Higher Spin Conformal Symmetry for Matter Fields in 2 + 1 Dimensions

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Abstract

A simple realization of the conformal higher spin symmetry on the free 3d massless matter fields is given in terms of an auxiliary Fock module both in the flat and AdS$^3$ case. The duality between non-unitary field-theoretical representations of the conformal algebra and the unitary (singleton–type) representations of the 3d conformal algebra $sp(4, \mathbb{R})$ is formulated explicitly in terms of a certain Bogolyubov transform.

1 Introduction

The AdS/CFT correspondence [1, 2] suggests duality between theories of gravity in the bulk AdS$d$ space and $d − 1$ dimensional conformal theories at the boundary of AdS$d$. Some time ago a consistent theory of interacting massless fields of all spins in AdS$^4$ was developed [3] (see also [4, 5] for reviews and more references). As the higher spin gauge theory contains gravity and supergravity one may speculate that it should have some conformal dual theory exhibiting the infinite-dimensional AdS$^4$ higher spin symmetries as 3d conformal higher spin symmetries. In fact, the 3d conformal higher spin symmetry was identified long ago [6] with the AdS$^4$ higher spin algebra [7, 8], thus suggesting the higher spin AdS/CFT correspondence.

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the same reference [6] the set of 3d conformal higher spin gauge fields associated with the conformal higher spin algebra was introduced. However, 3d higher spin gauge fields do not propagate, i.e. their own dynamics is topological. To find a nontrivial boundary conformal theory one has to realize the 3d conformal higher spin symmetries on some matter fields. This is the goal of the present paper. We show that the conformal higher spin symmetry admits a natural realization on the 3d massless scalar and spinor. This realization is given in terms of an auxiliary Fock space dual to the singleton representation in the unitary Fock space [1] by a certain Bogolyubov transform. Let us note that the suggested realization is different from that used to describe 3d higher spin theories in [9, 10].

2 3d Conformal Higher Spin Algebra

Extending the usual correspondence between conformal symmetries in d dimensions and AdS symmetries in d + 1 dimensions, 3d conformal higher spin superalgebras were identified in [6] with the 4d AdS higher spin algebras [7, 8] according to the following construction. Consider the oscillators \( \hat{a}_\alpha \) and \( \hat{a}^{+\beta} \) with the commutation relations

\[
[\hat{a}_\alpha, \hat{a}^{+\beta}] = \delta_\beta^\alpha, \quad [\hat{a}_\alpha, \hat{a}_\beta] = [\hat{a}^{+\alpha}, \hat{a}^{+\beta}] = 0, \tag{2.1}
\]

where \( \alpha, \beta = 1, 2 \). The isomorphism \( so(3, 2) \sim sp(4, \mathcal{R}) \) allows one to use symplectic realization for 3d conformal (and AdS4) symmetries. The algebra \( sp(4, \mathcal{R}) \) admits the standard oscillator realization in terms of bilinears of the oscillators (2.1). The isomorphism \( so(3, 2) \sim sp(4, \mathcal{R}) \) is expressed by the relations

\[
\mathcal{L}_{nm} = \frac{1}{2} \epsilon_{nmk} \sigma^{k\alpha} \beta \mathcal{L} \alpha \beta, \quad \mathcal{L} \alpha \beta = \frac{1}{2} \left( \hat{a}_\alpha \hat{a}^{+\beta} + \hat{a}^{+\beta} \hat{a}_\alpha \right) - \frac{1}{4} \delta^\beta_\alpha \left( \hat{a}_\gamma \hat{a}^{+\gamma} + \hat{a}^{+\gamma} \hat{a}_\gamma \right),
\]

\[
\mathcal{P}_n = \sigma_n \alpha \beta \mathcal{P}_\alpha \beta, \quad \mathcal{P}_\alpha \beta = \frac{1}{2} \hat{a}_\alpha \hat{a}_\beta, \tag{2.2}
\]

\[
\mathcal{K}_n = \sigma_n \alpha \beta \mathcal{K}_\alpha \beta, \quad \mathcal{K}_\alpha \beta = \frac{1}{2} \hat{a}^{+\alpha} \hat{a}^{+\beta}, \mathcal{D} = \frac{1}{4} \left( \hat{a}_\alpha \hat{a}^{+\alpha} + \hat{a}^{+\alpha} \hat{a}_\alpha \right).
\]

Here \( n, m = 0, 1, 2 \) are fiber indices which are raised and lowered by the mostly minus Minkowski metric \( \eta_{nm} \) and \( \sigma_n \alpha \beta = \sigma_{n \beta \alpha} = (I, \tau_1, \tau_3)^{\alpha \beta} \) where \( \tau_1^{\alpha \beta}, \tau_3^{\alpha \beta} \) are the Pauli matrices. Spinorial indices \( \alpha, \beta \) are raised and lowered by \( \epsilon_{\alpha \beta} = -\epsilon_{\beta \alpha} (\epsilon_{12} = \epsilon_{12} = 1) \) as \( c^\alpha = \epsilon^{\alpha \beta} c_\beta, \ c_\beta = \epsilon_{\alpha \beta} c^\alpha (\epsilon_{\alpha \beta} \) is the symplectic form of \( sp(2, \mathcal{R}) \sim so(2, 1) \). \( \sigma_n \alpha \beta \) satisfy the following identities

\[
\sigma_n \alpha \beta \sigma_{n \alpha} \beta = 2 \eta_{nm}, \quad \sigma_n \alpha \beta \sigma^n \alpha \beta = \delta_\alpha^\alpha \delta_\beta^\beta + \alpha \leftrightarrow \beta, \tag{2.3}
\]

\(^3\)The terminology “higher spin” is therefore somewhat sloppy. It means that the higher spin gauge fields and symmetry parameters are tensors of higher ranks.
\[ \sigma_n^{\alpha} \sigma_m^{\beta \gamma} = \eta_{nm} \epsilon^{\alpha \gamma} - \epsilon_{nmk} \sigma^{k \alpha \gamma}. \]

\[ L^\alpha_{\beta}, \ P_{\alpha \beta}, \ K^{\alpha \beta} \text{ and } D \text{ identify, respectively, with the generators of the Lorentz rotations, Poincare translations, special conformal transformations and dilatations.} \]

The supergenerators \( Q^a \) with \( f^a \) the commuting variables for any polynomials and exponentials, this can easily be achieved by an appropriate redefinition of the variables.

