Dynamic Shortest Path and Transitive Closure Algorithms: A Survey

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Abstract. Algorithms which compute properties over graphs have always been of interest in computer science, with some of the fundamental algorithms, such as Dijkstra’s algorithm, dating back to the 50s. Since the 70s there has been interest in computing over graphs which are constantly changing, in a way which is more efficient than simple recomputing after each time the graph changes.

In this paper we provide a survey of both the foundational, and the state of the art, algorithms which solve either shortest path or transitive closure problems in either fully or partially dynamic graphs. We balance this with the known conditional lower bounds.

Keywords: Dynamic Graph Algorithms, Shortest Paths, Connectivity

1 Introduction

Graphs are one of the most fundamental and well studied data structures within computer science. Many problems of interest can be phrased in terms of graphs. In this work we focus on two problems over graphs; connectivity and shortest paths; Connectivity: Given two vertices within a graph, does there exist a path between the two vertices? If the graph is directed, asking if there is a path between vertex $u$ and vertex $v$ is called transitive closure. If a path is required both from $u$ to $v$ and from $v$ to $u$, the question is asking if $u$ and $v$ are in the same strongly connected component. If the graph is undirected, two vertices being connected and being in the same strongly connected component are equivalent.

Shortest Path: Given two vertices $u, v$ in a graph what is the shortest path from $u$ to $v$? A slight variant of the problem only requires the length of the shortest path to be returned and not the component itself. The problems described above are the exact variant; there is also an approximate variant where given $\alpha > 1$ a path of length $\alpha \cdot \Delta$ must be returned, where $\Delta$ is the length of the shortest path.

For both of these problems there are a few variants. In the single source variant, the queries will only be asked from a fixed start vertex. A similar variant can be considered for a single sink. In the $s$-$t$ variant, the only query asked will be between the vertices $s$ and $t$.

These two problems occur in a host of areas, including; databases [72], compilers [61] and VLSI design [17]. For example, in bioinformatics, shortest path algorithms have been used to identify genes associated with colorectal cancer [53]. Another example, is that if you represent the possible states of a Rubik’s cube as vertices in the graph, where an edge corresponds to a single move, then a shortest path algorithm will give an efficient way to solve any cube.

The problems defined above are given in a static setting; when the graph is defined, it will remain fixed for the entirety of the graph’s lifetime. However, in practice there may be times when the graph will change over the course of the algorithm. In these situations it is desirable not to have to start from scratch – dynamic graph algorithms are for these situations. There are three variants of dynamic graph algorithms; incremental algorithms which only allow edges to be inserted and not deleted, decremental algorithms which only allow edges to be deleted and fully dynamic algorithms which allow edges to both be inserted and deleted. Some algorithms on weighted graphs, also allow the edge weights to be changed.

Fully dynamic algorithms have a multitude of clear uses, as in many applications the graph will naturally change over time. Incremental algorithms are useful in scenarios where links appear but never disappear. For example, representing the graph of all people on a social network, where the edges are if two people have ever communicated. The road network tends to be constantly expanding, with it being extremely rare for a road to be permanently destroyed. The use of decremental algorithms, is

\footnote{In this work we will not consider the case where vertices can be inserted or deleted.}
slightly more subtle but they are of important theoretical interest. This is because they can be used as a building block in a variety of fully dynamic algorithms \cite{4,5,12,40}. Lots of decremental algorithms can be converted into incremental algorithms, while maintaining the same time complexities, such as \cite{13}. Decremental algorithms have even been used to give improved static graph algorithms \cite{56}.

The are other ways that a graph can be considered to be changing dynamically. For example, Albers and Schraflík \cite{7} consider graph colouring where the vertices of the graph arrive one at a time. We will not focus on this, or other models, in this survey but limit ourselves to the case where edges are added and removed.

Dynamic algorithms can trivially be constructed from the classic approach. If the algorithm does nothing when an edge is updated, it will have to explore the whole graph to answer queries. This gives a $O(1)$ update time and $O(m + n)$ query time algorithm for all pairs shortest path. However, if after each change the static algorithms are rerun, this results in a $O(n \cdot m + n^2)$ update time and $O(1)$ query time all pairs shortest path algorithm. The goal of research into dynamic graph algorithms is to do better than these trivial bounds. For incremental connectivity, a set union algorithm \cite{68} can be used to give a query and update time of $O(\alpha(n))$, where $\alpha$ is related to the inverse-Ackerman function and thus very slow growing. In this work we focus mainly on algorithms with a constant query time. However, there are a host of algorithms which do better than the trivial case without this property \cite{12,39,41,63}. For example, Roddity and Zwick \cite{63} give a fully dynamic $(1 + \epsilon)$-approximation algorithm for dynamic all pairs shortest paths with $\tilde{O}(\sqrt{n})$ update time and $O(t)$ query time. Hence, this algorithm provides a trade-off between update and query time.

Even and Shiloach \cite{33} designed the first decremental connectivity algorithm, which is more efficient than simply recomputing from scratch after the deletion of an edge. A similar result was independently discovered by Dinitz \cite{29}. The algorithm of Even and Shiloach will be the starting point for our survey, since it has had a great deal of influence in the literature, resulting in several variants and multiple algorithms being built on top of it \cite{13,41,42,49}. There have also been a host of algorithms which use techniques disjoint from those given by Even and Shiloach \cite{15,24,52,62}. In contrast to the efficient algorithms to solve these problems, there have been several lower bounds given in the literature \cite{2,45,51,59}. Table \ref{table:summary} summarises a large proportion of the results in the field, while the text focuses on a smaller proportion, to go into more detail on.

This paper focuses on surveying the theoretical aspects of dynamic graph algorithms for connectivity and shortest path. However, there has been previous work that compares these algorithms in practical scenarios \cite{8,21,22,27,35–37,48,74}. For example, Frigioni et al. \cite{37} implement and compare dynamic transitive closure algorithms, while Demetrescu and Italiano \cite{27} implement and compare dynamic all pairs shortest path algorithms.

In this paper, we have limited ourselves to the aforementioned two problems, on general graphs. However, there has been lots of work which considers other dynamic problems. These included how to embed a dynamically changing graph into the plane \cite{16}, topological ordering \cite{11,38}, matchings \cite{65} and min-cut \cite{69}. It is still an open question as to if fully dynamic max flow can be done faster than recomputation. There has also been work which gives more efficient solutions when restrictions are made to the graph, such as if the graph is planar \cite{13,34}, has bounded degree \cite{73} or is directed acyclic \cite{25,47}.

