

Abstract—Blowfish privacy is a recent generalisation of differential privacy that enables improved utility while maintaining privacy policies with semantic guarantees, a factor that has driven the popularity of differential privacy in computer science. This paper relates Blowfish privacy to an important measure of privacy loss of information channels from the communications theory community: min-entropy leakage. Symmetry in an input data neighbouring relation is central to known connections between differential privacy and min-entropy leakage. But while differential privacy exhibits strong symmetry, Blowfish neighbouring relations correspond to arbitrary simple graphs owing to the framework’s flexible privacy policies. To bound the min-entropy leakage of Blowfish-private mechanisms we organise our analysis over symmetrical partitions corresponding to orbits of graph automorphism groups. A construction meeting our bound with asymptotic equality demonstrates tightness.

Index Terms—Differential privacy, min-entropy leakage, graph symmetrisation.

I. INTRODUCTION

Differential privacy [1] has emerged as a leading measure of privacy loss across the machine learning, theoretical computer science, databases and computer security communities. Its success is due in large part to strong guarantees on the indistinguishability of input datasets based on releases of randomised mechanisms such as learned models [2], [3] and derived data structures [4] over sensitive data. This indistinguishability takes the form of mechanism response distribution smoothness over pairs of adjacent datasets—those that differ in one record. By relaxing the differential privacy adjacency relation, randomised mechanisms may achieve higher utility for the same privacy level on select pairs of datasets. The Blowfish framework, introduced by He et al. [5], attains this goal while maintaining meaningful privacy policies. As a result, Blowfish adjacency relations may lack the symmetry of those under differential privacy.

Our goal in this paper is to examine Blowfish privacy and its relationship to min-entropy leakage [6]—a leading notion of privacy in communications theory. Specifically we establish that bounded Blowfish privacy implies bounded min-entropy leakage. Together with previous work [5], [7]–[9] this completes the following strict hierarchy (with the converse implications not holding [5], [9]).

\[
\begin{align*}
\text{Differential privacy} & \Downarrow \\
\text{Blowfish privacy} & \Downarrow \\
\text{Information leakage}
\end{align*}
\]

The dataset adjacency relation can be viewed as a simple graph. Previous work bounding information leakage of differentially-private mechanisms exploits strong symmetry assumptions of this graph. While this corresponds to distance regularity and vertex transitivity, Blowfish-induced graphs can be arbitrary. As a result the challenge for analysing Blowfish privacy is one of graph symmetrisation. Our main bounds accomplish this by developing a new proof technique that organises the graph by vertex-transitive automorphism orbits. Beyond demonstrating a meaningful connection, we discuss implications of our results on understanding of both Blowfish and information-leakage frameworks.

Before describing our main results, we overview related work and describe necessary background material in differential privacy (Section II). Our presentation of Blowfish privacy is greatly simplified over the original exposition [5], and thus may be of independent interest.

A. Related Work

While differential privacy’s success is owed in large part to its worst-case guarantees, researchers have sought natural relaxations that: improve utility while maintaining semantic privacy guarantees, offering generic mechanisms, and permitting mechanism composition.

Approximate \((\epsilon, \delta)\)-differential privacy—the most well-known variant—relaxes pure \(\epsilon\)-differential privacy response distribution smoothness, on low-probability responses [11]. In so doing, it permits guarantees on privacy loss for highly concentrated mechanisms such as the Gaussian [10]. Citing the ensuing unbounded residual privacy risk on tails, Mironov [11] proposed Rényi differential privacy, based on Rényi divergence, to generalise (approximate) differential privacy while bounding tails of the privacy loss random variable. An alternate approach based on bounding all moments of the privacy loss variable, improving rates for composition of approximate DP, is concentrated differential privacy (CDP) [12].

Noting that pathological datasets can contribute to high query sensitivity, and so high utility loss, Hall et al. [13] introduced random differential privacy which requires response distribution smoothness to hold not on all datasets but
rather on i.i.d. datasets with high probability. Their framework permits analysis of mechanisms run on unbounded input data, for example, and permits private release under estimation of sensitivity of black-box functions [14].

Kifer and Machanavajjhala [15] proposed the Pufferfish privacy framework to provide privacy guarantees in the face of varying threat models. Notably the framework accounts for prior releases of non-differentially private information. Kifer and Machanavajjhala [16] demonstrate that without such an extension, large amounts of sensitive information may be leaked.

Inspired by Pufferfish, and a focus of this paper, is the Blowfish framework introduced by He et al. [3]. As detailed in Section II-A, the approach taken is for the defender to define a subset of data values to keep secret, as well as constraints on data already known publicly. These secrets and constraints together induce the adjacency relation on which response distribution smoothness is (relaxed) to hold on. In this way the generalised Blowfish threat model is parametrised by a semantic privacy policy. While Blowfish privacy adopts the smoothness criterion on response distributions of differential privacy, relaxations including Rényi DP and CDP are built on the same adjacency relationship.

The communications theory community have also developed frameworks for guaranteeing privacy, exemplified by the study of quantitative information flow [6] which characterises how information channels leak information with change to distributional entropy (viz., Section II-B). We continue to study the connections between quantitative information flow and differential privacy as initiated by Alvim et al. [7]. Other researchers have followed this thread of work also. For example Dwork et al. highlight early connections between differential privacy and relative entropy [17]. More recently, Issa et al. [18] situate local differential privacy [19] within a guessing framework designed for interpreting leakage definitions. In a celebrated result of practical significance, reformulating differential privacy as max-divergence admits an application of Azuma’s inequality to bound differential privacy of adaptive compositions of mechanisms [20]. Independent of Alvim et al., Barthe and Köpf developed bounds on the leakage of differentially-private mechanisms, without using the same symmetry properties leveraged by Alvim et al. For mechanisms acting on binary \( n \)-strings, they achieve a bound of \( n \log_2(e) \) [21, Corollary 2], which we recover in this paper. They then go on to improve their bound by exploiting specific structure of differential privacy that does not hold for Blowfish privacy in general. Our setting applies to more general input data, and makes fewer assumptions about the database neighbouring relation owing to the flexibility of Blowfish.

II. BACKGROUND

We next recall the Blowfish and information flow frameworks.

A. Blowfish Privacy

Adopting the language of differential privacy [1] from statistical databases, we consider a database \( D \) as comprising \( n \) records each taking a value in the set of values \( T \). As each record may for example represent a database system record or a dataset instance or labelled example, we refer to elements of \( T \) as tuples. Reflecting constraints on permissible database members—e.g., representing correlations known publicly and in particular by an adversary—databases are elements of some chosen \( \mathcal{I} \subseteq T^n \). Importantly, we do not assume that the data is independent or that it was generated by some stochastic process.

We define a secret graph \( G = (T, E) \) on the database constituent values, to be a simple graph with vertex set the tuple values \( T \). The edge set \( E \subseteq T \times T \) reflects which value pairs must be kept indistinguishable to the adversary.

**Definition 1** (Blowfish policies). A Blowfish policy \( P = (G, \mathcal{I}) \) comprises a secret graph \( G = (T, E) \) over database tuple values \( T \) and a (possibly constrained) set of permissible databases on \( n \) tuples, \( \mathcal{I} \subseteq T^n \).

