INDUCTIVE LS COCATEGORY AND LOCALISATION

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Abstract. In this paper we prove that the inductive cocategory of a nilpotent CW-complex of finite type \(X\), \(\text{indcocat} \ X\), is bounded above by an expression involving the inductive cocategory of the \(p\)-localisations of \(X\). Our arguments can be dualised to LS category improving previous results by Cornea and Stanley. Finally, we show that the inductive cocategory is generic for 1-connected \(H_0\)-spaces of finite type.

1. Introduction

In this note, we study the inductive cocategory of a pointed CW-complex \(X\), \(\text{indcocat} \ X\) \[1\]. We discuss the problem of finding an upper bound for \(\text{indcocat} \ X\) when \(\text{indcocat} \ X(\pi)\), the inductive cocategory of the \(\pi\)-localisation of \(X\), is known for every prime \(\pi\). In this work, \(X\) is thought of as an arithmetic square, that is, \(X\) is the homotopy pullback of the following arrows

\[
\begin{array}{ccc}
X(0) & \xrightarrow{(e(\pi)\pi)_0)} & (\Pi_p X(\pi))(0) \\
\downarrow & & \downarrow \\
\Pi_p X(\pi) & \xleftarrow{e(0)} & X(\pi)
\end{array}
\]

where \(e(\pi) : X \rightarrow X(\pi)\) denotes the \(\pi\)-localisation, \(\Pi_p X(\pi)\) the local expansion of \(X\), the arrow pointing to the right denotes the rationalisation of the \(\{e(\pi)\}_{\pi}\) and the arrow pointing to the left is the rationalisation of the local expansion of \(X\). This is always possible when \(X\) is nilpotent \[9, \text{Theorem 4.1, Example 2.6}\].

This problem has been first raised about \(\text{cat} \ X\), the Lusternik-Schnirelmann category of \(X\). Namely, Toomer \[26\] motivated by his Theorem 4 asserting that for simply connected CW-complexes \(\text{cat} \ X(\pi) \leq \text{cat} \ X\), raises the question (suggested by P. Hilton) of when \(\text{cat} \ X\) is the supremum of \(\text{cat} \ X(\pi)\) over all the primes. Stanley \[25\] improving a result of Cornea \[4\] shows that

\[
(1) \quad \text{cat} \ X \leq 2\sup\{\text{cat} \ X(\pi)\}
\]

for finite type 1-connected CW-complexes. This inequality is not known to be sharp for spaces verifying \(\sup\{\text{cat} \ X(\pi)\} = m > 1\). For \(m = 1\), Roitberg \[23\] constructs an infinite space such that \(X(\pi)\) is a co-\(H\)-space for every \(\pi\) (thus \(\text{cat} \ X(\pi) = 1\)) though \(\text{cat} \ X = 2\). In his (failed) attempt to show that \((1)\) is sharp for any \(m > 1\), Roitberg points out relevant examples of infinite spaces verifying \(\text{cat} \ X = m + 1\).

We believe that the stronger inequality

\[
(2) \quad \text{cat} \ X \leq \sup\{\text{cat} \ X(\pi)\} + 1
\]

holds for almost every “common” space. Moreover, if we add the assumption on \(X\) to be finite, no examples at all are known to verify that \((2)\) is sharp. Indeed, the
first author [7] proves that if \( \sup_p \{ \text{cat} X(p) \} = 1 \), then necessarily \( \text{cat} X = 1 \), so Roitberg example above cannot be reproduced for finite spaces.

In this paper we find an upper bound, for the inductive cocategory of nilpotent finite type \( CW \)-complexes, of the same flavor as [1].

**Theorem 1.1.** Let \( X \) be a finite type nilpotent \( CW \)-complex, then
\[
\text{indcocat} X \leq \sup_p \{ \text{indcocat} X_p \} \text{ and } \text{indcocat} X_0.
\]

The proof of this theorem is carried out in Section 3. Following the same lines and using [14, Theorem 4.2] it is immediate to prove (details are omitted) the following refinement of (1).

