Analytical controllability of deterministic scale-free networks and Cayley trees

Ming Xu,1, 2 Chuan-Yun Xu,1 Huan Wang,3 Cong-Zheng Deng,2 and Ke-Fei Cao1

1Center for Nonlinear Complex Systems, Department of Physics, School of Physics Science and Technology, Yunnan University, Kunming, Yunnan 650091, China
2School of Mathematical Sciences, Kaili University, Kaili, Guizhou 556011, China
3Department of Computer Science, Baoji University of Arts and Sciences, Baoji, Shaanxi 721016, China

According to the exact controllability theory, the controllability is investigated analytically for two typical types of self-similar bipartite networks, i.e., the classic deterministic scale-free network and Cayley trees. Due to their self-similarity, the analytical results of the exact controllability are obtained, and the minimum sets of driver nodes (drivers) are also identified by elementary transformations on adjacency matrices. For these two types of undirected networks, no matter their links are unweighted or weighted, the controllability of networks and the configuration of drivers remain the same, showing a characteristic of strong structural controllability. These results have implications for the control of real networked systems with self-similarity.

PACS number(s): 89.75.Hc, 02.30.Yy, 02.10.Ox

I. INTRODUCTION

The control of complex networks is of paramount importance in network science and engineering, which has received extensive attention in the past decade or so [1–7]. In control theory, the controllability property plays a pivotal role in many control problems. Generally speaking, a dynamical system is controllable if it can be driven from any initial state to any desired final state in finite time by a suitable choice of inputs [8, 9]. The theory of controllability is mathematically sound, and has been widely applied to engineering. However, it is difficult to apply the traditional controllability theory directly to complex networks. Here, the first challenge faced is how to find the minimum set of driver nodes (i.e., drivers) needed to fully control the whole network [2], which is a computationally prohibitive task for large networks by the traditional Kalman’s controllability rank condition [8, 10]. In recent years, controllability has become a hot research topic in the field of complex networks [4, 5, 11, 12, 13]. Liu et al. introduced a paradigm to study the structural controllability of an arbitrary complex directed network; this is an important progress in identifying a minimum set of drivers [4]. Another significant recent contribution was made by Yuan et al. [5]: according to the Popov–Belevitch–Hautus (PBH) rank condition [10, 11, 12, 13], they developed an exact controllability framework to determine the minimum set of drivers from the maximum multiplicity of eigenvalues of the network matrix by elementary column transformation. With this framework, the exact controllability has been studied for many networks such as Erdös–Rényi random networks, scale-free networks, small-world networks, simple regular networks, and some real networks [2]. Furthermore, some analytical results about the minimum number of drivers, which are determined by the algebraic multiplicity of the eigenvalue 0 of the coupling (adjacency) matrix, are obtained in ref. [3] for three typical regular fractal networks including the modified (1, 2)-tree network, the Peano network and the modified dual Sierpinski gaskets network.

In this paper, the exact controllability is explored for two typical types of self-similar bipartite networks including the classic deterministic scale-free network (DSFN) [19] and Cayley trees [20, 21]. Here, along the framework in ref. [3], we obtain both the minimum number and the minimum set of drivers using elementary row and column transformations on adjacency matrices. Since scale-free networks are abundant in nature and society, and Cayley trees can model the structure of dendrimers, a classic family of macromolecules; therefore, the study of the controllability for these two types of networks is of theoretical and practical significance.

This paper is arranged as follows. First, we briefly introduce the exact controllability theory and provide a sufficient condition for the expression to determine the minimum number of drivers for undirected bipartite networks. Second, the self-similarity in these two types of networks allows us to find some properties of the coupling matrix. By properly using elementary transformations of a matrix, we can find the minimum set of drivers and obtain analytical results of the controllability in these networks. Finally, the distribution characteristics of drivers and a robustness of the controllability to the link weights are discussed.

