DESCENT, FIELDS OF INVARIENTS, AND GENERIC FORMS VIA SYMMETRIC MONOIDAL CATEGORIES

EHUD MEIR

Abstract. Let $W$ be a finite dimensional algebraic structure (e.g. an algebra) over a field $K$ of characteristic zero. We study forms of $W$ by using Deligne’s Theory on symmetric monoidal categories. We construct a category $C_W$, which gives rise to a subfield $K_0 \subseteq K$, which we call the field of invariants of $W$. The category $C_W$ also enables us to construct a generic form $\tilde{W}$ over a commutative $K_0$ algebra $B_W$ (so that forms of $W$ are exactly the specializations of $\tilde{W}$). This generalizes some generic constructions for central simple algebras and for $H$-comodule algebras. We give some concrete examples arising from associative algebras and $H$-comodule algebras.

As an application, we also explain how can one use the construction to classify two cocycles on some finite dimensional Hopf algebras.

1. Introduction

Let $W$ be a finite dimensional algebraic structure (e.g. an algebra, a Hopf algebra, a comodule algebra, a module over a given algebra et cetera) defined over a field $K$ of characteristic zero (for the clarity of the exposition we will assume that $W$ is an algebra and that $K$ is algebraically closed). In this paper we will discuss the following questions:

1. Over what subfields $K_1$ of $K$ can $W$ be defined (i.e. over what subfields of $K$ does $W$ have a form)?
2. In case $W$ can be defined over a subfield $K_1$ of $K$, what are all the forms of $W$ over this subfield?

These questions are very hard in general. For example, the forms of $M_n(K)$ are all the central simple algebras of dimension $n^2$ over some subfield of $K$.

In this paper we will address these questions by using tools from the theory of symmetric monoidal categories, and by using a theorem of Deligne, which says that every symmetric monoidal category of a specific type is the representation category of an algebraic group. We will construct a symmetric rigid monoidal category $C_W$ (which we shall call the fundamental category of $W$). This category will be defined over a subfield $K_0$ of $K$. In a sense which will be explained later, $C_W$ will be the symmetric rigid monoidal category “generated by $W$” and the field $K_0$ will be the “field of invariants” of $W$. The category itself will be an invariant of the isomorphism type of $W$, and we will
show (see Lemma 5.7) that $W$ can be reconstructed from the following data: the category $\mathcal{C}_W$ together with a distinguished object (which corresponds to $W$) and a distinguished morphism (which corresponds to the multiplication $m : W \otimes W \to W$. In case $W$ is another algebraic structure, we might have more than one distinguished morphism).

We will then use Deligne’s Theory on symmetric monoidal categories to study the category $\mathcal{C}_W$. In Section 5 we will prove the following result:

**Proposition 1.1.** The category $\mathcal{C}_W$ is a $K_0$-form of the $K$-linear category $\text{Rep}_K - G$, where $G$ is the automorphism group of $W$. For any subfield $K_0 \subseteq K_1 \subseteq K$, we have a one to one correspondence between forms of $W$ over $K_1$ and isomorphism classes of fiber functors $F : \mathcal{C}_W \otimes_{K_0} K_1 \to \text{Vec}_{K_1}$.

Proposition 1.1 gives us a description of forms of $W$ in terms of fiber functors on the category $\mathcal{C}_W$ (by a fiber functor we mean here an exact symmetric monoidal functor, see Section 2). However, we would like to have a more concrete answer to questions 1 and 2. In Section 6 we will use Deligne’s Theory in order to construct a “generic form” of $W$, which specializes to all forms of $W$, and only to forms of $W$. More precisely, we will prove the following result:

**Proposition 1.2.** There exists a commutative $K_0$-algebra $B_W$ and a $B_W$-algebra $\tilde{W}$ such that:
1. The module $\tilde{W}$ is a free $B_W$-module of rank $\dim_K W$.
2. If $\phi : B_W \to K_1$ is a homomorphism of $K_0$-algebras from $B_W$ to a field $K_1$, then $W_\phi := \tilde{W} \otimes_{B_W} K_1$ is a form of $W$ over $K_1$.
3. Every form of $W$ is received in this way.
4. The algebra $B_W$ has no zero divisors. If the group $G$ is reductive, then the algebra $B_W$ can be chosen to be finitely generated.

Thus, $W$ will have a form over $K_1$ if and only if there is a homomorphism $B_W \to K_1$, and any form of $W$ over $K_1$ will be of the form $W_\phi$ for some homomorphism $\phi : B_W \to K_1$. In case $W$ is an algebraic structure which is not an algebra, then $\tilde{W}$ will be an algebraic structure of the same type.

The field $K_0$ contains elements which must be in any field over which $W$ has a form. We will prove in Section 7 the following characterization of $K_0$ in case the extension $K/K_0$ is algebraic.

**Proposition 1.3.** Assume that $K_0$ has a subfield $L$ such that $K/L$ is an algebraic extension (so in particular, $K/K_0$ is algebraic). Let $\Gamma := \text{Gal}(K/L)$. The group $\Gamma$ acts on the set of isomorphism classes of algebras of dimension $\dim(W)$ (this is just the Galois action on the structure constants). Denote by $H < \Gamma$ the stabilizer of the isomorphism class of $W$. Then $K^H = K_0$. 

The field $K_0$ also plays a role in studying polynomial identities of $W$. In Section 8 we will explain some connections between the category $\mathcal{C}_W$ and polynomial identities of $W$ (for this part only we need $W$ to be an algebra or an $H$-comodule algebra). We will explain how polynomial identities can be understood in the context of $\mathcal{C}_W$, and we will prove the following proposition:

**Proposition 1.4.** Let $W$ be an algebra or an $H$-comodule algebra of finite dimension over $K$, where $H$ is a finite dimensional Hopf algebra over a subfield $k \subseteq K$. Then all the identities of $W$ are already defined over $K_0$. (In case $W$ is an $H$-comodule algebra, we will necessarily have that $k \subseteq K_0$).

Finally, we will give some concrete examples in Sections 9, 10 and 11. In Section 9 we will consider a three dimensional associative algebra. We will describe $K_0$ in that case, and we will show that all the identities are already defined over a proper subfield of $K_0$.

In Section 10 we will consider the case where $W$ is the algebra $M_n(K)$. In this case, $K_0 = \mathbb{Q}$ and we will see that we can choose $B_W$ to be a localization of $K[M_n(K) \times M_n(K)]^{PGL_n}$, where the action of $PGL_n$ is by conjugation. This construction is not new. Amitsur was the first to construct a generic division algebra $R$ by using the polynomial identities of $M_n(K)$. Later, Procesi gave an alternative description of this algebra as the subalgebra of $M_n(K[x_{ij}])$ generated by generic matrices. Procesi introduced also the algebra $\overline{R}$, which is formed by joining to $R$ the traces of all elements in $R$ (which are central in $R$). The reader is referred to the paper [9] by Formanek for a survey on this.

The generic form we get here will be a localization of $\overline{R}$ by a central element.

In Section 11 we will study in detail the case of a twisted Hopf algebra. More precisely, let $H$ be a Hopf algebra defined over a subfield $k \subseteq K$, let $\alpha : H \otimes H \to K$ be a convolution invertible two cocycle, and let $W = \alpha H$ be the resulting twisted algebra. Then $W$ is an $H$-comodule algebra (the structure we will consider for $W$ here will be the multiplication in $W$ and the action of $H^*$ on $W$). Such $H$-comodule algebras can be thought of as the noncommutative analogue of a principal fiber bundle, where the group $G$ is replaced by the Hopf algebra $H$. (see e.g. [17]). They also coincide with the class of cleft Hopf Galois extensions of the ground field $k$. In [3] Aljadeff and Kassel studied such comodule algebra for general Hopf algebras (not necessarily finite dimensional ones) by means of polynomial $H$-identities. They have constructed an algebra $A^\alpha_H$, which is formed by taking the most general two cocycle cohomologous to $\alpha$. This algebra can be seen as a Hopf Galois extension of the commutative subalgebra $B^\alpha_H := (A^\alpha_H)^{coH}$. They have shown that if $\beta H$ is a form of $\alpha H$ over an extension $L$ of $k$, then there exists a homomorphism $\phi : B^\alpha_H \to L$ such that $A^\alpha_H \otimes_{B^\alpha_H} L \cong \beta H$, where
and that every homomorphism $B_H^\alpha \to L$ will give rise to a form of $^\alpha H$ if a certain integrality condition holds. In [13] this integrality condition was proved for finite dimensional Hopf algebras (among other cases). Thus, $A_H^\alpha$ is a generic form of $^\alpha H$. A generic form for a twisted group algebra was constructed in [2]. The nature of our construction of the generic form is different from that of $A_H^\alpha$. The main difference is the fact that here we will concentrate more on the algebra structure and the action of $H^*$ then on the fact that this algebra arises from a two cocycle. For a twisted group algebra, the construction we will give here will be very similar to the one which appears in [2]. The main difference is that our base ring will have smaller transcendence degree over the ground field. We will discuss the generic construction for group algebras, Taft algebras and products of Taft algebras. The nature of our construction of the fundamental category can help us classify all cocycles on $H$, where $H$ is a Taft algebra or a product of Taft algebras.

This paper is organized as follows: In Section 2 we give some preliminaries about structures and monoidal categories. In Section 3 we will construct the category $C_W$. In Section 4 we will describe the way the category $C_W$ behaves with respect to isomorphisms of structures and to field extensions. We will show that all forms of $W$ will give rise to equivalent categories. In Section 5 we will prove Proposition 1.1 and in Section 6 we will construct the generic form and prove Proposition 1.2. In Section 7 we will explain the connection of our construction to classical descent theory and we will prove Proposition 1.3. In Section 8 we will explain the relation with polynomial identities and prove Proposition 1.4. Finally, we will give examples in Sections 9, 10 and 11.

2. Preliminaries

2.1. Structures. Let $W$ be a finite dimensional vector space over a field $K$ (not necessarily algebraically closed) of characteristic zero. Let $x_i \in W^\otimes p \otimes (W^*)^\otimes q = W^{p,q}$ be a set of tensors. We call the pair $(W, \{x_i\})$ an algebraic structure or just structure. For example, if $W$ is an algebra then the algebra structure is given by the multiplication map $m : W \otimes W \to W$. This can be considered as a tensor $m \in W^{1,2}$ (we will use excessively the identification $\text{Hom}_K(V,W) \cong V^* \otimes W$ in what follows). If $W$ is a module over an algebra $A$, then for every $a \in A$ we have a tensor $T_a \in W^{1,1}$ which specifies the action of $a$.

If $x \in W^{p,q}$ we call $(p, q)$ the type of the tensor $x$. The type of a tensor is thus an element of $\mathbb{N}^2$ (we consider here 0 as a natural number). Let $(W, \{x_i\})$ and $(W', \{y_i\})$ be two structures such that the type of $x_i$
is the same as the type of \( y_i \) for each \( i \). If \( \psi : W \to W' \) is a linear isomorphism, then \( \psi \) induces a linear isomorphism \( \psi^{p,q} : W^{p,q} \to W'^{p,q} \). An isomorphism between the structures \((W, \{x_i\})\) and \((W', \{y_i\})\) is a linear isomorphism \( \psi : W \to W' \) such that \( \phi(x_i) = y_i \) for every \( i \). If \( L \) is an extension field of \( K \), then by extension of scalars we also get a structure over \( L \). This vector space will be \( W_L := W \otimes_K L \). Since \( W^{p,q}_L \) is naturally a subset of \( W^{p,q}_L \) we also get tensors which we denote by the same letter \( x_i \in W^{p,q}_{L,i} \). We write the extension of scalars structure as \((W_L, \{x_i\})\).

2.2. Monoidal categories. We will recall here some facts about monoidal categories. For more detailed discussion we refer the reader to Chapter VII of [14], Chapter XI of [11] and to the papers [4], [5], [6]. A monoidal category \( \mathcal{C} \) is a category equipped with a product (which we shall always denote by \( \otimes \))

\[
\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}.
\]

The category contains a unit object \( 1 \) with respect to that multiplication, and we have, for every \( X, Y, Z \in \text{Ob}\mathcal{C} \) functorial isomorphisms:

\[
\lambda_X : 1 \otimes X \to X,
\]

\[
\rho_X : X \otimes 1 \to X
\]

and \( \alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z) \).

These functorial isomorphisms should satisfy \( \rho_1 = \lambda_1 \) and should make the following two diagrams commute:

\[
\begin{array}{ccc}
(X \otimes 1) \otimes Y & \longrightarrow & X \otimes (1 \otimes Y) \\
\uparrow & & \uparrow \\
X \otimes Y & = & X \otimes Y \\
\end{array}
\]

\[
\begin{array}{ccc}
(X \otimes (Y \otimes Z)) \otimes W & \longrightarrow & X \otimes ((Y \otimes Z) \otimes W) \\
\uparrow & & \uparrow \\
(X \otimes (Y \otimes Z)) \otimes W & \longrightarrow & X \otimes (Y \otimes (Z \otimes W)) \\
\end{array}
\]

We will often omit the functorial isomorphism \( \alpha \) in our definitions and computations. This will make no harm, since by Mac Lane coherence the associativity isomorphisms can be inserted to every diagram in a unique way. A monoidal category is called symmetric if in addition we have a functorial isomorphism \( c_{X,Y} : X \otimes Y \to Y \otimes X \). This isomorphism should satisfy the symmetry condition \( c_{Y,X}c_{X,Y} = Id_{X \otimes Y} \) and the associativity coherence condition \( c_{X,Z}c_{Y,Z} = c_{X \otimes Y, Z} \). If \( X \) is an object of a symmetric monoidal category \( \mathcal{C} \), then a dual object of \( X \) is an object \( X^* \), equipped with two morphisms: \( ev_X : X^* \otimes X \to 1 \) and
coev}_X : 1 \to X \otimes X^* \text{ such that the following two morphisms are the identity morphisms:}

\[ X \xrightarrow{coev}_X \otimes \text{Id}_X X \otimes X^* \xrightarrow{\text{Id}_X \otimes ev}_X X \]
\[ X^* \xrightarrow{Id_X \otimes coev}_X X^* \otimes X \otimes X^* \xrightarrow{ev}_X \otimes \text{Id}_X X^*. \]

We say that \( C \) is rigid in case every object has a dual. In general monoidal categories we need to be more careful and define right and left duals. Since the category \( C \) is symmetric, we can avoid this distinction.

When \( C \) is rigid we have functorial isomorphisms for every \( X,Y,Z \in \text{Ob} C \):

\[ \text{Hom}_C(X^* \otimes Y,Z) \cong \text{Hom}_C(Y,X \otimes Z) \]
\[ \text{Hom}_C(Y \otimes X,Z) \cong \text{Hom}_C(Y,Z \otimes X^*). \]

If \( C \) and \( D \) are two monoidal categories, then a functor \( F : C \to D \) is called monoidal if it is equipped with functorial isomorphisms \( f_{X,Y} : F(X \otimes Y) \to F(X) \otimes F(Y) \) which is compatible with the associativity isomorphisms in \( C \) and in \( D \) (we refer to XI.4.1 in the book [11] for an exact definition). If \( C \) and \( D \) are symmetric monoidal categories and \( F : C \to D \) is a monoidal functor, then \( F \) is called symmetric in case the following diagram commutes:

\[ \begin{array}{ccc}
F(X \otimes Y) & \xrightarrow{f_{X,Y}} & F(Y \otimes X) \\
\downarrow & & \downarrow \\
F(X) \otimes F(Y) & \xrightarrow{f_{X,Y}} & F(Y) \otimes F(X).
\end{array} \]

In other words, \( F \) should “translate” the symmetry in \( C \) to the symmetry in \( D \). Notice that if \( C \) and \( D \) are rigid, we will get an isomorphism \( F(X^*) \cong F(X)^* \) without further restrictions on \( F \).

