Generical behavior of flows strongly monotone with respect to high-rank cones

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Abstract: We consider a $C^{1,\alpha}$ smooth flow in $\mathbb{R}^n$ which is “strongly monotone” with respect to a cone $C$ of rank $k$, a closed set that contains a linear subspace of dimension $k$ and no linear subspaces of higher dimension. We prove that orbits with initial data from an open and dense subset of the phase space are either pseudo-ordered or convergent to equilibria. This covers the celebrated Hirsch’s Generic Convergence Theorem in the case $k = 1$, yields a generic Poincaré-Bendixson Theorem for the case $k = 2$, and holds true with arbitrary dimension $k$. Our approach involves the ergodic argument using the $k$-exponential separation and the associated $k$-Lyapunov exponent (that reduces to the first Lyapunov exponent if $k = 1$.)

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1 Introduction

We are interested in the global dynamics of a semiflow $\Phi_t$ “monotone with respect to a cone” $C$ of rank-$k$ on a Banach space $X$. Here, a cone $C$ of rank $k$ is a closed subset of $X$ that contains a linear subspace of dimension $k$ and no linear subspaces of higher dimension. It is not a cone defined normally in the literature, but we adopt this definition in honor of the pioneering work of Fusco and Oliva [10] in a finite-dimensional space, and of Krasnoselskii et al. [28] in a Banach space $X$.

A convex cone $K$ (defined in the normal sense) does give rise to a cone of rank 1, $K \cup (-K)$. Therefore, the class of semiflows we consider includes the order-preserving (monotone) semiflows intensively studied since the series of groundbreaking work of Hirsch [15–20] and Matano [39,40]; see also monographs or surveys [21,55,65,66] for more details. There is however an essential difference between a convex cone $K$ and a high-rank cone due to the lack of convexity in the latter. The lack of convexity requires substantially new techniques for exploring the implication of monotonicity for the global and/or generic dynamics of the considered semiflows.

An important example of a semiflow monotone with respect to a cone of high-rank is the monotone cyclic feedback system (for example, [13,35,37]) arising from a wide range of cellular, neural and physiological control systems. A special class of a monotone cyclic feedback system with positive feedback generates a semiflow that is monotone in the sense of Hirsch (see, for example, [35]). However, a large class of monotone cyclic feedback systems are with negative feedback, such systems generate semiflows which are monotone with respect to a sequence of nested cones of even-ranks (see, e.g. [31,70]). Negative feedbacks embedded in a cyclic architecture are believed to be the underling source of oscillatory dynamic patterns. Indeed, a Poincaré-Bendixson theorem has been established for many monotone cyclic feedback systems with negative feedback even when time lags are involved in the feedback [34,36,37]. Other models arising from important applications, to which our generic convergence theory can be applied, include high dimensional competitive systems (see, for example, [2,17,21,25,41]); systems with quadratic cones and Lyapunov-like functions (see, for example, [49,67,68]); as well as systems with integer-valued Lyapunov functionals such as scalar parabolic equations on a circle (see, for example, [7,8,26,61,69,70]) and tridiagonal systems (see, for example, [6,38,63,64]).

For the sake of easy reference, in what follows, we call a semiflow, that is monotone with respect to a convex cone $K$ (that induces a 1-cone $K \cup (-K)$), a classical monotone semiflow. The core to the huge success of developing the theory and applications of global dynamics for a
classical monotone semiflow is the Generic Convergence Theorem, due to Hirsch. The Hirsch’s generic convergence theorem concludes that the set of all \( x \in X \), for which the omega-limit set \( \omega(x) \) of \( x \) satisfies \( \omega(x) \subset E \) (the set of equilibria), is generic (open-dense, residual) in \( X \).

Subsequent studies further establish that for a classical smooth strongly monotone systems, precompact semi-orbits are generically convergent to equilibria in the continuous-time case \([53, 54, 62]\) or to cycles in the discrete-time case \([22, 56, 71]\).

The Generic Convergence Theorem of Hirsch and its extensions are based on an observation that for a classical strongly monotone system, there are exactly two different kinds of nontrivial orbits: pseudo-ordered orbits and unordered orbits. Here, a nontrivial orbit: pseudo-ordered orbits and unordered orbits. Here, a nontrivial orbit \( O(x) := \{ \Phi_t(x) : t \geq 0 \} \) is pseudo-ordered if \( O(x) \) possesses one pair of distinct ordered-points \( \Phi_t(x) \) and \( \Phi_s(x) \) (i.e., \( \Phi_t(x) - \Phi_s(x) \in K \cup (-K) \)); otherwise, it is called unordered (see Definition 2.3 with \( C = K \cup (-K) \)).

The fundamental result in the classical strongly monotone semiflow theory, the Monotone Convergence Criterion, asserts that every pseudo-ordered precompact orbit converges monotonically to equilibria (see, for example, \([65, \text{Theorem 1.2.1}]\)). Further developments from this Monotone Convergence Criterion including the Nonordering of Limit Sets and Limit Set Dichotomy (See, for example, \([14, 21, 65]\)), provide the critical technical tools used to establish the Generic Convergence Theorem.

In comparison, the structure of an \( \omega \)-limit set of a pseudo-ordered semi-orbit for a semiflow strongly monotone with respect to a \( k(\geq 2) \)-cone \( C \) is much more complicated, due to the lack of convexity. Sanchez \([59, 60]\) addressed this problem and showed that the closure of any orbit in the \( \omega \)-limit set of a pseudo-ordered orbit is ordered with respect to \( C \). In particular, Sanchez \([59]\) obtained a Poincaré-Bendixson type result for pseudo-ordered orbits when \( k = 2 \): the \( \omega \)-limit set of a pseudo-ordered orbit containing no equilibria is a closed orbit. The work \([59, 60]\) for smooth finite dimensional flows, for which the \( C^1 \)-Closing Lemma was required, was recently extended in Feng, Wang and Wu \([11]\) to a semiflow on a Banach space without the smoothness requirement. It was showed in \([11]\) that the \( \omega \)-limit set \( \omega(x) \) of a pseudo-ordered semi-orbit admits a trichotomy, i.e., \( \omega(x) \) is either ordered; or \( \omega(x) \subset E \); or otherwise, \( \omega(x) \) possesses a certain ordered homoclinic property. Given the possibility of an ordered homoclinic structure in the \( \omega \)-limit set of an pseudo-ordered semi-orbit, we anticipate a number of difficulties one has to face to establish an extension of the Hirsch’s Generic Convergence Theorem for flows monotone with respect to \( k(\geq 2) \)-cone \( C \).

With \( k > 1 \), due to the loss of convexity, the “order”-relation defined by \( C \) (see Section 2) is
not a partial order since it is neither antisymmetric nor transitive. In addition, kernel tools such as the Nonordering of Limit Sets and Limit Set Dichotomy which have played a crucial role in proving Hirsch’s Generic Convergence Theorem are no longer valid. We need novel techniques to understand the generic dynamics without these technical tools and under the assumptions that the $\omega$-limit set of even a pseudo-ordered orbit may contain an ordered homoclinic structure.

In the present paper, we focus on the generic behavior of a flow in $\mathbb{R}^n$ strongly monotone with respect to $k(\geq 2)$-cone $C$. Before describing our approach and main results, we formulate the basic assumptions:

(FWW) The flow $\Phi_t$ on $\mathbb{R}^n$ is strongly monotone with respect to a $k$-cone $C$, is $C^{1,\alpha}$-smooth and its $x$-derivative $D_x\Phi_t$ satisfies that $D_x\Phi_t(C \setminus \{0\}) \subset \text{Int}C$ for $t > 0$.

We refer to Section 2 for more detailed definitions about strong monotonicity and Hölder continuity. In what follows, we assume (FWW) assumption holds. Let

$$Q = \{x \in \mathbb{R}^n : \text{the orbit } O(x) \text{ is pseudo-ordered}\}.$$

**Theorem A. (Generic Dynamics Theorem)** Let $D \subset \mathbb{R}^n$ be an open, bounded, and positively invariant set. Then the set $\{x \in D : x \in Q \text{ or } \omega(x) \text{ is a singleton}\}$ contains an open and dense subset of $D$.

This theorem, in a slightly stronger version, will be proved in Section 5 (Theorem 5.1). It concludes that, in finite-dimensional cases, generic orbits of a smooth flow strongly monotone with respect to $C$ are either pseudo-ordered or convergent to a single equilibrium. Needless to say that, if the rank $k = 1$, Theorem A in conjunction with the Monotone Convergence Criterion implies the celebrated Hirsch’s Generic Convergence Theorem.

In the special case where $k = 2$, together with the results obtained in our study [11], we have the following:

**Theorem B. (Generic Poincaré-Bendixson Theorem)** Let $k = 2$. Let $D \subset \mathbb{R}^n$ be an open, bounded, and positively invariant set. Then, for generic (open and dense) points $x \in D$, the $\omega$-limit set $\omega(x)$ containing no equilibria is a single closed orbit.

Theorem B (and its slightly stronger version Theorem 5.3) is titled as the Generic Poincaré-Bendixson Theorem. Note that while the Poincaré-Bendixson type result was obtained in [11,59] for just certain (i.e., pseudo-ordered) orbits, here we conclude the Poincaré-Bendixson property for generic orbits.
A critical ingredient of our approach is the smooth ergodic argument motivated by Poláčik and Tereščák [55]. According to Tereščák [70], the cone invariance condition of $D_x\Phi_t$ in (FWW) implies that the linear skew-product flow $(\Phi_t, D\Phi_t)$ admits a $k$-exponential separation along any compact invariant set $\Omega$ associated to the $k$-cone $C$. Roughly speaking, this property describes that the product bundle $\Omega \times \mathbb{R}^n$ admits a decomposition $\Omega \times \mathbb{R}^n = E \bigoplus F$ into two closed invariant subbundles of $(\Phi_t, D\Phi_t)$, one with $k$-dimensional fibres $\{E_x\}_{x \in \Omega}$, the other with $(n-k)$-dimensional fibres $\{F_x\}_{x \in \Omega}$ not containing a nonzero vector in $C$, such that for any $(x, v) \in \Omega \times \mathbb{R}^n$ with $v \notin F_x$, the $E$-component of $D_x\Phi_tv$ dominates the $F$-component as $t \to +\infty$ (see Definition 3.1 or its versions for random dynamics in [31, 32]). This $k$-exponentially separated continuous decomposition is also called dominated-splitting in Differentiable Dynamical Systems (see, for example, [3, 4, 51, 58]) and in control theory (see, for example, [5]).

When $C$ is a 1-cone and $\Omega$ is just a single point, the $k$-exponential separation is equivalent to the celebrated Krein-Rutman Theorem [29, 33, 48] (or Perron-Frobenius Theorem [12] in finite-dimensions). While, for strongly positive linear skew-product (semi)flows or random dynamical systems, the existence of 1-exponential separation associated to a convex cone $K$ can be found in [23, 43–47, 52, 57] with important applications. In particular, the 1-exponential separation plays a key role in proving the generic convergence to cycles for classical smooth strongly monotone discrete-time dynamical systems [56, 71].

