Generalized Similar Frenet Curves

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Abstract. The paper is devoted to differential geometric invariants determining a Frenet curve in $E^n$ up to a direct similarity. These invariants can be presented by the Euclidean curvatures in terms of an arc lengths of the spherical indicatrices. Then, these invariants expressed by focal curvatures of the curve. And then, we give the relationship between curvatures of evolute curve and shape curvatures. Moreover, these invariants is given the geometric interpretation.

1. Introduction

Curves are important for many areas of science. In Physics, the particle orbits as it moves is determined by using the curves. Especially, the curves have an important place in fractal science. In this field, fractal curves variation based on changing the similarities mapping on the same segments. In the human body, fingerprint is the remarkable example of the fractal curve. Fingerprint is occurs the similar curves. In nature, fractal curve which is called Koch curve is seen in the structure of snowflake. Koch curve is constructed by using the base curve and its similar curves.

Figure 1. The construction of Koch curve.

Euclidean Geometry can be described as a study of the properties of geometric figures. Only the properties which do not change under isometries deserve to be called geometric properties. A similarity of the Euclidean space $E^n$ is an automorphism of $E^n$ for which the ratio: distance between two arbitrary points to distance between the transformed points is a positive constant. This transformation preserves angles. In this study, we investigate preserve which properties of the curves under similarity transformation. The arc length parameter of the curve is not protected up to similarity transformation but arc length parameter of indicatrix curves

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are preserved. So, the curve is re-parameterized in terms of arc length parameters of indicatrix curves.

Encheva and Georgiev used spherical tangent indicatrix of the curve and its arc length parameter. As general of this study, we use all spherical indicatrix curves and their arc length parameters that is, we use its spherical images \( V_i \) which is the \( i-th \) Frenet vector field in \( E^n \). After, we calculated some differential-geometric invariants of curves up to direct similarities. The invariants are introduced shape curvatures. Then, we determine a curve which given the shape curvatures in the odd-dimensional and even-dimensional Euclidean space and this is illustrated with an example in \( E^3 \). Finally, a geometric interpretation of shape curvatures are given in \( E^3 \).

2. Preliminaries

In this section, we review some basic concepts on classical differential geometry of space curves in Euclidean \( n \)-space. For any two vectors \( x = (x_1, x_2, ..., x_n) \) and \( y = (y_1, y_2, ..., y_n) \in E^n \), \( x, y \) as the standard inner product. Let \( \alpha : I \subset \mathbb{R} \rightarrow E^n \) be a curve with \( \dot{\alpha}(t) \neq 0 \), where \( \dot{\alpha}(t) = \frac{d\alpha}{dt} \). We also denote the norm of \( x \) by \( \|x\| \).

The arc length parameters of curve \( \alpha \) is determined such that \( \|\alpha'(s)\| = 1 \), where \( \alpha'(s) = \frac{d\alpha}{ds} \). Let \( V_1, V_2, ..., V_n \) be a Frenet moving \( n \)-frame of the curve \( \alpha \). Then the following Frenet-Serret formula holds

\[
\begin{align*}
V'_1(s) &= \kappa_1(s)V_2(s) \\
V'_2(s) &= -\kappa_1(s)V_1(s) + \kappa_2(s)V_3(s) \\
V'_n(s) &= -\kappa_{n-1}(s)V_{n-1}(s)
\end{align*}
\]

where \( \kappa_1, \kappa_2, ..., \kappa_{n-1} \) are the curvatures of the curve \( \alpha \) at \( s \).

We will study the differential-geometric invariants of a curve in \( E^n \) with respect to the group \( Sim^+(\mathbb{R}^n) \) of all orientation-preserving similarities of \( \mathbb{R}^n \). Any such similarity \( F : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is called a direct similarity and can be expressed in the form

\[ F(x) = \lambda Ax + b \]

where \( x \in E^n \) is an arbitrary point, \( A \) is an orthogonal \( n \times n \) matrix, \( b \) is a translation vector and \( \lambda > 0 \) is a real constant.

