Square-full numbers
with an even number of prime factors

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Abstract: In this article, we study the functions $\omega(n)$ and $\Omega(n)$, where $n$ is an $s$-full number. For example, we prove that the square-full numbers with $\Omega(n)$ even are in greater proportion than the square-full numbers with $\Omega(n)$ odd. The methods used are elementary.

Keywords: Square-full numbers, Arithmetical functions $\omega(n)$ and $\Omega(n)$.

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1 Introduction and preliminary notes

Let us consider the prime factorization of a positive integer $n = q_1^{s_1} \cdots q_r^{s_r}$, where $q_i$ ($i = 1, \ldots, r$) ($r \geq 1$) are the different primes in the prime factorization and $s_i$ ($i = 1, \ldots, r$) are the multiplicities or exponents. We need the following well-known arithmetical functions: $\omega(n) = r$ that is the number of different prime factors in the prime factorization of $n$, $\Omega(n) = s_1 + \cdots + s_r$ that is the total number of prime factors in the prime factorization of $n$, $u(n) = q_1 \cdots q_r$ that denotes the kernel of $n$ and $w(n) = (q_1 + 1) \cdots (q_r + 1)$. Note that $w(n)$ is the sum of the positive divisors of the kernel of $n$.

The functions $\omega(n)$ and $\Omega(n)$ were studied by G. H. Hardy and S. Ramanujan in 1917 [6]. They obtained the following formulas

$$\sum_{n \leq x} \omega(n) = x \log \log x + M x + o(x),$$
\[ \sum_{n \leq x} \Omega(n) = x \log \log x + \left( M + \sum_p \frac{1}{p(p-1)} \right) x + o(x), \]

where \( M \) is Mertens’s constant. In the same paper they define the normal order of an arithmetical function and they prove that the normal order of \( \omega(n) \) and \( \Omega(n) \) is \( \log \log n \).

Let \( \Omega_p(x) \) be the number of positive integers \( n \) not exceeding \( x \) such that \( \Omega(n) \) is even and \( \Omega_i(x) \) the number of positive integers \( n \) not exceeding \( x \) such that \( \Omega(n) \) is odd. The following asymptotic formulas are well-known

\[ \Omega_i(x) = \frac{1}{2} x + o(x), \quad \Omega_p(x) = \frac{1}{2} x + o(x). \]

That is, these two sets of positive integers have density \( 1/2 \).

Let \( \omega_p(x) \) be the number of positive integers \( n \) not exceeding \( x \) such that \( \omega(n) \) is even and \( \omega_i(x) \) is the number of positive integers \( n \) not exceeding \( x \) such that \( \omega(n) \) is odd. Recently, R. Jakimczuk [10] proved that also these two sets of positive integers have density \( 1/2 \). That is

\[ \omega_i(x) = \frac{1}{2} x + o(x), \quad \omega_p(x) = \frac{1}{2} x + o(x). \]

A number is \( h \)-full if all the distinct primes in its prime factorization have multiplicity (or exponent) greater than or equal to \( h \). If \( h = 2 \) the numbers are called square-full. The square-full numbers were studied by P. Erdős and G. Szekeres [3] and many other authors. For example, P. T. Bateman and E. Grosswald [1], A. Ivić and P. Shiu (see [8] and [9]), S. W. Golomb [5], etc. Also, recently, R. Jakimczuk [12] studied the kernel of \( h \)-full numbers. See also the reference [2]. An elementary proof on the distribution of \( h \)-full numbers is established here.

In this article, we study the functions \( \Omega(n) \) and \( \omega(n) \) on the \( h \)-full numbers. In particular, on the square-full numbers. For example, between other results, we prove that the square-full numbers \( n \) with \( \Omega(n) \) even are in greater proportion than the square-full numbers \( n \) with \( \Omega(n) \) odd.

We shall need the following theorems on the distribution of square-free numbers. In this note a square-free number will be denoted \( q_1 \).

