LiDAR-based Control of Autonomous Rotorcraft for the Inspection of Pier-like Structures: Proofs

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Abstract—This is a complementary document to the paper presented in [1], where more detailed proofs are provided for some results. The main paper addresses the problem of trajectory tracking control of autonomous rotorcraft in operation scenarios where only relative position measurements obtained from LiDAR sensors are possible. The proposed approach defines an alternative kinematic model, directly based on LiDAR measurements, and uses a trajectory-dependent error space to express the dynamic model of the vehicle. An LPV representation with piecewise affine dependence on the parameters is adopted to describe the error dynamics over a set of predefined operating regions, and a continuous-time $\mathcal{H}_2$ control problem is solved using LMIs and implemented within the scope of gain-scheduling control theory. In this document, Section A presents the stability analysis of the attitude inner-loop presented in [1] Section II.B, whereas Section B presents a more detailed version of the stability and performance guarantees for the LPV system, extending the results presented in [1] Section IV.A.

APPENDIX A
INNER-LOOP DYNAMICS

This appendix presents a possible inner-loop stabilization method, within the framework of feedback linearization and Lyapunov stability methods, that results in a second-order linear model for the pitch and roll angular motion, as well as a first-order linear model for the yaw angular velocity.

Theorem 1 (Inner-loop Stability). Consider the control law given by

$$\dot{n}_{ext} = S(\omega) J_3 \omega + J_3 Q^{-1}(\lambda) (\dot{Q}(\lambda) \omega - K_2 (\lambda - e_3 e_3^T u_{IL}))$$

$$- \Pi_{e_3}^T \Pi_{e_3} (\lambda - u_{IL})$$ (1)

where $K_1 \in \mathbb{R}^2$ and $K_2 \in \mathbb{R}^3$ are positive definite diagonal matrices and the inner-loop input vector is denoted as $u_{IL} = [u_\phi \ u_\theta \ u_\psi]^T$, accounting for the desired roll angle, pitch angle, and yaw angular rate, respectively. Then, the resulting attitude dynamics is given by

$$\dot{\lambda} = -K_2 (\lambda - e_3 e_3^T u_{IL}) - \Pi_{e_3}^T K_1 \Pi_{e_3} (\lambda - u_{IL})$$ (2)

which, for constant inputs, guarantees that the equilibrium point $(\Pi_{e_3} \lambda, \lambda) = (\Pi_{e_3} u_{IL}, e_3 e_3^T u_{IL})$ is exponentially stable.

Proof. Recalling the rigid body dynamics, the angular motion equations can be written as

$$\begin{cases}
\dot{\omega} = J_3^{-1} [-S(\omega) J_3 \omega + n_{ext}] \\
\dot{\lambda} = Q(\lambda) \omega
\end{cases}$$ (3a,b)

Noting that the angular velocity can be defined as $\omega = Q^{-1}(\lambda) \dot{\lambda}$ and taking the time derivative of angular kinematics, the angular motion dynamics can be expressed as

$$\dot{\lambda} = \dot{Q}(\lambda) \omega + Q(\lambda) \dot{J}_3^{-1} [-S(\omega) J_3 \omega + n_{ext}]$$ (4)

Consider the desired roll angle, pitch angle, and yaw rate to be denoted as $u_\phi \in \mathbb{R}$, $u_\theta \in (-\pi/2, \pi/2)$, and $u_\psi \in \mathbb{R}$, respectively, indicating that they will be considered as inputs in the overall system. Let also the input vector of this inner-loop system be defined as $u_{IL} = [u_\phi \ u_\theta \ u_\psi]^T$.

Thus, a feedback linearizing control law can be designed as

$$n_{ext} = S(\omega) J_3 \omega + J_3 Q^{-1}(\lambda) (\dot{Q}(\lambda) \omega - K_2 (\lambda - e_3 e_3^T u_{IL}))$$

$$- \Pi_{e_3}^T K_1 \Pi_{e_3} (\lambda - u_{IL})$$ (5)

where $K_1 \in \mathbb{R}^{2 \times 2}$, $K_2 \in \mathbb{R}^{3 \times 3}$, $e_3 = [0 \ 0 \ 1]^T$, and as such $e_3 e_3^T e_{IL} = [0 \ 0 \ u_\psi]^T$, while $\Pi_{e_3} = [I_2 \ 0_{2 \times 1}]^T$ such that $\Pi_{e_3} u_{IL} = [u_\phi \ u_\theta]^T$. Replacing the control law (5) in the angular dynamics (4), it can be seen that the closed-loop dynamics results in

$$\dot{\lambda} = -K_2 (\lambda - e_3 e_3^T u_{IL}) - \Pi_{e_3}^T K_1 \Pi_{e_3} (\lambda - u_{IL})$$ (6)

