Decentralized Strategies for Finite Population Linear-Quadratic-Gaussian Games and Teams *

Bing-Chang Wang a, Huanshui Zhang a, Minyue Fu b, Yong Liang a

a School of Control Science and Engineering, Shandong University, Jinan 250061, P. R. China.
b the School of Electrical Engineering and Computing Science, University of Newcastle, NSW 2308, Australia.

Abstract

This paper is concerned with a new class of mean-field games which involve a finite number of agents. Necessary and sufficient conditions are obtained for the existence of the decentralized open-loop Nash equilibrium in terms of non-standard forward-backward stochastic differential equations (FBSDEs). By solving the FBSDEs, we design a set of decentralized strategies by virtue of two differential Riccati equations. Instead of the ε-Nash equilibrium in classical mean-field games, the set of decentralized strategies is shown to be a Nash equilibrium. For the infinite-horizon problem, a simple condition is given for the solvability of the algebraic Riccati equation arising from consensus. Furthermore, the social optimal control problem is studied. Under a mild condition, the decentralized social optimal control and the corresponding social cost are given.

Key words: Mean-field game, decentralized Nash equilibrium, finite population, non-standard FBSDE, weighted cost

1 Introduction

The mean-field game has drawn intensive research attention because it provides an effective theoretical scheme for analyzing the collective behavior of large population multiagent systems (MASs). This has found wide applications in various disciplines, such as economics, biology, engineering, and social science [5,9,11,14,41,47]. Mean-field games were initiated by two groups independently. Huang et al. designed an ε-Nash equilibrium for a decentralized strategy with discount costs based on a Nash certainty equivalence (NCE) approach [19]. Independently, Lasry and Lions introduced a mean-field game model and studied the well-posedness of coupled partial differential equation systems [26]. The NCE approach can be extended to cases with long run average costs [27] or with Markov jump parameters [44].

1.1 Literature review

Depending on the state-cost setup of a mean-field game, it can be classified into linear-quadratic-Gaussian (LQG) games or more general nonlinear ones. The LQG game is commonly adopted in mean-field studies because of its analytical tractability and close connection to practical applications. Relevant works include [4,16,19,27,32,43]. In contrast, a nonlinear mean-field game enjoys its modeling generality (see e.g. [10, 11, 21, 26]). Besides, depending on their system hierarchy, mean-field games can be classified into homogeneous, heterogeneous, or mixed. See [7, 17, 44] for mixed games.

Apart from noncooperative games, mean-field social optimal control has also drawn increasing attention recently. In a social optimum problem, all players cooperate to optimize the social cost—the sum of individual costs. Social optima are linked to a type of team decision [15] but with highly complex interactions. The work [20] studied social optima in mean-field LQG control, and provided an asymptotic team-optimal solution. Authors in [1] considered team-optimal control with finite population and partial information. For further literature, see [23] for socially optimal control for major-minor systems, [45] for the team problem with a Markov jump parameter as common random source, [37] for dynamic collective choice by finding a social optimum, [38] for stochastic dynamic teams and their mean-field limit.

1.2 Motivation and contribution

Most works on mean-field games and control focused on large/infinite population MASs and obtained approximate Nash equilibria. However, in many practical situ-
ations (such as oligopolistic markets), small population or moderate population systems are considered. How about the case of finite population? Which type of Nash equilibria can we obtain? In this paper, we investigate about the case of finite population? Which type of Nash equilibria? Small population age is known, i.e., the centralized or aggregate sharing programming approach. For the finite number of agents, the difficulty is circumvented ingeniously by taking mean field approximation.

In this paper, we construct decentralized strategies by solving nonstandard forward-backward stochastic differential equations (FBSDEs). For finite-population LQG games, we first derive necessary and sufficient conditions for the existence of the open-loop Nash equilibrium in terms of FBSDEs by variational analysis. Due to accessible information restriction, the decentralized Nash equilibrium is given by the conditional expectation of costates (solutions to adjoint equations). This leads to a set of nonstandard FBSDEs. The key step of strategy design is to solve these equations. We first construct auxiliary FBSDEs and establish the equivalent relationship for two set of FBSDEs. By decoupling auxiliary FBSDEs, we design a set of decentralized strategies in terms of two differential Riccati equations. Instead of the ε-Nash equilibrium, the set of decentralized strategies is shown to be an (exact) Nash equilibrium.

For the infinite-horizon problem, we give some criterion for solvability of the algebraic Riccati equation arising from consensus problems. Particularly, for the model of multiple integrators, agents can reach the mean-square consensus. Furthermore, the finite-population team problem is also studied, where the social cost is a weighted sum of individual costs. Under mild conditions, we give the decentralized social optimal control and the corresponding social cost.

1.3 Organization and Notation

The organization of the paper is as follows. In Section 2, we formulate the problems of finite-population mean-field game and control. In Section 3, decentralized Nash equilibria are designed for the finite- and infinite-horizon problems, respectively. Section 4 shows the connection between the proposed decentralized control and the results of classical mean-field games. In Section 5, we consider the team problem. Section 6 gives two numerical examples to verify results. Section 7 concludes the paper.

The following notation will be used throughout this paper. \( \| \cdot \| \) denotes the Euclidean vector norm or matrix spectral norm. For a vector \( z \) and a matrix \( Q, \| z \|_Q^2 = z^T Q z \), and \( Q \succ 0 \) (\( Q \preceq 0 \)) means that \( Q \) is positive definite (semi-positive definite). For two vectors \( x, y \), \( \langle x, y \rangle = x^T y \). \( C([0,T],\mathbb{R}^n) \) is the space of all \( \mathbb{R}^n \)-valued continuous functions defined on \( [0,T] \), and \( C_{\rho/2}([0,\infty),\mathbb{R}^n) \) is a subspace of \( C([0,\infty),\mathbb{R}^n) \) which is given by \( \{ f | \int_0^\infty e^{-\rho t}\| f(t) \|^2 dt < \infty \} \).

2 Problem Description

Consider an MAS with \( N \) agents. The \( i \)th agent evolves by the following stochastic differential equation (SDE):

\[
dx_i(t) = [Ax_i(t) + Bu_i(t) + f(t)] dt + [Cx_i(t) + Du_i(t) + \sigma(t)] dw_i(t),
\]

where \( x_i \in \mathbb{R}^n \) and \( u_i \in \mathbb{R}^r \) are the state and input of the \( i \)th agent, \( i = 1, \ldots, N \), \( f, \sigma \in C_{\rho/2}([0,\infty),\mathbb{R}^n) \) reflect the impact on each agent by the external environment. \( \{ w_i(t), i = 1, \ldots, N \} \) are a sequence of independent 1-dimensional Brownian motions on a complete filtered probability space \( (\Omega, \mathcal{F}, \{ \mathcal{F}_t \}_{t \geq 0} , \mathbb{P}) \). The cost function of agent \( i \) is given by
In this paper, we mainly study the following problems.

J_i(u_i, u_{-i}) = \mathbb{E} \int_0^\infty e^{-\rho t} \left\{ \|x_i(t) - \Gamma x^{(i)}(t)\|^2_{Q_i} + \|u_i(t)\|^2_R \right\} dt,

where Q \geq 0, R > 0 and \Gamma are constant matrices with appropriate dimension; the vector \eta \in C_{\rho/2}([0, \infty), \mathbb{R}^n) and \rho > 0 is a discount factor; x^{(i)}(t) = \sum_{j=1}^N s_j x_j(t) and u_{-i} = \{u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_N\}. Assume that the weight allocation satisfies:

(i) \alpha_j^{(N)} \geq 0, j = 1, \ldots, N;
(ii) \sum_{j=1}^N \alpha_j^{(N)} = 1.

Remark 2.1 The cost \( J_i \) represents the ith agent error of tracking an affine function of the weighted state average. Particularly, for the case \( \Gamma = I, \eta = 0 \), each agent will tend to track the weighted state average. The latter constitutes an optimization paradigm for the consensus problem of MAS [33].

Remark 2.2 Note that if \( \alpha_j^{(N)} = \frac{1}{N}, j = 1, \ldots, N \), then \( x^{(i)}(t) = x^{(N)}(t) = \sum_{j=1}^N \alpha_j^{(N)} x_j(t) \). The weighted summation \( x^{(i)} \) is a generalization of the population state average \( x^{(N)} \), which was commonly considered in the classical mean-field games [19], [5], [44]. For this weighted cost, different agents around an individual may affect differently the individual, and some agents are allowed to be more dominant. The weighted average interaction models appear in many practical issues such as selfish herd behavior of animals [36], lattice models in retailing service [6], deep structured teams [3] and social segregation phenomena [39]. See [22] for the NCE principle with weighted cost interactions.

We start with some definitions. Denote the filtration of agent \( i \) by the \( \sigma \)-subalgebra

\[ \mathcal{F}_t^i = \sigma(x_i(0), w_i(s), 0 \leq s \leq t), \quad i = 1, \ldots, N. \]

The admissible decentralized control set is given by

\[ U_{d,i} = \left\{ u_i \mid u_i(t) \text{ is adapted to } \mathcal{F}_t^i \right\}. \]

Definition 2.1 A set of strategies \( \{\tilde{u}_i \in U_{d,i}, i = 1, \ldots, N\} \) is said to be a decentralized Nash equilibrium if the following holds:

\[ J_i(\tilde{u}_i, \tilde{u}_{-i}) = \inf_{u_i \in U_{d,i}} J_i(u_i, \tilde{u}_{-i}), \quad i = 1, \ldots, N. \]

Furthermore, \( \{\tilde{u}_i \in U_{d,i}, i = 1, \ldots, N\} \) is a decentralized social optimal solution if the following holds:

\[ J_{soc}(\tilde{u}) = \inf_{u_i \in U_{d,i}, 1 \leq i \leq N} J_{soc}(u), \]

where \( J_{soc}(u) = \sum_{i=1}^N J_i(u) \) and \( u = (u_1, \ldots, u_N) \).

In this paper, we mainly study the following problems.

(G) Seek a set of decentralized Nash equilibrium strategies \( \{\tilde{u}_i \in U_{d,i}, i = 1, \ldots, N\} \) with respect to cost functions \( \{J_i, i = 1, \ldots, N\} \) for the system (1)-(2).

(S) Seek a set of decentralized social optimal strategies \( \{\tilde{u}_i \in U_{d,i}, i = 1, \ldots, N\} \) with respect to cost functions \( \{J_i, i = 1, \ldots, N\} \) for the system (1)-(2).

We make the assumption on initial states of agents.

A1) \( x_i(0) = x_{i0}, i = 1, \ldots, N, \) are mutually independent and have the same mathematical expectation \( \tilde{x}_0 \). There exists a constant \( c_0 \) such that \( \max_{1 \leq i \leq N} \mathbb{E}\|x_i(0)\|^2 < c_0 \).

3 Finite-Population Mean-field LQG Games

3.1 The finite-horizon problem

From now on, the time variable \( t \) may be suppressed when no confusion occurs. For the convenience of design, we first consider the following finite-horizon problem:

\[ \begin{aligned}
&(G') \quad dx_i = (Ax_i + Bu_i + f_i)dt + (Cx_i + Du_i + \sigma)dw_i, \\
&J_{t,T}(u_i, u_{-i}) = \mathbb{E} \int_0^T e^{-\rho t} \left[ \|x_i - \Gamma x^{(i)} - \eta\|^2_{Q_i} + \|u_i\|^2_R \right] dt.
\end{aligned} \]

We now obtain some necessary and sufficient conditions for the existence of decentralized Nash equilibrium strategies of Problem (G') by using variational analysis.

Theorem 3.1 (i) If Problem (G') admits a set of decentralized Nash equilibrium strategies \( \{\tilde{u}_i \in U_{d,i}, i = 1, \ldots, N\} \), then the following FBSDE system admits a set of adapted solutions \( (\tilde{x}_i, \tilde{\lambda}_i, \tilde{\beta}^j_i, i = 1, \ldots, N) \):

\[ \begin{aligned}
&d\tilde{x}_i = (A\tilde{x}_i + B\tilde{u}_i + f_i)dt + (C\tilde{x}_i + D\tilde{u}_i + \sigma)dw_i, \\
d\tilde{\lambda}_i = -\left[(A - \rho I)^T \tilde{\lambda}_i + C^T \tilde{\beta}^j_i \right]dt + \sum_{j=1}^N \tilde{\beta}^j_i dw_j \\
&\quad + \left[(I - \alpha_i^{(N)})^T Q(\tilde{x}_i - \Gamma x^{(i)} - \eta)\right]dt, \\
\tilde{x}_i(0) = x_{i0}, \quad \tilde{\lambda}_i(T) = 0, \quad i = 1, \ldots, N.
\end{aligned} \]

with

\[ \tilde{u}_i = -R^{-1} \left(B^T \mathbb{E}[\tilde{x}_i | \mathcal{F}_t^i] + D^T \mathbb{E}[\tilde{\beta}^j_i | \mathcal{F}_t^i]\right). \]

(ii) If the equation system (3) admits a set of solutions \( (\tilde{x}_i, \tilde{\lambda}_i, \tilde{\beta}^j_i, i = 1, \ldots, N) \), then Problem (G') has a set of decentralized Nash equilibrium strategies \( \tilde{u}_i, i = 1, \ldots, N \), which is given by (4).

Proof. See Appendix A.

