NODAL CURVES AND POSTULATION OF GENERIC FAT POINTS ON SURFACES

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ABSTRACT. Let $X$ be a smooth projective surface. Here we study the postulation of a general union $Z$ of fat points of $X$, when most of the connected components of $Z$ have multiplicity 2. This problem is related to the existence of "good" families of curves on $X$, with prescribed singularities, most of them being nodes, and to the cohomology of suitable line bundles on blowing ups of $X$. More precise statements are obtained in the case $X = \mathbb{P}^2$.

INTRODUCTION

Let $X$ be an algebraic surface, defined over an algebraically closed field of characteristic 0; let $m > 0$ be an integer and let $P \in X$. The $(m-1)$-th infinitesimal neighbourhood of $P$ in $X$ will be denoted by $mP$, hence $mP$ has $(I_{P,X})^m$ as ideal sheaf. Often $mP$ is called a fat point; $m$ is the multiplicity of $mP$ and $h^0(mP, O_{mP}) = m(m+1)/2$ is called its degree or its length. If $s, m_1, \ldots, m_s$ are positive integers and $P_1, \ldots, P_s$ are distinct points of $X$, the $0$-dimensional subscheme $W = \bigcup_{1 \leq i \leq s} m_i P_i$ of $X$ is called a multi-jet of $X$, with multiplicity $\max \{m_i\}$, type $(s; m_1, \ldots, m_s)$ and length $h^0(W, O_W)$. For a fixed type $(s; m_1, \ldots, m_s)$, the set of all multi-jets of type $(s; m_1, \ldots, m_s)$ on $X$, is an integral variety, of dimension $2s$; hence we may speak of the general multi-jet of type $(s; m_1, \ldots, m_s)$.

In this paper, we study the postulation of a $0$-dimensional general subscheme $Z$ of a smooth complex projective surface, under the assumption that "many" of the connected components of $Z$ are fat points of multiplicity 2. This study is a key tool for the understanding of families of curves with prescribed singularities, many of them being nodes, on a smooth surface (see e.g. [2] and [3]). This study gives also cohomological results for suitable line bundles on certain blowing ups of $X$ ([4]). Our result 0.1 below is related with the study of such families, in the blowing up of $\mathbb{P}^2$ at $r$ general points.

We state all our results in the introduction, the proofs will be given in section 1. They use a very powerful lemma ([1], Lemma 2.3) which is a key improvement of the so-called Horace method, used in [4] for this type of problems.

\textbf{Theorem 0.1.} Fix positive integers $t, r, d_1, \ldots, d_r$ and $e$; set $d_j = 2$ for $r < j \leq r+e$. Set $m = \max \{d_i\}_{1 \leq i \leq r+e}$. Assume $(t+2)(t+1)/2 \geq 1 + \sum_{1 \leq j \leq r+e} d_j(d_j+1)/2$.
and \(e \geq (m-1)(t-1)/2\). Then for a general multi-jet \(Z := \bigcup_{1 \leq j \leq r+e} d_j P_j\) of type \((r + e; d_1, \ldots, d_{r+e})\) in \(P^2\), we have \(h^1(P^2, I_Z(t)) = 0\).

Here is a generalization of theorem 0.1 to the case of an arbitrary smooth projective surface.

**Theorem 0.2.** Let \(X\) be a smooth projective surface. Fix integers \(t > 0, r \geq 0, d_j \geq 0 \ 1 \leq j \leq r\) and \(e\); set \(d_j = 2\) for \(r < j \leq r + e\) and \(m := \max\{d_i\}_{1 \leq i \leq r+e}\); assume \(e \geq (m-1)(t-1)/2\). Fix \(H \in \text{Pic}(X)\), with \(H\) very ample and spanned. Assume \(h^1(X, H^{\otimes j}) = 0\) for all \(j > 0\) and:

1. \(2(\sum_{1 \leq i \leq r} \max\{d_i - t + j, 0\}) + 2m \leq h^0(X, H^{\otimes j}) - h^0(X, H^{\otimes j-1})\) for all \(2 \leq j \leq t\).
2. \(h^0(X, H^{\otimes j}) \geq h^0(X, H) + \sum_{1 \leq j \leq r+e} d_j(d_j + 1)/2\).