Considering that $u_{IL} \in U_{IL} \subset \mathbb{R}^3$ and define a state vector as $x_{IL} = [\Pi_{e_3} \lambda \ \dot{\lambda}]^T \in \mathbb{R}^3$, the resulting system is a linear time-invariant (LTI) system of the form

$$x_{IL} = A_{IL} x_{IL} + B_{IL} u_{IL}$$

where the system matrices are given by

$$A_{IL} = \begin{bmatrix} 0_{2 \times 2} & \Pi_{e_3} \\ -\Pi_{e_3}^T K_1 & -K_2 \end{bmatrix}$$

$$B_{IL} = \begin{bmatrix} 0_{2 \times 3} \\ \Pi_{e_3}^T K_1 \Pi_{e_3} + K_2 e_3 e_3^T \end{bmatrix}$$

It can be seen that matrix $A_{IL}$ is Hurwitz for any positive definite matrices $K_1$ and $K_2$, and, therefore, the system is locally input-to-state stable, noting that the local part of the
result is a direct consequence of the domain of the Euler angles not being $\mathbb{R}^3$.

Considering the matrices $K_1 = \text{diag}(k_\phi, k_\theta)$ and $K_2 = \text{diag}(k_\phi', k_\theta')$, the system described in (6) can also be written as

$$
\begin{align}
\dot{\phi} &= -k_\phi \phi - k_\phi (\phi - u_\phi) & (7a) \\
\dot{\theta} &= -k_\theta \theta - k_\theta (\theta - u_\theta) & (7b) \\
\dot{\psi} &= -k_\psi (\psi - u_\psi) & (7c)
\end{align}
$$

It can be seen that, with constant inputs and the change of variables $\dot{\phi} = \phi - u_\phi$, $\dot{\theta} = \theta - u_\theta$, and $\dot{\psi} = \psi - u_\psi$, the resulting autonomous system is exponentially stable. Thus, it can be concluded that the state variables $\phi$, $\theta$, and $\psi$ converge exponentially to the constant inputs $u_\phi$, $u_\theta$, and $u_\psi$, respectively.

\[\square\]

APPENDIX B

CONTROLLER SYNTHESIS

In this section, an LMI approach is used to tackle the continuous-time state feedback $H_2$ synthesis problem for polytopic LPV systems. Consider a general LPV system of the form

$$
\begin{align}
\dot{x} &= A(\xi)x + B_w(\xi)w + B(\xi)u & (8a) \\
z &= C(\xi)x + D(\xi)w + E(\xi)u & (8b)
\end{align}
$$

where $x$ is the state, $u$ is the control input, $z$ denotes the error signal to be controlled, and $w$ denotes the exogenous input signal. The system is parameterized by $\xi$, which is a possibly time-varying parameter vector and belongs to the convex set $E^j = \text{co}(\xi_0^j)$. Here, the operator $\text{co}(\cdot)$ denotes the convex hull of the elements of the argument set, $E_0^j = \{\xi_1, \ldots, \xi_{n_j}\}$, where $\xi_1$ to $\xi_{n_j}$ are the vertices of a polytope. It is also noted that the controller synthesis presented in the following subsection will only be valid for a specific operating region, here represented by $E_0^j \subset E$.

Applying the static state feedback law given by $u = Kx$ to (3) results in the closed-loop system given by

$$
\begin{align}
T_{zw}(\xi) := \begin{cases}
\dot{x} = A(\xi)x + B_w(\xi)w & (9a) \\
z = C(\xi)x + D(\xi)w & (9b)
\end{cases}
$$

where $T_{zw}(\xi)$ denotes the resulting closed-loop operator from the disturbance input $w$ to the performance output $z$, and the system matrices are defined as $A_1(\xi) = A(\xi) + B(\xi)K$, $B_2(\xi) = B_w(\xi)$, $C_1(\xi) = C(\xi) + E(\xi)K$ and $D_1(\xi) = D(\xi)$. The closed-loop system can be characterized in terms of quadratic stability using the following definition, where $X \succ 0$ denotes that matrix $X$ is positive definite.

**Definition 1** (Quadratic stability, [2]). The system is said to be quadratically stable if there exists a matrix $X \succ 0$ such that $A_1^TX + XA_1 \prec 0$ is satisfied for all $\xi \in E^j$.

