Quantum Computers and Unstructured Search:
Finding and Counting Items with an Arbitrarily Entangled Initial State

A. Carlini and A. Hosoya

Department of Physics, Tokyo Institute of Technology, Oh-Okayama, Meguro-ku, Tokyo 152, Japan
e-mail: carlini@th.phys.titech.ac.jp ; ahosoya@th.phys.titech.ac.jp

Abstract

Grover’s quantum algorithm for an unstructured search problem and the Count algorithm by Brassard et al. are generalized to the case when the initial state is arbitrarily and maximally entangled. This ansatz might be relevant with quantum subroutines, when the computational qubits and the environment are coupled, and in general when the control over the quantum system is partial.

1 Introduction

In the recent years quite a significant progress has been made in the theory of quantum computation [1-3], both at the theoretical and experimental level [4-5]. In the quest for quantum algorithms, in particular, after the discovery by Shor [6] of an algorithm for factoring integers (which achieves an exponential speed up compared to the best classical algorithm currently known), one of the main successes has been Grover’s algorithm for the unstructured database search [7]. Grover considered the problem to find a ‘good’ file, represented as the state \( |g> \), out of \( N \) files \( |a> \); \( a = 0 \ldots N - 1 \). The algorithm starts with the preparation of a flat superposition of all states \( |a> \), i.e.

\[
|\psi_0> \equiv W|0> \equiv \frac{1}{\sqrt{N}} \sum_{a=0}^{N-1} |a>,
\]  

(1)

1 Assuming, without loss of generality, that \( N \) is a power of two.
where $W$ is the Walsh-Hadamard transform, and assumes that there is an oracle which evaluates the function $H(a)$, s.t. $H(g) = 1$ for the 'good' state $|g>$, and $H(b) = 0$ for the 'bad' states $|b>$ (i.e., the remaining states in the set of all the a's). The unitary transformation for the 'search' of $|g>$ is usually defined [7-8] in terms of the operator $G_H = -WS_0WS_H$, with the inversion operators $S_0 \equiv I - 2|0><0|$ and $S_H \equiv I - 2\sum_g |g><g|$.

Iterating $G_H$ for $n \approx O[\sqrt{N}]$ times on (1) then produces a state whose amplitude is peaked around the searched item $|g>$. Classically, it would take of the order of $O[N]$ steps on the average to find the same element $g$, so that Grover’s quantum method achieves a square root speed up compared to its classical analogue. Subsequently, Grover’s algorithm has been extended to the case when there are $t$ 'good' items $|g>$ to be searched or when the number of 'good' items is not known in advance [8-11]. The number of steps required in these cases is of the order of $O[\sqrt{N/t}]$, again a square root improvement with respect to the classical algorithms. Then, Brassard et al. [12] combined Grover’s operator and Shor’s quantum Fourier transform in an algorithm that counts the number of 'good' items present in the flat superposition (1) with an exponential precision and a success probability exponentially close to one. It has then been shown that Grover’s algorithm is optimal [13-14], that a single (complex) oracle query in Grover’s algorithm might suffice in certain cases for finding the 'good' states [15-18], the algorithm has been exploited in NP-structured search problems [19-20], its Hamiltonian formulation [21] and robustness discussed [22]. Although Grover’s algorithm was originally devised assuming that the starting state is to be prepared in the flat superposition form (1), subsequent work [8, 11, 23] showed that this condition can actually be relaxed and that one can also work with the general initial pure state $|\psi_0>$, i.e. replace $W$ by an arbitrary unitary transformation $U$, and then use the operator $G_H = -US_0U^{-1}S_H$. It was also shown [24] that $G_H$ can be interpreted in terms of a rotation of $|\psi_0>$ towards the vector representing the 'good' states, $|w> = \sum_g |g>/\sqrt{t}$, which is a product of two reflections in the two-dimensional space spanned by $|\psi_0>$ and $|w>$. An explicit calculation for the case when the amplitudes of the initial superposition of states are arbitrary and unknown complex numbers was then made by Biham et al. [25], who found that one can still express the optimal measurement time and the maximal probability of success in a closed and exact form which depends only on the averages and the variances of the initial amplitude distribution of states.\(^3\)

