Minimum Cost Homomorphisms to Semicomplete Multipartite Digraphs

Gregory Gutin∗ Arash Rafiey† Anders Yeo‡

Abstract

For digraphs D and H, a mapping \( f : V(D) \rightarrow V(H) \) is a homomorphism of D to H if \( uv \in A(D) \) implies \( f(u)f(v) \in A(H) \). For a fixed directed or undirected graph H and an input graph D, the problem of verifying whether there exists a homomorphism of D to H has been studied in a large number of papers. We study an optimization version of this decision problem. Our optimization problem is motivated by a real-world problem in defence logistics and was introduced very recently by the authors and M. Tso.

Suppose we are given a pair of digraphs D, H and a positive integral cost \( c_i(u) \) for each \( u \in V(D) \) and \( i \in V(H) \). The cost of a homomorphism \( f \) of D to H is \( \sum_{u \in V(D)} c_{f(u)}(u) \). Let H be a fixed digraph. The minimum cost homomorphism problem for H, MinHOMP(H), is stated as follows: For input digraph D and costs \( c_i(u) \) for each \( u \in V(D) \) and \( i \in V(H) \), verify whether there is a homomorphism of D to H and, if it does exist, find such a homomorphism of minimum cost. In our previous paper we obtained a dichotomy classification of the time complexity of MinHOMP(H) for H being a semicomplete digraph. In this paper we extend the classification to semicomplete k-partite digraphs, \( k \geq 3 \), and obtain such a classification for bipartite tournaments.

1 Introduction

In our terminology and notation, we follow [II]. In this paper, directed (undirected) graphs have no parallel arcs (edges) or loops. The vertex (arc) set of a digraph G is denoted by \( V(G) \) (\( A(G) \)). The vertex (edge) set of an undirected graph G is denoted by \( V(G) \) (\( E(G) \)). A digraph D obtained from a complete k-partite (undirected) graph G by replacing every

∗Corresponding author. Department of Computer Science, Royal Holloway University of London, Egham, Surrey TW20 0EX, UK, gutin@cs.rhul.ac.uk and Department of Computer Science, University of Haifa, Israel

†Department of Computer Science, Royal Holloway University of London, Egham, Surrey TW20 0EX, UK, arash@cs.rhul.ac.uk

‡Department of Computer Science, Royal Holloway University of London, Egham, Surrey TW20 0EX, UK, anders@cs.rhul.ac.uk
an input digraph $D$ edge $xy$ of $G$ with arc $xy$, arc $yx$, or both $xy$ and $yx$, is called a semicomplete $k$-partite digraph (or, semicomplete multipartite digraph when $k$ is immaterial). The partite sets of $D$ are the partite sets of $G$. A semicomplete $k$-partite digraph $D$ is semicomplete if each partite set of $D$ consists of a unique vertex. A $k$-partite tournament is a semicomplete $k$-partite digraph with no directed cycle of length 2. Semicomplete $k$-partite digraphs and its subclasses mentioned above are well-studied in graph theory and algorithms, see, e.g., [1].

For excellent introductions to homomorphisms in directed and undirected graphs, see [8, 10]. For digraphs $D$ and $H$, a mapping $f : V(D) \rightarrow V(H)$ is a homomorphism of $D$ to $H$ if $uv \in A(D)$ implies $f(u)f(v) \in A(H)$. A homomorphism $f$ of $D$ to $H$ is also called an $H$-coloring of $G$, and $f(x)$ is called a color of $x$ for every $x \in V(D)$. We denote the set of all homomorphisms from $D$ to $H$ by $\text{HOM}(D, H)$.

For a fixed digraph $H$, the homomorphism problem $\text{HOM}(H)$ is to verify whether, for an input digraph $D$, there is a homomorphism of $D$ to $H$ (i.e., whether $\text{HOM}(D, H) \neq \emptyset$). The problem $\text{HOM}(H)$ has been studied for several families of directed and undirected graphs $H$, see, e.g., [8, 10]. The well-known result of Hell and Nešetřil [9] asserts that $\text{HOM}(H)$ for undirected graphs is polynomial time solvable if $H$ is bipartite and it is NP-complete, otherwise. Such a dichotomy classification for all digraphs is unknown and only partial classifications have been obtained; see [10]. For example, Bang-Jensen, Hell and MacGillivray [2] showed that $\text{HOM}(H)$ for semicomplete digraphs $H$ is polynomial time solvable if $H$ has at most one cycle and $\text{HOM}(H)$ is NP-complete, otherwise.

