Abstract
We introduce a new class of random variables and their distributions — the class of logarithmic Lambert W random variables (or simply log-Lambert W random variables) for a specific family $F$ of continuous distributions with support on the nonnegative real axis. In particular, we present the basic characteristics of the exact distribution of log-Lambert W random variables for chi-squared distribution, and a generalization, which naturally appears in the statistical inference based on the likelihood of normal random variables. More generally, the class of log-Lambert W random variables is also related to the exact distribution of the Kullback-Leibler $I$-divergence in the exponential family with gamma distributed observations. By simple examples we illustrate their applicability of the suggested random variables and their distributions for the exact (small sample) statistical inference on model parameters based on normally distributed observations.

Keywords: logarithmic Lambert W random variables, exact likelihood based inference

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1. Introduction

Originally Goerg (Goerg (2011, 2012)) introduced the Lambert $W \times F$ random variables (RVs) and families of their distributions as a useful tool for modeling skewed and heavy tailed distributions. In particular, for a continuous location-scale family of $rX \sim F_\theta$, parametrized by a vector $\theta$, Goerg defined a class of location-scale Lambert $W \times F_\theta$ random variables

$$Y = \{U \exp(\gamma U)\} \sigma_X + \mu_X, \quad \gamma \in \mathbb{R},$$

parametrized by the vector $(\theta, \gamma)$, $\mu_X$ and $\sigma_X$ are the location and scale parameters, and $U = (X - \mu_X)/\sigma_X$.

The inverse relation to (1) can be obtained via the multivalued Lambert W function, namely, by the branches of the inverse relation of the function $z = u \exp(u)$, i.e., the Lambert W function satisfies $W(z) \exp(W(z)) = z$, for more details see, e.g., Corless et al. (1996).

Here we formally introduce a class of related but different (transformed) RVs and their distributions, the logarithmic Lambert $W \times F_\theta$ RVs for a specific family of distributions $F_\theta$ defined on the nonnegative real axis.

We shall focus on the family of the so-called log-Lambert $W \times$ chi-squared distributions, which naturally appear in the statistical likelihood based inference of normal RVs. As we shall illustrate later, a specific type of such RVs plays an important role. In particular, the random variable

$$Y = (Q_\nu - \nu) - \nu \log \left( \frac{Q_\nu}{\nu} \right),$$

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and its generalization

\[ Y = (Q_v - a) - c \log \left( \frac{Q_v}{b} \right), \tag{3} \]

where \( Q_v \) is a chi-squared distributed random variable with \( v \) degrees of freedom, \( Q_v \sim \chi_v^2 \), and \( a, b, c \) are further parameters. The random variable \( Q_v \) will be denoted here as the standard log-Lambert \( W \times \chi^2 \) random variable.

Stehlík (Stehlík (2003)) studied related RVs (and their distributions) to derive the exact distribution of the Kullback-Leibler I-divergence in the exponential family with gamma distributed observations. In particular, he derived the cumulative distribution function (CDF) and the probability density function (PDF) of the transformed gamma RVs, which are directly related to the here considered log-Lambert \( W \times \chi^2 \) RVs. Stehlík showed that the I-divergence from the observed vector to the canonical parameter can be decomposed as a sum of two independent RVs with known distributions. Since it is related to the likelihood ratio statistics, Stehlík also calculated the exact distribution of the likelihood ratio tests and discussed the optimality of such exact tests. Recently Stehlík et al. (2014) applied the exact distributions. Since it is related to the likelihood ratio statistics, Stehlík also calculated the exact distribution of the cumulative distribution function (CDF) and the probability density function (PDF) of the transformed gamma RVs, which are directly related to the here considered log-Lambert W analysis and practical importance could benefit from transformation statistics, as it was considered and illustrated in log-Lambert W random variables for certain families of distributions

Let \( X \) be a continuous RV with support on the nonnegative real axis with the probability distribution depending on a (vector) parameter \( \theta \), i.e. \( X \sim F_\theta \), where \( F \) indicates a family of distributions. Here we shall consider the following class of transformed RVs,

\[ Y = g_\theta(X) = \theta_1 - \theta_2 \log(X) + \theta_3 X, \tag{4} \]

where \( g_\theta(\cdot) \) is a strictly convex log-linear transformation on \( x \geq 0 \) for real parameters \( \theta = (\theta_1, \theta_2, \theta_3) \), where \( \theta_1 \in \mathbb{R} \), \( \theta_2 > 0 \), and \( \theta_3 > 0 \). The support of \( Y \) is the set \( y \in (y_{\min}, \infty) \), where

\[ y_{\min} = g_\theta(x_{\min}) = \theta_1 + \theta_2 - \theta_1 \log \left( \frac{\theta_2}{\theta_3} \right), \tag{5} \]

with \( x_{\min} = \theta_2/\theta_3, 0 < x_{\min} < \infty \). Note that

\[ \frac{Y - \theta_1}{\theta_2} = -\log \left( X \exp \left( \frac{\theta_1}{\theta_2} X \right) \right). \tag{6} \]

Therefore, the random variable \( Y \), defined by \( 4 \), will be called the log-Lambert \( W \) random variable associated with the distribution \( F_\theta \) (the minus) log-Lambert \( W \times F_\theta \) RV, and the distribution of \( Y \) will be denoted by \( Y \sim LW (F_\theta, \theta) \).

For illustration, let \( X \sim \chi^2 \), then we shall denote the corresponding log-Lambert \( W \times \chi^2 \) RV and its distribution by \( Y \sim LW (\chi^2, \theta) \). In particular, the RV \( 2 \) can be expressed in this parametrization as \( Y = g_\theta(Q_v) \) with \( Q_v \sim \chi^2 \) and \( \theta = (\nu(\log(\nu) - 1), \nu, 1) \), and consequently with \( x_{\min} = \theta_2/\theta_3 = \nu \) and \( y_{\min} = \theta_1 + \theta_2 - \theta_1 \log(\theta_2/\theta_3) = \nu(\log(\nu) - 1) + \nu - \nu \log(\nu) = 0 \).

