Reducible higher–spin multiplets in flat and AdS spaces and their geometric frame–like formulation

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Abstract

We consider the frame–like formulation of reducible sets of totally symmetric bosonic and fermionic higher–spin fields in flat and AdS backgrounds of any dimension, that correspond to so-called higher–spin triplets resulting from the string–inspired BRST approach. The explicit relationship of the fields of higher–spin triplets to the higher–spin vielbeins and connections is found. The gauge invariant actions are constructed including, in particular, the reducible (i.e. triplet) higher–spin fermion case in $AdS_D$ space.
1 Introduction

This paper is an essentially extended version of the contribution to the volume dedicated to the 60th birthday anniversary of Joseph Buchbinder, our colleague and friend, who, among other important subjects in his fruitful scientific carrier, made an extensive contribution to the theory of higher–spin fields.
The minimal approach to the description of massless higher–spin fields, developed originally by Fronsdal [1, 2] and Fang and Fronsdal [3, 4] for the generic massless fields in four dimensional Minkowski [1, 3] and anti–de Sitter space [2, 4], is usually referred to as the metric–like formalism because it is a natural generalization [5] of the metric formulation of the linearized gravity (i.e., massless spin two). The construction of gauge invariant actions for single (irreducible) massless higher–spin fields in this approach requires these fields to be double–traceless in the bosonic case [1, 2] or triple–gamma–traceless [3, 4] in the fermionic case.

The frame-like formulation of massless higher–spin gauge fields, that generalizes the Cartan formulation of gravity, is also available both in Minkowski [6, 7] and anti–de–Sitter [8, 9, 10, 11] spaces. In this approach, higher–spin fields are described by differential forms that carry irreducible representations of the fiber Lorentz group. In the spin–two case this approach reproduces the linearized Cartan gravity with the one-form frame field or vielbein \( e^a = dx^m e_m^a \) carrying a vector representation (index \( a \)) of the Lorentz group. For higher spins, the frame-like fields are rank \( s - 1 \) symmetric traceless tensors \( e^{a_1...a_{s-1}} = dx^m e_m^{a_1...a_{s-1}} \). Higher-spin gauge symmetry parameters \( \xi^{a_1...a_{s-1}} \) are rank \( s - 1 \) traceless symmetric tensors in the both approaches.

Both metric-like and frame-like approaches are geometric, although in a slightly different fashion, extending Riemann and Cartan geometries, respectively. As in the standard case of gravity, the frame-like approach [6, 7, 8, 9, 10, 11] is more general than the metric-like [1, 2, 3, 4, 5]. The latter is a particular gauge of the former. Moreover, in the fermionic case, the frame-like approach is the only one working at the interaction level. In fact, the frame-like approach, which operates in terms of differential forms and connections, has a greater power. It allows, in particular via the unfolded dynamics approach [12] (see [13] for more detail), to introduce higher-spin interactions and to uncover deep geometric structures that underly the higher–spin theory and are likely to deviate from the standard concepts of Riemann geometry in the strong field regime. (For more detail on the higher–spin theory see recent reviews [13, 14, 15, 16, 17, 18] and references therein).

If, instead of a single spin, a set of different spins is considered, their dynamics can be described in terms of tensor fields that are less constrained than in the single higher–spin case or even unconstrained. An example of such a system is provided by the so called triplet systems of massless higher–spin fields which naturally appear in the process of truncation of the open string spectrum in the tensionless limit [19, 20, 21, 22, 23, 24, 25] (see also [26] for further developments and e.g. [27, 28, 29, 30] for other aspects of the tensionless string limit and higher spin theory). So one can regard the triplets as fields which manifest their origin from massive higher–spin fields of the tensionful string.

The geometrical nature of triplet fields, i.e. their relation to higher–spin counterparts of metric (or vielbein) and connection, has not been clarified yet. Moreover, neither equations of motion nor the action for fermionic triplets in AdS space have been constructed. This hinders the study of the relation of the fermionic triplets to string states in AdS backgrounds and corresponding applications.

A purpose of this paper is to reconsider these problems using the frame–like approach. Upon establishing the geometrical meaning of the triplet fields and finding their proper gauge transformations both in Minkowski and in \( AdS_D \), we construct the Lagrangian description of the bosonic and fermionic triplets in flat and AdS backgrounds. We present two descriptions of AdS triplet systems. The formulation of Section 3, which uses \( O(1, D - 1) \) Lorentz tensors
as in [6] for irreducible fields, is relatively simple but is only implicitly gauge invariant while another one considered in Section 7, as in [11] for irreducible fields, is manifestly gauge and $\text{AdS}_D$ invariant (i.e. invariant under the $\text{AdS}_D$ isometry group $O(2, D - 1)$) but requires a somewhat more involved action. Note that the relaxed systems of fields that contain the triplet systems along with the so-called partially massless fields [31] in the frame–like formalism [32] were considered recently by Alkalaev in [33] within the $\text{AdS}_D$ covariant frame-like approach. We extend these results by constructing actions and formulating conditions that sort out the quantum-mechanically inconsistent (nonunitary in anti - de Sitter, or tachionic in de Sitter space) partially massless fields.

As we have mentioned, to describe the dynamics of a field of a single higher–spin in the frame–like formulation one should impose traceless conditions on higher–spin vielbeins and connections [6, 7, 8, 9, 10, 11, 14, 13]. We shall show that these conditions can be relaxed in such a way that the higher–spin vielbein becomes unrestricted, while the higher–spin connections are subject to weaker traceless constraints. In the integer–spin case the weaker conditions result in a Lagrangian system that describes the set of fields of spins $s, s - 2, s - 4, \ldots, 3$ or 2, depending on whether $s$ is odd or even. The system formulated in terms of frame-like differential forms does not describe lower–spin massless fields with spins $s \leq 1$. These can be included by adding to the action their kinetic terms formulated in terms of the so–called Weyl zero–forms as discussed in [34] for the scalar case and in [35] for the spin one case. Analogously, in the half–integer spin case, the system under consideration describes the set of fields of spins $s, s - 1, s - 2, \ldots, 3/2$. The description a massless spin 1/2 field also needs inclusion of zero-forms. To simplify consideration we discard the analysis of the lower–spin fields in this paper. Let us stress that we consider the reducible fermionic systems both in flat and in $\text{AdS}$ spaces, that is important for the analysis of a relationship of higher–spin theories with superstrings.

The higher–spin triplets described in the metric–like formulation have the same physical state contents (modulo the lowest spin states 0, 1 and 1/2), and indeed we find the correspondence between the triplet fields and components of the higher–spin vielbein and connection of the higher–spin system with relaxed trace constraints. A transparent geometrical structure of the frame–like formulation allows us to construct relatively simple actions for the bosonic and, in particular, fermionic triplet fields both in flat and in $\text{AdS}$ backgrounds which should be useful for their applications, e.g. for studying interactions of triplets.

The trace constraints on the dynamical fields and gauge symmetry parameters, which single out the irreducible higher–spin fields, are algebraic (free of derivatives of fields) and therefore harmless. If desired, they can be easily removed by introducing Lagrange multipliers along with Stueckelberg fields and symmetries to make the higher–spin gauge fields and parameters traceful. Several versions of the unconstrained Lagrangian formulation of higher–spin fields have been proposed in the literature using different approaches (see e.g. [36, 37, 38, 39, 23, 40, 41, 42, 43, 44, 45, 46]). We shall also show how imposing constraints on the (gamma)–trace of the higher–spin vielbeins to be pure gauge reduces the higher–spin systems under consideration to the frame–like versions of the unconstrained formulations of single higher–spin fields considered in [42, 45].

The triplet systems of higher-spin fields resulting from string theory in the tensionless limit by construction carry the sets of states that are appropriate for the description of massive higher–spin fields. In other words, the study of these sets may shed some light on a mechanism
of higher–spin symmetry breaking resulting in the generation of mass in Higher–Spin Theory, which is the necessary step in establishing a relationship of the Higher–Spin Gauge Theory with String Theory. In particular, the results of this work are expected to help to obtain frame–like versions of the string–inspired BRST formulation of massless and massive higher–spin fields (see e.g. [18] for a recent review and references) as well as of the gauge invariant (Stueckelberg) description of the massive fields analogous to the metric–like approach by Zinoviev [47, 48] both in the bosonic and fermionic cases. Note that the BRST version of the frame-like unfolded formulation of massless higher–spin fields was considered in [46] for the Minkowski case and in [49] for the AdS case.

A closely related problem for the future study is to figure out what might be an algebraic structure (higher–spin symmetry) that underlies the relaxed higher–spin multiplets considered in this paper. In this respect, an encouraging result is that the relaxed higher–spin systems under consideration can be singled out from the appropriate oscillator algebras by imposing natural conditions which are invariant under the $\text{AdS}_D$ symmetry algebra $\mathfrak{o}(2, D − 1)$. These conditions pick up only those representations of $\mathfrak{o}(2, D − 1)$ that correspond to standard (unitary) systems of massless fields, sorting out the non–unitary systems that describe the partially massless fields [31] in the frame–like formalism [32].

The results of this paper can be useful for the further study of higher–spin triplets and their generalization to mixed–symmetry fields in particular in AdS backgrounds (on various aspects of mixed–symmetry fields see e.g. [19, 50, 51, 52, 37, 53, 54, 55, 43, 44] and references therein), in various contexts of higher–spin theory and its relation to string theory.

For the structure of the paper see the Table of Contents.

2 Frame–like action for bosonic higher–spin fields in flat space–time

In the frame–like formulation (we refer the reader to [6, 8, 14, 13] for details) a massless symmetric field of an integer spin $s$ in flat space–time of dimension $D$ is described by the higher–spin vielbein one–form

$$e^{n_1...n_{s−1}} = dx^m e_{m; n_1...n_{s−1}},$$ (2.1)

by the one–form connection

$$\omega^{n_1...n_{s−1},p} = dx^m \omega_{m; n_1...n_{s−1},p},$$ (2.2)

and by so-called extra fields that do not contribute to the free action and field equations but control higher–spin gauge symmetries and play a role at the interaction level [35, 11]. In (2.1) and (2.2) the indices $n_1...n_{s−1}$ are symmetrized, and $\omega^{n_1...n_{s−1},p}$ has the symmetry properties of the Young tableau $Y(s − 1, 1)$, i.e. its totally symmetric part in tangent–space indices

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1 In flat space–time we shall not distinguish between the world and tangent–space indices. Both kinds of indices will be denoted by lower case Latin letters. World indices will be separated from the tangent–space ones by ‘;’.

2 Sets of symmetric tangent–space indices are separated by comma. $Y(s − 1, 1)$ means that the Young tableau has $s − 1$ cells in the first row and one cell in the second row.
vanishes
\[
\omega^{(n_1 \ldots n_{s-1} p)} := \frac{1}{s} (\omega^{n_1 \ldots n_{s-1} p} + \omega^{p \ldots n_{s-1} n_1} + s - 2 \text{ terms}) = 0. \tag{2.3}
\]

The brackets ( ) and [ ] denote, respectively, the symmetrization and anti–symmetrization of indices with the unit weight.

The connection \(\omega^{n_1 \ldots n_{s-1} p}\) is an auxiliary field provided that, like in the case of the Einstein gravity, we impose the zero torsion condition
\[
T^{n_1 \ldots n_{s-1}} \equiv d e^{n_1 \ldots n_{s-1}} - (s - 1) dx^q \omega^{n_1 \ldots n_{s-1} p} \eta_{pq} = 0. \tag{2.4}
\]

The dynamical degrees of freedom of a massless higher–spin field are contained in the higher–spin vielbein (2.1) which also describes pure gauge degrees of freedom because of the presence of the higher–spin gauge symmetries. In particular, the torsion (2.4) is invariant under the following gauge transformations of the vielbein and connection
\[
\delta e^{n_1 \ldots n_{s-1}} = d\xi^{n_1 \ldots n_{s-1}} - (s - 1) dx^q \xi^{n_1 \ldots n_{s-1} p} \eta_{pq}, \tag{2.5}
\]
\[
\delta \omega^{n_1 \ldots n_{s-1} p} = d\xi^{n_1 \ldots n_{s-1} p} - (s - 2) dx^q \xi^{n_1 \ldots n_{s-1} p q r} \eta_{q r}. \tag{2.6}
\]

The gauge parameters \(\xi^{n_1 \ldots n_{s-1}}, \xi^{n_1 \ldots n_{s-1} p}\) and \(\xi^{n_1 \ldots n_{s-1} p q r}\) are symmetric in each group of indices \(n\) and \(p\). In addition, \(\xi^{n_1 \ldots n_{s-1} p}\) and \(\xi^{n_1 \ldots n_{s-1} p q r}\) have the symmetry properties of the Young tableaux \(Y(s - 1, 1)\) and \(Y(s - 1, 2)\), respectively, which means, like in (2.3), that the symmetrization of any \(s\) indices gives zero. Note that the gauge symmetry parameter \(\xi^{n_1 \ldots n_{s-1} p q r}\) is associated with the first extra field connection \(\omega^{n_1 \ldots n_{s-1} p q r}\) that has analogous symmetry properties in the tangent indices.

Note that so far we have not imposed traceless conditions either on the higher–spin vielbein and connection or on the gauge parameters.

We would like to derive the zero torsion condition (2.4) from an action together with dynamical field equations on the physical components of \(e^{n_1 \ldots n_{s-1}}\). We construct such an action by analogy with the frame formulation of the action for (linearized) gravity.

The free higher–spin action has the following simple first–order form
\[
S = \int_{M^D} d \varepsilon_{a_1 \ldots a_{D-3}} dx^{a_{D-3} p q r} (d e^{n_1 \ldots n_{s-2} p} - \frac{s - 1}{2} dx^m \omega^{n_1 \ldots n_{s-2} p m} \omega^{n_1 \ldots n_{s-2} q r}) \omega^{n_1 \ldots n_{s-2} q r}. \tag{2.7}
\]

This action is a straightforward generalization of the 4d action of [6]. It has the important property that its part bilinear in the higher–spin connection \(\omega^{n_1 \ldots n_{s-1} m}\) is symmetric under the exchange of the product factors, \(i.e.,\)
\[
d \varepsilon_{a_1 \ldots a_{D-3}} dx^m \varepsilon_{a_{D-3} p q r} (\omega^{n_1 \ldots n_{s-2} p m} \omega^{n_1 \ldots n_{s-2} q r} - \omega^{n_1 \ldots n_{s-2} p m} \omega^{n_1 \ldots n_{s-2} q r}) = 0, \tag{2.8}
\]

provided that \(\omega\) is Young projected as in eq. (2.3) and, in addition, is subject to the trace constraint
\[
\eta_{n_1 m} \omega^{n_1 \ldots n_{s-1} m} = 0. \tag{2.9}
\]

Indeed, this can be easily seen by using the identity
\[
\varepsilon_{a_1 \ldots a_{D-3}} e^{a_{D-3} bcd} e^{a_1 \ldots e^{a_{D-2}} = \frac{3}{D - 2} \delta^f_b \varepsilon_{a_1 \ldots a_{D-2}} e^{a_1 \ldots e^{a_{D-2}}}, \tag{2.10}
\]
where in the case under consideration the background vielbein $e^a$ is flat, i.e. $e^a = dx^a$ in Cartesian coordinates. The identity (2.10) is a particular case of the generic identity

$$\varepsilon_{a_1\ldots a_{D-p}b_1\ldots b_p} e^{a_1} \ldots e^{a_{D-p}} e^f = \frac{(-1)^{(p-1)(D-p+1)}}{D-p+1} p \delta^f_{[b_1} \varepsilon_{b_2\ldots b_p]} a_1\ldots a_{D-p+1} e^{a_1} \ldots e^{a_{D-p+1}},$$

which (for different $p$) have been used when checking the gauge invariance of the actions considered in this paper.

Thus, the part of (2.7) bilinear in $\omega$ consists of three terms, that contain $\eta_{mpq}$, $\eta_{mq}$ and $\eta_{mr}$, respectively. The first term vanishes by the trace condition (2.9), while the other two are symmetric, either manifestly or on account of the Young projection property (2.3).

Note that a consequence of (2.9) is that the trace of $\eta$ is pure gauge in view of the gauge transformations (2.5). Hence, the action (2.7) does not describe fields of spin one.

Let us stress that the trace $\eta_{n_1n_2} \omega^{n_1n_2n_3\ldots n_{s-1},m}$ is non-zero. Therefore, the condition (2.9) is weaker than the conventional trace constraint of the frame–like formulation of a single higher–spin field which corresponds to Fronsdal theory [1] and requires all traces in tangent indices to be zero. We shall call eq. (2.9) the relaxed traceless condition.

Note that the vielbein $e^{n_1\ldots n_{s-1}}$ remains traceful. Here we should point out, however, that in the case of the odd integer spins $s = 2k+1$ the fully trace part of $e^{n_1\ldots n_{s-1}}$, i.e. $e^{n_1\ldots n_{2k}} \eta_{n_1n_2} \ldots \eta_{n_{2k-1}n_{2k}}$, does not contribute to the action (2.7) because of its differential form structure. Technically, the reason for this is that the one–form associated with the spin–1 field does not have external (tangent–space) indices required for the construction of the action as an integral of a differential form. Thus, the action (2.7) does not describe fields of spin one.

In the case of the even integer spins $s = 2k$, the total trace component of the higher–spin vielbein $e^{n_1\ldots n_{2k+1}} \eta_{n_1n_2} \ldots \eta_{n_{2k+1}n_{2k+1}}$ describes the conformal mode of the spin two field, which is pure gauge in view of the gauge transformations (2.5). Hence, the action (2.7) does not describe scalar fields either.

To include the spin 0 and spin 1 fields into the system one should add to the action (2.7) the corresponding Klein–Gordon and Maxwell terms. This can be achieved by adding the spin–one and spin–zero kinetic terms formulated in terms of the so-called Weyl zero-forms as discussed in [34] for the scalar case and in [35] for the spin–one case.

By virtue of (2.8) and (2.9), the general local variation of the action (2.7) can be presented in the following two forms, which are equivalent up to total derivatives,

$$\delta S = \int_{M^D} dx^{a_1} \ldots dx^{a_{D-3}} \varepsilon_{a_1\ldots a_{D-3}}^{pqr} \delta T^{n_1\ldots n_{s-2}p} \omega^{n_1\ldots n_{s-2}q,r}$$

$$= \int_{M^D} dx^{a_1} \ldots dx^{a_{D-3}} \varepsilon_{a_1\ldots a_{D-3}}^{pqr} (T^{n_1\ldots n_{s-2}p} \delta \omega^{n_1\ldots n_{s-2}q,r} - \delta e^{n_1\ldots n_{s-2}p} d\omega^{n_1\ldots n_{s-2}q,r}),$$

where the torsion $T^{n_1\ldots n_{s-1}}$ is defined in the left hand side of (2.4).

The first form of the variation is convenient for the identification of the gauge symmetry of the action. We notice that whereas the torsion $T$ is invariant under arbitrary unrestricted gauge
transformation, the relaxed traceless condition (2.9) requires gauge symmetry parameters to obey the relaxed trace constraints as well

\[ \eta_{n_1 m} \xi^{n_1 \ldots n_{s - 1}, m} = 0, \quad \eta_{n_1 m} \xi^{n_1 \ldots n_{s - 1}, m^l} = 0. \]  

(2.14)

The second line of (2.13) yields the field equations for \( \omega \) and \( e \), which follow from

\[ \delta^n_b \delta^r_d \partial_{[m} T_{n_{s - 2}^{n_1 \ldots n_{s - 2}}} b \omega_{r; n_1 \ldots n_{s - 2} c, d} = 0, \]  

(2.15)

\[ \delta^n_b \delta^r_d \delta e_{m_{n_1 \ldots n_{s - 2}}} b \partial_{n_1 \omega_{r; n_1 \ldots n_{s - 2} c, d}} = 0. \]  

(2.16)

As we explain in Appendix, the equation (2.15) is equivalent to the zero–torsion condition (2.4), modulo its full trace in the tangent space indices in the case of spin \( s = 2k + 1 \). There is no condition on the full trace of the torsion, since, as we have explained above, the full trace of the higher–spin vielbein does not contribute to the action.

