EXTENSION OF A DIOPHANTINE TRIPLE WITH THE PROPERTY $D(4)$

MARIJA BLIZNAC TREBJEŠANIN

Abstract. In this paper we give an upper bound on the number of extensions of a triple to a quadruple for the Diophantine $m$-tuples with the property $D(4)$ and confirm the conjecture of uniqueness of such extension in some special cases.

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1. Introduction

Definition 1.1. Let $n \neq 0$ be an integer. We call the set of $m$ distinct positive integers a $D(n)$-$m$-tuple, or $m$-tuple with the property $D(4)$, if the product of any two of its distinct elements increased by $n$ is a perfect square.

One of the most interesting and most studied questions is how large those sets can be. In the classical case, first studied by Diophantus, when $n = 1$, Dujella has proven in [10] that a $D(1)$-sextuple does not exist and that there are at most finitely many quintuples. Over the years many authors improved the upper bound for the number of $D(1)$-quintuples and finally He, Togbé and Ziegler in [20] have given the proof of the nonexistence of $D(1)$-quintuples. To see details of the history of the problem with all references one can visit the webpage [9].

Variants of the problem when $n = 4$ or $n = -1$ are also studied frequently. In the case $n = 4$ similar conjectures and observations can be made as in the $D(1)$ case. In the light of that observation, Filipin and the author have proven that $D(4)$-quintuple also doesn’t exist.

In both cases $n = 1$ and $n = 4$, conjecture about uniqueness of extension of a triple to a quadruple with a larger element is still open. In the case $n = -1$, a conjecture of nonexistence of a quadruple is studied, and for the survey of a problem one can see [6].

A $D(4)$-pair can be extended with a larger element $c$ to form a $D(4)$-triple. The smallest such $c$ is $c = a + b + 2r$, where $r = \sqrt{ab} + 4$ and such triple is often called a regular triple, or in the $D(1)$ case it is also called an Euler triple. There are infinitely many extensions of a pair to a triple and they can be studied by finding solutions to a of a Pellian equation

$$bs^2 - at^2 = 4(b - a),$$  

(1)
where $s$ and $t$ are positive integers defined by $ac + 4 = s^2$ and $bc + 4 = t^2$.

For a $D(4)$-triple $\{a, b, c\}$, $a < b < c$, we define

$$d_{\pm} = d_{\pm}(a, b, c) = a + b + c + \frac{1}{2} \left( abc \pm \sqrt{(ab + 4)(ac + 4)(bc + 4)} \right),$$

and it is easy to check that $\{a, b, c, d_{\pm}\}$ is a $D(4)$-quadruple, which we will call a regular quadruple, and if $d_\neq 0$ then $\{a, b, c, d_{\neq}\}$ is also a regular $D(4)$-quadruple with $d_\neq < c$.

**Conjecture 1.2.** Any $D(4)$-quadruple is regular.

Results which support this conjecture in some special cases can be found for example in [13], [1], [17], [18] and some of those results are stated in the next section and will be used as known results.

In [19] Fujita and Miyazaki approached this conjecture in the $D(1)$ case differently – they examined how many possibilities are there to extend a fixed Diophantine triple with a larger integer. They improved their result from [19] further in the joint work [7] with Cipu where they have shown that any triple can be extended to a quadruple in at most 8 ways.

In this paper we will follow the approach and ideas from [19] and [7] to prove similar results for extensions of a $D(4)$-triple. Usually, the numerical bounds and coefficients are slightly better in the $D(1)$ case, which can be seen after comparing Theorem 1.4 and [19, Theorem 1.5]. To overcome this problem we have made preparations similar as in [5] by proving a better numerical lower bound on the element $b$ in an irregular $D(4)$-quadruple and still many results needed considering and proving more special cases.

Let $\{a, b, c\}$ be a $D(4)$-triple which can be extended to a quadruple with an element $d$. Then there exist positive integers $x, y, z$ such that

$$ad + 4 = x^2, \quad bd + 4 = y^2, \quad cd + 4 = z^2.$$

By expressing $d$ from these equations we get the following system of generalized Pellian equations

$$cx^2 - az^2 = 4(c - a), \quad (2)$$
$$cy^2 - bz^2 = 4(c - b). \quad (3)$$

There exists only finitely many fundamental solutions $(z_0, x_0)$ and $(z_1, y_1)$ to these Pellian equations and any solution to the system can be expressed as $z = v_m = w_n$, where $m$ and $n$ are non-negative integers and $v_m$ and $w_n$ are recurrence sequences defined by

$$v_0 = z_0, \quad v_1 = \frac{1}{2} \left( sz_0 + cx_0 \right), \quad v_{m+2} = sv_{m+1} - v_m,$$
$$w_0 = z_1, \quad w_1 = \frac{1}{2} \left( tz_1 + cy_1 \right), \quad w_{n+2} = tw_{n+1} - w_n.$$

The initial terms of these equations were determined by Filipin in [15, Lemma 9] and one of the results of this paper is improving that Lemma
by eliminating the case where \( m \) and \( n \) are even and \(|z_0|\) is not explicitly determined.

**Theorem 1.3.** Suppose that \( \{a, b, c, d\} \) is a \( D(4) \)-quadruple with \( a < b < c < d \) and that \( w_m \) and \( v_n \) are defined as before.

i) If equation \( v_{2m} = w_{2n} \) has a solution, then \( z_0 = z_1 \) and \(|z_0| = 2 \) or \(|z_0| = \frac{1}{2}(cr - st)\).

ii) If equation \( v_{2m+1} = w_{2n} \) has a solution, then \(|z_0| = t \), \(|z_1| = \frac{1}{2}(cr - st)\) and \( z_0z_1 < 0 \).

iii) If equation \( v_{2m} = w_{2n+1} \) has a solution, then \(|z_1| = s \), \(|z_0| = \frac{1}{2}(cr - st)\) and \( z_0z_1 < 0 \).

iv) If equation \( v_{2m+1} = w_{2n+1} \) has a solution, then \(|z_0| = t \), \(|z_1| = s\) and \( z_0z_1 > 0 \).

Moreover, if \( d > d_+ \), case ii) cannot occur.

Also we improved a bound on \( c \) in the terms of \( b \) for which an irregular extension might exist.

**Theorem 1.4.** Let \( \{a, b, c, d\} \) be a \( D(4) \)-quadruple and \( a < b < c < d \). Then

i) if \( b < 2a \) and \( c \geq 890b^4 \) or

ii) if \( 2a \leq b \leq 12a \) and \( c \geq 1613b^4 \) or

iii) if \( b > 12a \) and \( c \geq 39247b^4 \)

we must have \( d = d_+ \).

**Theorem 1.5.** Let \( \{a, b, c, d\} \) be a \( D(4) \)-quadruple and \( a < b < c < d_+ < d \). Then any \( D(4) \)-quadruple \( \{e, a, b, c\} \) with \( e < c \) must be regular.

For a fixed \( D(4) \)-triple \( \{a, b, c\} \), denote by \( N \) a number of positive integers \( d > d_+ \) such that \( \{a, b, c, d\} \) is a \( D(4) \)-quadruple. The next theorem is proven as in [7] and similar methods yielded analogous result.

**Theorem 1.6.** Let \( \{a, b, c\} \) be a \( D(4) \)-triple with \( a < b < c \).

i) If \( c = a + b + 2r \), then \( N \leq 3 \).

ii) If \( a + b + 2r \neq c < b^2 \), then \( N \leq 7 \).

iii) If \( b^2 < c < 39247b^4 \), then \( N \leq 6 \).

iv) If \( c \geq 39247b^4 \), then \( N = 0 \).

This implies next corollary.

**Corollary 1.7.** Any \( D(4) \)-triple can be extended to a \( D(4) \)-quadruple with \( d > \max\{a, b, c\} \) in at most 8 ways. A regular \( D(4) \)-triple \( \{a, b, c\} \) can be extended to a \( D(4) \)-quadruple with \( d > \max\{a, b, c\} \) in at most 4 ways.

We can apply the previous results on triples when \( c \) is given explicitly in the terms of \( a \) and \( b \) which gives us a slightly better estimate on a number of extensions when \( b < 6.85a \).

**Proposition 1.8.** Let \( \{a, b\} \) be a \( D(4) \)-pair with \( a < b \). Let \( c = c_\nu \) given by

\[
c = c_\nu = \frac{4}{ab} \left\{ \left( \frac{\sqrt{b} + \sqrt{a}}{2} \right)^2 \left( \frac{r + \sqrt{ab}}{2} \right)^{2\nu} + \left( \frac{\sqrt{b} - \sqrt{a}}{2} \right)^2 \left( \frac{r - \sqrt{ab}}{2} \right)^{2\nu} - \frac{a + b}{2} \right\}
\]
where \( \tau \in \{1, -1\} \) and \( \nu \in \mathbb{N} \).

i) If \( c = c_{\tau}^{1} \) for some \( \tau \), then \( N \leq 3 \).

ii) If \( c \leq c_{1}^{+} \), then \( N \leq 6 \).

iii) If \( c = c_{2}^{-} \) and \( a \geq 2 \) then \( N \leq 6 \) and if \( a = 1 \) then \( N \leq 7 \).

iv) If \( c \geq c_{3}^{+} \) or \( c \geq c_{4}^{-} \) and \( a \geq 35 \) then \( N = 0 \).

**Corollary 1.9.** Let \( \{a, b, c\} \) be a \( D(4) \)-triple. If \( a < b \leq 6 \cdot 85a \) then \( N \leq 6 \).

## Preliminary results about elements of a \( D(4) \)-tuple

First we will list some known results.

**Lemma 2.1.** Let \( \{a, b, c\} \) be a \( D(4) \)-triple and \( a < b < c \). Then \( c &= a + b + 2r \) or \( c > \max\{ab + a + b, 4b\} \).

*Proof.* This follows from [15, Lemma 3] and [11, Lemma 1]. \( \square \)

The next lemma can be proven similarly as [20, Lemma 2].

**Lemma 2.2.** Let \( \{a, b, c, d\} \) be a \( D(4) \)-quadruple such that \( a < b < c < d \) and \( b > 10^5 \).

*Proof.* This result extends the result from [4, Lemma 2.2] and [2, Lemma 3] and is proven similarly by using Baker-Davenport reduction as described in [12]. For the computation we have used Mathematica 11.1 software package on the computer with Intel(R) Core(TM) i7-4510U CPU @2.00-3.10 GHz processor and it took approximately 170 hours to check all possibilities. \( \square \)

From [17, Theorem 1.1] we have a lower bound on \( b \) in the terms of element \( a \).

**Lemma 2.3.** If \( \{a, b, c, d\} \) is a \( D(4) \)-quadruple such that \( a < b < c < d_{+} < d \) then \( b \geq a + 57\sqrt{a} \).

The next Lemma gives us possibilities for the initial terms of sequences \( v_{m} \) and \( w_{n} \), and will be improved by Theorem 1.3.

**Lemma 2.4.** [15, Lemma 9] Suppose that \( \{a, b, c, d\} \) is a \( D(4) \)-quadruple with \( a < b < c < d \) and that \( w_{m} \) and \( v_{n} \) are defined as before.

i) If equation \( v_{2m} = w_{2n} \) has a solution, then \( z_{0} = z_{1} \) and \( |z_{0}| = 2 \) or \( |z_{0}| = \frac{1}{2}(cr - st) \) or \( z_{0} < 1.608a^{-5/14}c^{9/14} \).

ii) If equation \( v_{2m+1} = w_{2n} \) has a solution, then \( |z_{0}| = t, |z_{1}| = \frac{1}{2}(cr - st) \) and \( z_{0}z_{1} < 0 \).

iii) If equation \( v_{2m} = w_{2n+1} \) has a solution, then \( |z_{1}| = s, |z_{0}| = \frac{1}{2}(cr - st) \) and \( z_{0}z_{1} < 0 \).

iv) If equation \( v_{2m+1} = w_{2n+1} \) has a solution, then \( |z_{0}| = t, |z_{1}| = s \) and \( z_{0}z_{1} > 0 \).
Remark. From the proof of [15, Lemma 9] we see that case $v_{2m} = w_{2n}$ and $z_0 < 1.608a^{-5/14}c^{9/14}$ holds only when $d_0 = \frac{z_0^2 - 4}{c}$, $0 < d_0 < c$ is such that \{a, b, d_0, c\} is an irregular D(4)-quadruple. As we can see from the statement of Theorem 1.3, this case will be proven impossible and only cases where $z_0$ is given in the terms of a triple \{a, b, c\} will remain.

Using the lower bound on $b$ in an irregular quadruple from Lemma 2.2 we can slightly improve [15, Lemma 1].

