Symmetries of a class of Nonlinear Third Order Partial Differential Equations

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Abstract
In this paper we study symmetry reductions of a class of nonlinear third order partial differential equations
\[
\Delta \equiv u_t - \epsilon u_{xxx} + 2\kappa u_x = uu_{xxx} + \alpha uu_x + \beta u u_{xx}, \tag{1.1}
\]
where \(\epsilon, \kappa, \alpha\) and \(\beta\) are arbitrary constants. Three special cases of equation (1) have appeared in the literature, up to some rescalings. In each case the equation has admitted unusual travelling wave solutions: the Fornberg-Whitham equation, for the parameters \(\epsilon = 1, \alpha = -1, \beta = 3\) and \(\kappa = \frac{1}{2}\), admits a wave of greatest height, as a peaked limiting form of the travelling wave solution; the Rosenau-Hyman equation, for the parameters \(\epsilon = 0, \alpha = 1, \beta = 3\) and \(\kappa = 0\), admits a “compacton” solitary wave solution; and the Fuchssteiner-Fokas-Camassa-Holm equation, for the parameters \(\epsilon = 1, \alpha = -3\) and \(\beta = 2\), has a “peakon” solitary wave solution.

A catalogue of symmetry reductions for equation (1) is obtained using the classical Lie method and the nonclassical method due to Bluman and Cole.

1 Introduction
In this paper we are concerned with symmetry reductions of the nonlinear third order partial differential equation given by
\[
\Delta \equiv u_t - \epsilon u_{xxx} + 2\kappa u_x = uu_{xxx} + \alpha uu_x + \beta u u_{xx} = 0, \tag{1.1}
\]
where \(\epsilon, \kappa, \alpha\) and \(\beta\) are arbitrary constants. Three special cases of (1.1) have appeared recently in the literature. Up to some rescalings, these are: (i), the
The Fornberg-Whitham (FW) equation
\[ u_t - u_{xxt} + u_x = uu_{xxx} - uu_x + 3u_x u_{xx} \]  
was used to look at qualitative behaviours of wave-breaking [58]. It admits a wave of greatest height, as a peaked limiting form of the travelling wave solution [28],
\[ u(x,t) = A \exp \left(-\frac{1}{2}|x - \frac{4}{3}t|\right), \]
where \( A \) is an arbitrary constant.

The Rosenau-Hyman (RH) equation
\[ u_t = uu_{xxx} + uu_x + 3u_x u_{xx}. \]  
models the effect of nonlinear dispersion in the formation of patterns in liquid drops [54]. It also has an unusual solitary wave solution, known as a “compacton”,
\[ u(x,t) = \begin{cases} -\frac{8}{3}c \cos^2 \left(\frac{1}{4}(x - ct)\right), & \text{if } |x - ct| \leq 2\pi, \\ 0, & \text{if } |x - ct| > 2\pi. \end{cases} \]
These waves interact producing a ripple of low amplitude compacton-anticompacton pairs.

The Fuchssteiner-Fokas-Camassa-Holm (FFCH) equation
\[ u_t - u_{xxt} + 2\kappa u_x = uu_{xxx} - 3uu_x + 2u_x u_{xx}, \]  
first arose in the work of Fuchssteiner and Fokas [25,27] using a bi-Hamiltonian approach; we remark that it is only implicitly written in [27] — see equations (26e) and (30) in this paper — though is explicitly written down in [25]. It has recently been rederived by Camassa and Holm [9] from physical considerations as a model for dispersive shallow water waves. In the case \( \kappa = 0 \), it admits an unusual solitary wave solution
\[ u(x,t) = A \exp (-|x - ct|), \]
where \( A \) and \( c \) are arbitrary constants, which is called a “peakon”. A Lax-pair [9] and bi-Hamiltonian structure [27] have been found for the FFCH equation [1.4] and so it appears to be completely integrable. Recently the FFCH equation [1.4] has attracted considerable attention. In addition to the aforementioned, other studies include [10,21,22,23,24,26,32,41,46].

The FFCH equation [1.4] may be thought of as an integrable modification of the regularized long wave (RLW) equation [7,47]
\[ uu_x + uu_t - u_x = 0, \]  
i.e.,
\[ u_{xxt} + uu_x - u_t - u_x = 0, \]  
(iii), the Fuchssteiner-Fokas-Camassa-Holm equation [9,10,25,27] for the parameters \( \epsilon = 1, \alpha = -1, \beta = 2 \).
sometimes known as the Benjamin-Bona-Mahoney equation. However, in contrast to (1.4), the RLW equation (1.5) is thought not to be solvable by inverse scattering (cf., [42]); its solitary wave solutions interact inelastically (cf., [37]) and only has finitely many local conservation laws [45]. However physically it has more desirable properties than the celebrated Korteweg-de Vries (KdV) equation

\[ u_t + u_{xxx} + 6u_x = 0, \]  

which was the first equation to be solved by inverse scattering [31]. We remark that two other integrable variants of the RLW equation (1.5) are

\[ u_{xxt} + 2uu_t - u_x \partial_x^{-1} u_t - u_t - u_x = 0, \]  

where \((\partial_x^{-1} f)(x) = \int_x^\infty f(y) \, dy\), which was introduced by Ablowitz, Kaup, Newell and Segur [2], and

\[ u_{xxt} + uu_t - u_x \partial_x^{-1} u_t - u_t - u_x = 0, \]  

which was discussed by Hirota and Satsuma [34]. We also note that (1.4), with \(\kappa = \frac{1}{2}\), (1.5), (1.7) and (1.8) all have the same linear dispersion relation \(\omega(k) = -k/(1 + k^2)\) for the complex exponential \(u(x,t) \sim \exp\{i(kx + \omega(k)t)\}\).

Recently, Gilson and Pickering [32] have shown that no equation in the entire class of equations (1.1) will satisfy the necessary conditions of either the Painlevé PDE test due to Weiss, Tabor and Carnevale [57] or the Painlevé ODE test due to Ablowitz, Ramani and Segur [3,4] to be solvable by inverse scattering. However, the integrable FFCH equation (1.4) does possess the “weak Painlevé” property (cf., [49,50]), as does the FW equation (1.2).

All these special travelling wave solutions are essentially exponential solutions, or sums of exponential solutions, and thus would suggest some sort of linearity in the differential equation. This is discussed by Gilson and Pickering [32], who show that (1.3), with \(\alpha \neq 0\) and \(\beta(1 + \beta) \neq 0\), can be written as

\[ (\beta u_x + u \partial_x + \epsilon \partial_t) (u_{xx} - \mu^2 u - 2\kappa / \beta) = 0, \]  

where \(\partial_x \equiv \partial / \partial x\), \(\partial_t \equiv \partial / \partial t\) and \(\mu^2 = -\alpha/(1 + \beta)\), provided that \(\epsilon \alpha + \beta + 1 = 0\), which includes the FFCH equation (1.4). For the travelling wave reduction,

\[ u = w(z), \quad z = x - ct, \]

the resulting ordinary differential equation is

\[ (2\kappa - c) w' + \epsilon cw''' - w w''' - \alpha w' - \beta w' w'' = 0, \]  

where \(\prime \equiv \partial / \partial z\), which also may be factorised as

\[ \left[ \beta w' + (w - \epsilon c) \frac{d}{dz} \right] (w'' - \mu^2 w + \gamma) = 0, \]  

provided that

\[ \mu^2 = -\frac{\alpha}{1 + \beta}, \quad \beta(1 + \beta) \gamma - 2\kappa(1 + \beta) + c(1 + \beta + \alpha \epsilon) = 0. \]
This includes all three special cases (1.2)–(1.4); since $\beta (1+\beta)$ is strictly non-zero in these three cases then a suitable $\gamma$ can always be found.

