CONCERNING THE WAVE EQUATION ON ASYMPTOTICALLY
EUCLIDEAN MANIFOLDS

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ABSTRACT. We obtain KSS, Strichartz and certain weighted Strichartz esti-
mate for the wave equation on \((\mathbb{R}^d, g)\), \(d \geq 3\), when metric \(g\) is non-trapping
and approaches the Euclidean metric like \(\langle x \rangle^{-\rho}\) with \(\rho > 0\). Using the KSS
estimate, we prove almost global existence for quadratically semilinear wave
equations with small initial data for \(\rho > 1\) and \(d = 3\). Also, we establish the
Strauss conjecture when the metric is radial with \(\rho > 1\) for \(d = 3\).

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1. INTRODUCTION AND MAIN RESULTS

This paper is devoted to the study of the semilinear wave equation on asymp-
totically Euclidean non-trapping Riemannian manifolds. We shall obtain almost
global existence for quadratic semilinear wave equations with small data, and show
that the Strauss conjecture holds in this setting, in dimension \(d = 3\).

In Minkowski space, the quadratically semilinear wave equation has been thor-
oughly studied. Global existence is known in dimension \(d \geq 4\) for small initial data
(see Klainerman and Ponce [19] and references therein). Almost global existence in
dimension \(d = 3\) for small data was shown by John and Klainerman in [11]. Almost
global means that the life time of a solution is at least \(\exp(c/\delta)\) with some \(c > 0\),
where \(\delta\) is the size of the initial data in some appropriate Sobolev space. Note that,
in dimension \(d = 3\), Sideris [27] has proved that global existence does not hold in
general (see also John [10]).
In [14], Keel, Smith and Sogge gave a new proof of the almost global existence result in dimension 3 using estimates (known as KSS estimates) of the form (1.1)

\[(\ln(2 + T))^{-1/2} \| \langle x \rangle^{-1/2} u' \|_{L^2([0,T] \times \mathbb{R}^3)} \lesssim \| u'(0, \cdot) \|_{L^2(\mathbb{R}^3)} + \int_0^T \| F(s, \cdot) \|_{L^2(\mathbb{R}^3)} ds, \]

and a certain Sobolev type estimate due to Klainerman (see [18]). Here \( u \) solves the wave equation \( \Box u = F \) in \([0, +\infty) \times \mathbb{R}^d \) and \( u' = (\partial_t u, \partial_x u) \). Earlier versions of (1.1) appeared before. The first ones appear to be due to Morawetz [26] and Strauss [33], who proved somewhat weaker versions of (1.1). See also [17]. In [14], existence results for the non-trapping obstacle case were also obtained. In [15], Strauss [33] extended this philosophy to the almost global existence for the quadratic semilinear wave equation.

Recently, Bony and Häfner [3] obtained a weaker version of KSS estimate in the current setting and proved the long time existence for quadratic semilinear wave equations with small data. In the present paper, by using results of Metcalfe-Sogge [23], we are able to prove the full KSS estimate, and hence the almost global existence for the quadratic semilinear wave equation.

Recently, in Minkowski space, Fang and Wang [6] and Hidano-Metcalfe-Smith-Sogge-Zhou [9] proved the Strauss conjecture with low regularity for \( d = 2, 3, 4 \), by using a weighted Strichartz estimate of the form

\[(1.2) \quad \| x^{\frac{1}{2} - \frac{1}{r}} \cdot \frac{\partial}{\partial x} \langle x \rangle^{-\gamma} u \|_{L^r_t \cap L^\infty_x} \lesssim \| u(0, \cdot) \|_{H^\gamma} + \| \partial_x u(0, \cdot) \|_{H^\gamma - 1} + \| F \|_{L^1_t H^\gamma - 1}, \]

for \( \gamma \in (1/2 - 1/r, d/2 - 1/r) \), \( r \in [2, \infty) \), where we use the notation

\[\| f \|_{L^r_t L^\infty_x} = \left( \int_0^\infty \| f(t, |x|) \|_{L^r_x} |x|^{d-1} d|x| \right)^{1/r} \]

with \( L^r_x \) denoting the \( L^r \)-norm on \( S^{n-1} \) with respect to the standard measure. In this paper, we obtain a somewhat weaker version of this estimate in this general setting, which suffices for us to establish the Strauss conjecture for \( d = 3 \) (when the metric is radial).

Using ideas from Burq [4], Metcalfe [22], Smith-Sogge [30] and Hidano-Metcalfe-Smith-Sogge-Zhou [9], we can also use the local energy decay estimates to prove global Strichartz estimates in this setting. We should point out, though, that the idea that, in many situations, local energy estimates can be used to prove global Strichartz estimates occurs in many other works. The first seems to be that of Journé, Soffer and Sogge [12] who proved global Strichartz estimates for Schrödinger operators with potential using local energy estimates (local smoothing) for \( e^{it\Delta} \). Staffilani and Tataru [32] extended this philosophy by considering more general perturbations of \( \Delta \), and more recently Metcalfe and Tataru [24] used the philosophy that local energy estimates imply Strichartz estimates to handle (small) metric perturbations of \( \Box = \partial_t^2 - \Delta \). Thus, in many ways, some of the techniques employed in this paper are not novel, since they have been used in many earlier works. A slight novelty, though, might be that we obtain our global estimates by combining local energy estimates (in this case due to Bony and Häfner [3]) with global Strichartz estimates not involving \( \Delta \), but rather small perturbations of the Laplacian.
us to prove the standard mixed-norm Strichartz estimates are due to Metcalfe and Tataru [24].

Let us now state our precise results. We consider asymptotically Euclidean manifolds \((\mathbb{R}^d, g)\) with \(d \geq 3\) and
\[
g = \sum_{i,j=1}^{d} g_{ij}(x) dx^i dx^j.
\]
We suppose \(g_{ij}(x) \in C^\infty(\mathbb{R}^d)\) and, for some \(\rho > 0\),
\begin{equation}
\forall \alpha \in \mathbb{N}^d \quad \partial_\alpha^\rho (g_{ij} - \delta_{ij}) = O(\langle x \rangle^{-|\alpha| - \rho}),
\end{equation}
with \(\delta_{ij} = \delta^{ij}\) being the Kronecker delta function. We also assume that
\begin{equation}
\text{(H2)} \quad g \text{ is non-trapping.}
\end{equation}

Let \(g(x) = (\det(g))^{1/2}\). The Laplace–Beltrami operator associated with \(g\) is given by
\[
\Delta_g = \sum_{ij} \frac{1}{g^2} \partial_i g^{ij} g^{2j},
\]
where \(g^{ij}(x)\) denotes the inverse metric. Note \(-\Delta_g\) is self-adjoint non-negative on \(L^2(\mathbb{R}^d, g^2 dx)\), while \(P = -g\Delta_g g^{-1}\) is self-adjoint non-negative on \(L^2(\mathbb{R}^d, dx)\). Let \(\Omega = \Omega_{k,\ell} := x_k \partial_\ell - x_\ell \partial_k\) be the rotational vector fields. We consider the following semilinear wave equation
\begin{equation}
\square_g u = Q(u'), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d
\end{equation}
\begin{align*}
u(0, x) &= U_0(x), & \partial_t u(0, x) &= U_1(x), & x &\in \mathbb{R}^d.
\end{align*}

Here \(\square_g = \partial_t^2 - \Delta_g\) and \(Q(u')\) is a quadratic form in \(u' = (\partial_t u, \partial_x u)\). One of our main results is the following theorem.

**Theorem 1.1.** Assume hypotheses [H1] and [H2] with \(\rho > 1\). Suppose \(u_0, u_1 \in C_0^\infty(\mathbb{R}^3)\), and
\begin{equation}
\sum_{|\alpha| + |\beta| \leq 5} \|\partial_\alpha^\beta \Omega^\rho u_0\|_{L_2^2} + \sum_{|\alpha| + |\beta| \leq 4} \|\partial_\alpha^\beta \Omega^\rho u_1\|_{L_2^2} \leq \delta.
\end{equation}
For \(\delta\) small enough, the problem (1.3) has a unique almost global solution \(u \in C^\infty([0, T_\delta] \times \mathbb{R}^d)\) with \(T_\delta = \exp(c/\delta)\) for some \(c > 0\).

The main ingredient of the proof are estimates of type (1.1). Let us therefore consider the corresponding linear equation. Let \(u\) be solution of
\begin{equation}
(\partial_t^2 - \Delta_g) u(t, x) = F(t, x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d
\end{equation}
\begin{align*}
u(0, x) &= U_0(x), & \partial_t u(0, x) &= U_1(x), & x &\in \mathbb{R}^d.
\end{align*}

With the notation
\[
A_\mu(T) = \begin{cases}
\ln(2 + T)^{-1/2} & \mu = 1/2, \\
1 & \mu > 1/2,
\end{cases}
\]
we have the following KSS estimate.
Theorem 1.2. Assume that \([\text{H1}]\) and \([\text{H2}]\) hold with \(\rho > 0\) and let \(\mu \geq 1/2\), \(d \geq 3\). For all \(\varepsilon > 0\), the solution of (1.5) satisfies
\[
A_\mu(T) \left( \| (x)^{-\mu} u \|_{L^2_x L^2_t} + \| (x)^{-\mu-1} u \|_{L^2_x L^1_t} \right) \lesssim \| u'(0, \cdot) \|_{L^2_x} + \| (x)^{\frac{1}{2} + \varepsilon} F \|_{L^2_x L^1_t},
\]
where we use \(L^2_t\) to denote \(L^2_{t \in [0, T]}\).

We remark that our estimate (1.6) agrees with ones in Bony and H"afner [3] when \(\mu > 1/2\), while for \(\mu = 1/2\) they are slightly stronger since we obtain the sharp bounds with \(A_{1/2}(T) = (\ln(2 + T))^{-1/2}\), as opposed to the bounds of \(T^{-\varepsilon}\), \(\varepsilon > 0\), in [3]. This improvement allows us to obtain the almost global existence results alluded to before. On the other hand, our proof is very similar to that of Bony and H"afner [3] and papers that preceded it, starting with [14]. A slight point of departure is that we combine local energy estimates (due to Bony and H"afner [3]) not with global KSS estimates for \(\Delta\) but rather for small metric perturbations of \(\Delta\) (which are due to Metcalfe and the first author [30]).

To prove the nonlinear theorem, we need to get higher order estimates. For this purpose, let us put \(Z = \{\partial_t, \partial_x, \Omega\}, Y = \{\partial_x, \Omega\}, X = \{\partial_x\}.\) Then, we have

Theorem 1.3. Assume that \([\text{H1}]\) and \([\text{H2}]\) hold with \(\rho > 1\). Let \(N \geq 0\) and \(\mu \geq 1/2\). The solution of \((1.5)\) satisfies
\[
\sup_{0 \leq t \leq T} \sum_{1 \leq k + j \leq N + 1} \| \partial^k_t \partial^j_x u(t, \cdot) \|_{L^2_x} + \sum_{|\alpha| \leq N} A_\mu(T) \| (x)^{-\mu} (Z^\alpha u) \|_{L^2_x L^1_t} \lesssim \sum_{|\alpha| \leq N} \| (Y^\alpha u)'(0, \cdot) \|_{L^2_x} + \sum_{|\alpha| \leq N} \int_0^T \| Z^\alpha F(s, \cdot) \|_{L^2_x} ds.
\]

Note that the estimate (1.6) can be viewed as the local energy decay estimate for \(\Box_g\). The local Strichartz estimates for variable coefficient wave equations have been studied extensively; see e.g. Kapitanski [13], Mockenhaupt-Seeger-Sogge [25], Smith [29], Bahouri-Chemin [1] [2], Tataru [31] [35] [37]. And recently, Metcalfe and Tataru have obtained global Strichartz estimates involving small perturbations of the Minkowski metric in [24]. As we mentioned before, by combining these with local energy estimates we shall prove global Strichartz estimates for \(\Box_g\).