\(^2\) In fact $S_H$ can be easily implemented as an $|a>$-'controlled’ unitary transformation $U_H$ by tensoring $|\psi_0>$ with an extra ancilla qubit $|e> = \frac{|0> - |1>}{\sqrt{2}}$, such that $U_H|a> = |e> = |a> + H(a) \mod 2>$, thus obtaining $U_H|\psi_0> = \sum_{a=0}^{N-1} |a> (-1)^{H(a)}|e>$. Furthermore, the ‘inversion about the average’ operator can also be compactly written as $U_{\psi_0} = WS_0W = I - 2|\psi_0><\psi_0|$.\(^3\) Further discussion for the case of an arbitrary (non entangled) initial state and/or an arbitrary unitary transform in Grover’s search can be found in Pati (1998) and
One of the main resources and ingredients of quantum computation lies, however, not only in the possibility of dealing with arbitrary complex superpositions of qubits, but also in the massive exploitation of quantum entanglement (see, e.g., refs. [28-29]). In particular, it is not obvious to which extent one can directly use Grover’s algorithm when the states to be searched are nontrivially coupled with the states of another system over which we cannot have a complete control, i.e. when the initial (normalized) superposition is given by

$$|\psi\rangle \equiv \frac{1}{\sqrt{N}} \sum_{a=0}^{N-1} |a\rangle |f_a\rangle,$$

(2)

where $f_a$ is an arbitrary mapping (not necessarily one to one). In fact, in Grover’s algorithm the application of any unitary transformation acting on the computational states $|a\rangle$ would automatically affect also and nontrivially the $|f_a\rangle$’s, by producing a complicated mixing of the original entanglements. An interesting case may arise, for instance, when the system $|f_a\rangle$ represents the environment. Of course, one might trace out (measure) the ‘environment’ and simply work with the $|a\rangle$ subsystem, which after the measurement of the $|f_a\rangle$ subsystem will collapse into a generic mixed state, i.e. an incoherent superposition of sectors of pure states (each characterized by a different value of $f_a$), and then simply study the dynamics in each of these sectors using the methods described, e.g., in ref. [25]. The problem with this approach is that sectors with different $f_a$’s will generally have different numbers of ‘good’ states, and thus require different computational time to find the ‘good’ states (in some cases failing, for the sectors where there are no ‘good’ states). One can give an upper bound on the success probability of the algorithm, which cannot exceed the expectation value of the operator $I_a \sum_{g,g'} |f_g\rangle\langle f_{g'}|$, however, for generic situations in principle one could do much worse and there is no explicit scheme telling what one should expect. Alternatively, one might consider the setting in which Alice prepares the state (2), where the function $f_a$ is complicated and requires a lengthy computation, and sends it to Bob. Bob then wants to access the information contained in the $|f_g\rangle$’s associated with certain marked states $|g\rangle$, under the promise by Alice that there are $t$ ‘good’ $|g\rangle$’s, but without having an a priori ‘local’ knowledge about the $f_a$’s or how to compute them (except, possibly, the knowledge about certain ‘global’ properties such as the first order momenta of the $|f_a\rangle$’s), and the simplest way to do that is to enhance the amplitude of the $|g\rangle$’s using Grover’s algorithm first and then read the associated $|f_a\rangle$. Finally, in number theory one might want to count (for example, when testing the Prime Number theorem, see ref. [30]) the

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4 We take the normalization conditions $\sum_{a=0}^{N-1} <a|a> \equiv N$, $<a|a> = \delta_{aa'}$, $\sum_{a=0}^{N-1} <f_a|f_a> \equiv N$ and $\sum_g <f_g|f_g> \equiv N_1$, with $N_1 \leq N$ and, in general, $<f_a|f_b> \neq \delta_{ab}$. 

