A procedure for loss-optimising default definitions across simulated credit risk scenarios

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Abstract

A new procedure is presented for the objective comparison and evaluation of default definitions. This allows the lender to find a default threshold at which the financial loss of a loan portfolio is minimised, in accordance with Basel II. Alternative delinquency measures, other than simply measuring payments in arrears, can also be evaluated using this optimisation procedure. Furthermore, a simulation study is performed in testing the procedure from ‘first principles’ across a wide range of credit risk scenarios. Specifically, three probabilistic techniques are used to generate cash flows, while the parameters of each are varied, as part of the simulation study. The results show that loss minima can exist for a select range of credit risk profiles, which suggests that the loss optimisation of default thresholds can become a viable practice. The default decision is therefore framed anew as an optimisation problem in choosing a default threshold that is neither too early nor too late in loan life. These results also challenge current practices wherein default is pragmatically defined as ‘90 days past due’, with little objective evidence for its overall suitability or financial impact, at least beyond flawed roll rate analyses or a regulator’s decree.

Keywords— Decision Analysis; Credit Loss; Loan Delinquency; Default Definition

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1 Introduction

Consumer credit has exponentially grown over the last few decades, largely spurred by the introduction of the credit card during the 1950s, as discussed in Thomas (2009, pp. 2–3). Retail credit is currently estimated at $13 trillion for the US market, which largely consists of mortgages, credit cards, personal loans, vehicle financing, overdrafts and other revolving loans for the individual, as reported in The Board of Governors of the Federal Reserve System (US) (2018). For perspective, consumer debt in 2007 was 40% greater than total industry debt ($9.2 trillion) and more than double total corporate debt ($5.8 trillion). Although greatest in the USA, consumer debt in other countries are not far behind, e.g., the United Kingdom had debt levels in 2007 at £1.4 trillion – a staggering £400 billion growth within the span of a mere three years. While Canada’s consumer debt is estimated at $666 billion, this figure also constituted 110% of total annual household income. In fact, this trend of debt levels exceeding household income is true for quite a few countries for the last twenty years, of which a few examples are shown in Fig. 1.

![Fig. 1. Consumer household debt-to-income over annual periods by country, including Australia (AUS), Canada (CAN), Finland (FIN), Greece (GRC), Netherlands (NLD), Norway (NOR), and the United States of America (USA). Reproduced from OECD (2018).](image)

This credit growth, as argued in Thomas (2009, pp. 1–6) and Thomas (2010), could not have been possible without a degree of automation, historically facilitated by statistical decision-making models otherwise known as application credit scorecards. These models rendered consistent approve/decline credit decisions that enabled greater application volumes whilst keeping default risk aligned with a lender’s risk appetite. This is mainly achieved by only approving those applications with a predicted probability of default within a desired limit. Constructing these scorecards involves finding a statistical relationship between a set of borrower-specific characteristics and the successful (or failed) repayment outcome over time, using historical data. Naturally, the literature on credit scoring is considerable, e.g., Hand and Henley (1997), Hand (2001), Thomas, Edelman and Crook (2002, pp. 2–6, 41–86), Siddiqi (2005), Crook, Edelman and Thomas (2007), Hao, Alam and Carling (2010), and Louzada, Ara and

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The advent of these automated models did, however, call for a more methodical manner of measuring default before predicting the risk thereof. This includes capturing the development of loan delinquency over time, most notably using the accountancy-based number of payments in arrears, as calculated from the number of days past due. Specifically, the unpaid portion of an instalment is aged into several increasingly severe bins given the time elapsed: 30 days, 60 days, 90 days, and so forth, as discussed in Cyert, Davidson and Thompson (1962). Using these resulting arrears categories, banks commonly specified three payments (or 90 days) in arrears as their point of default, long before the introduction of the Basel II Capital Accords, which standardised default definitions to some degree. This so-called ‘threshold’ is often pragmatically informed by managerial discretion, though supported by some analysis, and generally ranges between 30–180 days, depending on data availability and the type of product, as discussed in Thomas et al. (2002, pp. 123–124). However, the direct financial implications of any chosen definition are not readily known, nor accounted for when deciding the point of default. Therefore, the pursuit of scorecard modelling excellence becomes questionable when the constructed response variable itself, i.e., the binary ‘good/bad’ risk class that results from an applied default definition, is inherently arbitrary, as argued in Hand (2001).

According to Finlay (2010, pp. 11–13), should an account accrue sufficient arrears (despite increased collection efforts), then the lender rather pursues debt recovery, including selling off any assets underlying the original credit agreement. This is to say that every unpaid instalment erodes trust between bank and borrower but only up to a certain point, as characterised by the default definition. This study attempts to frame this point (and finding it) as a mathematical optimisation problem such that ‘default’ occurs neither too early nor too late during loan life. Too strict a threshold will marginalise accounts that would have resumed repayment, had the bank not been too brash in its default decision. Conversely, too lenient a threshold may prove naive in tolerating increasing arrears at the cost of liquidity risk. Moreover, profitability ought to be used as the basis for this optimisation, as argued and discussed in section 2 of this study. A procedure for this loss optimisation is then presented in section 3 along with a simulation framework for testing this procedure across various credit risk profiles. In particular, three probabilistic techniques are used to generate cash flows according to set parameters, which are then varied as part of the study. Finally, the simulation results are discussed in section 4, which demonstrate that the loss optimisation of default thresholds is a viable strategy for a select range of credit risk profiles.

2 A defining background on loan default

The estimation of the frequency of any event in a given sample fundamentally depends on the definition of the event. This is to say that while loan ‘default’ lies intrinsic to credit risk (and its estimation), the phenomenon thereof certainly has many definitions, both historically and in modern times. These definitions typically vary by product, customer type, and bank, e.g., filing for bankruptcy, unfulfilled claims, negative net present values, overdrawing beyond an agreed credit limit, as well as becoming three instalments in arrears, as discussed in Van Gestel and Baesens (2009 pp. 203–212) and Baesens, Rösch and Scheule (2016 pp. 137–138). The Basel II Capital Accords also standardised default definitions to some extent upon its introduction, while still leaving room for the lender’s discretion. Specifically, paragraph 452 of the Basel Committee on Banking Supervision (2006 defines ‘default’

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1 For credit lines, e.g., credit cards and overdrafts, ‘payments in arrears’ are technically irrelevant since there are no amortising instalments. However, the number of days by which a facility is in excess of an agreed limit, can still be aged into these arrears categories. The ‘instalment’ is simply the amount required to recover from this overdrawn status.
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as one of the following two conditions. Firstly (and perhaps more commonly-known), the obligor has reached 90 days past due (or three payments in arrears) on a material loan balance, or has been in excess of an advised credit limit for 90 days. Alternatively, the bank considers, in its opinion, that the obligor is unlikely to repay its obligations in full, without the necessary intervention of the bank, e.g., liquidating any collateral. To help inform this opinion, Basel II also includes a few reasonable indicators of ‘default’, which are often promulgated verbatim by a particular country’s regulator, e.g., Regulation 67 of the Banks Act of South Africa (2012, pp. 1201–1202) that defines ‘default’ exactly the same way as in Basel II. At a minimum, these indicators include:

1. The bank assigns a non-accrued status to the debt, thereby no longer charging interest;
2. The bank writes down a portion of the debt, or raises a specific provision, since it believes credit quality has significantly deteriorated;
3. The bank resolves to sell the debt at a material economic loss;
4. The bank files for the obligor’s bankruptcy;
5. The bank agrees to restructure the debt, which likely results in an overall reduced financial obligation;
6. The obligor files for bankruptcy (or is placed therein), which will likely either delay or circumvent repayment.

