HAANTJES ALGEBRAS OF THE LAGRANGE TOP

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Abstract. A symplectic–Haantjes manifold and a Poisson–Haantjes manifold for the Lagrange top are studied and a set of Darboux-Haantjes coordinates are computed. Such coordinates are separation variables for the associated Hamilton-Jacobi equation.

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1. Introduction

In this article, we propose a new theoretical framework for dealing with integrable mechanical systems, and we illustrate it on the classical example of the Lagrange top. Precisely, we discuss such Hamiltonian system in the framework of the theory of Haantjes algebras, very recently introduced in [27]. To this aim, the new geometric notion of Poisson–Haantjes (PH) manifold is proposed, as a natural extension of the notion of symplectic-Haantjes (ωH) manifold, already introduced in [26]. The main idea is to replace the symplectic structure of a ωH manifold with a Poisson (not invertible) bivector that fulfills a suitable algebraic compatibility condition with a Haantjes algebra of operators on the tangent bundle of the manifold. Besides, the dynamical notion of Magri-Haantjes chain [26] is generalized, to be adapted to the novel geometrical notion of PH manifold. These new structures can be used to describe bi-Hamiltonian chains of vector fields for Gelfand-Zakarevich (GZ) manifolds [10]. When such structures can be reduced to the symplectic leaves of the Poisson bivector, a ωH structure is obtained and a set of Darboux-Haantjes coordinates can be computed. Such coordinates are separation variables for the Hamilton-Jacobi equation of the Hamiltonian systems belonging to a Magri–Haantjes chain. Here, we detail the Lagrange top, that is a GZ system of corank 2, whilst the discussion about the stationary flows of the KdV hierarchy (a GZ system of corank 1) will appear elsewhere.
The notion of $PH$ structures, inspired, from one side, by the theory of Poisson–Nijenhuis structures [18, 13], from the other side by the notion of Haantjes manifolds [14, 15, 16, 17]. In our opinion, the new theory provides us with a very flexible and unifying theoretical framework for dealing with the integrability and separability properties of Hamiltonian systems, and represents a formulation that completes the one offered by the Poisson–Nijenhuis geometry.

The paper is organized as follows. After a review, in Section 2, of the main algebraic structures needed in this work, we recall in Section 3 the concept of Haantjes algebras. In Section 4, we present the new notion of Poisson–Haantjes manifolds and of the related Magri–Haantjes chains. In Section 5, we apply the theory to the real and complex Lagrange top.

2. Nijenhuis and Haantjes torsion

The natural frames $(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$ of local charts $(x_1, \ldots, x_n)$ in a differentiable manifold, being obviously integrable, can be characterized in a tensorial manner as eigen-distributions of a suitable class of $(1,1)$ tensor fields, i.e., the ones with vanishing Haantjes tensor. In this section, we review some basic results concerning the theory of such tensors. For a more complete treatment, see the original papers [11, 24], the related ones [25, 9], and the nice recent review [12].

Let $M$ be a differentiable manifold and $L: TM \to TM$ be a $(1,1)$ tensor field, i.e., a field of linear operators on the tangent space at each point of $M$.

**Definition 1.** The Nijenhuis torsion of $L$ is the skew-symmetric $(1,2)$ tensor field defined by

\[(1) \quad T_L(X, Y) := L^2[X, Y] + L[X, LY] - L([X, LY] + [LY, X]),\]

where $X, Y \in TM$ and $[ , ]$ denotes the commutator of two vector fields.

In local coordinates $x = (x_1, \ldots, x_n)$, the Nijenhuis torsion can be written in the form

\[(2) \quad (T_L)_{ij}^k = \sum_{\alpha=1}^{n} \left( \frac{\partial L_i^\alpha}{\partial x^j} L_j^{\alpha} - \frac{\partial L_i^\alpha}{\partial x^k} L_k^{\alpha} + \frac{\partial L_i^\alpha}{\partial x^\beta} L_j^{\alpha} - \frac{\partial L_i^\alpha}{\partial x^\beta} L_j^{\alpha} \right),\]

amounting to $n^2(n - 1)/2$ independent components.

**Definition 2.** The Haantjes torsion associated with $L$ is the $(1,2)$ tensor field defined by

\[(3) \quad H_L(X, Y) := L^2T_L(X, Y) + T_L(LX, LY) - L\left( T_L(X, LY) + T_L(LX, Y) \right).\]

The skew-symmetry of the Nijenhuis torsion implies that the Haantjes tensor is also skew-symmetric. Its local expression is

\[(4) \quad (H_L)_{ij}^k = \sum_{\alpha, \beta=1}^{n} \left( L_i^\alpha L_j^\beta (T_L)_{ij}^\delta + (T_L)_{ij}^\alpha L_j^\beta L_k^\delta - L_i^\alpha (T_L)_{ij}^\alpha L_j^\beta + (T_L)_{ij}^\alpha L_j^\beta L_k^\delta \right).\]

We shall consider a case, in which the computation of the Haantjes torsion will be particularly simple [26].
Proposition 3. Let $L$ be a smooth field of operators. If there exists a local chart $\{U,(x_1,\ldots,x_n)\}$ where $L$ takes the diagonal form

\begin{equation}
L(x) = \sum_{i=1}^{n} l_i(x) \frac{\partial}{\partial x_i} \otimes dx_i ,
\end{equation}

then the Haantjes tensor of $L$ vanishes.

