SPECTRAL PROPERTIES OF A CONFORMABLE BOUNDARY VALUE PROBLEM ON TIME SCALES

ZEKI CEYLAN

Abstract. We study a self-adjoint conformable dynamic equation of second order on an arbitrary time scale $\mathbb{T}$. We state an existence and uniqueness theorem for the solutions of this equation. We prove the conformable Lagrange identity on time scales. Then, we consider a conformable eigenvalue problem generated by the above-mentioned dynamic equation of second order and we examine some of the spectral properties of this boundary value problem.

1. Introduction

We study the self-adjoint conformable dynamic equation of second order

\begin{equation}
Lx = 0, \quad \text{where } Lx(t) = (px^{\Delta_{\alpha}})_{\Delta_{\alpha}}(t) + q(t)x^{\sigma}(t)
\end{equation}

on an arbitrary time scale $\mathbb{T}$. Throughout we assume that $p,q \in C_{rd}$ and $p(t) \neq 0$ for all $t \in \mathbb{T}$.

Continuous conformable calculus is a natural extension of the usual calculus and has yielded several articles such as [1–6]. Some follow-up papers related to Sturm-Liouville equations for conformable calculus include [7–9].

Recently, researchers have started to deal with studies relating to conformable calculus on time scales (see [10–16]).

A conformable derivative on time scales was first introduced in [10] by the formula

\begin{equation}
f^{\Delta_{\alpha}}(t) = \begin{cases} 
\frac{f(\sigma(t)) - f(t)}{t^{1-\alpha}}, & \sigma(t) > t, \\
\lim_{s \to t} \frac{f(t) - f(s)}{t - s} t^{1-\alpha}, & \sigma(t) = t.
\end{cases}
\end{equation}

Note that, if $f$ is $\Delta-$differentiable at a right scattered point $t \in \mathbb{T}^\kappa_{[0,\infty)}$ [17], then $f$ is $\alpha-$ conformable differentiable and for the above definition we have

\date{February, 2020.}
\textit{2000 Mathematics Subject Classification.} 34N05, 26A33, 34K08.
\textit{Key words and phrases.} Time scales, Conformable derivative, Boundary value problems.
(1.3) \[ f^\Delta(t) = t^{1-\alpha} f^\Delta(t) \]

where \( f^\Delta(t) = \frac{f^\sigma(t) - f(t)}{\sigma(t) - t} \).

The following formula [10, Theorem 4, iv] will be needed in the sequel:

(1.4) \[ f(\sigma(t)) = f(t) + \mu(t)t^{\alpha-1}f^\Delta(t). \]

Assume \( f, g : \mathbb{T} \to \mathbb{R} \) are conformable differentiable of order \( \alpha \). Then, if \( f \) and \( g \) are continuous, then the product \( fg : \mathbb{T} \to \mathbb{R} \) is conformable differentiable with

(1.5) \[ (fg)^\Delta = f^\Delta g + (f\sigma)g^\Delta = f^\Delta (g\sigma) + fg^\Delta, \]

if \( f \) and \( g \) are continuous, then \( \frac{f}{g} \) is conformable differentiable with

(1.6) \[ \left( \frac{f}{g} \right)^\Delta = \frac{f^\Delta g - fg^\Delta}{g(g\sigma)}, \]

valid at all points \( t \in \mathbb{T}^\kappa \) for which \( g(t)g(\sigma(t)) \neq 0 \).

This paper consists of five sections. After this Introduction part, we will state an existence and uniqueness theorem in Section 2. In Section 3, we will recall the definition of \( \alpha \)-Wronskian and we prove the conformable Lagrange identity after defining the Lagrange bracket of two functions. In Section 4, we will consider a conformable boundary value problem and we will investigate some of its spectral properties after proving the Green's theorem. Finally, we conclude the paper in Section 5 by giving some remarks about the current paper.

2. A Conformable Dynamic Equation

In this section, we will investigate the self-adjoint conformable dynamic equation (1.1). First, we will state a theorem concerning the existence and uniqueness of solutions of initial value problems for \( Lx(t) = f(t) \). Then, we will suggest a method to construct the solutions of (1.1).

