Regularity of multi-parameter Fourier integral operator

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Abstract

We study a family of Fourier integral operators by allowing their symbols to satisfy a multi-parameter differential inequality. We extend the sharp $L^p$-theorem obtained by Seeger, Sogge and Stein to product spaces.

1 Introduction

In this paper, we consider a Fourier integral operator defined by

$$\mathcal{F} f(x) = \int_{\mathbb{R}^n} f(y) \Omega(x, y) dy,$$

$$\Omega(x, y) = \int_{\mathbb{R}^n} e^{2\pi i (\Phi(x, \xi) - y \cdot \xi)} \sigma(x, \xi) d\xi. \quad (1.1)$$

The symbol function $\sigma(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ has a compact support in $x$. On the other hand, the phase function $\Phi(x, \xi)$ is real, homogeneous of degree 1 in $\xi$ and smooth for every $x$. Moreover, it satisfies the non-degeneracy condition

$$\det \left[ \frac{\partial^2 \Phi}{\partial x \partial \xi} \right](x, \xi) \neq 0 \quad (1.2)$$

at $\xi \neq 0$ on the support of $\sigma(x, \xi)$.

For more background of $\mathcal{F}$, we refer to the classical reference by Sogge [6].

Throughout, we regard $C$ as a generic constant depending on its subindices.

We say $\sigma \in S^m$ if

$$\left| \partial^\alpha_\xi \partial^\beta_x \sigma(x, \xi) \right| \leq C_{\alpha \beta} (1 + |\xi|)^m \left( \frac{1}{1 + |\xi|} \right)^{|\alpha|} \quad (1.3)$$

for every multi-indices $\alpha, \beta$.

For $\sigma \in S^0$, $\mathcal{F}$ defined as (1.1)-(1.2) is bounded on $L^2(\mathbb{R}^n)$ as shown by Eskin [9] and Hörmander [7]. In contrast to this $L^2$-result, it is well known that $\mathcal{F}$ of order zero is not bounded on $L^p(\mathbb{R}^n)$ if $p \neq 2$.

The optimal $L^p$-estimate was first investigated by Duistermaat and Hörmander [8] and then by Colin de Verdière and Frisch [10], Brenner [11], Peral [12], Miyachi [13], Beals [14] and eventually obtained by Seeger, Sogge and Stein [1].
Theorem One: Seeger, Sogge and Stein, 1991
Let $\mathcal{F}$ defined as (1.1)-(1.2). Suppose $\sigma \in S^m$ for $-(n-1)/2 < m \leq 0$. We have
\[ \|\mathcal{F}f\|_{L^p(\mathbb{R}^n)} \leq C_p \|\sigma\| \|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < \infty \]
whenever
\[ \left|\frac{1}{2} - \frac{1}{p}\right| \leq \frac{-m}{n - 1}. \]

Remark One This result is sharp. Consider a $(x, y) \in C^\infty_c (\mathbb{R}^n \times \mathbb{R}^n)$ where $a(x) \not= 0$ for $|x| = 1$ and $b(y) \equiv 1$ for $|y| < 1$. Define
\[ \sigma(x, y, \xi) = a(x)b(y)(1 + |\xi|)^m, \quad \Phi(x, \xi) = x \cdot \xi + |\xi|. \] (1.4)
Then $\mathcal{F}$ given by (1.1) is not bounded on $L^p(\mathbb{R}^n)$ if $\left|\frac{1}{2} - \frac{1}{p}\right| > -m/(n - 1)$, $(1 - n)/2 \leq m \leq 0$.
Regarding estimates can be found at 6.13, chapter IX in the book of Stein [5].

Now, define $\sigma \in S^m$ if
\[ \left|\partial_\xi^\alpha \partial_\lambda^\beta \sigma(x, \xi)\right| \leq C_{\alpha \beta} (1 + |\xi|)^m \prod_{i=1}^{n} \left(\frac{1}{1 + |\xi|}\right)^{a_i} \] (1.5)
for every multi-indices $\alpha, \beta$.

We give an extension of Theorem One by considering the Fourier integral operator $\mathcal{F}$ with a symbol $\sigma \in S^m$ satisfying the differential inequality in (1.5). The study of such operators that commute with a multi-parameter family of dilations dates back to the time of Jessen, Marcinkiewicz and Zygmund. Over the several past decades, a number of pioneering results have been accomplished, for example by Robert Fefferman [17], Fefferman and Stein [18], Chang and Fefferman [20], Cordoba and Fefferman [19] and Müller, Ricci and Stein [21].

Theorem Two Let $\mathcal{F}$ defined as (1.1)-(1.2). Suppose $\sigma \in S^m$ for $-(n-1)/2 < m \leq 0$. We have
\[ \|\mathcal{F}f\|_{L^p(\mathbb{R}^n)} \leq C_p \|\sigma\| \|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < \infty \] (1.6)
whenever
\[ \left|\frac{1}{2} - \frac{1}{p}\right| \leq \frac{-m}{n - 1}. \] (1.7)

In the next section, we sketch the proof of Theorem Two by developing a new framework where the frequency space is decomposed into an infinitely many dyadic cones. Every consisting partial operator whose symbol has a support in one of these dyadic cones is essentially an one-parameter Fourier integral operator, satisfying the desired regularity.

2 Cone decomposition on frequency space

We write $\xi = (\tau, \lambda) \in \mathbb{R} \times \mathbb{R}^{n-1}$. By symmetry, it is suffice to consider $|\tau| = \max_{i=1,2,...,n} |\xi_i|$. Without loss of generality, fix $\tau = \xi_n$. Moreover, $\ell$ denotes an $(n - 1)$-tuple $(\ell_1, \ell_2, \ldots, \ell_{n-1})$ where $\ell_i, i = 1, 2, \ldots, n - 1$ are non-negative integers.
Let \( \varphi \) be a smooth bump-function on \( \mathbb{R} \) such that
\[
\varphi(t) = 1 \quad \text{for} \quad |t| \leq 1, \quad \varphi(t) = 0 \quad \text{for} \quad |t| > 2. \tag{2.1}
\]
Define \( \delta(\xi) = \prod_{i=1}^{n-1} \delta(\xi_i) \) and \( \delta_i(\xi) = \varphi(2^{\ell} \lambda_i t) - \varphi(2^{\ell+1} \lambda_i t) \), \( i = 1, 2, \ldots, n-1 \).

Observe that \( \delta(\xi) \) is supported in the dyadic cone
\[
\Lambda = \{(\tau, \lambda) \in \mathbb{R} \times \mathbb{R}^{n-1} : 2^{-\ell} < \frac{|\lambda_i|}{|\tau|} < 2^{-\ell+1}, \quad i = 1, 2, \ldots, n-1 \}. \tag{2.3}
\]

From (2.2)-(2.3) and direct computation, we find
\[
\left| \partial^\alpha_\tau \partial^\beta_\lambda \delta(\tau, \lambda) \right| \leq C_\alpha \beta \left( \frac{1}{|\tau|} \right)^{n-1} \prod_{i=1}^{n-1} \left( \frac{1}{|\lambda_i|} \right)^{\beta_i} \tag{2.4}
\]
for every multi-indices \( \alpha, \beta \).

Define the partial operator
\[
\tilde{M}_\ell f(x) = \int_{\mathbb{R}^n} f(y) \Omega(x, y) dy,
\]
where
\[
\Omega(x, y) = \int_{\mathbb{R}^n} e^{2\pi i (\phi(x, \xi) - y \cdot \xi)} \sigma(x, \xi) \delta(\xi) d\xi.
\]
Lemma One Suppose $\sigma \in S^m$ for $-n/2 < m \leq 0$. We have

$$\|\delta_{\ell}f\|_{L^2(\mathbb{R}^n)} \leq C_{\sigma} \|f\|_{H_{\sigma}(\mathbb{R}^n)} \quad \text{for} \quad \frac{-m}{n} = \frac{1}{p} - \frac{1}{2},$$

(2.6)

$$\|\delta_{\frac{1}{2}}f\|_{L^2(\mathbb{R}^n)} \leq C_{\sigma} \|f\|_{H_{\sigma}(\mathbb{R}^n)} \quad \text{for} \quad \frac{-m}{n} = \frac{1}{2} - \frac{p-1}{p}.$$  

(2.7)

In section 3, we prove Lemma One together with the $L^2$-estimate:

$$\|\delta_{\ell}f\|_{L^2(\mathbb{R}^n)} \leq C_{\sigma} \|f\|_{L^2(\mathbb{R}^n)}, \quad \sigma \in S^0.$$  

(2.8)

Our main objective is to conclude

$$\|\delta_{\ell}f\|_{L^1(\mathbb{R}^n)} \leq C_{\sigma} \|f\|_{H_{\sigma}(\mathbb{R}^n)}, \quad \|\delta_{\ell}f\|_{BMO(\mathbb{R}^n)} \leq C_{\sigma} \|f\|_{L^\infty(\mathbb{R}^n)}$$

(2.9)

for $\sigma \in S^{-\frac{n-1}{2}}$.

From (2.8) and (2.9), we can then finish the proof of Theorem Two by carrying out an interpolation argument set out at 4.9, chapter IX in the book of Stein [5].

Furthermore, by the duality between $H^1$ and BMO spaces, as investigated by Fefferman [3], the second norm inequality in (2.9) is equivalent to

$$\|\delta_{\ell}^*f\|_{H^1(\mathbb{R}^n)} \leq C_{\sigma} \|f\|_{H_{\sigma}(\mathbb{R}^n)}, \quad \sigma \in S^{-\frac{n-1}{2}}.$$

(2.10)

Let $a$ be an $H^1$-atom associated to a ball $B_r(x_0)$ centered on some $x_0 \in \mathbb{R}^n$ with radius $r > 0$. In order to obtain (2.9), it is suffice to have

$$\int_{\mathbb{R}^n} |\delta_{\ell}a(x)| dx \leq C_{\sigma} \phi, \quad \int_{\mathbb{R}^n} |\delta_{\ell}^*a(x)| dx \leq C_{\sigma} \phi, \quad \sigma \in S^{-\frac{n-1}{2}}.$$  

(2.11)

See the characterization of $H^1$-Hardy space established by Fefferman and Stein [4].

Consider a subset $S_{\sigma}(x_0) \subset \mathbb{R}^n$, so-called the region of influence, satisfying

$$|S_{\sigma}(x_0)| \leq C_{\sigma} r.$$  

(2.12)

Let $\delta_{\ell}$ defined in (2.5). By using Schwartz inequality, we find

$$\int_{S_{\sigma}(x_0)} |\delta_{\ell}a(x)| dx \leq |S_{\sigma}(x_0)|^\frac{1}{2} \|\delta_{\ell}a\|_{L^2(\mathbb{R}^n)}$$

(2.13)

$$\leq C_{\sigma} r^\frac{1}{2} \|\delta_{\ell}a\|_{L^2(\mathbb{R}^n)}.$$

By applying Lemma One, (2.6) implies

$$\|\delta_{\ell}a\|_{L^2(\mathbb{R}^n)} \leq C_{p, \sigma} \|a\|_{L^p(\mathbb{R}^n)} \quad \text{for} \quad \frac{1}{p} = \frac{1}{2} + \frac{n-1}{2n}.$$  

(2.14)
Note that $\|a\|_{L^p(\mathbb{R}^n)} \leq \|B_r(x_o)\|^{-1 + \frac{1}{p}}$ because $|a(x)| \leq |B_r(x_o)|^{-1}$ and $a$ is supported inside $B_r(x_o)$.