The authors of [6] introduced an optimization problem on $H$-colorings for undirected graphs $H$, $\text{MinHOM}(H)$ (defined below). The problem is motivated by a problem in defence logistics. In our previous paper [5], we obtained a dichotomy classification for the time complexity of $\text{MinHOM}(H)$ for $H$ being a semicomplete digraph. In this paper, we extend that classification to obtain a dichotomy classification for semicomplete $k$-partite digraphs $H$, $k \geq 3$. We also obtain a classification of the complexity of $\text{MinHOM}(H)$ when $H$ is a bipartite tournament. The case of arbitrary semicomplete bipartite digraphs appears to be significantly more complicated (see the last paragraph of Section 4). Our main approach to proving polynomial time solvability and NP-hardness of special cases of $\text{MinHOM}(H)$ is the use of various reductions to/from the maximum weight independent set problem.

Suppose we are given a pair of digraphs $D, H$ and a positive integral cost $c_i(u)$ for each $u \in V(D)$ and $i \in V(H)$. The cost of a homomorphism $f$ of $D$ to $H$ is $\sum_{u \in V(D)} c_{f(u)}(u)$. For a fixed digraph $H$, the minimum cost homomorphism problem $\text{MinHOM}(H)$ is formulated as follows. For an input digraph $D$ and costs $c_i(u)$ for each $u \in V(D)$ and $i \in V(H)$, verify whether $\text{HOM}(D, H) \neq \emptyset$ and, if $\text{HOM}(D, H) \neq \emptyset$, find a homomorphism in $\text{HOM}(D, H)$ of minimum cost. The maximum cost homomorphism problem $\text{MaxHOM}(H)$ is the same problem as $\text{MinHOM}(H)$, but instead of minimization we consider maximization. Let $M$ be an integral constant larger than any cost $c_i(u)$, $u \in V(D)$, $i \in V(H)$. Then the cost $c'_i(u) = M - c_i(u)$ is positive for each $u \in V(D)$, $i \in V(H)$. Due to this transformation,
the problems MinHOMP($H$) and MaxHOMP($H$) are equivalent.

For a digraph $G$, if $xy \in A(G)$, we say that $x$ dominates $y$ and $y$ is dominated by $x$ (denoted by $x \rightarrow y$). The outdegree $d_G^+(x)$ (indegree $d_G^-(x)$) of a vertex $x$ in $G$ is the number of vertices dominated by $x$ (that dominate $x$). For sets $X, Y \subseteq V(G)$, $X \rightarrow Y$ means that $x \rightarrow y$ for each $x \in X$, $y \in Y$, but no vertex of $Y$ dominates a vertex in $X$. A set $X \subseteq V(G)$ is independent if no vertex in $X$ dominates a vertex in $X$. A $k$-cycle, denoted by $C_k$, is a directed simple cycle with $k$ vertices. A digraph $H$ is an extension of a digraph $D$ if $H$ can be obtained from $D$ by replacing every vertex $x$ of $D$ with a set $S_x$ of independent vertices such that if $xy \in A(D)$ then $uv \in A(H)$ for each $u \in S_x$, $v \in S_y$. For subsets $X, Y$ of $V(G)$, $X \times Y = \{xy : x \in X, y \in Y\}$.

The underlying graph $U(G)$ of a digraph $G$ is the undirected graph obtained from $G$ by disregarding all orientations and deleting one edge in each pair of parallel edges. A digraph $G$ is connected if $U(G)$ is connected. The components of $G$ are the subdigraphs of $G$ induced by the vertices of components of $U(G)$. A digraph $G$ is strongly connected if there is a path from $x$ to $y$ for every ordered pair of vertices $x, y \in V(G)$. A strong component of $G$ is a maximal induced strongly connected subdigraph of $G$. A digraph $G'$ is the dual of a digraph $G$ if $G'$ is obtained from $G$ by changing orientations of all arcs.

The rest of the paper is organized as follows. In Section 2 we give all polynomial time solvable cases of MinHOMP($H$) when $H$ is semicomplete $k$-partite digraph, $k \geq 3$, or a bipartite tournament. Section 3 is devoted to a full dichotomy classification of the time complexity of MinHOMP($H$) when $H$ is a semicomplete $k$-partite digraph, $k \geq 3$. A classification of the same problem for $H$ being a bipartite tournament is proved in Section 4.

## 2 Polynomial Time Solvable Cases

The following definitions and theorem were given in [5]. The homomorphic product of digraphs $D$ and $H$ is an undirected graph $D \otimes H$ defined as follows: $V(D \otimes H) = \{u_i : u \in V(D), i \in V(H)\}$, $E(D \otimes H) = \{uv_i : uv \in A(D), ij \notin A(H)\} \cup \{u_iu_j : u \in V(D), i \neq j \in V(H)\}$. Let $\mu = \max\{c(v) : v \in V(D), j \in V(H)\}$. We define the cost of $u_i$, $c(u_i) = c_i(u) + \mu|V(D)|$. For a set $X \subseteq V(D \otimes H)$, we define $c(X) = \sum_{x \in X} c(x)$.

