Magnetic operations: a little fuzzy mechanics?

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Received 31 January 2011
Accepted for publication 16 August 2011
Published 13 September 2011
Online at stacks.iop.org/PhysScr/84/045008

Abstract

We examine the behaviour of charged particles in homogeneous, constant and/or oscillating magnetic fields in the non-relativistic approximation. A special role of the geometric centre of the particle trajectory is elucidated. In the quantum case, it becomes a ‘fuzzy point’ with non-commuting coordinates, an element of non-commutative geometry that enters into the traditional control problems. We show that its application extends beyond the usually considered time-independent magnetic fields of the quantum Hall effect. Some simple cases of magnetic control by oscillating fields cause the stability maps to differ from the traditional Strutt diagram. The elementary mathematical results help explain the structure of the obtained solutions.

PACS numbers: 03.65.Sq, 42.50.Dv

(Some figures in this article are in colour only in the electronic version.)

1. Introduction

Present-day quantum theories offer some visions of a new, mathematically possible reality that have not been experimentally detected so far. Thus, the idea of supersymmetry looks for particles in superposed states of ‘being fermion’ and ‘being boson’ and for the corresponding invariance group, with consequences for elementary particles, strings and quantum cosmology [1–6]. Other models propose to avoid the field singularities by assuming that the very structure of the physical space forbids exact point localizations. In some of them (Connes et al [7], Madore [8] and Bellisard et al [14]), the physical points have non-commuting coordinates [7–19]. According to Doplicher et al [20, 21], this might explain why exact position measurements in Heisenberg’s microscope cannot produce microscopic black holes (even if the reality of such a danger is an open problem). In the simplest model of the non-commutative plane (Madore [8]), the points $(x, y)$ fulfil $[x, y] = i\kappa \neq 0$, preventing as well the singularity creation.

Although all these ideas are just free associations (in an almost psychoanalytic sense), no better justified is the traditional hypothesis that our space is indeed a continuum of the ‘exact points’. In fact, if adopting Wigner’s observation about the ‘unreasonable power of mathematics’ [22], the mathematical models, if correct, must appear somewhere in nature. This indeed happens, but with one amendment. Nature might ‘use’ our models in its own way, without caring about the author’s intentions. Thus, while the problem of boson–fermion supersymmetry waits to be solved (still no trace of the Higgs boson, no gravitinos, etc), the same mathematical structure appeared in supersymmetric quantum mechanics (Witten [23, 24]) helping to solve exactly a class of spectral problems (Duplij et al [6]). While there is still no sign of strings, branes or extra dimensions, analogous mathematical structures permit us to understand certain biological phenomena better. In fact, the ‘clones’ of almost all unborn structures start invading the present-day physics forming a little-great science in its own right (see ‘big-bang in test tube’, etc [25, 26]).

Below, we shall discuss the similar status of the ‘non-commutative points’ that appear in the quantum Hall effect. We shall show that the idea works not only for the static fields but also for time-dependent ones, even though the more
fundamental hypothesis concerning the granular structure of the space itself is still waiting to be confirmed (or forgotten?).

This paper is organized as follows. In sections 2 and 3, we outline the familiar phenomena of circular and drifting motions in homogeneous, constant magnetic fields. Although extremely simple, they form an explicit (or implicit) basis for more sophisticated approaches.

The less known phenomena of non-circular closed evolution trajectories (‘evolution loops’) caused by time-dependent quadratic Hamiltonians and/or magnetic fields are considered in sections 4 and 5. It is shown that the loop centres form again a simple but realistic model for some elementary facts of non-commutative geometry. If non-vanishing, they show a drifting behaviour generalizing that of strictly circular motions (see proposition 1). In section 6 we illustrate this effect for Schrödinger’s particle in a periodic pattern of rectangular magnetic pulses, proving that the non-commutative structure of the loop centres can be experimentally controlled by the length and intensities of the applied pulses.

In section 7, we describe the control of charged particles by soft magnetic pulses of cylindrical symmetry, modulated by one or two sinusoidally varying amplitudes, granting the existence of some highly symmetric but non-circular evolution loops. In contrast to the circular cases described in sections 2 and 3, they resist drifting under the influence of constant external forces. Some analogies to the anomalous in sections 6 evolution loops. In contrast to the circular cases described for some elementary facts of non-commutative geometry. If non-vanishing, they show a drifting behaviour generalizing that of strictly circular motions (see proposition 1). In section 6 we illustrate this effect for Schrödinger’s particle in a periodic pattern of rectangular magnetic pulses, proving that the non-commutative structure of the loop centres can be experimentally controlled by the length and intensities of the applied pulses.

In section 7, we describe the control of charged particles by soft magnetic pulses of cylindrical symmetry, modulated by one or two sinusoidally varying amplitudes, granting the existence of some highly symmetric but non-circular evolution loops. In contrast to the circular cases described in sections 2 and 3, they resist drifting under the influence of constant external forces. Some analogies to the anomalous electric resistance of the electron gas in two dimensions (2D) might be considered. We also show that on the thresholds of the stability zones, the unitary evolution operators generated by our soft, sinusoidal pulses, during the field periods, can imitate the results on the δ kicks of the external (oscillator) potential or else can cause state transformation inverse to the free evolution. Applications to the new techniques of quantum control seem imminent.

2. Charged particles in homogeneous magnetic fields: drifting trajectories

We start from the well-known facts. The time-independent magnetic fields \( \mathbf{B} \) in an open domain of \( \mathbb{R}^3 \) can be described by a class of the vector potentials \( \mathbf{A}(x) \) with \( \mathbf{B}(x) = \text{rot} \mathbf{A} = \nabla \times \mathbf{A}(x) \). If \( \mathbf{B}(x) \) is homogeneous, \( \mathbf{B} = (0, 0, B) \) (for convenience, let \( \mathbf{B} \) define the \( z \)-axis in \( \mathbb{R}^3 \)), then one of the natural choices of \( \mathbf{A}(x) \) is

\[
\mathbf{A}(x) = \frac{1}{2} \mathbf{B} \times \mathbf{x} = \frac{1}{2} B \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix},
\]

interpretable as the vector potential created by a homogeneous current density on the cylindrical surface. The particle motion along the \( z \)-direction is then free; hence, we shall be interested only in the motion trajectories on the \( x \)- and \( y \)-planes. In the non-relativistic approximation, the Hamiltonian of a classical point particle of charge \( e \) and mass \( m \) is

\[
H_\beta = \frac{1}{2m} \left( \mathbf{p} - \frac{e}{c} \mathbf{A} \right)^2 = \frac{1}{2m} \left[ (p_x + \beta y)^2 + (p_y - \beta x)^2 \right],
\]

or equivalently

\[
H_\beta = \frac{1}{2m} \left[ \mathbf{p}^2 + \beta \mathbf{e}^2 \mathbf{x}^2 \right] - (\beta/m) M_\beta, \tag{3}
\]

where \( \beta = \frac{eB}{2mc}; \quad \mathbf{x} = (x, y) \) and \( \mathbf{p} = (p_x, p_y) \) are the generalized momenta, and \( M_\beta = \mu p_x - yp_y \). The well-known shape of the motion trajectories is most easily derived from the Hamiltonian in the form (2). The first pair of canonical equations defines the interrelation of the generalized and kinetic momenta \( \mathbf{m} \mathbf{v} \):

\[
\frac{dx}{dt} = \frac{\partial H_\beta}{\partial p_x} = \frac{1}{m} (p_x + \beta y) \Rightarrow p_x = \frac{m}{\beta} v_x - \beta y, \tag{4}
\]

\[
\frac{dy}{dt} = \frac{\partial H_\beta}{\partial p_y} = \frac{1}{m} (p_y - \beta x) \Rightarrow p_y = \frac{m}{\beta} v_y + \beta x
\]

and the second pair yields the proper dynamical equations:

\[
\frac{dp_x}{dt} = -\frac{\partial H_\beta}{\partial x} = \frac{\beta}{m} (p_y - \beta x), \quad \frac{dp_y}{dt} = -\frac{\partial H_\beta}{\partial y} = -\frac{\beta}{m} (p_x + \beta y). \tag{5}
\]

Both (4) and (5) immediately imply the existence of two conservative quantities:

\[
X = \frac{x}{2} + \frac{p_y}{2\beta} = x + \frac{mv_y}{2\beta} = x + \frac{v_y}{\omega}, \tag{6}
\]

\[
Y = \frac{y}{2} - \frac{p_x}{2\beta} = y - \frac{mv_x}{2\beta} = y - \frac{v_x}{\omega}, \tag{7}
\]

with \( \frac{d}{dt} X = \frac{d}{dt} Y = 0 \); hence,

\[
\frac{d}{dt} \begin{pmatrix} x - X \\ y - Y \end{pmatrix} = \frac{2\beta}{m} \begin{pmatrix} y - Y \\ -(x - X) \end{pmatrix}.
\]

so each charged particle just rotates around a fixed centre \( X = (X, Y) \) with a constant (cyclotron) frequency \( \omega = \frac{2eB}{mc} \):

\[
\begin{pmatrix} x(t) - X \\ y(t) - Y \end{pmatrix} = \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix} \begin{pmatrix} x(0) - X \\ y(0) - Y \end{pmatrix}. \tag{9}
\]

