Super-Activation of Zero-Error Capacity of Noisy Quantum Channels

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We study various super-activation effects in the following zero-error communication scenario: One sender wants to send classical or quantum information through a noisy quantum channel to one receiver with zero probability of error. First we show that there are quantum channels of which a single use is not able to transmit classical information perfectly yet two uses can. This is achieved by employing entangled input states between different uses of the given channel and thus cannot happen for classical channels. Second we exhibit a class of quantum channel with vanishing zero-error classical capacity such that when a noiseless qubit channel or one ebit shared entanglement are available, it can be used to transmit $\log_2 d$ noiseless qubits, where $2d$ is the dimension of input state space. Third we further construct quantum channels with vanishing zero-error classical capacity when assisted with classical feedback can be used to transmit both classical and quantum information perfectly. These striking findings not only indicate both the zero-error quantum and classical capacities of quantum channels satisfy a strong super-additivity beyond any classical channels, but also highlight the activation power of auxiliary physical resources in zero-error communication.

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I. INTRODUCTION

The notion of zero-error capacity was introduced by Shannon in 1956 to characterize the ability of noisy channels to transmit classical information with zero probability of error [1]. Since Shannon’s seminal work, the study of this notion and the related topics has grown into a vast field called zero-error information theory [2]. The main motivation is partly due to the following facts: (1) In many real-world critical applications no errors can be tolerated; (2) In practice, the communication channel can only be available for a finite number of times; (3) Deep connections to other research fields such as graph theory and communication complexity theory have been established [2][4][5][6]. These works indicate that unlike the ordinary capacity, computing the zero-error capacity of classical channels is essentially a combinatorial optimization problem about graphs, and is extremely difficult even for very simple graphs. Despite the fact that numerous interesting and important results have been reported (see [2] for an excellent review), the theory of zero-error capacity is still far from complete even for classical channels.

The generalization of zero-error capacity to quantum channels is somewhat straightforward but nontrivial as the input states of the channel may be entangled between different uses, and the information transmitted may be classical or quantum. At least two notions of zero-error capacity of quantum channels exist: one is the zero-error classical capacity, the least upper bound of the rates at which one can send classical information perfectly through a noisy quantum channel, denote $C^{(0)}$. If replacing classical information with quantum information in the definition of $C^{(0)}$, we have another notion $Q^{(0)}$, the zero-error quantum capacity. A careful study of these generalizations will not only help us to exploit new features of quantum information, but also be useful in building highly reliable communication networks. The notion of $Q^{(0)}$ has been extensively investigated in the context of quantum-error correction. In this paper we mainly focus on $C^{(0)}$ of which little was known. A few preliminary works have been done towards to a better understanding of the zero-error classical capacity of quantum channels. In particular, some basic properties of $C^{(0)}$ of quantum channels were observed in [7]. Later, it was shown that the zero-error classical capacity for quantum channels is in general also extremely difficult to compute [8]. However, in these works the only allowable input states for channels were restricted to be product states and entangled uses of the channel were prohibited. Consequently, many of the properties of this notion is similar to the classical case and it was not clear what kind of role the additional quantum resources such as entanglement will play in zero-error communication.

In a recent work it was demonstrated that the zero-error classical capacity of quantum channels behaves dramatically different from the corresponding classical capacity [9]. More precisely, it was shown that in the so-called multi-user communication scenario, there is noisy quantum channel of which one use cannot transmit any
classical information perfectly yet two uses can. To achieve this, one needs to encode the classical message using entangled states as input and thus to make two uses of the channel entangled. This is a purely quantum effect that cannot happen for any classical channels. Furthermore, it cannot be observed under the assumptions of Refs. [4, 5] where only product input states between different uses are allowed. One drawback of the channel constructed in [6] is that we have at least two senders or two receivers and require the senders or the receivers to perform local operations and classical communication (LOCC) only. This LOCC restriction is a reasonable assumption in practice as it captures the fact that the quantum communication among the senders or the receivers would be relatively expensive. If this local requirement is removed, one use of these channels are able to transmit classical information perfectly. Thus a major open problem left is to ask whether there is quantum channel with only one sender and one receiver enjoying the same property.

II. MAIN RESULTS

The purpose of this paper is to further develop the theory of zero-error capacity for quantum channels. Our first main result (Theorem 1) is an affirmative answer to the above open problem. More precisely, we show by an explicit construction that there does exist quantum channel \( \mathcal{G} \) with one sender and one receiver such that one use of \( \mathcal{G} \) cannot transmit classical information perfectly while two uses of \( \mathcal{G} \) can transmit at least one bit without any error. Fig. 1 demonstrates our construction. In our construction we don’t construct \( \mathcal{G} \) directly. Instead, we construct two quantum channels \( \mathcal{E} \) and \( \mathcal{F} \) such that both of them cannot transmit classical information perfectly by a single use while can transmit at least one bit if employed jointly. This confirms the usefulness of entangled input for perfect transmission of classical information.

![Diagram](image)

FIG. 1: \( \mathcal{E} \) is a noisy quantum channel from Alice to Bob. With one use of \( \mathcal{G} \), Alice cannot transmit classical information to Bob perfectly. Interestingly, by using \( \mathcal{G} \) twice, Alice can transmit a classical bit “\( \phi \)” perfectly to Bob. To do so, Alice carefully encodes the bit “\( \phi \)” into a bipartite entangled state \( |\phi_b\rangle \) and applies \( \mathcal{G} \) twice. By decoding the output state \( \mathcal{G}^{\otimes 2}(|\phi_b\rangle \langle \phi_b|) \), Bob can perfectly recover the bit “\( \phi \)”.

Similar to the previous work [4], our main tool is the notion of unextendible bases (or equivalently, completely entangled subspaces) [10, 11, 12, 13, 15]. The key ingredient in our construction is to partition a bipartite Hilbert space into two orthogonal subspaces which are both completely entangled, or equivalently, unextendible. This kind of partitions has been found before [3, 4, 12] and has been demonstrated very useful in quantum information theory [3, 14, 15, 10]. However, all these previous partitions are not sufficient for our purpose. Additional requirements make the construction rather difficult and tricky.

Our second main result (Theorem 2) is to show that both the zero-error quantum and classical capacities of noisy quantum channels are strongly super-additive. This is achieved by introducing a class of special quantum channels which can be treated as the generalizations of retro-correctible channels [19]. It was known that the zero-error capacity of classical channels are super-additive in the following sense [4, 5]. There are \( \mathcal{N}_0 \) and \( \mathcal{N}_1 \) such that \( C(0)(\mathcal{N}_0 \otimes \mathcal{N}_1) > C(0)(\mathcal{N}_0) + C(0)(\mathcal{N}_1) \). This is very different from the ordinary classical capacity of classical channels, which is always additive. However, any classical channels \( \mathcal{N}_0 \) and \( \mathcal{N}_1 \) satisfying the super-additivity must have the ability to transmit classical information perfectly, that is \( C(0)(\mathcal{N}_0) > 0 \) and \( C(0)(\mathcal{N}_1) > 0 \). It remains unknown whether the above super-additivity still holds if one of the quantum channels are with vanishing zero-error capacity. Here we show that for quantum channels such type of stronger super-additivity can exist. Actually, we show that there are quantum channels \( \mathcal{E} \) and \( \mathcal{F} \) such that \( C(0)(\mathcal{E}) = 0 \), \( Q(0)(\mathcal{F}) = C(0)(\mathcal{F}) = 1 \), but \( Q(0)(\mathcal{E} \otimes \mathcal{F}) = \log_2 d > C(0)(\mathcal{E}) + C(0)(\mathcal{F}) = 1 \), where \( 2d \) is dimension of the input state space of \( \mathcal{E} \). The channel \( \mathcal{F} \) can be chosen as a noiseless qubit channel. If we are only concerned with zero-error classical capacity, then \( \mathcal{E} \) can be made entanglement-breaking (Theorem 2). Furthermore, if a \( 2 \times 2 \) maximally entangled state is shared between the sender and the receiver or allowing two-way classical communication that is independent from the message sending from the main protocol, one use of \( \mathcal{E} \) can be used to send \( \log_2 d \) noiseless qubits. This type of \( \mathcal{E} \) has the following weird property: It is not able to communicate any classical information perfectly; however, with a small amount of auxiliary resources (such as one noiseless qubit channel, or one ebit, or two-way classical communication independent from the messages sending through main protocol), the channel acts as a noiseless quantum channel with large perfect quantum capacity (achieving zero-error quantum capacity \( \log_2 d \)). Intuitively, the hiding zero-error communication ability of channel can be activated by these auxiliary resources.

