Information-Theoretic Equivalence of Entropic Multi-Marginal Optimal Transport: A Theory for Multi-Agent Communication

Shuchan Wang
swangshuchan@gmail.com

Abstract

In this paper, we propose our information-theoretic equivalence of entropic multi-marginal optimal transport (MOT). This equivalence can be easily reduced to the case of entropic optimal transport (OT). Because OT is widely used to compare differences between knowledge or beliefs, we apply this result to the communication between agents with different beliefs. Our results formally prove the statement that entropic OT is information-theoretically optimal given by Wang et al. [2020] and generalize it to the multi-agent case. We believe that our work can shed light on OT theory in future multi-agent teaming systems.

1 Introduction

After Shannon’s groundbreaking work [Shannon, 1948], information theory has been playing a central role in engineering communication. Shannon’s work focused on the technical problems of communication, which, as defined by Weaver in [Weaver, 1953], studies “how accurately can the symbols of communication be transmitted”. With the rise of artificial intelligence (AI), further problems beyond Shannon’s have received increasing attention. Weaver posed his semantic problem in [Weaver, 1953], “the semantic problems are concerned with the identity, or satisfactorily close approximation, in the interpretation of meaning by the receiver, as compared with the intended meaning of the sender.” Later, Wiener pointed out a problem of agent communication in [Wiener, 1960], “if the communication between these two agencies as to the nature of this purpose is incomplete, it must only be expected that the results of this cooperation will be unsatisfactory.”

In this paper, we are interested in the communication between agents with individual beliefs, where each agent’s belief is described by a probability measure for a specific set of events. Unlike Shannon’s engineering communication, our work focuses more on the use of formal information-theoretic tools to study communication in cognitive science and AI. Following Weaver’s semantic problem, we assume in our problem that the sender conveys a sequence of hypotheses of the events through a message. The receiver infers its hypotheses based on this message and its own belief. We further follow Wiener’s agent communication problem and assume that the purpose of communication is given by a cost function of the system.

The principal contributions are summarized as follows: In Section 2, we formulate the cooperative communication problem of Wang et al. [2020] formally in a stationary memoryless system. We prove the statement in [Wang et al., 2020] that entropic OT is information-theoretically optimal in our formulation and generalize it to MOT in Theorem 1. It leads to a fundamental limit of a multi-agent teaming system in Figure 2. We further show in Proposition 1 that entropic MOT can also be interpreted as a process of Bayesian inference under certain assumptions. In Section 3, we show that our results can be extended to multi-hop communication if the entropic MOT has the graphical structures in [Haastr et al., 2021]. We then present an experimental study of our main results in Section 4 and compare our results with previous studies in Section 5.
1.1 Related Studies

1.1.1 Semantic Communication

One of the latest advances in the communication beyond Shannon’s is called semantic communication \cite{Srinati and Barbarossa, 2021, Lan et al., 2021, Kountouris and Pappas, 2021}, which targets Weaver’s semantic problem. In the words of Weaver in \cite{Wiener, 1960}, these works study “how precisely the transmitted symbols convey the desired meaning”. As a broadly defined problem, different approaches are given in various works, such as goal-oriented approach \cite{Goldreich et al., 2012} and deep learning approach \cite{Xie et al., 2021}.

1.1.2 Cooperative Communication

Cooperative communication is a concept introduced in cognitive sciences and robotics \cite{Wang et al., 2020, Yuan et al., 2022}. It focuses more on how the sender and receiver understand each other, rather than on Weaver’s problem of transmitting symbols. Cooperative communication is closely linked to many other fields, e.g., explainable AI \cite{Edmonds et al., 2019}, linguistics \cite{Goodman and Lassiter, 2015}, inverse reinforcement learning \cite{Wiener, 1960} and pedagogy \cite{Shafto et al., 2014, Yang et al., 2022}. Wiener’s problem in \cite{Wiener, 1960} plays an important role in these works and it is also called value alignment problem in \cite{Hadfield-Menell et al., 2016, Yuan et al., 2022}.

1.1.3 Optimal Transport

There is no doubt that OT is getting increasing attention in the machine learning community and mathematics community; see, e.g., \cite{Montavon et al., 2016, Arjovsky et al., 2017, Zhang et al., 2018, Liu et al., 2021, Cordero-Erausquin, 2017, Bolley et al., 2018}. Entropy OT, given in \cite{Cuturi, 2013, Benamou et al., 2015}, is a variant of the original OT for computational efficiency. It was later found that entropy OT is equivalent to another classic problem called Schrödinger bridge \cite{Leonard, 2014} and has many desirable properties \cite{Feydy et al., 2019}. Recently, Bayesian interpretations of entropic OT and entropic MOT are given in \cite{Wang et al., 2020, Haasler et al., 2021}. The works \cite{Conforti, 2019, Bai et al., 2020, Wang et al., 2022} investigated the functional inequality perspective of entropic OT. In the context of cooperative communication, entropic OT was used to model the process of pedagogy \cite{Wang et al., 2020}.

1.2 Notations

\( \mathbb{N} \) is the set of natural numbers \{0, 1, 2, ...\}. \( \mathbb{R} \) is the set of real numbers. \( \mathbb{R}^n \) is the \( n \)-dimension Euclidean space. \( \mathbb{R}_+ \) denotes the set \( \{x \in \mathbb{R} : x \geq 0\} \). Let \( \mathcal{X} \) be a Polish space, i.e., a separable complete metrizable space. We write an element \( x \in \mathcal{X} \) in lower-case letters and a random variable \( X \) on \( \mathcal{X} \) in capital letters. Let a sequence \( k := (k_1, k_2, \ldots, k_m) \), we define \( k^{-j} \) as \( k \) to remove the \( j \)-th element, i.e., \( k^{-j} := (k_1, \ldots, k_{j-1}, k_{j+1}, \ldots, k_m) \), \( \forall j \in \{1, 2, \ldots, m\} \). \( X_{1:m} \) and \( X_{1:m}^n \) denote sequences of random variables with the sizes of \( m \) and \( m \times n \). \( X^\otimes n \) denotes a \( n \)-length sequence of i.i.d. random variables. We denote \( \mathcal{P}(\mathcal{X}) \) as the set of all probability measures on \( \mathcal{X} \). We denote \( \Pi\{\{P_{X_j}\}_{j=1}^n\} \) as the set of all probability measures supported on \( X_1 \times X_2 \times \cdots \times X_m \), with the marginal measures \( \{P_{X_j}\}_{j=1}^n \). \( \Pi \) is the push-forward of a measure \( P \) on its \( j \)-th marginal.