If \( K \) is any field, a \( K \)-linear category is an abelian category in which all the hom sets are \( K \)-vector spaces and in which the composition is \( K \)-bilinear. Our main focus in this paper will be on categories \( C \) which are \( K \)-linear symmetric rigid monoidal categories. We will further assume that \( \text{End}_C(1) = K \). The general example to keep in mind is the following: let \( G \) be an affine algebraic group over \( K \). Then the category \( C = \text{Rep}_K - G \) of finite dimensional representations of \( G \) is such a category. Indeed, this category is abelian. If \( V \) and \( W \) are two \( G \)-representations then \( V \otimes W \) is also a representation, by the diagonal action, and \( V^* \) is a representation by the dual action: \( g \cdot f = f(g-). \)

The tensor identity \( 1 \) is the trivial one dimensional representation. In fact, all the examples which we will encounter in this paper will be forms of \( \text{Rep}_K - G \) (we will prove this in Section 5).

If we take \( G \) to be the trivial group we get the category \( \text{Vec}_K \) of finite dimensional \( K \)-vector spaces. An exact symmetric tensor functor \( F : C \to \text{Vec}_K \) (or more generally, \( F : C \to B - \text{mod} \) where \( B \) is some commutative algebra) is also called a fiber functor. For example, if
If \( \mathcal{C} \) is a symmetric monoidal category over \( K \), we can carry a lot of the constructions usually done in \( \text{Vec}_K \) in \( \mathcal{C} \). For example, we can still talk about algebras inside \( \mathcal{C} \): an algebra will be an object \( A \in \mathcal{C} \) together with a morphism \( m : A \otimes A \to A \). The algebra \( A \) is said to be associative in case the two compositions \( A \otimes A \otimes A \to A \) are equal. It is said to be commutative if \( m = mc_{A,A} \).

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If \( X \) is any object of \( \mathcal{C} \), we can still construct the tensor algebra \( T(X) \) and its maximal commutative quotient \( \text{Sym}(X) \). These algebras are formed as infinite direct sums of objects of \( \mathcal{C} \). Therefore, they will not necessarily be contained in \( \mathcal{C} \), but in a bigger category, \( \text{Ind}(\mathcal{C}) \) (See Section 2.2 of [4] for an exact definition. This will not make much difference for us). If \( A \) is an associative commutative algebra with unit, \( D \) is a tensor-invertible object inside \( \mathcal{C} \) (so that \( \text{ev} : D \otimes D^* \to 1 \) is an isomorphism), and we have a nonzero map \( f : D \to A \), then we can form the localization \( A_f \) in the following way: Let \( A := A \otimes \text{Sym}(D^*) \). We have two maps \( 1 \to A \). The first is the one that sends \( 1 \) to the identity of \( A \). The second one is given as the composition \( 1 \xrightarrow{\text{coev}} D \otimes D^* \xrightarrow{f \otimes 1} A \otimes D^* \subseteq \tilde{A} \).

We consider the difference of these two maps \( \tilde{f} : 1 \to \tilde{A} \) and we define \( A_f := A/\langle \tilde{f} \rangle \) where \( \langle \tilde{f} \rangle \) is the ideal generated by \( \tilde{f} \). For example, if \( \mathcal{C} = \text{Rep}_K - G \), then \( D \) is a one dimensional object, \( f(D) = K \cdot \hat{f} \) for some \( \hat{f} \in A \), and the localization \( A_f \) is the same as \( A_{\hat{f}} \). We will sometimes write \( A_f \) for \( A_{\hat{f}} \).

A module \( M \) over an associative algebra \( A \) will be an object of \( \mathcal{C} \) together with a morphism \( m_M : A \otimes M \to M \) such that the usual module axioms hold. If \( A \) is an associative commutative algebra in \( \mathcal{C} \), then the category \( A - \text{mod} \) of \( A \) modules inside \( \mathcal{C} \) is again a symmetric monoidal category. Indeed, the tensor product of two \( A \)-modules is given by \( M \otimes_A N := \text{Coker}(g : M \otimes A \otimes N \to M \otimes N) \), where \( g \) is given by \( m_M \otimes 1_N - 1_M \otimes m_N \) (we use here the fact that left and right modules are the same over a commutative algebra). The objects \( M \otimes_A N \) is again an \( A \)-module.

3. Construction of the fundamental category

Let \( K \) be a field of characteristic zero and let \( (W, \{ x_i \}) \) be a finite dimensional structure over \( K \). We are going to construct the fundamental category \( \mathcal{C}_W \) of \( W \). We will construct this category as an ascending sequence (or a direct limit) of categories:

\[
\mathcal{C}_W = \bigcup_{i \geq 0} \mathcal{C}_i.
\]
All the categories $C_i$ will be additive, and all morphisms in $C_i$ will have kernels and cokernels in $C_{i+1}$ (this will ensure us that $C$ will be abelian). Moreover, for each $i$ we will have a symmetric tensor functor $F_i : C_i \to Vec_{K}$, and their union will give us a functor $F : C_W \to Vec_{K}$.

3.1. Construction of the first category. We will begin by constructing a pre-additive category, $C_0$ together with an additive tensor functor $F_0 : C_0 \to Vec_{K}$. We then define $C_1$ to be the additive envelope of $C_0$, (we will give later a definition of the additive envelope). We will show that $F_0$ extends to an additive tensor functor $F_1 : C_1 \to Vec_{K}$. The idea is that $C_0$ will be the tensor category “freely generated” by $W$ and $W^*$, and the morphisms will be everything that arises from the tensors (or the “structure”) $\{x_i\}$. We begin with defining a collection of $\mathbb{Q}$-subspaces $X^{p,q} \subseteq W^{p,q}$ for every pair $(p,q) \in \mathbb{N}^2$. These subspaces will not necessarily be $K$-subspaces. However, we will see that $\mathbb{Q} \subset X^{0,0} \subseteq K$ might be a proper intermediate field, and that $X^{p,q}$ will be a vector space over $X^{0,0}$ for every $(p,q)$.

We give the following definition:

**Definition 3.1.** $X^{p,q} \subseteq W^{p,q}$ is the smallest collection of $\mathbb{Q}$-subspaces which satisfy the following conditions:

1. $x_i \in X^{p,q}$
2. the identity map $Id_W \in End_K(W)$ is contained in $X^{1,1}$.
3. The concatenation of $X^{p,q}$ with $X^{r,s}$ is contained in $X^{p+r,q+s}$.
4. If $ev_{p,q} : W^{p,q} \to W^{p-1,q-1}$ is the map which evaluates the first copy of $W^*$ on the first copy of $W$ then $ev(X^{p,q}) \subseteq X^{p-1,q-1}$.
5. For any $\sigma \in S_p$ and $\tau \in S_q$ we have that $(\sigma, \tau)(X^{p,q}) = X^{p,q}$, where the action is given by permuting the tensor factors.
6. If $0 \neq x \in X^{0,0} \subseteq W^{0,0} = K$, then also $x^{-1} \in X^{0,0}$ (the inversion is made in $K$).

It is clear that this collection exists, and it is clear how to construct it. We just start with the $\mathbb{Q}$-vector spaces generated by the $x_i$’s, and preform some closure operations. Notice also that by Condition 6, $X^{0,0}$ is a subfield of $K$.

We now construct a pre-additive category $C_0$ in the following way: the object set of our category will be $\mathbb{N}^2 = \{(p,q)\}$. The morphism groups will be $Hom_{C_0}((p,q), (a,b)) = X^{a+q,b+p}$. Composition is defined in the following way: we have $X^{a+q,b+p} \subseteq W^{a+q,b+p} = Hom_K(W^{p,q}, W^{a,b})$, so if $f : (p,q) \to (a,b)$ and $g : (a,b) \to (c,d)$ then we can form the composition $gf : W^{p,q} \to W^{c,d}$ of maps of vector spaces. By Conditions 3 and 4 we have that $gf \in X^{p+d,q+c}$, so this is well defined. Notice that by Condition 2 we have the identity maps, and that all the morphism groups are in fact vector spaces over $X^{0,0}$. The category $C_0$ is a symmetric rigid monoidal category, and we have a symmetric rigid monoidal additive functor $F_0 : C_0 \to Vec_{K}$ given by $F_0((p,q)) = W^{p,q}$.
Indeed, the tensor product of \((p, q)\) with \((a, b)\) will be \((p + a, q + b)\), and the dual of \((p, q)\) will be \((q, p)\).

If \(\mathcal{D}\) is any preadditive category, we can form an additive category, \(\text{add}(\mathcal{D})\) (the additive envelope of \(\mathcal{D}\)), by simply adding finite direct sums to \(\mathcal{D}\) (objects of \(\text{add}(\mathcal{D})\) are \(n\)-tuples of objects of \(\mathcal{D}\) (for some natural \(n\)), and morphisms are given by matrices of morphisms. See also the exercises in Section 6.2 of [14]).

We define \(\mathcal{C}_1 = \text{add}(\mathcal{C}_0)\). The functor \(F_0\) can be extended in a unique way to a functor \(F_1 : \mathcal{C}_1 \to \text{Vec}_K\). The category \(\mathcal{C}_1\) is still a symmetric rigid monoidal category over \(\text{Vec}_{K_{0,0}}\), and \(F_1\) is still a symmetric rigid monoidal additive functor. We would like to construct an abelian category out of \(\mathcal{C}_1\). For this, we will need to add kernels and cokernels to \(\mathcal{C}_1\). In the next subsection we will explain how to expand a category by adding to it objects and morphisms.

### 3.2. Expansion of categories.

Let \(\mathcal{D}\) be a small additive category (so the class of objects and the class of morphisms in \(\mathcal{D}\) are sets), and let \(F : \mathcal{D} \to \text{Vec}_K\) be an additive faithful functor. We can thus think of \(\text{Hom}_D(A, B)\) as an abelian subgroup of \(\text{Hom}_{\text{Vec}_K}(F(A), F(B))\) for every \(A, B \in \text{ob}\mathcal{D}\). Let \(X\) be some set. Assume that for every \(x \in X\), \(V_x\) is a finite dimensional \(K\)-vector space. Let \(Y = (Y_{A,B})_{A,B \in \text{ob}\mathcal{D}, x \in X}\) where \(Y_{A,B} \subseteq \text{Hom}_{\text{Vec}_K}(F(A), F(B))\) be a collection of linear transformations (if \(A = x \in X\) then by \(F(A)\) we mean \(V_x\)). We are going to define a new category \(\tilde{\mathcal{D}}_{(X,Y)}\), and an extension of the functor \(F\) from this category to \(\text{Vec}_K\).

We begin with defining a category \(\tilde{\mathcal{D}}\) (\(\mathcal{D}_{(X,Y)}\) will be the additive envelope of this category). The objects of \(\tilde{\mathcal{D}}\) are \(\text{ob}\mathcal{D} = \text{ob}\mathcal{D} \sqcup X\) (disjoint union). We first extend \(F\) to the objects of \(\tilde{\mathcal{D}}\) by defining \(F(x) = V_x\). We define the morphisms in \(\tilde{\mathcal{D}}\) in the following way:

1. For \(A, B \in \text{ob}\mathcal{D}\), we have that \(\text{Hom}_D(A, B) \subseteq \text{Hom}_{\mathcal{D}}(A, B)\).
2. All the morphisms in \(Y\) are also morphisms in \(\mathcal{D}\).
3. All the identity maps \(V_x \to V_x\) are in \(\tilde{\mathcal{D}}\).
4. The collection is closed under compositions. That is, since we think of \(\text{Hom}_{\mathcal{D}}(A, B)\) as a subgroup of \(\text{Hom}_{\text{Vec}_K}(F(A), F(B))\), we demand that if \(T_1 \in \text{Hom}_{\mathcal{D}}(A, B) \subseteq \text{Hom}_{\text{Vec}_K}(F(A), F(B))\) and \(T_2 \in \text{Hom}_{\mathcal{D}}(B, C) \subseteq \text{Hom}_{\text{Vec}_K}(F(B), F(C))\), then their composition \(T_2T_1 \in \text{Hom}_{\text{Vec}_K}(F(A), F(C))\) is contained in \(\text{Hom}_{\mathcal{D}}(A, C)\).

In other words, we can think of the objects of \(\mathcal{D}\) as vector spaces over \(K\), and we can think of the morphisms between them as \(K\)-linear maps, by using the faithful functor \(F\). We then just add some new vector spaces and new linear transformation, and make sure that they...
will still form a preadditive category, by adding the identity morphisms, and by closing the morphism subgroups under compositions.

We define $D_{(X,Y)} := \text{add}(D)$. The category $D_{(X,Y)}$ is an additive category, and we have an extension of $F$ to an additive faithful functor $D_{(X,Y)} \rightarrow \text{Vec}_K$ which we shall denote by the same letter. One important instance of this is the following: Assume that $D$ is equal to:

Let $\sigma \in T_{k+1}$ and $\tau \in T_{k+2}$ be such that $\sigma \neq \tau$. Then we call the resulting category the localization of $D$ at $F$, and denote it by $F^{-1}D$.

3.3. Back to the construction of $C$. We will now use the construction of the previous subsection in order to add kernels and cokernels to $C_1$. In order to do so, we need to overcome a certain difficulty: if $f : A \rightarrow B$ is a morphism in $C_1$, then it is clear how to define $\text{Hom}_C(X, \text{Ker}(f))$ and how to define $\text{Hom}_C(\text{Coker}(f), Y)$, just by their universal properties. However, it is not so clear how to define $\text{Hom}_C(\text{Ker}(f), X)$ and $\text{Hom}_C(Y, \text{Coker}(f))$. In order to overcome this obstruction, we will show that any kernel in our category is isomorphic with a cokernel and vice versa.

We begin with proving the following linear algebra lemma:

**Lemma 3.2.** Let $T : U \rightarrow V$ be a linear map between two finite dimensional vector spaces over $K$. Let $k$ be a positive integer. Then the image of the map:

\[ K_T : (V^*)^k \otimes U^{k+1} \rightarrow U \]

\[ f_1 \otimes f_2 \otimes \cdots \otimes f_k \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_{k+1} \mapsto \sum_{\sigma \in S_{k+1}} (-1)^{\sigma} f_1(T(v_{\sigma(1)}))f_2(T(v_{\sigma(2)})) \cdots f_k(T(v_{\sigma(k)}))v_{\sigma(k+1)} \]

is equal to:

1. $U$, in case $k < \text{rank}(T)$.
2. $\text{Ker}(T)$ in case $k = \text{rank}(T)$ and
3. $0$ in case $k > \text{rank}(T)$.