Remark 3.1 The definition of adapted solutions arises in solving backward SDEs. The adapted solution to the backward SDE in (3) is a sequence of adapted stochastic processes \( (\tilde{\lambda}_i, \tilde{\beta}^j_i, i = 1, \ldots, N) \), and it is the terms \( \tilde{\beta}^j_i, j = 1, \ldots, N \) that correct the possible “nonadaptiveness” caused by the backward nature of (3). See [29] for more details of adapted solutions.

3.1.1 Homogeneous weights

We first consider the case \( \alpha_i^{(N)} = \frac{1}{N} \).
Lemma 3.1 For any $j \neq i$, the following holds:

$$E[\hat{x}_j(r)|F_i^r] = E[\hat{x}_j(r)] = E[\hat{x}_i(r)].$$  \hspace{1cm} (5)

Proof. Note that $\hat{x}_i$ is adapted to $F_i^r$, $i = 1, \ldots, N$. Since $w_i$, $i = 1, \ldots, N$, are mutually independent, by A1 then $\hat{x}_i$, $i = 1, \ldots, N$, are independent of each other. Since all agents have the same parameters, we obtain (5). \hspace{1cm} \Box

By Lemma 3.1, we have

$$E[\hat{x}^N(r)|F_i^r] = \frac{1}{N} \hat{x}_i + \sum_{j \neq i} E[\hat{x}_j] = \frac{1}{N} \hat{x}_i + \frac{N-1}{N} E[\hat{x}_i].$$

It follows from (3)-(4) that \begin{align}

\rho_{K_N} &= K_N + A^T K_N + K_N A - (B^T K_N + D^T K_N C)^T
\times \Upsilon_N^{-1} (B^T K_N + D^T K_N C) + C^T K_N C
\times \left( I - \frac{1}{N} \Gamma \right)^T Q \left( I - \frac{1}{N} \Gamma \right), \quad K_N(T) = 0, \quad (7)

\rho_{\Pi_N} &= \Pi_N + A^T \Pi_N + \Pi_N A - \Pi_N B \Upsilon_N^{-1} B^T \Pi_N
\times \left( I - \frac{1}{N} \Gamma \right)^T Q \left( I - \frac{1}{N} \Gamma \right), \quad \Pi_N(T) = 0, \quad (8)

$$

\rho_{s_N} = s_N + \left[ A - B \Upsilon_N^{-1} B^T (K_N + \Pi_N) + D^T K_N C \right] s_N
\times \left( I - \frac{1}{N} \Gamma \right)^T Q, \quad s_N(T) = 0, \quad (9)

with \Upsilon_N \triangleq R + D^T K_N D.

Lemma 3.2 If (7)-(9) have a solution, respectively, then (3) admits a set of adapted solutions.

Proof. Let

$$\tilde{s}_N(t) = E[\hat{\lambda}_i(t)|F_i^t] - K_N(t) \hat{x}_i(t) - \Pi_N(t) E[\hat{x}_i(t)],$$  \hspace{1cm} (10)

where $K_N(t)$ and $\Pi_N(t)$ satisfy (7) and (8), respectively. Here, $\tilde{s}_N(t) \in \mathbb{R}^n$ may depend on $i$. However, we will show later that $\tilde{s}_N(t)$ is actually independent of $i$. It follows by (10) that $E[\hat{\lambda}_i(t)] = (K_N(t) + \Pi_N(t)) E[\hat{x}_i(t)] + \tilde{s}_N(t)$.

By Itô's formula, we obtain

$$dE[\hat{\lambda}_i|F_i^t] = K_N \hat{x}_i dt + K_N [(A \hat{x}_i + B \tilde{u}_i + f) dt]
+ (C \hat{x}_i + D \tilde{u}_i + \sigma) d\omega_i + \Pi_N E[\hat{x}_i] dt + \Pi_N [A E[\hat{x}_i] + B E[\tilde{u}_i] + f] dt + \tilde{s}_N dt.$$  \hspace{1cm} (11)

Comparing this with (6) and equating the $d\omega_i$ terms, it follows that $E[\hat{\lambda}_i|F_i^t] = K_N(C \hat{x}_i + D \tilde{u}_i + \sigma)$. By (4) and (10), we have

$$R \tilde{u}_i + B^T (K_N \hat{x}_i + \Pi_N E[\hat{x}_i] + \tilde{s}_N)$$

$$+ D^T K_N (C \hat{x}_i + D \tilde{u}_i + \sigma) = 0,$$

which leads to

$$\tilde{u}_i = - (R + D^T K_N D)^{-1} [(B^T K_N + D^T K_N C) \hat{x}_i + B^T \Pi_N E[\hat{x}_i] + B^T \tilde{s}_N + D^T K_N \sigma].$$  \hspace{1cm} (12)

Then

$$E[\tilde{u}_i] = - \Upsilon^{-1} [(B^T K_N + B^T \Pi_N + D^T K_N C) E[\hat{x}_i]
+ B^T \tilde{s}_N + D^T K_N \sigma].$$  \hspace{1cm} (13)

Applying (12)-(13) to (6) and (11), and comparing them, we obtain that $\tilde{s}_N$ satisfies (9). Here, the $\tilde{x}_i$ terms equalling 0 is obtained from (7), and the $E[\hat{x}_i]$ terms equalling 0 from (8). From (10), FBSDE (6) is solvable. By Proposition 3.1, (3) admits a set of adapted solutions.

\Box

By Lemma 3.2 and (12), we obtain the following decentralized strategies for $N$ agents,

$$\tilde{u}_i = - \Upsilon^{-1} [(B^T K_N + D^T K_N C) \hat{x}_i + B^T \Pi_N E[\hat{x}_i]
+ B^T \tilde{s}_N + D^T K_N \sigma], \quad i = 1, \ldots, N,$$

where $K_N, \Pi_N, \tilde{s}_N$ are given by (7)-(9), and $E[\hat{x}_i]$ obeys

$$dE[\hat{x}_i] = A - B \Upsilon_N^{-1} B^T (K_N + \Pi_N + D^T K_N C)$$

$$\times E[\hat{x}_i] dt - B \Upsilon_N^{-1} (B \tilde{s}_N + D^T K_N \sigma) dt + f dt,$$

$$E[\hat{x}_i(0)] = \tilde{x}_0.$$  \hspace{1cm} (15)
Remark 3.2 In previous works (e.g., [19], [27]), the mean-field term $x^{(N)}$ in cost functions is first substituted by a deterministic function $\tilde{x}$. By handling the fixed-point equation, $\tilde{x}$ is obtained, and the decentralized control is constructed. Here, we first obtain the solvability condition of decentralized control by virtue of variational analysis, and then design decentralized control laws by tackling FBSDEs and conditional Hamiltonians. Note that in this case $s_n$ and $\mathbb{E}[\tilde{x}_i]$ are decoupled and no fixed-point equation is needed.

Next, we study whether that above proposed strategy gives a decentralized Nash equilibrium. For the analysis, we introduce the following assumption.

**A2** Equation (8) admits a solution in $C([0, T], \mathbb{R}^{n \times n})$.

**Theorem 3.2** Let A1-A2 hold. Then for Problem $(G^r)$, the set of control laws $\{\tilde{u}_1, \cdots, \tilde{u}_N\}$ given by (14) is a decentralized Nash equilibrium, i.e.,

$$ J_i, T(\tilde{u}_i, \tilde{u}_-i) = \inf_{u_i \in \mathcal{U}_{u_i}} J_i, T(u_i, \tilde{u}_-i). $$

**Proof.** Note that (7) admits a solution in $C([0, T], \mathbb{R}^{n \times n})$ since $\left(I - \frac{1}{N}\Gamma\right)^T Q \left(I - \frac{1}{N}\Gamma\right) \geq 0$. Under A2, (9) has a solution in $C([0, T], \mathbb{R}^{n \times n})$. By Lemma 3.2, (3) admits a set of adapted solutions, which together with Theorem 3.1 completes the proof of the theorem. $\square$

**Remark 3.3** In this paper, we consider the case $Q \geq 0$ and $R > 0$. Indeed, even if $Q$ and $R$ are indefinite, under some convex conditions, we may design the decentralized strategies and obtain the optimality result. (See e.g., [40])

By [29, P. 48], A2 holds if and only if

$$ \det \begin{bmatrix} I & e^{At} \end{bmatrix} > 0, \quad \forall t \in [0, T], $$

where $\tilde{A}_N \triangleq A - B\tilde{Y}_N^{-1}(B^T K_N + D^T K_N C)$ and

$$ A \triangleq \begin{bmatrix} A_N - \frac{1}{N} I & -B\tilde{Y}_N^{-1} B^T \\ \frac{N}{N-1} \left(I - \frac{1}{N}\Gamma\right)^T Q \left(I - \frac{1}{N}\Gamma\right) & -\left(A_N - \frac{1}{N} I\right)^T \end{bmatrix}. $$

Particularly, if $\Gamma = I$, we have the following result.

**Proposition 3.2** If $\Gamma = I$, then A2 holds necessarily. $\square$

**Proof.** See Appendix A.

### 3.1.2 Heterogeneous weights

We now consider the situation that some agent plays a dominant role in the system. Specifically, the weights in $x^{(i)}$ are taken as $\alpha_1 = \alpha$, and $\alpha_i = \frac{\alpha}{N}$, $i = 2, \cdots, N$, $0 < \alpha < 1$. For simplicity, consider the case where $f = \sigma = \eta = 0$.

Note that $\mathbb{E}[\tilde{x}_2] = \cdots = \mathbb{E}[\tilde{x}_N] \neq \mathbb{E}[\tilde{x}_1]$. By Lemma 3.1, we obtain that for $i = 1$, $\mathbb{E}[\tilde{x}^{(i)}|F^i_t] = \alpha \tilde{x}_1 + (1 - \alpha)\mathbb{E}[\tilde{x}_j] (j \neq 1)$, and $i \neq 1$,

$$ \mathbb{E}[\tilde{x}^{(i)}|F^i_t] = \frac{1 - \alpha}{N-1} \tilde{x}_i + \alpha \mathbb{E}[\tilde{x}_1] + \frac{N-2}{N-1}(1 - \alpha)\mathbb{E}[\tilde{x}_i]. $$

It follows from (3) that

$$ d\mathbb{E}[\tilde{x}_i] = (AE[\tilde{x}_i] + BE[\tilde{u}_i])dt, \quad \mathbb{E}[\tilde{x}(0)] = \tilde{x}_0, $$

$$ i = 1, \cdots, N, $$

$$ d\mathbb{E}[\tilde{x}^{(i)}|F^i_t] = -\left[(A - \rho I)^T \mathbb{E}[\tilde{x}^{(i)}|F^i_t] + C^T \mathbb{E}[\tilde{x}^{(i)}|F^i_t] \right]dt $$

$$ + (I - \alpha \Gamma)^T Q (I - \alpha \Gamma) \tilde{x}_1 - (1 - \alpha \Gamma) \mathbb{E}[\tilde{x}_1])dt $$

$$ + \mathbb{E}[\tilde{\beta}_i^{(i)}|F^i_t]dw_i, \quad \mathbb{E}[\tilde{x}^{(i)}|F^i_t] = \tilde{x}_0. $$

$$ d\mathbb{E}[\tilde{x}^{(i)}|F^i_t] = -\left[(A - \rho I)^T \mathbb{E}[\tilde{x}^{(i)}|F^i_t] + C^T \mathbb{E}[\tilde{x}^{(i)}|F^i_t] \right]dt $$

$$ + (I - \frac{1 - \alpha}{N-1}\Gamma)^T \mathbb{E}[\tilde{x}^{(i)}|F^i_t]dt $$

$$ + \mathbb{E}[\tilde{\beta}_j^{(i)}|F^i_t]dw_j, \quad \mathbb{E}[\tilde{x}^{(i)}|F^i_t] = \tilde{x}_0, \quad j = 2, \cdots, N. $$

We now decouple (16) by using the idea of Lemma 3.2 (see also [29,33]). Let

$$ \mathbb{E}[\tilde{x}^{(i)}|F^i_t] = \tilde{K}_N^{(i)} \tilde{x}_1 + \tilde{\Pi}_N^{(i)} \mathbb{E}[\tilde{x}_1] + \tilde{\Pi}_N^{1, j} \mathbb{E}[\tilde{x}_j], \quad j \neq 1, $$

where $\tilde{K}_N^{(i)}, \tilde{\Pi}_N^{1, i}, \tilde{\Pi}_N^{1, j} \in \mathbb{R}^{n \times n}$. Then by Itô’s formula,

$$ d\mathbb{E}[\tilde{x}^{(i)}|F^i_t] = \tilde{K}_N^{(i)} \tilde{x}_1 dt + \tilde{\Pi}_N^{(i)} [(A \tilde{x}_1 + D \tilde{u}_1) dt $$

$$ + (C \tilde{x}_1 + D \tilde{u}_1) dw_i] + \tilde{\Pi}_N^{(i)} \mathbb{E}[\tilde{x}_1] $$

$$ + \tilde{\Pi}_N^{1, j} (AE[\tilde{x}_1] + BE[\tilde{u}_1]) + \tilde{\Pi}_N^{1, j} \mathbb{E}[\tilde{x}_j] ] dt $$

$$ + \tilde{\Pi}_N^{1, j} (AE[\tilde{x}_j] + BE[\tilde{u}_j]) dt. $$

Comparing this with (16), it follows that $\mathbb{E}[\tilde{\beta}_i^{(i)}|F^i_t] = \tilde{K}_N^{(i)} (C \tilde{x}_1 + D \tilde{u}_1)$. By (4), we have

$$ R \tilde{u}_i + B^T (\tilde{K}_N^{(i)} \tilde{x}_1 + \tilde{\Pi}_N^{(i)} [\mathbb{E}[\tilde{x}_1] + \tilde{\Pi}_N^{1, j} \mathbb{E}[\tilde{x}_j]]) $$

$$ + D^T \tilde{K}_N^{(i)} (C \tilde{x}_1 + D \tilde{u}_1) = 0. $$

This leads to

$$ \tilde{u}_i = - (\tilde{T}_N^{(i)})^{-1} [(B^T \tilde{K}_N^{(i)} + D^T \tilde{K}_N^{(i)} C) \tilde{x}_1 $$

$$ + B^T \tilde{\Pi}_N^{1, i} \mathbb{E}[\tilde{x}_1] + B^T \tilde{\Pi}_N^{1, j} \mathbb{E}[\tilde{x}_j]], \quad j \neq 1, $$

where $\tilde{T}_N^{(i)} \triangleq R + D^T \tilde{K}_N^{(i)} D$.

