Plasma Frequency Shift Due to a Slowly Rotating Compact Star

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Abstract

We investigate the effects of a slowly rotating compact gravitational source on electron oscillations in a homogeneous electrically neutral plasma in the absence of an external electric or magnetic field. Neglecting the random thermal motion of the electrons we assume the gravitoelectromagnetic approximation to the general theory of relativity for the gravitational field. It is shown that there is a shift in the plasma frequency and hence in the dielectric constant of the plasma due to the gravitomagnetic force. We also give estimates for the difference in the frequency of radially transmitted electromagnetic signals for typical compact star candidates.

1 Introduction

Compact stars consist mostly of a degenerated plasma which close to the surface of the star forms the so called ion crust of densely packed ions and relatively free electrons. Moreover a highly ionized plasma constitutes the stellar atmosphere formed as a result of ionization due to the increase in the mean collisional rate of the atoms constituting the star’s atmosphere [1]. Outside the ion crust the plasma co-rotating with the star is rather dilute and highly conducting [2]. A plasma, both in a degenerate form and as a dilute and highly conducting medium, behaves as an oscillating system having a characteristic frequency called the plasma frequency. In particular the dielectric constant of a medium is determined by the plasma frequency, which in turn determines the transmission and reflection of electromagnetic radiation for the plasma [3].

On the other hand compact stars possess very strong gravitational fields, so that general relativistic effects are important for an adequate description of the
phenomenon occurring in vicinity of these stars. Usually general relativistic effects are not directly observable but are manifest in an indirect way, for example via interaction with the magnetic field \cite{3, 4}, in the accretion of matter \cite{6, 7, 8, 9}, and other material and radiative processes occurring in vicinity of the star (e.g. Ref. \cite{2}). An investigation of these effects is of interest for various astrophysical processes as well as for testing the general theory of relativity \cite{10}.

In this paper we investigate gravitomagnetic effects of a slowly rotating compact gravitational source on the electron oscillations in a homogeneous electrically neutral plasma. We assume the absence of any electric or magnetic field and neglect the random thermal motion of the electrons. In the next section we present the gravitoelectromagnetic approximation to the general theory of relativity particularly for the geodesic equation. In section 3 we formulate the equations of motion for the electron oscillations in the gravitational field of the star and study the effects on plasma frequency. It found that there is a reduction in the plasma frequency due to the gravitomagnetic force. We then estimate in section 4 the effects of the variation in the dielectric constant on the electromagnetic waves propagating radially through the plasma surrounding the compact gravitational source. In conclusion we summarize the main results of the paper and discuss their relevance to observation. Throughout we use the gravitational units $G = 1 = c$ unless mentioned otherwise.

2 The Gravitoelectromagnetic Approximation

The field equations of general theory of relativity, for a slowly rotating gravitational source, bear a remarkable formal similarity with the fundamental equations of classical electromagnetism \cite{11, 12}. Particularly in Einstein’s theory of gravitation the trajectory of a test particle in vicinity of a gravitational source is given by the geodesic equation:

$$\frac{d^2 x^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{ds} \frac{dx^\gamma}{ds} = 0;$$

(1)

For sufficiently weak gravitating systems, such as the compact stars, a linearization of the metric can be adequately assumed, where

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta},$$

(2)

and where $\eta_{\alpha\beta} = \text{diag}(-1, -1, -1, 1)$ is the Minkowski metric tensor and $h_{\alpha\beta}$ is the perturbation to the metric such that $h_{\alpha\beta} \ll 1$ and $x^\alpha \equiv (x^i, x^0) = (r, t)$ are the position coordinates of test particle, with time $t$ being the affine parameter, the geodesic equation takes the form

$$\frac{d^2 r}{dt^2} = G + v \times H,$$

(3)

where

$$G = -\nabla \varphi, \quad H = \nabla \times 4a,$$

(4)
and

\[ \varphi = - \iiint \frac{\rho}{r} dV, \quad \mathbf{a} = \iiint \frac{\mathbf{v} \times \mathbf{r}}{r} dV. \]  

(5)

Assuming to the first order of approximation, the star to be a slowly rotating sphere of homogeneous mass \( M \) and radius \( R \), we have

\[ \mathbf{G} = -M\mathbf{\hat{e}}_r/r^2, \]  

(6)

which is the gravitoelectric force given by the Newtonian gravitational force per unit mass, and \( \mathbf{\hat{e}}_r \) is a unit vector in the radial direction. The term \( m(\mathbf{v} \times \mathbf{H}) \) is the gravitomagnetic force where \( \mathbf{H} \) is given by

\[ \mathbf{H} = -\frac{12}{5}MR^2(\Omega \mathbf{r}/r^5 - \frac{1}{3} \frac{\Omega}{r^3}), \]  

(7)

\( \Omega \) being the angular frequency vector of the gravitational source, and \( \mathbf{v} \) is velocity of the test particle. The approximation, called the gravitoelectromagnetic approximation to the general theory of relativity, is generally valid for compact astrophysical sources. The independence of the gravitomagnetic potential \( \mathbf{a} \) from a particular frame and particular coordinate system has been demonstrated [13]. Physically it can be interpreted as ‘gravitomagnetic current’ induced in the vicinity of the gravitational source due to its rotation.

