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Global transmission dynamic of SIR model in the time of SARS-CoV-2

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\begin{abstract}
This current work studies a new mathematical model for SARS-CoV-2. We show how immigration, protection, death rate, exposure, cure rate and interaction of infected people with healthy people affect the population. Our model is SIR model, which has three classes including susceptible, infected and recovered respectively. Here, we find the basic reproduction number and local stability through jacobean matrix. Lyapunvo function theory is used to calculate the global stability for the problem under investigation. Also a nonstandard finite difference sachem (NSFDS) is used to simulate the results.
\end{abstract}

1. Introduction

It was reported in December 2019, that a new kind of virus named Corona has affected badly the Wuhan city in China. The concerned disease was later named COVID-19 and announced a pandemic throughout the globe by WHO (see [1,3–7]). Firstly, the said virus and its resultant outbreak hit the Wuhan city and later it affects almost the whole world. It took the lives of hundreds of thousands lives throughout the world. It is hard to find a single view about the origin of the origin of the said virus, for example, it is due to seafood market; migration of people from people one place to other places by transmission from animals to human or may be it is due to human to human interaction. Currently it has almost devastated everything around the world. Social life, health, economic, education almost every field of the life have been affected badly. Researchers of the field of health, policy makers of the countries and health field are puzzled how to tackle with this deadly outbreak. They all have their own point of views observing the situations. They are trying hard to at least minimize the number of deaths due to this outbreak. People infected with this pandemic experiences a mild respiratory problems. Fever, dry cough, throat infection, and tiredness are the symptoms of this disease. People may have these symptoms; nasal infection, aches, and sore throat.

Investigation of dynamics of real world phenomenon through mathematical models is well known area. Particularly, the transmission of diseases which are very important to understand. A number of mathematical models of infectious diseases have been analyzed in the field of mathematics we refer few as [8–11,14]. The area of modeling can recently be widened to non-integer order and nonlocal derivatives of fractional order see [12–15]. Mathematical models are used to understand the Dynamics of transmission of a specific diseases and it can be helpful and to present a palm to save people from lost. Following this technique and with the help of mentioned tools will help to make a strategy to have control on the spread of disease or to eliminate it thoroughly. The recently discovered virus and its resultant outbreak has been studied from different point of views from last few months (see [2,11,14,16,19–23]). Inspired from the above discussion, it is the migration which has played a major role in the spreading of the virus in the societies. It took a short period of almost there months that the said virus spread in almost every part of the world.

2. Derivation of the model

In this section of manuscript, we formulate our new model for (SARS-CoV-2) in the form (1). We split the whole population $N(t)$ into three classes Susceptible $S(t)$, Infected $I(t)$ and Recovered $R(t)$ class to form system (1) as

\begin{equation}
\begin{array}{c}
\frac{dS(t)}{dt} = -\beta S(t)I(t) + \gamma S(t)R(t) - \sigma S(t) \\
\frac{dI(t)}{dt} = \beta S(t)I(t) - (\gamma + \nu) I(t) \\
\frac{dR(t)}{dt} = \nu I(t) - \gamma R(t)
\end{array}
\end{equation}

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Table 1

Parameters and their description of the system.

| Notation | Description of Parameters |
|----------|---------------------------|
| $S(t)$  | Susceptible class         |
| $I(t)$  | Infected class            |
| $R(t)$  | Recovered class           |
| $\mu$   | Corona death rate         |
| $\delta$| Natural death             |
| $\beta$ | Birth rate                |
| $b$     | Contact rate              |
| $c$     | Immigrant to Infected     |
| $a$     | Immigrant to Susceptible  |
| $e$     | Recovery rate             |

The sum of all equations of model (1), we get

$$\frac{dS(t)}{dt} = \beta + a - \delta S(t) - bS(t)I(t)$$

$$\frac{dI(t)}{dt} = bS(t)I(t) + c - (\mu + \delta + e)I(t)$$

$$\frac{dR(t)}{dt} = eI(t) - \delta R(t).$$

Equation of the system (1) shows

From (2), we get

From above, we can see that if $N > N_0$, then $\frac{dN(t)}{dt} < 0$ yields

We consider the existence of equilibrium for the system (1). Disease free equilibrium exists for some values of the variables in (1) denoted by $\mathcal{E}_0 = (S^*, 0, 0)$.

