Budgeted Dominating Sets in Uncertain Graphs

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Abstract

We study the Budgeted Dominating Set (BDS) problem on uncertain graphs, namely, graphs with a probability distribution \( p \) associated with the edges, such that an edge \( e \) exists in the graph with probability \( p(e) \). The input to the problem consists of a vertex-weighted uncertain graph \( G = (V, E, p, \omega) \) and an integer budget (or solution size) \( k \), and the objective is to compute a vertex set \( S \) of size \( k \) that maximizes the expected total domination (or total weight) of vertices in the closed neighborhood of \( S \). We refer to the problem as the Probabilistic Budgeted Dominating Set (PBDS) problem. In this article, we present the following results on the complexity of the PBDS problem.

1. We show that the PBDS problem is NP-complete even when restricted to uncertain trees of diameter at most four. This is in sharp contrast with the well-known fact that the BDS problem is solvable in polynomial time in trees. We further show that PBDS is \( \mathcal{W}[1] \)-hard for the budget parameter \( k \), and under the Exponential time hypothesis it cannot be solved in \( n^{o(k)} \) time.

2. We show that if one is willing to settle for \((1 - \epsilon)\) approximation, then there exists a PTAS for PBDS on trees. Moreover, for the scenario of uniform edge-probabilities, the problem can be solved optimally in polynomial time.

3. We consider the parameterized complexity of the PBDS problem, and show that Uni-PBDS (where all edge probabilities are identical) is \( \mathcal{W}[1] \)-hard for the parameter pathwidth. On the other hand, we show that it is FPT in the combined parameters of the budget \( k \) and the treewidth.

4. Finally, we extend some of our parameterized results to planar and apex-minor-free graphs.

Our first hardness proof (Thm. 1) makes use of the new problem of \( k \)-\text{Subset} \( \Sigma - \Pi \) \text{-Maximization} (\( k \)-SPM), which we believe is of independent interest. We prove its NP-hardness by a reduction from the well-known \( k \)-SUM problem, presenting a close relationship between the two problems.

2012 ACM Subject Classification  Mathematics of computing \( \rightarrow \) Graph algorithms

Keywords and phrases  Uncertain graphs, Dominating set, NP-hard, PTAS, treewidth, planar graph.

1 Introduction

Background and Motivation. Many optimization problems in network theory deal with placing resources in key vertices in the network so as to maximize coverage. Some practical
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Contexts where such coverage problems occur include placing mobile towers in wireless networks to maximize reception, assigning emergency vehicle centres in a populated area to guarantee fast response, opening production plants to ensure short distribution lines, and so on. In the context of social networks, the problem of spreading influencers so as to affect as many of the network members as possible has recently attracted considerable interest.

Coverage problems may assume different forms depending on the optimized parameter. A basic “full coverage” variant is the classical dominating set problem, which asks to find a minimal vertex set \( S \) such that each vertex not in \( S \) is dominated by \( S \), i.e., is adjacent to at least one vertex in \( S \). In the dual budgeted dominating set (BDS) problem, given a bound \( k \) (the budget), it is required to find a set \( S \) of size at most \( k \) maximizing the number of covered vertices. Over vertex weighted graphs, the goal is to maximize the total weight of the covered vertices, also known as the domination. It is this variant that we’re concerned with here.

Traditionally, coverage problems involve a fixed network of static topology. The picture becomes more interesting when the network structure is uncertain, due to potential edge connections and disconnections or link failures. Pre-selection of resource locations at the design stage becomes more challenging in such partial-information settings.

In this work, we study the problem in one of the most fundamental settings, where the input is a graph whose edges fail independently with a given probability. The goal is to find a \( k \)-element set that maximizes the expected (1-hop) coverage (or domination). Our results reveal that the probabilistic versions of the coverage problem are significantly harder than their deterministic counterparts, and analyzing them require more elaborate techniques.

An uncertain graph \( G \) is a triple \( (V, E, p) \), where \( V \) is a set of \( n \) vertices, \( E \subseteq V \times V \) is a set of \( m \) edges, and the function \( p : E \to [0,1] \) assigns a probability of existence to each edge in \( E \). So an \( m \)-edge uncertain graph \( G \) represents a probability space consisting of \( 2^m \) graphs, sometimes called possible worlds, derived by sampling each edge \( e \in E \) independently with probability \( p(e) \). For \( H = (V, E' \subseteq E) \), the event of sampling \( H \) as a possible world, denoted \( H \subseteq G \), occurs with probability \( \Pr(H \subseteq G) = \prod_{e \in E'} p(e) \prod_{e \in E \setminus E'} (1 - p(e)) \). The notion of possible worlds dates back to Leibniz and possible world semantics (PWS) is well-studied in the modal logic literature, beginning with the work of Kripke.

Our work focuses on budgeted dominating sets on vertex-weighted uncertain graphs, i.e., the Probabilistic Budgeted Dominating Set (PBDS) problem. The input consists of a vertex-weighted uncertain graph \( G = (V, E, p, \omega) \), with a weight function \( \omega : V \to \mathbb{Q}^+ \) and an integer budget \( k \). Set \( p(uv) = 1 \) for every \( v \). For a vertex \( u \) and a set \( S \subseteq V \), denote by \( \Pr(u \sim S) = 1 - \prod_{v \in S} (1 - p(uv)) \) the probability that \( u \in S \) or \( u \) is connected to some vertex in \( S \). For sets \( S_1, S_2 \subseteq V \), the expected coverage (or domination) of \( S_1 \) by \( S_2 \) is defined as \( C(S_1, S_2) = \sum_{v \in S_1} \omega(v) \Pr(v \sim S_2) \). The PBDS problem aims to find a set \( S \) of size \( k \) that maximizes \( C(V, S) \) over the possible worlds. Its decision version is defined as follows.

| Probabilistic budgeted dominating set (PBDS) |
|---------------------------------------------|
| **Input:** A vertex-weighted uncertain graph \( G = (V, E, p, \omega) \), an integer \( k \) and a target domination value \( t \). |
| **Question:** Is there a set \( S \subseteq V \) of size at most \( k \) such that \( C(V, S) \geq t \)? |

Our Results and Discussion. The budgeted dominating set problem is known to have a polynomial time solution on trees. A natural question is if the same applies to the probabilistic version of the problem. We answer this question negatively, showing the following.
Theorem 1. The PBDS problem is NP-hard on uncertain trees of diameter 4. Furthermore, (i) the PBDS problem on uncertain trees is \( W[1] \)-hard for the parameter \( k \), and (ii) an \( n^{o(k)} \) time solution to PBDS will falsify the Exponential time hypothesis.

In order to prove the theorem, we introduce the following problem.

**SUBSET \( \Sigma - \Pi \) MAXIMIZATION (\( k\)\(-SPM\))**

**Input:** A multiset \( A = \{(x_1, y_1), \ldots, (x_N, y_N)\} \) of \( N \) pairs of positive rationals, an integer \( k \), and a rational \( t \).

**Question:** Is there a set \( S \subseteq [N] \) of size exactly \( k \) satisfying \( \sum_{i \in S} x_i - \prod_{i \in S} y_i \geq t \) ?

To establish the complexity of the \( k\)\(-SPM \) problem, we present a polynomial time reduction from \( k\)-SUM to \( k\)\(-SPM \), thereby proving that both \( k\)\(-PBDS \) and \( k\)\(-SPM \) are NP-hard. Moreover, Downey and Fellows [24] showed that the \( k\)-SUM problem is \( W[1] \)-hard, implying that if \( k\)-SUM has an FPT solution with parameter \( k \), then the \( W \) hierarchy collapses. This provides our second hardness result.

Theorem 2. The \( k\)-SUM problem is \( W[1] \)-hard for the parameter \( k \). Furthermore, any \( N^{o(k)} \) time solution to \( k\)-SUM falsify the Exponential time hypothesis.

The \( k\)-SUM problem can be solved easily in \( \tilde{O}(n^{\lceil k/2 \rceil}) \) time. However, it has been a long-standing open problem to obtain any polynomial improvement over this bound [11] [48]. Patrascu and Williams [49] showed an \( n^{o(k)} \) time algorithm for \( k\)-SUM falsifies the famous *Exponential time hypothesis* (ETH). Hence, our polynomial time reductions also imply that any algorithm optimally solving \( k\)-PBDS or \( k\)-SPM must require \( n^{\Omega(k)} \) time unless ETH fails.

Theorem 3. Under the \( k\)-SUM conjecture, for any \( \varepsilon > 0 \), there does not exist an \( n^{\lceil k/2 \rceil - \varepsilon} \) time algorithm to PBDS problem on vertex-weighted uncertain trees.

An intriguing question is whether the \( k\)-SPM is substantially harder than \( k\)-SUM. For the simple scenario of \( k = 2 \), the 2-SUM problem has an \( O(n \log n) \) time solution. However, it is not immediately clear whether the 2-SPM problem has a truly sub-quadratic time solution (i.e., \( O(n^{2-\varepsilon}) \) time for some \( \varepsilon > 0 \)). We leave this as an open question. This is especially of interest due to the following result.

Theorem 4. Let \( 1 \leq c < 2 \) be the smallest real such that 2-SUM problem has an \( \tilde{O}(n^c) \) time algorithm. Then, there exists an \( \tilde{O}(dn^{c(k/2)+1}) \) time algorithm for optimally solving \( k\)-PBDS on trees with arbitrary edge-probabilities, for some constant \( d > 0 \).

Given the hardness of \( k\)-PBDS on uncertain trees, it is of interest to develop efficient approximation algorithms. Clearly, the expected neighborhood size of a vertex set is a submodular function, and thus it is known that the greedy algorithm yields a \((1 - 1/e)\)-approximation for the PBDS problem in general uncertain graphs [40] [47]. For uncertain trees, we improve this by presenting a fully polynomial-time approximation scheme for PBDS.

Theorem 5. For any integer \( k \), and any \( n \)-vertex tree with arbitrary edge probabilities, a \((1 - \varepsilon)\)-approximate solution to the optimal probabilistic budgeted dominating set (PBDS) of size \( k \) can be computed in time \( \tilde{O}(k^2 \varepsilon^{-1} n^2) \).

We also consider a special case that the number of distinct probability edges on the input uncertain tree is bounded above by some constant \( \gamma \).

Theorem 6. For any integer \( k \), and an \( n \)-vertex tree \( T \) with at most \( \gamma \) edge probabilities, an optimal solution for the PBDS problem on \( T \) can be computed in time \( \tilde{O}(k^{(\gamma+2)} n) \).
We investigate the complexity of PBDS on bounded treewidth graphs. The hardness construction on bounded treewidth graphs is much more challenging. Due to this inherent difficulty, we focus on the uniform scenario, where all edge probabilities $p(e)$ are identical. We refer to this version of the problem as Uni-PBDS. We show that for any $0 < q < 1$, the Uni-PBDS problem with edge-probability $q$ is $W[1]$-hard for the pathwidth parameter of the input uncertain graph $G$. In contrast, the BDS problem (when all probabilities are one) is FPT when parameterized by the pathwidth of the input graph.

**Theorem 7.** Uni-PBDS is $W[1]$-hard w.r.t. the pathwidth of the input uncertain graph.

Then, we consider the Uni-PBDS problem with combined $k$ and treewidth parameters. We show that the Uni-PBDS problem can be formulated as a variant of the Extended Monadic Second order (EMS) problem due to Arnborg et al. [6], to derive an FPT algorithm for the Uni-PBDS problem parameterized by the treewidth of $G$ and $k$.

**Theorem 8.** For any integer $k$, and any $n$-vertex uncertain graph of treewidth $w$ with uniform edge probabilities, $k$-Uni-PBDS can be solved in time $O(f(k,w)n^2)$, and thus is FPT in the combined parameter involving $k$ and $w$. Furthermore, $f(k,w)$ is $kO(w)$.

Finally, using the structural property of dominating sets from Fomin et al. [29], we derive FPT algorithms parameterized by the budget $k$ in apex-minor-free graphs and planar graphs.

**Theorem 9.** For any integer $k$, and any $n$-vertex weighted planar or apex-minor free graph, the Uni-PBDS problem can be solved in time $2^{O(\sqrt{k \log k})}n^{O(1)}$.

**Related Work.** Uncertain graphs have been used in the literature to model the uncertainty among relationships in protein-protein interaction networks in bioinformatics [7], road networks [8, 38] and social networks [23, 10, 53, 55]. Connectivity [9, 10, 54, 59, 52, 51], network flows [28, 32], structural-context similarity [56], minimum spanning trees [27], coverage [16, 35, 36, 45, 44], and community detection [11, 50] are well-studied problems on uncertain graphs. In particular, budgeted coverage problems model a wide variety of interesting combinatorial optimization problems on uncertain graphs. For example, the classical facility location problem [37, 41] is a variant of coverage. As another example, in a classical work, Kempe, Kleinberg, and Tardos [40] study influence maximization problem as an expected coverage maximization problem in uncertain graphs. They consider the scenario where influence propagates probabilistically along relationships, under different influence propagation models, like the Independent Cascade (IC) and Linear Threshold (LT) models, and show that choosing $k$ influencers to maximize the expected influence is NP-hard in the IC model. The coverage problem in the presence of uncertainty was studied extensively also in sensor placement and w.r.t. the placement of light sources in computer vision. A special case of the budgeted coverage problem is the Most Reliable Source (MRS) problem, where given an uncertain graph $G = (V, E, p)$, the goal is to find a vertex $u \in V$ such that the expected number of vertices in $u$’s connected component is maximized. To the best of our knowledge, the computational complexity of MRS is not known, but it is polynomial time solvable on some specific graph classes like trees and series-parallel graphs [12, 20, 21, 22, 33]. Domination is another special kind of coverage and its complexity is very well-studied. The classical dominating set (DS) problem is known to be $W[2]$-hard in general graphs [24], and on planar graphs it is fixed parameter tractable with respect to the size of the dominating-set as the parameter [34]. Further, on $H$-minor-free graphs, the dominating-set problem is solvable in subexponential time [4, 14]. It also admits a linear kernel on $H$-minor-free graphs and graphs of bounded expansion [3, 20, 30, 31, 34]. On graphs of treewidth bounded by
w, the classical dynamic programming approach \cite{15} can be applied to show that the DS problem is FPT when parameterized by w. The Budgeted Dominating Set (BDS) problem is known to be NP-hard \cite{43} as well as \textbf{W}[1]-hard for the budget parameter \cite{24}. Furthermore, a subexponential parameterized algorithm is known for BDS on apex-minor-free graphs \cite{29}. The treewidth-parameterized FPT algorithm for the dominating-set problem can be adapted to solve the BDS problem in time $O(3^w kn)$. In particular, for trees there exists a linear running time algorithm. PBDS was studied as Max-Exp-Cover-1-RF \cite{5} in the survey paper \cite{46}, and given a dynamic programming algorithm on a nice tree decomposition with runtime $2^{O(w\Delta)} n^{O(1)}$, where $\Delta$ is the maximum degree of $G$. The question whether PBDS has a treewidth parameterized FPT algorithm remained unresolved; it is settled in the negative in this work.

2 Preliminaries

Consider a simple undirected graph $G = (V, E)$ with vertex set $V$ and edge set $E$, and let $n = |V|$ and $m = |E|$. Given a vertex subset $S \subseteq V$, the subgraph induced by $S$ is denoted by $G[S]$. For a vertex $v \in V$, $N(v)$ denotes the set of neighbors of $v$ and $N[v] = N(v) \cup \{v\}$ is the closed neighborhood of $v$. Let $deg(v)$ denote the degree of the vertex $v$ in $G$. A vertex subset $S \subseteq V$ is said to be a dominating set of $G$ if every vertex $u \in V \setminus S$ has a neighbor $v \in S$. For an integer $r > 0$, a vertex subset $S \subseteq V$ is said to be an $r$-dominating set of $G$ if for every vertex $u \in V \setminus S$ there exists a vertex $v \in S$ at distance at most $r$ from $u$. A graph $H$ is said to be an apex if it can be made planar by the removal of at most one vertex. A graph $G$ is said to be apex-minor-free if it does not contain as its minor some fixed apex graph $H$. All planar graphs are apex-minor-free as they do not contain as minor the apex graphs $K_{3,3}$ and $K_5$. The notations $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{N}$ denote, respectively, the sets of real, rational, and natural numbers (including 0). For integers $a \leq b$, define $[a, b]$ to be the set $\{a, a + 1, \ldots, b\}$, and for $b > 0$ let $[b] = [1, b]$.

Other than this, we follow standard graph theoretic and parameterized complexity terminology \cite{15 19 25}.

