ON THE RATIONALITY CONJECTURE OF SOME FINITE CW-COMPLEXES

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Abstract. In this paper, we establish the rationality conjecture raised in [6] for any \((r - 1)\)-connected \((r \geq 2)\) \(kr\)-dimensional CW-complex \(X\) \((k \geq 2)\) having a unique spherical cohomology class \(u \in \tilde{H}^r(X, \mathbb{Z})\) such that \(u^k \neq 0\). Next, we illustrate (topologically) our result by giving the minimal cell structure of such a CW-complex whose cohomology is a truncated polynomial algebra.

1. Introduction and statement of the results

Throughout this paper, unless stated otherwise, \(X\) will denote an \((r - 1)\)-connected finite CW-complex \(X\) \((r \geq 2)\) and \(\mathbb{K}\) an arbitrary field.

Very recently, M. Farber et al. have posed in [6], the following rationality conjecture

**Conjecture 1.0.1.** The \(TC\)-generating function \(F_X(x) = \sum_{n=1}^{\infty} TC_{n+1}(X)x^n\) is a rational function with a single pole of order 2 at \(x = 1\). More explicitly, there exists \(P \in \mathbb{Z}[X]\) such that:

\[
F_X(x) := \sum_{n=1}^{\infty} TC_{n+1}(X)x^n = \frac{P(x)}{(1-x)^2}.
\]

To make precise terms of \(F_X(x)\), recall that the sectional category (or Schwartz’s genus [14]) of a fibration \(F \to E \xrightarrow{p} B\) is the minimal integer \(k \geq 0\) denoted \(secat(p)\) such that there exists an open cover \(B = U_0 \cup U_1 \cup \ldots \cup U_k\) with the property that over each set \(U_i\), the fibration admits a continuous section. \(secat(-)\) is indeed an homotopy invariant. Special cases useful for us are the (higher) topological complexities \(TC_n(X) = secat(\pi_n)\) \((n \geq 2)\) introduced firstly by M. Farber in [4] for \(n = 2\) and generalized for any \(n \geq 2\) by Y. Rudyak in [13] (see also [1]). Here

\[
\pi_n : P(X) \to X^n, \quad \gamma \mapsto (\gamma(0), \gamma(\frac{1}{n}), \ldots, \gamma(\frac{n-1}{n}), \gamma(1))
\]
is the fibrational substitute for the iterated diagonal map $\Delta_n : X \to X^n$. In fact, $\pi_n$ describes a motion planning algorithm of a physical system moving in its configuration space $X$ between any two positions (the input and the output) but having to reach $n - 2$ intermediate stats. Referring to [6] Lemma 1], if $X$ satisfies (1.0.1) then for all $n$ large enough $TC_{n+1}(X) - TC_n(X)$ becomes constant. The invariants $TC_n(X)$ are usually difficult to determine and are often approximated by the algebraic invariants cup-length and zero-divisors-cup-length defined respectively as follows:

$$\text{cup}_k(X) = \sup\{ j \geq 1, \exists a_1, a_2, \ldots, a_j \in \check{H}^*(X, \mathbb{K}) \text{ such that } a_1 a_2 \ldots a_j \neq 0 \}$$

and

$$\text{zcl}_n(X, \mathbb{K}) = \sup\{ j \geq 1, \exists a_1, \ldots, a_j \in \text{Ker}(\sim_{n,\mathbb{K}}) \text{ such that } a_1 \ldots a_j \neq 0 \}$$

where $\sim_{n,\mathbb{K}} : H^*(X, \mathbb{K})^{\otimes n} \to H^*(X, \mathbb{K})$ denotes the graded algebra multiplication generalizing the standard cup-product $u \sim 1_{\mathbb{K}} = \sim_{2,\mathbb{K}}$. These will be subsequently used.

Our goal in this paper is to study the conjecture (1.0.1) for any $(r - 1)$-connected finite CW-complex $(r \geq 2)$ satisfying the following condition:

$$(1) \quad \dim X = kr, \text{ and } \check{H}^r(X, \mathbb{Z}) \cong \mathbb{Z}u, \text{ with } u^k \neq 0 \text{ some } k \geq 2.$$ Using the Universal Coefficients Theorem and the Hurewicz Theorem we obtain

$$\check{H}^r(X, \mathbb{Z}) \cong \text{Hom}_\mathbb{Z}(\check{H}_r(X, \mathbb{Z}), \mathbb{Z}) \xrightarrow{\text{can}} \text{Hom}_\mathbb{Z}(\pi_r(X), \mathbb{Z}).$$

It results that, if $a \in \check{H}_r(S^r, \mathbb{Z})$ and $[f] \in \pi_r(X) \cong \mathbb{Z}$ are generating homology and homotopy classes respectively, then $<_r f^*(u), a > = < u, f_*(a) > \neq 0$ and consequently, $f^*(u) \neq 0$. Thus $u \in \check{H}^r(X, \mathbb{Z})$ is indeed a spherical cohomology class [9] Lemma 3.1.

To establish our main result, we have to determine the characteristics of the fields $\mathbb{K}$ for which $u^k_{\mathbb{K}}$ is non-zero. Then, we specify among the latter, those that realize equalities $TC_n(X) = \text{zcl}_n(X, \mathbb{K}) \ (\forall n \geq 2)$. From this last condition, there arises the integers $\lambda_{(3,k)} = \sum_{0 \leq i < k} (-1)^i (C_k^i)^3$ which depend only on $k$.

