Geometric phase: an indicator of entanglement

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Received 9 February 2012
Published online 29 June 2012 – © EDP Sciences, Società Italiana di Fisica, Springer-Verlag 2012

Abstract. Using a kinematic approach we show that the non-adiabatic, non-cyclic, geometric phase corresponding to the radiation emitted by a three level cascade system provides a sensitive diagnostic tool for determining the entanglement properties of the two modes of radiation. The nonunitary, noncyclic path in the state space may be realized through the same control parameters which control the purity/mixedness and entanglement. We show analytically that the geometric phase is related to concurrence in certain region of the parameter space. We further show that the rate of change of the geometric phase reveals its resilience to fluctuations only for pure Bell type states. Lastly, the derivative of the geometric phase carries information on both purity/mixedness and entanglement/separability.

1 Introduction

Geometric phase\(^1\) and topological phases have provided impetus\(^4\) to quantum information processing. Of the two, geometric phase is easier for implementation. Both these approaches may be designed to realize fault tolerance\(^5\) in addition to resilience to decoherence\(^6,7\). Simple quantum gates using geometric phase have been demonstrated experimentally in the nuclear magnetic resonance set up\(^8\). Geometric phase has been studied in its entirety encompassing various aspects of evolution including adiabatic\(^9\), cyclic, unitary and subsequently the generalization to non-adiabatic\(^10,11\), non-cyclic\(^12–14\), and non-unitary\(^13–15\) evolution using either a dynamic or a kinematic approach\(^15,16\). Non-unitary evolution in the context of open systems has also been reported\(^17–22\). These studies have been carried out both for pure and mixed states\(^23–25\). On the other hand an operational definition of the geometric phase has been given by Sjöqvist et al.\(^26\). There have been further studies on the relationship of the geometric phase of an entangled system and its sub-system\(^27–31\). Recently, a proposal for studying the Pancharatnam phase and its relation to non-local quantum correlations has been suggested\(^32\) in the two-photon interferometric set up.

Holonomic quantum computation and entanglement require the evolution of qubits in parameter space by the control of parameters which are physically feasible. Both geometric phase and entanglement are important tools in quantum information processing. Atom photon interaction provides common ground for exploring the relationship between geometric phase and entanglement with the added advantage of the existence of control parameters for manipulating the two photon states. In this paper we study the non-adiabatic, non-cyclic, non-unitary evolution of the geometric phase of the two-photon state corresponding to the two modes emitted by a three level cascade system interacting with two driving fields\(^33\). The two photon state is in general a mixed state, however it may also be prepared in a pure state by proper control. We consider the evolution in the state space by varying the control parameters namely, the driving field strength and detuning.

2 The model

The scheme considered here corresponds to a three level cascade system interacting with two coherent fields which address the only two allowed dipole transitions \(i \leftrightarrow i+1\), \(i = 1, 2\) with energy separation given by \(\omega_L\). Two counter propagating (Doppler free geometry) driving fields of nearly equal frequencies \(\omega_{L1}\) and \(\omega_{L2}\) and respective strengths \(\Omega_1\) and \(\Omega_2\) are resonant with these two transitions. The decay constants of the energy levels \(|3\rangle\) and \(|2\rangle\) are indicated by \(\Gamma_3\) and \(\Gamma_2\) respectively. The parameters \(\Delta_1, \Delta_2\) refer to the detunings of the driving fields. This scheme may be realized, for example, in \(^{87}\text{Rb}\) vapor with the corresponding energy levels \(5S_{1/2}, 5P_{3/2}\) and \(5D_{5/2}\) and has been used by Banacloche et al.\(^34\) for studying electromagnetically induced transparency. The parameters used for the numerical computation correspond to this scheme which we label as scheme I. The driven three
This class of two photon states is in a smaller sub-space photon state in the Fock space and its density matrix is the atomic indices. It has been further shown \[33\], that the