Moreover, $-1 + \frac{1}{p} = -\frac{1}{2} + \frac{n-1}{2n} = -\frac{1}{2n}$.

Together with (2.14), we have

$$\int_{\mathbb{R}^n} |\vec{a}(x)| \, dx \leq C_{p, \sigma} \phi \prod_{i=1}^{n-1} 2^{-(\frac{n-1}{2n})} \|a\|_{L^p(\mathbb{R}^n)}$$

$$\leq C_{p, \sigma} \phi \prod_{i=1}^{n-1} 2^{-(\frac{n-1}{2n})} = C_{p, \sigma} \phi \prod_{i=1}^{n-1} 2^{-(\frac{n-1}{2n})}.$$

Clearly, from (2.15), we conclude

$$\int_{\mathbb{R}^n} |\vec{a}(x)| \, dx \leq C_{p, \sigma} \phi \prod_{i=1}^{n-1} 2^{-(\frac{n-1}{2n})} \|a\|_{L^p(\mathbb{R}^n)}$$

$$\leq C_{p, \sigma} \phi \prod_{i=1}^{n-1} 2^{-(\frac{n-1}{2n})}.$$

On the other hand, by applying Lemma One, (2.7) implies

$$\|\vec{a}'\|_{L^2(\mathbb{R}^n)} \leq C_{p, \sigma} \phi \prod_{i=1}^{n-1} 2^{-(\frac{n-1}{2n})} \|a\|_{L^p(\mathbb{R}^n)}.$$  \hspace{\fill} (2.17)

for $\frac{1}{p} = \frac{1}{2} + \frac{n-1}{2n}$.

The region of influence associated to $\vec{a}'$ is denoted by $\mathcal{S}_{\sigma}'(x_o)$ satisfying

$$|\mathcal{S}_{\sigma}'(x_o)| \leq C_{\sigma} r.$$  \hspace{\fill} (2.18)

By repeating the estimate in (2.13)-(2.16) and using (2.17) instead of (2.14), we find

$$\int_{\mathbb{R}^n \setminus \mathcal{S}_{\sigma}'(x_o)} |\vec{a}(x)| \, dx \leq C_{\sigma} \phi.$$  \hspace{\fill} (2.19)

Therefore, our task can be completed if we show

$$\int_{\mathbb{R}^n \setminus \mathcal{S}_{\sigma}(x_o)} |\vec{a}(x)| \, dx \leq C_{\sigma} \phi, \quad \int_{\mathbb{R}^n \setminus \mathcal{S}_{\sigma}'(x_o)} |\vec{a}(x)| \, dx \leq C_{\sigma} \phi$$

(2.20)

for $\sigma \in \mathbb{S}^{n-1}$.

In section 4, we give a heuristic estimate for (2.20) by assuming that the kernel of the partial operator satisfies certain majorization properties. These properties are accumulated into Lemma Two.

In section 5, we construct a second dyadic decomposition in analogue to the framework of Seeger, Sogge and Stein [1]. The frequency space is asserted as an union of geometric cones, denoted by $\Gamma_j$ whose central directions $\xi_j$ are almost uniformly distributed on $\mathbb{S}^{n-1}$ with a grid length approximately equal to $2^{-j/2}$, $j \geq 0$. In particular, we shall study the intersection $\Gamma_j \cap \Lambda_j \cap \{2^{-j} \leq \|\xi\| \leq 2^{j+1}\}$.

In section 6, we explicitly define $\mathcal{S}_{\sigma}(x_o)$ and $\mathcal{S}_{\sigma}'(x_o)$. The size estimates in (2.12) and (2.18) hold respectively.

We prove Lemma Two in the last section.
3 Proof of Lemma One and the L²-boundedness of \( \tilde{\gamma} \)

First, we show that \( \tilde{\gamma} \) of order zero is bounded on \( L^2(\mathbb{R}^n) \). By applying Plancherel theorem, our assertion reduces to

\[
Sf(x) = \int_{\mathbb{R}^n} e^{2\pi i \Phi(x, \xi)} \sigma(x, \xi) f(\xi) d\xi
\]  

(3. 1)

whose adjoint operator is

\[
S^*f(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \Phi(x, \xi)} \tilde{\sigma}(x, \xi) f(x) dx.
\]  

(3. 2)

Let \( c \) be a small positive constant. We define an narrow cone as follows: suppose \( \xi \) and \( \eta \) belong to a same narrow cone and \( |\eta| \leq |\xi| \). By writing \( \eta = \rho \xi + \eta^\perp \) for \( 0 \leq \rho \leq 1 \) and \( \eta^\perp \) perpendicular to \( \xi \), we require \( |\eta^\perp| \leq c\rho|\xi| \). The value of \( c \) depends on \( \Phi \).

Clearly, we can decompose the frequency space for which \( S \) or \( S^* \) can be written as a finite sum of partial operators. Each one of them has a symbol supported in such an narrow cone.

Note that our decomposition has no affection to the differentiation property \( w.r.t \) of \( \sigma(x, \xi) \).

Recall the estimate given at 3.1.1, chapter IX of Stein [5]. We have

\[
\left| \nabla_x (\Phi(x, \xi) - \Phi(x, \eta)) \right| \geq c_\Phi |\xi - \eta|
\]  

(3. 3)

whenever \( \xi \) and \( \eta \) belong to a same narrow cone.

Consider

\[
S^* S f(\xi) = \int_{\mathbb{R}^n} f(\eta) \Xi^\#(\xi, \eta) d\eta
\]  

(3. 4)

where

\[
\Xi^\#(\xi, \eta) = \int_{\mathbb{R}^n} e^{2\pi i (\Phi(x, \eta) - \Phi(x, \xi))} \sigma(x, \eta) \tilde{\sigma}(x, \xi) dx.
\]  

(3. 5)

**Remark 3.1** In order to prove the \( L^2 \)-boundedness of \( S \) for \( \sigma \in S^0 \), we assume that \( \sigma(x, \xi) \) has a support inside an narrow cone defined as above. Moreover, it only satisfies the differential inequality \( w.r.t \) \( x \) inside (1. 5).

Recall that \( \sigma(x, \xi) \) has a compact support in \( x \). Hence that \( \Xi^\#(\xi, \eta) \) is bounded in norm.

By using (3. 3), an \( N \)-fold integration by parts \( w.r.t \) \( x \) gives

\[
\left| \Xi^\#(\xi, \eta) \right| \leq c_\Phi \left| \xi - \eta \right|^{-N} \int_{\mathbb{R}^n} e^{2\pi i (\Phi(x, \eta) - \Phi(x, \xi))} \nabla_x^N (\sigma(x, \eta) \tilde{\sigma}(x, \xi)) dx
\]  

(3. 6)

for \( \xi \neq \eta \).

Together with (3. 6), we have

\[
\left| \Xi^\#(\xi, \eta) \right| \leq c_\sigma \Phi N \left( \frac{1}{1 + |\xi - \eta|} \right)^N
\]  

(3. 7)

for every \( N \geq 1 \).
To conclude the $L^2$-boundedness of $S$, we write
\[
\|S^*S f\|_{L^2(\mathbb{R}^n)} = \left\{ \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(\eta) \Xi^\#(\xi, \eta) d\eta \right|^2 d\xi \right\}^{\frac{1}{2}}
\]
\[
= \left\{ \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(\xi - \zeta) \Xi^\#(\xi, \xi - \zeta) d\zeta \right|^2 d\xi \right\}^{\frac{1}{2}} \quad (\zeta = \xi - \eta)
\]
\[
\leq C \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} \left| f(\xi - \zeta) \right|^2 \left| \Xi^\#(\xi, \xi - \zeta) \right|^2 d\zeta \right\}^{\frac{1}{2}} d\xi \quad \text{by Minkowski integral inequality}
\]
\[
\leq C_\sigma \Phi \eta N \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} \left| f(\xi - \zeta) \right|^2 \left( \frac{1}{1 + |\zeta|^2} \right)^{2N} d\zeta \right\}^{\frac{1}{2}} d\xi \quad \text{by (3. 7)}
\]
\[
= C_\sigma \Phi \eta N \left\| f \right\|_{L^2(\mathbb{R}^n)} \int_{\mathbb{R}^n} \left( \frac{1}{1 + |\zeta|^2} \right)^N d\zeta
\]
\[
\leq C_\sigma \Phi \left\| f \right\|_{L^2(\mathbb{R}^n)} \quad \text{for $N$ sufficiently large.} \tag{3. 8}
\]

Next, we begin to prove (2. 6) in **Lemma One**.

Suppose $\sigma \in S^m$ for $-n/2 < m < 0$. Let $\delta_t(\xi)$ defined in (2. 2). We write
\[
\begin{align*}
\tilde{\delta}_t f(x) &= \int_{\mathbb{R}^n} e^{2\pi i \Phi(x, \xi)} \sigma(x, \xi) \tilde{\delta}_t(\xi) \tilde{f}(\xi) d\xi \\
&= \int_{\mathbb{R}^n} e^{2\pi i \Phi(x, \xi)} \sigma(x, \xi) \left(1 + |\xi|^2\right)^{-\frac{m}{2}} \left[ \tilde{\delta}_t(\xi) \tilde{f}(\xi) \left(1 + |\xi|^2\right)^{\frac{m}{2}} \right] d\xi \\
&= \prod_{i=1}^{n-1} 2^{\left\lfloor \frac{m}{2} \right\rfloor} c_i \int_{\mathbb{R}^n} e^{2\pi i \Phi(x, \xi)} \sigma(x, \xi) \left(1 + |\xi|^2\right)^{-\frac{m}{2}} \left[ \tilde{\delta}_t(\xi) \tilde{f}(\xi) \prod_{i=1}^{n-1} 2^{-\left\lfloor \frac{m}{2} \right\rfloor} c_i \left(1 + |\xi|^2\right)^{\frac{m}{2}} \right] d\xi \\
&= \prod_{i=1}^{n-1} 2^{\left\lfloor \frac{m}{2} \right\rfloor} c_i \int_{\mathbb{R}^n} e^{2\pi i \Phi(x, \xi)} \sigma(x, \xi) \left(1 + |\xi|^2\right)^{-\frac{m}{2}} T_t f(\xi) d\xi. \tag{3. 9}
\end{align*}
\]

Observe that $\sigma(x, \xi)(1 + |\xi|^2)^{-\frac{m}{2}} \in S^0$. By using the $L^2$-boundedness of $\tilde{\delta}_t$, it is suffice to prove (2. 6) for $T_t$ defined implicitly in (3. 9). We have
\[
(T_t f)(x) = \int_{\mathbb{R}^n} f(y) K_t(x - y) dy, \tag{3. 10}
\]
\[
K_t(x) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \delta_t(\xi) \prod_{i=1}^{n-1} 2^{-\left\lfloor \frac{m}{2} \right\rfloor} c_i \left(1 + |\xi|^2\right)^{\frac{m}{2}} d\xi.
\]