For the Minkowski case, it is known ([16], [20]) that we have Strichartz estimates if \((s, q, r)\) is admissible, i.e.,
\[
\frac{1}{q} \leq \min \left( \frac{1}{2}, \frac{d - 1}{2}, \frac{1}{2} \left( 1 - \frac{1}{r} \right) \right), \quad (q, r) \neq (2, \infty), (\infty, \infty), \quad s = d \left( \frac{1}{2} - \frac{1}{r} \right) - \frac{1}{q}.
\]
Our global Strichartz estimates for \(\Box_g\) are the following.

Theorem 1.4 (Global Strichartz estimate). Assume that \([\text{H1}]\) and \([\text{H2}]\) hold with \(\rho > 0\), \(d \geq 3\) and \(s \in [0, 1] \quad (s \in (0, 1) \text{ if } d = 3)\). The solution of \((1.5)\) satisfies
\[
\| u \|_{L^q_t L^r_x} \lesssim \| u_0 \|_{H^s} + \| u_1 \|_{H^{s-1}} + \| F \|_{L^1_t H^{s-1}},
\]
for any admissible \((s, q, r)\) with \(q > 2, r < \infty\).

The proofs of Theorem 1.2 and Theorem 1.3 follow a similar strategy. We first use results from [23] and [24] to show that we can construct a metric \(g\) which agrees with \(g\) near infinity and has the property that the bounds in these two theorems are valid if \(\Box_g\) is replaced by \(\Box_g\). Then, by adapting arguments from [3] and [30],
we can use these estimates along with the local energy decay estimates for \( \square_g \) (see Lemma 2.6 below) to show that \( \square_g \) satisfies the same global estimates as its compact perturbation \( \square_{\bar{g}} \).

Now let us describe the weighted Strichartz estimate and its application to Strauss conjecture in this general setting. Let \( p > 1 \),

\[
s_c = \frac{d}{2} - \frac{2}{p-1}, \quad s_{ab} = \frac{1}{2} - \frac{1}{p}.
\]

The equation that we shall consider is

\[
\begin{cases}
(\partial_t^2 - \Delta_g)u(t,x) = F_p(u(t,x)), & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d \\
u(0,x) = u_0(x), & \partial_t u(0,x) = u_1(x), & x \in \mathbb{R}^d,
\end{cases}
\]

We shall assume that the nonlinear term behaves like \(|u|^p\), and so we assume that

\[
\sum_{0 \leq j \leq 1} \|u_j\|_{\partial_i F_p(u)} \lesssim \|u\|_p.
\]

See [31], §4.4 for a discussion about how \( s \geq s_{ab} \) is needed for local existence, while \( s_c \) is critical for global existence.

We can now state our existence theorem for (1.9). Due to some technical difficulties, we are only able to deal with only the case where \( g^{ij}(x) = h(|x|)^{\delta ij} \) for some function \( h \).

**Theorem 1.5.** Assume that \( g^{ij}(x) = h(|x|)^{\delta ij} \) for some function \( h \). Let \( \text{H1} \) and \( \text{H2} \) hold with \( \rho > d \), \( d = 3 \) and \( p > p_c = 1 + \sqrt{2} \). Then for any \( \epsilon > 0 \) such that

\[
s = s_c - \epsilon \in (s_{ab}, 1/2),
\]

there is an \( \delta > 0 \) depending on \( p \) so that (1.9) has a global solution satisfying

\[
\sum_{|\alpha| \leq 2} \|Y^\alpha u_0\|_{\dot{H}^s} + \|Y^\alpha u_1\|_{\dot{H}^{s-1}} < \delta
\]

with \( 0 < \delta < \delta_0 \).

Existence results of this type when \( \Delta_g = \Delta \) are a celebrated result of John [10]. Subsequently, Strauss conjectured that for dimensions \( d \geq 2 \) the critical exponent for small data global existence for equations of the form (1.9) (when \( \Delta_g = \Delta \)) should be the positive root of the equation \( (d - 1)p^2 - (d + 1)p - 2 = 0 \). This conjecture was settled for the Minkowski space case in [7], [8], [21], [28] and [38]. See [31] for further discussion.

As in the case of (1.3), the main ingredient of the proof are estimates of type (1.2). If we consider the corresponding linear equation (1.5), then we have the following estimate, where the metric is not restricted to the special case where \( g^{ij}(x) = h(|x|)^{\delta ij} \).

**Theorem 1.6.** Assume that \( \text{H1} \) and \( \text{H2} \) hold with \( \rho > 0 \), \( d = 3 \), \( 2 < q \leq \infty \) and \( s \in (s_{ab}(q), 1) \). For all \( \epsilon, \eta > 0 \) small enough, the solution of (1.5) satisfies

\[
\|\langle x \rangle^{-\frac{d+1}{2} - s - \epsilon} u\|_{L^q_{t,x}} + \|\langle x \rangle^{-\frac{d+1}{2} - s - \epsilon} u\|_{L^q_{t,x}} \lesssim \|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-1}} + \|F\|_{L^1_t \dot{H}^{s-1}}
\]
We should comment on the hypotheses in the existence theorems. First, because of the various commutator terms that arise in the proofs we are, at present, only able to handle semilinear terms in the existence theorems involving quadratic nonlinearities, as opposed the the quasilinear case (for $\Box_g = \partial_t^2 - \Delta$) treated in [13]. For similar reasons, in our results involving the Strauss conjecture, due to difficulties in dealing with commutators involving the $\Omega$ vector fields and $\Box_g$ we have to assume that the metric $g$ is spherically symmetric. For similar reasons, although the linear estimates just require the hypothesis that $\rho > 0$ currently our techniques require the assumption that $\rho > 1$ in the hypotheses of the nonlinear theorems. We do not know, however, what the natural assumption regarding $\rho$ should be for the latter, though.

2. KSS Estimates

In this section, we give the proof of the KSS estimates. First, we will need the following lemmas, where we denote $\tilde{\partial}_x := \partial_x g^{-1}$.

**Lemma 2.1** (Theorem 1.3 and Proposition 4.6 in [3]). Assume that $[H_1]$ and $[H_2]$ hold with $\rho > 0$, then for all $\epsilon > 0$, the solution of the equation $(\tilde{\partial}_x^2 + P)u = F$ satisfies

$$\| \langle x \rangle^{-\frac{1}{2} - \epsilon} (\tilde{\partial}_x P^{1/2}) u \|_{L^2(\mathbb{R} \times \mathbb{R}^d)} \lesssim \| (\tilde{\partial}_x P^{1/2}) u(0, \cdot) \|_{L^2(\mathbb{R}^d)} + \| \langle x \rangle^{-\frac{1}{2} + \epsilon} F \|_{L^2(\mathbb{R} \times \mathbb{R}^d)}.$$

**Remark 2.1.** In fact, from the proof of Proposition 4.4 and 4.6 in [3], we also have

$$\| \langle x \rangle^{-\frac{1}{2} - \epsilon} u \|_{L^2(\mathbb{R} \times \mathbb{R}^d)} \lesssim \| (\tilde{\partial}_x P^{1/2}) u(0, \cdot) \|_{L^2(\mathbb{R}^d)} + \| \langle x \rangle^{\frac{1}{2} + \epsilon} F \|_{L^2(\mathbb{R} \times \mathbb{R}^d)}.$$

**Lemma 2.2** (Theorem 5.1 in [23]). Let $\Box_h = \partial_t^2 - \Delta + h^{\alpha \beta}(t, x) \partial_\alpha \partial_\beta, h^{\alpha \beta} = h^{\beta \alpha}$ and $\sum |h^{\alpha \beta}| \leq \delta$. Then if $\delta > 0$ is small enough, $d \geq 3$, the solution to the equation $\Box_h u = F$ satisfies

$$\langle \ln(2 + T) \rangle^{-1} \| \langle x \rangle^{-1/2} (|u'| + \frac{|u|}{\langle x \rangle}) \|_{L^2([0, T] \times \mathbb{R}^d)} + \| \langle x \rangle^{-1/2 - \epsilon} (|u'| + \frac{|u|}{\langle x \rangle}) \|_{L^2([0, T] \times \mathbb{R}^d)}$$

$$\lesssim \| u'(0, \cdot) \|_{L^2(\mathbb{R}^d)} + \int_0^T \int \left( |u'| + \frac{u}{\langle x \rangle} \right) \left( |F| + (|h'| + \frac{h}{\langle x \rangle}) |u'| \right) \text{d}x \text{d}t$$

for any $\epsilon > 0$.

To obtain higher order estimates, we will need the following.

**Lemma 2.3** (Lemma B.13, 4.1 and 4.2 in [3]). For all $-3/2 \leq \tilde{\mu} < \mu \leq 3/2$, we have

$$\| \langle x \rangle^{-\tilde{\mu}} \tilde{\partial}_t u \|_{L^2(\mathbb{R}^d)} \lesssim \| \langle x \rangle^{-\tilde{\mu}} P^{1/2} u \|_{L^2(\mathbb{R}^d)};$$

$$\| \langle x \rangle^{-\mu} P^{1/2} u \|_{L^2(\mathbb{R}^d)} \lesssim \sum_{\ell=1}^{d} \| \langle x \rangle^{-\tilde{\mu}} \tilde{\partial}_\ell u \|_{L^2(\mathbb{R}^d)}.$$

Also, for $u \in H^1(\mathbb{R}^d)$,

$$\| P^{1/2} u \|_{L^2(\mathbb{R}^d)} \lesssim \| \nabla g^{-1} u \|_{L^2(\mathbb{R}^d)} \lesssim \| P^{1/2} u \|_{L^2(\mathbb{R}^d)}.$$
We also need a lemma which says that the homogeneous spaces defined by $P$ and $-\Delta$ are essentially the same. In what follows, “remainder terms”, $r_j$, $j \in \mathbb{N}$, will denote a smooth function such that

$$\partial_x^p r_j(x) = O(\langle x \rangle^{-p-j-|\alpha|}).$$

**Lemma 2.4.** If $s \in [-1, 1]$, then

$$\|u\|_{\dot{H}^s} \simeq \|P^{s/2}u\|_{L^2}.  \tag{2.8}$$

If $s \in [0, 1]$,

$$\|\partial_j u\|_{\dot{H}^{-s}} \lesssim \|P^{1/2}u\|_{\dot{H}^{-s}},  \tag{2.9}$$

$$\|P^{1/2}u\|_{\dot{H}^{-s}} \lesssim \sum_j \|\partial_j u\|_{\dot{H}^{-s}}.  \tag{2.10}$$

Moreover, we have for $s \in (0, 2]$ and $1 < q < d/2$,

$$\|P^{s/2}u\|_{L^q} \lesssim \|u\|_{\dot{H}^{s,q}}.  \tag{2.11}$$

**Proof.** For the first estimate (2.8), by interpolation and duality, we need only to prove the estimate for the special case where $s = 1$, i.e.

$$\|\nabla u\|_{L^2} \simeq \|\nabla g^{-1}u\|_{L^2}.  \tag{2.9}$$

In fact,

$$\|\nabla g^{-1}u\|_{L^2} \lesssim \|\nabla g\|_{L^2} \lesssim \|\nabla g^{-1}\|_{L^2} \lesssim \|\nabla g^{-1}\|_{L^\infty} \|\nabla u\|_{L^2}.$$  

By the hypotheses (1.11) and the ellipticity of $P$, we know that $\text{Image}(g) \subset (\delta, \delta^{-1})$, $|\partial g|, |\partial g^{-1}| = O((x)^{-1-\rho}) \in L^d$, $|\partial^2 g^{-1}| = O((x)^{-2-\rho}) \in L^{d/2}$, for some $\delta > 0$. So we know that

$$\|\nabla h u\|_{L^2} \lesssim \|\nabla u\|_{L^2}$$

whenever $h = g$ or $h = g^{-1}$.

To prove (2.9), we note first that by the first inequality (2.8),

$$\|\partial_j f\|_{\dot{H}^{-1}} = \|\partial_j g^{-1} f\|_{\dot{H}^{-1}} \lesssim \|g^{-1} f\|_{L^2} \lesssim \|f\|_{L^2} \lesssim \|P^{1/2} f\|_{\dot{H}^{-1}}.  \tag{2.10}$$

Thus the inequality (2.9) follows from interpolation with (2.6).

