin-Martínez et al., 2016]. In this work, the aggregator is a self-interest market participant who has the goal of participating in the real-time wholesale market in order to maximize its revenue. To do so, the aggregator considers each individual prosumer demand and supply situation and proposes a personalized price to buy its excess supply or provide the prosumer its energy deficiency at each time-step in a Real-Time Market (RTM).

On the other hand, each prosumer receives a price from the aggregator and responds optimally by considering its demand preferences and supply situations over a horizon. We assume that the aggregator can anticipate the reaction of the prosumers. This price oriented setup falls into the category of bilevel optimization problems [Colson et al., 2007] and Stackelberg games [Von Stackelberg, 2010], where the lower level problem and the upper level problem are the problems related to the prosumers and the aggregator, respectively.

Bilevel optimization problems have extensively used to model and solve energy systems problems [Dempe et al., 2015]. The initial work [Hobbs et al., 2000] models strategic offering of a dominant generating firm as a bilevel optimization problem, where at the upper level a generator firm maximizes its profit and at the lower level a system operator maximizes social welfare or minimizes total system cost. This problem is rewritten as a Mathematical Programming with Equilibrium Constraints (MPEC) and solved by a penalty interior point algorithm. More recent works (e.g., Zugno et al. 2013) focus on the aggregator and prosumers problem. The state-of-the-art approach to solve these types of problems is to reformulate the bilevel optimization problem as a Mixed-Integer Programming (MIP).

Both the MPEC and MIP based methods are computationally expensive. One of the main challenges to implement an RTM is the computational efficiency. For an RTM the time intervals are in the order of a few minutes [Vlachos and Biskas, 2013]. Therefore, new computational tools are needed for the aggregator’s real-time control over the prosumers and its participation in the RTM. Ghamkhari et al. 2016 has addressed the computational efficiency of the dominant firm’s strategic offering by introducing a convex relaxation for the bilevel optimization problem and has found a close to optimal solution. However, to the best of our knowledge, no study has been done on finding the global optimum of a bilevel optimization problem by solving a convex one in the field of prosumers integration in the wholesale energy markets.

In this paper, we define the problem of economic optimization of an aggregator and its corresponding prosumers for participation in an RTM over a time horizon as a bilevel optimization problem. The aggregator represents the prosumers to participate in the wholesale market in a real-time scenario. This problem, in general, is nonconvex [Luo et al., 1996]. We show that a subset of the set of global minimizers for the nonconvex problem can be obtained as the solution of a certain convex optimization problem. The convex problem has two main advantages. On the one hand, a convex formulation is attractive in real-time applications since the computation time is linear in the number of variables. On the other hand, off-the-shelf software packages can be used to solve the problem. In addition, replacing a bilevel optimization problem by a convex one is a key step toward decentralized or distributed algorithms [Bertsekas, 1999]. This work is a continuation of the preliminary study by the authors [Shomalzadeh et al., 2020] which dealt with a simple static model for balancing markets.

The paper is organized as follows. In Section 2, we define the aggregator and prosumers problems as a bilevel optimization problem. The results toward introducing a convex optimization problem for the bilevel one comes in Section 3. Finally, the paper closes with the conclusions in Section 4.

Notation

We denote the set of real numbers by \( \mathbb{R} \), \( n \)-vectors by \( \mathbb{R}^n \) and \( m \times n \) matrices by \( \mathbb{R}^{m \times n} \). Throughout the paper, the inequalities for vectors are meant entrywise. The \( n \)-vectors of ones is denoted by \( \mathbf{1}_n \). For vectors \( x_i \in \mathbb{R}^n \) with \( i = 1, 2, \ldots, k \), we write \( \text{col}(x_1, x_2, \ldots, x_k) \) to denote the vector \( [x_1^T \ x_2^T \ \cdots \ x_k^T]^T \). The \( m \times m \) identity matrix is denoted by \( \mathbf{I}_m \). For a matrix \( M \in \mathbb{R}^{m \times n} \) and index sets \( \alpha \subseteq \{1, 2, \ldots, m\} \), \( \beta \subseteq \{1, 2, \ldots, n\} \), the notation \( M_{\alpha \beta} \) denotes the matrix \( (M_{ij})_{i \in \alpha, j \in \beta} \). If \( \alpha = \{1, 2, \ldots, m\} \), then we write \( M_{\bullet \beta} \) and if \( \beta = \{1, 2, \ldots, n\} \), then we write \( M_{\alpha \bullet} \). A symmetric matrix \( M = M^T \in \mathbb{R}^{m \times m} \) is said to be positive semidefinite if \( x^T M x \geq 0 \) for all \( x \in \mathbb{R}^m \) and positive definite if \( x^T M x > 0 \) for all \( 0 \neq x \in \mathbb{R}^m \). The symmetric square root of a positive definite matrix \( M \) is denoted by \( M^{\frac{1}{2}} \). For a vector \( v \in \mathbb{R}^n \), we write \( \text{diag}(v) \) for the diagonal matrix with diagonal entries \( v_1, v_2, \ldots, v_n \).
Let \( f : \mathbb{R}^n \to \mathbb{R} \) and \( S \subseteq \mathbb{R}^n \). Consider the optimization problem

\[
\text{OP} : \quad \min_{x} f(x) \quad \text{(1a)}
\]
\[
\text{subject to } \quad x \in S. \quad \text{(1b)}
\]

We say that \( \bar{x} \) is feasible for \( \text{OP} \) if \( \bar{x} \in S \). Also, we define the set of global optima for \( \text{OP} \) as

\[
\text{MIN}(\text{OP}) = \{ x^* \in S \mid f(x^*) \leq f(x) \forall x \in S \}. \quad \text{(2)}
\]

2 Problem Statement

In this section, we define a market model and platform for an aggregator and the prosumers under its contract to participate in an RTM with the grid, i.e., the wholesale market. Here, the role of the aggregator is to act as an intermediary agent between the prosumers and the grid to facilitate the energy transactions. We consider the case where each prosumer can generate energy through some RESs with zero cost. Example of such energy sources are solar panels and wind turbines. Moreover, each prosumer’s demand is elastic at each time-step. The aggregator goal is to propose the prosumers with a personalized price to deal their surplus or shortage energy with the grid in an optimal way. The advantages of a personalized price over a unique price have been addressed in many recent research (see e.g., Tushar et al. 2014, Yang et al. 2018). Next, we explain the problem setting and market structure in detail.

