Interface development for the nonlinear degenerate multidimensional reaction–diffusion equations II: fast diffusion versus absorption

Ugur G. Abdulla and Amna Abu Weden

Abstract. This paper presents a full classification of the short-time behavior of the solution and the interfaces in the Cauchy problem for the nonlinear second order singular parabolic PDE
\[ u_t - \Delta u^m + bu^\beta = 0, \quad x \in \mathbb{R}^N, 0 < t < T \]
with nonnegative initial function \( u_0 \) such that
\[ \text{supp } u_0 = \{|x| < R\}, \quad u_0 \sim C(R - |x|)^\alpha, \quad \text{as } |x| \to R - 0, \]
where \( 0 < m < 1, b, \beta, C, \alpha > 0 \). Depending on the relative strength of the fast diffusion and absorption terms the problem may have infinite (\( \beta \geq m \)) or finite (\( \beta < m \)) speed of propagation. In the latter case, the interface surface \( t = \eta(x) \) may shrink, expand or remain stationary depending on the relative strength of the fast diffusion and strong absorption terms near the boundary of support, expressed in terms of the parameters \( m, \beta, \alpha, \) and \( C \). In all cases we prove the existence or non-existence of the interfaces, explicit formula for the interface asymptotics, and local solution near the interface or at infinity.

Mathematics Subject Classification. Primary 35K55, 35K57, 35K65; Secondary 35R35, 35K10, 35K59.

Keywords. Nonlinear degenerate parabolic equations, Reaction–diffusion equations, Interfaces, Nonlinear diffusion, Weak solutions, Comparison theorem.

1. Introduction

Consider the Cauchy problem (CP) for the singular PDE:
\[ Lu = u_t - \Delta u^m + bu^\beta = 0, \quad x \in \mathbb{R}^N, 0 < t < T, \]
\[ u(x,0) = u_0(x), \quad x \in \mathbb{R}^N, \]  

(2)

where \( 0 < m < 1, \ b > 0, \ \beta > 0 \). Equation (1) is a singular parabolic equation arising in various applications in fluid mechanics, plasma physics, population dynamics, etc. as a mathematical model of singular diffusion in the presence of absorption of energy [1–4]. Let \( u_0 \in C(\mathbb{R}^N; \mathbb{R}^+) \) be radially symmetric with \( \text{supp} \ u_0 = \overline{B_R} \)

where \( B_R := \{ x \in \mathbb{R}^N, \ |x| < R \} \), and

\[ u_0(x) \sim C(R - |x|)^\alpha \text{ as } |x| \to R - 0 \]  

(3)

for some \( C > 0, \alpha > 0 \). In particular, we consider the model case

\[ u_0(x) = C(R - |x|)^\alpha, \quad x \in \mathbb{R}^N \]  

(4)

where \( \kappa_+ = \max\{\kappa; 0\} \). Solution of the CP (1), (2) is understood in a weak sense (Definition 4.1, [5]).

Second order nonlinear degenerate and singular parabolic PDEs have a well-established theory. Well-posedness of main boundary value problems and the Cauchy problem, and local regularity properties of weak solutions are well known [3,6–22]. We refer to [23,24] for the complete list of references. The existence and uniqueness of the solution to the CP (1)–(4) is proved in [15,16]. The general theory of boundary value problems in non-cylindrical domains with non-smooth boundary manifolds under minimal regularity assumptions on the boundaries is developed in [25–28].

One of the key problems of the qualitative theory of degenerate and singular parabolic equations is understanding the smoothness and evolution properties of interfaces. The importance is twofold: first, the existence of interfaces demonstrates more relevance for applications to the contrast of infinite speed of propagation property of the linear diffusion equation. Second, the singularities of the weak solutions are primarily concentrated along the interfaces, or along the zero level set due to degeneration or singularity of the equation. The major difficulty in finding explicit asymptotics of the interfaces is hidden in the fact that in most cases the behavior of solutions near the interfaces is nonuniform in the sense of singular perturbation theory. That is to say, the dominant balance as \( t \to 0 \) between the terms in (1) on manifolds that approach the boundary of support on the initial space depending on how they do so. In order to find the explicit asymptotics of the interfaces and local solutions, it is vital to overcome this difficulty. In particular general theory in non-cylindrical non-smooth domains was motivated by the problem of the evolution of interfaces. In papers [29,30] a method was developed for the solution of the interface problem for the one-dimensional nonlinear reaction–diffusion equations (1). The idea of the method is based on the identification of the boundary layer near the interface, where the asymptotics of the solution is uniform, derivation of the asymptotics of the solution along the boundary of the layer through rescaling, and construction of the sharp super- and subsolutions via comparison theorems in non-cylindrical domains with non-smooth boundaries. The last step requires the application of the general theory of
boundary-value problems in non-cylindrical domains with boundary surfaces which has the same kind of behavior as the interface. In many cases, this may be nonsmooth and characteristic. By using the new method, and primarily by applying the general theory of reaction–diffusion equations in non-cylindrical domains with non-smooth boundaries [28] the full classification for the initial development of interfaces was presented in [29,30] for the slow ($m > 1$) and fast ($0 < m < 1$) diffusion cases respectively. The method turned out to be very effective and was recently applied to solve the interface problem for the $p$-Laplacian type reaction–diffusion equations in [31,32], and for the reaction–diffusion equations with double degenerate diffusion in [33]. In a recent paper [5] the method is generalized to solve the interface problem in a multi-dimensional case for the reaction–diffusion equation (1) with $m > 1$.

An alternative approach for the analysis of the interfaces based on the energy methods is pursued in [22,34].

The nature of the fast diffusion ($0 < m < 1$) is qualitatively different from the slow diffusion. Solution of the fast diffusion equation has an infinite speed of propagation, meaning that the solution of (1), (2) with $b = 0, 0 < m < 1$, is positive everywhere for $t > 0$, if the initial function $u_0 \in C(\mathbb{R}^N; \mathbb{R}^+) \text{ is positive somewhere.}$ On the contrary, solution of (1), (2) with $m > 1$ always possesses a finite speed of propagation property, meaning that it is compactly supported for any $t > 0$ [2]. If $0 < m < 1$ and $b > 0$, a finite vs. infinite speed of propagation for the CP (1)–(3) is an outcome of the two competing forces: fast diffusion pushing the boundary of the support of the initial function towards infinity vs. the absorption term pushing it back towards the origin. As a result of this competition, there is a finite or infinite speed of propagation according to $\beta < m$ or $\beta \geq m$ accordingly. Hence, when $\beta < m$ there exists a finite interface as a boundary of the support of the solution. Interface emerging from the sphere $\partial B_R \times \{t = 0\}$ may expand, shrink or remain stationary depending of the relative strength of the fast diffusion and strong absorption forces near $\partial B_R \times \{t = 0\}$. Following [5], for all $x \in B_R$ near the boundary define the interface surface as

$$t = \eta_-(x) := \sup\{\tau : u(x, t) > 0, 0 < t < \tau\}.$$  

If $\eta_-(x)$ is defined and finite for all $x$ such that $0 \ll |x| < R$ and

$$\eta_-(x) = o(1) \text{ for } |x| \rightarrow R - 0,$$  

then we say that the interface initially shrinks at $\partial B_R$. For all $x \in B_R^c = \{|x| > R\}$ near the boundary define the interface surface as

$$t = \eta_+(x) := \inf\{\tau \geq 0 : u(x, t) > 0, \quad \tau < t < \tau + \epsilon \text{ for some } \epsilon > 0\}.$$  

If $\eta_+(x)$ is defined, positive and finite for all $x$ such that $R < |x| \ll +\infty$ and satisfies (5), then we say that the interface initially expands at $\partial B_R$.

The decisive factor to identify the direction of the movement of the finite interface in the battle between the fast diffusion and strong absorption is asymptotics of the initial function near $\partial B_R \times \{t = 0\}$, expressed via parameters $\alpha$ and $C$ in (3). If $\alpha > \frac{2}{m - \beta}$, then the absorption fully dominates over the diffusion, and the interface shrinks. In this case, the local solution and the
shrinking interface coincide with the solution of the Eq. (1) without diffusion term. If \( \alpha < \frac{2}{m-\beta} \), then diffusion manages to overcome the full domination of the absorption, and the interface expands. If \( \alpha = \frac{2}{m-\beta} \), then the diffusion and absorption are in balance, and the outcome of their competition is decided by the constant \( C \). There is a critical value \( C_* \), such that the interface expands or shrinks according to \( C > C_* \) or \( C < C_* \) respectively.

In the case when \( \beta \geq m \), fast diffusion dominates the absorption term near \( \partial B_R \times \{ t = 0 \} \) and pushes the interface to infinity by generating infinite speed of propagation. The behavior of the solution at infinity is independent of parameters \( \alpha \) and \( C \) and dependent on \( m \) and \( \beta \) only. There are three regimes with different decay rates at infinity. If \( \beta \geq 1 \), then diffusion fully dominates over absorption at infinity, and the decay rate of the solution at infinity coincides with that of the solution to the fast diffusion equation. It is power-type decay with critical spacial decay power being \( \frac{2}{m-1} \). If \( m < \beta < 1 \), then the absorption term gains more power at infinity, and the balance between diffusion and absorption at infinity enforces faster power-type decay with spacial decay power being \( \frac{2}{m-\beta} \). Finally, in the case when \( \beta = m \), the absorption term resists even stronger at infinity and enforces an exponential decay.

The goal of this paper is to pursue the full classification of the short-time behavior of the solution to the Cauchy Problem (1)–(3) in the fast diffusion case, and prove the outlined results on the existence or non-existence of the interfaces, the short time behavior of the interfaces \( \eta_{\pm} \), and local solution near \( \eta_{\pm} \) or at infinity in terms of the parameters \( m, \beta, b, C, \alpha \).