Note that by (2.9),

$$\|P f, h\| = |\langle P f, h \rangle| = |\langle g^2 g^{ij} \partial_j f, \partial_i h \rangle| \lesssim \sum_j \|\partial_j f\|_{\dot{H}^{-1}} \|P^{1/2} h\|_{\dot{H}^{-s}}.$$

This gives (2.10).

For the last inequality, let $a^{ij} = g^2 g^{ij}$ and $P_1 = \partial_i a^{ij} \partial_j$, then $P_g = g^{-1} P_1$, $a^{ij} \in L^\infty$ and $\partial_i a^{ij} \in L^d$. Denoting $D = \sqrt{\Delta}$, then for $1 < q < d$

$$\|P_1 D^{-2} u\|_{L^q} \lesssim \|\partial_i a^{ij} \partial_j D^{-2} u\|_{L^q} + \|a^{ij} D^{-2} \partial_i \partial_j u\|_{L^q} \lesssim \|u\|_{L^q}.$$  

Thus if $1 < q < d/2$,

$$\|P u\|_{L^q} = \|g^{-1} P_g g^{-1} u\|_{L^q} \lesssim \|P_1 g^{-1} u\|_{L^q} \lesssim \|D^2 g^{-1} u\|_{L^q} \lesssim \|u\|_{\dot{H}^{2,q}}.$$  

Consequently, the last inequality in the lemma follows from interpolating with the trivial estimate where $s = 0$.

Using this we can obtain analogues of Lemma 2.1 and Remark 2.1 involving $-\Delta_g$.  


Corollary 2.5. Assume that $[H_1]$ and $[H_2]$ hold with $\rho > 0$, then for all $\epsilon > 0$, the solution of the equation $(\partial_t^2 - \Delta g)u = F$ satisfies

$$
(\partial_t^2 - \Delta g)u = F
$$

Proof. Let $v = gu$ and $G = gF$, then we have

$$(\partial_t^2 - g\Delta g^{-1})v = G.$$ 

Then by Lemma 2.4, Remark 2.4 and 2.6 in Lemma 2.3 we have

$$
\|\langle x \rangle^{-\frac{1}{2}} \partial_t x u\|_{L^2_x} + \|\langle x \rangle^{-\frac{1}{2}} u\|_{L^2_x} \lesssim \|\partial_t x u(0, \cdot)\|_{L^2} + \|\langle x \rangle^{\frac{1}{2} + \epsilon} F\|_{L^2_x}.
$$

This completes the proof.

We now establish the local energy decay estimates for the $\square_g$.

Lemma 2.6 (Local Energy Decay). For the linear equation (1.3), if $F(t, x) = 0$ for $|x| > R$ with $R$ fixed, then for fixed $\beta \in C_0^\infty(\mathbb{R}^d)$, we have

$$
\beta u \|_{L^2_{t, H^1}} \lesssim \|u_0\|_{H^1} + \|u_1\|_{L^2_x} + \|F\|_{L^2_{t, L^2}}.
$$

Moreover, if $F \equiv 0$ and $s \in [0, 1]$, then

$$
\beta u \|_{L^2_{t, H^s}} \lesssim \|u_0\|_{H^s} + \|u_1\|_{H^{s-1}}.
$$

Proof. By (2.12), noting the support property of the forcing term, we know that

$$
\|\langle x \rangle^{-1/2 - \epsilon}(\partial_t, \partial_x)u\|_{L^2_{t, x}} + \|\langle x \rangle^{-3/2 - \epsilon} u\|_{L^2_{t, x}} \lesssim \|u_0\|_{H^1} + \|u_1\|_{L^2_x} + \|F\|_{L^2_{t, L^2}}.
$$

Thus

$$
\|\beta u\|_{L^2_{t, H^1}} \lesssim \|\langle x \rangle^{-1/2 - \epsilon}(u, \partial_x u)\|_{L^2_{t, x}} \lesssim \|u_0\|_{H^1} + \|u_1\|_{L^2_x} + \|F\|_{L^2_{t, L^2}}.
$$

In the case $F = 0$, let $v = gu$, then we have $(\partial_t^2 - g\Delta g^{-1})v = 0$. Note that $\|gu\|_{H^1} \lesssim \|u\|_{H^1}$, this gives $\|gu\|_{H^{s-1}} \lesssim \|u\|_{H^s}$, by duality. Thus we have by (2.1),

$$
\|\beta u\|_{L^2_{t, H^s}} \lesssim \|\beta v\|_{L^2_{t, L^2}} \lesssim \|\langle x \rangle^{-1/2 - \epsilon} v\|_{L^2_{t, x}} \lesssim \|\partial_t v(0, \cdot)\|_{L^2} + \|\partial_x v(0, \cdot)\|_{H^{-1}} \lesssim \|u_0\|_{H^2} + \|u_1\|_{H^{s-1}}.
$$

This completes the proof by interpolation with (2.13).

Now we are ready to give the proof of the KSS estimates presented in Theorem 1.2. In the proof, we shall cut the solution into two parts: a spatially localized part and the part near spatial infinity. For the localized part, we can use the local energy decay in Lemma 2.6 while for the part near infinity, we can view the metric as small perturbation of the Minkowski metric and use Lemma 2.2.

To this end, we introduce a cutoff function $\phi \in C_0^\infty$ which equals 1 in the unit ball $B_1$ and support in $B_2$ and let $\phi_R(x) = \phi(x/R)$. Let $v = \phi_R u$ and $w = (1 - \phi_R) u$ with $R > 1$ to be determined later, then

$$
\square_g v = [-\Delta_g, \phi_R] u + \phi_R F := f + \phi_R F,
$$

$$
\square_g w = [-\Delta_g, \phi_R] u + (1 - \phi_R) F := -f + (1 - \phi_R) F.
$$
By (2.12), we have
\begin{equation}
\|f\|_{L^2_t L^2_x} \lesssim \|\phi_2 R \partial_x u\|_{L^2_t L^2_x} + \|\phi_2 R u\|_{L^2_t L^2_x} \lesssim \|u_0\|_{H^1} + \|u_1\|_{L^2} + \|\langle x \rangle^{1/2 + \epsilon} F\|_{L^2_t L^2_x}.
\end{equation}

Note that \(v\) is compactly supported in \(x\), so Lemma 2.6 applies, i.e.
\begin{equation}
\|v\|_{L^2_t H^1_x} + \|\partial_t v\|_{L^2_t L^2_x} \lesssim \|u_0\|_{H^1} + \|u_1\|_{L^2} + \|\langle x \rangle^{1/2 + \epsilon} F\|_{L^2_t L^2_x}.
\end{equation}

If we define
\begin{equation}
\tilde{g}^{ij} = (1 - \phi_{R/2})g^{ij} + \phi_{R/2}\delta^{ij},
\end{equation}
then
\begin{equation*}
\tilde{\Delta} = \sum_{ij} \tilde{g}^{-2} \partial_i \tilde{g}^{ij} \partial_j = \Delta + h^{ij} \partial_i \partial_j + b^i \partial_i + c
\end{equation*}
where \(h^{ij} = (1 - \phi_{R/2})(g^{ij} - \delta^{ij})\), \(b^i = (1 - \phi_{R/4})r^i_1\) and \(c = (1 - \phi_{R/4})r^2_2\), with the \(r_j\) satisfying the bounds in (2.7). Thus since, \(\tilde{\Delta} w = \Delta w, w\) satisfies
\begin{equation}
(\tilde{\partial}_t^2 - \Delta - h^{ij})w = \Box w + (b^i \partial_i + c)w = (1 - \phi_R)F - f + (b^i \partial_i + c)w := G.
\end{equation}

Note that by (11) and choosing \(R \gg 1\) large enough, \(h^{ij}\) will be small enough so that we can apply Lemma (2.2) with \(\epsilon < \rho/4\),
\begin{align*}
\text{RHS of (2.3)} &\lesssim \|w'(0, \cdot)\|^2_{L^2_x(R^4)} + \int_0^T \int \left( |w'| + \frac{w}{|x|} \right) \left( |\sigma| + |H'| + \frac{h}{|x|} \right) |w'| \, dx \, dt \\
&\lesssim \|w'(0, \cdot)\|^2_{L^2_x(R^4)} + \|\langle x \rangle^{-1/2-\epsilon} \left( |w'| + \frac{w}{|x|} \right) \|_{L^2_t L^2_x} + \|\langle x \rangle^{1/2+\epsilon} G\|_{L^2_t L^2_x} \\
&\lesssim \|w'(0, \cdot)\|^2_{L^2_x(R^4)} + \|\langle x \rangle^{-1/2-\epsilon} \left( |w'| + \frac{w}{|x|} \right) \|_{L^2_t L^2_x} + \frac{1}{\epsilon} \|\langle x \rangle^{1/2+\epsilon} G\|_{L^2_t L^2_x} \\
&\lesssim \|w'(0, \cdot)\|^2_{L^2_x(R^4)} + 2\epsilon \|\langle x \rangle^{-1/2-\epsilon} \left( |w'| + \frac{w}{|x|} \right) \|_{L^2_t L^2_x} + \frac{1}{\epsilon} \|\langle x \rangle^{1/2+\epsilon} G\|_{L^2_t L^2_x}.
\end{align*}
Since we know from Lemma (2.2)
\begin{align*}
\|\langle x \rangle^{-1/2-\epsilon} \left( |w'| + \frac{w}{|x|} \right) \|_{L^2_t L^2_x} &\lesssim \|\langle x \rangle^{-1/2-\epsilon} |w'| + \langle x \rangle^{-3/2-\epsilon} |u| \|_{L^2_t L^2_x} \lesssim \|u'(0)\|_{L^2} + \|\langle x \rangle^{1/2+\epsilon} F\|_{L^2_t L^2_x}, \\
\|\langle x \rangle^{1/2+\epsilon} (b^i \partial_i + c)w \|_{L^2_t L^2_x} &\lesssim \|\langle x \rangle^{-1/2+\epsilon} - \rho \left( |w'| + \frac{w}{|x|} \right) \|_{L^2_t L^2_x} \lesssim \|\langle x \rangle^{-1/2-\epsilon} \left( |w'| + \frac{w}{|x|} \right) \|_{L^2_t L^2_x},
\end{align*}
and
\begin{align*}
\text{RHS of (2.3)} &\lesssim \|u_0\|_{H^1} + \|u_1\|_{L^2} + \|\langle x \rangle^{1/2+\epsilon} F\|_{L^2_t L^2_x}.
\end{align*}

Applying Lemma (2.2) and (2.16), we get finally that
\begin{align*}
\ln((2 + T))^{-1/2} \|\langle x \rangle^{-1/2} (|w'| + \frac{w}{|x|}) \|_{L^2_t L^2_x} &\lesssim \|u_0\|_{H^1} + \|u_1\|_{L^2} + \|\langle x \rangle^{1/2+\epsilon} F\|_{L^2_t L^2_x},
\end{align*}
Therefore, we can use Lemma (2.2) to see that \(w\) also satisfies the bounds in (11.6), which completes the proof of Theorem 1.2.
3. Higher Order KSS Estimates

In this section, we prove the KSS estimates involving high order derivatives as stated in Theorem 1.3

We first give the KSS estimate for \( P \). Consider the equation

\[
\begin{aligned}
\left\{ \begin{array}{l}
(\partial_t^2 + P)v(t, x) = G(t, x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d \\
u(0, x) = v_0(x), \quad \partial_t v(0, x) = v_1(x), \quad x \in \mathbb{R}^d.
\end{array} \right.
\end{aligned}
\]

Recall that if we let \( v = gu \) and \( G = gF \), then

\[
(\partial_t^2 - \Delta_g)u = F \Leftrightarrow (\partial_t^2 + P)v = G.
\]

Thus we have also the following KSS estimate

\[
A_\mu(T) \left( \|\langle x \rangle^{-\mu}v'\|_{L^2_{\tau, x}} + \|\langle x \rangle^{-\mu-1}v\|_{L^2_{\tau, x}} \right) \lesssim \|v'(0, \cdot)\|_{L^2_\tau} + \|\langle x \rangle^{1/2+\epsilon}G\|_{L^2_{\tau, x}}
\]

for \( \mu \geq 1/2 \) and \( \epsilon > 0 \).

