Infrared finite semi-inclusive cross section in two dimensional type 0B string theory

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ABSTRACT: D-instanton induced S-matrix in type 0B string theory in two dimensions suffers from infrared divergences. This can be traced to the fact that these processes produce low energy rolling tachyon states that cannot be regarded as linear combination of finite number of closed string states. We compute semi-inclusive cross sections in this theory where we allow in the final state a fixed set of closed strings carrying given energies and any number of other closed string states carrying the rest of the energy. The result is infrared finite and agrees with the results in the dual matrix model, described by non-relativistic fermions moving in an inverted harmonic oscillator potential. In the matrix model the role of ‘any number of other closed string states’ is played by a fermion hole pair on opposite sides of the potential barrier.

KEYWORDS: D-Branes, String Field Theory

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1 Introduction and summary

Non-critical string theories in two dimensions provide toy models for critical string theory where many of the computational tools in string theory can be tested [1–4]. Of these, type 0B string theory provides a non-perturbatively consistent string theory [5–7]. Its dual matrix model description is a theory of free fermions moving in an inverted harmonic oscillator potential, with energy levels filled up to a fermi level that lies below the maximum of the potential. The closed string states represent excitations of the fermi sea involving low energy fermion hole pairs, with the parity even excitations describing NS sector states and parity odd excitations describing R-sector states. On the other hand, single fermions of the matrix model represent rolling tachyon configurations [8, 9] on unstable D-branes of the theory [5, 6, 10–12].

In this theory, perturbative amplitudes for external closed strings involve reflection of the fermion hole pair excitations near the fermi sea from the potential barrier. The transmission through the barrier are non-perturbative in the string coupling, and represent D-instanton effects. The effect of a single D-instanton is to transmit a single fermion or a single hole across the barrier. The corresponding final state cannot be interpreted as a conventional closed string since it involves a pair of fermion and hole on opposite sides of the potential barrier. In the string theory computation of closed string S-matrix, this is reflected in the fact that single D-instanton (or anti-D-instanton) mediated processes are infrared divergent [13]. In fact these divergences exponentiate and make the amplitude vanish. On the other hand, a D-instanton – anti-D-instanton induced process involves either the transmission of a fermion and a hole across the barrier, or a non-perturbative contribution to the reflection amplitude of a fermion or a hole. In either case, the final state represents fermion hole pair excitations on the same side of the barrier, and therefore can be interpreted as a regular closed string state, On the string theory side this is reflected in the fact that the amplitudes induced by a D-instanton anti-D-instanton pair are infrared
finite, Explicit computation of these amplitudes yield results in perfect agreement with the predictions of the matrix model [14, 15].

In this paper we analyze the single (anti-) D-instanton induced amplitudes in more detail. In particular we show that even though the infrared divergences in the exponent make the S-matrix of a fixed set of external closed string states vanish, we have infrared finite semi-inclusive cross section where we sum over final states containing a fixed number of closed strings with given energies and arbitrary number of other closed string states carrying the rest of the energy. This can then be compared to the matrix model result for a similar semi-inclusive cross section. However in the matrix model computation, we can replace the ‘arbitrary number of other closed string states’ in the final state by an additional fermion hole pair on opposite sides of the potential barrier, since single (anti-) D-instanton induced processes produce such fermion hole pair. The agreement between the two computations suggests that single fermion or hole excitations in the matrix model, representing rolling tachyon configuration on the unstable D0-brane of type 0B string theory, can be regarded as a collection of infinite number of closed strings, as expected from the rules of bosonization.

For convenience of the reader, we shall now describe our main result. Instead of working with NSNS and RR sector states, we work with right and left sector states, given respectively by the sum and difference of the NSNS and RR sector states. In the matrix model language these are represented by fermion hole pair excitations on the right and the left of the potential barrier. We start with a single incoming right sector closed string carrying energy \( \omega_1 \) and compute the semi-inclusive cross section for producing a final state containing \( r \) right sector closed string states of energies in the range \((e_1, e_1 + \Delta e_1), \ldots, (e_r, e_r + \Delta e_r)\), \( l \) left sector closed string states of energies in the range \((e'_1, e'_1 + \Delta e'_1), \ldots, (e'_l, e'_l + \Delta e'_l)\) and any number of other closed string states. If we denote a final state satisfying these requirements by \( |n \rangle \) and denote by \( M_1(\omega_1, n) \) the transition amplitude for this process, then for infinitesimal \( \Delta e_i, \Delta e'_i \) we have

\[
\sum_n' M_1(\omega_1, n)M_1(\omega_2, n)^* = \left\{ \prod_{i=1}^r \frac{\Delta e_i}{e_i} \right\} \left\{ \prod_{i=1}^l \frac{\Delta e'_i}{e'_i} \right\} \delta (\omega_1 - \omega_2) \frac{1}{\pi} \sinh \left( 2\pi \left( \omega_1 - \sum_{i=1}^r e_i - \sum_{i=1}^l e'_i \right) \right) \times \cosh \left( 2\pi \left( \omega_1 + \sum_{i=1}^l e'_i - \sum_{i=1}^r e_i \right) \right),
\]

where the ′ on the sum on the left hand side is a reminder that we sum over only a restricted set of final states. Some salient features of this formula are as follows:

1. (1.1) represents the contribution to the semi-inclusive cross section due to a single D-instanton or a single anti-D-instanton. As long as \( l \geq 1 \), i.e. the final state contains at least one left sector closed string, this is the dominant contribution to the cross section. However for \( l = 0 \) there is also a perturbative contribution, not shown here, that dominates the result.
2. Even though the final formula (1.1) is free from infrared divergence, the intermediate steps of the calculation in string theory suffer from infrared divergences. We regulate the infrared divergences by putting a lower cut-off on the spatial momentum. The matrix model side of the calculation is free from infrared divergence at all steps. The difference can be traced to the fact that in the matrix model we use the free fermion-hole basis for the part of the final state representing ‘any number of other closed string states’. This allows us to include in the final state fermion and hole states on opposite sides of the potential barrier. In contrast, a finite number of closed string states in string theory describes only fermion hole pairs on the same side of the potential.

3. In string theory, free fermion and hole states are represented by rolling tachyon solution on unstable D0-brane [5, 6, 10–12]. The agreement between the string theory and matrix model results for the semi-inclusive cross section suggests that we should be able to represent these rolling tachyon configurations as infinite collection of closed strings.

4. Single (anti-) D-instanton contribution to the total cross section, where we sum over all final states, is given by setting \( l = r = 0 \) in (1.1), and yields a finite answer. This was already computed in [14].

5. Naively one might expect that if we integrate (1.1) over the final state energies \( e_i \) and \( e'_i \) and divide the result by the symmetry factor \( l!r! \), we shall get part of the total cross section that has at least \( l \) left sector closed string states and \( r \) right sector closed string states carrying any energy. However this is infrared divergent from the \( e_i \approx 0 \) and / or \( e'_i \approx 0 \) region and would contradict the finiteness of the total cross section. The resolution of this puzzle is provided by the fact that if the final state had \( p \) additional left sector closed strings and \( q \) additional right sector closed strings, then the computation of the total cross section should include a factor of \( 1/\{l!(q+r)!\} \). However (1.1) only includes a factor of \( 1/\{p!q!\} \). Therefore simple integration of (1.1) and division by \( l!r! \) will overestimate the actual result by a factor of \( \{p+1\}\{q+r\} \). For \( p \to \infty \) or \( q \to \infty \), this is an infinite factor.

