Detrended fluctuation analysis as a regression framework: Estimating dependence at different scales

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We propose a novel framework combining detrended fluctuation analysis with standard regression methodology. The method is built on detrended variances and covariances and it is designed to estimate regression parameters at different scales and under potential non-stationarity and power-law correlations. Selected examples from physics, finance and environmental sciences illustrate usefulness of the framework.

PACS numbers: 05.10.-a, 05.45.-a, 05.45.Tp
Keywords: detrended fluctuation analysis, regression, scales, time series analysis

Detrended fluctuation analysis (DFA) was introduced in early 1990s [13] as a method for analyzing fractal properties of underlying data. The method was later majorly popularized in the long-range correlations [4, 5] and multi-fractal analyses [6]. Recently, DFA has been generalized for long-range cross-correlations analysis [7, 10] as well as an examination of correlations between non-stationary series [11, 12]. The method has been applied and utilized across wide range of disciplines ranging from physiology to cardiology, DNA analysis and neurology to (hydro)meteorology, economics and finance, engineering and environment and many others [13–21]. Here, we propose a novel framework based on detrended fluctuation analysis which allows for a regression analysis of possibly non-stationary and long-range dependent data at different scales.

When studying dependence between two series, one often considers a linear model in its simplest form

\[ Y = \alpha + X \beta + u \]

where \( Y \) is a dependent (response) variable, \( X \) is an independent (impulse) variable, \( u \) is an error-term, and parameters \( \alpha \) and \( \beta \) represent relationship between \( X \) and \( Y \). Estimation of parameter \( \beta \) then becomes a crucial point of empirical studies across disciplines. Contrary to frequently used correlation coefficient, \( \beta \) is not normalized so that it displays an actual effect of variable \( X \) on variable \( Y \). The standard regression analysis uses the (ordinary) least squares method for estimation of the parameter of interest \( \beta \) as

\[
\hat{\beta}_{LS} = \frac{\sum_{t=1}^{T} (x_t - \bar{x})(y_t - \bar{y})}{\sum_{t=1}^{T} (x_t - \bar{x})^2} \sim \frac{\sigma_{XY}}{\sigma_X^2},
\]

where \( \bar{x} = \frac{1}{T} \sum_{t=1}^{T} x_t \) and \( \bar{y} = \frac{1}{T} \sum_{t=1}^{T} y_t \). Variance of the estimator is obtained using the residuals \( \hat{u}_t = y_t - x_t \hat{\beta}_{LS} \) as

\[
\text{var} (\hat{\beta}_{LS}) = \frac{\sum_{t=1}^{T} \hat{u}_t^2}{\sum_{t=1}^{T} (x_t - \bar{x})^2} \sim \frac{1}{T - 2} \frac{\sigma_u^2}{\sigma_X^2}
\]

and it illustrates accuracy of the estimated parameter. Variance can be further utilized in hypothesis testing. To describe quality of the model, coefficient of determination \( R^2 \) defined as

\[
R^2 = 1 - \frac{\sum_{t=1}^{T} \hat{u}_t^2}{\sum_{t=1}^{T} (y_t - \bar{y})^2} \sim 1 - \frac{\sigma_u^2}{\sigma_Y^2}
\]

and ranging between 0 and 1 is utilized. \( R^2 \) quantifies a proportion of variance of \( Y \) explained by \( X \) and thus a higher value of \( R^2 \) signifies a better information content of \( X \) in explaining \( Y \). On the right-hand side of Eqs. [13] we translate the standard notation into estimated variances and covariances using the \( \hat{\sigma} \) notation. Evidently, the whole framework is based on estimated variances of \( X, Y \) and \( u \) and covariance between \( X \) and \( Y \). We employ the same idea using the detrended fluctuation analysis methodology, which we now shortly recall.

For time series \( x_t \), we construct a profile as \( X_t = \sum_{j=1}^{T} (x_t - \bar{x}) \) which is split into non-overlapping boxes of length (scale) \( s \). In each box between \( j \) and \( j + s - 1 \), the linear (or in practice any other) fit of a time trend \( \hat{X}_{k,j} \) is constructed for \( j \leq k \leq j + s - 1 \). Fluctuation function \( f^2_X(s,j) \) is then defined for each box of length \( s \) as

\[
f^2_X(s,j) = \frac{\sum_{k=j}^{j+s-1} (X_k - \hat{X}_{k,j})^2}{s - 1}.
\]

