Harmonic Bilocal Fields Generated by
Globally Conformal Invariant Scalar Fields

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Abstract

The twist two contribution in the operator product expansion of $\phi_1(x_1)\phi_2(x_2)$ for a pair of globally conformal invariant, scalar fields of equal scaling dimension $d$ in four space–time dimensions is a field $V_1(x_1,x_2)$ which is harmonic in both variables. It is demonstrated that the Huygens bilocality of $V_1$ can be equivalently characterized by a “single–pole property” concerning the pole structure of the (rational) correlation functions involving the product $\phi_1(x_1)\phi_2(x_2)$. This property is established for the dimension $d = 2$ of $\phi_1, \phi_2$. As an application we prove that any GCI scalar field of conformal dimension 2 (in four space–time dimensions) can be written as a (possibly infinite) superposition of products of free massless fields.

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1 Introduction

Global Conformal Invariance (GCI) of Minkowski space Wightman fields yields rationality of correlation functions [NT01]. This result opens the way for a nonperturbative construction and analysis of GCI models for higher dimensional Quantum Field Theory (QFT). The first studied cases were the theories generated by a scalar field \( \phi(x) \) of (low) integral dimension \( d > 1 \). (The case \( d = 1 \) corresponds to a free massless field with a vanishing truncated 4-point function \( w^\text{tr}_4 \).) The cases \( 2 \leq d \leq 4 \), which give rise to non-zero \( w^\text{tr}_4 \) were considered in [NST02, NST03, NRT05].

The main purpose in these papers was to study the constraints for the 4-point correlation (= Wightman) functions coming from the Wightman (= Hilbert space) positivity. This was achieved by using the conformal partial wave expansion. An important technical tool in this expansion is the splitting of the Operator Product Expansion (OPE) into different twist contributions (see (2.11)). Each term gives a nonrational contribution to the complete rational 4-point function. It is therefore remarkable that the sum of the leading, twist two, conformal partial waves (corresponding to the contributions of all conserved symmetric traceless tensors in the OPE of basic fields) can be proven in certain cases to be a rational function. This means that the twist two part in the OPE of two fields \( \phi \) is convergent in such cases to a bilocal field, \( V_1(x_1, x_2) \), which is our first main result in the present paper. Here “bilocal” means Huygens locality with respect to both arguments. Proving bilocality exploits the bounds on the poles due to Wightman positivity, and the conservation laws for twist two tensors which imply that the bilocal fields are harmonic in both arguments.

Trivial examples of harmonic bilocal fields are given by bilinear free field constructions of the form: \( \varphi(x_1)\varphi(x_2) \), or: \( \bar{\psi}(x_1)\gamma_\mu(x_1-x_2)^\mu\psi(x_2) \). A major purpose of this paper is to explore whether harmonic twist two fields can exist which are not of this form, and whether they can be bilocal. Moreover, we show that the presence of a bilocal field \( V_1 \) completely determines the...
structure of the theory in the case of a scaling dimension $d = 2$. The first step towards the classification of $d = 2$ GCI fields was made in [NST02] where the case of a unique scalar field was considered. Here we extend our study to the most general case of a theory generated by an arbitrary (countable) set of $d = 2$ scalar fields.

The paper is organized as follows. Section 2 contains a review of relevant results concerning the theory of GCI scalar fields. A new result is the strengthening of the pole bounds on truncated correlation functions, implied by the cluster property (Eq. (2.7)).

In Sect. 3 we study conditions for the existence of the harmonic bilocal field $V_1(x_1, x_2)$. We prove that the bilocality of $V_1(x_1, x_2)$ is equivalent to the single pole property (SPP) of the correlation functions involving the manifestly bilocal field $U(x_1, x_2)$ (2.8), which presents the nonvacuum part in the OPE of $\phi_1(x_1)\phi_2(x_2)$ whose twist expansion starts with $V_1(x_1, x_2)$. This nontrivial condition qualifies a premature claim in [BNRT07] that bilocality is automatic.

Indeed, the SPP is trivially satisfied for all correlations of free field constructions of harmonic fields with other (products of) free fields, due to the bilinear structure of $V_1$. Thus any violation of the SPP is a clear signal for a nontrivial field content of the model. Moreover, the SPP will be proven from general principles for an arbitrary system of $d = 2$ scalar fields (the case studied in [BNRT07]). Yet, although the pole structure of $U(x_1, x_2)$ turns out to be highly constrained in general by the conservation laws of twist two tensor currents, the SPP does not follow for fields of higher dimensions, as illustrated by a counter-example of a 6-point function of $d = 4$ scalar fields involving double poles (Sect. 3.5).

The existence of $V_1(x_1, x_2)$ in a theory of dimension $d = 2$ fields allows to determine the truncated correlation functions up to a single parameter in each of them. This is exploited in Sect. 4, where an associative algebra structure of the OPE of $d = 2$ scalar fields and harmonic bilocal fields is revealed.

2 Properties of GCI scalar fields

2.1 Structure of correlation functions and pole bounds

We assume throughout the validity of the Wightman axioms for a QFT on the $D = 4$ flat Minkowski space–time $M$ (except for asymptotic completeness) – see [SW]. Our results can be, in fact, generalized in a straightforward way to any even space–time dimension $D$. The condition of GCI in the Minkowski space is an additional symmetry condition on the correlation functions of the theory [NT01]. In the case of a scalar field $\phi(x)$, it asserts that the correlation functions of $\phi(x)$ are invariant under the substitution

$$\phi(x) \mapsto \det \left( \frac{\partial g}{\partial x} \right)^4 \phi(g(x)),$$

(2.1)
where \( x \mapsto g(x) \) is any conformal transformation of the Minkowski space, \( \frac{\partial g}{\partial x} \) is its Jacobi matrix and \( d > 0 \) is the scaling dimension of \( \phi \). An important point is that the invariance of Wightman functions \( \langle 0 | \phi(x_1) \cdots \phi(x_n) | 0 \rangle \) under the transformation (2.1) should be valid for all \( x_k \in M \) in the domain of definition of \( g \) (in the sense of distributions). It follows that \( d \) must be an integer in order to ensure the singlevaluedness of the prefactor in (2.1). Thus, GCI implies that only integral anomalous dimensions can occur.

The most important consequences of GCI in the case of scalar fields \( \phi_k(x) \) of dimensions \( d_k \) are summarized as follows.

(a) **Huygens Locality** ([NT01, Theorem 4.1]). Fields commute for non-light–like separations. This has an algebraic version:

\[
[(x_1 - x_2)^2]^N [\phi_1(x_1), \phi_2(x_2)] = 0
\]  
(2.2)

for a sufficiently large integer \( N \).

(b) **Rationality of Correlation Functions** (cf. [NT01, Theorem 3.1]). The general form of Wightman functions is:

\[
\langle 0 | \phi_1(x_1) \cdots \phi_n(x_n) | 0 \rangle = \sum_{\{\mu_{jk}\}} \prod_{j<k} (\rho_{jk})^{\mu_{jk}},
\]  
(2.3)

where here and in what follows we set

\[
\rho_{jk} := (x_{jk} - i 0 e_0)^2 = (x_{jk})^2 + i 0 x_{jk}^0, \quad x_{jk} := x_j - x_k;
\]  
(2.4)

the sum in Eq. (2.3) is over all configurations of integral powers \( \{\mu_{jk} = \mu_{kj}\} \) subject to the following conditions:

\[
\sum_{j \neq k} \mu_{jk} = -d_k,
\]  
(2.5)

and pole bounds \( \mu_{jk} > -\left[ \frac{d_j + d_k}{2} + \frac{\delta_{d_j d_k} - 1}{2} \right] \). Equation (2.5) follows from the conformal invariance under (2.1); the pole bounds express the absence of non-unitary representations in the OPE of two fields [NT01, Lemma 4.3]. Under these conditions the sum in (2.3) is always finite and there are a finite number of free parameters for every \( n \)-point correlation function. We shall refer to the form (2.3) as a **Laurent polynomial** in the variables \( \rho_{jk} \).

(c) The **truncated** Wightman functions \( \langle 0 | \phi_1(x_1) \cdots \phi_n(x_n) | 0 \rangle_{tr} \) are of the same form like (2.2) but with pole degrees \( \mu_{jk}^{tr} \) bounded by

\[
\mu_{jk}^{tr} > -\frac{d_j + d_k}{2}.
\]  
(2.6)

This is due to the standard cluster property in QFT applied to \( \langle 0 | \phi_1(x_1) \cdots \phi_n(x_n) | 0 \rangle_{tr} \) (cf. [NT01, Corollary 4.4]). Furthermore, the cluster property implies the following generalization of (2.6):

\[
\sum_{j,k \in K; j<k} \mu_{jk}^{tr} > -\frac{1}{2} \sum_{k \in K} d_k,
\]  
(2.7)
for every non-empty proper subset $K$ of $\{1, \ldots, n\}$. This generalization (which is proven in a similar way as (2.6)) will be used in establishing the single pole property for $d = 2$.

2.2 Twist expansion of the OPE and bi–harmonicity of twist two contribution

The most powerful tool provided by GCI is the explicit construction of the OPE of local fields in the general (axiomatic) framework.

Let $\phi_1(x)$ and $\phi_2(x)$ be two GCI scalar fields of the same scaling dimension $d$ and consider the operator distribution

$$U(x_1, x_2) = (\rho_{12})^{d-1} \left( \phi_1(x_1) \phi_2(x_2) - \langle 0 | \phi_1(x_1) \phi_2(x_2) | 0 \rangle \right). \tag{2.8}$$

As a consequence of the pole bounds (2.6), $U(x_1, x_2)$ is smooth in the difference $x_{12}$. This is to be understood in a weak sense for matrix elements of $U$ between bounded energy states. Obviously, $U(x_1, x_2)$ is a Huygens bilocal field in the sense that

$$\left[ (x_1 - x)^2 (x_2 - x)^2 \right]^N U(x_1, x_2), \psi(x) = 0 \tag{2.9}$$

for every field $\psi(x)$ that is Huygens local with respect to $\phi_k(x)$. Then, one introduces the OPE of $\phi_1(x_1) \phi_2(x_2)$ by the Taylor expansion of $U$ in $x_{12}$

$$U(x_1, x_2) = \sum_{n=0}^{\infty} \sum_{\mu_1, \ldots, \mu_n = 0}^3 x_{12}^{\mu_1} \cdots x_{12}^{\mu_n} X_n^{\mu_1 \cdots \mu_n}(x_2), \tag{2.10}$$

where $X_n^{\mu_1 \cdots \mu_n}(x_2)$ are Huygens local fields. We can consider the series (2.10) as a formal power series, or as a convergent series in terms of the analytically continued correlation functions of $U(x_1, x_2)$. We will consider at this point the series (2.10) just as a formal series. (See also [BN06] for the general case of constructing OPE via multilocal fields in the context of vertex algebras in higher dimensions.)

Since the prefactor in (2.8) transforms as a scalar density of conformal weight $(1 - d, 1 - d)$ then $U(x_1, x_2)$ transforms as a conformal bilocal field of weight $(1, 1)$. Hence, the local fields $X_n^{\mu_1 \cdots \mu_n}$ in (2.10) have scaling dimensions $n + 2$ but are not, in general, quasiprimary. One can pass to an expansion in quasiprimary fields by subtracting from $X_n^{\mu_1 \cdots \mu_n}$ derivatives of lower dimensional fields $X_n'_{\mu_1' \cdots \mu_n'}$. The resulting quasiprimary fields $O_k^{\mu_1 \cdots \mu_\ell}$ are traceless tensor fields of rank $\ell$ and dimension $k$. The difference

$$k - \ell \quad ("\text{dimension} - \text{rank}") \tag{2.11}$$

is called twist of the tensor field $O_k^{\mu_1 \cdots \mu_\ell}$. Unitarity implies that the twist is non-negative [M77], and by GCI, it should be an even integer. In this way one can reorganize the OPE (2.10) as follows

$$U(x_1, x_2) = V_1(x_1, x_2) + \rho_{12} V_2(x_1, x_2) + (\rho_{12})^2 V_3(x_1, x_2) + \cdots, \tag{2.12}$$

Quasiprimary fields transform irreducibly under conformal transformations.
Harmonic Bilocal Fields

where $V_{\kappa}(x_1, x_2)$ is the part of the OPE (2.10) containing only twist $2\kappa$ contributions. Note that Eq. (2.12) contains also the information that the twist $2\kappa$ contributions contain a factor $(\rho_{12})^{\kappa-1}$ (i.e. $V_{\kappa}$ are “regular” at $x_1 = x_2$), which is a nontrivial feature of this OPE (obtained by considering 3-point functions). Thus, the expansion in twists can be viewed as a light-cone expansion of the OPE.

