The wave equation on axisymmetric stationary black hole backgrounds

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Abstract. Understanding the behaviour of linear waves on black hole backgrounds is a central problem in general relativity, intimately connected with the nonlinear stability of the black hole spacetimes themselves as solutions to the Einstein equations—a major open question in the subject. Nonetheless, it is only very recently that even the most basic boundedness and quantitative decay properties of linear waves have been proven in a suitably general class of black hole exterior spacetimes. This talk will review our current mathematical understanding of waves on black hole backgrounds, beginning with the classical boundedness theorem of Kay and Wald on exactly Schwarzschild exteriors and ending with very recent boundedness and decay theorems (proven in collaboration with Igor Rodnianski) on a wider class of spacetimes. This class of spacetimes includes in particular slowly rotating Kerr spacetimes, but in the case of the boundedness theorem is in fact much larger, encompassing general axisymmetric stationary spacetimes whose geometry is sufficiently close to Schwarzschild and whose Killing fields span the null generator of the horizon.

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1. THE PROBLEM

Let \((\mathcal{M}, g)\) be a black hole spacetime, for instance Schwarzschild or Kerr\(^1\), but more generally, a spacetime whose geometry is “near” one of the above. We will understand the meaning of “near” further down, so the reader may for now wish to fix \((\mathcal{M}, g)\) as precisely Schwarzschild or Kerr.

Let \(\Sigma\) denote an arbitrary Cauchy surface\(^2\) for \((\mathcal{M}, g)\). It is known that for suitably regular initial data \(\Psi, \Psi'\) prescribed on \(\Sigma\) for the wave equation

\[\Box_g \psi = 0,\]

there exists a unique solution \(\psi\) defined globally on \(\mathcal{M}\).

The problem of interest here is:

**Problem.** Understand the quantitative boundedness and decay properties of \(\psi\) in the closure \(\mathcal{D}\) of the domain of outer communications of \((\mathcal{M}, g)\).

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\(^1\) We refer the reader to standard texts \([17, 27]\) for a discussion of these spacetimes.

\(^2\) For the purpose of this talk, all spacetimes are globally hyperbolic. In particular, by the term “Kerr spacetime” we mean the globally hyperbolic subset of maximally extended Kerr consisting of the development of a partial Cauchy hypersurface with two asymptotically flat ends.
Below is a Penrose diagram indicating the region of interest in the case of Schwarzschild.

We restrict in fact to

\[ \mathcal{D} = \overline{\mathcal{I}^+(\mathcal{I}_A^-)} \cap \mathcal{J}^-(\mathcal{I}_A^+) \]

where the closure refers to the topology of \( \mathcal{M} \), and where \( \mathcal{I}_A^\pm \) denote a pair of connected components of \( \mathcal{I}^\pm \), respectively, with a common limit point. We are thus interested in understanding the behaviour of \( \psi \) up to and including the event horizon \( \mathcal{H} = \{ r = 2M \} \).

“Quantitative” in the statement of our problem means we want to estimate the size of \( \psi \) in \( \mathcal{D} \) in terms of quantities depending only on a suitable norm of the data \( \Psi, \Psi' \) on \( \Sigma \).

The above problem is one of the most basic questions to pose about black hole spacetimes, and is in fact intimately related to the non-linear stability problem of the Kerr family as a family of solutions of the Einstein equations (see Section 5). Not surprisingly then, the problem has been the object of much study in general relativity, beginning with the work of Regge and Wheeler [25]. Nonetheless, until the past 5 years, the only result known for general solutions of the Cauchy problem—i.e. solutions not restricted by symmetry assumptions or support assumptions—was the uniform boundedness of \( \psi \) in \( \mathcal{D} \), in the very special case of Schwarzschild. This celebrated theorem of Kay and Wald is reviewed in Section 2.1.

The rest of the talk will then review the recent progress in this area, which has now allowed for a satisfactory answer to our motivating problem, not only for Schwarzschild itself, but for spacetimes \((\mathcal{M}, g)\) suitably “near” Schwarzschild, including the important Kerr and Kerr-Newman families (for small parameters \( a, Q \)). The main elements central to our understanding of the problem can be summarised by the following:

1. A new, more robust proof of Kay and Wald’s theorem making use of the red-shift effect, ensuring good control at the horizon. (See Sections 2.4–2.7). The proof turns out to be stable to a large class of perturbations of the Schwarzschild metric, not however to Kerr! (See Section 2.8.)

2. A proof of quantitative decay bounds for solutions of (11) on Schwarzschild. The main difficulties are (i) understanding and quantifying the phenomenon of trapping associated with the photon sphere, (ii) finding the analogue for Schwarzschild of the conformal energy current used to prove energy decay in Minkowski space and

\[^3\text{Without loss of generality, we can restrict to this set as opposed to } J^+(\mathcal{I}^-) \cap \mathcal{J}^-(\mathcal{I}^+). \text{ Astrophysical black holes, of course, have only one asymptotically flat end.}\]
(iii) relating this to the red-shift effect, recovering decay near the horizon. (See Section 3.)

3. The discovery that superradiant frequencies are not trapped for axisymmetric stationary spacetimes sufficiently near Schwarzschild, allowing for a boundedness theorem for all such spacetimes without a detailed understanding of trapping. This class of spacetimes includes Kerr and Kerr-Newman for $a \ll M, Q \ll M$, but is in fact much more general. (See Section 4.1.)

4. Quantifying the trapping phenomenon on Kerr itself by frequency-localised versions of the virial identities used in Schwarzschild. In view of the robustness of the other aspects of the decay proof on Schwarzschild, this yields a proof of decay for solutions to (1) on Kerr for $a \ll M$. (See Section 4.2.)

The above serves also as an outline for the bulk of the talk. Let us emphasize that the results outlined here do not close the book on this subject. What is the situation for higher spin? What are the least amount of assumptions on the geometry which yield quantitative decay? What happens when the condition $a \ll M$ is relaxed? What is the relation with the non-linear stability of the background solutions themselves? We end with remarks about future directions in Section 5.

2. UNIFORM BOUNDEDNESS ON SCHWARZSCHILD

2.1. The Kay–Wald theorem

The first definitive theorem in the direction of our motivating problem is the following celebrated uniform boundedness result for solutions of (1) on Schwarzschild exteriors.

**Theorem.** (Kay–Wald [12], 1987) Let $(\mathcal{M}, g_M)$ be Schwarzschild with parameter $M > 0$, $\mathcal{D}$ as above the closure of its domain of outer communications, $\Sigma$ a Cauchy surface for $\mathcal{M}$ and $\psi$ the unique solution of the wave equation (1) on $\mathcal{M}$ with sufficiently regular initial data $\Psi, \Psi'$ on $\Sigma$, decaying appropriately near spatial infinity $i^0$. Then there exists a $D$ depending only on the data such that

$$|\psi| \leq D$$

holds in $\mathcal{D}$.