\( H(\cdot), h(\cdot), I(\cdot; \cdot) \) denote entropy, differential entropy, mutual information, respectively. \( \exp(\cdot) \) is the epigraph of a function \( f \). All the logarithms are the logarithms to the base 2. We use \( \exp() \) to denote the element-wise exponential to the base 2. \( \sum() \) denotes the sum for all the elements of vectors, matrices, and tensors. \( 1 \) denotes an array with all the elements equal to 1.

2 Main Results

We start by stating that our communication problem is under an asymptotic setting in a stationary memoryless system, i.e., the number of events that occur \( n \rightarrow \infty \). We say that a sequence \( X^n \) is i.i.d. if and only if \( X^n \in \mathcal{T}_e(n)(X) \) asymptotically almost surely, where \( \mathcal{T}_e(n)(X) \) is \( \epsilon \)-typical set for a given probability measure \( P_X \); see formal definition in \cite{El Gamal and Kim, 2011} Section 2.4. It is from the fact that the typical set is the smallest probable set \cite{Cover, 1999} Theorem 3.3.1. For
To study the properties of the system in Figure 2, we first give a new definition of entropic MOT.

The cost of the system is widely used for this knowledge transfer [Courty et al., 2017]. Solving many tasks [Dosovitskiy et al., 2020, Brown et al., 2020, Lee et al., 2022]. We also note that knowledge into domain-specific knowledge through its consultants. A similar and long-standing factors, e.g., the total profit and the privacy of the clients. The cost can be a combination of different consultants are dispatched to various clients by agent 1. The sender 1 with belief $P_{X_1}$ encodes its hypotheses $X^n_{1}$ into a message $\omega_{j1,j2} \in \Omega_{j1,j2}$ using an encoding function $f_{j1,j2,n} : \mathcal{T}^{(n)}_e(X_1^j) \rightarrow \Omega_{j1,j2}$, where $\Omega_{j1,j2} := \{1, 2, \cdots, 2^{m_{j1,j2}}\}$ and $R_{j1,j2}$ is called the rate of the communication link. The receiver 2 with belief $P_{X_2}$ receives $\omega_{j1,j2}$ and interprets it as its hypotheses $X^n_{j2}$ using a decoding function $g_{j1,j2,n} : \Omega_{j1,j2} \rightarrow \mathcal{T}^{(n)}_e(X_2^j)$, as illustrated in Figure 1.

$$\begin{align*}
X^n_{j1} & \xrightarrow{\text{Agent } j_1} f_{j1,j2,n}(X^n_{j1}) \in \{1, \cdots, 2^{nR_{j1,j2}}\} & \xrightarrow{\text{Agent } j_2} X^n_{j2}
\end{align*}$$

Figure 1: Communication link between agents $j_1$ and $j_2$.

In this paper, we first study a centralized system with only one sender in Figure 2. The system consists $m$ agents with the beliefs $\{P_{X_i}\}_{i=1}^m$. We default agent 1 to be the sender and it has communication links with the rest of the agents. The received hypotheses can be written as

$$X^n_{2:m}(X^n_{1}) = (g_{12,n}(f_{12,n}(X^n_{1})), \cdots, g_{1m,n}(f_{1m,n}(X^n_{1}))) = (X^n_{2}, \cdots, X^n_{m}).$$

The strategy of communication is defined by the encoding and decoding functions $\{f_{j1,j2,n}, g_{j1,j2,n}\}_{j=1}^m$. The cost of the system $c_n := \frac{1}{n} \sum_{i=1}^n c(x_1, x_2, \cdots, x_m)$, where $c : \mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_m \rightarrow \mathbb{R}_+$ is a given cost function upper bounded by $c_{\text{max}}$. Our goal is to minimize the expectation of the cost, $\forall X^n_i \in \mathcal{T}^{(n)}_e(X_j)$, by finding an optimal strategy. Further assume that agent 1 communicates with each agent $j$ with rate $R_{1j}$, and the total communication rate $R' := \sum_{j=2}^m R_{1j}$.

$$\begin{align*}
X^n_{1} \xrightarrow{\text{Agent 1}} f_{12,n}(X^n_{1}) \in \{1, \cdots, 2^{nR_{12}}\} & \xrightarrow{\text{Agent 2}} X^n_{2} \\
& \xrightarrow{\text{Agent 3}} \cdots \\
& \xrightarrow{\text{Agent } m} X^n_{m}
\end{align*}$$

Figure 2: Diagram of a centralized communication system.

We next give an example of our problem in Figure 2.

**Example 1.** Imagine that agent 1 is a consulting firm and agents 2 to $m$ are its consultants. The consultants are dispatched to various clients by agent 1. Agent 1 communicates the market information $X^n_{1}$ privately with each agent using a message. Agent $j$ receives the message and transfers it into the specific information $X^n_{j}$ of its client’s business. This specific business has a certain belief $P_{X_j}$, which may be related to its budget, target customers, etc. The cost can be a combination of different factors, e.g., the total profit and the privacy of the clients.

