A new definition of $t$-entropy for transfer operators

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In the series of articles [1, 2, 3, 4, 5, 6] there have been established the variational principles for the spectral radii of weighted shift and transfer operators generated by an arbitrary dynamical system. These principles are based on the Legendre duality and the main role here is played by a newly introduced dynamical invariant — $t$-entropy, which gives the explicit form of the Legendre dual object to the logarithm of the spectral radii of operators in question. The description of $t$-entropy is not elementary and its calculation is rather sophisticated. In the present article we give a new definition of $t$-entropy that makes it more explicit and essentially simplifies the process of its calculation.

The article consists of two sections. In Section 1 we consider $t$-entropy for the model example of transfer operators associated with continuous dynamical systems. The new definition of $t$-entropy is introduced here in Theorem 2. In Section 2 we discuss the general $C^*$-dynamical situation. To illustrate similarity and difference between the objects considered in the model and general situations we present here a number of examples and finally introduce the general new definition of $t$-entropy in Theorem 10.

1 A new definition of $t$-entropy for continuous dynamical systems

Let us consider a Hausdorff compact space $X$. We denote by $C(X)$ the algebra of continuous real-valued functions on $X$ equipped with the uniform norm. Let $\alpha : X \to X$ be a continuous mapping. This mapping generates the dynamical system with discrete time, which will be denoted by $(X, \alpha)$.

\[ \text{Keywords: transfer operator, variational principle, } t\text{-entropy} \]

\[ \text{2000 MSC: 37A35; 47B37; 47C15} \]
A linear operator $A : C(X) \to C(X)$ is called a transfer operator for the dynamical system $(X, \alpha)$ if

a) $A$ is positive (that is it maps nonnegative functions to nonnegative) and
b) it satisfies the homological identity

$$A(f \circ \alpha \cdot g) = fAg, \quad f, g \in C(X). \quad (1)$$

We denote by $M \subset C^*(X)$ the set of all linear positive normalized functionals on $C(X)$ (that is linear functionals that take nonnegative values on nonnegative functions and are equal to 1 on the unit function). By the Riesz theorem these functionals are bijectively identified with regular Borel probability measures on $X$, and so all the elements of $M$ will be referred to as measures.

A measure $\mu \in M$ is called $\alpha$-invariant if

$$\mu(f \circ \alpha) = \mu(f)$$

for all functions $f \in C(X)$. This is equivalent to the identity

$$\mu(\alpha^{-1}(G)) = \mu(G)$$

for all Borel subsets $G \subset X$. The collection of all $\alpha$-invariant measures from $M$ will be denoted by $M_\alpha$.

Recall that a continuous partition of unity on $X$ is a finite set $D = \{g_1, \ldots, g_k\}$ consisting of nonnegative functions $g_i \in C(X)$ satisfying the identity $g_1 + \cdots + g_k \equiv 1$.

According to [6], $t$-entropy is the functional $\tau(\mu)$ on $M$ defined by the formulae

$$\tau(\mu) := \inf_{n \in \mathbb{N}} \frac{\tau_n(\mu)}{n}, \quad \tau_n(\mu) := \inf_D \tau_n(\mu, D), \quad (2)$$

$$\tau_n(\mu, D) := \sup_{m \in M} \sum_{g \in D} \mu(g) \ln \frac{m(A^n g)}{\mu(g)} \cdot (3)$$

The second infimum in (2) is taken over all the continuous partitions of unity $D$ on $X$. If we have $\mu(g) = 0$ for a certain element $g \in D$, then we set the corresponding summand in (3) to be zero independently of the value $m(A^n g)$. And if there exists an element $g \in D$ such that $A^n g = 0$ and simultaneously $\mu(g) > 0$, then we set $\tau(\mu) = -\infty$.

Given a transfer operator $A$ we define a family of operators $A_\varphi : C(X) \to C(X)$ depending on the functional parameter $\varphi \in C(X)$ by means of the formula

$$A_\varphi f = A(e^{\varphi} f).$$

Evidently, all the operators of this family are transfer operators as well. Let us denote by $\lambda(\varphi)$ the logarithm of the spectral radius of $A_\varphi$, that is

$$\lambda(\varphi) = \lim_{n \to \infty} \frac{1}{n} \ln \|A_\varphi^n\|.$$

The principal importance of $t$-entropy is clearly demonstrated by the following Variational Principle.

