MULTIPARTICLE SUSY QUANTUM MECHANICS AND THE REPRESENTATIONS OF PERMUTATION GROUP.

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Abstract. The method of multidimensional SUSY Quantum Mechanics is applied to the investigation of supersymmetrical $N$–particle systems on a line for the case of separable center-of-mass motion. New decomposition of the superhamiltonian into block-diagonal form with elementary matrix components is constructed. Matrices of coefficients of these minimal blocks are shown to coincide with matrices of irreducible representations of permutation group $S_N$, which correspond to the Young tableaux $(N - M, 1^M)$. The connections with known generalizations of $N$–particle Calogero and Sutherland models are established.

1. Introduction

One of the most natural generalizations of the standard 1-dimensional Super-symmetrical Quantum Mechanics (SUSY QM) concerns the systems in the spaces of arbitrary dimension $d$. It was shown for such systems that superhamiltonian is a matrix $(2^d \times 2^d)$ block-diagonal operator with $(d + 1)$ components on the diagonal. These components of the superhamiltonian are Schrödinger-type operators with matrix $(C^d_n \times C^d_n)$ potentials $(C^d_n – \text{binomial coefficients}, n = 0, 1, ..., d)$. Supersymmetry of the system leads to important SUSY intertwining relations between neighbouring components of the superhamiltonian and provides definite connections between their spectra and eigenfunctions. More definitely, for each component its spectrum consists of the eigenvalues which coincide with a part of eigenvalues of neighbouring components of superhamiltonian. Corresponding eigenfunctions are connected with each other by the action of supercharge operators (see details in [2]). This approach was successfully used for some 2- and 3-dimensional physical systems.

It would be interesting to apply this method for the systems with another possible interpretation of several degrees of freedom in the superhamiltonian. Namely, it seems
to be useful in the description of supersymmetrical systems of $N$ interacting quantum particles on a line.

The supersymmetric generalization of a known exactly solvable [4], [5] $N$-particle Calogero model was considered for the first time in the paper [6], where its spectrum was found (see also papers [7], [8], [9], [10], [11]). In the paper [6] the hypothesis was put forward (but not proved), that the (super)Calogero, Sutherland and some other models possess shape-invariance [12], which could help us find the spectrum of the models.

In the present paper the most general variety of supersymmetrical $N$-particle systems on a line will be considered using the generalization of method [2]. The only restriction, introduced in Sect. 2, is the condition of separability of center-of-mass motion (CMM) [7] in the superpotential which seems to be very natural for such systems. Introducing usual bosonic Jacobi coordinates and their fermionic analogues, we derive the superhamiltonian and SUSY intertwining relations for systems with separable CMM. This block-diagonal superhamiltonian has the same matrix dimension $2^N \times 2^N$, as in [2], but more detailed structure: $2N$ blocks $C^M_{N-1} \times C^M_{N-1}$ instead of $N+1$ blocks $C^N_N \times C^N_N$ in [2].

In Sect. 3 the internal structure of the blocks on the diagonal of the superhamiltonian is considered. It is shown that for any $M$ the coefficients $B^{(M)}_{ij}$ in matrix potentials coincide with the matrices of irreducible representation of permutation group $S_N$, which is characterized by the Young tableau $(N - M, 1^M)$. This statement provides that these matrix potentials are elementary blocks of the superhamiltonian, i.e. they cannot be further decomposed into the block-diagonal form. At the end of Sect. 3 the SUSY intertwining relations are built in terms of Jacobi coordinates, using the Clebsh-Gordan coefficients for the corresponding irreducible representations of $S_N$. Two examples are considered in Sect. 4. For the case $N = 3$ with particular choice of superpotential our approach gives a part of the spectrum of $2 \times 2$ matrix Hamiltonian. The class of superpotentials corresponding to $N$-particle models with pairwise interactions (including Calogero and Sutherland models) is considered in the second example. The connections with known generalizations [9], [13], [8], [11] of $N$-particle Calogero and Sutherland models are established. The proof of the Theorem of Sect. 3 can be found in Appendix.

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\footnote{a However, some kind of shape-invariance for the Calogero model was constructed in [8] using the Dunkl operators [14].}
2. Systems with a separable centre-of-mass motion

The supersymmetric quantum system for arbitrary number of dimensions \( N \) consists of the superhamiltonian and the supercharges:

\[
H_S = \frac{1}{2} (-\Delta + \sum_{i=1}^{N} (\partial_i W)^2 - \Delta W) + \sum_{i,j=1}^{N} \psi_i^+ \psi_j \partial_i \partial_j W; \quad \Delta \equiv \sum_{i=1}^{N} \partial_i \partial_i; \quad \partial_i \equiv \partial/\partial x_i; \quad (1)
\]

\[
Q^\pm \equiv \frac{1}{\sqrt{2}} \sum_{j=1}^{N} \psi^\pm_j (\pm \partial_j + \partial_j W); \quad (2)
\]

with the algebra

\[
[Q^+, Q^-] = 0, \quad (3)
\]

\[
(Q^+)^2 = (Q^-)^2 = 0, \quad (4)
\]

\[
[H_S, Q^\pm] = 0, \quad (5)
\]

where \( \psi_i, \psi_i^+ \) are fermionic operators:

\[
\{\psi_i, \psi_j\} = 0, \quad \{\psi_i^+, \psi_j^+\} = 0, \quad \{\psi_i, \psi_j^+\} = \delta_{ij}. \quad (6)
\]

The dynamics of a SUSY QM system is determined by a superpotential \( W \), depending on \( N \) real coordinates \( (x_1, \ldots, x_N) \).

For \( N \)-particle systems on a line it is natural to consider potentials with a separable centre-of-mass motion (CMM). Therefore in this paper we restrict ourselves by considering the superpotentials:

\[
W(x_1, \ldots, x_N) = w(x_1, \ldots, x_N) + W_C \left( \frac{x_1 + \ldots + x_N}{\sqrt{N}} \right); \quad \sum_{j=1}^{N} \partial_j w(x_1, \ldots, x_N) = 0, \quad (7)
\]

allowing a separation of CMM motion ( \( w(x_1, \ldots, x_N) \) does not depend on \( x_1 + \ldots + x_N \)).