2.1 Prosumer’s Problem

The main source of energy supply for a prosumer is its renewable energy units. Due to uncertain and uncontrollable nature of RESs, there might be a mismatch between supply and demand at each time-step. Each prosumer has two options to cancel this mismatch. One is to trade with the wholesale market through the aggregator. The other option is to use its demand elasticity. Therefore, the prosumer needs to find a trade-off between these two possible options for its optimal strategy. Before providing a mathematical formulation for the prosumer, we elaborate on demand elasticity.

We say that the demand of each prosumer is elastic if:

1. Each prosumer has a preference for its demand at each time-step.
2. Altering the demand from its preferred value causes dissatisfaction for the prosumer. Here, we model this dissatisfaction using a quadratic function.
3. Each prosumer has a lower bound and an upper bound for its demand at each time-step.
4. Total demand of each prosumer in a specific time period is constant.

As explained before, the prosumer goal is to find a trade-off between two possible options to minimize its cost and maximize its comfort. We model this problem as an optimization problem. We define the set of prosumers by \( \{1, 2, \ldots, n\} \) and the set of time-steps by \( \{1, 2, \ldots, K\} \). Then, prosumer \( i \in \{1, 2, \ldots, n\} \) at time-step \( k \in \{1, 2, \ldots, K\} \) has three decision variables: its demand \( h_i(k) \), the energy it sells to (buy form) the grid \( y^+_i(k) \) \((y^-_i(k))\). For the \( i \)th prosumer, we consider the following optimization problem:

\[
\text{PP}_i : \quad \min_{h_i(k), y^+_i(k), y^-_i(k)} \quad \sum_{k=1}^{K} \frac{1}{2} q_i(k)(h_i(k) - h_i^0(k))^2 + x^+_i(k) y^+_i(k) - x^-_i(k) y^-_i(k) \quad \text{(3a)}
\]
\[
\text{subject to } \quad y^+_i(k) - y^-_i(k) + h_i(k) = s_i(k) \quad \forall k \in \{1, \ldots, K\} \quad \text{(3b)}
\]
\[
y^+_i(k), y^-_i(k) \geq 0 \quad \forall k \in \{1, \ldots, K\} \quad \text{(3c)}
\]
\[
h_i(k) \leq h_i(k) \leq \bar{h}_i(k) \quad \forall k \in \{1, \ldots, K\} \quad \text{(3d)}
\]
\[
\sum_{k=1}^{K} h_i(k) = h^\text{tot}_i \quad \text{(3e)}
\]

where \( x^+_i(k), (x^-_i(k)) \) is the proposed price by the aggregator to buy energy from (sell energy to) the prosumer at time-step \( k \). \( s_i(k) \geq 0 \) is the generated energy by the prosumer at time-step \( k \), which assumed to be known, and \( q_i(k) > 0 \) is the dissatisfaction parameter for the prosumer. Moreover, \( h^0_i(k) \geq 0, h_i(k) \geq 0 \) and \( \bar{h}_i(k) \geq 0 \) are the preferred value, lower bound and upper bound for the demand \( h_i(k) \), respectively. The parameter \( h^\text{tot}_i \) is the total demand for the prosumer over the period \( k = 1 \) to \( k = K \).
In (3a), the first term models the dissatisfaction the prosumer experiences by changing its demand from the preferred value. The second term is the cost of buying energy from the grid through the aggregator and the third term is the revenue the prosumer can obtain by selling energy through the aggregator. The constraint (3b) indicates that the total demand should be equal to the total supply for each prosumer at each time-step. The constraints (3c) and (3d) specify the lower bound and upper bound for the decision variables. Finally, (3e) captures the assumption that the total demand over a period is constant.

**Assumption 1.** The sum of preferred values \( h^0_i(k) \) over the period \( k = 1 \) to \( k = K \) is equal to \( h^\text{tot}_i \), i.e.,

\[
\sum_{k=1}^{K} h^0_i(k) = h^\text{tot}_i. \tag{4}
\]

Form (3b), we can write \( h_i(k) \) as

\[
h_i(k) = s_i(k) - (y_i^+(k) - y_i^-(k)). \tag{5}
\]

Thus, the variable \( h_i(k) \) can be eliminated from the problem \( \text{PP}_i \) and we can rewrite it as the following optimization problem:

\[
\begin{align*}
\text{PP}_i': & \quad \min_{\forall k \in \{1, 2, \ldots, K\}} \sum_{k=1}^{K} \frac{1}{2} q_i(k) (y_i^+(k) - y_i^-(k))^2 + c_i(k) (y_i^+(k) - y_i^-(k)) \\
& \quad + x_i^-(k)y_i^-(k) - x_i^+(k)y_i^+(k) + \frac{1}{2} q_i(k)(h^0_i(k))^2 \\
& \text{subject to} \quad y_i^+(k), y_i^-(k) \geq 0 \quad \forall k \in \{1, \ldots, K\} \tag{6a} \\
& \quad s_i(k) - \bar{h}_i(k) \leq y_i^+(k) - y_i^-(k) \leq s_i(k) - h_i(k) \quad \forall k \in \{1, \ldots, K\} \tag{6b} \\
& \quad - \sum_{k=1}^{K} (y_i^+(k) - y_i^-(k)) = h^\text{tot}_i - \sum_{k=1}^{K} s_i(k) \tag{6c}
\end{align*}
\]

where \( c_i(k) = q_i(k)(h^0_i(k) - s_i(k)) \). Moreover, since the optimization problems \( \text{PP}_i' \)'s are independent, we can add them and rewrite them in a vector form. To do so, we define the following vectors:

\[
\begin{align*}
q &= \text{col}(q_1(1), q_1(2), \ldots, q_n(K)), \quad c = \text{col}(c_1(1), c_1(2), \ldots, c_n(K)), \\
h^0 &= \text{col}(h^0_1(1), h^0_1(2), \ldots, h^0_n(K)), \quad \bar{h} = \text{col}(\bar{h}_1(1), \bar{h}_1(2), \ldots, \bar{h}_n(K)), \\
\bar{h} &= \text{col}(\bar{h}_1(1), \bar{h}_1(2), \ldots, \bar{h}_n(K)), \quad s = \text{col}(s_1(1), s_1(2), \ldots, s_n(K)), \\
y^+ &= \text{col}(y^+_1(1), y^+_1(2), \ldots, y^+_n(K)), \quad y^- = \text{col}(y^-_1(1), y^-_1(2), \ldots, y^-_n(K)), \\
x^+ &= \text{col}(x^+_1(1), x^+_1(2), \ldots, x^+_n(K)), \quad x^- = \text{col}(x^-_1(1), x^-_1(2), \ldots, x^-_n(K)), \\
h^\text{tot} &= \text{col}(h^\text{tot}_1, h^\text{tot}_2, \ldots, h^\text{tot}_n).
\end{align*}
\]