The conservative quantities \( X, Y \) represent hidden symmetries of the system [10, 13, 14]. The expression for \( X, Y \) in terms of the generalized momenta might seem peculiar, but is natural in terms of velocities (kinetic momenta) (4). It shows that the radius \( \rho \) of each rotating trajectory depends only on its (constant) velocity scalar \( v = |\mathbf{v}| = \sqrt{(v_x^2 + v_y^2)} \):

\[
\rho^2 = (x - X)^2 + (y - Y)^2 = \frac{v^2}{\omega^2}. \tag{10}
\]

However, to find the system response to the external forces, most convenient are expressions (6) and (7) in terms of \( p_x, p_y \). Indeed, if the circulating charge (2)–(9) is affected by an additional potential \( V(x) \), the Hamiltonian becomes \( \hat{H} = H_\beta + V(x) \) and since the motion centre is conserved by \( H_\beta \), the canonical equations for \( X, Y \) are reduced to

\[
\frac{dX}{dt} = \{X, V\} = \frac{1}{2} \left\{ \frac{p_x}{\beta}, V(x, y) \right\} = -\frac{1}{2\beta} \frac{\partial V}{\partial y} = \frac{1}{2\beta} F_y, \tag{11}
\]
In the simplest case, if the trajectory (9) is affected by a constant force $F$, the well-known though counter-intuitive effect is that the rotation centre starts to drift in the direction orthogonal to $F$. Curiously, the exact solutions of (12) exist also for the elastic potentials $V(x) = a|x|^2$. For $a < 0$, $|a| < \frac{F^2}{2}$, they illustrate the surprising phenomenon of a charged particle trapped by the repulsive centre (figure 1).

A less elementary form of the same effect, in the presence of crossed electric and magnetic fields, is discussed in [27], where the KAM theorem [28] is applied to show that the existence of an additional repellent obstacle (in the form of a disc) can interrupt the rectilinear drifting, trapping the charged particle, which, instead of being rejected, ‘obsessively returns’ to the repelling obstacle. Below, we shall not pretend to deepen this line of thought but, instead, we shall focus our attention on the corresponding quantum systems.

3. The quantum case: the ‘fuzzy points’

‘Pure quantum states are objective but not real’, Hans Primas, citing Heisenberg [33].

The quantum equivalents of (1)–(12) are so widely studied that we focus attention just on one particular aspect. We shall use the same symbol $H_\beta$ to denote the quantum Hamiltonian (2) and (3), although now $x_\beta$, $p_\beta$ will mean the quantum observables, $[x_\beta, p_\beta] = i\hbar$. By commutating the Hamiltonian (2) with $x_{\beta_1}$, $p_{\beta_1}$, one obtains the same two pairs of equations (4) and (5), although now they concern Heisenberg’s operators $x(t)$, $p_{\beta_{1}}(t)$, $y(t)$ and $p_{\beta_{2}}(t)$. In complete analogy to the classical case, the formulae (4) and (5) show the existence of a pair of conservative observables $X$, $Y$ given by (6) and (7).

The works on quantum systems (2) and (3) dedicate much care to the non-commuting kinetic momenta [7, 29]. It does not escapes attention either that the same phenomenon affects the coordinates of some abstract ‘space localizations’ such as the rotation centre of Heisenberg’s trajectory (see Avron et al [30] and Dodonov et al [31]):

\[
X = \frac{1}{2} x + \frac{1}{2\beta} p_y, \tag{13}
\]

\[
Y = \frac{1}{2} y - \frac{1}{2\beta} p_x. \tag{14}
\]

While the instantaneous particle coordinates $x(t)$, $y(t)$ at any fixed moment time of $t$ are still commuting observables, this is no longer true for the rotation centre $X = (X, Y)$, which becomes a ‘fuzzy point’ [8]:

\[
[X, Y] = -\frac{i}{\beta} - \frac{i}{m_0}. \tag{15}
\]

Note that ‘fuzzy localization’ is an abstract concept (resembling a ‘grin without the cat’ of Lewis Carroll [32]). There is simply nothing there, commuting or not. However, by paraphrasing Heisenberg [33], the fuzzy centre (13) and (14) is ‘not real but is objective’.

Another curious aspect, resembling some open gravity problems [34–36], concerns the ‘surface’ of the 2D orbits (9) and (10). Indeed

\[
\pi \rho^2 = \pi (x - X)^2 = \frac{\pi}{4\beta^2} [(p_x + \beta y)^2 + (p_y - \beta x)^2] = \frac{\pi}{(2\beta)^2} H_\beta. \tag{16}
\]

Hence, the surface of the orbit is not only conserved but also quantized: it is proportional to the Hamiltonian and cannot change continuously (for an equivalent observation, see e.g. [37]).

As it seems, this is not the first time the magnetic fields provide an imitation of still unchecked theories. The suggestive ideas of supersymmetry are still not verified in particle physics. However, due to the anomalous relation between the spin and orbital magnetic moments of the electron, its energy levels in a static, homogeneous magnetic field reproduce the supersymmetric spectrum.

Figure 1. Three generically similar phenomena. (a) The rectilinear drift along the $y$-axis under a constant force $F$ in the $x$-direction. (b) The circular motion trapped by a repulsive oscillator. (c) The precession of a gyroscope.
Hence, many authors conclude that ‘the supersymmetry exists in nature’ [38–41]. Quite similarly, there is no evidence that physical particles move in non-commuting spaces, but the properties of the rotation center (15) might support the statement that ‘the non-commutative positions exist in nature’.

It might thus be interesting to note that the phenomenon is not limited to the circular trajectories in static magnetic fields. It appears as well in more general physical scenarios, including time-dependent oscillator potentials and magnetic fields.

4. Time-dependent oscillators: non-circular loops

The evolution problem for variable oscillator potentials has a notable past. The first systematic studies were presented during 1967–69 by Lewis and Riesenfeld [42, 43] and then by Malkin and Man’ko [44], inspiring ample research on quantum evolution. The group theoretical approach to the Baker–Campbell–Hausdorff (BCH) problem [45–48] for quadratic Hamiltonians was noted by from the time-dependent mass (see the Hamiltonian of generalized oscillator or magnetic Hamiltonians that would parallel research, there was hardly any constant in the other authors; for an ample review, see Dodonov [49] (for a generalization, see Zhang et al [50]). In 1976, Yuen noted the possibility of solving the evolution equation for the squeezed photon states [51]. For massive charged particles, the operations induced by variable oscillators were studied by Ma and Rhodes [52], Royer [53], Brown and Carson [54], Combescure [55], Wolf [56] and other authors; for an ample review, see Dodonov [57].

The links with coherent states are immanent [58–61]. In parallel research, there was hardly any constant in the generalized oscillator or magnetic Hamiltonians that would not be replaced by its time-dependent analogue, starting from the time-dependent mass (see the Hamiltonian of Caldirola–Kanai [62–64]) up to the variable dielectric or magnetic permeability [65–68].

The possibility of closed non-circular trajectories for the time-dependent oscillator Hamiltonians was noted by Malkin and Man’ko [44]. The extremely simple cases of the closed evolution caused by sudden δ(t)-shocks of oscillator potentials are described in [69–72]. Thus, for Schrödinger’s particle in 1 space dimension the evolution loops can be produced by sequences of oscillator pulses and free evolution intervals corresponding to the elementary cases of the BCH formula, e.g.

\[
e^{-i\tau \frac{\partial}{\partial \tau}} e^{-i \frac{\tau}{2} \frac{\partial^2}{\partial \tau^2}} \cdots e^{-i \frac{\tau}{2} \frac{\partial^2}{\partial \tau^2}} e^{-i \tau \frac{\partial}{\partial \tau}} \equiv 1
\]  

we put \( h = m = 1 \), illustrated by the evolution diagram:

\[
\begin{array}{c}
\text{\( \tau \)} \\
\text{\( \theta \)} \\
\text{\( \sigma \)} \\
\end{array}
\]

where the vertices symbolize the shocks of the elastic potential (the corresponding numbers mean the pulse amplitudes) and the sides correspond to the ‘rest intervals’ of the free evolution. The equivalence sign \( \equiv \) in (17) means operator proportionality, i.e. \( U(\tau, \sigma) = U \exp(i\varphi) \). The many-vertex analogues of (17) as well as the more general ‘kicked systems’ and their non-singular equivalents are described in [70, 71, 73–81].

