Linear Hamiltonians on homogeneous Kähler manifolds of coherent states

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Abstract

Representations of coherent state Lie algebras on coherent state manifolds as first order differential operators are presented. The explicit expressions of the differential action of the generators of semisimple Lie groups determine for linear Hamiltonians in the generators of the groups first order differential equations of motion with holomorphic polynomials coefficients. For hermitian symmetric manifolds the equations of motion are matrix Riccati equations. It is presented the simplest example of the non-symmetric space $SU(3)/S(U(1) \times U(1) \times U(1))$ where the polynomials describing the equations of motion have the maximum degree 3.

1 Introduction

In references [3, 4] it was shown that the differential action of the generators of the groups on coherent state manifolds which have the structure of hermitian symmetric spaces can be written down as a sum of two terms, one a polynomial $P$, and the second one a sum of partial derivatives times some polynomials $Q$-s, the degree of polynomials being less than 3. Our investigations on the differential action of the generators has been extended from hermitian groups acting on hermitian symmetric spaces to semisimple Lie groups acting on coherent state manifolds which admit a Kähler structure, and explicit formulas for the polynomials $P$ and $Q$-s have been given [6]. Similar investigation has been done in [13]. Explicit formulas for the simplest example of a compact non-symmetric coherent state manifold, $SU(3)/S(U(1) \times U(1) \times U(1))$, where the degree of the polynomial is already 3, have been also obtained [6]. We have formulated the problem of the differential action of the generators of the so called coherent state (shortly, CS) groups [21, 22, 23, 28] in [8]. These are groups whose quotient with the stationary groups are manifolds which admit a holomorphic embedding in a projective Hilbert space. This class of groups contains all compact groups, all simple hermitian groups, certain solvable groups and also mixed groups as the semidirect product of the Heisenberg group and the symplectic group [28].

The coherent states are a useful tool of investigation of quantum and classical systems [32]. It was shown in [3, 4] that a linear Hamiltonian in the generators of the groups implies equivalent quantum and classical evolution. It was proved that for Hermitian symmetric spaces the evolution equation generated by Hamiltonians which are...
linear in the generators of the group is a matrix Riccati equation. It is interesting to see how it looks like the corresponding equation of motion generated by linear Hamiltonians for CS-manifolds. This question is the main topic of the present investigation.

Another field of possible applications is the determination of the Berry phase [35] on CS-manifolds. In [3, 4] there were presented explicit expressions for the Berry phase for the complex Grassmann manifold. These results were used further [7] for explicit calculation of the symplectic area of geodesic triangles on the complex Grassmann manifold.

The paper is laid out as follows. In the first part we recall some facts about realization of coherent state Lie algebras by differential operators (cf. [8]). §2 contains the definition of CS-orbits, in the context of Lisiecki [21, 22, 23] and Neeb [28]. The geometry of coherent state manifolds for compact groups was previously considered in [2]. §3 deals with the so called Perelomov’s CS-vectors. In §4 we construct the space of functions on which the differential operators will act. In §5 we study the representations of Lie algebras of CS-groups by differential operators. §6 deals with the semisimple case. §6.1 recalls some standard facts about the semisimple Lie algebras. Perelomov’s coherent state vectors for semisimple Lie groups are defined in §6.2 §6.3 recalls the results established in [6]. The example of $SU(3)/S(U(1) \times U(1) \times U(1))$ is presented in §6.4.

In the second part the equations of motion associated to linear Hamiltonians in the generators of the groups are investigated. Some known facts established in references [3, 4] are recalled in §7. The next sections present firstly the example of equations of motion generated by linear Hamiltonians in the generators of the oscillator group (§8.1) and on the Riemann sphere and its non-compact dual (§8.2). The case of the complex Grassmann manifold is summarized in §8.3. The last example in §8.4 presents the equations of motions generated by the differential operators of §6.4.