The $k$-exponential separation is one of the major tools in the proof of our main results. In particular, the $k$-exponential separation allows us to employ the linearization to examine the behavior of orbits of $\Phi_t$ near a compact invariant set $\Omega$. Another critical tool we use is the $k$-Lyapunov exponent of $x \in \Omega$, defined as

$$\lambda_{kx} = \limsup_{t \to +\infty} \frac{\log m(D_x\Phi_t|_{E_x})}{t},$$

where $m(D_x\Phi_t|_{E_x}) = \inf_{v \in E_x \cap \mathcal{S}} \|D_x\Phi_tv\|$ is the infimum norm of $D_x\Phi_t$ restricted to $E_x$. Properties of $k$-exponential separation and $\lambda_{kx}$ will be described in Section 3, among which is an important characterization of a point in $k$-cone $C$ in terms of its projections to $E$ and $F$ of the $k$-exponential separation (Lemma 3.5).

The Multiplicative Ergodic Theorem (see, for example, [24, 30, 50, 72]), ensures that $\lambda_{kx}$ is actually the limit, not just the superior limit, for “most” points $x \in \Omega$. Such points for which $\lambda_{kx}$ is the limit are said to be regular; and other points are said to be irregular. According to the sign of the $k$-Lyapunov exponents of the regular/irregular points on any given $\omega$-limit set $\omega(x)$, we develop our technical proofs for main Theorems into three cases in Section 4. Firstly,
we show that if \( \omega(x) \) contains a regular point \( z \) such that \( \lambda_k z \leq 0 \), then either \( x \in Q \) or \( \omega(x) \) is a singleton (Theorem 4.2). Secondly, we prove that if \( \lambda_k z > 0 \) for any \( z \in \omega(x) \), then \( x \) is highly unstable (Lemma 4.4), and belongs to the closure \( \overline{Q} \) (Theorem 4.5). Thirdly, we show that if \( \lambda_k \tilde{z} > 0 \) for any regular point \( \tilde{z} \in \omega(x) \) but there is an irregular point \( z \) with \( \lambda_k z \leq 0 \), then \( x \in Q \) (Theorem 4.6).

The smooth ergodic argument of Poláčik and Tereščák [56] was used in our proof. However, for \( k(\geq 2) \)-cones, the loss of convexity of \( C \) prevents the utilization of the order-topology (that is valid only for 1-cone) to estimate the evolution of the orbits near \( \omega(x) \). We overcome such a difficulty by Lemma 3.5 that provides a characterization of the point in \( C \) in terms of its projections to the bundles \( E \) and \( F \) associated with the \( k \)-exponential separation. We should mention that for the flows monotone with respect to a 1-cone, some ergodic theory arguments were used in [56] to exclude the possibility of the third case. This case may happen when \( k > 1 \), so we need to deal with this case by estimating the \( t \)-derivative of \( \Phi_t \) at the irregular points.

Our generic convergence and generic Poincaré-Bendixson theorems hold for any flow satisfying (FWW). We illustrate these general results in Section 6 by an application to an \( n \)-dimensional ODE system with a quadratic cone. R. A. Smith [67, 68] proved a Poincaré-Bendixson theorem for this system by assuming the existence of a certain infinite quadratic Lyapunov function, and Sanchez [59] established the connections between Smith’s results and the smooth systems strongly monotone with respect to some cones of rank-2. Our general theory allows us to establish a generic Poincaré-Bendixson Theorem for such a high-dimensional system even if a quadratic Lyapunov function can not be constructed.

2 Notations and Preliminary Results

We start with some notations and a few definitions. Let \( \mathbb{R}^n \) be the \( n \)-dimensional Euclidean space with a norm \( ||\cdot|| \). A flow on \( \mathbb{R}^n \) is a continuous map \( \Phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) with: \( \Phi_0 = \text{Id} \) and \( \Phi_t \circ \Phi_s = \Phi_{t+s} \) for \( t, s \in \mathbb{R} \). Here, \( \Phi_t(x) = \Phi(t, x) \) for \( t \in \mathbb{R} \) and \( x \in \mathbb{R}^n \) and \( \text{Id} \) is the identity map on \( \mathbb{R}^n \). A flow \( \Phi_t \) on \( \mathbb{R}^n \) is \( C^{1,\alpha} \)-smooth if \( \Phi|_{\mathbb{R} \times \mathbb{R}^n} \) is a \( C^{1,\alpha} \)-map (a \( C^1 \)-map with a locally \( \alpha \)-Hölder derivative) with \( \alpha \in (0, 1] \). The derivatives of \( \Phi_t \) with respect to \( x \), at \( (t, x) \), is denoted by \( D_x \Phi_t(x) \).

Let \( x \in \mathbb{R}^n \), the orbit of \( x \) is denoted by \( O(x) = \{ \Phi_t(x) : t \geq 0 \} \). An equilibrium (also called a trivial orbit) is a point \( x \) for which \( O(x) = \{ x \} \). Let \( E \) be the set of all equilibria of \( \Phi_t \). A nontrivial orbit \( O(x) \) is said to be a \( T \)-periodic orbit for some \( T > 0 \) if \( \Phi_T(x) = x \). The
ω-limit set ω(x) of x ∈ R^n is defined by ω(x) = \bigcap_{s \geq 0} \bigcup_{t \geq s} \Phi_t(x). If O(x) is bounded, then ω(x) is nonempty, compact, connected and invariant.

A subset D is called positively invariant if Φ_t(D) ⊂ D for any t ≥ 0, and is called invariant if Φ_t(D) = D for any t ∈ R. Given a subset D ⊂ R^n, the orbit O(D) of D is defined as O(D) = \bigcup_{x \in D} O(x). A subset D is called ω-compact if O(x) is bounded for each x ∈ D and \bigcup_{x \in D} ω(x) is bounded. Clearly, D is ω-compact provided that the orbit O(D) is bounded.

Now we define k-cones of R^n.

**Definition 2.1.** A closed set C ⊂ R^n is called a cone of rank-k (abbr. k-cone) if the following are satisfied:

i) For any v ∈ C and l ∈ R, lv ∈ C;

ii) max\{dim W : C ⊃ W linear subspace\} = k.

Moreover, the integer k(≥ 1) is called the rank of C.

A k-cone C ⊂ R^n is said to be solid if its interior IntC ≠ ∅; and C is called k-solid if there is a k-dimensional linear subspace W such that W \{0\} ⊂ IntC. Given a k-cone C ⊂ R^n, we say that C is complemented if there exists a k-codimensional space H^c ⊂ R^n such that H^c ∩ C = \{0\}.

For two points x, y ∈ R^n, we say that x and y are ordered, denoted by x ≈ y, if x − y ∈ C. Otherwise, x, y are called to be unordered. The pair x, y ∈ X are said to be strongly ordered, denoted by x ≈ y, if x − y ∈ IntC.

**Definition 2.2.** A flow Φ_t on R^n is called monotone with respect to a k-solid cone C if

Φ_t(x) ∼ Φ_t(y), whenever x ∼ y and t ≥ 0;

and Φ_t is called strongly monotone with respect to C if Φ_t is monotone with respect to C and

Φ_t(x) ≈ Φ_t(y), whenever x ≠ y, x ∼ y and t > 0.

Throughout this paper, we always impose the following assumptions:

(FWW) The flow Φ_t is C^{1,α}-smooth and strongly monotone with respect to a k-cone C. Moreover, the x-derivative D_xΦ_t satisfies D_xΦ_t(C \{0\}) ⊂ IntC for t > 0.

We note that, due to the lack of convexity, the cone invariance condition in (FWW) does not have to imply Φ_t is strongly monotone with respect to k(≥ 2)-cone C.
**Definition 2.3.** A nontrivial orbit $O(x)$ is called pseudo-ordered (also called of Type-I), if there exist two distinct points $\Phi_{t_1}(x), \Phi_{t_2}(x)$ in $O(x)$ such that $\Phi_{t_1}(x) \sim \Phi_{t_2}(x)$. Otherwise, $O(x)$ is called unordered (also called of Type-II).

Hereafter, we let $Q = \{x \in \mathbb{R}^n : \text{the orbit } O(x) \text{ is pseudo-ordered}\}$. Clearly, $Q$ is an open subset of $\mathbb{R}^n$ provided that $\Phi_t$ is strongly monotone with respect to $C$. The following co-limit Lemma, which will be useful in our proof, is due to [11, Lemma 4.3].

**Lemma 2.4.** Assume that $\Phi_t$ is strongly monotone with respect to $C$. If $x_1 \sim x_2$ and there is a sequence $t_n \to \infty$ such that $\Phi_{t_n}(x_1) \to z$ and $\Phi_{t_n}(x_2) \to z$, then one has $z \in Q \cup E$.

## 3 $k$-Exponential Separation and $k$-Lyapunov Exponents

In the first part of this section, we will focus on a crucial tool in our approach called the $k$-exponential separation along any compact invariant set associated to the $k$-cone $C$.

Let $G(k, \mathbb{R}^n)$ be the Grassmanian of $k$-dimensional linear subspace of $\mathbb{R}^n$, which consists of all $k$-dimensional linear subspaces in $\mathbb{R}^n$. $G(k, \mathbb{R}^n)$ is a completed metric space by endowing the **gap metric** (see, for example, [27, 30]). More precisely, for any nontrivial closed subspaces $L_1, L_2 \subset \mathbb{R}^n$, define that

$$d(L_1, L_2) = \max \left\{ \sup_{v \in F \cap S} \inf_{u \in F \cap S} |v - u|, \sup_{v \in F \cap S} \inf_{u \in E \cap S} |v - u| \right\},$$

where $S = \{v \in \mathbb{R}^n : \|v\| = 1\}$ is the unit ball. For a solid $k$-cone $C \subset X$, we denote by $\Gamma_k(C)$ the set of $k$-dimensional subspaces inside $C$, i.e.,

$$\Gamma_k(C) = \{L \in G(k, \mathbb{R}^n) : L \subset C\}.$$

Let $K \subset \mathbb{R}^n$ be a compact subset and $S_t$ be a flow on $K$. Let $\{R^t_x, x \in K, t \in \mathbb{R}\}$ be a family of bounded linear maps on $\mathbb{R}^n$ satisfying: (i). the map $(t, x) \mapsto R^t_x : K \times \mathbb{R} \to L(\mathbb{R}^n)$ is continuous; (ii). $R^t_{x+t} = R^t_{S_t x} \circ R^t_x$ for all $t_1, t_2 \in \mathbb{R}$ and $x \in K$. The pair of families of maps $(S_t, R^t)$ is called a vector bundle (or linear skew-product) flow on $K \times \mathbb{R}^n$. In particular, $(\Phi_t, D\Phi_t)$ is a linear skew-product flow on $K \times \mathbb{R}^n$.