The outline of the paper is as follows. In Sect. 2 we formulate the main results. Theorems 1, 2, 3, 4, 5, 6 of Sect. 2 present full classification of the short-time behavior and asymptotics of the solution and interfaces expressed in respective six regions of the parameter space \((\alpha, \beta)\). Some technical details of the main results are outlined in Sect. 2.1. In Sect. 3 we prove the main results.

2. Main results

Throughout this section, we assume that \( u \) is a unique weak solution of the CP (1)–(2). There are six different subcases, as shown in Fig. 1. The main results are outlined below in Theorems 1, 2, 3, 4, 5, 6 corresponding directly to the cases (1)–(6) respectively. We collect in Appendix the expressions for all the constants appearing below in Theorems 1, 2, 3, 4, 5, 6, and in Sect. 2.1.

**Theorem 1.** If \( 0 < \beta < m \) and \( 0 < \alpha < \frac{2}{m-\beta} \), then there is a finite speed of propagation, and the interface initially expands with

\[
\left( \frac{R - |x|}{\zeta_2} \right)^{\frac{2(1-\beta)}{m-\beta}} \leq \eta_+(x) \leq \left( \frac{R - |x|}{\zeta_1} \right)^{\frac{2(1-\beta)}{m-\beta}}, \quad R \leq |x| \leq R + \gamma,
\]

for some \( \gamma > 0 \).
Theorem 2. If $0 < \beta < m$, $\alpha = \frac{2}{m-\beta}$, then there is a finite speed of propagation, and the interface expands or shrinks according to $C > C_*$ or $C < C_*$, where

$$C_* = \left[ b(m-\beta)^2/(2m(m+\beta)) \right]^{1/(m-\beta)}$$

If $C > C_*$, then the interface manifold $t = \eta_+(x)$ satisfies

$$\eta_+(x) \sim \left( \frac{R - |x|}{\zeta_+} \right)^{2(1-\beta)/(m-\beta)} \text{ as } |x| \to R^+,$$

while if $C < C_*$, then the interface manifold $t = \eta_-(x)$ satisfies

$$\eta_-(x) \sim \left( \frac{R - |x|}{\zeta_-} \right)^{2(1-\beta)/(m-\beta)} \text{ as } |x| \to R^-,$$

where $\zeta_+ \in [\zeta_3, \zeta_*]$, $\zeta_- \in [\zeta_5, \zeta_*]$, $\zeta_* = \zeta_*(C, m, \beta, b) \geq 0$ according to as $C \leq C_*$. For arbitrary $\rho > \zeta_*$ there exists $h(\rho) > 0$ such that

$$u(x, t) \bigg|_{|x|=R-\rho t^{\frac{m-\beta}{2(1-\beta)}}} \sim h(\rho) t^{-\frac{1}{\beta}} \text{ as } t \downarrow 0.$$

Theorem 3. If $0 < \beta < m$, $\alpha > \frac{2}{m-\beta}$, then there is a finite speed of propagation, and the interface initially shrinks with

$$\eta_-(x) \sim \left( \frac{R - |x|}{l_*} \right)^{\alpha(1-\beta)} \text{ as } |x| \to R^-.$$

where $l_* = C^{-\frac{1}{\beta}} (b(1-\beta))^{\frac{1}{\alpha(1-\beta)}}$. For $\forall l > l_*$ the solution satisfies the asymptotic formula

$$u(x, t) \bigg|_{|x|=R-l t^{\frac{1}{\alpha(\beta-\gamma)}}} \sim \{C^{1-\beta} l^{\alpha(1-\beta)} - b(1-\beta)\} t^{-\frac{1}{\beta}} \text{ as } t \downarrow 0.$$

**Figure 1.** Classification of different cases in the $(\alpha, \beta)$ plane for the short-time behavior of the interfaces, and local solution near interfaces and at infinity in the problem (1)–(3)
Theorem 4. If $\beta = m$, then there is an infinite speed of propagation and there exists $\exists \delta > 0$ such that for $\forall$ fixed $t \in (0, \delta]$,
\[
\log u(x, t) \sim -b^2 m^{-1} |x|, \quad \text{as } |x| \to +\infty. \quad (12)
\]

Theorem 5. If $m < \beta < 1$ then there is an infinite speed of propagation and there exists $\exists \delta > 0$ such that for arbitrary fixed $t \in (0, \delta]$,
\[
u(x, t) \sim C_* |x|^{2/(m-\beta)} \quad \text{as } |x| \to +\infty. \quad (13)
\]

Theorem 6. If $\beta \geq 1$ then there is an infinite speed of propagation, and
\[
\limsup_{|x| \to +\infty} u(x, t)|x|^{2/(1-m)} \leq Dt^{1/(1-m)}, \quad \text{for } t > 0, \quad (14)
\]
with
\[
D = (2m(m + 1(1-m)^{-1})^{1/(1-m)},
\]
whereas for some $\delta > 0$ and for arbitrary $\nu < \frac{2}{m-1}$ we have
\[
\lim_{|x| \to +\infty} u(x, t)|x|^{-\nu} = +\infty, \quad 0 < t \leq \delta \quad (15)
\]

If a space dimension $n$ satisfies the restriction
\[
n < \frac{2}{1-m} \quad (16)
\]
then we have
\[
\lim_{t \to 0} \lim_{|x| \to +\infty} u(x, t)t^{1/(1-m)} |x|^{2/(1-m)} = \gamma D, \quad (17)
\]
with some $\gamma$ satisfying
\[
\left(\frac{2 - (1-m)n}{1+m}\right)^{1/(1-m)} \leq \gamma \leq 1. \quad (18)
\]

Remark 1. The results of Theorem 6 is valid for the fast diffusion equation ($b = 0$), and for the Eq. (1) with $b < 0, \beta \geq 1$. Proof given in Sect. 3 can be applied to the case $b = 0$ without any changes, and only slight modifications are needed in the case $b < 0, \beta \geq 1$. It should be also noted that if $n = 1$, then (18) implies $\gamma = 1$, and the explicit asymptotic formula (17) coincides with the one proved in [30].

2.1. Details of the main results

Here we outline sharp local estimates of solutions that imply the explicit asymptotics in the context of Theorems 1, 2, 3, 4, 5, 6.

Technical details of Theorem 1: The solution $u$ satisfies
\[
C_1 t^{1/(1-n)} (\zeta - \zeta_1)^{2/(m-\beta)} \leq u(x, t) \leq C_* t^{1/(1-n)} (\zeta - \zeta_2)^{2/(m-\beta)}, \quad 0 < t \leq \delta. \quad (19)
\]
where $\zeta = (R - |x|)t^{-m/(m-\beta)}$, the left-hand side of (19) is valid for $|x| > R$, while the right-hand side is valid for $|x| \geq R + lt^{m/(m-\beta)}$. The estimation (7) is a
On the other side, for arbitrary $\rho \in \mathbb{R}$, there exists a positive number $f(\rho)$ depending on $C, m$ and $\alpha$ such that

$$u(x, t) \bigg|_{|x|=R-\rho t^{2+\alpha(1-m)}} \sim f(\rho)t^{\frac{\alpha}{2+\alpha(1-m)}} \text{ as } t \downarrow 0^+. \quad (20)$$

In fact, $f : \mathbb{R} \to \mathbb{R}^+$ solves the nonlinear ODE problem

$$\begin{cases}
d^{m}f^{m}/d\xi^{2} + \frac{1}{2+\alpha(1-m)}\xi d f/d\xi - \frac{\alpha}{2+\alpha(1-m)}f = 0, & \xi \in \mathbb{R} \\
f \sim C\xi^\alpha \text{ as } \xi \to +\infty, & f(\xi) = o(|\xi|^\alpha) \text{ as } \xi \downarrow -\infty.
\end{cases} \quad (21)$$

It is a shape function of the self-similar solution of the 1d Cauchy Problem

$$w_t = w_{yy}, \quad y \in \mathbb{R}, \quad 0 < t < +\infty \quad (22)$$

$$w_0(y) = C(y)^{\alpha}, \quad y \in \mathbb{R} \quad (23)$$

Using rescaling we can find the dependence of $f$ on $C$ [29]:

$$f(\rho) = C^{\frac{2}{2+\alpha(1-m)}} f_0(C^{\frac{1-m}{2+\alpha(1-m)}} \rho), \quad f_0(\rho) = w_0(\rho, 1), \quad (25)$$

where $w_0$ and $f_0$ are corresponding solutions of (22), (23), and (21), respectively, with the constant $C = 1$. In particular, we have

$$f(0) = C^{\frac{2}{2+\alpha(1-m)}} A_0, \quad A_0 = A_0(\alpha, m) := f_0(0). \quad (26)$$

Sharp two-sided estimates for $w$ and $f$ are given in [30].

The formula (20) demonstrates that for $\forall \rho > 0$ in the region $R \leq |x| < R + \rho t^{2+\alpha(1-m)}$, $t \downarrow 0$, diffusion dominates over the absorption, and the local asymptotics of the solution coincides with that of the diffusion equation (1) with $b = 0$. However, the estimation (19) demonstrates that the domination of the diffusion fails in the region $R + lt^{\frac{m-\beta}{2+\alpha(1-m)}} \leq |x| < \infty$, $t \downarrow 0$, and the balance between the absorption and diffusion terms in this region forces the finite speed of propagation and drives the interface as expressed in (7).

**Technical details of Theorem 2:** If $C > C_*$ we have an estimate (19) for all $|x| > R$ and with $C_1, \zeta_1$ and $\zeta_2$ replaced with $C_2, \zeta_3$ and $\zeta_4$ respectively; This implies an expanding interface satisfying (7) with $\zeta_1$ and $\zeta_2$ replaced with $\zeta_3$ and $\zeta_4$.

