FIXED POINTS OF MAPS ON THE SPACE OF RATIONAL FUNCTIONS

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ABSTRACT. Given integers \( s, t \), define a function \( \phi_{s, t} \) on the space of all formal series expansions by \( \phi_{s, t}(\sum a_n x^n) = \sum a_{sn+t} x^n \). For each function \( \phi_{s, t} \), we determine the collection of all rational functions whose Taylor expansions at zero are fixed by \( \phi_{s, t} \). This collection can be described as a subspace of rational functions whose basis elements correspond to certain \( s \)-cyclotomic cosets associated with the pair \((s, t)\).

1. INTRODUCTION

Let \( \mathcal{R} \) denote the space of rational functions with complex coefficients. The Taylor expansion at \( x = 0 \) of \( R \in \mathcal{R} \) can be written as a Laurent series, i.e.,

\[
R(x) = \sum_{n \gg -\infty} a_n x^n
\]

where \( n \gg -\infty \) denotes the fact that the coefficients vanish for large negative \( n \). For \( s, t \in \mathbb{Z} \), define the map \( \phi_{s, t} : \mathcal{R} \rightarrow \mathcal{R} \) by

\[
\phi_{s, t}(\sum a_n x^n) = \sum a_{sn+t} x^n.
\]

Denote the standard \( s \)-th root of unity throughout this paper by \( \omega_s = e^{2\pi i/s} \). When \( s \) is positive, consider the restriction \( \phi_{s, t} : \mathcal{R} \rightarrow \mathcal{R} \). One can rewrite this map explicitly without the use of series expansions:

\[
\phi_{s, t}(R(x)) = \left( \frac{1}{s} \right) x^{-s/t} \sum_{j=0}^{s-1} \omega_s^{-jt} R(\omega_s^j x^{1/s}).
\]

Indeed, if \( R(x) = \sum a_n x^n \), then \( R(\omega_s^j x^{1/s}) = \sum a_n \omega_s^{jn} x^{n/s} \), and so the coefficient of \( x^{(sn+t)/s} \) in the summation \( \sum a_n \omega_s^{jn} x^{n/s} \) is \( a_{sn+t} \omega_s^{j(sn+t)} \). Therefore, the coefficient of \( x^n \) in \( \left( \frac{1}{s} \right) x^{-s/t} \sum_{j=0}^{s-1} \omega_s^{-jt} R(\omega_s^j x^{1/s}) \) is

\[
\left( \frac{1}{s} \right) \sum_{j=0}^{s-1} \omega_s^{-jt} a_{sn+t} \omega_s^{j(sn+t)} = \left( \frac{1}{s} \right) \sum_{j=0}^{s-1} a_{sn+t} = a_{sn+t}.
\]

The map \( \phi_{2, 1} \) can be used in a general procedure for the exact integration of rational functions, as described in [2]. Dynamical properties of \( \phi_{2, 1} \), including kernels of the iterates, dynamics of subclasses of rational functions, and fixed points are discussed in [11]. The purpose of this paper is to generalize one of the results in [11] by classifying, for each pair of integers \( s, t \), the collection of all rational functions that are fixed by \( \phi_{s, t} \). If \( s \) is an integer such that \( s \leq 1 \), then \( 0 \) is the only rational function fixed by \( \phi_{s, t} \), unless, of course, \((s, t) = (1, 0)\), in which case \( \phi_{s, t} \) is the identity. When \( s \geq 2 \), however, the story is much more interesting.

2. CYCLOTOMIC COSETS

In this section, we assume throughout that \( s \geq 2, 0 \leq t \leq s - 2 \), and \( R \in \mathcal{R} \) such that

\[
\phi_{s, t}(R(x)) = R(x).
\]

Given these restrictions on \( s \) and \( t \), it follows that \( |t/(s-1)| < 1 \). Thus, if \( n \leq -1 \), then \( n < -t/(s-1) \), and so \( sn + t < n \). Assuming that \( R(x) \) is fixed by \( \phi_{s, t} \), we have that \( a_{sn+t} = a_n \) for all \( n \). Thus, if \( a_n \) is nonzero for any negative value of \( n \), then there are infinitely many nonzero coefficients of negative powers of \( x \), contradicting the assumption that \( R(x) \) is of the form given in equation (1.3).