Since the twist decomposition of the fields is conformally invariant then each $V_{\kappa}$ will be behave, at least infinitesimally, as a scalar $(\kappa, \kappa)$ density under conformal transformations.

Every $V_{\kappa}$ is a complicated (formal) series in twist $2\kappa$ fields and their derivatives:

$$V_{\kappa}(x_1, x_2) = \sum_{\ell=0}^{\infty} \sum_{\mu_1 \ldots \mu_\ell} K_{\mu_1 \ldots \mu_\ell}^\kappa (x_{12}, \partial x_2) O^\ell_{\mu_1 \ldots \mu_\ell}(x_2),$$  \hspace{1cm} (2.13)

where $K_{\mu_1 \ldots \mu_\ell}^\kappa (x_{12}, \partial x_2)$ are infinite formal power series in $x_{12}$ with coefficients that are differential operators in $x_2$ acting on the quasiprimary fields $O$. The important point here is that the series $K_{\mu_1 \ldots \mu_\ell}(x_{12}, \partial x_2)$ can be fixed universally for any (even generally) conformal QFT. This is due to the universality of conformal 3-point functions. The explicit form of $K_{\mu_1 \ldots \mu_\ell}(x_{12}, \partial x_2)$ can be found in [DMPPT, DO01] (see also [NST03]).

Thus, we can at this point consider $V_{\kappa}(x_1, x_2)$ only as generating series for the twist $2\kappa$ contributions to the OPE of $\phi(x_1)\phi(x_2)$ but we still do not know whether these series would be convergent and even if they were, it would not be evident whether they would give bilocal fields. In the next section we will see that this is true for the leading, twist two part under certain conditions, which are automatically fulfilled for $d = 2$.

The higher twist parts $V_{\kappa}$ ($\kappa > 1$) are certainly not convergent to Huygens bilocal fields, since their 4-point functions, computed in [NST03], are not rational.

The major difference between the twist two tensor fields and the higher twist fields is that the former satisfy conservation laws:

$$\partial_{x^{\mu_1}} O_{\mu_1 \ldots \mu_\ell}^{\ell+2}(x) = 0 \quad (\ell \geq 1).$$  \hspace{1cm} (2.14)

This is a well known consequence of the conformal invariance of the 2-point function and the Reeh–Schlieder theorem. It includes, in particular, the conservation laws of the currents and the stress–energy tensor. It turns out that $V_1(x_1, x_2)$ encodes in a simple way this infinite system of equations.

**Theorem 2.1.** ([NST03]) *The system of differential equations (2.14) is equivalent to the harmonicity of $V_1(x_1, x_2)$ in both arguments (bi–harmonicity) as a formal series, i.e.,

$$\Box_{x_1} V_1(x_1, x_2) = 0 = \Box_{x_2} V_1(x_1, x_2).$$*
The proof is based on the explicit knowledge of the $K$ series in (2.13) and it is valid even if the theory is invariant under infinitesimal conformal transformations only.

The separation of the twist two part in (2.12) amounts to a splitting of $U$ of the form

$$U(x_1, x_2) = V_1(x_1, x_2) + \rho_{12} \tilde{U}(x_1, x_2).$$  

(2.15)

This splitting can be thought in terms of matrix elements of $U(x_1, x_2)$ expanded as a formal power series according to (2.10). It is unique by virtue of Theorem 2.1, due to the following classical Lemma:

**Lemma 2.2.** ([BT77, BN06]) Let $u(x)$ be a formal power series in $x \in \mathbb{C}^4$ (or, $\mathbb{C}^D$) with coefficients in a vector space $V$. Then there exist unique formal power series $v(x)$ and $\tilde{u}(x)$ with coefficients in $V$ such that

$$u(x) = v(x) + x^2 \tilde{u}(x)$$  

(2.16)

and $v(x)$ is harmonic in $x$ (i.e., $\Box_x v(x) = 0$). (2.16) is called the harmonic decomposition of $u(x)$ (in the variable $x$ around $x = 0$), and the formal power series $v(x)$ is said to be the harmonic part of $u(x)$.

### 3 Bilocality of twist two contribution to the OPE

Let us sketch our strategy for studying bilocality of $V_1(x_1, x_2)$. The existence of $V_1(x_1, x_2)$ as a Huygens bilocal field can be established by constructing its correlation functions. On the other hand, every correlation function $\langle V_1(x_1, x_2) \rangle$ of $V_1$ is obtained (originally, as a formal power series in $x_{12}$) under the splitting (2.15). It thus appears as a harmonic decomposition of the corresponding correlation function $\langle U(x_1, x_2) \rangle$ of $U$:

$$\langle U(x_1, x_2) \rangle = \langle V_1(x_1, x_2) \rangle + \rho_{12} \langle \tilde{U}(x_1, x_2) \rangle.$$  

(3.1)

Note that we should initially treat the left hand side of (3.1) also as a formal power series in $x_{12}$ in order to make the equality meaningful. It is important that this series is always convergent as a Taylor expansion of a rational function in a certain domain around $x_1 = x_2$ in $M_C^{x_2}$, for the complexified Minkowski space $M_C = M + iM$, according to the standard analytic properties of Wightman functions. We shall show in Sect. 3.1 that this implies the separate convergence of both terms in the right hand side of (3.1). Hence, the key tool in constructing $V_1$ are the harmonic decompositions

$$F(x_1, x_2) = H(x_1, x_2) + \rho_{12} \tilde{F}(x_1, x_2)$$  

(3.2)

This short-hand notation stands for $\langle 0 | \phi_3(x_3) \cdots \phi_k(x_k) V_1(x_1, x_2) \phi_{k+1}(x_{k+1}) \cdots \phi_n(x_n) | 0 \rangle$, here and in the sequel.
of functions $F(x_1, x_2)$ that are analytic in certain neighbourhoods of the diagonal $\{x_1 = x_2\}$.

Recall that $H$ in (3.2) is uniquely fixed as the harmonic part of $F$ in $x_1$ around $x_2$, due to Lemma 2.2. This is equivalent to the harmonicity $\Box_{x_1} H(x_1, x_2) = 0$. On the other hand, according to Theorem 2.1 we have to consider also the second harmonicity condition on $H$, $\Box_{x_2} H(x_1, x_2) = 0$, i.e., $H$ is the harmonic part in $x_2$ around $x_1$. This leads to some “integrability” conditions for the initial function $F(x_1, x_2)$, which we study in Sect. 3.2.

Next, to characterize the Huygens bilocality of $V_1$, we should have rationality of its correlation functions $\langle V_1(x_1, x_2) \rangle$, which is due to a straightforward extension of the arguments of [NT01, Theorem 3.1]. But we have started with the correlation functions of $U$, which are certainly rational. Hence, we should study another condition on $U$, namely that its correlation functions have a rational harmonic decomposition. We show in Sect. 3.3 that this is equivalent to a simple condition on the correlation functions of $U$, which we call “Single Pole Property” (SPP).

In this way we establish in Sect. 3.4 that $V_1$ always exists as a Huygens bilocal field in the case of scalar fields of dimension $d = 2$. However, for higher scaling dimensions one cannot anymore expect that $V_1$ is Huygens bilocal in general. This is illustrated by a counter example, involving the 6-point function of a system of $d = 4$ fields, given at the end of Sect. 3.5.

3.1 Convergence of harmonic decompositions

To analyze the existence of the harmonic decomposition of a convergent Taylor series we use the complex integration techniques introduced in [BN06].

Let $M_\mathbb{C} = M + iM$ be the complexification of Minkowski space, which in this subsection is assumed to be $D$-dimensional, and $E = \{x : (ix^0, x^1, \ldots, x^{D-1}) \in \mathbb{R}^D\}$ its Euclidean real submanifold, and $S^{D-1} \subset E$ the unit sphere in $E$. We denote by $\|\cdot\|$ the Hilbert norm related to the fixed coordinates in $M_\mathbb{C}$: $\|x\|^2 := |x^0|^2 + \cdots + |x^{D-1}|^2$.

Let us also introduce for any $r > 0$ a real compact submanifold $M_r$ of $M_\mathbb{C}$:

$$M_r = \{\zeta \in M_\mathbb{C} : \zeta = r e^{i\theta} w, \ \theta \in [0, \pi], w \in S^{D-1}\} \quad (3.3)$$

(note that $\theta \in [\pi, 2\pi]$ gives another parameterization of $M_r$). Then there is an integral representation for the harmonic part of a convergent Taylor series.

**Lemma 3.1.** (cf. [BN06, Sect. 3.3 and Appendix A]) Let $u(x)$ be a complex formal power series that is absolutely convergent in the ball $\|x\| < r$, for some $r > 0$, to an analytic function $U(x)$. Then the harmonic part $v(x)$ of $u(x)$ (around $x = 0$), which is provided by Lemma 2.2, is absolutely convergent for

$$|x^2| + 2r \|x\| < r^2. \quad (3.4)$$

The analytic function $V(x)$ that is the sum of the formal power series $v(x)$
has the following integral representation:

$$V(x) = \int \frac{d^{D}z}{2\pi^{2}} \left[ \frac{1 - \frac{x^{2}}{2}}{(z - x)^{2}} \right] Dz \quad \mathcal{G}_{1} = \int \frac{d^{D}z}{2\pi^{2}} |i\pi| S^{D-1},$$

(3.5)

where \( r' < r \), \(|x^{2}| + 2r' ||x|| < r'^{2} \), and the (complex) integration measure \( d^{D}z \) is obtained by the restriction of the complex volume form \( d^{D}z (= dz^{0} \wedge \cdots \wedge dz^{D-1}) \) on \( M_{\mathcal{C}} (\cong \mathbb{C}D) \) to the real \( D \)-dimensional submanifold \( M_{\mathcal{R}} \) (3.3), \( r' > 0 \).

Proof. Consider the Taylor expansion in \( x \) of the function \( (1 - \frac{x^{2}}{2})/[(z - x)^{2}] \) and write it in the form (cf. [BN06, Sect. 3.3])

$$\frac{1 - \frac{x^{2}}{2}}{(z - x)^{2}} = \sum_{\ell = 0}^{\infty} (z^{2})^{-\frac{D}{2} - \ell} H_{\ell}(z, x), \quad H_{\ell}(z, x) = \sum_{\mu} h_{\ell\mu}(z) h_{\ell\mu}(x),$$

(3.6)

where \( \{h_{\ell\mu}(u)\} \) is an orthonormal basis of harmonic homogeneous polynomials of degree \( \ell \) on the sphere \( S^{D-1} \). This expansion is convergent for

$$|x^{2}| + 2|z \cdot x| < |z^{2}|$$

(3.7)

since its left-hand side is related to the generating function for \( H_{\ell} \):

$$\frac{1 - \lambda^{2} x^{2} y^{2}}{(1 - 2\lambda x \cdot y + \lambda^{2} x^{2} y^{2})^{\frac{D}{2}}} = \sum_{\ell = 0}^{\infty} \lambda^{\ell} H_{\ell}(x, y),$$

(3.8)

the expansion (3.8) being convergent for \( \lambda \leq 1 \) if \(|x^{2} y^{2}| + 2|x \cdot y| < 1 \). Then if we fix \( r' < r \) and \( z \) varies on \( M_{\mathcal{R}} \), a sufficient condition for (3.7) is \(|x^{2}| + 2r' ||x|| < r'^{2} \) (since \( \sup_{w \in S^{D-1}} |w \cdot x| = ||x|| \)).