Table 2

Parameters and their description of the system.

| Parameters | Physical Description | Numerical Value of the Parameter |
|------------|----------------------|---------------------------------|
| $S_0$      | Susceptible compartment | 222.002105 Millions [17]     |
| $I_0$      | Infected compartment | 0.537176 Millions [17]     |
| $R_0$      | Recovered compartment | 0.482 Millions [17]         |
| $\mu$      | Death rate due to corona | 0.02 [17]                        |
| $\delta$   | Natural death | 0.0062 [17]                        |
| $\beta$    | Birth rate | 0.6 (assumed)                        |
| $b$        | Contact rate | 0.0002 (assumed)                      |
| $c$        | Immigrant to Infected | [0.00001, 0.00091, 0.00097] (assumed) |
| $a$        | Immigrant to Susceptible | [0.00211, 0.0011, 0.00031] (assumed) |
| $e$        | Recovery rate | 0.003 (assumed)                        |

The reproductive number $\mathcal{R}_0$ provides the transmission dynamics information about the disease. From $\mathcal{R}_0$, we look how the disease id spread in population and we can control it from this. The method of finding $\mathcal{R}_0$ is below. Let $X(t) = (S,I)$, then form system (1), one has

$$\frac{dX(t)}{dt} = \gamma - \mathcal{F}.$$
\[ \mathcal{F}' = \left( \begin{array}{cc} \beta + a - \delta S(t) & 0 \\ c + (\mu + \delta + e)I(t) & 0 \end{array} \right). \]

We get

\[ FV^{-1} = \left( \begin{array}{cc} \frac{bS}{\delta} & 0 \\ 0 & 0 \end{array} \right). \]

We get the expression for \( \mathcal{R}_0 \) as

\[ \mathcal{R}_0 = \frac{b(\beta + a)}{\delta}. \] (4)

To compute the basic reproduction number we obtained \( \mathcal{R}_0 = 3.13 \) from the parameters used in Table 2. This show that the COVID-19 that occurred in Pakistan is not well controlled by govt of Pakistan. We have the following theorem on the basis of (4).

**Theorem 1.**
- [[i]] If \( \mathcal{R}_0 \leq 1 \), then there is no positive equilibrium of model (1);
- [[ii]] If \( \mathcal{R}_0 > 1 \), then there is a unique positive equilibrium \( \mathcal{E}^* = (S^*(t), I^*(t), R^*(t)) \) of the model (1) called the endemic equilibrium.

### 3. Local stability

We reduced our model (1) to obtained the result for local stability as

\[ \begin{align*}
\frac{dS(t)}{dt} & = \beta + a - \delta S(t) - bS(t)I(t) \\
\frac{dI(t)}{dt} & = bS(t)I(t) + c - (\mu + \delta + e)I(t).
\end{align*} \] (5)

Subject to initial condition

\[ S(0) = S_0 \geq 0, I(0) = I_0 \geq 0. \]

Here we established the coming result.

**Theorem 2.** At \( \mathcal{E}^0 \), the disease free equilibrium of the model (1) is locally asymptotically stable under the condition \( \mathcal{R}_0 < 1 \).