Gingrich et al. (1999).
number of primes \( k \) less than a given integer \( N \), and then need the quantum (entangled) superposition \( \sum_{k=0}^{N-1} |k > | f(k) > \), where \( f \) is a ‘primality flag’ for \( k \) (i.e., \( f = 1 \) for \( k \) prime, and \( f = 0 \) for \( k \) composite), which itself maybe the result of a previous lengthy quantum subroutine and which we want to reuse in further computations.

The aim of the present paper is thus to generalize Grover’s methods to the case when the initial state is given by the entangled superposition (2).

2 Finding Good States

The problem is to find the ‘good’ states \( |g> \), promised to be in number \( t \), from the initial normalized entangled superposition (2). By defining the remaining or ‘bad’ states as \( |b> \), in number \( N-t \), and applying Grover’s unitary transformation \( G_H \equiv -WS_0WS_H \) on the state \( |\psi> \) of eq. (2), it is easy to show (by induction) that the \( n\)-th iteration of \( G_H \) on \( |\psi> \) produces the state

\[
G_H^n|\psi> = \frac{1}{\sqrt{N}} \left\{ \left( \sum_g |g> |f_g> \right) + (-1)^n \left( \sum_b |b> |f_b> \right) \right\} - \frac{2}{N} \left( \sum_g |g> |X^{(n)}> + (\sum_b |b> |Y^{(n)}> \right) \right\}. 
\]

(3)

Adopting a compact matrix notation, i.e. by substituting for \( |X^{(n)}> \rightarrow X_n \) and \( |Y^{(n)}> \rightarrow Y_n \), writing \( \vec{Z}_n \equiv (X_n,Y_n) \) and defining the matrix \( M \equiv \cos 2\theta (I + \sigma_x) + i\sigma_y \) (with the angle \( \sin^2 \theta \equiv t/N \)), the states \( |X^{(n)}> \) and \( |Y^{(n)}> \) can be seen to satisfy the recurrence relations

\[
\vec{Z}_n = M \vec{Z}_{n-1} + \vec{C}_n, 
\]

(4)

which depend only on the number of ‘good’ items via the angle \( \theta \) and the following initial ‘average’ states

\[
|\vec{G}(0)> \equiv \frac{\sum_g |f_g>}{t} ; \quad |\vec{B}(0)> \equiv \frac{\sum_b |f_b>}{N-t}
\]

(5)

via the quantities \( |C^{(n)}> \equiv t|\vec{G}(0)> + (-1)^n (N-t)|\vec{B}(0)> \), with the substitution \( |C^{(n)}> \rightarrow C_n \) and writing \( \vec{C}_n \equiv C_n(1,1) \) (subject to the the initial condition \( X_1 = Y_1 = C_1 \)). Eq. (4) can then be solved using standard techniques and, substituting back in eq. (3) we finally get for the \( n\)-th iteration of \( G_H \) on the entangled state \( |\psi> \)
\[
G_H^n |\psi > = \frac{1}{\sqrt{N}} \left\{ \sum_g |g > \left[ |f_g > - \frac{\sin 2n\theta}{\sin 2\theta} (\tan n\theta \sin 2\theta) |\bar{G}(0) > - 2 \cos^2 \theta |\bar{B}(0) > \right] \right\} + \sum_b |b > \left[ (-1)^n |f_b > - \sin 2n\theta \sin 2\theta (\tan n\theta) |\bar{B}(0) > \right] \right\}
\equiv \frac{1}{\sqrt{N}} \left\{ \sum_g |g > |f_g(n) > + \sum_b |b > |f_b(n) > \right\}.
\]

As we can already note from expression (6), similarly to the case of the original Grover's algorithm acting on an initial flat superposition of states, also in the presence of entanglements \(G^n_H|\psi >\) is periodic in \(n\) with period \(\pi/\theta\), and one can anticipate that a Fourier analysis can still be performed in order to find an estimate of \(\theta\) (see next section).

