A Note on Large Time Behavior of Velocity in the Barotropic Compressible Navier–Stokes Equations

Fei Jiang\textsuperscript{a,b,*}

\textsuperscript{a}College of Mathematics and Computer Science, Fuzhou University, Fuzhou, 361000, China.
\textsuperscript{b}Institute of Applied Physics and Computational Mathematics, Beijing, 100088, China.

Abstract

Recently, for periodic initial data with initial density allowed to vanish, Huang and Li\textsuperscript{1} establish the global existence of strong and weak solutions for the two-dimensional compressible Navier–Stokes equations with no restrictions on the size of initial data provided the bulk viscosity coefficient is $\lambda = \rho^\beta$ with $\beta > 4/3$. Moreover, the large-time behavior of the strong and weak solutions are also obtained, in which the velocity gradient strongly converges to zero in $L^2$ norm. In this note, we further point out that the velocity strongly converges to an equilibrium velocity in $H^1$ norm, in which the equilibrium velocity is uniquely determined by the initial data. Our result can also be regarded a correction for the result of large-time behavior of velocity in \textsuperscript{2}.

Keywords: Navier–Stokes equations, strong solution, weak solution, large time behavior.

2000 MSC: 35Q35, 76D03

1. Introduction

In this note, we are concerned with the two-dimensional barotropic compressible Navier–Stokes equations which read as follows:

\begin{align}
\partial_t \rho + \text{div}(\rho \mathbf{v}) &= 0, \quad (1.1) \\
\partial_t (\rho \mathbf{v}) + \text{div}(\rho \mathbf{v} \otimes \mathbf{v}) + \nabla P(\rho) &= \mu \Delta \mathbf{v} + \nabla((\lambda + \mu)\text{div}\mathbf{v}), \quad (1.2)
\end{align}

where $\rho$ and $\mathbf{v}$ represent the density and velocity respectively, and the pressure $P$ is given by

$$P(\rho) = a \rho^\gamma, \quad \gamma > 1.$$ 

Here $a = e^S > 0$ is the constant determined by the entropy constant $S$, and $\gamma \geq 1$ the adiabatic constant. Values of $\gamma$ have their own physical significance, and are also take important part in the existence of solutions (see \textsuperscript{3–6} for example). The viscosity coefficients satisfy the following hypothesis:

$$\mu = \text{constant}, \quad \lambda(\rho) = b \rho^\beta, \quad b > 0, \quad \beta > 0.$$ 

As in \textsuperscript{1}, we consider the Cauchy problem with the given initial density $\rho_0$ and the given initial momentum $\mathbf{m}_0$, which are periodic with period 1 in each space direction $x_i, i = 1, 2$, i.e., functions defined on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. We require that

$$\rho(x, 0) = \rho_0(x), \quad \rho \mathbf{v}(x, 0) = \mathbf{m}_0(x), \quad x = (x_1, x_2) \in \mathbb{T}^2.$$ 

*Corresponding author: Tel +86 15001201710.

Email address: jiangfei0591@163.com (Fei Jiang)
There is a huge literature concerning the theory of strong and weak solutions for the system of the multidimensional compressible Navier–Stokes with constant viscosity coefficients. The local existence and uniqueness of classical solutions are known in [7, 8] in the absence of vacuum and recently, for strong solutions also, in [9–11] for the case that the initial density need not be positive and may vanish in open sets. The global classical solutions were first obtained by Matsumura and Nishida [12] for initial data close to a non-vacuum equilibrium in some Sobolev space $H^s$. Later, Hoff [13] studied the problem for discontinuous initial data. For the existence of solutions for large data, the major breakthrough is due to Lions [14] (see also Feireisl [15, 16]), where he obtained global existence of weak solutions, defined as solutions with finite energy, when the exponent $\gamma$ is suitably large. The main restriction on initial data is that the initial energy is finite, so that the density is allowed to vanish initially. Recently, Huang, Li and Xin [17] established the global existence and uniqueness of classical solutions to the Cauchy problem for the isentropic compressible Navier–Stokes equations in the three-dimensional space with smooth initial data which are of small energy but possibly large oscillations; in particular, the initial density is allowed to vanish, even has compact support.

However, there are few results regarding global strong solvability for equations of multidimensional motions of viscous gas with no restrictions on the size of initial data. One of the first ever ones is due to Vaigant–Kazhikhov [18] who obtained a remarkable result which can be stated that the two-dimensional system (1.1)–(1.2) admits a unique global strong solution for large initial data away from vacuum provided $\beta > 3$. Lately, Perepelitsa [2] proved the global existence of a weak solution with uniform lower and upper bounds on the density, as well as the decay of the solution to an equilibrium state in a special case that

$$\beta > 3, \; \gamma = \beta,$$

when the initial density is away from vacuum. Very recently, Jiu, Wang and Xin [19] consider classical solutions and removed the condition that the initial density should be away from vacuum in Vaigant-Kazhikhov [18] but still under the same condition that $\beta > 3$ as that in [18]. No long after, Huang and Li establish the global existence of strong and weak solutions provided $\beta > 4/3$ and $\gamma > 1$.

