In this article, we studied the system of (2+1) dimensional Dirac equation in time-dependent noncommutative phase-space. Exactly, we investigated the analytical solution of the corresponding system by the Lewis-Riesenfeld invariant method based on the construction of the Lewis-Riesenfeld invariant. Knowing that we obtained the time-dependent Dirac Hamiltonian of the problem in question from a time-dependent Bopp-Shift translation, then it used to set the Lewis-Riesenfeld invariant operators. Thereafter, the obtained results used to express the eigenfunctions that lead to determining the general solution of the system.

**Keywords**: Lewis-Riesenfeld invariant method; time-dependent Bopp-Shift translation; Bopp’s Shift; time-dependent Dirac equation; time-dependent noncommutative phase-space
I. INTRODUCTION

It is known that Heisenberg suggested the opinion of noncommutative (NC) space-time in 1930, and in 1947, Snyder presented it [1, 2] to the necessity to regularizing the divergence of the quantum field theory. Then, in recent years, noncommutative geometry (NCG) became very interesting for studying several physical problems, and it became clear that there is a strong connection between NCG and string theories. Studies of this geometric type and its involvement have been incorporated with important physical concepts and tools, and have been useful in highlighting various fields of physics, particularly in matrix theory (matrix model BFSS (1997)) [3]. NCG involved also in the description of quantum gravity theories [4], Aharonov-Bohm effect [5], Aharonov-Casher effect [6], etc [7]. Knowing that the origins of NCG related to the investigations for topological spaces of C∗-algebras of functions. Later this type of geometry was theorized by A. Connes and others in 1985 [8–12] by studying and defining a cyclical cohomology. It has been shown that the differential calculus on manifolds had a NC equivalent. Next, the NCG found great encouragement through several mathematical results such as of K-theory of C∗-algebras, Gelfand-Naïmark theorem on the C∗-algebras, characterizations of commutative von Neumann algebras, cyclic cohomology of the C∞(M) algebra, relations between Dirac operators and Riemannian metrics, Serre-Swan theorem, etc. The idea of phase-space noncommutativity is largely motivated by the foundations of quantum mechanics through the canonical quantization.

It is easy to apply the phase-space noncommutativity using the ordinary product with Weyl operators (Weyl-Wigner maps) [12], or by replacing the ordinary product with the Moyal-Weyl product (∗-product) in the functions and actions of our systems [13, 14], also the Bopp-shift linear transformations [15, 16], and the Seiberg-Witten maps [8, 10, 17].

Studying physics within NCG has attracted a lot of interest in recent years, because noncommutativity is necessary when considering the low-energy efficiency of D-brane with a background magnetic field, and also in a tiny scale of strings or in conditions of the very high energy, the effects of noncommutativity may appear. Besides, one of the strong motivations of NCG, is to obtain a coherent mathematical framework in which it would be possible to describe quantum gravitation. For all these reasons and advantages, we carry out this work in NC formalism.

A large variety of scientific papers concerning time-dependent systems were interested in the time-dependent harmonic oscillator, or in time-dependent linear potentials, but in our current work, to be more specific we report the...
time-dependent Background of the NC phase-space. We consider a time-dependent Bopp-shift translation to transform the system to a time-dependent NC one, then due to the LR invariant method we obtain the LR invariant and its eigenstates to solve our system equations.

II. TIME-DEPENDENT NONCOMMUTATIVITY

In the theory of NCG space may not commute anymore (i.e. \( AB \neq BA \)). In a \( d \) dimensional time-dependent NC phase-space let us consider the operators of coordinates and momentum \( x_j^{nc} \) and \( p_k^{nc} \) respectively. These NC operators satisfy the deformed commutation relations

\[
\begin{align*}
[x_j^{nc}, x_k^{nc}] &= i\Theta_{jk}(t) \\
[p_j^{nc}, p_k^{nc}] &= i\eta_{jk}(t), \\
[x_j^{nc}, p_k^{nc}] &= i\hbar\epsilon^{jkl}\delta_{jk}
\end{align*}
\]

the effective Planck constant being

\[
\hbar^{eff} = \hbar \left( 1 + \frac{\Theta_{jk}}{4\hbar^2} \right),
\]

where \( \Theta_{jk} \ll 1 \) is the consistency condition in the usual quantum mechanics. \( \delta_{jk} \) is the identity matrix, and \( \Theta_{jk}, \eta_{jk} \) are real constant antisymmetric \( d \times d \) matrices.

