Lax Pair Covariance and Integrability of Compatibility Condition

S. B. Leble,
Faculty of Applied Physics and Mathematics
Technical University of Gdansk,
ul. G.Narutowicza, 11/12 80-952, Gdansk-Wrzeszcz, Poland,
email leble@mifgate.pg.gda.pl

and

Kaliningrad State University, Theoretical Physics Department,
Al.Nevsky st.,14, 236041 Kaliningrad, Russia.

Abstract

We continue to study Lax (L-A, U-V) pairs (LP) joint covariance with respect to Darboux transformations (DT) as a classification principle. The scheme is based on a comparison of general expressions for the transformed coefficients of LP and its Frechet derivative. We use the compact expressions of the DT via some version of nonabelian Bell polynomials. It is shown that one more version of Bell polynomials, so-called binary ones, form a convenient basis for the invariant subspaces specification. Some non-autonomous generalization of KdV and Boussinesq equations are discussed in the context. Trying to pick up restrictions at minimal operator level we consider Zakharov-Shabat - like problem. The subclasses that allow a DT symmetry (covariance at the LP level ) are considered from a point of view of dressing chain equations. The case of classic DT and binary combinations of elementary DT are considered with possible reduction constraints of Mikhailov type (generated by an automorphism). Examples of Liuville - von Neumann equation for density matrix are referred as illustrations .

1 Introduction

If a pair of linear problems is simultaneously covariant with respect to a DT, it generates Backlund transformations of the corresponding compatibility condition. In the context of such integrability the joint covariance principle [1] may be considered as a classification scheme origin. In this paper we examine realizations of such scheme looking for possible covariant form and appropriate basis with a simplest transformation properties. It is important to note that the proof of the covariance theorems for the linear operators incorporates the so-called (generalized) Miura transformation having the form of (generalized) Riccati equation. We give and examine here the explicit form of the equality in both general and stationary case,
look [2] as well. It gives additional nonlinear equation that is automatically solved by DT theory and used for generation of t-chain equations [3]. We show how the form of the covariant operator may be found while some kind of Frechet derivatives of the operator coefficients and the transforms are compared.

Further, it is clear that the choice of DT type defines the class of covariant operators. The ”classic” DT generalization for the polynomials of a differentiation operator results in transformation formulas first obtained in Matveev papers [4], the form of DT we use below is from [2]. It is also possible to do it in an abstract way: i.e. to define elementary DT (eDT) for any projector (idempotent) element in a differential ring or module [4]. The abstract differentiation is naturally included in the scheme [1]. The appropriate Zakharov - Shabat (ZS) problem which contains some elements as ”potentials” and ”wave functions” (WF) is studied. The results open the way for DT theory applications in arbitrary matrix dimension (including infinite - dimensional or strictly operator case). It is shown [6] that the sequence of $n$ such transformations in $n \times n$ matrix case with usual differentiation by a parameter give the standard dressing formulas of ZS problem [7].

In refs. [6], [6] eDT for direct and conjugate problems were used to obtain the binary DT (bDT) for matrices and in [6] - for three projectors with applications to N-wave interaction, a step from [3] upto the general case [10]. The symmetric form of the resulting expressions for potentials and wave functions (WF) make almost obvious the heredity of reduction restrictions [8] and underlying automorphisms [11] of a generic ZS problems. In [12] an application to some operator problem (Liuville - von Neumann equation) is studied.

In the Sec. 2 we examine the general non-Abelian version of the operator transformations constructed by means of the above-mentioned Bell polynomials generalization. The resulting transformation formulas do not contain double summation that is convenient for the further analysis as in the context of classification as on the way of dressing chain reformulations [13], [14], [15] and solution generation. In the last part of the Sec. 2 we list few such polynomials. We show then the equation for the intermediate (sigma) function that naturally appear during the Darboux theorem proof. This equation generalizes Riccati equation (Miura transformation) for the celebrated KdV theory for general dynamics - it may be called generalized Burgers equation.

In the next Sec 3 we concentrate our attention on the algorithmic derivation of Lax pairs starting from the idea of covariance. We use more simple nonabelian case to show main features of the scheme. The point we demonstrate is based on a comparison of the formal Taylor expansion of the transformed operator (by means of Frechet derivative notion) of a Lax pair and its formal DT expression. The comparison gives differential equations which solutions produce the explicit expressions for the Lax pair coefficients.

Sec 4 introduces binary Bell polynomials and study the transformation of them via the differential forms incorporated. We also prove invariance of some combinations of such BBP with respect to DT.

Next section starts from covariance of generalized ZS equation. Two versions of dressing chain are constructed for stationary spectral problems with nonabelian spectral parameter. We are trying to pick up relations that do not depend on matrix representation [17]. Simple examples are studied by closures of the chain equations on the operator level.

The section 6 is devoted to the nonabelian Lax pairs from the point of view of bDT covari-
The Abelian version of binary transform (commuting "matrix elements") reproduces results of the dressing method based on the matrix Riemann-Hilbert problem [7]. Here we mainly treat the dressing chain equations, sending readers to [18], where the equation

\[-iX_t = [X, h(X)], \quad (1.1)\]
h(X) - analytical function, is studied. Further generalizations for essentially nonabelian functions (e.g. \(h(X) = XA + AX, [A, x] \neq 0\), [12]) are considered in [19], where abundant set of integrable equations is listed. The list is in a partial correspondence with [21], and give direct link to solutions via the dressing chains. The papers also contain examples of "self-scattering solutions" with discussions of possible applications [20].

2 Darboux transformations in terms of Bell polynomials.

The solution of the problem of a polynomial linear differential operator left and right division produces a version of the Darboux theorem. The theorem was reformulated for the classical DT

\[\psi[1] = D\psi - \sigma \psi, \quad (2.1)\]
in terms of some generalization of the Bell polynomials [2] that give compact expressions for transformed coefficients of the operator. We reproduce here the results with links to further applications, namely with equations for the element "\(\sigma\)" that appear inside the theorem proof.

