Falling chains

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The one-dimensional fall of a folded chain with one end suspended from a rigid support and a chain falling from a resting heap on a table is studied. Because their Lagrangians contain no explicit time dependence, the falling chains are conservative systems. Their equations of motion are shown to contain a term that enforces energy conservation when masses are transferred between subchains.

We show that Cayley’s 1857 energy nonconserving solution for a chain falling from a resting heap is incorrect because it neglects the energy gained when a transferred link leaves a subchain. The maximum chain tension measured by Calkin and March for the falling folded chain is given a simple if rough interpretation. Other aspects of this falling folded chain are briefly discussed.

I. INTRODUCTION

A folded flexible heavy chain is suspended from a rigid support by its two ends placed close together. One end is then released in the manner of a bungee fall, while the stationary arm gets longer. Calkin and March have noted that the system is conservative, “there being no dissipative mechanisms.” Energy conservation allows them to describe the one-dimensional motion of the falling chain simply and transparently in the continuum limit where the link length goes to zero: As the chain falls, energy conservation concentrates the entire mechanical energy in the still falling arm. When the mass of the falling arm finally vanishes at the end of the fall, both its falling velocity \( v \) and its falling acceleration diverge to infinity.\(^1\) This phenomenon of energy concentration is similar to that occurring in the crack of the whip. In the words of Bragg\(^2,3\), a shock “wave runs down the cord and carries energy to the lash at the end,” where the velocity diverges to infinity in the continuum limit\(^4,5\).

Calkin and March\(^1\) went on to measure the falling motion of an actual chain 2 m long containing 81 links. They found that the physical chain does indeed fall faster than free fall, and that the continuum model accurately describes the experimental chain motion except near the end of the chain fall. Their measurement of the chain tension \( T \) at the fixed support of the chain is particularly interesting. The theoretical chain tension given by the continuum model contains a term proportional to \( v^2 \) of the falling velocity. It therefore increases without limit as the theoretical value of \( v \) becomes infinite at the end of the fall. Calkin and March\(^1\) found that the experimental tension increases only to a maximum value of about 25\( Mg \), where \( M \) is the total mass of the chain. This maximum tension is of course far in excess of the maximum value of only 2\( Mg \) expected when the falling end is falling freely, thus demonstrating beyond doubt that the folded chain indeed falls faster than \( g \). We shall explain in Sect.IV that it is the finite size of the link that prevents \( T \) from going to the infinite value predicted by the continuum model.

In many older textbooks on mechanics,\(^6–8\) the falling arm is incorrectly described as freely falling and is brought to rest by inelastic impacts at the fold of the chain. The kinetic energy loss in a completely inelastic collision is real. It was first described by Lazare Carnot,\(^9–11\) father of the Sadi Carnot of thermodynamics. The effect is called Carnot’s energy loss or Carnot’s theorem in Sommerfeld’s book on mechanics.\(^9\) The effect of an impulse alone on a dynamical system was treated correctly by an additional term by Lagrange.\(^12,13\) In contrast, Hamel\(^14\) obtained the correct solution for the falling chain by assuming energy conservation. We shall show that energy conservation results because the Carnot energy loss caused by a transferred mass absorbed by the receiving subchain is counterbalanced by the energy gained when the mass leaves the “emitting” subchain.

The Calkin–March observation\(^1\) that the folded chain falls faster than \( g \) was subsequently confirmed experimentally by Schagerl et al.\(^15\) Photographic evidence can also be found in Ref. 16. The correct solution of the motion of the falling folded chain by energy conservation has been included in some recent textbooks on classical dynamics.\(^17–19\)

Schagerl et al.\(^15\) were unaware of the measurement of Calkin and March.\(^1\) The results of their measurements\(^15\) came as a surprise to them because they had concluded by theoretical arguments that the chain fell only as fast as \( g \), and that the total mechanical energy was not conserved.\(^20,21\) In these earlier papers, the authors rejected Hamel’s energy conserving solution.\(^14\) and claimed that there was dissipation caused by the inelastic but momentum-conserving impacts at the fold of the chain. They justified their treatment by citing Sommerfeld’s use of Carnot’s energy loss in another falling chain problem,\(^9\) which we will describe in the following.