II. EXACT CONTROLLABILITY THEORY

Consider a network of $N$ nodes described by the following system of linear ordinary differential equations [4, 5]:

$$\dot{x} = Ax + Bu,$$

where $x = (x_1, x_2, \ldots, x_N)^T$, $u = (u_1, u_2, \ldots, u_M)^T$, $A \in \mathbb{R}^{N \times N}$, and $B \in \mathbb{R}^{N \times M}$ denote, respectively, the vector of the states of $N$ nodes, the vector of $M$ controllers, the coupling (adjacency) matrix of the network, and the input (control) matrix. According to the PBH
rank condition, system (1) is fully controllable if and only if
\[
\text{rank}(cI_N - A, B) = N \tag{2}
\]
is satisfied for any complex number \(c\), where \(I_N \in \mathbb{R}^{N \times N}\) is the identity matrix. Thus, the minimum number \(N_D\) of independent drivers required to control the whole network, which is defined by \(N_D \equiv \min \{\text{rank}(B)\} \tag{4}\), can be deduced as \([1]\):
\[
N_D = \max_i \{\mu(\lambda_i)\}, \tag{3}
\]
where \(\mu(\lambda_i) = N - \text{rank}(\lambda_i I_N - A)\) stands for the geometric multiplicity of the eigenvalue \(\lambda_i\) of matrix \(A\). For a symmetric coupling matrix \(A\), Eq. (3) can be written as follows \([1]\):
\[
N_D = \max_i \{\delta(\lambda_i)\}, \tag{4}
\]
where \(\delta(\lambda_i)\) is the algebraic multiplicity of the eigenvalue \(\lambda_i\), which is also the eigenvalue degeneracy of matrix \(A\).

Especially, for a large sparse undirected network, in which the number of links is of the same order as the number of nodes in the limit of large \(N\) \([22]\), in the absence of self-loops or with a small fraction of self-loops, the maximum geometric (or algebraic) multiplicity of the coupling matrix occurs at the eigenvalue \(\lambda = 0\) with high probability \([22, 23]\), which yields a simplified expression for \(N_D\) \([2]\):
\[
N_D = \max_i \{1, N - \text{rank}(A)\}. \tag{5}
\]
Since Eq. (4) is not always true for sparse undirected networks, for example, in a ring network with \(N\) nodes, whose eigenvalues are given as \(\lambda_i = 2\cos(2\pi(i - 1)/N)\) \((i = 1, 2, \ldots, N)\), it is known that \(N_D = 2 \tag{6}\), whereas \(\max \{1, N - \text{rank}(A)\} = \max \{1, 0\} = 1 \neq N_D\) when \(N\) is a prime number; hence, if a sufficient condition is given for Eq. (4), it will be convenient to use Eq. (5) for determining \(N_D\).

As we have known, a graph (network) \(\Gamma\) is bipartite if and only if for each eigenvalue \(\lambda\) of \(\Gamma\), \(-\lambda\) is also an eigenvalue, with the same multiplicity \([24]\). So, for an undirected bipartite network, if the following sufficient condition
\[
\delta(0) \geq \frac{N}{3} \tag{6}
\]
is satisfied, then
\[
N_D = \max_i \{\delta(\lambda_i)\} = \delta(0) \tag{7}
\]
and Eq. (4) hold.

How can we find the minimum set of drivers for an undirected network when Eq. (7) holds? According to the exact controllability theory \([9]\), the control matrix \(B\) to ensure full control should meet the condition (Eq. (2)) by substituting 0 for the complex number \(c\), as follows:
\[
\text{rank}[\lambda I_N - A, B] = N, \tag{8}
\]
which means that the \(N\) rows of \([-A, B]\) is linearly independent. To find the drivers from \(B\) to satisfy Eq. (8), we should select a maximal linearly independent group from all row vectors of \(A\), and the redundant rows would correspond to the drivers. Note that the configuration of drivers is not unique because there are many possible choices of maximal linearly independent groups. Nevertheless, the minimum number of drivers is unchanged.

Elementary transformations of matrices play a crucial role in finding the drivers. There are three types of elementary row transformations of a matrix \([25]\): interchanging rows \(i\) and \(j\), denoted by \(R_i \leftrightarrow R_j\); multiplying a row (say, the \(j\)th row) by a nonzero number \(k\), denoted by \(kR_j\); and adding \(k\) times the \(j\)th row to the \(i\)th row, denoted by \(R_i + kR_j\), where \(i \neq j\). Similarly, we can also define elementary column transformations by changing “row” into “column” (the corresponding symbol is changed from “\(R\)” to “\(C\)”, respectively). Note that the rank of a matrix remains unchanged by elementary matrix transformations. Furthermore, if matrix \(A_1\) can be obtained from matrix \(A_2\) by a series of elementary operations except \(R_i \leftrightarrow R_j\), then the redundant rows in \(A_1\) correspond with that in \(A_2\). Namely, row \(i\) of \(A_2\) is redundant \(\iff\) row \(i\) of \(A_1\) can become a redundant row.

