ON A SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS OF MONGE-KANTOROVICH TYPE

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Abstract. We consider a system of PDEs of Monge-Kantorovich type arising from models in granular matter theory and in electrodynamics of hard superconductors. The existence of a solution of such system (in a regular open domain \( \Omega \subset \mathbb{R}^n \)), whose construction is based on an asymmetric Minkowski distance from the boundary of \( \Omega \), was already established in [G. Crasta and A. Malusa, The distance function from the boundary in a Minkowski space, to appear in Trans. Amer. Math. Soc.]. In this paper we prove that this solution is essentially unique. A fundamental tool in our analysis is a new regularity result for an elliptic nonlinear equation in divergence form, which is of some interest by itself.

1. Introduction

Let \( \rho : \mathbb{R}^n \to \mathbb{R} \) be a \( C^2 \) gauge function, i.e. a convex and positively 1-homogenous function, of class \( C^2 \) in \( \mathbb{R}^n \setminus \{0\} \). In this paper we are concerned with the system of partial differential equations

\[
\begin{align*}
- \text{div}(v D\rho(Du)) &= f \quad \text{in } \Omega, \\
\rho(Du) &\leq 1 \quad \text{in } \Omega, \\
\rho(Du) &= 1 \quad \text{in } \{v > 0\},
\end{align*}
\]

complemented with the conditions

\[
\begin{align*}
u \geq 0, \quad v \geq 0 \quad &\text{in } \Omega, \\
u = 0 \quad &\text{on } \partial \Omega.
\end{align*}
\]

Here \( \Omega \subset \mathbb{R}^n \) is a bounded domain of class \( C^2 \) and \( f \geq 0 \) is a bounded continuous function in \( \Omega \). A solution of this system is a pair \((u, v)\) of nonnegative functions, with \( u \) Lipschitz continuous in \( \Omega \) and \( v \) bounded and continuous in \( \Omega \), satisfying the following additional conditions: (a) \( u = 0 \) on \( \partial \Omega \); (b) \( \rho(Du) \leq 1 \) almost everywhere in \( \Omega \); (c) \( u \) is a viscosity solution of \( \rho(Du) = 1 \) in the open set \( \{v > 0\} \); (d) \( v \) is a solution of the first equation in (1) in the sense of distributions (see Definition \[\text{(i)}\] below).

This system of PDEs arises in some different situations. For example, the functions \( u \) and \( v \) can be interpreted respectively as the magnetic field and the power dissipation in a cylindrical hard superconductor of cross-section \( \Omega \) exposed to an external magnetic field linearly increasing in time (see e.g. \[\text{(ii)}\]). Moreover, in the case \( \rho(\xi) = |\xi| \), (1)-(2) gives the stationary solutions of models in granular matter theory (see \[\text{(iii)}\]). Another application concerns the existence of solutions to nonconvex minimum problems in calculus of variations (see \[\text{(iv)}\]). Finally, Bouchitté and Buttazzo \[\text{(v)}\] have studied a more general system in order to describe optimal solutions of some shape optimization problems.

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The results presented in this paper are an extension of those proved in [3], where the case $\rho(\xi) = |\xi|$ was considered. An explicit solution to (1)-(2) was constructed in [11]. In order to describe this solution, we need some additional notation. Let $d_{\Omega} : \overline{\Omega} \to \mathbb{R}$ be the Minkowski distance from the boundary $\partial\Omega$, defined by
\[
d_{\Omega}(x) = \inf_{y \in \partial\Omega} \rho^0(x - y), \quad x \in \overline{\Omega},
\]
where $\rho^0$ is the polar function of $\rho$. It is well-known that $d_{\Omega}$ is Lipschitz continuous in $\overline{\Omega}$, and it is the unique viscosity solution of $\rho(Dd_{\Omega}) = 1$ in $\Omega$ vanishing on $\partial\Omega$ (see [3]). In [11] it was shown that there exists a bounded continuous function $v_f : \Omega \to [0, +\infty)$, whose explicit expression depends on $f$, $\rho$ and on the geometry of $\Omega$ (see Section 4), such that the pair $(d_{\Omega}, v_f)$ is a solution to (1)-(2).

The aim of this paper is to show that this solution is essentially unique. More precisely, we shall show that if $(u, v)$ is a solution to (1)-(2), then $v = v_f$ and $u = d_{\Omega}$ in $\Omega_f = \{x \in \Omega; \; v_f(x) > 0\}$. The proof of this uniqueness result is based on several ingredients. Some of them are an adaptation to our setting of arguments developed in [1], [7], [10], [3]. A key point of the uniqueness proof consists in showing that, if $(d_{\Omega}, v)$ is a solution to (1)-(2), then $v$ vanishes on the singular set $\Sigma$ of $d_{\Omega}$ (see Proposition 5 below). In this respect, we use here a blow-up argument introduced by Evans and Gangbo for the case $\rho(\xi) = |\xi|$ (see [1], Section 7, and [3]), which in turn relies on the regularity of the solutions to the classical Laplace equation $\Delta u = 0$ in an open set $A \subset \mathbb{R}^n$. In our setting the classical Laplace equation is replaced by
\[
-\text{div}(D\rho(Du)) = 0 \quad \text{in} \; A.
\]
Since the function $a(\xi) := D\rho(\xi)$ is defined and positively 0-homogeneous in $\mathbb{R}^n \setminus \{0\}$, no standard regularity result can be applied. In Section 3 we prove that, if $\rho \in C^2(\mathbb{R}^n \setminus \{0\})$ and $u$ is a Lipschitz continuous solution of (3) satisfying $\rho(Du) = 1$ almost everywhere in $A$, then $u$ is of class $C^{1,\alpha}$ locally in $A$ (see Theorem 5.1). Thanks to this regularity result, the blow-up argument of Evans and Gangbo still works in our setting (see Proposition 6.7).

2. Notation and Preliminaries

2.1. Basic notation. The standard scalar product of two vectors $x, y \in \mathbb{R}^n$ is denoted by $(x, y)$, and $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^n$. Given two vectors $v, w \in \mathbb{R}^n$, the symbol $v \otimes w$ will denote their tensor product, i.e. the linear application from $\mathbb{R}^n$ to $\mathbb{R}^n$ defined by $(v \otimes w)(x) = v \cdot w(x)$.

By $S^{n-1}$ we denote the set of unit vectors of $\mathbb{R}^n$, and by $\mathcal{M}_k$ the set of $k \times k$ square matrices. We shall denote by $(e_1, \ldots, e_n)$ the standard basis of $\mathbb{R}^n$. The closed segment joining $x \in \mathbb{R}^n$ to $y \in \mathbb{R}^n$ will be denoted by $[x, y]$, while $(x, y)$ will denote the same segment without the endpoints.

As customary, $B_r(x_0)$ and $\overline{B}_r(x_0)$ are respectively the open and the closed ball centered at $x_0$ and with radius $r > 0$.

Given $A \subset \mathbb{R}^n$, we shall denote by $\text{Lip}(A)$, $C(A)$, $C_b(A)$ and $C^k(A)$, $k \in \mathbb{N}$ the set of functions $u : A \to \mathbb{R}$ that are respectively Lipschitz continuous, continuous, bounded and continuous, and $k$-times continuously differentiable in $A$. (Here and thereafter $\mathbb{N}$ will denote the set of nonnegative integers.) Moreover, $C^\infty(A)$ will denote the set of functions of class $C^k(A)$ for every $k \in \mathbb{N}$, while $C^{k,\alpha}(A)$ will be the set of functions of class $C^k(A)$ with Hölder continuous $k$-th partial derivatives with exponent $\alpha \in [0, 1]$.

A bounded open set $A \subset \mathbb{R}^n$ (or, equivalently, its closure $\overline{A}$ or its boundary $\partial A$) is of class $C^k$, $k \in \mathbb{N}$, if for every point $x_0 \in \partial A$ there exists a ball $B = B_r(x_0)$ and a one-to-one mapping $\psi : B \to D$ such that $\psi \in C^k(B)$, $\psi^{-1} \in C^k(D)$, $\psi(B \cap A) \subseteq \{x \in \mathbb{R}^n; \; x_n > 0\}$,
2.2. Convex geometry. By $\mathcal{K}_0^n$ we denote the class of nonempty, compact, convex subsets of $\mathbb{R}^n$ with the origin as an interior point. We shall briefly refer to the elements of $\mathcal{K}_0^n$ as convex bodies. The polar body of a convex body $K \in \mathcal{K}_0^n$ is defined by

$$K^0 = \{ p \in \mathbb{R}^n; \langle p, x \rangle \leq 1 \forall x \in K \}.$$  

We recall that, if $K \in \mathcal{K}_0^n$, then $K^0 \in \mathcal{K}_0^n$ and $K^00 = (K^0)^0 = K$ (see [18, Thm. 1.6.1]).

Given $K \in \mathcal{K}_0^n$ we define its gauge function as

$$\rho_K(\xi) = \inf \{ t \geq 0; \xi \in tK \}.$$  

It is easily seen that

$$\rho_{K^0}(\xi) = \sup \{ \langle \xi, p \rangle : p \in K \},$$  

i.e. the gauge function of the polar set $K^0$ coincides with the support function of the set $K$. Let $0 < c_1 \leq c_2$ be such that $\overline{B}_{c_2}(0) \subseteq K \subseteq \overline{B}_{c_1}(0)$. Upon observing that $\xi/\rho_K(\xi) \in K$ for every $\xi \neq 0$, we get

$$c_1|\xi| \leq \rho_K(\xi) \leq c_2|\xi|, \quad \forall \xi \in \mathbb{R}^n.$$  

We say that $K \in \mathcal{K}_0^n$ is of class $C^2_+$ if $\partial K$ is of class $C^2$ and all the principal curvatures are strictly positive functions on $\partial K$. In this case, we define the $i$-th principal radius of curvature at $x \in \partial K$ as the reciprocal of the $i$-th principal curvature of $\partial K$ at $x$. We remark that if $K$ is of class $C^2_+$, then $K^0$ is also of class $C^2_+$ (see [18, p. 111]).