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We write $R$ in the form

$$R(x) = \sum_{n=0}^{\infty} f(n)x^n$$

to emphasize the fact that the coefficients can be interpreted as the images of a generating function $f : \mathbb{N} \to \mathbb{C}$. Since $R(x)$ is fixed by $\phi_{s,t}$, it follows that

$$f(n) = f(sn + t)$$

for all integers $n$. The following result, which was proven on page 202 of [4], elucidates the relationship between the generating function $f$ of the coefficients of the Taylor expansion of $R(x)$ and the representation of $R(x)$ as a quotient of polynomials.

**Lemma 2.1.** Let $q_1, q_2, \ldots, q_d$ be a fixed sequence of complex numbers, $d \geq 1$, and $q_d \neq 0$. The following conditions on a function $f : \mathbb{N} \to \mathbb{C}$ are equivalent:

1. $\sum_{n \geq 0} f(n)x^n = \frac{P(x)}{Q(x)}$ where, $Q(x) = 1 + q_1x + q_2x^2 + q_3x^3 + \cdots + q_dx^d$.
2. For $n \gg 0$,

$$f(n) = \sum_{i=1}^{J} P_i(n)\lambda_1^n,$$

where $1 + q_1x + q_2x^2 + q_3x^3 + \cdots + q_dx^d = \prod_{i=1}^{J}(1 - \lambda_i x)^{d_i}$, the $\lambda_i$'s are distinct, and $P_i(n)$ is a polynomial in $n$ of degree less than $d_i$.

In this section, we construct a collection of rational functions that are fixed by $\phi_{s,t}$, and in the next section we use the above lemma to justify that this collection spans the subspace of $\mathcal{R}$ consisting of all rational functions that are fixed by $\phi_{s,t}$.

The description of all the fixed points of $\phi_{s,t}$ requires the notion of *cyclotomic cosets*: given $n$, $r \in \mathbb{N}$ with $r \geq 1$ such that $r$ and $s$ are relatively prime, we define

$$C_{s,r,n} = \{s^in \mod r : i \in \mathbb{Z}\}$$

as a finite set called the $s$-cyclotomic coset of $n$ mod $r$. We will characterize the fixed points $\phi_{s,t}$ using cyclotomic cosets with a special property. To describe this property, first define

$$\beta_{s,t}(k) = t\left(\frac{s^k - 1}{s - 1}\right)$$

for which we have the following recursive formula:

$$\beta_{s,t}(j + 1) = s\beta_{s,t}(j) + t.$$

**Definition 2.2.** A positive integer $r$ is called *distinguished* with respect to the pair $(s,t)$ if $r$ and $s$ are relatively prime and

$$r \mid \beta_{s,t}(\text{Ord}(s;r)),$$

where $\text{Ord}(s;r)$ represents the smallest positive integer $i$ such that $s^i \equiv 1 \mod r$. We say $r = 0$ is distinguished with respect to $(s,t)$ if and only if $t = 0$. We denote the set of integers distinguished with respect to $(s,t)$ by $\Omega(s,t)$.

**Proposition 2.3.** For each pair $(s,t)$, the set $\Omega(s,t)$ is infinite.

**Proof.** Since $\Omega(s,t) \subset \Omega(s,1)$, we need only show that $\Omega(s,1)$ is infinite. Let $r$ be a positive integer such that $\gcd(r, s(s-1)) = 1$. If $\alpha = \text{Ord}(s;r)$, then $s^\alpha \equiv 1 \mod r$; that is, $r \mid s^{\text{Ord}(s;r)} - 1$. Since $s^{\text{Ord}(s;r)} - 1$ is a multiple of $(s-1)$, and $r$ is relatively prime to $(s-1)$, it follows that $r(s-1) \mid s^{\text{Ord}(s;r)} - 1$. Thus, $r \mid s^{\text{Ord}(s;r) - 1} = \beta_{(s,1)}(\text{Ord}(s;r))$, and so $r$ is distinguished with respect to $(s,1)$. \qed
For example, consider

\[ \Omega(3, 1) = \{1, 4, 5, 7, 10, 11, 13, 14, 17, 19, 20, 23, 25, 28, 29, 31, 34, 35, 37, 38, \ldots \}. \]

