PROPOSAL FOR REALIZATION OF A TOFFOLI GATE VIA CAVITY-ASSISTED ATOMIC COLLISION

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Abstract

Cavity QED is a versatile tool to explore small scale quantum information processing. Within this setting, we describe a particular protocol for implementing a Toffoli gate with Rydberg atoms and a cavity field. Our scheme uses both resonant and non resonant interactions, and in particular a cavity assisted atomic collision. The experimental feasibility of the protocol is carefully analyzed with the help of numerical simulations and takes into account the decoherence process. Moreover, we show that our protocol is optimal within the constraints imposed by the experimental setting.

1 Introduction

The recent development of quantum information processing has shed new light on complexity and communication theory. It is now widely accepted that some problems are solved more efficiently by quantum computers than by their classical analogues [1, 2]. This matter of fact has triggered in the past years a lot of studies on theoretical and practical aspects of quantum computation. In particular, finding universal sets of gates for processing quantum information is of paramount importance: the possibility of implementing universal gates is a key requirement for building interesting quantum information processing devices. A lot of efforts have been dedicated to this question, leading to major theoretical results (for a review see [3]).

One of the next challenges that need to be overcome is to design robust implementations of these sets of gates. Several constructions have already been proposed and
realized for various experimental settings [3,5,8]. These results currently serve as benchmarks for other implementations and do provide deep insights into the ability of the chosen devices to manipulate quantum systems efficiently. In this article, we describe a protocol for realizing the Toffoli gate — a three qubits “control-control-NOT” — in the Cavity QED context (CQED). This gate, together with one qubit rotations, form a universal set for quantum computing [1]. We will see that the combination of techniques used makes our protocol optimal in term of number of interactions within the constraints of our experimental setting. It is thus less resources demanding than the standard implementation as a sequence of control-NOT gates [9]. Finally we estimate the performance of our protocol by taking into account imprecisions as well as decoherence effects during the protocol.

2 Cavity-QED toolbox

In this paragraph we review briefly the different techniques used for implementing the Toffoli gate. Further experimental details concerning our particular setting are exposed with great care in [10]. In our protocol, quantum information will be stored in circular Rydberg states of highly excited Rubidium atoms and in the cavity field. The very long lifetime of these atoms together with the transition frequencies of the chosen transitions allow to deal with those complex atoms as if they had only three levels noted $|g\rangle$, $|e\rangle$ and $|i\rangle$. Transformations between those three states can be driven coherently by a classical microwave pulse adjusted nearly resonantly to the proper transition frequency. For example, if the classical field is tuned with respect to $|i\rangle \leftrightarrow |g\rangle$, a $\pi$ pulse will transform any pure quantum state of the form $\alpha |e\rangle + \beta |g\rangle$ into $\alpha |e\rangle + \beta |i\rangle$. Indeed, those transformations are the analogues of the well known one qubit gates for our three level systems. In order to process quantum information in a non-trivial way, we also need an equivalent for the two-qubit gates. Within the CQED context, this involves a coupling between the atoms and a high-$Q$ Niobium superconducting microwave cavity. The frequency of the mode inside the cavity can be adjusted in and out of resonance with the $|g\rangle \leftrightarrow |e\rangle$ transition. Thus, we can consider two distinct approaches for processing quantum information. The first one, which has been throughly used in recent experiments [11], relies on a resonant interaction with a single mode of the cavity. As an example, for a well chosen interaction time, an initial atom-field state of the form $(\alpha |g\rangle + \beta |e\rangle) |0\rangle$ will evolve into $|g\rangle (\alpha |0\rangle + \beta |1\rangle)$. This transformation is called $\pi$-Rabi rotation. Similarly, continuing the interaction for an equal amount of time leads to the state $(\alpha |g\rangle - \beta |e\rangle) |0\rangle$. In the perspective of information processing it corresponds to transferring the information from the atom into the cavity and back to the atom. Hence, the field inside the cavity acts as a temporary quantum memory. The second approach for building atom field interactions, referred to as cavity assisted atomic collision, has been proposed [12,13] and experimentally tested [14] more recently. In this setting, two atoms enter the cavity at the same time and follow an evolution conditioned upon the state of the cavity field. More precisely, the cavity is detuned from the $|e\rangle \leftrightarrow |g\rangle$ transition. In this regime, where the detuning $\delta$ is much larger than the atom-field coupling constant $\Omega$, the effective Hamiltonian can be derived using second order perturbation theory:

$$H_c = \lambda \left( |e_1\rangle \langle e_1| a a^+ - |g_1\rangle \langle g_1| a^+ a + |e_2\rangle \langle e_2| c a a^+ - |g_2\rangle \langle g_2| a^+ a + |e_1\rangle \langle g_1| \otimes |g_2\rangle \langle e_2| + |g_1\rangle \langle e_1| \otimes |e_2\rangle \langle g_2| \right),$$

where $\lambda = \Omega^2/4\delta$, and where the operators $a$ and $a^+$ are the annihilation and creation operators for the cavity field. In the above formula, we see that any exchange of energy
between the field and the atoms present in the cavity is forbidden, but this still allows conditional dynamics upon the photon number in the cavity mode. This interaction will be our main tool to perform the Toffoli gate in the CQED setting.

