ON THE MINIMAL PENALTY FOR MARKOV ORDER ESTIMATION

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We show that large-scale typicality of Markov sample paths implies that the likelihood ratio statistic satisfies a law of iterated logarithm uniformly to the same scale. As a consequence, the penalized likelihood Markov order estimator is strongly consistent for penalties growing as slowly as \( \log \log n \) when an upper bound is imposed on the order which may grow as rapidly as \( \log n \). Our method of proof, using techniques from empirical process theory, does not rely on the explicit expression for the maximum likelihood estimator in the Markov case and could therefore be applicable in other settings.

1. Introduction. For the purposes of this paper, a Markov chain is a discrete time stochastic process \( (X_k)_{k \geq 1} \), taking values in a state space \( A \) of finite cardinality \( |A| < \infty \), such that the conditional law of \( X_k \) given the past \( X_1, \ldots, X_{k-1} \) depends on the most recent \( r \) states \( X_{k-r}, \ldots, X_{k-1} \) only. The smallest number \( r \) for which this assumption is satisfied is called the order of the Markov chain. It is evident that the order of a Markov chain determines the most parsimonious representation of the law of the process. Thus estimation of the order from observed data is a problem of practical interest, which moreover raises interesting mathematical questions at the intersection of probability, statistics and information theory.

Denote by \( P(x_1:n) \) the probability of the sequence \( x_1:n \in A^n \) under the law \( P \), and denote by \( \Theta^r \) the collection of all laws of Markov chains whose order is at most \( r \). As the parameter spaces \( \Theta^r \subset \Theta^{r+1} \) are increasing, the naive maximum likelihood estimate of the order \( \hat{r}_n = \arg\max_r \sup_{P \in \Theta^r} P(x_1:n) \) fails to be consistent. Instead, we introduce the penalized likelihood order estimator

\[
\hat{r}_n = \arg\max_{0 \leq r < \kappa(n)} \left\{ \sup_{P \in \Theta^r} \log P(x_1:n) - \pen(n,r) \right\},
\]

where \( \pen(n,r) \) is a penalty function and \( \kappa(n) \) is a cutoff function. The estimator is called strongly consistent if \( \hat{r}_n \to r^* \) \( P^* \)-a.s. as \( n \to \infty \) whenever the law of the observations \( P^* \) is the law of a Markov chain whose order is \( r^* \). We aim to understand which penalties and cutoffs yield a strongly consistent estimator.

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Results of this type date back to Finesso [3], who considers the case where the order $r^*$ of the Markov chain $P^*$ is known \textit{a priori} to be bounded above by some constant $r^* < K$. In this setting, Finesso shows that the penalty and cutoff

$$\text{pen}(n, r) = C|A|^r \log \log n, \quad \kappa(n) = K$$

yield a strongly consistent order estimator for a sufficiently large constant $C$ (by [1], p. 592, it suffices to choose $C > 2|A|$). It can be argued from the law of iterated logarithm for martingales that a penalty of this form is the minimal penalty that achieves strong consistency, so that the result is essentially optimal (in the sense that the probability of underestimation of the order is minimized). However, the requirement imposed by the knowledge of an \textit{a priori} upper bound on the order is a significant drawback and is unrealistic in many applications.

Order estimation in the absence of an upper bound has been investigated, for example, by Kieffer [5]. However, the penalty used there is significantly larger than the minimal penalty in the case of an \textit{a priori} upper bound. Kieffer’s conjecture that the well known BIC penalty $\text{pen}(n, r) = \frac{1}{2}|A|^r(|A| - 1) \log n$ yields a strongly consistent order estimator was proved by Csiszár and Shields [3]. The best result to date, due to Csiszár [2], shows that the penalty and cutoff

$$\text{pen}(n, r) = c|A|^r \log n, \quad \kappa(n) = \infty$$

yield a strongly consistent order estimator for any choice of the constant $c > 0$. However, this penalty is still larger than the minimal penalty obtained by Finesso in the case of an \textit{a priori} upper bound on the order. These results raise a basic question [2, 3]: is the $\log \log n$ growth of the penalty the necessary price to be paid for the lack of a prior upper bound on the order, or is the minimal possible penalty $\log \log n$ already sufficient for consistency in the absence of a prior upper bound?

1.1. Results of this paper. The purpose of this paper is twofold.

First, we will show that a penalty of order $\log \log n$ does indeed suffice for consistency of the Markov order estimator, provided we impose a cutoff of order $\kappa(n) \sim \log n$. Remarkably, this is precisely the same cutoff as is required to establish the consistency of minimum description length (MDL) order estimators [2], of which the BIC penalty is an approximation. As the $\log \log n$ penalty is much smaller than the BIC penalty for large $n$, this constitutes a significant improvement over previous results. However, the basic question posed above is only partially resolved, as our results fall short of establishing consistency of the $\log \log n$ penalty in the absence of a cutoff $\kappa(n) = \infty$ as is done in [2, 3] for the BIC penalty.

Second, we introduce a new approach for proving consistency of order estimators in the absence of a prior upper bound on the order. The techniques used in previous work [2, 3] rely heavily on rather delicate explicit computations which
explore the availability of a closed form expression for the maximum likelihood estimator in the Markov case. In contrast, our method of proof, which uses techniques from empirical process theory [6,7], is entirely different and can be applied much more generally. The present approach could therefore provide a possible starting point for extending the results of Csiszár and Shields to problems where an explicit expression for the maximum likelihood is not available, such as the challenging problem of order estimation in hidden Markov models (see [1], Chapter 15).

1.2. Comparison with the approach of Csiszár and Shields. A direct consequence of our main result is that the penalty and cutoff
\[ \text{pen}(n,r) = C^*|A|^r \log \log n, \quad \kappa(n) = \alpha^* \log n \]
with suitable constants \( C^* \) and \( \alpha^* \), where \( \alpha^* \) depends on the observation law \( P^* \), yield a strongly consistent penalized likelihood estimator (in order to obtain a strongly consistent order estimator which does not require prior knowledge of \( P^* \) it suffices to choose \( \kappa(n) = o(\log n) \)). The upper bound \( \kappa(n) = \alpha^* \log n \) is inherited directly from the \textit{large scale typicality} property which plays a central role also in [2,3]. Our main result states that if large scale typicality holds with an upper bound \( r < \kappa(2n) \) on the order, then the likelihood ratio statistic satisfies a law of iterated logarithm uniformly for \( r < \kappa(n) \) (the details are in the following section). Strong consistency of the penalized likelihood order estimator then follows directly.

It is instructive to make a comparison with the approach of [2,3] for the penalty \( \text{pen}(n,r) = c|A|^r \log n \). The proof of strong consistency in this setting consists of two parts. First, large-scale typicality is used to prove strong consistency of the estimator with cutoff \( \kappa(n) = \alpha^* \log n \). Next, a separate argument is employed to show that the larger orders \( r \geq \alpha^* \log n \) are negligible. Our result improves the first part of the proof, as we show that the conclusion already holds for the smaller penalty \( \text{pen}(n,r) = C^*|A|^r \log \log n \). However, the second part of the proof is missing in our setting, and it is unclear whether such a result could in fact be established. The resolution of this problem should effectively identify the minimal penalty for Markov order estimation in the absence of a cutoff.

Let us also note that the first part of the proof in [2] makes use of a sort of truncated law of iterated logarithm for the empirical transition probabilities of the Markov chain. However, the result in [2] implies that the likelihood ratio statistic grows as \( \log \log n \) only for orders as large as \( \log \log n \), while the bound grows as \( \log n \) for orders as large as \( \log n \). Our main result shows that such a bound is not the best possible, resolving in the negative a question posed in [2], p. 1621.

1.3. Organization of the paper. In Section 2, we set up the notation to be used throughout the paper and state our main results. In Section 3, we reduce the proof
of our main result to the problem of establishing a suitable deviation bound. The requisite deviation bound is proved in Section 4. The proof is based on an extension of a maximal inequality of van de Geer [7], which can be found in the Appendix.

2. Main results. Let us fix once and for all the alphabet $A$ of finite cardinality $|A| < \infty$ and the canonical space $\Omega = A^\infty$ endowed with its Borel $\sigma$-field and coordinate process $(X_k)_{k \geq 1}$ ($X_k(\omega) = \omega(k)$ for $\omega \in \Omega$). We will write $x_{m:n}$ for a sequence $(x_m, \ldots, x_n) \in A^{n-m+1}$. Moreover, for any probability measure $P$ on $\Omega$, we will write $P(x_{m:n})$ and $P(x_{m:n} \mid x_{r:s})$ instead of $P(X_m = x_m, \ldots, X_n = x_n)$ and $P(X_{r:s} = x_{r:s}, X_{m:n} = x_{m:n})$, respectively, whenever no confusion can arise.

A Markov chain is defined by a probability measure $P$ such that for some $r \geq 0$

$$P(x_1:n) = P(x_{1:r}) \prod_{i=r+1}^n P(x_i \mid x_{i-r:i-1}) \quad \text{for all } n \geq r, \ x_{1:n} \in A^n.$$  

We will always presume that our Markov chains are time homogeneous:

$$P(X_i = x_{r+1} \mid X_{i-r:r-1} = x_{1:r}) = P(x_{r+1} \mid x_{1:r}) \quad \text{for all } i > r, \ x_{1:r+1} \in A^{r+1}.$$  

We denote by $\Theta^r$ the set of all probability measures that satisfy these conditions for the given value of $r$ ($\Theta^0$ is the class of all i.i.d. processes). Note that $\Theta^r \subset \Theta^{r+1}$ for all $r$. The order of a Markov chain $P$ is the smallest $r \geq 0$ such that $P \in \Theta^r$.

Throughout the paper we fix a distinguished Markov chain $P^*$ of order $r^*$, representing the true probability law of an observed process. We assume that $P^*$ is stationary and irreducible. On the basis of a sequence of observations $x_{1:n}$ we obtain an estimate $\hat{r}_n$ of the true order $r^*$ by maximizing the penalized likelihood

$$\hat{r}_n = \arg\max_{0 \leq r < \kappa(n)} \left\{ \sup_{0 \leq r < \kappa(n)} \log P(x_{1:n}) - \text{pen}(n, r) \right\},$$

where $\text{pen}(n, r)$ is a penalty function and $\kappa(n)$ is a cutoff function. If $\hat{r}_n \xrightarrow{n \to \infty} r^*$ strongly $P^*$-a.s.,

the estimator is called strongly consistent.

Remark 2.1. As discussed in [3], the assumption that $P^*$ is irreducible is necessary for the order estimation problem to be well posed, while stationarity of $P^*$ entails no loss of generality. In particular, the latter claim follows from the fact that any irreducible Markov chain $P$ is absolutely continuous with respect to a stationary Markov chain $P_s$ with the same transition probabilities, so that strong consistency under $P_s$ automatically holds under $P$ also.
Define for any sequence $a_{1:r} \in A^r$ and $n \geq 1$ the random variable

$$N_n(a_{1:r}) = \sum_{i=r+1}^{n} 1_{x_{i-r:i-1} = a_{1:r}},$$

that is, $N_n(a_{1:r})$ is the number of times the sequence $a_{1:r}$ appears as a subsequence of $x_{1:n-1}$. By the ergodic theorem, the approximation $N_n(a_{1:r})/(n-r) \approx P^*(a_{1:r})$ holds for large $n$. The large scale typicality property essentially requires that this approximation holds uniformly for all $a_{1:r}$ with $r < \rho(n)$. As in [2, 3], this idea plays an essential role in the proof of our main result.