density matrix equation \[36\],

\[
\rho^{n}(t) = \frac{1}{N} \left( \sum_{i} |\alpha_{i}^{0}|^{2} \sum_{n} |\alpha_{i}^{0}|^{2} \sum_{n} |\alpha_{i}^{0}|^{2} \sum_{n} |\alpha_{i}^{0}|^{2} \right)
\]

where \(N\) is the normalization factor. The solution for the complex coefficient, \(\alpha_{i}^{n}(t)\) may be obtained by solving the density matrix equation \[36\],

\[
\text{i} \hbar \dot{\rho}^{A} = [H, \rho^{A}] - \frac{\text{i} \hbar}{2} \{\Gamma, \rho^{A}\}
\]

where \(\Gamma\) is the relaxation matrix which incorporates the phenomenological decay constants corresponding to the finite lifetimes of the atomic system and \(H\) is the Hamiltonian given by

\[
H = \frac{1}{2} \left( \omega_{1} \sigma_{i}^{+} + \omega_{2} \sigma_{2}^{+} \right) + \hbar \Omega_{1} (e^{-\text{i} \omega_{0} t} \sigma_{i}^{+} + h.c.) + \hbar \Omega_{2} (e^{-\text{i} \omega_{0} t} \sigma_{2}^{+} + h.c.)
\]

in the interaction picture and in the rotating wave approximation. The Rabi frequency is given by \(\Omega_{i} = -\frac{1}{\pi} \mu_{i} E_{i};\ i = 1, 2\), corresponding to the two driving field strengths. \(\sigma_{i}^{+} = \{i+1\} \langle i\rangle;\ \sigma_{i}^{-} = \{i\} \langle i+1\rangle;\ i = 1, 2\) and \(\sigma_{2}^{+} = \{i+1\} \langle i+1\rangle - \{i\} \langle i\rangle\) are the atomic transition operators.

3 Geometric phase

The Pancratenam relative phase for a pure state is defined as

\[
\alpha(t) = \text{Arg} (\langle \Psi(0) | \Psi(t) \rangle).
\]

The two photon state under consideration, on the other hand is in general a mixed state. We follow here the kine-
matic approach of Tong et al. \[15\] for determining the geo-
metric phase for non-cyclic, non-unitary, non-adiabatic evolution of mixed states. We briefly recapitulate their

procedure here: a general mixed state density matrix \(\rho(t)\) is subject to purification by the introduction of an ancilla

as

\[
|\Psi(t)\rangle = \sum_{k} \sqrt{\lambda_{k}(t)} |\phi_{k}(t)\rangle \otimes |\delta_{k}\rangle; t \in [0, \tau].
\]

The Pancratenam relative phase given in equation (5) reduces to the geometric phase when the parallel transport condition is satisfied. Tong et al. \[15\] impose the parallel transport condition for the mixed state to be \(\langle \phi_{k}(t) | d/dt | \phi_{k}(t) \rangle = 0, k = 1 \ldots N\) corresponding to the \(N\) eigenstates. In the scheme under consideration \(N = 1, 2, 3\).

The geometric phase for the mixed state, \(\rho^{m}(t)\), satisfying the parallel transport conditions assumes the form

\[
\gamma_{g}(\tau) = \text{Arg} \left[ \sum_{k} \sqrt{\lambda_{k}(\tau) \lambda_{k}(0)} \langle \phi_{k}(0) | \phi_{k}(\tau) \rangle \times e^{-\int_{t_{0}}^{t} \langle \phi_{k}(\tau') | d/dt | \phi_{k}(\tau') \rangle dt'} \right]
\]
where \( \lambda_i(\tau) \) are the eigenvalues and \( \phi_i(\tau) \) are the corresponding eigenvectors. Here the evolution is considered over the parameter space consisting of \( \{\Delta, X\} \) defined in the next section.