Lemma 2.5. Let $(z, x)$ and $(z, y)$ be positive solutions of (2) and (3). Then there exist solutions $(z_0, x_0)$ of (2) and $(z_1, y_1)$ of (3) in the ranges

$$1 \leq x_0 < \sqrt{s + 2} < 1.00317\sqrt{ac},$$

$$1 \leq |z_0| < \sqrt{c\sqrt{c}\sqrt{b}} < 0.05624c,$$

$$1 \leq y_1 < \sqrt{t + 2} < 1.00011\sqrt{bc},$$

$$1 \leq |z_1| < \sqrt{c\sqrt{c}\sqrt{b}} < 0.003163c,$$

such that

$$z\sqrt{a} + x\sqrt{c} = (z_0\sqrt{a} + x_0\sqrt{c}) \left(\frac{s + \sqrt{ac}}{2}\right)^m,$$

$$z\sqrt{b} + y\sqrt{c} = (z_1\sqrt{b} + y_1\sqrt{c}) \left(\frac{t + \sqrt{bc}}{2}\right)^n.$$

This Lemma can now be used to get a lower bound on $d$ in the terms of the elements of a triple \{a, b, c\}.

Lemma 2.6. Suppose that \{a, b, c, d\} is a D(4)-quadruple with $a < b < c < d_+ < d$ and that $w_m$ and $v_n$ are defined as before. If $z \geq w_n$, $n = 4, 5, 6, 7$, then

$$d > k \cdot b^{n-1.5}c^{n-0.5}$$

where $k \in \{0.249970, 0.249974, 0.249969, 0.249965\}$ respectively.

If $z \geq v_m$, $n = 4, 5, 6$, then

$$d > l \cdot a^{m-1.5}c^{n-0.5}$$

where $l \in \{0.243775, 0.242245, 0.240725\}$ respectively.

Proof. In the proof [15, Lemma 5] it has been shown that

$$\frac{c}{2x_0}(s - 1)^{m-1} < v_m < cx_0s^{m-1},$$

$$\frac{c}{2y_1}(t - 1)^{n-1} < w_n < cy_1t^{n-1}.$$

We use that $bc > 10^{10}$ for $d > d_+$ and $d = \frac{z_0^2 - 4}{c}$ to obtain desired inequalities. □
We know a relation between \( m \) and \( n \) if \( v_m = w_n \).

**Lemma 2.7.** [15, Lemma 5] Let \( \{a, b, c, d\} \) be \( D(4) \)-quadruple. If \( v_m = w_n \) then \( n-1 \leq m \leq 2n+1 \).

But we can also prove that a better upper bound holds using more precise argumentation and the fact from [14, Lemma 8] that \( c < 7b^{11} \).

**Lemma 2.8.** If \( b > c^\varepsilon \) and \( 1 \leq \varepsilon < 12 \), then
\[
m < \frac{\varepsilon + 1}{0.999\varepsilon} n + 1.5 - 0.4 \frac{\varepsilon + 1}{0.999\varepsilon}.
\]

*Proof.* If \( v_m = w_n \) then
\[
2.00634^{-1}(ac)^{-1/4}(s-1)^{m-1} < 1.000011(bc)^{1/4}t^{n-1}.
\]
Since \( c > b > 10^5 \) we can easily check that \( s-1 > 0.9968a^{1/2}c^{1/2} \) and \( b^{1/2}c^{1/2} < t < 1.0001b^{1/2}c^{1/2} \), so the previous inequality implies
\[
(s-1)^{m-3/2} < 2.00956t^{n-0.5} < t^{n-0.4}.
\]
On the other hand, the assumption \( b < c^{1/\varepsilon} \) yields \( t < 1.0001c^{\frac{\varepsilon+1}{\varepsilon}} \) and
\[
s-1 > 0.9968c^{1/2} > (0.99t)^{\frac{\varepsilon+1}{\varepsilon}} > t^{0.999\frac{\varepsilon+1}{\varepsilon}}.
\]
So we observe that an inequality
\[
0.999(m-1.5)\frac{\varepsilon}{\varepsilon+1} < n - 0.4
\]
must hold, which proves our statement. \( \square \)

We have the next Lemma from the results of [16, Lemma 5] and [14, Lemma 5].

**Lemma 2.9.** If \( v_m = w_n \) has a solution, then \( 6 \leq m \leq 2n+1 \) and \( n \geq 7 \) or case \( |z_0| < 1.608a^{-5/14}b^{9/14} \) from Lemma 2.4 holds and either \( 6 \leq m \leq 2n+1 \) or \( m = n = 4 \).

**Lemma 2.10.** Assume that \( c \leq 0.243775a^{2.5}b^{3.5} \). If \( z = v_m = w_n \) for some even \( m \) and \( n \) then \( d > 0.240725a^{4.5}c^{5.5} \).

*Proof.* Let us assume that \( z_0 \notin \{2, \frac{1}{2}(cr-st)\} \), i.e. there exists an irregular \( D(4) \)-quadruple \( \{a, b, d_0, c\} \), \( c > d_0 \) but then Lemma 2.6 implies \( c > 0.243775a^{2.5}b^{3.5} \), a contradiction. So, we must have \( z_0 \in \{2, \frac{1}{2}(cr-st)\} \) and by Lemma 2.9 we know that \( \max\{m, n\} \geq 6 \). Statement now follows from Lemma 2.6. \( \square \)

By using an improved lower bound on \( d \) in an irregular quadruple from Lemma 2.6 we can prove next result in the same way as [15, Lemma 9].

**Lemma 2.11.** If \( v_m = w_n \) for some even \( m \) and \( n \) and \( |z_0| \notin \{2, \frac{1}{2}(cr-st)\} \) then \( |z_0| < 1.2197b^{-5/14}c^{9/14} \).
Similarly as in [19], by using Lemma 2.5 we can prove some upper and lower bounds on $c$ in the terms of smaller elements depending on the value of $z_0$.

**Lemma 2.12.** Set

$$\tau = \frac{\sqrt{ab}}{r} \left(1 - \frac{a + b + 4/c}{c}\right), \quad (< 1).$$

We have that

i) $|z_0| = \frac{1}{2}(cr - st)$ implies $c < ab^2\tau^{-4}$,

ii) $|z_1| = \frac{1}{2}(cr - st)$ implies $c < a^2b\tau^{-4}$,

iii) $|z_0| = t$ implies $c > ab^2$,

iv) $|z_1| = s$ implies $c > a^2b$,

and ii) and iii) cannot occur simultaneously when $d > d_+$. 

Next lemma can be easily proved by induction.

**Lemma 2.13.** Let $\{v_{z_0,m}\}$ denote a sequence $\{v_m\}$ with an initial value $z_0$ and $\{w_{z_1,n}\}$ denote a sequence $\{w_n\}$ with an initial value $z_1$. It holds that $v_{\frac{1}{2}(cr-st),m} = v_{-t,m+1}$, $v_{-\frac{1}{2}(cr-st),m+1} = v_{t,m}$ for each $m \geq 0$ and $w_{\frac{1}{2}(cr-st),n} = w_{-s,n+1}$, $w_{-\frac{1}{2}(cr-st),n+1} = w_{s,n}$ for each $n \geq 0$.

For the proof of Theorem 1.6 we will use the previous Lemma. It is obvious that if we "shift" a sequence as in Lemma 2.13 the initial term of the new sequence would not satisfy the bounds from Lemma 2.5 since the original sequence did. In the next Lemma we will prove some new lower bounds on $m$ and $n$ when $|z_0| = t$ and $|z_1| = s$, but without assuming bounds on $z_0$ and $z_1$ from Lemma 2.5. Since Filipin has proven that $n \geq 7$ in any case $(z_0, z_1)$ which appears when "shifting" sequences as in Lemma 2.13 we can consider that bound already proven. Even though the following proof is analogous to the proof of [7, Lemma 2.6], there is more to consider in the $D(4)$ case which is why we present the proof in detail.

**Proposition 2.14.** Let $\{a, b, c\}$ be a $D(4)$-triple, $a < b < c$, $b > 10^5$ i $c > ab + a + b$. Let us assume that the equation $v_m = w_n$ has a solution for $m > 2$ such that $m \equiv n \equiv 1 \pmod{2}$, $|z_0| = t$, $|z_1| = s$, $z_0z_1 > 0$. Then $\min\{m, n\} \geq 9$.

**Proof.** It is easy to see that $v_1 = w_1 < v_3 < w_3$. If we show that

i) $w_3 < v_5 < w_5 < v_9$ and $v_7 \neq w_5$,

ii) $v_7 < w_7 < v_{13}$ and $v_9 \neq w_7 \neq v_{11}$,

from Lemma 2.7 we see that it leads to a conclusion that $\min\{m, n\} \geq 9$. 

Let $(z_0, z_1) = (\pm t, \pm s)$. We derive that
\[
\begin{align*}
w_3 & = \frac{1}{2}(cr \pm st)(bc + 1) + cr \\
w_5 & = \frac{1}{2}(cr \pm st)(b^2c^2 + 3bc + 1) + cr(bc + 2) \\
w_7 & = \frac{1}{2}(cr \pm st)(b^3c^3 + 5b^2c^2 + 6bc + 1) + cr(b^2c^2 + 4bc + 3)
\end{align*}
\]
and
\[
\begin{align*}
v_5 & = \frac{1}{2}(cr \pm st)(a^2c^2 + 3ac + 1) + cr(ac + 2) \\
v_7 & = \frac{1}{2}(cr \pm st)(a^3c^3 + 5a^2c^2 + 6ac + 1) + cr(a^2c^2 + 4ac + 3) \\
v_9 & = \frac{1}{2}(cr \pm st)(a^4c^4 + 7a^3c^3 + 15a^2c^2 + 10ac + 1) + cr(a^3c^3 + 6a^2c^2 + 10ac + 4) \\
v_{11} & = \frac{1}{2}(cr \pm st)(a^5c^5 + 9a^4c^4 + 28a^3c^3 + 35a^2c^2 + 15ac + 1) + cr(a^4c^4 + 8a^3c^3 + 21a^2c^2 + 20ac + 5) \\
v_{13} & = \frac{1}{2}(cr \pm st)(a^6c^6 + 11a^5c^5 + 45a^4c^4 + 84a^3c^3 + 70a^2c^2 + 21ac + 1) + cr(a^5c^5 + 10a^4c^4 + 36a^3c^3 + 56a^2c^2 + 35ac + 6)
\end{align*}
\]
Since $a < b$, $c > ab$, and sequences $v_m$ and $w_n$ are increasing, it is easy to see that $w_3 < v_5 < v_7$, $v_5 < v_7 < v_9 < v_{11}$, $v_7 < v_7$, $w_5 < v_9$ and $w_7 < v_{13}$. It remains to prove that $v_7 \neq w_5$ and $v_9 \neq w_7 \neq v_{11}$.

From the Lemma 2.9 and the explanation before this Lemma, we consider that $v_7 \neq w_5$ is already proven, but it is not hard to follow the following proof to prove this case also.

First, we want to prove that $v_9 \neq w_7$. Let us assume contrary, that $v_9 = w_7$. We have for $(z_0, z_1) = (\pm t, \pm s)$ that
\[
cr(a^4c^4 + 9a^3c^3 + 27a^2c^2 + 30ac - b^3c^3 - 7b^2c^2 - 14bc + 2) = \pm st(a^4c^4 + 7a^3c^3 + 15a^2c^2 + 10ac - b^3c^3 - 5b^2c^2 - 6bc).
\]
a) **Case** $z_0 = -t$

Since $cr > st$ we have
\[
\begin{align*}
a^4c^4 + 9a^3c^3 + 27a^2c^2 + 30ac - b^3c^3 - 7b^2c^2 - 14bc + 2 & < \\
& < a^4c^4 + 7a^3c^3 + 15a^2c^2 + 10ac - b^3c^3 - 5b^2c^2 - 6bc
\end{align*}
\]
which leads to
\[
a^3c < b^2. \tag{4}
\]
Since $c > 10^5$ we have $b > 316a$. On the other hand, we easily see that
\[
cr - st > \frac{4c^2 - 4ac - 4bc - 16}{2cr} > \frac{2ab}{r} > 1.99r,
\]
which can be used to prove
\[ v_9 < \frac{1}{2} (cr - st)(a^4c^4 \left( 2.0051 + \frac{7}{ac} \right) + 15a^2c^2 + 10ac + 1) + cr(6a^2c^2 + 10ac + 4), \]
\[ w_7 > \frac{1}{2} (cr - st)(b^3c^3 + 15a^2c^2 + 10ac + 1) + cr(6a^2c^2 + 10ac + 4). \]

So
\[ b^3 < 2.006a^4c. \quad (5) \]

Combining equations (4) and (5) we see that \( b < 2.006a \), which is in a contradiction with \( b > 316a \).

b) Case \( z_0 = t \)

Since \( cr > st \) we have
\[ a^4c^4 + 9a^3c^3 + 27a^2c^2 + 30ac - b^3c^3 - 7b^2c^2 - 14bc + 2 < \]
\[ < -a^4c^4 - 7a^3c^3 - 15a^2c^2 - 10ac + b^3c^3 + 5b^2c^2 + 6bc \]
which leads to
\[ 2a^4c^4 + 16a^3c^3 + 42a^2c^2 + 40ac + 2 < 2b^3c^3 + 12b^2c^2 + 20bc. \quad (6) \]

If \( 16a^3c^3 < 12b^2c^2 \) then \( c < 0.75 \frac{b^2}{a^2} \). On the other hand, if \( 16a^3c^3 \geq 12b^2c^2 \) then \( 2a^4c^4 < 2b^3c^3 \). In each case, inequality
\[ a^4c < b^3 \quad (7) \]
holds. Since \( c > 10^5 \), \( b > 46a \) and since \( c > ab \) we have \( b > a^{5/2} \) and \( c > a^{7/2} \). It is easily shown that \( cr + st > 632r^2 \) and we can use it to see that
\[ \frac{1}{2} (cr + st)(a^4c^4 + 7a^3c^3) + cr \cdot a^3c^3 < \frac{1}{2} (cr + st)a^4c^4 \left( 1 + \frac{2}{632ar} + \frac{7}{ac} \right), \]
and get an upper bound on \( v_9 \). Now, from \( v_9 = w_7 \) we have
\[ b^3 < 1.001a^4c. \quad (8) \]

Notice that \( c > b^31.001-1a^{-4} > 0.999b^{7/5} > 9.99 \cdot 10^6 \).