**Definition 2.2.** The process $P^*$ is said to satisfy the large-scale typicality property with cutoff $\rho(n)$ if there exists a constant $\eta < 1$ such that

$$\left| \frac{1}{P^*(a_{1:r})} \frac{1}{n-r} - 1 \right| < \eta$$

for all $a_{1:r} \in A^r$ with $P^*(a_{1:r}) > 0$, $r < \rho(n)$ eventually as $n \to \infty$ $P^*$-a.s.

We are now ready to state the main result of this paper, which can be viewed as a law of iterated logarithm for the likelihood ratio statistic. A similar result was established in [4], Lemma 3.4.1 for the case of a fixed order $r > r^*$. Our key innovation is that here the result holds uniformly over the order $r^* < r < \kappa(n)$, where $\kappa(2n)$ is a cutoff for which the large-scale typicality property holds.

**Theorem 2.3.** Let $\kappa(n) \leq n/4$ be an increasing function, such that the process $P^*$ satisfies the large-scale typicality property with cutoff $\kappa(2n)$. Then there is a nonrandom constant $C_0 > 0$ (depending only on $\eta$) such that

$$\sup_{r^* < r < \kappa(n)} \frac{1}{|A|^r} \left\{ \sup_{P \in \Theta^r} \log P(x_{1:n}) - \sup_{P \in \Theta^{r^*}} \log P(x_{1:n}) \right\} \leq C_0 \log \log n$$

eventually as $n \to \infty$ $P^*$-a.s.

The following sections are devoted to the proof of this result. As a corollary, we obtain the following conclusion for the order estimation problem.

**Corollary 2.4.** There exist constants $C^*$ and $\alpha^*$, where $\alpha^*$ depends on $P^*$, such that any penalty and cutoff that satisfy eventually as $n \to \infty$

$$\text{pen}(n, r) = |A|^r f(n) \log \log n, \quad \kappa(n) \leq \alpha^* \log n,$$

yield a strongly consistent Markov order estimator.
PROOF. First, it is easy to see ([3], Proposition A.1) that \( P^\ast \)-a.s.

\[
\limsup_{n \to \infty} \frac{1}{n} \left\{ \sup_{P \in \Theta^r} \log P(x_{1:n}) - \sup_{P \in \Theta^r^\ast} \log P(x_{1:n}) \right\} \leq -C
\]

for some constant \( C > 0 \) and all \( r < r^\ast \). As \( \text{pen}(n,r)/n \to 0 \) as \( n \to \infty \), this implies that \( P^\ast \)-a.s. we have eventually as \( n \to \infty \)

\[
\sup_{P \in \Theta^r} \log P(x_{1:n}) - \text{pen}(n,r) < \sup_{P \in \Theta^r^\ast} \log P(x_{1:n}) - \text{pen}(n,r^\ast) \quad \forall r < r^\ast.
\]

As \( \kappa(n) \geq r^\ast \) for \( n \) sufficiently large, this shows that \( \liminf_{n \to \infty} \hat{r}_n \geq r^\ast \) \( P^\ast \)-a.s.

On the other hand, it is shown in [2, 3] that the large-scale typicality property holds with cutoff \( \kappa(2n) \leq \alpha^\ast \log 2n \) for some constant \( \alpha^\ast \) which depends on \( P^\ast \) (the constant \( \eta \) in Definition 2.2 may be fixed arbitrarily). By Theorem 2.3,

\[
\sup_{r^\ast < r < \kappa(n)} \frac{1}{\text{pen}(n,r)} \left\{ \sup_{P \in \Theta^r} \log P(x_{1:n}) - \sup_{P \in \Theta^r^\ast} \log P(x_{1:n}) \right\} \leq \frac{|A| - 1}{2|A|}
\]

eventually as \( n \to \infty \) \( P^\ast \)-a.s., provided \( C^\ast \) is chosen sufficiently large. Note that

\[
\frac{1}{\text{pen}(n,r) - \text{pen}(n,r^\ast)} = \frac{1}{|A|^r} \frac{|A|^r - |A|^r^\ast}{|A|^r - |A|^r^\ast} \leq \frac{1}{\text{pen}(n,r)} \frac{|A|}{|A| - 1}
\]

for all \( r > r^\ast \), so we find that \( P^\ast \)-a.s. we have eventually as \( n \to \infty \)

\[
\sup_{P \in \Theta^r} \log P(x_{1:n}) - \text{pen}(n,r) < \sup_{P \in \Theta^r^\ast} \log P(x_{1:n}) - \text{pen}(n,r^\ast)
\]

for all \( r^\ast < r < \kappa(n) \). Thus \( \limsup_{n \to \infty} \hat{r}_n \leq r^\ast \) \( P^\ast \)-a.s.

\( \square \)

REMARK 2.5. The proofs of large-scale typicality in [2, 3] actually establish a slightly stronger result, where the constant \( \eta \) in Definition 2.2 is replaced by \( n^{-\beta} \) for some \( \beta > 0 \). This improvement is not needed for Theorem 2.3 to hold.

REMARK 2.6. Theorem 2.3 states that the constant \( C_0 \) depends only on the value of \( \eta \) in Definition 2.2. Unfortunately, the constants obtained by our method of proof are expected to be far from optimal; one can read off a value for \( C_0 \) of order \( 10^6 \) in the proof of Theorem 2.3, which is likely excessively large.

REMARK 2.7. It is not difficult to establish that there is a constant \( C \) such that

\[
\frac{1}{n} \left\{ \sup_{P \in \Theta^r} \log P(x_{1:n}) - \sup_{P \in \Theta^r^\ast} \log P(x_{1:n}) \right\} \leq C
\]
for all $n$ and $r$. It follows that
\[
\sup_{r > (\log |A|)^{-1} \log n} \frac{1}{\text{pen}(n, r)} \left\{ \sup_{P \in \Theta^r} \log P(x_1:n) - \sup_{P^* \in \Theta^{r^*}} \log P(x_1:n) \right\} \leq \frac{|A| - 1}{2|A|}
\]
eventually as $n \to \infty$. In order to obtain a version of Corollary 2.4 with $\kappa(n) = \infty$, the key difficulty is therefore to deal with orders in the range $\alpha^* \log n \leq r \leq (\log |A|)^{-1} \log n$. It is an open question whether it is possible to close this gap.

3. Reduction to a deviation bound. The proof of Theorem 2.3 consists of two steps. In this section, we will prove the result assuming that the likelihood ratio statistic satisfies a certain deviation bound. The requisite deviation bound, which is stated in the following Proposition, will be proved in the next section.

**Proposition 3.1.** Define $F_n = G_n \cap G_{2n}$, where $G_n$ denotes the event
\[
\left\{ \frac{1}{P^*(a_{1:r})} \frac{N_n(a_{1:r})}{n - r} - 1 \leq \eta \text{ for all } a_{1:r} \in A^r \text{ with } P^*(a_{1:r}) > 0, \ r < \rho(n) \right\},
\]
with $\rho(n)$ increasing and $\rho(n) \leq n/2$. Then there exist constants $C_1, C_1', C_2 > 0$, which can be chosen to depend only on $\eta$, such that
\[
P^* \left[ F_n \cap \max_{i = n, \ldots, 2n} \left\{ \sup_{P \in \Theta^r} \log P(x_1:i) - \log P^*(x_1:i|1_{1:r}) \right\} \geq \epsilon \right] \leq C_1' e^{-\epsilon/C_1}
\]
for all $n \geq 1, r^* < r < \rho(n)$, and $\epsilon \geq C_2 |A|^r$.

Conceptually, this result can be understood as follows. It is well known in classical statistics that, in “regular” cases, the likelihood ratio statistic
\[
\sup_{P \in \Theta^r} \log P(x_1:n) - \log P^*(x_1:n)
\]
converges weakly as $n \to \infty$ to a $\chi^2$-distributed random variable. Therefore, we expect the likelihood ratio statistic to possess exponential tails at least for large $n$. Proposition 3.1 provides a precise nonasymptotic description of this phenomenon.

We now prove Theorem 2.3 presuming that Proposition 3.1 holds.

**Proof of Theorem 2.3.** We clearly need only consider sequences $x_{1:n}$ with
\( \mathbb{P}^*(x_{1:n}) > 0 \). We begin with some straightforward estimates:

\[
\sup_{r^* < r < \kappa(n)} \frac{1}{|A|^r} \left\{ \sup_{p \in \Theta^r} \log \mathbb{P}(x_{1:n}) - \sup_{p \in \Theta^{r^*}} \log \mathbb{P}(x_{1:n}) \right\}
\]

\[
\leq \sup_{r^* < r < \kappa(n)} \frac{1}{|A|^r} \left\{ \sup_{p \in \Theta^r} \log \mathbb{P}(x_{1:n}) - \log \mathbb{P}^*(x_{1:n}) \right\}
\]

\[
= \sup_{r^* < r < \kappa(n)} \frac{1}{|A|^r} \left\{ \sup_{p \in \Theta^r} \log \mathbb{P}(x_{1:n}) - \log \mathbb{P}^*(x_{1:n}|x_{1:r}) - \log \mathbb{P}^*(x_{1:r}) \right\}
\]

\[
\leq \sup_{r^* < r < \kappa(n)} \frac{1}{|A|^r} \left\{ \sup_{p \in \Theta^r} \log \mathbb{P}(x_{1:n}) - \log \mathbb{P}^*(x_{1:n}|x_{1:r}) \right\} + C,
\]

for a constant \( C \) independent of \( n \) and \( x_{1:n} \). Here we have used that for any irreducible (and time homogeneous) Markov chain \( \mathbb{P}^* \), there exists a constant \( 0 < \lambda < 1 \) such that \( \mathbb{P}^*(x_{1:r}) > \lambda^r \) whenever \( \mathbb{P}^*(x_{1:r}) > 0 \), so that

\[
\sup_{r^* > r^*} \frac{-\log \mathbb{P}^*(x_{1:r})}{|A|^r} \leq C := \log(1/\lambda) \sup_{r^* > r^*} \frac{r}{|A|^r} < \infty.
\]

We conclude that it suffices to prove

\[
\sup_{r^* < r < \kappa(n)} \frac{1}{|A|^r} \left\{ \sup_{p \in \Theta^r} \log \mathbb{P}(x_{1:n}) - \log \mathbb{P}^*(x_{1:n}|x_{1:r}) \right\} \leq C_0 \log \log n
\]

eventually as \( n \to \infty \) \( \mathbb{P}^* \)-a.s. Define for simplicity

\[
\Delta_{i,r} = \sup_{p \in \Theta^r} \log \mathbb{P}(x_{1:i}) - \log \mathbb{P}^*(x_{1:i}|x_{1:r}).
\]