If $C < C_*$ the left-hand side of (19) holds for $|x| > R + \rho t^{\frac{m-\beta}{2+\alpha(1-m)}}$, with $\rho > \zeta_*$, while the right-hand side holds for all $|x| > R$, and the constants $C_1, C_*, \zeta_1$ and $\zeta_2$ are replaced with $(1 + \epsilon)C_*, C_3, \zeta_5$ and $\zeta_6$ respectively; this implies a shrinking interface satisfying

$$\left(\frac{R - |x|}{\zeta_5}\right)^{2\frac{(1-\beta)}{m-\beta}} \leq \eta_-(x) \leq \left(\frac{R - |x|}{\zeta_6}\right)^{2\frac{(1-\beta)}{m-\beta}}, \quad R \leq |x| \leq R + \gamma, \quad (27)$$

In the special case of initial function given by (4), the upper bounds in estimations (19) are global in time with $\delta = \infty$. 

---

**NoDEA Interface development for the nonlinear Page 7 of 22 38**
Precise values of the constant $\zeta_*$ and the function $h$ are associated with the one-dimensional Cauchy Problem

$$w_t = w_{yy}^m - bw^\beta, \quad y \in \mathbb{R}, \quad 0 < t < +\infty \quad (28)$$

$$w_0(y) = C(y)\frac{2}{m+\beta}, \quad y \in \mathbb{R} \quad (29)$$

There exists a unique solution of the problem (28), (29), which is of self-similar form

$$w(y,t) = t^{\frac{1}{1-\beta}} h(\zeta), \quad \text{where} \quad \zeta = yt^{-\frac{m-\beta}{2(1-\beta)}},$$

and the shape function $h$ solves the nonlinear ODE problem

$$\begin{cases} 
\frac{1}{1-\beta}h - \frac{m-\beta}{2(1-\beta)} \zeta h' - (h^m)'' + bh^\beta = 0, \quad \zeta \in \mathbb{R}. \\
h(\zeta) \sim C\zeta^{\frac{2}{m-\beta}} \quad \text{as} \quad \zeta \uparrow +\infty, \quad h(\zeta) = o(|\zeta|^{\frac{2}{m-\beta}}) \quad \text{as} \quad \zeta \downarrow -\infty.
\end{cases}$$

There exists a finite interface $\zeta_*$ such that $\zeta_* = \zeta_*(C, m, \beta, b)$ such that $\zeta_* \gtrless 0$ according to as $C \lesssim C_*$, and

$$h(\zeta) > 0, \quad \zeta_* < \zeta < +\infty; \quad h(\zeta) \equiv 0, \quad \zeta \leq \zeta_* \quad (32)$$

In particular, $A_1 := h(0) > 0$ if $C > C_*$. If $C = C_*$, the initial function (29) is a unique solution of the Cauchy problem (28), (29), and the formula (30) reproduces it with $h(\zeta) = C_*(\zeta)\frac{2}{m-\beta}$ and $\zeta_* = 0$.

If $C < C_*$ we have an estimation [30]

$$0 < h(\zeta) < C_*\zeta^{\frac{2}{m-\beta}}, \quad \text{for} \quad \zeta_* < \zeta < \infty. \quad (33)$$

Sharp two-sided estimates for $w$ and $h$ near the interface, and $\zeta_*$ are given in [30].

**Technical details of Theorem 4:** For arbitrary $M > 0$ and $\epsilon > 0$, $\exists \delta = \delta(\epsilon, M) > 0$ such that

$$t^{1/(1-m)} \phi(x) \leq u(x,t) \leq (t^\epsilon)^{1/(1-m)} \phi(x) \quad \text{for} \quad R \leq |x| < +\infty, \quad 0 \leq t \leq \delta, \quad (34)$$

where $\phi(x)$ is a solution to the elliptic PDE problem

$$\begin{cases} 
\mathcal{L}\phi := -\Delta \phi^m + b\phi^m + \frac{1}{1-m}\phi = 0, \quad |x| > R, \\
\phi|_{|x|=R} = M, \quad \phi|_{|x|\rightarrow +\infty} = 0.
\end{cases} \quad (35)$$

Solution $u$ also satisfies the asymptotic property (20), where $f$ be a positive solution of the nonlinear ODE problem (21). It expresses the fact that near $\partial B_R \cap \{t = 0\}$ the diffusion fully dominates the absorption and generates an infinite speed of propagation by pushing the interface to infinity. However, the asymptotic formula (12), and the estimation (34) express the fact that the absorption term gains more power at infinity, and the balance between diffusion and absorption generates exponential decay at infinity instead of power-like decay intrinsic for the fast diffusion equation.

**Technical details of Theorem 5:** If $m < \beta < 1$, the following global upper bound holds:

$$u(x,t) \leq C_* (|x| - R)\frac{2}{m-\beta}, \quad |x| > R, \quad 0 < t < \infty \quad (36)$$
Matching asymptotics at infinity is given with the following local lower estimate: \( \forall \, \epsilon > 0 \exists \, \delta > 0 \) such that
\[
u(x, t) \geq C_*(1 - \mu)t^{1/(\beta_m(n - \beta)m)}(\zeta_7 - \zeta)^{2-m/\beta}\,, \quad |x| \geq R, \; 0 < t \leq \delta, \tag{37}\]
where \( 0 < \mu < 1 \) is an arbitrary number, and the length of the time interval \( \delta \) is independent of \( \mu \).

**Technical details of Theorem 6:** The following global upper bound is valid in all cases:
\[
u(x, t) \leq D t^{1/(1 - m)}(|x| - R)^{2/m - \beta}\,, \quad |x| > R, \; 0 < t < \infty \tag{38}\]
If \( \beta \geq 1 \) then for \( \forall \, \epsilon > 0 \) there exists \( \exists \, \delta > 0 \) such that
\[
u_\delta t^{2+\alpha/(1-n-m)}(\xi + \xi_1)^{2/(m-\beta)} \leq \nu(x, t) \leq \nu_0 t^{2+\alpha/(1-n-m)}(\xi + \xi_2)^{2/(m-\beta)}\,, \quad |x| \geq R,
\quad 0 \leq t \leq \delta \tag{39}\]
where \( \xi = (|x| - R)t^{-1/(1-n-m)} \), and the left-hand side only valid for spacial dimension \( n \) restricted by the bound (16), while the right-hand side holds for all \( n \geq 1 \). In general, for arbitrary \( \epsilon > 0 \) there exists \( \exists \, \delta > 0 \) such that the following lower estimate is valid in all space dimensions:
\[
u(x, t) \geq C_\delta t^{1/\beta}(\zeta_8 - \zeta)^{2/(m-\beta)}\,, \quad |x| > R, \; 0 < t \leq \delta \tag{40}\]
where \( \bar{\beta} \in (m, 1) \), and \( \bar{\zeta} = (R - |x|)t^{-\bar{m}(1-\bar{\beta})} \).

### 3. Proofs of the main results

**Proof of Theorem 1.** The proof of the asymptotic formula (20) is identical with the proof of Lemma 4.4 of [5]. Applying (20) with \( \rho = 0 \), for \( \forall \, \epsilon > 0 \) we can find a \( \delta > 0 \) such that
\[
u_{\epsilon, \delta} t^{1/(1 - m)}t^{2+\alpha/(1-n-m)} \leq \nu(x, t) \leq \nu_{0, \epsilon} t^{2+\alpha/(1-n-m)} t^{2+\alpha/(1-n-m)}\,, \quad |x| = R, 0 < t \leq \delta, \tag{41}\]
(see (26)). Consider a function
\[
u(x, t) = t^{1/m}f_1(\zeta)\,, \quad \text{with} \quad f_1(\zeta) = C_0(\zeta - \zeta_0)^{2/(m-\beta)}\,, \tag{42}\]
and \( C_0 > 0 \), \( \zeta_0 < 0 \) are some constants to be selected. Our goal is to estimate the operator \( L\nu \) in
\[D = \{(x, t) : |x| > R, 0 < t \leq \delta\}.
\]
For \( \zeta_0 < \zeta < 0 \) we have
\[
u_{\delta} = bg^\beta \left\{ 1 - \left(\frac{C_0}{C_*}\right)^{m-\beta} - \frac{C_0^{1-\beta}\zeta_0(\zeta - \zeta_0)^{2-m/\beta}}{b(1 - \beta)} \right\} + \frac{2m(n-1)}{b(m-\beta)} t^{2/(1-n)} C_0^{m-\beta}(\zeta - \zeta_0) \tag{43}\]
For \( \forall \, \epsilon > 0 \) we choose \( C_0 \) and \( \zeta_0 \) such that
\[
u_{\epsilon, \delta} = 1 - \left(\frac{C_0}{C_*}\right)^{m-\beta} + \frac{C_0^{1-\beta}}{b(1 - \beta)}(-\zeta_0)^{2/(m-\beta)} = -\epsilon \tag{44}\]
Moreover, the sharpest upper bound for the interface will be achieved by minimizing $\zeta_0$ under the condition (44). This problem is solved in [30], and the required selection is achieved by choosing $\zeta_0 = \zeta_1, C_0 = C_1$. Having made this selection, from (76) it follows that for sufficiently small $\delta$ we have
\begin{equation}
Lg \leq 0 \text{ for } R < |x| < R - \xi_0 t^{\frac{m-\beta}{2m+\alpha}}(1-m), \quad 0 < t \leq \delta, \tag{45a}
\end{equation}
\begin{equation}
Lg = 0 \text{ for } |x| > R - \xi_0 t^{\frac{m-\beta}{2m+\alpha}}(1-m), \quad 0 < t \leq \delta. \tag{45b}
\end{equation}
From (45) it follows that $g$ is a weak subsolution of (1) in D. Since $\frac{1}{1-\beta} > \frac{\alpha}{2m+\alpha(1-m)}$ from (41) it follows that for all sufficiently small $\delta$ we have
\begin{equation}
g_{|x|=R} = C_0(-\xi_0)\frac{1}{m-\beta} t^{\frac{1}{1-\beta}} \leq A_0(C - \epsilon)\frac{2}{2+\alpha(1-m)} t^{\frac{\alpha}{2+\alpha(1-m)}} \leq u_{|x|=R}, \quad \text{for } 0 \leq t \leq \delta \tag{46a}
\end{equation}
\begin{equation}
g_{t=0} = u_{|x|=0}, \quad \text{for } |x| \geq R. \tag{46b}
\end{equation}
From (45), (46) it follows that $g$ is a weak subsolution of the Cauchy–Dirichlet problem for (1) in D. From the comparison theorem ([26]), the left-hand side of (19) and the right-hand side of (7) follows.