We denote the image of the curve \( \alpha \) under the direct similarity \( F \) by the curve \( \bar{\alpha} \), i.e., \( \bar{\alpha} = F \circ \alpha \). Then, the curve \( \bar{\alpha} \) can be expressed as

\[ F \circ \alpha(t) = F(\alpha(t)) = \lambda A\alpha(t) + b. \]

The arc length functions of \( \alpha \) and \( \bar{\alpha} \) are

\[
\begin{align*}
\bar{s}(t) &= \int_{t_0}^{t} \left\| \frac{d\alpha(u)}{du} \right\| du \\
\bar{s}(t) &= \int_{t_0}^{t} \left\| \frac{d\bar{\alpha}(u)}{du} \right\| du = \lambda s(t).
\end{align*}
\]

Let \( \{\bar{V}_1, \bar{V}_2, ..., \bar{V}_n, \bar{\kappa}_1, \bar{\kappa}_2, ..., \bar{\kappa}_{n-1} \} \) be a Frenet apparatus of the curve \( \bar{\alpha} \). Since \( \frac{ds}{d\bar{s}} = \frac{1}{\lambda} (=\text{const.}) \) the curvatures of the curve \( \bar{\alpha} \) are given by

\[ \bar{\kappa}_i = \frac{1}{\lambda} \kappa_i(s), \quad i = 1, 2, ..., n - 1. \]

We obtain \( \kappa_i ds = \bar{\kappa}_i d\bar{s} \).
3. Expressed with respect to arc length parameter \( \sigma \) of the \( V_i \)-indicatrix curve of the curve \( \alpha \)

In this section, we give some characterizations of the curve \( \alpha \) by using the arc length parameters of its \( V_i \) indicatrix curve.

Let \( \gamma(\sigma_i) = V_i(s) \) be the spherical \( V_i \) indicatrix of the curve \( \alpha \) and be \( \sigma_i \) an arc length parameter of the curve \( \gamma \). Then the curve \( \alpha \) admits a reparametrization by \( \sigma_i \)

\[
\alpha = \alpha(\sigma_i) : I \subset \mathbb{R} \to \mathbb{E}^n.
\]

It is clear that

\[
d\sigma_i = \sqrt{(\kappa_{i-1}(s))^2 + (\kappa_i(s))^2} \, ds, \quad d\sigma_i = \frac{1}{\sqrt{(\kappa_{i-1})^2 + (\kappa_i)^2}} \frac{d}{ds}.
\]

Hence, \( d\sigma_i = \sqrt{(\kappa_{i-1})^2 + (\kappa_i)^2} \, ds \) is invariant under the group of the direct similarities of \( \mathbb{E}^n \).

Let \( V_1, V_2, \ldots, V_n \) be a Frenet frame field along the curve \( \alpha \) parameterized by the arc length parameter \( \sigma_i \) of its \( V_i \)-indicatrix curve. Then the structure equations of the curve \( \alpha \) are given by

\[
\frac{d\alpha}{d\sigma_i} = \frac{1}{\sqrt{(\kappa_{i-1})^2 + (\kappa_i)^2}} V_1(s),
\]

\[
\frac{d}{d\sigma_i}(V_1, V_2, \ldots, V_{n-1}, V_n)^T = K(V_1, V_2, \ldots, V_{n-1}, V_n)^T,
\]

where,

\[
K = \begin{pmatrix}
0 & \frac{\kappa_i}{\sqrt{(\kappa_{i-1})^2 + (\kappa_i)^2}} & \cdots & 0 & 0 \\
\frac{\kappa_i}{\sqrt{(\kappa_{i-1})^2 + (\kappa_i)^2}} & 0 & \cdots & 0 & 0 \\
0 & -\frac{\kappa_2}{\sqrt{(\kappa_{i-1})^2 + (\kappa_i)^2}} & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & -\frac{\kappa_{n-1}}{\sqrt{(\kappa_{i-1})^2 + (\kappa_i)^2}} \\
0 & 0 & \cdots & -\frac{\kappa_{n-1}}{\sqrt{(\kappa_{i-1})^2 + (\kappa_i)^2}} & 0
\end{pmatrix}
\]

\[
\left\{ \frac{1}{\sqrt{(\kappa_{i-1})^2 + (\kappa_i)^2}} V_1(s), \frac{1}{\sqrt{(\kappa_{i-1})^2 + (\kappa_i)^2}} V_2(s), \ldots, \frac{1}{\sqrt{(\kappa_{i-1})^2 + (\kappa_i)^2}} V_n(s) \right\}
\]

are orthogonal \( n \)-frame of the curve \( \alpha(\sigma_i) \).

We take \( \tilde{\kappa} = \frac{d}{ds} \left( \frac{1}{\sqrt{(\kappa_{i-1})^2 + (\kappa_i)^2}} \right) \) and \( \tilde{\kappa}_j = \frac{\kappa_j}{\sqrt{(\kappa_{i-1})^2 + (\kappa_i)^2}} \), \( j = 1, 2, \ldots, n-1 \).
Then we obtain this equation
\[ \frac{d}{d\sigma_i} \left( \frac{1}{\sqrt{(\kappa_{i-1})^2 + (\kappa_i)^2}} V_1, ..., \frac{1}{\sqrt{(\kappa_{i-1})^2 + (\kappa_i)^2}} V_n \right)^T = \tilde{K} \left( \frac{1}{\sqrt{(\kappa_{i-1})^2 + (\kappa_i)^2}} V_1, ..., \frac{1}{\sqrt{(\kappa_{i-1})^2 + (\kappa_i)^2}} V_n \right)^T \]

Remark 1. For \( i = 1 \) (that is, \( \sigma_i = \sigma \)) equation (3.3) coincidence with equation (2.6) in [8].