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The useful realization \([11]\). One extends (2.1) by adding a new generating element \( \hat{k} \) for the standard relation between spin and statistics counts oddness of a number of \( \hat{\alpha} \) spinor indices

\[ \hat{k}^2 = \mathbb{I}, \quad \{ \hat{k}, \hat{a}_\alpha \} = \{ \hat{k}, \hat{a}^+_{+ \alpha} \} = 0. \tag{2.4} \]

The supergenerators \( Q_{j \alpha} \), \( S_j^\alpha \) and the \( u(1) \) charge \( J \) are

\[ Q_{j \alpha} = \frac{1}{\sqrt{2}}(i \hat{k})^j \hat{a}_\alpha, \quad S_j^\alpha = \frac{1}{\sqrt{2}}(i \hat{k})^j \hat{a}^+_{+ \alpha}, \quad j = 0 \text{ or } 1, \tag{2.5} \]

\[ J = \frac{1}{4i} \hat{k}. \tag{2.6} \]

Let the (associative) enveloping algebra of the relations (2.1), (2.4) be denoted \( AK_2 \). The higher spin Lie superalgebra \( hgl(1; 1|4) \) is defined as the same linear space endowed with the product law defined via the (anti)commutators in \( AK_2 \), \( [\hat{f}, \hat{g}] = \hat{f}\hat{g} - (\hat{f}(\hat{g}) - (-1)^{\pi(\hat{f})\pi(\hat{g})} \hat{g}\hat{f}) \). The canonical \( Z_2 \) grading chosen in accordance with the standard relation between spin and statistics counts oddness of a number of spinor indices

\[ \hat{f}(-\hat{a}^+, -\hat{a}, \hat{k}) = (-1)^{\pi(\hat{f})} \hat{f}(\hat{a}^+, \hat{a}, \hat{k}). \tag{2.7} \]

Left Fock module of the algebras \( hgl(1; 1|4) \) and \( AK_2 \) can be defined by the relations

\[ \hat{a}_\alpha |0\rangle = 0, \quad \hat{k} |0\rangle = |0\rangle. \tag{2.8} \]

The basis vectors are

\[ \hat{a}^+_{+ \alpha_1} \ldots \hat{a}^+_{+ \alpha_l} |0\rangle. \tag{2.9} \]

In practice, instead of working with the operator realization of the algebra \( AK_2 \), it is convenient to use its star product version

\[ (f \star g)(a^+, a) = \frac{1}{\pi^4} \int d^2 u d^2 v d^2 u^+ d^2 v^+ \exp(2v_\alpha u^{+ \alpha} - 2u_\alpha v^{+ \alpha}) f(a^+ + u^+, a + u) g(a^+ + v^+, a + v), \tag{2.10} \]

with \( f(a^+, a) \) and \( g(a^+, a) \) being functions (polynomials or formal power series) of the commuting variables \( a^{+ \alpha} \) and \( a_{\beta} \).
This formula describes the associative algebra with the defining relations
\[
[a_\alpha, a^{+\beta}]_* = \delta_\beta^\alpha, \\
[a_\alpha, a_\beta]_* = [a^{+\alpha}, a^{+\beta}]_* = 0.
\] (2.11)

The star product defined this way describes the product of symmetrized polynomials of oscillators in terms of symbols of operators and realizes the subalgebra \(A_2 \subset AK_2\) spanned by the \(\hat{k}\) - independent elements. The full algebra \(AK_2\) is defined for the functions of the form \(f(a^+, a, k) = f_0(a^+, a) + f_1(a^+, a)k\) by the formula (2.10) along with
\[
k \ast f(a^+, a, k) = f(-a^+, -a, k) \ast k = f(-a^+, -a, k)k, \quad k \ast k = 1. \tag{2.12}
\]

The algebra \(AK_2\) admits the involution \(\mu\) defined by the relations
\[
\mu(a_\alpha) = i a_\alpha, \quad \mu(a^{+\alpha}) = ia^{+\alpha}, \quad \mu(k) = k. \tag{2.13}
\]

Any involution \(\mu\) of an associative algebra \(A\) induces a conjugation
\[
\sigma(f) = -(i)^\pi(f) \mu(f) \tag{2.14}
\]
of the Lie superalgebra \(L_A\) built from \(A\) by virtue of (anti)commutators. The algebra \(hu(1; 1|4)\) is the real form of the algebra \(hgl(1; 1|4)\) singled out by the conjugation (2.14), i.e. a generic element of \(hu(1; 1|4)\)
\[
f^R(a^+, a, k) = f_0^R(a^+, a) + f_1^R(a^+, a)k
\]
satisfies
\[
f_0^R(a^+, a) = -(i)^\pi(f_0^R) f_0^R(-ia^+, -ia), \quad f_1^R(a^+, a) = -(i)^\pi(f_1^R) f_1^R(ia^+, ia), \tag{2.15}
\]
where bar denotes the complex conjugation of the power series expansion coefficients. Note that the superconformal generators (2.2), (2.5), (2.6) satisfy the reality conditions \(\sigma(f) = f\) and thus belong to \(hu(1; 1|4)\).

Fock representation of \(AK_2\) is its left module \(F^l\) spanned by the vectors of the form
\[
f \ast |0\rangle \langle 0|, \tag{2.16}
\]
where \(|0\rangle \langle 0|\) is the projector to the vacuum space. It is well-known (see e.g. [12, 13]) that Fock projectors admit the exponential realization in the star product algebra. The Fock vacuum satisfying (2.8) is
\[
|0\rangle \langle 0| = 2(1 + k) \exp(-2a_\alpha a^{+\alpha}). \tag{2.17}
\]

5Recall that involution is an involutive semilinear antiautomorphism, i.e. \(\mu(f \ast g) = \mu(g) \ast \mu(f)\), \(\mu^2 = 1\), \(\mu(a f) = \bar{a} \mu(f)\) (\(\bar{a}\) is complex conjugated to \(a\)). A conjugation is an involutive semilinear automorphism, i.e. \(\sigma(f \ast g) = \sigma(f) \ast \sigma(g)\), \(\sigma^2 = 1\), \(\sigma(\alpha f) = \bar{\alpha} \sigma(f)\). Any conjugation \(\sigma\) singles out a real form of a complex algebra by the condition \(\sigma(f) = f\).
Indeed, it is easy to see that

\[ a_\alpha^*|0\rangle\langle 0| = |0\rangle\langle 0|a^{+\alpha} = 0, \quad k^*|0\rangle\langle 0| = |0\rangle\langle 0|k = |0\rangle, \quad |0\rangle\langle 0|0\rangle\langle 0| = |0\rangle\langle 0|. \]