It can be seen that testing for stability or solving the synthesis problem without any further result, involves an infinite number of LMIs. Thus, several different structures for LPV systems have been proposed which reduce the problem to that of solving a finite number of LMIs.

In this section, an affine polytopic description is adopted, which can also be used to model a wide spectrum of systems and, as shown in the results presented in [2], is an adequate choice for the system at hand.

**Definition 2** (Affine polytopic LPV system). The system (5) is said to be a polytopic LPV system if the system matrix

$$
P(\xi) = \begin{bmatrix} A(\xi) & B_w(\xi) & B(\xi) \\ C(\xi) & D(\xi) & E(\xi) \end{bmatrix}
$$

verifies $P(\xi) \in \text{co}(P_1, \ldots, P_n)$ for all $\xi \in E^j$, where

$$
P_i = \begin{bmatrix} A_i & B_{w_i} & B_i \\ C_i & D_i & E_i \end{bmatrix}
$$

for all $i = 1, \ldots, n_j$. Moreover, if $E^j$ is a polytopic set, such as $E^j = \text{co}(\xi_0^j)$, $\xi_0^j = \{\xi_1, \ldots, \xi_{n_j}\}$, and $P(\xi)$ depends affinely on $\xi$, then $P_i = P(\xi_i)$ for all $i = 1, \ldots, n_j$, i.e., the vertices of the parameter set can be uniquely identified with the vertices of the system.

This polytopic structure used with the following lemma, enables the use of a powerful set of results.

**Proposition 2** ([3, Proposition 1.19]). Let $f : E^j \to \mathbb{R}$ be a convex function defined on the convex set $E^j = \text{co}(\xi_0^j)$. Then, for some $\gamma \in \mathbb{R}$, $f(\xi) \leq \gamma$ for all $\xi \in E^j$ if and only if $f(\xi) \leq \gamma$ for all $\xi \in E_0^j$.

Thus, the quadratic stability of an affine polytopic LPV system can be efficiently established if there exists a matrix $X \succ 0$ such that $A_1^TX + XA_1 \prec 0$ is satisfied for all $\xi \in E_0^j$.

The $H_2$ synthesis problem can be described as that of finding a control matrix $K$ that stabilizes the closed-loop system and minimizes the $H_2$-norm of $T_{zw}(\xi)$, denoted by $\|T_{zw}(\xi)\|_{H_2}$. It is assumed that matrix $D(\xi) = 0$ in order to guarantee that $\|T_{zw}(\xi)\|_{H_2}$ is finite for every internally stabilizing and strictly proper controller. The following theorem is used for controller design and relies on results available in [3] and [5], after being rewritten for the case of polytopic LPV systems. In the following, $\text{tr}(\cdot)$ denotes the trace of the argument matrix.

**Theorem 3** (Polytopic stability). If there are real matrices $X = X^T \succ 0$, $Y \succ 0$, and $W$ such that

$$
\begin{align}
A(\xi)X + XA^T(\xi) + B(\xi)W + W^TB^T(\xi)B_w(\xi) & \prec 0 & (11a) \\
Y & \succ 0 & (11b) \\
\text{tr}(Y) & \leq \gamma^2 & (11c)
\end{align}
$$

for all $\xi \in E_0^j$, where the static feedback controller is defined as $K = WX^{-1}$, then, the closed-loop system is quadratically stable and there exists an upper-bound $\gamma$ for the continuous-time $H_2$-norm of the closed-loop operator $T_{zw}(\xi)$ for all $\xi \in E^j$, i.e.,

$$
\|T_{zw}(\xi)\|_{H_2} \leq \gamma, \forall \xi \in E^j.
$$

**Proof.** Using Proposition 2 and assuming an affine polytopic LPV system, it can be seen that satisfying the LMI system for all $\xi \in E_0^j$ is equivalent to satisfying the same system for all
The proof that the LMI system (11) implies (12) can be obtained from the definition of $\mathcal{H}_2$-norm of $T_{zw}(\xi)$. Let the transfer function matrix of the closed-loop operator $T_{zw}(\xi)$ be denoted as $T_{iw}(s)$ for some parameter vector $\xi_i \in \mathcal{E}_0^j$, and defined as

$$
T_{iw}(s) = C_c(\xi_i) \left[ sI - A_c(\xi_i) \right]^{-1} B_c(\xi_i) + D_c(\xi_i) = [C(\xi_i) + E(\xi_i) K] \left[ sI - A(\xi_i) - B(\xi_i) K \right]^{-1} B_w(\xi_i) + D(\xi_i).
$$