Based on [6] Theorem 1], we establish the following

**Proposition 1.0.2.** Let $X$ be an $(r - 1)$-connected finite CW-complex satisfying (1). Then $X$ verifies (1.0.1) provided there exists a field $\mathbb{K}$ with characteristic zero or a prime number not dividing $\lambda_{(3,k)}$ (resp. $2\lambda_{(3,k-1)}$) when $k$ is even (resp. when $k$ is odd)

As a immediate consequence of Dirichly’s Theorem, we state our main result as follows:

**Theorem 1.0.3.** Any $(r - 1)$-connected finite CW-complex satisfying (1) verifies the conjecture (1.0.1).

Using similar arguments we obtain the following (cf. Remark 3.2.1 below for the notion of $r$-admissible fields):
Theorem 1.0.4. If $X$ is an $(r - 1)$-connected $kr$-dimensional CW-complex whose cohomology over any choice of $r$-admissible field $\mathbb{K}$ is a truncated polynomial algebra, i.e. $H^*(X, \mathbb{K}) \cong \mathbb{K}[v]/(v^{k+1})$, then $X$ verifies the conjecture (1.0.1).

Examples of spaces verifying the above theorem are those satisfying in addition to (1) the condition: $\dim H^*(X, \mathbb{K}) = k$ for every $\mathbb{K}$ with $\text{char}(\mathbb{K}) \in \mathcal{P}(i)$ (resp. $\text{char}(\mathbb{K}) \in \mathcal{P}(ii)$) (cf. §2 for the definitions of $\mathcal{P}(i)$ and $\mathcal{P}(ii)$).

In §2, we determine fields $\mathbb{K}$ for which $u^k_\mathbb{K} \neq 0$. §3 is devoted to proofs of our main results and in §4 we describe minimal structures of $(r - 1)$-connected $kr$-dimensional CW-complexes such that $H^*(X, \mathbb{K}) \cong \mathbb{K}[u_\mathbb{K}]/(u^{k+1}_\mathbb{K})$ for any choice of $r$-admissible field $\mathbb{K}$. We end the paper by §5 by giving a range of examples of $kr$-dimensional CW-complexes of the form $K = S^r \cup_{\beta_1} e^{2r} \cup \ldots \cup_{\beta_l} e^{kr}$ which satisfy (1.0.1).

2. Fields satisfying $u^k_\mathbb{K} \neq 0$

Throughout this section, $X$ will denote an $(r - 1)$-connected finite CW complex satisfying condition (1).

Recall that the cup-product can be defined in terms of the Alexander-Whitney diagonal approximation:

$$\cup: H^p(X, G_1) \otimes H^q(X, G_2) \to H^{p+q}(X, G_1 \otimes G_2)$$

given, in general, for any groups $G_1$ and $G_2$ by:

$$(a \cup b)(\sigma) = a(\sigma_{[[u_0, v_1, \ldots, v_p]]}) \otimes b(\sigma_{[[u_0, v_{p+1}, \ldots, v_{p+q}]]}).$$

Therefore, by putting in (2) $G_1 = \mathbb{Z}$ and $G_2 = \mathbb{K}$; any field and $u_\mathbb{K} := u \sim 1_\mathbb{K}$, we obtain $u^k_\mathbb{K} = u^k \sim 1_\mathbb{K}$.

Thereafter, we will make use of the identification:

$$H^{kr}(X, \mathbb{Z}) = \mathbb{Z}^{m_k} \oplus (\mathbb{Z}/p_1^{a_1} \mathbb{Z}) \oplus (\mathbb{Z}/p_2^{a_2} \mathbb{Z}) \oplus \ldots \oplus (\mathbb{Z}/p_\theta^{a_\theta} \mathbb{Z})$$

and will consider the family:

$$\mathcal{P}_{u^k} := \{p_1, p_2, \ldots, p_\theta(k)\}.$$

Next, although $u \in H^r(X, \mathbb{Z})$ has infinite additive order, it may exist $m \in \mathbb{N}^*$ such that $mu^{l+1} = 0$ for some $2 \leq l \leq k - 1$ (i.e. in the chain level, $mu^{l+1} = \delta v$ for some $v \in C^{(l+1)r+1}(X)$). Thus, we should discuss two cases depending on the additive order $o(u^k)$ of $u^k$. In all what follows, for any group $G$, its free part will be denoted by $\text{Free}(G)$.

(i) If $o(u^k)$ is infinite so that $\text{Free}(H^{lr}(X, \mathbb{Z})) \neq 0$ for each $1 \leq l \leq k$, condition (1) implies $u^k_\mathbb{K} \neq 0$ for all fields $\mathbb{K} \in \mathcal{P}\setminus \mathcal{Q}_{u^k}$, where $\mathcal{Q}_{u^k} := \{q_1, q_2, \ldots, q_\theta(k)\}$ is such that $u^k = q_1^{f_1} q_2^{f_2} \ldots q_\theta(k)^{f_\theta(k)} w; w \in \mathbb{Z}^{m_k}$.

(ii): If $o(u^k)$ is finite, so that $mu^{l+1} = 0$ for some $m \in \mathbb{N}^*$ and $2 \leq l \leq k - 1$.

Hence, $mu^k = 0$ and therefore $o(u^k) =: p_1^{\beta_1} p_2^{\beta_2} \ldots p_\theta(k)^{\beta_\theta(k)}/m$ such that for any $1 \leq i \leq \theta(k)$, we have $0 \leq \beta_i \leq \alpha_i$ and at least one $\beta_i > 0$.

Thus, on one hand, for $\mathcal{Q}_{u^k}$ being defined similarly as $\mathcal{Q}_{u^k}$, condition
Lemma 3.0.1. \( u^l \neq 0 \) for all \( K \) such that \( \text{char}(K) \in \mathcal{P} \setminus \mathcal{Q}_{u^l} \), where \( \mathcal{P} \) stands for the set of all prime numbers. On the other one, for any field whose characteristic lies to \( \mathcal{P} \setminus \mathcal{P}_{u^l} \); \( u^{l+1} \neq 0, \ldots, u^k \neq 0 \).