Numerical Approximation. When analyzing our polynomial reductions, we employ numerical analysis techniques to bound the error in numbers obtained as products of an exponential and the root of an integer. We use the following well-known bound on the error in approximating an exponential function by the sum of the lower degree terms in the series expansion.

\textbf{Lemma 10.} \cite{15} For $z \in [-1, 1]$, $e^z$ can be approximated using the Lagrange remainder as

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots + \frac{z^Q}{Q!} + R_Q(z)$$

where $|R_Q(z)| \leq e/(Q + 1)! \leq 1/2^Q$.

We use the following lemma for bounding the error in multiplying approximate values.

\textbf{Lemma 11.} For any set $\{d_1, \ldots, d_k\}$ of $k$ reals in the range $[0, 1]$,

$$\prod_{i \in [k]} (1 - d_i) \geq 1 - \sum_{i \in [k]} d_i.$$  

\textbf{Proof.} The proof is by induction on $k$. The base case of $k = 1$ trivially holds. For any two reals $a, b \in [0, 1]$, $(1 - a)(1 - b) \geq 1 - (a + b)$. Applying this result iteratively yields that for
any $k \geq 1$, if $\prod_{i \in [k-1]} (1 - d_i) \geq 1 - (\sum_{i \in [k-1]} d_i)$, then

$$\prod_{i \in [k]} (1 - d_i) \geq \left( 1 - \left( \sum_{i \in [k-1]} d_i \right) \right) (1 - d_k) \geq 1 - \sum_{i \in [k]} d_i.$$ 

The claim follows.

Tree Decomposition. A Tree decomposition of an undirected graph $G = (V, E)$ is a pair $(T, X)$, where $T$ is a tree whose set of nodes is $X = \{ X_i \subseteq V \mid i \in V(T) \}$, such that

1. for each edge $u \in V$, there is an $i \in V(T)$ such that $u \in X_i$,
2. for each edge $uv \in E$, there is an $i \in V(T)$ such that $u, v \in X_i$, and
3. for each vertex $v \in V$ the set of nodes $\{ i \mid v \in X_i \}$ forms a subtree of $T$.

The width of a tree decomposition $(T, X)$ equals $\max_{i \in V(T)} |X_i| - 1$. The treewidth of a graph $G$ is the minimum width over all tree decompositions of $G$.

A tree decomposition $(T, X)$ is nice if $T$ is rooted by a node $r$ with $X_r = \emptyset$ and every node in $T$ is either an insert node, forget node, join node or leaf node. Thereby, a node $i \in V(T)$ is an insert node if $i$ has exactly one child $j$ such that $X_j = X_j \cup \{ v \}$ for some $v \notin X_j$; it is a forget node if $i$ has exactly one child $j$ such that $X_j = X_j \setminus \{ v \}$ for some $v \in X_j$; it is a join node if $i$ has exactly two children $j$ and $h$ such that $X_j = X_h$; and it is a leaf node if $X_i = \emptyset$. Given a tree decomposition of width $w$, a nice tree decomposition of width $w$ and $O(w n)$ nodes can be obtained in linear time $[22]$.

A tree decomposition $(T, X)$ is said to be a path decomposition if $T$ is a path. The pathwidth of a graph $G$ is minimum width over all possible path decompositions of $G$. Let $pw(G)$ and $tw(G)$ denote the pathwidth and treewidth of the graph $G$, respectively. The pathwidth of a graph $G$ is one lesser than the minimum clique number of an interval supergraph $H$ which contains $G$ as an induced subgraph. It is well-known that the maximal cliques of an interval graph can be linearly ordered so that for each vertex, the maximal cliques containing it occur consecutively in the linear order. This gives a path decomposition of the interval graph. A path decomposition of the graph $G$ is the path decomposition of the interval supergraph $H$ which contains $G$ as an induced subgraph. In our proofs we start with the path decomposition of an interval graph and then reason about the path decomposition of graphs that are constructed from it.

3 Hardness Results on Trees

3.1 $k$-SPM hardness

We first show that the $k$-SUBSET $\Sigma - \Pi$ MAXIMIZATION $(k$-SPM) problem is NP-hard by a reduction from the $k$-SUM problem. Let $\langle X, k \rangle$ with $X = \{ x_1, \ldots, x_N \}$ be an instance of the $k$-SUM problem. Let $L = 1 + \max_{i \in [N]} |x_i|$.

Denote by $\langle A, k, t \rangle$ an instance of the $k$-SPM problem. Given an instance $\langle X, k \rangle$ of $k$-SUM, we compute the array $A(X) = \{ (\tilde{x}_i, \tilde{y}_i) \mid i \in [N] \}$ of the $k$-SPM problem as follows. For $1 \leq i \leq N$, set $\tilde{x}_i := (L + x_i)/(kL)$.

Let $Q = 3 \log_2(kL)$. For $i \in [N]$, define $y_i = e^{\tilde{x}_i/(kL)}$, and let $\tilde{y}_i$ be a rational approximation of $y_i$ that is computed using Lemma $[10]$ such that $0 \leq y_i - \tilde{y}_i \leq 1/2^Q$. The new instance of the $k$-SPM problem is $\langle A(X), k, t = 0 \rangle$.

Observe that for each $i \in [N]$, $\tilde{y}_i \geq y_i - 1/2^Q \geq e^{-1/k} - 1/(L^2) \geq 1/2$, for $k \geq 3$. Thus, the elements of $A(X)$ are positive rationals. The next lemma provides a crucial property of any set $S$ of vertices of size $k$.
Lemma 12. Let $\lambda = (2kL)^{-2}$. For each $S \subseteq [N]$ of size $k$,

$$0 \leq \prod_{i \in S} y_i - \prod_{i \in S} \tilde{y}_i \leq \lambda.$$ 

Proof. Let $\alpha = 1/(kL)^3$. We have:

$$\prod_{i \in S} y_i - \prod_{i \in S} \tilde{y}_i \leq \prod_{i \in S} y_i - \prod_{i \in S} (y_i - \alpha)$$

$$= \prod_{i \in S} y_i \left(1 - \prod_{i \in S} \left(1 - \frac{\alpha}{y_i}\right)\right)$$

$$\leq \prod_{i \in S} y_i \left(\sum_{i \in S} \frac{\alpha}{y_i}\right)$$

$$\leq e^{\sum_{i \in S} x_i/(kL)} \alpha k^{1/k} \leq \alpha k^2 \leq \frac{1}{(kL)^3},$$

where the second inequality is obtained by Lemma 11. The claim follows.

We now establish the correctness of the reduction.

Theorem 13. The $k$-SUM problem is polynomial-time reducible to $k$-SPM.

Proof. Let $M = \sum_{i \in [N]} (x_i + L)$. Define a real valued function $F(z) = z - e^{-1+z}$ with domain $[0, M/(kL)]$. Observe that $F(1) = 0$ and the derivative is $F'(1) = 0$. The function $F(\cdot)$ is clearly concave, which indicates that:

(i) $F(z) \leq 0$, for each $z \in [0, M/(kL)]$,

(ii) $F(z)$ obtains its unique maximum at $z = 1$, where its value is 0, and

(iii) When restricted to the values in the set $\{z/(kL) \mid z \in [0, M] \text{ is an integer}\}$, $F(z)$ obtains its second largest value at $z = 1 - 1/(kL)$.

For any $S \subseteq [N]$, denote $z_S = \sum_{i \in S} \tilde{x}_i$. For a set $S \subseteq [N]$ of size $k$, we have:

$$\sum_{i \in S} \tilde{x}_i - \prod_{i \in S} y_i = z_S - e^{\sum_{i \in S} x_i/(kL)} = z_S - e^{\sum_{i \in S} \tilde{x}_i/(kL) - 1/k}$$

$$= z_S - e^{-1} e^{z_S/(kL)} = F(z_S).$$

By combining Lemma 12 and Eq. 2, we obtain the following.

$$F(z_S) \leq \sum_{i \in S} \tilde{x}_i - \prod_{i \in S} \tilde{y}_i \leq F(z_S) + \lambda.$$ 

On the other hand, for any set $S \subseteq [N]$ of size $k$ for which $F(z_S) < 0$, we have $F(z_S) \leq F(1 - 1/(kL)) = (1 - 1/(kL) - e^{-1/(kL)})$. Further,

$$1 - \frac{1}{kL} - e^{-1/(kL)} \leq (1 - \frac{1}{kL}) - \frac{1}{2(kL)^2} - \frac{1}{6(kL)^2} \leq -\frac{1}{4(kL)^2} = -\lambda.$$ 

So, for a set $S$, $\sum_{i \in S} \tilde{x}_i - \prod_{i \in S} \tilde{y}_i \geq 0$ if and only if $\sum_{i \in S} \tilde{x}_i = 1$, or equivalently $\sum_{i \in S} x_i = 0$. It follows that $\langle X, k \rangle$ is a yes instance of the $k$-SUM problem if and only if $\langle A(X), k, t = 0 \rangle$ is a yes instance of the $k$-SPM problem. The time to compute $\tilde{x}_i$ and $\tilde{y}_i$ from $x_i$ is polynomial in $Q \cdot \log_2(kL)$, for $1 \leq i \leq N$. Thus, the time-complexity of our reduction is $N \cdot \log_2(kL)$, which is at most polynomial in $N$ as long as $L = 2^{O(N)}$. Hence, the $k$-SUM problem is polynomial-time reducible to the $k$-SPM problem.

Proof of Theorem 2. The reduction given in the proof of Theorem 13 is a parameter preserving reduction for the parameter $k$. That is, the parameters in the instances of the $k$-SUM and the $k$-SPM problem are same in values and the constructed instance of the $k$-SPM problem is of size polynomial in the input size of the $k$-SUM instance. Thus, the
In order to prove our claim, we provide a reduction from the $k$-SUM problem is known to be $W[1]$-hard for the parameter $k$ \cite{2,21}, the $k$-SPM problem is also $W[1]$-hard for the parameter $k$. Further, it is known that under the Exponential time hypothesis (ETH), there cannot exist an $o(N^k)$ time solution for the $k$-SUM problem \cite{40}, so it follows that under ETH there is no $o(N^k)$ time algorithm for $k$-SPM as well.

### 3.2 Hardness of PBDS on Uncertain Trees

In this subsection, we show the hardness results for the PBDS problem on trees, establishing Theorem 1. In order to achieve this, we present a polynomial time reduction from $k$-SPM to PBDS on unweighted trees.

**Proof of Theorem 1.** In order to prove our claim, we provide a reduction from $k$-SPM to PBDS. Given an instance $\langle A = ((x_1, y_1), \ldots, (x_N, y_N)), k, t \rangle$ of the $k$-SPM problem, where $t$ is a rational, an equivalent instance of the PBDS problem is constructed as follows. Let $n = N^2 + N + 1$. Construct an uncertain tree $T = (V, E, p)$, where the vertex set $V$ consists of three disjoint sets, namely, $A = \{a_0\}$, $B = \{b_1, \ldots, b_N\}$, and $C = \{c_{11}, c_{12}, \ldots, c_{NN}\}$. (see Figure 1). Note that the uncertain tree $T$ is considered to be unweighted or unit weight on the vertices. The vertex $a_0$ is connected by edges to the vertices in $B$. For each $1 \leq i \leq n$, the vertex $b_i$ is connected by edges to the vertices $c_{i1}, \ldots, c_{iN}$. Let $X_{max} = \max\{x_1, x_2, \ldots, x_N\}$ and $Y_{max} = \max\{y_1, y_2, \ldots, y_N\}$. To complete the construction, define the probability function $p : E \rightarrow [0, 1]$ as follows:

$$p(v \bar{v}) = \begin{cases} 
   r_i = 1 - (y_i)/(X_{max} \cdot Y_{max}), & \text{if } v \bar{v} = a_0b_i \text{ for } 1 \leq i \leq N, \\
   q_i = x_i/(X_{max} \cdot Y_{max})^k, & \text{if } v \bar{v} = b_ib_{c_{i1}} \text{ for } 1 \leq i \leq N, \\
   1, & \text{otherwise.}
\end{cases}$$

Since $x_i, y_i \geq 0$ for each $1 \leq i \leq n$, we have that $p(v, \bar{v}) \in [0, 1]$ is rational for every $(v, \bar{v}) \in E$. This completes the construction of the instance for the PBDS problem. We show that the given instance $\langle A, k, t \rangle$ is a yes instance of $k$-SPM if and only if $T$ has a set $S$ of size $k$ such that $C(V, S) \geq 1 + (N - 1)k + t/(X_{max}Y_{max})^k$.

![Figure 1 Illustration of the lower bound of Theorem 1. Here $p_i = 1 - y_i/(X_{max} \cdot Y_{max})$ and $q_i = x_i/(X_{max} \cdot Y_{max})^k$ for $i \in [N]$.](image)

Let $S_{opt}$ be a set of size $k$ maximizing $C(V, S_{opt})$ in $T$. We show $S_{opt} \subseteq B$. Assume, to the contrary, that there exists some $z \in S_{opt}$ satisfying $z \notin B$. Consider $i \in [N]$ such that none of the vertices $c_{i1}, \ldots, c_{iN}$ lie in $S_{opt}$. Such $i$ must exist since $|S_{opt}| = k$. If $z \in C$, then replacing $z$ by $b_i$ results in a set $S' = S_{opt} \setminus \{z\} \cup \{b_i\}$ such that $C(V, S') \geq C(V, S_{opt}) + N - 3$, contradicting the optimality of $S_{opt}$. Hence, $S_{opt}$ must be contained in $A \cup B$. In this case, $z$ must be $a_0$. Now the set $S' = S_{opt} \setminus \{z\} \cup \{b_i\}$ is such that $C(V, S') \geq C(V, S_{opt}) + N/2 - 2$. Since $N \geq 6$, this contradicts the optimality of $S_{opt}$. It follows that $S_{opt} \subseteq B$. 

Next, consider a set \( S \subseteq B \) of size \( k \), and let \( I_S = \{ i \in [N] \mid b_i \in S \} \). We have

\[
C(V, S) = (1 - \prod_{i \in I_S} (1 - p(a_0, b_i))) + (\sum_{i \in I_S} (p(b_i, c_{01}) + N - 1))
= 1 + (N - 1)k + \sum_{i \in I_S} x_i/(X_{\text{max}} \cdot Y_{\text{max}})^k - \prod_{i \in I_S} y_i/(X_{\text{max}} \cdot Y_{\text{max}})
= 1 + (N - 1)k + (X_{\text{max}} \cdot Y_{\text{max}})^{-k} (\sum_{i \in I_S} x_i - \prod_{i \in I_S} y_i).
\]

This formulation of the coverage function shows that the given instance \((A, k, t)\) is a yes instance of \( k\)-SPM if and only if \( T \) has a set \( S \) of vertices of size \( k \) such that \( C(V, S) \geq 1 + (N - 1)k + t/(X_{\text{max}} Y_{\text{max}})^k \). Thus, the \( k\)-SPM problem is reduced in polynomial time to PBDS on unweighed trees.

It remains to prove NP-hardness, W[1]-hardness, and \( n^{o(k)} \) lower-bound under ETH. Note that the above reduction is a parameterized preserving reduction for the parameter \( k \).

That is, the parameter \( k \) in the \( k\)-SPM problem is the solution size (also called \( k \)) parameter for the PBDS problem. Since the \( k\)-SPM problem (i) is NP-hard, (ii) is W[1]-hard for the parameter \( k \), and (iii) cannot have time complexity \( n^{o(k)} \) under the Exponential time hypothesis (by Theorem 2), it follows that the same hardness results hold for PBDS as well.

Therefore, the PBDS problem on uncertain trees (i) is NP-hard, (ii) is W[1]-hard for the parameter \( k \), and (iii) cannot have time complexity \( n^{o(k)} \) if ETH holds true.

The \( k\)-SUM conjecture \( \cite{1} \) \( \cite{48} \) states that the \( k\)-SUM, for the parameters \( N \) and \( k \), requires at least \( N^{\lceil k/2 \rceil - \epsilon(1)} \) time.

**Conjecture 14 (\( k\)-SUM Conjecture).** There do not exist a \( k \geq 2 \), an \( \epsilon > 0 \), and an algorithm that succeeds (with high probability) in solving \( k\)-SUM in \( N^{\lceil k/2 \rceil - \epsilon} \) time.