(2.18)

The right Fock module \( F^r \) of \( AK_2 \) is spanned by the vectors \(|0\rangle\langle 0|f \). The important properties of the Fock vacuum \(|0\rangle\langle 0|\) are that it is Lorentz invariant

\[ \mathcal{L}_\alpha^\beta \ast |0\rangle\langle 0| = |0\rangle\langle 0| \ast \mathcal{L}_\alpha^\beta = 0 \]

and has the definite conformal weight \( 1/2 \)

\[ \mathcal{D} \ast |0\rangle\langle 0| = |0\rangle\langle 0| \ast \mathcal{D} = \frac{1}{2}|0\rangle\langle 0|. \]

(2.19)

Note, that the \( \mu \)-conjugated Fock module \( \mu(|0\rangle\langle 0|) = 2(1 + k) \exp(2a_\alpha a^{+\alpha}) \) is different from \(|0\rangle\langle 0|\). Moreover, it belongs to a distinct sector of the star product algebra because \(|0\rangle\langle 0| \ast \mu(|0\rangle\langle 0|) = \infty \). This fact is not important from the perspective of the present paper in which only free fields are considered (i.e., matter field modules are not multiplied), but it should be taken into account when thinking of a nonlinear theory describing interactions that would exhibit higher spin conformal symmetries.

\section{3 Conformally Invariant Vacua}

Let \( \omega(x) \) be a 1-form taking values in the higher spin algebra \( AK_2 \), i.e. \( \omega \) is the generating function of the conformal higher spin gauge fields

\[ \omega(x) = \sum_{q=0,1} \sum_{l,r=0}^\infty \frac{1}{l!r!} \omega_q(x)_{\alpha_1...\alpha_l, \beta_1...\beta_r} a^{+\alpha_1} ... a^{+\alpha_l} a_{\beta_1} ... a_{\beta_r} (k)^q. \]

(3.1)

The zero-curvature equation

\[ d\omega = \omega \wedge \ast \omega \]

(3.2)

is invariant under the higher spin conformal gauge transformations

\[ \delta \omega = d\epsilon - [\omega, \epsilon]_s, \]

(3.3)

where \( \epsilon(a^+, a, k|x) \) is an infinitesimal gauge symmetry parameter and \( d = dx^{\underline{m}} \frac{\partial}{\partial x^{\underline{n}}} \) (underlined indices \( \underline{m}, \underline{n} = 0, 1, 2 \) are used for the components of differential forms).

Any vacuum solution \( \omega_0 \) of the equation (3.2) breaks the local higher spin symmetry to its stability subalgebra with the infinitesimal parameters \( \epsilon_0(a^+, a, k|x) \) satisfying the equation

\[ d\epsilon_0 - [\omega_0, \epsilon_0]_s = 0. \]

(3.4)
The consistency of this equation is guaranteed by (3.2).

Locally, the equation (3.2) admits a pure gauge solution

\[ \omega_0 = -g^{-1} \ast dg. \]  

(3.5)

Here \( g(a^+, a, k|x) \) is some invertible element of the algebra \( AK^2 \), i.e. \( g^{-1} \ast g = g \ast g^{-1} = 1 \). For \( \omega_0 \) (3.3) one finds that the generic solution of (3.4) is

\[ \epsilon_0(x) = g^{-1}(x) \ast \xi \ast g(x), \]  

(3.6)

where \( \xi(a^+, a, k) = \xi_0(a^+, a) + \xi_1(a^+, a)k \) is an arbitrary \( x \)-independent element playing a role of the initial data for the equation (3.4)

\[ \epsilon_0(a^+, a, k|x)|_{x = x_0} = \xi(a^+, a, k) \]  

(3.7)

for such a point \( x_0 \) that \( g(x_0) = 1 \). Therefore \( hu(1; 1|4) \) is indeed the global symmetry algebra that leaves invariant the vacuum solution. It contains the \( N = 2 \) global conformal supersymmetry algebra spanned by the generators (2.2), (2.5), (2.6).

As usual, the gravitational fields (i.e., frame and Lorentz connection) are associated with the generators of translations and Lorentz rotations in the Poincare or \( \text{AdS} \) subalgebras of the conformal algebra. For the flat Minkowski space one can choose

\[ \omega_0 = \omega_f = \frac{1}{2} d x^\alpha \sigma_\alpha^{\alpha\beta} a_\alpha a_\beta \]  

(3.8)

thus setting the Lorentz connection equal to zero. Here \( \sigma_\alpha^{\alpha\beta} = \sigma_\alpha^{\alpha\beta} \). The function \( g_f \) that gives rise to the flat gravitational field (3.8) is

\[ g_f = \exp(-\frac{x^{\alpha\beta}}{2} a_\alpha a_\beta), \]  

(3.9)

where we use the notation

\[ x^{\alpha\beta} = x^\alpha \sigma_\alpha^{\alpha\beta}, \quad x^\alpha = \frac{1}{2} \sigma_\alpha^{\alpha\beta} x^{\alpha\beta}. \]  

(3.10)

Within the oscillator realization of the 3d conformal algebra (2.2), the embedding of the \( \text{AdS}_3 \) algebra \( o(2, 2) \subset o(3, 2) \) can be realized as follows

\[ L_\alpha^\beta = a_\alpha a^+\beta - \frac{1}{2} \delta_\alpha^\beta a_\gamma a^{+\gamma}, \]  

(3.11)

\[ P_\alpha^\beta = \frac{1}{2} (a_\alpha a_\beta + \frac{\lambda^2}{4} a_\alpha^+ a_\beta^+) \].