The consulting firm in Example 1 can be thought of as a generalist agent that transfers general knowledge into domain-specific knowledge through its consultants. A similar and long-standing question in the field of AI is how to scale the diverse knowledge into a generalist agents capable of solving many tasks [Dosovitskiy et al., 2020, Brown et al., 2020, Lee et al., 2022]. We also note that OT is widely used for this knowledge transfer [Courty et al., 2017].

To study the properties of the system in Figure 2, we first give a new definition of entropic MOT.
Definition 1 (Information-constrained MOT). Assume that all spaces are Polish. Given a lower semi-continuous cost function $c(x_1, x_2, \cdots, x_m): \mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_m \to \mathbb{R}_+$ and $R \in \mathbb{R}_+$, the information-constrained MOT is defined as

$$
\mathcal{W}_c(\{P_{X_j}\}_{j=1}^m; R) := \inf_{P_{X_{1:m}} \in \Pi(\{P_{X_j}\}_{j=1}^m)} \int c(x_1, x_2, \cdots, x_m) \, dP_{X_{1:m}},
$$

where

$$
\Pi(\{P_{X_j}\}_{j=1}^m; R) := \{P \in \Pi(\{P_{X_j}\}_{j=1}^m) : C(X_{1:m}) \leq R\},
$$

and the total correlation [Watanabe 1996] of $m$ agents is defined as

$$
C(X_{1:m}) := \int \log \frac{dP_{X_{1:m}}}{\prod_{j=1}^m dP_{X_j}} \, dP_{X_{1:m}},
$$

(1) is a linear minimization over a convex, weakly compact set [Pass 2012, Wang et al. 2022], thus the solution exists.

Remark 1. The total correlation $C$ quantifies the amount of dependence among a set of different variables. In our case, the denominator $\prod_{j=1}^m dP_{X_j}$ in (3) is fixed, because the marginals are given. Hence, $C(X_{1:m})$ is equivalent to the negative of the entropy $H(X_{1:m})$, which is used for regularization in [Benamou et al. 2015]. We will show later that the Lagrange dual problem of (1) is exactly the entropic MOT in [Benamou et al. 2013].

In Theorem 1 we show that the information-constrained MOT gives the minimal achievable cost of the system in Figure 2.

Theorem 1. Consider the system in Figure 2 agent 1 holds a randomly drawn i.i.d. sequence of hypotheses $X_1^n \in \mathcal{T}^{(n)}_e(X_1)$ and communicates it with others. If the total rate $R' \geq R$, there exists a strategy $\{\{f_{ij,n}, g_{ij,n}\}\}_{j=1}^m$ such that $E[c_n(X_{1:m})] \rightarrow \mathcal{W}_c(\{P_{X_j}\}_{j=1}^m; R)$, as $n \to \infty$. If $R' < R$, $E[c_n(X_{1:m})] \geq \mathcal{W}_c(\{P_{X_j}\}_{j=1}^m; R)$.

Proof. See Appendix A.

Remark 2 (On Theorem 1). Theorem 1 states that the more symbols agents communicate, the better they cooperate. It is because $\mathcal{W}_c(\{P_{X_j}\}; R)$ is monotonic non-increasing w.r.t. $R$ according to Lemma 2. We further note that the joint distribution of the hypotheses $X_{1:m}$ follows the solution of (1) if we use the strategy $\{\{f_{ij,n}, g_{ij,n}\}\}_{j=1}^m$ included in the proof. Thus, we say that the system in Figure 2 with this strategy is the information-theoretical equivalence of (1).

MOT is a generalization of OT, which increases the number of the marginals from 2 to $m$. Let $m = 2$, then (1) becomes the definition of entropic OT given in [Cuturi 2013], which is written as follows,

$$
\mathcal{W}_c(\{P_{X_j}\}_{j=1}^2; R) := \inf_{P \in \Pi(\{P_{X_{1:2}}\})} \int c(x_1, x_2) \, dP,
$$

(4) where the constraint of total correlation in (2) reduces to $I(X_1; X_2) \leq R$. From Theorem 2 we know that (4) gives the minimal achievable cost of the point-to-point communication in Figure 2. This formally proves the statement in [Wang et al. 2020] that entropic OT is information-theoretically optimal.

The intuition of the proof of Theorem 1 is very simple. Without communication, $X_1^n, X_2^n, \cdots, X_m^n$ are i.i.d. in the typical set

$$
\mathcal{A} := \mathcal{T}^{(n)}_e(X_1) \times \mathcal{T}^{(n)}_e(X_2) \times \cdots \times \mathcal{T}^{(n)}_e(X_m),
$$

whose cardinality

$$
|\mathcal{A}| = 2^n \sum_{j=1}^m H(X_j),
$$

To reduce the cost, we want the agents to cooperate with a joint distribution $dP_{1:m}$, i.e., to let $X_{1:m}^n \in \mathcal{T}^{(n)}_e(X_{1:m})$, whose cardinality $|\mathcal{T}^{(n)}_e(X_{1:m})| = 2^n H(X_{1:m})$. From the definition of the total correlation, we have

$$
\frac{|\mathcal{A}|}{|\mathcal{T}^{(n)}_e(X_{1:m})|} = 2^n \sum_{j=1}^m H(X_j) - nH(X_{1:m}) = 2^n C(X_{1:m}).
$$
It means that we need \( nC(X_{1:m}) \) bits to determine a sequence in \( T_c^{(n)}(X_{1:m}) \) from the rather uncertain set \( A \). Due to the property of the total correlation from \( (25) \), these bits can be separated and located over \( m - 1 \) communication links.

Furthermore, if we let an arbitrary agent be the sender, the joint distribution of \( X_{1:m} \) would always be the same solution of \( (1) \). Thus, each agent can be either a sender or a receiver with the same cost, which is a bidirectional communication similar to the one in [Yuan et al., 2022].

