Rational points on cubic surfaces and AG codes from the Norm–Trace curve

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Abstract
In this paper, we derive general bounds for the number of rational points on a cubic surface defined over \( \mathbb{F}_q \), which constitute an extension of a result due to Weil. Exploiting these bounds, we are able to give a complete characterization of the intersections between the Norm–Trace curve over \( \mathbb{F}_{q^3} \) and the curves of the form \( y = ax^3 + bx^2 + cx + d \), generalizing a previous result by Bonini and Sala and providing more detailed information about the weight spectrum of one-point AG codes arising from such curve.

Keywords Norm–Trace curve · AG code · Weight spectrum · Cubic surfaces

Mathematics Subject Classification 14G50 · 11T71 · 94B27

1 Introduction
Many of the best performing algebraic codes are known to be Algebraic Geometry (AG) codes, which arise from algebraic varieties over finite fields. Among them, the most studied codes are those arising from algebraic curves, that were introduced by Goppa in the ’80s; see [17, 18] for a detailed description.

Let \( \mathcal{X} \) be an algebraic curve defined over the finite field with \( q \) elements \( \mathbb{F}_q \). The parameters of codes arising from \( \mathcal{X} \) strictly depend on some geometrical properties of the curve. In general, curves with many \( \mathbb{F}_q \)-rational places with respect to their genus give rise to long AG codes with good parameters. For this reason, maximal curves, that is, curves attaining
the Hasse–Weil upper bound, have been widely investigated in the literature; see [2, 6, 30, 34, 36, 38].

In general, the determination of the weight spectrum of a code $C$ (i.e. the set of the possible weights of $C$) is a very difficult task, see for instance [25].

For AG codes, it is possible to derive information about their weight spectrum by the study of the intersection of the base curve $X$ and low degree curves, as done in [3, 5, 11, 28, 29].

The Norm–Trace curves are a natural generalization of the celebrated Hermitian curve to any extension field $\mathbb{F}_{q^r}$, and their codes have been widely studied; see [3, 8, 15, 16, 31].

In this paper, we focus on the intersection between the Norm–Trace curve over $\mathbb{F}_{q^3}$ and curves of the form $y = Ax^3 + Bx^2 + Cx + D$, giving the proof of a result which turns out to be the exact statement related to a previous conjecture [8]. In addition to this, we partially deduce the weight spectrum of its one-point codes in the place at the infinity.

In order to obtain these results, we translate the problem of finding the planar intersection between the cubic Norm–Trace curve and the above mentioned curves into that of counting the number of $\mathbb{F}_{q^3}$-rational points of certain cubic surfaces, which is a well-known topic in algebraic geometry in positive characteristic, see [10, 27, 37]. Hence, we focus on the investigation of the latter problem, giving the proof of a bound for the number of rational points on a cubic surface defined over $\mathbb{F}_{q^3}$ which constitutes an extension of a classical result by Weil.

In Sect. 2, we collect some preliminary algebraic geometry definitions and results, together with a brief introduction to the Norm–Trace curve and AG codes.

In Sect. 3, we present general results on the number of $\mathbb{F}_{q^3}$-rational points on cubic surfaces defined over $\mathbb{F}_{q^3}$, that are stated in detail in Theorem 3.1 and summarized in Theorem 3.2. These results end up constituting an extension of a theorem by Weil and we obtain them exploiting results on cubic surfaces over finite fields and techniques of birational geometry over algebraically non-closed fields.

In Sects. 4 and 5, we analyze cubic surfaces arising from the intersection between the Norm–Trace curve and curves of the form $y = Ax^3 + Bx^2 + Cx + D$ over $\mathbb{F}_{q^3}$, noting that Corollary 4.3 is implied by Theorem 3.1.

Finally, in Sect. 6, we use the results obtained in the previous sections to investigate the weight spectrum of a certain family of one-point codes arising from the cubic Norm–Trace curve.

## 2 Preliminaries

In this section, we collect some preliminary definitions and results that will be useful throughout the paper.

### 2.1 Some results on cubic surfaces

We list here some results on cubic surfaces that will be useful later on in the paper, see [21] and [13] for more details. In order to state the first result, we start by recalling the definition of Del Pezzo surface.

**Definition 2.1** A Del Pezzo surface is a non-degenerate irreducible surface of degree $d$ in $\mathbb{P}^d$ that is not a cone and not isomorphic to a projection of a surface of degree $d$ in $\mathbb{P}^{d+1}$.

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Theorem 2.2 [10] Let $S \subset \mathbb{P}^3(\mathbb{k})$ be a singular irreducible cubic surface defined on the field $\mathbb{k}$. Let $\overline{S} = S(\overline{\mathbb{k}})$ be the surface defined by $S$ over $\overline{\mathbb{k}}$, the algebraic closure of $\mathbb{k}$. Let $\delta$ be the number of isolated double points of $S$. Then $\delta \leq 4$ and $S$ is birationally equivalent (over $\mathbb{k}$) to

1. $\mathbb{P}^2(\mathbb{k})$ if $\delta = 1, 4$;
2. A smooth Del Pezzo surface of degree $4$ if $\delta = 2$;
3. A smooth Del Pezzo surface of degree $6$ if $\delta = 3$.

The following result concerns instead the structure of the singular locus of an irreducible cubic surface with non-isolated singularities.

Proposition 2.3 Let $S \subset \mathbb{P}^3(\mathbb{k})$ be an irreducible cubic surface with non-isolated singularities over a field $\mathbb{k}$ of positive characteristic. Then all the singular points of $S$ lie on a double line $\ell$.

Proof We start by observing that, by a result due to Bertini and proved in positive characteristic by Nakai in [32], a general hyperplane $L$ in $\mathbb{P}^3(\mathbb{k})$ cuts on $S$ an irreducible cubic curve $C$ whose singular points are exactly the singular points of $S$ lying on the hyperplane $L$.

Since the curve $C$ is an irreducible cubic curve, it cannot have more than one singular double point, which means that on a general hyperplane must lie at most one singular point of the surface $S$ (see also [9, Section 2]). This leads to conclude that the singular set of $S$ is a line $\ell$, which is double since $S$ is a cubic irreducible surface. Indeed, if $\ell$ were a line of multiplicity higher than two, then $S$ would have to be the union of three (distinct or coincidental) planes through the line $\ell$ (see [1]).

If there were a singular point $Q$ of $S$ not lying on $\ell$, then the plane through $Q$ and $\ell$ would be a component of $S$, which is impossible since the surface is irreducible. Hence, there are no singular points of $S$ not lying on $\ell$ and we have our desired result. \qed

Finally, we state a result on the number of lines that pass through a point on a surface defined over a finite field and are contained in the surface.

Let $q = p^h$, where $p$ is a prime and $h > 0$ an integer, and denote with $\mathbb{F}_q$ the finite field with $q$ elements.

Theorem 2.4 [12] Let $Y \subset \mathbb{P}^3$ be a surface of degree $d$ defined over $\mathbb{F}_q$ and $P \in Y(\mathbb{F}_q)$. Then one of the following holds:

(a) $Y$ contains a plane defined over $\mathbb{F}_q$.
(b) $Y$ contains a cone over a plane curve defined over $\mathbb{F}_q$ with center at $P$.
(c) $|\{ l \subset \mathbb{P}^3 \mid l \text{ is a line such that } P \in l \subset Y \}| \leq d(d - 1)$.

2.2 Blow-ups

We introduce now the definition of blow-up of $\mathbb{P}^n$ at a point (see [19, Chapter 7] and [20, Paragraph 1.4] for more details).
Definition 2.5 Let \( \phi : \mathbb{P}^n \rightarrow \mathbb{P}^{n-1} \) be the rational map given by projection from a point \( P \in \mathbb{P}^n \) and let \( \tilde{\mathbb{P}}^n \subset \mathbb{P}^n \times \mathbb{P}^{n-1} \) be its graph. The projection map \( \pi : \tilde{\mathbb{P}}^n \rightarrow \mathbb{P}^n \) is called the blow-up of \( \mathbb{P}^n \) at the point \( P \).

Away from \( P \), the map \( \pi \) projects \( \tilde{\mathbb{P}}^n \) isomorphically to \( \mathbb{P}^n \). Over \( P \), the fiber is instead isomorphic to \( \mathbb{P}^{n-1} \). Sometimes we refer to the variety \( \tilde{\mathbb{P}}^n \) as the blow-up.

More generally, we have the following definition.

Definition 2.6 Let \( X \subset \mathbb{P}^n \) be a quasi-projective variety and \( P \in X \) a point. Let \( \tilde{X} \subset X \times \mathbb{P}^{n-1} \) be the graph of the projection map of \( X \) to \( \mathbb{P}^{n-1} \) from \( P \). The map \( \pi : \tilde{X} \rightarrow X \) is called the blow-up of \( X \) at \( P \). The inverse image \( \pi^{-1}(P) \subset \tilde{X} \) is called the exceptional divisor of the blow-up.

Definition 2.7 Let \( \tilde{X} \rightarrow X \) be the blow-up of \( X \) at \( P \); then, for \( X \subset \mathbb{P}^n \) closed, the proper transform \( \tilde{X} \) of \( X \) in \( \tilde{X} \) is the closure of the inverse image \( \pi^{-1}(X \setminus \{P\}) \) of the complement of \( P \) in \( X \).