Due to the relevance of the Haantjes (Nijenhuis) vanishing condition, we propose the following definition.

Definition 4. A Haantjes (Nijenhuis) operator is a field of operators whose Haantjes (Nijenhuis) tensor identically vanishes.

It has been proved in the following proposition that a single Haantjes operator generates an algebra of Haantjes operators over the ring of smooth functions on $M$. This is not the case for a Nijenhuis operator $N$ since a polynomial in $N$ with coefficients $a_j \in C^\infty(M)$, is not necessarily a Nijenhuis operator.

Proposition 5. [5]. Let $L$ be a Haantjes operator in $M$. Then for any polynomial in $L$ with coefficients $a_j \in C^\infty(M)$, the associated Haantjes tensor vanishes, i.e.

\begin{equation}
H_L(X,Y) = 0 = H(\sum_j a_j(x)L_j)(X,Y) = 0 .
\end{equation}

Proof. See Corollary 3.3, p. 1136 of [5].

3. HAANTJES ALGEBRAS

In this section we recall the notion of Haantjes algebra and the class of cyclic Haantjes algebras, very recently introduced in [27].

Definition 6. A Haantjes algebra of rank $m$ is a pair $(M,H)$ which satisfies the following conditions:

- $M$ is a differentiable manifold of dimension $n$;
- $H$ is a set of Haantjes operators $K_i : TM \to TM$ that generates
  - a free module of rank $m$ over the ring of smooth functions on $M$
  \begin{equation}
  H(fK_i + gK_j)(X,Y) = 0 , \quad \forall X,Y \in TM , \forall f,g \in C^\infty(M) ;
\end{equation}
  - a ring w.r.t. the composition operation
  \begin{equation}
  H(K_i,K_j)(X,Y) = H(K_j,K_i)(X,Y) = 0 , \quad \forall K_i,K_j \in H , \quad \forall X,Y \in TM ,
  \end{equation}

The assumption (7), (8), ensure that the set $H$ is an associative algebra of Haantjes operators. In addition, if

\begin{equation}
K_i,K_j = K_iK_j \quad \forall K_i,K_j \in H ,
\end{equation}

the algebra $H$ will be said an Abelian Haantjes algebra.

Remark 7. The conditions of Definition 6 are apparently very demanding and difficult to solve. However, a class of natural solutions is given, in a local chart $\{U,x = (x_1,\ldots,x_n)\}$, by each operator of the form

\begin{equation}
K = \sum_{i=1}^{n} l_i(x) \frac{\partial}{\partial x_i} \otimes dx_i .
\end{equation}
The diagonal operators $K$ have their Haantjes tensor vanishing and satisfy the differential compatibility condition (7) by virtue of Proposition 3. Moreover, they form a commutative ring, therefore they fulfill Eqs. (8). In fact, such operators generate an algebraic structure that we shall call a diagonal Haantjes algebra.

A particular but especially relevant class of Haantjes algebras is given by the ones generated by a single Haantjes operator $L : TM \mapsto TM$. In fact, one can construct directly a Haantjes algebra $\mathcal{L}$, of rank $\leq n = \dim(M)$, by choosing as a set of generators the first $(n-1)$ powers of $L$ together with $L^0 := I$.

\[ \mathcal{L}(L) := \text{Span}\{I, L, L^2, L^{n-1}\}. \]

We shall call these algebras cyclic Haantjes algebras. Their rank is equal to the degree of the minimal polynomial of $L$.

A natural question is to establish when a given Haantjes algebra can be generated by a single Haantjes operator, giving rise to a cyclic Haantjes algebra. This problem has been investigated in [27] starting from the following

**Definition 8.** Let $(M, \mathcal{H})$ be a Haantjes algebra of rank $m$. An operator $L$ will be called a cyclic generator of $\mathcal{H}$ if

\[ \mathcal{H} \equiv \mathcal{L}(L) \]

The basis

\[ B_{\text{cyc}} = \{I, L, L^2, L^{m-1}\} \]

will be called a cyclic basis of $\mathcal{H}$ and allows us to represent each Haantjes operator $K \in \mathcal{H}$ as a polynomial field in $L$ of degree at most $(m-1)$, i.e.

\[ K = p_K(x, L) = \sum_{i=0}^{m-1} a_i(x) L^i, \]

where $a_i(x)$ are smooth functions in $M$.

4. Poisson–Haantjes manifolds

In order to deal with GZ systems, we need to extend the notion of symplectic-Haantjes manifold $(\omega \mathcal{H})$ already introduced in [26]. Here we propose the new notion of Poisson–Haantjes ($PH$) manifold.

As usual, the transposed operator $K^T : T^*M \rightarrow T^*M$ is defined as the transposed linear map of $K : TM \rightarrow TM$, with respect to the natural pairing between a vector space and its dual space.