There are two types of algorithm which can be considered; randomised or deterministic, both of which will be discussed in this paper. In general randomised algorithms tend to be more efficient than deterministic algorithms. However, in the dynamic graph scenario, the randomised algorithms come with a large disadvantage. The randomised algorithms tend to assume that the adversary is both oblivious to the randomness used by the algorithm and non-adaptive in their queries. The non-adaptivity is required because otherwise it is possible that queries would allow the adversary learn about the randomness used by the algorithm and then act upon this information. Without this assumption the algorithms tend to lose their benefit. Using the nice example from \cite{13}. If the goal is to maintain a set of approximately $\sqrt{n}$ vertices called centers such that every vertex is at most $\sqrt{n}$ away from a center, in the static case this can be achieved by simply choosing $\sqrt{n} \cdot \log n$ centers uniformly at random. In the deterministic setting this does not work because the adversary can simply choose to disconnect all the centers (while leaving the rest of the graph intact), by deleting a suitable subset of edges. In the randomised setting with oblivious adversaries the randomised solution once again holds. Hence, one of the desirable goals for dynamic algorithms, is to construct deterministic algorithms with the same time complexity as their randomised counterparts.

\footnote{Here we define an adversary as a user of the algorithm who is trying to get the algorithm to run in the worst possible time.}
1.1 Related Work

Recently Madkour et al. [25] published a survey on shortest path algorithms and fitted algorithms into a framework to aid relating algorithms. They give a section providing an overview of dynamic algorithms. However, due to the wide subject area, they only touch on dynamic algorithms, while we can give considerably more detail.

Demetrescu and Italiano [26] provided a survey of dynamic shortest path and transitive closure algorithms. The authors abstract out some of the combinatorial, algorithmic and data structure techniques used in the literature and then present a subset of the known algorithms within this unifying framework. However, there has been a host of work published since the survey was published over a decade ago.

1.2 Outline

In Section 2 we give some preliminaries. Section 3 gives the algorithm of Even and Shiloach [33] and some variants. Section 4 gives algorithms which are built upon the algorithm of Even and Shiloach, while Section 5 gives algorithms built from other techniques. In Section 6 we describe the known lower bounds on these problems, which reduce to other well studied problems. We conclude in Section 7 with some open problems.

2 Preliminaries

In this section we give the notation that will be used throughout this paper, along with the definitions of problems which the given algorithms will try to solve.

2.1 Notation

Let $G = (V, E)$ be a (possibly directed) graph. We denote $n = |V|$ and $m = |E|$. If the graph is weighted, we use the weight function $\ell(u, v)$ for $u, v \in V$ to give the weight of the edge, between $u$ and $v$. Given $U \subseteq V$, we define $G[U] = (U, E')$ to be the graph with vertices from $U$ and $E' = E \cap U^2$, i.e. all edges from $G$ that start and end in $U$.

A series of vertices $p = (v_0, \ldots, v_k)$ is said to be a path between $u$ and $v$, of length $k$ if $v_0 = u, v_k = v$ and $(v_i, v_{i+1}) \in E \forall 0 \leq i < k$. The weight of a path $\ell(p)$ is defined as the sum of the weight of all the edges in the path. A path $p$ is called a shortest path between $u$ and $v$ if $\ell(p) \leq \ell(p')$ for all other paths $p'$ between $u$ and $v$.

Given three variables $n_1, n_2, n_3$ we say a function $f(n_1, n_2, n_3) = \tilde{O}(n_1^{c_1} \cdot n_2^{c_2} \cdot n_3^{c_3})$ if there exists a constant $\epsilon > 0$ such that $f(n_1, n_2, n_3) = O(n_1^{c_1- \epsilon} \cdot n_2^{c_2} \cdot n_3^{c_3} + n_1^{c_1- \epsilon} \cdot n_2^{c_2- \epsilon} \cdot n_3^{c_3} + n_1^{c_1} \cdot n_2^{c_2- \epsilon} \cdot n_3^{c_3- \epsilon})$. It can be defined analogously for an arbitrary number of parameters.

2.2 Definitions

Given a graph $G$, a dynamic algorithm is one which allows changes to the graph intermixed with the queries, while a static algorithm does not allow changes to the graph. An algorithm which only allows edge deletion is called decremental, an algorithm which only allows edge insertion is called incremental, while one which allows both is called fully dynamic.

The type of queries allowed determine which problem the algorithm can solve. In this work we consider two types of problem: shortest path and transitive closure. We define both in the static setting but the algorithms are trivially extended to the dynamic setting by allowing the relevant changes to the graph.

**Definition 1 (Transitive Closure Problem).** Given a directed graph $G = (V, E)$, the transitive closure problems require an algorithm to answer questions of the form:

- **Transitive Closure** “Given $u, v \in V$, is there a path from $u$ to $v$?”
- **Single Source Transitive Closure** “Given $u \in V$, is there a path from the fixed vertex $s$ to $u$?”
- **Single Sink Transitive Closure** “Given $u \in V$, is there a path from $u$ to the fixed vertex $t$?”
- **s-t Transitive Closure** “Is there a path from the fixed vertex $s$ to the fixed vertex $t$?”

**Note 1.** When the graph is not directed transitive closure is referred to as connectivity.

**Definition 2 (Shortest Path Problem).** Given a directed graph $G = (V, E)$, the shortest path problems requires an algorithm to answer questions of the form:
All Pairs Shortest Paths “Given \( u, v \in V \), return the shortest path from \( u \) to \( v \)”

Single Source Shortest Paths “Given \( u \in V \), return the shortest path from the fixed vertex \( s \) to \( u \)”

Single Sink Shortest Paths “Given \( u \in V \), return the shortest path from \( u \) to the fixed vertex \( t \)”

s-t Shortest Paths “Return the shortest path from the fixed vertex \( s \) to the fixed vertex \( t \)”

Note 2. A variation of the shortest path problem only requires the algorithm to return the length of the shortest path, instead of the actual path.

Definition 3 (Strongly Connected Component Problem). Given a directed graph \( G = (V, E) \), the strongly connected component problem requires an algorithm to answer the question:

“Given \( u, v \in V \) are \( u \) and \( v \) in the same strongly connected component?”

where a set of vertices \( v_1, \ldots, v_k \) are said to form a strongly connected component \( C \) if for all \( v_i, v_j \in C \) there is a path from \( v_i \) to \( v_j \) and a path from \( v_j \) to \( v_i \).

Similar source and sink variants can also be defined.

Note 3. If the graph is undirected, two vertices being connected is equivalent to them being in the same strongly connected component.

\[ \text{Fig. 1. An example graph (left) with the corresponding shortest path graph } H_A \text{ from } A \text{ (right).} \]

Several of the algorithms need the notion of a shortest path graph from source \( s \). This is defined below and an example is given in Figure 1.

Definition 4 (Shortest Path Graph). Given a graph \( G \) and a source \( s \), the shortest path graph \( H_s \) is defined as the union of all shortest paths in \( G \) starting from \( s \).