We next make four preliminary definitions that lift secret tuple pairs to secret database pairs. For databases \( D, D' \in \mathcal{I} \) the total difference is the set of tuples which differ between \( D \) and \( D' \), in particular it is the set of triples \( (i, u, v) \) which indicate that the \( i \)th tuples in \( D, D' \) are \( u \) and \( v \) respectively:

\[
\Delta_T(D, D') = \{(i, u, v) \in [n] \times T^2 \mid u = D_i, v = D_i', u \neq v\}
\]

The secret difference between \( D \) and \( D' \) is the subset of the total difference for which \( u \) and \( v \) are kept secret under Blowfish policy \( P \):

\[
\Delta_S(D, D') = \{(i, u, v) \in \Delta_T(D, D') \mid (u, v) \in E\}
\]

**Definition 2** (Minimally secretly different). A pair of databases \( D, D' \) are secretly different if they have a non-empty secret difference, i.e., \( \Delta_S(D, D') \neq \emptyset \). Two databases \( D, D' \in \mathcal{I} \) are minimally secretly different under Blowfish policy \( P \) if both

(a) (secretly different) \( \Delta_S(D, D') \neq \emptyset \); and

(b) (no closer intermediate database) There exists no secretly different \( D'' \in \mathcal{I} \) (i.e., with \( \Delta_S(D, D'') \neq \emptyset \)) satisfying either

(i) (smaller secret difference) \( \Delta_S(D, D'') \subseteq \Delta_S(D, D') \)

or

(ii) (same secret difference, smaller total difference) \( \Delta_S(D, D'') = \Delta_S(D, D') \) and \( \Delta_T(D, D'') \subseteq \Delta_T(D, D') \).

**Definition 3** (Database adjacency graph). A Blowfish policy \( P = (G, \mathcal{I}) \) induces a database adjacency graph with vertex set \( \mathcal{I} \). Two databases \( D, D' \in \mathcal{I} \) are adjacent in this graph—i.e., \( D \sim D' \)—if and only if they are minimally secretly different.

This definition describes the pairs of databases over which we require a Blowfish private mechanism to have a smooth response distribution. In the differential privacy setting we are concerned with pairs of databases which differ in a single tuple. The definition of minimally secretly different generalises

\(^1\)We use \( \subseteq \) and \( \supseteq \) to denote proper subset and proper superset respectively.
this concept for the Blowfish privacy setting where we have a restricted set of permissible databases $\mathcal{I}$ and secret values $(u, v) \in E$.

In the case where the set of permissible databases is unconstrained, i.e., $\mathcal{I} = \mathcal{T}^n$, the database adjacency relationship simplifies to one more analogous to differential privacy.

**Theorem 1.** Suppose $\mathcal{I} = \mathcal{T}^n$ and $D, D' \in \mathcal{I}$, then $D \sim D'$ if and only if $D$ and $D'$ differ only on a single index $i \in [n]$ and the values $D_i$ and $D'_i$ are to be kept secret. That is, $D \sim D'$ if and only if $\Delta_T(D, D') = \Delta_S(D, D') = \{(i, u, v)\}$.

**Proof:** ($\Rightarrow$) Assume for $i \in [n]$ and $u, v \in T$ we have $\Delta_T(D, D') = \Delta_S(D, D') = \{(i, u, v)\}$. Definition 2 (a) holds as $\Delta_S(D, D') \neq \emptyset$. Definition 2 (b) must hold since $\Delta_T(D, D')$ and $\Delta_S(D, D')$ are singleton sets and hence have no non-empty proper subsets.

($\Rightarrow$) Assume $D \sim D'$, so $\Delta_T(D, D') \supseteq \Delta_S(D, D') \neq \emptyset$. We must show that $\Delta_T(D, D') = |\Delta_S(D, D')| = 1$. Since $\Delta_T(D, D') \neq \emptyset$ there must exist $i \in [n], u, v \in T$ such that $(i, u, v) \in \Delta_T(D, D')$. Assume for the sake of contradiction that there is $j \in [n] \setminus \{i\}$ and $x, y \in T$ such that $(j, x, y) \in \Delta_T(D, D')$, i.e., assume that $\Delta_T(D, D') > 1$. Consider $D''$ which only differs from $D$ at $i$, where $i = v$. So $\Delta_S(D, D'') = \{(i, u, v)\} \neq \emptyset$ and then either $\Delta_S(D, D'') \subseteq \Delta_S(D, D')$ or $\Delta_S(D, D'') = \Delta_S(D, D')$ with $\Delta_T(D, D'') \subseteq \Delta_T(D, D')$. So Definition 2 (b) doesn’t hold, contradicting $D \sim D'$. Hence $|\Delta_T(D, D')| \leq 1$. And so since $0 < |\Delta_S(D, D')| \leq |\Delta_T(D, D')| \leq 1$ we have that $\Delta_T(D, D') = \Delta_S(D, D') = \{(i, u, v)\}$.

It is clear that we recover the differential privacy adjacency relationship if $\mathcal{I} = \mathcal{T}^n$ and the secret graph is a clique.

**Example 1.** An example introduced by He et al. [3, Section 3.1] is the distance threshold secret. For $\mathcal{T}$ with the metric $d$ and some $\theta \in \mathbb{R}$, the distance threshold secrets graph is $G_\theta = (\mathcal{T}, E)$, with $(u, v) \in E$ if $d(u, v) \leq \theta$. Figures 7 (a) and (b) show secret and adjacency graphs respectively for a simple example. Applications of distance threshold secrets include data on age and salary.

**Definition 4** (Blowfish privacy). Let $\epsilon > 0$ and $P = (G, \mathcal{I})$ be a policy with induced database adjacency graph $(\mathcal{I}, \sim)$. A randomised mechanism $K$ is said to be $(\epsilon, P)$-Blowfish private if, for all $D, D' \in \mathcal{I}$, $D \sim D'$ and all measurable $S \subseteq \text{range}(K)$, mechanism $K$ satisfies

$$
\Pr(K(D) \in S) \leq \exp(\epsilon) \cdot \Pr(K(D') \in S) .
$$

Note that differential privacy is a special case of Blowfish privacy where the secret graph $G$ is a complete graph over $\mathcal{T}$ i.e., $E = \mathcal{T}^2$; permissible datasets are unconstrained $\mathcal{I} = \mathcal{T}^n$; and as a result, $\sim$ reduces to the usual neighbouring relation from differential privacy.

**B. Quantitative Information Flow**

Quantitative information flow [6] models an information-theoretic channel as a triple $(X, Z, K)$. Representing channel input and output, $X$ and $Z$ are discrete random variables (viz., Remark 1) over the domains $X = \{x_1, \ldots, x_\ell\}$ and $Z = \{z_1, \ldots, z_\ell\}$ respectively. $K$ represents the channel matrix conditional probabilities $K_{i,j} = \Pr(Z = z_j | X = x_i)$. And if the prior distribution $\pi$ over $X$ is such that $\pi_i = \Pr(X = x_i)$, then the joint probability distribution over $X$ and $Z$ factors as $p(x_i, z_j) = \Pr(X = x_i) \Pr(Z = z_j | X = x_i) = \pi_i K_{i,j}$.

The vulnerability of random variable $X$ is defined by $V(X) = \max_{i \in [\ell]} \Pr(X = x_i)$, representing the worst-case probability that an adversary can correctly guess the value of $X$ in a single try. Similarly, the conditional vulnerability representing the probability of an adversary correctly guessing $X$ in a single try after observing $Z$, is defined by $V(X|Z) = \sum_{i \in [\ell]} \max_{j \in [\ell]} \Pr(X = x_i) \Pr(Z = z_j | X = x_i)$.

Measured as information, vulnerability is equivalent to the min-entropy $H_\infty(X) = -\log V(X)$ of $X$, and the conditional min-entropy $H_\infty(X|Z) = -\log V(X|Z)$ of $X$ given $Z$. We use the notation $H^X_\infty(Z)$ to refer to the min-entropy $H_\infty(X|Z)$ for the channel matrix $K$ when the channel matrix in question is not clear from the context. Information leakage (or min-entropy leakage) is the difference between the min-entropy before and after observing the output $Z$, i.e., $I_\infty(X; Z) = H_\infty(X) - H_\infty(X|Z)$. We will make use of a simplification of the min-entropy of channel matrices under uniform prior.

**Lemma 1.** Let $(X, Z, K)$ be an information-theoretic channel, with $X, Z$ random variables over domains $X$ and $Z$ respectively. $K$ is the $\ell \times p$ channel matrix. If $X$ has the uniform distribution over $X$ then, $H_\infty(X|Z) = -\log \frac{1}{\ell} \sum_{j=1}^p \max_{i \in [\ell]} K_{i,j}$, i.e., the information leakage of the channel is equal to the sum of the column maxima of $K$. 