We can estimate

\[
\mathbb{P}^* \left[ F_{2^n} \cap \max_{2^n \leq i \leq 2^{n+1}} \frac{1}{\log \log i} \sup_{r^* < r < \kappa(i)} \frac{\Delta_{i,r}}{|A|^r} \geq C_0 \right]
\]

\[
\leq \mathbb{P}^* \left[ F_{2^n} \cap \max_{2^n \leq i \leq 2^{n+1}} \sup_{r^* < r < \kappa(2^{n+1})} \frac{\Delta_{i,r}}{|A|^r} \geq C_0 \log \log 2^n \right]
\]

\[
\leq \sum_{r^* < r < \kappa(2^{n+1})} \mathbb{P}^* \left[ F_{2^n} \cap \max_{2^n \leq i \leq 2^{n+1}} \Delta_{i,r} \geq C_0 |A|^r \log \log 2^n \right],
\]

where we used that \( \kappa(n) \) is increasing. Now let \( F_n \) be defined as in Proposition 3.1 for \( \rho(n) = \kappa(2n) \). Then there exist \( C_1, C'_1 \) such that for all \( n \) sufficiently large,

\[
\mathbb{P}^* \left[ \max_{2^n \leq i \leq 2^{n+1}} \Delta_{i,r} \geq C_0 |A|^r \log \log 2^n \right] \leq C'_1 e^{-C_0 |A|^r \log \log 2^n / C_1}
\]
for all \( r^* < r < \kappa(2^n+1) \). Therefore

\[
P^* \left[ F_{2^n} \cap \max_{2^n \leq i \leq 2^{n+1}} \frac{1}{\log \log i} \sup_{r^* < r < \kappa(i)} \frac{\Delta_{i,r}}{|A|^r} \geq C_0 \right] \leq C'_1 e^{-C_0 \log \log 2/C_1} n - C_0/C_1 \]

for \( n \) sufficiently large. Thus for any choice of \( C_0 > C_1 \), we find that

\[
\sum_{n=1}^{\infty} P^* \left[ F_{2^n} \cap \max_{2^n \leq i \leq 2^{n+1}} \frac{1}{\log \log i} \sup_{r^* < r < \kappa(i)} \frac{\Delta_{i,r}}{|A|^r} \geq C_0 \right] < \infty.
\]

By the Borel-Cantelli lemma,

\[
F_{2^n} \cup \max_{2^n \leq i \leq 2^{n+1}} \frac{1}{\log \log i} \sup_{r^* < r < \kappa(i)} \frac{\Delta_{i,r}}{|A|^r} < C_0 \text{ eventually as } n \to \infty \text{ P}^* \text{-a.s.}
\]

But by large-scale typicality with cutoff \( \kappa(2n) \), we know that \( F_{2^n} \) must hold eventually as \( n \to \infty \) \( P^* \)-a.s. The result follows immediately.

**Remark 3.2.** The proof of Theorem 2.3 shows that the large-scale typicality property is in fact only needed along an exponentially increasing subsequence of times \( t_n = 2^n \), so that the assumption of the Theorem can be weakened slightly. However, the weaker assumption does not ultimately appear to lead to better results than the full large-scale typicality assumption (for example, note that the proof of large-scale typicality in [3] already utilizes such a subsequence).

**Remark 3.3.** Theorem 2.3 could be improved by employing the blocking procedure along the subsequence \( t_n = \gamma^n \) for arbitrary \( \gamma > 1 \). In this manner, one can establish that the result is still valid under the weaker assumption that the large-scale typicality property holds with cutoff \( \kappa(\gamma n) \) for some \( \gamma > 1 \). However, this does not appear to lead to a substantially different conclusion for the order estimation problem. In order to keep the notation and proofs as transparent as possible we have restricted our results to the case \( \gamma = 2 \), but the necessary modifications for the case of arbitrary \( \gamma > 1 \) are easily implemented.

**4. Proof of Proposition 3.1.** The longest part of the proof of Theorem 2.3 consists of the proof of Proposition 3.1. To establish this result, we adapt an approach using techniques from empirical process theory [6, 7] that was originally developed to obtain rates of convergence for nonparametric maximum likelihood
estimators in the i.i.d. setting. At the heart of the proof of Proposition 3.4 lies an extension of a maximal inequality for families of martingales under bracketing entropy conditions, due to van de Geer \cite{van1989}, Theorem 8.13. The extension of this result that is needed for our purposes is developed in the Appendix.

4.1. Preliminary computations. Any measure $P \in \Theta^r$ is uniquely determined by its initial probability $P(x_{1:r})$ and its transition probability $P(x_{r+1}|x_{1:r})$. It is easily seen that the measure which maximizes the log-likelihood $\log P(x_{1:n})$ of $P \in \Theta^r$ assigns unit probability to the observed initial path $x_{1:r}$. Thus for $r > r^*$

$$\sup_{P \in \Theta^r} \log P(x_{1:n}) - \log P^*(x_{1:n}|x_{1:r}) = \sup_{P \in \Theta^r} \sum_{i=r+1}^{n} \log \left( \frac{P(x_i|x_{i-r:i-1})}{P^*(x_i|x_{i-r:i-1})} \right).$$

The family of functions $\log(P(x_i|x_{i-r:i-1})/P^*(x_i|x_{i-r:i-1})) (P \in \Theta^r)$ is $P^*$-a.s. uniformly bounded from above but not from below. To avoid problems later on, we apply a standard trick. For any $P \in \Theta^r$, define

$$\tilde{P}(x_i|x_{i-r:i-1}) = \frac{P(x_i|x_{i-r:i-1}) + P^*(x_i|x_{i-r:i-1})}{2}.$$ 

Thus $\tilde{P}$ is a Markov chain whose transition probabilities are an equal mixture of the transition probabilities of $P$ and $P^*$ (the initial probabilities of $\tilde{P}$ are irrelevant for our purposes and need not be defined). By concavity of the logarithm, we find

$$\sup_{P \in \Theta^r} \log P(x_{1:n}) - \log P^*(x_{1:n}|x_{1:r}) \leq 2 \sup_{P \in \Theta^r} \sum_{i=r+1}^{n} \log \left( \frac{\tilde{P}(x_i|x_{i-r:i-1})}{P^*(x_i|x_{i-r:i-1})} \right).$$

It therefore suffices to obtain a deviation bound for the right hand side of this expression, whose summands are $P^*$-a.s. uniformly bounded above and below.

4.2. Peeling. The first part of the proof of Proposition 3.1 aims to reduce the problem to a deviation inequality for martingales. To this end we employ a peeling device from the theory of weighted empirical processes.

Define the natural filtration $\mathcal{F}_n = \sigma\{X_1, \ldots, X_n\}$. For any $P \in \Theta^r$, we define

$$M^P_n = \sum_{i=r+1}^{n} \left\{ \log \left( \frac{\tilde{P}(x_i|x_{i-r:i-1})}{P^*(x_i|x_{i-r:i-1})} \right) - E^* \left[ \log \left( \frac{\tilde{P}(x_i|x_{i-r:i-1})}{P^*(x_i|x_{i-r:i-1})} \right) \right| \mathcal{F}_{i-1} \right\},$$

which is a martingale (under $P^*$) by construction. It is easily seen that

$$M^P_n = \sum_{i=r+1}^{n} \log \left( \frac{\tilde{P}(x_i|x_{i-r:i-1})}{P^*(x_i|x_{i-r:i-1})} \right) + D^P_n,$$
where we have defined
\[ D_n^P = - \sum_{i=r+1}^{n} \sum_{a_i \in A} P^*(a_i|x_{i-r:i-1}) \log \left( \frac{\hat{P}(a_i|x_{i-r:i-1})}{P^*(a_i|x_{i-r:i-1})} \right). \]

We also define for any \( P, P' \in \Theta^r \) the quantity
\[ H_n(P, P') = \sum_{i=r+1}^{n} \sum_{a_i \in A} \left( \hat{P}(a_i|x_{i-r:i-1})^{1/2} - \hat{P}'(a_i|x_{i-r:i-1})^{1/2} \right)^2. \]

Note that \( \sqrt{H_n(P, P')} \) defines a random distance on \( \Theta^r \). As we will see below, the role of the set \( F_n \) (and hence the large-scale typicality assumption) in the proof of Proposition [37] is that it allows us to control this random distance.

**Lemma 4.1.** For any \( \epsilon > 0, n \geq 1 \) and \( r > r^* \)
\[
P^* \left[ F_n \cap \max_{i=n, \ldots, 2n} \left\{ \sup_{P \in \Theta^r} \log P(x_{1:i}) - \log P^*(x_{1:i}) \right\} \geq \epsilon \right]
\]
\[
\leq \sum_{k=0}^{\infty} P^* \left[ F_n \cap \sup_{P \in \Theta^r} 1_{H_n(P, P') \leq 2k\epsilon} \max_{i=n, \ldots, 2n} M_i^P \geq 2^{k+1}\epsilon \right].
\]

**Proof.** From the discussion above, it is clear that
\[
P^* \left[ F_n \cap \max_{i=n, \ldots, 2n} \left\{ \sup_{P \in \Theta^r} \log P(x_{1:i}) - \log P^*(x_{1:i}) \right\} \geq \epsilon \right]
\]
\[
\leq P^* \left[ F_n \cap \max_{i=n, \ldots, 2n} \sup_{P \in \Theta^r} \sum_{\ell=r+1}^{i} \log \left( \frac{\hat{P}(x_{\ell}|x_{\ell-r:r-1})}{P^*(x_{\ell}|x_{\ell-r:r-1})} \right) \geq \frac{\epsilon}{2} \right]
\]
\[
= P^* \left[ F_n \cap \max_{i=n, \ldots, 2n} \sup_{P \in \Theta^r} \left\{ M_i^P - D_i^P \right\} \geq \frac{\epsilon}{2} \right].
\]

Now note that as \( - \log x \geq 2 - 2\sqrt{x} \) for \( x > 0 \),
\[
D_i^P \geq 2 \sum_{i=r+1}^{n} \sum_{a_i \in A} P^*(a_i|x_{i-r:i-1}) \left( 1 - \frac{\hat{P}(a_i|x_{i-r:i-1})^{1/2}}{P^*(a_i|x_{i-r:i-1})^{1/2}} \right) = H_n(P, P^*).
\]

Therefore, we can estimate
\[
P^* \left[ F_n \cap \max_{i=n, \ldots, 2n} \left\{ \sup_{P \in \Theta^r} \log P(x_{1:i}) - \log P^*(x_{1:i}) \right\} \geq \epsilon \right]
\]
\[
\leq P^* \left[ F_n \cap \max_{i=n, \ldots, 2n} \sup_{P \in \Theta^r} \left\{ M_i^P - H_i(P, P^*) \right\} \geq \frac{\epsilon}{2} \right]
\]
\[
\leq P^* \left[ F_n \cap \sup_{P \in \Theta^r} \left\{ \max_{i=n, \ldots, 2n} M_i^P - H_n(P, P^*) \right\} \geq \frac{\epsilon}{2} \right].
\]
We now partition the space \( \Theta^r \) into an inner ring \( \{ \mathbf{P} \in \Theta^r : H_n(\mathbf{P}, \mathbf{P}^*) \leq \varepsilon \} \) and a collection of concentric rings \( \{ \mathbf{P} \in \Theta^r : 2^{k-1}\varepsilon \leq H_n(\mathbf{P}, \mathbf{P}^*) \leq 2^k\varepsilon \} \) (note that this is a random partition, as the quantity \( H_n(\mathbf{P}, \mathbf{P}^*) \) depends on the observed path). Applying the union bound gives the estimates

\[
\mathbf{P}^* \left[ F_n \cap \max_{i=n, \ldots, 2n} \left\{ \sup_{\mathbf{P} \in \Theta^r} \log \mathbf{P}(x_{1:i}) - \log \mathbf{P}^*(x_{1:i}|x_{1:r}) \right\} \geq \varepsilon \right]
\]

\[
\leq \mathbf{P}^* \left[ F_n \cap \sup_{\mathbf{P} \in \Theta^r} \left\{ \max_{i=n, \ldots, 2n} M_{1}^{\mathbf{P}} - H_n(\mathbf{P}, \mathbf{P}^*) \right\} 1_{H_n(\mathbf{P}, \mathbf{P}^*) \leq \varepsilon} \geq \frac{\varepsilon}{2} \right]
\]

\[
+ \sum_{k=1}^{\infty} \mathbf{P}^* \left[ F_n \cap \sup_{\mathbf{P} \in \Theta^r} \left\{ \max_{i=n, \ldots, 2n} M_{1}^{\mathbf{P}} - H_n(\mathbf{P}, \mathbf{P}^*) \right\} \times 1_{2^{k-1}\varepsilon \leq H_n(\mathbf{P}, \mathbf{P}^*) \leq 2^{k}\varepsilon} \geq \frac{\varepsilon}{2} \right]
\]

\[
\leq \sum_{k=0}^{\infty} \mathbf{P}^* \left[ F_n \cap \sup_{\mathbf{P} \in \Theta^r} 1_{H_n(\mathbf{P}, \mathbf{P}^*) \leq 2^{k}\varepsilon} \max_{i=n, \ldots, 2n} M_{1}^{\mathbf{P}} \geq 2^{k-1}\varepsilon \right].
\]

The proof is complete. \(\square\)

4.3. Control of \( H_n \). Our next task is to control the quantity \( H_n(\mathbf{P}, \mathbf{P}') \). First, we show that on the event \( F_n \) the quantity \( H_n \) is comparable to

\[
H(\mathbf{P}, \mathbf{P}') = \sum_{a_{1:r+1} \in A^{r+1}} \mathbf{P}^*(a_{1:r}) \left( \mathbf{P}'(a_{r+1}|a_{1:r})^{1/2} - \mathbf{P}'(a_{r+1}|a_{1:r})^{1/2} \right)^2,
\]

which is a nonrandom squared distance on \( \Theta^r \).