Let $\mathbb{E}[\tilde{x}^{(i)}|F^i_t] = \tilde{K}_N^{(i)} \tilde{x}_1 + \tilde{\Pi}_N^{1, i} \mathbb{E}[\tilde{x}_1] + \tilde{\Pi}_N^{1, j} \mathbb{E}[\tilde{x}_j], 2 \leq j \leq N$. 


\( N \), where \( \bar{K}_N^i, \Pi_N^{i1}, \Pi_N^{i2} \in \mathbb{R}^{n \times n} \). Then by Itô’s formula,

\[
d\mathbb{E}[\xi_j | F_t] = \dot{\Pi}_N^{i1}(\mathbb{E}[\hat{x}_j] + B \mathbb{E} \hat{u}_j) dt + \Pi_N^{i1}(\mathbb{E}[\dot{\hat{x}}_j] + B \mathbb{E}[\dot{\hat{u}}_j] + D \mathbb{E} \hat{u}_j) dt.
\]

Comparing this with (16), it follows that \( \mathbb{E}[\beta_j^1 | F_t] = \bar{K}_N^i(C \hat{x}_j + D \hat{u}_j) \). This together with (A.2) gives

\[
\begin{align*}
R \hat{u}_j + B^T(\bar{K}_N^i \hat{x}_j + \Pi_N^{i1} \mathbb{E}[\hat{x}_j]) + D^T \bar{K}_N^i(\mathbb{E}[\hat{u}_j] + D \hat{u}_j) &= 0.
\end{align*}
\]

This leads to

\[
\dot{u}_j = -(\bar{T}_N^j)^{-1}[(B^T \bar{K}_N^i + D^T \bar{K}_N^i C) \hat{x}_j + B^T \bar{\Pi}_N^{i1} \mathbb{E}[\hat{x}_j] + B^T \bar{\Pi}_N^{i2} \mathbb{E}[\hat{x}_j]],
\]

where \( \bar{T}_N^j \triangleq R + D^T \bar{K}_N^i D, 2 \leq j \leq N \). Applying (18) and (21) to (16)-(17) and comparing them, it follows that

\[
\begin{align*}
\rho \bar{K}_N^i &= \dot{K}_N^i + A^T \bar{K}_N^i + \bar{K}_N^i A - (B^T \bar{K}_N^i + D^T \bar{K}_N^i C) \bar{T}_N^j(\bar{T}_N^j)^{-1} B^T \bar{\Pi}_N^{i1} C + C^T \bar{K}_N^i C \\
&\quad + (I - \alpha \Gamma)^T Q(I - \alpha \Gamma), \\
\bar{K}_N^i(T) &= 0. \tag{22}
\end{align*}
\]

\[
\begin{align*}
\rho \bar{\Pi}_N^{i1} &= \dot{\Pi}_N^{i1} + A^T \bar{\Pi}_N^{i1} + \bar{\Pi}_N^{i1} A - \bar{\Pi}_N^{i1} B(\bar{T}_N^j)^{-1} B^T \bar{\Pi}_N^{i1} C + B^T \bar{\Pi}_N^{i1} C \\
&\quad - \bar{\Pi}_N^{i1} B(\bar{T}_N^j)^{-1} B^T \bar{\Pi}_N^{i1} C + B^T \bar{\Pi}_N^{i2} C + C^T \bar{\Pi}_N^{i1} C \\
&\quad - \bar{\Pi}_N^{i1} B(\bar{T}_N^j)^{-1} B^T \bar{\Pi}_N^{i1} C, \\
\bar{\Pi}_N^{i1}(T) &= 0. \tag{23}
\end{align*}
\]

\[
\begin{align*}
\rho \bar{\Pi}_N^{i2} &= \dot{\Pi}_N^{i2} + A^T \bar{\Pi}_N^{i2} + \bar{\Pi}_N^{i2} A - (B^T \bar{K}_N^i + D^T \bar{K}_N^i C) \bar{T}_N^j(\bar{T}_N^j)^{-1} B^T \bar{\Pi}_N^{i1} C \\
&\quad + B^T \bar{\Pi}_N^{i1} C + C^T \bar{\Pi}_N^{i2} C + B^T \bar{\Pi}_N^{i2} C \\
&\quad + (1 - \alpha)(I - \alpha \Gamma)^T Q(I - \alpha \Gamma), \\
\bar{\Pi}_N^{i2}(T) &= 0. \tag{24}
\end{align*}
\]

In the above, (22) is obtained by equating the \( \hat{x}_j \) terms; (23) by equating the \( \mathbb{E}[\hat{x}_j] \) terms and (24) by equating the \( \mathbb{E}[\hat{x}_j] \) terms. After strategies (18) and (21) are applied, comparing (16) and (19), it follows that

\[
\rho \bar{K}_N^i = \dot{K}_N^i + A^T \bar{K}_N^i + \bar{K}_N^i A - (B^T \bar{K}_N^i + D^T \bar{K}_N^i C)^T \bar{T}_N^j(\bar{T}_N^j)^{-1} (B^T \bar{K}_N^i + D^T \bar{K}_N^i C) + C^T \bar{K}_N^i C \\
+ (I - \frac{1 - \alpha}{N - 1} \Gamma)^T Q(I - \frac{1 - \alpha}{N - 1} \Gamma) = 0, \\
\bar{K}_N^i(T) = 0, \tag{25}
\]

\[
\rho \bar{\Pi}_N^{i1} = \dot{\Pi}_N^{i1} + A^T \bar{\Pi}_N^{i1} + \bar{\Pi}_N^{i1} A - \bar{\Pi}_N^{i1} B(\bar{T}_N^j)^{-1} B^T \bar{\Pi}_N^{i1} C \\
- (B^T \bar{K}_N^i + D^T \bar{K}_N^i C)^T(\bar{T}_N^j)^{-1} B^T \bar{\Pi}_N^{i1} C \\
- \bar{\Pi}_N^{i1} B(\bar{T}_N^j)^{-1} B^T \bar{\Pi}_N^{i1} C + \bar{\Pi}_N^{i1} B(\bar{T}_N^j)^{-1} B^T \bar{\Pi}_N^{i1} C \\
- \bar{\Pi}_N^{i1} B(\bar{T}_N^j)^{-1} B^T \bar{\Pi}_N^{i1} C - \bar{\Pi}_N^{i1} B(\bar{T}_N^j)^{-1} B^T \bar{\Pi}_N^{i2} C \\
- \frac{N - 2}{N - 1} (1 - \alpha)(I - \frac{1 - \alpha}{N - 1} \Gamma)^T Q \bar{\Pi}_N^{i1}(T) = 0. \tag{26}
\]

**Theorem 3.3** *Assume that A1 holds. For Problem (G’), if (23)-(24) and (26)-(27) admit solutions, respectively, then the set of strategies given by (18) and (21) is a decentralized Nash equilibrium.*

**Proof.** Note that \( (I - \alpha \Gamma)^T Q(I - \alpha \Gamma) \geq 0 \) and \( (I - \frac{1 - \alpha}{N - 1} \Gamma)^T Q(I - \frac{1 - \alpha}{N - 1} \Gamma) \geq 0 \). Then (22) and (25) admit solution \( \bar{K}_N^i \geq 0 \) and \( \bar{K}_N^i \geq 0 \), respectively. If (23)-(24) and (26)-(27) admit solutions, respectively, then by the derivation above, we obtain that (3) admits a set of adapted solutions. By Theorem 3.1, the set of strategies in (18) and (21) is a decentralized Nash equilibrium. \( \square \)

### 3.2 The infinite-horizon problem

In this section, we consider the infinite-horizon problem with homogeneous weights (\( \alpha_i^{(N)} = \frac{1}{N} \)). Based on the analysis in Section 3.1, we may design the following decentralized control for Problem (G):

\[
\begin{align*}
\hat{u}_i(t) &= -\mathcal{Y}_N^{-1}(B^T K_N + D^T K_N C) \hat{x}_i(t) \\
&\quad + B^T (P_N - K_N \mathbb{E}[\hat{x}_i(t)] + B^T s_N(t)) \\
&\quad + D^T K_N \sigma(t), \quad t \geq 0, \quad i = 1, \ldots, N,
\end{align*}
\]
where \( K_N \) and \( P_N \) satisfy

\[
\rho K_N = A^T K_N + K_N A - (B^T K_N + D^T K_N C)\hat{Y}_N^{-1} K_N C + (I - \frac{1}{N}\Gamma)^T Q(I - \frac{1}{N}\Gamma),
\]

\[
\rho P_N = A^T P_N + P_N A - (B^T P_N + D^T K_N C)\hat{Y}_N^{-1} K_N C + (I - \frac{1}{N}\Gamma)^T Q(I - \frac{1}{N}\Gamma),
\]

(29)

and \( s_N \in C_{\rho/2}([0, \infty), \mathbb{R}^n) \) is determined by

\[
\rho s_N = \dot{s}_N + [A - BY_N^{-1}(B^T P_N + D^T K_N C)]s_N + [C - DY_N^{-1}(B^T P_N + D^T K_N C)]K_N \sigma + P_N f - (I - \frac{1}{N}\Gamma)^T Q \eta.
\]

(31)

Example 3.1 Consider a two-dimensional system with \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \), \( C = 0 \) and \( \Gamma = I \). Denote \( \bar{a} = a - \frac{\bar{f}}{2} \) and \( \bar{d} = d - \frac{\bar{f}}{2} \). We have \( |\lambda - (A - \frac{\bar{f}}{2}I)| = \lambda^2 - (\bar{a} + \bar{d})\lambda + \bar{a}\bar{d} - \bar{b}c \). Then \( A - \frac{\bar{f}}{2}I \) has eigenvalues on the imaginary axis if and only if \( \bar{a} + \bar{d} = 0 \) and \( \bar{a}\bar{d} - \bar{b}c \geq 0 \). Thus, if \( \bar{a} + \bar{d} \neq 0 \) or \( \bar{a}\bar{d} - \bar{b}c < 0 \), then \( A - \frac{\bar{f}}{2}I \) has no eigenvalues on the imaginary axis, which with \( \Gamma = I \) and \( C = 0 \) implies that \( A4) \) holds. Particularly, if \( A = 0 \), then \( A4) \) holds.

The next theorem characterizes the performance of the decentralized strategies.

Theorem 3.4 Assume \( A1), A3), A4) \) hold and \( N \) is sufficiently large such that \( I - \frac{1}{N}\Gamma \) is nonsingular. For Problem (G), the set of strategies \( \{\bar{u}_1, \cdots, \bar{u}_N\} \) given by (28) is a decentralized Nash equilibrium, i.e., for any \( i = 1, \cdots, N \), \( J_i(\bar{u}_i, \bar{u}_{-i}) = \inf u_i \in U_i J_i(u_i, \bar{u}_{-i}) \).

Proof. See Appendix B. \( \square \)

3.2.1 The model of noisy multiple integrators

For the case \( A = C = 0 \), \( \Gamma = B = D = I \), and \( f = \eta = 0 \), the system (1)-(2) reduces to the model of noisy multiple integrators. Specifically, agent \( i \) evolves by

\[
x_i(t) = u_i dt + \bar{u}_i dw_i, \quad i = 1, \cdots, N,
\]

(32)

and the cost function is given by

\[
J_i = \mathbb{E} \int_0^\infty e^{-\alpha t} \{\|x_i(t)\|_Q^2 + \|u_i(t)\|_R^2\} dt,
\]

(33)

where \( x_i \in \mathbb{R}^n \), \( u_i \in \mathbb{R}^m \), \( Q > 0 \) and \( R > 0 \).

By Proposition 3.3, it can be verified that \( A3)-A4) \) hold and (29) admits a solution \( K_N > 0 \). Furthermore, we have the following result.

Proposition 3.4 (i) (30) admits a unique \( \rho \)-stabilizing solution \( P_N = 0 \);

(ii) (31) admits a unique bounded solution \( s_N(t) \equiv 0 \);

(iii) \( \mathbb{E}[\bar{x}_i(t)] \equiv \bar{x}_0 \).