It is then possible to show, following methods similar to those used in ref. [25] and after some elementary algebra, that the 'variances' of the distribution of the amplitudes of the initial entangled state are constants of the motion, i.e.

\[
\sigma_G^2(n) = \frac{\sum_g \| (|f_g(n) > - |\bar{G}(n) > ) \|^2}{N - t} = \sigma_G^2(0) \equiv \sigma_G^2,
\]

\[
\sigma_B^2(n) = \frac{\sum_b \| (|f_b(n) > - |\bar{B}(n) > ) \|^2}{N - t} = \sigma_B^2(0) \equiv \sigma_B^2,
\]

where \(\|x\|^2 \equiv <x|x>\), and \(|\bar{G}(n) > \equiv [\sum_g |f_g(n) > ]/t\) and \(|\bar{B}(n) > \equiv [\sum_b |f_b(n) > ]/(N - t)\) are the 'averages' at time \(n\).

Using this fact, one can calculate the probability of picking up a 'good' item after \(n\) iterations of \(G_H\) over the initial entangled state \(|\psi >\), defined by \(P(n) \equiv \sum_g <f_g(n)|f_g(n) >\), as

\[
P(n) \equiv P_{AV} - \Delta P \cos 2(2n\theta - \phi_R) e^{-2\phi_I}
\]

\[
P_{AV} \equiv 1 - \Delta P - N\sigma_B^2 \cos^2 \theta
\]

\[
\Delta P \equiv \frac{N}{2} \cos^2 \theta \left[ <B(0)|B(0) > + \tan^2 \theta <G(0)|G(0) > \right],
\]

where the complex angle \(\phi \equiv \phi_R + i\phi_I\) is defined by the formula

\[
\exp[2i\phi] \equiv \frac{2 <F_0(0)|F_0(0) >}{| <F_0(0)|F_0(0) > + | <F_0(0)|F_0(0) > |},
\]

with \(|F_0(0) > \equiv |\bar{B}(0) > \equiv \pm i \tan \theta |\bar{G}(0) >\). As it can be easily seen from eqs. (8) and (9), the probability \(P(n)\) only depends on the first order momenta.
of the distribution of the amplitudes of the initial entangled state, i.e. upon the initial ‘averages’ \( \langle G^{(0)} \rangle \), \( \langle B^{(0)} \rangle \) and the initial ‘variance’ \( \sigma_B^2 \), and on the number of ‘good’ items \( t \). This is similar to the case of a non entangled initial state with arbitrary amplitudes [25]. The probability \( P(n) \) is maximized, \( P_{MAX} = P_{AV} + \Delta P e^{-2\phi t} \), at the times

\[
n_j = \frac{\pi(2j+1)}{2 + \phi_R}/2\theta \quad ; \quad j \in \mathbb{Z}, \tag{10}
\]

and the minimum number of iterations required to achieve the maximal probability of success \( P_{MAX} \) is \( n_0 \). In particular, for the case of \( \theta \approx \sqrt{t/N} \ll 1 \), we have that \( n_0 \approx O(\sqrt{N/t}) \), which is the same as in Grover’s original algorithm. Moreover, one can almost be certain to find a ‘good’ item \( |g> \) only provided that (for \( \theta \ll 1 \)) the initial ‘variance’ of the amplitudes of the ‘bad’ states is small enough, i.e., if \( N\sigma_B^2 \equiv \varepsilon \ll 1 \) then we have that \( P_{MAX} \approx 1 - \varepsilon \approx O(1) \), independently of the values of the initial ‘averages’ \( \langle G^{(0)} \rangle \), \( \langle B^{(0)} \rangle \). A more detailed analysis of the possible outcomes when the ‘averages’ and the ‘variances’ of the amplitudes of the initial entangled state \( |\psi> \) are arbitrary and unknown (e.g., best average number of steps necessary to find the ‘good’ states \( |g> \), etc.) can be done following the same lines as in ref. [25]. It is straightforward to show that the model of an initial state with arbitrary complex amplitudes described in ref. [25] can be recovered as a subcase of our algorithm provided one makes the substitutions \( |f_g> = k_i|0> \) and \( |f_b> = l_i|0> \) (with \( k_i \) and \( l_i \) arbitrary complex phases). This includes, in particular, Grover’s original ansatz [7] for the choice \( k_i = l_i = 1 \).