We proceed now to the derivation of the right-hand side of (19), which requires some more delicate estimates. First, we establish the rough upper estimation (38). By applying the comparison theorem it follows that the solution $u$ satisfies the following upper estimate
\begin{equation}
u(x, t) = \text{Dt}^{\frac{1}{1-m}}(-x_1 - R)^{\frac{2}{m-1}} \text{in } \Omega = \{(x, t) \in \mathbb{R}^n \times \mathbb{R}^+ : -\infty < x_1 < -R\}. \tag{47}
\end{equation}
Indeed, the function on the right hand side of (47) is an explicit solution of (1) with $b = 0$, and therefore it is a supersolution of (1) with $b > 0$. Moreover, it vanishes on $\partial \Omega \cap \{t = 0\}$, and positive infinity at $\partial \Omega \cap \{y_1 = -R\}$. In particular, (47) implies
\begin{equation}
u_{|x_2 = \cdots = x_n = 0} \leq \text{Dt}^{\frac{1}{1-m}}(-x_1 - R)^{\frac{2}{m-1}} \text{for } -\infty < x_1 < -R, \quad t > 0. \tag{48}
\end{equation}
Since $u$ is radially symmetric, from (48), (38) follows.

Next, we are going to use (38) to estimate solution along the manifolds $\Gamma : |x| = R + lt^{\frac{m-\beta}{2m+\alpha}}(1-m), \ t \downarrow 0$, which form a boundary layer where diffusion and absorption are balanced and the interface is generated. From (38) it follows that
\begin{equation}
u(x, t)_{|\Gamma} \leq \text{Dt}^{\frac{2}{m-1}}t^{\frac{1}{1-m}}, \quad 0 \leq t < \infty. \tag{49}
\end{equation}
Using (49) we are going to establish an accurate upper bound in a boundary layer
\begin{equation}G_{l, \delta} = \{(x, t) : |x| > R + lt^{\frac{m-\beta}{2m+\alpha}}(1-m), \ 0 < t \leq \delta\},
\end{equation}
where the constant $l > 0$ is on our account. Consider a function $g$ as in (42). By choosing $C_0 = C_*$, for $\zeta_0 < \zeta < -l$ we have
\begin{equation}\begin{aligned}
Lg &= bg\left\{-\frac{C_*^{1-\beta}}{b(1-\beta)} \zeta_0 (\zeta - \zeta_2)^{\frac{2m-\beta}{m-\beta}} + \frac{2m(n-1)}{b(m-\beta)} |x|^{-\frac{m-\beta}{m-\beta}} C_*^{m+\beta}(\zeta - \zeta_0)\right\} \geq 0
\end{aligned}
\end{equation}
Hence, we have
\[ Lg \geq 0 \text{ for } R + lt^{\frac{m-\beta}{2(1-\beta)}} < |x| < R - \zeta_0 t^{\frac{m-\beta}{2(1-\beta)}}, \quad 0 < t \leq \delta, \]  
(50a)
\[ Lg = 0 \text{ for } |x| > R - \zeta_0 t^{\frac{m-\beta}{2(1-\beta)}}, \quad 0 < t \leq \delta. \]  
(50b)

From (50) it follows that \( g \) is a weak supersolution of (1) in \( G_{l,\delta} \). Comparing \( u \) and \( g \) on the parabolic boundary of \( G_{l,\delta} \), and by using (49) we have
\[ u|_{t=0} = g|_{t=0} = 0, \quad \text{for } |x| \geq R, \]  
(51a)
\[ u|_{\Gamma} \leq D_l \frac{2}{m-\beta} t^{\frac{1}{1-\beta}} = C_* (-l - \zeta_0) \frac{2}{m-\beta} t^{\frac{1}{1-\beta}} = g|_{\Gamma}, \]  
(51b)
provided that the constants \( \zeta_0 \) and \( l \) are chosen such that to satisfy
\[ \zeta_0 < -l < 0, \quad D_l \frac{2}{m-\beta} = C_* (-l - \zeta_0) \frac{2}{m-\beta}. \]  
(52)

Moreover, the sharpest lower bound for the interface will be achieved by maximizing \( \zeta_0 \) under the condition (52). This problem is solved in [30], and the required selection is achieved by choosing \( \zeta_0 = \zeta_2, l = l_0 \). Having made this selection, and by applying the comparison theorem [26] in \( G_{l,\delta} \), the upper bound of (19) and left-hand side of (7) follow.

Proof of Theorem 2. All the claims of Theorem 2 are proved in Theorem 2.4 of [5] if \( 1 \leq m < 2 - \beta \), and the proofs can be extended to the case \( \beta < m < 2 - \beta \) without any changes. In particular, the upper bounds in the corresponding estimations (19) are proved in [5], and they are global in time for the special case of initial function given by (4). We only need to prove the lower bounds in corresponding estimations (19) for the solution \( u \) of the CP (1)–(3). Assume \( C > C_* \). Consider the function \( g \) from (42) with the constants \( C_0 > 0, \zeta_0 < 0 \) on our account. Applying asymptotic estimation (9) with \( \rho = 0, \) for \( \forall \epsilon > 0 \) we can find a \( \delta > 0 \) such that
\[ (A_1 - \epsilon)t^{\frac{1}{1-\beta}} \leq u(x, t) \leq (A_1 + \epsilon)t^{\frac{1}{1-\beta}}, \quad |x| = R, \quad 0 < t \leq \delta, \]  
(53)
where \( A_1 = h(0) > 0 \). We have
\[ g|_{|x|=R} = C_0 (-\zeta_0) \frac{2}{m-\beta} t^{\frac{1}{1-\beta}} = (A_1 - \epsilon)t^{\frac{1}{1-\beta}} \leq u|_{|x|=R}, \quad \text{for } 0 \leq t \leq \delta \]  
(54a)
\[ g|_{t=0} = u|_{t=0} = 0, \quad \text{for } |x| \geq R, \]  
(54b)
provided that
\[ C_0 = (A_1 - \epsilon)\zeta_0 \frac{2}{m-\beta}. \]  
(55)

We estimate \( Lg \) for \( \zeta_0 < \zeta < 0 \) as in (76). If \( C_0 \) and \( \zeta_0 \) satisfy (44), then (45) would be satisfied for all sufficiently small \( \delta \). At this point we select \( C_0 = C_2 \) and \( \zeta_0 = \zeta_3 \) to satisfy both (45), (55) to make \( g \) a weak subsolution of the Cauchy–Dirichlet problem for (1) in \( D \). From the comparison theorem [26], the left-hand side of (19) follows with \( C_1 \) and \( \zeta_1 \) replaced with \( C_2 \) and \( \zeta_3 \) respectively. This implies the right-hand side of (7) with \( \zeta_1 \) replaced with \( \zeta_3 \).