The zero torsion condition

\[ (s - 1) \omega_{[n; 1 \ldots n_{s - 1}, b} \eta_{m] b} = \partial_{[m} e_{n]; n_1 \ldots n_{s - 1} } \]  

(2.17)

expresses the higher–spin connection in terms of the first derivatives of the higher–spin vielbein up to the Stueckelberg gauge transformations (2.6). In the case of odd \( s = 2k + 1 \), \( e_{m_{n_1 \ldots n_{s - 1}}} \) in (2.17) stands for the part of the vielbein whose total trace is zero \( e_{m_{n_1 \ldots n_{2k}}} \eta_{a_1 a_2} \cdots \eta_{a_{2k - 1} a_{2k}} = 0 \). In view of the relation (2.17) the equations which follow from eq. (2.16), namely

\[ \delta^n_b \partial^c \omega_{d_{n_1 \ldots n_{s - 2}}[c, d]} + \partial_d \omega_{[b_{n_1 \ldots n_{s - 2}}} [m, d] + \partial_{(b \omega_{d_{n_1 \ldots n_{s - 2}})]d, m]} = 0 \]  

(2.18)

are the dynamical (second–order) equations of motion of the higher–spin vielbein field.

Let us analyze the field content of the model.

2.1 Fronsdal case

Let us first consider the case of an irreducible massless field. Following [6] we impose on the higher–spin vielbein and connection the strongest trace constraints

\[ \eta_{n_1 n_2} \tilde{e}^{n_1 \ldots n_{s - 1}} = 0, \quad \eta_{n_1 n_2} \tilde{\omega}^{n_1 \ldots n_{s - 1}, m} = 0, \]  

(2.19)

where we use \( \tilde{e} \) and \( \tilde{\omega} \) for the traceless fields to distinguish them from the relaxed \( e \) and \( \omega \). Note that the condition (2.9) follows from (2.3) and (2.19), but not vice versa.

The parameters of the gauge transformations (2.5) and (2.6) of the traceless \( \tilde{e} \) and \( \tilde{\omega} \) are also traceless

\[ \eta_{n_1 n_2} \tilde{\xi}^{n_1 \ldots n_{s - 1}} = 0, \quad \eta_{n_1 n_2} \tilde{\xi}^{n_1 \ldots n_{s - 1}, m} = 0, \quad \eta_{n_1 n_2} \tilde{\xi}^{n_1 \ldots n_{s - 1}, m^l} = 0. \]  

(2.20)

Using the gauge transformations (2.5) and (2.6) with the parameters \( \tilde{\xi} \) one can gauge fix to zero the respective “antisymmetric” parts of the components of the vielbein \( \tilde{e} \) and of the connection \( \tilde{\omega} \). Then, taking into account the zero torsion condition (2.17), we see that \( \omega \) is the auxiliary field and all the physical degrees of freedom are contained in the symmetric part of the vielbein

\[ s \tilde{e}_{(n_1, n_2 \ldots n_{s - 1})} := \tilde{\phi}_{n_1 \ldots n_{s - 1} }. \]  

(2.21)
which is double traceless because the vielbein $\tilde{e}_{n_1\cdots n_{s-1}}$ is traceless in the indices $n_1 \cdots n_{s-1}$.

The remaining local symmetry is then just that of the Fronsdal metric–like formulation of a single symmetric bosonic higher–spin field in flat space–time [1] with the completely symmetric traceless parameter $\tilde{\xi}_{n_1\cdots n_{s-1}}$.

If we now substitute the connection $\tilde{\omega}$ with its expression (2.17) in terms of the symmetric and double traceless field (2.21) into the action (2.7), the resulting action will be quadratic in the derivatives of $\tilde{\phi}_{n_1\cdots n_s}$ and will be invariant under the local transformations

$$\delta \tilde{\phi}_{n_1\cdots n_s} = s \partial_{(n_1} \tilde{\xi}_{n_2\cdots n_{s-1})}$$

(2.22)

with the traceless gauge parameters $\tilde{\xi}_{n_1\cdots n_{s-1}}$. In [57] it was shown that, up to a normalization, any such action is equivalent to the Fronsdal action for a spin $s$ massless gauge field

$$S = \int d^Dx \left( \frac{1}{2} \tilde{\phi}_{m_1\cdots m_s} \mathcal{F}_{m_1\cdots m_s} - \frac{1}{8} s(s-1) \tilde{\phi}_{n}^{nm_{m_3\cdots m_s}} \mathcal{F}_{pm_3\cdots m_s}^p \right)$$

(2.23)

where

$$\mathcal{F}_{m_1\cdots m_s}(x) \equiv \partial^2 \tilde{\phi}_{m_1\cdots m_s} - s \partial_{(m_1} \partial^{n} \tilde{\phi}_{m_2\cdots m_s)n} + \frac{s(s-1)}{2} \partial_{(m_1} \partial_{m_2} \tilde{\phi}_{m_3\cdots m_s)n}$$

(2.24)

is the so-called Fronsdal operator.

2.2 Triplet case

Let us now analyze the field content of the model described by the action (2.7) with the tracefree higher–spin vielbein and with the higher–spin connection subject to the relaxed traceless condition (2.9). By representing the vielbein as a sum of traceless (lower rank) symmetric tensors, one can see that the action (2.7) is actually the sum of the actions for the traceless vielbeins $\tilde{e}_{a_1\cdots a_{t-1}}$ and connections $\tilde{\omega}^{a_1\cdots a_{t-1}b}$ with $t$ taking even or odd values ($t = 2, 4, \cdots, s$ or $t = 3, 5, \cdots, s$) depending on whether $s$ is even or odd,

$$S = \sum_{k=1}^{\lfloor \frac{s}{2} \rfloor} \alpha(t, D) \int_{MD} dx^{a_1} \cdots dx^{a_{D-3}} \varepsilon_{a_1\cdots a_{D-3}pqrr} (d \tilde{e}^{n_1\cdots n_{t-2}p} - \frac{t-1}{2} dx^m \tilde{\omega}^{n_1\cdots n_{t-2}p;m}) \tilde{\omega}_{n_1\cdots n_{t-2}q;r},$$

(2.25)

where $t = 2k$ or $t = 2k + 1$, $\lfloor \frac{s}{2} \rfloor$ denotes the integral part of $\frac{s}{2}$ when $s$ is odd, $\alpha(t, D)$ are constants which depend on space–time dimension $D$ and the rank $t$ (spin) of the tensor fields. Thus, the sum in (2.7) is taken over even $t = 2, 4, \cdots, s - 2$, $s$ or odd $t = 3, 5, \cdots, s - 2$, $s$ depending whether $s$ is even or odd. Each of the terms of the sum (2.25) with a given $t$ is gauge invariant under the transformations analogous to eqs. (2.5), (2.6) but with the traceless parameters (2.20).

As explained in Subsection 2.1, for a given $t$ each term of (2.25) describes a single free massless field of spin $t$. Thus the action (2.25), and hence (2.7), describes the family of massless fields of even integer spins $t = 2, 4, \cdots, s - 2$, $s$ and of odd integer spins $t = 3, 5, \cdots, s - 2$, $s$.

These field contents are similar to the field contents of the higher–spin triplets [20, 21, 23, 24] (except for the presence in the latter of the fields of the lowest spins 0 and 1). Let us now establish the relationship between the triplet fields and the components of the higher–spin vielbein $e$ and connection $\omega$, thus clarifying the geometrical meaning of the former.
Recall that the higher–spin triplet is described by the following three symmetric tracefull tensor fields of rank $s$, $s - 1$ and $s - 2$

\[ \Phi_{n_1 \ldots n_s}, \quad C_{n_1 \ldots n_{s-1}}, \quad D_{n_1 \ldots n_{s-2}}. \]

On the mass shell these fields satisfy the following equations

\[ C_{n_1 \ldots n_{s-1}} = \partial_m \Phi^m_{n_1 \ldots n_{s-1}} - (s - 1) \partial_{(n_{s-1}} D_{n_1 \ldots n_{s-2})}, \]
\[ \Box \Phi_{n_1 \ldots n_s} = s \partial_{(n_s} C_{n_1 \ldots n_{s-1})}, \quad \Box := \partial_m \partial^m, \]
\[ \Box D_{n_1 \ldots n_{s-2}} = \partial_m C^m_{n_1 \ldots n_{s-2}}. \]

Eqs. (2.26)–(2.28) are invariant under the gauge transformations

\[ \delta \Phi_{n_1 \ldots n_s} = s \partial_{(n_s} \xi_{n_1 \ldots n_{s-1})}, \]
\[ \delta C_{n_1 \ldots n_{s-1}} = \Box \xi_{n_1 \ldots n_{s-1}} \]
\[ \delta D_{n_1 \ldots n_{s-2}} = \partial_m \xi^m_{n_1 \ldots n_{s-2}} \]

where the unconstrained parameter $\xi_{n_1 \ldots n_{s-1}}$ is completely symmetric.

Let us now compare the gauge transformations (2.29)–(2.31) with the gauge transformations (2.5) and (2.6) of the higher–spin vielbein and connection. Using the transformation (2.5) with the parameter $\xi^{n_1 \ldots n_{s-1}, p}$ one can gauge away the part of the vielbein $e_{(n_1 \ldots n_{s-1}}$ that corresponds to the hook Young tableau of $\xi^{n_1 \ldots n_{s-1}, p}$, i.e. the part that satisfies the ‘antisymmetry’ condition $e_{(p; n_1 \ldots n_{s-1})} = 0$ and is subject to the relaxed trace constraint similar to (2.14). Upon imposing this gauge fixing condition the vielbein splits into two completely symmetric tensors of rank $s$ and $s - 2$

\[ e_{(n_s; n_1 \ldots n_{s-1})} \quad \text{and} \quad \frac{s - 2}{s} \eta^{n_{s-1}p} (e_{p; n_1 \ldots n_{s-2}n_{s-1}} - e_{(n_1; n_2 \ldots n_{s-2})n_{s-1}}). \]

Under the gauge symmetry (2.5), (2.14) $e_{(n_s; n_1 \ldots n_{s-1})}$ and $e_{p; n_1 \ldots n_{s-2}n_{s-1}} \eta^{n_{s-1}p}$ transform in the following way

\[ \delta e_{(n_s; n_1 \ldots n_{s-1})} = \partial_{(n_s} \xi_{n_1 \ldots n_{s-1})}, \quad \delta e_{p; n_1 \ldots n_{s-2}n_{s-1}} \eta^{n_{s-1}p} = \partial_m \xi^m_{n_1 \ldots n_{s-2}}. \]

The comparison of eq. (2.33) with (2.29) and (2.31) suggests that the fields $\Phi$ and $D$ of the triplet are just the symmetric components of the higher–spin vielbein

\[ \Phi_{n_1 \ldots n_s} = s e_{(n_s; n_1 \ldots n_{s-1})}, \quad D_{n_1 \ldots n_{s-2}} = e_{p; n_1 \ldots n_{s-2}n_{s-1}} \eta^{n_{s-1}p}. \]

It remains only to identify the field $C$. To this end let us have a look at the zero torsion condition (2.17). In (2.17) we first symmetrize the index $n$ with $n_1, \ldots, n_{s-1}$ and then take the trace of $n$ with $m$. In view of eqs. (2.3) and (2.9) we thus get

\[ (s - 1) \omega^m_{m; n_1 \ldots n_{s-1}}, \]

where $\Phi$ and $D$ are defined in (2.34). Comparing (2.35) with (2.26) we see that the triplet field $C$ is actually composed of the trace of the higher–spin connection and the divergence of the higher–spin vielbein

\[ C_{n_1 \ldots n_{s-1}} = (s - 1) \omega^m_{m; n_1 \ldots n_{s-1}}, \]

\[ + \partial^m e_{m; n_1 \ldots n_{s-1}}. \]
We have thus identified the fields of the higher-spin triplet as components of the higher-spin vielbein and connection of the frame-like formulation with the relaxed trace constraints.

The comment on the lowest spin fields (i.e. the scalar and the vector) is now in order. As we have already mentioned, these fields are not contained in the frame-like action (2.7). In the case of the even integer spin \( s = 2k \) the complete trace component of the vielbein, which could be the scalar field, is in fact the pure gauge conformal component of the spin two field in the system. Indeed, as we explained, upon the partial gauge fixing of the local transformations (2.5) the higher-spin vielbein splits into two completely symmetric tensors (2.32). The maximal trace in all the indices of the rank \( s - 2 \) tensor in (2.32), which might be an independent scalar field, is identically zero. Hence the scalar field component of the vielbein is contained only in its rank-\( s \) symmetric part (i.e. it is the full trace of \( e_{(n_k; n_1 \ldots n_{s-1})} \)) and can be gauged away by a corresponding residual local transformation (2.33). This is similar to the case of gravity where the trace of the vielbein (or the metric) is a pure gauge scalar component.

In the case of the odd integer spin \( s = 2k + 1 \), as we have explained earlier, the spin 1 part of the vielbein does not enter the action (2.7).

As a result, the zero torsion condition (2.17) and its consequence (2.35), which defines the field \( C \) (2.36), are applicable only to the components of the vielbein whose complete trace in the tangent space indices \( n_1, \ldots, n_{s-1} \) is zero (i.e. do not contain the spin 1 field).

To include the scalar and the vector field into the above scheme one should add to the action (2.7) corresponding kinetic terms. As we have mentioned, a systematic way to do this is to use the zero-forms in the so-called twisted adjoint representation of the higher-spin algebra. (see e.g. [34] for the spin zero case).

To conclude this section we show that the zero torsion condition (2.17) and the dynamical field equations (2.18) indeed imply the equations of motion (2.26)–(2.28) of the triplet higher-spin fields defined by eqs. (2.34)–(2.36). To perform this consistency check, the following relations between the derivatives of connection components and the triplet fields (2.34) and (2.36) are useful

\[
2\partial_t \omega_{m;n_1 \ldots n_{s-2}}^{[l,m]} = -\frac{1}{s-1} (\Box D_{n_1 \ldots n_{s-2}} - \partial^m C_{mn_1 \ldots n_{s-2}}), \quad (2.37)
\]

\[
\partial_m \omega_{(n_1;n_2 \ldots n_s)},^m - \omega_{(n_1 \omega^m; n_2 \ldots n_s),m} = \frac{1}{s(s-1)} (\Box \Phi_{n_1 \ldots n_s} - s \partial_{(n_1} C_{n_2 \ldots n_s)}), \quad (2.38)
\]

\[
\partial_t \omega_{(n_1;n_2 \ldots n_{s-2})m}^{m,l} - \omega_{(n_1 \omega^m; n_2 \ldots n_{s-2})l,m} = \frac{1}{(s-1)(s-2)} \eta^{n_{s-1} n_s} (\Box \Phi_{n_1 \ldots n_s} - s \partial_{(n_1} C_{n_2 \ldots n_s)} - \partial^{n_{s-2}} C_{n_1 \ldots n_{s-1}}), \quad (2.39)
\]

Note that the right hand sides of (2.37)–(2.39) are proportional to the left hand sides of the triplet field equations (2.27) and (2.28).

Let us now take the trace of eq. (2.18) multiplying it with \( \delta^b_m \)

\[
\delta^b_m \left( \delta^m_n \partial^c \omega_{n_1 \ldots n_{s-2}}^{[c,d]} + \partial_d \omega_{(b;n_1 \ldots n_{s-2})}^{[m,d]} + \partial_{[b} \omega_{d;n_1 \ldots n_{s-2}]}^{[m,m]} \right)
\]

10
\[ \frac{2(D+s-4)}{s-1} \partial_c \omega_{d;n_1\ldots{n_{s-2}}}[c,d] + \frac{s-2}{s-1} (\partial_d \omega_{(n_1;n_2\ldots{n_{s-2}})c,d} - \partial_{(n_1 \omega_{n_2\ldots{n_{s-2}}})}c,d) = 0. \] 

In view of (2.37)–(2.39), eq. (2.40) takes the form
\[ (D + s - 2) \left( \Box D_{n_1\ldots{n_{s-2}}} - \partial^n C_{mn_1\ldots{n_{s-2}}} \right) = \eta^{n_{s-1}n_s} \left( \Box \Phi_{n_1\ldots{n_s}} - s \partial_{(n_1 C_{n_2\ldots{n_s})}} \right). \] 

Now let us in (2.18) symmetrize the index \( m \) with the indices \( b, n_1, \ldots, n_{s-2} \). The result is
\[ 2\eta_{(mb} \partial_c \omega^{d; n_1\ldots{n_{s-2}})[c,d] + \frac{s}{s-1} \left( \partial_d \omega_{(b;n_1\ldots{n_{s-2}}m),d} - \partial_{(b \omega^{d; n_1\ldots{n_{s-2}}m},d) \right) = 0. \] 

To arrive at eq. (2.42) we used the relation
\[ (s - 1) \omega_{(b;n_1\ldots{n_{s-2}}d,m) = -\omega_{(b;n_1\ldots{n_{s-2}}m)},d \] 

which is a consequence of the symmetry property (2.3) of the connection. By virtue of the relations (2.37) and (2.38), eq. (2.42) takes the form
\[ \eta_{(n_1n_2} \left( \Box D_{n_3\ldots{n_s}} - \partial^n C_{n_3\ldots{n_s})m} \right) = \frac{1}{(s - 1)} \left( \Box \Phi_{n_1\ldots{n_s}} - s \partial_{(n_1 C_{n_2\ldots{n_s})}} \right). \] 

Comparing eqs. (2.41) and (2.44) we conclude that their left and right hand sides must vanish separately thus producing the triplet field equations (2.27) and (2.28).

To recapitulate, we have shown that, up to a subtlety regarding the spin-0 and spin-1 field, the higher-spin system described by the frame-like action (2.7) for the unconstrained vielbein and the connection subject to the relaxed trace constraint (2.9) is equivalent to the higher-spin triplet. The triplet fields \( \Phi, C \) and \( D \) have been thus endowed with a geometrical meaning to be certain components of the higher-spin vielbein and connection. By singling out these components in the action (2.7) and partially solving the zero-torsion condition (2.17) one should be able to reduce action (2.7) to the triplet actions of [24].

We shall now extend the results of this section to the AdS background.

### 3 Frame-like action for bosonic higher-spin fields in AdS\(_D\)

The AdS space is described by the vielbein \( e^a = dx^m e^a_m \) and the connection \( \omega^{ab} = dx^m \omega^{ab}_m \) which satisfy the following torsion and constant curvature conditions
\[
T^a := de^a + \omega^a_b e^b := \nabla e^a = 0, \tag{3.1}
\]
\[
R^{ab}(\omega) := d\omega^{ab} + \omega^a_c \omega^{cb} = -\Lambda e^a e^b, \tag{3.2}
\]
where \( \nabla = d + \omega \) is the \( O(1, D - 1) \) covariant differential and \( \Lambda \) is the negative (‘cosmological’) constant determining the AdS curvature. The indices from the beginning of the Latin alphabet now denote the tangent space indices rotated by the local \( O(1, D - 1) \) Lorentz transformations. The indices \( m, n, \ldots \) from the middle of the alphabet denote curved world indices.
The frame–like action for a system of higher–spin fields which generalizes to AdS the flat space action (2.7) has the following form

\[
S = \int_{\text{AdS}} e^{a_1} \ldots e^{a_{D-3}} \varepsilon_{a_1 \ldots a_{D-3} cd} \left[ (\nabla e^{b_1 \ldots b_{s-2} c} - \frac{s-1}{2} e^{b_1 \ldots b_{s-2} c, k}) \omega_{b_1 \ldots b_{s-2} d, f} + \Lambda \frac{s(D+s-4)}{2(s-1)(D-2)} e^{b_1 \ldots b_{s-2}} e^{d b_1 \ldots b_{s-2}} - \Lambda \frac{(s-2)(s-3)}{2(D-2)(s-1)} e^{cb_1 \ldots b_{s-4}} e^{d j b_1 \ldots b_{s-4}} e^i c f \right].
\]

(3.3)

We observe that, apart from covariantized derivatives, the action (3.3) differs from the flat space action (2.7) by the last two mass–like terms proportional to the AdS curvature. Note that the last term in (2.7) contains the trace of the higher–spin vielbein. As is well known, in AdS space such terms are required to keep a number of physical states of the higher–spin field equal to that of the massless field, i.e. to preserve gauge symmetries. Hence, the coefficients in front of these terms are fixed by the requirement of the invariance of this action under the gauge transformations of the higher–spin vielbein and connection whose form we shall discuss in the next two subsections.