Our last main result is to study the role of classical feedback in zero-error communication. As pointed out by Shannon, for classical channels, the classical feedback cannot increase the ordinary channel capacity but may increase the zero-error capacity [1]. However, a necessary condition for such a feedback improvement is that the channel should be able to communicate classical in-
formation perfectly, i.e., with non-vanishing zero-error capacity. It is of great interest to ask that whether this requirement can be removed for quantum channels. Surprisingly, this answer is yes. Specifically, we construct a quantum channel with a two-dimensional input state space and vanishing zero-error classical capacity such that when assisted with classical feedback enables perfect transmission of classical and quantum information (Theorem 4). In other words, the zero-error capacity of quantum channels can be activated from 0 to positive by classical feedback. This remarkable phenomenon, demonstrates that the zero-error communication ability of a quantum channel may be recovered when assisted with classical feedback.

We notice that very recently several important super-activation effects about different type of capacities of quantum channels, namely quantum capacity, classical capacity, and the private capacity, were discovered [17, 18, 20, 21]. Clearly, these results are incomparable to ours due to the special zero-error transmission requirement.

III. NOTATIONS AND DEFINITIONS

Let Alice be the sender with state space $\mathcal{H}_A$, and let Bob be the receiver with output state space $\mathcal{H}_B$. A quantum channel $\mathcal{E}$ is a completely positive map from $\mathcal{B}(\mathcal{H}_A)$ to $\mathcal{B}(\mathcal{H}_B)$ that can be written into the form $\mathcal{E}(\rho) = \sum_{k=1}^N E_k \rho E_k^\dagger$, where $\{E_k : 1 \leq k \leq N\}$ is the set of Kraus operators and the completeness condition $\sum_{k=1}^N E_k^\dagger E_k = I_A$ is satisfied. A super-operator is a completely positive map for which the completeness condition doesn’t need to be satisfied. For simplicity, sometimes we identify a super-operator $\mathcal{E}$ with Kraus operators $\{E_k : 1 \leq k \leq n\}$ by $\mathcal{E} = \{E_k : 1 \leq k \leq n\}$.

A given quantum channel can be used for zero-error communication as follows: Alice starts with $|0\rangle$, and encodes a message $k$ into a quantum state $\rho_k \in \mathcal{B}(\mathcal{H}_A)$ by a quantum operation $\mathcal{E}_k$, say $\rho_k = \mathcal{E}_k(|0\rangle\langle 0|)$. Bob receives $\mathcal{E}(\rho_k)$, and decodes the message $k$ by suitable quantum operations. Define $\alpha(\mathcal{E})$ to be the maximum integer $N$ with which there exist a set of states $\rho_1, \ldots, \rho_N \in \mathcal{B}(\mathcal{H}_A)$ such that $\mathcal{E}(\rho_1), \ldots, \mathcal{E}(\rho_N)$ can be perfectly distinguished by Bob. It follows from the linearity of super-operators that a set $\{\rho_k : k = 1, \ldots, N\}$ achieving $\alpha(\mathcal{E})$ can be assumed without loss of generality to be orthogonal pure states. In [8] $\alpha(\mathcal{E})$ was termed as the quantum clique number of $\mathcal{E}$. Intuitively, one use of $\mathcal{E}$ can be used to transmit $\log_2 \alpha(\mathcal{E})$ bits of classical information perfectly. When $\alpha(\mathcal{E}) = 1$ it is clear that by a single use of $\mathcal{E}$ Alice cannot transmit any classical information to Bob with zero probability of error.

The zero-error classical capacity of $\mathcal{E}$, $C^{(0)}(\mathcal{E})$, is defined as follows:

$$C^{(0)}(\mathcal{E}) = \sup_{k \geq 1} \frac{\log_2 \alpha(\mathcal{E}^{\otimes k})}{k}.$$  

(1)

If we are concerned with the transmission of quantum information instead of classical information, the notion of zero-error quantum capacity can be similarly introduced. Let $\alpha^{q}(\mathcal{E})$ be the maximum integer $k$ so that there is a $k$-dimensional subspace $\mathcal{H}_A' \subseteq \mathcal{H}_A$ can be perfectly transmitted through $\mathcal{E}$. That is, there is a recovery trace-preserving quantum channel $\mathcal{R}$ from $\mathcal{B}(\mathcal{H}_B)$ to $\mathcal{B}(\mathcal{H}_A')$ such that $(\mathcal{R} \circ \mathcal{E})(|\psi\rangle\langle \psi|) = |\psi\rangle\langle \psi|$ for any $|\psi\rangle \in \mathcal{H}_A'$. Clearly, the quantity $\log_2 \alpha^{q}(\mathcal{E})$ represents the optimal number of intact qubits one can send by a single use of $\mathcal{E}$. The zero-error quantum capacity of $\mathcal{E}$, $Q^{(0)}(\mathcal{E})$, is defined as follows:

$$Q^{(0)}(\mathcal{E}) = \sup_{k \geq 1} \frac{\log_2 \alpha^{q}(\mathcal{E}^{\otimes k})}{k}.$$  

(2)

In the following discussion, we mainly focus on the properties of $\alpha(\mathcal{E})$ and $C^{(0)}(\mathcal{E})$.

We will frequently employ the notion of unextendible bases (UB). Although this notion can be defined on arbitrary multipartite state space (see Ref. [12]), for our purpose here it suffices to focus on matrix spaces. Let $S$ be a set of matrices on $\mathcal{B}(\mathcal{H}_d)$. $S$ is said to be a UB if $S^\perp$ contains no rank-one matrix; otherwise $S$ is said to be extendible. Clearly, when $S$ is a UB, any nonzero matrix in $S^\perp$ with rank at least two. In this case we say $S^\perp$ is completely entangled. If $S$ is a UB and can be spanned by rank-one matrix only, we say $S$ an unextendible product bases (UPB). The properties of UB, in particular UPB, have been extensively studied in literature. We just mention two of them here. The first one is that the tensor product of two UPB is again another UPB [22]. The second one is that if the dimension of a matrix subspace $S$ is small enough, say $\dim(S) < 2d - 1$, $S$ is always extendible [10].

IV. THE QUANTUM CLIQUE NUMBER $\alpha(\cdot)$ IS STRONGLY SUPER-MULTIPLICATIVE

Suppose that $\mathcal{E}$ is classical (a so-called memoryless stationary channel), that is, $\mathcal{E} = \sum_k (|k\rangle \langle k|) \rho_k$ for some states $\rho_k$ diagonalized under the computational basis $\{|k\rangle\}$. Then $\alpha(\mathcal{E}) = 1$ if and only if for all pairs of $k$ and $l$, $\rho_k \rho_l \neq 0$. Thus $\alpha(\mathcal{E}) = 1$ if and only if $\alpha(\mathcal{E}^{\otimes k}) = 1$ for any $k$. Therefore, $C^{(0)}(\mathcal{E}) = 0$ if and only if $\alpha(\mathcal{E}) = 1$. In fact, we can prove that for all entanglement-breaking channel $\mathcal{E}$ of the form $\mathcal{E}(\rho) = \sum_k \text{tr}(M_k^\dagger M_k \rho) \rho_k$, where $\{M_k\}$ is a generalized measurement satisfying $\sum_k M_k^\dagger M_k = I$, it always holds that $\alpha(\mathcal{E}) = 1$ implies that $C^{(0)}(\mathcal{E}) = 0$. (See Corollary 1 below for a proof)

We will show that for quantum channels it would be very different. Let $\mathcal{E} = \sum_{k=1}^n E_k \cdot E_k^\dagger$, where $E_k^\dagger E_k = I_A$. Let us define

$$\mathcal{K}(\mathcal{E}) = \text{span}\{E_k^\dagger E_l : 1 \leq k, l \leq n\}.$$  

(3)

$\mathcal{K}(\mathcal{E})$ plays an important role in determining the properties of zero-error capacity, mainly due to the following useful lemma:
Lemma 1. Let $\mathcal{E} = \{E_k : 1 \leq k \leq n\}$ be a quantum channel. Then $\alpha(\mathcal{E}) > 1$ if and only if $\mathcal{K}(\mathcal{E})$ is extendible, i.e., $\mathcal{K}^\perp(\mathcal{E})$ contains a rank-one matrix.

