Neutrino Spin Oscillations in Gravitational Fields

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Abstract—We study neutrino spin oscillations in black hole backgrounds. In the case of a charged black hole, the maximum frequency of oscillations is a monotonically increasing function of the charge. For a rotating black hole, the maximum frequency decreases with increasing angular momentum. In both cases, the frequency of spin oscillations decreases as the distance from the black hole grows. As a phenomenological application of our results, we study a simple bipolar neutrino system which is an interesting example of collective neutrino oscillations. We show that the precession frequency of the flavor pendulum as a function of the neutrino number density will be higher for a charged non-rotating black hole as compared with a neutral rotating one.

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1. INTRODUCTION

Neutrino physics is an active area of research with important implications for particle physics, cosmology and astrophysics. Cosmological implications of neutrinos include lepto/baryogenesis and possible connections to the dark sector of the universe [1]. The phenomenon of neutrino oscillations can explain solar and atmospheric neutrino problems. It also provides the first experimental evidence for physics beyond the Standard model since it requires a nonzero mass for neutrinos.

The effects of gravitational fields on neutrino oscillations have been studied in the literature. In [2], neutrino flavor oscillations in Kerr-Newman spacetime have been studied. The authors of [3] investigated the effect of a quantum gravity-induced minimal length on neutrino oscillations. Neutrino optics in gravitational fields has been studied in [4]. Interaction of neutrinos with an external field provides one of the factors required for a transition between helicity states. In [5] neutrino spin oscillations have been studied in the Schwarzschild background which describes the gravitational field of an uncharged and non-rotating black hole.

In this paper, we study neutrino spin oscillations in the Reissner-Nordström (RN) and Kerr backgrounds which describe the gravitational fields of charged non-rotating and neutral rotating black holes, respectively.

2. NEUTRINO SPIN OSCILLATIONS IN THE REISSNER-NORDSTRÖM (RN) METRIC

The RN metric of a charged non-rotating black hole reads

\[ dr^2 = A^2 dt^2 - A^{-2} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2, \]
\[ A = \sqrt{1 - \frac{2M}{r} + \frac{Q^2}{r^2}}. \]

Here \( Q \) and \( M \) are the charge and mass of the black hole, respectively, and we use the natural units \( h = c = 1 \). The vierbein four-velocity components are [5]

\[ u^a = (\gamma A, U_r A^{-1}, U_\theta, U_\phi \sin \theta), \]

where

\[ U^\mu = (U^0, U_r, U_\theta, U_\phi), \quad U^0 = \frac{dt}{d\tau} = \gamma, \]
\[ U_r = \frac{dr}{d\tau}, \quad U_\theta = \frac{d\theta}{d\tau}, \quad U_\phi = \frac{d\phi}{d\tau}. \]

\( U^0 \) is the four-velocity of a particle in its geodesic path, which is related to \( u^a \) by \( u^a = e^a_\mu U^\mu \). The four-velocity \( U^\mu \) is related to the world velocity of a particle through \( \vec{U} = \gamma \vec{V} \), where \( \gamma = \frac{dt}{d\tau} \) and \( \tau \) is the proper time. The nonzero vierbein vector components are

\[ e^0_t = A, \quad e^1_r = A^{-1}, \]
\[ e^2_\theta = r, \quad e^3_\phi = r \sin \theta. \]

To study the spin evolution of a particle in a gravitational field, we calculate \( G_{ab} = (\vec{E}, \vec{B}) \) which is
It should be stressed that the electric and magnetic fields in (9) are not (real) electric and magnetic fields but only their analogues.

Using Eqs. (4), (5), (6), (8), and (9), we have the following forms for the electric and magnetic fields:

$$\vec{E} = \left( -\frac{\gamma}{2}, \frac{2Q^2}{r^3}, 0, 0 \right),$$

$$\vec{B} = \left( \gamma v_{\varphi} \cos \theta, -\gamma v_{\varphi} A \sin \theta, \gamma A v_{\theta} \right).$$

(10)

Since the gravitational field around a charged non-rotating black hole is symmetric, we can consider the analogue of the electromagnetic field tensor. It is defined as

$$G_{ab} = e_{a\mu\nu}e^\mu_b U^\nu, \quad \text{(6)}$$

where \(e_{a\mu\nu}\) are covariant derivatives of vierbein vectors given by

$$e_{a\mu\nu} = \frac{\partial e_{a\mu}}{\partial x^\nu} - \Gamma^\lambda_\mu_\nu e_{a\lambda}. \quad \text{(7)}$$