Assume now that \( C < C_* \). Consider the function \( g \) from (42) for \( \zeta_0 < \zeta < \rho \) with the constants \( C_0 > 0, \zeta_0 > 0, \rho > 0 \) on our account. We select \( \rho > \zeta_* \) (see (32)). Our goal is to demonstrate that the function \( g \) with appropriately chosen constants \( C_0 \) and \( \zeta_0 \) is a low bound for the solution \( u \) in a region...
G_{-\rho,\delta}$, where the balance between the diffusion and absorption generates the shrinking interface. By selecting $C_0 = C_*(1 + \epsilon), \epsilon > 0$ from (76) we deduce that for all $\zeta_0 < \zeta < \rho$  
\[
Lg = \frac{b \gamma}{(1 + \epsilon)^{m-\beta}} \left\{ 1 - \left( \frac{C_0}{C_*} \right)^{m-\beta} \left[ \frac{C_0^{1-\beta} \zeta_0 (\zeta - \zeta_0)}{b(1-\beta)} + \frac{2m(n-1)}{b(m-\beta)} \right] \right\} 
\leq \frac{b \gamma}{(1 + \epsilon)^{m-\beta}} \left\{ 1 - (1 + \epsilon)^{m-\beta} \right\} 
\leq \frac{b \gamma}{(1 + \epsilon)^{m-\beta}} \left\{ 1 - (1 + \epsilon)^{m-\beta} \right\}.
\]
Hence, for all sufficiently small $\delta$ we have  
\[
Lg \leq 0 \text{ for } R - \rho t \frac{m-\beta}{2(1-\beta)} < |x| < R - \zeta_0 t \frac{m-\beta}{2(1-\beta)}, \quad 0 < t \leq \delta,
\]
\[
Lg = 0 \text{ for } |x| > R - \zeta_0 t \frac{m-\beta}{2(1-\beta)}, \quad 0 < t \leq \delta.
\]
and therefore, $g$ is a weak subsolution of (1) in $G_{-\rho,\delta}$. Applying asymptotic estimation (9) for $\forall \epsilon > 0$ we can find a $\delta > 0$ such that  
\[
u(x, t) \geq \left( h(\rho)(1-\epsilon) \right) t^{\frac{1}{(1-\beta)}}, \quad |x| = R - \rho t \frac{m-\beta}{2(1-\beta)}, \quad 0 < t \leq \delta,
\]
Comparing $g$ and $u$ on the parabolic boundary of $G_{-\rho,\delta}$ we deduce that  
\[
g|_{|x|=R-\rho t \frac{m-\beta}{2(1-\beta)}} = C_*(1 + \epsilon)(\rho - \zeta_0) \frac{\chi_{m-\beta}}{t^\frac{1}{(1-\beta)}} \leq \left( h(\rho)(1-\epsilon) \right) t^{\frac{1}{(1-\beta)}} \leq u|_{|x|=R-\rho t \frac{m-\beta}{2(1-\beta)}}, \quad \text{for } 0 \leq t \leq \delta
\]
\[
g|_{t=0} = u|_{t=0} = 0, \quad \text{for } |x| \geq R,
\]
provided the $\zeta_0$ is chosen as  
\[
\zeta_0 = \zeta_5 := \rho - \left( \frac{1-\epsilon}{1+\epsilon} \right) h(\rho) \frac{\chi_{m-\beta}}{C_*}.
\]
Note that from (33) it follows that $\zeta_5 > 0$ for all sufficiently small $\epsilon$. Hence, from (57), (59) it follows that $g$ is the weak supersolution of the Cauchy–Dirichlet problem for (1) in $G_{-\rho,\delta}$. From the comparison theorem, the left-hand side of (19) follows with $C_1$ and $C_1$ replaced with $C_*(1 + \epsilon)$ and $\zeta_5$ respectively. This implies the right-hand side of (7) with $\zeta_1$ replaced with $\zeta_5$.

All the claims of Theorem 3 are proved in Theorem 2.4 and Lemma 4.6 of [5] in the case $1 \leq m < 2 - \beta$. The proofs can be extended to the case $\beta < m$ without any changes.

**Proof of Theorem 4.** Applying asymptotic estimation (20) with $\rho = 0$, for $\forall \epsilon > 0$ we can find a $\delta > 0$ such that (41) is satisfied with $A_0 = f(0) > 0$. Consider a solution of the PDE (1) of the form  
\[
u_\epsilon(x, t) = (t + \epsilon)^{\frac{1}{m}} \phi(x),
\]
where $\phi(x)$ is a solution of the elliptic PDE problem (35). We have  
\[
Lu_\epsilon = (t + \epsilon)^{\frac{m}{m}} \mathcal{L}\phi(x) = 0, \quad \text{in } D.
\]
Since \((1 - m)^{-1} > \alpha(2 + \alpha(1 - m))^{-1}\), from (20) it follows that for arbitrary \(\epsilon > 0\) and \(M > 0\) we can find a \(\delta > 0\) such that
\[
\begin{align*}
    u_0|_{|x|=R} &\leq u|_{|x|=R} \leq u_\epsilon|_{|x|=R}, \quad \text{for } 0 \leq t \leq \delta, \quad (63a) \\
    u_0|_{t=0} & = u|_{t=0} = u_\epsilon|_{t=0}, \quad \text{for } |x| \geq R. \quad (63b)
\end{align*}
\]

Applying the comparison theorem in \(D\), from (62), (63), the estimate (34) follows. From (34) it follows that the asymptotic of the solution at infinity is defined by the asymptotic of the solution of the elliptic PDE problem (35) which expresses the balance between diffusion and absorption.

Since elliptic problem (35) is invariant with respect to rotation in space, a solution is radially symmetric, and (35) reduces to the following ODE problem for \(\psi(r) = \phi(x), \; r = |x|:\)
\[
\begin{align*}
    \mathcal{L}\psi & = -\left(\psi^m\right)'' - \frac{n-1}{r}\left(\psi^m\right)' + b\psi^m + \frac{1}{1-m}\psi = 0, \quad R < r < +\infty. \\
    \psi(R) & = M, \quad \psi(+\infty) = 0. \quad (64)
\end{align*}
\]

Consider functions
\[
\psi_\epsilon(r) = M \exp\left(\frac{(b + \epsilon)\frac{1}{m}}{r}(R - r)\right), \quad \epsilon \geq 0. \quad (65)
\]

We have
\[
\mathcal{L}\psi_0 = \frac{n-1}{r}M^m b^{\frac{1}{m}} e^{b^{\frac{1}{m}}(R-r)} + \frac{M}{1-m} e^{\frac{1}{m}}(R-r) > 0, \quad R < r < +\infty, \quad (66)
\]

and therefore, \(\psi_0\) is a supersolution of the problem (64), and the corresponding function \(u_\epsilon\) from (61) with \(\epsilon > 0\) and \(\phi(x) = \psi_0(|x|)\) is a supersolution of the PDE (1) in \(D\). Applying the comparison theorem in \(D\), from (63) we derive the following upper bound for the solution \(u:\)
\[
    u(x,t) \leq M(t + \epsilon)\frac{1}{1-m} e^{\frac{b}{m}(R-|x|)}, \quad \text{on } D \quad (67)
\]

To establish a sharp lower bound we consider \(\psi_\epsilon\) with \(0 < \epsilon < 1\). We have
\[
\begin{align*}
    \mathcal{L}\psi_\epsilon & = M^m e^{(b+\epsilon)\frac{1}{m}(R-r)} \left\{ -\epsilon + \frac{n-1}{r} (b + \epsilon) \frac{1}{2} + \frac{M^{1-m}}{1-m} e^{(b+\epsilon)\frac{1}{m}(R-r)} \right\} \\
    & < M^m e^{(b+\epsilon)\frac{1}{m}(R-r)} \left\{ -\epsilon + \frac{n-1}{r} (b + 1) \frac{1}{2} + \frac{M^{1-m}}{1-m} \right\}. \quad (68)
\end{align*}
\]

Let us define
\[
    R_\epsilon := 4(n-1)(b+1)\frac{1}{2}\epsilon^{-1}, \quad M_\epsilon := \left(\frac{\epsilon(1-m)}{4}\right)^{\frac{1}{1-m}}.
\]

From (68) it follows that
\[
    \mathcal{L}\psi_\epsilon < -M^m e^{(b+\epsilon)\frac{1}{m}(R-r)} \frac{\epsilon}{2} < 0, \quad \text{for } R_\epsilon < |x| < \infty. \quad (69)
\]

Hence, \(\psi_\epsilon\) is a subsolution of the problem (64), with \(R\) and \(M\) replaced with \(R_\epsilon\) and \(M_\epsilon\) respectively, and the corresponding function \(u_0\) from (61) with \(\phi(x) = \psi_\epsilon(|x|)\) is a subsolution of the PDE (1) in \(D_\epsilon = D \cap \{|x| > R_\epsilon\}\).
Applying the comparison theorem in $D_\epsilon$, from (63) we derive the following lower bound for the solution $u$:

$$u(x, t) \geq M_\epsilon t^{\frac{1}{1-m}} \exp\left(\frac{(b + \epsilon)^{1/2}}{m}(|x| - R_\epsilon)\right) \text{ in } D_\epsilon.$$ 

(70)

Applying log to (67) and (70) we have

$$\log M_\epsilon t^{\frac{1}{1-m}} + \frac{(b + \epsilon)^{1/2}}{m}(|x| - R_\epsilon) \leq \log u(x, t) \leq \log M(t + \epsilon)^{\frac{1}{1-m}} + \frac{b^{1/2}}{m}(|x| - R)$$

(71)

in $D_\epsilon$. Dividing (71) by $|x|$, and passing to limit as $|x| \to \infty$ we derive

$$-\frac{(b + \epsilon)^{1/2}}{m} \leq \liminf_{|x| \to +\infty} \frac{\log u(x, t)}{|x|} \leq \limsup_{|x| \to +\infty} \frac{\log u(x, t)}{|x|} \leq -\frac{b^{1/2}}{m}$$

(72)

Passing to the limit as $\epsilon \to 0$, from (72), (12) follows. \hfill \Box

**Proof of Theorem 5.** To prove the global estimate (36), first note that the following upper bound trivially holds:

$$u(x, t) \leq C_* (R - x_1)^{\frac{2}{m-\beta}}, \quad -\infty < x_1 \leq -R, \quad (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}, \quad 0 < t < +\infty,$$ 

(73)

Indeed, the function on the right-hand side is a positive solution of the PDE (1) in $\Omega = \{x, t\} \in \mathbb{R}^n \times \mathbb{R}^+ : x_1 < -R\}$, and is $+\infty$ for $x_1 = -R$. Therefore, (73) easily follows from the comparison theorem for the Cauchy–Dirichlet problem for the PDE (1) in $\Omega$. In particular, from (73) we deduce

$$u|_{x_2=\ldots=x_n=0} \leq C_* (R - x_1)^{\frac{2}{m-\beta}}, \quad -\infty < x_1 \leq -R, \quad 0 < t < +\infty.$$ 

(74)

Since $u$ is radially symmetric, from (74), the upper bound (36) follows.