The rest of the paper is organized as follows. In section 2 we review recent results of [14, 15] on D-instanton corrections to two dimensional type 0B string theory amplitudes. In section 3 we study the unitarity of the D-instanton anti-D-instanton induced amplitude. This requires studying the single (anti-) D-instanton induced amplitudes and regulating the infrared divergences in these amplitudes. This analysis was done earlier in [14] using dimensional regularization. We use a lower cut-off on the spatial momentum to regulate the infrared divergences and also formulate the problem using the language of string field theory that makes the validity of Cutkosky rules and unitarity manifest. This also allows us to generalize the analysis to compute semi-inclusive cross sections. This is carried out explicitly in section 4, yielding the result (1.1). In section 5 we compute the same semi-inclusive cross section in the matrix model and show that the result agrees with the string theory results, even though the sum over states that we use in the matrix model looks different from the
sum over states that we perform in string theory. We end in section 6 by speculating on possible application to quantum electrodynamics in four dimensions.

2 Review

The world-sheet theory of non-critical type 0B string theory in two dimensions has a scalar describing the time direction, its world-sheet superpartner Majorana fermion, super-Liouville theory with central charge \(27/2\) and the usual \(b,c,\beta,\gamma\) ghost system. Physical closed string states in this theory are two scalars \(\phi_{NS}\) and \(\phi_R\) from the NSNS sector and the RR sector respectively. We shall denote their vertex operators by \(V_{NS}\), \(V_R\) and work in the \(\alpha' = 2\) unit as in [14].

This theory is expected to be dual to a matrix model. The simplest description of this model is provided by the theory of non-interacting, non-relativistic fermions, each moving under an inverted harmonic oscillator potential, and the energy levels are filled up to a fermi level that is a height \(\mu\) below the maximum of the potential [5–7]. \(\mu\) is inversely proportional to the string coupling \(\alpha_s\). The asymptotic closed string states are incoming and outgoing fermion hole pairs on the right of the potential and the left of the potential. These can be identified to the fields \(\chi_R\) and \(\chi_L\) given by sum and difference of \(\phi_{NS}\) and \(\phi_R\) [5–7]. We shall call them right and left sector closed string states respectively and, following [14], normalize them such that their vertex operators are given by \(W_R = V_{NS} + V_R\) and \(W_L = V_{NS} - V_R\) respectively. The parity symmetry of the inverted harmonic oscillator potential translates to the \((-1)^F\) symmetry of the type 0B string theory under which the RR sector states change sign.

Let us consider the 2-point amplitude where the incoming and outgoing states are right sector closed strings of energy \(\omega_1\) and \(\omega_2\) respectively. Instanton – anti-instanton contribution to this amplitude is given by [14]:

\[
\mathcal{M}_2(\omega_1, \omega_2) = e^{-2S_D} \exp \left[ \int_0^\infty \frac{dt}{2t} \left\{ -2 + 2e^{2\pi i\frac{1}{2} \left( \frac{1}{2} - \frac{1}{2} \left( \frac{\Delta x}{\pi} \right)^2 \right)} \right\} \right] \\
(\pi P_1 + i\omega_2 x_1 - e^{-\pi P_1 + i\omega_2} x_2) (\pi P_2 - i\omega_2 x_2 - e^{-\pi P_2 - i\omega_2} x_1), \quad \omega^F \equiv -i\omega,
\]

where \(x_1, x_2\) are the positions of the D-instanton along the Euclidean time direction and \(\Delta x = x_1 - x_2\). Here \(P_i\) denotes the Liouville momentum carried by the \(i\)-th particle and the on-shell condition is \(\omega_i = P_i\). \(S_D\) is the action of a single D-instanton, so that the \(e^{-2S_D}\) term can be regarded as the result of summing over arbitrary number of disk partition function with either D-instanton or anti-D-instanton boundary condition. The last factor of the first line represents the exponential of the annulus partition function, with the first term inside the curly bracket representing the contribution from the annulus with both boundaries on the instanton or both boundaries on the anti-instanton and the second term inside the curly bracket representing the contribution from the annulus with one boundary lying on the instanton and the other boundary lying on the anti-instanton. The first factor in the second line is the contribution of the disk one point function of \(W_L\) associated with the incoming state, with the two terms representing the contribution from the disks with
instanton and anti-instanton boundary conditions respectively. Similarly the second factor in the second line is the contribution of the disk one point function of $W_L$ associated with the outgoing state, with the two terms representing the contribution from the disks with anti-instanton and instanton boundary conditions respectively.

For fixed $x_1, x_2$, the integral over $t$ has no divergence in the $t \to 0$ limit since the term inside the curly bracket vanishes in this limit. The divergence at large $t$ are associated with the open string channel and can be resolved using open string field theory. This gives a finite result \[15\]:

$$\exp\left[\int_0^\infty dt \frac{-2 + e^{2\pi t}}{2t}\right] = \frac{1}{4\pi^2} \int dx_1 dx_2 \frac{1}{(\Delta x)^2 - 4\pi^2}, \quad (2.2)$$

with the understanding that the integration over the zero modes $x_1, x_2$ should be done at the end, after including the contribution from the disk amplitudes given in the last line of (2.1). Furthermore, unitarity demands that we use the principal value prescription for dealing with the singularities at $\Delta x = \pm 2\pi$ \[16\]. Substituting (2.2) into (2.1) we get \[14\]

$$M_2(\omega_1, \omega_2) = e^{-2S_D} \frac{1}{4\pi^2} \int dx_1 dx_2 \frac{1}{(x_1 - x_2)^2 - 4\pi^2} \left( e^{\pi P_1 + i\omega E x_1} - e^{-\pi P_1 + i\omega E x_2} \right) \left( e^{\pi P_2 - i\omega E x_2} - e^{-\pi P_2 - i\omega E x_1} \right). \quad (2.3)$$

We can perform the integration over $x_1, x_2$ by changing variables to the center of mass coordinate $x_1 + x_2$ and the relative coordinate $\Delta x = x_1 - x_2$. The integration over $\Delta x$ may be done by closing the contour at infinity in the upper / lower half plane. While doing this we need to keep in mind that the analytic continuation from Euclidean energy $\omega^E$ to the Lorentzian energy $\omega = i\omega^E$ has to be done via the first or third quadrant of the complex $\omega$ plane. Therefore positive $\omega_1, \omega_2$ requires us to start with positive $\omega_1^E, \omega_2^E$. After doing the integration over $\Delta x$, we can analytically continue the energies to Lorentzian values $\omega_1, \omega_2$. During this analytic continuation we also rotate the contour of integration over $x_1 + x_2$ so as to keep the combination $\omega x$ fixed. At the end we are left with the integration over the center of mass coordinate along the real time axis, and this integral produces a factor of $2\pi i \delta(\omega_1 - \omega_2)$. The $i$ arises from having to express the integration over the center of mass location along the imaginary time axis in terms of integration along the real time axis. The final result is \[14\]

$$M_2(\omega_1, \omega_2) = -e^{-2S_D} \frac{1}{2\pi} \delta(\omega_1 - \omega_2) \cosh(2\pi \omega_1) \sinh(2\pi \omega_1). \quad (2.4)$$

At this order we also have contribution from the two instanton processes and two anti-instanton processes, but these vanish due to infrared divergence in the closed string channel ($t \to 0$ limit in the integral in the exponent). For this reason we shall not consider these contributions. Similarly the contribution to the closed string S-matrix due to single D-instanton or single anti-D-instanton also vanish due to infrared divergence in the exponent

\[Note that the computation in \[15\] was done in the $\alpha' = 1$ unit. The result quoted in (2.2) is the translation of that result to the $\alpha' = 2$ unit.\]
of the normalization constant. Therefore the instanton – anti-instanton contribution is apparently the leading instanton contribution to the closed string S-matrix, and we write,

$$S(\omega_1, \omega_2) = \omega_1 \delta(\omega_1 - \omega_2) + \mathcal{M}_2(\omega_1, \omega_2).$$  \hspace{1cm} (2.5)

Note that we have ignored the perturbative contribution to the S-matrix since they will not play any role in our analysis. The $\omega_1$ in the definition of the identity matrix on the right hand side indicates that the sum over states is performed with the integration measure $d\omega/\omega$. We shall see in section 4 that this is the correct choice for the normalization of the states that we have chosen.