Similarly as in Goerg (2011), we can define the log-Lambert \( W \) RVs \( Y \sim LW (F_\theta, \theta) \) for other commonly used families of distributions \( F_\theta \) with support on positive real axis. For example, for the gamma and the inverse gamma distribution with the parameters \( \alpha \) and \( \beta \) we get \( Y \sim LW (\Gamma(\alpha, \beta), \theta) \) and \( Y \sim LW (\text{inv}\Gamma(\alpha, \beta), \theta) \), respectively, and for
the Fischer-Snedecor $F$ distribution with $\nu_1$ and $\nu_2$ degrees of freedom, we get $Y \sim LW(F_{\nu_1,\nu_2}, \theta)$. In all these cases $\theta = (\theta_1, \theta_2, \theta_3)$, $\theta_1 \in \mathbb{R}$, $\theta_2 > 0$, and $\theta_3 > 0$.

Application of the Lambert W function provides the explicit inverse transformation to (4). This can be directly used to determine the exact distribution of $Y$, given the distribution of $X$. The cumulative distribution function (CDF) of $Y$ (here denoted by $\text{cdf}_Y \equiv \text{cdf}_{LW(F_\theta)}$), i.e. $\text{cdf}_{LW(F_\theta)}(y) = \Pr(Y \leq y | Y \sim LW(F_\theta, \theta))$, is given by

$$\text{cdf}_{LW(F_\theta)}(y) = \text{cdf}_{F_\theta}(\chi^3_U(y)) - \text{cdf}_{F_\theta}(\chi^3_L(y)), \quad (7)$$

where $\text{cdf}_{F_\theta}(x) = \Pr(X \leq x | X \sim F_\theta)$ is the CDF of the RV $X$, and $\chi^3_L(y)$ and $\chi^3_U(y)$ are the two distinct real solutions of the equation $y = g_\theta(x)$. In particular, $\chi^3_L(y)$ and $\chi^3_U(y)$ are the solutions on the intervals $(0, x_{\text{sm}})$ and $(x_{\text{sm}}, \infty)$, respectively, given by

$$\chi^3_L(y) = -\frac{\theta_1}{\theta_3} W_0 \left(-\frac{\theta_1}{\theta_2} \exp \left(-\frac{y - \theta_1}{\theta_2} \right) \right), \quad \text{and} \quad \chi^3_U(y) = -\frac{\theta_1}{\theta_3} W_{-1} \left(-\frac{\theta_1}{\theta_2} \exp \left(-\frac{y - \theta_1}{\theta_2} \right) \right), \quad (8)$$

where $W_0(\cdot)$ and $W_{-1}(\cdot)$ are the two real valued branches of the multivalued Lambert W function, i.e. such function that $z = W(z) \exp(W(z))$, for more detailed discussion see e.g. Corless et al. (1996) and Štehlik (2003). Fast numerical implementations of the Lambert W function are available in standard software packages such as MATLAB, R, MATHEMATICA or MAPLE.

Based on the properties of the Lambert W function, see e.g. Lemma 1 in Štehlik (2003), note that

$$\frac{d}{dy} \chi^3_L(y) = \frac{\chi^3_L(y)}{\theta_2 - \theta_3 \chi^3_L(y) - \theta_2}, \quad \text{and} \quad \frac{d}{dy} \chi^3_U(y) = \frac{\chi^3_U(y)}{\theta_3 \chi^3_U(y) - \theta_2}. \quad (9)$$

If $X \sim F_\theta$ is a continuous RV, then, from (7) and (9), we get that the probability density function (PDF) of $Y \sim LW(F_\theta, \theta)$ is given by

$$\text{pdf}_{LW(F_\theta, \theta)}(y) = \frac{\chi^3_L(y)}{\theta_2 - \theta_3 \chi^3_L(y) - \theta_2} \text{pdf}_{F_\theta}(\chi^3_L(y)) + \frac{\chi^3_U(y)}{\theta_3 \chi^3_U(y) - \theta_2} \text{pdf}_{F_\theta}(\chi^3_U(y)), \quad (10)$$

where $\text{pdf}_{F_\theta}(x)$ denotes the PDF of the RV $X \sim F_\theta$.

For completeness, by $q_{1-\alpha}$ we shall denote the $(1 - \alpha)$-quantile of the distribution $LW(F_\theta, \theta)$, i.e. such value $q_{1-\alpha}$ that $\Pr(Y \leq q_{1-\alpha} | Y \sim LW(F_\theta, \theta)) = 1 - \alpha$, so

$$q_{1-\alpha} = q_{LW(F_\theta, \theta)}(1 - \alpha) \equiv \text{cdf}_{LW(F_\theta, \theta)}^{-1}(1 - \alpha), \quad (11)$$

where $q_{LW(F_\theta, \theta)}(\cdot)$ denotes the quantile function (QF) of the distribution $LW(F_\theta, \theta)$. In general, an analytical solution for $q_{LW(F_\theta, \theta)}(\cdot)$ is not available.

3. Distribution of log-Lambert W random variables for family of chi-squared distributions

Here we consider the log-Lambert W RV $Y \sim LW(\chi^2_\nu, \theta)$. Recognizing that the CDF of $\chi^2_\nu$ RV can be expressed by help of incomplete gamma function, directly from (7) we get

$$\text{cdf}_{LW(\chi^2_\nu, \theta)}(y) = \frac{1}{\Gamma \left(\frac{\nu}{2} \right)} \Gamma \left(\frac{y}{2}, \chi^3_L(y), \chi^3_U(y)\right), \quad (12)$$

where $\Gamma(\cdot)$ is the gamma function, and $\Gamma(a, x) = \int_0^x t^{a-1} \exp(-t) \, dt$ is the generalized incomplete gamma function.