Let us start from the linear operator

\[L = \sum_{n=0}^{N} a_n D^n,\]

and the evolution equation (flow)

\[\psi_t = L\psi. \quad (2.2)\]

Here the operator D may be a differentiation by some variable and \(\psi_t\) is the derivative with respect to another one (see [3] for generalizations). The transformation of the solutions of the equation is taken in the standard form (2.1), where \(\sigma = (D\phi)\phi^{-1}\) with a different solution of (2.2) incorporated \(\phi_t = L\phi\). We would now promote the convenient formulation of Matveev theorem [4].

**Theorem.** The coefficients of the resulting operator

\[L[1] = \sum_{n=0}^{N} a_n[1] D^n\]

are defined by

\[a_N[1] = a_N,\]
and for all other \( n \), by

\[
a_n[1] = a_n + \sum_{k=n+1}^{N} [a_k B_{k,k-n} + ((Da_k) - \sigma a_k)B_{k-1,k-1-n}]
\]

that yields a covariance principle.

It means that the function \( \psi[1] \) is a solution of the equation

\[
\psi_t[1] = L[1] \psi[1],
\]

where \( L[1] \) has the same structure and order as \( L \). The proof of this statement incorporates the equation that links \( \sigma \) and coefficients of the operator \( L \). It may be derived inside a factorization theory \([2]\) and will be useful further

\[
\sigma_t = Dr + [\sigma, r], \quad r = \sum_{k=1}^{N} a_k B_k, \quad (2.4)
\]

where \( B_k \) are nonabelian Bell polynomials (see, e.g. \([22]\)). In the stationary case one has

\[
\sigma_t = \phi_t \phi^{-1} - \phi_x \phi^{-1} \phi_t \phi^{-1} = 0. \quad (2.5)
\]

The functions \( B_{m,n} \) are introduced in \([3]\).

We reproduce here the definition and some statements about them.

**Definition**

\[
B_{n,0}(\sigma) = 1, \quad n = 0, 1, 2, \ldots,
\]

and recurrence relations

\[
B_{n,k}(\sigma) = B_{n-1,k}(\sigma) + DB_{n-1,k-1}(\sigma), \quad k = 1, n - 1, \quad n = 2, 3, \ldots \quad (2.6)
\]

\[
B_{n,n}(\sigma) = DB_{n-1,n-1}(\sigma) + B_n(\sigma), \quad n = 1, 2, \ldots
\]

define the generalized Bell polynomials \( B_{m,n} \).

The following formula is extracted,

\[
B_{n,n-k+1}(\sigma) = \sum_{i=k}^{n} \binom{n}{k} B_{n-i}(\sigma) D^{i-k}\sigma, \quad k = 1, n, \quad n = 0, 1, 2, \ldots;
\]

\[
B_{n+1}(\sigma) = \sum_{i=0}^{n} B_{n-i}(\sigma) D^i\sigma, \quad n = 0, 1, 2, \ldots \quad (2.7)
\]

it gives the link between standard (nonabelian) Bell polynomials and the generalized ones:

\[
B_{n+1}(\sigma) = \sum_{i=0}^{n} B_{n,i}(v) D^{n-i}\sigma, \quad n = 0, 1, 2, \ldots
\]

Evaluation of the first three generalized Bell polynomials by the definition gives

\[
B_{n,1}(\sigma) = \sigma; \quad B_{n,2}(\sigma) = \sigma^2 + n D\sigma; \quad B_{n,3}(\sigma) = \sigma^3 + n \sigma' \sigma + (n - 1) \sigma D\sigma + \binom{n}{2} D^2\sigma;
\]
Towards the classification scheme. Joint covariance of L-A pairs

The basis of the formalism we introduce is elaborated starting from [1] and the compact formulas with the generalized differential (Bell) polynomials from the previous section. Note again, that it is valid for nonabelian entries as well, the coefficients $a_n$, solutions of the equation (1) $\phi$ and $\psi$ we consider as matrices or operators. Let us however start from the scalar case.

To prepare the explicit expressions for such work and show details we would setup a couple of examples of the theory. Let us deliver a very simple analysis for better understanding of the sense of the integrability notion we introduce. First of all we notice that the "elder" coefficients, with $n = N$ and $n = N-1$ are transformed almost trivially. It follows that in general the functions rather do not play the role of potentials or unknown functions for a nonlinear equation of a compatibility condition.

If $N=2$, the general transforms (2.3) reduces to

$$
\begin{align*}
a_2[1] &= a_2 = a(x,t), \\
a_1[1] &= a_1(x,t) + Da(x,t) \\
a_0[1] &= a_0 + Da_1(x,t) + 2a(x,t)D\sigma + \sigma Da(x,t)
\end{align*}
$$

(3.1)

Keep in mind that we touch hear only the scalar (abelian, more precise) case. One can see that the explicit form of the transformations really shows a difference between the coefficients $a(x,t), a_1(x,t)$ that transform without solutions account and the $a_0 = u(x,t)$ to be an unknown function of a forthcoming nonlinear equation that we call potential in the context of Lax (L-A-pair) representation. One may easily recognize the KdV case here. Namely when $a = const, a_1 = 0$, $a_0$ plays the role of the only unknown function of the KdV equation. So we may formulate an

observation: The abelian case $N = 2$ is the first nontrivial example of covariant operators set with coefficients $a_{1,2}$ that depends only on $x$ and additional parameter (say $t$) but its transforms contain the only functions $a_{1,2}$, hence, to be called trivial. The transformation $(DT)$ for $u$ is given by the last equation of (3.1) and depends on both $a_{1,2}$ and solutions of the equation (2.2) via $\sigma$.

