The evaluation of loop amplitudes via differential equations

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Abstract

The evaluation of loop amplitudes via differential equations and harmonic polylogarithms is discussed at an introductory level. The method is based on evolution equations in the masses or in the external kinematical invariants and on a proper choice of the basis of the transcendental functions. The presentation is pedagogical and goes through specific one-loop and two-loop examples in order to illustrate the general elements and ideas.

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1 Introduction

The evaluation of radiative corrections in quantum field theory is notoriously a hard task and various methods have been proposed in decades to accomplish it, such as Feynman parameters, dispersion relations, low-momentum expansions, etc. In the past few years, a new method has been developed, which is based on (i) the reduction of the amplitudes to a minimal set of scalar integrals called master integrals and (ii) their evaluation by means of differential equations in the masses or in the external kinematical invariants; the differential equations are then solved using a proper basis of special functions, the harmonic polylogarithms [1, 2, 3, 4, 5]. Our aim is to present a simple introduction to this method. Even though the latter has been used to do multi-loop calculations, it can also be used to reproduce standard one-loop results. Indeed, we are going to make use mostly of one-loop examples to describe the general elements and ideas. As we shall see, computations which are rather laborious with older techniques become much simpler with the differential equations and with the harmonic polylogarithms. The plan of this note is the following.

In sec. 2 we describe the tensor decomposition, i.e. how the evaluation of tensor integrals, coming directly from the application of Feynman rules, can be reduced to that of scalar integrals. This step is well known to many people and, strictly speaking, it does not belong to the method under discussion; it is included for completeness.

In sec. 3 we describe two widely-used schemes to transform the dependent scalar amplitudes generated with the previous step into a smaller set of linearly-independent ones. We present both one-loop and two-loop examples.

In sec. 4 we derive and solve the so-called integration-by-parts identities, which allow to reduce the independent amplitudes to a (much) smaller subset, the so-called master integrals. The two main methods of solutions are discussed by means of simple examples.

In sec. 5 we describe the method of the differential equations to analytically evaluate the master integrals. We consider a couple of one-loop examples which exhibit many of the general properties of the method.

In sec. 6 we overview the main ideas and results of the harmonic polylogarith theory, including also the extension of the basis function set to describe amplitudes with threshold at $s = 4m^2$.

2 Feynman diagrams

The evaluation of virtual corrections to a cross section begins with the application of Feynman rules to the relevant diagrams. Delicate points are typically the inclusion of the correct multiplicity factors, the signs of fermion loops and, whenever gauge interactions are present, a convenient gauge choice. Nowadays this step can be done in an automated way.

Let us consider for instance the top contribution to Higgs production by gluon fusion, i.e. the process

\[ g + g \rightarrow H. \]

(1)

The Feynman amplitude reads:

\[ \mathcal{M} = \epsilon_\mu(p_1) \epsilon_\nu(p_2) T^{\mu\nu}(p_1, p_2), \]

(2)

where $\epsilon_\mu(p_1)$ and $\epsilon_\nu(p_2)$ are the polarizations of the gluons with momenta $p_1^2 = 0$ and $p_2^2 = 0$ and $T^{\mu\nu}(p_1, p_2)$ is the following tensor:

\[ T^{\mu\nu}(p_1, p_2) = -\frac{4}{9} e^2 N_c \frac{m_t}{2 m_W} \int \frac{d^4 k}{(2\pi)^4} \text{Tr} \left[ \gamma^\mu \frac{\hat{k} + \hat{p}_1 + m_t}{(k + p_1)^2 - m_t^2} \frac{\hat{k} - \hat{p}_2 + m_t}{(k - p_2)^2 - m_t^2} \gamma^\nu \frac{\hat{k} + m_t}{k^2 - m_t^2} \right], \]

(3)
where $N_c$ is the color factor, $g$ the SU(2) coupling and $n$ the dimension of the space-time. By evaluating the trace, the tensor comes out to be of the form:

\[
T^{\mu\nu}(p_1, p_2) = \frac{d^n k}{(2\pi)^n} \frac{N^{\mu\nu}(p_1, p_2, k)}{[(k + p_1)^2 - m_1^2] [(k - p_2)^2 - m_2^2] [k^2 - m_l^2]},
\]

where $N^{\mu\nu}(p_1, p_2, k)$ is a tensor depending on the external momenta $p_1$ and $p_2$ as well as on the loop momentum $k$: many possible tensor structures are possible with 3 momenta,

\[
g^{\mu\nu}, \quad k^\mu p_1^\nu, \quad p_1^\mu k^\nu, \quad k^\mu p_2^\nu, \quad p_2^\mu k^\nu, \quad p_1^\mu p_2^\nu, \quad p_1^\mu p_2^\nu, \quad \cdots
\]

The following reduction is convenient. According to relativistic invariance, the tensor can be parametrized as

\[
T^{\mu\nu}(p_1, p_2) = p_1^\mu p_2^\nu T_1(q^2) + p_2^\mu p_1^\nu T_2(q^2) + p_1^\mu p_2^\nu T_3(q^2) + p_2^\mu p_1^\nu T_4(q^2) + g^{\mu\nu}(p_1 \cdot p_2) T_5(q^2)
\]

where $q = p_1 - p_2$ is the Higgs momentum and $\epsilon_{\mu\nu\rho\sigma}$ is the antisymmetric tensor. The last form factor $T_5$ is related to parity violation of weak interactions. Gauge invariance implies:

\[
p_1^\mu T_{\mu\nu} = p_2^\nu T_{\mu\nu} = 0,
\]

which imply, in turn:

\[
T_1 = T_2 = 0 \quad \text{and} \quad T_5 = -T_4.
\]

The above relations can be used as checks of the computation. The form factors $T_i$ can be derived from the tensor $T_{\mu\nu}$ by means of projectors. We have for instance:

\[
T_5 = \frac{1}{(n - 2)p_1 \cdot p_2} \left[ g^{\mu\nu} - \frac{p_2^\mu p_1^\nu}{p_1 \cdot p_2} \right] T_{\mu\nu}.
\]