Further, as the PDF of the $\chi^2_\nu$ RV is

$$\text{pdf}_{\chi^2_\nu}(x) = \frac{2^{-\frac{x}{2}}}{\Gamma \left(\frac{\nu}{2} \right)} x^{\frac{\nu}{2}-1} \exp \left(-\frac{x}{2} \right), \quad (13)$$

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generating function (MGF) into a power series. Note that the gamma function. Hence, the first four basic moment characteristics of this distribution are:

3.1. The standard log-Lambert $W$ distribution

As mentioned above and as illustrated by examples in the next section, the central role in the likelihood based inference for normally distributed data plays the RV

Thus directly from (10) we get

\[
\text{pdf}_{LW_t, \theta}(y) = \frac{2^{-\frac{1}{2}}}{\Gamma\left(\frac{z}{2}\right)} \left(\frac{x_t^y(y)}{\theta_2 - \theta_3 x_t^y(y)} \theta_3 x_t^y(y) - \theta_2 \right) + \frac{2^{-\frac{1}{2}}}{\Gamma\left(\frac{z}{2}\right)} \left(\frac{x_t^y(y)}{\theta_2 - \theta_3 x_t^y(y)} \theta_3 x_t^y(y) - \theta_2 \right)
\]  

(14)

The characteristic function (CF) of $Y \sim LW(\psi, \theta)$, can be derived by algebraic manipulations directly from its definition, i.e. $\text{cf}_{LW_t, \theta}(t) = E(\exp(itY)) = E(\exp(it\psi(X)))$ where $X \sim \chi^2(\nu)$, and is given by

\[
\text{cf}_{LW_t, \theta}(t) = \frac{2^{-\frac{1}{2}}}{\Gamma\left(\frac{z}{2}\right)} \left(\frac{\exp\left(i\theta_\nu\psi\left(\frac{y}{2}\right)\right)}{\left(\frac{z}{2} - i\theta_\nu\psi\right)}\right)^\nu.
\]  

(15)

The cumulants $(\kappa_j, j = 1, 2, \ldots)$ of $Y \sim LW(\chi^2_\nu, \theta)$ are readily obtained by expanding the logarithm of the moment generating function (MGF) into a power series. Note that

\[
\text{mgf}_{LW_t, \theta}(t) = \text{cf}_{LW_t, \theta}(-it) \quad \text{for} \quad t < \frac{1}{2} \min\left(\frac{\nu}{\theta_2 - \theta_3}\right).
\]  

(16)

Thus

\[
\kappa_1 = \theta_1 - \theta_2 \log(2) + \nu\theta_3 - \theta_2 \psi^{(0)}\left(\frac{y}{2}\right),
\]

\[
\kappa_j = 2^{j-1}\Gamma(j-1)\theta_3^{j-1}(-j\theta_2 + (j-1)\nu\theta_3) + (-1)^j\theta_2^j\psi^{(j-1)}\left(\frac{y}{2}\right),
\]  

(17)

for $j = 2, 3, \ldots$, and $\psi^{(m)}(\cdot)$ is the $m$-th order polygamma function, i.e. the $(m+1)$-derivative of the logarithm of the gamma function. Hence, the first four basic moment characteristics of this distribution are:

\[
\text{mean} = \kappa_1 = \theta_1 - \theta_2 \log(2) + \nu\theta_3 - \theta_2 \psi\left(\frac{y}{2}\right),
\]  

(18)

\[
\text{variance} = \kappa_2 = 2\theta_3 (-2\theta_2 + \nu\theta_3) + \theta_2^2 \psi^{(1)}\left(\frac{y}{2}\right),
\]  

(19)

\[
\text{skewness} = \frac{\kappa_3}{\kappa_2^3} = \frac{4\theta_3^2 (-3\theta_2 + 2\nu\theta_3) - \theta_2^3 \psi^{(2)}\left(\frac{y}{2}\right)}{(2\theta_3 (-2\theta_2 + \nu\theta_3) + \theta_2^2 \psi^{(1)}\left(\frac{y}{2}\right))^3},
\]  

(20)

\[
\text{kurtosis} = \frac{\kappa_4}{\kappa_2^4} = \frac{16\theta_3^3 (-4\theta_2 + 3\nu\theta_3) + \theta_2^3 \psi^{(3)}\left(\frac{y}{2}\right)}{(2\theta_3 (-2\theta_2 + \nu\theta_3) + \theta_2^2 \psi^{(1)}\left(\frac{y}{2}\right))^4}.
\]  

(21)

3.1. The standard log-Lambert $W \times \chi^2_\nu$ distribution

As mentioned above and as illustrated by examples in the next section, the central role in the likelihood based inference for normally distributed data plays the RV

\[
Y_\nu = (Q_\nu - \nu) - \nu \log\left(\frac{Q_\nu}{\nu}\right),
\]  

(22)

where $Q_\nu \sim \chi^2_\nu$. RV $Y_\nu \sim LW(\chi^2_\nu, \theta)$, is the special case of RV (4) with $\theta = (\nu(\log(\nu) - 1), \nu, 1)$. We shall call it the standard log-Lambert $W \times \chi^2_\nu$ RV with $\nu$ degrees of freedom.

Then, directly from (15) we get the characteristic function of the standard log-Lambert $W \times \chi^2_\nu$ RV as

\[
\text{cf}_{Y_\nu}(t) = \frac{2^{-\frac{1}{2}}}{\Gamma\left(\frac{z}{2}\right)} \left(\frac{\exp\left(-i\theta_\nu\psi\left(\frac{y}{2}\right)\right)}{\left(\frac{z}{2} - i\theta_\nu\psi\right)}\right)^\nu.
\]  

(23)
and consequently, for \( \nu \to \infty \), we get the convergence of \( Y_\nu \) (in distribution) to the chi-squared distribution with 1 degree of freedom, i.e.

\[
Y_\nu \xrightarrow{\nu \to \infty} \chi^2_1,
\]

for more details see Appendix A.

