Closed Formulas for the Sums of Cubes of Generalized Fibonacci Numbers: Closed Formulas of $\sum_{k=0}^{n} W_{3k}$ and $\sum_{k=1}^{n} W_{3k}$

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Author’s contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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ABSTRACT

In this paper, closed forms of the sum formulas for the cubes of generalized Fibonacci numbers are presented. As special cases, we give summation formulas of Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal and Jacobsthal-Lucas numbers.

Keywords: Fibonacci numbers; Lucas numbers; Pell numbers; Jacobsthal numbers; sum formulas.

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1 INTRODUCTION

The Fibonacci and Lucas sequences are very well-known examples of second order recurrence sequences. The Fibonacci numbers are perhaps most famous for appearing in the rabbit breeding problem, introduced by Leonardo de Pisa in 1202 in his book called Liber Abaci. The Fibonacci and Lucas sequences are a source of many nice and interesting identities. The sequence of Fibonacci numbers $\{F_n\}$ is defined by $F_n = F_{n-1} + F_{n-2}, \quad n \geq 2, \quad F_0 = 0, \quad F_1 = 1.$

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Theorem 2.1. The following theorem presents some summing formulas of generalized Fibonacci numbers with

and the sequence of Lucas numbers \( \{L_n\} \) is defined by

\[
L_n = L_{n-1} + L_{n-2}, \quad n \geq 2, \quad L_0 = 2, \quad L_1 = 1.
\]

The Fibonacci numbers, Lucas numbers and their generalizations have many interesting properties and applications to almost every field. Horadam [1] defined a generalization of Fibonacci sequence, that is, he defined a second-order linear recurrence sequence \( \{W_n(a, b; r, s)\} \), or simply \( \{W_n\} \), as follows:

\[
w_n = rW_{n-1} + sW_{n-2}; \quad W_0 = a, \quad W_1 = b, \quad (n \geq 2) \quad (1.1)
\]

where \( W_0, W_1 \) are arbitrary complex numbers and \( r, s \) are real numbers, see also Horadam [2], [3] and [4]. Now these generalized Fibonacci numbers \( \{W_n(a, b; r, s)\} \) are also called Horadam numbers. The sequence \( \{W_n\}_{n \geq 0} \) can be extended to negative subscripts by defining

\[
W_{-n} = -\frac{r}{8}W_{-(n-1)} + \frac{1}{8}W_{-(n-2)}
\]

for \( n = 1, 2, 3, \ldots \) when \( s \neq 0 \). Therefore, recurrence (1.1) holds for all integer \( n \).

For some specific values of \( a, b, r \) and \( s \), it is worth presenting these special Horadam numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 1) are used for the special cases of \( r, s \) and initial values.

| Name of sequence | Notation: \( W_n(a, b; r, s) \) | OEIS: [5] |
|------------------|----------------------------------|-----------|
| Fibonacci        | \( F_n \) = \( W_n(0, 1; 1, 1) \) | A000045   |
| Lucas            | \( L_n \) = \( W_n(2, 1; 1, 1) \) | A000032   |
| Pell             | \( P_n \) = \( W_n(0, 1; 2, 1) \) | A000129   |
| Pell-Lucas       | \( Q_n \) = \( W_n(2, 2; 1, 1) \) | A002203   |
| Jacobsthal       | \( J_n \) = \( W_n(0, 1; 1, 2) \) | A001045   |
| Jacobsthal-Lucas | \( j_n \) = \( W_n(2, 1; 1, 2) \) | A014551   |

The evaluation of sums of powers of these sequences is a challenging issue. Two pretty examples are

\[
\sum_{k=0}^{n} F_k^3 = \frac{1}{2}(-F_{n+2}^3 - 3F_{n+1}^3 + 3F_{n+2}^2F_{n+1} + 1)
\]

and

\[
\sum_{k=0}^{n} P_k^3 = \frac{1}{14}(-2P_{n+2}^3 - 16P_{n+1}^3 + 9P_{n+2}^2P_{n+1} - 3P_{n+1}^2P_{n+2} + 2).
\]

In this work, we derive expressions for sums of third powers of generalized Fibonacci numbers. We present some works on sum formulas of powers of the numbers in the following Table 2.

| Name of sequence | sums of second powers | sums of third powers | sums of powers |
|------------------|-----------------------|----------------------|---------------|
| Generalized Fibonacci | [6,7,8,9,10,11] | [12,13] | [14,15,16] |
| Generalized Tribonacci | [17] |                      |               |
| Generalized Tetranacci | [18,19] |                      |               |

2 SUM FORMULAS OF GENERALIZED FIBONACCI NUMBERS WITH POSITIVE SUBSCRIPTS

The following theorem presents some summing formulas of generalized Fibonacci numbers with positive subscripts.