From Proposition 2.3 we see that \( \Omega(3, 1) \) contains the arithmetic sequences \( \{6n + 1\} \) and \( \{6n + 5\} \). With a little more effort, one can show that \( \Omega(3, 1) \) also contains the arithmetic sequences \( \{24n + 4\}, \{24n + 10\}, \{24n + 14\} \), and \( \{24n + 20\} \). The smallest integer in \( \Omega(3, 1) \) not contained in any of these sequences is 40. Moreover, a calculation shows that \( 96n + 40 \), for \( 0 \leq n \leq 5 \) is in \( \Omega(3, 1) \), but \( 616 = 96 \cdot 6 + 40 \) is not in \( \Omega(3, 1) \). An interesting question of further study is whether the sets \( \Omega(s, t) \) have a nice characterization. For example, we might ask whether they can be written as a (possibly infinite) union of arithmetic sequences, as is the case for \( \Omega(2, 1) \), which consists precisely of all odd natural numbers. However, the example \( \Omega(3, 1) \) suggests that this may not be the case in general.

A generating set for the collection of fixed points of \( \phi_{s, t} \) will be indexed by \( s \)-cyclotomic cosets \( C_{s, r, n} \) where \( r \) is distinguished with respect to \((s, t)\). Note that by computing

\[ \phi_{s, t} \left( \frac{1}{1 - \lambda x} \right) = \frac{\lambda^t}{1 - \lambda^s x} \]

we acquire the following formula for the iterates of \( \phi_{s, t} \):

\[ \phi_{s, t}^{(k)} \left( \frac{1}{1 - \lambda x} \right) = \frac{\lambda^{\delta_{s, t}(k)}}{1 - \lambda^{s \cdot k} x}. \]

For \( r \geq 1 \) and \( n \in \mathbb{N} \), define

\[ \psi_{s, t, r, n}(x) = \sum_{j=1}^{\text{Ord}(s, r)} \frac{\omega^n_{r, s, t}(j)}{1 - \omega^n_{r, s, t} x} = \sum_{j=1}^{\text{Ord}(s, r)} \phi_{s, t}^{(j)} \left( \frac{1}{1 - \omega^n_{r, s, t} x} \right). \]

Note that if \( n = 0 \), then \( \psi_{s, t, r, 0}(1) = 1/(1 - x) \). If \( t = 0 \), then \( r = 0 \) is distinguished with respect to \((s, t)\), and we define

\[ \psi_{s, 0, 0, n}(x) = 1. \]

**Proposition 2.4.** If \( r \) is distinguished with respect to \((s, t)\), then \( \psi_{s, t, r, n}(x) \) is fixed by \( \phi_{s, t} \).

**Proof.** If \( r > 1 \) is distinguished with respect to \((s, t)\), then

\[ \phi_{s, t}^{\text{Ord}(s, r) + 1} \left( \frac{1}{1 - \omega^n_{r, s, t} x} \right) = \phi_{s, t} \left( \phi_{s, t}^{\text{Ord}(s, r)} \left( \frac{1}{1 - \omega^n_{r, s, t} x} \right) \right) = \phi_{s, t} \left( \frac{1}{1 - \omega^n_{r, s, t} x} \right), \]

and so

\[ \phi_{s, t}(\psi_{s, t, r, n}(x)) = \phi_{s, t} \left( \sum_{j=1}^{\text{Ord}(s, r)} \phi_{s, t}^{(j)} \left( \frac{1}{1 - \omega^n_{r, s, t} x} \right) \right) = \phi_{s, t} \left( \sum_{j=1}^{\text{Ord}(s, r)} \phi_{s, t}^{(j+1)} \left( \frac{1}{1 - \omega^n_{r, s, t} x} \right) \right) = \psi_{s, t, r, n}(x). \]

Thus \( \psi_{s, t, r, n}(x) \) is fixed by \( \phi_{s, t} \). Since constants are fixed by \( \phi_{s, 0} \), it follows that \( \psi_{s, 0, 0, n} \) is fixed by \( \phi_{s, 0} \). Since \( r = 0 \) is distinguished only with respect to \( t = 0 \), we have shown the result holds in all possible cases. \( \square \)

### 3. The Space of Fixed Points of \( \phi_{s, t} \)

We now classify all the fixed points of \( \phi_{s, t} \) for all integers \( s, t \). To do so, we first demonstrate a bijective correspondence between fixed points of \( \phi_{s, t} \) and \( \phi_{s, t+u(s-1)} \) where \( u \) is an arbitrary integer.