Definition 1. Let \( \alpha : I \subset \mathbb{R} \to \mathbb{R}^n \) be a Frenet space curve parameterized by an arc length parameter \( \sigma_i \) of its \( V_i \)-indicatrix curve. The functions
\[ \tilde{\kappa}(\sigma_i) = -d \sqrt{(\kappa_{i-1})^2 + (\kappa_i)^2} \quad \text{and} \quad \tilde{\kappa}_j(\sigma_i) = \frac{\kappa_j}{\sqrt{(\kappa_{i-1})^2 + (\kappa_i)^2}}, \quad j = 1, 2, ..., n-1 \]
are called shape curvatures of the curve \( \alpha \).

Proposition 1. Let \( \alpha(\sigma_i) : I \subset \mathbb{R} \to \mathbb{R}^n \) Frenet curve the orthogonal frame is
\[ \left\{ \frac{1}{\sqrt{(\kappa_{i-1})^2 + (\kappa_i)^2}} V_1, ..., \frac{1}{\sqrt{(\kappa_{i-1})^2 + (\kappa_i)^2}} V_n \right\}. \]
The functions
\[ \tilde{\kappa}(\sigma_i) = -d \sqrt{(\kappa_{i-1})^2 + (\kappa_i)^2} \quad \text{and} \quad \tilde{\kappa}_j(\sigma_i) = \frac{\kappa_j}{\sqrt{(\kappa_{i-1})^2 + (\kappa_i)^2}}, \quad j = 1, 2, ..., n-1 \]
are differential geometric invariants determining the curve \( \alpha \) up to a direct similarity.

Proof. Let \( \{ V_1, \tilde{V}_2, ..., \tilde{V}_n, \tilde{\kappa}_1, \tilde{\kappa}_2, ..., \tilde{\kappa}_{n-1} \} \) be a Frenet apparatus of the curve \( \tilde{\alpha} = F \circ \alpha \). The shape curvatures of the curve \( \tilde{\alpha} \) are given
\[ \tilde{\kappa}(\sigma_i) = -d \sqrt{(\tilde{\kappa}_{i-1})^2 + (\tilde{\kappa}_i)^2} \sqrt{(\kappa_{i-1})^2 + (\kappa_i)^2} \]
By using the \( \tilde{\kappa}_i = \frac{1}{\lambda}(\kappa_i(s)) \), \( i = 1, 2, ..., n-1 \) and \( \frac{ds}{d\tilde{\sigma}_i} = \frac{1}{\lambda} (= \text{const.}) \) we have
\[ \tilde{\kappa}(\sigma_i) = \kappa(\sigma_i) \]
and are obtained analogously

\[
\tilde{\kappa}_j(\sigma_i) = \frac{\tilde{\kappa}_j}{\sqrt{(\tilde{\kappa}_i-1)^2 + (\tilde{\kappa}_i)^2}}
\]

\[
\tilde{\kappa}_j(\sigma_i) = \tilde{\kappa}_j(\sigma_i)
\]

The proof is completed. \(\square\)

**Remark 2.** For \(i = 1\) (that is, \(\sigma_i = \sigma\)) \(\tilde{\kappa}\) and \(\tilde{\kappa}_j\) \((j = 2, \ldots, n-1)\) coincidence with \(\tilde{\kappa}_1\) and \(\tilde{\kappa}_i\) \((i = 2, 3, \ldots, n-1)\) respectively in [8].

4. The relation between the curve \(\alpha\) and its evolute curve

**Definition 2.** Let \(\beta : I \subset \mathbb{R} \to \mathbb{E}^3\) be a unit speed Frenet with Serret-Frenet apparatus \(\{\kappa_1, \kappa_2, V_1, V_2, V_3\}\) and be an evolute curve of \(\alpha\). Then the following equality

\[
\beta(s) = \alpha(s) + m_1(s)V_2(s) + m_2(s)V_3(s)
\]

where \(m_1(s) = \frac{1}{\kappa_1(s)}\), \(m_2(s) = \frac{1}{\kappa_1(s)} \cot \left( \int \kappa_2(s)ds \right)\).