Next, we discuss decision-making, in order to find the optimal strategy before communication. From the proof, we notice that the strategy is given by the solution of \( (1) \). Therefore, we propose the diagram of decision-making in Figure [3], in which a centralized controller exists. The controller collects the beliefs of the agents, solves \( (1) \), and distributes the strategy to the agents.

![Figure 3: Decision-making before communication.](image-url)

Now the central problem in decision-making becomes solving \( (1) \). Because it is numerically intractable on infinite supports, we first change the notations into the discrete version. For every \( j \in \{1, 2, \ldots, m\} \), let \( P_{X_j} \) be a discrete measure on a finite support \( \{x_{j,k_j}\}_{k_j=1}^{K_j} \), i.e.,

\[
dP_{X_j} = \sum_{k_j=1}^{K_j} \mu_j(k_j) \delta_{x_{j,k_j}},
\]

where \( \mu_j \in \mathbb{R}^{K_j} \) and \( \text{sum}(\mu_j) = 1 \). Then the joint probability can be written as \( dP_{X_{1:m}} = \sum_k P(k) \delta_{x_k} \), where \( P \in \mathbb{R}^{K_1 \times \cdots \times K_m} \) is a \( m \)-dimensional array indexed as \( P(k) \) for \( k := (k_1, \ldots, k_m) \in \{1, \ldots, K_1\} \times \cdots \times \{1, \ldots, K_m\} \), and \( x_k := (x_{1,k_1}, \ldots, x_{m,k_m}) \).

The cost function can be written as \( c(x_1, \ldots, x_m) = \sum_k c(k) \delta_{x_k} \), where \( c \in \mathbb{R}_+^{K_1 \times \cdots \times K_m} \) is also a \( m \)-dimensional array. The information-constrained MOT now can be written as

\[
\min_{P} \langle c, P \rangle, \text{ s.t. } \Pi_j P = \mu_j, \forall j, \sum_{j=1}^{m} H(\mu_j) - H(P) \leq R,
\]

where \( \langle c, P \rangle := \sum_k c(k) P(k) \), the push-forward on the \( j \)-th marginal \( \Pi_j P(k_j) := \sum_{j-} P(k) \), and by definition, the total correlation is represented by entropy. From the Lagrange duality theorem [Luenberger, 1997, Theorem 1, pp. 224–225], we know that \( (6) \) is equivalent to the following optimization problem,

\[
\max_{\lambda \geq 0} \min_{P: \Pi_j P = \mu_j, \forall j} \langle c, P \rangle + \lambda \left( \sum_{j=1}^{m} H(\mu_j) - H(P) - R \right).
\]

It is easy to see that, \( \forall R \geq 0 \), there always exists a corresponding \( \lambda \) solving the outer maximization of \( (7) \). Therefore, as long as we sweep out all the possible \( \lambda \), we can always find the corresponding \( \lambda \) to a specific \( R \). Thus, we only need to solve the inner minimization problems using the corresponding \( \lambda \), which is equivalent to

\[
\min_{P: \Pi_j P = \mu_j, \forall j} \langle c, P \rangle - \lambda H(P).
\]

Benamou et al. introduced a method called iterative Bregman projection in [Benamou et al., 2015] to solve \( (8) \). We then state their results. Let \( \gamma^{(0)} = \exp(-c/\lambda) \). Let \( l \in \mathbb{N} \) and \( j \in \{1, 2, \ldots, m\} \).

Then, the iterative Bregman projection can be written as

\[
\gamma^{(ml+j)}(k) = \frac{\mu_j(k_j)}{\Pi_j(\gamma^{(ml+j-1)})(k_j)} \gamma^{(ml+j-1)}(k).
\]
When \( l \to +\infty \), \( \gamma^{(mk+j)} \) converges to \( P^* \), which is the solution of \( (8) \) [Bregman, 1967].

Based on the above, we propose our Algorithm 1 for the optimal decision-making in Figure 3, where line 3 can be the coding scheme included in the proof of Theorem 1.

Algorithm 1 Optimal Decision-Making

Input: \( c, \mu_1, \mu_2, \ldots, \mu_m, R \)

Output: \( \{(f_{1,j,n}, g_{1,j,n})\}_{j=1}^m \)

1. Search the corresponding \( \lambda \) of \( R \)
2. Solve \( P^* \) of \( (8) \) using iterative Bregman projection
3. Generate \( \{(f_{1,j,n}, g_{1,j,n})\}_{j=1}^m \) based on \( P^* \)

Recall that our problem is a bidirectional communication in Remark 2. Thus, each agent \( j \) has a conditional communication plan \( P_{X_{m+1}^{j+1}|X_1} \), which is a conditional probability describing how it transmits its hypothesis \( X_j \) to other agents. The decision-making in Figure 3 can also be thought of as how to find the optimal conditional communication plans. In the works on cooperative inference [Yang et al., 2018, Wang et al., 2019, 2020], Bayesian inference is used to obtain these optimal communication plans. We show in the next proposition that, under certain assumptions, the iterative Bregman projection can also be interpreted as Bayesian inference.