3 Description of the protocol

Before going into the details of the protocol, recall that the Toffoli gate is a three qubits gate. Its action on the computational basis of the three qubits is to perform a “control-control-NOT”. Thus, only two basis vectors are affected by this evolution:

\[
\begin{align*}
|1\rangle_1 |1\rangle_2 |0\rangle_3 &\rightarrow |1\rangle_1 |1\rangle_2 |1\rangle_3, \\
|1\rangle_1 |1\rangle_2 |1\rangle_3 &\rightarrow |1\rangle_1 |1\rangle_2 |0\rangle_3,
\end{align*}
\]

where the control qubits are the first two ones, and the target is the last one. Taking into account the specific CQED setting, we chose an implementation with one cavity mode and two atoms. The photon number states of the cavity mode will be denoted \(|n_c\rangle\), while the energy levels of the atoms will be written as \(|i_{c,t}\rangle, |g_{c,t}\rangle, |e_{c,t}\rangle\) (the subscripts \(c\) and \(t\) are short hands for control and target). The relation between those states and the computational basis of the Toffoli gate is summarized below (obvious normalization factors have been omitted):

\[
\begin{array}{c|c|c|c}
\text{Computational} & \text{Control 1} & \text{Control 2} & \text{Target} \\
\text{basis} & \text{(cavity mode)} & \text{(Rb atom)} & \text{(Rb atom)} \\
\hline
|0\rangle & |1_c\rangle & |i_c\rangle & |g_t\rangle + |e_t\rangle \\
|1\rangle & |0_c\rangle & |g_c\rangle & |g_t\rangle - |e_t\rangle
\end{array}
\]

For sake of simplicity we will concentrate on the realization of the gate assuming that the preparation of the cavity has been already performed.

Since all our manipulations are done coherently, it is sufficient to describe the quantum evolution for basis vectors of the whole system (ie, our three qubits). The protocol can be decomposed into three distinct phases: encoding, atomic collision, and decoding. The encoding and decoding operations only involve resonant interactions between the cavity mode (control 1) and a single Rb atom (control 2). Decoding is performed by applying the encoding evolutions in reverse order. The heart of the evolution is the cavity assisted atomic collision which realizes the equivalent of the Toffoli gate on the encoded quantum information. More precisely, it realizes a control phase gate between the encoded information of the two control qubits and the target qubit. Thus, the first atom (control 2) is sent at a relatively low speed into the cavity. After the encoding, it collides with the faster moving second atom (target). When the second atom has left the cavity, the first one interacts with the cavity mode to accomplish the decoding step.

The encoding starts when the atom identified as Control 2 (\(A_c\)) is sent into the cavity. First, it interacts resonantly with the cavity field and undergoes a \(\pi\)-Rabi rotation. In terms of the basis vectors given in Eq. (3), only one of them is affected by the evolution:

\[
|1_c\rangle |g_c\rangle \rightarrow - |0_c\rangle |e_c\rangle.
\]

At this point, for a generic input state of the Toffoli gate, all three levels of the control atom can be populated: the quantum information which was initially stored into two separated qubits is now spread over one qubit (the cavity field) and one qutrit (the Rb atom). The state obtained in Eq. (4) almost corresponds to the needed preparation of the cavity and control atom before the atomic collision: a microwave pulse tuned to the \(|g_c\rangle \leftrightarrow |i_c\rangle\) transition, and corresponding to a basis rotation exchanging \(|i_c\rangle\) and \(|g_c\rangle\), completes this first step of the protocol.
The main part of the protocol can now be achieved: the cavity assisted atomic collision. Thus, the cavity is set far from resonance with the $|e\rangle \leftrightarrow |g\rangle$ transition, such that the interaction Hamiltonian is given by Eq. (3). The target atom ($A_t$) then enters the cavity and interacts with the field and the control atom. The evolution of the basis states is easily computed by diagonalizing the effective Hamiltonian. After an interaction time $t_{\text{col}} = \pi/\lambda$, all states remain unaffected except the following ones:

$$
|0_c\rangle |i_e\rangle (|g_c\rangle + |e_c\rangle) \rightarrow |0_c\rangle |i_e\rangle (|g_c\rangle - |e_c\rangle)
$$

$$
|0_c\rangle |i_e\rangle (|g_c\rangle - |e_c\rangle) \rightarrow |0_c\rangle |i_e\rangle (|g_c\rangle + |e_c\rangle).
$$

(5)

All operations done before the atomic collision can be considered as the preparation of the quantum systems such that they undergo the proper overall evolution described by the Hamiltonian of Eq. (5). Hence, to complete the whole protocol, we only need to undo all the preparatory steps: after $A_t$ left the cavity (recall $A_i$ is the fast moving atom), we apply the resonant pulse on $A_c$ exchanging $|i_e\rangle$ and $|g_c\rangle$, followed by a $\pi$-Rabi rotation to extricate from $A_c$ the information initially contained in the cavity.