**Proof of Theorem 3.** Consider the uncertain tree \( T \) constructed in the proof of Theorem 1. We set \( n_0 = 0 \). Modify the original construction of \( T \) by deleting the \( N - 2 \) vertices: \( c_{i3}, c_{i4}, \ldots, c_{iN} \), and setting \( \omega_{c_{i2}} = N - 1 \), for \( 1 \leq i \leq N \). Thus the tree contains exactly \( n = 3N + 1 \) vertices. Now, \( k\)-SUM is reducible to \( k\)-SPM, and \( k\)-SPM is reducible to PBDS, both in polynomial time, and, moreover the parameter \( k \) remains unaltered and the size of problem grows by at most constant factor. This shows that, for \( \epsilon > 0 \), an \( n^{\lceil k/2 \rceil - \epsilon} \) time algorithm to weighted PBDS implies an \( N^{\lceil k/2 \rceil - \Omega(\epsilon)} \) time algorithm to \( k\)-SUM, thereby, falsifying the \( k\)-SUM conjecture.

Note that since the PBDS problem is NP-hard on trees, it is also para-NP-hard \( \cite{15} \) \( \cite{25} \) for the treewidth parameter.

4. **Hardness of Uni-PBDS for the pathwidth parameter**

In this section, we show that even for the restricted case of uniform probabilities, the Uni-PBDS problem is W[1]-hard for the pathwidth parameter, and thus also for treewidth (Theorem 4). This is shown by a reduction from the Multi-Colored Clique problem to the Uni-PBDS problem. It is well-known that the Multi-Colored Clique problem is W[1]-hard for the parameter solution size \( \cite{24} \).

**Multi-Colored Clique**

**Input:** A positive integer \( k \) and a \( k \)-colored graph \( G \).
**Parameter:** \( k \)
**Question:** Does there exist a clique of size \( k \) with one vertex from each color class?

Let \((G = (V, E), k)\) be an input instance of the Multi-Colored Clique problem, with \( n \) vertices and \( m \) edges. Let \( V = (V_1, \ldots, V_k) \) denote the partition of the vertex set \( V \) in the
input instance. We assume, without loss of generality, $|V_i| = n$ for each $i \in [k]$. For each $1 \leq i \leq k$, let $V_i = \{u_{i,\ell} \mid 1 \leq \ell \leq n\}$.

### 4.1 Gadget based reduction from Multi-Colored Clique

Let $(G, k)$ be an instance of the MULTI-COLORED CLIQUE problem. For any probability $0 < p < 1$, and for any integer $f$ such that $f > \max\{kn, n + k^2/p\}$, our reduction constructs an uncertain graph $G$. The output of the reduction is an instance $(G', k', t')$ of the Uni-PBDS problem where each edge has probability\( p \) and \( t' = (kn + m)(n + 1)f + np + 1 + 2(1 - (1 - p)^n)) + 4(1 - (1 - p)^n+1) \). In the presentation below, we show that this choice of $k'$ and $t'$ ensures that there is a set of size $k'$ with expected domination at least $t'$ in $G$ if and only if $G$ has a multi-colored clique of size $k$.

We first construct a gadget graph to represent the vertices and edges of the input instance of the MULTI-COLORED CLIQUE problem. We construct two types of gadgets, $D$ and $I$ in the reduction, illustrated in Figure\( \Box \). The gadget $I$ is the primary gadget and $D$ is a secondary gadget used to construct $I$. When we refer to a gadget, we mean the primary gadget $I$ unless the gadget $D$ is specified. For each vertex and edge in the given graph, our reduction has a corresponding gadget. The gadget $D$ is defined as follows.

**Gadget of type $D$.** Given a pair of vertices $u$ and $v$, the gadget $D_{u,v}$ consists of vertices $u$, $v$, and $f$ additional vertices. The vertices $u$ and $v$ are made adjacent to every other vertex. We refer to the vertices $u$ and $v$ as heads, and remaining vertices of $D_{u,v}$ as tails, and $u$ are $v$ are said to be connected by the gadget $D_{u,v}$.

**Observation 15.** The pathwidth of a gadget of type $D$ is 2.

**Gadget of type $I$.** We begin the construction of the gadget with $2n$ vertices partitioned into two sets where each partition contains $n$ vertices. Let $A = \{a_1, \ldots, a_n\}$ and $C = \{c_1, \ldots, c_n\}$ be this partition. For each $i \in [n]$, vertices $a_i$ and $c_i$ are connected by the gadget $D_{a_i,c_i}$. Let $h_a$ and $h_c$ be two additional vertices connected by the gadget $D_{h_a,h_c}$. The vertices in the sets $A$ and $C$ are made adjacent to $h_a$ and $h_c$, respectively. This completes the construction of the gadget. In the reduction, a gadget of type $I$ is denoted by the symbol $I$ along with an appropriate subscript based on whether the gadget is associated with a vertex or an edge.

**Claim 16.** The pathwidth of a gadget of type $I$ is at most 4.

**Proof.** We observe that the removal of the vertices $h_a$ and $h_c$ results in a graph in which for each $i \in [n]$, there is a connected component consisting $a_i$ and $c_i$ which are the heads of a gadget of type $D$. Each component is a gadget of type $D$ and from Observation 15 is of pathwidth 2. Let $(T', X')$ be the path decomposition of $I - \{h_a, h_c\}$ with width 2. Thus adding $h_a$ and $h_c$ into all the bags of the path decomposition $(T', X')$ gives a path decomposition for the gadget $I$, and thus the pathwidth of the gadget $I$ is at most 4.

**Description of the reduction.** For $1 \leq i < j \leq k$, let $E_{i,j} = \{xy \mid x \in V_i, y \in V_j\}$ be the set of edges with one end point in $V_i$ and the other in $V_j$ in $G$. For each $1 \leq i < j \leq k$, the graph $G$ has an induced subgraph $G_i$ corresponding to $V_i$, and has an induced subgraph $G_{i,j}$ for the edge set $E_{i,j}$. We refer to $G_i$ as a vertex-partition block and $G_{i,j}$ as an edge-partition block. Inside block $G_i$, there is a gadget of type $I$ for each vertex in $V_i$, and in the block $G_{i,j}$, there is a gadget for each edge in $E_{i,j}$. For a vertex $u_{i,j}, I_x$ denotes the gadget corresponding
to \(u_{i, \ell}\) in the partition \(V_i\), and for an edge \(e\), \(I_e\) denotes the gadget corresponding to \(e\). The blocks are appropriately connected by connector vertices which are defined below.

We start by defining the structure of a block denoted by \(B\). The definition of the block applies to both the vertex-partition block and the edge-partition block. A block \(B\) consists of gadgets and additional vertices as follows (See Figure 3).

The block \(B\) corresponding to the vertex-partition block \(G_i\) for any \(i \in [k]\) is described as follows: for each \(\ell \in [n]\), add a gadget \(I_\ell\) to the vertex-partition block \(G_i\), to represent the vertex \(u_{i, \ell} \in V_i\). In addition to the gadgets, we add \(n + 1\) vertices to the block \(B\) described as follows: Let \(F(B) = \{b_1, \ldots, b_n, d_i\}\) be the set of additional vertices that are added to the block \(B\). For each \(\ell \in [n]\), the vertices in the set \(C\) of the gadget \(I_\ell\) in the block \(B\) are made adjacent to \(b_\ell\). For each \(\ell \in [n]\), the vertices in the set \(A\) of the gadget \(I_\ell\) in the block \(B\) are made adjacent to \(d_i\).

The block \(B\) corresponding to the edge-partition block \(G_{i,j}\) for any \(1 \leq i < j \leq k\) is described as follows: for each \(e \in E_{i,j}\), add a gadget \(I_e\) in the edge-partition block \(G_{i,j}\), to represent the edge \(e\). In addition to the gadgets, we add \(|E_{i,j}| + 1\) vertices to the block \(B\) described as follows: Let \(F(B) = \{b_e | e \in E_{i,j}\} \cup \{d_{i,j}\}\) be the set of additional vertices that are added to the block \(B\). For each \(e \in E_{i,j}\), the vertices in the set \(C\) of the gadget \(I_e\) in the block \(B\) are made adjacent to \(b_e\). For each \(e \in E_{i,j}\), the vertices in the set \(A\) of the gadget \(I_e\) in the block \(B\) are made adjacent to \(d_{i,j}\).

The blocks defined above are connected by the connector vertices described next. These connector vertices are used to connect the edge-partition blocks and vertex-partition blocks, and thus ensure that each edge in \(G\) is appropriately represented in \(G\). Let \(R = \{r_{i,j}^i, s_{i,j}^i, r_{i,j}^j, s_{i,j}^j | 1 \leq i < j \leq k\}\) be the connector vertices. The blocks are connected based on the cases described below. The connections involving the \(I\) gadgets in two vertex-partition blocks and an \(I\) gadget in an edge-partition block is illustrated in Figure 4. First, we describe the connection of vertex-partition blocks corresponding \(V_i\) and \(V_j\) to the appropriate connector vertices. Following this, we describe the connection of the two vertex-partition blocks to the edge-partition block corresponding to \(E_{i,j}\) through the appropriate connector vertices.

For each \(i \in [k]\), each \(i < j \leq k\) and each \(\ell \in [n]\),

- for each \(1 \leq t \leq \ell\), \(a_t\) in the gadget \(I_\ell\) of \(G_i\) is made adjacent to \(s_{i,j}^t\), and...
for each $\ell \leq t \leq n$, $a_t$ in the gadget $I_\ell$ of $G_i$ is made adjacent to the vertex $r_{i,j}^t$.

For each $i \in [k]$, each $1 \leq j < i$ and each $\ell \in [n]$,

- for each $1 \leq t \leq \ell$, $a_t$ in the gadget $I_\ell$ of $G_i$ is made adjacent to the vertex $s_{i,j}^t$, and
- for each $\ell \leq t \leq n$, $a_t$ in the gadget $I_\ell$ of $G_i$ is made adjacent to the vertex $r_{i,j}^t$.

Now, we describe the edges to connect the $I$ gadgets in the vertex-partition blocks $G_i$ and $G_j$ and to the appropriate $I$ gadgets in the edge-partition block $G_{i,j}$. For each $1 \leq i < j \leq k$, and for each $e = u_{i,x}u_{j,y} \in E_{i,j}$,

- for each $1 \leq t \leq x$, $a_t$ in the gadget $I_x$ of $G_{i,j}$ is made adjacent to the vertex $r_{i,j}^t$, and
- for each $x \leq t \leq n$, $a_t$ in the gadget $I_x$ of $G_{i,j}$ is made adjacent to the vertex $s_{i,j}^t$.
- for each $1 \leq t \leq y$, $a_t$ in the gadget $I_y$ of $G_{i,j}$ is made adjacent to the vertex $r_{i,j}^t$, and
- for each $y \leq t \leq n$, $a_t$ in the gadget $I_y$ of $G_{i,j}$ is made adjacent to the vertex $s_{i,j}^t$.

This completes the construction of the graph $G$ with $O(mn^2)$ vertices and $O(mn^3)$ edges.

\textbf{Claim 17.} The pathwidth of a block $B$ is at most 6.

\textbf{Proof.} Without loss of generality, assume that the block $B$ is a vertex partition block $G_i$ for any $i \in [n]$. If we remove the vertex $d_i$ from the block $B$, then the resulting graph is a disjoint collection of gadgets of type $I$ with an additional vertex. See Figure 3 for an
illustration. By Claim[16] the pathwidth of a gadget is 4. Therefore, for each \( \ell \in \{n\}, \) adding the additional vertex \( b_\ell \) to all bags of the path decomposition of the gadget \( T_\ell \) gives a path decomposition for the connected component containing the gadget. Thus, each connected component is of pathwidth at most 5. Let \( (T', X') \) be a path decomposition of \( B - \{d_1\} \) with pathwidth 5. Thus, adding \( d_1 \) into all bags of \( (T', X') \) gives a path decomposition for the block \( B \), and thus the pathwidth of the block is at most 6.

The following lemma bounds the pathwidth of the graph \( G \) by a polynomial in \( k \).

**Lemma 18.** The pathwidth of the graph \( G \) is at most \( 4k^2 + 6 \).

**Proof.** Removal of the connector vertices \( R \) from \( G \) results in a collection of disjoint blocks. By Claim[17] the pathwidth of a block is 6. Let \( (T', X') \) be a path decomposition of \( G - R \) with pathwidth 6. Therefore, adding all connector vertices to the path decomposition \( (T', X') \) gives a path decomposition for the graph \( G \) with pathwidth at most \( 4k^2 + 6 \).

4.2 Properties of a feasible solution for the Uni-PBDS instance \((G, k', t')\) output by the reduction

We start with the observation that in a reduced instance, the expected domination achieved by a set of size \( k' \) is at most \( t' \). We then prove properties of a feasible solution for the instance \((G, k', t')\).

**Observation 19.** The maximum expected domination that can be achieved by any vertex set of size \( k' \) in \( G \) is \( t' \).

Let \( S \) be a feasible solution for the Uni-PBDS instance \((G, k', t')\). We state a set of canonical properties of the set \( S \) of size \( k' \) and which achieves the maximum value of \( C(V(G), S) \geq t' \). All the observations below follow crucially from the fact that \( |S| = k' = (n + 1)(kn + m) \) and \( C(V(G), S) \geq t' = (kn+m)((n+1)fp+np+1+2(1-(1-p)^n))+4(k^2)(1-(1-p)^{n+1}). \)

- Observe that the vertices in the sets \( A, C \) and \( \{h_a, h_c\} \) are of degree at least \( f \). This is because they are all heads in a gadget of type \( D \). Thus, the vertices of \( A, C \), and \( \{h_a, h_c\} \) have degree greater than all the other vertices in \( G \). We refer to these vertices as high degree vertices and to the other as low degree vertices.
- There are \( kn + m \) gadgets of type \( I \) and each gadget has \( n \) vertices in the sets \( A \) and \( C \), respectively. Therefore, the number of high degree vertices in \( G \) is \( 2(n+1)(kn + m) \), and \( k' = (n + 1)(kn + m) \). Also, the number of \( D \) type gadgets is \( k' \). In the following points, we show that from each gadget exactly one head should be in \( S \).
- If the set \( S \) contains a low degree vertex, then it is possible to replace it with a high degree vertex which is not in \( S \). Since the edge probabilities are all identical, the resulting expected domination does not decrease.
- Tails of the gadgets of type \( D \) are vertices with degree two, and thus are low-degree vertices. Therefore, the set \( S \) does not contain any tails.
- Let \( B \) be a block in \( G \). The vertices in the set \( F(B) \) have degree max\{\( mn, n^2 \)\}, and are low-degree vertices. Thus, we can assume that \( S \) does not contain any vertex in \( F(B) \).
- The connector vertices are also low degree vertices. We conclude that \( S \) does not contain a connector vertex.
- There are \( kn + m \) gadgets of type \( I \) and \( k' = (n + 1)(kn + m) \). Therefore, \( S \) contains \( n + 1 \) vertices from each gadget of type \( I \). Based on the observations above, it follows that \( S \) contains vertices from \( A, C \), and \( \{h_a, h_c\} \) from each gadget \( I \) of type \( I \).
For every gadget of type $\mathcal{D}$, at least one of the head vertex must be in $S$. Suppose there exists a gadget $D$ such that both heads are not in $S$, then we cannot dominate the tail vertices of $D$. Since the number of gadgets of type $\mathcal{D}$ in $\mathcal{G}$ is the same as $k'$, there exists a gadget as $D'$ such that both heads are in $S$. Let us consider the set $S'$ obtained by replacing a head $\alpha$ in $V(D') \cap S$ by a head $\beta$ in $D$. Then, we get $C(V(\mathcal{G}), S') \geq C(V(\mathcal{G}), S) + fp - (f + n + k^2)(p - p^2)$ where $fp - (f + n + k^2)(p - p^2) > 0$ since $f > (n + k^2)/p$. This contradicts Observation 19 by which $C(V(\mathcal{G}), S) = t'$ is the maximum value possible by a set of size $k'$.

Since $S$ achieves an expected domination of $t'$, it follows that each gadget of type $\mathcal{I}$ selects exactly either $A \cup \{h_i\}$ or $C \cup \{h_a\}$ to achieve a part of the first term in the expression for $t'$. Further, the additional term of $2(1 - (1 - p)^n)$ for each gadget comes from covering the vertex named $d$ in a block $B$ which is adjacent to the set $A$ of each gadget in $B$, and a vertex named $b$ which is adjacent to the set $C$.

We formalize the observations below.

\> **Claim 20.** For each tail vertex $x$ in $\mathcal{G}$, $N(x) \cap S$ is non empty.

\> **Claim 21.** For every block $B$ in $\mathcal{G}$, and each gadget $I$ of type $\mathcal{I}$ in $B$, $S \cap V(I) \subseteq A \cup C \cup \{h_a, h_c\}$.