Theorem 2.1. For \( n \geq 0 \) we have the following formulas: If \( (r + s - 1)(rs - s^3 + 1)(r + a - rs + r^2 + s^2 + 1) \neq 0 \) then
Proof. Using the recurrence relation

\[ \sum_{k=0}^{n} W_k^3 = \frac{\Delta_1}{(r+s-1)(r+s-r^2+s^2+1)} \]

where

\[ \Delta_1 = (-s^3 + 2sr - 1)W_{n+2}^3 - (r^4 + 3r^2s^2 - r^3s + 2rs + r^3 + s^3 - 1)W_{n+1}^3 \\
+ 3s(r+s^2)W_{n+1}W_{n+2}^3 - 3rs^2(r-s-1)W_{n+2}^3 + (2rs + s^3 - 1)W_1^3 \\
+ (r^4 + 3r^2s^2 - r^3s + 2rs + r^3 + s^3 - 1)W_0^3 - 3s(r+s^2)W_0^2W_1 + 3r^2s(r-s-1)W_0^2W_1 \]

\[ \sum_{k=0}^{n} W_k^2 W_{k+1} = \frac{\Delta_2}{(r+s-1)(r+s-r^2+s^2+1)} \]

where

\[ \Delta_2 = -(r^2 - 1)W_{n+2}^3 - r^3(r-s-1)W_{n+1}^3 + s(2r^3 - s^3 + 1)W_{n+2}^2 W_{n+1} \\
- (r^3 + 3r^2s + 2rs + r^3 + s^3 - 1)W_{n+1}^2 W_{n+2} + r(r-s-1)W_1^3 \\
+ rs^2(r-s-1)W_0^3 - s(2r^3 - s^3 + 1)W_0^2 W_1 + (-2r^4 + 2r^2s + 2rs + r^3 + s^3 - 1)W_0^2 W_1. \]

\[ \sum_{k=0}^{n} W_k^2 W_{k+1} = \frac{\Delta_3}{(r+s-1)(r+s-r^2+s^2+1)} \]

where

\[ \Delta_3 = r(r+s^2)W_{n+2}^3 - r^3(r+s)W_{n+1}^3 - (3r^3 + r^3 + s^3 - 1)W_{n+2}^2 W_{n+1} \\
+ s^2(2r^3 - s^3 + 1)W_{n+1}^2 W_{n+2} - r(r+s^2)W_1^3 - r^3(r+s)W_0^3 \\
+ (3r^4 + 3r^2s + 3r^2s + 3s^3 - 1)W_1^2 W_0 - s^2(2r^3 - s^3 + 1)W_0^2 W_1. \]

Proof. Using the recurrence relation

\[ W_{n+2} = rW_{n+1} + sW_n \]

i.e.

\[ sW_n = W_{n+2} - rW_{n+1} \]

we obtain

\[ s^3 W_n = W_{n+2}^3 - 3rW_{n+2}^2 W_{n+1} + 3r^2 W_{n+1} W_{n+2} - r^3 W_{n+1} \]

and so

\[ s^3 W_n = W_{n+2}^3 - 3rW_{n+2}^2 W_{n+1} + 3r^2 W_{n+1} W_{n+2} - r^3 W_{n+1} \]

\[ s^3 W_{n-1} = W_{n+1}^3 - 3rW_{n+1} W_n + 3r^2 W_n^2 W_{n+1} - r^3 W_n^3 \]

\[ s^3 W_{n-2} = W_n^3 - 3rW_n^2 W_{n-1} + 3r^2 W_{n-1}^2 W_n - r^3 W_{n-1}^3 \]

\[ \vdots \]

\[ s^3 W_3 = W_4^3 - 3rW_4^2 W_3 + 3r^2 W_3^2 W_4 - r^3 W_3^3 \]

\[ s^3 W_2 = W_3^3 - 3rW_3^2 W_2 + 3r^2 W_2^2 W_3 - r^3 W_2^3 \]

\[ s^3 W_1 = W_2^3 - 3rW_2^2 W_1 + 3r^2 W_1^2 W_2 - r^3 W_1^3 \]

\[ s^3 W_0 = W_1^3 - 3rW_1^2 W_0 + 3r^2 W_0^2 W_1 - r^3 W_0^3 \]

If we add the above equations by side by, we get

\[ s^3 \sum_{k=0}^{n} W_k^3 = (W_{n+2}^3 + W_{n+1}^3 - W_0^3 - \sum_{k=0}^{n} W_k^3) - 3r(W_{n+2}^2 W_{n+1} - W_1^3 W_0 + \sum_{k=0}^{n} W_{n+1} W_k) \tag{2.1} \]

\[ + 3r^2(W_{n+1} W_{n+2} - W_0^2 W_1 + \sum_{k=0}^{n} W_k^2 W_{k+1}) - r^3(W_{n+1} - W_0^3 + \sum_{k=0}^{n} W_k^2). \]
Next we calculate \( \sum_{k=0}^{n} W^2_{k+1} W_k \). Again, using the recurrence relation

\[
W_{n+2} = r W_{n+1} + s W_n
\]

i.e.

\[
s W_n = W_{n+2} - r W_{n+1}
\]

we obtain

\[
s W^2_{n+1} W_n = W^2_{n+1} W_{n+2} - r W^3_{n+1}
\]

and so

\[
s W^2_{n+1} W_n = W^2_{n+1} W_{n+2} - r W^3_{n+1}
\]

\[
s W^2_{n+1} W_n = W^2_{n+1} W_{n+2} - r W^3_{n+1}
\]

\[
s W^2_{n+1} W_n = W^2_{n+1} W_{n+2} - r W^3_{n+1}
\]

\[
\vdots
\]

\[
s W^2_{k} W_2 = W^2_{k} W_4 - r W^3_{3}
\]

\[
s W^2_{k} W_1 = W^2_{k} W_3 - r W^2_{3}
\]

\[
s W^2_{k} W_0 = W^2_{k} W_2 - r W^2_{3}
\]