By applying the projector to both sides of eq. 4 and taking it inside the loop integral, one obtains:

\[
T_5 = \frac{N_5(k, p_1, p_2)}{(2\pi)^n \frac{N_5(k, p_1, p_2)}{[(k + p_1)^2 - m_1^2] [(k - p_2)^2 - m_2^2] [k^2 - m_l^2]},
\]

where:

\[
N_5(k, p_1, p_2) = \frac{1}{(n - 2)p_1 \cdot p_2} \left[ g^{\mu\nu} - \frac{p_2^\mu p_1^\nu}{p_1 \cdot p_2} \right] N_{\mu\nu}(k, p_1, p_2).
\]

We have now a scalar numerator instead of a tensor one, depending only on invariants:

\[
N_5(k, p_1, p_2) = P(k^2, k \cdot p_1, k \cdot p_2),
\]

where $P$ is a polynomial:

\[
2 \quad \text{This is true is all local quantum field theories.}
\]

\[
\sum_{l,r,s=0}^{n_{max}} \sum_{l,r,s=0}^{n_{max}} a_{lrs} \left( k^2 \right)^l (k \cdot p_1)^r (k \cdot p_2)^s
\]

and $a_{lrs}$ are known constants. $n_{max}$ is the maximum number of invariants and depends on the interaction; typically, in one-loop computations, $n_{max} = 1, 2$. By using eqs. 10, 12 and 13, we obtain for the form factor $T_5$:

\[
T_5 = \sum_{l,r,s=0}^{n_{max}} a_{lrs} \int \frac{d^n k}{(2\pi)^n} \frac{(k^2)^l (k \cdot p_1)^r (k \cdot p_2)^s}{[(k + p_1)^2 - m_1^2] [(k - p_2)^2 - m_2^2] [k^2 - m_l^2]},
\]
3 Independent amplitudes

As we have seen, the calculations of the radiative corrections to a generic process can be reduced to the evaluation of scalar integrals, that in a $2 \to 1$ process according to eq. (14), are of the form

$$\int \frac{d^n k}{(2\pi)^n} \frac{(k^2)^l (k \cdot p_1)^r (k \cdot p_2)^s}{D_1 D_2 D_3},$$

(15)

where we have defined:

$$D_1 = k^2 - a,$$

$$D_2 = (k + p_1)^2 - a,$$

$$D_3 = (k - p_2)^2 - a$$

(16)

with $a = m^2$, a generic mass squared.

The above amplitudes are linearly dependent on each other and it is convenient to reduce them to a smaller set of linearly independent ones. We express the kinematical invariants in terms of the denominators by means of the following formulas — the so-called rotation:

$$k^2 = D_1 + a,$$

$$k \cdot p_1 = \frac{1}{2} [D_2 - D_1],$$

$$k \cdot p_2 = \frac{1}{2} [-D_3 + D_1].$$

(17)

In general, in one-loop amplitudes, denominators form a basis for the invariants. There are two different schemes to implement the rotations given in eqs. (17):

1. auxiliary diagram scheme;
2. shift scheme.

1) According to the first method, one uses eqs. (17) inside eq. (14), to obtain:

$$T_5 = \sum_{l,r,s=0}^{n_{max}} b_{lrs} \text{Topo}(1-l,1-r,1-s),$$

(18)

where $b_{lrs}$ are known constants and we have defined:

$$\text{Topo}(n_1, n_2, n_3) = \int \frac{d^n k}{(2\pi)^n} \frac{1}{D_1^{n_1} D_2^{n_2} D_3^{n_3}}.$$  

(19)

The amplitudes above constitute the linearly independent ones in this scheme. Let us note that a propagator $D_i$ originally present in the denominator may be cancelled by a term $D_i, D_i^2, D_i^3, ...$ in the numerator. In diagrammatic language, that means that internal line $i$ is shrunk to a point. The reduction to independent amplitudes then generates a pyramid of subdiagrams of the original diagram, in which any subset of internal lines is contracted to a point. For clarity’s sake, one has to evaluate amplitudes of the form

$$\frac{1}{D_1 D_2 D_3}, \frac{D_1}{D_2 D_3}, \frac{D_1^2}{D_2 D_3}, ... , \frac{1}{D_2 D_3}, \frac{D_2}{D_1 D_3}, ... , \frac{1}{D_3}, \frac{D_3}{D_1 D_2}, ... , \frac{1}{D_1}, \frac{D_1}{D_2}, ... \frac{1}{D_3}.$$  

(20)
Let us now consider the alternative scheme for the reduction to independent amplitudes. The shift scheme makes use of the substitutions (17) only when there is an effective cancellation of denominators. In other words, the “allowed” substitutions are:

\[
\frac{k^2}{D_1} \rightarrow 1 + \frac{a}{D_1}, \\
\frac{k \cdot p_1}{D_1 D_2} \rightarrow \frac{1}{2} \left[ \frac{1}{D_1} - \frac{1}{D_2} \right], \\
\frac{k \cdot p_2}{D_1 D_3} \rightarrow \frac{1}{2} \left[ -\frac{1}{D_1} + \frac{1}{D_3} \right].
\]

This way we are left with independent amplitudes of the following kinds:

1. amplitudes containing only denominators:

\[
\frac{1}{D_1 D_2 D_3}: \text{ scalar vertex,} \\
\frac{1}{D_1 D_2}, \frac{1}{D_1 D_3}, \frac{1}{D_2 D_3}: \text{ scalar bubbles,} \\
\frac{1}{D_1}, \frac{1}{D_2}, \frac{1}{D_3}: \text{ scalar tadpoles.}
\]

The bubble diagrams are obtained shrinking any one of the internal lines to a point, while tadpoles are obtained shrinking any pair of internal lines;

