The 2+1 Dimensional $Q\bar{Q}$ Potential in Fourth Order

Color-Dielectric Transverse Lattice QCD

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Abstract

The 2+1 dimensional (2 space dimensions and 1 time dimension) Transverse Lattice is constructed using the Color Dielectric formalism. The dynamics of the Color Dielectric link fields are not known from first principles, so an ansatz for the effective action is made, which is truncated at fourth order in the fields. Since the Transverse Lattice is a Light-Front field theory, it lacks manifest rotational invariance. This lack of manifest rotational invariance is exploited to determine the couplings in the effective action by calculating the $Q\bar{Q}$ Potential, an observable that is very sensitive to rotational symmetry. This calculation lays the groundwork for a fully 3+1 dimensional treatment of the Transverse Lattice.
I. INTRODUCTION

Theory and phenomenology both have an important place in helping us understand the nature of the strong interaction. The very best phenomenological models strip away the superfluous aspects of the theory and focus solely on the mechanisms which underly the physics at hand. Rigorous theory, however, provides the crucial framework upon which our basic physical understanding rests. In order to perform specific calculations, however, approximations must be made to the rigorous theory of the strong interaction. Choosing what approximations to make and determining the affect of these approximations are some of the challenges in performing such calculations.

The Color Dielectric Formulation of the Transverse Lattice provides a means of bridging the gap between rigorous theory and phenomenology. The Transverse Lattice is a Light Front field theory in which the Light Front Coordinates, \( x^\pm = \frac{1}{\sqrt{2}} (x^0 \pm x^3) \), are continuous variables and the remaining two space-time coordinates, \( x^{1,2} \), are discrete quantities. The longitudinal dynamics (the \( x^\pm \) directions) are provided by an instantaneous interaction established by t’Hooft in the limit of a large number of quark colors, \( N_c \), in \( 1+1 \) dimensional theories. The transverse dynamics are mediated by link fields which share some similarity to the link fields in Euclidean Lattice simulations. An important difference is that these link fields are not the localized gluon fields developed in the usual Euclidean Lattice framework. Instead, they represent an average over a number of these localized gluon fields. It is for this reason that such a formulation of the Transverse Lattice is labeled a Color Dielectric formulation.

The Color Dielectric formulation is analogous to the dielectric approximation in electromagnetic materials where the electromagnetic field is smoothed by taking its spatial average. The localized gluonic fields, like the microscopic electromagnetic field in a dielectric, can vary significantly over short distance scales. Color Dielectric theory averages over many of these fields, and these new averaged link fields are the effective degrees of freedom of the theory. If the average is a spatial average the link fields can account for (relatively) long distance physics using comparatively few degrees of freedom.
In this paper, the Transverse Lattice is used to calculate the static $Q\bar{Q}$ Potential in 2+1 dimensions. The dynamics of the Color Dielectric link fields are not known from first principles, so an ansatz for the interactions of the link fields is made with the couplings left as free parameters. These parameters are tuned to reproduce the rotational symmetry of the $Q\bar{Q}$ Potential. In this way, the dynamics of the Color Dielectric link fields can be determined.

II. FORMULATING THE TRANSVERSE LATTICE HAMILTONIAN

The Transverse Lattice was first formulated by Bardeen and Pearson [1, 2]. In those papers, the hamiltonian is formulated in the limit of a large number of quark colors. This has the effect of limiting the number of terms which contribute to the effective hamiltonian. The large-$N_c$ limit also provides a means of systematically excluding quark pair production since quark pair production mechanisms arise from non-planar diagrams. In [2] the effective action was truncated at second order in the link fields. Although the results were acceptable for a first study, there are two important reasons to examine terms which are fourth order in the link fields. The first reason is obvious: including fourth order terms is less restrictive. In an ideal world, we would allow all orders in the link fields to contribute. Truncating the expression for the effective action is necessary to do practical computations, but the later we truncate, the closer we will be to an “exact” hamiltonian. The second reason is that fourth order terms are necessary to model 3+1 dimensions. The canonical plaquette interaction is the simplest interaction which distinguishes two link fields strung end to end in a straight line versus two link fields end to end going around a corner on the lattice. The plaquette interaction term is fourth order in the link fields. For this reason, and for consistency, we choose to include the fourth order terms which contribute in our approximation. These terms are represented by,

\begin{equation}
H^{(4)} = \lambda_1 Tr \left[ M_{i,j,x} M_{i+1,j,x} \bar{M}^\dagger_{i+1,j,x} \bar{M}^\dagger_{i,j,x} \right] \\
+ \lambda_2 Tr \left[ M_{i,j,x} M_{i+1,j,y} \bar{M}^\dagger_{i+1,j,y} \bar{M}^\dagger_{i,j,x} \right] \\
+ \lambda_p Tr \left[ M_{i,j,x} M_{i+1,j,y} \bar{M}^\dagger_{i+1,j+1,x} \bar{M}^\dagger_{i,j,y} \right] 
\end{equation}