**Theorem 2.1.** If \( f \in C_{rd}, t_0 \in \mathbb{T}, \) and \( x_0, x_0^\alpha \) are given constants, then the initial value problem

(2.1) \[ Lx(t) = f(t), \quad x(t_0) = x_0, \quad x^\Delta(t_0) = x_0^\alpha \]

has a unique solution that exists on the whole time scale \( \mathbb{T} \).

**Proof.** Define \( y(t) = p(t)x^\Delta(t) \). From here we have

(2.2) \[ x^\Delta(t) = \frac{y(t)}{p(t)}. \]
With the help of (1.1), (1.4), (1.3) and (2.1) we have
\[ (2.6) \]

Let \( z(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \). From (1.3) We may compute
\[ (2.3) \]
\[ (2.4) \]
\[ (2.5) \]
This is a system of dynamic equations whose components are rd-continuous. Therefore by Thorem 5.24 [18] we observe that a unique solution exists.

Now, let us suggest a method to construct solutions of the self-adjoint conformable dynamic equation (1.1). To do this, first, assume that \( x \) is a nontrivial solution of (1.1). Then
\[ (2.3) \]
\[ (2.4) \]
are satisfied. We call (2.3) and (2.4) the Prüfer transformation.

**Theorem 2.2.** If \( x \) is a nontrivial solution of (1.1) and if \( \varphi \) and \( \varphi \) are defined by (2.3) and (2.4), then the equations
\[ (2.5) \]
\[ (2.6) \]
are satisfied.

Proof. Using the product rule for (2.3) yields
\[ \varrho^\Delta \alpha (\sin \varphi) = (\rho \sin \varphi) = x^\Delta \alpha = \frac{1}{p}(px^\Delta \alpha) = \frac{1}{p} \rho \cos \varphi, \]
while doing the same for (2.4) implies
\[ \varrho^\Delta \alpha (\cos \varphi) = (\rho \cos \varphi) = \frac{1}{p}(px^\Delta \alpha) = \frac{1}{p} \rho \cos \varphi, \]
where we have also used that \( x \) is a solution of (1.1) and got help from equation (1.4). Hence we obtain the two equations
\[
\begin{align*}
(2.7) & \quad \varrho^\Delta \alpha (\sin \varphi) + \varrho(\sin \varphi) = \frac{1}{p} \rho \cos \varphi, \\
(2.8) & \quad \varrho^\Delta \alpha (\cos \varphi) + \varrho(\cos \varphi) = -q \rho \sin \varphi - \frac{\mu q t^{\alpha - 1}}{p} \rho \cos \varphi.
\end{align*}
\]
We now multiply (2.7) by \( \sin \varphi \) and (2.8) by \( \cos \varphi \) and add the resulting equations to obtain (2.5). To verify (2.6), we multiply (2.7) by \( \cos \varphi \) and (2.8) by \( -\sin \varphi \) and add the resulting equations. Dividing the obtained equation by \( \varrho \) directly yields (2.6). \( \square \)

Observe that the conformable dynamic equation (2.6) for \( \varphi \) is independent of \( \varrho \). Of course it might be difficult to solve this equation, but once a solution of (2.6) is obtained, the linear conformable dynamic equation (2.5) for \( \varrho \) is readily solved.

3. Conformable Lagrange Identity on Time Scales

In this section, we collect some knowledge needed in the rest of this paper. First, we introduce the definition of \( \alpha \)-Wronskian and give one of its properties, then we define the Lagrange bracket of two functions and prove the conformable Lagrange identity on time scales.

Definition 3.1. If \( x, y : T \rightarrow \mathbb{R} \) are conformable differentiable on \( T^\kappa \), then we define the \( \alpha \)-Wronskian of \( x \) and \( y \) by
\[ W_\alpha(x, y)(t) = \det \begin{pmatrix} x(t) & y(t) \\ x^\Delta \alpha(t) & y^\Delta \alpha(t) \end{pmatrix} \]
for \( t \in T^\kappa \).

The following lemma will be used in the proof of Theorem 3.4.