If we add the above equations by side by, we get

\[
s \sum_{k=0}^{n} W^2_{k+1} W_k = (W^2_{n+1} W_{n+2} - W^2_{0} W_1 + \sum_{k=0}^{n} W^2_{k+1} W_k) - r(W^3_{n+1} - W^3_{0} + \sum_{k=0}^{n} W^3_{k}). \quad (2.2)
\]

Next we calculate \( \sum_{k=0}^{n} W^2_{k} W_{k+1} \). Again, using the recurrence relation

\[
W_{n+2} = r W_{n+1} + s W_n
\]

i.e.

\[
s W_n = W_{n+2} - r W_{n+1} \Rightarrow s^2 W^2_n = W^2_{n+2} + r^2 W^2_{n+1} - 2r W_{n+2} W_{n+1}
\]

we obtain

\[
s^2 W^2_{n+1} W_{n+1} = W^2_{n+2} W_{n+1} + r^2 W^3_{n+1} - 2r W^2_{n+2} W_{n+1}
\]

and so

\[
s^2 W^2_{n+1} W_{n+1} = W^2_{n+2} W_{n+1} + r^2 W^3_{n+1} - 2r W^2_{n+2} W_{n+1}
\]

\[
s^2 W^2_{n+1} W_{n+1} = W^2_{n+2} W_{n+1} + r^2 W^3_{n+1} - 2r W^2_{n+2} W_{n+1}
\]

\[
\vdots
\]

\[
s^2 W^2_{3} W_3 = W^2_{3} W_3 + r^2 W^2_{3} - 2r W^2_{3} W_4
\]

\[
s^2 W^2_{2} W_2 = W^2_{2} W_2 + r^2 W^2_{2} - 2r W^2_{2} W_3
\]

\[
s^2 W^2_{1} W_1 = W^2_{1} W_1 + r^2 W^2_{1} - 2r W^2_{1} W_2
\]

If we add the above equations by side by, we get

\[
s^2 \sum_{k=0}^{n} W^2_{k} W_{k+1} = (W^2_{n+2} W_{n+1} - W^2_{0} W_0 + \sum_{k=0}^{n} W^2_{k+1} W_k) + r^2 (W^3_{n+1} - W^3_{0} + \sum_{k=0}^{n} W^3_{k}). \quad (2.3)
\]

\[
-2r(W^2_{n+1} W_{n+2} - W^2_{0} W_1 + \sum_{k=0}^{n} W^2_{k} W_{k+1})
\]

Then, solving the system (2.1)-(2.2)-(2.3), the required results of (a),(b) and (c) follow.
Taking $r = s = 1$ in Theorem 2.1 (a) and (b), we obtain the following proposition.

**Proposition 2.1.** If $r = s = 1$ then for $n \geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n} W_k^3 = \frac{1}{2}(-W_{n+2}^3 - 3W_{n+1}^3 + 3W_{n+2}^2 W_{n+1} + W_1^3 + 3W_0^3 - 3W_0^2 W_0)$.

(b) $\sum_{k=0}^{n} W_k^2 W_{k+1} = \frac{1}{2}(W_{n+2}^2 W_{n+1} - W_{n+1}^2 W_{n+2} - W_1^2 W_0 + W_0^2 W_1)$.

(c) $\sum_{k=0}^{n} W_k^2 W_{k+1} = \frac{1}{2}(W_{n+2}^3 + W_{n+1}^3 - 2W_{n+2}^2 W_{n+1} + W_{n+1}^2 W_{n+2} - W_1^3 - W_0^3 + 2W_1^2 W_0 - W_0^2 W_1)$.

From the above proposition, we have the following corollary which gives sum formulas of Fibonacci numbers (take $W_n = F_n$ with $F_0 = 0$, $F_1 = 1$).

**Corollary 2.2.** For $n \geq 0$, Fibonacci numbers have the following properties:

(a) $\sum_{k=0}^{n} F_k^3 = \frac{1}{2}(-F_{n+2}^3 - 3F_{n+1}^3 + 3F_{n+2}^2 F_{n+1} + 1)$.

(b) $\sum_{k=0}^{n} F_k^2 F_{k+1} = \frac{1}{2}(F_{n+2}^2 F_{n+1} - F_{n+1}^2 F_{n+2})$.

(c) $\sum_{k=0}^{n} F_k^2 F_{k+1} = \frac{1}{2}(F_{n+2}^3 + F_{n+1}^3 - 2F_{n+2}^2 F_{n+1} + F_{n+1}^2 F_{n+2} - 1)$.

Taking $W_n = L_n$ with $L_0 = 2$, $L_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Lucas numbers.

**Corollary 2.3.** For $n \geq 0$, Lucas numbers have the following properties:

(a) $\sum_{k=0}^{n} L_k^3 = \frac{1}{2}(-L_{n+2}^3 - 3L_{n+1}^3 + 3L_{n+2}^2 L_{n+1} + 19)$.

(b) $\sum_{k=0}^{n} L_k^2 L_{k+1} = \frac{1}{2}(L_{n+2}^2 L_{n+1} - L_{n+1}^2 L_{n+2} + 2)$.

(c) $\sum_{k=0}^{n} L_k^2 L_{k+1} = \frac{1}{2}(L_{n+2}^3 + L_{n+1}^3 - 2L_{n+2}^2 L_{n+1} + L_{n+1}^2 L_{n+2} - 9)$.