Before turning to the main conceptual difficulty of the proof of the above theorem, let us make some general remarks. The proofs of all theorems to be discussed in this talk use “energy type estimates” to control square integral quantities of $\psi$ and its derivatives; pointwise bounds are retrieved at the last stage from these energy integrals and a Sobolev inequality.5

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4 Defined naturally relative to the geometric symmetries and hidden symmetries of Kerr
5 The centrality of energy bounds in the study of the wave equation arises from the fact that estimates of square integral quantities are the only estimates for solutions $\psi$ of (1) (in more than one spatial dimension) which do not lose derivatives.
2.2. Energy currents and vector fields

Energy estimates for (1) have a very geometric origin which is intimately related to its Lagrangian structure. Let us briefly explain.

2.2.1. Energy currents constructed from vector field multipliers

Associated to the Lagrangian for (1) is the so called energy-momentum tensor

\[ T_{\mu\nu}(\psi) = \partial_\mu \psi \partial_\nu \psi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \psi \partial_\beta \psi. \]

For a solution \( \psi \) to (1), \( T_{\mu\nu} \) is divergence-free, i.e.

\[ \nabla^\mu T_{\mu\nu} = 0. \] (2)

Given any vector field \( V \), we may associate to it two currents:

\[ J^V_\mu(\psi) = T_{\mu\nu}(\psi) V^\nu, \quad K^V(\psi) = 2\pi^\mu_{\nu} V T_{\mu\nu}(\psi), \]

where \( \pi^\mu_{\nu} = \frac{1}{2} V^\mu;^\nu \) is the so called deformation tensor of \( V \). The relation (2) yields

\[ K^V(\psi) = \nabla^\mu J^V_\mu(\psi) \]

for solutions \( \psi \) of (1). Thus, by the divergence theorem If \( \Sigma_1 \) and \( \Sigma_2 \) are homologous hypersurfaces bounding a spacetime region \( \mathcal{B} \), we have

\[ \int_{\Sigma_2} J^V_\mu(\psi) n^\mu_{\Sigma_2} + \int_{\mathcal{B}} K^V(\psi) = \int_{\Sigma_1} J^V_\mu(\psi) n^\mu_{\Sigma_1}. \] (3)

When \( V \) is timelike and \( \Sigma_i \) is spacelike, then \( J^V_\mu(\psi) n^\mu_{\Sigma_2} \geq 0 \), and in fact controls the spacetime gradient of \( \psi \). If \( V \) is in addition Killing, then \( K^V = 0 \), and (3) would provide an estimate for the solution on \( \Sigma_2 \) from knowledge of the solution on \( \Sigma_1 \) ("data"). Even when \( V \) is not Killing, \( K^V \) can sometimes be treated as an error term.

One can also turn the identity (3) on its head, and think about it as a way to estimate

\[ \int_{\mathcal{B}} K^V \]

from the boundary terms. (Think about the classical virial theorem . . . ) This is particularly useful when the boundary terms are controlled by a controlled energy say, and when \( K^V(\psi) \geq 0 \) and controls derivatives of \( \psi \).

Both uses of (3) will arise in what follows.

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6 For instance, the well-posedness for (1) can be proven by using (3) for an arbitrary timelike \( V \).
### 2.2.2. Vector fields as commutators and higher order currents

In order to obtain pointwise bounds via energy control, one must consider “higher order energies”.

Let us first consider the case where $W$ is a vector field in the Lie algebra of isometries of $g$. Then if $\psi$ satisfies (1), then so does $W\psi$. More generally, if $W_1, \ldots, W_k$ are in the Lie algebra, and $\psi$ satisfies (1), then so does $W_1 \cdots W_k \psi$. Given a multiplier vector field $V$, we may thus consider the $k + 1$’th order currents $J^V_{\mu}(W_1 \cdots W_k \psi)$, and $K^V(W_1 \cdots W_k \psi)$.

Again, $\nabla^\mu J^V_{\mu} = K^V$, and (3) allows for proving higher-order energy estimates.

If $W_1, \ldots, W_k$ are not in the Lie algebra, one obtains an identity

\[ \nabla^\mu J^V_{\mu}(W_1 \cdots W_k \psi) = K^V(W_1 \cdots W_k \psi) + V^\mu \partial_\mu(W_1 \cdots W_k \psi)F^{W_1 \cdots W_k} \]

where $K^V, J^V_{\mu}$ are defined as before, and $F^{W_1 \cdots W_k}$ is a current of order less than or equal to $k + 1$. The above identity upon integration again allows for estimation of the higher order energy $J^V_{\mu}(W_1 \cdots W_k \psi)$. For a fundamental application of considering Lorentz boosts as commutators for proving decay for solutions of (1) on Minkowski space, see [20].

For a more general discussion of the origin of these identities for general Lagrangian theories and their relation to hyperbolicity, see the beautiful discussion in [7].

### 2.3. The Kay–Wald proof

Let us turn now to the proof of the Kay–Wald theorem, so as to see the main difficulty. In Kay and Wald’s proof, the only vector field used as a multiplier is

\[ T = \frac{\partial}{\partial t}, \]

where $t$ is a Schwarzschild coordinate (in which the metric takes the Schwarzschild form $-(1-2M/r)dt^2 + (1-2M/r)^{-1}dr^2 + r^2d\gamma^2$ in the interior of $\mathcal{D}$). Recall that the vector field $T$ extends to a Killing field on all $\mathcal{M}$, is timelike in the interior of $\mathcal{D}$, and null on its boundary $\mathcal{H}^+ \cup \mathcal{H}^-$, vanishing on the sphere of bifurcation $\mathcal{H}^+ \cap \mathcal{H}^-$. 

Without loss of generality, we may assume that our Cauchy surface $\Sigma$ intersected with $\mathcal{D}$ is as depicted below by $\Sigma_0$:

We may define a regular coordinate system $(r, t^*)$ in

\[ \mathcal{R} \cong \mathcal{D} \cap J^+(\Sigma_0) \]
such that \( \Sigma_0 \) corresponds to \( t^* = 0 \) and \( T \) still corresponds to \( \partial_{r^*} \). We may define then \( \Sigma_\tau = \{ t^* = \tau \} \). Note that \( \Sigma_\tau = \varphi_\tau(\Sigma_0) \), where \( \varphi_\tau \) denotes the one-parameter group of diffeomorphisms generated by \( T \).

Let us apply the energy identity of \( J^T \) in the region bounded by \( \Sigma_0 \), \( \Sigma_\tau \), and the corresponding piece of \( \mathcal{H} \). On \( \Sigma_\tau \), one has

\[
J^T_\mu(\psi)n^\mu_{\Sigma_\tau} \approx \left( (1 - 2M/r)(\partial_r \psi)^2 + (\partial_r \psi)^2 + |\nabla \psi|^2 \right)
\]

where \( | \cdot | \), \( \nabla \) denote here the induced norm and connection in the \( \text{SO}(3) \) group orbits. Note the degeneration of the \( \partial_r \) derivative at \( \Sigma_\tau \cap \mathcal{H} \). This arises because \( T \) becomes null on \( \mathcal{H} \).

