A SPECIALITY THEOREM FOR CURVES IN $\mathbb{P}^5$ CONTAINED IN NOETHER-LEFSCHETZ GENERAL FOURFOLDS

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Abstract. Let $C \subset \mathbb{P}^r$ be an integral projective curve. We define the speciality index $e(C)$ of $C$ as the maximal integer $t$ such that $h^0(C, \omega_C(-t)) > 0$, where $\omega_C$ denotes the dualizing sheaf of $C$. In the present paper we consider $C \subset \mathbb{P}^5$ an integral degree $d$ curve and we denote by $s$ the minimal degree for which there exists a hypersurface of degree $s$ containing $C$. We assume that $C$ is contained in two smooth hypersurfaces $F$ and $G$, with $\text{deg}(F) = n > k = \text{deg}(G)$. We assume additionally that $F$ is Noether-Lefschetz general, i.e. that the 2-th Néron-Severi group of $F$ is generated by the linear section class. Our main result is that in this case the speciality index is bounded as $e(C) \leq d s n k + s + n + k - 6$. Moreover equality holds if and only if $C$ is a complete intersection of $T := F \cap G$ with hypersurfaces of degrees $s$ and $d s n k$.

Keywords: Complex projective curve; speciality index; arithmetic genus; linkage; Castelnuovo - Halphen Theory.

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1. Introduction

Let $C \subset \mathbb{P}^r$ be an integral projective curve. We define the speciality index $e(C)$ of $C$ as the maximal integer $t$ such that $h^0(C, \omega_C(-t)) > 0$, where $\omega_C$ denotes the dualizing sheaf of $C$. The speciality index of a space curve is a fundamental invariant which turned out to be crucial in many issues of projective geometry. For instance, in the papers [7], [8] and [9], such an invariant has been proved to be very useful in the study of projective manifolds of codimension 2.

In [13] Gruson and Peskine prove the following theorem concerning the speciality index of an integral space curve (see also [14]):

Theorem 1.1 (Speciality Theorem). Let $C \subset \mathbb{P}^3$ be an integral degree $d$ curve not contained in any surface of degree $< s$. Then we have:

$$e(C) \leq \frac{d}{s} + s - 4.$$ 

Moreover equality holds if and only if $C$ is a complete intersection of surfaces of degrees $s$ and $\frac{d}{s}$.

In our previous work [2], we prove an extension of this theorem to curves in $\mathbb{P}^5$:

Theorem 1.2. [2, Theorem B] Let $C \subset \mathbb{P}^5$ be an integral degree $d$ curve not contained in any surface of degree $< s$, in any threefold of degree $< t$, and in any
fourfold of degree \( < u \). Assume \( d \gg s \gg t \gg u \geq 1 \). Then we have:

\[
e(C) \leq \frac{d}{s} + \frac{s}{t} + \frac{t}{u} + u - 6.
\]

Moreover equality holds if and only if \( C \) is a complete intersection of hypersurfaces of degrees \( u, \frac{1}{u}, \frac{1}{t} \) and \( \frac{1}{s} \).

Unfortunately, it seems hard to find a generalization of Gruson-Peskine Speciality Theorem without the assumptions \( d \gg s \gg t \gg u \geq 1 \) and to prove a sharp version of the Speciality Theorem for curves in \( \mathbb{P}^5 \).

In this paper we adopt a somewhat different strategy and prove a sharp version of the Speciality Theorem for curves in \( \mathbb{P}^5 \) under the assumption that the curve is contained in a smooth hypersurface with a nice behaviour from the point of view of Noether-Lefschetz theory (compare with Definition 2.2). More precisely, what we are going to do is to assume that \( C \) is contained in a smooth hypersurface having the 2-th Néron-Severi group generated by the linear section class. The main results of this paper are collected in the following Theorem.

**Theorem 1.3.** Let \( C \subset \mathbb{P}^5 \) be an integral degree \( d \) curve. Assume that \( C \) is contained in two smooth hypersurfaces \( F \) and \( G \), with \( \text{deg}(F) = n > k = \text{deg}(G) \). Assume additionally that \( F \) is Noether-Lefschetz general, i.e. that the 2-th Néron-Severi group of \( F \) is generated by the linear section class.

1. If \( C \) is not contained in any hypersurface of degree \( < s \), then we have:

\[
e(C) \leq \frac{d}{s}n + s + n + k - 6.
\]

2. If \( C \) is contained in a hypersurface of degree \( s < k \), then the inequality above still holds true. Moreover, equality holds if and only if \( C \) is a complete intersection of \( T := F \cap G \) with hypersurfaces of degrees \( s \) and \( \frac{d}{s}n + k \).