![Diagram](image-url)
C. Differential Privacy Implies Bounded Information Leakage

Alvim et al. [9] consider a differentially-private mechanism $\mathcal{K}$ as an information-theoretic channel $(\mathcal{X}, \mathcal{Z}, K)$ with $\mathcal{X} = \mathcal{I}$ the set of permissible databases, $\mathcal{Z} = \text{range}(\mathcal{K})$ the mechanism’s response space, and $K$ the $\ell \times p$ channel matrix with $K_{i,j} = \Pr(K(x_i) = z_j)$. They established that the differential privacy of $\mathcal{K}$ implies an upper bound on the information leakage for the corresponding channel. They also demonstrate that this implication does not go the other way: a channel with known information leakage does not necessarily satisfy $\epsilon$-differential privacy for any $\epsilon$.

Note that when discussing channel matrices we will often refer to elements of the input and output sets by their indices, e.g., writing $x_i \sim x_h$ as $i \sim h$. A release mechanism $\mathcal{K}$ with corresponding channel matrix $K$ being $(\epsilon, P)$-Blowfish private is equivalent to the statement that, for all $i, h \in [\ell]$ and all $j \in [p]$ such that $i \sim h$,

$$\exp(-\epsilon) \leq \frac{K_{i,j}}{K_{h,j}} \leq \exp(\epsilon) \quad .$$

(1)

Remark 1. Like Alvim et al. [9], we assume channels with discrete input and output spaces which correspond to discrete data and responses. Rounding due to finite precision in floating-point implementations of private mechanisms can cause low-probability responses to become zero-probability [22], violating differential/Blowfish privacy. It is therefore regarded best practice that privacy analysis of mechanisms require discrete response distributions [20, Remark 2.1]. We assume suitably discretised distributions.

III. MAIN RESULTS

In this section we present and discuss Theorem 2 and Main Theorem 3 which bound the min-entropy and information leakage of Blowfish-private mechanisms. Proofs for these results are given in Section IV.

Maximum information leakage is attained for a uniform prior over input $\mathcal{X}$ [23]. As a result, we can assume a uniform prior in order to derive a general upper bound on information leakage for $\mathcal{K}$, holding when the random variables $X$ and $Z$ have any distribution over $\mathcal{X}$ and $\mathcal{Z}$.

Theorem 2 (Min-entropy of Blowfish-private mechanisms). Let $\epsilon > 0$ and $P$ be a Blowfish policy. Let $(\mathcal{X}, \mathcal{Z}, K)$ be the channel which corresponds to a mechanism $\mathcal{K}$ satisfying $(\epsilon, P)$-Blowfish privacy. If $X$ has the uniform distribution then,

$$H_\infty(X|Z) \geq -\log \left( \frac{1}{\ell} \sum_{t=1}^{q} \exp(\epsilon d_t) \right) \quad ,$$

where $\ell = |\mathcal{X}|$, $q$ is the number of connected components of database adjacency graph $(\mathcal{X}, \sim)$ and, for $t \in [q]$, $d_t$ is the $t$th connected component’s diameter, i.e., the maximal shortest-path distance between any pair of vertices in the component.

Motivating examples of Blowfish adjacency graphs from the literature [5, 24] are frequently connected or have $q \ll \ell$.

Example 2. Revisiting the $G_\theta$ secret graph of Example 7 the induced adjacency graph $(\mathcal{X}, \sim)$ is connected unless $T$ contains consecutive values $u, v$ such that $d(u, v) \geq \theta$.

Main Theorem 3 (Information leakage of Blowfish-private mechanisms). Let $\epsilon > 0$ and $P$ be a Blowfish policy. Let $(\mathcal{X}, \mathcal{Z}, K)$ be the channel which corresponds to a $(\epsilon, P)$-Blowfish-private mechanism $\mathcal{K}$. Then there is an upper bound on the information leakage of $K$,

$$I_\infty(X; Z) \leq \log \left( \sum_{t=1}^{q} \exp(\epsilon d_t) \right) \quad ,$$

(2)

where $q$ is the number of connected components of $(\mathcal{X}, \sim)$ and $d_t$ is the diameter of the $t$th connected component for $t \in [q]$. Note that this result holds for all prior distributions on $\mathcal{X}$.

Recall here that $(\mathcal{X}, \sim)$ is the database adjacency graph, with $\mathcal{X} = \mathcal{I}$. In the case that $(\mathcal{X}, \sim)$ is connected, (2) simplifies to $I_\infty(X; Z) \leq \epsilon d$, where $d = \text{Diam}(\mathcal{X}, \sim)$. As expected, increasing the level of Blowfish privacy (by decreasing $\epsilon$) pushes down the bound on information leakage.

As discussed in Example 7 the differential privacy case corresponds to $\mathcal{I} = \mathcal{T}^n$ and a complete secret graph on $\mathcal{T}$. Hence the database adjacency graph $(\mathcal{X}, \sim)$ is connected with diameter $n$, and the bound simplifies to $\epsilon n$. For an unconstrained set of databases $\mathcal{I} = \mathcal{T}^n$ and a connected secret graph, the diameter of $(\mathcal{X}, \sim)$ is given by $n$ times the diameter of the secret graph. A larger diameter of the secret graph, and hence a larger diameter for the database adjacency graph in this case, arises when there are fewer pairs of values to be kept secret.

Fewer secret value pairs allows our mechanism to attain the same level of Blowfish privacy (i.e., the same $\epsilon$) while adding less perturbation to the response. In other words, when we are concerned about revealing differences between a smaller set of values, there is a smaller set of responses over which the channel’s probability distribution must be smooth.

While Blowfish privacy measures only the level of privacy on values to be kept indistinguishable, min-entropy and related privacy loss measures do not encapsulate such fine-grained policies. That is, revealing information about “secrets” and “non-secrets” impacts the information leakage equally. The increase in Main Theorem 3’s bound corresponding to fewer pairs of secrets while holding $\epsilon$ fixed is consistent with this difference between the definitions.

Fig. 2: Upper bound on Example 7’s information leakage for $\mathcal{T} = [4]$ as $n$ is varied, $\epsilon$ held constant.
Figure 2 plots the relationship between \( n \) and (b) the diameter of the induced database adjacency graph into vertex-transitive automorphism orbits. Theorem 4). We therefore symmetrise by organising the graph of this graph symmetry \([9]\). The challenge in proving our main transitivity. Previous work has focused only on differentia l latitudes, or times of the day. Blowfish privacy in the comple te complete secret graph.

\[
\begin{align*}
\Delta_S(D, D') \subseteq \Delta_T(D, D') = \emptyset \quad \text{and that Definition 2a does not hold, so } D \text{ and } D' \text{ are not neighbouring databases. Also } \text{since } u = v, (u, v) \notin E_A. \text{ In case (b) we have } \Delta_T(D, D') = \{1, u, v\}, \text{ but since } (u, v) \notin E_A \text{ we have } (u, v) \notin E \text{ and so } \Delta_S(D, D') = \emptyset. \text{ Again, Definition 2a does not hold and so } D \text{ and } D' \text{ are not neighbouring databases.}
\end{align*}
\]

In case (c) we have \( \Delta_T(D, D') = \{1, u, v\} \). Since \( (u, v) \in E_A \) we have \( (u, v) \in E \) and so \( \Delta_S(D, D') = \{1, u, v\} \) as well. Now Definition 2a is satisfied. To show \( D \sim D' \) we need to demonstrate that there does not exist \( D'' \) with \( \Delta_S(D, D'') \neq \emptyset \) satisfying Definition 2a.i or Definition 2a.ii. Suppose there exists \( D'' \in \mathcal{I} \) with \( \Delta_S(D, D'') \neq \emptyset \). The maximum size of \( \Delta_T(D, D'') \) is 1 since \( \mathcal{I} \) is the set of databases with 1 tuple, so \( |\Delta_T(D, D'')| \leq 1 \). Also since \( \Delta_S(D, D'') \neq \emptyset \), \( 1 \leq |\Delta_S(D, D'')| \). Combining these properties, along with the fact that \( \Delta_S(D, D'') \subseteq \Delta_T(D, D'') \) we have \( 1 \leq |\Delta_S(D, D'')| \leq |\Delta_T(D, D'')| \leq 1 \). So \( \Delta_S(D, D'') \) must not be a proper subset of \( \Delta_T(D, D'') \) and \( \Delta_T(D, D'') \) must not be a proper subset of \( \Delta_T(D, D') \); thus neither Definition 2a.i nor Definition 2a.ii are satisfied. So \( D \) and \( D' \) are not adjacent in \( (\mathcal{I}, \sim) \).