**Lemma 3.1.** For all integers \( s, t, u \), the rational function \( R(x) \) is a fixed point of \( \phi_{s, t} \) iff \( x^{-u} R(x) \) is a fixed point of \( \phi_{s, t+u(s-1)} \).

**Proof.** Using equation (1.6), one can show directly that for any integers \( s, t, u \),

\[ \phi_{s, t}(R(x)) = x^{u} \phi_{s, t+(s-1)u}(x^{-u} R(x)), \]
and so

\[ \phi_{s,t}(R(x)) = R(x) \Leftrightarrow x^n \phi_{s,t+(s-1)u}(x^{-u}R(x)) = R(x) \]

\[ \Leftrightarrow \phi_{s,t+(s-1)u}(x^{-u}R(x)) = x^{-u}R(x). \]

Given this correspondence, we only have to compute the fixed points of \( \phi_{s,t} \) in case \( 0 \leq t \leq s - 2 \). Once this is accomplished, to compute the fixed points of \( \phi_{s,t} \) for arbitrary \( t \), we only need to find \( t', u \) such that \( 0 \leq t' \leq s - 2 \) and \( t = t' + u(s - 1) \), and then use the correspondence. The following result provides the missing component of this scheme, thus allowing us to compute the fixed points \( \phi_{s,t} \) for any integers \( s \) and \( t \).

**Proposition 3.2.** Suppose \( s \geq 2 \) and \( 0 \leq t \leq s - 2 \). A rational function is fixed by \( \phi_{s,t} \) if and only if it is a linear combination of the functions \( \psi_{s,t,r,n}(x) \) where \( r \) is distinguished with respect to \( (s,t) \) and \( n \) is relatively prime to \( r \).

**Proof.** We showed in Proposition 2.4 that if \( r \) is distinguished with respect to \( (s,t) \), then \( \psi_{s,t,r,n}(x) \) is fixed by \( \phi_{s,t} \), and so every linear combination of such functions must be fixed by \( \phi_{s,t} \).

To prove the converse, we consider a rational function \( R(x) \) fixed by \( \phi_{s,t} \), and express it as

\[ R(x) = C(x) + \frac{P(x)}{Q(x)} \]

where \( C(x), P(x), Q(x) \) are polynomials such that \( P(x) \) and \( Q(x) \) are relatively prime with \( \deg P(x) < \deg Q(x) \). Our first goal is to show that the poles of \( R(x) \) must be simple. We write

\[ \frac{P(x)}{Q(x)} = \sum_{n=0}^{\infty} f(n)x^n \]

where \( f(n) \) is the generating function for \( P(x)/Q(x) \). Since \( f(n) = f(sn + t) \), we have by Lemma 2.1

\[ f(sn + t) = \sum P_i(sn + t)\lambda_i^{s}\lambda_i^t\]

and so

\[ Q(x) = \prod_{i=1}^{J}(1 - \lambda_i x)^{d_i} = \prod_{i=1}^{J}(1 - \lambda_i^s x^{c_i}), \]

and so

\[ \{\lambda_1, \ldots, \lambda_J\} = \{\lambda_1^s, \ldots, \lambda_J^s\}. \]

Thus the set \( \{\lambda_1, \ldots, \lambda_J\} \) is permuted by the map \( z \mapsto z^s \), and so each \( \lambda_i \) is a primitive \( r_j \)-th root of unity where \( r_j \) is a positive integer. Moreover, since \( \{\lambda_1, \ldots, \lambda_J\} \) is permuted by the map \( z \mapsto z^s \), it follows that for each \( 1 \leq j \leq J \), there exists a positive integer \( \ell_j \) such that \( \lambda_j^{\ell_j} = \lambda_j \) (after applying the map \( z \mapsto z^s \) multiple times). Therefore, \( \lambda_j^{\ell_j-1} = 1 \), and so \( r_j | s\ell - 1 \). Thus \( r_j \) and \( s \) are relatively prime.