Then, the vector form can be written as

\[
\begin{align*}
\text{PP} : & \quad \min_{y^+, y^-} \frac{1}{2} (y^+ - y^-)^T Q (y^+ - y^-) + c^T (y^+ - y^-) \tag{8a} \\
& \quad + (x^-)^T y^- - (x^+)^T y^+ \\
& \text{subject to} \quad y^+, y^- \geq 0 \tag{8b} \\
& \quad \ell \leq y^+ - y^- \leq u \tag{8c} \\
& \quad E(y^+ - y^-) = d \tag{8d}
\end{align*}
\]

where we have the following parameters:

\[
\begin{align*}
Q &= \text{diag}(q), \quad c = Q(h^0 - s), \quad \ell = s - \bar{h}, \\
u &= s - \bar{h}, \quad E = -I_n \otimes 1^T_K, \quad d = h^\text{tot} - Es. \tag{9}
\end{align*}
\]

Note that \( \otimes \) denotes the Kronecker product.

The prices \( x^+_i(k) \) and \( x^-_i(k) \) are proposed by the aggregator. In this work, the aggregator acts as a self-interest agent which has the ability to anticipate the reaction of the prosumers. Therefore, knowing the reaction of the prosumers, the aggregator sets the prices to maximize its revenue as an intermediary player between the grid and the prosumers. In the next subsection, we elaborate on the aggregator’s problem as a bilevel optimization problem.
2.2 Aggregator’s Problem

The aggregator receives two prices from the grid for each time-step. The price \( p^+(k) \) is the price for selling energy to grid and the price \( p^-(k) \) is the price for buying energy from the grid at \( k \)-th time-step. Having these prices and the ability of the aggregator to anticipate the reaction of the prosumers allow the aggregator to propose prices \( x^+(k) \) and \( x^-(k) \) to the prosumers in an optimal way. The bilevel optimization below models this problem for the aggregator.

\[
\text{AP} : \quad \max_{x^+, x^-, y^+, y^-} \quad (p^+-x^+)^T y^+ - (p^- - x^-)^T y^- \tag{11a}
\]

subject to

\[
\begin{align*}
x^+, x^- & \geq 0 \quad \text{(11b)} \\
(y^+, y^-) & \in \text{MIN(PP)} \quad \text{(11c)}
\end{align*}
\]

where \( p^+ = \mathbb{1}_n \otimes \text{col}(p^+(1), p^+(2), \ldots, p^+(K)) \) and \( p^- = \mathbb{1}_n \otimes \text{col}(p^-(1), p^-(2), \ldots, p^-(K)) \). The first term in (11a) corresponds to aggregator’s revenue from selling energy to the grid. The second term models the aggregator’s cost for buying energy from the grid.

In this paper, we consider a scenario where \( p^+ = -p^- = p \) and the aggregator proposes prices \( x^+ \) and \( x^- \) such that \( x^+ = -x^- = x \). Therefore, we can rewrite the optimization problems AP and PP based on the new decision variables \( x \) and \( y = y^+ - y^- \) as the minimization problems BLP and LLP, respectively.

\[
\text{BLP} : \quad \min_{x, y} \quad (x - p)^T y \tag{12a}
\]

subject to

\[
\begin{align*}
x & \geq 0 \quad \text{(12b)} \\
y & = \text{MIN( LLP)} \quad \text{(12c)}
\end{align*}
\]

Here the decision vector \( x \in \mathbb{R}^m \) is the proposed prices of the aggregator and the parameter vector \( p \in \mathbb{R}^m \) is the prices of selling to and buying from the grid. The prosumers’ reactions \( y \) to the proposed prices are the solution of the optimization problem LLP.

\[
\text{LLP} : \quad \min_y \quad \frac{1}{2} y^T Q y + (c - x)^T y \tag{13a}
\]

subject to

\[
\begin{align*}
\ell \leq y & \leq u \quad \text{(13b)} \\
E y & = d \quad \text{(13c)}
\end{align*}
\]

The vector \( y \in \mathbb{R}^m \) is the decision variable for LLP. The vectors and matrices \( c, \ell, u \in \mathbb{R}^m, d \in \mathbb{R}^n, Q \in \mathbb{R}^{m \times m}, E \in \mathbb{R}^{n \times m} \) are parameters for LLP as defined in (9) and (10). Moreover, \( x \in \mathbb{R}^m \) is the decision variable for the aggregator and the prosumers has no control over it. It should be noted that \( m = nK \) and \( \text{rank } E = n \leq m \). We assume that there exists \( y \) which satisfies (13b)-(13c). Since \( Q = \text{diag}(q) \) is positive definite, LLP is a strictly convex quadratic optimization problem and hence has always a unique optimal solution, i.e., the set MIN(LLP) is a singleton.

Bilevel optimization problems are in general nonconvex and have combinatorial nature. Many algorithms and approaches have been developed to solve different classes of bilevel problems. Recent surveys on bilevel optimization can be found in [Dempe and Zemkoho, 2020] and [Luo et al., 1996]. In contrast to existing methods that deal with rather more general bilevel optimization problems, our focus here is to exploit the particular structure of (12) in order to introduce a convex optimization problem which has the same global optimum as the bilevel one. The next section investigates the conditions under which the global optimal solution of the optimization (12) can be found by solving a convex problem.