Let us introduce the operator:

\[
\tilde{K}_{ij} \equiv \psi_i^+ \psi_j + \psi_j^+ \psi_i - \psi_i^+ \psi_i - \psi_j^+ \psi_j + 1 = 1 - (\psi_i^+ - \psi_j^+)(\psi_i - \psi_j) = \tilde{K}_{ji} = (\tilde{K}_{ij})^\dagger, \quad (8)
\]

\[b\] Here and below the indices \( i, j, k, \ldots \) range from 1 to \( N \).

\[c\] The usefulness of the factor \( 1/\sqrt{N} \) will be explained later.

\[d\] The equation (2) is equivalent to the condition: \( \sum_k \partial_k (\partial_i - \frac{1}{N} \sum_l \partial_l) w = 0 \) for every \( i = 1, \ldots, N \).
which plays the role of the fermionic permutation operator
\[ \hat{K}_{ij} \psi_i^+ = \psi_j^+ \hat{K}_{ij}, \]
\[ \hat{K}_{ij} \psi_k^+ = \psi_k^+ \hat{K}_{ij}, \quad k \neq i, j. \]

In the fermionic Fock space
\[ \psi_1^+ \ldots \psi_M^+ |0 > \equiv |i_1 \ldots i_M >; \quad \psi_i |0 > = 0 \quad i, i_1 \ldots i_M, M \leq N \]
this operator acts as:
\[ \hat{K}_{ij} |i_1 \ldots i \ldots j \ldots i_M > = |i_1 \ldots j \ldots i \ldots i_M > \]
\[ \hat{K}_{ij} |i_1 \ldots i \ldots i_M > = |i_1 \ldots j \ldots i_M > \quad \text{for} \quad i_1 \ldots i_M \neq i, j. \]

Let us rewrite \( H_S \) for the superpotentials (7) using \( \hat{K}_{ij} \). We will take into account
the following equations:
\[ \sum_{i,j=1}^{N} \psi_i^+ \psi_j \partial_i \partial_j w = \frac{1}{2} \sum_{i,j=1}^{N} \hat{K}_{ij} \partial_i \partial_j w = \frac{1}{2} \sum_{i \neq j}^{N} \hat{K}_{ij} \partial_i \partial_j w + \frac{1}{2} \sum_{i=1}^{N} \partial_i^2 w, \]
\[ \sum_{i,j=1}^{N} \psi_i^+ \psi_j \partial_i \partial_j W_C = \frac{1}{N} \left( \sum_{i=1}^{N} \psi_i^+ \right) \left( \sum_{j=1}^{N} \psi_j \right) W_C'' , \]
\[ \sum_{j=1}^{N} \left( \partial_j (w + W_C) \right)^2 = \sum_{j=1}^{N} \left( \partial_j w \right)^2 + (W_C')^2 \]
to obtain
\[ H_S = -\frac{1}{2} \Delta + \frac{1}{2} \sum_{j=1}^{N} (\partial_j w)^2 + \frac{1}{2} \sum_{i \neq j}^{N} \hat{K}_{ij} \partial_i \partial_j w + \]
\[ + \frac{1}{2} (W_C')^2 - W_C'' + \frac{1}{N} \left( \sum_{i=1}^{N} \psi_i^+ \right) \left( \sum_{j=1}^{N} \psi_j \right) W_C''. \]

For the superpotentials (8) with a separable CMM it is natural to go to the well-known Jacobi coordinates:
\[ y_b = \frac{1}{\sqrt{b(b+1)}} (x_1 + \ldots + x_b - bx_{b+1}) \]
\[ y_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} x_i, \]
\[ \text{From this moment on, the variables denoted by letters } a, b, c, \ldots \text{ range from } 1 \text{ to } N - 1. \]
or \( y_k = \sum_{l=1}^{N} R_{kl} x_l \), where the matrix \( R \) is determined by (17). The derivatives are connected by the same matrix: \( \partial/\partial y_k = \sum_{l=1}^{N} R_{kl} \partial/\partial x_l \), because \( R \) is an orthogonal matrix.

For the supersymmetric systems it is natural to introduce also the fermionic analogues of the Jacobi variables:

\[
\phi_b = \frac{1}{\sqrt{b(b+1)}}(\psi_1 + \ldots + \psi_b - b\psi_{b+1});
\]

\[
\phi_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \psi_i,
\]

or \( \phi_k = \sum_{l=1}^{N} R_{kl} \psi_l \), where the matrix \( R \) is the same as in (17). These variables also satisfy the standard anticommutation relations:

\[
\{ \phi_k, \phi_l \} = 0, \quad \{ \phi_k^+, \phi_l^+ \} = 0, \quad \{ \phi_k, \phi_l^+ \} = \delta_{kl}.
\]

In terms of the Jacobi variables the supercharges (2) can be rewritten as:

\[
Q^\pm = Q_C^\pm + q^\pm;
\]

\[
Q_C^\pm \equiv \frac{1}{\sqrt{2}} \phi_N^\pm \left( \pm \frac{\partial}{\partial y_N} + W_C' \right); \quad q^\pm \equiv \frac{1}{\sqrt{2}} \sum_{b=1}^{N-1} \phi_{b}^\pm \left( \pm \frac{\partial}{\partial y_b} + \frac{\partial}{\partial y_b} W \right).
\]

Because

\[
\{ Q_C^\pm, q^\pm \} = 0,
\]

the superhamiltonian \( H_S \), acting in a \( N \)-dimensional superspace, describes two non-interacting supersymmetric quantum systems:

\[
H_S = \{ Q^+, Q^- \} = \{ q^+, q^- \} + \{ Q_C^+, Q_C^- \} \equiv h + H_C,
\]

where

\[
h = \frac{1}{2} \sum_{b=1}^{N-1} \left( -\frac{\partial^2}{\partial y_b^2} + \left( \frac{\partial w}{\partial y_b} \right)^2 - \frac{\partial^2 w}{\partial y_b^2} \right) + \sum_{b,c=1}^{N-1} \phi_{b}^+ \phi_c \frac{\partial^2 w}{\partial y_b \partial y_c},
\]

\[
H_C = \frac{1}{2} \left( -\frac{\partial^2}{\partial y_N^2} + (W_C')^2 - W_C'' \right) + \phi_N^+ \phi_N W_C''
\]

are \((N-1)\)- and 1-dimensional SUSY Hamiltonians, respectively.