In all what follows we put 
\[
\mathcal{P}(i) = \{0\} \cup \mathcal{P} \setminus \mathcal{Q}_{u^l} \text{ and } \mathcal{P}(ii) = \mathcal{P} \setminus (\mathcal{P}_{u^l} \cup \mathcal{Q}_{u^l}).
\]

As a Consequence, we have established the following

**Lemma 2.0.1.** \( \dim \text{Span}\{u_k, u^2_k, \ldots, u^k_k\} = k \) if and only if \( \text{char}(K) \in \mathcal{P}(i) \) or else \( \text{char}(K) \in \mathcal{P}(ii) \).

3. **Proofs of our main results**

The following lemma will clarify the behavior of \( zcl_n(X, K) \) towards \( \text{char}(K) \) and the parity of the nil-potency order \( k \). For any \( n \geq 3 \) we put:
\[
A_1 = u_K \otimes 1 \otimes \cdots \otimes 1, \ A_2 = 1 \otimes u_K \otimes 1 \otimes \cdots \otimes 1, \ldots, A_n = 1 \otimes \cdots \otimes 1 \otimes u_K.
\]

\[
\xi_{n,k} = \left( \prod_{i=2}^{n} (A_1 - A_i) (A_2 - A_3) \right)^k
\]
and
\[
\mu_{n,k} = \xi_{n,k-1}[(A_1 - A_2)(A_1 - A_3) \cdots (A_1 - A_n)] = \xi_{n,k}/(A_2 - A_3).
\]

**Lemma 3.0.1.** Let \( K \in \mathcal{P}(i) \) (resp. \( \mathcal{P}(ii) \)). Then, for any \( n \geq 3 \):

(i) \( \xi_{n,2k} = \lambda_{(3,2k)} u^k_k \otimes \cdots \otimes u^k_k \) and \( \xi_{n,2k+1} = 0 \).

(ii) \( \mu_{n,2k+1} (A_1 - A_n) = 2(-1)^{n-1} \lambda_{(3,2k)} u^k_k \otimes \cdots \otimes u^k_k \) where:
\[
\lambda_{(3,2k)} = 2 \left( \sum_{0 \leq i \leq k} (-1)^i (C^i_{2k})^3 \right) + (C^k_{2k})^3.
\]

**Proof.** Notice first the following inductive formulas for any \( n \geq 3, k \geq 2 \) and \( l \geq 2 \):
\[
\xi_{n+1,k} = \xi_{n,k} (A_1 - A_{n+1})^k \text{ and } \xi_{n,k+1} = \xi_{n,k} \xi_{n,l}.
\]
Now, since \( u^l_k = 0, \forall j > k \), a straightforward calculus gives
\[
\xi_{n,k} = \lambda_{(n,k)} u^k_k \otimes \cdots \otimes u^k_k
\]
(some constant \( \lambda_{(n,k)} \in \mathbb{Z} \)). Thus, using again \( u^l_k = 0, \forall j > k \) we deduce that in the relation \( \xi_{n+1,k} = \xi_{n,k} (A_1 - A_{n+1})^k \), the term \((-1)^k A_{n+1}^k\) is the only one to be retained from teh equality \( (A_1 - A_{n+1})^k = \sum_{i=0}^{k} (-1)^{k-i} C_i^k A_1^i A_{n+1}^{k-i} \).

Hence:
\[
\xi_{n+1,k} = (-1)^k \xi_{n,k} A_{n+1}^k = (-1)^k \lambda_{(n,k)} u^k_k \otimes \cdots \otimes u^k_k
\]
(with \( n+1 \) factors \( u^k_k \)) so \( \lambda_{(n+1,k)} = (-1)^k \lambda_{(n,k)} = \ldots = (-1)^{(n-2)} k \lambda_{(3,k)} \).

Consequently, we have:
\[
\lambda_{(n,k)} = (-1)^{(n-1)} k \lambda_{(3,k)}, \forall n \geq 3.
\]
Now, by using Newton’s formula for each term in
\[
\xi_{3,k} = (-1)^k(A_1 - A_2)^k(A_3 - A_2)^k(A_2 - A_3)^k,
\]
we get the deterministic coefficient:
\[
\lambda_{(3,k)} = \sum_{0 \leq i \leq k} (-1)^i(C^i_k)^3.
\]
Thus,
\[
\lambda_{(3,2k)} = 2[\sum_{0 \leq i \leq k} (-1)^i(C^i_{2k})^3] + (C^k_{2k})^3 \quad \text{and} \quad \lambda_{(n,2k+1)} = 0.
\]

Next, consider \(\mu_{(n,k)} = \xi_{(n,k-1)}[(A_1 - A_2)(A_1 - A_3)\ldots(A_1 - A_n)].\) Formulas lying coefficients and roots in \(Q(A_1) = (A_1 - A_2)(A_1 - A_3)\ldots(A_1 - A_n)\) and the constraints \(u_k^j = 0, \forall j > k\) lead us to take only into account the constant coefficient \(q_0 = (-1)^{n-1}A_2A_3\ldots A_n\) as well as that of \(A_1,\) i.e. that of \(q_1 = (-1)^{n-2}\sum_{2 \leq i_1 < \ldots < i_{n-2} \leq n} A_{i_1}\ldots A_{i_{n-2}}.\) Therefore,
\[
\mu_{(n,k)} = (-1)^{n-1}\lambda_{(n,k-1)}[u_k^{k-1} \otimes u_k^k \ldots \otimes u_k^k - \sum_{j=2}^{j=n} u_k^j \otimes \ldots \otimes u_k^j \otimes u_k^{j-1} \otimes u_k^k \ldots \otimes u_k^k]
\]
where \(u_k^{j-1}\) means that this factor is in the \(j\)-th place. It is then immediate that
\[
\mu_{(n,2k+1)}(A_1 - A_n) = 2(-1)^{n-1}\lambda_{(3,2k)}u_k^k \otimes \ldots \otimes u_k^k. \quad \square
\]

