All partitions have small parts -
Gallai-Ramsey numbers of bipartite graphs

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Abstract

Gallai-colorings are edge-colored complete graphs in which there are no rainbow triangles. Within such colored complete graphs, we consider Ramsey-type questions, looking for specified monochromatic graphs. In this work, we consider monochromatic bipartite graphs since the numbers are known to grow more slowly than for non-bipartite graphs. The main result shows that it suffices to consider only 3-colorings which have a special partition of the vertices. Using this tool, we find several sharp numbers and conjecture the sharp value for all bipartite graphs. In particular, we determine the Gallai-Ramsey numbers for all bipartite graphs with two vertices in one part and initiate the study of linear forests.

1 Introduction

Ramsey numbers have been a hot topic in mathematics for decades now due to their intrinsic beauty, wide applicability, and overwhelming difficulty despite somewhat misleadingly simple statements. The notion of “order amid chaos” defines the entire concept while applications to many different

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areas of mathematics and other sciences drive the study of new directions and extensions. See [19] for a dynamic survey of known small Ramsey numbers and [20] for a dynamic survey of applications of Ramsey Theory.

Recall that the Ramsey number \( R(p, q) \) is the minimum integer \( n \) such that, in every coloring of the edges of the complete graph on \( n \) vertices using red and blue, there is either a red clique of order \( p \), or a blue clique of order \( q \). For more general graphs \( G \) and \( H \), let \( R(G, H) \) denote the minimum integer \( n \) such that, in every coloring of the edges of the complete graph on \( n \) vertices using red and blue, there is either a red copy of \( G \) or a blue copy of \( H \).

We consider edge-colorings of complete graphs which contain no rainbow triangles. This restricted class of colorings is particularly interesting due to the powerful structure provided by the following result of Gallai.

**Theorem 1** ([3, 12, 14]). In any edge-coloring of a complete graph with no rainbow triangle, there exists a partition of the vertices into at least two parts (called a Gallai partition or G-partition for short) such that, there are at most two colors on the edges between the parts, and only one color on the edges between each pair of parts.

In light of this result, we say that a colored complete graph with no rainbow triangle is a Gallai coloring (or G-coloring for short). Closely related to results in [9], Fox et al. posed a conjecture about monochromatic complete graphs. In order to concisely state their conjecture, we must present some definitions.

Given a graph \( H \), the \((k\text{-colored})\) **Gallai-Ramsey number** \( gr_k(K_3 : H) \) is defined to be the minimum integer \( n \) such that every \( k\)-coloring (using all \( k \) colors) of the complete graph on \( n \) vertices contains either a rainbow triangle or a monochromatic copy of \( H \).

We refer to the survey of rainbow generalizations of Ramsey Theory [10, 11] for more information on this topic and a complete list of known results involving Gallai-Ramsey numbers. We are now able to state the conjecture of Fox et al.

**Conjecture 1** ([9]). For \( k \geq 1 \) and \( p \geq 3 \),

\[
gr_k(K_3 : K_p) = \begin{cases} 
(R(p, p) - 1)^{k/2} + 1 & \text{if } k \text{ is even,} \\
(p - 1)(R(p, p) - 1)^{(k-1)/2} + 1 & \text{if } k \text{ is odd.}
\end{cases}
\]

The case where \( p = 3 \) was actually verified first in 1983 by Chung and Graham [5] and then again over the years in different contexts.
Theorem 2 \([15, [13]]\). For \(k \geq 1\),

\[
gr_k(K_3 : K_3) = \begin{cases} 
5^{k/2} + 1 & \text{if } k \text{ is even,} \\
2 \cdot 5^{(k-1)/2} + 1 & \text{if } k \text{ is odd.}
\end{cases}
\]

The next case of this conjecture was recently proven in [16].

Theorem 3 \([16]\). For \(k \geq 1\),

\[
gr_k(K_3 : K_4) = \begin{cases} 
17^{k/2} + 1 & \text{if } k \text{ is even,} \\
3 \cdot 17^{(k-1)/2} + 1 & \text{if } k \text{ is odd.}
\end{cases}
\]

The landscape for finding monochromatic bipartite graphs is very different. Where non-bipartite monochromatic graphs yield exponential functions of the number of colors \(k\), as seen in the previous results, bipartite monochromatic graphs yield linear functions of \(k\) (see Theorem 9 below). In this work, we therefore consider the Gallai-Ramsey question for bipartite graphs, particularly complete bipartite graphs. Given a bipartite graph \(H\), let \(s(H)\) be the order of the smallest part of the bipartition of \(H\). Our main tool, which allows us to prove several results for different classes of bipartite graphs, is the following reduction theorem.