If we consider \( v_9 = w_7 \) modulo \( c^2 \) and use the fact that \( st(cr - st) \equiv 16 \) (mod \( c \)) it yields a congruence
\[ 2r(cr - st) \equiv 16(6b - 10a) \pmod{c}. \]

Since \( 2r(cr - st) < 4.01c \) and \( 16(6b - 10a) < 96b < 96 \cdot (1.001a^4c)^{1/3} < 97c^{5/7} < c \), we have that one of the equations
\[ kc = 2r(cr - st) - 16(6b - 10a), \quad k \in \{0, 1, 2, 3, 4\} \]
must hold.
If \( k = 0 \), we have \( 2r(cr - st) = 16(6b - 10a) \). The inequality \( 2r(cr - st) > 3.8r^2 > 3.8ab \) implies \( 96b > 16(6b - 10a) > 3.8ab \) so \( a \leq 25 \). Now, we have

\[
\frac{cr - st}{cr} > \frac{2c^2 - 2ac - 2bc - 8}{2c} > \frac{2\left(\frac{b^3}{25} + 4\right) - (2b + 10 + 1)}{\sqrt{25b + 4}}
\]

\[
> \frac{b(0.00000256b^2 - 2) - 51}{b^{1/2}\sqrt{26}} > 0.0000005b^{5/2},
\]

so \( 16(6b - 10a) > 2\sqrt{ab} + 4 \cdot 0.0000005b^{5/2} \). For each \( a \in [1, 25] \), we get from this inequality a numerical upper bound on \( b \) which is in a contradiction with \( b > 10^5 \).

If \( k \neq 0 \), we have a quadratic equation in \( c \) with possible solutions

\[
c_\pm = \frac{-B \pm \sqrt{B^2 - 4AE}}{2A}
\]

where \( A = r^2(16 - 4k) + k^2 > 0 \), \( B = -(32(2r^2 - 6)(6b - 10a) + 16r^2(a + b)) < 0 \) and \( E = 64(4(6b - 10a)^2 - r^2) > 0 \).

If \( k \leq 3 \), \( A > 4r^2 \) and \( c_\pm < \frac{B}{A} < \frac{32-2r^2+6b+16r-32b}{4r^2} = 100b \). Since \( c > ab + a + b \), we have \( a \leq 98 \) and from \( b^3 < 1.001a^4c < 100.1a^4b \) we have \( b < 10^{11/2}a^2 < 96089 \) which is in a contradiction with \( b > 10^5 \).

In only remains to check if \( k = 4 \). But in this case we express \( b \) in the terms of \( a \) and \( c \) and get

\[
b_\pm = \frac{-B \pm \sqrt{B^2 + 4AE}}{2A}
\]

where \( A = 400ac - 9216 > 0 \), \( B = c(16a^2 - 640a + 832) + 64a + 30720 \) and \( E = 16c^2 + 1216ac + 25600a^2 - 256 > 0 \).

We have \( B^2 + 4AE > 4AE > 25536ac^3 \), so it is not hard to see that \( b_\pm < 0 \). Also, \( B < 0 \) when \( a \in [2, 38] \), and \( B > 0 \) otherwise.

When \( B > 0 \), \( b_+ < \frac{\sqrt{B^2 + 4AE}}{2A} < \frac{\sqrt{256a^4c^2 + 400ac^2(16c + 2000a)}}{798ac} \) and since \( a < c^{2/7} \), we get \( b_+ < \frac{84a^{1/2}c^{3/2}}{798ac} < 0.106c^{1/2}a^{-1/2} \). On the other hand, \( b_+ > \frac{\sqrt{25536ac^3 - 16c^2}}{800ac} > 0.18c^{1/2}a^{-1/2} \), which is a contradiction with the previous inequality. In the last case, when \( B < 0 \), we have \( |B| < 7232c \) and get similar contradiction.

Now it only remains to show that \( v_{11} \neq w_7 \). Let us assume contrary, that \( v_{11} = w_7 \). We have for \((z_0, z_1) = (\pm t, \pm s)\) that

\[
\begin{align*}
&c r (a^5c^5 + 11a^4c^3 + 44a^3c^3 + 77a^2c^3 + 55ac - b^3c^3 - 7b^2c^2 - 14bc + 4) = \\
&\mp st (a^5c^5 + 9a^4c^3 + 28a^3c^3 + 35a^2c^2 + 15ac - b^3c^3 - 5b^2c^2 - 6bc). 
\end{align*}
\]
a) **Case** $z_0 = -t$

Since $cr > st$ we have

\[
a^5c^5 + 11a^4c^4 + 44a^3c^3 + 77a^2c^2 + 55ac - b^3c^3 - 7b^2c^2 - 14bc + 4 < \\
< a^5c^5 + 9a^4c^4 + 28a^3c^3 + 35a^2c^2 + 15ac - b^3c^3 - 5b^2c^2 - 6bc
\]

which leads to

\[
a^4c^2 < b^2.
\]

But, this is a contradiction with $c > ab$ and $a \geq 1$.

b) **Case** $z_0 = t$

Here we have

\[
a^5c^5 + 11a^4c^4 + 44a^3c^3 + 77a^2c^2 + 55ac - b^3c^3 - 7b^2c^2 - 14bc + 4 < \\
< -a^5c^5 - 9a^4c^4 - 28a^3c^3 - 35a^2c^2 - 15ac + b^3c^3 + 5b^2c^2 + 6bc
\]

which gives us an inequality $a^5c^2 < b^3$. Together with $ab < c$, we get

\[
b > a^{7/3}, \quad b > 2154a, \quad c > a^{10/3}.
\]

If we consider the equality $v_{11} = w_7$ modulo $c^2$, and use the fact that $st(cr - st) \equiv 16 \pmod{c}$ we have that a congruence

\[
4r(cr - st) \equiv 16(6b - 15a) \pmod{c}
\]

must hold. On the other hand, from $v_{11} = w_7$, as in [7], we can derive that

\[
b^3 < 1.001a^5c^2.
\]

Also, since we can show that $4r(cr - st) < 8.02c$ and $16(6b - 15a) < 2.1c$, we have

\[
kc = 4r(cr - st) - 16(6b - 15a), \quad k \in \{2, -1, 0, 1, 2, \ldots, 8\}.
\]

In the case $k = 0$, we have $7.6ab < 4r(cr - st) = 16(6b - 15a) < 96b$ which means that $a \leq 12$. We can express $c$ as a solution of quadratic equation and get

\[
c_\pm = \frac{-B \pm \sqrt{B^2 - 4AD}}{2A}
\]

where $A = r^2$, $B = -(13b - 29a)r^2$ and $D = 4((6b - 15a)^2 - r^2)$. It is easy to see that

\[
c_\pm < \frac{|B|}{A} < \frac{13br^2}{r^2} = 13b
\]

so $b < 1.001a^5 \cdot 13^2 = 169.169a^5$ and we get $a \geq 4$ since $b > 10^5$. For $D(4)$-pairs in the range $a \in [4, 12]$, $b \in [10^{5}, 169.169 \cdot a^5]$, we check by computer explicitly that there is no integer $c_\pm$ given by this equation.

In the case where $k \neq 0$, we observe equality $kc = 4r(cr - st) - 16(6b - 15a)$ and express $c$ as

\[
c_\pm = \frac{-B \pm \sqrt{B^2 - 4AD}}{2A},
\]
where \( A = (4r^2 - k)^2 - 4r^2ab > 12a^2b^2, \) \( B = -(32(4r^2 - k)(6b - 15a) - 16r^2(a + b)) > -800r^2b \) and \( D = 64(4(6b - 15a)^2 - r^2) > 0. \) We see that
\[
c \in \frac{|B|}{A} < \frac{800r^2b}{12a^2b^2} < \frac{400}{3a} < 134
\]
which is in contradiction with \( c > b > 10^5. \)

\[\Box\]

**Corollary 2.15.** Let \( c > ab + a + b, \) \( v_m = w_n, \) \( n > 2, \) and let one of the cases (ii) – (iv) or case (i) where \( |z_0| = \frac{1}{2}(cr - st) \) from Lemma 2.4 hold. Then \( \min\{m, n\} \geq 8. \)

**Proof.** It follows from Proposition 2.14 and Lemma 2.13. \[\Box\]

### 3. Proof of Theorem 1.3

From now on, when we assume that \( z = v_m = w_n \) has a solution for some \( D(4)\)-triple \( \{a, b, c\}, \) we are usually considering only solutions where \( d = \frac{a^2}{c} > d_+. \)

**Lemma 3.1.** Let us assume that \( c > 0.243775a^2b^3/5 \) and \( z = v_m = w_n \) has a solution for \( m \) and \( n, n \geq 4. \) Then \( m \equiv n \pmod{2} \) and

i) if \( m \) and \( n \) are even and \( b \geq 2.21a, \) then
\[
n > 0.45273b^{-9/28}c^{5/28},
\]
and if \( b < 2.21a \) then
\[
n > \min\{0.35496a^{-1/2}b^{-1/8}c^{1/8}, 0.177063b^{-11/28}c^{3/28}\};
\]

ii) if \( m \) and \( n \) are odd, then
\[
n > 0.30921b^{-3/4}c^{1/4}.
\]

**Proof.** Since from \( b > 10^5 \) and \( c > 0.243775a^2b^3/5 \) we have \( c > ab^2 \) and \( \tau^{-1} < b/a, \) cases i) and ii) from Lemma 2.12 cannot hold. So, we see that only options from Lemma 2.4 are i), when \( |z_0| = 2 \) or \( |z_0| < 1.219b^{-5/14}c^{9/14}, \) and iv). In each case we have \( m \equiv n \pmod{2}. \)

First, let us observe the case \( |z_0| < 1.219b^{-5/14}c^{9/14}, \) \( z_0 = z_1 \) and \( m \) and \( n \) even. Since \((z_0, x_0)\) satisfies an equality
\[
cl_0^2 - ax_0^2 = 4(c - a)
\]
we have
\[
x_0^2 \leq \frac{a}{c} \cdot 1.21972b^{-5/7}c^{9/7} + 4 \left(1 - \frac{a}{c}\right) < 1.7109ac^{2/7}b^{-5/7},
\]
where we have used the estimate \( c^2b^{-5} > 0.243775^2b^2 > 5.9426 \cdot 10^8. \) So,
\[
x_0 < 1.30802a^{1/2}b^{-5/14}c^{1/7}.
\]
Similarly, we get
\[
y_1 < 1.21972b^{1/7}c^{1/7}.
\]
From [15, Lemma 12] we have that the next congruence holds
\[ a\varepsilon_0 m^2 - b\varepsilon_1 n^2 \equiv t\varepsilon_1 n - s\varepsilon_0 m \pmod{c}. \tag{11} \]

From \( b > 10^5 \) and \( c > 0.243775b^{3.5} \) we have \( c > b^{3.377} \), so we can use \( \varepsilon = 3.377 \) in the inequality from Lemma 2.8 and get
\[ m < 1.2975n + 0.9811. \]
This implies that the inequality \( m < 1.34n \) holds for every possibility \( m \) and \( n \) even except for \((m, n) = (6, 4)\), which we will observe separately.

Now we observe the case where \( b \geq 2.21a \) and let us assume contrary, that \( n \geq 0.45273b^{-9/28}c^{5/28} \). Then from \( c > b > 10^5 \) we have
\[ am^2|z_0| < a \cdot 1.342 \cdot 0.45273^2 b^{-9/14} c^{5/14} \cdot 1.2197 b^{-5/14} c^{9/14} \]
\[ < a \cdot 0.4489 b^{-1} c < \frac{c}{4}, \]
and from inequalities (9) and (10) we also have
\[ t\varepsilon_1 n < (bc + 4)^{1/2} 1.2197 c^{1/7} \cdot 0.45273 b^{-9/28} c^{5/28} \]
\[ < \frac{2.21 b^{1/2} c^{1/2}}{c^{9/28} b^{5/28}} \cdot \frac{c}{4} = \frac{2.21}{c^{5/28} c^{9/14} c^{5/14} c^{9/14} c^{9/14} c^{9/14} c^{9/14} c^{9/14} c^{9/14}} < \frac{c}{4}, \]
and similarly
\[ s\varepsilon_0 m < (ac + 4)^{1/2} \cdot 1.34 n \cdot x_0 < 0.79361 a b^{-19/28} c^{23/28} < \frac{c}{4}. \]

In the case \((m, n) = (6, 4)\) we can prove the same final inequalities. So, from congruence (11) we see that an equality
\[ a\varepsilon_0 m^2 + s\varepsilon_0 m = b\varepsilon_1 n^2 + t\varepsilon_1 n \tag{12} \]
must hold.