3.1. Proof of Proposition 1.0.2. The following theorem will be subsequently used:

Theorem 3.1.1. \([39]\): For any \(s\)-connected CW-complex \(X,\) any field \(\mathbb{K}\) and any integer \(n \geq 2\) we have
\[
cup_n(X) \leq \text{cat}_{LS}(X) \leq \frac{\dim X}{s + 1} \quad \text{and} \quad \text{zcl}_n(X,\mathbb{K}) \leq T C_n(X) \leq \frac{n \dim X}{s + 1}.
\]

Proof. By the above theorem, \((r - 1)\)-connectedness with \(r \geq 2\) and dimension \(kr\) of \(X\) with \(k \geq 2\) imply that \(\text{cat}_{LS}(X) \leq k\) and \(T C_n(X) \leq nk\) for every \(n \geq 2.\) We continue considering the following steps (we fix \(\mathbb{K}\) a field satisfying hypothesis of Proposition 1.0.2):

1. By Lemma 2.0.1 we have \(u_k^j \neq 0,\) hence, \(k \leq \text{cup}_n(X) \leq \text{cat}_{LS}(X)\) \([39]\) and so \(\text{cup}_n(X) = k = \text{cat}_{LS}(X).\)
2. If \(n = 2\) one shows easily (for any \(\mathbb{K}\)) that \(\xi_{(2,k)} = (A_1 - A_2)^{2k} \neq 0,\) thus, \(2k \leq \text{zcl}_2(X,\mathbb{K})\) and therefore \(\text{zcl}_2(X,\mathbb{K}) = 2k = T C_2(X).\)
3. Assume for the rest that \(n \geq 3:\)
   (i) if \(k\) is even, \(k \geq 2,\) using Lemma 3.0.1, we see that
   \[
   \xi_{(n,k)} \neq 0 \leftrightarrow \lambda_{(n,k)} \neq 0 \leftrightarrow \lambda_{(n,k)} \wedge \text{char} (\mathbb{K}) = 1
   \]
in which case we have $nk \leq zcl_n(X, K)$. Therefore, $zcl_n(X, K) = nk = TC_n(X)$.

(ii) if $k$ is odd, the same argument using $\mu_{(n,k)}(A_1 - A_n)$ gives us the same equality $zcl_n(X, K) = nk = TC_n(X)$.

To conclude, it remain to use [6, Theorem 1].

3.2. Proof of Theorem 1.0.3.

Proof. Proposition 1.0.2 is satisfied for any field whose characteristic lies to $P_{(i)}$ or else $P_{(ii)}$ and does not divide the integer $\lambda_{(3,k)}$ or $\lambda_{(3,k-1)}$ depending on the parity of $k$. It suffice then to make use of Dirichely’s theorem by which we may fix such a field.

Next, to establish Theorem 1.0.4 recall that a field $K$ is said $r$-admissible if there exists an $(r-1)$-connected CW-complex $X$ such that $H^*(X, K) \cong K[v]/(v^{k+1})$. Referring to [17, Theorem 2] and [10, Theorem 4L.9 and the remark just after] we note on one hand that if $char(K) = p$, a prim number, then the nil-potency order $k$ of the indeterminate $v$ should be greater than $p - 1$ and on the other we resume in the following remark complete results concerning this notion:

Remark 3.2.1. (a): Let $K$ be a field of characteristic zero or a prim $p$. Then, $K$ is $r$-admissible

1. for every even $r \geq 2$ if characteristic is zero.
2. for $r \in \{2, 4\}$ if $p \geq 2$.
3. for $r = 8$ if $p = 2$ or an arbitrary odd prim of the form $p = 4s + 1$ some $s \geq 1$.
4. for every even $r \notin \{2, 4, 8\}$ if $p$ is an arbitrary odd prim of the form $p = \frac{r}{2}s + 1$ some $s \geq 1$.

(b): According to Dirichly’s theorem, there are infinitely many primes in any arithmetic progression $p = \frac{r}{2}s + 1$, that is, to each integer $r \geq 2$ it is associated infinitely many $r$-admissible fields.

3.2.1. Proof of Theorem 1.0.4. Notice first that with hypothesis of Theorem 1.0.4 and Remark 3.2.1 above, Lemma 2.0.1 and Lemma 3.0.1 are satisfied for any choice of $r$-admissible field $K$ when we replace $v$ by $u_K$.

Proof. It suffice to use successively similar arguments as in proves of Proposition 1.0.2 and Theorem 1.0.3.

4. Minimal cell structure of CW-complexes with truncated polynomial cohomology

Our goal in this section is to give a "simultaneously" example of spaces $X$ satisfying (1) and having a truncated polynomial algebra cohomology over any choice of $r$-admissible field $K$, i.e. $H^*(X, K) \cong K[u_K]/(u_K^{k+1})$. These (as it is mentioned in the introduction) form a particular class among those
given by Theorem [1,0.3] Simple examples of such spaces are $\mathbb{C}P^k$ and $\mathbb{H}P^k$ [10 Corollary 4L. 10]. Notice however that real projective spaces $\mathbb{R}P^{2k}$ and $\mathbb{R}P^{2k+1}$ do not satisfy (1) since $H^*(\mathbb{R}P^{2k}, \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(2\alpha, \alpha^{k+1})$; $|\alpha| = 2$ and $H^*(\mathbb{R}P^{2k-1}, \mathbb{Z}) \cong \mathbb{Z}[\alpha, \beta]/(2\alpha, \alpha^{k+1}, \beta^2, \alpha\beta)$; $|\alpha| = 2$ and $|\beta| = 2k + 1$ [10].