3 ES Trees

One of the foundational pieces of work was the ES tree by Even and Shiloach [33], which solves the decremental connectivity problem, on unweighted, undirected graphs. The algorithm has constant query time and \( O(q + m \cdot n) \) total update time, for \( q \) queries. Hence for \( q > n \), the algorithm outperforms the naive solution of rerunning a static algorithm after each edge deletion (which has total runtime \( O(q \cdot m) \)). This was the first algorithm to beat the naive solution. It has since had many generalisations [15, 39, 49]. We begin by discussing the original algorithm before giving the generalisations.

3.1 Original [33]

In this section we will discuss the algorithm by Even and Shiloach [33], before giving an example of how it behaves on a small graph.

Given a graph \( G = (V, E) \), the algorithm stores an array which states which connected component each vertex is in. This array can be used to answer connectivity queries in \( O(1) \) time and thus all that is required is to show that the array can be maintained in \( O(m \cdot n) \) total time.

The algorithm will construct a shortest path graph \( H \), starting from an arbitrary root vertex \( r \). We construct a distance oracle \( d \) such that \( d(r) = 0 \). Any vertex \( v \) at distance \( i \) from \( r \) is assigned \( d(v) = i \). This can be calculated using a Breadth First Search. \( u \) is said to be a witness of \( v \), if the edge \((u,v)\) is
in $H$ and $d(u) = i - 1$. If there is a component of the graph which is not connected to $r$, choose a vertex $r'$, assign it $d(r') = 1$ by adding an artificial edge $(r, r')$ and continue with the BFS, repeating until the whole graph is within the structure. This can be achieved in $O(m + n)$ time.

When an edge $(u, v)$ is removed, there are two cases: edges $u$ and $v$ continue to belong to the same connected component, or they now belong to different connected components. We will run two processes in parallel, one to deal with each of these cases. Each process is discussed in turn. The processes will 'race', so the first one to finish, will terminate the other one. Thus, we only need to consider the time complexity of the process when it finds the event (if the graph is still connected or not) that it was looking for.

**Process One** checks whether removing $(u, v)$ disconnects the two components and handles this case. It does so by calling two Depth First Searches (DFS) from $u$ and $v$, on the graph $G$. If either DFS finds the other vertex, **Process One** stops because they are still connected. However, if one of the DFSs finish without finding the other vertex, the two vertices have become disconnected. The smaller component (the one whose DFS finished first) is given a new component name in the array. Since each time the smaller component is renamed, using a charging argument on the edges, it can be shown that the total time complexity for this process is $O(m \cdot \log m)$.

**Process Two** handles the second case where removing $(u, v)$ does not disconnect the two components and maintains the shortest path graph $H$. **Process Two** runs in parallel with **Process One** and starts with the assumption that we are in case two. As we will discuss, its actions will be reversed if **Process One** determines that we are in-fact in case one.

If $(u, v) \in G \setminus H$ then $(u, v)$ is simply removed from $G$, hence we only need to consider the case where $(u, v) \in H$. If $d(u) = d(v)$ then removing the edge $(u, v)$ does not change the connected components. Therefore, assume, without loss of generality $d(v) = d(u) + 1$. The function can only be at most one different for $d(u)$ and $d(v)$ since it is defined as the distance to $r$ and there was an edge $(u, v)$. If there is another witness $w$ for $v$, simply remove $(u, v)$ from both $H$ and $G$. We will now consider the case where $u$ was the final witness for $v$.

When $(u, v)$ is removed from $H$, $v$ must increase $d(v)$ by at least one. As a side effect of removing $v$, anything rooted at $v$ will also need to be reinserted into $H$. Starting with $i = d(v)$, repeat the following. For each $w \in V$ with $d[w] = i$ and no incoming edges, remove all outgoing edges from $w$ and increment $d(w)$ by 1. For each $w$ incremented, if there exists a $y$ such that $d(y) = i$ and $(y, w) \in E$, insert the edge into $H$, thus adding $w$ back into $H$. Then increment $i$ and repeat until all vertices have been added back to the tree.

Clearly if removing $(u, v)$ does not disconnect the graph, then this process will terminate. If $(u, v)$ does disconnect the graph, then **Process One** can detect this and cancel **Process Two** which can reset the data structure to its original state and simply mark the edge $(u, v)$ as artificial. If the graph remains connected the algorithm runs in total time $O(m \cdot n)$ - each time an edge is processed one of its ends drops by a level. Since $d(v) < n \forall v \in V$, each edge can be processed at most $O(n)$ times. This gives the desired time complexity.

**Figure 2** gives an example of the construction, on a given graph. The left hand column shows how the graph $G$ changes over a sequence of deletions. The middle column shows $H$ as it corresponds to $G$, when it is created with $A$ as the root vertex. The dashed lines represent edges which are in the original graph but sit in the same level of the $H$. The right hand column shows when only a shortest path tree $T$ is stored instead of the shortest path graph, this will be discussed in more detail below.

**Figure 3** shows how the data structures change upon the deletion of the edge $(A, B)$ – the final edge deletion in Figure 2. In the graph $G$ the edge $(A, B)$ is removed and nothing else changes, hence, the figure focuses on the steps undergone by $H$ and $T$ (which are identical up until the final step). The first step is to remove the edge $(A, B)$ from the data structure. As $B$, which is in level 1, has no edges connecting it to level 0 $(A)$, it is dropped by a level. The data structure $H$ directly has access to this information, while $T$ has to use the adjacency information of $G$. At the next step (looking at level 2), both $B$ and $D$ need to be considered. The vertex $B$ has no edges to the level above and thus drops a level. In $H$, $D$ is connected to level 1 and thus is done, while in $T$ it is not connected, so the adjacency information is checked and the edge $(C, D)$ is added to $T$. Finally, $B$ would be checked in level 3 and is connected to level 2 and thus the process completes.

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3 Here parallel refers to running the two processes in an interleaved fashion.

4 **Process Two** is left to finish, since it must maintain its internal data structure.

5 **Process One** will not succeed and thus will not be discussed here.
Fig. 2. An example of the data structures under the following series of deletions: \((A, D), (B, C), (A, B)\). The left hand column represents the original graph \(G\), the middle column is the data structure \(H\), and the right hand column is the data structure \(T\) which utilises the reduced memory technique.

3.2 Generalisation

In this section we discuss how the above can be generalised to other forms of graph or problem.

**Directed graph** A similar approach can be taken for single source transitivity \[39\]. We now only look at the connected component the source \(s\) is in and construct the ES tree using it as the root. The ES tree will now only contain vertices in the same connected component as \(s\) and will not contain any artificial edges. Other than this minor modification the algorithm behaves as previously described.