Some incomplete versions of the loop process may prove to be of interest. In fact, whenever an evolution loop contains a δ-pulse of the attractive oscillator potential, as in (17), the rest of the process must imitate the effect of a repulsive elastic pulse. Furthermore, if any evolution loop contains an interval of free evolution \( e^{-i\tau \frac{\partial}{\partial \tau}} (\tau > 0) \), it means that the rest must be equivalent to its inverse, thus suggesting the techniques of reverting the free propagation. The importance of the closed dynamical processes for the general control operations was recognized in [69, 72, 74, 75, 78].

All these models depend on some idealizations. Thus, the repulsive oscillator kicks cannot be straightforwardly simulated by the magnetic fields (in fact, even the attractive ones can hardly be engineered!). Moreover, to describe the time-dependent potentials in a few but macroscopic areas such as, e.g., an ion trap, one typically uses the non-relativistic laboratory approximation, disregarding the little delays needed to propagate the potential inputs all over the trap surfaces (telegraphist’s equations) or in its interior. Since these delays in typical experiments are insignificant, the laboratory approximation works very well and, indeed, is implicit in all papers postulating the time-dependent external parameters (e.g. [42, 44, 58, 65, 70, 71, 82]).

Once this approximation is adopted, one can see that the fuzzy centres are not restricted to the orthodox Hall effect with a fixed magnetic background. They arise for arbitrary evolution loops generated by the static or time-dependent quadratic Hamiltonians.

5. Loops of quadratic Hamiltonians: stability and drifting

To show this, we shall consider the Hermitian Hamiltonians \( H(\tau) \) defined by time-dependent quadratic forms \( H(\tau) = \sum_{i=1}^{2s} \sum_{j=1}^{2s} h_{ij}(\tau) q_i q_j \) (with \( h_{ij}(\tau) = h_{ji}(\tau) \in \mathbb{R} \)) of a complete set \( q_1, \ldots, q_{2s} \) of any number of the canonical observables, \( x_i, p_i \) \( (i = 1, \ldots, s) \) in a certain Hilbert space \( \mathcal{H} ; [x_i, x_j] = [p_i, p_j] = 0, [x_i, p_j] = i\hbar \delta_{ij} \). An agreeable property of the quadratic Hamiltonians is that even though \( H(\tau) \) are unbounded, if the coefficients \( h_{ij}(\tau) \) are non-singular and piecewise continuous, the corresponding unitary evolution operators \( U(\tau, \tau) \) are well defined by the operator equations:

\[
\frac{d}{d\tau} U(\tau, \tau) = -i H(\tau) U(\tau, \tau), \quad \frac{d}{d\tau} U(\tau, \tau) = i U(\tau, \tau) H(\tau),
\]  

with \( U(\tau, \tau) = 1 \) and

\[
U(\tau, \tau) U(\tau, \sigma) = U(\tau, \sigma)
\]  

(for detailed results and amendments, see Simon [83] and Hagedorn et al [84]). As is well known, for the quadratic \( H(\tau) \), the time-dependent Heisenberg’s observables \( x_i(\tau), p_i(\tau) \) are linear combinations of the initial \( x_i, p_i \). For
convenience, we shall denote by \( \mathbf{q} \) the vector-column of \( 2s \) (dimensionless) observables:

\[
\mathbf{q} = \begin{bmatrix} q_1 \\ \vdots \\ q_{2s} \end{bmatrix} = \begin{bmatrix} \mathbf{x} \\ \mathbf{p} \end{bmatrix}.
\]

(20)

One therefore has

\[
\mathbf{q}(t) = U(t, 0)^{-1} \mathbf{q} U(t, 0) = \upsilon(t) \mathbf{q},
\]

(21)

where \( \upsilon(t) = \upsilon(t, 0) \) is a dimensionless \( 2s \times 2s \) evolution matrix yielding simultaneously the classical and quantum trajectories.

If \( \mathbf{q} \) generates all observables in \( \mathcal{H} \), then not only does the evolution operator \( U(t) \) determine the matrix \( \upsilon(t) \), but conversely the matrix (21) determines the unitary \( U(t) \) up to a numerical phase factor. Indeed, should \( U \) and \( U' \) be two unitary operators generating the same transformation of the canonical variables \( \mathbf{q} \), i.e. \( U' \mathbf{q} U = U' \mathbf{q} U' \), then \( U' U^{-1} = \mathbf{q} U U' \mathbf{q}^{-1} \), and hence, \( U' U^{-1} \) would commute with all canonical variables \( \mathbf{x} \) and \( \mathbf{p} \) and their functions. However, if the algebra spanned by the canonical variables \( \mathbf{q} \) is irreducible in \( \mathcal{H} \), then any operator that commutes with all of them must be just a c-number. Since \( U \) and \( U' \) are unitary, this criterion can only be a phase factor, i.e. \( U' U^{-1} = e^{i \varphi} \) implying \( U' = e^{i \varphi} U \). The concrete value of \( \varphi \) is relevant for the transformation of the state vectors but not for that of Heisenberg’s variables, so we shall simply denote

\[
U' = e^{i \varphi} U \implies U' \equiv U.
\]

(22)

This equivalence becomes quite essential in quantum control. In fact, another agreeable property of the quadratic Hamiltonians is that the linear transformation defining the evolution of the canonical observables (i.e. the evolution matrix \( \upsilon(t) \)) is exactly the same on classical and quantum levels. Hence, the properties of the evolution operators \( U(t) \) can be read as well from the classical motion trajectories. This includes not only the average position and momenta of any wave packet (which follow the classical motion orbits), but also their quadratic deviations and higher order statistical moments, thus allowing non-perturbative solutions of the continuous BCH problem [49, 85–87] for the evolution (18). In particular, an evolution loop occurs if after a certain time interval (for convenience, let it be \( [0, T] \)) all canonical (classical and/or quantum) variables \( \mathbf{q} \) return to their initial values:

\[
\mathbf{q}(T) = U(T, 0)^{-1} \mathbf{q} U(T, 0) = \mathbf{q} \iff \upsilon(T) = \mathbb{1}_s,
\]

(23)

so that, in the sense of (22),

\[
U(T, 0) = e^{i \varphi} \mathbb{1} \equiv 1 \quad (\varphi \in \mathbb{R}),
\]

(24)

implying also a loop of all other observables \( \upsilon(t) \) that do not depend explicitly on time in Schrödinger’s frame, i.e. \( \upsilon(T) = U(T, 0)^{-1} \mathbf{A} U(T, 0) = \mathbf{A} \). The value of \( \varphi \in \mathbb{R} \) (i.e. the geometric phase for the loop process), though interesting in itself [72, 88], does not affect the results of our present argument.

Below, we shall consider a Hamiltonian \( H(t) \) varying periodically, i.e. \( H(t + T) = H(t) \), in a finite or infinite \( t \)-domain. We denote for simplicity \( U(t, 0) = U(T + t, T) = U(t) \). We assume, moreover, that the evolution produces a loop in \( [0, T] \) and in the subsequent periodicity intervals. Then, any Heisenberg’s observable \( \tilde{A}(t) = U(t)^{-1} \mathbf{A} U(t) \) in any periodicity interval \( [\tau, \tau + T] \) admits the time average

\[
\tilde{A} = \frac{1}{T} \int_{\tau}^{\tau + T} A(t) \, dt,
\]

(25)

independent of \( \tau \), defining a certain global characteristic of the loop process. (Indeed, the change of \( \tau \) in (25) means just that the same contributions \( A(t) \, dt \) are summed up along the same closed cycle, rearranging only the summation order.)

In particular, applying (25) for the time-dependent variables \( \upsilon_j(t) \), one obtains the coordinates \( \tilde{X}_j \) of the ‘loop centre’. For \( H(t) \) quadratic, they are as well linear in the initial \( q_1, \ldots, q_{2s} \):

\[
\tilde{X}_j = \mathbf{\tilde{X}}_j = \tilde{a}_{j1} q_1 + \cdots + \tilde{a}_{js} q_s \quad (j = 1, \ldots, s),
\]

(26)

generalizing the already described rotation centres \( \tilde{X}_j \) and \( \tilde{X}_{kl} \) in 2D. A more general concept might also be useful.

**Definition.** For the quadratic, periodic Hamiltonians, even if the evolution in \( [0, T] \) does not close to a loop, one might apply (26) defining the Floquet point \( \mathbf{X}(\tau) = \frac{1}{T} \int_0^T X(t) \, dt \) (although, in general, \( \mathbf{X}(\tau) \) will depend on \( \tau \)).