Using Eqs. (5) and (7) to calculate the covariant derivatives of vierbein vectors, we have:

$$e_{0r;t} = -\frac{1}{2} \left( 2M + \frac{2Q^2}{r^3} \right) A^{-1},$$

$$e_{1r;t} = \frac{1}{2} \left( 2M + \frac{2Q^2}{r^3} \right) A,$n

$$e_{1\theta;\theta} = -r + 2M - \frac{Q^2}{r},$$

$$e_{1\varphi;\varphi} = \sin^2 \theta A^{-1} \left( -r + 2M - \frac{Q^2}{r} \right),$$

$$e_{2r;\theta} = 1, \quad e_{1\varphi;\varphi} = -r \sin \theta \cos \theta,$n

$$e_{3r;\varphi} = \sin \theta, \quad e_{3\theta;\varphi} = r \cos \theta. \quad \text{(8)}$$

Using Eqs. (4), (5), (6), (8), and (9), we have the following forms for the electric and magnetic fields:

$$\vec{E} = \left( -\frac{\gamma}{2}, \frac{2Q^2}{r^3}, 0, 0 \right),$$

$$\vec{B} = \left( \gamma v_{\varphi} \cos \theta, -\gamma v_{\varphi} A \sin \theta, \gamma A v_{\theta} \right).$$

(10)

Since the gravitational field around a charged non-rotating black hole is symmetric, we can consider neutrino motion in the equatorial plane \(\theta = \pi/2\) (hence \(U_\theta = 0\)). The four-velocity in the vierbein frame is not constant in the general case, but it can be shown that for circular orbits \(du^a/d\tau = 0\), which means that the velocity four-vector of neutrinos is constant with respect to the vierbein frame. We assume that the motion takes place in a circular orbit with constant radius \(r\) \((U_r = dr/d\tau = 0\). It is important to note that not all orbits with arbitrary radius are stable. We concentrate on stable orbits. The geodesic equation of a particle in a gravitational field is given by [6]

$$\frac{d^2x^\mu}{dq^2} + \Gamma^\mu_\sigma_\nu \frac{dx^\sigma}{dq} \frac{dx^\nu}{dq} = 0,$$

(11)

where the variable \(q\) parameterizes the particle’s world line. From Eqs. (1), (2), and (11), we can calculate the values of \(v_\varphi\) and \(\gamma^{-1}\):

$$\gamma^{-1} = \sqrt{1 - \frac{3M}{r} + \frac{2Q^2}{r^2}}, \quad \text{(12)}$$

$$v_\varphi = \sqrt{\frac{M}{r^3} + \frac{Q^2}{r^4}}. \quad \text{(13)}$$

Neutrino spin precession is given by the expression \(\Omega = \vec{G}/\gamma\), where the vector \(\vec{G}\) is defined as follows [7]:

$$\vec{G} = \frac{1}{2} \left[ \vec{B} + \frac{1}{1 + u^0} \left[ \vec{E} \times \vec{u} \right] \right]. \quad \text{(14)}$$

Substituting (3), (10), (12), (13), (14) and \(\vec{U} = \gamma \vec{V}\) into \(\Omega = \vec{G}/\gamma\), we obtain the only nonzero frequency component:

$$\Omega_2 = -\frac{1}{4M} \sqrt{1 - \frac{3M}{r} + \frac{2Q^2}{r^2}}$$

$$\times \sqrt{\frac{4M^3}{r^3} - \frac{4M^2Q^2}{r^4}} = -\frac{v_\varphi}{2}\gamma^{-1}. \quad \text{(15)}$$

In Fig. 1 we show \(2|\Omega_2| \leq M\) versus \(r/2M\) for different values of \(\alpha = Q/M\). For \(\alpha = 0\) our results reduce to that in the case of the Schwarzschild metric [5].
We stop at \( \alpha = 1 \) since the RN metric with \( Q > M \) does not describe a black hole and a naked singularity occurs at \( r = 0 \).