Proof. Necessity: $\alpha(\mathcal{E}) > 1$ implies there are pure states $|\psi_0\rangle$ and $|\psi_1\rangle$ such that $\mathcal{E}(|\psi_0\rangle) = 0$. Substituting Kraus sum representation of $\mathcal{E}$ into

$$
\text{tr}(\mathcal{E}^t(|\psi_0\rangle\langle\psi_0|\mathcal{E}(|\psi_1\rangle\langle\psi_1|)) = 0,
$$

we have that $\text{tr}(\mathcal{E}_l^tE_l|\psi_0\rangle\langle\psi_0|E_l|\psi_1\rangle\langle\psi_1|) = 0$ for any $1 \leq k, l \leq n$. In other words, $\mathcal{K}(\mathcal{E})$ is extendible. Reversing the above arguments we can easily verify the sufficiency.

Combining the properties of UB mentioned above, we have the following immediate corollary.

**Corollary 1.** Let $\mathcal{E} = \{E_k : 1 \leq k \leq n\}$ be a quantum channel with input state space $\mathcal{B}(\mathcal{H}_d)$. Then we have i) If $n < \sqrt{2d-1}$, then $\alpha(\mathcal{E}) > 1$; ii) If $\mathcal{K}(\mathcal{E})$ is spanned by a set of rank-one matrices, then $\alpha(\mathcal{E}) = 1$ implies $\mathcal{K}^0(\mathcal{E}) = 0$. In particular, any entanglement-breaking channel satisfies this property.

For any quantum channel $\mathcal{E}$ with a set of Kraus operators $\{E_k : 1 \leq k \leq n\}$ and input state space $\mathcal{B}(\mathcal{H}_d)$, one can readily verify that $\mathcal{K}(\mathcal{E})$ contains: a) $\mathcal{K}^\perp(\mathcal{E}) = \mathcal{K}(\mathcal{E})$; and b) $I_d \in \mathcal{K}(\mathcal{E})$. A somewhat surprising fact is that for a given matrix subspace $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H}_d)$, these two properties guarantee the existence of a quantum channel $\mathcal{E}$ such that $\mathcal{K}(\mathcal{E}) = \mathcal{M}$. Here we define $\mathcal{M}^\perp = \{\mathcal{M}^\perp : \mathcal{M} \in \mathcal{M}\}$.

**Lemma 2.** Let $\mathcal{M}$ be a matrix subspace of $\mathcal{B}(\mathcal{H}_d)$. Then there is a quantum channel $\mathcal{E}$ from $\mathcal{B}(\mathcal{H}_d)$ to $\mathcal{B}(\mathcal{H}_d)$ for some integer $d'$ such that $\mathcal{K}(\mathcal{E}) = \mathcal{M}$ if and only if $\mathcal{M}^\perp = \mathcal{M}$ and $I_d \in \mathcal{M}$.

Proof. Necessity is trivial. We only prove sufficiency. First it is easy to see that when $\mathcal{M}^\perp = \mathcal{M}$, we can choose a Hermitian basis for $\mathcal{M}$. Actually, for any matrix $\mathcal{M} \in \mathcal{M}$, we know that $\mathcal{M}^\perp \in \mathcal{M}$. On the other hand, $\mathcal{M}$ and $\mathcal{M}^\perp$ can be spanned by two Hermitian matrices $\mathcal{M} + \mathcal{M}^\perp$ and $i(\mathcal{M} - \mathcal{M}^\perp)$. So we can choose a Hermitian basis for $\mathcal{M}$, say $\{M_1, \ldots, M_n\}$.

Second we show this basis can be made positive definite. Let us choose a positive real number $\varepsilon$ and consider $F_k = I_d + sM_k$. Since $M_k$ is Hermitian, for sufficiently small $s$, all $F_k$ can be made positive definite. Consider $F_0 = I_d - t\sum_{k=1}^n F_k$. Similarly, choose $t$ sufficiently small we can guarantee that $F_0$ is positive definite. So we have a set of positive definite matrices $\{F_k : 0 \leq k \leq n\}$ such that $\sum_{k=0}^n F_k = I_d$ and span $\{F_k : 0 \leq k \leq n\} = \mathcal{M}$.

Third, for each operator $F_k$, we will construct a super-operator $\mathcal{E}_k$ from $\mathcal{B}(\mathcal{H}_d)$ to $\mathcal{B}(\mathcal{H}_d^{(k)})$, where $\mathcal{H}_d^{(k)}$ and $\mathcal{H}_d^{(j)}$ are pairwise orthogonal for $0 \leq k \neq j \leq n$. Take the spectral decomposition of $M_k = \sum_{j=1}^d m_j^{(k)} |\psi_j^{(k)}\rangle\langle\psi_j^{(k)}|$ and let $\{|j\rangle^{(k)} : 1 \leq j \leq d\}$ be an orthonormal basis for $\mathcal{H}_d^{(k)}$. Define a super-operator $\mathcal{E}_k = A_k \cdot A_k^\perp$, where

$$
A_k = \sum_{j=1}^d \sqrt{m_j^{(k)}} |j\rangle^{(k)} \langle\psi_j^{(k)}|.
$$

It is clear that $A_k$ is from $\mathcal{B}(\mathcal{H}_d)$ to $\mathcal{B}(\mathcal{H}_d^{(k)})$ and $A_k A_k = I_d$. Now the desired quantum operation $\mathcal{E}$ is given by the sum of $\mathcal{E}_k$, namely $\mathcal{E} = \sum_{k=0}^n A_k \cdot A_k^\perp$. The output space $\mathcal{H}_d^{(n+1)d} = \bigoplus_{k=0}^n \mathcal{H}_d^{(k)}$. To prove that $\mathcal{K}(\mathcal{E}) = \mathcal{M}$ one only needs to notice that $A_k A_k = \delta_{k0} M_k$.

The above lemma greatly simplifies the study of zero-error classical capacity of noisy quantum channels. It enables us to focus on the matrix subspaces satisfying two very easily grasped conditions. Some remarks are as follows:

(i) The condition b) ensures that a trace-preserving super-operator can be found. For our purpose here, we only need there is a positive definite matrix $M$. Then a super-operator $\mathcal{E}$ with Kraus operators $\{A_k\}$ such that $\sum_k A_k A_k = M$ can be similarly constructed. Based on $\mathcal{E}$ we can further construct a trace-preserving quantum operation $\mathcal{E}'$ with Kraus operations $\{A_k M^{-1/2}\}$. It is easy to check that $\alpha(\mathcal{E}) = \alpha(\mathcal{E}')$. (Here we assume $\alpha(\cdot)$ is also defined for any super-operator $\mathcal{E} = \sum_k E_k \cdot E_k^\perp$ such that $\sum_k E_k^\dagger E_k$ is positive definite)

(ii) None of the conditions a) and b) can be further relaxed. This can be seen from a one-dimensional matrix spanned by a Hermitian matrix with both negative and positive eigenvalues.

(iii) In general $\mathcal{M}$ itself may not satisfy conditions a) and b). However, sometimes we may find two non-singular matrices $E$ and $F$ so that $\mathcal{M}' = EMF$ satisfies conditions a) and b). The extendibility of $\mathcal{M}'$ remains the same as that of $\mathcal{M}$. That is, for any matrix subspace $\mathcal{M}'$, $\mathcal{M} \otimes \mathcal{M}'$ is extendible if and only if $\mathcal{M}' \otimes \mathcal{M}'$ is extendible.