4 Relation between geometric phase and entanglement

In order to get a physical understanding of the dependence of the two photon state on the free parameters, we consider first, the density matrix of the ideal system (scheme II) near two photon resonance. The first order perturbation of the two photon state in the parameter \( \Delta \) in the neighborhood of \( \Delta = 0 \) has the simple form

\[
\rho = \begin{pmatrix}
S^2 & \Delta \mathcal{C}^2 & -SC(1-i\gamma_21)\Delta \\
\Delta \mathcal{C}^2 & 0 & -\Delta \mathcal{C}S \\
-SC(1+i\gamma_21)\Delta & -\Delta \mathcal{C}S & \mathcal{C}^2
\end{pmatrix}
\]

where \( S = \sin(X) \) and \( \mathcal{C} = \cos(X) \). We have defined the dimensionless parameters as \( X = \tan^{-1}(\Omega_1/\Omega_2), \Delta = \Delta/\sqrt{\Omega_1^2 + \Omega_2^2} \) and \( \gamma_21 = \Gamma_2/2\sqrt{\Omega_1^2 + \Omega_2^2} \). The density matrix is first order in the parameter \( \Delta \). The reason for choosing the state around this point will become clear shortly. Consider the spectral resolution of the density matrix

\[
\rho = \sum_i \lambda_i |\psi_i\rangle \langle \psi_i|; \ i = 1, 3
\]

where \( \lambda_i \) are the eigenvalues and \( |\psi_i\rangle \) are the corresponding eigenstates of the density matrix. The steady state eigenvalues at \( \Delta = 0 \) is given by \( \{0, 0, 1\} \), which means it is a pure state. While in the real system (scheme I) the largest eigenvalue is always \( <1 \). However, for any given \( \Omega_1, \Omega_2 \), purity is still maximum for \( \Delta = 0 \) with the largest eigenvalue being maximum at this point. A comparison of the eigenvalues for the ideal and real system is illustrated in Figure 2. The role of the metastable state thus, is to introduce an admixture, due to decoherence, to an otherwise pure state. Since the atomic decay constants are fixed in a real system, the only remaining free parameter for controlling purity is the two photon detuning \( \Delta \). The state around the point \( \Delta = 0, X \) may be approximated to a pure state when there is no decoherence. This is the reason for choosing to study the evolution near two photon resonance. In the ideal case (scheme II) the pure state at \( \Delta = 0 \) is given by \( |\psi_{\text{pure}}\rangle = -\sin(X)|00\rangle + \cos(X)|11\rangle \); \( \sin(X) = \Omega_1/\sqrt{\Omega_1^2 + \Omega_2^2} \). The concurrence for this pure state is \( \mathcal{C} = 2\sin(X)\cos(X) \) and is maximum for \( X = \pi/4 \) or \( \Omega_1 = \Omega_2 \) which we call the bell state regime since this corresponds to the Bell state \((|00\rangle + |11\rangle)/\sqrt{2}\). On the other hand concurrence is very small when \( \Omega_1 \ll \Omega_2 \). The state is separable in this region and we call this the separable state regime. A non-perturbative evaluation of the distribution of concurrence in the parameter space for the system with very small decoherence is illustrated in Figure 3a. It is clear that \( \mathcal{C} \) is maximum in the region where the ratio of the field strengths is approximately one while the state becomes less entangled as the ratio between \( \Omega_1 \) and \( \Omega_2 \) differs from one. We would like to comment here that in Figure 3a the maximum value of concurrence \( <1 \), since \( \Gamma_3 \) is assumed to be \( 10^{-4} \) instead of 0. Thus the parameter \( X \) which is a function of \( \Omega_1, \Omega_2 \) is the control parameter for entanglement. Having established that \( \Delta \) and \( X \) are the control parameters for purity and entanglement...