Most of these indicators (items 1–4) are retrospective in that they denote ‘default’ as a result of certain ex post actions taken by a bank. However, these actions are only reasonably pursued after a bank has already resolved that continuing the credit agreement is of little financial benefit. In other words, the trust between bank and borrower has already eroded beyond a certain point, likely as a result of persistent non-payment. Consider that if reaching this particular point already reflects ‘default’ in itself, then these specific default indicators do not signal ‘default’ as much as they merely reaffirm what a bank already considers to be obvious. This suggests the fallacy of circular reference, or petitio principii, on the premise of using these indicators in defining default when they themselves are deduced by presumably the same default criteria. Lastly, items 5–6 ought to be considered more as possible predictors of default, rather than indicating definite default at a certain point in time – even though default is reasonably likely for those cases in practice.

Apart from Basel II, the three main external rating agencies (Moody’s, Standard & Poor’s, and Fitch) use their own but not too dissimilar default definitions in guiding investment decisions on a wide range of counterparts, as discussed in Van Gestel and Baesens (2009, pp. 115–117, 149–151, 208–209). While the original intent was to indicate investment-grade debt securities (mainly government bonds), modern ratings cover a much larger spectrum of companies and banks. In particular, Moody’s seeks to capture events that change the relationship between bondholder and issuer, as its philosophy of default. All three agencies signal ‘default’ when interest and/or capital portions go unpaid, although a variable grace period apply: one day for Moody’s, 10–30 days for the others. Also, Moody’s does not consider technical defaults (e.g., covenant violations), while S&P does not consider the dividends that are due from preferred stock as ‘financial obligations’ and, as such, do not count missed dividend payments as defaults. Despite these nuanced differences amongst the three agencies, the aim of specifying a default definition is the same as that of Basel II, which is to find a certain ‘point of no return’ at which most delinquent accounts will remain delinquent and not recover.

In addition to using their discretion, lenders also perform a statistical exercise called a roll rate analysis to help choose a default definition, as explained in Siddiqi (2005, pp. 33–42). This is best described as a cross-tabulation of observed transition rates amongst pre-binned arrears categories across a chosen outcome period (e.g., 12 months).
The principle is to set ‘default’ to the category at which most accounts do not recover (or ‘cure’) to less severe categories, i.e., finding the ‘point of no return’. However, the choice of outcome period affects these roll rates significantly, which typically varies between 6–24 months in practice, as discussed in Thomas et al. (2002, pp. 91) and Van Gestel and Baesens (2009, pp. 101–102). In fact, the work of Kennedy, Mac Namee, Delany, O’Sullivan and Watson (2013) experimented with different outcome periods in predicting default (using a constant definition thereof). Classifier accuracy degraded as the outcome period lengthened, though shorter periods also gave volatile default frequencies (due to seasonal effects). Moreover, too short a window may not adequately capture curing rates due to maturity, while an overly long window may become divorced from current market conditions, or may simply require more data than available. Longer windows may also ignore oscillations between defaulting and curing, as discussed in Kelly and O’Malley (2016).

Using different default definitions were first explored in Harris (2013b) and Harris (2013a), wherein the model accuracy of support vector machines predicting default are studied whilst employing various default definitions. However, while optimising accuracy is certainly worthwhile, the implications of variable default definitions for overall profitability are less clear. Moreover, the work of Hand and Henley (1997) argues that a lender is mainly concerned with profitability when making a credit decision, and not as much with model accuracy. To that point, loan profitability also depends on factors other than delinquency, e.g., the market response to a lender’s risk-based pricing (higher interest rates for riskier borrowers), as explored in Phillips (2013). Another factor is the amount of loss provisions raised today in covering expected credit losses tomorrow, as explained in Van Gestel and Baesens (2009, pp. 38-74) and Finlay (2010, pp. 167–169). The recent introduction of IFRS 9, reviewed in Novotny-Farkas (2016), Xu (2016), Cohen, Edwards Jr et al. (2017) and Skoglund (2017), also aligns this loss estimation closer to capital reservation, which are meant to absorb unexpected losses. In both cases, quantifying credit risk consists of estimating default risk (probability of default, or PD), loss risk (loss given default, or LGD), and exposure risk (exposure at default, or EAD), as thoroughly discussed in Thomas (2009, pp. 289–293), Van Gestel and Baesens (2009 chap. 4–6), and Baesens et al. (2016 chap. 5–11). Naturally, these three components rely on a consistent and immutable default definition, similar to loan pricing, with variable default definitions largely unstudied in literature.

Although the exact reasons for retail credit defaults are innumerable (e.g., job loss, marital breakdown, financial naivety, fraud), they are crudely grouped into either fraud (‘won’t pay’) or financial distress (‘can’t pay’), as explored in Thomas (2009, pp. 282) and Bravo, Thomas and Weber (2015). Modelling the exact reason and its underlying causes is challenging in practice since lenders rarely keep record of defaulting reasons. Instead, a more tangible approach is to consider whether an impaired ability to repay is either persistent or temporary. Sufficient patience on the lender’s part may allow certain financially-distressed borrowers enough time to recover and resume their loan repayments, at the cost of accruing arrears and increased liquidity risk. On the other hand, too long a period may prove naive with the lender partially recovering monies (if at all) at an opportunity cost. Specifying a default threshold therefore serves as a margin of tolerance towards accruing arrears before pursuing debt recovery instead, which aligns with the five-phase credit management model from Finlay (2010, pp. 11-13).

In fact, an arrears-based default definition was intrinsic to the double hurdle PD model developed in Moffatt (2005). Both the payments in arrears (the ‘first’ hurdle) and the arrears amount itself (the ‘second’ hurdle) was used in classifying overall ‘default’. This recognises that not all defaults are equal in their financial impact. The recent work of Kelly and McCann (2016) also supports this notion, using mortgage defaults from the Irish market to model so-called ‘deep defaults’ (360+ days in arrears). A legal peculiarity during 2009-2013 made it extremely difficult for Irish lenders to liquidate defaulted mortgages, which led to an artificially high level of arrears. Banks had little
recourse but to relax their risk aversion and adapt to so-called ‘deeper’ defaults as the new norm. If nothing else, this particular instance casts doubt on both the meaning and supposed severity of the default decision itself as well as using 90 days past due as the point of default.

In summary, varying the outcome period in a roll rate analysis will give different transition rates amongst arrears categories, simply due to sampling. This presents additional uncertainty as certain outcome periods may obscure idiosyncratic features of the loan portfolio. For example, it would be difficult to decide whether a particularly low curing rate (estimated across a certain outcome period) is attributable to the risk profile of borrowers, a shift in market conditions, or simply too short an outcome period – without conducting additional analysis. Furthermore, the mere possibility of curing from default injects uncertainty into a chosen default definition, which is supposedly the point at which the relationship between bank and borrower ultimately crumbles away. Finally, there is little objective evidence for the presupposed profit-optimality of using Basel II’s 90 days past due as a sacrosanct default definition. For these reasons, a new approach to finding optimal default definitions is deemed necessary, based on profitability instead of model accuracy or roll rate analyses.