Furthermore, if $1 + \beta + \alpha \epsilon = 0$ and $\epsilon \neq 0$, then (1.1) with $\kappa = 0$ possesses the “peakon” solution

$$u(x, t) = A \exp \left( -\epsilon^{-1/2}|x - ct| \right),$$

where $A$ and $c$ are arbitrary constants. More generally, if $\alpha/(1 + \beta) < 0$, $1 + \beta + \alpha \epsilon \neq 0$ and $\kappa \neq 0$, then (1.1) possesses the solution

$$u(x, t) = A \exp \left\{ - \left( - \frac{\alpha}{1 + \beta} \right)^{1/2} |x - ct| \right\}, \quad c = \frac{2(1 + \beta)\kappa}{1 + \beta + \alpha \epsilon},$$

where $A$ is an arbitrary constant. If $\alpha/(1 + \beta) > 0$, $\beta \neq -1$ and $\alpha \beta \neq 0$, then (1.1) possesses the “compacton” solution

$$u(x, t) = \frac{2[2(1 + \beta)\kappa - (1 + \beta + \alpha \epsilon)c]}{\alpha \beta} \cos^2 \left\{ \frac{1}{2} \left( \frac{\alpha}{1 + \beta} \right)^{1/2} (x - ct) \right\},$$

where $c$ is an arbitrary constant.

The classical method for finding symmetry reductions of partial differential equations is the Lie group method of infinitesimal transformations. As this method is entirely algorithmic, though often both tedious and virtually unmanageable manually, symbolic manipulation programs have been developed to aid the calculations. An excellent survey of the different packages available and a description of their strengths and applications is given by Hereman [33] (see also his contribution in this volume). In this paper we use the MACSYMA package `symmgrp.max` [11] to calculate the determining equations.

In recent years the nonclassical method due to Bluman and Cole [8] (in the sequel referred to as the “nonclassical method”), sometimes referred to as the “method of partial symmetries of the first type” [56], or the “method of conditional symmetries” [35], and the direct method due to Clarkson and Kruskal [15] have been used to generate many new symmetry reductions and exact solutions for several physically significant partial differential equations that are not obtainable using the classical Lie method (cf., [13] and the references therein). The nonclassical method is a generalization of the classical Lie method, whereas the direct method is an ansatz-based approach which involves no group theoretic techniques. Nucci and Clarkson [43] showed that for the Fitzhugh-Nagumo equation the nonclassical method is more general than the direct method, since they demonstrated the existence of a solution of the Fitzhugh-Nagumo equation, obtainable using the nonclassical method but not using the direct method. Subsequently Olver [45] (see also [6,48]) has proved the general result that for a scalar equation, every reduction obtainable using the direct method is also obtainable using the nonclassical method. Consequently we use the nonclassical method in this paper rather than the direct method.
Symmetry reductions and exact solutions have several different important applications in the context of differential equations. Since solutions of partial differential equations asymptotically tend to solutions of lower-dimensional equations obtained by symmetry reduction, some of these special solutions will illustrate important physical phenomena. In particular, exact solutions arising from symmetry methods can often be used effectively to study properties such as asymptotics and “blow-up” (cf., [29,30]). Furthermore, explicit solutions (such as those found by symmetry methods) can play an important role in the design and testing of numerical integrators; these solutions provide an important practical check on the accuracy and reliability of such integrators (cf., [5,55]).

Classical symmetries of differential equations are found in practice by a two-step process. The first involves finding the determining equations for the infinitesimals of the group action. These determining equations form an overdetermined, linear system of partial differential equations. The second step involves integrating this system. The first step is entirely algorithmic, and has been implemented in all the commercial symbolic manipulation languages (cf., [33]). The second step involves heuristic integration procedures which have been implemented in some symbolic manipulation programs and are largely successful, though not infallible. Commonly, the overdetermined systems to be solved are simple, and heuristic integration is both fast and effective. However, there are three areas where heuristics can break down (cf., [39] for further details and examples).

1. **Arbitrary parameters and functions.** If the partial differential equation whose symmetries are sought involves arbitrary parameters, such as (1.1) or more generally, arbitrary functions, heuristics yield usually the general solution, and miss those special cases of the parameters and arbitrary functions where additional symmetries exist.

2. **Termination.** Heuristic algorithms are not guaranteed to terminate, and may become trapped in infinite loops for some examples.

3. **Too difficult to solve.** The system may not be solvable by the heuristic. The heuristic will then attempt to represent the general solution in terms of functions satisfying certain conditions, but may give up before a useful representation is obtained.

These problems are addressed by use differential Gröbner bases (DGBs) which we describe below.

The method used to find solutions of the determining equations in the non-classical method is that of DGBs, defined to be a basis $\mathcal{B}$ of the differential ideal generated by the system such that every member of the ideal pseudo-reduces to zero with respect to $\mathcal{B}$. This method provides a systematic framework for finding integrability and compatibility conditions of an overdetermined system of partial differential equations. It avoids the problems of infinite loops in reduction processes and yields, as far as is currently possible, a “triangulation” of the system from which the solution set can be derived more easily [16,40,51,52].
In a sense, a DGB provides the maximum amount of information possible using elementary differential and algebraic processes in finite time.

In pseudo-reduction, one must, if necessary, multiply the expression being reduced by differential (non-constant) coefficients of the highest derivative terms of the reducing equation, so that the algorithms used will terminate [40]. In practice, such coefficients are assumed to be non-zero, and one needs to deal with the possibility of them being zero separately. These are called singular cases.

The triangulations of the systems of determining equations for infinitesimals arising in the nonclassical method in this paper were all performed using the MAPLE package diffgrob2 [38]. This package was written specifically to handle nonlinear equations of polynomial type. All calculations are strictly ‘polynomial’, that is, there is no division. Implemented there are the Kolchin–Ritt algorithm using pseudo-reduction instead of reduction, and extra algorithms needed to calculate a DGB (as far as possible using the current theory), for those cases where the Kolchin–Ritt algorithm is not sufficient [40]. The package was designed to be used interactively as well as algorithmically, and much use is made of this fact here. It has proved useful for solving many fully nonlinear systems [16–19].

In the following sections we shall consider the cases $\epsilon = 0$ and $\epsilon \neq 0$, when we set $\epsilon = 1$ without loss of generality, separately because the presence or lack of the corresponding third order term is significant. In §2 we find the classical Lie group of symmetries and associated reductions of (1.1). In §3 we discuss the nonclassical symmetries and reductions of (1.1) in the generic case. In §4 we consider special cases of the the nonclassical method in the so-called $\tau = 0$; in full generality this case generates a single equation which is considerably more complex than our original equation! In §5 we discuss our results.