The Hamiltonian \( h \) acting in the fermionic Fock space:

\[
\phi_{b_1}^+ \ldots \phi_{b_M}^+ |0>; \quad b_i < b_j \text{ for } i < j; \quad M < N,
\]
generated by fermionic creation operators $\phi_b^+$, conserves the corresponding fermionic number. Therefore, in the basis (23) it has a block-diagonal form:

$$h = \text{diag}(h^{(0)}, ..., h^{(N-1)})$$

where matrix operator $h^{(M)}$ of dimension $C^M_{N-1} \times C^M_{N-1}$ is the component of $h$ in the subspace with fixed fermionic number $M$.

In the same basis the supercharge $q^+$ changes the fermionic number from $M$ to $M + 1$ and has the following under-diagonal structure:

$$q^+ = \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 \\
q_{(0,1)}^+ & 0 & \cdots & 0 & 0 \\
0 & q_{(1,2)}^+ & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & q_{(N-2,N-1)}^+
\end{pmatrix}.$$  (24)

Similarly, $q^- = (q^+)^\dagger$ is an over-diagonal matrix with nonzero elements $q_{(-1,M)}^+ = (q_{(M+1,M)}^-)^\dagger$.

Analogously, $H_C$ has a diagonal form $H_C = \text{diag}(H_C^{(0)}, H_C^{(1)})$ in a basis $(|0>, \phi_N^+|0>)$ and conserves the number of fermions $\phi_N$. In this case $H_C^{(0)}, H_C^{(1)}$ are scalar (non-matrix) Hamiltonians. The one-dimensional supercharges $Q_C^+, Q_C^-$ are a partial case of (24) with off-diagonal components $Q_{C(0,1)}^+, Q_{C(1,0)}^-$. Superinvariance of the superhamiltonian corresponds, in components, to the intertwining relations (2) which can now be decomposed as:

$$hq^+ = q^+h \iff h^{(M+1)}q_{(M,M+1)}^+ = q_{(M,M+1)}^+ h^{(M)}$$  (25)

$$q^- h = h q^- \iff q_{(M+1,M)}^- h^{(M+1)} = h^{(M)} q_{(M+1,M)}^-$$  (26)

$$H_C Q_C^+ = Q_C^+ H_C \iff H_C^{(1)} Q_{C(0,1)}^+ = Q_{C(0,1)}^+ H_C^{(0)}$$  (27)

$$Q_C H_C = H_C Q_C^- \iff Q_{C(1,0)}^- H_C^{(1)} = H_C^{(0)} Q_{C(1,0)}^-$$  (28)

These intertwining relations lead to the important connections between spectra and eigenfunctions of “neighbouring” Hamiltonians, whose fermionic numbers differ by 1. In particular, $q_{(M,M+1)}^+ (q_{(M,M+1)}^-)$ maps eigenfunctions of $h^{(M)}$ onto those of $h^{(M+1)}(h^{(M-1)})$ with the same energy (see details in 2).

In the total fermionic Fock space

$$\phi_{b_1}^+ \cdots \phi_{b_M}^+ |0>, \quad \phi_{b_1}^+ \cdots \phi_{b_M}^+ \phi_N^+ |0> ; \quad M < N$$  (29)

the superhamiltonian $H_S$ commutes with the operators $\sum_{b = 1}^{N-1} \phi_b^+ \phi_b$ and $\phi_N^+ \phi_N$ and therefore conserves the fermionic numbers of both $\phi_b$ and $\phi_N$, separately. Therefore, in this basis it has a block-diagonal form, too:

\footnote{Let us note that the eigenfunctions of $h^{(M)}$ and $h^{(M+2)}$ are not connected directly by supercharges $q^\pm$, contrary to the hypothesis of the paper \cite{2} in the context of Calogero-like models.}
\[ H_S = \begin{pmatrix}
  h^{(0)} + H_C^{(0)} \\
  & \ddots \\
  & & h^{(N-1)} + H_C^{(0)} \\
  & & & h^{(0)} + H_C^{(1)} \\
  & & & & \ddots \\
  & & & & & h^{(N-1)} + H_C^{(1)}
\end{pmatrix}
\] (30)

where \( h^{(M)} \), \( H_C^{(0),(1)} \) were determined above.

It is important that, due to (19), in the intertwining relations (25)-(28) one can replace \( h^{(M)} \), \( H_C^{(0),(1)} \) by the components (30) of \( H_S \).

3. Internal structure of the components of the superhamiltonian

In the Eq. (16) the superhamiltonian \( H_S \) was written in the coordinates \((x_i, \psi_j)\) in terms of the fermionic permutation operator \( \hat{K}_{ij} \). In this section the structure of the blocks of \( H_S \) (30) in the basis (29) will be investigated. In this basis the components of \( H_S \) have the form:

\[ H_S^{(M)} = -\frac{1}{2} \Delta + \frac{1}{2} \sum_{j=1}^{N} \left( \frac{\partial w}{\partial x_j} \right)^2 + \frac{1}{2} \sum_{i \neq j}^{N} B_{ij}^{(M)} \partial_i \partial_j w + \frac{1}{2} \left( (W_C')^2 \mp W_C'' \right), \] (31)

where the matrices \( B_{ij}^{(M)} \) represent\(^9\) the operator \( \hat{K}_{ij} \) in the Fock subspace with fermionic number \( M \). Signs \( \mp \) in (31) correspond to the components \((h^{(M)} + H_C^{(0),(1)})\) of the superhamiltonian. From this moment on we will consider the components of \( H_S \) only in the form (31), i.e., in terms of the coordinates \((x_i, \phi_i^+)\). We prefer \( \phi_i^+ \) to \( \psi_i^+ \), because in terms of \( \phi_i^+ \) the block structure of \( H_S \) is more detailed\(^2\)(2\(N\) blocks instead of \( N+1 \)). The variables \( x_i \) are preferable to \( y_i \), because \( x_i \) represent the coordinates of physical particles. It is especially important in the cases when the particles are identical or the interaction is pairwise.

We move on to determining the matrices\(^h\) \( B_{ij}^{(M)} \). The relations (9),(10) can be rewritten as

\(^9\) It can be checked that \( \hat{K}_{ij} \) conserves the fermionic numbers of both \( \phi_b \) and \( \phi_N \), so it has the same block-diagonal structure in the basis (29) as \( H_S \).