Throughout the paper we shall assume that

$$K \in \mathcal{K}_0^n \text{ is of class } C^2_+.$$  

Since $K$ will be kept fixed, from now on we shall use the notation $\rho = \rho_K$ and $\rho^0 = \rho^0_K$.

We collect here some known properties of $\rho$ and $\rho^0$ that will be frequently used in the sequel.

**Theorem 2.1.** Let $K$ satisfy (3). Then the following hold:

(i) The functions $\rho$ and $\rho^0$ are convex, positively 1-homogeneous in $\mathbb{R}^n$, and of class $C^2$ in $\mathbb{R}^n \setminus \{0\}$. As a consequence,

$$\rho(t\xi) = t\rho(\xi), \quad D\rho(t\xi) = D\rho(\xi), \quad D^2\rho(t\xi) = \frac{1}{t}D^2\rho(\xi),$$

$$\rho^0(t\xi) = t\rho^0(\xi), \quad D\rho^0(t\xi) = D\rho(\xi), \quad D^2\rho^0(t\xi) = \frac{1}{t}D^2\rho^0(\xi),$$

for every $\xi \in \mathbb{R}^n \setminus \{0\}$ and $t > 0$. Moreover

$$\langle D\rho(\xi), \xi \rangle = \rho(\xi), \quad \langle D\rho^0(\xi), \xi \rangle = \rho^0(\xi), \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}$$  

$$D^2\rho(\xi)\xi = 0, \quad D^2\rho^0(\xi)\xi = 0, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$  

(ii) For every $\xi, \eta \in \mathbb{R}^n$, we have

$$\rho(\xi + \eta) \leq \rho(\xi) + \rho(\eta), \quad \rho^0(\xi + \eta) \leq \rho^0(\xi) + \rho^0(\eta),$$

and equality holds if and only if $\xi$ and $\eta$ belong to the same ray, that is, $\xi = \lambda\eta$ or $\eta = \lambda\xi$ for some $\lambda \geq 0$.

(iii) The eigenvalues of the second differential $D^2\rho$ at $\nu \in S^{n-1}$ are 0 (with corresponding eigenvector $\nu$) and the principal radii of curvature of $\partial K^0$ at the unique point $p \in \partial K^0$ at which $\nu$ is attained as an outward normal vector. Symmetrically, the eigenvalues of $D^2\rho^0$
at $\nu \in S^{n-1}$ are 0 (with corresponding eigenvector $\nu$) and the principal radii of curvature of $\partial K$ at the unique point $p \in \partial K$ at which $\nu$ is attained as an outward normal vector.

3. Distance from the Boundary

Throughout the paper, we shall assume that

\begin{equation}
\Omega \subset \mathbb{R}^n \text{ is a nonempty, bounded, open connected set of class } C^2.
\end{equation}

Let us define the function

\begin{equation}
d_\Omega(x) = \inf_{y \in \partial \Omega} \rho^0(x - y), \quad x \in \overline{\Omega},
\end{equation}

that measures the distance from the boundary $\partial \Omega$ to a point $x \in \overline{\Omega}$ in the Minkowski norm associated to the polar function $\rho^0$ of $\rho$. Since $\partial \Omega$ is a compact subset of $\mathbb{R}^n$ and $\rho^0$ is a continuous function, the infimum in the definition of $d_\Omega$ is achieved. We shall denote by $\Pi(x)$ the set of projections of $x$ onto $\partial \Omega$, that is

\begin{equation}
\Pi(x) = \{ y \in \partial \Omega; \ d_\Omega(x) = \rho^0(x - y) \}, \quad x \in \overline{\Omega}.
\end{equation}

By abuse of notation, when $\Pi(x) = \{x_0\}$ then we shall use $\Pi(x)$ to indicate the point $x_0$.

It is well-known that $d_\Omega$ is a viscosity solution of the Hamilton-Jacobi equation

\begin{equation}
\rho(Du) = 1 \quad \text{in } \Omega.
\end{equation}

More precisely, it is the unique viscosity solution of (12) satisfying the boundary condition $u = 0$ on $\partial \Omega$. Moreover, $d_\Omega(x) > 0$ for every $x \in \Omega$, $d_\Omega \in \text{Lip}(\overline{\Omega})$, and $\rho(Du(x)) = 1$ for a.e. $x \in \Omega$ (see [1, 10]).

We say that $x \in \Omega$ is a regular point of $\Omega$ if $\Pi(x)$ is a singleton. We say that $x \in \Omega$ is a singular point of $\Omega$ if $x$ is not a regular point. We denote by $\Sigma \subseteq \Omega$ the set of all singular points of $\Omega$. It is well-known that $d_\Omega$ is differentiable at every regular point of $\Omega$ (see [1, 10]; see also Theorem 3.1(i) below).

From now on, for every $x_0 \in \partial \Omega$ we shall denote by $\kappa_1(x_0), \ldots, \kappa_{n-1}(x_0)$ the principal curvatures of $\partial \Omega$ at $x_0$, and by $\nu(x_0)$ the inward normal unit vector to $\partial \Omega$ at $x_0$. We extend these functions to $\overline{\Omega} \setminus \Sigma$ by setting

$$
\nu(x) = \nu(\Pi(x)), \quad \kappa_i(x) = \kappa_i(\Pi(x)), \quad i = 1, \ldots, n - 1, \quad x \in \overline{\Omega} \setminus \Sigma.
$$

We collect in the following theorem all the results proved in [10] that are relevant to the subsequent analysis.

**Theorem 3.1.** Let $\Omega$ and $K$ satisfy respectively (i) and (ii). Then the following hold.

(i) $\Sigma \subset \Omega$, and the Lebesgue measure of $\Sigma$ is zero.

(ii) Let $x \in \Omega$ and $x_0 \in \Pi(x)$. Then, for every $z \in [x_0, x)$, $d_\Omega$ is differentiable at $z$ and

\begin{equation}
Dd_\Omega(z) = \frac{\nu(x_0)}{\rho(\nu(x_0))}.
\end{equation}

(iii) The function $d_\Omega$ is of class $C^2$ on $\overline{\Omega} \setminus \Sigma$.

**Proof.** See Remark 4.16, Corollary 6.9, Lemma 4.3 and Theorem 6.10 in [10].

At any point $x_0 \in \partial \Omega$ there is a unique inward “normal” direction $p(x_0)$ with the properties $\Pi(x_0 + tp(x_0)) = \{x_0\}$ and $d_\Omega(x_0 + tp(x_0)) = t$ for $t \geq 0$ small enough (see [10, Remark 4.5]). More precisely, these properties hold true for $t \in [0, \tau(x_0))$, where $\tau(x_0)$ is the normal distance to the cut locus $\Sigma$, defined below (see [10, Proposition 4.8]). It can be proved that $p(x_0) = D\rho(\nu(x_0))$ (see [17, Lemma 2.2] and [10, Proposition 4.4]).
From Theorem 3.1(ii) and the positive 0-homogeneity of $D\rho$ it is clear that $D\rho(\nu(x_0)) = D\rho(Dd\Omega(x_0))$. Summarizing, given $x \in \bar{\Omega}$ we have that
\begin{equation}
    x_0 \in \Pi(x) \iff x = x_0 + t D\rho(Dd\Omega(x_0)) \text{ for some } t \in [0, \tau(x_0)],
\end{equation}
and, in such case, $d\Omega(x) = t$.

The above considerations motivate the following definition.

**Definition 3.2.** The normal distance to cut locus of a point $x \in \bar{\Omega}$ is defined by
\begin{equation}
    \tau(x) = \begin{cases} 
        \min\{t \geq 0 : x + t D\rho(Dd\Omega(x)) \in \Sigma\}, & \text{if } x \in \bar{\Omega} \setminus \Sigma, \\
        0, & \text{if } x \in \Sigma.
    \end{cases}
\end{equation}
The cut point $m(x)$ of $x \in \bar{\Omega} \setminus \Sigma$ is defined by $m(x) = x + \tau(x) D\rho(Dd\Omega(x))$.

**Proposition 3.3.** Let $\Omega$ satisfy (1). Then $\tau$ is continuous in $\bar{\Omega}$. Furthermore, there exists $\mu > 0$ such that $\tau(x_0) \geq \mu$ for every $x_0 \in \partial \Omega$.

*Proof.* See [10], Lemma 4.1 and Theorem 6.7. \(\square\)

From Theorem 3.1(iii), the function $d\Omega$ is of class $C^2$ on $\bar{\Omega} \setminus \Sigma$. We can then define the function
\begin{equation}
    W(x) = -D^2\rho(Dd\Omega(x)) D^2d\Omega(x), \quad x \in \bar{\Omega} \setminus \Sigma.
\end{equation}
For any $x_0 \in \partial \Omega$, let $T_{x_0}$ denote the tangent space to $\partial \Omega$ at $x_0$. If $x \in \bar{\Omega} \setminus \Sigma$ and $\Pi(x) = \{x_0\}$, we set $T_x = T_{x_0}$. Observe that, by (13) and (12), we have $D^2\rho(Dd\Omega(x)) \nu(x) = 0$. Then, for every $v \in T_x$, we have $W(x) v = 0$. Hence, we can define the map
\begin{equation}
    \overline{W}(x) : T_x \to T_x, \quad \overline{W}(x) w = W(x) w,
\end{equation}
that can be identified with a linear application from $\mathbb{R}^{n-1}$ to $\mathbb{R}^{n-1}$.

We shall use the following results (see [10], Lemmas 4.10 and 5.1).

**Lemma 3.4.** Let $x_0 \in \partial \Omega$. Then
\begin{equation}
    \det[I - t W(x_0)] = \det[I_{n-1} - t \overline{W}(x_0)]
\end{equation}
for every $t \in \mathbb{R}$, and both determinants are strictly positive for every $t \in [0, \tau(x_0))$.