**Lemma 4.2.** There exist constants \( C_3, C_4 \) such that for any \( n \geq 1 \), we have

\[
H_{2n}(\mathbf{P}, \mathbf{P}') \leq C_3 H_n(\mathbf{P}, \mathbf{P}')
\]

and

\[
(n-r) C_4^{-1} H(\mathbf{P}, \mathbf{P}') \leq H_n(\mathbf{P}, \mathbf{P}') \leq (n-r) C_4 H(\mathbf{P}, \mathbf{P}')
\]

for all \( \mathbf{P}, \mathbf{P}' \in \Theta^r \) and \( r^* < r < \rho(n) \) on the event \( F_n \).

**Proof.** It is easily seen that for any \( n \geq 1 \)

\[
H_n(\mathbf{P}, \mathbf{P}') = \sum_{a_{1:r+1} \in A^{r+1}} N_n(a_{1:r}) \left( \mathbf{P}'(a_{r+1}|a_{1:r})^{1/2} - \mathbf{P}'(a_{r+1}|a_{1:r})^{1/2} \right)^2.
\]
On the event $F_n$, we have by construction

$$ (1 - \eta) P^*(a_{1:r}) \leq \frac{N_n(a_{1:r})}{n - r} \leq (1 + \eta) P^*(a_{1:r}) $$

and

$$ (1 - \eta) P^*(a_{1:r}) \leq \frac{N_{2n}(a_{1:r})}{2n - r} \leq (1 + \eta) P^*(a_{1:r}) $$

for all $a_{1:r} \in A^r$ and $r < \rho(n)$. Here we have used that $\rho(n) \leq \rho(2n)$ as $\rho(n)$ is presumed to be increasing. In particular, we have

$$ N_{2n}(a_{1:r}) \leq \frac{1 + \eta}{1 - \eta} \frac{2n - r}{n - r} N_n(a_{1:r}) \leq 4 \frac{1 + \eta}{1 - \eta} N_n(a_{1:r}), $$

where we have used that $n - r > n/2$ as $r < \rho(n) < n/2$. The result follows directly provided we choose $C_3, C_4$ (depending only on $\eta$) sufficiently large. \qed

Next, we control the quantity $H_n(P, P^*)$ in terms of the “Bernstein norm” needed in order to apply the results developed in the Appendix. As in the Appendix, we define the function $\phi(x) = e^x - x - 1$.

**Lemma 4.3.** Define for any $P \in \Theta^r$, $r > r^*$ and $n \geq 1$

$$ R_n^P = 8 \sum_{i=r+1}^n \mathbb{E}^*[\phi\left(\frac{1}{2}\left|\log\left(\frac{\tilde{P}(x_i|x_{i-r:i-1})}{P^*(x_i|x_{i-r:i-1})}\right)\right|\right] | \mathcal{F}_{i-1}] . $$

Then $R_n^P \leq 8H_n(P, P^*)$ for any $P \in \Theta^r$, $r > r^*$ and $n \geq 1$.

**Proof.** Note that $\log(\tilde{P}(x_i|x_{i-r:i-1})/P^*(x_i|x_{i-r:i-1})) \geq -\log(2)$. By [1], Lemma 7.1, we have $\phi(|x|) \leq (e^x - 1)^2$ for any $x \geq -\log(2)/2$. Therefore

$$ R_n^P \leq 8 \sum_{i=r+1}^n \mathbb{E}^* \left[ \left( \frac{\tilde{P}(x_i|x_{i-r:i-1})^{1/2}}{P^*(x_i|x_{i-r:i-1})^{1/2}} - 1 \right)^2 | \mathcal{F}_{i-1} \right] $$

$$ = 8 \sum_{i=r+1}^n \mathbb{E} \left[ \sum_{a_i \in A} P^*(a_i|x_{i-r:i-1}) \left( \frac{\tilde{P}(a_i|x_{i-r:i-1})^{1/2}}{P^*(a_i|x_{i-r:i-1})^{1/2}} - 1 \right)^2 \right] . $$

The result follows immediately. \qed

Together with Lemma 7.1, we obtain the following.
COROLLARY 4.4. Define for any $\sigma > 0$ the ball

$$\Theta'(\sigma) = \{ P \in \Theta' : H(P, P^*) \leq \sigma \}.$$ 

Then for any $\varepsilon > 0$, $n \geq 1$ and $r^* < r < \rho(n)$

$$P^* \left[ F_n \cap \max_{i=n, \ldots, 2n} \left\{ \sup_{P \in \Theta'} \log P(x_{1;i}) - \log P^*(x_{1;i} | x_{1:r}) \right\} \geq \varepsilon \right]$$

$$\leq \sum_{k=0}^{\infty} P^* \left[ F_n \cap \sup_{P \in \Theta'(C_42^k\varepsilon/(n-r))} 1_{R_{2^n-1} \leq C_32^{k+1}\varepsilon} \max_{i \leq 2n} M_i^P \geq 2^{k+1}\varepsilon \right].$$

The proof is straightforward and is therefore omitted.

4.4. Control of the bracketing entropy. We have now reduced the proof of Proposition 3.1 to the problem of estimating the summands in Corollary 4.4. We aim to do this by applying Proposition A.2 in the Appendix with $\Theta \subseteq \Theta'$.

$$\xi_i^P = \begin{cases} \log(\tilde{P}(x_{1|x_{i-r;i-1}})/P^*(x_{1|x_{i-r;i-1}})) & \text{for } i > r, \\ 0 & \text{for } i \leq r, \end{cases}$$

and $K = 2$. To this end, the main remaining difficulty is to estimate the bracketing entropy of Definition A.1. This is our next order of business.

LEMMA 4.5. Given $c > 0$, there exists $C_5 > 0$ depending only on $c$ such that

$$\log N(2n, \Theta'(\sigma), F_n, 2, \delta) \leq |A|^{r+1} \log \left( \frac{C_5\sqrt{(2n-r)\sigma}}{\delta} \right)$$

for all $n \geq 1$, $r^* < r < \rho(n)$, $\sigma > 0$ and $0 < \delta \leq c\sqrt{(2n-r)\sigma}$.

PROOF. Fix $n \geq 1$, $r^* < r < \rho(n)$, $\sigma > 0$ and $0 < \delta \leq c\sqrt{(2n-r)\sigma}$ throughout the proof. We begin by defining the family of functions

$$T_\beta = \{ p : A^{r+1} \rightarrow [0, \infty) : P^*(a_{1;r})^{1/2}p(a_{1;r+1})^{1/2} \in \beta \exists \forall a_{1:r+1} \in A^{r+1} \},$$

where $\beta > 0$ is to be determined in due course. We claim that for any $P \in \Theta'$, there exist $\lambda^P, \gamma^P \in T_\beta$ such that for all $a_{1:r+1} \in A^{r+1}$ with $P^*(a_{1;r}) > 0$