We use for the scalar product the convention: $\langle \lambda x, y \rangle = \bar{\lambda} \langle x, y \rangle$, $x, y \in H, \lambda \in \mathbb{C}$.

## 2 Coherent state representations

Let us consider the triplet $(G, T, \mathcal{H})$, where $T$ is a continuous, unitary representation of the Lie group $G$ on the separable complex Hilbert space $\mathcal{H}$. Let us denote by $\mathcal{H}^\infty$ the dense subspace of $\mathcal{H}$ consisting of those vectors $v$ for which the orbit map $G \to \mathcal{H}, g \mapsto T(g).v$ is smooth. Let us pick up $e_0 \in \mathcal{H}^\infty$ and let the notation $e_{g,0} := T(g).e_0, g \in G$. We have an action $G \times \mathcal{H}^\infty \to \mathcal{H}^\infty, g.e_0 := e_{g,0}$. When there is no possibility of confusion, we write just $e_g$ for $e_{g,0}$. Let us denote by $[\cdot] : \mathcal{H}^\ast := \mathcal{H} \setminus \{0\} \to \mathbb{P}(\mathcal{H}) = \mathcal{H}^\ast / \sim$ the projection with respect to the equivalence relation $[\lambda x] \sim [x], \lambda \in \mathbb{C}^\ast, x \in \mathcal{H}^\ast$. So, $[\cdot] : \mathcal{H}^\ast \to \mathbb{P}(\mathcal{H}), [v] = \mathbb{C}v$. The action $G \times \mathcal{H}^\infty \to \mathcal{H}^\infty$ extends to the action $G \times \mathbb{P}(\mathcal{H}^\infty) \to \mathbb{P}(\mathcal{H}^\infty), g.[v] := [g.v]$.

Let us now denote by $H$ the isotropy group $H := G_{[e_0]} := \{ g \in G | g.e_0 \in \mathbb{C}e_0 \}$. We shall consider generalized coherent states on complex homogeneous manifolds $M \cong G/H$, imposing the restriction that $M$ be a complex submanifold of $\mathbb{P}(\mathcal{H}^\infty)$. 

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Definition 1. a) The orbit \( M \) is called a CS-orbit if there exists a holomorphic embedding \( \iota : M \hookrightarrow \mathbb{P}(\mathcal{H}^\infty) \). In such a case \( M \) is also called a CS-manifold.

b) \((T, \mathcal{H})\) is called a CS-representation if there exists a cyclic vector \( 0 \neq e_0 \in \mathcal{H}^\infty \) such that \( M \) is a CS-orbit.

c) The groups \( G \) which admit CS-representations are called CS-groups, and their Lie algebras \( \mathfrak{g} \) are called CS-Lie algebras.

The \( G \)-invariant complex structures on the homogeneous space \( M = G/H \) can be introduced in an algebraic manner. For \( X \in \mathfrak{g} \), where \( \mathfrak{g} \) is the Lie algebra of the Lie group \( G \), let us define the unbounded operator \( dT(X) \) on \( \mathcal{H} \) by \( dT(X).v := d/dt|_{t=0} T(\exp tX).v \) whenever the limit on the right hand side exists. We obtain a representation of the Lie algebra \( \mathfrak{g} \) on \( \mathcal{H}^\infty \), the derived representation, and we denote \( X.v := dT(X).v \) for \( X \in \mathfrak{g}, v \in \mathcal{H}^\infty \). Extending \( dT \) by complex linearity, we get a representation of the complex Lie algebra \( \mathfrak{g}_C \) on the complex vector space \( \mathcal{H}^\infty \). Lemma XV.2.3 p. 651 in [28] and Prop. 4.1 in [23] determine when a smooth vector generates a complex orbit in \( \mathbb{P}(\mathcal{H}^\infty) \). We denote by \( B := < \exp_{\mathfrak{g}_C} b > \) the Lie group corresponding to the Lie algebra \( \mathfrak{b} \), with \( b := \overline{b(e_0)} \), where \( b(v) := \{ X \in \mathfrak{g}_C : X.v \in \mathbb{C}v \} = (\mathfrak{g}_C)_v \).