2. bubble diagrams with irreducible numerators:

\[
\frac{k \cdot p_2}{D_1 D_2}, \frac{(k \cdot p_2)^2}{D_1 D_2}, \frac{k \cdot p_1}{D_1 D_3}, \frac{(k \cdot p_1)^2}{D_1 D_3}, \cdots,
\]

and amplitudes which do not contain anymore \(D_1\), i.e. the denominator containing the loop momentum squared \(k^2\):

\[
\frac{k^2}{D_2 D_3}, \frac{k \cdot p_1}{D_2 D_3}, \frac{k \cdot p_2}{D_2 D_3}, \frac{(k^2)^2}{D_2 D_3}, \frac{k^2 k \cdot p_1}{D_2 D_3}, \frac{k^2 k \cdot p_2}{D_2 D_3}, \cdots
\]

No simplification is possible for the amplitudes in (24) as any cancellation in eq. (21) is feasible only if \(D_1\) is present. We then make a shift of the loop momentum \(k\) in order to reproduce a denominator containing \(k^2\), such as for instance:

\[
k \rightarrow k - p_1,
\]

so that

\[
D_2 \rightarrow D_1, \quad D_3 \rightarrow D_4 = (k - p_1 - p_2)^2 - a.
\]

The shift introduces therefore the new denominator \(D_4\), not initially present in the diagram. Since \(D_4\) contains both \(p_1\) and \(p_2\), one can express \(k \cdot p_2\) in terms of \(k \cdot p_1\) or \textit{vice versa}. Let us take the first choice:

\[
\frac{k \cdot p_2}{D_1 D_4} = \frac{1}{2D_4} - \frac{1}{2D_1} - \frac{k \cdot p_1}{D_1 D_4}.
\]

The amplitudes (24) are then transformed into amplitudes of the form:

\[
\frac{1}{D_1 D_4}, \frac{k \cdot p_1}{D_1 D_4}, \frac{(k \cdot p_1)^2}{D_1 D_4}, \frac{(k \cdot p_1)^3}{D_1 D_4}, \cdots + \text{ (tadpoles);}\]

These amplitudes as well as the following ones can be treated with a Passarino-Veltman reduction; we choose to use the general method valid also in the multi-loop case.
3. Tadpoles, involving:

- "final" amplitudes of the form:

\[
\frac{k \cdot p_1}{D_1}, \quad \frac{k \cdot p_2}{D_1}, \quad \frac{(k \cdot p_1)^2}{D_1}, \quad \frac{(k \cdot p_2)^2}{D_1}, \quad \frac{k \cdot p_1 k \cdot p_2}{D_1}; \quad \cdots \quad (29)
\]

- reducible amplitudes of the form:

\[
\frac{k^2}{D_2}, \quad \frac{k \cdot p_1}{D_2}, \quad \frac{k \cdot p_2}{D_2}, \quad \frac{(k^2)^2}{D_2}, \quad \frac{k^2 k \cdot p_1}{D_2}, \quad \frac{k^2 k \cdot p_2}{D_2}; \quad \cdots \quad (30)
\]

The amplitudes in the first line of the above expression are reduced by means of the shift in (25), while for the second line we make the shift \( k \to k + p_2 \); the resulting amplitudes are of the form (29).

The same paths of reduction to independent scalar amplitudes can be followed for multiloop corrections. As a 2-loop example, let us now consider the light-fermion correction to the process (11) consisting of a ladder diagram. The latter describes a gluon pair converting into a light quark pair which converts in turn into a pair of W’s or Z’s annihilating finally into a Higgs boson. The dependent scalar amplitudes are of the form:

\[
L = \int \frac{d^nk_1 \, d^nk_2 \, P(k_1^2, k_2^2, k_1 \cdot p_1, k_1 \cdot p_2, k_2 \cdot p_1, k_2 \cdot p_2, k_1 \cdot k_2)}{D_1 D_2 D_3 D_4 D_5 D_6}, \quad (31)
\]

where:

\[
\begin{align*}
D_1 &= k_1^2, \\
D_2 &= (k_1 + p_1)^2, \\
D_3 &= (k_1 - p_2)^2, \\
D_4 &= k_2^2, \\
D_5 &= (k_1 + k_2 + p_1)^2 - a, \\
D_6 &= (k_1 + k_2 - p_2)^2 - a.
\end{align*}
\]

(32)

The conversion to independent amplitudes is not straightforward in this case because there are six denominators and seven invariants. The denominators then do not form a basis for the invariants, as it happened in the one-loop case.

The solution to this problem, in the auxiliary diagram scheme, is to construct an auxiliary diagram with an additional, fictitious denominator linearly independent from the previous ones, such as for instance

\[
D_7 = (k_1 + k_2)^2. \quad (33)
\]

In diagrammatic language, we may say that we have “opened” the Higgs vertex: the auxiliary diagram is a planar double box in a forward configuration, i.e. with final momenta equal to the initial ones \( p_1 \) and \( p_2 \). After
the addition of the auxiliary denominator, the rotation is possible by means of the formulas:

\[
\begin{align*}
  k_1^2 &= D_1, \\
  k_2^2 &= D_4, \\
  k_1 \cdot p_1 &= \frac{1}{2} (D_2 - D_1), \\
  k_1 \cdot p_2 &= \frac{1}{2} (-D_3 + D_4), \\
  k_2 \cdot p_1 &= \frac{1}{2} (D_4 - D_2 + D_5 - D_7 + a), \\
  k_2 \cdot p_2 &= \frac{1}{2} (D_3 - D_4 - D_6 + D_7 - a), \\
  k_1 \cdot k_2 &= \frac{1}{2} (D_7 - D_1 - D_4). 
\end{align*}
\]

(34)

The numerator can then be expanded in powers of the denominators as:

\[
P(k_1^2, k_2^2, k_1 \cdot p_1, k_2 \cdot p_1, k_2 \cdot p_2, k_1 \cdot k_2) = \sum_{l_1,l_2,l_3,l_4,l_5,l_6,l_7=0}^{n_{\text{max}}} c_{l_1,l_2,l_3,l_4,l_5,l_6,l_7} D_{l_1}^{l_1} D_{l_2}^{l_2} D_{l_3}^{l_3} D_{l_4}^{l_4} D_{l_5}^{l_5} D_{l_6}^{l_6} D_{l_7}^{l_7} 
\]

