Type IIA Killing spinors and calibrations

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Abstract

We consider the dimensional reduction of eleven dimensional supergravity to type IIA in ten dimensions, and study the conditions for supersymmetry in terms of \( p \)-form spinor bi-linears of the supersymmetry parameter. For a bosonic solution to be supersymmetric these \( p \)-forms must satisfy a set of differential relations, which we derive in full; the supersymmetry variations of the dilatino give a set of algebraic relations which are also derived. These results are then used to provide the generalized calibration conditions for some of the basic brane solutions, we also follow up a suggestion of and Hackett-Jones and Smith and present a calibration condition for IIA supertubes. We find that a probe supertube satisfies this bound but does not saturate it, with the bound successfully accounting for the D0 charge of the supertube but not the string charge; we speculate that there should be a stronger calibration inequality than the one given.

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1 Introduction

The utility of bi-linears formed from Killing spinors has been known for some time, with them being used to present Bogomol’nyi bounds in supergravity [1], derive the full set of supersymmetric solutions to supergravities [2] and to give constraints on the manifolds used in dimensional reduction [3]. More recently this approach has been used to generate the supersymmetric solutions to higher dimensional supergravities [4][5][6][7], by solving the differential equations of the \( p \)-form spinor bi-linears rather than directly solving the Killing spinor equation; here we present the differential and algebraic equations for the spinor bi-linears of IIA supergravity. These \( p \)-forms also naturally lead one to the notion of \( G \)-structures, as the existence of globally defined \( p \)-forms reduces the frame bundle \( \text{SO}(1,D-1) \) to some \( G \)-sub-bundle, this can then be used to give classifications of supergravity solutions [8]-[19].

Calibrations [20] form an important part of string theory technology and our understanding has developed from the basic calibrations, which describe minimal volume submanifolds, to generalized calibrations [21] relevant to branes moving in background fluxes. In [22] it was shown how the differential relations of the \( p \)-form bi-linears in eleven dimensional supergravity could be combined in a natural way with the flat space supersymmetry algebra to give a proposal for the calibration bound of M2 and M5 branes. Applying these ideas to IIB supergravity in ten dimensions [23] lead to a proposal for the calibration bound of giant gravitons, the first example of a calibration for non-static branes. As an application of our results we consider a case suggested by [23] as another example of a solution which could lead to a calibration condition for non-static brane, the supertube [24], thought of as a configuration of D0 branes and parallel strings blown up to a cylindrical D2 brane. We propose such a bound but believe that it is not the full story as the bound is satisfied by a probe supertube but not saturated; the bound correctly captures the D0 charge but misses the string charge.

The outline of the paper is as follows. Section 2 gives the \( p \)-form spinor bi-linears that are allowed in IIA supergravity and relates them to those from eleven dimensions. The differential relations for these \( p \)-forms are then derived from the analogous 11D relations in section 3. The algebraic constraints which follow from the IIA Killing spinor equation are derived in section 4. The final section describes how calibration conditions are derived from the differential relations and the supersymmetry algebra, we show some simple examples of how this works and end with the supertube example. In the appendices we give our conventions in deriving IIA from eleven dimensions and also provide some of the basic IIA solutions which act as a check for our equations.

2 IIA from eleven dimensions

It is well known, [25][26][27] that IIA supergravity can be derived by dimensional reduction from the eleven dimensional supergravity of Cremmer-Julia-Scherk (CJS) [28]. This allows us to use the results of [9] to obtain the set of bi-linears in IIA. In order for our conventions to conform to that of [9] we show in the appendix both supergravities with their corresponding supersymmetry transformations.

When there is a single Killing spinor one can use the symmetry properties of the Dirac matrices in eleven dimensions to define the following spinor bi-linears (objects with a hat are eleven dimensional),

\[
\hat{K} = \bar{\hat{\epsilon}} \hat{\gamma}_a \hat{\epsilon} E^a, \quad (2.1)
\]

\[
\hat{\Omega} = \frac{1}{2} \bar{\hat{\epsilon}} \hat{\gamma}_{a_1 a_2} \hat{\epsilon} E^{a_1 a_2}, \quad (2.2)
\]
\[ \hat{\Sigma} = \frac{1}{5!} \hat{e} \hat{\Gamma}_{a_1 \ldots a_5} \eps \hat{E}^{a_1 \ldots a_5}. \]  

(2.3)

This is the full set, with the other rank \( p \)-forms either vanishing or being related by Hodge duality.