Theorem 4. Given a bipartite graph \(H\) and a positive integer \(R \geq R(H, H)\), if every \(G\)-coloring of \(K_R\) using only 3 colors, in which all parts of a \(G\)-partition have order at most \(s(H) - 1\), contains a monochromatic copy of \(H\), then

\[
gr_k(K_3 : H) \leq R + (s(H) - 1)(k - 2).
\]

Essentially, this result says that if we intend to prove that

\[
gr_k(K_3 : H) \leq R + (s(H) - 1)(k - 2),
\]

then it suffices to prove that every \(G\)-coloring of \(K_R\) using only 3 colors, in which all parts of a \(G\)-partition are small, contains a monochromatic copy of \(H\).

The remainder of this paper is organized as follows. Section 2 contains several known useful results along with some helpful lemmas which will be used later. Section 3 contains the proof of Theorem 4. The remaining sections use Theorem 4 to produce Gallai-Ramsey results for certain classes of bipartite graphs except Section 8 which contains our broad conjecture of the value for all bipartite graphs.
2 Preliminaries

We first state some known classical 2-color Ramsey numbers for complete bipartite graphs.

Theorem 5 ([2]). \( R(K_{2,3}, K_{2,3}) = 10 \).

Theorem 6 ([15]). \( R(K_{3,3}, K_{3,3}) = 18 \).

Theorem 7 ([7]). If \( 4n - 4 = 2^t k_1 \ldots k_s \) with \( t \geq 0 \), and either \( k_i = p_i^{r_i} + 1 \) where \( p_i^{r_i} \equiv 3 \pmod{4} \) is a prime, or \( k_i = 2(q_i^{s_i} + 1) \) where \( q_i^{s_i} \equiv 1 \pmod{4} \) is a prime power, then \( R(K_{2,n}, K_{2,n}) \geq 4n - 3 \) if \( t > 0 \), and \( R(K_{2,n}, K_{2,n}) \geq 4n - 4 \) otherwise.

We will also use the following result concerning monochromatic stars.

Theorem 8 ([14]). Any Gallai coloring of \( K_n \) contains a monochromatic star \( S_t \) with \( t \geq 2n/5 \).

Our first lemma provides the lower bound for our results on the Gallai-Ramsey number for bipartite graphs. Indeed, we believe this lower bound to be sharp for all connected bipartite graphs (see Conjecture 4).

Lemma 1. For a given connected bipartite graph \( H \) with Ramsey number \( R(H, H) = R \), we have

\[
gr_k(K_3 : H) \geq R + (s(H) - 1)(k - 2).
\]

Proof. With \( R = R(H, H) \), there exists a 2-colored complete graph \( G_2 \) on \( R - 1 \) vertices containing no monochromatic copy of \( H \). Given a colored complete graph \( G_{i-1} \), for each additional color \( i \) with \( 3 \leq i \leq k \), we add \( s(H) - 1 \) vertices with all incident edges colored in color \( i \) to create the graph \( G_i \). The resulting graph \( G_k \) is a \( k \)-colored complete graph on \( R - 1 + (s(H) - 1)(k - 2) \) vertices containing no monochromatic copy of \( H \) and no rainbow triangle. \(\square\)

Note that this construction, and therefore this lower bound, does not suffice if the monochromatic graph \( H \) is not bipartite. Indeed, it is easy to show that if \( H \) is not bipartite, then \( gr_k(K_3 : H) \) grows as an exponential function of \( k \), stated more precisely in the following result.

Theorem 9 ([13]). Let \( H \) be a fixed graph with no isolated vertices. If \( H \) is not bipartite, then \( gr_k(K_3 : H) \) is exponential in \( k \). If \( H \) is bipartite, then \( gr_k(K_3 : H) \) is linear in \( k \).
The following technical lemma will be used to eliminate many cases from our main results. It then suffices to consider only two possibilities: when a largest part of a G-partition is rather small or when this largest part is rather large.

**Lemma 2.** Given positive integers \( \ell, m, n \) where \( \ell \leq m \) and \( n \geq 3m - 2 \), let \( H = K_{\ell,m} \) be a complete bipartite graph and let \( G \) be a G-coloring of \( K_n \) with no monochromatic copy of \( H \). In any G-partition of \( G \), any largest part of the partition has order either at most \( \ell - 1 \) or at least \( n - 2\ell + 2 \).