Since the flux through the horizon is nonnegative

\[
J^T_\mu(\psi)n^\mu_{\mathcal{H}^+} \geq 0
\]

we have

\[
\int_{\Sigma_\tau} \left( (1 - 2M/r)(\partial_r \psi)^2 + (\partial_r \psi)^2 + |\nabla \psi|^2 \right) \leq B \int_{\Sigma_0} \left( (1 - 2M/r)(\partial_r \psi)^2 + (\partial_r \psi)^2 + |\nabla \psi|^2 \right).
\]

(5)

Commuting with \( T \), i.e. considering the current \( J^T(T\psi) \) and \( J^T(TT\psi) \), one obtains (5) with \( \psi \) replaced by \( T\psi \) and \( TT\psi \). An elliptic estimate and the Sobolev inequality suffices to show that if \( \lim_{x \to r} \Psi = 0 \), then

\[
(1 - 2M/r)^2 \leq \int_{\Sigma_0} \left( J^T_\mu(\psi) + J^T_\mu(T\psi) + J^T_\mu(TT\psi) \right) n^\mu_{\Sigma_0}
\]

in \( \mathcal{R} \).

The above argument is in fact completely standard and yields the statement of the theorem but where \( \psi \) is replaced by \( \sqrt{1 - 2M/r} \psi \). Thus, it provides no information about the behaviour of \( \psi \) along \( \mathcal{H} \) where \( r = 2M! \) Understanding the behaviour up to and including the horizon is thus the only real difficulty of this problem in the Schwarzschild case.

This difficulty is overcome by Kay and Wald by the following argument: One first notices that if there exists a \( \hat{\psi} \) satisfying (1) such that \( T\hat{\psi} = \psi \), then one can estimate \( \psi \) by a suitable Sobolev inequality from the energies of \( \hat{\psi} \) on \( \{ t = c \} \), providing one also commute with all angular momentum operators \( \Omega_\mu \). Such a \( \hat{\psi} \) can be constructed if it is assumed that \( \psi \) is not supported in a neighborhood of \( \mathcal{H}^+ \cap \mathcal{H}^- \), by inverting an elliptic operator on \( \Sigma_0 \). (This was in fact an earlier observation of Wald.) More generally, \( \hat{\psi} \) can be constructed if \( \psi \) decays suitably to 0 at \( \mathcal{H}^+ \cap \mathcal{H}^- \). But what to do in the general case where \( \psi \) is not assumed to have special behaviour at \( \mathcal{H}^+ \cap \mathcal{H}^- \)?

Here comes the second clever observation: Since one is only interested in the behaviour in \( \mathcal{R} \), one can replace \( \psi \) by a solution \( \hat{\psi} \) of (1) such that \( \psi = \hat{\psi} \) in \( \mathcal{R} \). Using the domain of dependence property for the wave equation, the discrete symmetry of (extended) Schwarzschild interchanging the two ends, and a preservation of symmetry
argument, one can construct a $\hat{\psi}$ with the desired behaviour at $\mathcal{H}^+ \cap \mathcal{H}^-$. One can then construct $\tilde{\psi}$ and continue as before. See [19] or [14] for more details.

2.4. A stable proof?

The above proof is remarkable, but fragile! It requires (i) the staticity property to construct $\hat{\psi}$, (ii) the spherical symmetry of Schwarzschild as one must commute (1) with $\Omega_i$, $i = 1, \ldots, 3$, and, finally, even (iii) the discrete symmetry of Schwarzschild. Is it really the case that a result so fundamental as boundedness must depend on all this special structure of Schwarzschild?

The difficulties of the proof arise because the set of multipliers and commutators are restricted to the Killing fields $T_i$ and $\Omega_i$. There is another important physical property of Schwarzschild which is not apparent from these alone: We discuss this in the next section.

2.5. The redshift effect

The redshift effect is one of the most celebrated aspects of black holes. It is classically described as follows: Suppose two observers, $A$ and $B$ are such that $A$ crosses the event horizon and $B$ does not. If $A$ emits a signal at constant frequency as he measures it, then the frequency at which it is received by $B$ is “shifted to the red”.

The consequences of this for the appearance of a collapsing star to far-away observers were first explored in the seminal paper of Oppenheimer-Snyder [23].

The red-shift effect as described above is a global one, and essentially depends only on the fact that the proper time of $B$ is infinite whereas the proper time of $A$ before crossing $\mathcal{H}^+$ is finite. In the case of the Schwarzschild black hole, there is a “local” version of this red-shift: If $B$ also crosses the event horizon but at advanced time later than $A$: 
then the frequency at which $B$ receives at his horizon crossing time is shifted to the red by a factor depending exponentially on the advanced time difference of the crossing points of $A$ and $B$.

The exponential factor is determined by the so-called surface gravity, a quantity that can in fact be defined for all so-called Killing horizons. This localised red-shift effect depends only on the positivity of this quantity.

2.6. The redshift as seen by vector fields

It turns out that the local red-shift effect can be captured by positivity properties in the energy identity of a suitably constructed vector field multiplier applied both to $\psi$ alone and to $\psi$ commuted with a suitably constructed vector field commutator.

**Proposition.** [10, 14] There exists a smooth vector field $N$, and two positive constants $0 < b < B$ such that $N$ is timelike and $\varphi_t$-invariant such that

$$bJ^N_\mu(\psi)N^\mu \leq K^N(\psi) \leq B J^N_\mu(\psi)N^\mu;$$

along $\mathcal{H}^+$, for all solutions $\psi$ of $\Box_g \psi = 0$.

A vector-field commutator version can be seen by

**Proposition.** [13, 14] Under the assumptions of the above theorem, let $Y = N - T$, and extend $T, Y$ to a null frame $T, Y, E_1, E_2$ on $\mathcal{H}^+$. If $\psi$ satisfies $\Box_g \psi = 0$, then for all $k \geq 1$.

$$\Box_g(Y^k \psi) = b_k Y^k \psi + \sum_{0 \leq |m| \leq k, 0 \leq m_4 < k} c_mE_1^{m_1}E_2^{m_2}T^{m_3}Y^{m_4} \psi$$

on $\mathcal{H}^+$, where $b_k > 0$.

These propositions apply in particular to the Schwarzschild metric, but in fact, their domain of validity is much more general: They apply to any stationary black hole with event horizon with positive surface gravity. See [14].

2.7. A stronger boundedness theorem

The above “positive terms”, $K^N$ and $b_k Y^k \psi$ can be viewed as exponential damping terms in the energy identities with $N$ as a multiplier, and more generally, $N$ as a multiplier applied to $\psi$ commuted with $Y^k$. Of course, these nice properties hold only near the horizon. Thus, to use these identities one must apply these estimates in conjunction with a statement giving good control away from the horizon. In the case of Schwarzschild, this good control follows from the first part of the argument described below, i.e. from application of $T$ to $\psi$, $T \psi$, $TT \psi$. This allows for a proof of the following stronger boundedness statement.