The fluctuation \( f^2_X(s,j) \) is further averaged over all boxes of length \( s \) to obtain

\[
F^2_X(s) = \frac{\sum_{j=1}^{T-s+1} f^2_X(s,j)}{T-s}.
\]

For bivariate series \( x_t \) and \( y_t \), the procedure is parallel and we get

\[
f^2_{XY}(s,j) = \frac{\sum_{k=j}^{j+s-1} (X_k - \hat{X}_{k,j})(Y_k - \hat{Y}_{k,j})}{s - 1}
\]

which is again averaged over all boxes with scale \( s \) to obtain

\[
F^2_{XY}(s) = \frac{\sum_{j=1}^{T-s+1} f^2_{XY}(s,j)}{T-s}.
\]
The scale-characteristic fluctuations $F_X^2(s)$ and $F_{XY}^2(s)$ can be seen as scale-dependent variance and covariance, respectively. We now utilize such correspondence to reformulate the standard regression framework. Estimator in Eq. 1 can be rewritten for a given scale $s$ as

$$\hat{\beta}^{DFA}(s) = \frac{F_{XY}^2(s)}{F_X^2(s)}$$

with a use of fluctuations defined in Eqs. 4-5. Using the estimated $\hat{\beta}^{DFA}(s)$, we obtain scale-specific residuals as

$$\tilde{u}_t(s) = y_t - x_t \hat{\beta}^{DFA}(s) - y_t - x_t \hat{\beta}^{DFA}(s)$$

with a mean value of zero. These are further plugged into the DFA procedure so that the fluctuation $F_X^2(s)$ can be used for estimating variance of $\hat{\beta}^{DFA}(s)$ via Eq. 2 as

$$\text{var} \left( \hat{\beta}^{DFA}(s) \right) = \frac{1}{T-2} \frac{F_{\tilde{u}^2}(s)}{F_X^2(s)}.$$  

Eq. 3 is then translated into the DFA framework as

$$R^2(s) = 1 - \frac{F_{\tilde{u}^2}(s)}{F_X^2(s)}.$$  

The whole standard regression framework in Eqs. 1-3 is thus transferred into a scale-dependent framework using the DFA methodology. Moreover, DFA provides some desirable statistical properties such as resistance to non-stationarity and trends which further enhance the proposed methodology \[5,7,12\].

**FIG. 1. Simulations results I.** Mean values (solid line, left axis) and root mean squared errors (dashed line, right axis) are shown for model $y_t = \alpha + x_t \beta + u_t$ with $\alpha = \beta = 1$, $x_t$ is defined as an ARFIMA process with changing parameter $d_x$ (x-axis) and $u_t$ is a standard Gaussian noise error-term. Each simulation has 1000 observations and 1000 series are generated for each setting. Results show that the DFA estimator of $\beta$ is unbiased (mean values range between 0.999 and 1.001) and its variance decreases with the memory strength.

In order to examine the utility of the newly introduced estimator, we study its properties in two non-stationary regression frameworks with $y_t = \alpha + x_t \beta + u_t$. Firstly, we show how the DFA estimator performs under various levels of long-range dependence in series $x_t$ and $y_t$. The former series is simulated as an ARFIMA process so that $x_t = \sum_{i=-\infty}^{i=\infty} a_i(d)x_{t-i}$ where $d$ is a fractional integration parameter and $a_i(d) = \frac{\Gamma(\nu-d)}{\Gamma(\nu-d)\Gamma(1+d)}$. Error-term $u_t$ is taken as a standard Gaussian noise so that the series $y_t$ has the same parameter $d$ as the series $x_t$. Fig. 1 shows the mean values and root mean squared errors of the DFA estimator for the series of length 1000 with parameter $d$ ranging between 0 and 1 with a step of 0.1. The estimator is averaged over scales between 10 and 100 with a step of 10, and the regression parameters are set to $\alpha = \beta = 1$. 1000 simulations are run for each setting. The estimator is unbiased regardless of the long-range dependence level. Moreover, the root mean squared error decreases with an increasing memory which is desirable. Secondly, we study how the estimator fares for long-range dependent error-terms $u_t$. To do so, we fix the memory parameter for the series $x_t$ to $d_x = 0.9$ and the error-term is generated as an ARFIMA process with $d_u$ ranging between 0
and 1 with a step of 0.1. The rest of the setting remains unchanged. In Fig. 2 we again report the mean values and root mean squared errors of the estimator as for the previous case. The DFA estimator is again remarkably stable and unbiased for different levels of memory in the error-terms. Even though variance of the estimator increases with $d_u$, which is expected due to an increasing weight of the error-term in the whole dynamics of $y_t$ as variance of the error-term increases with $d_u$, the overall performance remains excellent.

