DoF-Delay Trade-Off for the $K$-user MIMO Interference Channel With Delayed CSIT

Marc Torrellas, Adrian Agustin, and Josep Vidal

Abstract

The degrees of freedom (DoF) of the $K$-user multiple-input multiple-output (MIMO) interference channel are studied when perfect, but delayed channel state information is available at the transmitter side (delayed CSIT). Recent works have proposed schemes improving the state-of-the-art, but at the cost of long communication delays. This work proposes three linear precoding strategies. For each case, the achievable DoF for unbounded delay are derived, as well as its achievable DoF-delay trade-off. All strategies are based on the concept of interference alignment, and built upon three main ingredients: linear beamforming, user scheduling, and redundancy transmission. The approach consists in aligning the interference along the space-time domain at the non-intended receivers thanks to the wise use of delayed CSIT. Finally, the latter part of this work settles that all the proposed strategies work also for constant channels, by resorting to asymmetric complex signaling for the single-antenna case, as occurs for full CSIT. This removes the time-varying channels assumption common along all the literature on delayed CSIT.

Index Terms

Delayed Channel State Information, Interference Channel, MIMO, Degrees of freedom, Interference Alignment

I. INTRODUCTION

CHARACTERIZATION of the degrees of freedom (DoF, also known as the multiplexing gain) for interference networks have attracted many researchers during the last decade. They represent the scaling of channel capacity with respect to the signal-to-noise ratio (SNR) at the high SNR regime, and in the absence of known capacity expressions shed some light about the impact of different conditions on system capacity, e.g. available channel state information (CSI), number of transmit or receive antennas. In this context, interference alignment (IA) emerged a few years ago as a new tool for managing the signal dimensions (time, frequency, space) in pursuit of attaining the optimal DoF [5][6]. The concept consists on designing the transmitted signals in such a way that they are overlapped (or aligned) at the non-intended receivers. Therefore, the interference lies on a reduced dimensional subspace, releasing some dimensions to allocate desired signals which can be retrieved by means of zero-forcing (ZF) concepts.

The first application of IA appeared in the context of index coding in [5], while its application to wireless networks crystallized later on for the 2-user multiple-input multiple-output (MIMO) X-channel in [3] and for the $K$-user single-input single-output (SISO) interference channel (IC) with $K > 2$ in [4]. Surprisingly, this latter reference proposed a linear scheme providing each user half the cake as compared to the single-user case, i.e. a total of $\frac{K}{2}$ DoF are achieved over the network. Additionally, the authors showed that $\frac{Km}{2}$ DoF are achievable when each node is equipped with $m$ antennas. However, the results for the SISO case only apply for time-varying channels. Otherwise, asymmetric complex signaling (ACS) concepts have been shown as a tool for DoF boosting by exploiting improper Gaussian signaling [6]. Since then, IA has been used for studying many multi-user scenarios in combination with the well-known null-steering or ZF approach [7]. A very extensive survey of IA applications can be found in [8]. Of particular interest for this work is the characterization of the DoF of the MIMO IC for three users [9][10] and more than 3 users [11][12]. Moreover, there exists another type of IA in the literature, denoted as ergodic IA [13]. This strategy exploits opportunistic channel variations to perform IA, and performs better at low-medium SNR regime. Basically, it consists in using the channel once and then wait for a particular channel realization satisfying some conditions that allow to cancel the interference.

All IA-based and ZF-based schemes previously mentioned require perfect and instantaneous channel state information at the transmitters (CSIT), an assumption not always realistic in wireless cellular networks. For example, in frequency division duplexing systems, channels are usually estimated at the receivers through training, and then fed back to the transmitters, which incurs in delays and errors. The feedback error has been widely studied in the literature, and the main conclusion is that in order to preserve the DoF, the number of quantization bits should scale with the logarithm of the SNR [14]. On the other hand, it is usually assumed a block fading channel model, where channel remains constant in blocks of duration equal to the channel coherence time. Consequently, when the feedback delay is higher than the coherence time, the available CSIT is completely outdated. In such a case, all strategies based on full CSIT are no more effective to handle the interference.

The authors are with the Signal Theory and Communications Department at the Universitat Politècnica de Catalunya (UPC), Barcelona {marc.torrellas.socastro, adrian.agustin, josep.vidal}@upc.edu

This work has been supported by projects TROPIC FP7 ICT-2011-8-318784 (European Commission), MOSAIC TEC2010-19171/TCM, DISNET TEC2013-41315-R (Ministerio de Economía y Competitividad, Spanish Government and ERDFs), and 2014SGR-60 (Catalan Administration).

Part of this material was presented at the International Conference on Acoustics, Speech and Signal Processing, Florence, Italy, May 2014 along the papers [1] and [2].
Motivated by this, Maddah-Ali and Tse (MAT) introduced in \[15\] a new framework where IA concepts can be exploited even when the CSIT is completely outdated, referred to as delayed CSIT. Indeed, they assume perfect delayed CSIT, which is more realistic since during the time elapsed between transmissions, receivers can report a sufficient number of quantization bits. However, there are some drawbacks, e.g., since the current channels are not known, the effective rate at which symbols are sent is simply based on statistics and the particular topology/setting. Moreover, notice that the concept of reciprocity \[16\] no more holds for networks with delayed CSIT.

The MAT scheme was the first application of IA concepts using only delayed CSIT. Originally proposed for the \(K\)-user MISO broadcast channel (BC), the communication is carried out along \(K\) phases for transmitting \(b\) symbols per user. The two main ingredients of their approach are user scheduling and linear beamforming. On the one hand, users are served during phase \(p\) by groups of \(p\) users. On the other hand, linear combinations of all \(b\) symbols are sent along all the phases, working similarly to the automatic repeat request (ARQ) protocols, where the same message (or packet) is retransmitted until it can reliably be decoded at the receiver side. For example, during the first phase, users are served in a TDMA fashion, i.e., first the transmitter sends the symbols of user one, then symbols of user two, and so on. In this case, more symbols than receive antennas are transmitted, thus they cannot be linearly decoded. Note that it is usually assumed that channels across users are completely uncorrelated. Then, interestingly different and independent linear combinations (LCs) of the symbols are distributed both at the intended and non-intended receivers after the first phase. Those LCs of non-intended symbols are denoted as overheard interference, and the objective of the following phases is to exploit the delayed CSIT by retransmitting those pieces of information, which are desired at one receiver, and known at others. Therefore, more than one user can be simultaneously served after the first phase.

Inspired by the MAT scheme in \[15\], some works appeared for studying the interference channel with delayed CSIT. However, while the 2-user MIMO IC was completely characterized by Vaze et al. in \[17\], having \(K > 2\) users is an open problem whose contributions are next reviewed.

For the case \(M \geq KN\), Torrellas et al. proposed in \[11\] a two-phase precoding scheme providing \(\frac{2}{K+1}\) DoF per user on the \(K\)-user MISO IC. The first phase is developed in a TDMA fashion, whereas in the second phase only one pair of transmitters is simultaneously active. Then, from a high-level perspective it consists in the application of the MAT scheme tools (i.e., linear beamforming and user scheduling), to the IC, with only two phases. A different approach was proposed by Maleki et al. in \[19\] for the 3-user SISO IC. Their two-phase scheme, denoted as the retrospective interference alignment (RIA) scheme, provides \(\frac{3}{8}\) DoF per user. In contrast to the MAT scheme, no scheduling is applied, and all transmitters are active and interfering each other during all the communication time. The main innovation of \[19\] lies on including proper redundancy on the first phase transmission. This allows to project the received signal onto signal spaces where the desired signals are only interfered by one user. Thanks to this interference uncoupling, interference is aligned during the second phase by exploiting delayed CSIT. Two extensions followed: \[20\] and \[2\]. First, Maggi et al. proposed in \[20\] a generalization of the concept for \(K > 3\) users, even though their main conclusion was that it is preferable to consider only 3 active transmitter-receiver pairs and applying time-sharing. And second, Torrellas et al. extended in \[2\] the achievable scheme in \[19\] to the 3-user MIMO case, improving the state-of-the-art for certain antenna settings.

Combining all the ingredients in \[15\] and \[19\], i.e., beamforming, scheduling, and redundancy transmission, Abdoli et al. proposed in \[21\] a precoding scheme for the \(K\)-user SISO IC. This scheme will be referred hereafter as the beamforming, scheduling, redundancy (BSR) scheme. Developed in \(K\) phases, its sum DoF increase with the number of users \(K\), thus differing from the rule of thumb provided in \[20\]. Moreover, the authors conjectured that in contrast to the full CSIT case, the sum DoF of the IC with delayed CSIT collapse to a constant value as the number of users becomes asymptotically high, which seems to be a limitation of networks with delayed CSIT and distributed transmitters. More recently, a generalization of the BSR scheme has been proposed by Hao et al. in \[22\] for the \(K\)-user MISO IC, with transmitters equipped with \(K - 1\) or more antennas and single-antenna receivers. The achieved inner bounds outperform any proposed work for such setting, although at the cost of long communication delays.

Ergodic IA concepts have been also extended to the case of delayed CSIT in \[23\]. However, although the DoF results provided in \[23\] outperform some of the material presented here, it has some drawbacks. First, it is assumed that transmitters wait until channels satisfy some conditions and then transmit, thus entailing very long delays. And second, ergodic IA relies on channel variations, thus it only works for time-varying channels. In contrast, the results presented in this work are valid either for constant or time-varying channels.

Finally, it is worth pointing out that IA concepts have also been extended to the case of no CSIT. This kind of IA is labeled in the literature as Blind IA \[24\]. In such a case, proper channel variations are chosen for interference alignment. Although initially it was assumed that they appear naturally \[25\], i.e., by proper user selection, some recent works have shown that they can be manipulated or artificially constructed from a constant channel by means of reconfigurable antennas \[24\]. We do not consider Blind IA since it requires constant channels and reconfigurable antennas, which is not necessary for the proposed schemes.

\(^{1}\)This property states that if the number of antennas among the transmitters and receivers are switched, the DoF are preserved. Obviously, this property applies only when exactly the same CSI is available at both sides, e.g., with full CSIT.

\(^{2}\)This result was independently derived in the PhD Thesis \[18\].
Fig. 1. The $K$-user MIMO IC, with $(M, N)$ antennas at the transmitters and receivers, respectively. Solid lines define the intended signals, while dotted lines denote the interfering signals.

A. Contributions

This work studies the $K$-user $(M, N)$ MIMO IC with delayed CSIT, see Fig. 1, where transmitters and receivers are equipped with $M, N$ antennas, respectively. This paper subsumes our two previous conference papers [1][2] on this matter, and extends the proposed schemes to the general $K$-user MIMO case. The main contributions are next summarized:

- When $M \leq N$, new DoF inner bounds are provided by generalization of the RIA scheme in [19] to the $K$-user MIMO case. In contrast to the rule of thumb in [20], it is shown that considering $L \in \{3, 4, \ldots, K\}$ users simultaneously active may increase the attained DoF, where the optimal value of $L$ depends on each antenna setting and the total number of users $K$.

- When $M > N$, new DoF inner bounds are provided by generalization of the two-phase scheme in [1] to the $K$-user MIMO case. While in the original scheme the second phase was developed by rounds with only two active users, here groups of $G \in \{2, \ldots, K\}$ users are activated during the second phase, where the optimal value of $G$ depends on each antenna setting and the number of users $K$. According to this idea, we denote this scheme as the TDMA groups (TG) scheme.

- When $M \approx N$ and $K = 3$ users, new DoF inner bounds are obtained by generalization of the BSR scheme in [21] to the MIMO case. Moreover, this scheme also improves previous inner bounds when it is applied in a $K$-user MIMO IC combined with time-sharing concepts.

- A number of constraints are derived for each scheme to ensure feasibility. By collecting them, we formulated three DoF maximization problems. This allows deriving the optimal parameters (number of transmitted symbols and duration of the phases) as a function of each setting, i.e. number of users and antenna configuration. Moreover, this formulation allows to study the DoF-delay trade-off of proposed and state-of-the-art schemes. One of the main contributions is that for most cases the number of transmitted symbols and duration of the phases can be severely reduced for the sake of reducing delay without significant DoF losses.

- Usually the state-of-the-art on delayed CSIT assumes that channels are uncorrelated in time. The latter part settles that such assumption is not necessary, and the proposed schemes work even in case of delayed CSIT and constant channels, resorting to asymmetric complex signaling concepts for SISO.

B. Organization

The paper is organized as follows. Section II introduces the system model considered in this work. Next, Section III summarizes the main results: DoF inner bounds for the $K$-user MIMO IC with delayed CSIT with time-varying or constant channels. DoF inner bounds are attained by means of the RIA, TG, and BSR schemes, which are described in Sections IV, V, and VI respectively. The MIMO generalization of those schemes is obtained through the formulation of a DoF maximization problem, providing the best system parameters for each scheme given the number of users and antenna setting. Also, this formulation allows to study the DoF-delay trade-off of the proposed schemes in Section VII. Next, Section VIII addresses the analysis of delayed CSIT schemes under constant channels. Finally, conclusions and future work are drawn in Section IX.
As in all linear linear strategies designed for MIMO systems, the specific number of transmitter symbols, rounds, and slots depends on \( \rho \), and will be detailed later for each precoding strategy.

Each transmitter (TX) is equipped with \( M \) antennas, and wants to deliver \( b \) independent symbols to receiver (RX), equipped with \( N \) antennas. One of the key parameters defining this channel is its antenna ratio, defined as follows:

\[
\rho = \frac{M}{N}.
\]

As in all linear linear strategies designed for MIMO systems, the specific number of transmitter symbols, rounds, and slots depends on \( \rho \), and will be detailed later for each precoding strategy.