(iv) After we construct a set of positive semi-definite matrices $\{M_k\}$ such that $\sum_k M_k = I_d$ and span $\{M_k\} = \mathcal{M}$, we can use a more compact construction of the corresponding channel $\mathcal{E}$. To do this we introduce an auxiliary output system $\mathcal{H}_E$ and construct $\mathcal{E}$ from $\mathcal{B}(\mathcal{H}_d)$ to $\mathcal{B}(\mathcal{H}_d \otimes \mathcal{H}_E)$ as follows:

$$
\mathcal{E}(\rho) = \sum_{k=1}^N A_k \rho A_k^\dagger \otimes |k\rangle \langle k|,
$$

where $A_k = M_k^{1/2}$ is the positive root of $M_k$, and $|k\rangle$ is an orthonormal basis for $\mathcal{H}_E$. Intuitively, $\mathcal{H}_E$ can be treated as a friendly environment who also outputs its measurement outcome $k$ after the interaction. One can readily verify that $\mathcal{K}(\mathcal{E}) = \text{span}\{M_k : 1 \leq k \leq N\}$. Note that here the output of $\mathcal{H}_E$ is classical information so that a classical system is sufficient for our purpose here. This is an example of quantum communication with classical control.

The following lemma shows that the function of quantum clique number $\alpha(\cdot)$ is strongly super-multiplicative.
Lemma 3. There are noisy quantum channels $\mathcal{E}$ and $\mathcal{F}$ such that $\alpha(\mathcal{E}) = \alpha(\mathcal{F}) = 1$ and $\alpha(\mathcal{E} \otimes \mathcal{F}) > 1$.

Proof. By Lemmas 1 and 2, we only need to construct two unextendible matrix subspaces $S_0$ and $S_1$ both satisfy conditions (a) and (b), and $S_0 \otimes S_1$ are extendible.

Let $S_0$ be a matrix subspace spanned by the following matrix bases:

$$A_1 = |0\rangle \langle 0| + |1\rangle \langle 1|,$$
$$A_2 = |2\rangle \langle 2| + |3\rangle \langle 3|,$$
$$A_3 = |2\rangle \langle 0| - |0\rangle \langle 2|,$$
$$A_4 = |3\rangle \langle 0| + |0\rangle \langle 3|,$$
$$A_5 = |1\rangle \langle 3| + |3\rangle \langle 1|,$$
$$A_6 = \cos \theta |0\rangle \langle 1| + \sin \theta |2\rangle \langle 3| - |1\rangle \langle 2|,$$
$$A_7 = \cos \theta |1\rangle \langle 0| + \sin \theta |3\rangle \langle 2| - |2\rangle \langle 1|,$$
$$A_8 = \sin \theta |0\rangle \langle 1| - \cos \theta |2\rangle \langle 3| + \sin \theta |1\rangle \langle 0| - \cos \theta |3\rangle \langle 2|,$$

where $0 < \theta < \pi/2$ is a parameter. Let $U = |0\rangle \langle 0| - |1\rangle \langle 1| + |2\rangle \langle 2| - |3\rangle \langle 3|$, and let $S_1 = US_0^\perp$, where $S_0^\perp$ is the orthogonal complement via Hilbert-Schmidt inner product. More explicitly, $S_1$ is spanned by the following matrix bases:

$$A'_1 = |0\rangle \langle 0| + |1\rangle \langle 1|,$$
$$A'_2 = |2\rangle \langle 2| + |3\rangle \langle 3|,$$
$$A'_3 = |2\rangle \langle 0| + |0\rangle \langle 2|,$$
$$A'_4 = |3\rangle \langle 0| + |0\rangle \langle 3|,$$
$$A'_5 = |1\rangle \langle 3| + |3\rangle \langle 1|,$$
$$A'_6 = \cos \theta |0\rangle \langle 1| + \sin \theta |2\rangle \langle 3| - |1\rangle \langle 2|,$$
$$A'_7 = \cos \theta |1\rangle \langle 0| + \sin \theta |3\rangle \langle 2| - |2\rangle \langle 1|,$$
$$A'_8 = \sin \theta |0\rangle \langle 1| - \cos \theta |2\rangle \langle 3| + \sin \theta |1\rangle \langle 0| - \cos \theta |3\rangle \langle 2|,$$

We choose $S_1$ as $US_0^\perp$ instead of $S_0^\perp$ so that $S_1$ satisfies the Hermitian condition $S_1^\dagger = S_1$ and contains the identity matrix $I$. This is a key difference from the previous work [9]. By the above lemma, we can define quantum channels $\mathcal{E}$ and $\mathcal{F}$ such that $\mathcal{K}(\mathcal{E}) = S_0$ and $\mathcal{K}(\mathcal{F}) = S_1$.

For any $0 < \theta < \pi/2$, we will show that $S_0$ and $S_1$ satisfy the following useful properties:

(i) Both $S_0$ and $S_1$ are completely entangled and unextendible.

(ii) $S_0 \otimes S_1$ are extendible.

Property (ii) holds as $S_0 \otimes S_1$ is orthogonal to the following rank-one element $(I \otimes U)(|\Phi_4\rangle \langle \Phi_4|)$, where $|\Phi_4\rangle = (|00\rangle + |11\rangle + |22\rangle + |33\rangle)/2$.

We now prove Property (i). Let $|\psi\rangle \langle \phi|$ be a rank one matrix orthogonal to $S_0$, where $|\psi\rangle = \sum_{k=0}^3 a_k |k\rangle$ and $|\phi\rangle = \sum_{l=0}^3 b_l |l\rangle$. Then we have $\text{tr}(A_k |\psi\rangle \langle \phi|) = 0$ for $1 \leq k \leq 8$, that is,

$$a_0 b_0 + a_1 b_1 = 0,$$
$$a_2 b_2 + a_3 b_3 = 0,$$
$$a_2 b_0 - a_0 b_2 = 0,$$
$$a_3 b_0 + a_0 b_3 = 0,$$
$$a_1 b_3 + a_3 b_1 = 0,$$
$$\cos \theta a_0 b_1 + \sin \theta a_2 b_3 - a_1 b_2 = 0,$$
$$\cos \theta a_1 b_3 + \sin \theta a_3 b_2 - a_2 b_1 = 0.$$

Suppose that $a_0 b_0 \neq 0$. Assume without loss of generality that $a_0 = b_0 = 1$. Then

$$a_1 b_1 = -1, a_2 b_2 = -a_3 b_3, a_2 = b_2, a_3 = -b_3, a_1 b_3 = -a_3 b_1.$$

Substituting $a_2 = b_2$ and $a_3 = -b_3$ into $a_2 b_2 = -a_3 b_3$, we have $a_2^2 = a_3^2$. Similarly substituting $a_1 b_1 = -1$ and $b_3 = -a_3$ into $a_1 b_3 = -a_3 b_1$ we have $a_1^2 + a_1 = 0$. If $a_3 = 0$ then $a_2 = b_2 = b_3 = 0$. Hence $\cos \theta a_0 b_1 = \cos \theta a_1 b_3 + \sin \theta a_2 b_3 - a_1 b_2 = 0$, which is a contradiction as both $a_0$ and $b_1$ are nonzero. Thus $a_2^2 = -1$. By $a_1 b_1 = -1$ we know that $a_1 = b_1 = \pm i$. Substituting $b_2 = a_2$ and $b_3 = -a_3$ into the last equation we have $\sin \theta b_1 + \cos \theta a_2 a_3 + \sin \theta b_1 - \cos \theta a_3 a_2 = 0$. That is, $2 \sin \theta b_1 = 0$. Again a contradiction. Therefore $a_0 b_0 = 0$. Note that if $a_k b_l = 0$ and for a nonzero constant $\lambda$, $\lambda a_k b_l = a_k b_l$, then $a_k b_l = a_k b_l = 0$. Applying this inference rule many times, one concludes that all $a_k b_l = 0, 0 \leq k, l \leq 3$ in both $a_0 = 0$ and $b_0 = 0$ cases. Thus $|\psi\rangle \langle \phi| = 0$, and $S_0$ is unextendible. By the same technique, we can prove that $S_1$ is also unextendible.

Applying Lemma 1 we know that $\alpha(\mathcal{E}) = \alpha(\mathcal{F}) = 1$. On the other hand, by property (ii) we know that $\alpha(\mathcal{E} \otimes \mathcal{F}) \geq 2$. Actually, Alice can use $|\Psi_0\rangle = |\Phi_4\rangle$ and $|\Psi_1\rangle = (I \otimes U)|\Phi_4\rangle$ to encode “0” and “1”, respectively, and Bob can recover this bit by distinguish between $(\mathcal{E} \otimes \mathcal{F})(|\Psi_0\rangle \langle \Psi_0|)$ and $(\mathcal{E} \otimes \mathcal{F})(|\Psi_1\rangle \langle \Psi_1|)$, which are orthogonal by our construction.