As pointed out by one of the reviewers, a specific question for a practitioner is if using the usual rule of thumb, say \( n = 30 \) observations or \( \nu = 30 \) degrees of freedom, respectively, is a good enough approximation for application of the central limit theorem. Table 1 illustrates how the standard log-Lambert \( W \times \chi^2_\nu \) distribution, which is an exact null distribution of the likelihood-ratio statistic for testing the hypothesis about the variance parameter based on random sample from normal distribution, differs from the usual (asymptotic) \( \chi^2 \) approximation and also how fast is the convergence to \( \chi^2_1 \) for \( \nu \to \infty \) (for more details see the Example 1, below). Stehlík (2003) presents a detailed comparisons with the \( \chi^2 \)-asymptotic of the likelihood-ratio statistic, however in a different statistical model with independent observations from the exponential distribution.

### 3.2. Computing the distributions of linear combinations of independent log-Lambert \( W \times \chi^2_\nu \) random variables

The CDF, PDF and QF of the log-Lambert \( W \times \chi^2_\nu \) distribution can be numerically evaluated directly from (7), (10), and (11), by using suitable implementation of the Lambert W function.

Numerical evaluation of the distribution of a linear combination of independent log-Lambert \( W \times \chi^2_\nu \) RVs is based on methods similar to those discussed in Witkovský (2001a) and (2004), and is closely related to the method for computing the distribution of a linear combination of independent chi-squared RVs suggested by Imhof (1961), see also Davies (1980), and also related to the method for computing the distribution of a linear combination of independent inverted gamma variables suggested by Witkovský (2004). The procedure is based on the numerical inversion of the characteristic function, for more details see Gil-Pelaez (1951).

Consider thus the random variable \( Y = \sum_{j=1}^{k} \lambda_j Y_j \), a linear combination of independent log-Lambert \( W \) RVs \( Y_j \sim LW(\chi^2_{\nu_j}, \theta_j) \) with \( \nu_j \) degrees of freedom, parameters \( \theta_j = (\theta_{j1}, \theta_{j2}, \theta_{j3}) \), and real coefficients \( \lambda_j \), \( j = 1, \ldots, k \). Let \( \text{cf}_{Y_j}(t) \) denote the characteristic function of \( Y_j \). The characteristic function of \( Y \) is

\[
\text{cf}_Y(t) = \text{cf}_{Y_1}(\lambda_1 t) \cdots \text{cf}_{Y_k}(\lambda_k t),
\]

where

\[
\text{cf}_{Y_j}(\lambda t) = \frac{2^{-\frac{\nu_j}{2}} \exp(i \lambda t \theta_{j1}) \Gamma\left(\frac{\nu_j}{2} - i \lambda t \theta_{j2}\right)}{\Gamma\left(\frac{\nu_j}{2}\right)} \left(1 + i \lambda t \theta_{j3}\right)^{-\nu_j/2 - i \lambda t \theta_{j2}}.
\]

### Table 1: The \((1-\alpha)\)-quantiles of the standard log-Lambert \( W \times \chi^2_\nu \) distribution, i.e. the distribution of the RV \( Y_\nu \) with \( \nu \) degrees of freedom, computed for selected probabilities \( 1-\alpha \) and degrees of freedom \( \nu \). Note that for \( \nu \to \infty \) we get the chi-squared distribution with 1 degree of freedom.

| \( 1-\alpha \) | \( 1 \) | \( 2 \) | \( 3 \) | \( 5 \) | \( 10 \) | \( 20 \) | \( 30 \) | \( 100 \) | \( \infty \) |
|---|---|---|---|---|---|---|---|---|---|
| 0.7000 | 1.4145 | 1.2543 | 1.1951 | 1.1468 | 1.1103 | 1.0922 | 1.0862 | 1.0778 | 1.0742 |
| 0.7500 | 1.7308 | 1.5426 | 1.4713 | 1.4124 | 1.3677 | 1.3454 | 1.3380 | 1.3277 | 1.3233 |
| 0.8000 | 2.1306 | 1.9105 | 1.8245 | 1.7524 | 1.6974 | 1.6698 | 1.6607 | 1.6479 | 1.6424 |
| 0.8500 | 2.6605 | 2.4039 | 2.2993 | 2.2102 | 2.1415 | 2.1069 | 2.0953 | 2.0792 | 2.0723 |
| 0.9000 | 3.4254 | 3.1259 | 2.9968 | 2.8840 | 2.7956 | 2.7506 | 2.7356 | 2.7146 | 2.7055 |
| 0.9500 | 4.7606 | 4.4077 | 4.2418 | 4.0906 | 3.9683 | 3.9053 | 3.8841 | 3.8543 | 3.8415 |
| 0.9750 | 6.1137 | 5.7256 | 5.5301 | 5.3438 | 5.1806 | 5.0795 | 5.0406 | 5.0239 |
| 0.9900 | 7.9162 | 7.4984 | 7.2734 | 7.0470 | 6.8499 | 6.7441 | 6.7081 | 6.6570 | 6.6349 |
| 0.9950 | 12.477 | 12.022 | 11.754 | 11.456 | 11.035 | 10.9459 | 10.8635 | 10.8276 |
| 0.9990 | 17.0579 | 16.584 | 16.297 | 15.9575 | 15.5983 | 15.3792 | 15.3007 | 15.1868 | 15.1367 |
The distribution function (CDF) of $Y$, $\text{cdf}_Y(y) = \Pr(Y \leq y)$, is according to the inversion formula due to Gil-Pelaez
(1951) given by

$$\text{cdf}_Y(y) = \frac{1}{\pi} - \frac{1}{\pi} \int_0^\infty \Re \left( e^{-it} \text{cf}_Y(t) \right) dt,$$

and the PDF is given by

$$\text{pdf}_Y(y) = \frac{1}{\pi} \int_0^\infty \Re \left( e^{-it} \text{cf}_Y(t) \right) dt.$$

This approach can also be applied to compute the distributions of more general linear combinations of independent RVs, e.g. with $\chi^2_n$ and $LW(\chi^2_n, \theta)$ distributions.