3 Optimising default thresholds: a simulation study

The term ‘delinquency’ is interpreted as a measurable and variable quantity that signifies the severity of eroded trust between bank and borrower. A ‘delinquency measure’ $g$ should then reflect the extent of non-payment fundamentally based on a borrower owing $I_t > 0$ (instalment) though only repaying $R_t \geq 0$ (receipt) at a particular time $t$. The function $g$ then measures delinquency by quantifying the extent $I_t - R_t$ by which the borrower chips away at the communal trust. For this study, three different delinquency measures (see appendix) are used:

1. The popular accountancy-based number of payments in arrears, called the $Contractual \text{ Delinquency}$ (or $CD$-measure $g_1$);
2. The $Macaulay \text{ Duration}$ index-based measure (or $MD$-measure $g_2$) from Sah [2015], which is an index of the weighted average time to recover the capital portion of a loan;
3. A modified version of $g_2$, called the $Degree \text{ of Delinquency}$ (or $DoD$-measure $g_3$), which incorporates the sizes of disrupted cash flows in assessing delinquency.

Regardless of $g$, a procedure is developed in this section for finding the ‘best’ default threshold from a portfolio loss perspective. To test this procedure, a simulation framework is also described for generating various portfolios across several credit risk scenarios.

3.1. A procedure for optimising default thresholds

Consider a portfolio of $N$ loans, indexed by $i = 1, \ldots, N$, and let $g(i, t)$ denote the value of a particular measure $g \in \{g_1, g_2, g_3\}$ at periods $t = 0, \ldots, T_i$ with $T_i$ representing the contractual term of the $i^{th}$ account. Let $v_t^{(a)}$ and $v_t^{(b)}$ be standard actuarial discounting functions that respectively use an alternative risk-free interest rate and the client interest rate in discounting back $t$ periods. Let $R_t^i$ and $I_t^i$ be the receipt and expected instalment respectively
at time $t$ for the $i^{th}$ account. Then, let $R(i, t)$ be the summed historical receipts up to $t$, expressed as

$$R(i, t) = \sum_{l=0}^{t} R_i^l v_i^{(a)}.$$  \hfill (1)

For the remaining future instalments, let $O(i, t)$ denote the expected outstanding balance at $t$, defined as

$$O(i, t) = v_i^{(a)} \sum_{l=t+1}^{T_i} I_i^l v_i^{(b)} . \quad O(i, t) = 0 \quad \text{for } t = T_i .$$  \hfill (2)

To cater for arrears, let $A(i, t)$ be the historical and cumulative shortfall up to $t$ between instalments and receipts, given by

$$A(i, t) = \sum_{l=0}^{t} \left( I_i^l - R_i^l \right) v_i^{(a)}.$$  \hfill (3)

Financial loss can only be realised when the lender disposes of the impaired asset, regardless of the extent of impairment. Having breached the default threshold (signifying broken trust), the lender’s objective changes to collecting the maximum in the shortest time possible. However, changing the default threshold also implies a variable workout period between the default time and eventual resolution, i.e., curing from default or write-off. As a simplifying assumption for this study, the loan is immediately written-off at some rate upon entering ‘default’, regardless of its definition. As such, let $r_E \in [0, 1]$ be a loss rate applied on $O(i, t)$. Moreover, assume that $A(i, t)$ is also partly written-off though at another loss rate $r_A \in [0, 1]$. Using two different rates recognises that the recovery success may differ between these two components (expected balance and arrears). Finally, let $l(i, t)$ be the discounted loss assessed at $t$ and expressed as

$$l(i, t) = O(i, t) r_E + A(i, t) r_A .$$  \hfill (4)

In optimising default definitions, let $d \geq 0$ be the default threshold such that the $i^{th}$ account is considered as $(g, d)$-defaulting if and only if $g(i, t) \geq d$ at any particular time $t = 1, \ldots, T_i$. Accordingly, let $S_D$ be the subset of all $(g, d)$-defaulting accounts such that

$$S_D = \{ i \mid \exists t \in [0, T_i] : g(i, t) \geq d \} .$$  \hfill (5)

Since an account may enter and leave the ‘default’ state multiple times in reality, let $l_i^{(g,d)}$ be the earliest moment of default for the $i^{th} (g, d)$-defaulting account, defined as

$$l_i^{(g,d)} = \min \{ t : g(i, t) \geq d \}, \quad \forall \ i \in S_D .$$  \hfill (6)

Similarly, let $S_P$ be the subset of all accounts considered as $(g, d)$-performing such that

$$S_P = \{ i : g(i, t) < d \quad \forall \ t \in [0, T_i] \} .$$  \hfill (7)

The difference in assessing losses using Eq. 4 between a $(g, d)$-defaulting and a $(g, d)$-performing account is simply the time of assessment $t$, at either $t = l_i^{(g,d)}$ or $t = T_i$ respectively. Finally, the discounted total loss $L(g, d)$ of a
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given matured portfolio using the measure $g$ and a default threshold $d$, is defined as

$$L(g, d) = \sum_{i \in S_D} l(i, t_i^{(g,d)}) + \sum_{i \in S_P} l(i, T_i).$$  \hspace{1cm} (8)$$

Fig. 2. Illustrating the loss optimisation of default thresholds across several delinquency measures. As a result, Measure 3 is chosen as the best measure with its minimum loss attained at point $c$.

Losses can now be iteratively calculated across a range of thresholds $d \in D_g$ using a particular measure $g \in \{g_1, g_2, g_3\}$ with the loss model (Eq. 8), thereby forming a loss curve across $d$ for each $g$. To populate these thresholds in $D_g$, choose a sufficiently wide range of discrete thresholds $d = 0, \ldots, d_N$ using $g_1$ as a baseline. This $d_N$ is arbitrarily chosen as 60\% of the contractual term in balancing computation time against the desired width of the eventual loss curve. Next, the ranges of real-valued measures $g_2$ and $g_3$ are binned into the same number of thresholds using a combination of equal width discretisation and discretion. In summary, $D_g$ contains an equal number of thresholds for each measure $g$ in the loss optimisation, which is accompanied by two preparatory steps:

1. Delinquency must be measured for every account and across its history using $g \in \{g_1, g_2, g_3\}$;
2. A loss model $L(g, d)$ must be applied at every relevant default threshold $d \in D_g$.

Finally, losses are aggregated twice by finding the minimum each time: first by threshold $d$ for $g$ and then by measure $g$. This forms the basis of the loss optimisation procedure. Specifically, each resulting loss curve – one for each measure $g$ – can first be inspected to find the lowest loss as well as the associated threshold for each minimum loss. Secondly, these minima $m^{(g)}$, i.e., $\min L(g, d) = m^{(g)}$, coinciding at thresholds $d^{(g)}$, i.e., $\arg_d \min L(g, d) = d^{(g)}$, can then be compared to one another. The optimal measure $g^*$ is then the one that yielded the lowest loss $m^{(g^*)}$ at corresponding threshold $d^{(g^*)}$, i.e., $g^* = \arg_g \min m^{(g)}$, as illustrated in Fig. 2. Note that this procedure can also be used for loss-optimising the threshold using a single measure $g_1$. 