**Proposition 1.** Let \( \lambda = 1 \). Given an arbitrary conditional communication plan \( P_{X_{m+1}^{j+1}|X_1} \), we let the cost function satisfy the following,

\[
c(k) = -\log P_{X_{m+1}^{j+1}|X_1}(k).
\]

Let \( j \in \{1, 2, \ldots, m\} \), then the iterative Bregman projection \( (9) \) can be interpreted as

\[
P_{X_{m+1}^{j+1}|X_1}(k) = \frac{\mu_j(k_j)P_{X_{m+1}^{j+1}|X_1}(k)}{\Pi_{j+1}(P_{X_{m+1}^{j+1}|X_1}(k_j))},
\]

where we use the convention that \( X_{m+1} \) means \( X_1 \), and the joint distribution

\[
P_{X_{m+1}^{j+1}}(k) = \mu_j(k_j)P_{X_{m+1}^{j+1}|X_1}(k).
\]

**Proof.** We follow the steps of the iterative Bregman projection in the proof. Because \( \lambda = 1 \) and \( (10) \), the initialization of \( \gamma^{(0)} \) can be written as

\[
\gamma^{(0)} = \exp(-c/\lambda) = P_{X_{m+1}^{j+1}|X_1}.
\]

Because it is a conditional probability, we have

\[
\Pi_1(\gamma^{(0)})(k_1) = \Pi_1(P_{X_{m+1}^{j+1}|X_1})(k_1) = 1(k_1).
\]

Then, by applying \( (9) \), we obtain the joint probability

\[
\gamma^{(1)}(k) = P_{X_{m+1}^{j+1}}(k) = \mu_1(k_1)P_{X_{m+1}^{j+1}|X_1}(k),
\]

where the denominator in \( (9) \) vanishes because of \( (12) \). We borrow the denominator from the second step of \( (9) \), then \( (13) \) becomes

\[
\frac{\gamma^{(1)}(k)}{\Pi_2(\gamma^{(1)})(k_2)} = P_{X_{m+1}^{j+1}|X_2}(k) = \frac{\mu_1(k_1)P_{X_{m+1}^{j+1}|X_1}(k)}{\Pi_2(P_{X_{m+1}^{j+1}})(k_2)}.
\]

The first step of the proposition is now proved. It is easy to generalize to the \( j \)-th step.

**Remark 3 (On Proposition 1).** We observe that \( (13) \) has the form of Bayes’ theorem. Thus, the iterative Bregman projection becomes a Bayesian inference to sequentially update the conditional communication plan of agent \( j + 1 \) using the prior \( \mu_j \) and the previous agent’s conditional communication plan. This proposition generalizes the cooperative inference result in [Wang et al., 2020, Proposition 3] to the multi-agent case.
3 Multi-Hop Communication

In this section, we study a special kind of MOT on discrete supports, which is called MOT with graphical structures [Haasler et al., 2021]. In this case, the topology of communication links in Figure 2 can be changed into a multi-hop system, where some agents can act as both senders and receivers. Other settings are still the same as previously defined. It is trivial that a multi-hop system cannot outperform the centralized system in Figure 2 where all the information is available for agent 1. Therefore, agent 1 can perform the optimal communication. However, we show in this section that the minimal achievable cost of a multi-hop system is also given by (1) when the cost function has graphical structures.

We first introduce the factor graph [Wainwright et al., 2008]. A factor graph $G = (\Gamma, F, E)$ is an undirected bipartite graph, where $\Gamma$ denotes the original set of vertices, $F$ denotes the set of factor vertices, and $E$ is the set of the edges, joining only vertices $j \in \Gamma$ to factors $\alpha \in F$. In our case, $\Gamma$ is the set of the agents.

In our following discussion, we assume that the factor graphs are trees, i.e., they are acyclic. We say that the cost function has a graphical structure if there exists a factor graph $G = (\Gamma, F, E)$ such that the cost tensor can be decomposed as follows,

$$c(k) = \sum_{\alpha \in F} c_\alpha(k_\alpha),$$  \hspace{1cm} (14)

where $k_\alpha := \{k_j : j \in N(\alpha)\}$ denotes the index set of the agents in the neighbor of a factor $\alpha \in F$. Because the iterative Bregman projection (9) is a sequence of normalization over the individual marginals, the solution of (8) has the form [Haasler et al., 2021],

$$P^*(k) = \frac{1}{Z} \prod_{j \in F} \phi_j(k_j) \prod_{\alpha \in F} \psi_\alpha(k_\alpha),$$  \hspace{1cm} (15)

where $Z$ is a normalization constant, $\phi_j$ is called the local potential of vertex $j$, and $\psi_\alpha$ is called the factor potential of factor $\alpha$. It means that the dependencies between the random variables are local, which exist only inside the neighbors of the factors.

**Example 2.** We give an example of factor graph in Figure 4a, where $\Gamma = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $F = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. The neighbors $N(\alpha_1) = \{1, 2, 3, 4\}$, $N(\alpha_2) = \{1, 4, 7\}$, $N(\alpha_3) = \{2, 7\}$, $N(\alpha_4) = \{7, 8, 9\}$.

![Figure 4: The factor graph of a cost function and its corresponding communication scheme.](image)

Lemma 1. For two arbitrary vertices $j_1, j_n \in \Gamma$, there exists a unique $j_1 - j_n$ path in the factor graph.

Proof. From the definition of tree, we know that the graph is connected. Thus there exists at least one $j_1 - j_n$ path. If there are multiple paths, the graph is cyclic, which is against our assumption. Therefore, the uniqueness is proved.

Remark 4. Because the edges join only agents to factors, we can denote the unique path in Lemma 7 between $j_1$ and $j_n$ by $j_1 \alpha_{1,2} \alpha_{2,3} \cdots \alpha_{n-1,n}$, where $j_1, \ldots, j_n \in \Gamma, \alpha_1, \ldots, \alpha_{n-1} \in F, n \geq 2$. In this case, we call agent $j_{n-1}$ as a relay agent.
which is convex and monotonically non-increasing, as discussed in Lemma 2. We further plot the

Theorem 2. For any cost function with the structure of factor graph, there always exists a topology,
where the communication links only exist between the agents in the same neighbor \( N(\alpha), \forall \alpha \in F \),
and the minimal achievable cost with the total rate \( R \) is the same as the centralized system in Figure 2.

