Limit cycle theory of temporal current self-oscillations in sequential tunneling of superlattices

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A unified theory of the temporal current self-oscillations is presented. We establish these oscillations as the manifestations of limit cycles, around unstable steady-state solutions caused by the negative differential conductance. This theory implies that both the generation and the motion of an electric-field domain boundary are universal in the sense that they do not depend on the initial conditions. Under an extra weak ac bias with a frequency \( \omega_{ac} \), the frequency must be either \( \omega_{ac} \) or an integer fractional of \( \omega_{ac} \) if the tunneling current oscillates periodically in time, indicating the periodic doubling for this non-linear dynamical system.

Following the discovery of temporal current self-oscillations (TCSOs) in sequential tunneling of superlattices (SLs) under a dc bias [1–3], a large number of experimental and theoretical studies have focused on their origin and how these oscillations develop from steady-state solutions (SSSs). Experimentally, TCSOs have been observed in both doped and undoped SLs [1–3]. The oscillations can be induced by continuous illumination with laser light [3] or by a change in doping [1]. Recently, it has been shown that TCSOs can also be induced by applying an external magnetic field parallel to SL layers or varying the sample temperature [4].

Our current theoretical understanding of TCSOs is mainly from numerical studies [5]. Early works of Bonilla and his co-workers [5] established a correct model for TCSOs. Many numerical results and some analyses were also done. They simulated and reproduced many experimental results, including finding proper model parameters to simulate different experimental situations. Great progress in understanding of TCSOs had been made because of them. However, a simple physical picture did not appear in those early works. The understanding at the computational level is the first step, and deep insights can be only obtained when the general concepts and principles are found. There are also microscopic Green’s function calculations [6]. While the microscopic approach would be accurate if all the microscopic parameters and mechanisms were known, it remains a challenge to deduce the rules of macroscopic behavior from the microscopic details. Recently, a clear route to TCSOs developed from SSSs in sequential tunneling of SLs was proposed [7]: Due to the negative differential conductance (NDC) [8], a SSS is not stable. A limit cycle is generated around the unstable SSS because of the local repulsion and global attraction in the phase space. The system moves along the limit cycle, leading to a TCSO.

Unfortunately, more careful studies [9,10] show that the simple model used in reference 7 does not have a TCSO solution even though it gives the on-set of instability conditions of a SSS. Thus, the evidence for limit cycles in the TCSO regime is still lacking. In this letter we search for the fundamental concepts and general principles for TCSOs, based on the correct model of Bonilla and his co-workers [5]. We apply the concept of limit cycle, and explain TCSOs as the manifestations of limit cycles. Although numerical results are not our emphases in this paper, we show that they can be easily understood by using this concept. For example, the generation and motion of an EFD boundary do not depend on the initial conditions. An EFD-boundary does not necessarily start from the emitter, and end up in the collector. Furthermore, limit cycle concept gives us a power of prediction. We predict that the generation and motion of EFD boundaries are universal, and the frequency of a periodic motion under an ac bias must be either the ac bias frequency or its integer fractional. The limit cycle is a well-known concept in non linear physics. Thus, we find that TCSOs can be understood under the general concepts of non linear physics.

FIG. 1. Schematic illustration of an SL system. \( \mu_i \) is the local chemical potential of the \( i \)th well. \( \mu_L \) and \( \mu_R \) are the chemical potential of left-hand side and right-hand side electrodes, respectively. \( V_i \) is the bias on the \( i \)th barrier, and \( \mu_L - \mu_R = U \) is the external bias.

We consider a system consisting of \( N \) quantum wells as shown schematically in Fig. 1. An external bias \( U \) is applied between the two end wells. Current flows perpendicular to the SL layers. In the sequential tunneling, charge carriers are in local equilibrium within each well,
so that a chemical potential can be defined locally. The chemical potential difference between two adjacent wells is called bias \( V \) on the barrier between the two wells. A current \( I_i \) passes through the \( i^{th} \) barrier under a given bias \( V_i \). This current may depend on other parameters, such as doping \( N_D \). One of the results in reference 7 is that a SSS must be unstable if there are two or more barriers being in the NDC regime. It is worth pointing out that, although this instability result is obtained from an analysis of a simplified sequential tunneling model, it is generally true. Without losing generality, we assume that only barriers 1 and 2, which create well 2 shown in Fig. 1, are under a NDC regime for a SSS. Assume the chemical potential in well 2 increases a little bit due to a fluctuation. Then bias \( V_1 \) on barrier 1 decreases while \( V_2 \) increases. Because both of the barriers are under NDC regime, charge carriers flow more into well 2 through barrier 1 while less carriers flow out of it, leading to a further increase of the chemical potential in well 2. This drives the system away from the SSS, i.e. instability.