Note that the restriction of the map \( \pi \) to the proper transform \( \tilde{X} \) is exactly the blow-up of \( X \) at \( P \) (see [19, Chapter 7]).

We recall now the definitions of desingularization and minimal desingularization of a variety \( X \subset \mathbb{P}^n \).

Definition 2.8 Let \( \rho : Y \rightarrow X \) be a birational morphism and let \( \Sigma \subset X \) be the smallest closed subset of \( X \) outside of which \( \rho \) is an isomorphism, i.e., \( \rho : (Y \setminus \rho^{-1}(\Sigma)) \rightarrow (X \setminus \Sigma) \) is an isomorphism. Then the exceptional locus of \( \rho \) is \( \rho^{-1}(\Sigma) \).

Definition 2.9 A desingularization, or a resolution of singularities, of a variety \( X \) is a proper birational morphism \( \tilde{X} \rightarrow X \) from a non-singular variety \( \tilde{X} \).

Sometimes we refer to the variety \( \tilde{X} \) as the desingularization.

Definition 2.10 A desingularization \( \tilde{X} \) of \( X \) is minimal if it is such that no exceptional divisor of \( \tilde{X} \) is contained in the exceptional locus of \( \tilde{X} \rightarrow X \).

For Del Pezzo surfaces, the following result holds (see [24, Chapter 2, Proposition 6]).

Proposition 2.11 Let \( S \) be a singular Del Pezzo surface and \( \pi : \tilde{S} \rightarrow S \) a minimal desingularization. Then \( \tilde{S} \) is a weak Del Pezzo surface and the inverse image of the singular points of \( S \) is exactly the collection of \((-2)\)-curves on \( \tilde{S} \).

2.3 The Norm–Trace curve

Let \( q = p^h \), where \( p \) is a prime and \( h > 0 \) an integer, and denote with \( \mathbb{F}_q \) the finite field with \( q \) elements.

We recall that the norm \( N_{\mathbb{F}_q}^{\mathbb{F}_r} \) and the trace \( \text{Tr}_{\mathbb{F}_q}^{\mathbb{F}_r} \), where \( r \) is a positive integer, are functions from \( \mathbb{F}_q \) to \( \mathbb{F}_q \) such that

\[ N_{\mathbb{F}_q}^{\mathbb{F}_r}(x) = x^{q^r - q^0} \]
\[ \text{Tr}_{\mathbb{F}_q}^{\mathbb{F}_r}(x) = x^r \]
The number of planar intersections between $\mathcal{N}_r$ and cubic curves of the form $y = ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbb{F}_{q^r}$ and $a \neq 0$, is bounded by $q^2 + 7q + 1$, when the surface defined over $\mathbb{F}_{q^r}$ by the equation

$$X_0X_1X_2 = AX_0^2 + A^2X_1^2 + A^2X_2^2 + BX_0^2 + B^2X_1^2 + B^2X_2^2 + CX_0 + C^2X_1 + C^2X_2 + E$$

is irreducible, for $A, B, C, E \in \mathbb{F}_{q^r}$. 

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\[ \mathbb{N}^{\ell}_{q^r}(x) = x^{q^r-1} = x^{q^r-1} + q^{r-2} + \cdots + q \]  

and  

\[ \mathbb{T}^{\ell}_{q^r}(x) = x^{q^r-1} + x^{q^r-2} + \cdots + x + 1. \]

When $q$ and $r$ can be derived unequivocally from the context, we will omit the subscripts.

The Norm–Trace curve $\mathcal{N}_r$ is the curve defined over the affine plane $\mathbb{A}^2(\mathbb{F}_{q^r})$ by the equation

$$N(x) = T(y).$$

If $r = 2$ the curve $\mathcal{N}_r$ is smooth, while if $r \geq 3$ it can be easily seen that $\mathcal{N}_r$ has a singular point which is the point at the infinity $P_\infty$. It is then well-known that $\mathcal{N}_r$ has $q^{2r-1}$ affine places and a single place at the infinity; in fact, there is exactly one place centered at each affine point of $\mathcal{N}_r$ (these are all smooth points) and the point at the infinity is either smooth (in the case $r = 2$) and hence center of exactly one place, or it is singular and center of only one branch of the curve (if $r \geq 3$), so that $\mathcal{N}_r$ has a unique place at the infinity also in this case.

If $r = 2$, $\mathcal{N}_r$ coincides with the Hermitian curve, and this is the only case in which $\mathcal{N}_r$ is smooth, since, as noted above, for $r \geq 3$ it has a singularity in $P_\infty$.

Moreover, it is known that its Weierstrass semigroup in the place centered at $P_\infty$ is generated by \( \left( q r^{-1}, q r^{-1} \right) \), see [16]. Also, the automorphism group is determined by the following result.

**Theorem 2.12** [7] The automorphism group of $\mathcal{N}_r$, $\text{Aut}(\mathcal{N}_r)$, has order $q^{-1} (q^r - 1)$ and is a semidirect product $G \rtimes C$, where

\[
G = \{ (x, y) \mapsto (x, y + a) \mid T(a) = 0 \}
\]

\[
C = \{ (x, y) \mapsto (bx, b^{q^{-1}} y) \mid b \in \mathbb{F}_{q^r}^* \}.
\]

Our main aim is the study of the planar intersections (i.e. the intersections counted without multiplicity) between $\mathcal{N}_r$ and the cubic curves of the form $y = ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbb{F}_{q^r}$ and $a \neq 0$, in the case $r = 3$. The case $r = 2$ and $a = b = 0$ was investigated in [4] and the case $r = 2$ and $a = 0$ was completely investigated in [14, 28]. On the other hand, the case $r = 3$ and $a = 0$ was investigated in [8].

In [8], the authors claim the following result.

**Theorem 2.13** The number of planar intersections between $\mathcal{N}_3$ and cubic curves of the form $y = ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbb{F}_{q^3}$, $a \neq 0$, is bounded by $q^2 + 7q + 1$, when the surface defined over $\mathbb{F}_{q^3}$ by the equation

$$X_0X_1X_2 = AX_0^2 + A^2X_1^2 + A^2X_2^2 + BX_0^2 + B^2X_1^2 + B^2X_2^2 + CX_0 + C^2X_1 + C^2X_2 + E$$

is irreducible, for $A, B, C, E \in \mathbb{F}_{q^3}$. 

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Still, a proof of this result is not given in the paper. In this paper, we will obtain the result conjectured in Theorem 2.13, with an appropriate refinement of the statement, as a consequence (Corollary 4.3) of the results contained in Sect. 3 on the number of rational points on absolutely irreducible cubic surfaces defined over $\mathbb{F}_q$.

2.4 Algebraic geometry codes

In this section, we recall some basic facts on AG codes. For a detailed discussion, we refer to [35].

Let $\mathcal{X}$ be a projective curve over the finite field $\mathbb{F}_q$ and consider the function field $\mathbb{F}_q(\mathcal{X})$ of rational functions defined over $\mathbb{F}_q$. Denote with $\mathcal{X}(\mathbb{F}_q)$ the set of the $\mathbb{F}_q$-rational points of $\mathcal{X}$. A divisor $D$ on $\mathcal{X}$ can be seen as a finite sum $\sum_{P \in \text{supp}(D)} n_P P$, where the $n_P$’s are integers. For a function $f \in \mathbb{F}_q(\mathcal{X})$, $(f)$ denotes the divisor associated to $f$. A divisor $D$ is $\mathbb{F}_q$-rational if it coincides with its image $\sum_{P \in \text{supp}(D)} n_P P^q$ under the Frobenius map over $\mathbb{F}_q$, where if $P = (x, y)$ we mean $P^q = (x^q, y^q)$.

Given an $\mathbb{F}_q$-rational divisor $D = \sum^n_{i=1} n_IP_i$ on $\mathcal{X}$, its support is defined as $\text{supp}(D) = \{P_i : n_i \neq 0\}$.

The Riemann-Roch space associated with $D$ can be seen as a finite sum of $\mathbb{F}_q$-vector space of finite dimension $\ell(D)$. The exact dimension of this space can be computed using the Riemann-Roch theorem.

Consider now the divisor $D = \sum^n_{i=1} P_i$, where all the $P_i$’s are $\mathbb{F}_q$-rational. Let $G$ be another $\mathbb{F}_q$-rational divisor on $\mathcal{X}$ such that $\text{supp}(G) \cap \text{supp}(D) = \emptyset$. Consider the evaluation map

$$e_D : \mathcal{L}(G) \longrightarrow \mathbb{F}_q^n$$

$$f \mapsto e_D(f) = (f(P_1), \ldots, f(P_n)).$$

The map $e_D$ is $\mathbb{F}_q$-linear and it is injective if $n > \deg(G)$.

The AG code $C_{\mathcal{L}}(D, G)$, also called functional code, associated with the divisors $D$ and $G$ is defined as $C_{\mathcal{L}}(D, G) := e_D(\mathcal{L}(G)) = \{(f(P_1), \ldots, f(P_n)) \mid f \in \mathcal{L}(G) \} \subseteq \mathbb{F}_q^n$. Such a code is an $[n, \ell(G) - \ell(G-D), d]_q$ code, where $d \geq d = n - \deg(G)$ and $d$ is the so-called designed minimum distance (of such code).