Let $\{Y_x\}_{x \in K}$ be a family of $k$-dimensional subspaces of $\mathbb{R}^n$. We call $K \times (Y_x)$ a $k$-dimensional continuous vector bundle if the map $K \to G(k, \mathbb{R}^n); x \mapsto Y_x$ is continuous. The continuous bundle $K \times (Y_x)$ is invariant with respect to $(\Phi_t, D\Phi_t)$ if $D\Phi_t(x)Y_x = Y_{Fx}$ for any $x \in K$ and $t \geq 0$.
Definition 3.1. Let $K \subset \mathbb{R}^n$ be an invariant compact subset for $\Phi_t$. The linear skew-flow $(\Phi_t, D\Phi_t)$ admits a $k$-exponential separation along $K$ (for short, $k$-exponential separation), if there are $k$-dimensional continuous bundle $K \times (E_x)$ and $(n-k)$-dimensional continuous bundle $K \times (F_x)$ such that

(i) $\mathbb{R}^n = E_x \oplus F_x$, for any $x \in K$;
(ii) $D_x\Phi_tE_x = E_{\Phi_t(x)}$, $D_x\Phi_tF_x \subset F_{\Phi_t(x)}$ for any $x \in K$ and $t > 0$;
(iii) there are constants $M > 0$ and $0 < \gamma < 1$ such that
$$\|D_x\Phi_tw\| \leq M\gamma^t\|D_x\Phi_tv\|$$
for all $x \in K$, $w \in F_x \cap S$, $v \in E_x \cap S$ and $t \geq 0$, where $S = \{v \in \mathbb{R}^n : \|v\| = 1\}$.

Let $C \subset \mathbb{R}^n$ be a solid $k$-cone. If, in addition,
(iv) $E_x \subset \text{Int}C \cup \{0\}$ and $F_x \cap C = \{0\}$ for any $x \in K$,
then $(\Phi_t, D\Phi_t)$ is said to admit a $k$-exponential separation along $K$ associated with $C$.

The following proposition is essentially due to Tereščák [70].

Proposition 3.2. Assume (FWW). Then $(\Phi_t, D\Phi_t)$ admits a $k$-exponential separation along $K$ associated with $C$ along any invariant compact set $K$.

Proof. See Tereščák [70, Corollary 2.2]. One may also refer to Tereščák [70, Theorem 4.1] and the same argument in [42, Proposition A.1].

Hereafter, by virtue of Proposition 3.2, we will always assume that $(\Phi_t, D\Phi_t)$ admits a $k$-exponential continuous separation $\mathbb{R}^n = E_y \oplus F_y$ on a closed bounded set $K$ with respect to $C$, which satisfies (i)-(iv) in Definition 3.1.

We now present several critical properties of such $k$-exponential continuous separation with respect to $C$. Given any point $v \in \mathbb{R}^n$ and any subset $B \subset \mathbb{R}^n$, we let $d(v, B) = \inf_{w \in B} \|v - w\|$.

Lemma 3.3. There exists a constant $\delta' > 0$ such that
$$\{v \in \mathbb{R}^n : d(v, E_y \cap S) \leq \delta'\} \subset \text{Int}C \quad \text{for any } y \in K.$$

Proof. Let $M = \bigcup_{y \in K} (E_y \cap S)$. Clearly, $M \subset \text{Int}C$. We will prove that $M$ is compact. In fact, given any sequence $\{w_n\} \subset M$, there is a sequence $\{y_n\} \subset K$ such that $w_n \in E_{y_n} \cap S$ for every $n \geq 1$. Without loss of generality, we assume that $y_n \to y \in K$. Since the map $y \mapsto E_y \cap S$ is continuous, one can find a sequence $\{v_n\} \subset E_y \cap S$ such that $\|w_n - v_n\| \to 0$ as $n \to \infty$. Then the
compactness of $E_y \cap S$ implies that there is a subsequence $v_{n_k}$ of $v_n$ such that $v_{n_k} \to w \in E_y \cap S$. So, we have $w_{n_k} \to w \in M$; and hence, $M$ is compact. Therefore, one can find a $\delta' > 0$ such that $\{v \in \mathbb{R}^n : d(v, E_y \cap S) \leq \delta'\} \subset \text{Int}C$ for any $y \in K$.

Lemma 3.4. There exists a constant $\delta'' > 0$ such that $d(v, C) > \delta''$ for any $v \in \bigcup_{y \in K}(F_y \cap S)$.

Proof. Let $N = \bigcup_{y \in K}(F_y \cap S)$. By repeating the same argument in the proof of Lemma 3.3, one can show that $N$ is compact. Recall that $C$ is closed and $(F_y \cap S) \cap C = \emptyset$, for any $y \in K$. Then, for any $v \in N$, there exists an open ball $B_{\delta(v)}(v)$ with radius $\delta(v) > 0$ such that $B_{\delta(v)}(v) \cap C = \emptyset$. Due to the compactness of $N$, we can obtain a $\delta'' > 0$ such that $d(v, C) > \delta''$ for any $v \in \bigcup_{y \in \omega(x)}(F_y \cap S)$, completing the proof.

For each $y \in K$, we further denote by $P_y : X \mapsto E_y$ the natural projection along $F_y$, and $Q_y = I - P_y$ as the projection onto $F_y$. We have the following properties for the projections:

Lemma 3.5. (i) The projections $P_y$ and $Q_y$ are bounded uniformly for $y \in K$.

(ii) There exists a constant $C_1 > 0$ such that, if $v \in \mathbb{R}^n \setminus \{0\}$ satisfies $\|P_y(v)\| \geq C_1\|Q_y(v)\|$ for some $y \in K$, then $v \in \text{Int}C$.

(iii) For any $v \in C \setminus \{0\}$, there exists a constant $\delta_3 > 0$ such that $\|Q_y(v)\| \leq \delta_3\|P_y(v)\|$ for any $y \in K$.

Proof. (i). For any $y \in K$ and $v \in \mathbb{R}^n$, we write $v = P_y(v) + Q_y(v)$, where $P_y(v) \in E_y$ and $Q_y(v) \in F_y$. So, for any $v \in \mathbb{R}^n$ with $P_y(v) \neq 0$, one has

$$\frac{\|v\|}{\|P_y(v)\|} = \frac{\|Q_y(v)\|}{\|P_y(v)\|} + \frac{\|P_y(v)\|}{\|P_y(v)\|} \geq d\left(\frac{\|Q_y(v)\|}{\|P_y(v)\|}, E_y \cap S\right).$$

Note that $\frac{\|Q_y(v)\|}{\|P_y(v)\|} \in F_y$. Then it follows from Lemma 3.3 and $F_y \cap C = \{0\}$ that $\frac{\|v\|}{\|P_y(v)\|} \geq \delta'$. This implies that $\|P_y\| \leq \frac{1}{\delta'}$ for any $y \in K$. Hence, $\|Q_y\| \leq 1 + \frac{1}{\delta'}$ for any $y \in K$.

(ii). Let $C_1 = \frac{2}{\delta'}$, where $\delta' > 0$ is defined as in (i) and Lemma 3.3. For any $v \in \mathbb{R}^n \setminus \{0\}$ satisfying $\|P_y(v)\| \geq C_1\|Q_y(v)\|$ for some $y \in K$, it is clear that $P_y(v) \neq 0$ (otherwise, $v = 0$, a contradiction). So, we define the nonzero vector $w = v/\|P_y(v)\|$, and obtain

$$\frac{\|w\|}{\|w\| - \frac{P_y(v)}{\|P_y(v)\|}} \leq \frac{\|w\|}{\|w\|} - \frac{P_y(v)}{\|P_y(v)\|} + \frac{\|Q_y(v)\|}{\|P_y(v)\|} \leq \frac{\|w\|}{\|w\|} \cdot \frac{1}{\|w\|} + \frac{\|Q_y(v)\|}{\|P_y(v)\|} \leq \frac{\|Q_y(v)\|}{\|P_y(v)\|} \leq 2\frac{\|Q_y(v)\|}{\|P_y(v)\|}.$$
Noticing that \( d(\frac{w}{\|w\|}, E_y \cap S) \leq \frac{\|w\|}{P_y(v)} - \frac{P_y(v)}{\|P_y(v)\|} \), we have
\[
d(\frac{w}{\|w\|}, E_y \cap S^1) \leq 2\frac{\|Q_y(v)\|}{\|P_y(v)\|} = \frac{2}{C_1} = \delta'.
\]
It then follows from Lemma 3.3 that \( w \in \text{Int } C \). Therefore, \( v \in \text{Int } C \).

(iii). Let \( \delta_3 = \frac{1}{\delta''} \), where \( \delta'' > 0 \) is defined as in Lemma 3.4. Given any \( v \in C \setminus \{0\} \) and any \( y \in K \), we write \( v = P_y(v) + Q_y(v) \). So, by Lemma 3.4, we have
\[
\|P_y(v)\| = \|v - Q_y(v)\| \geq d(Q_y(v), C) \geq \|Q_y(v)\| \delta'',
\]
which entails that \( \|Q_y(v)\| \leq \delta_3\|P_y(v)\| \). This completes the proof.

In the second part of this section, we will introduce the \( k \)-Lyapunov exponent for \((\Phi_t, D\Phi_t)\) on \( K \) which admits the \( k \)-exponential separation \( K \times \mathbb{R}^n = E \oplus F \), where \( E = K \times (E_x) \) and \( F = K \times (F_x) \).

For each \( x \in K \), define the \( k \)-Lyapunov exponent as
\[
\lambda_{kx} = \limsup_{t \to +\infty} \frac{\log m(D_x \Phi_t |_{E_x})}{t},
\]
where \( m(D_x \Phi_t |_{E_x}) = \inf_{v \in E_x \cap S} \|D_x \Phi_t v\| \) is the infimum norm of \( D_x \Phi_t \) restricted to \( E_x \). A point \( x \in K \) is called a regular point if \( \lambda_{kx} = \lim_{t \to +\infty} \frac{\log m(D_x \Phi_t |_{E_x})}{t} \).

**Lemma 3.6.** Let \( x \in K \). Then

(i) If \( w \in F_x \setminus \{0\} \), then \( \lambda(x, w) \leq \lambda_{kx} + \log(\gamma) \), where \( \lambda(x, w) = \lim_{t \to +\infty} t^{-1} \log \|D_x \Phi_t v\| \).

(ii) Let \( x \) be a regular point. If \( \lambda_{kx} \leq 0 \), then there exists a number \( \beta \in (\gamma, 1) \) satisfying the following property: For any \( \epsilon > 0 \), there is a constant \( C_\epsilon > 0 \) such that
\[
\|D_{\Phi_{t_1}(x)} \Phi_{t_2} w\| \leq C_\epsilon e^{\epsilon t_1 \beta t_2} \|w\|
\]
for any \( w \in F_{\Phi_{t_1}(x)} \setminus \{0\} \) and \( t_1, t_2 > 0 \).

