Hyperelliptic Sigma Functions and Adler–Moser Polynomials

V. M. Buchstaber and E. Yu. Bunkova

Received June 18, 2021; in final form, June 18, 2021; accepted June 21, 2021

ABSTRACT. In a 2004 paper by V. M. Buchstaber and D. V. Leykin, published in “Functional Analysis and Its Applications,” for each $g > 0$, a system of $2g$ multidimensional heat equations in a nonholonomic frame was constructed. The sigma function of the universal hyperelliptic curve of genus $g$ is a solution of this system. In our previous work, published in “Functional Analysis and Its Applications,” explicit expressions for the Schrödinger operators that define the equations of this system were obtained in the hyperelliptic case.

In this work we use these results to show that if the initial condition of the system is polynomial, then its solution is uniquely determined up to a constant factor. This has important applications in the well-known problem of series expansion for the hyperelliptic sigma function. We give an explicit description of the connection between such solutions and the well-known Burchnall–Chaundy polynomials and Adler–Moser polynomials. We find a system of linear second-order differential equations that determines the corresponding Adler–Moser polynomial.

KEY WORDS: Schrödinger operator, polynomial Lie algebra, polynomial dynamical system, heat equation in a nonholonomic frame, differentiation of Abelian functions with respect to parameters, Adler–Moser polynomial, Burchnall–Chaundy equation, Korteweg–de Vries equation.

DOI: 10.1134/S0016266321030011

1. Introduction

Let $g \in \mathbb{N}$. Given a meromorphic function $f$ on $\mathbb{C}^g$, a vector $\omega \in \mathbb{C}^g$ is a period if $f(z + \omega) = f(z)$ for all $z \in \mathbb{C}^g$. If a meromorphic function $f$ has $2g$ independent periods in $\mathbb{C}^g$, then $f$ is called an Abelian function. Thus, an Abelian function is a meromorphic function on the complex torus $T^g = \mathbb{C}^g/\Gamma$, where $\Gamma$ is the lattice formed by the periods.

We work with the universal hyperelliptic curve of genus $g$ in the model

$$V_\lambda = \{(x, y) \in \mathbb{C}^2 : y^2 = x^{2g+1} + \lambda_4 x^{2g-1} + \lambda_6 x^{2g-2} + \cdots + \lambda_{4g} x + \lambda_{4g+2}\}.$$ 

Each curve is defined by specification of the parameters $\lambda = (\lambda_4, \lambda_6, \ldots, \lambda_{4g}, \lambda_{4g+2}) \in \mathbb{C}^{2g}$. Let $B \subset \mathbb{C}^{2g}$ be the subspace of parameters such that the curve $V_\lambda$ is nonsingular for $\lambda \in B$. Then we have $B = \mathbb{C}^{2g} \setminus \Sigma$, where $\Sigma$ is the discriminant hypersurface of the universal curve.

For each $\lambda \in B$, the set of periods of holomorphic differentials on the curve $V_\lambda$ generates a lattice $\Gamma_\lambda$ of rank $2g$ in $\mathbb{C}^g$. A hyperelliptic function of genus $g$ (see [1]–[3]) is a meromorphic function on $\mathbb{C}^g \times B$ such that, for each $\lambda \in B$, its restriction to $\mathbb{C}^g \times \lambda$ is an Abelian function. Here the torus $T^g$ is the Jacobian variety $\mathcal{J}_\lambda = \mathbb{C}^g/\Gamma_\lambda$ of the curve $V_\lambda$. We denote by $\mathcal{F}$ the field of hyperelliptic functions of genus $g$. Properties of this field, see [2] and [3].

We use the theory of hyperelliptic Kleinian functions (see [2], [4]–[6], and [7] for elliptic functions). Take coordinates $(z, \lambda)$ in $\mathbb{C}^g \times B \subset \mathbb{C}^{3g}$. Let $\sigma(z, \lambda)$ be the hyperelliptic sigma function (or the elliptic sigma function in the case of the genus $g = 1$). We denote $\partial_k = \partial/\partial z_k$. Following [1], [3], and [8], we use the notation

$$\zeta_k = \partial_k \ln \sigma(z, \lambda), \quad \varphi_{k_1, \ldots, k_n} = -\partial_{k_1} \cdots \partial_{k_n} \ln \sigma(z, \lambda),$$

where $n \geq 2$ and $k_s \in \{1, 3, \ldots, 2g - 1\}$. The functions $\varphi_{k_1, \ldots, k_n}$ give examples of hyperelliptic functions. The field $\mathcal{F}$ is the field of fractions of the polynomial ring $\mathcal{P}$ generated by the functions $\varphi_{k_1, \ldots, k_n}$, where $n \geq 2$ and $k_s \in \{1, 3, \ldots, 2g - 1\}$; see also [9].
Note that we denote the coordinates in $\mathbb{C}^g$ by $z = (z_1, z_3, \ldots, z_{2g-1})$. The indices of coordinates $z = (z_1, z_3, \ldots, z_{2g-1}) \in \mathbb{C}^g$ and of parameters $\lambda = (\lambda_4, \lambda_6, \ldots, \lambda_{4g}, \lambda_{4g+2}) \in \mathbb{C}^{2g}$ determine their weights; namely, $\text{wt } z_k = -k$ and $\text{wt } \lambda_k = k$. For suitable weights of all the other variables, all the equations in this paper are of homogeneous weight.

We note a property of the hyperelliptic sigma function $\sigma(z, \lambda)$ that holds also for the more general case of $(n, s)$-curves [10]; at $z = 0$ it can be expanded in a homogeneous series of degree $-\frac{1}{2}g(g + 1)$ in $z$ with coefficients polynomial in $\lambda$ as

$$\sigma(z, \lambda) = \sum_{|I|>0} \sigma_I(\lambda) z^I,$$

where $I = (i_1, i_3, \ldots, i_{2g-1}) \in \mathbb{Z}_2^g$, $z^I = z_1^{i_1} \cdots z_{2g-1}^{i_{2g-1}}$, and $\sigma_I(\lambda) \in \mathbb{Q}[\lambda]$. In this work we normalize the hyperelliptic sigma function by the condition

$$\sigma(z, \lambda) = z_1^{(g+1)/2} + \ldots.$$

In the case $g = 1$, we obtain the elliptic Weierstrass sigma function with $g_2 = -4\lambda_4$ and $g_3 = -4\lambda_6$:

$$\sigma(z, \lambda) = z_1 - \frac{g_2}{2} \frac{z_1^5}{5!} - 6g_3 \frac{z_1^7}{7!} + \ldots.$$

In the case $g = 2$, we obtain the hyperelliptic Kleinian sigma function

$$\sigma(z, \lambda) = z_1^3 - 3z_3 + \frac{1}{420} \lambda_4 z_1^7 + \frac{1}{4} \lambda_4 z_1^4 z_3 - \frac{1}{1890} \lambda_6 z_1^9 + \frac{1}{30} \lambda_6 z_1^6 z_3 + \frac{1}{2} \lambda_6 z_1^3 z_3^2 - \frac{1}{2} \lambda_6 z_3^3 + \ldots.$$

For $g = 1$ and $2$, there exist efficient algorithms for recursive calculation of the coefficients $\sigma_I(\lambda)$ in the series $\sigma(z, \lambda)$; see [11] and [12]. For any $g \geq 1$, the construction of the series $\sigma(z, \lambda)$ is known; see [13]. This construction can be implemented for any curve $V_\lambda$, including singular curves; see [14] and [11].

In the case of the most singular curve, namely, the curve

$$V_0 = \{(x, y) \in \mathbb{C}^2 : y^2 = x^{2g+1}\},$$

the sigma function is given by the homogeneous polynomial $\sigma(z, 0)$, which is called the rational limit of the sigma function.

Let us define the polynomial Lie algebra of vector fields tangent to the discriminant hypersurface $\Sigma$ in $\mathbb{C}^{2g}$. We denote it by $\mathcal{L}_L$. For this polynomial Lie algebra [15], the generators $\{L_0, L_2, L_4, \ldots, L_{4g-2}\}$ are the vector fields

$$L_{2k} = \sum_{s=2}^{2g+1} v_{2k+2,2s-2}(\lambda) \frac{\partial}{\partial \lambda_{2k}},$$

where $v_{2k+2,2s-2}(\lambda) \in P$ and $P$ is the ring of polynomials in $\lambda \in \mathcal{B} \subset \mathbb{C}^{2g}$. At a point $\lambda \in \mathcal{B}$ these vector fields determine a $2g$-dimensional nonholonomic frame.

The structure of a Lie algebra as a $P$-module with generators $1, L_0, L_2, L_4, \ldots, L_{4g-2}$ is determined by the polynomial matrices $V(\lambda) = (v_{2i,2j}(\lambda))$, where $i, j = 1, \ldots, 2g$, and $C(\lambda) = (c_{2i,2j}^{2k}(\lambda))$, where $i, j, k = 0, \ldots, 2g - 1$, such that

$$[L_{2i}, L_{2j}] = \sum_{k=0}^{2g-1} c_{2i,2j}^{2k}(\lambda)L_{2k}, \quad [L_{2i}, \lambda_{2q}] = v_{2i+2,2q-2}(\lambda), \quad [\lambda_{2q}, \lambda_{2r}] = 0.$$
Here $\lambda_q$ is the operator of multiplication by the function $\lambda_q$ in $P$.