We now compute a particular contribution to $S^\dagger S$:

$$\int \frac{d\omega}{\omega} S(\omega_2, \omega)^* S(\omega_1, \omega) = \int \frac{d\omega}{\omega} \delta(\omega_1 - \omega) \left[ \omega_1 - e^{-2S_D} \frac{1}{2\pi} \cosh(2\pi\omega_1) \sinh(2\pi\omega_1) \right] \delta(\omega_2 - \omega) \left[ \omega_2 - e^{-2S_D} \frac{1}{2\pi} \cosh(2\pi\omega) \sinh(2\pi\omega) \right]
$$

$$= \delta(\omega_1 - \omega_2) \left[ \omega_1 - e^{-2S_D} \frac{1}{\pi} \cosh(2\pi\omega_1) \sinh(2\pi\omega_1) + O(e^{-4S_D}) \right].$$  \hspace{1cm} (2.6)

Since this is not $\omega_1 \delta(\omega_1 - \omega_2)$, the S-matrix is apparently non-unitary. The perturbative contribution is unitary by itself and cannot help cancel this term. The interference term between $\mathcal{M}_2$ and the perturbative S-matrix has additional powers of string coupling and cannot contribute to this order. Therefore there must be additional contribution that has not been accounted for. The natural candidate is the contribution from single instanton or single anti-instanton. Even though we have argued that they vanish due to infrared divergences, let us tentatively denote by $\mathcal{M}_1(\omega, n)$ the single instanton (and single anti-instanton) contribution to the S-matrix for transition from a closed string state of energy $\omega$ to an arbitrary state $n$. Then unitarity demands that

$$\sum_n \mathcal{M}_1(\omega_2, n)^* \mathcal{M}_1(\omega_1, n) = \delta(\omega_1 - \omega_2) e^{-2S_D} \frac{1}{\pi} \cosh(2\pi\omega_1) \sinh(2\pi\omega_1).$$  \hspace{1cm} (2.7)

Since we have seen that the D-instanton or anti-D-instanton induced contribution to the closed string scattering amplitude vanishes, this poses an apparent conflict with unitarity [13, 17]. The resolution to the puzzle is simplest in the matrix model. There closed strings are represented by fermion-hole pair created on the same side of the potential, but the (anti-) D-instanton induced processes create a fermion hole pair on opposite sides of the potential [13]. Therefore, without including these in the final state we should not expect to get a unitary S-matrix. In string theory these are represented by low energy rolling tachyon configurations on unstable D0-branes [5, 6, 10, 11]. This suggests that in the sum over $n$ on the left hand side of (2.7) we must include these states besides the closed string states in order to restore unitarity.

This however is not the end of the story. With the unitary prescription for integrating over $\Delta x$, which in this case corresponds to using principal value prescription for integrating across the singularity at $\Delta x = \pm 2\pi$, the S-matrix in the closed string sector was shown to be unitary [16]. This will be in apparent conflict with the vanishing of $\mathcal{M}_1$ in the closed string sector. However, we can resolve this by considering the contribution from the D-instanton or anti-D-instanton induced processes.
sector. We shall see in section 3 that (2.7) holds for the closed string S-matrix if we include in the set $n$ the states with infinite number of low energy closed strings, without needing to sum over rolling tachyon states. This was already checked in [14] using a dimensional regularization scheme to regulate the infrared divergence of $M_1$. We shall use a lower cut-off on the Liouville momentum to regulate the infrared divergences and formulate the analysis in the language of string field theory which makes the proof of unitarity manifest by relating it to Cutkosky rules. This will also make the necessity of the principal value prescription for integration over $\Delta x$ clear. The result of this analysis can be interpreted as the statement that the rolling tachyon state can be regarded as a state made of infinite number of closed strings. The right hand side of (2.7) now gives the single instanton contribution to the total cross section for a single closed string of energy $\omega_1$ to scatter to any set of closed strings. The situation is very similar to what happens in quantum electrodynamics. There the probability of producing a set of charged states and a finite number of photons during a scattering process vanishes due to infrared divergences. However the inclusive cross section where we sum over all final states is non zero and is consistent with unitarity [18, 19].

3 Feynman diagram representation and Cutkosky rules

In order to understand how the Cutkosky rules lead to (2.7), we shall first formulate the computation of $M_2$ given in section 2 as a sum of Feynman diagrams of string field theory. Once this is done, the cuts of these Feynman diagrams, that keep the D-instanton induced vertex and the anti-D-instanton induced vertex on two sides of the cut, will give the contributions to the left hand side of (2.7). On the other hand, the sum over cuts where the D-instanton and the anti-D-instanton induced vertices are on the same side of the cut, will give $M_2 + M_2^\ast$. Since Cutkosky rules tell us that the sum over all the cuts of a diagram vanish, and since a general proof of this in a class of non-local theories that include (effective) string field theory was given in [20], we are led to (2.7). We shall also explicitly sum over the cut diagrams to verify (2.7). This will set up the framework for computing semi-inclusive cross section.

Explicit check of (2.7) was carried out in [14] where the authors use dimensional regularization scheme. Here we shall regularize the infrared divergences in the closed string channel by putting a sharp lower cut-off on the Liouville momentum. As explained above, the language of string field theory that we shall use will make unitarity manifest following the analysis of [16].

We express $M_2(\omega_1, \omega_2)$ given in (2.1) as,

$$M_2(\omega_1, \omega_2) = e^{-2S_D} \exp \left[ \int_\epsilon^\infty \frac{dt}{2t} \left\{ 2 + 2e^{-\pi t} \right\} - \int_{\epsilon}^\infty \frac{dt}{t} e^{-\pi t} \right.$$

$$+ \int_{0}^{\infty} \frac{dt}{t} \left\{ 2\pi t \left( \frac{1}{2} - \frac{1}{4}(\frac{\Delta x}{2\epsilon})^2 \right) - \Theta(\epsilon - t) \right\} \left. \right\}$$

$$(e^{\pi P_1 + i\omega_1} x_1 - e^{-\pi P_1 + i\omega_1} x_2)(e^{\pi P_2 - i\omega_2} x_2 - e^{-\pi P_2 - i\omega_2} x_1),$$

(3.1)
for some small positive number $\epsilon$ and an arbitrary positive number $h$. $\Theta$ is the Heaviside step function. Note that the $t$ integral in the second line diverges for $(\Delta x)^2 \leq 4\pi^2$. We need to define the integral for $(\Delta x)^2 > 4\pi^2$ and then analytically continue the result to $(\Delta x)^2 < 4\pi^2$, averaging over the contributions where $\Delta x$ goes around the singularity at $\pm 2\pi$ above and below the singularity in the complex plane. As explained in [16], this corresponds to a particular choice of integration contour in the path integral over open string fields, since $x_1$ and $x_2$ are modes of the open string. String field theory a priori does not fix the choice of contour, but a different choice of integration contour will lead to non-unitary amplitudes. Indeed, even an otherwise good quantum field theory can be made bad if in the path integral over fields we decide to integrate along a wrong choice of contour in the complex field space.