In [12], R. Uribe-Vargas found formulas which express the Euclidean curvatures in terms of the focal curvatures. Similarly, we may represent all differential-geometric invariants \(\tilde{\kappa}\) by the \(m_i\) \((i = 1, 2)\) curvatures and their derivatives. For \(i = 1\) we obtain that \(\tilde{\kappa}_1 = \frac{d}{ds}(\frac{1}{\kappa_1(s)})\) and \(\tilde{\kappa}_2 = \frac{\kappa_2}{\kappa_1}\).

**Proposition 2.** Let \(\alpha : I \to \mathbb{R}^3\) be a unit speed Frenet curve with constant invariants \(\tilde{\kappa}_1 \neq 0\) and \(\tilde{\kappa}_2 \neq 0\). Then, the \(m_i\) \((i = 1, 2)\) curvatures of \(\alpha\) are

\[
\tilde{\kappa}_1 = m'_1 \quad \text{and} \quad \tilde{\kappa}_2 = \frac{m_1(m'_1 m_2 - m_1 m'_2)}{m_1^2 + m_2^2}.
\]

**Proof.** Since \(\tilde{\kappa}_1 = \frac{1}{\kappa_1(s)}\) and \(\tilde{\kappa}_2 = \frac{\kappa_2}{\kappa_1}\) then we get

\[
m'_1(s) = \left( \frac{1}{\kappa_1(s)} \right)' = \tilde{\kappa}_1.
\]

\[
m'_2(s) = \left( \frac{1}{\kappa_1(s)} \cot \left( \int \kappa_2(s)ds \right) \right)'
\]

\[
= m_1 \frac{m_2}{m_1^2} - \frac{\kappa_2}{\kappa_1} \frac{1}{\sin^2 \left( \int \kappa_2(s)ds \right)}
\]

\[
\tilde{\kappa}_2 = \frac{m_1(m'_1 m_2 - m_1 m'_2)}{m_1^2 + m_2^2}.
\]

\(\square\)
5. The relation between the curve $\alpha$ and focal curve

Let $\alpha : I \subset \mathbb{R} \to \mathbb{R}^n$ be a unit speed Frenet curve. Suppose that all Euclidean curvatures of the curve $\alpha$ are nonzero for any $s \in I$. The curve $C_\alpha : I \subset \mathbb{R} \to \mathbb{R}^n$ consisting of the centers of the osculating spheres of the curve $\alpha$ is called the focal curve of $\alpha$. Then the focal curve $C_\alpha$ has a representation:

$$C_\alpha = \alpha(s) + f_1(s)V_2(s) + \ldots + f_{n-2}(s)V_{n-1}(s) + f_{n-1}(s)V_n(s)$$

where the functions $f_i(s)$, $i = 1, \ldots, n - 1$ are called focal curvature of the curve $\alpha$. In [12], R. Uribe-Vargas found equation between Euclidean curvatures and focal curvatures. Then by using this equation we can express the shape curvatures of $\alpha$ in terms of focal curvatures.

**Proposition 3.** Let $\alpha : I \subset \mathbb{R} \to \mathbb{R}^n$ be a space curve with all Euclidean curvatures different from zero. Then,

\[
\begin{align*}
\kappa_i & = \frac{d}{ds} \left( \frac{f_{i-2}f_{i-1}f_i}{\sqrt{((f_1f'_1 + f_2f'_2 + \cdots + f_{i-1}f'_{i-2})f_i)^2 + ((f_1f'_1 + f_2f'_2 + \cdots + f_{i-1}f'_{i-1})f_{i-2})^2}} \right), \\
\kappa_j & = \frac{f_{i-2}f_{i-1}f_i}{f_jf_j} \left( \frac{f_1f'_1 + f_2f'_2 + \cdots + f_{i-1}f'_{i-1}}{\sqrt{((f_1f'_1 + f_2f'_2 + \cdots + f_{i-2}f'_{i-2})f_i)^2 + ((f_1f'_1 + f_2f'_2 + \cdots + f_{i-2}f'_{i-1})f_{i-2})^2}} \right), \\
& \quad j = 1, \ldots, n - 1.
\end{align*}
\]

**Proof.** According to the first two theorems in [12], there are relations between the Frenet curvatures and focal curvatures as follows:

$$\kappa_i = \frac{f_1f'_1 + f_2f'_2 + \cdots + f_{i-1}f'_{i-1}}{f_1f_i}, \quad i = 2, 3, \ldots, n - 1.$$

By using the Eq. (5.3) and Eq. (5.3) we obtain the Eq. (5.3).

**Remark 3.** For $i = 1$ (that is $\sigma_i = \sigma$), the representation of the shape curvatures is given with Eq. (5.2) the same as representation of the shape curvatures is given with Eq. (4.2) in [9].