2 Classical symmetries

To apply the classical method we consider the one-parameter Lie group of infinitesimal transformations in $(x, t, u)$ given by

$$\begin{align*}
x^* &= x + \epsilon \xi(x, t, u) + O(\epsilon^2), \\
t^* &= t + \epsilon \tau(x, t, u) + O(\epsilon^2), \\
u^* &= u + \epsilon \phi(x, t, u) + O(\epsilon^2),
\end{align*}$$

(2.1)

where $\epsilon$ is the group parameter. Then one requires that this transformation leaves invariant the set

$$S_\Delta \equiv \{u(x, t) : \Delta = 0\}$$

(2.2)

of solutions of (1.1). This yields an overdetermined, linear system of equations for the infinitesimals $\xi(x, t, u), \tau(x, t, u), \phi(x, t, u)$. The associated Lie algebra is realised by vector fields of the form

$$v = \xi(x, t, u) \partial_x + \tau(x, t, u) \partial_t + \phi(x, t, u) \partial_u.$$

(2.3)
Having determined the infinitesimals, the symmetry variables are found by solving the characteristic equation

\[ \frac{dx}{\xi(x,t,u)} = \frac{dt}{\tau(x,t,u)} = \frac{du}{\phi(x,t,u)}. \]  

(2.4)

which is equivalent to solving the invariant surface condition

\[ \psi \equiv \xi(x,t,u)u_x + \tau(x,t,u)u_t - \phi(x,t,u) = 0. \]  

(2.5)

The set \( S_\Delta \) is invariant under the transformation (2.1) provided that

\[ \text{pr}^{(3)}(\Delta)|_{\Delta=0} = 0 \]

where \( \text{pr}^{(3)}(\Delta) \) is the third prolongation of the vector field \( \Delta \), which is given explicitly in terms of \( \xi, \tau \) and \( \phi \) (cf. [44]). This procedure yields the determining equations. There are two cases to consider.

2.1 \( \epsilon = 0 \)

In this case using the MACSYMA package `symmgrp.max` we obtain the following system of ten determining equations

\[
\begin{align*}
\tau_u &= 0, \quad \tau_x = 0, \quad \xi_u = 0, \quad u \phi_{uu} + \beta \phi_{uu} = 0, \quad 3u^2 \phi_{uu} + \beta u \phi_u - \beta \phi = 0, \\
3u \phi_{xu} - 3u \xi_{xx} + \beta \phi_x &= 0, \quad 3u \phi_{xuu} + 2 \beta \phi_{xu} - \beta \xi_{xx} = 0, \\
\tau_t - 3 \xi_x u + \phi &= 0, \quad \phi_{xxx} u + (\alpha u - 2 \kappa) \phi_x - \phi_t = 0, \\
3u^2 \phi_{xuu} + \beta u \phi_{xx} + 2 \kappa \phi - u^2 \xi_{xxx} + (2 \alpha u^2 - 4 \kappa u) \xi_x + u \xi_t &= 0.
\end{align*}
\]

(2.6)

Next applying the `reduceall` algorithm in the MAPLE package `diffgrob2` to this system yields

\[
\begin{align*}
(2 + \beta) \xi_{xx} &= 0, \quad (2 + \beta) [\alpha u \xi_{xt} + \xi_{tt} - 2 \kappa \xi_{xt}] = 0, \\
\xi_u &= 0, \quad \tau_x = 0, \quad \tau_u = 0, \\
2 \alpha u \xi_x + 2 \kappa \xi_x + \xi_t - 2 \kappa \tau_t &= 0, \quad (2 + \beta) [2 \kappa \phi + (2 \alpha u^2 - 4 \kappa u) \xi_x + u \xi_t] = 0.
\end{align*}
\]

This is simple enough to solve; there is no need to do the full Kochin-Ritt algorithm in this case. The output shows that there are three special values of the parameters, namely \( \alpha = 0, \beta = -2 \) and \( \kappa = 0 \), and combinations thereof. It transpires that the special case \( \beta = -2 \) is purely an artefact. For the three special cases (a) \( \alpha = 0, \kappa \neq 0 \), (b) \( \alpha \neq 0, \kappa = 0 \) and (c) \( \alpha = \kappa = 0 \), applying the `reduceall` algorithm of `diffgrob2` to (2.6) yields

(a) \( \alpha = 0, \kappa \neq 0 \)

\[
\begin{align*}
\xi_{xx} &= 0, \quad \xi_{tt} - 2 \kappa \xi_{xt} = 0, \quad \xi_u = 0, \\
\tau_x &= 0, \quad 2 \kappa \xi_x + \xi_t - 2 \kappa \tau_t = 0, \quad \tau_u = 0, \\
2 \kappa \phi + (2 \alpha u^2 - 4 \kappa u) \xi_x + u \xi_t &= 0.
\end{align*}
\]

(b) \( \alpha \neq 0, \kappa = 0 \)

\[
\begin{align*}
2 \alpha u \xi_x + \xi_t &= 0, \quad \xi_{tt} = 0, \quad \xi_u = 0, \\
\tau_x &= 0, \quad \tau_{tt} = 0, \quad \tau_u = 0, \\
2 \kappa \phi + (2 \alpha u^2 - 4 \kappa u) \xi_x + u \xi_t &= 0.
\end{align*}
\]
(c) \( \alpha = \kappa = 0 \) \( \xi_{xx} = 0, \ \xi_{t} = 0, \ \xi_{u} = 0, \)
\( \tau_{x} = 0, \ \tau_{tt} = 0, \ \tau_{u} = 0, \)
\( \phi - 3u\xi_{x} + u\tau_{t} = 0. \)

Hence we obtain the following infinitesimals:

**Case 2.1(i)** \( \alpha \neq 0 \) and \( \kappa \neq 0 \)
\( \xi = 2\kappa c_{3}t + c_{1}, \ \tau = c_{3}t + c_{2}, \ \phi = -c_{3}u. \) (2.7)

**Case 2.1(ii)** \( \alpha = 0 \) and \( \kappa \neq 0 \)
\( \xi = c_{3}x + 2\kappa(c_{4} - c_{3})t + c_{1}, \ \tau = c_{4}t + c_{2}, \ \phi = (3c_{3} - c_{4})u. \) (2.8)

**Case 2.1(iii)** \( \alpha \neq 0 \) and \( \kappa = 0 \)
\( \xi = c_{1}, \ \tau = c_{3}t + c_{2}, \ \phi = -c_{3}u. \) (2.9)

**Case 2.1(iv)** \( \alpha = 0 \) and \( \kappa = 0 \)
\( \xi = c_{3}x + c_{1}, \ \tau = c_{4}t + c_{2}, \ \phi = (3c_{3} - c_{4})u, \) (2.10)

where \( c_{1}, c_{2}, c_{3}, c_{4} \) are arbitrary constants.

Solving the invariant surface condition (2.5) yields four different canonical reductions:

**Reduction 2.1** \( \alpha \) and \( \kappa \) arbitrary. If \( c_{3} = c_{4} = 0 \) in (2.7)–(2.10) we may set \( c_{1} = c \) and \( c_{2} = 1 \) and we obtain the travelling wave reduction
\( u(x, t) = w(z), \quad z = x - ct, \)

where \( w(z) \) satisfies
\( ww'''' + \alpha ww' + \beta w'w'' + (c - 2\kappa)w' = 0. \)

This can be integrated to yield
\( ww'' + \frac{1}{2}(\beta - 1)(w')^{2} + \frac{1}{2}\alpha w^{2} + (c - 2\kappa)w = A, \)

where \( A \) is an arbitrary constant. Multiplying this by \( w^{\beta - 2}w' \) and integrating again yields
\( (w')^{2} + \frac{\alpha}{1 + \beta}w^{2} + \frac{2(2\kappa - c)}{\beta}w = \frac{2A}{\beta - 1} + Bw^{1 - \beta}, \) (2.11)

where \( B \) is an arbitrary constant, for \( \beta \neq -1, 0, 1. \) Generally if \( \beta \neq -1, 0, 1, \) then (2.11) is solvable using quadratures, though for certain special values of the parameters there are explicit solutions. For example (i), if \( \beta = -2 \) or \( \beta = -3, \) then (2.11) is solvable in terms of Weierstrass or Jacobi elliptic functions,
respectively, (ii), if $B = 0$, then (2.11) is solvable in terms of trigonometric functions, and (iii), if $c = 2\kappa$ and $\beta = 3$, then $w(z)$ can be expressed in terms of trigonometric functions via the transformation $w(z) = v^{1/2}$.