Lemma 3.2. If \( x, y : T \rightarrow \mathbb{R} \) are conformable differentiable on \( T^\kappa \), then
\[ W_\alpha(x, y)(t) = \det \begin{pmatrix} x'(t) & y'(t) \\ x^\Delta \alpha(t) & y^\Delta \alpha(t) \end{pmatrix} \]
holds for \( t \in T^\kappa \).
Proof. For \( t \in \mathbb{T}^\circ \), we have by equation (1.4)
\[
\det \begin{pmatrix} x^\sigma(t) & y^\sigma(t) \\ x^\Delta(t) & y^\Delta(t) \end{pmatrix} = \det \begin{pmatrix} x(t) + \mu(t)t^{\alpha - 1}x^\Delta(t) & y(t) + \mu(t)t^{\alpha - 1}y^\Delta(t) \\ x^\Delta(t) & y^\Delta(t) \end{pmatrix}
\]
\[
= \det \begin{pmatrix} x(t) & y(t) \\ x^\Delta(t) & y^\Delta(t) \end{pmatrix}
\]
\[
= W_\alpha(x,y)(t)
\]
which gives us the desired result. \( \square \)

Now, let us define the Lagrange bracket of two conformable functions, before proving the conformable Lagrange identity.

**Definition 3.3.** If \( x, y : \mathbb{T} \to \mathbb{R} \) are conformable differentiable on \( \mathbb{T}^\circ \), then the Lagrange bracket of \( x \) and \( y \) is defined by
\[
\{x;y\}(t) = p(t)W_\alpha(x,y)(t)
\]
for \( t \in \mathbb{T}^\circ \).

Define the set \( \mathbb{D} \) to be the set of all functions \( x : \mathbb{T} \to \mathbb{R} \) such that \( x^\Delta : \mathbb{T}^\circ \to \mathbb{R} \) is rd-continuous. A function \( x \in \mathbb{D} \) is then said to be a solution of (1.1) provided \( Lx(t) = 0 \) holds for all \( t \in \mathbb{T}^\circ \). The below-given theorem follows immediately.

**Theorem 3.4** (Conformable Lagrange Identity). If \( x, y \in \mathbb{D} \), then
\[
x^\sigma(t)Ly(t) - y^\sigma(t)Lx(t) = \{x;y\}^\Delta(t)
\]
holds for \( t \in \mathbb{T}^\circ \).

**Proof.** By the product rule (1.5), we have
\[
\{x;y\}^\Delta = \{xy^\Delta = px^\Delta y\}^\Delta
\]
\[
= x^\sigma(py^\Delta)\Delta + (py^\Delta)x^\Delta - y^\sigma(px^\Delta)\Delta - y^\Delta(px^\Delta)
\]
\[
= x^\sigma(py^\Delta)\Delta - y^\sigma(px^\Delta)\Delta
\]
\[
= x^\sigma(py^\Delta)\Delta - y^\sigma(px^\Delta)\Delta + qx^\sigma y^\sigma - qx^\sigma y^\sigma
\]
\[
= x^\sigma((py^\Delta)\Delta + qx^\sigma) - y^\sigma((px^\Delta)\Delta + qx^\sigma)
\]
\[
= x^\sigma Ly - y^\sigma Lx
\]
on \( \mathbb{T}^\circ \). \( \square \)

4. **A Conformable Boundary Value Problem and Green’s Function**

In this section, we consider a conformable boundary value problem of the form
\[
(4.1) \quad Lx + \lambda x^\sigma = 0, \quad R_\alpha(x) = R_\delta(x) = 0,
\]
where \( Lx = x^\Delta + qx^\sigma \) such that \( q : \mathbb{T} \to \mathbb{R} \) is rd-continuous, and
\[
R_\alpha(x) = \gamma_1 x(\rho(a)) + \gamma_2 x^\Delta(\rho(a)), \quad R_\delta(x) = \delta_1 x(\rho(b)) + \delta_2 x^\Delta(\rho(b))
\]
such that \( \gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{R} \) with \( (\gamma_1 + \gamma_2)(\delta_1^2 + \delta_2^2) \neq 0 \) hold.