**Theorem 1.2.** Let \( X \) be a finite type 1-connected \( CW \)-complex, then
\[
\text{cat} X \leq \sup_p \{ \text{cat} X_p \} + \text{cat} X_0.
\]

We recall here that \( X \) is an \( H_0 \)-space if its rationalisation is an \( H \)-space (thus \( \text{indcocat} X(0) = 1 \)). Dually, \( X \) is a co-\( H_0 \)-space if its rationalisation is a co-\( H \)-space (thus \( \text{cat} X(0) = 1 \)). From Theorems 1.1 and 1.2 we deduce:

**Corollary 1.3.** Let \( X \) be finite-type \( CW \)-complex. Then
\[
i) \text{ If } X \text{ is a nilpotent } H_0 \text{-space. Then } \text{indcocat} X \leq \sup_p \{ \text{indcocat} X(p) \} + 1.
\]
\[
ii) \text{ If } X \text{ is a 1-connected co-} H_0 \text{-space. Then } \text{cat} X \leq \sup_p \{ \text{cat} X(p) \} + 1.
\]

Since finite 1-connected \( H_0 \)-spaces verify also (2) (see [6]), one might expect that finite 1-connected co-\( H_0 \)-spaces could satisfy Corollary 1.3 i). The main obstruction to prove such a result is that we do not know the minimal models for Ganea cofibrations (see Remark 2.2). The upper bound given in Theorem 1.1 is sharp for infinite spaces the same as happened with [1]. In [22] Pan, inspired in the Roitberg’s example above-mentioned, shows an infinite space \( X \) such that all its localisations are \( H \)-spaces (thus \( \text{indcocat} X(p) = 1 \)) although \( X \) is not (\( \text{indcocat} X = 2 \)). If we add the assumption on \( X \) to be finite, Zabrodsky [27] proved for finite spaces that if \( X(p) \) is an \( H \)-space for every prime \( p \), then \( X \) is itself an \( H \)-space. Hence the inequality of Theorem 1.1 is not sharp for \( m = \sup_p \text{indcocat} = 1 \).

A problem somewhat related to the previous one was posed by McGibbon on his survey [19]. We recall that finite-type spaces not necessarily homotopic verifying \( X(p) \simeq Y(p) \) for every prime \( p \) are told to be in the same Mislin Genus. Hence Theorem 1.1 can be reread in terms of the Mislin Genus:

**Corollary 1.4.** Let \( X \) and \( Y \) be nilpotent finite type spaces in the same Mislin Genus. Then \( \text{indcocat} Y - \text{indcocat} X \leq r \) where \( r = \text{indcocat} Y(0) = \text{indcocat} X(0) \).

We recall that the inductive cocategory of a product is the supremum of the inductive cocategory of their factors [10, Theorem 2.7]. Using this result, we can prove:
Theorem 1.5. Let $X$ be a finite-type 1-connected $H_0$-space (with a finite number of homotopy groups or a finite number of homology groups). If $Y$ is in the Mislin genus of $X$, then $\text{indcocat } X = \text{indcocat } Y$.

Proof. According to Zabrodsky’s non cancellation result [27, Theorem 3.5] (see also [28]) there exists a product of (odd dimensional) spheres $S$, such that $X \times S = Y \times S$. Then

$$\text{indcocat } (X \times S) = \max \{\text{indcocat } X, \text{indcocat } S\}.$$ 

Since the inductive cocategory of an odd dimensional sphere is either 1 (for $S^1$, $S^3$, $S^7$) or 2 (otherwise) [15, p. 28], if this maximum is greater or equal to 3, then $\text{indcocat } X$ and $\text{indcocat } Y$ coincide. If the maximum is less than or equal to 2, we use [27, Corollary 2.9] asserting $\text{indcocat } X = 1$ if and only if $\text{indcocat } Y = 1$ to conclude. □

Notation. In what follows, all spaces we consider have the homotopy type of a $CW$-complex. By abuse of notation, we identify the maps between spaces and its homotopy classes. Finally, we understand $p$-localisation as in [16].

2. Inductive cocategory

The cocategory is presented by Ganea [10, 11, 12, 13] as the dual in an Eckmann-Hilton sense of the Lusternik Schnirelmann category. In some sense, it is a more natural algebraic invariant. The same way the nilpotency of a group $G$ measures how far is $G$ from being abelian, the cocategory of $X$ measures how far $X$ is from being an $H$-space. For example, the cocategory of the classifying space of a discrete group $G$ coincides with the nilpotency of $G$ [12, Theorem 2.7], and the cocategory of a space is an upper bound of the nilpotency of the space [12, Definition 2.2, Theorem 2.4].

Several notions of cocategory, other than Ganea’s, exist in literature (see e.g. [1, 17, 18]) though they are not known to be equivalent. In particular, the second author of this paper together with A. Murillo [21] have introduced the dual to the Whitehead approach of the LS category which has the advantage that is far more computable than the previous mentioned.