We shall show now that the behaviour of the loop affected by an additional force described in sections 2 and 3 is a typical phenomenon for all loop processes generated by any quadratic Hamiltonians.

**Proposition 1.** Suppose that a time-periodic Hamiltonian \( H(t) \equiv H(t + T) \), quadratic in the canonical observables \( \mathbf{q} \), generates a loop (24) in its periodicity intervals \( [nT, (n + 1)T], n = 0, 1, \ldots \). Then the precession or stability of the loop under an additional, constant force \( F \) depends on the loop centre \( \mathbf{X} = (X_1, \ldots, X_s) \). If its coordinates vanish, then the loop can change its shape but remains closed and stable. If the coordinates commute, then the centre is stable even if the trajectory is not. However, if \( \mathbf{X} \) is a ‘fuzzy point’ with non-commuting coordinates, \( [X_1, X_2] = i \kappa_{kl} \equiv 0 \), then the loop will show a drift with a constant velocity \( \sim v_\lambda = \kappa_{kl} F_l \) (summation convention) in a direction orthogonal to \( F \).

**Proof.** Suppose the perturbing force is \( F = (F_1, \ldots, F_s) \). The perturbed Hamiltonians then read

\[
\tilde{H}(t) = H(t) - F \mathbf{X},
\]

(27)

yielding the modified evolution operator \( \tilde{U}(t, 0) \) in the customary form

\[
\tilde{U}(t, 0) = U(t, 0) W(t),
\]

(28)

3 This does not mean that \( \tilde{A} \) is a constant of motion in a conventional sense. As an integral over the entire time interval \( [\tau, \tau + T] \), the observable \( \tilde{A} \) is not ‘local in time’. However, if the little evolution steps of \( A(t) \) in (25) obey their (different) instantaneous Hamiltonians, then (25) stays unchanged.
in which $U(t,0)$ is the evolution operator of the unperturbed loop (24), while $W(t)$ obeys the evolution equation in the interaction frame:

$$\frac{dW}{dt} = iFX(t)W(t); \quad W(0) = \mathbb{1},$$

(29)

where

$$x(t) = U(t,0)^{-1}xU(t,0)$$

(30)

are the time-dependent Heisenberg’s observables defined by the unspoiled loop evolution (18) and (24). Since the commutators $[x(t), x(t')]$ for any $t, t' \in \mathbb{R}$ are numbers, the operators $W(t)$ are given by the simplest case of the BCH formulae [85], [86], [87] in which only the integral of $x(t)$ matters:

$$W(t) = e^{i\chi(t)}e^{iF\int_0^t x(t')dt'},$$

(31)

and $\chi(t)$ is a real, c-number phase for each $t \in \mathbb{R}$. So,

$$\tilde{U}(T, 0) = U(T, 0)W(T) = e^{i\chi}e^{iTFX}, \quad \chi \in \mathbb{R},$$

(32)

implying

$$\tilde{U}(T, 0)^{\dagger}FXY\tilde{U}(T, 0) = e^{-iTFX}FXe^{iTFX} = FX,$$

(33)

but simultaneously

$$\tilde{U}(T, 0)^{\dagger}Y\tilde{U}(T, 0) = e^{-iTFY}FYe^{iTFX} = Y - iT[FX, Y]$$

$$= Y + \nu T,$$

(34)

for any $Y = n_{i}X_{i}$, where $\nu = n_{i}x_{i}F_{j}$ (summation convention). Hence, the (constant) external force $F$ cannot change the centre coordinate $FX$, but if $[FX, Y] \neq 0$ it can shift the combination $Y$ of the remaining ones. If the loop is $n$ times affected by the same $F$, then $FX$ is still unchanged, but $Y$ performs a cumulative drift:

$$\tilde{U}(nT, 0)^{\dagger}Y\tilde{U}(nT, 0) = W(T)^{n}YW(T)^{n} = e^{i\beta nTFX}Ye^{i\beta nTFX}$$

$$= Y + nTV,$$

(35)

with the same constant velocity $\nu$.

Note that in the above, it is not assumed that $T$ is the smallest period of $H(t)$, but only that it is a common period of $H(t)$ and of the loop phenomenon. In fact, a typical situation is that the loop occurs after several periods of $H(t)$ (compare section 7).

2. If $X$ identically vanishes, then $W(T)$ in (32) is just a phase factor and the loop is stable: it can be deformed but it will not be broken; neither will it precess under the influence of $F$.

The sense of this classification becomes obvious if one compares the traditional harmonic oscillator with the ‘magnetic oscillator’ $H_{\beta} (\beta \equiv \text{constant})$. Both admit circular orbits, however, the general elliptic orbit of the oscillator affected by an external force $F$ gets simply displaced in the direction of $F$ (where it remains stationary), while the orbit of $H_{\beta}$ starts drifting in the direction orthogonal to $F$. The key to this difference is the distinct nature of the classical/quantum centre $(X, Y)$ of both motions. While for the 2D oscillator the centre $X$ is exactly zero, the same centre for $H_{\beta}$ is a fuzzy point (13) and (14) implying the drifting trajectory.

In physical terms, the difference between both cases has some historical key. In fact, by reading (3) inversely: $H_{\text{osc}} = H_{\beta} + \frac{e}{2m}M_{c}$, one obtains a modern equivalent of an old idea: the description of the elliptic orbit of $H_{\text{osc}}$ as the superposition of two circular motions, for $H_{\beta}$ and $M_{c}$, i.e. the Ptolemaic picture of the oscillator trajectory. So, in a sense, ‘the epicycles are more stable than cycles’.

6. Landau fields: manipulating the fuzzy centre

As is already known, the magnetic operations can produce special effects such as rigid displacement, squeezing and distorted free evolution [29, 31, 57, 71]. Below, we shall be especially interested in the homogeneous magnetic fields in a fixed direction (e.g. of the $z$-axis), imitating the variable oscillator potentials in 2D [31, 70, 71, 77, 89]. The simplest physical conditions to approximate such fields arise in space domains surrounded by time-dependent currents, e.g. by solenoids of various forms (figures 2(a) and (b)).

If the surface currents do not depend on $z$, the motion in the $x, y$-plane decouples, leading to some typical vector potentials in 2D, e.g.

cylindrical: $A = \frac{B}{2} \begin{pmatrix} -y \\ x \end{pmatrix}$, \quad Landau: $A = B \begin{pmatrix} -y \\ 0 \end{pmatrix}$

(36)

If $B$ is static, both expressions (36) are exact and gauge equivalent. However, for variable $B = B(t)$ they still offer useful laboratory approximation, though neglecting the retarded field effects\(^4\). Below, we shall check the loop behaviour for the time-dependent quadratic Hamiltonian of Landau’s case:

$$H(t) = \frac{1}{2m} \left( p_{x} + \frac{eB(t)}{c}y \right)^{2} + p_{y}^{2}. $$

(37)

In the dimensionless variables with $t \to T$ and $\beta = \frac{eB(t)}{2mc}$ (where $T$ represent a time unit, $[x, p_{x}] = [y, p_{y}] = i$), it reads

$$H(t) = \frac{1}{2} \left( (p_{x} + 2\beta(t)y)^{2} + p_{y}^{2} \right).$$

(38)

\(^4\) In contrast to the case of the static $B$, the cylindrical and Landau potentials in (36) for variable $B(t)$ are no longer gauge equivalent, since they produce different electric fields, corresponding to distinct geometries of the distant sources; see Dodonov et al [31].
The canonical equations
\[
\frac{dx}{dt} = (p_x + 2\beta y), \quad \frac{dy}{dt} = p_y, \quad (39a)
\]
\[
\frac{dp_x}{dt} = 0, \quad \frac{dp_y}{dt} = -2\beta(p_x + 2\beta y). \quad (39b)
\]

Despite their apparent simplicity, require a computer study. We thus opted to approximate an arbitrary \(\beta(t)\) by a step function of \(n\) leading to the final result of \(2^n\) operations:

\[
\Gamma = \sum_{i=1}^{2n-1} \left( \frac{1}{\beta_i} - \frac{1}{\beta_{i+1}} \right)
\]

Within the loop condition \(\Gamma = 0\), the commutator (42) still admits various control options, producing the ‘fuzzy centres’ with distinct drifting capacities (see figure 3(a)). However, \([X, Y]\) may also vanish for some values of \(\beta\). This happens, e.g., for \(\beta_3 = -\beta_1, \beta_4 = -\beta_2\) when the quantum centre is non-trivial, but not fuzzy (see figure 3(b)), the case that cannot occur for the static field.