According to ref. \([4]\), the controllability of a network can be defined by the ratio of \(N_D\) to the network size \(N\), i.e.,
\[
n_D = \frac{N_D}{N}. \tag{9}
\]
In the following, we analytically derive \(N_D\) and \(n_D\) of two typical types of self-similar bipartite networks and consequently discuss their configurations of drivers.

III. DETERMINISTIC SCALE-FREE NETWORKS

The first DSFN was proposed by Barabási et al. \([19]\), which is a self-similar hierarchical network model, built in an iterative way. It is illustrated in Fig. 1 with empty circles and red filled circles respectively showing the hub and rim nodes, which has a bipartite structure. In this classic model, the total number of nodes and that of links (edges) in the \(g\)th step (generation) are counted as \(N_g = 3^g\) and \(E_g = 2(3^g - 2^g)\), respectively \([26]\), implying that this network is sparse when \(g\) is large enough. Thus, for the \(g\)th generation \(D_g\) of the DSFN, the minimum number \(N_D(D_g)\) of drivers can be determined conveniently and analytically.

Let \(A_g\) stand for the adjacency matrix of \(D_g\). Using the numbering shown in Fig. 1 the adjacency matrices are expressed by \([26]\):
\[
A_1 = \begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}, \tag{10}
\]
So, for any block diagonal matrix:

\[ \begin{bmatrix} A & 0 & \cdots & 0 \\ 0 & A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A \end{bmatrix} \]

and so forth. Here all blanks are zeros. Obviously, matrix \( A_1 \) can be transformed as:

\[ A_1 \xrightarrow{R_3-R_2 \quad C_3-C_2} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \]

and \( A_2 \) transformed as a block diagonal matrix:

\[ A_2 \rightarrow \begin{bmatrix} A_1 & A_1 \\ A_1 & A_1 \end{bmatrix} \]

by operations of \( R_9 - R_2, R_8 - R_2, R_6 - R_2, R_5 - R_2, C_9 - C_2, C_8 - C_2, C_6 - C_2, \) and \( C_5 - C_2. \) In the same way, it can be proven that \( A_g \) can also be transformed as a block diagonal matrix:

\[ A_g \rightarrow \begin{bmatrix} A_{g-1} & A_{g-1} \\ A_{g-1} & A_{g-1} \end{bmatrix} \]

So, for any \( g \geq 1, \) we have \( \text{rank}(A_g) = 3 \cdot \text{rank}(A_{g-1}) = \cdots = 3^{g-1} \cdot \text{rank}(A_1) = 2 \cdot 3^{g-1}, \) which implies \( \delta_g(0) = \text{rank}(A_g) = 3^g - \text{rank}(A_g) = 3^g - 1 = N_g/3. \) According to Eqs. (17) and (19), the minimum number \( N_D \) of drivers for \( D_g \) is given by

\[ N_D(D_g) = \max\{1, 3^g - \text{rank}(A_g)\} = 3^g - 1. \] (15)

The controllability \( n_D \) of the DSFN \( D_g \) is then calculated as a constant:

\[ n_D(D_g) = \frac{N_D(D_g)}{N_g} = \frac{3^g - 1}{3^g} = \frac{1}{3}. \] (16)

From transformations (12), (13) and (14), we can identify the redundant rows or drivers for the DSFN \( D_g \) (see Table I). For instance, from the transformed matrix (12) of \( A_1, \) we can determine the driver of \( D_1 \) to be node 3 because the third row is redundant. Of course, we can also choose node 2 as the driver of \( D_1 \) if we transform the second row of \( A_1 \) in (10) into a redundant row. Hence the minimum set of drivers for the DSFN is not unique. For other self-similar DSFNs generated by iteration, their controllability and drivers can also be investigated similarly.

### IV. CAYLEY TREES

Cayley trees are a classic model for dendrimers [20, 21]. Let \( C_{b,g} (b \geq 3, g \geq 0) \) denote the Cayley trees after \( g \) generations, which can be constructed in the following iterative fashion. Initially \( (g = 0), \) \( C_{b,0} \) is just a central node (the core). Next \((g = 1), b \) nodes are generated connecting the central node to form \( C_{b,1}, \) with the \( b \) single-degree nodes constituting the boundary nodes of \( C_{b,1}. \) For any \( g \geq 2, C_{b,g} \) is built from \( C_{b,g-1}: \) for each boundary node of \( C_{b,g-1}, b - 1 \) new nodes are created and linked to the boundary node. A specific Cayley tree \( C_{b,3} \) is shown in Fig. 2.