Then for a general multi-jet \(Z := \bigcup_{1 \leq j \leq r+e} d_j P_j\) of type \((r + e; d_1, \ldots, d_{r+e})\) in \(X\), we have \(h^1(X, I_Z(t)) = 0\).

Any reader of [3] will appreciate the extensions of theorem 0.1 and theorem 0.2 to the case in which we take \(r\) arbitrary 0-dimensional connected subschemes, instead of \(r\) multiple points (see e.g. the definition of (generalized) singularity scheme, introduced in [3], and its very effective use made there). We will do this now.

Let \(Z\) be a 0-dimensional connected subscheme of the germ \(A^2_0\) of the affine plane at \(O\) and let \(W\) be a 0-dimensional connected subscheme of a smooth projective surface \(X\); set \(P := W_{\text{red}}\). We will say that \(W\) is equivalent to \(Z\), or that \(W\) has type \(Z\), if there is a formal (or étale, or analytic if the base field is \(C\)) isomorphism of the germ \(A^2_0\) on the germ of \(X\) at \(P\), sending \(Z\) onto \(W\). The multiplicity \(\text{mult}_P(W)\) of \(W\) is the maximal integer \(m\) such that \(Z \subset mP\). Note that \(\text{mult}_P(W) = \text{mult}_O(Z)\) if \(Z\) and \(W\) are equivalent.

With these notations, the proofs of Theorems 0.1 and 0.2 give without any modification the following result:

**Theorem 0.3.** Fix positive integers \(t, r, e\) and the type \(Z_1, \ldots, Z_r\) of \(r\) 0-dimensional subschemes of the germ \(A^2_0\). Set \(m' := \max\{\text{mult}_0(Z_i)\}_{1 \leq i \leq r}\) and \(m := \max\{m', 2\}\). Assume \(m' + 1 \geq \sum_{1 \leq j \leq r} \text{length}(Z_i)\) and \(e \geq (m-1)(t-1)/2\).

Then for a general reunion \(Z \subset P^2\) of \(e\) double points and \(r\) subschemes \(W_1, \ldots, W_r\), with \(W_i\) equivalent to \(Z_i\) for every \(i\), we have \(h^1(P^2, I_Z(t)) = 0\).

**Theorem 0.4.** Let \(X\) be a smooth projective surface. Fix positive integers \(t, r, e\) and the type \(Z_1, \ldots, Z_r\) of \(r\) 0-dimensional subschemes of the germ \(A^2_0\). Set \(m' := \max\{\text{mult}_0(Z_i)\}_{1 \leq i \leq r}\) and \(m := \max\{m', 2\}\). Assume \(e \geq (m-1)(t-1)/2\). Fix \(H \in \text{Pic}(X)\) very ample and spanned and assume \(h^1(X, H^{\otimes j}) = 0\) for all \(j > 0\) and:

1. \(2(\sum_{1 \leq i \leq r} \max\{d_i - t + j, 0\}) + 2m \leq h^0(X, H^{\otimes j}) - h^0(X, H^{\otimes j-1})\) for all \(2 \leq j \leq t\).
2. \(h^0(X, H^{\otimes j}) \geq h^0(X, H) + \sum_{1 \leq j \leq r+e} d_j(d_j + 1)/2\).

Then for a general reunion \(Z \subset X\) of \(e\) double points and \(r\) subschemes \(W_1, \ldots, W_r\), with \(W_i\) equivalent to \(Z_i\) for every \(i\), we have \(h^1(P^2, I_Z(t)) = 0\).

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THE PROOFS

We will use several times the following easy form of the so-called Horace Lemma ([4]):

Lemma 1.1. Let $X$ be a smooth projective surface, $H \in \text{Pic}(X)$ an effective divisor and $Z$ a 0-dimensional subscheme of $X$. Let $W := \text{Res}_D(Z)$ be the residual scheme of $Z$ with respect to $D$, i.e. let $W \subset Z$ be the subscheme of $X$ with the conductor $(I_Z : I_D)$ as ideal sheaf. Set $L := H|_D \in \text{Pic}(X)$ and assume $H^1(X, I_W \otimes H(-D)) = H^1(D, I_{Z \cap D} \otimes L) = 0$. Then $H^1(X, I_Z \otimes H) = 0$.