The \( \text{AdS}_3 \) gravitational fields are identified with the 1-forms taking values in the \( \text{AdS}_3 \) algebra

\[ \omega_0 = \omega_{\text{AdS}_3} = dx^\alpha \left( \frac{1}{2} e_\alpha^{\alpha\beta}(x)(a_\alpha a_\beta + \frac{\lambda^2}{4} a_\alpha^+ a_\beta^+) + \omega_\alpha^{\alpha\beta}(x)(a_\alpha a^{+\beta} - \frac{1}{2} \delta_\alpha^\beta a_\gamma a^{+\gamma}) \right), \]  

(3.12)
where $e_n^{\alpha\beta}(x)$ and $\omega_n^{\alpha\beta}(x)$ are, respectively, the dreibein and Lorentz connection of $AdS_3$. A particular choice of the $AdS_3$ gravitational fields that solves the vacuum equations (3.2) and corresponds to the “stereographic” coordinates of $AdS_3$ is

$$e_n^{\alpha\beta} = \frac{4}{(4 + \lambda^2 x^2)^2} \left( (4 - \lambda^2 x^2) \delta^k_n + 4\lambda x^{m} e_n^{m\, k} + 2\lambda^2 x_n x^k \right) \sigma^k_{\alpha\beta},$$

and

$$\omega_n^{\alpha\beta} = -\frac{\lambda}{2} e_n^{\alpha\beta},$$

where

$$x^2 = \eta^{nm} x_n x_m = \frac{1}{2} x^{\alpha\beta} x_{\alpha\beta}. \quad (3.15)$$

The metric tensor of the $AdS_3$ is

$$g_{nm} = \frac{1}{2} e_n^{\alpha\beta} e_m^{\mu\nu} = 16 \frac{\eta_{nm}}{(4 + \lambda^2 x^2)^2}. \quad (3.16)$$

Let us mention that although the metric tensor is built from $\omega_0$ and, therefore, is invariant under the global symmetry transformations these symmetries are not necessarily associated with the Killing vectors of the metric tensor. This is only true for the Lorentz rotations and (Poincare or AdS) translations.

It is elementary to see that the representation (3.3) for the $AdS_3$ vacuum fields (3.12) - (3.14) is provided with the gauge function

$$g_{AdS_3} = \frac{1}{2} \sqrt{4 + \lambda^2 x^2} \exp \left( -\frac{x_{\alpha\beta}}{2} (a_\alpha a_\beta + \frac{\lambda^2}{4} a_\alpha^+ a_\beta^- - \lambda a_\alpha a_\beta) \right), \quad (3.17)$$

having the inverse

$$g_{AdS_3}^{-1} = \frac{1}{2} \sqrt{4 + \lambda^2 x^2} \exp \left( \frac{x_{\alpha\beta}}{2} (a_\alpha a_\beta + \frac{\lambda^2}{4} a_\alpha^+ a_\beta^- - \lambda a_\alpha a_\beta) \right). \quad (3.18)$$

Note that in the flat limit $\lambda \to 0$ one recovers the flat gauge function (3.9).

### 4 3d Conformal Field Equations

It was shown that the equations of motion for massless [14] and massive [13] fields in $AdS_3$ admit a formulation in terms of generating functions

$$C(y|x) = \sum_{l=0}^{\infty} \frac{1}{l!} C(x)_{\alpha_1...\alpha_l} \gamma^{\alpha_1} \cdots \gamma^{\alpha_l} \quad (4.1)$$
with the auxiliary spinor variables $y^\alpha$. The flat limit of the free equations of motion for the scalar and spinor massless fields of \[14\] has the form

$$dC(y|x) = \frac{1}{2} dx^a \sigma_\alpha^\beta \frac{1}{\partial y^\alpha \partial y^\beta} C(y|x). \quad (4.2)$$

This equation decomposes into two independent subsystems for even functions $C_e(-y|x) = C_e(y|x)$ and odd functions $C_o(-y|x) = -C_o(y|x)$ which describe the massless scalar and spinor respectively.

In \[9\] the spinor variables $y^\alpha$ were interpreted as generating elements of the AdS$_3$ higher spin algebra while the 0-form $C(y|x)$ took its values in the so-called twisted adjoint representation of this algebra. In \[10\] it was then shown that an appropriate modification of the formulation of \[9\] allows for a uniform description of both massless and massive matter fields. This formulation does not make manifest the conformal symmetries expected for the massless case, however. The key observation of this paper is that for the massless case the same equations (4.2) admit a different realization in the Fock space (2.16) that makes the higher spin conformal symmetries of the system manifest.

Let us introduce the Fock-space vector

$$|\Phi(a^+|x)\rangle = C(a^+|x) \ast |0\rangle \langle 0|, \quad C(a^+|x) = \sum_{l=0}^{\infty} \frac{1}{l!} c_{\alpha(l)}(x) a^{+\alpha}_1 \cdots a^{+\alpha}_l, \quad (4.3)$$

where 0-forms $c_{\alpha(l)}(x)$ are totally symmetric multispinors\[^6\]. The system of equations

$$d|\Phi\rangle - \omega \ast |\Phi\rangle = 0 \quad (4.4)$$

concisely encodes Klein-Gordon and Dirac equations for the scalar field $c(x)$ and spinor field $c_\alpha(x)$ provided that the equation (3.2) that guarantees the formal consistency of (4.4) is true. Indeed, the choice $\omega = \omega_f$ in the form (3.8) makes the equation (4.4) equivalent to (4.2). Let us note that the equations on the component fields $c_{\alpha(n)}(x)$ are Lorentz and scale invariant as a consequence of the Lorentz invariance (2.19) and definite scaling (2.20) of the vacuum $|0\rangle \langle 0|$.\[^8\]

Recall that, as shown in \[14\] for the AdS$_3$ case, the meaning of the equations (1.2) is that they impose the dynamical equations on the massless matter fields identified with the rank-0 and rank-1 spinors

$$c(x) = C(0|x), \quad (4.5)$$

$$c_\alpha(x) = \frac{\partial}{\partial a^{+\alpha}} C(a^+|x)|_{a^{+\alpha}=0} \quad (4.6)$$