To reduce these to ten dimensions we note from appendix A that the 11D and 10D vielbein are related by \( E^a = \exp(-\phi/3)\eps^a \). This takes us to the string frame, which is more natural in this context as the supersymmetry transformations (A.11), (A.12) have some degree of uniformity amongst the field strengths. With this we find

\[
\hat{K} = \exp(-2\phi/3)K + \exp(-\phi/3)XE^z, \\
\hat{\Omega} = \exp(-\phi)\Omega + \exp(-2\phi/3)\hat{K} \wedge E^z, \\
\hat{\Sigma} = \exp(-2\phi)\Sigma + \exp(-5\phi/3)Z \wedge E^z,
\]

(2.4)

(2.5)

(2.6)

where we have introduced the following 10D bi-lineras

\[
X = \hat{e} \hat{\Gamma}_z \eps, \\
K = \hat{e} \hat{\Gamma}_a \eps e^a, \\
\hat{K} = \hat{e} \hat{\Gamma}_a \hat{\Gamma}_z \eps e^a, \\
\Omega = \frac{1}{2} \hat{e} \hat{\Gamma}_{a_1 a_2} \eps e^{a_1 a_2}, \\
Z = \frac{1}{4!} \hat{e} \hat{\Gamma}_{a_1 \ldots a_4} \hat{\Gamma}_z \eps e^{a_1 \ldots a_4}, \\
\Sigma = \frac{1}{5!} \hat{e} \hat{\Gamma}_{a_1 \ldots a_5} \eps e^{a_1 \ldots a_5}.
\]

(2.7)

(2.8)

(2.9)

(2.10)

(2.11)

(2.12)

In doing this we have chosen to represent the two Majorana-Weyl spinor parameters, \( \eps^\pm \), of IIA by the single Majorana spinor \( \eps = \eps^+ + \eps^- \). An alternative set of bi-lineras could have been defined in terms of the Majorana-Weyl components, \( K^{++} = \hat{\eps}^+ \hat{\Gamma}_a \eps e^a \ldots \) but these can be constructed from linear combinations of (2.7-2.12) and so are equivalent. We shall find the former set more useful as \( \eps \) descends directly from \( \hat{\eps} \) allowing us to use the relations of [9] to derive differential equations for (2.7-2.12), which we now do.

### 3 Differential relations.

Now we have the set of \( p \)-forms we can derive the differential relations they must satisfy if the solution is to be supersymmetric. As described in [9] this is achieved by using the vanishing of the susy variation (A.4). One finds that the 11D bi-lineras must solve

\[
d\hat{K} = \frac{2}{3} \hat{\Omega} \wedge \hat{G} + \frac{1}{3} \hat{\Sigma} \ast \hat{\star} \hat{G}, \\
d\hat{\Omega} = \hat{K} \wedge \hat{\star} \hat{G}, \\
d\hat{\Sigma} = \hat{K} \wedge \hat{G} - \hat{\Omega} \wedge \hat{\star} \hat{G}.
\]

(3.13)

(3.14)

(3.15)

From these we can derive the analogous equations for the IIA \( p \)-forms, (2.7-2.12). We find the following,

\[
\exp(-\phi/3)d[\exp(\phi/3)X] = \frac{2}{3} \Omega \ast H + \frac{1}{3} \exp(\phi) \Sigma \ast \hat{\star} \hat{G}, \\
\exp(2\phi/3)d[\exp(-2\phi/3)K] = \frac{2}{3} \hat{K} \ast H + \frac{1}{3} \Sigma \ast H + \exp(\phi) \left[ \frac{2}{3} \Omega \ast \hat{G} - XF + \frac{1}{3} Z \ast \hat{G} \right]
\]

(3.16)

(3.17)
\[ d\bar{K} = K_J H, \]  
\[ \exp(\phi)d[\exp(-\phi)\Omega] = -X H + \exp(\phi) \left[ \bar{K} \wedge F + K_J \hat{G} \right], \]  
\[ \exp(\phi)d[\exp(-\phi)Z] = -\Omega \wedge H + \exp(\phi) \left[ 6K_J \star \hat{G} - \bar{K} \wedge \hat{G} \right], \]  
\[ \exp(2\phi)d[\exp(-2\phi)\Sigma] = K_J \star H + \exp(\phi) \left[ -Z \wedge dA + X \star \hat{G} - \Omega \wedge \hat{G} \right], \]

where the various field strengths are defined in appendix A.

An alternative route to these equations is to directly consider the derivative of the forms in (2.7-2.12) and use the vanishing of 10D gravitino variation (A.12) to replace the \( \nabla_m \epsilon \) terms.

An important result that also comes from the analysis of \( \nabla_a K_b \) is that
\[ \nabla_{\epsilon} K_{\epsilon} = \frac{1}{3} \delta_{ab} K_{\epsilon} d\phi = 0, \]
where the last equality follows from \( 1.23 \) to be derived in the next section. This tells us then that one of the vector bi-linears, \( K \), is in fact a Killing vector. That such a spinor bi-linear is Killing is also true in eleven dimensions [9] and in IIB theory [23], this has important consequences when it comes to constructing the calibration forms.

