Loops as sections in compact Lie groups

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Abstract

We prove that there does not exist any connected topological proper loop homeomorphic to a quasi-simple Lie group and having a compact Lie group as the group topologically generated by its left translations. Moreover, any connected topological loop homeomorphic to the 7-sphere and having a compact Lie group as the group of its left translations is classical. We give a particular simple general construction for proper loops such that the compact group of their left translations is direct product of at least 3 factors.

1. Introduction

H. Scheerer has clarified in [10] for which compact connected Lie groups $G$ and for which closed subgroups $H$ the natural projection $G \to G/H$ has a continuous section $\sigma$. If $G$ is a semisimple compact Lie group, then the image $\sigma(G/H)$ is not homeomorphic to a Lie group precisely if $G$ contains a factor locally isomorphic to $PSO_8(\mathbb{R})$. This is due to the fact that the group topologically generated by the left translations of the octonions of norm 1 is the group $SO_8(\mathbb{R})$. Hence any compact connected topological loop whose group topologically generated by the left translations is a compact Lie group is itself homeomorphic to a compact Lie group.

But it remained an open problem for which $\sigma$ the image $\sigma(G/H)$ determines a loop. This is the case if $\sigma(G/H)$ acts sharply transitively on $G/H$ what means that for given cosets $g_1H, g_2H$ there exists precisely one $z \in \sigma(G/H)$ such that the equation $zg_1H = g_2H$ holds. Continuous sections $\sigma$ with this property (they are called sharply transitive sections) correspond

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to topological loops \((L, \ast)\) (cf. [9], Proposition 1.21, p. 29) realized on \(G/H\) with respect to the multiplication \(xH \ast yH = \sigma(xH)yH\). The group topologically generated by the left translations of \((L, \ast)\) coincides with \(G\).

There are many examples of compact connected loops having a non-simple compact connected Lie group as the group topologically generated by their left translations (cf. [9], Theorem 16.7, p. 198 and Section 14.3, pp. 170-173). A particular simple general construction for proper loops such that the group generated by their left translations is the direct product of at least three factors is given in Section 3.

In contrast to this in this paper we prove that any connected topological loop \(L\) homeomorphic to a quasi-simple Lie group \(G\) and having a compact Lie group as the group topologically generated by its left translations must coincide with \(G\) (cf. Theorem 8). Similarly, any connected topological loop \(L\) homeomorphic to the 7-sphere and having a compact Lie group as the group topologically generated by its left translations is either the M"{o}bius loop \(O\) of octonions of norm 1 or the factor loop \(O/Z\), where \(Z\) is the centre of \(O\) (cf. Theorem 6).

2. Prerequisites

A set \(L\) with a binary operation \((x, y) \mapsto x \cdot y\) is called a loop if there exists an element \(e \in L\) such that \(x = e \cdot x = x \cdot e\) holds for all \(x \in L\) and the equations \(a \cdot y = b\) and \(x \cdot a = b\) have precisely one solution which we denote by \(y = a \backslash b\) and \(x = b/a\). The left translation \(\lambda_a : y \mapsto a \cdot y : L \rightarrow L\) is a bijection of \(L\) for any \(a \in L\).

The kernel of a homomorphism \(\alpha : (L, \circ) \rightarrow (L', \ast)\) of a loop \(L\) into a loop \(L'\) is a normal subloop \(N\) of \(L\), i.e. a subloop of \(L\) such that

\[
 x \circ N = N \circ x, \quad (x \circ N) \circ y = x \circ (N \circ y), \quad x \circ (y \circ N) = (x \circ y) \circ N
\]

holds for all \(x, y \in L\). A loop \((L, \cdot)\) is a product of two subloops \(L_1\) and \(L_2\) if any element \(x\) of \(L\) has a representation \(x = a \cdot b\), \(a \in L_1\) and \(b \in L_2\). A loop \((L, \cdot)\) is called a M"{o}bius loop if for all \(x, y, z \in L\) the identity \((x \cdot y) \cdot (z \cdot x) = [x \cdot (y \cdot z)] \cdot x\) holds.