Let \( N_b(i) \) denote the number of new nodes created at generation (iteration) \( i. \) It is easy to verify that

\[ N_b(i) = \begin{cases} 1, & i = 0; \\ b(b-1)^{i-1}, & i \geq 1. \end{cases} \] (17)
Thus, the total number of nodes in $C_{b,g}$ (network size of $C_{b,g}$) is
\begin{equation}
N_{b,g} = \sum_{i=0}^{g} N_b(i) = \frac{b(b-1)^g - 2}{b - 2},
\end{equation}
and the total number of edges in $C_{b,g}$ is $E_{b,g} = N_{b,g} - 1$. Both Eqs. (17) and (18) can be written recursively as 
$\gamma \geq 3$.

To find the drivers from $C_{3,g}$, reducing $A_{3,g}$ is needed. Let $\gamma$ represent the set $\{1, 2, \ldots, N_3(g-1)\}$, and $a_{i,j}$ the element at row $i$ and column $j$ of matrix $A_{3,g}$. To begin with, performing on $A_{3,g}$ a series of elementary row transformations: $R_k - a_{i,N_3,g-2+j} R_{N_3,g-1+j}$ where $i \in \gamma - \{N_3(g-1)\}$, $j \in \{1, 2, \ldots, N_3(g-1)\}$, we know that $A_{3,g}$ can be transformed as:
\begin{equation}
A_{3,g} \rightarrow \begin{bmatrix}
* & O & O & O \\
* & O & I_{N_3(g-1)} & O \\
O & O & I_{N_3(g-1)} & O \\
O & O & O & O
\end{bmatrix},
\end{equation}
where * denotes the entries that remain unchanged. Consequently, by a series of elementary column transformations similarly, $A_{3,g}$ can be further reduced as:
\begin{equation}
A_{3,g} \rightarrow \begin{bmatrix}
* & O & O & O \\
O & O & I_{N_3(g-1)} & O \\
O & I_{N_3(g-1)} & O & O \\
O & O & O & O
\end{bmatrix},
\end{equation}
Thus, we have $\text{rank}(A_{b,g}) = 2N_3(g-1) + \text{rank}(A_{b,g-2})$.

For $g \geq 2$, let $\alpha$ represent the set of nodes belonging to $C_{3,g-1}$, and $\beta$ the set of newly created nodes at the $g$th generation. Then, the $N_3 \times N_3$ square matrix $A_{3,g}$ ($\geq 2$) can be written in the following block form:
\begin{equation}
A_{3,g} = \begin{bmatrix}
A_{\alpha,\alpha} & A_{\alpha,\beta} \\
A_{\beta,\alpha} & A_{\beta,\beta}
\end{bmatrix} = \begin{bmatrix}
A_{3,g-1} & A_{\alpha,\beta} \\
A_{\beta,\alpha} & O
\end{bmatrix},
\end{equation}
where the matrix $A_{\alpha,\alpha}$ is just equal to $A_{3,g-1}$, $A_{\beta,\beta}$ is a zero matrix $O$ since there is no connection between any two nodes in $\beta$, and $A_{\alpha,\beta} = A_{\beta,\alpha}^\top$ can be explained by zero and identity matrices as:
\begin{equation}
A_{\beta,\alpha} = \begin{bmatrix}
O_{N_3(g-1) \times N_3,g-2} & I_{N_3(g-1)} \\
O_{N_3(g-1) \times N_3,g-2} & I_{N_3(g-1)}
\end{bmatrix}.
\end{equation}