Proof. For the model (32)-(33), (30) degenerates to \( \rho P_N = -P_N \hat{Y}_N^{-1} P_N \). It can be verified that \( P_N = 0 \) is a unique \( \rho \)-stabilizing solution. Note that (31) reduces to \( \dot{s}_N = \rho s_N \). This with \( s_N \in C_{\rho/2}([0, \infty), \mathbb{R}^n) \) implies \( s_N(t) \equiv 0 \) for any \( t \geq 0 \). Thus, we have \( \mathbb{E}[\bar{x}_i(t)] \equiv \bar{x}_0. \square \)

For the model (32)-(33), the decentralized strategies may be given as follows:

\[
\bar{u}_i(t) = -\hat{Y}_N^{-1} B^T K_N (\bar{x}_i(t) - \bar{x}_0), \quad i = 1, \cdots, N.
\]

(34)

Substituting (34) into (32), the closed-loop dynamics of agent \( i \) can be written as

\[
d\bar{x}_i(t) = -B^T K_N (\bar{x}_i(t) - \bar{x}_0) dt
\]

\[
-\hat{Y}_N^{-1} B^T K_N (\bar{x}_i(t) - \bar{x}_0) dw_i(t).
\]

(35)
It can be shown that all the agents can achieve mean-square consensus.

**Definition 3.1** In a multiagent system, the agents are said to reach the mean-square consensus if there exists a random variable $x^*$ such that $\lim_{t \to \infty} \mathbb{E}[|x_i(t) - x^*|^2] = 0.$

**Theorem 3.5** For the model (32)-(33), all the agents reach mean-square consensus. Specifically, under the strategy (34), there exist $c_1, c_2 > 0$ such that

$$\mathbb{E}[|\tilde{x}^{(N)}(t) - \tilde{x}_0|^2] \leq c_1 \epsilon^{-c_2 t}, \quad 1 \leq i \leq N.$$ 

**Proof.** See Appendix B. \qed

4 Comparison and Discussion

4.1 Comparison with classical mean-field games

We now review the classical results of mean-field games for comparison to this work. Consider the large-population case of Problem (G) with $\alpha_i^{(N)} = \frac{1}{N}$. By the mean-field (NCE) approach $[19, 42]$, the following decentralized strategies are obtained

$$u^*_i = -\mathcal{Y}^{-1}((B^T K + D^T K C)x_i + B^T f + D^T K \sigma),$$

where $i = 1, \ldots, N$, $\mathcal{Y} = R + D^T K D$ and $K$ is the unique solution of the differential equation

$$\rho K = A^T K + KA + C^T KC - (B^T K + D^T K C)^T \times \mathcal{Y}^{-1}(B^T K + D^T K C) + Q, \quad K(t) = 0,$$

where $\rho$ is determined by the fixed-point equation

$$\left\{ \begin{array}{l}
\frac{d\phi}{dt} = -[A - BY^{-1}(B^T K + D^T K C) - \rho I]^T \phi \\
- K f + QT(\tilde{x} + \eta), \\
\phi(T) = 0,
\end{array} \right. \quad \phi(T) = 0, \quad \phi(t) = \frac{d\tilde{x}}{dt} = [A - BY^{-1}(B^T K + D^T K C)]\tilde{x} - BKY^{-1}B^T f + f, \quad \tilde{x}(0) = \tilde{x}_0.
$$

Such set of decentralized strategies (36) is further shown to be an $\epsilon$-Nash equilibrium with respect to $U_{d,i}$, i.e.,

$$J_i(u_i, u_{-i}) \leq \inf_{u_i \in U_{d,i}} J_i(u_i, u_{-i} - \epsilon),$$

where $U_{d,i} = \{u_i, |u_i(t)| \leq \sigma \{ \cup_{i=1}^N F_i \} \}$ and $\epsilon = O(1/\sqrt{N}).$

Let $s(t) = \phi(t) - \Pi\tilde{x}(t)$, where $\Pi \in \mathbb{R}^{n \times n}$. Then

$$\dot{s} = \Pi [(A - BY^{-1}(B^T K + D^T K C))\tilde{x} - BKY^{-1}B^T(\Pi\tilde{x} + s)] + \dot{s}$$

$$= - (A - BY^{-1}(B^T K + D^T K C) - \rho I)^T(\Pi\tilde{x} + s) - K f + Q(\Pi\tilde{x} + \eta),$$

where the second equation follows from (38). Comparing the terms in the equation above gives

$$\rho \Pi = \Pi (A - BY^{-1}(B^T K + D^T K C))$$

$$+ (A - BY^{-1}(B^T K + D^T K C))^T \Pi$$

$$- \Pi BY^{-1}B^T K - Qf,$$

$$\rho s = \dot{s} + [A - BY^{-1}(B^T K + B^T \Pi + D^T K C)]^T s$$

$$+ (K + \Pi f) - Q\eta.$$  

(40)

We have the following result.

**Proposition 4.1** Assume that $\alpha_i^{(N)} = \frac{1}{N}$ for any $i = 1, \ldots, N$. Then A2 holds for all sufficiently large $N$ and $\|K_N\| + \|\Pi_N\| + \|s_N\| < \infty$ if and only if (39) admits a solution in $C([0, T], \mathbb{R}^{n \times n})$. Furthermore, we have

$$\|K_N - K\| + \|\Pi_N - \Pi\| + \|s_N - s\| = O(1).$$

(41)

**Proof.** If (39) admits a solution, then by the continuous dependence of the solution on the parameter $1/N$ (see e.g., [13]), there exists $N_0 \geq 1$ such that (8) admits a solution for all $N \geq N_0$ (i.e., A2 holds), and (41) is established. This implies $\|K_N\| + \|\Pi_N\| + \|s_N\| < \infty.$

On the other hand, if Assumption A2 holds for all sufficiently large $N$ and $\|K_N\| + \|\Pi_N\| + \|s_N\| < \infty$, then $(\Pi_N, N \geq 1)$ are uniformly bounded. Then there exists a subsequence $(\Pi_{N_k}, k \geq 1)$ such that $\Pi_{N_k}$ converges to $\Pi$ when $k \to \infty.$ It can be verified that $\Pi$ satisfies (39). Thus, (39) admits a solution. \qed

**Remark 4.1** The set of decentralized strategies (14) is an exact Nash equilibrium with respect to $U_{d,i}, i = 1, \ldots, N$. It is applicable for arbitrary number of agents.

In contrast, the set of decentralized strategies (36) is an asymptotic Nash equilibrium with respect to $U_{d,i}, i = 1, \ldots, N$. It is only applicable for the large population case. However, the control gains of (14) and (36) coincide for the infinite population case.

4.2 Computation of $\mathbb{E}[\tilde{x}_i]$ using consensus

The decentralized strategies (14) actually involves coupling between agents due to the fact that $\mathbb{E}[\tilde{x}_i]$ satisfies (15) which requires the averaged initial condition $\tilde{x}_0$.

For the infinite population case, the classical method is to compute $\tilde{x}_0 = \mathbb{E}[x_i(0)]$ by applying the statistical distribution of $x_i(0)$ or Monte-Carlo simulation. For the finite population case, we may use the average consensus algorithm to obtain $x^{(N)}(0).$ Note that the asymptotic behavior of $\mathbb{E}[\tilde{x}_i]$ is not affected by the initial $\tilde{x}_0$.

Suppose there exist local interactions among all agents. Let a graph $(V, E)$ be given, where $V = \{1, 2, \ldots, N\}$ is the set of vertices, and $E = V \times V$ is the set of edges. Denote the set of neighbors of agent $i$ by $N_i = \{j \in V | (i, j) \in E \}$. Given $x_i(0), i = 1, \ldots, N$, we may utilize the average consensus algorithm to obtain $x^{(N)}(0):$

$$y_i(k+1) = y_i(k) + \sum_{j \in N_i} l_{ij}(y_j(k) - y_i(k)), \quad y_i(0) = x_i(0),$$
where \( i = 1, \cdots, N \), and \( L = (\delta_{ij}) \) is the corresponding Laplacian matrix. If \((V, E)\) is connected or has a spanning tree, then \( y_i(k) \) will converge to \( x^{(N)}(t) \), as \( k \to \infty \). See e.g. [25, 34] for more details. Applying a belief propagation (BP)-like distributed algorithm, the consensus will be reached with a fast convergence rate [51].

5 Finite-population Mean-field LQG Teams

In this section, we study the mean-field LQG social control problem with a finite number of agents. Both finite-horizon and infinite-horizon problems will be discussed.

5.1 The finite-horizon problem

We first consider the finite-horizon social problem. 

\((S')\): Minimize social cost \( J_{soc,T}(u) = \sum_{i=1}^{N} \alpha_i^{(N)} J_{i,T} \), where

\[
J_{i,T}(u) = \mathbb{E} \int_0^T e^{-\gamma t} \left[ \|x_i - \Gamma x^{(N)} - \bar{\eta}\|^2_2 + \|u_i\|^2_2 \right] dt,
\]

\[
dx_i = (A x_i + B u_i + f) dt + (C x_i + D u_i + \sigma) dw_i.
\]

Denote

\[
Q_{\bar{\eta}} \triangleq \Gamma^T \bar{Q} + \bar{Q} \Gamma - \Gamma^T Q, \quad \bar{\eta} \triangleq Q_{\bar{\eta}} - \Gamma^T \bar{Q} \eta.
\]

**Theorem 5.1** The problem \((S')\) has a set of decentralized social optimal strategies \( \hat{u}_i \in U_{d,i} \), \( i = 1, \cdots, N \), if and only if the following equation system admits a set of solutions \( (\hat{x}_i, \hat{\lambda}_i, \hat{\beta}_i, i = 1, \cdots, N) \):

\[
\begin{aligned}
d\hat{x}_i &= (Ax_i + B \hat{u}_i + f) dt + (C x_i + D u_i + \sigma) dw_i, \\
d\hat{\lambda}_i &= -[(A - \rho I)^T \hat{\lambda}_i + C^T \hat{\beta}_i + Q \hat{x}_i - \Gamma \hat{x}^{(N)} - \bar{\eta}] dt \\
&\quad + \sum_{j=1}^{N} \hat{\beta}_i dw_j, \\
x_i(0) &= x_{i0}, \quad \hat{\lambda}_i(T) = 0, i = 1, \cdots, N,
\end{aligned}
\]

with

\[
\hat{u}_i = -R^{-1} \left( B^T \mathbb{E}[\hat{\lambda}_i | F_i^t] + C^T \mathbb{E}[\hat{\beta}_i | F_i^t] \right).
\]

**Proof.** See Appendix C. \( \square \)

For simplicity, consider the case \( \alpha_i^{(N)} = \frac{1}{N} \) later. Recall

\[
\mathbb{E}[\hat{x}^{(N)} | F_i^t] = \frac{1}{N} \hat{x}_i + \frac{N-1}{N} \mathbb{E}[\hat{x}_i].
\]

It follows from (42) that

\[
\begin{aligned}
d\mathbb{E}[\hat{x}_i] &= (A \mathbb{E}[\hat{x}_i] + B \mathbb{E}[\hat{u}_i] + f) dt, \quad \mathbb{E}[\hat{x}_i(0)] = \bar{x}_0, \\
d\mathbb{E}[\hat{\lambda}_i | F_i^t] &= -[(A - \rho I)^T \mathbb{E}[\hat{\lambda}_i | F_i^t] + (Q - \frac{1}{N} \Gamma \mathbb{E}[\hat{x}_i]) \hat{x}_i \\
&\quad - \frac{N-1}{N} \mathbb{E}[\hat{x}_i | \Gamma] dt - C^T \mathbb{E}[\hat{\beta}_i | F_i^t] dt \\
&\quad + \mathbb{E}[\hat{\beta}_i | F_i^t] dw_i, \quad \mathbb{E}[\hat{\lambda}_i(T) | F_i^t] = 0.
\end{aligned}
\]

Let \( \mathbb{E}[\hat{\lambda}_i(t) | F_i^t] = \hat{K}_N(t) \hat{x}_i(t) + \hat{\Pi}_N(t) \mathbb{E}[\hat{x}_i(t)] + \hat{s}_N(t) \), where \( \hat{K}_N(t), \hat{\Pi}_N(t) \in \mathbb{R}^{N \times N} \) and \( \hat{s}_N(t) \in \mathbb{R}^N \). By Itô’s formula, we obtain

\[
d\mathbb{E}[\hat{\lambda}_i(t) | F_i^t] = \hat{K}_N \hat{x}_i dt + \hat{\Pi}_N [(Ax_i + B \hat{u}_i + f) dt + (C \hat{x}_i + D \hat{u}_i + \sigma) dw_i] + \hat{\Pi}_N \mathbb{E}[\hat{x}_i] dt + \hat{\Pi}_N [A \mathbb{E}[\hat{x}_i] + B \mathbb{E}[\hat{u}_i] + f] dt + \hat{s}_N dt.
\]

Comparing this with (44), it follows that \( \mathbb{E}[\hat{\beta}_i(t) | F_i^t] = \hat{K}_N \hat{x}_i dt + \hat{\Pi}_N \mathbb{E}[\hat{x}_i] + \hat{s}_N dt + D^T \hat{K}_N \hat{\eta} \). By (43), we have

\[
\hat{u}_i = -\hat{\Upsilon}_N^{-1} \left( (B^T \hat{K}_N + D^T \hat{K}_N C) \hat{x}_i + B^T \hat{\Pi}_N \mathbb{E}[\hat{x}_i] + B^T \hat{s}_N + D^T \hat{K}_N \hat{\eta} \right),
\]

where \( \hat{\Upsilon}_N \triangleq R + D^T \hat{K}_N D \). Applying (46) to (42), it follows that

\[
\begin{aligned}
\rho \hat{K}_N &= \hat{K}_N + A^T \hat{K}_N + \hat{K}_N A + C^T \hat{K}_N C \\
&\quad - (B^T \hat{K}_N + D^T \hat{K}_N C) \hat{\Upsilon}_N^{-1} (B^T \hat{K}_N + D^T \hat{K}_N C) \\
&\quad + Q - \frac{1}{N} \Gamma \hat{K}_N(t) = 0, \quad \hat{\Upsilon}_N \hat{K}_N(t) = 0, \quad \hat{\Upsilon}_N \hat{\Pi}_N = \hat{\Pi}_N + A^T \hat{\Pi}_N + \hat{\Pi}_N A - \hat{\Pi}_N B \hat{\Upsilon}_N^{-1} B^T \hat{\Pi}_N \\
&\quad - \hat{\Pi}_N B \hat{\Upsilon}_N^{-1} (B^T \hat{K}_N + D^T \hat{K}_N C) - \frac{N - 1}{N} \Gamma \hat{\Pi}_N \\
&\quad - (B^T \hat{K}_N + D^T \hat{K}_N C) \hat{\Upsilon}_N^{-1} B^T \hat{\Pi}_N, \quad \hat{\Pi}_N(T) = 0, \quad \hat{\Pi}_N \hat{\Upsilon}_N^{-1} (B^T \hat{K}_N + D^T \hat{K}_N C) \hat{s}_N \\
&\quad + [C - D^T \hat{\Upsilon}_N^{-1} (B^T \hat{K}_N + \hat{\Pi}_N) + D^T \hat{K}_N \hat{\eta}] \hat{s}_N \\
&\quad + (\hat{K}_N + \hat{\Pi}_N) f - \bar{\eta}, \quad \hat{s}_N(T) = 0.
\end{aligned}
\]

We have the following result.