### 4 Algebraic relations.

Whilst it was possible to simply translate the differential relations of [9] into differential relations relevant to the IIA \( p \)-forms we also have a set of algebraic constraints coming from the vanishing of the susy variation of the dilatino, \( \epsilon A.11 \). To derive these we take \( \epsilon A.11 \) and act on the left with \( \tilde{\epsilon} \Gamma^i, \tilde{\epsilon} \Gamma^{ij}, \tilde{\epsilon} \Gamma^{ijk} \) and \( \tilde{\epsilon} \Gamma^{ijkl} \) to give
\[ 0 = K_{\epsilon} d\phi, \]
\[ 0 = d\phi \wedge K + \frac{1}{2} H_{\epsilon} \wedge \Sigma + \frac{1}{2} K_{\epsilon} \wedge H + \frac{1}{4} \exp(\phi) \left[ -3F_{\epsilon} \wedge Z + 3XF + \hat{G}_J \wedge Z - \Omega \wedge \hat{G} \right], \]
\[ 0 = (d \phi \wedge \Omega)^{ijkl} - \frac{3}{4} H_{bc} [i^{\epsilon} Z_{ijkl}]_{bc} + \frac{1}{2} X H^{ijkl} \]
\[ + \frac{1}{4} \exp(\phi) \left[ 3(F \wedge \hat{K})^{ijkl} + 3(F \wedge \hat{K})^{ijkl} - \frac{1}{2} \hat{G}_{bcd} [i^{\epsilon} \Sigma_{ijkl}]_{bcd} + (K_J \hat{G})^{ijkl} \right], \]
\[ 0 = (d \phi \wedge \Sigma)^{ijkl} - H_{bc} [i^{\epsilon} \Sigma_{ijkl}]_{bc} - \frac{1}{2} \hat{G}_{bcd} [i^{\epsilon} \Sigma_{ijkl}]_{bcd} \]
\[ + \exp(\phi) \left[ -3F_{b} [i^{\epsilon} Z_{ijkl}]_{bc} + \frac{1}{6} \hat{G}_{bcd} [i^{\epsilon} Z_{ijkl}]_{bcd} - \hat{G}_{d} [i^{\epsilon} \Omega_{ijkl}]_{d} \right]. \]

These are the full set of relations which can be derived from \( \epsilon A.11 \); if one hits \( \epsilon A.11 \) with more the four \( \Gamma \) matrices the resulting relations will be dual to the above equations.

Although we can get no more algebraic relations from \( \epsilon A.11 \) the fact that our \( p \)-forms are bi-linear implies they must have certain relations amongst each other. These can be constructed using Fiertz identities and they hold irrespective of supersymmetry. Although we will not present anything like an exhaustive list we give here two such examples to illustrate the point. From \( 9 \) we have
\[ K_{\epsilon} \hat{G} = 0, \]
\[ K_{\epsilon} \hat{G} = \frac{1}{2} \hat{G} \wedge \hat{\Omega} \]
\[ K_{\epsilon} \hat{G} = \frac{1}{2} \hat{G} \wedge \hat{\Omega}, \]
which we can convert into IIA language to get

\begin{align*}
    K_\alpha \tilde{K} &= 0, \\
    K_\alpha \Omega - X \tilde{K} &= 0,
\end{align*}

and

\begin{align*}
    XZ + K_\alpha \Sigma &= \frac{1}{2} \Omega \wedge \Omega, \\
    K_\alpha Z &= \Omega \wedge \tilde{K}.
\end{align*}

respectively.

We have now given the full set of relations, differential and algebraic, which must be satisfied by a supersymmetric solution of IIA supergravity. As a check that we have arrived at the correct set of equations appendix B provides some of the basic IIA solutions, giving the spinor bi-linears.

5 Calibration conditions.

5.1 strings

Following [22] we find out how to derive calibration conditions for some of the branes in IIA supergravity. We shall start with the simplest, namely the F1 string. The super-Poincaré algebra in eleven flat dimensions, with a probe M2 brane, can be dimensionally reduced to give the algebra in ten flat dimensions with a probe string,

\[ \{Q_\alpha, Q_\beta\} = (CT^\mu)_{\alpha\beta} P_\mu \pm (CT_\mu \Gamma_\zeta)_{\alpha\beta} Z^\mu, \]

where

\[ Z^\mu = \int dX^\mu, \]

and the integration is over the spatial direction of the string. If we now multiply (5.34) by \( \epsilon_0^\alpha \epsilon_0^\beta \), for constant \( \epsilon_0 \), we have that

\[ (Q \epsilon_0)^2 = K_0^\mu p_\mu \pm \tilde{K}_0^\mu \int dX^\mu = \int d\sigma K_0^\mu p_\mu \pm \int \tilde{K}_0. \]

Where we have introduced a momentum density, \( p_\mu \), and the notation \( K_0 \) and \( \tilde{K}_0 \) come from [248], applied to the constant spinor \( \epsilon_0 \), \( \sigma \) is the the spatial co-ordinate on the string world volume. By writing it in this form we see how this equation should be generalized to curved space. We expect the correction to the super-algebra to be a topological term reflecting the charge of the probe, so the aim is to take the flat space expression \( \int K_0 \) and write it as the integral of a closed form. Now we use the fact that \( K \) is a Killing vector and that the definition of the Lie derivative on forms,

\[ \mathcal{L}_K \alpha = d(K_\alpha) + K_\alpha d\alpha. \]

From (3.15) we have that \( d(K_\alpha H) \) and as \( H = dB \) then \( K_\alpha dH = 0 \) giving \( \mathcal{L}_K H = 0 \), so we may choose a gauge in which \( B \) mirrors the symmetry of its field strength in that \( \mathcal{L}_K B = 0 \). We may
therefore use this gauge and rewrite (3.18) as
\[ d(\tilde{K} + K \cdot B) = 0, \]
which gives us the closed form we were looking for. The proposal, therefore, for the curved version of (5.36) is
\[ (Q\epsilon)^2 = \int d\sigma K.p \pm \int (\tilde{K} + K \cdot B) \]  
(5.38)
where now we use the Killing spinor \( \epsilon \). This then leads to the calibration bound
\[ \int d\sigma K.p \geq \mp \int (\tilde{K} + K \cdot B), \]  
(5.39)
and a standard argument shows that a calibrated cycle (one for which the bound is saturated) minimizes \( \int d\sigma K.p \) in its homology class.