\[^6\]We follow the conventions of \[16\] convenient for the component analysis of complicated tensor structures: upper and lower indices denoted by the same letter should be first separately symmetrized and then the maximal possible number of them should be contracted; a number of indices can be indicated in brackets by writing e.g. $\alpha(l)$ instead of repeating $l$ times the index $\alpha$.\[^8\]
and express all highest spinors from (4.3) via higher order derivatives of \(c(x)\) and \(c_\alpha(x)\). In the simplest case of the flat space this can be seen as follows. Substituting \(\omega_f\) into (4.4) we obtain
\[
\partial_n c_\alpha(l) - \frac{1}{2} \sigma_n^{\alpha\alpha} c_\alpha(l+2) = 0. \tag{4.7}
\]
The first two equations for even \(l\), i.e. for \(l = 0\) and \(l = 2\), are
\[
\begin{align*}
\partial_n c - \frac{1}{2} \sigma_n^{\alpha\alpha} c_\alpha(2) &= 0, \tag{4.8} \\
\partial_n c_\alpha(2) - \frac{1}{2} \sigma_n^{\alpha\alpha} c_\alpha(4) &= 0. \tag{4.9}
\end{align*}
\]
From (4.8) we derive that \(c_\alpha(2) = \sigma_n^{\alpha\alpha} \partial_n c\). Substituting it into (4.9) and multiplying by \(\sigma_m^{\beta\beta}\) we obtain
\[
\sigma_n^{\alpha\alpha} \sigma_m^{\beta\beta} \partial_n \partial_m c = c_\alpha(2) \beta(2). \tag{4.10}
\]
The condition that \(c_\alpha(2) \beta(2)\) is totally symmetric is equivalent to the Klein-Gordon equation for \(c(x)\)
\[
\Box c(x) = 0. \tag{4.11}
\]
The equations for all other even values of \(l\) impose no further differential equations on \(c(x)\) just expressing highest multispinors via the highest derivatives of \(c(x)\). It is straightforward to see that the resulting expression for the even part \(C_e\) of \(C\) acquires the form
\[
C_e(a^+ | x) = \sum_{q=0}^{\infty} \frac{1}{(2q)!} \sigma_n^{\alpha_1\alpha} \cdots \sigma_n^{\alpha_q\alpha} \partial_{n_1} \cdots \partial_{n_q} c(x) a^{+\alpha} \cdots a^{+\alpha} \frac{\partial^{2q} c(x)}{2q}. \tag{4.12}
\]

The situation with the fermion is analogous. Starting from the equation (4.7) for \(l = 1\) we obtain
\[
c_\alpha(2) = \sigma_n^{\alpha\beta} \partial_n c_\alpha. \tag{4.13}
\]
Again, the condition that the third rank multispinor \(c_\alpha(2)\beta(2)\) is symmetric implies the Dirac equation for \(c_\alpha(x)\)
\[
\sigma_n^{\alpha\beta} \partial_n c_\beta(x) = 0. \tag{4.14}
\]
All other equations for odd values of \(l\) express highest spinors via derivatives of \(c_\alpha(x)\). For the odd part \(C_o\) of \(C\) we get
\[
C_o(a^+ | x) = \sum_{q=0}^{\infty} \frac{1}{(2q+1)!} \sigma_n^{\alpha_1\alpha} \cdots \sigma_n^{\alpha_q\alpha} \partial_{n_1} \cdots \partial_{n_q} c_\alpha(x) a^{+\alpha} \cdots a^{+\alpha} \frac{\partial^{2q+1} c_\alpha(x)}{2q+1}. \tag{4.15}
\]
We therefore conclude that, for the flat connection \(\omega_f\), the system (4.4) is equivalent to
\[
|\Phi(a^+ | x)\rangle = \sum_{q=0}^{\infty} \frac{1}{(2q)!} \sigma_n^{\alpha_1\alpha} \cdots \sigma_n^{\alpha_q\alpha} \partial_{n_1} \cdots \partial_{n_q} \left(c(x) + \frac{1}{2q+1} c_\alpha(x) a^{+\alpha} \right) a^{+\alpha} \cdots a^{+\alpha} |0\rangle \langle 0| \tag{4.16}
\]
along with the dynamical equations (4.11) and (4.14).

Analogously, for $\omega_{AdS_3}$ the system (4.3) is equivalent to the massless Klein-Gordon and Dirac equations in $AdS_3$

$$ (g^m_a D_n D_m - \frac{3}{4} \lambda^2) c(x) = 0, \tag{4.17} $$

$$ e^{\alpha \beta} D_n c_{\beta}(x) = 0 \tag{4.18} $$

along with

$$ |\Phi(\omega^+|x)\rangle = \sum_{q=0}^{\infty} \frac{1}{(2q)!} e^{\alpha a} \cdots e^{\alpha a} D_{a_1} \cdots D_{a_q} (c(x) + \frac{1}{2q + 1} c_{a}(x) a^{\alpha}) \underbrace{a^{\alpha} \cdots a^{\alpha}}_{2q} |0\rangle \langle 0|, \tag{4.19} $$

where $D_{a}$ is the $AdS_3$ Lorentz covariant derivative

$$ D_{a} c_{\alpha(l)} = \partial_{a} c_{\alpha(l)} + l \omega_{\alpha a} c_{\alpha(l-1)\beta}, \tag{4.20} $$

$$ D_{a} c_{\alpha a} - n \leftrightarrow m = 0. \tag{4.21} $$

Note that, as usual, we identify massless scalar and spinor in $AdS_3$ with the conformal fields. With this convention, the massless scalar field in $AdS_d$ has a non-zero mass-like term $m^2_0 = -(\lambda^2/4) d(d-2)$ being in agreement with (4.17) for $d = 3$.

5 3d Conformal Higher Spin Symmetries

The system of equations (3.2) and (4.4) is invariant under the infinite-dimensional local conformal higher spin symmetries of the form (3.3) and

$$ \delta |\Phi\rangle = \epsilon * |\Phi\rangle \tag{5.1} $$

($\epsilon = \epsilon(a^+, a, k|x)$). Once a particular vacuum solution $\omega = \omega_0$ is fixed, the local higher spin symmetry (3.1) breaks down to the global higher spin symmetry (3.6). Therefore the system (4.7) and its $AdS_3$ analog are invariant under the infinite-dimensional algebra $hu(1; 1|4)$ of global 3d conformal higher spin symmetries

$$ \delta |\Phi\rangle = \epsilon_0 * |\Phi\rangle \tag{5.2} $$

where $\epsilon_0$ satisfies the equation (3.4) with the flat or $AdS_3$ connection (3.8) or (3.12). Once the higher components $c_{\alpha(l)}$ are expressed via higher derivatives of the dynamical spin zero and spin 1/2 fields by (4.16) and (4.19), this implies the invariance of the massless Klein-Gordon and Dirac equations (4.11), (4.14) and (4.17), (4.18) under the conformal higher spin symmetries. Thus, the fact that massless Klein-Gordon and Dirac equations are reformulated in the form of the flatness conditions
and zero-curvature equation \[[3,2]\] makes higher spin conformal symmetries of these equations manifest.