Taking $r = 2$, $s = 1$ in Theorem 2.1 (a) and (b), we obtain the following proposition.

**Proposition 2.2.** If $r = 2$, $s = 1$ then for $n \geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n} W_k^3 = \frac{1}{4}(-2W_{n+2}^3 - 16W_{n+1}^3 + 9W_{n+2}^2 W_{n+1} - 3W_{n+1}^2 W_{n+2} + 2W_1^3 + 16W_0^3 - 9W_0^2 W_0 + 8W_0^2 W_1)$.

(b) $\sum_{k=0}^{n} W_k^2 W_{k+1} = \frac{1}{12}(-W_{n+2}^3 - W_{n+1}^3 + 8W_{n+2}^2 W_{n+1} - 12W_{n+1}^2 W_{n+2} + W_1^3 + W_0^3 - 8W_0^2 W_0 + 12W_0^2 W_1)$.

(c) $\sum_{k=0}^{n} W_k^2 W_{k+1} = \frac{3}{8}(3W_{n+2}^3 - 10W_{n+2}^2 W_{n+1} + 3W_{n+1}^3 + 8W_{n+1}^2 W_{n+2} - 3W_1^3 - 3W_0^3 + 10W_0^2 W_0 - W_0^2 W_1)$.

From the last proposition, we have the following corollary which gives sum formulas of Pell numbers (take $W_n = P_n$ with $P_0 = 0$, $P_1 = 1$).

**Corollary 2.4.** For $n \geq 0$, Pell numbers have the following properties:

(a) $\sum_{k=0}^{n} P_k^3 = \frac{1}{4}(-2P_{n+2}^3 - 16P_{n+1}^3 + 9P_{n+2}^2 P_{n+1} - 3P_{n+1}^2 P_{n+2} + 2)$.

(b) $\sum_{k=0}^{n} P_k^2 P_{k+1} = \frac{1}{12}(-P_{n+2}^3 + P_0^3 + 8P_{n+2}^2 P_{n+1} - 12P_{n+1}^2 P_{n+2} + 1)$.

(c) $\sum_{k=0}^{n} P_k^2 P_{k+1} = \frac{1}{12}(3P_{n+2}^3 - 10P_{n+2}^2 P_{n+1} + 3P_{n+1}^2 P_{n+2} - 3)$.

Taking $W_n = Q_n$ with $Q_0 = 2$, $Q_1 = 2$ in the last proposition, we have the following corollary which presents sum formulas of Pell-Lucas numbers.

**Corollary 2.5.** For $n \geq 0$, Pell-Lucas numbers have the following properties:

(a) $\sum_{k=0}^{n} Q_k^3 = \frac{1}{4}(-2Q_{n+2}^3 - 16Q_{n+1}^3 + 9Q_{n+2}^2 Q_{n+1} - 3Q_{n+1}^2 Q_{n+2} + 96)$.

(b) $\sum_{k=0}^{n} Q_k^2 Q_{k+1} = \frac{1}{12}(-Q_{n+2}^3 - Q_0^3 + 8Q_{n+2}^2 Q_{n+1} - 12Q_{n+1}^2 Q_{n+2} + 48)$. 
For $2.1$

**Theorem 2.1.** negative subscripts.

**Corollary 2.7.** Taking

**Proposition 2.3.** If $r = 1, s = 2$ then for $n \geq 0$ we have the following formulas:

**Corollary 2.6.** For $n \geq 0$, Jacobsthal numbers have the following properties:

**Corollary 2.7.** For $n \geq 0$, Jacobsthal-Lucas numbers have the following properties:

**3 SUM FORMULAS OF GENERALIZED FIBONACCI NUMBERS WITH NEGATIVE SUBSCRIPTS**

The following theorem presents some summing formulas of generalized Fibonacci numbers with negative subscripts.

**Theorem 3.1.** For $n \geq 1$ we have the following formulas: If $(rs-s^3+1)(r+s-1)(r+s-rs+r^2+s^2+1) \neq 0$ then

\[ \sum_{k=1}^{n} W_{rs}^3 k = \frac{\Delta_4}{(rs-s^3+1)(r+s-1)(r+s-rs+r^2+s^2+1)} \]

where

\[ \Delta_4 = (2rs+s^3-1)W_{rs}^3 n+1 + (rs+s^3+s^2+r^2-r^3+1)W_{rs}^3 n+2 + \frac{3s^2}{(rs-1)}W_{rs}^3 n+1W_{rs}^3 n+2 - \frac{3s^2}{(rs-1)}W_{rs}^3 n+2W_{rs}^3 n \]

\[ + (2rs+s^3-1)W_{rs}^3 n+1W_{rs}^3 n - (2rs+s^3-1)W_{rs}^3 n - 3rs(r+s^2)W_{rs}^3 n+1W_{rs}^3 n+2 + 3rs(r+s^2)W_{rs}^3 n+2W_{rs}^3 n+3 + 3rs(r+s^2)W_{rs}^3 n+3W_{rs}^3 n+4 - 3rs(r+s^2)W_{rs}^3 n+4W_{rs}^3 n+5 - 3rs(r+s^2)W_{rs}^3 n+5W_{rs}^3 n+6 \]