**Proof.** We will concentrate on case number 2. The other cases are easy to deduce. So assume that $k = \text{rank}(T)$. Let us write $U = \text{span}_K \{u_1, u_2, \ldots, u_k, u_{k+1}, \ldots, u_n \}$ where $u_{k+1}, \ldots, u_n$ span the kernel of $T$, and $T(u_1), T(u_2), \ldots, T(u_k)$ are linearly independent in $V$. Let $x = f_1 \otimes f_2 \otimes \cdots \otimes f_k \otimes u_1 \otimes u_2 \otimes \cdots \otimes u_{k+1}$. It is easy to see that $K_T(x) = 0$ is zero if $|\{i_1, \ldots, i_{k+1}\}| < k + 1$. Now, if two (or more) of the indices $\{i_j \}$ are bigger than $k$ then $K_T(x) = 0$ again, because every element in the sum will be zero. So the only possible way in which
Let us assume that this is \( j = k + 1 \). Then we have that:

\[
K_T(x) = \sum_{\sigma \in S_{k+1}} ((-1)^n f_1(T(u_{i_1})) f_2(T(u_{i_2})) \cdots f_k(T(u_{i_k})) u_{i_{k+1}} =
\]

\[
\sum_{\sigma \in S_k} (-1)^\sigma f_1(T(u_{i_1})) f_2(T(u_{i_2})) \cdots f_k(T(u_{i_k})) u_{i_{k+1}} \in \text{Ker}(T)
\]

And by taking the right element from \((V^*)^k\) we get all the elements of the kernel of \( T \) this way. Since every element in \((V^*)^k \otimes U^{k+1}\) is a linear combination of elements of the form of \( x \), we are done.  

**Remark 3.3.** Notice that a dual proof will reveal the fact that the image of \( f \) in \( V \) is also the kernel of some other morphism with source \( V \). Everything that we will prove in the sequel for kernel of morphisms can be duality for cokernels.

We shall denote from now on the map \( K_T \) to be the map which appears in the lemma in case \( k = \text{rank}(T) \). If \( T : U \to V \) is a morphism in \( \mathcal{C} \), we can consider \( F_1(T) \). Then \((V^*)^{\text{rank}(T)} \otimes U^{\text{rank}(T)+1}\) is an object of \( \mathcal{C} \), and the map \( K_T : (V^*)^{\text{rank}(T)} \otimes U^{\text{rank}(T)+1} \to U \) contained in \( \text{Hom}_{\mathcal{C}}((V^*)^{\text{rank}(T)} \otimes U^{\text{rank}(T)+1}, U) \). This is because the construction of \( K_T \) demands only the action of the symmetric group and the original map \( T \).

We are going to use now the results of Subsection 3.2. Let \( X \) be the set \( \text{Mor}_{\mathcal{C}} \times \{0, 1\} \). We define \( V_{(f,0)} = \text{Ker}(F(f)) \) and \( V_{(f,1)} = \text{Coker}(F(f)) \). Let \( Y \) be the following set of linear transformations: For any \( A \in \text{ob}\mathcal{C} \) and any \( f : B \to C \) we define

\[
Y_{A,(f,0)} = \{ g \in \text{Hom}_{\mathcal{C}}(A, B) | fg = 0 \}
\]

\[
Y_{(f,1),A} = \{ g \in \text{Hom}_{\mathcal{C}}(C, A) | gf = 0 \}
\]

We then get the category \( \mathcal{C}_{1(X,Y)} \). We localize this category at \( F_1 \), and we denote the result by \( \mathcal{C}_2 := F_1^{-1}\mathcal{C}_{1(X,Y)} \).

Some remarks are in order:

1. As we mentioned earlier, if \( f \) is any morphism in \( \mathcal{C} \), then we have an isomorphism between \( \text{Ker}(f) \) and \( \text{Im}(K_f) \). Since we can repeat the construction of \( K_f \), we actually get an isomorphism between \( \text{Coker}(K_{K_f}) \) and \( \text{Ker}(f) \). But when localizing at \( F \), we actually invert this map. This way, we can answer not only the obvious question, what should \( \text{Hom}(A, \text{Ker}(f)) \) be, but also what should be \( \text{Hom}(\text{Ker}(f), A) \) (since this will be isomorphic with \( \text{Hom}(\text{Coker}(K_{K_f}), A) \)).

2. The object \( \text{Ker}(f) \) is the kernel of \( f \) and \( \text{Coker}(f) \) is the cokernel of \( f \) in \( \mathcal{C}_2 \). The reason for that is the following: the morphism \( K_f \) induces a map in \( \text{Hom}_{\mathcal{C}_2}(\text{Coker}(K_{K_f}), B) \). When we compose this map with the inverse of \( \text{Coker}(K_{K_f}) \to \text{Ker}(f) \), we get a map \( \text{Ker}(f) \to B \) which is just the canonical inclusion. It is now easy to prove that this
map satisfies the desired universal property. The proof for the cokernel is similar. This means that all morphisms in $C_1$ have kernels and cokernels in $C_2$.

3. If a morphism $f : B \to C$ already has a kernel $Q$ in $C_1$, then we will get a morphism $Q \to \text{Ker}(f)$ which will become invertible after localization at $F$.

4. The category $C_2$ is not only additive, it is also still a symmetric rigid monoidal category. Indeed, there is a canonical way to extend the tensor product structure on $C_1$. Each object in $C_2$ is isomorphic to $\text{Ker}(f)$ for some $f \in \text{Mor}_C$ (objects of $C_1$, for example, are isomorphic to the kernels of the zero homomorphisms). We just need to know what is the tensor product $\text{Ker}(f_1) \otimes \text{Ker}(f_2)$, where $f_1 : B_1 \to C_1$ and $f_2 : B_2 \to C_2$. But this is of course the kernel of the map $1 \otimes f_2 + f_1 \otimes 1 : B_1 \otimes B_2 \to B_1 \otimes C_2 \oplus C_1 \otimes B_2$. Also, the dual of any kernel is a cokernel (and vice versa) so duality extends in a natural way to $C_2$.

So $C_2$ contains $C_1$, all arrows of $C_1$ have kernels and cokernels in $C_2$, and we have an extension of the functor $F_1$ to $F_2 : C_2 \to \text{Vec}_K$. We do not know however if all arrows in $C_2$ have kernels and cokernels in $C_2$. We overcome this problem by induction. We construct an expansion $C_3$ of $C_2$, the same way we constructed $C_2$ out of $C_1$. We continue doing this infinitely many times, and we take the union:

$$C_W := \bigcup_j C_j.$$  

Formally speaking, this is the direct limit in the category of small symmetric monoidal categories, but thinking of this as the union will be easier. Then it is clear that every arrow in $C_W$ has a kernel and a cokernel. Moreover, we can extend $F_2$ to $F_3 : C_3 \to \text{Vec}_K$, and the union of the functors will give us a functor $F : C_W \to \text{Vec}_K$.

Now, if $f : X \to Y$ is an injective morphism in $C_W$, then we have a canonical map $f : X \to \text{Ker} (\text{Coker}(f))$. But $F(f)$ is invertible in $\text{Vec}_K$, and therefore $f$ is an isomorphism in $C_W$. The same argument applies to show that if $f$ is surjective, we get an isomorphism between $\text{Coker}(\text{Ker}(f))$ and $Y$. Putting all these facts together, we find out that $C_W$ is an abelian category. We write $K_0 = \text{End}_C(W^{0,0})$. Since we added inverses to all morphisms which become invertible after applying $F$, we get that $K_0$ is a field. We call $K_0$ the field of invariants of $(W, \{x_i\})$.

In conclusion: the category $C_W$ is a $K_0$-linear abelian symmetric rigid monoidal category. We call $C_W$ the fundamental category of $(W, \{x_i\})$. In case $W$ is clear from the context, we just write $C$ for $C_W$. 

4. First properties of the category $\mathcal{C}$, the field of invariants and the functor $F$

We thus have a $K_0$-linear rigid symmetric monoidal category $\mathcal{C}$, together with a symmetric tensor functor $F : \mathcal{C} \to \text{Vec}_K$. We begin by studying the induced equivalence between the fundamental category of isomorphic structures.

Lemma 4.1. Let $(W, \{x_i\})$ and $(W', \{y_i\})$ be two structures, and let $\psi : W \to W'$ be an isomorphism between them (so $\psi(x_i) = y_i$ for every $i$). Then $\psi$ induces an equivalence of categories $\Psi : \mathcal{C}_W \to \mathcal{C}_{W'}$. If we write the fiber functors as $F$ and $F'$, then we have an induced isomorphism of tensor functors $\mu : F \cong F'$. In addition, the two fields of invariants are the same subfield of $K$, and the induced map $\mu : \text{End}_{\mathcal{C}_W}(1) \to \text{End}_{\mathcal{C}_{W'}}(1)$ is the identity map.

Proof. We write $(\mathcal{C}_W)_i$ and $(\mathcal{C}_{W'})_i$ for the chain of categories from the constructions of $\mathcal{C}_W$ and $\mathcal{C}_{W'}$ respectively. We write $F_W$ and $F_{W'}$ for the respective functors. Let us write $Y^{p,q}$ for the set of subspaces of $W^{p,q}$ formed during the construction of $(\mathcal{C}_{W'})_0$. Then $\psi$ induces isomorphisms $\psi^{p,q} : W^{p,q} \to W'^{p,q}$ and this restricts to $\psi^{p,q} : X^{p,q} \to Y^{p,q}$. This gives us already an equivalence of categories $\Psi_1 : (\mathcal{C}_W)_1 \to (\mathcal{C}_{W'})_1$. The isomorphism $\mu$ is given by

$$\mu = \psi^{p,q} : F_W((p,q)) = W^{p,q} \to W'^{p,q} = F_{W'}((p,q)).$$

Assume now that we have already defined $\psi : (\mathcal{C}_W)_i \to (\mathcal{C}_{W'})_i$. Then the new objects in $(\mathcal{C}_W)_{i+1}$ are kernels and cokernels of morphisms in $(\mathcal{C}_W)_i$. We define $\Psi(\text{Ker}(f))$ to be $\text{Ker}(\Psi(f))$ and similarly for cokernels. By induction, we have the following isomorphism:

$$F_W((\text{Ker}(f))) \cong \text{Ker}(F_W(f)) \cong \text{Ker}(F_{W'}\Psi(f)) \cong F_{W'}(\text{Ker}(\Psi(f))) = F_{W'}(\text{Ker}(f)).$$

A similar calculation for cokernels enables us to extend $\mu$.

There is only one way to define the action of $\Psi$ on the new morphisms in $(\mathcal{C}_W)_{i+1}$. Indeed, $\text{Hom}_{(\mathcal{C}_W)_{i+1}}(X,Y)$ is a subgroup of $\text{Hom}_K(F_W(X), F_W(Y))$. But $\mu$ gives us an isomorphism

$$\text{Hom}_K(F_W(X), F_W(Y)) \cong \text{Hom}_K(F_{W'}\Psi(X), F_{W'}\Psi(Y))$$

which restricts to an isomorphism

$$\text{Hom}_{(\mathcal{C}_W)_{i+1}}(X,Y) \cong \text{Hom}_{(\mathcal{C}_{W'})_{i+1}}(\Psi(X), \Psi(Y)).$$

By taking the limit we get the desired equivalence of categories, and the desired equivalence of functors $\mu$.

Assume that $K_0$ is the field of invariants of $W$ and $L_0$ is the field of invariants of $W'$. Then the above isomorphism shows us that $K_0$ and $L_0$ are the same subfield of $K$. In fact, the above equivalence of categories
Ψ will induce the identity map $K_0 = \text{End}_{C_W}(1) \to \text{End}_{C_{W'}}(1) = L_0$.

We next study how the construction of the fundamental category is affected by extension of scalars. We claim the following:

**Lemma 4.2.** Let $T$ be an extension field of $K$. Let $\mathcal{C}$ be the fundamental category of $(W, \{x_i\})$ and let $\tilde{\mathcal{C}}$ be the fundamental category of $(W \otimes_K T, \{x_i\})$. Then there is a natural equivalence of categories $G : \mathcal{C} \to \tilde{\mathcal{C}}$. Moreover, if $F : \mathcal{C} \to Vec_K$ and $\tilde{F} : \tilde{\mathcal{C}} \to Vec_T$ are the two tensor functors, and $i_{K,T} : Vec_K \to Vec_T$ is the extension of scalars functor, then we have a natural isomorphism of functors $i_{K,T}F \cong \tilde{F}G$.

**Proof.** We can think of both categories as subcategories of $Vec_T$ (because $Vec_K$ is also a subcategory of $Vec_T$). As such, we see that $\mathcal{C}_0$ and $\tilde{\mathcal{C}}_0$ are in fact the same subcategory (because they are both formed in the same way by the same tensors). The same holds for $\mathcal{C}_1$ and $\tilde{\mathcal{C}}_1$, and we can continue by induction and pass to the limit to show that $\mathcal{C}$ and $\tilde{\mathcal{C}}$ are in fact equal when considered as subcategories of $Vec_T$. This gives us the desired functor $G$, and the natural isomorphism of functors. It is also easy to see that the two inclusions

$$K_0 = \text{End}_{\mathcal{C}}(1) \to \text{End}_{Vec_K}(1) \to \text{End}_{Vec_T}(1) = T$$

$$K_0 = \text{End}_{\tilde{\mathcal{C}}}(1) \to \text{End}_{Vec_K}(1) \to \text{End}_{Vec_T}(1) = T$$

are the same.

The two lemmas have two corollaries:

**Corollary 4.3.** Assume that $K$ is a subfield of a field $T$, and that $L \subseteq T$ is another subfield. Assume that $W \otimes_K T$ has a form $W'$ over $L$. Then necessarily $K_0 \subseteq L$.

**Proof.** We assume that $(W' \otimes_L T, \{y_i\}) \cong (W \otimes_K T, \{x_i\})$. Let $\mathcal{D}$ be the fundamental category of $(W', \{y_i\})$, and let $\tilde{\mathcal{C}}$ be the fundamental category of $(W \otimes_K T, \{x_i\})$. Then as we have seen both $\mathcal{C}$ and $\mathcal{D}$ are equivalent to $\tilde{\mathcal{C}}$. Moreover, if $K_1 = \text{End}_{\mathcal{D}}(1)$, then the equivalence constructed in Lemma 4.2 will give us that the two subfields $K_1$ and $K_0$ of $T$ are equal. In particular $K_0 = K_1$ is a subfield of $L$.

**Corollary 4.4.** The fundamental categories of all the forms of $(W, \{x_i\})$ are equivalent.

In order to apply Deligne’s Theory, we will need to consider only symmetric tensor functors which are also exact. We claim the following lemma:

**Lemma 4.5.** The functor $F$ is exact. It is therefore a fiber functor.
Let us denote by $K$ \((\ker F(f))\) and \(C\ker F(f) = F(C\ker f)\).

Proof. We know that if $f : X \to Y$ is a morphism in $C$, then by construction $\ker F(f) = F(\ker f)$ and $\text{Coker}(F(f)) = F(\text{Coker}(f))$. In a similar way, since the image of $f$ is the kernel of the cokernel of $f$, we have that $\text{Im}(F(f)) = F(\text{Im}(f))$. This implies that $f$ is a mono (epi-) morphism if and only if $F(f)$ is a mono (epi-) morphism. Assume now that

$$0 \to X \overset{a}{\to} Y \overset{b}{\to} Z \to 0$$

is a short exact sequence in $C$. Then $F(a)$ is one to one, and $F(b)$ is onto. We have $\ker(b) = \text{Im}(a)$ and therefore $F(\ker(b)) = F(\text{Im}(a))$. But this implies that $\ker(F(b)) = \text{Im}(F(a))$, so we are done.

We next claim the following:

**Lemma 4.6.** If the set \(\{x_i\}\) is finite, then the transcendental degree of $K_0$ over $\mathbb{Q}$ is finite (it can as well be zero). In particular, $K_0$ is countable.