**Definition 9.** A Poisson–Haantjes manifold is a triple $(M, P, \mathcal{H})$ that satisfies the following conditions

i) $M$ is a differentiable manifold;

ii) $P : TM^* \rightarrow TM$ is a Poisson bivector in $M$;

iii) $\mathcal{H}$ is an Abelian Haantjes algebra;

iv) $(P, \mathcal{H})$ are algebraically compatible in the sense that $KP = PK^T, \forall K \in \mathcal{H}$.

As a consequence of the above conditions, we get the following simple proposition that turns out to be crucial for many results of the present theory.
Proposition 10. In a given a PH manifold, any composed operator, $K_i P, K_i P K_j^T$, $(K_\alpha - f(x)I)^T P, r \in \mathbb{N}$, is skew symmetric.

Remark 11. The class of $\omega H$ manifolds coincides with the class of PH manifolds of even dimension, with an invertible Poisson bivector. In fact, as in this case $\Omega = P^{-1}$ is a symplectic operator, the compatibility condition $KP = PK^T$ is equivalent to the compatibility condition $\Omega K = K^T \Omega$, required in $\omega H$ manifolds.

We show a paradigmatic example of PH manifold with a cyclic Haantjes algebra, that later will be used to describe a Haantjes algebra of the Lagrange top.

Example 12. Let $(M, P, N)$ be a Poisson–Nijenhuis (PN) manifold, that is, a manifold endowed with a Poisson bivector $P$ and a Nijenhuis operator $N$ that satisfy the following compatibility conditions

$$NP - P N^T = 0 \quad (13)$$

$$R(P, N)(\alpha, Y) = 0 \quad \forall \alpha \in T^* M, \forall Y \in TM,$$

where $R(P, N)$ is the $(2 + 1)$ tensor field defined in [18] by

$$R(P, N)(\alpha, Y) = L_{\alpha}(N)Y - P(L_Y(N^T \alpha) - L_{NY} \alpha), \quad (15)$$

$(L_Y$ denotes the Lie derivative with respect the vector field $Y$). Let us suppose that the Nijenhuis operator $N$ has its minimal polynomial of degree $m$. Then, the PN manifold $M$ has a standard PH structure, given by

$$(M, P, K_1 = I, K_2 = N, \ldots , K_m = N^{m-1})$$

with a Haantjes algebra of rank $m \leq \text{dim}(M)$. In fact, each Nijenhuis operator $N$ is also a Haantjes operator, therefore generates the cyclic Haantjes algebra $L(N)$. Moreover, the algebraic compatibility condition (13) assures that for all Haantjes operators

$$K = p_K(x, N) = \sum_{i=0}^{m-1} a_i(x) N^i,$$

the condition iv) of Def. 9 is fulfilled.

In addition, the differential condition (13) implies that

$$R(P, K)(\alpha, Y) - \left( \sum_{i=0}^{m-1} \left( X_i \wedge N^i Y - Y(a_i)N^i P \right) \right) \alpha = 0 \quad (16)$$

$\forall j \in \mathbb{N}, \forall \alpha \in T^* M, \forall Y \in TM$, where $X_i := P \text{d}a_i$ are the Hamiltonian vector fields with Hamiltonian functions $a_i$.

We also generalize the concept of Magri–Haantjes chain, introduced in [26] under the name of Lenard–Haantjes chains.

Definition 13. Let $(M, P, H)$ be a Poisson–Haantjes manifold. A function $H \in C^\infty(M)$ is said to generate a Magri–Haantjes chain of 1-forms if

$$d(K_i^T dH) = 0, \quad i = 1, \ldots , m,$$

for some basis $\{K_1, \ldots , K_m\}$ of $H$. The (locally) exact 1-forms $dH_i$ such that

$$dH_i = K_i^T dH \quad i = 1, \ldots , m,$$

are called the elements of the Magri–Haantjes chain, of length $m$, generated by $H$.

The relevance of Magri–Haantjes chains is due to the following
Lemma 14. Let \((M, P, \mathcal{H})\) be a Poisson–Haantjes manifold. The functions \(H_i\) whose differentials belong to all Magri-Haantjes chains generated by a single function \(H\) are in involution w.r.t. the Poisson bracket defined by \(P\). In fact,
\[
\{H_i, H_j\} = <dH_i, P dH_j> = <K^T_i dH, PK^T_j dH> = <dH, K^T_i PK^T_j dH >^{Prop. 10} = 0
\]

Definition 15. Let \((M, P, \mathcal{H})\) be a Poisson–Haantjes manifold. A vector field \(Y\) is said to generate a Magri–Haantjes chain of vector fields if the vector fields defined by
\[
Y_i := K_i Y, \quad i = 1, \ldots, m
\]
for some basis \(\{K_1, \ldots, K_m\}\), commute among each others.