\[ - (r^3+s^3+s^2+r^2-r^3+1)W_{rs}^3 n+1W_{rs}^3 n+2 + (r^3+s^3+s^2+r^2-r^3+1)W_{rs}^3 n+2W_{rs}^3 n+3 + (r^3+s^3+s^2+r^2-r^3+1)W_{rs}^3 n+3W_{rs}^3 n+4 - (r^3+s^3+s^2+r^2-r^3+1)W_{rs}^3 n+4W_{rs}^3 n+5 - (r^3+s^3+s^2+r^2-r^3+1)W_{rs}^3 n+5W_{rs}^3 n+6 . \]
(b) \[ \sum_{k=1}^{n} W_{2k+1} W_{-k} = \frac{\Delta_5}{(r^2 - s^2 + 1)(r^2 + s^2 + 1)} \]

where

\[ \Delta_5 = -r(r^2 + s^2)W_{-n+1}^3 - r(2r^3 - s^3 + 1)W_{-n}^3W_{-n+1} + r(r^2 + s^2)W_{-n+1}^3 + s^3(r + s^2)W_0^3 \]

\[ -3r^2 s^2 + r^3 + s^3 - 1)W_1^3 W_0 + s^2 (2r^3 - s^3 + 1)W_2^3 W_1. \]

(c) \[ \sum_{k=1}^{n} W_{-k}^3 W_{k+1} = \frac{\Delta_6}{(r^2 - s^2 + 1)(r^2 + s^2 + 1)} \]

where

\[ \Delta_6 = r(r^2 + s^2)W_{-n+1}^3 + r(2r^3 - s^3 + 1)W_{-n}^3 - s^3(r + s^2)W_0^3 \]

\[ + (2r^3 + r^3 s + 2rs + r^3 + s^3 - 1)W_{-n+1}^3W_{-n+1} - r(r^2 + s^2)W_1^3 \]

\[ - r^3 (r - 1) W_0^3 + s^3 (2r^3 - s^3 + 1)W_1^3 W_0 - (2r^3 + r^3 s + 2rs + r^3 + s^3 - 1)W_2^3 W_1. \]

Proof. Using the recurrence relation

\[ W_{-n+2} = r W_{-n+1} + s W_{-n} \Rightarrow W_{-n} = \frac{-r}{s} W_{-n+1} + \frac{1}{s} W_{-n+2} \]

i.e.

\[ s W_{-n} = W_{-n+2} - r W_{-n+1} \]

we obtain

\[ s^3 W_{-n} = W_{-n+2}^3 - 3r W_{-n+2} W_{-n+1} + 3r^2 W_{-n+2}^2 W_{-n+1} - r^3 W_{-n+1}^3 \]

and so

\[ s^3 W_{-n}^3 = W_{-n+3}^3 - 3r W_{-n+3} W_{-n+2} + 3r^2 W_{-n+3}^2 W_{-n+2} - r^3 W_{-n+2}^3 \]

\[ s^3 W_{-n+1}^3 = W_{-n+4}^3 - 3r W_{-n+4} W_{-n+3} + 3r^2 W_{-n+4}^2 W_{-n+3} - r^3 W_{-n+3}^3 \]

\[ \vdots \]

\[ s^3 W_{-2}^3 = W_{-3}^3 - 3r W_{-3} W_{-2} + 3r^2 W_{-2}^2 W_{-1} - r^3 W_{-1} \]

\[ s^3 W_{-1}^3 = W_{-4}^3 - 3r W_{-4} W_{-3} + 3r^2 W_{-3}^2 W_{-2} - r^3 W_{-2}^3 \]

If we add the above equations by side by, we get

\[ s^3 \left( \sum_{k=1}^{n} W_{-k}^3 \right) = \left( -W_{-n+1}^3 - W_{-n}^3 + W_{-n+2}^3 + W_0^3 + \sum_{k=1}^{n} W_{-k}^3 \right) \]

\[ = \left( -3r(1-W_{-n+1} W_{-n} + W_{-n+2}) + \sum_{k=1}^{n} W_{-k}^3 \right) \]

\[ + r^3 (1-W_{-n+1} W_{-n} + W_{-n+2}) + \sum_{k=1}^{n} W_{-k}^3 \]

\[ = -r^3 (W_{-n} + W_0^3 + \sum_{k=1}^{n} W_{-k}^3). \]
Next we calculate $\sum_{k=1}^{n} W_{-k+1}^2 W_{-k}$. Again using the recurrence relation

$$W_{-n+2} = r W_{-n+1} + s W_{-n} \Rightarrow W_{-n} = -\frac{r}{s} W_{-n+1} + \frac{1}{s} W_{-n+2}$$
i.e.

$$s W_{-n} = W_{-n+2} - r W_{-n+1}$$
we obtain

$$s W_{-n+1}^2 W_{-n} = W_{-n+1}^2 W_{-n+2} - r W_{-n+1}^3$$

and so

$$s W_{n+1}^2 W_{-n} = W_{n+1}^2 W_{n+2} - r W_{n+1}^3$$
$$s W_{n+2}^2 W_{-n+1} = W_{n+2}^2 W_{n+3} - r W_{n+2}^3$$
$$s W_{n+3}^2 W_{-n+2} = W_{n+3}^2 W_{n+4} - r W_{n+3}^3$$

$$\vdots$$
$$s W_{2}^2 W_{n-3} = W_{2}^2 W_{n-4} - r W_{2}^3$$
$$s W_{2}^2 W_{n-2} = W_{2}^2 W_{n-3} - r W_{2}^3$$
$$s W_{0}^2 W_{n-1} = W_{0}^2 W_{n-2} - r W_{0}^3$$

