INDUCED SUBGRAPHS AND TREE DECOMPOSITIONS
V. SMALL COMPONENTS OF BIG VERTICES

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ABSTRACT. In [1], Aboulker, Adler, Kim, Sintiari, and Trotignon conjectured that every graph with bounded maximum degree and large treewidth must contain, as an induced subgraph, a large subdivided wall, or the line graph of a large subdivided wall. This conjecture was recently proved by Korhonen [7], but the problem of identifying the obstacles to bounded treewidth in the general case (that is, without the bounded maximum degree condition) remains wide open. Examples of structures of large treewidth which avoid the “usual suspects” have been constructed by Sintiari and Trotignon [10], and by Davies [6]. In this note, we aim to better isolate the features of these examples that lead to large treewidth. To this end, we prove the following result. Let $G$ be a graph, and write $\gamma(G)$ for the size of a largest connected component in the graph induced by $G$ on the set of vertices of degree at least 3. If $\gamma(G)$ is small and the treewidth of $G$ is large, then $G$ must contain a large subdivided wall or the line graph of a large subdivided wall. This result is the best possible, in the sense that the conclusion fails if we replace 3 by any larger number in the definition of $\gamma(G)$, as evidenced by Davies’ example.

1. Introduction

All graphs in this paper are finite and simple. Let $G = (V(G), E(G))$ be a graph. For a set $X \subseteq V(G)$ we denote by $G[X]$ the subgraph of $G$ induced by $X$. In this paper, we use induced subgraphs and their vertex sets interchangeably. Let $v \in V(G)$. The open neighborhood of $v$, denoted by $N(v)$, is the set of all vertices in $V(G)$ adjacent to $v$. The closed neighborhood of $v$, denoted by $N[v]$, is $N(v) \cup \{v\}$. If $H$ is an induced subgraph of $G$ and $v \in V(G)$, then $N_H(v) = N(v) \cap H$ and $N_H[v] = N_H(v) \cup \{v\}$. A vertex $v$ is cubic if $|N(v)| = 3$.

For a graph $G = (V(G), E(G))$, a tree decomposition $(T, \chi)$ of $G$ consists of a tree $T$ and a map $\chi : V(T) \to 2^{V(G)}$ with the following properties:

(i) For every $v \in V(G)$, there exists $t \in V(T)$ such that $v \in \chi(t)$.
(ii) For every $v_1v_2 \in E(G)$, there exists $t \in V(T)$ such that $v_1, v_2 \in \chi(t)$.
(iii) For every $v \in V(G)$, the subgraph of $T$ induced by $\{t \in V(T) \mid v \in \chi(t)\}$ is connected.

For each $t \in V(T)$, we refer to $\chi(t)$ as a bag of $(T, \chi)$. The width of a tree decomposition $(T, \chi)$, denoted by $\text{width}(T, \chi)$, is $\max_{t \in V(T)} |\chi(t)| - 1$. The treewidth of $G$, denoted by $\text{tw}(G)$, is the minimum width of a tree decomposition of $G$.

Treewidth is a graph parameter of great algorithmic and structural interest, studied notably by Robertson and Seymour in their celebrated work on graph minors. To better understand the parameter, it is natural to investigate what graphs of large treewidth look like. Two very basic examples are the complete graph $K_t$, and the complete bipartite graph $K_{t,t}$, which have treewidth $t - 1$ and $t$ respectively. A slightly more involved example is the $(k \times k)$-wall $W_{k,k}$ with treewidth $k$, depicted in Figure 1 (see [2] for a precise definition).
This example plays an important role; one of the outcomes of Robertson and Seymour’s study was the Grid Theorem – a characterization of the unavoidable structures of large treewidth under the subgraph relation, which can be stated as follows:

**Theorem 1.1** ([9, 5]). There is a function \( f : \mathbb{N} \to \mathbb{N} \) such that every graph of treewidth at least \( f(k) \) contains a subdivision of \( W_{k \times k} \) as a subgraph.

Recent years have seen a steady and ongoing effort to identify the unavoidable structures of large treewidth under the induced subgraph relation (see for instance [1, 7, 10, 8], as well as the other papers in the current series). While the question in all its generality is far from settled, significant progress has been made on many fronts. One such milestone was conjectured by Aboulker, Adler, Kim, Sintiari, and Trotignon [1] and proved by Korhonen [7]. The result states that, when restricting our search to graphs of bounded maximum degree (which automatically forbids large complete graphs and complete bipartite graphs), the only obstacles to bounded treewidth are subdivided walls, and line graphs of subdivided walls (see [2] for the definition of the latter):

**Theorem 1.2** ([7]). For all \( k, \Delta > 0 \), there exists \( c = c(k, \Delta) \) such that every graph with maximum degree at most \( \Delta \) and treewidth more than \( c \) contains a subdivision of \( W_{k \times k} \) or the line graph of a subdivision of \( W_{k \times k} \) as an induced subgraph.