A number \( \lambda \in \mathbb{R} \) is called an eigenvalue of (4.1) provided there exists a non-trivial solution \( x \) of the conformable boundary value problem (4.1). Such an \( x \) is then called an eigenfunction corresponding to the eigenvalue \( \lambda \).
We define the inner product of \(x\) and \(y\) on \([\rho(a), b]\) by
\[
\langle x, y \rangle = \int_{\rho(a)}^{b} x(t) y(t) \Delta_{\alpha} t := \int_{\rho(a)}^{b} x(t) y(t) t^{\alpha - 1} \Delta t,
\]
and we say that \(x\) and \(y\) are orthogonal on \([\rho(a), b]\) provided \(\langle x, y \rangle = 0\). The norm of \(x\) is defined by
\[
\|x\| = \sqrt{\langle x, x \rangle}.
\]

The next theorem follows immediately from Theorem 3.4.

**Theorem 4.1 (Green’s Theorem).** If \(x, y \in \mathbb{D}\), then
\[
\langle x^\sigma, Ly \rangle - \langle y^\sigma, Lx \rangle = \{x; y\}(b) - \{x; y\}(a)
\]
holds.

**Proof.** This results from integrating both sides of the conformable Lagrange identity in Theorem 3.4 on \([a, b]\) and using the definition of the inner product. \(\square\)

It is easy to see that \(W_{\alpha}(x, y)(\rho(a)) = 0\) if \(R_{\alpha}(x) = R_{\alpha}(y) = 0\) and \(W_{\alpha}(x, y)(b) = 0\) if \(R_{b}(x) = R_{b}(y) = 0\). Indeed, from Definition 3.1 we have
\[
W_{\alpha}(x, y)(\rho(a)) = \det \begin{pmatrix} x(\rho(a)) & y(\rho(a)) \\ x^{\Delta_{\alpha}}(\rho(a)) & y^{\Delta_{\alpha}}(\rho(a)) \end{pmatrix} = \det \begin{pmatrix} -\gamma_{2} x^{\Delta_{\alpha}}(\rho(a)) & -\gamma_{2} y^{\Delta_{\alpha}}(\rho(a)) \\ \gamma_{1} x^{\Delta_{\alpha}}(\rho(a)) & \gamma_{1} y^{\Delta_{\alpha}}(\rho(a)) \end{pmatrix} = 0.
\]
The fact that \(R_{b}(x) = R_{b}(y) = 0\) implies \(W_{\alpha}(x, y)(b) = 0\) follows similarly.

Now, we shall provide a characterization for the eigenvalues of (4.1). For this, we denote the unique solutions (see Theorem 2.1) of the conformable initial value problem
\[
Lx + \lambda x^\sigma = 0, \ x(\rho(a)) = \gamma_{2}, \ x^{\Delta_{\alpha}}(\rho(a)) = -\gamma_{1}
\]
by \(x(\cdot, \lambda)\), where \(\lambda \in \mathbb{R}\), and we put \(\Lambda(\lambda) = R_{b}(x(\cdot, \lambda))\). With this notation in mind, we have the following.

**Theorem 4.2.** \(\lambda\) is an eigenvalue of (4.1) if and only if \(\Lambda(\lambda) = 0\).

**Proof.** If \(R_{b}(x(\cdot, \lambda)) = 0\), then \(x = x(\cdot, \lambda)\) satisfies
\[
Lx + \lambda x^\sigma = 0, \ R_{\alpha}(x) = R_{b}(x) = 0,
\]
i.e., \(\lambda\) is an eigenvalue of (4.1). Conversely, let \(\lambda \in \mathbb{R}\) be an eigenvalue of (4.1) with corresponding eigenfunction \(x\). Then because of the unique solvability of the conformable initial value problem (observe \(R_{\alpha}(x) = 0\)), the equation \(x = cx(\cdot, \lambda)\) holds with \(c = \frac{\gamma_{2} x_{\rho}(\rho(a)) - \gamma_{1} x^{\Delta_{\alpha}}(\rho(a))}{\gamma_{1}^{2} + \gamma_{2}^{2}}\). Hence \(R_{b}(x(\cdot, \lambda)) = 0\) which gives us the desired result. \(\square\)

The proof of Theorem 4.2 also shows that all eigenvalues of (4.1) are simple.
5. Conclusion

In this paper, we deal with a self-adjoint conformable dynamic equation of second order on an arbitrary time scale. We prove an existence and uniqueness theorem for the solutions of this equation and we suggest a method to construct these solutions via Prüfer transformation. Then we prove the conformable Lagrange identity. After that, we derive a conformable boundary value problem which consists of the above-mentioned conformable dynamic equation and boundary conditions. We prove Green’s theorem with the help of conformable Lagrange identity and we provide a characterization for the eigenvalues of this conformable boundary value problem. Presented results of this paper are generalizations of some results in [17] via conformable derivative.