Then, the $\mathcal{H}_2$-norm of $T_{iw}(s)$ is defined as

$$
\|T_{iw}\|_{\mathcal{H}_2}^2 = \frac{1}{2\pi} \text{tr} \left( \int_{-\infty}^{+\infty} T_{iw}^H(j\omega) T_{iw}(j\omega) \, d\omega \right)
$$

which using Parseval’s theorem can be rewritten as

$$
\|T_{iw}\|_{\mathcal{H}_2}^2 = \text{tr} \left( \int_{-\infty}^{+\infty} H_i^T(t) H_i(t) \, dt \right)
$$

where $H_i(t)$ is the impulse response matrix of $T_{iw}(s)$. Noting that the impulse response can be defined as

$$
H_i(t) = C_c(\xi_i) e^{A(\xi_i) t} B_c(\xi_i),
$$

it is possible to see that, after some algebraic manipulation,

$$
\|T_{iw}\|_{\mathcal{H}_2}^2 = \text{tr} \left( C_c(\xi_i) W_{ctr}(\xi_i) C_c^T(\xi_i) \right),
$$

where

$$
W_{ctr}(\xi_i) = \int_{0}^{+\infty} e^{A(\xi_i) t} B_c(\xi_i) B_c^T(\xi_i) e^{A^T(\xi_i) t} \, dt
$$

stands for the controllability Grammian, given by the symmetric positive definite solution of the Lyapunov equation

$$
A_c(\xi_i) W_{ctr}(\xi_i) + W_{ctr}(\xi_i) A_c^T(\xi_i) + B_c(\xi_i) B_c^T(\xi_i) = 0.
$$

It can be seen that, for each parameter vector $\xi_i \in \mathcal{E}_0^j$, $\gamma$ is an upper bound for the $\mathcal{H}_2$-norm of the closed-loop operator $T_{iw}$, if and only if there exists $X > 0$, $0 \prec W_{ctr}(\xi_i) \prec X$ such that

$$
A_c(\xi_i) X + X A_c^T(\xi_i) + B_c(\xi_i) B_c^T(\xi_i) \prec 0 \quad (13)
$$

and

$$
\text{tr} \left( C_c(\xi_i) X C_c^T(\xi_i) \right) < \gamma^2. \quad (14)
$$

Equation (13) can be rewritten as

$$
\begin{bmatrix}
A_c(\xi_i) X + X A_c^T(\xi_i) & X B_c(\xi_i) \\
B_c^T(\xi_i) X & -I
\end{bmatrix} \prec 0,
$$

that, using Schur complements, becomes

$$
\begin{bmatrix}
A_c(\xi_i) X + X A_c^T(\xi_i) & X B_c(\xi_i) \\
B_c^T(\xi_i) X & -I
\end{bmatrix} \prec 0,
$$

or equivalently, introducing the matrix $W = K X$ yields

$$
\begin{bmatrix}
A(\xi_i) X + X A^T(\xi_i) + B(\xi_i) W + W^T B^T(\xi_i) & B(\xi_i) \\
B^T(\xi_i) & -I
\end{bmatrix} \prec 0. \quad (15)
$$

Under the conditions of the theorem, it can be seen that (15) is satisfied for all $\xi_i \in \mathcal{E}_0^j$ and that (115) can be written as

$$
\begin{bmatrix}
Y & C_c(\xi_i) \\
C_c^T(\xi_i) & X^{-1}
\end{bmatrix} \succ 0,
$$

which, applying once again Schur complements, implies that

$$
\begin{bmatrix}
Y - C_c(\xi_i) X C_c^T(\xi_i) & 0 \\
0 & X^{-1}
\end{bmatrix} \succ 0
$$

and consequently

$$
\text{tr} \left( C_c(\xi_i) X C_c^T(\xi_i) \right) < \text{tr} (Y) < \gamma^2,
$$

also, for all $\xi_i \in \mathcal{E}_0^j$. Thus, (14) is implied and the bound on the $\mathcal{H}_2$-norm is established. As (13) is implied by the conditions of the theorem, it can be noted that

$$
A_c(\xi_i) X + X A_c(\xi_i)^T \prec -B_c(\xi_i) B_c^T(\xi_i) \preceq 0,
$$

for all $\xi_i \in \mathcal{E}_0^j$, implying the quadratic stability of the closed-loop and, thus, concluding the proof.

References

[1] B. J. Guerreiro, C. Silvestre, R. Cunha, and D. Cabecinhas, “Lidar-based control of autonomous rotorcraft for the inspection of pier-like structures,” *IEEE Transactions in Control Systems Technology*, 2017.

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