So, in all possible cases \( D, D' \in \mathcal{I} \) are minimally secretly different—and hence \( D \sim D' \) in \( (\mathcal{I}, \sim) \)—if and only if \( (u, v) \in E_A \). So the policy \( P \) induces a database adjacency graph \( (\mathcal{I}, \sim) \) which coincides with the arbitrary graph \( A = (V_A, E_A) \).

We next construct a family of channel matrices and an adjacency graph that asymptotically meet our bound with equality, and for which previous bounds \([7]–[9]\) do not hold. This demonstrates that the bound is tight in the limit. In particular, we describe an scenario for which a smaller upper bound would not hold.

**Theorem 5.** There exists a family of mechanisms \( K^{(\delta)} \), for \( \delta > 0 \), and a Blowfish policy \( P \), such that the Main Theorem 3 upper bound on information leakage is equal to the information leakage, asymptotically. Namely,

\[
\lim_{\delta \to 0} \frac{\log (\sum_{i=1}^{\infty} \exp(\epsilon(\delta) \cdot d_i))}{\mathcal{I}_\infty (X; K^{(\delta)}(X))} = 1,
\]

where \( K^{(\delta)} \) denotes the channel matrix for mechanism \( K^{(\delta)} \) and \( K^{(\delta)}(X) \) denotes its output random variable \( Z \), and \( \epsilon(\delta) \)
represents the Blowfish privacy level of \( K^{(\delta)} \) with respect to policy \( P \), \( q \) and \( d_1, \ldots, d_q \) are the number of connected components of \((\mathcal{I}, \sim P)\) and the corresponding component diameters. In particular, the policy \( P \)'s induced adjacency graph \((\mathcal{I}, \sim P)\) is neither vertex transitive nor distance regular.

**Proof:** Consider a fixed integer \( n > 1 \), and define the undirected graph \((\mathcal{X}, E_X)\) with nodes \( x_1, \ldots, x_{2n+2} \) as shown in Figure 5: one complete connected component \( \{x_1, \ldots, x_4\} \), and \( n - 1 \) complete connected components \( \{x_5, x_6\}, \ldots, \{x_{2n+1}, x_{2n+2}\} \), for a total of \( q = n \) connected components. Note that: component \( t \in [n] \) has diameter \( d_t = 1 \); the graph is not regular and so cannot be vertex transitive; and because there are 2 nodes at distance one from connected nodes \( x_1, x_2 \) but no nodes at distance one from connected nodes \( x_5, x_6 \), that the graph is not distance regular. By Theorem 4 there exists a Blowfish policy \( P = (G, I) \) such that the database adjacency graph \((\mathcal{I}, \sim) = (\mathcal{X}, E_X)\), where the permissible databases \( I \) coincide with the elements of \( \mathcal{X} \). As it is the database adjacency graph that directly impacts our bound on information leakage, we will not make further reference to details of \( P \).

For real \( \delta > 0 \), consider the block diagonal channel matrix \( K^{(\delta)} \), shown in Figure 4 with input variable \( X \) uniformly distributed on the vertex set of our constructed graph \( \mathcal{X} = \{x_1, \ldots, x_{2n+2}\} \), and output variable \( Z^{(\delta)} \) on finite space \( Z \) of cardinality \( 2n + 2 \). Each row of \( K^{(\delta)} \) is normalised by dividing by constant \( 4 + 2\delta \). By construction of the block structure and common normalising constants, the maximum ratio of any two elements within a column, between rows \( x_i \sim x_j \) is simply \( 1 + \delta \) and therefore the corresponding mechanism \( K^{(\delta)} \) preserves \((\varepsilon(\delta), P)\)-Blowfish privacy for \( \varepsilon(\delta) := \log(1 + \delta) \), independent of free parameter \( n \).

Since \( X \) is uniformly distributed over \( \mathcal{X} \) the information leakage of \( K^{(\delta)} \) is given by the log of the sum of column maxima,

\[
I_{\infty}(X; K^{(\delta)}(X)) = H_{\infty}(X) - H_{\infty}(X|Z^{(\delta)}) = \log \left( \frac{\sum_{j=1}^{2n+2} \max_{i \in [2n+2]} \Pr(X = x_i) C_{ij}}{\max_{i \in [2n+2]} \Pr(X = x_i)} \right)
\]

where \( C_{ij} \) denotes \( \Pr(Z^{\delta} = z_j|X = x_i) \). Putting these calculations together, noting that the limit of the ratio of logs is the ratio of logs of the limits since \( \log \) is a continuous function and the denominator is a positive constant in the limit,

\[
\lim_{\delta \downarrow 0} \frac{\log \left( \sum_{j=1}^{q} \exp(\varepsilon(\delta) \cdot d_j) \right)}{I_{\infty}(X; K^{(\delta)}(X))} = \lim_{\delta \downarrow 0} \frac{\log(n(1 + \delta))}{\log \left( \frac{4n(1 + \delta)}{4 + 2\delta} \right)} = \frac{\log(n)}{\log(n)} = 1 .
\]

As such we have constructed a family of mechanisms \( K^{(\delta)} \) which are \((\varepsilon(\delta), P)\)-Blowfish private for \( \varepsilon(\delta) := 1 + \delta \) and Blowfish policy \( P \) such that the upper bound from Main Theorem 3 is attained in the limit, as \( \delta \downarrow 0 \).

**IV. Graph Symmetrisation**

To prove Theorems 2 and 3 we perform matrix operations which maintain the \((\varepsilon, P)\)-Blowfish privacy and information leakage of the channel \( K \).

The first of these transformations (Lemma 6) takes the \( \ell \times p \) channel matrix \( K \) to an \( \ell \times \ell \) channel matrix \( K' \) such that each column attains its maximum value in the diagonal. This
matrix $K'$ satisfies $(\epsilon, P)$-Blowfish privacy and attains the same information leakage as $K$.

Second (viz., Lemma 7), this $K'$ is transformed into an $\ell \times \ell$ channel matrix $K''$, also satisfying $(\epsilon, P)$-Blowfish privacy and maintaining the same information leakage. Additionally all diagonal elements of $K''$ which are in the same orbit of some automorphism group $\Gamma$ over the adjacency graph are equal: for all if $i, h \in [\ell]$ members of the same $\Gamma$-orbit, $K''_{i, i} = K''_{h, h}$. A key property of these orbits is that they are vertex transitive.

In the specific case where $X$ has the uniform distribution over $\chi$, the properties of the partitions of $K''$ allow us to find a lower bound for min-entropy. Since information leakage achieves its maximum over the uniform distribution this allows us to bound information leakage for arbitrary priors over $\chi$.

a) Abstract algebra basics.: Before detailing our results, we list group-theoretic notation required for our proofs. We focus on the database adjacency graph $(\chi, \sim)$ induced by chosen Blowfish policy $P$. Note that the neighbouring relation $\sim$ imposes no restrictions on the graph: Theorem 4 demonstrates that any simple graph can be induced by $P$. In particular, $(\chi, \sim)$ need not be vertex transitive or distance regular unlike adjacency graphs under differential privacy [9]. We use $d(x_i, x_j)$ to denote the geodesic distance between $x_i$ and $x_j$ in $(\chi, \sim)$, i.e., the number of edges in a shortest path connecting $x_i$ and $x_j$.