On the other hand, writing \( u(z) = \sum_{k=0}^{\infty} u_{k}(z) \), where \( u_{k} \) are homogeneous polynomials of degree \( k \), we get by the absolute convergence of \( u(z) \) the relation (valid for \(|x^{2}| + 2r' ||x|| < r'^{2} \))

$$\sum_{k, \ell = 0}^{\infty} \int \frac{d^{D}z}{2\pi^{2}} \left[ \frac{1 - \frac{x^{2}}{2}}{(z - x)^{2}} \right] Dz \quad \mathcal{G}_{1} = \sum_{k, \ell = 0}^{\infty} \int \frac{d^{D}z}{2\pi^{2}} (z^{2})^{-\frac{D}{2} - \ell} H_{\ell}(x, z) u_{k}(z).$$

(3.9)

Noting next that in the parameterization (3.3) of \( M_{\mathcal{R}} \), we have \( d^{D}z \mid_{M_{\mathcal{R}}} = i r'^{D} e^{iD\vartheta} d\vartheta \wedge d\sigma(w) \), where \( d\sigma(w) \) is the volume form on the unit sphere, we obtain for the right hand side of (3.9):

$$\sum_{k, \ell = 0}^{\infty} \int_{0}^{\pi} \frac{\pi}{i\pi} e^{i\vartheta(k - \ell)} \int_{S^{D-1}} d\sigma(w) H_{\ell}(x, w) u_{k}(w).$$
Now if we write, according to Lemma 2.2,
\[ u_k(z) = \sum_{2j \leq k} \sum_{\mu} c_{k,j,\mu'} (z^2)^j \]
then we get by the orthonormality of \( h_{\ell,\mu}(w) \)
\[ \sum_{k,\ell=0}^{\infty} \sum_{2j \leq k} \sum_{\mu} \delta_{\ell,k-2j} \int_0^{\pi} \frac{d\vartheta}{i\pi} e^{i\vartheta(k-\ell)} c_{k,j,\mu'} h_{k-2j,\mu}(x) = \sum_{k=0}^{\infty} \sum_{\mu} c_{k,0,\mu} h_{k,\mu}(x) = v(x). \]
The latter proves both: the convergence of \( v(x) \) in the domain (3.4) (since \( r' < r \) was arbitrary) and the integral representation (3.5).

As an application of this result we will prove now

**Proposition 3.2.** For all \( n \) and \( k \), and for all local fields \( \phi_j \) \((j = 3, \ldots, n)\) the Taylor series
\[ \langle 0 \mid \phi_3(x_3) \cdots \phi_k(x_k) V_1(x_1, x_2) \phi_{k+1}(x_{k+1}) \cdots \phi_n(x_n) \mid 0 \rangle \]
in \( x_{12} \) converge absolutely in the domain
\[ \left( \|x_{12}\| + \sqrt{\|x_{12}\|^2 + |x_{12}^2|} \right) \left( \|x_{2j}\| + \sqrt{\|x_{2j}\|^2 + |x_{2j}^2|} \right) < |x_{2j}^2| \quad \forall j \quad (k = 3, \ldots, n). \]
They all are real analytic and independent of \( k \) for mutually nonisotropic points.

**Proof.** Let
\[ F_k(x_{12}, x_{23}, \ldots, x_{2n}) = \langle 0 \mid \phi_3(x_3) \cdots \phi_k(x_k) U(x_1, x_2) \phi_{k+1}(x_{k+1}) \cdots \phi_n(x_n) \mid 0 \rangle \]
be the correlation functions, analytically continued in \( x_{12} \).

As \( F_k \), which is a rational function, depends on \( x := x_{12} \) via a sum of products of powers \((x - x_{2j})^{-\mu_j}\) it has a convergent expansion in \( x \) for
\[ |x^2| + 2 |x \cdot x_{2j}| < |x_{2j}^2|. \]
If we want \( F_k \) to have a convergent Taylor expansion for \( \|x\| < r \) we get the following sufficient condition
\[ r^2 < |x_{2j}^2| - 2 \|x_{2j}\|. \]
By Lemma 3.1 we conclude that the series (3.10) is convergent for
\[ |x_{12}^2| + 2 \|x_{12}\| < r^2. \]
Combining both (sufficient) conditions (3.14) and (3.15) for \( r \) we find that they are compatible if \( \|x_{12}\| + \sqrt{\|x_{12}\|^2 + |x_{12}^2|} < \sqrt{\|x_{2j}\|^2 + |x_{2j}^2|} - \|x_{2j}\| \), which is equivalent to (3.11).
Note that one can also prove a similar convergence property for the correlation functions of several $V_1$.

**Remark 3.1.** The domain of convergence of (3.10) should be Lorentz invariant. Hence, (3.10) are convergent in the smallest Lorentz invariant set containing the domain (3.11). Such a set is determined by the values of the invariants $x_{12}^2$, $x_{2k}^2$ and $x_{12} \cdot x_{2k}$ and it turns out to be the set

$$|x_{12}^2|^\frac{1}{2} |x_{2k}^2|^\frac{1}{2} \leq |x_{12} \cdot x_{2k}| < \frac{( |x_{2k}^2|^\frac{1}{2} - |x_{12}^2|^\frac{1}{2})^2}{4}$$

or equivalently

$$\sqrt{|x_{12}^2||x_{2k}^2| + |x_{12} \cdot x_{2k}|^2} < \frac{( |x_{2k}^2|^\frac{1}{2} - |x_{12}^2|^\frac{1}{2})^2}{4}. \quad (3.16)$$

As a corollary of Proposition 3.2 we deduce that if $V_1(x_1, x_2)$ exists just as a field (i.e., as a distribution in all of $M \times M$), then it must be a Huygens bilocal field.

**Corollary 3.3.** Assume that there exists an operator valued distribution $V_1(x_1, x_2)$ defined over the domain of all fields $\phi_k(x)$ and leaving it invariant, whose matrix elements coincide with (3.10) on the domain (3.16). Then $V_1(x_1, x_2)$ is a Huygens bilocal field which is GCI of weight $(1, 1)$.

In other words, if $V_1(x_1, x_2)$ does not exist as a Huygens bilocal field then it cannot be defined as an operator valued distribution for all $x_1$ and $x_2$ either, since its correlation functions become multi-valued for large $x_{12}$.

### 3.2 A necessary and sufficient condition for bi–harmonicity

Now our objective is to find the harmonic decomposition of the rational functions $F(x_1, x_2)$ that depend on $x_1$ and $x_2$ through the intervals $\rho_{ik} = (x_i - x_k)^2$, $i = 1, 2$, $k = 3, \ldots, n$, for some additional points $x_3, \ldots, x_n$. The $F$’s, as correlation functions of $U(x_1, x_2)$, have the form

$$F(x_1, x_2) = \sum_{q=0}^{M} (\rho_{12})^q F_q(x_1, x_2) \equiv \sum_{q=0}^{M} (\rho_{12})^q F_q\left(\{\rho_{i,k}\}_{i,k}\neq\{1,2\}\right), \quad (3.17)$$

$$F_q(x_1, x_2) = \sum_{\{\mu_i\},\{\mu_j\}} C_{q,\{\mu_i\},\{\mu_j\}} \prod_{j=3}^{n} (\rho_{1j})^{\mu_{1j}} \prod_{j=3}^{n} (\rho_{2j})^{\mu_{2j}}, \quad (3.18)$$

where $M \in \mathbb{N}$ and $\mu_{1j}, \mu_{2j}$ ($j = 3, \ldots, n$) are integers $> -d$ such that $\sum_{j\geq3} \mu_{1j} = \sum_{j\geq3} \mu_{2j} = -1 - q$, and the coefficients $C_{q,\{\mu_i\},\{\mu_j\}}$ may depend on $\rho_{jk}$ ($j, k \geq 3$).

If $H$ is the harmonic part of $F$ in $x_{12}$, then the leading part $F_0$ (of order $(\rho_{12})^0$) is also the leading part of $H$. We shall now proceed to show that bi–harmonicity of $H$ (Theorem 2.1), together with the first principles of QFT including GCI, implies strong constraints on $F_0$. 

Harmonic Bilocal Fields
Proposition 3.4. Let $F_0(x_1,x_2)$ be as in (3.18), and let $H(x_1,x_2)$ be its harmonic part with respect to $x_1$ around $x_2$. Then $H$ is also harmonic with respect to $x_2$, if and only if $F_0$ satisfies the differential equation

\[(E_1D_2 - E_2D_1)F_0 = 0,\]  

(3.19)

where $E_1 = \sum_{i=3}^n \rho_{2i}\partial_{i1}$ (with $\partial_{jk} = \partial_{kij} = \frac{\partial}{\partial \theta_{jk}}$), $D_1 = \sum_{3 \leq j < k \leq n} \rho_{jk}\partial_{1j}\partial_{1k}$, and similarly for $E_2$ and $D_2$, exchanging $1 \leftrightarrow 2$.

Proof. By Proposition 3.2 (see also Remark 3.1) we can consider $H$ as a function in the $2n-3$ variables $\rho_{1i}$, $\rho_{2i}$ ($i \geq 3$) and $\rho_{12}$, analytic in some domain that includes $\rho_{12} = 0$.

Expanding $H = \sum_{q} (\rho_{12})^q H_q / q!$, the functions $H_q$ are homogeneous of degree $-1 - q$ in both sets of variables $\rho_{1i}$ and $\rho_{2i}$, and $H_0 = F_0$. To impose the harmonicity with respect to the variable $x_1$, we use the identity [NRT05, App. C]

\[\Box_{x_1} F = -4 \left[ \sum_{2 \leq i < j \leq n} \rho_{ij} \partial_{i1} \partial_{1j} F \right]_{\rho_{ij} = (x_i - x_j)^2},\]

(3.20)

valid for homogeneous functions of $\rho_{1i}$ of degree $-1$, to express the wave operator $\Box_{x_1}$ as a differential operator with respect to the set of variables $\rho_{1i}$ ($i \geq 2$). This yields the recursive system of differential equations

\[E_1 H_{q+1} = -D_1 H_q.\]  

(3.21)

Performing the same steps with respect to the variable $x_2$, one obtains

\[E_2 H_{q+1} = -D_2 H_q.\]  

(3.22)

Eq. (3.19) then arises as the integrability condition for the pair of inhomogeneous differential equations for $H_1$ (putting $q = 0$), observing that $E_2 E_1 - E_1 E_2 = \sum \rho_{1i} \partial_{i1} - \sum \rho_{2i} \partial_{2i}$ vanishes on $H_1$ by homogeneity.

Conversely, if (3.19) is fulfilled, then $H_1$ exists and satisfies $(D_1 E_2 - D_2 E_1)H_1 = -(D_1 D_2 - D_2 D_1)H_0 = 0$ because $D_1$ and $D_2$ commute. But this is equivalent to $(D_2 E_1 - D_1 E_2)H_1 = 0$, which is in turn the integrability condition for the existence of $H_2$, and so on. It follows that bi-harmonicity imposes no further conditions on the leading function $H_0 = F_0$. 

The differential equation (3.19) imposes the following constraints on the leading part $F_0$ of the rational correlation function $F$ (3.17):

**Corollary 3.5.** Assume that the function $F_0$ as in (3.18) satisfies the differential equation (3.19).

(i) If $F_0$ contains a “double pole” of the form $(\rho_{1i})^{\mu_{1i}}(\rho_{1j})^{\mu_{1j}}$ with $i \neq j$ and $\mu_{1i}$ and $\mu_{1j}$ both negative, then its coefficients must be regular in $\rho_{2k}$ ($k \neq i,j$).