During the \((p,r)\)th round, i.e. round \( r \) of phase \( p \), only a specific group of users denoted by \( A^{(p,r)} \), is served, thus RX obtains

\[
y^{(p,r)}_j = \sum_{i \in A^{(p,r)}} \mathbf{H}^{(p,r)}_{j,i} \mathbf{V}^{(p,r)}_i \mathbf{x}_i,
\]
where $y_j^{(p,r)} \in \mathbb{C}^{NS_p \times 1}$ is the vector containing the signals observed at the $j$th receiver, $x_i \in \mathbb{C}^{b \times 1}$ contains the $b$ uncorrelated unit-powered complex-valued data symbols intended to the $i$th receiver, $V_i^{(p,r)} \in \mathbb{C}^{MS_p \times b}$ is the precoding matrix carrying the signals desired by the $i$th user, designed subject to a maximum transmission power per user $\gamma$, and with $V_i^{(p,r)} = 0, \forall i \notin A^{(p,r)}$. Note that linear combinations of all $b$ symbols are transmitted during all phases, but receivers will not be able to decode them until the last phase either because the reduced number of receive antennas, or because of interference. Moreover, since the focus of this paper is on DoF analysis, all unit-powered noise terms will be omitted, thus $\gamma$ denotes also the signal-to-noise ratio.

The channel coefficients for each slot and each link between transmitter and receiver are described by an $N \times M$ matrix. Then, the channel matrix $H_{j,i}^{(p,r)} \in \mathbb{C}^{NS_p \times MS_p}$ in (4) is formed as the block diagonal composition of $S_p$ of such matrices, thus contains the channel gains from antennas of TX to RX during all time slots of the $(p,r)$th round.

Usually, most works on delayed CSIT assume a flat block fading channel model, i.e. channels are i.i.d. as $\mathcal{CN}(0,1)$, and completely uncorrelated in time and space. This will be the setting for all this work, except for Section [VIII] where the objective is to show that the proposed precoding schemes work without assuming time-varying channels.

After each phase RX$_j$ collects all the received signals and process them by means of the linear filter $U_j^{(p)} \in \mathbb{C}^{\beta_p \times N\tau_p}$, where $\beta_p$ and the design of those filters will be detailed for each case. The processed signal vector for phase $p$ writes as

$$z_j^{(p)} = U_j^{(p)} \text{stack} \left( y_{j,1}^{(p,1)}, \ldots, y_{j,Rj}^{(p,Rj)} \right).$$

Similarly, with the objective of retrieving $b$ linear combinations of its desired symbols, each receiver collects the signals along all the communication. Therefore, by grouping the magnitudes of the different rounds and phases the global-input output relationship is written in compact as

$$z_j = \text{stack} \left( z_j^{(1)}, \ldots, z_j^{(P)} \right) = \Omega_j \left[ x_1^T, \ldots, x_K^T \right]^T,$$

$$\Omega_j = U_j \left[ H_{j,1}V_1, \ldots, H_{j,K}V_K \right],$$

$$U_j = \text{bdiag} \left( U_j^{(1)}, \ldots, U_j^{(P)} \right),$$

$$H_{j,i} = \text{bdiag} \left( H_{j,i}^{(1)}, \ldots, H_{j,i}^{(P)} \right),$$

$$V_i = \text{stack} \left( V_i^{(1)}, \ldots, V_i^{(P)} \right),$$

$$H_{j,i}^{(p)} = \text{bdiag} \left( H_{j,i}^{(p,1)}, H_{j,i}^{(p,2)}, \ldots, H_{j,i}^{(p,R_p)} \right),$$

$$V_i^{(p)} = \text{stack} \left( V_i^{(p,1)}, V_i^{(p,2)}, \ldots, V_i^{(p,R_p)} \right),$$

where $\Omega_j$ is the signal space matrix [10], defining the subspaces occupied by the received signals at each receiver, $U_j$ is the composition of all per-phase receiving filters $U_j^{(p)}$ whose dimensions depend on each precoding scheme, $H_{j,i} \in \mathbb{C}^{N\tau \times M\tau}$, $V_j \in \mathbb{C}^{M\tau \times b}$, $H_{j,i}^{(p)} \in \mathbb{C}^{N\tau \times M\tau}$, and $V_i^{(p)} \in \mathbb{C}^{M\tau \times b}$.

All precoding and receiving filters are designed subject to a delayed CSIT model. Using the given formulation, this means that only the channels

$$\{H_{j,i}^{(p)}, p=1, \ldots, P; i=1, \ldots, K\},$$

are available at the transmitter side at the beginning of the phase $p$, whereas all CSI is instantaneously assumed to be known at the receiver side.

We analyze the normalized DoF per user, i.e. divided by the number of receive antennas, given by [26]

$$d_j^{(\text{in})} = \lim_{\gamma \to \infty} \frac{C_{S_j}^{(\gamma)}}{KN\tau \log_2 \gamma} \leq d_j^{(\text{out})},$$

(5)

where $C_{S_j}^{(\gamma)}$ denotes the sum capacity for SNR equal to $\gamma$, and $d_j^{(\text{out})}$ denotes the normalized DoF per user outer bound. For $\rho < \frac{1}{\alpha - 1}$, the DoF with full CSIT can be achieved without CSIT by applying zero-forcing concepts at the receiver only, see for example Section V.A of [9]. For the rest of cases, and for comparison purposes, we use the following DoF outer bound:

**Theorem 1 (DoF Outer bound [12], [15]):** For the $K$-user MIMO IC with delayed CSIT and antenna ratio $\rho$, the normalized DoF per user are bounded above by:

$$d_j^{(\text{out})} = \begin{cases} 
\frac{K - 1}{K} \rho & 1 \leq \rho < \alpha \\
\rho \frac{1}{\rho + 1} & \alpha \leq \rho < \frac{1}{\beta} \\
\frac{1}{\beta} + 1 & \rho \geq \frac{1}{\beta} 
\end{cases}$$

(6)
where \( \alpha = \frac{K - 2}{K^2 - 3K + 1} \) and \( \beta = \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{K} \).

**Proof:** The first two bounds follow by assuming full CSIT and applying the results in [12], since this cannot decrease the capacity of a network with delayed CSIT. Similarly, the other bound is based on the idea that cooperation can never hurt the DoF, thus the bounds for the 3-user BC with delayed CSIT in [15] can be applied here.

Fortunately, the achievable DoF can be written in a more handy way by using standard derivations [26]. Consider a receiving filter \( W_j \in \mathbb{C}^{b \times \tau} \) such that

\[
W_j U_j H_{j,i} V_i = 0, \quad \forall i \neq j, \tag{7}
\]

i.e. acting as a linear zero-forcing filter that projects the received signals onto the orthogonal-to-interference space, thus separating desired signals from interference. Then, defining the equivalent channel for RX \( j \) as

\[
H_j^{(\text{eq})} = W_j U_j H_{j,j} V_j, \tag{8}
\]

the normalized achievable DoF express as

\[
d_j^{(\text{in})} = \frac{1}{N\tau} \text{rank}(H_j^{(\text{eq})}) \leq \frac{b}{N\tau} \leq d_j^{(\text{out})}, \tag{9}
\]

where inequality (a) is satisfied with equality only if after projection the equivalent channel is rank \( b \). In other words, after projection each receiver should be able to retrieve \( b \) independent and free of interference LCs or observations of its desired symbols. Since usually the precoding matrices are designed to manage the interference, direct channels do not take part on the precoding matrix design. Therefore, it is conjectured that since channels are generic inequality (a) will be satisfied with equality with probability one. However, for some cases this is not always true, and a rigorous proof is required, as in [10].

Finally, notice that any scheme working for \( L < K \) users can be used for the \( K \)-user MIMO IC by turning off the \( K - L \) additional users, and applying time-sharing concepts. Let assume that one scheme provides \( d_j^{(\text{in})} \) DoF to each of \( L \) users along \( \tilde{\tau} \) slots. Then, the equivalent DoF per user and duration of the communication when it is used for the \( K \)-user case write as

\[
d_j^{(\text{in})} = \frac{L}{K} d_j^{(\text{in})}, \quad \tau = \left( \frac{K}{L} \right) \tilde{\tau}. \tag{10}
\]

### III. MAIN RESULTS

The main results of this work are summarized in Theorem 2, Theorem 3, and Theorem 4. The first two are next stated and illustrated by means of some examples:

**Theorem 2 (DoF Inner bound for 3 users):** For the 3-user MIMO IC with delayed CSIT and antenna ratio \( \rho \), the following DoF per user can be achieved:

\[
d_j^{(\text{in})} = \begin{cases} 
\frac{\rho^3}{2 - \rho} & \frac{1}{2} < \rho \leq \rho_{\text{BSR,1}} \\
\frac{2\rho^2}{5\rho^2 - 10\rho + 8} & \rho_{\text{BSR,1}} < \rho \leq \rho_{\text{BSR,2}} \\
\frac{6\rho}{3\rho + 10} & \rho_{\text{BSR,2}} < \rho \leq \frac{4}{5} \\
\frac{12}{31} & \rho \geq \frac{4}{5}
\end{cases}
\]

where

\[
\rho_{\text{BSR,1}} = \frac{1}{15} \left( 10 + 5^{2/3} \left( \sqrt{\frac{2}{3}} (3\sqrt{6} + 2) - \sqrt{\frac{2}{3}} (3\sqrt{6} - 2) \right) \right) \approx 0.7545 \tag{11}
\]

\[
\rho_{\text{BSR,2}} = \frac{1}{3} (5 - \sqrt{7}) \approx 0.7847 \tag{12}
\]

**Proof:** See Section [VI] describing the 3-user BSR scheme.
Fig. 3. Normalized DoF inner and outer bounds per user for the 3-user MIMO IC with delayed CSIT. Shaded regions identify where proposed inner bounds improve state-of-the-art.

Theorem 3 (DoF Inner bound for $K$ users): For the $K$-user MIMO IC with delayed CSIT and antenna ratio $\rho$, the following DoF per user can be achieved:

| $d_j^{(\text{in})}$ | $\rho$ | Scheme |
|----------------------|--------|--------|
| $\frac{\rho}{\rho + 1}$ | $\left(\frac{1}{K}, \rho_A(K)\right)$ | RIA |
| $\frac{1}{K} \max \left(\frac{L^2}{L^2 - 1}, \frac{(L - 1) \rho}{\rho + 1}\right)$ | $[\rho_A(L), \rho_A(L - 1)], L \in \{4 \ldots K\}$ | RIA |
| $\frac{9}{8K}$ | $\left(\frac{3}{5}, \rho_{\text{BSR,3}}\right)$ | 3-user BSR |
| $\frac{3}{K} \Gamma$ | $(\rho_{\text{BSR,3}}, \rho_{\text{BSR,4}}(K))$ | 3-user BSR |
| $\frac{\rho}{\rho + (K - 1)}$ | $(\rho_{\text{BSR,4}}(K), \rho_B(K))$ | TG |
| $\max \left(\frac{1 + \alpha(G) \cdot (G - 1)}{K + (1 + G \cdot (G - 2)) \cdot \left(\frac{K}{G}\right)}, \frac{G - 1}{K}, \frac{\rho}{\rho + (G - 2)}\right)$ | $[\rho_B(G), \rho_B(G - 1)], G \in \{3 \ldots K\}$ | TG |
| $\frac{2}{K + 1}$ | $(K, \infty)$ | |

where $\rho_A(L) = \frac{L}{L - 1}$, $\rho_{\text{BSR,3}} = \sqrt{249} - 15 \approx 0.7797$, $\Gamma$ denotes the DoF achieved for the 3-user MIMO IC, and stated in Theorem 2 applicable to the $K$-user case by means of time-sharing arguments, $\rho_{\text{BSR,4}}(K) = \frac{36(K - 1)}{31K - 36}$, $\alpha(G) = \frac{K - 1}{G - 1}$, and $\rho_B(G) = \frac{1 + \alpha(G) \cdot (G - 1)}{1 + \alpha(G) \cdot (G - 2)}$. Note that $\rho_A(3) = \frac{3}{5}$, and $\rho_B(2) = K$, both representing the extremal values of the range of application for the RIA and TG schemes, respectively.

**Proof:** Each DoF value is achieved by means of the precoding scheme indicated in the last column. The RIA scheme
Table of contents

1. Introduction

2. Notations

3. Problem Formulation

4. Inner Bounds

5. Outer Bounds

6. Numerical Results

7. Conclusion

References

Fig. 1. Normalized DoF inner and outer bounds per user for the 6-user MIMO IC with delayed CSIT. Shaded regions identify where proposed inner bounds improve state-of-the-art.

Fig. 2. Relative gap for $0 \leq \rho \leq 3$. The relative gap represents the relative distance from previous (dashed) and new (solid) inner bounds to the best known outer bounds.

No claim of optimality for the proposed inner bounds is stated, while it is worth pointing out that they outperform current inner bounds for many antenna settings. Moreover, for the region $\frac{1}{K-1} < \rho < \frac{K}{K-K-1}$, the RIA scheme gets close to the
best known DoF outer bound. To emphasize this, the relative gap for $K = 3, \ldots, 7$, $\rho < \frac{2}{5}$ is depicted in Fig. 5, defined as:

$$\text{gap} = \frac{d^{(\text{out})}_j - d^{(\text{in})}_j}{d^{(\text{out})}_j}.$$ 

The figure shows that for $\rho < \frac{1}{K-1}$, the DoF outer bound is attained. On the other hand, for the region $\frac{1}{K-1} < \rho < \frac{K}{K-1}$, the new inner bounds provide a much smaller relative gap as compared to the previous inner bounds. And finally, for $\frac{K}{K-1} < \rho < \frac{2}{5}$, the relative gap is significant for both previous and new inner bounds, which claims for the research of new and tighter outer bounds.

One may ask which of the previous results is applicable in case there is delayed CSIT, but the channel remains constant. In other words, are previous results applicable without assuming time-varying channels? In this regard, the following is stated:

**Theorem 4 (DoF Inner bound with delayed CSIT and constant channels):** All inner bounds proposed in Theorem 3 apply for the $K$-user MIMO IC with delayed CSIT, constant channels, and antenna ratio $\rho$.

**Proof:** See Section VIII.

---

**Fig. 6.** Transmission frame for the RIA scheme. Two single-round phases, where only $L$ out of the total $K$ users are served. By time-sharing concepts, the rest of users are considered on other frames.