**Proof.** Let \( H \) and \( G \) be as given and let \( H_1 \) be a largest part of a G-partition of \( G \). Suppose, for a contradiction, that \( \ell \leq |H_1| \leq n - 2\ell + 1 \).

First suppose that \( |H_1| \leq n - 2m + 1 \). Since \( H_1 \) is a part of a G-partition, every other vertex of \( G \) has one of only two colors on all edges to \( H_1 \). By the pigeonhole principle, there must be at least \( m \) vertices outside \( H_1 \), say a set \( S \), with all one color on edges to \( H_1 \). Since \( |H_1| \geq \ell \), \( S \cup H_1 \) contains a monochromatic copy of \( K_{\ell,m} \), a contradiction.

Finally suppose that \( n - 2m + 2 \leq |H_1| \leq n - 2\ell + 1 \). This means that there are at least \( 2\ell - 1 \) vertices in \( G \setminus H_1 \). Since \( n \geq 3m - 2 \), we get \( |H_1| \geq n - 2m + 2 \geq m \). By the pigeonhole principle, there is a set \( S \) of at least \( \ell \) vertices in \( G \setminus H_1 \) with the property that all edges between \( S \) and \( H_1 \) have the same color. Then \( S \cup H_1 \) induces a monochromatic graph containing \( K_{\ell,m} \), a contradiction completing the proof of Lemma 2. \( \square \)

## 3 Proof of Theorem 4

Recall that Theorem 4 stated as follows. Given a bipartite graph \( H \) and a given positive integer \( R \geq R(H,H) \), if we intend to prove that

\[
gr_k(K_3 : H) \leq R + (s(H) - 1)(k - 2),
\]

then it suffices to prove that every G-coloring of \( K_R \) using only 3 colors, in which all parts of the partition have order at most \( s(H) - 1 \), contains a monochromatic copy of \( H \).

**Proof.** Let \( H \) be the given bipartite graph with \( a = s(H) \) and \( b = |H| - a \). Let \( G \) be a G-coloring of \( K_n \) where \( n = R + (a - 1)(k - 2) \). Define a set \( T \) of vertices \( \{v_1, v_2, \ldots, v_T\} \) to be a largest set with the property that each vertex \( v_i \) has all except possibly at most \( a - 1 \) of its edges to \( G \setminus \{v_1, v_2, \ldots, v_i\} \) in a single color, with the extra restriction that if any of these edges has a different color, it must be an edge of the form \( v_i v_{i+\ell} \)
where $\ell \leq a - 1$. Note that $|T| \leq (a - 1)k$ since if $|T| \geq (a - 1)k + 1$, then there would exist a color, say $i$, such that at least $a$ vertices among the first $(a - 1)k + 1$ vertices of $T$ have all edges to the rest of the graph in color $i$. This yields a monochromatic $K_{a,b}$ in color $i$, which contains the desired bipartite graph $H$.

By Lemma 2, the largest part of any G-partition of $G$ has order either at least $n - 2a + 2$ or at most $a - 1$. For the proof of this lemma, it suffices to suppose the largest part $H_1$ has order at least $n - 2a + 2$. Then the vertices of $G \setminus H_1$ can be added to $T$, contradicting the maximality of $|T|$.

By Theorem 1, there is a G-partition of $G \setminus T$, say using colors red and blue. We now replace any vertices of $T$ (if they exist) that have red or blue edges to $G \setminus T$, replace all edges within the parts of the G-partition with a third color, and let $G'$ be the resulting 3-colored complete graph. Note that $|G'| \geq R$. Certainly there is a G-partition of $G'$ using colors red and blue with each part having order at most $a - 1$. Also note that a rainbow triangle and a monochromatic copy of $H$ can easily be avoided within $T$ so it suffices to consider the 3-coloring $G'$ of order $R$.

4 Bipartite Graphs With $s(H) = 2$

Recall that $s(H)$ is the order of the smallest part of the bipartition of $H$. For general bipartite graphs $H$ with $s(H) = 2$, we obtain the following very broad result.