By the next order example we would show more details especially when pairs of operators are analysed simultaneously. Especially it is important when both operators are functions of the only potential. Let us take $N = 3$.

$$
\begin{align*}
b_3[1] &= b_3, \\
b_2[1] &= b_2 + Db_3' \\
b_1[1] &= b_1 + Db_2 + 3b_3 D\sigma + \sigma Db_3 \\
b_0[1] &= b_0 + Db_1 + \sigma Db_2 + (\sigma^2 + (2D\sigma)) Db_3 + 3b_3(\sigma D\sigma + D^2\sigma)
\end{align*}
$$

(3.2)

Consider (3.1) and (3.2) as coefficients of a Lax pair operators both depending on the only variable $u$ simultaneously and suppose the coefficients of the operator are analytical.
functions of \( u \) and its derivatives with respect to \( x \). Now we change \( t \to y \) and \( L \to L_1 \) in the equation (2.2) corresponding to the case (3.1) and left the parameter \( t \) \((L \to L_2)\) for the case (2.5), forming a Lax pair

\[
\begin{align*}
\psi_y &= L_1 \psi, \\
\psi_t &= L_2 \psi.
\end{align*}
\]  

(3.3)  

(3.4)

To produce the KdV case generalization we go to a stationary in \( y \) solution of (3.3).

\[
\phi_y = \lambda \phi,
\]

where \( \lambda \) is a constant.

Let us also recall the KdV case. Then the stationary version of (2.4) for \( N = 2 \) is

\[
\sum_{0}^{2} a_n B_n = c,
\]

that reads as

\[
\sigma^2 + \sigma' + u = c = \text{const}.
\]  

(3.5)

Note that the equation (2.5) for \( N = 3 \) is still valid for this \( \sigma = \phi_x \phi^{-1} \), if \( \phi \) is a solution to the Lax pair system (3.3,3.4). If in the equations of (3.2) we restrict ourselves by the case of \( b_2 = 0 \) and \( b_3 = b = \text{const} \), we arrive at the second equation of the KdV Lax pair.

Next, returning to the general case and taking into account the triviality of \( b_3 = b(x,t) \) and \( b_2 \) in the above-mentioned sense, the first non-trivial potential is

\[
b_1 = F(u,u',...).\]

(3.6)

Suppose further that the covariance principle is valid, or, in shorten arguments notations, we address to an equation for this function \( F \).

\[
b_1[1] = F(u[1]) = F(u + Da_1 + 2aD\sigma + \sigma Da) = F(u) + Db_2 + 3bD\sigma + \sigma Db.
\]

(3.7)

Analyticity of \( F \) gives a possibility to use Taylor series expansion in the left side of (3.7)

\[
F(u[1]) = F(u) + Fu(2aD\sigma + Da_1 + \sigma Da) + F_{Du}(...) + ....
\]

(3.8)

In fact we compare the transform (3.7) with the Frechet differential (3.8) of the \( F \). The equation holds identically if the coefficients by the \( \sigma, D\sigma \) and the free term are the same. Introducing \( F_u = c(x,t) \) yields

\[
2ac = 3b,
\]

(3.9)

or

\[
F(u) = \frac{3bu}{2a}.
\]

(3.10)

with additional conditions

\[
cDa_1 = Db;
\]

(3.11)

\[
cDa = Db
\]

(3.12)
Plugging $c$ from (3.9) into (3.12), one go either to $3D(lna) = 2D(lnb)$ and, integrating, obtains

$$b = a^{3/2}c_1(t), \quad (3.13)$$

or to $Da = Db = 0$. In the last case (3.12) is valid with arbitrary $c$, or independent $b(t)$ and $a(t)$. While (3.11) yields the equation for $a_1$ for both cases

$$3Da_1 = - 2aDb/3b \quad (3.14)$$

with arbitrary $c_1(t)$.

Next conditions follow from the last equation of the system (3.2), i.e., if one introduces next analytical function $G$ and denote

$$b_0 = G(u, u', ...), \quad (3.15)$$

the transformed $b_0$ gives

$$G(u + Da_1 + 2aD\sigma + \sigma Da) = G(u) + G_u(Da_1 + 2aD\sigma + \sigma Da) + G_{Du} D(Da_1 + 2aD\sigma + \sigma Da) + ... \quad (3.16)$$

The DT transformation formula for the potential $u$ is obviously used. The DT for the last coefficient $b_0$, see (3.2), yields

$$b_0[1] = G(u) + Db_1 + \sigma Db_2 + (\sigma^2 + 2(D\sigma))Db + 3bD(\sigma^2/2 + D\sigma). \quad (3.17)$$

We should account now the general version of the Miura transformation (3.5) that has the form

$$\sum_0^2 a_n B_n = u + a_1\sigma + a(\sigma^2 + D\sigma) = \mu,$$

by which we would express the $\sigma^2$ in (3.17). Doing this and equalizing the expressions (3.16) and (3.17) yields

$$D^{3bu}/2a + \sigma Db_2 + ((\mu - u - a_1\sigma)/a + (D\sigma))Db + 3bD[(\mu - u - a_1\sigma)/2a + (D\sigma)] =$$

$$G_u(Da_1 + 2aD\sigma + \sigma Da) + G_{Du} D(Da_1 + 2aD\sigma + \sigma Da) \quad (3.18)$$