Theorem 1.3 turned out to be a consequence of much more general results stated in Theorem 3.2 and Theorem 4.1. They show that a sort of Speciality Theorem holds true for Cohen-Macaulay subschemes \( X \subset T \) of codimension 2 in any arithmetically Cohen-Macaulay and factorial variety \( T \).

2. Notations and preliminary results

In order to prove Theorem 1.3, in this section we gather some known properties and results, mainly borrowed from our previous works [3], [4] and [6].

**Notations 2.1.** Let \( X \subset \mathbb{P}^n \) be a smooth complete intersection of dimension \( 2i \geq 2 \). Denote by \( NS_i(X; \mathbb{Z}) \) be the \( i \)-th Néron-Severi group of \( X \), i.e. the image of the cycle map:

\[
NS_i(X; \mathbb{Z}) := \text{Im}(A_i(X) \to H_{2i}(X; \mathbb{Z}) \cong H^{2i}(X; \mathbb{Z})).
\]
Definition 2.2. Let $X \subset \mathbb{P}^n$ be a smooth complete intersection of dimension $2i \geq 2$. By the Lefschetz hyperplane section theorem we know that the homology group $H_{2i}(X; \mathbb{Z}) \cong H^{2i}(X; \mathbb{Z})$ is free. We will say that $X$ is Noether-Lefschetz general if the rank of $NS_i(X; \mathbb{Z})$ is one. In this case, the Lefschetz hyperplane section theorem also implies that $NS_i(X; \mathbb{Z})$ is generated by the linear section class $H^i$.

In [3], it can be found a proof for the following result:

Theorem 2.3. [3, Theorem 1.1] Let $F$ and $G$ be smooth hypersurfaces in $\mathbb{P}^{2m+1}$, with $\deg(F) = n > k = \deg(G)$, and set $X = F \cap G$. If $F$ is Noether-Lefschetz general then $\text{rk} NS_{m}(X) = 1$, and $NS_{m}(X)$ is generated by the linear section class.

Notations 2.4. (1) Let $Q \subset \mathbb{P}^n$ be an irreducible, reduced, non-degenerate projective variety of dimension $m + 1$, with isolated singularities. Let $Q_t$ be a general hyperplane section of $Q$. Let $U \subset \mathbb{P}^n$ be the open subset parametrizing smooth hyperplane sections of $Q$. The fundamental group $\pi_1(U)$ acts via monodromy on both $H^m(Q_t; \mathbb{Z})$ and $H^m(Q_t; \mathbb{Q})$. We denote by

$$H^m(Q_t; \mathbb{Q}) = \mathbb{I} \perp \mathbb{V}$$

the orthogonal decomposition given by the monodromy action on the cohomology of $Q_t$, where $\mathbb{I}$ denotes the invariant subspace.

(2) Denote by

$$i_k^*: H^m(Q_t; \mathbb{Q}) \to H^{2n-k}(Q_t; \mathbb{Z})$$

the map obtained composing the Gysin map $H_{k+2}(Q; \mathbb{Z}) \to H_k(Q_t; \mathbb{Z})$ with Poincaré duality $H_k(Q_t; \mathbb{Z}) \cong H^{2n-k}(Q_t; \mathbb{Z})$.

In [4], it can be found a proof for the following results:

Theorem 2.5. [4, Theorem 3.1] With notations as in 2.4, the vector subspace $\mathbb{V} \subset H^m(Q_t; \mathbb{Q})$ is generated, via monodromy, by standard vanishing cycles.

Corollary 2.6. [4, Corollary 3.7] The vector subspace $\mathbb{V} \subset H^m(Q_t; \mathbb{Q})$ is irreducible via monodromy action.

The results 2.5 and 2.6 concern rational cohomology. In the paper [6] they are used to prove similar results concerning integral cohomology:

Theorem 2.7. [6, Theorem 2.1] With notations as in 2.4 the following properties hold true.

1. For any integer $m < k \leq 2m$ the map $i_k^*$ is an isomorphism, the map $i_m^*$ is injective with torsion-free cokernel, and $H_{m+2}(Q; \mathbb{Z}) \cong \mathbb{I}$ via $i_m^*$.
2. For any even integer $m < k = 2i \leq 2m$ the map $i_k^* \otimes \mathbb{Z} \mathbb{Q}$ induces an isomorphism $NS_{i+1}(Q; \mathbb{Q}) \cong NS_i(Q_t; \mathbb{Q})$. 