### IV. RIA SCHEME ($M < N$)

This two-phase scheme is general for the $K$-user MIMO case, and proves Theorem 3 for $\rho < \rho_{\text{BSR,3}}$. Next section gives an intuition behind this strategy. Then, each of the two phases is built, and finally we present the optimization problem that provides the optimal system parameters for any antenna setting and number of users.

**A. Overview of the precoding strategy**

The transmission frame is depicted in Fig. 6, where in both phases only $L \leq K$ users are scheduled for the communication, with $L \in \{3, \ldots, K\}$.

The two phases are addressed in Sections IV-B and IV-C and denoted as the interference sensing (IS) phase, and the retro-
spective IA (RIA) phase, respectively. Linear beamforming and redundancy transmission constitute the two main ingredients. All $L$ users considered are active during two single-round phases, i.e:

$$R_1 = R_2 = 1, \quad |A^{(1,r)}| = |A^{(2,r)}| = L.$$ 

The objective of the IS phase is to sense the interference by precoding the transmitted signals with coefficients agreed before the communication. Thanks to the channel feedback, at the beginning of the RIA phase each transmitter is able to reconstruct the interference terms generated at the non-intended receivers. Then, LCs of desired signals may be delivered without causing additional interference, i.e. aligned with the interference generated during the first phase.

Next two sections describe the transmission scheme for a particular value of $L$. The methodology used to derive the optimal value of $L$, as well as the optimal system parameters for each antenna setting $\rho$ is addressed in Section IV-D. Table I shows the optimal system parameters for a given value of $L$, entailing two different antenna setting regimes: A.I = $\{\frac{1}{K-1} < \rho \leq \rho_a(L)\}$ and A.II = $\{\rho_a(L) < \rho \leq 1\}$. Note that for the regime A.II the achieved DoF are constant with respect to $M$, and equal to the achievable DoF for $\rho = \rho_a(L)$. Actually, this simply evidences that if a DoF value can be attained for $\rho = \rho_a(L)$, it is also achievable for $\rho > \rho_a(L)$. In particular, those cases may be tackled by scaling equally all the parameters and turning off enough transmit antennas to obtain the desired antenna ratio. Consequently, without loss of generalization, in what follows regime A.I is detailed only.

---

3This methodology might not be possible if parameters are limited to some value for the sake of e.g. low complexity or communication delay, as in Section VII.
B. Interference sensing phase

The first phase lasts for $S_1$ slots where transmitters have no CSI, thus they transmit with generic full-rank precoding matrices $V^{(1)}_i \in \mathbb{C}^{MS_i \times b}$ selected from a predetermined dictionary known by all nodes. As specified in Table II $b = MN$, $S_1 = N$. The signal processed by each receiver after the first phase writes as

$$ z^{(1)}_j = U^{(1)}_j H^{(1)}_{j,j} V^{(1)}_j x_j + U^{(1)}_j \left[ H^{(1)}_{j,I^1} V^{(1)}_{I^1} \cdots H^{(1)}_{j,I^1_{L-1}} V^{(1)}_{I^1_{L-1}} \right] \begin{bmatrix} x_{I^1} \\ \vdots \\ x_{I^1_{L-1}} \end{bmatrix}, $$

(15)

where $I^1 = \{1, \ldots, L\} \setminus \{j\}$, and $I^1_k$ is the $k$th index of the set $I^1$. Note that $NS_1 = N^2$ observations of symbols are obtained at the intended receiver, but polluted of interference. However, the parameters are designed such that $NS_1 > (L - 2)b$, thus there exists some redundancy on the received signals. This redundancy can be exploited to obtain processed signals such that the desired signals are interfered by only one user. In this regard, let define the receiving filter $U^{(1)}_j \in \mathbb{C}^{N \times NS_1}$, $\forall i \neq j$, with

$$ \varphi_0 = (L - 1)\varphi_1, $$

$$ \varphi_1 = NS_1 - (L - 2)b = N(N - (L - 2)M), $$

which consists of the composition of $L - 1$ linear filters $U^{(1)}_{j,i} \in \mathbb{C}^{N \times NS_1}$, $i \neq j$, defined such that

$$ U^{(1)}_{j,i} H^{(1)}_{j,k} V^{(1)}_k = 0, \quad k \neq \{i, j\} $$

$$ U^{(1)}_{j,i} H^{(1)}_{j,k} V^{(1)}_k \neq 0, $$

$$ U^{(1)}_j = \text{stack} \left( U^{(1)}_{j,1}, \ldots, U^{(1)}_{j,I^1_{L-1}} \right), $$

(16)

$$ U^{(1)}_j \left[ H^{(1)}_{j,I^1} V^{(1)}_{I^1} \cdots H^{(1)}_{j,I^1_{L-1}} V^{(1)}_{I^1_{L-1}} \right] = \text{diag} \left( T_{j,I^1}, \ldots, T_{j,I^1_{L-1}} \right), $$

where $T_{j,i} = U^{(1)}_{j,i} H^{(1)}_{j,i} V^{(1)}_i \in \mathbb{C}^{N \times b}$, $i \neq j$ represents the residual interference from TX$_i$ after applying the linear filter $U^{(1)}_{j,i}$, i.e. this processing together with the transmitted redundancy allows to uncouple the interference from the different sources at RX$_j$. Now, let define for each $i \neq j$ the subspace $T_{j,i} = \text{rspan} \left( T_{j,i} \right)$. Those subspaces represent the overheard interference which the signals of the second can be aligned with. Notice that they can be constructed using only delayed CSIT, thus transmitters will be able to construct them at the beginning of the second phase.

C. Retrospective Interference Alignment phase

The second phase lasts for $S_2 = M$ slots where the precoding matrix for TX$_i$ is designed to align the generated interference with the overheard interference at all non-intended receivers. In other words, each receiver should be able to remove the interference generated by $V^{(2)}_i$ using the overheard interference from the IS phase. Then, they are designed to satisfy the following set of constraints:

$$ \text{rspan} \left( H^{(2)}_{k,i} V^{(2)}_i \right) \subseteq T_{k,i}, \quad \forall k \neq i. $$

(17)

An easy way to ensure this without using full CSIT is to set

$$ V^{(2)}_i = \Sigma^{(2)}_i T^{(2)}_i, $$

(18)

$$ \text{rspan} \left( T^{(2)}_i \right) = T^{(2)}_i = \bigcap_{k \neq i} T_{k,i}, $$

(19)
where $\Sigma_i^{(2)} \in \mathbb{C}^{M \times \phi_2}$ is some arbitrary full rank matrix ensuring the transmit power constraint, and $T_i^{(2)} \in \mathbb{C}^\phi_2 \times b$ is some arbitrary matrix whose rows span the intersection subspace $T_i^{(2)}$ of dimension

$$\phi_2 = b - (L - 1)(b - \varphi_1) = N ((L - 1)N - L(L - 2)M),$$

(20)

derived using identity (1). The received signals along the whole communication at each receiver, can be more easily understood by writing the $j$th signal space matrix:

$$\Omega_j = \begin{bmatrix}
U_{j,i} H_{j,j}^{(1)} V_{j}^{(1)} & T_{j,i} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
U_{j,i_{\rho-1}} H_{j,j}^{(1)} V_{j}^{(1)} & 0 & \cdots & T_{j,i_{\rho-1}} \\\nH_{j,j}^{(2)} V_{j}^{(2)} & H_{j,j}^{(2)} V_{j}^{(2)} & \cdots & H_{j,j_{\rho-1}}^{(2)} V_{j}^{(2)}
\end{bmatrix},$$

where the dotted lines separate the blocks rows corresponding to each of the two phases. Note that combination of processed signals may be interpreted as row operations on the signal space matrix. Since precoding matrices satisfy conditions in (17), each interference term generated during the second phase is aligned with one of the overheard interference terms of the first phase. Therefore, all the second phase interference can be removed, and $N$ LCs of desired symbols free of interference are retrieved at each receiver per time slot, i.e. $NS_2 = MN = b$ LCs after all. In the next section, the constraints to be satisfied by all parameters for each antenna setting will be presented, including that all such $b$ LCs are linearly independent, and thus all desired symbols can be linearly decoded.

Finally, it is worth pointing out that unlike the BC case, the interference can only be aligned individually, i.e. two users cannot align their signals simultaneously at one receiver with the signals of one slot, since the transmitted signals travel through different channels. This is why the first phase interfering signals are uncoupled by means of the processing filter $U_{j}^{(1)}$, such that only one interference term affects the desired signals on the processed signal space. In terms of the signal space matrix, this means that block columns corresponding to interference should have at most one non-zero element per block row.

D. System parameters optimization

Optimal system parameters for each antenna setting and number of users are derived next. First, the optimal value of $L$ can be found by exhaustive evaluation of the expressions in Theorem 3. Since for high values of $K$ there will be many regions, the following algorithm is provided to alleviate the search for the optimal $L$ to only two candidates:

**Algorithm 1: $L$ solver**

| Step 1 | $x := \frac{1}{2} \left( (1 + \rho^{-1}) + \sqrt{(1 + \rho^{-1})^2 + 4} \right)$ |
|-------|----------------------------------------------------------------|
| Step 2 | $y := \left\lfloor x \right\rfloor \frac{\rho}{\rho + 1}, \; z := \frac{1}{K} \left\lceil x^2 \right\rceil$ |
| Step 3 | $L(\rho) = \begin{cases} 
K & x \geq K \\
3 & x < 3 \\
3 < x < K, \; y > z & \\
[x] & \text{otherwise}
\end{cases}$ |

The motivation behind each of these different steps is next explained. First, the real number $x$ is the positive solution of inverting the definition of $\rho_\lambda(L)$, defined in Theorem 3. Then, since the inner bound is a piecewise function, $x$ represents the value of $L$ between two steps. For this reason, using the ceil and floor functions the two closest integers are selected as candidates, evaluating the achievable DoF for each of them. Finally, the best integer value $L$ is chosen taking into account the extreme cases.
Assuming a particular value for $L$, we formulate the following DoF optimization problem:

$$
\mathcal{P}_1 : \begin{array}{ll}
\text{maximize} & \frac{L}{KN} \frac{b}{S_1 + S_2} \\
\text{s.t.} & MS_1 \geq b \\
& NS_1 > (L - 2)b \\
& NS_2 \geq b \\
& LMS_2 \geq b \\
& L\varphi_2 \geq b,
\end{array}
$$

(21a)

(21b)

(21c)

(21d)

(21e)

(21f)

with $\varphi_2 = (L - 1)NS_1 - L(L - 2)b$. This problem provides the optimal values for $b$, $S_1$, and $S_2$ when the RIA scheme is employed. The objective function corresponds to the number of symbols divided by the channel uses, and a factor due to time-sharing and DoF normalization. On the other hand, the following four constraints are introduced to ensure linear feasibility:

1) Transmit rank during the IS phase (21b): During the first phase, $MS_1$ linear combinations of the $b$ symbols are transmitted using $M$ antennas, and during $S_1$ slots. Then, for linear decodability of the desired symbols, no more symbols than the number of transmit dimensions can be sent.

2) IS phase redundancy (21c): After the first phase, the linear filters $U_{j,i} \in \mathbb{C}^{\varphi_1 \times NS_1}$ in (16) are applied assuming some redundancy has been transmitted, with $\varphi_1 = NS_1 - (L - 2)b$. Then, we force $\varphi_1 > 0$ or, equivalently, (21c).

3) Receiver space-time dimensions (21d): Each receiver should have enough space-time dimensions to allocate all the desired and interference signals without space overlapping. First, notice that the interference received during the IS phase occupies at most $NS_1$ dimensions. This subspace remains the same after the RIA phase, since all the interference generated during the RIA phase is aligned. On the other hand, the desired signals occupy at most $b$ dimensions at each receiver. Hence, we must have

$$
\underbrace{b}_{\text{desired dim.}} + \underbrace{NS_1}_{\text{interference dim.}} \leq \underbrace{NS_1 + NS_2}_{\text{total dimensions}}
$$

4) Rank of desired signals after zero-forcing (21f): For ease of exposition, the signal space matrix $\Omega_j$ at each receiver is here rewritten:

$$
\Omega_j = \begin{bmatrix}
U_{j,x_1} \mathbf{H}_{j,j}^{(1)} \mathbf{V}_j^{(1)} & & \cdots & & 0 \\
\vdots & & \ddots & & \vdots \\
U_{j,x_{L-1}} \mathbf{H}_{j,j}^{(1)} \mathbf{V}_j^{(1)} & 0 & \cdots & T_{j,x_{L-1}} \\
\mathbf{H}_{j,j}^{(2)} \mathbf{V}_j^{(2)} & \mathbf{H}_{j,j}^{(2)} \mathbf{V}_j^{(2)} & \cdots & \mathbf{H}_{j,x_{L-1}}^{(2)} \mathbf{V}_j^{(2)}
\end{bmatrix} \uparrow \varphi_1 \uparrow \varphi_1 \uparrow NS_2
$$

where the block rows corresponding to the first phase have $\varphi_1$ rows each, whereas the block row of the second phase has $NS_2$ rows. Now, recall that the precoding matrices $\mathbf{V}_j^{(2)}$ lie on a subspace of dimension $t = \min(MS_2, \varphi_2) < \varphi_1$, see (18)-(20). Then, if the interference is to be removed, each of the $L - 1$ block rows corresponding to the IS phase must be projected onto the corresponding subspace of dimension $t$ and linearly combined with the block row of the second phase. This is done by means of the linear filter $\mathbf{W}_j$, obtaining

$$
\text{rank}(\mathbf{W}_j U_{j,j} \mathbf{V}_j) = \min(L, \min(MS_2, \varphi_2), b).
$$

Since any linear precoding scheme requires $\text{rank}(\mathbf{W}_j U_{j,j} \mathbf{V}_j) \geq b$, this yields to

$$
L \cdot \min(MS_2, \varphi_2) \geq b \Rightarrow \begin{cases}
LMS_2 \geq b \\
L\varphi_2 = L((L - 1)NS_1 - L(L - 2)b) \geq b
\end{cases}
$$