It is seen that the maximum frequency of oscillations increases with \( \alpha \). It is also seen that \( \Omega_2 \) has its largest value at \( Q = M \). The \( \alpha = 0 \) curve is in agreement with the result obtained for the Schwarzschild metric [5]. We also see that curves for different values of \( \alpha \) coincide at large \( r \). This indicates that the effect of the black hole charge diminishes at large distances.

In Fig. 2 we show the neutrino transition probability \( P(t) = \sin^2(\Omega_2 t) \).

3. NEUTRINO SPIN OSCILLATIONS IN THE KERR METRIC

The Kerr metric [8] describes the space-time geometry in the vicinity of a rotating black hole. The Kerr metric is usually written in Boyer-Lindquist coordinates, which is not diagonal. One can transform it to the standard (diagonal) form with an appropriate change of variables [9]:

\[
ds^2 = -C dt^2 + F dr^2 + R d\varphi^2 + H d\theta^2,
\]

where

\[
H = \rho^2, \quad C = \frac{\Delta}{H}, \quad R = \frac{S}{H}, \quad F = H, \quad \Delta = r^2 + a^2 - 2Mr,
\]

\[
a = \frac{J}{M}, \quad \rho^2 = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 + a^2 - 2Mr, \quad S = (r^2 + a^2)^2 \sin^2 \theta.
\]

The vierbein vectors and their inverses are:

\[
e_\rho^0 = (\sqrt{C}, 0, 0, 0), \quad e_\rho^1 = (0, \sqrt{F}, 0, 0),
\]

\[
e_\rho^2 = (0, 0, \sqrt{H}, 0), \quad e_\rho^3 = (0, 0, 0, \sqrt{R}),
\]

\[
e_\rho^0 = (C^{-1/2}, 0, 0, 0), \quad e_\rho^1 = (0, F^{-1/2}, 0, 0),
\]

\[
e_\rho^2 = (0, 0, R^{-1/2}, 0), \quad e_\rho^3 = (0, 0, 0, R^{-1/2}).
\]

The nonzero components of covariant derivatives of the vierbein vectors can be found using Eqs. (7) and (19):

\[
e_{\rho\theta\tau}, e_{\rho\theta\varphi}, e_{\rho\varphi\tau}, e_{\rho\varphi\varphi}, e_{\tau\theta\tau}, e_{\tau\theta\varphi}, e_{\tau\varphi\tau}, e_{\tau\varphi\varphi}, e_{\varphi\theta\tau}, e_{\varphi\theta\varphi}, e_{\varphi\varphi\tau}, e_{\varphi\varphi\varphi}.
\]

Using Eqs. (4), (6), (9), (19), and (20), we arrive at the following expressions for the analogues of the electric and magnetic fields:

\[
\vec{E} = \left( \frac{\gamma}{\sqrt{F}} e_{\rho\varphi\tau}, \frac{\gamma}{\sqrt{H}} e_{\rho\theta\tau}, 0 \right),
\]

\[
\vec{B} = \left( -e_{\varphi\varphi\tau} e_\theta^\rho U_\varphi, -e_{\varphi\varphi\varphi} e_\theta^\rho U_\varphi, - [e_{\theta\varphi} e_\rho^\theta U_\varphi + e_{\theta\rho} e_\rho^\theta U_\varphi]. \right)
\]

The trajectories of particles and photons in the Kerr metric have been studied in [10]. For motion in the equatorial plane, the equations of motion (11) are reduced to

\[
\frac{d^2\varphi}{dp^2} + 2r \frac{d^2r}{dp^2} - 2hr \frac{dr}{dp} = 0,
\]

\[
\frac{d^2t}{dp^2} + 2r t \frac{dt}{dp} = 0,
\]

\[
\frac{d^2r}{dp^2} + \Gamma_r \left( \frac{dt}{dp} \right)^2 + \Gamma_{r\varphi} \left( \frac{d\varphi}{dp} \right)^2 = 0.
\]

Solving these equations, we find:

\[
\gamma^{-1} = \sqrt{\frac{(2M - r)\xi}{r[a^2M - r(-2M + r)^2]}},
\]

\[
\xi = -2Ma^2 + r(6M^2 - 5Mr + r^2);
\]

\[
v_\varphi = \sqrt{\frac{M}{(-Ma^2 + r(-2M + r)^2)}}.
\]