If we add the above equations by side by, we get

$$s \sum_{k=1}^{n} W_{-k+1}^2 W_{-k} = (-W_{-n}^2 W_{n+1} + W_{0}^2 W_{1} + \sum_{k=1}^{n} W_{2}^2 W_{-k+1}) - r(-W_{-n}^3 + W_{0}^3 + \sum_{k=1}^{n} W_{3}^3). \tag{3.2}$$

Next we calculate $\sum_{k=1}^{n} W_{-k+1}^2 W_{-k}$. Again using the recurrence relation

$$W_{-n+2} = r W_{-n+1} + s W_{-n}$$
i.e.

$$s W_{-n} = W_{-n+2} - r W_{-n+1}$$
we obtain

$$s^2 W_{-n}^2 = W_{n+1}^2 - 2r W_{n+2} W_{-n+1} + r^2 W_{n+1}^2 \Rightarrow s^2 W_{-n}^2 W_{n+1} = W_{n+2}^2 W_{-n+1} - 2r W_{n+1}^2 W_{-n+2} + r^2 W_{n+1}^3$$

and so

$$s^2 W_{n+1}^2 W_{n+1} = W_{n+2}^2 W_{n+1} - 2r W_{n+1}^2 W_{n+2} + r^2 W_{n+1}^3$$
$$s^2 W_{n+2}^2 W_{n+2} = W_{n+3}^2 W_{n+2} - 2r W_{n+2}^2 W_{n+3} + r^2 W_{n+2}^3$$
$$s^2 W_{n+3}^2 W_{n+3} = W_{n+4}^2 W_{n+3} - 2r W_{n+3}^2 W_{n+4} + r^2 W_{n+3}^3$$

$$\vdots$$
$$s^2 W_{2}^2 W_{n-2} = W_{2}^2 W_{n-2} - 2r W_{2}^2 W_{n-3} + r^2 W_{2}^3$$
$$s^2 W_{2}^2 W_{n-1} = W_{2}^2 W_{n-1} - 2r W_{2}^2 W_{0} + r^2 W_{2}^3$$
$$s^2 W_{0}^2 W_{0} = W_{0}^2 W_{0} - 2r W_{0}^2 W_{1} + r^2 W_{0}^3$$
If we add the above equations by side by, we get

\[ s^2 \sum_{k=1}^{n} W_k^2 W_{-k+1} = (-W_{-n+1} W_n + W_0^2 W_0 + \sum_{k=1}^{n} W_k^2 W_{-k}) \] (3.3)

\[-2r(-W_{-n} W_{-n+1} + W_0^2 W_1 + \sum_{k=1}^{n} W_k^2 W_{-k+1}) + r^2(-W_{-n} + W_0^3 + \sum_{k=1}^{n} W_k^3).\]

Then, solving the system (3.1)-(3.2)-(3.3), the required results of (a),(b) and (c) follow.

Taking \( r = s = 1 \) in Theorem 3.1 (a) and (b), we obtain the following proposition.

**Proposition 3.1.** If \( r = s = 1 \) then for \( n \geq 1 \) we have the following formulas:

(a) \( \sum_{k=1}^{n} W_k^3 = \frac{1}{2}(W_{-n+1}^3 + 3W_n^3 - 3W_{-n+1}^2 W_n - W_0^3 - 3W_0 + 3W_0^2 W_0). \)

(b) \( \sum_{k=1}^{n} W_k^2 W_{-k+1} = \frac{1}{2}(-W_{-n}^3 + 3W_n^3 - 2W_{-n+1}^2 W_n + W_0^3 + W_0^2 - 2W_0^2 W_0 + W_0^2 W_1). \)

(c) \( \sum_{k=1}^{n} W_k^2 W_{-k} = \frac{1}{2}(-W_{-n+1} W_n + W_0^2 W_0 + 2W_0^2 W_0 - W_0^2 W_1). \)

From the above proposition, we have the following corollary which gives sum formulas of Fibonacci numbers (take \( W_n = F_n \) with \( F_0 = 0, F_1 = 1 \)).

**Corollary 3.2.** For \( n \geq 1 \), Fibonacci numbers have the following properties.

(a) \( \sum_{k=1}^{n} F_k^3 = \frac{1}{2}(F_{-n+1}^3 + 3F_n^3 - 3F_{-n+1}^2 F_n - 1). \)

(b) \( \sum_{k=1}^{n} F_k^2 F_{-k+1} = \frac{1}{2}(-F_{-n}^3 + 3F_{-n+1}^2 F_{-n} - F_{-n+1}^3 - 1). \)

(c) \( \sum_{k=1}^{n} F_k^3 F_{-k+1} = \frac{1}{2}(-F_{-n+1} F_{-n} + 3F_n^2 F_{-n+1} + F_{-n}^3). \)

Taking \( W_n = L_n \) with \( L_0 = 2, L_1 = 1 \) in the last proposition, we have the following corollary which presents sum formulas of Lucas numbers.