From the relativistic point of view, Hamiltonian (37) is inexact, since the retarded effects are missing. Note, however, that the non-relativistic Landau’s potential (36) with \(B = B(t)\) is the limit for \(\frac{1}{c} \to 0\) of an exact, relativistic expression in the form of a finite difference

\[
A(t, x) = \left[ \begin{array}{c} A(t, x) \\ 0 \end{array} \right], \quad A(t, x) = \frac{1}{2} \frac{G(t - \frac{x}{c}) - G(t + \frac{x}{c})}{\frac{x}{c}} \to -G'(t)y \quad (43)
\]

Figure 2. (a) Cylindrical geometry. (b) Landau geometry.

As an elementary example, we did it for four-step loops, obtaining

\[
[X, Y] = -\frac{i\pi}{4T} \left[ \frac{1}{\beta_1|\beta_1|} + \frac{1}{\beta_2|\beta_2|} + \frac{1}{\beta_3|\beta_3|} + \frac{1}{\beta_4|\beta_4|} \right]. \quad (42)
\]

Of two plane Landau pulses propagating in opposite directions, where \(G(t)\) is a continuous, real function with a bounded, piecewise continuous derivative \(G'(t) = B(t)\), modelling the time-dependent magnetic fields for \(\frac{1}{c} \to 0\). So, the non-relativistic Hamiltonian (37) with the time-dependent \(B(t)\) is indeed the first step of the Einstein–Infeld–Hoffman (EIH) approximation commonly used in general relativity to describe the slow motions in limited space domains [91]. In a laboratory of size \(\approx 1\ m\) the \(y\)-dependent delays do not exceed \(\delta t = \frac{10m}{c} \approx \frac{1}{8} \times 10^{-8}\ s\) and the approximation is almost perfect (errors invisible in our figures 3(a) and (b)). We thus conclude that in not too huge magnetic traps the ‘fuzzy centres’ of the Hamiltonian (37) describe correctly the principal part of the drifting mechanism.

7. Cylindrical geometry: the symmetry phenomena

In turn, we shall examine some curious effects induced by softly pulsating fields of cylindrical geometry. The
non-relativistic Hamiltonian becomes

\[ H(t) = \frac{1}{2m} \left[ p^2 + \left( \frac{eB(t)}{2c} \right)^2 x^2 \right] - \left( \frac{eB(t)}{2mc} \right) M, \]  

(44)

In contrast to the ‘homogeneous Schrödinger’s case’ (see [14]), equation (44) represents a 2D ‘Aristotelian world’, whose symmetry centre \( y = 0 \) is distinguished by the circular electric fields (see also [31]). In principle, \( B(t) \) might be arbitrary but, below, we shall be interested mostly in the harmonic and biharmonic fields:

\[ B(t) = B_0 + B_1 \sin(\omega t), \]

(45)

\[ B(t) = B_1 \sin(\omega t) + B_2 \sin(2\omega t), \]

(46)
of period \( T = \frac{2\pi}{\omega} \). Following the commonly applied approximation \([44, 57, 92]\) we adopt the semiclassical picture (Thomson rather than Compton [93]). Our design is quite unsophisticated compared with the time-dependent mass \([62, 63]\), the variable material constants \([67, 68]\) or non-trivial perturbative terms of Combescure [94]. However, what precisely happens with the microparticles in this simple scenario?

To describe ample classes of similar evolution processes, it is practical to introduce the dimensionless time, \( t' = \frac{t}{\sqrt{\pi}} \), field \( \beta = \frac{2\pi}{\sqrt{\pi}} B \) and canonical variables \( [x, p_x] = [y, p_y] = i \). The field oscillations now become

\[ \beta(t) = \beta_0 + \beta_1 \sin(2\pi t) + \beta_2 \sin(4\pi t), \]

(47)

where \( \beta_0 = 0 \) corresponds to the harmonic and \( \beta_0 = 0 \) to the biharmonic cases (45) and (46), respectively. The rescaled Hamiltonian is

\[ H(t) = \frac{1}{2} \left[ \left( p^2 + \beta(t)^2 x^2 \right)^2 - \beta(t) M \right] \]

(48)

with \( H_{\text{osc}} \) representing the time-dependent ‘magnetic oscillator’ and \( M \) the rotation generator. Since \( H_{\text{osc}}(t) \) and \( M \) commute, \( U(t) \) factorizes into two commuting unitary operators, \( U(t) = U_{\text{osc}}(t) U_{\text{rot}}(t) \), where

\[ \frac{dU_{\text{osc}}}{dt} = -iH_{\text{osc}}(t)U_{\text{osc}}(t), \quad U_{\text{osc}}(0) = \mathbb{1}, \]

(49)

while

\[ U_{\text{rot}}(t) = e^{-i\Delta t \beta(t) dq/\hbar M}, \]

(50)

produces just the rotations \( r(t) \) between the canonical pairs \( x, p_x \) and \( y, p_y \). The canonical transformation defining the evolution matrix \( u(t) \) can be split into two steps: \( q \rightarrow q_{\text{osc}}(t) \rightarrow q(t) \), implemented by \( U_{\text{osc}}(t) \) and \( U_{\text{rot}}(t) \), respectively. The operation \( U_{\text{osc}}(t) \) is reducible, affecting separately both canonical pairs \( x, p_x \) and \( y, p_y \), which evolve simultaneously according to the same \( 2 \times 2 \) matrix further denoted by \( b(t) \), i.e.

\[ U_{\text{osc}}(t) \begin{pmatrix} x \cr p_x \end{pmatrix} = \begin{pmatrix} x \cr p_x \end{pmatrix}, \quad U_{\text{osc}}(t) \begin{pmatrix} y \cr p_y \end{pmatrix} = \begin{pmatrix} y \cr p_y \end{pmatrix}, \]

(51)

and the same for \( y, p_y \). By differentiating both sides of (51) in agreement with (49) and using the canonical commutation rules, one sees that \( b(t) \) is determined by the differential matrix equation:

\[ \frac{db}{dt} = \Lambda(t)b(t), \quad \Lambda(t) = \begin{pmatrix} 0 & 1 \\ 0 & -\beta(t)^2 \end{pmatrix}; \quad b(0) = \mathbb{1}, \]

(52)

which is at the bottom of all quantum control problems for time-dependent oscillator Hamiltonians \([42–44, 53, 70, 71, 95]\). Once having \( b(t) \), one immediately constructs \( u_{\text{osc}}(t) \), as the simple pair of two \( b(t) \)-cells. In turn, multiplying \( u_{\text{osc}}(t) \) by the \( 4 \times 4 \) matrix \( r(t) \) rotating all variables by \( \gamma(t) = \left( \begin{array}{c} 0 \\ \beta(t) \end{array} \right) \) around the \( z \)-axis, one obtains the complete \( 4 \times 4 \) evolution matrix

\[ u(t) = r(t) \begin{pmatrix} x \cr p_x \cr y \cr p_y \end{pmatrix} = \begin{pmatrix} x \\ p_x \\ y \cr p_y \end{pmatrix}, \]

(53)

Here the role of \( b(t) \) is especially relevant for the periodically repeated field patterns, since the one-period-evolution step \( b(T) \) decides the bounded (stable) character of the motion, or its capacity to produce the parametric resonance \([96, 97]\). In our case (47), \( T = 1 \). Representing the canonical evolution, \( b(t) \) is simplectic, including the Floquet matrix \( b(1) \), which permits us to classify the motions. Since \( \text{Det}[b(1)] = 1 \), its eigenvalues depend on just a single (real) trace invariant. The characteristic equation

\[ D(\lambda) = \text{Det}(\lambda I - b(1)) = \lambda^2 - \lambda \text{ Tr } b(1) + 1 = 0 \]

(54)

has two non-vanishing roots

\[ \lambda_{\pm} = \frac{1}{2} \text{ Tr } b(1) \pm i\sqrt{\Delta}, \quad \Delta = 1 - \frac{1}{2}[\text{Tr } b(1)]^2. \]

(55)
Figure 4. The types of Floquet operations $U(1)$ generated by the harmonic case of (47) on the $(\beta_0, \beta_1)$ plane (the map of Delgado). The clear areas are the stability zones, in which the evolution loops can occur for $\text{Tr} b(1) = \pm 2 \cos \frac{2\pi}{n}$, under the subsidiary condition $\varepsilon(1) = 1$ (the curves $\text{Tr} b(1) = 0, \pm 1$ marked in red).

Figure 5. The classical trajectories in the $x, y$-plane illustrating the evolution loops. (a) For $\beta(t) = -\frac{\pi}{5} - 1.152 \sin (2\pi t)$, closing after 15 periods. In blue, the deformation under the constant force $F = (-1.5, 0)$. (b) The evolution loop for $\beta(t) = \frac{\pi}{8} - 0.815 \sin (2\pi t)$ closing after 24-field periods. In green, the deformation by $F = (1, 0)$.

with $\lambda_+\lambda_- = 1$, distinguishing three possible types of motion:

I. Stability area. If $|\text{Tr} b(1)| < 2$, then $\lambda_+, \lambda_-$ are two different complex eigenvalues with $|\lambda_+| = |\lambda_-| = 1$ (phase factors) of the form $\lambda_+ = e^{i\phi}, \lambda_- = e^{-i\phi}, \phi \in \mathbb{R}$. The $b(1)$ shows oscillating behaviour of $b(n) = b(1)^n, n \in \mathbb{N}$ defining the bounded trajectories.