Before stating the exciting result of Huang and Li, we explain the notations and conventions used throughout this paper. We denote

$$\int f \, dx = \int_{\mathbb{T}^2} f \, dx, \quad \bar{f} = \frac{1}{|\mathbb{T}^2|} \int f \, dx.$$

For $1 \leq r \leq \infty$, we also denote the standard Lebesgue and Sobolev spaces as follows:

$$L^r = L^r(\mathbb{T}^2), \; W^{s,r} = W^{s,r}(\mathbb{T}^2), \; H^s = W^{s,2}.$$

Then, we state the Huang and Li’s result concerning the global existence and large-time behavior of strong solutions as follows:

**Theorem 1.1.** Assume that

$$\beta > 4/3, \; \gamma > 1, \tag{1.3}$$

and that the initial data $(\rho_0, m_0)$ satisfy that for some $q > 2$,

$$0 \leq \rho_0 \in W^{1,q}, \; \bar{\rho}_0 > 0, \; v_0 \in H^1, \; m_0 = \rho_0 v_0.$$
Then the problem (1.1)–(1.2) has a unique global strong solution \((\rho, v)\) satisfying

\[
\begin{align*}
\rho &\in C([0, T], W^{1,1}), \quad \rho_t \in L^\infty(0, T; L^2), \\
v &\in L^\infty(0, T; H^1) \cap L^{(q+1)/q}(0, T; W^{2,q}), \\
t^{1/2}v &\in L^2(0, T; W^{2,q}), \quad t^{1/2}v_t \in L^2(0, T; H^1), \\
\rho v &\in C([0, T], L^2), \quad \sqrt{\rho} v_t \in L^2(\mathbb{T}^2 \times (0, T)),
\end{align*}
\]

for any \(0 < T < \infty\). Moreover, if

\[
\beta > 3/2, \quad 1 < \gamma < 3(\beta - 1),
\]

there exists a constant \(C\) independent of \(T\) such that

\[
\begin{align*}
\sup_{0 \leq t \leq T} \|\rho(\cdot, t)\|_{L^\infty} &\leq C, \\
\sup_{0 \leq t \leq T} \|v(\cdot, t)\|_{H^1} &\leq C,
\end{align*}
\]

and the following large-time behavior holds:

\[
\lim_{t \to \infty} (\|\rho - \bar{\rho}_0\|_{L^p} + \|\nabla v\|_{L^2}) = 0,
\]

for any \(p \in [1, \infty)\).

**Remark 1.1.** The results above can be found in [1, Theorem 1.1], except for the estimate (1.6). Fortunately we can obtain (1.6) by [1, Proposition 3.5].

The result (1.7) above indicates that the density \(\rho(t)\) strongly converges to the equilibrium density \(\bar{\rho}_0\) in \(L^p\) norm as \(t \to \infty\). We naturally propose an interesting question of whether there exists an equilibrium velocity \(v_s\) such that the velocity \(v(t)\) strongly converges \(v_s\) in some norm as \(t \to \infty\). In this note, we give the positive result. Next, we state our result, which will be proved in Section 2.

**Theorem 1.2.** Assume that the strong solution \((\rho, v)\) is provided by Theorem 1.1. If \((\rho, v)\) satisfies (1.6) and (1.7), then

\[
\lim_{t \to \infty} \|v - v_s\|_{H^1} = 0,
\]

where

\[
\frac{1}{\bar{\rho}_0 |T^2|} \int \rho_0 v_0 \, dx \text{ is a constant vector.}
\]
2. Proof of Theorem 1.2

In this section, we start to prove Theorem 1.2. First, exploiting (1.6) and the fact that $H^1 \hookrightarrow L^2$ is compact, we have that for any sequence $\{t_n\}_{n=1}^{\infty} \subset (0, \infty)$, there exists a subsequence $\{t_{n_m}\}_{m=1}^{\infty} \subset \{t_n\}_{n=1}^{\infty}$, such that

\[
\begin{align*}
v(t_{n_m}) &\rightharpoonup v_M \text{ weakly in } H^1, \\
v(t_{n_m}) &\rightarrow v_M \text{ strongly in } L^2, \\
n_m &\rightarrow \infty \text{ as } m \rightarrow \infty.
\end{align*}
\]

Thanks to the condition (1.7), we see that $\lim_{t \rightarrow \infty} \|\nabla v(t)\| = 0$, so $v_M$ must be a constant vector.