**Theorem 2.1** Let $D$ and $H$ be digraphs. Then there is a homomorphism of $D$ to $H$ if and only if the number of vertices in a largest independent set of $D \otimes H$ equals $|V(D)|$. If $HOM(D, H) \neq \emptyset$, then a homomorphism $h \in HOM(D, H)$ is of maximum cost if and only if $I = \{x_{h(x)} : x \in V(D)\}$ is an independent set of maximum cost.

Let $TT_k$ denote the acyclic tournament on $k$ vertices. In [5] we used Theorem 2.1 in order to prove the following result:

**Theorem 2.2** MWHOMP($TT_k$) and MCHOMP($TT_k$) are polynomial time solvable.
Here we prove another, somewhat more involved, corollary of Theorem 2.1. A digraph $D$ is transitive if $xy, yz \in A(D)$ implies $xz \in A(D)$ for all pairs $xy, yz$ of arcs in $D$. A digraph $D$ is the transitive closure of a digraph $H$ if $V(D) = V(H)$ and $D$ is has the minimum number of arcs with respect to the following properties: $D$ is transitive and $A(H) \subseteq A(D)$. It is easy to see the uniqueness of the transitive close of a digraph. A graph is a comparability graph if it has an orientation, which is transitive. Let $k \geq 3$ and let $TT^-_k$ be a digraph obtained from $TT^-_k$ by deleting the arc from the source to the sink.

**Theorem 2.3** MWHOMP($TT^-_k$) and MCHOMP($TT^-_k$) are polynomial time solvable.

**Proof:** Let $V(TT^-_k) = \{1, 2, \ldots, k\}$ and let $A(TT^-_k) = \{ij : 1 \leq i < j \leq k \; j - i < k - 1\}$. Observe that $HOM(D, H) = \emptyset$ unless $D$ is acyclic. Since we can verify that $D$ is acyclic in polynomial time (for example, by deleting vertices of indegree 0), we may assume that $D$ is acyclic. We will furthermore assume that $D$ has no isolated vertices as these can be assigned an optimal color greedily. Let $S = \{s : d^-_D(s) = 0\}$ and let $T = \{t : d^+_D(t) = 0\}$. Let $P(x, y)$ denote the set of paths from a vertex $x$ to a vertex $y$ in $D$. Let $D'$ be a digraph obtained from $D$ by adding the following set of arcs:

$$\{xy : xy \in V(D) \times V(D) - (A(D) \cup (S \times T)), \; P(x, y) \neq \emptyset\}.$$  

Note that $D'$ is the transitive closure of $D$, where we delete all arcs from $S$ to $T$ which are not present in $D$. One can find $D'$ in polynomial time using the depth first search or breadth first search. If $h \in HOM(D, TT^-_k)$ and $xy \in A(D') - A(D)$ then $x \not\in S$ or $y \not\in T$, which implies that $h(x) \neq 1$ or $h(y) \neq k$, which again implies that $h(x)h(y) \in A(TT^-_k)$. This implies that $HOM(D, TT^-_k) = HOM(D', TT^-_k)$.

Observe that for every $TT^-_k$-coloring $h$ of $D$ and each $s \in S$, $t \in T$, $z \in V(D) - (S \cup T)$, we have $h(s) < k$, $h(t) > 1$ and $1 < h(z) < k$. Consider

$$G = D' \otimes TT^-_k - (\{s_k : \; s \in S\} \cup \{t_1 : \; t \in T\} \cup \{z_1, z_k : \; z \in V(D) - (S \cup T)\}).$$

By Theorem 2.1 and the above observations, $HOM(D, H) \neq \emptyset$ if and only if the number of vertices in a largest independent set of $G$ equals $|V(D)|$. Moreover, a homomorphism $h \in HOM(D', TT^-_k)$ is of maximum cost if and only if $I = \{x_{h(x)} : \; x \in V(D)\}$ is an independent set of maximum cost.

If we prove that $G$ is a comparability graph, we will be able to find a largest independent set in $G$ and an independent set of $G$ of maximum cost in polynomial time. So, it remains to find an orientation of $G$, which is a transitive digraph.