Remark 16. Let us note that, thanks to the compatibility condition iv) in Definition 9, to every Magri-Haantjes chain of 1-forms generated by a function \(H\) corresponds a Magri-Haantjes chain of Hamiltonian vector fields \(X_H = PdH\_i\) generated by \(X_H = PdH\). Moreover, the Hamiltonian vector fields belonging to different chains generated by the same Hamiltonian vector field \(X_H\) commute among each others.

In [26], it has been shown that, given a \(\omega\mathcal{H}\) manifold and a function \(H\), the existence of a Magri–Haantjes chain generated by \(H\) is equivalent to the Frobenius integrability of the co-distribution
\[
K^T_i dH \quad i = 1, \ldots m.
\]
In this paper, we limit ourselves to exhibit the example of the Magri-Haantjes chain of the Lagrange top, leaving the finding of the conditions assuring the existence of Magri-Haantjes chains in \(P\mathcal{H}\) manifolds to future work.

5. The Lagrange top

The classical Lagrange top is a heavy symmetric top, that is, a symmetric rigid body with a fixed point \(O\), immersed in the uniform gravity field \(\mathbf{g}\). It admits different geometric formulations in the framework of the bi–Hamiltonian theory [22, 23, 30].
5.1. Euler angles. In the phase space $M = T^*(SO(3))$ one can choose as local coordinates the classical Euler angles and conjugate momenta $(\varphi, \theta, \psi, p_\varphi, p_\theta, p_\psi)$. In such a chart, the Hamiltonian function of the Lagrange top takes the form

$$H = \frac{1}{2A} \left( p_\theta^2 + \frac{(p_\varphi - p_\psi \cos \theta)^2}{\sin^2 \theta} + \frac{1}{c^2} p_\psi^2 \right) + \mu g a \cos \theta$$

and the Hamilton equations of the motion are

$$\dot{\varphi} = \frac{1}{A} \frac{p_\varphi - p_\psi \cos \theta}{\sin^2 \theta}$$
$$\dot{\theta} = \frac{1}{\dot{\Phi}}$$
$$\dot{\psi} = \frac{1}{cA} p_\theta$$
$$\dot{p}_\varphi = 0$$
$$\dot{p}_\theta = -\frac{\partial}{\partial \theta} \left( \frac{(p_\varphi - p_\psi \cos \theta)^2}{\sin^2 \theta} + \mu g a \sin \theta \right)$$
$$\dot{p}_\psi = 0,$$

where $A$ and $cA$ are, respectively, the inertia momenta w.r.t. every axis in the equatorial plane and the symmetry axis, $\mu$ is the mass of the top, $g$ the gravity acceleration, $\alpha$ the coordinate of the mass center $G$ along the symmetry axis. It is evident from the Hamiltonian function and the equations of the motion that the Lagrange top admits the three integrals of motion

$$H_1 = H, \quad H_2 = p_\varphi, \quad H_3 = p_\psi,$$

that are, the energy and the components of the angular momentum along the vertical and the symmetry axis, respectively. Moreover, it is well known that the coordinates $(\varphi, \theta, \psi, p_\varphi, p_\theta, p_\psi)$ are separation variables for the Hamilton-Jacobi equation associated to $H$. From our point of view, it is worth of interest to show that even this very classical system, as well as every Hamiltonian separable system, can be described in the framework of $\omega H$ Haantjes manifolds, according to Theorem 57 of [26]. A Haantjes algebra for the Lagrange top can be easily computed, whose basis is

$$K_1 = I$$
$$K_2 = \frac{A \sin^2 \theta}{p_\varphi - p_\psi \cos \theta} \left( \frac{\partial}{\partial \varphi} \otimes d\varphi + \frac{\partial}{\partial p_\varphi} \otimes dp_\varphi \right)$$
$$K_3 = -\frac{A \sin^2 \theta}{\cos \theta(p_\varphi - p_\psi \cos \theta)} \left( \frac{\partial}{\partial \theta} \otimes d\theta + \frac{\partial}{\partial p_\theta} \otimes dp_\theta \right).$$

In fact, the action of the (transpose of the) Haantjes operators $K_1, K_2, K_3$ on the gradient of the Hamiltonian function $(19)$ produces the Magri–Haantjes chain of the (gradients of the) three integrals of motion

$$K_1^T dH = dH_1, \quad K_2^T dH = dH_2, \quad K_3^T dH = dH_3.$$

The fact that the Haantjes operators take a diagonal form in the Euler chart $(\varphi, \theta, \psi, p_\varphi, p_\theta, p_\psi)$, means that the Haantjes algebra is diagonal and the Euler coordinates are Darboux-Haantjes coordinates for it.
More interesting and hard to solve is the problem to construct a Haantjes algebra directly in the physical coordinates, and, afterwards determining a set of Darboux–
Haantjes coordinates that are separation variables according to Theorem 59 of [26].
In the next Section, we will show how to proceed in this example, starting with the
tri-Hamiltonian formulation of the Lagrange top and reducing it to the symplectic
leaves of one of its Poisson bivectors.