3 Main Results

In this section, we will show that the set of global optima for a specific convex optimization problem is a subset of the set of global optima for the optimization BLP, under some assumptions on the parameters of the problem. Before dealing with the the optimization problem BLP, we consider two variations of this optimization problem. First, we only consider a lower bound on \( y \) in (13b). Later, we will consider an upper bound on \( y \). Then, we come back to BLP (12) to introduce a convex optimization problem which can be used to find a subset of MIN(BLP). Finally, we comment on the restrictions of the proposed convex optimization for the RTM platform.
3.1 Lower Bound on \( y \)

Consider the following bilevel optimization problem for which the decision variable \( y \) has only a lower bound:

\[
\text{BLP1 : } \min_{x,y} \phi(x, y) \tag{14a}
\]

subject to

\[
x \geq 0 \tag{14b}
\]

\[
y = \text{MIN}(\text{LLP1}) \tag{14c}
\]

where \( \phi : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R} \) is given by

\[
\phi(x, y) = (x - p)^T y, \tag{15}
\]

and LLP1 is as

\[
\text{LLP1 : } \min_y \frac{1}{2} y^T Ry + (c - x)^T y \tag{16a}
\]

subject to

\[
\ell \leq y \tag{16b}
\]

\[
Fy = d. \tag{16c}
\]

Here \( R \in \mathbb{R}^{m \times m} \) is positive definite and not necessarily diagonal and \( F \in \mathbb{R}^{n \times m} \) has full row rank. Assume that there exists \( \bar{y} \) satisfying (16b)-(16c) Since LLP1 is a convex optimization problem, we can write the following necessary and sufficient KKT conditions to characterize (14c):

\[
Ry + c - x + FT \lambda - \mu = 0, \tag{17}
\]

\[
Fy = d, \tag{18}
\]

\[
0 \leq \mu \perp y - \ell \geq 0 \tag{19}
\]

where \( \mu \in \mathbb{R}^m \) and \( \lambda \in \mathbb{R}^n \) are dual variables for the constraints (16b) and (16c), respectively. The dual variable \( \lambda \) can be eliminated from KKT conditions (17)-(19). First, we solve \( y \) from (17) as

\[
y = R^{-1}(x + \mu - c - FT \lambda), \tag{20}
\]

and then substitute \( y \) in (18):

\[
FR^{-1}(x + \mu - c - FT \lambda) = d. \tag{21}
\]

By substituting \( \lambda \) in (20), we can write (17)-(19) as

\[
y = M(x + \mu) + r, \tag{23}
\]

\[
0 \leq \mu \perp y - \ell \geq 0 \tag{24}
\]

where

\[
M = R^{-1} - R^{-1} FT (FR^{-1} FT)^{-1} FR^{-1}, \tag{25}
\]

\[
r = R^{-1} FT (FR^{-1} FT)^{-1} d - Mc. \tag{26}
\]

**Lemma 1.** The matrix \( M \) is a positive semidefinite matrix.

**Proof.** Clearly \( M \) is the Schur complement of

\[
X = \begin{bmatrix} R^{-1} - R^{-1} FT (FR^{-1} FT)^{-1} FR^{-1} & R^{-1} FT \\ FR^{-1} & FR^{-1} FT \end{bmatrix} \tag{27}
\]

with respect to \( FR^{-1} FT \). Since \( X = \begin{bmatrix} R^{-\frac{1}{2}} & \cdot \\ FR^{-\frac{1}{2}} & \cdot \end{bmatrix} \begin{bmatrix} R^{-\frac{1}{2}} & \cdot \\ FR^{-\frac{1}{2}} & \cdot \end{bmatrix} \), \( X \) is positive semidefinite. It follows from [Zhang, 2006, Theorem 1.12] that \( M \) is also positive semidefinite. \( \square \)

As a result of (23) and (24), BLP1 can be rewritten as the following optimization problem:

\[
\text{SLP1 : } \min_{x,y} \phi(x, y) \tag{28a}
\]

subject to \( (x, y) \in S_1 \) \( \tag{28b} \)

where

\[
S_1 = \{(x, y) \mid x \geq 0, y = M(x + \mu) + r, y \geq \ell, \mu^T (y - \ell) = 0 \text{ for some } \mu \geq 0\}. \tag{29}
\]

Since neither \( \phi \) nor \( S_1 \) is convex, the optimization problem SLP1 is a nonconvex one. Nevertheless, a subset of the global minimizers of SLP1 can be captured by a convex optimization problem.
Theorem 1. Suppose that $\ell \leq 0$. Consider the optimization problem

$$\text{CVX1} : \begin{array}{ll}
\min_{x,y} & \phi(x,y) \\
\text{subject to} & (x,y) \in C_1
\end{array}$$

where

$$C_1 = \{(x,y) \mid x \geq 0, y = Mx + r, y \geq \ell\}.$$  \hfill (31)

Then, $\text{MIN(CVX1)} \subseteq \text{MIN(SLP1)}$. Furthermore, the optimization problem $\text{CVX1}$ is convex.

To prove the theorem above, we need some auxiliary results. The following lemma plays an essential role in the proof of Theorem 1.

Lemma 2. Consider the sets $S'_1 = \{(x,y) \mid x \geq 0, y = M(x + \mu) + r', y \geq \ell, \mu^T(y - \ell) = 0 \text{ for some } \mu \geq 0\}$ and $C'_1 = \{(x,y) \mid x \geq 0, y = Mx + r', y \geq \ell\}$ where $r' \in \mathbb{R}^m$ is an arbitrary vector. Suppose that $\ell \leq 0$. Then, for any $(\bar{x}, \bar{y}) \in S'_1$ there exists $(\hat{x}, \hat{y}) \in C'_1$ such that $\phi(\hat{x}, \hat{y}) \leq \phi(\bar{x}, \bar{y})$.

Proof. Let $(\bar{x}, \bar{y}) \in S'_1$. Therefore, there exists $\bar{\mu} \geq 0$ such that

$$\bar{x} \geq 0, \quad \bar{y} = M(\bar{x} + \bar{\mu}) + r', \quad \bar{y} \geq \ell, \quad \bar{\mu}^T(\bar{y} - \ell) = 0.$$  \hfill (32)

Take $\hat{x} = \bar{x} + \bar{\mu}$. Then, $\hat{x} \geq 0, M\hat{x} + r' = \bar{y} \geq \ell$ and thus $(\hat{x}, \bar{y}) \in C'_1$.