Proof. We have seen that $K_0$ is contained in every field over which $(W, \{x_i\})$ has a form. Choose an arbitrary basis $(w_j)$ for $W$ over $K$. Denote the dual basis by $(w^j)$. Each tensor $x_i \in W^n$ can be written as a linear combination of tensor products of $w_j$'s and $w^j$'s. Let us denote by $K_1$ the subfield of $K$ which is generated by all the coefficients appearing in all these linear combinations. Then since the number of tensors is finite, $K_1$ has a finite transcendental degree over $\mathbb{Q}$. The transcendental degree of $K_0$ over $\mathbb{Q}$ is thus finite as well.

The next lemma will be used in Section 8 to prove that all polynomial identities of $W$ are already defined over $K_0$ (in case $W$ is an algebra or a comodule algebra).

**Lemma 4.7.** Let $X, Y$ be two objects of $C$. Let $f_1, f_2, \ldots, f_n \in \text{Hom}_C(X, Y)$ be non-zero morphisms. Assume that $F(f_1), \ldots, F(f_n)$ are linearly dependent over $K$. Then $f_1, \ldots, f_n$ are linearly dependent over $K_0$.

Proof. By using the isomorphism $\text{Hom}_C(X, Y) \cong \text{Hom}_C(1, X^* \otimes Y)$, we can reduce to the case $X = 1$. If $n = 2$, then we have that $F(f_1) = \lambda F(f_2)$ for some $\lambda \in K \setminus \{0\}$. Both $f_1$ and $f_2$ then induces isomorphisms $\hat{f}_1$ and $\hat{f}_2$ between $1$ and a subobject $Y$ of $Y$ (this subobject is the image of $f_1$ and of $f_2$ in $Y$). The composition $\hat{f}_2^{-1} f_1 \in \text{End}_C(1)$ is then multiplication by $\lambda$. But this implies that $\lambda \in K_0$, as desired.

In case $n > 2$ we proceed in the following way: assume that the set $\{F(f_1), \ldots, F(f_n)\}$ is linearly dependent over $K$. By induction, we can assume that no proper subset is linearly dependent over $K$. We can thus write $F(f_n) = \sum \lambda_i F(f_i)$ with $\lambda_i \in K$ are nonzero. Consider now the $(n - 1)$th exterior power of $Y$, $\bigwedge^{n-1} Y$. We have the nonzero vector $y = f_1 \wedge f_2 \cdots \wedge f_{n-1} \in \text{Hom}_C(1, \bigwedge^{n-1} Y)$. For every $i$ we also have the vector $y_i = f_1 \wedge f_2 \cdots \wedge f_{i-1} \wedge f_n \wedge f_{i+1} \cdots \wedge f_{n-1}$. But it is easy to
see that $F(y_i) = \lambda_i F(y)$, and therefore, by the case $n = 2$, $\lambda_i \in K_0$ for every $i$, as desired. □

5. THE CATEGORY $\mathcal{C}$ AS A FORM OF A REPRESENTATION CATEGORY, AND A PROOF OF PROPOSITION [11]

From this section onwards we will assume that the field $K$ is algebraically closed. Since we can always extend scalars to the algebraic closure, this will not be very restrictive. On the other hand, it will be very useful when we will apply Deligne’s Theory.

Let $G$ be the automorphism group of $(W, \{x_i\})$. That is

$G = \{ g \in GL(W) | \forall i g(x_i) = x_i \}$

(we have used here the fact that we have an induced action of $GL(W)$ on $W^{p,q}$). Notice that $G$ is a closed subgroup of $GL(W)$ with respect to the Zariski Topology. Our goal is to prove that $\mathcal{C}$ is a form of $Rep_K G$.

We will begin by recalling the following result of Deligne (see Theorem 1.12 and Theorem 7.1 in [5], and also Theorem 0.6 in [4] for a more general statement):

Theorem 5.1. Let $\mathcal{C}$ be a symmetric rigid tensor category, tensor-generated by finitely many objects, over an algebraically closed field $K$ of characteristic zero. Assume that for every object $X \in \mathcal{C}$ we have $\bigwedge^n X = 0$ for some $n > 0$. Then there exists a unique (up to isomorphism) fiber functor $F : \mathcal{C} \rightarrow Vec_K$. Moreover, the group $G := Aut_\otimes(F)$ is an affine algebraic group over $K$, and we have a natural equivalence of $K$-linear symmetric tensor categories $\mathcal{C} \rightarrow Rep_K G$. The equivalence is given in the following way: if $X$ is an object of $\mathcal{C}$ then the action of $G = Aut_\otimes(F)$ on $F$ makes $F(X)$ a $G$-representation in a natural way.

Definition 5.2. We call $G$ the fundamental category of $\mathcal{C}$ (see also Section 8 of [5]).

In order to apply Deligne’s Theorem, we will begin with the following definition:

Definition 5.3. Let $K \subseteq T$ be a field extension. Assume that $K' \subseteq T$, is another subfield which contains $K_0$. By composing with $i_{K,T} : Vec_K \rightarrow Vec_T$ we get a symmetric tensor functor $i_{K,T} : \mathcal{C} \rightarrow Vec_T$. Then the category $\mathcal{C} \otimes_{K_0} K'$ is the category we get from $\mathcal{C}$ by adding the morphisms $K' \subseteq End_{Vec}(1)$ to $End_{\mathcal{C}}(1)$. To say it in a different way: in the construction of $X^{p,q}$ in Section 3 we add all the elements of $K'$ to $X^{0,0}$.

Remark 5.4. Deligne and Milne (see [6]) defined the extension of scalars in a different way: they considered $K'$ as an algebra in $\mathcal{C}$ (which is possible because $K'$ is an algebra in $Vec_{K_0}$ and $Vec_{K_0}$ is embedded in $\mathcal{C}$ by using the unit object), and they then considered the category
$K' - \text{mod}$ of $K'$ modules inside $\mathcal{C}$. It is possible to show that the two definitions will give equivalent categories.

Notice that if $K' \subseteq K'' \subseteq T$, then there is a one to one correspondence between fiber functors $\mathcal{C} \to Vec_{K''}$ and fiber functors $\mathcal{C} \otimes_{K_0} K' \to Vec_{K''}$. Indeed, $\mathcal{C}$ can be seen as a subcategory of $\mathcal{C} \otimes_{K_0} K'$, so we can restrict functors, and every functor $\mathcal{C} \to Vec_{K''}$ can be extended uniquely to a functor $\mathcal{C} \otimes_{K_0} K'$.

The following proposition is the first half of Proposition 1.1.

**Proposition 5.5.** The category $\mathcal{C}$ is a $K_0$-form of $Rep_K - G$. In other words, we have an equivalence of symmetric monoidal $K$-linear categories $\mathcal{C} \otimes_{K_0} K \to Rep_K - G$.

**Proof.** The functor $F: \mathcal{C} \to Vec_K$ extends naturally to $F_K: \mathcal{C} \otimes_{K_0} K \to Vec_K$. Let us write $\tilde{G} = \text{Aut}(F_K)$. By Tannaka reconstruction we know that we have an equivalence between $\mathcal{C} \otimes_{K_0} K$ and $Rep_K - \tilde{G}$. The equivalence is given in the following way: if $U$ is an object of $\mathcal{C} \otimes_{K_0} K$, then $F_K(U) \in Vec_K$ is a vector space, and if $g \in \tilde{G}$, then we have $g_U: F_K(U) \to F_K(U)$. This furnishes a structure of a $\tilde{G}$-representation on $F_K(U)$. We would like now to determine $\tilde{G}$. First of all, notice that the map $\tilde{G} \to GL(F_K((1,0)))$ is one to one. This is due to the following reason: if $g \in \tilde{G}$ acts trivially on $F_K((1,0))$ then by the fact that $F_K$ is a tensor functor, $g$ acts trivially on all of $\mathcal{C}_1$. It then acts trivially on all of $\mathcal{C}$, because all the other objects of $\mathcal{C}$ are derived from $\mathcal{C}_1$ either as kernels or as cokernels. It thus follows that $g = 1$. We can thus consider $\tilde{G}$ as a subgroup of $GL(F_K((1,0))) = GL(W)$. But if $g \in \tilde{G}$ then it follows that $g$ fixes all the tensors $x_i$ (because the way we have constructed $X^{pv}$ at the beginning). Conversely, if $g$ fixes all the $x_i$’s, then it follows easily that all the vectors in $X^{pv}$ for any $p$ and $q$ are $g$-invariant, and therefore $g$ induces an action on all the objects of $\mathcal{C}_1$, and then also on all the objects of $\mathcal{C}$, because they are derived from the objects of $\mathcal{C}_1$ by construction of kernels and cokernels. So $\tilde{G} = G$, and we have the desired result.

**Remark 5.6.** Due to the construction of $\mathcal{C}$ the field $K_0$ must be contained in every field over which $W$ has a form. In case $W$ has a form over $K_0$ itself, $K_0$ is usually referred to as a field of definition for $W$. However, there are cases in which $W$ will not have a form over $K_0$ (see Section 10). We do see that even though $W$ might not have a form over $K_0$, the category of representations of the automorphism group of $W$ will always have one.

The above proposition shows us how we can reconstruct $W$ out of $\mathcal{C}$ and some additional data. Indeed, we can think of $W$ as the object $(1,0)$ of $\mathcal{C}$, and the tensors $x_i$ can be considered as morphisms $x_i \in Hom_\mathcal{C}(1,(1,0)^{op} \otimes ((1,0)^*)^{op})$. The equivalence $\mathcal{C} \cong Rep_K - G$ gives
us a fiber functor $F : \mathcal{C} \to Vec_K$, and this gives us the structure $(F((1,0)), \{F(x_i)\})$, which is isomorphic with $(W, \{x_i\})$. We have used here the fact that $K$ is algebraically closed, and therefore there exists only one fiber functor on $\mathcal{C}$ (up to equivalence). We record our result in the following lemma:

**Lemma 5.7.** The structure $(W, \{x_i\})$ can be reconstructed from the following data: the category $\mathcal{C}$, the object $(1,0)$ and the morphisms $x_i \in \text{Hom}_\mathcal{C}(1, (1,0)^{\otimes p} \otimes ((1,0)^*)^{\otimes q})$.

The next result gives us the connection between forms of $(W, \{x_i\})$ and fiber functors. It finished the proof of Proposition 1.1.

**Proposition 5.8.** Let $K_0$ be contained in a field $K'$. There is a one to one correspondence between forms of $(W, \{x_i\})$ over $K'$ and fiber functors $\mathcal{C} \to Vec_{K'}$.

*Proof.* Let $T$ be an algebraically closed field containing both $K$ and $K'$. Assume that $F' : \mathcal{C} \to Vec_{K'}$ is a fiber functor. Let $W' := F'((1,0))$. Then $W'$ is a vector space over $K'$, and if $x_i$ is a tensor of type $(p_i, q_i)$, then we have the tensors $y_i := F'(x_i) \in W'^{p_i,q_i}$ (as mentioned earlier, we can think of the tensors $x_i$ as morphisms inside $\mathcal{C}$). We need to prove that $(W', \{y_i\})$ is indeed a form of $(W, \{x_i\})$. In other words, we need to prove that $(W' \otimes_{K'} T, \{y_i\}) \cong (W \otimes_K T, \{x_i\})$.

The functors $F$ and $F'$ induce two fiber functors $\mathcal{C} \otimes_{K_0} T \to Vec_T$. Since $T$ is algebraically closed, we know from Theorem 5.1 that they are isomorphic. But an isomorphism between them will induce an isomorphism $(W' \otimes_{K'} T, \{y_i\}) \cong (W \otimes_K T, \{x_i\})$ as required. In the other direction, assume that $(W', \{y_i\})$ is a form of $(W, \{x_i\})$. Then we can construct the fundamental category $\mathcal{D}$ of $(W', \{y_i\})$. But we have seen that this category depends only on the extension of scalars of $W'$ to $T$. Therefore, we have an equivalence of $K_0$-linear rigid symmetric tensor categories $\mathcal{D} \cong \mathcal{C}$. Since $W'$ induces a fiber functor from $\mathcal{D}$ to $Vec_{K'}$, we get a functor from $\mathcal{C}$ to $Vec_{K'}$ as required. \hfill $\square$

6. **Construction of the generic form, and a proof of Proposition 1.2**

By the work of Deligne we know that if $K_0$ is an algebraically closed field, then $\mathcal{C}$ is necessarily the representation category of some algebraic group. However, $K_0$ is usually not algebraically closed. In this section we will use Deligne’s theory, in order to deduce Proposition 1.2 and construct the generic form $\tilde{W}$.

In order to prove the proposition, we will follow the original proof of Deligne. We will study algebras and modules inside the category $Ind(\mathcal{C})$, and we will explain how can we use them in order to construct fiber functors. Let then $A$ be a commutative algebra inside $Ind(\mathcal{C})$. We have a natural exact tensor functor $F_A : \mathcal{C} \to A - \text{mod}$ given by
$F_A(X) = A \otimes X$, where the action of $A$ is on the first tensor factor. We have

$$\text{Hom}_{A-mod}(F_A(X), M) \cong \text{Hom}_C(X, M).$$

In particular, $\text{Hom}_{A-mod}(A, A) \cong \text{Hom}_C(1, A)$. We denote the last algebra by $B$. Notice that this is an actual $K_0$-algebra. However, we can also view it as a subalgebra of $A$. Indeed, the algebra $B \otimes 1$ is an algebra inside $C$, and can be considered as a subobject (and in fact a subalgebra) of $A$. Notice that if $F : C \to \text{Vec}_{K'}$ is any fiber functor, then $F(B)$ will just be the extension of scalars $B \otimes_{K_0} K'$. Let us denote by $B - mod$ the category of $B$-modules (when $B$ is considered as an algebra in $\text{Vec}_{K_0}$, not in $C$, and we also consider only modules in $\text{Vec}_{K_0}$). We have the following lemma:

**Lemma 6.1.** Assume that $W \otimes A$ is isomorphic with $A^n$ for some $n$. Then the functor $F_B : C \to B - mod$ given by $F_B(X) = \text{Hom}_C(1, F_A(X))$ is a tensor functor.

**Proof.** We have a natural morphism $F_B(X) \otimes F_B(Y) = \text{Hom}_C(1, X \otimes A) \otimes \text{Hom}_C(1, Y \otimes A) \to \text{Hom}_C(1, X \otimes Y \otimes A) = F(X \otimes Y)$. This morphism factors through $F_B(X) \otimes_B F_B(Y)$. We will write the resulting morphism as $\beta_{X,Y} : F_B(X) \otimes_B F_B(Y) \to F_B(X \otimes Y)$. Our goal is to prove that $\beta_{X,Y}$ is an isomorphism for every $X$ and $Y$. We will prove it by induction on the subcategories $C_i$. Since $W \otimes A \cong A^n$ we also have that $W^* \otimes A \cong A^n$, and $W^{p,q} \otimes A$ will also be a free $A$-module of finite rank for every $p$ and $q$. Since the functor

$$A - mod \to B - mod$$

is a tensor functor when restricted to the subcategory of $A$-modules of the form $\oplus A$, we have that $\beta_{X,Y}$ is an isomorphism for $X, Y \in C_1$ (since the objects of $C_1$ are direct sums of the objects of $C_0$).