In some studies concerning the NC parameters as in the experiment by “Nesvizhevsky et al” [50, 51], we note that \( \Theta \approx 10^{-30}m^2 \), \( \eta \approx 1,76.10^{-61}Kg^2m^2s^{-2} \). Other bounds exist. For example \( \Theta \approx 4.10^{-40}m^2 \) when assuming the natural units, \( \hbar = c = 1 \) [52]. As well as when taking into account that the experimental energy resolution is related to the uncertainty principle because of the finite lifetime of the neutron, this leads to obtaining \( \eta \approx 10^{-67}Kg^2m^2s^{-2} \) (kind of a correction). These obtained results including of the experiment by “Nesvizhevsky et al”, allow us to evaluate the consistency condition of the NC model \( \left| \frac{\Theta_{jk}}{4\hbar^2} \right| \leq 10^{-24} \). But if we consider the modifications introduced by noncommutativity over \( h \) value (the precision is about \( 10^{-9} \)), which are at least about 24 orders of magnitude smaller than its value, with considering the corrected bounds of \( \eta \), we have \( \left| \frac{\Theta_{jk}}{4\hbar^2} \right| \leq 10^{-29} \) [53]. These values agree with the higher limits on the basic scales of coordinate and momentum. These limits will be suppressed if the used magnetic field in the experiment is weak, about \( B \approx 5mG \).

As long as the system in which we investigate the effects of noncommutativity is 2 dimensional, we restrict ourselves to the following NC algebra

\[
\begin{align*}
[x_j^{nc}, x_k^{nc}] &= i\Theta e^{\gamma t}\epsilon_{jk} \\
[p_j^{nc}, p_k^{nc}] &= i\eta e^{-\gamma t}\epsilon_{jk}, \\
[x_j^{nc}, p_k^{nc}] &= i\hbar^{eff}\delta_{jk}
\end{align*}
\]

we have \( \epsilon_{12} = -\epsilon_{21} = 1, \epsilon_{11} = \epsilon_{22} = 0, \) and \( \Theta, \eta \) are real-valued with the dimension of \( \text{length}^2 \) and \( \text{momentum}^2 \), respectively.

While the space coordinates and momentum are fuzzy and fluid [54], they can not be localized, unless for minus infinite times. The parameters \( \Theta, \eta \) represent the fuzziness and \( \gamma \) represents the fluidity of the space. The above equation is the relation of the ordinary NCG except that NC structure constants are considered as exponentially increasing functions with the evolution of time. Certainly, there are a multitude of other possibilities, such as \( \Theta(t) = \Theta\cos(\gamma t), \eta(t) = \eta\sin(\gamma t) \).

The new deformed geometry can be described by the operators

\[
\begin{align*}
x_1^{nc} &= x^{nc} = x - \frac{1}{\hbar}\Theta e^{\gamma t}p_y, \\
x_2^{nc} &= y^{nc} = y + \frac{1}{\hbar}\Theta e^{\gamma t}p_x, \\
p_1^{nc} &= p_x^{nc} = p_x + \frac{1}{\hbar}\eta e^{-\gamma t}y, \\
p_2^{nc} &= p_y^{nc} = p_y - \frac{1}{\hbar}\eta e^{-\gamma t}x.
\end{align*}
\]

When \( \gamma = 0 \), the time-dependency in the structure of NC parameters vanishes. In Addition, for \( \Theta = \eta = 0 \), the NCG reduces to commutative one, yonder the coordinates \( x_j \) and the momentum \( p_k \) satisfy the ordinary canonical commutation relations

\[
\begin{align*}
[x_j, x_k] &= 0 \\
[p_j, p_k] &= 0, \\
[x_j, p_k] &= i\hbar\delta_{jk}
\end{align*}
\]
III. (2+1) D EXPLICITLY TIME-DEPENDENT DIRAC EQUATION AND ITS INVARIANT OPERATOR

A. (2+1) D Dirac equation in time-dependent noncommutative phase-space

In presence of an electromagnetic four-potential \( A_\mu = (A_0, A_i) \), the Dirac equation in (2+1) d is given by

\[
(\epsilon \alpha_i (p_i - \frac{\hbar}{i} A_i (x)) + e A_0 (x) + \beta mc^2) |\psi\rangle = i \hbar \frac{\partial}{\partial t} |\psi\rangle,
\]  

(III.1)

with \( |\psi\rangle \) is the Dirac wave function, and \( p_j = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} \) is the momentum. The Dirac matrices \( \alpha_j, \beta \)

\[
\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad \alpha_1 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \alpha_2 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \beta = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathcal{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

(III.2)

satisfy the following anticommutation relations

\[
\{ \alpha_i, \alpha_j \} = 2 \delta_{ij}, \quad \{ \alpha_i, \beta \} = 0 \text{ with } \alpha_i^2 = \beta^2 = 1.
\]

(III.3)

