SCHATTEN $p$-NORM INEQUALITIES RELATED TO AN EXTENDED OPERATOR PARALLELOGRAM LAW

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Abstract. Let $C_p$ be the Schatten $p$-class for $p > 0$. Generalizations of the parallelogram law for the Schatten 2-norms have been given in the following form: If $A = \{A_1, A_2, \ldots, A_n\}$ and $B = \{B_1, B_2, \ldots, B_n\}$ are two sets of operators in $C_2$, then

$$\sum_{i,j=1}^{n} \|A_i - A_j\|^2 + \sum_{i,j=1}^{n} \|B_i - B_j\|^2 = 2 \sum_{i,j=1}^{n} \|A_i - B_j\|^2 - 2 \left( \sum_{i=1}^{n} (A_i - B_i) \right)^2.$$  

In this paper, we give generalizations of this as pairs of inequalities for Schatten $p$-norms, which hold for certain values of $p$ and reduce to the equality above for $p = 2$. Moreover, we present some related inequalities for three sets of operators.

1. Introduction

Suppose that $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded linear operators on a separable complex Hilbert space $\mathcal{H}$ endowed with an inner product $\langle \cdot, \cdot \rangle$. Let $A \in \mathcal{B}(\mathcal{H})$ be a compact operator and let $\{s_j(A)\}$ denote the sequence of decreasingly ordered singular values of $A$, i.e. the eigenvalues of $|A| = (A^* A)^{1/2}$. The Schatten $p$-norm ($p$-quasi-norm, resp.) for $1 \leq p < \infty$ ($0 < p < 1$, resp.) is defined by

$$\|A\|_p = \left( \sum_{j=1}^{\infty} s_j^p(A) \right)^{1/p}.$$  

For $p > 0$, the Schatten $p$-class, denoted by $C_p$, is defined to be the two-sided ideal in $\mathcal{B}(\mathcal{H})$ of those compact operators $A$ for which $\|A\|_p$ is finite. Clearly

(1.1) $$\| |A|^2\|_{p/2} = \|A\|_p^2$$

for $p > 0$. In particular, $C_1$ and $C_2$ are the trace class and the Hilbert-Schmidt class, respectively. For $1 \leq p < \infty$, $C_p$ is a Banach space; in particular the triangle inequality holds. For $0 < p < 1$, the quasi-norm $\|\cdot\|_p$ does not satisfy the triangle inequality, but
In this paper, we give a generalization of the equality \((1.4)\) and \(\|A\|_p^p + \|B\|_p^p \leq \|A + B\|_p^p\). For more information on the theory of Schatten \(p\)-norms the reader is referred to [8, Chapter 2].

It follows from [7, Corollary 2.7] that for \(A_1, \ldots, A_n, B_1, \ldots, B_n \in \mathbb{B}(\mathcal{H})\)
\[
(1.2) \sum_{i,j=1}^{n} \|A_i - A_j\|^2 + \sum_{i,j=1}^{n} \|B_i - B_j\|^2 = 2 \sum_{i,j=1}^{n} \|A_i - B_j\|^2 - 2 \left\| \sum_{i=1}^{n} (A_i - B_i) \right\|^2,
\]
which is indeed a generalization of the classical parallelogram law:
\[
|z + w|^2 + |z - w|^2 = 2|z|^2 + 2|w|^2 \quad (z, w \in \mathbb{C}).
\]