Let $\varphi$ be the smooth bump-function defined in (2. 1). Consider
\[
\phi_j(\xi) = \varphi(2^{-j}||\xi||) - \varphi(2^{-j+1}||\xi||), \quad j \in \mathbb{Z}. \tag{3. 11}
\]
Recall from (3.10)-(3.11), we have

\[ K_\ell(x) = \prod_{i=1}^{n-1} 2^{-(\frac{\bar{\tau}}{\tau})_i} \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \delta_\ell(\xi) \left(1 + |\xi|^2\right)^{\frac{\bar{\lambda}}{2}} d\xi \]

\[ = \prod_{i=1}^{n-1} 2^{-(\frac{\bar{\tau}}{\tau})_i} \sum_{j \in \mathbb{Z}} \int_{\mathbb{R} \times \mathbb{R}^{n-1}} e^{2\pi i (\tau + w \cdot \lambda)} \delta_\ell(\tau, \lambda) \phi_j(\tau, \lambda) \left(1 + \tau^2 + |\lambda|^2\right)^{\frac{\bar{\lambda}}{2}} d\tau d\lambda. \]  

(3.12)

Note that \( \phi_j(\xi) \) is supported in the dyadic annuli \( 2^{j-1} \leq |\xi| \leq 2^{j+1} \). On the other hand, \( \delta_\ell(\xi) \) defined in (2.2) is supported in the dyadic cone \( \Lambda_\ell \) given in (2.3). We have \( |\tau| \leq \mathbb{C}^j \) and \( |\lambda| \leq \mathbb{C} 2^{-\ell} \), \( i = 1, 2, \ldots, n - 1 \) so that

\[ |\text{supp}\delta_\ell(\tau, \lambda)\phi_j(\tau, \lambda)| \leq \mathbb{C} 2^j \prod_{i=1}^{n-1} 2^{-\ell_i}. \]  

(3.13)

Recall \( \delta_\ell(\xi) \) satisfying the differential inequality in (2.4). We have

\[ \left| \partial_\tau^M \delta_\ell(\tau, \lambda)\phi_j(\tau, \lambda) \left(1 + \tau^2 + |\lambda|^2\right)^{\frac{\bar{\lambda}}{2}} \right| \leq \mathbb{C}_M \left(1 + \tau^2 + |\lambda|^2\right)^{\frac{\bar{\lambda}}{2}} \left(\frac{1}{|\tau|}\right)^M \]

\[ \leq \mathbb{C}_M 2^{jm} 2^{-jM}, \]

\[ \left| \partial_{\lambda_i}^N \delta_\ell(\tau, \lambda)\phi_j(\tau, \lambda) \left(1 + \tau^2 + |\lambda|^2\right)^{\frac{\bar{\lambda}}{2}} \right| \leq \mathbb{C}_{N_i} \left(1 + \tau^2 + |\lambda|^2\right)^{\frac{\bar{\lambda}}{2}} \left(\frac{1}{|\lambda_i|}\right)^{N_i} \]

\[ \leq \mathbb{C}_{N_i} 2^{jm} 2^{-(j-\ell_i)N_i} \]

for every \( M \geq 1 \) and \( N_i \geq 1, i = 1, 2, \ldots, n - 1 \).

Let \( N = N_1 + N_2 + \cdots + N_{n-1} \). An \( M+N \)-fold integration by parts \( w.r.t \ (\tau, \lambda) \) gives

\[ \prod_{i=1}^{n-1} 2^{-(\frac{\bar{\tau}}{\tau})_i} \int_{\mathbb{R} \times \mathbb{R}^{n-1}} e^{2\pi i (\tau + w \cdot \lambda)} \delta_\ell(\tau, \lambda) \phi_j(\tau, \lambda) \left(1 + \tau^2 + |\lambda|^2\right)^{\frac{\bar{\lambda}}{2}} d\tau d\lambda \]

\[ \leq \mathbb{C}_{MN} \prod_{i=1}^{n-1} 2^{-(\frac{\bar{\tau}}{\tau})_i} (2|z|^{-M} \prod_{i=1}^{n-1} |w_i|^{-N_i}) \]

\[ \left| \int_{\mathbb{R} \times \mathbb{R}^{n-1}} e^{2\pi i (\tau + w \cdot \lambda)} \partial_\tau^M \prod_{i=1}^{n-1} \partial_{\lambda_i}^N \delta_\ell(\tau, \lambda)\phi_j(\tau, \lambda) \left(1 + \tau^2 + |\lambda|^2\right)^{\frac{\bar{\lambda}}{2}} d\tau d\lambda \right| \]

\[ \leq \mathbb{C}_{MN} \prod_{i=1}^{n-1} 2^{-(\frac{\bar{\tau}}{\tau})_i} \left(2^{jm} 2^{j-\ell_i} \right) (2|z|)^{-M} \prod_{i=1}^{n-1} \left(2^{-\ell_i}|w_i|\right)^{-N_i} \]

by (3.13)-(3.14)

\[ = \mathbb{C}_{MN} 2^{\left(\frac{nm}{n-1}\right)} (2|z|)^{-M} \prod_{i=1}^{n-1} 2^{(j-\ell_i)\left(\frac{nm}{n-1}\right)} \left(2^{-\ell_i}|w_i|\right)^{-N_i}. \]
We choose

\[ M = 0 \text{ if } |z| \leq 2^{-j} \quad \text{or} \quad M = 1 \text{ if } |z| > 2^{-j}; \]

\[ N_i = 0 \text{ if } |w_i| \leq 2^{-j+i \ell_i} \quad \text{or} \quad N_i = 1 \text{ if } |w_i| > 2^{-j+i \ell_i}, \quad i = 1, 2, \ldots, n - 1. \]  

From (3.12) and (3.15), we have

\[ |\mathcal{K}_\ell(z, w)| \leq C_{M, N} \sum_j 2^j \left( \frac{\alpha m}{n} \right) \left( 2^j |z| \right)^{-M} \prod_{i=1}^{n-1} 2^{(j-\ell_i) \left( \frac{\alpha m}{n} \right) \left( 2^j |w_i| \right)^{-N_i}} \]

\[ \leq C_{M, N} \left\{ \sum_j 2^j \left( \frac{\alpha m}{n} \right) \left( 2^j |z| \right)^{-M} \right\} \prod_{i=1}^{n-1} \left\{ \sum_j 2^{(j-\ell_i) \left( \frac{\alpha m}{n} \right) \left( 2^j |w_i| \right)^{-N_i}} \right\} \]

\[ = C \left\{ \sum_{|z| \leq 2^{-j}} 2^j \left( \frac{\alpha m}{n} \right) \left( 2^j |z| \right)^{-1} + \sum_{|z| > 2^{-j}} 2^j \left( \frac{\alpha m}{n} \right) \left( 2^j |z| \right)^{-1} \right\} \]

\[ \prod_{i=1}^{n-1} \left\{ \sum_{|w_i| \leq 2^{-j+i \ell_i}} 2^{(j-\ell_i) \left( \frac{\alpha m}{n} \right) \left( 2^j |w_i| \right)^{-1}} + \sum_{|w_i| > 2^{-j+i \ell_i}} 2^{(j-\ell_i) \left( \frac{\alpha m}{n} \right) \left( 2^j |w_i| \right)^{-1}} \right\} \quad \text{by (3.16)} \]

\[ \leq C \left\{ \left( \frac{1}{|z|} \right)^{\frac{\alpha m}{n}} + \left( \frac{1}{|z|} \right) \sum_{|z| > 2^{-j}} 2^j \left( \frac{\alpha m}{n} \right) \left( 2^j |z| \right)^{-1} \right\} \prod_{i=1}^{n-1} \left\{ \left( \frac{1}{|w_i|} \right)^{\frac{\alpha m}{n}} + \left( \frac{1}{|w_i|} \right) \sum_{|w_i| > 2^{-j+i \ell_i}} 2^{(j-\ell_i) \left( \frac{\alpha m}{n} \right) \left( 2^j |w_i| \right)^{-1}} \right\} \quad (m < 0) \]

\[ \leq C \left( \frac{1}{|z|} \right)^{\frac{\alpha m}{n}} \prod_{i=1}^{n} \left( \frac{1}{|w_i|} \right)^{\frac{\alpha m}{n}} . \]

Let \(- \frac{m}{n} = \frac{1}{p} - \frac{1}{2}\). By applying the Hardy-Littlewood-Sobolev inequality [15]-[16] on every coordinate subspace and carrying out an iteration argument \(^1\) by using Minkowski integral inequality, we have

\[ \|T_\ell f\|_{L^2(\mathbb{R}^n)} = \left\{ \int_{\mathbb{R} \times \mathbb{R}^{n-1}} \left( f * \mathcal{K}_\ell \right)^2 (z, w) dz dw \right\}^{\frac{1}{2}} \]

\[ \leq C \left\{ \int_{\mathbb{R} \times \mathbb{R}^{n-1}} \left\{ \int_{\mathbb{R} \times \mathbb{R}^{n-1}} |f(u, v)| \left( \frac{1}{|z - u|} \right)^{\frac{\alpha m}{n}} \prod_{i=1}^{n-1} \left( \frac{1}{|w_i - v_i|} \right)^{\frac{\alpha m}{n}} dudv \right\}^2 dz dw \right\}^{\frac{1}{2}} \quad \text{by (3.17)} \]

\[ \leq C_p \|f\|_{L^p(\mathbb{R}^n)} . \]

Now, we turn to (2.7) in Lemma One.