More examples are among CW-complexes of the form $L = S^{m_1} \cup \beta_{m_1} e_n \cup \ldots \cup \beta_{m_k} e_n$, with $n_{i+1} - 1 \geq n_i \geq 2$. Indeed, I. M. James associated to any $\beta_k \in \pi_{nk-1}(L)$ such that $n_k = n_{k-1} + n_1$ its generalized Hopf invariant $h_k([\beta_k])$ to be the integer $m_k$ satisfying $x_k x_{k-1} = m_k x_k$. Here $x_i$ stands for the generating element in $H^*(L \cup \beta_k e_n, \mathbb{Z})$. He then showed that this correspondence defines an isomorphism $h_k : \pi_{nk-1}(L) \to \mathbb{Z}$. As a particular case, if $n_1 = r \geq 2$ and $n_{i+1} = n_i + n_1$ for each $1 \leq i \leq k - 1$, we get $K' = S^r \cup \beta_2 e^{2r} \cup \ldots \cup \beta_k e^{(k-1)r}$ and $h_k : \pi_{rk-1}(K') \to \mathbb{Z}$ is an homomorphism extending the usual Hopf invariant $H = h_2 : \pi_{2r-1}(S^r) \to \mathbb{Z}$. Thereafter, we denote $K = S^r \cup \beta_2 e^{2r} \cup \ldots \cup \beta_k e^{kr}$ obtained from $K'$ and $\beta_k$ via the relation $x_k x_{k-1} = h_k([\beta_k]) x_k := m_k x_k$ ($k \geq 3$). Now, by using the long cohomology exact sequences of pairs

$$(S^r \cup \beta_2 e^{2r} \cup \ldots \cup \beta_k e^{kr}, S^r \cup \beta_2 e^{2r} \cup \ldots \cup \beta_{i+1} e^{(i+1)r})$$

we exhibit a family of isomorphisms

$$H^{ir}(S^r \cup \beta_2 e^{2r} \cup \ldots \cup \beta_{i+1} e^{(i+1)r}) \cong H^{ir}(S^r \cup \beta_2 e^{2r} \cup \ldots \cup \beta_i e^{ir})$$

$(2 \leq i \leq k - 1)$. Knowing that $H^r(S^r \cup \beta_2 e^{2r}) \cong H^*(S^r) \cong \mathbb{Z} x_1$ we may identify each $x_i$ with its antecedent, hence, if we put $H([\beta_2]) = h_2([\beta_2]) = m_2$, we see by induction that:

$$m_2 \ldots m_{k-1} m_k x_k = x_1^k.$$ Consequently, every CW-complex $K = S^r \cup \beta_2 e^{2r} \cup \ldots \cup \beta_k e^{kr}$ whose attaching maps are such that $m_2 \ldots m_{k-1} m_k \neq 0$ satisfies (1) and $o(x_1^k)$ is infinite since the one on $x_k$ is (due to the hypothesis $r \geq 2$). Moreover, (cf. §2), $Q_{x_1^l} = \{q_i \in \mathcal{P}, : q_i/m_2 \ldots m_{l-1} m_k\}$ and $P(i) = \mathcal{P} \setminus Q_{x_1^l}$.

Next, put $L = K \cup \beta_{l+1} e^{(l+1)r+1}$ (some $l \leq k$) such that (in cellular homology) the cell $e^{(l+1)r+1}$ satisfies

$$\delta(e^{(l+1)r+1}) = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_{\theta(k)}^{\alpha_{\theta(k)}} e^{(l+1)r}.$$ It is clear that $L$ satisfies (1) with $x_1^{l+1}$ and consequently $x_1^k$ of finite order, and $Q_{x_1^l} = \{q_i \in \mathcal{P}, : q_i/m_2 \ldots m_{l-1} m_k\}$ and $P_{(ii)} = \mathcal{P} \setminus (P_{x_1^l} \cup Q_{x_1^l})$.

Our main result in this section is given with the following terms already used in [10] Proposition 4C.1:

$\tilde{e}^n$ will denote the pair of cells $(e^{n+1}, e^n, \delta(e^{n+1}) = q^e \epsilon^n)$ where $q^e$ is the additive order of the homology (hence the cohomology) class induced by a summand $\mathbb{Z}/q\mathbb{Z}$.

**Theorem 4.0.1.** Let $X$ be a finite $kr$-dimensional CW-complex satisfying (1) with cohomology a truncated polynomial algebra $H^*(X, \mathbb{K}) \cong \mathbb{K}[u_k]/(u_k^{k+1})$ for any choice of $r$-admissible field $\mathbb{K}$. 
If $o(u^k)$ is infinite, then $X$ is homotopy equivalent to a CW-complex $K = S^r \cup_{u^2} e^{2r} \cup \ldots \cup_{u^k} e^{kr}$ such that $h_2(\eta_2)h_3(\eta_3) \ldots h_k(\eta_k) = q_1^{f_1} q_2^{f_2} \ldots q_i^{f_i(k)}$ where $q_i \in \mathbb{Q}_{u^i}$.