The complex structure on \( M \) is induced by an embedding in a complex manifold, \( i_1 : M \cong G/H \hookrightarrow G_C/B \). We consider such manifolds which admit a holomorphic embedding \( i_2 : G_C/B \hookrightarrow \mathbb{P}(\mathcal{H}^\infty) \). Then the embedding \( \iota = i_1 \circ i_2, \iota : M \hookrightarrow \mathbb{P}(\mathcal{H}^\infty) \) is a holomorphic embedding, and the complex structure comes as in Theorem XV.1.1 and Proposition XV.1.2 p. 646 in [28].

3 Coherent state vectors

Now we construct what we call Perelomov’s generalized coherent state vectors, or simply CS-vectors, based on the CS-homogeneous manifolds \( M \cong G/H \).

We denote also by \( T \) the holomorphic extension of the representation \( T \) of \( G \) to the complexification \( G_C \) of \( G \), whenever this holomorphic extension exists. In fact, it can be shown that in the situations under interest in this paper, this holomorphic extension exists [23] [27]. Then there exists the homomorphism \( \chi_0(\lambda) : H \rightarrow T, (\chi : B \rightarrow \mathbb{C}^*) \), such that \( H = \{ g \in G | e_g = \chi_0(g)e_0 \} \) (respectively, \( B = \{ g \in G_C | e_g = \chi(g)e_0 \} \)), where \( T \) denotes the torus \( T := \{ z \in \mathbb{C} | |z| = 1 \} \).

For the homogeneous space \( M = G/H \) of cosets \( \{ gH \} \), let \( \lambda : G \rightarrow G/H \) be the natural projection \( g \mapsto gH \), and let \( o := \lambda(1) \), where \( 1 \) is the unit element of \( G \). Choosing a section \( \sigma : G/H \rightarrow G \) such that \( \sigma(o) = 1 \), every element \( g \in G \) can be written down as \( g = \tilde{g}(g)h(g) \), where \( \tilde{g}(g) \in G/H \) and \( h(g) \in H \). Then we have \( e_g = e^{i\alpha(h(g))}e_{\tilde{g}(g)} \), where \( e^{i\alpha(h(g))} = \chi_0(h) \). Now we take into account that \( M \) also admits an embedding in \( G_C/B \). We choose a local system of coordinates parametrized by \( z_g \) (denoted also simply \( z \), where there is no possibility of confusion) on \( G_C/B \). Choosing a section \( G_C/B \rightarrow G_C \) such that any element \( g \in G_C \) can be written as \( g = \tilde{g}_b(g) \),
where \( \tilde{g}_b \in G_C/B \), and \( b(g) \in B \), we have \( e_g = \Lambda(g)e_{z_0} \), where \( \Lambda(g) = \chi(b(g)) = e^{i\alpha(h(g))}(e_{z_0}, e_{z_0})^{-\frac{1}{2}} \).

Let us denote by \( m \) the vector space orthogonal to \( h \) of the Lie algebra \( g \), i.e. we have the vector space decomposition \( g = h + m \). Even more, it can be shown that the vector space decomposition \( g = h + m \) is \( \text{Ad} \) \( H \)-invariant. The homogeneous spaces \( M \cong G/H \) with this decomposition are called reductive spaces (cf. [28]) and it can be proved that the CS-manifolds are reductive spaces (cf. [8]). So, the tangent space to \( M \) at \( o \) can be identified with \( m \). Recall that for CS-groups the CS-representations are highest weight representations and the vector \( e_0 \) is a primitive element of the generalized parabolic algebra \( b \) (cf. [28]).