Next, we analytically derive the solution of problem $\mathcal{P}_1$. For any given value of $b$, the objective function in (21a) is strictly decreasing with $S_1$ and $S_2$, i.e. their optimum values are their minimum feasible values. Therefore, since $S_2$ appears in (21d) and (21e) only, its optimum value $S_2^*$ is given by

$$
S_2^* = \left[ b \max\left( \frac{1}{N}, \frac{1}{ML} \right) \right]
$$
This establishes two regions, with the threshold \( \rho = \frac{1}{L} \). However, it can be seen that taking \( S_2^* = \left\lceil \frac{b}{M} \right\rceil \) and solving the problem produces a DoF value which is always outperformed by taking \( S_2^* = \left\lceil \frac{b}{N} \right\rceil \) and increasing the value of \( L \). Hence, we definitely take

\[
S_2^* = \left\lceil \frac{b}{N} \right\rceil. \tag{22}
\]

On the other hand, the optimum value of \( S_1 \) is set to satisfy one of the constraints (21b), (21c), and (21f) with equality:

\[
S_1^* = \left\lceil \max\left( \frac{b}{M}, \frac{b+1}{N}, \frac{b}{NL} (L^2 - L - 1) \right) \right\rceil = b \cdot \max\left( \frac{1}{M}, \frac{1}{N L} (L^2 - L - 1) \right). \tag{23}
\]

While in Section VII a maximum-value constraint for \( b \) will be included, here the problem is solved for unbounded \( b \), i.e. it is simply chosen such that all parameters are integer values. Accordingly, one optimal solution is specified in Table I. Note that low values of \( G \) relax the number of receivers where the transmitted signals should be aligned, but also increase the number of rounds. This is a trade-off that should be balanced by the optimal value of \( G \). The derivation of the optimal system parameters, as well as \( G \) is deferred to Section V-D and presented in Table II. For each value of \( G \), it can be seen that there exist two different antenna setting regimes: B.I = \( \{1 < \rho \leq \rho_B(G)\} \) and B.II = \( \{\rho > \rho_B(G)\} \), with \( \rho_B(G) = \frac{1 + \alpha(G)}{\left(\frac{G}{G-1}\right)} \). Following similar arguments as for the RIA scheme, only the case B.I will be addressed in the sequel, and a particular value for \( G \) is assumed. Therefore, for ease of notation simply

\[
\alpha \triangleq \alpha(G) = \frac{K - 1}{G - 1}
\]

will be used during the following two sections.
TABLE II
SYSTEM PARAMETERS FOR TG SCHEME AS A FUNCTION OF $\rho$

| B.I = $\{1 < \rho \leq \rho_B(G)\}$ | $b$ | $S_1$ | $S_2$ | $d^{(u)}_j$ |
|---------------------------------|-----|------|------|--------|
| $\alpha MN$                     | $\alpha N$ | $M - N$ | $\frac{G}{\rho_B(G-	au_j)}$ |
| B.II = $\{\rho > \rho_B(G)\}$  | $(1 + \alpha \cdot (G - 1))N$ | $1 + \alpha \cdot (G - 2)$ | $1$ | $\frac{1 + \alpha \cdot (G - 1)}{K + (G(G-2)+1)(\frac{\rho_B}{G})}$ |

B. Orthogonal Transmission phase

Each TX sends linear combinations of its $b = \alpha MN$ symbols during the $S_1 = \alpha N$ time slots of the $(1, i)$th round, thus RX obtains

$$y^{(1,i)}_j = H^{(1,i)}_{j,i} V^{(1,i)}_i x_i,$$

(26)

where $H^{(1,r)}_{j,r} \in \mathbb{C}^{NS_1 \times MS_1}$, and the precoding matrices $V^{(1,r)}_i \in \mathbb{C}^{MS_1 \times b}$ are chosen as some generic full-rank matrices. Since no redundancy was transmitted ($b < NS_1$), none per-phase receiving filter is applied, i.e. equivalently we have $U^{(1)}_j = I_{N\tau_1}$. Moreover, we define

$$T^{(1,i)}_{j,i} = H^{(1,i)}_{j,i} V^{(1,i)}_i$$

as the overheard interference generated by TX at RX, with $\dim(T^{(1,i)}_{j,i}) = NS_1 = \alpha N^2$. Note that each receiver obtains $NS_1 = \alpha N^2$ observations of the desired symbols, as well as $\alpha N^2(K - 1)$ linear combinations of overheard interference, and since $NS_1 < b$, linear decodability is not possible yet.

C. Retrospective Interference Alignment phase

The objective of the RIA phase is to exploit the overheard interference, i.e. the subspaces $T^{(1,i)}_{j,i}$ available at the non-intended receivers, to construct signals that can be canceled even without knowing the current CSI. The design pursues that for each round $r$ of the second phase, the transmitted signals are aligned at all the $G$ receivers in $A^{(2,r)}$. For this reason, the optimal value of $G$ depends on each antenna setting and the total number of users $K$.

According to this objective, the signal transmitted during the $(2, r)$th round by each active transmitter $i \in A^{(2,r)}$ should satisfy the following set of constraints:

$$\text{rspan} \left(H^{(2,r)}_{k,i} V^{(2,r)}_i \right) \subseteq T^{(2,r)}_{k,i}, \forall k \in A^{(2,r)} \setminus \{i\}.$$

(27)

This can be ensured by setting

$$V^{(2,r)}_i = \Sigma^{(2,r)}_i T^{(2,r)}_i,$$

(28)

$$\text{rspan} \left(T^{(2,r)}_i \right) = T^{(2,r)}_i,$$

where $\Sigma^{(2,r)}_i \in \mathbb{C}^{MS_2 \times \varphi}$ is some arbitrary full rank matrix ensuring the transmit power constraint, and $T^{(2,r)}_i \in \mathbb{C}^{\varphi \times b}$ is a matrix whose rows lie on the intersection subspace of dimension

$$\varphi = (G - 1)NS_1 - (G - 2)b$$

$$= \alpha N (N(G - 1) - (G - 2)M).$$

(29)

In order to illustrate how the signals are received and aligned, the signal space matrix at each receiver for the case $K = 4$, ...
Remark: Each receiver obtains $\rho > \rho \cdot 1 \geq \alpha (\alpha (x-1))$, with $\rho \geq \frac{b}{\alpha (x-1)}$, $\alpha (x) = (K_{x-1})$.

Finally, by combining the $\alpha$ interference from the OT phase. Now, recall that $\rho \cdot \alpha \cdot \alpha$ linear combinations retrieved from the OT phase with that obtained during this phase, each receiver obtains $b = \alpha \cdot MN$ LCs of its desired symbols.

Thanks to conditions in (27), all the interference captured during the RIA phase can be removed using the overheard interference from the OT phase. Now, recall that $\alpha \cdot \alpha$ represents the number of groups of the RIA phase to which each user belongs. Therefore, the RIA phase provides $\alpha \cdot \alpha$ extra observations of the desired symbols. Finally, by combining the $\alpha \cdot \alpha$ linear combinations retrieved from the OT phase with that obtained during this phase, each receiver obtains $b = \alpha \cdot MN$ LCs of its desired symbols.

Remark: It can be seen that when $\rho < \rho (G)$ only a subspace of dimension $N(M - N) < \alpha \cdot \alpha$ of $\alpha \cdot \alpha$ is revealed to each receiver. This is in contrast with the case $\rho > \rho (G)$ where the entire subspaces $\alpha \cdot \alpha$ must be delivered to RX, in order to obtaining a sufficient number of observations, and thus ensure linear decodability.

D. System parameters optimization

Given a value of $\rho$, the optimal value of $G$ for the TG scheme may be obtained by means of the steps described in Algorithm 2. The philosophy here is similar to the one in Algorithm 1, and thus its description omitted to avoid redundancy. The parameters, e.g. number of symbols $b$ and number of slots per round $S_1$, $S_2$, given $G$, $K$, and $\rho$, are derived by means of the following DoF optimization problem:

$$\mathcal{P}_2 : \text{maximize } \frac{b}{\alpha}$$

$$s.t. \quad MS_1 \geq b$$

$$NS_1 < b$$

$$NS_2 \leq (G-1)NS_1 - (G-2)b$$

$$\alpha (G) \cdot S_2 \geq b.$$  

While the objective function corresponds to number of symbols delivered per user divided by the duration of the communication, and normalized, the different constraints imposed to ensure linear feasibility are next described:

1) Transmit rank during the OT phase [31b]: During the first phase, $MS_1$ linear combinations of the $b$ symbols are transmitted using $M$ antennas, and during $S_1$ slots. Then, for linear decodability of the desired symbols, no more symbols than the number of transmit dimensions can be sent, thus we force $MS_1 \geq b$. 

\[
G = 3 \text{ is next shown:}
\]

\[
\Omega_j = \begin{bmatrix}
T_{j,1} & 0 & 0 & 0 \\
0 & T_{j,2} & 0 & 0 \\
0 & 0 & T_{j,3} & 0 \\
0 & 0 & 0 & T_{j,4}
\end{bmatrix}
\]

(30)

- $\mathcal{P}_2 : \text{maximize } \frac{b}{\alpha}$
- $s.t. \quad MS_1 \geq b$
- $NS_1 < b$
- $NS_2 \leq (G-1)NS_1 - (G-2)b$
- $\alpha (G) \cdot S_2 \geq b.$
2) Need of RIA phase (31c): Since the first phase provides $NS_1$ interference-free linear observations of the desired symbols, we force $NS_1 < b$.

3) Non-redundant RIA phase (31d): The precoding matrices for each round of the second phase lie on a subspace of dimension $\varphi$, see (28) and (29), and they are used during $S_2$ slots. Then, to avoid redundancy on the received signals, we force that no more than $\varphi$ linear combinations are obtained at the receivers, i.e. $S_2 < \varphi$.

4) Linear combinations at the end of the transmission (31e): Each round of the first phase provides $NS_1$ LCs of desired symbols to each receiver, while each round of the second phase $\min (M S_2, NS_2, \varphi) = NS_2$, which follows from $M > N$ the previous constraint. Hence, since each user is active during $\alpha (G)$ rounds of the RIA phase the number of interference-free linear combinations of desired symbols obtained at the end of the transmission are $NS_1 + \alpha (G) \cdot NS_2$, and they should be enough for linearly decoding the $b$ desired symbols.

This problem will be handled as problem $P_1$. First, $S_2$ is removed by setting it to its minimum feasible integer value, i.e.

$$S_2^* = \left\lceil \frac{1}{\alpha (G)} \left( \frac{b}{N} - S_1 \right) \right\rceil,$$

dictated by (31e). Then, (31d) forces that:

$$(G - 1) NS_1 - (G - 2)b \geq N \left[ \frac{1}{\alpha (G)} \left( \frac{b}{N} - S_1 \right) \right] \geq N \frac{1}{\alpha (G)} \left( \frac{b}{N} - S_1 \right),$$

Therefore, $S_1$ may be written as follows:

$$S_1^* = \left\lceil b \cdot \max \left( \frac{1}{M}, \frac{1}{N} + \frac{1}{\alpha (G)} \cdot (G - 1) \right) \right\rceil,$$

where B.I and B.II follow from choosing one of the two values above, with the threshold given by $\rho_B (G) = \frac{1 + \alpha (G) \cdot (G - 1)}{1 + \alpha (G) \cdot (G - 2)}$.

VI. 3-USER BSR SCHEME ($M \approx N$)

The three-phase scheme proposed in [21] for the 3-user SISO IC is generalized to the 3-user MIMO case, proving Theorem 2. Moreover, Theorem 3 for $(\rho_{BSR.3}, \rho_{BSR.4}(K))$ follows from applying this scheme together with time-sharing concepts. Next section gives an intuition behind this strategy. Then, each of the phases is built, and finally we present the optimization problem that provides the optimal system parameters for any antenna setting and number of users.

A. Overview of the precoding strategy

This approach is designed according to the three ingredients exploited so far: linear beamforming, user scheduling, and redundancy transmission. For this reason, it is denoted as the Beamforming, Scheduling, Redundancy scheme. The first and third phases will be labeled as the IS and RIA phases, as those phases for the RIA scheme. In a similar manner, the objective is to sense the interference for the former and to transmit signals do not causing additional interference for the latter. This is achieved by exploiting only linear beamforming and redundancy transmission, thus all users are active during those phases. But, a hybrid phase developed by pairs is introduced as the second phase. The objective of the hybrid phase is twofold. First, each transmitter based on channel feedback reconstructs the overhead information created at each receiver during the IS phase to deliver desired linear combinations of symbols. Second, some redundancy is sent in order to create the overhead interference terms that will be used during the last phase. Hence, all the three ingredients are mixed up in this phase in pursuit of DoF maximization. According to all these ideas, we have:

$$R_1 = R_3 = 1, \quad R_2 = \left( \frac{3}{2} \right) = 3,$$

$$\left| A^{(1,r)} \right| = \left| A^{(3,r)} \right| = 1, \quad \left| A^{(2,r)} \right| = 3,\quad \left| A^{(m)} \right| = \frac{12}{17}$$

which is also summarized in Fig. 8. The optimal system parameters are derived in Section VI-E and specified in Table III. Recall that $\rho_{BSR.1} \approx 0.7545$, $\rho_{BSR.2} \approx 0.7847$, see [11] and [12], which means that regimes C.II and C.III require $M, N > 10$. Moreover, it can be seen that this scheme is always outperformed by the RIA scheme for regime C.I. Therefore, the most significant finding in this case is that the DoF inner bound for SISO ($d_{(m)}^{(n)} = \frac{12}{17}$) is valid whenever $\rho \geq \frac{1}{b}$. Consequently, next sections focus on regime C.IV for simplicity on the description.
TABLE III
SYSTEM PARAMETERS FOR THE BSR SCHEME AS A FUNCTION OF $\rho$

| $b$ | $S_1$ | $S_2$ | $S_3$ | $d^{(n)}$ |
|-----|-------|-------|-------|----------|
| C.I = $\{ \frac{1}{4} < \rho \leq \rho_{BSR,1} \}$ | $M^2$ | $M^2$ | $M(N - M)$ | $2(N - M)^2$ | $\frac{\rho^3}{2 - \rho}$ |
| C.II = $\{ \rho_{BSR,1} < \rho \leq \rho_{BSR,2} \}$ | $2M^2N$ | $2M^2N$ | $2N(N - M)$ | $5M^2 - 6MN + 2N^2$ | $\frac{2\rho^2}{\frac{5}{3} - 5\rho + \frac{8}{3}}$ |
| C.III = $\{ \rho_{BSR,2} < \rho < \frac{1}{2} \}$ | $6MN$ | $6N$ | $4N - 3M$ | $4(3M - 2N)$ | $\frac{6\rho}{\frac{5}{3} - 5\rho}$ |
| C.IV = $\{ \rho \geq \frac{1}{2} \}$ | $12N$ | $15$ | $4$ | $4$ | $\frac{12}{31}$ |

Fig. 8. Transmission frame for the BSR scheme. Each phase $p$ has $\binom{K}{p}$ rounds of $S_p$ slots. Active groups $A^{(p,r)}$ are represented for each round of the three phases.