**Proposition 5.1** Equations (47)-(48) admit solutions in \( C([0, T], \mathbb{R}^{N \times N}) \).

**Proof.** See Appendix C. \( \square \)

Note that by Proposition 5.1, (47)-(49) admit a solution, respectively. Then we obtain the decentralized strategy (46), where \( \hat{K}_N, \hat{\Pi}_N, \hat{s}_N \) are given by (47)-(49). The following theorem gives the performance of the proposed decentralized strategy above.

**Theorem 5.2** Let A1) hold. Then for Problem \((S')\), the set of control strategies \{\( \hat{u}_1, \cdots, \hat{u}_N \)\} given by (46) is a decentralized social optimal solution, and the corresponding social cost is given by

\[
J_{soc,T}(\hat{u}) = \sum_{i=1}^{N} \mathbb{E} \left\{ \|x_{i0} - \mathbb{E}[x_{i0}]\|^2_{\hat{K}_N(0)} + \|\mathbb{E}[x_{i0}]\|^2_{\hat{\Pi}_N(0)} + 2 \hat{\eta}_N^T(0) \mathbb{E}[x_{i0}] + N \hat{q}_N^T \right\}.
\]
where

\[
q^N_T = \int_0^T e^{-\rho t} \left[ \|\sigma\|^2_{K_N} + \|\sigma\|^2_{P_N} + \|\eta\|^2_Q - \|\dot{B}^T s_N + D^T \dot{K}_N \sigma\|^2_{\tilde{N}^{-1}_N} + 2 s_N f \right] dt. \tag{51}
\]

**Proof.** See Appendix C. \qed

**Remark 5.1** The work [46] investigated mean-field social control for the case \( C = D = 0 \). When the classical mean-field social controller \( \{u_i^* = -\bar{Y}^{-1} B^T \bar{K} x_i + (P - K) \bar{x} + s, i = 1, \ldots, N\} \) is applied, the corresponding social cost is given by

\[
J_{soc,T}(\bar{u}) = \sum_{i=1}^{N} \mathbb{E}\left\{ \left\| x_{i0} - x^{(N)}(0) \right\|_{K(0)}^2 + \left\| x^{(N)}(0) \right\|_{P(0)}^2 + 2 s_{T(0)} x^{(N)}(0) \right\} + Nq_T + N\epsilon_T,
\]

where

\[
q_T = \int_0^T e^{-\rho t} \left[ \|\sigma\|^2_{K} + \|\sigma\|^2_{P} - \|\dot{B}^T s\|^2_{\tilde{N}^{-1}} + 2 s T f \right] dt,
\]

\[
\epsilon_T = \mathbb{E} \int_0^T e^{-\rho t} \left[ \|\dot{B}^T [P - K] [x(N) - \bar{x}]\|^2 \right] \tilde{N}^{-1} dt.
\]

Here, \( K \) satisfies (37) and \( P \) satisfies

\[
\rho P = \dot{P} + A^T P + PA - PBR^{-1}B^T P + (I - \Gamma)^T Q(I - \Gamma), \quad P(T) = 0.
\]

Compared to (50), residual term \( \epsilon_T \) vanishes as \( N \to \infty \).

### 5.2 The Infinite-Horizon Problem

Based on the discussion above, we may design the following decentralized strategy:

\[
\dot{u}_i = -\bar{Y}_N^{-1} \left[ (B^T \bar{K}_N + D^T \bar{K}_N C) \bar{x}_i + B^T (\bar{P}_N - \bar{K}_N) \bar{\varepsilon}_i \right] + B^T \dot{s}_N + D^T \dot{K}_N \sigma, \quad i = 1, \ldots, N,
\]

where \( \bar{Y}_N = R + D^T \bar{K}_N D \) and \( \bar{K}_N, \bar{P}_N, \bar{s}_N \) are given by

\[
\rho \bar{K}_N = A^T \bar{K}_N + \bar{K}_N A - (B^T \bar{K}_N + D^T \bar{K}_N C)^T \bar{Y}_N^{-1} \left( B^T \bar{K}_N + D^T \bar{K}_N C \right) + CT \bar{K}_N C + Q - \frac{1}{N} \bar{Q}_r,
\]

\[
\rho \bar{P}_N = A^T \bar{P}_N + \bar{P}_N A - (B^T \bar{P}_N + D^T \bar{K}_N C)^T \bar{Y}_N^{-1} \left( B^T \bar{P}_N + D^T \bar{K}_N C \right) + C^T \bar{K}_N C + Q - \bar{Q}_r,
\]

\[
\rho \bar{s}_N = \bar{s}_N + [A - B \bar{Y}_N^{-1} (B^T \bar{P}_N + D^T \bar{K}_N C)] \bar{s}_N + \bar{P}_N f - \bar{\eta} + [C - D \bar{Y}_N^{-1} (B^T \bar{P}_N + D^T \bar{K}_N C)] \bar{K}_N \sigma.
\]

For further analysis, we introduce the assumption:

**A5** (54) admits a unique \( \rho \)-stabilizing solution.

Note

\[
Q - \frac{1}{N} \bar{Q}_r = \frac{N - 1}{N} Q + \frac{1}{N} (I - \Gamma)^T Q(I - \Gamma).
\]

From A3) and [50, Theorem 4.1], we obtain that \( [A - \frac{\rho}{2} I, C, \sqrt{Q(I - \Gamma)}] \) is exactly detectable and (53) admits a unique \( \rho \)-stabilizing solution. Define

\[
\bar{M}_r = \begin{bmatrix} \bar{A}_N - \frac{\rho}{2} I & B \bar{Y}_N^{-1} B^T \\ C^T N + (Q - \bar{Q}_r) - \bar{A}_N^T + \frac{\rho}{2} I \end{bmatrix}.
\]

The following proposition provides some conditions to ensure that A5) holds.

**Proposition 5.2** Assume that A3) holds. Then A5) holds if and only if \( \bar{M}_r \) has no eigenvalues on the imaginary axis. Particularly, if (i) \( [A - \frac{\rho}{2} I, C, \sqrt{Q(I - \Gamma)}] \) is exactly detectable or (ii) \( \Gamma = I, C = 0 \) and \( A - \frac{\rho}{2} I \) has no eigenvalues on the imaginary axis, then A5) holds.

**Proof.** Since A3) holds, by [31], A5) holds if and only if \( \bar{M}_r \) has no eigenvalues on the imaginary axis. Particularly, if \( [A - \frac{\rho}{2} I, C, \sqrt{Q(I - \Gamma)}] \) is exactly detectable, then from A3) and [50, Theorem 4.1], (54) admits a unique \( \rho \)-stabilizing solution. Note \( Q - \bar{Q}_r = 0 \) for \( \Gamma = I \). If \( C = 0 \), and \( A - \frac{\rho}{2} I \) has no eigenvalues on the imaginary axis, then \( \bar{M}_r \) has no eigenvalues on the imaginary axis, which further implies that A5) holds. \qed

**Remark 5.2** For the case \( \Gamma = I \) and \( \eta = 0 \), we have \( (I - \frac{1}{N})^T Q(I - \Gamma) = Q - \bar{Q}_r = 0 \), which gives that (50) and (54) have the same solutions. On other hand,

\[
(I - \frac{1}{N})^T Q(I - \Gamma) = \frac{(N - 1)^2}{N^2} Q \neq \frac{N - 1}{N} Q = Q - \frac{1}{N} \bar{Q}_r.
\]

This implies the solutions to (29) and (53) are different somewhat. Thus, the social solution and the game solution are slightly different for the finite-population consensus problem. However, the two solutions coincide for the infinite population case.

Similar to Theorem 5.2, we obtain the following result.

**Theorem 5.3** Let A1), A3) and A5) hold. Then for Problem (S), the set of control laws \( \{u_1, \ldots, u_N\} \) given by (52) is a decentralized social optimal solution.

### 5.3 Extension to the case with state coupling

Consider the case that agents \( i = 1, \ldots, N \) evolves by

\[
dx_i = (Ax_i + Bu_i + Gx^{(N)} + f)dt + (Cx_i + \sigma)dw_i, \tag{56}
\]
with the social cost

\[ J_{soc,	ext{T}}(u) = \sum_{i=1}^{N} \mathbb{E} \int_{0}^{T} e^{-\rho t} (|x_{i} - \Gamma x^{(N)}| - \eta|_{Q}^{2} + |u_{i}|_{R}^{2}) dt. \]  

(57)

By a similar derivation for Theorem 5.1, we obtain that the problem (56)-(57) has a set of team-optimal strategies \( \bar{u}_{i} \in U_{d,i}, i = 1, \cdots, N \) if and only if the FBSDEs admit a solution \((\hat{x}_{i}, \hat{\lambda}_{i}, \hat{\beta}_{i}, i = 1, \cdots, N)\):

\[
\begin{aligned}
&d\hat{x}_{i} = (A\hat{x}_{i} + B\hat{u}_{i} + G\bar{x}^{(N)} + f)dt + (C\hat{x}_{i} + \sigma)dw_{i}, \\
&d\hat{\lambda}_{i} = -\left\{ [A - \rho I]^{T}\hat{\lambda}_{i} + CT\hat{\beta}_{i} + G^{T}\bar{x}^{(N)} + Q\hat{x}_{i} \right\} dt + \sum_{j=1}^{N}\hat{\beta}_{j}dw_{j}, \\
&\hat{x}_{i} = x_{i0}, \quad \hat{\lambda}_{i}(T) = 0, \quad i = 1, \cdots, N,
\end{aligned}
\]

and furthermore the optimal control laws are given by

\[ \bar{u}_{i} = -R^{-1}B^{T}\mathbb{E}[\hat{\lambda}_{i}^{(T)}], \quad i = 1, \cdots, N. \]

Note that for \( 1 \leq i \neq j \leq N \),

\[ \mathbb{E}[\bar{x}^{(N)}|F_{i}^{T}] = \frac{1}{N}\mathbb{E}[\bar{x}_{i}|F_{i}^{T}] + \frac{N-1}{N}\mathbb{E}[\bar{x}_{j}|F_{j}^{T}], \]

where \( F_{i}^{T} = \sigma(x_{i}(0), w_{i}(s), 0 \leq s \leq t) \). We have

\[
\begin{aligned}
d\mathbb{E}[\bar{x}_{i}|F_{i}^{T}] &= \left\{ (A + \frac{G}{N})\mathbb{E}[\bar{x}_{i}|F_{i}^{T}] - BR^{-1}B^{T}\mathbb{E}[\bar{x}_{i}|F_{i}^{T}] + f \right\} dt + \frac{N-1}{N}\mathbb{E}[\bar{x}_{i}|F_{i}^{T}] + \frac{G}{N}\mathbb{E}[\bar{x}_{i}|F_{i}^{T}] \\
d\mathbb{E}[\bar{x}_{j}|F_{j}^{T}] &= \left\{ (A + \frac{N-1}{N}G)\mathbb{E}[\bar{x}_{i}|F_{i}^{T}] + \frac{G}{N}\mathbb{E}[\bar{x}_{i}|F_{i}^{T}] - BR^{-1}B^{T}\mathbb{E}[\bar{x}_{i}|F_{i}^{T}] + f \right\} dt, \\
d\mathbb{E}[\hat{\lambda}_{i}|F_{i}^{T}] &= -\left\{ [A - \rho I + \frac{G}{N}]^{T}\mathbb{E}[\hat{\lambda}_{i}|F_{i}^{T}] + CT\mathbb{E}[\hat{\beta}_{i}|F_{i}^{T}] \right\} dt + \frac{N-1}{N}G\mathbb{E}[\bar{x}_{i}|F_{i}^{T}] + \frac{G}{N}\mathbb{E}[\bar{x}_{i}|F_{i}^{T}] \\
&\quad + \left\{ (Q - \frac{N-1}{N}Qf)\mathbb{E}[\bar{x}_{i}|F_{i}^{T}] - \eta \right\} dt + \mathbb{E}[\hat{\beta}_{i}|F_{i}^{T}]dw_{i}, \\
\end{aligned}
\]

\[
\mathbb{E}[\bar{x}_{i}(0)|F_{i}^{T}] = x_{i0}, \mathbb{E}[\bar{x}_{i}|F_{i}^{T}] = \bar{x}_{i0}, \mathbb{E}[\hat{\lambda}_{i}(T)|F_{i}^{T}] = 0, \quad \mathbb{E}[\hat{\beta}_{i}(T)|F_{i}^{T}] = 0, \quad 1 \leq i \neq j \leq N.
\]