(b) Assume that $o(u^k)$ is finite and $l \geq 2$ is the greatest power of $u$ with infinite order, then $X$ is homotopy equivalent to a CW-complex $K = L \cup L'$ where

- $L = S^r \cup_{u^2} e^{2r} \cup \ldots \cup_{u^k} e^{kl}$ such that $h_2(\eta_2)h_3(\eta_3) \ldots h_l(\eta_l) = q_1^{g_1} q_2^{g_2} \ldots q_{l-1}^{g_{l-1}}$ with $q_{i-1} \in \mathbb{Q}_{u^i}$ and $0 \leq g_i \leq f_i$ and

- $L'$ is formed, for each $p_i \in \mathbb{P}_{u^i}$, by $k - l$ cells: an $e_i^{j_{r-1}}$ or else an $e_i^{j_{r-1}}$ for each $l + 1 \leq j \leq k - 1$, each of these cells is of finite additive order $p_i^{\alpha_i} (\alpha_i \geq \alpha_i)$ and an $e_i^{j_{r-1}}$ of finite additive order $p_i^{\alpha_i}$.

Proof. We note at the beginning that, by Lemma 2.0.1, every $r$-admissible field is indeed in $\mathbb{P}_{(ii)}$ or else in $\mathbb{P}_{(ii)}$. This explains notations in the statement of the theorem.

(a) Assume that $o(u^k)$ is infinite. By hypothesis on the cohomology of $X$ the UCT with $\mathbb{Q}$-coefficients, condition (1) and Remark 3.2.1 imply that $H_{ir}(X, \mathbb{Z}) \cong \mathbb{Z}$ for each $0 \leq i \leq k$. Therefor $X$ is homotopy equivalent to the CW-complex $K = S^r \cup_{u^2} e^{2r} \cup \ldots \cup_{u^k} e^{kr}$ whose attaching maps satisfy the condition $h_2(\eta_2)h_3(\eta_3) \ldots h_k(\eta_k) = q_1^{f_1} q_2^{f_2} \ldots q_i^{f_i(k)}$ where $q_i \in \mathbb{Q}_{u^i}$.

(b) Assume that $o(u^k)$ is finite and let $l \geq 2$ be the greatest power of $u$ with infinite order. First of all, since the powers $u^i$, $1 \leq i \leq l$ are in the free part of $H^*(X, \mathbb{Z})$, we should introduce cells $e^0, e^1, \ldots, e^r$ in the minimal structure of $X$. These form $L = S^r \cup_{u^2} e^{2r} \cup \ldots \cup_{u^k} e^{kl}$ such that $h_2(\eta_2)h_3(\eta_3) \ldots h_l(\eta_l) = q_1^{g_1} q_2^{g_2} \ldots q_{l-1}^{g_{l-1}}$ with $q_{i-1} \in \mathbb{Q}_{u^i}$ and $0 \leq g_i \leq f_i$.

Now, by Lemma 2.0.1 in order to realize hypothesis on the cohomology of $X$, the cells $e^{l+1}, \ldots, e^{kr}$ should be introduced for every $p = char(\mathbb{K}) \in \mathbb{P}_{(ii)} = \mathbb{P} \backslash (\mathbb{P}_{u^i} \cup \mathbb{Q}_{u^i})$. Hence, in order to rich the (full dimension) $\dim H^r(X, \mathbb{K}) = k$, the UCT:

$$H^r(X, \mathbb{K}) \cong Hom(H_{ir}(X, \mathbb{Z}), \mathbb{K}) \oplus Ext(H_{ir-1}(X, \mathbb{Z}), \mathbb{K})$$

imposes to add (for each $p_i \in \mathbb{P}_{(ii)}$):

- $k - l - 1$-generator cells $e_i^{j_{r-1}}$ or else $e_i^{j_{r-1}}$ ($l + 1 \leq j \leq k$) and its corresponding relater-cells $e_i^{j_{r+1}}$ or else $e_i^{j_{r}}$ satisfying $\delta(e_i^{j_{r+1}}) = p_i^{a_i} e_i^{j_{r-1}}$ or else $\delta(e_i^{j_{r}}) = p_i^{a_i} e_i^{j_{r-1}}$ with $a_i \geq a$.

- a cell of degree $rk$ which should came from the Ext term, that is from a pair $(e^{rk}, e^{rk-1})$ such that $\delta(e_i^{rk}) = p_i^{a_i} e_i^{rk-1}$. We then obtain $L'$ as it is stated in the theorem.

□
5. Examples and a final remark

To have an idea on sequences of prime numbers introduced in Proposition 1.0.2, we give in the table below (Table 1) all prime factors of $\lambda_{(3,k)}$ for $2 \leq k \leq 40$. This is possible thanks to the Scientific Workplace 5.5 software:

Table 1. List of prime factors of $\lambda_{(3,k)}$ up to 40

| $k$ | prime factors of $\lambda_{(3,k)}$ |
|-----|-----------------------------------|
| 2   | 2, 3                              |
| 4   | 2, 3, 5                           |
| 6   | 2, 3, 5, 7                        |
| 8   | 2, 3, 5, 7, 11                    |
| 10  | 2, 3, 7, 11, 13                   |
| 12  | 2, 3, 7, 11, 13, 17               |
| 14  | 2, 3, 5, 11, 13, 17, 19           |
| 16  | 2, 3, 5, 11, 13, 17, 19, 23       |
| 18  | 2, 3, 5, 11, 13, 17, 19, 23, 29   |
| 20  | 2, 3, 5, 7, 11, 13, 17, 19, 23, 29 |
| 22  | 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31 |
| 24  | 2, 3, 5, 7, 13, 17, 19, 23, 29, 31 |
| 26  | 2, 3, 5, 7, 17, 19, 23, 29, 31, 37 |
| 28  | 2, 3, 5, 7, 17, 19, 23, 29, 31, 37, 41 |
| 30  | 2, 3, 5, 11, 17, 19, 23, 29, 31, 37, 41, 43 |
| 32  | 2, 3, 5, 7, 11, 17, 19, 23, 29, 31, 37, 41, 43, 47 |
| 34  | 2, 3, 5, 7, 11, 19, 23, 29, 31, 37, 41, 43, 47 |
| 36  | 2, 3, 5, 7, 11, 13, 19, 23, 29, 31, 37, 41, 43, 47, 53 |
| 38  | 2, 3, 5, 7, 11, 13, 23, 29, 31, 37, 41, 43, 47, 53 |
| 40  | 2, 3, 5, 7, 11, 13, 23, 29, 31, 37, 41, 43, 47, 53, 59 |