**Remark 3.5.** Let $x_0 \in \partial \Omega$. We recall that $D\rho(\nu(x_0))$ is an eigenvector of $W(x_0)$ with corresponding eigenvalue zero (see [10], Lemma 4.18). On the other hand, from Lemma 3.4 we deduce that a number $\kappa \neq 0$ is an eigenvalue of $W(x_0)$ if and only if it is an eigenvalue of $\overline{W}(x_0)$.

Although the matrix $\overline{W}(x_0)$ is not in general symmetric, its eigenvalues are real numbers, and so its eigenvectors are real (see [10], Remark 5.3). The eigenvalues of $\overline{W}(x_0)$ have an important geometric interpretation.

**Definition 3.6 (\(\rho\)-curvatures).** Let $x_0 \in \partial \Omega$. The principal $\rho$-curvatures of $\partial \Omega$ at $x_0$, with respect to the Minkowski norm $d\Omega$, are the eigenvalues $\kappa_1(x_0) \leq \cdots \leq \kappa_{n-1}(x_0)$ of $\overline{W}(x_0)$. The corresponding eigenvectors are the principal $\rho$-directions of $\partial \Omega$ at $x_0$.

Up to now we have analyzed some properties of the matrices $W(x_0)$ and $\overline{W}(x_0)$ at points $x_0 \in \partial \Omega$. Now we are interested in the evolution of these matrices along the transport ray starting from $x_0$. 
Lemma 3.7. Let $x_0 \in \partial \Omega$, and define

$$V(t) = W(x_0 + t D\rho(Dd\Omega(x_0))) = \nabla W(x_0 + t D\rho(Dd\Omega(x_0))),$$

for $t \in [0, \tau(x_0))$. Then

$$V(t) [I - t W(0)] = V(0), \quad \nabla(t) [I_{n-1} - t \nabla(0)] = \nabla(0),$$

for every $t \in [0, \tau(x_0))$. Furthermore, $\text{Tr} V(t) = \text{Tr} \nabla(t)$ for every $t \in [0, \tau(x_0))$. 

Proof. Let us consider the principal coordinate system at $x_0$, i.e. the coordinate system such that $x_0 = 0$, $e_n = \nu(x_0)$ and $e_i$ coincides with the $i$-th principal direction of $\partial \Omega$ at $x_0$, $i = 1, \ldots, n - 1$. Let $X: \mathcal{U} \to \mathbb{R}^n$ be a local parametrization of $\partial \Omega$ in a neighborhood of $x_0 = 0$. The relation (13) can be written as

$$Dd\Omega(X(y) + t D\rho(Dd\Omega(X(y)))) = \frac{N(y)}{\rho(N(y))}, \quad y \in \mathcal{U}, \ t \in [0, \tau(X(y))),$$

where $N(y) = \nu(X(y))$. Recall that

$$N(0) = e_n, \quad \frac{\partial N}{\partial y_i}(0) = -\kappa_i e_i, \quad i = 1, \ldots, n - 1,$$

where $\kappa_1, \ldots, \kappa_{n-1}$ are the principal curvatures of $\partial \Omega$ at $x_0$. Differentiating (20) with respect to $y_i$ at $y = 0$ and using (21) we obtain, for every $i = 1, \ldots, n - 1$,

$$D^2 d\Omega(x_0 + t D\rho(Dd\Omega(x_0))) [I + t D^2 \rho(Dd\Omega(x_0)) D^2 d\Omega(x_0)] e_i = -\xi_i,$$

where $\xi_i = \kappa_i [\rho(\nu) e_i - \langle D\rho(\nu), e_i \rangle \nu] / \rho(\nu)^2$, and $\nu = \nu(x_0)$. Differentiating (20) with respect to $t$ at $y = 0$ we get

$$D^2 d\Omega(x_0 + t D\rho(Dd\Omega(x_0))) D\rho(Dd\Omega(x_0)) = 0.$$

Let us apply $D^2 \rho(Dd\Omega(x_0 + t D\rho(Dd\Omega(x_0)))) = D^2 \rho(Dd\Omega(x_0))$ to both sides of equations (22) and (23). Recalling the definition (14) of $W$, we obtain the relations

$$W(x_0 + t D\rho(Dd\Omega(x_0))) [I - t W(x_0)] e_i = \xi_i, \quad i = 1, \ldots, n - 1,$$

$$W(x_0 + t D\rho(Dd\Omega(x_0))) D\rho(Dd\Omega(x_0)) = 0,$$

where $\xi_i = D^2 \rho(Dd\Omega(x_0)) \kappa_i$. We have that, for every $t \in [0, \tau(x_0))$,

$$V(t) [I - t W(0)] e_i = \xi_i, \quad i = 1, \ldots, n - 1,$$

for every $v \in \mathbb{R}^n$, hence the second identity in (13) is also satisfied, and $\text{Tr} V(t) = \text{Tr} \nabla(t)$.

Remark 3.8. Let $\nabla(t)$ be the function defined in (13). By definition, the eigenvalues $\kappa_1, \ldots, \kappa_{n-1}$ of $\nabla(0)$ are the principal $\rho$-curvatures of $\partial \Omega$ at $x_0$, and the corresponding eigenvectors $w_1, \ldots, w_{n-1}$ are the principal $\rho$-directions of $\partial \Omega$ at $x_0$. From the identity (13) we obtain that

$$\nabla(t) (1 - t \kappa_i) w_i = \kappa_i w_i, \quad i = 1, \ldots, n - 1, \ t \in [0, \tau(x_0)).$$
Since for every $t \in [0, \tau(x_0))$ the point $x_0 + t D\rho(Dd\Omega(x_0))$ belongs to $\overline{\Omega \setminus \Sigma}$, then $1 - t \tilde{\kappa}_i > 0$ (see [10, Lemma 5.4]), and hence the eigenvalues of $\nabla(t)$ are

$$\tilde{\kappa}_i(t) = \frac{\tilde{\kappa}_i}{1 - t \tilde{\kappa}_i}, \quad i = 1, \ldots, n - 1,$$

with corresponding eigenvectors $w_1, \ldots, w_{n-1}$.

**Proposition 3.9.** For every $x_0 \in \partial \Omega$ let us define the function

$$M_{x_0}(s, t) = \exp \left( - \int_s^t \text{Tr} \nabla(x_0 + \sigma D\rho(Dd\Omega(x_0))) \, d\sigma \right),$$

for $s, t \in [0, \tau(x_0))$. Then

$$M_{x_0}(s, t) = \frac{\det \nabla(x_0 + s D\rho(\nu(x_0)))}{\det \nabla(x_0 + t D\rho(\nu(x_0)))} = \frac{\det [I_{n-1} - t \nabla(x_0)]}{\det [I_{n-1} - s \nabla(x_0)]} = \prod_{i=1}^{n-1} \frac{1 - t \tilde{\kappa}_i(x_0)}{1 - s \tilde{\kappa}_i(x_0)}$$

for every $s, t \in [0, \tau(x_0))$, where $\tilde{\kappa}_1(x_0), \ldots, \tilde{\kappa}_{n-1}(x_0)$ are the principal $\rho$-curvatures of $\partial \Omega$ at $x_0$.

**Proof.** Let $\nabla(t), t \in [0, \tau(x_0)),$ be the function defined in (18). From Lemmas 3.4 and 3.7 we have

$$\nabla(t) = \nabla(0) [I_{n-1} - t \nabla(0)]^{-1}, \quad \forall t \in [0, \tau(x_0)).$$

This implies that the matrix-valued function $\nabla(t)$ satisfies the differential equation

$$\nabla'(t) = \nabla(t)^2, \quad \forall t \in [0, \tau(x_0)),$$

and hence the function $\delta(t) = \det \nabla(t)$ is a solution of the differential equation $\delta'(t) = \text{Tr}[\nabla'(t)] \delta(t)$ in $[0, \tau(x_0))$. Now, the first equality in (26) follows from the fact that

$$\delta(t) = \delta(s) \exp \left( \int_s^t \text{Tr} \nabla(\sigma) \, d\sigma \right) = \delta(s)/M_{x_0}(s, t),$$

for every $s, t \in [0, \tau(x_0))$. The second equality follows from (27). The last equality is a direct consequence of the fact that $\tilde{\kappa}_1(x_0), \ldots, \tilde{\kappa}_{n-1}(x_0)$ are the eigenvalues of the matrix $\nabla(0)$. \qed

**Proposition 3.10.** The function $M_{x_0}(s, t)$, defined in (27), is jointly continuous with respect to $x_0 \in \partial \Omega$ and $s, t \in [0, \tau(x_0))$. Furthermore,

$$0 \leq M_{x_0}(s, t) \leq \prod_{i=1}^{n-1} (1 + T \overline{K}_-), \quad \forall x_0 \in \partial \Omega, \ 0 \leq s \leq t < \tau(x_0),$$

where

$$T = \max\{\tau(x); \ x \in \partial \Omega\},$$

$$\overline{K}_- = \max\{\overline{\kappa}_i(x)_-; \ x \in \partial \Omega, \ i = 1, \ldots, n - 1\},$$

being $[a]_- = \max\{0, -a\}$ the negative part of a real number $a$. 

Proof. The continuity of $M_{x_0}(s,t)$ follows from its definition and the continuity of $\tilde{k}_i$. The estimate (28) follows from the representation formula (26) and the estimate
\[
\frac{1-t\tilde{k}_i(x_0)}{1-s\tilde{k}_i(x_0)} \leq 1 + (t-s)[\tilde{k}_i(x_0)]_-
\]
that holds for every $x_0 \in \partial \Omega$ and $0 \leq s \leq t < \tau(x_0)$.