To our purpose of $p$-localising we need the functoriality of the cofiber-fiber construction of the inductive cocategory of Ganea [13, Definition 6.1, Remark 6.2]:

Definition 2.1. The $n$-th Ganea cofibration of $X$, $X \xrightarrow{q_n} G_nX \xrightarrow{} C_nX$, is defined inductively as follows: $q_0$ is the cofibration $X \xrightarrow{q_0} CX \xrightarrow{} \Sigma X$. Next consider $F$ the fibre of $G_{n-1}X \xrightarrow{} C_{n-1}X$ and factor $q_{n-1}$ through $F$ to get a map $X \xrightarrow{} F$. The associated cofibration to that map is by definition the $n$-th
Ganea cofibration of \( X \). This can be better viewed in the following diagram:

\[
\begin{array}{c}
X \\
\downarrow q_n \\
G_n X \\
\downarrow \iota_n \\
G_{n-1} X \\
\vdots \\
\downarrow q_0 \\
\Sigma X \\
\end{array}
\]

Then, \( \text{indcocat} \, X \) is the least integer \( n \) for which \( q_n \) has a homotopy retraction.

**Remark 2.2.** The reason for the cocategory not being as popular among specialists as the LS category relies in the fact that the homotopy type of the cofibers \( C_n(X) \) in the construction above are not known, whether in the dual fiber-cofiber construction, we do know that \( F_n(X) = \Omega^{n+1} X \) (see [13, Remark 3.5]).

In [26, Theorem 7] Toomer explains how the \( p \)-localisation of 1-connected spaces of finite type behaves nicely with respect the cofiber-fiber construction. Considering the generalization of Whitehead Theorem in [8], the result can be extended to nilpotent spaces:

**Theorem 2.3.** Let \( X \) be a nilpotent space of finite type. Then \( (G_n X)(p) \simeq G_n(X(p)) \), and as a consequence if \( \text{indcocat} \, X \leq n \), then \( \text{indcocat} \, X(p) \leq n \).

The idea of the proof is to consider the following diagram

\[
\begin{array}{c}
X(p) \\
\downarrow (q_n(p)) \\
(G_n X)(p) \\
\downarrow l_p \\
G_n(X(p)) \\
\end{array}
\]

where \((q_n)(p)\) is the \( p \)-localisation of \( q_n \), the \( n \)-th Ganea cofibration, \( q_n(X(p)) \) is the Ganea cofibration for \( X(p) \), and \( l_p \) exists by the universal property of \( p \)-localisation. In order to show that \( l_p \) is an homotopy equivalence, we have to use [8, Theorem 3.1, Example 4.3] where conditions are given for an homological equivalence to be an homotopical equivalence. Then, it suffices to consider a retraction of \( q_n \) and \( p \)-localise it to conclude. Conversely:

**Lemma 2.4.** Let \( X \) be a nilpotent finite type space. Then the \( n \)-th Ganea cofibration of \( X \) admits a homotopy retraction if and only if for every prime \( p \) there exists \( r_p \) a homotopy retraction of the \( n \)-th Ganea cofibration of \( X(p) \) such that \( (r_p)_0 = r_0 \).
**Proof.** Consider arithmetic squares as we mentioned at the beginning of this paper:

![Diagram](image)

By a standard argument, the induced (dotted) map is a retraction of $q_n$ up to a self-homotopy equivalence of $X$. In fact, if we take their composition and we $p$-localise, we obtain $\text{Id}_{X(p)}$ for every prime $p$. Since $X$ is nilpotent of finite type, we have that this composition is a self-homotopy equivalence of $X$. Composing with the inverse of this equivalence provides the desired retraction. □

3. **Homotopy retractions of the Ganea cofibration**

The aim of this section is to provide a way of comparing two different retractions of the Ganea cofibration without using the homotopy type of the cofiber $C_n(X)$ (see Remark 2.2, as Ganea did in [14, Section 4]). In fact, using this result of Ganea, it is immediate to prove the refinement of (1) in Theorem 1.2.

We use the monodromy action associated to a fibration to compare retractions. We have included at the end of the paper an appendix that describes results on actions that we need in next the proposition.