**Theorem 1.1.** Let \( Q_1(x) \) be the number of square-free numbers not exceeding \( x \), we have

\[ Q_1(x) = \sum_{q_1 \leq x} 1 = \frac{6}{\pi^2} x + o(x). \]

Let \( Q_p(x) \) be the number of square-free \( n \) not exceeding \( x \) such that \( \Omega(n) = \omega(n) \) is even and let \( Q_i(x) \) be the number of square-free \( n \) not exceeding \( x \) such that \( \Omega(n) = \omega(n) \) is odd. We have (prime number theorem)

\[ Q_p(x) = \frac{1}{2} \frac{6}{\pi^2} x + o(x), \quad Q_i(x) = \frac{1}{2} \frac{6}{\pi^2} x + o(x). \]

**Proof.** See [7, chapter XVIII].
In this note a square-free multiple of the different and fixed primes \( q_1, \ldots, q_s \), that is multiple of the square-free \( q_1q_2 \cdots q_s \), will be denoted \( q_{q_1q_2}q_3 \).

**Theorem 1.2.** Let \( Q_{q_1 \cdots q_s}(x) \) be the number of square-free not exceeding \( x \) multiple of the different and fixed primes \( q_1, q_2, \ldots, q_s \), we have

\[
Q_{q_1q_2 \cdots q_s}(x) = \sum_{q_1q_2 \cdots q_s \leq x} 1 = \frac{6}{\pi^2} \prod_{i=1}^{s} \frac{1}{q_i + 1} x + o(x).
\]

**Proof.** See [11].

Let \((MP)_{q_1 \cdots q_s}(x)\) be the number of square-free \( n \) not exceeding \( x \) multiple of \( q_1 \cdots q_s \) such that \( \Omega(n) = \omega(n) \) is even. On the other hand, let \((MI)_{q_1 \cdots q_s}(x)\) be the number of square-free \( n \) not exceeding \( x \) multiple of \( q_1 \cdots q_s \) such that \( \Omega(n) = \omega(n) \) is odd. We have the following theorem.

**Theorem 1.3.** The following asymptotic formulas hold.

\[
(MP)_{q_1 \cdots q_s}(x) = \frac{1}{2} \frac{6}{\pi^2} \prod_{i=1}^{s} \frac{1}{q_i + 1} x + o(x),
\]

\[
(MI)_{q_1 \cdots q_s}(x) = \frac{1}{2} \frac{6}{\pi^2} \prod_{i=1}^{s} \frac{1}{q_i + 1} x + o(x).
\]

**Proof.** See [10].

**Theorem 1.4.** If \( \alpha > 0 \) the following two series of positive terms are convergent

\[
\sum_{n=1}^{\infty} \frac{1}{w(n)n^\alpha} \quad \sum_{n=1}^{\infty} \frac{1}{u(n)n^\alpha}
\]

and besides the following two equations hold

\[
\sum_{n=1}^{\infty} \frac{1}{w(n)n^\alpha} = \prod_{p} \left( 1 + \frac{1}{(p+1)(p^\alpha - 1)} \right),
\]

\[
\sum_{n=1}^{\infty} \frac{1}{u(n)n^\alpha} = \prod_{p} \left( 1 + \frac{1}{p(p^\alpha - 1)} \right),
\]

where the notation \( \prod_{p} \) means that the product runs over all positive primes \( p \).

**Proof.** We have

\[
\sum_{n=1}^{\infty} \frac{1}{w(n)n^\alpha} = \prod_{p} \left( 1 + \frac{1}{(p+1)p^\alpha} + \frac{1}{(p+1)(p^\alpha)^2} + \frac{1}{(p+1)(p^\alpha)^3} + \cdots \right)
\]

\[
= \prod_{p} \left( 1 + \frac{1}{(p+1)p^\alpha \left( 1 - \frac{1}{p^\alpha} \right)} \right) = \prod_{p} \left( 1 + \frac{1}{(p+1)(p^\alpha - 1)} \right).
\]
Now, the product
\[ \prod_p \left( 1 + \frac{1}{(p+1)(p^\alpha - 1)} \right) \]
converges to a positive number, since the series of positive terms
\[ \sum_p \frac{1}{(p+1)(p^\alpha - 1)} \]
clearly converges. The theorem is proved.