**Theorem.** [14] Let $(\mathcal{M}, g_M)$ be Schwarzschild and $\Sigma_\tau$ as above. Then there exists a constant $C$ depending only on $M, \Sigma_0$ such that for all $\psi$ satisfying $\Box_g \psi = 0$, the following
\begin{align*}
|n_{\tau} \Psi|_{L^2(\Sigma)} + |\nabla_{\Sigma} \psi|_{L^2(\Sigma)} & \leq C \left( |n_{\tau} \Psi|_{L^2(\Sigma)} + |\nabla_{\Sigma} \psi|_{L^2(\Sigma)} \right).
\end{align*}

Moreover, for all \( m \geq 0 \), the \( m \)'th order pointwise bounds
\[
\sum_{0 \leq m_1 + m_2 \leq m} |\nabla_{\Sigma}^{(m_1)} n_{\tau}^{(m_2)} \Psi| \leq C Q_m
\]
hold in \( \mathcal{R} \), where \( Q_m \) is an appropriate norm on initial data.

The above theorem is stronger than the Kay and Wald statement in that it proves the uniform boundedness of an “energy” which does not degenerate in local coordinates on the horizon\(^7\). Moreover, this boundedness is proven for arbitrary higher order energies, leading to pointwise bounds for arbitrary derivatives, including transversal derivatives to the horizon. It is interesting to remark that the Kay–Wald argument cannot prove the uniform boundedness of these transversal derivatives.

### 2.8. Perturbing the metric

The above proof now is much more robust. In fact, it can be perturbed to nearby metrics as long as one retains \( \mathcal{H}^+ \) as a null boundary and \( T \) as Killing and causal:

**Theorem.**\(^{[14]}\) Let \( \mathcal{R}, T \) be as before, and let \( g \) be a metric on \( \mathcal{R} \) sufficiently close to Schwarzschild such that \( T \) is Killing and causal on \( \mathcal{R} \), and \( \mathcal{H}^+ \) is null with respect to \( g \). Then the statement of the previous theorem applies verbatim.

In view of the remarks at the end of Section 2.6, it follows that one may weaken the assumption “\( g \) sufficiently close to Schwarzschild”, replacing it with the assumption that the geometry is that of a black hole with positive surface gravity. This and the remaining assumptions are then in particular satisfied by all the classical static electrovacuum black holes (Reissner-Nordström-de Sitter, etc.) Moreover, one need not assume that \( T \) is Killing, merely that \( \pi_T^{\mu\nu} \) decays appropriately in \( \tau \). See \([14]\) for details.

**What about Kerr?** Unfortunately, for all \( a \neq 0 \), the stationary vector field \( T \) of the Kerr metric \( g_{M,a} \) is no longer causal in the interior of \( \mathcal{D} \) and thus \( g_{M,a} \) does not satisfy the assumptions of the above Theorem! In particular, \([13]\) does not hold and we can thus no longer a priori infer the uniform boundedness of
\[
\int_{\Sigma_\tau} J^T_{\mu} (\psi) n^\mu_{\tau}
\]
from the energy identity of \( J^T \).

The part of \( \mathcal{D} \) where \( T \) is spacelike is known as the ergoregion, and the associated behaviour of waves is known as superradiance. The test-particle manifestation of this fact is the celebrated Penrose process. See \([27]\) for a nice discussion.

\(^7\) That is to say, the energy computed by a \( \varphi_t \)-invariant family of freely falling observers is proven bounded.
The above suggests that it may be difficult to prove boundedness alone, and that of necessity one must try to prove more than boundedness at the same time, i.e. decay.

3. DECAY ON SCHWARZSCHILD

Before contemplating discussing decay for solutions to (1) on Kerr, we must first understand how such results can be proven on Schwarzschild. Some non-quantitative results, i.e. decay without a rate [26], scattering and asymptotic completeness statements [2], have been known for some time. In view of our motivation in the problem of non-linear stability of the background spacetime (see Section 5), we are here interested exclusively in quantitative statements: rates of decay depending only on the size of initial data.

3.1. The pointwise and energy decay theorem

To talk about energy decay on Schwarzschild, one must introduce a different type of foliation.

Let $\Sigma$ be the Cauchy hypersurface as before (say coinciding with a surface \{t = c\} for all sufficiently large $r$), and let $\tilde{\Sigma}$ now be a hypersurface with $\tilde{\Sigma} \subset J^+(\Sigma)$ such that $\tilde{\Sigma} \cap J^+ \neq \emptyset$, and $\tilde{\Sigma}$ meets $\mathcal{I}^+$ appropriately, and define $\tilde{\Sigma}_0 = \tilde{\Sigma} \cap \partial$, $\tilde{\Sigma}_\tau = \varphi_\tau(\tilde{\Sigma}_0)$ for $\tau \geq 1$.

**Theorem.** [10] Let $(\mathcal{M}, g_M)$ be Schwarzschild with parameter $M$, let $\Sigma, \tilde{\Sigma}, \partial$ as above, and let $\Omega_i$ denote the angular momentum operators. Then there exists a constant $C$ depending only on $M$, $\Sigma$ and $\tilde{\Sigma}$ such that for all $\psi$ satisfying $\Box_g \psi = 0$, the following holds:

$$
|n_{\Sigma_\tau} \psi|_{L^2(\Sigma_\tau)} + |\nabla_{\Sigma_\tau} \psi|_{L^2(\Sigma_\tau)} \leq C \tau^{-1} \sum_{|m| \leq 3} (r |n_{\Sigma} \Omega^m \psi|_{L^2(\Sigma)} + r |\nabla_{\Sigma} \Omega^m \psi|_{L^2(\Sigma)}).
$$

(8)

Moreover, the pointwise decay rates

$$
|\sqrt{r} \psi| \leq C Q \tau^{-1}, \quad |r \psi| \leq C Q \tau^{-1/2}
$$

(9)

hold, where $Q$ is an appropriate norm on initial data.

One can in fact show decay for non-degenerate energies of arbitrary order, and pointwise decay for arbitrary derivatives of $\psi$, including derivatives transverse to the horizon. See [14].
An independent proof of similar decay rates away from the horizon but weaker decay rates along the horizon was given by Blue and Sterbenz [5].

3.2. Trapping

Before turning to the proof of the above theorem, let us point out a central feature of its statement: The energy decay estimate (8) “loses” derivatives, that is to say, one needs control of more derivatives initially on Σ to estimate the energy later on Στ. This is an essential aspect of the problem and has to do with trapping, i.e. the fact that there are null geodesics neither crossing the event horizon nor approaching null infinity. These in fact asymptote to the so-called photon sphere at \( r = 3M \):

![Diagram of photon sphere](image)

which is itself spanned by null geodesics.

A rigorous study of the geometric optics approximation easily shows that one can construct a sequence of solutions to (1) with fixed initial energy, such that the energy concentrates near such a trapped null geodesic for longer and longer time. This sequence shows that an estimate of the form (8) cannot hold without losing derivatives.

3.3. The vector fields

The proof of decay uses multipliers constructed from 4 different vector fields.

3.3.1. The vector field \( T \)

We have already discussed the use of this in the context of the Kay and Wald theorem.