\(^1\)See section 6 of [22] for example.
Consider
\[ S_\ell f(x) = \int_{\mathbb{R}^n} e^{2\pi i \Phi(x, \xi)} \sigma(x, \xi) \delta_\ell(\xi) f(\xi) d\xi \quad (3.19) \]
where
\[ S^* f(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \Phi(x, \xi)} \overline{\sigma(x, \xi)} \overline{\delta_\ell(\xi)} f(x) dx. \quad (3.20) \]

Recall \( \overline{\delta_\ell} \) defined in (2.5). We have
\[ \overline{\delta_\ell} f(x) = \int_{\mathbb{R}^n} f(y) \left\{ \int_{\mathbb{R}^n} e^{2\pi i (x - \Phi(y, \xi))} \sigma(y, \xi) \overline{\delta_\ell}(\xi) d\xi \right\} dy \quad (3.21) \]
\[ = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} S^*_\ell f(\xi) d\xi. \]

We prove (2.7) by showing
\[ \| \overline{S^*_\ell f} \|_{L^2(\mathbb{R}^n)} \leq C_{p, \sigma} \sigma \prod_{i=1}^{n-1} 2 \big( \frac{m}{n} \big) \| f \|_{L^p(\mathbb{R}^n)} \quad \text{for} \quad \frac{-m}{n} = \frac{1}{p} - \frac{1}{2}. \quad (3.22) \]

By using Plancherel theorem, it is suffice to obtain (3.22) for \( S^*_\ell \). Note that
\[ \| S^*_\ell f \|_{L^2(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} S_\ell S^*_\ell f(x) f(x) dx \quad (3.23) \]
\[ \leq \| S_\ell S^*_\ell f \|_{L^p(\mathbb{R}^n)} \| f \|_{L^p(\mathbb{R}^n)} \quad \text{by H"older inequality}. \]

We aim to show
\[ \| S_\ell S^*_\ell f \|_{L^p(\mathbb{R}^n)} \leq C_{p, \sigma} \sigma \prod_{i=1}^{n-1} 2 \big( \frac{m}{n} \big) \| f \|_{L^p(\mathbb{R}^n)} \quad \text{for} \quad \frac{-2m}{n} = \frac{1}{p} - \frac{p-1}{p}. \quad (3.24) \]

From (3.19)-(3.20), we have
\[ S_\ell S^*_\ell f(x) = \int_{\mathbb{R}^n} f(y) \overline{\xi}(x, y) dy \quad (3.25) \]
where
\[ \overline{\xi}(x, y) = \int_{\mathbb{R}^n} e^{2\pi i (\Phi(x, \xi) - \Phi(y, \xi))} \sigma(x, \xi) \delta_\ell(\xi) \overline{\sigma}(y, \xi) \overline{\delta_\ell}(\xi) d\xi. \quad (3.26) \]

Write
\[ \nabla_\xi (\Phi(x, \xi) - \Phi(y, \xi)) = \left[ \frac{\partial^2 \Phi}{\partial x \partial \xi} \right] (x, \xi)(x - y) + O \left( |x - y|^2 \right). \quad (3.27) \]

Note that \( \Phi(x, \xi) \) satisfies the non-degeneracy condition in (1.2). From (3.27), we have
\[ \left| \nabla_\xi (\Phi(x, \xi) - \Phi(y, \xi)) \right| \geq C_{\Phi} |x - y| \quad (3.28) \]
for \( x, y \) sufficiently close.
Remark 3.2. Recall that \(\sigma(x, \xi)\) has a compact support in \(x\). By using a smooth partition of unity, we can write it as a finite sum of symbol functions. Each one of them has a sufficiently small \(x\)-support depending on \(\Phi\). We make this assumption for the remaining section. Note that it does not affect the differentiation w.r.t \(\xi\) for \(\sigma(x, \xi)\).

Consider
\[
x = \mathcal{U}^{-1}x' = \left(z, 2^{\ell_1}w_1', \ldots, 2^{\ell_{n-1}}w_{n-1}'\right), \quad y = \mathcal{U}^{-1}y' = \left(u, 2^{\ell_1}v_1', \ldots, 2^{\ell_{n-1}}v_{n-1}'\right)
\]
(3.29)
and \(\xi = \mathcal{U}\xi' = \left(\tau, 2^{-\ell_1}\lambda_1', \ldots, 2^{-\ell_{n-1}}\lambda_{n-1}'\right)\).

By definition of \(\delta_\ell(\xi)\) in (2.2), we have \(\delta_\ell(\mathcal{U}\xi') = \delta_\ell(\xi')\). Let \(\phi_j(\xi)\) defined in (3.11). We write
\[
\Xi_j^\delta(x,y) = \Xi_j^\delta\left(\mathcal{U}^{-1}x', \mathcal{U}^{-1}y'\right) = \prod_{i=1}^{n-1} 2^{-\ell_i} \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} e^{2\pi i (\Phi(\mathcal{U}^{-1}x', \mathcal{U}\xi') - \Phi(\mathcal{U}^{-1}y', \mathcal{U}\xi'))} \sigma(\mathcal{U}^{-1}x', \mathcal{U}\xi') \delta_\ell(\mathcal{U}^{-1}y', \mathcal{U}\xi') |\delta_\ell(\xi')\phi_j(\xi')| d\xi'.
\]
(3.30)
Note that \(2^{\ell_i - 1} \leq |\xi'| < 2^{\ell_i + 1}\) for \(\xi'\) in the support of \(\phi_j(\xi')\). Recall \(\sigma \in S^m\) satisfying the differential inequality in (1.5). We have
\[
\left|\partial_{\xi'}^\alpha \sigma(\mathcal{U}^{-1}x', \mathcal{U}\xi') \delta_\ell(\mathcal{U}^{-1}y', \mathcal{U}\xi') |\delta_\ell(\xi')\phi_j(\xi')\right| \leq \mathcal{C}_\alpha 2^{-|\alpha|} 2^m
\]
(3.31)
for every multi-index \(\alpha\).

Moreover,
\[
\det \left[\frac{\partial^2 \Phi}{\partial x \partial \xi}\right](x, \xi) = \det \left[\frac{\partial^2 \Phi}{\partial x' \partial \xi'}\right](\mathcal{U}^{-1}x', \mathcal{U}\xi').
\]
(3.32)
Indeed, \(2^{\ell_i}\) appears at the \(i\)-th column of \(\left[\frac{\partial^2 \Phi}{\partial x \partial \xi}\right](\mathcal{U}^{-1}x', \mathcal{U}\xi')\). On the other hand, \(2^{-\ell_i}\) appears at every \(i\)-th row of \(\left[\frac{\partial^2 \Phi}{\partial x \partial \xi}\right](\mathcal{U}^{-1}x', \mathcal{U}\xi')\) respectively.

From (3.27)-(3.28) and (3.32), we find
\[
\left|\nabla_{\xi'} \left(\Phi(\mathcal{U}^{-1}x', \mathcal{U}\xi') - \Phi(\mathcal{U}^{-1}y', \mathcal{U}\xi')\right)\right| \geq \mathcal{C}_\Phi |x' - y'|.
\]
(3.33)
By using (3.31) and (3.33), an \(M + N\)-fold integration by parts w.r.t \(\xi'\) shows
\[
\prod_{i=1}^{n-1} 2^{-\ell_i} \left|\int_{\mathbb{R}^n} e^{2\pi i (\Phi(\mathcal{U}^{-1}x', \mathcal{U}\xi') - \Phi(\mathcal{U}^{-1}y', \mathcal{U}\xi'))} \sigma(\mathcal{U}^{-1}x', \mathcal{U}\xi') \delta_\ell(\mathcal{U}^{-1}y', \mathcal{U}\xi') |\delta_\ell(\xi')\phi_j(\xi')| d\xi'\right|
\]
\[
\leq \mathcal{C}_{\Phi M N} \prod_{i=1}^{n-1} 2^{-\ell_i} \left(2^{2m(j-1)}\right)^{\left(2^j |z| - |u|\right)} \left(2^j |w_i - v_i|\right)^{-M-N}
\]
\[
\leq \mathcal{C}_{\Phi M N} \left(2^{2m}\right) \left\{2^j \prod_{i=1}^{n-1} 2^{-\ell_i}\right\} \left(2^j |z| - u\right)^{-M} \prod_{i=1}^{n-1} \left(2^j |w_i - v_i|\right)^{-N_j}
\]
\[
= \mathcal{C}_{\Phi M N} \prod_{i=1}^{n-1} 2^j \left(\frac{2^{2m}}{n}\right)^{\ell_i} \left(2^j |z| - u\right)^{-M} \prod_{i=1}^{n-1} \left(2^{j-\ell_i}\right)^{\left(\frac{2^{2m}}{n}\right)} \left(2^{j-\ell_i}|w_i - v_i|\right)^{-N_j}.
\]
(3.34)
We choose
\[ M = 0 \text{ if } |z - u| \leq 2^{-j} \quad \text{or} \quad N = 1 \text{ if } |z - u| > 2^{-j}, \]
\[ N_i = 0 \text{ if } |w_i - v_i| \leq 2^{-j+i\ell} \quad \text{or} \quad N_i = 1 \text{ if } |w_i - v_i| > 2^{-j+i\ell}, \quad i = 1, 2, \ldots, n-1. \] (3.35)

From (3.30) and (3.34), we have
\[
\left| \Xi^b_{\ell}(x, y) \right| \leq C_{\Phi, M, N} \sum_{j \in \mathbb{Z}} \prod_{i=1}^{n-1} 2 \left( \frac{2m}{n} \right) \theta_i \left\{ 2 \left( \frac{2m + n}{n} \right) \left( 2 |z - u| \right)^{-M} \prod_{i=1}^{n-1} 2 \left( j - \ell_i \right) \left( \frac{n+2m}{n} \right) \left( 2 \left| w_i - v_i \right| \right)^{-N_i} \right\}
\]
\[ \leq C_{\Phi, M, N} \prod_{i=1}^{n-1} 2 \left( \frac{2m}{n} \right) \theta_i \left\{ \sum_{|z - u| \leq 2^{-j}} 2 \left( \frac{2m + n}{n} \right) \left( 2 |z - u| \right)^{-M} \prod_{i=1}^{n-1} 2 \left( j - \ell_i \right) \left( \frac{n+2m}{n} \right) \left( 2 \left| w_i - v_i \right| \right)^{-N_i} \right\} 
\]
\[ = C_{\Phi} \prod_{i=1}^{n-1} 2 \left( \frac{2m}{n} \right) \theta_i \left\{ \sum_{|z - u| \leq 2^{-j}} 2 \left( \frac{2m + n}{n} \right) + \sum_{|z - u| > 2^{-j}} 2 \left( \frac{2m + n}{n} \right) \left( 2 |z - u| \right)^{-1} \right\}
\]
\[ \prod_{i=1}^{n-1} \left\{ \sum_{|w_i - v_i| \leq 2^{-j+i\ell_i}} 2 \left( j - \ell_i \right) \left( \frac{n+2m}{n} \right) \left( 2 \left| w_i - v_i \right| \right)^{-1} \right\} \quad \text{by (3.35)}
\]
\[ \leq C_{\Phi} \prod_{i=1}^{n-1} 2 \left( \frac{2m}{n} \right) \theta_i \left\{ \left( \frac{1}{|z - u|} \right)^{\frac{n+2m}{n}} + \left( \frac{1}{|z - u|} \right) \sum_{|z - u| > 2^{-j}} 2 \left( \frac{2m}{n} \right) \right\}
\]
\[ \prod_{i=1}^{n-1} \left( \frac{1}{|w_i - v_i|} \right)^{\frac{n+2m}{n}} + \sum_{|w_i - v_i| > 2^{-j+i\ell_i}} 2 \left( \frac{2m}{n} \right) \right\} \quad (m < 0)
\]
\[ \leq C_{\Phi} \prod_{i=1}^{n-1} 2 \left( \frac{2m}{n} \right) \theta_i \left\{ \left( \frac{1}{|z - u|} \right)^{\frac{n+2m}{n}} \prod_{i=1}^{n-1} \left( \frac{1}{|w_i - v_i|} \right)^{\frac{n+2m}{n}} \right\}. \quad (3.36)
\]

Let \(-\frac{2m}{n} = \frac{1}{p} - \frac{p-1}{p}\). By applying Hardy-Littlewood-Sobolev inequality [15]-[16] on every coordinate subspace and using Minkowski integral inequality, we have
\[
\|S_\ell S_{\ell'}^c f\|_{L^p(R^n)} = \left\{ \int_{R^n} \left\| \int_{R^n} f(y) \Xi^b_{\ell}(x, y) dy \right\|_{L^p(R^n)} \right\}^{\frac{p}{p-1}} dx \quad \text{by (3.25)}
\]
\[ \leq C_{\Phi} \prod_{i=1}^{n-1} 2 \left( \frac{2m}{n} \right) \theta_i \left\{ \int_{R^n} \left\| \int_{R^n} f(u, v) \left( \frac{1}{|z - u|} \right)^{\frac{n+2m}{n}} \prod_{i=1}^{n-1} \left( \frac{1}{|w_i - v_i|} \right)^{\frac{n+2m}{n}} du dv \right\|_{L^p(R^n)} \right\}^{\frac{p}{p-1}} dx \]
\[ \leq C_{\Phi} \prod_{i=1}^{n-1} 2 \left( \frac{2m}{n} \right) \theta_i \|f\|_{L^p(R^n)}. \quad (3.37)
\]
Recall Remark 3.2. By using Minkowski inequality and (3. 37), we obtain (3. 24) as desired.