**Proof.** (i). Since \((\Phi_t, D\Phi_t)\) admits a \( k \)-dimensional exponential separation \( K \times \mathbb{R}^n = E \oplus F \), we have
\[
\|D_x \Phi_t w\| \leq M_\gamma^t \|w\| \|D_x \Phi_t v\|
\]
for any \( w \in F_x \setminus \{0\}, v \in E_x \setminus \{0\} \) and \( t > 0 \), which implies \( \|D_x \Phi_t w\| \leq M_\gamma^t \|w\| \cdot m(D_x \Phi_t |_{E_x}) \).

Therefore,
\[
\lambda(x, w) = \limsup_{t \to +\infty} \frac{\log \|D_x \Phi_t w\|}{t} \leq \limsup_{t \to +\infty} \frac{\log (M_\gamma^t \|w\| \cdot m(D_x \Phi_t |_{E_x}))}{t}
\]
\[
= \limsup_{t \to +\infty} \frac{\log m(D_x \Phi_t |_{E_x})}{t} + \log(\gamma) \leq \lambda_{kx} + \log(\gamma).
\]
(ii). Take a number $\beta \in (\gamma, 1)$ and fix any $\varepsilon \in (0, \log(\frac{2}{\beta}))$. Since the point $x$ is regular, for any $\varepsilon > 0$, there exists some $T^*_1 > 0$ such that

$$\lambda_{kx} - \varepsilon \leq \frac{\log m(D_x \Phi_t | E_x)}{t} \leq \lambda_{kx} + \varepsilon$$

for any $t \geq T^*_1$, and hence,

$$\frac{m(D_x \Phi_{t_1+t_2} | E_x)}{m(D_x \Phi_{t_1} | E_x)} \leq e^{\lambda_{kx} t_2} e^{\varepsilon t_1} e^{\frac{\varepsilon}{2} t_2}$$

for any $t_1 \geq T^*_1$ and $t_2 \geq 0$. Note also that

$$m(D_x \Phi_{t_1} | E_x) \geq m(D_{\Phi_{t_1}(x)} \Phi_{t_2} | E_{\Phi_{t_1}(x)} \cdot m(D_x \Phi_{t_1} | E_x) \quad \text{for } t_1, t_2 \geq 0.$$ 

Then, together with the $\kappa$-exponentially separated property, we have

$$\|D_{\Phi_{t_1}(x)} \Phi_{t_2} w\| \leq M \gamma^{t_2} \|w\| \cdot m(D_{\Phi_{t_1}(x)} \Phi_{t_2} | E_{\Phi_{t_1}(x)})$$

$$\leq M \gamma^{t_2} \|w\| \cdot \frac{m(D_x \Phi_{t_1+t_2} | E_x)}{m(D_x \Phi_{t_1} | E_x)} \leq M \gamma^{t_2} \|w\| e^{\lambda_{kx} t_2} e^{\varepsilon t_1} e^{\frac{\varepsilon}{2} t_2}$$

for any $t_1 \geq T^*_1$, $t_2 \geq 0$ and $w \in F_{\Phi_{t_1}(x)} \setminus \{0\}$. Recall that $\lambda_{kx} \leq 0$. Then $\|D_{\Phi_{t_1}(x)} \Phi_{t_2} w\| \leq M e^{\varepsilon t_1} (e^{\frac{\varepsilon}{2} \gamma})^{t_2} \|w\|$. Then, we obtain

$$\|D_{\Phi_{t_1}(x)} \Phi_{t_2} w\| \leq M e^{\varepsilon t_1} \beta^{t_2} \|w\|, \quad \text{for any } t_1 \geq T^*_1, t_2 \geq 0 \text{ and } w \in F_{\Phi_{t_1}(x)} \setminus \{0\}. \quad (3.2)$$

Now, for $t_1 \in [0, T^*_1]$, we define $\chi^1 \varepsilon = \max_{0 \leq t_2 \leq T^*_1-t_1} \|D_{\Phi_{t_1}(x)} \Phi_{t_2}\|$. By the smoothness of $\Phi_t$, $\chi^1 \varepsilon < +\infty$. Let also $\chi^2 \varepsilon = \chi^1 \varepsilon \cdot \max_{0 \leq t_2 \leq T^*_1-t_1} \{e^{-\varepsilon t_1} \beta^{t_2}\}$ and $\chi^3 \varepsilon = M \chi^1 \varepsilon (e^{\gamma-1} T^*_1)$. Then, the following properties hold:

(P1) $\|D_{\Phi_{t_1}(x)} \Phi_{t_2} w\| \leq \chi^2 \varepsilon e^{\varepsilon t_1} \beta^{t_2} \|w\|$ for any $0 \leq t_1 \leq T^*_1$, $0 \leq t_2 \leq T^*_1-t_1$ and $w \in F_{\Phi_{t_1}(x)} \setminus \{0\}$;

(P2) For any $0 \leq t_1 \leq T^*_1$, $t_2 > T^*_1-t_1$ and $w \in F_{\Phi_{t_1}(x)} \setminus \{0\}$, one has

$$\|D_{\Phi_{t_1}(x)} \Phi_{t_2} w\| = \|D_{\Phi_{T^*_1}(x)} \Phi_{T^*_1-t_1} D_{\Phi_{t_1}(x)} \Phi_{T^*_1-t_1} w\| \leq \chi^3 \varepsilon e^{\varepsilon t_1} \beta^{t_2} \|w\|.$$ 

Therefore, combining with (3.2), we obtain that $\|D_{\Phi_{t_1}(x)} \Phi_{t_2} w\| \leq \max\{M, \chi^2 \varepsilon, \chi^3 \varepsilon\} \cdot e^{\varepsilon t_1} \beta^{t_2} \|w\|$ for any $w \in F_{\Phi_{t_1}(x)} \setminus \{0\}$ and $t_1, t_2 > 0$. The proof is complete with

$$C\varepsilon = \begin{cases} \max\{M, \chi^2 \varepsilon, \chi^3 \varepsilon\}, & \varepsilon \in (0, \log(\frac{2}{\beta})) \\ \max\{M, \chi^2 \log(\frac{2}{\beta}), \chi^3 \log(\frac{2}{\beta})\}, & \varepsilon \in [\log(\frac{2}{\beta}), \infty). \end{cases} \quad (3.3)$$

\[\square\]

**Remark 3.7.** When $k = 1$ in (3.1), $\lambda_{kx}$ naturally reduces to $\lambda_{1x} = \limsup_{t \to +\infty} \frac{\log \|D_x \Phi_t v_x\|}{t}$, where $\|v_x\| = 1$ with $E_x = \text{span}\{v_x\}$. In general, $\lambda_{1x}$ is referred as the first (or principal) Lyapunov exponent (see, for example, [56]).
4 \hspace{1em} k\text{-Stability and Pseudo-ordered Orbits}

We will investigate in this section the generic behavior of the orbits for the smooth flow $\Phi_t$ under the fundamental assumption (FWW).

For this purpose, throughout this section we fix an open set $D \subset \mathbb{R}^n$ that is $\omega$-compact, and focus on the types of the orbits starting from $D$. For any $x \in D$, $(\Phi_t, D\Phi_t)$ admits a $k$-exponential continuous separation $\omega(x) \times \mathbb{R}^n = E \oplus F$ with respect to $C$. Motivated by Poláčik and Tereščák [56], we will classify the dynamics for the orbit of $x$ according to the $k$-Lyapunov exponents on $\omega(x)$. To be more specific, we firstly show that if $\omega(x)$ contains a regular point $z$ such that $\lambda_{kz} \leq 0$, then either $x \in Q$ or $\omega(x)$ is a singleton (see Theorem 4.2). Secondly, we prove that if $\lambda_{kz} > 0$ for any $z \in \omega(x)$, then $x$ is highly unstable (see Lemma 4.4), and meanwhile, it belongs to the closure $\overline{Q}$ (see Theorem 4.5). Finally, we show that if $\lambda_{k\tilde{z}} > 0$ for any regular points $\tilde{z} \in \omega(x)$ but there is an irregular point $z$ with $\lambda_{kz} \leq 0$, then $x \in \overline{Q}$ (see Theorem 4.6).

We begin with the following lemma, which extends the nonlinear dynamics nearby an equilibrium, the linearization of which is governed by the Perron-Frobenius Theorem, to that nearby a regular point.

**Lemma 4.1.** Let $x$ be a regular point. If $\lambda_{kx} \leq 0$, then there exists an open neighborhood $V$ of $x$ such that for any $y \in V$, one of two following properties holds:

(a) $\|\Phi_t(x) - \Phi_t(y)\| \to 0$ as $t \to +\infty$;

(b) There exists $T > 0$ such that $\Phi_T(x) - \Phi_T(y) \in C$; an hence, $\Phi_t(x) - \Phi_t(y) \in \text{Int}C$ for any $t > T$.

**Proof.** In order to prove this lemma, we first assert that there exists an open neighborhood $V'$ of $x$ such that for any $y \in V'$, one of two following properties must hold:

(a') $\|\Phi_n(x) - \Phi_n(y)\| \to 0$ as $n \to +\infty$;

(b') There exists $N \in \mathbb{N}$ such that $\Phi_N(x) - \Phi_N(y) \in \text{Int}C$.

Before proving this assertion, we will show how it implies this lemma. In fact, fix any $y \in V$ and suppose (b) does not hold. Then (b') does not hold for $y$. By virtue of the assertion, one has $\lim_{n \to +\infty} \|\Phi_n(x) - \Phi_n(y)\| = 0$ as $n \to +\infty$. For any $t \geq 0$, we write $t = n + \tau$ with $n \in \mathbb{N}$ and $\tau \in [0, 1)$. Then

$$\|\Phi_t(x) - \Phi_t(y)\| = \|\Phi_\tau \circ \Phi_n(x) - \Phi_\tau \circ \Phi_n(y)\| \leq \sup_{s \in [0, 1]} \|D\Phi_s(B)\| \cdot \|\Phi_n(x) - \Phi_n(y)\|.$$
Here, $\mathcal{B} = \{z \in \mathbb{R}^n : \|z\| \leq 1\}$. Therefore, one has $\|\Phi_t(x) - \Phi_t(y)\| \to 0$ as $n \to \infty$, that is, (a) holds. Then, we complete the proof of this lemma.

It remains to prove the assertion. Recall that the linear skew product flow $(\Phi, D\Phi)$ on $\overline{O(x)} \times \mathbb{R}^n$ admits $k$-exponential continuous separation as $\overline{O(x)} \times \mathbb{R}^n = \overline{O(x)} \times (E_y) \oplus \overline{O(x)} \times (F_y)$ with $E_y \subset \text{Int} C \cup \{0\}$ and $F_y \cap C = \{0\}$ for any $y \in \overline{O(x)}$. Let $P_y$ be the continuous projection onto $E_y$ along $F_y$ and $Q_y = I - P_y$.