There are several extensions of parallelogram law among them we could refer the interested reader to [2, 3, 6, 9, 10]. Generalizations of the parallelogram law for the Schatten \(p\)-norms have been given in the form of the celebrated Clarkson inequalities (see [4] and references therein). Since \(C_2\) is a Hilbert space under the inner product \(\langle A, B \rangle = \text{tr}(B^*A)\), it follows from an equality similar to \((1.2)\) stated for vectors of a Hilbert space (see [7, Corollary 2.7]) that if \(A_1, \ldots, A_n, B_1, \ldots, B_n \in C_2\), then
\[
(1.3) \sum_{i,j=1}^{n} \|A_i - A_j\|^2 + \sum_{i,j=1}^{n} \|B_i - B_j\|^2 = 2 \sum_{i,j=1}^{n} \|A_i - B_j\|^2 - 2 \left\| \sum_{i=1}^{n} (A_i - B_i) \right\|^2.
\]
In [7] a joint operator extension of the Bohr and parallelogram inequalities is presented. In particular, it follows from [7, Corollary 2.3] that if \(A_1, \ldots, A_n, B_1, \ldots, B_n \in \mathbb{B}(\mathcal{H})\), then
\[
(1.4) \sum_{1 \leq i < j \leq n} |A_i - A_j|^2 + \sum_{1 \leq i < j \leq n} |B_i - B_j|^2 = \sum_{i,j=1}^{n} |A_i - B_j|^2 - \left\| \sum_{i=1}^{n} (A_i - B_i) \right\|^2.
\]
In this paper, we give a generalization of the equality \((1.3)\) for the Schatten \(p\)-norms \((p > 0)\). First we present similar consideration for three sets of operators. In addition, we provide pairs of complementary inequalities that reduce to \((1.3)\) for the certain value \(p = 2\).

2. Schatten \(p\)-norm inequalities

To accomplish our results, we need the following lemma which can be deduced from [1, Lemma 4] and [8, p. 20]:

**Lemma A.** Let \(A_1, \ldots, A_n \in C_p\) for some \(p > 0\). If \(A_1, \ldots, A_n\) are positive, then
\[
(2.1) n^{p-1} \sum_{i=1}^{n} \|A_i\|^p_p \leq \left\| \sum_{i=1}^{n} A_i \right\|^p_p \leq \sum_{i=1}^{n} \|A_i\|^p_p
\]
for \(0 < p \leq 1\) and the reverse inequalities hold for \(1 \leq p < \infty\).
Let us define a constant $D_A$ for a set of operators $A = \{A_1, A_2, \ldots, A_n\}$ as follows:

$$D_A := \sum_{i=1}^{n} \delta(A_i)$$

where $\delta(A_i) = \begin{cases} 1 & (A_i \neq 0) \\ 0 & (A_i = 0) \end{cases}$.

If there exists $1 \leq i \leq n$ with $A_i = 0$, then Lemma A is refined as follows:

$$D_A^{p-1} \sum_{i=1}^{n} \|A_i\|_p^{p} \leq \left\| \sum_{i=1}^{n} A_i \right\|_p^{p} \leq \sum_{i=1}^{n} \|A_i\|_p^{p}$$

for $0 < p \leq 1$ and the reverse inequalities hold for $1 \leq p < \infty$.

We also put $A - B := \{A_i - B_j : 1 \leq i, j \leq n\}$ for sets of operators $A = \{A_1, A_2, \ldots, A_n\}$ and $B = \{B_1, B_2, \ldots, B_n\}$. Then we remark that $0 \leq D_{A-B} \leq n^2$.

Now we give our main results that involve three sets of operators.

Theorem 2.1. Let $A = \{A_1, A_2, \ldots, A_n\}$, $B = \{B_1, B_2, \ldots, B_n\}$, $C = \{C_1, C_2, \ldots, C_n\} \subset C_p$ for some $p > 0$. Then

$$\sum_{i,j=1}^{n} \|A_i - A_j\|_p^p + \sum_{i,j=1}^{n} \|B_i - B_j\|_p^p + \sum_{i,j=1}^{n} \|C_i - C_j\|_p^p$$

$$\geq \left( D_A^{n-2} \sum_{i,j=1}^{n} \|A_i - B_j\|_p^p + D_B^{n-2} \sum_{i,j=1}^{n} \|B_i - C_j\|_p^p + D_C^{n-2} \sum_{i,j=1}^{n} \|C_i - A_j\|_p^p \right)$$

$$- \left( \left\| \sum_{i=1}^{n} (A_i - B_i) \right\|_p^p + \left\| \sum_{i=1}^{n} (B_i - C_i) \right\|_p^p + \left\| \sum_{i=1}^{n} (C_i - A_i) \right\|_p^p \right)$$

for $0 < p \leq 2$ and the reverse inequality holds for $2 \leq p < \infty$.