The equation (3.18) gives for the coefficients: by $D^2\sigma$

$$G(Du)2a = 3b, \quad (3.19)$$

by $D\sigma$, taking (3.19) into account

$$G_u2a + 9b(Da)/2a = Db - 3ba_1/2a, \quad (3.20)$$

by $\sigma$

$$G_u Da + \frac{3b}{2a} D^2a(x, t) = Db_2 - a_1/a - 3bD(a_1/2a), \quad (3.21)$$

and the free term is

$$D^{3bu}/2a + ((\mu - u)/a)Db + 3bD[(\mu - u)/2a] = G_u Da_1 + 3b(D^2a_1)/2a. \quad (3.22)$$
From (3.19) and (3.20) we express $G_u$:

$$G_u = Db/2a - 3ba_1/4a^2 - 9b(Da)/4a^2. \quad (3.23)$$

If $G_u$ is nonzero, from (3.21) follows

$$(Db/2a - 3ba_1/4a^2 - 9b(Da)/4a^2)Da + \frac{3b}{2a}D^2a = Db_2 - a_1/a - 3bD(a_1/2a)$$

The free term (3.22) gives

$$\frac{uDb}{2a} + \mu(Db/a - 3bDa/2a^2) = (Db/2a - \frac{3ba_1}{4a^2} - \frac{9bDa}{4a^2})Da_1(x,t) + 3b(D^2a_1)/2a. \quad (3.24)$$

When $u$ is linear independent from $\sigma$ and derivatives, and we do not account higher terms in the Frechét differential, the only choice $Db = 0$ kills the term with $u$, and (3.24) simplifies

$$D^2a_1 - a_1(Da_1)/2a = 0.$$

The condition $Da = 0$ as the corollary of (3.12) is taken into account. The equation (3.21) also simplifies

$$Db_2 - a_1/a - 3b(Da_1)/2a = 0$$

and integration gives the expression for $b_2$.

Another possibility is $G_u = 0$. It gives $9b(Da)/2a = Db - 3ba_1/2a$ instead of (3.23). The free term transforms as

$$\frac{uDb}{2a} + \mu(Db/a - 3bDa/2a^2) = +3b(D^2a_1)/2a. \quad (3.25)$$

and by the same reasons gives the conditions $Db = Da = 0$. It further means $a_1 = 0$ and, finally, from (3.21),

$$Db_2 = 0.$$

Hence this case contain KdV equation with the possible $a(t), b(t), b_2(t)$.

**Remark 1.** The results for one isolated equation (3.3) contain rather wide class of coefficients (in a comparison with the joint covariance of (3.3) and (3.4). Namely, the $a, a_1$ are arbitrary functions of $x, t$. It may be useful for construction of potentials and solutions (e.g. special functions) for the linear Schrödinger and evolution equations of the one-dimensional quantum mechanics [23]. The KdV case may be described separately (denote further $Df = f'$):

$$G_u\sigma' + G_u'\sigma'' = 3b(1 - a)u'/4a^2 + 3b\sigma''/4a$$

The only choice is possible, if we consider $\sigma, \sigma', u'$ as independent variables

$$G_u = 0,$$

$$G_u' = 3b/4a$$

or, taking into account the condition of zero coefficient by $u'$, $a = 1$, one arrives at

$$G(u, u', ...) = 3bu'/4.$$
The result leads directly to one of equivalent Lax pairs for the KdV equation.

Now we would consider the next natural example with interchanged equations defining a spectral problem and evolution. Hence we should start from the equation (3.3) for the third order spectral operator, using (3.4) as evolution with $N=2$.

We would restrict ourselves by the case of $b_2 = 0$ and the autonomous $b_3 = 1$, $b_1 = \text{const} = b$, $b_0 = u$. The DT yields

$$b_1[1] = b_1 + 3b\sigma'$$
$$b_0[1] = b_0 + b_1 + 3b(\sigma_x\sigma + \sigma_{xx}).$$

As it was shown by analysis of the third order operator, see (3.21-25), the covariant spectral problem has the form

$$\psi_{xxx} + b\psi_x + G\psi = \lambda\psi$$

(3.28)

The second (evolution) equation of the case is:

$$\psi_t = -\psi_{xx} - w\psi$$

(3.29)

If one consider (3.3) (specified in (3.26) and (3.27)) or (3.28,29) as the Lax pair equations both with coefficients depending on the only variable $w$. Suppose again that the coefficients of the operators are analytical functions of $w$ and its derivatives (or integrals) with respect to $x$. If one wants to save the form of the standard DT for the variable $w$ (potential) the analysis just similar to the previous example give for the constant $\alpha$

$$b = 3w/2 + \alpha.$$ 

(3.30)

Compare with the formula (3.25), or $w[1] = w + 2\sigma_x$. Then the transformation for the potential $w$ follows from the last equation of the system (3.27), i.e.,

$$G[1] = G + 3w_x/2 + 3(\sigma^2/2 + \sigma_x)_x.$$ 

(3.31)

we see that further analysis is necessary due to the possible constraint (reduction) existence. Then, again similar to the previous case of KdV the covariant equations (3.27,28) are accompanied by the following (Burgers) equation.

$$\sigma_t = -(\sigma^2 + \sigma_x)_x - w_x$$

(3.32)

for the problem (3.28) and

$$\sigma^3 + 3\sigma_x\sigma + \sigma_{xx} + b\sigma + u = \text{const},$$

(3.33)

see (2.5), compare with (3.5) that was Riccati equation (stationary version) for the second order spectral problem corresponding to KdV. If one would use the equation (3.32) in (3.31), the time-derivative of $w$ appear. Moreover, the further analysis shows that the case we study need to widen the functional dependence in $u$, namely we should include not only derivatives of $w$ with respect to $x$, but integrals (inverse derivatives) as arguments of the potential. So, for the equation (3.28) let us introduce next analytical function $G$ and denote

$$G = G(\partial^{-1}w, w, w_x, ...\partial^{-1}w_t, w_t, w_{tx}, ...),$$

(3.34)
the first terms of Taylor series for (3.34) read

\[ G(w + 2\sigma_x) = G(w) + G_{w_x}2\sigma_{xx} + G_{\partial^{-1}w_t}2\sigma_t + ..., \]  

(3.35)

where we show only terms of further importance. The DT transform (24) after substitution of (3.31) gives

\[ G(w) + 3w_x/2 + 3(-\sigma_t - w_x)/2 + 3\sigma_{xx}/2. \]  

(3.36)

Equalizing (3.36) and (3.35) one obtains

\[ G_{w_x} = 3/4; G_{\partial^{-1}w_t} = -3/4, \]

That leads to the exact form of the Lax pair for the Boussinesq equation from [2] for the choice of \( \alpha = -3/4. \)

Remark 2. As one could see we cut the Fréchet differential formulas on the level that is necessary for the minimal flows. The account of higher terms lead to higher flows (higher KdV, for example).