For brevity, we write $x_n = \Phi_n(x)$, $y_n = \Phi_n(y)$ and $g_n(z) = \Phi_1(x_n + z) - \Phi_1(x_n) - D_{x_n} \Phi_1 z$. Recall that $\Phi_t$ is $C^{1,\alpha}$ and $\overline{O(x)}$ is bounded. Then

$$g_n(z) = O(\|z\|^{1+\alpha})$$

as $z \to 0$ uniformly for $n \in \mathbb{N}$. Let $z_n = y_n - x_n$, then

$$z_{n+1} = D_{x_n} \Phi_1 z_n + g_n(z_n).$$

By virtue of Lemma 3.5 (ii), the assertion follows from the following statement: There exists an open neighborhood $\mathcal{W}$ of $0$ such that if $z_0 \in \mathcal{W}$, then either

(a") $z_n \to 0$ as $n \to +\infty$; or else

(b") There is an $n$ such that $\|P_{x_n} z_n\| \geq C_1 \|Q_{x_n} z_n\|$.

Here, $C_1$ is defined in Lemma 3.5 (ii).

To prove this statement and the existence of $\mathcal{W}$, we assume Case (b") does not hold for any $n \geq 0$, which means $C_1 \|Q_{x_n} z_n\| \geq \|P_{x_n} z_n\|$ for any $n \geq 0$. By utilizing $\lambda_{kx} \leq 0$, we will show that $Q_{x_n} z_n \to 0$ if $z \in \mathcal{W} \doteq \{z \in \mathbb{R}^n: \|Q_0 z_0\| < \xi\}$ with $\xi > 0$ sufficiently small to be defined later. This then implies $z_n \to 0$ as $n \to +\infty$.

For brevity, we hereafter rewrite $D_{x_n} \Phi_1$, $P_{x_n}$ and $Q_{x_n}$ as $D_n$, $P_n$ and $Q_n$, respectively. By the invariance of exponential separation, we have $D_n P_n = P_{n+1} D_n$ and $D_n Q_n = Q_{n+1} D_n$. So

$$P_{n+1} z_{n+1} = D_n P_n z_n + P_{n+1} g_n(z_n)$$

and

$$Q_{n+1} z_{n+1} = D_n Q_n z_n + Q_{n+1} g_n(z_n)$$

for any $n \geq 0$. It then follows that

$$Q_n z_n = T(n, 0)Q_0 z_0 + \sum_{k=0}^{n-1} T(n, k+1) Q_{k+1} g_k(z_k)$$

for $n \in \mathbb{N}^+$, where $T(n, m) = D_{n-1} \circ D_{n-2} \circ \cdots \circ D_m$ for $n > m$ and $T(m, m) = id_X$.
Recall that \(g_n(z) = O(\|z\|^{1+\alpha})\) as \(z \to 0\) uniformly for \(n = 0, 1, 2, \cdots\). Then there exist constants \(r > 0\) and \(C_2 > 0\) such that \(\|g_n(z)\| \leq C_2 \|z\|^{1+\alpha}\) for any \(n \geq 0\) if \(\|z\| < r\). Together with \(C_1\|Q_n z_n\| \geq \|P_n z_n\|\), one has that \(\|z_n\| \leq r\) if \(\|Q_n z_n\| \leq \frac{r}{1+C_1}\). Hence, one has that

\[
\|g_n(z_n)\| \leq C_2\|z_n\|^{1+\alpha} \leq C_2(1+C_1)^{1+\alpha}\|Q_n z_n\|^{1+\alpha}
\]  

(4.1)

whenever \(\|Q_n z_n\| \leq \frac{r}{1+C_1}\) for \(n \geq 0\).

Recall that \(x\) is a regular point with \(\lambda_{kx} \leq 0\). Then Lemma 3.6(ii) yields that there exists a constant \(\beta \in (\gamma, 1)\) such that for any \(\epsilon > 0\), we can find a number \(C_\epsilon > 0\) satisfying

\[
\|T(n,k)\| \leq C_\epsilon \epsilon^{\alpha k} \beta^{\alpha n-k}\|w\|
\]

for any \(w \in F_{zk}\) with \(n \geq k \geq 0\).

Without loss of generality, we assume \(C_\epsilon \geq 1\) and take \(\epsilon > 0\), \(\eta \in (\beta, 1)\) such that \(\epsilon^\beta \eta^\alpha < 1\). By Lemma 3.5(i), one can find an upper bound \(C_3 > 0\) such that \(\|Q_n\| \leq C_3\) for any \(n \geq 0\). Let \(C_4 = C_\epsilon(1+C_1)^{1+\alpha}C_2\epsilon\beta\eta^{-1}\) and define the following neighborhood of 0

\[
\mathcal{W} = \{z \in \mathbb{R}^n : \|Q_0 z_0\| < \xi\},
\]

where \(\xi = \frac{\rho}{2C_\epsilon}\) with \(0 < \rho \leq \min\{\frac{r}{1+C_1}, (\frac{\rho - \beta}{2C_\epsilon})^{-\alpha}\}\).

We now claim that \(\|Q_n z_n\| \leq \rho \eta^n\) for any \(n \geq 0\), where \(z_0 \in \mathcal{W}\). Clearly, \(\|Q_0 z_0\| \leq \frac{\rho}{2C_\epsilon} < \rho\). We will prove the claim by induction. Suppose that \(\|Q_k z_k\| \leq \rho \eta^k \leq \rho\) for any \(k = 0, 1, \cdots, n-1\), it suffices to prove \(\|Q_n z_n\| \leq \rho \eta^n\). Noticing that \(\rho \leq \frac{r}{1+C_1}\). The inequality (4.1) yields that \(\|g_k(z_k)\| \leq C_2(1+C_1)^{1+\alpha}\|Q_k z_k\|^{1+\alpha}\) for any \(0 \leq k \leq n-1\). Consequently,

\[
\|Q_n z_n\| \leq \|T(n,0)Q_0 z_0\| + \sum_{k=0}^{n-1}\|T(n,k+1)Q_{k+1} g_k(z_k)\|
\]

\[
\leq C_\epsilon \beta^n \|Q_0 z_0\| + \sum_{k=0}^{n-1} C_\epsilon \epsilon^{(k+1)\beta} \beta^{\alpha n-k-1}\|Q_{k+1}\| \|g_k(z_k)\|
\]

\[
\leq \frac{\beta^n}{2} \rho + C_\epsilon \left(\sup_{0 \leq k \leq n-1}\|Q_{k+1}\|\sum_{k=0}^{n-1}\|\epsilon^{(k+1)\beta}\beta^{\alpha n-k-1}C_2(1+C_1)^{1+\alpha}\|Q_k z_k\|^{1+\alpha}\right)
\]

\[
\leq \rho \cdot \frac{\beta^n}{2} + \rho \cdot C_4 \sum_{k=0}^{n-1} (\epsilon^{\beta} \eta^\alpha)^k \beta^{\alpha n-k-1} \eta^{k+1} \rho^\alpha.
\]

Recall that \(0 < \epsilon^\beta \eta^\alpha < 1\). One has

\[
\|Q_n z_n\| \leq \eta^n\left[\frac{\beta}{2} \epsilon^{\beta} \eta^\alpha + \rho C_4 \cdot \rho^\alpha \cdot \frac{\eta}{\eta - \beta}\right].
\]

Together with \(\beta < \eta < 1\) and \(\rho \leq (\frac{\eta - \beta}{2C_\epsilon})^{-\alpha}\), we obtain that \(\|Q_n z_n\| \leq \rho \cdot \eta^n\). Thus, we have proved the claim and hence \(z_n \to 0\) as \(n \to \infty\). \(\square\)
Theorem 4.2. Assume that there exists a regular point \( z \in \omega(x) \) satisfying \( \lambda_{kz} \leq 0 \). Then either \( x \in Q \), or \( \omega(x) = \{z\} \) is a singleton.

Proof. Without loss of generality, we assume that \( x \) is not an equilibrium. By Lemma 4.1, there exists an open neighborhood \( V \) of \( z \) such that for any \( y \in V \), one of following properties holds:

\( (a) \, \|\Phi_t(z) - \Phi_t(y)\| \to 0 \) as \( t \to +\infty \).

\( (b) \, \) There is \( T > 0 \) such that \( \Phi_T(z) - \Phi_T(y) \in C \).

Now, we define the set \( \Gamma = \{t \geq 0 : \Phi_t(x) \in V\} \). Since \( z \in \omega(x) \), it is clear that \( \Gamma \neq \emptyset \).

If \( \Phi_{t_0}(x) \) satisfies property \( (b) \) for some \( t_0 \in \Gamma \), then there is some \( T > 0 \) such that \( \Phi_T(z) - \Phi_{T+t_0}(x) \in C \). Furthermore, by the strong monotonicity of \( \Phi_t \), one can find a \( T_0 > 0 \) and an open neighborhood \( U \) of \( z \) and \( V \) of \( \Phi_{t_0}(x) \), respectively such that \( \Phi_{T_0+t_0}U \approx \Phi_{T_0}V \). Recall that \( z \in \omega(x) \). Then, take \( t_1 > t_0 \) such that \( \Phi_{t_1}(x) \in U \setminus \{\Phi_{t_0}(x)\} \). Consequently, we have \( \Phi_{t_1+T_0}(x) \approx \Phi_{t_0+T+t_0}(x) \), which implies \( O(x) \) is pseudo-ordered.

If \( \Phi_t(x) \) satisfies property \( (a) \) for any \( \tau \in \Gamma \), then for any \( \tau_1, \tau_2 \in \Gamma \) with \( \tau_1 < \tau_2 \) satisfying \( \Phi_{\tau_1}(x) \neq \Phi_{\tau_2}(x) \), one has \( \|\Phi_{\tau_1+t}(x) - \Phi_t(x)\| \to 0 \) and \( \|\Phi_{\tau_2-t}(x) - \Phi_t(x)\| \to 0 \) as \( t \to +\infty \). So \( \|\Phi_{\tau_1+t}(x) - \Phi_{\tau_2+t}(x)\| \to 0 \) as \( t \to +\infty \). Fix any \( u \in \omega(x) \), there is a sequence \( \{s_k\}_{k=1}^{\infty} \) such that \( s_k \to +\infty \) and \( \Phi_{s_k}(x) \to u \) as \( k \to +\infty \). Hence, \( \|\Phi_{\tau_1+s}(x) - \Phi_{\tau_2+s}(x)\| \to 0 \) as \( k \to +\infty \), which implies that \( u = \Phi_{\tau_2-\tau_1}(u) \). By the arbitrariness of \( u \), it follows that \( \omega(x) \) consists of \( (\tau_2 - \tau_1) \)-periodic points.