Following reference 5, the dynamics of the system is governed by the discrete Poisson equations

\[
k(\n_i - \n_{i-1}) = n_i - N_D, \quad i = 1, 2, \ldots N, \tag{1}
\]

and the current continuity equations

\[
J = k \frac{\partial V_i}{\partial t} + I_i, \quad i = 0, 1, 2, \ldots N \tag{2}
\]

where \( k \) depends on the SL structure and its dielectric constant. \( n_i \) is the electric charge in the \( i^{th} \) well. In Eq. (1), a same doping in all wells is assumed. \( I_i \) is, in general, a function of \( V_i \) and \( n_i \). It can be shown [9] that all SSSs are stable if \( I_i \) is a function of \( V_i \) only. On the other hand, a SSS may be unstable [5] if one chooses \( I_i = n_i v(V_i) \), where \( v \) is a phenomenological drifting velocity which is, for simplicity, assumed to be a function of \( V_i \) only. The constraint equation for \( V_i \) is

\[
\sum_{i=0}^{N} V_i = U. \tag{3}
\]

Previous studies [5] proved that this model is capable of describing TCSOs. To close the equations, a proper boundary condition is needed. It is proper to assume a constant \( n_0 \), \( n_0 = \delta N_D \) with \( \delta \) as a model parameter, if the carrier density in the emitter is much larger than those in wells, and its change due to a tiny tunneling current is negligible. However, it is mathematically equivalent to other boundary conditions used in literature [5]. Our goal is to show that the limit cycle is one of the most important features in this widely studied model.

According to our theory, NDC is essential for TCSOs. Thus, a TCSO can only occur when there is a negative differential velocity in \( v(V) \) [5]. There are many ways of choosing it. One can assume \( v \) being a sum of a series of Lorentzian functions if NDC is due to the resonance tunneling between discrete quasi-localized states of wells. One may also choose a piecewise linear function in order to make an analytic investigation easy. We shall assume \( v(V) \) as the sum of two Lorentzian functions, \( v(V) = 0.0081/[(V/E-1)^2 + 0.01] + 0.36/[(V/E-2.35)^2 + 0.18] \). This \( v \) has two peaks at \( V = E \) and \( V = 2.35E \). Each peak is in the low EFD while \( V = 3.5E \) is in the high EFD here. Clearly, one obtains a closed isolated curve indicating a limit cycle. The current oscillation period, the inverse of system intrinsic frequency \( \omega_0 \), is the time that the system needs to move around the cycle once. The inset is the phase diagram of the TCSOs in \( U-N_D \) plane with all other parameters unchanged, where \( N_D \) is in units of \( kE \). The shadowed area corresponds to the TCSO regime. Of course, the diagram depends on the values of \( \delta \), \( N \), and function of \( v \).

Although it is known that TCSOs are accompanied by the motion of EFD boundaries, how EFD boundaries are generated and propagate inside SLSs were debated [5]. According to the limit cycle picture, the charge accumulation (depletion) is responsible to the creation of an EFD boundary. Charge carriers are accumulated (depleted) in a particular well because of an imbalance of carriers flowing in and flowing out. This imbalance is caused by NDC as we argued early. Thus an EFD boundary can

\[
\begin{align*}
\text{FIG. 2. The trajectory of the system in phase plane } V_5-V_{38} \text{ in a TCSO regime. The closed curve is the projection of the limit cycle in the plane. The bias is in units of } E. \text{ The inset is the phase diagram in } U-N_D \text{ plane with all other parameters unchanged, where } N_D \text{ is in units of } kE. \text{ The system inside the shadowed area will have a TCSO solution while it has a static current-voltage characteristic outside this area.}
\end{align*}
\]

2
start at any well and oscillates inside a SL. Furthermore, the limit cycle picture means that EFD boundaries should not depend on the initial conditions, but are completely determined by the limit cycles around unstable SSSs.