Substituting (3) and (21) in Eq. (14) using (23) and denoting \( a = 2yM = J/M, r = 2Mk \), the nonzero component \( \Omega_2 \) is found to be

\[
\Omega_2 = \sqrt{\frac{1}{KM - K^2M} \sqrt{\frac{k^2}{-k + k^2 + y^2}} \times \left[ k(-3\beta - 2\zeta) - 2k^3(\beta + \zeta)
\right.
\]

\[
+ k^2(3\beta + 4\zeta) + y^2(2\beta + \zeta)] \left[ 8\beta k^4 M \times \sqrt{\frac{-2k + 4k^2 - 2k^3 + y^2}{(-1 + k)kM(-k + k^2 + y^2)}}(\beta + \zeta) \right]^{-1},
\]

\[
\zeta = \sqrt{\frac{(-1 + k)(3k - 5k^2 + 2k^3 - 2y^2)}{k(2k - 4k^2 + 2k^3 - y^2)}},
\]

\[
\beta = \sqrt{(k - 1)/k}.
\]

In Fig. 3 we show \( 2|\Omega_2|M \) as a function of \( r/2M \) for \( 0 \leq y \leq 0.5 \). We stop at \( y = 0.5 \) since the Kerr metric does not describe a black hole at \( a > M \) and a naked singularity occurs at \( r = 0 \).

It is seen that the maximum frequency decreases with increasing \( J \). It is also seen that the frequency has its largest value at \( J = 0 \). Also, at long distances all the curves approach a common value. This indicates that the effect of rotation diminishes at large \( r \). We note that the \( y = 0 \) case reproduces the result for the Schwarzschild metric [5]. Figure 4 shows the transition probability \( P(t) \).
This implies that the precession frequency $\Omega$ of the flavor pendulum as a function of the neutrino number density will be higher for a charged black hole.

Similarly, it is seen from Fig. 4 that

$$ T_{J=0} < T_{J>0}, $$

(27)

which results in:

$$ \epsilon_t^{J=0} < \epsilon_t^{J>0}. $$

(28)

This implies that the precession frequency $\Omega$ of the flavor pendulum as a function of the neutrino number density will be higher for a non-rotating black hole.

One comment is in order at this point. Collective oscillations are known to happen for neutrinos that leave a supernova mainly along radial trajectories. On the other hand, we have studied spin oscillations of neutrinos trapped in circular orbits. To know the reason, we have provided below three components of the frequency for the Reissner-Nordström metric:

$$ \Omega_1 = \frac{G_1}{\gamma} = \frac{1}{2\gamma} \left[ B_1 + \frac{1}{1 + u_0} \left( E \times \bar{u} \right)_1 \right] 
= \frac{1}{2} v_\varphi \cos \theta, $$

$$ \Omega_2 = \frac{G_2}{\gamma} = \frac{1}{2\gamma} \left[ B_2 + \frac{1}{1 + u_0} \left( E \times \bar{u} \right)_2 \right] 
= \frac{v_\varphi}{2} \sin \theta \left[ -A + \frac{\gamma}{2(1 + \gamma A)} \left( \frac{2M}{r^2} - \frac{2Q^2}{r^3} \right) \right], $$

$$ \Omega_3 = \frac{G_3}{\gamma} = \frac{1}{2\gamma} \left[ B_3 + \frac{1}{1 + u_0} \left( E \times \bar{u} \right)_3 \right] 
= \frac{v_\varphi}{2} \left[ A - \frac{\gamma}{2(1 + \gamma A)} \left( \frac{2M}{r^2} - \frac{2Q^2}{r^3} \right) \right]. $$

(29)

We see that there will be no spin oscillations in the case of a radial trajectory. It is also pointed out in [5] that there is no spin flip for radial motion in the Schwarzschild metric. On the other hand, in the Kerr metric spin oscillations occur for both radial and circular trajectories. Therefore spin oscillations, which can be considered as neutrino-antineutrino oscillations for Majorana neutrinos is a relevant effect for trapped neutrinos.

5. CONCLUSION

In this paper we have investigated neutrino spin oscillations in the gravitational fields of charged and rotating black holes. We have also analyzed the dependence of the oscillation frequency on the orbital radius. In the case of a charged black hole, the maximum frequency is a monotonically increasing function of the charge. For a rotating black hole, the maximum frequency decreases with increasing angular momentum. We have also briefly studied the effects of charge and rotation of a black hole on a bipolar neutrino system.
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