We now continue by induction. Assume that $\beta_{X,Y}$ is an isomorphism for every $X, Y \in \text{ob}C_i$. All the objects in $C_{i+1}$ are formed as kernels and cokernels of morphisms in $C_i$. We have already seen that every cokernel is a kernel (and vice versa). So if $X \in \text{ob}C_{i+1}$ then there exists an exact sequence of the form

$$0 \to X \to Q \to W$$

where $Q, W \in \text{ob}C_i$. Assume that $Y$ is another object of $C_i$. Then we also have the short exact sequence

$$0 \to X \otimes Y \to Q \otimes Y \to W \otimes Y.$$
By the induction hypothesis we know that \( \beta_{Q,Y} \) and \( \beta_{W,Y} \) are isomorphisms. An easy diagram chasing shows that \( \beta_{X,Y} \) is also an isomorphism. In a similar way, we can now prove that \( \beta_{X,Y} \) is an isomorphism for \( X,Y \in \text{ob} \mathcal{C}_{i+1} \) and we are done. \( \square \)

Notice, however, that the resulting functor \( F: \mathcal{C} \to \mathcal{B} \text{mod} \) might fail to be exact. We have the following lemma:

**Lemma 6.2.** Assume that every short exact sequence \( 0 \to X \to Y \to Z \to 0 \) splits after tensoring with \( A \). Then the functor \( F_A \) is exact.

**Proof.** This follows from the fact that the functor \( F_A \) is exact, and that any additive functor between abelian categories is exact when restricted to split exact sequences. \( \square \)

In the original work of Deligne, he constructed an algebra \( A_D \) which satisfies the requirements of the two lemmas above. This algebra \( A_D \) will be a tensor product of localizations of symmetric algebras. It might be an infinite tensor product (see section 2.10-2.11 of [1]). We get a fiber functor \( F_B: \mathcal{C} \to B \text{mod} \). If \( \phi: B \to K_1 \) is a homomorphism from \( B \) to a field \( K_1 \) of characteristic zero, then we can compose with the resulting functor \( \phi: B \text{mod} \to \text{Vec}_{K_1} \) to get a fiber functor from \( \mathcal{C} \) to \( \text{Vec}_{K_1} \)

**Definition 6.3.** We say that a commutative algebra \( A \) inside \( \mathcal{C} \) is a classifying algebra if:
1. It satisfies the conditions in Lemmas 6.1 and 6.2
2. Every fiber functor from \( \mathcal{C} \) to some extension field \( K_1 \) of \( K_0 \) arises from some homomorphism from \( B \) to \( K_1 \)

**Proposition 6.4.** Assume that \( A \) satisfies the conditions of Lemmas 6.1 and 6.2, and that \( A \) is a tensor product of localizations of symmetric algebras. Then \( A \) also satisfies condition 2 of the definition above, and therefore \( A \) is a classifying algebra.

**Proof.** Let \( F': \mathcal{C} \to \text{Vec}_{K_1} \) be a fiber functor. Write \( A \) as

\[
A = \bigotimes_{i} \text{Sym}(M_i)_{f_i}
\]

where \( M_i \) are objects of \( \mathcal{C} \) and \( f_i \) are polynomials in \( \text{Sym}(M_i) \). Then we have that \( F'(\text{Sym}(M_i)_{f_i}) = \text{Sym}(F'(M_i))_{F'_{(f_i)}} \). The second algebra is a localization of a symmetric algebra over \( K_1 \). We thus have a \( K_1 \) homomorphism \( \text{Sym}(F'(M_i))_{F'_{(f_i)}} \to K_1 \). This is because \( K_1 \) is infinite, and therefore almost all homomorphisms \( \text{Sym}(F'(M_i)) \to K_1 \) will extend to any finite localization. By taking the tensor product, we
get a homomorphism \( \phi : F'(A) \to K_1 \). This homomorphism restricts to \( F'(B) = B \otimes K_1 \) and therefore to \( B \). The resulting homomorphism (which we denote by the same letter) \( \phi : B \to K_1 \) gives rise to a fiber functor \( F'' : C \to Vec_{K_1} \). Moreover, the two functors are equivalent. Indeed, the equivalence is defined in the following way: We have \( F''(X) = Hom_C(1, X \otimes A) \otimes_B K_1 \). We have a natural map

\[
Hom_C(1, X \otimes A) \to Hom_K(1, F'(X) \otimes F'(A)) \cong F'(X) \otimes F'(A) \xrightarrow{Id \otimes \phi} F'(X).
\]

This map factors through \( Hom_C(1, X \otimes A) \otimes_B K_1 \), and we get a natural transformation \( F'' \to F' \). By Deligne (see Section 2.7 in [5]) we know that any natural tensor transformation between two fiber functors is an equivalence, so we are done. \( \square \)

**Remark 6.5.** In most cases which will be of interest for us the group \( G \) will be reductive, and the category \( C \) will be semisimple. The conditions of Lemma 6.2 will then be satisfied automatically.

We would like to construct a concrete example of a classifying algebra in case \( G \) is reductive. Assume that the dimension of \( W \) is \( n \). We take \( n \) copies of \( W, W_1, W_2, \ldots, W_n \) and \( n \) copies of \( W^*, W_1^*, \ldots, W_n^* \). We write \( Id_{i,j} \in W_i \otimes W_j^* \) for the canonical element representing the identity map. We write \( A = Sym(W_1 \oplus W_2 \oplus \cdots \oplus W_n \oplus W_1^* \oplus \cdots \oplus W_n^*)/(Id_{i,j} - \delta_{i,j})_{i,j} \) We claim the following proposition:

**Proposition 6.6.** If \( G \) is reductive then the algebra \( A \) is a classifying algebra for \( C \).

**Proof.** For every \( i = 1, \ldots, n \) we have a map \( W \to A \) given by the inclusion of \( W \) as \( W_i \) in \( A \), and we have a map \( 1 \to W \otimes A \) given by the coevaluation of \( W \), \( 1 \to W \otimes W_i^* \). By extension of scalars we get maps \( \rho_i : W \otimes A \to A \) and \( \nu_i : A \to W \otimes A \). The direct sum of these maps will give us maps \( \rho : W \otimes A \to A^n \) and \( \nu : A^n \to W \otimes A \). Applying the original fiber functor \( F \) reveals the fact that these two maps are mutually inverse to each other and that the algebra \( A \) is non-zero. (we use here the fact that \( F \) extends naturally to \( Ind(C) \), and that it is faithful). We thus see that \( A \) satisfies the conditions of Lemma 6.2. Since \( G \) is reductive, \( A \) satisfies the conditions of Lemma 6.4 trivially. We notice that \( A \) can be seen as a localization of a symmetric algebra. Indeed, \( A \) is equal to the localization of the subalgebra \( Sym[(W_1)^*, \oplus(W_2)^* \oplus \cdots \oplus (W_n)^*] \) by the determinant polynomial. More explicitly: we have an identification of \( (W_1)^* \otimes (W_2)^* \otimes \cdots \otimes (W_n)^* \) with \( (W^*)^{\otimes n} \). The action of \( S_n \) on the last vector space gives us a one dimensional sub-object inside \( (W_1)^* \otimes (W_2)^* \otimes \cdots \otimes (W_n)^* \). This sub-object will correspond to the determinant polynomial. It follows from 6.4 that \( A \) is a classifying algebra. It follows that \( B \otimes_{K_0} K \) is the subalgebra of \( G \)-invariants of \( A \otimes K \). The algebra \( B \otimes_{K_0} K \) is a finitely generated algebra, because \( G \) is reductive. It then follows that \( B \) itself is finitely generated over \( K_0 \). \( \square \)
Proof of Proposition 1.2. Let now $A$ be a classifying algebra for $C$. We write $B_W = \text{Hom}_C(1, A)$. We then have a fiber functor $F_B : C \to B_W - \text{mod}$. Consider the $B$-module $\widetilde{W} := F_B((1, 0))$. Since $W \otimes A \cong A^n$, we have automatically that $\widetilde{W} \cong B_W^n$ as $B_W$-modules. For every $i$, $x_i$ can be considered as a morphism in $\text{Hom}_C((q_i, 0), (p_i, 0))$. We can therefore consider $F(x_i) : \widetilde{W}^\otimes_{\mathbb{R}} \to \widetilde{W}^\otimes_{\mathbb{R}}$. This will give us the structure on $\widetilde{W}$. Now, if $\phi : B_W \to K_1$ is a homomorphism of rings, then we can consider the composition $\phi F_B : C \to Vec_{K_1}$, which is a fiber functor. We then have that $\phi F_B((1, 0)) = \widetilde{W} \otimes_{B_W} K_1$ is a form of $W$. In the other direction, every form $W'$ of $W$ over $K_1$ will induce $F' : C \to Vec_{K_1}$. By Proposition 6.4, this gives rise to a homomorphism $B_W \to K_1$. Finally, in order to prove that $B_W$ has no zero divisors, it is enough to prove that $B_W \otimes_{K_0} K$ has none. But $B_W \otimes_{K_0} K = F(B_W)$ is a subalgebra of $F(A)$. Since the algebra $F(A)$ is the localization of a symmetric algebra, it is integral, and the same is true for $B_W$ as desired. \qed

7. The action of the Galois group, and a proof of Proposition 1.3

In this section we study the case in which $K/K_0$ is an algebraic extension. More generally, assume that $L \subseteq K_0$ is a subfield, and that $K/L$ is algebraic. We write $\Gamma = \text{Gal}(K/L)$. For $\gamma \in \Gamma$, we write $\gamma W$ for the vector space $W$ with the twisted action of $K$:

$$x \cdot v = \gamma(x)v.$$ 

The tensors $\{x_i\}$ will give us tensors $\{\gamma x_i\}$ on $\gamma W$. If $\{e_j\}$ is a basis for $W$, then we can write every tensor as a $K$-linear combination of tensor products of elements from the basis with elements from the dual basis. The vector space $\gamma W$ will then have the same basis, and the tensors $\{\gamma x_i\}$ will be the tensors given by the action of $\gamma$ on the coefficients of the original tensors.

It is possible that the two structures $(W, \{x_i\})$ and $(\gamma W, \{\gamma x_i\})$ will not be isomorphic. For example, if $G$ is a finite group, and $W = K^\alpha G$ is a $G$-graded algebra, then $\gamma W$ will be the twisted group algebra $K^{\gamma(\alpha)} G$. These two graded algebras need not be isomorphic. However, if $(W, \{x_i\}) \cong (W', \{y_i\})$ then $(\gamma W, \{\gamma x_i\}) \cong (\gamma W', \{\gamma y_i\})$. We thus have an action of $\Gamma$ on all isomorphism classes of structures $(W, \{x_i\})$ where $W$ is a vector space of dimension $n$ and $\{x_i\}$ is a family of tensors of types $(p_i, q_i)$.

Let now $\gamma \in \Gamma$. We consider the structure $\gamma W$. Since $C_W$ is a $K_0$-linear category, we can twist scalars, and get a new category $\gamma C_W$ which is $\gamma^{-1}(K_0)$ linear. We claim the following lemma:

Lemma 7.1. We have an equivalence of categories $E_\gamma : \gamma C_W \cong C_{\gamma W}$ which takes $W$ to $\gamma W$. Moreover, the two functors

$$\gamma F_W : \gamma C_W \to \gamma Vec_K \to Vec_K$$ and
By Galois correspondence, we know that if \( K \) is an equivalence of \( \gamma Vec_K \) is the category of \( K \)-vector spaces, in which the action of \( K \) on the hom spaces is twisted by \( \gamma \). The functor \( \gamma : \gamma Vec_K \to Vec_K \) is the functor which takes a vector space \( W \) to \( \gamma W \).

**Proof.** The proof is very similar to the proof of Lemma 4.1. We construct the equivalence step by step, starting from \( C \). The category \( \gamma Vec_K \) is the category of \( K \)-vector spaces, in which the action of \( K \) on the hom spaces is twisted by \( \gamma \). The functor \( \gamma : \gamma Vec_K \to Vec_K \) is an equivalence of \( K \)-linear categories. By the previous lemma we have an equivalence between \( C_W \) and \( \gamma C_W \), which sends \( W \) to \( \gamma W \) and \( x_i \) to \( \gamma x_i \). This implies that \( W \cong \gamma W \), as desired. On the other hand, assume that \( \psi : W \cong \gamma W \). The idea is that any \( x \in K_0 \) is an invariant of the isomorphism type of \( W \). The corresponding invariant for \( \gamma W \) will be \( \gamma^{-1}(x) \). But if \( W \cong \gamma W \) then it must hold that \( x = \gamma^{-1}(x) \).

Let us write it in a more accurate way: First, consider \( X^{0,0} \). It is generated over \( \mathbb{Q} \) by taking tensor product of the \( x_i \) tensors, applying permutation on the tensor factor and applying the pairing. Such an element will clearly be an invariant of the isomorphism type of \( W \). Since \( W \cong \gamma W \) we get that \( X^{0,0} \subseteq K \) is invariant under \( \gamma \). Now let us prove by induction that for every \( i \) we have that \( End_{\gamma}(1) \) is \( \gamma \)-invariant. The endomorphism ring \( End_{\gamma}(1) \) will be generated by endomorphisms of the form

\[
1 \to X_1 \to X_2 \to \cdots \to X_l \to 1
\]

for some objects \( X_j \) in \( C \). All these objects are direct sums of kernels and cokernels of morphisms in \( C_{l-1} \). We can form the same composition in the fundamental category of \( \gamma W \). If this composition equals \( x \in K_0 \) for \( W \), it will equal to \( \gamma^{-1}(x) \in \gamma^{-1}(K_0) \) for \( \gamma W \). But since \( W \cong \gamma W \) it follows that \( x = \gamma^{-1}(x) \) as desired. We have that \( K_0 = \bigcup_i End_{\gamma}(1) \) so \( H \) fixes \( K_0 \) and we are done.

We will now explain how the classical setting of descent theory appears in the construction we have here. Let \( L \subseteq K_1 \subseteq K \) be an intermediate field. Let \( H_1 \subset \Gamma \) be the stabilizer of \( K_1 \). Assume that \( W \) has a form over \( K_1 \). Then \( W \) has a basis with respect to which all the structure constants are \( K_1 \). In particular, \( \gamma W \cong W \) for every \( \gamma \in H \).

An additive map \( \phi : W \to W \) is called \( \gamma \)-linear if it satisfies

\[
\forall x \in K \phi(\gamma(x) \cdot w) = x\phi(w).
\]

We consider the group \( \tilde{H} \) of all additive maps \( \phi : W \to W \) such that \( \phi(x_i) = x_i \) for every tensor \( x_i \) which are \( \gamma \)-linear for some \( \gamma \in H_1 \). We
have a short exact sequence

\[ 1 \to G \to \tilde{H} \to H_1 \to 1 \]

where \( G \) is the automorphism group of the structure \( W \). This sequence splits if and only if \( W \) has a form over \( K_1 \). Moreover, the different forms of \( W \) correspond to different splittings (where two splitting are considered to be equivalent if they differ by conjugation by an element of \( G \)).