$$\lambda^P(a_{1;r+1}) \leq P(a_{r+1}|a_{1:r}) \leq \gamma^P(a_{1;r+1})$$

and

$$\gamma^P(a_{1;r+1})^{1/2} - \lambda^P(a_{1;r+1})^{1/2} \leq \frac{\beta}{P^*(a_{1;r})^{1/2}}.$$
Indeed, this follows immediately by setting
\[ \lambda^P(a_{1:r+1}) = \left( \frac{\beta^{-1} P^*(a_{1:r})^{1/2} P(a_{r+1}|a_{1:r})^{1/2}}{\beta^{-1} P^*(a_{1:r})^{1/2}} \right)^2, \]
\[ \gamma^P(a_{1:r+1}) = \left( \frac{\beta^{-1} P^*(a_{1:r})^{1/2} P(a_{r+1}|a_{1:r})^{1/2}}{\beta^{-1} P^*(a_{1:r})^{1/2}} \right)^2 \]
for all \( a_{1:r+1} \in \mathcal{A}^{r+1} \) with \( P^*(a_{1:r}) > 0 \). Therefore \( P^* \)-a.s.
\[ \Lambda^P_i := \log \left( \frac{\lambda^P(x_i|x_{i-r-1})}{\lambda^* P(x_i|x_{i-r-1})} \right) \leq \xi^P_i \leq \log \left( \frac{\gamma^P(x_i|x_{i-r-1})}{\gamma^* P(x_i|x_{i-r-1})} \right) =: \Upsilon^P_i \]
for all \( P \in \Theta^r \), \( i > r \) (we set \( \Lambda^P_i = \Upsilon^P_i = 0 \) for \( i \leq r \)), where we have defined \( \gamma^P(x_i|x_{i-r-1}) = \frac{\gamma^P(x_i|x_{i-r-1}) + P^*(x_i|x_{i-r-1})}{2} \) and \( \lambda^P(x_i|x_{i-r-1}) = \frac{\lambda^P(x_i|x_{i-r-1}) + P^*(x_i|x_{i-r-1})}{2} \). Moreover, we can estimate
\[
8 \sum_{i=1}^{2n} \mathbb{E} \left[ \phi \left( \frac{\Upsilon^P_i - \Lambda^P_i}{2} \right) \right] \mathcal{F}_{i-1} \leq 4 \sum_{i=1}^{2n} \mathbb{E} \left[ \left( \frac{\gamma^P(x_i|x_{i-r-1})^{1/2} - \lambda^P(x_i|x_{i-r-1})^{1/2}}{\lambda^P(x_i|x_{i-r-1})^{1/2} - 1} \right)^2 \right] \mathcal{F}_{i-1} \]
\[
\leq 8 \sum_{i=r+1}^{2n} \sum_{a_i \in \mathcal{A}_{r+1}} (\gamma^P(a_i|x_{i-r-1})^{1/2} - \lambda^P(a_i|x_{i-r-1})^{1/2})^2 \]
\[
\leq 4 \sum_{a_{1:r+1} \in \mathcal{A}^{r+1}} N_{2n}(a_{1:r+1}) \left( \gamma^P(a_{1:r+1})^{1/2} - \lambda^P(a_{1:r+1})^{1/2} \right)^2 \]
\[
\leq 4 \beta^2 \sum_{a_{1:r+1} \in \mathcal{A}^{r+1}} N_{2n}(a_{1:r+1}) \frac{N_{2n}(a_{1:r+1})}{P^*(a_{1:r+1})},
\]
where we have used that \( \phi(x) \leq (e^x - 1)^2/2 \) for \( x \geq 0 \) and \( \Upsilon^P_i \), Lemma 4.2. As in the proof of Lemma 4.2, we find that for any \( P \in \Theta^r \)
\[ 8 \sum_{i=1}^{2n} \mathbb{E} \left[ \phi \left( \frac{\Upsilon^P_i - \Lambda^P_i}{2} \right) \right] \mathcal{F}_{i-1} \leq 4 C_4 (2n - r) |\mathcal{A}|^{r+1} \beta^2 \]
on the event \( F_n \) (as \( r < \rho(n) \) by assumption). Therefore, if we choose
\[ \beta = \frac{\delta}{\sqrt{4 C_4 (2n - r) |\mathcal{A}|^{r+1}}}, \]
then \( \{(\Lambda^P_i, \Upsilon^P_i)_{1 \leq i \leq 2n} \}_{P \in \Theta^r} \) is a \((2n, \Theta^r, F_n, 2, \delta)\)-bracketing set. To complete the proof we must estimate the cardinality of this set.
We approach this problem through a well known geometric device. We can represent any function from $\mathbb{A}^{r+1}$ to $\mathbb{R}$ as a vector in $\mathbb{R}^{|\mathbb{A}|^{r+1}}$ in the obvious fashion. In particular, for any $p : \mathbb{A}^{r+1} \to \mathbb{R}$, denote by $\iota[p]$ the representative in $\mathbb{R}^{|\mathbb{A}|^{r+1}}$ of the function $\bar{p}(a_{1:r+1}) = \mathbf{P}^{*}(a_{1:r})^{1/2} p(a_{1:r+1})^{1/2}$. Then by [7], Lemma 4.2
\[
\iota[\Theta^r(\sigma)] \subseteq B(0, 4\sqrt{\sigma}) \cap \mathbb{R}_{+}^{|\mathbb{A}|^{r+1}}, \quad x_0 = \iota[\mathbf{P}^{*}(a_{r+1}|a_{1:r})],
\]
where $B(x, h)$ denotes the Euclidean ball in $\mathbb{R}^{|\mathbb{A}|^{r+1}}$ with center $x$ and radius $h$. On the other hand, we clearly have $\iota[T_\beta] = (\beta \mathbb{Z}_+)^{|\mathbb{A}|^{r+1}} \subseteq \mathbb{R}^{|\mathbb{A}|^{r+1}}$. Define for any $x, x' \in \mathbb{R}^{|\mathbb{A}|^{r+1}}$ with $x' > x$ the cube $[x, x'] := \{ \tilde{x} \in \mathbb{R}^{|\mathbb{A}|^{r+1}} : x \leq \tilde{x} \leq x' \}$. Let
\[
\Xi_\beta := \{ x \in (\beta \mathbb{Z}_+)^{|\mathbb{A}|^{r+1}} : [x, x + \beta 1] \cap B(x_0, 4\sqrt{\sigma}) \neq \emptyset \},
\]
where $1 \in \mathbb{R}^{|\mathbb{A}|^{r+1}}$ denotes the vector all of whose entries are one. Then clearly
\[
\iota[\Theta^r(\sigma)] \subseteq B(x_0, 4\sqrt{\sigma}) \cap \mathbb{R}_{+}^{|\mathbb{A}|^{r+1}} \subseteq \bigcup_{x \in \Xi_\beta} [x, x + \beta 1],
\]
and, in particular, it is easily established from our previous computations that
\[
N(2n, \Theta^r(\sigma), F_\beta, 2, \delta) \leq |\Xi_\beta|.
\]
Now suppose that $x' \in [x, x + \beta 1]$ for some $x \in \Xi_\beta$. Then there is an $x'' \in [x, x + \beta 1]$ such that $x'' \in B(x_0, 4\sqrt{\sigma})$. In particular, we have $\|x' - B(x_0, 4\sqrt{\sigma})\|_{\infty} \leq \beta$, and therefore $\|x' - B(x_0, 4\sqrt{\sigma})\|_2 \leq |\mathbb{A}|^{(r+1)/2} \beta$, for every $x' \in [x, x + \beta 1], x \in \Xi_\beta$. We conclude that
\[
\bigcup_{x \in \Xi_\beta} [x, x + \beta 1] \subseteq B(x_0, 4\sqrt{\sigma} + |\mathbb{A}|^{(r+1)/2} \beta).
\]
Therefore, we can estimate
\[
|\Xi_\beta| \beta^{|\mathbb{A}|^{r+1}} = \text{vol} \left( \bigcup_{x \in \Xi_\beta} [x, x + \beta 1] \right) \leq \text{vol} \left( B(x_0, 4\sqrt{\sigma} + |\mathbb{A}|^{(r+1)/2} \beta) \right) = (4\sqrt{\sigma} + |\mathbb{A}|^{(r+1)/2} \beta)^{|\mathbb{A}|^{r+1}} \text{vol}(B(0, 1)).
\]
But from [7], p. 249 we have the estimate
\[
\text{vol}(B(0, 1)) \leq \left( \frac{\sqrt{2\pi e}}{|\mathbb{A}|^{(r+1)/2}} \right)^{|\mathbb{A}|^{r+1}}.
\]
Substituting the expression for $\beta$ and rearranging, we find that
\[
|\Xi_\beta| \leq \left( \frac{(8\sqrt{C_4} + c)\sqrt{2\pi e}}{\delta} \sqrt{(2n - r))\sigma} \right)^{|\mathbb{A}|^{r+1}},
\]
where we have used that $\delta \leq c\sqrt{(2n - r))\sigma}$. The proof is easily completed. \qed
4.5. **End of the proof.** To complete the proof of Proposition 3.1, it remains to put together the results obtained above with Proposition A.2 in the Appendix.

**Proof of Proposition 3.1.** In the following, we will always apply Lemma 4.5 and Proposition A.2 with the same constants \(c, c_0, c_1 > 0\). The appropriate values of these constants will be determined below. We will also fix \(n \geq 1\), \(r^* < r < \rho(n)\) and \(\varepsilon \geq C_2 |A|^r\), with the constant \(C_2\) to be determined.

To apply Corollary 4.4, we invoke Proposition A.2 with \(K = 2\), \(\alpha = 2^{k-1} \varepsilon\), and \(R = C_3 2^{k+3} \varepsilon\) (fixing \(k \geq 0\) for the time being). We find that

\[
P^* \left[ F_n \cap \sup_{P \in \Theta^r(\mathcal{C}_4 2^{k} \varepsilon/(n-r))} 1_{P_{2n} \leq C_2 2^{k+3} \varepsilon} \max_{i \leq 2n} M_i^P \geq 2^{k-1} \varepsilon \right] \leq 2 \exp \left( -\frac{2^{k-5} \varepsilon}{C_3 C_2 (c_1 + 1)} \right),
\]

provided that \(c_0^2 \geq C^2 (c_1 + 1)\) and

\[
c_0 \int_0^{\sqrt{C_3 2^{k+3} \varepsilon}} \sqrt{\log N(2n, \Theta^r(\mathcal{C}_4 2^{k} \varepsilon/(n-r)), F_n, 2, u)} \, du \leq 2^{k-1} \varepsilon \leq c_1 C_3 2^{k+2} \varepsilon.
\]

To ensure that the second inequality holds, it suffices to choose \(c_1 = (8C_3)^{-1}\), and the condition on \(c_0\) is satisfied by choosing \(c_0 = C \sqrt{(8C_3)^{-1} + 1}\). To simplify the first inequality, choose \(c = \sqrt{8C_3/C_4}\). Then the variable \(u\) in the integral satisfies

\[
u \leq \sqrt{C_3 2^{k+3} \varepsilon} \leq c \sqrt{(2n-r) C_4 2^{k} \varepsilon/(n-r)},
\]

so by Lemma 4.3 it suffices to ensure that

\[
2^{k-1} \varepsilon \geq |A|^{(r+1)/2} C \sqrt{(8C_3)^{-1} + 1} \int_0^{\sqrt{C_3 2^{k+3} \varepsilon}} \sqrt{\log \left( \frac{(4C_4)^{1/2} C_5 2^{k} \varepsilon}{v} \right)} \, dv,
\]

where we have used that \(r < \rho(n) \leq n/2\) implies \((2n-r)/(n-r) \leq 4\). Defining

\[
C_6 := \int_0^{\sqrt{8C_3}} \sqrt{\log \left( \frac{(4C_4)^{1/2} C_5}{v} \right)} \, dv < \infty,
\]

a simple change of variables shows that the above inequality is equivalent to

\[
2^{k-1} \varepsilon \geq |A|^{(r+1)/2} C_6 C \sqrt{(8C_3)^{-1} + 1} \sqrt{2^{k} \varepsilon},
\]

or, equivalently,

\[
2^{k} \varepsilon \geq 4C_6^2 C^2 ((8C_3)^{-1} + 1)|A|^{r+1}.
\]
But this is always satisfied if we choose $C_2 = 4C_0^2C^2((8C_3)^{-1} + 1)|A|$.

With these choices of $c, c_0, c_1, C_2$, we have thus shown that by Corollary 4.4

$$
P^* \left[ F_n \cap \max_{i=n, \ldots, 2n} \left\{ \sup_{P \in \Theta} \log P(x_{1:i}) - \log P^*(x_{1:i}) \right\} \geq \varepsilon \right] \leq 2 \sum_{k=0}^{\infty} \exp \left[ -\frac{2^k \varepsilon}{2^k C^2 (C_3 + 1/8)} \right] \leq C_1' \exp \left[ -\frac{\varepsilon}{C_1} \right]
$$

with

$$
C_1 = 2^5 C^2 (C_3 + 1/8), \quad C_1' = \frac{2}{1 - e^{-C_2/2^k C^2 (C_3 + 1/8)}},
$$

where we have used $\varepsilon \geq C_2$. This completes the proof. \qed

**APPENDIX A: A MAXIMAL INEQUALITY FOR MARTINGALES**

The purpose of this Appendix is to obtain a deviation bound on the supremum of an uncountable family of martingales, extending a result of van de Geer [7].

We work on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_i\}_{i \geq 0}, \mathbb{P})$. We are given a parameter set $\Theta$ and a collection $(\xi^\theta_i)_{i \geq 1}, \theta \in \Theta$ of random variables such that $\xi^\theta_i$ is $\mathcal{F}_i$-measurable for all $i, \theta$. This setting will be presumed throughout the Appendix. In the following we will frequently use the function $\phi(x) = e^x - x - 1$.

**Definition A.1.** Let $n \in \mathbb{N}, F \in \mathcal{F}, K > 0$ and $\delta > 0$ be given. A finite collection $\{\Lambda^j_i, \Upsilon^j_i\}_{1 \leq i \leq n, 1 \leq j \leq N}$ of random variables is called a $(n, \Theta, F, K, \delta)$-bracketing set if $\Lambda^j_i, \Upsilon^j_i$ are $\mathcal{F}_i$-measurable for all $i, j$, and for every $\theta \in \Theta$, there is a $1 \leq j \leq N$ (the map $\theta \mapsto j$ is nonrandom) such that $\mathbb{P}$-a.s.

$$
\Lambda^j_i \leq \xi^\theta_i \leq \Upsilon^j_i \quad \text{for all } i = 1, \ldots, n
$$

and such that

$$
2K^2 \sum_{i=1}^{n} \mathbb{E} \left[ \phi \left( \frac{\left| \Upsilon^j_i - \Lambda^j_i \right|}{K} \right) \right] \leq \delta^2 \quad \text{on } F.
$$

We denote as $\mathcal{N}(n, \Theta, F, K, \delta)$ the cardinality $N$ of the smallest $(n, \Theta, F, K, \delta)$-bracketing set (log $\mathcal{N}(n, \Theta, F, K, \delta)$ is called the bracketing entropy).