5.2. Euler–Poisson coordinates. An alternative formulation of the Lagrange top
is based on the Euler-Poisson equation that are, roughly speaking, the equation of
the motion projected onto the comoving reference frame ($\vec{e}$ is based on the Euler-Poisson equation that are, roughly speaking, the equation of
the phase space $M := \{ m | m = (\vec{e}, \gamma) \}$, where $\vec{e} = \omega_1 \vec{e}_1 + \omega_2 \vec{e}_2 + \omega_3 \vec{e}_3$ is the angular velocity of the top, $\gamma = \gamma_1 \vec{e}_1 + \gamma_2 \vec{e}_2 + \gamma_3 \vec{e}_3$ the vertical unit vector (the Poisson vector), $G - O = a \vec{e}_3$ the vector of the center of mass and $J = \text{diag}(A, A, cA)$ the
principal inertia matrix. In such a formulation, the equations of the motion $\dot{m} = X_L(m)$ are the Euler equations coupled with the Poisson equations, so that the
Lagrange vector field is given by

$$X_L(m) = \begin{bmatrix}
(1 - c) \omega_2 \omega_3 - \gamma_2 \\
-(1 - c) \omega_3 \omega_1 + \gamma_1 \\
0 \\
\gamma_2 \omega_3 - \gamma_3 \omega_2 \\
\gamma_3 \omega_1 - \gamma_1 \omega_3 \\
\gamma_1 \omega_2 - \gamma_2 \omega_1
\end{bmatrix},$$

where the normalization $\mu g a / A = 1$ has been chosen.

5.2.1. The tri-Hamiltonian formulation of the Lagrange top. The vector field (21)
admits a tri-Hamiltonian formulation (see [22] and reference therein) w.r.t. the
three non invertible Poisson bivectors

$$P_0 = \begin{bmatrix} 0 & B \\ B & C \end{bmatrix}, \quad P_1 = \begin{bmatrix} -B & 0 \\ 0 & \Gamma \end{bmatrix}, \quad P_2 = \begin{bmatrix} T & R \\ -R^T & 0 \end{bmatrix},$$

where

$$B = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & c \omega_3 & -\omega_2 \\ -c \omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix},$$

$$\Gamma = \begin{bmatrix} 0 & \gamma_3 & -\gamma_2 \\ -\gamma_3 & 0 & \gamma_1 \\ \gamma_2 & -\gamma_1 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & -\gamma_3 & \gamma_2 \\ \gamma_3 & 0 & -\gamma_1 \\ -\gamma_2 / c & \gamma_1 / c & 0 \end{bmatrix},$$

$$T = \begin{bmatrix} 0 & -c \omega_3 & \omega_2 / c \\ c \omega_3 & 0 & -\omega_1 / c \\ -\omega_2 / c & \omega_1 / c & 0 \end{bmatrix},$$

and the Hamiltonian functions $(h_0, h_1, h_2)$

$$dh_0 \quad dh_1 \quad dh_2$$

$$\xrightarrow{P_0} \xrightarrow{P_1} \xrightarrow{P_2} X_L$$

defined by

$$h_0 = \frac{1}{2} F_4 + (c - 1) F_1 F_3, \quad h_1 = -F_3 - (c - 1) F_1 F_2, \quad h_2 = F_2.$$
The functions $F_1, F_2, F_3, F_4$ are the integrals of motion given by

$$F_1 = \omega_3, \quad F_2 = \frac{1}{2}(\omega_1^2 + \omega_2^2 + c \omega_3^2) - \gamma_3,$$

$$F_3 = \omega_1 \gamma_1 + \omega_2 \gamma_2 + c \omega_3 \gamma_3, \quad F_4 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2.$$

The three Poisson bivectors $(P_0, P_1, P_2)$ generate three Poisson pencils

$P_0 - \lambda P_1, \quad P_1 - \lambda P_2, \quad P_0 - \lambda P_2,$

that possess two polynomial Casimir functions each \[23\]. Below, we concentrate on the Casimir function of the Poisson pencil $P_0 - \lambda P_1$.

5.2.2. A Gelfand–Zakharevich system of co–rank 2. The Poisson pencil $P_0 - \lambda P_1$ possess two polynomial Casimir functions $H(\lambda)^{(1)} = H_0^{(1)} = F_1$ and $H(\lambda)^{(2)} = H_0^{(2)} \lambda^2 + H_1^{(2)} \lambda + H_2 = \frac{E}{\omega} \lambda^2 - F_3 \lambda + F_2$, of length 1 and 3 respectively. They can be represented graphically in the following way

\[
\begin{align*}
\text{d}F_1 &\quad \text{d}F_4/2 &\quad \text{d}(-F_3) &\quad \text{d}(F_2) \\
P_0 &\quad P_1 &\quad P_0 &\quad P_1 &\quad P_0 &\quad P_1 &\quad P_0
\end{align*}
\]

The vector fields $X_1, X_2$ are bi–Hamiltonian vector fields as

\[
X_i = P_0 \text{d}H_i^{(2)} = P_1 \text{d}H_i^{(2)}, \quad i = 1, 2.
\]

Moreover, the vector field \[21\] of the Lagrange top can be formulated as

\[
X_L = X_1 - (c - 1)F_1 X_2,
\]

therefore defining a system of Gelfand–Zakharevich type of co–rank 2.