We consider the magnetic field \( \vec{B} \) along z-direction, and it is defined in terms of the symmetric potential

\[
A_i = \frac{B}{2} (-y, x, 0), \quad \text{with } A_0 = 0,
\]

(III.4)

most research about time-dependent systems concerns the presence of an electric field. Differently, in our current work, we do not rely on the electric field. Using Eq.(III.4), then the Hamiltonian of the system becomes

\[
\mathcal{H}(x, y, p_x p_y) = \epsilon \alpha_1 p_x + \epsilon \alpha_2 p_y + e \alpha_1 \frac{B}{2} y - e \alpha_2 \frac{B}{2} x + \beta mc^2.
\]

(III.5)

Achieving the NCG in the Dirac Hamiltonian (III.5) as follows

\[
\mathcal{H}(x^{nc}, y^{nc}, p_x^{nc}, p_y^{nc}) = \epsilon \alpha_1 p_x^{nc} + \epsilon \alpha_2 p_y^{nc} - e \alpha_2 \frac{B}{2} x^{nc} + e \alpha_1 \frac{B}{2} y^{nc} + \beta mc^2,
\]

(III.6)

by applying Eq.(II.4), we necessarily express the new NC Hamiltonian using the commutative variables \( \{ x, y, p_x, p_y \} \), and by assuming that \( \hbar = c = 1 \) (natural units) to simplify the calculations, then we obtain

\[
\mathcal{H}^{nc}(x, y, p_x p_y, t) = \alpha_1 (1 + \frac{eB}{4} \Theta e^{\gamma t}) p_x - \alpha_2 (\frac{eB}{2} + \frac{\eta}{2} e^{-\gamma t}) x + \alpha_2 (1 + \frac{eB}{4} \Theta e^{\gamma t}) p_y + \alpha_1 (\frac{eB}{2} + \frac{\eta}{2} e^{-\gamma t}) y + \beta m.
\]

(III.7)

The time-dependent Dirac equation in NC phase-space is giving by

\[
i \frac{\partial}{\partial t} |\bar{\psi}(t)\rangle = \mathcal{H}^{nc}(t) |\bar{\psi}(t)\rangle,
\]

(III.8)

where \( |\bar{\psi}(t)\rangle \) is the Dirac NC wave function.

B. The construction of the Lewis-Riesenfeld invariants

To solve Eq.(III.8), we use the LR invariant method, which assumes the existence of a quantum-mechanical invariant \( I(t) \) which satisfies

\[
\frac{dI(t)}{dt} = -i [I(t), \mathcal{H}^{nc}(t)] + \frac{\partial I(t)}{\partial t} = 0,
\]

(III.9)
with
\[i \frac{\partial}{\partial t} \langle \psi(t) | I(t) \rangle = \mathcal{H}^{nc}(t) \langle \psi(t) | I(t) \rangle.\]  
(III.10)

The Eq.(III.9) is called the invariance condition for the dynamical invariant operator \(I(t)\), which is a Hermitian operator

\[I(t) = I^+(t).\]  
(III.11)

Assuming that

\[I(t) = A_1(t)p_x + B_1(t)x + A_2(t)p_y + B_2(t)y + C(t),\]  
(III.12)

with \(A_1(t), B_1(t), A_2(t), B_2(t), C(t)\) are time-dependent matrices. The substitution of Eqs.(III.12, III.6) into Eq.(III.9), and using the properties of the commutation relations, lead to

\[[I, \mathcal{H}^{nc}] + i \frac{\partial I}{\partial t} = [A_1p_x, \mathcal{H}^{nc}] + [B_1x, \mathcal{H}^{nc}] + [A_2p_y, \mathcal{H}^{nc}] + [B_2y, \mathcal{H}^{nc}] + [C, \mathcal{H}^{nc}] + i \frac{\partial I}{\partial t} = 0,\]  
(III.13)

for simplicity we take \(f_\phi(t) = 1 + \frac{i}{\hbar}B \Theta e^{\gamma t}\) and \(f_\eta(t) = \frac{e^B}{2} + \frac{\hbar^2}{2} e^{-\gamma t}\), which are not matrices, then we have