It follows from the analysis of [15] that up to corrections of order $\epsilon$, the contribution from the first integral in the exponent of (3.1) is given by replacing $\left(\Delta x\right)^2 - 1$ by $h$ in (2.2):

$$\exp \left[ \int_{\epsilon}^{\infty} \frac{dt}{2t} \left\{ -2 + 2e^{-\pi t} \right\} \right] = \frac{1}{16\pi^2 h} \int dx_1 dx_2 + \mathcal{O}(\epsilon),$$

with the understanding that the integrations over $x_1, x_2$ are to be performed after including the rest of the contribution. The contribution from the second integral in the exponent of (3.1) is given by,

$$\exp \left[ -\int_{\epsilon}^{\infty} \frac{dt}{t} e^{-\pi t} \right] = \exp \left[ \gamma_E + \ln(\pi h) + \mathcal{O}(\epsilon) \right] = \pi \epsilon h e^{\gamma_E} \left( 1 + \mathcal{O}(\epsilon) \right),$$

where $\gamma_E$ is the Euler constant. In the last integral in the exponent of (3.1) we change variable to $s = 1/(2t)$ and write this as

$$\exp \left[ \int_{0}^{\infty} \frac{dt}{t} \left\{ e^{2\pi i \left( \frac{1}{2} - \frac{1}{2} \left( \Delta x \right)^2 \right) - \Theta(\epsilon - t)} \right\} \right]$$

$$= \exp \left[ \int_{0}^{\infty} \frac{ds}{s} \left\{ e^{\pi s^{-1} \left( \frac{1}{2} - \frac{1}{2} \left( \Delta x \right)^2 \right) - \Theta \left( s - \frac{1}{2\epsilon} \right)} \right\} \right]$$

$$= \exp \left[ 4 \int_{0}^{\infty} ds \int d^2 k_E e^{-2\pi s k_E^2} \left\{ e^{-i\omega E \Delta x (\cosh^2(\pi P) + \sinh^2(\pi P)) - \Theta \left( s - \frac{1}{2\epsilon} \right)} \right\} \right],$$

where $k_E = (\omega^E, P)$ with $-\infty < \omega^E < \infty$, $0 \leq P < \infty$. It is easy to see that after doing the integral over $k_E$, we reproduce the expression in the second line of (3.4). Physically $\omega^E$ represents Euclidean energy and $P$ represents Liouville momentum. We can now exchange the order of integration and do the $s$ integral to write,

$$\exp \left[ \int_{0}^{\infty} \frac{dt}{t} \left\{ e^{2\pi i \left( \frac{1}{2} - \frac{1}{2} \left( \Delta x \right)^2 \right) - \Theta(\epsilon - t)} \right\} \right]$$

$$= \exp \left[ 2 \pi \int \frac{d^2 k_E}{k_E^2} \left\{ e^{-i\omega E \Delta x (\cosh^2(\pi P) + \sinh^2(\pi P)) - e^{-\pi k_E^2/\epsilon}} \right\} \right].$$

Note that if in (3.5) we try to carry out the integral over $P$ first, the integral diverges for large $P$ due to the presence of the $(\cosh^2(\pi P) + \sinh^2(\pi P))$ term in the integrand. Since
the original expression that we started with was finite for \((\Delta x)^2 > 4\pi^2\), this divergence can be attributed to the exchange of the order of integration over \(s\) and \(k_E\). This problem can be avoided if we follow the prescription that the integration over \(\omega^E\) has to be done before the integration over \(P\). In that case we can easily check, via closing the contour at infinity in the complex \(\omega^E\) plane and picking up residues at \(\omega^E = \pm iP\) for negative / positive \(\Delta x\), that the result of \(\omega^E\) integration produces a factor proportional to \(e^{-|\Delta x|P}\). The \(P\) integral now converges for \(|\Delta x| > 2\pi\) and produces the original result. Therefore from now on it will be understood that the \(\omega^E\) integration needs to be done before integration over \(P\).

We can now substitute (3.2), (3.3) and (3.5) into (3.1) to write

\[
M_2(\omega_1, \omega_2) = e^{-2S_0} \frac{\epsilon}{16\pi^3} e^{\gamma E} \int dx_1 dx_2
\]

\[
\exp\left[\frac{2}{\pi} \int \frac{d^2k_E}{k_E^2} e^{-\pi k_E^2/\epsilon} e^{-\omega^E(x_1-x_2)} \left\{ \cosh^2(\pi P) + \sinh^2(\pi P) \right\} \right]
\]

\[
\exp\left[\frac{-1}{\pi} \int \frac{d^2k_E}{k_E^2} e^{-\pi k_E^2/\epsilon} \right] \exp\left[\frac{-1}{\pi} \int \frac{d^2k_E}{k_E^2} e^{-\pi k_E^2/\epsilon} \right]
\]

\[
\exp\left[\frac{2}{\pi} \int \frac{d^2k_E}{k_E^2} \left(1 - e^{-\pi k_E^2/\epsilon} \right) e^{-\omega^E(x_1-x_2)} \left\{ \cosh^2(\pi P) + \sinh^2(\pi P) \right\} \right]
\]

\[
(e^{\pi P_1+i\omega^E_1 x_1} - e^{-\pi P_1+i\omega^E_1 x_2})(e^{\pi P_2-i\omega^E_2 x_2} - e^{-\pi P_2-i\omega^E_2 x_1}).
\]

The proof of equivalence of (3.1) and (3.6) holds for real \(\omega^E_1, \omega^E_2\), i.e. imaginary \(\omega_1, \omega_2\). In this case the integration contour over the momenta \(k = (\omega, P) = (i\omega^E, P)\) are taken to be along real \(\omega^E\), i.e. imaginary \(\omega\) axis; \(P\) is always kept real and positive. To compute \(M_2(\omega_1, \omega_2)\) for real \(\omega_1, \omega_2\), we need to deform the external energies from the imaginary axis to the real axis via the first quadrant of the complex \(\omega\) plane. During this deformation the poles of the propagators may approach the integration contours and we need to deform the integration contours over the internal energies \(\omega = i\omega^E\) to avoid the poles, keeping the end-points fixed at \(\pm i\infty\) so as to make use of the \(e^{-\pi k^2/\epsilon} = e^{-\pi k_E^2/\epsilon}\) factor to make the integral converge at large momentum [20]. Using the relation \(d\omega^E = -id\omega\), we can replace \(d^2k_E/k_E^2\) by \(-id^2k/(k^2 - i\epsilon)\) where the \(i\epsilon\) in the denominator essentially encodes the contour deformation prescription described above [20].

Note that (3.6) is free from infrared divergences in the closed string channel, i.e. free from divergences from the \(k_E \approx 0\) region. Therefore if we put a lower limit \(\eta\) on the integration range of the Liouville momentum \(P\), then the result is finite in the \(\eta \to 0\) limit. But now we can split the integral into sum of terms each of which could diverge in the \(\eta \to 0\) limit, manipulate them appropriately and then combine the results before taking the \(\eta \to 0\) limit. With the understanding that we have a lower cut-off \(\eta\) on \(P\), we rewrite (3.6) as,

\[
M_2(\omega_1, \omega_2) = e^{-2S_0} \frac{\epsilon}{16\pi^3} e^{\gamma E} \int dx_1 dx_2
\]

\[
\exp\left[\frac{2}{\pi} \int \frac{d^2k_E}{k_E^2} e^{-\pi k_E^2/\epsilon} e^{-i\omega^E(x_1-x_2)} \left\{ \cosh^2(\pi P) + \sinh^2(\pi P) \right\} \right]
\]

\[
\exp\left[\frac{-1}{\pi} \int \frac{d^2k_E}{k_E^2} e^{-\pi k_E^2/\epsilon} \right] \exp\left[\frac{-1}{\pi} \int \frac{d^2k_E}{k_E^2} e^{-\pi k_E^2/\epsilon} \right]
\]

\[
\exp\left[\frac{2}{\pi} \int \frac{d^2k_E}{k_E^2} \left(1 - e^{-\pi k_E^2/\epsilon} \right) e^{-i\omega^E(x_1-x_2)} \left\{ \cosh^2(\pi P) + \sinh^2(\pi P) \right\} \right]
\]

\[
(e^{\pi P_1+i\omega^E_1 x_1} - e^{-\pi P_1+i\omega^E_1 x_2})(e^{\pi P_2-i\omega^E_2 x_2} - e^{-\pi P_2-i\omega^E_2 x_1}).
\]
We can give this an interpretation in terms of Feynman diagrams by introducing a set of D-instanton induced vertices in the effective closed string field theory [16]. In writing down the expressions for these vertices, we shall use both the Euclidean momenta $k_E = (\omega^E, P)$ and the Lorentzian momenta $k = (\omega, P)$ with the understanding that $\omega = i\omega^E$.