Remark 3 In the second example we obviously restrict the description by the autonomous case within the special choice of DT and do not consider the class of gauge equivalent equations. The general case with account of higher derivatives will be presented elsewhere in forthcoming publications.

We would finish with the

**Theorem.** The Darboux covariant Lax pair (3.3,4) with the stationary equation (3.3) where the operator \( L_1 \) is given by (3.1) has the following coefficients:

\[ a_2 = a(t), \]
\[ a_1 = a_1(x,t), \]

is a solution of the equation \( Da_1 - a_1^2/4a = c_2(t) \) with arbitrary \( c_2, \)

\[ a_0 = u. \]

The evolution (3.4), which form is presented by (3.2), contains

\[ b_3 = b(t), \]
\[ b_2 = \int a_1 dx/a + 3ba_1/2a, \]
\[ b_1 = 3b/2a, \]
\[ b_0 = -3ba_1 u/4a^2 + 3b(Du)/2a. \]

The pair produce the generalization of the KdV equation

\[ L_{1t} = [L_1, L_2] \]

that is solved by standard DT.
4 Binary Bell polynomials under Darboux transformation

The binary Bell polynomials (BBP) \( Y \) that we are going to use are defined in terms of the exponential differential polynomials:

\[
Y_{mx,nt}(f) \equiv e^{-f} \partial_x^m \partial_t^n e^f
\]

by means of the link between \( Y \)-polynomials and the standard Hirota expressions [24]:

\[
D_p x D_q t G' \cdot G \equiv (\partial_x - \partial_x')^p (\partial_t - \partial_t')^q G'(x, t)G(x', t') |_{x'=x, t'=t}.
\]

(4.2)

The connection is given by the identity:

\[
Y_{mx,nt}(v, w) \equiv \exp\left[\frac{v-w}{2}\right] \exp\left[\frac{-v-w}{2}\right] \partial_x^m \partial_t^n \exp\left[\frac{v+w}{2}\right] \cdot \exp\left[\frac{w-v}{2}\right]
\]

(4.3)

They inherit the easily recognizable partitional structure of the Bell polynomials (BP):

\[
Y_x(v) = v_x, \quad Y_{2x}(v, w) = w_{2x} + v^2_x, \quad Y_{x,t}(v, w) = w_{xt} + v_x v_t,
\]

\[
Y_{3x}(v, w) = v_{3x} + 3 v_x w_{2x} + v^3_x, \ldots
\]

(4.4)

The expression of the BBP contains the structure we studied in previous sections. Therefore the covariance of the expressions can be examined in similar way. To check the conjecture about covariance of a BBP one should confirm the coincidence of the expressions of such polynomial generated by (4.3) and transformations of functions (2.3). Let, as the simplest example, consider the second order polynomial from the point of view of preceding analysis reproducing the addition formula from [25]. We get

\[
Y_{2x}(v, v + q) = P_0 B_2(v) + P_{2x} B_0.
\]

Here the symbols \( Y_{2x}, P_{2x}(q) \) are also defined in [24], [25] and \( B_n \) are usual BP.

\[
Y_{2x}(v, v + q) = \psi^{-1} L \psi = \psi^{-1}(\psi_{xx} + q_{2x} \psi)
\]

Thus, as it was shown in the Sec. 3, the polynomial contains D-covariant expression for the operator \( L \) (\( N=2 \)) with the potential \( q_{2x} \). It means that the spectral problem for KdV case may be expressed by means of the BBP, or

\[
Y_{2x}(v, v + q) = \lambda.
\]

(4.5)

This simple example shows how to find the basis in which it is convenient to make the transformations. It turns out easier than the direct test and allows to forward the business for arbitrary \( N \).

The general expression for the BBP, transformed like (2.3) [24] is the addition formula

\[
Y_{Nx}(v, v + q) = \sum_{p=0}^{E(N/2)} \binom{N}{2p} P_{2px} Y_{(N-2p)x}(v)
\]

(4.6)
where $E(N/2)$ is the integer part of $N/2$. After the logarithmic linearization (16),

$$\mathcal{Y}_{N,x}(v, v + q) = \sum_{m=N-2E(N/2)}^{E(N/2)} \binom{N}{N-m} P_{(N-m)x}(q) D^m \psi = \sum_{m=0}^{N} a^P_m D^m \psi, \quad (4.7)$$

the expression now coincide by the form with the right-hand side of a Lax equation with the operator $L$. We see the coefficients of the DO in (4.7), which DT is determined by (2.3). The extracted coefficients are

$$a^P_m = \binom{n}{N-m} P_{(N-m)x}(q),$$

if $N - m$ is even, otherwise $a^P_m = 0$ (for odd $N - m$. Or in other words, $a^P_m$ are nonzero only in the set of $m \in \{N, N - 2, ..., N - 2E(n/2)\}$. Substituting the coefficients from (4.7) with $m = N - 2p$ into (2.3), one obtains

$$a^P_m[1] = a^P_m + \sum_{j=m+1}^{N} t \binom{N}{N-j} q P_{(N-j)x}(q)[B_{j,j-m} - \sigma B_{j-1,j-1-m}]$$

$$+ P_{(N-j)x}(q)'B_{j-1,j-1-m}]. \quad (4.8)$$

The terms in the sum of (4.8) are nonzero also if $N - j = 2p$ is even, it is marked by the prime in the sum. The definition, convenient formulas and explicit expressions to the generalized Bell polynomials $B_{n,m}$ are given in the Sec. 2.