**Proposition 3.1.** Let $A \overset{a}{\rightarrow} X \overset{j}{\rightarrow} C$ be a cofibration, $F_j \overset{i}{\rightarrow} X$ be the inclusion of the homotopy fiber of $X \overset{j}{\rightarrow} C$, and $F \overset{p}{\rightarrow} E \overset{i}{\rightarrow} B$ be a fibration. Then, for any $X \overset{g}{\rightarrow} F$ and for any retraction $A \overset{a}{\rightarrow} X \overset{r}{\rightarrow} A$, the following holds:

$$ig = i gar \iff gi = gar$$

**Proof.** For any space $Y$, let $+ \,$ denote the group operation in $[Y, \Omega B]$ and let $\vdash \,$ denote the $[Y, \Omega B]$-action on $[Y, F]$ (see Appendix 1). Since $ig = i gar$, by Theorem 4.1 there exists an element $\epsilon \in [X, \Omega B]$ such that $g = (\epsilon \vdash gar)$. Moreover, $ra = 1_A$, and therefore

$$ga = (\epsilon \vdash gar) a = (\epsilon a \vdash gar a) = (\epsilon a \vdash ga).$$

Using again that $ra = 1_A$, we deduce that $(\epsilon - ear) a = 0$, hence there exits an element $\psi \in [C, \Omega B]$ such that $\psi j = \epsilon - ear$, that is, $\epsilon = \psi j + ear$. 


Therefore $g = (\psi j + ear) \vdash gar$ and (notice that $j\iota = 0$)

$$
\begin{align*}
g\iota &= (\psi j + ear)\iota \\
&= (\psi j + ear)\iota \vdash gar\iota \\
&= (ear \iota \vdash gar\iota) \\
&= (ea \iota \vdash ga\iota) \\
&= gar\iota
\end{align*}
$$

We can now prove the main result in this section. Let $\iota^n_m$ denote the composition $\iota_{m+1} \iota_{m+2} \cdots \iota_m : G_n X \to G_m X$ where $\iota_k$ is as in Definition 2.1. Then,

**Theorem 3.2.** Let $X$ be a space such that $\text{indcocat} X \leq k$. Then, for any $m \geq 0$ and any $n \geq k + m$, the following diagram is homotopy commutative

$$
\begin{array}{ccc}
G_n X & \xrightarrow{\iota^n_m} & G_m X \\
\downarrow{\iota^n_k} & & \downarrow{\iota^n_m} \\
G_k X & & G_m X \\
\downarrow{\iota_k} & & \downarrow{\iota_m} \\
X & & X
\end{array}
$$

where $r_k$ is an arbitrary retraction of the $k$-th Ganea cofibration.

**Proof.** The proof follows by induction. Since $G_0 X = CX$, the result is trivial for $m = 0$. Now, $n - 1 \geq k$, and we apply Proposition 3.1 to the $(n - 1)$-th Ganea cofibration, to the fibration $G_{m+1} X \xrightarrow{\iota^{m+1}_m} G_m X \xrightarrow{j_m} C_m$, to $g = \iota^{n-1}_{m+1}$, and to $r = r_k \iota^{n-1}_k$ as a retraction of $q_{n-1}$. That is, to the following diagram:

$$
\begin{array}{ccc}
G_n X & \xrightarrow{\iota^{n-1}_{m-1}} & G_{n-1} X \\
\downarrow{\iota^{n-1}_m} & & \downarrow{j_{n-1}} \\
G_{n-1} X & & G_n X \\
\downarrow{j_m} & & \downarrow{\iota^{n-1}_m} \\
C_{n-1} & & C_n
\end{array}
$$

Then,

$$
\iota^{m+1}_{m-1} \iota^{n-1}_m = \iota^{m+1}_{m-1} q_{n-1} r_k \iota^{n-1}_k \\
\iff \iota^{n-1}_m = \iota^{n-1}_{m-1} q_{n-1} r_k \iota^{n-1}_k
$$

or equivalently

$$
\iota^{n-1}_m = q_m r_k \iota^{n-1}_k \\
\iff \iota^{n}_m = q_{m+1} r_k \iota^{n}_k
$$

Using this result we can now prove Theorem 1.1.
Theorem 1.1. Let \( m = \sup_p \{ \text{indcocat } X(p) \} \) and let \( k = \text{indcocat } X(0) \). Choose, for every prime \( p \), an arbitrary retraction of the \( m \)-th Ganea cofibration
\[
\{ \tau_p : G_m(X(p)) \to X(p) \mid p \text{ prime} \}
\]
and fix \( r_k : G_k(X_0) \to X(0) \) a retraction for \( X(0) \). Following Theorem 3.2 applied to \( X(0) \) for \( n = k + m \), we have the the following equality
\[
\iota^n_m = q_m r_k l^n_k,
\]
where maps and notations are the same as in [3]. Notice that maps in this case are rational. Hence,
\[
(\tau_p)(0) \iota^n_m = (\tau_p)(0) q_m r_k l^n_k = r_k l^n_k,
\]
since by Theorem 2.3 \( (\tau_p)(0) q_m = Id_{X(0)} \). So, the rationalisations of the considered retractions are homotopically equivalent when composed with \( \iota^n_m \) where \( n = m + k \).