The experimental observation\(^15\) that the free end of the falling folded chain falls faster than \( g \) might have led the authors of Ref. 15 to conclude that the motion of the falling chain is non-unique, because “it is important to note that for the folded string itself there exist more solutions which fulfill the balance of linear momentum (but do not conserve the mechanical energy).”\(^15\) This non-uniqueness is the paradox referred to in the title of their paper.\(^15\)

The conclusion that non-unique solutions exist is clearly untenable because whether the chain is energy-conserving or not, its equation of motion is a lin-
ear differential equation with a unique solution for a given set of initial conditions. Hence the experimental observation\textsuperscript{1,15,16} of faster than $g$ fall proves that the motion cannot be the freely falling, energy nonconserving one. Thus there is no paradox.

A review article by Irschik and Holl\textsuperscript{22} mentions the same erroneous interpretation that for the falling folded chain, momentum is conserved but mechanical energy is not conserved. These authors knew of the experimental work in Ref. 15 but not that of Ref. 1. In a previous paper on Lagrange’s equations, Irschik and Holl\textsuperscript{22} were puzzled by the result of Ref. 15 because they thought that the string tension at the base of the falling arm ($N$ in their Eq. (6.22)) should vanish, and therefore the arm should fall freely. They realized that this conclusion is not consistent with the observation of Ref. 15.

We shall show that the erroneous conclusion of energy loss comes from the neglect of the energy gained when the transferred mass at the fold of the chain leaves the falling arm. This energy gain is the time reverse of the Carnot energy loss incurred when the transferred mass is received by the stationary arm of the folded chain.

There is another falling chain problem for which the consensus is that the total mechanical energy is not conserved. The steady fall of a stationary chain resting on a table link by link through a hole in the table appears to have been first studied by Arthur Cayley in 1857.\textsuperscript{22,24} He treated the motion as a continuous-impact problem leading to a nonconservative system and an acceleration of $g/3$. Cayley’s falling chain problem appears as Problem I.7 of Sommerfeld,\textsuperscript{9} where the connection to Carnot’s energy loss is explicitly stated. It can also be found in Refs. 18,25–30. Note that Problem 9-15 in Ref. 29 has been rewritten in Ref. 30 without any mention of energy loss. However, the solutions given in the instructor’s manuals\textsuperscript{31,32} are identical.

The only dissent we have found of this common consensus is that energy is not conserved is in the recent paper by de Sousa and Rodrigues.\textsuperscript{33} They first describe the falling folded chain by using a Newton equation for the two variable mass subchains that contains the gravitational force but no chain tension. They obtain the wrong or energy-nonconserving solution with acceleration $a = g$. They then solve the problem of the chain falling from a resting heap in a different way by assuming energy conservation. This assumption yields the right solution, as we shall show in the following. Their solution is the only correct solution we have been able to find in the literature for the chain falling from a heap on a table.

In Sec. III we shall show specifically that the transfer of a link from the heap to the falling subchain is the same energy conserving process that operates in the falling folded chain, namely an exoergic mass emission followed by a counterbalancing endoergic mass absorption. We will see that Cayley and others considered only half of a two-step mechanical process that is energy-conserving as a whole.

Given the brief history of falling chains sketched here, it is interesting to determine unambiguously when a mechanical system such as a falling chain is energy conserving. The answer was already given in 1788 by Lagrange.\textsuperscript{12} In modern terminology using the Lagrangian $\mathcal{L}(x,y)$ and the Hamiltonian $\mathcal{H}(x,p)$, two conditions must be satisfied for the mechanical energy $E$ to be conserved: $E = \mathcal{H}$ and $\frac{\partial \mathcal{L}}{\partial t} = 0$. Consequently

$$\frac{dE}{dt} = \frac{d\mathcal{H}}{dt} = -\frac{\partial \mathcal{L}}{\partial t} = 0,$$

as we shall discuss in in Sec. II. We shall also write the condition $E = \mathcal{H}$ in the original form given by Lagrange,\textsuperscript{12} who referred to kinetic energies as “live forces” (“forces vives”). These conditions are well known and can be found in most textbooks on analytical mechanics, but they have been too infrequently applied on actual physics problems.