Define an orientation $G^*$ of $G$ as follows: $A(G^*) = A_1 \cup A_2 \cup A_3$, where

$$A_1 = \{y_iz_j : \; yz \in A(D'), \; i \geq j\}, \quad A_2 = \{x_ix_j : \; x \in V(D), \; i > j\},$$

$$A_3 = \{tksl : \; st \in A(D), \; s \in S, \; t \in T\}.$$ 

Let $u_iv_j, v_jw_l \in A(G^*)$. We will prove that $u_iw_l \in A(G^*)$. Consider three cases covering all possibilities without loss of generality.
Case 1: $u = v = w$. Since $u_iu_j, u_ju_l \in A_2$, we have $i > j > l$ and $u_iu_l \in A_2$.

Case 2: $u = v \neq w$. Then $u_iu_j \in A_2$, $u_jw_l \in A_1$ and, thus, $i > j \geq l$ and $uw \in A(D')$ implying $u_iw_l \in A_1$.

Case 3: $|\{u, v, w\}| = 3$. Since $u_iv_j, v_jw_l \in A_1$, we have $uv, vw \in A(D')$ and thus $uw \in A(D')$. Also, $i \geq j \geq l$. We conclude that $u_iw_l \in A_1$. ☐

Lemma 2.4 Suppose that $\text{MinHOMP}(H)$ is polynomial time solvable then, for each extension $H'$ of $H$, $\text{MinHOMP}(H')$ is also polynomial time solvable.

Proof: Recall that we can obtain $H'$ from $H$ by replacing every vertex $i \in V(H)$ with a set $S_i$ of independent vertices. Consider an $H'$-coloring $h'$ of an input digraph $D$. We can reduce $h'$ into an $H$-coloring of $D$ as follows: if $h'(u) \in S_i$, then $h(u) = i$.

Let $u \in V(D)$. Assign $\min\{c_j(u): j \in S_i\}$ to be a new cost $c_i(u)$ for each $i \in V(H)$. Observe that we can find an optimal $H$-coloring $h$ of $D$ with the new costs in polynomial time and transform $h$ into an optimal $H'$-coloring of $D$ with the original costs using the obvious inverse of the reduction described above. ☐

In [5], we proved that $\text{MinHOMP}(H)$ is polynomial time solvable when $H = \tilde{C}_k$, $k \geq 2$. Combining this results with Theorems 2.2, 2.3 and Lemma 2.4 we immediately obtain the following:

Theorem 2.5 If $H$ is an extension of $TT_k$, $\tilde{C}_k$ or $TT_k^{-}$ ($k \geq 3$), then $\text{MinHOMP}(H)$ and $\text{MaxHOMP}(H)$ are polynomial time solvable.

Theorem 2.6 Let $H$ be an acyclic bipartite tournament. Then $\text{MinHOMP}(H)$ and $\text{MaxHOMP}(H)$ are polynomial time solvable.

Proof: A pair of vertices in $H$ is called similar if they have the same set of out-neighbors and the same set of in-neighbors; $H$ is simple if it has no similar vertices. Let $V_1, V_2$ be the partite sets of $H$, let $D$ be an input digraph and let $c_i(x)$ be the costs, $i \in V(H), x \in V(D)$. Observe that if $D$ is not bipartite, then $\text{HOM}(D, H) = \emptyset$, so we may assume that $D$ is bipartite. We can check whether $D$ is bipartite in polynomial time using, e.g., the breadth first search [3]. Let $U_1, U_2$ be the partite sets of $D$.

To prove that we can find a minimum cost $H$-coloring of $D$ in polynomial time, it suffices to show that we can find a minimum cost $H$-coloring $f$ of $D$ such that $f(U_1) = V_1$ and $f(U_2) = V_2$. Indeed, if $D$ is connected, to find a minimum cost $H$-coloring of $D$ we can choose from a minimum cost $H$-coloring $f$ with $f(U_1) = V_1$ and $f(U_2) = V_2$ and a minimum cost $H$-coloring $h$ of $D$ with $h(U_1) = V_2$ and $h(U_2) = V_1$. If $D$ is not connected, we can find a minimum cost $H$-coloring of each component of $D$ separately.

To force $f(U_1) = V_1$ and $f(U_2) = V_2$ for each $H$-coloring $f$, it suffices to modify the costs such that it is too expensive to assign any color from $V_j$ to a vertex in $U_{3-j}$, $j = 1, 2$. 

5
Let $M = |V(D)| \cdot \max\{c_i(x) : i \in V(H), x \in V(D)\} + 1$ and let $c_i(x) := c_i(x) + M$ for each pair $x \in U_j, i \in V_{3-j}, j = 1, 2$.

We consider the following two cases.