On the other hand, from equation \( a\varepsilon_0^2 - c\varepsilon_0^2 = 4(a - c) \), since \(|z_0| \neq 2\) in this case, and \( c \mid (z_0^2 - 4) \) we have \( z_0^2 \geq c + 4 \). Let us assume that \( z_0^2 < \frac{5c}{a} \), then we would have \( c(x_0^2 - 9) + 4a < 0 \) and since \( c > 4a \) we must have \( x_0 = 2 \) and \(|z_0| = 2\), which is not our case. So, here we have \( z_0^2 \geq \max\{c + 4, \frac{5c}{a}\} \).

Now,
\[ 0 \leq \frac{s\varepsilon_0}{a|z_0|} - 1 = \frac{4a^2}{a|z_0|(|s\varepsilon_0 + a|z_0|)} - \frac{2a^2}{a^2 z_0} + \frac{2ac - 2a^2}{a^2 z_0^2} \]
\[ < 2 \left( \frac{1}{ac} + 4 \left( 1 - \frac{a}{c} \right) \frac{1}{a^2 z_0^2} \right) + \left( \frac{1}{z_0} \left( \frac{2c}{a} - 2 \right) \right) \]
\[ < \frac{18}{5ac} + \left( \frac{2}{5} - \frac{2a}{5c} \right) \leq 0.000036 + 0.4 = 0.400036, \]
and similarly
\[ 0 \leq \frac{t\varepsilon_1}{b|z_0|} - 1 < 0.00004. \]
When \( z_0 > 0 \), i.e. \( z_0 > 2 \), we have
\[
\begin{align*}
bz_0n(n + 1) < bz_0n(n + \frac{ty_1}{bz_0}) = az_0m(m + \frac{s\tau_0}{az_0}) < az_0m(m + 1.400036).
\end{align*}
\]

Since \( m \geq 4 \) we have from the previous inequality that \( 2.21n(n + 1) < 1.35001m^2 < 1.35001(1.2975n + 0.9811) \) must hold, but then we get \( n < 1 \), an obvious contradiction.

On the other hand, if \( z_0 < 0 \), i.e. \( z_0 < -2 \), we similarly get
\[
am(m - 1) > bn(n - 1.00004).
\]
Since, \( m < 1.2975n + 0.9811 \) and \( b > 2.21a \) we have \( n \leq 6 \). If \( b \geq 2.67a \), we would have \( n < 4 \). So, it only remains to observe the case \( 2.21a \leq b < 2.67a \) and \( n \leq 6 \). In that case we have \( c > 0.243775a^2c^{3.5} > 0.03419b^{5.5} \), so we can put \( \varepsilon = 5.2 \) in Lemma 2.8 and get \( m < 1.1936n + 1.0226 \). Inserting in the inequality (13) yields that only the case \((m, n) = (4, 4)\) remains, but it doesn’t satisfy equation (13).

Now, let us observe the case where \( b < 2.21a \). We again consider the congruence (11), which after squaring and using \( z_0^2 \equiv t^2 \equiv s^2 \equiv 4 \pmod{c} \) yields a congruence
\[
4((am^2 - bn^2)^2 - y_1^2n^2 - x_0^2m^2) \equiv -2tsx_0y_1mn \pmod{c}.
\]
(14)
Let us denote \( C = 4((am^2 - bn^2)^2 - y_1^2n^2 - x_0^2m^2) \), and (14) multiplied by \( s \) and by \( t \) respectively shows that
\[
\begin{align*}
Cs & \equiv -8tx_0y_1mn \pmod{c}, \\
Ct & \equiv -8sx_0y_1mn \pmod{c}.
\end{align*}
\]
(15)(16)
Now, assume that \( n \leq \min\{0.35496a^{-1/2}b^{-1/8}c^{1/8}, 0.177063b^{-11/28}c^{3/28}\} \). Then also \( n \leq 0.45273b^{-9/28}c^{5/28} \) holds, so we again have an equality in the congruence (11), i.e.
\[
az_0m^2 + sx_0m = bz_0n^2 + ty_1n.
\]
It also holds \( x_0^2 < y_1^2 < 1.4877b^{2/7}c^{2/7} \). Since \( b < 2.21a \) we have \( c > 0.04991b^{5.5} \), so we can take \( \varepsilon = 5.239 \) in Lemma 2.8 and we get \( m < 1.1921n + 1.0232 \), and since we know \( m, n \geq 4 \) and \( m \) and \( n \) even, we also have \( m < 1.34n \) from this inequality. This yields
\[
|Cs| < |Ct| = |4t((am^2 - bn^2)^2 - (y_1^2n^2 + x_0^2m^2))| < \max\{4tb^2m^4, 8ty_1^2m^2\} < \max\{12.896718b^2n^2\sqrt{bc} + 4, 21.370513b^{2/7}c^{2/7}n^2\sqrt{bc} + 4\}.
\]
On the other hand, we have from our assumption on \( n \) that
\[
12.896718b^2n^2\sqrt{bc} + 4 < 12.896718 \left(\frac{b}{a}\right)^2 0.35496^4 \cdot 1.000001c^{1/2} < c,
\]
and
\[
21.370513b^{2/7}c^{2/7}n^2\sqrt{bc} + 4 < 21.370513 \cdot 0.177063^2 \cdot 1.000001c < c,
\]
so, \(|Cs| < |Ct| < c\). On the other hand, \(8s x_0 y_1 mn < 8t x_0 y_1 mn\) and
\[
8t x_0 y_1 mn < 8y_1^2 \cdot 1.34 n^2
\]
\[
< 8 \cdot 1.000001 b^{1/2} c^{1/2} 1.4877 b^{3/7} c^{2/7} \cdot 1.34 \cdot 0.177063 b^{-11/14} c^{3/14}
\]
\[
< 0.5 c < c,
\]
So, in congruences (15) and (16) we can only have
\[
Cs = kc - 8t x_0 y_1 mn, \quad Ct = lc - 8s x_0 y_1 mn, \quad k, l \in \{0, 1\}.
\]
If \(k = l = 0\), we would have \(s = t\), which is not possible. When \(k = l = 1\),
we get \(c = 8(t + s)x_0 y_1 mn < 0.5 c + 0.5 c < c\), a contradiction. In the case
\(k = 0\) and \(l = 1\) we get \(cs = 8(s^2 - t^2)x_0 y_1 mn < 0\), and in the case \(k = 1\)
and \(l = 0\), we have \(ct = 8(t^2 - s^2)x_0 y_1 mn\), so \(c(t - s) < 8(t^2 - s^2)x_0 y_1 mn\),
which leads to a contradiction as in the case \(k = l = 1\).
So, \(n > \min\{0.35496a^{-1/2} b^{-1/8} c^{1/8}, 0.177063 b^{-11/28} c^{3/28}\}\). Let us assume that \(n \leq 0.30921 b^{-3/4} c^{1/4}\). In this case we have \(m < 1.2975 n + 0.9811\) and since \(m \geq n \geq 5\) both odd we also have \(m < 1.47 n\). Notice that
\(2t (am(m+1)-bn(n+1)) < 2bm(m+1)\) holds. Also, from \(c > 0.243775 b^{3/5} > 7.7 \cdot 10^{16}\) we have \(b < 0.243775^{-2/7} c^{2/7}\). So it suffice to observe that
\[
2bm(m+1) < 2\sqrt{bc} \cdot 1.000001b \cdot 1.21 m^2
\]
\[
< 5.22944 b^{3/2} c^{1/2} < 5.22944 \cdot 0.243775^{-2/7} c^{2/7} \cdot 2^{3/2} c^{1/2}
\]
\[
< 7.82711 c^{13/14} < 0.5 c
\]
\[
2rt (m-n) < 2\sqrt{ab} \left(1+\frac{1}{\sqrt{ab}}\right) \sqrt{bc} \left(1+\frac{1}{\sqrt{bc}}\right) (0.2962n + 0.85194)^2
\]
\[
< 2a^{1/2} b^{1/2} c^{1/2} \cdot 1.00318 \cdot 0.51^{1/2} \cdot 0.30921 b^{-3/2} c^{1/2}
\]
\[
< 0.0499 (a/b)^{1/2} c
\]
which means that we have equalities in congruences (17) and (18) and implies
\[
\pm \frac{2rs}{t} (n - m) = \pm \frac{2rt}{s} (n - m).
\]
Since \(s \neq t\), only possibility is \(n = m\), but then \(2t (am(m+1)-bn(n+1)) = 0\)
implies \(a = b\), a contradiction. So, our assumption for \(n\) was wrong and
\(n > 0.30921 b^{-3/4} c^{1/4}\) when \(n\) and \(m\) are odd. \(\square\)

Various altered versions of Rickert’s theorem from [23] and results derived
from it are often used when considering problems of \(D(1)-m\)-tuples and \(D(4)-m\)-tuples. In this paper we will use one of the results from [2] and we will
give, without proof since it is similar as in the other versions, a new version which improves this result in some cases.

**Lemma 3.2.** [2, Lemmas 6 and 7] Let \( \{a, b, c, d\} \) be a \( D(4) \)-quadruple, \( a < b < c < d \) and \( c > 308.07a' b (b - a)^2 a^{-1} \), where \( a' = \max \{4a, 4(b - a)\} \). Then

\[
n < \frac{2 \log(32.02a'd^2c^2) \log(0.026ab(b - a)^{-2}c^2)}{\log(0.00325a(a')^{-1}b^{-1}(b - a)^{-2}c) \log(bc)}.\]

Combining results from [2, Lemma 7] and [8, Lemma 3.3] we can prove a generalization of [5, Lemma 7].

**Lemma 3.3.** Let \( \{a, b, c, d\} \) be a \( D(4) \)-quadruple, \( a < b < c < d \) and \( c > 59.488a'b(b - a)^2a^{-1} \), where \( a' = \max \{4a, 4(b - a)\} \). If \( z = v_m = w_n \) for some \( m \) and \( n \) then

\[
n < \frac{8 \log(8.40335 \cdot 10^{13}a^{1/2}(a')^{1/2}b^2c) \log(0.20533a^{1/2}b^{1/2}(b - a)^{-1}c)}{\log(bc) \log(0.016858a(a')^{-1}b^{-1}(b - a)^{-2}c)}.\]

**Proposition 3.4.** Let \( \{a, b, c, d\} \) be a \( D(4) \)-quadruple such that \( a < b < c < d \) and that equation \( z = v_m = w_n \) has a solution for some \( m \) and \( n \). If \( b \geq 2.21a \) and \( c > \max \{0.24377a^2b^{3.5}, 2.3b^5\} \) or \( b < 2.21a \) and \( c > 1.1b^{7.5} \) then \( d = d_+ \).

**Proof.** Let us assume that \( d > d_+ \). Since \( c > 0.24377a^2b^{3.5} \) in both cases we can use Lemma 3.1, and as in the proof of that Lemma, we have that \( m \) and \( n \) have the same parity. Also, since in any case \( c > 2.3b^5 > 308.07a'b(b - a)^2a^{-1} \) we can use Lemmas 3.2 and 3.3, but Lemma 3.2 will give better results in these cases.

Let us assume that \( m \) and \( n \) are even and \( b \geq 2.21a \). Then we have 

\[
n > 0.45273b^{-9/28} c^{5/28}, \quad a' = \max \{4a, 4(b - a)\} = 4(b - a), \quad a < b - a < \frac{b}{2.21}.\]

We estimate

\[
32.02a'd^2c^2 = 32.02a' \cdot 4(b - a)b^4c^2 < 57.955b^6c^2;
\]

\[
0.026ab(b - a)^{-2}c^2 < \frac{0.026}{2.21} \frac{b^2}{1.21^2} \frac{b^2}{b^{-2}} \cdot \frac{b^{-2}c^2}{0.0393c^2},
\]

\[
0.00325a(a')^{-1}b^{-1}(b - a)^{-2}c > 0.00325 \cdot 4^{-1}(b - a)^{-3}b^{-1}c > 0.0008125b^{-4}c.
\]

Now, from Lemma 3.2 we have an inequality

\[
0.45273b^{-9/28} c^{5/28} < \frac{2 \log(57.955b^6c^2) \log(0.0393c^2)}{\log(bc) \log(0.0008125b^{-4}c)},
\]

where the right hand side is decreasing in \( c \) for \( b > 0, c > 2.3b^5 \) so we can observe the inequality in which we have replaced \( c \) with \( 2.3b^5 \) which gives us \( b < 19289 \), a contradiction with \( b > 10^5 \).