Single sink transitivity problems can be answered in a similar manner, by reversing the direction of all edges before constructing the ES tree. Transitivity for arbitrary vertex pairs can be answered by storing an ES tree per vertex.

We use \(\text{out}(x)\) to denote the ES tree with root \(x\) (single source) and \(\text{in}(x)\) to denote the ES tree with root \(x\) on the graph with all the edge directions reversed (single sink).

**Shortest path** To answer single source shortest path problems \[39\], an ES tree can be created using the source \(s\) as the root of the tree. However, the ES tree will only be constructed for the connected component \(s\) is in, it will not contain any artificial edges or vertices not in the same connected component. The distance between \(s\) and a queried vertex \(u\) is then simply \(d(u)\) and the path can be constructed using the ES tree.

Single sink shortest path problems can be answered in a similar manner, by reversing the direction of all edges before constructing the ES tree. All pairs shortest paths can be answered by storing an ES tree per vertex.
Fig. 3. An example of the steps taken by the algorithm when the edge \((A, B)\) is deleted from the example graph. The left hand column represents the data structure \(H\) and the right hand column is the data structure \(T\) which utilises the reduced memory technique.

**Weighted graph** The above descriptions can be adjusted to deal with weighted graphs \([19]\). Again we will focus on a single connected component and can boost the result by storing multiple ES trees. The function \(d\) now represents the distance of a vertex from the root. Firstly we will only store the graph up to a given depth \(\Delta\). This will result in a total update time complexity of \(O(m \cdot \Delta)\). Note, when \(\Delta = n\) we get back the original result.

Firstly we now may not have that \(d(v) = d(u) + 1\) since \(\ell(u, v)\) might be greater than 1. In fact, we will have \(d(v) = d(u) + \ell(u, v)\), for \((u, v) \in E\). Therefore, when a vertex \(w\) is moved down from \(i\) to layer \(i + 1\) we won’t be looking at edges that have the other end in layer \(i\) but we will be looking for a \(y\) such that \(y \in H, (y, w) \in E\) and seeing if \(d(y) + \ell(y, w) = i + 1\). Note that this does not change the time complexity. It is straightforward to see the result still holds.

We use \(\text{out}(x, \Delta)\) to represent the ES tree when the distance is restricted to \(\Delta\), \(\text{in}(x, \Delta)\) is defined similarly.

**Limited Insertions** While ES trees do not provide a fully dynamic algorithm, Bernstein and Roditty \([15]\) show how the algorithm can handle a very specific form of insertions. If the insertion of an edge \((u, v)\) does not decrease the distance between the root of the tree and \(v\), then it can be supported by the ES Tree. The total update time is now \(O(m' \cdot \Delta + d)\) where \(m'\) is the maximum number of edges ever in \(G\) and \(d\) is the total number of edge changes. The \(O(d)\) arises because \(O(1)\) work must be spent per edge change. This property will become useful in Sect. 4.1 to construct a more efficient decremental shortest path algorithm.

This can also be used to increase the weight of a given edge, by inserting the edge, with the new weight, and then deleting the original edge from the ES tree \([19]\).
3.3 Reduced Memory

King and Thorup [50] show how to reduce the space for ES trees, as well as several other algorithms [23, 49]. The technique allows the memory (beyond the input) to be reduced from \( O(m) \) to \( O(n) \). Firstly assume an ordering on the vertices. With such an ordering in place, the algorithm can be tweaked as follows.

Instead of storing all of \( H \), the edge \((x, v)\) is stored such that \( x \) is first in the vertex ordering of all vertices with the property that \((x, v) \in H\). Therefore, instead of storing all of \( H \) we are storing a tree \( T \), which has memory requirement \( O(n) \). It just remains to show that the tree can be updated without changing the time complexity.

An edge deletion \((u, v)\) will only matter if it was in \( T \), if not it is simply removed from \( G \). If \((u, v) \in T\) a new edge must be found, the edges leading to \( v \) can be scanned, in order, starting at \( u \), stopping if \((x, v)\) is found such that \( d(x) + \ell(x, v) = d(v)\) and adding \((x, v) \) to \( T \). If not the value of \( d(v) \) is incremented. As before, each edge into \( v \) is only considered once for each value of \( d(v) \). Hence, the running time remains the same.

Several other works have also given space saving techniques [16, 23] but these only allow the distance of the shortest path to be given and can not produce the path. Thus they will not be discussed in detail here.

4 Algorithms Built upon ES Trees

In this section we describe some of the algorithms which use ES trees as a building block to solve the discussed dynamic algorithms. While there are a host more, such as [12, 14, 43], we discuss a subset which made significant progress or contain interesting ideas.

4.1 Approximate Decremental Single Source Shortest Paths [13]

In this section we discuss an approximate algorithm for the decremental single source shortest path problem, for unweighted and undirected graphs, by Bernstein and Chechik [13]. This is the first deterministic algorithm which manages to have a total update time better than the \( O(m \cdot n) \) of ES trees. The algorithm is a \((1 + \epsilon)\) approximation with a total update time of \( \tilde{O}(n^2) \).

Here we will formally describe an algorithm that gives a time complexity of \( O(n^{2.5}) \) and then discuss how to improve this to the stated bound.

A vertex \( v \in V \) is called heavy if it has degree at least \( \sqrt{n} \) and light otherwise. A path between two vertices can contain at most \( 5\sqrt{n} \) vertices - intuitively, no two heavy vertices can share a common neighbour else there would be a shorter path. Since a heavy vertex has \( \sqrt{n} \) neighbours the result follows. This will be discussed formally below.

The algorithm works by storing two ES trees one which will store short paths and one which will approximate the long paths. For short paths we will simply create a ES tree on the original graph from source \( s \) up to depth \( 5\sqrt{n}e^{-1} \). The remainder of this explanation will discuss long paths.

Let \( H \) be the set of heavy vertices in \( G \). The auxiliary graph \( G' \) is the graph with an additional vertex \( c \) per connected component in \( G[H] \) which is connected to each vertex in the connected component with weight \( 1/2 \). The light vertices then have all their edges added to the graph. The graph \( G' \) has at most \( n^{1.5} \) edges; 1 per heavy vertex and at most \( \sqrt{n} \) per light vertex. We will now show that this graph provides the following bounds:

\[
\text{dist}_{G'}(s, v) \leq \text{dist}_G(s, v) \leq \text{dist}_{G'}(s, v) + 5\sqrt{n}
\]

The lowerbound follows from the observation that given an \( s-t \) path in \( G \), we can construct an \( s-t \) path in \( G' \) as follows. For each \((u, v)\) if either \( u \) or \( v \) are light then \((u, v) \in E' \) and can be added to the path. If \((u, v)\) are both heavy then they must be in the same connected component so \((u, c), (c, v)\) can be added to the path. Since these two edges have weight half the pair have the same weight as the original path. Thus the path in \( G' \) has the same weight as the path in \( G \). Note that this path may not be simple.