As examples of the potential utility of the method, we present three applications to real-world datasets. First, we study a relationship between daily air temperature and relative humidity in London, UK, between years 2000 and 2012 (4651 observations). An increasing temperature is expected to increase a moisture holding capacity of air and thus decrease its relative humidity. Fig. 3 shows an estimated effect of temperature on relative humidity for scales between 10 and 1150 (approximately a quarter of the time series length) days. Expectedly, the effect is negative. However, a strong variation across scales is uncovered. The effect is weak for low scales but its strength increases for higher scales. After approximately quarter of a year, the effect reaches values around unity and even though some further variation is visible, the effect remains fairly close to one. The effect is thus found to be rather cumulative than instantaneous and it takes several months before the temperature changes fully translate into relative humidity. An increase of one degree Celsius in mean daily temperature is accompanied by an eventual decrease of one percentage point in relative humidity. Narrow confidence intervals suggest high reliability of the estimates.

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Second, we analyze a relationship between stock returns and related stock indices, which forms a building
block of the capital asset pricing model (CAPM) in finance. We focus on two competing companies and their stocks – Apple, Inc. (traded on NASDAQ, USA) and Samsung Electronics Co. Ltd. (traded on KOSPI, South Korea). The CAPM relationship stems in a simple model
\[ r_{i,t} = \alpha + \beta r_{M,t} + u_t \]
where \( r_{i,t} \) and \( r_{M,t} \) are returns of a stock and a relevant stock index, respectively. Both parameters - \( \alpha \) and \( \beta \) - have important implications in classical financial economics. The former one is standardly taken as a measure of under- or over-pricing of the stock, and the latter one is a measure of systematic risk due to overall market conditions. We study the systematic risk (market betas) of Apple and Samsung stocks with respect to their respective markets on weekly data between January 2000 and August 2014 (764 observations). Fig. 4 uncovers that both stocks are very tightly connected to their market indices with \( \beta \) very close to one. Apple stocks seem to be more aggressive than Samsung stocks as the former \( \beta \) remains above one more frequently. Variation of the systematic risk parameters across scales is much weaker than for the previous case and confidence intervals are much wider mainly due to high volatility of financial returns.

And third, we focus on elasticity between ethanol and corn prices. Corn is a primary producing factor of ethanol in the USA and as such, its price changes are mirrored in ethanol price. We study a standard elasticity model
\[ \log(P_{E,t}) = \alpha + \beta \log(P_{C,t}) + u_t \]
where \( P_{E,t} \) and \( P_{C,t} \) are ethanol and corn prices, respectively. The logarithmic specification allows for interpretation of the \( \beta \) coefficient as elasticity, i.e. one percent change in corn is accompanied by \( \beta \% \) change in ethanol. Fig. 5 presents the results for daily series between January 2007 and March 2014 (1821 observations). Elasticity varies strongly across scales. For low scales, i.e. in a short run, elasticity remains low yet still positive and increases with an increasing scale. The highest sensitivity of ethanol to changes in corn are reported for scales between approximately half a year and year and a half. Therefore, ethanol reacts to corn very strongly in a long run as one percent change in corn is reflected in ethanol price by between 0.5 and 0.8 percent change. Narrow confidence intervals again support the reported results.

In conclusion, we introduce a new framework of examining relationship between two variables by merging standard regression least squares with detrended fluctuation analysis. The connection not only allows for studying connection between variables at different scales but also provides related standard errors and coefficients of determination. Moreover, the method is constructed to work even for non-stationary and long-range correlated data. The detrended fluctuation analysis regression opens a new area of research in various empirically oriented disciplines and it also delivers a complete framework not necessarily restricted to DFA itself. Other methods for studying long-range correlations such as detrending moving averages, height correlation analysis and others can be easily implemented into the framework as well.

Support from the Czech Science Foundation under project No. 14-11402P is gratefully acknowledged.

FIG. 5. Elasticity between ethanol and corn. Estimated elasticity varies considerably for different scales. Weak effect going from corn to ethanol is found at low scales with an increasing tendency towards higher scales. Ethanol is the most elastic at scales between approximately 100 and 350 trading days. Scale dependent estimates are represented by a black curve and 95% confidence intervals are shown using the grey curves.