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3.2. An overview of simulation techniques

A real-world portfolio inherently suffers from censoring insofar that delinquent loans are only kept on the balance sheet up to a certain point, as controlled by the bank’s policies. Alternatively, a simulation-based approach is more conducive to studying threshold optimisation from ‘first principles’ since a whole range of credit risk profiles and associated assumptions can be simulated, contrasted by a real-world portfolio’s single profile. Moreover, some delinquent accounts will simply never recover in reality, which implies a continuous stream of zeros in their receipts up to a certain point, as controlled by the bank’s policies. Alternatively, a simulation-based approach is more conducive to studying threshold optimisation from ‘first principles’ since a whole range of credit risk profiles and associated assumptions can be simulated, contrasted by a real-world portfolio’s single profile. Moreover, some delinquent accounts will simply never recover in reality, which implies a continuous stream of zeros in their receipts. Therefore, the so-called episodic defaults technique is also used wherein $p_D = 50\%$ is the given probability of default, i.e., half the portfolio is bound to have a default episode by design. Let $l_j$ be the number of consecutive non-payments to be simulated for the $j^{th}$ delinquent account within the defaulting-segment. This episode length $l_j \in [1, k]$ is sampled from the uniform distribution up to $k$, coinciding with $(k, g_1)$-truncation. When applying $(k, g_1)$-truncation, accounts will only cure if they had less than $k$ consecutive non-payments, as a limiting condition. Thereafter, the starting point $o_j \in [1, t_c - l_j]$ of the episode is also sampled from the uniform distribution up to $t_c - l_j$, which is to say the entire episode must fit within the remaining loan life. Finally, each element $R_t$ of the $j^{th}$ delinquent account is then populated with either $I$ or 0, expressed as

$$R_t = \begin{cases} I & \text{if } u_t < b \\ 0 & \text{otherwise} \end{cases}$$

Despite its simplicity, random defaults do not feasibly generate periods of consecutive non-payments followed by resumed payment, which frequently occurs in practice. Therefore, the so-called episodic defaults technique is also used wherein $p_D = 50\%$ is the given probability of default, i.e., half the portfolio is bound to have a default episode by design. Let $l_j$ be the number of consecutive non-payments to be simulated for the $j^{th}$ delinquent account within the defaulting-segment. This episode length $l_j \in [1, k]$ is sampled from the uniform distribution up to $k$, coinciding with $(k, g_1)$-truncation. When applying $(k, g_1)$-truncation, accounts will only cure if they had less than $k$ consecutive non-payments, as a limiting condition. Thereafter, the starting point $o_j \in [1, t_c - l_j]$ of the episode is also sampled from the uniform distribution up to $t_c - l_j$, which is to say the entire episode must fit within the remaining loan life. Finally, each element $R_t$ of the $j^{th}$ delinquent account is then simulated as

$$R_t = \begin{cases} 0 & \text{if } o_j \leq t \leq (o_j + l_j) \\ I & \text{otherwise} \end{cases}$$

For the actual simulation study, consider $N = 10,000$ standard amortising loan accounts that are indexed by $i = 1, \ldots, N$, with a fixed contractual term of $t_c = 60$ months, a fixed effective annual interest rate of 20%, and a fixed principal amount such that the level instalment is $I = 100$ at every period $t = 1, \ldots, t_c$. An effective annual risk-free rate of 7% is used in discounting, which is realistic for the South African market. Let the maximum loan size be $L_M = 5,000$ and let $r_E = 40\%$ and $r_A = 70\%$ with the rationale that losses on arrears ought to be penalised more than losses on expected balances. This is due to the latter being a decreasing quantity while the former increases over time for a continuously delinquent loan.

In simulating the receipt vector $R$ of each loan account, three probabilistic techniques are now described. As a basic technique (called random defaults), let $u_t \in [0, 1]$ be a randomly generated number at every period $t = 1, \ldots, t_c$ and let $b$ be the probability of payment, i.e., $P(R_t = I) = b$ with $I$ denoting the level instalment. Note that $b = 80\%$ is chosen as a default value, though this is later varied. Each element $R_t$ within $R$ is then populated with either $I$ or 0, expressed as

$$R_t = \begin{cases} 1 & \text{if } u_t < b \\ 0 & \text{otherwise} \end{cases}$$
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Realistically, an account may experience multiple default episodes during its life, though the previous episodic technique only gives one such episode. Therefore, the Markovian defaults technique is also defined wherein $X_t \in \{P, D\}$ denotes a random variable that can assume one of two states at each period $t$; the state $P : R_t = 1$ (paid) and the state $D : R_t = 0$ (delinquent). Then, let $X_1, X_2, \ldots$ be a sequence of random variables that form a discrete-time first-order Markov chain. For simplicity, assume that every accounts starts off in the paying state, which implies that the initial state probabilities are $P(X_1 = P) = 1$ and $P(X_1 = D) = 0$. Subsequently, the given transition probabilities between states at future time $t + 1$, conditional on the current state at time $t$, are denoted by the transition matrix as

$$
\begin{bmatrix}
P_{PP} & P_{PD} \\
P_{DP} & P_{DD}
\end{bmatrix} = \begin{bmatrix}
P(X_{t+1} = P \mid X_t = P) & P(X_{t+1} = D \mid X_t = P) \\
P(X_{t+1} = P \mid X_t = D) & P(X_{t+1} = D \mid X_t = D)
\end{bmatrix} = \begin{bmatrix}
P_{PP} & 1 - P_{PP} \\
1 - P_{DD} & P_{DD}
\end{bmatrix}. \tag{12}
$$

4 Simulation results of loss optimisation

Following the aforementioned simulation approach, the parameters of each technique are now varied while optimising the default thresholds towards the lowest loss. The simulation results are grouped below by technique.

4.1 Random defaults

![Fig. 3. Losses (as a proportion of summed principals) across thresholds $d$ by measure $g \in \{g_1, g_2, g_3\}$ using the random defaults technique. In (a), simulated loans are $(4, g_1)$-truncated, while they are $(6, g_3)$-truncated in (b). The zoomed plots show that global minima occur at or near the truncation point, $d = k$, in both cases.](image)

In using this technique, $(k, g)$-truncation is applied to control the simulation and to serve as a sanity check. Intuitively, one expects that the lowest loss across default thresholds $d$ to coincide wherever $d = k$, since simulated receipts are zeroed after having breached $k$. As an illustration, $(4, g_1)$ is first applied in Fig. 3a, which shows the lowest loss to occur at $d = 4$ for $g_1$. Note that $k$ is arbitrarily set, though varied later. However, the choice of $g \in \{g_1, g_2, g_3\}$ when applying $(k, g)$-truncation also introduces bias in the timing of these simulated non-payments. Specifically,
the lowest loss (across all curves) is biased towards the curve of the same \( g \) used in truncation. As shown in Fig. 3b, when using \((6, g_3)\)-truncation instead, minimum loss now occurs approximately at \( d = k = 6 \) for \( g_3 \).

In general, minimum losses ought to occur wherever \( d = k \) when \((k, g)\)-truncating simulated receipts. This is largely confirmed in Fig. 4 wherein various portfolios are generated in succession using different truncation parameters \( k = 1, \ldots, 10 \). As a result, loss minima occur consistently at the truncation point \( d = k \). Each increasing value of \( k \) also yielded a smaller minimum loss as a result of the overall lessening truncation effect. Since receipts are truncated less frequently as \( k \) increases, portfolios exhibit less delinquency, which explains both lower loss curves and lower loss minima. Although not shown, this behaviour is also true for \( g_2 \) and \( g_3 \) with associated \((k, g)\)-truncation, though minima only occur approximately at \( d = k \pm \eta \) with the discrepancy \( \eta \) becoming greater as \( k \) increases.