Proof. For the converse part, it directly follows from the proof in Appendix A. Next, we show
the achievability. We define a process such that only one communication link is active at each
time \( t \in \{1, \cdots, m-1\} \). We use \( \Gamma^{(t)} \) to denote the set of agents that have received hypotheses.
If there exist a triple \((j_t, j'_t, \alpha_t)\) such that \( j_t \notin \Gamma^{(t)}, j'_t \notin \Gamma^{(t)}, \alpha_t \in F \), and \( j_t, j'_t \in N(\alpha_t) \),
we let agent \( j'_t \) be the local transmitter in \( N(\alpha_t) \) and send a message to \( j_t \). We use the same
coding scheme in Appendix A to encode and decode the hypotheses \( X^n_{j_t} \). That is, we generate i.i.d.
\( X^n_{j_t} \). Because agent \( j'_t \) has the information \( X^n_{\Gamma^{(t)} \cap N(\alpha_t)} \), we encode \( X^n_{j_t} \) by searching the existing
\( (X^n_{j_t}, X^n_{\Gamma^{(t)} \cap N(\alpha_t)}) \) inside the joint typical set \( T^{(n)}_c \) given by \( P^* \). Because of \( (15) \), the random
variables form a Markov chain \( X_{j_1} - X_{j_2} - \cdots - X_{j_t} \) for a path in Lemma 1. Therefore, we have
\( I(X_{\Gamma^{(t)} \cap N(\alpha_t)}; X_{j_t}) \) from the definition of the mutual information, which is the
rate of this communication link. From Lemma 1, we know that this process can always cover all the
agents. The hypotheses \( X_{\Gamma^{(t)} \cap N(\alpha_t)} \) follows the solution of \( (1) \) and the sum of rates \( \sum_{t=1}^{m-1} I(X_{\Gamma^{(t)} \cap N(\alpha_t)}; X_{j_t}) = C(X_{1:m}) \). The achievability is proved.

Remark 5 (On Theorem 2). In Figure 4b, we give the corresponding topology of communication
links of Example 2. The hypotheses of each agent can be propagated along the directed graph from
agent 1 with the minimal achievable cost.

4 Experimental Results

In this section, we present our experimental results for Theorem 1. We study a system with 4 agents.
The supports of the hypotheses are given in Figure 5 by vertices, where a different color indicates that
the support belongs to a different agent. The beliefs of the agents are uniform on these supports.
The cost function \( c(k) := \sum_{j_1, j_2, j_3 \neq j_2} |x_{j_1, k_1} - x_{j_2, k_2}| \), where \( |\cdot| \) denotes the Euclidean distance
in Figure 5.

We use the edges in Figure 5 to represent the optimal joint probabilities for different rates. The density
of an edge is proportional to the joint probability of the two vertices connected by the edge.
The graph is sparser with a larger rate, which means that the relationship between the two sets of
hypotheses is more deterministic. The relationship between the cost and the rate is given in Figure 6
which is convex and monotonically non-increasing, as discussed in Lemma 2. We further plot the
relationship between the sum of the rates and the individual rates in Figure 7.
5 Comparison with Previous Studies

It is not difficult to see that the form of Theorem 1 and the techniques in its proof are very similar to Shannon’s rate-distortion theory [Shannon et al., 1959]. However, since the definitions of these two communication problems are inherently different, we do not overly relate our results to rate-distortion theory. New properties are observed in our agent communication problem, e.g., the total correlation finds its information-theoretic meaning, and the communication is bidirectional.

On the other hand, our result shows that entropic MOT finds the optimal coupling between the typical sets with a given rate. This is similar to the classical OT theory, which is to find the optimal coupling between two probability measures [Villani, 2008]. The coupling between the typical sets may be further viewed as a value alignment, similar to the one in [Yuan et al., 2022].

Next, we compare our work with various studies in agent communication. The communication in our paper is to achieve a specific purpose defined by a cost function. It is similar to the semantic communication with a referee in [Goldreich et al., 2012] or with a referential function in [Newcomb, 1953; Jakobson and Sebeok, 1960]. It is different from Shannon’s communication, where the communication is to transmit and recover some given information. In our Example 1, $X^*$ can also be a proposal to encourage all agents to cooperate, which does not necessarily contain information in Shannon’s sense.
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Before the proof, we first introduce several lemmas.

**Lemma 2.** \( f(R) := W_c(\{P_{X_j}\}; R) \) is convex and monotonically non-increasing w.r.t. \( R \).

**Proof.** To prove that \( f(R) \) is convex, we only need to show that \( \text{epi} f \) is a convex set. We assume that there are two points \( (R_0, f_0(R_0)), (R_1, f_1(R_1)) \) \( \in \text{epi} f \), which are given by two joint distributions \( dP_0 \) and \( dP_1 \) respectively. Let \( P_{\lambda} = \lambda P_0 + (1 - \lambda)P_1, \forall \lambda \in [0, 1] \). We know that \( \int c dP_{\lambda} = \lambda f_0 + (1 - \lambda)f_1 \). From Remark [1], we know that \( C_{P_{\lambda}}(X_{1:m}) \), which is the total correlation with the joint distribution \( dP_{\lambda} \), is equivalent to \(-H(X_{1:m})\). Thus, we have

\[
C_{P_{\lambda}}(X_{1:m}) \leq \lambda R_0 + (1 - \lambda)R_1,
\]

which is from the concavity of the entropy. Because \( P_{\lambda} \) also has the same marginals \( \{X_j\} \) and \([1]\) is a minimization problem, we know that \( (C_{P_{\lambda}}(X_{1:m}), \int c dP_{\lambda}) \in \text{epi} f \). Hence, \((\lambda R_0 + (1 - \lambda)R_1, \int c dP_{\lambda}) \in \text{epi} f \), which means that \( \text{epi} f \) is a convex set.