\> **Claim 22.** For every block $B$ in $\mathcal{G}$, and each gadget $I$ of type $\mathcal{I}$ in $B$, either $A \cup \{h_c\} = S \cap V(I)$ or $C \cup \{h_a\} = S \cap V(I)$.

\> **Claim 23.** In every block $B$ in the graph $\mathcal{G}$, there exists a unique gadget $I$ such that $A \cup \{h_c\} = S \cap V(I)$.

\> **Claim 24.** If the set $S$ satisfies Claim 23, then

$$C(V(\mathcal{G}) \setminus R, S) = (kn + m)((n + 1)fp + n + np + 1 + 2(1 - (1 - p)^n)).$$

**Proof.** Let $B$ be a block in $\mathcal{G}$. Let $I$ be a gadget of type $\mathcal{I}$ in $B$. Either the set $A$ or the set $C$ in $I$ is in $S$. Further, in every gadget of type $\mathcal{D}$ in $I$, exactly one head is in $A$ and another head is in $C$. The other two heads of a gadget of type $\mathcal{D}$ are the vertices in the set $\{h_a, h_c\}$. Therefore, in every gadget of type $\mathcal{D}$ in the graph $\mathcal{G}$, exactly one of the heads is in $S$. For each pair $(u, v)$ such that there is a gadget $D_{u,v}$ in $I$, the expected domination of the set $V(D_{u,v}) \setminus \{u, v\}$ by the set $S$ is $fp$. The gadget $I$ has $n + 1$ gadgets of type $\mathcal{D}$. The expected domination of $V(I)$ by the set $S$ is given as follows:

$$C(V(I), S) = \sum_{(u,v) \in I} C(V(D_{u,v}) \setminus \{u, v\}, S) + C(A \cup C \cup \{h_a, h_c\}, S)$$

$$= (n + 1)fp + (n + 1) + np + (1 - (1 - p)^n)$$

Then, the expected domination of $V(B)$ by the set $S$ is the sum of the expected domination contributed by the gadgets of type $\mathcal{I}$ and the domination due to $F(B)$ by the set $S$. There exists a unique gadget $I$ in the block $B$ such that the vertex set $A$ is added to $S$. In the remaining gadgets, the vertex set $C$ is added to $S$. We compute the value $C(V(B), S)$ based on type of the block $B$ Let $B$ be a vertex-partition block $\mathcal{G}_i$ for some $i \in [k]$. All vertices in $F(\mathcal{G}_i)$ except $b_{xi}$ have $n$ neighbors in $S$. Therefore, the value $C(V(\mathcal{G}_i), S)$ is given as follows:

$$C(V(\mathcal{G}_i), S) = \sum_{\ell \in [n]} C(V(\mathcal{I}_\ell), S) + C(F(\mathcal{G}_i), S)$$

$$= n((n + 1)fp + (n + 1) + np + (1 - (1 - p)^n)) + n(1 - (1 - p)^n)$$
Let $B$ be an edge-partition block $\mathcal{G}_{i,j}$ for some $1 \leq i, j \leq k$. All vertices in $F(\mathcal{G}_{i,j})$ except $b_{u_{i,j}, u_{j,i}}$ have $n$ neighbors in $S$. Therefore, the value $C(V(\mathcal{G}_{i,j}), S)$ is given as follows:

$$C(V(\mathcal{G}_{i,j}), S) = \sum_{e \in E_{i,j}} C(V(I_e), S) + C(F(\mathcal{G}_{i,j}), S)$$

$$= |E_{i,j}|((n + 1)fp + (n + 1) + np + (1 - (1 - p)^n)) + |E_{i,j}|((1 - (1 - p)^n))$$

Finally, the expected domination of $V(G) \setminus R$ by the set $S$ is computed as follows:

$$C(V(G) \setminus R, S) = \sum_{i \in [k]} C(V(G_i), S) + \sum_{1 \leq i < j \leq k} C(V(\mathcal{G}_{i,j}), S)$$

$$= \sum_{i \in [k]} n((n + 1)fp + (n + 1) + np + 2(1 - (1 - p)^n))$$

$$+ \sum_{1 \leq i < j \leq k} |E_{i,j}|((n + 1)fp + (n + 1) + np + 2(1 - (1 - p)^n))$$

$$= (m + kn)((n + 1)fp + (n + 1) + np + 2(1 - (1 - p)^n))$$

Hence the claim is proved.

\[ \triangleright \] Claim 25. If the set $S$ satisfies Claim 23 then for each $1 \leq i < j \leq k$,

$$C(\{r_{i,j}, s_{i,j}\}, S) \leq 2(1 - (1 - p)^{n+1}).$$

**Proof.** Since $S$ satisfies Claim 23, it follows that for each block there is a unique gadget $I$ in the block such that the set $A$ inside the gadget $I$ is contained in $S$. For the vertex partition block $\mathcal{G}_i$, let $I_{e_i}$ be this unique gadget. Clearly, $x_i$ is a vertex in $V_i$. Similarly, for the edge partition block $\mathcal{G}_{i,j}$, let $I_{u_{i,j}}$ be the corresponding unique gadget. It is clear that $u_{i,j} \in E_{i,j}$. Let $A$ be the union of the sets $A$ in the above mentioned gadgets. By construction of $\mathcal{G}_i$, the neighbors of the vertices $r_{i,j}^i$ and $s_{i,j}^i$ in $S$ are subsets of the set $A$. More precisely, $|N(s_{i,j}^i) \cap \hat{A}| = x_i + n - z + 1$ and $|N(r_{i,j}^i) \cap \hat{A}| = n - x_i + 1 + z$. Therefore,

$$C(\{r_{i,j}^i, s_{i,j}^i\}, S) = (1 - (1 - p)^{n+1+x_i-z}) + (1 - (1 - p)^{n+1+z-x_i}).$$

We consider two cases based on the values $x_i$ and $z$. First, consider the case $x_i \neq z$. Let $q = x_i - z > 0$.

$$C(\{r_{i,j}^i, s_{i,j}^i\}, S) = (1 - (1 - p)^{n+1+q}) + (1 - (1 - p)^{n+1-q})$$

$$= 2 - 2(1 - p)^{n+1}((1 - p)^q + (1 - p)^{-q})$$

$$< 2 - 2(1 - p)^{n+1} = 2(1 - (1 - p)^{n+1})$$

Next, we consider $x_i = z$. Then,

$$C(\{r_{i,j}^i, s_{i,j}^i\}, S) = (1 - (1 - p)^{n+1}) + (1 - (1 - p)^{n+1}) = 2(1 - (1 - p)^{n+1}).$$

The claim follows.

\[ \triangleright \]

### 4.3 Equivalence between multi-colored clique and Uni-PBDS

\[ \triangleright \] Lemma 26. If $(G, k)$ is a YES-instance of the MULTI-COLORED CLIQUE problem, then $(\mathcal{G}, k', t')$ is a YES-instance of the Uni-PBDS problem.
Lemma 27. If \((G, k', t')\) is a YES-instance of the Uni-PBDS problem, then \((G, k)\) is a YES-instance of the Multi-Colored Clique problem.
Proof. Let S be a feasible solution to the instance \((G, k', t')\) of the Uni-PBDS problem. For each \(i \in [k]\), let \(I_{x_i}\) be the unique gadget for some \(x_i \in [n]\), for which the set \(A\) in \(I_{x_i}\) is in \(S\). For each \(1 \leq i < j \leq k\), let \(I_{u_i,x_i,u_j,x_j}\) be the unique gadget for some \(u_i,x_i,u_j,x_j \in E_{i,j}\), for which the set \(A\) in \(I_{u_i,x_i,u_j,x_j}\) is in \(S\). The existence of such gadgets are ensured by the Claim \(\text{(23)}\). Let \(K = \{u_i,x_i \mid i \in [k]\}\). We show that the set \(K\) is a clique in \(G\) as follows.

Observe that we picked one vertex from each partition \(V_i\) for \(i \in [k]\). Next, we show that for each \(1 \leq i < j \leq k\), there is an edge \(u_i,x_i,u_j,x_j \in E(G)\). Let \(i,j\) such that \(i < j\). By Claim \(\text{(23)}\),

\[
C(V(G) \setminus R, S) = (m + kn)((n + 1)fp + (n + 1) + np + 2(1 - (1 - p)^n))
\]

\[
= t' - 4\left(\binom{k}{2}\right)(1 - (1 - p)^{n+1}).
\]

Since \(C(V(G), S) \geq t'\) and \(C(V(G) \setminus R, S) = t' - 4\left(\binom{k}{2}\right)(1 - (1 - p)^{n+1})\), the expected domination of \(R\) by \(S\) is at least \(4\left(\binom{k}{2}\right)(1 - (1 - p)^{n+1})\). There are \(2\binom{k}{2}\) disjoint pairs of connector vertices in the graph \(G\). By Claim \(\text{(23)}\), each pair of connectors contributes at most an expected domination of value \(2(1 - (1 - p)^{n+1})\). It follows that for each pair of connector vertices the expected domination by \(S\) is equal to \(2(1 - (1 - p)^{n+1})\). Consequently, for each \(i < j \in [k]\), each of the two pairs of connector vertices connecting the blocks \(G_i, G_{i,j}\) and \(G_j\), contributes \(2(1 - (1 - p)^{n+1})\) to the expected domination only if \(I_{u_i,x_i,u_j,x_j}\), and \(I_{u_j,x_j}\) are the unique gadgets in which the set \(A\) of the gadgets are subsets of \(S\). It follows that \(u_i,x_i,u_j,x_j \in E(G)\).

Hence, the set \(K\) forms a clique in \(G\).

Given an instance \((G, k)\) of \textsc{Multi-Colored Clique}, the instance \((G, k')\) is constructed in polynomial time where \(k'\) and \(t'\) are polynomial in input size. By Lemma \(\text{(18)}\) the pathwidth of \(G\) is a quadratic function of \(k\). Finally, by Lemmas \(\text{(26)}\) and \(\text{(27)}\) the Uni-PBDS instance \((G, k', t')\) output by the reduction is equivalent to the \textsc{Multi-Colored Clique} instance \((G, k)\) that was input to the reduction. Since \textsc{Multi-Colored Clique} is known to be \(W[1]\)-hard for the parameter \(k\), it follows that the Uni-PBDS problem is \(W[1]\)-hard with respect to the pathwidth parameter of the input graph. This complete the proof of Theorem \(\text{(7)}\).

5 PBDS on Trees: PTAS and Exact Algorithm

In this section, we present our algorithmic results for the PBDS problem on trees. Throughout this section, assume \(T\) is rooted at some vertex \(r\). For each \(x \in V\), denote by \(\text{par}(x)\) the parent of \(x\) in \(V\), and by \(T(x)\) the subtree of \(T\) rooted at \(x\).

5.1 PTAS for PBDS on Trees

For each \(v \in V\) and each \(b \in [0, k]\), define \(Y_v(\text{par}, \text{curr}, b)\) to be the optimal value of \(C(V(T(v)), S)\) where \(\text{par}\) and \(\text{curr}\) are boolean indicator variables that, respectively, denote whether or not \(\text{par}(v)\) and \(v\) are in \(S\), and \(b\) denotes the number of descendants of \(v\) in \(S\). Formally, \(Y_v(\text{par}, \text{curr}, b)\) is represented as follows:

\[
\arg \max \left\{ \sum_{x \in T(v)} C(x, S) \mid S \subseteq V, |S \cap (T(v) \setminus v)| = b, \text{curr} = I_{v \in S}, \text{par} = I_{\text{par}(v) \in S} \right\}
\]

The main idea behind our PTAS is to use the rounding method. Instead of computing \(Y_v\), we compute its approximation, represented as \(\tilde{Y}_v\). This is done in a bottom-up fashion, starting from leaf nodes of \(T\). For each \(x \in V\), define \(\delta(x)\) to be \(|Y_x - \tilde{Y}_x|\). Throughout our algorithm, we maintain the invariant that \(\tilde{Y}_x \leq Y_x\), for every \(x \in V\).
We now present an algorithm to compute $\widehat{Y}$. Since $Y_v$ is easy to compute for a leaf $v$, we set $\widehat{Y}_v = Y_v$. For a leaf $x$, $Y_x(par, curr, b)$ is (i) undefined if $b \neq 0$, (ii) $\omega_x$ if $curr = 1, b = 0$, (iii) $\omega_x p_{(par(x), x)}$ if $par = 1, curr = 0, b = 0$, and (iv) zero otherwise. Consider a non-leaf $v$. Let $z_1, \ldots, z_t$ be $v$’s children in $T$, and $z_0$ be $v$’s parent in $T$ (if exists). Let $L(\beta)$, for $\beta \geq 0$, denote the collection of all integral vectors $\sigma = (b_1, curr_1, \ldots, b_t, curr_t)$ of length $2t$ satisfying (i) $curr_i \in \{0, 1\}$ and $b_i \geq 0$, for $i \in [1, t]$, and (ii) $\sum_{i \in [1, t]} (b_i + curr_i) = \beta$. In our representation of $\sigma$ as $(b_1, curr_1, \ldots, b_t, curr_t)$, the term $curr_i$ corresponds to the indicator variable representing whether or not $z_i$ lies in our tentative set $S$, and $b_i$ corresponds to the cardinality of $S \cap (V(T(z_i)) \setminus z_i)$. Further, for $i \in [1, t]$, let $L_i(\beta)$ be the collection of those vectors $\sigma = (b_1, curr_1, \ldots, b_t, curr_t) \in L(\beta)$ that satisfy $b_j, curr_j = 0$ for $j > i$.

For a given $curr, par, b \geq 0$, we now explain the computation of $\widehat{Y}_v(par, curr, b)$. Assume that we have already computed the approximate values $\widehat{Y}_{z_i}(i \in [1, t])$ corresponding to $v$’s children in $T$. Setting $W = \max_{u \in V} \omega_u$, and using the scaling factor $M = eW/n$, let

$$A(\sigma) = \begin{cases} 
\omega_v, & \text{if curr=1,} \\
\frac{W}{M} \left( 1 - (1-par \cdot p(z_0, v)) \cdot \prod_{i \in [1, t]} (1 - p(z_i, v)) \right), & \text{otherwise,}
\end{cases}$$

$$B(\sigma) = \sum_{i \in [1, t]} b_i \widehat{Y}_{z_i}(curr, curr, b_i),$$

$$\widehat{Y}_v(par, curr, b) = \max_{\sigma \in L(\beta)} \left( A(\sigma) + B(\sigma) \right).$$

In order to efficiently compute $\widehat{Y}_v$, we define the notion of preferable vectors. For any two vectors $\sigma_1, \sigma_2 \in L(\beta)$, we say that $\sigma_1$ is preferable over $\sigma_2$ (and write $\sigma_1 \geq \sigma_2$) if both (i) $A(\sigma_1) \geq A(\sigma_2)$, and (ii) $B(\sigma_1) \geq B(\sigma_2)$. For $i \in [1, t]$, let $L_i^*(\beta)$ be a maximal subset of $L_i(b)$ such that $\sigma_1 \geq \sigma_2$ for any two vectors $\sigma_1, \sigma_2 \in L_i^*(\beta)$.

Define $\phi_v = \left| \{ A(\sigma) \mid \sigma \in L(\beta), \ for \ \beta \in [0, k] \} \right|$. The following observation is immediate by the definition of $L_i^*(\beta)$.

**Observation 28.** For each $i \in [1, t]$ and $\beta \in [0, k]$, $|L_i^*(\beta)| \leq \phi_v$.

In order to compute $\widehat{Y}_v(par, curr, b)$, we explicitly compute and store $L_i^*(\beta)$, for $1 \leq i \leq t$. The set $L_i^*(\beta)$ is quite easy to compute. Let $\sigma_1 = (\beta, 0, 0, \ldots, 0)$ and $\sigma_2 = (\beta - 1, 1, 0, \ldots, 0)$ be the only two vectors lying in $L_1(\beta)$. Then $L_i^*(\beta)$ is that vector among $\sigma_1$ and $\sigma_2$ that maximizes the sum $A(\sigma) + B(\sigma)$.

The lemma below provides an iterative procedure for computing the sets $L_i^*(\beta)$, for $i \geq 2$.

**Lemma 29.** For every $i, \beta \geq 1$, the set $L_i^*(\beta)$ can be computed from $L_{i-1}^*(\beta)$ in time $O(\beta + \sum_{\alpha \in [0, \beta]} |L_{i-1}^*(\alpha)|)$.