To show explicitly how this energy conservation enters in the mass transfer between subchains, we begin in Sec. III with the standard force equation of motion for a variable mass system\textsuperscript{9,33–36}. For the special case where no external forces act on these subchains, we show explicitly that the mass transfer is made up of an exoergic mass emission followed by an endoergic mass absorption when the transferred mass sticks inelastically to the receiving arm. We also find that the complete process of mass transfer conserves mechanical energy when the transferred mass has the velocity given to it by Lagrange’s equation of motion. Hence Lagrange’s formulation gives both the simplest and the most complete description of the motion of both falling chains.

There is an important practical difference between the two falling chains, however. The link-by-link mass transfer of the falling folded chain is automatically guaranteed at the fold of the chain, but is difficult to realize for a real chain falling from a resting heap. The folded chain always falls in more or less the same way, but the motion of the resting heap depends on its geometry. More than one link at a time might be set into motion as the chain falls, and some of them might even be raised above the table before falling off it. These complications make it difficult to check the idealized theoretical result by an actual measurement. We therefore concentrate on the falling folded chain in the rest of the paper. In Sec. IV we give a simple-minded interpretation of the maximal chain tension measured by Calkin and March.\textsuperscript{1} Then we explain in Sec. V how to understand the total loss of kinetic energy at the moment the chain reaches full extension, and why the chain rebounds against its support afterward. In Sec. VI we pay tribute to Lagrange’s formulation of classical mechanics.

II. THE LAGRANGIAN AND THE HAMILTONIAN

Figure 1 shows the folded chain when its falling end has fallen a distance $x$. The chain is flexible, and has mass...
Hamiltonian in the idealized one-dimensional treatment is

\[ L(x, v) = \frac{\mu}{4}(L - x)v^2 + MgX, \]

where \( v = \dot{x} \) and

\[ X = \frac{m_Lx_L + mRx_R}{M} = \frac{1}{4L}(L^2 + 2Lx - x^2) \]

is its center of mass (CM) position measured in the downward direction. Here \( m_L \) is the mass, \( b_L \) is the length and \( x_L \) is the CM position of the left arm, while the corresponding quantities for the right arm are \( m_R, b_R \) and \( x_R \):

\[
\begin{align*}
m_i &= \mu b_i, \\
b_L &= \frac{1}{2}(L + x) \quad (4a) \\
b_R &= \frac{1}{2}(L - x) \quad (4b) \\
x_L &= \frac{1}{4}(L + x) \quad (4c) \\
x_R &= \frac{1}{4}(L + 3x). \quad (4d)
\end{align*}
\]

The parameters in the Lagrangian are time-independent and hence \( \partial L / \partial t = 0 \).

The Hamiltonian of the falling folded chain is

\[ \mathcal{H}(x, p_R) = p_Rv - L(x, v) = \frac{p_R^2}{2m_R} - MgX = E. \]

The canonical momentum,

\[ p_R = \frac{\partial L}{\partial v} = m_Rv, \]

is the momentum of the right arm. Hence Eq. (1) is satisfied and the system is conservative.