3 General results on the number of rational points on cubic surfaces

In this section, we will exploit well-known results on cubic surfaces over finite fields and techniques of birational geometry over algebraically non-closed fields to obtain some general results on the number of rational points on a projective cubic surface $\mathcal{S}$ defined over a finite field $\mathbb{F}_q$. In addition to being theoretically interesting, these results will be useful later on in the paper, in particular in Sect. 4 where we will use them to bound the intersections between the Norm–Trace curve over $\mathbb{F}_q$ and the curves of the form $y = ax^3 + bx^2 + cx + d$.

For brevity, when referring to a surface $\mathcal{S}$, in what follows we will usually just write irreducible instead of absolutely irreducible, since we will see all the surfaces as elements of $\mathbb{P}^3(\mathbb{F}_q)$. 
3.1 Main result

Let $S(\mathbb{F}_q)$ and $\overline{S}(\mathbb{F}_q)$ denote, respectively, the set of affine and projective $\mathbb{F}_q$-rational points of a projective cubic surface $S$; then clearly $|S(\mathbb{F}_q)| \leq |\overline{S}(\mathbb{F}_q)|$. The main result of the section is the following theorem.

**Theorem 3.1** Consider a projective irreducible cubic surface $S$ defined over $\mathbb{F}_q$ and assume it is not a cone over a smooth irreducible cubic plane curve. Then the following bounds for the number $|\overline{S}(\mathbb{F}_q)|$ hold.

1. $S$ has non-isolated singularities:
   
   (1) If $S$ has non-isolated singularities and it is a cone over a singular absolutely irreducible cubic plane curve, then
   
   $$|\overline{S}(\mathbb{F}_q)| \leq q^2 + 2q + 1.$$
   
   (2) If $S$ has non-isolated singularities and it is not a cone, then
   
   $$|\overline{S}(\mathbb{F}_q)| \leq q^2 + 7q + 1.$$

2. $S$ has only isolated singularities:
   
   (1) If $S$ has one isolated $\mathbb{F}_q$-rational singular point, then
   
   $$|\overline{S}(\mathbb{F}_q)| \leq q^2 + 6q + 1.$$
   
   (2) If $S$ has two isolated and conjugate $\mathbb{F}_q$-rational singular points, then
   
   $$|\overline{S}(\mathbb{F}_q)| \leq q^2 + 3q + 1.$$
   
   (3) If $S$ has three isolated and conjugate $\mathbb{F}_q$-rational singular points, then
   
   $$|\overline{S}(\mathbb{F}_q)| \leq q^2 + q + 1.$$
   
   (4) If $S$ has four isolated and conjugate $\mathbb{F}_q$-rational singular points, then
   
   $$|\overline{S}(\mathbb{F}_q)| = q^2 + q + 1.$$

Note that, if $S$ is irreducible, it cannot be a cone over a reducible plane curve, otherwise it would be reducible as well.

In particular, considering the classification of cubic surfaces in positive characteristic, if $S$ is an irreducible cubic surface with isolated singularities, which is not a cone over a smooth plane cubic curve, then its singular points are double points, as it is noted in [33]. Then, by Theorem 2.2, it follows that the cases treated in Theorem 3.1 really cover all the possible cases of singularities of projective irreducible cubic surfaces defined over $\mathbb{F}_q$ and that are not a cone over an elliptic curve. Therefore, Theorem 3.1 implies the following result.
Theorem 3.2 Consider a projective irreducible cubic surface $S$ defined over $\mathbb{F}_q$ and assume it is not a cone over a smooth irreducible cubic plane curve. Then

$$|S(\mathbb{F}_q)| \leq |\tilde{S}(\mathbb{F}_q)| \leq q^2 + 7q + 1.$$ 

Theorem 3.2 constitutes an extension of the proposition stated below and due to Weil.

Proposition 3.3 (Weil) Let $S$ be a smooth projective irreducible cubic surface over $\mathbb{F}_q$, then

$$|S(\mathbb{F}_q)| \leq |\tilde{S}(\mathbb{F}_q)| \leq q^2 + 7q + 1.$$ 

In fact, this result is a consequence of the following two theorems by Weil.

Theorem 3.4 (Weil) Let $S$ be a surface defined over a finite field $\mathbb{F}_q$. If $S \otimes \overline{\mathbb{F}}_q$ is birationally trivial, then

$$|\tilde{S}(\mathbb{F}_q)| = q^2 + q\text{Tr}(\varphi^*) + 1,$$

where $\varphi$ denotes the Frobenius endomorphism and $\text{Tr}(\varphi^*)$ denotes the trace of $\varphi$ in the representation of $\text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ on $\text{Pic}(S \otimes \overline{\mathbb{F}}_q)$.

Theorem 3.5 ([27], Theorem 23.1) Let $S$ be a smooth projective irreducible cubic surface over $\mathbb{F}_q$, then the number of points of $\tilde{S}(\mathbb{F}_q)$ is exactly

$$|\tilde{S}(\mathbb{F}_q)| = q^2 + \eta q + 1$$

where $\eta \in \{-2, -1, 0, 1, 2, 3, 4, 5, 7\}$.

Our goal in this subsection is to prove Theorem 3.1. We will discuss separately (in Sect. 3.2) the cases in which $S$ is a cone over a smooth absolutely irreducible cubic plane curve or $S$ is reducible, which are not comprised in Theorem 3.1; we assume henceforth that $S$ is not a cone over a smooth absolutely irreducible cubic plane curve and consider only the irreducible case.

For the proof of Theorem 3.1, we will also use techniques of birational geometry over algebraically non-closed fields. In particular, for the cases in which $S$ has only isolated singularities, we first make some observations on the minimal resolution of the singularities of $S$ and the number of its projective $\mathbb{F}_q$-rational points, which are collected in the following two remarks.

Remark 3.6 Let $\sigma : \tilde{S} \longrightarrow S$ be the minimal resolution of the singularities of $S$, then $\tilde{S}$ is a weak Del Pezzo surface of degree 3. As the minimal resolution is unique, the surface $\tilde{S}$ and the morphism $\sigma$ are both defined over $\mathbb{F}_q$ and all the non-singular $\mathbb{F}_q$-rational points of the surface $S$ are preserved by $\sigma$. Moreover, Theorem 3.4 is applicable to $\tilde{S}$, as any weak Del Pezzo surface is birationally trivial over $\overline{\mathbb{F}}_q$.

Remark 3.7 As $\tilde{S}$ is obtained from $S$ by blowing up the singular points, we have the following inequality:
\[ |\tilde{S}(\mathbb{F}_q)| \leq |S(\mathbb{F}_q)|. \]

This holds since on the exceptional divisor there might be more \( \mathbb{F}_q \)-rational points. In fact, the inverse image of the singular points of \( S \) is exactly the collection of \((-2)\)-curves on \( \tilde{S} \) (see Proposition 2.11). Hence, for instance, if \( S \) has an ordinary double point defined over \( \mathbb{F}_q \) and we blow up this point, then we have a \((-2)\)-curve defined over \( \mathbb{F}_q \) on the exceptional divisor (and hence we find \( q + 1 \) rational points on it).

**Proof of Theorem 3.1** We prove the theorem by considering separately each of the cases.

1. \( S \) has non-isolated singularities.
   
   (1) If \( S \) is a cone over a singular absolutely irreducible cubic plane curve, then take \( P \) the vertex of the cone and choose an arbitrary plane in \( \mathbb{P}^3(\mathbb{F}_q) \) not containing \( P \). We can think of the cone as the union of the lines between \( P \) and a singular absolutely irreducible cubic curve lying on this plane. Hence, if the cubic curve has \( m \mathbb{F}_q \)-rational points, the obtained cone has \( mq + 1 \mathbb{F}_q \)-rational points (see [24, Chapter 2, Section 4]). This is easy to see as \( \mathbb{F}_q \)-rational points of the cone lie on lines through \( P \) and \( \mathbb{F}_q \)-rational points of the curve, so we count \( q \) point for every line (all \( \mathbb{F}_q \)-rational points except the vertex \( P \)) and finally we add \( P \). Then, due to well-known results, see [22, Chapter 11] for details, a singular absolutely irreducible cubic curve in \( \mathbb{P}^2(\mathbb{F}_q) \) can have at most \( q + 2 \mathbb{F}_q \)-rational points, and this yields the desired result
   
   \[ |\tilde{S}(\mathbb{F}_q)| \leq (q + 2)q + 1 = q^2 + 2q + 1. \]

   (2) If \( S \) is not a cone, then Theorem 3.4 gives the desired bound. To see this, notice first that \( S \) has no isolated singularities. This is established from the classification of cubic surfaces (see [33] and [21, proof of Proposition 6.4]), from which it is actually known even more on the singular locus of such a surface (see Proposition 2.3). Therefore, as \( S \) is the anti-canonical model of a degree-3 weak Del Pezzo surface (see [24, Chapter 2, Section 4]), then by Theorem 3.4 we have that
   
   \[ |\tilde{S}(\mathbb{F}_q)| \leq q^2 + 7q + 1 \]

   as the lattice \( \text{Pic}(S \otimes \overline{\mathbb{F}_q}) \) has rank 7.

2. \( S \) has only isolated singularities.
   