As an example we could consider a string probe in the background of a stack of strings whose solution is given by (B.2). There we find that
\[ \tilde{K} + K \cdot B = dx, \]  
(5.40)
with \( dx \) being the spatial direction of the string, this is clearly closed. With these relations we can check to see if a string probe in the background of a multi-string solution (Appendix B) saturates the calibration bound. For this we identify \( K.p \) with the Hamiltonian density of the probe, which we can calculate from the string action
\[ S_{F1} = -\int d^2\sigma \sqrt{-\gamma} - \int \mathcal{P}[B]. \]  
(5.41)
Where \( \gamma_{\mu\nu} \) is the induced metric on the world-volume of the probe and \( \mathcal{P}[B] \) is the pull back of the spacetime field, \( B \), to the string world-volume. We shall look at a probe string oriented in the same way as the background strings, the \( t-x \) plane as in (B.21). For the world-volume co-ordinates \( (\tau, \sigma) \) we choose the natural gauge \( \tau = t, \sigma = x \) which then leads to the energy density \( \mathcal{H} = 1 = K.p \). Using (5.40) we see that the string probe saturates the calibration bound (5.39).

### 5.2 D2-branes

For D2-branes the situation is slightly more complicated as we have been unable to find the general calibration condition for any given background fields, but we can look at the bound in any specific case. As an example we look at the bound for a probe D2 in the background of a stack of gravitating D2-branes, (B.3). In this case we see that \( H = 0, F = 0 \) so (A.9) gives \( \tilde{G} = dC \) and (3.19) shows \( \mathcal{L}_K dC = d(K \cdot dC) + K \cdot ddC = 0. \) Thus, \( K \) represents a symmetry of the field strength \( dC \), in which case we may pick a gauge where \( C \) also has this symmetry and choose \( \mathcal{L}_K C = 0. \) In that gauge then we have \( \mathcal{L}_K C = d(K \cdot C) + K \cdot dC = 0 \) and (3.19) gives us the following closed 2-form,
\[ d[\exp(-\phi)\Omega + K \cdot C] = 0. \]  
(5.42)
The calibration bound again comes from the supersymmetry algebra. The terms relevant for D2-branes in the flat 10D algebra are.
\[ \{Q_\alpha, Q_\beta\} = (CT^\mu)_{\alpha\beta} P_\mu \pm (CT_\mu)_{\alpha\beta} Z^{\mu\nu}, \]  
(5.43)
where
\[ Z^{\mu\nu} = \int dX^\mu \wedge dX^\nu. \]  
(5.44)
Going through the same procedure as the probe string we are led to suggest the following calibration bound for a probe D2 in the background of a stack of D2-branes,

$$\int d^2 \sigma K.p \geq \pm \int (\exp(-\phi)\Omega + K.C).$$  \hspace{1cm} (5.45)

Note that the D2-brane stack given in B.3 has

$$\exp(-\phi)\Omega + K.C = dx^{12} + dy^{12} + dy^{34} + dy^{56},$$  \hspace{1cm} (5.46)

where $dx^{12}$ is the spatial orientation of the stack and $dy$ are transverse directions. Again, this is clearly closed. As for the string, we can check to see if a D2 probe in the background of a stack of D2-branes, appendix B, saturates the calibration bound (5.45). We shall orient our probe in the same direction as the stack, $(t, x^1, x^2)$, and choose a gauge where the world-volume co-ordinates match the spacetime ones, $(\tau = t, \sigma^1 = x^1, \sigma^2 = x^2)$. Taking the action for a D2 brane

$$S_{D2} = -\int \exp(-\phi)\sqrt{-\gamma} - \int \mathcal{P}[C].$$  \hspace{1cm} (5.47)

allows us to calculate the energy density to be $\mathcal{H} = 1 = K.p$. Now, using (5.46), we see that the calibration bound (5.45) is saturated. In fact, (5.47) is not the full world volume action of the D2-brane as there can be a Born-Infeld field, $F_{BI}$, living on the world-volume which changes the action to

$$S = -\int \exp(-\phi)\sqrt{-\gamma + \mathcal{F}} + \int (\mathcal{P}[C] + \mathcal{P}[A] \wedge \mathcal{F}),$$  \hspace{1cm} (5.48)

where $\mathcal{F}$ is the gauge invariant world-volume field strength, $\mathcal{F} = \mathcal{P}[B] + F_{BI}$. This world-volume field strength must be reflected in the calibration condition, and as $F_{BI}$ is already a closed two-form we would anticipate that (5.46) should become

$$\int d^2 \sigma K.p \geq \pm \int (\exp(-\phi)\Omega + K.C + F_{BI}).$$  \hspace{1cm} (5.49)

This type of term has already been seen in the M5 brane calibration conditions, \cite{[22][29]}, and we shall see evidence in the next section that it should be there.

