Spin-density (charge) separation is a remarkable feature predicted for one-dimensional interacting spin-1/2 fermions \(^1\) and widely searched in condensed matter systems. Consequently, its investigation in atomic systems could be of interest to different fields of physics. Unlike the higher-dimensional fermionic systems, where elementary excitations normally carry both spin and density (charge) degree of freedom, the collective excitations of the one-dimensional Fermi system separate into two distinct modes, spin and density waves due to the fact that the interaction in one-dimensional systems lead to a Luttinger liquid state with bosonic excitations \(^2\). This behaviour is a hallmark of collective effects caused by interactions. For bosons, using two-components, corresponding to two hyperfine states of cold atoms \(^2\) allows us to study the (iso)spin waves as the relative spatial oscillations of the two-components. Till now, the study of spin-density separation has been limited to one-dimensional fermions \(^3\) and bosons \(^1\) \(^2\) since, both these systems are believed to belong to the same Luttinger liquid class, which leads to the spin-density separation. Contrary to expectations, in this Letter we will show that for bosons spin-density separation not only exists in one-dimensional systems but also in higher dimensions since for bosons, excitations are always collective excitations in all dimensions due to the presence of the (quasi)condensate fraction and that the Luttinger liquid approach is not essential to describe the spin-density separation in bosons. We also compute the dynamical structure factor which reveals distinct features of spin density separation in all dimensions.

We start with the Lagrangian density for two-component Bose gas at zero temperature:

\[
\mathcal{L} = \frac{i}{2} \sum_{i=1,2} (\varphi_i^* \partial_t \varphi_i - \varphi_i \partial_t \varphi_i^*) - \frac{1}{2m} (\nabla_r \varphi_i)^2 - \mu_i n_i \\
- \frac{1}{2} \sum_{i,j=1,2} g_{d,ij} n_i n_j,
\]

where \(\varphi_i = \varphi_i(r,t), i = 1, 2\) is the field representing two different Bose particles, \(r\) is the space coordinate, \(t\) is the real time, and here we set \(\hbar = 1\). Also \(\mu_i, n_i = |\varphi_i|^2\) is the chemical potential and the particle density of the \(i^{th}\) component and \(g_{d,ij} > 0\) is the repulsive effective atom-atom interaction between the \(i^{th}\) and \(j^{th}\) components.

For a 3D Bose gas, \(g_3 = 4\pi a_\text{a}/m \ [6,7]\). Here \(a_\text{a}\) is the 3D scattering length. For lower dimensional Bose gas in a 3D trap with longitudinal harmonic trapping frequency \(\omega_\perp, g_2 = 4\pi/(m \ln n_\text{a}_2) \ [8,10]\) with the 2D scattering length given as \(a_2 = 7.41 e^{-\sqrt{\pi/\omega_\perp m}} \ [11]\) and \(g_1 = 2\omega_\perp a_3\) with \(a_3 < 1/\sqrt{m\omega_\perp} \ [12]\). The behavior of the system depends crucially on the dimensionless parameter \(\gamma_d(\gamma_d') \equiv m_g\text{d}^{1-2/d}(m_g\text{d}^{1-2/d})\). For the gas to be weakly interacting, we must have \(\gamma_d(\gamma_d') < 1\). The chemical potential \(\mu_i\) of the \(i^{th}\) component is determined by the condition \(\sum_{ij} g_{d,ij} n_i = \mu_i n_i\).

To understand the low-energy excitations in two-component Bose gas, one can derive a low-energy effective hydrodynamical Lagrangian that contains only modes related to the low-energy excitations \([3,13]\). We write the Boson field \(\varphi_i\) in the terms of the number density \(n_i\) and the phase \(\theta_i\) as \(\varphi_i = n_i e^{i\theta_i}\). In the weak coupling regime the phase changes slowly in space while the density fluctuates fast \([4]\), therefore one can integrate out the high energy fast density fluctuation \([13]\) to obtain the effective hydrodynamic action. We introduce the density fluctuation \(\delta n_i = n_i - \bar{n} + \delta n_i\). In terms of the new basis, \(\delta n_p(\sigma) = (\delta n_1 \pm \delta n_2)/\sqrt{2}\) and \(\theta_p(\sigma) = (\theta_1 \pm \theta_2)/\sqrt{2}\), the action obtained from the Lagrangian density can be rewritten as

\[
S = -\int d^4 x dt \sum_{\lambda=\rho,\sigma} \left[ (\bar{n}_\lambda + \delta n_\lambda) \partial_t \theta_\lambda \right.
\]