Let us observe the case when \( m \) and \( n \) are even and \( 2a < b < 2.21a \). Then

\[
a' = 4(b - a), \quad \frac{b}{2} < b - a < \frac{121}{221}b.\]

Denote

\[
F \in \{0.35496a^{-1/2}b^{-1/8}c^{3/8}, 0.177063b^{-11/28}c^{3/28}\} > \{0.50799b^{-9/16}, 0.17888b^{23/56}\}.
\]
Then by Lemma 3.2
\[ F < \frac{2 \log(35.0627b^6c^2) \log(0.052c^2)}{\log(bc) \log(0.0022397b^{-3}c)}, \]
and the right hand side of this inequality is decreasing in \( c \) for \( b > 0 \), \( c > 1.1b^{7.5} \), and for each possibility for \( F \) we get \( b < 722 \) and \( b < 81874 \) respectively, a contradiction in either case.

In the case when \( m \) and \( n \) are even and \( b < 2a \), we have \( a' = 4a \) and \( 57 < b-a < \frac{b}{2} \) and with \( F \) defined as before, we have \( F > \{0.35921b^{9/16}, 0.17888b^{23/56}\} \) and
\[ F < \frac{2 \log(128.088b^6c^2) \log(0.0000081b^2c^2)}{\log(bc) \log(0.00325b^{-3}c)}. \]
Again, right hand side is decreasing in \( c \) for \( c > 1.1b^{7.5} \). We get \( b < 1396 \) for the first choice for \( F \), and \( b < 98413 \) for the second, again, a contradiction.

It remains to consider the case when \( m \) and \( n \) are odd. If \( b \geq 2a \) we have \( a' = 4(b-a) \) and \( c > 2.3b^5 \), so similarly as before \( 32.02ad'c^2 < 64.04b^6c^2, \)
\[ 0.026ab(b-a)^{-2}c^2 < 0.052c^2, \]
\[ 0.00325a(a')^{-1}b^{-1}(b-a)^{-2}c > 0.0008125b^{-4}c. \]
In this case we have \( n > 0.30921b^{-3/4}c^{1/4} \) so by Lemma 3.2 we observe an inequality
\[ 0.30921b^{-3/4}c^{1/4} < \frac{2 \log(64.04b^6c^2) \log(0.052c^2)}{\log(bc) \log(0.0008125b^{-4}c)}, \]
and since the right hand side is decreasing in \( c \) for \( c > 2.3b^5 \) we get \( b < 97144 \), a contradiction.

If \( b < 2a \) and \( c > 1.1b^{7.5} \) we observe \( 0.30921b^{-3/4}c^{1/4} < \frac{2 \log(128.088b^6c^2) \log(0.0000081b^2c^2)}{\log(bc) \log(0.00325b^{-3}c)} \)
and since the right hand side is decreasing in \( c \) for \( c > 1.1b^{7.5} \) we get \( b < 48 \), a contradiction.

\[ \Box \]

**Proof of Theorem 1.3.** Let us assume that \( d = d_+ \). If \( n \geq 3 \) then
\[ z \geq w_3 > \frac{c}{2y_1}(t-1)^2 > \frac{c}{2.000022 \sqrt{bc}} \cdot 0.999bc \]
\[ > 0.499b^{3/4}c^{7/4} > 0.499bc^{6/4} > 157bc \]
where we have used Lemma 2.5 and \( bc > 10^{10} \) from Lemma 2.2. On the other hand, when \( d = d_- \) we have \( z = \frac{1}{2}(cr + st) < cr < cb \), which is an obvious contradiction. So, when \( d = d_+ \), we must have \( n \leq 2 \). Also, since \( a < b < c < d \), from the proof of [15, Lemma 6] we see that when \( n \leq 2 \),
only possibility is \( d = d_+ \) when \((m, n; z_0, z_1) \in \{(1, 1; t, s), (1, 2; t, \frac{1}{2}(st - cr)), (2, 1; \frac{1}{2}(st - cr), s), (2, 2; \frac{1}{2}(st - cr), \frac{1}{2}(st - cr))\}\).

Let us assume that \( d > d_+ \). Then from Lemma 2.9 we have \( n \geq 4 \). Here we only observe a possibility when \( m \) and \( n \) are even and \( z_0 = z_1 \notin \{2, \frac{1}{2}(cr - st)\} \). Then from [15] we know that for \( d_0 = \frac{a^2 - 4}{c} < c \), \( D(4) \)-quadruple \( \{a, b, d_0, c\} \) is irregular.

Denote \( \{a_1, a_2, a_3\} = \{a, b, d_0\} \) such that \( a_1 < a_2 < a_3 \).

If \( a_3 \leq 0.234775a_1^{2.5}a_2^{3.5} \) holds then by Lemma 2.10 we also have an inequality \( c > 0.240725a_1^{4.5}a_3^{5.5} \geq 0.240725a_1^{4.5}b^{5.5} \). If \( b > 2.21a \), since \( b > 10^5 \), this inequality implies \( c > 76a_1^{4.5}b^{5.5} \). On the other hand, if \( b \leq 2.21a \), we have \( a \geq 45249 \) and \( c > 0.240725 \cdot 2.21^{-2}a^{2.5}b^{7.5} > 10a_1b^{7.5} \). We see that in either case we can use Proposition 3.4 and conclude \( d = d_+ \), i.e. we have a contradiction with the assumption \( d > d_+ \).

It remains to observe a case \( a_3 > 0.234775a_1^{2.5}a_2^{3.5} \). If \( b = a_2 \) then from Lemma 2.6 we get
\[
c > 0.249979b^{2.5}(0.243775b^{3.5})^{3.5} > 0.00178799b^{14.75}
\]
and it is easy to see that we again have conditions of Proposition 3.4 satisfied and can conclude \( d = d_+ \) is the only possibility. On the other hand, if \( b = a_3 \), then \( b > 0.243775a_1^{2.5}a_3^{3.5} > 0.243775 \cdot 10^{17.5} \) since \( a_2 > 10^5 \), so by Lemma 3.1 we need to consider two cases, first when \( a_2 \geq 2.21a_1 \) then \( n' > 0.45273a_2^{-9/28}b^{5/28} > 0.45273(0.243775^{-2/7}b^{2/7})^{-9/28}b^{5/28} > 0.3976b^{17/16} > 11 \) so by Lemma 2.6 we have \( z' > w_6 \) and \( c > 0.249969a_1^{4.5}b^{5.5} \) which, as before, yields a contradiction when Proposition 3.4 is applied. Second case is when \( a_2 < 2.21a_1 \). If \( n' \geq 6 \), we have \( c > 0.249969a_1^{4.5}b^{5.5} > b^{7.5} \), since \( a_1 > a_2^{2/5} > 10^{5/2} \) and get a same conclusion as before. If \( n < 6 \), by Lemma 2.9 we see that we have \( m' \) and \( n' \) even, so \( n' = 4 \) and since \( b > 0.0335a_2^6 > a_2^{5.7} \) from Lemma 2.8 we have \( m' < 1.1766n' + 1.0294 \), i.e. \( m' = 4 \). From the proof of [14, Lemma 5] we know that \( m' = n' = 4 \) can hold only when \( |z'_0| < 1.2197(b')^{-5/14}(c')^{9/14} \), i.e. we have a \( 0 < d'_0 < b \) such that \( \{a, d'_0, d_0, b\} \) is irregular \( D(4) \)-quadruple and we can use the same arguments to prove that such quadruple cannot exist by Proposition 3.4 or we have a new quadruple with \( 0 < d''_0 < d'_0 < b \). Since this process cannot be repeated infinitely, for some of those quadruples in the finite process we must have \( n > 6 \) and contradiction with Proposition 3.4.

The last assertion of Theorem 1.3 follows from Lemma 2.12. \( \square \)

4. PROOFS OF THEOREMS 1.4 AND 1.5

Analogously as [4, Proposition 3.1] and [5, Lemma 16] a more general result for the lower bound on \( m \) can be proven.

Lemma 4.1. Let \( \{a, b, c, d\} \) be a \( D(4) \)-quadruple with \( a < b < c < d \) for which \( v_{2m} = w_2n \) has a solution such that \( z_0 = z_1 = \pm 2 \) and \( Lm \geq m \geq n \geq 2 \) and \( m \geq 3 \), for some real number \( L > 1 \).

Suppose that \(a \geq a_0, b \geq b_0, c \geq c_0, b \geq \rho a\) for some positive integers \(a_0, b_0, c_0\) and a real number \(\rho > 1\). Then
\[
m > ab^{-1/2}c^{1/2},
\]
where \(\alpha\) is any real number satisfying both inequalities
\[
\alpha^2 + (1 + 2b_0^{-1}c_0^{-1})\alpha \leq 1
\]
and
\[
4 \left(1 - \frac{1}{L^2}\right) \alpha^2 + \alpha(b_0(\lambda + \rho^{-1/2}) + 2c_0^{-1}(\lambda + \rho^{1/2})) \leq b_0
\]
with \(\lambda = (a_0 + 4)^{1/2}(\rho a_0 + 4)^{-1/2}\).
Moreover, if \(c^r > \beta b\) for some positive real numbers \(\beta\) and \(\tau\) then
\[
m > \alpha \beta^{1/2}c^{(1-\tau)/2}.
\]

**Lemma 4.2.** Let us assume that \(c > b^4\) and \(z = v_m = w_n\) has a solution for some positive integers \(m\) and \(n\). Then \(m \equiv n \pmod{2}\) and \(n > 0.5348b^{-3/4}c^{1/4}\).

**Proof.** As in the proof of Lemma 3.1 we can see that \(m \equiv n \pmod{2}\) when \(c > b^4\), and when \(m\) and \(n\) are even, the only possibility is \(|z_0| = 2\). By Theorem 1.3 we need to consider two cases.

1) Case \(m\) and \(n\) are even, \(|z_0| = |z_1| = 2\).

From Lemma 2.8 we have \(m < 1.252n + 0.9995\), and since from Lemma 2.9 we have \(m \geq 6\) we also have \(n \geq 6\) and \(m < 1.4n\). Using Lemma 4.1 yields \(m > 0.4999998b^{-1/2}c^{1/2}\), i.e.
\[
n > 0.35714b^{-1/2}c^{1/2}.
\]

2) Case \(m\) and \(n\) odd, \(|z_0| = t, |z_1| = s\).

Congruences (17) and (18) from the proof of Lemma 3.1 hold. Let us assume contrary, that \(n < 0.5348b^{-3/4}c^{1/4}\). Then
\[
2t|am(m + 1) - bn(n + 1)| < 2tbn^2 < 2.00004b^{3/2}c^{1/2}n^2 < 0.57204c,
\]
\[
2rt(m - n) < 0.8rtn < 0.800032bc^{1/2}n < 0.42786c,
\]
which means that we have equality in those congruences, so a contradiction can be shown as in the proof of Lemma 3.1.

**Proof of Theorem 1.4.** Let us assume that \(d > d_+\). We prove this as in Proposition 3.4

i) Case \(b < 2a\) and \(c \geq 890b^4\).

Since \(a' = 4a\) and \(b > 105\), inequality
\[
308.07a'b(b - a)^2a^{-1} < 1233b^3 < b^4 < c
\]
holds, so we can use Lemma 3.2 and Lemma 4.2 and observe inequality
\[
0.5348b^{-3/4}c^{1/4} < \frac{2\log(128.08b^6c^2)\log(0.0000081b^2c^2)}{\log(bc)\log(0.00325b^{-3/4}c)}
\]
and since the right hand side is decreasing in \( c \) for \( c > 890b^4 \) we get \( b < 99887 \) a contradiction with \( b > 10^5 \).

ii) Case \( 2a \leq b \leq 12a \) and \( c \geq 1613b^4 \).

Here we have \( a' = 4(b - a) \) and \( \frac{b}{2} \leq b - a \leq \frac{11}{12}b \), so

\[
308.07a'b(b - a)^2a^{-1} < 11400b^3 < b^4 < c.
\]

We observe an inequality

\[
0.5348b^{-3/4}c^{1/4} < \frac{2\log(58.71b^6c^2)}{\log(bc)} \log(0.052c^2) \log(0.0000879035c^3)
\]

which, after using that the right hand side is decreasing in \( c \) for \( c > 1613b^4 \), yields \( b < 99949 \), a contradiction.

iii) Case \( b > 12a \) and \( c \geq 52761b^4 \).

Let us first assume that \( c \geq 52761b^4 \). Here we have \( a' = 4(b - a) \) and \( \frac{b}{12} < b - a < b \), so

\[
308.07a'b(b - a)^2a^{-1} < 1233b^4 < c.
\]

Here we use Lemma 3.3 to obtain inequality

\[
0.5348b^{-3/4}c^{1/4} < \frac{8\log(8.4035 \cdot 10^{13}b^6c)}{\log(bc)} \log(0.002579c^2) \log(0.0042145b^{-4}c)
\]

which, after using that the right hand side is decreasing in \( c \) for \( c > 52761b^4 \), yields \( b < 99998 \), a contradiction.

Now, assume that \( 39247b^4 < c < 52761b^4 \). We can modify the method in the next way. For \( a = 1 \) we only have to notice that the right hand side in the inequality in the Lemma 3.3 is decreasing in \( c \) and insert lower bound on \( c \) and calculate an upper bound on \( b \) from the inequality.