For the upper bound let \( L \) be the set of light vertices on the shortest path in \( G' \) and let \( X \) be the set of non light edges on the path (so either heavy or a center \( c \)). We want to show that \( |X| < 5\sqrt{n} \), therefore showing that ignoring the heavy edges doesn’t cost too much. Let \( Y \) be every 5th element of \( X \), therefore \( |Y| \geq \frac{|X|}{5} \). We know that \( \text{Ball}(X, v, 2) \geq \sqrt{n} \) for \( v \in Y \), since either \( v \) is heavy or adjacent
to a heavy vertex. For \( v, w \in Y \) we know \( Ball(X, v, 2) \cap Ball(X, w, 2) = \emptyset \) otherwise there would be a shorter path (since \( v \) and \( w \) are at least distance 5 away from each other on the path). Thus we know \( \left| \bigcup_{v \in Y} Ball(X, v, 2) \right| \geq \sqrt{n}Y \) but since the graph contains at most \( n \) vertices we get the desired result.

An ES tree can be stored for the original graph \( G \) up to distance \( 5\sqrt{ne^{-1}} \) to respond to short edge queries, while the ES tree up to distance \( n \) on \( G' \) allows us to respond to longer path queries. Both run in time \( O(n^{3.5}) \), the first due to its bounded depth and the second because it is sparse.

We now need to show that the distances can be maintained under edge deletions. Any edge incident to a light vertex is easy to maintain as it is in the original graph. Deleting an edge can cause a vertex to go from heavy to light (but this can only happen once). When this happens, all of its edges must be added to the auxiliary graph. The slightly trickier case is when an edge is deleted between two heavy vertices. A data structure which maintains connectivity information in dynamic graphs can be used to maintain the auxiliary graph. When edge \((u, v)\) is deleted it must be checked that \((u, v)\) are still in the same connected component. If yes nothing changes. Else these two edges now need to be connected to different centers \( c_u, c_v \) instead of the same center \( c \). This is done by choosing the smallest center and moving all vertices adjacent to it over to a newly created center. Hence, the graph can be maintained.

To reduce the time complexity from \( \tilde{O}(n^{2.5}) \) to \( \tilde{O}(n^2) \), instead of having two ES trees (a “heavy one” and a “light one”), \( \tilde{O}(\log n) \) heaviness thresholds can be used to handle \( O(\log n) \) ranges of distance queries and returning the minimum of the \( O(\log n) \) queries.

The algorithm can be trivially converted to the incremental setting, with the same time complexity.

### 4.2 Fully Dynamic Transitivity [39]

Henzinger and King [39] give the first fully dynamic transitive close algorithm, along with a decremental algorithm. Both are Monte Carlo algorithms. The fully dynamic algorithm has either; query time \( \tilde{O}(\frac{n}{\log n}) \) and update time \( \tilde{O}(\bar{m} \cdot \sqrt{n} \cdot \log^2 n + n) \), or query time \( \tilde{O}(\frac{n}{\log n}) \) and update time \( \tilde{O}(n \cdot \bar{m} \cdot \frac{\sqrt{\log n}}{\mu} \cdot \log^2 n) \) where \( \bar{m} \) is the average number of edges in the graph and \( \mu \) is the exponent for matrix multiplication. Note that, unlike the other algorithms given, these algorithms do not have a constant query time.

The deletions only algorithm takes in a user defined parameter \( r \) and for \( i = 1, \ldots, \log r \) stores a set of \( \min(2^r \cdot \log n, n) \) distinguished vertices \( S_i \). For each distinguished vertex \( x \) maintain ES trees \( \text{in}(x, \frac{x}{2^r}) \) and \( \text{out}(x, \frac{x}{2^r}) \), where \( x \in S_i \). Then \( \text{out}(x) \) can be defined as the union of all \( \text{out}(x, \frac{x}{2^r}) \) where \( x \in S_i \), with \( \text{in}(x) \) being defined similarly. For each vertex \( v \in V \) also maintain \( \text{in}(v, \frac{v}{2^r}) \) and \( \text{out}(v, \frac{v}{2^r}) \).

Given a query \((u, v)\), test if \( v \in \text{in}(u, \frac{u}{2^r}) \), if yes return true. Else see if there exists a distinguished vertex \( x \) such that \( u \in \text{in}(x, \frac{x}{2^r}) \) and \( v \in \text{out}(x, \frac{x}{2^r}) \). If this is the case answer yes, else answer no.

If the path is of length less than \( \frac{x}{2^r} \) then the answer will always be correct, otherwise it will be correct with high probability.

The two fully dynamic algorithms use similar techniques of using the deletion only data structure discussed above and suitably keeping track of inserted edges. Thus only one of the two will be discussed here. The intuition is that you store the deletion only data structure, for \( r = \frac{n}{\log n} \), along with storing \( \text{in}(x, n) \) and \( \text{out}(x, n) \) each time an edge is inserted. These extra structures, along with the deletion only data structure are updated each time an edge is removed. To answer a query \((u, v)\) check the deletion only data structure and check if \( u \in \text{in}(x, n) \) and \( v \in \text{out}(x, n) \) for every newly inserted edge. After every \( \sqrt{n} \) updates to the graph the deletion only data structure is rebuilt. This gives the desired result.

### 4.3 Fully Dynamic All Pairs Shortest Paths and Transitivity [49]

King [49] gives the first fully dynamic algorithms for all pairs shortest paths in directed graphs with positive integer weights less than \( b \). Three algorithms are given; a \((2 + \epsilon)\) approximation, a \((1 + \epsilon)\) approximation and an exact algorithm with amortized update times; \( \tilde{O}(\frac{n^2 \log^2 n}{\log \log n}) \), \( \tilde{O}(\frac{2n^2 \log^3 (kn)}{\epsilon^2}) \) and \( \tilde{O}(n^{2.5} \sqrt{b \cdot \log n}) \) respectively. They also give a fully dynamic transitive closure algorithm with update time \( \tilde{O}(n^2 \cdot \log n) \). The update times are amortized over \( \tilde{O}(\frac{n}{\mu}) \) operations.