Let \( \hat{\lambda}_{i} = \hat{K}_{N}\hat{x}_{i} + \hat{\Pi}_{N}\hat{x}^{(N)} + \hat{s}_{N} \). We have

\[
\begin{aligned}
\mathbb{E}[\hat{\lambda}_{i}|F_{i}^{T}] &= (\hat{K}_{N} + \frac{\hat{\Pi}_{N}}{N})\mathbb{E}[\bar{x}_{i}|F_{i}^{T}] + \frac{N-1}{N}\hat{\Pi}_{N}\mathbb{E}[\bar{x}_{j}|F_{j}^{T}] + \hat{s}_{N}, \\
\mathbb{E}[\hat{\beta}_{i}|F_{i}^{T}] &= \frac{\hat{\Pi}_{N}}{N}\mathbb{E}[\bar{x}_{i}|F_{i}^{T}] + (\hat{K}_{N} + \frac{N-1}{N}\hat{\Pi}_{N})\mathbb{E}[\bar{x}_{j}|F_{j}^{T}] + \hat{s}_{N}.
\end{aligned}
\]

By applying the four-step scheme [29], we obtain the following social optimal control

\[ \bar{u}_{i} = -R^{-1}B^{T} [(\hat{K}_{N} + \frac{\hat{\Pi}_{N}}{N})\mathbb{E}[\bar{x}_{i}|F_{i}^{T}] \\
+ \frac{N-1}{N}\hat{\Pi}_{N}\mathbb{E}[\bar{x}_{j}|F_{j}^{T}] + \hat{s}_{N}], \]

where

\[ \rho\hat{K}_{N} = \hat{K}_{N} + A^{T}\hat{K}_{N} + \hat{K}_{N}A + Q - \hat{K}_{N}BR^{-1}B^{T}\hat{K}_{N} - C^{T}(\hat{K}_{N} + \frac{\hat{\Pi}_{N}}{N})C, \quad \hat{K}_{N}(T) = 0, \]

\[ \rho\hat{\Pi}_{N} = \hat{\Pi}_{N} + A^{T}\hat{\Pi}_{N} + \hat{\Pi}_{N}A - \hat{K}_{N}BR^{-1}B\hat{\Pi}_{N} - \hat{K}_{N}BR^{-1}B^{T}(\hat{K}_{N} + \hat{\Pi}_{N}) + G^{T}(\hat{K}_{N} + \hat{\Pi}_{N}) \\
+ (\hat{K}_{N} + \hat{\Pi}_{N})G - QT, \quad \hat{\Pi}_{N}(T) = 0, \]

\[ \rho\hat{s}_{N} = \hat{s}_{N} + (A + G - B - R^{-1}B(\hat{K}_{N} + \hat{\Pi}_{N}))^{T}\hat{s}_{N} - \eta \\
+ (\hat{K}_{N} + \hat{\Pi}_{N})f + C^{T}(\hat{K}_{N} + \frac{\hat{\Pi}_{N}}{N})\sigma, \quad \hat{s}_{N}(T) = 0. \]

6 Numerical Examples

In this section, we give two numerical examples to illustrate the properties of proposed decentralized strategies. Consider Problem (G) for 6 agents with single-integrator dynamics and additive noise (i.e., \( A = 0 \) and \( C = D = 0 \)). Take the parameters as \( B = Q = R = \Gamma = 1, f(t) = \eta(t) = 0, \sigma = 0.1, \rho = 0.2, \) and \( \alpha_{(N)} = \frac{1}{N} \). The initial states of 6 agents are taken independently from a normal distribution \( N(5, 1) \). Note that \( B \neq 0, \) and \( Q > 0 \). Then A1 and A3) hold. By Proposition 3.3, A4) holds.

Under the strategy (28), the trajectories of \( \mathbb{E}[\bar{x}_{i}] \) and \( \bar{x}^{(N)} \) are shown in Fig. 1. It can be seen that \( \mathbb{E}[\bar{x}_{i}] \) and \( \bar{x}^{(N)} \) do not coincide well, but \( \mathbb{E}[\bar{x}_{i}] \) attains the mean of \( \bar{x}^{(N)} \). This is different from classical mean-field games, where \( \mathbb{E}[\bar{x}_{i}] \) and \( \bar{x}^{(N)} \) coincide as agent number is large.
parameters as \( B = Q = R = D = \Gamma = 1, f(t) = \sigma(t) = \eta(t) = 0, \rho = 0.2 \). The initial states of 6 agents are taken independently from a normal distribution \( N(5, 1) \). Note that \( B = D \neq 0 \), and \( Q > 0 \). Then A1) and A3) hold. By Proposition 5.2, A5) holds.

Under the proposed control strategy (52), the trajectories of \( \dot{x}_i, i = 1, \ldots, 6 \) are shown in Fig. 3. It can be seen that under multiplicative noise, all the agents can achieve consensus strictly, which verifies the result of Theorem 3.5. The trajectories of \( E[\dot{x}_i] \) and \( \dot{x}^{(N)} \) are shown in Fig. 4. It can be seen that \( \dot{x}^{(N)} \) and \( E[\dot{x}_i] \) coincide well when the time is sufficiently long. This is different from the case with additive noise, which is shown in Fig. 1. Fig. 5 shows the performance comparison of the proposed social strategy and the classical mean-field controller. When \( N \) is not very large, it gives superior performance using the proposed social strategy (46) than the classical mean-field controller (36). The two performances merge when \( N \) is sufficiently large.

Consider Problem (S) for 6 single-integrator agents with multiplicative noise (i.e., \( A = C = 0 \)). Take the
of two differential Riccati equations by decoupling non-
standard FBSDEs. The proposed decentralized strategies
were further shown to be a Nash equilibrium and a
social optimal solution, respectively. For infinite-horizon
problems, we gave some criteria for the solvability of
algebraic Riccati equations arising from consensus. For
future investigation, it would be interesting to generalize
the results to more complicated situations, such as mixed
games with a major player or leader-follower games, and
nonlinear games with a finite number of agents.

A Proofs for Section 3.1

Proof of Theorem 3.1. (i) Suppose \{\hat{u}_i, i = 1, \ldots, N\}
is a set of decentralized Nash equilibrium strategies
of Problem (G’), and \{\hat{x}_i, i = 1, \ldots, N\} are the

corresponding states of agents, i.e., they satisfy (3). Let
\{\hat{\lambda}_i, \hat{\beta}_i, i, j = 1, \ldots, N\} be a set of adapted solutions
to the second equation of (3). For any \hat{u}_i \in U_{d;i} and
\hat{\theta} \in \Theta (\hat{\theta} \neq 0), let \hat{v}_i = \hat{u}_i + \hat{\theta} \hat{v}_i.
Denote by \hat{x}^i the solution of the following perturbed state equation

\[ dx^i = [Ax^i + B(\hat{u}_i + \hat{\theta} \hat{v}_i) + f]dt + [Cx^i + D(\hat{u}_i + \hat{\theta} \hat{v}_i) + \sigma]d\hat{w}_i, \quad x^i(0) = x_{i0}, \quad i = 1, 2, \ldots, N. \]  

Let \hat{y}_i = (\hat{x}^i - \hat{x}_i)/\hat{\theta}. It can be verified that \hat{y}_i satisfies

\[ dy^i = (Ay^i + Bv^i)dt + (Cy^i + Dv^i)dv^i, \quad y^i(0) = 0. \]  

Then by Itô’s formula, for any \hat{y}_i \in U_{d;i},

\[ 0 = \mathbb{E}[\hat{\lambda}_i(T), e^{-\hat{\theta}T}y_i(T)] - \mathbb{E}[\hat{\lambda}_i(0), y_i(0)] \]

\[ = \mathbb{E} \int_0^T e^{-\hat{\theta}t} \left[ \left( - (I - \alpha^{(N)}_i)T \right) Q(\hat{x}_i - \hat{x}^{(\alpha)}_i - \eta), y_i \right] + \left( B^T \hat{\lambda}_i + D^T \hat{\beta}_i, \hat{v}_i \right) dt. \]  

(A.2)

We have

\[ J_{i,T}(\hat{u}_i, \hat{v}_i) - J_{i,T}(\hat{u}_i, \hat{v}_i) = 2\hat{\theta}I_1 + \hat{\theta}^2 I_2 \]  

(A.3)

where \hat{v}_i = (\hat{u}_1, \ldots, \hat{u}_{i-1}, \hat{u}_{i+1}, \ldots, \hat{u}_N), and

\[ I_1 = \mathbb{E} \int_0^T e^{-\hat{\theta}t} \left[ \left( Q(\hat{x}_i - \hat{x}^{(\alpha)}_i + \eta), (I - \alpha^{(N)}_i)T \right) y_i \right] + \left( \hat{R}^i, \hat{v}_i \right) dt, \]

\[ I_2 = \mathbb{E} \int_0^T e^{-\hat{\theta}t} \left[ \left( I - \alpha^{(N)}_i \right) y_i \right]_Q^2 + \left( \hat{v}_i \right)_B^2 dt. \]

From (A.2), one can obtain that

\[ I_1 = \mathbb{E} \int_0^T e^{-\hat{\theta}t} \left[ \left( Q(\hat{x}_i - \hat{x}^{(\alpha)}_i + \eta), (I - \alpha^{(N)}_i)T \right) y_i \right] + \left( \hat{R}^i, \hat{v}_i \right) dt + \mathbb{E} \int_0^T e^{-\hat{\theta}t} \left[ \left( - (I - \alpha^{(N)}_i)T \right) Q(\hat{x}_i - \hat{x}^{(\alpha)}_i - \eta), y_i \right] + \left( B^T \hat{\lambda}_i + D^T \hat{\beta}_i, \hat{v}_i \right) dt \]

\[ = \mathbb{E} \int_0^T \left( \hat{R}^i + B^T \hat{\lambda}_i + D^T \hat{\beta}_i, \hat{v}_i \right) dt. \]

Note that \hat{u}_i, \hat{v}_i \in U_{d;i}. By the smoothing property of conditional mathematical expectation,

\[ I_1 = \mathbb{E} \int_0^T \left( \hat{R}^i + B^T \mathbb{E}[\hat{\lambda}_i|F_t^i] + D^T \mathbb{E}[\hat{\beta}_i|F_t^i], \hat{v}_i \right) dt. \]  

(A.4)

Since \( Q \geq 0 \) and \( R > 0 \), we have \( I_2 \geq 0 \). From

(A.3)-(A.4), the fact that \{\hat{u}_i, i = 1, \ldots, N\} is a Nash
equilibrium strategy implies \( I_1 = 0 \), which is equivalent to

\[ \hat{u}_i = -R^{-1}B^T \mathbb{E}[\hat{\lambda}_i|F_t^i] - R^{-1}D^T \mathbb{E}[\hat{\beta}_i|F_t^i]. \]

Thus, we have the optimality system (3). This implies that (3) admits an adapted solution \((\hat{x}_i, \hat{\lambda}_i, \hat{\beta}_i)\).

(ii) If (3) admits a solution \((\hat{x}_i, \hat{\lambda}_i, \hat{\beta}_i)\), then it can be verified that \( I_1 = 0 \), which with (A.3) implies that \{\hat{u}_i, i = 1, \ldots, N\} in (4) is a set of Nash strategies. □

Proof of Proposition 3.2. When \( \Gamma = I, (7) \) and (8) are
simplified as

\[ \rho K_N = K_N + A^T K_N + K_N A - (B^T K_N + D^T K_N C)^T \]

\[ \times \left[ (B^T K_N + D^T K_N C) + C^T K_N C \right], \]

\[ + \frac{(N - 1)^2}{N^2} Q, \quad K_N(T) = 0, \]  

(A.5)

\[ \rho \Pi_N = \Pi_N + A^T \Pi_N + \Pi_N A - \Pi_N B Y_N^{-1} B^T \Pi_N \]

\[ - \Pi_N B Y_N^{-1} (B^T K_N + D^T K_N C) - (B^T K_N + D^T K_N C)^T Y_N^{-1} B^T \Pi_N \]

\[ - \frac{(N - 1)^2}{N^2} Q, \quad \Pi_N(T) = 0. \]  

(A.6)

Note \( Q \geq 0 \). We obtain that (A.5) admits a solution \( K_N \geq 0 \) (40). Let \( P_N = K_N + \Pi_N \). Then the equation

\[ \rho P_N = \hat{P}_N + A^T P_N + P_N A - (B^T P_N + D^T K_N C)^T Y_N^{-1} \]

\[ \times (B^T P_N + D^T K_N C) + C^T K_N C, \quad P_N(T) = 0 \]  

(A.7)

admits a unique solution \( P_N \geq 0 \). Since \( K_N \geq 0 \),
we obtain that (A.6) admits a solution, which further implies A2) holds. □
Proofs for Section 3.2

Proof of Proposition 3.3. For the case \( \Gamma = I \), (29)-(30) can be written as

\[
\rho K_N = A^T K_N + K_N A - (B^T K_N + D^T K_N C)^T Y_N^{-1} \\
\times (B^T K_N + D^T K_N C) + C^T K_N C + \left(\frac{N-1}{N}\right)^2 Q, \\
\rho P_N = A^T P_N + P_N A - (B^T P_N + D^T K_N C)^T Y_N^{-1} \\
\times (B^T P_N + D^T K_N C) + C^T K_N C.
\]

(B.1)

(B.2)

Since A3) holds, then (B.1) admits a stabilizing solution. From [31], (B.2) admits a stabilizing solution if and only if \( A - \frac{\rho}{2} I \) has no eigenvalues on the imaginary axis. For the case \( C = 0 \), \( \mathcal{M}_I \) has no eigenvalues on the imaginary axis if and only if \( A - \frac{\rho}{2} I \) has no eigenvalues in the imaginary axis. Then the theorem follows.

\[ \square \]

To prove Theorem 3.4, we first provide a lemma, which shows uniform stability of the closed-loop systems.