It is then natural to ask whether forbidding those four “usual suspects” – \( K_t, K_{t,t} \), subdivisions of \( W_{k,k} \), and line graphs of subdivisions of \( W_{k,k} \) – suffices to guarantee bounded treewidth without any degree restrictions. The answer is no, as evidenced by the so-called layered wheels constructed by Sintiari and Trotignon [10]. The authors of [10] noted that those layered wheels have a number of vertices exponential in the treewidth, in contrast to the usual suspects, where the number is polynomial. Accordingly, they conjectured that perhaps a weaker conclusion holds when forbidding the usual suspects, namely that the treewidth should be logarithmic rather than bounded. Alas, this conjecture is also false. This follows from a more recent example due to Davies [6], which we briefly describe in Section 2 for the sake of completeness.

To motivate the result from our present note, our starting point is the remark that Sintiari and Trotignon’s layered wheel construction contains many vertices of large degree. More specifically, the eponymous layers consist of long paths where vertices have large degree. What if we were to disallow that from happening, by limiting the size of a connected component induced by the vertices of large degree? For a graph \( G \) and \( \Delta \in \mathbb{N} \), let us denote by \( \gamma_\Delta(G) \) the maximum size of a connected component of the graph induced by \( G \) on the set of vertices whose degree in \( G \) is at least \( \Delta \). From a quick examination, Davies’ examples have \( \gamma_\Delta(G) \) at most 1 for all \( \Delta \geq 4 \). In other words, no two vertices of degree 4 or more are adjacent, yet the treewidth is unbounded, and the usual suspects are absent. While \( \gamma_\Delta(G) \) does not seem to tell us much about the treewidth when \( \Delta > 4 \), there is still hope that \( \gamma_3(G) \) might give us some insight. And indeed, it turns out that bounding it is enough to eliminate all obstacles to bounded treewidth which are not our usual suspects. From now on, we will omit the subscript, and write \( \gamma(G) \) to mean \( \gamma_3(G) \). Our result, which we show in Section 3, is the following:
Theorem 1.3. For all \( k, \gamma > 0 \), there exists \( c = c(k, \gamma) \) such that every graph \( G \) with \( \gamma(G) \leq \gamma \) and treewidth more than \( c \) contains a subdivision of \( W_{k \times k} \) or the line graph of a subdivision of \( W_{k \times k} \) as an induced subgraph.

2. Davies’ example

In this section, we describe Davies’ [6] example of graphs of large treewidth which have large girth (thus avoiding \( K_3 \) and \( K_{2,2} \)) and which do not have a subdivision of \( W_{5,5} \) or a line graph thereof as an induced subgraph.

For integers \( g \) and \( n \), we construct a graph \( G_{n,g} \) on \( gn^2 + n \) vertices as follows. We start with \( n \) vertex-disjoint paths \( P_1, \ldots, P_n \) each on \( gn \) vertices, and with no edges between them. We label the vertices of the path \( P_i \) as \( x_{i,1}, \ldots, x_{i,gn} \). Each path \( P_i \) is the concatenation of \( n \) subpaths \( P_{i,1}, \ldots, P_{i,n} \), each on \( g \) vertices. To obtain \( G_{n,g} \), we add \( n \) vertices \( v_1, \ldots, v_n \) and for each \( i \) and \( j \), we connect \( v_i \) to \( x_{j,g(i-1)+1} \) (that is, we connect \( v_i \) to the first vertex of each of the paths \( P_{1,i}, P_{2,i}, \ldots, P_{n,i} \)).

By construction, \( G_{n,g} \) has girth \( 2g + 4 \). Moreover, it can be shown that \( G_{g,n} \) does not contain a subdivision of \( W_{5,5} \) or the line graph of a subdivision of \( W_{5,5} \) as an induced subgraph. Finally, contracting \( P_i \cup \{v_i\} \) to a vertex for each \( i \) yields a \( K_n \) minor, showing that \( G_{n,g} \) has treewidth at least \( n \). The graph \( G_{4,3} \) is depicted in Figure 2.