For \( \partial_\xi^2 \Omega^\beta v \), we will use induction on |\( \beta \)| to give the proof. First, for |\( \beta \)| = 0, we have

\[
(\partial_t^2 + P)\partial_\xi^2 v = \partial_\xi^2 G + [P, \partial_\xi]\alpha v.
\]

Locally, by ellipticity, we have

\[
\sum_{|\alpha|=2} \|\partial_\xi^2 f\|_{L^2_{|\alpha| \leq R}} \lesssim \sum_{|\beta| \leq 1} \|\partial_\xi^2 f\|_{L^2_{|\beta| \leq 2R}} + \|P f\|_{L^2_{|\beta| \leq 2R}},
\]

and so, by an induction argument, and (2.4) and (2.5) in Lemma 2.3

\[
\|\partial_\xi^2 f\|_{L^2_{|\alpha| \leq R}} \lesssim \sum_{2j+1 \leq |\alpha|} \|\partial_\xi P^j f\|_{L^2_{|\alpha| \leq 2R}} + \sum_{2j \leq |\alpha|} \|P f\|_{L^2_{|\alpha| \leq 2R}}
\]

\[
\lesssim \|\langle x \rangle^{-2} f\|_{L^2} + \sum_{j \leq |\alpha|-1} \|\langle x \rangle^{-1} \partial_\xi P^{j/2} f\|_{L^2}.
\]

Thus by the KSS estimates (3.3), (2.6) and (2.10),

\[
\sum_{|\alpha| \leq N+1} \|\partial_\xi^2 v\|_{L^2_{|\alpha| \leq R}} \lesssim \sum_{j \leq N} \|\langle x \rangle^{-1} \partial_\xi P^{j/2} v\|_{L^2_{\tau, x}} + \|\langle x \rangle^{-2} v\|_{L^2}
\]

\[
\lesssim \sum_{j \leq N} \left( \|\partial_\xi P^{j/2} v_0\|_{L^2} + \|P^{j/2} v_1\|_{L^2} + \|\langle x \rangle^{1/2+\epsilon/2} P^{j/2} G\|_{L^2_{\tau, x}} \right)
\]

\[
\lesssim \sum_{\beta \leq N} \left( \|\partial_\xi^2 v_0\|_{H^1} + \|\partial_\xi^2 v_1\|_{L^2} + \|\langle x \rangle^{1/2+\epsilon} \partial_\xi G\|_{L^2_{\tau, x}} \right);
\]

Using again (3.3), we have

\[
A_\mu(T) \sum_{|\alpha| \leq N} \|\langle x \rangle^{-\mu} ((\partial_\xi^2 v)' + \frac{[\partial_\xi^2 v]}{\langle x \rangle})\|_{L^2_{\tau, x}} \lesssim \sum_{|\alpha| \leq N} \left( \|\partial_\xi^2 v_0\|_{H^1} + \|\partial_\xi^2 v_1\|_{L^2} + \|\langle x \rangle^{1/2+\epsilon} \partial_\xi G\|_{L^2_{\tau, x}} + \|\langle x \rangle^{1/2+\epsilon} [P, \partial_\xi^2 v]\|_{L^2_{\tau, x}} \right)
\]

for \( \mu \geq 1/2 \) and \( \epsilon > 0 \). Note that

\[
\|P, \partial_\xi^2 v\| \lesssim \langle x \rangle^{-\rho-1} \sum_{|\gamma| \leq |\alpha|+1} \langle \partial_\xi^2 v \rangle + \langle x \rangle^{-\rho-2} |v|,
\]
if choose $\varepsilon > 0$ small enough so that $1/2 + \varepsilon - \rho - 1 < -1/2 - 3\varepsilon$ and $R > 1$,

$$\sum_{|\alpha| \leq N} \| (x)^{1/2+\varepsilon} [P, \partial_x^\alpha v] \|_{L^2_{x,s}} \leq \varepsilon \sum_{|\alpha| \leq N} \| (x)^{-1/2-\varepsilon} (|\partial_x^\alpha v| + |\partial_x^\alpha v(x)|) \| + \sum_{|\alpha| \leq N+1} \| \partial_x^\alpha v \|_{L^2_{x,s}}.$$ 

So the first term in the right hand side of the inequality can be absorbed by (3.5) with $\mu = 1/2 + \varepsilon$. Thus for $\rho > 0$, $\mu \geq 1/2$ and $\varepsilon > 0$,

$$A_\mu(T) \sum_{|\alpha| \leq N} \| (x)^{-\mu} ((\partial_x^\alpha v)^\prime)^2 \|_{L^2_{x,s}} \lesssim \sum_{|\alpha| \leq N} \left( \| \partial_x^\alpha v_0 \|_{H^1} + \| \partial_x^\alpha v_1 \|_{L^2} + \| (x)^{1/2+\varepsilon} \partial_x^\alpha G \|_{L^2_{x,s}} \right).$$

Now we claim that for $\rho > 1$, $\mu \geq 1/2$ and $\varepsilon > 0$, we have

$$A_\mu(T) \sum_{|\alpha|+|\beta| \leq N} \| (x)^{-\mu} ((\partial_x^\alpha \partial_x^\beta v)^\prime)^2 \|_{L^2_{x,s}} \lesssim \sum_{|\alpha|+|\beta| \leq N} \left( \| \partial_x^\alpha \partial_x^\beta v_0 \|_{H^1} + \| \partial_x^\alpha \partial_x^\beta v_1 \|_{L^2} + \| (x)^{1/2+\varepsilon} \partial_x^\alpha \partial_x^\beta G \|_{L^2_{x,s}} \right).$$

We use induction on $|\beta|$ to give the proof. Assume the estimate (3.7) is true for $|\beta| \leq k$, then

$$(\partial_x^k + P)\Omega v = \Omega G + [P, \Omega] v$$

and we are reduced to estimate

$$\sum_{|\alpha|+|\beta| \leq N-1, |\beta| \leq k} \| (x)^{1/2+\varepsilon} \partial_x^\alpha \partial_x^\beta [P, \Omega] v \|_{L^2_{x,s}}$$

by (3.7). Note that

$$[P, \Omega] v = \sum_{|\gamma| \leq 2} r_{\gamma} v,$$

where $r_j$ are functions satisfying (2.7). Thus if $\rho > 1$ and $\mu > 0$ small enough such that $\rho > 1 + 2\varepsilon$,

$$\sum_{|\alpha|+|\beta| \leq N-1, |\beta| \leq k} \| (x)^{1/2+\varepsilon} \partial_x^\alpha \partial_x^\beta [P, \Omega] v \|_{L^2_{x,s}} \lesssim \sum_{|\alpha|+|\beta| \leq N, |\beta| \leq k} \| (x)^{1/2+\varepsilon-\rho} (|\partial_x^\alpha \partial_x^\beta v|) \|_{L^2_{x,s}} \lesssim \sum_{|\alpha|+|\beta| \leq N, |\beta| \leq k} \| (x)^{-1/2-\varepsilon} (|\partial_x^\alpha \partial_x^\beta v|) \|_{L^2_{x,s}} \lesssim \text{(3.7)}.$$

Since $\partial_t$ commute with $\partial_x^k + P$, we can conclude that for $\rho > 1$, $\mu \geq 1/2$ and $\varepsilon > 0$, we have

$$A_\mu(T) \sum_{j+|\alpha|+|\beta| \leq N} \| (x)^{-\mu} (|\partial_x^\alpha \partial_x^\beta v|) \|_{L^2_{x,s}} \lesssim \sum_{j+|\alpha|+|\beta| \leq N} \left( \| (\partial_x^\alpha \partial_x^\beta v(0, \cdot)) \|_{L^2} + \| (x)^{1/2+\varepsilon} \partial_x^\alpha \partial_x^\beta G \|_{L^2_{x,s}} \right).$$

(3.8)
Combining the energy estimates, we get the following estimates

\[
\sup_{0 \leq t \leq T} \sum_{1 \leq k+j \leq N+1} \left\| \partial_t^k P^{1/2} v(t, \cdot) \right\|_{L^2_x} + \sum_{|\alpha| \leq N} A_\mu(T) \left\| \langle x \rangle^{-\mu} \left| \left( Z^\alpha v \right) \right|_{L^2_x} \right\|_{L^2_t L^2_x} \\
\lesssim \sum_{|\alpha| \leq N} \left( \left\| \langle Z^\alpha v \rangle (0, \cdot) \right\|_{L^2_x} + \int_0^T \left\| Z^\alpha G(s, \cdot) \right\|_{L^2_x} ds \right).
\]

for the solution to the equation (3.1). For the \( (Z^\alpha v)'(0, \cdot) \) part, if it has \( \partial_t^j \) component with \( j \geq 2 \), we can use the equation to reduce it to the case \( \partial_t^{j-2} \), with an additional term

\[
\sum_{|\alpha| \leq N-1} \left\| Z^\alpha G(0, \cdot) \right\|_{L^2_x} \lesssim \sum_{|\alpha| \leq N-1} \left\| Z^\alpha G(t, \cdot) \right\|_{L^2_x W^{1,1}} \lesssim \sum_{|\alpha| \leq N} \left\| Z^\alpha G(t, \cdot) \right\|_{L^2_t L^2_x}.
\]

This means that

\[
\sup_{0 \leq t \leq T} \sum_{1 \leq k+j \leq N+1} \left\| \partial_t^ P^{1/2} v(t, \cdot) \right\|_{L^2_x} + \sum_{|\alpha| \leq N} A_\mu(T) \left\| \langle x \rangle^{-\mu} \left| \left( Z^\alpha v \right) \right|_{L^2_x} \right\|_{L^2_t L^2_x} \\
\lesssim \sum_{|\alpha| \leq N} \left\| (Y^\alpha v)'(0, \cdot) \right\|_{L^2_x} + \sum_{|\alpha| \leq N} \int_0^T \left\| Z^\alpha G(s, \cdot) \right\|_{L^2_x} ds.
\]

(3.9)

Turning back to the equation (1.5), let \( v = gu \) and \( G = gF \), then \((\partial_t^2 + P^2)v = G\) with \( v_0 = gu_0, v_1 = gu_1 \). Note that

\[
\partial_x u = g^{-1} \partial_x gu - g^{-1}(\partial_x g)u = g^{-1} \partial_x v - g^{-2}(\partial_x g)v,
\]

we have

\[
\sum_{|\alpha| \leq N} \left\| \langle x \rangle^{-\mu} \left| \left( Z^\alpha u \right) \right|_{L^2_x} \right\|_{L^2_t L^2_x} \lesssim \sum_{|\alpha| \leq N} \left\| \langle x \rangle^{-\mu} \left| \left( Z^\alpha v \right) \right|_{L^2_x} \right\|_{L^2_t L^2_x}.
\]

This concludes the proof of Theorem 1.3

4. Almost Global Existence

In this section we shall prove one of our main existence theorems, Theorem 1.1. The proof will be similar to that of Keel, Smith and Sogge for the Minkowski case (see [14]). We start with the now standard Sobolev estimate (see [18]).

**Lemma 4.1.** Suppose that \( h \in C^\infty(\mathbb{R}^3) \). Then, for \( R > 1 \),

\[
\| h \|_{L^\infty(R/2 \leq |x| \leq R)} \lesssim R^{-1} \sum_{|\alpha| \leq 2} \| Y^\alpha h \|_{L^2(R/4 \leq |x| \leq 2R)}.
\]

(4.1)

We now define the bilinear form \( \tilde{Q} \) by \( \tilde{Q}(u', u') = Q(u') \). The following estimate for the nonlinear part will be crucial.