Proof of Theorem 0.1. If $t \leq 2$ the result is trivial, hence we may assume $t \geq 3$. We have $m < t$ because otherwise $e \geq (t - 1)^2 / 2$ and one cannot have $(t + 1)(t + 2)/2 \geq 1 + t(t + 1)/2 + 3e$.

Fix a line $D \subset \mathbb{P}^2$. Take a general multi-jet $W$ of type $(r; d_1, \ldots, d_r)$ with length($D \cap W$) $\leq t + 1$ and length($D \cap W$) as large as possible. Set $s := t + 1 - \text{length}(D \cap W)$ and let $J$ be the union of $W$, $e - [s/2]$ general double points of $D$ and $[s/2]$ general double points supported on $D$. Note that $t \leq \text{length}(D \cap J) \leq t + 1$ and that $[s/2] \leq (m - 1)/2$. Let $x$ be the number of connected components of $J$, with support on $D$; we have $x \geq 2$ because $m < t$ and $t + 1 - \text{length}(D \cap W) < m$, by the maximality of $\text{length}(D \cap W)$. Let $m'$ be the maximum of the multiplicities of the fat points of $J \cap (\mathbb{P}^2 - D)$ and $e'$ be the number of double points of $J \cap (\mathbb{P}^2 - D)$; we have $e' \geq e - [s/2]$. If $m' < m$, since $s \leq t - 1$ with strict inequality when $\text{length}(D \cap J) = t$, then we have $e' \geq e - (t - 1)/2 > (t - 1)(m - 2)/2 \geq (t - 2)(m' - 1)/2$. If $m' = m$, then $s \leq (m - 1)/2$, with strict inequality if $\text{length}(D \cap J) = t$; thus $e' \geq e - (s - 1)/2 > (t - 2)(m' - 1)/2$.

First assume $\text{length}(D \cap J) = t + 1$, i.e. $s$ even. By construction we have $h^0(D, I_{D \cap J}(t)) = h^0(D, I_{D \cap J}(t)) = 0$. Let $G := \text{Res}_D(J)$ be the residual scheme of $J$ with respect to $D$. By Lemma 1.1 and semicontinuity, it is sufficient to show that $h^1(\mathbb{P}^2, I_G(t - 1)) = 0$. $G$ contains at least $e' \geq (t - 2)(m' - 1)/2$ double points; it is not a general multi-jet, because some of the points of its support are forced to be contained in $D$. But $\text{length}(D \cap G) = \text{length} (J \cap D) - x = t + 1 - x \leq t - 1$ and we will be able to continue, exploiting again the same line, if we know how to handle the case in which $\text{length}(D \cap J) = t$, i.e. $s$ is odd, for at the next step we may meet such situation.

Assume $\text{length}(D \cap J) = t$. We take a general $P \in D$ and set $E := J \cup \{P\}$. Let $p$ be the length 2 subscheme of $D$ with $p_{\text{red}} = \{P\}$; the scheme $p$ is the second simple residue of $P$ with respect to $D$, in the sense of [1], Definition 2.2. Note that $J$ is a general multi-jet of type $(r + e - 1; d_1, \ldots, d_r, 2, \ldots, 2)$, containing $J \cap D$. Set $G' := \text{Res}_D(J) \cup p$; we have $h^0(D, I_{D \cap E}(t)) = h^1(D, I_{D \cap E}(t)) = 0$.