The identity \( d\mathcal{H}/dt = -\partial L/\partial t \) used in Eq. (1) follows from the relation

\[ \frac{\partial \mathcal{H}}{\partial x} \frac{dx}{dt} + \frac{\partial \mathcal{H}}{\partial p} \frac{dp}{dt} = 0. \]

These two terms cancel each other because the total time derivatives satisfy the canonical equations of motion of Hamilton\textsuperscript{37,38}

\[
\begin{align*}
\dot{x} &= \frac{\partial \mathcal{H}}{\partial p} \quad (8a) \\
\dot{p} &= -\frac{\partial \mathcal{H}}{\partial x}. \quad (8b)
\end{align*}
\]

Equation (1) can also be obtained without using the Hamiltonian. We start with \( E = 2K - L \), where \( K \) is the kinetic energy, and write

\[ \frac{dE}{dt} = \frac{d}{dt}(2K) - \left( \frac{\partial L}{\partial x} \dot{x} + \frac{\partial L}{\partial v} \dot{v} + \frac{\partial L}{\partial t} \right). \]

The second term on the right can be written in terms of \( \partial L/\partial v \) by using Lagrange's equation of motion\textsuperscript{39}

\[ \frac{\partial L}{\partial x} = \frac{d}{dt} \left( \frac{\partial L}{\partial v} \right) \]

to simplify \( dE/dt \) to the form obtained by Lagrange\textsuperscript{40}

\[ \frac{dE}{dt} = \frac{d}{dt} \left( 2K - v \frac{\partial L}{\partial v} \right) - \frac{\partial L}{\partial t}. \]

Thus two conditions are needed for \( E \) to be conserved: \( \partial L/\partial t = 0 \) and \( v\partial L/\partial v = 2K \). The second condition is equivalent to the requirement \( E = \mathcal{H} \).

By using energy conservation, the squared velocity at position \( x \) is found to be \textsuperscript{5,14,17,19}

\[ v^2 = 2gx \frac{1 - \frac{x}{L}}{1 - \frac{x}{L}}, \]

where \( \hat{x} = x/L \). A Taylor expansion for small \( \hat{x} \),

\[ \hat{x}^2 \approx 2gx[1 + \frac{1}{2}(\hat{x} + \hat{x}^2 + \ldots)], \]

shows that the falling chain falls faster than free fall right from the beginning. Its falling speed then increases monotonically beyond free fall, and reaches infinity as \( \hat{x} \to 1 \).

We can obtain from Eq. (10) Lagrange’s equation of motion for the falling folded chain:

\[ m_Rg - \frac{1}{4}\mu v^2 = \ddot{p}_R = m_R\dot{v} + \dot{m}_Rv. \]

We can then verify by direct substitution that the energy-conserving solution (12) satisfies Eq. (14). Equation (14) can also be solved directly for \( v^2 \) by using the identity

\[ \dot{v} = \frac{1}{2} \frac{dv^2}{dx} \]

to change it into a first-order inhomogeneous differential equation for \( v^2(x) \).

Lagrange’s equation (14) is particularly helpful in understanding the problem conceptually because it uniquely defines the chain tension \( -\mu v^2/4 \) that acts upward on the bottom of the right arm at the point \( B_R \) shown in Fig. 1. This tension comes from the \( x \) dependence of the kinetic energy and serves the important function of enforcing energy conservation. The mistake made in the erroneous energy-nonconserving solution is to omit this term. We shall explain in the next section why this tension points up and not down, as might be expected naively.

It is interesting to apply our analysis to a chain falling from a resting heap on a table through a hole in it because this situation is even more transparent. Let \( x \) be the falling distance, now measured from the table. The
falling chain is described by
\[ \mathcal{L}(x, v) = \frac{\mu}{2}xv^2 + \frac{1}{2}\mu x^2 \]  
(16a)
\[ p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = \mu xv \]  
(16b)
\[ \mathcal{H} = \frac{p_x^2}{2\mu x} - \frac{\mu g x^2}{2} = E, \]  
(16c)
where the subscript \( x \) refers to the falling part of the chain of length \( x \). Because the Lagrangian \( \mathcal{L} \) is not explicitly time-dependent, we again find \( \partial \mathcal{L}/\partial t = 0 \) and a conservative system. Energy conservation can be written in the form
\[ E = \frac{1}{2} \mu v^2 - gx = 0. \]  
(17)
The resulting solution,\(^{33}\)
\[ v^2 = gx, \]  
(18)
shows that the acceleration of the falling chain is \( g/2 \), not the value \( g/3 \) of Cayley’s energy-nonconserving chain.