For more illustration of the technique let us take again the minimal polynomials and evaluate the r.h.s of (4.7) and (4.8). First take an even $N$ and begin with

$$a_{N-2}[1] = a_{N-2} + a_N(B_{N,2} - \sigma B_{N-1,1}) = a_{N-2} + N\sigma'.$$

We have taken into account that $a_N = P_0 = 1$ Compare with the addition formula

$$P_{2x}(q + 2\sigma) = \binom{2}{0} P_{2x}(q) + \binom{2}{2} P_{2x}(2\sigma) = q_{2x} + 2v_{2x} = q_{2x} + 2\sigma'.$$

We see that in the case of $N = 2$ the polynomial is invariant.

The example of odd polinomial $N = 3$ yields

$$\mathcal{Y}_{3x}(v, v + q) = \mathcal{Y}_{3x}(v) + 3q_{2x}v_{x} = \psi^{-1}(D^3 + 3q_{2x} D)\psi \quad (4.9)$$

with the known transform of the operator inside (4.9). There are two possible scopes of an incorporation of the polynomials into Lax pairs. Either it belongs to evolution equation or results in a spectral problem of third order and leads to SK or Boussinesq equations.

Let us first comment the KdV case, so the spectral problem

$$\mathcal{Y}_{2x}(v, v + q) = \lambda. \quad (4.10)$$

introduces a potential $q_{2x}$ to be a dependent variable for the KdV equation. We should rewrite now the evolution equation in terms of binary Bell polynomials. Starting from (4.10) one notice that the corresponding coefficient of (4.10) are defined by the equation (4.10):

$$a^P_3 = 1, \quad a^P_1 = 3q_{2x}^{(3)}, \quad 12$$
\[ a_2^P = a_0^P = 0. \]

The transforms of them is determined by (4.9) or one may also use the explicit expressions from (4.5). We should write, for example,

\[ 3a_1^P[1] = 3a_1^P + 3\sigma'. \]

It follows that the potential (dependent variable) should be chosen as

\[ q^{(3)}_{xx} = q_{xx}/2 \quad (4.11) \]

to fit the transformation law for \( q_{xx} \). If one examine the transformation of the coefficients of the general operator (e.g.(4.6)), one notices that the "youngest" coefficient \( b_0 \) is changed by necessity. The addition depends on a potential and \( \sigma \). So the zero value of \( a_0^P \) could not be generally preserved. We arrive to the

**Observation:** the isolated \( Y_{3x} \) is not generally covariant.

Therefore there is a stright necessity to incorporate different polynomials to support the covariance. So we take the combination

\[ c_3 Y_{3x}(v, v + q^{(3)}) + c_2 Y_{2x}(v, v + q^{(2)}) + c_1 Y_{1x} \quad (4.12) \]

with different potentials \( q_i \). If we restrict our choice by the unique potential that enter the spectral problem (4.5) and note that the choice of \( q^{(3)} = q/2 \) is fixed by (4.11), we get the link of \( q^{(1)} \) and \( q \) from (4.5) that gives

\[ q^{(2)} = q + 3c_3q_x/(4c_2) \quad (4.13) \]

it means that constants \( c_i \) are arbitrary and influence only the form of the Lax pair and the compatibility condition.

**Proposition.** The combination (4.12) by the condition (4.13) is covariant with respect to DT. If \( c_2 = c_3 = 1 \), the compatibility condition of (4.5) - (4.12) is equivalent to the equation

\[ u_t = 3u_{xx}/2 + u_{xxx}/4 \]

that is nothing but one of KdV forms.

Many of other integrable equations may be incorporated into the scheme to be developped.

## 5 The example of nonabelian Zakharov-Shabat problem

Here we outline the general scheme of the dressing chain derivation in the nonabelian case. Let us start from the evolution

\[ a_0\Psi + a_1D\Psi = \Psi_t \quad (5.1) \]

that contain the simplest form of the polynomial \( L(D) \)-operator. The case is the nontrivial example of a general equation (2.3) with non-abelian entries. We would respect \( \Psi \) or other (necessary for DT construction) solution \( \Phi \) as operators. The equation enter to the Lax
pair of integrable nonlinear equations (e.g. - Davey-Stewartson) while its stationary versions produce Nonlinear Schrödinger case. It is also interesting by itself: quantum one-dimensional Pauli or Dirac equations after multiplication by the appropriate matrix takes the form of the equation (5.1).

The potential $a_0$ may be expressed in the terms of $\sigma$ from the stationary version of the equation (2.4,5), when $\sigma_t = 0$. Namely, introducing the iteration index $i$, we have the link

$$Da_0^i + [a_0^i, \sigma_i] = -Da_1 - [a_1, \sigma_i]$$

(5.2)

the connection is linear, but contains the commutator. Let us denote $ad_{\sigma} x = [\sigma, x]$. Then

$$a_0^i = (D - ad_{\sigma_i})^{-1}\{-Da_1 - [a_1, \sigma_i]\}$$

(5.3)

The existence of the inverse operator in (5.3) need some restriction for the expression in $\{,\}$ brackets, the expression should not belong to the kernel of the operator $D + ad_{\sigma_i}$. In the subspace, where the Lie product is zero, the equation (5.2) simplifies. The DT is also simple.

$$a_0^{i+1} = a_0^i + [a_1, \sigma_i]$$

(5.4)

Note that $a_1$ is not transformed due to the general formulas (2.3). Substituting the link (5.3) for $i,i+1$ into (5.4) one arrives to the chain equations. One also could express matrix elements of $a_0$ in terms of the elements of the matrix $\sigma$ and plug into the Darboux transform (5.4) separately.