3. Proof of Theorem 1.3

Given a cubic vertex \( x \) of \( G \) with \( N_G(x) = \{a_1, a_2, a_3\} \), replacing \( x \) by a triangle means removing \( x \) and adding three pairwise adjacent vertices \( x_1, x_2, x_3 \) such that \( N_{G \setminus \{x_1,x_2,x_3\}}(x_i) = \{a_i\} \). Let \( W \) be a subdivision of a wall. A scrambling of \( W \) is a graph obtained from \( W \) by replacing some of its cubic vertices with triangles (in particular, the line graph of a subdivision of \( W \) is a scrambling of \( W \)).

Our proof uses two results. The first is a Ramsey-type result shown in [1]. In our terminology, it may be stated as follows:

**Proposition 3.1** ([1]). Let \( k \in \mathbb{N} \). There exists \( K \in \mathbb{N} \) such that any scrambling of a subdivision of the wall \( W_{K,K} \) contains as an induced subgraph either a subdivision of the wall \( W_{k,k} \), or the line graph of a subdivision of the wall \( W_{k,k} \).

The second is a general result regarding the way 3 vertices may connect with each other, shown in [4]:

**Proposition 3.2** ([4]). Let \( x_1, x_2, x_3 \) be three distinct vertices of a graph \( G \). Assume that \( H \) is a connected induced subgraph of \( G \setminus \{x_1, x_2, x_3\} \) such that \( V(H) \) contains at least one neighbor
of each of $x_1, x_2, x_3$, and that $V(H)$ is minimal subject to inclusion. Then, one of the following holds:

(i) For some distinct $i,j,k \in \{1,2,3\}$, there exists $P$ that is either a path from $x_i$ to $x_j$ or a hole containing the edge $x_i x_j$ such that
   - $V(H) = V(P) \setminus \{x_i, x_j\}$, and
   - either $x_k$ has two non-adjacent neighbors in $H$ or $x_k$ has exactly two neighbors in $H$ and its neighbors in $H$ are adjacent.
(ii) There exists a vertex $a \in V(H)$ and three paths $P_1, P_2, P_3$, where $P_i$ is from $a$ to $x_i$, such that
   - $V(H) = (V(P_1) \cup V(P_2) \cup V(P_3)) \setminus \{x_1,x_2,x_3\}$, and
   - the sets $V(P_1) \setminus \{a\}, V(P_2) \setminus \{a\}$ and $V(P_3) \setminus \{a\}$ are pairwise disjoint, and
   - for distinct $i,j \in \{1,2,3\}$, there are no edges between $V(P_i) \setminus \{a\}$ and $V(P_j) \setminus \{a\}$, except possibly $x_i x_j$.
(iii) There exists a triangle $a_1a_2a_3$ in $H$ and three paths $P_1, P_2, P_3$, where $P_i$ is from $a_i$ to $x_i$, such that
   - $V(H) = (V(P_1) \cup V(P_2) \cup V(P_3)) \setminus \{x_1,x_2,x_3\}$, and
   - the sets $V(P_1), V(P_2)$ and $V(P_3)$ are pairwise disjoint, and
   - for distinct $i,j \in \{1,2,3\}$, there are no edges between $V(P_i)$ and $V(P_j)$, except $a_i a_j$ and possibly $x_i x_j$.

We start with an intermediate lemma which can be seen as a corollary of Proposition 3.2.

**Lemma 3.3.** Let $G$ be a graph, let $X$ be a set of vertices of $G$ inducing a connected subgraph, and let $G'$ be the graph obtained from $G$ by contracting $X$ to a single vertex $x$, and deleting the parallel edges which were created in the process. Suppose that $x$ has exactly 3 neighbors $x_1, x_2, x_3$ in $G'$, and that $x_1, x_2, x_3$ each have exactly one neighbor among the vertices from $X$. Then $G$ contains an induced subgraph isomorphic to either:

a) a subdivision of $G'$ in which the subdivided edges are incident to $x$, or

b) a subdivision of $G'$ in which the subdivided edges are incident to $x$ and the vertex corresponding to $x$ has been replaced with a triangle.

