On affine functions with respect to some means

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Abstract. The purpose of the present paper is to investigate the functional equation
\[ M(f(x), g(y)) = h(N(x, y)), \]
where \( f, g \) and \( h \) are self-mappings of a real interval \( I \) and \( M, N : I^2 \to I \) are functions. In particular, we will show that under appropriate assumptions imposed on the functions \( M, N \) the local boundedness of \( f \) implies the continuity of \( g \).

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1. Introduction: basic definitions and auxiliary lemmas

In this paper \( I \) will always denote a non-degenerate interval contained in \( \mathbb{R} \).

A function \( M : I^2 \to I \) is called a mean on \( I \) if
\[ \min\{x, y\} \leq M(x, y) \leq \max\{x, y\}, \quad x, y \in I. \]

A mean \( M \) is called a strict mean if
\[ \min\{x, y\} < M(x, y) < \max\{x, y\}, \quad x, y \in I, \quad x \neq y. \]

Now, let \( M \) be a mean on \( I \) and let \( f : I \to I \). We say that \( f \) is affine with respect to \( M \) (or shortly \( M \)-affine) if
\[ f(M(x, y)) = M(f(x), f(y)), \quad x, y \in I. \]

Let us also introduce the following notation. For nonempty sets \( X, Y, Z \), \( u \in X, \; v \in Y \) and a function \( F : X \times Y \to Z \), we define functions \( F_u : Y \to Z \), \( F^v : X \to Z \) by the formulas:
\[ F_u(y) := F(u, y), \quad y \in Y; \]
\[ F^v(x) := F(x, v), \quad x \in X. \]
The present paper is devoted to the following functional equation:

\[ M(f(x), g(y)) = h(N(x, y)), \quad x, y \in I, \]  

(1.1)

where \( M \) and \( N \) are given functions, \( M, N : I^2 \to I \), whereas \( f, g \) and \( h \) are unknown functions, \( f, g, h : I \to I \). Our results are analogous to the ones of Ng [1], who has investigated the functional equation

\[ f(x) + g(y) = h(T(x, y)), \]

where the unknown functions \( f, g \) and \( h \) act on a connected or a locally connected topological space \( X \), and \( T : X^2 \to X \) is a given function. The methods we apply to (1.1) are modifications of those used by Ng in [1].

In particular, our results cover the case of \( L \)-affine functions, where \( L \) is a logarithmic mean, i.e.

\[ L(x, y) := \frac{x - y}{\ln x - \ln y}, \quad x \neq y, \]

\[ x, x = y, \]

which was investigated by Matkowski in [2].

Let us start with a technical lemma.

**Lemma 1.** Let \( M : I^2 \to I \) be a function such that for all \( u, v \in I \) the mapping \( M_u \) is strictly increasing, and the mapping \( M^v \) is increasing and continuous. Then, for all \( s, S, v_0 \in I \) such that \( s \leq S \) and \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that for all \( u, \bar{u}, v \in I \) satisfying \( M(u, v) = M(\bar{u}, v_0) \), \( \bar{u} - u \leq \delta \) and \( s \leq u \leq S \), we have \( v - v_0 \leq \varepsilon \).

**Proof.** Suppose (in search of a contradiction) that there exist \( s, S, v_0 \in I \) such that \( s \leq S \) and

\[ \exists_{\varepsilon > 0} \forall_{\delta > 0} \exists_{u, \bar{u}, v \in I} \quad (M(u, v) = M(\bar{u}, v_0), \bar{u} - u \leq \delta, s \leq u \leq S, v - v_0 > \varepsilon). \]

In particular, for every \( n \in \mathbb{N} \) there exist \( u_n, \bar{u}_n, v_n \in I \) fulfilling the conditions:

\[ M(u_n, v_n) = M(\bar{u}_n, v_0), \bar{u}_n - u_n \leq \frac{1}{n}, s \leq u_n \leq S, v_n - v_0 > \varepsilon. \]

Observe that the sequence \( (u_n)_{n \in \mathbb{N}} \) has to have a convergent subsequence, thus we may assume that the sequence is convergent to some \( u \in I \).

If \( u_n + \frac{1}{n} \in I \) for infinitely many \( n \in \mathbb{N} \), then we deduce the following inequalities:

\[ M(u_n, v_0 + \varepsilon) \leq M(u_n, v_n) = M(\bar{u}_n, v_0) \leq M \left( u_n + \frac{1}{n}, v_0 \right). \]

On letting \( n \) tend to infinity, we get that \( M(u, v_0 + \varepsilon) \leq M(u, v_0) \), which contradicts the fact that \( M_u \) is strictly increasing.

