Reduced phase space quantization

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We examine two singular Lagrangian systems with constraints which apparently reduce the phase space to a 2-dimensional sphere and a 2-dimensional hyperboloid. Rigorous constraint analysis by Dirac’s method, however, gives 2-dimensional open disc and an infinite plane with a hole in the centre respectively as the reduced phase spaces. Upon canonical quantisation the classical constraints show up as restrictions on the Hilbert space.

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I. INTRODUCTION

In the study of quantization of classical systems one must start with two essential things, namely, a phase space for the system and a dynamical principle. This principle may be a classical Hamiltonian derived from a Lagrangian or from a set of hyperbolic field equations. It may also be some quantum requirement such as annihilation of unphysical states by constraint operators. In an interesting paper Radhika Vathsan considered quantization of a 4-dimensional phase space with canonical coordinates \((q^1, q^2, p_1, p_2)\) with \textit{a priori} chosen constraints

\[
\phi \equiv q^1 q^2 + q^2 p_1^2 + p_2^2 - R^2 = 0 \\
\chi \equiv p_3 = 0
\]

using geometric and Dirac method of quantization. In her analysis these constraints do not follow from any Lagrangian. \(\phi\) is arbitrarily assumed to be a first class constraint and \(\chi\) is chosen as a gauge fixing condition. The reduced phase space turns out to be 2-dimensional sphere \(S^2\) of radius \(R/2\).

In the present paper we reanalyse quantization of the above system using Dirac’s method. Dirac’s method starts with a singular Lagrangian which inherently contains the constraints. Whether the constraints are first or second class follows in a straightforward manner from the analysis without any arbitrariness. We choose here two Lagrangians, the first of which reproduces the same set of constraints as Radhika Vathsan as a pair of second class constraints. The second example gives similar looking constraints but with a minus sign for the \((q^2)^2\) term. We do rigorous constraint analysis and then quantize canonically.

II. CONSTRAINT ANALYSIS AND QUANTIZATION

Consider the Lagrangian

\[
L = \frac{q^1 q^2}{4 q^2} - q^2 \left( q^1 q^2 + \frac{q^2}{3} - R^2 \right)
\]

(1)

excluding the line \(q^2 = 0\) on the configuration space. We solve for canonically conjugate momenta to get

\[
p_1 = \frac{q^1}{2 q^2} \\
p_2 = 0
\]

The second equation is a primary constraint

\[
\phi_1 \equiv p_2 = 0
\]

The Hamiltonian is given by

\[
H = q^2 p_1^2 + p_2 v_2 + q^2 \left( q^1 q^2 + \frac{q^2}{3} - R^2 \right)
\]

(3)

where \(v_2\) is unknown Lagrange multiplier. By evolving \(\phi_1\) and setting it to zero

\[
\{p_2, H\} = 0
\]

we get a secondary constraint

\[
\phi_2 \equiv q^1 q^2 + q^2 p_1^2 + p_2^2 - R^2 = 0
\]

(4)

Further evolution of \(\phi_2\) determines \(v_2\)

\[
v_2 = 0
\]

(5)

There are no further constraints. We, therefore, obtain two constraints \(\phi_1, \phi_2\) with non-zero Poisson bracket between them and so are second class constraints. To get the reduced phase space the extra degrees of freedom corresponding to these constraints must be completely removed. The Dirac bracket is defined by

\[
\{f, g\}_D = \{f, g\} - \{f, \phi_i\} (C^{-1})_{ij} \{\phi_j, g\}
\]

(6)

for any two classical observables \(f(q, p), g(q, p)\). The matrix \(C\) is

\[
C = \begin{pmatrix}
0 & -2 q^2 \\
2 q^2 & 0
\end{pmatrix}
\]

where

\[
C_{ij} = \{\phi_i, \phi_j\}
\]

The basic Dirac brackets are
\[ \{q^i, p_i\}_D = 1 \]
\[ \{\dot{q}^i, p_i\}_D = -\frac{q^i}{q^2} \]
\[ \{q^i, q^j\}_D = -\frac{p_i}{q^2} \]

(7)

The rest are zero.