Let \( M = \text{lcm}(r_1, \ldots, r_J) \) and for \( a \in \mathbb{N} \), define

\[ R_a = \{m \in \mathbb{N} : m \equiv a \mod M\}. \]

Let \( f_a = f \big|_{R_a} \) be the restriction of the function \( f : \mathbb{N} \rightarrow \mathbb{C} \) to the set \( R_a \). Then

\[ f_a(a + jM) = \sum_{i=1}^{J} P_i(a + jM)\lambda_i^{a+jM} = \sum_{i=1}^{J} P_i(a + jM)\lambda_i^a, \]

and so each \( f_a \) has a representation as a polynomial in the variable \( j \) since \( \lambda_i^a \) is constant on the set \( R_a \). We denote the natural extension of this map to an element of the polynomial ring \( \mathbb{C}[j] \) by \( F_a \). Note that the restriction of \( F_a \) to \( \mathbb{N} \) need not be \( f \) in general. Our goal is to prove that each \( F_a \) is a constant function, with corresponding constant denoted by \( c_a \). Once this is shown, we have

\[ \frac{P(x)}{Q(x)} = \sum_{n=0}^{\infty} f(n)x^n = \sum_{a=0}^{M-1} c_a \sum_{j=0}^{\infty} x^{a+jM} = \sum_{a=0}^{M-1} c_a x^a \frac{1}{1 - x^M}, \]
and so \( P(x)/Q(x) \) is a rational function with only simple poles, as desired.

It remains to show that each polynomial map \( F_a : \mathbb{C} \to \mathbb{C} \) is a constant function. For each positive integer \( n \), define

\[
(3.6) \quad S_n = \{ \beta_{s,t}^{(j)}(n) : j \in \mathbb{N} \}.
\]

We say that \( a \) has an infinite cross-section if \( R_a \cap S_n \) is an infinite set for some \( n \in \mathbb{N} \). We proceed by considering two cases, depending on whether \( a \) has an infinite cross-section or not

**Case 1:** Suppose \( a \) has an infinite cross-section, i.e., \( R_a \cap S_n \) is an infinite set. Since \( f(j) = f(sj + t) \) for all \( j \in \mathbb{N} \), \( F_a \) is constant on \( R_a \cap S_n \). Since \( R_a \cap S_n \) is an infinite set, \( F_a \) is a constant polynomial.

**Case 2:** Suppose \( a \) does not have an infinite cross-section, i.e., \( R_a \cap S_n \) is finite for all positive integers \( n \). Then \( R_a \cap S_n \) must be nonempty for infinitely many values of \( n \). Since there are only finitely many distinct sets of the form \( R_b \), it follows that for each \( S_n \), there exists \( b \in \mathbb{N} \) such that \( R_b \cap S_n \) is infinite. Moreover, since there are only finitely many choices for \( R_b \), there is at least one \( b \in \mathbb{N} \) such that there exist infinitely many values of \( n \) where \( R_a \cap S_n \) is nonempty and \( R_b \cap S_n \) is infinite. Since \( b \) has an infinite cross-section, an application of Case 1 demonstrates that the restriction of \( f \) to \( R_b \) is the constant function \( c_b \). Since \( f \) is constant on each \( S_n \), the restriction of \( f \) to \( S_n \) is the constant \( c_b \). Thus \( F_a \) achieves the value \( c_b \) infinitely many times, and so \( F_a \) must be a constant polynomial.

Thus in either case, we have that \( F_a \) is a constant polynomial, and so the poles of \( R \) must be simple. Using this fact, we can decompose \( R(x) \) using partial fractions:

\[
(3.7) \quad R(x) = C(x) + \sum_{j=1}^{J} \frac{\alpha_j}{1 - \lambda_j x}.
\]

Via (2.8), an application of \( \phi_{s,t} \) yields

\[
(3.8) \quad R(x) = \phi_{s,t}(R(x)) = \phi_{s,t}(C(x)) + \sum_{j=1}^{J} \frac{\alpha_j \lambda_j^t}{1 - \lambda_j^t x}.
\]

Each rational function has a unique decomposition, and since \( \phi_{s,t} \) maps polynomials to polynomials,

\[
(3.9) \quad C(x) = \phi_{s,t}(C(x))
\]

and

\[
(3.10) \quad \sum_{j=1}^{J} \frac{\alpha_j}{1 - \lambda_j x} = \sum_{j=1}^{J} \frac{\alpha_j \lambda_j^t}{1 - \lambda_j^t x} = \phi_{s,t} \left( \sum_{j=1}^{J} \frac{\alpha_j}{1 - \lambda_j x} \right).
\]

If \( t > 0 \), it is easy to see that no nonzero polynomial is fixed by \( \phi_{s,t} \), in which case \( C(x) = 0 \). If \( t = 0 \), then the only polynomials fixed by \( \phi_{s,t} \) are constant, and so \( C(x) \) is a constant multiple of \( \psi_{s,0,0,n} = 1 \).