To demonstrate the correctness of our picture, we locate numerically the position of the EFD boundary in the calculation that gives Fig. 2. Figure 3 is the evolution of the boundary. It reaches a stable oscillating state quickly. One can see that the EFD boundary oscillating between wells 26 and 37 in the SL of total 40 wells. The inset is the field distribution across the SL at points a, b, c, and d in Fig. 3. The bias \( V_3 \) of the low (high) EFD moves up and down as the EFD boundary oscillates inside the SL. The stable oscillation state, a manifestation of a limit cycle, does not change when different initial conditions are used. In this sense both generation and motion of an EFD boundary are universal. Although early numerical calculations [5] might have already implied that an EFD boundary can start in an interior well and oscillate inside a SL. It may not be easy for theories like that of reference 5 to explain this universal property.

![Well Number vs t](image1.png)

**FIG. 3.** The time evolution of an EFD boundary. The parameters are the same as those for Fig. 2. The boundary is oscillating between well 26 and well 37. The stable oscillation does not depend on the initial conditions. The inset is the electric field distribution at points a, b, c, and d in Fig. 3.

Except a few of attempts [11,12] which were mainly on the numerical aspects dealing with non-periodic time-dependent tunneling current, most theoretical studies have not considered TCSOs under the influence of an ac bias. The limit cycle theory offers a way of analyzing the ac bias effect. Applying a small extra ac bias, it affects TCSOs through perturbing the system trajectories. In the case that the limit cycles still exist and are stable, or in other words, the tunneling current can oscillate periodically with frequency \( \omega \), the ac bias on the system should return to its starting value after the system completes one-round motion on the limit cycle. It means \( \omega_{ac}/\omega = n = integer \). A weak ac bias cannot greatly change the evolution trajectory of the system in the phase space. The consequences are as follows: a) Current oscillation frequency \( \omega \) cannot be much greater than the intrinsic frequency. Thus, at high ac bias frequency (\( \omega_{ac} >> \omega_0 \)), it is impossible for the bias to deform the limit cycle slightly such that the time for the system moving around the cycle once to be the same as the period of the ac bias. b) In a case that the system cannot deform itself to match \( \omega_{ac} \), a trajectory may become a closed curve after several turns in the phase space. Therefore, \( \omega_0/\omega = m = integer \). Indeed, figure 4 is the limit cycles in phase plane \( E_3-E_5 \) under an extra ac bias \( V_{ac} \cos(\omega_{ac}t) \) with \( V_{ac} = 0.44E \) and \( \omega_{ac} = 2\omega_0 \) (dot line), \( 3\omega_0 \) (solid line) while the rest of parameters remain the same as those for Figs. 2-3. The right (left) inset is the current-time curve for \( \omega_{ac} = 2\omega_0 \) (3\( \omega_0 \) after the current oscillation becomes stable. The Fourier transformation shows the current oscillation frequency \( \omega \) being \( \omega_0 \). This is the solution of \( \omega_{ac}/\omega = n = integer \) and \( \omega_0/\omega = m = integer \) for the smallest possible \( n \) (=2, 3) and \( m \) (=1).

![Current oscillation](image2.png)

**FIG. 4.** The limit cycles in phase plane \( E_3-E_5 \) under an extra weak ac bias \( V_{ac} \cos(\omega_{ac}t) \) with \( \omega_{ac} \) being \( 2\omega_0 \) and \( 3\omega_0 \). The inset on the right (left) is the current-time curve for \( \omega_{ac} = 2\omega_0 \) (3\( \omega_0 \) after the current oscillation becomes stable. The Fourier transformation shows the current oscillation frequency \( \omega \) being \( \omega_0 \). This is the solution of \( \omega_{ac}/\omega = n = integer \) and \( \omega_0/\omega = m = integer \) for the smallest possible \( n \) (=2, 3) and \( m \) (=1).