B. Interfering sensing phase

The first phase lasts for $S_1$ slots where transmitters have no CSI, thus they transmit with generic full-rank precoding matrices $\mathbf{V}_i^{(1)} \in \mathbb{C}^{M S_1 \times b}$ selected from a predetermined dictionary known by all nodes. As specified in Table III, $b = 12N$, $S_1 = 15$. Similarly to the RIA scheme, we define the receiving filter $\mathbf{U}_i^{(1)} \in \mathbb{C}^{2N \times N S_1}$ where $i \neq j$, with

$$\varphi_1 = \min \left( (N S_1 - b) b \right) = 3N,$$

which consists of the composition of two linear filters $\mathbf{U}_i^{(1)} \in \mathbb{C}^{2N \times N S_1}$, $i \neq j$, defined such that (16) is satisfied. Then, $\mathbf{T}_{j,i} = \mathbf{U}_j^{(1)} \mathbf{H}_j^{(1)} \mathbf{V}_i^{(1)} \in \mathbb{C}^{b \times b}$, $i \neq j$ is defined as a residual interference from TX$_i$ after applying the linear filter $\mathbf{U}_j^{(1)}$, and subspaces $\tilde{\mathbf{T}}_{j,i} = \text{span} (\mathbf{T}_{j,i})$.

C. Hybrid phase

The transmission is developed by pairs, where each pair transmits during $S_2 = 4$ slots. The objective of this phase is twofold. First, each transmitter exploits the overhead information at each receiver during the IS phase to deliver desired linear combinations of symbols, similarly to the second phase of the TG scheme (Section V.C) when $G = 2$. Second, each transmitter sends some redundancy in order to create overhead interference that will be seized during the last phase.

Consider the $(2, r)$th round, with active users $A^{(2,r)} = \{i, j\}$. The transmitted signals are designed such that

$$\text{span} \left( \mathbf{H}_{i,j}^{(2,r)} \mathbf{V}_{j}^{(2,r)} \right) \subseteq \mathbf{T}_{i,j}, \quad \text{span} \left( \mathbf{H}_{j,i}^{(2,r)} \mathbf{V}_{i}^{(2,r)} \right) \subseteq \mathbf{T}_{j,i},$$

thus the precoding matrices are set to

$$\mathbf{V}_{j}^{(2,r)} = \Sigma_{j}^{(2,r)} \mathbf{T}_{j,i}, \quad \mathbf{V}_{i}^{(2,r)} = \Sigma_{i}^{(2,r)} \mathbf{T}_{i,j},$$

where $\Sigma_{i}^{(2,r)}, \Sigma_{j}^{(2,r)} \in \mathbb{C}^{M S_2 \times 3}$ are some arbitrary full rank matrices ensuring the transmit power constraint. For each active pair, $N S_2 = 4N$ LCs of symbols are received, although the rank of the transmitted signals is

$$\text{rank} (\mathbf{V}_{i}^{(2,r)}) = \min \left( M S_2, \text{dim}(\tilde{\mathbf{T}}_{i,j}) \right) = \varphi_1 = 3N,$$

thus there exists some redundancy on the received signals. In this case the per-phase receiving filters are defined as follows:

$$\mathbf{U}_1^{(2)} = \text{bdiag} \left( \mathbf{I}, \text{stack} (\mathbf{U}_{1,2}^{(2)}, \mathbf{U}_{1,3}^{(2)}) \right),$$
$$\mathbf{U}_2^{(2)} = \text{bdiag} \left( \mathbf{I}, \text{stack} (\mathbf{U}_{2,1}^{(2)}, \mathbf{U}_{2,3}^{(2)}) \right),$$
$$\mathbf{U}_3^{(2)} = \text{bdiag} \left( \text{stack} (\mathbf{U}_{3,1}^{(2)}, \mathbf{U}_{3,2}^{(2)}), \mathbf{I} \right).$$
where \( U_{j,i}^{(2)} \in \mathbb{C}^{p_2 \times N S_2} \), with
\[
\varphi_2 = \min (N S_2 - \varphi_1, \varphi_1) = N.
\]

Note that the received signal is modified only for the round where all transmitted signals are interference. The objective of this processing is to obtain signal spaces where the desired signals is interfered by only one user, which will be useful to align the interference during the last phase. For example, the processed signal at the first receiver for the (2, 3)th round writes as:
\[
z_{1}^{(2,3)} = \begin{bmatrix} z_{1,1}^{(2,3)} \\ z_{1,2}^{(2,3)} \end{bmatrix} = H_{1,2}^{(2,3)} V_{2}^{(2,3)} + H_{1,3}^{(2,3)} V_{3}^{(2,3)}
\]
where \( z_{1}^{(2,3)} \) is defined as
\[
F_{k,i}^{(j)} = U_{j,i}^{(2)} H_{j,i}^{(2,r)} V_{r}^{(2,r)} = U_{j,i}^{(2)} H_{j,i}^{(2,r)} \Sigma_{r}^{(2,r)} T_{k,i},
\]
\[
\mathcal{F}_{k,i}^{(j)} = \text{rspan}(F_{k,i}^{(j)}) \subset \mathcal{T}_{k,i},
\]
i.e. \( \mathcal{F}_{k,i}^{(j)} \) is the remaining contribution of \( V_{r}^{(2,r)} \) at RX\(_j\) after suppressing the signal corresponding to user \( k \), i.e. the other active transmitter during the (2, \( r \))th round. Moreover, it represents the subspace of \( \mathcal{T}_{k,i} \) (completely known at RX\(_k\)) that is known thanks to this phase at RX\(_j\). For a better reader’s understanding, let us write the signal space matrix obtained at RX\(_1\) after this phase:
\[
\Omega_{1}^{(2)} = \begin{bmatrix}
U_{1,2}^{(1)} H_{1,1}^{(1)} V_{1}^{(1)} & T_{1,2} & 0 \\
U_{1,3}^{(1)} H_{1,1}^{(1)} V_{1}^{(1)} & 0 & T_{1,3}
\end{bmatrix}
\]
where the dotted lines separate the signals corresponding to each phase, and \( \Omega_{j}^{(p)} \) collects the rows of the signal space matrix \( \Omega_{j} \) up to phase \( p \).

Finally, the number of interference-free LC of desired signals each receiver can retrieve after this phase is summarized. On the one hand, since at each receiver the signals of each round occupy \( N S_2 = 4N \) dimensions, and the interference has rank \( \varphi_1 = 3N \) only, there exists almost surely a \( \varphi_2 \)-dimensional subspace where interference can be projected to. Then, from the two pairs \( 2 \cdot \min (\varphi_1, \varphi_2) = 2N \) LCs are obtained. On the other hand, since precoding matrices are designed to align the interference (conditions in (38)), RX\(_j\) will be able to combine the first phase processed signals with the second phase received signals to cancel the interference. Consequently, \( 2 \varphi_2 = 6N \) additional interference-free LCs of desired signals are retrieved, and only \( b - 8N = 4N \) more LCs are required for ensuring linear decodability.

### D. RIA phase

The third phase lasts for \( S_3 = 4 \) slots, where all users are active. The objective is to design the transmitted signals based on the information commonly known at the non-intended receivers after the first two phases. The precoding matrices for this phase are constructed as follows:
\[
V_{1}^{(3)} = \Sigma_{1}^{(3)} \begin{bmatrix} F_{1,2}^{(3)} \\ F_{1,3}^{(3)} \end{bmatrix}, \quad V_{2}^{(3)} = \Sigma_{2}^{(3)} \begin{bmatrix} F_{2,1}^{(3)} \\ F_{2,3}^{(3)} \end{bmatrix}, \quad V_{3}^{(3)} = \Sigma_{3}^{(3)} \begin{bmatrix} F_{3,1}^{(3)} \\ F_{3,2}^{(3)} \end{bmatrix}
\]
where \( \Sigma_{1}^{(3)} \in \mathbb{C}^{N S_3 \times 2 \times p_2} \). This design ensures that all the generated interference is already known at both non-intended receivers, thus receivers will be able to remove it. Moreover, each receiver observes \( N S_3 = 4N \) linear combinations of the transmitted signals of rank
\[
\text{rank}(V_{i}^{(3)}) = \dim \left( \mathcal{F}_{j,i}^{(k)} + \mathcal{F}_{k,i}^{(j)} \right) = \dim \left( \mathcal{F}_{j,i}^{(k)} \right) + \dim \left( \mathcal{F}_{k,i}^{(j)} \right) = 2 \varphi_2 = 2N, \ i \neq j \neq k.
\]
Then, the same idea as for the second phase applies here: some redundancy is transmitted in order to apply zero-forcing concepts at the receiver. Following the same notation as before, two linear filters \( U_{j,i}^{(3)} \in \mathbb{C}^{2 \times NS}, j \neq i \), are applied at each receiver, with

\[
\varphi_3 = \min \left( NS_3 - 2\varphi_2, 2\varphi_2 \right) = 2N,
\]

For brevity and clarity, the final signal space matrix at RX\(_1\) is next shown:

\[
\Omega_1 =
\begin{bmatrix}
U_{1,2}^{(3)} H_{1,1}^{(3)} V_1^{(1)} & T_{1,2} & 0 \\
U_{1,3}^{(3)} H_{1,1}^{(3)} V_1^{(2)} & 0 & T_{1,3} \\
0 & H_{1,2}^{(2,1)} V_2^{(1)} & 0 \\
0 & 0 & H_{1,3}^{(2,2)} V_3^{(2)} \\
U_{1,2}^{(3)} H_{1,1}^{(3)} V_1^{(3)} & U_{1,2}^{(3)} H_{1,2}^{(3)} V_2^{(3)} & 0 \\
0 & U_{1,3}^{(3)} H_{1,3}^{(3)} V_3^{(3)} & 0
\end{bmatrix},
\]

where the signals received during the RIA phase are processed using \( U_{1,2}^{(3)} \) and \( U_{1,3}^{(3)} \), see the last two blocks rows. Now it is easy to see that all the interference is aligned. For example, consider the 1st, 5th and 7th block rows. Since

\[
\begin{align*}
\text{rspan} & \left( U_{1,2}^{(3)} H_{1,2}^{(3)} V_2^{(3)} \right) \subseteq \text{rspan} \left( V_2^{(3)} \right), \\
\text{rspan} & \left( V_2^{(3)} \right) \subseteq \mathcal{F}_3^{(3)} + \mathcal{F}_1^{(3)}, \\
\mathcal{F}_3^{(3)} & \subseteq \mathcal{T}_1^{(3)},
\end{align*}
\]

the signals corresponding to the 1st and 5th block row can be used to remove the interference from the signals represented by the 7th block row. Then, \( 2\varphi_2 \) LCs of desired signals are retrieved. Following similar arguments for rows 2nd, 6th, and 8th, \( 2N \) extra LCs are obtained. Combining the \( 4\varphi_2 = 4N \) LCs of desired signals obtained from this phase with the \( 8N \) LCs from previous phases, each receiver obtains enough LCs for linearly decode all of its \( b = 12N \) desired symbols.

**E. System parameters optimization**

The parameters for the BSR scheme are derived by means of the following DoF optimization problem:

\[
\mathcal{P}_3 : \text{maximize } \frac{b}{b + 4\varphi_1 + 5\varphi_2 + \varphi_3} \quad \text{s.t. } \begin{align*}
\rho(\varphi_1 + b) & \geq b \\
4\varphi_1 & \geq b \\
\rho(\varphi_1 + \varphi_2) & \geq \varphi_1 \\
\varphi_2 & \leq \varphi_1 \\
2(\varphi_1 + \varphi_2) & < b \\
\rho(\varphi_3 + 2\varphi_2) & \geq 2\varphi_2 \\
2(\varphi_1 + \varphi_2 + \varphi_3) & \geq b \\
\varphi_3 & \leq 2\varphi_2
\end{align*}
\]

formulated in terms of \( \varphi_1 > 0, i = 1, 2, 3 \), where the number of slots can be retrieved by applying the following change of variables:

\[
\varphi_1 = NS_1 - b, \quad \varphi_2 = NS_2 - \varphi_1, \quad \varphi_3 = NS_3 - 2\varphi_2.
\]

While the objective function corresponds to \( \frac{b}{N^2} \) in terms of the new variables, the constraints imposed to ensure linear feasibility are next described:
1) Transmit rank during the IS phase (54b): Similarly to other schemes, $MS_1 \geq b$ is imposed to ensure the transmit rank.

2) Linear combinations on the system (54c): After the first phase processing, $4\varphi_1$ linear combinations of the symbols of each user are distributed along the receivers: $2\varphi_1$ at the intended receiver (known coupled with interference), and $\varphi_1$ at each non-intended receiver. Then, since the rest of phases are just retransmissions, a necessary condition is that at least obtaining all of them the $b$ desired symbols should be linearly decodable.

3) Transmit rank during the hybrid phase (54d): Written in terms of the new variables, it is forced $MS_2 \geq \varphi_1$, since the rank of the transmitted signals during each second phase round is equal to $\varphi_1$, see (40).