Let \(L\) be a topological space. Then \((L, \cdot)\) is a topological loop if the maps \((x, y) \mapsto x \cdot y, (x, y) \mapsto x \backslash y, (x, y) \mapsto y/x : L^2 \rightarrow L\) are continuous. If only the multiplication and the left division are continuous, then the loop \(L\) is called almost topological. An almost topological loop \(L\) is a topological loop if the group generated by the left translations of \(L\) is a connected Lie group (see [9], Corollary 1.22). A loop \(L\) is almost differentiable if \(L\) is a differentiable manifold and the multiplication and the left division are differentiable.
Let $G$ be a compact connected Lie group, let $H$ be a connected closed subgroup of $G$ containing no non-trivial normal subgroup of $G$ and $G/H = \{xH, x \in G\}$. Let $\sigma : G/H \to G$ be a continuous map with $\sigma(H) = 1 \in G$ such that the set $\sigma(G/H)$ is a system of representatives for $G/H$ which generates $G$ and operates sharply transitively on $G/H$ which means that to any $xH$ and $yH$ there exists precisely one $z \in \sigma(G/H)$ with $zxH = yH$. Then the multiplication on the factor space $G/H$ given by $xH \cdot yH = \sigma(xH)yH$, respectively the multiplication on the set $\sigma(G/H)$ given by $x \cdot y = \sigma(xyH)$ yields a compact topological loop having $G$ as the group topologically generated by the left translations $xH \mapsto \sigma(xH)$, respectively $x \mapsto \sigma(xH)$.

If $L$ is a compact topological loop such that the group topologically generated by all left translations of $L$ is a compact connected Lie group, then the set $\{\lambda_a, a \in L\}$ forms a sharply transitive section $\sigma : G/Ge \to G$ with $\sigma(\lambda_a Ge) = \lambda_a$, where $Ge$ is the stabilizer of $e \in L$ in $G$.

If the section $\sigma : G/H \to G$ is differentiable, then the loop $L$ is almost differentiable.

A quasi-simple compact Lie group is a compact Lie group $G$ containing a normal finite central subgroup $N$ such that the factor group $G/N$ is simple. A semisimple connected compact group $G$ is a Lie group containing a normal finite central subgroup $N$ such that the factor group $G/N$ is a direct product of simple Lie groups. A connected compact Lie group is an almost direct product of compact semisimple Lie groups if its universal covering (cf. [2], Appendix A) is a direct product of simply connected quasi-simple Lie groups.

For connected and locally simply connected topological loops there exist universal covering loops (cf. [3], [4], [6], IX.1). This yields the following lemma:

**Lemma 1.** The universal covering loop $\tilde{L}$ of a connected and locally simply connected topological loop $L$ is simply connected and $L$ is isomorphic to a factor loop $\tilde{L}/N$, where $N$ is a central subgroup of $L$.

### 3. Loops corresponding to products of groups

Let $G = K \times P \times S$ be a group, where $K$ is a group, $P$ is a non-abelian group, $g : K \to S$ is a map which is not a homomorphism such that $g(1) = 1$, the set $\{(k, 1, g(k)); k \in K\}$ generates the group $K \times \{1\} \times S$ and $S$ is isomorphic to a subgroup of $P$ having with the centre of $P$ trivial intersection. Hence there is a monomorphism $\varphi$ from $S$ into $P$ and we may assume $H = \{(1, x, x); x \in S\}$. Moreover, we put $M = \{(k, lg(k), g(k)); k \in K, l \in P\}$.

Every element $(a, b, c) \in G$ may be uniquely decomposed as $(a, b, c) = (a, bc^{-1}, 1)(1, c, c)$ with $(1, c, c) \in H$. Since for all $a \in K, b \in P$ there are
unique elements \( m = (a, bg(a), g^{-1}(a)) \in M \) and \( h = (1, g(a)^{-1}, g(a)^{-1}) \in H \) such that \( (a, b, 1) = mh \) the set \( M \) determines the section \( \sigma : G/H \to G; (x, y, 1)H \mapsto \sigma((x, y, 1)H) = (x, yg(x), g(x)). \) Since for given \( a_1, a_2 \in K, b_1, b_2 \in P \) the equation

\[
(k, lg(k), g(k))(a_1, b_1, 1) = (a_2, b_2, 1)(1, d, d)
\]