\(^h\) Let us stress that \( B_{ij}^{(M)} \) is not a matrix element, but a whole matrix of dimension \( C_N^M \times C_N^M \).
\[ \hat{K}_{ij} \psi^+_k = \sum_l T_{(ij)lk} \psi^+_l \hat{K}_{ij}, \]  

\[ T_{(ij)lk} \equiv \delta_{lk} - \delta_{li} \delta_{kj} - \delta_{lj} \delta_{ki} + \delta_{li} \delta_{kj} + \delta_{lj} \delta_{ki}. \]  

Applying this commutation rule to a state \( |1\rangle \) \( M \) times, we get:

\[ \hat{K}_{ij} \psi^+_{k_1} \ldots \psi^+_{k_M} |0> = \sum_{l_1, \ldots, l_M} T_{(ij)l_1k_1} \ldots T_{(ij)l_Mk_M} \psi^+_{l_1} \ldots \psi^+_{l_M} |0> . \]  

From the partial case of \( (34) \) with \( M = 1 \) one can see that \( T_{(ij)lk} \) is a matrix element of the permutation operator \( \hat{K}_{ij} \) between one-fermionic states.

Thus, \( \hat{K}_{ij} \) realizes a tensor representation of rank \( M \) of the symmetric group \( S_N \) of permutations of \( \psi^+_i \) on the states \( |1\rangle \) with fixed fermionic number \( M \). These states are obviously antisymmetric.

Substituting \( \psi^+_i = \sum_k R_{kl} \phi^+_k \) into \( (34) \), one obtains:

\[ \hat{K}_{ij} \phi^+_1 \ldots \phi^+_m |0> = \sum_{m_1, \ldots, m_M} \tilde{T}_{(ij)m_1n_1} \ldots \tilde{T}_{(ij)m_Mn_M} \phi^+_{m_1} \ldots \phi^+_{m_M} |0> , \]  

where

\[ \tilde{T}_{(ij)mn} \equiv \sum_{k,l} R_{mk} T_{(ij)kl} R_{nl}. \]  

Note that \( \hat{K}_{ij} \phi^+_N = \phi^+_N \hat{K}_{ij} \), so it is enough to consider \( \hat{K}_{ij} \) in the subspace

\[ \phi^+_{b_1} \ldots \phi^+_{b_M} |0> ; \ M < N. \]  

It is therefore possible to rewrite \( (33) \) as

\[ \hat{K}_{ij} \phi^+_{a_1} \ldots \phi^+_{a_M} |0> = \sum_{b_1, \ldots, b_M} \tilde{T}_{(ij)b_1a_1} \ldots \tilde{T}_{(ij)b_Ma_M} \phi^+_{b_1} \ldots \phi^+_{b_M} |0> . \]  

Thus, in the space spanned by the states \( (37) \) the operators \( K_{ij} \) and matrices \( B_{ij}^{(M)} \) also realize a tensor representation of rank \( M \) of the symmetric group \( S_N \) of permutations of \( \psi^+_i \) (the states \( (37) \) are also antisymmetric).

In the Appendix we prove by induction the

**Theorem:** the representation \( (38) \) of the group \( S_N \) of permutations of \( \psi^+_i \) is irreducible and corresponds to the Young tableau \[ (N - M, 1, \ldots, 1) \equiv (N - M, 1^M). \]

\[ \text{It should be stressed that } \hat{K}_{ij} \text{ in } (35) \text{ permutes } \psi^+_k, \text{ not } \phi^+_k. \]

\[ \text{The standard notation } (\lambda) \text{ for a Young tableau containing } \lambda_i \text{ cells in the } i\text{-th line is } (\lambda_1, \ldots, \lambda_n); \text{ if the tableau contains } m \text{ identical lines with } \mu \text{ cells, it is denoted by } (\ldots, \mu^m, \ldots). \]
It is clear that the $2N$ blocks $(h^{(M)} + H^{(0),(1)}_{C})$ in (30) are "subblocks" of the $N + 1$ components of $H_{S}$ that would be obtained if we developed a standard supersymmetric formalism $[2]$ for $H_{S}$ in the coordinates $x_{i}$, $\psi^{+}_{i}$. The natural question is whether there exist even smaller "subblocks", or the blocks (31) are "elementary". Because for any $M$, due to the statement of the Theorem, the matrices $B^{(M)}_{ij}$ realize an irreducible representation of $S_{N}$, they cannot be simultaneously made block-diagonal by a change of basis. In general case, when all their coefficients $\partial_{i}\partial_{j}w$ are independent, it means that $H^{(M)}_{S}$ cannot in turn be made block-diagonal by any change of basis. Thus, the blocks (31) are "elementary".

According to (25)-(28), these elementary blocks of $H_{S}$ are intertwined by the components $q^{+}_{(M,M+1)}$, $q^{+}_{(M,M+1)}$, $Q^{+}_{(0,1)}$, $Q^{+}_{(1,0)}$ of supercharges. Let us now consider the properties of $q^{+}_{(M,M+1)}$, $q^{+}_{(M,M+1)}$ in the context of permutation group $S_{N}$.

Being the components of

$$q^{+} = \sum_{b=1}^{N-1} \phi^{+}_{b} a^{+}_{b}; \quad a^{+}_{b} = \frac{1}{\sqrt{2}}(\partial/\partial y_{b} + \partial w/\partial y_{b}),$$

(39)

the operators $q^{+}_{(M,M+1)}$ map the eigenfunctions $\Psi^{(M)}$ of the subspace with fermionic number $M$ into the eigenfunctions $\Psi^{(M+1)}$ of the subspace with fermionic number $M + 1$. Let $\Psi^{(M)}$ be the components of $\Psi^{(M)}$ in some basis of the subspace (37). The exact form of the basis is unimportant, though it is necessary that the matrices $B^{(M)}_{ij}$ and the Clebsh-Gordan coefficients introduced below be defined in this basis too. Because the dimension of the subspace (37) is $C^{2M}_{N-1}$, $\nu$ ranges from 1 to $C^{2M}_{N-1}$.