For every $x \in \overline{\Omega} \setminus \Sigma$ let us define the function
\[
M_x(t) = \exp \left( -\int_0^t \text{Tr} \, \overline{W} (x + \sigma \, D\rho(Dd\Omega(x)) \, d\sigma) \right), \quad t \in [0,\tau(x)],
\]
where $\overline{W}$ is the matrix defined in (17). For an explicit computation of $M_x$ it can be of some aid to recall that
\[
\text{Tr} \, \overline{W}(x) = \text{Tr} \, W(x) = -\text{Tr} \left[ D^2 \rho(Dd\Omega(x)) \, D^2d\Omega(x) \right]
\]
for every $x \in \overline{\Omega} \setminus \Sigma$ (see Lemma 3.7). Given $x \in \overline{\Omega} \setminus \Sigma$, let $\Pi(x) = \{x_0\}$. By (14) we have that
\[
\overline{W}(x + \sigma \, D\rho(Dd\Omega(x))) = \overline{W}(x_0 + (d\Omega(x) + \sigma) \, D\rho(Dd\Omega(x))), \quad \sigma \in [0,\tau(x)),
\]
which implies the relation
\[
M_x(t) = M_{x_0}(d\Omega(x),d\Omega(x) + t), \quad \forall t \in [0,\tau(x)).
\]
From the identity (26) we have that
\[
M_x(t) = \prod_{i=1}^{n-1} \frac{1-(d\Omega(x) + t) \tilde{k}_i(x)}{1-d\Omega(x) \tilde{k}_i(x)}
\]
where $\tilde{k}_i(x) := \tilde{k}_i(x_0)$, $i = 1, \ldots, n-1$, are the $\rho$-curvatures of $\partial \Omega$ at $x_0$.

4. Existence of solutions

In this section we recall the existence result for system (1)-(2) proved in [10, Thm. 7.2]. The rigorous meaning of solution is the following.

Definition 4.1. Let $\Omega \subset \mathbb{R}^n$ satisfy (1) and let $f \in C_b(\Omega)$ be a nonnegative function. A solution of system (1)-(2) is a pair $(u,v)$ of functions satisfying the following properties:

1. $u \in \text{Lip}(\overline{\Omega})$, $v \in C_b(\Omega)$, $u,v \geq 0$ in $\Omega$;
2. $u = 0$ on $\partial \Omega$, $\rho(Du) \leq 1$ a.e. in $\Omega$, and $u$ is a viscosity solution of
   \[
   \rho(Du) = 1 \quad \text{in } \{v > 0\};
   \]
3. $v$ is a solution in the sense of distributions of $-\text{div}(v \, D\rho(Du)) = f$ in $\Omega$, that is
   \[
   \int_\Omega v(x) \, (D\rho(Du(x)), D\varphi(x)) \, dx = \int_\Omega f(x) \, \varphi(x) \, dx
   \]
   for every $\varphi \in C_c^\infty(\Omega)$.

Remark 4.2. Since $v \in L^\infty(\Omega)$ and $D\rho(Du) \in [L^\infty(\Omega)]^n$, by a standard density argument (14) holds for every test function $\varphi$ in the Sobolev space $W_0^{1,1}(\Omega)$.

For $f \in C_b(\Omega)$, let us define the function
\[
v_f(x) = \begin{cases} \int_0^{\tau(x)} f(x + t \, D\rho(Dd\Omega(x))) \, M_x(t) \, dt, & \text{if } x \in \overline{\Omega} \setminus \Sigma, \\ 0, & \text{if } x \in \Sigma. \end{cases}
\]
Theorem 4.3 (Existence). Let $\Omega \subset \mathbb{R}^n$ satisfy (1) and let $f \geq 0$ be a bounded continuous function in $\Omega$. Then, the pair $(d_\Omega, v_f)$ is a solution to (1)-(2) in the sense of Definition 4.1.

A rigorous proof of Theorem 4.3 was given in [10, Theorem 7.2]. In Section 6 we shall prove that the pair $(d_\Omega, v_f)$ is essentially the unique solution to problem (1)-(2). In order to gain some insight in the representation formula (35), a formal derivation of (35) might be in order.

Assume that $(d_\Omega, v_f)$ is a solution of (1)-(2), and that $v \in C^1(\Omega \setminus \Sigma)$, with $v$ vanishing on $\Sigma$. Outside $\Sigma$, the equation $-\text{div}(v D\rho(Du)) = f$ is satisfied pointwise, that is
\[ v(x) \text{ Tr } W(x) - \langle Dv(x), D\rho(Dd_\Omega(x)) \rangle = f(x), \]
where $W(x) = -D^2\rho(Dd_\Omega(x)) D^2d_\Omega(x)$. Furthermore, from Lemma 3.7 we have that $\text{Tr } W(x) = \text{Tr } W(x)$, where $W(x)$ is the matrix defined in (17).

Let $x \in \Omega \setminus \Sigma$, and define $\bar{v}(t) = v(x + t D\rho(Dd_\Omega(x)))$, $t \in [0, \tau(x)]$. The function $\bar{v}(t)$ satisfies the following linear differential equation
\[ \bar{v}'(t) = \left[ \text{Tr } W(x + t D\rho(Dd_\Omega(x))) \right] \bar{v}(t) - f(x + t D\rho(Dd_\Omega(x))) \]
in $[0, \tau(x)]$, supplemented by the boundary condition
\[ \bar{v}(\tau(x)) = 0 \]
since $x + \tau(x) D\rho(Dd_\Omega(x)) \in \Sigma$. The solution of this Cauchy problem, evaluated at $t = 0$, gives
\[ v(x) = \bar{v}(0) = \int_0^{\tau(x)} f(x + t D\rho(Dd_\Omega(x))) M_x(t) dt, \]
that is, the solution $v(x)$ has to be the function defined in formula (35).

With this heuristic in mind, our aim will be to prove that, if $(u, v)$ is a solution to (1)-(2), then $(d_\Omega, v_f)$ is a solution too (see Lemma 6.3(ii)), and that if $(d_\Omega, v_f)$ is a solution to (1)-(2), then $v$ must vanish on $\Sigma$ (see Proposition 6.7). The first goal will be achieved using the same arguments proposed in [5], whereas the second one needs a new regularity result for solutions of elliptic equations, which seems to be of some interest by itself, and that will be proved in the following section.

5. A REGULARITY RESULT

The aim of this section is to prove the following regularity result.

Theorem 5.1. Assume that $\rho$ is the gauge function of a convex body $K$ satisfying (1). Let $A \subset \mathbb{R}^n$ be an open bounded set, and let $u \in W^{1, \infty}(A)$ be a solution in the sense of distributions of the equation
\[ -\text{div}(D\rho(Du)) = 0 \quad \text{ in } A, \]
that is
\[ \int_A \langle D\rho(Du(x)), D\varphi(x) \rangle \, dx = 0 \]
for every $\varphi \in C^\infty_c(A)$. If in addition
\[ \rho(Du(x)) = 1 \quad \text{a.e. in } A, \]
then $u \in C^{1,\alpha}_{\text{loc}}(A)$.

We recall the standard regularity result about solutions to the equation
\[ -\text{div}(a(Du)) = 0 \quad \text{ in } A, \]
(see [14, §4.6], [13, §8.2]).
Theorem 5.2. Assume that the vector-valued function $a$ belongs to $C^1(A)$ and satisfies the following growth conditions: there exist $p > 1$, and $\alpha_0, \beta_0 > 0$ such that for every $\xi \in \mathbb{R}^n$

\begin{align}
|a(\xi)| + (1 + |\xi|^2)\frac{1}{2}|Da(\xi)| &\leq \alpha_0(1 + |\xi|^2)^{\frac{p-1}{2}}, \\
\langle Da(\xi), w \rangle &\geq \beta_0(1 + |\xi|^2)^{\frac{p-2}{2}}|w|^2, \quad \forall w \in \mathbb{R}^n.
\end{align}

Then every solution $u \in W^{1,\infty}(A)$ of (39) belongs to $C^{1,\alpha}_{loc}(A)$.

In our case, $\rho$ is a positively 1-homogeneous function of class $C^2$ in $\mathbb{R}^n \setminus \{0\}$ satisfying the bounds (4), and then $D\rho$ is a positively 0-homogeneous function of class $C^1$ in $\mathbb{R}^n \setminus \{0\}$, but in general $\rho$ is not even differentiable at the origin. Moreover, (40) cannot be verified near the origin. Hence we have no chance to apply directly the standard regularity results to solutions of equation (39).

An easy trick in order to have the “right” growth is to consider the function

\begin{align}
\gamma(\xi) &= \frac{1}{2} \rho(\xi)^2, \quad \xi \in \mathbb{R}^n.
\end{align}

We have that

\begin{align}
D\gamma(\xi) &= \begin{cases} \rho(\xi) D\rho(\xi), & \text{if } \xi \neq 0, \\
0, & \text{if } \xi = 0. \end{cases}
\end{align}

Clearly, a function $u \in W^{1,\infty}(A)$ satisfying (37) is a solution to (36) if and only if it is a solution to

\begin{align}
\begin{cases} -\text{div}(D\gamma(Du)) = 0 & \text{in } A, \\
\rho(Du) = 1 & \text{a.e. in } A. \end{cases}
\end{align}

Moreover, $\gamma(\xi)$ is a positively 2-homogeneous function of class $C^2$ in $\mathbb{R}^n \setminus \{0\}$, hence $D^2\gamma(\xi)$ is 0-homogeneous and continuous in $\mathbb{R}^n \setminus \{0\}$. In particular the matrix–valued function $D^2\gamma$ is bounded in $\mathbb{R}^n \setminus \{0\}$, and

\begin{align}
\|D^2\gamma\| \leq c_5 = \max_{i,j=1,\ldots,n} \max\{|D^2_{ij}\gamma(\xi)|, \quad \xi \in S^{n-1} \}.
\end{align}

The following positive constants will be used throughout this section:

\begin{align}
c_3 &= \max\{|D\rho(\xi)|; \quad \xi \in \mathbb{R}^n \setminus \{0\}\} = \max\{|D\rho(\nu)|; \quad \nu \in S^{n-1}\}, \\
c_4 &= \max\{|D^2\rho(\nu)|; \quad \nu \in S^{n-1}\}, \\
r^0 &= \min\{r^0_i(p); \quad p \in \partial K^0, \quad i = 1,\ldots,n-1\}, \\
R^0 &= \max\{r^0_i(p); \quad p \in \partial K^0, \quad i = 1,\ldots,n-1\}
\end{align}

where $r^0_i(p) \leq \ldots \leq r^0_{n-1}(p)$ are the principal curvatures of $\partial K^0$ at $p$. We remark that $r^0 > 0$ since $K^0$ is of class $C^2$.