(ii) $F_0$ cannot contain a “triple pole” of the form $(\rho_{1i})^{\mu_{1i}}(\rho_{1j})^{\mu_{1j}}(\rho_{1k})^{\mu_{1k}}$ with $i,j,k$ all different and $\mu_{1i}$, $\mu_{1j}$, $\mu_{1k}$ all negative.

The same hold true, exchanging $1 \leftrightarrow 2$. 

Proof. Pick any variable, say \( \rho_{2k} \), and decompose \( F_0 = \sum_{r \geq -p}(\rho_{2k})^r f_r \) as a Laurent polynomial in \( \rho_{2k} \). The differential equation (3.19) turns into the recursive system

\[
\left( \rho_{1k} \sum_{i<j} \rho_{ij} \partial_{1i} \partial_{1j} - \sum_{i,j \neq k} \rho_{2i} \rho_{kj} \partial_{1i} \partial_{2j} \right) r \cdot f_r = X_r f_{r-1} + Y f_r
\]

of differential equations for the functions \( f_r \) which are Laurent polynomials in the remaining variables. The precise form of the polynomial differential operators \( X_r \) and \( Y \) does not matter. Assume the lowest power \(-p\) of \( \rho_{2k} \) to be negative. For \( r = -p \), the right-hand-side vanishes. Because the term \( \rho_{ij} \partial_{1i} \partial_{1j} \) on the left-hand-side would produce a singularity that cannot be cancelled by any other term, \( f_{-p} \) cannot have a “double pole” in any pair of variables \( \rho_{1i}, \rho_{1j} \) with \( i \neq j \) and \( i, j \neq k \). This property passes recursively to all \( f_r \) with \( r < 0 \), because also the right-hand-side never can contain such a pole. This implies that a double pole in a pair of variables \( \rho_{1i}, \rho_{1j} \) with \( i \neq j \) cannot multiply a term that is singular in \( \rho_{2k} \) unless \( k = i \) or \( k = j \), proving (i).

If the coefficient of the double pole were singular in \( \rho_{1k}, k \neq i, j \), then the resulting double pole in the pair \( \rho_{1i}, \rho_{1k} \) resp. \( \rho_{1j}, \rho_{1k} \) would imply regularity also in \( \rho_{2j} \) resp. \( \rho_{2i} \). Hence the coefficient of a triple pole must be regular in all \( \rho_{2m} \), which contradicts the total homogeneity \(-1\) of \( F_0 \) in these variables. This proves the statement (ii). \( \square \)

3.3 A necessary and sufficient condition for bilocality

Definition 3.1. (“Single Pole Property”, SPP) Let \( f(x_1, \ldots, x_n) \) be a Laurent polynomial in the variables \( \rho_{ij} \), i.e., regarded as a function of \( x_1 \) only, it is a finite linear combination of functions of the form

\[
\prod_{j \geq 2} (\rho_{1j})^{\mu_{1j}} \equiv \prod_{j \geq 2} [(x_1 - x_j)^2]^{\mu_{1j}},
\]

where \( \mu_{1j} \) \((j \geq 2)\) are integers and the coefficients may depend on the parameters \( \rho_{jk} \) \((j, k \geq 2)\). Then \( f \) is said to satisfy the single pole property with respect to \( x_1 \) if it contains no terms for which there are \( j \neq k \) \((j, k \geq 2)\) such that both \( \mu_{1j} \) and \( \mu_{1k} \) are negative.

The significance of SPP stems from the fact that the harmonic parts \( H \) of \( F_0 \), i.e., the correlation functions of \( V_1 \), are again Laurent polynomials if and only if \( F_0 \) satisfies the SPP. Namely, if \( H \) is a harmonic Laurent polynomial, the same argument as in [NRT05, Lemma C.1] (using the representation (3.20) of the wave operator) shows that \( H \) fulfils the SPP with respect to \( x_1 \), and so does \( F_0 \), because it is the leading part of order \((\rho_{12})^0\) of \( H \). The converse is an immediate consequence of Lemma 3.6 (allowing for a relabelling and multiple counting of the points \( x_3, \ldots, x_n \), which are not required to be distinct).
Lemma 3.6. Let \( n \geq 4 \). Every finite linear combination of monomials of the form
\[
g_n(x_1) = \prod_{i=4}^{n} \frac{\rho_{1i}}{(\rho_{13})^{n-2}} \equiv \prod_{i=4}^{n} \frac{(x_1 - x_i)^2}{[(x_1 - x_3)^2]^{n-2}}
\] (3.24)
has a rational harmonic decomposition in \( x_1 \) around \( x_2 \)
\[
g_n(x_1) = h_n(x_1) + (x_1 - x_2)^2 \cdot \tilde{g}_n(x_1)
\] (3.25)
i.e., \( h_n \) is harmonic with respect to \( x_1 \) and \( \tilde{g}_n \) is regular at \( x_1 = x_2 \), and both \( h_n \) and \( \tilde{g}_n \) are rational. More precisely, \( (\rho_{13})^{n-2}(\rho_{23})^{n-3}h_n \) is a homogeneous polynomial of total degree \( 2(n-3) \) in the variables \( \{\rho_{ij} : 1 \leq i < j\} \), which is separately homogeneous of degree \( n-3 \) in the variables \( \{\rho_{1i} : i \geq 2\} \) and in the variables \( \{\rho_{12}, \rho_{2i} : i \geq 3\} \).

Proof. It is convenient to introduce the variables
\[
t_i = \frac{\rho_{1i} \rho_{23}}{\rho_{13} \rho_{2i}}, \quad s_i = \frac{\rho_{12} \rho_{3i}}{\rho_{13} \rho_{2i}}, \quad u_{ij} = \frac{\rho_{12} \rho_{23} \rho_{ij}}{\rho_{13} \rho_{2i} \rho_{2j}} \quad (4 \leq i < j \leq n). \] (3.26)
We claim that \( h_n(x_1) \) is of the form
\[
h_n(x_1) = \left( \prod_{i=4}^{n} \frac{\rho_{2i}}{\rho_{23}} \right) \frac{f_n(t_i, s_i, u_{ij})}{\rho_{13}}, \] (3.27)
where \( f_n \) are polynomials of degree \( n - 3 \) such that \( f_n(t_i, s_i = 0, u_{ij} = 0) = \prod_{i=4}^{n} t_i \). Because all \( s_i \) and \( u_{ij} \) contain a factor \( \rho_{12} \), these properties ensure that \( \tilde{g}_n \) given by \( (g_n - h_n)/\rho_{12} \) is regular in \( \rho_{12} \).

Using again the identity (3.20) for the wave operator, and transforming this into a differential operator with respect to the set of variables (3.26), we find
\[
\square_{x_1} h_n(x_1) = -4 \left( \prod_{i=4}^{n} \frac{\rho_{2i}}{\rho_{23}} \right) \frac{\rho_{23}}{(\rho_{13})^2 \rho_{12}} \cdot D f_n(t_i, s_i, u_{ij}), \] (3.28)
where \( D \) is the differential operator
\[
D = (1 + t \partial_t + s \partial_s + u \partial_u)(s \partial_t + s \partial_s + u \partial_u) - (s \partial_s + u \partial_u) \partial_t - u \partial_u \partial_t \] (3.29)
with shorthand notations for degree-preserving operators
\[
t \partial_t = \sum_{i=4}^{n} t_i \partial_{t_i}, \quad s \partial_t = \sum_{i=4}^{n} s_i \partial_{t_i}, \quad s \partial_s = \sum_{i=4}^{n} s_i \partial_{s_i}, \quad u \partial_u = \sum_{4 \leq i < j \leq n} u_{ij} \partial_{u_{ij}}
\]
and degree-lowering operators
\[
\partial_t = \sum_{i=4}^{n} \partial_{t_i}, \quad u \partial_u \partial_t = \sum_{4 \leq i < j \leq n} u_{ij} \partial_{t_i} \partial_{t_j}.
\]
To solve the condition $Df_n = 0$ for harmonicity, we make an ansatz

$$f_n(t_i, s_i, u_{ij}) = \sum_{K \subset N} g^{(n)}_K(s_k, u_{kl}) \prod_{i \in N \setminus K} (t_i - s_i),$$

where $N \equiv \{4, \ldots, n\}$, $g^{(n)}_K$ are polynomials in the variables $s_k, u_{kl}$ ($k, l \in K$) only, and $g^{(n)}_\emptyset = 1$. Then the harmonicity condition $Df_n = 0$ is equivalent to the recursive system

$$(n - 2 - |K| + \Delta) \Delta g^{(n)}_K = \Delta \sum_{k \in K} g^{(n)}_K \{k\} + \sum_{k,l \in K, k < l} (u_{kl} - s_k - s_l) g^{(n)}_K \{k,l\},$$

where $|K|$ is the number of elements of the set $K$ and the differential operator $\Delta = s \partial_s + u \partial_u$ measures the total polynomial degree $r$ in $s_k$ and $u_{kl}$. Since one can divide by $(n - 2 - |K| + r)r$ if $r > 0$, there is a unique polynomial solution such that $g^{(n)}_K(s_k = 0, u_{kl} = 0) = 0$ ($K \neq \emptyset$), and $g^{(n)}_K$ is of order $\leq |K|$. So $f_n$ is of order $n - 3$. (Explicitly, the first three functions are $f_3 = 1$, $f_4 = t_4 - s_4$ and $f_5 = (t_4 - s_4)(t_5 - s_5) + \frac{1}{2}(u_{45} - s_4 - s_5)$.) An inspection of the recursion also shows that all possible factors $\rho_{2i}$ in the denominators of the arguments of $f_n$ cancel with the factors in the prefactor in (3.27), thus $h_n$ can have poles only in $\rho_{13}$ and $\rho_{23}$ of the specified maximal degree. This proves the Lemma.

The upshot of the previous discussion is

**Theorem 3.7.** The field $V_1(x_1, x_2)$ weakly converges on bounded energy states to a bilocal field which is conformal of weight $(1, 1)$, if and only if the leading parts $F_0$ of the Laurent polynomials $F$ (3.17) satisfy the “single pole property” (Def. 3.1) with respect to both $x_1$ and $x_2$. In this case, the formal series $H$ converge to Laurent polynomials in $(x_i - x_j)^2$ subject to the same pole bounds, specified in Theorem 2.1, as $F$.

**Proof.** We know already that if $V_1$ is a bilocal field, then its correlation functions $H$ are Laurent polynomials of the form (2.3), and that this implies the SPP for $F_0$ with respect to $x_1$ and $x_2$. Conversely, if the SPP holds for $F_0$ with respect to $x_1$ and $x_2$, then $H$ are Laurent polynomials by Lemma 3.6, and hence $V_1$ is relatively Huygens bilocal with respect to the fields $\phi_i$. There is a general argument [B60] that this already implies local commutativity of $V_1$ with itself. We want to give an explicit argument in the present case.

All the previous remains true when in (3.10) or (3.17) a product of fields $\phi_k(x_k, x_{k+1})$ is replaced by $U(x_k, x_{k+1})$. By assumption, and because $U$ is bilocal, the contributions of order $(\rho_{k,k+1})^0$ to the correlation functions of $U(x_k, x_{k+1})$ fulfil the SPP with respect to $x_k$ and $x_{k+1}$. By Lemma 3.6, this property is preserved upon the passage to the harmonic parts with respect to $x_1$ and $x_2$. One may therefore continue in the same way with $x_k, x_{k+1}$, and eventually find that all mixed correlation functions of $\phi$’s and $V_1$’s converge to rational functions. By this convergence we conclude that all products of $\phi$’s and $V_1$’s converge on the vacuum, and this then defines $V_1$ as a Huygens bilocal field, since its matrix elements will satisfy Huygens locality.
The conformal properties of $V_1$ follow from the preservation of the homogeneity and the pole degrees in the harmonic decomposition, as guaranteed by Lemma 3.6.