Then the operator $q^{+}_{(M,M+1)}$ takes a matrix form:

$$\Psi^{(M+1)} = (q^{+}_{(M,M+1)})_{\mu}^{(M+1)} \Psi^{(M)}_{\mu} = \sum_{\nu=1}^{C^{2M}_{N-1}} (q^{+}_{(M,M+1)})_{\mu\nu} \Psi^{(M)}_{\nu}.$$  

(40)

In the same way,

$$(\phi^{+}_{b}\Psi^{(M)})_{\mu} = \sum_{\nu=1}^{C^{2M}_{N-1}} (\phi^{+}_{b})_{\mu\nu} \Psi^{(M)}_{\nu}.$$  

(41)

Therefore (33) can be rewritten as

$$(q^{+}_{(M,M+1)})_{\mu\nu} = \sum_{b=1}^{N-1} a^{+}_{b} (\phi^{+}_{b})_{\mu\nu}.$$  

(42)

We know that $\phi^{+}_{b}$ is transformed under permutations of $\psi^{+}_{i}$ as an irreducible representation of $S_{N}$ with a Young tableau $(N - 1, 1)$. Therefore, the matrix element
$(\phi_b^+)^\mu\nu = (M \mu, 1 b | M + 1 \nu)$, where $(M \nu, 1 b | M + 1 \mu)$ are Clebsh-Gordan coefficients which correspond to the transition \cite{k}: $(N - M, 1^M) \times (N - 1, 1) \to (N - M - 1, 1^{M+1})$. In the notations of \cite{16} the symbols $M, M + 1$ in the Clebsh-Gordan coefficients correspond to the representations of $S_N$ with the Young tableaux $(N - M, 1^M), (N - M - 1, 1^{M+1}), (N - 1, 1)$, respectively. Finally,

$$\left( q_{(M, M+1)^+}^+ \right)_{\mu\nu} = \sum_{b=1}^{N-1} a_b^+ \cdot (M \nu, 1 b | M + 1 \mu). \quad (43)$$

Similarly,

$$\left( q_{(M+1, M)^-}^- \right)_{\rho\sigma} = \sum_{b=1}^{N-1} a_b^- \cdot (M + 1 \sigma, 1 b | M \rho); \quad a_b^- = (a_b^+)^\dagger \quad (44)$$

In the $x_i$ coordinates, $a_b^\pm = \sum_{l=1}^N R_{bl} A_l^\pm$ where $A_l^\pm \equiv \frac{1}{\sqrt{2}}(\pm \partial_t + \partial_x^l)$. Substituting $H_S^{(M)}$ and (43)-(44) into the intertwining relations (23)-(26), one obtains:

$$\sum_{b=1}^{N-1} \sum_{\mu=1}^{C^{M+1}_{N-1}} \sum_{l=1}^N H_{\rho\mu}^{(M+1)} R_{bl} A_l^+ (M \sigma, 1 b | M + 1 \mu) = \sum_{c=1}^{N-1} \sum_{\nu=1}^{C^M_{N-1}} \sum_{m=1}^N \sum_{l=1}^N \sum_{\epsilon=1}^{N-1} \sum_{\delta=1}^{C^{M+1}_{N-1}} H_{\rho\mu}^{(M)} R_{cm} A_m^- (M + 1 \epsilon, 1 c | M \delta). \quad (45)$$

Now it follows from (15)-(16) that the eigenfunctions $\Psi^{(M)}$ and $\Psi^{(M+1)}$ are connected by

$$\Psi^{(M+1)}_{\mu} = \sum_{b=1}^{N-1} \sum_{\nu=1}^{C^M_{N-1}} R_{bl} A_l^+ (M \nu, 1 b | M + 1 \mu) \Psi^{(M)}_{\nu}; \quad (47)$$

$$\Psi^{(M)}_{\rho} = \sum_{c=1}^{N-1} \sum_{m=1}^N \sum_{\nu=1}^{C^{M+1}_{N-1}} R_{cm} A_m^- (M + 1 \sigma, 1 c | M \rho) \Psi^{(M+1)}_{\nu}. \quad (48)$$

The relations (27),(28) remain unchanged, because the supercharges $Q^+_{(0,1)}, Q^-_{(1,0)}$ are scalar operators.

$k\times$ denotes the interior product of Young tableaux.
4. Examples.

4.1. 3-particle supersymmetrical system

In the case of $N = 2$ the superhamiltonian $h$ corresponds to the standard one-dimensional SUSY QM. For the simplest non-trivial case of $N = 3$ the superhamiltonian \(^{(30)}\) consists of six components: four scalar and two $2 \times 2$ matrix Schrödinger-type operators. Their spectra and eigenfunctions are connected with each other due to SUSY intertwining relations.

Let us consider a simple system, generated by the superpotential

$$W = -\ln \left( 3 + a \sum_{j<k}^3 (x_j - x_k)^2 \right) - \frac{a^2}{2} \sum_{i=1}^3 x_i^2; \quad a > 0. \quad (49)$$

This superpotential allows the separability of CMM \((7)\) with

$$W_C = -\frac{a^2}{2} y_3^2; \quad (50)$$

$$w = -\ln \left( 3 + a \sum_{j<k}^3 (x_j - x_k)^2 \right) - \frac{a^2}{6} \sum_{j<k}^3 (x_j - x_k)^2 \quad (51)$$

Two of scalar blocks \((31)\) of the corresponding superhamiltonian will then take the form of 3-dimensional harmonic oscillator with well-known spectrum:

$$H^{(0)}_S = \frac{1}{2} \left( \sum_{i=1}^3 \partial_i^2 + a^2 \sum_{i=1}^3 x_i^2 \right) \pm \frac{a}{2}. \quad (52)$$

Two other scalar components of \((30)\) are not exactly solvable:

$$H^{(2)}_S = \frac{1}{2} \left( -\Delta + a^2 \sum_{i=1}^3 x_i^2 \right) - \frac{36a}{(3 + a \sum_{j<k}^3 (x_j - x_k)^2)^2} \pm \frac{a}{2}. \quad (53)$$

Nevertheless, SUSY intertwining relations \((15) - (16)\) give us the opportunity to find the part\(^{\dagger}\) of spectrum of both matrix components \((31)\):

$$H^{(1)}_S = \frac{1}{2} \left( -\Delta + a^2 \sum_{i=1}^3 x_i^2 \right) - \frac{18a + 12a^2 \sum_{i<j}^3 B_{ij}^{(1)} (x_i - x_j)^2}{(3 + a \sum_{j<k}^3 (x_j - x_k)^2)^2} \pm \frac{a}{2}, \quad (54)$$

\(^{\dagger}\)In some sense, this situation resembles so called quasi-exactly-solvable models \((17)\), for which only a part of eigenstates and eigenfunctions is known.
where matrices $B^{(1)}_{ij}$ realize a simple irreducible representation of group $S_3$:

$$B^{(1)}_{12} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}; \quad B^{(1)}_{23} = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}; \quad B^{(1)}_{13} = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}.$$  