Now, we define index sets $\alpha \subseteq \{1, 2, \ldots, m\}$ and $\tilde{\alpha} = \{1, 2, \ldots, m\} \setminus \alpha$ such that $\bar{\mu}_\alpha = 0$ and $\bar{\mu}_{\tilde{\alpha}} > 0$. Then, based on (32), we have

$$\hat{x}_\alpha = \bar{x}_\alpha, \quad \hat{x}_{\tilde{\alpha}} > \bar{x}_{\tilde{\alpha}}, \quad \bar{y}_\alpha = \ell_\alpha, \quad \bar{y}_{\tilde{\alpha}} = \ell_{\tilde{\alpha}}.$$  \hfill (33)

Since $\ell \leq 0$, we have

$$\hat{x}_\alpha = \bar{x}_\alpha \Rightarrow (\hat{x}_\alpha - p_\alpha)^T\bar{y}_\alpha = (\bar{x}_\alpha - p_\alpha)^T\bar{y}_\alpha,$$

$$\hat{x}_{\tilde{\alpha}} > \bar{x}_{\tilde{\alpha}} \quad \text{and} \quad \bar{y}_\alpha = \ell_\alpha \leq 0 \Rightarrow (\bar{x}_\alpha - p_\alpha)^T\bar{y}_\alpha \leq (\bar{x}_{\tilde{\alpha}} - p_{\tilde{\alpha}})^T\bar{y}_{\tilde{\alpha}},$$  \hfill (35), (36)

and consequently $\phi(\hat{x}, \bar{y}) \leq \phi(\bar{x}, \bar{y})$. \hfill \Box

Now, we are ready to prove Theorem 1.

Proof of Theorem 1. Let $(x^*, y^*) \in \text{MIN(CVX1)}$ and $(\bar{x}, \bar{y}) \in S_1$. It follows from Lemma 2, with the choices $r' = r$, $S'_1 = S_1$ and $C'_1 = C_1$, that there exists $(\hat{x}, \bar{y}) \in C_1$ such that

$$\phi(\hat{x}, \bar{y}) \leq \phi(\bar{x}, \bar{y}).$$  \hfill (37)

Therefore, we see that

$$\phi(x^*, y^*) \leq \phi(\hat{x}, \bar{y}) \leq \phi(\bar{x}, \bar{y}).$$  \hfill (38)

This means that $\phi(x^*, y^*) \leq \phi(\hat{x}, \bar{y})$ for all $(\bar{x}, \bar{y}) \in S_1$. Since $C_1 \subseteq S_1$, $(x^*, y^*) \in S_1$ and hence $(x^*, y^*) \in \text{MIN(SLP1)}$. Thus, we can conclude that $\text{MIN(CVX1)} \subseteq \text{MIN(SLP1)}$. Note that $C_1$ is a polyhedron. Furthermore, $\phi$ is convex since $M$ is positive semidefinite due to Lemma 1. Therefore, CVX1 is a convex optimization problem. \hfill \Box

### 3.2 Upper Bound on $y$

Next, we consider the following bilevel problem for which the decision variable $y$ has only an upper bound:

$$\text{BLP2} : \begin{array}{ll}
\min_{x,y} & \phi(x,y) \\
\text{subject to} & x \geq 0 \\
& y = \text{MIN(LLP2)}
\end{array}$$

where LLP2 is given by

$$\text{LLP2} : \begin{array}{ll}
\min_y & \frac{1}{2}y^T R y + (c - x)^T y \\
\text{subject to} & y \leq u \\
& F y = d
\end{array}$$

\hfill (40a), (40b), (40c)
Assume that there exists $\bar{y}$ which satisfies (40b)-(40c). In a procedure similar to what we had for BLP1, we can show that BLP2 can be written as the following optimization problem using the KKT conditions:

$$\text{SLP2} : \quad \min_{x,y} \phi(x,y)$$
subject to \( (x,y) \in S_2 \)

where

$$S_2 = \{ (x,y) \mid x \geq 0, \ y = M(x - \nu) + r, \ y \leq u, \ \nu^T (u - y) = 0 \text{ for some } \nu \geq 0 \},$$

and $M$ and $r$ are as in (25) and (26). Note that $\nu$ is the dual variable for the constraint (40b).

Again here we want to find sufficient conditions such that at least a global optimum of SLP2 can be found using a convex optimization problem. The main assumption here is on the structure of $M$. The following definitions elaborate on this specific structure.

**Definition 1.** A matrix $N \in \mathbb{R}^{k \times k}$ is called

- a Z-matrix if its off-diagonal entries are nonpositive.
- an M-matrix if it is an Z-matrix and the real part of its eigenvalues are nonnegative.

**Remark 1.** In particular, a positive semidefinite matrix is an M-matrix if its off-diagonal entries are nonpositive.

Now, we can state the main result concerning the optimization problem SLP2.

**Theorem 2.** Suppose that $M$ is an M-matrix, $u \geq 0$ and $u > r$. Consider the following optimization problem:

$$\text{CVX2} : \quad \min_{x,y} \phi(x,y)$$
subject to \( (x,y) \in C_2 \)

where

$$C_2 = \{ (x,y) \mid x \geq 0, \ y = Mx + r, \ y \leq u \}.$$

Then, $\text{MIN(CVX2)} \subseteq \text{MIN(SLP2)}$. Furthermore, the optimization problem CVX2 is convex.

The following results are needed to prove Theorem 2.

**Lemma 3.** Consider the following optimization problem:

$$\text{SLP2}' : \quad \min_{x,y} \phi(x,y)$$
subject to \( (x,y) \in S'_2 \)

where

$$S'_2 = \{ (x,y) \mid x \geq 0, \ y = M(x - \nu) + r, \ y \leq u, \ \nu^T (u - y) = 0 \text{ for some } \nu \geq 0 \}.$$

Suppose that $u \geq 0$. Then, for any $(\bar{x}, \bar{y}) \in S_2$ there exists $(\hat{x}, \hat{y}) \in S'_2$ such that $\phi(\hat{x}, \hat{y}) \leq \phi(\bar{x}, \bar{y})$.

**Proof.** Let $(\bar{x}, \bar{y}) \in S_2$. Therefore, there exists $\nu \geq 0$ such that

$$\bar{x} \geq 0, \quad \bar{y} = M(\bar{x} - \nu) + r \leq u, \quad \nu^T (u - \bar{y}) = 0. \tag{47}$$

We define index sets $\alpha \subseteq \{1, 2, \ldots, m\}$ and $\bar{\alpha} = \{1, 2, \ldots, m\} \setminus \alpha$ such that $\nu_\alpha = 0$ and $\nu_{\bar{\alpha}} > 0$. Then, (47) can be rewritten based on $\alpha$ and $\bar{\alpha}$ as

$$\bar{x}_\alpha \geq 0, \quad \bar{y}_\alpha = M_{\alpha \alpha} \bar{x}_\alpha + M_{\bar{\alpha} \alpha} (\bar{x}_{\bar{\alpha}} - \bar{y}_{\bar{\alpha}}) + r_\alpha \leq u_\alpha, \quad \nu_\alpha = 0, \tag{48}$$

$$\bar{y}_{\bar{\alpha}} = M_{\alpha \bar{\alpha}} \bar{x}_\alpha + M_{\bar{\alpha} \bar{\alpha}} (\bar{x}_{\bar{\alpha}} - \bar{y}_{\bar{\alpha}}) + r_{\bar{\alpha}} = u_{\bar{\alpha}}, \quad \nu_{\bar{\alpha}} > 0. \tag{49}$$