The first term represents the interaction between two link fields laid end to end in a
FIG. 1: Schematic view of the terms in the effective action. (A) represents the link fields in a straight line interacting, (B) represents the link fields around a corner interacting, and (C) is the plaquette interaction.

straight line. The second term accounts for the interaction between two link fields which form a corner. The final term is the canonical plaquette interaction with the link fields being multiplied in order around a simple plaquette in the lattice. In addition to these fourth order interactions, a “contact” interaction was introduced which acts only on the links at the ends of the “string” connecting the quarks. The coupling associated with this interaction is labeled $\lambda_c$. The couplings $\lambda_1$, $\lambda_2$, $\lambda_p$, and $\lambda_c$ are free parameters in the theory which must be determined. They are free parameters since the detailed dynamics of the effective degrees of freedom represented by the link fields are not known from first principles. In [?], Lorentz symmetry was used as the benchmark for tuning the couplings. As we shall see, another excellent observable with which to tune these parameters is the static $Q\bar{Q}$ Potential.

III. CALCULATING THE $Q\bar{Q}$ POTENTIAL

The $Q\bar{Q}$ Potential is calculated using the same method as in our previous work [? ]. The Discrete Light Cone Quantization (DLCQ) formalism is exploited for the transverse directions [? ? ]. The hamiltonian is constructed for a Fock space which has two essential approximations. The first is that only Fock states with total momentum less than some cut-off are considered. This approximation is necessary to give the problem a finite size, making it amenable to computational solution. The second approximation is the minimal
string approximation in which the color sources are connected by a link field string of the shortest possible length in lattice units. This is an ad hoc approximation, however, in hindsight it appears to not affect the results of the calculation to any great degree. For a system with two link fields between the quarks, the eigenvalue equation is found to be,

\[
P^- \psi(p_1, p_2) = \left( \frac{p_1 + p_2}{2v^+} + \frac{\mu^2}{2p_1} + \frac{\mu^2}{2p_2} \right) \psi(p_1, p_2)
+ G^2 \int_0^{\infty} dq \frac{(q + p_1) \left[ \psi(p_1, p_2) - \psi(q, p_2) e^{i(p_1 - q)x^-/2} \right]}{2\sqrt{p_1 q(q - p_1)^2}}
+ G^2 \int_0^{\infty} dq \frac{(q + p_2) \left[ \psi(p_1, p_2) - \psi(p_1, q) e^{i(q - p_2)x^-/2} \right]}{2\sqrt{p_2 q(q - p_1)^2}}
+ G^2 \int_0^{p_1 + p_2} dq \frac{(q + p_1)(p_1 + 2p_2 - q)}{4\sqrt{p_1 p_2 q(p_1 + p_2 - q)(q - p_1)^2}}
\times \left[ \psi(p_1, p_2) - \psi(q, p_2) \right]
+ G^2 \frac{\pi}{4\sqrt{p_1 p_2}} \psi(p_1, p_2)
+ \lambda_1 \int_0^{p_1 + p_2} dq \frac{\psi(p_1, p_2) + \psi(q, p_2)}{\sqrt{p_1 p_2 q(p_1 + p_2 - q)}}
+ \lambda_c \int_0^{p_1 + p_2} dq \frac{\psi(p_1, p_2)}{v^+(p_1 p_2)}
+ \lambda_c \int_0^{p_1 + p_2} dq \frac{\psi(q, p_2) e^{i(q - p_2)x^-/2}}{v^+ \sqrt{p_1 p_2 q(p_1 + p_2 - q)}}
+ \lambda_c \int_0^{p_1 + p_2} dq \frac{\psi(p_1, q) e^{i(p_1 - q)x^-/2}}{v^+ \sqrt{p_1 p_2 q(p_1 + p_2 - q)}}
\]

The momentum cut-off, along with the “plus” component of the four-velocity of the quark/anti-quark pair \((v^+)\), must be carefully extrapolated to infinity. By considering the asymptotic dependence of the momentum-space wave function in our eigenvalue equation, this dependence of the energy on the cut-off, \(\Lambda\), is found to be,

\[
E \sim -\frac{1}{\Lambda^4}
\quad (2)
\]

The dependence on \(v^+\) is assumed to be of the form,
\[ E \sim \left( \frac{1}{v^2} \right)^p \]

\( p \) is found empirically from the data for each configuration. Typically, \( p \) is around 0.20 when extrapolating the lowest energy eigenvalue corresponding to the ground state \( Q\bar{Q} \) Potential.

As was discussed in previous work \([?]\), the \( Q\bar{Q} \) Potential is an important observable for us to study. One of the reasons is the lack of manifest rotational invariance in Light Front field theories\([?]\). Because rotational invariance is not a manifest symmetry of the Light Front hamiltonian, it is sensitive to the the dynamics included in the hamiltonian. If the dynamics of the hamiltonian are incorrect, rotational invariance will not be observed and this will be obvious from the shape of the \( Q\bar{Q} \) Potential. From Euclidean Lattice Monte Carlo Simulations \([?]\) the \( Q\bar{Q} \) Potential is known to be a linear function of separation with a slope, \( \sigma \), known as the string tension. If the \( Q\bar{Q} \) Potential were to depend on orientation of the quarks (i.e. not rotationally invariant) then the string tension would be different for these different orientations. In such a case the \( Q\bar{Q} \) Potential would not be a very good line: the points would be scattered in a wedge between the maximum and minimum string tensions.