2 Main results

Let \( h \geq 2 \) be an arbitrary but fixed positive integer. A number is \( h \)-full if all the distinct primes in its prime factorization have multiplicity (or exponent) greater than or equal to \( h \). That is, the number \( q_1^{s_1} \cdots q_r^{s_r} \) is \( h \)-full if \( s_i \geq h \) \((i = 1, \ldots, r)\) \((r \geq 1)\). We shall denote a general \( h \)-full number \( n_h \). If \( h = 2 \), the numbers are called square-full. The \( h \)-kernel of the \( h \)-full number \( n_h \) we define in the form \((u(n_h))^h\) and the \( h \)-remainder in the form \((u(n_h))^h\). Note that the \( h \)-remainder is 1 if and only if the \( h \)-full number is of the form \((q_1 \cdots q_r)^h\).

Let \( A_h(x) \) be the number of \( h \)-full numbers not exceeding \( x \).

**Theorem 2.1.** Let \( h \geq 2 \) be an arbitrary but fixed positive integer. The following asymptotic formula holds
\[ A_h(x) = \sum_{n_h \leq x} 1 = \frac{6}{\pi^2} C_{0,h} x^{\frac{1}{h}} + o \left( x^{\frac{1}{h}} \right), \]  
(1)
where
\[ C_{0,h} = \sum_{n=1}^{\infty} \frac{1}{w(n) n^{\frac{1}{h}}} = \prod_p \left( 1 + \frac{1}{(p+1)(p^\alpha - 1)} \right)^{w(1) = 1}. \]  
(2)

**Proof.** Let us consider the prime factorization of a positive integer \( a \geq 2 \)
\[ a = q_1^{s_1} q_2^{s_2} \cdots q_t^{s_t}, \]
where \( q_1, q_2, \ldots, q_t \) are the different primes in the prime factorization of \( a \). We put
\[ a' = q_1 q_2 \cdots q_t \]
and
\[ a'' = (q_1 + 1)(q_2 + 1) \cdots (q_t + 1). \]
If \( a = 1 \), then we put \( a' = a'' = 1 \).

Therefore, we have (see Theorem 1.1 and Theorem 1.2)
\[ \sum_{q_i \leq x} 1 = \frac{6}{\pi^2} a'' x + o(x). \]  
(3)
Let us consider the set \( H \) of all \( h \)-full numbers \( n_h \) not exceeding \( x \). Now, let us consider the set \( T_a \) of all \( h \)-full numbers \( n_h \) not exceeding \( x \) with the same \( h \)-remainder \( a \), that is, \( T_a = \{ n_h : n_h \leq x, v_h(n_h) = a \} \). Note that if \( a_1 \neq a_2 \) we have \( T_{a_1} \cap T_{a_2} = \emptyset \), that is, the sets \( T_{a_1} \) and \( T_{a_2} \) are disjoint. Suppose that \( A_x \) (depending on \( x \)) is the greatest \( h \)-remainder among the numbers in the set \( H \). Then

\[
\bigcup_{a=1}^{A_x} T_a = H.
\]

Therefore, the sets \( T_a \) are partitions of the set \( H \). Note that some \( T_a \) can be empty.

The set of the \( h \)-kernel of the numbers in the set \( T_a \) will be denoted by \( S_a \). Hence,

\[
S_a = \left\{ q_h^a : q_h^a \leq \frac{x}{a} \right\} = \left\{ q_h^a : q_h^a \leq \frac{x/(1/h)}{a/(1/h)} \right\}.
\]  (4)

The series \( \sum_{a=1}^{\infty} \frac{1}{a^\pi} \) converges (see Theorem 1.4). Hence

\[
\sum_{a=1}^{\infty} \frac{1}{a^\pi} = C_{0,h}.
\]  (5)

We choose \( B \) such that (see Theorem 1.4)

\[
\sum_{a=B+1}^{\infty} \frac{1}{a^\pi} < \epsilon
\]  (6)

and

\[
\frac{\pi^2}{6} \sum_{a=B+1}^{\infty} \frac{1}{a^\pi} < \epsilon.
\]  (7)