If \( u_n + \frac{1}{n} \in I \) holds only for finitely many \( n \in \mathbb{N} \), then \( u = S \in I \) is the right endpoint of the interval \( I \). Then we have:

\[ M(u_n, v_0 + \varepsilon) \leq M(u_n, v_n) = M(\bar{u}_n, v_0) \leq M(u, v_0) \]
and letting \( n \) tend to infinity, we get that \( M(u, v_0 + \varepsilon) \leq M(u, v_0) \), which contradicts again the fact that \( M_u \) is strictly increasing. \( \Box \)

The proof of the next lemma is analogous and therefore we omit it here.

**Lemma 2.** Let \( M : I^2 \to I \) be a function such that for all \( u, v \in I \) the mapping \( M_u \) is strictly increasing, and the mapping \( M^v \) is increasing and continuous. Then, for all \( s, S, v_0 \in I \) such that \( s \leq S \) and \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that for all \( u, \bar{u}, v \in I \) satisfying \( M(u, v) = M(\bar{u}, v_0) \), \( u - \bar{u} \leq \delta \) and \( s \leq \bar{u} \leq S \), we have \( v_0 - v \leq \varepsilon \).

In the proof of our main result we also need the following lemma:

**Lemma 3.** [3] Let \( X \) be a connected and locally connected space, \( \theta : X \to \mathbb{R} \) a continuous function and \( t_1, t_2 \in \theta(X) \), \( t_1 < t_2 \). Then there exists a connected subset \( B \subseteq \theta^{-1}((t_1, t_2)) \) such that \( \theta(B) = (t_1, t_2) \).

## 2. Main results

Now let us state and prove our main result:

**Theorem 1.** Let \( M, N : I^2 \to I \) be functions such that for all \( u, v \in I \) the mapping \( M_u \) is strictly increasing, the mapping \( M^v \) is strictly increasing and continuous, and the mappings \( N_u, N^v \) are continuous. Assume that a triple \((f, g, h) : I \to I^3\) is a solution of (1.1) and there exists a subinterval \( I_0 \subseteq I \) such that \( f \) is nonconstant on \( I_0 \) and \( f(I_0) \subseteq [s, S] \) for some \( s, S \in I \). Then \( g \) is continuous.

**Proof.** Choose \( x_1, x_2 \in I_0 \) such that \( f(x_1) \neq f(x_2) \). From (1.1) and the strict monotonicity of \( M^g(y) \) it follows that \( N(x_1, y) \neq N(x_2, y) \) for all \( y \in I \).

Fix \( y_0 \in I \) arbitrarily. We will prove the continuity of \( g \) at \( y_0 \) using Lemmas 1 and 2 for \( v_0 = g(y_0) \). Let \( \varepsilon > 0 \); there exists a \( \delta > 0 \) such that the conditions of Lemmas 1 and 2 hold. Denote \( l_1 := N(x_1, y_0) \) and \( l_2 := N(x_2, y_0) \). We may assume that \( l_1 < l_2 \). According to Lemma 3 applied to \( X = I_0 \), \( \theta = N_{y_0} \), \( t_1 = l_1 \), \( t_2 = l_2 \) there exists an interval \( B \subseteq I_0 \) such that \( N(B, y_0) = (l_1, l_2) \). The function \( f \) is bounded on \( I_0 \) and \( B \subseteq I_0 \), so \( \sup\{f(x) : x \in B\} < +\infty \). Hence, \( f(x_0) \geq f(x) - \delta \) for some \( x_0 \in B \) and for all \( x \in B \).

Let \( V := \{y \in I : l_1 < N(x_0, y) < l_2\} \). It is easy to notice that \( y_0 \in V \) and \( V \) is open, since \( N_{x_0} \) is continuous. For arbitrary \( y \in V \) we have \( N(x_0, y) \in (l_1, l_2) = N(B, y_0) \), so there exists an \( x \in B \) such that \( N(x_0, y) = N(x, y_0) \). By virtue of (1.1) we get \( M(f(x_0), g(y)) = M(f(x), g(y_0)) \). Apply Lemma 1 to \( v_0 := g(y_0) \), \( L \), \( \bar{u} := f(x) \), \( u := f(x_0) \), \( v := g(y) \) to get that \( g(y) - g(y_0) \leq \varepsilon \) for \( y \in V \).
Since \( N(x_0, y_0) \in (l_1, l_2) = N(B, y_0) \) we can find points \( x_3, x_4 \in B \) fulfilling \( N(x_3, y_0) < N(x_0, y_0) < N(x_4, y_0) \). Define the set \( W \) by
\[
W := \{ y \in I : N(x_3, y) < N(x_0, y_0) < N(x_4, y) \}.
\]
One can check that \( W \) is a neighborhood of \( y_0 \). Moreover, for each \( y \in W \) the set \( N^y(\{\min\{x_3, x_4\}, \max\{x_3, x_4\}\}) \) is an interval which contains the points \( N(x_3, y) \) and \( N(x_4, y) \). Therefore, the point \( N(x_0, y_0) \), which lies between them, belongs to that set. Thus, for every \( y \in W \) there exists a point \( x \) between \( x_3 \) and \( x_4 \), which fulfills \( N(x_0, y_0) = N(x, y) \). But such a point \( x \) must belong to \( B \), since \( B \) is connected and \( x_3, x_4 \in B \). From (1.1) we get \( M(f(x_0), g(y_0)) = M(f(x), g(y)) \). From Lemma 2 it follows that \( g(y_0) - g(y) < \varepsilon \) for \( y \in W \).