We refer to the (full) automorphism group of $(\chi, \sim)$ by $\text{Aut}(\chi, \sim)$, and consider $\Gamma \subseteq \text{Aut}(\chi, \sim)$ to be an automorphism (sub)group of $(\chi, \sim)$. For $u \in \chi$, denote the stabiliser of $u$ in $\Gamma$ by $\Gamma_u = \{ \sigma \in \Gamma | \sigma(u) = u \}$. Additionally, denote the $\Gamma$-orbit of $u$ in $\chi$ by $\Gamma(u) = \{ \gamma(u) | \gamma \in \Gamma \} \subseteq \chi$.

If $\Gamma(u) = \chi$ for some (and hence all) $u \in \chi$ then $\Gamma$ is said to be transitive on $\chi$ and $\chi$ is said to be $\Gamma$-vertex transitive. We say that $(\chi, \sim)$ is vertex transitive if it is $\text{Aut}(\chi, \sim)$-vertex transitive.

For $u, v \in \chi$ the following notation is introduced in [9] p.28 to indicate the set of automorphisms in $\Gamma$ taking $u$ to $v$, i.e., $\Gamma_{u \rightarrow v} = \{ \sigma \in \Gamma | \sigma(u) = v \}$. Note that $\Gamma_{u \rightarrow v}$ is not a group unless $u = v$, in which case $\Gamma_{u \rightarrow v} = \Gamma_{u \rightarrow u} = \Gamma_u$.

Additional graph-theoretic results used in the proof of Lemma 7 are introduced next.

A. Technical Symmetrisation Lemmata

Lemmata 2, 5 establish group-theoretic facts about the automorphism groups of undirected graphs that are used in Section 4B where we prove that transformations of the channel matrix have well understood effects on the level of Blowfish-privacy and conditional min-entropy. Lemmata 4 and 5 rely on the orbit-stabiliser theorem, a textbook result in group theory (e.g., see Dixon and Mortimer [25, Theorem 1.4A]).

Lemma 2. Let $(\chi, \sim)$ be an undirected graph. Let $u, v \in \chi$. Let $\Gamma \in \text{Aut}(\chi, \sim)$. Let $\sigma \in \Gamma_{u \rightarrow v}$. Let $\sigma \Gamma_u$ denote the left coset $\{ \sigma \circ \gamma | \gamma \in \Gamma_u \}$. Then,

$$\Gamma_{u \rightarrow v} = \sigma \Gamma_u$$

Proof: For $\gamma \in \Gamma_u$ we have $\sigma \gamma(u) = \sigma(\gamma(u)) = \sigma(u) = v$, so $\sigma \circ \gamma \in \Gamma_{u \rightarrow v}$, so $\sigma \Gamma_u \subseteq \Gamma_{u \rightarrow v}$.

Conversely, for $\gamma \in \Gamma_{u \rightarrow v}$ we have $\sigma(u) = \gamma(u) = \sigma(\gamma(u)) = \sigma^{-1}(v)$. So $\sigma^{-1} \circ \gamma(u) = \sigma^{-1}(\gamma(u)) = \sigma^{-1}(v) = u$, and as such $\sigma^{-1} \circ \gamma \in \Gamma_u$. Hence $\sigma \circ \sigma^{-1} \circ \gamma = \gamma \in \sigma \Gamma_u$. So $\Gamma_{u \rightarrow v} \subseteq \sigma \Gamma_u$.

Since $\Gamma_{u \rightarrow v} \subseteq \Gamma_{u \rightarrow u}$ and $\Gamma_{u \rightarrow v} \subseteq \sigma \Gamma_u$, we have shown that $\Gamma_{u \rightarrow v} = \sigma \Gamma_u$.

Lemma 3. Let $(\chi, \sim)$ be an undirected graph. Let $u, v \in \chi$. Let $\Gamma \in \text{Aut}(\chi, \sim)$. Let $\sigma \in \Gamma_{u \rightarrow v}$. Let $\gamma \in \Gamma \sigma$ denote the right coset $\{ \gamma \circ \sigma | \gamma \in \Gamma_u \}$. Then,

$$\Gamma_{u \rightarrow v} = \Gamma \sigma \Gamma_u$$

Proof: For $\gamma \in \Gamma_u$ we have $\gamma(\sigma(u)) = \gamma(v) = v$, so $\gamma \circ \sigma \in \Gamma\sigma \Gamma_u$, and as such $\gamma \circ \sigma \in \Gamma_{u \rightarrow v}$. Consider $\gamma \in \Gamma_{u \rightarrow v}$. We have $\gamma(u) = \gamma(\sigma(u)) = \gamma(v) = v$, so $\gamma \circ \sigma^{-1} \in \Gamma\sigma \Gamma_u$ and $\gamma \circ \sigma^{-1} = \gamma \in \Gamma\sigma \Gamma_u$. So $\Gamma\sigma \Gamma_u \subseteq \Gamma_{u \rightarrow v}$.

Since $\Gamma \sigma \Gamma_u \subseteq \Gamma_{u \rightarrow v}$ and $\Gamma_{u \rightarrow v} \subseteq \Gamma \sigma \Gamma_u$, we have shown that $\Gamma_{u \rightarrow v} = \Gamma \sigma \Gamma_u$.

Lemma 4. Let $(\chi, \sim)$ be an undirected graph. Let $u, v \in \chi$. Let $\Gamma \in \text{Aut}(\chi, \sim)$. Then,

$$|\Gamma_{u \rightarrow v}| = |\Gamma_u| = \frac{|\Gamma|}{|\Gamma_u|} = \frac{|\Gamma|}{|\Gamma(v)|} = |\Gamma(v)|$$

Proof: We start by citing the orbit-stabiliser theorem. For any $u \in \chi$ we have

$$|\Gamma_u| = \frac{|\Gamma|}{|\Gamma(u)|}$$

From Lemma 2 we have, $|\Gamma_{u \rightarrow v}| = |\sigma \Gamma_u| = |\Gamma_u|$. Similarly, from Lemma 3 we have $|\Gamma_{u \rightarrow v}| = |\Gamma \sigma \Gamma_u| = |\Gamma \sigma|$. So applying the orbit-stabiliser theorem, we have

$$|\Gamma_{u \rightarrow v}| = |\Gamma_u| = \frac{|\Gamma|}{|\Gamma(u)|} = \frac{|\Gamma|}{|\Gamma(v)|}$$

establishing the result.

Lemma 5. Let $(\chi, \sim)$ be an undirected graph. Let $u, v \in \chi$. Let $\Gamma \in \text{Aut}(\chi, \sim)$. If $\Gamma_{u \rightarrow v} \neq \emptyset$ then,

$$|\Gamma_u| = |\Gamma(v)|$$

Proof: Let $\sigma \in \Gamma_{u \rightarrow v}$. Let $\gamma \in \Gamma_u$. So $\gamma(u) = u$, and $\sigma(\gamma(u)) = \sigma(u) = v$. Also $\sigma^{-1}(v) = u$, so $\sigma \circ \gamma \circ \sigma^{-1}(v) = \sigma(\gamma(\sigma^{-1}(v))) = \sigma(\gamma(u)) = v$. So for any $\sigma \in \Gamma_{u \rightarrow v}$ and $\gamma \in \Gamma_u$ we have $\sigma \circ \gamma \circ \sigma^{-1} \in \Gamma_{u \rightarrow v}$. So $\sigma \Gamma_u \sigma^{-1} \subseteq \Gamma_{u \rightarrow v}$.