The MATLAB implementation of the algorithms for computing CDF, PDF, CF, cumulants and QF of the log-Lambert $W$ RVs (resp. their linear combinations) is currently available at
http://www.mathworks.com/matlabcentral/fileexchange/46754-lambertwchi2,
the MATLAB Central File Exchange. In future, these algorithms will become a part of a more general MATLAB suite of programs (under development) to calculate the tail probabilities (including CDF, PDF, and QF) of a linear combination of RVs in one of the following classes: (1) class of symmetric RVs containing normal, Student’s $t$, uniform and triangular distributions, and (2) class of RVs with support on positive real axis, e.g., the chi-squared and inverse gamma distributions, see
http://sourceforge.net/projects/tailprobabilitycalculator/.

4. Examples

For illustration, here we present simple examples of the likelihood based inference for normally distributed data, where the distribution of the likelihood ratio test statistic under the null hypothesis can be expressed using the log-Lambert $W$ RVs.

4.1. Example 1: Distribution of the LRT statistic for testing a single variance component

Let $S^2$ be the estimator of the variance parameter $\sigma^2$ (e.g. the restricted maximum likelihood estimator (REML) of $\sigma^2$, based on a random sample from normally distributed data, $\hat{Y} \sim N(0, \sigma^2 I)$), such that $\hat{S}^2 \sim \chi^2_\nu$. The PDF of the RV $S^2$ can be directly derived from the PDF of the chi-squared distribution with $\nu$ degrees of freedom (13). So the log-likelihood function is

$$\loglik(\sigma^2 | S^2) = \text{const} + \left( \frac{\nu}{2} - 1 \right) \log \left( \frac{\nu S^2}{\sigma^2} \right) - \frac{1}{2} \left( \frac{\nu S^2}{\sigma^2} \right) + \log \left( \frac{\nu}{\sigma^2} \right),$$

and the (log-) likelihood-ratio test statistic (LRT) for testing $H_0 : \sigma^2 = \sigma^2_0$ vs. alternative $H_A : \sigma^2 \neq \sigma^2_0$ is

$$l_{\text{lrt}} = -2 \left( \sup_{\hat{H}_0} \loglik(\sigma^2 | S^2) - \sup \loglik(\sigma^2 | S^2) \right) = -2 \left( \loglik(\sigma^2_0 | S^2) - \loglik(\hat{\sigma}^2 | S^2) \right),$$

where $\hat{\sigma}^2 = S^2$ is the REML estimator of $\sigma^2$. From that,

$$l_{\text{lrt}} = -2 \left( \left( \frac{\nu}{2} - 1 \right) \log \left( \frac{\nu S^2}{\sigma^2_0} \right) - \frac{1}{2} \left( \frac{\nu S^2}{\sigma^2_0} \right) + \log \left( \frac{\nu}{\sigma^2} \right) \right) - \left( \left( \frac{\nu}{2} - 1 \right) \log \left( \frac{\nu S^2}{\sigma^2} \right) - \frac{1}{2} \left( \frac{\nu S^2}{\sigma^2} \right) + \log \left( \frac{\nu}{\sigma^2} \right) \right) \right)$$

$$= \frac{\nu S^2}{\sigma^2_0} - \nu \log \left( \frac{1}{\nu} \frac{\nu S^2}{\sigma^2_0} \right) \frac{Q_\nu}{\nu} + \nu \log \left( \frac{Q_\nu}{\nu} \right),$$

where $Q_\nu \sim \chi^2_\nu$. That is, under the null hypothesis $H_0$, the LRT statistic has the standard log-Lambert $W \times \chi^2_\nu$ distribution, $l_{\text{lrt}} \sim LW(\chi^2_\nu, \theta)$ with $\theta = (\nu \log(\nu - 1), \nu, 1)$.

Based on that, the $(1 - \alpha)$-confidence interval for the parameter $\sigma^2$, say $c_{\text{LRT}}$, obtained by inverting the LRT, can be expressed as

$$c_{\text{LRT}} = \left\{ \sigma^2 : \frac{\nu S^2}{\sigma^2} - \nu \log \left( \frac{1}{\nu} \frac{\nu S^2}{\sigma^2} \right) \leq Q_{1-\alpha} \right\} = \left\{ \sigma^2 : \frac{\nu S^2}{\chi^2_\nu(1-\alpha)} \leq \sigma^2 \leq \frac{\nu S^2}{\chi^2_\nu(\nu)} \right\},$$

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where \( q_{1-\alpha} \) denotes the \((1-\alpha)\)-quantile of the RV \( (Q_\alpha-\nu) - \nu \log(\frac{\nu}{\nu - 1}) \), i.e. \( LW(\chi_\nu^2, \theta) \) distribution with \( \theta = (\nu(\log(\nu) - 1), \nu, 1) \), and the limits \( x^0_{\alpha}(q_{1-\alpha}) \) and \( x^1_{\alpha}(q_{1-\alpha}) \) are defined by [5]. The minimum length confidence interval for \( \sigma^2 \), say \( c_{ML} \), can be expressed, e.g., as

\[
c_{ML} = \left\{ \sigma^2 : \frac{vS^2}{\sigma^2} - v - (v + 2) \log \left( \frac{vS^2}{\sigma^2} \right) \leq q_{1-\alpha} \right\} = \left\{ \sigma^2 : \frac{vS^2}{\hat{x}^0_{\alpha}(q_{1-\alpha})} \leq \sigma^2 \leq \frac{vS^2}{\hat{x}^1_{\alpha}(q_{1-\alpha})} \right\},
\]

(33)

where \( \hat{q}_{1-\alpha} \) is the \((1-\alpha)\)-quantile of the RV \( (Q_\alpha-\nu) - (v+2) \log(\frac{\nu}{\nu - 1}) \), i.e. \( LW(\chi_\nu^2, \hat{\theta}) \) distribution with \( \hat{\theta} = ((v + 2) \log(\nu) - v, v + 2, 1) \), see also Tate and Kleij [1969] and Juola [1993].