Here we only note that, as in the flat case, the higher–spin vielbein is unconstrained (modulo the subtlety that it does not contain the spin–1 field, as was discussed in detail in Section 2), while the variation of the action (3.3) with respect to the higher–spin connection produces the zero torsion condition

\[
T^{a_1 \ldots a_{s-1}} = 0 \iff (s-1)\omega_{n; a_1 \ldots a_{s-1}, b} e_{m|b} = \nabla_{[m} e_{n]} a_1 \ldots a_{s-1},
\]

(3.4)

provided that the higher–spin connection obeys the relaxed traceless condition

\[
\eta_{a_1 b} \omega^{a_1 a_2 \ldots a_{s-1}, b} = 0.
\]

(3.5)

The dynamical field equation of the higher–spin vielbein in AdS gets modified by the contribution of the terms proportional to the AdS curvature \(\Lambda\) and acquires the form

\[
\left( \nabla_{n} \omega_{r;(a_1 \ldots a_{s-2}} c, d} - \Lambda \frac{s(D+s-1)}{(s-1)(D-2)} e^{d}_{n} e_{r;(a_1 \ldots a_{s-2}) c} + \Lambda \frac{(s-2)(s-3)}{2(D-2)(s-1)} e^{d}_{n} e_{r;(a_3 \ldots a_{s-2}) c} \eta_{a_1 a_2} \right) e^{[m}_{b} e^{n}_{c} e^{r]} = 0.
\]

(3.6)

### 3.1 Fronsdal case

The frame–like action for irreducible massless fields in \(AdS_D\) originally proposed in [9] was, in fact, the first action for symmetric massless fields in \(D > 4\).\(^3\) The action constructed in [9] is manifestly gauge invariant due to the use of higher connections called extra fields, which however do not contribute to the free field equations. A version of this approach which is manifestly \(o(2, D - 1)\) (rather than \(o(1, D - 1)\)) invariant was later proposed in [11]. In Section 7 we shall demonstrate how this approach works in the case of the relaxed higher–spin multiplets being the main subject of this paper. In this section we shall use an alternative

\(^3\)The metric–like formulation of Fronsdal was originally proposed in [1, 3] for the case of \(D = 4\). It turns out that the coefficients in front of different terms of the action are independent of \(D\) in Minkowski space but those of mass–like terms are \(D\)-dependent in \(AdS_D\).
approach in which the gauge symmetry is not manifest but the analysis is simpler since the set of gauge fields is free from the extra fields.

Imposing the conventional traceless conditions on the higher–spin vielbein and connection (as above we distinguish between the traceless and traceful quantities by putting tildes on the former)

\[ \eta_{a_1 a_2} e^{a_1 a_2 \ldots a_{s-1}} = 0, \quad \eta_{a_1 a_2} \bar{\omega}^{a_1 a_2 \ldots a_{s-1}, b} = 0, \]  

(3.7)

the action (3.3) reduces to

\[ S = \int_{\text{AdS}} e^{a_1} \cdots e^{a_{D-3}} \xi_{a_1 \ldots a_{D-3} c d f} \left[ (\nabla \bar{\epsilon}^{b_1 \ldots b_{s-2} c} - \frac{s-1}{2} \epsilon_k \bar{\omega}^{b_1 \ldots b_{s-2} c, k}) \bar{\omega}^{b_1 \ldots b_{s-2} d, f} + \Lambda \frac{s(D+s-4)}{2(s-1)(D-2)} \bar{\epsilon}^{b_1 \ldots b_{s-2} d} \bar{\epsilon}^{b_{s-2} e} \right]. \]  

(3.8)

It is invariant under the following gauge transformations of the higher–spin vielbein and connection

\[ \delta \bar{\epsilon}^{a_1 \ldots a_{s-1}} = \nabla_\xi \bar{\epsilon}^{a_1 \ldots a_{s-1}} - (s-1) \epsilon^c \bar{\epsilon}^{a_1 \ldots a_{s-1}, b} \eta_{b c}, \]  

(3.9)

\[ \delta \bar{\omega}^{a_1 \ldots a_{s-1}, b} = \nabla_\xi \bar{\omega}^{a_1 \ldots a_{s-1}, b} - (s-2) \epsilon^c \bar{\omega}^{a_1 \ldots a_{s-1}, b d} \eta_{c d} - \Lambda \left( \epsilon^b \bar{\omega}^{a_1 \ldots a_{s-1}} - \epsilon^{a_1} \bar{\xi}^{a_2 \ldots a_{s-1}, b} \right), \]  

(3.10)

where the parameters \( \bar{\xi}^{a_1 \ldots a_{s-1}} \) and \( \bar{\xi}^{a_1 \ldots a_{s-1}, b} \) are traceless and the parameter \( \bar{\xi}^{a_1 \ldots a_{s-1}, b d} \) satisfies the following trace conditions \(^4\)

\[ \eta_{a_1 a_2} \bar{\xi}^{a_1 a_2 \ldots a_{s-1}, b_{1} b_{2}} = \frac{2\Lambda}{(s-1)(s-2)} \bar{\xi}^{a_{s-1} \ldots a_{s-1}, b_{1} b_{2}}, \]  

(3.11)

\[ \eta_{a_1 b_{1}} \bar{\xi}^{a_1 a_2 \ldots a_{s-1}, b_{1} b_{2}} = -\frac{\Lambda}{s-1} \bar{\xi}^{a_{s-1} \ldots a_{s-1}, b_{1} b_{2}}. \]

Note that the second relation is a consequence of the first one by virtue of the Young symmetry properties of \( \bar{\xi}^{a_1 \ldots a_{s-1}, b d} \).

In the flat limit \( \Lambda \to 0 \), eqs. (3.9)–(3.11) reduce to the corresponding gauge transformations discussed in Subsection 2.1.

Action (3.8) describes in AdS space the dynamics of a single massless field of spin \( s \). Upon solving \( \bar{\omega}^{a_1 \ldots a_{s-1}, b} \) in terms of \( \bar{\epsilon}^{a_1 \ldots a_{s-1}} \) and partially fixing local higher–spin symmetry (3.9), (3.10) the action (3.8) gives the generalization to any dimension of the \( \text{AdS}_4 \) Fronsdal action [2] for the double traceless field \( \phi^{a_1 \ldots a_s} := e^{m(a_1} \bar{\epsilon}_{m; a_2 \ldots a_s)}. \)

### 3.2 Triplet case

If the full traceless condition is not imposed, the higher–spin connection only satisfies the relaxed trace condition (3.5) while the higher–spin vielbein remains unconstrained. Then the action (3.3) describes in AdS space a system of free massless fields of descending spins \( s-2, s-4, \ldots, 3 \) or 2 depending on whether \( s \) is odd or even. The analysis and the proof is the same as

\(^4\)The transformations (3.10) and the trace conditions (3.11) can be obtained from the \( O(2, D-1) \)–covariant expressions (7.12)–(7.15) of Section 7 in the standard gauge (7.5).
in the flat case (see Subsection 2.2). The only difference is that the gauge transformations of the higher–spin vielbein and connection which leave the action (3.3) invariant and which have the same form as eqs. (3.9) and (3.10) contain the unconstrained parameter \( \xi^{a_1 \ldots a_{s-1}, b} \) satisfies the relaxed traceless condition
\[
\xi^{a_1 \ldots a_{s-1}, b} \eta_{a_1 b} = 0 ,
\] (3.12)
while the parameter \( \xi^{a_1 \ldots a_{s-1}, bd} \) is subject to the following relaxed constraint
\[
(s - 1) \eta_{bc} \xi^{a_1 \ldots a_{s-1} b, cd} = \Lambda \left( \eta^{d(a_1 \xi^{a_2 \ldots a_{s-2} b}_{b})} - \xi^{a_1 \ldots a_{s-2} d} \right) ,
\] (3.13)
instead of being traceless as in the flat space case (see eq. (2.14)) or ‘partially’ traceless in the Fronsdal AdS case (see eq. (3.11)). We shall also use the following consequences of (3.13) and of the Young symmetry \( Y (s - 1, 2) \) of \( \xi^{a_1 \ldots a_{s-1}, bc} \)
\[
\eta_{bc} \xi^{a_1 \ldots a_{s-1}, bc} = \Lambda \left( \xi^{a_1 \ldots a_{s-1} - \eta(a_1 a_2 - \xi^{a_3 \ldots a_{s-1} b}_{b}) \right) ,
\] (3.14)
\[
\eta_{bc} \xi^{a_1 \ldots a_{s-3}, a_{s-2} d} = \frac{2 \Lambda}{(s - 1)(s - 2)} \left( \xi^{a_1 \ldots a_{s-2} d} - \eta^{d(a_1 \xi^{a_2 \ldots a_{s-2} b}_{b})} \right) .
\] (3.15)

Let us now identify the AdS higher–spin triplet in terms of components of the higher–spin vielbein and connection. In the AdS space the bosonic higher–spin triplet is defined (in our notation and convention) by the following equations [23, 24]
\[
C_{n_1 \ldots n_{s-1}} = \nabla_m \Phi^m_{n_1 \ldots n_{s-1}} - (s - 1) \nabla_{n_{s-1}} D_{n_1 \ldots n_{s-2}} ,
\] (3.16)

\[
\Box \Phi_{n_1 \ldots n_s} = s \nabla (n_s C_{n_1 \ldots n_{s-1}}) + \Lambda \left[ (s - (s - 2)(D + s - 3)) \Phi_{n_1 \ldots n_s} \right.
\] (3.17)
\[
\left. + 2s(s - 1) g(n_{1n_2} (\Phi_{n_3 \ldots n_{s}}) m l m l - 4 D_{n_1 \ldots n_{s-2}}) \right] ,
\]
\[
\Box D_{n_1 \ldots n_{s-2}} = \nabla_m C^m_{n_1 \ldots n_{s-2}} - \Lambda \left[ (s(D + s - 2) + 6) D_{n_1 \ldots n_{s-2}} \right.
\] (3.18)
\[
\left. - 4 \Phi_{n_1 \ldots n_{s-2} ml} m l - (s - 2)(s - 3) g(n_{1n_2} D_{n_3 \ldots n_{s-2}}) m l m l \right] ,
\]
where \( \Box := \nabla_m \nabla^m \) and \( g_{mn} = e^a_m e^b_n \eta_{ab} \) is the AdS metric.

The equations (2.26)–(2.28) are invariant under the following gauge transformations
\[
\delta \Phi_{n_1 \ldots n_s} = s \nabla (n_s \xi_{n_1 \ldots n_{s-1}})
\] (3.19)
\[
\delta D_{n_1 \ldots n_{s-2}} = \nabla_m \xi^m_{n_1 \ldots n_{s-2}} ,
\] (3.20)
\[
\delta C_{n_1 \ldots n_{s-1}} = \Box \xi_{n_1 \ldots n_{s-1}} - \Lambda (D + s - 3) (s - 1) \xi_{n_1 \ldots n_{s-1}}
\] (3.21)
\[
+ (s - 1)(s - 2) \Lambda g(n_{1n_2} \xi_{n_3 \ldots n_{s-1}}) m l m l ,
\]
where the parameter \( \xi_{n_1 \ldots n_{s-1}} \) is completely symmetric and traceful.

As in the flat case, we can identify the fields \( \Phi_{n_1 \ldots n_s} \) and \( D_{n_1 \ldots n_s} \) with the completely symmetric part and a trace part of the higher–spin vielbein \( e^{a_1 \ldots a_{s-1}} \), respectively,
\[
\Phi_{n_1 \ldots n_s} = s e^{(n_s; n_1 \ldots n_{s-1})} \quad D_{n_1 \ldots n_{s-2}} = e^{p; n_1 \ldots n_{s-2} n_{s-1}} g^{n_{s-1}} ,
\] (3.22)
where the tangent space indices of the higher–spin vielbein have been converted into the ‘curved’ world indices with the use of the AdS vielbein $e^a_m$. The field $C_{n_1 \ldots n_{s-1}}$ is then identified by analyzing the zero torsion condition (3.4) and has the form similar to (2.36), namely

$$C_{n_1 \ldots n_{s-1}} = (s - 1) \omega_{m;n_1 \ldots n_{s-1},m} + \nabla^m e_{m;n_1 \ldots n_{s-1}}. \quad (3.23)$$

As in the flat space case, one can show that the gauge transformations (3.19)–(3.21) and the equations of motion (3.16)–(3.18) of the triplet fields follow, respectively, from the transformations (3.9)–(3.10) and the equations of motion (3.4)–(3.6). For instance, the last two terms in the variation of the field $C$ (3.21) come from the terms of the gauge variation of the higher–spin connection which are proportional to $\Lambda$ (see eqs. (3.10) and (3.14)).

By singling out the fields (3.22) and (3.23) in the action (3.3) and partially solving the zero–torsion condition (3.4) one should be able to reduce the action (3.3) to the AdS triplet actions of [24] for $s \geq 2$. As has been already explained in the case of flat space–time, the scalar and the vector fields are not part of the triplet spectrum in our formulation, but they can be included into the model by adding corresponding terms to the AdS action (3.3).

4 Frame–like action for fermionic higher–spin fields in flat space–time

The frame-like formulation of irreducible higher–spin fermions was originally proposed in $D = 4$ Minkowski space in [6, 7] solely in terms of the frame-like fields, then extended to $AdS_4$ using the formalism of two-component spinors in [8], where also the extra field connections were introduced, and then to $AdS_D$ with any $D$ in [10]. The difference compared to the bosonic case is that the fermionic field equations are of the first order and hence the free action does not contain auxiliary fields.

The flat space higher–spin field strengths (curvatures) are of the same form as in the bosonic case [10]

$$R_{a_1 \ldots a_{s-3}}^{b_1 \ldots b_t} = d\psi_{a_1 \ldots a_{s-3}}^{b_1 \ldots b_t} - (s - t - \frac{3}{2}) e^c \psi_{a_1 \ldots a_{s-3}}^{b_1 \ldots b_t c}. \quad (4.1)$$

where $\psi^{\alpha}_{a_1 \ldots a_{s-3}} b_1 \ldots b_t = dx^a \psi^n_{n, a_1 \ldots a_{s-3}} b_1 \ldots b_t$ is a one-form connection (with respect to the index $n$) and a rank $s - \frac{3}{2} + t$ tensor-spinor ( $0 \leq t \leq s - \frac{3}{2}$ and $\alpha$ being a (usually implicit) index associated with a spinor representation of $Spin(1, D-1)$). The field strengths (4.1) are manifestly invariant under the gauge transformations

$$\delta \psi_{a_1 \ldots a_{s-3}}^{b_1 \ldots b_t} = d\xi_{a_1 \ldots a_{s-3}}^{b_1 \ldots b_t} - (s - t - \frac{3}{2}) e^c \xi_{a_1 \ldots a_{s-3}}^{b_1 \ldots b_t c}. \quad (4.2)$$

5In a generic D–dimensional space–time the spinors are of the Dirac type. In even dimensions one can restrict spinors to be Weyl and in certain dimensions, e.g. $D = 3, 4, 6, 10$ and 11, one can consider Majorana or symplectic Majorana tensor–spinors. Note, however, that in the even–dimensional AdS spaces the Weyl condition cannot be imposed due to the presence of mass–like terms (see Section 5). In the case of the Dirac and Weyl spinors the actions which we consider below implicitly contain the hermitian conjugate part, which we shall skip for brevity. The addition of the hermitian conjugate part makes the first–order Lagrangian for Dirac fermions real exactly (i.e. not only up to a total derivative).
As in the bosonic case, the symmetry properties of the fermionic higher-spin fields and of
the gauge parameters are governed by the Young tableaux. The tensor–spinor fields with \( t \geq 1 \)
are of extra type and will not participate in the description of the free higher-spin fermionic
system. They play an important role, though, in the construction of the consistent interacting
higher-spin theory [35, 58, 13].

Let us consider the following first order action for the fermionic higher-spin field

\[
S = i \int_{M^D} e^{a_1 \ldots e^{a_{D-3}}} \varepsilon_{a_1 \ldots a_{D-3}pqr} (\bar{\psi}_{d_1 \ldots d_{s-\frac{3}{2}}} \gamma^{pqr} \psi^{d_1 \ldots d_{s-\frac{3}{2}}} + \alpha \bar{\psi}_{d_1 \ldots d_{s-\frac{3}{2}}} \gamma^p \psi^{d_1 \ldots d_{s-\frac{3}{2}}}) , (4.3)
\]

where \( \alpha \) is a constant parameter and

\[
\gamma^{pqr} = \frac{1}{6} ((\gamma^p \gamma^q \gamma^r - \gamma^q \gamma^p \gamma^r) + \text{two cyclic permutations of } p, q, r). \quad (4.4)
\]

The value of the parameter \( \alpha = -6(s - \frac{3}{2}) \) is fixed by requiring the invariance of the action
(4.3) under the gauge transformations (4.2). Indeed, the action is manifestly gauge invariant
provided that it can be reformulated in the following form

\[
S = i \int_{M^D} e^{a_1 \ldots e^{a_{D-3}}} \varepsilon_{a_1 \ldots a_{D-3}pqr} (\bar{R}_{d_1 \ldots d_{s-\frac{3}{2}}} \gamma^{pqr} \psi^{d_1 \ldots d_{s-\frac{3}{2}}} + \alpha \bar{R}_{d_1 \ldots d_{s-\frac{3}{2}}} \gamma^p \psi^{d_1 \ldots d_{s-\frac{3}{2}}}) . (4.5)
\]

and (up to a total derivative) as

\[
S = i \int_{M^D} e^{a_1 \ldots e^{a_{D-3}}} \varepsilon_{a_1 \ldots a_{D-3}pqr} (\bar{R}_{d_1 \ldots d_{s-\frac{3}{2}}} \gamma^{pqr} \psi^{d_1 \ldots d_{s-\frac{3}{2}}} + \alpha \bar{R}_{d_1 \ldots d_{s-\frac{3}{2}}} \gamma^p \psi^{d_1 \ldots d_{s-\frac{3}{2}}}). \quad (4.6)
\]

In view of the definition of the curvature (4.1) with \( t = 0 \), the equivalence of (4.3) and (4.5),
(4.6) requires that the connections \( \psi_{a_1 \ldots a_{s-\frac{3}{2}} b} \) and \( \bar{\psi}_{a_1 \ldots a_{s-\frac{3}{2}} b} \) do not contribute, respectively,
to the action (4.5) and (4.6). Note that in the form (4.5) the action is manifestly invariant
under the gauge variations of \( \psi \) (4.2) (with \( t = 0,1 \)), while in the form (4.6) it is manifestly
gauge invariant under the gauge variation of the Dirac conjugate connections \( \bar{\psi} \).

Thus, the possibility of rewriting the same action (4.3) both in the form (4.5) and (4.6)
implies constraints on the connection \( \psi^{a_1 \ldots a_{s-\frac{3}{2}} b} \) and the corresponding gauge parameter
\( \xi^{a_1 \ldots a_{s-\frac{3}{2}} b} \). To find these constraints we use the identity (2.10) to present the \( \psi^{a_1 \ldots a_{s-\frac{3}{2}} b} \)
dependent part of the action (4.5) in the form

\[
X = -\frac{s - \frac{3}{2}}{D - 2} i \int_{M^D} e^{a_1 \ldots e^{a_{D-2}}} \varepsilon_{a_1 \ldots a_{D-3}pqr} \left(3 \bar{\psi}_{a_1 \ldots a_{s-\frac{3}{2}}} \gamma^p \gamma^q \gamma^r \psi^{a_1 \ldots a_{s-\frac{3}{2}}} - 2 \eta^{pq} \gamma^p \psi^{a_1 \ldots a_{s-\frac{3}{2}}}, r \right) + \alpha \left(\bar{\psi}_{a_1 \ldots a_{s-\frac{3}{2}}} \gamma^p \psi^{a_1 \ldots a_{s-\frac{3}{2}}}, r + \bar{\psi}_{a_1 \ldots a_{s-\frac{3}{2}}} \gamma^q \psi^{a_1 \ldots a_{s-\frac{3}{2}}}, r - \frac{1}{s - \frac{3}{2}} \bar{\psi}_{a_1 \ldots a_{s-\frac{3}{2}}} \gamma^p \psi^{a_1 \ldots a_{s-\frac{3}{2}}}, q \right) \right) \quad (4.7)
\]

Setting

\[
\alpha = -6(s - \frac{3}{2}), \quad (4.8)
\]

we find that \( X \) vanishes provided that

\[
\gamma_b \psi^{a_1 \ldots a_{s-\frac{3}{2}} b} = 0, \quad \psi^{a_1 \ldots a_{s-\frac{3}{2}} b} = 0. \quad (4.9)
\]
These conditions provide a fermionic analog of the relaxed traceless condition (2.9). Note that the fermionic higher–spin vielbein $\psi_{m_1 \ldots n_s - \frac{3}{2}}$ remains unconstrained.