To find the drivers from $C_{3,g}$, reducing $A_{3,g}$ is needed. Let $\gamma$ represent the set $\{1, 2, \ldots, N_3\}$, and $a_{i,j}$ the element at row $i$ and column $j$ of matrix $A_{3,g}$. To begin with, performing on $A_{3,g}$ a series of elementary row transformations: $R_i - a_{i,N_3,g-2+j} R_{N_3,g-1+j}$ where $i \in \gamma - \{N_3,g-1+j\}$, $j \in \{1, 2, \ldots, N_3(g-1)\}$, we know that $A_{3,g}$ can be transformed as:
\begin{equation}
A_{3,g} \rightarrow \begin{bmatrix}
* & O & O & O \\
* & O & I_{N_3(g-1)} & O \\
O & O & I_{N_3(g-1)} & O \\
O & O & O & O
\end{bmatrix},
\end{equation}
where * denotes the entries that remain unchanged. Consequently, by a series of elementary column transformations similarly, $A_{3,g}$ can be further reduced as:
\begin{equation}
A_{3,g} \rightarrow \begin{bmatrix}
A_{3,g-2} & O & O & O \\
O & O & I_{N_3(g-1)} & O \\
O & I_{N_3(g-1)} & O & O \\
O & O & O & O
\end{bmatrix},
\end{equation}
Thus, we have $\text{rank}(A_{b,g}) = 2N_3(g-1) + \text{rank}(A_{b,g-2})$.

Note that $\text{rank}(A_{b,0}) = 0$ and $\text{rank}(A_{b,1}) = 2$. So, if $g \geq 2$ is even, then
\begin{equation}
\text{rank}(A_{b,g}) = 2N_3(g-1) + 2N_3(g-3) + \ldots + 2N_3(g-3) + \text{rank}(A_{b,0})
\end{equation}
\begin{equation}
= 2(b-1)^g - 2 \frac{(b-1)^g - 2}{b - 2};
\end{equation}
if $g \geq 3$ is odd, then
\begin{equation}
\text{rank}(A_{b,g}) = 2N_3(g-1) + 2N_3(g-3) + \ldots + 2N_3(g-3) + \text{rank}(A_{b,1})
\end{equation}
\begin{equation}
= 2(b-1)^g - 2 \frac{(b-1)^g - 2}{b - 2}.
\end{equation}
Thus, for any $g \geq 0$, we always have
\begin{equation}
\text{rank}(A_{b,g}) = 2 \frac{(b-1)^g - 2}{b - 2}.
\end{equation}
Therefore, the algebraic multiplicity of eigenvalue 0 for $C_{b,g}$ can be deduced as
\begin{equation}
\delta_g(0) = N_3, \quad \text{rank}(A_{b,g}) = (b-1)^g,
\end{equation}
which implies $\delta_g(0) > N_{b,g}/3$. According to Eqs. (11) and (15), we obtain
\begin{equation}
N_D(C_{b,g}) = \delta_g(0) = (b-1)^g.
\end{equation}
The controllability $n_D$ of the Cayley tree $C_{b,g}$ can be given by

$$n_D(C_{b,g}) = \frac{N_D(C_{b,g})}{N_{b,g}} = \frac{(b-1)^g(b-2)}{b(b-1)^g - 2}$$

with its thermodynamic limit

$$\lim_{g \to \infty} n_D(C_{b,g}) = \lim_{g \to \infty} \frac{(b-1)^g(b-2)}{b(b-1)^g - 2} = \frac{b - 2}{b}$$

for any given $b \geq 3$.

In Fig. 3 we show the analytical results of the controllability $n_D$ of Cayley trees $C_{b,g}$ with $b = 3, 4$ and $5$, respectively. We see that the controllability measure $n_D$ decreases monotonically and approaches rapidly its limit as $g$ increases, and the analytical results obtained from Eq. (31) are in exact agreement with the numerical results based on Eq. (11). Here, when $g$ increases, although both the minimum number $N_D(C_{b,g})$ of drivers and the network size $N_{b,g}$ increase exponentially, the ratio of the increase of drivers to that of nodes is constant for two adjacent $g$ values: $\Delta N_D(C_{b,g})/\Delta N_{b,g} = (b-2)/b$, thus leads to the thermodynamic limit of $n_D$ as shown in Eq. (32) and Fig. 3.

For $C_{b,g}$ ($b \geq 3, g \geq 0$), let $\beta_{b,j} = \{N_{b,j-1}+1, N_{b,j-1}+2, \ldots, N_{b,j}\}$ ($j \leq g$) denote an ordered set of newly created nodes at the $j$th generation. Obviously, the number of elements of $\beta_{b,j}$ is $|\beta_{b,j}| = N_{b,j}$. By removing the first $N_b(j-1)$ elements from $\beta_{b,j}$, we can get another set denoted by $\beta_{b,j} = \{N_{b,j-1}+N_b(j-1)+1, N_{b,j-1}+N_b(j-1)+2, \ldots, N_{b,j}\}$ with $(b-2)N_b(j-1)$ elements. For example, $\beta_{b,0} = \beta_{b,0} = \{1\}$, $\beta_{b,1} = \{2, 3, 4\} - \{2\} = \{3, 4\}$, and $\beta_{b,1} = \{3, 4, \ldots, b+1\}$. From transformations (13) and (24), we can thus identify the redundant rows or drivers for (dendrimers modeled by) Cayley trees (see Table I). Here, the choice of $\beta_{b,j}$ is not unique: one can also, for instance, obtain $\beta_{b,j}^-$ by removing the last $N_b(j-1)$ elements from $\beta_{b,j}$.