3.3.2. Trapping and the vector field \( X \)

In the obstacle problem on Euclidean space, trapping is often “captured” by the bulk term of the identity (3) for multipliers corresponding to well chosen vector fields \( X = f(r)\partial_r \). Soffer and collaborators in their pioneering [21, 3] were the first to pursue the programme of constructing such vector fields to capture the trapping phenomenon in Schwarzschild. The programme was first successfully completed in [10] and [4], but using spherical harmonic decompositions. The multiplier to be discussed here, the first not to require such decompositions, was constructed in [12].
Le us recall first so-called Regge-Wheeler coordinates \((r^*, t)\), where \(r^*\) is defined by
\[
r^* = r + 2M \log(r - 2M) - 3M - 2M \log M.
\] (10)

The current “capturing” trapping is actually a higher order current, involving also commutation, and takes the form
\[
J^X_\mu(\psi) = eN^\mu(\psi) + J^X_\mu(\psi) + \sum_i J^{X_{b,w}}_\mu(\Omega_i \psi)
\]
\[
- \frac{1}{2} \frac{r(f^b)'(r - 2M)}{f^b(r - 2M)} \left( \frac{r - 2M}{r^2} - \frac{(r^* - \alpha - \alpha^{1/2})}{\alpha^2 + (r^* - \alpha - \alpha^{1/2})^2} \right) X^b_\mu \psi^2.
\] (11)

Here, \(N\) is as in Section 2.6, \(X^a = f^a \partial_r\), \(X^b = f^b \partial_r\), the modified current \(J^{X,w}_\mu\) is defined by
\[
J^{X,w}_\mu = X^\nu T^\mu_{\nu} + \frac{1}{8} w_\mu (\psi^2) - \frac{1}{8} (\partial_\mu w) \psi^2,
\] (12)

and
\[
f^a = - \frac{C_a}{\alpha r^2} + \frac{c_a}{r^3}, \quad f^b = \frac{1}{\alpha} \left( \tan^{-1} \frac{r^* - \alpha - \alpha^{1/2}}{\alpha} - \tan^{-1} (-1 - \alpha^{-1/2}) \right),
\] (13)

\[
w^b = \frac{1}{8} \left( (f^b)' + 2 \frac{r - 2M}{r^2} f^b \right),
\]
and \(e, C_a, c_a, \alpha\) are positive parameters which must be chosen accordingly. With these choices, one can show (after some computation) that the divergence \(K^X = \nabla^\mu J^X_\mu\) controls in particular
\[
\int_{S^2} K^X(\psi) \geq b \chi \int_{\mathcal{H}^+} J^N_\mu(\psi) n^\mu,
\] (14)

where \(\chi\) is non-vanishing but decays (polynomially) as \(r \to \infty\), and the integration is over any \(SO(3)\) orbit. Note that in view of the normalisation (10) of the \(r^*\) coordinate, \(X^b = 0\) precisely at \(r = 3M\). The left hand side of the inequality (14) controls also second order derivatives which degenerate however at \(r = 3M\). We have dropped these terms. It is actually useful for applications that the \(J^{X,a}_\mu(\psi)\) part of the current is not “modified” by a function \(w^a\), and thus \(\psi\) itself does not occur in the boundary terms. That is to say
\[
|J^X_\mu(\psi)n^\mu| \leq B \left( J^N_\mu(\psi)n^\mu + \sum_{i=1}^3 J^{X_i}_\mu(\Omega_i \psi)n^\mu \right).
\] (15)

On the event horizon \(\mathcal{H}^+\), we have a better one-sided bound
\[
-J^X_\mu(\psi)n^\mu_{\mathcal{H}^+} \leq B \left( J^T_\mu(\psi)n^\mu_{\mathcal{H}^+} + \sum_{i=1}^3 J^{T_i}_\mu(\Omega_i \psi)n^\mu_{\mathcal{H}^+} \right).
\] (16)

For details of the construction, see [12].
In view of (14), (15) and (16), together with our previous boundedness theorem of Section 2.7, one obtains in particular the estimate
\[
\int_{J^+(\Sigma')} \chi J^N_{\nu}(\psi)n^\nu_{\Sigma'} \leq B \int_{\Sigma'} \left( J^N_{\mu}(\psi) + \sum_{i=1}^{3} J^N_{\mu}(\Omega_i \psi) \right) n^\mu_{\Sigma'},
\]
for some nonvanishing \(\phi_t\)-invariant function \(\chi\) which decays polynomially as \(r \to \infty\). Such estimates are known as integrated decay.

For a sketch of yet another construction yielding an estimate (17) which degenerates however on \(\mathcal{H}^+\), see [22].

3.3.3. The vectorfield \(Z\)

To turn this integrated decay into decay of energy as in the statement (8), one introduces a current \(J^{Z,w}\) (of the form (12)) associated to a vector field \(Z\) defined by
\[
u^2 \partial_u + v^2 \partial_v
\]
where \(u = t - r^*, v = t + r^*,\) and
\[
w = \frac{2tr^*(1 - 2M/r)}{r}.
\]
In the case of Minkowski space \((M = 0)\), the divergence \(K^{Z,w} = 0\), while
\[
\int_{J^+(\Sigma')} J^{Z,w}_{\mu} n^\mu \geq b \int_{J^+(\Sigma')} u^2(\partial_u \psi)^2 + v^2(\partial_v \psi)^2 + (u^2 + v^2)|\nabla \psi|^2.
\]
The identity (3) yields the boundedness of the left hand side above, and thus, in view of the weights on the right hand side of (19), this yields decay of energy as in (8).

In the case of Schwarzschild, a similar relation to (19) holds (with an extra factor of \((1 - 2M/r)\)). But now the error term \(K^{Z,w} \neq 0\), in fact the best one can estimate is
\[
-K^{Z,w} \geq BtJ^N_{\mu} n^\mu
\]
in a region \([r_1, R_2]\) for some \(R_2 > r_1 > 2M\).

The error term on the right hand side of (20) at first seems problematic, but it can in fact be absorbed by a simple iteration argument given only the integrated decay estimate (17). Thus one retrieves energy decay statements on Schwarzschild exactly analogous to the case of Minkowski space, but now “losing” derivatives, in view of the use of (17) to absorb the error term above.

Note that a related method of absorbing the error term on the right hand side of (20) was independently attained in the paper [5] referred to previously.

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8 This degeneration can be overcome by adding the energy identity of the current \(N\) of Section 2.6
9 Note that the current \(J^{Z,w}_{\mu}\) is related to the conformal covariance properties of the wave equation on Minkowski space.
10 using also the considerations of Section 3.3.4 below
3.3.4. The vectorfield $N$

The above does not give proper control at the horizon. For this, one must return to the vector field $N$ of Section 2.6. It turns out that the calculation (6), in conjunction with the bounds obtained away from the horizon, allows one to extend the energy decay and pointwise decay results to the horizon. For details, see [10, 12] or [14].

3.4. Commutation and Sobolev inequalities

To achieve pointwise control (9) from (8), we commute with $\Omega_i$ and apply Sobolev inequalities. See [10].