The reason for the difference can be seen in Lagrange’s equation of motion
\[ m_x g + \frac{1}{2} \mu \dot{v}^2 = \dot{p}_x = m_x \dot{v} + \dot{m}_x v. \]  
(19)

In the incorrect treatment, the downward tension \( \mu \dot{v}^2/2 \) that comes from the \( x \) dependence of the kinetic energy of the falling chain is missing.

With or without the chain tension term, the differential equation (19) describes a system undergoing a constant acceleration \( \ddot{v} = a \). Hence \( v^2 = 2ax \). The differential equation can then be reduced term by term to the algebraic equation,
\[ g + sa = a + 2a, \]  
(20)
giving
\[ a = \frac{g}{3 - s} \]  
(21)
A switching function \( s = 1 \) or \( 0 \) has been added to the second term on the left in Eq. (20). Hence the solution is \( a = g/2 \) for \( s = 1 \) with the chain tension, and \( a = g/3 \) for \( s = 0 \) without the chain tension.

We see that the Lagrangian approach gives a straightforward way of generating the correct equations of motion in a situation that is confusing.

III. MASS TRANSFER BETWEEN SUBCHAIRS

We now clarify how the falling chain transfers mass from one subchain to the other. Assume that subchain 2 of mass \( m_2 + \Delta m \) and velocity \( \mathbf{v}_2 \) transfers a small mass \( \Delta m \) at velocity \( \mathbf{u} \) to subchain 1 of mass \( m_1 - \Delta m \) and velocity \( \mathbf{v}_1 \). The transferred mass is related to the subchain masses as
\[ \Delta m = \Delta m_1 = -\Delta m_2. \]  
(22)
At the receiving subchain 1, the initial and final momenta are
\[ \mathbf{p}_{1i} = (m_1 - \Delta m)\mathbf{v}_1 + \mathbf{u}\Delta m, \]  
(23a)
\[ \mathbf{p}_{1f} = m_1(\mathbf{v}_1 + \Delta \mathbf{v}_1), \]  
(23b)
where we have included the momentum of the transferred mass \( \Delta m \) in the initial state, for the sake of notational simplicity. The total momentum change,
\[ \Delta \mathbf{p}_1 = \mathbf{p}_{1f} - \mathbf{p}_{1i} = m_1 \Delta \mathbf{v}_1 + \Delta m(\mathbf{v}_1 - \mathbf{u}), \]  
(24)
on receiving the transferred mass \( \Delta m \) can be associated with an impulse \( \mathbf{F}_1 \Delta t \) received from an external force
\[ \mathbf{F}_1 \equiv \frac{d\mathbf{p}_1}{dt} = \frac{d}{dt}(m_1 \mathbf{v}_1) - \frac{dm_1}{dt}. \]  
(25)
This variable mass equation of motion holds whether or not the system is conservative.

In a similar way, we can show that subchain 2 on emitting the transferred mass experiences an external force
\[ \mathbf{F}_2 \equiv \frac{d\mathbf{p}_2}{dt} = \frac{d}{dt}(m_2 \mathbf{v}_2) - \frac{dm_2}{dt}. \]  
(26)
Note how these well-known “rocket” equations take the same form whether the rocket is discharging or absorbing masses. Because the total chain mass \( M = m_1 + m_2 \) is constant, the sum of these variable mass equations is just the simple equation
\[ \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = \dot{\mathbf{P}}, \]  
(27)
for the center of mass of the entire chain. The internal forces due to mass transfer always cancel out for any choice of \( \mathbf{u} \) when the total mass \( M \) is constant.\(^{41}\)