1. Single D-instanton induced $n$-point vertex $\bullet$ with external closed strings of momenta $k_1 = (\omega_1, P_1), \cdots, k_n = (\omega_n, P_n)$:

$$\int dx_1 \cdots dx_n \prod_{i=1}^{n} e^{-\pi k_i^2/(2\epsilon)} \left( \frac{\epsilon}{16\pi^3 e^{\gamma_E}} \right) \prod_{i=1}^{n} e^{-\pi k_i^2/(2\epsilon)} \left( \frac{\epsilon}{16\pi^3 e^{\gamma_E}} \right) \left( \sigma_i \cosh \pi P_i \right) \left( \frac{\sigma_i \cosh \pi P_i}{\sinh \pi P_i} \right),$$

where $\cosh(\pi P)$ refers to RR-sector states, $\sinh(\pi P)$ refers to NSNS sector states and $\sigma_i$ takes value +1 if the $i$-th state is incoming and −1 if the $i$-th state is outgoing.

2. Single anti-D-instanton induced $n$-point vertex $\circ$ with external closed strings of momenta $k_1 = (\omega_1, P_1), \cdots, k_n = (\omega_n, P_n)$:

$$\int dx_1 \cdots dx_n \prod_{i=1}^{n} e^{-\pi k_i^2/(2\epsilon)} \left( \frac{\epsilon}{16\pi^3 e^{\gamma_E}} \right) \prod_{i=1}^{n} e^{-\pi k_i^2/(2\epsilon)} \left( \frac{\epsilon}{16\pi^3 e^{\gamma_E}} \right) \left( \sigma_i \cosh \pi P_i \right) \left( \frac{\sigma_i \cosh \pi P_i}{\sinh \pi P_i} \right).$$

3. D-instanton – anti-D-instanton induced composite $n$-point vertex $\boxdot$ with external closed strings of momenta $k_1 = (\omega_1, P_1), \cdots, k_n = (\omega_n, P_n)$:

$$\int dx_1 \cdots dx_n \prod_{i=1}^{n} e^{-\pi k_i^2/(2\epsilon)} \left( \sigma_i \cosh \pi P_i \right) \left( \frac{\epsilon}{16\pi^3 e^{\gamma_E}} \right) \prod_{i=1}^{n} e^{-\pi k_i^2/(2\epsilon)} \left( \frac{\epsilon}{16\pi^3 e^{\gamma_E}} \right) \left( \sigma_i \cosh \pi P_i \right) \left( \frac{\sigma_i \cosh \pi P_i}{\sinh \pi P_i} \right).$$

To go from the first expression to the second expression, we change variables to $x = (x_1 + x_2)/2$, $\Delta x = (x_1 - x_2)$ and then do the $x$ integral by changing variables to $x = iy$, $\omega_E = -i\omega_i$. This gives a factor of $2\pi i \delta \left( \sum_{i} \sigma_i \omega_i \right)$. Due to the $(1 - e^{-\pi k_E^2/\epsilon})$ factor in the integrand that vanishes at $k_E^2 = 0$, this vertex has no singularity from $k_E^2 = 0$ even when the external momenta are Lorentzian, i.e. when the $\omega_i$’s are real. This justifies declaring this as a single composite vertex. The apparent ultraviolet divergence of the $k_E$ integral in the last line can be avoided for $|\Delta x| > 2\pi$ by doing the $\omega^E$ integration before the $P$ integration. The result will have a singularity at $\Delta x = \pm 2\pi$ and the integrand has been continued to $|\Delta x| < 2\pi$ via analytic continuation. If we want the interaction vertex (3.10) to correspond to a real term in the effective action, then we need to use the ‘unitary prescription’ for integrating over $\Delta x [16]$, which in this case translates to the principal value prescription.
We now claim that $\mathcal{M}_{2}(\omega_{1}, \omega_{2})$ given in (3.7) can be regarded as a sum of contributions from the Feynman diagrams shown in figure 1. Since the external incoming and outgoing states are NS+R sector scalars, for the D-instanton vertex given in (3.8) they couple via terms proportional to $\cosh(\pi P_{1}) + \sinh(\pi P_{1}) = e^{\pi P_{1}}$ for the incoming state and $-\cosh(\pi P_{2}) + \sinh(\pi P_{2}) = -e^{-\pi P_{2}}$ for the outgoing state. On the other hand, for the anti-D-instanton vertex given in (3.9), they couple via terms proportional to $-\cosh(\pi P_{1}) + \sinh(\pi P_{1}) = -e^{-\pi P_{1}}$ for the incoming state and $\cosh(\pi P_{2}) + \sinh(\pi P_{2}) = e^{\pi P_{2}}$ for the outgoing state. Therefore the contribution from figure 1(a) is given by:

$$e^{-2S_{D}} \epsilon \frac{\epsilon^{\gamma_{E}}}{16 \pi^{3}} \prod_{i=1}^{n} \frac{d\omega_{i} \, d\tilde{P}_{i}}{2\pi} \frac{8\pi i}{k^{2} - i\epsilon} \frac{8\pi i}{\omega^{2} - P^{2} + i\epsilon} e^{-\pi \omega_{i}^{2}/\epsilon} 2\pi i \delta \left( \omega_{1} - \sum_{i=1}^{n} \omega_{i} \right) 2\pi i \delta \left( \omega_{2} - \sum_{i=1}^{n} \omega_{i} \right) \left\{ \cosh^{2}(\pi \tilde{P}_{1}) + \sinh^{2}(\pi \tilde{P}_{1}) \right\} e^{\pi P_{1} + \pi P_{2}}$$

$$\left( \prod_{i=1}^{m} \left( -\frac{1}{2} \right) \frac{d\tilde{\omega}_{i} \, d\tilde{P}_{i}}{2\pi} \frac{8\pi i}{k_{i}^{2} + i\epsilon} e^{-\pi k_{i}^{2}/\epsilon} \right) \left( \prod_{i=1}^{p} \left( \frac{1}{2} \right) \frac{d\tilde{\omega}_{i} \, d\tilde{P}_{i}}{2\pi} \frac{8\pi i}{k_{i}^{2} + i\epsilon} e^{-\pi k_{i}^{2}/\epsilon} \right).$$

Note the factors of $1/2$ in the integrands in the last line – these are the correct combinatoric factors associated with the propagators with both ends on the same vertex. The minus sign...
comes from the vertex factors, since an internal line of momentum \((\omega, P)\) joining the same
vertex generates \(- \cosh^2(\pi P) + \sinh^2(\pi P) = -1\).

(3.13) can be identified to the following contribution from (3.7):

1. Pick the term
   \[ e^{\pi P_1 + \pi P_2} e^{i\omega_1 E x_1 - i\omega_2 E x_2} \]
   (3.14)
   from the last line of (3.7).

2. Pick \(n\) factors of
   \[ \frac{2}{\pi} \int \frac{d^2 k_E}{k_E^2} e^{-\pi k_E^2/\epsilon} \left\{ e^{-i\omega E (x_1 - x_2)} (\cosh^2(\pi P) + \sinh^2(\pi P)) \right\} \]
   (3.15)
   from the expansion of the exponential in the second line of (3.7).

3. Pick \(m\) factors of
   \[ -\frac{1}{\pi} \int \frac{d^2 k_E}{k_E^2} e^{-\pi k_E^2/\epsilon} \]
   (3.16)
   from the expansion of the first term in the third line of (3.7).

4. Pick \(p\) factors of
   \[ -\frac{1}{\pi} \int \frac{d^2 k_E}{k_E^2} e^{-\pi k_E^2/\epsilon} \]
   (3.17)
   from the expansion of the second term in the third line of (3.7).