\[
+ \frac{\bar{n}(\nabla_r \theta_\lambda)^2}{2m} + \frac{\nabla_r \delta n_\lambda}{8m\bar{n}} + \frac{g_{d,\lambda}}{2} (\delta n_\lambda)^2, \]

where \(\bar{n}_p(\bar{n}) = \sqrt{2} n(0)\) and \(g_{d,\rho(\sigma)} = g(1 \pm \alpha)\) with
density and the spin waves, centered at $v_{\rho}d$ and $v_{\sigma}q$, respectively. The one dimensional structure factor is found to be a delta function, while the two- and three dimensional DSF is broadened because of Beliaev damping. In two dimension the DSF for the density wave is broader compared to that of the spin waves, while the width of the three-dimensional peaks remains the same. There is no specific reason why the width of the 2D peaks is larger than both the 1D and 3D peaks, this depending on the choice of the parameters.

\[ \alpha = g_d'/g_d. \]

For $\alpha < 1$, after performing two Gaussian integrals, the effective action has the form 

\[
S_{\text{eff}} = \int d^4x dt \sum_{\lambda=\rho,\sigma} \frac{\chi_\lambda}{2} \left( (\partial_t \phi_\lambda)^2 - v_\lambda^2 |\nabla_x \phi_\lambda|^2 \right),
\]

where $\phi_\rho(\sigma) = e^{i\theta_\rho(\sigma)}$, $\chi_\rho(\sigma) = 1/g_{d,\rho(\sigma)}$ is the density (spin) compressibility and $v_{\rho(\sigma)} = \sqrt{g_{d,\rho(\sigma)}}/m$ is the sound velocity of the density/spin mode. Here we assumed that the fields $\theta_{\rho(\sigma)}$ vary slowly in space and we have dropped the $\nabla_{\min}$ term. The effective action \ref{eq:effective-action} describes the low-energy excitations of two sound waves with linear dispersions $\omega_{\rho(\sigma)} = v_{\rho(\sigma)} k$. The bosons split into two gapless modes, namely density mode and spin mode, propagating with different velocities. The density wave propagates faster than the spin wave, which can be seen by the relation $v_\rho/v_\sigma \approx \sqrt{(1-\alpha)/(1+\alpha)}$. In this regime the energy gap of the lowest excitation above the ground state is zero. Such systems have a diverging length scale determining the exponential decay of equal time correlators in the ground state, which defines the quantum critical behavior. Therefore the systems for $\alpha < 1$ lie at the quantum critical points \ref{fig:critical-points}.

\vspace{0.5cm}

The meaning of the low-energy effective Lagrangian \ref{eq:effective-action} is that the bosonic system separates into two independent degrees of freedom, i.e. spin and density. Unlike in fermionic one-dimensional systems, we do not need the bosonization method to obtain the spin-density separation, the only thing we need is the (quasi)condensate density $\bar{n}$ to have fluctuations around. This can be fulfilled in all dimensions at zero temperature for bosonic systems. The Bogoliubov energy dispersion relation of one-component interacting Bose gas is $\epsilon(k) = \sqrt{(k^2/2m)^2 + g_{d,\rho}(\sigma)\bar{n}k^2/m}$ \ref{fig:dsf}. For the two component Bose gas, replacing the interaction $g_d$ with $g_{d,\rho(\sigma)}$, we obtain two branches of the excitations

\[
\epsilon_{\rho(\sigma)}(k) = \sqrt{(k^2/2m)^2 + g_{d,\rho(\sigma)}(1+\alpha)\bar{n}k^2/m},
\]

which is in agreement with the result obtained by the semiclassical method \ref{fig:dsf}. From the dispersion relations \ref{fig:dsf} we can define the chemical potential for the density and spin waves as $\mu_{\rho(\sigma)} = g_{d,\rho(\sigma)}\bar{n}$.