Thus we have $|\sigma \Gamma_u \sigma^{-1}| = |\Gamma_u| \leq |\Gamma|$. Similarly, by exchanging the roles of $u$ and $v$ above, we get that $|\Gamma(v)| \leq |\Gamma|$. By equating the sizes of $\Gamma_u$ and $\Gamma(v)$, we have $|\Gamma_u| = |\Gamma(v)|$.

B. Channel Matrix Transformations

We now develop the matrix transformations discussed in the sketch above. First we transform channel matrix $K$ to attain column maxima along the diagonal.

Lemma 6. Let $K$ be an $\ell \times p$ channel matrix such that $K$ satisfies $(\epsilon, P)$-Blowfish privacy. Then there exists an $\ell \times \ell$ matrix $K'$ such that:

(a) $K'$ is a channel matrix, i.e., $K'_{i,j} \in [0, 1]$ and $\sum_{h=1}^{\ell} K'_{i,h} = 1$ for all $i, j \in [\ell]$;
(b) Each column \( j \in [\ell] \) has a maximum in the diagonal, \( K'_{i,j} = \max_{x \in [\ell]} K'_{i,j} \).
(c) \( K' \) satisfies \((\epsilon, P)\)-Blowfish privacy, \( K'_{i,j} \leq e^\epsilon K'_{h,j} \) for all \( i, j, h \in [\ell] \) such that \( i \sim h \); and
(d) If \( X \) has the uniform distribution over \( \mathcal{X} \) then the conditional min-entropies for channel matrices \( K, K' \) are equal, i.e., \( H_{\infty}^K(X|Z) = H_{\infty}^{K'}(X|Z) \).

Proof: First we assume that \( K \) is an \( \ell \times p \) matrix, with \( \ell \leq p \). This assumption is without loss of generality as we can append all-zero columns until this condition is satisfied, corresponding to augmenting the output set \( Z \) with elements of probability 0. That is, \( Z' = Z \cup \{z_{p+1}, z_{p+2}, \ldots, z_{\ell}\} \) holds with \( \Pr(Z = z_i) = 0 \) for all \( i \in \{p+1, \ldots, \ell\} \).

From here the proof of [9] Lemma 7] is sufficient. The definition of \( \epsilon \)-differential privacy in this proof is identical to the definition of \((\epsilon, P)\)-Blowfish privacy, except for the structure of the adjacency relation \( \sim \). As [9] Lemma 7] makes no assumptions about this relation the same arguments go through for \((\epsilon, P)\)-Blowfish private channel \( K \).

We next transform the channel matrix such that the diagonal corresponds to the maximum within column orbits. We construct the transformed channel matrix \( K'' \) by replacing the probability of response \( j \) given \( i \) given by entry \( (i,j) \), with the average of the entries in \( K' \) in the orbit of the edge \( (i,j) \) under the database adjacency graph’s automorphism group \( \Gamma \).

**Lemma 7.** Let \( K' \) be an \( \ell \times \ell \) channel matrix satisfying the conditions in Lemma [9] and \( (\mathcal{X}, \sim) \) be the adjacency graph. Let \( \Gamma \) be a subgroup of \( \text{Aut}(\mathcal{X}, \sim) \) and \( K'' \) the matrix defined by:

\[
K''_{i,j} = \frac{1}{|\Gamma|} \sum_{\sigma \in \Gamma} K'_{\sigma(i),\sigma(j)}, \quad i,j \in [\ell].
\]

Then \( K'' \) has the following properties:

(a) \( K'' \) is a channel matrix, i.e., \( K''_{i,j} \in [0,1] \) and

\[
\sum_{h=1}^{\ell} K''_{i,h} = 1 \quad \text{for all } i,j \in [\ell].
\]

(b) Each diagonal entry of \( K'' \) is the maximum in its column \( j \in [\ell] \): \( K''_{i,j} = \max_{k \in [\ell]} K''_{i,k} \), moreover \( K''_{i,i} = K'_{i,i} \) whenever \( i, h \in [\ell] \) are in the same \( \Gamma \)-orbit on \( X \); if in addition all diagonal entries of \( K' \) are equal (and hence are maximum entries of \( K' \)), then so too are all diagonal entries of \( K'' \).

(c) \( K'' \) satisfies \((\epsilon, P)\)-Blowfish privacy, \( K''_{i,j} \leq e^\epsilon K'_{h,j} \) for all \( i, j, h \in [\ell] \) such that \( i \sim h \); and

(d) \( H_{\infty}^{K''}(X|Z) = H_{\infty}^{K'}(X|Z) \) if \( X \) has the uniform distribution over \( \mathcal{X} \).

Proof: For \( i \in [\ell] \) we have,

\[
\sum_{j=1}^{\ell} K''_{i,j} = \frac{1}{|\Gamma|} \sum_{\sigma \in \Gamma} K'_{\sigma(i),\sigma(j)} = \frac{1}{|\Gamma|} \sum_{\sigma \in \Gamma} K'_{i,k} = \frac{1}{|\Gamma|} \sum_{\sigma \in \Gamma} 1 = 1.
\]

The second equality holds since for \( \sigma \in \Gamma \), we have \( \{\sigma(1), \ldots, \sigma(\ell)\} = [\ell] \) and the second last equality holds because \( K' \) is a channel matrix by Lemma 6(a). This proves property (a).

Let \( j, h \in [\ell] \) and \( \sigma \in \Gamma \). Since \( K' \) satisfies the conditions of Lemma 6 the maximum entry in its \( \sigma(j) \)th column must be \( K'_{\sigma(j),\sigma(j)} \). So,

\[
K''_{j,j} = \frac{1}{|\Gamma|} \sum_{\sigma \in \Gamma} K'_{\sigma(j),\sigma(j)} \geq \frac{1}{|\Gamma|} \sum_{\sigma \in \Gamma} e^{\epsilon} K'_{\sigma(h),\sigma(j)} = e^{\epsilon} K''_{h,j}.
\]

Hence each diagonal entry of \( K'' \) is the maximum in its column. For \( i \in [\ell] \) we have,

\[
K''_{i,i} = \frac{1}{|\Gamma|} \sum_{\sigma \in \Gamma} K'_{\sigma(i),\sigma(i)} = \frac{1}{|\Gamma|} \sum_{k \in [\ell]} K'_{i,k} = \frac{1}{|\Gamma|} \sum_{k \in [\ell]} |\Gamma_{i,k}| K'_{i,k}.
\]

From Lemma 5 we have \( |\Gamma_{i,k}|/|\Gamma| = 1/|\Gamma(i)| \), hence

\[
K''_{i,i} = \frac{1}{|\Gamma(i)|} \sum_{k \in [\ell]} K'_{i,k} = \frac{1}{|\Gamma(i)|} \max_{h,j \in [\ell]} K'_{h,j} = \max_{h,j \in [\ell]} K'_{h,j}.
\]

Therefore \( K''_{i,i} = \sum_{k \in [\ell]} K'_{i,k}/|\Gamma(i)| \). Hence if \( i, h \in [\ell] \) such that \( \Gamma(i) = \Gamma(h) \), i.e., \( i \) and \( h \) are in the same orbit, then \( K''_{i,i} = K''_{h,h} \). Furthermore, if all diagonal entries of \( K' \) are equal (and thus equal to the maximum element of \( K' \)), then so too are all diagonal entries of \( K'' \), for \( i \in [\ell] \):

\[
K''_{i,i} = \frac{1}{|\Gamma(i)|} \sum_{k \in [\ell]} K'_{i,k} = \frac{1}{|\Gamma(i)|} \max_{h,j \in [\ell]} K'_{h,j} = \max_{h,j \in [\ell]} K'_{h,j}.
\]

This establishes the last part of property (b).