4) Bounded redundancy during the hybrid phase and need of RIA phase (54e) and (54f): After the hybrid phase, each receiver is able to retrieve $\varphi_1 + \varphi_2$ interference-free LCs of desired symbols from each of the two rounds where desired LCs of signals are sent. First, exploiting the redundancy on the received signals due to $\varphi_2 = NS_2 - \varphi_1 > 0$, $\min(\varphi_2, \varphi_1)$ linear combinations can be retrieved by zero-forcing concepts. Then, we force (54e), since having $\varphi_2 > \varphi_1$ does not provide additional LCs. This constraint bounds the value of $S_2$, and it is also imposed by $\mathcal{F}_{k,j} \subset \mathcal{T}_{k,i}$, as assumed in (44).

On the other hand, $\varphi_1$ LCs are obtained through RIA concepts, by projecting the signals of the corresponding round of the hybrid phase onto a subspace of dimension $\varphi_1$, and combining them with the IS phase processed signals. Consequently, at the end of the hybrid phase $2(\varphi_1 + \varphi_2)$ independent observations are obtained. (54f) ensures that still some extra LCs are required, and thus RIA phase is necessary.

5) Transmit rank during the RIA phase (54g): Written in terms of the new variables, it is forced $MS_3 \geq 2\varphi_2$, see (39).

6) Linear combinations at the end of the transmission (54h) and bounded redundancy during the RIA phase (54i): The signal received during the RIA phase is processed to decouple the interference, see (50). Those processed signals combined with the rest of available overheard interference provide $2 \cdot \min(\varphi_3, 2\varphi_2)$ extra observations of the desired symbols. First, (54i) is forced to bound the value of $S_3$, and because in this case more redundancy does not provide additional LCs. Second, the number of interference-free LCs of desired signals each receiver is able to retrieve at the end of the transmission is equal to $2 \cdot (\varphi_1 + \varphi_2 + \varphi_3)$, and it should be enough to linearly decode all the $b$ desired symbols.

The problem $\mathcal{P}_3$ in (54) is next solved. Before proceeding, let us introduce the following proposition:

**Proposition 1:** Consider the following two linear inequalities:

$$x + by \geq cz, \quad (56)$$
$$dx + ey \leq fz, \quad (57)$$

where $\{a, b, c, d, e, f\}$ are positive given parameters, and $\{x, y, z\}$ represent unknown variables. Then, any solution satisfying both inequalities also satisfies:

$$cdx + cey \leq fax + fby. \quad (58)$$

This trivial proposition is useful because it allows to suppress variables from linear constraints. Actually, it is the basis of the Fourier-Motzkin Elimination method, see [27].

Consider the application of Proposition 1 to (54e), (54f), and (54i), such that variable $\varphi_3$ is removed. This leads to the following two constraints:

$$2(\varphi_1 + 3\varphi_2) \geq b, \quad (59)$$
$$\varphi_2 (1 - 2\rho) \geq 0, \quad (60)$$

where the second constraint forces $\rho \geq \frac{1}{2}$. Now, let us apply again the proposition to (54e), (54i), and the new constraint (59) in order to remove $\varphi_2$. Again, two new constraints are procuded:

$$8\varphi_1 \geq b$$
$$\varphi_1 (1 - 2\rho) \geq 0,$$

which are loose with respect to the rest of constraints. Then, the value of $\varphi_1$ is completely determined by (54e) and (54c), as follows:

$$\varphi_1 = b \max \left\{ \frac{1}{4}, \frac{1 - \rho}{\rho} \right\}, \quad (61)$$

thus establishing two regions: $\rho \geq \frac{4}{5}$ and $\rho < \frac{4}{5}$. For a given value of $\varphi_1$, the optimal $\varphi_2$ is decided according to (54e) and (59), as follows:

$$\varphi_2 = \max \left( \varphi_1 \frac{1 - \rho}{\rho} \cdot \frac{1}{6} (b - 2\varphi_1) \right). \quad (62)$$
Finally, the optimal value of $\varphi_3$ is set according to

$$\varphi_3 = \max \left( 2\varphi_2 \frac{1 - \rho}{\rho} \frac{b}{2} - \varphi_1 - \varphi_2 \right).$$  \hfill (63)

It can be checked that the control constraints (54) and (55) are always satisfied following these rules. The values in Table III are obtained by inverting the change of variables and taking the value of $b$ such that $S_1$, $S_2$, and $S_3$ are integer values.

VII. DoF-Delay Trade-off

The precoding schemes exploiting delayed CSIT require multi-phase transmissions. For some settings, this entails long communication delays, as well as transmitting a high number of transmitted symbols, thus increasing the complexity of the encoding/decoding operation at transmitters/receivers. This section studies the DoF-delay trade-off of the proposed and some state-of-the-art schemes. Thanks the DoF-delay trade-off analysis, two main insights are concluded:

- The supremacy in terms of achievable DoF of one scheme with respect to another can vary depending on the complexity that is allowed. This occurs for example when $\rho = 1$, $K = 3$, where the RIA scheme may outperform the BSR scheme.
- The communication delay can be highly alleviated without high DoF penalties. The balance between optimal (but usually large) parameters and maximum DoF provided by the proposed and previous schemes may be also drawn from our results.

The DoF-delay trade-off of the proposed schemes is studied by means of two methodologies. In the following three sections, the trade-off is analyzed by limiting the maximum number of symbols per user that can be transmitted to $B_{\text{max}}$. Exploiting the formulation of the system parameters optimization, the following constraint has simply to be included into the DoF optimization problems:

$$b \leq B_{\text{max}}.$$

The case where this constraint is omitted or, equivalently, $B_{\text{max}} \rightarrow \infty$, will be hereafter denoted as the unbounded case. For each scheme, a simplified version of the DoF optimization problem for finite $B_{\text{max}}$ is provided. Then, at least two examples are evaluated for each case, one for $K = 3$ and one for $K = 6$, which are useful to benchmark one of the values of $\rho$ for Fig. 3 and 4 as a function of $L$.

On the other hand, in Section VII-D the DoF-delay trade-off is studied by limiting the order of the transmitted symbols. This alternative is proposed in order to compare the proposed schemes with the BSR scheme in [21] and its extension to MISO in [22] for $K > 3$. The derivation of DoF optimization problems for those cases remains as future work.

An order-$m$ symbol refers to a supersymbol which is desired or available at $m$ receivers, either to remove interference or because it contains the symbols intended to that receiver. For example, during the second phase it can be interpreted that the RIA and TG schemes transmit order-$L$ and order-$G$ symbols, respectively. While we only work with order-1 DoF, the literature following this framework denotes by $d_j^{(m)}$ the DoF for order-$m$ messages, i.e. the efficiency of transmitting order-$m$ symbols through the network, and they are formulated in a recursive way as follows:

$$d_j^{(m)} = f \left( d_j^{(m+1)} \right), \quad m = 1 \ldots K - 1,$$

$$d_j^{(1)} \equiv d_j,$$

$$d_j^{(K)} = 1. \hfill (66)$$

In other words, the efficiency of transmitting order-$m$ symbols depends on the efficiency of transmitting order-$(m+1)$ symbols, whereas the last phase, when order-$K$ symbols are transmitted, is developed in a TDMA fashion. Inspired by this formulation, the precoding schemes in [21] and [22] will be constrained to maximum order $\Theta$ by forcing:

$$d_j^{(\Theta)} = 1. \hfill (67)$$

A. RIA scheme

A closed-form solution for $S_1$ and $S_2$ was obtained in Section IV-D see (22) and (23). For unbounded $b$, the value of $L$ was obtained given $\rho$ and $K$ by means of Algorithm 1. However, for finite $B_{\text{max}}$ the optimal value of $L$ becomes a function of $B_{\text{max}}$. In this regard, the achievable DoF for the RIA scheme write as follows:

$$d_j(B_{\text{max}}) = \frac{1}{KN} \max \left( f_{\text{RIA},1}(B_{\text{max}}), f_{\text{RIA},2}(B_{\text{max}}) \right), \hfill (68)$$

$$f_{\text{RIA},1}(x) = \max_{b \leq x, L} \frac{bL}{\left\lfloor \frac{b}{M} \right\rfloor + \left\lfloor \frac{b}{N} \right\rfloor},$$

$$f_{\text{RIA},2}(x) = \max_{b \leq x, L} \frac{bL}{\left\lfloor \frac{b}{M} \frac{L^2 - L - 1}{L} \right\rfloor + \left\lfloor \frac{b}{N} \right\rfloor}. \hfill (70)$$
where $f_{\text{RIA},1}(x)$ and $f_{\text{RIA},2}(x)$ represent the achievable DoF for each side of the stepping function. Since the value of $L$ depends on $B_{\text{max}}$, it is not possible to derive a threshold as $\rho_A(L)$. Then, we maximize w.r.t. $L$ and $b$, and then just take the maximum between the two sides of the stepping function.

The maximization problem for finite $B_{\text{max}}$ has been solved for the two cases: $(M, N, K) = (4, 7, 3)$, and $(M, N, K) = (3, 4, 6)$, where the solutions follow the expressions given in (68)–(69). The achievable DoF w.r.t. the communication delay are depicted in Fig 9 (top for $B_{\text{max}} = 1 \ldots b^*$, where $b^*$ denotes for each setting the optimal value of $b$ for the unbounded case. Moreover, the DoF achieved without the need of CSIT are also included for comparison. First, notice that since $L$ is defined in the set \{3, ..., $K$\}, the only possible value for the first setting is $L = 3$. In such a case, since $\rho < \rho_A(3) = \frac{2}{3}$, it follows

$$d_j(B_{\text{max}}) = \frac{1}{KN} f_{\text{RIA},1}(B_{\text{max}}).$$

The more interesting conclusion from the figure is that the number of required slots can be dramatically reduced without high DoF penalties. In particular, the number of time slots may be halved (from 11 to 5), while 94% of the maximum DoF are attained (from 0.3636 to 0.3429). In contrast, for the setting $(M, N, K) = (2, 5, 6)$ the value of $L$ changes as a function of $B_{\text{max}}$, as highlighted in the figure. Notice that in this case the number of slots required to outperform TDMA is huge, and DoF gains are insignificant.

The reader may have noticed that the cases with $\rho > \frac{2}{3}$ have been omitted. In this regard, two additional examples will be shown for the RIA scheme in Section VII-C, deferred to that section in order to compare together the RIA and BSR schemes performance for limited $B_{\text{max}}$.

### B. TG scheme

Closed form solutions for $S_1^*$ and $S_2^*$ were found in Section V-D see (32) and (34), next restated for reader’s convenience. Note that $S_2^*$ depends on the value taken for $S_1^*$, which depend on the antenna ratio and $B_{\text{max}}$. In this case, the achievable DoF for a given $B_{\text{max}}$ write as follows:

$$d_j(B_{\text{max}}) = \max_{b \leq B_{\text{max}}, G} \frac{1}{N} KS_1^* + \frac{b}{K} S_2^*,$$  

$$S_1^* = \left[ b \cdot \max \left( \frac{1}{M}, \frac{1 + \alpha(G) \cdot (G - 2)}{N} \right) \right],$$

$$S_2^* = \left[ \frac{1}{\alpha(G)} \left( \frac{b}{N} - S_1^* \right) \right], \quad \alpha(G) = \frac{K - 1}{G - 1}.$$  

Two settings are simulated and shown in Fig 9 (middle): $(M, N, K) = (7, 5, 3)$, and $(M, N, K) = (4, 1, 6)$. While the curves have been obtained by solving the problem $P_2$ in (31), one can check that they follow the expressions in (eq:TG-1)–(71). For comparison purposes, in addition to the TDMA performance, the scheme in (17) for the 2-user IC has been considered. This scheme is applied to the $K$-user case by means of time-sharing, which dramatically increases the communication delay. In order to obtain its performance for different values of $B_{\text{max}}$, a DoF maximization problem has been formulated. The problem is very similar to the TG scheme with $G = 2$, and thus omitted.

Both figures show the DoF gains provided by the wise use of delayed CSIT w.r.t. no CSIT by increasing the duration of the communication $\tau$. Two remarkable observations can be drawn, one for each setting. For the setting $(M, N, K) = (7, 5, 3)$ the DoF attained using delayed CSIT for both strategies are similar for the unbounded case. However, this is at the cost of a high communication delay for the scheme in (17). If otherwise $\tau$ is reduced, then the TG scheme clearly outperforms any other strategy.

On the other hand, for the setting $(M, N, K) = (4, 1, 6)$, it can be observed that the unbounded case requires $\tau = 75$ slots, while similar DoF gains can be obtained using only $\tau = 27$ slots, and also outperforming any other scheme. This is one of the main conclusions obtained from our analysis: while the best DoF are attained using a high number of time slots, usually one solution with reduced number of time slots can be found without high DoF penalties. The reader may have noticed that no case with $\rho > K - 1$ has been considered, where the scheme in (22) surpasses the proposed TG scheme. One example for $K = 6$ will be addressed in Section VII-D.

### C. BSR scheme

The performance of the BSR scheme is compared with the RIA scheme for $K = 3$ users. Since the region of most interest for this scheme is $\rho > \frac{4}{5}$, we consider two representative antenna settings: $(M, N) = (4, 5)$, and SISO $(M = N = 1)$. In this
Fig. 9. Achievable DoF of the proposed schemes vs duration of the transmission $\tau$ for different values of $B_{\text{max}}$. The DoF achieved without the need of CSIT or using previous schemes in the literature are also depicted for comparison purposes.
The performance for the two settings is depicted in Fig. 9-bottom. The most remarkable result is that whenever $B_{\text{max}}$ is below $b^*$, the RIA outperforms the BSR scheme. Moreover, notice that for the unbounded case a similar DoF performance (from 0.387 to 0.375) is obtained for RIA w.r.t. the BSR scheme with only a quarter of the number of slots (from 31 to 8).

VIII. ACHIEVABLE DOF FOR CONSTANT CHANNELS

The literature on delayed CSIT always assume that channel feedback incurs a delay larger than channel coherence time, i.e. the current channel is completely uncorrelated w.r.t. the channel that has been reported. However, this assumption is not always realistic in practice, where the transmitter has no way to know the current channel coefficients. In this regard, this section studies the extreme case where the channel is constant, but transmitter is not aware of this, and performs a delayed-CSIT strategy anyways. Then, the next sections prove Theorem 4, stating that all results so far also apply for constant channels.