Now we only have left to show that the second summand in (3.7) is a linear combination of functions of the form \( \psi_{s,t,r,n} \). To do this, we begin by showing that each \( r_k \) is distinguished with respect to \((s,t)\). We have already shown that \( r_k \) and \( s \) are relatively prime for each \( k \). Using (2.9), multiple iterations of \( \phi_{s,t} \) to (3.10) yield

\[
(3.11) \quad \sum_{j=1}^{J} \frac{\alpha_j}{1 - \lambda_j x} = \phi_{s,t}^{(\text{Ord}(s,r_k))} \left( \sum_{j=1}^{J} \frac{\alpha_j}{1 - \lambda_j x} \right) = \sum_{j=1}^{J} \frac{\alpha_j \lambda_j^{s_{s,t}^{(\text{Ord}(s,r_k))}}}{1 - \lambda_j^{s_{s,t}^{(\text{Ord}(s,r_k))} x}}.
\]

The term corresponding to \( j = k \) in the first of these three expressions is

\[
(3.12) \quad \frac{\alpha_k}{1 - \lambda_k x}.
\]
and the corresponding term in the last of these three expressions is

\[ \frac{\alpha_k \lambda_k^{\beta_{s,t}(\text{Ord}(s;r_k))}}{1 - \lambda_k^{\text{Ord}(s;r_k)} x} = \frac{\alpha_k \lambda_k^{\beta_{s,t}(\text{Ord}(s;r_k))}}{1 - \lambda_k x}. \]

Thus

\[ \lambda_k^{\beta_{s,t}(\text{Ord}(s;r_k))} = 1. \]

Therefore, \( r_k \mid \beta_{s,t}(\text{Ord}(s;r_k)) \), and so \( r_k \) is distinguished with respect to \((s,t)\).

Now that we’ve shown that each \( r_k \) is distinguished with respect to \((s,t)\), group terms in the sum

\[ \sum_{j=1}^{J} \frac{\alpha_j}{1 - \lambda_j x} \]

according to the orbits of the map \( z \mapsto z^s \) on the set \( \{ \lambda_1, \ldots, \lambda_J \} \). Since \( r_k \mid \beta_{s,t}(\text{Ord}(s;r_k)) \) for each \( k \), we know that the sum of terms in \[ \text{(3.16)} \]

\[ \mathcal{O}(k) = \sum_{i=1}^{m} \phi_{s,t}^{(i)} \left( \frac{\alpha_k}{1 - \lambda_k x} \right), \]

where \( m \) is the length of the orbit of \( \lambda_k \) under the map \( z \mapsto z^s \). That is, \( m \) is the smallest positive integer such that \( \lambda_k^m = 1 \), and so \( m = \text{Ord}(s;r_k) \). Moreover, \( \lambda_k \) is a primitive \( r_k \)-th root of unity, and so it must be of the form \( \lambda_k = (\omega_{r_k})^n \) for some \( n \in \mathbb{N} \) such that \( r_k \) and \( n \) are relatively prime. Thus

\[ \mathcal{O}(k) = \alpha_k \left( \sum_{i=1}^{m} \phi_{s,t}^{(i)} \left( \frac{1}{1 - \omega_{r_k}^n x} \right) \right) = \alpha_k \psi_{s,t,r_k,n}(x), \]

and so \[ \text{(3.15)} \], and hence \[ \text{(3.1)} \], is a linear combination of rational functions of the form \( \psi_{s,t,r,n} \).