The following extends a result of van de Geer [7], Theorem 8.13.

**Proposition A.2.** Fix $K > 0$, and define for all $i \geq 0$

$$
M^\theta_i = \sum_{\ell=1}^{i} \xi^\theta_{\ell} - \mathbb{E} [\xi^\theta_{\ell} | \mathcal{F}_{\ell-1}], \quad R^\theta_i = 2K^2 \sum_{\ell=1}^{i} \mathbb{E} \left[ \phi \left( \frac{\left| \xi^\theta_{\ell} \right|}{K} \right) \right] \mathcal{F}_{\ell-1}.
$$
There is a universal constant $C > 0$ such that for any $n \in \mathbb{N}$, $R < \infty$ and $F \in \mathcal{F}$

$$
P \left[ F \cap \sup_{\theta \in \Theta} 1_{R_n^\theta \leq R} \max_{i \leq n} M_i^\theta \geq \alpha \right] \leq 2 \exp \left[ -\frac{\alpha^2}{C^2(c_1 + 1)R} \right]
$$

for any $\alpha, c_0, c_1 > 0$ such that $c_0^2 \geq C^2(c_1 + 1)$ and

$$
c_0 \int_0^{\sqrt{R}} \sqrt{\log N(n, \Theta, F, K, u)} \, du \leq \alpha \leq c_1 R.
$$

[For example, the choice $C = 100$ works.]

**Remark A.3.** Throughout, all uncountable suprema should be interpreted as essential suprema under the measure $P$. Thus measurability problems are avoided.

For our purposes, the key improvement over [7], Theorem 8.13 is that the bound in this result is given for $\max_{i \leq n} M_i^\theta$ rather than $M_n^\theta$. This is essential in order to employ the blocking procedure in the proof of Theorem 2.3. Rather than repeat the proof of [7], Theorem 8.13 here with the necessary modifications, we take the opportunity to obtain a more general result from which Proposition A.2 follows.

**Theorem A.4.** Fix $K > 0$, and define for all $i \geq 0$

$$
M_i^\theta = \sum_{\ell=1}^{i} \{ \xi_\ell^\theta - \mathbb{E}[\xi_\ell^\theta | \mathcal{F}_{\ell-1}] \}, \quad R_i^\theta = 2K^2 \sum_{\ell=1}^{i} \mathbb{E} \left[ \phi \left( \frac{|\xi_\ell^\theta|}{K} \right) \bigg| \mathcal{F}_{\ell-1} \right].
$$

Then we have for any $n \in \mathbb{N}$, $R < \infty$, $F \in \mathcal{F}$ and $x > 0$

$$
P \left[ F \cap \sup_{\theta \in \Theta} 1_{R_n^\theta \leq R} \max_{i \leq n} M_i^\theta \geq 16 \mathcal{H} + 32\sqrt{Rx} + 16Kx \right] \leq 2 e^{-x},
$$

where we have written

$$
\mathcal{H} = K \log N(n, \Theta, F, K, \sqrt{R}) + 4 \int_0^{\sqrt{R}} \sqrt{\log N(n, \Theta, F, K, u)} \, du.
$$

Before we proceed, let us prove Proposition A.2 using Theorem A.4.

---

\[\text{A closer look at the proof of [7], Theorem 8.13 reveals a few inconsistencies which are corrected here. For example, equation (A.12) in [7] seems to presuppose that } X \geq 0 \text{ on an event } A \text{ implies that } P[X|A] \geq 0 \text{ on } A, \text{ which need not be the case. The bracketing condition given in [7], Definition 8.1 therefore seems too weak to give the desired result. Similarly, the version of Bernstein’s inequality given as [7], Lemma 8.9 does not appear to be the one used in the proof of Theorem 8.13.}\]
PROOF OF PROPOSITION A.2. Let $\alpha = \sqrt{C^2(c_1 + 1)R}$ and assume that the given bounds on $\alpha$ hold. Then we can estimate

$$x = \frac{\alpha^2}{C^2(c_1 + 1)R} \leq \frac{c_1 R}{K} \times \frac{\alpha}{C^2(c_1 + 1)R} \leq \frac{\alpha}{C^2K}, \quad \alpha = (\sqrt{\alpha})^2 \leq \frac{c_1 R\alpha}{K}.$$

On the other hand, as $N(n, \Theta, F, K, \delta)$ is nonincreasing, we have

$$c_0 \sqrt{R \log N(n, \Theta, F, K, \sqrt{R})} \leq c_0 \int_0^{\sqrt{R}} \sqrt{\log N(n, \Theta, F, K, u)} \, du \leq \alpha.$$

Applying Theorem A.4, we find that

$$P \left[ F \cap \sup_{\theta \in \Theta} 1_{R_n^\theta \leq R} \max_{i \leq n} M_i^\theta \geq \left\{ \frac{16c_1}{c_0^2} + \frac{64}{c_0}\sqrt{C^2(c_1 + 1)} + \frac{32}{C^2} + \frac{16}{C^2} \right\} \alpha \right] \leq 2 \exp \left[ -\frac{\alpha^2}{C^2(c_1 + 1)R} \right].$$

But using $c_0^2 \geq C^2(c_1 + 1) \geq C^2$, we can estimate

$$\frac{16c_1}{c_0^2} + \frac{64}{c_0} + \frac{32}{\sqrt{C^2(c_1 + 1)}} + \frac{16}{C^2} \leq \frac{32}{C^2} + \frac{96}{C} \leq 1$$

for $C$ sufficiently large (e.g., $C = 100$).

The remainder of the Appendix is devoted to the proof of Theorem A.4. It should be emphasized that the approach taken here is entirely standard in empirical process theory: the notion of bracketing entropy for martingales and the proof of the requisite form of Bernstein’s inequality follows van de Geer [7], while the relatively transparent proof of Theorem A.4 closely follows the proof given by Massart [6], Theorem 6.8 in the i.i.d. setting. The full proofs are given here for completeness. Note also that we have made no effort to optimize the constants in the proof (the constants are necessarily somewhat larger than those obtained in [6] due to the presence of the additional maximum $\max_{i \leq n} M_i^\theta$).

A.1. A variant of Bernstein’s inequality. The following result is a variant of Bernstein’s inequality for martingales. It slightly improves on [7], Lemma 8.11 in that we do not assume that $E[\xi_i | \mathcal{F}_{i-1}] = 0$ for all $i$ (though it appears that this version is implicitlyused in the proof of [7], Theorem 8.13).
**Proposition A.5.** Let \((\xi_i)_{i \geq 1}\) be a sequence of random variables such that \(\xi_i\) is \(\mathcal{F}_i\)-measurable for all \(i\), and define the martingale

\[ M_j = \sum_{i=1}^{j} \{\xi_i - \mathbb{E}[\xi_i|\mathcal{F}_{i-1}]\} \quad \text{for all } j \geq 0. \]

Fix \(K > 0\), and let \((Z_j)_{j \geq 0}\) be predictable (i.e., \(Z_j\) is \(\mathcal{F}_{j-1}\)-measurable) such that

\[ \sum_{i=1}^{j} \mathbb{E}[|\xi_i|^m|\mathcal{F}_{i-1}] \leq m!K^m Z_j \quad \text{for all } m \geq 2, j \geq 0. \]

Then we have for all \(\alpha > 0\) and \(Z > 0\)

\[ \mathbb{P}[M_j \geq \alpha \text{ and } Z_j \leq Z \text{ for some } j] \leq \exp\left[-\frac{\alpha^2}{2K(\alpha + 2KZ)}\right]. \]

**Proof.** Given \(\lambda^{-1} > K\) we define the process \((S_j)_{j \geq 0}\) as \(S_j = e^{\lambda M_j - Z_j}\), where \(Z_j^\lambda = \sum_{i=1}^{\lambda j} \mathbb{E}[\phi(\lambda|\xi_i|)|\mathcal{F}_{i-1}]\). Using \(1 + x \leq e^x\), we find

\[ \frac{S_j}{S_{j-1}} = e^{\lambda_j - \mathbb{E}[\lambda_j|\mathcal{F}_{j-1}] - \mathbb{E}[\phi(\lambda|\xi_j)|\mathcal{F}_{j-1}]} \leq \frac{\{1 + \phi(\lambda_j) + \lambda_j\}e^{-\mathbb{E}[\lambda_j|\mathcal{F}_{j-1}]} + \mathbb{E}[\phi(\lambda|\xi_j)|\mathcal{F}_{j-1}] + \mathbb{E}[\mathbb{E}[\lambda_j|\mathcal{F}_{j-1}]]}{1 + \mathbb{E}[\phi(\lambda|\xi_j)|\mathcal{F}_{j-1}]} \leq 1. \]

Now using the basic property \(\phi(x) \leq \phi(|x|)\) and \(1 + x \leq e^x\), we have

\[ \mathbb{E} \left[ \frac{S_j}{S_{j-1}} \big| \mathcal{F}_{j-1} \right] \leq e^{-\mathbb{E}[\lambda_j|\mathcal{F}_{j-1}]} \left(1 + \frac{\mathbb{E}[\mathbb{E}[\lambda_j|\mathcal{F}_{j-1}]]}{1 + \mathbb{E}[\phi(\lambda|\xi_j)|\mathcal{F}_{j-1}]} \right) \leq e^{-\mathbb{E}[\lambda_j|\mathcal{F}_{j-1}]} \{1 + \mathbb{E}[\lambda_j|\mathcal{F}_{j-1}]\} \leq 1. \]

Thus \(S_j\) is a positive supermartingale. To proceed, define the stopping time

\[ \tau = \min\{j : M_j \geq \alpha \text{ and } Z_j \leq Z\}. \]

Then \(\{M_j \geq \alpha \text{ and } Z_j \leq Z \text{ for some } j\} = \{\tau < \infty\}\). Moreover, as \(\lambda^{-1} > K\)

\[ Z_j^\lambda = \sum_{\ell=2}^{\infty} \frac{\lambda^\ell}{\ell!} \sum_{i=1}^{\ell} \mathbb{E} \left[ |\xi_i|^{\ell} \big| \mathcal{F}_{i-1} \right] \leq Z_j \sum_{\ell=2}^{\infty} (\lambda K)^{\ell} = \frac{\lambda^2 K^2}{1 - \lambda K} Z_j \quad \text{for all } j. \]

Therefore \(Z_\tau^\lambda \leq \lambda^2 K^2 Z_\tau/(1 - \lambda K)\), and we can estimate

\[ S_\tau = e^{\lambda M_\tau - Z_\tau} \geq e^{\lambda M_\tau - \lambda^2 K^2 Z_\tau/(1 - \lambda K)} \geq e^{\lambda \alpha - \lambda^2 K^2 Z/(1 - \lambda K)} \quad \text{on } \{\tau < \infty\}. \]

We obtain, using the supermartingale property,

\[ \mathbb{P}[\tau < \infty] \leq \mathbb{E}[1_{\{\tau < \infty\}}e^{\lambda^2 K^2 Z/(1 - \lambda K) - \lambda \alpha} S_\tau] \leq e^{\lambda^2 K^2 Z/(1 - \lambda K) - \lambda \alpha}. \]