For \( \omega_{ac}/\omega_0 \) being a non-integer rational number, \( \omega_0 \) should be different from \( \omega_0 \) according to the rules of \( \omega_{ac}/\omega = n = integer \) and \( \omega_0/\omega = m = integer \). For example, \( \omega_{ac}/\omega_0 = 1/q \) with \( q = integer \), then the limit cycle can deform itself in such a way that it becomes a closed curve after \( q \) turns in the phase space, corresponding to \( n = 1 \) and \( m = q \). In this case, the current oscillates with \( \omega_{ac} \). The solid line in figure 5 is the numerical results of the limit cycle in phase plane \( V_5-V_38 \) for \( \omega_{ac} = \omega_0/3 \) while the rest of parameters are kept the same as those for Figs. 2 to 4. Indeed, the limit cycle, which does not depend on the initial conditions, makes \( q = 3 \) turns in the phase plane as expected. The left inset is the corresponding tunneling current evolution curve. The Fourier transformation of the current evolution indeed shows \( \omega = \omega_{ac} \). For \( \omega_{ac}/\omega_0 = 2.5 \), our rules predict the \( \omega = \omega_0/2 = \omega_{ac}/5 \), corresponding to \( n = 5 \).
and \( m = 2 \). This result is verified by the numerical calculation as displayed in figure 5 (dot line). As before, all other parameters are kept the same as those for Figs. 2 to 4. The right inset is the corresponding tunneling current evolution curve. The limit cycle is a two-turn closed curve in phase plane \( V_5-V_{38} \). The Fourier transformation of the current evolution demonstrates that the current oscillates with frequency \( \omega_0/2 = \omega_{ac}/5 \). We would like to make following remarks. a) \( \omega = \omega_{ac} \) for \( \omega_{ac} = \omega_0/q \) cannot be true for all \( q \) because \( \omega \) should approach \( \omega_0 \) in the limit \( \omega_{ac} \rightarrow 0 \). b) We have considered only periodic responses of the tunneling current. It does not rule out other more complicated behaviors. In fact, there are reports [11–13] of chaotic current-time behavior under a combined dc and ac biases. c) Vary external parameters, the size and shape of a limit cycle should change in general. Thus \( \omega_0 \) can shift. As a consequence, \( \omega_{ac}/\omega = n \) may be quite robust against \( \omega_{ac} \). When \( \omega_{ac} \) is very close to \( \omega_0 \), it may be possible for the limit cycle to deform itself in such a way that \( \omega_0 \) shifts to \( \omega_{ac} \). In this case, the current shall oscillate with frequency \( \omega_{ac} \). Our preliminary results indeed show so. How much \( \omega_0 \) can shift depends on the magnitude of \( V_{ac} \) and \( \omega_{ac} \). The detail analysis in this regime will be our future direction in this project.

![Diagram](image)

FIG. 5. The limit cycles in phase plane \( V_5-V_{38} \) under an extra weak ac bias \( V_{ac}\cos(\omega_{ac}t) \) with \( \omega_{ac} \) being \( \omega_0/3 \) (solid line) and \( 2.5\omega_0 \) (dot line). The inset on the left (right) is the curve of current vs. time for \( \omega_{ac} = \omega_0/3 \) (\( =2.5\omega_0 \)). All other parameters are the same as those for Fig. 4. The current frequency equals to \( \omega_{ac} \) (\( \omega_{ac}/5 \)) for \( \omega_{ac} = \omega_0/3 \) (\( =2.5\omega_0 \)).

In summary, we show that the trajectories of SLs in their phase spaces are closed curves when TCSOs occur. These closed curves do not depend on the initial conditions. In other words, the closed trajectories are isolated. Thus we show numerically the existence of the limit cycles, and TCSOs can be understood as the manifestations of limit cycles. Just like many other non-linear dynamical systems, TCSOs are governed by the properties of the unstable SSSs. According to this theory, the generation and motion of an EFD boundary are also the properties of unstable SSSs. An EFD boundary does not necessarily need to start from the emitter. It can start from an interior well of a SL, and it then oscillates inside the SL. They are universal in the sense that they do not depend on the initial conditions. This universal property may not be so obvious in previous theories [5]. We have also investigated the effects of a small extra external ac bias on TCSOs. We find that the tunneling current will oscillate periodically when the ac bias frequency \( \omega_{ac} \) is commensurate with the system intrinsic frequency \( \omega_0 \). The current frequency equals either \( \omega_{ac} \) or \( \omega_{ac}/n \), where \( n \) is an integer, showing periodic doubling which is a general phenomenon in non linear dynamical systems.

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