The first technical tool is to prove that $\gamma$ satisfies some growth conditions similar to (39) and (40) with $p = 2$.

Lemma 5.3. Let $\gamma$ be the function defined in (41). Then

\begin{align}
|D\gamma(\xi)| &\leq c_2 c_3 |\xi|, \\
\langle D\gamma(\xi), \xi \rangle &\geq c_4^2 |\xi|^2.
\end{align}
for every $\xi \neq 0$. Here $c_1$, $c_2$, $c_3$ are the constants defined in (41) and (42). Moreover there exists a constant $c_6 > 0$, independent of $\xi \neq 0$, such that

$$\langle D^2\gamma(\xi)w, w \rangle \geq c_6|w|^2,$$

for every $w \in \mathbb{R}^n$.

**Proof.** Since $\gamma$ is 2–homogeneous, by Euler’s formula $\langle D\gamma(\xi), \xi \rangle = 2\gamma(\xi)$, and then

$$\langle D\gamma(\xi), \xi \rangle = \rho(\xi)^2 \geq c_1^2|\xi|^2.$$

On the other hand, by (42),

$$|D\gamma(\xi)| \leq \rho(\xi)|D\rho(\xi)| \leq c_2c_3|\xi|.$$

It remains to prove that there exists a constant $c_6 > 0$ such that

$$\langle D^2\gamma(\xi)w, w \rangle \geq c_6|w|^2, \quad \forall w \in \mathbb{R}^n, \forall \xi \neq 0.$$

Since $\gamma$ is a convex function, of class $C^2$ in $\mathbb{R}^n \setminus \{0\}$, we have that the quadratic form $w \mapsto \langle D^2\gamma(\xi)w, w \rangle$ is positive semidefinite for every $\xi \neq 0$. We shall show that, in fact, it is positive definite uniformly with respect to $\xi \neq 0$.

Fixed $\xi \neq 0$ and using the notation $\nu = \xi/|\xi|$, we have

$$D^2\gamma(\xi) = D^2\gamma(\nu) = \rho(\nu) D^2\rho(\nu) + D\rho(\nu) \otimes D\rho(\nu),$$

so that

$$\langle D^2\gamma(\xi)w, w \rangle = \langle D^2\gamma(\nu)w, w \rangle = \rho(\nu) \langle D^2\rho(\nu)w, w \rangle + \langle D\rho(\nu), w \rangle^2$$

for every $w \in \mathbb{R}^n$.

Fixed $w \in \mathbb{R}^n$, let us denote by $\lambda = \langle \nu, w \rangle$, and by $\overline{w}$ the projection of $w$ on the orthogonal space $\mathcal{L}^{\perp}(\nu)$ to $\nu$, so that $w = \overline{w} + \lambda\nu$, $\langle \overline{w}, \nu \rangle = 0$, and $|w|^2 = |\overline{w}|^2 + \lambda^2$.

From Theorem 2.1 we have that $D^2\rho(\nu)\nu = 0$ and $D^2\rho(\nu)$ is positive definite in $\mathcal{L}^{\perp}(\nu)$ with

$$\langle D^2\rho(\nu)\overline{w}, \overline{w} \rangle \geq r^0|\overline{w}|^2,$$

for every $\overline{w} \in \mathcal{L}^{\perp}(\nu)$, where $r^0 > 0$ is the constant defined in (43). Hence we obtain

$$\langle D^2\rho(\nu)w, w \rangle = \langle D^2\rho(\nu)\overline{w}, \overline{w} \rangle \geq r^0|\overline{w}|^2,$$

for every $w \in \mathbb{R}^n$. On the other hand, by the 1–homogeneity of $\rho$ we have

$$\langle D\rho(\nu), \overline{w} + \lambda\nu \rangle = \langle D\rho(\nu), \overline{w} \rangle + \lambda\rho(\nu).$$

Hence we get the inequality

$$\langle D^2\gamma(\xi)w, w \rangle \geq r^0\rho(\nu)|\overline{w}|^2 + \langle D\rho(\nu), \overline{w} \rangle^2 + \lambda^2\rho(\nu)^2 + 2\lambda\rho(\nu) \langle D\rho(\nu), \overline{w} \rangle,$$

for every $w = \overline{w} + \lambda\nu \in \mathbb{R}^n$, $\overline{w} \in \mathcal{L}^{\perp}(\nu)$.

It remains to prove that there exists $c_6 > 0$ independent of $\nu$ such that

$$r^0\rho(\nu)|\overline{w}|^2 + \langle D\rho(\nu), \overline{w} \rangle^2 + \lambda^2\rho(\nu)^2 + 2\lambda\rho(\nu) \langle D\rho(\nu), \overline{w} \rangle \geq c_6 \left(|\overline{w}|^2 + \lambda^2\right).$$

We have that

$$\frac{r^0}{2}\rho(\nu)|\overline{w}|^2 + \langle D\rho(\nu), \overline{w} \rangle^2 \geq \left(1 + \frac{r^0}{2}\frac{\rho(\nu)}{|D\rho(\nu)|^2}\right)\langle D\rho(\nu), \overline{w} \rangle^2$$

$$\geq \left(1 + \frac{r^0c_1}{2c_3}\right)\langle D\rho(\nu), \overline{w} \rangle^2$$
where $c_1 > 0$ and $c_3 > 0$ are defined respectively in (4) and (13). If $c = 1 + \frac{n}{2c_1}$, from (50) we get

$$
\int_0^\infty \rho(\nu) \|\mathbf{w}\|^2 + \langle D\rho(\nu), \mathbf{w} \rangle^2 + \lambda_2^2 \rho(\nu)^2 + 2\lambda\rho(\nu) \langle D\rho(\nu), \mathbf{w} \rangle
$$

$$
\leq \frac{\lambda_2^2}{2} \rho(\nu)^2 + c \left( \langle D\rho(\nu), \mathbf{w} \rangle + \frac{\lambda\rho(\nu)}{c} \right)^2 + \left( 1 - \frac{1}{c} \right) \lambda_2^2 \rho(\nu)^2
$$

$$
\leq \frac{\lambda_2^2}{2} \rho(\nu)^2 + \left( 1 - \frac{1}{c} \right) c_5^3 \lambda_2^2 \geq c_6 \left( \|\mathbf{w}\|^2 + \lambda^2 \right),
$$

where $c_5 = \min \left( \frac{\lambda_2^2}{2}, \left( 1 - \frac{1}{c} \right) c_5^3 \right) > 0$.

Thanks to the estimates (41), (17), and (18) we can now prove that a solution $u$ to the first equation in (13) belongs to $H^2_{\text{loc}}(A)$. This part of the proof is based on a standard argument in regularity theory (see e.g. [13, §8.2]). The only point that should be stressed concerns the regularity of $\gamma$. Namely, $\gamma$ is of class $C^2$ in $\mathbb{R}^n \setminus \{0\}$, but in general the positively 0-homogeneous function $D^2\gamma$ is not even defined at the origin. In our case this is not a real problem, since the condition $\rho(Du(x)) = 1$ guarantees that $Du(x)$ stays always outside a ball centered at the origin.

**Lemma 5.4.** Every solution $u \in W^{1,\infty}(A)$ of (13) belongs to $H^2_{\text{loc}}(A)$.

**Proof.** The function $u$ solves

$$
\int_A \langle D\gamma(Du(x)), D\varphi(x) \rangle \, dx = 0
$$

for every $\varphi \in C^\infty_c(A)$. Let us fix $\varphi$, a coordinate direction $e_i$, and $h \in \mathbb{R}$ such that $\varphi_h(x) = \varphi(x - he_i)$ has compact support in $A$. Choosing $\varphi_h$ as a test function in (51) and making a change of variables in the integral we obtain

$$
\int_A \langle D\gamma(Du(x + he_i)), D\varphi(x) \rangle \, dx = 0.
$$

Taking the difference between (51) and (52), we get

$$
\int_A \langle D\gamma(Du(x + he_i)) - D\gamma(Du(x)), D\varphi(x) \rangle \, dx = 0.
$$

On the other hand we have that $\rho(Du(x)) = \rho(Du(x + he_i)) = 1$ for a.e. $x \in A$. Then the function $\alpha(t) = D\gamma((1 - t)Du(x) + tDu(x + he_i))$ is continuous in the interval $[0,1]$. In addition, either $\alpha \in C^1([0,1])$, or there exists $t_0 \in (0,1)$ such that $\alpha$ is of class $C^1$ and with bounded derivatives in $[0,1] \setminus \{t_0\}$. Hence $\alpha(t)$ is a Lipschitz function, and

$$
D\gamma(Du(x + he_i)) - D\gamma(Du(x))
$$

$$
= \int_0^1 \frac{d}{dt} [D\gamma((1 - t)Du(x) + tDu(x + he_i))] \, dt
$$

$$
= \int_0^1 D^2\gamma((1 - t)Du(x) + tDu(x + he_i))[Du(x + he_i) - Du(x)] \, dt,
$$

for a.e. $x \in A$. Finally, the matrix

$$
L_h(x) = \int_0^1 D^2\gamma((1 - t)Du(x) + tDu(x + he_i)) \, dt
$$

is well defined and, for $w \in \mathbb{R}^n$, the integrals of the kind

$$
\langle L_h(x)w, w \rangle = \int_0^1 \langle D^2\gamma((1 - t)Du(x) + tDu(x + he_i))w, w \rangle \, dt
$$

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satisfy for a.e. \( x \in A \)

\[
(55) \quad c_6 |w|^2 \leq \langle L_h(x)w, w \rangle, \quad \forall w \in \mathbb{R}^n,
\]

\[
(56) \quad \|L_h(x)\| \leq c_5,
\]

where \( c_6 \) and \( c_5 \) are the positive constants defined in Lemma 5.3 in (44).