### 3.4 The case of dimension 2

Let us consider now the case of scalar fields $\phi_k$ of dimension 2. We claim that in this case, Corollary 3.5 is sufficient to establish the SPP, Definition 3.1. Hence we conclude by Theorem 3.7 that the twist two bi–harmonic fields $V(x_1, x_2)$ are indeed bilocal fields.

To prove our claim, we use that by (2.6), $\mu_{ij} \geq -1$, for any two and by (2.7),

$$\mu_{ij} + \mu_{ik} + \mu_{jk} \geq -2,$$

for any three entries in a truncated $n$-point correlation function. Suppose that $\mu_{12} = -1$ and the leading part $F_0$ of the corresponding correlation of the bi-local field $U(x_1, x_2)$ contains a double pole $(x_{13}^2)^{-1}(x_{14}^2)^{-1}$, i.e. $\mu_{13} = \mu_{14} = -1$. Then (3.30) implies that $\mu_{23}$ and $\mu_{24}$ cannot be negative, while Corollary 3.5 implies that the remaining powers $\mu_{2i}$ ($i \geq 5$) cannot be negative. This contradicts the total degree $-1$ in the variables $x_{2i}^2$ ($i \geq 3$).

### 3.5 A $d = 4$ 6-point function violating the SPP

We proceed with an example of 6-point function violating the SPP in the case of two $d = 4$ GCI scalar fields $L_i(x)$ such that the bi-local field $U(x_1, x_2)$ obtained from $L_1(x_1)L_2(x_2)$ has a non-zero skew–symmetric part. Let $L$ be any linear combination of $L_1$ and $L_2$.

The following admissible contribution to the truncated part of the 6-point function $\langle 0|U(x_1, x_2)L(x_3)L(x_4)U(x_5, x_6)|0\rangle$ clearly violates the SPP:

$$F_0(x_1, x_2) = A_{12} A_{56} \left[ \frac{p_{15}p_{26}p_{34} - 2p_{15}p_{23}p_{46} - 2p_{15}p_{24}p_{36}}{p_{13}p_{14}p_{23}p_{24} \cdot p_{34} \cdot p_{35}p_{45}p_{46}} \right], \tag{3.31}$$

where $A_{ij}$ stands for the antisymmetrization in the arguments $x_i$, $x_j$. It is admissible as a truncated 6-point structure because $(\rho_{12}\rho_{56})^{-3}F_0$ obeys all the pole bounds of Sect. 2 for a correlation $\langle 0|L_1(x_1)L_2(x_2)L(x_3)\rightarrow L_1(x_5)L_2(x_6)|0\rangle^{tr}$ of six fields of dimension $d = 4$.

On the other hand, $F_0$ satisfies the differential equation

$$E_1D_2 - E_2D_1)F_0(x_1, x_2) = 0 \tag{3.32}$$

(and similar in the variables $x_5$ and $x_6$), ensuring that $F_0$ is the leading part of a bi–harmonic function, analytic in a neighborhood of $x_1 = x_2$ and $x_5 = x_6$, representing a contribution to the twist two 6-point function $\langle 0|V_1(x_1, x_2)L(x_3)L(x_4)V_1(x_5, x_6)|0\rangle$, of which $F_0$ is the leading part. This function cannot be a Laurent polynomial in the $\rho_{ij}$ by our general argument that the leading part of a bi–harmonic Laurent polynomial cannot satisfy the SPP. Hence the twist two field $V_1(x_1, x_2)$ cannot be bilocal.
The resulting contribution to the conserved local current 4-point function \( \langle 0 | J_\mu(x_1) L(x_3) L(x_4) J_\nu(x_5) | 0 \rangle \) is obtained through \( J_\mu(x) = i(\partial_\mu - \partial_\nu) V_1(x, y) |_{x=y} \). It also satisfies the pertinent pole bounds. This structure is rational as it should, because only the leading part \( F_0 \) contributes. In fact, while the 6-point structure involving the harmonic field cannot be reproduced by free fields due to its double pole, the resulting 4-point structure does arise as one of the three independent connected structures contributing to 4-point functions involving two Dirac currents \( \bar{\psi}_c \gamma^\mu \psi_b \) and two Yukawa scalars \( \varphi : \bar{\psi}_c \psi_d \) (allowing for internal flavours \( a, b, \ldots \)).

4 The theory of GCI scalar fields of scaling dimension \( d = 2 \)

The scaling dimension \( d = 2 \) is the minimal dimension of a GCI scalar field for which one could expect the existence of nonfree models. It turns out however, that in this case the fields can be constructed as composite fields of free, or generalized free, fields. Namely, we will establish the following result.

**Theorem 4.1.** Let \( \{ \Phi_m(x) \}_{m=1}^\infty \) be a system of real GCI scalar fields of scaling dimension \( d = 2 \). Then it can be realized by a system of generalized free fields \( \{ \psi_m(x) \} \) and a system of independent real massless free fields \( \{ \varphi_m(x) \} \), acting on a possibly larger Hilbert space, as follows:

\[
\Phi_m(x) = \sum_{j=1}^{\infty} \alpha_{m,j} \psi_j(x) + \frac{1}{2} \sum_{j,k=1}^{\infty} \beta_{m,j,k} : \varphi_j(x) \varphi_k(x) :, \tag{4.1}
\]

where \( \alpha_{m,j} \) and \( \beta_{m,j,k} = \beta_{m,k,j} \) are real constants such that \( \sum_{j=1}^{\infty} \alpha_{m,j}^2 < \infty \) and \( \sum_{j,k=1}^{\infty} \beta_{m,j,k}^2 < \infty \). Here, we assume the normalizations \( \langle 0 | \varphi_j(x_1) \varphi_k(x_2) | 0 \rangle = \delta_{jk} (\rho_{12})^{-1} \), \( \langle 0 | \psi_j(x_1) \psi_k(x_2) | 0 \rangle = \delta_{jk} (\rho_{12})^{-2} \).

The proof of Theorem 4.1 is given at the end of Sect. 4.2. The main reason for this result is the fact that in the \( d = 2 \) case the harmonic bilocal fields exist and furthermore, they are Lie fields. It was originally recognized in [NST02], [BNRT07] under the assumption that there is a unique field \( \phi \) of dimension 2. We are extending here the result to an arbitrary system of \( d = 2 \) GCI scalar fields.

4.1 Structure of the correlation functions

We consider a GCI QFT generated by a set of hermitean (real) scalar fields. We denote by \( \mathcal{F} \) the real vector space of all GCI real scalar fields of scaling dimension 2 in the theory. (Note that the space \( \mathcal{F} \) may be larger than the linear span of the original system of \( d = 2 \) fields of Theorem 4.1.) We shall find in this section the explicit form of the correlation functions of the fields from \( \mathcal{F} \).
Theorem 4.2. Let $\phi_1(x), \ldots, \phi_n(x) \in \mathcal{F}$ then their truncated n-point functions have the form
\[
\langle 0|\phi_1(x_1) \cdots \phi_n(x_n)|0\rangle^{\text{tr}} = \frac{1}{2^n} \sum_{\sigma \in S_n} c^{(n)}(\phi_{\sigma_1}, \ldots, \phi_{\sigma_n}) \left(\rho_{\sigma_1 \sigma_2} \cdots \rho_{\sigma_n \sigma_1}\right)^{-1},
\]
(4.2)
where $c^{(n)}$ are multilinear functionals $c^{(n)} : \mathcal{F}^{\otimes n} \rightarrow \mathbb{R}$ with the symmetry $c^{(n)}(\phi_1, \ldots, \phi_n) = c^{(n)}(\phi_n, \phi_1, \ldots, \phi_{n-1})$ ($\sigma \in S_n$).

Before we prove the theorem, let us first illustrate it on the example of the free field realization (4.1). In this case one finds
\[
c^{(2)}(\Phi_{m_1}, \Phi_{m_2}) = \sum_{j=1}^{\infty} \alpha_{m_1,j} \alpha_{m_2,j} + \sum_{j,k=1}^{\infty} \beta_{m_1,j,k} \beta_{m_2,j,k}
\]
\[
\equiv \sum_{j=1}^{\infty} \alpha_{m_1,j} \alpha_{m_2,j} + \text{Tr} \beta_{m_1,j} \beta_{m_2,j},
\]
\[
c^{(n)}(\Phi_{m_1}, \ldots, \Phi_{m_n}) = \text{Tr} \beta_{m_1} \cdots \beta_{m_n} \quad \text{for} \quad n > 2
\]
(4.3)
($\beta_m = (\beta_{m,j,k})_{j,k}$). If there contribute no generalized free fields, then the stress-energy tensor is the sum of the stress-energy tensors for all free fields $\varphi_k$ contributing to (4.1). Hence if infinitely many of the coefficients $\beta_k$ are non-zero, then the stress-energy tensor does not exist as an operator–valued distribution. Yet it exists (after smearing) as a quadratic form between states generated by the fields $\Phi_m$ from the vacuum. The existence of a stress-energy tensor with similar properties also for the generalized free fields has been demonstrated in [DR03].

Proof of Theorem 4.2. We first recall the general form (2.3) of the truncated correlation function with pole bounds (2.6) that read in this case: $\mu_{j,k}^{\text{tr}} \geq -1$. We claim that the nonzero contributing terms in Eq. (2.3) have for every $j = 1, \ldots, n$ exactly two negative $\mu_{j,k}^{\text{tr}}$ or $\mu_{k,j}^{\text{tr}}$ for some $k = k_1, k_2$ different from $j$. This is actually equivalent to the SPP, Definition 3.1, in the $d = 2$ case, since there is only one admissible negative value for the pole degrees. But in Sect. 3.4 we have already established SPP in the $d = 2$ case.

Having proven the above stronger version of SPP for the correlation functions of the fields $\phi_1, \ldots, \phi_n$ we conclude by the cluster property Eq. (2.7) that $\langle 0|\phi_1(x_1) \cdots \phi_n(x_n)|0\rangle^{\text{tr}}$ is a linear combination of terms like those in (4.2) (i.e., cyclic products of propagators) with some coefficients $c_{\sigma}(\phi_1, \ldots, \phi_n)$ depending on the permutations $\sigma \in S_n$ and on the fields $\phi_j$ (multilinearly). Locality, i.e. $\langle 0|\phi_1(x_1) \cdots \phi_n(x_n)|0\rangle^{\text{tr}} = \langle 0|\phi_{\sigma_1}(x_{\sigma_1}) \cdots \phi_{\sigma_n}(x_{\sigma_n})|0\rangle^{\text{tr}}$, then implies $c_{\sigma'}(\phi_1, \ldots, \phi_n) = c_{\sigma}(\phi_{\sigma_1}, \ldots, \phi_{\sigma_n})$ ($\sigma, \sigma' \in S_n$), so that $c_{\sigma}(\phi_1, \ldots, \phi_n) = c^{(n)}(\phi_{\sigma_1}, \ldots, \phi_{\sigma_n})$ for some $c^{(n)} : \mathcal{F}^{\otimes n} \rightarrow \mathbb{R}$. The equalities $c^{(n)}(\phi_1, \ldots, \phi_n) = c^{(n)}(\phi_n, \ldots, \phi_1) = c^{(n)}(\phi_n, \phi_1, \ldots, \phi_{n-1})$ are again due to locality.

This completes the proof of Theorem 4.2. \qed
As we already know by the general results of the previous section, the harmonic bilocal field exist in the case of fields of dimension \( d = 2 \). Moreover, the knowledge of the correlation functions of the \( d = 2 \) fields allows us to find the form of the correlation functions of the resulting bilocal fields. This yields an algebraic structure in the space of real (local and bilocal) scalar fields, which we proceed to display.

Let us introduce together with the space \( \mathcal{F} \) of \( d = 2 \) fields also the real vector space \( \mathcal{V} \) of all harmonic bilocal fields. We shall consider \( \mathcal{F} \) and \( \mathcal{V} \) as built starting from our original system of \( d = 2 \) fields \( \{ \Phi_m \} \) of Theorem 4.1, by the following constructions.