**Case 1:** $H$ is simple. Observe that the vertices of $H$ can be labeled $i_1, i_2, \ldots, i_p$ such that $i_k$ is the only vertex of in-degree zero in $H - \{i_1, i_2, \ldots, i_{k-1}\}$. Thus, $H$ is a spanning subdigraph of $TT_p$ with vertices $i_1, i_2, \ldots, i_p$ ($i_si_t \in A(TT_p)$ if and only if $s < t$). Observe that

$$\{f \in HOM(D, H) : f(U_j) = V_j, j = 1, 2\} = \{f \in HOM(D, TT_p) : f(U_j) = V_j, j = 1, 2\}.$$ 

Thus, to solve $\text{MinHOMP}(H)$ with the modified costs it suffices to solve $\text{MinHOMP}(TT_p)$ with the same costs. We can solve the latter in polynomial time by Theorem 2.2.

**Case 2:** $H$ is not simple. Then $H$ is an extension of an acyclic simple bipartite tournament. Thus, we are done by Case 1 and Lemma 2.4.

\[\diamond\]

### 3 Classification for semicomplete $k$-partite digraphs, $k \geq 3$

The following lemma allows us to prove that $\text{MaxHOMP}(H)$ and $\text{MinHOMP}(H)$ are NP-hard when $\text{MaxHOMP}(H')$ and $\text{MinHOMP}(H')$ are NP-hard for an induced subdigraph $H'$ of $H$.

**Lemma 3.1** [5] Let $H'$ be an induced subdigraph of a digraph $H$. If $\text{MaxHOMP}(H')$ is NP-hard, then $\text{MaxHOMP}(H)$ is also NP-hard.

The following lemma is the NP-hardness part of the main result in [5].

**Lemma 3.2** Let $H$ be a semicomplete digraph containing a cycle and let $H \not\in \{\vec{C}_2, \vec{C}_3\}$. Then $\text{MinHOMP}(H)$ and $\text{MaxHOMP}(H)$ are NP-hard.

The following lemma was proved in [7].

**Lemma 3.3** Let $H_1$ be a digraph obtained from $\vec{C}_3$ by adding an extra vertex dominated by two vertices of the cycle and let $H$ be $H_1$ or its dual. Then $\text{HOMP}(H)$ is NP-complete.

We need two more lemmas for our classification.

**Lemma 3.4** Let $H'$ be given by $V(H') = \{1, 2, 3, 4\}, A(H') = \{12, 23, 34, 14, 24\}$ and let $H$ be $H'$ or its dual. Then $\text{MaxHOMP}(H)$ and $\text{MinHOMP}(H)$ are NP-hard.

6
Proof: We reduce the maximum independence set problem (MISP) to MinHOMP($H$).
Let $G$ be an arbitrary graph. We construct a digraph $D$ from $G$ as follows: every vertex of $G$ belongs to $D$ and, for each pair $x, y$ of non-adjacent vertices of $G$, we add to $D$ new vertices $u = u(x, y)$ and $v = v(x, y)$ together with arcs $ux, uv, vy$. (No edge of $G$ is in $D$.) Let $n$ be the number of vertices in $D$. Let $x, y$ be a non-adjacent pair of vertices in $G$ and let $u = u(x, y), v = v(x, y)$. We set $c_3(x) = c_3(y) = c_4(u) = c_4(v) = 1$ for $i = 1, 2, 3,$ $c_4(x) = c_4(y) = n + 1$ and $c_1(x) = c_2(y) = c_4(u) = c_4(v) = n^2 + n + 1$ for $j = 1, 2$.

Consider a minimum cost $H$-coloring $h$ of $D$. Let $x, y$ be a pair of non-adjacent vertices in $G$ and let $u, v$ be the vertices added to $D$ due to $x, y$ such that $ux, uv, vy$ are arcs of $D$. Due to the values of the costs, $h$ can assign $x, y$ only colors 3 and 4 and $u, v$ only colors 1, 2, 3. The coloring can assign $u$ either 1 or 2. If $u$ is assigned 1, then $v, y, x$ must be assigned 2, 3 and 4, respectively. If $u$ is assigned 2, then $v, y, x$ must be assigned 3, 4 and (3 or 4), respectively. In both cases, only one of the vertices $x$ and $y$ can receive color 3. Since $h$ is optimal, the maximum number of vertices in $D$ that it inherited from $G$ must be assigned color 3. This number is the maximum number of independent vertices in $G$. Since MISP is NP-hard, so is MinHOMP($H$).

Lemma 3.5 Let $H$ be given by $V(H) = \{1, 2, 3, 4\}$, $A(H) = \{12, 23, 31, 34, 41\}$. Then MaxHOMP($H$) and MinHOMP($H$) are NP-hard.