The difference in the system model between constant and time-varying channels is that all block diagonal compositions of channels are simplified to Kronecker products. Let $\tilde{H}_{j,i} \in \mathbb{C}^{N \times M}$ denote the channel between TX$_i$ and RX$_j$ for all $\tau$ slots of the communication, since the channels are constant. Then, we have

$$H_{j,i} = I_\tau \otimes \tilde{H}_{j,i}.$$
It is instructive to particularize it to the SISO case, where channels become scaled identity matrices, i.e:

$$H_{j,i} = \mathbf{I}_r \otimes \tilde{h}_{j,i} = \tilde{h}_{j,i} \mathbf{I}_r,$$  \hspace{1cm} (75)

with a particular structure that presents lower diversity than MIMO channels.

### A. RIA scheme

We show that the RIA scheme described in Section [IV] fails for the SISO case if channels are constant and $L = 3$. Next section will show that using asymmetric complex signaling makes this scheme to work. Similar arguments allow to show that for the rest of antenna settings it works with probability one.

During the first phase of the RIA scheme, all transmitters are active, using predetermined precoding matrices $\mathbf{V}_i^{(1)} \in \mathbb{C}^{5 \times 3}$, and interfering to all users. The received signal is processed using the per-phase linear filters $\mathbf{U}_{i,j} \in \mathbb{C}^{2 \times 5}$, such that the desired signals in the the processed signal are only mixed with interference from another user. Consider the signal space matrix for the signals received during the first phase:

$$\Omega_i^{(1)} = \begin{bmatrix} \mathbf{U}_{i,i+1} h_{i,i} \mathbf{V}_i^{(1)} & \mathbf{U}_{i,i+1} h_{i,i+1} \mathbf{V}_i^{(1)} & 0 \\ \mathbf{U}_{i,i-1} h_{i,i} \mathbf{V}_{i-1}^{(1)} & 0 & \mathbf{U}_{i,i+1} h_{i,i-1} \mathbf{V}_{i-1}^{(1)} \end{bmatrix},$$  \hspace{1cm} (76)

where indices in this section are assumed to be in the set $\{1, 2, 3\}$, applying the modulo-3 operation only if necessary. Notice that matrices $\mathbf{U}_{i,j}$ satisfy

$$\text{rspan} \left( \mathbf{U}_{i,j} \right) = \text{null} \left( \text{span} \left( \mathbf{V}_k^{(1)} \right) \right), \forall i,j \neq k, i \neq j,$$  \hspace{1cm} (77)

i.e. $\mathbf{U}_{i,j}$ removes the interference generated at $\text{RX}_i$ by user $k \neq j$, but not the interference from user $j$. Due to definition (77), there are only three different per-phase filters. Indeed, they correspond to the null space of each $\mathbf{V}_i^{(1)}$, which will be denoted as $\bar{\mathbf{V}}_i^{(1)} \in \mathbb{C}^{2 \times 5}$ for ease of description. Accordingly, the signal space matrix for the whole communication writes as

$$\Omega_i = \begin{bmatrix} h_{i,i} \mathbf{V}_{i-1}^{(1)} & h_{i,i+1} \mathbf{V}_{i-1}^{(1)} & 0 \\ h_{i,i} \mathbf{V}_{i+1}^{(1)} & 0 & h_{i,i-1} \mathbf{V}_{i+1}^{(1)} \\ h_{i,i} \Sigma_i^{(2)} \mathbf{I}_3 & h_{i,i+1} \Sigma_i^{(2)} \mathbf{I}_3 & h_{i,i-1} \Sigma_i^{(2)} \mathbf{I}_3 \\ h_{i,i} \Sigma_i^{(2)} \mathbf{I}_3 & h_{i,i+1} \Sigma_i^{(2)} \mathbf{I}_3 & h_{i,i-1} \Sigma_i^{(2)} \mathbf{I}_3 \\ h_{i,i} \Sigma_i^{(2)} \mathbf{I}_3 & h_{i,i+1} \Sigma_i^{(2)} \mathbf{I}_3 & h_{i,i-1} \Sigma_i^{(2)} \mathbf{I}_3 \end{bmatrix},$$  \hspace{1cm} (78)

where the precoding matrices for the second phase are computed following (18) and (19), here repeated for reader’s convenience:

$$\mathbf{V}_i^{(2)} = \Sigma_i^{(2)} \mathbf{T}_i^{(2)},$$

$$\text{rspan} \left( \mathbf{T}_i^{(2)} \right) = \mathbf{T}_i^{(2)} = \mathbf{T}_{i+1,i} \cap \mathbf{T}_{i-1,i} ,$$

where $\mathbf{T}_{i,i} \in \mathbb{C}^{2 \times 3}$, with $\mathbf{T}_{i+1,i} = h_{i+1,i} \mathbf{V}_{i-1}^{(1)} \mathbf{V}_i^{(1)}$ and $\mathbf{T}_{i-1,i} = h_{i-1,i} \mathbf{V}_{i+1}^{(1)} \mathbf{V}_i^{(1)}$. This design allows that the interference generated during the RIA phase to be aligned with the IS phase overheard interference at both non-intended receivers. Now, since $\mathbf{T}_{i+1,i}$ and $\mathbf{T}_{i-1,i}$ are independent, its intersection will be of dimension one with probability one. Then, there exist two vectors $\mathbf{\vartheta}_i, \overline{\mathbf{\vartheta}}_i \in \mathbb{C}^{2 \times 1}$ such that $\mathbf{T}_i^{(2)}$ can be written as

$$\mathbf{T}_i^{(2)} = \vartheta_i^T \bar{\mathbf{V}}_{i-1}^{(1)} \mathbf{V}_i^{(1)} \mathbf{V}_i^{(2)} = \overline{\vartheta}_i^T \bar{\mathbf{V}}_{i+1}^{(1)} \mathbf{V}_i^{(2)},$$  \hspace{1cm} (79)

where $\doteq$ is short for equality of row spans. Notice that $\mathbf{\vartheta}_i$ and $\overline{\mathbf{\vartheta}}_i$ correspond to the vectors that project $\bar{\mathbf{V}}_{i-1}^{(1)} \mathbf{V}_i^{(1)}$ and $\bar{\mathbf{V}}_{i+1}^{(1)} \mathbf{V}_i^{(1)}$ to its intersection subspace, respectively. The following lemma states a key property satisfied by these vectors:

**Lemma 1:** If the vectors $\mathbf{\vartheta}_i, \overline{\mathbf{\vartheta}}_i, i = 1, 2, 3$ are computed satisfying the properties in (79), then $\mathbf{\vartheta}_i \doteq \overline{\mathbf{\vartheta}}_{i+1}$.

**Proof:** Only the proof for $i = 1$ will be shown. The proof for $i = 2, 3$ follows the same steps thus it is omitted. First, notice that (79) for $i = 1, 2$ can be written as follows:

$$\mathbf{\vartheta}_1^T \bar{\mathbf{V}}_3^{(1)} - \mathbf{\vartheta}_1^T \bar{\mathbf{V}}_2^{(1)} \subset \bar{\mathbf{V}}_1^{(1)} \Rightarrow \mathbf{\vartheta}_1^T \bar{\mathbf{V}}_3^{(1)} - \mathbf{\vartheta}_1^T \bar{\mathbf{V}}_2^{(1)} = \lambda \bar{\mathbf{V}}_1^{(1)},$$  \hspace{1cm} (80)

$$\mathbf{\vartheta}_2^T \bar{\mathbf{V}}_3^{(1)} - \mathbf{\vartheta}_2^T \bar{\mathbf{V}}_1^{(1)} \subset \bar{\mathbf{V}}_2^{(1)} \Rightarrow \mathbf{\vartheta}_2^T \bar{\mathbf{V}}_3^{(1)} - \mathbf{\vartheta}_2^T \bar{\mathbf{V}}_1^{(1)} = \varphi \bar{\mathbf{V}}_2^{(1)}.$$  \hspace{1cm} (81)
for some $\lambda, \varphi \in \mathbb{C}^{2 \times 1}$, which is equivalent to

$$
\begin{bmatrix}
\lambda^T, \varphi_1^T, \varphi_1^T
\end{bmatrix}
\begin{bmatrix}
\mathbf{V}_1^{(1)} \\
\mathbf{V}_2^{(1)} \\
-\mathbf{V}_3^{(1)}
\end{bmatrix} = 0, \quad
\begin{bmatrix}
\varphi_2^T, \varphi^T, \varphi_2^T
\end{bmatrix}
\begin{bmatrix}
\mathbf{V}_1^{(1)} \\
\mathbf{V}_2^{(1)} \\
-\mathbf{V}_3^{(1)}
\end{bmatrix} = 0. \quad (82)
$$

Hence, $\varphi_1$ and $\varphi_2$ are the last two components of any vector lying on the null space of the $6 \times 5$ full rank matrix on the right hand side. Since it has dimension one, the last two components will always be proportional, thus $\varphi_1 \approx \varphi_2$.

Linear feasibility requires that the rank of the equivalent channel is equal to the number of transmitted symbols. This will be settled in the negative for user one, while non-feasibility for the rest of users may be similarly proved. In this regard, consider its equivalent channel:

$$
H_1^{(\text{eq})} = W_1 U_1 H_1 V_1 = W_1 \begin{bmatrix}
h_{1,1} \mathbf{V}_3^{(1)} \mathbf{V}_1^{(1)} \\
-\mathbf{V}_2^{(1)} \mathbf{V}_1^{(1)} \\
\mathbf{I}_4
\end{bmatrix}, \quad (83)
$$

where (a) is just a remainder of the definition (8) for the sake of reader’s convenience, and (b) simply writes the equivalent channel as the first block column block rows of the signal space matrix (containing the desired signals) multiplied by the receiving filter $W$. The objective of this filter is to remove the interference by combining the rows of the signal space matrix. One simple solution is

$$
W_1 = \begin{bmatrix}
\Sigma_2^{(2)} \varphi_2^T \\
\Sigma_3^{(2)} \varphi_3^T \\
\mathbf{I}_4
\end{bmatrix}, \quad (84)
$$

thus the equivalent channel in (83) writes as

$$
H_1^{(\text{eq})} = h_{1,1} \Sigma_2^{(2)} \varphi_2^T \mathbf{V}_3^{(1)} \mathbf{V}_1^{(1)} + h_{1,1} \Sigma_3^{(2)} \varphi_3^T \mathbf{V}_2^{(1)} \mathbf{V}_1^{(1)} + h_{1,1} \Sigma_2^{(2)} \varphi_3^T \mathbf{V}_3^{(1)} \mathbf{V}_1^{(1)}, \quad (85)
$$

where it can be seen that the first and last terms are proportional according to Lemma 1. Moreover, the last term can be written as $h_{1,1} \Sigma_2^{(2)} \varphi_2^T \mathbf{V}_3^{(1)} \mathbf{V}_1^{(1)}$ due to definition (79), which is also proportional to the last term based on Lemma 1. Consequently, the equivalent channel has rank one, and the three desired symbols cannot be retrieved.

B. RIA scheme with ACS

As for the full CSIT case [6][10], the application of asymmetric complex signaling concepts enables the feasibility of the RIA scheme either for constant or time-varying channels also for the SISO case. To the best of the authors knowledge, this is the first claim that improper signaling may be useful for precoding schemes using delayed CSIT. This section provides a sketch of the proof, omitted due to redundancy with the cited references.

In case of using asymmetric complex signaling, the channel can be modeled in terms of real magnitudes (see [10]), such that 2b real symbols are transmitted to each user along 2τ slots, and the channel model in (75) translates to

$$
H_{j,i} = I_\tau \otimes |\tilde{h}_{j,i}| \Phi_{j,i} = |\tilde{h}_{j,i}| \Phi_{j,i} \in \mathbb{R}^{2\tau \times 2\tau}, \quad (86)
$$

where $\phi_{j,i}$ is the phase of the complex channel gain $\tilde{h}_{j,i}$, and

$$
\Phi_{j,i} = \begin{bmatrix}
\cos (\phi_{j,i}) & -\sin (\phi_{j,i}) \\
\sin (\phi_{j,i}) & \cos (\phi_{j,i})
\end{bmatrix} \in \mathbb{R}^{2 \times 2}, \quad (87)
$$

$$
\Phi_{j,i} = I_\tau \otimes \hat{\Phi}_{j,i}. \quad (88)
$$

Matrices $\Phi_{j,i}$ break the diagonal structure of channel matrices. This is of interest because in previous section the same interference was generated at both unintended receivers thereby the same per-phase filter was used to remove it, see (77). Nonetheless, in this case different per-phase filters should be used, thus the connections among vectors $\theta_1, \varphi_1$ stated by Lemma 1 no more hold, and feasibility is ensured for any channel realization. Similar arguments apply to the MIMO case.
C. TG scheme

We review the foundations of this scheme, proposed in Section V for \( M > N \), in order to show that it also works for constant channels. During the OT phase transmitters are scheduled in a TDMA fashion. Therefore, for each \( RX_j \) obtains

\[
\begin{align*}
\mathbf{y}_{j}^{(1,r)} &= \mathbf{H}_{j,r}^{(1,r)} \mathbf{V}_{r}^{(1,r)} \mathbf{x}_r = \left( \mathbf{I}_{S_l} \otimes \tilde{\mathbf{H}}_{j,i} \right) \mathbf{V}_{r}^{(1,r)} \mathbf{x}_r, \\
\mathbf{T}_{j,i} &= \left( \mathbf{I}_{S_l} \otimes \tilde{\mathbf{H}}_{j,i} \right) \mathbf{V}_{r}^{(1,i)},
\end{align*}
\]

where the precoding matrices \( \mathbf{V}_{i}^{(1,i)} \in \mathbb{C}^{M S_l \times b} \) are chosen to be some generic full-rank matrices, with \( \mathbf{V}_{i}^{(1,r)} = \mathbf{0} \) for \( r \neq i \). Since \( M > N \), and \( NS_1 < b \) by design, it is easy to see that all ranks are preserved even for constant channels, i.e. rank(\( \mathbf{T}_{j,i} \)) = \( NS_1 \), \( \forall i \), and all such pieces of overheard interference generate generic subspaces \( T_{j,i} \).