**Proof.** Initialize $L_i^*(\beta)$ to $\emptyset$. At each stage, maintain the list $L_i^*(\beta)$ sorted by the values $A(\cdot)$, and reverse-sort by the values $B(\cdot)$. Our algorithm to compute $L_i^*(\beta)$ involves the following steps.

1. For each $curr \in \{0, 1\}$ and $b \in [0, \beta]$, first compute a set $P_{b, curr}$ obtained by replacing the values $b_i$ and $curr_i$ in each $\sigma \in L_{i-1}^*(\beta - (curr + b))$ by $b$ and $curr$ respectively. Let $P = \bigcup_{b \in [0, \beta], curr \in \{0, 1\}} P_{b, curr}$.
2. For each $\sigma \in P$, check in $O(|\log |P| |)$ time if there is a $\sigma' \in L_i^*(\beta)$ that is preferred over $\sigma$ (i.e. $\sigma' \geq \sigma$). If no such $\sigma'$ exists, then (a) add $\sigma$ to $L_i^*(\beta)$, and (b) remove all those $\sigma''$ from $L_i^*(\beta)$ that are less preferred than $\sigma$, that is, $\sigma'' < \sigma$. 

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**18 Budgeted Dominating Sets in Uncertain Graphs**
Lemma 30.

In order to prove the first inequality, we first show that
\[
\tilde{O}(\beta + |P| \log |P|) \text{ which is at most } \tilde{O}(\beta + \sum_{\alpha \in [0,\beta]} |L_{\alpha - 1}(\alpha)|).
\]

Next we now prove its correctness. Consider a \( \sigma = (b_1, curr_1, \ldots, b_t, curr_t) \in L_t(\beta) \). It suffices to show that if \( \sigma \notin P_{b_t, curr_t} \), then there exists a \( \sigma' \in P_{b_t, curr_t} \) satisfying \( \sigma' \geq \sigma \).

Let \( \sigma_0 \) be obtained from \( \sigma \) by replacing \( b_i, curr_i \) with 0. Since \( \sigma \notin P_{b_t, curr_t} \), it follows that \( \sigma_0 \notin L_{t-1}(\beta - (b_i + curr_i)) \). So there must exist a vector \( \sigma'_0 = (b'_1, curr'_1, \ldots, b'_t, curr'_t) \in L_{t-1}(\beta - (b_i + curr_i)) \) satisfying \( A(\sigma'_0) \geq A(\sigma_0) \) and \( B(\sigma'_0) \geq B(\sigma_0) \). Let \( \sigma' \) be the vector obtained from \( \sigma'_0 \) by replacing \( b'_i, curr'_i \) with \( b_i, curr_i \). It can be easily verified from Eq. (3) and (4), that \( A(\sigma') \geq A(\sigma) \) and \( B(\sigma') \geq B(\sigma) \). Since the constructed \( \sigma' \) indeed lies in \( P_{b_t, curr_t} \), the proof follows.

The following claim is an immediate corollary of Lemma 29.

Lemma 30. The value of \( \tilde{O}(par, curr, b) \), for any \( par, curr \in \{0, 1\} \) and \( b \in [0, k] \), is computable in \( \tilde{O}(b \cdot \deg(v) \cdot \phi_v) \) time, given the values of \( \tilde{Y}_i \) for \( i \leq t \).

Proof. Observe that \( \tilde{O}(par, curr, b) = \max_{\sigma \in L(\beta)} (A(\sigma) + B(\sigma)) \), where \( A(\sigma) \) and \( B(\sigma) \) are as defined in Eq. (3) and (4). By Observation 28 and Lemma 29, the total computation time of the set \( L_i(\beta) \) is at most \( \tilde{O}(b \cdot \deg(v) \cdot \phi_v) \), which is equal to \( \tilde{O}(b \cdot \deg(v) \cdot \phi_v) \).

Lemma 29 implies that starting from leaf nodes, the values \( \tilde{O}(par, curr, b) \) can be computed in bottom-up manner, for each valid choice of triplet \( (par, curr, b) \) and each \( x \in V \), in total time \( \tilde{O}(k^2 n \cdot \max_{x \in V} \phi_v) \). We now prove \( \phi_v = O(\epsilon^{-1} n) \). If \( curr = 1 \), then \( A(\sigma) \) takes only one value. If \( curr = 0 \), then the value of \( A(\sigma) \) is a multiple of \( M \) and is also bounded above by \( W \). This implies that the number of distinct values \( A(\sigma) \) can take is indeed bounded by \( W/M = O(\epsilon^{-1} n) \).

Proposition 31. Computing \( \tilde{O}_x \) for all \( x \in V \) takes in total \( \tilde{O}(k^2 n \cdot \max_{x \in V} \phi_v) = \tilde{O}(k^2 \epsilon^{-1} n^2) \) time.

5.2 Approximation Analysis of PTAS on Trees

We provide here the approximation analysis of the \((1 - \epsilon)\)-bound. Let

\[
S_{opt} = \arg \max_{S \subseteq V : |S| = k} C(V, S) = \arg \max_{S \subseteq V : |S| = k} \sum_{x \in S} \omega_x \Pr(x \sim S),
\]

\[
\hat{S}_{opt} = \arg \max_{S \subseteq V : |S| = k} \left( \sum_{x \in S} \omega_x \Pr(x \sim S) + \sum_{x \in V \setminus S} M \left[ \frac{\omega_x \Pr(x \sim S)}{M} \right] \right).
\]

Observe that \( \max \{ \hat{Y}_t(0, 0, k), \hat{Y}_t(0, 1, k-1) \} = C(V, S_{opt}) \) and \( \max \{ \hat{Y}_t(0, 0, k), \hat{Y}_t(0, 1, k-1) \} = C(V, \hat{S}_{opt}) \). The following lemma proves that \( \hat{S}_{opt} \) indeed achieves a \((1 - \epsilon)\)-approximation bound.

Lemma 32. \((1 - \epsilon) C(V, S_{opt}) \leq C(V, \hat{S}_{opt}) \leq C(V, S_{opt}) \).

Proof. In order to prove the first inequality, we first show that \( C(V, S_{opt}) - C(V, \hat{S}_{opt}) \leq \epsilon C(V, S_{opt}) \).

\[
C(V, S_{opt}) - C(V, \hat{S}_{opt}) = \max_{|S| = k} \left( \sum_{x \in V} \omega_x \Pr(x \sim S) - \sum_{x \in S} \omega_x \Pr(x \sim S) - \sum_{x \in V \setminus S} M \left[ \frac{\omega_x \Pr(x \sim S)}{M} \right] \right)
\]

\[
\leq (n - k)M \leq \epsilon C(V, S_{opt}).
\]

Next, for each \( x \in V \) and \( S \subseteq V \), we have \( M |M^{-1} \omega_x \Pr(x \sim S)| \leq \omega_x \Pr(x \sim S) \), thereby implying that \( C(V, \hat{S}_{opt}) \leq C(V, S_{opt}) \). This completes our proof.
For any integer $k$, any $n$-vertex tree $T$ with arbitrary edge probabilities, and for every $\epsilon > 0$, a $(1 - \epsilon)$ approximate solution can be computed in time $O(k^2 \epsilon^{-1} n^2)$. This follows from Proposition 34 and Lemma 32. We thus prove Theorem 5.

### 5.3 Linear-time algorithm for Uni-PBDS on Trees

We next establish our result for the scenario of Uni-PBDS on trees (Theorem 6). In fact, this result holds for a somewhat broader scenario, wherein, for each vertex $x$, the cardinality of $\text{PROB}_x = \{ p_x \mid \epsilon \text{ is incident to } x \}$ is bounded above by some constant $\gamma$.

**Proof of Theorem 6.** Observe that the only place where approximation was used in our PTAS was in bounding the number of distinct values that can be taken by $A(\sigma)$ in Eq. (3). In order to obtain an exact solution for the bounded probabilities setting, the only modification performed in our algorithm is to redefine $A(\sigma)$ as follows.

$$A(\sigma) = \omega_v \cdot I_{(\text{curr}=1)} + \omega_v \left(1 - (1 - \text{par} \cdot p(z_i,v)) \cdot \prod_{i \in [1,t]} (1 - p(z_i,v))\right) \cdot I_{(\text{curr}\neq 1)}.$$

It can be verified that the algorithm correctly computes $f_x$ at each step, that is, $\delta(x)$ is essentially zero. The time it takes to compute $f_v(\text{par}, \text{curr}, b)$, for a non-leaf $v$, crucially depends on the cardinality of $\{ A(\sigma) \mid \sigma \in \mathcal{T}_t(b) \}$, where $t$ is the number of children of $v$ in $T$. Observe that the number of distinct values $A(\sigma)$ can take is at most $b^{[\text{PROB}_v]} = O(k^t)$. This along with Lemma 31 implies that the total runtime of our exact algorithm is $O(k^{t+2} n)$. ▶

### 5.4 Solving PBDS optimally on general trees

Let $c \geq 1$ be the smallest real such that 2-SPM problem has an $O(N^c)$ time algorithm. We will show that in such a case, $k$-PBDS can be solved optimally on trees with arbitrary probabilities in $O((\delta N)^{c(k/2)+1})$ time, for a constant $\delta > 0$.

For any node $v \in T$, let $T'_v$, for $1 \leq i \leq \deg(v)$, represent the components of the subgraph $T \setminus \{v\}$. We start with the following lemma which is easy to prove using a standard counting argument.

**Lemma 33.** For any set $S$ of size $k$ in $T$, there exist a node $v \in T$ and an index $q \in [1, \deg(v)]$ such that the cardinalities of the sets $S \cap \bigcup_{i \leq q} T'_v$, $S \cap T'_v$ and $S \cap \bigcup_{i \geq q} T'_v$ are all bounded by $k/2$.

**Proof.** We first show that there exists a node $v$ in $T$ satisfying the property $|S \cap T'_v| \leq k/2$, for each $i \in [1, \deg(v)]$. Consider a node $u \in T$. If $u$ satisfies the above mentioned property then we are done. Otherwise, there exists an index $j \in [1, \deg(u)]$ for which $|S \cap T'_u| \geq k/2$. This implies the number of elements of $S$ lying in $\{u\} \cup \bigcup_{i=1}^{j-1} T'_u \cup \bigcup_{i=1}^{\deg(u)} T'_u$ is at most $k/2$. In such a case we replace $u$ by its $j$th neighbor. Repeating the process eventually leads to the required node $v$.

Now, let $q \in [1, \deg(v)]$ be the smallest integer for which $S \cap \bigcup_{i \leq q} T'_v$ is larger than $k/2$. Then, $S \cap \bigcup_{i \leq q} T'_v$ and $S \cap \bigcup_{i \geq q} T'_v$ are both bounded by $k/2$, by definition of $q$. Also, $S \cap T'_v$ is bounded by $k/2$ due to the choice of $v$. ▶

For the rest of this section, we refer to a tuple $(v, q)$ satisfying the conditions stated in Lemma 33 as a valid pair. Let us suppose we are provided with a valid pair $(v, q)$. For sake of convenience, we assume that $T$ is rooted at node $v$. Let $U_0 = T'_v$, $U_1 = \bigcup_{i \leq q} T'_v$, and
Lemma 34. Consider the following 2-SPMs obtained by two equal partitions of $Q$ in linear time to the following equivalent

This takes in total $O(d)$ where $d$ integers in the range $[0, k/2]$ of integers in the range $[0, k/2]$ of $S$ solution $S$ contained in $S$. Assuming $(v, q)$ is a valid pair, and $v$ is not contained in the optimal $S$, the solution $S$ to $k$-PBDS is the union $S_0 \cup S_1 \cup S_2$ of the tuple $(S_0, S_1, S_2) \in U_0 \times U_1 \times U_2$ that maximizes

$$C(U_0, S_0) + C(U_1, S_1) + C(U_2, S_2) + \omega_v \left( \prod_{i \in [1, n]} (1 - p(v, x_i)) \right) \prod_{j \in [1, m]} (1 - p(v, y_j)) \right)$$

where $d$ is an indicator variable denoting whether or not $z \in S_0$, and $|S_1|, |S_2|, |S_3|$ are integers in the range $[0, k/2]$ that sum up to $k$.

Define $\Gamma$ to be set of all quadruples $(d, c_0, c_1, c_2)$ comprising of integers in the range $[0, k/2]$ such that $d \in \{0, 1\}$ and $c_0 + c_1 + c_2 = k$. For each $\gamma = (d, c_0, c_1, c_2) \in \Gamma$, let

$$L_\gamma^1 = \left\{ \left. \frac{C(U_1, S_1)}{\omega_v (1 - d \cdot p(v, z))}, \prod_{i \in [1, n]} (1 - p(v, x_i)) \right| S_1 \subseteq U_1 \text{ is of size } c_1 \right\}$$

$$L_\gamma^2 = \left\{ \left. \frac{C(U_2, S_2)}{\omega_v (1 - d \cdot p(v, z))}, \prod_{j \in [1, m]} (1 - p(v, y_j)) \right| S_2 \subseteq U_2 \text{ is of size } c_2 \right\}$$

$$Z_\gamma = \max \left\{ C(U_0, S_0) \mid S_0 \subseteq U_0 \text{ is of size } c_0, \text{ and } d = I_{z \in S_0} \right\}$$

So, the maximization considered in Eq. (6) is equivalent to the following optimization:

$$\max_{\gamma = (d, c_0, c_1, c_2) \in \Gamma} \omega_v + Z_\gamma + \left( \omega_v (1 - d \cdot p(v, z)) \right) \left( a + \bar{a} - b \bar{b} \right).$$

(7)

In the next lemma we show that optimizing the above expression is equivalent to solving $|\Gamma| = O(k^2)$ different 2-SPM problems (each of size $O(n^{k/2})$).

Lemma 34. Let $A = ((a_1, b_1), \ldots, (a_N, b_N))$ and $\bar{A} = ((\bar{a}_1, \bar{b}_1), \ldots, (\bar{a}_N, \bar{b}_N))$ be two arrays. Then, solving the maximization problem $\max_{i, j \in [0, n]} (a_i + \bar{a}_j, b_i - \bar{b}_j)$, can be transformed in linear time to the following equivalent 2-SPM:

$$L = ((Q + a_1, R b_1), \ldots, (Q + a_N, R b_N), (-Q + \bar{a}_1, R^{-1} \bar{b}_1), \ldots, (-Q + \bar{a}_N, R^{-1} \bar{b}_N)),$$

where $Q = \max_{i, j \in [0, n]} (b_i + 2 \max_{i, j} (a_i + \bar{a}_j)$ and $R = \sqrt{4Q^2 \min_{i} (b_i)^2}$.

Proof. Consider the following 2-SPMs obtained by two equal partitions of $L$:

$L^1 = ((Q + a_1, R b_1), \ldots, (Q + a_N, R b_N))$, and

$L^2 = (-Q + \bar{a}_1, R^{-1} \bar{b}_1), \ldots, (-Q + \bar{a}_N, R^{-1} \bar{b}_N))$. 

Lemma 35. It suffices to show that the time to compute \( L^1 \), \( L^2 \), which is strictly less than \(-Q\). Similarly, the optimal value of \( L^2 \) is bounded above by \( 2Q + 2 \max_i \min_j a_{ij} \), which is again strictly less than \(-Q\).

Now the answer to the optimization problem max\(_{i,j}\) \((a_{ij} + a_{ji} - b_{ij}b_{ji})\) is at least \(-Q\). This clearly shows that the solution to \( L \) cannot be obtained by its restrictions \( L^1 \) and \( L^2 \). Hence, the maximization problem max\(_{i,j}\) \((a_{ij} + a_{ji} - b_{ij}b_{ji})\) is equivalent to solving the 2-SPM \( L \).

Proof of Theorem 4. The time to compute \( L^1 \), \( L^2 \), for a given \( \gamma \), is \( \tilde{O}(n^{k/2}) \). By transformation presented in Lemma 34, it follows that the total time required to optimize the expression in Eq. (7) is \( kO(1) \cdot n^{c(k/2)} \), which is at most \( O((\delta n)^{c(k/2)+1}) \), for some constant \( \delta \geq 1 \). Now recall that Eq. (7) provides an optimal solution assuming \((v, q)\) is a valid pair, and \( v \) is not contained in optimal \( S \). Even when \((v, q)\) is a valid pair, and \( v \) is contained in the optimal \( S \), the time complexity turns out to be \( O(k^2 \cdot n^{k/2}) \). Iterating over all choices of pair \((v, q)\) incurs an additional multiplicative factor of \( n \) in the runtime. ▶

6 Parameterization based on graph structure

In this section, we state our results on structural parameterizations of the Uni-PBDS problem. First, following the approach of Arnborg et al. [6], we formulate the MSOL formula for the Uni-PBDS problem where the quantifier rank of the formula is \( O(k) \). This indeed yields an FPT algorithm for the Uni-PBDS problem parameterized by budget \( k \) and the treewidth of the input graph.