   (1) If \( S \) has one isolated \( \mathbb{F}_q \)-rational singular point, consider the tangent cone to \( S \) at \( P \). This contains exactly \( q + 1 \mathbb{F}_q \)-rational lines through \( P \) and these lines either intersect \( S \) only at \( P \) or are contained in \( S \). Moreover, note that the lines through \( P \) contained in the surface are all contained in the tangent cone. Hence, we have the following estimate:
   
   \[ |\tilde{S}(\mathbb{F}_q)| = q^2 + q + 1 - (q + 1) + hq + 1 = q^2 + hq + 1. \]

   This follows as:

   - we count one \( \mathbb{F}_q \)-rational point on each line through \( P \) that is not contained in the tangent cone;
• denoting by \( h \) the number of lines through \( P \) contained in \( S \), we count \( q \frac{h}{q} \)-rational points on each of these \( h \) lines;
• finally, we count the point \( P \).

By Theorem 2.4, we have that

\[ h \leq |\{I \subset \mathbb{P}^3 \mid I \text{ is a line such that } P \in I \subset S\}| \leq 6 \]

hence we have the bound

\[ |\mathcal{S}(\mathbb{F}_q)| \leq q^2 + 6q + 1. \]

(2) If \( S \) has two isolated and conjugate \( \mathbb{F}_q \)-rational singular points, it turns out that we can adopt the same argument used in [8], that comes from the investigation of the possible curves obtained from the intersection between \( S \) and the book of planes through the line between the singularities. In fact, by using more carefully such counting argument, we obtain a refinement of [8, Proposition 5.9] that gives the bound \( |\mathcal{S}(\mathbb{F}_q)| = q^2 + 3q + 2 \). Then, by noticing that the number \( |\mathcal{S}(\mathbb{F}_q)| \) should be congruent to 1 modulo \( q \), we can refine this bound further and we have

\[ |\mathcal{S}(\mathbb{F}_q)| \leq q^2 + 3q + 1. \]

(3) If \( S \) has three isolated and conjugate \( \mathbb{F}_q \)-rational singular points \( P_1, P_2 \) and \( P_3 \), we proceed as follows. Note that we can obtain a birational map from the surface \( S \) to a quadric surface \( Q \) in the following way: we first consider the map given by blowing up the points \( P_1, P_2, P_3 \) and \( Q \) on \( S \), where \( Q \) is a non-singular point on the surface. Then, we compose this map with the contraction of the proper transforms of the following curves, lying on the hyperplanes defined by all the possible sets of three points among \( P_1, P_2, P_3 \) and \( Q \):

• three conjugate lines defined by the three possible ways of choosing a pair of points among \( P_1, P_2 \) and \( P_3 \);
• three conjugate conics passing through \( Q \) and a pair of the three singular conjugate points.

None of these curves is defined over \( \mathbb{F}_q \), but the point \( Q \) is \( \mathbb{F}_q \)-rational; hence, we find a \((-1)\)-curve defined over \( \mathbb{F}_q \) on the exceptional divisor and there are \( q + 1 \frac{1}{q} \)-rational points on this curve. Therefore, we have that

\[ |\mathcal{S}(\mathbb{F}_q)| = |\mathcal{Q}(\mathbb{F}_q)| - q. \]

Since \( |\mathcal{Q}(\mathbb{F}_q)| \leq q^2 + 2q + 1 \) (see [23, Section 15.3]), we then have that

\[ |\mathcal{S}(\mathbb{F}_q)| = |\mathcal{Q}(\mathbb{F}_q)| - q \leq q^2 + q + 1. \]

If, instead, we blow up \( P_1, P_2, P_3 \) and \( Q \) on \( S \), where \( Q \) is another singular point on the surface, then we obtain a birational map from \( S \) to \( \mathbb{P}^2(\mathbb{F}_q) \). This map is obtained, as above, by composing the map given by the blow-ups of the points \( P_1, P_2, P_3 \) and \( Q \) and the contraction of the proper transforms of the following curves:

• three conjugate lines defined by the three possible ways of choosing a pair of points among \( P_1, P_2 \) and \( P_3 \);
• three conjugate lines defined by \( Q \) and one among \( P_1, P_2 \) and \( P_3 \).
Similarly to the case in which \( Q \) was non-singular, none of these curves is defined over \( \mathbb{F}_q \), but the point \( Q \) is \( \mathbb{F}_q \)-rational; hence, we find a \((-2)\)-curve defined over \( \mathbb{F}_q \) on the exceptional divisor and there are \( q + 1 \) \( \mathbb{F}_q \)-rational points on this curve. Therefore, we have that

\[
|\tilde{\mathbb{S}}(\mathbb{F}_q)| = |\mathbb{P}^2(\mathbb{F}_q)| - q = q^2 + 1 \leq q^2 + q + 1.
\]

(4) If \( S \) has four isolated and conjugate \( \mathbb{F}_q \)-rational singular points, we firstly note that, if \( P_1, P_2, P_3 \) and \( P_4 \) are the four singular points on \( S \), then no three among them are collinear. In fact, suppose without loss of generality that \( P_1, P_2, P_3 \) are collinear: then a general plane through \( P_1, P_2, P_3 \) would meet the surface in a cubic curve with three collinear double points, which means that the only possibility is that the curve is union of a double line \( \ell \) (the line joining the three double points) and another line \( r \) (see [10]). In this case, the whole line \( \ell \) would be double on the surface, which is impossible since by hypothesis \( S \) has only isolated singularities.

Let \( \sigma : \tilde{S} \rightarrow S \) be the minimal resolution of the singularities of \( S \). As noted in Remark 3.6, \( \tilde{S} \) is a weak Del Pezzo surface of degree 3 defined over \( \mathbb{F}_q \). Then, we have seen that the singular points on \( S \) lie in general position and any line through a pair of them is contained in the surface \( S \). We have that the proper transform of each line through a pair of singular points on \( \tilde{S} \) is a \((-1)\)-curve. Moreover, the set of lines through pairs of singular points is defined over \( \mathbb{F}_q \), hence if we consider the contraction \( \tilde{S} \rightarrow X \) of the proper transforms of these lines, we have that it is defined over \( \mathbb{F}_q \). This means that \( X \) is a Del Pezzo surface over \( \mathbb{F}_q \) of degree 9, and the only degree 9 Del Pezzo surface is \( \mathbb{P}^2(\mathbb{F}_q) \). Hence,

\[
|\tilde{\mathbb{S}}(\mathbb{F}_q)| = |\tilde{\mathbb{S}}(\mathbb{F}_q)| = |\tilde{\mathbb{X}}(\mathbb{F}_q)| = q^2 + q + 1.
\]

This follows as, since the singular points \( P_1, P_2, P_3 \) and \( P_4 \) of \( S \) are conjugate, any line passing through a pair of them is not defined over \( \mathbb{F}_q \) and therefore there are no extra \( \mathbb{F}_q \)-rational points on the exceptional divisor (but only four \( \mathbb{F}_q \)-rational points corresponding to \( P_1, P_2, P_3 \) and \( P_4 \)).

\( \square \)

**Remark 3.8** In the proof of the bound for the case in which \( S \) has one isolated \( \mathbb{F}_q \)-rational singular point, Theorem 2.4 can be exploited as \( S \) is by hypothesis absolutely irreducible and not a cone over a smooth absolutely irreducible cubic plane curve. Moreover, by the classification of cubic surfaces in positive characteristic (see [33]), we also know that \( S \) cannot contain a cone over a smooth absolutely irreducible cubic plane curve. Since we are supposing that \( S \) has only isolated singularities, it cannot even contain a cone over a non-singular curve, otherwise it would have a double line.

**Remark 3.9** In Theorem 3.1, the bound for the case in which \( S \) has four isolated and conjugate \( \mathbb{F}_q \)-rational singular points is actually an equality, as it is stated and shown in the proof. Concerning the other cases in which \( S \) has only isolated singularities, it can be seen that the obtained bounds are sharp.

- If \( S \) has one isolated \( \mathbb{F}_q \)-rational singular point: consider a conic in \( \mathbb{P}^2(\mathbb{F}_q) \) and the blow-up of six \( \mathbb{F}_q \)-rational points on this conic. Then, we obtain a singular cubic surface with an ordinary double point, given by the contraction of the proper transform of the conic. The number of \( \mathbb{F}_q \)-rational points of this surface attains the bound \( q^2 + 6q + 1 \).
• If $S$ has two isolated and conjugate $\mathbb{F}_q$-rational singular points: consider the blow-up of $\mathbb{P}^2(\mathbb{F}_q)$ at 6 points, the point of intersection of two conjugate lines, four conjugate points lying on these lines and an additional point. Considering the proper transforms of the two conjugate lines chosen, we then have that these are $(-2)$-curves in the blow-up and their contraction gives a cubic surface with two singularities that are ordinary double points and whose number of $\mathbb{F}_q$-rational points attains the bound $q^2 + 3q + 1$.

• If $S$ has three isolated and conjugate $\mathbb{F}_q$-rational singular points: consider three conjugate $\mathbb{F}_q$-rational lines in $\mathbb{P}^2(\mathbb{F}_q)$ and the blow-up of the intersection points of these lines and an additional point such that there are three geometric points lying over it, each of them lying on one of the chosen lines. Then, the contraction of the proper transforms of the three considered lines gives a cubic surface with three conjugate ordinary double points and $q^2 + q + 1$ $\mathbb{F}_q$-rational points.