Now, let us recall that the precoders for each round of the RIA phase, see (28), are linear combinations of \( \mathbf{T}_{i}^{(2,r)} \), obtained as a basis of

\[
\text{rspan} \left( \mathbf{T}_{i}^{(2,r)} \right) = \mathbf{T}_{i}^{(2,r)} = \bigcap_{k \in \mathcal{A}^{(2,r)} \setminus \{i\}} \mathcal{T}_{k,i}
\]

which will also preserve the rank. Therefore, we conclude that this scheme does not require the time-varying channels assumption, since each receiver can acquire enough linear combinations of desired symbols even in case of constant channels.

D. 3-user BSR scheme

The first phase of this scheme is similar to that for the RIA scheme. In contrast, there are three phases and the second phase is developed by pairs. Feasibility is easy to show for MIMO channels, whereas the SISO setting fails. Since the scheme delivers exactly 12 LCs of the \( b = 12 \) desired symbols to each receiver, by simply showing that some of those LCs are linearly dependent is sufficient to show the no feasibility. In this regard, next we show that not all LCs delivered during the first round of the second phase are linearly independent. Consider the signal space matrix for the second phase, particularized for this case:

\[
\mathbf{\Omega}_1^{(2)} = \begin{bmatrix}
\mathbf{h}_{1,1} \mathbf{V}_{3}^{(1)} & \mathbf{T}_{1,2} & \mathbf{0} \\
\mathbf{h}_{1,1} \mathbf{V}_{2}^{(1)} & \mathbf{0} & \mathbf{T}_{1,3} \\
\mathbf{h}_{1,1} \mathbf{\Sigma}_{1}^{(2,1)} \mathbf{T}_{2,1} & \mathbf{h}_{1,2} \mathbf{\Sigma}_{2}^{(2,1)} \mathbf{T}_{1,2} & \mathbf{0} \\
\mathbf{h}_{1,1} \mathbf{\Sigma}_{1}^{(2,2)} \mathbf{T}_{3,1} & \mathbf{0} & \mathbf{h}_{1,1} \mathbf{\Sigma}_{3}^{(2,2)} \mathbf{T}_{1,3} \\
\mathbf{0} & \mathbf{F}_{3,2}^{(1)} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{F}_{2,3}^{(1)}
\end{bmatrix},
\]

where the same notation as for the RIA case has been used, and in this case we have \( \mathbf{T}_{2,1} = \mathbf{h}_{2,1} \mathbf{V}_{3}^{(1)} \mathbf{V}_{1}^{(1)} \in \mathbb{C}^{3 \times 12} \).

Two methods for delivering LCs of desired symbols were used in the second phase. First, recall that \( \mathbf{\Sigma}_{2}^{(2,1)} \in \mathbb{C}^{4 \times 3} \), thus zero-forcing the interference received during the first round of the second phase, RX1 obtains

\[
\lambda^T \mathbf{h}_{1,1} \mathbf{\Sigma}_{1}^{(2,1)} \mathbf{T}_{2,1} \mathbf{x}_1
\]

for some \( \lambda \in \mathbb{C}^{4 \times 1} \) that satisfies \( \lambda^T \mathbf{\Sigma}_{2}^{(2,1)} = \mathbf{0} \). Clearly, such LC of desired signals lies on \( \text{rspan}(\mathbf{T}_{2,1}) \).

On the other hand, four LCs of desired signals may be obtained by combining the IS phase received signals with the signals received during the first round of the hybrid phase:

\[
\left( \mathbf{h}_{1,2} \mathbf{\Sigma}_{2}^{(2,1)} \mathbf{h}_{1,1} \mathbf{V}_{3}^{(1)} \mathbf{V}_{1}^{(1)} + \mathbf{h}_{1,1} \mathbf{\Sigma}_{1}^{(2,1)} \mathbf{T}_{2,1} \right) \mathbf{x}_1 = \left( \frac{\mathbf{h}_{1,2}}{\mathbf{h}_{2,1}} \mathbf{\Sigma}_{2}^{(2,1)} + \mathbf{\Sigma}_{1}^{(2,1)} \right) \mathbf{h}_{1,1} \mathbf{T}_{2,1} \mathbf{x}_1.
\]

Those LCs form a basis of the three-dimensional subspace \( \text{rspan}(\mathbf{T}_{2,1}) \), thus actually would provide only three independent desired LCs of desired symbols. However, since the LC obtained by the first method lies also in \( \text{rspan}(\mathbf{T}_{2,1}) \), after this round user one acquires only three instead of four independent desired LCs, and linear feasibility is discarded.

Nonetheless, this problem can be fixed by exploiting asymmetric complex signaling, since the per-phase receiving filters for the second phase are distinct across users, similarly to what occurs for the RIA scheme. Then, the BSR scheme can be made feasible even for SISO constant channels.
IX. CONCLUSION

The DoF-delay trade-off has been studied for the $K$-user MIMO IC with delayed CSIT. Three basic tools are envisioned for designing precoding strategies using delayed CSIT: linear beamforming, user scheduling, and redundancy transmission. In this regard, this work proposes three precoding strategies, and evaluate them as a function of the antenna ratio $\rho$.

For $\rho < 1$, the RIA scheme initially proposed for the 3-user SISO IC ($\rho = 1$) has been generalized to the $K$-user MIMO case. This scheme exploits linear beamforming and redundancy transmission. In contrast to a previous conjecture, our results show that state-of-the-art DoF can be improved by considering $L \geq 3$ active pairs. Moreover, we have shown that for the region $\frac{1}{K-1} < \rho < \frac{K}{K^2-K-1}$ our proposed inner bound using the RIA scheme gets very close to the best known outer bound.

Moreover, we have generalized the BSR scheme for 3 users from SISO to MIMO, which combines the three tools: linear beamforming, user scheduling, and redundancy. This scheme provides the best achievable DoF when the number of antennas at the transmitter and receiver are similar ($\rho \approx 1$) not only for the 3-user MIMO IC, but also for the $K$-user MIMO IC by applying time-sharing concepts. Nevertheless, a MIMO generalization for $K > 3$ users remains open.

In case the transmitter has more antennas than the receiver ($\rho > 1$), we propose the TG scheme improving state-of-the-art for $1 < \rho < K-1$. Linear beamforming and user scheduling are carefully designed for DoF boosting, where the first phase is carried out orthogonally among users, whereas the second phase is developed in groups of $G \leq K$ users. The proper value of $G$ lies on the trade-off between the constraints imposed by interference alignment, and the increase on the number of rounds, in turn depending on the antenna ratio $\rho$ and the number of users $K$.

The DoF-delay trade-off of the proposed schemes either by limiting the number or the order of the transmitted symbols. The first method builds upon the formulation of the parameters of each scheme (number of transmitted symbols and duration of the phases) as the solution of a DoF constrained maximization problem, and as a function of the number of users and the antenna ratio. In this regard, the analysis shows that although the BSR scheme and its extensions attain the best DoF values, this is at the cost of long transmission delays, which increases the complexity both at the transmitter and the receiver.

Finally, the latter part of this work has concluded that the time-varying channels assumption, which is common along all the literature on delayed CSIT, is indeed not necessary. This implies that delayed CSIT strategies can be used even if the channel remains constant, which could be the case since the transmitter does not actually know the current channel coefficients. Nonetheless, we have claimed that the particular SISO case requires asymmetric complex signaling concepts when channels are constant. A rigorous proof has been omitted due to similarity with references.

Many possible lines of future work remain open for this channel. On the one hand, a MIMO generalization of the BSR scheme for $K$ users may lead to tighter DoF results, although may be impractical. On the other hand, there is a lack of tight outer bounds. This would be interesting to deepen on which is the actual gap between the proposed inner bounds and the optimal DoF. Finally, the formulation presented in this paper seems to be a good starting point for deriving precoding strategies for the assymmetric MIMO IC, i.e. when not all transmitters and receivers have the same number of antennas. In a similar way, it would be interesting to study not only the DoF per user or sum DoF, but also the DoF region for this channel.

REFERENCES

[1] M. Torrellas, A. Agustin, and J. Vidal, “On the Degrees of freedom of the $K$-user MISO Interference Channel with imperfect delayed CSIT,” IEEE ICASSP, pp. 1155–1159, May 2014.

[2] M. Torrellas, A. Agustin, and J. Vidal, “Retrospective Interference Alignment for the 3-user MIMO Interference Channel with delayed CSIT,” IEEE ICASSP, May 2014.

[3] M.A. Maddah-Ali, A.S. Motahari, and A.K. Khandani, “Communication Over MIMO X Channels: Interference Alignment, Decomposition, and Performance Analysis,” IEEE Trans. Inf. Theory, vol. 54, pp. 3457–3470, Aug. 2008.

[4] V.R. Cadambe and S.A. Jafar, “Interference Alignment and Degrees of Freedom of the $K$-user Interference Channel,” IEEE Trans. Inf. Theory, vol. 54, pp. 3425–3441, Aug. 2008.

[5] Y. Birk and T. Kol, “Informed-source coding-on-demand (ISCOD) over broadcast channels,” in IEEE INFOCOM, Mar. 1998.

[6] V.R. Cadambe, S.A. Jafar, and C. Wang, “Interference Alignment With Asymmetric Complex Signaling - Settling the Host-Madsen-Nosratinia Conjecture,” IEEE Trans. Inf. Theory, vol. 56, pp. 4552–4565, Sept. 2010.

[7] Q.H. Spencer, A.L. Swindlehurst, and M. Haardt, “Zero-forcing methods for downlink spatial multiplexing in multiuser MIMO channels,” IEEE Trans. Signal Process., vol. 52, no. 2, pp. 461–471, Feb. 2004.

[8] S.A. Jafar, “Interference Alignment: A New Look at Signal Dimensions in a Communication Network,” Found. and Trends in Commun. and Inf. Theory, Jul. 2011.

[9] C. Wang, T. Gou, and S.A. Jafar, “Subspace Alignment Chains and the Degrees of Freedom of the Three-User MIMO Interference Channel,” IEEE Trans. Inf. Theory, vol. 60, pp. 2432–2479, May 2014.

[10] M. Torrellas, A. Agustin, J. Vidal, and O. Munoz-Medina, “The DoF of the 3-User ($p, p+1$) MIMO Interference Channel,” IEEE Trans. on Commun., vol. 62, pp. 3842–3853, Nov. 2014.

[11] A. Ghasemi, A.S. Motahari, and A.K. Khandani, “Interference alignment for the $K$-user MIMO interference channel,” in IEEE ISIT, Jun. 2010.

[12] C. Wang, H. Sun, and S.A. Jafar, “Genie Chains: Exploring Outer Bounds on the Degrees of Freedom of MIMO Interference Networks,” ArXiv e-prints, vol. arXiv:1404.2258v1 [cs.IT], Apr. 2014.

[13] B. Nazer, M. Gastpar, S.A. Jafar, and S. Vishwanath, “Ergodic Interference Alignment,” IEEE Trans. Inf. Theory, vol. 58, pp. 6355–6371, Oct. 2012.

[14] N. Jindal, “MIMO Broadcast Channels With Finite-Rate Feedback,” IEEE Trans. Inf. Theory, vol. 52, pp. 5045–5060, Nov. 2006.

[15] M.A. Maddah-Ali and D. Tse, “Completely Stale Transmitter Channel State Information is Still Very Useful,” IEEE Trans. Inf. Theory, vol. 58, pp. 4418–4431, Jul. 2012.

[16] K. Gomadam, V.R. Cadambe, and S.A. Jafar, “Approaching the Capacity of Wireless Networks through Distributed Interference Alignment,” in IEEE GLOBECOM, Nov. 2008.

[17] C.S. Vaze and M.K. Varanasi, “The Degrees of Freedom Region and Interference Alignment for the MIMO Interference Channel With Delayed CSIT,” IEEE Trans. Inf. Theory, vol. 58, pp. 4396–4417, Jul. 2012.
[18] A. Ghasemi, *Interference Management in MIMO Wireless Networks*, Ph.D. thesis, Dept. Elect. and Comp. Eng., Waterloo Univ., Ontario, Canada, 2013.
[19] H. Maleki, S.A. Jafar, and S. Shamai, “Retrospective Interference Alignment Over Interference Networks,” *J. Sel. Topics Signal Process.*, Jun. 2012.
[20] L. Maggi and L. Cottatellucci, “Retrospective interference alignment for interference channels with delayed feedback,” in *IEEE WCNC*, Apr. 2012, pp. 453–458.
[21] M. Abdoli, A. Ghasemi, and A. Khandani, “On the Degrees of Freedom of K-User SISO Interference and X Channels with Delayed CSIT,” *IEEE Trans. Inf. Theory*, vol. 59, pp. 6542–6561, Jun. 2013.
[22] Chenxi Hao and Bruno Clerckx, “Degrees-of-Freedom of the K-User MISO Interference Channel with Delayed Local CSIT,” *ArXiv e-prints*, vol. 1502.07123 [cs.IT], Feb. 2015.
[23] M.G. Kang and W. Choi, “Ergodic Interference Alignment With Delayed Feedback,” *IEEE Signal Process. Lett.*, May 2013.
[24] Gou T., Wang C., and S.A. Jafar, “Aiming Perfectly in the Dark-Blind Interference Alignment Through Staggered Antenna Switching,” *IEEE Trans. Signal Process.*, Jun. 2011.
[25] S.A. Jafar, “Exploiting Channel Correlations - Simple Interference Alignment Schemes with no CSIT,” *IEEE Globecom*, pp. 1–5, Dec. 2010.
[26] T.M. Cover and J.A. Thomas, *Elements of Information Theory*, Wiley, 2006.
[27] G. Dantzig and B. Eaves, “Fourier-Motzkin elimination and its dual,” *J. of Combinatorial Theory*, vol. 14, pp. 288 – 297, May 1973.