II. The threshold (separatrix) is characterized by $|\text{Tr} b(1)| = 2$. The Floquet matrix $b(1)$ here has two coinciding eigenvalues $\lambda_+ = \lambda_- = \pm 1$. If $b(1)$ is diagonalizable, then once again $b(t)$ trajectories are bounded but, if not, they can show a weak parametric resonance growing in arithmetic but not geometric progression.

III. If $|\text{Tr} b(1)| > 2$, then $b(1)$ has a pair of real eigenvalues, $\lambda_+, \lambda_- \neq 0, \lambda_- = \frac{1}{\lambda_+}$, of which at least one has the absolute value $> 1$. The trajectories show strongly resonant behavior of the squeezing type.

Below, the instability zones II or III will be called of type $(+)$ if $\text{Tr} b(1) \geq 2$ and of type $(−)$ if $\text{Tr} b(1) \leq −2$.

In the best-known case of ion traps [92], the solutions $b(t)$ are given by the Mathieu functions and the resonance borders form the well-known Strutt diagram (see, e.g., [95, 98]). Due to its familiar shape, many studies illustrating the effects of the time-dependent elastic potentials stick to the Mathieu scheme (see, e.g., Paul’s trap [92], quantum tomography [58], etc). The need for wider stability designs was pointed out by Glauber (see the statement in Baseia et al [99]). Indeed, some distinct stability cases were considered for finite dimensional state spaces [100–102] or else for the rotating magnetic [103–105] or electric...
Suppose that for a pair of amplitudes \((\theta, q)\) the trajectory closes with \(q\) joining the subsequent points areas where
\[
\beta = \text{rotation on both classical and quantum levels.}
\]

\[\lambda_{\pm} = e^{\pm \frac{2\pi i}{n}} \neq 1, \quad \lambda^n_{\pm} = 1, \quad (56)\]
going to oscillatory motions close after the \(n\) periods of \(H(t)\) forming the twin loops in both 2D subspaces \((x, p_x)\) and \((y, p_y)\). In order to generate the loop of four canonical variables, one must ensure that the rotation \(r(1)\) simultaneously closes. The harmonic component \(\beta_1 \sin(2\pi t)\) does not contribute to \(r(1)\). The only condition is that the constant intensity \(\beta_0\) should rotate the canonical variables by \(\pm 2\pi \sigma\) after some \(m\) repetitions \((k, m = 1, 2, \ldots)\). This distinguishes the sequence of straight lines \(\beta_0 = \pm \frac{2\pi}{m}\) on the stability map of figure 4. Their interactions with the loop curves yield the amplitude pairs \((\beta_0, \beta_1)\) generating the loop phenomena for all four canonical variables after a finite number of \(mn\) repetitions (see figure 5).

While the existence of the harmonic loops is known \([89]\), their extremely regular, kaleidoscopic forms have some more implications. One of them is the exact vanishing of the ‘operator centres’ \(X = (X, Y)\).

**Proposition 2.** Suppose that for a pair of amplitudes \((\beta_0, \beta_1)\) the evolution loop of the oscillatory part \(b(t)\) closes for \(t = n\), while the rotation \(r(t)\) yields a certain non-trivial angle \(\theta = \left(\frac{k}{m}\right) 2\pi\), where \(\frac{k}{m}\) is rational but not an integer (no matter whether \(k\) is smaller or greater than \(1\)). Then, the loop obtained for \(t = mt\) \(m = nm\) by simultaneous closing of both the oscillatory and rotational motions has the trivial centre on both classical and quantum levels.

**Proof.** In fact, for the rotation \(r(t)\) breaks the oscillatory loop at \(t = \tau\), marking a new end point \(q_0\) rotated by \(\theta\) with respect to the initial \(q\). The \(q_0\), in turn, becomes the initial point for the next fragment of the trajectory, which again does not close, but is just the \(\theta\)-rotated version of the previous one, ending up at \(q_2\), etc. As a result, the whole trajectory is the sum of similar fragments, generated between the time moments \(t, 2t, \ldots\) joining the subsequent points \(q, q_0, q_2, \ldots\), each one just the \(\theta\)-rotated version of the previous one, until finally, at \(t = mt\), the trajectory closes with \(q_{2n} = q\). As the sum of the \(\theta\)-rotated steps, the whole loop is invariant under the \(\theta\)-rotation and so is the loop centre \(X\) defined by \((25)\). However, the only vector (with numerical or operator components) invariant under a non-trivial rotation around the coordinate centre is \(X = 0\).

Consistent with section 5, this ensures the stability of the harmonic loops which resist drifting (see figure 5). It looks almost like an elementary analogue of the anomalous resistance observed in 2D \([109]\) (the attempts at

![Figure 6. Biharmonic evolution loop for \(\beta(t) = \frac{\pi}{4} \sin(2\pi t) + 9.966 \sin(4\pi t)\) on the \(x, y\)-plane. The loop closes after six-field periods.](image-url)

overestimating the analogy might be misleading, yet the effect is clean at the level of non-relativistic exact solutions.)

### 7.2. The biharmonic case

In turn, we shall consider the biharmonic fields \((46)\) or \((47)\) with \(\beta_0 = 0\), defined by pairs of the dimensionless amplitudes \(\beta_1\) and \(\beta_2\). Their agreeable property is that at each \(t = nT = n\) all rotations cancel, and simultaneously, the field amplitudes \(\beta(n) = 0\). Henceforth, the evolution operators \(U(nT) = U(n)\) are reduced to the purely oscillatory part; the corresponding one-period evolution matrix \((51)\) reduces to a pair of \(b(1)\) cells. Moreover, the physical sense of the corresponding evolution in the time moments \(nT = n (n \in \mathbb{Z})\) is not affected by the difference between the canonical and kinetic momenta.

The stability areas determined by the computer scanning of \(\text{Tr} b(1)\) (see \([110]\)) form now a new two-parameter map (see figure 7).

As before, they are densely populated by curves \(\text{Tr} b(1) = \pm 2 \cos \left(\frac{2\pi n}{m}\right)\), whose points \((\beta_1, \beta_2)\) generate loops, though now \((\beta_1, \beta_2)\) do not need to fulfill any extra condition as the rotations \(r(1)\) automatically cancel for \(t = n \in \mathbb{Z}\). A loop that closes up after six field periods is drawn in figure 6. Note its symmetry under the parity reflection, which grants the exact vanishing of \(X\). However, parity is not the only mechanism ensuring this.

**Proposition 3.** Consider a periodic sequence of field pulses \(\beta(t) = \beta(t + 1)\), with \(\beta(k) = 0\), for \(k \in \mathbb{Z}\), and moreover, \(\int_{0}^{1} \beta(t) dt = 0\) (i.e. the rotations cancel in each full period \(T = 1\) and its multiples). Then any evolution loop which closes for the first time after a certain number \(n > 1\) of the \(\beta\)-periods has the vanishing centre \(X = 0\).

**Proof.** Within our assumptions the eigenvalues \(\lambda_{\pm}\) of \(b(1)\) are the \(n\)th roots of unity, \(\lambda^n_{\pm} = \lambda^n_{\pm} = 1\). If now \(\lambda_+ \neq \lambda_-\), then \(b(1)\) is diagonalizable, with complex eigenvalues, and so is the \(4 \times 4\) matrix \(u(1)\). If \(\lambda_+ = \lambda_-\) the only possibilities are \(\lambda_{\pm} = \lambda_{\pm} = 1\) and \(\lambda_{\pm} = \lambda_{\pm} = -1\). Should then \(b(1)\) be non-diagonalizable, it could not produce the closed process, contrary to our assumption. Hence, the cell must be trivial.
Figure 7. The map of stability, resonance areas and the separatix borders of types (+) and (−) on the intensity diagram $\beta_1$, $\beta_2$, scanned by integrating the matrix equation (52) for the biharmonic $b(1)$ in the periodicity interval $[0,1]$. The separatix borderlines host exceptional matrices of types (61) and (62). The branches yielding the oscillator kicks are brown, and the ones representing the distorted free evolution are green. The approximate values of the distorted evolution time $\tau$ and the simulated oscillator kicks $a$ are given by the sequences of numbers along the corresponding borderlines. Note the negative $\tau$ on two sections of the internal green branch. The points A and B, marked blue and green, represent the field amplitudes generating the space trajectories of figures 6 and 8, respectively.