In the case of the Lie algebra $L_L$, explicit expressions for the matrix $V(\lambda)$ can be found in Section 4.1 of [16] (see also [15] and Lemma 3.1 in [12]). The elements of this matrix are given by the following formulas. For convenience, we assume that $\lambda_s = 0$ for all $s \not\in \{0, 4, 6, \ldots, 4g, 4g + 2\}$ and $\lambda_0 = 1$. Let $k, m \in \{1, \ldots, 2g\}$. If $k \leq m$, then we set

$$v_{2k,2m}(\lambda) = \sum_{s=0}^{k-1} 2(k + m - 2s)\lambda_s \lambda_{2(k+m-s)} - \frac{2k(2g - m + 1)}{2g + 1} \lambda_{2k} \lambda_{2m},$$

and if $k > m$, then we set $v_{2k,2m}(\lambda) = v_{2m,2k}(\lambda)$. The structure polynomials $c_{2i,2j}^2(\lambda)$ are described in Theorem 2.5 of [14].

The vector field $L_0$ is the Euler vector field; namely, since $\text{wt} \lambda_{2k} = 2k$, we have

$$[L_0, \lambda_{2k}] = 2k \lambda_{2k}, \quad [L_0, L_{2k}] = 2k L_{2k}.$$ 

This determines the weights of the vector fields $L_k$, namely, $\text{wt} L_{2k} = 2k$.

The classical Lie–Witt algebra $W_0$ (see [17]) over the field $\mathbb{C}$ of complex numbers is generated by operators $l_{2i}$, where $i = 0, 1, 2, \ldots$, with the commutation relations

$$[l_{2i}, l_{2j}] = 2(j - i) l_{2(i+j)}.$$

With respect to the bracket $[\cdot, \cdot]$ the Lie–Witt algebra $W_0$ is generated by the three operators $l_0$, $l_2$, and $l_4$. The graded polynomial Lie algebra $L_L$ over $P$ is a deformation of the Lie–Witt algebra $W_0$. It is also generated by only three operators, $L_0$, $L_2$, and $L_4$. The following relation holds (see Lemma 3.3 in [18]):

$$[L_2, L_{2k}] = 2(k - 1)L_{2k+2} + \frac{4(2g - k)}{2g + 1} (\lambda_{2k+2} L_0 - \lambda_4 L_{2k-2}).$$

Now we introduce Schrödinger operators. We consider the space $\mathbb{C}^{3g}$ with coordinates $(z, \lambda)$ and let $C(z, \lambda)$ denote the ring of differentiable functions in $z$ and $\lambda$. We set

$$Q_{2k} = L_{2k} - H_{2k}, \quad k = 0, 1, \ldots, 2g - 1,$$

where

$$H_{2k} = \frac{1}{2} \sum (\alpha_{ab}^{(k)}(\lambda) \partial_a \partial_b + 2\beta_{ab}^{(k)}(\lambda) z_a \partial_b + \gamma_{ab}^{(k)}(\lambda) z_a z_b) + \delta^{(k)}(\lambda),$$

the summation is over odd $a$ and $b$ from 1 to $2g - 1$, and $\alpha_{ab}^{(k)}(\lambda)$, $\beta_{ab}^{(k)}(\lambda)$, $\gamma_{ab}^{(k)}(\lambda)$, and $\delta^{(k)}(\lambda)$ are polynomials in $\lambda$.

**Definition 1.1.** The system of equations

$$Q_{2k} \psi = 0$$

for $\psi = \psi(z, \lambda)$ is called the system of heat equations in the nonholonomic frame $L_k$. The operators $Q_{2k}$ are called Schrödinger operators.

In [14] a solution to the following problem was given.

**Problem 1.1.** Find sufficient conditions on $\{\alpha^{(i)}(\lambda), \beta^{(i)}(\lambda), \gamma^{(i)}(\lambda), \delta^{(i)}(\lambda)\}$ for the operators (1) to give a representation of the Lie algebra $L_L$ in the ring of operators on $C(z, \lambda)$.
As shown in [14], the system of heat equations with Schrödinger operators \( Q_{2k} \) that give a solution to Problem 1.1 determines the hyperelliptic sigma function \( \sigma(z, \lambda) \). This makes it possible to construct the theory of hyperelliptic Kleinian functions starting from such a system. In what follows, we denote by \( Q_{2k} \) the Schrödinger operators that give a solution to Problem 1.1. A construction of these operators is given in [14].

In this work we show that, for any solution \( \psi(z, \lambda) \) of (3) such that the expression \( \psi(z, 0) \) is a polynomial, this polynomial coincides with the rational limit of the sigma function \( \sigma(z, 0) \) up to a constant factor. The condition \( \psi(1, 0, \ldots, 0) = 1 \) normalizes this constant factor.

By Theorem 2.6 in [14], if \( \psi(z, \lambda) \) is an entire function such that \( \psi(z, 0) = \sigma(z, 0) \), then it coincides with the hyperelliptic sigma function \( \sigma(z, \lambda) \). Therefore, we obtain the following result.

**Theorem 1.2.** If an entire function \( \psi(z, \lambda) \) is a solution of the system (3) of heat equations with Schrödinger operators \( Q_{2k} \) that give a solution to Problem 1.1 and \( \psi(z, 0) \) is a polynomial with \( \psi(1, 0, \ldots, 0) = 1 \), then \( \psi(z, \lambda) \) is the hyperelliptic sigma function.

Let us note that, for the sigma function \( \sigma(z, \lambda) \) of a nonsingular curve \( \mathcal{V}_\lambda \) on the Jacobian variety of this curve, the Abelian function

\[
\varphi_{1,1}(z, \lambda) = -\frac{\partial^2}{\partial z_1^2} \ln \sigma(z, \lambda)
\]

determines a solution of the form \( u(z) = 2\varphi_{1,1}(z, \lambda) \) of the Korteweg–de Vries (KdV) hierarchy; see [6]. In [19] the inverse problem was posed and solved. Namely, in the notation of [19], the equation

\[
2\partial_z^2 \log f = -u,
\]

where \( u \) is a solution of the stationary \( g \)-KdV equation, was considered. The problem of supplementing (4) with natural conditions under which its solution is unique was solved. This problem is deeply connected with the problem of expressing the sigma function in terms of tau functions [20]. In [21] tau functions were introduced by using Eq. (4).

In [22] the so-called Adler–Moser polynomials were introduced. They give solutions to the stationary \( g \)-KdV hierarchy. The construction of these polynomials uses a recurrent sequence of inhomogeneous first-order differential equations with polynomial coefficients, which was introduced by Burchnall and Chaundy in [23]. The remarkable property of this sequence is that it has polynomial solutions. These solutions naturally arise in a number of problems (see [24]) and are called the Burchnall–Chaundy polynomials.

The rational limit \( \sigma(z, 0) \) also determines a solution \( -2\partial_z^2 \ln \sigma(z, 0) \) of the KdV hierarchy (the proof uses the fact that all coefficients \( \sigma_1(\lambda) \) of the series \( \sigma(z, \lambda) \) are polynomials in \( \lambda \)). This naturally leads to the problem of describing a relationship between the polynomial \( \sigma(z, 0) \) and Adler–Moser polynomials [22]. We give a solution to this problem more precise than the result presented in [25].

The work is organized as follows.

In Section 2 we give explicit formulas for the operators \( H_{2k} \) that were found in [18].

In Section 3 we give the corresponding examples in the cases \( g = 1, 2, 3 \), and 4.

Section 4 contains results aimed at proving Theorem 1.2. For \( k = 0, 1, \ldots, 2g - 1 \), we set

\[
\hat{H}_{2k} = H_{2k}|_{\lambda=0}.
\]

These second-order linear differential operators act on functions in \( z = (z_1, \ldots, z_{2g-1}) \). A direct verification shows that the operators \( -\hat{H}_{2k} \) determine a representation of the Lie–Witt algebra, namely, \( [\hat{H}_2, \hat{H}_{2k-2}] = 2(k - 2)\hat{H}_{2k} \).

The main result of Section 4 is Theorem 4.6: For each genus \( g \), any polynomial solution \( \psi(z) \) of the system

\[
\hat{H}_0 \psi(z) = 0, \quad \hat{H}_2 \psi(z) = 0, \quad \hat{H}_4 \psi(z) = 0
\]

(5)
coincides with the rational limit of the sigma function up to a multiplicative constant.

In Section 5 we introduce differential operators $A_{2k}$ with $k \geq 0$. These operators act on the ring of functions of an infinite number of variables $z_1, z_3, \ldots, z_{2k-1}, \ldots$. Note that all $A_{2k}$ for $k > 0$ are operators of second order. It follows directly from the formulas for $A_{2k}$, $k \geq 0$, that if $z_{2s-1} = 0$ and $\partial_{2s-1} = 0$ for $s > g$, then $A_{2k} = -\hat{H}_{2k}$. We show that the Lie algebra with generators $A_{2k}$, $k \geq 0$, over the field $\mathbb{Q}$ of rational numbers coincides with the Lie–Witt algebra $W_0$ generated by the operators $A_0$, $A_2$, and $A_4$.

In Section 6 we consider the problem of constructing the Lie algebra of derivations of $\mathcal{F}$. This Lie algebra has $3g$ generators: $\mathcal{L}_{2k-1}$, where $k = 1, \ldots, g$, and $\mathcal{L}_{2k}$, where $k = 0, \ldots, 2g - 1$. Here $\mathcal{L}_{2k-1} = \partial/\partial z_{2k-1}$. The operators $\mathcal{L}_{2k}$ contain, as summands, the operators $L_{2k}$ of differentiation over the parameters $\lambda$. In general form, the method for constructing the operators $\mathcal{L}_{2k}$ is given in [26] and [27]. Here we specify the explicit form of the operators $\mathcal{L}_0$, $\mathcal{L}_2$, and $\mathcal{L}_4$ for all $g \geq 1$. The cases $g = 1, 2, 3, 4$ are considered in detail.

In Section 7 we describe the construction of polynomial dynamical systems that correspond to the differential operators $\mathcal{L}_{2k}$. We focus on the relationship with the KdV equation.