If $k = 2i = m$ and the orthogonal complement $\mathbb{V}$ of $I \otimes_{\mathbb{Z}} \mathbb{Q}$ in $H^n(Q; \mathbb{Q})$ is not of pure Hodge type $(m/2, m/2)$, then $NS_i(Q; \mathbb{Z}) \subseteq I$, and the map $i_n \otimes_{\mathbb{Z}} \mathbb{Q}$ induces an isomorphism $NS_{i+1}(Q; \mathbb{Q}) \cong NS_i(Q; \mathbb{Q})$.

One of the main ingredients of the proof of Theorem 1.3 is the following Proposition.

**Proposition 2.8.** Let $F \subset \mathbb{P}^5$ be a Noether-Lefschetz general hypersurface and let $G \subset \mathbb{P}^5$ be a smooth hypersurface with $k := \deg G < d := \deg F$. Define $T := F \cap G$. Then we have:

1. the threefold $T$ is factorial with isolated singularities;
2. if $\deg T \geq 4$ then the general hyperplane section $S := H \cap T$ is a Noether-Lefschetz general surface.

**Proof.** (1) The threefold $T$ has at worst finitely many singularities by [10, Proposition 4.2.6]. Furthermore, $T$ is factorial by Theorem 2.3.

(2) Combining Theorem 2.5, Corollary 2.6 and Theorem 2.7, the proof runs similarly as the classical one (compare with the proof of [5, Theorem 3.2]). Indeed, denote by $U \subset \mathbb{P}^5$ the affine open subset parametrizing smooth hyperplane sections of $T$. The fundamental group $\pi_1(U)$ acts via monodromy on both $H^2(S; \mathbb{Z})$ and $H^2(S; \mathbb{Q})$.

As in 2.4, consider the orthogonal decomposition $H^2(S; \mathbb{Q}) = \mathbb{I} \perp V$, where $\mathbb{I}$ is the $\pi_1(U)$-invariant cohomology (compare also with [5, Notations 3.1 (ii)]). By Theorem 2.5 and Corollary 2.6 we know that the vanishing cohomology $V$ is a $\pi_1(U)$-irreducible module generated by the standard vanishing cycles. On the other hand, Theorem 2.7 implies that the $\pi_1(U)$-invariant part of $H^2(S; \mathbb{Z}) \cong H_2(S; \mathbb{Z})$ is the image of the Gysin map:

$$I \cap H_2(S; \mathbb{Z}) = \text{Im}(H_4(T; \mathbb{Z}) \xrightarrow{n} H_2(S; \mathbb{Z}))$$

(here $u \in H^2(T, T - S; \mathbb{Z})$ denotes the orientation class [12, §19.2]). By point (1) $T$ is factorial, hence the subspace $\mathbb{I}$ is generated by the hyperplane class. But then $V$ is not of pure Hodge type because $\deg T \geq 4$. By irreducibility, the image of $NS_1(S; \mathbb{Z})$ in $V$ vanishes. This implies that the Néron-Severi group $NS_1(S; \mathbb{Z})$ is $\pi_1(U)$-invariant and [11] says that $S$ is Noether-Lefschetz general. □

3. **Proof of Theorem 1.3 (2)**

**Definition 3.1.** Let $X$ be a Cohen-Macaulay projective scheme. We define the speciality index $e_X$ of $X$ as the maximal integer $t$ such that $h^0(X, \omega_X(-t)) > 0$, where $\omega_X$ denotes the dualizing sheaf of $X$.

The proof of Theorem 1.3 (2) rests on the following much more general result:

**Theorem 3.2 (Speciality theorem for aCM varieties).** Let $T \subset \mathbb{P}^n$ be an arithmetically Cohen-Macaulay (aCM for short), factorial and subcanonical variety with $\dim T = m \geq 3$ and $\omega_T \cong O_T(t)$. 
Let $G \subset T$ be an integral divisor. Since $T$ is factorial and $aCM$, we have $G = \tilde{G} \cap T$ with $\tilde{G} \subset \mathbb{P}^n$ a projective hypersurface of some degree $g$. Let $X \subset G$ be a Cohen-Macaulay scheme of codimension two in $T$. Then

$$e_X \leq \frac{\deg(X)}{\deg(T)} + g + t$$

and the equality holds iff $X$ is a complete intersection $X = T \cap \tilde{G} \cap H$, with $\deg(H) = \deg(X)$. 