The action (4.3) or (4.5) and (4.6) is invariant under the gauge transformations (4.2) (with $t = 0, 1$) provided that the gauge parameters $\xi^{a_1 \ldots a_{s - \frac{3}{2}} b}$ satisfy the constraints analogous to (4.9)

$$\gamma_b \xi^{a_1 \ldots a_{s - \frac{3}{2}} b} = 0, \quad \xi^{a_1 \ldots a_{s - \frac{3}{2}} b} = 0.$$  

(4.10)

The variation of the action (4.3) with respect to $\psi$ produces the following equations of motion

$$\frac{1}{s - \frac{3}{2}} \gamma^m r \partial_r \bar{\psi}_{q; a_1 \ldots a_{s - \frac{3}{2}}} = \gamma^m \partial_r \bar{\psi}_{(a_1; a_2 \ldots a_{s - \frac{3}{2})} r - \gamma^q \partial_r \bar{\psi}_{q; a_2 \ldots a_{s - \frac{3}{2}} \delta^m_{a_1}}$$

$$- \gamma^m \partial_r \bar{\psi}_{(a_1; a_2 \ldots a_{s - \frac{3}{2})}^m + \gamma^r \partial_r \bar{\psi}_{r; a_2 \ldots a_{s - \frac{3}{2}} \delta^m_{a_1}}$$

(4.11)

$$- \gamma^m \partial_r \bar{\psi}_{a \ldots a_{s - \frac{3}{2}}} q + \gamma^d \partial_r \bar{\psi}_{m; a \ldots a_{s - \frac{3}{2}}} q.$$  

4.1 Fang–Fronsdal case

Let us now consider the case in which the fermionic higher–spin vielbein, the connections and the gauge parameters are required to be gamma–transversal (or gamma–traceless) and hence traceless in all tangent space indices

$$\gamma^c \bar{\psi}_{a_1 \ldots a_{s - \frac{3}{2}}} b_1 \ldots b_c = 0, \quad \gamma^c \bar{\psi}_{a_1 \ldots a_{s - \frac{3}{2}}} c, b_1 \ldots b_c = 0,$$  

(4.12)

$$\gamma^c \bar{\xi}_{a_1 \ldots a_{s - \frac{3}{2}}} b_1 \ldots b_c = 0, \quad \gamma^c \bar{\xi}_{a_1 \ldots a_{s - \frac{3}{2}}} c, b_1 \ldots b_c = 0.$$  

(4.13)

As we shall now demonstrate, in this case the action (4.3) is the frame–like counterpart [6, 7] of the Fang–Fronsdal action [3] for a single fermionic field of a half–integer spin $s$ in flat space.

The gamma–transversal higher–spin vielbein $\bar{\psi}_{m; n_1 \ldots n_s - \frac{3}{2}}$ contains the irreducible Lorentz tensor-spinors $\bar{\psi}$ described by the following gamma-transversal Young tableaux

$$\begin{array}{c}
\begin{array}{c}
\square \otimes \square \quad s - \frac{3}{2} = \square \quad s - \frac{3}{2} \oplus \square \quad s - \frac{3}{2} \oplus \square \quad s - \frac{3}{2} \oplus \square \quad s - \frac{3}{2}.
\end{array}
\end{array}$$  

(4.14)

The first tableau of length $s$ on the right hand side of (4.14) describes the totally symmetric and gamma–transversal part $\bar{\psi}_{(m; n_1 \ldots n_s - \frac{3}{2})}$, the second and third tableaux of the length $s - \frac{3}{2}$ and $s - \frac{5}{2}$, respectively, corresponds to the contractions $\gamma^m \bar{\psi}_{m; n_1 \ldots n_s - \frac{3}{2}}$ and $\eta^{mk} \bar{\psi}_{m; n_1 \ldots n_s - \frac{3}{2}} k$, respectively. The hook tableau corresponds to the irreducible (gamma–transversal) part of $\bar{\psi}$ that satisfies $\bar{\psi}_{(m; n_1 \ldots n_s - \frac{3}{2})} = 0$.

In virtue of the gauge transformations of the higher–spin vielbein (with gamma–traceless parameters)

$$\delta \bar{\psi}^{n_1 \ldots n_s - \frac{3}{2}} = d\xi^{n_1 \ldots n_s - 1} - (s - \frac{3}{2}) e^m \xi^{n_1 \ldots n_s - \frac{3}{2}} b \eta_{mb},$$  

(4.15)

the hook part of the higher–spin vielbein field can be gauge fixed to zero by the appropriate choice of the parameter $\xi^{a_1 \ldots a_{s - \frac{3}{2}} b}$. As a result, the remaining part of the vielbein amounts to
the combination of three totally symmetric gamma-transversal tensor–spinors of rank \( s - \frac{1}{2} \), \( s - \frac{3}{2} \) and \( s - \frac{5}{2} \) which are equivalent to the Fang–Fronsdal symmetric tensor–spinor field \( \Psi_{\alpha_1 \cdots \alpha_{s - \frac{1}{2}}} \) that satisfies the triple gamma–traceless condition

\[
\gamma^l \gamma^m \gamma^p \Psi_{lmpn} = \eta^{lm} \gamma^p \Psi_{lmpn} = 0 .
\]

The remaining local symmetry is the gauge invariance of the Fang–Fronsdal metric–like formulation in flat space–time with the completely symmetric gamma–transversal tensor–spinor parameter \( \tilde{\xi}_{m_1 \cdots m_{s - \frac{3}{2}}} \)

\[
\gamma^n \tilde{\xi}_{m_1 \cdots m_{s - \frac{5}{2}}} = 0 .
\]

Thus, the action (4.3) or (4.5) with the fields restricted by the conditions (4.12) is equivalent [6, 7, 10] to the Fang-Fronsdal action.

### 4.2 Triplet case

Let us now consider the case in which the fermionic higher–spin vielbein \( \psi^{a_1 \cdots a_{s - \frac{3}{2}}} \) and the gauge parameter \( \xi^{a_1 \cdots a_{s - \frac{5}{2}}} \) are unconstrained while the parameter \( \xi^{a_1 \cdots a_{s - \frac{5}{2}},b} \) of the gauge transformation (4.2) (for \( t = 0 \)) is constrained by the relaxed conditions

\[
\gamma_b \xi^{a_1 \cdots a_{s - \frac{3}{2}},b} = 0 , \quad \xi^{a_1 \cdots a_{s - \frac{5}{2}},b} = 0 \implies [\gamma^c, \gamma^d] \xi^{a_1 \cdots a_{s - 2c,d}} = 0 .
\]

In order to figure out what is the field content of the model in this case we observe that, the one-form fermionic field \( \psi_{a_1 \cdots a_{s - \frac{3}{2}}} = dx^m \psi_{m a_1 \cdots a_{s - \frac{3}{2}}} \) is composed of the tensor-spinors characterized by the following unrestricted (i.e. gamma–traceful) Young tableaux

\[
\begin{array}{c}
\square \otimes \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\square \end{array}
\end{array}
\end{array}
\end{array}
\overset{s - \frac{3}{2}}{=} \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\square \end{array}
\end{array}
\end{array}
\end{array}
\overset{s - \frac{1}{2}}{\oplus} \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\square \end{array}
\end{array}
\end{array}
\end{array}
\overset{s - \frac{5}{2}}{=}
\end{array}
\end{array} .
\]

On the other hand, the parameter \( \xi_{a_1 \cdots a_{s - 1}} \) of the Stueckelberg gauge symmetry that satisfies (4.18) has the following components

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\square \end{array}
\end{array}
\end{array}
\end{array}
\overset{s - \frac{3}{2}}{=}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\square \end{array}
\end{array}
\end{array}
\end{array}
\overset{s - \frac{1}{2}}{\oplus}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\square \end{array}
\end{array}
\end{array}
\end{array}
\overset{s - \frac{5}{2}}{=}
\end{array}
\end{array} \quad \left( \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\square \end{array}
\end{array}
\end{array}\right)
\]

where the subtracted (factored out) tensors take into account the two conditions (4.18). As a result, we find that, upon gauge fixing to zero the pure gauge part of \( \psi_{a_1 \cdots a_{s - \frac{3}{2}}} \) associated with the Stueckelberg symmetry, the remaining components of the fermionic field are described by the sum of the following unrestricted Young tableaux

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\square \end{array}
\end{array}
\end{array}
\end{array}
\overset{s - \frac{3}{2}}{=}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\square \end{array}
\end{array}
\end{array}
\end{array}
\overset{s - \frac{1}{2}}{\oplus}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\square \end{array}
\end{array}
\end{array}
\end{array}
\overset{s - \frac{5}{2}}{=}
\end{array}
\end{array} .
\]

Each term in (4.21) describes unconstrained totally symmetric spinor–tensors of ranks \( s - \frac{1}{2} \), \( s - \frac{3}{2} \) and \( s - \frac{5}{2} \), respectively. Decomposing this set of fields into Lorentz irreducible gamma–traceless components, we get the set of Fang-Fronsdal massless fields of the half–integer spins descending from \( s \) down to 3/2. Note that, analogously to the fields of spin one and zero in the bosonic case, the spin–1/2 field (being a zero–form) is not described by the action (4.3) and should be treated separately.
Up to this subtlety, the field content of the model under consideration is the same as that of the fermionic higher–spin triplets \([23, 24]\). Since both models describe free fields, there should be a relation between them.

To find this relation let us look at the form of the equations and gauge transformations which define the triplet of unconstrained fermionic higher–spin fields \(\Psi_{m_1 \cdots m_{s-\frac{1}{2}}}\), \(\chi_{m_1 \cdots m_{s-\frac{3}{2}}}\) and \(\lambda_{m_1 \cdots m_{s-\frac{5}{2}}}\) in flat space–time \([23, 24]\). Their equations of motion are

\[
\gamma^n \partial_n \Psi_{m_1 \cdots m_{s-\frac{1}{2}}} = (s - \frac{1}{2}) \partial(m_1 \chi_{m_2 \cdots m_{s-\frac{1}{2}}}) ,
\]

\[
\partial^n \Psi_{nm_2 \cdots m_{s-\frac{1}{2}}} = \left( s - \frac{3}{2} \right) \partial(m_2 \chi_{m_3 \cdots m_{s-\frac{1}{2}}}) = \gamma^n \partial_n \chi_{m_2 \cdots m_{s-\frac{1}{2}}} ,
\]

\[
\gamma^n \partial_n \lambda_{m_1 \cdots m_{s-\frac{5}{2}}} = (s - \frac{5}{2}) \partial^n \chi_{nm_1 \cdots m_{s-\frac{5}{2}}} .
\]

These equations are invariant under the following unconstrained gauge transformations of the fields

\[
\delta \Psi_{m_1 \cdots m_{s-\frac{1}{2}}} = (s - \frac{1}{2}) \partial(m_1 \xi_{m_2 \cdots m_{s-\frac{1}{2}}}) ,
\]

\[
\delta \chi_{m_1 \cdots m_{s-\frac{3}{2}}} = \gamma^n \partial_n \xi_{m_2 \cdots m_{s-\frac{3}{2}}} ,
\]

\[
\delta \lambda_{m_1 \cdots m_{s-\frac{5}{2}}} = \partial^n \xi_{nm_1 \cdots m_{s-\frac{5}{2}}} .
\]

The form of the gauge transformations (4.25)–(4.27) prompts us that the fermionic triplet fields are related to the components of the fermionic higher–spin ‘vielbein’ \(\psi_{nm_1 \cdots m_{s-\frac{1}{2}}}\) as follows

\[
\Psi_{m_1 \cdots m_{s-\frac{1}{2}}} = (s - \frac{1}{2}) \psi_{nm_1 \cdots m_{s-\frac{1}{2}}} ,
\]

\[
\chi_{m_1 \cdots m_{s-\frac{3}{2}}} = \gamma^n \psi_{nm_1 \cdots m_{s-\frac{3}{2}}} ,
\]

\[
\lambda_{m_1 \cdots m_{s-\frac{5}{2}}} = \eta^n \psi_{nm_1 \cdots m_{s-\frac{5}{2}}} .
\]

Upon this identification the fermionic triplet field equations of motion (4.22)–(4.24) follow from eqs. (4.11). As such, upon substituting eqs. (4.28)–(4.30) for corresponding components of the fermionic frame–like field into the action (4.3) and gauge fixing to zero its Stueckelberg symmetry, one will reduce eq. (4.3) to the fermionic triplet action of \([24]\) in flat space–time.

## 5 Frame–like action for fermionic higher–spin fields in \(\text{AdS}_D\)

In \(\text{AdS}_D\) space the gauge transformations (4.2) (for \(t = 0\)) of the dynamical fermionic field \(\psi_{a_1 \cdots a_{s-\frac{3}{2}}}^{\alpha}\) are modified as follows \(^7\)

\[
\delta \psi_{a_1 \cdots a_{s-\frac{3}{2}}}^{\alpha} = \mathcal{D} \xi_{a_1 \cdots a_{s-\frac{3}{2}}}^{\alpha} - (s - \frac{3}{2}) e^b \xi_{a_1 \cdots a_{s-\frac{3}{2}} b} ,
\]

\(^7\)The form of the gauge transformations (4.2) of the higher rank extra fields (with \(t \geq 1\)) is more involved. It is not needed for our consideration, however.
where following [4] the generalized covariant differential $\mathcal{D}$ is defined as the sum of the conventional AdS covariant differential $\nabla$ and the term $i\sqrt{-\Lambda} e^a \gamma_a$, namely,

$$\mathcal{D} = \nabla + i\sqrt{-\Lambda} e^a \gamma_a. \quad (5.32)$$

The exterior differential (5.32) is actually covariant with respect to the AdS isometry group $\text{Spin}(2, D - 1)$. It is defined in such a way that its square vanishes when acting on spinor differential forms

$$\mathcal{D}^2 \chi^\alpha = 0 \quad (5.33)$$

and it acts as $\nabla^2$ on the tensor differential forms

$$\mathcal{D}^2 T_{a_1 \cdots a_t} = \nabla^2 T_{a_1 \cdots a_t} = -t \Lambda e^{(a_1} e_b T^{a_2 \cdots a_t)b}. \quad (5.34)$$

Thus, in virtue of eq. (5.33), $\mathcal{D}^2$ acts on the tensor–spinors in the same way as on the tensors, i.e.

$$\mathcal{D}^2 \psi^{a_1 \cdots a_t} = -t \Lambda e^{(a_1} e_b \psi^{a_2 \cdots a_t)b}. \quad (5.35)$$

Note also that

$$\mathcal{D} \gamma_a = -i\sqrt{-\Lambda} e^b [\gamma_b, \gamma_a] = -i\sqrt{-\Lambda} e^b \gamma_{ba}. \quad (5.36)$$

Eqs. (5.33)–(5.36) are useful when checking the gauge invariance of the action for the fermionic higher–spin fields in AdS under the transformations (5.31).

### 5.1 Fang-Fronsdal case in AdS

As in the flat–space, the fermionic dynamical higher–spin field $\bar{\psi}^{a_1 \cdots a_s - \frac{d}{2}}$ in AdS is subject to the gamma–trace condition

$$\gamma^c \bar{\psi}_{a_1 \cdots a_s - \frac{d}{2}} = 0. \quad (5.37)$$

For this condition to be compatible with the gauge transformations (5.31) the gauge parameters must obey the following constraints

$$\gamma^c \tilde{\xi}_{a_1 \cdots a_s - \frac{d}{2}} = 0, \quad (s - \frac{3}{2}) \gamma^c \tilde{\xi}_{a_1 \cdots a_{s-2} c b} = i \sqrt{-\Lambda} \gamma_b^c \tilde{\xi}_{a_1 \cdots a_{s-2} c b} = -i \sqrt{-\Lambda} \tilde{\xi}_{a_1 \cdots a_{s-2} c b} \quad (5.38)$$

$$\Rightarrow \gamma^c \bar{\psi}_{a_1 \cdots a_{s-2} c b} = i \sqrt{-\Lambda} \bar{\xi}_{a_1 \cdots a_{s-2} c b}.$$

The action for $\gamma$–traceless $\bar{\psi}_{a_1 \cdots a_{s-\frac{d}{2}}}$ in AdS, that is invariant under (5.31) with the parameters satisfying (5.38) has the following form

$$S = i \int_{M_D} e^{a_1} \cdots e^{a_{D-3}} e_{a_1 \cdots a_{D-3} p q r} \left[ \bar{\psi}^{d_1 \cdots d_s - \frac{d}{2}} \gamma^{p q r} \mathcal{D} \bar{\psi}^{d_1 \cdots d_s - \frac{d}{2}} - 6(s - \frac{3}{2}) \bar{\psi}^{d_1 \cdots d_s - \frac{d}{2}} p \gamma^q \mathcal{D} \bar{\psi}^{d_1 \cdots d_s - \frac{d}{2}} r + 3i \sqrt{-\Lambda} (s - \frac{3}{2}) \left( e^r \bar{\psi}^{d_1 \cdots d_s - \frac{d}{2}} \gamma^{p q r} \bar{\psi}^{d_1 \cdots d_s - \frac{d}{2}} + 2(s - \frac{3}{2}) e^p \bar{\psi}^{d_1 \cdots d_s - \frac{d}{2}} \bar{\psi}^{r d_1 \cdots d_s - \frac{d}{2}} \right) \right]. \quad (5.39)$$

The last two “mass–like” terms in (5.39) are proportional to the square root of the cosmological constant (which is also present in the covariant differential $\mathcal{D}$ (5.32)). These terms insure the gauge invariance of the higher–spin system in AdS.
5.2 Fermionic triplets in AdS

Let us now consider the form of the action in AdS space for the fermionic higher–spin fields \( \psi^{a_1 \cdots a_{s-\frac{3}{2}}} \) which are not subject to the gamma–trace condition, i.e. describe fermionic triplets. By now the action and the equations of motion for the fermionic triplets have been unknown. To demonstrate that such an action and equations of motion do exist, we first consider the simplest case of the reducible field of spin \( \frac{5}{2} \).

5.2.1 Spin–\( \frac{5}{2} \) example

The one–form tensor–spinor field under consideration is the gamma–traceful field \( \psi^a = dx^m \psi^a_m \). Its gauge transformations have the form

\[
\delta \psi^a = D \xi^a - e^b \xi^{a,b},
\]

where the parameter \( \xi^a \) is gamma–traceful, while the antisymmetric parameter \( \xi^{a,b} = -\xi^{b,a} \) is required to satisfy the following relation

\[
\gamma^b \xi^{a,b} = -i \sqrt{-\Lambda} \gamma^{ab} \xi_b = i \sqrt{-\Lambda} (\xi^a - \gamma^a \gamma^b \xi_b).
\]

The condition (5.41), which reduces to the corresponding eq. (5.38) in the gamma–traceless case, ensures that the gamma trace of \( \psi^a \) transforms as a divergence, i.e. as a Rarita–Schwinger field of spin 3/2,

\[
\delta (\gamma^a \psi^a) = D (\gamma^a \xi^a).
\]

The action for the field \( \psi^a \) which is invariant under the transformations (5.40)–(5.42) has the following form

\[
S = i \int_{M^D} e^{a_1 \cdots a_{D-2} \varepsilon_{a_1 \cdots a_{D-3} b c d}} \left[ \bar{\psi} f \gamma^{b c d} D \psi^f - 6 \bar{\psi} \gamma^c D \psi^d + \frac{3i \sqrt{-\Lambda}}{D-2} (e^d \bar{\psi} f \gamma^{b c} \psi^f - 2 e^b \bar{\psi} c \psi^d + 2 e^d (\bar{\psi} f \gamma^f) \gamma^b \psi^c - e^d (\bar{\psi} f \gamma^f) \gamma^{b c} (\gamma_i \psi^i)) \right].
\]

One can see that in comparison with the action (5.39) for a single spin–5/2 field, the action (5.43) contains two more terms which depend on the gamma–trace of \( \psi^a \). It can be shown that by splitting \( \psi^a \) into the gamma-traceless and gamma-trace parts

\[
\psi^a = \bar{\psi}^a - \frac{1}{D} \gamma^a \bar{\psi}, \quad \gamma^a \bar{\psi}^a = 0, \quad \bar{\psi} = \gamma^a \psi^a,
\]

the action (5.43) splits into the direct sum of the actions for the single spin–5/2 field \( \bar{\psi}^a \) and the spin–3/2 field \( \bar{\psi} \) in a way similar to the bosonic case (see Subsection 2.2). As mentioned above, the spin–1/2 field does not appear in our construction. The above example is the simplest fermionic “triplet” (actually the doublet) of fields in AdS space

\[
\Psi_{ab} = 2 \psi_{(b,a)}, \quad \chi_a = \gamma^b \psi_{b,a}.
\]

Their gauge transformations are

\[
\delta \Psi_{ab} = 2 D_{(b} \xi_{a)}, \quad \delta \chi_a = \gamma^b D_b \xi_a - i \sqrt{-\Lambda} \gamma^a \gamma^b \xi_b.
\]
5.2.2 Generic higher–spin fermion triplets in AdS

The gamma–traceful one–form tensor–spinor field $\psi^{a_1\cdots a_{s-\frac{3}{2}}}$ describing the fermionic triplet in AdS space undergoes the gauge transformations

$$\delta \psi^{a_1\cdots a_{s-\frac{3}{2}}} = \mathcal{D} \xi^{a_1\cdots a_{s-\frac{3}{2}}} - (s - \frac{3}{2}) e_b \xi^{a_1\cdots a_{s-\frac{3}{2}} b}$$

(5.47)

with the unconstrained parameter $\xi^{a_1\cdots a_{s-\frac{3}{2}}}$ and the Stueckelberg parameter $\xi^{a_1\cdots a_{s-\frac{3}{2}} b}$ satisfying the Young tableau property, $\xi^{(a_1\cdots a_{s-\frac{3}{2}}, b)} = 0$, the relaxed traceless condition (as in the case of the bosonic triplets)

$$\xi^{a_1\cdots a_{s-\frac{3}{2}} b} \eta_{bc} = 0$$

(5.48)

and the following relation

$$\gamma_b \xi^{a_1\cdots a_{s-\frac{3}{2}} b} = - i \sqrt{-\Lambda} \gamma_{a_1 b} \xi^{a_2\cdots a_{s-\frac{3}{2}} b} .$$

(5.49)

Eq. (5.49) reduces to (5.38) if the parameter $\xi^{a_1\cdots a_{s-\frac{3}{2}}}$ was gamma–traceless and ensures that the gamma–trace of $\psi^{a_1\cdots a_{s-\frac{3}{2}}}$ transforms as a spin–$(s-1)$ field, i.e. similarly to (5.47) with $s \rightarrow s - 1$.