In Section 8 we give examples of such systems in the case of the genus $g = 3$.

In Section 9 we describe the connection of polynomial solutions of system (5) with the Burchall–Chaundy polynomials ([23], [24]) and Adler–Moser polynomials ([22], [25]).

2. Explicit Expressions for Schrödinger Operators That Determine Hyperelliptic Sigma Functions

The construction of the operators $Q_{2k}$ in [14] uses the condition (see Eq. (1.3) in [14]) that the commutator $[Q_{2i}, Q_{2j}]$ is determined by a formula over $P$ with the same coefficients as the formula for $[L_{2i}, L_{2j}]$. In other words, the Lie algebra generated by the operators $Q_{2i}$ with $i = 0, 1, \ldots$ is yet another realization of a deformation of the Lie–Witt algebra $W_0$. To effectively describe this Lie algebra, one needs to obtain explicit formulas for $Q_0$, $Q_2$, and $Q_4$. These formulas were found in [18]. For $Q_{2k} = L_{2k} - H_{2k}$, we obtain

$$H_0 = \sum_{s=1}^{g} (2s-1) z_{2s-1} \partial_{2s-1} - \frac{g(g+1)}{2},$$

$$H_2 = \frac{1}{2} \partial_1^2 + \sum_{s=1}^{g-1} (2s-1) z_{2s-1} \partial_{2s+1} - \frac{4}{2g+1} \lambda_4 \sum_{s=1}^{g-1} (g-s) z_{2s+1} \partial_{2s-1}$$

$$+ \sum_{s=1}^{g} \left( \frac{2s-1}{2} \lambda_{4s} - \frac{2(g-s+1)}{2g+1} \lambda_4 \lambda_{4s-4} \right) z_{2s-1}^2,$$

$$H_4 = \partial_1 \partial_3 + \sum_{s=1}^{g-2} (2s-1) z_{2s-1} \partial_{2s+3} + \lambda_4 \sum_{s=1}^{g-1} (2s-1) z_{2s+1} \partial_{2s+1} - \frac{6}{2g+1} \lambda_6 \sum_{s=1}^{g-1} (g-s) z_{2s+1} \partial_{2s-1}$$

$$+ \sum_{s=1}^{g} \left( (2s-1) \lambda_{4s+2} - \frac{3(g-s+1)}{2g+1} \lambda_6 \lambda_{4s-4} \right) z_{2s-1}^2$$

$$+ \sum_{s=1}^{g-1} (2s-1) \lambda_{4s+4} z_{2s-1} z_{2s+1} - \frac{g(g-1)}{2} \lambda_4.$$

Here $\lambda_s = 0$ for all $s \notin \{0, 4, 6, \ldots, 4g, 4g+2\}$ and $\lambda_0 = 1$. 
Lemma 2.1 (Lemmas 3.1 and 4.2 in [18]). For Schrödinger operators, the coefficients in (2) are

\[
\alpha_{a,b}^{(k)}(\lambda) = 1 \text{ if } a + b = 2k \text{ and } a, b \in 2\mathbb{N} + 1,
\]

\[
\alpha_{a,b}^{(k)}(\lambda) = 0 \text{ if } a + b \neq 2k \text{ and } a, b \in 2\mathbb{N} + 1,
\]

\[
\beta_{a,b}^{(k)}(\lambda) \text{ is a linear function in } \lambda,
\]

\[
\gamma_{a,b}^{(k)}(\lambda) \text{ is a quadratic function in } \lambda,
\]

\[
\delta^{(k)}(\lambda) = \left( -\frac{1}{4}(2g - k + 1)(2g - k) + \frac{1}{2} \left( g + \left\lfloor \frac{k + 1}{2} \right\rfloor - k \right) \left( g - \left\lfloor \frac{k + 1}{2} \right\rfloor \right) \right) \lambda_{2k}.
\]

Corollary 2.2 (Corollary 4.4 in [18]). For \( k = 3, 4, 5, \ldots, 2g - 1 \),

\[
Q_{2k} = \frac{1}{2(k - 2)}[Q_2, Q_{2k-2}] - \frac{2(2g - k + 1)}{(k - 2)(2g + 1)}(\lambda_{2k}Q_0 - \lambda_4 Q_{2k-4}).
\]

This relation recursively defines the operators \( Q_{2k} \) for \( k = 3, 4, 5, \ldots, 2g - 1 \) and yields explicit expressions for these operators.

3. Explicit Formulas in the Case of \( g = 1, 2, 3, 4 \)

In this section, to illustrate the results of Section 2, we give the operators \( H_0, H_2, \) and \( H_4 \) for \( g = 1, 2, 3, 4 \). The explicit forms of all operators \( H_{2k} \) for these \( g \) is given in [18].

3.1. The Schrödinger operators for the genus \( g = 1 \). In this case, the explicit formulas for \( \{H_{2k}\} \) in (1) are

\[
H_0 = z_1 \partial_1 - 1, \quad H_2 = \frac{1}{2} \partial_1^2 - \frac{1}{6} \lambda_4 z_1^2.
\]

3.2. The Schrödinger operators for the genus \( g = 2 \). In this case, the explicit formulas for \( \{H_{2k}\} \) in (1) are

\[
H_0 = z_1 \partial_1 + 3z_2 \partial_3 - 3,
\]

\[
H_2 = \frac{1}{2} \partial_1^2 - \frac{4}{5} \lambda_4 z_3 \partial_1 + z_1 \partial_3 - \frac{3}{10} \lambda_4 z_1^2 + \left( \frac{3}{2} \lambda_8 - \frac{2}{5} \lambda_4^2 \right) z_3^2,
\]

\[
H_4 = \partial_1 \partial_3 - \frac{6}{5} \lambda_6 z_3 \partial_1 + \lambda_4 z_3 \partial_3 - \frac{1}{5} \lambda_6 z_1^2 + \lambda_8 z_1 z_3 + \left( 3 \lambda_{10} - \frac{3}{5} \lambda_4 \lambda_6 \right) z_3^2 - \lambda_4.
\]

3.3. The Schrödinger operators for the genus \( g = 3 \). In this case, the explicit formulas for \( \{H_{2k}\} \) in (1) are

\[
H_0 = z_1 \partial_1 + 3z_3 \partial_3 + 5z_5 \partial_5 - 6,
\]

\[
H_2 = \frac{1}{2} \partial_1^2 - \frac{8}{7} \lambda_4 \partial_3 + \left( z_1 - \frac{4}{7} \lambda_4 z_5 \right) \partial_3 + 3z_3 \partial_5 - \frac{5}{14} \lambda_4 z_1^2 + \left( \frac{3}{2} \lambda_8 - \frac{4}{7} \lambda_4^2 \right) z_3^2 + \left( \frac{5}{2} \lambda_{12} - \frac{2}{7} \lambda_4 \lambda_8 \right) z_5^2,
\]

\[
H_4 = \partial_1 \partial_3 - \frac{12}{7} \lambda_6 z_3 \partial_1 + \left( \lambda_4 z_3 - \frac{6}{7} \lambda_6 z_5 \right) \partial_3 + (z_1 + 3 \lambda_4 z_5) \partial_5 - \frac{2}{7} \lambda_6 z_1^2 + \lambda_8 z_1 z_3 + \left( 3 \lambda_{10} - \frac{6}{7} \lambda_4 \lambda_6 \right) z_3^2 + 3 \lambda_1 z_3 z_5 + \left( 5 \lambda_{14} - \frac{3}{7} \lambda_6 \lambda_8 \right) z_5^2 - 3 \lambda_4.
\]
3.4. The Schrödinger operators for the genus $g = 4$. In this case, the explicit formulas for $\{H_{2k}\}$ in (1) are

$$
H_0 = z_1 \partial_1 + 3z_3 \partial_3 + 5z_5 \partial_5 + 7z_7 \partial_7 - 10,
H_2 = \frac{1}{2} \partial_1^2 + z_1 \partial_1 + 3z_3 \partial_3 + 5z_5 \partial_5 - \frac{4}{9} \lambda_4 (3z_3 \partial_1 + 2z_5 \partial_3 + z_7 \partial_5) - \frac{7}{18} \lambda_4 z_1^2
+ \left(\frac{3}{2} \lambda_8 - \frac{2}{3} \lambda_4^2\right) z_3^2 + \frac{5}{2} \lambda_12 - \frac{4}{9} \lambda_4 \lambda_8) z_5^2 + \left(\frac{7}{2} \lambda_16 - \frac{2}{9} \lambda_4 \lambda_{12}\right) z_7^2,
$$

$$
H_4 = \partial_1 \partial_3 + z_1 \partial_5 + 3z_3 \partial_7 + \lambda_4(z_3 \partial_3 + 3z_5 \partial_5 + 5z_7 \partial_7) - \frac{2}{3} \lambda_6 (3z_3 \partial_1 + 2z_5 \partial_3 + z_7 \partial_5)
- \frac{1}{3} \lambda_6 z_1^2 + \lambda_8 z_1 z_3 + \left(3 \lambda_{10} - \lambda_4 \lambda_6\right) z_3^2 + 3 \lambda_12 z_3 z_5 + \left(5 \lambda_{14} - \frac{2}{3} \lambda_6 \lambda_8\right) z_5^2
+ 5 \lambda_16 z_5 z_7 + \left(7 \lambda_{18} - \frac{1}{3} \lambda_6 \lambda_{12}\right) z_7^2 - 6 \lambda_4.
$$

4. Adler–Moser Polynomials and the Rational Limit of Sigma Functions

Let $\psi(z, \lambda)$ be a solution of the system (3) of heat equations. We have

$$
L_{2k} \psi(z, \lambda) = H_{2k} \psi(z, \lambda)
$$

for $k = 0, 1, \ldots, 2g - 1$. Note that from the form of the operators $L_{2k}$ it follows that, for $\lambda = 0$, we have $L_{2k} \psi(z, 0) = 0$ for all $k$. Thus, the equations

$$
H_{2k} \psi(z, 0) = 0
$$

hold for $k = 0, 1, \ldots, 2g - 1$.