**Proof.** At \( \mathcal{E}^0 \), we find Jacobian Matrix as

\[ J^0 = \left( \begin{array}{cc} -\delta & -bS_0 \\ 0 & \mathcal{R}_0 - 1 \end{array} \right). \]

The auxiliary equation of \( J^0 \) is given by

\[ k_1 \lambda^2 + k_2 \lambda + k_3 = 0, \]

where

\[ \begin{align*}
k_1 & = (\delta + b)(\delta + c) + (\mu + \delta + e)(1 - \mathcal{R}_0) > 0 \\
k_2 & = (\delta + b)(\delta + c)(1 + (\mu + \delta + e)(1 - \mathcal{R}_0)) > 0 \\
k_3 & = (\delta + b)(\delta + c)(\mu + \delta + e)(1 - \mathcal{R}_0) > 0.
\end{align*} \]

We have

\[ \begin{align*}
k_{k_1} - k_1 & = (\delta + b)(\delta + c)((\mu + \delta + e)^2 + (\delta + b)(\delta + c)((\mu + \delta + e) + 1)(1 - \mathcal{R}_0) \\
& > 0.
\end{align*} \] (6)

(6) show that the “Routh-Hurwitz criteria is satisfied” as \( k_1 > 0, k_2 > 0, k_3 > 0 > 0 > 0 > 0 > 0 \), if \( \mathcal{R}_0 < 1 \), which show the system (1) is locally asymptotically stable at \( \mathcal{E}^0 \). \( \square \)

**Theorem 3.** Under the condition \( \mathcal{R}_0 > 1 \), at \( \mathcal{E}^* \) model (1) is locally asymptotically stable, otherwise unstable.

**Proof.** The Jacobian matrix of the model (5), at the endemic equilibrium \( \mathcal{E}^* \) is

\[ J_1 = \left( \begin{array}{ccc} -\delta - b\ell(t) & -bS(t) \\
\ell(t) & \delta S(t) - (\mu + \delta + e) \end{array} \right). \]

After some operations on matrix \( J_1 \), we get

\[ J_1 = \left( \begin{array}{ccc} -\delta - b\ell(t) & -bS(t) \\
\ell(t) & \delta \end{array} \right). \]

We calculated \( \det(J_1) \) and trace of \( J_1 \) is given below

\[ \text{tr}J_1 = - (b\ell(t) + 2\delta + \mu + e) < 0. \]

and

\[ \detJ_1 = (\delta + b\ell(t))(\mu + \delta + e) + 2b\delta S(t) > 0. \]

The determinant of \( J_1 \) show that the “endemic equilibrium” of the system (5) at \( \mathcal{E}^* \) has negative real part. Thus, with condition \( \mathcal{R}_0 > 1 \), one gets that the endemic equilibrium \( \mathcal{E}^* \) of system (5) is locally asymptotically stable. \( \square \)

### 4. Global stability

In this section of our work, we study global stability of model (1). For global stability, we constructed “Lyapunov function” for diseases free and endemic equilibria.

**Theorem 4.** The disease free equilibrium of the system (1) is globally asymptotically stable if \( \mathcal{R}_0 < 1 \).

**Proof.** To prove this required result, we construct a Lyapunov function as following:

\[ \Psi = a_1(S(t) - S_0) + a_2I(t). \] (7)

such that \( a_1, a_2 > 0 \) are constant may be computed later. With respect to time \( t \) taking derivative of (7) with, we have

\[ \frac{d\Psi}{dt} = a_1(\beta + a - \delta S(t) - bS(t)I(t)) + a_2(\delta S(t)I(t) + c - (\delta + \mu + e)I(t)). \]

We get

\[ \frac{d\Psi}{dt} = bS(t)I(t)(\alpha_2 - a_1) + a_1(\beta + a) - a_2\delta S(t) + a_2(c - (\delta + \mu + e)I(t)). \]

Let assume \( a_1 = a_2 = 1 \), we get finally

\[ \frac{d\Psi}{dt} = - (\delta N(t) - (\beta + a + c)) = (\mu + e)I(t) < 0. \]

Hence “globally asymptotically stable” for system (1) with \( \mathcal{R}_0 < 1 \) has reached. \( \square \)

**Theorem 5.** “The endemic equilibrium \( \mathcal{E}^* \) of model (1) is stable globally asymptotically if \( \mathcal{R}_0 > 1 \).”