Let us denote \( X := dT(X), X \in \mathcal{U}(g_C) \), where \( \mathcal{U} \) denotes the universal enveloping algebra. Let \( g(g) = \exp X, \tilde{g}(g) \in G/H, X \in m, e_{\tilde{g}(g)} = \exp(X)e_0 \). Let us remember again Theorem XV.1.1 p. 646 in [28]. Note that \( T_o(G/H) \cong g/h \cong g_C/b \cong (b + b)/b \cong b/h_C \), where we have a linear isomorphism \( \alpha : g/h \cong g_C/b, \alpha(X + h) = X + b \) (cf. [28]). We can take instead of \( m \subset g \) the subspace \( m' \subset g_C \) complementary to \( b \), or the subspace of \( b \) complementary to \( h_C \). If we choose a local canonical system of coordinates \( \{z_\alpha\} \) with respect to the basis \( \{X_\alpha\} \) in \( m' \), then we can introduce the vectors

\[
e_z = \exp(\sum_{X_\alpha \in m'} z_\alpha X_\alpha).e_0 \in \mathcal{H}. \tag{3.1}
\]

We get

\[
e_\sigma(z) = T(\sigma(z)), \quad z \in M, \tag{3.2}
\]

and we prefer to choose local coordinates such that

\[
e_\sigma(z) = N(z)e_z, \quad N(z) = (e_z, e_z)^{-1/2}. \tag{3.3}
\]

Equations (3.1), (3.2), and (3.3) define locally the coherent vector mapping

\[
\varphi : M \to \bar{\mathcal{H}}, \quad \varphi(z) = e_z, \tag{3.4}
\]

where \( \bar{\mathcal{H}} \) denotes the Hilbert space conjugate to \( \mathcal{H} \). We call the vectors \( e_z \in \bar{\mathcal{H}} \) indexed by the points \( z \in M \) Perelomov’s coherent state vectors.

4 The symmetric Fock space \( \mathcal{F}_{\mathcal{H}} \)

We have considered homogeneous CS-manifolds \( M \cong G/H \) whose complex structure comes from the embedding \( i_1 : M \hookrightarrow G_C/B \). We have chosen a section \( \sigma : G_C/B \to G_C \), and \( G_C \) can be regarded as a complex analytic principal bundle \( B \xrightarrow{i} G_C \xrightarrow{\lambda} G_C/B \).

Let us introduce the function \( f'_\psi : G_C \to \mathbb{C}, f'_\psi(g) := (e_g, \psi), g \in G, \psi \in \mathcal{H} \). Then \( f'_\psi(gb) = \chi(b)^{-1}f'_\psi(g), g \in G_C, b \in B \), where \( \chi \) is the continuous homomorphism of the isotropy subgroup \( B \) of \( G_C \) in \( \mathbb{C}^* \). The coherent states realize the space of holomorphic
global sections $\Gamma^{\text{hol}}(M, L_\chi) = H^0(M, L_\chi)$ on the $G_\mathcal{C}$-homogeneous line bundle $L_\chi$ associated by means of the character $\chi$ to the principal $B$-bundle (cf. [30]). The holomorphic line bundle is $L_\chi := M \times_\chi \mathbb{C}$, also denoted $L := M \times_B \mathbb{C}$ (cf. [12, 36]).

The local trivialization of the line bundle $L_\chi$ associates to every $\psi \in \mathcal{H}$ a holomorphic function $f_\psi$ on an open set in $M \hookrightarrow G_\mathcal{C}/B$. Let the notation $G_S := G_\mathcal{C} \setminus S$, where $S$ is the set $S := \{g \in G_\mathcal{C}|\alpha_g = 0\}$, and $\alpha_g := (e_g, e_0)$. $G_S$ is a dense subset of $G_\mathcal{C}$. We introduce the function $f_\psi : G_S \hookrightarrow \mathbb{C}$, $f_\psi(g) = f_\psi(z_g) = \frac{e_{z_g, \psi}(\bar{e}_{z_g, e_0})}{e_{z_g, e_0}}$, where $(e_{z_g, e_0}) \neq 0$, and also the coherent state map $\varphi : M \rightarrow \mathcal{H}^\infty$, $\varphi(z) = e_z, z \in \mathcal{V}_0$, where the canonical coordinates $z = (z_1, \ldots, z_n)$ constitute a local chart on $\mathcal{V}_0 := M_S \rightarrow \mathbb{C}^n$, such that $0 = (0, \ldots, 0)$ corresponds to $\{B\}$. Note also that $\mathcal{V}_0 \equiv M \setminus \Sigma_0$, where $\Sigma_0 := \lambda(S)$ is the set of points of $M$ for which the coherent state vectors are orthogonal to $e_0 \in \mathcal{H}$, called polar divisor of the point $z = 0$ (cf. [3]).