We claim that by [1], Lemma 2.3, to prove 0.1 it is sufficient to prove that $h^1(\mathbb{P}^2, I_{G'}(t - 1)) = 0$; since we will use the claim also to prove 0.2, 0.3 and 0.4, we want to give some details concerning the proof and translate the notations of [1], Lemma 2.1, in our situation. Set $\alpha := h^0(\mathbb{P}^2, I_{G'}(t - 1)) - \text{length}(G')$; the vanishing of $h^1(\mathbb{P}^2, I_{G'}(t - 1))$ is equivalent to the fact that $\alpha \geq 0$ and that for the union, $A$ of $\alpha$ general points of $\mathbb{P}^2$, we have $h^0(\mathbb{P}^2, I_{A \cup \{A\}}(t - 1)) = 0$. In the notations of the statement of [1], Lemma 2.3, we may take $Z_0 = J \cup A$, $L = O_{\mathbb{P}^2}(t)$, $H = D$, $r = h^0(D, O_D(t)) - \text{length}(D \cap J) = 1$ (hence the integer $r$ appearing in [1] is not our integer $r$) and $Q_1 = P$, i.e. $Q_1$ is a general point of $D$; hence we obtain the claim.
$G'$ is not a multi-jet, but since we want to exploit again $D$ for Lemma 1.1 and $G' \cap D$ is an effective divisor on $D$ with multiplicity 2 at $P$, this is not a problem and we may repeat the construction. To obtain $H^1(\mathbb{P}^2, I_{G'}(t-1)) = 0$, we use in an essential way that $x \geq 2$ in the following argument: since $P \in D$ and $\text{length}(p) = \text{length}(\{P\}) + 1$, we have $\text{length}(D \cap G') = 2 + \text{length}(\text{Res}_D(J) \cap D) = 2 + \text{length}(J \cap D) - x = 2 + t - x \leq t = h^0(D, \mathcal{O}_D(t-1))$. Alternatively, we may be sure that $J \cap D$ is not connected (i.e. that $x \geq 2$) if we impose that at each step we add at least a double point; if however at the previous step we added a double point, then at this step we are not forced to add a double point, say $2Q$ (except if $s > 2$), because the residual scheme $\{Q\} = \text{Res}_D(2Q)$ of 2Q is one connected component of $J$ and obviously not the unique one, when $t \geq 2$; this alternative proof is useful for 0.2, 0.3 and 0.4.

To prove $h^1(\mathbb{P}^2, I_{G'}(t-1)) = 0$, we continue with the same procedure, moving some points to $D$ and taking the residue with respect to $D$; in the residue, the contribution of $p$ disappears, hence we will never have more than one 0-dimensional component which is not a multiple point and this component (if any) will be a length 2 subscheme of $D$. Then we continue using the line $D$ to apply Lemma 1.1, each time with respect to $\mathcal{O}_{\mathbb{P}^2}(t')$, with a lower integer $t'$. In this way, we finally reduce 0.1 to a maximal rank assertion for $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ and a 0-dimensional subscheme $A \subset \mathbb{P}^2$. To conclude, it is sufficient to prove that $A$ is either empty or a reduced point. This is true because:

$$h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(t)) \geq h^0((\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(0)) + \sum_{1 \leq j \leq r+e} d_j(d_j + 1)/2. \quad \Box$$

**Proof of Theorem 0.2.** Fix $D \in |H|$, with $D$ smooth and irreducible. Since $H$ is very ample, we may find such $D$ passing through a general point $P$ of $X$ and tangent to an arbitrary tangent vector to $X$ at $P$. Set $L := H|_D$. Since $h^1(X, H^{\otimes j}) = 0$ for every $j > 0$, we have $h^0(X, H^{\otimes j+1}) = h^1(X, H^{\otimes j}) + h^0(D, L^{\otimes j+1})$ for every $j > 0$.

We do not want to assume the vanishing of $H^1(X, \mathcal{O}_X)$ and this explains why, in the statement of 0.2, we are forced to add the term $h^0(X, H)$ in equation (2).

the postulation of a general multi-jet on $D$ is as good as possible, i.e. for every integer $j > 0$ and any datum $(x, m_1, \ldots, m_x)$, then for a general multi-jet $Z$ on $D$, with datum $(x, m_1, \ldots, m_x)$, the restriction map $H^0(D, L^{\otimes j}) \to H^0(Z, L^{\otimes j}_Z)$ has maximal rank (see [1], Proposition 7.2).

We repeat verbatim the proof of 0.1. Call $G(t-j)$ the 0-dimensional scheme that we obtain after $t - j$ steps and set $Z(j) := D \cap (\text{Res}_D(G(t-j)))$. By the weak form of one of the assumptions in the statement of 0.2 (i.e. equation (1), without the term $2m$ in the left hand side) we have $\text{length}(Z(j)) \leq h^0(D, L^{\otimes j-1})$ and hence the construction is possible, even if at one step we add a second residue, supported at a point of $D$. The condition on the integer $\text{length}(D \cap G')$ appearing in the proof of 0.1 is satisfied because we added the term $2m$ in the left hand side of equation (1). \( \boxed{} \)

One should compare Theorem 0.1 and Theorem 0.2 with the very general paper [1], Theorem 1.1 and Corollary 1.2. After [1], the only justification for these kind of result is given by being very explicit.
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