\[\{A_1, \alpha_1 f_\phi\} p_x^2 + \{A_2, \alpha_2 f_\phi\} p_y^2 - \{B_1, \alpha_1 f_\phi\} x^2 + \{B_2, \alpha_1 f_\phi\} y^2 + \{[A_1, \alpha_2 f_\phi] + [A_2, \alpha_1 f_\phi]\} p_x p_y + \{[A_1, \alpha_1 f_\eta] + [B_2, \alpha_1 f_\phi]\} y p_x + \{[A_1, \alpha_2 f_\eta] + [C, \alpha_2 f_\phi] + i\frac{\partial A_1}{\partial t}\} p_x + \{[A_2, \beta m] + [C, \alpha_2 f_\phi] + \frac{\partial A_2}{\partial t}\} p_y + \{[B_1, \alpha_2 f_\phi] - [A_2, \alpha_2 f_\phi]\} x p_y + \{[B_1, \alpha_1 f_\eta] - [B_2, \alpha_2 f_\phi]\} x y + \{[B_2, \alpha_1 f_\eta] + [B_2, \alpha_2 f_\phi]\} y p_y + i\{A_1 \alpha_2 f_\phi + iB_1 \alpha_1 f_\eta + iA_2 \alpha_1 f_\eta + iB_2 \alpha_2 f_\phi - i[B_1, \alpha_1 f_\phi] - i[B_2, \alpha_2 f_\phi] + [C, \beta m] + i\frac{\partial C}{\partial t}\} = 0.\]  
(III.14)

Then, to satisfy Eq.(III.9), and always taking advantage of the properties of commutation relations, with \(p_i p_j = \delta_{ij}, \; x_i p_j = p_j x_i\) if \(i \neq j \in \{1, 2\}\), else \(p_x x = xp_x - i, \; p_y y = yp_y - i\). We demand

\[[A_1, \alpha_1 f_\phi] = 0,\]  
(III.15)

\[[A_2, \alpha_2 f_\phi] = 0,\]  
(III.16)

\[[B_1, \alpha_2 f_\eta] = 0,\]  
(III.17)

\[[B_2, \alpha_1 f_\eta] = 0,\]  
(III.18)

\[[A_1, \alpha_1 f_\phi] + i\frac{\partial A_1}{\partial t} = 0,\]  
(III.19)

\[[A_2, \alpha_2 f_\phi] + i\frac{\partial A_2}{\partial t} = 0,\]  
(III.20)

\[[B_1, \alpha_2 f_\eta] - [C, \alpha_2 f_\phi] + i\frac{\partial B_1}{\partial t} = 0,\]  
(III.21)

\[[B_2, \alpha_2 f_\phi] + i\frac{\partial B_2}{\partial t} = 0,\]  
(III.22)

\[[A_1, \alpha_2 f_\phi] + [A_2, \alpha_1 f_\phi] = 0,\]  
(III.23)

\[[B_1, \alpha_1 f_\phi] - [A_1, \alpha_2 f_\phi] = 0,\]  
(III.24)
\[ [B_1, \alpha_2 f_\theta] - [A_2, \alpha_2 f_\eta] = 0, \quad (\text{III.25}) \]
\[ [B_1, \alpha_1 f_\eta] - [B_2, \alpha_2 f_\eta] = 0, \quad (\text{III.26}) \]
\[ [B_2, \alpha_1 f_\eta] + [A_1, \alpha_1 f_\eta] = 0, \quad (\text{III.27}) \]
\[ [A_2, \alpha_1 f_\eta] + [B_2, \alpha_2 f_\eta] = 0, \quad (\text{III.28}) \]

\[ iA_1 \alpha_2 f_\eta + iB_1 \alpha_1 f_\theta - iA_2 \alpha_1 f_\eta + iB_2 \alpha_2 f_\theta - i \{ [B_1, \alpha_1 f_\theta] + [B_2, \alpha_2 f_\theta] \} + [C, \beta m] + i \frac{\partial C}{\partial t} = 0. \quad (\text{III.29}) \]

From the relations \((\text{III.15} - \text{III.18})\), and as long as from Eq.\((\text{III.15})\), we have
\[ A_1 = a_0(t) + a_1(t) \alpha_1 + a_2(t) \alpha_4^2 + a_3(t) \alpha_3^3 + a_4(t) \alpha_4^4 + \ldots = a_0(t) + a'_1(t) \alpha_1, \text{ with } a'_1(t) = a_1(t), \quad (\text{III.30}) \]
therefore, we obtain
\[ A_1 = a_1 + a_2 \alpha_1, \quad (\text{III.31}) \]
\[ A_2 = a_3 + a_4 \alpha_2, \quad (\text{III.32}) \]
\[ B_1 = b_1 + b_2 \alpha_2, \quad (\text{III.33}) \]
\[ B_2 = b_3 + b_4 \alpha_1. \quad (\text{III.34}) \]