Theorem 10. Let $H$ be a bipartite graph with $s(H) = 2$ and $R(H, H) = R$. Then for any integer $k \geq 2$, we have

$$gr_k(K_3 : H) = R + (k - 2).$$

Proof. The lower bound follows from Lemma 1. For the upper bound, let $G$ be a G-coloring of $K_n$ where $n = R + k - 2$ containing no monochromatic copy of $K_{2,m}$. By Theorem 1 it suffices to consider a 3-coloring $G'$ of $K_R$ with a G-partition in which all parts have order 1. Since there are only two colors in the G-partition, this is, in fact, a 2-coloring of $K_R$. By the definition of $R$, this contains a monochromatic $K_{2,m}$, a contradiction to complete the proof.

Using Theorem 4 the proof of Theorem 10 and therefore we have the following corollary.

Corollary 11. Let $R_{2,m} = R(K_{2,m}, K_{2,m})$. For $k \geq 2$ and $m \geq 3$, we have

$$gr_k(K_3 : K_{2,m}) = R_{2,m} + (k - 2).$$
5 Complete Bipartite Graph $K_{3,m}$

For complete bipartite graphs $H$ with $s(H) = 3$, we obtain the following results.

**Theorem 12.** For $k \geq 3$, we have

$$gr_k(K_3 : K_{3,3}) = 2k + 14.$$<br>

More generally, for $K_{3,m}$, we have the following result.

**Theorem 13.** For $k \geq 3$ and $m \geq 3$, we get

$$R(K_{3,m}, K_{3,m}) + 2(k - 2) \leq gr_k(K_3 : K_{3,m}) \leq \max\{(6m - 2), R(K_{3,m}, K_{3,m})\} + 2(k - 2).$$

Although Theorem 12 is actually a corollary of Theorem 13, we include the proof of Theorem 12 since it deals with explicit values.

Since $2^{(3m - 1)/(3 + m)} \leq R(K_{3,m}, K_{3,m}) \leq 8m - 2$ from [18] and [17], the sharpness of this result in general depends on the Ramsey number. The only small value of $m \geq 3$ for which $R(K_{3,m}, K_{3,m})$ is known is when $m = 3$ (see Theorem 4). Since $R(K_{3,3}) = 18 \geq 6m - 2$, the bounds in Theorem 13 are equal, establishing the conclusion of Theorem 12 as noted above. Otherwise, the general relationship between $R(K_{3,m}, K_{3,m})$ and $6m - 2$ is not clear.

**Proof of Theorem 12.** The lower bound follows from Lemma 1. For the upper bound, suppose $G$ is a $G$-coloring of $K_n$ using at most $k$ colors where $n = 2k + 14$ and suppose $G$ contains no monochromatic $K_{3,3}$. By Theorem 4 we may consider a 3-coloring $G'$ of $K_{18}$ with a $G$-partition in which all parts have order at most 2. In particular, since $|G'| = 18$, this means that there are at least 9 parts in the $G$-partition.

Let $R$ be the reduced graph of the $G$-partition of $G'$. Since $R(C_4, C_4) = 6$, if there at least 5 parts of the $G$-partition each containing at least 2 vertices, then the reduced graph corresponding to these parts (along with an additional part if we have only 5 of order 2) contains a monochromatic $C_4$. Such a subgraph corresponds to a monochromatic complete bipartite subgraph with at least 3 vertices in each part, a contradiction. We may therefore assume that there are at most 4 parts of this $G$-partition which have order 2 while all the rest have order 1. On the other hand, since $|G'| = 18$, if the $G$-partition had only parts of order 1, then $G'$ would simply be a 2-coloring and the result would follow from $R(K_{3,3}) = 18$. We
may therefore assume that the number of parts in the partition \( t \) satisfies \( 14 \leq t \leq 17 \) and there are between 1 and 4 parts of order 2.

Let \( A \) be a part of order 2. There are at least 16 vertices remaining in \( G' \setminus A \) so at least 8 of them must have a single color on all edges to \( A \), say blue. Let \( B \) be a set of 8 vertices with all blue edges to \( A \) (chosen so that if \( B \) contains a vertex of a part of the G-partition, then \( B \) contains all vertices of that part). Let \( v_1, v_2, v_3 \) and \( v_4 \) be four vertices in \( G' \setminus (A \cup B) \).