In the above construction, $\mathcal{E}$ and $\mathcal{F}$ are not identical. However, using the direct sum construction [24], we can find a quantum channel enjoying similar property. Now we are ready to present our main result:

Theorem 1. There is a classical of quantum channels $\mathcal{G}$ such that $\alpha(\mathcal{G}) = 1$ and $\alpha(\mathcal{G}^\otimes 2) > 1$. Hence $C^{(0)}(\mathcal{G}) \geq 0.5$.

Proof. The idea is to take $\mathcal{G}$ as the direct sum of $\mathcal{E}$ and $\mathcal{F}$, say $\mathcal{G} = \mathcal{E} \oplus \mathcal{F}$. More explicitly,

$$\mathcal{G}(\rho) = \mathcal{E}(P_0 \rho P_0) + \mathcal{F}(P_1 \rho P_1),$$

where $P_0$ and $P_1$ are the projections on the input state spaces of $\mathcal{E}$ and $\mathcal{F}$, respectively, and $P_0 + P_1$ is the projection of the whole input state space for $\mathcal{G}$. The function of $\mathcal{G}$ can be understood as follows: for any input state $\rho$, we first perform a projective measurement
\(\{P_0, P_1\}\). If the outcome is 0, then we apply \(\mathcal{E}\) to the resulting state; otherwise we apply \(\mathcal{F}\). It is clear that 
\[
\alpha(\mathcal{G}) = \max\{\alpha(\mathcal{E}), \alpha(\mathcal{F})\}. 
\]
Furthermore, we have
\[
\alpha(\mathcal{G} \otimes 2) = \max\{\alpha(\mathcal{E} \otimes 2), \alpha(\mathcal{E} \otimes \mathcal{F}), \alpha(\mathcal{F} \otimes \mathcal{E}), \alpha(\mathcal{F} \otimes 2)\}. 
\]
For channels \(\mathcal{E}\) and \(\mathcal{F}\) constructed above, we have \(\alpha(\mathcal{G}) = 1\) and \(\alpha(\mathcal{G} \otimes 2) \geq \alpha(\mathcal{E} \otimes \mathcal{F}) > 1\).

Based on our previous work about UB \[26\], we know that any channel \(\mathcal{G}\) with the property in Theorem 1 should be at least with a 4-dimensional input space. It remains unknown how to construct quantum channel with similar property and with smaller input and output dimensions.

V. A CLASS OF SPECIAL QUANTUM CHANNELS

The construction in Lemma 2 suggests us to consider a special class of quantum channels, which can be treated as a generalization of retro-correctible channels introduced in \[19\]. Consider a quantum channel \(\mathcal{E}\) from \(\mathcal{B}(\mathcal{H}_c \otimes \mathcal{H}_d)\) to \(\mathcal{B}(\mathcal{H}_c' \otimes \mathcal{H}_d')\) as follows:
\[
\mathcal{E} = \sum_{k=1}^{N} \mathcal{E}_k \otimes \mathcal{F}_k, 
\]
where both \(\mathcal{E}_k\) and \(\mathcal{F}_k\) are super-operators for each \(1 \leq k \leq N\). Usually we choose \(\{\mathcal{F}_k\}\) to be a set of quantum channels and \(\{\mathcal{E}_k\}\) is a set of super-operators such that \(\sum_{k=1}^{N} \mathcal{E}_k\) is trace-preserving.

![Diagram](image)

**FIG. 2:** Internal realization of a controlled communication channel \(\mathcal{E}\): 1) Perform a measurement \(\{\mathcal{E}_k\}\) to the control input system \(\mathcal{H}_c\); 2) If the measurement outcome is \(k\), apply \(\mathcal{F}_k\) to the data input \(\mathcal{H}_d\); 3) Output both the control and the data inputs to \(\mathcal{H}_c'\) and \(\mathcal{H}_d'\), respectively. Here the input dimensions \(c\) and \(d\) are not required to be the same as the output dimensions \(c'\) and \(d'\), respectively.

Imposing special constraints on \(\mathcal{E}_k\) and \(\mathcal{F}_k\), we can construct some quantum channels with desirable properties. In particular, if the receiver Bob can distinguish between \(\{\mathcal{E}_k : 1 \leq k \leq N\}\), he will be able to determine the quantum operation performed on the data input exactly. Thus the net effect of the channel \(\mathcal{E}\) will reduce to some of \(\mathcal{F}_k\). In the case that \(\mathcal{F}_k\) has a large amount of classical or quantum capacity, the above channel will also have large capacity. Symmetrically, if Bob can distinguish between \(\mathcal{F}_k\) then he will be able to know the measurement operator performed by the environment, and then is able to correct the errors.

For example, if we choose \(d' = c' = c = n\), and \(d = 1\), and choose \(\mathcal{F}_k\) to be the unitary (isometry) \(|k\rangle\langle 0|\) from \(\mathcal{H}_d\) to \(\mathcal{H}_d'\), and let control operators \(\mathcal{E}_k\) be a set of generalized measurement \(\{A_k : 1 \leq k \leq n\}\) from \(\mathcal{H}_c\) to \(\mathcal{H}_c'\), then we have the following channel:
\[
\mathcal{E}(\rho \otimes |0\rangle\langle 0|) = \sum_{k=1}^{n} (A_k \otimes |k\rangle\langle 0|)(\rho \otimes |0\rangle\langle 0|)(A_k^\dagger \otimes |0\rangle\langle k|). 
\]
If we ignore the one-dimensional data input, the above channel can be simplified as follows:
\[
\mathcal{E}(\rho) = \sum_{k=1}^{n} A_k \rho A_k^\dagger \otimes |k\rangle\langle k|. 
\]
This is precisely the channel we introduced in the previous section, where a similar interpretation has been presented.

Another special case is that \(\{\mathcal{F}_k\}\) or \(\{\mathcal{E}_k\}\) are not distinguishable in general, but would be distinguishable if an entangled state \(|\Phi\rangle\rangle\) is provided. That is, the set of \(\{(|I \otimes \mathcal{E}_k)(|\Phi\rangle\rangle)\}\) is distinguishable. Intuitively, \(\{\mathcal{E}_k\}\) cannot be distinguished means that the channel is very noisy. So the capacity without any assistance would be generally small. However, supplying additional resources such as shared entanglement will greatly improve the capacity. The class of retro-correctible channels introduced by Bennett et al. \[14\] is a typical example.

VI. SUPER-ADDITIVITY OF ZERO-ERROR CLASSICAL AND QUANTUM CAPACITIES

It was known that the zero-error classical capacity of classical channels are super-additive in the following sense \[3, 4\]: there are classical channels \(N_0\) and \(N_1\) such that \(C^{(0)}(N_0 \otimes N_1) > C^{(0)}(N_0) + C^{(0)}(N_1)\). This is very different from the ordinary capacity, which is always additive. However, any classical channels \(N_0\) and \(N_1\) satisfying the super-additivity must have the ability to transfer classical information perfectly, that is \(C^{(0)}(N_0) > 0\) and \(C^{(0)}(N_1) > 0\). It remains unknown whether the above super-additivity still holds if one or two quantum channels are with vanishing zero-error capacity. Here we will show that both \(C^{(0)}\) and \(Q^{(0)}\) satisfy a stronger type of super-additivity. Let’s consider \(C^{(0)}\) first.

**Theorem 2.** There is an entanglement-breaking channel \(\mathcal{E}\) on \(\mathcal{B}(\mathcal{H}_{2d})\) such that \(C^{(0)}(\mathcal{E}) = 0\) and \(C^{(0)}(I_2 \otimes \mathcal{E}) = \log_2 d \gg C^{(0)}(I_2) + C^{(0)}(\mathcal{E}) = 1\), where \(I_2\) is one qubit noiseless quantum channel.