From Eqs.\((\text{III.19} - \text{III.22})\) and with the same manner, supposing that \(C\) is written in terms of \(\alpha_1, \alpha_2\) and \(\beta\) as follows
\[ C = c_1 + c_2 \alpha_1 + c_3 \alpha_2 + c_4 \beta, \quad (\text{III.35}) \]
where \(a_j, b_j\) and \(c_j\) (with \(j = 1, \ldots, 4\)) are supposed to be time-dependent arbitrary functions. Substituting Eqs.\((\text{III.35}, \text{III.31})\) into Eq.\((\text{III.19})\) and Eqs.\((\text{III.35}, \text{III.32})\) into Eq.\((\text{III.20})\), taking into consideration Eq.\((\text{III.3})\) yield
\[ \frac{\partial a_1}{\partial t} = 0, \quad \frac{\partial a_3}{\partial t} = 0, \quad a_2 = a_4 = c_2 = c_3 = c_4 = 0, \quad (\text{III.36}) \]

thereafter, substituting Eqs.\((\text{III.35}, \text{III.33})\) into Eq.\((\text{III.21})\) and Eqs.\((\text{III.35}, \text{III.34})\) into Eq.\((\text{III.22})\), taking into consideration Eq.\((\text{III.3})\) yield
\[ \frac{\partial b_1}{\partial t} = 0, \quad \frac{\partial b_3}{\partial t} = 0, \quad b_2 = b_4 = c_2 = c_4 = 0. \quad (\text{III.37}) \]

From the Eqs.\((\text{III.36}, \text{III.37})\) we note that \(a_1, a_3, b_1, b_3\) are time-independent constant. We have
\[ A_1 = a_1, \quad A_2 = a_3, \quad B_1 = b_1, \quad B_2 = b_3, \quad C = c_1. \quad (\text{III.38}) \]

In addition, from Eqs.\((\text{III.25}, \text{III.27})\), and assuming that there exist \(\chi(t), \varphi(t)\), which are time-dependent matrices, with \([\chi(t), \alpha_2] = [\varphi(t), \alpha_1] = 0\). The time-dependency may appear as follows
\[ b_1 f_\theta - a_3 f_\eta = \chi(t) \]
\[ b_3 f_\theta + a_1 f_\eta = \varphi(t). \quad (\text{III.39}) \]

Now, substituting Eq.\((\text{III.38})\) into Eq.\((\text{III.29})\) and using Eq.\((\text{III.39})\) give us
\[ \frac{\partial c_1}{\partial t} = - \{ a_1 f_\eta + b_3 f_\theta \} \alpha_2 - \{ b_1 f_\theta - a_3 f_\eta \} \alpha_1, \quad (\text{III.40}) \]
using system of relations \((\text{III.39})\), we find
\[ \frac{\partial c_1}{\partial t} = 0 \text{ and } \chi = \varphi = 0. \quad (\text{III.41}) \]

Last but not least, the dynamical invariant \((\text{III.12})\) of time-dependent NC Dirac equation can be written as follows
\[ I = a_1 p_x + b_1 x + a_3 p_y + b_3 y + c_1, \quad (\text{III.42}) \]
we inferred that Eq.\((\text{III.9})\) is verified and \(c_1\) should be a constant. We may note also that all the spin-dependent parts which are proportional to \(\alpha_3, \beta\) disappear. Which means that \(I\) has no spin-dependency, but it is proportional to the matrix of identity in the spinor of space.
C. Eigenvalues and eigenstates of $I$ and $\mathcal{H}(t)$

Supposing that the invariant in general $I(t)$ is a complete set of eigenfunctions $|\phi(\lambda, k)\rangle$ (in this subsection, the analysis is not concerning only on time-independent invariants), with $\lambda$ being the corresponding eigenvalue (spectrum of the operator), and $k$ represents all other necessary quantum numbers to specify the eigenstates. The eigenvalues equation is written as

$$I(t) |\phi(\lambda, k)\rangle = \lambda |\phi(\lambda, k)\rangle,$$  \hspace{1cm} \text{(III.43)}

where $|\phi(\lambda, k)\rangle$ are an orthogonal eigenfunctions

$$\langle \phi(\lambda, k) | \phi(\lambda', k') \rangle = \delta_{\lambda\lambda'} \delta_{kk'}.$$  \hspace{1cm} \text{(III.44)}

According to Eq.(III.11), the eigenvalues are real and not time-dependent. Deriving Eq.(III.43) in time, we find

$$\frac{\partial I}{\partial t} |\phi(\lambda, k)\rangle + I \frac{\partial}{\partial t} |\phi(\lambda, k)\rangle = \frac{\partial \lambda}{\partial t} |\phi(\lambda, k)\rangle + \lambda \frac{\partial}{\partial t} |\phi(\lambda, k)\rangle,$$  \hspace{1cm} \text{(III.45)}

we apply Eq.(III.9) over the eigenfunctions $|\phi(\lambda, k)\rangle$, we have

$$i \frac{\partial I}{\partial t} |\phi(\lambda, k)\rangle + I \mathcal{H}^{mc} |\phi(\lambda, k)\rangle - \mathcal{H}^{mc} \lambda |\phi(\lambda, k)\rangle = 0,$$  \hspace{1cm} \text{(III.46)}