5.2.3. The reduction of the Poisson pencil. Without loss of generality, we fix a Poisson bivector inside the Poisson pencil, say $P_1$, and, in order to getting rid of its Casimir functions, we perform a reduction to its symplectic leaves

\[
S_1 := \{ F_1 = C_1, F_4 = C_4 \}.
\]

To this aim, it is convenient to introduce complex coordinates in $M$ adapted to such a reduction \[22\]

\[
x_1 = -c \omega_3 + i \omega_2, \quad x_2 = \gamma_3 - i \gamma_2, \quad y_1 = \omega_1, \quad y_2 = -\gamma_1, \quad F_1, \quad F_4,
\]

in which the Poisson bivectors take the following simple form

$$P_0 = \begin{bmatrix}
\bar{P}_0 & 0 & 2X_1^1 & 2X_2^1 & 2X_3^1 & 2X_4^1 \\
0 & 0 & 2X_1^2 & 2X_2^2 & 2X_3^2 & 2X_4^2 \\
0 & 0 & 2X_1^3 & 2X_2^3 & 2X_3^3 & 2X_4^3 \\
0 & 0 & 2X_1^4 & 2X_2^4 & 2X_3^4 & 2X_4^4 \\
-2X_1^3 & -2X_2^3 & -2X_3^3 & -2X_4^3 & 0 & 2X_1^5 \\
-2X_1^2 & -2X_2^2 & -2X_3^2 & -2X_4^2 & 0 & 2X_1^5
\end{bmatrix}, \quad \bar{P}_0 = -i \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & -x_1 \\
0 & -1 & 0 & 0 \\
-1 & x_1 & 0 & 0
\end{bmatrix}.$$
It is evident, that the Poisson bivector $P_0$ cannot be restricted to $S_1$, unlike $P_1$. So, we perform a reduction procedure introduced in [7]. It is a highly non trivial generalization of the Marsden–Ratiu method [19] and consists in an ingenious deformation of the Poisson bivector $P_0$, by means of a suitable family of vector fields transversal to the symplectic leaves of $P_1$. Such a deformation assures that the deformed Poisson pencil shares the same axis with the old one and can be restricted to $S_1$.

5.2.4. Deformation. We choose the two vector fields

$$Z_1 = \frac{\partial}{\partial F_1}, \quad Z_2 = 2 \frac{\partial}{\partial F_4}$$

normalized as

$$Z_i(H_0^{(j)}) = \delta_i^j, \quad i,j = 1,2.$$  

As they fulfill the equations

$$\mathcal{L}_{Z_i}(P_1) = 0 \quad \mathcal{L}_{Z_i}(P_0) = [Z_i, X_1] \wedge Z_2 \quad i = 1,2,$$

they can deform the Poisson bivector $P_0$ into the new Poisson bivector

$$Q := P_0 - X_1 \wedge Z_2, \quad Q = \begin{bmatrix} \hat{P}_0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & x_2 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -x_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

that can be restricted to $S_1$ and its restriction is $\hat{Q} = \hat{P}_0$.

5.2.5. Haantjes algebra. There are different manners of endowing the manifold $M$ with a Haantjes algebra compatible with $P_1$. In this section, we limit ourselves to show one of them, leaving a more general discussion to future work. In order to construct a Haantjes algebra for the bi–Hamiltonian chain (22), which will be preserved after the restriction to $S_1$, we look for a Nijenhuis operator $N$ that plays the role of a cyclic generator for a Haantjes algebra of low rank, that is $2 \leq m \leq 3$. Precisely, we require that $N$:

i) factorizes the deformed Poisson bivector

$$Q = N P_1 ;$$

ii) has its restriction $\tilde{N}$ to $S_1$ equal to

$$\tilde{N} = Q \hat{P}_1^{-1} = \hat{P}_0 \hat{P}_1^{-1} ;$$
iii) is a cyclic generator of a Haantjes algebra that provides the following Magri–Haantjes chain of vector fields generated by \(X_1\)

\[
K_i X_1 = X_i , \quad i = 1, 2, 3 ,
\]

where \(X_3 = 0\).

In other words, denoted with \(\Phi\) the immersion of \(S_1\) in \(M\), and with \(\Phi_*\) and \(\Phi^*\) its pullback and pushforward respectively, we search for an operator \(N : TM \rightarrow TM\) that solves the system

\[
\begin{align*}
N P_1 &= P_0 - X_1 \wedge Z_2 \\
\Phi_* N \Phi^* &= \tilde{P}_0 \tilde{P}_1^{-1} \\
(d_1 I + e_1 N + f_1 N^2) X_1 &= X_1 \\
(d_2 I + e_2 N + f_2 N^2) X_1 &= X_2 \\
(d_3 I + e_3 N + f_3 N^2) X_1 &= X_3 \\
\tau (N) &= 0 ,
\end{align*}
\]

together with the unknown functions \((d_i, e_i, f_i), \ i = 1, 2, 3\). Such a system can be decoupled in order to firstly determine the unknown functions \((d_i, e_i, f_i)\). In fact, applying both terms of Eq. \((30)\) to the gradients of the bi–Hamiltonian chain \((29)\), and taking into account the fact that the Hamiltonian functions \(H_j\) are integrals of motion for \(X_1\), one finds that the bi-Hamiltonian vector fields \((33)\) must fulfill the system

\[
\begin{align*}
NX_j = X_{j+1} + \frac{1}{2}(H_j^2) Z_2 - Z_2(H_j^2) X_1 , \quad X_0 := 0 , \quad j = 0, 1, 2 .
\end{align*}
\]