**Corollary 3.3.** For \( n \geq 1 \), Lucas numbers have the following properties.

(a) \( \sum_{k=1}^{n} L_k^3 = \frac{1}{2}(L_{-n+1}^3 + 3L_n^3 - 3L_{-n+1}^2 L_n - 19). \)

(b) \( \sum_{k=1}^{n} L_k^2 L_{-k+1} = \frac{1}{2}(-L_{-n}^3 + L_{-n+1}^3 + 2L_{-n+1}^2 L_n - L_n^2 L_{-n+1} + 9). \)

(c) \( \sum_{k=1}^{n} L_k^3 L_{-k+1} = \frac{1}{2}(-L_{-n+1}^3 L_n + L_n^2 L_{-n+1} + 2). \)

Taking \( r = 2, s = 1 \) in Theorem 3.1 (a) and (b), we obtain the following proposition.

**Proposition 3.2.** If \( r = 2, s = 1 \) then for \( n \geq 1 \) we have the following formulas:

(a) \( \sum_{k=1}^{n} W_k^3 = \frac{1}{2}(2W_{-n+1}^3 + 16W_n^3 - 9W_{-n+1}^2 W_n + 3W_n^2 W_{-n+1} - 2W_0^3 - 16W_0^2 + 9W_0^2 W_0 - 10W_0^2 W_1 + 8W_0^2 W_1). \)

(b) \( \sum_{k=1}^{n} W_k^2 W_{-k+1} = \frac{1}{2}(-3W_{-n+1}^3 + 10W_n^3 - 8W_{-n+1}^2 W_n + 3W_n^2 W_{-n+1} + 3W_0^3 - 10W_0^2 W_0 + 8W_0^2 W_1). \)

(c) \( \sum_{k=1}^{n} W_k^2 W_{-k} = \frac{1}{2}(W_{-n+1}^3 + W_n^3 + 12W_{-n+1}^2 W_n - W_n^2 W_{-n+1} + W_0^3 - 10W_0^2 + 12W_0^2 W_1). \)

From the last proposition, we have the following corollary which gives sum formulas of Pell numbers (take \( W_n = P_n \) with \( P_0 = 0, P_1 = 1 \)).
Corollary 3.4. For \( n \geq 1 \), Pell numbers have the following properties.

(a) \( \sum_{k=1}^{n} P_{2k} = \frac{1}{14} (2P_{3n+1} + 16P_{n} - 9P_{n+1}^2 + 3P_{2n}^2 - P_{n-1} + 2) \).

(b) \( \sum_{k=1}^{n} P_{2k+1} = \frac{1}{14} (-3P_{3n+1} + 3P_{n} + 10P_{n+1}^2 + P_{n-1} - 8P_{2n}^2 + 3P_{n+1} - 3) \).

(c) \( \sum_{k=1}^{n} P_{2k}P_{n+k} = \frac{1}{14} (P_{n+1}^2 + P_{n} - 8P_{n-1}^2 + 12P_{n+1} - P_{2n+1} - 1) \).

Taking \( W_n = Q_n \) with \( Q_0 = 2, Q_1 = 2 \) in the last proposition, we have the following corollary which presents sum formulas of Pell-Lucas numbers.

Corollary 3.5. For \( n \geq 1 \), Pell-Lucas numbers have the following properties.

(a) \( \sum_{k=1}^{n} Q_{2k} = \frac{1}{14} (2Q_{3n+1} + 16Q_{n} - 9Q_{n+1}^2 + 3Q_{2n}^2 - Q_{n-1} + 96) \).

(b) \( \sum_{k=1}^{n} Q_{2k+1}Q_{n+k} = \frac{1}{14} (-3Q_{3n+1} + 3Q_{n} + 10Q_{n+1}^2 + Q_{n-1} - 8Q_{2n}^2 + 3Q_{n+1} - 32) \).

(c) \( \sum_{k=1}^{n} Q_{2k}Q_{n+k} = \frac{1}{14} (Q_{n+1}^2 + Q_{n} - 8Q_{n-1}^2 + 12Q_{n+1}^2 - Q_{2n+1} - 48) \).

Taking \( r = 1, s = 2 \) in Theorem 3.1 (a), (b) and (c), we obtain the following proposition.

Proposition 3.3. If \( r = 1, s = 2 \) then for \( n \geq 1 \) we have the following formulas:

(a) \( \sum_{k=1}^{n} W_{2k} = \frac{1}{14} (-11W_{3n+1}^2 + 30W_{2n+1}^2 + 18W_{n+1}^2 + 11W_1^2 + 18W_0^2 - 30W_2^2 + W_0^2) \).

(b) \( \sum_{k=1}^{n} W_{2k+1}W_{n+k} = \frac{1}{14} (5W_{3n+1}^2 - 20W_{2n+1}^2 + 20W_{n+1}^2 - W_1^2 + 48W_0^2 + 20W_2^2 + 20W_0^2 W_1) \).

(c) \( \sum_{k=1}^{n} W_{2k}W_{n+k} = \frac{1}{14} (-8W_{3n+1}^2 + 10W_{2n+1}^2 + 18W_{n+1}^2 + 8W_1^2 + 10W_2^2 W_0 - 18W_0^2 W_1) \).

From the last proposition, we have the following corollary which gives sum formula of Jacobsthal numbers (take \( W_n = J_n \) with \( J_0 = 0, J_1 = 1 \)).

Corollary 3.6. For \( n \geq 1 \), Jacobsthal numbers have the following properties:

(a) \( \sum_{k=1}^{n} J_{2k} = \frac{1}{14} (-11J_{3n+1} + 30J_{2n+1} - 12J_{n+1} + J_n + 11) \).

(b) \( \sum_{k=1}^{n} J_{2k+1}J_{n+k} = \frac{1}{14} (5J_{3n+1} + 20J_{2n+1} - 20J_{n+1} - J_n - J_{n+1} - 5) \).