(a) If \( \phi_1(x), \phi_2(x) \in \mathcal{F} \) then introducing in accord with Eq. (2.8), the bilocal \((1,1)\)-field \( U(x_1, x_2) = x_2^2 \left[ \phi_1(x_1)\phi_2(x_2) - \langle 0 | \phi_1(x_1)\phi_2(x_2) | 0 \rangle \right] \) we consider the harmonic decomposition \( U(x, y) = V_1(x, y) + (x - y)^2 \tilde{U}(x, y) \) for the above bilocal field. We denote \( V_1(x, y) \) by \( \phi_1 \ast \phi_2 \); this defines a bilinear map \( \mathcal{F} \otimes \mathcal{F} \to \mathcal{V} \).

(b) If now \( v(x, y) \in \mathcal{V} \) then \( v'(x, y) := v(y, x) \) also belongs to \( \mathcal{V} \) and \( \gamma(v)(x) := \frac{1}{2} v(x, x) \) is a field from \( \mathcal{F} \).

(c) If \( v(x, y), v'(x, y) \in \mathcal{V} \) then there is a harmonic bilocal field

\[
(v \ast v')(x, y) := \text{w-lim}_{x' \to y'} (x' - y')^2 \left( v(x, x') v'(y', y) - \langle 0 | v(x, x') v'(y', y) | 0 \rangle \right).
\]

(4.4)

The existence of the above weak limit (i.e., a limit within correlation functions) will be established below together with the independence of \( x' = y' \) and the regularity of the resulting field for \( (x - y)^2 = 0 \).

(d) If \( v(x, y) \in \mathcal{V} \) and \( \phi(x) \in \mathcal{F} \) then we can construct the following bilocal field belonging to \( \mathcal{V} \):

\[
(v \ast \phi)(x, y) := \text{w-lim}_{x' \to y} (x' - y)^2 \left( v(x, x') \phi(y) - \langle 0 | v(x, x') \phi(y) | 0 \rangle \right).
\]

(4.5)

where again the existence of the limit and the regularity for \( (x - y)^2 = 0 \) will be established later.

One can define similarly a product \( \phi \ast v \in \mathcal{V} \), but it would then be expressed as: \( (v' \ast \phi)' \).

To summarize, we have three bilinear maps: \( \mathcal{F} \otimes \mathcal{F} \to \mathcal{V}, \mathcal{V} \otimes \mathcal{V} \to \mathcal{V}, \mathcal{V} \otimes \mathcal{F} \to \mathcal{V}, \mathcal{V} \to \mathcal{F} \). Applying these maps we construct \( \mathcal{F} \) and \( \mathcal{V} \) inductively, starting from our original system of \( d = 2 \) fields, given in Theorem 4.1, and at each step of this inductive procedure, we establish the existence of the above limits in (c) and (d). In fact, we shall establish this together with the structure of the truncated correlation functions for the fields in \( \mathcal{F} \) and \( \mathcal{V} \).

\footnote{Since we shall use the notion of truncated correlation functions also for bilocal fields let...}
Before we state the inductive result it is convenient to introduce the vector space
\[ \hat{\mathcal{A}} = \mathcal{F} \times \mathcal{V} \] (4.6)
and endow it with the following bilinear operation
\[ (\phi_1, v_1) * (\phi_2, v_2) := (0, \phi_1 * \phi_2 + v_1 * v_2 + v_1 * \phi_2 + (v_2^t * \phi_1)^t), \] (4.7)
and with the transposition
\[ (\phi, v)^t := (\phi, v^t). \] (4.8)
The spaces $\mathcal{F}$ and $\mathcal{V}$ will be considered as subspaces in $\hat{\mathcal{A}}$. Thus, the new operation $*$ in $\hat{\mathcal{A}}$ combines the above listed three operations. We shall see later that $\hat{\mathcal{A}}$ is actually an associative algebra under the product (4.7). We note that the transposition $t$ (4.8) is an antiinvolution with respect to the product: $(q_1 * q_2)^t = q_2^t * q_1^t$, for every $q_1, q_2 \in \hat{\mathcal{A}}$.

**Proposition 4.3.** There exist multilinear functionals
\[ c^{(N)} : \hat{\mathcal{A}}^\otimes N \to \mathbb{R} \] (4.9)
such that if we take elements $q_1, \ldots, q_{n+m} \in \hat{\mathcal{A}} : q_k := v_k(x_{k[0]}, x_{k[1]}) \in \mathcal{V}$, where $[\varepsilon]$ stands for a $\mathbb{Z}/2\mathbb{Z}$-value and $k = 1, \ldots, n$, and $q_k := \phi_{k-n}(x_k) \in \mathcal{F}$ for $k = n + 1, \ldots, n + m$, then the truncated correlation functions can be written in the following form:
\[ \langle 0 | v_1(x_{1[0]}, x_{1[1]}) \cdots v_n(x_{n[0]}, x_{n[1]}) \phi_1(x_{n+1}) \cdots \phi_m(x_{n+m}) | 0 \rangle^{\text{tr}} \]
\[ = \frac{1}{2(n+m)} \sum_{\sigma \in S_{n+m}} \sum_{(\varepsilon_1, \ldots, \varepsilon_n) \in (\mathbb{Z}/2\mathbb{Z})^n} K_{\sigma, \varepsilon} T_{\sigma, \varepsilon}(x_{1[0]}, \ldots, x_{n[1]}, x_{n+1}, \ldots, x_{n+m})^{-1}. \] (4.10)

Here: $K_{\sigma, \varepsilon}$ are coefficients given by
\[ K_{\sigma, \varepsilon} := c^{(n+m)}(q_{\varepsilon_1}^{[0]}, \ldots, q_{\varepsilon_{n+m}}^{[1]}), \]
where we set $\varepsilon_{n+1} = \ldots = \varepsilon_{n+m} = 0$, and $q^{[0]} := q, q^{[1]} := q^t$ (for $q \in \hat{\mathcal{A}}$); the terms $T_{\sigma, \varepsilon}$ are the following cyclic products of intervals
\[ T_{\sigma, \varepsilon} = (x_{\sigma_{n+m}} - x_{\sigma_1})^2 \prod_{k=1}^{n-1} (x_{\sigma_k[1+\varepsilon_k]} - x_{\sigma_{k+1}[\varepsilon_{k+1}]^2})^2 \times (x_{\sigma_n[1+\varepsilon_n]} - x_{\sigma_{n+1}})^2 \prod_{k=1}^{m-1} (x_{\sigma_{n+k}} - x_{\sigma_{n+k+1}}) \] (4.11)
It follows by Eq. (4.10) that the limits in the steps (c) and (d) above are well defined.
Before the proof let us make some remarks. First, we used the same notation $c^{(n)}$ as in Theorem 4.2 since the above multilinear functionals are obviously an extension of the previous, i.e., Eq. (4.10) reduces to Eq. (4.2) for $m = 0$. Let us also give an example for Eq. (4.10) with $n = m = 1$:

$$(0|v(x_1, x_2)\phi(x_3)|0) = \frac{1}{4} \left( c^{(2)}(v, \phi) \left( \rho_{23} \rho_{31} \right)^{-1} + c^{(2)}(v', \phi) \left( \rho_{13} \rho_{32} \right)^{-1} + c^{(2)}(\phi, v) \left( \rho_{31} \rho_{23} \right)^{-1} + c^{(2)}(\phi, v') \left( \rho_{32} \rho_{13} \right)^{-1} \right). \quad (4.12)$$

As one can see, $c^{(n)}$ (as well as $c^{(n)}$ of Theorem 4.2) possess a cyclic and an inversion symmetry:

$$c^{(n)}(q_1, \ldots, q_n) = c^{(n)}(q_n, q_1, \ldots, q_{n-1}) = c^{(n)}(q_1, \ldots, q_1). \quad (4.13)$$

This is the reason for choosing the prefactors in Eqs. (4.2) and (4.10) (the inverse of the orders of the symmetry groups).

**Proof of Proposition 4.3.** According to our preliminary remarks it is enough to prove that Eq. (4.10) is consistent with the operations $\mathcal{F} \otimes \mathcal{F} \overset{s}{\rightarrow} \mathcal{V}$, $\mathcal{V} \otimes \mathcal{V} \overset{s}{\rightarrow} \mathcal{V}$ and $\mathcal{V} \overset{s}{\rightarrow} \mathcal{F}$.

Starting with $\mathcal{F} \otimes \mathcal{F} \overset{s}{\rightarrow} \mathcal{V}$ one should prove that any truncated correlation function $\langle \phi_1(x_1) \phi_2(x_2) \rangle^{tr}$ given by Eq. (4.10) yields a harmonic decomposition: $\rho_{12} \langle \phi_1(x_1) \phi_2(x_2) \rangle^{tr} = \langle (\phi_1 * \phi_2)(x_1, x_2) \rangle^{tr} + \rho_{12} R(x_1, x_2)$, with a correlation function $\langle (\phi_1 * \phi_2)(x_1, x_2) \rangle^{tr}$ given by Eq. (4.10) and a rational function $R$ regular at $\rho_{12} = 0$. This gives us relations of the type

$$c^{(n+2)}(q_1, \ldots, \phi_1, \phi_2, \ldots, q_n) = c^{(n+1)}(q_1, \ldots, \phi_1 * \phi_2, \ldots, q_n). \quad (4.14)$$

Next, having correlation functions of type $\langle v_1(x_1, x_2) v_2(x_3, x_4) \rangle^{tr}$ or $\langle v(x_1, x_2) \phi(x_3) \rangle^{tr}$ of the form (4.10), one verifies that the limits (4.4) and (4.5) exist within these correlation functions, and they yield expressions for $\langle (v_1 * v_2)(x_1, x_4) \rangle^{tr}$ and $\langle (v * \phi)(x_1, x_3) \rangle^{tr}$ consistent with (4.10). As a result we obtain again relations between the $c$’s:

$$c^{(n+2)}(q_1, \ldots, v_1, v_2, \ldots, q_n) = c^{(n+1)}(q_1, \ldots, v_1 * v_2, \ldots, q_n), \quad c^{(n+2)}(q_1, \ldots, v, \phi, \ldots, q_n) = c^{(n+1)}(q_1, \ldots, v * \phi, \ldots, q_n). \quad (4.15)$$

Finally, one verifies that setting $x_1 = x_2$ in $\langle v(x_1, x_2) \rangle^{tr}$ we obtain the correlation functions $\langle \gamma(v)(x_1) \rangle^{tr}$ with the relation

$$c^{(n+1)}(q_1, \ldots, v + v'), \ldots, q_n) = 2 c^{(n+1)}(q_1, \ldots, \gamma(v), \ldots, q_n). \quad (4.16)$$

This completes the proof of Proposition 4.3 as well as the proof that the products $\mathcal{V} \otimes \mathcal{V} \overset{s}{\rightarrow} \mathcal{V}$ and $\mathcal{V} \otimes \mathcal{F} \overset{s}{\rightarrow} \mathcal{V}$ are well defined. □
4.2 Associative algebra structure of the OPE

Note that Eqs. (4.14), (4.15) read (under (4.7))

\[ c^{(n)}(q_1, \ldots, q_k, q_{k+1}, \ldots, q_n) = c^{(n)}(q_1, \ldots, q_k * q_{k+1}, \ldots, q_n) . \]  \hspace{1cm} (4.17)

This implies that the bilinear operation \(*\) on \(\hat{A}\) is an associative product.

Indeed, consider the element \(q := (q_1 * q_2) * q_3 - q_1 * (q_2 * q_3)\) for \(q_1, q_2, q_3 \in \hat{A}\). By (4.7) \(q\) is a bilocal field. Equation (4.17) implies that all \(c\)'s in which \(q\) enters vanish and hence, by Eq. (4.10) \(q\) has zero correlation functions with all other fields, including itself. But then this (bilocal) field is zero by the Reeh–Schlieder theorem, since its action on the vacuum will be identically zero.