Now, since \( V \) is open and \( x \) is not an equilibrium, one can find \( \tau_0 \in \Gamma \) and \( \epsilon > 0 \) satisfying:

\( (i) \, [\tau_0, \tau_0 + \epsilon] \subset T; \) \( (ii) \, \Phi_{s_1}(x) \neq \Phi_{s_2}(x) \) for any \( s_1, s_2 \in [\tau_0, \tau_0 + \epsilon] \) with \( s_1 \neq s_2 \).

By repeating the argument in the previous paragraph, one can obtain that \( \omega(x) \) consists of \( s \)-periodic point for any \( s \in [0, \epsilon] \). Thus, \( \omega(x) \) consists of equilibria. In particular, \( z \) is an equilibrium. Recall that \( \Phi_\tau(x) \) satisfies property \( (a) \) for \( \tau \in \Gamma \). Then one has \( \|\Phi_{\tau+t}(x) - z\| = \|\Phi_{\tau+t}(x) - \Phi_t(z)\| \to 0 \) as \( t \to +\infty \). Therefore, one has \( \omega(x) = \{z\} \), which completes the proof. \( \square \)

Now we will discuss the case that \( \lambda_{kz} > 0 \) for all point \( z \in \omega(x) \). Before going further, we here present the following two technical lemmas.

Lemma 4.3. Let \( \delta_3 > 0 \) be defined in Lemma 3.5(iii). If \( \lambda_{kz} > 0 \) for any \( z \in \omega(x) \), then there is a locally constant (hence bounded) function \( \nu(z) \) on \( \omega(x) \) such that

\( (i) \, \|D_2 \Phi_{\nu(z)}w_F\| < \frac{1}{2\delta_3} \|D_2 \Phi_{\nu(z)}w_E\|, \)

\( (ii) \, \|D_2 \Phi_{\nu(z)}w_E\| > 4(1 + \delta_3) \)
for any $z \in \omega(x)$ and any unit vector $w_E \in E_z, w_F \in F_z$.

**Proof.** Since $(\Phi_t, D\Phi_t)$ admits a $k$-dimensional exponential continuous separation along $\omega(x)$ as $\omega(x) \times \mathbb{R}^n = \omega(x) \times (E_y) \oplus \omega(x) \times (F_y)$, it is clear that there is a number $T > 0$ such that

$$\|D_2\Phi_tw_F\| < \frac{1}{200}\|D_2\Phi_tw_E\|$$

for any $w_E \in E_z \cap S$, $w_F \in F_z \cap S$ and any $t > T$.

Moreover, by the definition of $\lambda_{kz}$, for each $z \in \omega(x)$, there is a sequence $t_n \to +\infty$ such that

$$\|D_2\Phi_{t_n}w_E\| > e^{\frac{\lambda_{kz}}{200}t_n}$$

for any $w_E \in E_z \cap S$. Since $\lambda_{kz} > 0$, one can find an integer $N_z > 0$ such that $\|D_2\Phi_{t_n}w_E\| > 4(1 + \delta_3)$ for any $t_n > N_z$ and $w_E \in E_z \cap S$.

Therefore, for each $z \in \omega(x)$, one can associate with a number $\nu(z) \geq \max\{T, N_z\} > 0$ such that $\|D_2\Phi_{\nu(z)}w_F\| < \frac{1}{200}\|D_2\Phi_{\nu(z)}w_E\|$ and $\|D_2\Phi_{\nu(z)}w_E\| > 4(1 + \delta_3)$ for any $w_E \in E_z \cap S$, $w_F \in F_z \cap S$ and $z \in \omega(x)$. Moreover, together with the compactness of $\omega(x)$, the continuity of $z \mapsto D_2\Phi_{\nu}$ implies that one can further take such $\nu(z)$ as a locally constant (hence bounded) function. This completes the proof.

**Lemma 4.4.** Assume that $\lambda_{kz} > 0$ for any $z \in \omega(x)$. There exists a constant $\delta > 0$ such that

$$\limsup_{t \to +\infty} \|\Phi_t(y) - \Phi_t(x)\| \geq \delta,$$

whenever $y$ satisfies $y \neq x$ and $y \sim x$.

**Proof.** Let $m$ be an upper bound of the function $\nu(\cdot)$ on $\omega(x)$ defined in Lemma 4.3. Due to the compactness of $\omega(x)$, one can choose a $\delta_0 > 0$ so small that $\|D_u\Phi_{\nu} - D_v\Phi_{\nu}\| \leq \frac{1}{2}$ for any $u, v \in \text{Co}(u \in \mathbb{R}^n : d(u, \omega(x)) \leq 1)$ with $\|u - v\| \leq \delta_0$ and any $\nu \in [0, m]$, where “Co” means the convex hull.

Let $\delta = \delta_0/2 > 0$. We will show that such $\delta$ satisfies this lemma. Suppose that one can find some $y \neq x$ with $y \sim x$ such that $\limsup_{t \to +\infty} \|\Phi_t(y) - \Phi_t(x)\| < \delta$. Then there exists an $N_1 > 0$ such that $\|\Phi_t(y) - \Phi_t(x)\| < \delta$ for any $t \geq N_1$. Since $\Phi_t(x)$ is attracted to $\omega(x)$, one can choose $z_t \in \omega(x)$ such that $\|\Phi_t(x) - z_t\| \to 0$ as $t \to \infty$. Hence, there exists a number $N_2 > N_1$ such that

$$\|\Phi_t(y) - \Phi_t(x)\| \leq \delta$$

and $\|\Phi_t(x) - z_t\| \leq \delta$, for any $t \geq N_2$. (4.2)

Let also $\tau_1 = 1$ and $\tau_{k+1} = \tau_k + \nu(z_{\tau_k})$ for $k = 1, 2, \cdots$. Denoted by $y_{\tau_k} = \Phi_{\tau_k}(y)$ and $x_{\tau_k} = \Phi_{\tau_k}(x)$. Then we have that

$$y_{\tau_{k+1}} - x_{\tau_{k+1}} = D_{x_{\tau_k}} \Phi_{\nu(z_{\tau_k})} (y_{\tau_k} - x_{\tau_k})$$

$$+ \int_0^1 [D_{x_{\tau_k} + s(y_{\tau_k} - x_{\tau_k})} \Phi_{\nu(z_{\tau_k})} - D_{x_{\tau_k}} \Phi_{\nu(z_{\tau_k})}] (y_{\tau_k} - x_{\tau_k}) ds.$$
By (4.2), one can find a positive number \( N > 0 \) such that \( \| y_{r_k} - x_{r_k} \| \leq \delta \) and \( \| x_{r_k} - z_{r_k} \| \leq \delta \) for any \( k \geq N \). Hence, \( \| x_{r_k} + s(y_{r_k} - x_{r_k}) - z_{r_k} \| \leq 2\delta = \delta_0 \) for any \( s \in [0, 1] \) and \( k \geq N \). As a consequence, we have

\[
\| \int_0^1 [D_{x_{r_k} + s(y_{r_k} - x_{r_k})} \Phi_{\nu(z_{r_k})} - D_{z_{r_k}} \Phi_{\nu(z_{r_k})}] (y_{r_k} - x_{r_k}) ds \| < \frac{1}{2} \| (y_{r_k} - x_{r_k}) \| \quad (4.3)
\]

for any \( k \geq N \).

On the other hand, since \( y_{r_k} - x_{r_k} \in C \setminus \{ 0 \} \), Lemma 3.5(iii) directly entails that

\[
\| y_{r_k} - x_{r_k} \| \leq (1 + \delta_3)\| P_{r_k}(y_{r_k} - x_{r_k}) \| \quad \text{and} \quad \frac{\| Q_{r_k}(y_{r_k} - x_{r_k}) \|}{\| P_{r_k}(y_{r_k} - x_{r_k}) \|} \leq \delta_3 \quad (4.4)
\]

for any \( k \geq 1 \). It then follows from Lemma 4.3(i)-(ii) and (4.4) that

\[
\| D_{z_{r_k}} \Phi_{\nu(z_{r_k})} (y_{r_k} - x_{r_k}) \| \geq \| D_{z_{r_k}} \Phi_{\nu(z_{r_k})} P_{r_k}(y_{r_k} - x_{r_k}) \| - \| D_{z_{r_k}} \Phi_{\nu(z_{r_k})} Q_{r_k}(y_{r_k} - x_{r_k}) \|
\]

\[
\geq \| D_{z_{r_k}} \Phi_{\nu(z_{r_k})} P_{r_k}(y_{r_k} - x_{r_k}) \| - \| D_{z_{r_k}} \Phi_{\nu(z_{r_k})} Q_{r_k}(y_{r_k} - x_{r_k}) \| \cdot \left[ 1 - \frac{\| D_{z_{r_k}} \Phi_{\nu(z_{r_k})} Q_{r_k}(y_{r_k} - x_{r_k}) \|}{\| D_{z_{r_k}} \Phi_{\nu(z_{r_k})} P_{r_k}(y_{r_k} - x_{r_k}) \|} \right]
\]

\[
\geq \frac{1}{2} \| D_{z_{r_k}} \Phi_{\nu(z_{r_k})} P_{r_k}(y_{r_k} - x_{r_k}) \| \geq 2(1 + \delta_3)\| P_{r_k}(y_{r_k} - x_{r_k}) \|
\]

\[
\geq 2 \| y_{r_k} - x_{r_k} \|
\]

for any \( k \geq N \). Together with (4.3), this implies that

\[
\| y_{r_{k+1}} - x_{r_{k+1}} \| \geq \frac{3}{2} \| y_{r_k} - x_{r_k} \|
\]

for any \( k \geq N \). This contradicts \( \limsup_{t \to +\infty} \| \Phi_t(y) - \Phi_t(x) \| < \delta \). Thus, we have proved the Lemma.

\[ \square \]

**Theorem 4.5.** If \( \lambda_{kz} > 0 \) for any \( z \in \omega(x) \), then we have \( x \in \bar{Q} \).

*Proof.* Choose any sequence \( \{ x_n \}_{n=1}^{\infty} \subset \mathbb{R}^n \) approaching \( x \) with \( x_n \sim x \). Since \( x \in D \) and \( D \) is open and \( \omega \)-compact, \( \bigcup_{n \geq 1} \omega(x_n) \) is a nonempty compact set. Define \( \omega_c(x) = \bigcap_{n \geq 1} \bigcup_{m \geq n} \omega(x_m) \).

Then, it is clear that \( \omega_c(x) \) is a nonempty and compact subset which is invariant with respect to \( \Phi_t \). Moreover, \( q \in \omega_c(x) \) if and only if there are two subsequences \( n_k \to +\infty \) and \( q_{n_k} \in \omega(x_{n_k}) \) such that \( q_{n_k} \to q \) as \( k \to \infty \).