In addition, we show that the Uni-PBDS problem is FPT for the budget parameter on apex-minor-free graphs. In particular, we show that, for any integer \( k \), and any \( n \)-vertex weighted apex-minor free graph with uniform edge probability, the Uni-PBDS problem can be solved in time \((2^{O(\sqrt{k} \log k)} n^{O(1)})\).

6.1 Results on Planar and Apex Minor-Free graphs

We present here a subexponential time algorithm to solve the Uni-PBDS problem on planar and apex minor-free graphs. The algorithm is based on the technique due to Fomin et al. [29] used in the subexponential algorithm for the partial cover problem, and the claim proved in Theorem 8 and Theorem 14. Let \( H \) be a given apex graph. Then our input is an instance \((G = (V, E, p, \omega), k)\) of the PBDS problem where \( G = (V, E) \) is an \( H \)-minor-free graph. Let \( \sigma = (v_1, v_2, \ldots, v_n) \) be an ordering of the vertices in non-increasing order of their expected coverage, that is, \( C(V, v_1) \geq C(V, v_2) \geq \cdots \geq C(V, v_n) \). For \( 1 \leq i \leq n \), let \( V_i = \{v_1, \ldots, v_i\} \) and \( \hat{G}_i = G[V_i] \). Let \( S_{opt} \) be an optimal solution and \( i \) be the largest index of a vertex in \( S_{opt} \). The following lemma states a crucial property of \( S_{opt} \).

Lemma 35. \( S_{opt} \) is a 3-dominating set for \( \hat{G}[N[V_i]] \).

Proof. It suffices to show that \( S_{opt} \) is a 2-dominating set for \( G_i \). The proof is by contradiction. Suppose \( S_{opt} \) is not a 2-dominating set for \( G_i \), then there exists a vertex \( v_j \in V_i \), with \( j < i \), such that \( N_{G_i}[S_{opt}] \cap N_{G_i}[v_j] = \emptyset \). Let \( S' = (S_{opt} \setminus \{v_i\}) \cup \{v_j\} \). We know that \( C(V, v_j) \geq C(V, v_i) \), and \( v_j \) is not 2-dominated by \( S \). Thus,

\[
C(V, S') = C(V, S_{opt} \setminus \{v_i\}) + C(V, v_j) \geq C(V, S_{opt}) - C(V, v_i) + C(V, v_j) \geq C(V, S_{opt}).
\]

Clearly, \( S' \) is also an optimal solution and \( S' \) is lexicographically smaller than \( S_{opt} \). This contradicts the fact that \( S_{opt} \) is the lexicographically least solution. Therefore, the set \( S_{opt} \) must be a 2-dominating set for \( G_i \), and thus also a 3-dominating set for \( G_i \). ▶
We use the following structural property on apex-minor-free graphs from Fomin et al. [29].

Lemma 36 (Fomin et al. [29]). If an apex-minor-free graph $G$ has an $r$-dominating set of size $k$ then the treewidth of the graph $G$ is at most $(c \cdot r \cdot \sqrt{k})$, where $c$ is a constant dependent only the size of the apex graph.

We next use the following lemma, to compute an approximation to treewidth of prescribed-minor-free graphs.

Lemma 37 (Demaine et al. [17]). For each fixed $H$, there is a polynomial time algorithm, which for any $H$-minor free graph $G$ computes a tree decomposition of width $\delta$ times the treewidth of $G$, where $\delta$ is a constant.

Algorithm 1, presented next, solves the PBDS problem on apex minor free graphs.

Algorithm 1: FPT for the PBDS problem in apex-minor-free graphs

Data: An uncertain $H$-minor-free graph $G = (V, E, p, \omega)$, and an integer $k$, where $H$ is an apex graph.

1. For each $j \in [n]$, compute the $\delta$-approximate treewidth $tw_j$ of $G[V'_j]$ using Lemma 57.
2. Let $i = \max\{j \mid tw_j \leq 3c \cdot \delta \cdot \sqrt{k}\}$.
3. Compute a tree decomposition $(T, X)$ of $G[V'_i]$ using Lemma 57.
4. Run the FPT algorithm in Theorem 8 on the instance $(G[V'_i], p, w, k)$ with tree decomposition $(T, X)$ and output the solution.

Theorem 38. For any integer $k$, and any $n$-vertex weighted apex-minor free graph $G$ with uniform edge probability, the Uni-PBDS problem can be solved in time $(2^{O(\sqrt{k} \log k)}n^{O(1)})$.

7 Uni-PBDS problem parameterized by the treewidth and budget $k$

7.1 MSOL formulation of the Uni-PBDS problem

We show that an extension of Courcelle’s theorem due to Arnborg et. al. [9] results in an FPT algorithm for the combined parameters treewidth and $k$. This is obtained by expressing the Uni-PBDS problem as a monadic second order logic (MSOL) formula (see [13, 14]) of length $O(k)$. The following MSOL formulas are used in the MSOL formula for the Uni-PBDS problem. The upper case variables (with subscripts) take values from the set of subsets of $V$, and the lower case variables take values from $V$.

- The vertex set $S$ contains $d$ elements.

  $$\text{SIZE}_d(S) = \exists x_1 \exists x_2 \cdots \exists x_d \forall y (y \in S \rightarrow \bigvee_{i=0}^{d} (y = x_i))$$

- Given a vertex set $X$ and a vertex $x$, there exists a set $S \subseteq X$ of size $d$, and for each vertex $y$ in $S$, $y$ is a neighbor of $x$.

  $$\text{INC}_d(x, X) = \exists S(\text{SIZE}_d(S) \land \forall y ((y \in S \rightarrow y \in X) \land (y \in S \rightarrow \text{adj}(x, y))))$$
The sets $X, Y_0, Y_1, \ldots, Y_k$ partition the vertex set $V$. 

$$\text{PART}(X, Y_0, Y_1, \ldots, Y_k) = \forall x \left( (x \in X \lor \bigvee_{i=0}^{k} x \in Y_i) \land \bigwedge_{i=0}^{k} \neg (x \in X \land x \in Y_i) \land \bigwedge_{i \neq j} \neg \left( x \in Y_i \land x \in Y_j \right) \right)$$

Now we define the MSOL formula for the Uni-PBDS problem. The formula expresses the statement that $V$ can be partitioned into $X$ and $V \setminus X$, and $V \setminus X$ can be partitioned into $k + 1$ sets $Y_0, Y_1, \ldots, Y_k$ such that for each set $Y_i$ and each vertex $y$ in $Y_i$, $y$ has $i$ neighbors in $X$.

$$\text{Uni-PBDS} = \exists X \exists Y_0 \exists Y_1 \cdots \exists Y_k \left( \text{PART}(X, Y_0, Y_1, \ldots, Y_k) \land \forall x \forall y \left( (y \in Y_0 \land x \in X) \rightarrow \neg (\text{adj}(x, y)) \right) \land \forall y \left( \bigwedge_{i=1}^{k} (y \in Y_i \rightarrow \text{INC}_i(y, X)) \right) \right)$$

**Lemma 39.** The quantifier rank of $\text{Uni-PBDS}$ is $O(k)$.

**Proof.** There are $k + 2$ initial quantifiers for the sets $X, Y_0, Y_1, \ldots, Y_k$. For two MSOL formulas $\phi$ and $\psi$ with quantifier rank $\text{qr}(\phi)$ and $\text{qr}(\psi)$, respectively, $\text{qr}(\phi \land \psi) = \text{qr}(\phi \lor \psi) = \max\{\text{qr}(\phi), \text{qr}(\psi)\}$. Therefore, $\text{qr}(\text{Uni-PBDS})$ is bounded as follows:

$$\begin{align*}
\text{qr}(\text{Uni-PBDS}) &= k + 2 + \max\{\text{qr}(\text{PART}), 1 + \text{qr}(\text{INC})\} \\
&= k + 2 + \max\{1, 1 + \max\{\text{qr}(\text{SIZE}), 2\}\} \\
&\leq k + 2 + k = 2k + 3 = O(k)
\end{align*}$$

We now show that the $\text{Uni-PBDS}$ problem is fixed-parameter tractable in parameters $k$ and treewidth by expressing the maximization problem on the MSOL formula as a minor variation of extended monadic second-order extremum problem as described by Arnborg et. al. [6].

**Proof of Theorem** For each $0 \leq i \leq k$, define the weight function $w^i$ associated with the set variable $Y_i$ as follows: for each $v \in V(G)$, $w^i_v = \left(1 - (1 - p)^i\right)w(v)$. The difference between the weight function in [6] and our problem is that in their paper $w(v)$ is considered to be constant value, for all vertices, for the set variable $Y_i$. Observe, however, that the running time of their algorithm does not change as long as $w^i_v$ can be computed in polynomial time, which is the case in our definition. Therefore, our maximization problem is now formulated as a variant of the EMS maximization problem in [6]:

$$\text{Maximize } \sum_{u \in X} w(u) + \sum_{i=0}^{k} \sum_{v \in Y_i} w^i_v \cdot y^i_v \text{ over partitions } (X, Y_0, Y_1, \ldots, Y_k) \text{ satisfying } \text{Uni-PBDS}$$

Using Theorem 5.6 in [6] along with the additional observation, we make, that $w^i_v$ can be efficiently computed, an optimal solution for the Uni-PBDS problem can be computed in time $f(\text{qr}(\text{Uni-PBDS}), w) \cdot \text{poly}(n)$, where $f(\text{qr}(\text{Uni-PBDS}), w)$ is a function which does not
The uncertain graph induced by the vertices $X_i$ at state $s$. A state $C$ over all instances $\langle G, w \rangle$ at a node specified by state $s$ is proved. The coverage problem at state $s$ is to be considered as $\omega$. For each state $s$, the DP formulation gives a recursive definition of the values $Val_i[s]$ and $Sol_i[s]$. $Sol_i[s]$ is a subset $S$ of $X_i^+$ that achieves the optimum coverage for $G_i$ and it is denoted by $C_i$. The expected domination function $C$ over the graph $G_i$ is denoted by $C_i$. We refer to the expected optimization as coverage in the presentation below.

Definitions and Notation. We first set up some definitions and notation required for the DP formulation. Let $(T, X)$ be a nice tree decomposition of the graph $G_i$ rooted at node $r$. For a node $i \in V(T)$, let $T_i$ be the subtree rooted at $i$ and $X_i^+ = \cup_{j \in V(T_i)} X_j$. The uncertain graph induced by the vertices $X_i^+$ is $G_i[X_i^+]$. The expected domination function $C_i$ over the graph $G_i$ is denoted by $C_i$. We refer to the expected domination as coverage in the presentation below.

For each node $i \in T$, we compute two tables $Sol_i$ and $Val_i$. The rows of both tables are indexed by 4-tuples which we refer to as states. $S_i$ denotes the set of all states associated with node $i$. For a state $s$, the DP formulation gives a recursive definition of the values $Sol_i[s]$ and $Val_i[s]$. $Sol_i[s]$ is a subset $S$ of $X_i^+$ that achieves the optimum coverage for $G_i$ and satisfies additional constraints specified by the state $s$. $Val_i[s]$ is the value $C_i(X_i^+, Sol_i[s])$. A state $s$ at the node $i$ is a tuple $(b, \gamma, \alpha, \beta)$, where

- $0 \leq b \leq k$ is an integer and specifies the size of $Sol_i[s]$.
- $\gamma : X_i \rightarrow \{0, 1\}$ is an indicator function for the vertices of $X_i$. This specifies the constraint that $\gamma^{-1}(1) \subseteq Sol_i[s]$ and $\gamma^{-1}(1) \cap Sol_i[s] = \emptyset$. We use $A$ to denote $\gamma^{-1}(1)$ and the state will be clear from the context.
- $\alpha : X_i \rightarrow [0, k]$ is a function. The constraint is that for each vertex $u \in \gamma^{-1}(0)$, $u$ should have $\alpha(u)$ neighbors in $Sol_i[s]$.
- $\beta : X_i \rightarrow [0, k]$ is a function. The constraint is that for each $u \in \gamma^{-1}(0)$, the weight of $s$ in this state is to be considered as $\omega_u(u) = \omega(u)(1 - p)^{\beta(u)}$.

The coverage problem at state $s$: The PBDS instance at state $s$ is $(G_i = (X_i^+, E(X_i^+), p, \omega_u))$ and budget $b$. $\omega_u$ is defined as follows: For each $u \in X_i^+$,

$$\omega_u(u) = \begin{cases} (1 - p)^{\beta(u)} \omega(u) & \text{if } u \in X_i \text{ and } c(u) = 0, \\ \omega(u) & \text{otherwise}, \end{cases}$$

In the following presentation the usage of $G_i$ and $C_i$ will always be at a specific state which will be made clear in the context. $Sol_i[s]$ is a subset of $X_i^+$ which satisfies all the constraints specified by state $s$ and $Val_i[s] = C_i(X_i^+, Sol_i[s])$ is the maximum value among $C_i(X_i^+, S)$ over all $S \subseteq X_i^+$ and $|S| \leq b$. In other words it is the optimum solution for Uni-PBDS problem on instance $(G_i, b)$ and it satisfies the constraints specified by $s$. A state $s = (b, \gamma, \alpha, \beta)$ at a node $i$ is said to be invalid if there is no feasible solution that satisfies the constraints specified by $s$, and $Sol_i[s] = undefined$ and $Val_i[s] = undefined$. If there is a feasible
Lemma 40. The correctness follows from the fact that the graph $G[X_i^+]$ can be computed in constant time.

Next, we consider the second case that $\gamma(v) = 1$. Let $D_v = N(v) \cap \gamma^{-1}(0)$, that is the set of neighbors of $v$ which are not to be selected in $\text{Sol}_i[s]$. Indeed, these vertices must be considered as their contribution to the expected coverage will increased due to the introduction of solution, the state is called valid.

**State induced at a node in $T$ by a set:** For a set $D \subseteq V$ of size $k$, we say that $D$ induces a state $s = (b, \gamma, \alpha, \beta)$ at node $i$ and $s$ is defined as follows:

- $b = \left| D \cap X_i^+ \right|$.
- The function $\gamma : X_i \to \{0, 1\}$ is defined as follows- for each $u \in D \cap X_i$, $\gamma(u) = 1$ and $\gamma(u) = 0$, for each $u \in D \setminus X_i$.
- The functions $\alpha, \beta : X_i \to [0, k]$ are defined as follows- for each $u \in \gamma^{-1}(0)$, $\alpha(u) = |N(u) \cap X_i^+ \cap D|$, and $\beta(u) = |N(u) \cap (V \setminus X_i^+) \cap D|$.

Depending on $\alpha$ and $\beta$ that there can be different states induced by a set $D$.

**7.2.1 Recursive definition of $\text{Sol}_i$ and $\text{Val}_i$**

For each node $i \in V(T)$ and $s = (b, \gamma, \alpha, \beta) \in S_i$, we show how to compute $\text{Sol}_i[s]$ and $\text{Val}_i[s]$ from the tables at the children of $i$. $\text{Sol}_i[s]$ and $\text{Val}_i[s]$ are recursively defined below and we prove a statement on the structure of an optimal solution based on the type of the node $i$ in $T$. These statements are used in Section 7.2.2 to prove the correctness of the bottom-up evaluation.

**Leaf node:** Let $i$ be a leaf node with bag $X_i = \emptyset$. The state set $S_i$ is a singleton set with a state $s = (0, \emptyset \to \{0, 1\}, \emptyset \to [k], \emptyset \to [k])$. Therefore, $\text{Sol}_i[s] = \emptyset$ and $\text{Val}_i[s] = 0$. This can be computed in constant time.

**Lemma 40.** The table entries for the state $s$ at a leaf node is computed optimally.

**Proof.** The correctness follows from the fact that the graph $G_i$ is a null graph. Thus, for a null graph and a valid state $s$, empty set with coverage value zero is the only optimal solution.