the scalar product of Eq.(III.46) by $\langle \phi(\lambda', k') |$ is

$$i \left( \langle \phi(\lambda', k') | \frac{\partial I}{\partial t} \phi(\lambda, k) \right) + \left( \lambda' - \lambda \right) \left\langle \phi(\lambda', k') | \mathcal{H}^{mc} | \phi(\lambda, k) \right\rangle = 0,$$  \hspace{1cm} \text{(III.47)}

which implies

$$\left\langle \phi(\lambda', k') | \frac{\partial I}{\partial t} \phi(\lambda, k) \right\rangle = 0,$$  \hspace{1cm} \text{(III.48)}

the scalar product of Eq.(III.45) by $\langle \phi(\lambda', k') |$ is

$$\left\langle \phi(\lambda', k') | \frac{\partial I}{\partial t} \phi(\lambda, k) \right\rangle = \frac{\partial \lambda}{\partial t},$$  \hspace{1cm} \text{(III.49)}

from Eq.(III.48), the Eq.(III.49) shows that

$$\langle \phi(\lambda', k') | \frac{\partial I}{\partial t} \phi(\lambda, k) \rangle = \frac{\partial \lambda}{\partial t} = 0.$$  \hspace{1cm} \text{(III.50)}

While the eigenvalues are time-independent, the eigenstates should be time-dependent.

In order to find the link between the eigenstates of the invariant $I(t)$ and the solutions of the relativistic Dirac equation, firstly, we start with writing the motion equation of $|\phi(\lambda, k)\rangle$, so that using Eq.(III.45) and Eq.(III.50), we obtain

$$\frac{\partial I}{\partial t} |\phi(\lambda, k)\rangle = (\lambda - I) \frac{\partial}{\partial t} |\phi(\lambda, k)\rangle,$$  \hspace{1cm} \text{(III.51)}

by using the scalar product with $\langle \phi(\lambda', k') |$, and taking Eq.(III.47) to eliminate $\langle \phi(\lambda', k') | \frac{\partial I}{\partial t} | \phi(\lambda, k) \rangle$, then we obtain

$$i \left( \langle \phi(\lambda', k') | (\lambda - \lambda') \frac{\partial}{\partial t} \phi(\lambda, k) \right) = (\lambda - \lambda') \left\langle \phi(\lambda', k') | \mathcal{H}^{mc} | \phi(\lambda, k) \right\rangle,$$  \hspace{1cm} \text{(III.52)}

for $\lambda' \neq \lambda$, we deduce

$$i \left\langle \phi(\lambda', k') | \frac{\partial}{\partial t} \phi(\lambda, k) \right\rangle = \left\langle \phi(\lambda', k') | \mathcal{H}^{mc} | \phi(\lambda, k) \right\rangle,$$  \hspace{1cm} \text{(III.53)}
then we deduce immediately that \( |\phi(\lambda, k)\rangle \) satisfy the Dirac equation, that is to say \( |\phi(\lambda, k)\rangle \) are particular solutions of Dirac equation.

It is assumed that, a phase has been taken, but it still always possible to multiply it by an arbitrary time-dependent phase factor, which means that we can define a new set of \( I(t) \) eigenstates linked to our overall by a time-dependent gauge transformation, and

\[
|\phi(\lambda, k)\rangle_I = e^{i \alpha(\lambda, t)} |\phi(\lambda, k)\rangle,
\]  

(III.54)

where \( \alpha(\lambda, t) \) is a real time-dependent function arbitrarily chosen called LR phase. \( |\phi_\lambda(x, y, t)\rangle_\alpha \) are eigenstates of \( I(t) \) which are orthonormal and associated with \( \lambda \). By putting Eq. (III.54) in Eq. (III.53) and using Eq. (III.44), we find

\[
\frac{\partial \alpha(\lambda, k)}{\partial t} \delta_{\lambda', k'} = \left\langle \phi(\lambda', k') \right| i \frac{\partial}{\partial t} - \mathcal{H}^{nc} |\phi(\lambda, k)\rangle.
\]  

(III.55)

All the eigenstates of the invariant are also solutions of the time-dependent Dirac equation, it was shown in [46] that its general solution is done by

\[
|\tilde{\psi}(t)\rangle = \sum_{\lambda, k} C_{\lambda, k} e^{i \alpha(\lambda, k, t)} |\phi(\lambda, k, t)\rangle,
\]  