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**Fig. 2.** (Color online) Variation of the eigenvalues with \( \Delta \). The solid lines correspond to scheme II while the dashed lines correspond to scheme I. Parameter values are (a) and (b) \( \Omega_1 = 6, \Omega_2 = 6 \) and (c) and (d) \( \Omega_1 = 3, \Omega_2 = 6 \).

**Fig. 3.** (Color online) Variation of concurrence in the parameter space \((\Delta, X)\). (a) Ideal system (scheme II) and (b) real system (scheme I) with \( \Omega_2 = 6 \).
we now show that the geometric phase is related to entanglement.

We evaluate the geometric phase in the space spanned by the parameters \(\{\Delta, X\}\) in the neighborhood of \(\chi_0 = (0, X)\) where \(\lambda_i(0, X) = 0, \forall X, i = 1, 2\) and \(\lambda_3(0, X) = 1, \forall X\). In other words we need to evaluate the geometric phase of the pure state at \(\chi = (\delta, X + dX)\) which is essentially the Pancharatnam phase given by

\[
\gamma_g(\chi) = \arg[(\psi_3(\chi_0)|\psi_3(\chi))] \tag{10}
\]

here \(|\psi_3\rangle\) corresponds to the pure eigenstate and \(\chi = \chi_0 + d\chi = (\delta, X + dX)\) is any point chosen infinitesimally close to \(\chi_0\). Taylor expanding \(\gamma_g\) near \(\chi_0\) and retaining up to second order in \(\delta\) and \(dX\) we obtain

\[
\gamma_g(\chi) = \arg \left[ 1 - \frac{dX^2}{2} + \beta \delta^2 - i\gamma_{21}\cos(X)^2\delta + \frac{i\gamma_{21}\cos(X)\sin(X)\delta}{2} dX \right] \\
= \arctan \left[ -i\gamma_{21}\cos(X)^2\delta / \left(1 - \frac{dX^2}{2} + \beta \delta^2\right) \right] \\
\tag{11}
\]

where

\[
\beta = (1/8)\cos(X)^4(4 + 16\gamma_{21}^2 - (5 + 8\gamma_{21}^2)\cos(2X) + \cos(4X)) + \frac{C^2((1 + 8\gamma_{21}^2)\cos(2X) + \cos(4X))}{16}
\]

and \(C\) is the concurrence at \((0, X)\). We have thus, shown that the geometric phase \(\gamma_g(\chi)\) at \(\chi_0\) depends on the concurrence \(C\) at \(\chi_0\) for the ideal system (scheme II).

We next evaluate the derivative \(\gamma'_g\) of the geometric phase with respect to the two photon detuning \(\Delta\) which is shown in Figure 4. To be more specific, we evaluate \(\gamma'_g\) in the neighborhood of \((\Delta = 0, X)\). We have already mentioned that when \(X = 0\), concurrence \(C = 0\) and we have a separable state. Let us examine the behavior of geometric phase around this point. The rate of change of the geometric phase in the neighborhood of \((\Delta = 0, X = 0)\) varies rapidly for small variations (Fig. 4a) \(\delta\) along the \(y\) axis and seems to be unchanged for small variations \(dX\) along the \(x\) axis. On the other hand when \(X = \pi/4\) concurrence \(C = 1\), which is a Bell state. Here, the rate of variation of the geometric phase is relatively very slow for small perturbation in both \(\delta\) and \(dX\). This is illustrated in Figure 4b. While in the case of \(C = 1\), the total change of the slope, \(\gamma'_g\), is from \(-1.4\) to \(-2.7\) (Fig. 4b), in the case of \(C = 0\) it changes from \(-3.0\) to \(-14.0\) (Fig. 4a) for the same range of values of \(\delta\) and \(dX\). Thus the geometric phase seems to change very slowly in the vicinity of maximum entanglement. Faster the sweep of the geometric phase or larger the rate of change of geometric phase implies weaker the entanglement. In other words, stability of the geometric phase in any region of the parameter space seems to indicate the state has maximum entanglement in this region. In the general mixed case, however, it is difficult to show analytically the relation between entanglement and geometric phase. We therefore illustrate the relation through numerical simulations.