Supposing that the line bundle $L_\chi$ is already very ample, $\mathcal{F}_{\mathcal{H}}$ is defined as the set of functions corresponding to sections such that $\{f \in L^2(M, L) \cap \mathcal{O}(M, L)|(f, f)_{\mathcal{F}_{\mathcal{H}}} < \infty\}$ with respect to the scalar product

$$(f, g)_{\mathcal{F}_{\mathcal{H}}} = \int_M \tilde{f}(z)g(z) d\nu_M(z, \bar{z}),$$

where $d\nu_M(z, \bar{z})$ is the invariant measure $\frac{dw_M(z, \bar{z})}{(e_z, e_\bar{z})}$, and $d\mu_M(z, \bar{z})$ represents the $G$-invariant Radon measure on $M$. It can be shown that the space of functions $\mathcal{F}_{\mathcal{H}}$ identified with $L^2,\text{hol}(M, L_\chi)$ is a closed subspace of $L^2(M, L_\chi)$ with continuous point evaluation (cf. [31]).

Note that eq. (4.1) is nothing else than the Parseval overcompletness identity [9]:

$$(\psi_1, \psi_2) = \int_{M=G/K} (\psi_1, e_z)(e_{\bar{z}}, \psi_2) d\nu_M(z, \bar{z}), \quad (\psi_1, \psi_2) \in \mathcal{H}. \quad (4.2)$$

Let us now introduce the map

$$\Phi : \mathcal{H}^* \rightarrow \mathcal{F}_{\mathcal{H}}, \Phi(\psi) := f_\psi, f_\psi(z) = \Phi(\psi)(z) = (\varphi(z), \psi)_{\mathcal{H}} = (e_z, \psi)_{\mathcal{H}}, \quad z \in \mathcal{V}_0, \quad (4.3)$$

where we have identified the space $\mathcal{H}$ complex conjugate to $\mathcal{H}$ with the dual space $\mathcal{H}^*$ of $\mathcal{H}$.

In fact, our supposition that $L_\chi$ is already a very ample line bundle implies the validity of eq. (4.2) (cf. Theorem XII.5.6 p. 542 in [28], Remark VIII.5 in [24], and Theorem XII.5.14 p. 552 in [28]).

It can be seen that the group-theoretic relation (4.2) on homogeneous manifolds fits into Rawnsley’s global realization [33] of Berezin’s coherent states on quantizable Kähler manifolds [9].

It can be defined a function $K$, $K : M \times \mathcal{M} \rightarrow \mathbb{C}$, which on $\mathcal{V}_0 \times \mathcal{V}_0$ reads

$$K(z, \bar{w}) := K_w(z) = (e_z, e_{\bar{w}})_{\mathcal{H}}. \quad (4.4)$$
Taking into account (5.3) and supposing that eq. (4.2) is true, it follows that the function $K$ (4.4) is a reproducing kernel. We have:

**Proposition 1.** Let $(T, \mathcal{H})$ be a CS-representation and let us consider the Perelomov’s CS-vectors defined in (3.1)-(3.3). Suppose that the line bundle $L$ is very ample. Then

i) The function $K : M \times M \to \mathbb{C}$, $K(z, \overline{w})$ defined by equation (4.4) is a reproducing kernel.