Here we give the transitive closure algorithm, since the others follow a similar strategy. The algorithm works by keeping \( k = \lceil \ln n \rceil \) forests \( F^1, \ldots, F^k \) where each \( F_i \) contains \( \text{in}^i(v, 2) \) and \( \text{out}^i(v, 2) \) for each \( v \in V \). The count between two vertices \( \text{count}^i(u, w) \) is the number of vertices \( v \) such that \( u \in \text{in}^i(v, 2) \) and \( w \in \text{out}^i(v, 2) \). The list \( \text{list}^i(u, w) \) contains all such vertices. The forests (and the graphs they are built upon) are defined recursively. The edges \( E' = \{(u, w) \mid \text{count}^{i-1}(u, v) > 0 \} \) and then the forests are the in and out ES trees on top of this. Inserting an edge requires adding it at the bottom layer and adjusting the edge sets and forests as required. Deleting the edge requires deleting it from the lowest layer and
the recursively deleting edges in higher layers where the count has gone from positive to zero (i.e. where there is no longer a path). It follows that \( u, w \in V \) are connected in \( G \) if and only if \( \text{count}^k(u, w) > 0 \), resulting in a transitive closure algorithm.

### 4.4 Decremental Single Source Shortest Paths

Henzinger et al. [41] present a near linear algorithm for \((1 + \epsilon)\) decremental single-source shortest path for unweighted graphs, with total update time \( \mathcal{O}(m^{1+o(1)} \log \epsilon^{-1}) \). The algorithm works by maintaining a sparse \( (d, \epsilon)\)-hop set (introduced by Cohen [19]). This allows the distance between any two vertices to be \((1 + \epsilon)\)-approximated using at most \( d \) edges. To maintain the hop set, under deletions, the authors introduce a *monotone bounded-hop Even Shiloach tree*.

The high-level idea of the algorithm is to create a hop set and then defining the shortcut graph as the original graph plus the edges from the hop set. This process is the repeated on the resulting graph up until a suitable depth. Intuitively this works because while the graph gains more edges, the number of hops is being constrained. The final algorithm is slightly more complex, where it has to contain active and inactive vertices. Inactive vertices are those which it would be too expensive to maintain an ES-tree for but the authors show that not constructing these trees will not change the result. See the paper for the formal definition of active and inactive vertices.

### 4.5 Decremental Single Source Shortest Paths

Henzinger et al. [42] give a \((1 + \epsilon)\)-approximation decremental single-source shortest path algorithm for directed graphs with total update time \( \mathcal{O}(m \cdot n^{0.984}) \) and constant update time. Here we will describe the \( s-t \) reachability algorithm, and the authors show how it can be extended to single source reachability and shortest paths.

Given a set of vertices \( H \) called a hub, the hub distance between \( u \) and \( v \) is the shortest distance between \( u \) and \( v \) such that one of the vertices in the path belongs to \( H \). A path between \( u \) and \( v \) is called a \( h \)-hop if it contains at most \( h \) edges. The \( h \)-hop \( u-v \) path union graph is the graph created by taking the union of all the \( h \)-hops between \( u \) and \( v \). The goal of the algorithm is to maintain reachability while \( \text{dist}(s, t) \leq h \) for some parameter \( h \). The algorithm then maintains the hub distance between \( s \) and \( t \) while the hub distance is less than \( h \) and maintains the distance between \( s \) and \( t \) in the path union graph when the distance becomes greater than \( h \).

For each \( v \in H \) \( \text{dist}(s, v) \) and \( \text{dist}(v, t) \) are maintained using ES-trees of depth \( h \). Maintaining the distance between \( s \) and \( t \) in the path union graph varies depending on the algorithm given in the paper. The authors give algorithms for sparse graphs, dense graphs and other graphs. One way to do this is maintaining an ES tree over the path union graph.

The authors later improved upon this [43] for the single source reachability case, with total update time \( \mathcal{O}(m \cdot n^{0.98+o(1)}) \). The algorithm works by extending to multiple layers of path union graphs and hubs from the previous algorithm.

### 5 Algorithms Using Alternate Techniques

In this section we discuss some other algorithms which do not utilise ES trees. While there is a host of work, such as [28],[58],[60], we focus on a particular subset which contain interesting concepts.

#### 5.1 Approximate Decremental All Pairs Shortest Paths

Bernstein and Roditty [15] give the first decremental algorithms to beat the \( \mathcal{O}(m \cdot n) \) update time of ES trees. These algorithms beat the time complexity only when the graph is “not too sparse”. They present two algorithms for shortest path problems on unweighted and undirected graphs. Both of the algorithms are randomised.

The first algorithm is a \((1 + \epsilon)\) approximation algorithm for single source shortest path, with an expected total update time of \( \tilde{\mathcal{O}}(n^{2+O(\sqrt{\frac{1}{\epsilon} \cdot \log n})}) \). The second algorithm returns all pairs shortest distances with total expected update time of \( \tilde{\mathcal{O}}(n^{2+\frac{2}{k}+O(\sqrt{\frac{1}{\epsilon} \cdot \log n})}) \) for any fixed integer \( k \) and the stretch of the returned distances is at most \( 2 \cdot k - 1 + \epsilon \).

Both algorithms are created using a similar technique. Existing algorithms are run on a sparse subgraph of the graph (e.g. spanner or emulator) which is being queried, instead of the graph itself. This
technique has been used previously to construct more efficient static algorithms. However, it is more complex in the decremental only setting. The issue arises, in that a deletion from the original graph, can cause an insertion into the emulator. The authors resolve the issue by showing that the insertions will be limited and “suitably well behaved”. Given this they are able to show that existing decremental algorithms can support the required insertions, giving the desired result.

5.2 Dynamic All Pairs Shortest Paths [24]

Demetrescu and Italiano [24] give a deterministic, fully dynamic algorithm for APSP on directed graphs with non-negative real-valued edge weights. The algorithm has amortized update time $O(n^2 \cdot \log^3 n)$ and worst case constant time query time. We begin by discussing the algorithm which can only increase the weight edges and has amortized update time $O(n^2 \cdot \log n)$. This was the first algorithm to do better than simple recomputation.

An important definition for this problem is that of a local shortest path. A local shortest path is a path where all proper subpaths are shortest paths. Note this does not require that the path itself is a shortest path. The algorithm works by storing priority queues $P_{x,y}$ of locally shortest paths between $x$ and $y$. Distance queries are answered by returning elements from the correct priority queue. It remains to show that the priority queues can be updated efficiently.

The paper shows that a graph can have at most $m \cdot n$ locally shortest paths (assuming unique shortest paths) and that at most $O(n^2)$ paths can stop being locally shortest per edge weight increase. Using this result the data structures can be maintained in the claimed time.

To upgrade the algorithm to be fully dynamic in the claimed time, the authors use historically shortest paths. At current time $t$ such that the given path had last been updated at time $t' \leq t$, a path is called historical if it has been a shortest path in the interval $[t',t]$. A path is locally historical if it contains a single vertex or all proper subpaths are historical. The algorithm has an additional step to keep the number of historical paths at a reasonable level. If there are either too few or too many, it slows the algorithm. Otherwise the algorithms behaves in a similar manner.