Proof. We only prove the case when $0 < p \leq 2$. The other case can be proved by a similar argument.

We have

$$\sum_{i,j=1}^{n} \|A_i - A_j\|_p^p + \sum_{i,j=1}^{n} \|B_i - B_j\|_p^p + \sum_{i,j=1}^{n} \|C_i - C_j\|_p^p$$

$$+ \left( \left\| \sum_{i=1}^{n} (A_i - B_i) \right\|_p^p + \left\| \sum_{i=1}^{n} (B_i - C_i) \right\|_p^p + \left\| \sum_{i=1}^{n} (C_i - A_i) \right\|_p^p \right)$$

$$= 2 \left( \sum_{1 \leq i < j \leq n} \|A_i - A_j\|_p^p + \sum_{1 \leq i < j \leq n} \|B_i - B_j\|_p^p + \sum_{1 \leq i < j \leq n} \|C_i - C_j\|_p^p \right)$$
\[
\begin{align*}
\sum_{i,j=1}^{n} (A_i - B_j)^2 + \sum_{i,j=1}^{n} (B_i - C_j)^2 + \sum_{i,j=1}^{n} (C_i - A_j)^2 & \\
\geq D_{A-B}^{\frac{n-1}{2}} \sum_{i,j=1}^{n} ||A_i - B_j||_p^{p/2} + D_{B-C}^{\frac{n-1}{2}} \sum_{i,j=1}^{n} ||B_i - C_j||_p^{p/2} + D_{C-A}^{\frac{n-1}{2}} \sum_{i,j=1}^{n} ||C_i - A_j||_p^{p/2} \\
= D_{A-B}^{\frac{n-2}{p-2}} \sum_{i,j=1}^{n} ||A_i - B_j||_p^p + D_{B-C}^{\frac{n-2}{p-2}} \sum_{i,j=1}^{n} ||B_i - C_j||_p^p + D_{C-A}^{\frac{n-2}{p-2}} \sum_{i,j=1}^{n} ||C_i - A_j||_p^p.
\end{align*}
\]

So we have the desired inequality (2.3). \hfill \Box

The next result can be regarded as a generalization of (1.3).
Proposition 2.2. Let $A_1, \ldots, A_n, B_1, \ldots, B_n \in C_p$ for some $p > 0$. Then

$$\sum_{i,j=1}^{n} \|A_i - A_j\|_p^p + \sum_{i,j=1}^{n} \|B_i - B_j\|_p^p \geq 2n^{p-2} \sum_{i,j=1}^{n} \|A_i - B_j\|_p^p - 2 \left\| \sum_{i=1}^{n} (A_i - B_i) \right\|_p^p$$

for $0 < p \leq 2$ and the reverse inequality holds for $2 \leq p < \infty$.

Proof. We only prove the case where $0 < p \leq 2$. Putting $C_i = 0$ for $i = 1, 2, \ldots, n$, we have

$$D_{B-C}^{p-2} \sum_{i,j=1}^{n} \|B_i - C_j\|_p^p = nD_{B}^{p-2} \sum_{i=1}^{n} \|B_i\|_p^p, \quad D_{C-A}^{p-2} \sum_{i,j=1}^{n} \|C_i - A_j\|_p^p = nD_{A}^{p-2} \sum_{i=1}^{n} \|A_i\|_p^p.$$