**Proof.** Consider a general hypersurface $P$ of degree $p \gg 0$ containing $X$. Denote by $Y$ the scheme $T \cap \tilde{G} \cap P$ which we are going to consider as a complete intersection in $T$. Following Peskine-Szpiro [17], we consider the scheme $R$ residual of $X$ with respect to $Y$ (compare also with [11, §2]).

The Noether Linkage Sequence [11, Proposition 2.3] inside $T$ looks like

$$0 \to I_Y \to I_R \to \omega_X \otimes \omega_Y^{-1} \to 0,$$

and can be written as

$$0 \to I_Y \to I_R \to \omega_X(-t - g - p) \to 0$$

(all the ideal sheaves are meant to be defined in $T$). Recall that

$$h^0(\omega_X(-e)) \neq 0$$

($e := e_X$). Since $T$ is $aCM$ and $Y$ is a complete intersection in $T$ of type $(g, p)$, the short exact sequence

$$0 \to \mathcal{O}_T(-g - p) \to \mathcal{O}_T(-g) \oplus \mathcal{O}_T(-p) \to I_Y \to 0$$

implies

$$\cdots \to H^1(\mathcal{O}_T(l-g) \oplus \mathcal{O}_T(l-p)) \to H^1(I_Y(l)) \to H^2(\mathcal{O}_T(l-g-p)) \to \cdots \forall l$$

hence

$$h^1(I_Y(t + g + p - e)) = 0.$$ 

Combining (2), (4) and (5), we see that there exists a hypersurface $S$ of degree $s = t + g + p - e$ containing $R$ and not containing $Y$. But $G$ is integral and $Y' = G \cap S$ is a complete intersection, in $T$, containing $R$. Set $Y' = R \cup R'$ the corresponding, possibly algebraic, linkage. But then

$$\deg(R') + \deg(R) = \deg(T)gs, \quad \deg(X) + \deg(R) = \deg(T)gp$$

and by a simple computation, we find

$$\deg(R') = \deg(X) - \deg(T)g(e - t - g) \geq 0$$

and the first statement follows.

Suppose now the equality holds. Then we have

$$\deg(X) = \deg(T)g(e - t - g) = \deg(T)g(p - s).$$

and the scheme $R'$ is empty. Furthermore, we have that $R = Y' = G \cap S$ is a complete intersection with $\omega_R \simeq \mathcal{O}_R(t + g + s)$. 

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Coming back to the Noether Linkage Sequence (2)

\[ 0 \to \mathcal{I}_Y \to \mathcal{I}_X \to \omega_R \otimes \omega_Y^{-1} \to 0 \]

we find

\[ (6) \quad 0 \to \mathcal{I}_Y \to \mathcal{I}_X \to \mathcal{O}_R(s-p) \to 0. \]

Similarly as above, the short exact sequence

\[ 0 \to \mathcal{O}_T(-g-s) \to \mathcal{O}_T(-g) \oplus \mathcal{O}_T(-s) \to \mathcal{I}_R \to 0 \]

implies

\[ \cdots \to H^1(\mathcal{O}_T(l-g) \oplus \mathcal{O}_T(l-s)) \to H^1(\mathcal{I}_Y(l)) \to H^2(\mathcal{O}_T(l-g-s)) \to \cdots \quad \forall l \]

and

\[ h^1(\mathcal{I}_R(p-s)) = 0. \]

Hence there is a hypersurface \( H \) of degree \( h = p-s \) containing \( X \) and not containing \( Y \). Finally, since \( G \) is integral and \( \text{deg}(X) = \text{deg}(T)g(p-s) \) we conclude that \( X = G \cap H \).  \( \square \)

Proof of Theorem 1.3 (2). It suffices to apply Theorem 3.2 to the complete intersection \( T := F \cap G \), which is aCM with \( \dim T = 3 \) and \( \omega_T \simeq \mathcal{O}_T(n + k - 6) \), and factorial in view of Proposition 2.8.  \( \square \)

4. Proof of Theorem 1.3 (1)

The proof of Theorem 1.3 (1) rests on the following much more general result:

**Theorem 4.1.** Let \( T \subset \mathbb{P}^n \) be an aCM, factorial variety with \( \dim T = m \geq 3 \) and \( \omega_T \simeq \mathcal{O}_T(t) \). Assume moreover that \( T \) is smooth in codimension 2 and that the very general surface section of \( T \) is factorial. Let \( X \subset T \) be a C.M. subscheme of codimension 2 which is generically complete intersection. If \( h^0(\mathcal{I}_X,T(h-1)) = 0 \) and \( h > 0 \) then

\[ e_X \leq \frac{\deg(X)}{\deg(T)h} + h + t. \]

The main idea in the proof of 4.1 which goes back to the work of Hartshorne, is to construct a rank two reflexive sheaf on \( T \) having a section vanishing in \( X \) (see e.g. [16] and [1]).