V. DISCUSSION AND CONCLUSIONS

From Table I we can see that, although these self-similar networks analyzed are all constructed by iteration, their exact controllability $n_D$ can be quite different. The controllability may vary with parameters in some situations such as in Cayley trees (see Eq. (24) and Fig. 3) or be just a constant such as in the DSFN (see Eq. (16)).

The configuration of drivers is related to the degree distribution. For the DSFN $D_g$, the drivers all come from the rim nodes, which implies that the drivers are likely located on the low-degree nodes. For Cayley trees $C_{b,g}$ ($b \geq 3, g \geq 2$), the nodes can be divided into two parts: $\alpha_{b,g-1} = \{1, 2, \ldots, N_{b,g-1}\}$ (the set of nodes of $C_{b,g-1}$) and $\beta_{b,g} = \{N_{b,g-1}+1, N_{b,g-1}+2, \ldots, N_{b,g}\}$; obviously, $\alpha_{b,g} = \alpha_{b,g-1} \cup \beta_{b,g}$. We know that the degree of any node in $\beta_{b,g}$ is 1, and the proportion of drivers in $\beta_{b,g}$ is $n_{D,1}(\beta_{b,g}) = |\beta_{b,g}^+|/|\beta_{b,g}| = (b-2)/(b-1)$. While the degree of every node in $\alpha_{b,g-1}$ is $b \geq 3$, and correspondingly, the proportion of drivers in $\alpha_{b,g-1}$ is $n_{D,1}(\alpha_{b,g-1}) = (b-1)^{g-1} - |\beta_{b,g}^-|/|\alpha_{b,g-1}| = (b-1)^{g-2}(b-2)/(b-1)^{g-1} - 2$. To compare them, we introduce a degree preference index defined as their ratio:

$$p_{b,1}(g) = \frac{n_{D,1}(\alpha_{b,g-1})}{n_{D,1}(\beta_{b,g})} = \frac{(b-1)^{g-1}}{b(b-1)^{g-1} - 2} \leq \frac{1}{b-1},$$

with the limit $\lim_{g \to \infty} p_{b,1}(g) = 1/b$. Obviously, $p_{b,1}(g)$ is always smaller than 1, which means that the proportion of drivers among high-degree nodes, $p_{b,1,b}$, is lower (see Fig. 4). Moreover, from the perspective of the transformations of the coupling matrix, the rows and columns of low-degree nodes are more easily eliminated. Therefore, the drivers in both models tend to avoid the high-degree nodes (or hubs), which is in agreement with ref. 4.

In fact, our study can be extended to the DSFN and Cayley trees with arbitrary nonzero link weights. It is worth noting that the results remain exactly the same because the weight values do not affect the structure and elementary transformations of adjacency matrices. That is, the exact controllability and the distribution of drivers are independent of link weights in these two types of networks. In other words, they are robust to the variation of link weights, showing a characteristic of strong structural controllability $^{21, 22}$. A possible reason for this robustness lies in that the self-similar structures of adjacency matrices are always preserved under elementary transformations. Such a robustness would be of significance, as the link weights for most real networks are either unknown or known only approximately.

All in all, the controllability of self-similar networks can be predicted by applying the exact controllability.
theory. In particular, for undirected bipartite networks, we provide a sufficient condition (5) for the expression (Eq. 5) of the minimum number $N_D$ of drivers. So, the controllability can be completely determined by the rank of the coupling matrix, which can be analytically derived by elementary matrix transformations because of the self-similarity of the network. Furthermore, using these transformations properly, we can identify the drivers (driver nodes) and reveal their distribution characteristics. In this paper, two prototypical self-similar bipartite networks, including the DSFN and Cayley trees, have been explored to validate our analytical results. From our research experience, exploring structural features of coupling matrices by elementary transformations, which can determine not only the controllability but also the drivers’ distributions, will contribute to a deeper understanding for the control of real systems.