Next we put both constraints equal to zero. \(\phi_1 = 0\) eliminates \(p_2\) and from \(\phi_2 = 0\) we eliminate \(q^2\).

\[ q^2 = \pm \sqrt{R^2 - p_1^2 - q^2} \]

(8)

There is \pm sign ambiguity in \(q^2\) which corresponds to the fact that the configuration space we started with consisted of two disconnected parts given by \(q^2 > 0\) and \(q^2 < 0\). This amounts to residual freedom in obtaining the phase space even after imposing all constraint conditions. For each choice of sign for \(q^2\) eqn(8) gives

\[ p_1^2 + q^2 = R^2 - q^2 \]

(9)

or,

\[ p_1^2 + q^2 < R^2 \]

(10)

The completely reduced phase space is, therefore, a disc of radius \(R\) without the boundary. Choosing \(q^2\) to be positive the reduced Hamiltonian is

\[ H = \frac{2}{3} \left( R^2 - p_1^2 - q^2 \right)^{3/2} \]

(11)

The equations of motion are

\[ \dot{q}^i = \{q^i, H\} = \frac{\partial H}{\partial p_i} = 2q^i p_i \]

\[ \dot{p}_i = \{p_i, H\} = -\frac{\partial H}{\partial q^i} = -2q^i q^j \]

(12)

We proceed to canonically quantise the system. Hilbert space \(\mathcal{H}\) consists of configuration space wave-functions \(\psi(q)\) (index 1 has been dropped from \(q^i\)) which are square integrable in the interval \(-R < q < R\). Observables are self-adjoint operators on \(\mathcal{H}\). \(q\) and \(p\) go over to position and momentum operators

\[ q^i \rightarrow \hat{q} \]

\[ \hat{q} \psi(q) = q \psi(q) \]

(13)

\[ p_1 \rightarrow \hat{p} \equiv -i\hbar \frac{\partial}{\partial q} \]

(14)

and \(\hat{q}, \hat{p}\) satisfy the commutation relation

\[ [\hat{q}, \hat{p}] = i\hbar \]

(15)

The evolution of the system is generated by the Hamiltonian operator

\[ \hat{H} = \frac{2}{3} \left( R^2 - \hat{p}^2 - \hat{q}^2 \right)^{3/2} \]

(16)

which must be self-adjoint for the evolution to be unitary.

Further, the operator \(\left( R^2 - \hat{p}^2 - \hat{q}^2 \right)\) must have positive eigenvalues for \(\hat{H}\) to be positive-definite. This requirement is due to the classical constraints showing up at the quantum level.

Consider next the Lagrangian

\[ L = \frac{\dot{q}^2}{4q^2} - q^2 \left( \frac{q^2}{3} - R^2 \right) \]

(17)

The constraints for this system are

\[ \chi_1 \equiv p_2 = 0 \]

\[ \chi_2 \equiv q^2 - q^2 + p_1^2 + p_2^2 - R^2 = 0 \]

(18)

Constraint analysis is straightforward. The reduced phase space is obtained from the inequality

\[ p_1^2 + q^2 > R^2 \]

(19)

\(q^2\) has two branches. For each choice the reduced phase space is 2-dimensional infinite plane with a hole of radius \(R\) at the centre, where we have restricted \(q^2\) to be positive. The reduced Hamiltonian is

\[ H = \frac{2}{3} \left( p_1^2 + q^2 - 2R^2 \right)^{3/2} \]

(20)

The system can then be quantized. The Hilbert space consists of square-integrable functions on real line \(\mathbb{R}\) excluding the interval \([-R, R]\). The evolution will be generated by the Hamiltonian

\[ \hat{H} = \frac{2}{3} \left( \hat{p}^2 + \hat{q}^2 - R^2 \right)^{3/2} \]

As in the earlier case the self-adjointness and positive definiteness of the Hamiltonian will restrict the Hilbert space.

**III. CONCLUSION**

We have discussed two singular systems with non-trivial reduced phase spaces. For the first system the reduced phase space is not \(S^2\) as it appears to be but an open disc of radius \(R\). If we include the boundary we can map all points on it to the south pole of \(S^2\) with radius.
The reduced space would then be $S^2$. However, this can be done only if the singularity at $q^2 = 0$ is a coordinate singularity and can be removed by appropriate choice of coordinates. This is not the case for us since the Lagrangian we chose is essentially singular at $q^2 = 0$. For the second system we find that the reduced phase is 2-dimensional plane with a hole at the center. This Lagrangian is again singular at $q^2 = 0$ which cannot be removed by any coordinate transformation. Canonical quantization reveals restrictions on the Hilbert spaces of the two systems. These restrictions are manifestations of the classical constraints at the quantum level.

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