**Proof.** On construction of Lyapunov function as

\[ \Theta = (\delta + b)(S(t) - S^*) + (\delta + b)I(t). \] (8)

Take the derivative (8) w.r.t \( t \) along with system (5),

\[ \frac{d\Theta}{dt} = (\delta + b)S(t) + (\delta + b)I(t). \]

Putting the values from (1) yields

\[ \frac{d\Theta}{dt} = (\delta + b)(\delta + a - \delta S(t) - bS(t)I(t)) + (\delta + b)(bS(t)I(t) + c - (\delta + \mu + e)I(t)). \]

After some rearrangement, one has

\[ \frac{d\Theta}{dt} = - (\delta + b)(\delta S(t) + (\delta + \mu + e)I(t)) < 0. \]
Thus $\frac{d\mathcal{R}_0}{dt} < 0$, “The endemic equilibrium” $\mathcal{E}^*(t)$ of the model (1) is “Globally asymptotically stable”, which show that $\mathcal{R}_0 > 1$. □

5. Discussion about the numerical analysis

In this part of our manuscript, we study simulation for model (1) by using the values of Table 2. From first January up to next 300 days, we simulate the considered model (1) for Pakistan a.

Here, using NSFD [16,18] to rewrite the system in difference equations form. Hence from the first equation of system (1) via NSFD one has

$$\frac{dS(t)}{dt} = \beta + a - bS(t)I(t) - \mu S(t). \quad (9)$$

Thank to NSFD, (9) yields

$$\frac{S_{j+1} - S_j}{h} = \beta + a - bS_jI_j(t) - \delta S_j(t). \quad (10)$$

Like (10), Using NSFD scheme, we decomposed the system (1) as follow

$$S_{j+1} = S_j + h(\beta + a - bS_jI_j(t) - \delta S_j(t))$$
$$I_{j+1} = I_j + h(bS_jI_j(t) + c - (\delta + \mu + e)I_j(t))$$
$$R_{j+1} = R_j + h(eI_j(t) - \delta R_j(t)). \quad (11)$$

By using NSFD scheme, developed in (11). Using the values from 2, we plotted the system (1).

Using real data, we testify our system (1) for the real data of Pakistan with used Table 2 for the parameters values for the 1 January 2021. Using the NSFD scheme, we simulate the model corresponding to different values in the Table 2. In Fig. 1–3, we investigate dynamic of transmission of SARS-CoV-2 in Pakistan at different values of $a, c$. During the 1st January 2021 in the present of immigration rate the susceptible population dynamic is given in Fig. 1. When the rate of immigration is high the decline in the host population of susceptible is decreasing. This yields that they are exposing to infection and hence higher the immigration rate faster the growth rate of infected population and vice versa. From the recovery SARS-CoV-2 more death will occurs. In the recovery compartment the growth is different under various immigrant rates. The concerned dynamical behavior is provided Fig. 2 and Fig. 3. The recovered population is also growing with faster speed when immigration rate is low. This is due to the people getting ride from infection. As the recovery rate is high in Pakistan as majority of population is consisted on young people between 20 and 45 years. Hence bringing change in certain parameters of the considered model, significant change can be observed in the dynamics of respective compartments. Further the adopted numerical scheme is good to use to simulate mathematical model.

6. Conclusion

To overcome the pandemic the migration should be decreased for the sake of saving humanity. Also the immigration of exposed population to infected community increase the infection. Isolation of infected one is the best option to secured the healthy community. It is necessary to judge the spread and model with various parameters for proper supervision. The proper treatment of this pandemic is to keep infected away from healthy people. High internal defense system aids to get healthy soon while the low internal defense system need more attention. This is model is just an indication to see the interaction of different compartments and its dynamics by vary the values of certain parameters. Since precautionary measures are not strict in well populated country like Pakistan which cause faster growth of infection in the second wave of COVID-19.

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Declaration of Competing Interest

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