Let \( i, h \in [\ell] \) be such that \( i \sim h \). First note that for all \( \sigma \in \Gamma \), \( \sigma(i) \sim \sigma(h) \) from the definition of an automorphism. Also note that \( K'' \) satisfies \((\epsilon, P)\)-Blowfish privacy, so for all \( j \in [\ell] \) we have \( K''_{\sigma(i),\sigma(j)} \leq e^{\epsilon} K'_{\sigma(h),\sigma(j)} \). So,

\[
K''_{i,j} = \frac{1}{|\Gamma|} \sum_{\sigma \in \Gamma} K'_{\sigma(i),\sigma(j)} \leq \frac{1}{|\Gamma|} \sum_{\sigma \in \Gamma} e^{\epsilon} K'_{\sigma(h),\sigma(j)} = e^{\epsilon} K''_{h,j}.
\]

Therefore \( K'' \) satisfies \((\epsilon, P)\)-Blowfish privacy and property (c).

We now prove the final property (d), which is rather more involved than the previous three properties of \( K'' \). Denote \( r \) the number of \( \Gamma \)-orbits on \( \mathcal{X} \). Let \( \mathcal{X}_1, \ldots, \mathcal{X}_r \) be these orbits, with cardinalities \( c_1, \ldots, c_r \). Note that \( \{\mathcal{X}_1, \ldots, \mathcal{X}_r\} \) is then a partition of \( \mathcal{X} \), and that for each \( s \in [r] \) we have that \( \mathcal{X}_s \) is \( \Gamma \)-vertex transitive. Fix \( i_s \in \mathcal{X}_s \), the generator of \( \mathcal{X}_s \) so that \( \Gamma(i_s) = \mathcal{X}_s \). Then \( c_s = |\mathcal{X}_s| = \frac{1}{|\Gamma(i_s)|} \) by the orbit-stabiliser theorem.

Denote by \( \Gamma_{i_s} \) the set of left cosets of \( \Gamma_{i_s} \) in \( \Gamma \). Choose \( \{\sigma_{s_1}, \ldots, \sigma_{s_c}\} \) to be a set of representatives of \( [\Gamma : \Gamma_{i_s}] \), so that \( \Gamma = \cup_{s=1}^c \sigma_{s} \Gamma_{i_s} \). For all \( t \in [c_s] \) denote \( i_{st} = \sigma_{st}(i_s) \). For all \( \gamma \in \Gamma_{i_s} \) and \( t \in [c_s] \) we have \( (\sigma_{st} \circ \gamma)(i_s) = \sigma_{st}(i_s) \), so

\[
\mathcal{X}_s = \Gamma(i_s) = \bigcup_{t=1}^{c_s} \{(\sigma_{st} \circ \gamma)(i_s) : \gamma \in \Gamma_{i_s}\} = \bigcup_{t=1}^{c_s} \{i_{st} \} = \bigcup_{t=1}^{c_s} \{i_{st}, \ldots, i_{st}^*\}.
\]
Since $X_s$ is $\Gamma$-vertex transitive, $\sum_{\sigma \in \Gamma} K_\sigma(j), \sigma(j)$ is independent of the choice of $j \in X_s$. Fixing $k \in X_s$, for all $j \in X_s$ we have,

$$\sum_{j \in X_s} \sum_{\sigma \in \Gamma} K'_\sigma(j), \sigma(j) = c_s \sum_{\sigma \in \Gamma} K'_\sigma(k), \sigma(k)$$

$$= c_s \sum_{t=1}^{|\Gamma_k - i^*_s|} \sum_{i \in \Gamma_k - i^*_s} K'_{i^*_s, i^*_s}.$$  

From Lemma 4 we know that $|\Gamma_k - i^*_s| = |\Gamma_{i^*_s}|$, and so,

$$c_s \sum_{t=1}^{|\Gamma_{i^*_s}|} \sum_{i \in \Gamma_{i^*_s}} K'_{i^*_s, i^*_s} = c_s |\Gamma_{i^*_s}| \sum_{t=1}^{|\Gamma_{i^*_s}|} K'_{i^*_s, i^*_s}$$

$$= c_s |\Gamma_{i^*_s}| \sum_{t=1}^{|\Gamma_k - i^*_s|} \sum_{i \in \Gamma_k - i^*_s} K'_{i^*_s, i^*_s}.$$  

Since $i^*_s = \sigma_s(i_s)$, we know that $\Gamma_{i^*_s} \neq \emptyset$. Thus Lemma 5 yields $|\Gamma_{i^*_s}| = |\Gamma_{i_s}|$. Also recall that $c_s = |\Gamma|/|\Gamma_{i_s}|$ and $\mathcal{X} = \{i^*_s, \ldots, i^*_s^n\}$. Therefore,

$$c_s |\Gamma_{i^*_s}| \sum_{t=1}^{|\Gamma_k - i^*_s|} \sum_{i \in \Gamma_k - i^*_s} K'_{i^*_s, i^*_s}$$

$$= |\Gamma| \sum_{t=1}^{|\Gamma_{i^*_s}|} \sum_{i \in \mathcal{X}_i} K'_{i^*_s, i^*_s}.$$  

So $\sum_{j \in X_s} \sum_{\sigma \in \Gamma} K'_\sigma(j), \sigma(j) = |\Gamma| \sum_{j \in X_s} K'_{j, j}$. Now consider,

$$\sum_{j=1}^{\ell} K''_{j, j} = \sum_{s=1}^{r} \sum_{j \in X_s} K''_{j, j}$$

$$= \sum_{s=1}^{r} \sum_{j \in X_s} \left( \frac{1}{|\Gamma|} \sum_{\sigma \in \Gamma} K'_\sigma(j), \sigma(j) \right)$$

$$= \sum_{s=1}^{r} \sum_{j \in X_s} K'_{j, j}$$

$$= \ell K'_{j, j}.$$  

So the sum of the diagonals of $K''$ is equal to the sum of diagonals of $K'$. We know from Lemma 1 that $H^K_\infty(X|Z)$ is a function of the sum of the maximum entries in each column of $K$. In addition, we know that both $K'$ and $K''$ attain a maximum for each column in the diagonal from Lemmas 6(b) and 7(b). Therefore we have both

$$H^K''_\infty(X|Z) = - \log \frac{1}{\ell} \sum_{j=1}^{\ell} \max_{i} K''_{j, i}$$

$$= - \log \frac{1}{\ell} \sum_{j=1}^{\ell} K''_{j, j}.$$  

We have shown that $\sum_{j=1}^{\ell} K''_{j, j} = \sum_{j=1}^{\ell} K'_{j, j}$, and as such $H^K_\infty(X|Z) = H^K''_\infty(X|Z)$. So property (d) is satisfied by $K''$.  

C. Proof of Theorem 2

Let $\epsilon, P$ satisfy the theorem’s conditions, $(X, Z, K)$ be an $(\epsilon, P)$-Blowfish-private channel. Assume $X$ is uniformly distributed over $\mathcal{X}$. From Lemmas 6 and 7 we know that we can transform the $\ell \times p$ channel matrix $K$ into an $\ell \times \ell$ channel matrix $K''$ satisfying Lemma 7’s conditions.