Proof: We will reduce the maximum independent set problem to MinHOMP($H$). However before we do this we consider a digraph $D_{gadget}(u, v)$ defined as follows: $V(D_{gadget}(u, v)) = \{x, y, u', u, v', v, c_1, c_2, \ldots , c_{12}\}$ and

\[A(D_{gadget}(u, v)) = \{xy, xc_1, yc_6, u'u, c_6u', u'v', v'c_1, c_2c_3, c_3c_4, \ldots , c_{11}c_{12}, c_{12}c_1\}\]

Observe that in any homomorphism $f$ of $D_{gadget}(u, v)$ to $H$ we must have $f(c_1) = 1$, by the existence of $x$ and $y$. This implies that $(f(c_1), f(c_2), \ldots , f(c_{12}))$ has to coincide with one of the following two sequences:

\[(1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 2, 3) \text{ or } (1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4)\]

If the first sequence is the actual one, then we have $f(c_6) = 3, f(u') \in \{1, 4\}, f(u) \in \{1, 2\}, f(c_{11}) = 2, f(v') = 3$ and $f(v) \in \{1, 4\}$. If the second sequence is the actual one, then we have $f(c_6) = 2, f(u') = 3, f(u) \in \{1, 4\}, f(c_{11}) = 3, f(v') \in \{1, 4\}$ and $f(v) \in \{1, 2\}$. So in both cases we can assign both of $u$ and $v$ color 1. Furthermore by choosing the right sequence we can color one of $u$ and $v$ with color 2 and the other with color 1. However we cannot assign color 2 to both $u$ and $v$ in a homomorphism.

Let $G$ be a graph. Construct a digraph $D$ as follows. Start with $V(D) = V(G)$ and, for each edge $uv \in E(G)$, add a distinct copy of $D_{gadget}(u, v)$ to $D$. Note that the vertices in $V(G)$ form an independent set in $D$ and that $|V(D)| = |V(G)| + 16|E(G)|$.

Let all costs $c_i(t) = 1$ for $t \in V(D)$ apart from $c_1(p) = 2$ for all $p \in V(G)$. Clearly, a minimum cost $H$-coloring $h$ of $D$ must aim at assigning as many vertices of $V(G)$ in $D$ a
color different from 1. However, if \( pq \) is an edge in \( G \), by the arguments above, \( h \) cannot assign colors different from 1 to both \( p \) and \( q \). However, \( h \) can assign colors different from 1 to either \( p \) or \( q \) (or neither). Thus, a minimum cost homomorphism of \( D \) to \( H \) corresponds to a maximum independent set in \( G \) and vice versa (the vertices of a maximum independent set are assigned color 2 and all other vertices in \( V(G) \) are assigned color 1).

\[ \diamond \]

**Theorem 3.6** Let \( H \) be a semicomplete \( k \)-partite digraph, \( k \geq 3 \). If \( H \) is an extension of \( TT_k \), \( C_3^\circ \) or \( TT_p^- \) (\( p \geq 4 \)), then \( \text{MinHOMP}(H) \) and \( \text{MaxHOMP}(H) \) are polynomial time solvable. Otherwise, \( \text{MinHOMP}(H) \) and \( \text{MaxHOMP}(H) \) are NP-hard.

**Proof:** We assume that \( P \neq \text{NP} \) as otherwise this theorem is of no interest. Since \( H \) is a semicomplete \( k \)-partite digraph, \( k \geq 3 \), if \( H \) has a cycle, then there can be three possibilities for the length of a shortest cycle in \( H \): 2, 3 or 4. Thus, we consider four cases, the above three cases and the case when \( H \) is acyclic.

**Case 1:** \( H \) has a 2-cycle \( C \). Let \( i, j \) be vertices of \( C \). The vertices \( i, j \) together with a vertex from a partite set different from those where \( i, j \) belong to form a semicomplete digraph with a 2-cycle. Thus, by Lemmas 3.1 and 3.2 \( \text{MinHOMP}(H) \) and \( \text{MaxHOMP}(H) \) are NP-hard.