Therefore, we have (see (3), (4), (5) and (6))

\[
A_h(x) = \sum_{a=1}^{A(x)} \left( \sum_{q_h^{a'} \leq x/(1/h)/a} 1 \right) = \sum_{a=1}^{B} \left( \sum_{q_h^{a'} \leq x/(1/h)/a} 1 \right) + \sum_{a=B+1}^{A(x)} \left( \sum_{q_h^{a'} \leq x/(1/h)/a} 1 \right)
\]

\[
= \sum_{a=B+1}^{A(x)} \left( \sum_{q_h^{a'} \leq x/(1/h)/a} 1 \right) = \sum_{a=1}^{B} \left( \frac{1}{a^\pi} \frac{6}{\pi^2} x^{1/h} \right) + o \left( x^{1/h} \right)
\]

\[
= \sum_{a=B+1}^{A(x)} \left( \sum_{q_h^{a'} \leq x/(1/h)/a} 1 \right) = \frac{6}{\pi^2} x^{1/h} \left( \sum_{a=1}^{B} \frac{1}{a^\pi} \right) + o \left( x^{1/h} \right)
\]

\[
= \sum_{a=B+1}^{A(x)} \left( \sum_{q_h^{a'} \leq x/(1/h)/a} 1 \right) = \frac{6}{\pi^2} x^{1/h} \left( C_{0,h} - \frac{6}{\pi^2} x^{1/h} \left( \sum_{a=B+1}^{\infty} \frac{1}{a^\pi} \right) \right)
\]

\[
+ o(1) \frac{6}{\pi^2} x^{1/h} + \sum_{a=B+1}^{A(x)} \left( \sum_{q_h^{a'} \leq x/(1/h)/a} 1 \right).
\]  (8)
Equation (8) can be written in the form

\[
\frac{A_h(x)}{\pi^x} - C_{0,h} = - \left( \sum_{a=B+1}^{\infty} \frac{1}{a^{x/2}} \right) + o(1) + \frac{\sum_{a=B+1}^{A(x)} \left( \sum_{q_0 \leq x^{(1/h)}} 1 \right)}{\frac{6}{\pi^x} x^{1/2}}. \tag{9}
\]

We have (see (8) and (7))

\[
0 \leq \sum_{a=B+1}^{A(x)} \left( \sum_{q_0 \leq x^{(1/h)}} 1 \right) \leq \sum_{a=B+1}^{A(x)} \left( \sum_{q_0 \leq x^{(1/h)}} a^{x/2} \right) \leq \sum_{a=B+1}^{A(x)} \left( \frac{x^{(1/h)}}{a^{x/2}} \right)
\leq x^{1/2} \sum_{a=B+1}^{A(x)} \frac{1}{a^{x/2}} \leq \frac{6}{\pi^2} x^{1/2} \sum_{a=B+1}^{\infty} \frac{1}{a^{x/2}} \\
\leq \epsilon \frac{6}{\pi^2} x^{1/2}. \tag{10}
\]

We choose \(x_0\) such that if \(x \geq x_0\) then \(|o(1)| < \epsilon\) in equation (9). Equations (9), (6) and (10) give

\[
\frac{A_h(x)}{\pi^x} - C_{0,h} \leq 3\epsilon.
\]

Therefore, since \(\epsilon\) is arbitrarily small, we have

\[
\lim_{x \to \infty} \frac{A_h(x)}{\pi^x} = C_{0,h}.
\]

That is (1). The theorem is proved. \(\square\)