The proof is completed by choosing \(\lambda^{-1} = K + 2K^2 Z/\alpha\). \qed
Corollary A.6. Let $(\xi_i)_{1 \leq i \leq n}$ be a sequence of random variables such that \( \xi_i \) is \( \mathcal{F}_i \)-measurable for all \( i \), and fix \( K > 0 \). Define \((M_j)_{0 \leq j \leq n}\) and \((R_j)_{0 \leq j \leq n}\) as

\[
M_j = \sum_{i=1}^{j} \{ \xi_i - E[\xi_i | \mathcal{F}_{i-1}] \}, \quad R_j = 2K^2 \beta \sum_{i=1}^{j} E \left[ \frac{\Phi(\xi_i)}{K} \right] |\mathcal{F}_{i-1} |.
\]

Then we have for all \( \alpha > 0 \) and \( R > 0 \)

\[
P \left[ \max_{j \leq n} M_j \geq \alpha \text{ and } R_n \leq R \right] \leq \exp \left[ -\frac{\alpha^2}{2(K\alpha + R)} \right].
\]

If in addition \( \|\xi_i\|_\infty \leq 3U \) for all \( i \), then for all \( \alpha > 0 \) and \( R > 0 \)

\[
P \left[ \max_{j \leq n} M_j \geq \alpha \text{ and } R_n \leq R \right] \leq \exp \left[ -\frac{\alpha^2}{2(U\alpha + R)} \right].
\]

Proof. To obtain the first inequality, note that for any \( m \geq 2 \) and \( j \geq 0 \)

\[
\sum_{i=1}^{j} E [ |\xi_i|^m | \mathcal{F}_{i-1} ] \leq \sum_{m=2}^{\infty} \sum_{i=1}^{j} E [ |\xi_i|^m | \mathcal{F}_{i-1} ] = \frac{R_j}{2K^2}.
\]

We can therefore apply Proposition A.5 with \( Z_j = R_j/2K^2 \). For the second inequality, note that \( \|\xi_i\|_\infty \leq 3U \) implies that for all \( m \geq 2 \) and \( j \geq 0 \)

\[
\sum_{i=1}^{j} E [ |\xi_i|^m | \mathcal{F}_{i-1} ] \leq (3U)^{m-2} \sum_{i=1}^{j} E [ |\xi_i|^2 | \mathcal{F}_{i-1} ] \leq (3U)^{m-2} R_j \leq \frac{m!U^m R_j}{2U^2},
\]

where we used that \( m! \geq 2 \times 3^{m-2} \) for \( m \geq 2 \). We can therefore apply Proposition A.5 with \( Z_j = R_j/2U^2 \). It remains to use that \( R_j \) is nondecreasing. \( \square \)

A.2. Maximal inequalities for finite sets. The following result allows us to control finite families of random variables that satisfy a Bernstein-type deviation inequality. A sharper form of this result can be obtained using an estimate on the moment generating function of the random variables, see [3], Lemma 2.3, but we do not have such an estimate for the maximum \( \max_{i \leq n} M_i^d \). Throughout the remainder of the Appendix, we define \( E^A[X] = E[1_AX]/P[A] \) for any event \( A \in \mathcal{F} \).

Lemma A.7. Let \( X_1, \ldots, X_N \) be random variables such that

\[
P[|X_i| \geq \alpha] \leq \exp \left[ -\frac{\alpha^2}{2(K\alpha + R)} \right] \text{ for all } 1 \leq i \leq N.
\]

Then we have for any event \( A \in \mathcal{F} \)

\[
E^A \left[ \max_{i=1, \ldots, N} |X_i| \right] \leq \sqrt{8R \log \left( 1 + \frac{N}{P[A]} \right)} + 2K \log \left( 1 + \frac{N}{P[A]} \right).
\]
PROOF. Let $\psi(x)$ be a Young function. Then
\[
\psi\left(\frac{E^A [\max_{i \leq N} |X_i|]}{\max_{i \leq N} \|X_i\|_\psi}\right) \leq E^A \left[\max_{i \leq N} \psi \left(\frac{|X_i|}{\|X_i\|_\psi}\right)\right] \\
\leq \sum_{i \leq N} E^A \left[\psi \left(\frac{|X_i|}{\|X_i\|_\psi}\right)\right] \leq \frac{1}{P[A]} \sum_{i \leq N} E \left[\psi \left(\frac{|X_i|}{\|X_i\|_\psi}\right)\right] \leq \frac{N}{P[A]},
\]
where $\| \cdot \|_\psi$ denotes the Orlicz norm. Therefore
\[
E^A \left[\max_{i=1,\ldots,N} |X_i|\right] \leq \psi^{-1}\left(\frac{N}{P[A]}\right) \max_{i=1,\ldots,N} \|X_i\|_\psi.
\]
To proceed, note that for $1 \leq i \leq N$
\[
P[|X_i|_{|X_i| \leq R/K} \geq \alpha] = P[R/K \geq |X_i| \geq \alpha] \leq \exp \left[-\frac{\alpha^2}{4R}\right],
\]
\[
P[|X_i|_{|X_i| \geq R/K} \geq \alpha] = P[|X_i| \geq \alpha \lor R/K] \leq \exp \left[-\frac{\alpha}{4}\right].
\]
By [\text{[3]}], Lemma 2.2.1, $\|X_i 1_{|X_i| \leq R/K}\|_{\psi_2} \leq \sqrt{8R}$ and $\|X_i 1_{|X_i| \geq R/K}\|_{\psi_1} \leq 8K$ for all $i$, where $\psi_p(x) = e^{xp} - 1$. The proof is easily completed. \hfill $\Box$

\textbf{Corollary A.8.} Let $(\xi^h_i)_{1 \leq i \leq n}$, $h = 1, \ldots, N$ be random variables such that $\xi^h_i$ is $\mathcal{F}_i$-measurable for all $i, h$. Fix $K > 0$, and define
\[
M^h_j = \sum_{i=1}^j \{\xi^h_i - E[\xi^h_i | \mathcal{F}_{i-1}]\}, \quad R^h_j = 2K^2 \sum_{i=1}^j E \left[\phi \left(\frac{|\xi^h_i|}{K}\right) | \mathcal{F}_{i-1}\right].
\]

Then we have
\[
E^A \left[\max_{h=1,\ldots,N} 1_{R^h_j \leq R} \max_{j \leq n} M^h_j\right] \leq \sqrt{8R \log \left(1 + \frac{N}{P[A]}\right) + 8K \log \left(1 + \frac{N}{P[A]}\right)}
\]
for any event $A \in \mathcal{F}$. If in addition $\|\xi^h_i\|_{\infty} \leq 3U$ for all $i, h$, then
\[
E^A \left[\max_{h=1,\ldots,N} 1_{R^h_j \leq R} \max_{j \leq n} M^h_j\right] \leq \sqrt{8R \log \left(1 + \frac{N}{P[A]}\right) + 8U \log \left(1 + \frac{N}{P[A]}\right)}
\]
for any event $A \in \mathcal{F}$. \hfill $\Box$

\textbf{Proof.} Apply the previous lemma with $X_h = 1_{R^h_j \leq R} \max_{j \leq n} M^h_j$. Note that as $M^h_0 = 0$, certainly $X_h \geq 0$. Therefore $X_h = |X_h|$, and the requisite tail bounds are obtained immediately from Corollary A.8 above. \hfill $\Box$
A.3. Proof of Theorem A.4. We now proceed to the proof of Theorem A.4. We follow closely the proof given by Massart [3]. Theorem 6.8 in the i.i.d. setting. The general approach, by means of a chaining device with bracketing with adaptive truncation, is standard in empirical process theory.

Before we proceed to the proof, let us define the function

$$\Phi(x) := 16H + 32\sqrt{R}x + 16Kx,$$

where $H$ is as defined in Theorem A.4. We claim that in order to prove the Theorem, it actually suffices to prove the estimate

$$E^A \left[ \sup_{\theta \in \Theta} 1_{R^\theta_n \leq R} \max_{i \leq n} M^\theta_i \right] \leq \Phi \left( \log \left( 1 + \frac{1}{P[A]} \right) \right)$$

for any event $A \subseteq F$. Indeed, if this is the case, then choosing

$$A = F \cap \left\{ \sup_{\theta \in \Theta} 1_{R^\theta_n \leq R} \max_{i \leq n} M^\theta_i \geq \Phi(x) \right\},$$

allows us to estimate

$$\Phi(x) \leq E^A \left[ \sup_{\theta \in \Theta} 1_{R^\theta_n \leq R} \max_{i \leq n} M^\theta_i \right] \leq \Phi \left( \log \left( \frac{2}{P[A]} \right) \right),$$

from which the conclusion of Theorem A.4 is immediate. We therefore concentrate without loss of generality on obtaining the above estimate.

PROOF OF THEOREM A.4. We fix $n \in \mathbb{N}$, $K, R < \infty$, $F \in \mathcal{F}$ and $A \subseteq F$ throughout the proof. Define $\delta_j = 2^{-j} \sqrt{R}$ and $N_j = N(n, \Theta, F, K, \delta_j)$ for $j \geq 0$. We assume that $N_j < \infty$ for all $j$, otherwise there is nothing to prove. Therefore, for each $j$, we can choose a collection $B_j = \{(\Lambda_i^{j,\theta}, \Upsilon_i^{j,\theta})_{1 \leq i \leq n} \}_{\rho=1,\ldots,N_j}$ that satisfies the conditions of Definition A.1, and these will remain fixed throughout the proof. In particular, for every $j, \theta$, there exists $\rho(j, \theta)$ such that

$$\Lambda_i^{j,\rho(j,\theta)} \leq \xi_i^\theta \leq \Upsilon_i^{j,\rho(j,\theta)} \quad \text{for all } i = 1, \ldots, n.$$

For notational simplicity, we will write

$$\Pi_i^{j,\theta} = \Upsilon_i^{j,\rho(j,\theta)}, \quad \Delta_i^{j,\theta} = \Upsilon_i^{j,\rho(j,\theta)} - \Lambda_i^{j,\rho(j,\theta)}.$$

At the heart of the proof is a chaining device: we introduce the telescoping sum

$$\xi_i^\theta = \{ \xi_i^\theta - \Pi_i^{\tau_i^\theta,\theta} \wedge \Pi_i^{\tau_i^\theta-1,\theta} \} + \{ \Pi_i^{\tau_i^\theta,\theta} \wedge \Pi_i^{\tau_i^\theta-1,\theta} - \Pi_i^{\tau_i^\theta-1,\theta} \} + \sum_{j=1}^{\tau_i^\theta-1} \{ \Pi_i^{j,\theta} - \Pi_i^{j-1,\theta} \} + \Pi_i^{0,\theta},$$
where by convention $\Pi_i^{-1,\theta} = \Pi_i^{0,\theta}$. The length of the chain is chosen adaptively:

$$\tau_i^\theta = \min \{ j \geq 0 : \Delta_i^{j,\theta} > a_j \} \land J.$$ 

The levels $a_j > 0$ and $J \geq 1$ will be determined later on (we will choose $a_j$ to control the second term in Corollary A.8, and we will ultimately let $J \to \infty$).

It will be convenient to split the chain into three parts:

\begin{align}
\xi_i^{\theta} &= \Pi_i^{0,\theta} + \\
(A.1) & \sum_{j=0}^{J} (\xi_i^{\theta} - \Pi_i^{j,\theta} \land \Pi_i^{j-1,\theta}) 1_{\tau_i^\theta = j} + \\
(A.2) & \sum_{j=1}^{J} \left\{ (\Pi_i^{j,\theta} \land \Pi_i^{j-1,\theta} - \Pi_i^{j-1,\theta}) 1_{\tau_i^\theta = j} + (\Pi_i^{j,\theta} - \Pi_i^{j-1,\theta}) 1_{\tau_i^\theta > j} \right\}.
\end{align}

Denote by $b_i^{j,\theta}$ the summands in (A.1) by $c_i^{j,\theta}$ the summands in (A.2), and define the martingales $A_i^{\theta} = \sum_{i=1}^{J} (\Pi_i^{0,\theta} - E[\Pi_i^{0,\theta}|F_{\ell-1}])$, $B_i^{j,\theta} = \sum_{i=1}^{J} (b_i^{j,\theta} - E[b_i^{j,\theta}|F_{\ell-1}])$, and $C_i^{j,\theta} = \sum_{i=1}^{J} (c_i^{j,\theta} - E[c_i^{j,\theta}|F_{\ell-1}])$. We will control each martingale separately.