5.3 Decremental Strongly Connected Components [52]

Łącki [52] gives a deterministic algorithm for strongly connected components, with total update time $O(m \cdot n)$. The algorithm works by reducing the problem to solving connectivity in a set of directed acyclic graphs. The algorithm begins by removing all vertices that are not reachable from the given source $s$, which takes $O(mn)$ time. When an edge $(u,v) \in E$ is deleted, the graph becomes disconnected if $v$ loses its last in edge. Not only may this cause $v$ to become disconnected, it may also cause children of $v$ to become disconnected. The algorithm works using this observation. It starts by taking the vertex $v$ and if it is has no other incoming edges it declares that $v$ has become disconnected. If $v$ has become disconnected, it must recur on the all children $x$ of $v$, removing the edge $(v,x)$ since that no longer connects $x$ to the source. The runtime is linear in the number of newly disconnected vertices and their incident edges. A similar algorithm can be given for the reachability of a sink by reversing all the edges.

The strongly connected components algorithm works by splitting a vertex $d$ into two $d_1$ and $d_2$ where all edges $(u,d) \in E$ get replaced with $(u, d_1)$ and edges $(d,v) \in E$ get replaced with $(d_2, v)$. Then the condensation of the resulting graph is calculated. This graph is denoted $G_d$. This graph is a DAG and therefore reachability from $d_2$ and to $d_1$ can be maintained using the above algorithm. As soon as one of the reachability sets is different from the vertex set then the graph has stopped being strongly connected. To handle deletion of edges not in $G_d$ this must be applied recursively to each strongly connected component, this is stored as a tree.

Roddity [62] improves on the preprocessing and worst case update time of the above algorithm to $O(m \cdot \log n)$, without changing the query time or total update time. This is achieved by giving a new preprocessing algorithm which generates the tree containing the strongly connected components in $O(m \cdot \log n)$. The update function is the original one by Łącki, with the adjustment that if it takes over $O(m \cdot \log n)$ time then the process is terminated and the tree is simply built again from scratch.

6 Limitations

In this section we discuss work that attempts to give lowerbounds on the complexity of these algorithms [2] [15] [51] [59]. They achieve this by giving a reduction to a well studied problem. If these dynamic problems can be solved “too efficiently” then it would result in a more efficient algorithm for a problem which
is well studied and conjectured that no algorithm can beat a certain threshold. Here we detail the result of Henzinger et al. [45] who use online boolean matrix-vector multiplication as their underlying problem. This work is for undirected and unweighted graphs. We chose to discuss the work of Henzinger et al. because lots of the results given subsume previously known results [2,51,59].

We begin by defining the online boolean matrix-vector multiplication problem that they reduce all the problems to, to construct lower bounds.

**Definition 5 (Online Boolean Matrix-Vector Multiplication Problem (OMv)).** Given an integer $n$ and Boolean matrix $M$ answer the following question over $n$ rounds: Given an $n$-dimensional column vector $v_i$ compute $Mv_i$. The algorithm must output the result before being given the next column vector.

The OMv problem was chosen because it has been well studied [54,57,60,67,71]. It is widely believed that (up to logarithmic factors) the OMv problem can not be solved more efficiently than $O(n^3)$. This led to the formalisation of the OMv conjecture which is given below.

**Definition 6 (Online Boolean Matrix-Vector Multiplication Conjecture).** For any constant $\epsilon > 0$, there is no $O(n^{3-\epsilon})$ time algorithm that solves OMv with an error probability of at most $\frac{1}{3}$.

For the remainder of the paper, the authors construct lowerbounds for a host of problems by assuming the OMv conjecture to be true. We will describe the lowerbounds for the problems we are interested in below. However, many more lowerbounds are given, including bounds for: triangle detection, connectivity, Langerman’s problem, Erickson’s problem, approximate diameter and densest subgraph.

**Theorem 1 (Corollary 4.2).** For any $n$ and $m = O(n^2)$ unless the OMv conjecture fails, there is no partially dynamic algorithm for $s$-$t$ shortest path with polynomial preprocessing time, total update time $u(m) = \tilde{O}(n^2)$ and query time $q(m) = \tilde{O}(n)$ that has error probability of at most $\frac{1}{3}$.

This theorem, in particular, shows why it was many years for an improvement to be given to the algorithm of Even and Shiloach [33]. If $m \in O(n^2)$ then the two bounds are matching. This shows why it is important to consider approximate algorithms instead of exact algorithms. Since the result is for exact algorithms, considering approximate algorithms could be a way to skirt around this. However, Henzinger et al. also provide some bounds for approximate algorithms.

**Theorem 2 (Corollary 3.10).** Unless the OMv conjecture fails, there is no partially dynamic algorithm for $(3-\epsilon)$-$s$-$t$ shortest path with preprocessing time $p(n) = \text{poly}(n)$, worst update time $u(m) = \tilde{O}(\sqrt{n})$ and query time $q(m) = \tilde{O}(n)$ that has error probability of at most $\frac{1}{4}$.

**Note 4.** The theorem also holds for fully dynamic algorithms with amortized update time.

We conclude this section by giving a few more results which are also of relevance.

**Theorem 3 (Corollary 3.4).** For any $n$ and $m \leq n^2$ unless the OMv conjecture fails, there is no partially dynamic algorithm for Strongly Connected Components or Transitivity with preprocessing time $p(m) = \text{poly}(m)$, worst update time $u(m) = \tilde{O}(\sqrt{m})$ and query time $q(m) = \tilde{O}(m)$ that has error probability of at most $\frac{1}{4}$.

**Theorem 4 (Corollary 4.8).** For any $n$ and $m = \Theta(n^{\frac{1}{2}})$, and constant $\delta \in (0, \frac{1}{2}]$, unless the OMv conjecture fails, there is no partially dynamic algorithm for the problems listed below with preprocessing time $p(m) = \text{poly}(m)$, total update time $u(m) = \tilde{O}(m \cdot n)$ and query time $q(m) = \tilde{O}(m^3)$ that has error probability of at most $\frac{1}{4}$. The problems are:

- Single Source Shortest Path
- All Pairs Shortest Path (2 vs 4)
- Transitive closure

The 2 vs 4 variant of the APSP problem requires you to distinguish if the shortest path between the given vertices is less than, or equal to, two or if it is greater than, or equal to, four. This shows that even decision variants of these problems can’t be solved “too efficiently”.

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7 Conclusion and Open Problems

In this paper we survey, both the foundational and state of the art, algorithms for computing shortest paths and connectivity on graphs which are constantly changing. Table 1 summarises the time complexities of all the algorithms discussed in this paper, while Table 2 gives the known lower bounds. We now provide a brief conclusion and describe some of the open problems.