Then, by Lemma 2.4 \( \text{indcocat } X \leq m + k \). □

4. Appendix: the monodromy action of a fibration

The aim of this appendix is to define the monodromy action associated to a fibration in a way it extends the classical monodromy action associated to covering maps, and to dualise the contents of [2, p. 442]. That is, given a fibration \( F \to E \to B \), and an arbitary space \( X \), we define an action of the group \([X, \Omega B]\) on the set \([X,F]\), and relate the orbits of this action with the Barrat-Puppe exact sequence associated to the fibration. Although we could not find any reference addressing the contents of this section, we claim no original result in this part.

Let \( p : E \to B \) be an arbitrary map. The homotopy fiber of \( p \) is the following pullback
\[
\begin{array}{ccc}
F &=& PB \times_B E \\
\pi_2 \downarrow & & \downarrow p \\
PB & \cong & B \\
\pi_1 \downarrow & & \downarrow \epsilon_1 \\
E & \to & B
\end{array}
\]
where \( F = \{(w, e) \mid w(1) = p(e)\} \). Let \( X \) be an arbitrary space and \( \bullet \) denote the adjunction of paths, then there exists an action of the group \([X, \Omega B]\) on \([X,F]\) (the monodromy action)
\[
[X, \Omega B] \times [X,F] \to [X,F]
\]
\[
(\epsilon, g) \mapsto \epsilon \bullet g
\]
given by
\[
(\epsilon \bullet g)(x) = (\epsilon \bullet (g', g''))(x) = (g'(x) \bullet (\epsilon(x), g''(x))),
\]
where \( g' = \pi_1 g \), and \( g'' = \pi_2 g \).

Notice this action is natural by construction, that is given \( f : X \to Y \) then
\[
(\epsilon \bullet g)f = ((\epsilon f) \bullet (gf)).
\]
The following result describes the \([X, \Omega B]\)-orbits in \([X,F]\).
Theorem 4.1. Let $F = PB \times_B E \xrightarrow{i} E \xrightarrow{p} B$ be a fibration and let the Barrat Puppe exact sequence:

$$[X, \Omega B] \xrightarrow{i_*} [X, F] \xrightarrow{p_*} [X, E] \xrightarrow{0} [X, B].$$

Then, for $g_1, g_2 \in [X, F]$ we have that $i_*(g_1) = i_*(g_2)$ if and only if there exists $\epsilon \in [X, \Omega B]$ such that $(\epsilon \circ g_1) = g_2$.

Proof. Let us write $g_i = (g_i', g_i'')$, for $i = 1, 2$. Since $i_* (g_1) = i_*(g_2)$, we have that $g_1'' \simeq_B g_2''$ with $H : X \times I \rightarrow E$. Define $\epsilon : X \rightarrow \Omega B$ by

$$\epsilon(x) = g_1'(x)^{-1} \cdot (pH_x)^{-1} \cdot g_2'(x).$$

We now prove that $\epsilon \circ g_1$ is homotopic to $g_2$. By definition we have

$$(\epsilon \circ g_1)(x) = \left( g_1'(x) \cdot (g_1'(x)^{-1} \cdot (pH_x)^{-1} \cdot g_2'(x)) \right).$$

which is clearly homotopic to the map defined by

$$\phi(x) = \left( (pH_x)^{-1} \cdot g_2'(x), g_1''(x) \right).$$

Finally, $\phi$ is homotopic to $g_2$ by the homotopy $G = (G', G'') : X \times I \longrightarrow F$ defined by

$$G'(x, s)(t) = (pH_x)^{-1}((1 - s)t) \cdot g_2'(x)$$

$$G''(x, s) = H_x(s).$$

Notice that $G$ is well defined over the fiber, since $p(G''(x, s)) = G'(x, s)(1)$.

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