**Proof.** Consider the quantum channel
\[
\mathcal{E} = \mathcal{E}_0 \otimes \mathcal{F}_0 + \mathcal{E}_1 \otimes \mathcal{F}_1, 
\]
where $\mathcal{E}_0 = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$, $\mathcal{E}_1 = \{|0\rangle\langle 1|, |1\rangle\langle 0|\}$, $\mathcal{F}_0 = \{|k\rangle\langle k| : 1 \leq k \leq d\}$ and $\mathcal{F}_1 = \{|\bar{k}\rangle\langle \bar{k}| : 1 \leq k \leq d\}$. In particular, $\{|k\rangle\}$ and $\{|\bar{k}\rangle\}$ are two orthonormal bases such that $\langle k|\bar{k}\rangle \neq 0$. By choice we have that $\mathcal{F}_0(\rho) \perp \mathcal{F}_1(\sigma) = 0$ if and only if $\rho = 0$ or $\sigma = 0$. It is also clear that $\alpha(\mathcal{F}_0) = \alpha(\mathcal{F}_1) = d$, and the set of input states can be chosen as $\{|k\rangle : 1 \leq k \leq d\}$. These facts will be useful later.

First we show that $C(\mathcal{E}) = 0$. Clearly $\mathcal{E}$ is an entanglement breaking channel as it has a set of rank-one Kraus operators. Thus it suffices to show that $\alpha(\mathcal{E}) = 1$. Take an input state $|\psi\rangle = |\psi_0\rangle + |\psi_1\rangle$ and calculate

$$
\mathcal{E}(\psi) = \langle \psi| \mathcal{F}_0(\psi_0) + \mathcal{F}_1(\psi_1) \rangle + |\psi_1\rangle \langle \psi_0 + \psi_1| \langle \psi_0 - \psi_1|, \langle \psi_0 + \psi_1| \langle \psi_0 + \psi_1| + \mathcal{F}_0(\psi_0 + \psi_1) + \mathcal{F}_1(\phi_0 - \phi_1).
$$

If $\mathcal{E}(\psi)$ and $\mathcal{E}(\phi)$ are orthogonal, we should have

$$
\mathcal{F}_0(\psi_0)\mathcal{F}_1(\phi_1) = 0, \quad \mathcal{F}_1(\psi_1)\mathcal{F}_0(\phi_0) = 0, \quad \mathcal{F}_0(\psi_0 + \psi_1)\mathcal{F}_1(\phi_0 - \phi_1) = 0, \quad \mathcal{F}_1(\psi_0 - \psi_1)\mathcal{F}_0(\phi_0 + \phi_1) = 0.
$$

From the first equation we know that $\psi_0 = 0$ or $\psi_1 = 0$. Without loss of generality, assume that $\psi_0 = 0$. It follows from the second equation that $\phi_0 = 0$ or $\phi_1 = 0$. However this would imply that both $\psi_0 + \psi_1 = \psi_1$ and $\phi_0 - \phi_1 = -\phi_1$ are nonzero, thus the third equation cannot hold. With that we complete the proof of $\alpha(\mathcal{E}) = 1$.

The next step is to show that if a noiseless qubit channel $\mathcal{I}_2$ is supplied between Alice and Bob, Alice can send $d$ messages perfectly to Bob using $\mathcal{I}_2 \otimes \mathcal{E}$. Let $|\Phi_2\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. The key here is that $\mathcal{E}_0$ and $\mathcal{E}_1$ are distinguishable by $\{|\Phi_2\rangle\}$ in the sense that

$$
\rho_0 = (\mathcal{I}_2 \otimes \mathcal{E}_0)(|\Phi_2\rangle) = (|00\rangle\langle 00| + |+1\rangle\langle +1|)/2
$$

and

$$
\rho_1 = (\mathcal{I}_2 \otimes \mathcal{E}_1)(|\Phi_2\rangle) = (|10\rangle\langle 10| + |-1\rangle\langle -1|)/2
$$

are orthogonal. If Alice encodes message $k$ into $|\Phi_2\rangle \otimes |k\rangle$ and transmits it to Bob via $\mathcal{I}_2 \otimes \mathcal{E}$, the received states by Bob are

$$
\{(\rho_0 \otimes |k\rangle\langle k| + \rho_1 \otimes |\bar{k}\rangle\langle \bar{k}|) : 1 \leq k \leq d\},
$$

which are mutually orthogonal. That completes the proof of $\alpha(\mathcal{I}_2 \otimes \mathcal{E}) \geq d$.

It is easy to see that the role of the noiseless qubit channel $\mathcal{I}_2$ can be replaced with a pre-shared $2 \otimes 2$ maximally entangled state $|\Phi_2\rangle$ between Alice and Bob. To encode the message $k$, Alice simply sends $|k\rangle$ together with her half of entangled state to Bob. The received states by Bob are the same as above.

With a more careful analysis we can easily see that $\rho_0$ and $\rho_1$ are locally distinguishable in the following way: Bob performs a projective measurement according to $\{|0\rangle, |1\rangle\}$, and then sends outcome $b$ to Alice. If $b = 0$ then Alice measures her particle using the same basis, otherwise using diagonal basis $\{|+\rangle, |-\rangle\}$. The outcomes $00, +1$ correspond to $\rho_0$, while $10, -1$ correspond to $\rho_1$. So a more economic way to achieve the perfect transmission is that: Alice locally prepares a Bell state $|\Phi_2\rangle$ and then send $|k\rangle$ and one half of $|\Phi_2\rangle$ to Bob. Bob feedbacks his measurement outcome on the control qubit to Alice. Based on Bob’s information, Alice performs the measurement on the left half of $|\Phi_2\rangle$ and forwards the measurement outcome to Bob. Bob will then know which of $\mathcal{F}_0$ and $\mathcal{F}_1$ is performed on the data input and can perfectly decode the message $k$. Note here we use two-way classical communication which is usually not allowable. However, from the above analysis we can see these communications are independent from the message $k$ we send in our main protocol. To summarize, we have the following

**Corollary 2.** For the quantum channel $\mathcal{E}$ constructed in above theorem, we have 1) $C^{(0)}_{Q_{\text{eit}}} \geq \log_2 d \gg C^{(0)}(\mathcal{E}) = 0$, where the subscript means one ebit available; 2) $C^{(0)}(\mathcal{E}) \geq \log_2 d \gg C^{(0)}(\mathcal{E}) = 0$, where the subscript 2 denotes the two-way classical communication independent of the message sending through the main protocol.

So far we haven’t touched the zero-error quantum capacity yet. Using a similar construction, we can prove the strong super-additivity of $Q^{(0)}$. A somewhat surprising fact is that even for quantum channel with vanishing zero-error classical capacity, the super-activation effect remains possible. Actually we have the following

**Theorem 3.** There is quantum channel $\mathcal{E}$ with input state space $\mathcal{B}(\mathcal{H}_{2d})$ such that $C^{(0)}(\mathcal{E}) = 0$ and $Q^{(0)}(\mathcal{I}_2 \otimes \mathcal{E}) = \log_2 d \gg Q^{(0)}(\mathcal{I}_2) + Q^{(0)}(\mathcal{E}) = 1$, where $\mathcal{I}_2$ is the noiseless qubit channel.

**Outline of Proof.** Consider the following quantum channel:

$$
\mathcal{E} = \frac{1}{\sqrt{N}} \sum_{k=1}^N (\mathcal{E}_{\text{inv}} \otimes I_d + \mathcal{E}_{\text{inv}} \otimes U_k)
$$

where $\mathcal{E}_{\text{inv}} = \{|k\rangle\langle \psi_{k0}|, \mathcal{E}_{\text{inv}} = \{|k\rangle\langle \psi_{k1}|, \{\psi_{k0}, |\psi_{k1}\rangle\}$ is an orthogonal basis for $\mathcal{H}_d$. “*” is the complex conjugate according to $\{|0\rangle, |1\rangle\}$, and $\{U_k\}$ is a set of unitary operations on $\mathcal{H}_d$. The function of $\mathcal{E}$ can be understood as follows: First, randomly choose an integer $k \in \{1, \ldots, N\}$, and perform a projective measurement $\{|\psi_{k0}\rangle, |\psi_{k1}\rangle\}$ on the control input qubit. If the outcome is 0 no action to the data input; otherwise perform $U_k$. 


Second, output the classical information $k$ but keep the measurement outcome hidden. This is exactly one special instance of retro-correctible channel \[12\].