**Remark 3.10** In the case in which $S$ has two isolated and conjugate $\mathbb{F}_q$-rational singular points, by using more carefully the counting arguments exploited in [8], we also get the lower bound

$$|\mathcal{S}(\mathbb{F}_q)| \geq q^2 - 10q + 15,$$

which is sharper than the one obtained in [8] but can possibly be improved further by applying more sophisticated techniques.

**Remark 3.11** Another approach to the determination of the bounds shown in Theorem 3.1 could be via explicit computations. However, the methods used above in the proof provide more precise bounds than those that can be obtained with explicit computations and also allow more concise arguments.

### 3.2 Missing cases

In this final part of the section, we discuss the two remaining cases not comprised in Theorem 3.1, i.e., $S$ being a cone over a smooth irreducible cubic plane curve or $S$ being reducible. We investigate these two cases for completeness, and for the future applications to AG codes. Moreover, the bounds obtained in this section hold for the number of projective $\mathbb{F}_q$-rational points of $S$.

If the $\mathbb{F}_q$-rational surface $S$ is a cone over a smooth absolutely irreducible cubic plane curve $\mathcal{E}$, as seen in the proof of Theorem 3.1, case 1.(1), we have that the $\mathbb{F}_q$-rational points on $S$ are $mq + 1$, where $m$ is the number of $\mathbb{F}_q$-rational points of $\mathcal{E}$.

Since the cubic curve in this case is a smooth irreducible cubic plane curve, by well-known results in [22, Chapter 11] we know that it has at most $q + 2\sqrt{q} + 1$ $\mathbb{F}_q$-rational points in $\mathbb{P}^2(\mathbb{F}_q)$; this yields the following result.

**Proposition 3.12** Let $S$ be a cubic surface defined over $\mathbb{F}_q$. If $S$ is a cone over a smooth absolutely irreducible cubic plane curve $\mathcal{E}$, then

$$|\mathcal{S}(\mathbb{F}_q)| \leq (q + 2\sqrt{q} + 1)q + 1 = q^2 + 2q\sqrt{q} + q + 1.$$ 

Finally, if $S$ is reducible, then three possible situations can happen:
(1) $S$ is the union of a non-singular quadric surface and a plane;
(2) $S$ is the union of a quadric cone and a plane;
(3) $S$ is the union of three planes.

(1) If $S$ is the union of a non-singular quadric surface and a plane, from well-known results on the number of rational points on quadric surfaces in $\mathbb{P}^3(\mathbb{F}_q)$ (see [22, Theorem 5.2.6]) the following results hold for the number of $\overline{\mathbb{F}}_q$-rational points on $S$.

- If $S$ is the union of a hyperbolic quadric and a secant plane (which cuts a conic on the quadric), then

$$\left|\tilde{S}(\mathbb{F}_q)\right| = q^2 + 2q + 1 + q^2 = 2q^2 + 2q + 1$$

as the number of $\mathbb{F}_q$-rational points on the quadric surface is $q^2 + 2q + 1$ and the number of $\tilde{\mathbb{F}}_q$-rational points on the plane is $q^2 + q + 1$, but only $q^2$ points need to be counted as the remaining $q + 1$ are those lying in the intersection and hence have already been counted.

- If $S$ is the union of a hyperbolic quadric and a tangent plane at a point of the quadric, then

$$\left|\tilde{S}(\mathbb{F}_q)\right| = q^2 + 2q + 1 + q^2 - q = 2q^2 + q + 1$$

as the number of $\mathbb{F}_q$-rational points on the quadric surface is $q^2 + 2q + 1$ and the number of $\tilde{\mathbb{F}}_q$-rational points on the plane is $q^2 + q + 1$, but only $q^2 - q$ points need to be counted on the plane. This is due to the fact that the plane cuts on the quadric a pair of $\overline{\mathbb{F}}_q$-rational lines, intersecting in a point. Hence, the $2q + 1$ points of the plane lying on these two lines have already been counted.

- If $S$ is the union of an elliptic quadric and a secant plane (which cuts a conic on the quadric), then

$$\left|\tilde{S}(\mathbb{F}_q)\right| = q^2 + 1 + q^2 = 2q^2 + 1.$$  

This follows for analogous reasons as in the previous case of the hyperbolic quadric, noticing that the number of $\overline{\mathbb{F}}_q$-rational points on an elliptic quadric is $q^2 + 1$.

- If $S$ is the union of an elliptic quadric and a tangent plane at a point of the quadric, then

$$\left|\tilde{S}(\mathbb{F}_q)\right| = q^2 + 1 + q^2 + q = 2q^2 + q + 1$$

as the number of $\overline{\mathbb{F}}_q$-rational points on the quadric surface is $q^2 + 1$ and the number of $\tilde{\mathbb{F}}_q$-rational points on the plane is $q^2 + q + 1$, but only $q^2 + q$ points need to be counted as the point of tangency has already been counted.

(2) If $S$ is the union of a quadric cone and a plane, as a quadric cone has $q^2 + q + 1$ $\mathbb{F}_q$-rational points, then the following results hold.

- If the plane intersects the cone only in the vertex, then

$$\left|\tilde{S}(\mathbb{F}_q)\right| = q^2 + q + 1 + q^2 + q = 2q^2 + 2q + 1$$

as the vertex has already been counted.
• If the plane cuts either a non-degenerate conic or a line on the cone, then
  \[|\mathcal{S}(\mathbb{F}_q)| = q^2 + q + 1 + q^2 = 2q^2 + q + 1\]
as the \(q + 1\) points on the intersection curve have already been counted.

• If the plane cuts a pair of intersecting lines on the cone, then
  \[|\mathcal{S}(\mathbb{F}_q)| = q^2 + q + 1 + q^2 - q = 2q^2 + 1\]
as the \(2q + 1\) points lying on the intersection lines have already been counted (these are indeed \(q\) points on each line plus the point where the two lines intersect).

(3) If \(S\) is the union of three planes, then we have the following cases instead.

• If \(S\) is the union of three planes defined over \(\mathbb{F}_q\) and intersecting in a common line, then
  \[|\mathcal{S}(\mathbb{F}_q)| = q^2 + q + 1 + q^2 + q^2 = 3q^2 + q + 1\]
as the \(\mathbb{F}_q\)-rational points on the intersection line have to be counted only once.

• If \(S\) is the union of three planes defined over \(\mathbb{F}_q\) and intersecting in three distinct lines, then
  \[|\mathcal{S}(\mathbb{F}_q)| = q^2 + q + 1 + q^2 + q^2 - q = 3q^2 + 1.\]
This follows as: we count first the \(q^2 + q + 1\) points on the first plane \(\pi\), then we count \(q^2\) points on the second plane \(\rho\) (as the \(q + 1\) points on the line \(\pi \cap \rho\) have already been counted) and finally we count \(q^2 - q\) points on the third plane \(\tau\) as the points on the lines \(\pi \cap \tau\) and \(\rho \cap \tau\) have already been counted (note that these two lines intersect in one point).

• If \(S\) is the union of a plane defined over \(\mathbb{F}_q\) and two conjugate planes intersecting the \(\mathbb{F}_q\)-rational plane in two distinct lines, then
  \[|\mathcal{S}(\mathbb{F}_q)| = q^2 + q + 1 + q = q^2 + 2q + 1.\]
This follows as: we count first the \(q^2 + q + 1\) points on the \(\mathbb{F}_q\)-rational plane, then we count \(q\) points on the \(\mathbb{F}_q\)-rational line which is the intersection of the two conjugate planes. Note that we count only \(q\) points on this line as it intersects the \(\mathbb{F}_q\)-rational plane at a point, which has hence already been counted.

• If \(S\) is the union of a plane defined over \(\mathbb{F}_q\) and two conjugate planes, all intersecting in a common (\(\mathbb{F}_q\)-rational) line, then
  \[|\mathcal{S}(\mathbb{F}_q)| = q^2 + q + 1.\]
This follows as the line containing the \(\mathbb{F}_q\)-rational points on the two conjugate planes lies on the \(\mathbb{F}_q\)-rational plane.

• If \(S\) is the union of three conjugate planes intersecting in a common line, then
  \[|\mathcal{S}(\mathbb{F}_q)| = q + 1\]
as the only \(\mathbb{F}_q\)-rational points are exactly those lying on the common intersection line.
• If $S$ is the union of three conjugate planes intersecting in three distinct lines, then

$$|\tilde{S}(\mathbb{F}_q)| = 1$$

as the only possibility is that the lines are concurrent in a point, which is the unique $\mathbb{F}_q$-rational point of the surface.

4 Cubic surfaces from intersections of algebraic curves

We start now to investigate the intersection over $\mathbb{F}_q$ of the Norm–Trace curve $\mathcal{N}_3$ with the curve defined by

$$y = \mathcal{A}(x) = A_3x^3 + A_2x^2 + A_1x + A_0$$

where $A_3 \neq 0$ and $A_i \in \mathbb{F}_q$, for $i = 0, 1, 2, 3$.