(35)

where \( c_{l_1,l_2,l_3,l_4,l_5,l_6,l_7} \) are known constants. By inserting the expansion in eq. (35), one obtains independent amplitudes to be computed, containing formally only denominators, of the form:

\[
\text{Topo}(n_1, n_2, n_3, n_4, n_5, n_6, n_7) = \int \frac{d^n k_1}{(2\pi)^n} \frac{d^n k_2}{(2\pi)^n} \frac{1}{D_{l_1}^{l_1} D_{l_2}^{l_2} D_{l_3}^{l_3} D_{l_4}^{l_4} D_{l_5}^{l_5} D_{l_6}^{l_6} D_{l_7}^{l_7}} 
\]

(36)

with \( n_i \leq 1 \) for \( i = 1 \ldots 6 \) and \( n_7 \leq 0 \). Note that the auxiliary denominator \( D_7 \) appears only in the numerator while the standard denominators \( D_i (i = 1 \ldots 6) \) appear both in the denominator and in the numerator.

As for the case of the auxiliary diagram scheme, also the shift method can be extended to the multi-loop case in a straightforward way.

4 Integration by parts identities

The virtual corrections to a cross section involve, in general, the evaluation of a large number of independent amplitudes — in the case of massive two-loop computations hundreds if not thousands. In the past all these amplitudes had to be individually computed [8]. It is however possible to reduce by a large amount the number of amplitudes to be computed by using integral identities [9]. In sec. 4.1 we discuss the derivation of the identities, while in secs. 4.2 and 4.3 we present two methods for their solution. We work in the auxiliary diagram scheme; the discussion in the shift scheme is completely analogous.

4.1 Derivation of the identities

Let us begin with the simplest case, that of the one-loop tadpole. According to the divergence theorem:

\[
\int d^n k \left. \frac{\partial}{\partial k_\mu} \frac{k_\mu}{D_1^{n_1}} \right|_{D_1^{n_1}} = \int_{S_{\infty}} d^{n+1} s \frac{k_\mu}{D_1^{n_1}} = 0,
\]

(37)

where

\[
D_1 = k^2 - a.
\]

(38)
$S_\infty$ is a sphere of infinite radius in momentum space and $ds^\mu$ is a surface element. The flux integral actually vanishes only for $n_1 > n/2$, but we will analytically continue eq. (37) to all the $(n, n_1)$ space.

By explicitly performing the derivative and re-expressing the result in terms of independent amplitudes by means of the relation (see previous section)

$$k^2 = D_1 + a,$$

we obtain the following integration-by-parts (ibp) identity:

$$(n - 2n_1)T(n_1) - 2an_1T(n_1 + 1) = 0,$$

where we have defined:

$$T(n_1) = \int d^n k \frac{1}{D_{n_1}^1}.$$  

By introducing the identity operator $I$ and the plus and minus operators,

$$I T(n_1) = T(n_1),$$

$$1^\pm T(n_1) = T(n_1 \pm 1),$$

the ibp identity can be written as:

$$[(n - 2n_1)I - 2an_11^+] T(n_1) = 0.$$  

Let us now consider as a less trivial case: a bubble with one massive line,

$$B(n_1, n_2) = \int d^n k \frac{1}{D_{n_1}^{n_1} D_{n_2}^{n_2}},$$

where

$$D_1 = k^2 - a,$$

$$D_2 = (k + p)^2.$$  

The integration-by parts identities are derived according to:

$$\int d^n k \frac{\partial}{\partial k_\mu} \frac{v_\mu}{D_{n_1}^{n_1} D_{n_2}^{n_2}} = 0,$$

where $v_\mu = k_\mu$ or $p_\mu$. We have therefore two identities for each set of indices $(n_1, n_2)$:

$$(n - n_1 - 2n_2)I - n_1 \left[ a(1 + x) + 2^- \right] 1^+ = 0,$$

$$(n_1 - n_2)I + n_1 \left[ a(1 - x) - 2^- \right] 1^+ + n_2 \left[ a(1 + x) + 1^- \right] 2^+ = 0,$$

where $x = -p^2/a$ and all the operators are intended to be applied to $B(n_1, n_2)$. Three different kinds of operators do appear in the identities:

$$I, \quad i^+, \quad i^+ j^- \quad (i \neq j = 1, 2).$$

The generalization to multi-loop multi-leg amplitudes is obvious. In the case for instance of the two-loop ladder diagram of the previous section, we have:

$$\int d^n k_1 d^n k_2 \frac{\partial}{\partial k_\mu_j} \frac{v_\mu}{D_{k_1}^{n_1} D_{k_2}^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5} D_6^{n_6} D_7^{n_7}} = 0,$$

where $j = 1, 2$ and $v = k_1, k_2, p_1, p_2$ is any one of the loop or external momenta. We have eight identities for any choice of the indices. Let us note that, in general, the identities are not all independent on each other.
4.2 Symbolic solution

Once the identities have been generated, the next step is to solve them in a convenient way \[9\]. Let us begin with the tadpole identity \[10\]. We can solve it with respect to the amplitude with the greater index:

$$ T(n_1+1) = \frac{n-2n_1}{2a n_1} T(n_1). \quad (51) $$

If we assume for instance \(T(1)\) to be known, we can determine from the above equation \(T(2), T(3), T(4), \ldots\), so that we can write:

$$ T(k) = a(k) T(1), \quad (52) $$

where \(a(k)\) is a known coefficient and \(k\) is an integer. We can also solve eq. \[10\] with respect to the amplitude with the smaller index:

$$ T(n_1) = \frac{2a n_1}{n-2n_1} T(n_1+1). \quad (53) $$

By setting \(n_1 = 0\) we obtain \(T(0) = 0\), and hence \(T(n_1) = 0\) for \(n_1 < 0\), as well known from elementary quantum field theory computations. The conclusion is that the tadpole topology has one master integral, which can be taken as \(T(k)\) with \(k\) a positive integer.