The action, that is invariant under the transformations (5.47)–(5.49), has the following form

$$S = i \int_{M^D} e^{a_1} \cdots e^{a_{D-3}} \varepsilon_{a_1\cdots a_{D-3}abc} \left[ \bar{\psi}_{d_1\cdots d_{s-\frac{3}{2}}} \gamma^{abc} \mathcal{D} \psi^{d_1\cdots d_{s-\frac{3}{2}}} - 6(s - \frac{3}{2}) \bar{\psi}_{d_1\cdots d_{s-\frac{3}{2}}} a \gamma^b \mathcal{D} \psi^{d_1\cdots d_{s-\frac{3}{2}} c} + \frac{3i}{D-2} \sqrt{-\Lambda (s-\frac{3}{2})} \left( e^c \bar{\psi}_{d_1\cdots d_{s-\frac{3}{2}}} \gamma^{ab} \psi^{d_1\cdots d_{s-\frac{3}{2}}} + 2(s - \frac{3}{2}) e^a \bar{\psi}_{d_1\cdots d_{s-\frac{3}{2}}} \psi^{cd_1\cdots d_{s-\frac{3}{2}}} \right) ight] + \frac{3i}{D-2} \sqrt{-\Lambda (s-\frac{3}{2})} \left( 2 e^c \left( \bar{\psi}_{d_1\cdots d_{s-\frac{3}{2}}} \gamma^f \gamma^{d_1\cdots d_{s-\frac{3}{2}}} \right) - 6i \bar{\psi}_{d_1\cdots d_{s-\frac{3}{2}}} \gamma_i \left( \gamma^f \psi^{c_{d_1\cdots d_{s-\frac{3}{2}}} \gamma^f} \right) \right).$$

(5.50)

It has one more (the last) term in comparison with the action (5.43) for the $s = \frac{5}{2}$ triplet.

The AdS analogues of the flat–space fermionic triplet fields of [23, 24] are extracted from $\psi^{a_1\cdots a_{s-\frac{3}{2}}} = e^b \psi_{b a_1\cdots a_{s-\frac{3}{2}}}$ analogously to (4.28)–(4.30)

$$\Psi^{a_1\cdots a_{s-\frac{3}{2}}} = (s - \frac{1}{2}) \psi^{(a_1; a_2\cdots a_{s-\frac{3}{2}})} ,$$

(5.51)

$$\chi^{a_1\cdots a_{s-\frac{3}{2}}} = \gamma^b \psi_{b a_1\cdots a_{s-\frac{3}{2}}},$$

(5.52)

$$\lambda^{a_1\cdots a_{s-\frac{3}{2}}} = \eta^{b c} \psi_{b c a_1\cdots a_{s-\frac{3}{2}}}. $$

(5.53)

Their gauge transformations are easily obtained from eqs. (5.47)–(5.49)

$$\delta \Psi^{a_1\cdots a_{s-\frac{3}{2}}} = (s - \frac{1}{2}) D(a_1, \xi_{a_2\cdots a_{s-\frac{3}{2}}}) ,$$

(5.54)
\[
\delta \chi_{a_1 \cdots a_{s-\frac{3}{2}}} = \gamma^b D_b \xi_{a_1 \cdots a_{s-\frac{3}{2}}} - \left( s - \frac{3}{2} \right) i \sqrt{-\Lambda} \gamma^{(a_1}_b \xi_{a_2 \cdots a_{s-\frac{3}{2})b}}, \tag{5.55}
\]
\[
\delta \lambda_{a_1 \cdots a_{s-\frac{3}{2}}} = D^b \xi_{ba_1 \cdots a_{s-\frac{3}{2}}} \equiv \nabla^b \xi_{ba_1 \cdots a_{s-\frac{3}{2}}} + \frac{i \sqrt{-\Lambda}}{2} \gamma^b \xi_{ba_1 \cdots a_{s-\frac{3}{2}}}, \tag{5.56}
\]
and the equations of motion, which generalize to AdS space eqs. (4.22)–(4.24), follow from the action (5.50).

Note that the gauge transformations (5.55) and (5.56) of the fields \(\chi\) and \(\lambda\) of the fermionic triplet in AdS differ from those given in [42] by terms proportional to the gamma–trace of the gauge parameter \(\gamma^b \xi_{ba_1 \cdots a_{s-\frac{3}{2}}}\). We assume that this is a reason which have not allowed previous authors to obtain the Lagrangian description of the fermionic triplets in AdS.

To recapitulate, in the frame–like formulation the fermionic triplet is described by the unconstrained fermionic higher–spin vielbein \(dx^m \psi_{m; a_1 \cdots a_{s-\frac{3}{2}}}\) subject to the gauge transformations (5.47) with the Stueckelberg parameters \(\xi_{a_1 \cdots a_{s-\frac{3}{2}}, b}\) satisfying the relaxed (gamma)–trace constraints (5.48) and (5.49). Upon eliminating the Stueckelberg degrees of freedom and splitting the components of the spinor-tensor \(\psi_{b; a_1 \cdots a_{s-\frac{3}{2}}}\) into its triplet constituents (5.51)–(5.53) one can reduce the action (5.50) to an action which describes the fermionic triplets in AdS in the metric–like formulation.

### 6 Relation to unconstrained formulations of irreducible higher–spin fields

Let us now demonstrate how the triplet systems discussed in the previous sections can be reduced to corresponding irreducible fields of spin \(s\) without imposing conventional (gamma)–trace constraints on the fields and gauge parameters. The consideration below applies both to the flat space–time and to the AdS background.

We observe that the action \(S^{irr}\) for the irreducible spin \(s\) system results from the action \(S^{red}\) for the reducible (triplet) system by adding the Lagrange multiplier term

\[
S^{irr} = S^{red} + \int l_{a_1 \cdots a_{s-3}} e^{a_1 \cdots a_{s-3}c} \, c, \tag{6.57}
\]

where \(l_{a_1 \cdots a_{s-3}} = dx^{m_1} \cdots dx^{m_{D-1}} l^{a_1 \cdots a_{s-3}}_{m_1 \cdots m_{D-1}}(x)\) is a (frame–like) differential \((D - 1)\)–form Lagrange multiplier, which is assumed to be gauge invariant. The Lagrange multiplier term in (6.57) is not invariant under the full relaxed gauge symmetry transformations, but only under those with traceless parameters. The full relaxed gauge invariance can be restored, however, by making the following substitution in the action (6.57)

\[
\begin{align*}
\alpha^{a_1 \cdots a_{s-3}} & \rightarrow \alpha^{a_1 \cdots a_{s-3}} - \nabla \alpha^{a_1 \cdots a_{s-3}} + (s - 1) e^b \beta^{a_1 \cdots a_{s-3}, b}, \\
\omega_c^{a_1 \cdots a_{s-3}, b} & \rightarrow \omega_c^{a_1 \cdots a_{s-3}, b} - \nabla \beta^{a_1 \cdots a_{s-3}, b} + \Lambda (e^b \alpha^{a_1 \cdots a_{s-3}} - e^{a_1} \alpha^{a_2 \cdots a_{s-3}} b),
\end{align*} \tag{6.58}
\]

where \(\alpha^{a_1 \cdots a_{s-3}}\) and \(\beta^{a_1 \cdots a_{s-3}, m}\) are zero–form Stueckelberg fields (i.e. compensators) (the latter having the symmetry of the Young tableau \(Y(s - 3, 1)\)). To make the final action compatible
with the transformation rules (2.5) and (2.14) in flat space or (3.9) and (3.12) in AdS, the compensator fields are endowed with the following transformation laws

\[ \delta \alpha^{a_1 \cdots a_{s-3}} = \xi^{a_1 \cdots a_{s-3}} b, \]
\[ \delta \beta^{a_1 \cdots a_{s-3}, b} = \xi^{a_1 \cdots a_{s-3}} c b. \]

(6.59)  

(6.60)

Let us stress that since the \( S^{\text{red}} \) is invariant under the full relaxed higher-spin gauge transformations, it obviously remains intact under the substitution (6.58), \textit{i.e.} does not contain the compensator fields. Thus, the resulting compensator action for the irreducible field of spin \( s \) is

\[ S^{\text{irr}} = S^{\text{red}} + \int l_{a_1 \cdots a_{s-3}} (e_b a_1 \cdots a_{s-3} c - \nabla_b \alpha^{a_1 \cdots a_{s-3}} + (s - 1) \beta^{a_1 \cdots a_{s-3}, b}). \]

(6.61)

By construction the action (6.61) is of the first order in derivatives and is invariant under the relaxed gauge transformations similar to those of the reducible triplet system.

The compensator field \( \alpha^{a_1 \cdots a_{s-3}} \) is nothing but the one considered for the spin–3 case already by Schwinger [59] and for arbitrary spin in [39, 23, 40, 24, 42, 45], while the field \( \beta^{a_1 \cdots a_{s-3}, b} \) is a new one, it “compensates” the Stueckelberg gauge transformations of the trace of the higher-spin vielbein. (If we imposed the gauge in which the compensator fields are zero, the system would have reduced to the conventional Fronsdal case.) The Lagrange multiplier \( l_{a_1 \cdots a_{s-3}} \) is the frame–like counterpart of the gauge–invariant Lagrange multipliers of the unconstrained formulation of [42]. More precisely, up to the coefficients, the Lagrange multipliers \( \lambda^{a_1 \cdots a_{s-2}} \) and \( \lambda^{a_1 \cdots a_{s-4}} \) of [42] are

\[ \lambda^{a_1 \cdots a_{s-2}} = \varepsilon^{b_1 \cdots b_{D-1}} (a_1 p^{a_2 \cdots a_{s-2}} b_1 \cdots b_{D-1}), \quad \lambda^{a_1 \cdots a_{s-4}} = \varepsilon^{b_1 \cdots b_{D-1}} (a_1 p^{a_2 \cdots a_{s-4}} b_1 \cdots b_{D-1}). \]

(6.62)

The constraint on the higher–spin vielbein that follows from the action (6.61)

\[ e_b a_1 \cdots a_{s-3} c = \nabla_b \alpha^{a_1 \cdots a_{s-3}} - (s - 1) \beta^{a_1 \cdots a_{s-3}, b} \]

(6.63)

implies that the trace of the higher–spin vielbein is pure gauge, \textit{i.e.} it can be set to zero by gauge fixing the compensators to zero using (6.59) and (6.60).

Let us now show that eq. (6.63) reduces to corresponding equations of [42]. To this end, we symmetrize all the indices of (6.63). In view of the Young symmetry property \( \beta^{(a_1 \cdots a_{s-3}, b)} = 0 \), the result is

\[ e_{(b a_1 \cdots a_{s-3}) c} = \nabla^{(b} \alpha^{a_1 \cdots a_{s-3})}. \]

(6.64)

Using the triplet field redefinition (2.34) (or (3.22)) we can rewrite eq. (6.64) in the form

\[ D^{a_1 \cdots a_{s-2}} - \frac{1}{2} \Phi^{a_1 \cdots a_{s-2} c} = - \frac{(s - 2)}{2} \nabla^{(a_1} \alpha^{a_2 \cdots a_{s-2})}. \]

(6.65)

Let us now contract the index \( b \) in (6.63) with one of the indices \( a_i \). We get another condition that

\[ e_{b(a_1 \cdots a_{s-4} c} = \nabla_b \alpha^{a_1 \cdots a_{s-4}} b, \]

(6.66)

or, in view of eqs. (2.34) and (3.22),

\[ D^{a_1 \cdots a_{s-4} b} = \nabla_b \alpha^{a_1 \cdots a_{s-4} b}. \]

(6.67)
Up to a field rescaling, eqs. (6.65) and (6.67) are the ones which appeared in minimal unconstrained formulations \[24, 42, 45\] of the irreducible higher–spin fields (for non–minimal formulations see e.g. \[36, 37, 38, 46\]).

The fermionic case can be considered in a similar fashion. The additional requirement that the gamma–trace of the higher–spin field $\psi_{m; a_{1}...a_{s-\frac{3}{2}}}^{a_{s-\frac{5}{2}}}$ is a pure gauge, i.e. the constraint that should be added to the triplet action (4.3) or (5.50) with the Lagrange multiplier to reduce the fermionic triplet to the irreducible field of spin $s$ is

$$\gamma^{a_{s-\frac{3}{2}}} \psi_{m; a_{1}...a_{s-\frac{3}{2}}^{a_{s-\frac{5}{2}}} = D_{m} \alpha_{a_{1}...a_{s-\frac{3}{2}}} - (s - \frac{3}{2}) \beta_{a_{1}...a_{s-\frac{3}{2}}}^{m},} \tag{6.68}$$

(with $\alpha_{a_{1}...a_{s-\frac{3}{2}}}$ and $\beta_{a_{1}...a_{s-\frac{3}{2}}}^{m}$ being fermionic compensators). By construction it is of first order in derivatives.

We have thus, obtained the frame–like version of the unconstrained Lagrangian formulations of the irreducible higher–spin fields considered in \[42, 45\].

7 AdS covariant formalism for bosonic HS fields

The description of the higher–spin fields in AdS space considered in the previous sections was not manifestly invariant under the higher–spin gauge transformations. Also it was manifestly invariant only under the Lorentz subgroup $O(1, D-1)$ of the full AdS isometry group $O(2, D-1)$. To make both the AdS isometry and higher–spin gauge symmetries manifest it is convenient to use the formalism a la MacDowell, Mansouri, Stelle and West \[60, 61\] (MMSW). We shall introduce only basic ingredients of this formulation which are required for our purposes and refer the reader to \[11, 13\] for further details.

7.1 Basic definitions

The $AdS_D$ space is described by the vielbein $e^{a} = dx^{m} e_{m}^{a}$ and the connection $\omega^{ab} = dx^{m} \omega_{m}^{ab}$ which satisfy the zero torsion and constant curvature conditions (3.1) and (3.2). To make the $O(2, D-1)$ AdS isometry symmetry manifest we unify $e^{a}$ and $\omega^{ab}$ into a connection $\Omega^{AB}$ valued in the algebra $o(2, D-1)$

$$\Omega^{AB} := (\omega^{ab}, \sqrt{-\Lambda} e^{a}), \quad i.e. \quad e^{a} = \frac{1}{\sqrt{-\Lambda}} \Omega^{a0}, \tag{7.1}$$

where the capital Latin indices $A, B, \cdots = (0', a) = (0', 0, 1, \cdots, D-1)$ correspond to the vector representation of $o(2, D-1)$ acting in a $D + 1$ dimensional vector space with the invariant metric $\eta_{AB} = (+, +, -, \cdots, -)$, and the index $0'$ denotes the second time–like direction in this space. Recall that the cosmological constant $\Lambda$ is negative in the AdS case.

The connection $\Omega^{AB}$ that satisfies the zero curvature equation

$$\mathcal{R} := d\Omega + \Omega^{2} = 0, \tag{7.2}$$

promotes the rigid isometry symmetry $O(2, D-1)$ to the local one.
By construction eq. (7.2) is equivalent to the relations (3.1) and (3.2) satisfied by the AdS torsion and curvature.

Because of the zero curvature condition (7.2) it is convenient to work with the exterior covariant derivative associated with the connection Ω, that squares to zero in virtue of (7.2)

\[ \mathcal{D} = d + \Omega, \quad \mathcal{D}^2 = 0. \tag{7.3} \]

Note that the exterior covariant derivative (5.32) which we used to describe the fermionic fields in AdS in Section 5 is nothing but the covariant derivative (7.3) in the spinor representation of \( Spin(2, D-1) \) with the spin connection \( \frac{1}{2} \Omega^{AB} \Gamma_A \Gamma_B \). The matrices \( \Gamma_A \) are Dirac matrices corresponding to the group \( Spin(2, D-1) \). The Dirac matrices \( \gamma^a \) corresponding to \( Spin(1, D-1) \) are related to \( \Gamma_A \) as follows

\[ \gamma^a = i \Gamma^a \Gamma^0. \]

Another ingredient of the MMSW formulation is the so called compensator vector field \( V^A(x) \) satisfying the normalization condition

\[ V^AV^B\eta_{AB} = -\frac{1}{\Lambda}. \tag{7.4} \]

The extension of the symmetry from the local Lorentz group \( O(1, D-1) \) to \( O(2, D-1) \) brings about \( D \) more local symmetry parameters, which can be regarded as coordinates for the coset space \( O(2, D-1)/O(1, D-1) \). The role of the field \( V^A(x) \) is similar to that of the Goldstone fields. It compensates the action of these additional local symmetries and thus maintains intact the number of the physical degrees of freedom of the model. Using local \( O(2, D-1) \) transformations of \( V^A \) one can choose the gauge

\[ V^A = \frac{1}{\sqrt{-\Lambda}} \delta^A_0. \tag{7.5} \]

which breaks the local symmetry \( O(2, D-1) \) down to \( O(1, D-1) \). The one–form

\[ E^A = \mathcal{D} V^A \tag{7.6} \]

is the \( O(2, D-1) \)–covariant vielbein. It reduces to \( e^a \) in the gauge (7.5).

In the AdS-covariant formulation, the dynamics of massless higher–spin fields is described [11] by the generalized connection one–form

\[ \Omega^{A_1 \cdots A_{s-1}, B_1 \cdots B_{s-1}}(x) = dx^m \Omega^{A_1 \cdots A_{s-1}, B_1 \cdots B_{s-1}}_m \quad (A,B = 0', 0, 1 \cdots, D-1), \tag{7.7} \]

that takes values in the \( O(2, D-1) \)–module described by the two–row rectangular Young tableau of length \( s-1 \) and, hence, satisfies the symmetry conditions

\[ \Omega^{A_1 \cdots A_{s-1}, B_1 \cdots B_{s-1}}_m = \Omega^{(A_1 \cdots A_{s-1}), B_1 \cdots B_{s-1}}_m = \Omega^{A_1 \cdots A_{s-1}, (B_1 \cdots B_{s-1})}_m, \tag{7.8} \]

\[ \Omega^{(A_1 \cdots A_{s-1}, A_s) B_2 \cdots B_{s-1}}_m = 0. \tag{7.9} \]

As a consequence of eqs. (7.8) and (7.9), the higher–spin connection is (anti)symmetric with respect to the interchange of the two groups of symmetrically indices:

\[ \Omega^{A_1 \cdots A_{s-1}, B_1 \cdots B_{s-1}}_m = (-1)^{s-1} \Omega^{B_1 \cdots B_{s-1}, A_1 \cdots A_{s-1}}_m. \tag{7.10} \]
The linearized higher–spin curvature associated with this connection is
\[ \mathcal{R}^{A_1\cdots A_{s-1},B_1\cdots B_{s-1}} = \mathcal{D} \Omega^{A_1\cdots A_{s-1},B_1\cdots B_{s-1}} \]
\[ = d\Omega^{A_1\cdots A_{s-1},B_1\cdots B_{s-1}} + (s - 1) \Omega^{(A_1 \Omega^{A_2\cdots A_{s-1})C,B_1\cdots B_{s-1}} - (s - 1) \Omega^{A_1\cdots A_{s-1},C(B_2\cdots B_{s-1}) \Omega^{B_1)}}, \]
(7.11)
where \( \Omega^{AB} \) is the AdS background \( O(2,D - 1) \) spin connection (7.1).