5 Numerical results

We present below the numerical results for the real (scheme I) system shown in Figure 1.

We evaluate the three eigenvalues and eigenvectors numerically up to all orders in the two photon detuning \(\Delta\) and determine the geometric phase as a weighted sum of the individual phase corresponding to each eigenstate. Since we are addressing hyperfine transitions the eigenvalues are always non-degenerate in the parameter regimes of interest, in other words the hyperfine separations are larger compared to the Rabi frequency of the applied fields.

All the system parameters are rendered dimensionless by scaling with the atomic lifetime \(\gamma \approx 1\) MHz. We restrict our study to the parameter regime \(\Delta, \Omega \leq 6.0\) which is the lifetime of the first excited state. We would like to mention here that in the numerical program, \(\gamma_g\) has been initialized to zero for minimum values of the parameter \(\Delta_1\).
In the previous section, for the scheme I, we have seen that the state is dominantly entangled when $\Omega_1 \approx \Omega_2$ and separable otherwise. Even in the case of the real system (scheme I), we may identify three regions in the parameter space: (i) when $\Omega_1 \ll \Omega_2$, $\Delta_{1,2} = 0$ the two photon state is dominantly in a pure state with very weak entanglement; (ii) when $\Omega_1 \approx \Omega_2$, $\Delta_{1,2} = 0$ the state is almost pure Bell state of the type $|00\rangle - |11\rangle/\sqrt{2}$; and (iii) when $\Omega_1 \gg \Omega_2$, $\Delta_{1,2} = 0$ the state is mixed with weak entanglement. Non-zero detuning ($\Delta \neq 0$) further affects both purity and entanglement. A distribution of concurrence as a function of the two photon detuning $\Delta$ and $\Omega_1 - \Omega_2$ is shown in Figure 3b for the real system with decoherence. The state seems to be show maximum entanglement near $\Omega_1 \approx \Omega_2$ and shows weaker entanglement for $\Omega_1 \ll \Omega_2$ and $\Omega_1 \gg \Omega_2$. It is clear that even in the presence of decoherence the qualitative features of entanglement distribution is similar even though there are quantitative changes. This is not very surprising given the fact that there is no unique entanglement measure for mixed states.

We illustrate in detail the variation of the geometric phase in the neighborhood of $\Delta_1 = 0$ with $\Delta_2 = 0$. This is effectively varying the two photon detuning $\Delta$. Figures 5, 5a, 5b show the variation of $\gamma_2$ and the rate of change of $\gamma_2$, respectively with $\Delta_1$ near $\Delta_1 = 0$ for $\Omega_1 = \Omega_2 = 6$. The main feature we observe is that the geometric phase varies slowly near the region of entanglement which we showed to be true in the case of the ideal system earlier (Fig. 3b). Observe that $\gamma_2$ is constant in the region $-0.25 < \Delta_1 < 0.25$ (Figs. 5a, 5b). This is substantiated by the vanishing of the derivative in this region thus indicating that the geometric phase in this case is stable under small perturbations in the neighborhood of $\Delta_1 = 0$. The numerical values reveal [33] that the population is non-zero correlation which is typical of a Bell like state. Again in Figures 5g, 5h which corresponds to the parameter values $\Omega_1 = \Omega_2 = 6$ and $\Delta_2 = 3$, the geometric phase $\gamma_2$ does not change much and the derivative is nearly zero in the region $-0.5 < \Delta_1 < 0$. The entanglement distribution in these two cases is nearly the same with the two photon state being an almost pure Bell state. We have verified that the variation of the density matrix elements in these two cases is less than 2%. In the case of Figures 5c, 5d, the parameter values are $\Omega_1 = 3$, $\Omega_2 = 6$, $\Delta_2 = 0$. Observe that rate of variation of $\gamma_2$ is more rapid compared to (b) and (h) indicating an overall weaker entanglement. The slope decreases to a minimum at $\Delta_1 = 0$, implying that the state is maximally entangled here. However, $\gamma_2$ is never constant in this regime which means it is not stable under perturbation in the parameter $\Delta$. A look at the density matrix elements reveals [33] that the state is an almost pure state but with weak entanglement.