### 4.2. Example 2: Distribution of the LRT statistic for testing normal linear regression model parameters

Let \( Y \sim N(\beta \nu, \sigma^2 I) \) be an \( n \)-dimensional normally distributed random vector with a non-stochastic full-ranked \((n \times k)\)-design matrix \( X \), parameters \( \beta \in \mathbb{R}^k \) and \( \sigma^2 > 0 \). Here the log-likelihood function is

\[
\loglik(\beta, \sigma^2 | Y) = -\frac{n}{2} \log(2\pi) - \frac{n}{2} \log(\sigma^2) - \frac{1}{2\sigma^2} (Y - X\beta)^T (Y - X\beta),
\]

(34)

and the LRT statistic for testing \( H_0 : (\beta, \sigma^2) = (\beta_0, \sigma_0^2) \), vs. alternative \( H_A : (\beta, \sigma^2) \neq (\beta_0, \sigma_0^2) \), is given by

\[
\text{lrt} = -2 \left( \loglik(\hat{\beta}, \hat{\sigma}^2 | Y) - \loglik(\beta_0, \sigma_0^2 | Y) \right) = \frac{1}{\sigma_0^2} (Y - X\beta_0)^T (Y - X\beta_0) - n \log\left( \frac{\sigma_0^2}{\hat{\sigma}^2} \right) - n,
\]

(35)

where \( \hat{\beta} = (X^T X)^{-1} X^T Y \) and \( \hat{\sigma}^2 = \frac{1}{n} (Y - X\hat{\beta})^T (Y - X\hat{\beta}) \), such that \( Q_\nu = \frac{\nu\hat{\sigma}^2}{\sigma_0^2} \sim \chi_\nu^2 \), with \( n = n - k \), and independent of \( Q_\nu \sim \chi_\nu^2 \). That is, under the null hypothesis \( H_0 \), the LRT statistic is distributed as a linear combination (sum) of two independent RVs with \( \chi_\nu^2 \) and \( LW(\chi_\nu^2, \theta) \) distributions, respectively, where \( \nu = n - k \) and \( \theta = (n(\log(n) - 1), n, 1) \). For more details, see e.g. Choudhuri et al. [2001] and Chvostekova and Witkovsky [2009].

### 4.3. Example 3: Distribution of the (restricted) LRT statistic for testing canonical variance components

Consider a normal linear model with two variance components, \( Y \sim N(X\beta, \sigma_1^2 V + \sigma_2^2 I_n) \), where \( Y \) is an \( n \)-dimensional normally distributed random vector, \( X \) is a known \((n \times k)\)-design matrix, \( \beta \) is a \( k \)-dimensional unknown vector of fixed effects, \( V \) is a known \( n \times n \) positive semi-definite matrix, \( I_n \) is the \( n \times n \) identity matrix, and \( \sigma_1^2 \geq 0, \sigma_2^2 > 0 \) are the variance components — the parameters of interest.

The (restricted) LRT methods are based on distribution of the maximal invariant \( \tilde{Y} = B^T Y \), where \( B \) is an arbitrary matrix, such that \( B B^T = I_n - XX^T \) (here \( X^T \) denotes the Moore-Penrose g-inverse of \( X \)) and \( B^T B = I_{n-rank(X)} \). Hence, \( \tilde{Y} \sim N(0, \Sigma) \), where \( \Sigma = \sigma_1^2 W + \sigma_2^2 D = \sum_{i=1}^r (\sigma_1^2 \Theta_i + \sigma_2^2 D_i) \), is a spectral decomposition with the eigenvalues \( \Theta_1 \geq \cdots \geq \Theta_r \geq 0 \) and their multiplicities \( \nu_i \), \( \nu = \sum_{i=1}^r \nu_i \), \( D_i \) are mutually orthogonal symmetric matrices such that \( D_1 D_i = D_i D_1 = 0 \) for \( i \neq j \), and \( I_n = \sum_{i=1}^r D_i \). Here we consider the problem of testing hypothesis about canonical variance components \( \theta = (\theta_1, \ldots, \theta_r) \), where \( \theta_i = \sigma_1^2 \Theta_i + \sigma_2^2 \epsilon_i \), \( i = 1, \ldots, r \). Namely, we consider testing the hypothesis \( H_0 : \theta = \theta_0 \), vs. alternative \( H_A : \theta \neq \theta_0 \).

Let \( U_i = \tilde{Y}^T D_i \tilde{Y} \), according to Ohlsen et al. [1976], the following holds true: \( U = (U_1, \ldots, U_r) \) is a minimal sufficient statistic for the parameters \( (\sigma_1^2, \sigma_2^2) \), and \( U_i/\sigma_1^2 \Theta_i + \sigma_2^2 = U_i/\theta_i \equiv Q_{\theta_i} \sim \chi_{\nu_i}^2 \), \( i = 1, \ldots, r \) are mutually independent chi-squared RVs with \( \nu_i \) degrees of freedom. Thus, for specific values of the canonical parameter \( \theta_0 = (\theta_{01}, \ldots, \theta_{0r}) \) and the minimal sufficient statistic \( U = (U_1, \ldots, U_r) \), the (restricted) log-likelihood function can be expressed as

\[
\loglik(\theta_0 | U) = -\frac{r}{2} \log(2\pi) - \frac{1}{2} \sum_{i=1}^r \nu_i \log(\theta_{0i}) - \frac{1}{2} \sum_{i=1}^r \frac{U_i}{\theta_{0i}},
\]