**Lemma 4.2.** We have

\[
\sum_{|\alpha| \leq 4} \| Z^\alpha \tilde{Q}(u', v') \|_{L^2(\mathbb{R}^3)} \lesssim \left( \sum_{|\alpha| \leq 4} \| \langle x \rangle^{-1/2} Z^\alpha u' \|_{L^2(\mathbb{R}^3)}^2 \right) \left( \sum_{|\alpha| \leq 4} \| \langle x \rangle^{-1/2} Z^\alpha v' \|_{L^2(\mathbb{R}^3)}^2 \right).
\]
Proof. We clearly have the pointwise bound:

$$|Z^{\delta} \tilde{Q}(u', v')(s, x)| \lesssim \left( \sum_{|\alpha| \leq 4} |Z^\alpha u'(s, x)| \right) \left( \sum_{|\alpha| \leq 2} |Z^\alpha v'(s, x)| \right) + \left( \sum_{|\alpha| \leq 4} |Z^\alpha v'(s, x)| \right) \left( \sum_{|\alpha| \leq 2} |Z^\alpha u'(s, x)| \right).$$

We need only to estimate the first term. Using Lemma 4.1 for a given $R = 2^j$, $j \geq 0$, we get

$$\|Z^{\delta} \tilde{Q}(u', v')\|_{L^2([|x| \leq [2^j, 2^{j+1}]])}^2 \lesssim 2^{-2j} \sum_{|\alpha| \leq 4} \|Z^\alpha u'\|_{L^2([|x| \leq [2^j, 2^{j+1}]])}^2 \sum_{|\alpha| \leq 4} \|Z^\alpha v'\|_{L^2([|x| \leq [2^{j-1}, 2^{j+2}]])}^2$$

$$\lesssim \sum_{|\alpha| \leq 4} \|\langle x \rangle^{-1/2} Z^\alpha u'\|_{L^2([|x| \leq [2^j, 2^{j+1}]])}^2 \sum_{|\alpha| \leq 4} \|\langle x \rangle^{-1/2} Z^\alpha v'\|_{L^2([|x| \leq [2^{j-1}, 2^{j+2}]])}^2$$

$$\lesssim \sum_{|\alpha| \leq 4} \|\langle x \rangle^{-1/2} Z^\alpha u'\|_{L^2([|x| \leq [2^j, 2^{j+1}]])}^2 \sum_{|\alpha| \leq 4} \|\langle x \rangle^{-1/2} Z^\alpha v'\|_{L^2(\mathbb{R}^3)}^2.$$}

We also have the bound

$$\|Z^{\delta} \tilde{Q}(u', v')\|_{L^2([|x| < 1])}^2 \lesssim \sum_{|\alpha| \leq L} \|Z^\alpha u'\|_{L^2([|x| < 2])}^2 \sum_{|\alpha| \leq L} \|Z^\alpha v'\|_{L^2([|x| < 2])}^2.$$

Summing over $j$ gives the lemma. □

Proof of Theorem 1.1. We follow [14]. Let $u_{-1} = 0$. We define $u_k$, $k \in \mathbb{N}$ inductively by letting $u_k$ to solve

$$\begin{cases} \square_g u_k = Q(u_{k-1}',) \\
 u(0, x) = u_0(x), \quad \partial_t u(0, x) = u_1(x). \end{cases}$$

For $T > 0$, we denote

$$M_k(T) = \sup_{0 \leq t \leq T} \sum_{1 \leq i + j \leq 5} \|\partial_t^j P^{3/2} g u_k\|_{L^2(\mathbb{R}^3)} + \sum_{|\alpha| \leq 4} (\ln(2+T))^{-1/2} \|\langle x \rangle^{-1/2} Z^\alpha u_k\|_{L^2([0, T] \times \mathbb{R}^3)}.$$

Using Theorem 1.3, we see that there exists a constant $C_0$ such that

$$M_0(T) \leq C_0 \delta,$$

for any $T$. We claim that, for $k \geq 1$, we have

$$M_k(T_k) \leq 2C_0 \delta,$$

for $\delta$ sufficiently small and $T_k$ appropriately chosen later. We will prove this inductively. Assume that the bound holds for $k - 1$. By Theorem 1.3, we have, for $\delta$
small enough,
\[ M_k(T_δ) \leq C_0 δ + C \sum_{|α| \leq 4} \int_0^{T_δ} \left\| Z^α Q(u_{k-1})(s, \cdot) \right\|_{L^2(\mathbb{R}^d)} ds \]
\[ \leq C_0 δ + C \sum_{|α| \leq 4} \int_0^{T_δ} \left\| \langle x \rangle^{-1/2} Z^α u'_{k-1} \right\|^2_{L^2(\mathbb{R}^d)} ds \]
\[ \leq C_0 δ + C \ln(2 + T_δ) M^2_{k-1}(T_δ) \]
\[ \leq C_0 δ + C \ln(2 + T_δ)(2C_0 δ)^2, \]

where we have also used Lemma 4.2 and the induction hypothesis. Then, to prove (4.3), it is enough to have
\[ C_0 δ + C \ln(2 + T_δ)(2C_0 δ)^2 \leq 2C_0 δ \iff 4CC_0 \ln(2 + T_δ) δ \leq 1. \]

Therefore, we can set \( T_δ = \exp(c/δ) \) and \( c \) small enough.

To show that the sequence \( u_k \) converges, we estimate the quantity
\[ A_k(T) = \sup_{0 \leq t \leq T} \sum_{1 \leq i + j \leq 5} \left\| \partial_x^{i} t^{1/2} g(u_k - u_{k-1}) \right\|_{L^2(\mathbb{R}^d)} \]
\[ + \sum_{|α| \leq 4} \left( \ln(2 + T_δ) \right)^{-1/2} \left\| \langle x \rangle^{-1/2} Z^α (u'_k - u'_{k-1}) \right\|_{L^2(\mathbb{R}^d)} \]

It is clearly sufficient to show
\[ A_k(T) \leq \frac{1}{2} A_{k-1}(T). \]

Using Lemma 4.2 and repeating the above arguments, we obtain
\[ A_k(T_δ) \leq \tilde{C} \sum_{|α| \leq 4} \int_0^{T_δ} \left\| \langle x \rangle^{-1/2} Z^α (u'_{k-1} - u'_{k-2}) \right\|_{L^2(\mathbb{R}^d)} \]
\[ \times \sum_{|α| \leq 4} \left( \left\| \langle x \rangle^{-1/2} Z^α u'_{k-1} \right\|_{L^2(\mathbb{R}^d)} + \left\| \langle x \rangle^{-1/2} Z^α u'_{k-2} \right\|_{L^2(\mathbb{R}^d)} \right) ds \]
\[ \leq \tilde{C} \ln(2 + T_δ)(M_{k-1}(T_δ) + M_{k-2}(T_δ)) A_{k-1}(T_δ). \]

Using (4.3), the above inequality leads to (4.5) if \( δ \) is small enough.

5. Global Strichartz Estimates

In this section we shall prove Theorem 1.4, the global Strichartz estimates for \( \Box_\mathfrak{g} \). The proof will follow closely the arguments of [30], [3] and [9], which dealt with compact perturbations of the Minkowski metric or obstacle case.

Recall that we have equivalence (3.2) between the wave equations involving \( P \) and \( -\Delta_\mathfrak{g} \). We need only to give the proof for the case of following equation,
\[ \left\{ \begin{array}{l}
(\partial_t^2 + P)u(t, x) = F(t, x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d \\
u(0, x) = u_0(x), \quad \partial_t u(0, x) = u_1(x), \quad x \in \mathbb{R}^d.
\end{array} \right. \]

As in [9], we will need the following four ingredients: local energy decay for \( \partial_t^2 + P \), local Strichartz estimates for \( \partial_t^2 + P \), global Strichartz estimates for \( \partial_t^2 + \tilde{P} \) with \( \tilde{g} = g \) in \(|x| \geq R \) for some \( R > 0 \), and an estimate for \( \partial_t^2 + \tilde{P} \) like Proposition 2.1 in [9].
Let us give first the estimates for $\partial_t^2 + \tilde{P}$, where $\tilde{g}$ is defined by (2.17), i.e.,

$$\tilde{g}^{ij} = (1 - \phi_{R/2})g^{ij} + \phi_{R/2}\delta^{ij}.$$  

Then

$$\tilde{P} = -\sum_{ij} \frac{1}{\tilde{g}}\partial_i\tilde{g}^{ij}\partial_j\frac{1}{\tilde{g}} = -\partial_i\tilde{g}^{ij}\partial_j + b^i\partial_i + c$$

where $b^i = -2\tilde{g}^{-1}\partial_i(\tilde{g}\tilde{g}^{ij}) - \tilde{g}\tilde{g}^{ij}\partial_j\tilde{g}^{-1}$ and $c = \tilde{g}^{-1}\partial_i(\tilde{g}^{ij}\partial_j\tilde{g}).$

Consider the wave equation

\begin{equation}
(\partial_t^2 + \tilde{P})u = 0, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d
\end{equation}

\begin{equation}
\begin{cases}
u(0, x) = u_0(x), & \partial_t u(0, x) = u_1(x), \quad x \in \mathbb{R}^d.
\end{cases}
\end{equation}

We denote

$$A_j = \mathbb{R} \times \{2^j \leq |x| \leq 2^{j+1}\}, \quad A_{<j} = \mathbb{R} \times \{|x| \leq 2^j\}.$$ 

By hypotheses (H1), we can see that if $R \gg 1$ so that $R^{-\rho} \leq \epsilon,$

$$\sum_{j \in \mathbb{Z}} \sup_{A_j} |x|^2|\nabla^2 g^{ij}(x)| + |x||\nabla g^{ij}(x)| + |g^{ij}(x) - \delta^{ij}| \lesssim \sum_{j \in \mathbb{R}(R)} \sup_{A_j} |x|^{-\rho} \lesssim R^{-\rho} \leq \epsilon$$

and, for the lower order terms,

\begin{equation}
\sum_{j \in \mathbb{Z}} \sup_{A_j} |x|^2|\nabla b^i(x)| + |x||b^i(x)| \lesssim \sum_{j \in \mathbb{R}(R)} \sup_{A_j} |x|^{-\rho} \leq \epsilon,
\end{equation}

\begin{equation}
\sum_{j \in \mathbb{Z}} \sup_{A_j} |x|^2|c(t, x)|^2 \lesssim \sum_{j \in \mathbb{R}(R)} \sup_{A_j} |x|^{-2\rho} \leq \epsilon.
\end{equation}

Thus we can apply Theorem 6 in Metcalfe-Tataru [24] to get the following

**Proposition 5.1** (Global Strichartz Estimates for Small Perturbation). Let $d \geq 3$, $s \in [0, 1]$ ($s \in (0, 1)$ if $d = 3$). The solution of (5.2) satisfies

\begin{equation}
\|u\|_{L_t^1L_x^\infty} \lesssim \|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-1}},
\end{equation}

for any admissible $(s, q, r)$ with $q > 2, r < \infty$. In other words, we have

\begin{equation}
\|e^{it\tilde{P}^{1/2}}f\|_{L_t^1L_x^\infty} \lesssim \|f\|_{\dot{H}^s}.
\end{equation}

As a corollary, we can get an estimate for $\partial_t^2 + \tilde{P}$ like Proposition 2.1 in [9].