5.1. Examples. The following theorem that we recall in our notations ($h_r^i$ is denoted above by $h_i$) is paramount to get examples of CW-complexes of the form $K = S^r \cup_{\beta_2} e^{2r} \cup_{\beta_3} \ldots \cup_{\beta_k} e^{kr}$, satisfying the conjecture (1.0.1):

Theorem 5.1.1. [2 Thorem 1]

$$\text{Im}(h_r^i) = \begin{cases} \mathbb{Z}, & \text{if } r = 2, 4, 8 \text{ and } i = 2 \\ \mathbb{Z}, & \text{if } r = 2 \text{ and } i \text{ a prime number} \\ i\mathbb{Z}, & \text{otherwise} \end{cases}$$

Example 5.1.2. (1) For $r = 2$ and $k = 2$, the space $K = S^2 \cup_{\beta_2} e^4$, where $\beta_2 : S^3 \rightarrow S^2$ is the Hopf map, that is such that $h_2(\beta_2) = 1$, clearly satisfies Proposition 1.0.2 for any field of characteristic $p \notin \{2, 3\}$. 


(2) For \( r = 2 \) and \( k = 3 \), let \( \beta_3 = [i_2, [i_2, i_2]] : S^5 \to S^2 \cup _{\beta_2} e^4 \) satisfying \( h_3(\beta_3) = 3 \). The space
\[
K = S^2 \cup _{\beta_2} e^4 \cup _{\beta_3} e^6
\]
satisfies hypothesis of Proposition 1.0.2 for any field of characteristic \( p \notin \{2, 3\} \).

(3) For \( r = 4 \) and \( k = 3 \), let \( \beta_2 : S^7 \to S^4 \) be the Hopf map, that is such that \( h_2(\beta_2) = 1 \), and \( \beta_3 = [i_4, [i_4, i_4]] : S^{11} \to S^4 \cup _{\beta_2} e^8 \) satisfying \( h_3(\beta_3) = 3 \). The space
\[
K = S^4 \cup _{\beta_2} e^8 \cup _{\beta_3} e^{12}
\]
satisfies hypothesis of Proposition 1.0.2 for any field of characteristic \( p \notin \{2, 3\} \).

(4) For \( r = 4 \) and \( k = 4 \), in addition to \( \beta_2 \) and \( \beta_3 \) of the above item, consider \( \beta_4 : S^{15} \to S^4 \cup _{\beta_2} e^8 \cup _{\beta_3} e^{12} \), provided by Theorem 5.1.1, such that \( h_4(\beta_4) = 4 \). Hence, the space
\[
K = S^4 \cup _{\beta_2} e^8 \cup _{\beta_3} e^{12} \cup _{\beta_4} e^{16}
\]
satisfies hypothesis of Proposition 1.0.2 for any field of characteristic \( p \notin \{2, 3, 5\} \).

(5) For \( r = 8 \) and \( k = 7 \), let \( \beta_2 : S^{15} \to S^8 \) be the Hopf map, that is with \( h_2(\beta_2) = 1 \) and consider the maps \( \beta_3 : S^{24} \to S^8 \cup _{\beta_2} e^{16} \), \( \beta_4 : S^{31} \to S^8 \cup _{\beta_2} e^{16} \cup _{\beta_3} e^{24} \), \( \beta_5 : S^{39} \to S^8 \cup _{\beta_2} e^{16} \cup _{\beta_3} e^{24} \cup _{\beta_4} e^{32} \), \( \beta_6 : S^{47} \to S^8 \cup _{\beta_2} e^{16} \cup _{\beta_3} e^{24} \cup _{\beta_4} e^{32} \cup _{\beta_5} e^{40} \cup _{\beta_6} e^{48} \), \( \beta_7 : S^{55} \to S^8 \cup _{\beta_2} e^{16} \cup _{\beta_3} e^{24} \cup _{\beta_4} e^{32} \cup _{\beta_5} e^{40} \cup _{\beta_6} e^{48} \cup _{\beta_7} e^{56} \), provided by Theorem 5.1.1, such that \( h_3(\beta_3) = 3 \), \( h_4(\beta_4) = 4 \), \( h_5(\beta_5) = 5 \), \( h_6(\beta_6) = 6 \) and \( h_7(\beta_7) = 7 \). Hence, the space
\[
K = S^8 \cup _{\beta_2} e^{16} \cup _{\beta_3} e^{24} \cup _{\beta_4} e^{32} \cup _{\beta_5} e^{40} \cup _{\beta_6} e^{48} \cup _{\beta_7} e^{56}
\]
satisfies hypothesis of Proposition 1.0.2 for any field of characteristic \( p \notin \{2, 3, 5, 7\} \).