We can choose $\hat{x}, \hat{\nu}$ based on $\bar{x}, \bar{\nu}$ as

$$\hat{x}_\alpha = \bar{x}_\alpha \geq 0, \quad \hat{\nu}_\alpha = \bar{\nu}_\alpha = 0, \tag{51}$$

$$\hat{x}_{\bar{\alpha}} = \begin{bmatrix} \bar{x}_{\bar{\alpha}} - \bar{\nu}_{\bar{\alpha}} \\ 0 \end{bmatrix} \quad \text{if} \quad \bar{x}_{\bar{\alpha}} - \bar{\nu}_{\bar{\alpha}} \geq 0,$$

$$\hat{\nu}_{\bar{\alpha}} = \begin{bmatrix} 0 \\ \bar{\nu}_{\bar{\alpha}} - \bar{x}_{\bar{\alpha}} \end{bmatrix} \quad \text{if} \quad \bar{x}_{\bar{\alpha}} - \bar{\nu}_{\bar{\alpha}} < 0. \tag{52}$$

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We argue that this system has at least one solution. To see this, take
\( \alpha \), we have the following implications:
\[
\hat{x}_\alpha = \bar{x}_\alpha \implies (\hat{x}_\alpha - p_\alpha)^T \bar{y}_\alpha = (\bar{x}_\alpha - p_\alpha)^T \bar{y}_\alpha, \tag{54}
\]
\[
\hat{x}_\alpha < \bar{x}_\alpha \quad \text{and} \quad \bar{y}_\alpha = u_\alpha \geq 0 \implies (\hat{x}_\alpha - p_\alpha)^T \bar{y}_\alpha \leq (\bar{x}_\alpha - p_\alpha)^T \bar{y}_\alpha \tag{55}
\]
which conclude that \( \phi(\hat{x}, \bar{y}) \leq \phi(\bar{x}, \bar{y}) \). \hfill \Box

The set \( S'_2 \) is a nonconvex set due to complementarity terms. In what follows, we will show that under some conditions on \( M, u \) and \( r \), the set \( S'_2 \) is equal to the polyhedral set \( C_2 \) in CVX2.

Let \( (\bar{x}, \bar{y}) \in S'_2 \). Therefore, there exists \( \bar{\nu} \geq 0 \) such that
\[
\bar{x} \geq 0, \quad \bar{y} = M(\bar{x} - \bar{\nu}) + r \leq u, \quad \bar{\nu}^T (u - \bar{y}) = 0, \quad \bar{\nu}^T \bar{x} = 0. \tag{56}
\]
We define index sets \( \alpha \subseteq \{1, 2, \ldots, m\} \) and \( \bar{\alpha} = \{1, 2, \ldots, m\} \setminus \alpha \) such that \( \bar{\nu}_\alpha = 0 \) and \( \bar{\nu}_{\bar{\alpha}} > 0 \). Then, the following implications follow from (56):
\[
\bar{\nu}_\alpha = 0 \implies \bar{x}_\alpha \geq 0 \quad \text{and} \quad \bar{y}_\alpha = M_{\alpha\alpha} \bar{x}_\alpha - M_{\alpha\bar{\alpha}} \bar{\nu}_{\bar{\alpha}} + r_\alpha \leq u_\alpha, \tag{57}
\]
\[
\bar{\nu}_{\bar{\alpha}} > 0 \implies \bar{x}_{\bar{\alpha}} = 0 \quad \text{and} \quad \bar{y}_{\bar{\alpha}} = M_{\bar{\alpha}\alpha} \bar{x}_{\bar{\alpha}} - M_{\bar{\alpha}\bar{\alpha}} \bar{\nu}_{\bar{\alpha}} + r_{\bar{\alpha}} = u_{\bar{\alpha}}. \tag{58}
\]
Consequently, we can write the following system of (in)equalities for \( S'_2 \):
\[
\begin{align*}
\bar{y}_\alpha &= M_{\alpha\alpha} \bar{x}_\alpha - M_{\alpha\bar{\alpha}} \bar{\nu}_{\bar{\alpha}} + r_\alpha \leq u_\alpha, \quad \bar{x}_\alpha \geq 0, \\
\bar{y}_{\bar{\alpha}} &= M_{\bar{\alpha}\alpha} \bar{x}_{\bar{\alpha}} - M_{\bar{\alpha}\bar{\alpha}} \bar{\nu}_{\bar{\alpha}} + r_{\bar{\alpha}} = u_{\bar{\alpha}}, \quad \bar{\nu}_{\bar{\alpha}} > 0.
\end{align*} \tag{59}
\]

Next, we elaborate on some tools to work with linear (in)equalities.

**Lemma 4** (Cottle et al. 2009, Theorem 3.8.3). Let \( N \in \mathbb{R}^{k \times k} \) be a positive semidefinite matrix. Then, for every index set \( \beta \subseteq \{1, 2, \ldots, k\} \), the inequality system
\[
N_{\beta\beta} \xi \leq 0, \quad \xi \leq 0
\]
has a nonzero solution \( \xi \).

**Lemma 5** (Cottle et al. 2009, Theorem 2.7.9). Let \( A \in \mathbb{R}^{k_1 \times k_2} \) and \( b \in \mathbb{R}^{k_1} \) be given. Exactly one of the following statements holds:

i) There exists \( w \geq 0 \) such that \( A w \leq b \).

ii) There exists \( z \geq 0 \) such that \( A^T z \geq 0 \) and \( b^T z < 0 \).

The following lemma provides sufficient conditions for (59) and \( S'_2 \) to be feasible for the index set \( \alpha = \{1, 2, \ldots, m\} \) and infeasible \( \bar{\alpha} \neq \{1, 2, \ldots, m\} \).

**Lemma 6.** Suppose that \( M \) is an M-matrix and \( u > r \). Then, \( S'_2 = C_2 \), i.e., the system (59) is only feasible when \( \nu = 0 \).