These Hamiltonians (52)-(53) are intertwined (see (45), (46)) by the supercharges (43),(44), where

$$A^\pm_l = \frac{1}{\sqrt{2}} \left( \pm \partial_l - \frac{2a(3x_l - \sqrt{3}y_3)}{3 + a \sum_{j<k}(x_j - x_k)^2} - a(x_l - \sqrt{3}y_3) \right). \quad (55)$$

The Clebsh-Gordan coefficients can be found, for example, in [16]. In the case of $N = 3$ one can write them explicitly:

$$\begin{align*}
(0 \nu, 1 b|1 \mu) &= (1 \mu, 1 b|0 \nu) = \delta^1_\mu \delta^2_b + \delta^2_\mu \delta^1_b \quad \text{where } \nu = 1; \\
(2 \sigma, 1 b|1 \rho) &= (1 \rho, 1 b|2 \sigma) = \delta^1_\rho \delta^2_b - \delta^2_\rho \delta^1_b \quad \text{where } \sigma = 1. \quad (56)
\end{align*}$$

In these expressions $\nu = 1$ and $\sigma = 1$ means that these indices span the basis of the representations of $S_3$, corresponding to the Young tableaux (3) and (1^3), respectively. These representations are one-dimensional.

The generalization onto higher $N$ is straightforward. Let us note, that the above approach can also be applied without any change to the systems, which are not symmetric under permutations of $x_i$.

4.2. $N$-particle supersymmetric systems with a pairwise interaction.

If we are interested in the scalar and matrix Hamiltonians $H^{(M)}_S$ with a pairwise interaction\textsuperscript{m}, it is necessary (but insufficient) to consider superpotentials such that\textsuperscript{n}

$$\partial_i \partial_j w = f(x_i - x_j); \ i \neq j, \quad (57)$$

where $f(x)$ is some real function. It means that

$$w = \sum_{i<k}^N U(x_i - x_j) + \sum_{j=1}^N h(x_j), \quad (58)$$

where $U(x), h(x)$ are also real functions.

\textsuperscript{m} We restrict ourselves here by considering the superpotentials $w$ that are symmetric under permutations of $x_i$, though this approach admits direct generalization to non-symmetric superpotentials with separable CMM.

\textsuperscript{n} We imply here that $W_C = 0$, except for the well-known Calogero model (see below).
We will restrict ourselves further by considering only the case of
\[ w = \sum_{i<j}^N U(x_i - x_j); \quad U(x) = U(-x). \] (59)

For such \( w \) components \( H_S^{(M)} \) (see (61)) have the form
\[
H_S^{(M)} = \frac{1}{2} \left[ -\Delta + \sum_{i\neq l}^N (U'_l)^2 + \sum_{i\neq l_1 \neq l_2 \neq i} U'_{l_1} U'_{l_2} - \sum_{i \neq j}^N B_{ij}^{(M)} U''_{ij} \right],
\] (60)
where \( U_{ij} \equiv U(x_i - x_j). \)

These matrix Hamiltonians are intertwined by relations (45), (46), where
\[ A_i^\pm = \frac{1}{\sqrt{2}} (\pm \partial_i + \sum_{k \neq l}^N U'(x_l - x_k)). \] (61)

For \( H_S^{(M)} \) to be pairwise, it is also necessary [5], [7], that
\[
sum_{i \neq l_1 \neq l_2 \neq i} U'_{l_1} U'_{l_2} \text{ decompose into a sum of pairwise terms. Therefore there should exist a real function } v_0(x) :
\]
\[ U'(A)U'(B) + U'(A)U'(C) + U'(B)U'(C) = v_0(A) + v_0(B) + v_0(C). \] (62)

Then the scalar term in (60) has the form:
\[
\frac{1}{2} \sum_{i \neq l}^N \left[ (U'(x_i - x_l))^2 + v_0(x_i - x_l) \right] \equiv \sum_{i \neq l}^N V(x_i - x_l).
\]

All solutions of (62) were found by Calogero [18]. They are given in the Table (See [7]). So, the components (60) are now pairwise and take the form:
\[
H_S^{(M)} = \frac{1}{2} \left[ -\Delta + \sum_{i \neq l}^N V(x_i - x_l) - \sum_{i \neq j}^N B_{ij}^{(M)} U''_{ij} \right].
\] (63)

Let us note, that to obtain standard Calogero model [4] \( (U(x) = ax^2/2 + b \ln x) \), we have to add into (63) the terms:
\[
\frac{1}{2}(W_C' + W_C'') = \frac{1}{2}(a^2N^2y_N^2 \mp aN),
\]
corresponding to nonzero \( W_C(y_N) = \frac{1}{2}aNy_N^2 \).

In the cases of superpotentials from the Table, which correspond to Calogero and Sutherland models, spectrum of the superhamiltonian \( H_S \) was obtained in [8], [12]. For the same superpotentials, components (63) of the superhamiltonian coincide with (also exactly solvable) matrix generalizations of Calogero and Sutherland models [9], [11] in partial case of the Young tableaux \((N-M,1^M)\). These tableaux were obtained in the Theorem in the present paper (see Appendix).
Table

| $U(x)$                              | $V(x)$                              | $U''(x)$                             |
|-------------------------------------|-------------------------------------|--------------------------------------|
| $\frac{1}{2}ax^2 + b \ln(x)$       | $\frac{b(b+1)}{x^2} + \frac{3}{2}a^2x^2 + 3ab + a$ | $a - \frac{b}{x^2}$                  |
| $a|x|$                              | $a^2 \text{sgn}^2 x + a\delta(x) - \frac{1}{3}a^2$ | $2a\delta(x)$                       |
| $a \ln \sin x$                    | $\frac{a(a-1)}{\sin^2 x} - \frac{4}{3}a^2$     | $-\frac{a}{\sin^2 x}$               |
| $a \ln \sinh x$                   | $\frac{a(a-1)}{\sinh^2 x} + \frac{2}{3}a^2$    | $-\frac{a}{\sinh^2 x}$              |
| $\ln |\theta_1 (\frac{\pi x}{2\omega_1} | (\frac{\pi x}{2\omega_2})|$ | $(a + \frac{1}{2})P(x) + \frac{\zeta(\omega)}{\omega}$ | $aP(x) - \frac{\zeta(\omega)}{\omega}$ |

$\zeta(x)$ is the Weierstrass $\zeta$ function; $\zeta'(x) = -P(x)$ and $\omega$ is half-period \cite{5}.

5. **Appendix**

In this Appendix, we will prove the following

**Theorem**: the operator $\hat{K}_{ij}$ realizes on the states $\{|\psi_i^+\rangle\}$ the irreducible representation of the group $S_N$ of permutations of $\psi_i^+$, corresponding to the Young tableau $(N - M, 1^M)$.