If we denote by \( \Delta_h u(x) = \frac{u(x+h) - u(x)}{h} \), by (54) equation (53) can be rewritten as

\[
(57) \quad \int_A \langle L_h(x)D\Delta_h u(x), D\varphi(x) \rangle \, dx = 0
\]

for every \( \varphi \in C^\infty_0(A) \) and for every \( |h| < \text{dist}(\text{supp}\varphi, \partial A) \). By a density argument, we have that (57) remains valid for every test function in \( H^1_0(A') \), where \( A' \) is an open set compactly contained in \( A \), and for every \( |h| < \text{dist}(A', \partial A) \). Hence we can choose \( \varphi = \eta^2 \Delta_h u \) as test function in (57), where \( \eta \in C^\infty_0(A) \) is defined in the following way: given \( x_0 \in A \) and \( r > 0 \) such that the ball \( B_r(x_0) \) is compactly contained in \( A \), we require that \( 0 \leq \eta(x) \leq 1 \) in \( A \), \( \eta(x) = 0 \) in \( A \setminus B_{2r}(x_0) \), \( \eta(x) = 1 \) in \( B_r(x_0) \), and there exists \( m > 0 \) such that \( |D\eta(x)| \leq m/r \) in \( A \).

With this choice of the test function, (57) becomes

\[
(58) \quad \int_A \langle L_h(x)D\Delta_h u(x), D\Delta_h u(x) \rangle \eta^2(x) \, dx
\]

\[
= -2 \int_A \langle L_h(x)D\Delta_h u(x), D\eta(x) \rangle \eta(x)D\Delta_h u(x) \, dx,
\]

for every \( |h| < \text{dist}(x_0, \partial A) - 2r \). Recalling (55) we have that

\[
(59) \quad c_6 \int_A |D\Delta_h u(x)|^2 \eta^2(x) \, dx \leq \int_A \langle L_h(x)D\Delta_h u(x), D\Delta_h u(x) \rangle \eta^2(x) \, dx.
\]

On the other hand, by (56) and Young’s inequality we obtain that there exists a constant \( C > 0 \) such that

\[
(60) \quad \left| \int_A \langle L_h(x)D\Delta_h u(x), D\eta(x) \rangle \eta(x)D\Delta_h u(x) \, dx \right|
\]

\[
\leq C \left( \varepsilon \int_A |D\Delta_h u(x)|^2 \eta^2(x) \, dx + \frac{1}{\varepsilon} \int_A |D\eta(x)|^2 |\Delta_h u(x)|^2 \, dx \right),
\]

for every \( \varepsilon > 0 \). Choosing \( \varepsilon \) small enough, from (58), (59), (60), and the estimates of \( |D\eta| \), we get

\[
(61) \quad \int_{B_r(x_0)} |D\Delta_h u(x)|^2 \, dx \leq \frac{M}{r^2} \int_{B_{2r}(x_0)} |\Delta_h u(x)|^2 \, dx,
\]

which implies, by a standard argument, that \( u \in H^2_{\text{loc}}(A) \) (see Lemmas 7.23 and 7.24 in [12], or [13, §8.1]).

Proof of Theorem 5.4. We already know that a solution \( u \in W^{1,\infty}(A) \) of (37)–(36) is also a solution to (43). Fixed \( \varphi \in C^\infty_0(\Omega) \), we can choose \( \frac{\partial \varphi}{\partial x_i} \) as test function in the weak formulation (50), obtaining

\[
(62) \quad \int_A \left\langle D\gamma(Du(x)), D\frac{\partial \varphi}{\partial x_i}(x) \right\rangle \, dx = 0
\]

Moreover, by Lemma 5.4, the function \( u \) belongs to \( H^2_{\text{loc}}(A) \). Then an integration by parts leads

\[
(63) \quad \int_A \left\langle D^2\gamma(Du(x))D\frac{\partial u}{\partial x_i}(x), D\varphi(x) \right\rangle \, dx = 0.
\]
for every \( \varphi \in C^\infty_0(\Omega) \). Then the partial derivative \( \frac{\partial u}{\partial x_i} \) is a bounded solution of the linear elliptic equation
\[
-\text{div}(D^2\gamma(Du(x))Dv) = 0 \quad \text{in} \quad A,
\]
where the matrix \( D^2\gamma(Du(x)) \) satisfies the hypothesis of the De Giorgi–Nash regularity result (see [14, §3.14], Theorem 14.1, [12, Theorem 8.22]). Hence we can conclude that the partial derivatives of \( u \) are locally Hölder continuous.

**Remark 5.5.** Let us define the function \( \gamma_p(\xi) = \rho(\xi)^p, \quad p > 1 \). Clearly a function \( u \in W^{1,\infty}(A) \) is a solution to (64) if and only if it is a solution to
\[
\begin{cases}
-\text{div}(D\gamma_p(Du(x))) = 0 & \text{in} \quad A, \\
\rho(Du(x)) = 1 & \text{in} \quad A
\end{cases}
\]
where, as usual, the first equation is interpreted in the sense of distributions and the second in viscosity sense. Arguing as in the proof of Lemma 5.3, it can be proved that there exists a positive constant \( c_p \) such that, for every \( \xi \neq 0 \),
\[
|D\gamma_p(\xi)| \leq c_p^{-1}c_3|\xi|^{p-1},
\]
\[
\langle D\gamma_p(\xi), \xi \rangle \geq c_p^2|\xi|^p,
\]
\[
\langle D^2\gamma_p(\xi)w, w \rangle \geq c_p|\xi|^2|w|^2, \quad \forall w \in \mathbb{R}^n.
\]
We remark that, if \( p > 2 \), then \( \gamma_p \) is of class \( C^2 \) on \( \mathbb{R}^n \), and the estimates above hold for every \( \xi \in \mathbb{R}^n \). On the other hand, if \( p > 2 \), we cannot obtain an estimate of the type (40) near the origin, due to the \( p \)-homogeneity of the function \( \gamma_p \), which implies the \( (p-2) \)-homogeneity of \( D^2\gamma_p(\xi) \). For this reason we have considered the case \( p = 2 \).

6. **Uniqueness**

This section is devoted to the proof of the following uniqueness result.

**Theorem 6.1.** Let \( (u, v) \) be a solution of system (1)–(2) in the sense of Definition 4.1. Then \( v = v_f \), where \( v_f \) is the function defined in (23), and \( u = d_\Omega \) in \( \Omega_f = \{ x \in \Omega; \ v_f(x) > 0 \} \).

The proof of Theorem 6.1 is essentially based on the techniques developed in [11, 14, 17]. We will first prove the uniqueness of the first component of the solution of system (1)–(2). More precisely, we will show that if \( (u, v) \) is a solution of system (1)–(2), then \( u = d_\Omega \) in \( \Omega_f := \{ x \in \Omega : \ v_f(x) > 0 \} \) (see Proposition 6.4 below).

Let us consider the functional \( \Phi : H^1_0(\Omega) \times L^2_+(\Omega) \rightarrow \mathbb{R} \), where \( L^2_+(\Omega) = \{ z \in L^2(\Omega); \ z \geq 0 \} \), defined by
\[
\Phi(w, z) = -\int_\Omega f(x)w(x) \, dx + \int_\Omega z(x) \left( \rho(Dw(x)) - 1 \right) \, dx.
\]

**Lemma 6.2.** If \( (u, v) \) is a solution of system (1)–(2), then \( (u, v) \) is a saddle point of \( \Phi \), in the sense that
\[
\Phi(u, z) \leq \Phi(u, v) \leq \Phi(w, v) \quad \forall (w, z) \in H^1_0(\Omega) \times L^2_+(\Omega).
\]

**Proof.** Since \( (u, v) \) is a solution of (1)–(2), then
\[
\int_\Omega v(x) \left( \rho(Du(x)) - 1 \right) \, dx = 0
\]
and
\[
\int_\Omega z(x) \left( \rho(Du(x)) - 1 \right) \, dx \leq 0, \quad \forall z \in L^2_+(\Omega).
\]
Hence, for any $z \in L^2_{+}(\Omega)$ we have

$$\Phi(u,v) = -\int_{\Omega} f(x) u(x) \, dx$$

(67)

$$\geq -\int_{\Omega} f(x) u(x) \, dx + \int_{\Omega} z(x) \left[ \rho(Du(x)) - 1 \right] \, dx = \Phi(u,z).$$

Moreover, by the convexity of $\rho$, for any $w \in H^1_0(\Omega)$ we have

$$\rho(Dw(x)) - \rho(Du(x)) \geq \langle D\rho(Du(x)), Dw(x) - Du(x) \rangle$$

(68)

for a.e. $x \in \Omega$. By Remark 4.2 we can choose $\varphi = w - u \in H^1_0(\Omega)$ as test function in (34), obtaining

$$-\int_{\Omega} f(x)(w(x) - u(x)) \, dx + \int_{\Omega} v(x) \langle D\rho(Du(x)), Dw(x) - Du(x) \rangle \, dx = 0.$$ 

(69)

Collecting together (67) and (69) we get the conclusion. \hfill \Box

In what follows we shall use the set of functions

$$\text{Lip}_\rho^1(\Omega) := \{ w \in \text{Lip}(\Omega); \ w = 0 \ \text{on} \ \partial\Omega, \ \rho(Dw) \leq 1 \ \text{a.e. in} \ \Omega \}.$$ 

(70)

It can be checked that $w \in \text{Lip}_\rho^1(\Omega)$ if and only if $w \in \text{Lip}(\Omega), \ w = 0 \ \text{on} \ \partial\Omega,$ and

$$w(x) - w(y) \leq \rho^0(x - y) \ \ \forall x, y \in \overline{\Omega}, \ \text{with} \ [x, y] \subset \overline{\Omega}$$

(71)

(see [16, Chap. 6]).