The proof of the estimate (37) is similar to the proof of lower bound in Theorem 1. Consider the function $g$ from (42) with

$$f(\zeta) = C_0 (\zeta_0 - \zeta)^{\frac{4}{m-\beta}}, \quad -\infty < \zeta < 0.$$ 

(75)

Our goal is to estimate the operator $Lg$ in $D$. We have

$$Lg = bg^\beta \left\{ 1 - \left(\frac{C_0}{C_*}\right)^{m-\beta} + \frac{C_0^{1-\beta} \zeta_0 (\zeta_0 - \zeta)^{\frac{2}{m-\beta}}}{b(1-\beta)} + \frac{2m(n-1)}{b(m-\beta)} t^{\frac{m-\beta}{m-\beta-1}} C_0^{m-\beta} (\zeta_0 - \zeta) \right\}$$

$$\leq bg^\beta \left\{ 1 - \left(\frac{C_0}{C_*}\right)^{m-\beta} + \frac{C_0^{1-\beta}}{b(1-\beta)} \zeta_0^{\frac{2(1-\beta)}{m-\beta}} \right\}.$$ 

(76)

We choose $\forall \mu > 0$, and select $C_0 = (1 - \mu) C_*$ and $\zeta_0 = \zeta_7$ to achieve

$$1 - \left(\frac{C_0}{C_*}\right)^{m-\beta} + \frac{C_0^{1-\beta}}{b(1-\beta)} \zeta_0^{\frac{2(1-\beta)}{m-\beta}} = 0,$$ 

(77)

and therefore

$$Lg \leq 0, \quad \text{for } |x| > R, \quad 0 < t < +\infty.$$ 

(78)
Since \((1 - \beta)^{-1} > \alpha(2 + \alpha(1 - m))^{-1}\) for all sufficiently small \(\delta > 0\) we have
\[
 u|_{x=R} \geq A_0(C - \epsilon) \frac{t^{\frac{2}{2+\alpha(1-m)}}}{t^{\frac{2}{2+\alpha(1-m)}}} t^{\frac{\alpha}{2+\alpha(1-m)}} \geq C_*(1 - \mu) \frac{t^{\frac{2}{2+\alpha(1-m)}}}{t^{\frac{1}{2+\alpha(1-m)}}} = g|x|=R, \; 0 < t \leq \delta, 
\]
(79a)
\[
 u|_{t=0} = g|_{t=0} = 0, \; \text{for} \; |x| \geq R.
\]
(79b)
Moreover, it is essential to note that the smallness of \(\delta\) is independent of \(\mu\). From (78), (79) it follows that \(g\) is a subsolution of the Cauchy–Dirichlet problem on \(D\), and from the comparison theorem we deduce the estimate (37).

From (36) and (37) it follows that
\[
 C_*(1 - \mu) \left( \frac{\zeta t^{\frac{m-\beta}{2}} + |x| - R}{|x|} \right)^{\frac{m-\beta}{m-\gamma}} \leq u(x,t)|x|^{\frac{2}{m-\gamma}} \leq C_* \left( \frac{|x| - R}{|x|} \right)^{\frac{m-\beta}{m-\gamma}},
\]
for \(|x| > R, \; 0 < t \leq \delta\). Passing to limit as \(|x| \to +\infty\) we get
\[
 C_*(1 - \mu) \leq \liminf_{|x| \to +\infty} u(x,t)|x|^{\frac{2}{m-\gamma}} \leq \limsup_{|x| \to +\infty} u(x,t)|x|^{\frac{2}{m-\gamma}} \leq C_*, \; \text{for} \; 0 < t \leq \delta
\]
(81)
Finally, passing to the limit as \(\mu \to 0\), from (81), the desired asymptotic (13) follows.

**Proof of Theorem 6.** The proof of the asymptotic formula (20) is identical with the proof of Lemma 4.4 of [5]. Let \(u_\epsilon\) be a solution of the CP (1), (4) with \(C\) replaced with \(C + \epsilon\). Applying (20) with \(\rho = 0\), for \(\forall \epsilon > 0\) we can find a \(\delta > 0\) such that
\[
 u|_{x=R} \leq A_0(C + \epsilon/2)^{\frac{2}{2+\alpha(1-m)}} t^{\frac{\alpha}{2+\alpha(1-m)}} \leq u_\epsilon|_{x=R}, \; 0 < t \leq \delta, 
\]
(82a)
\[
 u|_{t=0} = u_\epsilon|_{t=0} = 0, \; \text{for} \; |x| \geq R.
\]
(82b)
Applying the comparison theorem on \(D\), from (82) it follows that
\[
 u \leq u_\epsilon \; \text{on} \; D
\]
(83)
Changing the variable \(y = x + \bar{x}\) with \(\bar{x} = (R, 0, ..., 0)\), the function \(v_\epsilon(y,t) = u_\epsilon(y - \bar{x},t)\) solves the CP
\[
 \begin{cases} 
 v_t - \Delta v^\epsilon + bv^\beta = 0, \; y \in \mathbb{R}^N, & t > 0 \\
 v(x,0) = (C + \epsilon)(R - |y - \bar{x}|)^\alpha_+, \; y \in \mathbb{R}^N 
\end{cases}
\]
(84)
Since
\[
 (R - |y - \bar{x}|)^\alpha_+ \leq (y_1)^\alpha_+, \; y \in \mathbb{R}^N
\]
(85)
from the comparison theorem, it follows that
\[
 v_\epsilon(y,t) \leq w_\epsilon(y,t) \; \text{for} \; y \in \mathbb{R}^N, 0 < t < \infty,
\]
(86)
where \(w_\epsilon\) is a solution of the CP
\[
 \begin{cases} 
 w_t - \Delta w^\epsilon + bw^\beta = 0, \; y \in \mathbb{R}^N, & t > 0 \\
 w(x,0) = (C + \epsilon)(y_1)^\alpha_+, \; y \in \mathbb{R}^N 
\end{cases}
\]
(87)
Since the problem (87) is uniquely solvable, the solution is only $y_1$ and $t$ dependent:

$$w_\epsilon(y,t) = w_\epsilon(y_1,t),$$

and it solves the 1d CP

$$\begin{cases}
w_t - (w^m)y_{1y_1} + bw^{\beta} = 0, & y_1 \in \mathbb{R}, \ t > 0 \\
w(y_1,0) = (C + \epsilon)(y_1)^\alpha_+ & y_1 \in \mathbb{R}
\end{cases}$$ (88)

If $\beta \geq 1$, the following estimate is proved in [30]:

$$w_\epsilon(y_1,t) \leq C_6 t^{\frac{m-2}{m+\alpha(1-m)}} \left(\xi_2 - y_1 t^{-\frac{1}{m+\alpha(1-m)}}\right)^{\frac{2}{m-1}}, \ -\infty < y_1 < 0, \ 0 \leq t \leq \delta.$$ (89)

From (86) and (89) we deduce the following upper bound for $u_\epsilon$ in $x_1t$-plane:

$$u_\epsilon|\{x_2=\ldots=x_n=0\} \leq C_6 t^{\frac{m-2}{m+\alpha(1-m)}} \left(\xi_2 - (x_1 + R) t^{-\frac{1}{m+\alpha(1-m)}}\right)^{\frac{2}{m-1}}, \ -\infty < x_1 < -R, \ 0 \leq t \leq \delta.$$ (90)

Since $u_\epsilon$ is radially symmetric, through spacial rotation from (90) it follows the estimate

$$u_\epsilon(x,t) \leq C_6 t^{\frac{m-2}{m+\alpha(1-m)}} \left(\xi_2 + (|x| - R) t^{-\frac{1}{m+\alpha(1-m)}}\right)^{\frac{2}{m-1}}, \ |x| > R, \ 0 \leq t \leq \delta.$$ (91)

From (83) and (91), the right-hand side of the estimate (39) follows in the case $\beta \geq 1$. Note that the upper bound (38) holds for all $b \geq 0$ as it is demonstrated above in the proof of Theorem 1.

To prove the lower bound in (39), consider a function

$$g(x,t) = t^{\frac{-m-2}{m+\alpha(1-m)}} f(\xi), \ \text{with} \ f(\xi) = C_0 (\xi + \xi_0)^{\frac{2}{m-1}},$$ (92)

where $C_0 > 0$ and $\xi_0 > 0$ are some constants to be selected. We have

$$Lg = \frac{t^{\frac{m-2}{m+\alpha(1-m)}} f(\xi)}{2m(n-1)(2 + \alpha(1-m)) C_0^{m-1}} \left[\alpha + \frac{2\xi}{(1-m)(\xi + \xi_0)} - \frac{2m(1+m)(2 + \alpha(1-m)) C_0^{m-1}}{(1-m)^2} \right]$$

$$+ \frac{t^{\frac{m-2}{m+\alpha(1-m)}}}{1-m} \frac{2m(n-1)(2 + \alpha(1-m)) C_0^{m-1}}{1-m} (\xi + \xi_0)$$

$$+ b(2 + \alpha(1-m)) t^{\frac{-m-2}{m+\alpha(1-m)}} C_0^{\beta-1} (\xi + \xi_0)^{\frac{2(1-\beta)}{1-m}}.$$ (93)