**Remark 2.2.** If \(h = 2\) then it is well-known that the constant can be written in terms of the Riemann zeta function \(\zeta(s)\), that is, the value of the constant is \(\zeta(3/2)/\zeta(3)\). This can be obtained from our formulas (16) and (17), since

\[
\frac{6}{\pi^2} C_{0,2} = \prod_p \left( 1 - \frac{1}{p^2} \right) \left( 1 + \frac{1}{(p+1)(p^{1/2} - 1)} \right) \]

\[
= \prod_p \left( 1 - \frac{1}{p^2} + \frac{p^{1/2} + 1}{p^{3/2}} \right) = \prod_p \left( 1 + \frac{1}{p^{3/2}} \right) = \prod_p \left( \frac{1}{1 - p^{-s/2}} \right) \]

\[
= \frac{\zeta(3/2)}{\zeta(3)} = 2.1732543125...
\]

See [4, page 112].
Let $\omega_{p,h}(x)$ be the number of $h$-full numbers $n_h$ not exceeding $x$ such that $\omega(n_h)$ is even and let $\omega_{i,h}(x)$ be the number of $h$-full numbers $n_h$ not exceeding $x$ such that $\omega(n_h)$ is odd. We have the following theorem.

**Theorem 2.3.** The following asymptotic formulas hold.

\[
\omega_{p,h}(x) = \frac{1}{2} \frac{6}{\pi^2} C_{0,h,x}^{\frac{1}{h}} + o\left(\frac{x^{\frac{1}{h}}}{x}\right),
\]

\[
\omega_{i,h}(x) = \frac{1}{2} \frac{6}{\pi^2} C_{0,h,x}^{\frac{1}{h}} + o\left(\frac{x^{\frac{1}{h}}}{x}\right).
\]

**Proof.** The proof of (11) is the same as the proof of Theorem 2.1. Equation (3) is replaced by (Theorem 1.1 and Theorem 1.3)

\[
\sum_{\omega(q,a)\equiv 0 \pmod{2}} 1 = \frac{1}{2} \frac{6}{\pi^2} \frac{1}{a''} x + o(x).
\]

If $a = 1$ we put $a' = a'' = 1$. The proof of (12) is by difference using (11) and Theorem 2.1 or using the equation

\[
\sum_{\omega(q,a')\equiv 1 \pmod{2}} 1 = \frac{1}{2} \frac{6}{\pi^2} \frac{1}{a''} x + o(x).
\]

The theorem is proved. \qed

Let $\Omega_{h,r}(x)$ be the number of $h$-full numbers $n_h$ not exceeding $x$ such that $\Omega(n_h) \equiv r \pmod{h}$ ($r = 0, \ldots, h - 1$). We have the following theorem.

**Theorem 2.4.** The following asymptotic formulas hold.

\[
\Omega_{h,r}(x) = \frac{6}{\pi^2} C_{0,h,r} x^{\frac{1}{h}} + o\left(x^{\frac{1}{h}}\right) \quad (r = 0, \ldots, h - 1),
\]

where the constants $C_{0,h,r}$ are given by the series

\[
C_{0,h,r} = \sum_{\Omega(n)\equiv r \pmod{h}} \frac{1}{w(n)} \frac{1}{n^{\frac{1}{h}}} \quad (r = 0, \ldots, h - 1)
\]

and

\[
\sum_{r=0}^{h-1} C_{0,h,r} = C_{0,h}.
\]

**Proof.** Since the total number of prime factors in the $h$-kernel is multiple of $h$, the proof is the same as the proof of Theorem 2.1, where we consider only the $h$-remainder $a$ such that $\Omega(a) \equiv r \pmod{h}$. If $a = 1$ we put $a' = a'' = 1$ and $\Omega(a) = \Omega(1) = 0$, therefore $\Omega(1) \equiv 0 \pmod{h}$. The theorem is proved. \qed
Let $\Omega_{p,h}(x)$ be the number of $h$-full numbers $n_h$ not exceeding $x$ such that $\Omega(n_h)$ is even and let $\Omega_{i,h}(x)$ be the number of $h$-full numbers $n_h$ not exceeding $x$ such that $\Omega(n_h)$ is odd. We have the following theorem.