In the special cases $\beta = -1, 0, 1$ we obtain the equations

$$(w')^2 + \alpha w^2 \ln w + 2(2\kappa - c)w = Bw^2 - A,$$

$$(w')^2 + \alpha w^2 + 2(2\kappa - c)\ln w = Bw - 2A,$$

$$(w')^2 + \alpha w^2 + 2(2\kappa - c)w = B - A \ln w,$$

respectively, with $A$ and $B$ arbitrary functions. If the coefficient of $\ln w$ in these equations is zero, then $w(z)$ is expressible in terms of elementary functions, otherwise in terms of quadratures.

**Reduction 2.2** $\alpha \neq 0$, $\kappa$ arbitrary. If $c_3 \neq 0$ in (2.8) and (2.10) we may set $c_3 = 1$, $c_1 = c$ and $c_2 = 0$, without loss of generality, and obtain the reduction

$$u(x, t) = w(z)t^{-1}, \quad z = x - c \ln t - 2\kappa t,$$

where $w(z)$ satisfies

$$ww''' + \beta w'w'' + \alpha w' + cw' + w = 0.$$

Also if $c_3 = 0$ and $c_4 \neq 0$ in (2.8) we may set $c_4 = 1$, $c_1 = c$ and $c_2 = 0$, without loss of generality, and obtain the same reduction (2.12).

**Reduction 2.3** $\alpha = 0$, $\kappa$ arbitrary. If $c_3 \neq 0$ and $c_4 \neq 0$ in (2.8) and (2.10), we may set $c_3 = m + \frac{1}{3}$, $c_4 = 1$ and $c_1 = c_2 = 0$, without loss of generality, and obtain the reduction

$$u(x, t) = w(z)e^{3mt}, \quad z = (x - 2\kappa t)e^{-m}t^{-1/3},$$

where $w(z)$ satisfies

$$ww''' + \beta w'w'' + (m + \frac{1}{3})zw' - 3mw = 0.$$

**Reduction 2.4** $\alpha = 0$, $\kappa$ arbitrary. If $c_3 \neq 0$ and $c_4 = 0$ in (2.8) and (2.10), we may set $c_3 = m$, $c_1 = 2\kappa$ and $c_2 = 1$, without loss of generality, and obtain the reduction

$$u(x, t) = w(z)e^{3mt}, \quad z = (x - 2\kappa t)e^{-mt},$$

where $w(z)$ satisfies

$$ww''' + \beta w'w'' + mzw' - 3mw = 0.$$
2.2 $\epsilon = 1$

In this case we obtain the following system of eleven determining equations:

\[
\begin{align*}
\tau_u &= 0, \quad \tau_x = 0, \quad \xi_u = 0, \quad \phi_{uu} = 0, \quad 2\phi_{xu} - \xi_{xx} = 0, \\
\beta(u\phi_u - \phi + \xi_t) &= 0, \quad \phi + ur_t - u\xi_x - \xi_t = 0, \\
3u\phi_{xu} + \phi_{tu} + \beta\phi_x - 3u\xi_{xx} - 2\xi_{xt} &= 0, \\
u\phi_{xxu} + \phi + ur_t - 3\xi_xu - \xi_t &= 0, \quad (2.13) \\
u\phi_{xxx} + \phi_{xxt} - \phi_t + (\alpha u - 2\kappa)\phi_x &= 0, \\
2\kappa\phi &= [2\kappa - (\alpha + 1)u]\xi_t.
\end{align*}
\]

As in the previous case, we apply the reduceall algorithm in the MAPLE package diffgrob2, to this system, which yields

\[
\begin{align*}
\xi_x &= 0, \quad (\alpha + 1)\xi_t = 0, \quad \xi_u = 0, \\
\tau_x &= 0, \quad (\alpha + 1)\tau_t = 0, \quad \tau_u = 0, \\
2\kappa\phi &= [2\kappa - (\alpha + 1)u]\xi_t.
\end{align*}
\]

This shows that there are two special values of the parameters, namely $\alpha = -1$ and $\kappa = 0$. For the three special cases (a) $\alpha = -1, \kappa \neq 0$, (b) $\alpha \neq -1, \kappa = 0$ and (c) $\alpha = -1, \kappa = 0$, applying the reduceall algorithm of diffgrob2 to (2.13) yields

\[
\begin{align*}
\text{(a)} \quad &\alpha = -1, \quad \kappa \neq 0 \\
&\xi_x = 0, \quad \xi_t = 0, \quad \xi_u = 0, \\
&\tau_x = 0, \quad \tau_t = 0, \quad \tau_u = 0, \\
&\phi = \xi_t. \\
\text{(b)} \quad &\alpha \neq -1, \quad \kappa = 0 \\
&\xi_x = 0, \quad \xi_t = 0, \quad \xi_u = 0, \\
&\tau_x = 0, \quad \tau_t = 0, \quad \tau_u = 0, \\
&\phi = -ur_t. \\
\text{(c)} \quad &\alpha = -1, \quad \kappa = 0 \\
&\xi_x = 0, \quad \xi_t = 0, \quad \xi_u = 0, \\
&\tau_x = 0, \quad \tau_t = 0, \quad \tau_u = 0, \\
&\phi - u\xi_t + ur_t = 0.
\end{align*}
\]

Hence we obtain the following infinitesimals:

**Case 2.2(i) $\alpha \neq -1, \kappa \neq 0$**

\[
\begin{align*}
\xi &= c_3t + c_1, \quad \tau = \frac{(1 + \alpha)c_3t}{2\kappa} + c_2, \quad \phi = c_3\left[1 - \frac{(1 + \alpha)u}{2\kappa}\right]. \quad (2.14)
\end{align*}
\]

**Case 2.2(ii) $\alpha = -1, \kappa \neq 0$**

\[
\begin{align*}
\xi &= c_3t + c_1, \quad \tau = c_2, \quad \phi = c_3. \quad (2.15)
\end{align*}
\]
Case 2.2(iii) $\alpha \neq -1, \kappa = 0$

$$\xi = c_1, \quad \tau = c_3 t + c_2, \quad \phi = -c_4 u.$$ \hfill (2.16)

Case 2.2(iv) $\alpha = -1, \kappa = 0$

$$\xi = c_3 t + c_1, \quad \tau = c_4 t + c_2, \quad \phi = c_3 - c_4 u,$$ \hfill (2.17)

where $c_1, c_2, c_3, c_4$ are arbitrary constants.

There are four canonical reductions.