This completes the overall protocol and leads to:

$$
|1_c\rangle |i_e\rangle (|g_c\rangle + |e_c\rangle) \rightarrow |1_c\rangle |i_e\rangle (|g_c\rangle - |e_c\rangle)
$$

$$
|1_c\rangle |i_e\rangle (|g_c\rangle - |e_c\rangle) \rightarrow |1_c\rangle |i_e\rangle (|g_c\rangle + |e_c\rangle)
$$

$$
|0_c\rangle |i_e\rangle (|g_c\rangle + |e_c\rangle) \rightarrow |0_c\rangle |i_e\rangle (|g_c\rangle - |e_c\rangle)
$$

$$
|0_c\rangle |i_e\rangle (|g_c\rangle - |e_c\rangle) \rightarrow |0_c\rangle |i_e\rangle (|g_c\rangle + |e_c\rangle).
$$

(6)

which, in turn, exactly corresponds to the Toffoli gate in the computational basis of Eq. (5). The pulses and interactions sequence are summarized in Fig. 1.
4 Discussion

The scheme presented here proposes to use the atomic collision as the key interaction in the making of the Toffoli gate. The most fundamental reason that lead to this choice is that using only resonant interactions would not allow us perform the gate in this very specific one mode CQED setting: it has been shown [5], that resonant interactions can be used to design CNOT gates and hence lead to universal quantum computation. However, the implementations of the Toffoli gate using CNOT gates together with one qubit gates presented in [9] would require to address the qubits separately between each CNOT gate. Our particular setting forbids such addressing to take place inside the cavity. Hence, to implement the circuits of [9], we would need to have more than one cavity. The counterpart of using the atomic collision is have to encode and decode the quantum information of the control qubits. In this proposal, each of these steps is accomplished by a single interaction involving two quantum systems, and an interaction involving one quantum system and one classical system. It is easily shown that this cannot be made simpler, and hence that our proposal is optimal within the restrictions imposed by the experimental setting.

Let us now discuss the practical feasibility of this proposal. The cavity-field coupling constant is $\Omega/2\pi = 50\,\text{kHz}$ [10]. For the atomic collision, we need to detune the cavity from the $|e\rangle \leftrightarrow |g\rangle$ transition by an amount $\delta \gg \Omega$. This can be done by applying an external electric field to the mirrors. Here we choose $\delta/2\pi = 4\Omega$, which gives an interaction Hamiltonian well approximated by Eq (2) with $\lambda = \Omega^2/4\delta$. With these figures, the interaction time to realize a $\pi$-Rabi rotation is $2 \times 10^{-5}$s and for the atomic collision $1.25 \times 10^{-4}$s. The time required to perform the interaction with the classical microwave cavity is negligible. This imposes a velocity of the atoms of order $50\,\text{m.s}^{-1}$. This value can be reached by means of simple atomic beam techniques with transverse laser cooling. The total interaction time with the cavity field is of order $0.18\,\text{ms}$, still much smaller than the cavity lifetime (1ms). Decoherence effects due to the loss of a photon should then be relatively small. We present in Fig.2 the result of a numerical simulation to estimate the fidelity of the gate. Dissipation effects during the atomic collision have been accounted for using quantum jumps. The fidelity is plotted for various strength of imperfections in the classical pulses. For a strength of $3\%$ — ie, for the achievable precision in the current setting — and for cavity lifetime of 1ms, the fidelity of the gate is 70%. Thus, the practical realization of the Toffoli gate through this protocol is not out of reach and could set an interesting benchmark for comparing the efficiency of quantum information processing approaches using CQED. Moreover, we can see that even small improvements on either the precision of the pulses or the cavity lifetime result in achieving a fidelity of nearly 90%, thus making this realization of the gate very attractive for analyzing its experimental behavior.

We now return to the preparation of the cavity field, before the above protocol takes place. This can be done in full generality by sending a Rydberg atom ($A_p$) containing the desired quantum information, and by transferring its state to the cavity through a $\pi$-Rabi rotation. The protocol is then started when this ancillary atom leaves the cavity. The retrieval is accomplished by transferring the state of the cavity back into an atom ($A_c$) initially in the $|g\rangle$ state. Thus, the result of the Toffoli gate is contained in the state of the atoms $A_c$, $A_t$ and $A_r$. This state, and hence the behavior of the gate, can be easily analyzed by using standard quantum tomography techniques: this requires only requires single atom rotations and accumulation of statistics.
5 Conclusion

We have presented a realistic scheme for implementing the Toffoli gate with currently available CQED techniques. We have shown that using a cavity assisted collision instead of only resonant interactions makes our scheme optimal given the restrictions of the experimental setting. It thus enlarges the range of possible applications of CQED to process information for realizing basic logical operations [5, 11, 14] as well as more complicated protocols [15, 16, 17, 18]. The estimated achievable fidelity (around 70% for current imprecision and decoherence levels) ensures that the behavior of the gate can be tested experimentally. The corresponding experimental results could be compared to analogous quantities for other sets of universal gates and thus provide a deeper insight into the quantum information processing capability of the CQED setting with non resonant interactions.

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