Finally, it is also interesting to note that, for \( n = n_j \), we get \( \sqrt{N} G_{HI}^{(0)} |\psi> = \sum_g |g> \langle f_g > \rangle (1 + (-1)^j \sin \phi_R) |G^{(0)}> + \langle -1 \rangle^j \cos \phi_R \cot \theta |B^{(0)}> \). Of course, unitarity prevents one to naively get only the exact contribution from the initial ‘unperturbed’ entangled states \( |f_g> \in G_{HI}^{(0)} |\psi> \). However, as some elementary algebra can show, it is still possible, for instance in the case of a large enough number of ‘good’ items \( g \), i.e. for \( t/N \leq O(1) \), to make the amplitude contribution coming from the other entangled states \( |G^{(0)}> \) and

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\[\text{The probability of finding the ‘good’ states does not change with the number of trials} \ n \ \text{(besides the trivial cases of} \ \theta = \pi/2, \ t = N, \ \theta = t = 0 \ \text{or, e.g., when the function} \ f_a \ \text{is one to one, since then the reduced density matrix} \ \rho_a = Tr_f |\psi> <\psi| = \sum_a |a><a|, \ \text{on which Grover’s operator acts, is totally mixed, i.e.} \ \rho_a = I, \ \text{and so} \ \text{invariant under any unitary transformation) when} \ |< F_f^{(0)} |F_f^{(0)} > | = 0, \ \text{since then} \ \text{we get} \ P(n) = P_{AV} = 1 - N\sigma_B^2 \cos^2 \theta = \text{const.} \ \text{The probability of finding the ‘good’ states can be small, i.e.} \ P(n) \approx \delta \ll 1, \ \text{if the initial variance} \ \sigma_B^2 \approx (1 - \delta)/(N - t) \ \text{(} \ \sigma_B^2 \approx N^{-1} \ \text{for} \ t \ll N). \ \text{Finally, the algorithm completely fails only when one of the previous conditions holds and there is a fine tuning} \ N\sigma_B^2 \cos^2 \theta = 1, \ \text{for which} \ P(n) = \text{const} = 0. \]
|\tilde{B}(0)\rangle$ is relatively small compared to that of $|f_g\rangle$, if $j$ is even and provided that $\langle \tilde{G}(0)|\tilde{G}(0)\rangle \simeq \langle \tilde{B}(0)|\tilde{B}(0)\rangle$.

3 Conclusions

We have shown how to generalize one of the most useful algorithms discovered in quantum computing so far, i.e. Grover’s algorithm for the search into an unstructured database, in the most general case when this algorithm is to be applied on a generic initial state in an unknown, arbitrary and maximally entangled superposition of qubits. This situation might arise, for example, when the computational qubits are nontrivially entangled with the environment, when one wants to search or count marked items in subroutines part of larger quantum networks, or more in general when part of the quantum system is not accessible. In particular, we have seen that even in this general case, the dynamics of the quantum entangled system is periodic, with the period depending on the number of ‘good’ items, and fixed by the first order momenta of the amplitudes of the initial state. Furthermore, the search algorithm still generically needs $O(\sqrt{N/t})$ iterations to sort one of the ‘good’ items, and the maximum probability to obtain such ‘good’ items can be made close to one, provided that the initial ‘variance’ of the amplitude distribution of the ‘bad’ states is small enough. In the appendix we also generalized the COUNT algorithm and showed that one can preserve a good success probability and a high accuracy in determining the number of ‘good’ items even if the initial state is entangled, provided that some conditions are satisfied by the ‘averages’ of the amplitude distribution of the initial state (for example, for the choice $\langle \tilde{G}(0)|\tilde{G}(0)\rangle \geq \langle \tilde{B}(0)|\tilde{B}(0)\rangle > \pi^2/(8\sqrt{2})$). These results were not obvious a priori and constitute a non trivial feature for quantum computational states with entanglement.