Let us now solve the bubble identities \[17\] and \[18\]. It is convenient to introduce the sum of the indices

$$ \Sigma = n_1 + n_2. \quad (54) $$

The plus operators \(1^+\) and \(2^+\) increase \(\Sigma\) by one, while the identity \(I\) and the plus-minus operators \(1^+2^-\) and \(2^+1^-\) keep \(\Sigma\) unchanged. Let us assume that this topology has one master integral, which we take as \(B(1,1)\), having \(\Sigma = 2\) — this will be proved \textit{a posteriori}. A general amplitude, with \(n_1 \geq 1\) and \(n_2 \geq 1\), has \(\Sigma \geq 2\). That means we have to reduce \(\Sigma\) by solving the above identities with respect to the plus operators. The first equation is solved with respect to \(1^+\):

$$ 1^+ = \frac{n-n_1-2n_2}{an_1(1+x)} I - \frac{1}{a(1+x)} 1^+2^- . \quad (55) $$

With this equation we can shift the first index \(n_1 > 1\) down to the value \(n_1 = 1\). Let us remark that it is impossible to go further because the coefficients have \(n_1\) in the denominator and then become singular. Similarly, the second equation can be used to shift the second index \(n_2\) down to one:

$$ 2^+ = \frac{n_2-n_1}{an_2(1+x)} I - \frac{n_1(1-x)}{n_2(1+x)} 1^+ \quad + \quad \frac{n_1}{an_2(1+x)} 1^+2^- \quad - \quad \frac{1}{a(1+x)} 1^-2^+. \quad (56) $$

Because of the presence of the minus operators, amplitudes with one of the indices equal to zero such as \(B(1,0), B(0,1), B(2,0), \ldots\), are encountered. These amplitudes have one of the internal lines shrunk to a point and are therefore tadpoles, whose reduction has already been discussed. By recursively using eqs. \[55\] and \[56\] we can reduce any amplitude \(B(n_1,n_2)\) with \(n_1 \geq 1\) and \(n_2 \geq 1\) to \(B(1,1)\) + (tadpoles):

$$ B(n_1,n_2) = c(n_1,n_2) B(1,1) + d(n_1,n_2) T(1), \quad (57) $$

where \(c(n_1,n_2)\) and \(d(n_1,n_2)\) are known functions. We have thus proved that the one-mass bubble has one master integral.

In some cases, amplitudes of a given topology can be reduced to subtopologies, i.e. to amplitudes with less internal lines. Let us consider as specific example a vertex diagram representing the annihilation of two massless particles with momenta \(p_1\) and \(p_2\) into a virtual particle with momentum \(q = p_1 + p_2\):

$$ V(n_1,n_2,n_3) = \int d^4 k \left( \left[ (k-p_2)^2 - a \right]^{n_1} \left[ (k+p_1)^2 \right]^{n_2} \left[ k^2 \right]^{n_3} \right) \quad (58) $$
with $p_1^2 = 0$ and $p_2^2 = 0$. There is a massive line “on a side”, connecting one of the massless particles with the virtual one. The following two ibp identities are easily derived:

\[
(n - n_1 - n_2 - 2n_3)I - n_1(a + 3^-)1^+ - n_22^+3^- = 0, \tag{59}
\]

\[
(-n_2 + n_3)I + n_1(-xa - 2^- + 3^-)1^+ + n_22^+3^- - n_33^+2^- = 0, \tag{60}
\]

where $x = -q^2/a$. By solving the first equation with respect to the $1^+$ operator and substituting the solution into the second equation, one obtains:

\[
a \left[(-n_2 + n_3) - (n - n_1 - n_2 - 2n_3)x\right]I + n_1a \left[-2^- + (1 + x)3^-\right]1^+ + n_2a(1 + x)2^+3^- - n_3a3^+2^- = 0. \tag{61}
\]

The above equation does not contain anymore plus operators, bringing unknown amplitudes, but only the identity and the plus-minus operators. By setting $n_1 = n_2 = n_3 = 1$ it is immediately seen that the basic amplitude $V(1, 1, 1)$ is expressed in terms of amplitudes having one of the indices zero, i.e. of bubbles: we succeeded in the above-mentioned reduction.

### 4.3 Laporta method

This method has been originally introduced in [10] and has since then been widely used for the evaluation of 2-loop 3-point and 4-point functions in a variety of mass and kinematical configurations [3, 11, 12, 5]. The idea is that of replacing explicit values for the indices $n_i = \cdots -1, 0, 1 \cdots$ in the ibp identities. This way a system of linear equations is generated, whose unknowns are the amplitude themselves. In the simple case of the tadpole, for instance, one generates a system of equations of the form:

\[
\begin{align*}
2aT'(2) - (n - 2)T'(1) &= 0, \\
4aT'(3) - (n - 4)T'(2) &= 0, \\
6aT'(4) - (n - 6)T'(3) &= 0, \\
&\vdots \quad \vdots \quad \vdots \\
2k aT'(k + 1) - (n - 2k)T'(k) &= 0.
\end{align*}
\tag{62}
\]

The system is then solved with the method of elimination of variables of Gauss. One has to decide which amplitudes have to be solved first. In the above example, one could solve first for $T'(k)$, then for $T'(k-1)$, and so on. In general, a good criterion is the following [13]:

- We solve first for the amplitudes with the largest number of denominators. More formally, we define the recursive parameter:

  \[
  \Sigma_1 = \sum_i \theta(n_i), \tag{63}
  \]

  where the step function is defined as $\theta(u) = 1$ if $u > 0$ and zero otherwise, and we solve first for the amplitudes with the greatest $\Sigma_1$;

- Among the amplitudes with the same number of denominators, i.e. with the same value of $\Sigma_1$, we solve first for those ones with the greatest sum of the indices of the denominators,

  \[
  \Sigma_2 = \sum_i n_i \theta(n_i); \tag{64}
  \]

- finally, among the amplitudes with the same values for $\Sigma_1$ and $\Sigma_2$, we solve first for the amplitudes with the largest number of $D_i$ in the numerator – in the shift scheme, that is the largest number of irreducible numerators:

  \[
  \Sigma_3 = \sum_i n_i \theta(-n_i). \tag{65}
  \]
Many amplitudes have, in general, the same values of \((\Sigma_1, \Sigma_2, \Sigma_3)\): the choice of the amplitude can be random. Furthermore, a given amplitude appears, in general, in various equations: also the choice of the equation can be random.