Etingof and Gelaki (see [7]) have classified forms of tensor categories (see also Section 3 in [6]). They have discussed semi-simple categories, but their result also applies to our case. Their result, for the special case of the category \( \mathcal{C} = \text{Rep}_K - G \) (where \( K \) is algebraically closed) can be understood in the following way: Let \( \mathcal{C} = \text{Rep}_K - G \). For \( \mathcal{C} \) to have a form over \( K_1 \), it is necessary that for every \( \gamma \in H_1 \) the categories \( \gamma \mathcal{C} = \text{Rep}_K - \gamma G \) will be equivalent. If this is the case, then it is necessary that we can choose equivalences \( \nu_{\gamma} : \gamma \mathcal{C} \to \mathcal{C} \) in such a way that the two functors \( \gamma_{1,2} \mathcal{C} \to \gamma_{1} \mathcal{C} \to \mathcal{C} \) and \( \gamma_{1,2} \mathcal{C} \to \mathcal{C} \) will be isomorphic. If this is the case, then we should be able to choose isomorphism between the functors which will satisfy another coherence condition.

Since we are interested in forms of \( \mathcal{C} = \text{Rep}_K - G \) as a symmetric tensor category, and since over an algebraically closed field each such category has a unique fiber functor into \( \text{Vec}_K \), this classification can also be described in the following way: For \( \mathcal{C} \) to have a form over \( K_1 \), it is necessary that for every \( \gamma \in H_1 \) the groups \( G \) and \( \gamma G \) will be isomorphic. If this is the case, then it is necessary that we can choose isomorphisms \( \rho_{\gamma} : G \to \gamma G \) in such a way that the automorphism \( \rho_{\gamma,1,2} (\rho_{\gamma})_1 \rho_{\gamma,2} : G \to G \) will be inner (this is because the automorphisms of \( G \) for which the resulting functors \( \text{Rep}_K - G \to \text{Rep}_K - G \) are equivalent to the identity are the inner ones). This gives us a short exact sequence

\[ 1 \to G/Z(G) \to \tilde{H}_1 \to H_1 \to 1. \]

The category \( \mathcal{C} \) has a form over \( K_1 \) if and only if this extension can be lifted to an extension of the form

\[ 1 \to G \to \tilde{H}_1 \to H_1 \to 1, \]

and different forms will correspond to different equivalence classes of extensions. A careful verification shows that the extension \( \tilde{H}_1 \) we get here is equivalent to the extension we have described above, arising from the descent problem of \( W \).

8. Relation to polynomial identities

In this section we assume that our object \( W \) is an associative algebra or an \( H \)-comodule algebra, where \( H \) is some finite dimensional Hopf algebra. A polynomial identity of \( W \) is a noncommutative polynomial
f(X_1, X_2, \ldots, X_n) such that \( f(v_1, v_2, \ldots, v_n) = 0 \) for any \( v_1, v_2, \ldots, v_n \in W \). For example, if \( W \) is a commutative algebra, then \( f(X_1, X_2) = X_1X_2 - X_2X_1 \) is a polynomial identity of \( W \). Another example for polynomial identity is the well known Amitsur-Levitsky identity: if \( W = M_n(K) \), then the polynomial

\[
 f(X_1, \ldots, X_{2n}) = \sum_{\sigma \in S_{2n}} (-1)^\sigma X_{\sigma(1)}X_{\sigma(2)} \cdots X_{\sigma(2n)}
\]

is an identity, and \( W \) does not have polynomial identities of lower degree then \( 2n \). Notice that both the Amitsur-Levitsky identity and the commutation identity are multilinear polynomials (that is- in all monomials every variable appears exactly once). In characteristic zero it is possible to prove that all polynomial identities are derived from multilinear identities, and therefore we will focus on them. We can think of a polynomial identity in the following way: Let us denote by \( m : W \otimes W \rightarrow W \) the multiplication on \( W \). We write \( m^{n-1} : W^{\otimes n} \rightarrow W \) for the iterated multiplication. The symmetric group \( S_n \) acts on \( \text{Hom}_W(W^{\otimes n}, W) \). We can therefore look on the sub \( S_n \)-module of \( \text{Hom}_W(W^{\otimes n}, W) \) generated by \( m^{n-1} \). A polynomial identity of degree \( n \) will then be the same as a relation of the form

\[
 \sum_{\sigma \in S_n} a_\sigma \sigma \cdot m^{n-1}
\]

where \( a_\sigma \in K \). Indeed, such a relation corresponds to the polynomial identity \( \sum_{\sigma \in S_n} a_\sigma X_{\sigma(1)} \cdots X_{\sigma(n)} \).

Let now \( H \) be a finite dimensional Hopf algebra defined over a sub-field \( k \subseteq K \). An \( H \)-comodule algebra can be thought of as an algebra \( W \) together with an action of \( H^* \), such that

\[
 \forall f \in H^* \, f(a \cdot b) = f_1(a) \cdot f_2(b).
\]

We recall here the definition of \( H \)-identities from [12] (this definition is slightly different from the one in [3]). For every \( i \), let \( X_i^H \) be a copy of the vector space \( H \). We will denote the element in \( X_i^H \) which corresponds to \( h \) by \( X_i^h \). The tensor algebra \( T = T(\oplus_i X_i^H) \) is an \( H \)-comodule algebra, where the coaction is given on the generators by:

\[
 \rho(X_i^h) = X_i^{h_1} \otimes h_2.
\]

an element \( P \in T \) is a graded identity of \( W \) if for every homomorphism \( \phi : T \rightarrow W \) of \( H \)-comodule algebras it holds that \( \phi(P) = 0 \). We would like to write the identities as linear relations on morphisms in our category. Since \( H \) is finite dimensional, it is known that \( H \) is isomorphic with \( H^* \) when considered as a left \( H^* \)-module (or a right \( H \)-comodule). A canonical choice of a basis element will be the left integral \( l \) of \( H \), which is unique up to a nonzero scalar. Any homomorphism of \( H \)-algebras \( T \rightarrow W \) is uniquely defined by its restriction to \( \oplus_i X_i^H \). Its restriction to \( X_i^H \) will be a map of \( H \)-comodules, and therefore it
will be uniquely defined by the image of $X^l_i$. Let us write $\{f^j\}$ for a basis of $H^\ast$. An identity will thus be a noncommutative polynomial $P$ in the variables $f^j \cdot X^l_i$, which will vanish upon any instance of $X^l_i \mapsto v_i \in W$. A multilinearization shows that this identity is equivalent to a noncommutative polynomial in the variables $\{f^j \cdot X^l_i\}$, in which every monomial contains $X^l_i$ exactly once.

The algebra $((H^\ast)^{op})^\otimes n$ acts on $\text{Hom}_C(W^\otimes n, W)$ by its action on the tensor factors of $W^\otimes n$. The group $S_n$ acts on the same space as before-by permuting the tensor factors of $W^\otimes n$. Together, we get an action of the crossed product algebra $H_n := ((H^\ast)^{op})^\otimes n \ast S_n$ where the action of $S_n$ is by permuting the tensor factors of $((H^\ast)^{op})^\otimes n$. We conclude this discussion in the following lemma:

**Lemma 8.1.** Let $\{t_i\}$ be a basis for $H_n$ over $k$. An $H$-polynomial identity of $W$ is equivalent to a linear relation of the form

$$\sum_i a_i t_i \cdot m^{n-1} = 0.$$  

The conclusion of this is that both regular polynomial identities and $H$-polynomial identities can be understood as linear relations between morphisms in the category $C_W$. We are now ready to prove Proposition 1.4.

**Proof of proposition 1.4**. Assume that $W$ is an $H$-comodule algebra over a field $K$, and that the Hopf algebra $H$ is defined already over a subfield $k$ of $K$. We have seen that $H$-polynomial identities correspond to linear combinations of morphisms in $C_W$ over $K$. We have proved in Lemma 4.7 that if morphisms in $C_W$ are linearly dependent over $K$, then they are linearly dependent already over $K_0$. This finishes the proof. \[\square\]

Notice that polynomial identities give us in general less information on the algebra than the category $C_W$. Indeed, the polynomial identities of $W$ and of $W \oplus W$ are the same, and therefore the polynomial identities cannot define the isomorphism type of the algebra. The polynomial identities do define the algebra if one make some extra assumptions on the algebra. In [1] Aljadeff and Haile proved that if $W$ is a simple $H$-comodule algebra where $H = kG$ is a group algebra, then the identities of $W$ determine $W$. In [12] Kassel proved that $H$-identities can be used to distinguish between isomorphism classes of different Hopf Galois extensions of the ground field for the Taft algebras $H_{n,2}$ and for the Hopf algebras $E(n)$.

We give here an example: Let $n$ be a natural number, and let $G = C_n \times C_n$ be generated by $g$ and $h$. Let $\alpha$ be the two cocycle on $G$ defined by $\alpha(g^i h^j, g^k h^l) = \zeta^{ik}$ where $\zeta$ is a primitive $n$-th root of one. Then the polynomial identity $X_h X_g = \zeta X_g X_h$ is defined over $\mathbb{Q}(\zeta)$, and we will prove in Section 11 that $\mathbb{Q}(\zeta) = K_0$. In the next section we will
see an example for an associative algebra in which all the polynomial identities are already defined over a proper subfield of $K_0$.

9. Example: The field of definition of an associative algebra

Let $W$ be the following three dimensional algebra defined over $\mathbb{Q}(a)$ (where $a \neq 1$ can be algebraic or transcendental over $\mathbb{Q}$):

$$W = \text{span}\{x, y, z\}$$

$$xz = zx = yz = zy = z^2 = 0$$

$$z = x^2 = y^2 = xy = a^{-1}yx$$

Notice that $W$ does not have a unit. We will construct certain objects and morphisms in the category $\mathcal{C}_W$, and we will prove that $a \in K_0$. This will prove that $K_0 = \mathbb{Q}(a)$, since the algebra $W$ is defined over that field.

We denote the multiplication map by $m : W \otimes W \to W$. Consider first the subobject $W^2 = \text{Im}(m) = \text{span}(z)$. The multiplication induces a map $W/W^2 \otimes W/W^2 \to W^2$. This gives us two morphisms $W/W^2 \to W^2 \otimes (W/W^2)^*$. If we denote the dual basis $\{\bar{x}, \bar{y}\}$ of $W/W^2$ by $\{e, f\}$, we get that these maps are invertible and given by:

$$T_1(\bar{x}) = e \otimes z + f \otimes z$$

$$T_1(\bar{y}) = ae \otimes z + f \otimes z$$

and

$$T_2(\bar{x}) = e \otimes z + af \otimes z$$

$$T_2(\bar{y}) = e \otimes z + f \otimes z.$$ 

The composition $T_2^{-1}T_1$ will give us a morphism $W/W^2 \to W/W^2$. A direct calculation shows that the trace of this morphism is $a + 1$. This implies that $a \in K_0$ as desired.

The element $a$ cannot be seen via the polynomial identities of $W$. Indeed, since $W$ is a nilpotent algebra of rank 3, any monomial of rank $\geq 3$ will be an identity, and in degree 2 there is no $b$ such that the polynomial $f(X_1, X_2) = X_1X_2 + bX_2X_1$ is an identity. All the polynomial identities are therefore already defined over $\mathbb{Q}$.

10. Example: Central Simple Algebras

10.1. A splitting field for a central simple algebra. Let $D$ be a central simple algebra of dimension $n^2$ over a field $k$ of characteristic zero. The algebra $D$ splits over an algebraic extension $L$ of $k$ if and only if $D \otimes_k L$ has a representation of dimension $n$ over $L$. Assume then that $L$ is such an extension, and that $V$ is such a representation. For every $d \in D$, we have a tensor $x_d \in V \otimes V^*$ which gives the action of $d$ on $V$. We construct the category for $(V, \{x_d\})$. The field $K_0$ must include $k$ (since the traces of the tensors $x_d$ are in $K_0$), and will in fact
coincide with it. The group $G$ will be $\mathbb{G}_m$, the multiplicative group, since the only elements in $GL(V)$ commuting with all the $x_d$ tensors will be scalar multiplications. The resulting short exact sequence we will get (see Section 7)

$$1 \to \mathbb{G}_m \to \tilde{G} \to \Gamma \to 1$$

(where $\Gamma = Gal(L/k)$) will correspond to an element in $H^2(\Gamma, \mathbb{G}_m)$ which is the class of $[D]$ in $Br(L/k)$. As a classifying algebra we can take

$$A = k[V \oplus V^*]/(f(v) = 1).$$

The invariant subalgebra will then be

$$B = \text{Hom}_C(1, A) = k[D]/(\text{tr}(d) = 1, \text{rank}(d) = 1).$$

A point in $B$ will give us an element $e$ in $D$ which is an idempotent and for which $\dim(De) = n$. The representation $De$ will then be the desired representation.

10.2. Central simple algebras and generic division algebras.

Let $n$ be a natural number, and let $W = M_n(\mathbb{Q})$ be the $n \times n$ matrix algebra over $\mathbb{Q}$. Let us denote by $m$ the multiplication map $m : W \otimes W \to W$. The fundamental category of $(W, \{m\})$ will give us a generic form of the matrix algebra. This will in fact give us a localization of the generic division algebra which appears in the work of Procesi (see [9]). The group $G$ here will be $\text{PGL}_n$, which is reductive. We will therefore receive a finitely generated commutative $\mathbb{Q}$-algebra $B$, and a $B$-algebra $\tilde{W}$ which is free of rank $n^2$ as a $B$-module, with the following property: for every homomorphism $\phi : B \to K$ from $B$ into a field $K$ of characteristic zero the algebra $W_\phi = W \otimes_B K$ will be a central simple $K$-algebra of dimension $n^2$, and every central simple algebra will be received in this way.

Before constructing explicitly the generic division algebra, let us look on some morphisms in $C_W$. The multiplication $m$ is an element in $\text{Hom}_{C_W}(W \otimes W, W) \subseteq W \otimes W^* \otimes W^*$. By pairing $W$ with one of the copies of $W^*$ we get an element in $\text{Hom}_{C_W}(W, 1) \subseteq W^*$. A direct calculation shows that this element will be the trace of the (left or right) regular representation of $W$ (this will be true for any algebra). By multiplying by $\frac{1}{n}$ we get the usual trace. Now, if $t$ is a natural number, and $\sigma \in S_t$ is given by $\sigma = (i_1, i_2, \ldots, i_r)(j_1, j_2, \ldots, j_s)\ldots$ then we have the morphism $T_\sigma \in \text{Hom}_{C_W}(W^t, 1)$ given by

$$T_\sigma(X_1 \otimes X_2 \ldots, X_t) = \text{tr}(X_{i_1}X_{i_2} \ldots X_{i_r})\text{tr}(X_{j_1}X_{j_2} \ldots X_{j_s})\ldots$$

By the work [16] of Procesi we know that these are all the $\text{PGL}_n$-invariants.

We would like to construct a classifying algebra for $W$. We will take the classifying algebra to be a localization of $\text{Sym}[W^* \oplus W^*]$. The algebra $\text{Sym}[W^* \oplus W^*]$ will not be a classifying algebra itself, because
it will have too many points. However, it will be a classifying algebra after we localize by a specific polynomial. We begin with recalling Lemma 14 from [8]

Lemma 10.1. Let $M_1, M_2, \ldots, M_{n^2}$ be $n^2 \times n$ matrices over a field $K$ of characteristic zero. Then they will form a basis for $M_n$ if and only if

$$f(M_1, \ldots, M_{n^2}) = \sum_{\sigma \in S_{n^2}} (-1)^{\sigma} \text{tr}(M_{\sigma(1)}) \text{tr}(M_{\sigma(2)}M_{\sigma(3)}M_{\sigma(4)}) \cdots \text{tr}(M_{\sigma(n^2-2n^2+2)} \cdots M_{\sigma(n^2)})$$

is nonzero.