**Reduction 2.5** $\alpha$ and $\kappa$ arbitrary. If in (2.14)–(2.17) $c_3 = c_4 = 0$, we may set $c_1 = c$ and $c_2 = 1$ without loss of generality. Thus we obtain the reduction

$$u(x, t) = w(z) + c, \quad z = x - ct,$$

where $w(z)$ satisfies

$$ww''' + \beta w' w'' + \alpha w w' = [2\kappa - (1 + \alpha)c]w'.$$

This can be integrated to yield

$$ww'' + \frac{1}{2}(\beta + 1)(w')^2 + \frac{1}{2}\alpha w^2 = [2\kappa - (1 + \alpha)c]w + A,$$

where $A$ is an arbitrary constant. Then multiplying through by $w^{\beta-2}w'$ and integrating again yields

$$(w')^2 + \frac{2\alpha w^2}{\beta + 1} = \frac{2[2\kappa - (1 + \alpha)c]w}{\beta} + \frac{2A}{\beta - 1} + Bw^{1-\beta},$$ \hfill (2.18)

provided that $\beta \neq -1, 0, -1$. Generally if $\beta \neq -1, 0, 1$, then (2.18) is solvable using quadratures, though for certain special values of the parameters, there are explicit solutions. For example (i), if $\beta = -2$ or $\beta = -3$, then (2.18) is solvable in terms of Weierstrass or Jacobi elliptic functions, respectively, (ii) if $B = 0$, then (2.18) is solvable in term of trigonometric functions, and (iii) if $(1 + \alpha)c = 2\kappa$ and $\beta = 3$, then $w(z)$ can be expressed in terms of trigonometric functions via the transformation $w(z) = v^{1/2}$.

In the special cases $\beta = -1, 0, 1$ we obtain the following equations,

$$(w')^2 + 2\alpha w^2 \ln w = -2[2\kappa - (1 + \alpha)c]w - A + Bw^2,$$
$$ (w')^2 + 2\alpha w^2 = -2[2\kappa - (1 + \alpha)c]w \ln w - 2A + Bw^2,$$
$$ (w')^2 + 2\alpha w^2 = -2[2\kappa - (1 + \alpha)c]w + 2A \ln w + Bw^2,$$

respectively, where $A$ and $B$ are arbitrary constants. If the coefficient of $\ln w$ in these equations is zero, then $w(z)$ is expressible in terms of elementary functions, otherwise in terms of quadratures.
Reduction 2.6 \( \alpha \neq -1, \kappa \) arbitrary. If \( c_3 \neq 0 \) in (2.14), we may set \( c_3 = 1, c_2 = 0 \) and \( c_1 = 2\kappa c/(1 + \alpha) \), without loss of generality. Thus we obtain the reduction

\[
u(x,t) = \frac{w(z) + c}{t} + \frac{2\kappa}{1 + \alpha}, \quad z = x - \frac{2\kappa t}{1 + \alpha} - \frac{c}{c} \ln t,
\]

(2.19)

where \( w(z) \) satisfies

\[
w'''' + \beta w'''w' - w''' + \alpha w + (\alpha + 1)cw' + w + c = 0.
\]

(2.20)

If \( c_3 \neq 0 \) in (2.16) we may set \( c_3 = 1, c_1 = c \) and \( c_2 = 0 \) to obtain the reduction (2.19) with \( \kappa = 0 \).

Reduction 2.7 \( \alpha = -1, \kappa \neq 0 \). If \( c_3 \neq 0 \) in (2.15) then we set \( c_3 = m, c_1 = 0 \) and \( c_2 = 1 \), without loss of generality. Thus we obtain the reduction

\[
u(x,t) = w(z) + mt, \quad z = x - \frac{1}{2}mt^2,
\]

(2.21)

where \( w(z) \) satisfies

\[
w'''' + \beta w'''w' - w''' - 2\kappa w' - m = 0,
\]

which may be integrated to yield

\[
w''' + \frac{1}{2}(\beta - 1)(w')^2 - \frac{1}{2}w^2 - 2\kappa w - mz = A,
\]

(2.22)

where \( A \) is an arbitrary constant.

Reduction 2.8 \( \alpha = -1, \kappa = 0 \). If \( c_3 \neq 0 \) and \( c_4 \neq 0 \) in (2.17) we may set \( c_3 = m, c_4 = 1, c_1 = c \) and \( c_2 = 0 \), without loss of generality. Thus we obtain the reduction

\[
u(x,t) = \frac{w(z) + c}{t} + m, \quad z = x - mt - c \ln t,
\]

(2.23)

where \( w(z) \) satisfies

\[
w'''' + \beta w'''w' - w''' - 2\kappa w' + w + c = 0.
\]

(2.24)

3 Nonclassical symmetries (\( \tau \neq 0 \))

In the nonclassical method one requires only the subset of \( S_\Delta \) given by

\[
S_{\Delta,\psi} = \{ u(x,t) : \Delta(u) = 0, \psi(u) = 0 \},
\]

(3.1)

where \( S_\Delta \) is defined in (2.2) and \( \psi = 0 \) is the invariant surface condition (2.5), to be invariant under the transformation (2.1). The usual method of applying the nonclassical method (e.g. as described in [35]), involves applying the
prolongation \( \mathrm{pr}^{(3)} v \) to the system composed of (1.1) and the invariant surface condition (2.5) and requiring that the resulting expressions vanish for \( u \in S_{\Delta, \psi} \), i.e.
\[
\mathrm{pr}^{(3)} v(\Delta)|_{\Delta=0, \psi=0} = 0, \quad \mathrm{pr}^{(1)} v(\psi)|_{\Delta=0, \psi=0} = 0. \tag{3.2}
\]
It can well known that the latter vanishes identically when \( \psi = 0 \) without imposing any conditions upon \( \xi, \tau \) and \( \phi \). To apply the method in practice we advocate the algorithm described in [18] for calculating the determining equations, which avoids difficulties arising from using differential consequences of the invariant surface condition (2.5).

In the canonical case when \( \tau \neq 0 \) we set \( \tau = 1 \) without loss of generality. We proceed by eliminating \( u_t \) and \( u_{xxx} \) in (1.1) using the invariant surface condition (2.5) which yields
\[
\begin{align*}
\epsilon \xi_{xxx} - u_{xxx} + 3\epsilon \xi_{xx} u_{xx} - \beta u_{xx} - \epsilon \phi_u u_{xx} + 2\epsilon \xi_x u_{xx} \\
+ \epsilon \xi_{uu} u^3 - \epsilon \phi_u u^2 + 2\epsilon \xi_{uu} u^2 - \alpha \phi u - 2\epsilon \phi_{xx} u + 2\kappa u_x \\
+ \epsilon \xi_{xxx} u_x - \epsilon \phi_{xxx} x - \xi_{xxx} - 0. \tag{3.3}
\end{align*}
\]
We note that this equation now involves the infinitesimals \( \xi \) and \( \phi \) that are to be determined. Then we apply the classical Lie algorithm to (3.3) using the third prolongation \( \mathrm{pr}^{(3)} v \) and eliminating \( u_{xxx} \) using (3.3). It should be noted that the coefficient of \( u_{xxx} \) is \( (\xi - \epsilon u) \). Therefore, if this is zero the removal of \( u_{xxx} \) using (3.3) is invalid and so the next highest derivative term, \( u_{xx} \), should be used instead. We note again that this has a coefficient, \( \beta - 3 \), and so that in the case \( \xi = u \) one needs to calculate the determining equations for the cases \( \beta \neq 3 \) and \( \beta = 3 \) separately. Continuing in this fashion, there is a cascade of cases to be considered. In the remainder of this section, we consider these cases in turn. First, however, we discuss the case given by \( \epsilon = 0 \).