At first, it is necessary to prove the Lemma, corresponding to the partial case of the Theorem for $M = 1$:

**Lemma**: the operator $\hat{K}_{ij}$ (and matrices $\tilde{T}_{ij}$ \cite{44}), acting on the states $\phi_0^+|0\rangle$, realize the irreducible representation of the group $S_N$ of permutations of $\psi_i^+$, corresponding to the Young tableau $(N - 1, 1)$.

Proof: by induction on $N$.

For $N = 2$ we have only one Jacobi coordinate: $\phi_1^+ = \frac{1}{\sqrt{2}}(\psi_1^+ - \psi_2^+)$. Obviously, the state $\phi_1^+|0\rangle$ is transformed as a representation of $S_2$, with a Young tableau $(1^2)$.

Let us suppose that $\phi_i^+|0\rangle, \ldots, \phi_{N-2}^+|0\rangle$ form a representation of the group $S_{N-1}$ of permutations of $\psi_i^+$; $i' < N$, corresponding to a Young tableau $(N - 2, 1)$. It is to be proved that $\phi_1^+|0\rangle, \ldots, \phi_N^+|0\rangle$ form a representation of the group $S_N$ of permutations of $\psi_i^+$, corresponding to a Young tableau $(N - 1, 1)$.

(i) We follow the method from the book \cite{49} of construction of the irreducible representations of $S_N$ when the representations of $S_{N-1}$ are known\cite{51}. It uses the fact

\[ \text{Here and below, the (physical) coordinates } x_i, \psi_i \text{ of the first } N - 1 \text{ particles are denoted by indices } i', j', k', \ldots \text{ Let us stress that they are not the Jacobi coordinates } y_a, \phi_a. \]

\[ \text{In } \cite{49} \text{ another numeration of basis vectors was used.} \]
that arbitrary permutation $\hat{K}_{ij}$ of $\psi^+_i, \psi^+_j$ is a combination of the permutations $\hat{K}_{i'j'}$ where $i', j' < N$ and $\hat{K}_{N-1,N}$. The only nontrivial case here is

$$\hat{K}_{j'N} = \hat{K}_{N-1,N} \hat{K}_{j',N-1} \hat{K}_{N-1,N}. \quad (64)$$

The method of [16], applied to a representation with a Young tableau $(N-1,1)$ (we do not reproduce here the long proof) yields the following result: the subgroup $S_{N-1}$ of $S_N$ consisting of the permutations $\hat{K}_{i'j'}$ is realized in this representation by $(N-1) \times (N-1)$ matrices

$$\begin{pmatrix} U_{(i'j')} & 0 \\ 0 & 1 \end{pmatrix}, \quad (65)$$

where $U_{(i'j')}$ are the matrices of representation of $S_{N-1}$ corresponding to a Young tableau $(N-2,1)$; their components are $U_{(i'j')ab}$, but here and below we will not write the indices $a, b$ for brevity. The permutation $\hat{K}_{N-1,N}$ is realized in this representation by the $(N-1) \times (N-1)$ matrix

$$\begin{pmatrix} 1 & \cdots & 1 \\ \frac{1}{N-1} & \frac{\sqrt{N(N-2)}}{N-1} & -\frac{1}{N-1} \end{pmatrix}. \quad (66)$$

(ii) It is easy to show that in the terms of the Jacobi variables the permutations of $\psi^+_i$ are realized by the following matrices:

$$\begin{pmatrix} \tilde{T}_{(i'j')}^{(N-1)} & 0 \\ 0 & 1 \end{pmatrix}, \quad (67)$$

where $\tilde{T}_{(i'j')}^{(N-1)}$ is equal to $\tilde{T}_{(ij)}$ from the previous step of induction. Because of the assumption of induction, $\tilde{T}_{(i'j')}^{(N-1)} = U_{(i'j')}$. The permutation $\hat{K}_{N-1,N}$ is realized (see (33)) by the following matrix: $\tilde{T}_{(N-1,N)bc} \equiv \sum_{j,l=1}^{N} R_{bj} T_{(N-1,N)jl} R_{cl}$. Substituting $T_{(N-1,N)jl}$ from (33) ($R$ is the same as above) we obtain:

$$\tilde{T}_{(N-1,N)} = \begin{pmatrix} 1 & \cdots & 1 \\ \frac{1}{N-1} & \frac{\sqrt{N(N-2)}}{N-1} & -\frac{1}{N-1} \end{pmatrix} \quad (68)$$
Permutation $\hat{K}_{bN}$ can be decomposed into a combination of these. Thus, the proof of the Lemma is completed.

Now we can prove the Theorem by induction on $M$.

For $M = 1$ the statement of the Theorem is satisfied due to Lemma.

Let us suppose that, for some fixed $M$, on the states $\phi_{b_1}^+ \ldots \phi_{b_M}^+|0>$ the irreducible representation of the group $S_N$ of permutations of $\psi^+_i$ with $i \leq N$, corresponding to the Young tableau $(N - M, 1^M)$, is realized.

Then we use this assumption to prove that on the states $\phi_{b_1}^+ \ldots \phi_{b_{M+1}}^+|0>$ the irreducible representation of the group $S_N$, corresponding to the Young tableau $(N - M - 1, 1^{M+1})$, is realized.

(i) As we have mentioned above, (37) is transformed under permutations from $S_N$ as an antisymmetric tensor. Such tensor can be provided with a Young tableau, describing the symmetry of its indices, namely, $[1^M]$ (we deliberately use different brackets).

The assumption of induction affirms that (37) transforms under $S_N$ as an irreducible representation with a Young tableau $(N - M, 1^M)$.

Let us consider a tensor product of the states (37) and $\phi_{b_{M+1}}^+|0>$ and show that it contains $\phi_{b_1}^+ \ldots \phi_{b_{M+1}}^+|0>$.

The tensor product will contain the tensors whose index structure corresponds to the Young tableaux, contained in the so-called exterior product of Young tableaux, corresponding to the index structure of the states (37) and $\phi_{b_{M+1}}^+|0>$, namely, $[1^M] \otimes [1]$. If some Young tableau $[D]$ is contained in $[1^M] \otimes [1]$, then, the corresponding tensor representation of $S_N$ is contained in the tensor product of the tensor representation of $S_N$ corresponding to (37) and one corresponding to $\phi_{b_{M+1}}^+|0>$. But these representations can also be considered as irreducible representations of $S_N$ with Young tableaux $(N - M, 1^M)$ and $(N - 1, 1)$. The tensor representations from the tensor product may also be decomposed into a direct sum of irreducible representations of $S_N$ corresponding to some Young tableaux. The last Young tableaux form a so-called interior product: $(N - M, 1^M) \times (N - 1, 1)$.