It follows from Theorem 2.1 that

$$0 \leq \sum_{i,j=1}^{n} \|A_i - A_j\|_p^p + \sum_{i,j=1}^{n} \|B_i - B_j\|_p^p - D_{A-B}^{p-2} \sum_{i,j=1}^{n} \|A_i - B_j\|_p^p$$

$$- n \left( D_{B}^{p-2} \sum_{i=1}^{n} \|B_i\|_p^p + D_{A}^{p-2} \sum_{i=1}^{n} \|A_i\|_p^p \right)$$

$$+ \left( \left\| \sum_{i=1}^{n} (A_i - B_i) \right\|_p^p + \sum_{i=1}^{n} \|B_i\|_p^p + \sum_{i=1}^{n} \|A_i\|_p^p \right)$$

$$= \sum_{i,j=1}^{n} \|A_i - A_j\|_p^p + \sum_{i,j=1}^{n} \|B_i - B_j\|_p^p - 2D_{A-B}^{p-2} \sum_{i,j=1}^{n} \|A_i - B_j\|_p^p + 2 \left\| \sum_{i=1}^{n} (A_i - B_i) \right\|_p^p$$

$$+ \left( D_{A-B}^{p-2} \sum_{i,j=1}^{n} \|A_i - B_j\|_p^p - \left\| \sum_{i=1}^{n} (A_i - B_i) \right\|_p^p \right)$$

$$+ \left( \left\| \sum_{i=1}^{n} A_i \right\|_p^p - nD_{A}^{p-2} \sum_{i=1}^{n} \|A_i\|_p^p \right) + \left( \left\| \sum_{i=1}^{n} B_i \right\|_p^p - nD_{B}^{p-2} \sum_{i=1}^{n} \|B_i\|_p^p \right).$$

Here the inequality (2.2) implies

$$D_{A-B}^{p-2} \sum_{i,j=1}^{n} \|A_i - B_j\|_p^p - \left\| \sum_{i=1}^{n} (A_i - B_i) \right\|_p^p$$

$$= D_{A-B}^{p-2} \sum_{i,j=1}^{n} \|A_i - B_j\|_p^{p/2} - \sum_{i=1}^{n} \|A_i - B_i\|_p^{p/2} \leq 0$$
by relation (1.1). Due to $nD_{A}^{p-2}$ is no less than 1, we deduce from Lemma A that

$$\left\| \sum_{i=1}^{n} A_i \right\|_p^p - nD_{A}^{p-2} \sum_{i=1}^{n} \left\| A_i \right\|_p^p \leq \left\| \sum_{i=1}^{n} A_i \right\|_p^p - \sum_{i=1}^{n} \left\| A_i \right\|_p^p \leq 0.$$ 

Similarly, we have $\left\| \sum_{i=1}^{n} B_i \right\|_p^p \leq nD_{B}^{p-2} \sum_{i=1}^{n} \left\| B_i \right\|_p^p$. It therefore implies that

$$\sum_{i,j=1}^{n} \left\| A_i - A_j \right\|_p^p + \sum_{i,j=1}^{n} \left\| B_i - B_j \right\|_p^p \geq 2D_{A-B}^{p-2} \sum_{i,j=1}^{n} \left\| A_i - B_j \right\|_p^p - 2 \left\| \sum_{i=1}^{n} (A_i - B_i) \right\|_p^p \geq 0 \quad (\text{by } n^2 \geq D_{A-B}(\geq 0)).$$

Thus we obtain the desired inequality. \(\square\)

**Corollary 2.3.** Let $A_1, \ldots, A_n, B_1, \ldots, B_n \in \mathcal{C}_p$ for some $p > 0$ and $\sum_{i=1}^{n} A_i = \sum_{i=1}^{n} B_i$. Then

$$\sum_{i,j=1}^{n} \left\| A_i - A_j \right\|_p^p + \sum_{i,j=1}^{n} \left\| B_i - B_j \right\|_p^p \geq 2n^{p-2} \sum_{i,j=1}^{n} \left\| A_i - B_j \right\|_p^p$$

for $0 < p \leq 2$ and the reverse inequality holds for $2 \leq p < \infty$.