For $\alpha = 1(g_d = g_d')$, only one Gaussian integral can be performed in action \ref{eq:effective-action} giving the gapless density wave with linear dispersion. However, one obtains a quadratic dispersion for the spin-wave excitations, in agreement with SU(2) symmetry \ref{fig:dsf}. This effect can also be seen from the Bogoliubov excitations $\epsilon_{\sigma} = \sqrt{(k^2/2m)^2 + 2g_{\sigma}nk^2/m} + \epsilon_{\rho} = k^2/2m$ by replacing $g_d,\rho = 2g_d$ and $g_{d,\sigma} = 0$ in \ref{fig:dsf}. In this case, due to the SU(2) symmetry, the eigenstates are classified according to their total spin $S$ ranging from 0 to $N/2$, and according to recent result by Eisenberg and Lieb \ref{fig:dsf}, the ground state is fully polarized ($S = N/2$). In one dimension, the ground state is described by Lieb-Liniger(LL) model of one-component interacting Bose gas \ref{fig:dsf}, for which the elementary excitations in the weak-coupling regime are density waves \ref{fig:dsf}, and the system is ferromagnetic.

\vspace{0.5cm}

In the case of $\alpha > 1(g_d < g_d')$, we found $g_{d,\rho} < 0$. This implies, $v_\sigma = \sqrt{|g_{d,\rho}|}/m$ in the long wave length limit is imaginary. The spin waves become unstable and damped out in the thermodynamic limit. Therefore we obtain a phase separation of the two-component Bose gas \ref{fig:dsf}.

\vspace{0.5cm}

The dynamical structure factor (DSF) of many-body system is defined as follows

\[
S_{\rho(\sigma)}(q, \omega) = \int d^4x dt e^{i(\omega t - qx)} \langle \delta n_{\rho(\sigma)}(x, t)\delta n_{\rho(\sigma)}(0, 0) \rangle,
\]

where $\langle \cdots \rangle$ can be calculated using path integral with the effective action. Experimentally, one can measure the dynamical structure factor using Bragg spectroscopy \ref{fig:dsf}.

For $\alpha < 1$, from the action \ref{eq:effective-action} one can get the equation of motion for $\delta n_{\rho(\sigma)}$ as $\delta n_{\rho(\sigma)}(x, t) = -1/g_{d,\rho(\sigma)}\partial_t \theta_{\rho(\sigma)}(x, t)$. From the quadratic Lagrangian density, the DSF \ref{eq:dsf} can be obtained as

\[
S_{\rho(\sigma)}(q, \omega) = \frac{\chi_{\rho(\sigma)}v_{\rho(\sigma)}^2q^2}{\omega^2 - \omega_{\rho(\sigma)}^2},
\]

where $\omega_{\rho(\sigma)}(q) = v_{\rho(\sigma)}q + i\Gamma_{\rho(\sigma)}(q)$ with the quasiparticle decay rate $\Gamma_{\rho(\sigma)}(q)$.

In order to obtain the DSF, one has to find the compressibility $\chi_{\rho(\sigma)}$, velocity $v_{\rho(\sigma)}$ and decay rate $\Gamma_{\rho(\sigma)}(q)$ in terms of the dimensionless parameters $\alpha_{d,\rho(\sigma)}$. Using
the macroscopic argument, the compressibility $\chi_{\rho(\sigma)}$ is related to the energy $E(g_{\rho(\sigma)},n)$ as $\chi_{\rho(\sigma)}^{-1} = \frac{1}{V} \frac{\partial^2 E}{\partial n^2}$ with the constant system size: $V = L^d$ and density: $n = N/V$. Similarly, the sound velocity can also be obtained using the macroscopic energy spectrum as $v_{\rho(\sigma)} = \left( \frac{V}{mn} \frac{\partial E}{\partial n} \right)^{1/2}$ with constant particle number $N$ \[21\]. The way to obtain the ground state energy spectrum is diverse and depends on the dimension. As indicated by Beliaev \[22\], the dimensional dependent decay rate $\Gamma_{\rho(\sigma)}(q)$ is caused by the process of a long wave-length phonon decaying into two phonons and it can be calculated for small momenta using the formula \[8\]

$$
\Gamma_{\rho(\sigma)}(q) = \frac{9v_{\rho(\sigma)}}{128\pi^2 \hbar m} \int d^d k |q||k| |q-k| \delta(\epsilon_{\rho(\sigma)}(q) - \epsilon_{\rho(\sigma)}(k) - \epsilon_{\rho(\sigma)}(q-k)).
$$

(7)