Since \( \lambda_{kz} > 0 \) for any \( z \in \omega(x) \), it follows from Lemma 4.4 that for each \( n \geq 1 \), there exist some \( p_n \in \omega(x) \) and \( q_n \in \omega(x_n) \) such that \( p_n \sim q_n \) and \( \| p_n - q_n \| \geq \delta \). Choose a subsequence \( n_k \to \infty \), if necessary, such that \( p_{n_k} \to p \in \omega(x) \) and \( q_{n_k} \to q \in \omega_c(x) \) as \( k \to +\infty \); and hence, one can further find a subsequence \( s_k \to \infty \) such that \( \Phi_{s_k}(x_{n_k}) \to q \) as \( k \to +\infty \).
Clearly, \( p \sim q \) and \( \|p - q\| \geq \delta \). Since \( \Phi_t \) is strongly monotone with respect to \( k \)-cone \( C \), we have \( \Phi_1(p) \approx \Phi_1(q) \). Choose some open neighborhoods \( U \) and \( V \) of \( \Phi_1(p) \) and \( \Phi_1(q) \), respectively, such that \( U \approx V \) and \( U \cap V = \emptyset \). Then \( \Phi_{s_k+1}(x_{n_k}) \in V \) for all \( k \) sufficiently large. Recalling that \( x_n \to x \), we can take a number \( T > 0 \) s.t. \( \Phi_{1+T}(x_{n_k}) \in U \) for all \( k \) sufficiently large. Consequently, \( \Phi_{1+T}(x_{n_k}) \approx \Phi_{1+s_k}(x_{n_k}) \) for all large \( k \). This entails that the orbit \( O(x_{n_k}) \) is pseudo-ordered, i.e., \( x_{n_k} \in Q \) for all large \( k \), which implies that \( x \in \bar{Q} \). \( \square \)

Motivated by Theorems 4.2 and 4.5, we define the set of regular points on \( \omega(x) \) as:

\[
\omega_0(x) = \{ z \in \omega(x) : z \text{ is a regular point} \}. \tag{4.5}
\]

Due to the Multiplicative Ergodic Theorem (cf. [24, Theorem 2.1]) and the similar argument in [72, Proposition 4.1], one has \( \omega_0(x) \) is non-empty. Moreover, it is easy to see that any equilibrium in \( \omega(x) \) is regular, and hence, is contained in \( \omega_0(x) \).

**Theorem 4.6.** Assume that \( \lambda_{k \varepsilon} > 0 \) for any \( \varepsilon \in \omega_0(x) \). If there exists some \( z \in \omega(x) \setminus \omega_0(x) \) such that \( \lambda_{k \varepsilon} \leq 0 \), then \( x \in \bar{Q} \).

**Proof.** For any \( y \in \omega(x) \), we define a vector \( v_y := \frac{d}{dt}|_{t=0} \Phi_t(y) \) in \( \mathbb{R}^n \). Clearly, the map \( y \mapsto v_y \) is continuous. Moreover, we have \( D_y \Phi_t(v_y) = v_{\Phi_t(y)} \) for all \( t \in \mathbb{R} \); and hence, \( v_y = 0 \) if and only if \( y \) is an equilibrium.

Since \( z \in \omega(x) \setminus \omega_0(x) \), we have \( v_z \neq 0 \). Then

\[
\lambda(z, v_z) = \limsup_{t \to +\infty} \frac{\log \|D_z \Phi_t(v_z)\|}{t} = \limsup_{t \to +\infty} \frac{\log \|v_{\Phi_t(z)}\|}{t}. \tag{4.6}
\]

Recall that \( y \mapsto v_y \) is continuous and \( \omega(x) \) is compact. Then \( \|v_{\Phi_t(z)}\| \) is bounded uniformly for any \( t \geq 0 \). This implies that \( \lambda(z, v_z) \leq 0 \). We will consider the following two cases: (A1). \( \lambda(z, v_z) = 0 \); (A2). \( \lambda(z, v_z) < 0 \), respectively.

(A1). If \( \lambda(z, v_z) = 0 \), then by \( \lambda_{k \varepsilon} \leq 0 \) and Lemma 3.6 (i), one has \( v_z \notin F_z \). So, one can write \( v_z \) as \( v_z = \alpha v + \beta w \), where \( v \in E_z \setminus \{0\} \), \( w \in F_z \) and \( \alpha \neq 0 \). By the exponential separation property and Lemma 3.5(ii), there is a constant \( T > 0 \) such that \( D_z \Phi_t(v_z) \in \text{Int} C \) for any \( t > T \). This implies \( z \in Q \). Together with the strong monotonicity of \( \Phi \), one can find two different constants \( \tau_1, \tau_2 > 0 \) such that \( \Phi_{\tau_1}z \approx \Phi_{\tau_2}z \). Hence, there exist open neighborhoods \( U_1 \) and \( U_2 \) of \( \Phi_{\tau_1}z \) and \( \Phi_{\tau_2}z \), respectively such that \( U_1 \approx U_2 \) and \( U_1 \cap U_2 = \emptyset \). Since \( z \in \omega(x) \), we obtain that \( x \in Q \).

(A2). If \( \lambda(z, v_z) < 0 \), then (4.6) implies that there exists some \( M > 0 \) such that \( \|v_{\Phi_t(z)}\| \leq Me^{\lambda(z, v_z)t/2} \) for any \( t > 0 \) sufficiently large. Therefore, \( \omega(z) \) only consists of equilibria, which
implies that \( \omega(z) \subset \omega_0(x) \). Based on our assumption, we have \( \lambda_{k\tilde{z}} > 0 \) for any \( \tilde{z} \in \omega(z) \). By virtue of Theorem 4.5, we obtain that \( z \in \overline{Q} \).

Since \( z \in \omega(x) \), we choose a sequence \( t_n \to \infty \) such that \( \Phi_{t_n}(x) \to z \) as \( n \to \infty \). Now we define

\[
C_x = \{ y \in D : y \neq x \text{ and } y \sim x \}.
\]

Clearly, \( C_x \) is nonempty, since \( x \in D \), \( D \) is open and \( C \) is \( k \)-solid. Moreover, \( C_x \) is \( \omega \)-compact because \( D \) is \( \omega \)-compact. So, if there exists some \( y \in C_x \) with a subsequence of \( \{t_n\}_{n=1}^{\infty} \), still denoted by \( \{t_n\}_{n=1}^{\infty} \), such that \( \Phi_{t_n}(y) \to z \), then by Lemma 2.4, one has either \( z \) is an equilibrium or \( z \in Q \). Recall that \( z \notin \omega_0(x) \). So \( z \in Q \). Hence, we again obtain \( x \in Q \) and we have done.

On the other hand, if for any \( y \in C_x \), there is a subsequence \( t_{nk}^{y} \to \infty \) of \( \{t_n\}_{n=1}^{\infty} \) such that \( \Phi_{t_{nk}^{y}}(y) \to z_y \neq z \) as \( k \to \infty \), then it is clear that \( z_y \sim z \) for any \( y \in C_x \). Since \( \lambda_{k\tilde{z}} > 0 \) for any \( \tilde{z} \in \omega(z) \), Lemma 4.4 (for \( \omega(z) \)) implies that there is a \( \delta > 0 \) such that

\[
\limsup_{t \to +\infty} \| \Phi_t(z_y) - \Phi_t(z) \| \geq \delta
\]

for any \( y \in C_x \). As a consequence, for each \( y \in C_x \) (hence, for each \( z_y \)), we can choose without loss of generality a sequence \( s_n^y \to \infty \) such that

(P1). \( \Phi_{s_n^y}(z_y) \to z_y \) and \( \Phi_{s_n^y}(z) \to z_y^y \) as \( n \to +\infty \); and

(P2). \( z_y^y \sim z_y \) with \( \| z_y^y - z_y \| \geq \delta \).

Based on this, we further claim that, for each \( y \in C_x \), there is a sequence \( \{\tau_n^y\}_{n=1}^{\infty} \) such that \( \Phi_{\tau_n^y}(x) \to z_y^y \in \omega(x) \) and \( \Phi_{\tau_n^y}(y) \to z_y \in \omega(y) \) as \( n \to +\infty \) satisfying \( z_y^y \sim z_y \) and \( \| z_y^y - z_y \| \geq \delta \).

Before proving this claim, we first show how it implies \( x \in \overline{Q} \). Take any a sequence \( \{x_k\}_{k=1}^{\infty} \subset C_x \) with \( x_k \to x \). For each \( x_k \), we utilize the claim to obtain \( z_{x_k}^y \in \omega(x) \) and \( z_{x_k} \in \omega(x_k) \) satisfying \( z_{x_k}^y \sim z_{x_k} \) and \( \| z_{x_k} - z_{x_k}^y \| \geq \delta \). Then, due to the \( \omega \)-compactness of \( C_x \), one can repeat the argument in the last two paragraphs of the proof of Theorem 4.5 to obtain that there is a subsequence \( \{k_l\}_{l=1}^{\infty} \) such \( x_{k_l} \in Q \) for all \( l \) sufficiently large. Consequently, we have \( x \in \overline{Q} \), which completes the proof.

Finally, it remains to prove the claim. Due to (P2), it is clear that \( z_y^y \sim z_y \) and \( \| z_y^y - z_y \| \geq \delta \). So, we only need to show the existence of the sequence \( \{\tau_n^y\}_{n=1}^{\infty} \). To this end, we observe (P1), that is, \( \Phi_{s_n^y}(z_y) \to z_y \) and \( \Phi_{s_n^y}(z) \to z_y^y \) as \( n \to +\infty \). So, one can find a subsequence \( \{N_1(n)\}_{n=1}^{\infty} \) of positive integers such that

\[
\| \Phi_{s_n^{N_1(n)}}(z) - z_y^y \| < \frac{1}{2n} \quad \text{and} \quad \| \Phi_{s_n^{N_1(n)}}(z_y) - z_y \| < \frac{1}{2n}
\]
for every \( n \geq 1 \). Recall also that \( \Phi_{t_N}^y(x) \to z \) and \( \Phi_{t_N}^y(y) \to z_y \) as \( k \to +\infty \). Then, for each \( n \), one can further choose an integer \( N_2(n) > 1 \) such that
\[
\|\Phi_{s_N}^y(\Phi_{t_N}^y(x)) - \Phi_{s_N}^y(z)\| < \frac{1}{2n},
\]
\[
\|\Phi_{s_N}^y(\Phi_{t_N}^y(y)) - \Phi_{s_N}^y(z_y)\| < \frac{1}{2n}
\]
for every \( n \geq 1 \). Hence, we have
\[
\|\Phi_{s_N}^y(x) - z\| < \frac{1}{n} \quad \text{and} \quad \|\Phi_{s_N}^y(y) - z_y\| < \frac{1}{n}
\]
for every \( n \geq 1 \). Let \( \tau_N^y = s_N^y + t_N^y \) for \( n \geq 1 \). This establishes the existence of \( \{\tau_N^y\} \) and the claim verifies.

\[\square\]

5 Generic behavior and Poincaré-Bendixson Theorem

Based on our discussion in the previous sections, we can now describe in this section the generic behaviors of the flow \( \Phi_t \) strongly monotone flow with respect to \( k \)-cone \( C \) (see Theorem 5.1 or Theorem A), which concludes that generic (open and dense) orbits are either pseudo-ordered or convergent to equilibria.