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A Counting Good States

The algorithm COUNT, introduced by Brassard et al. [12] for the case of an initial flat superposition of states, essentially exploits Grover’s unitary operation $G_H$, already discussed in the previous section, and Shor’s Fourier operation $F$ for extracting the periodicity of a quantum state, defined as

$$F|a> \equiv \frac{1}{\sqrt{k}} \sum_{c=0}^{k-1} e^{2i\pi ac/k}|c>.$$  \hspace{1cm} (A.1)

Then, the COUNT algorithm can be summarized by the following sequence of operations: 1) $(W|0>) (W|0>) = \sum_{m=0}^{P-1} |m> \sum_{a=0}^{N-1} |a> \quad \rightarrow \quad (F \otimes I) [\sum_{m=0}^{P-1} |m> G_H^{m} (\sum_{a=0}^{N-1} |a>)] \quad \rightarrow \quad$ measure $|m>$. 

The main idea at the core of the algorithm is that, since the amplitude of the set of the good states $|g>$ after $m$ iterations of $G_H$ on $|a>$ is a periodic function of $m$, the estimate of such a period by use of the Fourier analysis and the measurement of the ancilla qubit $|m>$ will give information on the size $t$ of this set, on which the period itself depends. The parameter $P$ determines both the precision of the estimate $t$ and the computational complexity of the COUNT algorithm (which requires $P$ iterations of $G_H$ [12]).

Let us now discuss how one can use the previous results, obtained for the generalized Grover’s algorithm when the initial state is arbitrarily entangled, to the case of the COUNT algorithm. We start by tensoring the state $|\psi>$ with log $P$ ancilla qubits set to $|0>$, act on these qubits with $W$, obtaining $|\psi_1> \equiv \sum_{m=0}^{P-1} |m> |\psi> / \sqrt{P}$, with an $|m->$ ‘controlled’ Grover operation $G_H^m$ on the state $|\psi>$ and with $F$ on $|m>$, thus getting

$$|\psi_f> \equiv \frac{\sum_{m,n=0}^{P-1} e^{2\pi i mn/P} |n>}{\sqrt{P}} G_H^m |\psi>.$$  \hspace{1cm} (A.2)

As in the standard COUNT algorithm, requiring that the time needed to compute the repeated Grover operations $G_H^m$ is polynomial in log $k$, leads to the choice $P \simeq O(poly(\log k))$ in eq. (A.2). Explicitly summing over $n$ in eq. (A.2), after some elementary algebra we get

$$|\psi_f> \equiv \frac{1}{\sqrt{N}} \left[ |0> |A> + |P/2> |B> + \frac{1}{2N} \sum_{m=0}^{P-1} |m> (\phi_m^+ s_m^+ |C_+ >.$$

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6 We assume, without loss of generality, that $k$ is a power of 2.

7 Without loss of generality, we assume that $P$ is a power of 2.
where we have introduced the following quantities

\[
\begin{align*}
\phi_m^\pm &\equiv \frac{\sin \pi (m \pm f)}{P \sin \pi (m \pm f)/P} ; \\
\phi_m &\equiv e^{i\pi(m \pm f)(1-1/P)} ; \\
f &\equiv \frac{P\theta}{\pi},
\end{align*}
\]

(A.4)

with \(0 \leq f \leq P/2\), and \(|A\), \(|B\) and \(|C_\pm\) are certain mutually orthogonal states.

At this point one can rewrite formula (A.3), similarly to how it is explained in ref. [12], in the general case when \(f\) is not an integer, distinguishing three possible cases. In particular, in the most general case in which \(1 < f < P/2 - 1\) we have

\[
|\psi_f\rangle = |f^-\rangle |a_1\rangle + |P-f^-\rangle |b_1\rangle + |f^+\rangle |c_1\rangle \\
+ |P-f^+\rangle |d_1\rangle + |R_1\rangle,
\]