**Proposition 5.2.** Let $d \geq 3$, $s \in [0, 1]$ ($s \in (0, 1)$ if $d = 3$). The solution $u$ of $(\partial_t^2 + \tilde{P})u = \beta F$ satisfies

\begin{equation}
\|u\|_{L_t^1L_x^\infty} \lesssim \|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-1}} + \|F\|_{L_t^2H^{s-1}},
\end{equation}

for any admissible $(s, q, r)$ with $q > 2, r < \infty.$

**Proof.** Applying (2.14) in this setting, we know that for any $s \in [0, 1],$

$$\|\beta e^{it\tilde{P}^{1/2}}f\|_{L_t^1H^s} \lesssim \|f\|_{\dot{H}^s}.$$  

By duality, this is equivalent to say that for $s \in [-1, 0],$

$$\| \int e^{-it\tilde{P}^{1/2}}\beta F(t, \cdot)dt\|_{\dot{H}^s} \lesssim \|f\|_{L_t^1H^s}.$$
Proof. In Lemma 2.4 and (5.10),
\[ \| e^{i(t-s)\tilde{P}^{1/2}} \tilde{P}^{-1/2} \beta F(s, \cdot) ds \|_{L^q_t L^r_x} \lesssim \| e^{-i\alpha \tilde{P}^{1/2}} \tilde{P}^{-1/2} \beta F(s, \cdot) ds \|_{\dot{H}^s} \lesssim \| e^{-i\alpha \tilde{P}^{1/2}} \beta F(s, \cdot) ds \|_{\dot{H}^{s-1}} \lesssim \| F \|_{L^2_t H^{s-1}}. \]

Since \( q > 2 \), we can apply the Christ-Kiselev lemma [5] to conclude the proof. \( \square \)

Now we give the local Strichartz estimates for the \( \partial_t^2 + P \). Consider the wave equation
\[ (5.9) \quad \begin{cases} (\partial_t^2 + P)u(t, x) = 0, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d \\ u(0, x) = u_0(x), & \partial_t u(0, x) = u_1(x), \quad x \in \mathbb{R}^d. \end{cases} \]

The local Strichartz estimates for the variable coefficient wave equations was studied extensively, see e.g. Kapitanski [13], Mockenhaupt-Seeger-Sogge [25], Smith [29], Bahouri-Chemin [1] [2], Tataru [34], [35], [37]. In particular, we have

**Lemma 5.3** (Theorem 1.1 in Tataru [37]). If \((\partial_t, \partial_x)^\alpha a^{ij}(t, x) \in L^1_{t \in [0,1]} L^\infty_x\) for any \( |\alpha| = 2 \) and \((\partial_t^2 - \partial_x a^{ij}(t, x) \partial_x)u = F\), then
\[ \| D^{1-s} u \|_{L^q_{t \in [0,1]} L^r_x} \lesssim \| u' \|_{L^\infty_{t \in [0,1]} L^2_x} + \| F \|_{L^1_{t \in [0,1]} L^2_x} \]
for any admissible \((s, q, r)\) with \( r < \infty \).

Note that we can write
\[ \partial_t^2 + P = \partial_t^2 - \partial_x g^{ij} \partial_j + b^i \partial_i + c \]
with \( b^i = O((x)^{-1-\rho}) \) and \( c = O((x)^{-2-\rho}) \). Moreover, for \( a^{ij} = g^{ij} \) and \( |\alpha| = 2 \), we have \( \partial_x^2 a^{ij}(x) = O((x)^{-2-\rho}) \in L^1_{t \in [0,1]} L^\infty_x. \) Then for the equation \( (5.9) \), we have
\[ \| D^{1-s} u \|_{L^q_{t \in [0,1]} L^r_x} \lesssim \| u' \|_{L^\infty_{t \in [0,1]} L^2_x} + \| b^i \partial_i u + cu \|_{L^1 L^2_x} \]
\[ \lesssim \| u' \|_{L^\infty_{t \in [0,1]} L^2_x} \]
\[ \lesssim \| u_0 \|_{\dot{H}^1} + \| u_1 \|_{L^2} \]
(5.10)
by energy estimates and Lemma 5.3. Moreover, we can prove the following

**Proposition 5.4** (Local Strichartz Estimates). Let \( s \in [0,1], (s, q, r) \) admissible with \( r < \infty \), and \( u \) be the solution to the equation \( (5.9) \), then
\[ \| u \|_{L^q_{t \in [0,1]} L^r_x} \lesssim \| u_0 \|_{\dot{H}^s} + \| u_1 \|_{\dot{H}^{s-1}}. \]

**Proof.** Since \((\partial_t^2 + P)u = 0\), we have \((\partial_t^2 + P)^{s-1}u = 0\). Thus by (2.11) and (2.8) in Lemma 2.4 and (5.10),
\[ \| u \|_{L^q_{t \in [0,1]} L^r_x} \lesssim \| P^{s-1} u \|_{L^q_{t \in [0,1]} L^r_x} \]
\[ \lesssim \| D^{1-s} P^{s-1} u \|_{L^q_{t \in [0,1]} L^r_x} \]
\[ \lesssim \| P^{s-1} u_0 \|_{\dot{H}^s} + \| P^{s-1} u_1 \|_{L^2} \]
\[ \lesssim \| u_0 \|_{\dot{H}^s} + \| u_1 \|_{\dot{H}^{s-1}}, \]
if \((s, q, r)\) is admissible, \(s \in [0, 1]\) and \(r < \frac{d}{1 + s}\), i.e., \(s > 1 - \frac{d}{r}\). Since \((s, q, r)\) is admissible, then if \(d = 3\), we must have \(q > 2\).

\[
s = \frac{d}{2} - \frac{d}{r} - \frac{1}{q} > 1 - \frac{d}{r}
\]

Similarly, if \(d \geq 4\), we have \(q \geq 2\),

\[
s = \frac{d}{2} - \frac{d}{r} - \frac{1}{q} \geq \frac{d - 1}{2} - \frac{d}{r} > 1 - \frac{d}{r}.
\]

This means that we have the local Strichartz estimates for all the cases where \((s, q, r)\) is admissible and \(s \in [0, 1]\).

Now we are ready to give the proof of the global Strichartz estimates for \(\partial_t^2 + P\).

**Proposition 5.5.** For the linear equation \((5.1)\), assume that

\[u_0(x) = u_1(x) = F(t, x) = 0\] when \(|x| > 2R\), then for any \(s \in [0, 1]\) and admissible \((s, q, r)\) with \(q > 2\) and \(r < \infty\), we have

\[
\|u\|_{L_t^2 L_x^r} \lesssim \|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-1}} + \|F\|_{L_t^2 \dot{H}^{s-1}}.
\]

**Proof.** One of the key ingredients in the proof is the following variant of Lemma \([2.6]\) which holds for all \(s \in [0, 1]\),

\[
(5.13) \quad \|\beta u\|_{L_t^2 \dot{H}^s \cap L_x^r} + \|\beta u_t\|_{L_t^2 \dot{H}^{s-1} \cap L_x^r} \lesssim \|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-1}} + \|F\|_{L_t^2 \dot{H}^{s-1}}.
\]

The \(L_t^2\) estimate with \(s = 1\) comes from \((2.13)\), then the estimates follow from Duhamel’s formula and duality.

To prove \((5.12)\), let us argue as before. Let \(v = \phi_3 R u\) and \(w = (1 - \phi_3 R) u\) with \(R \gg 1\) as in the definition of \(\tilde{g}\). Then \(w\) solves the wave equation for \(\partial_t^2 + \tilde{P}\),

\[
\begin{cases}
(\partial_t^2 + \tilde{P})w = (\partial_t^2 + P)w = [\phi_3 R, P]u \\
|w|_{t=0} = \partial_t w|_{t=0} = 0.
\end{cases}
\]

An application of Proposition \([5.4]\) shows that \(\|w\|_{L_t^2 L_x^r} \) is dominated by \(\|\beta u\|_{L_t^2 \dot{H}^s}\) if \(\beta\) equals one on the support of \(\phi_3 R\). Therefore, by \((5.13)\), \(\|w\|_{L_t^2 L_x^r} \) is dominated by the right side of \((5.12)\).

As a result, we are left with showing that if \(v = \phi_3 R u\) then

\[
(5.14) \quad \|v\|_{L_t^2 L_x^r} \lesssim \|u_0\|_{\dot{H}^s} + \|g\|_{\dot{H}^{s-1}} + \|F\|_{L_t^2 \dot{H}^{s-1}}.
\]

To do this, fix \(\varphi \in C_0^\infty((-1, 1))\) satisfying \(\sum_{j=-\infty}^{\infty} \varphi(t - j) = 1\). For a given \(j \in \mathbb{N}\), let \(v_j = \varphi(t - j) v\). Then \(v_j\) solves

\[
\begin{cases}
(\partial_t^2 + P)v_j = \varphi(t - j)[P, \phi_3 R]u - [\partial_t^2, \varphi(t - j)]\phi_3 R u + \varphi(t - j) F \quad &v_j(0, \cdot) = \partial_t v_j(0, \cdot) = 0, \\
v_j(0, \cdot) = \partial_t v_j(0, \cdot) = 0.
\end{cases}
\]

while \(v_0 = v - \sum_{j=1}^{\infty} v_j\) solves

\[
\begin{cases}
(\partial_t^2 + P)v_0 = \tilde{\varphi}[P, \phi_3 R]u - [\partial_t^2, \tilde{\varphi}]\phi_3 R u + \tilde{\varphi} F \\
v_0|_{t=0} = u_0, \quad \partial_t v_0|_{t=0} = u_1,
\end{cases}
\]

if \(\tilde{\varphi} = 1 - \sum_{j=1}^{\infty} \varphi(t - j)\) if \(t \geq 0\) and 0 otherwise.
If we then let $F_j = (\partial_t^2 + P) v_j$ be the forcing term for $v_j$, $j = 0, 1, 2, \ldots$, then, by (5.13), we have that
\[
\sum_{j=0}^{\infty} \| F_j \|_{L_t^2 H^{-s}}^2 \lesssim \| u_0 \|_{H^s}^2 + \| u_1 \|_{H^{s-1}}^2 + \| F \|_{L_t^2 H^{-s-1}}^2.
\]
By the local Strichartz estimates (5.11) and Duhamel’s formula, we get for $j = 1, 2, \ldots$
\[
\| v_j \|_{L_t^q L_x^r} \lesssim \int_0^\infty \| F_j(s, \cdot) \|_{H^{-s-1}} \, ds \lesssim \| F_j \|_{L_t^2 H^{-s-1}},
\]
using Schwarz’s inequality and the support properties of the $F_j$ in the last step. Similarly,
\[
\| v_0 \|_{L_t^q L_x^r} \lesssim \| u_0 \|_{H^s} + \| u_1 \|_{H^{s-1}} + \| F_0 \|_{L_t^2 H^{-s-1}}.
\]
Since $q, r \geq 2$, we have
\[
\| v \|_{L_t^q L_x^r}^2 \lesssim \sum_{j=0}^{\infty} \| v_j \|_{L_t^q L_x^r}^2
\]
and so we get
\[
\| v \|_{L_t^q L_x^r}^2 \lesssim \| u_0 \|_{H^s}^2 + \| g \|_{H^{s-1}}^2 + \| F \|_{L_t^2 H^{-s-1}}^2
\]
as desired, which finishes the proof. \[\square\]

**End of Proof of Theorem 1.4.** Recall that we are assuming that $(\partial_t^2 + P) u = 0$. By Proposition 5.5 we may also assume that the initial data for $u$ vanishes when $|x| < 3R/2$. We then fix $\beta \in C_c^\infty(\mathbb{R}^n)$ satisfying $\beta(x) = 1$, $|x| \leq R$ and $\beta(x) = 0$, $|x| > 3R/2$ and write
\[
u = \tilde{u} - v = (1 - \beta) \tilde{u} + (\beta \tilde{u} - v),
\]
where $\tilde{u}$ solves the Cauchy problem for $(\partial_t^2 + \tilde{P}) \tilde{u} = 0$ with initial data $(u_0, u_1)$. By the global Strichartz estimate (5.8), we can restrict our attention to $w = \beta \tilde{u} - v$. But
\[
(\partial_t^2 + P) w = [P, \beta] \tilde{u} \equiv G
\]
is supported in $R < |x| < 2R$, and satisfies
\[
(5.15) \int_0^\infty \| G(t, \cdot) \|_{H^{-s-1}}^2 \, dt \lesssim \| u_0 \|_{H^s}^2 + \| u_1 \|_{H^{s-1}}^2
\]
by Lemma 2.4. Note also that $w$ has vanishing initial data. Therefore, since Proposition 5.5 tells us that $\| w \|_{L_t^q L_x^r}^2$ is dominated by the left side of (5.15), the proof is complete. \[\square\]

6. **Weighted Strichartz Estimates**

Recall Lemma 2.1 and Remark 2.1, we have the following estimates
\[
\| \langle x \rangle^{-1/2 - \epsilon} e^{itP^{1/2}} f \|_{L_t^2 L_x^2} \lesssim \| f \|_{L_t^2},
\]
\[
\| \langle x \rangle^{-3/2 - \epsilon} e^{itP^{1/2}} f \|_{L_t^2 L_x^2} \lesssim \| f \|_{H_t^1},
\]
By interpolation, we get
\[
(6.1) \| |x|^{-\alpha} e^{itP^{1/2}} f \|_{L_t^2 L_x^2} \gtrsim \| \langle x \rangle^{-\alpha} e^{itP^{1/2}} f \|_{L_t^2 L_x^2} \lesssim \| f \|_{H_t^{s-1/2}}^2
\]
for any $\alpha \in [1/2, 3/2]$, $\epsilon > 0$ and $d \geq 3$. 