To avoid creating a blue \( K_{3,3} \), each vertex \( v_i \) shares at most 2 blue neighbors with \( A \), so each vertex \( v_i \) has at least 6 red edges to \( B \). Every pair of vertices \( v_i, v_j \) will then share at least 4 red neighbors in \( B \) and every triple of these vertices must share at least 2 common red neighbors in \( B \). Certainly if a triple shares at least 3 red neighbors, this would be a red \( K_{3,3} \), so this means that, in particular, \( v_1, v_2, v_3 \) must share exactly 2 common red neighbors in \( B \) and the red edges from these three vertices to \( B \) are strictly prescribed. More specifically, if \( b_1, \ldots, b_8 \) are the vertices of \( B \), we may assume that \( v_1 \) has green edges to \( b_1, b_2 \), \( v_2 \) has green edges to \( b_3, b_4 \), and \( v_3 \) has green edges to \( b_5, b_6 \). Then regardless of the choice of the 6 red edges from \( v_4 \) to \( B \), there are three vertices \( v_i, v_j, v_\ell \) which share at least 3 common red neighbors in \( B \), producing a red \( K_{3,3} \) for a contradiction.

\( \square \)

**Proof of Theorem 13.** The lower bound follows from Lemma 1. For the upper bound, suppose \( G \) is a G-coloring of \( K_n \) using at most \( k \) colors where

\[
n = \max\{(6m - 2), R(K_{3,m}, K_{3,m})\} + 2(k - 2)
\]

and suppose \( G \) contains no monochromatic \( K_{3,m} \). By Theorem 1 we may consider a 3-coloring \( G' \) of \( K_{n-2(k-2)} \) with a G-partition with colors red and blue in which all parts have order at most 2. In particular, since \( |G'| \geq 6m - 2 \), this means that there are at least \( 3m - 1 \) parts in this G-partition.

Since \( |G'| \geq R(K_{3,m}, K_{3,m}) \), there must exist at least one part \( X \) of this G-partition of order 2. First suppose there are at most \( 2m - 1 \) vertices in \( G' \setminus X \) with some color, say red, on edges to \( X \). This means that there are at least

\[
n - 2(k - 2) - 2 - (2m - 1) \geq 4m - 3
\]

vertices in \( G' \setminus X \) with all blue edges to \( X \). Let \( A \) be a set of \( 4m - 3 \) of these vertices. Let \( v_1, v_2, v_3 \) be three of the vertices in \( G' \setminus (X \cup A) \). In order to avoid creating a blue \( K_{3,m} \), \( v_i \) can have at most \( m - 1 \) blue edges to \( A \) so all remaining edges must be red. With \( |A| = 4m - 3 \), there must be a set of at least \( m \) vertices in \( A \) with all red edges to \( v_1, v_2, v_3 \), creating a red \( K_{3,m} \). This means that there is no color \( c \) (among red and blue) for which at most \( 2m - 1 \) vertices in \( G' \setminus X \) have color \( c \) on edges to \( X \).
Let \( A \) (and \( B \)) be the set of vertices in \( G \setminus X \) with blue (respectively red) edges to \( X \). Note that we have \(|A|, |B| \geq 2m\). To avoid creating a red \( K_{3,m} \), each vertex in \( A \) must have at most \( m - 1 \) red edges to \( B \). Symmetrically, each vertex in \( B \) must have at most \( m - 1 \) blue edges to \( A \). This means that there are a total of at most \((|A| + |B|)(m - 1)\) edges between \( A \) and \( B \). Since \(|A|, |B| \geq 2m\), we get

\[
(|A| + |B|)(m - 1) < |A| \cdot |B|,
\]
a contradiction, completing the proof.

### 6 General Complete Bipartite Graphs

For large complete bipartite graphs, the following was recently shown.

**Theorem 14** ([4]). For fixed integers \( k \geq 2 \) and \( m \geq 1 \), if \( \ell \to \infty \), then

\[
(1 - o(1))2^m n \leq gr_k(K_3 : K_{\ell,m}) \leq (2^m + 2^{m/2+1} + k)n + 4m^3.
\]

We obtain the following bounds on the Gallai-Ramsey numbers for all complete bipartite graphs.

**Theorem 15.** Given positive integers \( \ell, m \) where \( \ell \leq m \), let \( H = K_{\ell,m} \) and let \( R = R(H,H) \). Then

\[
R + (\ell - 1)(k - 2) \leq gr_k(K_3 : H) \leq (R + k - 3)(\ell - 1) + 1.
\]

**Proof.** The lower bound follows from Lemma [4]. For the upper bound, suppose \( G \) is a \( G \)-coloring of \( K_n \) using at most \( k \) colors where \( n = (R - 1)(m - 1) + (\ell - 1)(k - 2) \) and suppose \( G \) contains no monochromatic \( K_{\ell,m} \).