Derandomisation Currently the randomised algorithms tend to perform better than the deterministic algorithms. However, as discussed in the introduction randomised algorithms require a weaker adversary, who does not know the randomness and can not make adaptive queries. For real world scenarios, it is important to be able to remove these restrictions. This makes deterministic algorithms more desirable, since they do not have these restrictions. Hence, an important open question is if the randomised algorithms can be derandomised or if deterministic algorithms can be constructed with the same time complexities as their randomised counterparts.

Memory To have a constant query time for all pairs shortest path and transitive closure, at least $\Omega(n^2)$ memory is required for the lookup table. For connectivity and single source/sink shortest path only $\Omega(n)$ memory is required. However, lots of the dynamic algorithms require more space than this. Thus an important question is if existing algorithms can have their memory reduced to the lowerbound or if new algorithms can be designed meeting this bound. As mentioned above, there has been work moving algorithms in this direction [16, 23, 50].

Lower Bounds It is an important question to try and prove the online matrix vector multiplication conjecture, or equivalently conjectures from other lowerbound work. Since the lower bound techniques tend to work for both incremental and decremental algorithms, it is an interesting question to see if these can be bounded separately, to get tighter bounds. For example incremental single source connectivity can be solved in time $O(m \cdot \alpha(n))$ while the best known result for the decremental setting is $O(m \cdot n^{0.98+\omega(1)})$. In relation to the derandomisation question above, to show a seperation between deterministic and randomised algorithms, would require applying new techniques which only hold for the deterministic setting.

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All individual work posed their own open questions but here we try to focus on themes more than individual questions.
| Algorithm | Dynamic | \(u(m,n)\) | \(g(m,n)\) | Weighted | Directed | Time Complexity | Approximation |
|----------|---------|-----------|-----------|----------|----------|----------------|--------------|
| **Transitive Closure** | | | | | | | |
| Dec | \(m \cdot n^2\) | 1 | \(\times\) | \(\times\) | Total | | N/A |
| Fully | \(m\sqrt{n \cdot \log^2 n + n}\) | \(\frac{n \log n}{\log^2 n}\) | \(\times\) | \(\times\) | Amortized | | |
| Fully | \(n \cdot m \cdot \log^2 n\) | \(\frac{n \log n}{\log^2 n}\) | \(\times\) | \(\times\) | Amortized | | |
| Fully | \(n \cdot \log n\) | 1 | \(\underline{\times}\) | \(\checkmark\) | Amortized | | |
| **Strongly Connected Components** | | | | | | | |
| Dec | \(m \cdot n\) | 1 | \(\checkmark\) | \(\checkmark\) | Total | | N/A |
| Dec | \(m \cdot log n\) | 1 | \(\checkmark\) | \(\checkmark\) | Worst case (Total as above) | | |
| **Single Source Shortest Path** | | | | | | | |
| Dec | \(m \cdot n\) | 1 | \(\times\) | \(\times\) | Total | 1 | |
| Dec | \(n^{3+\epsilon}\) | 1 | \(\times\) | \(\times\) | Total | \(1 + \epsilon\) | |
| Dec | \(m \cdot n^{3+\epsilon}\) | 1 | \(Z^+_{\frac{3}{2}}\) | \(\checkmark\) | Amortized over \(\frac{m^{3/2}}{n}\) | 2 + \(\epsilon\) | |
| Dec | \(m \cdot n^{3+\epsilon}\) | 1 | \(Z^+_{\frac{3}{2}}\) | \(\checkmark\) | Amortized over \(\frac{m^{3/2}}{n}\) | 1 + \(\epsilon\) | |
| Dec | \(n \cdot n^{3+\epsilon}\) | 1 | \(Z^+_{\frac{3}{2}}\) | \(\checkmark\) | Amortized over \(\frac{m^{3/2}}{n}\) | 1 | |
| Dec | \(n^{3+\epsilon+\frac{1}{2}}\) | 1 | \(\times\) | \(\times\) | Total | \(2 \cdot k - 1 + \epsilon\) | |
| Dec | \(n^{3+\epsilon}\) | 1 | \(\underline{\times}\) | \(\checkmark\) | Amortized | 1 | |
| Dec | \(m^{3+\epsilon}\) | 1 | \(\times\) | \(\times\) | Amortized | 1 + \(\epsilon\) | |
| Dec | \(m \cdot n\) | 1 | \(\times\) | \(\times\) | Expected | 1 + \(\epsilon\) | |
| Dec | \(n^{3+\epsilon}\) | 1 | \(\times\) | \(\times\) | Amortized | 3 | |
| Dec | \(n^{3+\epsilon}\) | 1 | \(\times\) | \(\times\) | Amortized | 5 | |

**Table 1.** A summary of known upper bounds for various problems.

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Table 2. A summary of known lower bounds for various problems.

| Problem                        | \( p(m, n) \) | \( u(m, n) \) | \( q(m, n) \) | Assumption       | Remark | Citation |
|-------------------------------|----------------|----------------|----------------|------------------|--------|---------|
| Transitive Closure            | \((m-n)^{1+\varepsilon}\) | \((m-n)^{1-\varepsilon}\) | \(m^{1+\varepsilon}\) | BMM             | \(\delta \in (0, \frac{1}{2}], m = \Theta(n^{1+\varepsilon})\) | 43     |
|                               | poly           | \((m-n)^{1+\varepsilon}\) | \(m^{1+\varepsilon}\) | OMv             | \(\delta \in (0, \frac{1}{2}], m = \Theta(n^{1+\varepsilon})\) | 44     |
| Strongly Connected Components  | poly           | \((m-n)^{1-\varepsilon}\) | \(m^{1-\varepsilon}\) | OMv             | \(\delta \in (0, \frac{1}{2}], m = \Theta(n^{1+\varepsilon})\) | 45     |
| Single Source Shortest Path   | \((m-n)^{1+\varepsilon}\) | \((m-n)^{1-\varepsilon}\) | \(m^{1+\varepsilon}\) | BMM             | \(\delta \in (0, \frac{1}{2}], m = \Theta(n^{1+\varepsilon})\) | 46     |
|                               | poly           | \((m-n)^{1+\varepsilon}\) | \(m^{1+\varepsilon}\) | OMv             | \(\delta \in (0, \frac{1}{2}], m = \Theta(n^{1+\varepsilon})\) | 45     |
| All Pairs Shortest Path       | \((m-n)^{1+\varepsilon}\) | \((m-n)^{1-\varepsilon}\) | \(m^{1+\varepsilon}\) | BMM             | \(\delta \in (0, \frac{1}{2}], m = \Theta(n^{1+\varepsilon})\) | 43     |
|                               | poly           | \((m-n)^{1+\varepsilon}\) | \(m^{1+\varepsilon}\) | OMv             | \(\delta \in (0, \frac{1}{2}], m = \Theta(n^{1+\varepsilon})\) | 45     |

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