4 A heuristic estimate

Let $I \cup J = \{1, 2, \ldots, n - 1\}$ such that

$$0 \leq \ell_i \leq j/2 + 3, \quad i \in I, \quad \ell_i > j/2 + 3, \quad i \in J,$$

(4. 1)

$$J^\# = \{i \in J : \ell_i > j + 3\}, \quad J^\flat = \{i \in J : j/2 + 3 < \ell_i \leq j + 3\}$$

for every $j > 0$ and $\ell_i \geq 0, i = 1, 2, \ldots, n - 1$.

Their cardinalities are denoted by $|I|, |J|, |J^\#|$ and $|J^\flat|$ respectively.

Let $\varphi$ be the smooth bump-function given in (2. 1). Recall $\delta_\ell(\xi)$ defined in (2. 2)-(2. 3) and $\varphi_j(\xi)$ defined in (3. 11). Note that $\sum_\ell \delta_\ell(\xi) = \sum_j \varphi_j(\xi) \equiv 1$.

Define

$$\delta_\ell j(\xi) = \prod_{i \in I \cup J^\flat} \delta_\ell_i(\xi) \prod_{i \in J^\#} \sum_{\ell_i} \delta_\ell_i(\xi)$$

(4. 2)

$$= \prod_{i \in I \cup J^\flat} \delta_\ell_i(\xi) \prod_{i \in J^\#} \sum_{\ell_i} \varphi \left(2^\ell \frac{\lambda_i}{\tau} \right) - \varphi \left(2^{\ell+1} \frac{\lambda_i}{\tau} \right)$$

$$\prod_{i \in I \cup J^\flat} \delta_\ell_i(\xi) \prod_{i \in J^\#} \varphi \left(2^{j+4} \frac{\lambda_i}{\tau} \right) \quad j + 4 > 0,$$

$$\prod_{i \in I \cup J^\flat} \delta_\ell_i(\xi) \prod_{i \in J^\#} \varphi \left(\frac{\lambda_i}{\tau} \right) \quad j + 4 \leq 0.$$

Let $j > 0$. Observe that $\delta_\ell j(\xi)$ is supported in

$$\Lambda_{\ell j} = \left\{ (\tau, \lambda) \in \mathbb{R} \times \mathbb{R}^{n-1} : 2^{-\ell_i} < \frac{|\lambda_i|}{|\tau|} < 2^{-\ell_i+1}, \quad i \in I \cup J^\# \right\},$$

(4. 3)

and $0 < \frac{|\lambda_i|}{|\tau|} < 2^{-j-3}, \quad i \in J^\#$.

Moreover, by definition of $\delta_\ell(\xi)$ in (2. 2) and $\delta_\ell j(\xi)$ in (4. 2), we have

$$\sum_{\ell, \ell_1, j \geq j+3, \, i = 1, 2, \ldots, n-1} \delta_\ell j(\xi) \equiv 1.$$

(4. 4)

Consider

$$\Omega_{\ell j}(x, y) = \int_{\mathbb{R}^n} e^{2\pi i (\Phi(x, \xi) - y \cdot \xi)} \delta_\ell j(\xi) \varphi_j(\xi) \sigma(x, \xi) d\xi$$

(4. 5)

where $\varphi_j(\xi), \, j \in \mathbb{Z}$ is defined in (3. 11).
Let $a$ be an $H^1$-atom associated to the ball $B_r(x_0)$. Recall $\widehat{\gamma}$ defined in (1.1). We have

$$\int_{R^n \setminus C_r(x_0)} |\widehat{\gamma} a(x)| \, dx = \int_{R^n \setminus C_r(x_0)} \left| \int_{R^n} a(y) \sum_{\ell_j \leq j+3} \sum_{i=1,2,\ldots,n-1} \Omega_{\ell_j}(x,y) \, dy \right| \, dx$$

by (4.2) and (4.4)

$$\leq \int_{\text{supp } \{ a(y) \}} \left| \sum_{\ell_j \leq j+3} \sum_{i=1,2,\ldots,n-1} \Omega_{\ell_j}(x,y) \right| \, dy \right| \, dx$$

(4.6)

$$+ \sum_{j>0} \sum_{\ell_j \leq j+3} \int_{R^n \setminus C_r(x_0)} \int_{R^n} a(y) \Omega_{\ell_j}(x,y) \, dy \, dx.$$ 

Note that $\sum_{j \in \mathbb{N}} \phi_j(\xi)$ is supported inside the ball $|\xi| \leq 2$. From (4.4)-(4.5), we find the first term on the R.H.S of (4.6) bounded by $C_o$.

**Lemma Two** Suppose $\sigma \in S^{-\frac{m}{2}}$. For every $j > 0$, we have

$$\int_{R^n} |\Omega_{\ell_j}(x,y)| \, dx \leq C_o \Phi \prod_{i \in I} 2^{-\ell_i} \prod_{j \in J^p} 2^{-\ell_i} \prod_{j \in J^q} 2^{-\ell_j},$$

(4.7)

$$\int_{R^n} |\Omega_{\ell_j}(x,y) - \Omega_{\ell_j}(x,x_o)| \, dx \leq C_o \Phi 2 |y - x_o| \prod_{i \in I} 2^{-\ell_i} \prod_{j \in J^p} 2^{-\ell_j} \prod_{j \in J^q} 2^{-\ell_j}$$

(4.8)

and

$$\int_{R^n \setminus C_r(x_0)} |\Omega_{\ell_j}(x,y)| \, dx \leq C_o \Phi \frac{2^{-j}}{r} \prod_{i \in I} 2^{-\ell_i} \prod_{j \in J^p} 2^{-\ell_j} \prod_{j \in J^q} 2^{-\ell_j}, \quad y \in B_r(x_0)$$

(4.9)

whenever $2^j > r^{-1}$.

Consider $2^j \leq r^{-1}$. We write

$$\int_{R^n} a(y) \Omega_{\ell_j}(x,y) \, dy = \int_{B_r(x_0)} a(y) \left( \Omega_{\ell_j}(x,y) - \Omega_{\ell_j}(x,x_o) \right) \, dy$$

(4.10)

because $\int_{B_r(x_0)} a(y) \, dy = 0$ and $a$ is supported in $B_r(x_0)$.

By using (4.8) and (4.10), we find

$$\int_{R^n} \left| \int_{R^n} a(y) \Omega_{\ell_j}(x,y) \, dy \right| \, dx \leq \int_{B_r(x_0)} |a(y)| \left( \int_{R^n} |\Omega_{\ell_j}(x,y) - \Omega_{\ell_j}(x,x_o)| \, dx \right) \, dy$$

$$\leq C_o \Phi 2 |y - x_o| \prod_{i \in I} 2^{-\ell_i} \prod_{j \in J^p} 2^{-\ell_j} \prod_{j \in J^q} 2^{-\ell_j}$$

(4.11)

$$\leq C_o \Phi 2^j r \prod_{i \in I} 2^{-\ell_i} \prod_{j \in J^p} 2^{-\ell_j} \prod_{j \in J^q} 2^{-\ell_j} \quad y \in B_r(x_0).$$
By summing over all regarding ℓ and j’s, we have
\[
\sum_{2^i \leq r^{-1}} \sum_{\ell_i \leq r^{-3}, \ell_i = 1, 2, \ldots, \ell_n - 1} \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} a(y) \Omega_{\ell_i}(x, y) dy \right| dx
\]
\[
\leq C_\sigma \phi r \sum_{2^i \leq r^{-1}} 2^i \sum_{\ell_i \leq r^{-3}, \ell_i = 1, 2, \ldots, \ell_n - 1} \prod_{i \in I} 2^{-\ell_i} \prod_{j \in J} 2^{-\left(\frac{1}{2}\right) \ell_j} \prod_{j \in J} 2^{-\left(\frac{1}{2}\right) j}
\]
(4.12)
\[
\leq C_\sigma \phi r \sum_{2^i \leq r^{-1}} 2^i \prod_{j \in J} j2^{-\left(\frac{1}{2}\right) j}
\]
\[
\leq C_\sigma \phi r \sum_{2^i \leq r^{-1}} 2^i \leq C_\sigma \phi.
\]

For \(2^i > r^{-1}\), (4.9) implies
\[
\int_{\mathbb{R}^n \setminus \mathcal{C}_r(x_0)} \left| \int_{\mathbb{R}^n} a(y) \Omega_{\ell_i}(x, y) dy \right| dx \leq \int_{B_r(x_0)} |a(y)| \left\{ \int_{\mathbb{R}^n \setminus \mathcal{C}_r(x_0)} |\Omega_{\ell_i}(x, y)| dx \right\} dy
\]
\[
\leq C_\sigma \phi \frac{2^j}{r} \prod_{i \in I} 2^{-\ell_i} \prod_{j \in J} 2^{-\left(\frac{1}{2}\right) \ell_j} \prod_{j \in J} 2^{-\left(\frac{1}{2}\right) j}.
\]
(4.13)

By summing over all regarding ℓ and j’s, we have
\[
\sum_{2^i > r^{-1}} \sum_{\ell_i \leq r^{-3}, \ell_i = 1, 2, \ldots, \ell_n - 1} \int_{\mathbb{R}^n \setminus \mathcal{C}_r(x_0)} \left| \int_{\mathbb{R}^n} a(y) \Omega_{\ell_i}(x, y) dy \right| dx
\]
\[
\leq C_\sigma \phi r^{-1} \sum_{2^i > r^{-1}} 2^j \sum_{\ell_i \leq r^{-3}, \ell_i = 1, 2, \ldots, \ell_n - 1} \prod_{i \in I} 2^{-\ell_i} \prod_{j \in J} 2^{-\left(\frac{1}{2}\right) \ell_j} \prod_{j \in J} 2^{-\left(\frac{1}{2}\right) j}
\]
by (4.13)
(4.14)
\[
\leq C_\sigma \phi r^{-1} \sum_{2^i > r^{-1}} 2^j \prod_{j \in J} j2^{-\left(\frac{1}{2}\right) j}
\]
\[
\leq C_\sigma \phi r^{-1} \sum_{2^i > r^{-1}} 2^i \leq C_\sigma \phi.
\]

From (4.6), (4.12) and (4.14), we obtain the first inequality in (2.20).