**Case 2:** A shortest cycle \( C \) of \( H \) has three vertices \( i, j, l \) (\( ij, jl, li \in A(C) \)). If \( H \) has at least four partite sets, then \( \text{MinHOMP}(H) \) and \( \text{MaxHOMP}(H) \) can be shown be be NP-hard similarly to Case 1. Assume that \( H \) has three partite sets and that \( \text{MinHOMP}(H) \) and \( \text{MaxHOMP}(H) \) are polynomial time solvable. Let \( V_1 \), \( V_2 \) and \( V_3 \) be partite sets of \( H \) such that \( i \in V_1 \), \( j \in V_2 \) and \( l \in V_3 \). Let \( s \) be a vertex outside \( C \) and let \( s \in V_1 \). If \( s \) is dominated by \( j \) and \( l \) or dominates \( j \) and \( l \), then \( \text{MinHOMP}(H) \) and \( \text{MaxHOMP}(H) \) are NP-hard by Lemmas 3.1 and 3.3 a contradiction. If \( j \rightarrow s \rightarrow k \), then \( \text{MinHOMP}(H) \) and \( \text{MaxHOMP}(H) \) are NP-hard by Lemmas 3.1 and 3.3 a contradiction. Thus, \( k \rightarrow s \rightarrow j \). Similar arguments show that \( k \rightarrow V_1 \rightarrow j \). Let \( p \in V_2 - \{j\} \). Similar arguments show that \( p \rightarrow V_1 \rightarrow j \) and moreover \( V_3 \rightarrow V_1 \rightarrow j \). Again, similarly we can prove that \( V_3 \rightarrow V_1 \rightarrow V_2 \), i.e., \( H \) is an extension of \( C_3^\circ \).

**Case 3:** A shortest cycle \( C \) of \( H \) has four vertices \( i, j, s, t \) (\( i \rightarrow j \rightarrow s \rightarrow t \rightarrow i \)). Since \( C \) is a shortest cycle, \( i, s \) are belong to the same partite set, say \( V_1 \), and \( j, t \) belong to the same partite set, say \( V_2 \). Since \( H \) is not bipartite, there is a vertex \( l \) belonging to a partite set different from \( V_1 \) and \( V_2 \). Since \( H \) has no cycle of length 2 or 3, either \( l \) dominates \( V(C) \) or \( V(C) \) dominates \( l \). Consider the first case \( (l \rightarrow V(C)) \) as the second one can be tackled similarly. Let \( H' \) is the subdigraph of \( H \) induced by the vertices \( l, i, j, s \). Observe now that \( \text{MinHOMP}(H) \) and \( \text{MaxHOMP}(H) \) are NP-hard by Lemmas 3.1 and 3.3.

**Case 4:** \( H \) has no cycle. Assume that \( \text{MinHOMP}(H) \) and \( \text{MaxHOMP}(H) \) are polynomial time solvable, but \( H \) is not an extension of an acyclic tournament. The last assumption implies that there is a pair of nonadjacent vertices \( i, j \) and a distinct vertex \( l \) such that
Let $s$ be vertex belonging to a partite set different from the partite sets where $i$ and $l$ belong to. Without loss of generality, assume that at least two vertices in the set $\{i, j, l\}$ dominate $s$. If all three vertices dominate $s$, then by Lemmas 3.4 and 3.3, $\text{MinHOMP}(H)$ and $\text{MaxHOMP}(H)$ are NP-hard, a contradiction. Since $H$ is acyclic, we conclude that $\{i, l\} \rightarrow s \rightarrow j$. Let $V_1$ be the partite set of $i$ and $j$. Similar arguments show that for each vertex $t \in V(H) - V_1$, $i \rightarrow t \rightarrow j$. By considering a vertex $p \in V_1 - \{i, j\}$ and using arguments similar to the once applied above, we can show that either $p \rightarrow (V(H) - V_1)$ or $(V(H) - V_1) \rightarrow p$. This implies that we can partition $V_1$ into $V_1'$ and $V_1''$ such that $V_1' = (V(H) - V_1) \rightarrow V_2'$. This structure of $H$ implies that there is no pair $a, b$ of nonadjacent vertices in $V(D) - V_1$ such that $a \rightarrow c \rightarrow b$ for some vertex $c \in V(H)$. Thus, the subdigraph $H - V_1$ is an extension of an acyclic tournament and, therefore, $H$ is an extension of $TT_k^{-}$.

4 Classification for bipartite tournaments

The following lemma can be proved similarly to Lemma 3.3.

**Lemma 4.1** Let $H_1$ be given by $V(H_1) = \{1, 2, 3, 4, 5\}$, $A(H_1) = \{12, 23, 34, 41, 15, 35\}$ and let $H$ be $H_1$ or its dual. Then $\text{MaxHOMP}(H)$ and $\text{MinHOMP}(H)$ are NP-hard.

Now we can obtain a dichotomy classification for $\text{MaxHOMP}(H)$ and $\text{MinHOMP}(H)$ when $H$ is a bipartite tournament.

**Theorem 4.2** Let $H$ be a bipartite tournament. If $H$ is acyclic or an extension of a 4-cycle, then $\text{MinHOMP}(H)$ and $\text{MaxHOMP}(H)$ are polynomial time solvable. Otherwise, $\text{MinHOMP}(H)$ and $\text{MaxHOMP}(H)$ are NP-hard.