By the rational limit of the operators $H_{2k}$ we denote the operators

$$
\hat{H}_{2k} = H_{2k} |_{\lambda = 0}.
$$

From the form of the operators $H_{2k}$ it follows that

$$
(H_{2k} \psi)(z, 0) = \hat{H}_{2k} (\psi(z, 0)).
$$

Therefore, the function $\psi(z, 0)$ is a solution of the system of rational limit heat equations

$$
\hat{H}_{2k} \psi = 0,
$$

where $k = 0, 1, \ldots, 2g - 1$. We consider its subsystem for $k = 0, 1, 2$.

For a genus $g$, we denote by $m_g(z)$ a polynomial solution $m_g(z) = \psi(z)$ of the system

$$
\hat{H}_0 \psi(z) = 0, \quad \hat{H}_2 \psi(z) = 0, \quad \hat{H}_4 \psi(z) = 0
$$

with initial conditions $m_g(1, 0, \ldots, 0) = 1$.

**Example 4.1.** We have $m_1(z) = z_1$.

**Example 4.2.** We have $m_2(z) = z_1^3 - 3z_3$.

**Example 4.3.** We have $m_3(z) = z_1^6 - 15z_1^2 z_3 + 45z_1 z_5 - 45z_3^2$.

**Example 4.4.** We have $m_4(z) = z_1^{10} - 45z_1^7 z_3 + 315z_1^5 z_5 - 1575z_1^3 z_7 + 4725z_1^2 z_3 z_5 - 4725z_1 z_3 z_7 + 4725z_3 z_7 - 4725z_5^2$. 

185
Example 4.5. We have
\[ m_5(z) = z_1^{15} - 105z_1^{12}z_3 + 1260z_1^{10}z_5 + 1575z_1^9z_3^2 - 14175z_1^8z_7 + 14175z_1^7z_3z_5 - 33075z_1^6z_3^3 + 99225z_1^5z_9 - 297675z_1^4z_3z_7 - 297675z_1^3z_5^2 + 1488375z_1^4z_3^2z_5 - 992250z_1^3z_3^4 - 1488375z_1^3z_9z_5 + 1488375z_1^3z_5z_7 + 446125z_1^2z_3^2z_7 - 446125z_1^2z_3z_5^2 - 1488375z_1z_3^3z_5 + 446125z_1z_5z_9 - 446125z_1^2z_3^2 + 1488375z_3^3z_5 - 446125z_3^2z_9 + 8930250z_3z_5z_7 - 446125z_3^2z_5^3. \]

We will show further on that the polynomials \( m_g(z) \) satisfy the differential equation
\[ m'_{g+1}m_{g-1} - m_{g+1}m'_{g-1} = (2g + 1)m_g^2, \]
where the prime denotes differentiation with respect to \( z_1 \) and \( m_0(z) = 1 \). This coincides with the description of the polynomials \( \theta_k \) in [22], which were later called Adler–Moser polynomials. Their second logarithmic derivatives with respect to \( z_1 \) give rational solutions of the Korteweg–de Vries equation; see Section 9 for more detailed definitions and results.

Remark 4.6. In [22] the variables of the polynomials \( \theta_k(\tau_1, \ldots, \tau_k) \) were determined by the relation \( \tau_1 = z_1 \) and the normalization condition that the coefficient of \( z_1^{k(k-1)/2} \) in \( \theta_{k+1} \) is equal to \( \tau_{k+1} \). From Examples 4.1–4.5 we see that, for \( g = 1, 2, 3, 4, 5, \)
\[ m_g(z) = \theta_g(\tau_1, \ldots, \tau_g), \]
where \( \tau_1 = z_1, \tau_2 = -3z_3, \tau_3 = 45z_5, \tau_4 = -1575z_7, \) and \( \tau_5 = -33075z_3^3 + 99225z_9 \).

Lemma 4.1. The following expressions for rational limits hold:
\[ \hat{H}_0 = -\frac{g(g + 1)}{2} + \sum_{s=1}^{g} (2s - 1)z_{2s-1}\partial_{2s-1}, \]
\[ \hat{H}_2 = \frac{1}{2}\partial_1^2 + \sum_{s=1}^{g-1} (2s - 1)z_{2s-1}\partial_{2s+1}, \]
\[ \hat{H}_4 = \partial_1\partial_3 + \sum_{s=1}^{g-2} (2s - 1)z_{2s-1}\partial_{2s+3}. \]

This lemma is a direct corollary of the formulas of Section 2.

From Corollary 2.2 for \( k = 3, 4, 5, \ldots, 2g - 1 \) we obtain
\[ -2(k - 2)\hat{H}_{2k} = [\hat{H}_2, \hat{H}_{2k-2}]. \tag{6} \]

Lemma 4.2. The following expressions for rational limits hold:
\[ \hat{H}_{2k} = \frac{1}{2} \sum_{s=1}^{k} \partial_{2s-1}\partial_{2k+1-2s} + \sum_{s=1}^{g-k} (2s - 1)z_{2s-1}\partial_{2s+2k-1} \]
for \( k = 1, 2, 3, 4, 5, \ldots, 2g - 1 \). Here \( \partial_s = 0 \) if \( s > 2g - 1 \).

Proof. For \( k = 1 \) and \( k = 2 \), this formula holds by Lemma 4.1. For \( k = 3, 4, 5, \ldots, 2g - 1 \), it follows from (6). \( \square \)

This lemma leads us to the operators that we will consider in the next section of this paper.
Lemma 4.3. The conditions
\[ \hat{H}_0m_g(z) = \hat{H}_2m_g(z) = \hat{H}_4m_g(z) = 0 \]
and \( m_g(1, 0, \ldots, 0) = 1 \) determine the polynomial \( m_g(z) \) uniquely.

Proof. We set
\[ m_g(z) = \sum_{i_1, i_3, \ldots, i_{2g-1}} a(i_1, i_3, \ldots, i_{2g-1})z_1^{i_1}z_3^{i_3} \cdots z_{2g-1}^{i_{2g-1}}. \]

The equation \( \hat{H}_0m_g(z) = 0 \) implies
\[ a(i_1, i_3, \ldots, i_{2g-1}) = 0 \quad \text{for} \quad i_1 + 3i_3 + \cdots + (2g-1)i_{2g-1} \neq \frac{g(g+1)}{2}. \]

Let us find the coefficients with \( i_1 + 3i_3 + \cdots + (2g-1)i_{2g-1} = g(g+1)/2 \).

We have \( a(g(g+1)/2, 0, \ldots, 0) = 1 \), because \( m_g(1, 0, \ldots, 0) = 1 \). We find the other coefficients \( a(i_1, i_3, \ldots, i_{2g-1}) \) by induction on the weight
\[ \text{wt}(i_1, i_3, \ldots, i_{2g-1}) = i_1 + i_3 + \cdots + i_{2g-1}, \]
moving from the highest weight to lower ones. Among the coefficients of the same weight we find first the coefficients with \( i_{2g-1} \neq 0 \), then with \( i_{2g-2} = 0 \), \( i_{2g-3} \neq 0 \), and so on; at the \( k \)th step we find the coefficients \( a(i_1, i_3, \ldots, i_{2g-2k+1}, 0, \ldots, 0) \), where \( i_{2g-2k+1} \neq 0 \).

Consider the operator (see Lemma 4.2)
\[ \hat{H}_{2(g-k)} = \frac{1}{2} \sum_{s=1}^{g-k} \partial_{2s-1}\partial_{2g-2k-2s+1} + z_1\partial_{2g-2k+1} + \sum_{s=2}^{k} (2s-1)z_{2s-1}\partial_{2g-2k+2s-1}. \]

By (6) we have \( \hat{H}_{2(g-k)}m_g(z) = 0 \). The coefficient of
\[ z_1^{i_1+1}z_3^{i_3} \cdots z_{2g-2k+1}^{i_{2g-2k+1}-1} \]
in this equation gives a relation between \( a(i_1, i_3, \ldots, i_{2g-2k+1}, 0, \ldots, 0) \) and the coefficients \( a(j_1, j_3, \ldots, j_{2g-1}) \), where either
\[ \text{wt}(j_1, j_3, \ldots, j_{2g-1}) = \text{wt}(i_1, i_3, \ldots, i_{2g-2k+1}, 0, \ldots, 0) + 2 \]
or \( j_p = 1 \) for some \( p > 2g-2k+1 \). In fact, it gives an expression for
\[ i_{2g-2k+1}a(i_1, i_3, \ldots, i_{2g-2k+1}, 0, \ldots, 0) \]
as a polynomial with integer coefficients in \( a(j_1, j_3, \ldots, j_{2g-1}) \), where the conditions on \( (j_1, j_3, \ldots, j_{2g-1}) \) are given above. This provides the induction step and thus proves the lemma.

Corollary 4.4. There is no nonzero polynomial solution \( m_g^0(z) = \psi(z) \) to the system
\[ \tilde{H}_0\psi(z) = 0, \quad \tilde{H}_2\psi(z) = 0, \quad \tilde{H}_4\psi(z) = 0 \]
with condition \( m_g^0(1, 0, \ldots, 0) = 0 \).

Proof. For a solution \( m_g(z) \) of Lemma 4.3, the expression \( m_g(z) + m_g^0(z) \) gives another solution, as the problem is linear. This contradicts the statement of the lemma. \( \square \)
Corollary 4.5. The conditions
\[ \hat{H}_0 \psi(z) = 0, \quad \hat{H}_2 \psi(z) = 0, \quad \hat{H}_4 \psi(z) = 0 \]
determine the polynomial \( m_g^c(z) = \psi(z) \) uniquely up to a multiplicative constant.