In order to prove Theorem 4.1 we need some preliminary results. We recall the following result of R. Hartshorne:

**Lemma 4.2.** [16, Proposition 1.3] Let \( T \) be a normal scheme and let \( \mathcal{F} \) be a coherent sheaf defined on \( T \). Then \( \mathcal{F} \) is reflexive iff

1. \( \mathcal{F} \) is torsion-free;
2. \( \forall x \in T, \dim \mathcal{O}_x \geq 2 \implies \text{depth} \mathcal{F}_x \geq 2. \)

For the sake of completeness, we give a short proof of the following (maybe well known) result.
Lemma 4.3. Let \( T \subset \mathbb{P}^n \) be an aCM scheme such that \( m := \dim T \geq 3 \) and \( \omega_T \simeq \mathcal{O}_T(t) \). Let \( X \subset T \) be a Cohen-Macaulay subscheme of codimension 2. Then we have:

\[
\text{Ext}^1_T(\mathcal{I}_X,T(c),\mathcal{O}_T) \simeq H^0(X,\omega_X(-c-t)), \quad \forall c \in \mathbb{Z}.
\]

Proof. By applying the functor \( \text{Hom}_T(\cdot,\mathcal{O}_T) \) to the short exact sequence

\[
0 \to \mathcal{I}_X,T(c) \to \mathcal{O}_T(c) \to \mathcal{O}_X(c) \to 0
\]

we find

\[
\text{Ext}^1_T(\mathcal{O}_T(c),\mathcal{O}_T) \to \text{Ext}^1_T(\mathcal{I}_X,T(c),\mathcal{O}_T) \to \\
\text{Ext}^2_T(\mathcal{O}_X(c),\mathcal{O}_T) \to \text{Ext}^2_T(\mathcal{O}_T(c),\mathcal{O}_T).
\]

By Serre Duality, \( \omega_T \simeq \mathcal{O}_T(t) \) implies

\[
\text{Ext}^i_T(\mathcal{O}_T(c),\mathcal{O}_T) \simeq H^{m-i}(\mathcal{O}_T(-c-t)) = 0, \quad i = 1, 2
\]

where the last equality follows from the hypothesis that \( T \) is aCM of dimension \( \geq 3 \). Again by Serre Duality we have:

\[
\text{Ext}^1_T(\mathcal{I}_X,T(c),\mathcal{O}_T) \simeq \text{Ext}^2_T(\mathcal{O}_X(c),\omega_T(-t)) \simeq \\
H^{m-2}(T,\mathcal{O}_X(c+t)) \simeq H^{m-2}(X,\omega_X(c+t)) \simeq H^0(X,\omega_X(-c-t)).
\]

\( \square \)

Proposition 4.4. Let \( T \subset \mathbb{P}^n \) be an aCM variety such that \( m := \dim T \geq 3 \) and \( \omega_T \simeq \mathcal{O}_T(t) \). We assume additionally that \( T \) is smooth in codimension 2. For any pair \((X,\xi)\) with:

- \( X \subset T \) a Cohen-Macaulay, generically complete intersection subscheme of codimension two in \( T \),
- \( \xi \in H^0(\omega_X(-t-c)) \) generating almost everywhere,

there exists a rank two reflexive sheaf \( F \) on \( T \), with \( c_1(F) = cH \), \( c_2(F) = [X] \) (the fundamental cycle of \( X \)) and such that

\[
0 \to \mathcal{O}_T \to F \to \mathcal{I}_{X,T}(c) \to 0.
\]

Proof. The assertion concerning the Chern classes follows trivially from the rest of the statement so it suffices to prove the existence of a sequence like (7), with \( F \) reflexive.