Let $q$ be the number of connected components of $(\mathcal{X}, \sim)$. These components, $\{X(1), \ldots, X(q)\}$, partition $\mathcal{X}$. From Lemmas 1 and 7(b, d) we know that,

$$H^K_\infty(X|Z) = H^K''_\infty(X|Z)$$

$$= - \log \frac{1}{\ell} \sum_{j=1}^{q} \max_{i \in X(j)} K''_{j, i} = - \log \frac{1}{\ell} \sum_{j=1}^{q} K''_{j, j}.$$  

Let $t \in [q]$, and let $i, j, k \in X(t)$. For elements $i$ and $h$ in a connected component, the Blowfish privacy definition for a channel matrix (1) can be extended to $K''_{j, h} \leq e^{rd(i, j)} K''_{i, j}$. In particular, letting $h = j$, this yields $K''_{j, j} \leq e^{rd(i, j)} K''_{i, j}$. For each $t \in [q]$ select $i_t \in X(t)$. Also denote the diameter of the connected component $X(t)$ by $d_t = \text{Diam}(X(t)) := \max_{i,j \in X(t)} d(i, j)$. Now,

$$\sum_{j=1}^{q} K''_{j, j} = \sum_{t=1}^{q} \sum_{j \in X(t)} K''_{j, j}$$

$$\leq \sum_{t=1}^{q} \sum_{j \in X(t)} e^{rd(i_t, j)} K''_{i_t, j}$$

$$\leq \sum_{j=1}^{f} e^{rd_t} \sum_{t \in [q]} K''_{i_t, j}.$$  

Since $K''$ is a channel matrix $K''_{i_t, j} \geq 0$ for all $i, j \in [f]$, so $\sum_{j \in X(t)} K''_{i_t, j} \leq \sum_{j=1}^{f} K''_{i_t, j}$. Also since $K''$ is a channel matrix, $\sum_{j=1}^{f} K''_{i_t, j} = 1$. So,

$$\sum_{t=1}^{q} e^{rd_t} \sum_{j \in X(t)} K''_{i_t, j} \leq \sum_{t=1}^{q} e^{rd_t} \sum_{j=1}^{f} K''_{i_t, j} = \sum_{t=1}^{q} e^{rd_t}.$$  

Then we have $\sum_{j=1}^{f} K_{j, j} \leq \sum_{t=1}^{q} e^{rd_t}$, hence

$$H^K_\infty(X|Z) = - \log \frac{1}{\ell} \sum_{j=1}^{f} K''_{j, j} \geq - \log \frac{1}{\ell} \sum_{t=1}^{q} e^{rd_t}.$$  

D. Proof of Main Theorem 2

Assume $\epsilon$ and $P$ satisfy the conditions in the theorem statement. Let $(X, Z, K)$ be an $(\epsilon, P)$-Blowfish-private channel. Note that $X$ can have any prior over $\mathcal{X}$ and is not required to be uniformly distributed. Assume $X^{\text{uniform}}$ has the uniform distribution over $\mathcal{X}$. From Theorem 2 we know that $H^K_\infty(X^{\text{uniform}}|Z) \geq - \log \frac{1}{\ell} \sum_{t=1}^{q} e^{rd_t}$, with $q$ the number of connected components of $X$, $d_t$ the diameter of component $X(t)$. Observe that min-entropy of $X^{\text{uniform}}$ is $H^K_\infty(X^{\text{uniform}}) = - \log \max_{x \in X(p) = x} p(x) = - \log 1/\ell = \log \ell$. So,

$$H^K_\infty(X^{\text{uniform}}|Z) = H^K_\infty(X^{\text{uniform}}) - H^K_\infty(X^{\text{uniform}}|Z)$$

$$\leq \log \ell + \log \frac{1}{\ell} \sum_{t=1}^{q} e^{rd_t} = \log \sum_{t=1}^{q} e^{rd_t}.$$
Proposition 5.1] demonstrate that maximum leakage is attained over the uniform distribution, i.e., $I^K_\infty(X;Z) \leq I^K_\infty(X^{\text{uniform}};Z)$. Therefore

$$I^K_\infty(X;Z) \leq I^K_\infty(X^{\text{uniform}};Z) \leq \log \sum_{t=1}^q e^{dt}.$$  

E. Relationship Between the Graph Automorphism Groups

Our proof uses the (full) automorphism group for the database adjacency graph, i.e., $\text{Aut}(T,\sim)$. As discussed in Section II-A, the adjacency graph $(T,\sim)$ is defined in terms of the secret graph $(\mathcal{T},E)$: $D \sim D'$ if $D$ and $D'$ are minimally secretly different (Definition 2).

To expand on the relationship between these graphs we demonstrate the relationship between their automorphism groups.

**Theorem 6.** If all databases are permissible (i.e., $T = T^n$) and $\varphi_1,\ldots,\varphi_n \in \text{Aut}(\mathcal{T},E)$ are automorphisms for the secret graph then

$$\sigma(D) = \sigma((t_1,\ldots,t_n)) := (\varphi_1(t_1),\ldots,\varphi_n(t_n))$$

is an automorphism for the database adjacency graph.

**Proof:** Recall that we have $\Delta_T(D,D') = \{(i,u,v) \in [n] \times T^n \mid u = D_i, v = D_i', u \neq v\}$. Since each $\varphi_i$ is a permutation on $T$ for $u, v \in T$ we have $u = v$ if and only if $\varphi_i(u) = \varphi_i(v)$. Hence, $\Delta_T(D,D')$ and $\Delta_T(\sigma(D),\sigma(D'))$ are in bijection. Similarly, the secret difference is defined as $\Delta_S(D,D') = \{(i,u,v) \in \Delta_T(D,D') \mid (u,v) \in E\}$. If $(i,u,v) \in \Delta_S(D,D')$ then $D_i = u$, $D_i' = v$, and $(u,v) \in E$. Since $\varphi_i$ is a graph automorphism, $(\varphi_i(u),\varphi_i(v)) \in E$. So $(i,\varphi_i(u),\varphi_i(v)) \in \Delta_S(\sigma(D),\sigma(D'))$. Since the $\varphi_i$s are bijective, applying this argument in reverse demonstrates that $\Delta_S(D,D')$ and $\Delta_S(\sigma(D),\sigma(D'))$ are in bijection.

To show that $\sigma$ defines an automorphism for $(\mathcal{T},\sim)$ we need to show that $\sigma$ is a permutation on $T$, i.e., a bijection from $\mathcal{T}$ to $\mathcal{T}$ such that $D \sim D'$ implies $\sigma(D) \sim \sigma(D')$. The first point follows from the definition of $\sigma$ as the composition of element-wise permutations. The second point follows from the definition of $\sim$ (Definition 2), using the fact that $\Delta_T(D,D')$ and $\Delta_T(\sigma(D),\sigma(D'))$ are in bijection, as are $\Delta_S(D,D')$ and $\Delta_S(\sigma(D),\sigma(D'))$.

**Remark 2.** The graph automorphisms from Theorem 6 don’t generate all of $\text{Aut}(T,\sim)$. Consider again $T = T^n$ and functions $\sigma: \mathcal{T} \to \mathcal{T}$ which are permutations of the elements in $D$. That is, suppose $\pi: [n] \to [n]$ is a permutation, then $\sigma((t_1,\ldots,t_n)) = (t_{\pi(1)},\ldots,t_{\pi(n)})$ defines an automorphism for $\text{Aut}(T,\sim)$. This automorphism is not composed of element-wise transformations and thus cannot be generated by Theorem 6.

**Example 5.** Consider the secret graphs from Example 4 which are (a) the cyclic graph on $m$ vertices, $C_m$, and (b) the complete graph on $m$ vertices, $K_m$. The automorphism group $\text{Aut}(C_m)$ is isomorphic to the dihedral group $D_m$ of order $2m$, containing rotational and reflective symmetries of a regular $m$-gon. The automorphism group $\text{Aut}(K_m)$ is the symmetric group $\text{Sym}(m)$ of all bijective functions from $[m]$ to $[m]$.

V. Conclusions

This paper considers leading frameworks within two parallel threads of research on privacy-preserving aggregate or model release: min-entropy leakage within quantitative information flow and Blowfish privacy within differential privacy. The only known link between threads is a bound of Alvim et al. on min-entropy leakage by differential privacy which requires strong symmetry assumptions of the database adjacency graph, of distance regularity and vertex transitivity. Adjacency graphs under Blowfish privacy are arbitrary by design: the appealing property of Blowfish is its relaxation of adjacency for capturing public knowledge of underlying data. It is therefore interesting to understand how the graph structure of Blowfish semantic privacy policies—represented by adjacency graphs—bounds min-entropy leakage. We overcome this challenge by organising analysis around vertex-transitive automorphism groups. Our results relate these two important frameworks, and shed light on the structure of Blowfish privacy policies and their implications.

Noting that differential privacy and min-entropy leakage are well defined over continuous sets of tuples, while our results assume finite $T$, it is an interesting open question as to whether our symmetrisation argument extends relations on uncountable $T$.

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