The velocity \( \mathbf{u} \) of the mass transfer is not arbitrary, however. It too is determined uniquely by the chain tension term in Lagrange’s equation of motion. For the falling folded chain, the second term on the left-hand side of Eq.(14) gives
\[ \frac{v}{2} \frac{dm_R}{dt} = \frac{1}{4} \mu v^2, \]  
(28)
and for the chain falling from a resting heap, the second term on the left-hand side of Eq.(19) gives
\[ \frac{u}{2} \frac{dm_x}{dt} = \frac{1}{2} \mu \dot{v}^2. \]  
(29)
Thus \( u = v/2 \) for both falling chains. The two chains differ in that the falling folded chain has a fold in it, suggesting that the fold falling with the speed \( u = v/2 \) is the natural location of mass transfer. For the chain falling from a heap on a table, on the other hand, the
A related result occurs in elastic collisions where the internal forces are equal and opposite. As a result, “the kinetic energy lost in compression balances exactly the kinetic energy gained in restitution. This is sometimes called the third theorem of Carnot.”42 Because we know that the falling chains are conservative systems, it follows that the mass transfer taken as a whole constitutes a totally elastic collision.

One final point needs clarification. According to Eq. (14) the fold in the chain exerts an upward tension $T_{ii} = -\mu v^2/4$ on the right arm. The direction of this tension might appear counter-intuitive until it is realized that the rocket engine term $\dot{m} Rv = -\mu v^2/2$ on the right-hand side of Eq. (14) term can be moved to the left side, the force side, of the equation. In this position, the term carries a positive sign and represents a downward force that dominates the up-pointing tension. When added to the force of gravity, these two extra forces together gives a net downward force that causes the downward acceleration to exceed $g$.

For the chain falling from a heap, the situation is upside down and a time reverse of the falling folded chain. The mass transfer occurs at the top where the chain falls down link by link into the moving arm. The signs of both the chain tension and the rocket engine term are opposite to those in the falling folded chain because the falling arm is gaining mass. The rocket engine term, $\dot{m} R v = \mu v^2$, when moved to the left or force side of Eq. (19), dominates to give a net up-pointing braking force that prevents the falling chain from falling as fast as $g$. However, it is the chain tension term that pulls the chain down with an acceleration greater than $g/3$.

IV. THE CHAIN TENSION AT THE SUPPORT

The chain tension $T$ of the falling folded chain at the support $S$ of Fig. 1 can be calculated in the one-dimensional continuum model from Eqs. (27) and (12) using $F = M g - T$. The result,17-19

$$T(\dot{x}) = M g \frac{2 + 2\dot{x} - 3\dot{x}^2}{4(1 - \dot{x})},$$

is a positive monotonic function of $\dot{x}$ that increases to $\infty$ as $\dot{x} \to 1$.

Calkin and March1 studied experimentally the tension $T$ of a linked chain with $N = 81$ links. They measured a maximal tension of $25 M g$ as the chain approached the bottom. To understand this result within a simple theoretical framework, we shall assume that the theoretical tension (34) of the ideal chain with $N \to \infty$ holds until the last link remains standing upright. The chain tension at that moment is $T(79/81) = 11.1 M g$.

The tip of the last link will next fall a distance of $2\ell = L - x$, where $\ell = L/N$ is the link length. It does so by rotating about a pivot at the contact point between the last two links. This rotation can be separated into two steps: first a quarter turn to a horizontal position,
and then a second quarter turn to the hanging position at the bottom of its travel. To keep the chain center of mass falling straight down, the lower part of the left arm sways sideways to some maximal displacement after the first quarter turn, and then sways back at the second quarter turn. This sideway motion will not change the vertical tension.

In the first quarter turn, the falling chain tip is still above the pivot, meaning that fractions of the rotating link are still coming to rest against the left arm until the last link is horizontal. Hence the theoretical tension \((34)\) can be expected to hold until \(\ddot{x} = 80/81\), where \(T\) has almost doubled to \(21.2Mg\).