\( f(R) \) is monotonically non-increasing is simply from the fact that \( \Pi(\{P_{X_j}\}_{j=1}^m; R_1) \subset \Pi(\{P_{X_j}\}_{j=1}^m; R_2) \) if \( R_1 \leq R_2 \). Because \( f(R) \) is obtained from a minimization problem, it is immediately shown. \( \square \)

Then, we give a property of the total correlation. We first define the conditional total correlation.
Definition 2 (Conditional total correlation). The conditional total correlation of $X_{1:m}$ conditioning $Y$ is defined as follows.

$$
C(X_{1:m}|Y) := \int \log \frac{dP_{X_{1:m}|Y}}{\prod_{j=1}^{m} dP_{X_{j}|Y}} dP_{X_{1:m},Y}.
$$

(16)

The total correlation follows the chain rule.

Lemma 3 (Chain rule).

$$
C(X_{1:m}^{n}) := \int \log \frac{dP_{X_{1:m}^{n}}}{\prod_{j=1}^{m} dP_{X_{j}^{n}}} dP_{X_{1:m}^{n}}
= \sum_{i=1}^{n} C(X_{1:m,i}|X_{1:m,i-1},X_{1:m,i-2},\ldots,X_{1:m,1}).
$$

(17)

(18)

Proof. We first assume that $X_{1:m}^{n}$ is discrete. Then, we have

$$
C(X_{1:m}^{n})
= \sum_{j=1}^{m} H(X_{j}^{n}) - H(X_{1:m}^{n})
= \sum_{j=1}^{m} \sum_{i=1}^{n} H(X_{j,i}|X_{j,i-1},X_{j,i-2},\ldots,X_{j,1}) - \sum_{i=1}^{n} H(X_{1:m,i}|X_{1:m,i-1},X_{1:m,i-2},\ldots,X_{1:m,1})
= \sum_{i=1}^{n} C(X_{1:m,i}|X_{1:m,i-1},X_{1:m,i-2},\ldots,X_{1:m,1}).
$$

(19)

(20)

where (19) and (21) are from the definition of total correlation, (20) is from the chain rule of entropy. By substituting differential entropy for entropy, the discrete case can be easily adapted to the continuous case. \(\square\)

(Proof of Theorem 1). First, we prove the achievability. We follow the proof of rate-distortion theory in [Cover, 1999], which is for finite supports $X_{1}, \ldots, X_{n}$. The general case of countably infinite sets can be obtained using, e.g., the information spectrum methods in [Koga et al., 2013].

We give a realization of the coding scheme.

Generation of codebooks: Randomly generate $m-1$ rate distortion codebooks $C_{1j}$, $j \in \{2, \ldots, n\}$. Each $C_{1j}$ consists of $2^{nR_{1j}}$ sequences $X_{j}^{n}$ drawn i.i.d. $\sim \prod_{i=1}^{n} dP_{X_{j}}$, i.e., randomly select $X_{j}^{n} \in T_{\epsilon}^{(n)}(x_{j})$. Indexing these codewords by $\omega_{1j} \in \{1, 2, \ldots, 2^{nR_{1j}}\}$. Reveal all the codebooks to agent 1 and each $C_{1j}$ to agent $j$.

Encoding: For all $j \in \{2, \ldots, m\}$, $\epsilon > 0$ and a function $c(j): X_{1} \times \cdots \times X_{j} \rightarrow \mathbb{R}$, we say that a pair of sequences $(x_{1:j-1}^{n}, x_{j}^{n})$ is distortion $\epsilon$-typical if

$$
\left| -\frac{1}{n} \log dP_{X_{1:j-1}^{n}}(x_{1:j-1}^{n}) - H(X_{1:j-1}) \right| < \epsilon,
$$

$$
\left| -\frac{1}{n} \log dP_{X_{j}^{n}}(x_{j}^{n}) - H(X_{j}) \right| < \epsilon,
$$

$$
\left| -\frac{1}{n} \log dP_{X_{1:j}^{n}}(x_{1:j}^{n}) - H(X_{1:j}) \right| < \epsilon,
$$

$$
\left| c(j)(x_{1:j}^{n}) - \mathbb{E}[c(j)(X_{1:j})] \right| < \epsilon,
$$

13
where \( c_n^{(j)}(x_{1:m}^n) := \frac{1}{n} \sum_{i=1}^n c^{(j)}(x_{1,i}, \cdots, x_{j,i}) \) and \( c^{(m)} = c \).  The set of distortion typical sequences is denoted as \( \mathcal{T}_{c^{(j)},\epsilon}^{(n)} \).

We encode from \( X^n_2 \) to \( X^n_m \) sequentially. For \( j = \{2, \cdots, m\} \), we encode \( X^n_j \) by \( \omega_{1j} \) if there exists a \( \omega_{1j} \) such that \((X^n_{1:j-1}, X^n_j(\omega_{1j})) \in \mathcal{T}_{c^{(j)},\epsilon}^{(n)}\) with \( \epsilon \) sufficiently small. If there is more than one such \( \omega_{1j} \), send the least. If there is no such \( \omega_{1j} \), let \( \omega_{1j} = 1 \). Thus, \( nR_{1j} \) bits suffice to describe the index \( \omega_{1j} \) of the jointly typical codeword.

**Decoding:** For Decoder \( j \), the reproduced sequence is \( X_j^n(\omega_{1j}) \).

We use mathematical induction to show that \( \forall j = \{2, \cdots, m\}, \) \( P((X^n_{1:j-1}, X^n_j) \in \mathcal{T}_{c^{(j)},\epsilon}^{(n)}) \rightarrow 1 \) when \( n \rightarrow \infty \) and \( R_{1j} \geq I(X_{1:j-1}; X_j) \). For the first step of encoding and \( j = 2 \), we have \( P((X^n_1, X^n_2) \in \mathcal{T}_{c^{(2)},\epsilon}^{(n)}) \rightarrow 1 \) when \( n \rightarrow \infty \) and \( R_{12} \geq I(X_1; X_2) \), according to [Cover 1999 pp. 321-324]. Hence, the statement holds for the initial case.