ACKNOWLEDGMENTS

This work was supported by the National Natural Science Foundation of China (NSFC) (Grant No. 11365023). MX was supported by the Scientific and Technological Fund of Guizhou Province (Grant No. J-2013-2260), the Joint Fund of Guizhou Province (Grant Nos. LH-2014-7231 and J-LKK-2013-31), and the Natural Science Research Project of the Department of Education of Guizhou Province (Grant No. KY-2013-185). The authors would like to thank Prof. S.-L. Peng for his helpful discussions.

[1] S. H. Strogatz, Nature 410, 268 (2001).
[2] A. Lombardi and M. Hörnquist, Phys. Rev. E 75, 056110 (2007).
[3] S. Meyn, Control Techniques for Complex Networks (Cambridge University Press, New York, NY, 2008).
[4] Y.-Y. Liu, J.-J. Slotine, and A.-L. Barabási, Nature 473, 167 (2011).
[5] Z. Yuan, C. Zhao, Z. Di, W.-X. Wang, and Y.-C. Lai, Nat. Commun. 4, 2447 (2013).
[6] J. Li, Z. Yuan, Y. Fan, W.-X. Wang, and Z. Di, EPL (Europhys. Lett.) 105, 58001 (2014).
[7] Y.-C. Lai, Natl. Sci. Rev. 1, 339 (2014).
[8] R. E. Kalman, J. Soc. Indust. Appl. Math. Ser. A Control 1, 152 (1963).
[9] J.-J. E. Slotine and W. Li, Applied Nonlinear Control (Prentice Hall, Englewood Cliffs, NJ, 1991).
[10] W. J. Rugh, Linear System Theory, 2nd ed. (Prentice Hall, Upper Saddle River, NJ, 1996).
[11] G. Notarstefano and G. Parlangeli, IEEE Trans. Automat. Contr. 58, 1719 (2013).
[12] J. Ruths and D. Ruths, in Complex Networks IV, Proceedings of the 4th Workshop on Complex Networks CompleNet 2013, edited by G. Ghoshal, J. Poncela-Casasnovas, and R. Tolksdorf, Studies in Computational Intelligence Vol. 476 (Springer, Berlin, 2013), p. 185.
[13] S. Nie, X. Wang, H. Zhang, Q. Li, and B. Wang, PLoS ONE 9, e89066 (2014).
[14] G. Chen, Int. J. Control Autom. Syst. 12, 221 (2014).
[15] T. Jia and M. Pósfai, Sci. Rep. 4, 5379 (2014).
[16] Z. Yuan, C. Zhao, W.-X. Wang, Z. Di, and Y.-C. Lai, New J. Phys. 16, 103036 (2014).
[17] M. L. J. Hautus, Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen, Series A: Mathematical Sciences 72, 443 (1969).
[18] E. D. Sontag, Mathematical Control Theory: Deterministic Finite Dimensional Systems, 2nd ed., Texts in Applied Mathematics Vol. 6 (Springer, New York, NY, 1998).
[19] A.-L. Barabási, E. Ravasz, and T. Vicsek, Physica A 299, 559 (2001).
[20] C. Cai and Z. Y. Chen, Macromolecules 30, 5104 (1997).
[21] A. Julaiti, B. Wu, and Z. Zhang, J. Chem. Phys. 138, 204116 (2013).
[22] M. E. J. Newman, Phys. Rev. E 69, 066133 (2004).
[23] A. Papoulis and S. U. Pillai, Probability, Random Variables and Stochastic Processes, 4th ed. (McGraw-Hill, New York, NY, 2002).
[24] A. E. Brouwer and W. H. Haemers, Spectra of Graphs (Springer, New York, NY, 2012).
[25] J. H. M. Wedderburn, Lectures on Matrices, American Mathematical Society Colloquium Publications Vol. 17 (American Mathematical Society, Providence, RI, 1934).
[26] K. Iguchi and H. Yamada, Phys. Rev. E 71, 036144 (2005).
[27] C.-T. Lin, IEEE Trans. Automat. Contr. 19, 201 (1974).
[28] A. Chapman and M. Mesoabi, in 2013 American Control Conference (ACC) (17–19 June 2013, Washington, DC, USA) (IEEE, 2013), p. 6126.