![Fig. 4. Losses (as a proportion of summed principals) across thresholds \( d \) for the CD-measure \( g_1 \) with \((k, g_1)\)-truncation, using the random defaults technique. Several truncation points \( k = 1, \ldots, 10 \) are used, with the zoomed plot confirming that global minima in losses occur at each truncation point \( d = k \).](image)

The effect of different credit risk profiles during loss optimisation is simulated by varying the parameter \( b \) and generating portfolios accordingly, as shown in Fig. 5. Keeping \((6, g_1)\)-truncation as a benchmark, loss minima only occur at \( d = k = 6 \) for a certain range of \( 0.5 < b < 0.94 \). This suggests that loss minima may only exist for certain risk profiles in practice. Moreover, the two boundary cases of \( b = 0 \) and \( b = 1 \) in Fig. 5 also serve as a reasonableness test in that the loss minimum should occur at \( d = 0 \) for \( b = 0 \) since all receipts will be zero by design. Conversely, if there is no credit risk, i.e., \( b = 1 \), then zero losses should occur across all thresholds \( d > 0 \) since all receipts equal instalments. Credit risk scenarios can also be simulated by varying the loss rate \( r_A \) and generating associated portfolios, as shown in Fig. 6 using \( g_1 \) (though similar results hold for \( g_2 \) and \( g_3 \)). Decreasing values of \( r_A \) flattens the loss curve, which is sensible since arrears are penalised less. Conversely, increasing values of \( r_A \) causes a greater ‘bend’ at the chosen truncation point, while loss minima materialises at the same point only for a certain range of \( r_A \). This strengthens the previous result of loss optimisation across default thresholds being
viable only for portfolios with particular risk profiles – not too much risk nor too little risk.

Fig. 5. Losses (as a proportion of summed principals) across thresholds \( d \) for the CD-measure \( g_1 \) with \((6,g_1)\)-truncation, using the random defaults technique and several probabilities of payment \( b \in [0,1] \). The zoomed plot shows a smaller range of \( 0.65 \leq b \leq 0.91 \) where loss minima occur at the chosen truncation point.

4.2. Episodic defaults

Since this technique is tightly coupled with \((k,g_1)\)-truncation by design, portfolios are generated accordingly for \( k = 1, \ldots, 10 \), as shown in Fig. 7 for \( g_1 \) (with similar results for \( g_2 \) and \( g_3 \)). Clearly, the shapes of loss curves are different, even though loss minima are still found at each successive truncation point. Also note that accounts resume payment provided that the length of their default episode is less than \( k \). Then, longer default episodes (higher \( k \)) seems to absorb the loss specifically introduced by truncation itself, which is signified by flattening loss curves for \( d \geq k \). Since higher \( k \) also implies that truncation occurs less frequently, a greater proportion of accounts with default episode length less than \( k \) will resume payment. In turn, arrears stabilise as truncation becomes less likely, which explains the flattening slopes of loss curves for greater \( k \). In general then, small \( k \) implies shorter episode lengths but more truncated accounts, while large \( k \) means longer episode lengths but less truncated accounts. It seems this technique generates portfolios with an interesting trade-off between default episode length and truncation frequency, regarding their credit risk compositions. Since loss minima are still obtained at each truncation point as hoped, it suggests that loss-optimising default thresholds will likely remain viable in practice when facing portfolios with more interesting characteristics. This includes portfolios where truly delinquent accounts (as proxy for truncation) occur more frequently (e.g., unsecured lending), though accounts are also more prone to recover from shorter bouts of delinquency, and vice versa.
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Fig. 6. Losses (as a proportion of summed principals) across thresholds $d$ for the CD-measure $g_1$ with $(6, g_1)$-truncation, using the random defaults technique and several arrears loss rates $r_A \in [0, 1]$. The zoomed plot shows a smaller range of loss rates $0.62 \leq r_A \leq 1$ where loss minima occur at the chosen truncation point.

Fig. 7. Losses (as a proportion of summed principals) across thresholds $d$ for the CD-measure $g_1$ with $(k, g_1)$-truncation, using the episodic defaults technique with $p_D = 50\%$ and several truncation points $k = 1, \ldots, 10$. The zoomed plot shows that loss minima occur at each successive truncation point.
4.3. Markovian defaults

Using this technique, various credit risk contexts are simulated by substituting in a range of transition rates $P_{PP} \in [0, 1]$, while keeping $P_{DD} = 1$ constant at first, as shown in [Fig. 8a] for $g_1$ (though similar results hold for $g_2$ and $g_3$). For $P_{PP} = 0$, accounts are immediately absorbed into the delinquency state D since $P_{DD} = 1$, thereby yielding a minimum loss at $d = 0$. Similarly, for $P_{PP} = 1$, accounts will never leave the payment state P, which explains the zero loss curve across $d > 0$. As before, loss minima exist only for a limited range of $0.5 \leq P_{PP} \leq 0.988$ at $d = 1$, as a proxy for credit risk in practice. Curiously, loss minima at $d = 1$ amounts to an unintended (and rather strict) $(1, g_1)$-truncation effect due to specifying $P_{DD} = 1$. In this case, any laxer truncation using $k > 1$ will also be indiscernible in effect due to the superseding absorption of $P_{DD} = 1$. However, $(k, g)$-truncation ought to have a more visible effect when using more transient rates $P_{DD} \rightarrow 0$ later. Lastly, as another reasonableness check, the previous results in [Fig. 3] can be emulated using Markovian defaults by specifying $P_{PP} = 0.9$ and $P_{DD} = 0.6$, whilst using the same $(4, g_1)$ and $(6, g_3)$-truncation respectively.

The range of rates $P_{PP} \in [b_L, b_U]$ wherein loss minima occur at $d = 1$ depends on $P_{DD}$, based on the simple premise that it becomes less taxing to enter the delinquency state D, if one is also more prone to exit it again. To demonstrate this, three reference values for $P_{DD} \in \{0.85, 0.6, 0.4\}$ are experimentally chosen as boundary cases. Fixing $P_{DD} = 0.85$, the range $[b_L, b_U]$ shifts lower and narrows, eventually contracting to a single point as $P_{DD}$ is decreased, until it disappears when $P_{DD} < 0.35$. Moreover, loss minima at other thresholds $d > 1$ (without applying $(k, g_1)$-truncation) also appear for an upper non-overlapping range $b_U \leq P_{PP} < 1$ when $P_{DD} = 0.85$. As $P_{DD}$ is decreased, this range widens, e.g., becoming $0.7 \leq P_{PP} < 1$ when $P_{DD} = 0.6$. The Markov chain becomes increasingly unstable for certain $P_{PP}$ as $P_{DD}$ is decreased since accounts start switching rapidly between states P and D. Should this oscillation occur relatively early in loan life, then losses are higher owing to high expected outstanding balances. As a consequence, global loss maxima can start appearing when $P_{DD} \leq 0.6$ for certain $P_{PP}$, e.g., the range $0.35 \leq P_{PP} \leq 0.55$ when $P_{DD} = 0.4$ and no truncation.

By applying $(25, g_1)$-truncation and fixing $P_{DD} = 0.4$, the aforementioned ideas are clearly manifesting in [Fig. 8b]. Firstly, it shows a diminishing range $0.46 < P_{PP} < 0.53$ wherein loss minima at $d = 1$ are still found, owing to the majority being absorbed into delinquency. Secondly, an upper range $0.53 < P_{PP} \leq 0.65$ exhibits loss minima at the chosen truncation point $d = k$, which is purposefully set high to distinguish from the previous loss minima at $d = 1$. For $P_{PP} > 0.65$, fewer accounts become (or stay) delinquent, which explains both the waning strength of truncation and the flattening loss curve at $d \geq k$. Lastly, a small range $0.53 < P_{PP} < 0.59$ also has loss maxima, though the range expands significantly to $P_{PP} < 0.59$ when removing truncation. Although believed to be a simulation artefact, the existence of loss maxima suggests an interesting dilemma wherein greater losses are attained when choosing some middling $d$ than what would have been the case when choosing either an incredibly conservative threshold (e.g., $d = 0$) or remarkably naive threshold (e.g., $d = 40$). Loss maxima can reasonably occur in practice, e.g., real-world portfolios undergoing temporary macroeconomic stress, though further research is necessary.
Fig. 8. Losses (as a proportion of summed principals) across thresholds $d$ for the $CD$-measure $g_1$, using the Markovian defaults technique with several transition rates $P_{PP} \in [0, 1]$. In (a), $P_{DD} = 1$ is fixed with loss minima occurring for $0.5 \leq P_{PP} \leq 0.988$ at $d = 1$, highlighted in the zoomed plot. In (b), $P_{DD} = 0.4$ is fixed with $(25, g_1)$-truncation wherein the zoomed plot shows an even smaller range of rates $0.46 < P_{PP} < 0.53$ where loss minima still occur at $d = 1$. Lighter encircled points show rates $0.53 < P_{PP} \leq 0.65$ where loss minima occur at the chosen truncation point $d = k$. Darker encircled points show rates $0.53 < P_{PP} < 0.59$ where loss maxima occur at staggered thresholds.
5 Conclusion