In 3D, the dimensionless parameter $\gamma_3 = 4\pi a_0 n^{1/3}$. In this case, the requirement for a dilute gas $n a_0^3 \ll 1$ corresponds to the weak-coupling condition $\gamma_3 \ll 1$. The ground state energy was given for the first time by Lee et al. \[6\] as $E = N \tilde{n}^{2/3}/(2m) \left(1 + 16 \gamma_3^2/5 \pi^2\right)$. The ground state compressibility and velocity are given by $\chi_{\rho(\sigma)}^{-1} = g_3,\rho(\sigma)(1 + \frac{2}{\pi^2} \gamma_3,\rho(\sigma))^2)$ and $v_{\rho(\sigma)} = \sqrt{\frac{g_3,\rho(\sigma)}{m}} (1 + \frac{2}{\pi^2} \gamma_3,\rho(\sigma))^2)$, respectively. The decay rate for 3D system is obtained from eq(7): $\Gamma_{\rho(\sigma)}(q) = \Gamma(q) = \frac{9q^2}{128\pi^2 \hbar m}$ \[22\]. We can see that the decay rates for density and spin waves are equal and proportional to $q^5$. The DSF for $\omega > 0$ can be approximated as

$$
S_{\rho(\sigma)}(q,\omega) \approx \frac{\chi_{\rho(\sigma)} v_{\rho(\sigma)} \sqrt{\Gamma_{\rho(\sigma)}(q)}}{2 \left[ (\omega - v_{\rho(\sigma)}q)^2 + \Gamma_{\rho(\sigma)}(q)^2 \right]}
$$

(8)

In the Bragg scattering experiment, one should obtain two peaks centered at $v_{\rho(\sigma)}q$ for the cross section with the width $\Gamma(q)$.

For 2D Bose gas, renormalization-group analysis \[10\,23\] shows that the interaction of the 2D dilute gas is marginally irrelevant in a dilute limit specified by $\ln n a_0^2 \gg 1$. The corresponding ground state energy for a weak-interacting gas is given by $E = N \tilde{n}^{2/3}/(2m) \gamma_2(1 - C \gamma_2)$ where constant $C \ll 1$ is not universal but model-dependent due to the marginal interaction \[23\]. The compressibility and velocity for spin and density-wave excitations are $\chi_{\rho(\sigma)}^{-1} = g_2,\rho(\sigma)(1 - (C - \frac{3}{8}\pi^2) \gamma_2,\rho(\sigma))$ and $v_{\rho(\sigma)} = \sqrt{\frac{g_2,\rho(\sigma)}{m}} (1 - (C - \frac{3}{8}\pi^2) \gamma_2,\rho(\sigma))^{1/2}$. The Beliaev decay rate can be obtained by the integral \[7\]: $\Gamma_{\rho(\sigma)}(q) = \frac{\sqrt{\pi} \sqrt{\frac{g_2,\rho(\sigma)}{m}} q^3}{6 \pi^2}$. Therefore the DSF \[8\] has a broader width for density waves than spin waves.

In the case of one dimension, contrary to 2D and 3D systems, the weak coupling means that the system is in the high density regime because $\gamma_1 = m q^2 / \hbar$. In this regime, Lieb and Liniger \[19\] first gave the ground state energy as $E = \frac{N \tilde{n}^2}{2m} \gamma_1(1 - \frac{1}{\gamma_1 \sqrt{2}})$.

A few algebra leads to the compressibility and sound velocities as $\chi_{\rho(\sigma)}^{-1} = g_1,\rho(\sigma)(1 - \frac{1}{\gamma_1 \sqrt{2}} \gamma_1,\rho(\sigma))$ and $v_{\rho(\sigma)} = \sqrt{\frac{g_1,\rho(\sigma)}{m}} (1 - \frac{1}{\gamma_1 \sqrt{2}} \gamma_1,\rho(\sigma))^{1/2}$. For 1D, one obtains no decay rate. The reason is that the scenario for one phonon decaying into two phonons cannot exist due to the fact that energy conservation law in eq(7) cannot be fulfilled in 1D. Therefore two sharp peaks should be observed in the Bragg scattering experiments. Figure\[1\] illustrates and summarizes the results obtained above for the DSF in all the three dimensions.