(36)

and the (restricted) LRT statistic is given by

\[
\text{lrt} = -2 \left( \loglik(\theta_0 | U) - \loglik(\hat{\theta} | U) \right) = \sum_{i=1}^r \left\{ \left( \frac{U_i}{\theta_{0i}} - \nu_i \right) - \nu_i \log \left( \frac{1}{\nu_i} \frac{U_i}{\theta_{0i}} \right) \right\}
\]

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Table 2: The (different) eigenvalues $\varphi_i$ of the $W$ matrix from model (55), with their multiplicities $v_i$, together with the observed values of the sufficient statistics $U_i$, the true values of the canonical variance components $\theta_i^* = \varphi_i^2 \sigma_1^2 + \sigma_2^2$, and the hypothetical values of the parameters $\varphi_i^{lrt} = \varphi_i^2 \sigma_1^2 + \sigma_2^2$, $\varphi_i^{lrt} = 0 \times \varphi_i + \sigma_2^2$, and $\varphi_i^{lrt} = 1 \times \varphi_i + \sigma_2^2$, where $i = 1, \ldots, r, \sigma_1^2 = 0.1$ and $\sigma_2^2 = 1$.

| $\varphi_i$ | $v_i$ | $U_i$ | $\theta_i$ | $\theta_i^{lrt}$ | $\theta_i^{lrt}$ | $\theta_i^{lrt}$ |
|-------------|------|------|------------|-----------------|-----------------|-----------------|
| 19.24       | 1.00 | 0.65 | 2.92      | 0.65            | 2.92            | 1.00            |
| 17.04       | 1.00 | 17.12| 2.70      | 17.12           | 2.70            | 1.00            |
| 14.89       | 1.00 | 2.76 | 2.49      | 2.76            | 2.49            | 1.00            |
| 12.77       | 1.00 | 3.01 | 2.28      | 3.01            | 2.28            | 1.00            |
| 10.65       | 1.00 | 0.45 | 2.06      | 0.45            | 2.06            | 1.00            |
| 8.53        | 1.00 | 4.02 | 1.85      | 4.02            | 1.85            | 1.00            |
| 6.42        | 1.00 | 0.52 | 1.64      | 0.52            | 1.64            | 1.00            |
| 4.30        | 1.00 | 2.06 | 1.43      | 2.06            | 1.43            | 1.00            |
| 2.16        | 1.00 | 0.90 | 1.22      | 0.90            | 1.22            | 1.00            |
| 0.00        | 100.00 | 117.25 | 1.00 | 1.17 | 1.00 | 1.00 |

\[
\sum_{i=1}^{r} \left( \frac{\hat{\theta}_i}{\hat{\theta}_0} \right) - v_i \log \left( \frac{\hat{\theta}_i}{\hat{\theta}_0 v_i} \right) \right) \sim \sum_{i=1}^{r} \left( (Q_{vi} - v_i) - v_i \log \left( \frac{Q_{vi}}{v_i} \right) \right),
\] (37)

where $\hat{\theta} = (\hat{\theta}_1, \ldots, \hat{\theta}_r)$ is the REML estimator of $\theta$, $\theta = (\theta_1, \ldots, \theta_r)$ represents the true (unknown) vector of parameters, and $Q_{vi} \sim \chi^2_{v_i}$ are mutually independent RVs, $i = 1, \ldots, r$.

That is, under the null hypothesis $H_0 : \theta = \theta_0$, the restricted LRT statistic (37) is distributed as a linear combination of $r$ independent RVs with $\nu_i \chi^2_{v_i}$, $i = 1, \ldots, r$. In general, if $\theta \neq \theta_0$, the LRT statistic (37) is distributed as a linear combination of $r$ independent RVs with $\nu_i \chi^2_{v_i}$, $i = 1, \ldots, r$.

4.4. Example 4: Numerical example

In order to illustrate some of the numerical calculations required for testing hypothesis on canonical variance components based on the LRT statistic, as presented in Example 3, let us consider the following unbalanced one-way random effects ANOVA model, as a special case of a normal linear model with two variance components:

\[
Y_{ij} = \mu + b_i + \epsilon_{ij}, \quad i = 1, \ldots, G, \quad j = 1, \ldots, n_i,
\] (38)

where $Y = (Y_{i1}, \ldots, Y_{iG})^T$ is the $n \times 1$ vector of measurements, $n = \sum_{i=1}^{G} n_i$, $\mu$ represents the common mean, $b = (b_1, \ldots, b_G)^T$ is a vector of random effects, $b \sim N(0, \sigma_1^2 I_G)$, and $\epsilon = (\epsilon_{i1}, \ldots, \epsilon_{iG})^T$ is the $n \times 1$ vector of measurement errors, $\epsilon \sim N(0, \sigma_2^2 I_n)$.

In particular, for $G = 10$, and $n_1 = 2, n_2 = 4, n_3 = 6, \ldots, n_{10} = 20$, with $n = 110$, by spectral decomposition of the matrix $W = B^T I_B = \sum_{i=1}^{G} I_{Di}$, we get $r = 10$ different eigenvalues $\varphi_i$ with their multiplicities $v_i$, $i = 1, \ldots, r$. For the true values of the parameters $\mu = 0, \sigma_1^2 = 0.1$, and $\sigma_2^2 = 1$, we have generated the $n \times 1$ vector of observations $Y$ with the observed values of the sufficient statistics $U = (U_1, \ldots, U_r)$. The true values of the canonical variance components, $\theta^* = (\theta_1^*, \ldots, \theta_r^*)$ with $\theta_i^* = \sigma_1^2 \varphi_i^2 + \sigma_2^2$ were estimated by REML, $\hat{\theta} = (\hat{\theta}_1, \ldots, \hat{\theta}_r)$, where $\hat{\theta}_i = U_i/v_i$.