has the unique solution

\[
k = a_2a_1^{-1}, \quad l = b_2g(a_2a_1^{-1})b_1^{-1}g(a_2a_1^{-1})^{-1}
\]

with \( d = g(a_2a_1^{-1}) \in P, \) the set \( M \) acts sharply transitively on the left cosets \( \{(a, b, 1)H; \ a \in K, b \in P\}. \) Since the group \( H \) contains no normal subgroup of \( G \) the map \( \sigma \) corresponds to a loop \( L \) having the group \( G \) as the group generated by its left translations, the subgroup \( H \) as the stabilizer of \( e \in L \) and the set \( M \) as the set of all left translations of \( L \). The multiplication of \( L \) can be defined on the set \( \{(a, b, 1)H; \ a \in K, b \in P\} \) by

\[
(a_1, b_1, 1)H * (a_2, b_2, 1)H = (a_1a_2, b_1g(a_1)b_2g(a_1)^{-1}, 1)H
\]

and \( G = K \times P \times S \) is the group generated by the left translations of \( (L, *) \). Since \( (1, l_1, 1)H * (1, l_2, 1)H = (1, l_1l_2, 1)H \) for all \( l_1, l_2 \in P \) holds \( (N, *) = \{(1, l, 1)H; l \in P\}, * \) is a subgroup of \( (L, \ast) \) isomorphic to \( P \). As \( G \) is the direct product \( G = G_1 \times G_2 \) with \( G_1 = K \times \{1\} \times S \) and \( G_2 = \{1\} \times P \times \{1\} \) and \( \sigma(G/H) = M = M_1 \times G_2 \) with \( M_1 = \{(k, 1, g(k)); k \in K\} \subset G_1 \) it follows from Proposition 2.4 in \([9]\) p. 44, that the group \((N, \ast)\) is normal in the loop \((L, \ast)\). Moreover, for all \( k_1, k_2 \in K \) one has \( (k_1, 1, 1)H * (k_2, 1, 1)H = (k_1k_2, 1, 1)H. \) Hence \((K, \ast) = \{(k, 1, 1)H; k \in K\}, \ast\) is a subgroup of \((L, \ast)\) isomorphic to \( K \). Therefore the loop \((L, \ast)\) defined by \([1]\) is a semidirect product of the normal subgroup \((N, \ast)\) by the subgroup \((K, \ast)\).

The loop \( L \) is a group if the multiplication \([1]\) is associative, i.e.

\[
((a_1, b_1, 1)H * (a_2, b_2, 1)H) * (a_3, b_3, 1)H = (a_1, b_1, 1)H * ((a_2, b_2, 1)H * (a_3, b_3, 1)H).
\]

This identity holds if and only if for all \( a_i \in K \) and \( b_i \in P \) one has

\[
(a_1a_2a_3, b_1g(a_1)b_2g(a_1)^{-1}g(a_1a_2)b_3g(a_1a_2)^{-1}, 1)H = (a_1a_2a_3, b_1g(a_1)b_2g(a_2)b_3g(a_2)^{-1}g(a_1)^{-1}, 1)H
\]

or equivalently \( g(a_1)g(a_2)b_2g(a_2)^{-1}g(a_1)^{-1} = g(a_1a_2)b_3g(a_1a_2)^{-1}. \) This yields a contradiction since \( g \) is not a homomorphism. Therefore \( L \) is a proper loop.
If $K$ and $P$ are connected Lie groups and the function $g$ is continuous, then $L$ has continuous multiplication and left division (cf. [9], p. 29). Hence $L$ is a connected locally compact topological proper loop. If $g$ is differentiable, then $L$ is a connected almost differentiable proper loop (cf. [9], p. 32).

The constructed examples show the following

**Remark.** There exist proper loops with normal connected subgroups having a compact connected Lie group $G$ as the group topologically generated by the left translations if $G = G_1 \times G_2 \times G_3$, where $G_2$ is not a torus group and $G_3$ is isomorphic to a subgroup of $G_2$ having with the centre of $G_2$ trivial intersection.

The aim of the paper is to demonstrate that this is a typical situation for connected compact Lie groups $G$ being groups generated by the left translations of a proper loop.

4. Results

**Lemma 2.** Any one-dimensional connected topological loop having a compact Lie group as the group topologically generated by its left translations is the orthogonal group $SO_2(\mathbb{R})$.