This enables us to construct a classifying algebra. Indeed, We can take $\text{Sym}[W^* \oplus W^* \oplus \cdots \oplus W^*]_f$ (the coordinate algebra on the space of $n^2$ matrices, localized at the polynomial which says that they form a basis). We can also get a smaller classifying algebra: we consider $\text{Sym}[W^* \oplus W^*]$, the coordinate algebra for the space of two $n \times n$ matrices which we shall denote $X$ and $Y$. We write $M_{n+i+j} = X^iY^j$ for $0 \leq i, j \leq n-1$ and we define $D(X,Y) := f(M_1, \ldots, M_{n^2})$. The localization $A := \text{Sym}[W^* \oplus W^*]_D$ will thus give us a classifying algebra. The reason for this is the following: The algebra $A$ will have $W^* \oplus W^*$ in degree one. These two copies of $W^*$ will give us two morphisms $\phi_X, \phi_Y : 1 \to W \otimes A$ by the coevaluation of $W$. By using the multiplication in $W$, we get $n^2$ maps $\phi_{X^iY^j} : 1 \to W \otimes A$. We can extend scalars, take the direct sum and get a map $A^{n^2} \to W \otimes A$. The fact that $D$ is invertible implies that this map is invertible, and therefore $A$ is a classifying algebra for $C_W$. The resulting algebra $\tilde{W} = \text{Hom}_{C_W}(1, W \otimes A)$ is an Azumaya algebra which is free of rank $n^2$ over its center, $\text{Hom}_{C_W}(1, A) = \mathbb{Q}[W \oplus W]_{PGL_n}^{PGL_n}$. This algebra will specialize to any central simple algebra of dimension $n^2$ over any field of characteristic zero. This algebra is a localization of the algebra $\tilde{R}$ introduced by Procesi. See the paper [9] by Formanek for more details.

11. EXAMPLES: COMODULE ALGEBRAS

Let $H$ be a finite dimensional Hopf algebra over a subfield $k \subseteq K$. An $H$-comodule algebra over $K$ is an algebra $W$ equipped with a right coaction of $H$: $\rho : W \to W \otimes H$ which is also an algebra map. For example, if $H = kG$ is a group algebra, then an $H$-comodule algebra is a $G$-graded algebra. If $H = (kG)^*$, then an $H$-comodule algebra is an algebra $W$ equipped with an action of $G$ by algebra automorphisms. Comodule algebras play an important role in the theory of Hopf algebras. Of particular importance are comodule algebras of the form $\alpha H$, where $\alpha : H \otimes H \to K$ is some (convolution invertible) two cocycle on $H$. These algebras are identical with $H$ as $H$-comodules, and their
multiplication is given by the formula

\[ x \cdot_\alpha y = \alpha(x_1, y_1)x_2y_2. \]

The two cocycle condition on \( \alpha \) is equivalent to the associativity of this algebra. It is known that any comodule algebra which is isomorphic with \( H \) as an \( H \)-comodule is of this form.

We will now use the fundamental category to study comodule algebras. We will consider the following three cases: Group algebras, the Taft algebras, and product of Taft algebras. All our constructions can be generalized easily to the Hopf algebras \( E(n) \) and to the monomial Hopf algebras. We will explain, for \( H \) a Taft algebra or a product of Taft algebras, how can one classify all the cocycles on \( H \) by using the fundamental category (this classification is known. See for example [15] for the classification of two cocycles on Sweedler’s Hopf Algebra, which is the Taft Hopf algebra in dimension 4). In [3] Aljadeff and Kassel have constructed an algebra \( A^0_H \) which is a generic form of \( \alpha H \) (they have proved that it specialize only to the forms of \( \alpha H \) in [13] show that it specialize to any form of \( \alpha H \)). In [10] Iyer and Kassel have studied the algebra \( B^0_H = (A^0_H)\coH \) for Taft algebras, monomial Hopf algebras and the \( E(n) \) algebras. The construction we present here will give us a generic form over a basis of smaller Krull dimension (for the case of Taft algebras).

So let \( \alpha \) be a two cocycle on \( H \) with values in \( K \). The fundamental category \( C_W \) of \( W = \alpha H \) will thus be constructed from the following tensors:

1. The multiplication \( m : W \otimes W \rightarrow W \).
2. For every \( f \in H^* \) the action \( T_f : W \rightarrow W \).

In terms of the map \( \rho : W \otimes H \rightarrow W \), the map \( T_f \) is given by \( T_f = (Id \otimes f)\rho \). Let us determine first the fundamental group of the category. Since \( W \cong H \cong H^* \) as an \( H^* \)-module, the only maps \( W \rightarrow W \) which commute with the tensors \( T_f \) will be of the form \( h \mapsto g(h_1)h_2 \) (where we identify \( W \) with \( H \)). Now, such an element \( g \in H^* \) will commute with the multiplication if and only if \( g \) is a group like element in the dual of the twisted Hopf algebra \( \alpha H^\alpha \). So \( C_W \) is a \( K_0 \) form of \( Rep_K - G((\alpha H^\alpha \times)*) \). Notice in particular that this group is finite. Because we consider all tensors arising from the action of \( H^* \), we will have in particular the action by all the scalars in \( k \). This shows that \( k \) is necessarily contained in \( K_0 \), the field of invariants of \( C_W \). The field \( K_0 \) might be bigger though.

11.1. **Group algebras.** Let \( G \) be a finite group, and let \( H = \mathbb{Q}G \). A two cocycle on \( H \) with values in \( K \) will be the familiar object from group cohomology, namely a function \( \alpha : G \times G \rightarrow K^\times \) such that \( \alpha(x, y)\alpha(xy, z) = \alpha(y, z)\alpha(x, yz) \) for every \( x, y, z \in G \). Two two-cocycles \( \alpha \) and \( \beta \) are equivalent (or cohomologous) in case there is a function \( \lambda : G \rightarrow K^\times \) such that \( \alpha(x, y) = \beta(x, y)\lambda(x)\lambda(y)\lambda^{-1}(xy) \).
for every \(x, y \in G\). The twisted group algebra \(W = K^\alpha G\) has a basis \(\{U_g\}_{g \in G}\) and the multiplication is defined by \(U_x \cdot U_y = \alpha(x, y)U_{xy}\). The coaction of \(\mathbb{Q}G\) is given by \(\rho(U_g) = U_g \otimes g\). In other words, the action of \(e_g \in (\mathbb{Q}G)^*\) is the projection \(e_g : U_x \mapsto \delta_{x,g}U_x\).

We begin by describing some objects and morphisms in the category \(\mathcal{C}_W\). We will denote the image of \(e_g\) by \(W_g\). We have an isomorphism \(1 \rightarrow W_1\) given by sending \(1 \in K\) to the identity of \(W\). We will identify \(1\) and \(W_1\) henceforth. For every \(g \in G\), the restriction of the multiplication map \(W_g \otimes W_{g^{-1}} \rightarrow 1\) will give us an isomorphism between \(W_g^*\) and \(W_{g^{-1}}^*\). The coevaluation will then be

\[
\text{coev}_g : 1 \rightarrow W_g \otimes W_{g^{-1}}
\]

\(1 \mapsto U_g \otimes U_{g^{-1}}\).

So for every \(g, h \in G\), we have the following map in \(\mathcal{C}_W\):

\[
d_{g,h} : 1 \rightarrow W_g \otimes W_{g^{-1}} \otimes W_h \otimes W_{h^{-1}} \rightarrow W_g \otimes W_h \otimes W_{g^{-1}} \otimes W_{h^{-1}} \rightarrow W_{ghg^{-1}h^{-1}}
\]

which sends \(1\) to \(w_{g,h} = U_g U_h U_{g^{-1}} U_{h^{-1}} = c_{g,h} U_{ghg^{-1}h^{-1}}\) for some \(c_{g,h} \in K^\times\). This gives us an isomorphism in \(\mathcal{C}_W\) between \(1\) and \(W_{ghg^{-1}h^{-1}}\).

This means that \(w_{g,h}\) must be contained in any form of \(W\). This also means that if

\[
g_1 h_1 g_1^{-1} h_1^{-1} g_2 h_2 g_2^{-1} h_2^{-1} \cdots g_r h_r g_r^{-1} h_r^{-1} = 1 \in G,
\]

then the product \(w_{g_1, h_1} w_{g_2, h_2} \cdots w_{g_r, h_r} \in K^\times\) will be a scalar which will be an invariant of \(W\).

We can understand this invariant by the Hopf formula and the universal coefficients theorem (see Section 2 of \([2]\) for more details). Let \(F\) be the free group with generators \(x_g\), and let \(R\) be the kernel of the homomorphism \(F \rightarrow G \xrightarrow{x_g} g\). Then the Hopf formula says that \(M(G) := H_2(G, \mathbb{Z}) \cong [F, F] \cap R/[F, R]\) and that \(H^2(G, K^\times) \cong Hom_\mathbb{Q}(M(G), K^\times)\). The cocycle \(\alpha\) thus induces a homomorphism \(\tilde{\alpha} : [F, F] \cap R \rightarrow K^\times\) which vanishes on \([F, R]\). If \(t := [x_{g_1}, x_{h_1}] \cdots [x_{g_r}, x_{h_r}] \in [F, F] \cap R\) then a direct calculation shows that \(\tilde{\alpha}(t) = w_{g_1, h_1} \cdots w_{g_r, h_r}\).

We thus see that the image of \(\tilde{\alpha}\) is contained in \(K_0\). Since \(G\) is finite, \(\alpha\) is equivalent to a cocycle whose values are roots of unity. The image of \(\tilde{\alpha}\) is therefore generated by some root of unity \(\mu\).

We will now describe the generic form of \(W\). We will show that \(W\) has a form over \(\mathbb{Q}(\mu)\) and therefore \(K_0 = \mathbb{Q}(\mu)\). Notice that already in \(\mathcal{C}_W\) we can write \(W = (\bigoplus_{g \in G'} W_g) \oplus (\bigoplus_{g \notin G'} W_g)\), and the first direct summand is isomorphic with \(1^{G'}\). The group \(G/G'\) is finite abelian, and we can therefore write

\[
G/G' \cong \langle \bar{x}_1 \rangle \cdots \langle \bar{x}_r \rangle.
\]

We write \(n_i\) for the order of \(\bar{x}_i\). We consider the object \(M = W_{x_1}^* \oplus W_{x_2}^* \oplus \cdots \oplus W_{x_r}^*\). We let \(f_i\) be a basis element for \(W_{x_i}^*\) for every \(i\). We consider the localization \(A = Sym(M)_f\) where \(f = f_1 f_2 \cdots f_r\). Then \(A\) is a classifying algebra for \(W\). Indeed, \(A \cong A \otimes W_{x_i}\) for every \(i\). We
also have that $W_g \cong 1$ for every $g \in G'$, and therefore $A \otimes W_g \cong A$ for every $g \in G'$. Since the elements $x_i$ and the elements of $G'$ generate $G$, we can use the multiplication in $W$ to get an isomorphism $A \otimes W_g \cong A$ for every $g \in G$, and therefore $A \otimes W \cong A^G$ as desired. Since the category $\mathcal{C}_W$ is semisimple, $A$ is a classifying algebra.

For every $i$, we can write $U_{x_i} = c_i w_{g_i,h_i} \cdots w_{g_i,h_i}$ for some elements $g_i, h_i \in G$, $c_i \in K^\times$. We can change $U_{x_i}$ by a scalar and we can therefore assume that $c_i = 1$ for every $i$. The generic form $\bar{W}$ will then be generated by the elements $U_g U_h U_{g^{-1} h_{-1}}$ and the elements $f_i U_{x_i}$. The base ring will be $B_W = K_0[(f_i^n)^{\pm 1}]$. By taking the algebra generated by $U_{x_i}$ and $w_{g,h}$, we get a form for $W$ defined already over $\mathbb{Q}(\mu)$. This shows that $K_0 = \mathbb{Q}(\mu)$.

In [2] Aljadeff Haile and Natapov have defined an algebra $U_G$ which they call the universal $G$-graded algebra. They have defined the algebra using the polynomial graded identities of $W$, and have described it as the subalgebra of $W \otimes_K K[t^{\pm 1}]_{g \in G}$ generated over $\mathbb{Q}$ by the elements $U_g \otimes t_g$. The algebra $U_G$ will be the generic form associated to the classifying algebra $Sym(W^*)_f$ where $f = \prod_g f_g$. The resulting base ring will be bigger though. Since we chose $M$ instead of $W^*$, we get a smaller base algebra of smaller Krull dimension.

11.2. Taft Algebras. We begin by recalling the definition:

$$H_n = k < g, x > / (g^n - 1, x^n, gxg^{-1} - \zeta x)$$

where $\zeta$ is a primitive $n$-th root of unity. The comultiplication is given by $\Delta(g) = g \otimes g$ and $\Delta(x) = x \otimes 1 + g \otimes x$. The Hopf algebra $H$ is defined over $k := \mathbb{Q}(\zeta)$. Let $W = ^\alpha H$ be defined over $K$. We have an isomorphism of $H$-comodules $H \cong W$. We write the image of $h \in H$ by $\hat{h} \in W$. We are going to use the fundamental category of $W$ in order to classify all two $\hat{H}$-cocycles over $K$ up to equivalence.

We will do the following: we will use the maps we have in the category $\mathcal{C}_W$ in order to decompose $W$ as the direct sum of weight spaces with respect to some commutative subalgebra of $H^*$ and another commutative subalgebra of $W$. This decomposition will give us an invariant $b \in K$. We will then show that $b$ already defines the isomorphism type of $W$, construct a generic form and describe the different forms.

We begin by considering the element $\gamma \in H^*$ given by

$$\gamma(g^i x^j) = \zeta^i \delta_{ij,0}.$$ 

The element $\gamma$ generates the group of group-like elements in $H^*$. In particular, $\gamma : W \to W$ is an algebra map. We thus write $W = \bigoplus W_i$, where $W_i$ is the subspace upon which $\gamma$ acts by $\zeta^i$. Since the action of $\gamma$ does not depends on $\alpha$, we see that

$$W_i = \text{span}\{g^i x^j\}_{j}.$$
We now consider another element in $H^*$, namely $\xi$ which is given by

$$\xi(g^j x^j) = \delta_{j,1}.$$ 

Then one can show that $\Delta(\xi) = 1 \otimes \xi + \xi \otimes \gamma^{-1}$ and that $\gamma \xi \gamma^{-1} = \zeta \xi$. But this implies that $\xi(W_i) \subseteq W_{i+1}$. We consider now also $\text{Ker}(\xi) = T_1$. It is easy to see, again, since the action of $H^*$ does not depend on $\alpha$, that this is just the space spanned by the powers of $\tilde{g}$. Since $\gamma \xi \gamma^{-1} = \zeta \xi$, we have that $\gamma(T_1) = T_1$ and $W_i \cap T_1 = \tilde{K} \tilde{g}^i$. Now for every $i$ we have the map $\phi_i : W_i \cap T_1$.

We now consider another element in $\xi$, this implies that $\gamma \alpha \gamma^{-1} = \zeta \xi$. Since $\gamma \xi \gamma^{-1} = \zeta \xi$, we have that $\gamma(T_1) = T_1$ and $W_i \cap T_1 = \tilde{K} \tilde{g}^i$. Now for every $i$ we have the map $\phi_i : W_i \cap T_1$.