**Lemma 6.3.** If $(u,v)$ is a solution of system (1)–(2), then the following hold.

(i) $u = d_\Omega$ in supp$(f)$.

(ii) $(d_\Omega, v)$ is a solution of (1)–(2).

**Proof.** (i) By the maximality property of viscosity solutions we have that $u \leq d_\Omega$ in $\Omega$. On the other hand, by Lemma 6.2, $\Phi(u,v) \leq \Phi(w,v)$ for any $w \in \text{Lip}_\rho^1(\Omega)$. In addition, we have

$$\Phi(u,v) = -\int_{\Omega} f(x) u(x) \, dx,$$

$$\Phi(w,v) = -\int_{\Omega} f(x) w(x) \, dx + \int_{\Omega} v(x) \left[ \rho(Dw(x)) - 1 \right] \, dx$$

$$\leq -\int_{\Omega} f(x) w(x) \, dx.$$ 

Then

$$\int_{\Omega} f(x) w(x) \, dx \leq \int_{\Omega} f(x) u(x) \, dx \ \ \forall w \in \text{Lip}_\rho^1(\Omega).$$

Choosing $w = d_\Omega$, we obtain that $u = d_\Omega$ on supp$(f)$.

(ii) From (i) and Lemma 6.2 we have that

$$\Phi(d_\Omega, v) = \Phi(u,v) \leq \Phi(w,v). \ \ \forall w \in H^1_0(\Omega).$$


Hence, for every test function $\varphi \in C_c^\infty(\Omega)$ and every $h > 0$ we have that
\[
0 \leq \Phi(d\Omega + h \varphi, v) - \Phi(d\Omega, v)
= -h \int_{\Omega} f(x)\varphi(x) \, dx + \int_{\Omega} v(x) \left[ \rho(D(d\Omega(x) + h \varphi(x))) - \rho(Dd\Omega(x)) \right] \, dx.
\]
Since $\rho$ is convex we have
\[
\left| \frac{\rho(D(d\Omega(x) + h \varphi(x))) - \rho(Dd\Omega(x))}{h} \right| \leq \| D\rho \|_{\infty} \| D\varphi \|_{L^\infty(\Omega)}
\]
for a.e. $x \in \Omega$ and every $h > 0$, where
\[
\| D\rho \|_{\infty} := \sup_{\xi \neq 0} |D\rho(\xi)| = \max_{\xi \in S^{n-1}} |D\rho(\xi)|
\]
due to the positive 0-homogeneity of $D\rho$. Hence, from the differentiability of $\rho$ in $\mathbb{R}^n \setminus \{0\}$ and the dominated convergence theorem we get
\[
0 \leq - \int_{\Omega} f(x)\varphi(x) \, dx + \int_{\Omega} v(x) \langle D(\rho(Dd\Omega(x))), D\varphi(x) \rangle \, dx.
\]
Replacing $\varphi$ by $-\varphi$ we also get the opposite inequality. \hfill \Box

**Proposition 6.4.** If $(u, v)$ is a solution of system (1)–(2), then $u = d\Omega$ in the set $\Omega_f = \{ x \in \Omega; \, v_f(x) > 0 \}$, where $v_f$ is the function defined by (35).

**Proof.** Let $x \in \Omega_f \subseteq \Omega \setminus \Sigma$. By the definition (35) of $v_f$, and taking into account that $M_z(t) > 0$ for every $t \in [0, \tau(x))$ and $f \geq 0$, we deduce that there exists $t_0 \in (0, \tau(x))$ such that, at the point $x_0 = x + t_0 D\rho(Dd\Omega(x))$, one has $f(x_0) > 0$. Since
\[
d\Omega(x_0) = d\Omega(x) + t_0 = d\Omega(x) + \rho^0(x_0 - x),
\]
from this identity, Lemma 4(i), (71) and the inequality $u \leq d\Omega$ we get
\[
d\Omega(x) = d\Omega(x_0) - \rho^0(x_0 - x) = u(x_0) - \rho^0(x_0 - x) \leq u(x) \leq d\Omega(x),
\]
hence $u(x) = d\Omega(x)$. \hfill \Box

Now that the uniqueness of the first component of the solution of system (1)–(2) is proved, it remains to prove the uniqueness of the second one. In order to do so, we will first exhibit for such a function a representation formula on the set $\Omega \setminus \Sigma$ and then analyze its behavior on $\Sigma$.

**Proposition 6.5.** If $(d\Omega, v)$ is a solution of system (1)–(2), then for any $z_0 \in \Omega \setminus \Sigma$ and $\theta \in (0, \tau(z_0))$ we have
\[
v(z_0) - v(z_0 + \theta D\rho(Dd\Omega(z_0))) M_{z_0}(\theta)
= \int_{0}^{\theta} f(z_0 + t D\rho(Dd\Omega(z_0))) M_{z_0}(t) \, dt.
\]
**Proof.** Let $z_1 = z_0 + \theta D\rho(Dd\Omega(z_0))$, let $\Pi(z_0) = \{ x_0 \}$, and define $t_0 = d\Omega(z_0) = \rho^0(z_0 - x_0) + t_1 = d\Omega(z_1) = \rho^0(z_1 - x_0) = t_0 + \theta$. Let $Y: \mathcal{U} \to \mathbb{R}^n$, $\mathcal{U} \subset \mathbb{R}^{n-1}$ open, a local parametrization of $\partial\Omega$ in a neighborhood of $x_0$, such that $Y(0) = x_0$. Let $\Psi: \mathcal{U} \times \mathbb{R} \to \mathbb{R}^n$ be the map
\[
\Psi(y, t) = Y(y) + t D\rho(v(Y(y))), \quad (y, t) \in \mathcal{U} \times \mathbb{R}.
\]
Choose $r > 0$ such that $U(r) := \{ y \in \mathbb{R}^{n-1}; \, |y| \leq r \} \subset \mathcal{U}$, and
\[
D(r) := \{ \Psi(y, t); \, y \in U(r), \, t \in [t_0, t_1] \} \subset \Omega \setminus \Sigma.
The set $D(r)$ can be viewed as a tubular neighborhood of the segment $[z_0,z_1]$. Let us define

$$S_i(r) = \{\Psi(y,t_i) \cap U(r) \}, \quad i = 0,1,$$

and let $S_2(r)$ denote the lateral surface of $D(r)$, i.e.

$$S_2(r) = \{\Psi(y,t) \cap \partial U(r), \ t \in [t_0,t_1] \}.$$ 

All these surfaces are of class $C^1$ and are oriented with the outward normal with respect to $D(r)$.

For $\epsilon > 0$ small enough let $\psi_\epsilon : \mathbb{R} \to \mathbb{R}$ and $\eta_\epsilon : \mathbb{R}^{n-1} \to \mathbb{R}$ be the functions defined by

$$\psi_\epsilon(t) = \begin{cases} 0 & \text{if } t \leq t_0 \text{ or } t \geq t_1, \\ 1 & \text{if } t \in [t_0 + \epsilon, t_1 - \epsilon], \\ \frac{t-t_0}{\epsilon} & \text{if } t \in (t_0, t_0 + \epsilon), \\ \frac{t_1-t}{\epsilon} & \text{if } t \in (t_1 - \epsilon, t_1). \end{cases}$$

$$\eta_\epsilon(y) = \begin{cases} 0 & \text{if } |y| \geq r, \\ 1 & \text{if } |y| \leq r - \epsilon, \\ \frac{r-|y|}{\epsilon} & \text{if } r - \epsilon < |y| < r. \end{cases}$$

Let $\varphi_\epsilon$ be the function defined by

$$\varphi_\epsilon(x) := \begin{cases} \psi_\epsilon(t) \eta_\epsilon(y), & \text{if } \exists \ y \in U(r) \text{ and } t \in [t_0,t_1] \text{ s.t. } x = \Psi(y,t), \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that $\varphi_\epsilon$ belongs to $\text{Lip}($[\Omega]) and has support contained in $D(r)$, hence can be used as test function in (34).

It is plain that $\varphi_\epsilon$ converges monotonically to 1 in the interior of $D(r)$ as $\epsilon \to 0^+$, hence

$$\lim_{\epsilon \to 0^+} \int f \varphi_\epsilon \, dx = \int_{D(r)} f \, dx. \tag{74}$$

Let us compute the right-hand side of (34) when $\varphi = \varphi_\epsilon$ and $u = d_\Omega$. On $D(r)$ the test function $\varphi_\epsilon$ is defined by the relation

$$\varphi_\epsilon(\Psi(y,t)) = \psi_\epsilon(t) \eta_\epsilon(y), \quad y \in U(r), \ t \in [t_0,t_1].$$

Differentiating the relation above with respect to $t$ and recalling the definition (34) of $\Psi$ we obtain

$$\langle D\varphi(\Psi(y,t)), D\rho(Dd_\Omega(\Psi(y,t))) \rangle = \psi'_\epsilon(t) \eta_\epsilon(y), \quad y \in U(r), \ t \in [t_0,t_1].$$

Then, taking into account that $\psi'_\epsilon(t) = 0$ for $t \in (t_0 + \epsilon, t_1 - \epsilon)$, we get

$$\int v \langle D\rho(Dd_\Omega), D\varphi_\epsilon \rangle \, dx$$

$$= \int_{U(r)} \int_{t_0}^{t_1} v(\Psi) \langle D\rho(Dd_\Omega(\Psi)), D\varphi_\epsilon(\Psi) \rangle \det D\Psi \, dt \, dy$$

$$= \int_{U(r)} \int_{t_0}^{t_1} v(\Psi) \psi'_\epsilon(t) \eta_\epsilon(y) \det D\Psi \, dt \, dy$$

$$= I_0(\epsilon) + I_1(\epsilon) + I_2(\epsilon) + I_3(\epsilon), \tag{75}$$
where
\[
I_0(\epsilon) = \frac{1}{\epsilon} \int_{U(r-\epsilon)}^{t_0+\epsilon} v(\Psi) \det D\Psi \, dt \, dy,
\]
\[
I_1(\epsilon) = -\frac{1}{\epsilon} \int_{U(r-\epsilon)}^{t_1} \int_{t_1-\epsilon}^{t_0} v(\Psi) \det D\Psi \, dt \, dy,
\]
\[
I_2(\epsilon) = \frac{1}{\epsilon} \int_{U(r) \setminus U(r-\epsilon)}^{t_0+\epsilon} \int_{t_1-\epsilon}^{t_0} v(\Psi) \eta(z) \det D\Psi \, dt \, dy,
\]
\[
I_3(\epsilon) = -\frac{1}{\epsilon} \int_{U(r) \setminus U(r-\epsilon)}^{t_1} \int_{t_1-\epsilon}^{t_0} v(\Psi) \eta(z) \det D\Psi \, dt \, dy.
\]

Since \( v \) and \( D\Psi \) are bounded in \( D(r) \), an explicit computation leads to \( |I_2(\epsilon) + I_3(\epsilon)| \leq C\epsilon \).