By solving recursively such a system w.r.t. \(X_{j+1}\), only in terms of the vector field \(X_1\), a unique solution is found for the unknown functions \((d_i, e_i, f_i)\) in Eqs \((32), (33), (34)\). This solution can be written down in a compact form as

\[
\begin{bmatrix}
K_1 \\
K_2 \\
K_3
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
I \\
N \\
N^2
\end{bmatrix}.
\]

Summarizing, the operators \((37)\) are Haantjes operators that provide the Magri–Haantjes chain \((29)\) for any solution \(N\) of Eqs. \((30), (31)\) and \((35)\). A simple solution of Eqs. \((30)\) and \((31)\), which leaves invariant both \(TS_1\) and \(\text{Span}\{ Z_1, Z_2\}\), is

\[
N = \begin{bmatrix}
\begin{array}{c|ccc}
\hat{N} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}
\end{bmatrix} , \quad \hat{N} = \hat{P}_0 \hat{P}_1^{-1} =
\begin{bmatrix}
0 & \frac{1}{x_2} & 0 & 0 \\
1 & -\frac{x_2}{x_1} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{x_2}
\end{bmatrix}
\]

were \((N_6^5, N_6^6, N_5^6, N_6^5)\) are arbitrary functions to be determined requiring that the Nijenhuis torsion of \(N\) identically vanishes. However, we can postpone the solution of the PDE \((35)\), adding to the system \((30) - (35)\) a further algebraic request, namely that \(N\) satisfies the following Magri–Haantjes chain of 1-forms,

\[
(Z_2(-F_3) I + N)^T d(-F_3) = dF_2 , (Z_2(F_2) + Z_2(-F_3) N + N^2)^T d(-F_3) = 0 .
\]
Eqs. (39) are independent of Eqs. (32)–(34) as, due to the kernel of the Poisson operator, a Magri–Haantjes chain of Hamiltonian vector fields do not imply that the Magri–Haantjes chain of the gradients of their Hamiltonian functions holds true. Thus, substituting (38) into Eqs. (39), we find the unique solution

\[
N_5^5 = \frac{(c - 1)F_1 + x_1}{\Delta}, \quad N_5^6 = \frac{1}{2c} \frac{x_2}{x_2 \Delta},
\]

\[
N_6^5 = -2cx_2 \left((c - 1)(F_1^2 + F_1 x_1) - x_2\right),
\]

\[
N_6^6 = -\frac{x_3^2 + (c - 1)F_1 x_2^2 + 2x_1 x_2 + (c - 1)F_1 x_2}{x_2 \Delta},
\]

where \( \Delta = x_2^2 + (c - 1)F_1 x_1 + x_2 \). Now, we can verify that it satisfies also Eq. (35), therefore we have got a solution of the problem (27)–(29), which solves also Eqs. (39). It has its minimal polynomial of degree 2

\[
m_N(\lambda, x) = \lambda^2 + \frac{x_1}{x_2} \lambda - \frac{1}{x_2} = \lambda^2 + Z_2(-F_3)\lambda + Z_2(F_2),
\]

together with (39). It has an explicit form of degree 2

\[
m_N(\lambda, x) = \lambda^2 + \frac{x_1}{x_2} \lambda - \frac{1}{x_2} = \lambda^2 + Z_2(-F_3)\lambda + Z_2(F_2),
\]

where \( \Delta = x_2^2 + (c - 1)F_1 x_1 + x_2 \). Now, we can verify that it satisfies also Eq. (35), therefore we have got a solution of the problem (27)–(29), which solves also Eqs. (39). It has its minimal polynomial of degree 2

\[
m_N(\lambda, x) = \lambda^2 + \frac{x_1}{x_2} \lambda - \frac{1}{x_2} = \lambda^2 + Z_2(-F_3)\lambda + Z_2(F_2),
\]

therefore it is a cyclic generator of a Haantjes algebra of rank 2. Moreover, the coefficients of (40) coincides with the elements of the last row in the matrix of (37), consequently \( K_3 = 0 \).