(A.5)

where \(|R_1\rangle\) is an ’error’ term including all the other states in \(|\psi_f\rangle\) not containing the ancilla qubits \(|f^\pm\rangle\) and \(|P-f^\pm\rangle\) (with \(f^- \equiv [f] + \delta f\), \(f^+ \equiv f^- + 1\) and \(0 < \delta f < 1\)). After some easy algebra, one can show that the total probability amplitude in the first four terms is given by

\[
W_1 = [\sin^2 \theta < \hat{G}(0)|\hat{G}(0)> + \cos^2 \theta < \hat{B}(0)|\hat{B}(0)>] \Sigma_1,
\]

(A.6)

with \(\Sigma_1 \equiv (s^+_{f+})^2 + (s^-_{f-})^2 + (s^+_{P-f+})^2 + (s^-_{P-f-})^2\), and it can be shown that \(8/\pi^2 < \Sigma_1 \leq 1\). Similar calculations can be done for the cases when \(0 < f < 1\), when the probability of obtaining any of the states \(|0\rangle\), \(|1\rangle\) or \(|P-1\rangle\) is given by \(W_2 = \{N_1 + N[(\Sigma_2 - 1) \sin^2 \theta < \hat{G}(0)|\hat{G}(0)> + \Sigma_2 \cos^2 \theta < \hat{B}(0)|\hat{B}(0)>]\}/N\) (with \(\Sigma_2 \equiv (s^+_{0+})^2 + (s^-_{1-})^2 + (s^+_{P-1+})^2\) and \(8/\pi^2 < \Sigma_1 \leq 1\)), or when \(P/2 - 1 < f < P/2\), for which the probability to get any of the states \(|P/2\rangle\) or \(|P/2 \pm 1\rangle\) reads \(W_3 = 1 - \{N_1 - N[\Sigma_3 \sin^2 \theta < \hat{G}(0)|\hat{G}(0)> + (\Sigma_3 - 1) \cos^2 \theta < \hat{B}(0)|\hat{B}(0)>]\}/N\) (with \(\Sigma_3 \equiv (s^+_{P/2+})^2 + (s^-_{P/2-1+})^2\) and \(8/\pi^2 < \Sigma_3 \leq 1\)).

The final step of the COUNT algorithm consists in measuring the first ancilla qubit in the state \(|\psi_f\rangle\). As explained in ref. [12], if \(W_i \geq 1/2\) (for \(i = 1, 2\) or 3) then with high probability\(^8\) one can still be able to find one of the ancilla qubits \(|f_\pm\rangle\) or \(|P-f_\pm\rangle\), \(|0\rangle\), \(|1\rangle\) or \(|P-1\rangle\), \(|P/2 \pm 1\rangle\) or \(|P/2\rangle\), respectively, for the three cases, and, therefore, evaluate the number \(t\)

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\(^8\) By repeating the whole algorithm many times and using the majority rule, Brassard et al. (1998), and eventually using \(R\) ancilla qubits \(|m_1\rangle \ldots |m_R\rangle\) and acting with a ‘\(|m_1\rangle \ldots |m_R\rangle \text{-controlled} G^m_{ix}\) operation on the state \(|\psi\rangle\), Carlini et al. (1999).
of ‘good’ states from $\sin \theta = \sqrt{t/N}$ and eq. (A.4). For example, in the general case $1 < f < P/2 - 1$, the condition that $W_1 > 1/2$ is satisfied, e.g., for the choice of the initial ‘averages’ $\langle \tilde{G}^{(0)} | \tilde{G}^{(0)} \rangle \geq \langle \tilde{B}^{(0)} | \tilde{B}^{(0)} \rangle \geq \pi^2/(8\sqrt{2})$. Finally, although in general $f$ is not an integer and the measured $\tilde{f}$ will not match exactly the true value of $f$ but give the approximate estimate $\tilde{t} \equiv N \sin^2 \tilde{\theta}(\tilde{f})$, the error over $t$ for an entangled initial state will be $|\tilde{t} - t| \leq \pi N \left[ \frac{\pi}{P} + 2\sqrt{t/N} \right] / P$, i.e. the same as in ref. [12].