Passing to the limit in (72), by the continuity of \( v(\Psi) \) and \( D\Psi \) we obtain
\[
\lim_{\epsilon \to 0^+} \int_{
\Omega} v \langle D\Omega, D\varphi \rangle \, dx = \sum_{i=0}^{1} (-1)^i \int_{U(r)} v(\Psi(y,t_i)) \det D\Psi(y,t_i) \, dy.
\]

Recalling (74), we finally obtain
\[
\int_{D(r)} f(x) \, dx = \sum_{i=0}^{1} (-1)^i \int_{U(r)} v(\Psi(y,t_i)) \det D\Psi(y,t_i) \, dy.
\]

As a last step we want to pass to the limit as \( r \to 0^+ \). From the continuity of \( v \) and \( D\Psi \) we get
\[
\lim_{r \to 0^+} \frac{1}{\sigma(r)} \int_{U(r)} v(\Psi(y,t_i)) \det D\Psi(y,t_i) \, dy = v(z_i) \det D\Psi(0,t_i),
\]

where \( \sigma(r) = \omega_n r^{n-1} \) is the area of the ball with radius \( r > 0 \) in \( \mathbb{R}^{n-1} \). Finally
\[
\lim_{r \to 0^+} \frac{1}{\sigma(r)} \int_{D(r)} f(x) \, dx
\]
\[
= \lim_{r \to 0^+} \frac{1}{\sigma(r)} \int_{U(r)} \int_{t_0}^{t_1} f(\Psi(y,t)) \det D\Psi(y,t) \, dt \, dy
\]
\[
= \int_{t_0}^{t_1} f(\Psi(0,t)) \det D\Psi(0,t) \, dt
\]
\[
= \int_{0}^{\theta} f(z_0 + t D\rho(D\Omega(z_0))) \det D\Psi(0,t) \, dt.
\]

From Lemma 4.10 in [18] we have that
\[
\det D\Psi(0,t) = \sqrt{G} \rho(\nu(x_0)) \det(I_{n-1} - t \nabla(x_0)), \quad t \in [t_0,t_1],
\]

where \( G \) is the determinant of the matrix of the metric coefficients. Collecting together (77), (78), recalling the identity (70), and dividing by \( \det D\Psi(0,t_0) \) we obtain
\[
v(z_0) - v(z_1) \frac{\det(I_{n-1} - t_1 \nabla(x_0))}{\det(I_{n-1} - t_0 \nabla(x_0))} = \int_{0}^{\theta} f(z_0 + t D\rho(D\Omega(z_0))) \frac{\det(I_{n-1} - (t_0 + t) \nabla(x_0))}{\det(I_{n-1} - t_0 \nabla(x_0))} \, dt.
\]

The representation formula (72) now follows from (76) and the definition (32) of \( M_{z_0} \).
For the proof of Proposition 6.7 below we need two more technical ingredients. The first one is the regularity result proved in Theorem 5.1. The second one is the following convergence lemma due to H. Brezis (see [3, Theorem 1]).

**Lemma 6.6.** Let $\gamma : \mathbb{R}^n \rightarrow \mathbb{R}$ be a strictly convex function, satisfying the linear growth condition

$$\gamma(\xi) \geq c_0|\xi| - b_0, \quad \forall \xi \in \mathbb{R}^n,$$

for some positive constants $b_0$ and $c_0$. Let $(u_k)_k \subset [L^1(\Omega)]^n$ be a sequence of functions converging to $u \in [L^1(\Omega)]^n$ in the weak $L^1$ topology, and assume that $\gamma(u), \gamma(u_k) \in [L^1(\Omega)]^n$ for every $k \in \mathbb{N}$. If $\lim_{k \to \infty} \int_{\Omega} \gamma(u_k) dx = \int_{\Omega} \gamma(u) dx$, then $(u_k)_k$ converges to $u$ in the strong $L^1$ topology.

**Proposition 6.7.** If $(d_0, v)$ is a solution of system (1)–(2), then $v(x) = 0$ for every $x \in \Sigma$.

**Proof.** Since $v$ is a continuous function, it suffices to prove that $v = 0$ on $\Sigma$. Let us fix any $x_0 \in \Sigma$ and choose $\epsilon > 0$ sufficiently small such that $B_\epsilon(x_0) \subset \Omega$. Then, for any $x \in B_\epsilon(0)$ set

$$d_\epsilon(x) := \frac{d_0(x + \epsilon x) - d_0(x)}{\epsilon}, \quad v_\epsilon(x) := v(x + \epsilon x), \quad f_\epsilon(x) := f(x + \epsilon x).$$

By construction, for any $\epsilon > 0$ as above $d_\epsilon(0) = 0$ and

$$\rho(Dd_\epsilon(x)) = \rho(Dd_\epsilon(x + \epsilon x)) = 1 \quad \text{for a.e.} \ x \in B_\epsilon(0).$$

Hence, there exist a sequence $(\epsilon_j)_j, \epsilon_j \to 0^+$ and a Lipschitz function $d_0 : B_1(0) \to \mathbb{R}$ such that $(d_{\epsilon_j})_j$ converges to $d_0$ uniformly in $B_1(0)$. Moreover, since $\rho(Dd_{\epsilon_j}(x)) = 1$ in the viscosity sense in $B_1(0)$, by [1, Proposition 2.2] also $\rho(Dd_\epsilon(0)) = 1$ in the viscosity sense in $B_1(0)$, which gives $\rho(Dd_\epsilon(x)) = 1$ almost everywhere. Since $(Dd_{\epsilon_j})_j$ is bounded in $L^\infty$, we can also assume that it converges to $Dd_0$ in the weak $L^1$ topology.

From (13) in Lemma 6.6, we have that the function $\gamma(\xi) = \rho(\xi)^2/2$ is strictly convex in $\mathbb{R}^n$. Since $\gamma(Dd_{\epsilon_j}(x)) = 1/2$ and $\gamma(Dd_\epsilon(x)) = 1/2$ for a.e. $x \in B_1(0)$, we have that

$$\frac{1}{2} \int_{B_1(0)} \gamma(Dd_{\epsilon_j}(x)) dx = \frac{\omega_n}{2} = \int_{B_1(0)} \gamma(Dd_\epsilon(x)) dx.$$

Recalling that $(Dd_{\epsilon_j})_j$ converges to $Dd_0$ in the weak $L^1$ topology, from Lemma 6.6 we conclude that $(Dd_{\epsilon_j})_j$ converges to $Dd_0$ in the strong $L^1$ topology.

Finally, the functions $v_{\epsilon_j}$ and $f_{\epsilon_j}$ defined above uniformly converge to $v(x_0)$ and $f(x_0)$ respectively and the pair $(d_{\epsilon_j}, v_{\epsilon_j})$ solves

$$(79) \quad -\text{div}(v_{\epsilon_j} D\rho(Dd_{\epsilon_j})) = \epsilon_j f_{\epsilon_j} \quad \text{in} \ B_1(0)$$

in the sense of distributions, due to the fact that $(d_0, v)$ solves (1)–(4). Upon observing that $D\rho(Dd_{\epsilon_j})$ converges to $D\rho(Dd_0)$ in the strong $L^1$ topology, we can pass to the limit as $j \to \infty$ in (79), obtaining that $d_0$ is a weak solution of

$$(\text{79}) \quad -\text{div}(v(x_0) D\rho(Dd_0)) = 0 \quad \text{in} \ B_1(0).$$

Now, if $v(x_0) \neq 0$, then $d_0$ must be a solution to

$$\begin{cases}
-\text{div}(D\rho(Dd_0)) = 0 & \text{in} \ B_1(0), \\
\rho(Dd_0) = 1 & \text{a.e.} \ B_1(0).
\end{cases}$$

From Theorem 5.1 we have that $d_0 \in C^{1,\alpha}(B_1(0))$. On the other hand, $d_0$ cannot be differentiable in $x = 0$, because $d_0$ is the ‘blow up’ of the distance function around a singular point $x_0$. Hence $v(x_0) = 0$ and the proof is complete. \qed
The last two propositions allow us to prove Theorem 6.1 as a simple corollary. Indeed, we already know by Proposition 6.4 that if \((u,v)\) is a solution of system (1)–(2), then 
\[ u = d_\Omega \] on the set \(\Omega_f = \{v_f > 0\} \). So it only remains to prove that 
\[ v = v_f \] in \(\Sigma\), while Proposition 6.5 implies that for any \(z_0 \in \Omega \setminus \Sigma\) and \(\theta \in (0,\tau(z_0))\)
\[ v(z_0) - v(z_0 + \theta D\rho(Dd_\Omega(z_0))) M_{z_0}(\theta) = \int_0^\theta f(z_0 + t D\rho(Dd_\Omega(z_0))) M_{z_0}(t)\,dt . \]
Hence, letting \(\theta \to \tau(z_0)^-\) and using the continuity of \(v\) we obtain that 
\[ v(z_0) = v_f(z_0) . \]

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