(6) For \( r = 8 \) and \( k = 8 \) we consider \( \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7 \) given in the previous item and \( \beta_8 : S^{63} \to S^8 \cup _{\beta_2} e^{16} \cup _{\beta_3} e^{24} \cup _{\beta_4} e^{32} \cup _{\beta_5} e^{40} \cup _{\beta_6} e^{48} \cup _{\beta_7} e^{56} \) such that \( h_8(\beta_8) = 8 \). Hence, the space
\[
K = S^8 \cup _{\beta_2} e^{16} \cup _{\beta_3} e^{24} \cup _{\beta_4} e^{32} \cup _{\beta_5} e^{40} \cup _{\beta_6} e^{48} \cup _{\beta_7} e^{56} \cup _{\beta_8} e^{64}
\]
satisfies hypothesis of Proposition 1.0.2 for any field of characteristic \( p \notin \{2, 3, 5, 7, 11\} \).

(7) The above examples correspond to cases where \( r = 2, 4, 8 \). Now, by Theorem 5.1.1 when \( r \neq 2, 4, 8 \) we obtain a CW-complex \( K \) such that the product of its generalized Hopf invariants is \( h_2(\beta_2) \ldots h_k(\beta_k) = k! \). In order to illustrate the impact of \( \lambda_{3,k} \) in the process, we finish by fixing \( r = 6 \) and taking some arbitrary values of \( k \). The table below dresses fields characteristics to exclude when
\[
K = S^6 \cup _{\beta_2} e^{12} \cup _{\beta_3} \ldots \cup _{\beta_k} e^{6k} .
\]
Table 2. Finite characteristic $p$ not allowed, for some values of $k$ ($r = 6$)

| $k$ | finite characteristic $p$ not allowed |
|-----|--------------------------------------|
| 4   | 2, 3, 5                               |
| 5   | 2, 3, 5                               |
| 16  | 2, 3, 5, 7, 11, 13, 17, 19, 23        |
| 18  | 2, 3, 5, 7, 11, 13, 17, 19, 23        |
| 20  | 2, 3, 5, 7, 11, 13, 17, 19, 23, 29    |
| 22  | 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31|

5.2. Final remark. In this paper, we have treated the case where $u \in \tilde{H}^r(X, \mathbb{Z})$ in condition (1) is the unique spherical cohomology class (thus of infinite order). To achieve our approach in studying the rationality conjecture for finite CW-complexes $X$ satisfying the relations $zcl_n(X) = TC_n(X) \forall n \geq 2$, it remains, on the one hand, to deal with the case where $u \in \tilde{H}^r(X, \mathbb{Z})$ is of finite order (e.g. $\mathbb{R}P^n$) and, on the other hand, to consider more than one generating cohomology class in $\tilde{H}^r(X, \mathbb{Z})$ spherical or not. This is our main project for the future.

References

[1] I. Basabe, J. González, Y. B. Rudyak, and D. Tamaki, Higher topological complexity and its symmetrization, Algebr. Geom. Topol. Volume 14, Number 4 (2014), 2103-2124.

[2] J. H. Baues, Hopf Invariants For Reduced Products of Spheres, Proc. of The Ame. Math. Soc Volume S9, Number 1, August 1976.

[3] O. Cornea, G. Lupton, J. Oprea, D. Tanré, Lusternik-Schnirelmann Category, Mathematical Surveys and Monographs, Vol. 103 (2003).

[4] M. Farber, Topological complexity of motion planning, Discrete Comput. Geom. 29, 211–221 (2003) (MR1957228).

[5] M. Farber, Invitation to Topological Robotics, Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zurich (2008) (MR2455573).

[6] M. Farber, D. Kishimoto, and D. Stanley, Generating functions and topological complexity, Topology Appl. 278 (2020), 107235.

[7] M. Farber, S. Tabachnikov, and S. Yuzvinsky, Topological robotics: motion planning in projective spaces, Int. Math. Res. Not. 34, 1853 – 1870 (2003).

[8] J. González, M. Grant, and L. Vandembroucq, Hopf Invariants for sectional category with applications to topological robotics, Quarterly Journal of Mathematics, 70(4) (2019), 1209-1252. https://doi.org/10.1093/qmath/haz019.

[9] L. Guijarro, T. Schick, and G. Walschap, Bundles with spherical Euler class, Pacific J. of Math., Vol. 27, No. 2, (2002), 377-392. 19. 974.

[10] A. Hatcher, Algebraic Topology, Cambridge University Press, 2002.

[11] I. M. James, Note on cup-products, Proc. Amer. Math. Soc. 8 (1957), 374-383. MR 19. 974.

[12] D. Kishimoto and A. Yamaguchi On The Growth of Topological Complexity, arXiv 205.09847v1 [math.AT] 20 May 2020.
[13] Y. Rudyak, *On higher analogous of topological complexity*, Topology and its Applications Volume 157, Issue 5, 1 April 2010, Pages 916-920.
[14] A. Schwarz, *The genus of a fiber space*, Am. Math. Soc. Trans. 55, 49–140 (1966) (MR0154284).
[15] A. Shar, *The Homotopy Groups of Spaces whose Cohomology is a Truncated Polynomial Algebra*, Proceedings of the American Mathematical Society Volume 38, Number 1, March 1973.
[16] H. Toda, *p-primary components of homotopy groups IV. Compositions and toric constructions*, Memories of the College of Science, Univ. of Kyoto, A Vol. XXXII, Mathematics No. 2, 1959.
[17] H. Toda, *Note on Cohomology Ring of Certain Spaces*, Proceedings of the American Mathematical Society Vol. 14, No. 1 (Feb., 1963), pp. 89-95.
[18] J.H.C Whitehead, *A certain exact sequence*, Ann. Math. 52 (1950), 51-110, 2012, pp. 15-21.