3 Equations of Motion for Electron Oscillations

With a fixed ion background it is convenient to choose an orientation of the coordinates triplet \((x, y, z)\) such that the angular frequency vector \( \Omega \) is along the positive \( z \)-axis. Since the electron oscillations occur within a very small region of space we have, for the components of acceleration due to gravitoelectric force, \( \mathbf{G} \) to be of constant magnitude \( g \) along each direction \((x, y, z)\). In this case the components of \( \mathbf{H} \) are \((0, 0, H)\), therefore effective components of the gravitomagnetic force lie in the \( xy \)-plane. We take, without loss of generality, the \( xy \)-plane to be the equatorial plane of the star mainly because here the gravitomagnetic effects are of maximum magnitude for the given orientation [14, 15].

The equations of motion for the electron oscillations are:

\[ \frac{d^2x}{dt^2} = -\omega_p^2x - g - H \frac{dy}{dt}, \]  

(8)

\[ \frac{d^2y}{dt^2} = -\omega_p^2y - g + H \frac{dx}{dt}, \]  

(9)

where \( \omega_p = \sqrt{N_0e^2/m_e\epsilon_0} \) is the Newtonian plasma frequency, \( N_0 \) being the electron number density of the plasma per centimeter, \( e \) is the electronic charge, \( m_e \) is the electronic mass, and \( \epsilon_0 \) is the dielectric constant. Clearly \( H \) has dimensions of cycle per unit time i.e. of frequency.
To investigate the resonant frequency for the above system of coupled equations, we assume that the solutions to equations (8), and (9) can be expressed as
\[ x = a \exp(i\omega t) - \frac{g}{\omega^2} \] and
\[ y = b \exp(i\omega t) - \frac{g}{\omega^2} \] where \( \omega \) is the applied external frequency. The system can then be written as a single matrix equation
\[
\begin{bmatrix}
\omega^2 - \omega_p^2 & -i\omega H \\
i\omega H & \omega^2 - \omega_p^2
\end{bmatrix}
\begin{bmatrix}
a \\
b
\end{bmatrix}
= 0. \tag{10}
\]
For solution to exist equation (10) implies that,
\[
\left| \omega^2 - \omega_p^2 - i\omega H \right| = 0. \tag{11}
\]
Since \( H \ll \omega_p \), we expand the determinant and neglect terms involving squares and higher powers of \( H/\omega_p \). This gives the following expression for resonant frequency of the oscillating plasma
\[
\omega \approx \omega_p - H/2. \tag{12}
\]
Further if \( \chi \) is the angle between the position vector \( r \) and the angular frequency vector \( \Omega \), it follows from expression (7) that close to the surface of the star, i.e., at \( r \approx R \), \( H \) has magnitude given by
\[
H \equiv |H| \approx \mu \sqrt{1 + 3 \cos^2 \chi}, \tag{13}
\]
where \( \mu = (4GM/5Rc^2)|\Omega| \), \( \Omega \) being the magnitude of the angular frequency vector.

Substituting from expression (13) in (12) we obtain an expression relating plasma frequency \( \omega_p \) of the star’s atmosphere to the angle of inclination \( \chi \):
\[
\frac{\omega}{\omega_p} \approx 1 - \frac{\mu}{2\omega_p} \sqrt{1 + 3 \cos^2 \chi}, \quad \mu \ll \omega_p. \tag{14}
\]
In expression (14) we see that the shift in plasma frequency depends not only on the mass, radius and angular frequency of the compact star, via the parameter \( \mu \), but also on the angle of inclination \( \chi \). Here the parameter \( \mu \) determines the magnitude of the gravitomagnetic effect for different compact stellar sources. For a typical compact star \( \mu \) ranges from \( 0.1687 \times 10^{-3} Hz \) (for a typical white dwarf of mass \( 1M_\odot \), 1.989 \times 10^{30} kg, radius \( 7 \times 10^6 m \) and angular frequency \( 1 Hz \)) to \( 236.2932Hz \) (for a typical neutron star of mass \( 2M_\odot \), radius \( 1 \times 10^4 m \), and angular frequency \( 1kHz \)) with corresponding plasma frequency ranging approximately \( 5.65 \times 10^2Hz \) to \( 5.65 \times 10^6Hz \) or above. Given the parameters \( \mu \) and \( \omega_p \) the sft depend only on the angle of inclination \( \chi \). A plot between the plasma frequency shift and the angle \( \chi \); based on expression (14) for the cases of neutron star, pulsar, and white dwarf for the typical values of mass, radius and angular frequency is given in Fig. (1). Clearly the gravitomagnetic reduction in plasma frequency is maximum in the equatorial plane of the star.
Figure 1: Plots for the shift $\omega/\omega_p$ in the plasma frequency of a compact star atmosphere as a function of the angle of inclination $\chi$ of the plane of observation to the angular frequency vector for the case of a white dwarf ($M = 1M_\odot = 1.989 \times 10^{30} kg$, $R = 7 \times 10^6 m$, $\Omega = 1 Hz$, $\omega_p = 5.65 \times 10^2 Hz$), a pulsar ($M = 1.4M_\odot$, $R = 3 \times 10^4 m$, $\Omega = 30 Hz$, $\omega_p = 5.65 \times 10^4 Hz$), and a neutron star ($M = 2M_\odot$, $R = 1 \times 10^4 m$, $\Omega = 1 kHz$, $\omega_p = 5.65 \times 10^6 Hz$).