3.5. Price law tails?

In 1972, Price [24] put forth heuristic arguments suggesting that, decomposing $\psi$ into spherical harmonics $\psi_\ell$, each $\psi_\ell$ should asymptotically behave asymptotically like

\[
\psi_\ell (r, t) \sim C_\ell t^{-(3+2\ell)}. \tag{21}
\]

Related statements have indeed been proven in the case $\ell = 0$ (see [9]), but no statement of the form (21) has yet been shown for general $\ell$.

Recall that the our interest in the linear theory is motivated by the desire to understand the non-linear stability problem (See Section 5). For this, a statement of the form (21) would be essentially useless: The statement (21), even if true, would be completely non-quantitative, i.e. it would not give a bound for $\psi_\ell$ at “intermediate times” in terms of the size of initial data. In particular, the statement (21) would not “see” the trapping phenomenon and the associated loss of derivatives in the estimate (8).

One faces this non-quantitative aspect immediately when one tries to sum (21) over $\ell$ in order to yield a statement about $\psi$: A priori, the statement (21) is in fact completely compatible with

\[
\limsup_{t \to \infty} \psi (r, t) = \infty. \tag{22}
\]

4. Kerr

Now that we have a decay result for Schwarzschild, can we go back and retrieve this for Kerr?

Unfortunately, like the boundedness proof, our decay proof too is unstable, but for a different reason: The structure of trapping in Schwarzschild is very special. In particular, the construction of $X$ in Section 3.3.2 is based on the fact that the co-dimensionality of the set of trapped null geodesics manifests itself also in physical space in the following way: all such trapped geodesics approach the codimension-one hypersurface $r = 3M$. (Recall that the function $f_b$ of (13) vanishes precisely along this hypersurface.) See also [1] for a nice discussion of this issue.
Nonetheless, it turns out that using ideas from the decay proof, we can indeed perturb just the boundedness theorem for geometries $g$ “near” Schwarzschild, provided that $g$ retains two of the Killing fields of Schwarzschild ($T$ and $\Omega_1$ say), and a certain geometric property.

Unlike the theorem of Section 2.8, the class of spacetimes allowed will in particular include the Kerr case for $a \ll M$.

### 4.1. Uniform boundedness on axisymmetric stationary black hole exteriors

Let us first state the theorem

**Theorem.**\(^{[13]}\) Let $\mathcal{R}$ be as before, $g$ be a metric defined on $\mathcal{R}$, and let $T$ and $\Phi = \Omega_1$ be Schwarzschild Killing fields. Assume

1. $g$ is close to Schwarzschild in an appropriate sense
2. $T$ and $\Phi$ are Killing with respect to $g$
3. $\mathcal{H}^+$ is null with respect to $g$, and $T$ and $\Phi$ together span the null generator of $\mathcal{H}^+$.

Then the uniform boundedness theorem of Section 2.7 holds.

In particular, the theorem applies to Kerr for $|a| \ll M$, Kerr-Newman for $|a| \ll M$, $Q \ll M$, etc.

The heuristic idea of the proof of this result is actually quite simple. Consider a metric $g$ as described above, i.e. retaining the Killing fields $T$ and $\Phi$ of Schwarzschild, and suitable close to Schwarzschild.

Via the Fourier transform, we associate frequencies $\omega, k$ to the Killing fields $T$ and $\Phi$, where $\omega \in \mathbb{R}$, and $k \in \mathbb{Z}$. Suppose we could decompose

$$\psi = \psi_{\flat} + \psi_{\sharp} \quad (23)$$

where $\psi_{\flat}$ is supported in $\omega^2 \leq c k^2$ and $\psi_{\sharp}$ is supported in $\omega^2 \geq c k^2$.

The crucial observation is simply the following: For $c$ small enough, and for $g$ close enough to Schwarzschild, then in view of the geometric assumption 3. on the Killing fields, it follows that (i) there is no superradiance for $\psi_{\sharp}$, and (ii) there is no trapping for $\psi_{\flat}$.

That is to say, for appropriate choice of $c$, (i) the current $J_{\mu}^T(\psi_{\sharp})$ has a nonnegative flux through the horizon $\mathcal{H}$, and (ii) a variant of the $X$ vector field can be constructed, so that $K^X(\psi_{\flat})$ is nonnegative. In view of the absence of trapping, the current $K^X(\psi_{\flat})$ need not degenerate near $r = 3M$, and its construction is quite simple relative to Section 3.3.2 and moreover, completely stable to perturbation. In particular, it suffices to know that such a current can be constructed on Schwarzschild giving the required positivity properties in this frequency range.

Thus, the outline of the boundedness argument appears quite simple: Apply $T$ and $N$ to $\psi_{\sharp}$ as in the boundedness proof, and apply $T$, $N$, and $X$ to $\psi_{\flat}$ as in the decay proof to
obtain integrated decay (and thus in particular energy boundedness!) for $\psi$. This would in particular yield the non-degenerate energy boundedness statement for $\psi = \psi_{\uparrow} + \psi_{\downarrow}$. The pointwise estimates would then follow by commutation, in view also of Section 2.6.

To implement the above argument, however, is tricky: In order to decompose $\psi$ as in (23) one would in particular have to take the Fourier transform of $\psi$ in time. Yet a priori we have not shown that $\psi$ is even uniformly bounded. Thus we must replace $\psi$ with a cut-off version $\psi_{\infty} = \xi \psi$, where $\xi$ is a cutoff function in time, and apply the decomposition to $\psi_{\infty}$. generating error terms which must themselves be bounded. It is essential that one has at ones disposal a non-degenerate energy, as in the statement of theorem, to bound these error terms. This is accomplished via a bootstrap argument. See [13] for the details of the proof.

4.2. Decay for slowly rotating Kerr

The above argument for boundedness is relatively simple and robust because it circumvents the problem of understanding trapping: It sufficed to know that $\psi$ is not trapped. If one is to tackle the problem of decay, however, one has no choice but to come to terms with the structure of trapping in detail. Since the codimensionality of the trapping must be viewed in phase space, this suggests adapting our arguments, particularly the construction of $X$, to phase space. We shall be able to accomplish this, but at the expense of restricting to Kerr spacetimes, as opposed to the general class of Section 4.1.

4.2.1. The separation and the frequency-localised construction of $X$

There is a convenient way of doing phase space analysis in Kerr spacetimes, namely, as discovered by Carter [6], the wave equation can be separated. Walker and Penrose [28] later showed that both the complete integrability of geodesic flow and the separability of the wave equation have their fundamental origin in the presence of a Killing tensor. In fact, as we shall see, in view of its intimate relation with the integrability of geodesic flow, Carter’s separation of $\Box_g$ immediately captures the codimensionality of the trapped set.