In a full 3+1 dimensional calculation, the five couplings of the effective action would be tuned independently to obtain a rotationally invariant \( Q\bar{Q} \) Potential. The five couplings are \( \mu^2 \) (the link field mass), \( \lambda_1 \), \( \lambda_2 \), \( \lambda_p \), and \( \lambda_c \). In this 2+1 dimensional calculation, the plaquette interaction and the interaction between links that turn a corner are superfluous since there is only one discrete direction. Thus there are only three free parameters to tune in the hamiltonian: \( \mu^2 \), \( \lambda_1 \), and \( \lambda_c \). We found empirically that adjusting \( \mu^2 \) and \( \lambda_1 \) affected the global rotational invariance of the points while adjusting the contact coupling, \( \lambda_c \), most affected the intercept of the \( Q\bar{Q} \) Potential. As was shown in our previous work on the 2+1 Dimensional \( Q\bar{Q} \) Potential, its slope in our units should be \( G^2 \frac{\pi}{2} \).

The strong coupling limit provides us with an important point of comparison for our data. The strong coupling limit does not exhibit rotational invariance. In this limit \( (\mu^2 \to \infty) \), the \( Q\bar{Q} \) Potential has the form,
\[ V(x_\perp, x_L) = G^2 \frac{\pi}{2} (|x_\perp| + |x_L|) \] (4)

The data points in this limit would be scattered within a wedge. The lower slope would be \( G^2 \frac{\pi}{2} \) and the upper slope would be \( \sqrt{2} G^2 \frac{\pi}{2} \), corresponding to a separation which is “diagonal”, having equal transverse and longitudinal displacements. If the data points in our calculation stay close to the lower slope and well away from the upper slope indicated by the strong coupling limit, rotational invariance can be claimed.

IV. RESULTS AND ANALYSIS

Plots of the \( \bar{Q}Q \) Potential data are shown in Figures 2 and 3. In Figure 2 a set of data is shown which has some anisotropy. While it does not exhibit the uniform distribution of points characteristic of the strong coupling limit, the data points do not all lie upon the ideal line. Figure 3 represents the best-tuned data for the \( \bar{Q}Q \) Potential that we obtained. \( \chi^2 \)-analysis shows that this line has a slope very close to the expected slope of \( G^2 \frac{\pi}{2} \). The rotational invariance of this data is quite striking, thus we are very confident in the dynamics that were included in this incarnation of the Color Dielectric formulation of the Transverse Lattice.

The Tranverse Lattice is essentially an effective theory, but one which is grounded in the fundamental underlying dynamics of the point-like gluon fields. While no appeal has been made in this work to actually calculate the relationship between the link fields on the localized gluonic degrees of freedom, such a study is possible in principle. Determining this relationship would lend a rigor to this effective theory that most phenomenological models do not enjoy. Such a calculation could be performed in a pure glue (i.e. quenched) Euclidean Lattice Monte Carlo Simulation where a smearing transformation is performed on the Euclidean Lattice link fields. Determining the couplings for the effective action in this formalism holds great appeal since it does not rely on tuning them to fit data. Instead, the calculations could be used to test the validity of the theory or as predictions in their own right.

The couplings found from this study form a foundation on which to begin the more real-
FIG. 2: Plot of the poorly tuned $Q\bar{Q}$ Potential. The solid lines represent the maximum and minimum slopes for the strong coupling limit. The dashed line is the best fit through the data points. $\chi^2$ for the fit is 265. $\chi^2$ per degree of freedom is 13.3.

istic 3+1 dimensional calculations. In 3+1 dimensions, the plaquette coupling would need to be included along with a new coupling similar to $\lambda_1$, but acting on two links which turn a corner rather than go in a straight line. Work on the 3+1 dimensional $Q\bar{Q}$ Potential is currently underway by the authors. As computer storage becomes more affordable, another extension to this work can be considered: relaxing the minimal string approximation. Including such states would allow us to test the validity of the approximation as well as to relax the restriction on the Fock space. Including such transverse oscillations of the link field string could have important contributions for the excited state potentials. In addition, the excited state potentials themselves can also be extracted from the data. These potentials correspond to an excited gluonic string and can be interpreted as the potential experienced by the valence quarks in hybrid mesons. Comparing the excited state potentials to phenomenological models such as the Flux Tube Model [?] will give insight into the
FIG. 3: Plot of the finely tuned $Q\bar{Q}$ Potential. The solid lines represent the maximum and minimum slopes for the strong coupling limit. The dashed line is the best fit through the data points. $\chi^2$ for the fit is 2.13. $\chi^2$ per degree of freedom is 0.11.

phenomenology represented by these models. The suggested extensions to the present work, as stated above, demonstrate only a small fraction of the utility of the Transverse Lattice.