Lemma B.1 Assume that A1), A3), A4) hold and \( N \) is sufficiently large such that \( I - \frac{1}{N} \Gamma \) is nonsingular. Then (31) admits a unique solution \( s_N \in C_{\rho/2}(0, \infty, \mathbb{R}^n) \) and the following holds:

\[
\sum_{i=1}^{N} \mathbb{E} \int_0^\infty e^{-pt} \left( \|\bar{x}_i(t)\|^2 + \|\bar{\bar{u}}_i(t)\|^2 \right) dt < \infty.
\]

Proof. In view of A4), \( A - BT^{-1}Y_N^{-1}(B^T P_N + D^T K_N C) - \frac{\rho}{2} I \) is Hurwitz, and hence \( \int_0^\infty e^{-pt}(\mathbb{E} \bar{x}_i)^2 dt < \infty \). From an argument in [44, Appendix A], we obtain that (31) admits a unique solution \( s_N \in C_{\rho/2}(0, \infty, \mathbb{R}^n) \). Denote

\[
\tilde{A}_N \overset{\Delta}{=} A - BT^{-1}Y_N^{-1}(B^T K_N + D^T K_N C), \\
\tilde{C}_N \overset{\Delta}{=} C - DT^{-1}Y_N^{-1}(B^T K_N + D^T K_N C), \\
\tilde{f}_N \overset{\Delta}{=} f - BY_N^{-1}B^T((P_N - K_N)\mathbb{E}[\bar{x}_i] + s_N) + D^T P \sigma], \]

\[
\tilde{s}_N \overset{\Delta}{=} \sigma - DT^{-1}Y_N^{-1}B^T((P_N - K_N)\mathbb{E}[\bar{x}_i] + s_N) + D^T P \sigma.
\]

After the control (28) is applied, we have

\[
d\bar{x}_i(t) = [\tilde{A}_N \bar{x}_i(t) + \tilde{f}_N(t)] dt + [\tilde{C}_N \bar{x}_i(t) + \tilde{s}_N(t)] d\bar{w}_i(t). \quad (B.3)
\]

Note that \( N \) is sufficiently large such that \( I - \frac{1}{N} \Gamma \) is nonsingular. By A3) and [50], we obtain that \( [A - \frac{\rho}{2} I, C, \sqrt{(I - \frac{1}{N} \Gamma)^T Q (I - \frac{1}{N} \Gamma)}] \) is exactly detectable, and hence \( (\tilde{A}_N - \frac{\rho}{2} I, \tilde{C}_N) \) is mean-square stable. Let \( Y_N \) satisfy

\[
Y_N(\tilde{A}_N - \frac{\rho}{2} I) + (\tilde{A}_N - \frac{\rho}{2} I)^T Y_N + (\tilde{C}_N)^T Y_N \tilde{C}_N = -2I. \quad (B.4)
\]

From [50], we have \( Y_N > 0 \). By Itô’s formula and (B.3),

\[
\mathbb{E}[e^{-\rho t}\tilde{x}_i(T)Y_N\tilde{x}_i(T) - \tilde{x}_i(t)^2] dt] < \mathbb{E}[e^{-\rho t}\tilde{x}_i(T)Y_N\tilde{x}_i(T) - \tilde{x}_i(t)^2] dt]
\]

This with (28) completes the proof. \( \square \)

Proof of Theorem 3.4. Denote \( \tilde{u}_i = u_i - \bar{u}_i \) and \( \tilde{x}_i = x_i - \bar{x}_i \). Then \( \tilde{x}_i \) satisfies \( (\tilde{x}_i(0) = 0) \)

\[
d\tilde{x}_i = (A \tilde{x}_i + B \tilde{u}_i) dt + (C \tilde{x}_i + D \tilde{u}_i) dw_i. \quad (B.6)
\]

By Lemma B.1,

\[
\sum_{i=1}^{N} \mathbb{E} \int_0^\infty e^{-pt} \left( \|\tilde{x}_i(t)\|^2 + \|\tilde{u}_i(t)\|^2 \right) dt < \infty. \quad (B.7)
\]

From (2), we have \( J_i (u_i, \bar{u}_{i-1}) = J_i (\tilde{u}_i, \bar{u}_{i-1}) + \tilde{J}_i (\tilde{u}_i, \bar{u}_{i-1}) + \tilde{I}_i \), where

\[
\tilde{J}_i (\tilde{u}_i, \bar{u}_{i-1}) \overset{\Delta}{=} \mathbb{E} \int_0^\infty e^{-pt} \left[ \|\tilde{x}_i(t) - \frac{1}{N} \Gamma \tilde{x}_i(t)\|^2 + \|\tilde{u}_i(t)\|^2 \right] dt, \\
\tilde{I}_i = 2\mathbb{E} \int_0^\infty e^{-pt} \left[ \left( \tilde{x}_i(t) - \Gamma \tilde{x}(N)(t) - \eta(t) \right)^T \\
\times Q(\tilde{x}_i(t) - \frac{1}{N} \Gamma \tilde{x}_i(t)) + \tilde{u}_i^T(t) R \tilde{u}_i(t) \right] dt.
\]
By applying Itô’s formula with (29) and (30), we have
\[
0 = \lim_{T \to \infty} \sup_{T} \mathbb{E}\left[ e^{-\rho T} \bar{x}_i^T(T) \left( K_N \bar{x}_i(T) + (P_N - K_N) \mathbb{E}\left[ \bar{x}_i(T) + s_N(T) \right] \right) \right]
\]
\[
= \sum_{i=1}^{N} \mathbb{E} \int_{0}^{\infty} e^{-\rho t} \left\{ \bar{x}_i^T \left[ \left( A^T K_N + K_N A - \rho K_N + C^T K_N C \right) \right] \bar{x}_i \\
- \left( B K_N + D^T K_N C \right)^T \tilde{\Upsilon}^{-1}(B^T K_N + D^T K_N C) \bar{x}_i \\
- \left( A^T (P_N - K_N) + (P_N - K_N) A - \rho P_N \right) \bar{x}_i \right\} dt
\]
\[
= \mathbb{E} \int_{0}^{\infty} e^{-\rho t} \left\{ \bar{x}_i \left[ -\frac{1}{N} \Gamma \bar{x}_i \right] Q \frac{1}{N} \Gamma \bar{x}_i \\
- \frac{1}{N} \Gamma \frac{1}{N} \Gamma \bar{x}_i \right\} dt.
\]  (B.8)

Note \( \bar{u}_i, \bar{x}_i, \bar{u}_i, \bar{x}_i \) are adapted to \( F_t^i \). By the property of conditional expectation,
\[
\mathcal{I}_i = 2 \mathbb{E} \int_{0}^{\infty} e^{-\rho t} \left[ \bar{x}_i(t) - \Gamma \bar{x}_i^{(N)}(t) - \eta(t) \right] Q \left[ \bar{x}_i(t) - \Gamma \bar{x}_i^{(N)}(t) - \eta(t) \right] dt
\]
\[
= \mathbb{E} \int_{0}^{\infty} e^{-\rho t} \left[ \bar{x}_i(t) - \Gamma \bar{x}_i^{(N)}(t) - \eta(t) \right] Q \left[ \bar{x}_i(t) - \Gamma \bar{x}_i^{(N)}(t) - \eta(t) \right] dt.
\]
Comparing this with (B.8) leads to \( \mathcal{I}_i = 0 \). Thus, the theorem follows.

Proof of Theorem 5.1. (Necessity) Suppose \( \{ \hat{u}_i, i = 1, \ldots, N \} \) is a set of social optimal strategies, and \( \{ \hat{x}_i, i = 1, \ldots, N \} \) is the corresponding states of agents. Let \( \{ \lambda_i, \beta_i, i, j = 1, \ldots, N \} \) be a set of solutions to the second equation of (42). For any \( u_i \in U_{d,i} \) and \( \theta \in \mathbb{R} (\theta \neq 0) \), let \( \theta \) be the corresponding state under the control \( u_i \), \( i = 1, 2, \ldots, N \). Let \( y_i = \left( x_i^\theta - \hat{x}_i \right)/\theta \). It can be verified that \( y_i \) satisfies (A.1). Then by Itô’s formula, for any \( i = 1, \ldots, N \),
\[
0 = \sum_{i=1}^{N} \alpha_i^{(N)} \mathbb{E} \int_{0}^{T} e^{-\rho t} \left[ \left( \frac{1}{N} \Gamma \frac{1}{N} \Gamma \bar{x}_i^{(N)} - \eta(t) \right) \right] \left( \frac{1}{N} \Gamma \frac{1}{N} \Gamma \bar{x}_i^{(N)} - \eta(t) \right) dt.
\]  (C.1)

We have \( J_{soc,T}(u^\theta) - J_{soc,T}(\hat{u}) = 2\theta I_1 + \theta^2 I_2 \), where \( u^\theta = (u_1^\theta, \ldots, u_N^\theta), y(\alpha) = \frac{1}{\theta} \sum_{j=1}^{N} \alpha_j^{(N)} y_j \) and
\[
I_1 = \sum_{i=1}^{N} \alpha_i^{(N)} \mathbb{E} \int_{0}^{T} e^{-\rho t} \left[ \left( \frac{1}{N} \Gamma \frac{1}{N} \Gamma \bar{x}_i^{(N)} + \eta(t) \right) \right] \\
+ \langle \hat{\lambda}_i + D \hat{\beta}_i, v_i \rangle dt.
\]
\[
I_2 = \sum_{i=1}^{N} \alpha_i^{(N)} \mathbb{E} \int_{0}^{T} e^{-\rho t} \left[ \left( \frac{1}{N} \Gamma \frac{1}{N} \Gamma \bar{x}_i^{(N)} + \eta(t) \right) \right] ^2 + \| v_i \|^2 dt.
\]

Note that
where the second equation holds by the fact that \( \hat{u} \) is equivalent to \( \hat{\eta} \). From (C.2), it can be verified that

\[
I_1 = \sum_{i=1}^{N} \alpha_i^{(N)} \mathbb{E} \int_{0}^{T} e^{-pt} \left\langle R\hat{u}_i + B^T\hat{\lambda}_i + D^T\hat{\beta}_i, v_i \right\rangle dt
\]

\[
= \sum_{i=1}^{N} \alpha_i^{(N)} \mathbb{E} \int_{0}^{T} e^{-pt} \left\langle R\hat{u}_i + B^T\hat{\lambda}_i, F_i \right\rangle + D^T\mathbb{E} [\hat{\beta}_i | F_i], v_i \right\rangle dt.
\]

(C.2)

where the second equation holds by the fact that \( \hat{u}_i, v_i \) are adapted to \( F_i \). Since \( Q \geq 0 \) and \( R > 0 \), we have \( I_2 \geq 0 \). From (C.2), \( \hat{u} = (\hat{u}_1, \ldots, \hat{u}_N) \) is an optimal control of Problem (G') if and only if \( I_1 = 0 \), which is equivalent to \( \hat{u}_i = -R^{-1}B^T\mathbb{E}[\hat{\lambda}_i | F_i] - R^{-1}D^T\mathbb{E}[\hat{\beta}_i | F_i] \).

Thus, we have the following optimality system (42). This implies that (42) admits a solution \((\hat{x}_i, \hat{\lambda}_i, \hat{\beta}_i)\).