Proof. Let \( m_g^c(1, 0, \ldots, 0) = c \). By Corollary 4.4 we have \( c \neq 0 \) for a nonzero polynomial \( m_g^c(z) \), and thus we have \( m_g^c(z) = cm_g(z) \) for the solution \( m_g(z) \) of Lemma 4.3.

Theorem 4.6. For each genus \( g \), any polynomial solution \( \psi(z) \) of the system
\[ \hat{H}_0 \psi(z) = 0, \quad \hat{H}_2 \psi(z) = 0, \quad \hat{H}_4 \psi(z) = 0 \] (7)
coincides with the rational limit of the sigma function up to a multiplicative constant.

Proof. By Theorem 2.6 in [14] the function \( \sigma(z, \lambda) \) is a solution of the system (3) of heat equations. Thus, \( \sigma(z, 0) \) is a polynomial solution of the system (7), and by Corollary 4.5 it coincides with any other polynomial solution up to a multiplicative constant. \( \square \)

Theorem 4.7. For each genus \( g \), any polynomial solution \( \psi(z) \) of the system
\[ \hat{H}_0 \psi(z) = 0, \quad \hat{H}_2 \psi(z) = 0, \quad \hat{H}_4 \psi(z) = 0 \]
coincides with the corresponding Adler–Moser polynomial up to a multiplicative constant and a change of variables.

Proof. This result follows from the relationship between Adler–Moser polynomials, the Schur–Weierstrass polynomials, and rational limits of sigma functions. These results will be discussed in detail in Section 9; see also Theorem 6.3 in [25] and Exercise 7.35(c) in [28].

5. Lie Subalgebra of the Witt Algebra

For \( k = 0, 1, 2, \ldots \), we set
\[ A_{2k} = -\frac{1}{2} \sum_{s=1}^{k} \partial_{2s-1} \partial_{2k+1-2s} \sum_{s=1}^{\infty} (2s-1) z_{2s-1} \partial_{2s+2k-1}. \] (8)

Note that \( A_0 \) is the Euler vector field:
\[ A_0 = -\sum_{s=1}^{\infty} (2s-1) z_{2s-1} \partial_{2s-1}. \]

Given \( g \), we set \( z_{2s-1} \equiv 0 \) for \( s > g \); thus, for each \( g \), the sum in (8) is finite. Setting \( \partial_s = 0 \) for \( s > 2g - 1 \), we obtain the following relations between \( A_{2k} \) and rational limits of the operators \( \hat{H}_{2k} \):
\[ A_{2k} = -\hat{H}_{2k} \]
for \( k = 1, \ldots, 2g - 1 \) and
\[ A_0 = -\hat{H}_0 - \frac{g(g + 1)}{2}. \]

The following lemma shows that the operators \( A_{2k} \) are generators of the Lie subalgebra \( W_0 \) of the Witt algebra. In [17] the Lie subalgebra \( W_{-1} \) of the Witt algebra appears in a closely related construction.

Lemma 5.1. The following commutation relation holds:
\[ [A_{2i}, A_{2j}] = 2(j - i) A_{2(i+j)}. \]
In this section we give explicitly a part of a solution of the problem of constructing the Lie algebra of derivations of \( \mathcal{F} \), i.e., of finding \( 3g \) independent differential operators \( \mathcal{L} \) such that \( \mathcal{L} \mathcal{F} \subset \mathcal{F} \). The setting of the problem, as well as a general approach to the solution, can be found in [26] and [27]. An overview is given in [3]. In [29], [1], [8], and [30] an explicit solution of this problem was obtained for \( g = 1, 2, 3, 4 \).

For any \( g \), the operators \( \mathcal{L}_{2i-1} = \partial_{2i-1}, i \in \{1, \ldots, g\} \), belong to the set of generators of this Lie algebra. Here we give explicitly three of its other generators.

6. Derivations of the Field of Genus-\( g \) Hyperelliptic Functions

We have

\[
\sum_{s=1}^{i} (2s - 1) \partial_{2i+1-2s} \partial_{2s+2j-1} - \sum_{s=1}^{j} (2s - 1) \partial_{2j+1-2s} \partial_{2s+2i-1} \\
= \sum_{s=1}^{i} (2i - 2s + 1) \partial_{2s-1} \partial_{2i+2j-2s+1} - \sum_{s=1}^{j} (2j - 2s + 1) \partial_{2s-1} \partial_{2i+2j-2s+1} \\
= 2(i - j) \sum_{s=1}^{\min(i,j)} \partial_{2s-1} \partial_{2i+2j-2s+1} + \text{sign}(i - j) \sum_{s=1}^{\max(i,j)} (2s - 1) \partial_{2s-1} \partial_{2i+2j-2s+1} \\
+ \frac{1}{2} \text{sign}(i - j) \sum_{s=1}^{\max(i,j)} (2s - 1) \partial_{2s-1} \partial_{2i+2j-2s+1} \\
+ \frac{1}{2} \text{sign}(i - j) \sum_{s=1}^{\max(i,j)} (2s - 1) \partial_{2s-1} \partial_{2i+2j-2s+1} \\
= (i - j) \sum_{s=1}^{\min(i,j)} \partial_{2s-1} \partial_{2i+2j+1-2s} \\
\]

therefore,

\[
[A_{2i}, A_{2j}] = \sum_{s=1}^{i} (2s - 1) \partial_{2i+1-2s} \partial_{2s+2j-1} - \sum_{s=1}^{j} (2s - 1) \partial_{2j+1-2s} \partial_{2s+2i-1} \\
+ \sum_{s=1}^{\infty} (2s + 2i - 1) (2s - 1) \partial_{2s-1} \partial_{2i+2j-1} \\
- \sum_{s=1}^{\infty} (2s + 2j - 1) (2s - 1) \partial_{2s-1} \partial_{2s+2i+2j-1} \\
= (i - j) \sum_{s=1}^{i+j} \partial_{2s-1} \partial_{2i+2j+1-2s} + 2(i - j) \sum_{s=1}^{\infty} (2s - 1) \partial_{2s-1} \partial_{2s+2i+2j-1} \\
= 2(j - i) A_{2(i+j)}.
\]
Theorem 6.1. The operators

\[ \mathcal{L}_0 = L_0 - \sum_{s=1}^{g} (2s-1)z_{2s-1}\partial_{2s-1}, \]

\[ \mathcal{L}_2 = L_2 - \zeta_1 \partial_1 - \sum_{s=1}^{g-1} (2s-1)z_{2s-1}\partial_{2s+1} + \frac{4}{2g+1} \lambda_4 \sum_{s=1}^{g-1} (g-s)z_{2s+1}\partial_{2s-1}, \]

\[ \mathcal{L}_4 = L_4 - \zeta_3 \partial_1 - \zeta_1 \partial_3 - \sum_{s=1}^{g-2} (2s-1)z_{2s-1}\partial_{2s+3} \]

\[ - \lambda_4 \sum_{s=1}^{g-1} (2s-1)z_{2s+1}\partial_{2s+1} + \frac{6}{2g+1} \lambda_6 \sum_{s=1}^{g-1} (g-s)z_{2s+1}\partial_{2s-1} \]

belong to the Lie algebra of derivations of \( \mathcal{F} \).

Proof. The theorem follows directly from the explicit formulas given in Section 2 and Theorems 13 and 14 of [27]. □

Let us give, for illustration, formulas for the corresponding operators for \( g = 1, 2, 3, 4 \) (cf. [29], [1], [8], and [30]). They follow from formulas of Section 3.

6.1. Differential operators for the genus \( g = 1 \).

\[ \mathcal{L}_0 = L_0 - z_1 \partial_1, \quad \mathcal{L}_2 = L_2 - \zeta_1 \partial_1. \]

6.2. Differential operators for the genus \( g = 2 \).

\[ \mathcal{L}_0 = L_0 - z_1 \partial_1 - 3z_3 \partial_3, \quad \mathcal{L}_2 = L_2 - \zeta_1 \partial_1 + \frac{4}{5} \lambda_4 z_3 \partial_1 - z_1 \partial_3, \]

\[ \mathcal{L}_4 = L_4 - \zeta_3 \partial_1 - \zeta_1 \partial_3 + \frac{6}{5} \lambda_6 z_3 \partial_1 - \lambda_4 z_3 \partial_3. \]

6.3. Differential operators for the genus \( g = 3 \).

\[ \mathcal{L}_0 = L_0 - z_1 \partial_1 - 3z_3 \partial_3 - 5z_5 \partial_5, \]

\[ \mathcal{L}_2 = L_2 - \zeta_1 \partial_1 + \frac{4}{7} \lambda_4 z_3 \partial_1 - z_1 \partial_3 + \frac{4}{7} \lambda_4 z_5 \partial_3 - 3z_3 \partial_5, \]

\[ \mathcal{L}_4 = L_4 - \zeta_3 \partial_1 - \zeta_1 \partial_3 + \frac{12}{7} \lambda_6 z_3 \partial_1 - \lambda_4 z_3 \partial_3 + \frac{6}{7} \lambda_6 z_5 \partial_3 - z_1 \partial_5 - 3\lambda_4 z_5 \partial_5. \]