We get \( b < 999994 \), a contradiction. For \( a \geq 2 \), we modify estimate

\[
0.016858a(a')^{-1}b^{-1}(b - a)^{-2}c > 0.0042145b^{-4}c a_0, \text{ where } a_0 = 42 \text{ and get } b < 73454, \text{ a contradiction.}
\]

\[
\Box
\]

**Lemma 4.3.** If \( \{a, b, c, d\} \) is a \( D(4) \)-quadruple with \( a < b < c < d_+ < d \) then

\[
d > \min\{0.249965b^{5.5}c^{6.5}, 0.240725a^{4.5}c^{5.5}\}.
\]

**Proof.** From [16, Lemma 5] and Theorem 1.3 we have that \( m \geq 6 \) or \( n \geq 7 \) so Lemma 2.6 implies \( d > 0.249965b^{5.5}c^{6.5} \) or \( d > 0.240725a^{4.5}c^{5.5} \). \( \Box \)

**Proof of Theorem 1.5.** Let \( \{a, b, c, d\} \) be a \( D(4) \)-quadruple such that \( a < b < c < d_+ < d \) and let us assume contrary, that there exists \( e < c \) such that \( \{e, a, b, c\} \) is an irregular \( D(4) \)-quadruple. Then by Lemma 4.3 \( c > 0.240725a^{4.5}b^{5.5} > 0.240725 \cdot (10^9)^{1.5}b^4 > 52761b^4 \), where \( a' = \min\{a, e\} \geq 1 \) or \( c > 0.249965a^{5.5}c^{6.5} > 52761b^4 \). So, by Theorem 1.4 \( \{a, b, c, d\} \) must be a regular quadruple which is a contradiction. \( \Box \)
5. Proof of Theorem 1.6

In this section we aim to split our problem in several parts. We will consider separately the case when a triple \( \{a, b, c\} \) is regular, i.e. \( c = a+b+2r \), and when it is not regular and then \( c > ab + a + b \). In the latter case, we will consider solutions of the equation \( z = v = w \) without assuming that the inequalities from Lemma 2.5 hold when \(|z_0| \neq 2\). So Lemmas from this section will usually also address separately the case when \( c > ab + a + b \), more specifically, only this case when \((|z_0|, |z_1|) = (t, s) \) and \( z_0z_1 > 0 \), which can then be used to prove the results to all the other cases, except the case \(|z_0| = 2\), by using Lemma 2.13.

In the following pages, we will first introduce results concerning linear forms in three logarithms which will give us that there are at most 3 possible extensions of a triple to a quadruple for a fixed fundamental solution. Then we will use Laurent theorems to give further technical tools and finally prove Theorem 1.6.

This result has already been explained in the proof of Theorem 1.3. We state it again explicitly as we find it important to emphasize when \( d = d_+ \) is achieved. Statement of this Lemma follows notation and cases from Theorem 1.3.

**Lemma 5.1.** Let \( \{a, b, c\} \) be a \( D(4) \)-triple such that \( z = v = w \) has a solution \((m, n)\) for which \( d = d_+ = \frac{z^2 - 4}{c} \). Then only one of the following cases can occur:

i) \( (m, n) = (2, 2) \) and \( z_0 = z_1 = \frac{1}{2}(st - cr) \),

ii) \( (m, n) = (1, 2) \) and \( z_0 = t, z_1 = \frac{1}{2}(st - cr) \),

iii) \( (m, n) = (2, 1) \) and \( z_0 = \frac{1}{2}(st - cr), z_1 = s \),

iv) \( (m, n) = (1, 1) \) and \( z_0 = t, z_1 = s \).

Let \( \{a, b, c\} \) be a \( D(4) \) triple. We define and observe

\[
\Lambda = m \log \xi - n \log \eta + \log \mu,
\]

a linear form in three logarithms, where \( \xi = \frac{s + \sqrt{ac}}{2}, \eta = \frac{t + \sqrt{bc}}{2} \) and \( \mu = \frac{\sqrt{a(x_0 \sqrt{c} + x_0 \sqrt{a})}}{\sqrt{a(y_1 \sqrt{c} + z_1 \sqrt{b})}} \). This linear form and its variations were already studied before, for example in [15]. By Lemma 10 from that paper we know that

\[
0 < \Lambda < \kappa_0 \left( \frac{s + \sqrt{ac}}{2} \right)^{-2m},
\]

where \( \kappa_0 \) is a coefficient which is defined in the proof of the Lemma with

\[
\kappa_0 = \frac{(z_0 \sqrt{a} - x_0 \sqrt{c})^2}{2(c - a)}.
\]

**Lemma 5.2.** Let \((m, n)\) be a solution of the equation \( z = v = w \) and assume that \( m > 0 \) and \( n > 0 \). Then

\[
0 < \Lambda < \kappa_0 \xi^{-2(m-d)},
\]
where
i) \((\kappa, \delta) = (2.7\sqrt{ac}, 0)\) if the inequalities from Lemma 2.5 hold,
ii) \((\kappa, \delta) = (6, 0)\) if \(|z_0| = 2,\)
iii) \((\kappa, \delta) = (1/(2ab), 0)\) if \(z_0 = t,\)
iv) \((\kappa, \delta) = (2.0001b/c, 1)\) if \(z_0 = -t, b > 10^5\) and \(c > ab + a + b.\)

Proof. From Lemma 2.5 we get
\[
0 < x_0\sqrt{c} - z_0\sqrt{a} < 2x_0\sqrt{c} < 2.00634\sqrt{ac}\sqrt{c}.
\]
Inserting in the expression (24) yields \(\kappa_0 < 2.7\sqrt{ac}.\)
If \(|z_0| = 2,\) equation (2) yields \(x_0 = 2.\) Using \(c > 4a\) gives us the desired estimate.
If \(c > a + b + 2r\) and \(z_0 = t\) then \(x_0 = r\) and \(\kappa_0 = \frac{8(c-a)}{(\sqrt{\sqrt{r}}+\sqrt{c})^2} < \frac{1}{2ab}.\)
In the last case we observe that
\[
\kappa_0 \left(\frac{s + \sqrt{ac}}{2}\right)^2 < \frac{2r^2c}{c-a} \cdot \frac{1}{ac} = \frac{2b\left(1 + \frac{4}{c\sqrt{b}}\right)}{c\left(1 - \frac{2}{c}\right)} < 2.0001\frac{b}{c}
\]
where we have used that \(c > ab > 10^5a.\)

The next results is due to Matveev [22] and can be used to get a better lower bound on the linear form (22) than (23).

**Theorem 5.3** (Matveev). Let \(\alpha_1, \alpha_2, \alpha_3\) be a positive, totally real algebraic numbers such that they are multiplicatively independent. Let \(b_1, b_2, b_3\) be rational integers with \(b_3 \neq 0.\) Consider the following linear form \(\Lambda\) in the three logarithms:
\[
\Lambda = b_1 \log \alpha_1 + b_2 \log \alpha_2 + b_3 \log \alpha_3.
\]
Define real numbers \(A_1, A_2, A_3\) by
\[
A_j = \max\{D \cdot h(\alpha_j), |\log \alpha_j|\} \quad (j = 1, 2, 3),
\]
where \(D = [\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3 : \mathbb{Q}].\) Put
\[
B = \max \{1, \max\{(A_j/A_3)|b_j| : j = 1, 2, 3\}\}.
\]
Then we have
\[
\log |\Lambda| > -C(D)A_1A_2A_3 \log(1.5eD \log(eD) \cdot B)
\]
with
\[
C(D) = 11796480e^4D^2 \log(3^{5.5}e^{20.2}D^2 \log(eD)).
\]
Put \(\alpha_1 = \xi = \frac{s+\sqrt{ac}}{2},\) \(\alpha_2 = \mu = \frac{\sqrt{b}(x_0\sqrt{c}+2z_0\sqrt{a})}{\sqrt{c}(y_1\sqrt{c}+z_1\sqrt{b})}\) and \(\alpha_3 = \eta = \frac{t+\sqrt{bc}}{2}.\) We can easily show, similarly as in [14] or [3], that
\[
\frac{A_1}{A_3} < \frac{\log(1.001\sqrt{ac})}{\log(\sqrt{bc})} < 1.0001,
\]
and using similar arguments as in [19] yields \(h(\alpha_2) < \frac{1}{4} \log(P_2)\) where
\[
P_2 = \max\{b^2(c-a)^2, x_0^2abc^2, y_1^2abc^2, x_0y_1a^{1/2}b^{3/2}c^2\}.\]
First, we observe the case $c = a + b + 2r < 4b$, so the case $iii)$ in Lemma 2.12 cannot hold. Also, case $|z_1| = s$ and $|z_0| = \frac{1}{2}(cr - st)$ cannot occur since the same Lemma implies $c = a + b + 2r > a^2b$, i.e. $a = 1$, and this case can be eliminated with the same arguments as in [18]. Also, since $s = a + r$ and $t = b + r$, we have $\frac{1}{2}(cr - st) = 2$, so the only case is $|z_0| = |z_1| = x_0 = y_1 = 2$. As in [19] we easily get

$$A_2 = \max\{4h(\alpha_3), |\log(\alpha_3)|\} < 4 \log c.$$ 

Now we will observe the case when $c > ab + a + b$ and $(|z_0|, x_0, |z_1|, y_1) = (t, r; s, r)$ without assuming the inequalities from the Lemma 2.5. Here we have

$$x_0^2abc^2 = y_1^2abc^2 = (ab + 4)abc^2 < c^4$$

and

$$x_0y_1a^{1/2}b^{3/2}c^2 = r^2(ab)^{1/2}bc^2 < c^{9/2},$$

so in this case

$$A_2 < \frac{9}{2} \log c.$$ 

From Theorem 1.4 we know that $c < 39247b^4$ so $A_3 > 2 \log(0.071c^{5/8}) > \frac{5}{4} \log(0.014c)$ together with Lemma 2.7 implies

$$B = \max\{\frac{A_1}{A_3}, \frac{A_2}{A_3}, n\}$$

$$< \max\{m, 5.724, m + 1\} = \max\{5.724, m + 1\}.$$ 

Since by Lemma 2.9 and Theorem 1.3 we know that $m \geq 6$, we can take $B < m + 1$. Now we apply Theorem 5.3 which proves the next result.

**Proposition 5.4.** For $m \geq 6$ we have

$$\frac{m}{\log(38.92(m + 1))} < 2.7717 \cdot 10^{12} \log \eta \log c.$$ 

If we have a $D(4)$-quadruple $\{a, b, c, d\}$ then $z = \sqrt{cd + 4}$ is a solution of the equation $z = v_m = w_n$ for some $m, n$ and fundamental solutions $(z_0, z_1)$.

Let $\{a, b, c\}$ be a $D(4)$-triple and let us assume that there are 3 solutions to the equation $z = v_m = w_n$ which belong to the same fundamental solution. Let’s denote them with $(m_i, n_i)$, $i = 0, 1, 2$ and let us assume that $m_0 < m_1 < m_2$ and $m_1 \geq 4$. Denote

$$\Lambda_i = m_i \log \xi - n_i \log \eta + \log \mu.$$ 

As in [19] we use an idea of Okazaki from [3] to find a lower bound on $m_2 - m_1$ in the terms of $m_0$. We omit the proof since it is analogous to [19, Lemma 7.1].

**Lemma 5.5.** Assume that $v_{m_0}$ is positive. Then

$$m_2 - m_1 > \Lambda_0^{-1}\Delta \log \eta,$$
where
\[ \Delta = \begin{vmatrix} n_1 - n_0 & n_2 - n_1 \\ m_1 - m_0 & m_2 - m_1 \end{vmatrix} > 0. \]

In particular, if \( m_0 > 0 \) and \( n_0 > 0 \) then
\[ m_2 - m_1 > \kappa^{-1}(ac)^{m_0-\delta} \Delta \log \eta. \]

**Proposition 5.6.** Suppose that there exist 3 positive solutions \((x(i), y(i), z(i)), i = 0, 1, 2, \) of the system of Pellian equations (2) and (3) with \( z(0) < z(1) < z(2) \) belonging to the same class of solutions and \( c > b > 10^5. \) Put \( z(i) = v_m = w_n. \) Then \( m_0 \leq 2. \)

**Proof.** Let us assume contrary, that \( m_0 > 2. \) From Lemmas 2.9 and 5.1 we know that \( m_0 \geq 6 \) and \( m_2 > \kappa^{-1}(ac)^{6-\delta} \log \eta > (2.7 \sqrt{ac})^{-1}(ac)^6 \log \sqrt{bc} > c^5 \) by Lemmas 5.5 and 5.2. After observing that a left hand side in the inequality of Proposition 5.4 is increasing in \( m \) we get
\[ \frac{c^5}{\log(38.92(c^5 + 1))} < 2.7717 \cdot 10^{12} \log^2 c. \]

This inequality cannot hold for \( c > 10^5 \) so we conclude that \( m_0 \leq 2. \)

The next result is proven as in [5] with only some technical details changed, so we omit the proof.