5. Pick 1 in the expansion of the exponential in the fourth line of (3.7).

The integrations over \(x_1\) and \(x_2\) in (3.7) generate the two energy conserving delta functions
in (3.13). For this we need to rotate the \(\omega_i\)'s from the imaginary axis to the real axis via
the first quadrant and rotate the \(x_1, x_2\) integration contours in the opposite direction so
as to keep \(\omega x\) fixed. This produces the factors of \(i\) multiplying the delta functions. Also
\(\int d^2 k_E / k_E^2\) becomes \(d^2 k(-i/k^2)\) in Lorentzian variables.

The contributions from figure 1(b),(d) and (e) can be interpreted in the same way, except
that from the last line of (3.7) we pick respectively the terms proportional to
\(e^{-\pi(P_1 + P_2)}\), \(e^{\pi(P_1 - P_2)}\) and \(e^{\pi(P_2 - P_1)}\) instead of (3.14). Therefore the sum of these diagrams produces all
the terms in the expansion of (3.7), other than those obtained from the higher order terms
in the expansion of the exponential in the penultimate line of (3.7). The contribution from
figure 1(c) produces this contribution.\(^2\)

The Cutkosky rules tell us that the sum over all the cuts of the Feynman diagrams
of figure 1 vanishes. Of these the cuts that are fully to the left or fully to the right of

\(^2\)Note that the contribution from individual Feynman diagrams diverge in the \(\eta \to 0\) limit. However,
the finiteness of the original expression implies that the sum of all the Feynman diagrams with a fixed
number of total propagators is infrared finite. We shall implicitly follow this procedure even when some of
the propagators are cut, summing over all graphs with a fixed number of total propagators. The infrared
finite expression (3.26) that we get at the end for sum over cut diagrams should be regarded as a result of
organizing the sum over diagrams this way.
the diagrams, without cutting any internal propagators, produce the factor of $\mathcal{M}_2 + \mathcal{M}_2^\ast$. The other cuts are those of figure 1(a) and (b), cutting all the internal propagators that connect the D-instanton induced vertex to the anti-D-instanton induced vertex. These have been shown in figure 2(a) and (b) respectively. This gives $\mathcal{M}_1^\ast \mathcal{M}_1$, producing the unitarity relation $\mathcal{M}_1^\ast \mathcal{M}_1 = -(\mathcal{M}_2 + \mathcal{M}_2^\ast)$. Therefore as long as Cutkosky rules are applicable, (2.7) should hold. The original proof of Cutkosky rules using largest time equation [21, 22] and a different version given in perturbation theory [23] do not hold for non-local vertices, containing exponential in momenta, that we have in string field theory e.g. the $e^{-\pi k^2/2\epsilon}$ factors in (3.8)–(3.10). A perturbative proof that holds for these cases was given in [20]. Since the D-instanton induced vertices are of the same type, we can still make use of the proof given in [20]. Furthermore, the proof of Cutkosky rules given in [20] used manipulations of energy integration contour at fixed values of spatial components of the loop momenta. Therefore putting a lower cut-off on the Liouville momentum does not affect the proof of Cutkosky rules. The final ingredient in the proof was the reality of the action. The sum of D-instanton and anti-D-instanton induced vertices is manifestly real, but the reality of the composite vertex (3.10) requires that we use the unitary prescription for integrating over $\Delta x = x_1 - x_2$ [16]. Once this is done, the Cutkosky rules hold. Notwithstanding these general arguments, we shall now explicitly verify that the expression for $\mathcal{M}_1^\ast \mathcal{M}_1$ computed from the sum over cuts reproduces (2.7). Besides providing a check on the abstract arguments of [16, 20] for the validity of the cutting rules, this will be useful in computing the semi-inclusive cross section where in the sum over states on the left hand side of (2.7) we sum over only a subset of states.

The cut diagram shown in figure 2(a), obtained by replacing the $i/(-\hat{k}_i^2 + i\epsilon)$ by $2\pi\delta(\hat{k}_i^2)\Theta(\hat{\omega}_i)$, and complex conjugating the contribution from the right of the cut, is given by:

$$
e^{-2S_D} \frac{e^{-\xi}}{16\pi^3} \sum_{n=1}^{\infty} \sum_{m,p=0}^{\infty} \frac{1}{n!m!p!} \int \prod_{i=1}^{n} \left\{ 4d\hat{\omega}_i d\hat{P}_i \delta(\hat{k}_i^2) \Theta(\hat{\omega}_i) \left( \cosh^2(\pi \hat{P}_i) + \sinh^2(\pi \hat{P}_i) \right) \right\}$$

$$\int \prod_{i=1}^{m} \left\{ \left( -\frac{1}{\pi} \right) d\bar{\omega}_i d\bar{P}_i \frac{i}{-\bar{k}_i^2 + i\epsilon} e^{-\pi \bar{k}_i^2/\epsilon} \right\} \int \prod_{i=1}^{p} \left\{ \left( -\frac{1}{\pi} \right) d\bar{\omega}_i d\bar{P}_i \frac{i}{-\bar{k}_i^2 + i\epsilon} e^{-\pi \bar{k}_i^2/\epsilon} \right\}^*.$$  

Figure 2. Contributions to $\mathcal{M}_1^\ast \mathcal{M}_1$. The thin vertical lines represent cuts.
The sum over $m$ and $p$ easily exponentiates to
\[ \exp \left[ -\frac{2}{\pi} \int \frac{d^2k_E}{k_E^2} e^{-\pi k_E^2/\epsilon} \right]. \tag{3.19} \]

Defining $(u, v) = (P\sqrt{\pi/\epsilon}, \omega^E \sqrt{\pi/\epsilon})$ and noting that the limits of integration are $\eta \leq P < \infty$, $-\infty < \omega^E < \infty$, we can express (3.19) as
\[ \exp \left[ -\frac{2}{\pi} \int_{\eta\sqrt{\pi/\epsilon}}^{\infty} du \int_{-\infty}^{\infty} dv e^{-(u^2+v^2)} \right] = \exp \left[ \ln \frac{\pi\eta^2}{\epsilon} + \gamma_E + 2 \ln 2 + \mathcal{O}(\eta/\sqrt{\epsilon}) \right] = \frac{4\pi\eta^2}{\epsilon} e^{\gamma_E} \left( 1 + \mathcal{O}(\eta/\sqrt{\epsilon}) \right), \tag{3.20} \]
where we have organized the expansion so that we take the $\epsilon \to 0$ limit first before taking the $\eta \to 0$ limit.

The sum over $n$ in (3.18) takes the form:
\[ 2\pi\delta(\omega_1 - \omega_2)f(\omega_1), \]
\[ f(\omega) = \sum_{n=1}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{n} \left( \frac{2d\tilde{P}_i}{\tilde{P}_i} (\cosh^2(\pi \tilde{P}_i) + \sinh^2(\pi \tilde{P}_i)) \right) 2\pi\delta \left( \sum_{i=1}^{n} \tilde{P}_i - \omega \right). \tag{3.21} \]

To find $f(\omega)$ we first compute [14, 16]
\[ \int_{0}^{\infty} e^{-\nu \omega} f(\omega) \frac{d\omega}{2\pi} = \sum_{n=1}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{n} \left( \frac{2d\tilde{P}_i}{\tilde{P}_i} (\cosh^2(\pi \tilde{P}_i) + \sinh^2(\pi \tilde{P}_i)) e^{-\nu \tilde{P}_i} \right) = \exp \left[ \int \frac{2d\tilde{P}}{\tilde{P}} (\cosh^2(\pi \tilde{P}) + \sinh^2(\pi \tilde{P})) e^{-\nu \tilde{P}} \right] - 1. \tag{3.22} \]