**Proof.** It is proved in [9], Proposition 18.2. \hfill \square

**Lemma 3.** Let $G$ be a connected semisimple compact Lie group topologically generated by the left translations of a compact simply connected loop which is homeomorphic to a semisimple Lie group $K_1$. Let $H$ be the stabilizer of $e \in L$. Then one has $H = (H_1, \rho(H_1)) = \{(x, \rho(x)), x \in H_1\}$ and $G = K_1 \times \rho(H_1)$, where $\rho$ is a monomorphism and $H_1$ is a subgroup of $K_1$. Moreover, $H_1$ has with the centre $Z_1$ of $K_1$ a trivial intersection.

**Proof.** Since $G/H$ is homeomorphic to $K_1$ the group $G$ is homeomorphic to $K_1 \times H$. According to [10] or to Theorem 16.1 in [9], p. 195, the group $G$ has the form $G = K_1 \times K_2$ and the stabilizer $H$ of $e \in L$ is $H = (H_1, \rho(H_1))$, where $K_2$ is a Lie group isomorphic to $H$ and $\rho$ is a monomorphism. From this it follows that $G = K_1 \times \rho(H_1)$. If $Z_1 \cap H_1 \neq 1$, then $H$ has with the centre $Z$ of $G$ a non-trivial intersection. But this is a contradiction to the fact that $Z \cap H = 1$. \hfill \square

Lemma 3 yields

**Corollary 4.** Let $G = K_1 \times K_2$ be a compact semisimple Lie group such that $K_1$ is semisimple and let $H$ be a subgroup of $G$ such that $H = (H_1, \rho(H_1))$, 

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where $H_1$ is a subgroup of $K_1$ and $\rho$ is a monomorphism. If $K_1$ has a non-trivial centre $Z_1$ and $H_1 \cap Z_1 \neq 1$, then there exists no proper loop $L$ homeomorphic to $K_1$ such that $G$ is the group topologically generated by the left translations of $L$ and $H$ is the stabilizer of $e \in L$.

**Corollary 5.** There does not exist a connected topological proper loop $L$ homeomorphic to a covering of a product $K_1$ of the groups $SO_3(\mathbb{R})$ and having a compact semisimple Lie group $G = K_1 \times K_2$ as the group topologically generated by the left translations of $L$.

**Proof.** We may assume that $L$ is simply connected and hence $K_1$ is a direct product of groups isomorphic to $\text{Spin}_3(\mathbb{R})$. Then the stabilizer $H$ of $e \in L$ is the subgroup $H = (H_1, \rho(H_1))$, where $H_1$ is a subgroup of $K_1$ and $\rho$ is a monomorphism. The group $G$ topologically generated by the left translations of $L$ has the form $G = K_1 \times \rho(H_1)$ (cf. Lemma 3). The assertion follows from Corollary 4 because any subgroup $H_1$ intersects the centre of $K_1$ non-trivially.

**Theorem 6.** Let $L$ be a topological loop homeomorphic to the 7-sphere or to the 7-dimensional real projective space such that the group $G$ topologically generated by the left translations of $L$ is a compact Lie group. Then $L$ is one of the two 7-dimensional compact Moufang loops, $G$ is locally isomorphic to $PSO_8(\mathbb{R})$ and the stabilizer $H$ of $e \in L$ is isomorphic to $SO_7(\mathbb{R})$.

**Proof.** We may assume that $L$ is simply connected. Since $G$ is a compact Lie group using Proposition 2.4 in [7] and Ascoli’s Theorem, from IX.2.9 Theorem of [6] it follows that the loop $L$ has a left invariant uniformity. Therefore IX.3.14 Theorem in [6] yields that $L$ is the multiplicative loop $\mathcal{O}$ of octonions having norm 1. Then $G$ is isomorphic to $SO_8(\mathbb{R})$ and the stabilizer $H$ of $e \in L$ is isomorphic to $SO_7(\mathbb{R})$ (cf. [10]).