**Proposition 5.7.** Let \( a, b, c \) be integers with \( 0 < a < b < c \) and let \( a_1 = 4a(c - b), a_2 = 4b(c - a), N = abc^2, \) where \( z \) is a solution of the system of Pellian equations (2) and (3). Put \( u = c - b, v = c - a \) and \( w = b - a. \) Assume that \( N \geq 10^5a_2. \) Then the numbers
\[ \theta_1 = \sqrt{a + a_1/N}, \quad \theta_2 = \sqrt{1 + a_2/N} \]
satisfy
\[ \max \left\{ \left| \frac{\theta_1 - p_1}{q} \right|, \left| \frac{\theta_2 - p_2}{q} \right| \right\} > \left( \frac{512.01 a_1 a_2 u N}{a_1} \right)^{-1} q^{-\lambda} \]
for all integers \( p_1, p_2, q \) with \( q > 0, \) where \( a_1' = \max\{a_1, a_2 - a_1\} \) and
\[ \lambda = 1 + \frac{\log \left( \frac{256 a_1 a_2 u N}{a_1} \right)}{\log \left( \frac{0.02636 N^2}{a_1 a_2 (a_2 - a_1)uvw} \right)}. \]

**Lemma 5.8.** Let \((x(i), y(i), z(i))\) be positive solutions to the system of Pellian equations (2) and (3) for \( i \in \{1, 2, \} \), and let \( \theta_1, \theta_2 \) be as in Proposition 5.7 with \( z = z_1. \) Then we have
\[ \max \left\{ \left| \frac{\theta_1 - acy(z_1 y(2))}{abz(z_1 z(2))} \right|, \left| \frac{bcx(z_1 y(2))}{abz(z_1 z(2))} \right| \right\} < \frac{2c^{3/2}}{a^{3/2} z(2)}. \]
Proof. It is not hard to see that from Theorem ?? we have
\[ \left| \frac{1 + \frac{a_1}{N}}{\sqrt{\frac{b y(1) z(2)}{2 b \sqrt{b y(1) z(2)} - y(2) \sqrt{c}}} \right| = \frac{w(1) y(2) - y(2) \sqrt{c}}{2 b \sqrt{b y(1) z(2)}} \leq \frac{2 c^{3/2}}{b^{3/2}} z^{2} \]
and similarly
\[ \left| \frac{1 + \frac{a_2}{N}}{\sqrt{\frac{b y(1) z(2)}{2 b \sqrt{b y(1) z(2)} - y(2) \sqrt{c}}}} \right| < \frac{2 c^{3/2}}{a^{3/2}} z^{2} \]

\[ \Box \]

Proposition 5.9. Suppose that \{a, b, c, d_i\} are D(4)-quadruples with \( a < b < c < d_1 < d_2 \) and \( x(i), y(i), z(i) \) are positive integers such that \( ad_i + 1 = x_i^2 \), \( bd_i + 1 = y_i^2 \) and \( cd_i = z_i^2 \) for \( i \in \{1, 2\} \).

i) If \( n_1 \geq 8 \), then
\[ n_2 < \frac{(n_1 + 1)(1.3505 n_1 + 4.75675)}{0.479 n_1 - 3.82175} - 1.1. \]

More specifically, if \( n_1 = 8 \), \( n_2 < 2628 n_1 \), and if \( n_1 \geq 9 \) then \( n_2 < 83 n_1 \).

ii) If \( c \geq a + b + 1 \) and \( (z_0, z_1) = (t, s) \), \( z_0 z_1 > 0 \) and \( n_1 \geq 9 \) then
\[ n_2 < \frac{(n_1 + 1)(2.5147 n_1 + 5.11467)}{0.4853 n_1 - 3.85292} - 1 < 60 n_1. \]

Proof. Put \( N = a b^2 \), \( p_1 = a y(1) y(2) \), \( p_2 = b c x(1) x(2) \) and \( q = a b z(1) z(2) \) in Proposition 5.7 and Lemma 5.8. We get
\[ z^{2 - \lambda} < 4096 a^{\lambda - 3/2} b^{\lambda + 3} c^{7/2} z^{\lambda + 2} \]

We use estimates for fundamental solutions from Lemma 2.5 and the inequality from the proof of Lemma 2.6 which gives us
\[ 0.49999 - 0.99999^{n_1 - 1}(bc)^{\frac{n_1 - 1}{2} - \frac{1}{4}} c < w_{n_1} < 1.000011 - 1.00001^{n_1 - 1}(bc)^{\frac{n_1 - 1}{2} + \frac{1}{4}} c. \]

Since \( z(1) = w_{n_1} \), we use this inequality to show that
\[ 256 a_1 a_2 N \sqrt{\frac{a_1}{a_1}} < (1.000026 bc)^{n_1 + 3.5} \]
and
\[ 0.02636 N^2 \sqrt{\frac{a_2}{a_1}} (a_2 - a_1) uu w > (0.41 bc)^{2 n_1 - 4}, \]
where we have used the assumption \( n_1 \geq 8 \). So
\[ 2 - \lambda > \frac{0.4795 n_1 - 3.82175}{n_1 - 2}. \]

Now we can show that
\[ z^{0.4795 n_1 - 3.82175} < 4096^{n_1 - 2}(bc)^{4.0205 n_1 - 4} z^{(1)} \]
On the other hand
\[ z(1) > 1.2589^{n_1 - 1}(bc)^{0.49 n_1 - 0.24}. \]
By combining these inequalities we get
\[ z(2) < z(1) \]
where \( \sigma = \frac{3.5205n_1 + 1.75675}{2} = 3.5205n_1 + 3.75675 \) and

\[ n_2 \geq n_1 \sigma + 1.1(\sigma - 1) \]
If \( n_2 \geq n_1 \sigma + 1.1(\sigma - 1) \)
we would get a contradiction from
\[ z(2) z(1) = (\frac{2\sqrt{b}}{y(1)\sqrt{c} + z(1) \sqrt{b}})^{\sigma - 1} \xi^{n_2 - n_1} \frac{1 - A\xi^{-2n_1}}{(1 - A\xi^{-2n_1})^2} > 1, \]
where \( A = \frac{y(1)\sqrt{c} - z(1) \sqrt{b}}{y(1)\sqrt{c} + z(1) \sqrt{b}} \) and \( \xi = \frac{t + \sqrt{bc}}{2} \) as before. So, \( n_2 < n_1 \sigma + 1.1(\sigma - 1) \)
must hold, which proves the first statement.

The second statement is proven analogously by using an inequality
\[ 0.7435 \cdot (ab)^{-1/2} (t - 1)^{n_1 - 1} < w_{n_1} < 1.0001 (ab)^{1/2} (t^{-n_1 - 1}). \]
Notice that in this case we didn’t use Lemma 2.5 since we have explicit values \((z_0, z_1) = (t, s)\).

We now observe a linear form in two logarithms
\[ \Gamma = \Lambda_2 - \Lambda_1 = j \log \frac{s + \sqrt{ac}}{2} - k \log \frac{t + \sqrt{bc}}{2} = (m_2 - m_1) \log \frac{s + \sqrt{ac}}{2} - (n_2 - n_1) \log \frac{t + \sqrt{bc}}{2} \]
for which we know that \( \Gamma \neq 0 \) since it is not hard to show that \( \xi \) and \( \eta \) are multiplicatively independent.

From Lemma 5.2 we have that \( \Lambda_1, \Lambda_2 \in \left(0, \kappa(\xi)^{-2m_1}\right) \) so
\[ 0 < |\Gamma| < \kappa \xi^{-2m_1}. \]
We can now use Laurent’s theorem from [21] to find a lower bound on \( |\Gamma| \), similarly as in [5] and [19].

**Theorem 5.10 (Laurent).** Let \( \gamma_1 \) and \( \gamma_2 \) be multiplicatively independent algebraic numbers with \( |\gamma_1| \geq 1 \) and \( |\gamma_2| \geq 1 \). Let \( b_1 \) and \( b_2 \) be positive integers. Consider the linear form in two logarithms
\[ \Gamma = b_2 \log \gamma_2 - b_1 \log \gamma_1, \]
where \( \log \gamma_1 \) and \( \log \gamma_2 \) are any determinations of the logarithms of \( \gamma_1, \gamma_2 \) respectively. Let \( \rho \) and \( \mu \) be real numbers with \( \rho > 1 \) and \( 1/3 \leq \mu \leq 1 \). Set
\[ \sigma = \frac{1 + 2\mu - \mu^2}{2}, \quad \lambda = \sigma \log \rho. \]
Let $a_1$ and $a_2$ be real numbers such that
\[
a_i \geq \max\{1, \rho |\log \gamma_i| - \log |\gamma_i| + 2Dh(\gamma_i)\}, \quad i = 1, 2,
\]
and
\[
a_1a_2 \geq \lambda^2,
\]
where $D = [\mathbb{Q}(\gamma_1, \gamma_2) : \mathbb{Q}]/[\mathbb{R}(\gamma_1, \gamma_2) : \mathbb{R}]$. Let $h$ be a real number such that
\[
h \geq \max\left\{D \left(\log \left(\frac{b_1}{a_2} + \frac{b_2}{a_1}\right) + \log \lambda + 1.75\right) + 0.06, \lambda, \frac{D \log 2}{2}\right\} + \log \rho.
\]
\[
\text{Put}
\]
\[
H = \frac{h}{\lambda}, \quad \omega = 2 \left(1 + \sqrt{1 + \frac{1}{4H^2}}\right), \quad \theta = \sqrt{1 + \frac{1}{4H^2} + \frac{1}{2H}}.
\]
Then
\[
\log |\Lambda| \geq -C \left(h' + \frac{\lambda'}{\sigma}\right)^2 a'_1a'_2 - \sqrt{\omega\theta} \left(h' + \frac{\lambda'}{\sigma}\right) - \log \left(C' \left(h' + \frac{\lambda'}{\sigma}\right)^2 a'_1a'_2\right)
\]
with $C = C_0/\lambda^3 \sigma$ and $C' = \sqrt{C_0 \omega \theta / \lambda^6}$.

**Proposition 5.11.** If $z = v_{m_i} = w_{n_i}$ ($i \in \{1, 2\}$) has a solution, then
\[
\frac{2m_1}{\log \eta} < C_0'' \mu (\rho + 3)^2 h^2 + \frac{2\sqrt{\omega\theta}h + 2 \log \left(\sqrt{C_0'\omega \theta \lambda^{-3}(\rho + 3)^2}\right) + 4 \log h}{(\log(10^5))^2} + 1,
\]
where $\rho = 8.2$, $\mu = 0.48$

\[
C_0'' = \left(\frac{\omega}{6} + \frac{1}{2} \left[\frac{\omega^2}{9} + \frac{8\lambda \omega^{5/4} \theta^{1/4}}{3(\rho + 3)H^{1/2} \log(10^5)} + 4 \left(\frac{1}{a_1} + \frac{1}{a_2}\right) \frac{\lambda \omega}{H}\right]\right)^2,
\]
\[
h = 4 \log \left(\frac{2j}{\log \eta} + 1\right) + 4 \log \left(\frac{\lambda}{\rho + 3}\right) + 7.06 + \log \rho \text{ and } \sigma, \lambda, H, \omega, \theta \text{ are as in Theorem 5.10.}
\]

**Proof.** Similarly as in [5] we can take $a_i = (\rho + 3) \log \gamma_i$, $i = 1, 2$, $h = 4 \log \left(\frac{2j}{\log \gamma_1}\right) + 4 \log \left(\frac{\lambda}{\rho + 3}\right) + 7.06 + \log \rho$ which yields $C_0 < C_0'$ as defined in the statement of the proposition. Now, combining Theorem 5.10 and Lemma 5.2 finishes the proof. \(\square\)

**Proof of Theorem 1.6.** Let $\{a, b, c\}$ be a D(4)-triple. With $N(z_0, z_1)$ we will denote the number of nonregular solutions of the system (2) and (3), i.e. the number of integers $d > d_+$ which extend that triple to a quadruple and which correspond to the same fundamental solution $(z_0, z_1)$. From Proposition 5.6 we know that if we have 3 possible solutions $m_0, m_1, m_2$ with the same fundamental solution $(z_0, z_1)$ then $m_0 \leq 2$, so from Lemma 2.9 we know
that \( N(z_0, z_1) \leq 2 \) for each possible pair \((z_0, z_1)\) in Theorem 1.3. Also, from the same Theorem we know that the case when \( m \) is odd and \( n \) is even cannot occur when \( d > d_+ \). So, if we denote with \( N_{eo} \) the number of solutions \( d > d_+ \) when \( m \) is even and \( n \) is odd, and similarly for other cases, the number of extensions of a \( D(4) \)-triple to a \( D(4) \)-quadruple with \( d > d_+ \) is equal to
\[
N = N_{ee} + N_{eo} + N_{oo}.
\]

**Case iv)**
This follows from Theorem 1.4.

**Case i)**
Since \( c = a + b + 2r < 4b \), only case \(|z_0| = |z_1| = 2\) can hold as explained before Proposition 5.4. This implies
\[
N = N_{ee} = N(2, 2) + N(-2, -2).
\]
Let us prove that \( N(2, 2) \leq 1 \). Assume contrary, that \( N(z_0, z_1) = 2 \). Since \( \frac{1}{2}(st - cr) = 2 \), then \((m_0, n_0) = (2, 2)\) is a solution in this case. Beside that solution, there exist two more solutions \( z = v_{m_1} = w_{n_1}, (m_i, n_i)\), \( i = 1, 2\), such that \( 2 = m_0 < m_1 < m_2 \) and \( m_1 \geq 8 \) and \( n_1 \geq 8 \) by Lemma 2.9. From Lemmas 5.2 and 5.5 we have \( \kappa = 6 \), \( \Delta \geq 4 \) and \( m_2 - m_1 > \kappa^{-1}(ac)^{m_0} \log \eta, \) i.e.
\[
\frac{j}{\log \eta} = \frac{m_2 - m_1}{\log \eta} > \frac{4}{6}(ac)^2 > 6.66 \cdot 10^9.
\]
On the other hand, \( n_1 \geq 8 \) and Proposition 5.9 gives us
\[
m_2 \leq 2n_2 + 1 \leq 2 \cdot 2628n_1 \leq 5256m_1 + 5256 \leq 5913m_1,
\]
so \( j = m_2 - m_1 \leq 5912m_1 \). Using Proposition 5.11 yields \( \frac{j}{\log \eta} < 5.71 \cdot 10^8 \), which is a contradiction.