**Introduce node:** Let $i$ be an introduce node with child $j$ such that $X_i = X_j \cup \{v\}$ for some $v \not\in X_j$. Since $i$ is an introduce node, all the neighbors of $v$ in $G[X_i^+]$ are in $X_i$. Thus, $N(v) \cap X_i^+ = N(v) \cap X_i$. In case, $\alpha(v) \neq |N(v) \cap A|$, then, by definition of a solution at a state, the state $s$ does not have a feasible solution. Therefore, the state $s$ is invalid. Next consider that in state $s$, $\alpha(v) = |N(v) \cap A|$. We define the state $s_j$ and define $\text{Sol}_i[s]$ in terms of $\text{Sol}_j[s_j]$. The state $s_j$ differs based on whether $\gamma(v) = 0$ and $\gamma(v) = 1$. For the case $\gamma(v) = 0$ the solution to be computed for the state $s$ must not contain $v$, and for the case $\gamma(v) = 1$, solution to be computed must contain $v$.

In the case $\gamma(v) = 0$, define $s_j = (b, \gamma_j, \alpha_j, \beta_j)$ to be the state from node $j$ where the functions $\gamma_j : X_j \to \{0, 1\}$, $\alpha_j : X_j \to [0, k]$ and $\beta_j : X_j \to [0, k]$ are as follows: for each $u \in X_j$, $\gamma_j(u) = \gamma(u)$, $\alpha_j(u) = \alpha(u)$ and $\beta_j(u) = \beta(u)$. If the state $s_j$ is invalid then the state $s$ is also invalid. Therefore, we consider that the state $s_j$ is valid. Then the solution at state $s$ as follows:

\[
\text{Sol}_i[s] = \text{Sol}_j[s_j] \tag{9}
\]

and

\[
\text{Val}_i[s] = \text{Val}_j[s_j] + (1 - (1 - p)^{\alpha(v)}) \omega_s(v) \tag{10}
\]

Next, we consider the second case that $\gamma(v) = 1$. Let $D_v = N(v) \cap \gamma^{-1}(0)$, that is the set of neighbors of $v$ which are not to be selected in $\text{Sol}_i[s]$. Indeed, these vertices must be considered as their contribution to the expected coverage will increased due to the introduction
Lemma 41. Let \( s_j = (b - 1, \gamma_j, \alpha_j, \beta_j) \) be the state in the node \( j \) where the functions \( \gamma_j : X_j \to \{0, 1\} \), \( \alpha_j : X_j \to [0, k] \) and \( \beta_j : X_j \to [0, k] \) are defined as follows: for each \( u \in X_j \), \( \gamma_j(u) = \gamma(u) \), \( \alpha_j(u) = \begin{cases} \alpha(u) - 1 & \text{if } u \in D_v, \\ \alpha(u) & \text{otherwise,} \end{cases} \)

and

\[
\beta_j(u) = \begin{cases} \beta(u) + 1 & \text{if } u \in D_v, \\ \beta(u) & \text{otherwise.} \end{cases}
\]

Note the increase and decrease of \( \alpha \) and \( \beta \) at each \( u \in D_v \): this is to take care of the fact that a neighbor \( v \) has been introduced at node \( i \) and the aim is to compute a solution that contains \( v \). Therefore, at \( s_j \) for each \( u \in \gamma^{-1}(0) \) we consider \( \alpha_j(u) = \alpha(u) - 1 \) to reflect the fact that \( v \) is not in \( X_j \). To ensure that the solution computed at state \( s_j \) gives us the desired solution at \( s \), for each \( u \in \gamma^{-1}(0) \) we consider \( \beta_j(u) = \beta(u) + 1 \). Recall that this ensures that \( \omega_{s_j}(u) = (1 - p)\beta(u + 1) + \omega(u) = (1 - p)\omega_s(u) \). This is used in the coverage expressions in the proof of Lemma 11.

We now define \( \text{Sol}_i[s] \) and \( \text{Val}_i[s] \). If the state \( s_j \) is invalid then the state \( s \) also invalid. Therefore, we consider that the state \( s_j \) is valid. The solution for the state \( s \) is defined as follows:

\[
\text{Sol}_i[s] = \text{Sol}_j[s_j] \cup \{v\}
\]

and

\[
\text{Val}_i[s] = \text{Val}_j[s_j] + \sum_{u \in D_v} p(1 - p)^{\alpha(u)} - 1 \omega_s(u) + \omega(v)
\]

In both the cases, the state \( s_j \) can be computed in \( O(w) \) time.

\begin{itemize}
  \item \textbf{Lemma 41.} Let \( i \) be an introduce node in \( T \) and let \( s \) and \( s_j \) be as defined above. Let \( S \) be a solution of optimum coverage at the state \( s \), then \( S \setminus \{v\} \) gives the optimum coverage at the state \( s_j \) in the node \( j \).
\end{itemize}

\textbf{Proof.} Since \( S \) is solution at state \( s \), by definition \( S \) induces the state \( s \) at node \( i \). In this proof \( C_i(X_i^+, S) \) is the coverage at state \( s \) and \( C_j(X_j^+, S) \) is the coverage at state \( s_j \). The proof technique is to rewrite \( C_i(X_i^+, S) \) as a sum of \( C_j(X_j^+, S \setminus \{v\}) \) and an additional term that depends only on \( v \), its neighborhood and the state \( s \). This shows that \( S \setminus \{v\} \) induces the state \( s_j \) and attains the maximum coverage at \( s_j \). We consider two cases based on whether \( v \in A \) or not. In the case \( v \notin A \), first it follows that \( v \notin S \). Further, by definition we have \( C_i(X_i^+, S) = C_i(X_i^+ \setminus \{v\}, S) + C_i(v, S) \). Since \( s_j \) is identical to \( s \) except for the vertex \( v \) which is not in \( X_j \), it follows that for each \( u \in X_j \), \( \omega_j(u) = \omega_s(u) \). Therefore, \( C_j(X_j^+, S) = C_j(X_j^+ \setminus \{v\}, S) \). Further, the state induced by the solution \( S \) at node \( j \) is the state \( s_j \), and in this case \( S \setminus \{v\} \) is \( S \) itself. Therefore, the optimum coverage at \( s_j \) is achieved by \( S \setminus \{v\} \).

In the case \( v \in A \), it follows that \( v \in S \). Further, by definition of \( s_j \), and the fact that \( S \) induces the state \( s \), it follows that \( S \setminus \{v\} \) induces the state \( s_j \). The coverage value \( C_i(X_i^+, S) \) can be written as follows:

\[
C_i(X_i^+, S) = C_i(X_i^+ \setminus X_i, S) + C_i(X_i \setminus (D_v \cup \{v\}), S) + C_i(D_v, S) + C_i(v, S)
\]
The first equality follows by partitioning $X_I^+$ into four sets so that the coverage of each set by $S$ summed up gives the coverage of $X_I^+$ by $S$. In the second equality, the first two terms are re-written as coverage by $S \setminus \{v\}$ at node $j$, and $C_i(v, S)$ is re-written as $\omega_s(v)$. Since $i$ is an introduce node, the key facts used in this equality are that $X_i^+ \setminus X_i = X_i^+ \setminus X_j$ and $X_i \setminus (D_v \cup \{v\}) = X_j \setminus D_v$. The other key facts used are that $\gamma_j$ and $\gamma$ are identical on $X_j$, $\alpha_j$ and $\alpha$ are identical on $X_j^+ \setminus D_v$, and $\beta_j$ and $\beta$ are identical on $X_j^+ \setminus D_v$. The coverage value of $C_i(D_v, S)$ is now re-written as follows, using the key fact that for all $u \in D_v$, $\gamma(u) = 0$, and $|N(u) \cap S| = \alpha(u)$, and that the state induced by solution $S$ at node $i$ is the state $s$:

$$C_i(D_v, S) = \sum_{u \in D_v} C_i(u, S) = \sum_{u \in D_v} (1 - (1 - p)^{\vert N(u) \cap S \vert}) \omega_s(u) = \sum_{u \in D_v} (1 - (1 - p)^{\alpha(u)}) \omega_s(u)$$

$$= \sum_{u \in D_v} (1 - (1 - p)^{\alpha(u)})(1 - p)^{\beta(u)} \omega_s(u)$$

$$= \sum_{u \in D_v} \left( (1 - (1 - p)^{\alpha(u)-1})(1 - p)^{\beta(u)+1} + p(1 - p)^{\alpha(u)-1}(1 - p)^{\beta(u)} \right) \omega(u)$$

$$= \sum_{u \in D_v} \left( (1 - (1 - p)^{\alpha(u)-1})\omega_s(u) + \sum_{u \in D_v} p(1 - p)^{\alpha(u)-1} \omega_s(u) \right)$$

$$= \sum_{u \in D_v} C_i(u, S \setminus \{v\}) + \sum_{u \in D_v} p(1 - p)^{\alpha(u)-1} \omega_s(u)$$

$$= C_i(D_v, S \setminus \{v\}) + \sum_{u \in D_v} p(1 - p)^{\alpha(u)-1} \omega_s(u)$$

By substituting the value of $C_i(D_v, S)$ into $C_i(X_1^+, S)$, we get the following:

$$C_i(X_1^+, S) = C_i(X_1^+ \setminus D_v, S \setminus \{v\}) + C_i(D_v, S \setminus \{v\}) + \sum_{u \in D_v} p(1 - p)^{\alpha(u)-1} \omega_s(u) + \omega_s(v)$$

$$= C_i(X_1^+, S \setminus \{v\}) + \sum_{u \in D_v} p(1 - p)^{\alpha(u)-1} \omega_s(u) + \omega_s(v)$$

In this case also, we have rewritten $C_i(X_1^+, S)$ as sum of $C_j(X_j^+, S \setminus \{v\})$ and term dependent on $v$ and the state $s$. Thus, since $S$ is an optimal solution for state $s$ in $i$, it follows that $S \setminus \{v\}$ is an optimal solution for the state $s_j$ in $j$.

**Forget node:** Let $i$ be a forget node with child $j$ such that $X_i = X_j \setminus \{v\}$ for some $v \in X_j$. Since $i$ is a forget node, $N(v) \cap (V \setminus X_i^+) = \emptyset$, that is, all neighbors of $v$ in $G$ are in $G_i$. Further, for each vertex $u \in X_i$, $N[u] \cap X_i^+ = N[u] \cap X_j^+$. We define the state $s_j$ and define $\text{Sol}_i[s]$ in terms of $\text{Sol}_j[s_j]$. We consider all possible values of $\gamma(v)$, $\alpha(v)$, and $\beta(v)$ to define the state $s_j$. These values specify the different states in $j$. For each $z \in \{0, 1\}$, define $\gamma^z : X_j \to \{0, 1\}$ as follows: for each $u \in X_j$,

$$\gamma^z(u) = \begin{cases} \gamma(u) & \text{if } u \neq v, \\ z & \text{if } u = v. \end{cases}$$

The parameter $z$ specifies whether $v$ should be in the desired solution or not. For each $x \in [0, b]$, define $\alpha^x : X_j \to [0, k]$ as follows:

$$\alpha^x(u) = \begin{cases} \alpha(u) & \text{if } u \neq v, \\ x & \text{if } u = v. \end{cases}$$
In the case when $\gamma^*(v) = 0$, then the parameter $x$ specifies the number of neighbors of $v$ which should be in the desired solution. Define $\beta' : X_j \rightarrow [0, k]$ as follows:

$$
\beta'(u) = \begin{cases} 
\beta(u) & \text{if } u \neq v, \\
0 & u = v. 
\end{cases}
$$

For each $z \in \{0, 1\}$ and each $x \in [0, b]$, let $s^{z, x}$ denote the state $(b, \gamma^z, \alpha^x, \beta')$ in $j$. If for each $z \in \{0, 1\}$ and each $x \in [0, b]$, $s^{z, x}$ is invalid, then the state $s$ also invalid. Therefore, we consider that there exists a 2-tuple $z \in \{0, 1\}$ and each $x \in [0, b]$ such that the state $s^{z, x}$ is valid. Further, we define the following 2-tuple as follows:

$$
z', x' = \arg\max_{z \in \{0, 1\}, x \in [0, b]} \text{Val}_{1}[s^{z, x}]. \quad (13)
$$

Define $s_j = s'^{z, x'} = (b, \gamma^{z'}, \alpha^{x'}, \beta')$ and the solution at the state $s$ as follows:

$$
\text{Sol}_i[s] = \text{Sol}_j[s_j], \quad (14)
$$

and

$$
\text{Val}_i[s] = \text{Val}_j[s_j]. \quad (15)
$$

The state $s_j$ can be computed in $O(k)$ time.

**Lemma 42.** Let $i$ be a forget node in $T$, and let $s$ and $s_j$ be as defined above. If $S$ is a solution of optimum coverage at state $s$, then $S$ is a solution of optimum coverage at the state $s_j$ in node $j$.

**Proof.** Let $\hat{s}$ be the state induced by $S$ in the node $j$. In the following argument, the coverage $C_i(X_i^+, S)$ is considered at state $s$ and the coverage $C_j(X_j^+, S)$ is considered at state $\hat{s}$ in node $j$. Since $X_i^+ = X_j^+$, $C_i(X_i^+, S) = C_j(X_j^+, S) = C_j(X_j^+, S) = \text{Val}_j[s]$.

Let us define $z = |\{v\} \cap S|$ and $x = |N[z] \cap S|$. From Equation 13 it is clear that the $z', x'$ chosen is such that $\text{Val}_j[s^{z', x'}] \geq \text{Val}_j[s^{z, x}] = \text{Val}_j[s]$. Since $s_j$ is defined as $s'^{z', x'}$, $\text{Val}_j[s_j] \geq \text{Val}_j[s^{z', x'}] = \text{Val}_j[s] = C_j(X_j^+, S)$. Since $S$ is an optimal solution at state $s$, and since $X_i^+ = X_j^+$, it follows that $C_j(X_j^+, S) \geq \text{Val}_j[s_j]$. Therefore $\text{Val}_i[s] = C_i(X_i^+, S) = C_j(X_j^+, S) = \text{Val}_j[s_j]$. Hence the lemma.

**Join node:** Let $i$ be a join node with children $j$ and $h$ such that $X_i = X_j = X_h$. We define the states $s_j$ and $s_h$, and define $\text{Sol}_i[s]$ in terms of $\text{Sol}_j[s_j]$ and $\text{Sol}_h[s_h]$. $s_j$ and $s_h$ are selected from a set consisting of $O(k^\omega)$ elements.

To define $s_j$ and $s_h$, we observe that since $X_i = X_j = X_h$, $\gamma^{-1}(1)$ is contained in both $X_j$ and $X_h$. Therefore, the $\gamma$ gets carried over from $s$ to $s_j$ and $s_h$.

Next, we identify the candidate values for the budget in the two states $s_j$ and $s_h$. We know that in a solution $S$ which induces the state $s$ at node $i$, $|\gamma^{-1}(1)|$ vertices are in $X_i$ and $b - |\gamma^{-1}(1)|$ vertices are in $X_i \setminus X_i$. Since $X_i \setminus X_i$ can be partitioned into two sets $X_i^+ \setminus X_i$ and $X_i^- \setminus X_i$, we consider a parameter $z$ to partition the value $b - |\gamma^{-1}(1)|$. For each $0 \leq z \leq b - |\gamma^{-1}(1)|$, let $b_{j, z} = |\gamma^{-1}(1)| + z$ and $b_{h, z} = b - z$: we consider states at nodes $j$ and $h$ with budget $b_{j, z}$ and $b_{h, z}$, respectively. In other words, for each $0 \leq z \leq b - |\gamma^{-1}(1)|$, we search for a solution of size $b_{j, z}$ for a subproblem on $X_j^+$ which contains $\gamma^{-1}(1)$ and $z$ vertices from $X_j^+ \setminus X_j$. Symmetrically, in node $h$, we search for a solution of size $b_{h, z}$ for a subproblem on $X_h^+$ which contains $\gamma^{-1}(1)$ and $b - \gamma^{-1}(1) - z$ vertices from $X_h^+ \setminus X_h$. These
two solutions taken together gives a solution for size $b$ at state $s$ in node $i$. To ensure that the constraints specified by $\alpha$ and $\beta$ are met, we next consider appropriate candidate functions in $s_j$ and $s_h$.