**Proof.** We note $G[X]$ is a connected induced subgraph of $G \setminus \{x_1, x_2, x_3\}$, and $X$ contains at exactly one neighbor of each of $x_1, x_2, x_3$. Let $H$ be a minimal induced subgraph of $G$ with $V(H) \subseteq X$ satisfying these properties. Apply Proposition 3.2 to vertices $x_1, x_2, x_3$ of $G$ and $H$. We note that case (i) in the proposition cannot occur, since in that case, there is a vertex among $x_1, x_2, x_3$ with two neighbors in $X$. We look at each of the two remaining cases given by the proposition:

(ii) There exists a vertex $a \in V(H)$ and three paths $P_1, P_2, P_3$, where $P_i$ is from $a$ to $x_i$, such that
   - $V(H) = (V(P_1) \cup V(P_2) \cup V(P_3)) \setminus \{x_1,x_2,x_3\}$, and
   - the sets $V(P_1) \setminus \{a\}, V(P_2) \setminus \{a\}$ and $V(P_3) \setminus \{a\}$ are pairwise disjoint, and
   - for distinct $i,j \in \{1,2,3\}$, there are no edges between $V(P_i) \setminus \{a\}$ and $V(P_j) \setminus \{a\}$, except possibly $x_i x_j$.
   In this case, $G'[V(G) \setminus X \cup V(H)]$ is isomorphic to a subdivision of $G'$, and we are in case a).

(iii) There exists a triangle $a_1a_2a_3$ in $H$ and three paths $P_1, P_2, P_3$, where $P_i$ is from $a_i$ to $x_i$, such that
   - $V(H) = (V(P_1) \cup V(P_2) \cup V(P_3)) \setminus \{x_1,x_2,x_3\}$, and
   - the sets $V(P_1), V(P_2)$ and $V(P_3)$ are pairwise disjoint, and
   - for distinct $i,j \in \{1,2,3\}$, there are no edges between $V(P_i)$ and $V(P_j)$, except $a_i a_j$ and possibly $x_i x_j$. 


In this case, $G'[(V(G) \setminus X) \cup V(H)]$ is as in case b).

\textbf{Proof of Theorem 1.3.} Let $C_1, \ldots, C_m$ be the components of the graph induced by $G$ on the set of vertices whose degree in $G$ is at least three. Let $G'$ be the graph obtained from $G$ by contracting each $C_i$ to a vertex $c_i$ and deleting all the parallel edges formed by the contraction process. Then $\deg_{G}(v), \deg_{G'}(v) \leq 2$ for every $v \in V(G') \setminus \{c_1, \ldots, c_m\}$ and the set $\{c_1, \ldots, c_m\}$ is stable in $G'$.

Assume first that the treewidth of $G'$ is at most $\frac{7}{5}$. Let $(T, \chi')$ be a tree-decomposition of $G'$ of width $\text{tw}(G')$. Now for every $v \in T$ let $\chi'(v)$ be obtained from $\chi'(v)$ by replacing each $c_i \in \chi'(v)$ by the set $C_i$. It is easy to check that $(T, \chi')$ is a tree decomposition of $G$ of width at most $c$. Thus we may assume that $\text{tw}(G') > \frac{7}{5}$.

Let $K$ be as in Proposition 3.1. Choosing $c$ large enough, the Grid Theorem (Theorem 1.1) implies that $G'$ contains a subdivision of the $(K \times K)$-wall as a subgraph; denote this subgraph by $W'$. Since no two vertices of degree at least three in $G'$ are adjacent, it follows that $W'$ is an induced subgraph of $G'$.

Moreover, we claim that, if $c_i \in W'$ for some $i$, and $v \in V(W') \setminus \{c_1, \ldots, c_m\}$ is adjacent to $c_i$, then $v$ has exactly one neighbour in $C_i$. Indeed, it needs to have at least one by construction, and it cannot have two, since $v$ has degree at most two in $G$, and if both those neighbours were in $C_i$, then $v$ would have degree $1$ in $W'$ (a contradiction).

Now let $Z = V(W') \cap \{c_1, \ldots, c_m\}$, and let $W$ be the graph induced by $G$ on $V(W') \setminus Z \cup \bigcup_{i \in Z} C_i$. We apply Lemma 3.3 to $W$ and $W'$ repeatedly, in the following sense: we define a sequence $W = W_0, \ldots, W_r := W'$ where $W_s$ is obtained from $W_{s-1}$ by contracting the vertices of $C_i$ to a single vertex $x_{i,s}$. We then put $W_r := W'$. For $s = r - 1, \ldots, 0$, Lemma 3.3 then yields an induced subgraph $W^s_0$ of $W_s$ which is obtained from $W^s_{s+1}$ by subdividing some of the edges incident to $x_{i,s+1}$, then possibly replacing $x_{i,s+1}$ with a triangle. In the end, we obtain an induced subgraph $W^0_0$ of $W$ (and thus of $G$) that is a scrambling of a subdivision of $W'$ (and thus of $W_{K,K}$). Applying Proposition 3.1, we obtain that $G$ contains an induced subgraph either a subdivision of $W_{k \times k}$, or the line graph of a subdivision of $W_{k \times k}$. This completes the proof.

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\section*{References}

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