Thus, \( g \) is continuous at \( y_0 \). Since \( y_0 \) is arbitrarily chosen, \( g \) is continuous.

\(\square\)

Remark 1. Let us note that if \( I \) is compact, then the assumption of the boundedness of \( f \) by elements from \( I \) is obviously fulfilled. Similarly, if \( f \) is continuous in some point \( x_0 \) and \( f(x_0) \in intI \), then \( f \) is bounded on some neighborhood of \( x_0 \) by elements from \( I \).

Proof. If \( I = [a, b] \) and \( f : I \to I \), then the assumptions of the previous theorem are fulfilled for \( s = a \), \( S = b \), \( I_0 = I \).

If \( f \) is continuous in some point \( x_0 \) and \( f(x_0) \in intI \), then there exist \( s \), \( S \in intI \) such that \( s < f(x_0) < S \) and a non-degenerate interval \( I_0 \subseteq I \) such that \( x_0 \in I_0 \) and \( f(I_0) \subseteq [s, S] \).

In the following corollary we will show that under auxiliary assumptions imposed on functions \( M, N \) and if \( f = g \), then it is enough to assume that \( f \) is bounded from one side on some open subset of \( I \) by an element from \( I \).

Corollary 1. Let \( M, N : I^2 \to I \) be functions such that for all \( u, v \in I \) the mapping \( M_u \) is strictly increasing, the mapping \( M^v \) is strictly increasing and continuous and the mappings \( N_u, N^v \) are strictly increasing, continuous and \( N(x, x) = x \) for every \( x \in I \). If a pair \((f, h) : I \to I^2 \) is a solution of the equation
\[
h(N(x, y)) = M(f(x), f(y)), \quad x, y \in I \tag{2.1}
\]
and there exist a subinterval \( I_0 \subseteq I \) and a constant \( S \in I \) (or \( s \in I \)) such that \( f(I_0) \subseteq (-\infty, S) \) (or \( f(I_0) \subseteq [s, +\infty) \), respectively) and \( f \) is non-constant on \( I_0 \) and \( M^S \) is onto \( I \), then \( f \) is continuous.

Proof. We show that the function \( f \) fulfilling the assumptions of the corollary also fulfills the assumptions of Theorem 1.

Let \( f \) be bounded from above on an interval \( I_0 = [a, b] \subseteq I \) by a constant \( S \in I \) and let \( x_0 := \frac{1}{2}(a + b) \). We will prove that there exists a point \( x_1 \in (a, x_0) \) such that for every \( x \in (x_1, x_0) \) there exists \( y \in (a, b) \) such that \( N(x, y) = x_0 \). First, we show that there exists a pair \((x_1, y_1) \in (a, x_0) \times (a, b) \)
such that $N(x_1, y_1) = x_0$. If there were no such pair, then we would have $N(x, (a, b)) \subseteq (a, x_0)$ for each $x \in (a, x_0)$. We infer that for every $\hat{x} \in (x_0, b)$

$$x_0 = N(x_0, x_0) < N(x_0, \hat{x}) = \lim_{x \to x_0^-} N(x, \hat{x}) \leq x_0,$$

which is impossible. Let $(x_1, y_1) \in (a, x_0) \times (a, b)$ be a pair for which $N(x_1, y_1) = x_0$. For every $x \in (x_1, x_0)$ we get

$$N(x, x_0) < N(x_0, x_0) = x_0, \quad N(x, y_1) > N(x_1, y_1) = x_0.$$

It means that the interval $N(x, (a, b))$ contains a point greater than $x_0$ and a point smaller than $x_0$. Thus, there exists $y \in (a, b)$ such that $N(x, y) = x_0$.

Now let $x \in (x_1, x_0)$. We may find a $y \in (a, b)$ for which $N(x, y) = x_0$. Use Eq. (2.1) for the pair $(x, y)$ to get

$$h(x_0) = h(N(x, y)) = M(f(x), f(y)) \leq M(f(x), S).$$

Since $h(x_0) \in I$ and $M^S$ is a bijection from $I$ onto $I$, there exists exactly one $s \in I$ such that $M^S(s) = h(x_0)$. Observe that $s \leq f(x)$ for every $x \in (x_1, x_0)$. Thus, $f$ and $g = f$ satisfy the assumptions of Theorem 1.