The separation of the wave equation requires taking the Fourier transform with respect to time, and then expanding into oblate spheroidal harmonics. As before, taking the Fourier transform requires cutting off in time. Since this has essentially already been addressed in the previous section, let us pretend that this is not an issue, and that we may write

$$\hat{\psi}(\omega, \cdot) = \sum_{m, \ell} R_{m \ell}^{\omega}(r) S_{m \ell}(a\omega, \cos \theta) e^{im\phi^*},$$

where $S_{m \ell}$ are the oblate spheroidal harmonics with eigenvalues $\lambda_{m \ell}(\omega)$. The wave equation [1] then reduces to the following equation for $R_{m \ell}^{\omega}$:

$$\Delta \frac{d}{dr} \left( \Delta R_{m \ell}^{\omega} \right) + (a^2 m^2 + (r^2 + a^2)^2 \omega^2 - \Delta(\lambda_{m \ell} + a^2 \omega^2)) R_{m \ell}^{\omega} = 0.$$
Defining a coordinate $r^*$ by $\frac{dr^*}{dr} = \frac{r^2 + a^2}{\Delta}$, where $\Delta = r^2 - 2Mr + a^2$, and setting $u(r) = (r^2 + a^2)^{1/2}R_{m\ell}^\omega(r)$, then $u$ satisfies
\[
\frac{d^2}{(dr^*)^2}u + (\omega^2 - V_{m\ell}^\omega(r))u = 0
\]
where
\[
V_{m\ell}^\omega(r) = \frac{4Mr\omega - a^2m^2 + \Delta(\lambda_{m\ell} + \omega^2a^2)}{(r^2 + a^2)^2} + \frac{\Delta(3r^2 - 4Mr + a^2)}{(r^2 + a^2)^3} - \frac{3\Delta^2r^2}{(r^2 + a^2)^4}.
\]
Consider the following quantity
\[
Q = f \left( \left| \frac{du}{dr^*} \right|^2 + (\omega^2 - V)|u|^2 \right) + \frac{df}{dr^*} \text{Re} \left( \frac{du}{dr^*} \bar{u} \right) - \frac{1}{2} \frac{d^2f}{dr^*} |u|^2.
\]
Then, with the notation $' = \frac{d}{dr^*}$,
\[
Q' = 2f'|u|^2 - fV'|u|^2 - \frac{1}{2} f'''|u|^2.
\] (24)

The main difficulty is for $\psi$ supported in $|\omega| \geq \omega_1, \lambda_{m\ell} \geq \lambda_2$ large. An easy computation shows that for suitable choice of $\omega_1$, in this frequency range $V'$ has a unique simple zero. Let us denote the $r$-value of this zero by $r_{m\ell}^\omega$.

We now choose $f$ so that (i) $f' \geq 0$, (ii) $f \leq 0$ for $r \leq r_{m\ell}^\omega$ and $f \geq 0$ for $r \geq r_{m\ell}^\omega$, and (iii) $-fV' - \frac{1}{2} f''' \geq c > 0$.

Integrating the identity (24) and using that $u \to 0$ as $r \to \infty$ we obtain that for any compact set $K_1$ in $r^*$ and a certain compact set $K_2$ (which in particular does not contain $r = 3M$), there exists a positive constant $b > 0$ so that
\[
b \int_{K_1} (|u'|^2 + |u|^2)dr + b(\lambda_{m\ell} + \omega^2) \int_{K_2} |u|^2dr \leq (|u'|^2 + (\omega^2 - V)|u|^2)(r_+),
\]
where $r_+$ denotes the $r$-value of $\mathcal{H}^+$. Reinstating the dropped indices $m, \ell, \omega$, summing over $m, \ell$, integrating over $\omega$, and adding this estimate to an estimate for the remaining frequencies (which in fact need not degenerate near $r = 3M$), and finally adding a little bit of the estimate corresponding to $N$ (recall that the computation [6] is stable!), we obtain the analogue of (17) for $\psi$ (with $T\psi$ replacing $\Omega, \psi$).

This yields integrated decay for solutions to (11) on Kerr $g_{a,M}$ with small $a \ll M$.

4.2.2. The use of $N$ and $Z$

Once one has the integrated decay estimates, the other aspects of the proof of decay on Schwarzschild are stable to perturbation of the metric, modulo a loss in $\delta$ in the $\tau$
power of the rate of decay, where $\delta$ depends on the closeness to Schwarzschild. This requires, however, a refinement of the use of $N$, in view of the fact that the vector field $Z$ as defined in (18) fails to be $C^1$ on $\mathcal{H}^+$. A further issue arises in that one must commute with the Schwarzschild $\Omega_i$ to obtain the desired pointwise decay statements, and these are no longer Killing, generating errors. See [14] for details.

4.2.3. The statement of the theorem

Theorem. [14] Let $(\mathcal{M}, g_{a.M})$ be Kerr for $|a| \ll M$, $\mathcal{D}$ be the closure of its domain of dependence, let $\Sigma_0$ be the surface $\mathcal{D} \cap \{t^* = 0\}$, let $\Psi, \Psi'$ be initial data on $\Sigma_0$ such that $\psi \in H^s_{\text{loc}}(\Sigma)$, $\psi' \in H^s_{\text{loc}}(\Sigma)$ for $s \geq 1$, and $\lim_{r \to \infty} \psi = 0$, and let $\psi$ be the corresponding unique solution of $\Box_g \psi = 0$. Let $\varphi_\tau$ denote the 1-parameter family of diffeomorphisms generated by $T$, let $\tilde{\Sigma}_0$ be a spacelike hypersurface in $J^+(\Sigma_0)$ terminating on null infinity, and define $\tilde{\Sigma}_\tau = \varphi_\tau(\tilde{\Sigma}_0)$. Let $s \geq 3$ and assume

$$E_1 \doteq \int_{\Sigma_0} r^2 (J_{\mu}^n(\psi) + J_{\mu}^n(T \psi) + J_{\mu}^n(T T \psi)) n_0^\mu < \infty.$$  

Then there exists a $\delta > 0$ depending on $a$ (with $\delta \to 0$ as $a \to 0$) and a $B$ depending only on $\tilde{\Sigma}_0$ such that

$$\int_{\tilde{\Sigma}_\tau} J_{\mu}^n(\psi) n_0^\mu \leq B E_1 \tau^{-2+2\delta}.$$  

Now let $s \geq 5$ and assume

$$E_2 \doteq \sum_{|\alpha| \leq 2} \sum_{\Gamma = \{T,N,\Omega_i\}} \int_{\Sigma_0} r^2 (J_{\mu}^n(\Gamma^\alpha \psi) + J_{\mu}^n(\Gamma^\alpha T \psi) + J_{\mu}^n(\Gamma^\alpha TT \psi)) n_0^\mu < \infty$$

where $\Omega_i$ are the angular momentum operators corresponding to the related Schwarzschild metric $g_M$ on $\mathcal{D}$. Then

$$\sup_{\tilde{\Sigma}_\tau} \sqrt{r} |\psi| \leq B \sqrt{E_2} \tau^{-1+\delta}, \quad \sup_{\tilde{\Sigma}_\tau} r |\psi| \leq B \sqrt{E_2} \tau^{(-1+\delta)/2}.$$  

One can obtain decay for arbitrary derivatives, including transversal derivatives to $\mathcal{H}^+$, using additional commutation by $N$.

The above theorem was first announced at the Clay Summer School in Zürich in July 2008 and its proof appears in the lecture notes [14].