(III.56)

we remark that Eq. (III.56) is also spin-independent in its state. But maybe the spin-dependent part is entangled in the coefficient \( C \). \( |\phi(\lambda, k, t)\rangle \) are the orthonormal eigenstates of \( I(t) \), with \( C_{\lambda, k} \) are time-independent coefficients, which correspond to \( |\psi(0)\rangle \)

\[
C_{\lambda, k} = \langle \lambda, k | \psi(0)\rangle.
\]  

(III.57)

For a discrete spectrum of \( I(t) \), with \( \lambda = \lambda', k = k' \), and from Eq. (III.55) the LR phase is defined as

\[
\alpha(t) = \int_0^t \left\langle \phi(\lambda, k, t') \right| i \frac{\partial}{\partial t'} - \mathcal{H}^{nc} |\phi(\lambda, k, t')\rangle \, dt'.
\]  

(III.58)

But in the continuous spectrum case, the general expression of the phase is

\[
\frac{\partial \alpha(\lambda, k)}{\partial t} \left\langle \phi(\lambda', k', t') | \phi(\lambda, k, t) \right| = \left\langle \phi(\lambda', k', t') \right| i \frac{\partial}{\partial t'} - \mathcal{H}^{nc} |\phi(\lambda, k, t)\rangle,
\]  

(III.59)

where \( k \) is an index which varies continuously in the real values, thus

\[
\left\langle \phi(\lambda', k', t') | \phi(\lambda, k, t) \right| = \delta_{\lambda', \lambda} \delta(k - k'),
\]  

(III.60)

substituting Eq. (III.60) in Eq. (III.59) yields

\[
\alpha(t) = \int_0^t \left\langle \phi(\lambda, k', t') \right| i \frac{\partial}{\partial t'} - \mathcal{H}^{nc} |\phi(\lambda, k, t')\rangle \, dt' \, dk'.
\]  

(III.61)

Once found the expression of the phase \( \alpha(t) \), we can write the particular solution of our NC time-dependent Dirac equation (III.56).

We use for simplicity the notation of the discrete spectrum of \( I(t) \). We see that the eigenfunction of \( I(t) \) has the form of [55, 57]

\[
|\phi_{\lambda, k}(x, y, t)\rangle \propto |\lambda, k\rangle \exp \left[ i \left( \xi_1(t)x + \xi_2(t)y + \xi_3(t)x^2 + \xi_4(t)y^2 \right) \right],
\]  

(III.62)

where \( \xi_1(t), \xi_2(t), \xi_3(t), \xi_4(t) \) are arbitrary time-dependent functions.

By substituting Eq. (III.62) into Eq. (III.58) yields

\[
\alpha(t) = \vartheta - \int_0^t E^{nc} \, dt',
\]  

(III.63)

with

\[
\vartheta(x, y, t) = (\xi_1(0) - \xi_1(t)) x + (\xi_2(0) - \xi_2(t)) y + (\xi_3(0) - \xi_3(t)) x^2 + (\xi_4(0) - \xi_4(t)) y^2,
\]  

(III.64)

and \( E^{nc} \) is the eigenvalue of the Hamiltonian (III.7).

Finally, the solution of the NC Dirac equation (III.8) is [46]

\[
|\tilde{\psi}(t)\rangle = \sum_{\lambda, k} C_{\lambda, k} e^{i \left( \vartheta - \int_0^t E^{nc} \, dt' \right)} |\phi(\lambda, k, t)\rangle,
\]  

(III.65)
D. The exact form of the solutions of the problem

As agreed [55–57], the wave function of the NC Dirac equation is given by the following trial function

$$\hat{\psi}(x, y, t) = \mathcal{F}(t) \phi(x, y, t), \quad (III.66)$$

where $\mathcal{F}$ is a time-dependent vector of 2 components ($2 \times 1$)

$$\mathcal{F}(t) = \begin{pmatrix} F_1(t) \\ F_2(t) \end{pmatrix}, \quad (III.67)$$

as long as $I(t)$ is independent in time, Eq.(III.10) goes to Eq.(III.8). Then the substitution of Eq.(III.66) into Eq.(III.8), and using Eqs.(III.62, III.2) give

$$\left\{ \begin{array}{l}
\frac{\partial F_1}{\partial t} - F_1 \frac{\partial q_1}{\partial t} - F_2 \frac{\partial q_2}{\partial t} - F_1 \frac{\partial q_1}{\partial x} - F_2 \frac{\partial q_2}{\partial y} - F_1 \frac{\partial q_1}{\partial x^2} - F_2 \frac{\partial q_2}{\partial y^2} \\
\frac{\partial F_2}{\partial t} - F_1 \frac{\partial q_1}{\partial t} - F_2 \frac{\partial q_2}{\partial t} - F_1 \frac{\partial q_1}{\partial x} - F_2 \frac{\partial q_2}{\partial y} - F_1 \frac{\partial q_1}{\partial x^2} - F_2 \frac{\partial q_2}{\partial y^2} 
\end{array} \right\} = \left\{ \begin{array}{l}
m \\
\alpha_1 f_0 p_x - \alpha_2 f_0 q_x + \alpha_2 f_0 f_y + \alpha_1 f_0 y
\end{array} \right\}
\times \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \quad (III.68)$$