Lastly, we address the question whether geometric phase can distinguish between pure separable state and mixed state. As shown if Figure 3b, the distribution of concurrence for the real system does not seem to distinguish between the pure separable state and the mixed state. In fact this is true of other measures like negativity and

![Fig. 5. (Color online) Variation of the geometric phase and its derivative with the detuning parameter $\Delta_1$. Parameter values of (a) and (b) are $\Omega_1 = \Omega_2 = 6$, $\Delta_2 = 0$ (c) and (d) $\Omega_1 = 3$, $\Omega_2 = 6$, $\Delta_2 = 0$, (e) and (f) $\Omega_1 = 6$, $\Omega_2 = 3$, $\Delta_2 = 0$. (g) and (h) $\Omega_1 = \Omega_2 = 6$, $\Delta_2 = 3$, (i) and (j) $\Omega_1 = 1.5$, $\Omega_2 = 6$, $\Delta_2 = 0$.](image)
fidelity. However, the variation of geometric phase seems to show this distinction. For the parameter values $\Omega_1 = 6$, $\Omega_2 = 3$ the state is a mixed state [33] and the variation is illustrated in Figures 5e, 5f. Here $\gamma_\Omega$ is never a constant and the slope which corresponds to the rate of change of $\gamma_\Omega$ shows sharp fluctuations in the neighborhood of $\Delta_1 = 0$. While for a pure separable state $\Omega_1 = 1.5$, $\Omega_2 = 6$, the derivative shows a smooth variation as illustrated in Figures 5i, 5j. Thus geometric phase is unique in the sense that it carries information on both purity/mixedness and entanglement.

In Figure 6 we present the details of this plateau where $\gamma_\Omega$ is stable. The plateau in the center marked zero corresponds to the region where $\gamma_\Omega$ does not change. This is surrounded by region where $\gamma_\Omega$ changes by 1%. The negative sign indicates the decrease in $\gamma_\Omega$. The figure illustrates a total decrease of $\gamma_\Omega$ by 10%. Thus, in a given interval of the control parameter, faster the sweep of $\gamma_\Omega$ (with respect to the control parameter), weaker the entanglement and vice versa. This is consistent with the distribution of concurrence shown in Figure 3b.

### 6 Summary and conclusions

We have studied the evolution of the geometric phase in the general set up namely non-adiabatic, non-cyclic and non-unitary evolution of mixed states. The scheme corresponds to a Hamiltonian system consisting of interacting atom-photon system. We have identified the control parameters for manipulating the purity/mixedness and entanglement of the emitted two-photon states. We have been able to show the relation between the geometric phase and entanglement in terms of the control parameters analytically for an ideal system without decoherence. For the real system, we show that the variation of the geometric phase is correlated to the variation of the concurrence on the control parameters $\Omega_i$ and $\Delta_i$, $i = 1, 2$. We have shown that $\gamma_\Omega$ is robust under fluctuations of these control parameters only for pure Bell like states. We further show that the variation of the geometric phase is able to distinguish between pure separable and mixed states. These features provide a diagnostic tool for determining the entanglement of the two-photon state. It is interesting to note that such a connection, between entanglement and geometric phase, has also been found, recently [37], in another system, viz. Heisenberg spin chains undergoing unitary evolution.

One of us, S.N.S. thanks the Department of Science and Technology, India, for financial assistance through the WOS-A scheme. We would also like to thank Joseph Samuel and V. Ravishankar for useful discussions.

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