In the second and final quarter turn, the chain tip is below the pivot. The speed of the chain tip continues to increase, but now only by a freely falling rotation. The main consequence of this final quarter turn is to convert the vertical velocity \(\dot{x}\) to a slightly larger horizontal velocity as the chain tip reaches the bottom. At that moment, the chain tension \(T\) has increased by the weight \(Mg/N\) of the last link. Because this final increase is very small, our simple analysis yields a final result of about \(21Mg\), in rough agreement with experiment. The final swing of the rotating link is easily reproduced by a falling chain made up of paper clips.

We believe that the remaining discrepancy comes primarily from approximating the linear density \(\mu\) of the chain as uniform when it is not. The Calkin-March chain appears to be a common or standard link chain made up of straight interlinking oval links. At places where the links hook into each other, the linear density increases by at least a factor of two because all four sides of two links appear in cross section instead of the two sides of a single link. If we also count the bends of the links, we find a significant mass concentration at the linkages. Some of this mass concentration at the linkage for the last link should be allowed to produce some tension before the last link falls down from the horizontal position. Furthermore, this effect appears to be larger than any energy loss caused by possible slippage at the loose linkages of the chain.

The observed maximum chain tension of \(25Mg\) can be reproduced at \(\ddot{x} = 0.9896\), an increase of 0.0019 from the theoretical value of \(80/81 = 0.9877\). Each link in the chain has an inside length of \(\ell = 0.97''\). Hence the observed maximum tension is reproduced if we assume that an additional 0.15'' of the last link still produces tension according to the theoretical formula \((34)\) after it falls through the horizontal position.

We note that the link length used in the Calkin-March experiment\(^1\) matches that of the lightest proof coil chain manufactured by the Armstrong Alar Chain Corporation,\(^43\) but that the Armstrong chain is too heavy by a factor 1.75. The match would be good if the material diameter, that is, the diameter of the metal loop in the link, is decreased from the Armstrong chain value of \(d = (7/32)''\) to \((5/32)''\). For this estimated matter diameter \(d\), the extra length of 0.15'' needed to produce the additional tension is about 1.06\(d\). We leave it to the reader to determine if this is the correct way to analyze the discrepancy and if so, how the result of 1.06\(d\) can be accounted for theoretically.

Our simple interpretation is consistent with the general features obtained in the numerical simulation of a falling folded chain by Tomaszewski and Pieranski.\(^{44}\) They separate a chain of length \(L = 1\) m into 51 links of uniform linear density joined by smooth hinges. They solve the 51 coupled Lagrange’s equations numerically. They find a maximum velocity of about 21.5 m/s when the last link is falling. In our interpretation, the maximum velocity is expected to be \(v(50/51) = 22.4\) m/s, very close to the computed value. The numerical solution shows a significant amount of oscillation in the stationary left arm when the right arm is falling. This feature is not included in the simple one-dimensional treatment using only the falling distance \(x\). The loss of kinetic energy to oscillations in the left arm has the correct sign to account for the difference between the two theoretical maximal velocities.

In this connection we note that Calkin and March\(^1\) did not report any dramatic left-arm oscillations in their falling folded chain. We also do not find them in a falling folded chain of paper clips. A falling ball-chain, on the other hand, shows a wave-like vibration mostly in the lower half of the rebounding chain. This observed damping of the theoretical vibrations expected of the hinged-link model of Ref. 44 seems to suggest that the loose linkages in the physical chains do not transport energy readily to the transverse motion of the chain.

\section{V. The Last Hurrah}

For the idealized uniform and inextensible falling folded chain, we find its center of mass kinetic energy to be

\[
K_{\text{CM}} = Mg\frac{\ddot{x}(1-\dot{x})(2-\ddot{x})}{8}
\]  

\text{(35)}

in the one-dimensional continuum model. This CM kinetic energy increases from 0 at \(\ddot{x} = 0\) to a maximum value at \(\ddot{x} = 1 - 1/\sqrt{3}\) before decreasing to zero again at \(\ddot{x} = 1\). The work done against the chain tension \(T = Mg - F\), namely

\[
W(X) = \int_{L/4}^{X} T(X) dX = Mg \left( X - \frac{L}{4} \right) - K_{\text{CM}}.
\]  