In the case of $\alpha = 1$, the situation changes. For the density waves the dynamic structure factor remains the same as that in two-sound regime, while the DSF for spin-wave excitation alters due to the dramatic changing of the dispersion from linear to quadratic. In order to calculate the DSF one can use the effective Hamiltonian in the weak-coupling regime:

$$
H = \sum_p \epsilon_p a_p^\dagger a_p + \sum_p \epsilon_p b_p^\dagger b_p + g d \sqrt{\frac{\hbar}{8 \pi}} \sum_{k,q \neq 0} \sqrt{\frac{\epsilon_q}{\epsilon_k}} (a_q^\dagger + a_q) b_{k-q} b_k
$$

(9)

with the spectrum of free spin waves $\epsilon_p = p^2 / 2m$, the Bogoliubov spectrum $\epsilon_p = \sqrt{\epsilon_p^2 + 2 \mu \epsilon_p n}$ \[6\] and the chemical potential: $\mu_d = 2q \sqrt{\tilde{n}}$. Using $\delta n_q = \sqrt{n}(b^\dagger + b)$, the DSF can be related to the imaginary part of the Green function as $S(q,\omega) = \hbar \text{Im} G(q,\omega)$ where $G(q,\omega)$ is the single particle Green function of the spin operators $b_q$ and

![FIG. 2: Dynamic structure factor for SU(2) in symmetric Hamiltonian in 3D. Note that $\omega$ is in the units of $g_d n$ and $q$ is in the units of $\sqrt{n} g_d$. The DSF of the density waves varies linearly with $q$, while the DSF of the spin waves shows the quadratic dependence on $q$.](image)
Therefore the DSF for the spin waves reads

\[ S(q, \omega) = \frac{\bar{n}\Gamma_{d,\sigma}(q)}{(\omega - \frac{\pi}{2m_d})^2 + \Gamma_{d,\sigma}(q)^2}, \tag{10} \]

where the effective mass \( m^* \) is determined by the equation: \( m/m^*_d = (1 + 2/m^2 \delta \Sigma(p)/\delta p^2)(p=0) \) with the self energy defined as \( \delta \Sigma = G^{-1}(g_d) - G^{-1}(g_d = 0) \), and the decay rate: \( \Gamma_{d,\sigma}(q) = \text{Im} \delta \Sigma(q) \). To the second order diagram for the self energy \( \Sigma \), one obtains the inverse effective mass related to the dimensionless parameter \( \gamma_d = \mu_d N^{-2/3}d \) as \( m/m^*_d = 1 - \alpha_d \gamma_d^2/2 \) with \( \alpha_d = 2/3\pi, 1/2\pi, 1/8\pi \) for one, two, and three dimensions, respectively. The decay process depends on the spin-phonon interaction which requires the energy conservation: \( e_{q+k} + \epsilon_k = e_q \) with the spin momentum \( q \) and phonon momentum \( k \). For \( q < \sqrt{m\mu_d} \), this condition cannot be fulfilled, therefor \( \Gamma_{d,\sigma} = 0 \), i.e., \( S(q, \omega) = \delta(\omega - \omega_0) \). For \( q \gtrsim \sqrt{m\mu_d} \), an approximation can be obtained as follows: \( \Gamma_{d,\sigma}(\sqrt{m\mu_d}(1 + \delta)) = \beta_d \mu_d \delta^{3/2(d-1)} \) for \( \delta \ll 1 \) with \( \beta_d = 0, 1/8\pi, 2/3\pi \) for \( d = 1, 2, 3 \), respectively. The eqn. (10) shows the fact that the excitations for one dimension is a delta function due to the fact that the energy conservation relation for a particle emitting a phonon cannot be fulfilled, therefore \( \Gamma_{d,\sigma} = 0 \). In the same regime, the one-dimensional structure factors for spin waves in the two-sound regime \((\alpha < 1)\) show a linear dispersion (phonon like) and the DSF for all dimensions show two distinct peaks corresponding to the density and the spin waves, centered at \( v_p q \) and \( v_s q \), respectively. In the same regime, the one-dimensional structure factors are found to be delta functions, while the two- and three-dimensional DSF is broadened because of Beliaev damping. The spin waves show a quadratic dispersion in the SU(2) symmetric regime \((\alpha = 1)\) and the DSF for all dimensions also show two distinct peaks centered at \( v_p q \) and \( q^2/2m^*_d \). The spin wave is damped in the phase separated regime \((\alpha > 1)\) and there is only one peak corresponding to the density wave. These are interesting signatures of spin-density separation to look for using Bragg spectroscopy where, the response of the condensate to a two-photon Bragg pulse is measured[24]. The difference between spin and charge velocities allows us to have spin and charge wavepackets moving at different velocities. An optical potential generated by a laser tuned, e.g., between fine-structure levels of excited alkali states transfers momentum solely to the the spin waves, while an optical potential far detuned will act solely on the density waves[23]. One can also coherently excite the spin waves and the density waves simultaneously and then probe the two waves with a second laser pulse at a later time. Spin-charge separation manifests itself in a spatial separation of the spin and density wavepackets.

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