6. Self-Similar Frenet Curves

The curve $\alpha : I \subset \mathbb{R} \to \mathbb{R}^n$ is called self-similar if and only if all its invariants $\tilde{\kappa}, \tilde{\kappa}_1, \ldots, \tilde{\kappa}_{n-1}$ are constant.

$$K = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-\tilde{\kappa}_1 & 0 & \tilde{\kappa}_2 & 0 & \cdots & 0 & 0 \\
0 & -\tilde{\kappa}_2 & 0 & \tilde{\kappa}_3 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -\tilde{\kappa}_{n-2} & 0 & \tilde{\kappa}_{n-1} \\
0 & 0 & 0 & \cdots & 0 & -\tilde{\kappa}_{n-1} & 0
\end{pmatrix}$$
According to [2], the normal form of the matrix $K$ is given following as

$$
\begin{pmatrix}
0 & \lambda_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-\lambda_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_2 & \cdots & 0 & 0 & 0 \\
0 & 0 & -\lambda_2 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & \lambda_m & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & -\lambda_m & 0 \\
\end{pmatrix}
$$

Then the symmetric matrix $K^2$ has $m = n/2$ negative eigenvalues with multiplicity two: $-\lambda_1^2, -\lambda_2^2, \ldots, -\lambda_m^2$.

### 6.1 Self-Similar curves in Even-Dimensional Euclidean Spaces

In this section, we deal with self-similar curves. We can write a curve if we have constant shape curvatures of the curve. Besides, the curve $\alpha$ can be written arc length parameter $\sigma_i$ of its $V_i$-indicatrix curve.

**Theorem 1.** Let $\alpha : I \to \mathbb{R}^{2m}$ be a self-similar curve. The curve $\alpha$ parameterized by the general arc-length parameter $\sigma_i$ of its $V_i$-indicatrix curve. The curve $\alpha$ can be written,

$$
\alpha(\sigma_i) = \left( \frac{a_1}{b_1} e^{\tilde{\kappa} \sigma_i} \sin \theta_1, -\frac{a_1}{b_1} e^{\tilde{\kappa} \sigma_i} \cos \theta_1, \cdots, \frac{a_m}{b_m} e^{\tilde{\kappa} \sigma_i} \sin \theta_m, -\frac{a_m}{b_m} e^{\tilde{\kappa} \sigma_i} \cos \theta_m \right)
$$

where $b_j = \sqrt{\kappa^2 + \lambda_j^2}$, $\theta_j = \sqrt{\lambda_j^2 \sigma_i + \arccos \left( \frac{\sqrt{\lambda_j^2}}{\sqrt{\kappa^2 + \lambda_j^2}} \right)}$ for $j = 1, 2, \cdots, m$ and the real different nonzero numbers $a_1, a_2, \cdots, a_m$ are solution of the system

$$
\langle V_j, V_j \rangle = 1, \quad j = 1, 2, \cdots, m.
$$

**Proof.** We express in the form of a column vector $\omega(\sigma_i) = (V_1(\sigma_i), V_2(\sigma_i), \cdots, V_{2m-1}(\sigma_i), V_{2m}(\sigma_i))$ of unit vector fields $\{V_1(\sigma_i), V_2(\sigma_i), \cdots, V_{2m-1}(\sigma_i), V_{2m}(\sigma_i)\}$. From the solution of the differential equation $\frac{d}{d\sigma_i} \omega = K \omega$, we can calculate the unit vector fields $\{V_1(\sigma_i), V_2(\sigma_i), \cdots, V_{2m-1}(\sigma_i), V_{2m}(\sigma_i)\}$. Also, the first unit vector has

$$
V_1(\sigma_i) = (a_1 \cos(\lambda_1 \sigma_i), a_1 \sin(\lambda_1 \sigma_i), \cdots, a_m \cos(\lambda_m \sigma_i), a_m \sin(\lambda_m \sigma_i)).
$$

Because of $\langle V_1, V_1 \rangle = 1$, we obtain that $\sum_{j=1}^{m} (a_j)^2 = 1$.

The parametric equation of the curve $\alpha$ is given by $X = (x_1(\sigma), x_2(\sigma), \cdots, x_{2m-1}(\sigma), x_{2m}(\sigma))$.