### 5.3 supertubes

Now we come to a more substantial example, following a suggestion in \cite{[23]}, that of a supertube \cite{[24]}. This is a nice case for a number of reasons: it is sufficiently complex so as to excite all the field strengths, thereby proving a good check on our relations; from the point of view of the D2-brane there are world volume fields turned on, giving a check to the Born-Infeld term which did not follow from the flat space susy algebra; it is also not a static solution, unlike most other calibrated branes - except for the giant graviton \cite{[23]}.

The supergravity version of supertubes was presented in \cite{[30]} and is obtained by dimensionally reducing a solution found in \cite{[31]}, which describes the intersection of two rotating M5 branes and an M2 brane. We have given the IIA solution in appendix B.4.

As the supertube is to be considered as a D2 brane we shall be looking for a closed two-form living in the solution, composed from spinor bi-linears and the fields which are excited. The first thing to note is that while neither $d(K.C)$ nor $K.dC$ vanish, their sum does, giving $\mathcal{L}_R dC = 0$. As usual then, we may choose a gauge where $C$ matches the symmetry of its field strength, $\mathcal{L}_R C = 0$. 

For the supertube we have from appendix B.4 that $K \cdot A - \exp(-\phi) = -1$, using this and (3.18) shows that (3.19) leads to
\[
d \left[ \exp(-\phi)\Omega + K \cdot C + \bar{K} \wedge A + B \right] = 0, \tag{5.50}
\]
with the supertube solution giving
\[
\exp(-\phi)\Omega + K \cdot C + \bar{K} \wedge A + B = -dt \wedge dx, \tag{5.51}
\]
This therefore suggests that the calibration bound in this background becomes
\[
\int d^2\sigma K.p \geq \mp \int \left[ \exp(-\phi)\Omega + K \cdot C + \bar{K} \wedge A + B + F_{BI} \right], \tag{5.52}
\]
However, this has not taken into account the world-volume Born-Infeld field on the D2 and so, in analogy with the M5 calibration bound [29][22] we put forward the following bound
\[
\int d^2\sigma K.p \geq \mp \int \left[ \exp(-\phi)\Omega + K \cdot C + \bar{K} \wedge A + B + F_{BI} \right], \tag{5.53}
\]
with $F_{BI}$ being the Born-Infeld field strength. We are now in a position to test this bound by placing a probe supertube in the background. Following [30] we choose our supertube to be cylindrical and write the metric on $dy^2$ of (B.33) as
\[
dy^2 = dr^2 + r^2d\phi^2 + dp^2 + \rho^2d\Omega_5^2. \tag{5.54}
\]
The form of the harmonic functions is given in [30], they place the supertube at $r = R$ and $\rho = 0$ so it occupies the $t - x - \varphi$ direction, the one form $A$ is given by $A = \tilde{A}(r, \rho)d\varphi$. Now consider a probe D2 with the following Born-Infeld field strength
\[
F_{BI} = E_{BI} dt \wedge dx + B_{BI} dx \wedge d\varphi. \tag{5.55}
\]
The gauge invariant world volume field strength is
\[
\mathcal{F} = F_{BI} + \mathcal{P}[B] := \mathcal{E} dt \wedge dx + \mathcal{B} dx \wedge d\varphi. \tag{5.56}
\]
Now we may take the probe action
\[
S = -\int \exp(-\phi)\sqrt{-\text{det}(\gamma + \mathcal{F})} + \int (\mathcal{P}[C] + \mathcal{P}[A] \wedge \mathcal{F}), \tag{5.57}
\]
to find the following Lagrange density (in the usual physical gauge) [30]
\[
\mathcal{L} = -U^{1/2}V^{-1}\sqrt{V r^2(U^{-2} - \mathcal{E}^2) + U^{-1}(B - \tilde{A}\mathcal{E})^2 + V^{-1}(B - \tilde{A}\mathcal{E}) - B_{BI}}. \tag{5.58}
\]
To find the Hamiltonian we first need the conjugate of $E_{BI}$, $\mathcal{D}$.
\[
\mathcal{D} = \frac{\partial \mathcal{L}}{\partial E_{BI}} = \frac{\partial \mathcal{L}}{\partial \mathcal{E}} = \frac{U^{1/2}r^2\mathcal{E} + U^{-1/2}V^{-1}(B - \tilde{A}\mathcal{E})\tilde{A}}{\sqrt{V r^2(U^{-2} - \mathcal{E}^2) + U^{-1}(B - \tilde{A}\mathcal{E})^2}} - V^{-1}\tilde{A}, \tag{5.59}
\]
As a supertube has $E_{BI} = 1$ [30], then we find that the Hamiltonian is
\[
\int dxd\varphi \mathcal{H} = \int dxd\varphi [\mathcal{D}E - \mathcal{L}] = \int dxd\varphi [\mathcal{D} + B_{BI}], \tag{5.60}
\]
with \( D = R^2 / B_{BI} \). In terms of the calibration condition (5.53) we identify (5.60) with \( \int d^2 \sigma K.p \). On the right hand side of the bound we use (5.51) and (5.55) to get

\[
\int \left[ \exp(-\phi) \Omega + K \cdot C + \tilde{K} \wedge A + B + F_{BI} \right] = \int d\sigma d\phi B_{BI}.
\]  

(5.61)

So, while the bound is satisfied, it is not saturated. As noted in \[30\] \( D \) corresponds to the string charge of the probe and \( B_{BI} \) the D0 brane charge, so we see that the bound has successfully accounted for the D0 charge whilst missing the string charge. We believe that the supertube should saturate some calibration bound; if so, then there will be a stronger inequality than presented in (5.53).