According to Gauss method, we proceed with the progressive elimination of variables till all the equations have been used. The amplitudes which remain on the r.h.s. at the end are the master integrals. Within the scheme \((\Sigma_1, \Sigma_2, \Sigma_3)\) presented above, the master integrals typically involve amplitudes with unitary denominators \((n_i = 1)\) and irreducible numerators \((n_j = 0, -1, -2 \cdots)\). If we exchange \(\Sigma_2\) with \(\Sigma_3\), the master integrals typically involve amplitudes with denominators squared. In practise, one usually starts with a small linear system and looks at the master integrals coming from its solution. By enlarging the size of the system, a smaller or equal set of master integrals is obtained. The idea of the method is that there is a “critical mass” of equations, above which a complete reduction to the master integrals occurs. The reason for this is that, by enlarging the system, the number of equations grows faster than the number of unknowns \[3\]. The main virtue of this method is that it can be automated in a rather general way.

5 The method of differential equations

Differential equations in the masses or in the external kinematical invariants offer a general method for the calculation of master integrals. This method allows in principle to compute any loop amplitude which involves more than one scale\(^4\). In sec. \(5.1\) we sketch the derivation of the differential equation for the case of the one-mass bubble considered before, while in secs. \(5.2\) and \(5.3\) we describe the general method to solve the equation.

5.1 Generation of the equation

Let us begin with perhaps the simplest possible example: the bubble with one massive and one massless line,

\[
B(p^2) = \int d^n k \frac{1}{(k^2 - a)(k + p)^2},
\]

(66)

where

\[
d^n k = a^{2-n/2} \frac{d^n k}{i \pi^{n/2} \Gamma(3 - n/2)}.
\]

(67)

This diagram has a threshold at \(p^2 = m^2\). To obtain the differential equation, we take a derivative of the master integral with respect to the external invariant \(p^2\) using the formula:\(^5\)

\[
\frac{d}{dp^2} B(p^2) = \frac{1}{2\mu^2 \rho n} \frac{\partial}{\partial \rho} B(p^2).
\]

(68)

The partial derivative is taken inside the integral and produces various scalar amplitudes, which are reduced to the master integral itself by means of the methods described in the previous sections. The differential equation then closes on the master integral itself:

\[
\frac{d}{dx} B(x; \epsilon) = \left[ -\frac{1}{x} + \frac{1}{1 + x} \right] B(x; \epsilon) + \epsilon \left[ \frac{1}{x} - \frac{2}{1 + x} \right] B(x; \epsilon) + (1 - \epsilon) \left[ \frac{1}{x} - \frac{1}{1 + x} \right] T(\epsilon),
\]

(69)

where

\[
T(\epsilon) = \frac{1}{\epsilon}
\]

(70)

\(^4\)Bubble diagrams, vertex diagrams with two external particles on the light-cone, etc., having only massless propagators are then excluded. In all these cases, the differential equation gives only a dimensional, trivial information.

\(^5\)We could derive with respect to the mass squared \(a\) as well.
is the tadpole divided by $a$,
\[ \epsilon \equiv 2 - \frac{n}{2} \]  
(71)
and
\[ x \equiv -\frac{p^2}{a}. \]  
(72)
We have included a minus sign in the definition of $x$ so that the (simpler) euclidean region $p^2 < 0$ corresponds to $x > 0$. The presence of the threshold in $p^2 = a$ is reflected by the term $1/(1 + x)$ in the differential equation. Let us note that in the above derivation there is nothing specific about the one-loop case so the method extends trivially to the multiloop case.

5.2 Initial conditions

In order to obtain a unique value for the master integral, an initial condition has to be imposed to the general solution of the differential equation. That means we have to know the master integral in a given kinematical point $x$. Let us consider our example. Since $B(x; \epsilon)$ is regular for $x \to 0$, it holds:
\[ \lim_{x \to 0} x \frac{d}{dx} B(x; \epsilon) = 0. \]  
(73)
Multiplying both sides of eq. (69) by $x$, taking the limit $x \to 0$ and using eq. (73), one obtains:
\[ B(x = 0; \epsilon) = T(\epsilon). \]  
(75)
We have thus obtained the initial condition by studying the master integral close to zero momentum and using the differential equation itself.

5.3 Recursive solution in $\epsilon$

An efficient method to solve the differential equation for the master integral involves the $\epsilon$-expansion of the equation itself. Eq. (69) is of the general form:
\[ \frac{d}{dx} B(x; \epsilon) = A(x; \epsilon) B(x; \epsilon) + \Omega(x; \epsilon), \]  
(76)
where the coefficient of the unknown function is a polynomial of first order in $\epsilon$:
\[ A(x; \epsilon) = A_0(x) + \epsilon A_1(x), \]  
(77)
with
\[ A_0(x) = -\frac{1}{x} + \frac{1}{1 + x}, \]
\[ A_1(x) = \frac{1}{x} - \frac{2}{1 + x}. \]  
(78)
The main point is that $A(x; \epsilon)$ does not contain $1/\epsilon$ poles: this is true in general. $\Omega(x; \epsilon)$ is the known term of the differential equation and is associated to the tadpole — in general it is related to the subtopologies:
\[ \Omega(x; \epsilon) = \frac{1}{\epsilon} \Omega_{-1}(x) + \Omega_0(x) + \Omega_1(x) + \Omega_2(x) + \cdots, \]  
(79)
\[ B(x = 0; \epsilon) = \int d^d \tilde{k} \frac{1}{(k^2 - a) k^2} = \frac{1}{a} \int d^d \tilde{k} \frac{1}{k^2 - a} = \frac{1}{a} \int d^d \tilde{k} \frac{1}{k^2} = \frac{1}{a} \int d^d \tilde{k} \frac{1}{k^2 - a} = T(\epsilon). \]  
(74)
The integral of $1/k^2$ vanishes because the integrand is scaleless.