Then the Eq. (3.4) can write $\frac{d}{d\sigma_i} X = \frac{1}{\sqrt{\kappa_{i-1}^2 + \kappa_i^2}} V_1$. We can see easily that

$$
\sqrt{\kappa_{i-1}^2 + \kappa_i^2} = e^{-\tilde{\kappa} \sigma_i}. \quad \text{Hence, we have}
$$

$$
\frac{d}{d\sigma_i} x_{2j-1} = a_j e^{\tilde{\kappa} \sigma_i} \cos(\lambda_j \sigma_i) \quad \text{and} \quad \frac{d}{d\sigma_i} x_{2j} = a_j e^{\tilde{\kappa} \sigma_i} \sin(\lambda_j \sigma_i), \quad \text{for} \quad j = 1, 2, \cdots, m.
$$

Integrating the last equations, we obtain

$$
x_{2j-1} = \frac{a_j}{\kappa} e^{\tilde{\kappa} \sigma_i} \cos(\lambda_j \sigma_i) + \frac{\lambda_j}{\kappa} \lambda_j x_{2j}
$$
Using the Eq. (6.2) and Eq. (6.3), we get two equations:

\[ x_{2j} = \frac{a_j}{\kappa} e^{\kappa \sigma_i} \sin(\lambda_j \sigma_i) - \frac{\lambda_j}{\kappa} x_{2j-1}. \]

where

\[ \sin \theta_j = \frac{b_j}{\kappa} (\cos(\lambda_j \sigma_i) - \frac{\lambda_j}{b_j} \cos \theta_j), \]

\[ \cos \theta_j = -\frac{b_j}{\kappa} (\sin(\lambda_j \sigma_i) - \frac{\lambda_j}{b_j} \sin \theta_j). \]

From here, \( \theta_j \) can be found

\[ \theta_j = \lambda_j \sigma_i + \arccos \left( \frac{\lambda_j}{\sqrt{\kappa^2 + \lambda_j^2}} \right). \]

We can write from the equation \( \frac{d}{d\sigma_i} \varepsilon = K \varepsilon \)

\[ V_1(\sigma_i) = e^{-\kappa \sigma_i} \frac{d}{d\sigma_i} \alpha(\sigma_i) \]

\[ V_2(\sigma_i) = \frac{1}{\kappa_1} \frac{d}{d\sigma_i} V_1(\sigma_i) \]

\[ V_3(\sigma_i) = \frac{1}{\kappa_2} (-\kappa_1 V_1(\sigma_i) + \frac{d}{d\sigma_i} V_2(\sigma_i)) \]

\[ V_4(\sigma_i) = \frac{1}{\kappa_3} (\kappa_2 V_2(\sigma_i) + \frac{d}{d\sigma_i} V_3(\sigma_i)) \]

\[ \vdots \]

\[ V_m(\sigma_i) = \frac{1}{\kappa_{m-1}} (\kappa_{m-2} V_{m-2}(\sigma_i) + \frac{d}{d\sigma_i} V_{m-1}(\sigma_i)). \]

So, we show that by using an algebraic calculus

\[ \langle V_1, V_1 \rangle = 1 \Rightarrow \sum_{j=1}^{m} a_j^2 = 1 \]

\[ \langle V_2, V_2 \rangle = 1 \Rightarrow \sum_{j=1}^{m} a_j^2 \lambda_j^2 = \kappa_1^2 \]

\[ \langle V_3, V_3 \rangle = 1 \Rightarrow \sum_{j=1}^{m} a_j^2 (1 - \lambda_j^2) = \kappa_2^2 \]

\[ \langle V_4, V_4 \rangle = 1 \Rightarrow \sum_{j=1}^{m} a_j^2 \lambda_j^2 (\kappa_1^2 - \kappa_2^2) = \kappa_3^2 \kappa_4^2 \] and so on.

The proof is completed. \( \square \)

**Remark 4.** For \( i = 1 \) (that is \( \sigma_i = \sigma \)), the representation of the curve \( \alpha \) in even-dimensional Euclidean spaces is given with Eq. (6.11) the same as representation of the curve \( \alpha \) is given with Eq. (5.2) in [9].
6.2. Self-Similar Curves in Odd-Dimensional Euclidean Spaces

We can also express self-similar curves in $\mathbb{R}^{2m+1}$.

**Theorem 2.** Let $\alpha : I \rightarrow \mathbb{R}^{2m+1}$ be a self-similar curve. The curve $\alpha$ parameterized by the general arc-length parameter $\sigma$ of its $V_i-$indicatrix curve. The curve $\alpha$ can be written,

$$\alpha (\sigma) = \left( \frac{a_1}{b_1} e^{\tilde{\kappa}_1} \sin \theta_1, - \frac{a_1}{b_1} e^{\tilde{\kappa}_1} \cos \theta_1, \ldots, \frac{a_m}{b_m} e^{\tilde{\kappa}_m} \sin \theta_m, - \frac{a_m}{b_m} e^{\tilde{\kappa}_m} \cos \theta_m, a_{m+1} e^{\tilde{\kappa}_{m+1}} \right)$$