By Theorem [4] we may consider a 3-coloring \( G' \) of \( K_{(R-1)(\ell-1)+1} \) with a \( G \)-partition in which all parts have order at most \( \ell - 1 \). By the definition of \( R \), there are at most \( R - 1 \) parts so with each part having order at most \( \ell - 1 \), there can be at most \((R - 1)(\ell - 1)\) vertices in \( G' \), a contradiction.

### 7 Linear Forests

It is worth noting that the Gallai-Ramsey number for matchings is exactly the same as the Ramsey number for matchings (proven in [6]) since the sharpness example for the Ramsey number contains no rainbow triangle.
Corollary 16. For positive integers \( k, n_1, n_2, \ldots, n_k \) with \( n_1 = \max \{ n_i \} \), we have
\[
gr_k(K_3 : n_1P_2, n_2P_2, \ldots, n_kP_2) = R(n_1P_2, n_2P_2, \ldots, n_kP_2)
= n_1 + 1 + \sum_{i=1}^{k}(n_i - 1).
\]

For copies of \( P_3 \), the situation is not quite as clear. The 2-color Ramsey number was solved (in a more general form) in [8] with the following result.

Theorem 17 ([8]). For positive integers \( n_1 \geq n_2 \), we have
\[
R(n_1P_3, n_2P_3) = 3n_1 + n_2 - 1.
\]

We conjecture that this result extends to more general G-colorings in the following sense.

Conjecture 2. For positive integers \( k, n_1, n_2, \ldots, n_k \) with \( n_1 = \max \{ n_i \} \), we have
\[
gr_k(K_3 : n_1P_3, n_2P_3, \ldots, n_kP_3) = 2n_1 + 1 + \sum_{i=1}^{k}(n_i - 1).
\]

As a partial result, we prove the following.

Theorem 18. For positive integers \( k, n_1, n_2, \ldots, n_k \) with \( n_1 = \max \{ n_i \} \), we have
\[
2n_1 + 1 + \sum_{i=1}^{k}(n_i - 1) \leq gr_k(K_3 : n_1P_3, n_2P_3, \ldots, n_kP_3)
\leq 4(n_1 - 1) + \frac{9n_1 - 3}{2} \ln \left( \frac{3n_1}{2} - 1 \right) + 1 + (n_1 - 1)(k - 2).
\]

Moreover, when \( n_1 = 2 \), we have
\[
gr_k(K_3 : n_1P_3, n_2P_3, \ldots, n_kP_3) = 2n_1 + 1 + \sum_{i=1}^{k}(n_i - 1).
\]

Proof. If \( k = 1 \), the result is immediate and if \( k = 2 \), the result follows from Theorem 17 so suppose \( k \geq 3 \). For the lower bound, we follow the proof of Lemma 1. Let \( G_1 \) be a copy of \( K_{3n_1-1} \) colored entirely with color 1. Given \( G_i \), construct \( G_{i+1} \) by adding \( n_i - 1 \) vertices to \( G_i \) with all edges incident to
the new vertices having color \( i + 1 \). The resulting coloring \( G_k \) is a coloring of \( K_n \) where \( n = 2n_1 + \sum_{i=1}^{k} (n_i - 1) \) which contains no rainbow triangle and no monochromatic copy of \( n_i P_3 \) in color \( i \).

For the upper bound, let \( G \) be a G-coloring of \( K_{n'} \) where

\[
n' \geq 4(n_1 - 1) + \frac{9n_1 - 3}{2} \ln \left( \frac{3n_1}{2} - 1 \right) + 1 + (n_1 - 1)(k - 2).
\]

By Theorem 1 there is a G-partition of \( V(G) \). Choosing one vertex from each part of this partition produces a 2-colored complete graph as a subgraph of the original graph. Supposing that colors \( \text{red} \) and \( \text{blue} \) are the two colors appearing in the G-partition with \( n_{\text{red}} \geq n_{\text{blue}} \), this means that by Theorem 17 there are at most \( 3n_{\text{red}} + n_{\text{blue}} - 2 \) parts. Conversely, by Theorem 4 we may consider a 3-colored \( K_n \) where

\[
n \geq 4(n_1 - 1) + \frac{9n_1 - 3}{2} \ln \left( \frac{3n_1}{2} - 1 \right) + 1
\]

with a G-partition in which each part has order at most \( n_{\text{red}} - 1 \).