**Proof:** If $H$ is an acyclic bipartite tournament or an extension of a 4-cycle, then $\text{MinHOMP}(H)$ is polynomial time solvable by Theorems 2.6 and 2.5. We may thus assume that $H$ has a cycle $C$, but $H$ is not an extension of a cycle. We have to prove that $\text{MinHOMP}(H)$ and $\text{MaxHOMP}(H)$ are NP-hard.

Let $C$ be a shortest cycle of $H$. Since $H$ is a bipartite tournament, we note that $|V(C)| = 4$, so assume without loss of generality that $C = w_1w_2w_3w_4w_1$, where $w_1, w_3$ belong to a partite set $V_1$ of $H$ and $w_2, w_4$ belong to the other partite set $V_2$ of $H$.

Observe that any vertex in $V_1$ must dominate either $w_2$ or $w_4$ and must be dominated by the other vertex in $\{w_2, w_4\}$, as otherwise we are done by Lemmas 4.1 and 3.3. Analogously any vertex in $V_2$ must dominate exactly one of the vertices in $\{w_1, w_3\}$. Therefore, we may partition the vertices in $H$ into the following four sets.

\[
W_1 = \{v_1 \in V_1 : w_4 \rightarrow v_1 \rightarrow w_2\} \quad W_2 = \{v_2 \in V_2 : w_1 \rightarrow v_2 \rightarrow w_3\} \\
W_3 = \{v_3 \in V_1 : w_2 \rightarrow v_3 \rightarrow w_4\} \quad W_4 = \{v_4 \in V_2 : w_3 \rightarrow v_4 \rightarrow w_1\}
\]
If $q_2q_1 \in A(H)$, where $q_j \in W_j$ for $j = 1, 2$ then we are done by Lemma 4.1 (consider the cycle $q_1w_2w_3w_4q_1$ and the vertex $q_2$ which dominates both $q_1$ and $w_3$). Thus, $W_1 \rightarrow W_2$ and analogously we obtain that $W_2 \rightarrow W_3 \rightarrow W_4 \rightarrow W_1$, so $H$ is an extension of a cycle, a contradiction. \hfill \diamond$

To find a complete dichotomy for the case of semicomplete bipartite digraphs, one would need, among other things, to solve an open problem from [6]: establish a dichotomy classification for the complexity of MinHOMP$(H)$ when $H$ is a bipartite (undirected) graph. Indeed, let $B$ be a semicomplete bipartite digraph with partite sets $U, V$ and arc set $A(B) = A_1 \cup A_2$, where $A_1 = U \times V$ and $A_2 \subseteq V \times U$. Let $B'$ be a bipartite graph with partite sets $U, V$ and edge set $E(B') = \{uv : vu \in A_2\}$. Observe that MinHOMP$(B)$ is equivalent to MinHOMP$(B')$.

**Acknowledgement** Research of Gutin and Rafiey was supported in part by the IST Programme of the European Community, under the PASCAL Network of Excellence, IST-2002-506778.

**References**

[1] J. Bang-Jensen and G. Gutin, *Digraphs: Theory, Algorithms and Applications*, Springer-Verlag, London, 2000.

[2] J. Bang-Jensen, P. Hell and G. MacGillivray, The complexity of colouring by semicomplete digraphs. *SIAM J. Discrete Math.* 1 (1988), 281–298.

[3] T.H. Cormen, C.E. Leiserson, R.L. Rivest and C. Stein, *Introduction to Algorithms*, MIT, Cambridge, MA, 2nd Ed., 2001.

[4] M. Grötschel, L. Lovász, and A. Schrijver, Polynomial algorithms for perfect graphs. *Ann. Discrete Math.* 21 (1984), 325–356.

[5] G. Gutin, A. Rafiey and A. Yeo, Minimum Cost and List Homomorphisms to Semicomplete Digraphs. Submitted.

[6] G. Gutin, A. Rafiey, A. Yeo and M. Tso, Level of repair analysis and minimum cost homomorphisms of graphs. To appear in *Discrete Appl. Math.*

[7] W. Gutjahr, Graph colourings, PhD Thesis, Free University Berlin, 1991.

[8] P. Hell, Algorithmic aspects of graph homomorphisms, in ‘Survey in Combinatorics 2003’, London Math. Soc. Lecture Note Series 307, Cambridge U. Press, 2003, 239 - 276.

[9] P. Hell and J. Nešetřil, On the complexity of $H$-colouring. *J. Combin. Theory B* 48 (1990) 92-110.
[10] P. Hell and J. Nešetřil, *Graphs and Homomorphisms*. Oxford U. Press, Oxford, 2004.

[11] D. Kagaris and S. Tragoudas, Maximum weighted independent sets on transitive graphs and applications. *Integration, the VLSI journal* 27 (1999), 77–86.