**Proof.** Let \( \alpha = \{1, 2, \ldots, m\} \). The alternative system for (59) is as
\[
M_z \geq 0, \quad (u - r)^T z < 0, \quad z \geq 0 \tag{60}
\]
which clearly have no solutions since \( u - r \) is positive. Therefore, it follows from Lemma 5 that (59) has at least one solution.

Now let \( \alpha \neq \{1, 2, \ldots, m\} \) which means \( \bar{\alpha} \) is a nonempty set. The alternative system for (59) in this case is given by
\[
\begin{bmatrix} M_{\alpha\alpha} & M_{\alpha\bar{\alpha}} \\ -M_{\bar{\alpha}\alpha} & -M_{\bar{\alpha}\bar{\alpha}} \end{bmatrix} \begin{bmatrix} z_\alpha \\ z_{\bar{\alpha}} \end{bmatrix} \geq 0, \quad \begin{bmatrix} u_\alpha - r_\alpha \\ u_{\bar{\alpha}} - r_{\bar{\alpha}} \end{bmatrix}^T \begin{bmatrix} z_\alpha \\ z_{\bar{\alpha}} \end{bmatrix} < 0, \quad z_\alpha \geq 0. \tag{61}
\]
We argue that this system has at least one solution. To see this, take \( z_\alpha = 0 \). This leads to
\[
\begin{bmatrix} M_{\alpha\alpha} \\ -M_{\bar{\alpha}\bar{\alpha}} \end{bmatrix} z_{\bar{\alpha}} \geq 0, \quad (u_{\bar{\alpha}} - r_{\bar{\alpha}})^T z_{\bar{\alpha}} < 0. \tag{62}
\]
Since \( u > r \) and hence \( u \alpha > r \alpha \), (62) has a solution if and only if \( z_\alpha \leq 0 \) and \( z_\alpha \neq 0 \). Consequently, (62) has a solution if and only if the following has a solution:

\[
\begin{bmatrix}
M_{a\alpha}

-M_{a\alpha}
\end{bmatrix}
\begin{bmatrix}
z_\alpha

z_\alpha
\end{bmatrix}
\geq 0,
\begin{bmatrix}
z_\alpha

z_\alpha
\end{bmatrix}
\leq 0,
\begin{bmatrix}
z_\alpha

z_\alpha
\end{bmatrix}
\neq 0.
\]  

(63)

Since \( M \) is positive semidefinite due to Lemma 1, it follows from Lemma 4 that there exists a nonzero \( z_\alpha \) such that \(-M_{a\alpha}z_\alpha \geq 0 \) and \( z_\alpha \leq 0 \). As \( M \) is an M-matrix, \( M_{a\alpha} \) is nonpositive. Therefore, we have that \( M_{a\alpha}z_\alpha \geq 0 \) which concludes that (61) has at least one solution and hence (59) is infeasible for \( \alpha \neq \{1, 2, \ldots, m\} \).

After these preparations, we are in a position to prove Theorem 2.

**Proof of Theorem 2.** Based on Lemma 6, \( S'_2 = C_2 \) and hence optimization problems SLP' and CVX1 are the same. Therefore, \( \text{MIN}(\text{CVX}2) = \text{MIN}(\text{SLP}_2') \). Thus, it suffices to prove \( \text{MIN}(\text{SLP}_2') \subseteq \text{MIN}(\text{SLP}_2) \).

Let \((x^*, y^*) \in \text{MIN}(\text{SLP}_2') \). Then,

\[
\phi(x^*, y^*) \leq \phi(x, y) \quad \forall (x, y) \in S'_2.
\]  

(64)

Let \((\bar{x}, \bar{y}) \in S_2 \). Then, it follows from Lemma 3 that there exists \((\bar{x}, \bar{y}) \in S_2' \) such that

\[
\phi(\bar{x}, \bar{y}) \leq \phi(\bar{x}, \bar{y}).
\]  

(65)

Consequently, we obtain

\[
\phi(x^*, y^*) \leq \phi(\bar{x}, \bar{y}) \leq \phi(\bar{x}, \bar{y}).
\]  

(66)

This means that \( \phi(x^*, y^*) \leq \phi(\bar{x}, \bar{y}) \) for all \((\bar{x}, \bar{y}) \in S_2 \). It is clear that \( S'_2 \subseteq S_2 \). Therefore, \((x^*, y^*) \in S_2 \) and hence \((x^*, y^*) \in \text{MIN}(\text{SLP}_2) \). Thus, we can conclude \( \text{MIN}(\text{CVX}2) = \text{MIN}(\text{SLP}_2') \subseteq \text{MIN}(\text{SLP}_2) \). Furthermore, \( C_2 \) is a polyhedron and hence convex. Also, \( \phi \) is convex on \( C_2 \) since \( M \) is positive semidefinite due to Lemma 1. Therefore, CVX2 is a convex optimization problem.

**3.2.1 On \( M \) being an M-matrix:**

One of the assumptions in Theorem 2 is on the structure of the matrix \( M \). We know that \( M \) is the Schur complement of matrix \( X \) as mentioned in the proof of Lemma 1 and (27). Here, we discuss when \( M \) as in (25) is an M-matrix. The following lemma shows that the Schur complement of an M-matrix is also an M-matrix.

**Lemma 7** (Fiedler 2008, Theorem 5.13). Suppose that \( N \) is an M-matrix. Then, Schur complement of \( N \) with respect to a positive definite submatrix of \( N \) is also an M-matrix.

The following theorem provides sufficient conditions for \( M \) to be an M-matrix.

**Theorem 3.** Suppose that \( F \) is a nonpositive matrix and rows of \( F \) are orthogonal, i.e., \( F_i(F_j)^T = 0 \) for all \( i \neq j \). Then, \( M \) is an M-matrix if \( R \) is diagonal.

**Proof.** The matrix \( M \) is the Schur complement of matrix \( X \) given by (27). Since \( F \) is a matrix with orthogonal rows and \( R \) is positive definite and diagonal, \( FR^{-1}F^T \) and hence \( (FR^{-1}F^T)^{-1} \) are also positive definite diagonal matrices. Moreover, \( FR^{-1} \) is nonpositive which makes \( X \) an M-matrix. Consequently, \( M \) is also an M-matrix based on Lemma 7.