Basel II, often promulgated *verbatim* by a country’s financial regulator, allows the lender reasonable freedom to define their own default points. Yet many lenders pragmatically opt for the common ‘90 days past due’-definition and use it accordingly across collection operations, analytics, pricing, and risk modelling, amongst others. Moreover, the financial implications of a particular definition are not readily considered when choosing a default point. There is often little objective evidence for a definition’s overall suitability, beyond the questionable results from roll rate analyses or a regulator’s decree.

Therefore, a procedure is presented in this study for optimising the default threshold using profitability (or loss) as its base. To facilitate this optimisation, three delinquency measures are formulated (see appendix). Each measure $g$ is applied on the historical cash flows of all loan accounts within a portfolio. Thereafter, the loss $L(g, d)$ is calculated for each relevant default threshold $d$, resulting in a loss curve for each $g$. A loss curve can then be inspected to find the threshold at which the lowest loss occurs, which concludes the loss optimisation. This procedure also allows for the objective comparison and evaluation across multiple delinquency measures, should a lender wish to employ (or test) alternative measures.

In testing this procedure, a simulation study is conducted to generate loan portfolios across a wide range of credit risk scenarios. Three probabilistic techniques are used to generate cash flows, with each technique aiming for increased realism over the previous one. A simulation study allows for varying the parameters of each technique, thereby producing different risk profiles and portfolio characteristics. This is particularly useful for testing the loss optimisation procedure from ‘first principles’. Indeed, the simulation results show that loss minima can exist for a select range of credit risk profiles, which suggests that loss-optimising default thresholds can be a viable strategy in practice. These results also successfully frames the default decision as an optimisation problem in choosing a default threshold that is neither too early nor too late in loan life.

Future studies can focus on using real-world portfolio data in refining this procedure. Most real portfolios are censored (or ‘incomplete’) since the majority of loan accounts have not yet reached maturity, excluding written-off and settled accounts. This is a non-trivial challenge since the procedure was developed with ‘completed’ accounts in mind. Furthermore, the particular loss model, as used in the procedure, can be refined and made estimable from real-world loss experiences. As an example, the LGD can perhaps be restructured in a way that allows for estimating loss risk conditioned on a particular default definition, amongst other factors. This will intersect with the existing literature on credit loss modelling, which is currently enjoying greater research focus due to IFRS 9.

References

Baesens, B., Rösch, D. & Scheule, H. (2016). *Credit risk analytics: measurement techniques, applications, and examples in sas*. Hoboken, New Jersey: John Wiley & Sons.

Basel Committee on Banking Supervision. (2006). *International convergence of capital measurement and capital standards: a revised framework, comprehensive version*. Basel, Switzerland: Bank for International Settlements.

Bravo, C., Thomas, L. C. & Weber, R. (2015). Improving credit scoring by differentiating defaulter behaviour. *Journal of the Operational Research Society*, 66(5), 771–781.
A procedure for loss-optimising default definitions across simulated credit risk scenarios

Cohen, B. H., Edwards Jr, G. A. et al. (2017). The new era of expected credit loss provisioning. *BIS Quarterly Review*.

Crook, J. N., Edelman, D. B. & Thomas, L. C. (2007). Recent developments in consumer credit risk assessment. *European Journal of Operational Research, 183*(3), 1447–1465.

Cyert, R. M., Davidson, H. J. & Thompson, G. L. (1962). Estimation of the allowance for doubtful accounts by Markov chains. *Management Science, 8*(3), 287–303.

Finlay, S. (2010). *The management of consumer credit: theory and practice* (Second Edition). Hampshire, UK: Palgrave Macmillan.

Hand, D. J. (2001). Modelling consumer credit risk. *IMA Journal of Management Mathematics, 12*(2), 139–155.

Hand, D. J. & Henley, W. E. (1997). Statistical classification methods in consumer credit scoring: a review. *Journal of the Royal Statistical Society: Series A (Statistics in Society), 160*(3), 523–541.

Hao, C., Alam, M. & Carling, K. (2010). Review of the literature on credit risk modeling: development of the past 10 years. *Banks and Bank Systems, 5*(3), 43–60.

Harris, T. (2013a). Default definition selection for credit scoring. *Artificial Intelligence Research, 2*(4), 49–62.

Harris, T. (2013b). Quantitative credit risk assessment using support vector machines: broad versus narrow default definitions. *Expert Systems with Applications, 40*(11), 4404–4413.

Kelly, R. & McCann, F. (2016). Some defaults are deeper than others: understanding long-term mortgage arrears. *Journal of Banking & Finance, 72*, 15–27.

Kelly, R. & O’Malley, T. (2016). The good, the bad and the impaired: a credit risk model of the Irish mortgage market. *Journal of Financial Stability, 22*, 1–9.

Kennedy, K., Mac Namee, B., Delany, S. J., O’Sullivan, M. & Watson, N. (2013). A window of opportunity: assessing behavioural scoring. *Expert Systems with Applications, 40*(4), 1372–1380.

Louzada, F., Ara, A. & Fernandes, G. B. (2016). Classification methods applied to credit scoring: Systematic review and overall comparison. *Surveys in Operations Research and Management Science, 21*, 117–134.

Moffatt, P. G. (2005). Hurdle models of loan default. *Journal of the operational research society, 56*(9), 1063–1071.

Novotny-Farkas, Z. (2016). The interaction of the IFRS 9 expected loss approach with supervisory rules and implications for financial stability. *Accounting in Europe, 13*(2), 197–227.

OECD. (2018). *Household debt (indicator)*. The Organisation for Economic Co-operation and Development. Retrieved March 13, 2018, from [10.1787/f93b6469-en](https://doi.org/10.1787/f93b6469-en)

Phillips, R. (2013). Optimizing prices for consumer credit. *Journal of Revenue & Pricing Management, 12*(4), 360–377.

Sah, R. (2015). Loan recovery monitoring mechanism. *International Journal of Trade, Economics and Finance, 6*(1), 62.

Siddiqi, N. (2005). *Credit risk scorecards: developing and implementing intelligent credit scoring*. Hoboken, New Jersey: John Wiley & Sons.
A procedure for loss-optimising default definitions across simulated credit risk scenarios

Skoglund, J. (2017). Credit risk term-structures for lifetime impairment forecasting: a practical guide. *Journal of Risk Management in Financial Institutions, 10*(2), 177–195.

South Africa. (2012). *Banks Act (94/1990): Regulations relating to Banks (regulation gazette no. 35950)*. Government Gazette. Retrieved May 3, 2018, from [https://www.gov.za/sites/default/files/35950_rg9872_gon1029.pdf](https://www.gov.za/sites/default/files/35950_rg9872_gon1029.pdf)

The Board of Governors of the Federal Reserve System (US). (2018). *Households and Nonprofit Organisations; Home Mortgages; Liability, Level [HHMSDODNS]; Total Liabilities, Level [TLBSHNO]*. FRED, Federal Reserve Bank of St. Louis. Retrieved February 21, 2018, from [https://fred.stlouisfed.org/series/HHMSDODNS](https://fred.stlouisfed.org/series/HHMSDODNS)

Thomas, L. C. (2009). *Consumer credit models: pricing, profit and portfolios*. New York: Oxford University Press.