(c) \( \sum_{k=1}^{n} J_{2k}J_{n+k} = \frac{1}{14} (-8J_{3n+1} - 10J_{2n+1} + 18J_{n+1} + J_n + 11) \).

Taking \( W_n = J_n \) with \( j_0 = 2, j_1 = 1 \) in the last proposition, we have the following corollary which presents sum formulas of Jacobsthal-Lucas numbers.

Corollary 3.7. For \( n \geq 1 \), Jacobsthal-Lucas numbers have the following properties:

(a) \( \sum_{k=1}^{n} J_{2k}^2 = \frac{1}{14} (-11J_{3n+1} + 30J_{2n+1} - 12J_{n+1} + 143) \).

(b) \( \sum_{k=1}^{n} J_{2k+1}J_{n+k} = \frac{1}{14} (5J_{3n+1} + 20J_{2n+1} - 20J_{n+1} - 20J_{n+1} - 205) \).

(c) \( \sum_{k=1}^{n} J_{2k}J_{n+k} = \frac{1}{14} (-8J_{3n+1} - 10J_{2n+1} + 18J_{n+1} + 13) \).

4 CONCLUSION

Recently, there have been so many studies of the sequences of numbers in the literature and the sequences of numbers were widely used in many research areas, such as architecture, nature, art, physics and engineering. In this work, sum identities were proved. The method used in this paper can be used for the other linear recurrence sequences, too. We have written sum identities in terms of the generalized Fibonacci sequence, and then we have presented the formulas as special cases the corresponding identity for the Fibonacci, Lucas, Pell, Pell-Lucas,
Jacobsthal, Jacobsthal-Lucas numbers. All the listed identities in the corollaries may be proved by induction, but that method of proof gives no clue about their discovery. We give the proofs to indicate how these identities, in general, were discovered.

We can summarize the sections as follows:

- In section 1, we present some background about generalized Fibonacci numbers.
- In section 2, summation formulas have been presented for the generalized Fibonacci numbers with positive subscripts. As special cases, summation formulas of Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas numbers with positive subscripts have been given.
- In section 3, summation formulas have been presented for the generalized Fibonacci numbers with negative subscripts. As special cases, summation formulas of Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas numbers with negative subscripts have been given.

COMPETING INTERESTS
Author has declared that no competing interests exist.

REFERENCES
[1] Horadam AF. Basic properties of a certain generalized sequence of numbers. Fibonacci Quarterly. 1965;3(3):161-176.
[2] Horadam AF. A generalized fibonacci sequence. American Mathematical Monthly. 1961;68:455-459.
[3] Horadam AF. Special properties of the sequence $w_n(a, b, p, q)$. Fibonacci Quarterly. 1967;5(5):424-434.
[4] Horadam AF. Generating Functions for Powers of a Certain Generalized Sequence of Numbers. Duke Math. J. 1965;32:437-446.
[5] Sloane NJA. The on-line encyclopedia of integer sequences. Available: http://oeis.org/
[6] Čerin Z. Formulae for sums of Jacobsthal-Lucas numbers. Int. Math. Forum. 2007;2(40):1969-1984.
[7] Čerin Z. Sums of squares and products of jacobsthal numbers. Journal of Integer Sequences. 2007;10. Article 07.2.5, 2007
[8] Gnam A, Anitha B. Sums of squares jacobsthal numbers. IOSR Journal of Mathematics. 2015;11(6):62-64.
[9] Kiliç E, Taşçı D. The linear algebra of the pell matrix. Boletín de la Sociedad Matemática Mexicana. 2005;3(11).
[10] Kılıç E. Sums of the squares of terms of sequence $\{w_n\}$. Proc. Indian Acad. Sci. (Math. Sci.). 2008;118(1):27-41.
[11] Soykan Y. Closed formulas for the sums of squares of generalized fibonacci numbers. Asian Journal of Advanced Research and Reports. 2020;9(1):23-39. Available:https://doi.org/10.9734/ajarr/2020/v9i130212
[12] Frontczak R. Sums of cubes over odd-index fibonacci numbers. Integers. 2018;18.
[13] Wamiliana, Suharsono, Kristanto PE. Counting the sum of cubes for lucas and gibbonacci numbers. Science and Technology Indonesia. 2019;4(2):31-35.
[14] Chen L, Wang X. The power sums involving fibonacci polynomials and their applications. Symmetry. 2019;11. DOI:10.3390/sym11050635
[15] Frontczak R. Sums of powers of Fibonacci and Lucas numbers: A new bottom-up Approach. Notes on Number Theory and Discrete Mathematics. 2018;24(2):94-103.
[16] Prodinger H. Sums of powers of fibonacci polynomials. Proc. Indian Acad. Sci. (Math. Sci.). 2009;119(5):567-570.
[17] Raza Z, Riaz M, Ali MA. Some inequalities on the norms of special matrices with generalized tribonacci and generalized pell-padovan sequences. arXiv; 2015. Available:http://arxiv.org/abs/1407.1369v2
[18] Prodinger H, Selkirk SJ. Sums of squares of tetranacci numbers: A generating function approach; 2019. Available: http://arxiv.org/abs/1906.08336v1

[19] Schumacher R. How to sum the squares of the tetranacci numbers and the fibonacci m-step numbers. Fibonacci Quarterly. 2019;57:168-175.