On the other hand, define
\[
\Omega_{\ell_i}^*(x, y) = \int_{\mathbb{R}^n} e^{2\pi i (x \cdot \xi - \Phi(y, \xi))} \overline{\nu_{\ell_i}(\xi)} \overline{\nu_j(\xi)} \overline{\nu_j(y, \xi)} d\xi
\]
(4.15)
for the associated adjoint operator.

Remark 4.1 \(\Omega_{\ell_i}^*(x, y)\) satisfies (4.7)-(4.9) with \(\mathcal{C}_r(x_0)\) replaced by \(\mathcal{C}_r^*(x_0)\).

We prove the second inequality in (2.20) by repeating the estimate in (4.6)-(4.14) with \(\Omega_{\ell_i}(x, y)\) and \(\mathcal{C}_r(x_0)\) replaced by \(\Omega_{\ell_i}^*(x, y)\) and \(\mathcal{C}_r^*(x_0)\).
5 A second dyadic decomposition

For \( \xi = (\tau, \lambda) \in \mathbb{R} \times \mathbb{R}^{n-1} \), we denote
\[
S_{n-2}^{\infty} = \mathbb{S}^{n-1} \cap \{ (\tau, \lambda) \in \mathbb{R} \times \mathbb{R}^{n-1} : \tau = 0 \},
\]
\[
S_{i}^{n-2} = \mathbb{S}^{n-1} \cap \{ (\tau, \lambda) \in \mathbb{R} \times \mathbb{R}^{n-1} : \lambda_i = 0 \}, \quad i = 1, 2, \ldots, n - 1.
\]

(1) Every unit vector \((\xi_i, \xi_i^\perp) = (\pm 1, 0) \in \mathbb{R} \times \mathbb{R}^{n-1}, i = 1, 2, \ldots, n\) belongs to \(\{\xi_i^\nu\}_\nu\).

(2) For each \(i = 1, 2, \ldots, n\), a subset of \(\{\xi_i^\nu\}_\nu\) are equally distributed on \(S_i^{n-2} \subset \mathbb{R}^{n-1}\) with a grid length equal to \(c2^{-\nu/2}\) for \(1/\sqrt{2} \leq c \leq \sqrt{2}\).

(3) The remaining of \(\{\xi_i^\nu\}_\nu\) are equally distributed on \(S^{n-1} \setminus \bigcup_{i=1}^n S_i^{n-2}\) with the same grid length.

**Remark 5.1** From (1)-(3), there are at most a constant multiple of \(2^{(\nu + 1)/2}\) elements in \(\{\xi_i^\nu\}_\nu\).

**Remark 5.2** For every given \(\xi \in \mathbb{R}^n\), there exists a \(\xi_i^\nu\) such that
\[
\left| \frac{\xi_{\nu}}{\|\xi\|} - \xi_i^\nu \right| \leq 2^{-\nu/2}.
\]

Define
\[
\Gamma_j^\nu = \left\{ \xi \in \mathbb{R}^n : \left| \frac{\xi_{\nu}}{\|\xi\|} - \xi_i^\nu \right| \leq 3 \cdot 2^{-\nu/2} \right\}
\]
whose central direction is \(\xi_i^\nu\). We have
\[
\left| \Gamma_j^\nu \cap \{2^{j-1} \leq |\xi| < 2^{j+1} \} \right| \leq C 2^j 2^{(\nu + 1)/2}.
\]

Recall \(I \cup J^p \cup J^d = \{1, 2, \ldots, n - 1\}\) from (4. 1). Let \(\Lambda_{\ell_j}\) defined in (4. 3). We have
\[
\left| \Lambda_{\ell_j} \cap \{2^{j-1} \leq |\xi| < 2^{j+1} \} \right| \leq C 2^j \prod_{i \in I \cup J^p} 2^{j-\ell_i}.
\]

Note that \(\ell_i \leq j/2 + 3\) for \(i \in I\) and \(\ell_i > j/2 + 3\) for \(i \in J = J^p \cup J^d\). From (5. 3)-(5. 4), we find
\[
\left| \Gamma_j^\nu \cap \Lambda_{\ell_j} \cap \{2^{j-1} \leq |\xi| < 2^{j+1} \} \right| \leq C 2^j 2^{|I|/2} \prod_{i \in J^p} 2^{j-\ell_i}
\]
\[
= C 2^j 2^{(n-1-|I|)/2} \prod_{i \in J^p} 2^{j-\ell_i}.
\]

Recall \(\Lambda_{\ell}\) defined in (2. 3). Suppose \(\ell_i > j/2 + 3\) for some \(i = 1, 2, \ldots, n - 1\). We have \(|\lambda_i| \leq 2^{-\ell_i+1} < 2^{-j/2-1}\). Moreover, By definition of \(\Gamma_j^\nu\) in (5. 2), we have the following observation.
Remark 5.3 Let $\ell_i > j/2 + 3$ for some $i = 1, 2, \ldots, n - 1$. We have

$$\Lambda_\ell \subset \bigcup_{\nu} \xi_\nu^{\nu} \Gamma_j^\nu.$$  \hfill (5.6)

In particular, if $\ell_i > j/2 + 3$ for every $i = 1, 2, \ldots, n - 1$, we have

$$\Lambda_\ell \subset \bigcup_{\xi_\nu^{\nu} = (\tau, \lambda) = (\pm 1, 0)} \Gamma_j^\nu.$$  \hfill (5.7)

Figure 2: $(\tau, \lambda) \in \mathbb{R} \times \mathbb{R}$ and $\ell > j/2 + 3$.

From Remark 5.3, if $J$ is non-empty, $\Lambda_{\ell_j}$ can be covered by an union of $\Gamma_j^\nu$ whose central directions belong to $\bigcap_{i \in J} S_i^{n-2}$. We define the subset

$$V_{\ell_j} = \left\{ \nu : \xi_\nu^{\nu} \in S^{n-1} \cap \bigcap_{i \in J} S_i^{n-2}, \Gamma_j^\nu \cap \Lambda_{\ell_j} \neq \emptyset \right\}. $$  \hfill (5.8)

Let $\varphi$ defined in (2.1). Observe that

$$\varphi_j^\nu(\xi) = \varphi \left[ 2^{j/2} \left| \frac{\xi}{|\xi|} - \xi_j^\nu \right| \right] $$  \hfill (5.9)

is supported in the geometric cone $\Gamma_j^\nu$.

For every $\nu \in V_{\ell_j}$, we define

$$S_{\ell_j}^\nu(\xi) = \varphi_j^\nu(\xi) \left| \sum_{\nu \in V_{\ell_j}} \varphi_j^\nu(\xi) \right|.$$  \hfill (5.10)

Remark 5.4 Let $\Lambda_{\ell_j}$ defined in (4.3). From (1)-(3), there are at most a constant multiple of $2^{(n-1-|J|)/2} \prod_{i \in I} 2^{-\ell_i}$ many elements in $\{\xi_j^\nu\}_\nu$ such that $\xi_j^\nu \in S^{n-1} \cap \bigcap_{i \in J} S_i^{n-2}$ and $\Gamma_j^\nu \cap \Lambda_{\ell_j} \neq \emptyset$. 

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For every $\nu$ fixed, we consider a linear isometry: $\xi = L_\nu \eta$ where $L_\nu$ is an $n \times n$-matrix with $\det L_\nu = 1$. In particular, the $i$-th coordinate of $\eta$ is in the same direction of $\xi^\nu_j$ for some $i \in \{1, 2, \ldots, n\}$.

Denote $\eta^\nu_j = \left( \frac{\eta_j}{||\eta||}, 0 \right) \in \mathbb{R} \times \mathbb{R}^{n-1}$. We have

$$\xi^\nu_j = L_\nu \eta^\nu_j. \quad (5.11)$$

Furthermore, we require $\xi_i = \eta_i$ for every $i \in J$ as $\xi^\nu_j \in \bigcap_{j \in J} S^{n-2}_i$. In the special case of $J = \{1, 2, \ldots, n-1\}$, $L_\nu$ is the identity matrix so that $\eta_i = \tau$.

Remark 5.5 In the new coordinate system of $\eta \in \mathbb{R}^n$, there is NO definition for $I = \{1, 2, \ldots, n-1\} \setminus J$ such that $0 \leq \ell_i \leq j/2 + 3, i \in I$.

Let $\delta^\nu_{\ell_j}(\xi)$ defined in (5.10). From direct computation, we find

$$\left| \frac{\partial^{\alpha} \delta^\nu_{\ell_j}(L_\nu \eta)}{\partial \eta^\nu} \right| \leq C_\alpha 2^{\alpha(\frac{1}{2})} j |\eta|^{-|\alpha|} \quad (5.12)$$

for every multi-index $\alpha$.

Denote $r = ||\xi|| = ||\eta||$. For every $L_\nu \eta = \xi \in \Gamma^\nu_j$, the angle between $\eta$ and $\eta_i$ is bounded by $\arcsin(2 \cdot 2^{-j/2})$. By using polar coordinates, we have

$$\frac{\partial}{\partial \eta^\nu_i} = \left( \frac{\partial r}{\partial \eta^\nu_i} \right) \frac{\partial}{\partial r} + O \left( 2^{-j/2} \right) \cdot \nabla \eta^\nu_i. \quad (5.13)$$

Note that $\partial_r \delta^\nu_{\ell_j} = 0$ because $\delta^\nu_{\ell_j}(\xi) = \delta^\nu_{\ell_j}(L_\nu \eta)$ is homogeneous of degree zero in $\eta$. Together with (5.12) and (5.13), we have

$$\left| \frac{\partial^{\alpha} \delta^\nu_{\ell_j}(L_\nu \eta)}{\partial \eta^\nu_i} \right| \leq C_\alpha |\eta|^{-|\alpha|}, \quad \left| \frac{\partial^{\beta} \delta^\nu_{\ell_j}(L_\nu \eta)}{\partial \eta^\nu_i} \right| \leq C_\beta 2^{\beta j/2} |\eta|^{-|\beta|} \quad (5.14)$$

for every multi-indices $\alpha, \beta$.

6 Region of influence

Recall $I \cup J = \{1, 2, \ldots, n-1\}$ is defined in (4.1). From the previous section, we have $\eta^\nu_j = \left( \frac{\eta_j}{||\eta||}, 0 \right) \in \mathbb{R} \times \mathbb{R}^{n-1}$ for some $i \in \{1, 2, \ldots, n\}$ such that

$$\xi^\nu_j = L_\nu \eta^\nu_j \in S^{n-1} \cap \bigcap_{j \in J} S^{n-2}_i. \quad (6.1)$$

On the other hand, $\xi_i = \eta_i$ for every $i \in J$. Therefore, we must have $i \notin J$.