It is easy to see that $\mathcal{K}(\mathcal{E})$ is given by

$$\{\psi_{0}\otimes I_{d},\psi_{k}\otimes I_{d},|\psi_{k0}\rangle\langle \psi_{k1}|\otimes U_{k},|\psi_{k0}\rangle\langle \psi_{k0}| \otimes U_{k}^{\dagger}\},$$

where $1 \leq k \leq N$ and recall that $\psi_{k0} = |\psi_{k0}\rangle\langle \psi_{k0}|$. We can see that

$$\mathcal{K}^{-1}(\mathcal{E}) \subseteq \{I_{2} \otimes D : \text{tr}(D) = 0, D \in \mathcal{B}(\mathcal{H}_{d})\}.$$ 

We will show that by choosing $U_{k}$, $|\psi_{k0}\rangle$, $N$ appropriately, the above inequality holds with equality. To achieve this, we only need to choose $|\psi_{0k}\rangle$ and $U_{k}$ so that $\psi_{k0} \otimes I_{d}$ spans $\mathcal{B}(\mathcal{H}_{2}) \otimes I_{d}$, and $|\psi_{k0}\rangle\langle \psi_{k1}| \otimes U_{k}$ spans $\text{span}\{C \otimes \mathcal{B}(\mathcal{H}_{d}) : \text{tr}(C) = 0, C \in \mathcal{B}(\mathcal{H}_{2})\}$. This can be done easily as

$$\text{span}\{|\psi\rangle\langle \phi| : \langle \psi|\phi\rangle = 0\} = \{D : \text{tr}(D) = 0\}$$

and

$$\text{span}\{U : U^{\dagger}U = I_{d}\} = \mathcal{B}(\mathcal{H}_{d}).$$

Now it is clear that $\mathcal{K}(\mathcal{E})$ contains the following set of rank-one matrices:

$$\{|0\rangle\langle 1|, |1\rangle\langle 0|, |+\rangle\langle -|\} \otimes \{|k\rangle\langle l| : 1 \leq k, l \leq d\},$$

which is clearly a UPB as its orthogonal complement is a completely entangled subspace $I_{2} \otimes \mathcal{B}(\mathcal{H}_{d})$. Thus $\mathcal{K}(\mathcal{E}^{\otimes n})$ is unextendible for any $n \geq 1$, which follows $C^{(0)}(\mathcal{E}) = 0$.

The argument that the above channel is able to communicate quantum information perfectly is similar to the analysis for the retro-correctible channels. The key idea is that with the assistance of shared entanglement, the hidden measurement outcome can be revealed. Suppose that $|\Phi_{2}\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ is supplied to Alice and Bob. Then by inputting an arbitrary state $|\psi\rangle$ into the data slot, and Alice’s half of the $|\Phi_{2}\rangle$ into control slot, Bob receives the following output state

$$(I_{2} \otimes \mathcal{E})(|\Phi_{2}\rangle\langle \psi|) = 1/N \sum_{k=1}^{2N} (k|\psi_{k0}\rangle \otimes \psi + k|\psi_{k1}\rangle \otimes U_{k}\psi U_{k}^{\dagger}).$$

Note that $\bigcup_{k=1}^{N}\{|\psi_{k0}\rangle,|\psi_{k1}\rangle\}$ is an orthonormal basis for $\mathcal{H}_{N} \otimes \mathcal{H}_{2}$. Thus Bob can perfectly distinguish them by a projective measurement. If the outcome is $k0$, then the data output is $\psi$. If the outcome is $k1$, then the data output is $U_{k}\psi U_{k}^{\dagger}$. Applying $U_{k}^{\dagger}$ to the data output, we can recover $\psi$. That means $\mathcal{E}$ together with one ebit can be used to perfectly transfer a $d$-dimensional quantum system, or in other words, with entanglement-assisted zero-error quantum capacity at least $\log_{2}d$ qubits.

It is not difficult to see that the role of shared entanglement can be replaced by a noiseless qubit channel. That immediately implies $Q^{(0)}(I_{2} \otimes \mathcal{E}) \geq \log_{2}d > Q^{(0)}(I_{2}) + Q^{(0)}(\mathcal{E}) = 1$.

Notice further that $\bigcup_{k=1}^{N}\{|k\psi_{k0}\rangle,|k\psi_{k1}\rangle\}$ is an orthonormal basis for $\mathcal{H}_{N} \otimes \mathcal{H}_{2}$ that is LOCC distinguishable. So the assistance of two-way classical communications that are independent from the quantum information sending through the main protocol can be used to transmit $\log_{2}d$ noiseless qubits. The analysis is similar to the previous theorem and we omit the details here. \[\blacksquare\]

**Corollary 3.** There is quantum channel $\mathcal{E}$ with input state space $\mathcal{B}(\mathcal{H}_{2})$ such that $C^{(0)}(\mathcal{E}) = 0$ and 1) $C^{(0)}(I_{2} \otimes \mathcal{E}) \geq \log_{2}d > C^{(0)}(I_{2}) + C^{(0)}(\mathcal{E}) = 1$; 2) $Q^{(0)}(I_{2} \otimes \mathcal{E}) \geq \log_{2}d > Q^{(0)}(I_{2}) + Q^{(0)}(\mathcal{E}) = 1$; 3) $C^{(0)}_{s}(\mathcal{E}) \geq Q^{(0)}_{s}(\mathcal{E}) \geq \log_{2}d$, where the subscript 2 denotes two-way classical communications that are independent of the message sending through the main protocol; 4) $C^{(0)}_{1\text{-ebit}}(\mathcal{E}) \geq Q^{(0)}_{1\text{-ebit}}(\mathcal{E}) \geq \log_{2}d$, where the subscript means one ebit available.

The above corollary indicates the behaviors of zero-error capacity of quantum channels is very weird: There are quantum channels which have a large amount quantum capacity but with vanishing zero-error classical capacity. However, the channel can be unlocked for zero-error quantum communication if a small amount of additional resources such as two-way classical communication independent of the messages sending through the main protocol, shared entanglement, or a noiseless quantum channel is available.

**VII. CLASSICAL FEEDBACK ENABLES PERFECT TRANSMISSION OF INFORMATION**

In this section we will study the role of classical feedback. A well known result in classical information theory is that a noiseless classical feedback channel from Bob to Alice cannot increase the capacity of a classical channel. For quantum channels, it remains an open problem whether a classical feedback can strictly increase the capacity \[13\]. However, in the special case that a quantum channel with zero classical capacity it should be a constant channel, i.e., it sends any input quantum state to a fixed state. Clearly, the classical feedback cannot increase the capacity under this special assumption. The situation is very different for quantum capacity, which can be increased by classical feedback even the unassisted quantum capacity is zero. A typical example is the quantum erasure channel with erasure probability more than 0.5, which has vanishing quantum capacity but nonzero classical feedback assisted quantum capacity \[22\].

It was pointed out in \[1\] that for certain classical noisy channels a noiseless classical feedback channel from Bob to Alice may strictly increase the zero-error classical capacity. All these channels should satisfy $C^{(0)} > 0$. In other words, without any assistance they can be used to communicate classical information perfectly. Thus a question of interest is to ask whether this assumption can be removed. We provide an affirmative answer to this question as follows:
Theorem 4. There is quantum channel $\mathcal{E}$ such that $C^{(0)}(\mathcal{E}) = 0$ but $C^{(0)}_{\text{cfb}}(\mathcal{E}) > 0$ and $Q^{(0)}_{\text{cfb}}(\mathcal{E}) > 0$, where the subscript “cfb” represents classical feedback from Bob to Alice.