References

[1] T. Abdeljawad, On conformable fractional calculus, Journal of Computational and Applied Mathematics, 279(2015), 57-66.
[2] M. Abu Hammad, R. Khalil, Abel’s formula and Wronskian for conformable fractional differential equations, Internat. J. Diff. Equ. Appl., 13(2014), No. 3, 177-183.
[3] M. Abu Hammad, R. Khalil, Conformable fractional heat differential equations, Internat. J. Pure Appl. Math., 94(2014), No. 2, 215-221.
[4] H. Abu-Shaab, R. Khalil, Solution of some conformable fractional differential equations, Int. J. Pure Appl. Math., 103(2015), No. 4, 667-673.
[5] M. J. Lazlo, D. F. M. Torres, Variational calculus with conformable fractional derivatives, IEEE/CAA Journal of Automatica Sinica, 4(April 2017), No. 2.
[6] W. Rosa, J. Weberspil, Dual conformable derivative: Definition, simple properties and perspectives for applications, Chaos, Solitons and Fractals, 117(2018), 137-141.
[7] D. Anderson, R. I. Avery, Fractional-order boundary value problem with Sturm-Liouville boundary conditions, Electronic Journal of Differential Equations, 29(2015), 1-10.
[8] H. Batarfi, J. Losada, J. J. Nieto, W. Shammakh, Three-point boundary value problems for conformable fractional differential equations, Journal of Function Spaces, Volume 2015, Article ID 706383, 6 pages, doi:10.1155/2015/706383.
[9] B. P. Allahverdiev, H. Tuna, Y. Yalcinkaya, Conformable fractional Sturm-Liouville equation, Math. Meth. Appl. Sci., 42(2019), 3508-3526.
[10] N. Benkhettou, S. Hassani, D. F. M. Torres, A conformable fractional calculus on arbitrary time scales, Journal of King Saud University-Science, 28(2016), 93-98.
[11] M. Bohner, V. F. Hatipoğlu, Dynamic Cobweb models with conformable fractional derivatives, Nonlinear Anal., Hybrid Syst. 32(2019), 157-167.
[12] T. Gulsen, E. Yilmaz, S. Goktas, Conformable fractional Dirac system on time scales, J. Inequal. Appl., 2017:10, 2017.
[13] T. Gulsen, E. Yilmaz, H. Kemaloglu, Conformable fractional Sturm-Liouville equation and some existence results on time scales, Turk. J. Math. 42(2018), No. 3, 1348-1360.
[14] S. Rahmat, M. Rafi, A new definition of conformable fractional derivative on arbitrary time scales, Adv. Difference Equ., 2019:354, 2019.
[15] D. F. Zhou, X. X. You, A new fractional derivative on time scales, Adv. Appl. Math. Anal., 11(2016), No. 1, 1-9.
[16] C. Zhang, S. Sun, Sturm-Picone comparison theorem of a kind of conformable fractional differential equations on time scales, J. Appl. Math. Comput., 55(2017), 191-203, doi:10.1007/s12190-016-1032-9.
[17] M. Bohner, A. Peterson, Dynamic Equations on Time Scales, An Introduction with Applications, Birkhäuser, Boston Inc. Boston, MA, 2001.
[18] W. Kelley, A. Peterson, The Theory of Differential Equations Classical and Qualitative, Pearson Prentice Hall, Upper Saddle River, NT, 2004.
Mersin University, Institute of Science, Department of Mathematics, 33343, Mersin, Turkey.

Current address: Mersin University, Institute of Science, Department of Mathematics, 33343, Mersin, Turkey.

Email address: z2.ceylann@gmail.com