As already recalled, when we refer to a planar intersection (or simply intersection) of two curves lying in the affine space $\mathbb{A}^2(\mathbb{F}_q)$, we mean the number of points in $\mathbb{A}^2(\mathbb{F}_q)$ lying in both curves, disregarding multiplicity (see [8]). For the remaining part of this section, we exploit the same approach used in [8] to set the problem.

From now on, we will write $N$ and $T$ instead of $N_{\mathbb{F}_q}$ and $T_{\mathbb{F}_q}$, respectively. Moreover, throughout the paper we will always consider the curves (resp. surfaces) in the algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$, even when not stated explicitly. When we will need to consider smaller fields, we will point it out.

Substituting $y = \mathcal{A}(x)$ in the equation of $\mathcal{N}_3$, and exploiting the linearity of the trace function, we get

$$N(x) = T(A_3x^3) + T(A_2x^2) + T(A_1x) + T(A_0)$$

Consider now a normal basis $B = \{ \alpha, \alpha^q, \alpha^{q^2} \}$, for a suitable $\alpha \in \mathbb{F}_q$. We know that such a basis exists, see [26, Theorem 2.35]. The vector space isomorphism

$$\Phi_B : (\mathbb{F}_q)^3 \longrightarrow \mathbb{F}_q$$

$$\Phi_B((s_0, s_1, s_2)) = s_0\alpha + s_1\alpha^q + s_2\alpha^{q^2}$$

allows us to read the norm and the trace as maps from $(\mathbb{F}_q)^3$ to $\mathbb{F}_q$, considering $\tilde{N} = N \circ \Phi_B$ and $\tilde{T} = T \circ \Phi_B$. Let $T_i := T(A_ix^i)$ and $\tilde{T}_i := T_i \circ \Phi_B$, for $1 \leq i \leq 3$, then it is readily seen that $\tilde{N}$ and $\tilde{T}_i$ are homogeneous polynomials of degree, respectively, 3 and $i$ in $\mathbb{F}_q[x_0, x_1, x_2]$, $i = 0, 1, 2, 3$.

Therefore, we can rewrite (1) as

$$\tilde{N}(x_0, x_1, x_2) = \tilde{T}_3(x_0, x_1, x_2) + \tilde{T}_2(x_0, x_1, x_2) + \tilde{T}_1(x_0, x_1, x_2) + E$$

(2)

where $E = T(A_0)$. Equation (2) is the equation of a hypersurface of $\mathbb{A}^3(\overline{\mathbb{F}}_q)$, where $\overline{\mathbb{F}}_q$ is the algebraic closure of $\mathbb{F}_q$, and both RHS and LHS have degree 3.

All the considerations made so far lead to the fact that $\Phi_B^{-1}$ induces a correspondence between $\mathbb{F}_q[x]$ and $\overline{\mathbb{F}}_q[x_0, x_1, x_2]$, allowing us to substitute $x$ with $x_0\alpha + x_1\alpha^q + x_2\alpha^{q^2}$.

We exploit this relation to write the explicit equation of the surface defined by equation (2).
For simpler notations, from now on we consider the equation of the cubic curve $y = Ax^3 + Bx^2 + Cx + D$, $A = A_3, B = A_2, C = A_1, D = A_0$.

We have
\[
\tilde{T}_1 = C(x_0 \alpha + x_1 \alpha^q + x_2 \alpha^{q^2}) + C^q(x_0 \alpha^q + x_1 \alpha^{q^2} + x_2 \alpha) + C^{q^2}(x_0 \alpha^{q^2} + x_1 \alpha + x_2 \alpha^q) = x_0 T(aC) + x_1 T(\alpha C^q) + x_2 T(\alpha C),
\]
\[
\tilde{T}_2 = B(x_0 \alpha + x_1 \alpha^q + x_2 \alpha^{q^2})^2 + B^q(x_0 \alpha^q + x_1 \alpha^{q^2} + x_2 \alpha)^2 + B^{q^2}(x_0 \alpha^{q^2} + x_1 \alpha + x_2 \alpha^q)^2 = x_0^2 T(Ba^2) + x_1^2 T(Ba^{2q}) + x_2^2 T(Ba^{2q^2}) + 2x_0x_1 T(Ba^{q+1}) + 2x_0x_2 T(Ba^{q+1})
\]
\[
+ 2x_1x_2 T(Ba^{q+q}),
\]
\[
\tilde{T}_3 = A(x_0 \alpha + x_1 \alpha^q + x_2 \alpha^{q^2})^3 + A^q(x_0 \alpha^q + x_1 \alpha^{q^2} + x_2 \alpha)^3 + A^{q^2}(x_0 \alpha^{q^2} + x_1 \alpha + x_2 \alpha^q)^3 = x_0^3 T(Aa^3) + x_1^3 T(Aa^{3q}) + x_2^3 T(Aa^{3q^2}) + 3x_0^2 x_1 T(Aa^{q+2}) + 3x_0^2 x_2 T(Aa^{q^2+2})
\]
\[
+ 3x_0^2 x_2 T(Aa^{q+2q^2}) + 3x_0 x_1^2 T(Aa^{1+2q}) + 3x_0 x_2^2 T(Aa^{1+2q^2}) + 3x_1 x_2^2 T(Aa^{q+2q^2}),
\]
\[
\tilde{N} = (x_0 \alpha^{q^2} + x_1 \alpha + x_2 \alpha^q)(x_0 \alpha^q + x_1 \alpha^{q^2} + x_2 \alpha)(x_0 \alpha + x_1 \alpha^q + x_2 \alpha^q) = (x_0^3 + x_1^3 + x_2^3)N(\alpha) + (x_0^2 x_1 + x_1^2 x_2 + x_2^2 x_0) T(\alpha^{q+2})
\]
\[
+ (x_0^2 x_2 + x_1^2 x_0 + x_2^2 x_1) T(\alpha^{2q+1}) + x_0 x_1 x_2 (3N(\alpha) + T(\alpha^3)).
\]

Hence, we are now able to rewrite (2) as
\[
0 = -(x_0^3 + x_1^3 + x_2^3)N(\alpha) - (x_0^2 x_1 + x_1^2 x_2 + x_2^2 x_0) T(\alpha^{q+2}) - (x_0^2 x_2 + x_1^2 x_0 + x_2^2 x_1) T(\alpha^{2q+1})
\]
\[
- x_0 x_1 x_2 (3N(\alpha) + T(\alpha^3)) - x_0^3 T(Aa^3) + x_0^3 T(Aa^{3q}) + x_0^3 T(Aa^{3q^2})
\]
\[
+ 3x_0^2 x_1 T(Aa^{q+2}) + 3x_0^2 x_2 T(Aa^{q^2+2}) + 3x_0^2 x_2 T(Aa^{q^2+2q})
\]
\[
+ 3x_0 x_1^2 T(Aa^{1+2q}) + 3x_0 x_2^2 T(Aa^{1+2q^2}) + 3x_1 x_2^2 T(Aa^{q+2q^2})
\]
\[
+ x_0^2 T(Ba^2) + x_1^2 T(Ba^{2q}) + x_2^2 T(Ba^{2q^2}) + 2x_0 x_1 T(Ba^{q+1}) + 2x_0 x_2 T(Ba^{q+1})
\]
\[
+ 2x_1 x_2 T(Ba^{q+q}) + x_0 T(\alpha C) + x_1 T(\alpha C^q) + x_2 T(\alpha C^q) + E.
\]

Let $S_1$ be the surface defined by equation (3), and observe that it is defined over the field $\mathbb{F}_q$. Similarly to what was done in Sect. 3, we will denote with $S_1(\mathbb{F}_q)$ the set of its affine $\mathbb{F}_q$-rational points and with $S_1(\mathbb{F}_q)$ the set of its projective $\mathbb{F}_q$-rational points.

**Remark 4.1** By construction, $\mathbb{F}_q$-rational points of $S_1$ correspond to the intersections in $\mathbb{A}^2(\mathbb{F}_q)$ between $\mathcal{N}_3$ and the curve $y = Ax^3 + Bx^2 + Cx + D$. In other words, our algebraic manipulations proved that there exists $x \in \mathbb{F}_q$ such that $N(x) = T(Ax^3 + Bx^2 + Cx + D)$ if and only if exists $(x_0, x_1, x_2) \in (\mathbb{F}_q)^3$ satisfying (3) and its representation on the chosen normal basis is $x = x_0 \alpha + x_1 \alpha^q + x_2 \alpha^{q^2}$.

Note that equation (3) can be also rewritten as...
0 = -(x_0\alpha + x_1\alpha^q + x_2\alpha^q)(x_0\alpha^q + x_1\alpha^q + x_2\alpha^q) + A(x_0\alpha + x_1\alpha^q + x_2\alpha^q)^3 + A^q(x_0\alpha^q + x_1\alpha^q + x_2\alpha^q)^3 + C(x_0\alpha + x_1\alpha^q + x_2\alpha^q) + C^q(x_0\alpha^q + x_1\alpha^q + x_2\alpha^q) + E.

Consider now the affine change of variables in \( \mathbb{A}^3(\mathbb{F}_q) \) defined by

\[
\psi(x_0,x_1,x_2) = M(x_0,x_1,x_2) = (X_0,X_1,X_2)
\]

where \( M \) is the non-singular matrix

\[
M = \begin{pmatrix}
\alpha & \alpha^q & \alpha^q^2 \\
\alpha^q & \alpha^q^2 & \alpha \\
\alpha^q^2 & \alpha & \alpha^q 
\end{pmatrix}
\]

and let \( S_2 \) be the surface obtained from \( S_1 \) through \( \psi \) (i.e. \( S_2 = \psi(S_1) \)).