The existence of a sequence like (7) follows directly from Lemma 4.3. Since \( T \) is Cohen-Macaulay and smooth in codimension 2, it is also normal by Serre’s criterion. Then we may apply Lemma 4.2 in order to prove the reflexivity of \( F \). Further, since \( T \) is Cohen-Macaulay, both \( \mathcal{O}_T \) and \( \mathcal{I}_{X,T}(c) \) are torsion-free hence we only need to prove the second condition of Lemma 4.2. Fix a point \( x \) of codimension \( \geq 3 \) and denote by \( K \) the residue field at \( x \). Applying the functor \( \text{Hom}_{\mathcal{O}_x} (K, \cdot) \) to the sequence

\[
0 \to \mathcal{I}_{x,T} \to \mathcal{O}_{x,T} \to \mathcal{O}_{x,x} \to 0
\]
and recalling that both $T$ and $X$ are Cohen-Macaulay we have:

\[(8) \quad \text{Ext}^i_{\mathcal{O}_x}(\mathcal{K}, \mathcal{I}_{x,T}) = 0, \quad i \leq 2.\]

Applying the functor $\text{Hom}_{\mathcal{O}_x}(\mathcal{K}, \cdot)$ to the sequence

\[0 \to \mathcal{O}_{x,T} \to \mathcal{F}_x \to \mathcal{I}_{x,T}(c) \to 0\]

we see that the vanishing (8) implies:

\[\forall x \in X, \quad \text{dim} \mathcal{O}_x \geq 3 \implies \text{depth} \mathcal{F}_x \geq 2.\]

In order to conclude we need to prove:

\[\forall x \in X, \quad \text{dim} \mathcal{O}_x = 2 \implies \text{depth} \mathcal{F}_x \geq 2.\]

What we are going to do is to prove that $\mathcal{F}_x$ is a free module of rank two over $\mathcal{O}_x$, for any $x \in X$ such that $\text{dim} \mathcal{O}_x = 2$. In order to do this, we prove that $\mathcal{F}_x$ has homological dimension 0 ([19, IV]). Since $T$ is smooth in codimension 2, $\forall x \in X$ of codimension 2 the local ring $\mathcal{O}_x$ is regular of dimension 2. So it suffices to prove that

\[(9) \quad \text{Ext}^1_T(\mathcal{F}, \mathcal{O}_T)x = \text{Ext}^1_T(\mathcal{F}, \mathcal{O}_T)_x = 0.\]

From the sequence (6) we see that $\text{depth}(\mathcal{F}_x) \leq \text{depth}(\mathcal{I}_{x,T}) = 1$, the first inequality coming from ([19, IV p. 28]) and the last equality coming from the fact that $\mathcal{I}_{x,T}$ is complete intersection at $x$. So, in order to prove (9) we are left to show that $\text{Ext}^1_T(\mathcal{F}, \mathcal{O}_T)_x = 0$. Applying $\text{Hom}_T(\cdot, \mathcal{O}_T(c))$ to the sequence (7) we get:

\[(10) \quad 0 \to \mathcal{O}_T \to \mathcal{F}^*(c) \to \mathcal{O}_T \overset{\xi}{\to} \omega_X(-t) \to \text{Ext}^1_T(\mathcal{F}, \mathcal{O}_T(c)) \to 0\]

where we have taken into account the isomorphism $\text{Ext}^1_T(\mathcal{I}_{X,T}, \mathcal{O}_T) \simeq \omega_X(-t)$, which again follows from the fact that both $T$ and $X$ are Cohen-Macaulay and $\omega_T \simeq \mathcal{O}_T(t)$. Since $T$ is smooth in codimension 2, $\forall x \in X$ of codimension 2 the local ring $\mathcal{O}_x$ is regular of dimension 2. Furthermore, since $\xi$ generates almost everywhere and $X$ is generically complete intersection, the fourth map of the sequence (10) is an isomorphism at $x$ hence $\text{Ext}^1_T(\mathcal{F}, \mathcal{O}_T(c))_x \simeq 0$ and $\mathcal{F}_x$ is a free module of rank two over $\mathcal{O}_x$.

\[\square\]

**Lemma 4.5.** Let $C \subset \mathbb{P}^n$ be a smooth variety and $E$ a rank two vector bundle on $C$ having a section vanishing in the right dimension. If $c_1(E) < 0$ then $h^0(E) = 1$ and $h^0(E(-m)) = 0$, $\forall m > 0$.

**Proposition 4.6.** Let $T \subset \mathbb{P}^n$ be an aCM, factorial variety such that $m := \text{dim} T \geq 3$ and $\omega_T \simeq \mathcal{O}_T(t)$. We assume additionally that $T$ is smooth in codimension 2 and that the general hyperplane section of $T$ is factorial. Let $\mathcal{F}$ be a normalized (i.e. with $-1 \leq c_1(\mathcal{F}) \leq 0$) reflexive sheaf on $T$. If $d(c_1(\mathcal{F}) \cdot c_1(\mathcal{F})) > 4d(c_2(\mathcal{F}))$ then there exists $\alpha \leq 0$ such that $h^0(\mathcal{F}(\alpha)) \neq 0$. Furthermore, if $c_1(\mathcal{F}) = 0$ then $\alpha < 0$ hence we have $c_1(\mathcal{F}(\alpha)) < 0$.