**Control of $A^\theta$.** As $\phi$ is convex and nondecreasing, and as $|\Pi_i^{0,\theta} - \xi_i^\theta| \leq |\Delta_i^{0,\theta}|$,

$$\phi\left(\frac{|\Pi_i^{0,\theta}|}{2K}\right) \leq \phi\left(\frac{|\Pi_i^{0,\theta} - \xi_i^\theta|}{2K} + |\xi_i^\theta|\right) \leq \frac{1}{2} \phi\left(\frac{|\Delta_i^{0,\theta}|}{K}\right) + \frac{1}{2} \phi\left(\frac{|\xi_i^\theta|}{K}\right).$$

Using Definition A.1, we find that

$$R_n^{0,\theta} := 8K^2 \sum_{\ell=1}^{n} E\left[\phi\left(\frac{|\Pi_{\ell}^{0,\theta}|}{2K}\right)\bigg|F_{\ell-1}\right] \leq 2(\delta_0^2 + R) = 4R \quad \text{on } \{R_n^\theta \leq R\} \cap F.$$

Therefore

$$E^A\left[\sup_{\theta \in \Theta} 1_{R_n^\theta \leq R} \max_{i \leq n} A_i^{\theta}\right] \leq E^A\left[\sup_{\theta \in \Theta} 1_{R_n^\theta \leq 2(\delta_0^2 + R)} \max_{i \leq n} A_i^{\theta}\right] \leq \sqrt{32R \log \left(1 + \frac{N_0}{P[A]}\right)} + 16K \log \left(1 + \frac{N_0}{P[A]}\right)$$

by Corollary A.8, where we have used that $A \subseteq F$.

**Control of $B^\theta$.** Note that $b_i^{j,\theta} \leq 0$, so that

$$b_i^{j,\theta} - E[b_i^{j,\theta}|F_{\ell-1}] \leq E[(\Pi_i^{j,\theta} \land \Pi_i^{j-1,\theta} - \xi_i^\theta) 1_{\tau_i^\theta = j}|F_{\ell-1}] \leq E[\Delta_i^{j,\theta} 1_{\tau_i^\theta = j}|F_{\ell-1}].$$
Consider first the case that \( j < J \). When \( \tau_\ell^\theta = j \), we have \( \Delta_\ell^{j,\theta} > a_j \). Thus

\[
|b_\ell^{j,\theta} - \mathbb{E}[b_\ell^{j,\theta}|\mathcal{F}_{\ell-1}]| \leq \frac{1}{a_j} \mathbb{E}[|\Delta_\ell^{j,\theta}|^2|\mathcal{F}_{\ell-1}] \leq \frac{2K^2}{a_j} \mathbb{E} \left[ \phi \left( \frac{|\Delta_\ell^{j,\theta}|}{K} \right) |\mathcal{F}_{\ell-1} \right],
\]

where we have used \(|x|^2 \leq 2K^2 \phi(|x|/K)\). In particular,

\[
B_i^{j,\theta} \leq \frac{2K^2}{a_j} \sum_{\ell=1}^i \mathbb{E} \left[ \phi \left( \frac{|\Delta_\ell^{j,\theta}|}{K} \right) |\mathcal{F}_{\ell-1} \right] \leq \frac{\delta_i^2}{a_j} \quad \text{on } F,
\]

where we have applied Definition \( A.1 \). As \( A \subseteq F \), it follows that

\[
\mathbb{E}^A \left[ \sup_{\theta \in \Theta} 1_{R_n^\theta \leq R} \max_{i \leq n} B_i^{j,\theta} \right] \leq \frac{\delta_i^2}{a_j} \quad \text{for } j < J.
\]

Now consider the case \( j = J \). We can estimate

\[
B_i^{J,\theta} \leq \sum_{\ell=1}^i \mathbb{E}[|\Delta_\ell^{J,\theta}||\mathcal{F}_{\ell-1}] \leq \left[ \sum_{\ell=1}^i \mathbb{E}[|\Delta_\ell^{J,\theta}|^2|\mathcal{F}_{\ell-1}] \right]^{1/2} \leq \delta_J \sqrt{i} \quad \text{on } F,
\]

where we have applied the same computations as above. It follows that

\[
\mathbb{E}^A \left[ \sup_{\theta \in \Theta} 1_{R_n^\theta \leq R} \max_{i \leq n} B_i^{J,\theta} \right] \leq \delta_J \sqrt{n},
\]

where we have used that \( A \subseteq F \).

**Control of \( C^{\theta} \).** As \( \Pi_\ell^{j,\theta} - \Pi_{\ell-1}^{j,\theta} = \Pi_\ell^{j,\theta} - \xi_\ell^\theta + \xi_\ell^\theta - \Pi_{\ell-1}^{j,\theta} \), we have

\[
-\Delta_{\ell-1}^{j,\theta} \leq \Pi_\ell^{j,\theta} - \Pi_{\ell-1}^{j,\theta} \leq \Delta_\ell^{j,\theta}, \quad -\Delta_{\ell-1}^{j,\theta} \leq \Pi_\ell^{j,\theta} \land \Pi_{\ell-1}^{j,\theta} - \Pi_{\ell-1}^{j,\theta} \leq 0.
\]

Therefore

\[
-\Delta_\ell^{j,\theta} 1_{\ell \geq j} \leq c_\ell^{j,\theta} \leq \Delta_\ell^{j,\theta} 1_{\ell > j}.
\]

As \( \Delta_\ell^{j,\theta} \leq a_j \) whenever \( \tau_\ell^\theta > j \), we find that

\[
\|c_\ell^{j,\theta}\|_\infty \leq a_{j-1} \lor a_j.
\]

Moreover, as \(|c_\ell^{j,\theta}| \leq \Delta_\ell^{j-1,\theta} \lor \Delta_\ell^{j,\theta} \leq \Delta_\ell^{j-1,\theta} + \Delta_\ell^{j,\theta} \), we obtain using that \( \phi \) is convex and nondecreasing (in the same manner as above for the control of \( A^{\theta} \))

\[
R_n^{j,\theta} := 8K^2 \sum_{\ell=1}^n \mathbb{E} \left[ \phi \left( \frac{|c_\ell^{j,\theta}|}{2K} \right) |\mathcal{F}_{\ell-1} \right] \leq 2(\delta_j^2 + \delta_j^2) \quad \text{on } F,
\]
where we have used Definition A.1. As $A \subseteq F$, we can therefore estimate

$$
\mathbb{E}^A \left[ \sup_{\theta \in \Theta} 1_{R_n^\theta \leq R \max_{i \leq n} C^j_{i,\theta}} \right] \leq \mathbb{E}^A \left[ \sup_{\theta \in \Theta} 1_{R_n^\theta \leq 2(\delta_{j-1}^2 + \delta_j^2) \max_{i \leq n} C^j_{i,\theta}} \right].
$$

Now note that $C^j_{i,\theta}$ depends on $\theta$ only through the values of $\rho(0, \theta), \ldots, \rho(j, \theta)$. In particular, for fixed $j$, the supremum of $1_{R_n^\theta \leq 2(\delta_{j-1}^2 + \delta_j^2) \max_{i \leq n} C^j_{i,\theta}}$ as $\theta$ varies over $\Theta$ is in fact only the maximum over a finite collection of random variables, whose cardinality is bounded above by the quantity

$$
N_j := \prod_{p=0}^j N_p.
$$

We therefore obtain the estimate

$$
\mathbb{E}^A \left[ \sup_{\theta \in \Theta} 1_{R_n^\theta \leq R \max_{i \leq n} C^j_{i,\theta}} \right] \leq \sqrt{16(\delta_{j-1}^2 + \delta_j^2) \log \left( 1 + \frac{N_j}{P[A]} \right)} + \frac{8}{3} (a_{j-1} \vee a_j) \log \left( 1 + \frac{N_j}{P[A]} \right),
$$

where we have applied Corollary A.3.

**End of the proof.** Note that by construction

$$
M^\theta_i = A^\theta_i + \sum_{j=0}^J B^j_{i,\theta} + \sum_{j=1}^J C^j_{i,\theta}
$$

for all $i, \theta$. Collecting the above estimates gives

$$
\mathbb{E}^A \left[ \sup_{\theta \in \Theta} 1_{R_n^\theta \leq R \max_{i \leq n} M^\theta_i} \right] \leq \delta_j \sqrt{n} + \delta_0 \sqrt{32 \log \left( 1 + \frac{N_0}{P[A]} \right)} + 16K \log \left( 1 + \frac{N_0}{P[A]} \right) + \sum_{j=0}^{J-1} \frac{\delta_j^2}{a_j} + \sum_{j=1}^J \left\{ \delta_j \sqrt{80 \log \left( 1 + \frac{N_j}{P[A]} \right)} + \frac{8}{3} (a_{j-1} \vee a_j) \log \left( 1 + \frac{N_j}{P[A]} \right) \right\}.
$$

We aim to choose $a_j$ such that the $\log(1 + N_j/P[A])$ terms disappear. Set

$$
a_j = \delta_j \left( \frac{8}{3} \log \left( 1 + \frac{N_{j+1}}{P[A]} \right) \right)^{-1/2}.
$$
Then $a_j$ is decreasing with increasing $j$, so $a_{j-1} \lor a_j = a_{j-1}$ and

$$E^A \left[ \sup_{\theta \in \Theta} \mathbf{1}_{R_n^\theta \leq R} \max_{i \leq n} M_i^\theta \right] \leq \delta_j \sqrt{n} + 16K \log \left( 1 + \frac{N_0}{P[A]} \right) + 16 \sum_{j=0}^J \delta_j \sqrt{\log \left( 1 + \frac{N_j}{P[A]} \right)}.$$

We now estimate as follows:

$$\sum_{j=0}^J \delta_j \sqrt{\log \left( 1 + \frac{N_j}{P[A]} \right)} \leq \sum_{j=0}^J \delta_j \sqrt{\log \left( 1 + \frac{1}{P[A]} \right)} + \sum_{j=0}^J \delta_j \sum_{p=0}^j \sqrt{\log N_p},$$

and

$$\sum_{j=0}^J \delta_j \sum_{p=0}^j \log N_p \leq \sum_{p=0}^\infty \sqrt{\log N_p} \sum_{j=0}^J \delta_j \mathbf{1}_{p \leq j} \leq \sum_{p=0}^\infty \sqrt{\log N_p} \sum_{j=p}^\infty \delta_j = 4 \sum_{p=0}^\infty (\delta_p - \delta_{p+1}) \sqrt{\log N_p} \leq 4 \int_0^{\sqrt{R}} \sqrt{\log N(n, \Theta, F, K, u)} \, du.$$

We obtain

$$E^A \left[ \sup_{\theta \in \Theta} \mathbf{1}_{R_n^\theta \leq R} \max_{i \leq n} M_i^\theta \right] \leq \delta_J \sqrt{n} + \Phi \left( \log \left( 1 + \frac{1}{P[A]} \right) \right).$$

The result follows by letting $J \to \infty$. \qed

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