### 3.1 \( \epsilon = 0 \)

The first determining equation gives \( \xi_u = 0 \), and substituting this into the other seven determining equations yields
\[
\begin{align*}
\phi_{uu} u + \beta \phi_u & = 0, \quad 3\phi_{xx} u + 2\beta \phi_x - \beta \xi_x = 0, \\
3\phi_{uu} u^2 + \beta \phi_u u - \beta \phi & = 0, \quad 3\phi_{xx} u - 3\xi_{xx} u + \beta \phi_x = 0, \\
\phi_t u - \phi_{xxx} u^2 - \alpha \phi_x u^2 + 2\kappa \phi_x u + 3\xi_x u - \phi^2 & = 0, \\
3\phi_{xx} u^2 - \xi_{xxx} u^2 + 2\alpha \xi_x u^2 + \beta \phi_x u - 4\xi_k u + 3\xi_x u \\
+ \xi_t u + 2\kappa \phi - \xi \phi & = 0. \tag{3.4}
\end{align*}
\]
It is quite straightforward to solve these equations and so we obtain the following infinitesimals: (a), if \( \alpha \neq 0 \)
\[
\begin{align*}
(\text{a}) \quad \xi = 2\kappa + \frac{c_1}{t + c_2}, & \quad \phi = \frac{-u}{t + c_2}, \\
(ii) \quad \xi = c_1, & \quad \phi = 0,
\end{align*}
\]
and (b), if \( \alpha = 0 \)

\[
(i) \quad \xi = \frac{(c_1 + 1)x + 2\kappa(2c_1 - 1)t + c_2}{3(c_1t + c_3)}, \quad \phi = \frac{u}{c_1t + c_3},
\]

\[
(ii) \quad \xi = \frac{x + 4\kappa t + c_1}{3t + c_2}, \quad \phi = 0.
\]

These are all equivalent to classical infinitesimals. Hence in this case there are no new nonclassical symmetries.

### 3.2 \( \epsilon = 1 \)

As discussed in the preamble to this section, we must consider, in addition to the general case of the determining equations, each of the singular cases of the determining equations.

**Case 3.2.1 \( \xi \neq u \).** We can remove factors of \((\xi - u)\) from the determining equations, and we have then that \( \xi_u = 0 \). Reducing the remaining eight determining equations with respect to this, only the last six are non-zero:

\[
3\phi_{uu}u^2 - 6\xi\phi_{uu}u + \beta\phi_uu + 3\xi^2\phi_{uu} - \beta\xi\phi_u - \beta\phi + \beta\xi_x + \beta\xi_t = 0,
\]

\[
\phi_{uu}u - \xi\phi_{uu}u + \beta\phi_{uu} = 0,
\]

\[
\xi_x\phi_u - \beta\xi_x + \phi_{uu}u + \beta\phi_xu - \xi\phi_{uu}u - 5\xi\phi_{xx}u + 4\xi\phi_{tx}u + \phi_{tu}u
\]

\[\quad - \phi\phi_u + \xi_t\phi_u - \xi\phi_{tu} - \xi^2\phi_{xx} + 3\phi_{xy}u^2 - 3\xi\phi_{xx}u^2 - 2\xi^2u - 2\xi_xu
\]

\[\quad + 2\xi^2\phi_{xx} + 2" \xi - 2\xi\phi_x + 2\xi\phi_t = 0,
\]

\[
2\kappa\phi_x - \phi_tu + \alpha\phi_xu^2 - 2\kappa\phi_xu + \alpha\phi_xu + 2\phi_{uu}\phi_xu + \phi_{xxuu}u - 2\xi\phi_{uu}u
\]

\[\quad - \xi_x\phi_xu - \xi\phi_{xxu} + 3\xi\phi_x + 2\xi_x\phi + \xi\phi_{xx}u - \xi\phi_{xx}u
\]

\[\quad - \xi_t\phi + \phi^2 - \phi_{xx} + \phi_{xxuu}u^2 + \phi_{xxuu}u - \xi\phi_{xx}u + 3\xi\phi_{xx}u + \xi\phi_t = 0,
\]

\[
2\beta\phi_{xx}u - \xi_x\phi_{uu}u - \beta\xi_{xx}u + 2\phi_{uu}\phi_{uu}u + \beta\xi_{xx}u - 5\xi\phi_{xxu}u - \xi\phi_{uuuu}
\]

\[\quad + \phi_{uuu}u + \phi_{uu}u - \phi_{uu}u + \xi_t\phi_{uu} - \xi\phi_{tuu} + 3\phi_{xxuu}u^2 + 2\xi^2\phi_{xxuu}
\]

\[\quad - 2\phi\phi_{uu} + 2\xi\phi_{uu}u - 2\xi\phi_{xx} = 0,
\]

\[
4\xi_{xx}\phi_{uu} - 2\phi_{uu}\phi_xu - 2\phi_{xx}u - \xi\phi_{xx}u - 2\kappa\phi - 2\phi_{xxuu}u - 2\phi_{xxuu}u + \xi\phi
\]

\[\quad + \xi_{xxu}u + \xi_{xxuu}u^2 + \xi\phi_{xx} - 3\phi_{xxuu}u^2 + 2\phi_{xxu} - \xi\phi_{xx} + 2\xi^2\phi - 3\phi_{xxu}
\]

\[\quad - 2\xi\phi_{xx} - \alpha\xi_u + \alpha\phi + \xi\phi_{xx}u + 2\xi\phi_{xx}u - 3\xi\phi_{xx}u - \xi\phi_{uu}u + 2\xi\phi_{uu}\phi_x
\]

\[\quad + \xi\xi_{xx} + 4\xi\phi_{xxuu}u + \beta\phi_{xx} + 2\xi\phi_{xxuu} + 2\phi_{uu}\phi_{xxu} - 2\xi\phi_{xxu} - 2\xi\phi_{xxu} = \alpha\phi_{xx}u
\]

\[\quad + \alpha\phi_{xx}u + 2\xi\phi_{xxu} + 2\xi\phi - \xi_tu - \xi\phi_{xx}u - 2\phi\phi_{xxu} = 0.
\]

Reducing the fifth of these equations with respect to the fourth yields

\[
(\beta - 3) [(u - \xi)\phi_u - \phi + \xi\phi_x + \xi_s] = 0.
\]
If $\beta = 3$, then one easily finds via another route that the expression in the second bracket is necessarily zero. The equation for $\phi$ can be solved to give

$$\phi = F(x,t)(u - \xi) + \xi_x + \xi_t.$$  

When this is substituted into the remaining equations we can then take coefficients of powers of $u$ to be zero, and our problem is then easily solved. As in the $\epsilon = 0$ case discussed in §3.1 above, it is quite straightforward to solve the resulting equations. The complete solution set is

(a), if $\alpha \neq -1$

(i) $\xi = c_1, \ \phi = 0,$

(ii) $\xi = \frac{2\kappa}{1 + \alpha} - \frac{c_1}{t + c_2}, \ \phi = \frac{2\kappa - (1 + \alpha)u}{(1 + \alpha)(t + c_2)}.$  

(3.5)

(b), if $\alpha = -1$

$$\xi = c_1 t + c_2, \ \phi = c_1,$$  

(3.6)

(c), if $\alpha = -1$ and $\kappa = 0$

$$\xi = c_1 - \frac{c_3}{t + c_2}, \ \phi = \frac{c_1 - u}{t + c_2},$$  

(3.7)

(d), if $\beta = -1$ and $\alpha = 0$

$$\xi = c_1 x - 2c_1 \kappa t + c_2, \ \phi = 3c_1 u - 2c_1^2 x + 4c_1^2 \kappa t - 2c_1 c_2 - 2c_1 \kappa, \ \beta \neq 0.$$  

(3.8)

The infinitesimals (3.5)–(3.7) give rise to classical reductions, but (3.8) gives the following new nonclassical reduction.