where $b_j = \sqrt{\tilde{\kappa}_j^2 + \lambda_j^2}$, $\theta_j = \arccos \left( \frac{\sqrt{\lambda_j^2}}{\sqrt{\tilde{\kappa}_j^2 + \lambda_j^2}} \right)$ for $j = 1, 2, \ldots, m$ and the real different nonzero numbers $a_1, a_2, \ldots, a_{m+1}$. To calculate $a_1, a_2, \ldots, a_{m+1}$ we use these $m + 1$ equations

$$V_1(\sigma_i) = e^{-\tilde{\kappa}_i} \frac{d}{d\sigma_i} \alpha(\sigma_i)$$

$$V_2(\sigma_i) = \frac{1}{\tilde{\kappa}_1} \frac{d}{d\sigma_i} V_1(\sigma_i)$$

$$V_3(\sigma_i) = \frac{1}{\tilde{\kappa}_2} (-\tilde{\kappa}_1 V_1(\sigma_i) + \frac{d}{d\sigma_i} V_2(\sigma_i))$$

$$V_4(\sigma_i) = \frac{1}{\tilde{\kappa}_3} (\tilde{\kappa}_2 V_2(\sigma_i) + \frac{d}{d\sigma_i} V_3(\sigma_i))$$

$$\vdots$$

$$V_{m+1}(\sigma_i) = \frac{1}{\tilde{\kappa}_m} (\tilde{\kappa}_{m-1} V_{m-1}(\sigma_i) + \frac{d}{d\sigma_i} V_m(\sigma_i)).$$

$$\langle V_j, V_j \rangle = 1, \ j = 1, 2, \ldots, m + 1.$$

From here

$$\langle V_1, V_1 \rangle = 1 \Rightarrow \sum_{i=1}^{m} a_i^2 + \tilde{\kappa}_i^2 a_{m+1}^2 = 1$$

$$\langle V_2, V_2 \rangle = 1 \Rightarrow \sum_{i=1}^{m+1} a_i^2 \lambda_i^2 = \tilde{\kappa}_1^2$$

$$\langle V_3, V_3 \rangle = 1 \Rightarrow \sum_{i=1}^{m} a_i^2 (1 - \frac{\lambda_i^2}{\tilde{\kappa}_1^2})^2 + \tilde{\kappa}_2^2 a_{m+1}^2 = \tilde{\kappa}_2^2$$

$$\langle V_i, V_i \rangle = 1, i = 4, \ldots, m + 1,$$

**Proof.** The proof is the same as the proof of Theorem 2. □

**Remark 5.** For $i = 1$ (that is $\sigma_i = \sigma$), the representation of the curve $\alpha$ in odd-dimensional Euclidean spaces is given with Eq. (6.1) the same as representation of the curve $\alpha$ is given with Eq. (5.2) in [9].

**Example 1.** Let be shape curvature functions $\tilde{\kappa}_1 = \frac{3}{\sqrt{13}}$ and $\tilde{\kappa}_2 = \frac{2}{\sqrt{13}}$ of the curve $\alpha$ in $\mathbb{R}^3$. We can calculate the curve $\alpha$ (see Figure 2) corresponding to them

$$\alpha (\sigma_2) = \left( \frac{a_1}{b_1} e^{\tilde{\kappa}_2} \sin \theta_1, - \frac{a_1}{b_1} e^{\tilde{\kappa}_2} \cos \theta_1, a_2 e^{\tilde{\kappa}_2} \right)$$
where $\sigma_2$ is arc length parameter of its normal indicatrix curve. Then we use equations in odd dimensional space, an algebraic calculus shows that

$$\lambda_1^2 = \frac{5}{13}, \quad a_1 = \frac{3}{\sqrt{5}}, \quad a_2 = \frac{2}{\sqrt{5}}, \quad b_1 = \frac{\sqrt{36 + 5x^4}}{\sqrt{13}s^2}, \quad \theta_1 = \sqrt{\frac{5}{13} + \arccos \left( \frac{\sqrt{\frac{1}{13}}}{\sqrt{36 + 5x^4}} \right)}.$$ 

Figure 2. The curve $\alpha$.

7. Geometric Interpretation of Shape Curvatures in 3-Euclidean Spaces

In this section, we mentioned that geodesic curvatures of indicatrix curves are related to shape curvatures of the curve.