**Theorem 1** ([6], Theorem 5.6) Let $A : C(X) \to C(X)$ be a transfer operator for a continuous mapping $\alpha : X \to X$ of a Hausdorff compact space $X$. Then

$$\lambda(\varphi) = \max_{\mu \in M_\alpha} (\mu(\varphi) + \tau(\mu)), \quad \varphi \in C(X).$$

The next theorem presents a new definition of $t$-entropy.
Lemma 3 For any continuous partition of unity \( D \) on \( X \) and any pair of numbers \( n \in \mathbb{N} \), \( \varepsilon > 0 \) there exists a continuous partition of unity \( E \) such that for each pair of functions \( g \in D \) and \( h \in E \) the oscillation of \( A^n g \) on the support of \( h \) is less than \( \varepsilon \):

\[
\sup\{ A^n g(x) \mid h(x) > 0 \} - \inf\{ A^n g(x) \mid h(x) > 0 \} < \varepsilon.
\] (6)

Proof. Any function \( A^n g \) belongs to \( C(X) \). Therefore its range is contained in a certain segment \([a, b]\).

Evidently, there exists a continuous partition of unity \( \{f_1, \ldots, f_k\} \) on the segment \([a, b]\) such that the support of every its element is contained in a certain interval of the length less than \( \varepsilon \). Then the family \( E_g = \{f_1 \circ A^n g, \ldots, f_k \circ A^n g\} \) forms a continuous partition of unity on \( X \) and on the support of each its element the oscillation of \( A^n g \) is less than \( \varepsilon \). Now all the products \( \prod_{g \in D} h_g \), where \( h_g \in E_g \), form the desired partition of unity \( E \). \( \square \)

Now let us prove Theorem 2. Comparing (3) and (5) one sees that

\[
\tau'_n(\mu, D) \leq \tau_n(\mu, D).
\]

Therefore to prove (4) it is enough to verify the inequality

\[
\tau_n(\mu) \leq \tau'_n(\mu, D).
\]

Since in the case when \( \tau_n(\mu) = -\infty \) the latter inequality is trivial in what follows we assume that \( \tau_n(\mu) > -\infty \).

Let us fix a number \( n \in \mathbb{N} \), a continuous partition of unity \( D \) on \( X \) and a number \( \varepsilon > 0 \). For these objects there exists a continuous partition of unity \( E \) mentioned in Lemma 3. Consider one more continuous partition of unity consisting of the functions of the form \( g \cdot h \circ \alpha^n \), where \( g \in D \) and \( h \in E \). For this partition, by the definition of \( \tau_n(\mu) \), there exists a measure \( m \in M \) such that

\[
\tau_n(\mu) - \varepsilon \leq \sum_{g \in D} \sum_{h \in E} \mu(g \cdot h \circ \alpha^n) \ln \frac{m(A^n(g \cdot h \circ \alpha^n))}{m(g \cdot h \circ \alpha^n)}.
\]

From the homological identity it follows that \( A^n(g \cdot h \circ \alpha^n) = hA^n g \). Therefore, the latter inequality is equivalent to