Recall that Fang and Wang obtained the following Sobolev inequalities with angular regularity (Corollary 1.2 in [6])
\[(6.2) \quad \|x\|L^\infty_x L^{2n}_x \lesssim \|x\|H^{\frac{d}{2} - \alpha}_x \lesssim \|f\|\dot{H}^{\frac{d}{2} - \alpha}_x\]
for \(\alpha \in (1/2, d/2)\) and some \(\eta > 0\). Then by (2.8) in Lemma 2.4 we have
\[(6.3) \quad \|x\|L^\infty_x L^{2n}_x \lesssim \|e^{itP^{1/2}} f(x)\| L^\infty_x \lesssim \|f\|\dot{H}^{\frac{d}{2}}_x\]
for \(\alpha \in (1/2, 1]\) and some \(\eta > 0\).

If we interpolate between (6.1) and (6.3) we conclude that,
\[(6.4) \quad \|x\|L^\infty_x L^{2n}_x \lesssim \|e^{itP^{1/2}} f(x)\| L^\infty_x \lesssim \|f\|\dot{H}^{\frac{d}{2}}_x\]
for any \(\epsilon > 0, 2 < q \leq \infty\) and \(s \in (1/2 - 1/q, 1]\) with \(q > 0\) small enough. This implies Theorem 1.6 by Duhamel's formula and observation (3.2).

Using the weighted Strichartz estimates, we shall prove the Strauss conjecture in our setting, by adapting the arguments in Hidano-Metcalfe-Smith-Sogge-Zhou [9].

To this end, we define \(X = X_{s, \epsilon, q}(\mathbb{R}^d)\) to be the space with norm defined by
\[(6.5) \quad \|h\|_{X_{s, \epsilon, q}} = \|h\|L^\infty_{x, \infty} + \|x\|^{\frac{d}{2} - \frac{d_1}{2} - s - \epsilon} \leq \|h\|L^\infty_{x, \infty}\]
if \(d(1 - 1/q) = s\). Note that we have the embedding \(\dot{H}^s \subset X_{s, \epsilon, q}\) for \(s \in (1/2, 1]\) by Sobolev embedding and (6.3). By duality, we have
\[(6.6) \quad X_{s, \epsilon, q}^\prime \subset \dot{H}^{s-1}\]
for \(s \in (0, 1/2)\).

We also denote the space \(Y_{s, \epsilon, q}(\mathbb{R}^d)\) with norm
\[(6.7) \quad \|h\|_{Y_{s, \epsilon, q}} = \|\langle x\rangle^{\frac{d}{2} - s - \epsilon} h\|L^2_x\]
Note that by Remark 2.1 duality and interpolation and the homogeneous estimate (6.1), we have
\[(6.8) \quad \|u\|_{L^2_t Y_{s, \epsilon, q}} + \|u\|_{L^\infty_t H^s} + \|\partial_t u\|_{L^\infty_t H^{s-1}} \lesssim \|u_0\|\dot{H}^s + \|u_1\|\dot{H}^{s-1} + \|F\|_Y\]
for the solutions to the linear wave equation (5.1) and \(s \in [0, 1]\).

Then by (2.14) and energy estimate, if \((\partial_t^2 + P)u = 0\), we have
\[(6.9) \quad \|\phi u\|_{L^\infty_t H^s} \lesssim \|\phi u\|_{L^\infty_t H^s} + \|\phi u\|_{L^\infty_t H^{s-1}} \lesssim \|u_0\|\dot{H}^s + \|u_1\|\dot{H}^{s-1},\]
for any \(\phi \in C^\infty_\infty\), \(s \in [0, 1]\) and \(p \geq 2\). Thus by (6.7) and Christ-Kiselev lemma, we have
\[(6.10) \quad \|\phi u\|_{L^\infty_t H^s} \lesssim \|u_0\|\dot{H}^s + \|u_1\|\dot{H}^{s-1} + \|F\|_Y\]
for the solutions of the linear wave equation (5.1), \(s \in [0, 1]\) and \(p > 2\).

In conclusion, by (6.3), (6.7) and (6.8), we have
\[(6.11) \quad \|u\|_{L^\infty_t H^s \cap H^s_x \cap L^\infty_t Y_{s, \epsilon, q}} + \|\partial_t u\|_{L^\infty_t H^{s-1}} \lesssim \|u_0\|\dot{H}^s + \|u_1\|\dot{H}^{s-1} + \|F\|_Y\]
for the solutions to the linear wave equation (5.1), if \(p > 2, s \in (1/2 - 1/p, 1]\). Here, it will be useful to note that if \(d = 3, 4, p > p_c\), i.e. \(d - \frac{2}{p-1} > \frac{1}{2} - \frac{1}{p}\) and \(p > 1\), we can choose \(\epsilon > 0\), so that \(s = \frac{d}{2} - \frac{d + 1}{p} - \frac{2\epsilon}{p} \in (1/2 - 1/p, 1/2)\) and
\[(6.12) \quad p\left(\frac{d}{2} - \frac{d + 1}{p} - s - \epsilon\right) = -\left(\frac{d}{2} - (1-s)\right) = -s - \frac{d - 2}{2}.\]
Now we want to prove the higher order estimates for \((6.3)\). We claim that if \(g^{ij}(x) = h(|x|)\delta^{ij}\),

\[
\sum_{|\alpha|+|\beta|\leq 1} \left( \|\partial_x^\alpha \Omega^\beta u\|_{L_x^\infty H_t^{r-\rho} \cap L_x^\infty Y_{s,r} \cap L_x^\delta Y_{s,\epsilon}} + \|\partial_t \partial_x^\alpha \Omega^\beta u\|_{L_x^\infty H_t^{r-1}} \right) \lesssim \sum_{|\alpha|+|\beta|\leq 1} \left( \|\partial_x^\alpha \Omega^\beta u_0\|_{H^r} + \|\partial_x^\alpha \Omega^\beta u_1\|_{H_t^{r-1}} + \|\partial_t \partial_x^\alpha \Omega^\beta F\|_{L_x^\delta Y_{s,\epsilon}} \right)
\]

for the solutions to the linear wave equation \((6.1)\), if \(\rho > 1, p > 2, s \in (1/2 - 1/p, 1]\).

As in the proof of \((6.9)\), we need only to prove the higher order version of \((6.3)\), \((6.4)\) and \((6.8)\).

We first estimate \(L_x^2 Y_{s,\epsilon}\) part. By \((6.7)\), \((2.4)\) \((2.5)\) in Lemma \([2.3]\) and \((2.8)\) \((2.10)\) in Lemma \([2.4]\) for any \(s \in (1/2 - 1/p, 1]\),

\[
\sum_{|\alpha| \leq 1} \|\partial_x^\alpha u\|_{L_x^2 Y_{s,\epsilon}} \lesssim \sum_{|\alpha| \leq 1} \|\partial_x^\alpha u\|_{L_x^2 Y_{s,\epsilon}}
\]

\[
\lesssim \sum_{|\alpha| \leq 1} \left( \|P^{j/2} u\|_{L_x^2 Y_{s,\epsilon/2}} \right)
\]

\[
\lesssim \sum_{j \leq 1} \left( \|P^{j/2} u_0\|_{H^r} + \|P^{j/2} u_1\|_{H_t^{r-1}} + \|P^{j/2} F\|_{L_x^\delta Y_{s,\epsilon/2}} \right)
\]

\[
\lesssim \sum_{j \leq 1} \left( \|P^{(j+s)/2} u_0\|_{L_x^2} + \|P^{(j+s-1)/2} u_1\|_{L_x^2} \right) + \sum_{|\alpha| \leq 1} \|\partial_x^\alpha F\|_{L_x^\delta Y_{s,\epsilon}}
\]

Note that since \(g^{ij}(x) = h(|x|)\delta^{ij}\), \([P, \Omega^{ij}] = 0\). Thus

\[
\sum_{|\alpha| = 1} \|\Omega^\alpha u\|_{L_x^2 Y_{s,\epsilon}} \lesssim \sum_{|\alpha| = 1} \left( \|\Omega^\alpha u_0\|_{H^r} + \|\Omega^\alpha u_1\|_{H_t^{r-1}} + \|\Omega^\alpha F\|_{L_x^\delta Y_{s,\epsilon}} \right)
\]

In conclusion, we have for \(\rho > 0\),

\[
(6.12) \quad \sum_{|\alpha|+|\beta|\leq 1} \|\partial_x^\alpha \Omega^\beta u\|_{L_x^2 Y_{s,\epsilon}} \lesssim \sum_{|\alpha|+|\beta|\leq 1} \left( \|\partial_x^\alpha \Omega^\beta u_0\|_{H^r} + \|\partial_x^\alpha \Omega^\beta u_1\|_{H_t^{r-1}} + \|\partial_t \partial_x^\alpha \Omega^\beta F\|_{L_x^\delta Y_{s,\epsilon}} \right).
\]

We turn to the proof of the higher order estimate for \((6.3)\). Note that we assume \(g^{ij}(x) = h(|x|)\delta^{ij}\), then \(P = -h\Delta + r_1 \partial + r_2\), and hence by Hardy inequality and \((2.8)\) in Lemma \([2.4]\)

\[
\|\Delta u\|_{H^1} \lesssim \|\Delta u\|_{L_2} \lesssim \|P u\|_{L_2} + \|\partial u\|_{L_2} + \|r_2 u\|_{L_2} \lesssim \|P^{1/2} u\|_{H^1}.
\]

If we interpolate with \((2.9)\), we get that for \(s \in [0, 1]\),

\[
(6.13) \quad \|\partial u\|_{H^r} \lesssim \|P^{1/2} u\|_{H^r}.
\]
By (6.3), (6.13), (2.8) and (2.10) in Lemma 2.3 if $(\partial^2_t + P)u = 0$ and \( s \in (1/2, 1] \), we have the energy estimate,

\[
\sum_{|\alpha| \leq 1} \| \partial_x^\alpha u \|_{L^\infty_t H^s_x} \lesssim \sum_{j \leq 1} \| P^{j/2} u \|_{L^\infty_t H^s_x}
\lesssim \sum_{j \leq 1} \left( \| P^{j/2} u(0) \|_{H^s_x} + \| P^{j/2} \partial_t u(0) \|_{H^{s-1}_x} \right)
\lesssim \sum_{|\alpha| \leq 1} \left( \| \partial_x^\alpha u(0) \|_{H^s_x} + \| \partial_x^\alpha \partial_t u(0) \|_{H^{s-1}_x} \right)
\lesssim \sum_{|\alpha| \leq 1} \left( \| \partial_x^\alpha u(0) \|_{H^s_x} + \| \partial_x^\alpha \partial_t u(0) \|_{H^{s-1}_x} \right)
\]

and

\[
\sum_{|\alpha| \leq 1} \| x^{\frac{1}{2}-s} \partial_x^\alpha u \|_{L^\infty_{t,x} L^{2\times n}_{k+k+1,x}} \lesssim \sum_{|\alpha| \leq 1} \left( \| \partial_x^\alpha u(0) \|_{H^s_x} + \| \partial_x^\alpha \partial_t u(0) \|_{H^{s-1}_x} \right) + \| P, \partial_x u \|_{L^1_t H^{1/2}_x}
\lesssim \sum_{|\alpha| \leq 1} \left( \| \partial_x^\alpha u(0) \|_{H^s_x} + \| \partial_x^\alpha \partial_t u(0) \|_{H^{s-1}_x} \right) + \sum_{1 \leq |\beta| \leq 2} \| \partial_x^\beta u \|_{L^1_t H^{1/2}_x}
\lesssim \sum_{|\alpha| \leq 1} \left( \| \partial_x^\alpha u(0) \|_{H^s_x} + \| \partial_x^\alpha \partial_t u(0) \|_{H^{s-1}_x} \right) + \sum_{|\beta| \leq 1} \| \partial_x^\beta u \|_{L^1_t H^s_x}
\lesssim (1 + T) \sum_{|\alpha| \leq 1} \left( \| \partial_x^\alpha u(0) \|_{H^s_x} + \| \partial_x^\alpha \partial_t u(0) \|_{H^{s-1}_x} \right).
\]