6 Conclusion

In this paper we have described how one can construct \( p \)-forms from the Killing spinors of IIA supergravity using those of CJS supergravity in eleven dimensions. As is to be expected from dimensional reduction there are more of these bi-linear terms than in the parent theory, with one scalar, two vectors, a two-form, a four-form and a five-form, along with their Hodge duals. The set of differential relations satisfied by these spinor bi-linears was derived and shown to follow from the analogous relations in 11D, as given in \[9\]. Unlike the bi-linear terms of CJS supergravity, one of the killing spinor equations, the variation of the dilatino, gave a set of algebraic constraints, in concert with the algebraic constraints coming from Fiertz identities. We found the full set of these dilatino constraints and gave some examples of the Fiertz relations.

As an application of these results we considered a technique introduced in \[22\] for proposing calibration bounds. We used this to give the bound for a string in a general background, but for the D2 brane we could only find background-specific results. In particular we applied our equations to the supertube of \[30\] which lead to a putative calibration bound. A probe supertube was found to satisfy the bound, but not saturate it, leading to the suspicion that a stronger bound should exist. This stemmed from a deficiency of the technique in that world volume fields are not naturally accounted for.

Note added

After the completion of this work there appeared a pre-print by Cascales and Uranga \[32\] proposing another method for finding calibration bounds, we hope that their results when applied to IIA will strengthen the supertube bound found here.

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Appendices
A  D=11, 10 supergravities and conventions.

We follow the conventions of [9] in using \((-,+,+,...)\) as our spacetime signature with the alternating symbol 
\[ \epsilon_{012} = +1. \]

The inner product of \(q\)-forms with \(p(<q)\)-forms is
\[ (\alpha_p)_j \cdots \beta_{q-p} = (1/p!)\alpha^{b_1\cdots b_p} \beta_{b_1\cdots b_p \alpha_{a_1\cdots a_q-p}}, \] (A.1)

and the Hodge dual is defined by
\[ \star\alpha_{a_1\cdots a_D-p} = (1/p!)\epsilon_{a_1\cdots a_D-p \alpha_{b_1\cdots b_p}}. \] (A.2)

Flat indices are given by Roman characters \((m,n,...)\) or an underline, \(\underline{z}\), and curved indices are written with Greek letters \(\mu,\nu,...\). A hat on an index or field denotes it as an eleven dimensional object.

For the Dirac matrices we choose the basis where \(\Gamma_{012\cdots 3} = 1\) and the Majorana conjugate is given by \(\bar{\eta} = \eta^T \mathcal{C}\), with \(\mathcal{C}\) the charge conjugation matrix, chosen to be \(\mathcal{C} = \Gamma_0\).

The action of CJS supergravity is given by [28]
\[ L_{11} = \frac{1}{2\kappa^2} \left[ R - \frac{1}{2} \bar{\dot{\Psi}}_{\hat{\mu}} \Gamma_{\hat{\mu}\hat{n}\hat{p}} \hat{D}_{\hat{n}} \hat{D}_{\hat{p}} - \frac{1}{24} \hat{G}_{\hat{m}\hat{n}\hat{p}\hat{q}} \hat{G}^{\hat{m}\hat{n}\hat{p}\hat{q}} 
+ \frac{1}{12} \hat{G}_{\hat{m}1\cdots \hat{n}1\cdots \hat{p}1\cdots} \hat{G}_{\hat{n}1\cdots} \hat{C}_{\hat{p}1\cdots} + ... \right], \] (A.3)

where \(\hat{G} = d\hat{C}\) is the four form field strength for \(\hat{C}\). Writing the 11D vierbein as \(E_{\mu}^{\hat{m}}\) one has that supersymmetry requires
\[ \hat{\delta} \hat{\Psi}_{\hat{\mu}} = \hat{\nabla}_{\hat{\mu}} \hat{\epsilon} + \frac{1}{288} \left[ \hat{\Gamma}_{\hat{\mu}} \hat{G}_{\hat{m}\hat{n}\hat{p}\hat{q}} - 8 \hat{\delta}_{\hat{\mu}} \hat{G}_{\hat{m}\hat{n}\hat{p}\hat{q}} \right] \hat{\epsilon} \hat{G}_{\hat{m}\hat{n}\hat{p}\hat{q}} = 0. \] (A.4)