\[
\tau_n(\mu) - \varepsilon \leq \sum_{g \in D} \sum_{h \in E} \mu(g \cdot h \circ \alpha^n) \ln \frac{m(hA^n(g))}{m(g \cdot h \circ \alpha^n)}.
\] (7)
Now for each pair \( g \in D, h \in E \) choose a number \( y_{gh} \) satisfying two conditions
\[
m(hA^n g) = m(h) y_{gh}, \tag{8}
\]
\[
\inf \{ A^n g(x) \mid h(x) > 0 \} \leq y_{gh} \leq \sup \{ A^n g(x) \mid h(x) > 0 \}. \tag{9}
\]
Then inequality \((7)\) takes the form
\[
\tau_n(\mu) - \varepsilon \leq \sum_{g \in D} \sum_{h \in E} \mu(g \cdot h \circ \alpha^n) \ln \frac{m(h)y_{gh}}{\mu(g \cdot h \circ \alpha^n)}, \tag{10}
\]
which is equivalent to
\[
\tau_n(\mu) - \varepsilon \leq \sum_{g \in D} \sum_{h \in E} \mu(g \cdot h \circ \alpha^n) \ln \frac{y_{gh}}{\mu(g \cdot h \circ \alpha^n)} + \sum_{g \in D} \sum_{h \in E} \mu(g \cdot h \circ \alpha^n) \ln m(h). \tag{11}
\]
Let us consider separately the second summand in the right-hand side of \((11):\)
\[
\sum_{g \in D} \sum_{h \in E} \mu(g \cdot h \circ \alpha^n) \ln m(h) = \sum_{h \in E} \mu(h \circ \alpha^n) \ln m(h) = \sum_{h \in E} \mu(h) \ln m(h). \tag{12}
\]
Here in the left-hand equality we have exploited the fact that \( D \) is a partition of unity and in the right-hand equality we have used \( \alpha \)-invariance of \( \mu \). If we treat \( m(h) \) in \((12)\) as independent nonnegative variables satisfying the condition \( \sum_{h \in E} m(h) = 1 \) then the routine usage of Lagrange multipliers principle shows that the function \( \sum_{h \in E} \mu(h) \ln m(h) \) attains its maximum when \( m(h) = \mu(h) \). Evidently, the same is true for the right-hand sides in \((11)\) and \((10)\). Therefore,
\[
\tau_n(\mu) - \varepsilon \leq \sum_{g \in D} \sum_{h \in E} \mu(g \cdot h \circ \alpha^n) \ln \frac{\mu(h)y_{gh}}{\mu(g \cdot h \circ \alpha^n)}. \tag{13}
\]
Observe that estimates \((6)\) and \((9)\) imply
\[
\mu(h)y_{gh} \leq \mu(h(A^n g + \varepsilon)). \tag{14}
\]
Using \((13), (14)\), and the fact that \( E \) is a partition of unity and exploiting the concavity of logarithm we obtain the following relations:
\[
\tau_n(\mu) - \varepsilon \leq \sum_{g \in D} \sum_{h \in E} \mu(g \cdot h \circ \alpha^n) \ln \frac{\mu(h(A^n g + \varepsilon))}{\mu(g \cdot h \circ \alpha^n)} = \]
\[
= \sum_{g \in D} \mu(g) \sum_{h \in E} \mu(g \cdot h \circ \alpha^n) \ln \frac{\mu(h(A^n g + \varepsilon))}{\mu(g \cdot h \circ \alpha^n)} \leq \]
\[
\leq \sum_{g \in D} \mu(g) \ln \sum_{h \in E} \frac{\mu(h(A^n g + \varepsilon))}{\mu(g)} = \sum_{g \in D} \mu(g) \ln \frac{\mu(A^n g + \varepsilon)}{\mu(g)}. \]

By the arbitrariness of \( \varepsilon \) this implies
\[
\tau_n(\mu) \leq \sum_{g \in D} \mu(g) \ln \frac{\mu(A^n g)}{\mu(g)} = \tau'_n(\mu, D)
\]
and finishes the proof of Theorem 2. \( \square \)

Now let us proceed to the general \( C^\ast_n \)-dynamical setting.
2 The general case of $C^*$-dynamical systems

The general notion of $t$-entropy involves the so-called base algebra and a transfer operator for a $C^*$-dynamical system. Let us recall definitions of these objects (see [6]).

A base algebra $C$ is a selfadjoint part of a certain commutative $C^*$-algebra with an identity $1$. This means that there exists a commutative $C^*$-algebra $B$ with an identity $1$ such that

$$C = \{ b \in B \mid b^* = b \}.$$ 

A pair $(C, \delta)$, where $C$ is a base algebra and $\delta$ is its certain endomorphism such that $\delta(1) = 1$, is called a $C^*$-dynamical system.

Let $(C, \delta)$ be a $C^*$-dynamical system. A linear operator $A : C \to C$ is called a transfer operator (for $(C, \delta)$), if it possesses the following two properties

a) $A$ is positive (it maps nonnegative elements of $C$ into nonnegative ones);

b) it satisfies the homological identity

$$A((\delta f)g) = fAg \quad \text{for all } f, g \in C. \quad (15)$$

We denote by $M(C)$ the set of all positive normalized linear functionals on $C$. A functional $\mu \in M(C)$ is called $\delta$-invariant if for each $f \in C$ we have $\mu(f) = \mu(\delta f)$. The set of all $\delta$-invariant functionals from $M(C)$ will be denoted by $M_\delta(C)$.

By a partition of unity in the algebra $C$ we mean any finite set $D = \{g_1, \ldots, g_k\}$ consisting of nonnegative elements $g_i \in C$ satisfying the identity $g_1 + \cdots + g_k = 1$.