The \(N = 2\) conformal SUSY algebra \(osp(2|4, \mathcal{R})\) spanned by the elements \((2,2), (2,3)\) and \((2,4)\) is a finite-dimensional subalgebra of the 3d conformal higher spin symmetry algebra \(hu(1; 1|4)\). Including the \(u(1)\) factor generated by the constant element in \(hu(1; 1|4) = u(1) \oplus hsu(1; 1|4), u(1) \oplus osp(2|4, \mathcal{R})\) forms a maximal finite-dimensional subalgebra of \(hu(1; 1|4)\). Note that because of \((4.16)\) or \((4.19)\) and of the quantum-mechanical nonlocality of the star product \((2.10)\), the higher degree of \(\epsilon_0(a^+, a, k|x)\) as a polynomial of \(a_{\alpha}\) and \(a^{+\beta}\) is the higher space-time derivatives appear in the higher spin conformal transformations. This is a particular manifestation of the well-known fact that the higher spin symmetries mix higher derivatives of the dynamical fields.

The explicit form of the transformations can be obtained by the substitution of \((4.16)\) \((4.19)\) in \(AdS_3\) case into \((5.2)\). In practice, it is most convenient to evaluate the higher spin conformal transformations for the generating parameter

\[
\xi^j(a^+, a, k; h^+, h) = \xi \exp(a^{+\alpha}h_\alpha + a_\alpha h^{+\alpha})(k)^j,
\]

where \(\xi\) is an infinitesimal parameter and \(h_\alpha, h^{+\alpha}\) are spinor “sources”. The polynomial symmetry parameters can be obtained via differentiation of \(\xi^j(a^+, a, k; h^+, h)\) with respect to \(h_\alpha\) and \(h^{+\alpha}\). Using \((3.6), (3.17)\) and the star product \((2.10)\) we obtain upon evaluation of elementary Gaussian integrals

\[
\epsilon_0(a^+, a, k; h^+, h|x) = \xi \exp \left( \frac{1}{8+2\lambda x^2} \left( x^{\alpha\alpha} (8a_\alpha h_\alpha + 4\lambda a_\alpha h^{+\alpha} - 4\lambda a^{+\alpha} h_\alpha - 2\lambda^2 a^{+\alpha} h^{+\alpha}) + 8a^{+\alpha} h_\alpha + 8a_\alpha h^{+\alpha} + 4\lambda x^2 a^\alpha h_\alpha + \lambda^2 x^2 a^{+\alpha} h^{+\alpha} \right) \right)(k)^j.
\]

Substitution of \(\epsilon_0\) into \((5.2)\) gives the global higher spin conformal symmetry transformations induced by the parameter \(\xi^j(a^+, a, k; h^+, h)\)

\[
\delta|\Phi(a^+|x)\rangle = \xi \exp \left( \frac{1}{8+2\lambda x^2} \left( x^{\alpha\alpha} (4h_\alpha h_\alpha - \lambda^2 h^{+\alpha} h_\alpha - 4\lambda h_\alpha a^{+\alpha} - 2\lambda^2 h^{+\alpha} a_\alpha) + 8a^{+\alpha} h_\alpha + \lambda^2 x^2 a^{+\alpha} h^{+\alpha} + (4 - \lambda^2 x^2) h^{+\alpha} a_\alpha \right) \right)
\]

\[
C((-1)^j (a^{+\alpha} + \frac{1}{8+2\lambda x^2} (8 x^{\alpha\beta} h_\beta + 4\lambda x^{\alpha\beta} h^{+\beta} + 8 h^{+\alpha} - 4\lambda x^2 h^{\alpha}) |x\rangle |0\rangle |0\rangle).
\]

Setting \(\lambda = 0\) we obtain the flat space formula

\[
\delta|\Phi(a^+|x)\rangle = \xi \exp \left( \frac{1}{2} x^{\alpha\alpha} h_\alpha h_\alpha + a^{+\alpha} h_\alpha + \frac{1}{2} h^{+\alpha} h_\alpha \right) \right)
\]

\[
C((-1)^j (a^{+\alpha} + x^{\alpha\beta} h_\beta + h^{+\alpha}) |x\rangle |0\rangle |0\rangle).
\]

Differentiating with respect to the sources \(h_\alpha\) and \(h^{+\alpha}\) one derives explicit expressions for the particular global higher spin conformal transformations. For the transformations of the dynamical fields \((4.5), (4.6)\) these expressions further simplify. For example, the bosonic transformation of the dynamical scalar in the flat space is

\[
\delta c = \xi \exp \left( \frac{1}{2} x^{\alpha\alpha} h_\alpha h_\alpha + \frac{1}{2} h^{+\alpha} h_\alpha \right) \right)
\]

\[
C(x^{\alpha\beta} h_\beta + h^{+\alpha}) |x\rangle \right).
\]
with (4.12) substituted into the right hand side. Let us stress that such a compact form of the higher spin conformal transformations is a result of the reformulation of the dynamical equations in the unfolded form of the covariant constancy conditions, i.e. in terms of a flat section of the Fock bundle. Note that a “brutal force” search of the higher spin conformal transformations quickly gets very complicated (especially in the AdS₃ case).

For at most quadratic conformal superalgebra generators (2.2), (2.5), (2.6) one immediately obtains in the flat space

$$\mathcal{L}_{\alpha\beta}|\Phi(a^+|x)\rangle = \left( x^{\alpha\gamma} \partial^2 \frac{\partial}{\partial a^{+\alpha} \partial a^{+\gamma}} + a^{+\alpha} \frac{\partial}{\partial a^{+\alpha}} - \frac{1}{2} \delta_\alpha^\beta \left( x^{\gamma\gamma} \partial^2 \frac{\partial}{\partial a^{+\gamma} \partial a^{+\gamma}} + a^{+\gamma} \frac{\partial}{\partial a^{+\gamma}} \right) \right) |\Phi(a^+|x)\rangle,$$

$$\mathcal{P}_{\alpha\alpha}|\Phi(a^+|x)\rangle = \frac{1}{2} \partial^2 \frac{\partial}{\partial a^{+\alpha} \partial a^{+\alpha}} |\Phi(a^+|x)\rangle,$$