\text{(36)}

increases monotonically, reaching \(MgL/4\) at \(x = L\). Given the energy-conserving solution \((12)\) of the one-dimensional continuum model where the left arm remains at rest, it is clear that the change in potential energy given in Eq.\((36)\) appears as the kinetic energy of the right arm

\[
K_R = Mg\frac{\ddot{x}(2-\dot{x})}{4}.
\]  

\text{(37)}
Hence the work $W(X)$ done against friction is just

$$W(X) = K_R - K_{cm} = K_{int},$$

(38)

the internal kinetic energy of the falling arm not already included in $K_{cm}$. In a more detailed model where the motion of the left arm is also allowed, the excitation energy of the left arm will have to be included in the energy balance. The resulting $v^2$ will then differ from the value given in Eq. (12) for the one-dimensional continuum model.\textsuperscript{44}

At the moment the falling tip of the ideal one-dimensional chain turns over and straightens against the resting left arm, even this internal kinetic energy vanishes as the entire chain comes to rest at full extension. This resting state too has a simple explanation that is worth repeating: The act of straightening can be visualized as a completely inelastic Carnot collision in which the remaining mass $\Delta m = m_R$ of the right arm is transferred to the left arm of mass $m$. Momentum conservation in the laboratory requires that

$$p_f = (m + \Delta m)v = v\Delta m = p_i.$$

(39)

The resulting kinetic energy change in this totally inelastic collision is

$$\Delta K_{coll} = K_f - K_i = -\frac{1}{2}v^2\Delta m\left(\frac{m}{m + \Delta m}\right).$$

(40)

This analysis shows that in the limit $x \to L$ when the right-arm mass vanishes, all its remaining kinetic energy $K_R = MgL/4$ is converted into the internal potential energy of the momentarily resting chain in a single inelastic collision. For a perfectly inextensible chain suspended from a rigid support, $\Delta v$ must vanish, which means that the appropriate $m$ must be infinite, including not only the finite mass of the left arm but also the infinite mass of the support.

This description is not the end of the story for an actual falling folded chain. If the chain is an ideal spring, it will be stretched by an amount consistent with overall energy conservation as the final mass transfer takes place. This stored potential energy will be used to give the chain its kinetic energy on rebound. In actual chains the final rebound that follows Carnot’s energy loss should also appear, even though the rebound is not completely elastic. This grand finale is easily reproduced for a falling folded chain of paper clips.

VI. CONCLUSION

We conclude by paying homage to the genius of Lagrange whose formulation of classical mechanics helps us to decide definitively if a mechanical system is conservative. We have found that Lagrange’s equation of motion contains a unique description of what happens when masses are transferred between the two parts of a falling chain, a description that actually enforces energy conservation in the falling chain.

Joseph Louis Lagrange (1736–1813) was born Giuseppe Lodovico Lagrangia\textsuperscript{45} in Turin of Italian-French parents. He introduced purely analytic methods to replace the cumbersome geometrical arguments then commonly used in calculus. Using this algebraic method, he and his contemporary Leonhard Euler founded the calculus of variations as a special branch of mathematics where a function that minimizes an integral is to be constructed.\textsuperscript{46} In his masterpiece \textit{Mécanique Analytique} (1788),\textsuperscript{12} Lagrange discarded Newton’s geometrical approach and recast all of mechanics in algebraic form in terms of generalized coordinates whose motion satisfies a variational principle, the principle of virtual work. He emphasized in the preface that “No figures will be found in this work . . . only algebraic operations . . . ”\textsuperscript{47,48} He was one of the greatest mathematicians of the 18th century, perhaps its greatest.\textsuperscript{49} Truesdell, an admirer of Euler, faults the Lagrangian formulation for excessive abstractness that “conceals the main conceptual problems of mechanics.”\textsuperscript{50} However we have seen in this paper how Lagrange’s method gives definitive answers with unmatched ease, clarity, and elegance.

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FIG. 1: The falling folded chain.