References

[1] P. Benioff, Journ. Stat. Phys. 22 (1980) 563.
[2] D. Deutsch, Proc. Roy. Soc. London A400 (1985) 96.
[3] R.P. Feynman, Found. Phys. 16 (1986) 507.
[4] D.P. DiVincenzo, Science 270 (1995) 255.
[5] A. Steane, Rep. Prog. Phys. 61 (1998) 117.
[6] P.W. Shor, Algorithms for quantum computation: discrete logarithms and factoring, in: S. Goldwater (Ed.), Proceedings of the 35th Annual Symposium on Foundations of Computer Science, IEEE Computer Society Press, New York, 1994; SIAM Journ. Comput. 26 (1997) 1484.
[7] L.K. Grover, A fast quantum mechanical algorithm for database search, in: Proceedings of the 28th Annual Symposium on the Theory of Computing, ACM Press, New York, 1996; Phys. Rev. Lett. 79 (1997) 325.
[8] M. Boyer, G. Brassard, P. Hoyer and A. Tapp, Tight bounds on quantum searching, in: T. Toffoli et al. (Ed.), Proceedings of the 4th Workshop on Physics and Computation, New England Complex Systems Institute, Boston, 1996; Fortsch. Phys. 46 (1998) 493.
[9] C. Zalka, quant-ph/9902049, A Grover based quantum search of optimal order for an unknown number of marked elements (1999).
[10] G. Chen, S.A. Fulling and M.O. Scully, quant-ph/9909040, Grover’s algorithm for multiobject search in quantum computing (1999).
[11] L. Grover, quant-ph/9912001, Rapid sampling through quantum computing (1999).
[12] G. Brassard, P. Hoyer and A. Tapp, quant-ph/9805082, Quantum counting (1998).
[13] C.H. Bennett, E. Bernstein, G. Brassard and U. Vazirani, SIAM Journ. Comput. 26 (1997) 1510.
[14] C. Zalka, Phys. Rev. A60 (1999) 2746.

[15] D. P. Chi and J. Kim, quant-ph/9708005, Quantum database searching by a single query (1997).

[16] B.M. Terhal and J.A. Smolin, Phys. Rev. A58 (1998) 1822.

[17] L.K. Grover, Phys. Rev. Lett. 79 (1997) 4709.

[18] M. Grassl and T. Beth, quant-ph/9706052, On the complexity of quantum searching with complex queries (1997).

[19] N.J. Cerf, L.K. Grover and C.P. Williams, Phys. Rev. A61 (2000) 032303.

[20] E. Farhi and S. Gutmann, quant-ph/9711033, Quantum mechanical square root speed up in a structured search problem (1997).

[21] E. Farhi and S. Gutmann, quant-ph/9612026, An analog analogue of digital quantum computation (1996).

[22] G.L. Long, W.L. Zhang, Y.S. Li and L. Niu, Comm. Theor. Phys. 32 (1999) 335; Phys. Lett. A262 (1999), 27.

[23] L.K. Grover, Phys. Rev. Lett. 80 (1998) 4329; quant-ph/9711043, A framework for fast quantum mechanical algorithms (1997).

[24] R. Jozsa, quant-ph/9901021, Searching in Grover's algorithm (1999).

[25] E. Biham, O. Biham, D. Biron, M. Grassl and D.A. Lidar, Phys. Rev. A60 (1999) 2742.

[26] A.K. Pati, quant-ph/9807067, Fast quantum search algorithm and bounds on it (1998).

[27] R. Gingrich, C.P. Williams and N.J. Cerf, quant-ph/9904049, Generalized quantum search with parallelism (1999).

[28] R. Jozsa, quant-ph/9707034, Entanglement and quantum computation (1997).

[29] A. Peres, Superlattices Microstruct. 23 (1998) 373.

[30] G.H. Hardy and E.M. Wright, An Introduction to the Theory of Numbers, Clarendon Press, Oxford, 1938.

[31] A. Carlini and A. Hosoya, Phys. Rev. A62 (2000) 032312.