If $L$ is homeomorphic to the 7-dimensional real projective space, then the universal covering $\tilde{L}$ of $L$ is a Moufang loop homeomorphic to the sphere $S^7$. It follows from [9], p. 216, that the loop $L$ is a factor loop $\tilde{L}/N$, where $N$ is a central subgroup of $\tilde{L}$ of order 2. Lemma 1.33 in [9] yields that $L$ is the Moufang loop $\mathcal{O}/Z$, where $Z$ is the centre of the multiplicative loop of octonions having norm 1.

If $L$ is a topological loop homeomorphic to the 7-sphere and if we assume that the group $G$ topologically generated by the left translations of $L$ is a quasi-simple compact Lie group, then $G$ is isomorphic to $SO_8(\mathbb{R})$ and the stabilizer $H$ of $e \in L$ is isomorphic to $SO_7(\mathbb{R})$. This allows us to obtain the assertion of the previous theorem also in the following way. We identify the
set $G/H$ of the left cosets with the set $S$ of the left translations of the loop $O$. The section $\sigma : G/H \to G$ belonging to a topological loop $L$ has the form $\sigma(xH) = x\phi(x)$, where $x \in S$ and $\phi$ is a continuous map from $S$ to $H$. Since any two elements of $S$ are contained in a subgroup $D$ isomorphic to $\text{Spin}_3(\mathbb{R})$ the restriction of $\phi$ to $D$ is a homomorphism (Corollary 5). Hence $L$ is a diassociative Lie loop ([6], IX.6.42) and Theorem 16.10 in [9] yields the assertion of the previous theorem.

**Theorem 7.** Let $G$ be a compact Lie group which is the group topologically generated by the left translations of a proper topological loop $L$ homeomorphic to a connected semisimple compact Lie group. Then $G$ is a connected semisimple Lie group.

**Proof.** Since $L$ is connected also $G$ is connected. By Hofmann-Scheerer Splitting Theorem (cf. [2], p. 474) the group $G$ is isomorphic to a semidirect product $G = G' \rtimes T$, where $G'$ is the semisimple commutator subgroup of $G$ and $T$ is a torus. The group $G'$ is isomorphic to an almost direct product $G' = K_1 \cdots K_m$ of quasi-simple compact Lie groups. The loop $L$ is homeomorphic to a connected semisimple compact Lie group $K = K_1 \cdots K_s$ with $s \leq m$. We may assume that $L$ and hence also $K$ is simply connected. Since the universal covering $\tilde{G}'$ of $G'$ is the direct product of $K$ and the universal covering $\tilde{S}$ of $S = K_{s+1} \cdots K_m$, the group $G'$ is the direct product of $K$ and $S$. As $L$ is homeomorphic to the image of the section $\sigma : G/H \to G$, where $H$ is the stabilizer of $e \in L$, the set $\sigma(G/H)$ has the form $\{(x, \alpha(x))\}$, where $x \in K$ and $\alpha$ is a continuous mapping from $K$ into $S \times T$. The group $T = T_1 \times \cdots \times T_h$ is the direct product of one-dimensional tori $T_i$. Let $\pi$ be the projection from $S \times T$ into $T$ along $S$ and $\iota_i$ be the projection from $T$ into $T_i$ along the complement $\coprod_{j \neq i} T_j$. As $\sigma(G/H)$ is a compact connected homogeneous space and $T_i$ is a 1-sphere for all $i$ any $\iota_i \pi \alpha(K)$ is either constant or surjective. Since $\sigma(G/H)$ generates $G$ there exists one $i$ such that $\iota_i \pi \alpha(K)$ is different from $\{1\}$. As the group $T$ is the direct product of 1-dimensional tori $T_i$ the Bruschlinsky group $B$ (cf. [8], p. 47) of $K$ is not trivial. By Theorem 7.1 in [8], p. 49, $B$ is isomorphic to the first cohomology group $H^1(K)$. The graded cohomology algebra of the compact Lie group $K$ is the tensor product of the cohomology algebras $H^1(F_i)$ of the quasi-simple factors $F_i$ of $K$. Since $H^1(F_i)$ has no generators of degree 1 and 2 ([1], pp. 126-127) also the cohomology algebra $H^1(K)$ has no generators of degree 1 and 2. Hence the Poincare polynomial $\psi(K)$ has no linear and quadratic monomials, which is a contradiction. □

**Remark.** In contrast to the previous theorem a non semisimple compact Lie group may be the group topologically generated by the left translations of
a loop $L$ if $L$ is homeomorphic to a non semisimple compact connected Lie group.