ii) Let $\mathcal{F}_\mathcal{H}$ be the space $L^2_{\text{hol}}(M, L)$ endowed with the scalar product (4.1). Then $\mathcal{F}_\mathcal{H}$ is the reproducing kernel Hilbert space $\mathcal{H}_K \subset \mathbb{C}^M$ associated to the kernel $K$ (4.4).

iii) The evaluation map $\Phi$ defined in eqs. (4.3) extends to an isometry

\[
(\psi_1, \psi_2)_{\mathcal{H}_K} = (\Phi(\psi_1), \Phi(\psi_2))_{\mathcal{F}_\mathcal{H}} = (f_{\psi_1}, f_{\psi_2})_{\mathcal{F}_\mathcal{H}} = \int_M f_{\psi_1}(z) f_{\psi_2}(z) d\nu_M(z),
\]

and the overcompletness eq. (4.2) is verified.

5 Representations of coherent state Lie algebras by differential operators

We remember the definitions of the functions $f_\psi'$ and $f_\psi$, which allow to write down

\[
f_\psi(z) = (e_z, \psi) = \frac{(T(\overline{g})e_0, \psi)}{(T(g)e_0, e_0)}, \quad z \in M, \quad \psi \in \mathcal{H}.
\]

We get

\[
f_{T(\overline{g}), \psi}(z) = \mu(g', z) f_\psi(g^{-1}z),
\]

where

\[
\mu(g', z) = \frac{(T(g^{-1}g')e_0, e_0)}{(T(g)e_0, e_0)} = \frac{\Lambda(g^{-1}g)}{\Lambda(g)}.
\]

Recall that $T(g)e_0 = e^{i\alpha(g)}g = \Lambda(g)e_{z_0}$, where we have used the decompositions $g = \tilde{g}h$, $(G = G/H,H)$; $g = z_0b$ $(G_C = G_C/B.B)$. We have also the relation $\chi_0(h) = e^{i\alpha(h)}$, $h \in H$ and $\chi(b) = \Lambda(b)$, $b \in B$, where $\Lambda(g) = \frac{e^{i\alpha(g)}}{(e_{z_0}, e_{z_0})^{1/2}}$. We can also write down another expression for multiplicative factor $\mu$ appearing in eq. (5.2) using the CS-vectors:

\[
\mu(g', z) = \Lambda(g')e_{z_0} e_{z_0} = e^{i\alpha(g')} \frac{(e_{z_0}, e_{z_0})}{(e_{z_0}, e_{z_0})^{1/2}}.
\]

The following assertion is easily checked out:

**Remark 1.** Let us consider the relation (5.1). Then we have (5.2), where $\mu$ can be written down as in equations (5.3), (5.4). We have the relation $\mu(g, z) = J(g^{-1}, z)^{-1}$, i.e. the multiplier $\mu$ is the cocycle in the unitary representation $(T_{\mathcal{H}_K})$ attached to the positive definite holomorphic kernel $K$ defined by equation (4.4),

\[
(T_K(g).f)(x) := J(g^{-1}, x)^{-1}f(g^{-1}.x),
\]

(5.5)
Proposition 2. We can see that if \( A \), \( U \), and \( \Phi \) are three objects which correspond each to other:

\[
\begin{align*}
\mathfrak{g} &
\ni X 
\mapsto
\mathfrak{X} 
\ni \mathfrak{X} 
\mapsto
\mathfrak{A}_M 
\ni \mathfrak{A}_M 
\ni \mathfrak{D}_M, \\
\text{differential operator on } \mathfrak{F}_M.
\end{align*}
\]

We can see that \( \mathfrak{g} \ni X \mapsto \mathfrak{X} \in \mathfrak{A}_M \mapsto \mathfrak{X} \in \mathfrak{A}_M \subseteq \mathfrak{D}_M, \) differential operator on \( \mathfrak{F}_M. \) \( \tag{5.8} \)