Utilizing Corollary 2.3 with $B_i = 0$ ($1 \leq i \leq n$), we obtain the following result which is a refinement of [5, Corollary 2.3].

**Corollary 2.4.** Let $A_1, \ldots, A_n \in \mathcal{C}_p$ for some $p > 0$ such that $\sum_{i=1}^{n} A_i = 0$. Then

$$\sum_{i,j=1}^{n} \left\| A_i - A_j \right\|_p^p \geq 2n^{p-1} \sum_{i=1}^{n} \left\| A_i \right\|_p^p$$

for $0 < p \leq 2$ and the reverse inequality holds for $2 \leq p < \infty$.

Next, we have the following reverse inequalities of Proposition 2.2:
Proposition 2.5. Let $A_1, \ldots, A_n, B_1, \ldots, B_n \in \mathcal{C}_p$ for some $p > 0$. Then

$$2 \left( n^2 - n + 1 \right)^{\frac{2-p}{2}} \sum_{i,j=1}^{n} \|A_i - B_j\|_p^p - 2 \left\| \sum_{i=1}^{n} (A_i - B_i) \right\|_p^p$$

$$\geq \sum_{i,j=1}^{n} \|A_i - A_j\|_p^p + \sum_{i,j=1}^{n} \|B_i - B_j\|_p^p$$

for $0 < p \leq 2$ and the reverse inequality holds for $2 \leq p < \infty$.

Proof. We only consider the case when $0 < p \leq 2$. We have

$$\sum_{1 \leq i < j \leq n} \|A_i - A_j\|_p^p + \sum_{1 \leq i < j \leq n} \|B_i - B_j\|_p^p + \left\| \sum_{i=1}^{n} (A_i - B_i) \right\|_p^p$$

$$= \sum_{1 \leq i < j \leq n} ||A_i - A_j||_{p/2}^{p/2} + \sum_{1 \leq i < j \leq n} ||B_i - B_j||_{p/2}^{p/2} + \left\| \sum_{i=1}^{n} (A_i - B_i) \right\|_{p/2}^{p/2}$$

(by relation (1.1))

$$\leq \left( 2 \cdot \frac{n^2 - n}{2} + 1 \right)^{1 - \frac{p}{2}} \left\| \sum_{1 \leq i < j \leq n} |A_i - A_j|^2 \right\|_{p/2}^{p/2} + \sum_{1 \leq i < j \leq n} |B_i - B_j|^2 + \left\| \sum_{i=1}^{n} (A_i - B_i) \right\|_{p/2}^{p/2}$$

(by the first inequality of (2.1))

$$= (n^2 - n + 1)^{\frac{2-p}{2}} \left\| \sum_{i,j=1}^{n} |A_i - B_j|^2 \right\|_{p/2}^{p/2}$$

(by (1.4))

$$\leq (n^2 - n + 1)^{\frac{2-p}{2}} \sum_{i,j=1}^{n} ||A_i - B_j||_{p/2}^{p/2}$$

(by the second inequality of (2.1))

$$= (n^2 - n + 1)^{\frac{2-p}{2}} \sum_{i,j=1}^{n} ||A_i - B_j||_{p}^{p}$$

So we have the desired inequality. \qed

Remark 2.6. (i) By an easily calculation, we have the inequality $n^{p-2} < (2n^2 - n + 1)^{\frac{2-p}{2}} < (n^{p-2})^{-1}$ for $0 < p \leq 2$.

(ii) The values of $2n^{p-2}$, $2(n^2 - n + 1)^{\frac{2-p}{2}}$ of Propositions 2.2 and 2.5 are 2, if $p = 2$. So these results ensured the equality (1.3).

Finally we would like to give a problem for further research.
Problem 2.7. What the form of the identity is for the general case of $k$ sets of operators?

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