Thus, introducing the cartesian product \(\hat{A}\) (4.6) was not only convenient for combining three types of bilinear operations in one but also as a compact expression for the associativity (Eqs. (4.14), (4.15)). However, \(\hat{A}\) carries a redundant information due to the following relation:

\[ (-\gamma(v), \frac{1}{2}(v + v^t)) * q = 0 = q * (-\gamma(v), \frac{1}{2}(v + v^t)) \]  \hspace{1cm} (4.18)

for every \(v \in \mathcal{V}\) and \(q \in \hat{A}\). To prove (4.18) we point out first that it is equivalent to the identities \(v * \phi = \gamma(v) * \phi\) and \(v^t * v = v * \gamma(v)\) for \(v = v^t \in \mathcal{V}\) and any \(\phi \in \mathcal{F}\), \(v^t \in \mathcal{V}\). These identities can be established again first for the \(c\)'s, and then proceeding by using the Reeh–Schlieder theorem, as in the above proof of associativity.

Hence, the redundancy in \(\hat{A}\) is because we can identify symmetric bilocal fields \(v = v^t \in \mathcal{V}\) with their restrictions to the diagonal, \(\gamma(v) \in \mathcal{F}\), and this is compatible with the product \(*\). Let us point out that the restriction of the map \(\gamma\) to the \(t\)-invariant subspace \(\mathcal{V}_s := \{v \in \mathcal{V} : v = v^t\}\) is an injection into \(\mathcal{F}\). The latter follows from a simple analysis of the 4-point functions of \(v\) and the Reeh–Schlieder theorem: if \(v(x, y) = v(y, x)\) and \(\langle 0|v(x, x)v(y, y)|0\rangle = 0\) then \(\langle 0|v(x, x')v(y, y')|0\rangle = 0\). In this way we see that we can identify in \(\hat{A}\) the symmetric harmonic bilocal fields \(v = v^t\) with their restriction on the diagonal \(\gamma(v) \in \mathcal{F}\).

Formally, the above considerations can be summarized in the following abstract way. Let us introduce the quotient

\[ \mathcal{A} := \hat{A} / \{(-\gamma(v), \frac{1}{2}(v + v^t)) : v \in \mathcal{V}\} . \]  \hspace{1cm} (4.19)

It is an associative algebra according to Eq. (4.18). The involution \(t : \hat{A} \to \hat{A}\) can be transferred to an involution on the quotient (4.19) and we denote it by \(t\) as well. The spaces \(\mathcal{F}\) and \(\mathcal{V}\) are mapped into \(\mathcal{A}\) by the natural compositions \(\mathcal{F} \to \hat{A} \to \mathcal{A}\) and \(\mathcal{V} \to \hat{A} \to \mathcal{A}\). The injectivity of \(\gamma\) on \(\mathcal{V}_s\) implies that the maps \(\mathcal{F} \to \mathcal{A}\) and \(\mathcal{V} \to \mathcal{A}\) so defined are actually injections. Hence, we shall treat \(\mathcal{F}\) and \(\mathcal{V}\) also as subspaces of \(\mathcal{A}\). Furthermore, \(\mathcal{A}\)
becomes a direct sum of vector spaces
\[ \mathcal{A} = \mathcal{F} \oplus \mathcal{V}_a, \]  
\[ \{ q \in \mathcal{A} : q^t = q \} = \mathcal{F} \supseteq \mathcal{V}_a ( = \{ v \in \mathcal{V} : v^t = v \}), \]
\[ \{ q \in \mathcal{A} : q^t = -q \} = \mathcal{V}_a := \{ v \in \mathcal{V} : v^t = -v \}. \]

Hence, the \( t \)-symmetric elements of \( \mathcal{A} \) are identified with the \( d = 2 \) local fields, while the \( t \)-antisymmetric elements of \( \mathcal{A} \), with the antisymmetric, harmonic bilocal \((1,1)\) fields. (Neither \( \mathcal{F} \) nor \( \mathcal{V}_a \) are subalgebras of \( \mathcal{A} \).

To summarize, the associative algebra \( \mathcal{A} \) is obtained from \( \mathcal{A} \) by identifying the space \( \mathcal{V}_a \) of symmetric bilocal fields with its image \( \gamma(\mathcal{V}_a) \subseteq \mathcal{F} \).

For simplicity we will denote the equivalence class in \( \mathcal{A} \) of an element \( q \in \mathcal{A} \) again by \( q \). Also note that the \( c \)'s can be transferred as well, to multilinear functionals on \( \mathcal{A} \), since the kernel of the quotient \((4.19)\) is contained in the kernel of each \( c^{(n)} \) by \((4.16)\). We shall use the same notation \( c^{(n)} \) also for the multilinear functional \( c^{(n)} \) on \( \mathcal{A} \).

Example 4.1. Let us illustrate the above algebraic structures on the simplest example of a QFT generated by a pair of \( d = 2 \) GCI fields \( \Phi_1 \) and \( \Phi_2 \) given by normal a pair of two mutually commuting free massless fields \( \varphi_j \) : \( \Phi_1(x) = \frac{1}{2} (:\varphi_1^2(x) : - :\varphi_2^2(x) :) \) and \( \Phi_2(x) = \varphi_1(x) \varphi_2(x) \). Their OPE algebra involves a set of four independent harmonic bilocal fields \( v_{jk}(x_1, x_2) := :\varphi_j(x_1) \varphi_k(x_2) : (j, k = 1, 2) \), which satisfy \([v_{jk}(x_1, x_2)]^3 = v_{kj}(x_1, x_2) = v_{jk}(x_2, x_1)\). For instance, we have \( \Phi_1 \ast \Phi_2 = V_{12} - V_{21} \). Also note that \( \Phi_1 = \gamma(V_1) \) for \( V_1(x_1, x_2) = :\varphi_1(x_1) \varphi_1(x_2) : - :\varphi_2(x_1) \varphi_2(x_2) : \), etc.

By the associativity and Eq. \((4.17)\) we have
\[ c^{(n)}(q_1, \ldots, q_n) = c^{(2)}(q_1 \ast \cdots \ast q_{n-1}, q_n) \]  
\[ (4.21) \]
for \( q_1, \ldots, q_n \in \mathcal{A} \). Let us consider now \( c^{(2)} \) and define the following symmetric bilinear form on \( \mathcal{A} \):
\[ \langle q_1, q_2 \rangle := c^{(2)}(q_1^t, q_2). \]  
\[ (4.22) \]
First note that \( \mathcal{F} \) and \( \mathcal{V}_a \) are orthogonal with respect to this bilinear form: this is due to the fact that there is no nonzero triple point conformally invariant scalar function of weights \((2,1,1)\), which is antisymmetric in the second and third arguments. Next, we claim that \((4.22)\) is strictly positive definite.

This is a straightforward consequence of the Wightman positivity and the Reeh–Schlieder theorem (one should consider separately the positivity on \( \mathcal{F} \) and \( \mathcal{V}_a \)). In particular, \((4.22)\) is nondegenerate. By Eqs. \((4.13)\) and \((4.17)\) we have:
\[ \langle q_1 \ast q_2, q_3 \rangle = \langle q_2, q_4^t \ast q_3 \rangle \]  
\[ (4.23) \]
for all \( q_1, q_2, q_3 \in \mathcal{A} \).

\footnote{i.e., in the OPE \( \Phi_1(x_1)\Phi_2(x_2) \) there appears the antisymmetric bilocal field \( V_{12}(x_1, x_2) \) \( - V_{21}(x_1, x_2) \) that involves only odd rank conserved tensor currents in its expansion in local fields}
Let us introduce now an additional splitting of $F$. Denote by $F_0$ the kernel of the product, i.e.,

$$F_0 := \{ \psi \in F : \psi * q = 0 \forall q \in A \} \equiv \{ \psi \in F : q * \psi = 0 \forall q \in A \}$$

(4.24)

(the second equality is due to the identity $\phi * q = (q^t * \phi)^t$). Let $F_1$ be the orthogonal complement in $F$ of $F_0$ with respect to the scalar product (4.22):

$$F_1 := \{ \phi \in F : \langle \phi, \psi \rangle = 0 \forall \psi \in F_0 \}.$$  

(4.25)

The meaning of fields belonging to $F_0$ becomes immediately clear if we note that $c^{(n)}$ for $n \geq 3$ are zero if one of the arguments belongs to $F_0$ (this is due to Eq. (4.21)). Hence, all their truncated functions higher than two point are zero, i.e., the fields belonging to $F_0$ are \textit{generalized free} $d = 2$ fields. Furthermore, these fields commute with all other fields from $F_1$ and $\mathcal{V}_a \equiv A(1)$: this is because of the vanishing of $c^{(2)}(\psi, q)$ if $\psi \in F_0$ and $q \in F_1 \oplus \mathcal{V}_a$, as well as of all $c^{(n+1)}(\psi, q_1, \ldots, q_n)$ for $n \geq 2$ if $\psi \in F_0$ and $q_1, \ldots, q_n \in A$ (by (4.21) and (4.24)).

Clearly, $F_1 \oplus \mathcal{V}_a$ is a subalgebra of $A$: this follows from Eq. (4.23) with $q_3 \in F_0$ along with the definitions (4.24) and (4.25). Let us denote it by

$$\mathcal{B} := F_1 \oplus \mathcal{V}_a.$$  

(4.26)

We are now ready to state the main step towards the proof of Theorem 4.1.

\textbf{Proposition 4.4.} There is a homomorphism $\iota$ from the associative algebra $\mathcal{B}$ into the algebra of all Hilbert–Schmidt operators over some real separable Hilbert space, such that

$$c^{(n)}(q_1, \ldots, q_n) = \text{Tr} \left( \iota(q_1) \cdots \iota(q_n) \right),$$  

(4.27)

and $\iota(F)$ are symmetric operators while $\iota(V_a)$ are antisymmetric.

We shall give the proof of this proposition in the subsequent subsection. The main reason leading to it is that $\mathcal{B}$ becomes a real \textit{Hilbert algebra} with an \textit{integral} trace on it. Here we proceed to show how Theorem 4.1 can be proven by using the above results.

\textit{Proof of Theorem 4.1.} Let $\Phi_m = \Phi^0_m + \Phi^1_m$ is a decomposition of each field $\Phi_m$ according to the splitting $F = F_0 \oplus F_1$. Take an orthonormal basis $\psi_m$ in $F_0$ and let $\Phi^0_m = \sum_{j=1}^{\infty} \alpha_{m,j} \psi_j$, and $\beta_m = (\beta_{m,j,k})_{j,k}$ be the symmetric matrix corresponding to the Hilbert–Schmidt operator $\iota(\Phi^1_m)$ ($m = 1, 2, \ldots$). Then Eqs. (4.3) and (4.27) show that the constants $\alpha_{m,j}$ and $\beta_{m,j,k}$ so defined satisfy the conditions of Theorem 4.1. \hfill $\square$

\textbf{Remark 4.1.} In general, we have $F_1 \supseteq \mathcal{V}_s$. This is because the elements of $F_1$ correspond, by Proposition 4.4, to Hilbert–Schmidt symmetric operators and on the other hand, the elements of $\mathcal{V}$ are obtained, according to the inductive construction of Sect. 4.1, as products of elements of $F$ and will, hence, correspond to trace class operators.
4.3 Completion of the proofs

It remains to prove Proposition 4.4. We start with an equality of Cauchy–Schwartz type.