The definition of $t$-entropy introduced in the previous section in (2) and (3) can be carried over word by word to the case of $C^*$-dynamical systems. Namely, here $t$-entropy is the functional $\tau$ on $M(C)$ such that its value at any $\mu \in M(C)$ is defined by the following formulae

$$\tau(\mu) := \inf_{n \in \mathbb{N}} \frac{\tau_n(\mu)}{n}, \quad \tau_n(\mu) := \inf_D \tau_n(\mu, D), \quad (16)$$

$$\tau_n(\mu, D) := \sup_{m \in M(C)} \sum_{g \in D} \mu(g) \ln \frac{m(A^ng)}{\mu(g)}. \quad (17)$$

The infimum in (16) is taken over all the partitions of unity $D$ in the algebra $C$.

This $t$-entropy plays a crucial role in the corresponding variational principles for the spectral radii as for abstract transfer operators, so also for weighted shift operators in $L^p$-type spaces (see [6], Theorems 6.10, 11.2, 13.1 and 13.6).

To illustrate similarity and difference between the objects considered in this and the previous sections we present now a number of examples of $C^*$-dynamical systems and transfer operators that show, in particular, how far away from the continuous setting described in Section I one can move.

Example 4 Let $Y$ be a measurable space with a $\sigma$-algebra $\mathcal{A}$ and $\beta : Y \to Y$ be a measurable mapping. We denote by $(Y, \beta)$ the discrete time dynamical system generated by the mapping $\beta$ on the phase space $Y$. Let $C$ be any Banach algebra such that

a) $C$ consists of bounded real-valued measurable functions on $Y$,

b) it is supplied with the uniform norm,
c) it contains the unit function, and
d) it is $\beta$-invariant (that is $f \circ \beta \in C$ for all $f \in C$).

Clearly the mapping $\delta : C \to C$ given by $\delta(f) := f \circ \beta$ is an endomorphism of $C$ and therefore $(C, \delta)$ is a $C^*$-dynamical system with the base algebra $C$.

**Example 5** As a particular case of the base algebra in the previous example one can take the algebra of all bounded real-valued measurable functions on $Y$. We will denote this algebra by $B(Y)$.

**Example 6** Let $(Y, \mathfrak{A}, m)$ be a measurable space with a probability measure $m$, and let $\beta$ be a measurable mapping such that $m(\beta^{-1}(G)) \leq Cm(G)$, $G \in \mathfrak{A}$, where the constant $C$ does not depend on $G$. In this case one can take as a base algebra the space $L^\infty(Y, m)$ of all essentially bounded real-valued measurable functions on $Y$ with the essential supremum norm.

**Remark 7** 1) If, as in Example 5, $C = B(Y)$ then the elements of $M(C)$ can be naturally identified with finitely-additive probability measures on the $\sigma$-algebra $\mathfrak{A}$ by means of the equality $\mu(f) = \int_Y f \, d\mu$, $f \in B(Y)$.

2) If, as in Example 4 $C = L^\infty(Y, m)$ then $M(C)$ consists of finitely-additive probability measures on $\mathfrak{A}$ which are absolutely continuous with respect to $m$ (that is they are equal to zero on the sets of zero measure $m$).

3) In Example 4 the set $M_\delta(C)$ is the subset of $M(C)$ consisting of measures $\mu$ such that $\mu(\beta^{-1}(G)) = \mu(G)$ for each measurable set $G$.

4) It should be emphasized that in general given a specific functional algebra its endomorphism is not necessarily generated by a point mapping of the domain. For example, if $C = L^\infty(Y, m)$ then its endomorphisms are generated by set mappings that do not ‘feel’ sets of measure zero (see, for example, [7], Chapter 2). Thus not every endomorphism of $L^\infty(Y, m)$ is generated by a certain measurable mapping $\beta$ as in Example 6.

On the other hand, on the maximal ideals level any endomorphism is induced by a certain point mapping (for details see [6], Theorem 6.2). Therefore raising the apparatus of investigation to the $C^*$-algebraic level we not only essentially extend the field of its applications but additionally can always exploit point mappings in the study of transfer operators independently of their concrete origin (see in this connection the general description of transfer operators given in [6], Section 7).

The next example can be considered as a model example of transfer operators on $L^\infty$.