(Sufficiency) On the other hand, if (42) admits a solution \((\hat{x}_i, \hat{\lambda}_i, \hat{\beta}_i)\), then it can be verified that \( I_1 = 0 \), which implies \( \{\hat{u}_i, i = 1, \ldots, N\} \) is social optimal control. \( \square \)

**Proof of Proposition 5.1.** Note that

\[
Q - \frac{1}{N}Q_T = \frac{N - 1}{N}Q + \frac{1}{N}(I - \Gamma)^TQ(I - \Gamma) \geq 0.
\]

We obtain that (47) admits a solution \( \hat{K}_N \geq 0 \). Let \( \hat{P}_N = K_N + \Pi_N \). Then

\[
\rho \hat{P}_N = \hat{P}_N + A^T \hat{P}_N + \hat{P}_NA + C^T K_N C - (B^T \hat{P}_N + D^T K_N C) + (I - \Gamma)^T Q(I - \Gamma), \hat{P}_N(T) = 0
\]

admits a unique solution \( \hat{P}_N \geq 0 \). This further implies (48) that admits a solution. \( \square \)

**Proof of Theorem 5.2.** The proof for social optimality of (38) is similar to Theorem 3.2, and so we omit it here. Note that \( \hat{x}_i, \hat{u}_i \) are adapted to \( F_i \) and \( \mathbb{E} [\hat{x}_i | F_i] = \mathbb{E} [\hat{x}_i] = \mathbb{E} [\hat{x}^{(N)}] \). By direct calculations,

\[
J_{soc,T}(\hat{u}) = \sum_{i=1}^{N} \int_{0}^{T} e^{-pt} \mathbb{E} \left[ \left\| \hat{x}_i - \Gamma \hat{x}^{(N)} - \eta \right\|^2_Q + \left\| \hat{u}_i \right\|^2_R | F_i \right] dt
\]

\[
= \sum_{i=1}^{N} \int_{0}^{T} e^{-pt} \left[ \left\| (I - \frac{1}{N}\Gamma) \hat{x}_i \right\|^2_Q
\]

\[
- 2 \hat{x}_i^T(I - \frac{1}{N}\Gamma) \Gamma Q \frac{\Gamma}{N} \sum_{j \neq i} \mathbb{E} [\hat{x}_j] + \frac{1}{N^2} \mathbb{E} \left\| \Gamma \sum_{j \neq i} \hat{x}_j \right\|^2_Q
\]

\[
+ \left\| \eta \right\|^2_Q - 2\eta^T (I - \Gamma) \hat{x}_i + \left\| \hat{u}_i \right\|^2_R \right] dt.
\]

Note that

\[
\mathbb{E} \left\| \Gamma \sum_{j \neq i} \hat{x}_j \right\|^2_Q = \sum_{j \neq i} \mathbb{E} [\hat{x}_j^T \Gamma \Gamma Q \hat{x}_j]
\]

\[
= \sum_{j \neq i} \mathbb{E} [\hat{x}_j^T \Gamma \Gamma Q \hat{x}_j] + \sum_{j \neq i} \mathbb{E} [\hat{x}_j^T \Gamma \Gamma Q \Gamma \mathbb{E} [\hat{x}_k]]
\]

\[
= (N - 1) \mathbb{E} [\hat{x}_i^T \Gamma \Gamma Q \mathbb{E} [\hat{x}_i]]
\]

\[
+ (N - 1)(N - 2) \mathbb{E} [\hat{x}_i^T \Gamma \Gamma Q \mathbb{E} [\hat{x}_i]]
\]

Then we further have

\[
J_{soc,T}(\hat{u}) = \sum_{i=1}^{N} \int_{0}^{T} e^{-pt} \left[ \left\| (I - \frac{1}{N}\Gamma) \hat{x}_i \right\|^2_Q + \frac{N - 1}{N^2} \mathbb{E} [\hat{x}_i^T \Gamma \Gamma Q \mathbb{E} [\hat{x}_i]]
\]

\[
- 2 \frac{N - 1}{N} \hat{x}_i^T(I - \frac{1}{N}\Gamma) \Gamma Q \mathbb{E} [\hat{x}_i]
\]

\[
+ \frac{(N - 1)(N - 2)}{N^2} \mathbb{E} [\hat{x}_i^T \Gamma \Gamma Q \mathbb{E} [\hat{x}_i]] + \left\| \eta \right\|^2_Q
\]

\[
- 2\eta^T (I - \Gamma) \mathbb{E} [\hat{x}_i] + \left\| \hat{u}_i - \mathbb{E} [\hat{u}_i] \right\|^2_R
\]

\[
+ \left\| \mathbb{E} [\hat{u}_i] \right\|^2_R \right] dt.
\]

\[
= \sum_{i=1}^{N} \int_{0}^{T} e^{-pt} \left[ \left\| \hat{x}_i - \mathbb{E} [\hat{x}_i] \right\|^2_{Q - Q_T} + \mathbb{E} [\hat{x}_i] \right\|^2_{Q - Q_T}
\]

\[
+ \left\| \hat{u}_i - \mathbb{E} [\hat{u}_i] \right\|^2_R + \left\| \mathbb{E} [\hat{u}_i] \right\|^2_R \right] dt + N q_T^N
\]

\[
= \sum_{i=1}^{N} \int_{0}^{T} e^{-pt} \left[ \left\| \hat{x}_i - \mathbb{E} [\hat{x}_i] \right\|^2_{Q - Q_T} + \mathbb{E} [\hat{x}_i] \right\|^2_{Q - Q_T}
\]

\[
+ 2 \hat{x}_i^T (0) \mathbb{E} [\hat{x}_i(0)] + N q_T^N, + N q_T^N
\]
where $q^N_{t}$ is given by (51).

References

[1] Arabneydi, J., & Mahajan, A. (2015). Team-optimal solution of finite number of mean-field coupled LQG subsystems, in Proc. 54th IEEE CDC. Osaka, Japan, 5308-5313.

[2] Arabneydi, J., Malhamé, R. P., & Aghdam, A. G. (2020). Explicit sequential equilibria in linear quadratic games with arbitrary number of exchangeable players: A non-standard Riccati equation. https://arxiv.org/abs/1912.03931v1

[3] Arabneydi, J., & Aghdam, A. G. (2020). Deep structured teams with linear quadratic model: Partial equivariance and gauge transformation. https://arxiv.org/abs/1912.03951

[4] Bensoussan, A., Sung, K.C., Yam, S.C., & Yung, S. P. (2016). Linear-quadratic mean-field games. J. Optimization Theory & Applications, 169(2), 496-529.

[5] Bensoussan, A., Frehse, J., & Yam, P. (2013). Mean-field Games and mean-field Type Control Theory. Springer, New York.

[6] Blume, L. E. (1993). The statistical mechanics of strategic interaction. Games Econ. Behavior, 5, 387-424.

[7] Buckdahn, R., Li, J., & Peng, S. (2013). Nonlinear stochastic differential games involving a major player and a large number of collectivity acting minor agents. SIAM J. Control and Optimization, 52(1), 451-492.

[8] Caines, P. E., & Huang, M. (2018). Graphon mean-field games and the GMGF equations. Proc. the 57th IEEE CDC, Miami Beach, FL, 4129-4134.

[9] Caines, P. E., Huang, M., & Malhame, R. P. (2017). mean-field games. Handbook of Dynamic Game Theory, T. Basar and G. Zaccour Eds., Springer, Berlin.

[10] Carmona, R., & Delarue, F. (2013) Probabilistic analysis of mean-field games. SIAM J. Control Optim., 51(4), 2705-2734.

[11] Carmona, R., & Delarue, F. (2018). Probabilistic Theory of mean-field Games with Applications: I and II. Springer-Verlag.

[12] Charalambous, C. D., & Ahmed, N. U. (2017). Centralized versus decentralized optimization of distributed stochastic differential decision systems with different information structures-part I: A general theory. IEEE Trans. Autom. Control, 62(3), 1194-1209.

[13] Freiling, G., Jank, G., Lee, S.-R., & Abou-Kandil, H. (1996). On the dependence of the solutions of algebraic and differential game Riccati equations on the parameter $\mu^*$, Eur. J. Control, 2(1), 69-78.

[14] Gomes, D. A., & Saude, J. (2014). Mean-field games models—a brief survey. Dyn. Games Appl., 4(2), 110-154.

[15] Ho, Y. C. (1980). Team decision theory and information structures. in Proc. IEEE, 68(6), 644-654.

[16] Huang, J., & Huang, M. (2017). Robust mean-field linear-quadratic-Gaussian games with model uncertainty. SIAM J. Control and Optimization, 55(5), 2811-2840.

[17] Huang, M. (2010). Large-population LQG games involving a major player: the Nash certainty equivalence principle. SIAM J. Control and Optimization, 48(5), 3318-3353.

[18] Huang, M., Caines, P. E., & Malhamé, R. P. (2003). Individual and mass behaviour in large population stochastic wireless power control problems: Centralized and Nash equilibrium solutions. in Proc. 42nd IEEE CDC, Maui, HI, 98-103.

[19] Huang, M., Caines, P. E., & Malhamé, R. P. (2007). Large-population cost-coupled LQG problems with non-uniform agents: individual-mass behavior and decentralized $\varepsilon$-Nash equilibria. IEEE Trans. Autom. Control, 52(9), 1560-1571.

[20] Huang, M., Caines, P., & Malhame, R. (2012). Social optima in mean-field LQG control: centralized and decentralized strategies. IEEE Trans. Autom. Control, 57(7), 1736-1751.

[21] Huang, M., Malhamé, R. P., & Caines, P. E. (2006). Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle. Communication in Information and Systems, 6, 221-251.

[22] Huang, M., Malhamé, R. P., & Caines, P. E. (2010). The NCE (mean-field) principle with locality dependent cost interactions. IEEE Trans. Autom. Control, 55(12), 2799-2805.

[23] Huang, M., & Nguyen, L. (2016). Linear-quadratic mean-field teams with a major agent. Proc. 55th IEEE CDC, Las Vegas, NV, 6958-6963.

[24] Huang, M., & Zhou, M. (2020). Linear quadratic mean-field games: Asymptotic solvability and relation to the fixed point approach. IEEE Trans. Autom. Control, 65(4).

[25] Jadbabaie, A., Lin, J., & Morse, S. M. (2003). Coordination of groups of mobile autonomous agents using nearest neighbor rules. IEEE Trans. Autom. Control, 48(6), 988-1001.

[26] Lasry, J. M., & Lions, P. L. (2017). mean-field games. Japan J. Math., 2(1), 229-260.

[27] Li, T., & Zhang, J.-F. (2008). Asymptotically optimal decentralized control for large population stochastic multiagent systems. IEEE Trans. Autom. Control, 53(7), 1643-1660.

[28] Li, Z., Fu, M., Zhang, H., & Wu, Z. (2018). Finite number of mean-field optimal control for stochastic linear quadratic systems. preprint.

[29] Ma, J., & Yong, J. (1999). Forward-backward Stochastic Differential Equations and their Applications, Springer-Verlag, New York.

[30] Khalil, H. K. (2002). Nonlinear Systems, 3rd edition, Prentice Hall, Inc.

[31] Molinari, B. P. (1977). The time-invariant linear-quadratic optimal control problem. Automatica, 13(4), 347-357.

[32] Moon, J., & Basar, T. (2017). Linear quadratic risk-sensitive and robust mean-field games. IEEE Trans. Autom. Control, 62(3), 1062-1077.

[33] Nourian, M., Caines, P. E., Malhamé, R. P., & Huang, M. (2013). Nash, social and centralized solutions to consensus problems via mean-field control theory. IEEE Trans. Autom. Control, 58(3), 639-653.

[34] Olfati-Saber, R., Fax, J. A., & Murray, R. M. (2007). Consensus and cooperation in networked multi-agent systems. Proc. IEEE, 95(1), 215-233.

[35] Qi, Q., Zhang, H., & Wu, Z. (2019). Convex symmetric stochastic linear-quadratic mean-field optimal control problem. Automatica, 106, 1-11.

[36] Schmidl, A., & IEEE Trans. Autom. Control, 63(10), 3487-3494.

[37] Salhab, R., Ny, J. L., & Malhamé, R. P. (2018). Dynamic collective choice: Social optimal. IEEE Trans. Autom. Control, 63(10), 3487-3494.

[38] Sanjari, S. & Yuksel, S. (2019). Convex symmetric stochastic dynamic teams and their mean-field limit. Proc. 58th IEEE Annual Conference on Decision and Control, Nice, France, 4652-4657.
[39] Schelling, T. C. (1971). Dynamic models of segregation. J. Math. Soc., 1, 143-186.
[40] Sun, J., Li, X., & Yong, J. (2016). Open-loop and closed-loop solvabilities for stochastic linear quadratic optimal control problems. SIAM J. Control Optim., 54(5), 2274-2308.
[41] Wang, B.-C., & Huang, M. (2019). Mean field production output control with sticky prices: Nash and social solutions. Automatica, 100, 90-98.
[42] Wang, B.-C., Ni, Y.-H., & Zhang, H. (2019). Mean field games for multi-agent systems with multiplicative noises. International Journal of Robust and Nonlinear Control, 29, 6081-6104.
[43] Wang, B.-C., & Zhang, J.-F. (2012). Mean-field games for large-population multiagent systems with Markov jump parameters. SIAM J. Control and Optimization, 50(4), 2308-2334.
[44] Wang, B.-C., & Zhang, J.-F. (2012). Distributed control of multi-agent systems with random parameters and a major agent. Automatica, 48(9), 2093-2106.
[45] Wang, B.-C., & Zhang, J.-F. (2017). Social optima in mean-field linear-quadratic-Gaussian models with Markov jump parameters. SIAM J. Control and Optimization, 55(1), 429-456.
[46] Wang, B.-C., Zhang, H., & Zhang, J.-F. (2020). Mean field linear quadratic control: uniform stabilization and social optimality. Automatica, 121, article 109088.
[47] Weintraub, G., Benkard, C., & Van Roy, B. (2008). Markov perfect industry dynamics with many firms. Econometrica, 76(6), 1375-1411.
[48] Witsenhausen, H. S. (1968). A counterexample in stochastic optimum control. SIAM J. Control, 6, 131-147.
[49] Yong, J. (2013). Linear–quadratic optimal control problems for mean-field stochastic differential equations. SIAM J. Control Optim., 51(4), 2809-2838.
[50] Zhang, W., Zhang, H., & Chen, B. S. (2008). Generalized Lyapunov equation approach to state-dependent stochastic stabilization/detectability criterion. IEEE Trans. Autom. Control, 53(7), 1630-1642.
[51] Zhang, Z., Xie, K., Cai, Q., & Fu, M. (2019). A BP-like distributed algorithm for weighted average consensus. Proc. Asian Control Conference, Japan, 728-733.
This figure "hszhang.jpg" is available in "jpg" format from:

http://arxiv.org/ps/2206.05754v1
This figure "myfu.JPG" is available in "JPG" format from:

http://arxiv.org/ps/2206.05754v1