The higher–spin curvature is invariant under the local transformations with parameters \( \xi(x) \), that have the same symmetry properties as the higher–spin connection
\[ \delta\Omega^{A_1\cdots A_{s-1},C_1\cdots C_{s-1}} = \mathcal{D} \xi^{A_1\cdots A_{s-1},C_1\cdots C_{s-1}}. \]
(7.12)

The irreducible Lorentz components of the connection \( \Omega \) contain the higher–spin vielbein and Lorentz connection analogous to those in flat space–time (2.1) and (2.2) as well as all extra connections. They result from (7.7) by projecting \( \Omega \) along the compensator \( V^A \). For instance,
\[ \Omega^{A_1\cdots A_{s-1},C_1} := \Omega^{A_1\cdots A_{s-1},C_1\cdots C_{s-1}}V_{C_2}\cdots V_{C_{s-1}}, \]
(7.13)
contains the higher–spin vielbein
\[ E^{A_1\cdots A_{s-1}} := \Omega^{A_1\cdots A_{s-1},C_1\cdots C_{s-1}}V_{C_2}\cdots V_{C_{s-1}}, \]
(7.14)
as the most \( V \)-longitudinal components of \( \Omega \) and the higher–spin Lorentz connection as next to the most \( V \)-longitudinal components of \( \Omega \). In the gauge (7.5) they are
\[ e^{a_1\cdots a_{s-1}} \equiv E^{a_1\cdots a_{s-1}} := \frac{1}{(-\Lambda)^{s-2}} \Omega^{a_1\cdots a_{s-1},0'\cdots 0'}, \]
\[ \omega^{a_1\cdots a_{s-1},b} \equiv \Omega^{a_1\cdots a_{s-1},b} := \frac{1}{(-\Lambda)^{s-2}} \Omega^{a_1\cdots a_{s-1},b0'\cdots 0'}. \]
(7.15)
The gauge transformations of these fields, which follow from (7.12), are those given in eqs. (3.9) and (3.10) but with traceful gauge parameters since we have not imposed the trace constraints on the higher–spin connection (7.7) yet.

The other \( O(1,D - 1) \) tensor fields contained in \( \Omega^{A_1\cdots A_{s-1},B_1\cdots B_{s-1}} \), i.e. \( \omega^{a_1\cdots a_{s-1},b_1\cdots b_t} \) (with \( 2 \leq t \leq s - 1 \)) are the extra fields which play an important role in interacting higher–spin systems as shown in [35].

If the connection \( \Omega^{A_1\cdots A_{s-1},B_1\cdots B_{s-1}} \) and the parameters \( \xi^{A_1\cdots A_{s-1},B_1\cdots B_{s-1}} \) are traceless in the indices \( A \) and \( B \), they describe a single bosonic higher–spin field in \( AdS_D \) [11]. The corresponding \( O(1,D - 1) \)–covariant higher–spin vielbein, connections and gauge parameters satisfy the trace constraints discussed in Subsection 3.1.

### 7.2 Generating functions

In the previous subsection we have introduced main ingredients of the AdS covariant description of higher–spin fields characterized by a definite value of \( s \). It is however convenient, and actually indispensable when constructing higher–spin interactions, to work simultaneously with
the infinite set of spins \( s = 0, 1, \cdots, \infty \). To this end the formalism of generating functions is most appropriate.

The space of traceful rectangular two-row Young tableaux of the algebra \( gl(D + 1) \) can be conveniently described as the \( sp(2) \) invariant subspace \( \mathcal{V} \) of the space of polynomials \( f(Y) \) of the variables \( Y_i^A \) \((i = 1, 2, A = 0, 0', 1, \ldots D - 1) \) such that

\[
f(Y) = \sum_{n=0}^{\infty} f_{A_1 \cdots A_n, B_1 \cdots B_n} Y_1^{A_1} \cdots Y_1^{A_n} Y_2^{B_1} \cdots Y_2^{B_n},
\]

\[
f(Y) \in \mathcal{V} : (T_i^j - \frac{1}{2} \delta_j^k T_k^i) f(Y) = 0, \quad T_i^j = Y_i^A \frac{\partial}{\partial Y_j^A}.
\] (7.17)

Note that this system was recently described in analogous fashion in [33].

Since the conditions (7.17) are first order differential equations, \( \mathcal{V} \) is in fact the algebra with the pointwise product law in the \( Y \)-space, i.e., given two solutions \( f_1(Y) \) and \( f_2(Y) \) of (7.17), \( f_1(Y) f_2(Y) \) also solves the same condition. Note that the condition (7.17) requires in particular that

\[
\left( Y_1^A \frac{\partial}{\partial Y_1^A} - Y_2^A \frac{\partial}{\partial Y_2^A} \right) f(Y) = 0,
\] (7.18)

i.e. any polynomial in \( \mathcal{V} \) contains equal number of \( Y_1^A \) and \( Y_2^B \) as in (7.16). The coefficients \( f_{A_1 \cdots A_n, B_1 \cdots B_n} \) carry various \( gl(D + 1) \)–modules described by two-row rectangular Young tableaux (see also [33]). Note that, for a homogeneous polynomial of degree 2\( p \) (equivalently, for a rectangular two-row Young tableau of length \( p \)), the condition (7.17) can be rewritten in the form

\[
T_i^j f(Y) = p \delta_j^i f(Y).
\] (7.19)

A useful viewpoint is that the space \( \mathcal{V} \) is spanned by various functions of the elementary \( sp(2) \) invariant combinations \( T^{AB} = Y_i^A Y'^B_i \) where the \( sp(2) \) indices are raised and lowered by the \( sp(2) \) invariant symplectic form

\[
A^i = \varepsilon^{ij} A_j, \quad A_i = A^j \varepsilon_{ji},
\] (7.20)

i.e., all \( sp(2) \) indices are contracted among themselves. Clearly, the functions of this class form an algebra.

In terms of the generating functions (7.16)–(7.17) the higher–spin curvatures (7.11) and gauge transformations (7.12) take the following manifestly gauge invariant form

\[
\mathcal{R} = \mathcal{D} \Omega(Y),
\] (7.21)

\[
\delta \Omega(Y) = \mathcal{D} \xi(Y),
\] (7.22)

where

\[
\mathcal{D} = d + \Omega^{AB} Y_{Ai} \frac{\partial}{\partial Y_{A^B}^i}.
\] (7.23)

Irreducible two-row rectangular \( o(2, D - 1) \)–modules are described by polynomials with traceless coefficients \( f_{A_1 \cdots A_n, B_1 \cdots B_n} \), that, in addition to (7.17), satisfy the tracelessness conditions

\[
\frac{\partial^2}{\partial Y_i^A \partial Y_{A^B}^j} f(Y) = 0.
\] (7.24)
The class of functions with the relaxed traceless conditions which describe the AdS triplet system can be defined as the space $T \subset V$ spanned by various polynomials of the form (see also [33])

$$h(Y) = \sum_{p=0}^{\infty} (t(Y))^p h_p(Y),$$  \hspace{1cm} (7.25)

where

$$t(Y) = \eta_{ij}(Y)\eta^{ij}(Y) = 2 \det |\eta_{ij}(Y)|, \hspace{0.5cm} \eta_{ij}(Y) = Y_i^A Y_{Aj}, \hspace{0.5cm} \eta^{ij}(Y) = \epsilon^{il} \epsilon^{jk} \eta_{lk}(Y)$$  \hspace{1cm} (7.26)

and $h_p(Y)$ satisfy the conditions (7.17) and (7.24)

$$\frac{\partial^2}{\partial Y_i^A \partial Y_{Aj}} h_p(Y) = 0.$$  \hspace{1cm} (7.27)

Since, $t(Y)$ (7.26) is $sp(2)$ invariant, any element (7.25) belongs to the space $V$ of two-row rectangular Young tableaux. Although $h(Y)$ is not traceless, its $o(2, D-1)$ irreducible components $h_p(Y)$ all describe two–row rectangular traceless tensors.

Alternatively, the subspace $T \subset V$ can be described without explicit reference to the expansion (7.25) as the space of functions that satisfy the relaxed traceless condition

$$t(Y) \frac{\partial^2}{\partial Y_i^A \partial Y_{Aj}} h(Y) = \eta^{ij}(Y)\eta_{kl}(Y) \frac{\partial^2}{\partial Y_k^C \partial Y_{Cl}} h(Y).$$  \hspace{1cm} (7.28)

The key observation leading to this condition is that the result of action of $\frac{\partial^2}{\partial Y_i^A \partial Y_{Aj}}$ on any function of the form (7.25) is proportional to $\eta^{ij}(Y)$. It is then elementary to see that in this case (7.28) is true.

Thus, the condition (7.28) singles out only (and all) rectangular traceless two–row Young tableaux from the generic traceful two-row Young tableau. Correspondingly, if we consider a one-form connection $\Omega(Y)$ that takes values in $T$ and is a degree $2(s - 1)$ polynomial in $Y$

$$\Omega(Y) = \sum_{p=0}^{[\frac{s-2}{2}]} (t(Y))^p \Omega_p(Y),$$  \hspace{1cm} (7.29)

the traceless components $\Omega_p(Y)$ of this connection correspond to the set of fields of spins $s, s-2, s-4, \ldots, 3$ or 2. Thus, such an $\Omega(Y)$ describes a spin–$s$ triplet system (modulo the scalar and vector fields, as discussed in Section 3).

One can easily check that the components of $\Omega(Y)$ (7.29), which form the $2(s - 1)$ tensor (7.7)–(7.10), are related to the Lorentz covariant components of previous sections via projection along the compensator field $V^A$. Namely, the Lorentz irreducible components are singled out by the condition

$$\Omega^{(s-1,t)}(Y) = \Pi \left( V^{A_1} \frac{\partial}{\partial Y_{2}^{A_1}} \cdots V^{A_1} \frac{\partial}{\partial Y_{2}^{A_{s-1}-t}} \Omega(Y) \right),$$  \hspace{1cm} (7.30)

where $\Pi$ is the projector to the $V$–transversal part of $Y_i^A$

$$\Pi(f(Y)) = f(\Pi(Y)), \hspace{1cm} \Pi(Y_i^A) = Y_i^A + \Lambda V^A V_B Y_i^B.$$  \hspace{1cm} (7.31)
The higher-spin vielbein is the most $V$-longitudinal component of $\Omega$ with $t = 0$ in (7.30). The higher-spin auxiliary Lorentz-like connection has $t = 1$ while extra higher-spin connections have $t > 1$. In the gauge $V^A = \frac{1}{\sqrt{\Lambda}} \delta^A_0$ the resulting higher-spin vielbein and connection (7.15) (and corresponding gauge parameters) have the trace properties of the triplet system considered in Section 3, i.e., the higher-spin vielbein is traceful and the higher-spin connection satisfies the relaxed traceless condition (3.5). It can be also verified that the extra field

$$\omega^{a_1 \cdots a_{s-1}, b_1 b_2} = \frac{1}{(-\Lambda)^{s-2}} \Omega^{a_1 \cdots a_{s-1}, b_1 b_2 0' \cdots 0'}$$

with two $o(1,D-1)$ indices in the second row (and the corresponding Stueckelberg gauge parameter) satisfy the trace conditions (3.13)–(3.15). These are consistency checks of the relation of the AdS $o(2,D-1)$-covariant triplet construction under consideration with the $o(1,D-1)$-covariant description of the bosonic triplets of Section 3.

Let us note that beyond the space $T$, generic traceful rectangular two-row Young tableaux decompose into a set of irreducible $o(2,D-1)$ tensors that are not necessarily described by rectangular two-row Young tableaux (cf [33]). Since one-form connections valued in non-rectangular Young tableaux describe [32] so-called partially massless fields [31] which correspond to non-unitary representations of the $AdS_D$ algebra $o(2,D-1)$, it is important that these are ruled out of a quantum-mechanically consistent theory. In this respect, the relaxation of the tracelessness condition (7.24) to the relaxed (triplet) condition (7.28) is probably the maximal one within the class of fields that still correspond to a set of unitary massless fields described by the connections that take values in two-row rectangular traceless Young tableaux of $o(2,D-1)$.

### 7.3 Action

To formulate a manifestly gauge and $o(2,D-1)$-invariant action for the relaxed system we shall look for it in the form bilinear in the manifestly gauge invariant higher-spin curvatures (7.21).

It is convenient to use the version of the formalism of generating functions introduced in the previous subsection as proposed in [11, 56]. We describe a product of two curvatures as a state

$$\mathcal{R}(Y)\mathcal{R}(Z)|0\rangle$$

in a Fock space generated from the Fock vacuum $|0\rangle$ annihilated by the operators

$$\bar{Y}^A_i |0\rangle = \bar{Z}^A_i |0\rangle = 0.$$  

(7.33)

The annihilation operators $\bar{Y}^A_i$ and $\bar{Z}^A_i$ have the following commutation relations with $Y$ and $Z$

$$[\bar{Y}^A_i, Y^B_j] = \delta^B_i \delta^A_j, \quad [\bar{Z}^A_i, Z^B_j] = \delta^B_i \delta^A_j, \quad (7.34)$$

i.e., $\bar{Y}^A_i$ and $\bar{Z}^A_i$ are shorthand notations for $\frac{\partial}{\partial Y^A_i}$ and $\frac{\partial}{\partial Z^A_i}$, respectively.

We will look for the action of the form

$$S = \frac{1}{2} \int \langle 0 | F(Y, Z) \mathcal{R}(Y) \mathcal{R}(Z) |0\rangle,$$  

(7.35)
where $F$ is a $(D-4)$-form constructed from the one-form AdS background vielbein field $E^A = \mathcal{D}V^A$ and the compensator $V^A$

$$F(\bar{Y}, \bar{Z}) = e^{F_1...F_{D-4}ABCD}E_{F_1}...E_{F_{D-4}}V_A \bar{Y}_{B_1} \bar{Y}_{C_1} \bar{Z}_{D_1} \bar{Z}_{E_1} : \Phi(u, w, v) :.$$ \hfill (7.36)

In eq. (7.36) we use the following notation

$$u = V^C \bar{Y}_i^C V^D \bar{Z}_D^i,$$ \hfill (7.37)

$$w = \bar{Y}_D^i \bar{Z}_D^i,$$ \hfill (7.38)

$$v = \Delta_Y \Delta_Z, \quad \Delta_Y = \frac{1}{t(Y)} Y_k^C Y^{C_i} \bar{Y}_{D_k} \bar{Y}_i^D, \quad \Delta_Z = \frac{1}{t(Z)} Z_k^C Z^{C_i} \bar{Z}_{D_k} \bar{Z}_i^D.$$ \hfill (7.39)

Note that the operators $\Delta_Y, \Delta_Z$ and, hence, $v$ are well defined on the space $T$ of rectangular Young tableaux as one can easily check using the decomposition (7.25) and the property (7.19). Indeed, the operator $\Delta(Y)$ decreases by one unit a power of $t(Y)$ in the decomposition (7.25).

In particular, it gives zero when acting on the traceless polynomials that correspond to $p = 0$ in (7.25).

The normal ordering in (7.36) is defined such that $v$ acts before $u$ and $w$

$$: \Phi(u, w, v) := \sum_{p=0}^{\infty} \Phi_p(u, w)v^p.$$ \hfill (7.40)

A normal ordering prescription is required because $v$ does not commute with $u$ and $w$. In the sequel we omit the normal ordering symbol.

The respective roles of the variables $u, w$ and $v$ are as follows. The dependence of $\Phi$ on $u$ (7.37) takes care of the projection of the higher–spin curvatures along a certain number of $V^A$ similar to (7.30). The dependence of $\Phi$ on $w$ (7.38) controls the terms with different numbers of $O(2, D - 1)$ covariant contractions between the higher–spin curvatures. The dependence of $\Phi$ on $v$ (7.38) governs the coefficients for different irreducible fields in the reducible system. Since $v$ acts trivially on the traceless polynomials that correspond to the irreducible higher–spin system, the dependence on $v$ is irrelevant for their analysis, so in the irreducible case one can set $v = 0$.

The condition (7.28) on the vectors in the triplet space $T \otimes T$ is equivalent to

$$\bar{Y}_i^A Y^A j |\phi(Y, Z)\rangle = Y_i^A Y^A j \Delta_Y |\phi(Y, Z)\rangle, \quad \bar{Z}_i^A Z^A j |\phi(Y, Z)\rangle = Z_i^A Z^A j \Delta_Z |\phi(Y, Z)\rangle.$$ \hfill (7.41)

An important property of the construction is that if some $|h(Y, Z)\rangle$ satisfies the triplet condition (7.41), $\Delta_Y h(Y, Z), \Delta_Z h(Y, Z)$ and, hence, $vh(Y, Z)$ also does. This is a simple consequence of the fact that once $h(Y)$ has the form (7.25), then $\Delta_Y h(Y)$ also has this form.

The symmetry property of the action under the exchange of $\mathcal{R}(Y)$ and $\mathcal{R}(Z)$ implies that

$$\Phi(u, w, v) = \Phi(-u, -w, v).$$ \hfill (7.42)

Being constructed from the gauge invariant curvatures, the action (7.35) is manifestly invariant under the higher–spin gauge transformations. Consider a general variation of the action

$$\delta S = \int \langle 0 | F \mathcal{D} \delta \Omega(Y) \mathcal{R}(Z) | 0 \rangle.$$ \hfill (7.43)
Integrating by parts and taking into account that $F$ in (7.36) is constructed of manifestly $o(2, D - 1)$ covariant objects, we find that, in accordance with (7.6), nonzero contributions to the variation come only from the differentiation of the compensator field $V^A$ that enters $F$ both directly and via $u$ (7.37). The resulting expression, obtained with the help of the identity (2.11), has the form

$$\delta S = \frac{2}{D - 3} \int \langle 0|U \delta \Omega(Y) R(Z) |0\rangle,$$

where

$$U = \epsilon^{ABC} \bar{Y}_A \bar{Y}_B \bar{Z}_C \left( V^E \bar{Z}^i E_D \left( (D - 3) \Phi + 2u \frac{\partial \Phi}{\partial u} \right) + \left( \bar{Z}^G \bar{Y}_D + \bar{Y}_D V^F \bar{Z}^i \right) \right) + \bar{Y} \leftarrow \bar{Z}$$

and

$$\epsilon^{ABC} \equiv \epsilon^{F_1 \ldots F_{D-4}} \epsilon^{E F_1 \ldots E F_{D-4}}.$$

Our aim is to find such a function $\Phi(u, w, v)$ that the variation of the action is identically zero for all the fields in the allowed class except for the higher-spin vielbein and connection, identified, respectively with the $V$-longitudinal and $V$-transversal components of $(\lambda_2 \lambda_2)^{-2}(Y) |0\rangle$ (cf. eq. (7.30)). This condition, usually referred to as the extra field decoupling condition, guarantees that the action is free of higher derivatives carried by extra fields upon imposing appropriate constraints that express them in terms of derivatives of the higher-spin vielbein.

To this end it is helpful to use specific identities that hold as a consequence of the properties of the class of fields under consideration. The simplest of such identities follows from the condition (7.17) that the fields are $sp(2)$ singlets as they describe rectangular Young tableaux. Namely, from the identity

$$\langle 0|\epsilon^{ABC} Y_\lambda Y_\mu Y_\nu \Omega(Y) R(Z) |0\rangle = 0$$

for any $\Lambda(u, w, v)$, where we also use that $\bar{Y}_A \bar{Y}_B \bar{Z}_C$ commutes with $Z^A_i$, it follows that

$$\langle 0| - \frac{2}{D - 3} \epsilon^{ABC} \bar{Y}_A \bar{Y}_B \bar{Z}^i C \left( u \frac{\partial \Omega}{\partial u} V^F \bar{Z}^j + 2 \bar{Y}_F \bar{Z}^k \frac{\partial \Omega}{\partial w} V^G \bar{Z}^l \right) \delta \Omega(Y) R(Z) |0\rangle = 0.$$

A more complicated identity, that follows from the conditions (7.41), is

$$\langle 0| \frac{2}{D - 3} \epsilon^{ABC} \bar{Y}_A \bar{Y}_B \bar{Z}^i \left( \frac{\partial \Omega}{\partial u} \bar{Z}^j V^F \bar{Y}_F (v \frac{\partial W}{\partial w^4} - W) \right) + V^F \bar{Z}^j \left( 6 \frac{\partial^3}{\partial w^3} - 3 \frac{\partial^3}{\partial w^2 u} + u (2 \frac{\partial}{\partial w} - \frac{\partial}{\partial u}) \frac{\partial^3}{\partial u \partial w^2} \right) v W \right) \delta \Omega(Y) R(Z) |0\rangle = 0$$

for any $W(u, w, v)$.