To perform the dimensional reduction we write the standard triangular vierbein ansatz leading to a metric of the form
\[ ds^2 = \exp(-2\phi/3)ds^2 + \exp(4\phi/3)(dz + A)^2, \] (A.5)

which gives the action in ten dimensions in the string frame. The vielbeins are related by \(E^a = \exp(-\phi/3)e^a\), with \(E^z = \exp(2\phi/3)(dz + A)\). Such a reduction introduces a scalar field \(\phi\) and a one form \(A\), with field strength \(F = dA\), into the 10D spectrum. The three-form and four-form field strength are then decomposed as
\[ \hat{C} = C + B \wedge dz, \] (A.6)
\[ \hat{G} = \tilde{G} + H \wedge (dz + A), \] (A.7)

where we have defined
\[ H = dB, \] (A.8)
\[ \tilde{G} = dC - H \wedge A. \] (A.9)

The gravitino decomposes as
\[ \Psi_z = \frac{1}{3} \exp(\phi/6)\Gamma_z \lambda, \] (A.10)
\[ \Psi_m = \exp(\phi/6) \left[ \psi_m - \frac{1}{6} \Gamma_m \lambda \right], \]
\[ \hat{\epsilon} = \exp(-\phi/6)\epsilon, \]
which gives the following susy variations.

\[
\delta \lambda = \left[ \partial_a \phi \Gamma^a - \frac{1}{12} H_{abc} \Gamma^{abc} \Gamma^z \right] \epsilon - \frac{1}{8} \exp(\phi) \left[ 3F_{ab} \Gamma^{ab} \Gamma^z - \frac{1}{12} \tilde{G}_{abcd} \Gamma^{abcd} \right] \epsilon,
\]

(A.11)

\[
\delta \psi_m = D_m \eta - \frac{1}{8} H_{mbc} \Gamma^{bc} \Gamma^z \epsilon - \frac{1}{8} \exp(\phi) \left[ \frac{1}{2} F_{ab} \Gamma^{ab} \Gamma^m \Gamma^z - \frac{1}{4!} \tilde{G}_{abcd} \Gamma^{abcd} \Gamma^m \right] \epsilon.
\]

(A.12)

\section{Some IIA p-brane solutions.}

\subsection{D0-brane}

The D0 brane solution can be derived from the M-wave by dimensional reduction to give

\[
ds_{10}^2 = -U^{-\frac{1}{2}} dt^2 + U^\frac{3}{2} d\vec{x}^2,
\]

(B.13)

\[
A = U^{-1}(1 - U) dt,
\]

\[
\Rightarrow F = -U^{-2} dU \wedge dt,
\]

\[
\exp(\phi) = U^{\frac{3}{2}},
\]

\[
\epsilon = \exp(\phi/6) \hat{\epsilon} = U^{-\frac{1}{4}} \epsilon_0,
\]

where \( U \) is some harmonic function of \( x \) and \( \epsilon_0 \) satisfies the projection \( \Gamma^z \epsilon_0 = \epsilon_0 \). In order to work with a Killing spinor having a single degree of freedom we may also choose the compatible projections,

\[
\Gamma_{1234}\epsilon_0, = \Gamma_{1256}\epsilon_0, = \Gamma_{1278}\epsilon_0, = \Gamma_{1357}\epsilon_0 = \epsilon_0,
\]

(B.14)

which gives us the following bilinears

\[
X = U^{-\frac{1}{4}},
\]

(B.15)

\[
K = -U^{-\frac{1}{4}} \epsilon_0,
\]

(B.16)

\[
\tilde{K} = -U^{-\frac{1}{4}} \epsilon_0,
\]

(B.17)

\[
\Omega = -U^{-\frac{1}{4}} \epsilon_{00},
\]

(B.18)

\[
Z = U^{-\frac{1}{4}} [\epsilon_{1234} + ...],
\]

(B.19)

\[
\Sigma = -U^{-\frac{1}{4}} \epsilon_0 [\epsilon_{1234} + ...].
\]

(B.20)

\subsection{F1-string}

Taking the M2-brane in 11 dimensions and reducing to IIA gives the following string solution,

\[
ds_{10}^2 = U^{-1} ds^2(\mathbb{M}^2) + ds^2(\mathbb{E}^8),
\]

(B.21)

\[
H = -d(U^{-1}) \wedge Vol(\mathbb{M}^2),
\]

\[
B = -U^{-1} dt \wedge dx + dt \wedge dx,
\]

\[
\exp(\phi) = U^{-\frac{1}{4}},
\]

\[
\epsilon = \exp(\phi/6) \hat{\epsilon} = U^{-\frac{1}{4}} \epsilon_0,
\]

where we have chosen a gauge for \( B \) such that \( B \) vanishes asymptotically, \( U \) is some harmonic function on \( \mathbb{E}^8 \) and \( \Gamma_{012}\epsilon_0 = \epsilon_0 \). We may also make the following compatible projections so as \( \epsilon \) has only one degree of freedom,

\[
\Gamma_{023}\epsilon_0, = \Gamma_{045}\epsilon_0, = \Gamma_{067}\epsilon_0, = \Gamma_{012468}\epsilon_0, = \epsilon_0.
\]
We then find that the set of bi-linears is

\begin{align}
X &= 0, \\
K &= -U^{-\frac{1}{4}}e^0, \\
\tilde{K} &= U^{-\frac{1}{4}}e^1, \\
\Omega &= U^{-\frac{1}{2}}[e^{23} + e^{45} + e^{67} + e^{89}], \\
Z &= U^{-\frac{1}{2}}[-e^{3468} + ...] + U^{-\frac{1}{4}}e^{01} [e^{23} + e^{45} + e^{67} + e^{89}], \\
\Sigma &= U^{-\frac{1}{4}}[e^{1}(e^{2468} + ...) + e^{0}(e^{3456} + ...)].
\end{align}