This proves that \( N = N(2, 2) + N(-2, -2) \leq 3 \).

**Cases ii) and iii)**
Since \( c \neq a + b + 2r \), we have \( c > ab + a + b \). Let \( N'(z_0, z_1) \) denote a number of solutions of equation \( v_m = w_n \), with \( m > 2 \) and \( b > 10^3 \), but without assuming that inequalities from Lemma 2.5 hold. Then, by Lemma 2.13 and Proposition 2.14 we have
\[
N \leq N(-2, -2) + N(2, 2) + N'(z_0^+, z_1^-) + N'(z_0^-, z_1^+)
\]
where \((z_0^+, z_1^+) = \{(\frac{1}{2}(st - cr), \frac{1}{2}(st - cr)), (t, \frac{1}{2}(st - cr)), (t, \frac{1}{2}(st - cr), s), (t, s)\} \) and \((z_0^-, z_1^-) = (-z_0^+, -z_1^+)\).

It is not hard to see that the previous proof for \( N(2, 2) \leq 1 \) didn’t depend on the element \( c \), so it holds in this case too. We will now show that \( N'(-t, -s) \leq 2 \) which by Lemma 2.13 implies \( N'(z_0^-, z_1^-) \leq 2 \) and \( N'(t, s) \leq 2 \) for \( c < b^2 \) and \( N'(t, s) \leq 1 \) for \( c > b^2 \), which will prove our statements in these last two cases.
Let us assume contrary, that \( N'(-t, -s) \geq 3 \), i.e. there exist \( m_0, m_1, m_2, 2 < m_0 < m_1 < m_2 \). By Proposition 2.14 we know \( m_0 \geq 9 \). From Lemma 5.2 we have \( \Delta < 2.0001 \xi^{2(m-1)} \) which can be used in Lemma 5.5 to get

\[
m_2 > m_2 - m_1 > 1.999^8 \log \eta > 1.999^8 \log \sqrt{\xi}.
\]

Now, we can use Proposition 5.4 with \( B(m_2) = m_2 + 1, \log \eta < \log(c/2) \) which gives us an inequality

\[
\frac{m_2}{\log(3.892(m_2 + 1))} < 2.81 \cdot 10^{12} \log c \log(c/2).
\]

Left hand side is increasing in \( m_2 \) so we can solve inequality in \( c \) which yields \( c < 56 \), a contradiction with \( c > b > 10^5 \).

Now, let us prove that \( N'/(t, s) \leq 1 \). Assume contrary \( N'(t, s) \geq 2 \) and for some 3 solutions \( 1 < m_0 < m_1 < m_2 \) we also have \( 2 < m_1 \) (\( m_0 \) is associated with a regular solution). Then by Lemma 5.2, since \( \Delta \geq 1 \) and \( c > 4b \), \( b > 10^5 \), we get

\[
\frac{m_2 - m_1}{\log \eta} > 2ab(\sqrt{ac})^2 > 8b^2 > 8 \cdot 10^{10}.
\]

(25)

On the other hand, as in the proof of [19, Lemma 7.1] it can be shown that \( \frac{n_2 - n_1}{m_2 - m_1} > \frac{\log \xi}{\log \eta} \), which together with Proposition 5.9 implies

\[
\frac{m_2 - m_1}{\log \eta} < \frac{n_2 - n_1}{\log \xi} < \frac{f(n_1)n_1}{\log \xi},
\]

where \( f(n_1) = \frac{2.0294n_1 + 8.96759}{\frac{4}{3}n_1 - 4.80292} \left( 1 + \frac{1}{m_1} \right) \). These two inequalities yields \( n_1 > 9.34047 \cdot 10^{11} \) and \( f(n_1) \leq 4.1818 \).

From \( \log \xi \), we have

\[
\frac{m_1}{\log \eta} > \frac{m_2 - m_1}{f(n_1) \log \eta} - \frac{\log \mu}{\log \eta \log \xi} > \frac{m_2 - m_1}{f(n_1) \log \eta} - 1.
\]

So, we can use Proposition 5.11 and an inequality

\[
\frac{m_2 - m_1}{f(n_1) \log \eta} < \frac{C_1^\prime \mu h^2 \rho + 3)^2}{\pi^2 \sigma^2} + 2 \log \left( \frac{\sqrt{C_2^\prime \sigma \psi \lambda^{-3}(\rho + 3)^2)} + 4 \log h}{(\log(10^1))^2} \right) + 2
\]

yields \( \frac{m_2 - m_1}{\log \eta} < 152184 \) which is in a contradiction with (25). So we must have \( N'/(t, s) \leq 1 \).

It remains to prove that \( N'(-t, -s) \leq 1 \) for \( c > b^2 \). Again, let us assume contrary, that there are at least 2 solutions \( m_1 < m_2 \) besides a solution \((m_0, n_0) = (1, 1) \) (which gives \( d = d_-(a, b, c) \)). Then

\[
\frac{m_2 - m_1}{\log \eta} > 2.0001^{-\frac{c}{b}} \xi^{2(m_0 - 1)} \Delta > 1.99 \cdot 10^5.
\]

After repeating steps as in the previous case, we get \( f(n_1) < 4.1819 \) and Proposition 5.11 yields \( \frac{m_2 - m_1}{\log \eta} < 152184 \), a contradiction.

So, when \( c > b^2 \) we have \( N \leq 2 + 1 + 2 + 2 = 7 \) and when \( a + b + 2r \neq c < b^2 \) we have \( N \leq 2 + 1 + 2 + 1 = 6 \). □
6. Extension of a Pair

For completeness, to give all possible results similar to the ones in [19] and [7], we have also considered extensions of a pair to a triple and estimated a number of extensions to a quadruple in such cases. Extensions of a pair to a triple were considered in [2] for the $D(4)$ case. Baćić and Filipin have shown that a pair $\{a, b\}$ can be extended to a triple with a $c$ given by

$$c = c_{\nu}^{\pm} = \frac{4}{ab} \left\{ \left( \frac{\sqrt{b} \pm \sqrt{a}}{2} \right)^{2} \left( \frac{r_{\nu} + \sqrt{ab}}{2} \right)^{2\nu} + \left( \frac{\sqrt{b} \mp \sqrt{a}}{2} \right)^{2} \left( \frac{r_{\nu} - \sqrt{ab}}{2} \right)^{2\nu} - \frac{a+b}{2} \right\}$$

where $\nu \in \mathbb{N}$. These extensions are derived from the fundamental solution $(t_0, s_0) = (\pm 2, 2)$ of a Pell equation

$$at^{2} - bs^{2} = 4(a - b),$$

associated to the problem of extension of a pair to a triple. Under some conditions for the pair $\{a, b\}$ we can prove that these fundamental solutions are the only one. The next result is an improvement of [2, Lemma 1].

**Lemma 6.1.** Let $\{a, b, c\}$ be a $D(4)$-triple. If $a < b < 6.85a$ then $c = c_{\nu}^{\pm}$ for some $\nu$.

**Proof.** We follow the proof of [2, Lemma 1]. Define $s' = \frac{r_{\nu} - at}{2}$, $t' = \frac{r_{\nu} - bs}{2}$ and $c' = \frac{(s')^{2} - 4}{a}$. The cases $c' > b$, $c' = b$ and $c' = 0$ are the same as in the [2, Lemma 1] and yield $c = c_{\nu}^{\pm}$. It is only left to consider the case $0 < c' < b$. Here we define $r' = \frac{s' + at}{2}$ and $b' = \frac{(r')^{2} - 4}{a}$. If $b' = 0$ then it can be shown that $c' = c_{1}^{-}$ and $c = c_{\nu}^{-}$ for some $\nu$. We observe that $b' = d_{-}(a, b, c')$ so

$$b' < \frac{b}{ac'} < 6.85 \frac{c'}{c'.}$$

This implies $b'c' \leq 6$. If $c' = 1$, since $b' >$ and $b'c' + 4$ is a square, the only possibility is $b' = 5$. In that case, $a$ and $b$ extend a pair $\{1, 5\}$. Then

$$a = a_{\nu}^{\pm} = \frac{4}{3} \left\{ \left( \frac{\sqrt{5} \pm 1}{2} \right)^{2} \left( \frac{3 + \sqrt{5}}{2} \right)^{2\nu} + \left( \frac{\sqrt{5} \mp 1}{2} \right)^{2} \left( \frac{3 - \sqrt{5}}{2} \right)^{2\nu} - 3 \right\}$$

and $b = d_{+}(1, 5, a) = a_{\nu+1}^{\pm}$ for the same choice of $\pm$. Define $k := \frac{b}{a} = \frac{a_{\nu+1}^{\pm}}{a_{\nu}^{\pm}}$. It can be easily shown that $k \leq 8$ and decreasing as $\nu$ increases. Also,

$$\lim_{\nu \to \infty} \frac{a_{\nu+1}^{\pm}}{a_{\nu}^{\pm}} = \left( \frac{3 + \sqrt{5}}{2} \right)^{2} > 6.85,$$

which gives us a contradiction with the assumption $b < 6.85a$.

If $c' \in \{2, 3, 4, 6\}$ there is no $b'$ which satisfy needed conditions. The case $c' = 5$ gives $b' = 1$ which is analogous to the previous case. □

**Remark.** For a pair $(a, b) = (4620, 31680)$, where $b > 6.85a$, we have a solution $(s_0, t_0) = (68, 178)$ of Pellian equation (1), so it can be extended
with a greater element to a triple with $c \neq c_1^\pm$, for example $c = 146434197$. So, this result can’t be improved further more.

Proof of Proposition 1.8. We have


c_1^\pm = a + b \pm 2r,  \\
c_2^\pm = (ab + 4)(a + b \pm 2r) \mp 4r,  \\
c_3^\pm = (a^2b^2 + 6ab + 9)(a + b \pm 2r) \mp 4r(a^2b + 3),  \\
c_4^\pm = (a^3b^3 + 8a^2b^2 + 20ab + 16)(a + b \pm 2r) \mp 4r(a^2b^2 + 5ab + 6),  \\
c_5^\pm = (a^4b^4 + 10a^3b^3 + 35a^2b^2 + 50ab + 25) \mp 4r(a^3b^3 + 7a^2b^2 + 15ab + 10).

The aim is to use Theorem 1.6 and since $N = 0$ for $b < 10^5$ we can use the lower bound on $b$ (more precisely, on $b$ if $b < c$ and on $c$ otherwise).

Case $c_2^\pm$ implies that $\{a, b, c\}$ is a regular triple so $N \leq 3$.

If $a = 1$ then $c_2^\leq < b^2$ so the best conclusion is $N \leq 7$. On the other hand, if $c = c_2^\geq$ and $a^2 \geq b$ we have $c > b^2$ since $a + b - 2r \geq 1$ so $N = 0$. It remains to observe $b > a^2$. Then it can be shown that $r^2 < 0.004b^2$ and $c_2^\geq > 0.872ab^2$, so if $a \geq 2$ we can again conclude $N \leq 6$. Also if $c \geq c_2^\leq = (a + b)(ab + 4) + 2r(b + 2) > b^2$ we have $N \leq 6$. Observe that $c \geq c_5^\leq > a^4b^4$. If $b \leq 7104a$ then $c > \frac{1020}{1109}b^4 > 39263b^4$ so $N = 0$ by Theorem 1.6. If $b > 7104a$ then $a + b - 2r > 0.97b$ so $c_5^\geq > 0.97ba^4b^5 > 97000b^4$ so again $N = 0$.

Similarly, we observe $c_4^\leq > a^3b^3$, and if $b \leq 63a$ we get $N = 0$. If $b > 63a$ we have $c + b - 2r > 0.766b$ which will lead to $N = 0$ when $a \geq 38$. Cases $a \leq 37$ can be observed separately, and only $a \geq 35$ led to $N = 0$ and others to $N \leq 6$.

From Proposition 1.8 and Lemma 6.1 we can conclude the result of Corollary 1.9 after observing that $10^5 < b < 6.85a$ implies $a \geq 14599$.

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