Summing the variation (7.44) and (7.45) with the identities (7.47) and (7.49) and considering the terms in front of

$$\epsilon^{ABC} \bar{Y}_A \bar{Y}_B \bar{Z}_C V^E \bar{Z}^l, \quad \epsilon^{ABC} \bar{Y}_A \bar{Y}_B \bar{Z}_C \bar{Z}^i \bar{Z}^j \bar{Z}^k V^F \bar{Y}_E, \quad \epsilon^{ABC} \bar{Y}_A \bar{Y}_B \bar{Z}_C \bar{Z}^i \bar{Z}^j \bar{Z}^k V^E \bar{Y}_E.$$
we obtain the three conditions

\[(D - 3)\Phi + 2u \frac{\partial \Phi}{\partial u} + \frac{1}{2} w \frac{\partial \Phi}{\partial u} + u \frac{\partial \Lambda}{\partial u} - \left(6 \frac{\partial^3}{\partial w^3} - 3 \frac{\partial^3}{\partial u \partial w^2} + u(2 \frac{\partial}{\partial w} - \frac{\partial}{\partial u}) \frac{\partial^3}{\partial u \partial w^2}\right) v W = 2A(u, w, v),\]  

(7.50)

\[\frac{\partial \Phi}{\partial u} + W - v \frac{\partial^4}{\partial w^4} W = 0,\]  

(7.51)

\[\frac{\partial \Phi}{\partial u} + 2 \frac{\partial \Lambda}{\partial w} = 0,\]  

(7.52)

where \(A(u, w, v)\) determines the coefficients of the variation of the action.

To obey the extra field decoupling condition the coefficient function \(A(u, w, v)\) should only depend on \(uw\) and \(v\)

\[A(u, w, v) = \sum_{s=2}^{\infty} \sum_{p=0}^{\infty} A_{s,p}(uw)^{s-2-4p}(v)^p,\]  

(7.53)

where the dependence of \(A_{s,p}\) on \(s\) encodes relative coefficients for the triplet actions with different spins while the dependence on \(p\) encodes the relative coefficients of the irreducible spins within a given triplet system. Recall that the property that the maximal number (i.e., \(s - 2\)) of indices of the components \(\delta \Omega\) or \(R\) in (7.44) are contracted with the compensators \(V^4\) just implies that \(A(u, w, v)\) depends on \(uw\), thus ensuring that the action depends only on the higher-spin vielbein and connection.

The system of differential equations (7.50), (7.51) and (7.52) can be solved exactly. Indeed, (7.52) implies that

\[\Phi = 2 \frac{\partial \varphi}{\partial w}, \quad \Lambda = -\frac{\partial \varphi}{\partial u}.\]  

(7.54)

Setting also

\[W = \frac{\partial H}{\partial u}, \quad \varphi = \psi + \frac{1}{2} \frac{\partial^3 H}{\partial w^3} v\]  

(7.55)

we find by virtue of (7.51) that

\[H = -2 \frac{\partial \psi}{\partial w}\]  

(7.56)

and therefore everything is expressed in terms of \(\psi\) that has to satisfy the equation (7.50).

To analyze the resulting differential equation it is convenient to introduce the following integral transform

\[\psi(u, w, v) = \int d\sigma d\tau e^{\sigma u + \tau w} \tilde{\psi}(\sigma, \tau, v), \quad A(u, w, v) = \int d\sigma d\tau e^{\sigma u + \tau w} \tilde{A}(\sigma, \tau, v),\]  

(7.57)

where the integration measure is defined such that

\[\int d\sigma d\tau \sigma^p \tau^q = \delta_{p+1}^0 \delta_{q+1}^0, \quad p, q \in \mathbb{Z}.\]  

(7.58)

Clearly, this transform relates the power series expansions as follows

\[\phi(u, w) = \sum_{n,m=0}^{\infty} a_{n,m} u^n w^m \quad \longleftrightarrow \quad \tilde{\phi}(\sigma, \tau) = \sum_{n,m=0}^{\infty} n! m! a_{n,m} \sigma^{-n-1} \tau^{-m-1},\]  

(7.59)
thus adding (removing) factorials to the coefficients. Let us stress that, by analogy with the usual Cauchy integral, functions like $\tilde{\psi}(\sigma, \tau, v)$ and $\tilde{A}(\sigma, \tau, v)$, that are analytic in $\sigma$ and/or $\tau$, do not contribute under the integral (7.57). In the sequel, the equalities up to such terms will be denoted by $\simeq$. The functions expandable in strictly negative powers of $\sigma$ and $\tau$ will be called relevant, while those analytic in $\sigma$ and/or $\tau$ will be called irrelevant. Thus $\simeq$ is the equality modulo irrelevant functions.

By the transform (7.57), the equation (7.50) amounts to

$$
\left((1 - \tau^4)v\right)\left((D - 5)\tau + \frac{1}{2}\sigma - 2\sigma\tau \frac{\partial}{\partial \sigma} + \frac{1}{2}\sigma^2 \frac{\partial}{\partial \sigma} - \frac{1}{2}\sigma\tau \frac{\partial}{\partial \tau}\right)
+ 4\tau^4 \sigma v + (\sigma^3 \tau^3 - 2\tau^4 \sigma^2) v \frac{\partial}{\partial \sigma} \tilde{\psi}(\sigma, \tau, v) \simeq \tilde{A}(\sigma, \tau, v).$$

(7.60)

The following comment is now in order. The reason why we have chosen the action (7.35) with the function $F$ (7.36) that depends on the variables $u$ and $w$ is that this choice leads to the first-order equation (7.60). This choice of variables differs from that used by Alkalaev in [33], that has the advantage of being manifestly $sp(2)$ invariant, allowing to avoid using the identities (7.47) in the analysis. However, this is achieved at the cost that the variables of [33] are quartic in $\bar{Y}$ and $\bar{Z}$ (in our notation). This higher nonlinearity of the variables of [33] is expected to lead to nonlinearity of the identity (7.49) to be translated to a higher-order counterpart of the equation (7.60). Still, an interesting problem for the future study is to reconsider the problem using the variables of [33].

From eqs. (7.54)-(7.57) it follows that

$$\tilde{\Phi}(\sigma, \tau, v) \simeq 2\tau(1 - \tau^4v)\tilde{\psi}(\sigma, \tau, v).$$

(7.61)

The resulting equation on $\tilde{\Phi}$ is most conveniently formulated in terms of the variables

$$\mu = \sigma\tau, \quad \nu = 2\tau^2.$$ 

(7.62)

Using notation

$$\Phi'(\mu, \nu, v) = \tilde{\Phi}(\sigma, \tau, v), \quad A'(\mu, \nu, v) = \tilde{A}(\sigma, \tau, v),$$

(7.63)

the final equation is

$$\left(\frac{D - 5}{2} + L_0 + L_1\right)\Phi'(\mu, \nu, v) \simeq A'(\mu, \nu, v),$$

(7.64)

where

$$L_0 = \frac{\mu}{\nu} - \mu\left(\frac{\partial}{\partial \mu} + \frac{\partial}{\partial \nu}\right),$$

(7.65)

$$L_1 = \frac{1}{2}v \left(3\nu\mu + \mu^2(\mu - \nu) \frac{\partial}{\partial \mu}\right) \left(1 - \frac{1}{4}\nu^2 v\right)^{-1}.$$ 

(7.66)

In principle, it is not hard to solve the equation (7.64) with the strict equality instead of $\simeq$ (see Appendix II). However, this method is not most efficient just because the right hand side of (7.64) is known up to irrelevant terms. The formal solution obtained this way for the function $\tilde{A}(\sigma, \tau, v)$, that corresponds to (7.53), leads to the wrong physical solution with an unwanted contribution at the boundary of the “relevance region”. That this can happen
follows for example from the $\mu$ term in (7.64), that can map irrelevant functions to the relevant ones, thus giving a fake contribution at the boundary of the "relevance region". To get rid of these unwanted terms, one has to adjust an irrelevant right hand side of (7.64) such that it gives the relevant solution $\Phi^{8}$. The solution to this problem is not obvious, however. Hence, we proceed differently, by solving the equation (7.64) via an appropriate Ansatz directly in the relevant class.

Note that in terms of the variables $\mu$ and $\nu$, the relevant functions have the form

$$F^{\text{rel}}(\mu, \nu) = \mu^{-1} P(\mu^{-1}, \nu^{-1})$$  \hspace{1cm} (7.67)

for an entire function $P(x, y)$ (polynomial for a given triplet system).

The key observation is that the following identity is true for $a > -1$ and any $C(x, y)$

$$\left( L_0 + a \right) \left( \mu^{-1} \int_0^1 ds \int_0^1 dt (1-t)^a C(s(1-t)\mu^{-1} - (1-s)t\nu^{-1}, (1-s)\nu^{-1}) \right)$$

$$= \mu^{-1} \int_0^1 ds C(s\mu^{-1}, (1-s)\nu^{-1}) - \nu^{-1} \int_0^1 dt C(-t\nu^{-1}, \nu^{-1}),$$  \hspace{1cm} (7.68)

which follows from the following two elementary identities

$$L_0 \mu^{-1} = \mu^{-1}(L_0 + 1)$$  \hspace{1cm} (7.69)

and

$$\left( L_0 + a \right) \left( \int_0^1 ds \int_0^1 dt (1-t)^{(a-1)} C(s(1-t)\mu^{-1} - (1-s)t\nu^{-1}, (1-s)\nu^{-1}) \right) =$$

$$- \int_0^1 ds \int_0^1 dt \left( \frac{\partial}{\partial t} ((1-t)^a C(s(1-t)\mu^{-1} - (1-s)t\nu^{-1}, (1-s)\nu^{-1})) \right)$$

$$+ \frac{\mu}{\nu} \frac{\partial}{\partial s} \left( (1-s)C(s(1-t)\mu^{-1} - (1-s)t\nu^{-1}, (1-s)\nu^{-1)) \right).$$  \hspace{1cm} (7.70)

Now we observe that the second term on the right hand side of (7.68) is irrelevant because it is $\mu$ independent and, hence, $\sigma$–independent. (Note that this is just the irrelevant term to be added to make it possible to reconstruct an appropriate formal solution of (7.64) as discussed above.) Hence, the identity (7.68) gives

$$\left( L_0 + a \right) \left( \mu^{-1} \int_0^1 ds \int_0^1 dt (1-t)^a C(s(1-t)\mu^{-1} - (1-s)t\nu^{-1}, (1-s)\nu^{-1)) \right) = U(C)(\mu, \nu),$$  \hspace{1cm} (7.71)

where

$$U(C)(\mu, \nu) \equiv \mu^{-1} \int_0^1 ds C(s\mu^{-1}, (1-s)\nu^{-1}).$$  \hspace{1cm} (7.72)

Using that

$$\int_0^1 dt t^a (1-t)^b = \frac{a! b!}{(a + b + 1)!},$$  \hspace{1cm} (7.73)

\footnote{In practice, it is enough to get rid of the nonzero terms at the boundary of the set of irrelevant functions, \textit{i.e.}, those that are constants in $\sigma$ or $\tau$.}
we obtain
\[ C(\mu^{-1}, \nu^{-1}) = \sum_{n,m=0}^{\infty} c_{n,m} \mu^{-n} \nu^{-m} \quad \rightarrow \quad U(C)(\mu, \nu) = \sum_{n,m=0}^{\infty} \frac{n! m!}{(n + m + 1)!} c_{n,m} \mu^{-n-1} \nu^{-m}. \] (7.74)

The inverse transform to (7.72) can be written in the form
\[ C(\mu^{-1}, \nu^{-1}) = U^{-1}(A)(\mu^{-1}, \nu^{-1}) \equiv -\mu^2 \frac{\partial}{\partial \mu} \int d\sigma' d\tau' \frac{1}{\mu'} A((\mu^{-1} + \nu^{-1} \mu'^{-1})^{-1}, (\mu^{-1} \nu'^{-1} + \nu^{-1})^{-1}). \] (7.75)

Now we can write the solution of the equation
\[ (L_0 + a)\Phi_0(\mu, \nu) = A'_0(\mu, \nu) \] (7.76)
in the form
\[ \Phi_0(\mu, \nu) = (L_0 + a)^{-1} A'_0(\mu, \nu), \] (7.77)

where
\[ (L_0 + a)^{-1} A'_0(\mu, \nu) = \mu^{-1} \int_0^1 ds \int_0^1 dt (1 - t)^a U^{-1}(A)(s(1 - t)\mu^{-1} - (1 - s)t\nu^{-1}, (1 - s)\nu^{-1}). \] (7.78)

An important property of all maps under consideration is that they act within the class of relevant functions. Also it is clear that if \( A'_0(\mu, \nu) \) is a homogeneous function of degree \( n \), then the same is true for \( \Phi_0(\mu, \nu) \). This property manifests that the solutions for different spins, corresponding to different homogeneity degrees, are independent.

To obtain the formula for the solution of (7.64), that determines the coefficients of the action, in terms of the expansion in powers of \( \nu \), which is equivalent to the lower-spin expansion within the system of triplet fields, it remains to define precisely the multiplication law by \( \mu \) and \( \nu \) on the functions of the class (7.67). The rule is simply that the irrelevant terms should be discarded. This means that \( \mu P(\mu^{-1}, \nu^{-1}) \) should be replaced by \( \mu \circ P(\mu^{-1}, \nu^{-1}) = \mu(P(\mu^{-1}, \nu^{-1}) - P(0, \nu^{-1})) \). Equivalently, one can write
\[ \mu \circ P(\mu^{-1}, \nu^{-1}) = \int_0^1 dt P_{1,0}(t\mu^{-1}, \nu^{-1}), \quad \nu \circ P(\mu^{-1}, \nu^{-1}) = \int_0^1 dt P_{0,1}(\mu^{-1}, t\nu^{-1}), \] (7.79)

where
\[ P_{n,m}(x, y) = \frac{\partial^{n+m}}{\partial x^n \partial y^m} P(x, y). \] (7.80)

Successive application of this formula gives
\[ (\mu^n \nu^m) \circ P(\mu^{-1}, \nu^{-1}) = \frac{1}{nm} \int_0^1 dt (1 - t)^{n-1} \int_0^1 du (1 - u)^{m-1} P_{n,m}(t\mu^{-1}, u\nu^{-1}), \quad m, n > 0. \] (7.81)

The solution of the equation (7.64) can now be written in the form
\[ \Phi'(\mu, \nu, v) = \sum_{n=0}^{\infty} (-1)^n \left( L_0 + \frac{D - 5}{2} \right)^{-1} \left( L_1 \left( L_0 + \frac{D - 5}{2} \right)^{-1} \right)^n A'(\mu, \nu, v), \] (7.82)
where all multiplications with $\mu$ and $\nu$ contained in $L_1$ (7.66) should be understood as the $\circ$-multiplication (7.81). (Note that to work from the very beginning within the relevant class, one should replace usual multiplication by $\circ$ directly in (7.64), which however then becomes an integro-differential equation.)

To find the action for the triplet system we have to apply the formula (7.82) to $A'(\mu, v)$ because the functions $\tilde{A}(\sigma, \tau, v) = \sigma^p \tau^q \rho(v)$ with $p < q$ give rise to a trivial variation by virtue of the Young symmetry properties of the fields, while those with $p > q$ give rise to the actions that contain extra fields, thus leading to the field equations with higher derivatives. In accordance with (7.53), for the spin $s$ triplet system

$$\tilde{A}(\mu, v) = \mu^{1-s} \tilde{A}_s(v),$$

(7.83)

where the function $\tilde{A}_s(v)$ determines the coefficients of the action for the irreducible spin components in the triplet system with the highest spin $s$. The most convenient choice is $\tilde{A}_s(v) = \tilde{A}_s$, i.e.

$$\tilde{A}(\mu, v) = \mu^{1-s} A_s,$$

(7.84)

where $A_s$ is an overall normalization constant coefficient for the reducible spin–s system.

The absence of the extra fields in the triplet action (7.35), the Young–tableau structure and symmetry properties of its components allow us to conclude that in the gauge $V_A = \frac{1}{\sqrt{-\Lambda}} \delta^a_A$ it reduces to the bosonic triplet action (3.3).

### 7.4 Irreducible case

To illustrate the obtained result let us consider the example of an irreducible massless field. In this case, we should set $v = 0$ since $v$ acts trivially on the irreducible $\Omega$.

The covariant action for the irreducible case was obtained in [11] in the form

$$S = \frac{1}{2} \sum_{p=0}^{s-2} \frac{(s-1)!}{(p+1)!(d-5-2p)!} \alpha(s) 2^p \frac{(d-5)-p}{p!} F_{A_1 \ldots A_{s-1}} Y_{A_1} \ldots Y_{A_{s-1}} Y_{B_1} \ldots Y_{B_{s-1}}$$

(7.85)

with some overall spin-dependent normalization factor $\alpha(s)$. The coefficients in this action were determined in [11] from the extra field decoupling condition that guarantees that it properly describes a spin $s$ irreducible massless field.

The components of the higher–spin curvatures in (7.85) result from the following expansion of the higher–spin curvatures (7.21)

$$R(Y) = \sum_{s=1}^{\infty} \frac{1}{((s-1)!)^2} R_{A_1 \ldots A_{s-1}, B_1 \ldots B_{s-1}} Y_{A_1} \ldots Y_{A_{s-1}} Y_{B_1} \ldots Y_{B_{s-1}}.$$

(7.86)

To compare the action (7.35), (7.36) with (7.85) let us evaluate $\langle 0 | u^p w^{2s-p} R(Y) R(Z) | 0 \rangle$. Using the expansion (7.86) along with $u$ (7.37) and $w$ (7.38) in the form

$$u = -V_A V_B \left( \frac{\partial^2}{\partial Y_A \partial Z_B} - \frac{\partial^2}{\partial Y_A \partial Z_B} \right), \quad w = \left( \frac{\partial^2}{\partial Y_A \partial Z_A} - \frac{\partial^2}{\partial Y_A \partial Z_A} \right),$$

(7.87)
direct differentiation gives
\[
\langle 0 | u^p w^{2s'-p} R(Y) R(Z) | 0 \rangle = \sum_{n,m=0}^{\infty} (-1)^{p+s'} \delta(n + m - s') \frac{p!(2s' - p)!}{n!(p - n)!m!(2s' - p - m)!} V_{C_1} \ldots V_{2p-n} R_{C_{1-n}A_{1-n} C_{n+1-n} C_{p-n} B_1 \ldots B_{s'-p+n}} R_{C_{p-n+1} \ldots C_{2(p-n)-1} B_1 \ldots B_{s'-p+n}, A_1 \ldots A_m C_{2(p-n)+1} \ldots C_{2p-n}}. \tag{7.88}
\]

Then using repeatedly the Young properties of the higher-spin curvatures in the form
\[
R_{A_1 \ldots A_{s'-n}(B_1 \ldots B_n, B_{n+1} \ldots B_{s'-n+k}) C_1 \ldots C_{s'-n}} = (-1)^k \frac{n!(s'-n)!}{(n-k)!(s'-n-k)!} R_{B_1 \ldots B_{n+k}(A_1 \ldots A_{s'-n-k}, A_{s'-n-k+1} \ldots A_{s'-n}) C_1 \ldots C_{s'-n}} \tag{7.89}
\]
it is not difficult to obtain
\[
\langle 0 | u^p w^{2s'-p} R(Y) R(Z) | 0 \rangle = (-1)^p (p+1) \frac{(2s'-p)!}{s'!(s'-p)!} V_{C_1} \ldots V_{C_{2p}} R_{C_{p-1} \ldots C_{2p} A_1 \ldots A_{s'-p}, B_1 \ldots B_{s'}}. \tag{7.90}
\]

To apply this formula to the action (7.35) it remains to observe that the contraction of the indices \(A, B, C, D\) in the action (7.85) with the epsilon symbol does not change the Young symmetry properties with respect to the other indices carried by the higher-spin curvatures, shifting effectively the parameter \(s'\) by one unit.