At this point, we note that if \( n_1 = 2 \), each part has order 1, meaning that the graph is simply a 2-coloring and the sharp result follows from Theorem 17. It is also worthwhile to observe that we have already shown that \( n \) must be at most \( (3n_{\text{red}} + n_{\text{blue}} - 2)(n_{\text{red}} - 1) \) so

\[
gr_k(K_3 : n_1 P_3, n_2 P_3, \ldots, n_k P_3) < 4n_1^2 + (n_1 - 1)(k - 2),
\]

but our goal is closer to a linear bound as a function of \( n_1 \).

By Corollary 16 there is a monochromatic matching within the reduced graph of this G-partition. Given an integer \( t \geq 2 \), if we restrict our attention to those “large” parts of order at least \( \frac{3n_{\text{red}}}{2t} \), then if there were at least \( 3t - 1 \) such parts, Corollary 16 would guarantee the existence of a matching on \( t \) edges within the reduced graph of these “large” parts. With such a matching, it is easy to construct the desired monochromatic \( n_{\text{red}} P_3 \) as follows. For each matching edge, say with corresponding parts \( A \) and \( B \), select \( \frac{n_{\text{red}}}{2t} \) vertices in \( A \) (call this set \( A' \)) and \( \frac{n_{\text{red}}}{2t} \) vertices from \( B \) (call this set \( B' \)) to be the central vertices. For each selected vertex in \( A \), choose two previously unused vertices of \( B \) to construct a copy of \( P_3 \) and similarly for each vertex in \( B \), choose two previously unused vertices of \( A \) to construct a copy of \( P_3 \). With \( t \) edges within the matching, this results in \( t \frac{2n_{\text{red}}}{2t} = n_{\text{red}} \) copies of \( P_3 \) all in one color, as desired. See Figure 1 for an example of this construction. Thus, we arrive at the following fact.
Fact 1. For any integer \( t \geq 2 \), there can be at most \( 3t - 2 \) parts of order at least \( \frac{3n_{\text{red}}}{2t} \).

In particular, this means that there can be up to 4 parts of order \( n_{\text{red}} - 1 \) but the 5th part must have order at most \( \frac{3n_{\text{red}} - 1}{4} \). Similarly, the 8th largest part must have order at most \( \frac{3n_{\text{red}} - 1}{6} \) and so on. This means that

\[
R(n_{\text{red}} - 1, n_{\text{red}} - 1, \ldots, n_{\text{red}} - 1) \leq 4(n_{\text{red}} - 1) + \sum_{i=3}^{n_{\text{red}}} 3 \left( \frac{3n_{\text{red}} - 1}{2t - 2} \right) \leq 4(n_{\text{red}} - 1) + \frac{9n_{\text{red}} - 3}{2} \sum_{i=3}^{n_{\text{red}}} \frac{1}{t - 1} \leq 4(n_{\text{red}} - 1) + \frac{9n_{\text{red}} - 3}{2} \ln \left( \frac{3n_{\text{red}}}{2} - 1 \right) \leq 4(n_{1} - 1) + \frac{9n_{1} - 3}{2} \ln \left( \frac{3n_{1}}{2} - 1 \right),
\]

contradicting the assumed lower bound on \( n \) and completing the proof. \( \square \)

It appears as though the general multicolor classical Ramsey number for copies of \( P_3 \) is not known so we conjecture the following.

Conjecture 3. For positive integers \( k, n_1, n_2, \ldots, n_k \) with \( n_1 = \max\{n_i\} \), we have

\[
R(n_1P_3, n_2P_3, \ldots, n_kP_3) = gr_k(K_3 : n_1P_3, n_2P_3, \ldots, n_kP_3) = 2n_1 + 1 + \sum_{i=1}^{k} (n_i - 1).
\]
8 Conclusion and Further Discussion

Based on our results and, in fact, the entire literature of results on Gallai-Ramsey numbers for a rainbow triangle or monochromatic bipartite graph, we propose the following broad conjecture.

Conjecture 4. Given any connected bipartite graph $H$ with Ramsey number $R(H, H) = R$, we believe that

$$gr_k(K_3 : H) = R + (s(H) - 1)(k - 2).$$

In particular, this would mean that the lower bound provided by Lemma 1 is always sharp.

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