**Proposition 4.2** \( S_2 \) is a surface defined over \( \mathbb{F}_q \), has equation

\[
X_0X_1X_2 = AX_0^3 + A^qX_1^3 + A^q^2X_2^3 + BX_0^2 + B^qX_1^2 + B^q^2X_2^2 + CX_0 + C^qX_1 + C^q^2X_2 + E
\]

and preserves the multiplicities of the points and of the components of \( S_1 \). Moreover, the points on \( S_2 \) of the form \((\beta, \beta^q, \beta^q^2)\), \( \beta \in \mathbb{F}_q \), are in bijection with the \( \mathbb{F}_q \)-rational points on \( S_1 \).

**Proof** These properties come from straightforward computations combined with the fact that \( M \) is a non-singular affine transformation. \( \square \)

In order to estimate the intersection between \( N_3 \) and \( y = Ax^3 + Bx^2 + Cx + D \), our goal is the determination of an upper bound for the number of (affine) \( \mathbb{F}_q \)-rational points of \( S_1 \).

The results from Sect. 3 ensure that the following result holds, as a corollary of Theorem 3.2.

**Corollary 4.3** Consider the \( \mathbb{F}_q \)-rational cubic surface \( S_1 \) associated to the intersections between \( N_3 \) and \( y = Ax^3 + Bx^2 + Cx + D \), \( A \neq 0 \). If \( S_1 \) is absolutely irreducible and it is not a cone over an irreducible smooth plane cubic curve, then

\[
|S_1(\mathbb{F}_q)| \leq q^2 + 7q + 1.
\]

**Remark 4.4** Note that the results in [8] prove Corollary 4.3 under the assumption \( A = 0 \).

Even if this result is already established by the discussion in Sect. 3, below we treat separately the particular case \( B = C = 0 \). Studying this case is interesting since it gives explicit information on the reducibility of \( S_1 \), depending on its coefficients and the base field. Moreover, in the case \( \text{char} \( (\mathbb{F}_q) = 3 \), we find explicitly the form of the singular points that \( S_1 \) can have, which is not possible for the general case.

**Remark 4.5** Suppose that \( S_2 \) has a singular point \( \mathbb{F}_q \)-rational, i.e. a singular point of the shape \((u, u^q, u^q^2)\), \( u \in \mathbb{F}_q \). Then direct computations show that \( C = -3Au^2 - 2Bu + u^q + q \).
and \( E = 2u^3A + u^2B - 2u^{q+1} + 2u^{3q}A^q + u^{2q}B^q + 2u^{3q^2}A^q + u^{2q^2}B^q \). Suppose now that the line passing through \((u, u^q, u^{q^2})\) and \((x, y, z)\) is contained in \( S_2 \), then first note that \( x = u \) implies \((x, y, z) = (u, u^q, u^{q^2})\) (and the same holds for \( y = u^q \) and \( z = u^{q^2} \)). Then, this general line is contained in \( S_2 \) only if

\[
B(x - u)^3 + B^q(y - u^q)^2(x - u) + B^{q^2}(z - u^{q^2})^2(x - u) + 3A^q(y - u^q)^2(xu^q - yu) \\
+ 3A^{q^2}(z - u^{q^2} - zu) + (x - u)(xyu^{q^2} + xzu^{q^2} - 2xu^{q^2} + yu^{q^2} + z^{q^2}) = 0.
\]

Unfortunately, in general it is difficult to determine the number of possible solutions \((x, y, z)\) for general \( A, B \in \mathbb{F}_q \). Nevertheless, excluding some possible solutions can lead to the restriction of the possible shapes of \( S_1 \).

**Remark 4.6** If \( S_1 \) is a cone over a smooth absolutely irreducible cubic plane curve, we believe that it could be possible to obtain a better bound, in the same form as Corollary 4.3, writing explicitly the equation of such cone. Moreover, due to the variety of possible different equations of \( S_1 \), it is not even clear to us if there exist choices of parameters that realize this situation.

**Remark 4.7** The cases in which \( S_1 \) is reducible appear to be rare, and it is also unclear if they happen only for particular values of \( q \). Nevertheless, up to now, we are not able to characterize them uniquely. However, in Sect. 5, we will study a particular case for which we are able to obtain explicit information on the reducibility of \( S_1 \), depending on its coefficients and the base field.

**Problem 1** Characterize the reducibility of \( S_1 \), and when \( S_1 \) is a cone, using only relations on its coefficients and the base field.

### 5 \( S_1 \) irreducible with isolated singularities: case \( B = C = 0 \)

In this section, we will take into account the special case in which \( S_1 \) is irreducible, has only isolated singularities, and \( B = C = 0 \).

We investigate the singular points of \( S_2 \), which is equivalent to studying singular points of \( S_1 \), thanks to the map \( \psi \) defined in precedence. For this case of study, the equation of \( S_2 \) is

\[
X_0X_1X_2 = AX_0^3 + A^qX_1^3 + A^{q^2}X_2^3 + E
\]

and its affine singular points are the solutions to the following system:

\[
\begin{align*}
X_0X_1X_2 &= AX_0^3 + A^qX_1^3 + A^{q^2}X_2^3 + E \\
X_1X_2 &= 3AX_0^2 \\
X_0X_2 &= 3A^qX_1^2 \\
X_0X_1 &= 3A^{q^2}X_2^2
\end{align*}
\]

From this system of equations, it is immediately clear that \( E = 0 \) if and only if \((0, 0, 0)\) is a solution.

We distinguish now three different cases, depending on the characteristic of the field \( \mathbb{F}_q \).
Proposition 5.1 Let $B = C = 0$, and $\text{char}(\mathbb{F}_q) = 3$. Then the only possible singularities of $S_2$ are the following

- $(0, 0, 0)$ if and only if $E = 0$;
- $(0, 0, -\frac{E}{A^3}), (0, 0, -\frac{E}{A^3}, 0)$ and $\left(-\frac{E}{A^3}, 0, 0\right)$ if $E \neq 0$ and $\epsilon$ solution of $X^3 = -\frac{E}{A^3}$.

Proof In this case, system (4) becomes

$$\begin{cases} 
X_0X_1X_2 = AX^3_0 + A^qX^3_1 + A^{q^2}X^3_2 + E \\
X_1X_2 = 0 \\
X_0X_2 = 0 \\
X_0X_1 = 0 
\end{cases}$$

From the last three equations it follows that at least two among $X_0$, $X_1$ and $X_2$ have to be 0. Suppose $X_0 = X_1 = 0$, obtaining $A^qX^3_2 + E = 0$. If $E = 0$, the unique solution is $X_0 = X_1 = X_2 = 0$, otherwise we have

$$X^3_2 = -\frac{E}{A^{q^2}}$$

and this equation is solvable if and only if $-\frac{E}{A^{q^2}}$ is a cube in $\mathbb{F}_q$. Since $\text{char} (\mathbb{F}_q) = 3$, $A^{q^2}$ is always a cube in $\mathbb{F}_q$ and $E$ is always a cube since $x^3$ is a permutation of the field. This means that the above equation is always solvable and there is one possible value for $X_2$

$$X_2 = -\frac{E}{A^{q^2}}$$

where $E = \epsilon^3$. Hence, we have the solution $\left(0, 0, -\frac{E}{A^{q^2}}\right)$.

Iterating the above reasoning for the other two cases, we have our desired result. It is also immediate to see that there are no singular points at the infinity. \qed

Proposition 5.2 Let $B = C = 0$, and $\text{char}(\mathbb{F}_q) = 2$. Then the only possible singularity of $S_2$ is the point $(0, 0, 0)$.

Proof In this case, system (4) becomes

$$\begin{cases} 
X_0X_1X_2 = AX^3_0 + A^qX^3_1 + A^{q^2}X^3_2 + E \\
X_1X_2 = AX^3_0 \\
X_0X_2 = A^qX^3_1 \\
X_0X_1 = A^{q^2}X^3_2 
\end{cases} \quad (5)$$

Substituting, it is possible to see that $E$ must be equal to zero and that (5) is equivalent to

$$\begin{cases} 
X^3_0 = A^{q-1}X^3_1 \\
X^3_1 = A^{q^2}X^3_2 \\
X^3_0 = A^{q-1}X^3_2 
\end{cases} \quad (6)$$
Since we are looking for solutions with coordinates in \( \overline{\mathbb{F}}_q \), note that we have all the solutions of the form

\[
(\zeta_{3,i}A^\frac{q-1}{3}, X_2, \zeta_{3,j}A^\frac{q-2}{3} X_2, X_2)
\]

for \( X_2 \in \overline{\mathbb{F}}_q \) and \( \zeta_{3,i}, \zeta_{3,j} \) cubic roots of unity.

Considering now the first equation in (5), we have that \((\zeta_{3,i}A^\frac{q-1}{3}, X_2, \zeta_{3,j}A^\frac{q-2}{3} X_2, X_2)\) is a solution different from \((0, 0, 0)\) if and only if

\[
\zeta_{3,i}A^\frac{q^2-1}{3} \zeta_{3,j}A^\frac{q^2-2}{3} X_2^3 = A^q X_2^3 + A^q X_2^3 + A^q X_2^3
\]

So, we have solutions if and only if \( N(A) = 1 \).