5.2.6. Restriction to the symplectic leaves of \( P_1 \). From the previous steps it follows

**Proposition 17.** The deformed Poisson pencil \( Q - \lambda P_1 \) restricts to the symplectic leaves of \( P_1 \). Moreover, \( N, K_1, K_2 \), the Hamiltonian functions and the Hamiltonian vector fields \( (X_1, X_2) \) restrict as well. They endow the symplectic leaves of \( P_1 \) with a \( \omega_H \) structure and the Magri–Haantjes chain of exact forms, given by

\[
\hat{K}_1^T d(-F_3)|_{S_1} = d(-F_3)|_{S_1}, \quad \hat{K}_2^T d(-F_3)|_{S_1} = dF_2|_{S_1},
\]

in virtue of (39).

In particular, the relations (37) restrict to the Benenti relations

\[
\hat{K}_1 = \hat{I}, \quad \hat{K}_2 = \frac{x_1}{x_2} \hat{I} + \hat{N},
\]

as can be immediately seen from the analysis of the minimal polynomial of \( \hat{N} \) that is still equal to (40). Such relations are satisfied by the so-called \( L \)-systems [2, 3, 4], proved to be projections of Quasi–Bi–Hamiltonian [20, 21, 28] systems constructed in Riemannian manifolds [29].

5.2.7. Separation of variables. Let us construct in \( S_1 \) a set of Darboux-Haantjes (DH) coordinates for the \( \omega_H \) manifold \( (S_1, P_1, \hat{K}_1, \hat{K}_2) \). As the \( \omega_H \) structure, in this case, is equivalent to a \( \omega_N \) structure, as DH coordinates we can take just a set of Darboux-Nijenhuis coordinates [8]. To this aim, we proceed according to the Remark 70, Sect. 8 of [26]. Firstly, we choose as first two coordinates \( (\lambda_1, \lambda_2) \) just the two (double) eigenvalues of the Haantjes operator \( \hat{K}_2 \)

\[
\lambda_1 = \frac{x_1 - \sqrt{x_1^2 + 4x_2}}{2x_2}, \quad \lambda_2 = \frac{x_1 + \sqrt{x_1^2 + 4x_2}}{2x_2},
\]

as their gradients are eigenforms of \( \hat{K}_2^T \)

\[
\hat{K}_2^T d\lambda_1 = \lambda_2 d\lambda_1, \quad \hat{K}_2^T d\lambda_1 = \lambda_2 d\lambda_1.
\]
Let us note that \((\lambda_1, \lambda_2)\) are also the only eigenvalues of \(K_2 = Z_2(-F_3)I + N\), which, therefore, can be used to find half of the separation variables.

Further, we complete them with a pair of conjugate momenta
\[
\mu_1 = \frac{1}{\lambda_1}(\lambda_2 y_1 + y_2) \quad \mu_2 = \frac{1}{\lambda_2}(\lambda_1 y_1 + y_2)
\]
whose gradients are also eigenforms of \(\tilde{K}_2\)
\[
\tilde{K}_2^T d\mu_1 = \lambda_2 d\mu_1, \quad \tilde{K}_2^T d\mu_1 = \lambda_2 d\mu_2.
\]
The local chart \((\lambda_1, \lambda_2, \mu_1, \mu_2)\) so constructed is a Darboux chart for the Poisson operator \(\tilde{P}_1\) and a Haantjes chart for the Haantjes operator \(\tilde{K}_2\). In fact, in such chart, they take the form
\[
\tilde{P}_1 = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{bmatrix}, \quad \tilde{K}_2 = \begin{bmatrix}
\lambda_2 & 0 & 0 & 0 \\
0 & \lambda_1 & 0 & 0 \\
0 & 0 & \lambda_2 & 0 \\
0 & 0 & 0 & \lambda_1
\end{bmatrix}
\]
(To be more precise, in order to have a Darboux–Haantjes chart one should eliminate the factor \(i\) in the form of \(\tilde{P}_1\) above, by means of the map \(\lambda \mapsto i\lambda, \mu \mapsto \mu\).)

As a consequence of Theorem 59 in [26], the coordinates \((\lambda_1, \lambda_2, \mu_1, \mu_2)\) are separation variables for the functions \(F_2\) and \(F_3\). Furthermore, they are also separation variables for the (restriction to \(S_1\) of the) Hamiltonian function of the Lagrange top \(h_{1|S_1} = -F_3|_{S_1} - (c - 1)C_1F_2|_{S_1}\).

This fact can be proved by means of the Benenti test [1], or simply by observing that, thanks to Eq. (41), it holds true that
\[
dh_{1|S_1} = (-\dot{I} + (c - 1)C_1 \tilde{K}_2^T) dF_3|_{S_1}.
\]
Therefore the function \(h_{1|S_1}\) belongs to a Magri–Haantjes chain generated by \(F_3|_{S_1}\). Consequently, according to Theorem 59 in [26], also \(h_{1|S_1}\) is separable in any \(DH\) local chart.

### 6. Future Perspectives

It would be interesting to construct a Poisson–Haantjes algebra for the Poisson pencil \(P_2 - \lambda P_1\) of the Lagrange top, that unlike the pencil \(P_0 - \lambda P_1\), has two polynomial Casimir functions of the same length 2. Moreover, a Haantjes algebra for the stationary flows [6] of the Boussinesq hierarchy, that also are GZ systems of corank 2, should be worked out.

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