**Proof.** Let us denote by $S$ the general (smooth) surface section of $T$. Since $\mathcal{F} |_S$ is a normalized rank 2 vector bundle on $S$, Bogomolov’s theorem implies there exists $\alpha \leq 0$ such that $h^0(S, \mathcal{F}(\alpha) |_S) \neq 0$. Moreover, we can assume $\alpha < 0$ as soon
as $c_1(\mathcal{F}|_S) = 0$. Bogomolov’s theorem implies that a section of $\mathcal{F}|_S(\alpha)$ can be chosen in such a way that it vanishes in the right dimension. In any case we have $c_1(\mathcal{F}|_S(\alpha)) < 0$, so Lemma 4.3 above implies $h^0(S, \mathcal{F}(\alpha)|_S) = 1$.

Fix $C \subset S$ a general curve section of $\mathcal{T}$. We can assume that $C$ does not meet the zero locus of the general section of $\mathcal{F}|_S(\alpha)$ so Lemma 4.3 implies:

$$h^0(C, \mathcal{F}(\alpha)|_C) = 1 \quad \text{and} \quad h^0(C, \mathcal{F}(\beta)|_C) = 0 \ \forall \beta < \alpha.$$ 

Set

$$\mathcal{P} \simeq \mathbb{P}^{n-2} = \{ \pi \in \mathbb{G}(n - m + 2, \mathbb{P}^n) : C \subset \pi \} \subset \mathbb{G}(n - m + 2, \mathbb{P}^n),$$

denote by $\tilde{T} \subset T \times \mathcal{P}$ the incidence variety:

$$\tilde{T} = \{ (x, \pi) \in T \times \mathcal{P} : x \in \pi \cap T \}$$

and by $\phi : \tilde{T} \to T$, $\psi : \tilde{T} \to \mathcal{P}$ the natural maps.

Claim 1. $h^0(\psi^{-1}(p), \mathcal{F}(\alpha)|_{\psi^{-1}(p)}) = 1$, $\forall p \in \mathcal{P}$. As we have just said $h^0(\psi^{-1}(p), \mathcal{F}(\alpha)|_{\psi^{-1}(p)}) = 1$ for a very general $p \in \mathcal{P}$ so, by semicontinuity, $h^0(\psi^{-1}(p), \mathcal{F}(\alpha)|_{\psi^{-1}(p)}) \geq 1$, $\forall p \in \mathcal{P}$. In order to prove the Claim it is then sufficient to prove that $h^0(\psi^{-1}(p), \mathcal{F}(\alpha)|_{\psi^{-1}(p)}) < 2$, $\forall p \in \mathcal{P}$. Set $S' = \psi^{-1}(p)$ and assume by contradiction $h^0(S', \mathcal{F}(\alpha)|_{S'}) \geq 2$. From the short exact sequence

$$0 \to \mathcal{F}|_{S'}(\alpha - 1) \to \mathcal{F}|_{S'}(\alpha) \to \mathcal{F}|_C(\alpha) \to 0$$

and taking into account (11) we get $h^0(S', \mathcal{F}(\alpha - 1)|_{S'}) \neq 0$. Set $\overline{\alpha} := \min \{ \beta \in \mathbb{N} : h^0(S', \mathcal{F}(\beta)|_{S'}) \neq 0 \} \leq \alpha - 1$. From the short exact sequence

$$0 \to \mathcal{F}|_{S'}(\overline{\alpha} - 1) \to \mathcal{F}|_{S'}(\overline{\alpha}) \to \mathcal{F}|_C(\overline{\alpha}) \to 0$$

and by the definition of $\overline{\alpha}$ we find $h^0(C, \mathcal{F}|_C(\overline{\alpha})) \neq 0$ which contradicts (11) since $\overline{\alpha} < \alpha$. The claim is so proved.