We can sum up the situation as follows:

- if \( N(A) = 1 \), then there are more than 4 solutions to the system, in fact all triples of the form
  \[
  (\zeta_{3,i}A^\frac{q-1}{3}, X_2, \zeta_{3,j}A^\frac{q-2}{3} X_2, X_2)
  \]
  satisfy the system, for every value of \( X_2 \) in \( \overline{\mathbb{F}}_q \). However, this is not possible if the surface is irreducible (as it is in our case), due to Theorem 2.2.
- if \( N(A) \neq 1 \), there are no solutions to the system different from \((0, 0, 0)\) (which is a solution if and only if \( E = 0 \)).

\[\square\]

Considering what we have just found regarding the solutions to system (6), we conclude also that there are no singular points at the infinity. In fact, the only solution to the system in our case would be the point with coordinates \([0 : 0 : 0 : 0]\).

The same result can be obtained for the general case, we do not show the proof of this proposition since it is analogue to the one just written. This can be easily adapted doing some different reductions in system (4) and then noticing that \( N(A) = 27 \).

**Proposition 5.3** Let \( B = C = 0 \), and \( \text{char}(\mathbb{F}_q) \neq 2, 3 \). The only possible singularity of \( S_2 \) is the point \((0, 0, 0)\).

### 6 AG codes arising from Norm–Trace curves

We already know that \( \mathcal{N}_3 \) has \( N = q^5 \mathbb{F}_q \)-rational points in \( \mathbb{A}^2(\mathbb{F}_q) \), and that

\[
\mathcal{L}_{\mathbb{F}_q}(3q^2P_\infty) = \langle \{1, x, x^2, x^3, y, y^2, xy\} \rangle.
\]

Considering now the evaluation map

\[
ev : \mathcal{L}_{\mathbb{F}_q}(3q^2P_\infty) \longrightarrow (\mathbb{F}_q)^N
\]

\[
f = ay^2 + bxy + cy + dx^3 + ex^2 + fx + g \longmapsto (f(P_1), \ldots, f(P_N))
\]
the associated one-point code will be $C_=(D, 3q^2P_\infty) = \text{ev}(\mathcal{L}_{F_q}(3q^2P_\infty))$, where the divisor $D$ is the formal sum of all the $q^2$ rational affine points of $\mathcal{N}_3(\mathbb{F}_q)$. The weight of a codeword associated to the evaluation of a function $f \in \mathcal{L}_{F_q}(3q^2P_\infty)$ corresponds to

$$w(\text{ev}(f)) = |\mathcal{N}_3(\mathbb{F}_q)| - |\{\mathcal{N}_3(\mathbb{F}_q) \cap \{ay^2 + bxy + cy + dx^3 + ex^2 + fx + g = 0\}\}|.$$

Using the results obtained in the previous sections, we can give some bounds in a variety of cases.

- If $a = b = d = 0$ then we are in a case already studied in [8]. More specifically:

  1. If $c = 0$ then we have to consider the zeros of $ex^2 + fx + g$ that are points of $\mathcal{N}_3(\mathbb{F}_q)$.
     
     - (a) If $e = f = g = 0$ then $w(\text{ev}(f)) = 0$;
     - (b) if $e = f = 0$ and $g \neq 0$ then $w(\text{ev}(f)) = q^5$;
     - (c) if $e = 0$ and $f \neq 0$ then $w(\text{ev}(f)) = q^5 - q^2$;
     - (d) if $f \neq 0$ and $f^2 - 4eg = 0$ then $w(\text{ev}(f)) = q^5 - q^2$;
     - (e) otherwise $w(\text{ev}(f)) = q^5 - 2q^2$.

  2. On the other hand, if $c \neq 0$ then we have to consider the points of $\mathcal{N}_3(\mathbb{F}_q)$ that are zeros of $cy + ex^2 + fx + g$.

     - (a) If $e = f = g = 0$ then $w(\text{ev}(f)) = q^5 - 1$;
     - (b) if $e = f = 0$ and $g \neq 0$ then $w(\text{ev}(f)) = q^5 - q^2$;
     - (c) if $e = 0$ and $f \neq 0$ then, applying Bézout theorem, we have that $w(\text{ev}(f)) \geq q^5 - (q^2 + q + 1)$;
     - (d) otherwise, from what we said previously, $w(\text{ev}(f)) \geq q^5 - (q^2 + 7q + 1)$.

- If $a = b = 0$ and $d \neq 0$ then we can obtain some information from our results on intersections.

  1. If $c = 0$ then we have to consider the points of $\mathcal{N}_3(\mathbb{F}_q)$ that are zeros of $dx^3 + ex^2 + fx + g$.

     - (a) If $e = f = g = 0$ then $w(\text{ev}(f)) = q^5 - q^2$;
     - (b) otherwise $w(\text{ev}(f)) \geq q^5 - 3q^2$.

  2. If $c \neq 0$ then we have to consider the points of $\mathcal{N}_3(\mathbb{F}_q)$ that are zeros of $cy + dx^3 + ex^2 + fx + g$. We can do this exploiting our results on $\mathbb{F}_q$-rational points of the cubic surface obtained from the intersection of $\mathcal{N}_3$ and the cubic curve we are considering. As noted, we have different bounds according to the different shapes the surface $S_1$ assumes.

     - (a) If $S_1$ is absolutely irreducible and it is not a cone over an irreducible smooth plane cubic curve, then $w(\text{ev}(f)) \geq q^5 - (q^2 + 7q + 1)$;
     - (b) if there exist coefficients $c, d, e, f, g$ for which $S_1$ is a cone over a smooth absolutely irreducible cubic plane curve, then $w(\text{ev}(f)) \geq q^5 - (q^2 + 2q\sqrt{q} + 1)$;
     - (c) if there exist coefficients $c, d, e, f, g$ for which $S_1$ is reducible, then $w(\text{ev}(f)) \geq q^5 - (3q^2 + q + 1)$. 

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• If \( a \neq 0 \) or \( b \neq 0 \) unfortunately, from our results, we are not able to deduce information on the weights.

**Remark 6.1** These considerations about the weight spectrum of the code tell us that, despite the fact that the dimension of the code increases with respect to the one studied in [8], the lower weights of the weight spectrum, which are those that contribute more for the computation of the PUE (Probability of the Undetected Error), do not seem to have many variations.

**Remark 6.2** The results on the weights that we have obtained for the code \( C_{L}(D, 3q^2 P_{\infty}) \) hold for a more general class of AG codes. We gave lower bounds on the weights of such a code considering some monomials that are comprised in the considered basis of the Riemann-Roch space \( \mathcal{L}_{F_q}(3q^2 P_{\infty}) \). The same bounds hold for other codes on \( \mathcal{N}_3 \) associated to divisors \( D \) and \( G = kP_{\infty} \) with \( k \geq 3q^2 \). This follows because there exists a basis of \( \mathcal{L}_{F_q}(kP_{\infty}) \) (note that \( \mathcal{L}_{F_q}(3q^2 P_{\infty}) \subseteq \mathcal{L}_{F_q}(kP_{\infty}) \) for \( k \geq 3q^2 \)) that contains those monomials, when \( k \geq 3q^2 \). Hence, our results have impact on a vast range of codes arising from \( \mathcal{N}_3 \).

As noted above, our discussion does not cover all the possible cases since the basis of the Riemann-Roch space \( \mathcal{L}_{F_q}(3q^2 P_{\infty}) \) has also the monomials \( xy \) and \( y^2 \).

Using our approach, it seems difficult to study the intersections of \( \mathcal{N}_3 \) with curves with terms in \( xy \) or \( y^2 \), as the equation of the surface corresponding to the intersections would be much more complicated (for instance it can be a quartic surface). Therefore, it still remains an open problem to determine the weight spectrum of the code \( \mathcal{L}_{F_q}(3q^2 P_{\infty}) \).

**Problem 2** Determine the weight spectrum of the code \( \mathcal{L}_{F_q}(3q^2 P_{\infty}) \).

**Remark 6.3** Note that, if we consider the subspace \( H := \langle \{1, x, x^2, x^3, y\} \rangle \) of \( \mathcal{L}_{F_q}(3q^2 P_{\infty}) \) and the evaluation map

\[
eq \begin{align*}
\text{ev}_{\mathcal{L}_{F_q}(3q^2 P_{\infty})} : H &\longrightarrow (\mathbb{F}_q)^5 \\
(f = cy + dx^3 + ex^2 + fx + g) &\longmapsto (f(P_1), \ldots, f(P_N))
\end{align*}
\]

we still have that \( \text{ev}_{\mathcal{L}_{F_q}(3q^2 P_{\infty})}(H) \) is a linear code, for which our results on the weights apply (see the cases with \( a = b = 0 \) above).

**Acknowledgements** The authors would like to thank the anonymous referee for their helpful and constructive comments that highly contributed to improve and generalize the results of the paper. The research of M. Bonini was supported by the Irish Research Council, Grant No. GOIPD/2020/597. The results showed in this paper are included in L. Vicino’s MSc thesis (supervised by the first and the second author).

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