4 Gravitomagnetic Effects on the Dielectric Constant of the Plasma

To study the effects of plasma frequency shift on the dielectric constant of the plasma, let $\omega_p^{shift} \equiv \omega_p - H/2$, where $H$ is given by (9). Then in terms of plasma frequency we have the following expression for the dielectric constant $\varepsilon$ for the plasma:

$$\varepsilon = 1 - \left( \frac{\omega_p^{shift}}{\omega_{EM}} \right)^2 = 1 - \left( \frac{\omega_p}{\omega_{EM}} - \frac{\mu}{2\omega_{EM}} \sqrt{1 + 3 \cos^2 \chi} \right)^2,$$

where $\omega_{EM}$ is the angular frequency of an electromagnetic signal propagating through the plasma.

For an electromagnetic signal of some given angular frequency the dielectric constant (15) depends only on the angle $\chi$ for a compact gravitational source. Ploting this dependence of the constant $\varepsilon$ on the angle $\chi$ in Fig.(2) we find that the plasma medium becomes rarer as $\chi$ varies from 0 to $\pi/2$. Therefore if $\delta\omega_{EM}$ be the angular frequency range of the radially out-going waves, then from the usual conditions for the propagation of an electromagnetic signal in continuous media we have for the transmission and reflection the conditions in the field of a rotating gravitational source:

$$\delta\omega_{EM} < \omega_p^{shift}, \text{ reflection};$$
Figure 2: Plots for the refractive index $\varepsilon$ of a compact star atmosphere as a function of the angle of inclination $\chi$ of the plane of plasma oscillations to the angular frequency vector for the case of a white dwarf ($M = 1M_\odot = 1.989 \times 10^{30}kg$, $R = 7 \times 10^6m$, $\Omega = 1Hz$, $\omega_p = 5.65 \times 10^2Hz$), a pulsar ($M = 1.4M_\odot$, $R = 3 \times 10^4m$, $\Omega = 30Hz$, $\omega_p = 5.65 \times 10^4Hz$), and a neutron star ($M = 2M_\odot$, $R = 1 \times 10^4m$, $\Omega = 1kHz$, $\omega_p = 5.65 \times 10^6Hz$) for the propagation of an EM signal of angular frequency $10^3Hz$, $10^5Hz$, and $10^7Hz$ respectively.

\[
\delta \omega_{EM} > \omega_p^{shift}, \text{ transmission.} \tag{16}
\]

The shift in plasma frequency in the equatorial plane ($\chi = \pi/2$) is $\omega_p - \mu/2$ whereas in the plane orthogonal to it ($\chi = 0$) the shift is $\omega_p - \mu$. Denoting the frequency shift in the equatorial plane by $\omega_{p\parallel}$ and by $\omega_{p\perp}$ for the orthogonal plane, we find that the maximum difference in the frequencies transmitted through the plasma:

\[
|\omega_{p\perp}^{shift} - \omega_{p\parallel}^{shift}|_{max} = \frac{\mu}{2}. \tag{17}
\]

It is worth emphasizing here that since plasma oscillations and the gravitomagnetic force both are along the radial direction therefore even if the plasma co-rotates with the star (for instance close to the ion crust) the gravitomagnetic force will effect the oscillations of the plasma and hence the dielectric constant.

5 Conclusions

In this paper we have considered the effects of gravitomagnetic force on electron oscillations in a homogeneous plasma with a fixed ion background. It was found that there is a shift in the characteristic frequency and hence in the dielectric constant of the plasma surrounding the star. The shift depends on the intrinsic
parameters of the star, i.e., its mass and radius, and also on the component of the angular frequency vector of the star to the plane of oscillation of the electron. It was estimated that an electromagnetic wave of given frequency the shift in the dielectric constant results in changing the allowed range of frequencies transmitted through the plasma, by a maximum amount $\mu/2$ (as defined for expression (13) above).

For a plasma enveloping a white dwarf the predicted difference in the frequency of electromagnetic radiation emitted is very small (the magnitude of the parameter $\mu$ is about $0.1687 \times 10^{-3} \text{Hz}$). However for a typical pulsar ($\mu \approx 1.6540 \text{Hz}$) and especially neutron stars ($\mu \approx 236.2932 \text{Hz}$) the shift, though still small, may possibly be observed (for example in radiation spectra of various compact sources), especially when the effects of various contingencies (such as dispersion due to interstellar gases, effects due to the atmosphere of the Earth, etc.) are isolated.

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