Since $\xi(\xi + \xi_0)^{-1} \leq 1$, and

$$\frac{t^{\frac{m-2}{m+\alpha(1-m)}}}{|x|^2} (\xi + \xi_0) = \frac{|x|}{|x|} + \xi_0 \frac{t^{\frac{m-2}{m+\alpha(1-m)}}}{|x|} \leq 1 + R^{-1} \xi_0 t^{\frac{m-2}{m+\alpha(1-m)}}, \ \text{for} \ |x| \geq R,$$

from (93) it follows

$$Lg \leq t^{\frac{m-2}{m+\alpha(1-m)}(1-m)^{-1}} f(\xi) \left[1 - \frac{2 - (1-m)n (C_0)}{1+m} \left(\frac{D}{C_0}\right)^{m-1}ight]$$

$$+ 2R^{-1}m(n-1)C_0^{m-1} \xi_0 t^{\frac{1}{m+\alpha(1-m)}} + b(1-m)C_0^{\beta-1} \xi_0^{\frac{2(1-\beta)}{1-m}} t^{\frac{-m-2}{m+\alpha(1-m)}}.$$ (94)
By choosing 

\[ C_0 = C_5 := D\left(\frac{2 - (1 - m)n}{1 + m}\right)\frac{1}{m} \leq (1 + \epsilon)\frac{1}{m - 1} \]

from (94) it follows that for all sufficiently small \( \delta \)

\[ Lg \leq t^{2(1-\beta)}(1-m)^{-1}f(\xi)\left[-\epsilon + 2R^{-1}m(n - 1)C_5^{m-1}\xi_0 t^{2(1-\gamma)/(1-m)}\right] + b(1-m)C_5^{\beta-1}\xi_0^{2(1-\gamma)/(1-m)} t^{2+\alpha(\beta-m)/(1-m)} \leq 0, \text{ on } D, \]

(95)

and therefore \( g \) is a subsolution of the PDE (1) on \( D \). Applying (20) we have

\[ u|_{x=R} \geq A_0(C - \epsilon)\frac{2}{x^{\alpha(1-m)}t^{\alpha(1-m)}} = C_5\xi_0^{2(1-\gamma)/(1-m)} t^{2+\alpha(\beta-m)/(1-m)} = g|_{|x|=R}, \quad 0 < t \leq \delta, \]

(96a)

\[ u|_{t=0} = g|_{t=0} = 0, \quad \text{for } |x| \geq R. \]

(96b)

provided that the constant \( \xi_0 \) is chosen as

\[ \xi_0 = \xi_1 := (C_5/A_0)^{\frac{1-m}{2}}(C - \epsilon)\frac{m-1}{x^{\alpha(1-m)}} \]

(97)

Hence, by choosing \( C_0 = C_5 \) and \( \xi_0 = \xi_1 \), the function \( g \) becomes a subsolution of the Cauchy–Dirichlet problem for the PDE (1) on \( D \), and by applying the comparison theorem the left-hand side of the estimate (39) follows.

From lower bound of (39) and (38) it follows that

\[ C_5 \left(\left|\frac{x-R}{|x|}\right|^{\frac{1}{m-1}}\right)^{\frac{2}{m-1}} \leq u(x,t) t^{\frac{1}{m}} |x|^{\frac{2}{m-1}} \leq D \left(\left|\frac{x-R}{|x|}\right|^{\frac{2}{m-1}}\right), \text{ on } D \]

(98)

Passing to the limit as \(|x| \to \infty \) we derive

\[ C_5 \leq \liminf_{|x| \to +\infty} u(x,t) t^{\frac{1}{m}} |x|^{\frac{2}{m-1}} \leq \limsup_{|x| \to +\infty} u(x,t) t^{\frac{1}{m}} |x|^{\frac{2}{m-1}} \leq D, \quad 0 < t \leq \delta. \]

(99)

Now passing to the limit, first as \( t \downarrow 0 \), and finally as \( \epsilon \downarrow 0 \), from (99), (17) follows.

To prove the lower estimate (40), we fix arbitrary \( \tilde{\beta} \in (m, 1) \), and consider a function \( g \) from (42), with \( f \) chosen as in (75), and \( \beta \& \xi \) replaced with \( \tilde{\beta} \& \tilde{\zeta} = (R-|x|)^{-\frac{m-\tilde{\beta}}{2(1-\tilde{\beta})}} \) respectively. Estimating \( Lg \) in \( D \), instead of (76) we derive

\[
Lg = b\tilde{\beta}^2\left\{C_0^{\beta-\tilde{\beta}} t^{\frac{\tilde{\beta}}{1-\tilde{\beta}}} (\zeta_0 - \tilde{\zeta})^{\frac{2(\beta-\tilde{\beta})}{m-\tilde{\beta}}} - \left(C_0/C_*\right)^{m-\tilde{\beta}} + \frac{C_0^{1-\beta} \zeta_0 (\zeta_0 - \tilde{\zeta})^{\frac{2-m-\tilde{\beta}}{m-\tilde{\beta}}}}{b(1-\beta)} \right\} + \frac{2m(n-1)}{b(m-\beta)|x|} \cdot \frac{m-\tilde{\beta}}{2(1-\tilde{\beta})} C_0^{m-\tilde{\beta}} (\zeta_0 - \tilde{\zeta}) \leq b\tilde{\beta}^2\left\{C_0^{\beta-\tilde{\beta}} t^{\frac{\tilde{\beta}}{1-\tilde{\beta}}} \zeta_0^{\frac{2(\beta-\tilde{\beta})}{m-\tilde{\beta}}} - \left(C_0/C_*\right)^{m-\tilde{\beta}} + \frac{C_0^{1-\beta} \zeta_0^{\frac{2-m-\tilde{\beta}}{m-\tilde{\beta}}}}{b(1-\beta)} \right\},
\]

(100)
where
\[
\bar{C}_* = \left[ b(m - \bar{\beta})^2/(2m(m + \bar{\beta})) \right]^{1/(m - \bar{\beta})}
\]

Let \( \mu > 0 \) be arbitrary, and select \( C_0 = C_7 \) and \( \zeta_0 = \zeta_8 \) to achieve the conditions
\[
\left( \frac{C_7}{C_*} \right)^{m - \bar{\beta}} = \mu, \quad \frac{C_7^{1 - \bar{\beta}}}{b(1 - \bar{\beta})} \frac{2(1 - \bar{\beta})}{m - \bar{\beta}} = \frac{\mu}{2}.
\]
(101)

From (100), (101) it follows
\[
Lg \leq bg \bar{\beta}\left\{ \left( \frac{1}{2} b(1 - \bar{\beta}) \mu t \right)^{\frac{2 - \bar{\beta}}{1 - \bar{\beta}}} - \frac{\mu}{2} \right\} \leq 0, \quad \text{for } |x| \geq R, \quad 0 \leq t \leq \frac{1}{b(1 - \bar{\beta})} \left( \frac{\mu}{2} \right)^{\frac{1 - \bar{\beta}}{1 - \bar{\beta}}}
\]
(102)

Note that the time interval covers \( \mathbb{R}^+ \) as \( \mu \to 0 \). Hence, \( g \) is a subsolution of the PDE (1) on \( D \) for arbitrary \( \delta > 0 \), and for all sufficiently small \( \mu > 0 \). Since \( (1 - \bar{\beta})^{-1} > \alpha(2 + \alpha(1 - m))^{-1} \), we have
\[
u|_{|x|=R} \geq A_0(C - \epsilon) \frac{2}{\pi(1 - m)} t^{\frac{m}{2} - \frac{\alpha}{2}} - (1 - \bar{\beta})^{- \frac{1}{1 - \bar{\beta}}} t^{\frac{1}{1 - \bar{\beta}}} = g|_{|x|=R}, \quad 0 < t \leq \delta,
\]
(103a)
\[
u|_{t=0} = g|_{t=0} = 0, \quad \text{for } |x| \geq R.
\]
(103b)

Moreover, the time interval length \( \delta \) is only dictated by the first inequality in (103), while the middle inequality is valid in any time interval for all sufficiently small \( \mu > 0 \). Applying comparison theorem, from (102), (103), the estimate (40) follows. From (40) we derive
\[
u(x,t)|x|^{\frac{2}{m - \bar{\beta}}} \geq C_7 \left( \frac{\zeta_8 t^{\frac{m - \bar{\beta}}{2(1 - \bar{\beta})}} + |x| - R}{|x|} \right)^{2/m - \bar{\beta}}, \quad |x| > R, \quad 0 \leq t \leq \delta.
\]
(104)

Passing to limit as \( |x| \to +\infty \), we get
\[
\lim \inf_{|x| \to +\infty} \nu(x,t)|x|^{\frac{2}{m - \bar{\beta}}} \geq C_7, \quad 0 \leq t \leq \delta,
\]
(105)

and finally passing to limit as \( \mu \downarrow 0 \) we deduce
\[
\lim_{|x| \to +\infty} \nu(x,t)|x|^{\frac{2}{m - \bar{\beta}}} = +\infty, \quad 0 \leq t \leq \delta.
\]
(106)

Since \( \bar{\beta} \in (m, 1) \) is arbitrary, from (106), the claim (15) easily follows.
Author contributions UGA: Conceptualization, Methodology, Analysis, Project Administration, Supervision, Writing—review and editing; AAW: Investigation, Validation, Writing—original draft

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.