**Lemma 4.5.** Let $q_1, q_2 \in \mathcal{A}$ be such that each of them belongs either to $\mathcal{F}$ or to $\mathcal{V}_a$. Then we have

$$\langle q_1 * q_2, q_1 * q_2 \rangle^2 \leq \langle q_1 * q_1, q_1 * q_1 \rangle \langle q_2 * q_2, q_2 * q_2 \rangle. \quad (4.28)$$

**Proof.** Consider $\langle q_1 * q_1 + \lambda q_2 * q_2, q_1 * q_1 + \lambda q_2 * q_2 \rangle \geq 0$ and use that

$$\langle q_1 * q_1, q_1 * q_1 \rangle = \pm \langle q_1 * q_2, q_1 * q_2 \rangle$$

if each of $q_1, q_2$ belongs either to $\mathcal{F}$ or to $\mathcal{V}_a$. \qed

The space $\mathcal{B}$ (4.26) is a real pre–Hilbert space with a scalar product provided by (4.22). It is also invariant under the action of $t$ (actually the eigenspaces of $t$ are $\mathcal{F}_1$ and $\mathcal{V}_a$). The left action of $\mathcal{B}$ on itself gives us an algebra homomorphism

$$\iota : \mathcal{B} \to \text{Lin}_\mathbb{R} \mathcal{B} \quad (4.29)$$

of $\mathcal{B}$ into the algebra of all operators over $\mathcal{B}$. Moreover, the elements of $\mathcal{F}$ are mapped into symmetric operators and the elements of $\mathcal{V}_a$, into antisymmetric (this is due to (4.23)).

**Lemma 4.6.** Every element of $\mathcal{B}$ is mapped into a Hilbert–Schmidt operator.

**Proof.** Since $\mathcal{B}$ is generated by $\mathcal{F}_1$ (according to the inductive construction of $\mathcal{F}$ and $\mathcal{V}$ in Sect. 4.1) it is enough to show this for the elements of $\mathcal{F}_1$.

Let $\phi \in \mathcal{F}_1$ and consider the commutative subalgebra $\mathcal{B}_\phi$ of $\mathcal{B}$ generated by $\phi$. The algebra $\mathcal{B}_\phi$ is freely generated by $\phi$, i.e., is isomorphic to the algebra $\lambda \mathbb{R}[\lambda]$ of polynomials in a single variable $\lambda$ ($\leftrightarrow \phi$), since $\phi$ belongs to the orthogonal complement of $\mathcal{F}_0$ (4.24). For a $p(\lambda) \in \lambda \mathbb{R}[\lambda]$ we shall denote by $\phi[p]$ the corresponding element of $\mathcal{B}_\phi$. In particular,

$$\phi[p_1] * \phi[p_2] = \phi[p_1p_2]. \quad (4.30)$$

Setting

$$\phi^{*(n+1)} := \phi^{*n} * \phi, \quad c[\lambda^{n+1}] := c^{(2)}(\phi^{*n}, \phi) \equiv \langle \phi^{*n}, \phi \rangle \quad (4.31)$$

($\phi^{*n} := \phi$, $n \geq 1$) we obtain a positive definite functional over the algebra $\lambda^2 \mathbb{R}[\lambda] \cong \phi * \mathcal{B}_\phi$ (due to Eq. (4.23) and the positivity of $\langle \cdot, \cdot \rangle$ (4.22)).

Then, by the Hamburger theorem about the classical moment problem ([DS63, Chap. 12, Sect. 8]) we conclude that there exists a bounded positive Borel measure $d\mu(\lambda)$ on $\mathbb{R}$, such that

$$c[\lambda^2 p(\lambda)] = \int_{\mathbb{R}} p(\lambda) \, d\mu(\lambda) \quad (4.32)$$

for every $p(\lambda) \in \mathbb{R}[\lambda]$. Using this we can extend the fields $\phi[p](x)$ to $\phi[f](x)$ for Borel measurable functions $f$ having compact support with respect to $\mu$. 


The latter can be done in the following way. Fix \( \varepsilon \in (0, 1) \) and let \( g_1, \ldots, g_n \) be Schwartz test functions on \( M \). By Theorem 4.2 the correlators 
\[ \langle 0 | \phi^{p_1} [g_1] \cdots \phi^{p_n} [g_n] | 0 \rangle \]
depend polynomially on \( c^{(n)}(\phi^{p_1}, \ldots, \phi^{p_n}) = c[p_1 \cdots p_n] \) for all \( \{k_1, \ldots, k_j\} \subseteq \{1, \ldots, n\} \). But for every \( \varepsilon \in (0, 1) \) there exists a norm
\[ \|q\|_\varepsilon = A_\varepsilon \sup_{|\lambda| \leq \varepsilon} \frac{|q_k(\lambda)|}{\lambda^2} + B_\varepsilon \int_{\mathbb{R} \setminus (-\varepsilon, \varepsilon)} |q_k(\lambda)| d\mu(\lambda) \] (43.33)
on \( \lambda^2 \mathbb{R}[\lambda] \ni q(\lambda) \), where \( A_\varepsilon \) and \( B_\varepsilon \) are some positive constants, such that for every \( q_1, \ldots, q_m \in \lambda^2 \mathbb{R}[\lambda] \)
\[ |c[q_1(\lambda) \cdots q_m(\lambda)]| \leq m \prod_{k=1}^m \left\{ \int_{\mathbb{R}} \frac{|q_k(\lambda)|^m}{\lambda^2} d\mu(\lambda) \right\}^{\frac{1}{m}} \leq m \|q_k\|_\varepsilon. \]
Hence, \( |\langle 0 | \phi^{p_1} [g_1] \cdots \phi^{p_n} [g_n] | 0 \rangle| \leq C \prod_{k=1}^n \|p_k\|_\varepsilon \|g_k\|_S \) for some constant \( C \) and Schwartz norm \( \|\cdot\|_S \) (not depending on \( p_k \) and \( g_k \)). Since for every \( \varepsilon \in (0, 1) \) the Banach space \( L^1(\mathbb{R} \setminus \{(-\varepsilon, \varepsilon)\}, \mu) \) is contained in the completion of \( \lambda^2 \mathbb{R}[\lambda] \) with respect to the norms (43.33), we can extend the linear functional \( c[p(\lambda)] \) as well as the correlators \( \langle 0 | \phi^{p_1} [g_1] \cdots \phi^{p_n} [g_n] | 0 \rangle \) to a functional \( c[f(\lambda)] \) and correlators \( \langle 0 | \phi^{f_1} [g_1] \cdots \phi^{f_n} [g_n] | 0 \rangle \) defined for Borel functions \( f, f_1, \ldots, f_n \) compactly supported with respect to \( \mu \) in \( \mathbb{R} \setminus \{0\} \). Thus, we can extend the fields \( \phi^{p} \) by extending their correlators.

By the continuity we also have for arbitrary Borel functions \( f, f_k \), compactly supported in \( \mathbb{R} \setminus \{0\} \):
\[ \phi^{f_1} \ast \phi^{f_2} = \phi^{f_1 f_2}, \quad c^{(n)}(\phi^{f_1}, \ldots, \phi^{f_n}) = c[f_1 \cdots f_n], \]
\[ c[f] = \int_{\mathbb{R}} \frac{f(\lambda)}{\lambda^2} d\mu(\lambda) \] (43.34)
(cp. (43.32)), and \( c^{(n)} \) determine the correlation functions of \( \phi^{f_k} \) as in Theorem 4.2.

In particular, for every characteristic function \( \chi_S \) of a compact subset \( S \subset \mathbb{R} \setminus \{0\} \) we have \( \phi^{\chi_S} \ast \phi^{\chi_S} = \phi^{\chi_S} \). Hence, for such a \( d = 2 \) field we will have that all its truncated correlation functions are given by (4.2) with all normalization constants \( c^{(n)} \) equal to one and the same value \( c^{(2)}(\phi^{\chi_S}, \phi^{\chi_S}) \). Then, as shown in [NST02, Theorem 5.1], Wightman positivity requires this value to be a non-negative integer, i.e.,
\[ c^{(2)}(\phi^{\chi_S}, \phi^{\chi_S}) = c[\chi_S] = \int_{S} \frac{d\mu(\lambda)}{\lambda^2} \in \{0, 1, 2, \ldots\} \] (43.35)
(it is zero iff \( \phi^{\chi_S} = 0 \)). Hence, the restriction of the measure \( d\mu(\lambda)/\lambda^2 \) to \( \mathbb{R} \setminus \{0\} \) is a (possibly infinite) sum of atom measures of integral masses, each supported at some \( \gamma_k \in \mathbb{R} \setminus \{0\} \) for \( k = 1, \ldots, N \) (and \( N \) could be infinity). In particular, the measure \( \mu \) is supported in a bounded subset of \( \mathbb{R} \).
By Lemma 4.5 we can define $\iota(\phi[f])$ as a closable operator on $\mathcal{B}$ if $f$ is a Borel measurable function with compact support in $\mathbb{R}\setminus\{0\}$. It follows then that the projectors $\iota(\phi^{[x]}s)$, for a compact $S \subseteq \mathbb{R}\setminus\{0\}$, provide a spectral decomposition for $\iota(\phi)$ (in fact, $\iota(\phi[f]) = f(\iota(\phi))$). Thus, $\iota(\phi)$ has discrete spectrum with eigenvalues $\gamma_k$ ($k \in \mathbb{N}$), each of a multiplicity given by the integer $c^{(2)}(\phi^{(\gamma_k)}, \phi^{(\gamma_k)})$. Then $\iota(\phi)$ is a Hilbert–Schmidt operator since

$$\sum_{k=1}^{\infty} \gamma_k^2 c^{(2)}(\phi^{(\gamma_k)}, \phi^{(\gamma_k)}) = \sum_{k=1}^{\infty} \gamma_k^2 \int_{\{\gamma_k\}} \frac{d\mu(\lambda)}{\lambda^2} = \int_{\mathbb{R}\setminus\{0\}} d\mu(\lambda) < \infty$$

($\mu$ being a bounded measure).

The completion of the proof of Proposition 4.4 is provided now by the following corollary.

**Corollary 4.7.** For every $q_1, q_2 \in \mathcal{B}$ one has $c^{(2)}(q_1, q_2) = \text{Tr}(\iota(q_1)\iota(q_2))$.

*Proof.* If $q_1 = q_2 \in \mathcal{F}_1$ this follows from the proof of Lemma 4.6 and hence, by a polarization, for any $q_1, q_2 \in \mathcal{F}_1$. The general case can be obtained by using the facts that $\mathcal{B}$ is generated by $\mathcal{F}_1$ and $c^{(2)}$ has the symmetry $c^{(2)}(q_1 * q_2, q_3) = c^{(2)}(q_1, q_2 * q_3)$.

\section{Discussion. Open problems}

The main result of Sect. 4, the (generalized) free field representation of a system $\{\phi_a\}$ of GCI scalar fields of conformal dimension $d = 2$ (Theorem 4.1), is obtained by revealing and exploiting a rich algebraic structure in the space $\mathcal{F} \times \mathcal{V}$ of all $d = 2$ real scalar fields and of all harmonic bilocal fields of dimension $(1, 1)$. However, this structure is mainly due to the fact that we are in the case of lower scaling dimension: there is only one possible singular structure in the OPE (after truncating the vacuum part). One can try to establish such a result in spaces of spin–tensor bilocal fields (of dimension $(\frac{3}{2}, \frac{3}{2})$ or $(2, 2)$) satisfying linear (first order) conformally invariant differential equations (that again imply harmonicity). If these equations together with the corresponding pole bounds imply such singularities in the OPE, which can be “split” one would be able to prove the validity of free field realizations in such more general theories, too.

One may also attempt to study models, say in a theory of a system of scalar fields of dimension $d = 4$, without leaving the realm of scalar bilocal harmonic fields $V_1$ (of dimension $(1, 1)$). In [NRT05] there have been found examples of 6–point functions of harmonic bilocal fields, which do not have free field realizations. However, our experience with the $d = 2$ case shows that in order to complete the model (including the check of Wightman positivity for all correlation functions) it is crucial to describe the OPE in terms of some simple algebraic structure (e.g., associative, or Lie algebras).

On the other hand going beyond bilocal $V_1$’s is a true signal of non-triviality of a GCI model. Our analysis of Sect. 3 shows that this can be
characterized by a simple property of the correlation functions: the violation of the single pole property (of Sect. 3.3). From this point of view a further exploration of the example of Sect. 3.5 within a QFT involving currents appears particularly attractive.

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