We therefore have a second direct sum decomposition,

$$W_i = \oplus_j W_{i,j}$$

where $W_{i,j} = W_i \cap \text{Ker}(\phi_j)$ (to see that this is a direct sum we need to check that conjugation by $\tilde{g}$ stabilizes $W_i$, but this is immediate). A direct calculation shows that we have $\xi(W_{i,j}) \subseteq W_{i+1,j-1}$. We claim now the following:

**Lemma 11.1.** We have $\text{dim}_K(W_{i,j}) = 1$ for each $i$ and for each $j$.

**Proof.** We know that the kernel of $\xi$ is $\text{span}_K\{\tilde{g}^j\}_{i,j}$. We have that $\tilde{g}^i \in W_{i,0}$. Also, we know that as an $H$-comodule, $W_i$ is isomorphic to $H$. This means that $W_i$ is isomorphic to $H$ as an $H^*$-module. But this means that there are elements $t_{i,1} \in W_i$ such that $\xi(t_{i,1}) = \tilde{g}^i$. Now, $t_{i,1}$ will be a sum of an element $s_{i,1} \in W_{i-1,1}$ and an element in $T_1$. Since $T_1 \cap W_{i-1,1} = 0$, the element $s_{i,1}$ is well defined. We thus have, without loss of generality, that $t_{i,1} \in W_{i-1,1}$. We can now continue in a similar fashion: assuming that $n > 2$, there are also element $t_{i,2} \in W_i$ such that $\xi(t_{i,2}) = t_{i,1}$ (we use here the fact that for every $j < n$ we have that $\text{Ker}(\xi^j) = \text{Im}(\xi^{n-j})$). Again, $t_{i,2}$ will be the sum of an element in $W_{i-2,2}$ and an element in $T_1$. Since $T_1 \cap W_{i-2,2} = 0$, this element is uniquely defined. We thus assume, without loss of generality, that $t_{i,2} \in W_{i-2,2}$. We continue up to $n-1$ in this way. We thus got, for every $i = 0, \ldots, n-1$, a nonzero element $t_{i+j,j} \in W_{i,j}$ (indices are modulo $n$). So for each $i, j$ we have that $\text{dim}_K(W_{i,j}) \geq 1$. But then the dimensions already sum up to $n^2$, which is the dimension of the algebra $W$, so we must have an equality. $\square$

Notice that the last lemma says something stronger. We can deduce that the restriction of $\xi$ to $W_{i,j}$ gives us an isomorphism $\xi : W_{i,j} \rightarrow W_{i+1,j-1}$ for every $j \neq 0$. In particular, consider $1 \in W_{0,0}$. There is a unique element $t \in W_{-1,1}$ such that $\xi(t) = 1$. Since the multiplication
respects the double grading on $W$ we have that $t^n \in W_{0,0} = \text{span}_K \{1\}$. So $t$ is an invariant vector, and $t^n = b \in K$ is an invariant of the $H$-comodule algebra $W$. We have that $t^i \in W_{-i,i}$ for every $i < n$. We claim that $t^i \neq 0$ for every $i < n$. We have that $\xi(t^i) = (1 + \zeta + \cdots + \zeta^{i-1})t^{i-1}$, so the result follows easily by induction. This already implies that for every $i = 0, \ldots, n-1$, $t^i$ spans $W_{-i,i}$. The restriction to the subalgebra generated by $\tilde{g}$ will give us that $\tilde{g}^n = a \in K$ is non zero (because this is a two cocycle on the group $C_n$). We thus have that for every $i = 0, \ldots, n-1$ and every $j = 0, \ldots, n-1 \tilde{g}^j t^i$ spans $W_{j-i,i}$.

It follows that our algebra $W$ has a basis given by $\{\tilde{g}^j t^i\}_{i,j}$ subject to the relations

$$\tilde{g}^n = a, t^n = b,$$

$$\tilde{g} t \tilde{g}^{-1} = \zeta t.$$

The coaction of $H$ is given by:

$$\rho(\tilde{g}) = \tilde{g} \otimes g$$

$$\rho(t) = t \otimes g^{-1} + 1 \otimes g^{-1} x$$

Since $\tilde{g}$ and $x$ are generators of $W$, this determines $\rho$. This shows that the isomorphism type of $W$ depends only on the pair $(a, b)$. Moreover, it is not hard to show that for any pair $(a, b)$ we get an $H$-comodule algebra in this way. It is possible, however, that different values of $a$ will give us isomorphic algebras (we have already seen that $b$ depends only on the pair $(a, b)$). Indeed, if we change $\tilde{g}$ to be $x \tilde{g}$ for some $x \in K$, then we replace $a$ by $ax^n$. Since $K$ is assumed to be algebraically closed, we can assume without loss of generality that $a = 1$. Thus, over $K$ the equivalence classes of cocycles on $H$ are in one to one correspondence with $K$, and the field of invariants for the cocycle which corresponds to $b$ is $K_0 = \mathbb{Q}(\zeta, b)$.

We construct now a generic form to the algebra $W$ which corresponds to $(1, b)$. We take $A = \text{Sym}((W_{1,0})^*)_f$, where $f$ is a basis element of $(W_{1,0})^*$. The resulting base algebra will be $B_W = \mathbb{Q}(\zeta, b)[f^{\pm n}]$ (so it will be a Laurent polynomial ring in one variable $a := f^n$). The generic form will then be generated over $B_W$ by $\tilde{g}$ and $t$, subject to the relation written above and with the $H$-coaction written above. The only difference is that now $a$ will be a generic (invertible) element, and not a specific element of the ground field. It is easy to see that for any extension field $\mathbb{Q}(\zeta, b) \subseteq K_1$ the forms correspond to $(a, b)$ and $(a', b)$ will be isomorphic if and only if $a/a' \in (K^*)^n$.

11.3. Products of Taft Hopf algebras. We shall study now the same question for the tensor product of Taft Hopf algebras. Assume that for every $i = 1, \ldots, z$, $H_i$ is a Taft Hopf algebra of dimension $n_i^2$. Let $n = \text{l.c.m}(n_i)$, and let $\zeta$ be a primitive $n$-th root of unity. Then $H_1$ is generated by $g_i, x_i$, subject to the relations $g_i^{n_i} - 1 = x_i^{n_i} = 0, g_i x_i g_i^{-1} = \zeta^{\alpha_i} x_i$ where $\zeta^{\alpha_i}$ is a primitive $n_i$-th root of unity. The comultiplication
in \( H_i \) is given by \( \Delta(g_i) = g_i \otimes g_i \) and \( \Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i \). We write \( H = \bigotimes_{i=1}^{\infty} H_i \). We shall classify all algebras of the form \( \alpha H \) and describe the generic forms.

So let \( \alpha \) be a two cocycle on \( H \). To begin with, the restriction of \( \alpha \) to each of the algebras \( H_i \) will give us an \( H_i \) comodule algebra. By the results of the last subsection, we can assume that this algebra is generated by the elements \( \tilde{g}_i \) and \( t_i \) subject to the relations \( t_i^{\alpha_i} = b_i \), \( \tilde{g}_i^{t_i} = a_i \) and \( \tilde{g}_i t_i \tilde{g}_i^{-1} = \zeta^{\alpha_i} t_i \). The only thing that we need in order to understand \( \alpha H \) as well, is to understand how the subalgebras \( \alpha H_i \) commute with one another.

We begin by looking on the restriction of \( \alpha \) to the group algebra of \( G = \langle g_1, g_2, \ldots, g_k \rangle \). This is a finite abelian group, and we understand well the elements in \( H^2(G, K^\times) \). The invariants of \( \alpha|_G \) will be given by the scalars \( \zeta^{b_{ij}} \) such that \( \tilde{g}_i \tilde{g}_j = \zeta^{b_{ij}} \tilde{g}_j \tilde{g}_i \). Notice that \( \zeta^{b_{ij}} \) must be an \( n_i \) and \( n_j \) root of unity. In other words, \( n_i|b_{ij} \) and \( n_j|b_{ij} \). Notice also that \( b_{ii} = b_{ij} + b_{ji} = 0 \mod n \).

The cocycle \( \alpha|_G \) is completely determined by giving these scalars together with the scalars \( a_i \).

We have that \( \rho(t_j) = t_j \otimes g_j^{-1} + 1 \otimes g_j^{-1} x_j \) and \( \rho(\tilde{g}_i) = \tilde{g}_i \otimes g_i \) and therefore

\[
\rho(\tilde{g}_i t_j \tilde{g}_i^{-1}) = \tilde{g}_i t_j \tilde{g}_i^{-1} \otimes g_j^{-1} + 1 \otimes g_j g_j^{-1} x_j g_i^{-1} = \\
\tilde{g}_i t_j \tilde{g}_i^{-1} \otimes 1 + 1 \otimes g_j^{-1} x_j
\]

So if we write \( t_{ij} = \tilde{g}_i t_j \tilde{g}_i^{-1} - t_j \) we have that \( \rho(t_{ij}) = t_{ij} \otimes g_j^{-1} \) and therefore \( t_{ij} \in K \cdot \tilde{g}_j^{-1} \). By conjugating \( t_{ij} \) with \( \tilde{g}_j \) we see that \( \tilde{g}_j t_{ij} \tilde{g}_j^{-1} = \zeta^{t_{ij}} \). But since \( t_{ij} \in K \cdot \tilde{g}_j^{-1} \) we also have that \( \tilde{g}_j t_{ij} \tilde{g}_j^{-1} = t_{ij} \). This implies that \( t_{ij} = 0 \) and therefore \( \tilde{g}_i t_j \tilde{g}_i^{-1} = t_j \).

The last thing we need to understand is the commutation relations between \( t_i \) and \( t_j \) for \( i \neq j \). Consider the element \( t_i t_j - t_j t_i \). A direct calculation shows that \( \rho(t_i t_j - t_j t_i) = (t_i t_j - t_j t_i) \otimes g_j^{-1} g_j^{-1} \) and therefore \( t_i t_j - t_j t_i = \lambda_{ij} \tilde{g}_i \tilde{g}_j^{-1} \). We call the indices \( i \) and \( j \) connected, and we write \( i \sim j \) if \( \lambda_{ij} \neq 0 \). The fact that \( i \) and \( j \) are connected has some consequences:

**Lemma 11.2.** Let \( i \) and \( j \) be connected indices. Then it holds that \( b_{ij} = -a_i = a_j \mod n \) and \( b_{ik} + b_{jk} = 0 \mod n \) for every third index \( k \) which is not \( i \) nor \( j \).

**Proof.** We conjugate the equation \( t_i t_j - t_j t_i = \lambda_{ij} \tilde{g}_i \tilde{g}_j^{-1} \tilde{g}_j^{-1} \) by \( \tilde{g}_i, \tilde{g}_j \) and \( \tilde{g}_k \). The result follows from the fact that \( \lambda_{ij} \) is a nontrivial scalar. \( \square \)

We can now write the algebra \( W \) and the coaction explicitly: \( W \) is generated by the elements \( \tilde{g}_i, t_i \) subject to the following list of relations:

\[
\tilde{g}_i^{t_i} = a_i, t_i^{\alpha_i} = b_i \tilde{g}_i t_i \tilde{g}_i^{-1} = \zeta^{\alpha_i} t_i \\
\tilde{g}_i \tilde{g}_j = \zeta^{b_{ij}} \tilde{g}_j \tilde{g}_i, \quad \tilde{g}_i t_j \tilde{g}_i^{-1} = t_j \text{ for } i \neq j
\]
\[ t_it_j - t_jt_i = \lambda_{ij} \tilde{g}_i^{-1} \tilde{g}_j^{-1} \]
\[ \rho(\hat{g}_i) = \hat{g}_i \otimes g_i \]
\[ \rho(t_i) = t_i \otimes \hat{g}_i^{-1} + 1 \otimes \hat{g}_i^{-1} x_i. \]

Notice that \( \lambda_{ij} \) is not an invariant of \( W \). Indeed, changing \( \hat{g}_i \) to be \( \zeta^{c_i} \hat{g}_i \) will not change the scalars \( a_i \), \( b_i \) and \( \zeta^{b_{ij}} \) but will change \( \lambda_{ij} \) to be \( \lambda_{ij} \zeta^{-c_i} \). We have an action of \( G_2 = \prod_i \mathbb{Z}/n_i \) on the set of all tuples \((\lambda_{ij})\) where a generator of \( \mathbb{Z}/n_k \) sends \((\lambda_{ij})\) to \((\mu_{i,j,k} \lambda_{ij})\) where \( \mu_{i,j,k} = 1 \) if \( k \neq i \) and \( k \neq j \) and \( \mu_{i,j,k} = \zeta^{c_k} \) otherwise. One can show that \( W \) determines the orbit of \((\lambda_{ij})\) under the action of \( G_2 \). We summarize our discussion in the following proposition:

**Proposition 11.3.** The cocycle \( \alpha \) is determined (up to equivalence) by the scalars \( a_i, b_i, \zeta^{b_{ij}}, \lambda_{ij} \). The cocycle \( \alpha \) determines the scalars \( b_i \) and \( \zeta^{b_{ij}} \). The scalars \((\lambda_{ij})\) are determined up to an action of the group \( G_2 \). These scalars satisfy the relations: \( \zeta^{b_{ij}} \) is an \( n_i \) root of unity, \( \zeta^{b_{ij}+b_{ji}} = 1 \), \( \zeta^{b_{ii}} = 1 \), and if \( \lambda_{ij} \neq 0 \), then \( \zeta^{b_{ik}+b_{jk}} = 1 \) and \( \zeta^{b_{ij}} = \zeta^{c_i} \). Moreover, any such collection of scalars which satisfy the above relations will give us a cocycle on \( H \).

**Proof.** The fact that \( b_i \) and \( \zeta^{b_{ij}} \) are invariants of the cocycle \( \alpha \) follows from our discussion on group algebras and on Taft algebras. Since we wrote the algebra in terms of the scalars in the proposition, the collection of scalars \( b_i, a_i, \zeta^{b_{ij}}, \lambda_{ij} \) determines the equivalence class of the cocycle \( \alpha \).

In the other direction, if we have such a collection of scalars which satisfies the condition of the proposition, it is possible to construct an algebra with these relations and coaction. The only nontrivial part is to show that this algebra is really of the form \( ^aH \) (a priori, it is possible that the relations we have will define the trivial algebra, for example). The algebra can be constructed by Ore extensions and crossed product, and we can prove by induction on the number of factors \( z \) that the algebra is really of the form \( ^aH \). \( \Box \)

The last thing we need to do is to construct the generic form of \( ^aH \). As before, we will take \( M = \oplus_i (K \cdot \check{g}_i)^* \) and \( A = \text{Sym}(M)_f \) where \( f = \prod f_i \). \( (f_i) \) is the dual basis for \( \check{g}_i \) for the space \( K \cdot \check{g}_i \). Then the base algebra will be \( B = \mathbb{Q}(\zeta, b_i, \lambda_{ij})[a_1^{\pm1}, a_2^{\pm1}, \ldots, a_z^{\pm1}] \), where \( a_i = f_i^{n_i} \), and the generic form will be exactly the algebra written above (the only difference is that now \( a_i \) are generic elements, and not elements of the ground field). Notice that we can take even a smaller algebra: indeed, if \( i \sim j \) then the vector \( t_it_j - t_jt_i = \lambda_{ij} \check{g}_i^{-1} \check{g}_j^{-1} \) will be in any form. We can then take \( M = \oplus_i (K \cdot \check{g}_i)^* \) where we take only one index \( i \) from each equivalence class of the equivalence relation generated by \( \sim \).
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