---

In this simple case, the value of the bubble for $x = 0$ — equivalent to $p = 0$ — can also be obtained with partial fractioning:
\[ B(x = 0; \epsilon) = \int d^d \tilde{k} \frac{1}{(k^2 - a) k^2} = \frac{1}{a} \int d^d \tilde{k} \frac{1}{k^2 - a} = \frac{1}{a} \int d^d \tilde{k} \frac{1}{k^2} = \frac{1}{a} \int d^d \tilde{k} \frac{1}{k^2 - a} = T(\epsilon). \]  
(74)
where

\begin{align*}
\Omega_{-1}(x) &= \frac{1}{x} - \frac{1}{1+x}, \\
\Omega_0(x) &= \frac{1}{x} + \frac{1}{1+x}, \\
\Omega_1(x) &= 0, \\
\Omega_2(x) &= 0, \\
\cdots & \cdots \cdots \cdots .
\end{align*}

(80)

In our example \(\Omega(x; \varepsilon)\) has only two non-zero terms; in more complicated cases, \(\Omega(x; \varepsilon)\) contains higher-order poles and has an infinite number of positive powers of \(\varepsilon\). Let us now expand the (unknown) master integral in powers of \(\varepsilon\); since the known term contains at most a simple pole, we expect the same to be true for the MI:

\[ B(x; \varepsilon) = \frac{1}{\varepsilon} B_{-1}(x) + B_0(x) + \varepsilon B_1(x) + \varepsilon^2 B_2(x) + \cdots \]

(81)

Substituting the expansion (81) in eq. (76) and equating the coefficients of the powers of \(\varepsilon\), we obtain a series of chained differential equations:

\begin{align*}
\frac{d}{dx} B_{-1}(x) &= A_0(x) B_{-1}(x) + \tilde{\Omega}_{-1}(x), \\
\frac{d}{dx} B_0(x) &= A_0(x) B_0(x) + A_1(x) B_{-1}(x) + \Omega_0(x), \\
\frac{d}{dx} B_1(x) &= A_0(x) B_1(x) + A_1(x) B_0(x) + \Omega_1(x), \\
\cdots & \cdots \cdots \cdots , \\
\frac{d}{dx} B_k(x) &= A_0(x) B_k(x) + A_1(x) B_{k-1}(x) + \Omega_k(x), \\
\cdots & \cdots \cdots \cdots .
\end{align*}

(82)

The first equation, for the coefficient \(B_{-1}(x)\) of the simple pole, is the first one to be solved. Once \(B_{-1}(x)\) is known, we can insert its value in the second equation for \(B_0(x)\) and solve for the latter function, and so on. In more formal terms, we can redefine the known term as

\begin{align*}
\tilde{\Omega}_{-1}(x) &\equiv \Omega_{-1}(x), \\
\tilde{\Omega}_k(x; B_{-1}, \cdots, B_{k-1}) &\equiv A_1(x) B_{k-1}(x) + \Omega_k(x) \quad \text{for} \quad k \geq 0,
\end{align*}

(83)

and rewrite the system as:

\begin{align*}
\frac{d}{dx} B_{-1}(x) &= A_0(x) B_{-1}(x) + \tilde{\Omega}_{-1}(x), \\
\frac{d}{dx} B_0(x) &= A_0(x) B_0(x) + \tilde{\Omega}_0(x; B_{-1}), \\
\frac{d}{dx} B_1(x) &= A_0(x) B_1(x) + \tilde{\Omega}_1(x; B_{-1}, B_0), \\
\cdots & \cdots \cdots \cdots , \\
\frac{d}{dx} B_k(x) &= A_0(x) B_k(x) + \tilde{\Omega}_k(x; B_{-1}, \cdots, B_{k-1}), \\
\cdots & \cdots \cdots \cdots .
\end{align*}

(84)

The system (84) is solved with the method of variation of constants of Euler, which we now summarize. Let us first consider the associated homogeneous equation for \(B_k(x)\), i.e. the equation obtained by dropping the
An important point is that the above equation is the same for any $k$. This equation is solved with separation of variables:

$$\omega(x) = \exp \int A(x) \, dx.$$  \hfill (86)

In our example,

$$\frac{d}{dx} \omega(x) = \left(-\frac{1}{x} + \frac{1}{1+x}\right) \omega(x),$$  \hfill (87)

whose solution is:

$$\omega(x) = \frac{1+x}{x},$$  \hfill (88)

where we have taken equal to unity the integration constant.

The solution of the original, non-homogeneous equation is given by the following integral:

$$B_k(x) = \omega(x) \int K(x) \tilde{\Omega}(x; B_{-1}, \cdots, B_{k-1}) \, dx',$$  \hfill (89)

where the kernel $K$ is just the inverse of the homogeneous solution:

$$K(x) \equiv \frac{1}{\omega(x)}.$$  \hfill (90)

The simple pole of the one-mass bubble for instance is given by:

$$B_{-1} = \frac{1}{x} + \frac{1+x}{x} \int_0^x dx' \frac{x'}{1+x'} \left(\frac{1}{x'} - \frac{1}{1+x'}\right) = \frac{c - \frac{1}{x}}{x} + c = 1,$$  \hfill (91)

where in the last member we have imposed the initial condition (75).