**Theorem 2.5.** If $h$ is even, then

$$\Omega_{p,h}(x) = \frac{6}{\pi^2} D_{h,0} x^{\frac{1}{h}} + o\left(x^{\frac{1}{h}}\right),$$

$$\Omega_{i,h}(x) = \frac{6}{\pi^2} D_{h,1} x^{\frac{1}{h}} + o\left(x^{\frac{1}{h}}\right),$$

where the constants are given by the series

$$D_{h,0} = \sum_{\Omega(n) \equiv 0 \pmod{2}} \frac{1}{w(n)} \frac{1}{n^{\frac{1}{h}}},$$

$$D_{h,1} = \sum_{\Omega(n) \equiv 1 \pmod{2}} \frac{1}{w(n)} \frac{1}{n^{\frac{1}{h}}},$$

and

$$D_{h,0} + D_{h,1} = C_{0,h}.$$

If $h$ is odd, then

$$\Omega_{p,h}(x) = \frac{1}{2} \frac{6}{\pi^2} C_{0,h} x^{\frac{1}{h}} + o\left(x^{\frac{1}{h}}\right),$$

$$\Omega_{i,h}(x) = \frac{1}{2} \frac{6}{\pi^2} C_{0,h} x^{\frac{1}{h}} + o\left(x^{\frac{1}{h}}\right),$$

*(13)*

**Proof.** If $h$ is even, then the total number of prime factors in the $h$-kernel is even, therefore, the proof is the same as the proof of Theorem 2.4. If $h$ is odd, in the proof of equation (13) we consider two cases.

**Case 1.** $\omega(q_{a'}) \equiv 0 \pmod{2}$ and $\Omega(a) \equiv 0 \pmod{2}$.

**Case 2.** $\omega(q_{a'}) \equiv 1 \pmod{2}$ and $\Omega(a) \equiv 1 \pmod{2}$.

Hence, the theorem is proved. $\square$

If $h = 2$ (square-full numbers), we shall prove in the next theorem that $D_{2,0} > D_{2,1}$ and consequently the proportion of square-full numbers not exceeding $x$ with a total even number of prime factors is greater than the proportion of square-full numbers not exceeding $x$ with a total odd number of prime factors.

**Theorem 2.6.** The following inequality holds.

$$D_{2,0} > D_{2,1}.$$  

*(14)*
Proof. We have
\[ \sum_{n=1}^{\infty} \frac{1}{w(n)n} = \prod_p \left( 1 + \frac{1}{(p+1)p} + \frac{1}{(p+1)p^2} + \cdots \right) = \prod_p \left( \frac{1}{1 - \frac{1}{p^2}} \right) = \frac{\pi^2}{6}. \] (15)

Let us consider the pairs \((a, b)\): \((1, 1)\), \((2, 3)\), \((2, 5)\), \((2, 7)\), \((3, 5)\), \((2, 11)\), \((3, 7)\), \((2, 13)\).

Note that by Remark 2.2 we have
\[ \frac{6}{\pi^2} D_{2,0} + \frac{6}{\pi^2} D_{2,1} = \frac{6}{\pi^2} C_{0,2} = 2.1732543125... \] (16)

Now (see (16))
\[ \frac{6}{\pi^2} D_{2,0} > \frac{6}{\pi^2} \sum_{(a, b)} \left( \sum_{n=1}^{\infty} \frac{1}{w(abn^2)} \frac{1}{\sqrt{abn^2}} \right) \]
\[ > \left( \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{w(n)} \frac{1}{n} \right) \left( 1 + \sum_{(a, b) \neq (1,1)} \frac{1}{(a+1)(b+1)} \frac{1}{\sqrt{ab}} \right) \]
\[ > \frac{1}{2} \frac{6}{\pi^2} C_{0,2} = 1.086627... \]

since by (15) we have
\[ \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{w(n)} = 1. \]

Therefore (14) holds. The theorem is proved.

3 Conclusion

In this article we have studied the distribution of \(h\)-full numbers by use of an elementary method. By use of the same elementary method we have proved theorems on the functions \(\omega(n)\) and \(\Omega(n)\) defined on the sequence of \(h\)-full numbers. In particular, if \(h = 2\) then we have obtained that the square-full numbers with \(\Omega(n)\) even are in greater proportion than the square-full numbers with \(\Omega(n)\) odd.

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