Let us give more details of the construction in the stationary case. Let us denote $a_0 = u$, $a_1 = J$, choosing the case $DJ = 0$ and $\Psi, \Phi$ correspond $\lambda, \mu$. Note that there are two possibilities for stationary equations from nonabelian (5.1): either

$$\Psi_t = \lambda \Psi$$

or

$$\Psi_t = \Psi \lambda$$

(5.5)

and the first of them leads to the essentially trivial connection between solutions and potentials from the point of view of DT theory [21]. Namely, if

$$J\Phi_x + u\Phi = \mu\Phi,$$

then

$$\sigma = \Phi_x \Phi^{-1} = J^{-1}(\mu - u)$$

(5.6)

and DT do not contains eigen functions. It means, for example, that starting from constant potential one never obtain x-dependent $u$ by iterations.

In the second case one writes

$$\sigma = \Phi_x \Phi^{-1} = J^{-1}(\Phi\mu \Phi^{-1} - u)$$

and DT takes the following form

$$u^{i+1} = u^i - J^{-1}[s^i, J] + J^{-1}[u^i, J] = J^{-1}u^i J - J^{-1}[s^i, J],$$

(5.7)
where it is denoted \( s = \Phi \mu \Phi^{-1} \), here and further iteration number indices omitted. The potentials \( u^i \) may be excluded from the equation (2.4) for this case

\[
\sigma_t = Dr + [r, \sigma],
\]

with

\[
r = J\sigma + u = s \tag{5.8}
\]

The stationary case, after the plugging \( u^i \) from (5.8) and returning indices, gives

\[
s^{i+1} = s^i + J\sigma^{i+1} - \sigma^i J \tag{5.9}
\]

and the link of \( s \) with a \( \sigma \) is very simple

\[
Ds^i + [s^i, \sigma^i] \tag{5.10}
\]

The formal transformation that leads to the chain equations is similar to the (5.3), namely, after substitution

\[
\sigma^i = -ad_{s^i}^{-1}Ds^i
\]

into the equation (5.9).

Further progress in the development of this programme is connected with the choice of the additional algebraic structure over the field we consider. It can be useful for the concrete representation of solutions of the equation (5.10). For example, if the elements \( s^i, \sigma^i \) belong to a Lie algebra with structure constants \( C^\gamma_{\alpha \beta} \), then, after the choice of the basis \( e_\alpha \) one introduces the expansions

\[
s^i = \xi^i_\alpha e_\alpha \quad \text{and} \quad \sigma^i = \eta^i_\alpha e_\alpha.
\]

Plugging into (5.10) gives the differential equation

\[
D\xi^i_\alpha + C^\alpha_{\gamma \beta} \xi^i_\gamma \eta^i_\beta = 0.
\]

If one defines the matrix

\[
B_{\beta \alpha} = C^\alpha_{\beta \gamma} \xi^i_\gamma, \tag{5.11}
\]

then, outside of the kernel of \( B \)

\[
\eta^i_\beta = -B_{\beta \alpha}^{-1} \xi^i_\alpha t.
\]

By the definition of the Cartan subalgebra \( C \) the correspondent subspace does not contribute in the Lie product of (5.10).

Statement. If, further, \( J \) belongs to a module over the Lie algebra, \( Je_\alpha = J_{\beta \alpha} e_\beta \) and there exist an external involutive automorphism such that \( e_\alpha^+ = -e_\alpha, C^\gamma_{\alpha \beta} = C^\gamma_{\beta \alpha}, e_\alpha J = J_{\alpha \beta} e_\beta \). Then, the chain equation for the variables \( \xi^i_\alpha \) takes the form

\[
\xi^{i+1}_\alpha = \xi^i_\alpha - B_{\beta \gamma}^{-1} \xi^{i+1}_\gamma J_{\beta \alpha} + J_{\alpha \beta} B_{\beta \gamma}^{-1} \xi^i_\gamma t,
\]

where the matrix \( B \) is defined by (5.11) and the components \( e_\alpha \) outside of \( C \). Otherwise

\[
D\xi^i_\gamma = 0,
\]
if \( e_\gamma \in C \).

The system of differential equations is hence nonlinear as the matrix \( B \) depends on \( \xi_\gamma \). The related scheme is developed in [25].

Remark. The scheme may be generalized for a nonstationary equation (2.5). The equation (5.2) then have the additional term \( D\sigma_i^\gamma \) from the r.h.s.

6 Covariance with respect to Binary DT and Dressing Chains

In this section, as in previous, we do not touch directly the joint covariance with respect to binary DT (bDT) and classification problem, sending the reader to the publications [12], [18], [20] an, especially to most general [19], where the nobelium functions and rational dependence on parameter are admissible. Those papers are devoted to evolution in quantum statistics: the equation of Liuville-von Neumann is considered as compatibility condition of stationary ZS problem algebraic reduction and equation that introduce a time variable.