In particular, when \( k = 2 \), together with the results obtained in [11], we will further show that the generic orbit of \( \Phi_t \) satisfies the Poincaré-Bendixson Theorem (see Theorem 5.3 or Theorem B), that is, for generic (open and dense) points the \( \omega \)-limit set containing no equilibria is a single closed orbit. This result will be referred as the generic Poincaré-Bendixson Theorem for \( \Phi_t \).

Before we state our main theorems, we need more notations. We denote \( C_E \) as
\[
C_E = \{x \in \mathbb{R}^n : \text{the orbit } O(x) \text{ converges to equilibrium}\}.
\]
For any \( D \subset \mathbb{R}^n \), we recall that the orbit set of \( D \) is defined as \( O(D) = \bigcup_{x \in D} O(x) \).

**Theorem 5.1.** Assume that (FWW) hold. Let \( \mathcal{D} \subset \mathbb{R}^n \) be an open bounded set such that the orbit set \( O(\mathcal{D}) \) of \( \mathcal{D} \) is bounded. Then \( \text{Int}(Q \cup C_E) \) (interior in \( \mathbb{R}^n \)) is dense in \( \mathcal{D} \).

**Proof.** Given any \( \hat{x} \in \mathcal{D} \) and any neighborhood \( U \) of \( \hat{x} \) in \( \mathcal{D} \). If \( U \subset Q \cup C_E \), then one has \( \hat{x} \in \text{Int}(Q \cup C_E) \). Thus, we are done. So, we assume that there exists some \( x \in U \setminus (Q \cup C_E) \).

Before going further, we note that \( \mathcal{D} \) is \( \omega \)-compact because \( O(\mathcal{D}) \) is bounded. Thus, all the results in Section 4 hold for such \( x \). So, one of following three alternatives must occur:

(a) \( \lambda_{kz} \leq 0 \) for some point \( z \in \omega_0(x) \);
(b) $\lambda_{kz} > 0$ for any $z \in \omega_0(x)$, and there exists a $\tilde{z} \in \omega(x) \setminus \omega_0(x)$ such that $\lambda_{k\tilde{z}} \leq 0$;
(c) $\lambda_{kz} > 0$ for any $z \in \omega(x)$.

Here, $\omega_0(x)$ is the set of regular points in $\omega(x)$ defined in (4.5). Since $x \notin Q \cup C_E$, Theorem 4.2 directly yields that Case (a) cannot happen. For Case (c), it follows from Theorem 4.5 that $x \in Q$. So, one can choose some $y \in Q$ so close to $x$ that $y \in U$; and moreover, since $Q$ is open, we have $y \in Q = \text{Int}Q \subset \text{Int}(Q \cup C_E)$. For case (b), we can deduce from Theorem 4.6 that $x \in \bar{Q}$, which directly implies that there exists some $y$ in $U$ satisfying $y \in \text{Int}(Q \cup C_E)$.

By arbitrariness of $\bar{x}$ and $U$, we have proved that $\text{Int}(Q \cup C_E)$ is dense in $X$. 

**Remark 5.2.** Theorem 5.1 states that, for smooth flow $\Phi_t$ strongly monotone with respect to $k$-cone $C$, generic (open and dense) orbits are either pseudo-ordered or convergent to equilibria. If the rank $k = 1$, Theorem 5.1 automatically implies Hirsch’s Generic Convergence Theorem on $\mathbb{R}^n$ due to the Monotone Convergence Criterion.

Now we state the the **generic Poincaré-Bendixson Theorem** for the $\Phi_t$.

**Theorem 5.3.** Assume that (FWW) hold and $k = 2$. Let $\mathcal{D} \subset \mathbb{R}^n$ be an open bounded set such that the orbit set $\mathcal{O}(\mathcal{D})$ of $\mathcal{D}$ is bounded. Then, for generic (open and dense) points $x \in D$, the $\omega$-limit set $\omega(x)$ containing no equilibria is a single closed orbit.

**Proof.** Let $\mathcal{G} = \text{Int}(Q \cup C_E) \subset X$. By Theorem 5.1, $\mathcal{G}$ is open and dense in $X$. Now, given any $x \in \mathcal{G}$, if $\omega(x) \cap E = \emptyset$ then one has $x \in Q$. Consequently, it follows from [11, Theorem 5.1] that $\omega(x)$ consists of a periodic point, which completes the proof. 

**6 Applications to high-dimensional systems**

In this section, we demonstrate our general results by establishing the so-called **generic Poincaré-Bendixson Theorem** for high-dimensional autonomous systems of ordinary differential equations. This means that for a generic (open and dense) initial point in the phase space, the omega-limit set containing no equilibria must be a single closed orbit.

We consider a general autonomous system of ODEs

$$\dot{x} = F(x), \quad x \in \mathbb{R}^n, \quad (6.1)$$

in which $F$ is a $C^{1,\alpha}$-smooth vector field defined in $\mathbb{R}^n$. We denote by $\Phi_t(x)$ the flow generated by (6.1). System (6.1) is called dissipative if there is an open bounded set $\mathcal{B} \subset \mathbb{R}^n$ such that for each $x \in \mathbb{R}^n$ there is a $t_0 > 0$ such that $\Phi_t(x) \in \mathcal{B}$ for all $t \geq t_0$. 

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Let $C \subset \mathbb{R}^n$ be a 2-solid cone which is complemented. System (6.1) is said to be $C$-cooperative if the fundamental solution matrix $U^{pq}(t)$ of the linear system

$$\dot{U} = A^{pq}(t)U, \quad U(0) = I$$

satisfies the cone invariance condition

$$U^{pq}(t)(C \setminus \{0\}) \subset \text{Int}C, \quad \text{for all} \ t > 0. \tag{6.2}$$

Here, the associated matrix $A^{pq}(t)$ with $p, q \in \mathbb{R}^n$ is

$$A^{pq}(t) = \int_0^1 DF(s\Phi_t(p) + (1 - s)\Phi_t(q))ds.$$ 

Now, we give the following Poincaré-Bendixson Theorem for system (6.1).

**Theorem 6.1.** Assume that system (6.1) is $C^{1,\alpha}$-smooth, dissipative and $C$-cooperative. Then there is an open and dense subset $\mathcal{D} \subset \mathbb{R}^n$ such that for any $x \in \mathcal{D}$, if the $\omega$-limit set $\omega(x)$ contains no equilibrium then $\omega(x)$ is a periodic orbit.

**Proof.** Together with [59, Proposition 1] by Sanchez, it is clear that if system (6.1) is $C^{1,\alpha}$-smooth and $C$-cooperative, then the flow $\Phi_t(x)$ satisfies the assumption (FWW). Take any integer $i \geq 1$ and let $B_i$ be the open ball centered at the origin with radius $i$. Since system (6.1) is dissipative, the orbit set $\mathcal{O}(B_i)$ of $B_i$ is bounded. By Theorem 5.3, there is an open and dense set $D_i$ in $B_i$ from which the omega-limit set containing no equilibria is a closed orbit. Note that $\mathbb{R}^n = \cup_{i \geq 1} B_i$. Then $\mathcal{D} := \cup_{i \geq 1} D_i$ is an open and dense subset in $\mathbb{R}^n$. Moreover, for any $x \in \mathcal{D}$, if the $\omega$-limit set $\omega(x)$ contains no equilibrium then $\omega(x)$ is a periodic orbit. We have completed the proof. \qed

**Remark 6.2.** Theorem 6.1 will be referred to as the *generic Poincaré-Bendixson Theorem* for high-dimensional ODE systems. Based on this Theorem, we improve that the Poincaré-Bendixson type conclusion in [11, 59] is satisfied for generic orbits, instead of just certain (i.e., pseudo-ordered) orbits.

Finally, we specify a quadratic cone which is 2-solid and complemented. Let $P$ be a constant real symmetric non-singular matrix $n \times n$ matrix, with 2 negative eigenvalues and $(n - 2)$ positive eigenvalues. Then the set

$$C^-(P) = \{x \in \mathbb{R}^n : x^*Px \leq 0\}$$
is a 2-solid cone which is also complemented. Here $x^*$ denote the transpose of the vector $x \in \mathbb{R}^n$.

Assume that there exists a continuous function $\lambda : \mathbb{R}^n \to \mathbb{R}$ (not necessarily positive) such that the matrices

$$PDF(x) + (DF(x))^*P + \lambda(x)P < 0, \text{ for any } x \in \mathbb{R}^n,$$

where $DF(x)^*$ stands for the transpose of the Jacobian $DF(x)$ and “$<$” represents the usual order in the space of symmetric matrices (i.e., the matrices are negative definite). The following lemma is due to Sanchez [59,60].

**Lemma 6.3.** Assume that (6.3) holds. Then system (6.1) is $C^-(P)$-cooperative.

**Proof.** See Sanchez [59, Proposition 7] and his afterwards discussion in [59].

Combing with Theorem 6.1 and Lemma 6.3, we obtain the following generic Poincaré-Bendixson Theorem for high-dimensional system (6.1) with a quadratic cone:

**Corollary 6.4.** Assume that system (6.1) is $C^{1,\alpha}$-smooth, dissipative and satisfies (6.3). Then the conclusion of Theorem 6.1 holds.

**Remark 6.5.** In [67,68], R.A. Smith succeeded in establishing a Poincaré-Bendixson theorem for the high-dimensional system (6.1) by assuming that $F$ satisfies

$$(x - y)^* \cdot P \cdot [F(x) - F(y) + \lambda(x - y)] \leq -\epsilon |x - y|^2$$

for any $x, y \in \mathbb{R}^n$, where $\lambda, \epsilon > 0$ are positive constants and $|x - y|$ denote the Euclidean norm of the vector $x - y$. It states that any omega-limit set containing no equilibria must be a single closed orbit. If $F$ is of class $C^1$, by following the same arguments in Ortega and Sanchez [49, Remarks 1-2], one may obtain that (6.4) holds if and only if

$$PDF(x) + (DF(x))^*P + \lambda P \leq -\epsilon I, \text{ for any } x \in \mathbb{R}^n.$$  \hspace{1cm} (6.5)

In his proof [67,68], the matrix $P$ is employed to produce a quadratic Lyapunov function that helps to establish his Poincaré-Bendixson theorem.

Compared to (6.3), the assumption (6.5) in R. A. Smith’s work requires the matrices be negative definite in a uniform sense with respect to $x$, and $\lambda$ is a positive constant. Therefore, under the weaker assumption (6.3), the Lyapunov-function approach in [67,68] does not work any more. However, our Corollary 6.4 concludes that, generically, the Poincaré-Bendixson theorem still holds.
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