By Grauert’s theorem [15, Corollary 12.9], $\psi_* \phi^* \mathcal{F}(\alpha)$ is an invertible sheaf on $\mathcal{P}$. On the other hand, since $\phi^{-1}C = C \times \mathcal{P}$, we have

$$\psi_* \phi^* \mathcal{F}(\alpha)|_{\phi^{-1}C} \simeq H^0(C, \mathcal{F}(\alpha)|_C) \otimes \mathcal{O}_\mathcal{P} \simeq \mathcal{O}_\mathcal{P}.$$ 

Finally, the natural restriction $H^0(\psi^{-1}(p), \mathcal{F}(\alpha)|_{\psi^{-1}(p)}) \to H^0(C, \mathcal{F}(\alpha)|_C)$ is an isomorphism $\forall p \in \mathcal{P}$, so the natural map $\psi_* \phi^* \mathcal{F}(\alpha)) \to \psi_* (\phi^* (\mathcal{F}(\alpha))|_{\psi^{-1}(C)}) \simeq \mathcal{O}_\mathcal{P}$ is an isomorphism of invertible sheaves on $\mathcal{P}$. Then we have

$$H^0(\tilde{T}, \phi^* \mathcal{F}(\alpha)) = H^0(\mathcal{P}, \psi_* (\phi^* (\mathcal{F}(\alpha))) = H^0(\mathcal{P}, \mathcal{O}_\mathcal{P}) = \mathbb{C}.$$ 

We conclude by means of the projection formula, because $\phi_* \mathcal{O}_{\tilde{T}} \simeq \mathcal{O}_\mathcal{P}$. \hfill \qed

Remark 4.7. (1) By Lemma 4.3 the coefficient $\alpha$ arising in Proposition 4.6 is the least twist of $\mathcal{F}$ admitting a section.

(2) The proof of Proposition 4.6 shows that the zero locus of the section of $\mathcal{F}(\alpha)$ has the right dimension, because it does not meet the general curve $C$. 

Proof of Theorem 4.1. In this proof we closely follow [18].

By Proposition 4.1 there exists a normalized reflexive sheaf $\mathcal{F}$ (on $T$) such that

$$0 \to \mathcal{O}_T \to \mathcal{F}(k) \to \mathcal{I}_X(e-t) \to 0$$

$c_1(\mathcal{F}) = cH$, $c_2(\mathcal{F}) = [X] - (ck + k^2)H^2$ and $c + 2k = e - t)$. Set

- let $\alpha$ and $\beta$ be the smallest degrees of two independent generators of $H^0\mathcal{F}$ (compare with [18, p. 103]),
- $s = \min\{r : h^0(\mathcal{I}_{X,T}(r)) \neq 0\}$.

We distinguish two cases depending on whether the discriminant of $\mathcal{F}$ is $\leq 0$ or $> 0$.

$d(c_1(\mathcal{F}) - c_1(\mathcal{F})) = 4d(c_2(\mathcal{F}))$. This case is the simplest one because the expression $d(X) - d(T)h(e' - h - t)$ is the degree of the second Chern class of $\mathcal{F}(k - h)$. Since the discriminant is $\leq 0$, the second Chern class is always positive and we are done.

$d(c_1(\mathcal{F}) + c_1(\mathcal{F})) > 4d(c_2(\mathcal{F}))$. In this case Proposition 4.6 implies $\alpha \leq 0 (< 0$ if $c = 0)$. Furthermore, Remark 4.7 (2) says that the corresponding section vanishes in the right dimension. Then $d(c_2(\mathcal{F}(\alpha)) = d(c_2(\mathcal{F}(-\alpha - c)) \geq 0$ and the degree of the second Chern class is positive for any twist $\leq \alpha$ or $\geq -\alpha - c$. If $k = \alpha$ then $s = \beta + \alpha + c$ and the expression $d(X) - d(T)h(e' - h - t)$ is the degree of $c_2(\mathcal{F}(k - h)) = c_2(\mathcal{F}(h - \alpha - c))$ which is strictly positive since $h > 0$. So the inequality is proved and the equality cannot be attained. If $k \geq \beta$ then $s = \alpha + k + c$ and the expression $d(X) - d(T)h(e' - h - t)$ is the degree of $c_2(\mathcal{F}(k - h)) = c_2(\mathcal{F}(\alpha - (s - h)))$. So the inequality is proved and the equality can be attained only if $s = h$ and the degree of $c_2(\mathcal{F}(\alpha))$ vanishes.

Proof of Theorem 1.3 (1). It suffices to apply Theorem 4.1 to the complete intersection $T := F \cap G$, which is aCM with $dimT = 3$ and $\omega_T \simeq \mathcal{O}_T(n + k - 6)$. The hypotheses that $T$ is factorial and smooth in codimension 2 and that the very general surface section of $T$ is factorial follow from Proposition 2.8.

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