**Example 8** Let $(Y, \mathfrak{A})$ be a measurable space with a $\sigma$-finite measure $m$, and let $\beta$ be a measurable mapping such that for all measurable sets $G \in \mathfrak{A}$ the following estimate holds

$$m(\beta^{-1}(G)) \leq Cm(G),$$

where the constant $C$ does not depend on $G$. For example, if the measure $m$ is $\beta$-invariant one can set $C = 1$. Let us consider the space $L^1(Y, m)$ of real-valued integrable functions and the shift operator that takes every function $f \in L^1(Y, m)$ to $f \circ \beta$. Clearly the norm of this operator does not exceed $C$. The mapping $\delta f := f \circ \beta$ acts also on the space $L^\infty(Y, m)$ and it is an endomorphism of this space. As is known, the dual space to
$L^1(Y, m)$ coincides with $L^\infty(Y, m)$. Define the linear operator $A: L^\infty(Y, m) \to L^\infty(Y, m)$ by the identity
\[
\int_Y f \cdot g \circ \beta \, dm \equiv \int_Y (Af) g \, dm, \quad g \in L^1(Y, m).
\]

In other words, $A$ is the adjoint operator to the shift operator in $L^1(Y, m)$. If one takes as $g$ the index functions of measurable sets $G \subset Y$, then the latter identity takes the form
\[
\int_{\beta^{-1}(G)} f \, dm \equiv \int_G Af \, dm.
\]

Therefore $Af$ is nothing else than the Radon–Nikodim density of the additive set function $\mu_f(G) = \int_{\beta^{-1}(G)} f \, dm$. Evidently, the operator $A$ is positive and satisfies the homological identity
\[
A((\delta f) g) = f Ag, \quad f, g \in L^\infty(X, m).
\]

We see that $A$ is a transfer operator (for the $C^*$-dynamical system $(L^\infty(Y, m), \delta)$). And in the case when $m$ is $\beta$-invariant measure it is a conditional expectation operator.

**Remark 9** Recalling Remark 4 we have to stress that in general given a specific functional algebra and its endomorphism then a transfer operator is not necessarily associated with a point mapping of the domain.

We now present the $C^*$-dynamical analogue to Theorem 2.

**Theorem 10** For $\delta$-invariant functionals $\mu \in M_{\delta}(\mathcal{C})$ the following formula is true
\[
\tau(\mu) = \inf_{n, D} \frac{1}{n} \sum_{g \in D} \mu(g) \ln \frac{\mu(A^n g)}{\mu(g)}.
\] (18)

**Proof.** This theorem can derived from Theorem 2.

Indeed, to start with we observe that the Gelfand transform establishes an isomorphism between $\mathcal{C}$ and the algebra $C(X)$ of continuous real-valued functions on the maximal ideal space $X$ of $\mathcal{C}$. Therefore we can identify $\mathcal{C}$ with $C(X)$ mentioned above.

Moreover, under this identification the endomorphism $\delta$ mentioned in the definition of the $C^*$-dynamical system $(\mathcal{C}, \delta)$ takes the form
\[
[\delta f](x) = f(\alpha(x)),
\]
where $\alpha: X \to X$ is a uniquely defined continuous mapping (for details see [6], Theorem 6.2). Thus the $C^*$-dynamical system $(\mathcal{C}, \delta)$ is completely defined by the corresponding dynamical system $(X, \alpha)$.

In terms of the latter dynamical system the homological identity (15) for the transfer operator $A$ can be rewritten as (11).

Since we are identifying $\mathcal{C}$ and $C(X)$, the Riesz theorem implies that the set $M(\mathcal{C})$ can be identified with the set $M$ of all regular Borel probability measures on $X$ and the identification is established by means of the formula
\[
\mu(\varphi) = \int_X \varphi \, d\mu, \quad \varphi \in \mathcal{C} = C(X),
\] (19)
where $\mu$ in the right-hand part is a measure on $X$ assigned to the functional $\mu \in M(C)$ in the left-hand part.

Finally, if $\mu \in M_\delta(C)$ is a $\delta$-invariant functional then the corresponding measure $\mu$ in (19) is $\alpha$-invariant, that is

$$\mu(f) = \mu(f \circ \alpha), \quad f \in C(X).$$

Therefore the set $M_\delta(C)$ is naturally identified with the set $M_\alpha$ of all Borel probability $\alpha$-invariant measures on $X$.

Under all these identifications the desired result follows from Theorem 2. □

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