Hence we continue to study stationary version of the ZS problem (5.1)

\[ JD\Psi + u\Psi = z\Psi \]  \hspace{1cm} (6.1)

and dressing chains of bDT (see the Introduction, [21], compare with e.g. [11], the definition of which we use here). The algebraic reduction imply \( D\Psi = -\mu\Psi, D\mu = 0, \mu \) is scalar.\[ ]

After the reduction we use the straight corollary of (6.1), changing the notation as in [12], \( J \to A, u \to \rho \):

\[ \mu A\varphi + \rho \varphi = z_\mu \varphi, \] \hspace{1cm} (6.2)

and the conjugate problem

\[ \nu \chi A + \chi \rho = z_\nu \chi, \] \hspace{1cm} (6.3)

Let the projector \( P \) have a representation

\[ P = \varphi(\chi, \varphi)_p^{-1} \chi, \] \hspace{1cm} (6.4)

for a given constant idempotent \( p \) [10]. The bDT formula connects the elements (operator valued potentials - density matrix in the physical context),

\[ \rho[i + 1] = \rho[i] + (\mu_i - \nu_i)[P_i, A] \] \hspace{1cm} (6.5)

We would rewrite the conjugate spectral problems [12] in terms of \( P \), by means of the definition (6.4)

\( (\rho - \mu A)P = z_\mu P \)

\( P(\rho - \nu A) = z_\nu P \)

Each equation links the ”potential” \( \rho \) and the projector \( P \) that plays the role of the element \( \sigma \) of previous sections.

\[ \rho P = \mu AP + z_\mu P \]

\[ P\rho = \nu PA + z_\nu P. \] \hspace{1cm} (6.6)
The importance of both of them is obvious from the following corollaries

\[ P(\rho - \mu A)P = (\rho - \mu A)P \]
\[ P(\rho - \nu A)P = P(\rho - \nu A). \]

Let us also note that

\[ (\nu - \mu)PAP = (z_\mu - z_\nu)P \tag{6.7} \]

and

\[ P\rho P = \frac{\nu z_\mu - \mu z_\nu}{\nu - \mu} \tag{6.8} \]

One can check that the DT may be treated as similarity transformation

\[ \rho[i + 1] = \left(1 + \frac{\mu - \nu}{\nu} \right) \rho[i] \left(1 + \frac{\nu - \mu}{\mu} \right) \rho[i] \tag{6.9} \]

Multiplying (6.9) by \( P_{i+1} \) from the left side and by \( P_i \) by the right side, changing \( \rho[n+1] = \rho_n \) one obtains:

\[ P_{n+1}\rho_{n+1}P_n = P_{n+1}(1 + \frac{\mu_n - \nu_n}{\nu_n} P_n)\rho_n P_n(1 + f\nu_n - \mu_n \nu_n P_n) \]

or, using the links (6.6) for the appropriate \( n \),

\[ P_{n+1}(\nu_{n+1}A + z_{\nu,n+1})P_n = P_{n+1}(1 + \frac{\mu_n - \nu_n}{\nu_n} P_n)(\mu_n AP_n + z_{\mu,n} P_n)(1 + f\nu_n - \mu_n \nu_n P_n). \]

Simplifying, one arrives to the chain equation.

\[ (\nu_{n+1} - \nu_n)P_{n+1}AP_n + (z_{\nu,n+1} - z_{\nu,n})P_{n+1}P_n = 0. \tag{6.10} \]

or, similarly, after the right multiplication by \( P_{n+1} \) and left one by \( P_n \),

\[ (\mu_{n+1} - \mu_n)P_n AP_{n+1} + (z_{\mu,n+1} - z_{\mu,n})P_n P_{n+1} = 0. \tag{6.11} \]

These equations are recurrences and define the solution via parameters. The dependence of the parameters on the additional variable \( t \) is introduced by the ”master” equation (see [12]):

\[ tP_t = (AP - PAP)/\mu - (PA - PAP)/\nu. \tag{6.12} \]

The ”minimal” closure \( P_{n+1} = \lambda P_n = P_n \lambda \), is possible only if \( \lambda = 1 \) for the \( P_n \) is a projector. In this case

\[ P_{n+1} = P_n = P, \quad z_{\nu,n+1} = z_{\nu,n} = z_{\nu}. \]

So the equations (6.10,11) are valid automatically. Other simple closure

\[ P_{n+1} = a_n P_n + b_n \]

with \( a_n, b_n \in K \) leads to \( P_{n+1} = P_n - (1 - (-1)^n)/2. \)
Next possibility is when some element A do not commute with \( P_n \). For example

\[ P_{n+1} = P_n A, \quad z_{\mu,n+1} = \lambda_n z_\mu \]

that for (6.11) leads to

\[ (\lambda_n - 1) z_{\mu,n} S_n^2 = (\mu_n - \nu_n) S_n \]

where, \( S_n = P_n A \), is proportional to some projector, or, by recurrence, it means

\[ P_n = p_0(t) A^n \]

where the function \( p_0(t) \) is defined from (6.12).

Finally, we arrive to a conjecture:

The chain equations (6.11,12) may help to construct particular solutions of the compatibility condition for the Lax pair (see [18]) if each \( P_n \) is a solution of the equation (6.13).

7 Conclusion

We conclude that our approach

a) give a connection between a class of a bilinearizable NLPDE, and its Lax form. The class is formed by a certain combination of binary Bell polynomials that satisfy the covariance principle.

b) produce automatically a technique of integration of the equations that belong to this class. The underlying Darboux covariance of the both linear operators permits to generate a rich set of solutions. Between them are multisolitons, positons, periodic and rational solutions with abundant combinations that introduce their interaction.[

We add that a platform for a nonabelian generalization is prepared. The joint covariance principle application presumably cover the items of classification that appears inside most of symmetry approaches [29],[16] and contain the direct link to solutions via dressing chains - iterations of DT.

Mention also the generalization of the theory of small deformations of iterated transforms with respect to intermediate parameters that appear within the iteration procedure of bDT. Infinitesimal invariance generated by extended Backlund transformation was considered in [30]. The perturbation formulas allow to define and investigate generators of the corresponding group, that is a symmetry group of a given hierarchy associated with the ZS problem under consideration.

We continue to develop the dressing method for linear and soliton problems. The procedure of dressing includes both solutions and potentials of underlying linear system. The other reductions and generalizations may be introduced if in the ring there exist an involutive automorphism.

There are abundant possibilities for generalizations based on recent results of V.B. Matveev [27], where the DT transformation operations is defined via some automorphism at rings.

Acknowledgments
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