Recalling that the lower limit of $\tilde{P}$ integration is $\eta$, we get,
\[ \int_{0}^{\infty} e^{-\nu \omega} f(\omega) \frac{d\omega}{2\pi} = \exp[-2\gamma_E - \ln\{\eta^2(\nu^2 - 4\pi^2)\} + \mathcal{O}(\eta)] - 1 = e^{-2\gamma_E} \frac{1}{\eta^2(\nu^2 - 4\pi^2)} + \mathcal{O}(1/\eta). \tag{3.23} \]

Demanding that this holds for all $\nu$, we get
\[ f(\omega) = \frac{e^{-2\gamma_E}}{\eta^2} \sinh(2\pi \omega) + \mathcal{O}(1/\eta). \tag{3.24} \]

Substituting (3.20) and (3.24) into (3.18) we get the following expression for figure 2(a):
\[ 2\pi\delta(\omega_1 - \omega_2) e^{-2SD} \frac{\epsilon}{16\pi^3} e^{\gamma_E} \frac{4\pi\eta^2}{\epsilon} e^{\gamma_E} e^{-2\gamma_E} \sinh(2\pi \omega_1) e^{2\pi \omega_1}, \tag{3.25} \]
where we have used $P_1 = \omega_1 = \omega_2 = P_2$. The contribution from figure 2(b) has a similar form except that we have $e^{-2\pi \omega_1}$ instead of $e^{2\pi \omega_1}$. Taking the sum of the two cut diagrams we get:
\[ \sum_n M_1(\omega_1, n) M_1(\omega_2, n)^* = \delta(\omega_1 - \omega_2) e^{-2SD} \frac{1}{\pi} \sinh(2\pi \omega_1) \cosh(2\pi \omega_1). \tag{3.26} \]

This agrees with (2.7).
4 Semi-inclusive cross section

Our goal in this section will be to find the D-instanton induced semi-inclusive cross section for a right sector closed string state of energy $\omega_1$ to go into a set of $r$ right sector closed string states of energies in the range $(e_1, e_1 + \Delta e_1), \ldots, (e_r, e_r + \Delta e_r)$, $l$ left sector closed string states of energies in the range $(e'_1, e'_1 + \Delta e'_1), \ldots, (e'_l, e'_l + \Delta e'_l)$ and any number of other closed string states. $\Delta e_i$ and $\Delta e'_i$ are taken to be infinitesimal. To calculate this, we reexamine the expression (3.18) of the cut diagram associated with figure 2(a). Recalling the couplings of NS and R sector states to the D-instanton and anti-D-instanton induced vertices given in (3.8) and (3.9) respectively, and recalling that the right and left sector closed string states are given respectively by the sum and difference of the NS and R sector states, one finds that a right sector state propagating from the left to the right of the cut in figure 2(a) will have coupling proportional to $e^{-2\pi \hat{P}_1}$ and a left sector state propagating from the left to the right of the cut in figure 2(a) will have coupling proportional to $e^{2\pi \hat{P}_1}$. Writing $\cosh^2(\pi \hat{P}_1) + \sinh^2(\pi \hat{P}_1)$ as $(e^{2\pi \hat{P}_1} + e^{-2\pi \hat{P}_1})/2$ in (3.18) we see that the $e^{2\pi \hat{P}_1}/2$ factor can be traced to the propagation of a left-sector closed string and the $e^{-2\pi \hat{P}_1}/2$ factor can be traced to the propagation of a right sector closed string. Therefore to compute the desired semi-inclusive cross section, we need to,

1. restrict the integration over $r$ of the $\hat{\omega}_i$’s in (3.18) to the range $(e_1, e_1 + \Delta e_1), \ldots, (e_r, e_r + \Delta e_r)$ and replace the $\cosh^2(\pi \hat{P}_1) + \sinh^2(\pi \hat{P}_1)$ factor for these momenta by $e^{-2\pi \hat{P}_1}/2 = e^{-2\pi \epsilon_i}/2$,

2. restrict the integration over $l$ of the other $\hat{\omega}_i$’s in (3.18) to the range $(e'_1, e'_1 + \Delta e'_1), \ldots, (e'_l, e'_l + \Delta e'_l)$ and replace the $\cosh^2(\pi \hat{P}_1) + \sinh^2(\pi \hat{P}_1)$ factor for these momenta by $e^{2\pi \hat{P}_1}/2 = e^{2\pi \epsilon'_i}/2$,

3. and let the rest of the integrals run over the full range.

Also since there are $n!/(n-r-l)!$ ways of choosing the $r+l$ variables among the $n$ integration variables $\hat{\omega}_i$ whose integration ranges are restricted to $(e_1, e_1 + \Delta e_1), \ldots, (e_r, e_r + \Delta e_r)$, $(e'_1, e'_1 + \Delta e'_1), \ldots, (e'_l, e'_l + \Delta e'_l)$, we get an extra multiplicative factor of $n!/(n-r-l)!$ that converts the $1/n!$ in (3.18) to $1/(n-r-l)!$. We can now perform the sum over $n$, $m$ and $p$ as before, and the result takes the form of (3.25) with $\omega_1$ replaced by $\omega_1 - \sum_i e_i - \sum_i e'_i$ in the argument of $\sinh(2\pi \omega_1)$, multiplied by a factor of $\Delta e_i e^{-2\pi \epsilon_i}/\epsilon_i$ for each final state right sector closed string and a factor of $\Delta e'_i e^{2\pi \epsilon'_i}/\epsilon'_i$ for each final state left sector closed string.4

This gives

\[
2\pi \delta(\omega_1 - \omega_2) e^{-2S_{D}} \frac{1}{4\pi^2} \sinh \left( 2\pi \left( \omega_1 - \sum_{i=1}^{r} e_i - \sum_{i=1}^{l} e'_i \right) \right) e^{2\pi \omega_1} \times \left\{ \prod_{i=1}^{r} \frac{\Delta e_i e^{-2\pi \epsilon_i}}{\epsilon_i} \right\} \left\{ \prod_{i=1}^{l} \frac{\Delta e'_i e^{2\pi \epsilon'_i}}{\epsilon'_i} \right\}. \tag{4.1}
\]

\[\text{Since the integration measure in the sum over states is } \text{d}e/\epsilon, \text{ this justifies the definition of the identity matrix as } \omega_1 \delta(\omega_1 - \omega_2) \text{ as in (2.5).}\]
Similarly, the contribution to this semi-inclusive cross section from the cut diagram of figure 2(b) is given by:

\[ 2\pi \delta(\omega_1 - \omega_2) e^{-2SD} \frac{1}{4\pi^2} \sinh \left( 2\pi \left( \omega_1 - \sum_{i=1}^{r} e_i - \sum_{i=1}^{l} e'_i \right) \right) e^{-2\pi \omega_1} \]

\[ \times \left\{ \prod_{i=1}^{r} \frac{\Delta e_i 2\pi e_i}{e_i} \right\} \left\{ \prod_{i=1}^{l} \frac{\Delta e'_i e^{-2\pi e'_i}}{e'_i} \right\}. \quad (4.2) \]

Adding these two contributions we get,

\[ \sum'_{n} M_1(\omega_1, n) M_1(\omega_2, n)^* = e^{-2SD} \left\{ \prod_{i=1}^{r} \frac{\Delta e_i}{e_i} \right\} \left\{ \prod_{i=1}^{l} \frac{\Delta e'_i}{e'_i} \right\} \delta(\omega_1 - \omega_2) \frac{1}{\pi} \sinh \left( 2\pi \left( \omega_1 - \sum_{i=1}^{r} e_i - \sum_{i=1}^{l} e'_i \right) \right) \]

\[ \times \cosh \left( 2\pi \left( \omega_1 + \sum_{i=1}^{l} e'_i - \sum_{i=1}^{r} e_i \right) \right), \quad (4.3) \]

where \( \sum' \) on the left hand side denotes sum over all final states that contain \( r \) right sector closed string states of energy in the range \((e_1, e_1 + \Delta e_1), \cdots, (e_r, e_r + \Delta e_r)\), \( l \) left sector closed string states of energy in the range \((e'_1, e'_1 + \Delta e'_1), \cdots, (e'_l, e'_l + \Delta e'_l)\) and any number of other closed string states, with the restriction \( \omega_1 > \sum_i e_i + \sum_i e'_i \).