There is some additional interesting work in progress related to this section which should be noted: Tohaneanu et al. are pursuing a related approach to the integrated decay statement of Section 4.2.1, again relying on the red-shift estimates developed in [10, 13] (presented here in Section 2.6), but where the frequency localisation is carried out with the machinery of the pseudodifferential calculus. Andersson and Blue are pursuing

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11 Private communication from M. Tohaneanu who attended [14].
an alternative approach to the construction of Section 4.2.1 in terms of higher order currents, similar to the current (11) of Section 3.3.2 but where commutation with $\Omega_i$ is replaced by commutation with the so-called Carter operator.  

5. FUTURE DIRECTIONS AND THE NON-LINEAR STABILITY OF THE KERR FAMILY

The theorem of Section 4.2.3 does not close the book on this subject. It would be nice to obtain this result with the least possible assumptions on the geometry. For instance, what can be said about decay under the much more general assumptions of our boundedness result, the theorem of Section 4.1? It would be interesting also to obtain stronger decay rates in the interior. In the Kerr case, it is important to obtain results for the whole range $a < M$. Moreover, it is essential to understand boundedness and decay properties for higher spin (see below). Another interesting direction is to study spacetimes with cosmological constant (see [18]). For an extensive list of related open problems, see [14].

The most important future direction, however, and the main motivation for the problem considered in this talk is the stability of the Kerr family of spacetimes as solutions to the Cauchy problem for the Einstein vacuum equations

$$R_{\mu\nu} = 0.$$  

See [14] for a formulation. This latter problem is one of the main open problems in general relativity.

The role of linear theory for the understanding of the non-linear stability problem can be seen from the proof of the nonlinear stability of Minkowski space, first given in Christodoulou–Klainerman [8]. The proof of [8] required in particular a robust method of proving the results of Sections 4.1 and 4.2, not just for the wave equation (1), but for the spin-2 Bianchi system satisfied by the curvature tensor, and not just on a background which was exactly Minkowski, but for spacetimes sufficiently close to and decaying to Minkowski. Hence the importance of the open problems discussed above.

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12 Lecture of P. Blue, Stockholm, September 2009.
13 While no statement is known for general solutions in this range, the following pretty non-quantitative statement away from the horizon is given in [13, 18] for individual azimuthal modes $\psi_m$: If $\psi_m$ is not supported in a neighborhood of $i^0$ and $\mathcal{H}^+ \cap \mathcal{H}^-$, then for $r > r_+$, $\lim_{r \to r_+} \psi_m(r,t) = 0$. As in the heuristics of Price, this statement is of course compatible with (22) for the sum over $m$. 

REFERENCES

1. S. Alinhac *Energy multipliers for perturbations of Schwarzschild metric* preprint, 2008
2. A. Bachelot *Asymptotic completeness for the Klein-Gordon equation on the Schwarzschild metric*, Ann. Inst. H. Poincaré Phys. Théor. 16 (1994), no. 4, 411–441
3. P. Blue and A. Soffer *Semilinear wave equations on the Schwarzschild manifold. I. Local decay estimates*, Adv. Differential Equations 8 (2003), no. 5, 595–614
4. P. Blue and A. Soffer *Errata for “Global existence . . . Regge Wheeler equation”*, gr-qc/0608073
5. P. Blue and J. Sterbenz *Uniform decay of local energy and the semi-linear wave equation on Schwarzschild space* Comm. Math. Phys. 268 (2006), no. 2, 481–504
6. B. Carter *Hamilton-Jacobi and Schrödinger separable solutions of Einstein’s equations* Comm. Math. Phys. 10 (1968), 280–310
7. D. Christodoulou *The action principle and partial differential equations*, Ann. Math. Studies No. 146, 1999
8. D. Christodoulou and S. Klainerman *The global nonlinear stability of the Minkowski space* Princeton University Press, 1993
9. M. Dafermos and I. Rodnianski *A proof of Price’s law for the collapse of a self-gravitating scalar field*, Invent. Math. 162 (2005), 381–457
10. M. Dafermos and I. Rodnianski *The redshift effect and radiation decay on black hole spacetimes*, gr-qc/0512119
11. M. Dafermos and I. Rodnianski *The wave equation on Schwarzschild-de Sitter spacetimes*, arXiv:0709.2766v1 [gr-qc]
12. M. Dafermos and I. Rodnianski *A note on energy currents and decay for the wave equation on a Schwarzschild background*, arXiv:0710.0171v1 [math.AP]
13. M. Dafermos and I. Rodnianski *A proof of the uniform boundedness of solutions to the wave equation on slowly rotating Kerr backgrounds*, available online at http://arxiv.org/abs/0805.4309
14. M. Dafermos and I. Rodnianski *Lectures on black holes and linear waves*, to appear in Clay Lecture Notes, available online at http://arxiv.org/abs/0811.0354
15. F. Finster, N. Kamran, J. Smoller, S.-T. Yau *Decay of solutions of the wave equation in Kerr geometry* Comm. Math. Phys. 264 (2006), 465–503
16. F. Finster, N. Kamran, J. Smoller, S.-T. Yau *Erratum: Decay of solutions of the wave equation in Kerr geometry* Comm. Math. Phys., online first
17. S. W. Hawking and G. F. R. Ellis *The large scale structure of space-time* Cambridge Monographs on Mathematical Physics, No. 1. Cambridge University Press, London-New York, 1973
18. G. Holzegel *On the massive wave equation on slowly rotating Kerr-AdS spacetimes*, preprint 2009
19. B. Kay and R. Wald *Linear stability of Schwarzschild under perturbations which are nonvanishing on the bifurcation 2-sphere* Classical Quantum Gravity 4 (1987), no. 4, 893–898
20. S. Klainerman *Uniform decay estimates and the Lorentz invariance of the classical wave equation* Pure Appl. Math. 38 (1985), 321–332
21. I. Laba and A. Soffer *Global existence and scattering for the nonlinear Schrödinger equation on Schwarzschild manifolds* Helv. Phys. Acta 72 (1999), no. 4, 272–294
22. J. Metcalfe *Strichartz estimates on Schwarzschild space-times (joint work with D. Tataru, M. Tohaneanu)* Oberwolfach Reports 44 (2007), 8–11.
23. J. R. Oppenheimer and H. Snyder *On continued gravitational contraction* Phys. Rev. 56 (1939), 455–459
24. R. Price *Nonspherical perturbations of relativistic gravitational collapse. I. Scalar and gravitational perturbations* Phys. Rev. D (3) 5 (1972), 2419–2438
25. T. Regge and J. Wheeler *Stability of a Schwarzschild singularity* Phys. Rev. 108 (1957), 1063–1069
26. F. Twainy *The Time Decay of Solutions to the Scalar Wave Equation in Schwarzschild Background Thesis*. San Diego: University of California 1989
27. R. Wald *General relativity* University of Chicago Press, Chicago, 1984
28. M. Walker and R. Penrose *On quadratic first integrals of the geodesic equations for type 22 spacetimes* Comm. Math. Phys. 18 (1970), 265–274
29. B. Whiting *Mode stability of the Kerr black hole* J. Math. Phys. 30 (1989), 1301