Here the goal is to test the following null hypotheses: $H_{01} : \theta = \theta^{lrt}_1 = \theta^* = \sigma_1^2 \varphi_i^2 + \sigma_2^2$, as well as $H_{02} : \theta = \theta^{lrt}_2$ with $\theta^{lrt}_2 = 0 \times \varphi_i + \sigma_2^2$, and $H_{03} : \theta = \theta^{lrt}_3$ with $\theta^{lrt}_3 = 1 \times \varphi_i + \sigma_2^2$, for numerical values see Table 3.

The CDFs of the LRT statistic for testing the null hypotheses $H_{01}, H_{02},$ and $H_{03}$, respectively, that is $lr_1^{lrt}, lr_2^{lrt}$, and $lr_3^{lrt}$, are plotted in Figure 11 together with the CDF of the $\chi^2_{10}$ distribution, which is conventionally used as the approximate (asymptotic) distribution. Note that only $H_{01}$ is true, and so, only $lr_1^{lrt}$ has the correct null distribution given by (37).
Figure 1: The cumulative distribution functions (CDFs) of the LRT statistics for testing the null hypotheses $H_{01}$, $H_{02}$, and $H_{03}$, respectively. Here, the CDF of $lrt_{H_{01}}$ (the true null distribution) is plotted by solid line (blue), $lrt_{H_{02}}$ by dashed line (magenta), and $lrt_{H_{03}}$ by dashed-dotted line (red). The CDF of the $\chi^2_{10}$ distribution, which is conventionally used as the approximate (asymptotic) distribution, is plotted by dotted line (black).

For given observed values of the sufficient statistics, $U_i$, the observed value of the LRT statistic was $lrt_{H_{01}} = 7.3095$ ($lrt_{H_{02}} = 18.7350$, and $lrt_{H_{03}} = 10.6475$). The $(1 - \alpha)$-quantile of the null distribution, for $\alpha = 0.05$, is $q_{H_{01}}^{1-0.05} = 22.2689$. For comparison, the quantile of the $\chi^2_{10}$ distribution is $\chi^2_{10,1-0.05} = 18.3070$. Based on that, on significance level $\alpha = 0.05$, we cannot reject any of the hypotheses $H_{01}$, $H_{02}$, $H_{03}$. However, note that the hypothesis $H_{02}$ would be rejected if the approximate $\chi^2_{10}$ null distribution were used instead of the exact null distribution.

5. Conclusions

In this paper we introduce the class of the log-Lambert $W \times F$ random variables and their distributions. It includes, as special case, the class of log-Lambert $W \times \chi^2_\nu$ RVs, which naturally appears in statistical inference based on likelihood of normal RVs. A suite of MATLAB programs (implementation of algorithms for computing PDF, CDF, QF, CF, cumulants, and convolutions) is available at MATLAB Central File Exchange, [http://www.mathworks.com/matlabcentral/fileexchange/46754-lambertwchi2](http://www.mathworks.com/matlabcentral/fileexchange/46754-lambertwchi2).

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Appendix A. Limit distribution of the standard log-Lambert $W \times \chi^2_r$ random variables

To show that (24) holds true, note that the moment generating function (MGF) of the standard log-Lambert $W \times \chi^2_r$ RV with $r$ degrees of freedom is

$$\text{mgf}_{W_{r}}(t) = \frac{\Gamma\left(\frac{x}{2} - rt\right)}{\Gamma\left(\frac{x}{2}\right)} \left(\frac{t}{2}\right)^{rt}(1 - 2t)^{-\frac{x}{2}}$$

and, according to equation (1.4.24) in Lebedev (1963), for $x \in \mathbb{C}$, $|x| > 1$, $|\arg x| < \frac{\pi}{2}$, we have

$$\Gamma(x) = \sqrt{2\pi x^{r-\frac{1}{2}}} e^{-x}(1 + r(x))$$

where $|r(x)| \leq \frac{c}{x}$ for some real $c > 0$. Thus, for $\nu \to \infty$ and for all $t \in (0, \frac{1}{2})$, we get

$$\text{mgf}_{W_{r}}(t) = \frac{\Gamma\left(\frac{x}{2} - rt\right)}{\Gamma\left(\frac{x}{2}\right)} \left(\frac{t}{2}\right)^{rt}(1 - 2t)^{-\frac{x}{2}}$$

$$= \sqrt{2\pi \left(\frac{x}{2} - rt\right)^{\frac{x}{2}-\frac{1}{2}}} e^{-\frac{x}{2}rt(1 - 2t)^{-\frac{x}{2}}(1 + r(\frac{x}{2} - rt))}$$

$$= \left(\frac{x}{2} - rt\right)^{\frac{x}{2}-\frac{1}{2}} e^{\frac{x}{2}rt(1 - 2t)^{-\frac{x}{2}}(1 + r(\frac{x}{2} - rt))}$$

$$= \left[\frac{\nu(1 - 2t)}{\nu^2}\right]^{\frac{x}{2}-\frac{1}{2}} \left(\frac{x}{2} - rt\right)^{\frac{x}{2}-\frac{1}{2}} e^{\frac{x}{2}rt(1 - 2t)^{-\frac{x}{2}}(1 + r(\frac{x}{2} - rt))}$$

$$= (1 - 2t)^{\frac{x}{2}-\frac{1}{2}} \left(\frac{x}{2} - rt\right)^{\frac{x}{2}-\frac{1}{2}} e^{\frac{x}{2}rt(1 - 2t)^{-\frac{x}{2}}(1 + r(\frac{x}{2} - rt))}$$

$$= \frac{1}{\sqrt{1 - 2t}} \left(1 + r(\frac{x}{2} - rt)\right).$$

Consequently,

$$\lim_{\nu \to \infty} \text{mgf}_{W_{r}}(t) = \frac{1}{\sqrt{1 - 2t}}, \quad 0 \leq t < \frac{1}{2},$$

what coincides with the MGF of a chi-squared distribution with 1 degree of freedom.

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