So the Young tableaux obtained after decomposition of a tensor corresponding to $[D]$ are contained in $(N - M, 1^M) \times (N - 1, 1)$.

\[\text{Let us stress that every state (37) is described by two principally different Young tableaux: [1^M], denoting its index structure as antisymmetric tensor, and ($\lambda_1, \ldots, \lambda_N$), describing it as an irreducible representation of the symmetric group $S_N$ (the notations are taken from [16]).}\]

\[\text{\textsuperscript{r} denotes the exterior and interior products of Young tableaux, respectively.}\]
(ii) It is shown in [16] that for any Young tableau \([\lambda_1 \ldots \lambda_k]\)

\[ [\lambda_1 \ldots \lambda_k] \otimes [1] = \sum_{i=1}^{k} [\lambda_1, \ldots, \lambda_{i-1}, \lambda_i + 1, \lambda_{i+1} \ldots \lambda_k]. \tag{69} \]

So, for the state (37) we get:

\[ [1^M] \otimes [1] = [1^{M+1}] \oplus [2, 1^{M-1}]. \tag{70} \]

The state \(\phi_{b_1}^+ \ldots \phi_{b_{M+1}}^+ |0 >\) has an index structure described by the Young tableau \([1^{M+1}]\) and is a direct sum of irreducible representations of \(S_N\) corresponding to some Young tableaux. We will further denote the set of these tableaux as \((A)\). Taking into account the considerations from the point (ii), we can state that \((N - M, 1^M) \times (N - 1, 1) = (A) \oplus \ldots\).

(iii) It is known [16] that the interior product of every Young tableau with \((N - 1, 1)\) contains only the Young tableaux differing from the initial one by no more than the position of one cell. (Let us remind that all the Young tableaux describing the irreducible representations of \(S_N\) contain exactly \(N\) cells each.) Therefore, \((N - M, 1^M) \times (N - 1, 1)\) may contain only the following Young tableaux: \((N - M, 1^M), (N - (M + 1), 1^{M+1}), (N - M, 2, 1^{M-2}), (N - (M + 1), 2, 1^{M-1})\).

(iv) It is useful here to consider the state \(\phi_{b_1}^+ \ldots \phi_{b_{M+1}}^+ |0 >\). Its index structure corresponds to a Young tableau \([1^{N-1}]\). As a representation of \(S_N\), this state corresponds to a Young tableau \((1^N)\), because

\[ \phi_1^+ \ldots \phi_{N-1}^+ |0 > = \phi_1^+ \ldots \phi_{N-1}^+ \phi_N^+ |0 > = (-1)^{N-1} \phi_N \phi_1^+ \ldots \phi_{N-1}^+ |0 > = C \phi_1^+ \ldots \phi_{N-1}^+ |0 >, \tag{71} \]

where \(C\) is some nonzero constant. It is obvious that \(\hat{K}_{ij}\), acting upon (71), changes its sign.

(v) Let us consider a tensor product of \(\phi_{b_1}^+ \ldots \phi_{b_{M+1}}^+ |0 >\) with \(\phi_{b_{M+2}}^+ |0 >\), then once more a tensor product of the result with \(\phi_{b_{M+3}}^+ |0 >\), then once more a tensor product of the result with \(\phi_{b_{N-1}}^+ |0 >\), altogether \((N - M - 2)\) times. Reproducing the considerations from the point (i), especially the formulae (69),(70) we can see that this product contains a tensor with the index structure described by \([1^{N-1}]\):

\[ [1^M] \otimes [1] \otimes \ldots \otimes [1] = [1^{N-1}] \oplus \ldots. \tag{72} \]

In terms of irreducible representations of \(S_N\), (72) can be rewritten as: \((A) \times (N - 1, 1) \times \ldots \times (N - 1, 1) = (B) \oplus \ldots\) where \((B)\) denotes the Young tableaux corresponding to the state \(\phi_1^+ \ldots \phi_{N-1}^+ |0 >\), and \((A)\) is defined in (ii).
Let \((A)\) not contain the tableau \((N - (M + 1), 1^{M+1})\). As mentioned in point (iii), the interior product of every Young tableau with \((N - 1, 1)\) contains only the Young tableaux differing from the initial one by no more than the position of one cell. Therefore, the interior product of every Young tableau \((N - M - 2)\) times with \((N - 1, 1)\) contains only the Young tableaux differing from the initial one by no more than the position of \(N - M - 2\) cells. The rest three tableaux in (iii) differ from the initial one by more than \(N - M - 2\) cells because they all have less than \(M + 2\) cells in the first column. So, \((\ref{eq:2})\) is not satisfied, unless \((A)\) contains the tableau \((N - (M + 1), 1^{M+1})\).

(vi) The dimension of the space of states \(\phi_{b_1}^+ \ldots \phi_{b_{M+1}}^+ |0\rangle\) is equal to \(C_{N-1}^{M+1}\), because it is a dimension of a subspace with a fermionic number equal to \(M + 1\), when the total fermionic number is equal to \(N - 1\).

The dimension of the Young tableau \((N - (M + 1), 1^{M+1})\) (and hence the dimension of the corresponding irreducible representation of \(S_N\)) is also equal to \(C_{N-1}^{M+1}\), because the dimension is the number of different ways of placing the integer numbers from 1 to \(N\) consecutively into the \(N\) cells of the tableau, so that the number of the occupied cells not increase with the number of the line, and, placing each number into the line, we place it as close to its left end as possible.

At first, we place 1 into the corner cell. Then, the distribution of the numbers in the cells is determined unambiguously by deciding what numbers we place into the \(N - M - 2\) lateral cells. It can be done in \(\binom{N-1}{M-2} = C_{N-1}^{M+1}\) ways.

So, \((A)\) contains the tableau \((N - (M + 1), 1^{M+1})\) and nothing more. Therefore, the state \(\phi_{b_1}^+ \ldots \phi_{b_{M+1}}^+ |0\rangle\) corresponds to an irreducible representation of \(S_N\) with this Young tableau.

The proof is now completed.

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