If now $u(1)^n = \mathbb{I}$, then the whole matrix trajectory is invariant under the multiplication by $u(1)$ and so is the trajectory centre $\mathbf{X}$ which must fulfil $u(1) \mathbf{X} = \mathbf{X}$; but since $u(1)$ has no eigenvalue 1, the centre must vanish identically, implying the loop stability (the loop resists the external forces).

The group theoretical sense of the 2D loops can be noticed by reinterpreting the time variable $\theta = 2\pi t$ as the rotation angle in the plane. The quantity that returns to its initial value after a full $2\pi$-rotation is a tensor. An entity that changes its sign is a spinor. An entity that returns to its initial value only after a finite number of $2\pi$-rotations (such as e.g. the loops of figures 5 and 6) obeys a fractional representation of the rotation group in 2D [111–114].

7.3. The imperfections of the scheme

To estimate corrections it is convenient to return to physical units. For slow classical/quantum motions (low kinetic
momenta), the use of Schrödinger’s quantum mechanics seems to be justified, but some dissipative corrections must affect our semiclassical results. If the pulsating field is exactly periodic, the probability of quantum jumps is usually read from the quasi-energies of the Floquet Hamiltonian. The limitation of the Floquet paradigm is, however, that it predicts the radiation of the system in a stationary equilibrium with an infinity of field pulses (one virtual evolution history expected to jump into another virtual history [115–117]). For the low frequency of the applied fields and the evolution caused by a few pulse patterns, more reliable, in spite of all, seems the Abraham–Lorentz (A–L) radiative acceleration $a_{\text{rad}} = \sigma q$, where $\sigma = \frac{2\varepsilon}{\hbar}$ is the ‘characteristic time’ of the charged particle (see Jackson [118]). Taking as the model the electron with $\sigma \approx 2.12 \times 10^{-24}$ s, one has in dimensionless variables: $a_{\text{rad}} = \frac{q}{r}$; so, in the harmonic field frequency corresponding to long $\approx 1$ km RW the single operation period $T = \frac{2\pi}{\omega} \approx 3.3 \times 10^{-6}$ s brings very few A–L corrections $a_{\text{rad}} \approx 6.4 \times 10^{-18} \frac{q}{r}$ (though the evaluation changes for increasing $\omega$ [118]).

A different source of errors is the absence of the retarded terms in the vector potentials (36). Their relativistic form can be obtained by looking for the freely propagating fields of cylindrical symmetry:

$$A = \frac{1}{2} B(r, t) \begin{pmatrix} -y \\ x \end{pmatrix}, \quad r = \sqrt{x^2 + y^2}. \tag{57}$$

The d’Alembert equation $\Box A = 0$ then reduces to

$$\left[ \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - D \right] B(r, t) = 0,$$

where

$$D = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} + 2 \right) \tag{58}$$

(compare [119]), which admits an analytic solution $B(r, t) = B_0(t) + B_2(t) r^2 + B_4(t) r^4 + \cdots$ with the functional coefficients $B_n(t) = \frac{1}{\omega^2 (\sigma n + 1)^3} \left( \frac{\alpha}{\omega r} \right)^{2n} B(t)$, $n = 1, 2, \ldots$, defining the relativistic corrections in the solenoid interior. They show that the ‘laboratory Hamiltonian’ (44)–(48) is once again the first step of the EIH method [91] for the potential (57) and (58). In particular, the harmonic $B(t) = B \sin t$ contributes to (57) with $B(r, t) = \sin t B(r)$, where $B(r)$ fulfills

$$\left[ \frac{\omega^2}{c^2} + D \right] B(r) = 0 \tag{59}$$

equivalent to the Bessel equation for $r B(r)$. Its non-singular solution

$$B(r) = B_0 \begin{pmatrix} \frac{31}{2} \\ \frac{1}{2} \end{pmatrix} = B \left[ 1 - \frac{1}{112!} \left( \frac{\omega}{2 c} \right)^2 r^4 + \frac{1}{213!} \left( \frac{\omega}{2 c} \right)^4 r^8 - \cdots \right] \tag{60}$$

means that the validity of the non-relativistic ‘laboratory approximation’ (44)–(46) depends on the smallness of the inhomogeneous terms in (60), i.e. on the ratio between the effective size $\Delta r$ of each loop and the wavelength associated with the pulse frequency $\omega$, i.e. $\lambda = c T = c \omega^{1/2}$. If $\lambda$ is much greater than the loop size, $\Delta r / \lambda \ll 1$, then formula (57) describes the harmonic magnetic fields differing slightly from the homogeneous field (46), drawing with a good accuracy the regular loops of the quadratic Hamiltonians (44)–(46). Note that an analogous estimation (of the trajectory size versus the field inhomogeneity) works also in different circumstances, e.g. for atomic or molecular systems irradiated by coherent laser light. The typical atomic size (e.g. of widely studied Rb atoms) is in the range $10^{-8}$ cm ($\sim 1$ Å) but $\lambda$ of visible light is much greater (between 3800 and 7800 Å), which seems one of the reasons why the approximate description of the electric forces by a homogeneous, pulsating field $E \sim E_0 \sin(\omega t)$ gives such good results in the theory of Rabi rotations [120].

The question of whether the effects we described can have some analogues for other relativistic solutions (e.g. [121]) is open.

Until now, our exploration was limited to the stability areas with their typical ‘fauna’ and ‘flora’ of drifting and stable loops. However, the problems of quantum control unavoidably leads us across the borders.

8. The threshold: $\delta$-kicks and inverted free evolution

Since the rotations commute with the Hamiltonian (48), we shall again focus attention on the $b(t)$ cells of the evolution process (53).

One of the major control challenges is the extremely simple unitary operators and the corresponding $q, p$-transformation matrices:

$$e^{-i \omega \frac{\pi}{2}} \rightarrow \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix}, \quad e^{-i \frac{\tau}{2}} \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{61}$$

where $q, p$ mean $x$, $p_x$ or $y$, $p_y$. Can they be achieved by the oscillating field $\beta(t)$ in the periodicity interval [0, $T$]?

The left operation in (61) can be interpreted as a formal result of an infinitely strong pulse of an oscillatory potential applied within an infinitesimal time. The right one, if $\tau > 0$, $\tau \neq 1$, represents the distorted (slowed or accelerated) free evolution, but if $\tau < 0$, it yields a highly counter-intuitive effect: it inverts the free evolution of a wave packet, including the interference between the different packet components. No less provocative are the evolution operators and matrices:

$$Pe^{-i \omega \frac{\pi}{2}} \rightarrow - \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix}, \quad Pe^{-i \frac{\tau}{2}} \rightarrow - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{62}$$

i.e. (61) superposed with the parity reflection $P : q \rightarrow -q$, $p \rightarrow -p$. All matrices (61) and (62) belong to the resonance potential (case II of our classification). In principle, they can be generated by combinatorial field patterns [31, 71, 122]. However, could they be caused by a ‘soft persuasion’ instead of brutal force?

The computer scanning shows that (61) and/or (62) occur indeed on the separatix branches of figure 7 of type (+) or (−), respectively, the result that contained a puzzle. Although it is quite obvious that for $|\text{Tr}(b(1))| = 2$, the matrix $b(1)$ degenerates, this still does not imply that it must appear in the standard Jordan’s form (61) or (62) precisely in the $q, p$ basis.

As it turns out, the phenomenon occurs always when $\beta(t)^2$ in equation (52) is symmetric with respect to the centre
of the operation interval. To show this it is convenient to fix the time variable $t$ with $t = 0$ at the symmetry centre and then to consider the evolution matrix $b(t) = b(t, -t)$ generated in the expanding interval $[-t, t]$ (see [95]).

**Proposition 4.** Suppose that $\Lambda(t)$ is symmetric around the point $t = 0$, i.e. $\Lambda(-t) \equiv \Lambda(t)$. Then if at any $t \in \mathbb{R}$, $\text{Tr} b(t) = 2$, the matrix $b(t) = b(t, -t)$ must take one of the forms (61), but if $\text{Tr} b(t) = -2$, then $b(t)$ must take one of the forms (62).