The different candidate functions to obtain $\alpha_j$ and $\alpha_h$ in the states $s_j$ and $s_h$, respectively are based on the coverage of vertices in $\gamma^{-1}(0)$. For each $u \in \gamma^{-1}(0)$, $\alpha(u)$ and $\beta(u)$ are distributed between $s_j$ and $s_h$, respectively. The number of possible ways in which this can be done is defined by the following set $A$. Let

$$A = \{ \eta : \gamma^{-1}(0) \to [0, b] \mid \text{for each } u \in \gamma^{-1}(0), 0 \leq \eta(u) \leq \alpha(u) - |N(u) \cap \gamma^{-1}(1)| \}. $$

A function $\eta \in A$ specifies that for each $u \in \gamma^{-1}(0)$, $|N(u) \cap \gamma^{-1}(1)| + \eta(u)$ neighbors of $u$ from $X_j^+$ should be in the solution from the node $j$ and $\alpha(u) - \eta(u)$ neighbors of $u$ from $X_h^+$ should be in the solution from the node $h$. In particular, in the solution at node $j$, $\eta(u)$ neighbors of $u$ must be in $X_j^+ \setminus X_j$ and, in the solution at node $h$, $\alpha(u) - \eta(u) - |N(u) \cap \gamma^{-1}(1)|$ neighbors of $u$ must be in $X_h^+ \setminus X_h$. More precisely, for each $\eta \in A$, we define the following functions. Let $\alpha_{j, \eta} : X_j \to [0, k]$ such that for each $u \in X_j$,

$$\alpha_{j, \eta}(u) = \begin{cases} \alpha(u) & \text{if } \gamma(u) = 1, \\ |N(u) \cap \gamma^{-1}(1)| + \eta(u) & \text{if } \gamma(u) = 0. \end{cases}$$

Let $\alpha_{h, \eta} : X_h \to [0, k]$ such that for each $u \in X_h$,

$$\alpha_{h, \eta}(u) = \begin{cases} \alpha(u) & \text{if } \gamma(u) = 1, \\ \alpha(u) - \eta(u) & \text{if } \gamma(u) = 0. \end{cases}$$

Note that $N(u) \cap \gamma^{-1}(1)$ is counted in both the nodes $j$ and $h$, and we take care of this after identifying the candidates functions for $\beta_j$ and $\beta_h$. Recall, that at state $s$ in node $i$, for each $u \in \gamma^{-1}(0)$, $\text{Sol}_{i}[s]$ provides a coverage value for $u$ as if it has $\alpha(u) + \beta(u)$ neighbors in $\text{Sol}_{i}[s]$. Among these, exactly $\alpha(u)$ must be selected from $X_j^+$. To ensure that this constraint is met, we consider the following candidate functions for $\beta_j$ and $\beta_h$. For each $\eta \in A$, define $\beta_{j, \eta} : X_j \to [0, k]$ such that for each $u \in X_j$,

$$\beta_{j, \eta}(u) = \begin{cases} \beta(u) & \text{if } \gamma(u) = 1, \\ \beta(u) + \alpha(u) - |N(u) \cap \gamma^{-1}(1)| - \eta(u) & \text{if } \gamma(u) = 0. \end{cases}$$

Observe that $\alpha_{j, \eta}(u) = |N(u) \cap \gamma^{-1}(1)| - \eta(u)$ is subtracted from $\alpha(u) + \beta(u)$. Symmetrically, $\alpha_{h, \eta}(u)$ is subtracted from $\alpha(u) + \beta(u)$ to obtain a candidate $\beta_h$. Let $\beta_{h, \eta} : X_h \to [0, k]$ such that for each $u \in X_h$,

$$\beta_{h, \eta}(u) = \begin{cases} \beta(u) & \text{if } \gamma(u) = 1, \\ \beta(u) + \eta(u) & \text{if } \gamma(u) = 0. \end{cases}$$

Using the candidate values for budget, $\gamma$, $\alpha$, and $\beta$ at node $j$ and node $h$, we now specify the set of candidate states to be considered at nodes $j$ and $h$. For each $0 \leq z \leq b - |\gamma^{-1}(1)|$ and each $\eta \in A$, let $s_{j, z, \eta} = (b_{j, z}, \gamma, \alpha_{j, \eta}, \beta_{j, \eta})$ and $s_{h, z, \eta} = (b_{h, z}, \gamma, \alpha_{h, \eta}, \beta_{h, \eta})$.

Next, to write $\text{Sol}_{i}[s]$ in terms of coverage of $X_j^+$ and $X_h^+$, we need to identify vertices whose contribution the coverage would be over-counted when we take the union of $\text{Sol}_{i}[s_j]$ and
Lemma 43. Since $X_i = X_j = X_h$, two cases need to be handled. First, the vertices on $\gamma^{-1}(1)$ are counted twice, once in $X_j$ and once in $X_h$. Therefore, in the expansion of $C_i(X_j^+, S)$ in terms of the coverage at $X_j$ and $X_h$, we will have to subtract out $\sum_{u \in \gamma^{-1}(1)} \omega_u(u)$. Secondly, for each $\eta \in A$ and each $u \in \gamma^{-1}(0)$, the coverage of $u$ by $\gamma^{-1}(1)$ is counted twice, once in $X_j$ and once in $X_h$. To subtract this over-counting we introduce the following function $\lambda$ associated with the state $s$ at the join node $i$. It is an easy arithmetic exercise to verify that $\lambda(\eta, u)$ is the value that must be subtracted. For each $\eta \in A$ and for each $u \in \gamma^{-1}(0)$, let

$$\lambda(\eta, u) = \left((1 - p)^{a(u) - |N(u)|\gamma^{-1}(0)} - \gamma^{-1}(0) - (1 - p)^{q(u)} - (1 - p)^{a(u)} - 1\right)\omega_u(u).$$

Finally, we come to the recursive specification of $Sol_i[s]$. If for each $0 \leq z \leq b - |\gamma^{-1}(1)|$ and each $\eta \in A$, either $s_{j,z,\eta}$ or $s_{h,z,\eta}$ is invalid, then $s$ is also invalid. Therefore, we consider those values of $0 \leq z \leq b - |\gamma^{-1}(1)|$ and $\eta \in A$ such that both the states $s_{j,z,\eta}$ and $s_{h,z,\eta}$ are valid. Further, we define the following tuple:

$$z', \eta' = \arg \max \left\{ \begin{array}{l} Val_j[s_{j,z,\eta}] + Val_h[s_{h,z,\eta}] - \sum_{u \in \gamma^{-1}(0)} \lambda(\eta, u) \end{array} \right\}$$

Define $s_j = s_j(z', \eta') = (b_j, z', \gamma, \alpha_{j,\eta'}, \beta_{j,\eta'})$ and $s_h = s_h(z', \eta') = (b_h, z', \gamma, \alpha_{h,\eta'}, \beta_{h,\eta'})$. Then, the solution for the state $s$ is defined as follows:

$$Sol_i[s] = Sol_j[s_j] \cup Sol_h[s_h],$$

and

$$Val_i[s] = Val_j[s_j] + Val_h[s_h] - \sum_{u \in \gamma^{-1}(0)} \lambda(\eta', u) - \sum_{u \in \gamma^{-1}(1)} \omega_u(u).$$

The cardinality of the set $A$ is at most $O(k^w)$. This is the most dominant term in the size of the recursive definition. This is because for each fixed $\eta \in A$, $\alpha$ and $\beta$ are uniquely defined. Then, the states $s_j$ and $s_h$ can be computed in $O(k^{w+1}wn)$ time.

**Lemma 43.** Let $s$ be a state in the join node $i$ in $T$ and let $s_j$ and $s_h$ be as defined above. Let $S$ be a solution of optimum coverage at state $s$, then $C_i(X_j^+, S) \leq Val_i[s]$.

**Proof.** Let $\hat{s} = (\hat{b}, \hat{\gamma}, \hat{\alpha}, \hat{\beta})$ and $\check{s} = (\check{b}, \check{\gamma}, \check{\alpha}, \check{\beta})$ be the states induced at the nodes $j$ and $h$ by the set $S$, respectively. In this context, we consider the coverage functions $C_i(\cdot, \cdot)$, $C_j(\cdot, \cdot)$, and $C_h(\cdot, \cdot)$ are considered at states $s$, $\hat{s}$, and $\check{s}$, respectively.

Let $S_j = S \cap (X_j^+ \setminus X_j)$ and $S_h = S \cap (X_h^+ \setminus X_h)$. Let $z = |S_j|$ and $\eta : \gamma^{-1}(0) \rightarrow [0, b]$ such that for each $u \in \gamma^{-1}(0)$, $\eta(u) = N(u) \cap (X_j^+ \setminus X_j)$. We now define $C_i(X_j^+, S)$ in terms of $C_j(X_j^+, S)$, $C_h(X_h^+, S)$, and a subtracted term dependent on $z$, $\eta$, and $s$. This is done as follows and we ensure that the coverage is exactly counted.

$$C_i(X_j^+, S) = C_i(X_j^+ \setminus X_j, S) + C_i(X_j, S)$$

$$= C_i(\gamma^{-1}(1), S) + C_i(\gamma^{-1}(0), S) + C_i(X_j^+ \setminus X_j, S) + C_i(X_h^+ \setminus X_h, S) + C_i(\gamma^{-1}(0), S) + \sum_{u \in \gamma^{-1}(0)} C_i(u, S)$$

The first equality follows by the partition $X_j^+$ into $X_j^+ \setminus X_j$ and $X_j$. In the second equality, the set $X_j^+ \setminus X_j$ is partitioned into $X_j^+ \setminus X_j$ and $X_h^+ \setminus X_h$, and the set $X_j$ is partitioned in $\gamma^{-1}(0)$ and $\gamma^{-1}(1)$. The third equality follows from the fact that for each $u \in X_j^+ \setminus X_j$, $\omega_u(u) = \omega_u(u)$ and $N[u] \cap S = N[u] \cap S_j$. Similarly, for each $u \in X_h^+ \setminus X_h$, $\omega_u(u) = \omega_u(u)$.
and \(N[u] \cap S = N[u] \cap S_h\).

Next, we consider the term \(\sum_{u \in \gamma^{-1}(0)} C_i(u, S)\). For each \(u \in \gamma^{-1}(0)\), \(C_i(u, S)\) can be written as sum of coverages of \(u\) by the sets \(S_j\) and \(S_h\). In particular, here we carefully use the values of \(\hat{\delta}\) and \(\hat{\beta}\) at node \(j\) and \(\hat{\alpha}\) and \(\hat{\beta}\) at node \(h\). Further, each of the equations follows by simple arithmetic and the definition of \(C_i(u, S_j)\), \(C_i(u, S_h)\), \(\hat{\alpha}\), and \(\hat{\beta}\).

\[
C_i(u, S) = (1 - (1 - p)^{\alpha(u)})(1 - (1 - p)^{\beta(u)} \omega(u))
\]

\[
= \left(1 - (1 - p)^{\hat{\alpha}(u)} \right) \left(1 - (1 - p)^{\hat{\beta}(u)} \omega(u) \right) + \left( 1 - (1 - p)^{\hat{\delta}(u)} \right) \left(1 - (1 - p)^{\hat{\beta}(u)} \omega(u) \right)
\]

\[
= C_j(u, S_j) + C_h(u, S_h)
\]

\[
= C_j(u, S_j) + C_h(u, S_h)
\]

Using \(\hat{\delta}\) and \(\hat{\beta}\), it follows that \(C_j(X_j, S) = C_j(\gamma^{-1}(0), S_j) + C_j(\gamma^{-1}(1), S)\). Similarly, using \(\hat{\alpha}\) and \(\hat{\beta}\), it follows that \(C_h(X_h, S) = C_h(\gamma^{-1}(0), S_h) + C_h(\gamma^{-1}(1), S)\). Therefore,

\[
C_l(X_l, S) = C_j(X_j^+ \setminus X_j, S_j) + C_h(X_h^+ \setminus X_h, S_h) + C_j(X_j, S) + C_h(X_h, S)
\]

Putting together the terms corresponding to \(C_j(\cdot, \cdot)\) and \(C_h(\cdot, \cdot)\) it follows that

\[
C_l(X_l, S) = C_j(X_j^+, S) + C_h(X_h^+, S) - C_l(\gamma^{-1}(1), S) - \sum_{u \in \gamma^{-1}(0)} \lambda(\eta, u)
\]

Since \(\hat{s}\) and \(\hat{s}\) are states at nodes \(j\) and \(h\) induced by the state \(S\), we know that \(C_j(X_j^+, S) = Val_{j}[s]\) and \(C_h(X_h^+, S) = Val_{h}[s]\). Therefore,

\[
C_l(X_l^+, S) \leq Val_{j}[s] + Val_{h}[s] - \sum_{u \in \gamma^{-1}(1)} \omega_s(u) - \sum_{u \in \gamma^{-1}(0)} \lambda(\eta, u)
\]

Further, by the choice of \(\gamma', \eta'\) in Equation \[16\] we know that \(\hat{s}\) and \(\hat{s}\) do not result in a larger value than \(s_j\) and \(s_h\). Formally, we know that

\[
Val_{j}[s] + Val_{h}[s] - \sum_{u \in \gamma^{-1}(1)} \omega_s(u) - \sum_{u \in \gamma^{-1}(0)} \lambda(\eta, u) \leq Val_{l}[s]
\]

Hence the Lemma.
7.2.2 Bottom-Up Evaluation: Correctness of the DP Formulation

Correctness invariant. For a node $i$ and a valid state $s$ at $i$, the recursive definition in Section 7.2.1 ensures that

$$\text{Sol}_i[s] = \max_{D \subseteq V_i | |D| = k, D \text{ induces } s} C_i(X_i^+, D \cap X_i^+)$$

We prove this invariant by induction on the height of a node in the proof of the following theorem.

Theorem 44. The Uni-PBDS problem can be solved in time $2^{O(w \log k)} n^2$ where $w$ is treewidth of the input graph.

Proof. We first show that the bottom-up evaluation of the tables in $T$ maintains the correctness invariant.

Invariant: For each node $i$ in $T$, and for each state $s \in \mathcal{S}_i$, the correctness invariant is maintained for $\text{Sol}_i[s]$.

Proof of Invariant. The proof is by induction on the height of a node in $T$. Recall, that the height of a node $i$ in the rooted tree $T$ is the distance to the furthest leaf in the subtree rooted at $i$. The base case is when $i$ is a leaf node $T$ and height is 0 and the proof of the claim follows from Lemma [23]. Let us assume that the claim is true for all nodes in $T$ of height at most $\ell - 1 \geq 0$. We now prove that if the claim is true for all nodes of height at most $\ell - 1$, then it is true for a node of height $\ell$. Let $i$ be a node of height $\ell \geq 1$. Since $i$ is not a leaf node, its children are at height at most $\ell - 1$. Therefore, by the induction hypothesis, the correctness invariant is maintained at all the children of $i$. Now, we prove that the correctness invariant is maintained at node $i$. Let $s$ be a state in node $i$. Let $S$ be an optimum solution at state $s$ in node $i$. We show that $\text{Val}_i[s] = C_i(X_i^+, S)$. If $i$ is an introduce node then from Lemma [24] we know that the optimum coverage of $X_i^+$ is achieved by $S \setminus \{v\}$ at state $s_j$ at node $j$. Similarly, if $i$ is a forget node, then from Lemma [22] we know that the optimum coverage of $X_i^+$ is achieved by $S$ at state $s_j$ at node $j$. By the induction hypothesis in both these cases $\text{Sol}_i[s]$ is the set which achieves optimum coverage, and this proves that $\text{Sol}_i[s]$ is the optimum value at state $s$. Further, if $i$ is a join node, then from the description of the computation at a join node, we know that $\text{Val}_i[s]$ is recursively defined using $\text{Val}_i[s_j]$ and $\text{Val}_i[s_h]$ for an appropriate $s_j$ and $s_h$. By the induction hypothesis, $\text{Val}_i[s]$ and $\text{Val}_i[s_h]$ are the optimal values. Therefore, it follows from Lemma [25] that $C(X_i^+, S) \leq \text{Val}_i[s_i]$ and thus $\text{Val}_i[s]$ is the optimum value. Therefore, it follows from the induction hypothesis that the solution and value are correctly computed at state $s$ based on the correct values computed at $s_j$ and $s_h$. This completes the proof of the invariant.

Finally, at the root node $r$, the state set $\mathcal{S}_r$ is a singleton set with a state $s = (k, \emptyset \to \{0, 1\}, \emptyset \to [0, k], \emptyset \to [0, k])$. By the induction hypothesis, the solution and the value maintained at this state are indeed the set that achieves the optimum coverage and the value of the coverage, respectively. Finally, a node $i \in V(T)$ can have $k(2k + 2)^{w+3}$ states and each of them can be computed in time $O((k + 1)w^{w+1}n)$. Since the nice tree decomposition $(T, X)$ has $O(w n)$ nodes, the tables at the nodes in $T$ can be computed in time $O(w^2 (2k + 2k)w^8 n^2) = 2^{O(w \log k)} n^2$. This completes the proof of the theorem.

Acknowledgement. We thank an anonymous reviewer for pointing us to [9], yielding a shorter proof of the FPT algorithm for Uni-PBDS parameterized by treewidth and $k$.
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