Let $T_1$ be a torus of dimension $m \geq 1$, let $P$ be a connected semisimple compact Lie group and let $T_2$ be a torus of dimension $s$ with $1 \leq s \leq m$ such that there exists a monomorphism $\varphi : T_2 \to P$ with $\varphi(T_2) \cap Z(P) = \{1\}$, where $Z(P)$ is the centre of $P$. If $g : T_1 \to T_2$ is a continuous surjective mapping with $g(1) = 1$ which is not a homomorphism, then with the subgroup $H = \{(1, \varphi(x), x) ; x \in T_2\}$ of $G = T_1 \times P \times T_2$ as the stabilizer there exists according to Section 3 a proper connected loop $L$ homeomorphic to $T_1 \times P$ having the direct product $T_1 \times P \times T_2$ as the group topologically generated by the left translations of $L$.

**Theorem 8.** There does not exist any proper topological loop which is homeomorphic to a connected quasi-simple Lie group and has a compact Lie group as the group topologically generated by its left translations.

**Proof.** By Lemma 1 we may assume that $L$ is a proper loop homeomorphic to a simply connected quasi-simple compact Lie group $K_1$. Then the stabilizer $H$ of $e \in L$ has the form $H = (H_1, \rho(H_1)) = \{(x, \rho(x)) ; x \in H_1\}$, where $\rho$ is a monomorphism and the group $G$ topologically generated by the left translations of $L$ has the form $G = K_1 \times \rho(H_1)$ (cf. Lemma 3). Identifying the space $G/H$ with $K_1$ one has that the image $\sigma(K_1)$ of the section $\sigma : K_1 \to G$ intersects $H$ trivially. As $\rho$ is a monomorphism we may assume that $H = (H_1, H_1) = \{(x, x) ; x \in H_1\}$. The restriction of $\sigma$ to a one-dimensional torus subgroup $A$ of $H_1$ yields $\sigma(A) = \{(u, f(u))\}$, where $f$ is a continuous function. Since the compact loop $\sigma(A)$ is a group (cf. Lemma 2) the map $f$ is a homomorphism. It follows that $\sigma(A)$ has the form $\{(u, u^n)\}$ with fixed $n \in \mathbb{Z}$. Since $\{(u, u^n) ; u \in A \cong SO_2(\mathbb{R})\} \cap H = \{1\}$ the equation $x^n = x$, $x \in A$, can be satisfied only for $x = 1$. Equivalently, $x \mapsto x^{n-1}$ is an automorphism of $SO_2(\mathbb{R})$. Besides the identity the only non-trivial automorphism of the group $SO_2(\mathbb{R})$ is the map $x \mapsto x^{-1}$. Therefore we get $n \in \{0, 2\}$. Let $C_1$ be a 3-dimensional subgroup of $H_1$. Since any 3-dimensional compact loop which has a compact Lie group as the group topologically generated by its left translations is a group (cf. Corollary 3) $C = \sigma(C_1) = (C_1, \psi(C_1))$ is locally isomorphic to $SO_2(\mathbb{R})$ and $\psi$ is a homomorphism of $C_1$. Besides a homomorphism with finite kernel any continuous homomorphism is an automorphism induced by a conjugation with elements of the orthogonal group $O_2(\mathbb{R})$. Hence for no 1-dimensional subgroup $A$ of $C_1$ one can have $\sigma(A) = \{(x, x^2) ; x \in A\}$. As in compact groups the exponential map is surjective the compact group $C$ is the union of the one-dimensional connected subgroups $\sigma(A) = \{(x, 1) ; x \in A\}$. Hence $C$ has the form $(C_1, 1)$. Since the
3-dimensional subgroups of \( H_1 \) covers \( H_1 \) (cf. \([\text{[5]}],\) Propositions 6.45 and 6.46) for the continuous section \( \sigma \) one has \( \sigma(H_1) = (H_1,1) \).

Let \( B_i \) be a one-dimensional torus subgroup of \( K_1 \) such that \( \sigma(B_i) = (B_i,1) \). The union \( B = \bigcup B_i \) of the one-dimensional subgroups of \( K_1 \) forms a subgroup of \( K_1 \) containing \( H_1 \).