Appendix

Here we bring explicit values of the constants used in Sects. 2 and 3. (1) $0 < \beta < m, \ 0 < \alpha < (m - \beta)^{-1}$

$$C_* = \left[ b(m - \beta)^2 / (2m(m + \beta)) \right]^{1/(m-\beta)},$$

$$C_1 = \left( \frac{1 - \beta(1 + \epsilon)}{1 - m} \right) \frac{1}{\alpha} C_*,$$

$$\zeta_1 = -b^{m-1} \left( 2m \right)^{\frac{1}{2}} (m + \beta)^{\frac{1}{2}} (m - \beta)^{\frac{m-\beta}{2(1-\beta)}} \left( \frac{1 - m}{1 + \epsilon} \right)^{1-m} \frac{1}{\beta} C/C_* \left( 1 + \epsilon (1 - \beta) \right) \zeta_{1,2}.$$ 

$$l = \frac{m - \beta}{1 - \beta} \zeta_{1,2}. $$

(2) $\beta < m, \ \alpha = 2(m - \beta)^{-1}$

$$\zeta_3 = -(A_1 - \epsilon)^{\frac{m-1}{2}} (1 + b(1 - \beta)(1 + \epsilon)(A_1 - \epsilon)(m - \beta)^{-1})^{\frac{1}{2}} (m - \beta)^{-1}$$

$$\left( 2m(1 - \beta)(m + \beta) \right)^{\frac{1}{2}}$$

$$\zeta_4 = -(A_1 - \epsilon)^{\frac{m-\beta}{2}} C_*^{\frac{m-\beta}{2}}, \ C_2 = (A_1 - \epsilon) \zeta_3^{2/(m-\beta)}, \ A_1 = h(0)$$

$$\zeta_5 = \rho - \left( \frac{1 - \epsilon}{1 + \epsilon} \right)^{\frac{m-\beta}{2}}, \ \rho > \zeta_* > 0 \ (C < C_*)$$

$$l_2 = C^{\frac{m-\beta}{2}} \left[ b(1 - \beta)(1 - \delta, \Gamma)^{-1}(1 - \delta, \Gamma - (1 - \delta, \Gamma)^{-1}(C/C_*)^{m-\beta}) \right]^{\frac{m-\beta}{2(1-\beta)}}$$

$$\zeta_6 = \delta_* \Gamma_2, \ \Gamma = 1 - (C/C_*)^{(m-\beta)/2}, \ C_3 = C(1 - \delta_* \Gamma)^2/(\beta - m)$$

where $\delta_* \in (0, 1)$ satisfies

$$g(\delta_*) = \max_{[0;1]} g(\delta), \ g(\delta) = \delta^{2 - m}\beta / (m - \beta) \left[ 1 - \delta \Gamma - (1 - \delta \Gamma)^{-1}(C/C_*)^{m-\beta} \right]$$
\[ (5)-(6) \quad \beta > m \]

\[ D = [2m(m+1)(1-m)^{-1}]^{1/1-m}, \quad A_0 = f_0(0) > 0, \]

\[ \zeta_7 = \left[ b(1-\beta) \left( (1-\mu)^{m-\beta} - 1 \right) \right]^{m-\beta \over 2(1-\beta)} \left( (1-\mu)C^* \right)^{\beta-m \over 2}, \]

\[ C_5 = D \left( \frac{1}{1 + m} \right) \frac{1}{1-m} (1 + e)^{1 \over m-1}, \quad \xi_1 = (C_5/A_0)^{1-m \over 2} \left( C - \epsilon \right)^{m-1 \over 2}, \]

\[ \xi_2 = A_0^{m-1 \over 2} \left( C + \epsilon \right)^{m-1 \over 2}, \quad C_6 = \left[ \frac{2m(m+1)(2+\alpha(1-m))}{\alpha(1-m)^2} \right]^{1 \over m-1}, \quad C_7 = C^*_\mu^{m-\beta}, \quad \mu > 0, \quad \bar{\beta} \in (m,1), \]

\[ C^*_\mu = \left[ \frac{b(m-\beta)^2}{2m(m+\bar{\beta})} \right]^{1/(m-\beta)}, \quad \zeta_8 = C_7^{\beta-m \over 2} \left( \frac{1}{2} b(1-\bar{\beta}) \mu \right)^{m-\bar{\beta} \over 2}. \]

References

[1] Aris, R.: The Mathematical Theory of Diffusion and Reaction in Permeable Catalysts. Clarendon Press, Oxford (1975)

[2] Barenblatt, G.I.: On some unsteady motions of a liquid or a gas in a porous medium. Prikl. Mat. Mekh 16, 67–78 (1952)

[3] Kalashnikov, A.S.: Some problems of the qualitative theory of non-linear degenerate second-order parabolic equations. Russ. Math. Surv. 42, 169–222 (1987)

[4] Bear, J.: Dynamics of Fluids in Porous Media. Elsevier, New York (1972)

[5] Abdulla, U.G., Abuweden, A.: Interface development for the nonlinear degenerate multidimensional reaction-diffusion equations. Nonlinear Differ. Equ. Appl. NoDEA, 27, Article number: 3 (2020)

[6] Aronson, D.G., Peletier, L.A.: Large time behaviour of solutions of the porous medium equation in bounded domain. J. Differ. Equ. 39, 178–412 (1981)

[7] Aronson, D.G., Crandall, M.G., Peletier, L.A.: Stabilization of solutions of a degenerate nonlinear diffusion problem. Nonlinear Anal. TMA 6, 1001–1022 (1982)

[8] Caffarelli, L.A., Friedman, A.: Continuity of the density of a gas flow in a porous medium. Trans. Am. Math. Soc. 252, 99–113 (1979)

[9] Dibenedetto, E.: Continuity of weak solutions to a general porous media equation. Ind. Univ. Math. J. 32(1), 83–118 (1983)

[10] Alt, H.W., Luckhaus, S.: Quasilinear elliptic–parabolic differential equations. Mathematische Zeitschrift 183(3), 311–341 (1983)

[11] Lacey, A.A., Ockendon, J.R., Tayler, A.B.: Waiting-time solutions of a nonlinear diffusion equation. SIAM J. Appl. Math. 42, 1252–1264 (1982)
[12] Aronson, D.G., Caffarelli, L.A., Kamin, S.: How an initially stationary interface begins to move in porous medium flow. SIAM J. Math. Anal. 14(4), 639–658 (1983)

[13] Kersner, R.: Degenerate parabolic equations with general nonlinearities. Nonlinear Anal. TMA 4(6), 1043–1062 (1980)

[14] Benilan, P.H.: Solutions of the porous medium equation in $\mathbb{R}^N$ under optimal conditions of initial values. Indiana Univ. Math. J. 33, 51–87 (1984)

[15] Herrero, M.A., Pierre, M.: The Cauchy problem for $u_t = \Delta u^m$ when $0 < m < 1$. Trans. Am. Math. Soc. 291, 145–158 (1985)

[16] Tsutsumi, M.: On solutions os some doubly nonlinear degenerate parabolic equations with absorption. J. Math. Anal. Appl. 132, 187–212 (1988)

[17] Caffarelli, L.A., Wolanski, N.I.: $C^{1,\alpha}$ regularity of the free boundary for the N-dimensional porous media equation. Commun. Pure Appl. Math. 43, 885–902 (1990)

[18] Caffarelli, L.A., Vazquez, J.L., Wolanski, N.I.: Lipschitz continuity of solutions and interfaces of the N-dimensional porous medium equation. Indiana Univ. Math. J. 36(2), 373–401 (1987)

[19] Vazquez, J.L.: The interfaces of one-dimensional flows in porous media. Trans. Am. Math. Soc. 206, 787–802 (1984)

[20] Samarskii, A.A., Galaktionov, V.A., Kurdyumov, S.P., Mikhailov, A.P.: Blow-up in Problems for Quasilinear Parabolic Equations. Nauka, Moscow (1987)

[21] Otto, F.: The geometry of dissipative evolution equations: the porous medium equation. Commun. Partial Differ. Equ. 26(1–2), 101–174 (2001)

[22] Antontsev, S.N., Diaz, J.L., Shmarev, S.: Energy Methods for Free Boundary Problems: Applications to Nonlinear PDEs and Fluid Mechanics, vol. 48. Springer, Berlin (2012)

[23] DiBenedetto, E.: Degenerate Parabolic Equations. Universitext. Springer, New York (1993)

[24] Vazquez, J.L.: The Porous Medium Equation: Mathematical Theory. Oxford University Press, Oxford (2007)

[25] Abdulla, U.G.: On the Dirichlet problem for the nonlinear diffusion equation in non-smooth domains. J. Math. Anal. Appl. 260, 384–403 (2001)

[26] Abdulla, U.G.: Well-posedness of the Dirichlet problem for the non-linear diffusion equation in non-smooth domains. Trans. Am. Math. Soc. 357, 247–265 (2005)

[27] Abdulla, U.G.: Reaction–diffusion in nonsmooth and closed domains. Bound. Value Probl. 2007(1), 031261 (2007)

[28] Abdulla, U.G.: Reaction–diffusion in irregular domains. J. Differ. Equ. 164, 321–354 (2000)
[29] Abdulla, U.G., King, J.R.: Interface development and local solutions to reaction–diffusion equations. SIAM J. Math. Anal. 32, 235–260 (2000)

[30] Abdulla, U.G.: Evolution of interfaces and explicit asymptotics at infinity for the fast diffusion equation with absorption. Nonlinear Anal. Theory Methods Appl. 50, 541–560 (2002)

[31] Abdulla, U.G., Jeli, R.: Evolution of interfaces for the non-linear parabolic $p$-Laplacian type reaction–diffusion equations. Eur. J. Appl. Math. 28(5), 827–853 (2017)

[32] Abdulla, U.G., Jeli, R.: Evolution of interfaces for the non-linear parabolic $p$-Laplacian type reaction–diffusion equations. II. Fast diffusion vs. strong absorption. Eur. J. Appl. Math. 31(3), 385–406 (2019)

[33] Abdulla, U.G., Du, J., Prinkey, A., Ondracek, Ch., Parimoo, S.: Evolution of interfaces for the nonlinear double degenerate parabolic equation of turbulent filtration with absorption. Math. Comput. Simul. 153, 59–82 (2018)

[34] Shmarev, S., Vdovin, V., Vlasov, A.: Interfaces in diffusion–absorption processes in nonhomogeneous media. Math. Comput. Simul. 118, 360–378 (2015)

Ugur G. Abdulla
Analysis and PDE Unit
Okinawa Institute of Science and Technology
Okinawa 904-0495
Japan
e-mail: Ugur.Abdulla@oist.jp

Amna Abu Weden
Keiser University
Port Saint Lucie FL34987
USA
e-mail: amna.abuweden@keiseruniversity.edu

Received: 9 October 2022.
Accepted: 10 February 2023.