Thomas, L. C. (2010). Consumer finance: challenges for operational research. *Journal of the Operational Research Society, 61*(1), 41–52.

Thomas, L. C., Edelman, D. B. & Crook, J. N. (2002). *Credit scoring and its applications*. Philadelphia: SIAM: Monographs on Mathematical Modeling and Computation.

Van Gestel, T. & Baesens, B. (2009). *Credit risk management: basic concepts*. New York: Oxford University Press.

Xu, X. (2016). *Estimating lifetime expected credit losses under IFRS 9*. SSRN. Retrieved April 8, 2016, from [http://dx.doi.org/10.2139/ssrn.2758513](http://dx.doi.org/10.2139/ssrn.2758513)

6 Appendix – Measures of loan delinquency

Three mathematical operations are now presented as delinquency measures and discussed. Firstly, the popular number of payments/months in arrears, called the Contractual Delinquency (or CD-measure \( g_1 \)), is refined into a more robust measure. Secondly, a more concise algorithm is contributed that creates the Macaulay Duration index-based measure (or MD-measure \( g_2 \)) from Sah (2015), which is an index of the weighted average time to recover the capital portion of a loan. Finally, a modified version of the MD-measure is introduced, called the Degree of Delinquency (or DoD-measure \( g_3 \)), which incorporates the sizes of disrupted cash flows in assessing delinquency.

6.1. CD-measure \( g_1 \)

As a common measure, the unpaid portion of an amortising loan’s instalment is aged into a few increasingly severe bins, given the time elapsed: 30 days, 60 days, 90 days, and so forth, as discussed in Cyert et al. (1962). This is often converted to the number of payments in arrears (or arrears categories) simply by dividing the accumulated arrears at a particular point in time with the level instalment, followed by rounding this ratio upwards to an integer. However, this is quite stringent in that even a small difference \( I_t - R_t = \epsilon < 1 \) will increase the payments in arrears, purely due to rounding. Should the ratio instead be rounded to the nearest integer, then a change in this measure depends on whether the unrounded ratio is above or below 50%. This implied ‘threshold’ seems arbitrary and too fixed. Furthermore, this measure can potentially lag overall measurement when a significant overpayment is immediately followed by a severe underpayment the following month. Lastly, its construction quickly becomes cumbersome when the instalment is prime rate-linked and varies over time, which is common for secured lending.
Therefore, a more comprehensive variant, called the CD-measure, is presented here that circumvents these challenges. Let the receipt vector be \( \mathbf{R} = [R_0, R_1, \ldots, R_T] \) with its elements (or receipts) \( R_t \geq 0 \), and let the instalment vector be \( \mathbf{I} = [I_0, I_1, \ldots, I_T] \) with its elements \( I_t > 0 \). Both vectors are defined for a specific loan account across its discrete time periods \( t = 0, \ldots, T \), with \( t = 0 \) representing the origination point and \( T \) denoting the tenure (or current loan age). Note that \( T \) may exceed the contractual term \( t_c \), especially in cases of extreme delinquency. The repayment ratio \( h_t \in [0, \infty) \) is then defined as

\[
 h_t = \left( \frac{R_t}{I_t} \right) \quad \forall \ t = 1, \ldots, T \quad \text{and} \quad h_0 = 0 .
\] (13)

One can specify a certain threshold \( z \in [0, 1] \) for \( h_t \), above which an account at time \( t \) is considered current and beneath which it is considered delinquent. Note that \( z = 90\% \) is assumed in this study as an illustration. Next, a Boolean-valued decision function \( d_1(t) \in \{0, 1\} \) is defined for \( t = 1, \ldots, T \), using Iverson brackets \([a]\) that outputs 1 if the enclosed statement \( a \) is true, and 0 if false, as

\[
d_1(t) = [h_t < z].
\] (14)

Memory of past delinquency is introduced by defining another integer-valued function \( m(t) \in \{-1, 0, 1, \ldots\} \) for \( t = 1, \ldots, T \), which outputs the reduction in accrued delinquency (if any), as

\[
m(t) = \left( \frac{h_t}{z} \right) - 1 \left( 1 - d_1(t) \right) - d_1(t) \\
= \left( \frac{h_t}{z} \right) \left( 1 - d_1(t) \right) - 1.
\] (15)

This function \( m(t) \) gives the magnitude by which the measured delinquency at time \( t \) should be reduced (if at all) in catering for past delinquency. When overpaying, i.e., \( R_t > I_t \), the ratio between \( h_t \) and \( z \) in Eq. 15 signifies the total number of ‘payments’ by which accrued delinquency should be decreased. The floor is taken since \( g_1 \) should reflect payments in arrears and must therefore have a discrete scale. However, the currently owed instalment should be recognised first before reducing any accrued delinquency, by subtracting one instalment. For underpayment, i.e., \( R_t < zI_t \), the delinquency is increased by one payment, which resolves to \( m(t) = -1 \) when \( d_1(t) = 1 \).

To indicate previous cases of delinquency using \( g_1 \) at time \( t - 1 \), let \( d_2(t) \in \{0, 1\} \) be another Boolean-valued decision function for \( t = 1, \ldots, T \), which is defined using Iverson brackets again, as

\[
d_2(t) = [g_1(t - 1) = 0].
\] (16)

Finally, the reduction in delinquency \( m(t) \) at time \( t \) is subtracted from delinquency measured at the previous period \( t - 1 \), thereby giving the net number of payments in arrears respective to \( z \). The integer-valued CD-measure \( g_1(t) \geq 0 \) for \( t = 1, \ldots, T \) is then recursively expressed as

\[
g_1(t) = \max \left[ 0, \ d_1(t)d_2(t) + \left( 1 - d_2(t) \right) \left( g_1(t - 1) - m(t) \right) \right].
\] (17)

Note the necessary starting condition of \( g_1(0) = 0 \), since a newly-disbursed loan account cannot yet be delinquent.
6.2. MD-measure $g_2$

Recently introduced in Sah (2015), the Macaulay Duration Index is based on bond duration, i.e., the weighted average time to recover the capital portion of a loan. By comparing actual duration to expected duration, it incorporates interest rates and the time value of money of arrears amounts in assessing delinquency. Naturally, its output cannot be compared directly to the previous $g_1$ since both its scale and meaning differs. To ease the construction of $g_2$, a new algorithm is presented here.