Consider the rectangle

$$R^\nu_j(x_0) = \left\{ x \in \text{supp} \sigma : \left| \left( L_\nu^T x_0 - \nabla \eta \Phi \left( x, L_\nu \eta^\nu_j \right) \right)_i \right| \leq 4 \cdot 2^{-j}, \right.$$  

$$\left. \left\{ \sum_{i \notin J, i \notin J} \left( L_\nu^T x_0 - \nabla \eta \Phi \left( x, L_\nu \eta^\nu_j \right) \right)_i \right\}^{1/2} \leq 4 \cdot 2^{-j/2} \right\}. \quad (6.2)$$
Remark 6.1. There is no restriction for
\[
\left( L^T \nu x_0 - \nabla \eta \Phi \left( x, L^T \eta \nu \right)_i \right)_i, \quad i \in J. \tag{6.3}
\]
The set $\Omega_r(x_0)$ is defined by
\[
\Omega_r(x_0) = \bigcup_{2^{-j} \leq r} \left( \bigcup_{\nu : \xi_j \in \mathbb{S}^{n-1} \cap \bigcap_{i \in J} \mathbb{S}^2} R^\nu_j(x_0) \right). \tag{6.4}
\]
Note that there are at most a constant multiple of $2^{(n-1-\|\nu\|)/2}$ elements in $\{\xi_j\}_\nu$ for which $\xi_j \in \mathbb{S}^{n-1} \cap \bigcap_{i \in J} \mathbb{S}^2$. We have
\[
|\Omega_r(x_0)| \leq \sum_{2^{-j} \leq r} \sum_{\nu : \xi_j \in \mathbb{S}^{n-1} \cap \bigcap_{i \in J} \mathbb{S}^2} \left| R^\nu_j(x_0) \right|
\leq C_{\sigma, \Phi} \sum_{2^{-j} \leq r} \sum_{\nu : \xi_j \in \mathbb{S}^{n-1} \cap \bigcap_{i \in J} \mathbb{S}^2} 2^{-j(\|n-1-\|\nu\|)/2} 2^{-j} \quad \text{by (6.2) and Remark 6.2} \tag{6.5}
\leq C_{\sigma, \Phi} \sum_{2^{-j} \leq r} 2^{-j} \leq C_{\sigma, \Phi} r.
\]
For the associated adjoint operator $\hat{\gamma}^*$, we define
\[
^*R^\nu_j(x_0) = \left\{ x \in \text{supp} \sigma : \left| \left( L^T \nu x - \nabla \eta \Phi \left( x, L^T \eta \nu \right)_i \right)_i \right| \leq 4 \cdot 2^{-j}, \right\}
\]
\[
\left\{ \sum_{i+j \in J} \left( L^T \nu x - \nabla \eta \Phi \left( x, L^T \eta \nu \right)_i \right)_i^2 \right\}^{1/2} \leq 4 \cdot 2^{-j/2}, \right\}, \tag{6.6}
\]
whereas $x$ and $x_0$ are switched in (6.2). The corresponding region of influence is
\[
\Omega^*_r(x_0) = \bigcup_{2^{-j} \leq r} \left( \bigcup_{\nu : \xi_j \in \mathbb{S}^{n-1} \cap \bigcap_{i \in J} \mathbb{S}^2} ^*R^\nu_j(x_0) \right). \tag{6.7}
\]
Clearly, $\Omega^*_r(x_0)$ also satisfies the estimate in (6.5).

**Remark 6.2** With all preliminary estimates developed in Section 4 and 5, we are ready to prove Lemma Two in the following section. The same argument also applies to $\Omega^*_j(x, y)$ defined in (4.15) except that $\Omega_r(x_0)$ is replaced by $\Omega^*_r(x_0)$.

## 7 Proof of Lemma Two

Let $I \cup J = \{1, 2, \ldots, n-1\}$ and $J = J^b \cup J^\sharp$ defined in (4.1) where $0 \leq j_i \leq j/2 + 3, i \in I$, $j/2 + 3 < j_i \leq j + 3, i \in J^b$ and $j_i > j + 3, i \in J^\sharp$. 

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Note that $\xi = L_o \eta$ with det $L_o = 1$. In particular, we have $\xi_i = \eta_i$ for every $i \in J$. Moreover, there is an $i \not\in J$ such that $\eta_i = (\eta_i/|\eta_i|, 0) \in \mathbb{R} \times \mathbb{R}^{r-1}$ and $\xi_i = L_o \eta_i$ as shown in (5. 11).

Let $V_{ij}$ defined in (5. 8) and $\delta_{ij}(\xi)$ defined in (5. 9)-(5. 10).

Now, recall $\Omega_{ij}(x, y)$ from (4. 5). We have

$$\Omega_{ij}(x, y) = \sum_{\nu \in V_{ij}} \Omega^\nu_{ij}(x, y), \quad (7. 1)$$

where $\delta_{ij}(\xi)$ and $\phi_i(\xi)$ are defined in (4. 2) and (3. 11) respectively.

Consider

$$\Phi(x, L_o \eta) - y \cdot L_o \eta = (V_i \Phi(x, L_o \eta) - L_i^T y) \cdot \eta + \Psi(x, \eta), \quad (7. 2)$$

$$\Psi(x, \eta) = \Phi(x, L_o \eta) - V_i \Phi(x, L_o \eta) \cdot \eta.$$

We borrow the next result from 4.5, chapter IX of Stein [5]:

$$|\partial_\eta^\alpha \Psi(x, \eta)| \leq C_\alpha 2^{-|\alpha|j}, \quad |\partial_\eta^\beta \Psi(x, \eta)| \leq C_\beta 2^{-|\beta|j/2} \quad (7. 3)$$

for every multi-indices $\alpha, \beta$ whenever $2^{j-1} \leq |\eta| \leq 2^{j+1}$.

Rewrite

$$\Omega^\nu_{ij}(x, y) = \int_{\mathbb{R}^n} e^{2\pi i (\Phi(x, \xi) - y \cdot \xi)} \delta_{ij}(\xi) \delta_{ij}(\xi) \sigma(x, \xi) d\xi \quad (7. 4)$$

and

$$\Theta^\nu_{ij}(x, \eta) = e^{2\pi i \Psi(x, \eta)} \delta_{ij}(L_o \eta) \delta_{ij}(L_o \eta) \phi_i(L_o \eta) \sigma(x, L_o \eta). \quad (7. 5)$$

Observe that

$$L_o \eta = \xi \in \Gamma_j \cap \Lambda_j \cap \{2^{j-1} \leq |\xi| = |\eta| \leq 2^{j+1}\} \quad (7. 6)$$

for $\eta$ in the support of $\Theta^\nu_{ij}(x, \eta)$ where $\Gamma_j$ and $\Lambda_j(\xi)$ are defined in (5. 2) and (2. 3) respectively.

For $L_o \eta \in \Gamma_j \cap \{2^{j-1} \leq |\eta| \leq 2^{j+1}\}$, we have

$$2^{j-1} \leq |\eta| \leq 2^{j+1}, \quad |\eta| \leq C 2^{j/2}. \quad (7. 7)$$

On the other hand, for $\xi = (\tau, \lambda) \in \Lambda_j \cap \{2^{j-1} \leq |\xi| \leq 2^{j+1}\}$, we have

$$2^{j-1} \leq |\tau| \leq 2^{j+1}, \quad 2^{j-1-\ell_i} \leq |\lambda_i| \leq 2^{j+1-\ell_i}, \quad i = 1, 2, \ldots, n - 1. \quad (7. 8)$$

Write $(\tau, \lambda) = \xi = L_o \eta$ for which

$$\tau = a_m \eta_m + O(1) \cdot \eta^\dagger \quad (\xi_n = \tau)$$

$$\lambda_i = a_i \eta_i + O(1) \cdot \eta^\dagger, \quad i \in J, \quad \lambda_i = \eta_i, \quad i \not\in J,$$

where $a_i$ denotes the entry on the $i$-th row and the $i$-th column of $L_o$. 

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By putting together (7.7)-(7.8) and (7.9), we necessarily have

$$|a_m| \leq C \quad \text{and} \quad |a_i| \leq C 2^{-\ell_i}, \quad i \in I. \quad (7.10)$$

Let $\delta_{ij}(\xi)$ defined in (4.2). Moreover, recall $\delta_{ij}(\xi)$ from (2.2)-(2.3) satisfying the differential inequality in (2.4). Suppose $|\tau| = C 2^j$ and $|\lambda| = C 2^{j-\ell_i}$ as in (7.8). From direct computation, for every multi-indices $\alpha, \beta$, we have

$$\left| \partial^\alpha_{\lambda_i} \prod_{i \in I} \partial^\beta_{\lambda_i} \delta_{ij}(\tau, \lambda) \right| \leq C_{\alpha \beta} \left( \frac{1}{|\tau|} \right)^\alpha \prod_{i \in I} \left( \frac{1}{|\lambda_i|} \right)^\beta_i \quad (7.11)$$

where $\ell_i \leq j/2 + 3$ for $i \in I$, and

$$\left| \partial^\alpha_{\lambda_i} \prod_{i \in I} \prod_{i \in I'} \partial^\beta_{\lambda_i} \delta_{ij}(\tau, \lambda) \right| \leq C_{\alpha \beta} \left( \frac{1}{|\tau|} \right)^\alpha \prod_{i \in I} \left( \frac{1}{|\lambda_i|} \right)^\beta_i \prod_{i \in I'} 2^{|\beta_i|} \left( \frac{1}{|\tau|} \right)^\beta_i \quad (7.12)$$

Recall $\sigma \in S^{-\frac{n-1}{2}}$ satisfying the differential inequality in (1.5). Together with (7.10)-(7.11), by using the chain rule of differentiation, we find

$$|\partial^N_{\eta_i} \delta_{ij}(L, \eta) \sigma(x, L_{\cdot} \eta)| \leq C_N \left( \frac{1}{1 + |\eta|} \right)^{\frac{n-1}{2}} |\eta|^{-N} \quad (7.13)$$

$$\leq C_N 2^{-j(\frac{n-1}{2})} 2^{-jN}, \quad N \geq 0.$$ 

Consider $\eta_i$ for $i \not\in J$. From (1.5) and (7.10)-(7.11), by using the chain rule of differentiation, we find

$$|\partial^N_{\eta_i} \delta_{ij}(L, \eta) \sigma(x, L_{\cdot} \eta)| \leq C_N \left( \frac{1}{1 + |\eta|} \right)^{\frac{n-1}{2}} 2^{N(\frac{1}{2})} |\eta|^{-N} \quad (7.14)$$

$$\leq C_N 2^{-j(\frac{n-1}{2})} 2^{-jN/2}, \quad N \geq 0.$$ 

Now, we define the differential operator

$$\mathcal{D} = I + 2^j (\partial_{\eta_i})^2 + 2^j \sum_{i \neq i \neq j} (\partial_{\eta_i})^2 + \sum_{i \in I'} 2^{2(j-\ell_i)} (\partial_{\eta_i})^2 + \sum_{i \in I'} (\partial_{\eta_i})^2 \quad (7.15)$$

$$= I + 2^j (\partial_{\eta_i})^2 + 2^j \sum_{i \neq i \neq j} (\partial_{\eta_i})^2 + \sum_{i \in I'} 2^{2(j-\ell_i)} (\partial_{\eta_i})^2 + \sum_{i \in I'} (\partial_{\eta_i})^2. \quad (7.15)$$

Let $\Theta^\nu_{ij}(x, \eta)$ defined in (7.5). From (1.5), (5.14), (7.3), (7.12) and (7.13)-(7.14), we have

$$|\mathcal{D}^N \Theta^\nu_{ij}(x, \eta)| \leq C_N 2^{-j(\frac{n+1}{2})}, \quad N \geq 0. \quad (7.16)$$
On the other hand, by using (5.5), we have

\[ \left| \text{supp} \Theta_{\xi_j}^\nu(x, \eta) \right| \leq C 2^{2(n-1-|\mathcal{J}|)/2} \prod_{i \in \mathcal{J}^i} 2^{-\ell_i}, \quad (7.17) \]

Recall \( \Omega_{\xi_j}^\nu(x, y) \) defined in (7.4). Note that \( \Omega_{\xi_j}^\nu(x, y) \) has a same \( x \)-compact support of \( \sigma(x, \xi) \).