Direct comparison shows that the action (7.85) is reproduced by the function \(\Phi(u, w, 0)\) in (7.36) of the form
\[
\Phi(u, w, 0) = \sum_{p, s} \alpha(s) (-2)^p \left(\frac{D-5}{2} + p\right)! (s-p-2)! (s-2)! \frac{p!(2s-p)!}{p!(2(s-2)-p)!} u^p w^{2(s-2)-p}. \tag{7.91}
\]
The transform (7.57) gives
\[
\bar{\Phi}(\nu, \mu, 0) = \sum_{s=2}^{s=2} \sum_{p=0}^{\infty} \alpha(s) (-1)^p \left(\frac{D-5}{2} + p\right)! (s-p-2)! (s-2)! \mu^{-p-1} \nu^{-(s-p-2)}. \tag{7.92}
\]

On the other hand, choosing $\tilde{A}(\mu, 0)$, in accordance with (7.84), we obtain
\[
C(\mu^{-1}, \nu^{-1}) = U^{-1}(A') (\mu^{-1}, \nu^{-1}) = (s-1) A_s \mu^{2-s} \tag{7.93}
\]
and, by (7.78) with $a = \frac{D-5}{2}$,
\[
\check{\Phi}(\mu, \nu) = (s-1) A_s \mu^{-1} \int_0^1 du \int_0^1 dt (1-t) \frac{(D-5)}{2} (u(1-t) \mu^{-1} - (1-u) t \nu^{-1})^{s-2}. \tag{7.94}
\]
Using (7.73) this gives
\[
\check{\Phi}(\nu, \mu, 0) = \sum_{s=2}^{s=2} \sum_{p=0}^{\infty} (-1)^{s+p} A_s (-1)^p \left(\frac{D-5}{2} + p\right)! (s-p-2)! (\frac{D-5}{2} + s-2)! \mu^{-p-1} \nu^{-(s-p-2)}. \tag{7.95}
\]

We observe that this formula indeed coincides with (7.92) provided that
\[
A_s = (-1)^s \left(\frac{D-5}{2} + s-2\right)! (s-2)! \alpha(s). \tag{7.96}
\]

Expanding the expression (7.82) in powers of \(v\) one can systematically reconstruct the covariant action coefficients for the lower-spin components of the triplet systems.
Towards the AdS covariant formulation of the fermionic higher–spin fields

In conclusion of the main part of this paper we present basic ingredients of the AdS covariant description of the relaxed systems of fermionic massless symmetric higher–spin fields.

In the $o(2,D-1)$ covariant formalism, a single symmetric fermionic massless field of spin $s$ is described by a tensor–spinor one–form $\Psi_{A_1...A_{s-\frac{D-1}{2}},B_1...B_{s-\frac{D-1}{2}}}$ that has properties of a gamma–transversal two–row rectangular Young tableau

$$\Psi_{(A_1...A_{s-\frac{D-1}{2}},A_{s})B_2...B_{s-\frac{D-1}{2}}} = 0, \quad \Gamma^{A_1}\Psi_{A_1...A_{s-\frac{D-1}{2}},B_1...B_{s-\frac{D-1}{2}}} = 0.$$  \hfill (8.1)

This is because the decomposition of this $spin(2,D-1)$ irreducible tensor-spinor into the $spin(1,D-1)$ Lorentz irreducible tensor–spinors gives all gamma–transversal two–row Young tableaux, which is precisely the pattern of higher-spin connections introduced in [8, 10].

Like in the bosonic case, an unrestricted rectangular two–row tensor-spinor can be described by $\gamma$-dependent spinor

$$f_{\hat{\alpha}}(Y) = \sum_{n=0}^{\infty} f_{\hat{\alpha} A_1...A_n,B_1...B_n} Y_1^{A_1} \cdots Y_1^{A_n} Y_2^{B_1} \cdots Y_2^{B_n},$$  \hfill (8.2)

that satisfies the $sp(2)$ invariance condition (7.17) (here $\hat{\alpha}, \hat{\beta}$ are spinor indices of $o(2,D-1)$). Irreducible $o(2,D-1)$–modules are described by polynomials $f_{\hat{\alpha} A_1...A_n,B_1...B_n}$, that, in addition to (7.17), satisfy the gamma-transversality condition

$$\Gamma^{A\hat{\alpha}} \frac{\partial}{\partial Y^A} f_{\hat{\beta}}(Y) = 0.$$  \hfill (8.3)

The class of functions $T$ with the relaxed gamma–traceless conditions appropriate for the description of fermionic higher–spin triplets is formed by various polynomials of the form

$$f(Y) = \sum_{p=0}^{\infty} (\Gamma(Y))^p f_p(Y),$$  \hfill (8.4)

where $f_p(Y)$ satisfy the conditions (7.17) and (8.3) and

$$\Gamma(Y) \equiv \Gamma_i \Gamma^i, \quad \Gamma_i \equiv \Gamma^A Y_{Ai}.$$  \hfill (8.5)

Note that $(\Gamma(Y))^p$ in (8.4) is understood as the $p^{th}$ matrix power of $\Gamma_{\hat{\alpha}\hat{\beta}}(Y)$ (8.5). It is easy to see that the relation $\Gamma^A \Gamma^B + \Gamma^B \Gamma^A = 2\eta^{AB}$ implies

$$\Gamma_i \Gamma_j + \Gamma_j \Gamma_i = 2\eta_{ij}(Y), \quad \eta_{ij}(Y) = \eta_{AB} Y^A_i Y^B_j.$$  \hfill (8.6)

From here it follows that

$$(\Gamma(Y))^{2\hat{\beta}}_{\hat{\alpha}} = -2 t(Y) \delta^\beta_{\alpha},$$  \hfill (8.7)

where $t(Y) = \eta_{ij}(Y) \eta^{ij}(Y)$ was introduced in (7.26). Note also that

$$\eta_{ki} \eta^{kj} = -\eta_{i}^{k} \eta_{k}^{j} = \frac{1}{2} \delta^j_i \ t(Y).$$  \hfill (8.8)
Recall that the \( sp(2) \) indices are raised and lowered by the \( sp(2) \) symplectic forms according to (7.20).

Since \( i \) takes two values, \( \Gamma(Y) \) plays the role of the “\( \Gamma_3 \)-matrix”, i.e. it anticommutes with \( \Gamma_i \)

\[
\Gamma(Y) \Gamma_i + \Gamma_i \Gamma(Y) = 0, \quad (8.9)
\]

which together with (8.6) implies a useful relation

\[
\Gamma(Y) \Gamma^i = 2\eta^{ij}(Y) \Gamma_j. \quad (8.10)
\]

We observe that \( \Gamma(Y) \) satisfies the condition (7.17) because the indices in (8.5) are contracted in the \( sp(2) \) invariant way. Therefore, \( f(Y) \) of the form (8.4) satisfies the two-row Young symmetry condition (7.17).

It is not hard to see that the characteristic property of the space \( T \) (8.4) is that, taking into account (8.10), any its element satisfies the property

\[
f(Y) \in T : \quad \Gamma^A \frac{\partial}{\partial Y^A} f(Y) = \Gamma^i(Y) g(Y), \quad (8.11)
\]

where \( g(Y) \) is again a \( Y \)-polynomial of the type (8.4) (i.e. \( g(Y) \in T \)).

From eqs. (8.11) and (8.10) it follows that the relaxed gamma–transversality condition in \( T \) has the form

\[
\Gamma^A \frac{\partial}{\partial Y^A} f(Y) = -2\eta^{ij}(Y)\Gamma^A \frac{\partial}{\partial Y^A} f(Y) . \quad (8.12)
\]

Equivalently, it can be written in the form

\[
P^i_j \Gamma^A \frac{\partial}{\partial Y^A} f(Y) = 0, \quad (8.13)
\]

where

\[
P^i_j = \delta^i_j - \frac{1}{2t(Y)} \Gamma^i \Gamma_j = \frac{1}{2} \left( \delta^i_j - \frac{1}{t(Y)} \Gamma(Y) \eta^{ij}(Y) \right). \quad (8.14)
\]

is the projector, i.e. \( P^i_j P^j_k = P^i_k \).

Clearly, the constraint (8.13) is weaker than the gamma–transversality condition (8.3) satisfied by the irreducible higher–spin fields. As expected, eq. (8.13) singles out the reducible (triplet) systems of symmetric fermionic fields: all \( o(2, D - 1) \) irreducible components of a degree \( 2(s - 3/2) \) one–form connection \( \Psi(Y) \) describe two-row rectangular gamma-transversal tensor–spinors that correspond to the set of fermionic fields of spins \( s, s - 1, s - 2, \ldots, \frac{3}{2} \). Thus, the \( T \)–valued one–form spinors \( \Psi(Y) \) which are subject to the gauge transformations

\[
\delta \Psi(Y) = \mathcal{D} \xi(Y), \quad \mathcal{D} \mathcal{D} = 0 \quad (8.15)
\]

describe in the AdS \( o(2, D - 1) \)–covariant way the fermionic higher–spin triplets discussed in detail in Subsection 5.2.

The manifestly gauge and \( o(2, D - 1) \)–invariant analysis of the fermionic action is more involved than of the bosonic one. Even the case of irreducible higher–spin fermions is currently under investigation [62]. We therefore leave for the future consideration the formulation of the manifestly gauge and \( o(2, D - 1) \)–invariant action for the fermionic triplet system.


9 Conclusion

We have considered the frame–like Lagrangian formulation of free systems of bosonic and fermionic higher–spin fields in flat and AdS backgrounds of arbitrary dimension. We have shown that the higher–spin systems described by an unconstrained higher–spin vielbein and by the connections which are subject to weaker (gamma)–trace constraints than those required for the description of single Fronsdal and Fang–Fronsdal fields correspond to the higher–spin triplets whose fields are associated with certain components of the higher–spin vielbein and connection. We have thus endowed the triplet fields with a clear geometrical meaning. This allowed us to identify the appropriate form of the gauge transformations of the fermionic triplets in AdS space and construct the gauge invariant action which describes their dynamics. We have also shown how upon imposing the pure gauge constraints on the (gamma)–trace of the higher–spin vielbeins one reduces the triplet systems to the frame–like versions of the unconstrained formulations of single higher–spin fields considered in [42, 45].

An interesting direction of future research is the extension of the obtained results to the interacting level. An important related question is whether the conditions (7.24) and (8.12) have an algebraic meaning that would allow one to figure out what might be a higher–spin algebra underlying these reducible higher–spin multiplets.

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Appendix A

In this Appendix we show that the variation of the action (2.7) with respect to the connection $\omega$ results in the zero–torsion condition (2.4) both in the Fronsdal case and (modulo spin 1) in the case in which the higher–spin vielbein is traceful and the connection $\omega$ is subject to the relaxed trace condition (2.9).

In the Fronsdal case, the traceless higher–spin vielbein $\tilde{e}_{m; n_1 \ldots n_{s-1}}$ contains the irreducible Lorentz tensors described by the following Young tableaux

$$\square \otimes \begin{array}{c} s-1 \\ \end{array} = \begin{array}{c} s \\ \end{array} \oplus \begin{array}{c} s-2 \\ \end{array} \oplus \begin{array}{c} 1 \\ \end{array}^{s-1}. \quad (A.1)$$

The first tableau of length $s$ on the right hand side of (A.1) describes the totally symmetric and traceless part of $\tilde{e}$, the second tableau of the length $s-2$ corresponds to the traceless
\( \eta^{m_{n_{s-1}}} \tilde{e}_{m; n_1 \cdots n_{s-1}} \) and the hook tableau corresponds to the irreducible (traceless) part of \( \tilde{e} \) that satisfies \( \tilde{e}_{(m; n_1 \cdots n_{s-1})} = 0 \).

Because of the gauge transformation law of the higher-spin vielbein

\[
\delta \tilde{e}^{a_1 \cdots a_{s-1}} = d\tilde{\xi}^{a_1 \cdots a_{s-1}} - dx^m \tilde{\xi}^{a_1 \cdots a_{s-1}, b} \eta_{mb},
\]

the hook part of the vielbein can be gauged to zero by the gauge shift with the parameter \( \tilde{\xi}^{a_1 \cdots a_{s-1}, b} \). As a result, the remaining part of the vielbein is the combination of two totally symmetric traceless tensor of rank \( s \) and \( s - 2 \) equivalent to the double traceless Fronsdal field.

The fact that the zero torsion condition (2.17) is equivalent to the equation of motion (2.15) is deduced as follows.

The torsion tensor \( T_{mn; n_1 \cdots n_{s-1}} \) contains the irreducible (traceless) components describe by the following set of Young tableaux

\[
\otimes_{s-1} = \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}.
\]

Note that the last two tableaux describe the two irreducible parts of \( \eta_{nn_1} T_{mn; n_1 \cdots n_{s-1}} \).

On the other hand the connection \( \tilde{\omega}_{m; n_1 \cdots n_{s-1}, m} \) has the following decomposition

\[
\otimes_{s-1} = \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}.
\]

Observe that the two decompositions (A.3) and (A.4) differ by the first tableau on the right hand side of (A.4) which, however, is just the pure gauge part of the higher-spin connection which can be set to zero by the gauge shift

\[
\delta \tilde{\omega}_{n_1 \cdots n_{s-1}, m} = -\tilde{\xi}_{a_1 \cdots a_{s-1}, b}.
\]

As a result the torsion tensor has as many components as the higher-spin connection \( \tilde{\omega}^{a_1 \cdots a_{s-1}, m} \) modulo its pure gauge part. So the number of the independent field equations of \( \tilde{\omega} \) in (2.15) equals to the number of the components of the torsion.

**In the triplet case** in which the vielbein is unconstrained while the connection and gauge parameters satisfy the relaxed traceless conditions (2.9) and (2.14), the torsion tensor has the decomposition in terms of the *traceful* Young tableaux. It thus does not contain the last two terms in (A.3), namely,

\[
\otimes_{s-1} = \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}.
\]

The *traceful* Young tableau decomposition of the connection is

\[
\otimes_{s-1} = \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}.
\]

where / means that the last two diagrams, which take into account the relaxed traceless condition (2.9), must be subtracted from the first three.

The *traceful* Young tableau decomposition of the gauge parameter \( \xi_{a_1 \cdots a_{s-1}, bm} \) satisfying eq. (2.14) is

\[
\otimes_{s-1} = \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}.
\]
It is obtained by subtracting from the traceful Young tableau
\[ \begin{array}{c}
  \begin{array}{c}
    s \\
    s-1
  \end{array}
\end{array} \]  
the Young tableaux corresponding to the relaxed trace condition (2.14)
\[ \begin{array}{c}
  \begin{array}{c}
    s-2 \\
    s-1
  \end{array} \oplus \begin{array}{c}
    s-1 \\
    s-1
  \end{array} / \begin{array}{c}
    \Box
  \end{array}. \]  
\[ (A.10) \]

In (A.10), in the case of the odd spins, the Young tableau \[ \begin{array}{c}
  \begin{array}{c}
    a \cdot \cdot \cdot a
  \end{array}
\end{array} \] is subtracted, since it is not part of the traceless condition
\[ \eta^{a_{s-1}b} \xi_{a_{1} \cdots a_{s-1},bm} = 0, \]  
\[ (A.11) \]
which one can see by considering the symmetry of the full trace of (A.11) in the indices \[ a_{1} \cdots a_{s-3}. \]

Comparing (A.8) with (A.7) we conclude that the number of the components of the connection which are not gauged away by the Stueckelberg symmetry is the same as the number of the torsion components modulo an antisymmetric tensor field \[ F_{mn} \] corresponding to \[ \Box \], which is thus a pure gauge. This makes one more evidence to the fact stressed in the main text that our construction does not include the massless field of spin 1 whose field strength would be \[ F_{mn}. \]

\section*{Appendix B}

As an alternative to the approach explained in the main text, let us explain how to find the general solution of the equation (7.60). In this approach, however, the problem that remains to be solved is to find appropriate analytic functions \[ A(\sigma, \tau, v) \] such that the resulting solution \[ \tilde{\Phi}(\sigma, \tau, v) \] be free of the constant parts in \[ \sigma \] and \[ \tau. \]

The generic solution \[ \tilde{\psi}_{0}(\sigma, \tau, v) \] of the homogeneous first-order partial differential equation (7.60) with \[ \tilde{A}(\sigma, \tau, v) = 0 \] is
\[ \tilde{\psi}_{0}(\sigma, \tau, v) = \tau \left( 1 - \frac{2\tau}{\sigma} \right)^{-\frac{D+3}{2}} \left( 1 - \tau^{4}v \right)^{\frac{D-3}{2}} \tilde{\psi}(\xi, v), \]  
\[ (B.1) \]
where
\[ \xi = \frac{1 - \tau^{4}v}{\sigma \tau (1 - \frac{2\tau}{\sigma})} - \tau^{2}v \]  
\[ (B.2) \]
and \[ \tilde{\psi}(\xi, v) \] is an arbitrary function of its arguments.

For the first sight it may look as we have constructed a topological action with the trivial variation. This is not the case, however. As expected, the action that contains \[ \Phi \] defined by (B.1) via (7.54)-(7.56) is identically zero because the expansion of the function \[ \Phi \] in power series
\[ \Phi(u, w, v) = \int d\sigma d\tau e^{\sigma u + \tau w} \sum_{p, q, r} \tau^{p} \sigma^{q} v^{r} \]  
\[ (B.3) \]
contains only the terms with \( p > q \), \emph{i.e.} those with more \( \sigma \) than \( \tau \) in the denominator. All such terms do not contribute because of the Young property of fields. Indeed, they describe terms in which more than half of the indices of a tensor are contracted with the compensator contained in \( u \) (7.37). This implies the symmetrization over more than half of indices of the tensor described by a rectangular Young tableau, thus giving zero. Note that the dependence on \( v \) does not affect this argument because its application does not affect the Young symmetry property, mapping two-row Young tableaux to shorter two-row Young tableaux.

Thus, as expected, the solution of the homogeneous equation describes nothing. Setting

\[
\tilde{\psi}(\sigma, \tau, v) = \frac{\tau (\sigma \tau)^{\frac{D-5}{2}} (1 - \tau^4 v)^{\frac{D-9}{2}}}{(\sigma \tau - 2 \tau^2)^{\frac{D}{2}}} \chi(\xi, \tau, v),
\]

changing the variables from \( \sigma, \tau \) to \( \xi \) (B.2) and \( y \)

\[
y = 2 + \frac{1}{t^2 \xi},
\]

where \( v \) should be interpreted as a parameter, and using the relations

\[
\sigma \tau - 2 \tau^2 = \frac{1 - \frac{v}{(2 - y)^2 \xi^2}}{\xi(1 - \frac{v}{(2 - y)^2 \xi \tau})}, \quad \sigma \tau = \frac{-\frac{y}{2 - y} + \frac{v}{\xi^2 (2 - y)^2}}{\xi(1 - \frac{v}{(2 - y)^2 \xi \tau})},
\]

the inhomogeneous equation amounts to

\[
-(2 - y) \frac{\partial}{\partial y} \chi(\xi, \tau, v) = \frac{\xi(1 - \frac{v}{(2 - y)^2 \xi^2})(1 - \frac{v}{(2 - y)^2 \xi \tau})}{(-\frac{y}{2 - y} + \frac{v}{\xi^2 (2 - y)^2})^{\frac{D-3}{2}}} A(y, \xi, v).
\]

The appropriate solution of this equation that admits an expansion in integer negative powers of \( \sigma \) and \( \tau \) is

\[
\chi(\sigma, \tau, v) = \int_0^1 \frac{dt}{t^{2t - y}} \frac{\xi(1 - \frac{v}{(2 - t^4 y)^2 \xi \tau})(1 - \frac{v}{(2 - t^4 y)^2 \xi \tau})}{(-\frac{y}{2t - y} + \frac{v}{(2 - t^4 y)^2 \xi^2})^{\frac{D-3}{2}}} A(y t^{-1}, \xi, v).
\]

With the help of the relations (7.54)-(7.56) we obtain that

\[
\tilde{\Phi}(\sigma, \tau, v) = -2 \frac{(1 - \tau^4 v)^{\frac{D-7}{2}}}{\xi(2 - y)(1 - 2 \xi)^{\frac{D-7}{2}}} \chi(\sigma, \tau, v).
\]

It is also easy to reconstruct the functions \( \Lambda(\sigma, \tau, v) \) and \( W(\sigma, \tau, v) \).

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\(^9\)Recall that; because of the definition of the measure (7.58) only the terms with negative \( p \) and \( q \) contribute to eq. (B.3).
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