We can see that

**Proposition 2.** If \( \Phi \) is the isometry \( \Phi \mathfrak{g} \mathfrak{X} \mathfrak{A}_M \mathfrak{D}_M \), then \( \Phi \mathfrak{d} \mathfrak{T} (\mathfrak{g}_C) \mathfrak{X}^{-1} \subseteq \mathfrak{D}_1. \)
Proof. Let us consider an element in $\mathfrak{g}_C$ and his image in $D_M$ via the correspondence (5.8):

$$\mathfrak{g}_C \ni G \mapsto G \in D_M; \quad G_z(f_\psi(z)) = G_z(e_\bar{z}, \psi) = (e_\bar{z}, G\psi),$$

$$G = dT(G) = \frac{d}{dt}|_{t=0} T(\exp(tG)).$$

Remembering equation (5.2) and determining the derived representation, we get

$$G_z(f_\psi(z)) = (P_G(z) + \sum Q_G(z) \frac{\partial}{\partial z_i}) f_\psi(z); \quad (5.9)$$

where

$$P_G(z) = \frac{d}{dt}|_{t=0} \mu(\exp(tG), z); \quad Q_G(z) = \frac{d}{dt}|_{t=0} (\exp(-tG).z)_i.$$ 

Now we formulate the following assertion:

**Remark 2.** If $(G,T)$ is a CS-representation, then $A_M$ is a subalgebra of holomorphic differential operators with polynomial coefficients, $A_M \subset A_M \subset D_M$. More exactly, for $X \in \mathfrak{g}$, let us denote by $X := dT(X) \in A_M$, where the action is considered on the space of functions $\mathcal{F}_{\mathcal{H}}$. Then, for CS-representations, $X \in A_0 = A_0 \oplus A'_1$.

Explicitly, if $\lambda \in \Delta$ is a root and $G_\lambda$ is in a base of the Lie algebra $\mathfrak{g}_C$ of $G_C$, then his image $G_\lambda \in D_M$ acts as a first order differential operator on the symmetric Fock space $\mathcal{F}_{\mathcal{H}}$

$$G_\lambda = P_\lambda + \sum_{\beta \in \Delta_{m'}} Q_{\lambda,\beta} \partial_\beta, \quad \lambda \in \Delta, \quad (5.10)$$

where $P_\lambda$ and $Q_{\lambda,\beta}$ are polynomials in $z$, and $m'$ is the subset of $\mathfrak{g}_C$ which appears in the definition (3.1) of the coherent state vectors.

Actually, we don’t have a proof of this assertion for the general case of CS-groups. For the compact case, there exists the calculation of Dobaczewski [13]. For compact hermitian symmetric spaces it was shown [3] that degrees of the polynomials $P$ and $Q$-s are $\leq 2$ and similarly for the non-compact hermitian symmetric case [1]. Neeb [28] gives a proof of this Remark for CS-representations for the (unimodular) Harish-Chandra type groups. If $G$ is an admissible Lie group such that the universal complexification $G \to G_C$ is injective and $G_C$ is simply connected, then $G$ is of Harish-Chandra type (cf. Proposition V.3 in [24]). The derived representation (5.7) is obtained differentiating eq. (5.5), and we get two terms, one in $D_0$ and the other one in $D'_1$. A proof that the two parts are in fact $A_0$ and respectively $A'_1$ is contained in Prop. XII.2.1 p. 515 in [28] for the groups of Harish-Chandra type in the particular situation where the space $p^+$ in Lemma VII.2.16 p. 241 in [28] is abelian.
6 Representation of semisimple Lie groups by differential operators

6.1 Semisimple Lie groups and flag manifolds

We use standard notation referring to Lie algebras of a complex semisimple Lie group $G$. In this case $\Delta \equiv \Delta_s$, i.e. $\Delta_r = \{\emptyset\}$, i.e. all roots are semisimple.