**B.3 D2-brane**

Taking the smeared M2-brane in 11 dimensions and reducing to IIA gives the following brane solution,

\begin{align}
\bar{d}s^2_{10} &= U^{-\frac{1}{4}}ds^2(M^3) + U^{\frac{1}{2}}ds^2(E^7), \\
\bar{G} &= -d(C_3 - \text{Vol}(M^3)) = U^{-\frac{3}{4}}dU e^{012}, \\
C_3 &= -U^{-1}dt \wedge dx^1 \wedge dx^2 + dt \wedge dx^1 \wedge dx^2 = -U^{-\frac{1}{2}}(1 - U)e^{012}, \\
\exp(\phi) &= U^{\frac{1}{4}}, \\
\epsilon &= \exp(\phi/6)\dot{\epsilon} = U^{-\frac{1}{2}}\epsilon_0,
\end{align}

Where we have chosen a gauge for \( B \) such that \( C_3 \) vanishes asymptotically, \( U \) is harmonic on \( E^7 \) and \( \Gamma_0120 \epsilon_0 = \epsilon_0 \). If we also take the projections

\begin{align}
\Gamma_{034} \epsilon_0, = \Gamma_{056} \epsilon_0, = \Gamma_{078} \epsilon_0, = \Gamma_{013579} \epsilon_0, = \epsilon_0,
\end{align}

then we have that the spinor bi-linears become

\begin{align}
X &= 0, \\
K &= -U^{-\frac{1}{4}}e^0, \\
\tilde{K} &= U^{-\frac{1}{4}}e^1, \\
\Omega &= U^{-\frac{1}{2}}[e^{12} + e^{34} + e^{56} + e^{78}], \\
Z &= U^{-\frac{1}{2}}[e^{1468} + ...] + U^{-\frac{1}{2}}e^{01}[e^{12} + e^{34} + e^{56} + e^{78}] e^9, \\
\Sigma &= U^{-\frac{1}{4}}[e^{13579} + ... + e^{0}(e^{1234} + ...)].
\end{align}

**B.4 supertube**

The supergravity version of the flatspace supertube discovered in [24] was found by [30] to correspond to the dimensional reduction of an M-theory solution describing the intersection of two rotating M5 branes and an M2 brane. After dimensional reduction the solution is as follows,

\begin{align}
\bar{d}s^2_{10} &= -U^{-1}V^{-\frac{1}{2}}(dt - A)^2 + U^{-1}V^{\frac{1}{2}}dx^2 + V^{\frac{1}{2}}dy^2, \\
\exp(\phi) &= U^{-\frac{1}{2}}V^2, \\
\tilde{G} &= -V^{-1}e^{01}dA, \\
dC_3 &= -U e^{01}d(U^{-1}A) - U^{\frac{1}{2}}V^{-1}Ae^{1}d(U^{-1}A),
\end{align}
\[ C_3 = -e^{01}A, \quad (B.37) \]
\[ H = -U^{-1}e^{01}dU - U^{-\frac{1}{2}}V^{-\frac{1}{4}}e^1dA, \quad (B.38) \]
\[ B = (1 - U)e^{01} + U^\frac{1}{2}V^{-\frac{1}{4}}e^1A, \quad (B.39) \]
\[ A = V^{-1}(dt - A) - dt, \quad (B.40) \]
\[ F = dA = -U\frac{1}{2}V^{-\frac{7}{4}}dV^0 - V^{-1}dA, \quad (B.41) \]
\[ \epsilon = \exp(\phi/6)\dot{\epsilon} = U^{-\frac{1}{4}}V^{-\frac{3}{4}}\epsilon_0, \quad (B.42) \]
\[ \Gamma_{0\epsilon_0} = \Gamma_{0\epsilon_0} = \epsilon_0. \quad (B.43) \]

We are also free to make the following choice of \( \epsilon_0 \) in order to reduce the degrees of freedom in the spinor parameter to one.

\[ \Gamma_{2345}\epsilon_0 = \Gamma_{2367}\epsilon_0 = \Gamma_{2389}\epsilon_0 = \epsilon_0. \quad (B.44) \]

The bilinears are then

\[ X = U^{-\frac{1}{4}}V^{-\frac{1}{4}}, \quad (B.45) \]
\[ K = -U^{-\frac{1}{4}}V^{-\frac{3}{4}}e^0, \quad (B.46) \]
\[ \bar{K} = -U^{-\frac{1}{2}}V^{-\frac{1}{4}}e^1, \quad (B.47) \]
\[ \Omega = -U^{-\frac{1}{2}}V^{-\frac{1}{4}}e^{01}, \quad (B.48) \]
\[ Z = U^{-\frac{1}{4}}V^{-\frac{1}{4}}[e^{2345} + ...], \quad (B.49) \]
\[ \Sigma = U^{-\frac{1}{4}}V^{-\frac{1}{4}}[-e^0(e^{2345} + ...)]. \quad (B.50) \]

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