Let $\Delta_t = I_t - R_t$ be the difference between the instalment $I_t$ and the receipt $R_t$ at every time point $t = 0, \ldots, T$ of a loan account, including at disbursement $t = 0$ (to capture any account initiation fees). Considering the time value of money, let $\nu_j = (1 + r)^{-j}$ be a discounting function that uses a nominal monthly interest rate $r$. In addition, let $\delta$ be the continuously compounded rate with its nominal variant $\delta^{(n)} = \delta/p$ and with an annual compounding period $p$. Let $L_P$ denote the loan amount (or principal) that is to be amortised. While the Macaulay Duration is ordinarily calculated at origination as the weighted average time to recover cash flows, here it is recursively calculated instead at each subsequent period $t = 0, \ldots, T$ across the remaining $m$ instalments. Naturally, this expected duration quantity, denoted as $f_{ED}(t)$, tends towards zero over time as it nears the end of loan life, expressed as

$$ f_{ED}(t) = \sum_{m=t}^{T} \left( \frac{I_m \nu_{(m-t)}}{L_P} \right) \left( \frac{m-t}{p} \right) \quad \forall \ t = 0, \ldots, T . \quad (18) $$

However, Eq. 18 assumes that instalments $I$ are free of uncertainty. A marked difference is reasonably expected when substituting these instalments with the actual receipts $R$. Moreover, it becomes necessary to track the arrears balance as it develops (if it does) over the loan life. In line with Sah (2015), any arrears at any time are added to the last expected (contractual) instalment at $t = t_c$, since it represents the last contractual opportunity to repay any such arrears, short of the lender intervening and restructuring the loan. This last instalment is then recursively updated for each subsequent period $t$, denoted by the vector $I'$, which equals instalments $I$ at first. Likewise, the actual duration $f_{AD}(t)$ is also recursively calculated for each subsequent period $t$. This is illustrated using pseudo-code in Algorithm 1.

**Algorithm 1 Calculating $g_2$**

1. $I' := I$, where $I = [I_0, \ldots, I_T]$ and $T \leq t_c$
2. $f_{AD}(0) := f_{ED}(0)$
3. for $t = 0, \ldots, T$ do
   4. $I'_r := I'_r + \Delta_t \left( 1 + \frac{\delta^{(n)}}{p} \right)^{T-t}$, $\forall \ t = 1, \ldots, T$ $\Rightarrow$ such that $T \leq t_c$
   5. $f_{AD}(t) := \sum_{m=t}^{T} \left[ \frac{I'_m \nu_{(m-t)}}{L_P} \right] \left( \frac{m-t}{p} \right)$, $\forall \ t = 1, \ldots, T$ $\Rightarrow$ Add any arrears to $I'_{(T)}$
6. end for

Finally, the real-valued Macaulay Duration (MD) measure $g_2(t) \geq 0$ is then defined as the ratio between the actual duration and the expected duration at time points $t = 0, \ldots, T - 1$, which is expressed as

$$ g_2(t) = \frac{f_{AD}(t)}{f_{ED}(t)}. \quad (19) $$
6.3. **DoD-measure $g_3$**

From a cash flow perspective, an ideal delinquency measurement should penalise the non-payment of a larger loan’s instalment to a greater degree than that of a smaller loan’s instalment, given the relatively larger impact on a bank’s cash flow. Furthermore, the differences in risk concentration between a larger number of small loans versus a small number of larger loans should also be incorporated by the ideal delinquency measure. As a possible solution, the actual duration $f_{AD}(t)$ from Eq. 19 can be altered such that the eventual $g_2(t)$ is greater for larger loans than for smaller loans by defining an appropriate multiplier function.

Whilst refining $g_2$, note that it is only defined up to the contractual term $t_c$. However, delinquency can continue even past its contractual term $T \geq t_c$, likely due to persisting underpayment. Ignoring loan write-off policies for the moment, let $d_3(t) \in \{0, 1\}$ be a Boolean-valued decision function that returns 1 if the given time point $t$ precedes the contractual term $t_c$, and 0 otherwise. Using Iverson brackets, this is expressed as

$$d_3(t) = \left[ t \leq t_c \right].$$

(20)

When $t > t_c$, any arrears can clearly no longer be added to the last contractual instalment (since it has lapsed), as was added for $I'_{I}$ at $T = t$ when calculating $g_2$ in **Algorithm 1**. Instead, at least one more payment, albeit out-of-contract, can reasonably be expected at every subsequent period $t : t \geq t_c$ as long as collection efforts are actively pursued. Therefore, delinquency can now be computed up to time $T$ instead of the previous $T$, with $T$ either representing the contractual term $t_c$ when $t < t_c$, or becoming a moving target $T = t$ when $t \geq t_c$. Note that both $I$ and $R$ will incrementally expand with additional elements for as long as collection efforts continues past the contractual term. A revised algorithm is given in **Algorithm 2**.

**Algorithm 2 Calculating $g_3$**

1: $I' := I$, where $I = [I_0, \ldots, I_T]$ and $0 < t_c \leq T$
2: $T := t_c$
3: for $t = 0, \ldots, T$ do  
4: $\alpha := I'_{(T)}(t)$
5: $T := t_c + (1 - d_3(t))$  

$\triangleright$ This refers to the element at the $T^{th}$ position of $I'$  

$\triangleright$ $T$ is either equal to $t_c$ or to $t \geq t_c$
6: $I'_{(T)} := I'_{(T)} + \Delta_t \left(1 + \frac{\beta(t)}{p}\right)^{-t} + \alpha \left(1 - d_3(t)\right) \left(1 + \frac{\delta(t)}{p}\right), \quad \forall t = 1, \ldots, T$

$\triangleright$ Discounting periods, used in next two lines
7: $\beta(m) := m - t + 1 - d_3(t), \quad \forall t = 1, \ldots, T$
8: $f_{ED}(t) := \sum_{m=t}^{T} \left[ \left(\frac{l_m \psi_0(m)}{L_p}\right) \left(\frac{\delta(m)}{p}\right)\right], \quad \forall t = 0, \ldots, T$
9: $f_{AD}(t) := f_{ED}(t), \quad \forall t = 0$
10: $f_{AD}(t) := \sum_{m=t}^{T} \left[ \left(\frac{l_m \psi_0(m)}{L_p}\right) \left(\frac{\delta(m)}{p}\right)\right], \quad \forall t = 1, \ldots, T$
11: end for

Afterwards, let $\lambda(L_M, L_p, s)$ denote a multiplier function that inflates $f_{AD}(t)$ at the period $t$. Let $L_M$ denote the maximum loan size and let $s \in [0, 1]$ be a real-valued sensitivity that represents the ‘strength’ at which to apply this inflationary effect. Let $d_4(t) \in \{0, 1\}$ be another Boolean-valued decision function that returns 1 if there is currently any accrued delinquency at $t$, and 0 otherwise, defined using Iverson brackets as

$$d_4(t) = \left[ f_{AD}(t) > f_{ED}(t) \right].$$

(21)
As a simple example, this multiplier is defined as

$$\lambda(L_M, L_P, s) = s \left( 1 - \frac{L_M - L_P}{L_M} \right).$$

(22)

The inflated variant of $f_{AD}(t)$, denoted as $\tilde{f}_{AD}(t)$, is given by

$$\tilde{f}_{AD}(t) = f_{AD}(t) \left( d_4(t) \lambda(L_M, L_P, s) + 1 \right).$$

(23)

By including $d_4(t)$ into $\tilde{f}_{AD}(t)$ in Eq. 23, accrued delinquency will not be inflated when overpaying at some period $t$. Finally, the real-valued Degree of Delinquency (DoD) measure $g_3(t) \geq 0$ is defined for $t = 0, \ldots, T - 1$ and expressed as

$$g_3(t) = \frac{\tilde{f}_{AD}(t)}{f_{ED}(t)} = \left( \frac{g_2(t)}{f_{AD}(t)} \right) \tilde{f}_{AD}(t) = g_2(t) \left( d_4(t) \lambda(L_M, L_P, s) + 1 \right).$$

(24)

The sensitivity $s$, which is fixed in this study at $s = 100\%$ (though should ideally be optimised), represents a universal and intuitive lever at the lender’s disposal. Its adjustment can align with the lender’s particular risk appetite and tolerances. At $s = 0$, $g_3$ collapses back into $g_2$, though it purposefully resembles a more risk-adverse form of $g_2$ for $s > 0$. Delinquency values are more varied than those of $g_2$ due to the inherent sensitivity to loan principals by design.