$$\mathcal{K}^{\alpha\alpha}|\Phi(a^+|x)\rangle = \frac{1}{2} \left( x^{\alpha\alpha} + a^{+\alpha} a^{+\alpha} + 2 x^{\alpha\beta} a^{+\alpha} \frac{\partial}{\partial a^{+\beta}} + x^{\alpha\beta} x^{\alpha\beta} \frac{\partial^2}{\partial a^{+\beta} \partial a^{+\beta}} \right) |\Phi(a^+|x)\rangle,$$

$$\mathcal{D}|\Phi(a^+|x)\rangle = \frac{1}{2} \left( 1 + a^{+\alpha} \frac{\partial}{\partial a^{+\alpha}} + x^{\alpha\alpha} \frac{\partial^2}{\partial a^{+\alpha} \partial a^{+\alpha}} \right) |\Phi(a^+|x)\rangle,$$

$$\mathcal{Q}_{ja}|\Phi(a^+|x)\rangle = \frac{(ij)^j}{\sqrt{2}} \frac{\partial}{\partial a^{+\alpha}} |\Phi((-1)^ja^+|x)\rangle,$$

$$\mathcal{S}_j^a|\Phi(a^+|x)\rangle = \frac{(ij)^j}{\sqrt{2}} \left( a^{+\alpha} + x^{\alpha\beta} \frac{\partial}{\partial a^{+\beta}} \right) |\Phi((-1)^ja^+|x)\rangle,$$

$$\mathcal{J}|\Phi(a^+|x)\rangle = \frac{1}{4t} |\Phi(-a^+|x)\rangle.$$

Taking into account that the dynamical equations for $|\Phi(a^+|x)\rangle$ have the form (4.2) one can replace all second derivatives in $a^{+\alpha}$ by the space-time derivatives. This leads to the following standard expressions

$$\mathcal{L}_{nm}|\Phi(a^+|x)\rangle = \left( x_n \partial_n - x_n \partial_m + \frac{1}{2} \xi_{mnk} \sigma^{\kappa\beta}_\alpha a^{+\beta} \frac{\partial}{\partial a^{+\alpha}} \right) |\Phi(a^+|x)\rangle,$$

$$\mathcal{P}_n|\Phi(a^+|x)\rangle = \partial_n |\Phi(a^+|x)\rangle,$$

$$\mathcal{K}_n|\Phi(a^+|x)\rangle = \left( x_n + 2 x_n x^k \partial_k - x^2 \partial_n + \frac{1}{2} \sigma_{\alpha\alpha\beta} a^{+\alpha} a^{+\alpha} \frac{\partial}{\partial a^{+\beta}} \right) |\Phi(a^+|x)\rangle,$$

$$\mathcal{D}|\Phi(a^+|x)\rangle = \left( x^k \partial_k + \frac{1}{2} a^{+\alpha} \frac{\partial}{\partial a^{+\alpha}} \right) |\Phi(a^+|x)\rangle.$$

For the dynamical scalar and spinor fields (4.3) and (4.4) we find the expected results

$$\mathcal{L}_{nm}c = (x_m \partial_n - x_n \partial_m)c,$$

$$\mathcal{P}_n c = \partial_n c.$$
\[ K_n c = (x_n + 2 x_n x^k \partial_k - x^2 \partial_n) c , \]
\[ D c = (\frac{1}{2} + x^k \partial_k) c , \]
\[ Q_{j\alpha} c = (-i)^j c_{\alpha} , \]
\[ S_j^\alpha c = \frac{(-i)^j}{\sqrt{2}} x^{\alpha\beta} c_{\beta} , \]
\[ J c = -\frac{i}{4} c , \]
\[ (5.10) \]

\[ L_{nm} c_{\alpha} = ((x_m \partial_n - x_n \partial_m)\delta^\beta_{\alpha} + \frac{1}{2} \epsilon_{nmk} \sigma^{k\beta}_{\alpha}) c_{\beta} , \]
\[ P_n c_{\alpha} = \partial_n c_{\alpha} , \]
\[ K_n c_{\alpha} = (2 x_n + 2 x_n x^k \partial_k - x^2 \partial_n)\delta^\beta_{\alpha} - \epsilon_{nmk} x^m \sigma^{k\beta}_{\alpha}) c_{\beta} , \]
\[ D c_{\alpha} = (1 + x^k \partial_k) c_{\alpha} , \]
\[ (5.11) \]
\[ Q_{j\alpha} c_{\beta} = \frac{(i)^j}{\sqrt{2}} \sigma^m_{\alpha\beta} \partial_n c , \]
\[ S_j^\alpha c_{\beta} = \frac{(i)^j}{\sqrt{2}} \ ((1 + x^k \partial_k)\delta^\alpha_{\beta} + x_n \partial_m \epsilon_{nmk} \sigma_{k\beta}^\alpha) c , \]
\[ J c_{\alpha} = -\frac{i}{4} c_{\alpha} . \]

Note that the particular form of the dependence on the space-time coordinates \( x^{\alpha\beta} \) originates from the choice of the gauge function (3.9). The approach we use is applicable to any other coordinate system and conformally flat background (for example, AdS_3).

6 Field Theory - Singleton Duality Via Bogolyubov Transform

The formulation of the relativistic higher spin dynamics proposed in this paper operates in terms of the Fock module \( F \) (2.8) defined with respect to auxiliary oscillators associated with the 3d conformal superalgebra \( osp(2|4, \mathcal{R}) \) via (2.2), (2.5) and (2.6). This Fock module is analogous to the Fock representations of \( sp(4, \mathcal{R}) \) identified with the Dirac singletons [1]. The difference is that the singleton representation \( S \) is unitary while the Fock module \( F \) is not. In this section we show that our approach makes the duality between the two types of representations manifest by virtue of a certain Bogolyubov transform. This parallelism extends far enough. In particular, the Flato-Fronsdal theorem [17] that the tensor product of the two singleton representations is equivalent to the direct sum of all \( AdS_4 \) massless representations.
acquires a simple dynamical interpretation in the unfolded formulation of the higher spin dynamics.

The fact that the involution $\mu$ (2.13) maps the oscillators $a_\alpha$ and $a^{+\alpha}$ to themselves implies that the Fock module (4.3) is not unitary. This is in agreement with the fact that the vacuum $|0\rangle\langle 0|$ is Lorentz invariant (2.19) and, as a result, the module (4.3) decomposes into the infinite sum of the finite-dimensional representations of the noncompact 3d Lorentz algebra $o(2,1)$ identified with the component fields $c_{\alpha(l)}(x)$. (Recall that noncompact algebras do not admit finite-dimensional unitary representations.)