[1] C.K. Peng, S.V. Buldyrev, A.L. Goldberger, S. Havlin, M. Simons, and H.E. Stanley. Finite-size effects on long-range correlations: Implications for analyzing DNA sequences. Physical Review E, 47:3730–3733, 1993.
[2] C.K. Peng, S.V. Buldyrev, S. Havlin, M. Simons, H.E. Stanley, and A.L. Goldberger. Mosaic organization of
DNA nucleotides. Physical Review E, 49:1685–1689, 1994.

[3] C.K. Peng, S. Havlin, H.E. Stanley, and A.L. Goldberger. Quantification of scaling exponents and crossover phenomena in nonstationary heartbeat time series. Chaos, 5:82–87, 1995.

[4] S.V. Buldyrev, A.L. Goldberger, S. Havlin, R.N. Mantegna, M.E. Matsa, C.K. Peng, M. Simons, and H.E. Stanley. Long-range correlation properties of coding and noncoding DNA sequences: GenBank analysis. Physical Review E, 51:5084–5091, 1995.

[5] J.W. Kantelhardt, E. Koscielny-Bunde, H.H.A. Rego, S. Havlin, and A. Bunde. Detecting long-range correlations with detrended fluctuation analysis. Physica A, 295:441–454, 2001.

[6] J.W. Kantelhardt, S. Zschiegner, E. Koscielny-Bunde, A. Bunde, S. Havlin, and E. Stanley. Multifractal Detrended Fluctuation Analysis of Nonstationary Time Series. Physica A, 316:87–114, 2002.

[7] B. Podobnik and H.E. Stanley. Detrended cross-correlation analysis: A new method for analyzing two nonstationary time series. Physical Review Letters, 100:084102, 2008.

[8] B. Podobnik, D. Horvatic, A. Petersen, and H.E. Stanley. Cross-correlations between volume change and price change. PNAS, 106:22079–22084, 2009.

[9] W.X. Zhou. Multifractal detrended cross-correlation analysis for two nonstationary signals. Physical Review E, 77:066211, 2008.

[10] Z.Q. Jiang and W.X. Zhou. Multifractal detrending moving average cross-correlation analysis. Physical Review E, 84:016106, 2011.

[11] G.F. Zebende. DCCA cross-correlation coefficient: Quantifying level of cross-correlation. Physica A, 390:614–618, 2011.

[12] L. Kristoufek. Measuring correlations between nonstationary series with DCCA coefficient. Physica A, 402:291–298, 2014.

[13] H.E. Stanley, L.A.N. Amaral, A.L. Goldberger, S. Havlin, P.Ch. Ivanov, and C.K. Peng. Statistical physics and physiology: Monofractal and multifractal approaches. Physica A, 270:309–324, 1999.

[14] A. Bunde, S. Havlin, J.W. Kantelhardt, T. Penzel, J.H. Peter, and K. Voigt. Correlated and uncorrelated regions in heart-rate fluctuations during sleep. Physical Review Letters, 85:3736–3739, 2000.

[15] C.K. Peng, S.V. Buldyrev, A.L. Goldberger, S. Havlin, R.N. Mantegna, M. Simons, and H.E. Stanley. Statistical properties of DNA sequences. Physica A, 221:180–192, 1995.

[16] S.V. Buldyrev, N.V. Dokholyan, A.L. Goldberger, S. Havlin, C.K. Peng, H.E. Stanley, and G.M. Viswanathan. Analysis of DNA sequences using methods of statistical physics. Physica A, 249:430–438, 1998.

[17] T. Montez, S.-S. Poil, B.F. Jones, I. Manshanden, J.P.A. Verbunt, B.W. Van Dijk, A.B. Brussaud, A. Van Ooyen, C.J. Stam, P. Scheltens, and K. Linkenkaer-Hansen. Altered temporal correlations in parietal alpha and prefrontal theta oscillations in early-stage alzheimer disease. PNAS, 106:1614–1619, 2009.

[18] P. Talkner and R.O. Webber. Power spectrum and detrended fluctuation analysis: Application to daily temperatures. Physical Review E, 62:150–160, 2000.

[19] Y. Liu, P. Cizeau, M. Meyer, C.K. Peng, and H.E. Stanley. Correlations in economic time series. Physica A, 245:437–440, 1997.

[20] P.A. Varotsos, N.V. Sarlis, and E.S. Skordas. Long-range correlations in the electric signals that precede rupture. Physical Review E, 66:011902, 2002.

[21] R.O. Weber and P. Talkner. Spectra and correlations of climate data from days to decades. Journal of Geophysical Research: Atmospheres, 106:20131–20144, 2001.