then, we obtain

$$\frac{\partial F_1}{\partial t} - F_1 \frac{\partial q_1}{\partial t} - F_1 \frac{\partial q_2}{\partial t} - F_1 \frac{\partial q_1}{\partial x} - F_2 \frac{\partial q_2}{\partial y} - F_1 \frac{\partial q_1}{\partial x^2} - F_2 \frac{\partial q_2}{\partial y^2} = f_0 \mathcal{F}_2 p_x + i f_1 \mathcal{F}_2 p_y + f_2 \mathcal{F}_2 q_x + f_3 \mathcal{F}_2 q_y + m \mathcal{F}_1, \quad (III.69)$$

by solving the above system of equations, we find

$$\frac{\partial F_1}{\partial t} = -im F_1, \quad \frac{\partial F_2}{\partial t} = im F_1, \quad (III.70)$$

$$\mathcal{F}_1 \frac{\partial q_1}{\partial t} = -i \left\{ \frac{e B}{2} + \frac{\eta}{2} e^{-\gamma t} \right\} \mathcal{F}_2, \quad (III.71)$$

$$\mathcal{F}_1 \frac{\partial q_2}{\partial t} = -i \left\{ \frac{e B}{2} + \frac{\eta}{2} e^{-\gamma t} \right\} \mathcal{F}_2, \quad (III.72)$$

$$\frac{\partial \xi_1}{\partial t} = \frac{\partial \xi_2}{\partial t} = 0, \quad (III.73)$$

which lead to obtaining

$$\mathcal{F}_1 = e^{-imt+q_1}, \quad \mathcal{F}_2 = e^{imt+q_2}, \quad (III.74)$$

$$\frac{\partial \xi_1}{\partial t} = \frac{\partial \xi_2}{\partial t} = -i \left\{ \frac{e B}{2} + \frac{\eta}{2} e^{-\gamma t} \right\} e^{2mt+q_2-q_1}, \quad (III.75)$$

$$\xi_1 = i \xi_2 = -i \left\{ \frac{\kappa}{4l_B^2} e^{2mt} + \frac{\eta \kappa}{4im - 2\gamma} e^{(-\gamma+i2m)t} \right\}, \quad (III.76)$$

with $q_1, q_2$ and $\kappa = e^{q_2-q_1}$ are real constants, $l_B^{-1} = \sqrt{eB}$ is the magnetic length [58]. In commutative case ($\Theta = \eta = \gamma = 0$), then the above relations (III.74, III.76) return to that of general quantum mechanics

$$\xi_1(t) = i \xi_2(t) = -\frac{\kappa}{4l_B^2} e^{2mt}, \quad \phi(x, y, t) \mid_{\eta=\gamma=0} \sim e^{-i \frac{\kappa}{4l_B^2} e^{2mt}(x+i y) + o_1 x^2 + o_2 y^2}, \quad (III.77)$$

with $o_1, o_2$ are are real constants, and in $t = 0$

$$\xi_1(0) = i \xi_2(0) = -\frac{\kappa}{4l_B^2}, \quad \text{and} \quad \mathcal{F}_1 = \kappa \mathcal{F}_2 = e^{q_1}. \quad (III.78)$$
IV. CONCLUSION

In conclusion, the dynamics of the system of time-dependent NC Dirac equation has been analysed and formulated using LR invariant method. We introduced the time-dependent noncommutativity using a time-dependent Bopp-shift translation. Knowing that the NC structure constants postulated expanding exponentially with the evolution of time, and the time-dependency have a multitude of other possibilities.

We benefit from the dynamical invariant following the standard procedure allowed to construct and to obtain an analytical solution of the system. Having obtained the explicit solutions could help also to investigate and reformulate the modified version of Heisenberg’s uncertainty relations emerging from non-vanishing commutation relations (II.1).

The uncertainty for the observables $A, B$ has to satisfy the inequality $\Delta A \Delta B |\psi| \geq \frac{1}{2} |\langle \psi | [A, B] |\psi| \rangle|$ with $\Delta A |\psi|^2 = \langle \psi | A^2 |\psi| - \langle \psi | A |\psi| \rangle^2$ and the same for $B$ for any state. Depending on these results, we are planning to study the pair creation process, and investigate its implications in quantum optics.

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