6.4. Differential operators for the genus \( g = 4 \).

\[ \mathcal{L}_0 = L_0 - z_1 \partial_1 - 3z_3 \partial_3 - 5z_5 \partial_5 - 7z_7 \partial_7, \]

\[ \mathcal{L}_2 = L_2 - \zeta_1 \partial_1 - z_1 \partial_3 - 3z_3 \partial_3 - 5z_5 \partial_5 - 7z_7 \partial_7 + \frac{4}{9} \lambda_4 (3z_3 \partial_1 + 2z_5 \partial_3 + z_7 \partial_5), \]

\[ \mathcal{L}_4 = L_4 - \zeta_3 \partial_1 - \zeta_1 \partial_3 - z_1 \partial_5 - 3z_3 \partial_7 \]

\[ - \lambda_4 (3z_3 \partial_3 + 3z_5 \partial_5 + 5z_7 \partial_7) + \frac{2}{3} \lambda_6 (3z_3 \partial_1 + 2z_5 \partial_3 + z_7 \partial_5). \]

7. Polynomial Dynamical Systems and the Korteweg–de Vries Equation

Consider the complex linear space \( \mathbb{C}^{3g} \) with coordinates \( \{x_{i,j}\} \), where \( i \in \{1, 2, 3\} \) and \( j \in \{1, 3, \ldots, 2g - 1\} \). In the notation of Section 1, we define the map \( \mathbb{C}^g \times \mathcal{B} \to \mathbb{C}^{3g} \) by the relations

\[ (x_{1,j} \quad x_{2,j} \quad x_{3,j}) = (\varphi_{1,j}(z, \lambda) \quad \varphi_{1,1,j}(z, \lambda) \quad \varphi_{1,1,1,j}(z, \lambda)) \]
for all \( j \in \{1, 3, \ldots, 2g-1\} \). Under this map the rational functions on \( \mathbb{C}^{3g} \) correspond to hyperelliptic functions of genus \( g \) (see Section 5 in [8] for details). The map associates the differential operators \( \mathcal{L}_0, \mathcal{L}_2, \mathcal{L}_4, \) and \( \mathcal{L}_{2k-1}, k \in \{1, \ldots, g\} \), from Section 6 with polynomial vector fields \( \mathcal{D}_0, \mathcal{D}_2, \mathcal{D}_4, \) and \( \mathcal{D}_{2k-1}, k \in \{1, \ldots, g\} \), in \( \mathbb{C}^{3g} \). In the cases \( g = 1, 2, 3, 4 \), these vector fields are given explicitly in [1], [8], and [30].

For any genus \( g \), we have (see Eq. (22) in [8])

\[
\mathcal{D}_0 = \sum_j (j+1)x_{1,j} \frac{\partial}{\partial x_{1,j}} + (j+2)x_{2,j} \frac{\partial}{\partial x_{2,j}} + (j+3)x_{3,j} \frac{\partial}{\partial x_{3,j}}
\]

and (see Eq. (23) in [8])

\[
\mathcal{D}_1 = \sum_j x_{2,j} \frac{\partial}{\partial x_{1,j}} + x_{3,j} \frac{\partial}{\partial x_{2,j}} + 4(2x_{1,1}x_{2,j} + x_{2,1}x_{1,j} + x_{2,j+2}) \frac{\partial}{\partial x_{3,j}},
\]

where \( x_{2,2g+1} = 0 \).

Let us describe the graded homogeneous dynamical systems in \( \mathbb{C}^{3g} \) determined by these vector fields. Here we follow the approach of [1], where such a description was given in the case of the genus \( g = 2 \).

The dynamical system \( S_0 \), which corresponds to the Euler vector field \( \mathcal{D}_0 \), has the form

\[
\frac{\partial}{\partial \tau_0} x_{i,j} = (i+j)x_{i,j}, \quad i = 1, 2, 3, \quad j = 1, 3, \ldots, 2g-1.
\]

The dynamical system \( S_1 \) corresponding to the vector field \( \mathcal{D}_1 \) has the form

\[
\frac{\partial}{\partial \tau_1} x_{i,j} = x_{i+1,j}, \quad i = 1, 2, \quad j = 1, 3, \ldots, 2g-1,
\]

\[
\frac{\partial}{\partial \tau_1} x_{3,j} = 4(2x_{1,1}x_{2,j} + x_{2,1}x_{1,j} + x_{2,j+2}), \quad j = 1, 3, \ldots, 2g-1,
\]

where \( x_{2,2g+1} = 0 \). Relation (10) implies

\[
\frac{\partial}{\partial \tau_1} x_{3,1} = 4(3x_{1,1}x_{2,1} + x_{2,3}).
\]

The map (9) takes \( \partial/\partial \tau_1 \) to \( \partial_1 \) and \( x_{i,j} \) to the corresponding hyperelliptic \( \wp \)-functions. This gives the KdV equation

\[
4\partial_3 \wp_{1,1} = \partial_1 (\partial_1^2 \wp_{1,1} - 6\wp_{1,1}^2).
\]

The same result can be obtained directly from the algebraic relations between hyperelliptic \( \wp \)-functions; see Corollary 8 in [9].

Using results of Section 4, we see that in the rational limit as \( \lambda \to 0 \) this gives the KdV equation

\[
4\partial_3 \hat{\wp} = \partial_1 (\partial_1^2 \hat{\wp} - 6\hat{\wp}^2)
\]

for the function \( \hat{\wp} = -\partial_1 \partial_1 \ln \hat{\sigma} \), where \( \hat{\sigma} \) is the rational limit of the sigma function.

8. Examples: Polynomial Dynamical Systems Corresponding to Derivations of Hyperelliptic Functions of Genus \( g = 3 \)

The dynamical system \( S_0 \) corresponding to the Euler vector field \( \mathcal{D}_0 \) has the form

\[
\frac{\partial}{\partial \tau_0} x_{i,j} = (i+j)x_{i,j}, \quad i = 1, 2, 3, \quad j = 1, 3, 5.
\]
The dynamical system $S_1$ corresponding to the vector field $D_1$ has the form
\[
\frac{\partial}{\partial \tau_1} x_{i,j} = x_{i+1,j}, \quad i = 1, 2, \quad j = 1, 3, 5,
\]
\[
\frac{\partial}{\partial \tau_1} x_{3,j} = 4(2x_{1,1}x_{2,j} + x_{2,1}x_{1,j} + x_{2,j+2}), \quad j = 1, 3, 5,
\]
where $x_{2,7} = 0$.

The dynamical system $S_2$ corresponding to the vector field $D_2$ has the form
\[
\frac{\partial}{\partial \tau_2} x_{1,1} = \frac{12}{7} \lambda_4 + 2x_{1,1}^2 + 4x_{1,3},
\]
\[
\frac{\partial}{\partial \tau_2} x_{2,1} = 3x_{1,1}x_{2,1} + 5x_{2,3},
\]
\[
\frac{\partial}{\partial \tau_2} x_{3,1} = 3x_{2,1}^2 + 2x_{1,1}x_{3,1} + 6x_{3,3},
\]
\[
\frac{\partial}{\partial \tau_2} x_{1,3} = -\frac{8}{7} \lambda_4 x_{1,1} + 2x_{1,1}x_{1,3} + 6x_{1,5},
\]
\[
\frac{\partial}{\partial \tau_2} x_{2,3} = -\frac{8}{7} \lambda_4 x_{2,1} + 3x_{2,1}x_{1,3} + 7x_{2,5},
\]
\[
\frac{\partial}{\partial \tau_2} x_{3,3} = -\frac{8}{7} \lambda_4 x_{3,1} + 4x_{3,1}x_{1,3} + 3x_{2,1}x_{2,3} - 2x_{1,1}x_{3,3} + 8x_{3,5},
\]
\[
\frac{\partial}{\partial \tau_2} x_{1,5} = -\frac{4}{7} \lambda_4 x_{1,3} + 2x_{1,1}x_{1,5},
\]
\[
\frac{\partial}{\partial \tau_2} x_{2,5} = -\frac{4}{7} \lambda_4 x_{2,3} + 3x_{2,1}x_{1,5},
\]
\[
\frac{\partial}{\partial \tau_2} x_{3,5} = -\frac{4}{7} \lambda_4 x_{3,3} + 4x_{3,1}x_{1,5} + 3x_{2,1}x_{2,5} - 2x_{1,1}x_{3,5},
\]
where
\[
\lambda_4 = -3x_{1,1} + \frac{1}{2} x_{3,1} - 2x_{1,3}.
\]

The calculation of this system is based on results of [8].

9. Appendix. Hyperelliptic Schur–Weierstrass Polynomials, Adler–Moser Polynomials, and the Rational Limit of Sigma Functions

We denote by $e = (e_1, e_2, \ldots)$ the infinite vector with coordinates $e_k$. For convenience, we assume that $e_0 = 1$ and $e_k = 0$ for $k < 0$. We introduce the $g \times g$ matrices $E_g = (e_{g-2i+j+1})$, where $1 \leq i, j \leq g$. We set $\text{wt}(e_k) = -k$.

**Definition 9.1.** The genus-$g$ hyperelliptic Schur polynomial is the polynomial
\[
\text{Sh}_g(e) = \det E_g.
\]
This is a homogeneous polynomial of weight $\frac{1}{2}g(g+1)$ in the $2g - 1$ variables $(e_1, \ldots, e_{2g-1})$.

**Example 9.2.** $E_2 = \begin{pmatrix} e_2 & e_3 \\ 1 & e_1 \end{pmatrix}$, $\text{Sh}_2(e) = e_2 e_1 - e_3$.

**Example 9.3.** $E_3 = \begin{pmatrix} e_3 & e_4 & e_5 \\ e_1 & e_2 & e_3 \\ 0 & 1 & e_1 \end{pmatrix}$, $\text{Sh}_3(e) = e_3 e_2 e_1 + e_5 e_1 - e_4 e_1^2 - e_3^2$. 