It turns out that the collection of rational functions of the form \( \psi_{s,t,r,n} \) does not form a basis of fixed points. The lemma below shows that there is redundancy in the collection. Since cyclotomic cosets have many different representations, we must compare the ways in which points. The lemma below shows that there is redundancy in the collection. Since cyclotomic cosets have many different representations, we must compare the ways in which points.
Using Lemma 3.3 we can show that if two of functions of the form $\psi_{s,t,r,n}$ have a pole in common, then they are actually the same up to a scalar multiple. The following lemma leads us this result.

**Lemma 3.4.** Suppose $r_i$ is a positive integer that is distinguished with respect to $(s,t)$, and $n_i$ is a positive integer relatively prime to $r_i$ for $i = 1, 2$. If $\psi_{s,t,r_1,n_1}$ and $\psi_{s,t,r_2,n_2}$ have a pole in common, then $r_1 = r_2$ and $C_{s,r_1,n_1} = C_{s,r_2,n_2}$.

**Proof.** Note that $\psi_{s,t,r,n}$ has poles at $\omega_r^{nsj}$ for $0 \leq j \leq \text{Ord}(s;r)$; that is, $\psi_{s,t,r,n}$ has poles at $\omega_r^{-c}$ where $c \in C_{s,r,n}$. Suppose $\psi_{s,t,r_1,n_1}$ and $\psi_{s,t,r_2,n_2}$ have a pole in common; that is, $e^{-2\pi i c_1/r_1} = e^{-2\pi i c_2/r_2}$, where $c_i \in C_{s,r_i,n_i}$. Thus, $c_1/r_1 - c_2/r_2 \in \mathbb{Z}$. Without loss of generality, we can choose $1 \leq c_i < r_i$, in which case $0 < c_1/r_1 < 1$, and so $c_1/r_1 = c_2/r_2$. Since $\gcd(r_i, n_i) = 1$ and $c_i = s^j n_i \mod r_i$ for some $j \in \mathbb{N}$, it follows that $c_1$ and $r_1$ are relatively prime, and so $c_1 = c_2$ and $r_1 = r_2$. Therefore, $s^{j_1} n_1 = s^{j_2} n_2 \mod r$ (where $r = r_1 = r_2$), and so $C_{s,r,n_1} = C_{s,r,n_2}$. □

We now precisely describe the redundancy in the collection $\{\psi_{s,t,r,n}\}$ for fixed $s$ and $t$. We begin by defining an equivalence relation $\sim_{s,r}$ on $(C_{s,r,n} - \{0\})$ by $n_1 \sim_{s,r} n_2$ if $C_{s,r,n_1} = C_{s,r,n_2}$. Let $\Lambda_{s,r}$ be a collection of coset representatives (all chosen to be less than $r$) of $(C_{s,r,n} - \{0\})/\sim_{s,r}$. That is, $\Lambda_{s,r}$ is maximal set consisting of positive integers such that no two are in the same cyclotomic coset.

**Theorem 3.5.** Suppose $s \geq 2$ and $0 \leq t \leq s - 2$. The function $1/(1 - x)$ together with the collection of all $\psi_{s,t,r,n}$ where $r$ is distinguished with respect to $(s,t)$ and $n \in \Lambda_{s,r}$ form a basis for the set of all rational functions that are fixed points of $\phi_{s,t}$.

**Proof.** The case $n = 0$ corresponds to the function $1/(1 - x)$. We now consider the case $n > 0$. Given an integer $r$ that is distinguished with respect to $(s,t)$, and an integer $n$ that is relatively prime to $r$, there exists $n' \in \Lambda_{s,r}$ such that $C_{s,r,n} = C_{s,r,n'}$, in which case by Lemma 3.3 $\psi_{s,t,r,n}$ and $\psi_{s,t,r,n'}$ are scalar multiples of one another. Thus, by Proposition 3.2 this collection spans the space of rational functions fixed by $\phi_{s,t}$.

Suppose $\psi_{s,t,r_1,n_1}$ and $\psi_{s,t,r_2,n_2}$ have a pole in common where $n_i \in \Lambda_{s,r_i}$. Then by Lemma 3.3 $r_1 = r_2$ and $C_{s,r_1,n_1} = C_{s,r_2,n_2}$. Thus by the definition of $\Lambda_{s,r_1} = \Lambda_{s,r_2}$, $n_1 = n_2$. Therefore, none of the elements of the collection have a pole in common, and so no nontrivial linear combination of elements of this collection can be zero. □

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