**Reduction 3.1** If in (3.8), we set $c_1 \neq 0$ and $c_2 = 0$, without loss of generality, then we obtain

$$u(x,t) = w(z) \exp(3c_1 t) + c_1 z \exp(c_1 t) + 2\kappa, \ z = (x - 2\kappa t - 2\kappa/c_1) \exp(-c_1 t),$$

where $w(z)$ satisfies

$$ww'''' - w'w'' + c_1 zw' - 3c_1 w = 0.$$  

**Case 3.2.2** $\xi = u$, $\beta \neq 3$, $\beta \neq 1$. We generate five determining equations, the first of which is $\phi_{uu} = 0$. Thus $\phi$ is a linear function of $u$, and substituting this into the remaining four determining equations, we take coefficients of powers of $u$ to be zero. These equations are easily solved to give $\phi = 0$ provided that $\kappa = 0$ and $\alpha = -1$. The invariant surface condition and (1.1) are then solved to give the simple exact solution

$$u(x,t) = \frac{x + c_1}{t + c_2},$$  

15
where $c_1$ and $c_2$ are arbitrary constants.

**Case 3.2.3** $\xi = u$, $\beta = 1$. We consider here the case $\phi_{uu} \neq 0$, since taking $\phi_{uu} = 0$ yields the same solution as in Case 3.2.2 above. In this instance the remaining four determining equations are

$$
12\kappa - 2\phi_{xuu}u - 6\alpha u - 6u - 2\phi_{uu}u - 3\phi_{uu} - 4\phi_{xu} - 2\phi_{tuu} = 0,
$$

$$
\phi_{xu}\phi_{xx}u - \phi_{u}\phi_{xxx}u - \alpha\phi_{u}\phi_{x}u - \phi_{xu}u - 2\phi_{x}\phi_{xx} + \phi_{uu}\phi_{xx}
$$

$$
+ \phi_{tu}\phi_{xx} - 2\phi_{u}\phi_{xu}\phi_{x} + 2\phi_{xu}\phi_{x} + 2\phi_{x} - \phi_{xuu}u - \phi_{uu} - \phi_{xuu}u
$$

$$
- \phi_{xxt}\phi_{u} + \phi_{u}\phi_{u} - \phi_{tu} = 0,
$$

$$
\phi_{u}\phi_{xuu}u + 4\alpha\phi_{u}u - \phi_{uu}^2 - \phi_{tu}\phi_{uu} + \phi_{uu}\phi_{uu} + 6\phi - 4\phi_{xx}
$$

$$
+ \phi_{tuu}\phi_{u} - 4\phi_{xuu}u + 4\phi_{u} - 2\phi_{uu}\phi_{x} - 4\phi_{xuu}u + 2\phi_{uu}^2 - \phi_{uu}^3 - 8\phi_{uu}
$$

$$
- 2\phi - \phi_{xx}\phi_{uu}u - 4\phi_{xtu} = 0,
$$

$$
\phi_{u}\phi_{uu}u + \alpha\phi_{xx}u^2 - 2\phi_{u}\phi_{uu}\phi_{x} - 2\phi_{uu}\phi_{xuu} - 2\phi_{uu}\phi_{xxu}u + \alpha\phi_{tu}u
$$

$$
+ 2\phi_{xxu}\phi_{uu} - 2\kappa\phi_{uu}u + 2\phi_{tu}\phi_{uu} + \alpha\phi_{uu}u + 2\phi_{xxuu}u - 2\phi_{tuu}
$$

$$
+ 2\phi_{uu}^2 - 3\phi_{x}\phi_{xx} + 2\phi_{xxu}\phi_{x} - 2\phi_{xtu}\phi_{u} + 4\phi_{uu} - 2\phi_{tu}
$$

$$
- \alpha\phi_{uu} - 2\kappa\phi_{xu}u + \phi_{tu}u + \phi_{xu}u^2 + 2\phi_{xxt} - 2\phi_{t} = 0.
$$

Using the procedures in the package *diffgrob2* with an ordering designed to eliminate first derivatives with respect to $t$, then derivatives with respect to $x$, one can obtain several equations for derivatives of $\phi$ with respect to $u$ only. One can then continue to produce lower order and lower degree equations in the $u$-derivatives of $\phi$, using repeated cross-differentiation and reductions. For example, the “Direct Search” procedure in the *diffgrob2* manual, [38] may be used. This process suffers from expression swell. No termination of this process was observed by us within the computer memory available, and the expressions obtained contained thousands of summands! One of three results appear likely. Firstly, the process terminates with the highest derivative term being $\phi$ itself, yielding $\phi$ to be a function of $u$ alone (note that $x$ and $t$ do not appear explicitly in any of the determining equations). Inserting this into the determining equations, one must have that $\phi$ is constant, a contradiction to our standing assumption in this subcase. Secondly, the process may terminate with an inconsistency, and thirdly, the process may terminate but with such a large expression that the result is useless.

**Case 3.2.4** $\xi = u$, $\beta = 3$, $\phi_{u} \neq 0$. Four determining equations were obtained, the first of which is $\phi_{uu} = 0$, so we substitute $\phi = F(x, t)u + G(x, t)$ into the remaining three and require $F(x, t) \neq 0$. We find that there are no such solutions.

**Case 3.2.5** $\xi = u$, $\beta = 3$, $\phi_{u} = 0$ and not both $\kappa$ and $\alpha + 1$ are zero. One determining equation was obtained which was a polynomial in $u$ of degree two whose coefficients are functions of $x, t$ only, so the coefficients of powers of $u$
must be zero. These equations were easily simplified using the procedures in `diffgrob2` to yield,

\[ \kappa \neq 0, \quad \alpha = -1, \quad \phi = 0, \]  
\[ \kappa \neq 0, \quad \alpha = -1, \quad \phi = \frac{-2\kappa}{t + c_1}, \]  
\[ \kappa \text{ arbitrary}, \quad \alpha \neq -1, \quad \phi = c_1 \exp(\zeta) + c_2 \exp(-\zeta), \]

\[ \zeta = i\sqrt{\alpha} \left( x - \frac{2\kappa t}{1 + \alpha} \right). \]  

In (3.9) if we solve (1.1) and the invariant surface condition as a system of equations we find that the only solution is \( u(x, t) = c \), a constant.

In (3.10) we can solve (1.1) and the invariant surface condition to give the exact (canonical) solution

\[ u(x, t) = -2\kappa + x/t, \]

which cannot be realised by any of the previously found reductions, though it would not appear to be a particularly interesting solution. It is interesting to note that performing the `KolRitt` algorithm of `diffgrob2` on the system comprising the original equation with the invariant surface condition led to a simple calculation for \( u \). By contrast, the usual procedure of solving the invariant surface condition using the method of characteristics and inserting the result into the original equation to obtain the reduction was considerably more difficult due to the implicit nature of the reduction.