The proof when $f(I_0) \subseteq (s, \infty)$ follows in a similar fashion. \qed

On imposing stronger assumptions upon $N$ we may eliminate the assumption of non-constancy of $f$ on every interval.

**Theorem 2.** Let $M, N: I^2 \to I$ be mappings such that for all $u, v \in I$ the mapping $M_u$ is strictly increasing, the mapping $M^v$ is strictly increasing and continuous and the functions $N_u, N^v$ are continuous. Moreover, let $N$ be a symmetric strict mean. Assume that a triple $(f, g, h): I \to I^3$ is a solution of (1.1) and that there exist a subinterval $I_0 \subseteq I$ and $s, S \in I$ such that $f(I_0) \subseteq [s, S]$. Then $g$ is continuous.

**Proof.** Due to Theorem 1 it is enough to consider the case when $f$ is constant on some non-degenerate subinterval $P \subseteq I$. We prove that in this case $g$ is constant on the whole domain.

Let $c := f|_P$; we have

$$M(c, g(x)) = M(f(y), g(x)) = h(N(y, x)) = h(N(x, y))$$

$$= M(f(x), g(y)) = M(c, g(y)), \quad x, y \in P.$$ 

Since the function $M_c$ is strictly monotone, $g$ is constant on $P$, say $g|_P \equiv c_1$ for some $c_1 \in I$. Thus, for all $x, y \in P$ we get

$$h(N(x, y)) = M(f(x), g(y)) = M(c, c_1),$$

which means that $h|_{N(P \times P)}$ is constant and equal to $c_2 := M(c, c_1)$. On the other hand, $N$ is a mean, thus $N(P \times P) = P$. Thus, we have shown that $h|_P \equiv c_2$. 


Now we will verify that $f$ is constant on $\text{int}I$. Assume that $P$ is the maximal interval on which $f$ is constant and equals $c$ (if $Q$ is any interval such that $P \subseteq Q$ and $f|_Q \equiv c$ then $P = Q$); we will show that $\text{int}P = \text{int}I$. Let $\alpha, \beta \in \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ be chosen in such a way that $\text{int}I = (\alpha, \beta)$. Suppose that $\text{int}P \neq \text{int}I$. Then the left end-point of $P$ is a real number $a > \alpha$ or the right end-point of $P$ is a real number $b < \beta$. Without loss of generality we may assume that $b < \beta$. Choose $d$ so that $(d, b) \subseteq P$. We have $(b, b + \frac{1}{n}) \subseteq I$ for sufficiently large $n \in \mathbb{N}$. For such $n$ we may choose $y_n \in (b, b + \frac{1}{n})$ such that $f(y_n) \neq c$. The mean $N$ is strict, so $d < N(d, b) < b$. Moreover, we have $N(d, u) \in (d, b)$ for all points $u$ taken from some neighbourhood of $b$ because the function $N_d$ is continuous. Thus, $N(d, y_k) \in (d, b)$ for some $k \in \mathbb{N}$. On the other hand,

$$M(f(y_k), c_1) = M(f(y_k), g(d)) = h(N(y_k, d)) = c_2.$$ 

Thus, $f(y_k) = c$, because $M(c, c_1) = c_2$ and the function $M^{c_1}$ is strictly increasing. Thus, we get a contradiction.

Obviously, $g|_{\text{int}I} \equiv c_1$ and $h|_{\text{int}I} \equiv c_2$.

To complete the proof it remains to show that if any of the end-points of $I$ belongs to $I$, then in this edge $g$ is also equal to $c_1$. For example, if $\beta \in I$, then from Eq. (2.1) applied to $x \in (\alpha, \beta)$ and $y = \beta$ we get 

$$c_2 = h(N(x, \beta)) = M(f(x), g(\beta)) = M(c, g(\beta)),$$

thus, $g(\beta) = c_1$. If $\alpha \in I$, we proceed in a similar way. \hfill \Box

In what follows we apply the just proved theorems to $M$-affine functions. From results of Ng [1] it follows that an $M$-affine function defined on $I$ and bounded from both sides has to be continuous. From the previous results it follows that two-sided boundedness of $f$ may be weakened to one-side boundedness.

**Corollary 2.** If $M$ is a symmetric, strict mean such that the functions $M_u, M^v$ are continuous and strictly increasing for every $u, v \in I$ and $f$ is an $M$-affine function, then the local boundedness of $f$ from one side by an element from $I$ implies its continuity.

In particular, setting the logarithmic mean for $M$ we obtain a slightly more general result than those of Matkowski [2, Lemma 2].

**Corollary 3.** Assume that $I$ is contained in $(0, +\infty)$ and $L$ is the logarithmic mean. If $f : I \to I$ is $L$-affine, bounded at a point from one side by an element from $I$ then $f$ is continuous.

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