From (7.16)-(7.17), an \( N \)-fold integration by parts associated to \( \mathcal{D} \) shows that

\[ \left| \Omega_{\xi_j}^\nu(x, y) \right| \leq C_N 2^{-/2((n-1-|\mathcal{J}|)/2) \prod_{i \in \mathcal{J}^i} 2^{-\ell_i}, \]

\[ \left\{ 1 + 4\pi^2 \sum_{i \neq j \in \mathcal{J}} (\nabla \nu \Phi(x, L_v \eta_{\mathcal{J}}^\nu) - L_v^T y)^2_i + 4\pi^2 \sum_{i \in \mathcal{J}^i} (\nabla \nu \Phi(x, L_v \eta_{\mathcal{J}}^\nu) - L_v^T y)^2_i \right\}^{-N}. \]

(7.18)

Consider a local diffeomorphism

\[ \chi_\Phi : x \rightarrow \left( L_v^T \right)^{-1} \nabla \nu \Phi(x, L_v \eta_{\mathcal{J}}^\nu) \quad (7.19) \]

whose Jacobian is non-zero provided that \( \Phi \) satisfies the non-degeneracy condition (1.2).

Denote \( \mathcal{X} = \chi(x) = \left( L_v^T \right)^{-1} \nabla \nu \Phi(x, L_v \eta_{\mathcal{J}}^\nu) \). There are exactly \( n - 1 - |\mathcal{J}| \) many terms in the summation \( \sum_{i \neq j \in \mathcal{J}}. \) By using (7.18), we have

\[ \int_{\mathbb{R}^n} \left| \Omega_{\xi_j}^\nu(x, y) \right| dx \leq C_{\Phi, N} \int_{\mathbb{R}^n} 2^{-/2((n-1-|\mathcal{J}|)/2) \prod_{i \in \mathcal{J}^i} 2^{-\ell_i}, \]

\[ \left\{ 1 + 2^2 (X - y)^2 + \sum_{i \neq j \in \mathcal{J}} (X - y)^2_i + \sum_{i \in \mathcal{J}^i} 2^{2(\ell_i)} (X - y)^2_i + \sum_{i \neq j \in \mathcal{J}} (X - y)^2_i \right\}^{-N} dX \]

\[ \leq C_{\sigma, \Phi, N} \int \cdots \int_{\mathbb{R} \times \mathbb{R} \times \cdots} 2^{-/2((n-1-|\mathcal{J}|)/2) \prod_{i \in \mathcal{J}^i} 2^{-\ell_i}, \]

\[ \left\{ 1 + \sum_i Z_i^2 + \sum_{i \neq j \in \mathcal{J}} Z_i^2 + \sum_{i \in \mathcal{J}^i} Z_i^2 + \sum_{i \neq j \in \mathcal{J}} Z_i^2 \right\}^{-N} \]

\[ dZ_i \prod_{i \neq j \in \mathcal{J}} dZ_i \prod_{i \in \mathcal{J}^i} dZ_i \prod_{i \neq j \in \mathcal{J}} dZ_i \]

\[ Z_i = 2/(X - y)_i, \quad Z_i = 2^{1/2}(X - y)_i, \quad i \neq j, \quad Z_i = 2^{\ell_i}(X - y)_i, \quad i \in \mathcal{J}^i \text{ and } Z_i = (X - y)_i, \quad i \in \mathcal{J}. \]

\[ \leq C_{\sigma, \Phi} 2^{-/2((n-1)/2)} \quad \text{for } N \text{ sufficiently large.} \]

(7.20)
Recall from Remark 5.4. There are at most $\mathcal{C} 2^{(n-1-|J|)/2} \prod_{i \in I} 2^{-t_i}$ many elements in $\{x^j\}_\nu$ such that $\xi^j \in S^{n-1} \cap \bigcap_{i \in I} S_i^{n-2}$ and $\Gamma^j \cap \Lambda^j \neq \emptyset$. We thus have

$$\int_{\mathbb{R}^n} |\Omega_{\ell} (x, y)| \, dx \leq \sum_{\nu : \xi^j \in S^{n-1} \cap \bigcap_{i \in I} S_i^{n-2}, \Gamma^j \cap \Lambda^j \neq \emptyset} \int_{\mathbb{R}^n} |\Omega^\nu_{\ell} (x, y)| \, dx \quad \text{by (7.1)}$$

$$\leq \mathcal{C}_\sigma \Phi 2^{(n-1-|J|)/2} \prod_{i \in I} 2^{-t_i} 2^{-j(\frac{3}{2})}$$

$$\leq \mathcal{C}_\sigma \Phi \prod_{i \in I} 2^{-t_i} \prod_{i \in \mathcal{J}} 2^{-\left(\frac{\ell_i}{2}\right)}$$

$$= \mathcal{C}_\sigma \Phi \prod_{i \in I} 2^{-t_i} \prod_{i \in \mathcal{J}} 2^{-\left(\frac{\ell_i}{2}\right)} \prod_{i \in \mathcal{J}} 2^{-\left(\frac{\ell_i}{2}\right)} \quad (\ell_i \leq j + 1, i \in \mathcal{J}^b)$$

Observe that every $\partial_y$ acting on $\Omega^\nu_{\ell} (x, y)$ defined in (7.1) or (7.4) gains a factor of $\mathcal{C} 2^j$ whenever $2^{j-1} \leq |\xi| = |\eta| \leq 2^{j+1}$. By carrying out the same estimate in (7.6)-(7.21), we find

$$\int_{\mathbb{R}^n} |\nabla_y \Omega_{\ell} (x, y)| \, dx \leq \mathcal{C}_\sigma \Phi 2^j \prod_{i \in I} 2^{-t_i} \prod_{i \in \mathcal{J}} 2^{-\left(\frac{\ell_i}{2}\right)} \prod_{i \in \mathcal{J}} 2^{-\left(\frac{\ell_i}{2}\right)}.$$  \hspace{1cm} (7.22)

This further implies

$$\int_{\mathbb{R}^n} |\Omega_{\ell} (x, y) - \Omega_{\ell} (x, x_0)| \, dx \leq \mathcal{C}_\sigma \Phi 2^{|y - x_0|} \prod_{i \in I} 2^{-t_i} \prod_{i \in \mathcal{J}} 2^{-\left(\frac{\ell_i}{2}\right)} \prod_{i \in \mathcal{J}} 2^{-\left(\frac{\ell_i}{2}\right)}.$$  \hspace{1cm} (7.23)

Recall $\mathcal{Q}_r (x_0)$ defined in (6.2)-(6.4). Let $2^k \leq r^{-1} \leq 2^{k+1}$. For $x \in \mathbb{R}^n \setminus \mathcal{Q}_r (x_0)$, we either have

$$\left| \left( L^T y_0 - \nabla \eta \Phi (x, L_\eta y_0) \right)_i \right| \geq 2 \cdot 2^{-k}$$  \hspace{1cm} (7.24)

or

$$\left\{ \sum_{i \in \mathcal{J} \cap \mathcal{J}^b} \left( L^T y_0 - \nabla \eta \Phi (x, L_\eta y_0) \right)_i \right\}^{\frac{1}{2}} \geq 2 \cdot 2^{-k/2}.$$ \hspace{1cm} (7.25)

If $y \in B_r (x_0)$, then $|y - x_0| \leq 2^{-k}$. For every $2^j \geq r^{-1}$, we must have

$$2^{2^j} \left( L^T y - \nabla \eta \Phi (x, L_\eta y) \right)_i \geq 2^j \sum_{i \in \mathcal{J} \cap \mathcal{J}^b} \left( L^T y - \nabla \eta \Phi (x, L_\eta y) \right)_i$$

$$\geq 2^{2^{j+k}} \geq 2^{j-k}.$$  \hspace{1cm} (7.26)

Now, repeat the same estimates from (7.6) to (7.21), except that (7.18) is replaced by the following:
\[ |\Omega_{ij}^y(x, y)| \leq C_N 2^{-j(\frac{n+1}{2})} 2^{j/2(n-1-|\lambda|)/2} \prod_{i \in J^*} 2^{-\ell_i} \]

\[
\left\{ 1 + 4\pi^2 2^j \left( \nabla_\eta \Phi(x, L_\nu \eta_i^y) - L_\nu^T y \right)_i^2 + 4\pi^2 2^j \sum_{i \not \in J^*} \left( \nabla_\eta \Phi(x, L_\nu \eta_i^y) - L_\nu^T y \right)_i^2 \right\}^{-N} \\
+ 4\pi^2 \sum_{i \in J^*} 2^{2(j-\ell_i)} \left( \nabla_\eta \Phi(x, L_\nu \eta_i^y) - L_\nu^T y \right)_i^2 + 4\pi^2 \sum_{i \in J^*} \left( \nabla_\eta \Phi(x, L_\nu \eta_i^y) - L_\nu^T y \right)_i^2 \right\}^{-N} \]

\[
\leq C_N 2^{-j+k} 2^{-j(\frac{n+1}{2})} 2^{j/2(n-1-|\lambda|)/2} \prod_{i \in J^*} 2^{-\ell_i} \\
\left\{ 1 + 4\pi^2 2^j \left( \nabla_\eta \Phi(x, L_\nu \eta_i^y) - L_\nu^T y \right)_i^2 + 4\pi^2 2^j \sum_{i \not \in J^*} \left( \nabla_\eta \Phi(x, L_\nu \eta_i^y) - L_\nu^T y \right)_i^2 \right\}^{-N} \text{ by (7.26).} \]

We find

\[
\int_{\mathbb{R}^n \setminus C_8(x_0)} |\Omega_{ij}(x, y)| \, dx \leq C_{\sigma} \phi \frac{2^{-j}}{r} \prod_{i \in J^*} 2^{-\ell_i} \prod_{i \in J^*} 2^{-\ell_i} \prod_{i \in J^*} 2^{-\ell_i} \quad \text{(7.27)}
\]

for every \( y \in B_r(x_0) \) whenever \( 2^j \geq r^{-1} \).

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