Thus for any \( k \in \mathbb{Z} \),

\[
\sum_{|\alpha| \leq 1} \| x^{\frac{1}{2}-s} \partial_x^\alpha u(t) \|_{L^\infty_{t,x} L^{2\times n}_{k+k+1,x}} \lesssim \sum_{|\alpha| \leq 1} \left( \| \partial_x^\alpha u(k) \|_{H^s_x} + \| \partial_x^\alpha \partial_t u(k) \|_{H^{s-1}_x} \right)
\lesssim \sum_{|\alpha| \leq 1} \left( \| \partial_x^\alpha u(0) \|_{H^s_x} + \| \partial_x^\alpha \partial_t u(0) \|_{H^{s-1}_x} \right).
\]

Hence we get the higher order version of (6.3) if we combine the commutativity of \( P \) and \( \Omega \),

\[
\sum_{|\alpha| + |\beta| \leq 1} \| x^{\frac{1}{2}-s} \partial_x^\alpha \Omega^\beta u(t) \|_{L^\infty_{t,x} L^{2\times n}_{k+k+1,x}} \lesssim \sum_{|\alpha| + |\beta| \leq 1} \left( \| \partial_x^\alpha \Omega^\beta u_0 \|_{H^s_x} + \| \partial_x^\alpha \Omega^\beta u_1 \|_{H^{s-1}_x} \right)
\]

for the solution to the linear wave equation (6.11) with \( F = 0 \), and \( s \in (1/2, 1] \).

We need only to prove the higher order version of (6.5) now.

By (3.6), we know that if \((\partial_t^2 + P)u = 0\) and \( p > 1 \),

\[
\| \langle x \rangle^{-1/2-\epsilon} \partial_x u \|_{L^2_t H^s_x} \lesssim \sum_{|\alpha| = 2} \| \langle x \rangle^{-1/2-\epsilon} \partial_x^\alpha u \|_{L^2_t L^2_x} + \| \langle x \rangle^{-3/2-\epsilon} \partial_x^3 u \|_{L^2_t L^2_x} \lesssim \| \partial_x u(0) \|_{H^1_x} + \| \partial_t u(0) \|_{H^1_x},
\]

\[
\| \langle x \rangle^{-1/2-\epsilon} \partial_x u \|_{L^2_t L^2_x} \lesssim \| \partial_x u(0) \|_{L^2_x} + \| \partial_t u(0) \|_{L^2_x},
\]

and
Thus for \( s \in [0, 1] \),
\[
\| \phi \partial_x u \|_{L^2_t H^{-s}} \lesssim \| \partial u(0) \|_{H^2} + \| \partial_t u(0) \|_{H^{s-1}} + \sum_{|\alpha| \leq 1} \left( \| \partial_x^\alpha u(0) \|_{\dot{H}^s} + \| \partial_x^\alpha \partial_t u(0) \|_{\dot{H}^{s-1}} \right).
\]

Combining the energy estimates, (6.12) and Christ-Kiselev lemma, we know that \( (6.15) \)
\[
\sum_{|\alpha| + |\beta| \leq 1} \| \phi \partial_x^\alpha \partial_t^\beta u \|_{L^2_t H^s} \lesssim \sum_{|\alpha| + |\beta| \leq 1} \left( \| \partial_x^\alpha \partial_t^\beta u_0 \|_{\dot{H}^s} + \| \partial_x^\alpha \partial_t^\beta \partial_h u_1 \|_{\dot{H}^{s-1}} + \| \partial_x^\alpha \partial_t^\beta F \|_{L^2_t Y_{1-s, s}} \right)
\]
for the solutions to the linear wave equation (5.1), \( s \in [0, 1] \) and \( p > 2 \). This completes the proof of (6.11).

To conclude this section, let us point out that a similar estimate of (6.11) holds for the solution to (6.15), as in the end of Section 3. Precisely, if \( g^\gamma(x) = h(|x|) \delta^\gamma \), we have
\[
\sum_{|\alpha| + |\beta| \leq 1} \left( \| \partial_x^\alpha \partial_t^\beta u_0 \|_{L^2_t H^s} + \| \partial_x^\alpha \partial_t^\beta \partial_h u_1 \|_{L^2_t H^{s-1}} \right) \lesssim \sum_{|\alpha| + |\beta| \leq 1} \left( \| \partial_x^\alpha \partial_t^\beta u_0 \|_{\dot{H}^s} + \| \partial_x^\alpha \partial_t^\beta \partial_h u_1 \|_{\dot{H}^{s-1}} + \| \partial_x^\alpha \partial_t^\beta F \|_{L^2_t Y_{1-s, s}} \right)
\]
for the solutions to the linear wave equation (1.10), if \( \rho > 1, p > 2, s \in (1/2 - 1/p, 1] \).

7. Strauss Conjecture

In this section, we prove the Strauss conjecture in the setting where \( g^\gamma = h(|x|) \delta^\gamma, p > p_c = 1 + \sqrt{2} \) and \( d = 3 \), i.e., Theorem 1.5.

By (6.10), (6.6) and Duhamel’s formula, we have
\[
\sum_{|\alpha| + |\beta| \leq 1} \left( \| \partial_x^\alpha \partial_t^\beta u \|_{L^2_t H^s} + \| \partial_h \partial_x^\alpha \partial_t^\beta u \|_{L^2_t H^{s-1}} \right) \lesssim \sum_{|\alpha| + |\beta| \leq 1} \left( \| \partial_x^\alpha \partial_t^\beta u_0 \|_{\dot{H}^s} + \| \partial_x^\alpha \partial_t^\beta \partial_h u_1 \|_{\dot{H}^{s-1}} + \| \partial_x^\alpha \partial_t^\beta F \|_{L^2_t Y_{1-s, s}} \right)
\]
for the solutions to the linear wave equation (1.10), if \( \rho > 1, p > 2, s \in (1/2 - 1/p, 1/2) \).

Let us now see how we can use these estimates to prove Theorem 1.5. We assume Cauchy data \((u_0, u_1)\) satisfying the smallness condition (1.12), and let \( u^{(0)} \) solve the Cauchy problem (1.9) with \( F = 0 \). We iteratively define \( u^{(k)} \), for \( k \geq 1 \), by solving
\[
\begin{cases}
(\partial_t^2 - \Delta_p) u^{(k)}(t, x) = F_p(u^{(k-1)}(t, x)), & (t, x) \in \mathbb{R}_+ \times \Omega \\
u(0, \cdot) = u_0, & \partial_t u(0, \cdot) = u_1.
\end{cases}
\]

Our aim is to show that if the constant \( \delta > 0 \) in (1.12) is small enough, then so is
\[
M_k = \sum_{|\alpha| \leq 1} \left( \| Y^\alpha u^{(k)} \|_{L^2_t H^s} + \| \partial_t Y^\alpha u^{(k)} \|_{L^2_t H^{s-1}} \right)
\]
for every \( k = 0, 1, 2, \ldots \).
For \( k = 0 \), it follows by (7.1) that \( M_0 \leq C_0 \delta \), with \( C_0 \) a fixed constant. More generally, for some \( \eta_i \in (0, 1) \), (7.1) implies that
\[
M_k \leq C_0 \delta + C_0 \sum_{|\alpha| \leq 1} \left( \| x \|^{-\frac{d}{2} + 1 - \eta \alpha} F_p(u^{(k-1)}) \right)_{L^1_tL^1_x L^2_{x,s} (\mathbb{R}^+ \times \{|x| \geq R\})} + \| Y^\alpha F_p(u^{(k-1)}) \|_{L^1_t L^2_{x,s} (\mathbb{R}^+ \times \{|x| \leq R\})}.
\]
Note that our assumption (1.10) on the nonlinear term \( F \) implies that for small \( v \)
\[
\sum_{|\alpha| \leq 1} |Y^\alpha F_p(v)| \lesssim |v|^{p-1} \sum_{|\alpha| \leq 1} |Y^\alpha v|.
\]
Since the collection \( \mathcal{Y} \) contains vectors spanning the tangent space to \( S^2 \), by Sobolev embedding we have for any \( q \leq \infty \) and \( \eta_2 \in (0, 1) \)
\[
\| v(r) \|_{L^q_x} \lesssim \sum_{|\alpha| \leq 1} \| Y^\alpha v(r) \|_{L^{q+\eta_2}_x}.
\]
Consequently, for fixed \( t, r > 0 \)
\[
\sum_{|\alpha| \leq 1} \| Y^\alpha F_p(u^{(k-1)}(t, r)) \|_{L^{2-\eta_1}_x} \lesssim \sum_{|\alpha| \leq 1} \| Y^\alpha u^{(k-1)}(t, r) \|_{L^{1+\eta_2}_x}.
\]
By (6.10), the first summand in the right side of (7.2) is dominated by \( C_1 M_{k-1}^p \).
We next observe that, for each fixed \( t \), we have
\[
\sum_{|\alpha| \leq 1} \| Y^\alpha F_p(u^{(k-1)}(t, \cdot)) \|_{L^{q'-s}_{x,s} (|x| \leq R)} \lesssim \sum_{|\alpha| \leq 1} \| u \|^{p-1}_{L^s(|x| \leq R)} \| Y^\alpha u^{(k-1)}(t, \cdot) \|_{L^{q'}(|x| \leq R)},
\]
where
\[
1/q = 1/p - 1\left(\frac{1}{q'_{1-s}} - \frac{1}{q_s}\right) = \frac{1}{3(p-1)}.
\]
It follows by Sobolev embedding on \( \{|x| \leq R\} \) that
\[
\| u \|_{L^s(|x| \leq R)} \lesssim \sum_{|\alpha| \leq 1} \| Y^\alpha u \|_{L^{q'}(|x| \leq R)},
\]
since \( s \in \left(\frac{3}{2} - \frac{1}{p}, \frac{1}{2}\right) \subset \left[\frac{3}{2} - \frac{1}{p-1}, \frac{3}{2} - \frac{1}{p}\right], \) i.e. \( \frac{3}{q} = \frac{1}{p-1} \in [-1 + \frac{3}{q_s}, \frac{3}{q_{1-s}}]. \)

The second summand in the right side of (7.2) is thus also dominated by \( C_1 M_{k-1}^p \), and we conclude that \( M_k \leq C_0 \delta + 2C_0 C_1 M_{k-1}^p \). For \( \delta \) sufficiently small, then
\[
M_k \leq 2C_0 \delta, \quad k = 1, 2, 3, \ldots
\]
To finish the proof of Theorem 1.9 we need to show that \( u^{(k)} \) converges to a solution of the equation (1.9). For this it suffices to show that
\[
A_k = ||u^{(k)} - u^{(k-1)}||_{L^p_{x,s,p}}
\]
tends geometrically to zero as \( k \to \infty \). Since \( |F_p(v) - F_p(w)| \lesssim |v - w||v|^{p-1} + |w|^{p-1} \), the proof of (7.3) can be adapted to show that, for small \( \delta > 0 \), there is a uniform constant \( C \) so that
\[
A_k \leq CA_{k-1}(M_{k-1} + M_{k-2})^{p-1},
\]
which, by (7.3), implies that \( A_k \leq \frac{1}{2} A_{k-1} \) for small \( \delta \). Since \( A_1 \) is finite, the claim follows, which finishes the proof of Theorem 1.9.
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