**3.3 General Case**

Here, we consider the main problem, i.e. the optimization problem (12). We rewrite BLP by characterizing lower level problem based on KKT conditions as

\[
\text{SLP :} \quad \min_{x, y} \phi(x, y)
\]

subject to \( (x, y) \in S \)  

(67a)

(67b)

where

\[
S = \{ (x, y) \mid x \geq 0, \ y = M(x + \mu - \nu) + r, \ \ell \leq y \leq u, \ \mu^T(y - \ell) = 0, \ \nu^T(u - y) = 0 \text{ for some } \mu \geq 0 \text{ and } \nu \geq 0 \}.
\]  

(68)

Note that \( \mu \) and \( \nu \) are dual variables for \( \ell \leq y \) and \( y \leq u \) in (13b), respectively. Moreover, the dual variable for the constraint (13c) has been eliminated from KKT conditions in a similar way to (17) - (24). Also, \( M \) and \( r \) are as in (25) and (26), respectively with \( R = Q \) and \( F = E \). As a result, \( M \) is an M-matrix based on Theorem 3.

The theorem below indicates that there exists a convex optimization problem which can capture a subset of the set of global optima for SLP.
Theorem 4. Consider the following optimization problem:

\[
\text{CVX} : \quad \min_{x,y} \quad \phi(x,y)
\]

subject to \((x,y) \in C\) \hfill (69a)

where \(C = \{(x,y) \mid x \geq 0, \ y = Mx + r, \ \ell \leq y \leq u \}\). \hfill (70)

Suppose that \(\ell \leq 0, \ u \geq 0, \) and \(u > r\). Then, \(\text{MIN(CVX)} \subseteq \text{MIN(SLP)}\). Furthermore, the optimization problem \(\text{CVX}\) is convex.

Proof. Let \((\bar{x}, \bar{y}) \in S\). Therefore, there exist \(\bar{\mu} \geq 0\) and \(\bar{\nu} \geq 0\) such that

\[
\bar{x} \geq 0, \quad \bar{y} = M(\bar{x} + \bar{\mu} - \bar{\nu}) + r, \quad \ell \leq \bar{y} \leq u, \quad \bar{\mu}^T(\bar{y} - \ell) = 0, \quad \bar{\nu}^T(u - \bar{y}) = 0.
\]

Let \(\bar{y} = M(\bar{x} + \bar{\mu}) + r'\) where \(r' = r - M\bar{\nu}\). As a result, \((\bar{x}, \bar{y}) \in S_1\). From Lemma 2, there exists \((\hat{x}_1, \hat{y}) \in C_1\) such that

\[
\phi(\hat{x}_1, \hat{y}) \leq \phi(\bar{x}, \bar{y}).
\]

Moreover, as \(\bar{y}\) satisfies \(\bar{y} \leq u, \ \bar{\nu}^T(u - \bar{y}) = 0\) for \(\bar{\nu} \geq 0\) and \(r' = r - M\bar{\nu}\), we can conclude

\[(\hat{x}_1, \hat{y}) \in S_2 \cap \{(x,y) \mid y \geq \ell\}\] \hfill (73)

Thus, \((\hat{x}_1, \hat{y}) \in S_2\). Due to Lemma 3 and Lemma 6, there exists \((\hat{x}_2, \hat{y}) \in C_2\) such that

\[
\phi(\hat{x}_2, \hat{y}) \leq \phi(\hat{x}_1, \hat{y}).
\]

Furthermore, since \(\bar{y} \geq \ell\), \((\hat{x}_2, \bar{y}) \in C\). Then, (72) and (74) imply that

\[
\phi(\bar{x}, \bar{y}) \leq \phi(\hat{x}, \bar{y}) \quad \forall (\hat{x}, \bar{y}) \in S.
\]

Now, let \((x^*, y^*) \in \text{MIN(CVX)}\). That is,

\[
\phi(x^*, y^*) \leq \phi(x,y) \quad \forall (x,y) \in C.
\]

Since \((\hat{x}_2, \bar{y}) \in C\), (75) implies that

\[
\phi(x^*, y^*) \leq \phi(\hat{x}, \bar{y}) \quad \forall (\hat{x}, \bar{y}) \in S.
\]

Since \(C \subseteq S\), \((x^*, y^*) \in \text{MIN(SLP)}\) and hence \(\text{MIN(CVX)} \subseteq \text{MIN(SLP)}\). Furthermore, \(C\) is a polyhedral convex set. Also, \(\phi\) is convex on \(C\) since \(M\) is positive semidefinite due to Lemma 1. Therefore, CVX is a convex optimization problem. \(\square\)

3.4 Interpretation of Theorem 4 for RTM

Theorem 4 has certain hypotheses on the parameters \(\ell, u,\) and \(r\). Here, we discuss the implications of these hypotheses for the proposed RTM platform. It follows from Theorem 4 that the vectors \(\ell = s - h\) and \(u = s - h\) should be nonpositive and nonnegative, respectively. The vector \(s\) is the generated energy by RESs, the vector \(h\) is the lower bound of the prosumers’ demand and the vector \(\hat{h}\) is the upper bound for their demand. To have \(\ell \leq 0\) and \(u \geq 0\), the aggregator should ask the prosumers to set the upper bound of their demands \(h\) greater than or equal to their RESs’ capacity and also the lower bound for their demands \(\hat{h}\) equal to zero. Moreover, Theorem 4 states that \(u\) should be strictly greater than \(r\) where \(r\) is defined as in (26) with \(R = Q\) and \(F = E\). Considering Assumption 1, we can show that

\[
u > r \iff \hat{h}^0 > h.
\]

Therefore, to have \(u > r\), the aggregator should ask the prosumers to set their preferred values \(\hat{h}^0\) greater than the lower bound for their demand \(h\), i.e., \(\hat{h}^0 > 0\).

4 Conclusions

The problem of participation of the prosumers in the wholesale market through the aggregator has been widely studied in the literature. To represent the intrinsic hierarchy of this problem, we developed a market platform based on a bilevel optimization problem. Bilevel optimization are generally highly nonconvex and current approaches to deal with these problems are computationally expensive. To implement this market platform in real-time, we proposed a specific convex optimization problem and showed that each global minimizer of this convex problem are also a global minimizer for the original bilevel problem under some assumptions on the parameters.

While the proposed convex approach can reduce the computational time significantly in contrast to the state-of-the-art methods (e.g., MIP), the assumption that the aggregator has a centralized control over the prosumers may limit the applicability of the proposed method to large scale networks. An interesting important area of future research could be design of a decentralized or distributed control mechanism using the convex problem to tackle this issue.
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