Proof. The quantum channel constructed in Eq. 8 is exactly one such channel when $d = 2$. To see this, one only needs to show that one use of $\mathcal{E}$ together with back communication can generate a shared entangled state $|\Phi_d\rangle$. The protocol is as follows. First Alice prepares $|\Phi_2\rangle|\Phi_d\rangle$ and sends half of them to Bob. Second Bob measures the control output and feedbacks the outcome $k$. For the moment he has already known the shared entangled state between him and Alice should be one of $|\Phi_d\rangle$ or $(I \otimes U_k)|\Phi_d\rangle$. After receiving $k$, Alice performs a measurement according to $\{|\psi_{k0}\rangle, |\psi_{k1}\rangle\}$. If $k0$ is obtained, the final shared entangled state is $|\Phi_d\rangle$; if $k1$ is obtained, the final shared entangled state is $(I \otimes U_k)|\Phi_d\rangle$, and she only needs to perform $U_k^*$ to the left half of $|\Phi_d\rangle$, thus the final resulting state is again $|\Phi_d\rangle$. If $d = 2$, we already know that $\mathcal{E}$ together with this entangled state can be used to send one noiseless qubit. In total, two uses of $\mathcal{E}$ and classical feedback enable one noiseless qubit transmission. Therefore

$$C^{(0)}_{\text{cfb}}(\mathcal{E}) \geq Q^{(0)}_{\text{cfb}}(\mathcal{E}) \geq 0.5.$$ 

If in Eq. 8 we use a $d$-dimensional control input instead of a 2-dimensional one, we will know that $C^{(0)}(\mathcal{E}) = 0$ but

$$C^{(0)}_{\text{cfb}}(\mathcal{E}) \geq Q^{(0)}_{\text{cfb}}(\mathcal{E}) \geq 0.5 \log_2 d.$$

There is, however, a quantum channel with only a two-dimensional input state space enjoying the same property. Due to its simplicity, let us give a detailed analysis here. Consider the following quantum channel from $\mathcal{B}(\mathcal{H}_2)$ to $\mathcal{B}(\mathcal{H}_2 \otimes \mathcal{H}_4)$:

$$\mathcal{E}(\rho) = \sum_{k=1}^{4} E_k \rho E_k^\dagger \otimes |k\rangle\langle k|,$$

where $\{|k\rangle : 1 \leq k \leq 4\}$ is an orthonormal basis for auxiliary system.

Noticing that $\mathcal{K}(\mathcal{E}) = \text{span}\{|+\rangle\langle +|, |-\rangle\langle -|, |i_+\rangle\langle i_+|\}$ is a UPB, we have that $\alpha(\mathcal{E}^\otimes n) = 1$ for any $n \geq 1$. Thus $C^{(0)}(\mathcal{E}) = 0$. On the other hand, $K^\perp(\mathcal{E}) = \mathcal{Z}$. So if a maximally entangled state $|\Phi_2\rangle$ is shared between Alice and Bob, Alice can send one bit to Bob without any error. To do this, Alice first encodes “0” by applying $I$ and “1” by applying $Z$ to her half of the shared entangled state, respectively, and sends her half of $|\Phi_2\rangle$ to Bob. The received states by Bob are

$$\rho_0 = \sum_{k=1}^{4} (I \otimes E_k)|\Phi_2\rangle\langle \Phi_2| (I \otimes E_k)^\dagger \otimes |k\rangle\langle k|,$$

and

$$\rho_1 = \sum_{k=1}^{4} (I \otimes E_k Z)|\Phi_2\rangle\langle \Phi_2| (I \otimes E_k Z)^\dagger \otimes |k\rangle\langle k|,$$

respectively. By our assumption on $E_k$, $\rho_0$ and $\rho_1$ are orthogonal. Thus Bob can decode the bit perfectly.

Now the whole problem is reduced to generate a maximally entangled state between Alice and Bob using $\mathcal{E}$ and classical feedback only. Fortunately, this can be done as follows:

Step 1. Alice locally prepares $|\Phi_2\rangle$ and sends one half of $|\Phi_2\rangle$ to Bob through $\mathcal{E}$.

Step 2. Bob measures the auxiliary system according to $\{|k\rangle : 1 \leq k \leq 4\}$. If the outcome is $k$, Bob will know that an entangled state $|\Psi_k\rangle = \sqrt{2}(I \otimes E_k)|\Phi_2\rangle$ with Schmidt coefficient vector $(2/3, 1/3)$ is generated between him and Alice.

Step 3. Repeat steps 1 and 2 once more, Alice and Bob will share a state $|\Psi_k\rangle \otimes |\Psi_l\rangle$, with Schmidt coefficient vector $(4/9, 2/9, 2/9, 1/9)$. (However only Bob knows the exact form of $|\Psi_k\rangle$.)

Step 4. Bob feedbacks the measurement outcomes $k$ and $l$ to Alice. So Alice also knows the exact form of the shared entangled state between them.

Step 5. Bob and Alice transform the shared entangled state into a Bell state with standard form $|\Phi_2\rangle$. By Nielsen's theorem [23], this can be achieved with certainty as $(4/9, 2/9, 2/9, 1/9) < (1/2, 1/2)$. Furthermore, the transformation can be done using local measurements and classical communications from Bob to Alice only.

Combining the above discussions, we know that 3 uses together with back communication can transmit one bit perfectly from Alice to Bob. Thus $C^{(0)}_{\text{cfb}} \geq 1/3 > 0$. Moreover, we can send a qubit by sending two bits and consuming one ebit. Easily see that 8 uses of $\mathcal{E}$ can transmit one noiseless qubit. Hence $Q^{(0)}_{\text{cfb}} \geq 1/8$. 

\[\blacksquare\]
It seems that the retro-correctible channel $R_{2,2}$ introduced in [19] might enjoy the same property as above. However, we don’t know how to determine the value of $C^{(0)}(R_{2,2})$ and consequently, it remains unknown whether $C^{(0)}(R_{2,2})$ is vanishing or not.

**VIII. CONCLUSIONS AND DISCUSSIONS**

In sum, we have demonstrated that for a class of quantum channels, a single use of the channel cannot be used to transmit classical information with zero probability of error, while multiple uses can. This super-activation property is enabled by quantum entanglement between different uses, thus cannot be achieved by classical channels. We also have shown that additional resources such as classical communications independent of sending messages, shared entanglement, and noiseless quantum communication would be greatly improve the zero-error capacity for certain channels. In particular, both the zero-error classical capacity and zero-error quantum capacity are strongly super-additive even one of the channels is with vanishing zero-error classical capacity. Finally we construct a special class of quantum channels to show that the classical feedback enables perfect transmission of both classical and quantum information even when the quantum channel has vanishing zero-error classical capacity. These results suggest that a new quantum zero-error information theory would be highly desirable.

Many interesting problems remain open, and here we mention two of them. The first one is to show whether the following strongest super-additivity is possible: Find quantum channels $\mathcal{E}$ and $\mathcal{F}$ such that $C^{(0)}(\mathcal{E}) = C^{(0)}(\mathcal{F}) = 0$ and $C^{(0)}(\mathcal{E} \otimes \mathcal{F}) > 0$. According to Lemma 2, this is equivalent to find two matrix subspaces $S_0$ and $S_1$ such that 1) $I \in S_k$ and $S_k = S_k$; 2) $S_n^{(0)}$ are unextendible for any $n \geq 1$, $k = 0, 1$; and 3) $S_0 \otimes S_1$ are unextendible. The quantum channels presented in Lemma 3 may be eligible candidates. However, we are not able to answer this question at present as we don’t have a feasible way to check whether $S_n^{(0)}$ is extendible for $n > 1$. The second one is to study corresponding problems about the zero-error quantum capacity $Q^{(0)}$. In this case we don’t even know whether $\alpha^{(0)}(\cdot)$ is super-multiplicative. A result similar to Lemma 4 would be highly desirable. All these problems can be successfully solved for another notion of unambiguous capacity, which is a generalization of zero-error capacity by requiring the decoding process to be unambiguous.

**Note Added:** After the completion of this work, the author happened to know that Cubitt, Chen, and Harrow also independently obtained some super-activation results about the zero-error classical capacity which partially overlap with ours. More precisely, they employed the Choi-Jamiołkowski isomorphism between quantum channels and a class of bipartite mixed states to establish a theorem similar to Lemma 2 here. Then two quantum channels $\mathcal{E}$ and $\mathcal{F}$ with four-dimensional input state spaces such that $\alpha(\mathcal{E}) = \alpha(\mathcal{F}) = 1$ and $\alpha(\mathcal{E} \otimes \mathcal{F}) > 1$ were explicitly constructed. They further applied some powerful techniques from Algebraic Geometry to show that a pair of quantum channels satisfying the strongest super-additivity does exist, and thus solved the open problem mentioned above. (One of these techniques is a result about strongly unextendible bases that was previously proven and used in [26] to demonstrate a similar super-activation effect for unambiguous capacity of quantum channels.) Clearly, their remarkable result established the strongest type of super-additivity, which they termed as the super-activation of the asymptotic zero-error classical capacity of quantum channels. Interestingly, it is not difficult to show that all channels we constructed in Theorems 2-4 cannot be activated by any quantum channel $\mathcal{F}$ with $\alpha(\mathcal{F}) = 1$. Thus it is still a surprising fact that these channels do satisfy certain type of super-activation effects which are definitely impossible for any classical channel.

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