**Proposition 4.** Geodesic curvatures of indicatrix curves are invariant under the group $Sim^+(\mathbb{R}^3)$.

**Proof.** Let $\alpha : I \rightarrow \mathbb{R}^3$ be a Frenet curve with $\{V_1, V_2, V_3, \kappa_1, \kappa_2\}$ Frenet apparatus and $\vec{\kappa}_1, \vec{\kappa}_2$ be shape curvatures. Its indicatrix curve show that $\gamma : I \rightarrow S^2$ is a spherical curve with arc length parameter $\sigma_i$ ($i = 1, 2, 3$) of $\gamma$. Let us denote $t(\sigma_i) = \frac{d}{d\sigma_i} \gamma(\sigma_i)$ and we call $t(\sigma_i)$ a unit tangent vector of $\gamma$. We now set a vector $\rho(\sigma_i) = \gamma(\sigma_i) \wedge t(\sigma_i)$ along the curve $\gamma$. This frame is called the Sabban frame of $\gamma$ on $S^2$. Then we have the following spherical Frenet formulae of $\gamma$

$$\frac{d}{d\sigma_i} \begin{bmatrix} \gamma \\ t \\ \rho \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & \kappa_g \\ 0 & -\kappa_g & 0 \end{bmatrix} \begin{bmatrix} \gamma \\ t \\ \rho \end{bmatrix}$$

where $\kappa_g = \det(\gamma, t, \frac{dt}{d\sigma_i})$ is the geodesic curvature of $\gamma$ at $\gamma(\sigma_i)$.

Let $\bar{\alpha}$ be similar curve of $\alpha$ under the group $Sim^+(\mathbb{R}^3)$ and $\{\bar{V}_1, \bar{V}_2, \bar{V}_3, \bar{\kappa}_1, \bar{\kappa}_2, \bar{\kappa}_1, \bar{\kappa}_2\}$ be Frenet apparatus of the curve $\bar{\alpha}$. Its indicatrix curve $\bar{\gamma} : I \rightarrow S^2$ is a spherical curve with arc length parameter $\bar{\sigma}_i$ ($i = 1, 2, 3$) of $\bar{\gamma}$. The orthogonal frame $\{\bar{\gamma}(\bar{\sigma}_i), \bar{t}(\bar{\sigma}_i), \bar{\rho}(\bar{\sigma}_i)\}$ along $\bar{\gamma}$ is called the Sabban frame of $\bar{\gamma}$.

If the curve $\gamma$ is a tangent indicatrix curve with arc length parameter $\sigma_1$ then the following Frenet-Serret formulas hold

$$t(\sigma_1) = \frac{d}{d\sigma_1} \gamma(\sigma_1)$$
\[ \frac{d}{d\sigma_1} t(\sigma_1) = -\gamma(\sigma_1) + \frac{\kappa_2}{\kappa_1} \rho(\sigma_1). \]

So, the geodesic curvature \( \kappa_g = \frac{\bar{\kappa}_2}{\bar{\kappa}_1} \). Similarly, we calculate \( \bar{\kappa}_g = \frac{\bar{\kappa}_2}{\bar{\kappa}_1} \). Using the Eq. (1), we get \( \bar{\kappa}_g = \kappa_g \).

If the curve \( \gamma \) is a normal indicatrix curve with arc length parameter \( \sigma_2 \) then the following Frenet-Serret formulas hold

\[ t(\sigma_2) = \frac{d}{d\sigma_2} \gamma(\sigma_2), \]
\[ \frac{d}{d\sigma_2} t(\sigma_2) = -\gamma(\sigma_2) + \bar{\kappa}_2 \frac{d}{d\sigma_2} \left( \frac{\bar{\kappa}_2}{\bar{\kappa}_1} \right) \rho(\sigma_2). \]
\[ \kappa_g(\sigma_2) = \bar{\kappa}_2 \frac{d}{d\sigma_2} \left( \frac{\bar{\kappa}_2}{\bar{\kappa}_1} \right). \]

Similarly, we get \( \bar{\kappa}_g = \bar{\kappa}_2 \frac{d}{d\sigma_2} \left( \frac{\bar{\kappa}_2}{\bar{\kappa}_1} \right). \) So, \( \bar{\kappa}_g \) is equal to \( \kappa_g \).

If the curve \( \gamma \) is a binormal indicatrix curve with arc length parameter \( \sigma_3 \) then the following Frenet-Serret formulas hold

\[ t(\sigma_3) = \frac{d}{d\sigma_3} \gamma(\sigma_3), \]
\[ \frac{d}{d\sigma_3} t(\sigma_3) = -\gamma(\sigma_3) + \frac{\bar{\kappa}_1}{\bar{\kappa}_2} \rho(\sigma_2) \]
\[ \kappa_g(\sigma_3) = \frac{\bar{\kappa}_1}{\bar{\kappa}_2}. \]

Similarly, we get \( \bar{\kappa}_g = \frac{\bar{\kappa}_1}{\bar{\kappa}_2}. \) So, \( \kappa_g \) is equal to \( \bar{\kappa}_g \).

The proof is completed.
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