As another example, let us consider the bubble with two equal masses:

$$B(x; \epsilon) = \int d^n k \frac{1}{[k^2 - a][[(k + p)^2 - a]}.$$  \hfill (92)

This diagram has a threshold in $p^2 = 4m^2$. The differential equation reads:

$$\frac{d}{dx} B(x; \epsilon) = \left[-\frac{1}{2x} + \frac{1}{2(4+x)}\right] B(x; \epsilon) - \frac{\epsilon}{4+x} B(x; \epsilon) + (1-\epsilon) \left[\frac{1}{2x} - \frac{1}{2(4+x)}\right] T(\epsilon).$$  \hfill (93)

The solution of the associated homogeneous equation in four dimensions,

$$\frac{d}{dx} \phi(x) = \left[-\frac{1}{2x} + \frac{1}{2(4+x)}\right] \phi(x),$$  \hfill (94)

is:

$$\phi(x) = \sqrt{\frac{4+x}{x}}.$$  \hfill (95)

A first difference with respect to the one-mass case is that the term $1/(1+x)$ is replaced by the term $1/(4+x)$, as a consequence of the threshold in $p^2 = 4m^2$ instead of in $p^2 = m^2$. Another less trivial difference is that semi-integer coefficients appear in the homogeneous differential equation, leading to square roots in the solution $\phi(x)$.
6 Harmonic Polylogarithms

Let us write explicitly the formal solution of the set of differential equations considered in the previous section:

\[ B_{-1}(x) = \omega(x) \int_0^x dx' K(x') \Omega_{-1}(x'), \]
\[ B_0(x) = \omega(x) \int_0^x dx' A_1(x') \int_0^{x'} dx'' K(x'') \Omega_{-1}(x'') + \omega(x) \int_0^x dx' K(x') \Omega_0(x'), \]
\[ B_1(x) = \omega(x) \int_0^x dx' A_1(x') \int_0^{x'} dx'' A_1(x'') \int_0^{x''} dx''' K(x''') \Omega_{-1}(x''') \]
\[ + \omega(x) \int_0^x dx' A_1(x') \int_0^{x'} dx'' K(x'') \Omega_0(x'') + \omega(x) \int_0^x dx' K(x') \Omega_1(x'), \]
\[ \cdots \cdots \cdots \quad (96) \]

As is clearly seen from the above expressions, the solutions of the differential equations involve repeated integrations of products of the kernel \( K(x) \) and of coefficients of the differential equation itself: \( A_1(x), \Omega_{-1}(x), \Omega_0(x), \) etc. Natural representations of the solutions seem therefore repeated integrations of \( K(x) \) and of the elementary functions entering the differential equation under study. The idea behind the Harmonic Polylogarithms (HPLs) is simply that of giving a name to such repeated integrations \cite{4}. For the one-mass bubble, for instance, it is natural to define:

\[ g(-1; x) = \frac{1}{1 + x}, \]
\[ g(0; x) = \frac{1}{x}, \]
\[ g(1; x) = \frac{1}{1 - x}. \quad (97) \]

The harmonic polylogarithms of weight one are defined as integrals of the above functions:

\[ H(-1; x) = \int_0^x \frac{dx'}{1 + x'} = \log(1 + x), \]
\[ H(0; x) = \int_0^x \frac{dx'}{x'} = \log(x), \]
\[ H(1; x) = \int_0^x \frac{dx'}{1 - x'} = -\log(1 - x). \quad (98) \]

Note the slight asymmetry in the lower limit of integration of \( H(0; x) \) related to non-integrable singularity of \( 1/x \) in \( x = 0 \). Harmonic polylogarithms of higher weight \( w \) have the following integral recursive definition:

\[ H(a, \vec{w}; x) = \int_0^x g(a; x') H(\vec{w}; x') dx' \quad (99) \]

for \((a, \vec{w}) \neq (0, \vec{0}_w)\) and

\[ H(\vec{0}_w; x) = \frac{1}{w!} \log^w(x). \quad (100) \]

The index \( a \) takes the values 0, \( \pm 1 \) and \( \vec{w} \) is a string of \( w \) indices, each one taking the values 0, \( \pm 1 \). The vector \( \vec{0}_w \) is a string of 0 zeroes.

\[ ^{7}\text{The function } 1/(1 - x) \text{ is introduced for the closure under the transformation } x \to -x. \]
Using the above basis, the bubble with one mass reads:

\[
\begin{align*}
B_{-1} &= 1, \\
B_0 &= 2 - \left(1 + \frac{1}{x}\right) H(-1; x), \\
B_1 &= 4 - \left(1 + \frac{1}{x}\right) \left[2H(-1; x) + H(0, -1; x) - 2H(-1, -1; x)\right], \\
\cdots \cdots \cdots 
\end{align*}
\]  

Higher order terms in the \(\epsilon\)-expansion involve, as expected, harmonic polylogarithms of higher weight.

As far as the bubble with two masses is concerned, the above function set is not sufficient\(^8\). We add to the basis the functions \[5\]:

\[
\begin{align*}
f(-4; x) &= \frac{1}{4 + x}, \\
f(4; x) &= \frac{1}{4 - x}, \\
f(-r; x) &= \frac{1}{\sqrt{x(4 + x)}}, \\
f(r; x) &= \frac{1}{\sqrt{x(4 - x)}}.
\end{align*}
\]

The related harmonic polylogarithms of weight \(w \geq 1\) are defined analogously to the standard ones. By using this extended special function set, the two-mass bubble reads:

\[
\begin{align*}
B_{-1} &= 1, \\
B_0 &= 2 - \sqrt{\frac{x + 4}{4}} H(-r; x), \\
B_1 &= 4 - \sqrt{\frac{x + 4}{4}} \left[2H(-r; x) - H(-4, -r; x)\right], \\
\cdots \cdots \cdots 
\end{align*}
\]

The lesson is that loop diagrams are represented by complicated, special functions because they involve repeated integrations of simple basic functions. In other words, the complexity of the results originates solely from the repeated integrations.

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