5 Matrix model computation

We shall now see how to compute the semi-inclusive cross section in the matrix model. The computation in this case simplifies by noting that in the semi-inclusive cross section, the sum over ‘anything else’ can be taken in the fermionic basis, since the free fermions and holes form a complete basis of states.

We shall first illustrate this procedure by computing the contribution to the fully inclusive cross section induced by single instanton or single anti-instanton [13]. Since single instanton induces transmission of a fermion or a hole, and since the incoming closed string is a fermion hole pair, we can compute the inclusive cross section by summing over two final states: (1) the fermion is transmitted and the hole is reflected back and (2) the hole is transmitted and the fermion is reflected back. Let us denote by \( e' \) and \( e \) the energies of the transmitted and the reflected particle respectively in string units. Using the convention of [14] that the energy interval \( e \) in string theory corresponds to energy interval \( 2e \) in the matrix model, and that \( -\mu \) is the fermi energy of the matrix model, we see that a fermion carrying energy \( e \) in string units has energy \( -\mu + 2e \) in the matrix model and a hole carrying energy \( e \) in string units correspond to a hole at energy level \( -\mu - 2e \) in the matrix model. If we denote the \( T(x) \) and \( R(x) \) the reflection and transmission coefficient of a fermion carrying energy \( x \) in the matrix model, then the net contribution to the \( \sum_n M_1(\omega_1, n) M_1(\omega_2, n)^* \) is
given by:

\[
\int_0^\infty \frac{de}{2\pi} \int_0^\infty \frac{de'}{2\pi} 2\pi \delta(e + e' - \omega_1) 2\pi \delta(e + e' - \omega_2) \\
\times \left[ |T(-\mu + 2e')R(-\mu - 2e)|^2 + |T(-\mu - 2e')R(-\mu + 2e)|^2 \right].
\]  

(5.1)

The first term in the square bracket represents the contribution where the transmitted particle is a fermion, while the second term represents the contribution where the transmitted particle is a hole. In writing (5.1) we have used the result that the reflection and transmission coefficients of a hole are given by the complex conjugates of those of the fermion. To this order we can approximate \( T(x) \) and \( R(x) \) up to a phase by

\[
T(x) \simeq e^{\pi x}, \quad R(x) \simeq 1.
\]

(5.2)

Substituting this into (5.1) we get,

\[
\delta(\omega_1 - \omega_2) \frac{1}{\pi} e^{-2\pi \mu} \sinh(2\pi \omega_1) \cosh(2\pi \omega_1).
\]

(5.3)

This agrees with the string theory result (3.26) once we identify \( e^{-2S_D} \) with \( e^{-2\pi \mu} \). From now on ‘energy’ will always be understood as the energy measured in string units unless mentioned otherwise.

Next we consider the case of semi-inclusive cross section where the final state contains \( r \) right sector closed string states of energy in the range \((e_1, e_1 + \Delta e_1), \ldots, (e_r, e_r + \Delta e_r)\), \( l \) left sector closed string states of energy in the range \((e'_1, e'_1 + \Delta e'_1), \ldots, (e'_l, e'_l + \Delta e'_l)\) plus any other state. We can choose the basis of ‘any other states’ as fermion or hole states. We can compute this amplitude with the help of real time diagrams introduced in [17], except that here in the final state we allow free fermions and holes represented by open lines. A diagrammatic representation of the process under consideration is shown in figure 3. The basic process is that the initial closed string, containing the fermion hole pair, splits into a fermion and a hole, one on either side of the potential barrier, which then rearrange themselves to a set of closed strings and the final state fermion hole pair on the opposite sides of the barrier. The only non-trivial part of this diagram is the ‘interaction vertex’ \( P \), given by the product of the transmission coefficient of a fermion of energy \( e' + \sum_i e'_i \) and reflection coefficient of a hole of energy \( e + \sum_i e_i \) or vice versa. Leaving out the phase space factors \( \Delta e_i/e_i \) and \( \Delta e'_i/e'_i \), which have the same origin in string theory and the matrix model, we see from figure 3 that the contribution is given by an expression similar to (5.1)

---

5There is no energy dependent normalization in the phase space integration measure for non-relativistic fermions. To check the overall normalization in (5.1), we note that the leading identity matrix in \( S\dagger S \) comes from the term where both the fermion and the hole are reflected and we approximate the reflection coefficient \( R \) by 1. In this case the second line of (5.1) would be replaced by 1 and the integral in the first line gives \( \omega_1 \delta(\omega_1 - \omega_2) \) in agreement with (2.5).
Figure 3. Diagrammatic representation of a scattering process in which an incoming closed string state carrying energy $\omega_1$ splits into a set of right sector closed strings carrying energies $e_1, \ldots, e_r$, a set of left sector closed strings carrying energies $e'_1, \ldots, e'_l$ and a fermion hole pair, one on each side of the potential barrier. The time flows up, the thin lines denote the external closed strings and the thick lines denote fermions and holes. There are two distinct diagrams, one where the left sector state is a fermion and the right sector state is a hole and vice versa.

with $e$ replaced by $e + \sum_i e_i$ and $e'$ replaced by $e' + \sum_i e'_i$ in the integrand:

$$\int_0^\infty \frac{de}{2\pi} \int_0^\infty \frac{de'}{2\pi} 2\pi \delta \left( e + e' - \omega_1 + \sum_i e_i + \sum_i e'_i \right) 2\pi \delta \left( e + e' - \omega_2 + \sum_i e_i + \sum_i e'_i \right)$$

$$\times \left[ T \left( -\mu + 2e' + 2\sum_i e'_i \right) R \left( -\mu - 2e - 2\sum_i e_i \right) \right]^2$$

$$+ \left[ T \left( -\mu - 2e' - 2\sum_i e'_i \right) R \left( -\mu + 2e + 2\sum_i e_i \right) \right]^2.$$

(5.4)

Using (5.2) we can reduce this to,

$$\delta(\omega_1 - \omega_2) \frac{1}{\pi} e^{-2\pi \mu} \sinh \left( 2\pi \left( \omega_1 - \sum_i e_i - \sum_i e'_i \right) \right) \cosh \left( 2\pi \left( \omega_1 - \sum_i e_i + \sum_i e'_i \right) \right).$$

(5.5)

This is in perfect agreement with (4.3).

6 Discussion

Infrared divergences in the two dimensional type 0B string theory and those in four dimensional quantum electrodynamics share many common features. In both cases the infrared divergences make the usual S-matrix vanish, but the semi-inclusive cross section, where we allow in the final state arbitrary number of soft particles, is finite. In type 0B string theory the infrared divergence can be traced to the fact that the final state may be a
state containing fermion hole pair on opposite sides of the potential barrier, and this cannot be described as a collection of finite number of closed strings. Put another way, the final state may be in a different ‘charge sector’ compared to the initial state \cite{13, 14}. In quantum electrodynamics the vanishing of the S-matrix due to infrared divergence can be traced to the fact that the final state after the scattering is built on a different vacuum compared to the initial state \cite{24}. In type 0B string theory the infrared divergence in the S-matrix can be cured by allowing the final state to have a fermion hole pair on opposite sides of the potential barrier. In quantum electrodynamics the infrared divergences can be cured by using the Faddeev-Kulish states \cite{25–27}. This suggests that in quantum electrodynamics, the analog of the state containing fermion hole pair on opposite sides of the potential barrier may be related to the photon cloud in the Faddeev-Kulish states, together with a finite number of photons to balance energy and momentum. It will be interesting to explore this analogy in more detail.

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