The unitary Fock module of $sp(4,\mathbb{R})$ is built in terms of the oscillators

$$[\hat{b}_i^\pm, \hat{b}_j^\pm] = 0, \quad [\hat{b}_i^-, \hat{b}_j^+] = \delta_{ij}, \quad i, j = 1, 2,$$

satisfying the Hermitian conjugation conditions

$$(\hat{b}_i^\pm)^\dagger = \hat{b}_i^\mp.$$

The corresponding Fock vacuum $|0_u\rangle\langle 0_u|$ is defined according to

$$\hat{b}_i^- |0_u\rangle\langle 0_u| = 0, \quad |0_u\rangle\langle 0_u| \hat{b}_i^+ = 0.$$

The unitary left and right Fock modules built from the vacuum $|0_u\rangle\langle 0_u|$ identify with the supersingleton $S$ and its conjugate $\bar{S}$, respectively. (The supersingleton $S$ decomposes into two irreducible representations of the $sp(4,\mathbb{R})$ associated with the subspaces built from $|0_u\rangle\langle 0_u|$ with the aid of even and odd numbers of creation operators, called Rac and Di, respectively [17].)

The relationship between the two sets of oscillators is

$$\hat{b}_j^\pm = \frac{1}{\sqrt{2}} (\hat{a}_j \pm \hat{a}^{+j}),$$

$$\hat{a}_j = \frac{1}{\sqrt{2}} (\hat{b}_j^+ + \hat{b}_j^-), \quad \hat{a}^{+j} = \frac{1}{\sqrt{2}} (\hat{b}_j^+ - \hat{b}_j^-).$$

The unitary Fock vacuum can be realized in terms of the star product algebra (2.10) as

$$|0_u\rangle\langle 0_u| = 2(1 + k) \exp \left( -\delta^{ij}(a_i a_j - a_i^+ a_j^+) \right).$$

We therefore conclude that, in our approach, there is a natural duality between the field-theoretical module $F$ used in the unfolded formulation of the 3d conformal dynamics and unitary module $S$.

The idea that some duality of this kind takes place has been worked out earlier [18] for the 4d case and is very interesting in the context of the AdS/CFT correspondence. The new point is that it takes a simple form of the Bogolyubov transform (6.4), (6.5) in the framework of the unfolded formulation of the 3d conformal dynamics. Let us stress that the dependence on the space-time coordinates of the
elements of the field $|\Phi(x)\rangle$ is determined completely by the equation (4.4) in terms of its value at any fixed point $x_0$. This means that the module $|\Phi(x_0)\rangle$ contains the complete information on the on-mass-shell dynamics of the 3d conformal fields analogously to the fact that the singleton module contains the complete information on the (on-mass-shell) quantum states of the free theory. We believe that this phenomenon is quite general and the unfolded formulation of the dynamical systems in the form of some flatness (i.e., covariant constancy or zero-curvature) conditions will make the duality between the classical and quantum description of the dynamical systems manifest for general case.

The duality between unitary and unfolded formulations of the dynamical systems exhibiting higher spin symmetries admits an interesting extension to the $AdS_4$ higher spin gauge theories. As shown by Flato and Fronsdal [17] the tensor product of singleton representations amounts to the direct sum of the unitary representations of $sp(4,\mathbb{R})$ associated with the massless fields of all spins in $AdS_4$. In particular, a $AdS_4$ massless field of every spin $s = 0, 1/2, 3/2, \ldots$ appears in two copies in $S \otimes \bar{S}$. (See also [19] for the straightforward analysis). Remarkably, this is exactly the spectrum of the simplest (i.e., without non-Abelian Yang-Mills gauge symmetries) supersymmetric $AdS_4$ higher spin gauge theory built in [21, 3] based on $hu(1; 1|4)$. In [19] it was shown that this fact implies that $hu(1; 1|4)$ admits a unitary representation to be associated with the one-particle states of the quantum higher spin gauge theory. In [21] it was then shown that all consistent $4d$ higher spin gauge theories with various Yang-Mills groups admit unitary representations associated with certain tensor products of pairs of singletons.

Now, let us follow the field-theoretical picture. A particular basis in the tensor product $F^l \otimes F^r$ of the left and right Fock modules is spanned by the elements of the form

$$a^+_{\alpha_1} \ast \ldots \ast a^+_{\alpha_n} \ast |0\rangle \langle 0| \ast a^+_{\beta_1} \ast \ldots \ast a^+_{\beta_m}.$$ (6.7)

$F^l \otimes F^r$ can be identified with the algebra of endomorphisms of $F$. The star product algebra (2.10) also can be interpreted as the algebra of linear operators in $F$ and therefore it can be identified with $F^l \otimes F^r$. The point is that the $AdS_4$ higher spin gauge fields are the gauge fields (1-forms) taking values in this algebra. This provides an interesting dynamical realization of the Flato-Fronsdal theorem. Moreover, it has been shown recently [22] that, in agreement with the AdC/CFT correspondence [4], the set of the $AdS_4$ higher spin gauge fields is in one-to-one correspondence with the set of 3d conformal higher spin currents.

The important question of the explicit form of the AdS/CFT correspondence in
the framework of the higher spin gauge theories requires a more detailed analysis of
the higher spin dynamics and will be considered elsewhere. One relevant issue is that
the Fock-module realization of the 3d equations proposed in this paper is different
from the approach to the nonlinear 3d higher spin theory given in [1,2] where
the equation (4.4) was formulated in terms of the twisted adjoint representation
of the $AdS_3$ higher spin algebra realized in terms of the smaller set of oscillators
$y^a$ according to (4.1). As the formulation of [1,2] works both for massless [1] and
for massive matter fields [2] it has no manifest evidence of the conformal
higher spin symmetries. The approach developed in this paper therefore raises an
important problem of the search of a manifestly conformal theory describing higher
spin interactions of 3d massless matter, based on the Fock modules rather than on
the twisted adjoint representation. Let us note that this alternative is expected to
be analogous to the realization of the 2d higher spin matter system of [3,4] originally
formulated in terms of the Fock module. Also it is tempting to speculate that the
resulting models may have some relationship with the models of non-commutative
solitons [3,4] discussed recently in the context of the noncommutative phase of
superstring theory [4], which are realized in terms of Fock modules with respect to
the non-commutative space-time coordinates.

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