192
Now we consider each coordinate $e_k$ as the $k$th elementary symmetric polynomial in infinitely many variables $x = (x_1, x_2, \ldots)$. We set $\text{wt}(x_i) = -1$. The generating series for $e_k(x)$ is

$$E(x, t) = 1 + \sum_{k \geq 0} e_k(x)t^k = \prod_{i > 0}(1 + x_i t).$$

We denote by $\text{Sym}$ the ring of symmetric polynomials in $x$ over the field $\mathbb{Q}$ of rational numbers. There is an isomorphism of graded rings $\text{Sym} \cong \mathbb{Q}[e_1, e_2, \ldots]$. Another multiplicative homogeneous basis in $\text{Sym}$ is given by the Newton polynomials $p_1, \ldots, p_k, \ldots$, where

$$p_k = \sum_{i > 0} x_i^k.$$

We denote by $p = (p_1, p_2, \ldots)$ the infinite vector with coordinates $p_k$. The transition from the basis $\{e_k\}$ to the basis $\{p_k\}$ is described by the following relation between the generating series:

$$E(x, t) = \exp \mathcal{N}(x, t),$$

where $\mathcal{N}(x, t) = \sum_{n > 0} (-1)^{n-1} p_n(x)t^n/n$. We have $\text{wt}(p_k) = -k$.

One can express $e_k(x)$ as a polynomial in $p_1, \ldots, p_k$. It follows from (12) that

$$e_k(x) = \frac{1}{k!} p_k^k + \cdots + (-1)^{k-1} \frac{1}{k} p_k$$

and

$$\frac{\partial e_k(x)}{\partial p_n} = (-1)^{n-1} \frac{1}{n} e_{k-n}(x).$$

**Example 9.4.** $e_2(x) = \frac{1}{2} (p_1^2 - p_2)$.

**Example 9.5.** $e_3(x) = \frac{1}{3} (p_1^3 - 3p_2 p_1 + 2p_3)$.

The general Schur–Weierstrass polynomials were introduced in [25]. They correspond to a more general case of $(n, s)$-curves than that of hyperelliptic curves considered here.

**Definition 9.6.** The Schur polynomial $\text{Sh}_g(e)$, written as a polynomial in multiplicative generators $p_1, p_2, \ldots$, is called the genus-$g$ hyperelliptic Schur–Weierstrass polynomial $\text{ShW}_g(p)$.

**Lemma 9.1 (see [25]).** The Schur–Weierstrass polynomial $\text{ShW}_g(p)$ is a polynomial in $g$ variables $p_{2k-1}, k = 1, \ldots, g$.

**Example 9.7.** $\text{ShW}_2(p) = \frac{1}{3} (p_1^3 - p_3)$.

**Example 9.8.** $\text{ShW}_3(p) = \frac{1}{4} (p_1^6 - 5p_1^3 p_3 + 9p_1 p_5 - 5p_3^2)$.

We set $\text{ShW}_0(p) = 1$. We have $\text{ShW}_1(p) = p_1$. To describe properties of the sequence of polynomials $\text{ShW}_g(p), g = 0, 1, 2, \ldots$, we need some results about Wronskians. We will mainly follow the notation of [22]. Consider a sequence $\{\psi_k(x)\}$, where $k = 0, 1, \ldots$, of smooth functions in $x$, in which $\psi_0(x) = 1$ and $\psi_1(x) = x$. We set $D = d/dx$. The Wronskian of a set of functions $\psi_1, \psi_2, \ldots, \psi_k$ is the determinant $W_k = W_k(\psi_1, \ldots, \psi_k) = \det(D^{i-1} \psi_j), 1 \leq i, j \leq k$, of the Wronski matrix $(D^{i-1} \psi_j)$. We obtain a sequence of smooth functions $W_k = W_k(x), k = 1, 2, \ldots, W_1 = x$. We set $W_0 = 1$ and $f' = Df$.

**Lemma 9.2 ([22], Eq. (2.23)).** Let $\psi_j'' = \psi_{j-1}$. Then the sequence $W_k(\psi_1, \ldots, \psi_k), k = 0, 1, 2, \ldots$, of Wronskians satisfies the system of functional differential equations

$$W_{k+1} W_{k-1} - W_{k+1} W_{k-1}' = W_k^2.$$

**Definition 9.9.** The Burchall–Chaundy equations are the system of functional differential equations

$$\varphi_{k+1} \varphi_{k-1} - \varphi_{k+1} \varphi_{k-1}' = \varphi_k^2$$

with initial conditions $\varphi_0 = 1$ and $\varphi_1 = x$. 193
Burchnell and Chaundy showed in [23] that a sequence of smooth functions \( \varphi_k(x) \), \( k = 0, 1, 2, \ldots \), that gives a solution of system (14) with initial conditions \( \varphi_0 = 1 \) and \( \varphi_1 = x \) is polynomial and each polynomial \( \varphi_k \), where \( k > 1 \), has \( k - 1 \) free parameters. These equations arise naturally in various problems; see, e.g., [24]. In works on this topic the polynomials \( \varphi_k(x) \) are called the Burchnell–Chaundy polynomials.

**Lemma 9.3.** The sequence \( \text{ShW}_g(p) \) of Schur–Weierstrass polynomials considered as a sequence of polynomials \( \varphi_g(x) \), where \( x = p_1 \) and \( p_3, \ldots, p_{2g-1} \) are fixed, is the sequence of Burchnell–Chaundy polynomials.

**Proof.** Consider the polynomials \( e_k(x) \), \( k = 0, 1, 2, \ldots \), as polynomials in \( x = p_1 \) with fixed \( p_3, \ldots, p_{2k-1} \). According to (13), we have \( e'_k = e_{k-1} \). Therefore, the matrix \( E_g \) can be written as \((D^{k-1}e_{g-i+j}(x))\). Thus, the Schur–Weierstrass polynomial \( \text{ShW}_k(p) \) coincides with the Wronskian \( W_k(e_{2g-1}(x), \ldots, e_1(x)) \). Since \( e''_k = e_{k-2} \), the application of Lemma 9.2 completes the proof. \( \Box \)

**Definition 9.10.** A sequence of polynomials \( \theta_k(x), k = 0, 1, 2, \ldots \), where \( \theta_0(x) = 1 \) and \( \theta_1(x) = x = \tau_1 \), satisfying the system of functional differential equations

\[
\theta'_{k+1} \theta_{k-1} - \theta_{k+1} \theta'_{k-1} = (2k+1)\theta_k^2 \tag{15}
\]

is called an Adler–Moser sequence.

We set \( \mu_0 = \mu_1 = 1 \) and

\[
\mu_k = 3^{k-1} \cdot 5^{k-2} \cdots (2k - 1) = \prod_{j=1}^{k} (2k - 2j + 1)^j, \quad k > 1.
\]

A direct verification proves the following lemma.

**Lemma 9.4.** Let \( \varphi_k(x), k = 0, 1, 2, \ldots \), be a solution of the system of equations (14) with initial conditions \( \varphi_0 = 1 \) and \( \varphi_1 = x \). Then the sequence of functions \( \theta_k(x) = \mu_k \varphi_k(x) \) determines a solution of the system (15) with initial conditions \( \theta_0 = 1 \) and \( \theta_1 = x \).

In [22] it was shown that a general solution of the system of equations (15) has the form \( \theta_k = \theta_k(x, \tau_2, \ldots, \tau_k) \), where \( \tau_2, \ldots, \tau_k \) are free parameters not depending on \( x \).

From Eqs. (15) it follows that the choice of parameters \( \tau_k \), \( k \geq 2 \), becomes unambiguous if we fix the normalization

\[
\theta_k(x) = x^{k(k+1)/2} + \cdots, \quad k \geq 0, \tag{16}
\]

\[
\frac{\partial \theta_k(x, \tau_2, \ldots, \tau_k)}{\partial \tau_k} = x^{(k-2)(k-1)/2} + \cdots, \quad k \geq 2. \tag{17}
\]

This allows us to introduce the following notion.

**Definition 9.11.** Universal Adler–Moser polynomials are polynomials \( \theta_k(\tau_1, \ldots, \tau_k) \) in \( k \) independent variables, where \( k = 0, 1, 2, \ldots \), such that in \( x = \tau_1 \) they give a general solution of the system of equations (15) and satisfy (16) and (17).

The polynomials introduced by Adler and Moser in [22] are a specialization of the universal Adler–Moser polynomials from Definition 9.11.

Using formula (13) with \( k \geq 2 \), we obtain

\[
\frac{\partial \text{ShW}_k(p)}{\partial p_{2k-1}} = (-1)^{k+1} \text{ShW}_{k-2}(p).
\]
Theorem 9.5. The Schur–Weierstrass polynomials $\text{ShW}_k(p)$ are uniquely determined by the condition that they satisfy the system of equations (14) and the initial conditions

$$\text{ShW}_k(p) = \mu_k p_1^{k+1/2} + \ldots, \quad k \geq 0,$$

$$\frac{\partial \text{ShW}_k(p)}{\partial p_{2k-1}} = (-1)^{k+1} \frac{1}{2k-1} \mu_{k-2} p_1^{(k-2)(k-1)/2} + \ldots, \quad k \geq 2.$$

Proof. Using Lemma 9.4 and Definition 9.11, we obtain the theorem.

Corollary 9.6. The following expression holds:

$$\text{ShW}_g(p) = \frac{1}{\mu_g} \theta_g(\tau_1, \ldots, \tau_g), \quad \text{where} \quad \tau_k = (-1)^{k+1} \frac{1}{2k-1} \frac{\mu_k}{\mu_{k-2}} p_{2k-1}.$$

Example 9.12. $\tau_2 = -p_3$, $\tau_3 = 9p_5$.