Now, assume that the first \( j - 1 \) steps of encoding give \( P((X^n_{1:j-2}, X^n_{j-1}) \in \mathcal{T}_{c^{(j-1)},\epsilon}^{(n)}) \rightarrow 1 \). From [Cover 1999 Lemma 10.5.1], we can consider that \( X^n_{1:j-1} \) are drawn i.i.d. according to \( dP_{X_{1:j-1}} \) with probability \( 1 - \epsilon \). Hence, the condition of the step \( j - 1 \) is the same as the initial step with probability \( 1 - \epsilon \). Again from [Cover 1999 pp. 321-324], we have \( P((X^n_{1:j-2}, X^n_j) \in \mathcal{T}_{c^{(j)},\epsilon}^{(n)}) \rightarrow 1 \), if \( R_{1j} \geq I(X_{1:j-1}; X_j) \). Therefore, the statement holds and the cost satisfies the following,

\[
c_n(X_{1:m}^n) < \mathbb{E}[c(X_{1:m})] + \epsilon + \epsilon' c_{\max}, \tag{22}
\]

where \( c_{\max} \) is the max cost for any individual sequence and \( \epsilon' \) is the probability such that \((X_{1:m-1}^n, X_m^n) \notin \mathcal{T}_{c,\epsilon}^{(n)} \). Because \( \epsilon \) and \( \epsilon' \) can be sufficiently small, the expectation of the cost is close to \( \mathbb{E}[c(X_{1:m})] \).

Now, we go back to the optimization problem. Because the constraint set of (1) is strictly convex and the objective function is linear, thus there at least exists a solution with the total correlation \( C(X_{1:m}) \leq R \). We can decompose \( C(X_{1:m}) \) as follows,

\[
C(X_{1:m}) = \int \log \frac{dP_{X_{1:m}}}{dP_X} + \log \frac{dP_{X_{1:m-1}}}{\prod_{i=1}^{m-1} dP_X_i} dP_{X_{1:m}} \tag{23}
\]

\[
= I(X_m; X_{1:m-1}) + C(X_{1:m-1}) \tag{24}
\]

\[
= \sum_{j=1}^{m-1} I(X_{j+1}; X_{1:j}), \tag{25}
\]

where (24) is obtained by repeating (23) successively. Therefore, with the sum of the rates \( R' \geq R \geq C(X_{1:m}) \), there exists a sequence of the individual rates \((R_{12}, \cdots, R_{1m})\) which satisfies \( R_{1j} \geq I(X_{1:j-1}; X_j) \). Moreover, \( W_c(P_{X_j}; R) \) is achievable with this sequence of the individual rates \((R_{12}, \cdots, R_{1m})\) from (22). Now we finish the proof of the achievability.
Next, we prove the converse. Assume that the rate is \( R' \), we need to show that the cost \( \geq W_c(\{P_{X_j}\}; R), \forall R' \leq R \). We can show that

\[
\int c_n(x_1^n, \ldots, x_m^n) dP_{X_1^m}
\]

\[
:= \int \frac{1}{n} \sum_{i=1}^{n} c(x_{1,i}, \ldots, x_{m,i}) dP_{X_i^m}
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \int c(x_{1,i}, \ldots, x_{m,i}) dP_{X_i^m} \tag{26}
\]

\[
\geq \frac{1}{n} \sum_{i=1}^{n} W_c(\{P_{X_j}\}; C(X_{1:m,i})) \tag{27}
\]

\[
\geq W_c(\{P_{X_j}\}; \frac{1}{n} \sum_{i=1}^{n} C(X_{1:m,i})) \tag{28}
\]

\[
\geq W_c(\{P_{X_j}\}; \frac{1}{n} C(X_n^m)) \tag{29}
\]

\[
\geq W_c(\{P_{X_j}\}; R), \tag{30}
\]

where (26) is from the linearity of expectation, (27) is from (1), (28) is from Lemma 2. (29) follows from the fact that

\[
C(X_{1:m}^n) = \sum_{j=1}^{m-1} I(X^n_{j+1}; X_{1:j}^n)
\]

\[
\geq \sum_{j=1}^{m-1} H(X_{j+1}^n) - H(X_{1:j}^n|X_{j+1}^n, X_{1:j,1}) \tag{31}
\]

\[
\geq \sum_{j=1}^{m-1} \sum_{i=1}^{n} H(X_{j+1,i}) - H(X_{1:j,i}|X_{j+1,i}, X_{1:j,1,1}, \ldots, X_{1:j,1}) \tag{32}
\]

\[
\geq \sum_{j=1}^{m-1} \sum_{i=1}^{n} I(X_{j+1,i}; X_{1:j,i}) \tag{33}
\]

\[
= \sum_{i=1}^{n} C(X_{1:m,i}), \tag{34}
\]

where (31) and (34) are from (25), (32) is from the chain rule for entropy and the fact that \( X_{j+1}^n \) is i.i.d. and (33) is from the fact that conditioning reduces entropy. Then, using the fact that \( W_c(\{P_{X_j}\}; R) \) is non-increasing w.r.t. \( R \), (29) is obtained. (30) is obtained as follows

\[
nR \geq nR' := n \sum_{j=2}^{m} R_{1,j}
\]

\[
\geq \sum_{j=1}^{m-1} H(f_{1(j+1),n}(X^n_1)) \tag{35}
\]

\[
\geq \sum_{j=1}^{m-1} H(f_{1(j+1),n}(X^n_1)) - H(f_{1(j+1),n}(X^n_1)|X^n_{1:j}) \tag{36}
\]

\[
= \sum_{j=1}^{m-1} I(X^n_{j+1}; X_{1:j}^n)
\]

\[
= C(X_{1:m}^n), \tag{37}
\]
where (35) is from the fact that the range of \( f_{1(j+1),n} \) is at most \( 2^n R_{1(j+1)} \), (36) is from the fact that the conditional entropy is non-negative, (37) is from (25). Now we finish the proof of the converse.