Let \( F_i \) be a 1-dimensional torus subgroup of \( K_1 \) such that \( \sigma(F_i) \neq (F_i,1) \). Then one has \( \sigma(F_i) = \{ (x,x^n) ; x \in F_i \} \), where \( n \in \mathbb{Z} \setminus \{0\} \). Since any 1-dimensional subgroup of \( K_1 \) is contained in a 3-dimensional subgroup of \( K_1 \) locally isomorphic to \( SO_3(\mathbb{R}) \) (cf. \([5]\), Propositions 6.45 and 6.46) and by Corollary \([5]\) any 3-dimensional loop homeomorphic to a cover of \( SO_3(\mathbb{R}) \) is a group, besides a homomorphism with finite kernel we get that \( x \mapsto x^n \) is either an isomorphism or an anti-isomorphism of \( SO_3(\mathbb{R}) \). Hence one has \( n = 1 \) or \(-1 \). It follows that either \( \sigma(F_i) = \{ (x,x) ; x \in F_i \} \) or \( \sigma(F_i) = \{ (x,x^{-1}) ; x \in F_i \} \) for any 1-dimensional torus subgroup \( F_i \) of \( K_1 \) such that \( \sigma(F_i) \neq (F_i,1) \). The union \( F = \bigcup F_i \) of the 1-dimensional torus subgroups \( F_i \) of \( K_1 \) with \( \sigma(F_i) \neq (F_i,1) \) is isomorphic to the group \( \rho(H_1) \cong H_1 \).

The subgroups \( F \) and \( B \) yield a factorization of \( K_1 \) such that the intersection \( F \cap B \) is discrete which is a contradiction to the fact that \( K_1 \) is quasi-simple (cf. Theorem 4.6 in \([1]\), p. 145).

**Corollary 9.** Let \( L \) be a proper topological loop homeomorphic to a product of quasi-simple simply connected compact Lie groups and having a compact Lie group \( G \) as the group topologically generated by its left translations. Then \( G \) is at least 14-dimensional.

If \( \dim G = 14 \), then \( G \) is locally isomorphic to \( Spin_3(\mathbb{R}) \times SU_3(\mathbb{C}) \times Spin_3(\mathbb{R}) \) and \( L \) is homeomorphic to a group which is locally isomorphic to \( Spin_3(\mathbb{R}) \times SU_3(\mathbb{C}) \).

**Proof.** We assume that the loop \( L \) is simply connected. Then \( L \) is homeomorphic to the direct product \( K_1 \) of at least two quasi-simple simply connected factors (cf. Theorem \([5]\)). According to Theorem \([4]\) the connected group \( G \) is semisimple. Hence by Lemma \([5]\) the stabilizer \( H \) of \( e \in L \) has the form \( (H_1,\rho(H_1)) = \{ (x,\rho(x)), x \in H_1 \} \) and \( G = K_1 \times \rho(H_1) \), where \( H_1 \) is a subgroup of \( K_1 \) and \( \rho \) is a monomorphism. Since any subgroup of \( Spin_3(\mathbb{R}) \) intersects its centre not trivially according to Corollary \([5]\) and to the construction in Section 3 the group \( K_1 \) coincides with \( K \times P = Spin_3(\mathbb{R}) \times SU_3(\mathbb{C}) \), the subgroup \( H_1 \) has the form \( 1 \times Spin_3(\mathbb{R}) \) and \( \rho : H_1 \to S \) is an isomorphism. Therefore one has \( S = Spin_3(\mathbb{R}) \) and for the function \( g : K \to S \) one can choose the function \( x \mapsto x^n \) with \( n \in \mathbb{Z} \setminus \{-1,0,1\} \). Hence from the construction in Section 3 we have \( G = Spin_3(\mathbb{R}) \times SU_3(\mathbb{C}) \times Spin_3(\mathbb{R}) \). \( \Box \)

**Remark.** Euclidean and hyperbolic symmetric spaces correspond to global
differentiable loops (cf. [9], Theorem 11.8, p. 135). In contrast to this, compact simple symmetric spaces which are not Lie groups yield only local Bol loops $L$ since for $L$ the exponential map is not a diffeomorphism (cf. [9], Proposition 9.19, p. 115).

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