We set $\hat{\tau} = (\tau_2, \ldots, \tau_g)$, $g \geq 2$, and

$$u_g(x, \hat{\tau}) = -\frac{\partial^2}{\partial x^2} \ln \theta_g(x, \hat{\tau}).$$

Theorem 9.7 (cf. [22]). There is a uniquely determined change of variables $\hat{\tau} \rightarrow \hat{\tau}^*$: $\tau_k = b_k \tau_k^* + h_{2k-1}(\tau^*)$, where $b_k \in \mathbb{Q}$ and $h_{2k-1}(\tau^*)$ is a homogeneous polynomial of weight $2k-1$, such that the function $u_g(x, \hat{\tau}^*)$ satisfies the KdV hierarchy corresponding to the operator $L_g = \partial^2/\partial x^2 + u_g(x, \hat{\tau}^*)$.

Theorem 9.8 (see [31]). The change of variables $\hat{\tau} \rightarrow \hat{\tau}^*$ is described by the following relation between the generating series:

$$\sum_{i \geq 2} \frac{\tau_i}{\alpha_{2i-1}} t^{2i-1} = \text{th}\left( \sum_{i \geq 2} \tau_i^* t^{2i-1} \right), \quad (18)$$

where $\alpha_{2i-1} = (-1)^{i-1} 3^2 \cdots (2i-3)^2 (2i-1)$.

The hyperbolic tangent $\text{tanh}(t)$ is the exponential of the formal group over the ring of integers $\mathbb{Z}$ with the addition law

$$t_1 \oplus \text{tanh} \ t_2 = \frac{t_1 + t_2}{1 + t_1 t_2}.$$

Therefore, formula (18) can be rewritten as

$$\sum_{i \geq 2} \frac{\tau_i}{\alpha_{2i-1}} t^{2i-1} = \bigoplus_{i \geq 2} \text{tanh} \ Z_{2i-1}(t),$$

where $Z_{2i-1}(t) = \text{tanh}(\tau_i^* t^{2i-1})$.

We set $\hat{p} = (p_3, \ldots, p_{2g-1})$ and $\hat{z} = (z_3, \ldots, z_{2g-1})$ and denote by $\hat{\sigma}(z_1, \hat{z})$ the rational limit of the genus-$g$ hyperelliptic sigma function.

Theorem 9.9 (cf. [25]). There is a uniquely determined change of variables $\hat{p} \rightarrow \hat{z}$: $p_{2k-1} = \beta_k z_{2k-1} + q_{2k-1}(\hat{z})$, where $\beta_k \in \mathbb{Q}$ and $q_{2k-1}(\hat{z})$ is a homogeneous polynomial, such that

$$\hat{\sigma}(z_1, \hat{z}) = \text{ShW}_g(p_1, \hat{p}), \quad z_1 = p_1.$$

Acknowledgments

The authors are grateful to A. P. Veselov and V. N. Rubtsov for useful references to works on Adler–Moser polynomials and their generalizations.
References

[1] V. M. Buchstaber, “Polynomial dynamical systems and Korteweg–de Vries equation”, *Trudy Mat. Inst. Steklova*, 2016, 191–215; English transl.: *Proc. Steklov Inst. Math.*, 294 (2016), 176–200.

[2] V. M. Buchstaber, V. Z. Enolskii, and D. V. Leykin, *Multi-dimensional sigma-functions*, arXiv: 1208.0990.

[3] V. M. Buchstaber, V. Z. Enolskii, and D. V. Leykin, “Sigma-functions: old and new results”, *Integrable Systems and Algebraic Geometry*, vol. 2, LMS Lecture Note Series, no. 459, Cambridge Univ. Press, 2019, pp. 175–214.

[4] H. F. Baker, “On the hyperelliptic sigma functions”, *Amer. J. Math.*, 20:4 (1898), 301–384.

[5] V. M. Buchstaber, V. Z. Enolskii, and D. V. Leykin, “Kleinian functions, hyperelliptic Jacobians and applications”, *Reviews Math. Physics*, 10:2 (1997), 3–120.

[6] V. M. Buchstaber, V. Z. Enolskii, and D. V. Leykin, “Hyperelliptic Kleinian functions and applications”, *Solitons, Geometry and Topology: On the Crossroad*, Amer. Math. Soc. Transl., vol. 179, Adv. Math. Sci., no. 33, Providence, RI, 1997, pp. 1–34.

[7] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Cambridge Univ. Press, Cambridge, 1996.

[8] E. Yu. Bunkova, “Differentiation of genus 3 hyperelliptic functions”, *Europ. J. Math.*, 4:1 (2018), 93–112; arXiv:1703.03947.

[9] V. M. Buchstaber, “Multidimensional sigma functions and applications, Victor Enolski (1945–2019)”, *Notices Amer. Math. Soc.*, 67:11 (2020), 1756–1760.

[10] V. M. Buchstaber, D. V. Leykin, and V. Z. Enolskii, “σ-functions of (n,s)-curves”, *Uspekhi Mat. Nauk*, 54:3(327) (1999), 155–156; English transl.: *Russian Math. Surveys*, 54:3 (1999), 628–629.

[11] V. M. Buchstaber and D. V. Leykin, “Addition laws on Jacobian varieties of plane algebraic curves”, *Nonlinear dynamics, Trudy Mat. Inst. Steklova*, vol. 251, 2005, pp. 54–126; English transl.: *Proc. Steklov Inst. Math.*, 251:4 (2005), 49–120.

[12] J. C. Eilbeck, J. Gibbons, Y. Onishi, and S. Yasuda, *Theory of heat equations for sigma functions*, arXiv:1711.08395.

[13] A. Nakayashiki, “Sigma function as tau function”, *Int. Math. Res. Not.*, 2010, No. 3, 373–394.

[14] V. M. Buchstaber and D. V. Leykin, “Heat equations in a nonholomic frame”, *Funkt. Anal. Prilozhen.*, 38:2 (2004), 12–27; English transl.: *Functional Anal. Appl.*, 38:2 (2004), 88–101.

[15] V. M. Buchstaber and D. V. Leykin, “Polynomial Lie algebras”, *Funkt. Anal. Prilozhen.*, 36:4 (2002), 18–34; English transl.: *Functional Anal. Appl.*, 36:4 (2002), 267–280.

[16] V. I. Arnold, *Singularities of Caustics and Wave Fronts*, Mathematics and its Applications, no. 62, Kluwer Academic Publisher Group, Dordrecht, 1990.

[17] V. M. Buchstaber and A. V. Mikhailov, “Infinite-dimensional Lie algebras determined by the space of symmetric squares of hyperelliptic curves”, *Funkt. Anal. Prilozhen.*, 51:1 (2017), 4–27; English transl.: *Functional Anal. Appl.*, 51:1 (2017), 2–21.

[18] V. M. Buchstaber and E. Yu. Bunkova, “Sigma functions and Lie algebras of Schrödinger operators”, *Functional Anal. Appl.*, 54:4 (2020), 229–240.

[19] V. M. Buchstaber and S. Yu. Shorina, “The w-function of the KdV hierarchy”, *Geometry, topology, and mathematical physics*, Amer. Math. Soc. Transl. Ser. 2, no. 212, Amer. Math. Soc., Providence, RI, 2004, pp. 41–66.

[20] A. Nakayashiki, “On algebraic expressions of sigma functions for (n,s)-curves”, *Asian J. Math.*, 14:2 (2010), 175–212; arXiv:0803.2083.

[21] M. E. Kazarian and S. K. Lando, “Combinatorial solutions to integrable hierarchies”, *Uspekhi Mat. Nauk*, 70:3(423) (2015), 77–106; English transl.: *Russian Math. Surveys*, 70:3 (2015), 453–482.

[22] M. Adler and J. Moser, “On a class of polynomials connected with the Korteweg–de Vries equation”, *Comm. Math. Phys.*, 61:1 (1978), 1–30.
[23] J. L. Burchnall and T. W. Chaundy, “A set of differential equations which can be solved by polynomials”, Proc. London Math. Soc., 30:6 (1929–30), 401–414.
[24] A. P. Veselov and R. Willox, “Burchnall–Chaundy polynomials and the Laurent phenomenon”, J. Phys. A: Math. Theor., 48 (2015), 20; arXiv:1407.7394.
[25] V. M. Buchstaber, V. Z. Enolskii, and D. V. Leykin, “Rational analogs of abelian functions”, Funkts. Anal. Prilozhen., 33:2 (1999), 1–15; English transl.: Functional Anal. Appl., 33:2 (1999), 83–94.
[26] V. M. Buchstaber and D. V. Leykin, “Differentiation of Abelian functions with respect to parameters”, Uspekhi Mat. Nauk, 62:4(376) (2007), 153–154; English transl.: Russian Math. Surveys, 62:4 (2007), 787–789.
[27] V. M. Buchstaber and D. V. Leykin, “Solution of the problem of differentiation of Abelian functions over parameters for families of \((n, s)\)-curves”, Funkts. Anal. Prilozhen., 42:4 (2008), 24–36; English transl.: Functional Anal. Appl., 42:4 (2008), 268–278.
[28] R. P. Stanley, Enumerative Combinatorics, vol. 2, University Press, Cambridge, 1999.
[29] F. G. Frobenius and L. Stickelberger, “Über die Differentiation der elliptischen Functionen nach den Perioden und Invarianten”, J. Reine Angew. Math., 92 (1882), 311–337.
[30] V. M. Buchstaber and E. Yu. Bunkova, Differentiation of genus 4 hyperelliptic functions, arXiv: 1912.11379.
[31] A. du Crest de Villeneuve, “From the Adler–Moser polynomials to the polynomial tau functions of KdV”, J. Integrable Syst., 2:1 (2017), 012; arXiv:1709.05632.

V. M. Buchstaber
Steklov Mathematical Institute of Russian Academy of Sciences, Moscow, Russia
E-mail: buchstab@mi-ras.ru

E. Yu. Bunkova
Steklov Mathematical Institute of Russian Academy of Sciences, Moscow, Russia
E-mail: bunkova@mi-ras.ru

Translated by E. Yu. Bunkova