Next we shall show that the constant vector $v_M$ does not depend on the particular choice of subsequences. To this end, integrating the equation (1.2), we can deduce the momentum conservation

\[
\int \rho(t)v(t)dx = \int \rho_0 v_0 dx.
\]

Letting $t := t_m \rightarrow \infty$ in (2.3), and using (1.7) and (2.2), then we obtain

\[
\int \bar{\rho}_0 v_M dx = \lim_{t_m \rightarrow \infty} \int \rho(t_m)v(t_m)dx = \int \rho_0 v_0 dx,
\]

which yields

\[
v_M \equiv v_s := \frac{\int \rho_0 v_0 dx}{\bar{\rho}_0|T^2|}.
\]

Consequently, making use of the convergence of velocity in (1.7) and (2.2), we can conclude that for any sequence $\{t_n\}_{n=1}^{\infty} \subset (0, \infty)$, there exists a subsequence $\{t_{n_m}\}_{m=1}^{\infty} \subset \{t_n\}_{n=1}^{\infty}$, such that

\[
v(t_{n_m}) \rightarrow v_s \text{ strongly in } H^1 \text{ as } m \rightarrow \infty.
\]

Hence (1.8) holds, since the sequence $\{t_n\}_{n=1}^{\infty} \subset (0, \infty)$ is arbitrary. This completes the proof of Theorem 1.2.

References

[1] X. Huang, J. Li, Existence and Blowup Behavior of Global Strong Solutions to the Two-Dimensional Baratropic Compressible Navier-Stokes System with Vacuum and Large Initial Data, http://arxiv.org/abs/1205.5342v1 (2012).

[2] M. Perepelitsa, On the global existence of weak solutions for the Navier-Stokes equations of compressible fluid flows, SIAM. J. Math. Anal. 38 (2006) 1126–1153.

[3] S. Chandrasekhar, An Introduction to the Study of Stellar Structures, University of Chicago Press, 1938.

[4] W. W. Wang, F. Jiang, Z. S. Gao, Sequential stability of weak solutions in compressible self-gravitating fluids and stationary problem, Math. Meth. Appl. Sci. 35 (2012) 1014–1032.

[5] F. Jiang, Z. Tan, On radially symmetric solutions of the compressible isentropic self-gravitating fluid, Nonlinear Anal: TMA 72 (2010) 3463–3483.

[6] R. Guo, F. Jiang, J. Yin, A note on complete bounded trajectories and attractors for compressible self-gravitating fluid, Nonlinear Anal:TMA 75 (2012) 1933–1944.
[7] J. Nash, Le problème de Cauchy pour les équations différentielles d’un fluide général, Bull. Soc. Math. France. 90 (1962) 487–497.

[8] J. Serrin, On the uniqueness of compressible fluid motion, Arch. Rational. Mech. Anal. 3 (1959) 271–288.

[9] Y. Cho, H. J. Choe, H. Kim, Unique solvability of the initial boundary value problems for compressible viscous fluids, J. Math. Pures Appl. 83 (2004) 243–275.

[10] H. J. Choe, H. Kim, Strong solutions of the navier-stokes equations for isentropic compressible fluids, J. Differ. Eqs. 190 (2003) 504–523.

[11] I. Salvi, R.; Straškraba, Global existence for viscous compressible fluids and their behavior as $t \to \infty$, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 40 (1993) 17–51.

[12] A. Matsumura, T. Nishida, The initial value problem for the equation of motion of viscous and heat-conductive gases, J. Math. Kyoto. Univ. 20 (1980) 67–104.

[13] D. Hoff, Global existence of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data, J. Diff. Eqs. 120 (1995) 215–254.

[14] P. Lions, Mathematical Topics in Fluid Mechanics: Compressible models, Oxford University Press, USA, 1998.

[15] E. Feireisl, Dynamics of Viscous Compressible Fluids, Oxford University Press, 2003.

[16] E. Feireisl, A. Novotný, H. Petzeltová, On the existence of globally defined weak solutions to the Navier-Stokes equations, Journal of Mathematical Fluid Mechanics 3 (2001) 358–392.

[17] X. Huang, J. Li, Z. P. Xin, Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier-Stokes equations, Comm. Pure Appl. Math. 65 (2012) 549–585.

[18] V. A. Vaigant, K. A. V., On existence of global solutions to the twodimensional Navier-Stokes equations for a compressible viscous fluid, Sib. Math. J. 36 (1995) 1283–1316.

[19] Q. Jiu, Y. Wang, Z. P. Xin, Global well-posedness of 2D compressible Navier-Stokes equations with large data and vacuum, http://arxiv.org/abs/1202.1382 (2012).