In (3.11) we can again solve our problem to yield the exact (canonical) solution

\[ u(x, t) = \frac{-2\kappa}{1 + \alpha} \pm (c_0 + c_1 e^\kappa + c_2 e^{-\zeta})^{1/2}, \quad \zeta = i\sqrt{\alpha} \left( x - \frac{2\kappa t}{1 + \alpha} \right), \]

which is a special case of the travelling wave reduction 2.5.

Case 3.2.6 \( \xi = u, \beta = 3, \phi_u = 0, \kappa = 0, \alpha = -1 \). We are left simply with the determining equation \( \phi_{xx} - \phi = 0 \), which produces the following infinitesimal,

\[ \phi = g(t)e^x + h(t)e^{-x}, \]  

where \( g \) and \( h \) are arbitrary functions. Hence we have to solve the invariant surface condition

\[ uu_x + u_t = g(t)e^x + h(t)e^{-x}. \]  

It is straightforward to show that every solution of this equation is also a solution of (1.1).
4 Nonclassical ($\tau = 0$) and Direct Methods

In the canonical case of the nonclassical method when $\tau = 0$ we set $\xi = 1$ without loss of generality. We proceed by eliminating $u_x, u_{xx}, u_{xxx}$ and $u_{xxt}$ in (1.1) using the invariant surface condition (2.5) which yields

\[ u_t - \epsilon \phi_{uu} u_t - \epsilon \phi_{xx} u_t - \phi_x u - \phi_{uu} u - 2 \phi_{xu} u - 2 \phi_{uu} u u - \phi_{xu} u - 2 \phi_{uu} u u - \phi_{xu} u - 2 \phi_{uu} u u = 0, \]  

(4.1)

which involves the infinitesimal $\phi$ that is to be determined. As in the $\tau \neq 0$ case we apply the classical Lie algorithm to this equation using the first prolongation $pr^{(1)} v$ and eliminate $u_t$ using (4.1).

The equivalent approach using the direct method of Clarkson and Kruskal [15] is to consider the ansatz $u = U(x, t, w(t))$ and require that the result be ordinary differential equation for $w(t)$; see also [12,36]. It is straightforward to show that this yields the equivalent reductions.

**Case 4.1 $\epsilon = 0$.** The nonclassical method generates a single equation of 25 terms, without any singular solutions. Since this is difficult to solve explicitly, we seek polynomial solutions in $u$.

**Ansatz 1.** $\phi = F(x, t)$. In this case we obtain the following three exact solutions for (1.1) with $\epsilon = 0$:

\[ u(x, t) = \mu_2 [x - (2 \kappa - \beta \mu)t]^2 + \mu_0, \]

(4.2)

where $\mu_2$ and $\mu_0$ are arbitrary constants, provided that $\alpha = 0$,

\[ u(x, t) = \frac{(x - 2 \kappa t)^3}{12t} + \mu(x - 2 \kappa t) + \delta^{1/2}, \]

(4.3)

where $\delta$ is an arbitrary constant, provided that $\alpha = 0$ and $\beta = -1$, and

\[ u(x, t) = - \frac{x - 2 \kappa t}{\alpha t}, \]

(4.4)

provided that $\alpha \neq 0$.

**Ansatz 2.** $\phi = F(x, t) u^2 + G(x, t) u + H(x, t)$. In this case we obtain the following three exact solutions for (1.1) with $\epsilon = 0$:

\[ u(x, t) = \frac{1}{2} \sqrt{\alpha}(x - 2 \kappa t)]}, \]

(4.5)

where $\mu$ is an arbitrary constant, provided that $\beta = -3$,

\[ u(x, t) = A \exp\{\mu(x - 2 \kappa t)\}, \]

(4.6)

\[ u(x, t) = A \sec\{\frac{1}{2} \sqrt{\alpha}(x - 2 \kappa t)\}, \]

(4.7)

provided that $\beta \neq -1$, and
provided that $\beta = -3$.

**Case 4.2 $\epsilon = 1$.** In this case the nonclassical method generates a single equation of 150 terms, which has a singular solution if and only if

$$\phi \phi_u + \phi_x - u - 2\kappa/\beta = 0,$$

provided that $\alpha - \beta - 1 = 0$. We again seek polynomial solutions of $\phi$ using one ansatz.

**Ansatz 1.** $\phi = F(x,t)$. In this case we obtain three following three exact solutions for (1.1) with $\epsilon = 0$:

$$u(x,t) = \mu_2 [x - (2\kappa - \beta \mu)t]^2 + \mu_1 [x - (2\kappa - \beta \mu)t] + \mu_0, \quad (4.8)$$

where $\mu_2$, $\mu_1$ and $\mu_0$ are arbitrary constants, provided that $\alpha = 0$,

$$u(x,t) = \frac{(x - 2\kappa t)^3}{12t} + \frac{\mu_2(x - 2\kappa t)^2}{t} + \left( \frac{1 + 8\mu_2^2}{2t} + \mu_1 \right) (x - 2\kappa t) + \delta t^{1/2}$$

$$+ \frac{\mu_2(6 + 16\mu_2^2)}{3t} + 2\kappa + \mu_1 + \mu_2, \quad (4.9)$$

where $\mu_2$, $\mu_1$ and $\delta$ are arbitrary constants, provided that $\alpha = 0$ and $\beta = -1$, and

$$u(x,t) = -\frac{x - 2\kappa t}{\alpha t}, \quad (4.10)$$

provided that $\alpha \neq 0$.

**5 Discussion**

In this paper we have classified symmetry reductions of the nonlinear third order partial differential equation (1.1), which contains three special cases that have attracted considerable interest recently, using the classical Lie method and the nonclassical method due to Bluman and Cole [8]. The use of the MAPLE package `diffgrob2` was crucial in this classification procedure. In the classical case it identified the special cases of the parameters for which additional symmetries might occur whilst in the nonclassical case, the use of `diffgrob2` rendered a daunting calculation tractable and thus solvable.

In their recent paper, Gilson and Pickering [32] discuss the application of the Painlevé tests for integrability due to Ablowitz, Ramani and Segur [3,4] and Weiss, Tabor and Carnevale [57] to equation (1.1). In particular, they investigate the integrability of the ordinary differential equations arising from the travelling-wave reductions 2.1 and 2.5 above. It would be interesting to investigate the integrability of some of the ordinary differential equations arising from the other reductions derived in this paper using standard Painlevé analysis, “weak Painlevé analysis” [49,50] and “perturbative Painlevé analysis” [20],
though we shall not pursue this further here. Marinakis and Bountis [41] have also applied Painlevé analysis to the FFCH equation (1.4); an interesting aspect of their analysis is the use of a hodograph transformation. To conclude we remark that the RH equation (1.3) is a quasilinear partial differential equation of the form discussed by Clarkson, Fokas and Ablowitz [14]. It is routine to apply their algorithm, which involves a hodograph transformation, for applying the Painlevé PDE test to such quasilinear partial differential equations and show that (1.3) does not satisfy the necessary conditions to be solvable by inverse scattering.

Acknowledgments

We thank the editors for inviting us to write an article. We also thank the Program in Applied Mathematics, University of Colorado at Boulder, for their hospitality during our visit whilst some of this work was done. The research of PAC and ELM is supported by EPSRC (grant GR/H39420) and that of TJP by an EPSRC Postgraduate Research Studentship, which are gratefully acknowledged.

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