**Proof.** For the symmetric $\Lambda(t)$ given by (52) the evolution matrix $b = b(t, -t)$ fulfils

$$\frac{db}{dt} = \Lambda(t)b + b\Lambda(t) = (b_{21} - \beta^2 b_{12})I + (\text{Tr} b)\Lambda(t). \quad (63)$$

Hence, one easily shows that

$$\frac{d}{dt} \left[ b_{12}b_{21} - \frac{1}{4}(\text{Tr} b)^2 \right] = 0 \Rightarrow b_{12}b_{21} - \frac{1}{4}(\text{Tr} b)^2 = C = \text{const.} \quad (64)$$

The constant in (64) can be determined by shrinking the interval $[-t, t]$ to zero. For $t = 0$, the evolution matrix is trivial, $b = b(0, 0) = I$, $b_{12} = b_{21} = 0$, i.e. the constant in (64) is $C = -\frac{1}{4}(\text{Tr} b)^2 = -1$, implying

$$b_{12}b_{21} = \frac{1}{4}(\text{Tr} b)^2 - 1. \quad (65)$$

Hence, $b_{12}b_{21}$ must vanish whenever $\text{Tr} b = \pm 2$. So, either $b_{12}$ or $b_{21}$ must vanish, leading to one of Jordan’s forms (61) or (62) on borders of type $(+)$ or $(-)$. \hfill \square

**Corollary.** While under our symmetry assumptions the values $b_{12} = 0$ and $b_{21} = 0$ appear on the stability thresholds, neither can occur inside the squeezing areas. Indeed, equation (65) implies that for $|\text{Tr} b| > 2$ neither $b_{12}$ nor $b_{21}$ can vanish.

Note that proposition 4 applies to the biharmonic $\beta(t)$, odd with respect to the centres of the periodicity intervals $[n-\frac{1}{2}, n+\frac{1}{2}]$ thanks to the symmetry of $\beta(t)$. However, due to the existence of the constant field component, the vector potential $A$ does not vanish at the operation extremes, affecting the physical sense of (61) and (62) (see the remarks in Dodonov et al [31]).

The double period operation for the biharmonic field amplitudes $(\beta_1, \beta_2) = (2.40, 2.68)$ on the first separatrix branch. Under the repeated applications of the two-period pulse pattern, the Gaussian packet shrinks instead of expanding and its centre travels in the direction opposite to its initial velocity, simulating the incidents where the inverted free evolution with $\Delta t = n\tau \approx -0.3688\omega$ at $n = 2, 4, 6$. For an electron in the biharmonic field (46) of LRW frequency $\omega \approx 3 \times 10^7 Hz$, the magnetic pulse amplitudes $B_1$ and $B_2$ needed to perform the operation are of the order of magnitude of $\approx 0.1 G$, up to 100 G for $\omega$ of the medium RW; above 1000 G for microwaves; both $B_j$ multiplied by about 1800 in the proton operations.

The parity free effects $2a$ or $2\tau$ after two pulse periods (or $2\tau$ after $2n$ repetitions). Hence, one can observe the origin of the Floquet operators $U(1)$ with transient spectra described usually in purely abstract terms [84, 123]. To illustrate the details, our figure 7 reports the values of the ‘distorted time’ $\tau$ along the green branches and the amplitudes $a$ of the effective oscillator kicks along the first brown branch.

Our computer scanning shows that double period operations of type $(−)$ have exactly vanishing Floquet centres and so produce stable (resistant) effects, while operations of type $(+)\text{) define non-vanishing fuzzy centres and can be affected by constant external forces.}

As an example, we have chosen the case of the free evolution inversion for a point on the first (negative) separatrix branch of figure 7. The effects of the ‘retrospective operations’ after four double periods are illustrated by the centre of a wave packet recovering in the time instants $t = 2n\tau$ ($n = 1, 2, \ldots$) its past shapes (shrinking instead of spreading in the first four steps); its initial momentum (velocity) always recovered, but the subsequent positions show a sequence of shifts in the direction opposite to the initial velocity (figure 8).

In turn, the possibility of simulating the $\delta(t)$ pulses of the oscillator forces by the soft biharmonic fields (brown branches of figure 7) is no less essential: it saves the realistic sense of all operations programmed with the help of sharp oscillator pulses [72, 73, 80].
9. Dark areas: the squeezing

Below, we shall not comment on any particular squeezed states, but rather on the squeezing mechanism that can deform any wave packet, no matter whether it is Gaussian or not, and we shall be looking for the origin of this phenomenon in the structure of the full period evolution (Floquet) operator $U(1)$ and its matrix $u(1)$.

The squeezing phenomena were carefully examined in an ample sequence of papers by Dodonov et al [31, 51–54, 57, 66, 73, 95, 99, 103, 110, 122, 124]; so, our approach cannot dissent too much, though it can contribute with some more observations.

The use of the quadratic Hamiltonians with the time-dependent $B = B(t)$ is again justified as the first step of the known EIH method [91] for not too many experiment areas. For periodic $β(t) = β(t + 1)$ in the cylindrical geometry, information about the squeezing can be read from the $2 \times 2$ oscillatory matrix $b(1)$, even if the time-dependent fields do not distinguish any invariant ‘squeezing centre’ apart from $\mathcal{I}$. The same happens if the magnetic field vanishes at the beginning and at the end of the operation. If $|\text{Tr} b(1)| > 2$ (dark areas), then $\lambda_+\lambda_- = 1$, describe a squeezing on the canonical plane. For $\text{Tr} b(1) > 2$ (type $+$), the transformation is a squeezing sensu stricto with two positive coefficients $\lambda_+ > 0$, but if $\text{Tr} b(1) < −2$ (type $-$), then $b(1)$ yields the superposition of squeezing and the parity transformation. The squeezed or amplified canonical observables are defined just by the row eigenvectors of $b(1)$ (see also [95]).

Some authors were interested in the scale squeezing or expansion in which $q$ is squeezed at the cost of $p$ or vice versa [54, 122]. The cases of $b_{12} = 0$ and $b_{21} = 0$ are then relevant not only on the separatix branches, but also inside the squeezing areas. The value $b_{12} = 0$ in $b(1)$ implies the coordinate $q$ squeezed or expanded at the cost of some other canonical observable, while $b_{21} = 0$, the same for the canonical momentum. The coincidence $b_{12} = 0$ would represent the position $q$ (meaning $x$ and/or $y$) squeezed or amplified exactly at the cost of the corresponding momentum ($p_x$ or $p_y$). However, due to our proposition 4, none of these cases can happen if $β^2$ is symmetric around the operation centre, when the desired zeros of $b_{12}$ and/or $b_{21}$ are all located on the separatix, without ever penetrating inside the dark areas. Hence, scale transformations can be achieved only in the operation intervals with an asymmetric $β(t)^2$. In this last case, the role of the ‘fuzzy points’ again warrants attention. If the Floquet centre $X$ of the classical/quantum trajectory vanishes, then the squeezing operations will be immune to the constant external force $F$. However, should $X$ be fuzzy, then if the field patterns are repeated, the $q$-trajectory, apart from the sequence of squeezing operations, will show a drift analogous to the one described by proposition 1. So, the ‘fuzzy centre’ seems again an element of non-commutative geometry beyond the orthodox quantum Hall effects.

10. Open problems

What brings a surprise here is a contrast between an extremely simple operation scenario (just two harmonic frequencies!) and curious dynamical effects. Since our results are computed in the dimensionless field variables, it is not ruled out that their analogues can reappear in diverse orders of magnitude such as, e.g., the nodal points of the crossed electromagnetic waves [80, 125], etc.

The electron gas in 2D in the presence of harmonic or biharmonic fields shows a surprising behaviour in the form of a ‘giant’ [126] or vanishing, or even negative resistance [127, 128]. The effects are apparently due to the presence of material shells forming a sandwich on both sides of the motion plane; but discussions have not yet ended (see, e.g., Zudov et al [109]). One might ask whether the surprising phenomena have any elementary counterparts. Evidence for this was indeed shown in the case of electric ac fields [129]. An analogous role of the magnetic pulses (45) and (46) as not been proved, but cannot be a priori ruled out [130]. Could, e.g., the behaviour of the 2D loops with the vanishing ‘fuzzy centre’ be interpreted as a mesoscopic analogue of an ‘anomalous resistance’?

In turn, for electrons in oscillating fields of microwave frequency, the curious effects of the inverted free evolution or ‘soft oscillator kicks’ shown on our map require $B_1$, $B_2$ above 1000 G. The radiative friction in these frequency regimes can blur the effects, and the relativistic corrections (57) and (58) can affect the pulsating field homogeneity. However, even if imperfect, they illustrate the existence of the soft analogues of quantum control by sharp kicks [69, 70, 73–75, 78]. For particles in wider domains the possibility of moderating and even inverting the free propagation suggests some new control techniques [70, 73]. Could the distorted free evolution mean a modified effective time? Last but not least, could the analogous phenomena affect as well the polarized vacuum [131, 132]?

Acknowledgments

The authors are indebted to the interest shown by their colleagues at the Physics and Mathematics Departments of Cinvestav, México. BM acknowledges support from the Conacyt through project no. 49253-F. Eng. Erasmo Gómez is gratefully acknowledged for technical assistance.

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