- $\mathfrak{g}$ – complex semisimple Lie algebra
- $\mathfrak{t} \subset \mathfrak{g}$ – Cartan subalgebra
- $\mathfrak{b} = \mathfrak{t} + \mathfrak{b}_u$ – Borel subalgebra
- $\mathfrak{b}_u = \sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$ – the nilradical of $\mathfrak{b}$
- $\Sigma$ – root system for $(\mathfrak{g}, \mathfrak{t})$
- $\Sigma^+$ – a positive root system
- $\Psi$ – a simple root system for $\Sigma$

$\Sigma \ni \alpha = \sum_{\mu \in \Psi} n_\mu(\alpha) \mu$ – unique, $n_\mu \in \mathbb{N}$, $n_\mu(\alpha) \geq 0$ if $\alpha \in \Sigma^+$; $n_\mu(\alpha) \leq 0$ if $\alpha \in \Sigma^-$ $\Psi \supset \Phi^r = \{\alpha \in \Sigma; n_\mu(\alpha) = 0$ whenever $\mu \notin \Phi\}$ $\Phi^u = \{\alpha \in \Sigma; n_\mu(\alpha) > 0$ for some $\mu \notin \Phi\} = \Sigma^+ \setminus \{\Sigma^+ \cap \Phi^r\}$ $\mathfrak{p}_\Phi = \mathfrak{p}_\Phi^r + \mathfrak{p}_\Phi^u$ – parabolic subalgebras of $\mathfrak{g}$ corresponding to $\Psi \subset \Phi$

- $\mathfrak{p}_\Phi^u = \mathfrak{t} + \sum_{\alpha \in \Phi^u} \mathfrak{g}_\alpha$ – the unipotent part of $\mathfrak{p}_\Phi$
- $\mathfrak{p}_\Phi^u = \sum_{\alpha \in \Phi^u} \mathfrak{g}_\alpha$ – the unipotent part of $\mathfrak{p}_\Phi$

$\Delta_0 = \Phi^r; \ \Delta_- = -\Phi^u; \ \Delta_+ = \Phi^a$

$B = \{g \in G; \text{Ad}(g) \mathfrak{b} = \mathfrak{b}\}$ – Borel subgroup (maximal solvable)

$P = \{g \in G; \text{Ad}(g) \mathfrak{p} = \mathfrak{p}\}$ – parabolic subgroup (contains a Borel subgroup).

In the notation of Definitions VII.2.4 p. 234, VII.2.6 p. 236 and VII.2.22, p. 244 in [28] we have $\Phi^u \equiv \Delta^+_p$ and $\Phi^r \equiv \Delta_k$.

We remember the following facts:

1. Let $G$ be a complex semisimple Lie group, $G_u$ its compact real form. Then the isotropy group $K = G_u \cap P$ is connected and the compact simply connected Kähler manifold $M \approx G_u/K \approx G/P$ is an algebraic manifold (Hodge), called generalized complex flag manifold. $G$ is viewed as a group of holomorphic transformation on $M$.

2. Let $G$ be a homogeneous compact Kähler manifold. Then the projective space orbit of an extreme weight vector is an irreducible finite representation of $G^c$.

3. Let $G_0$ be a real form of $G$, $x \in G/P$. Then $K = G_0 \cap P = G_u \cap P$. $G_0(x)$ has a $G_0$-invariant Kähler metric. The Kähler orbit $G_0(x)$ is open in $G^c/P$.

For symmetric spaces $K$ is a maximal compact subgroup of $G$. Hermitian symmetric spaces correspond to centerless semisimple Lie groups, which verify the condition $j_\mathfrak{g}(j_\mathfrak{t}(t)) = \mathfrak{k}$ of Definition VII.2.15 p. 241 in [28] of quasihermitian groups, and $\mathfrak{p}_\Phi^u$ is abelian.
We also need the commutation relations in the Cartan-Weyl basis \[17\]

\[
\begin{align*}
[H_i, H_j] &= 0, \quad i = 1, \ldots, r, H_i \in \mathfrak{t}, \\
[H_i, E_\alpha] &= \alpha_i E_\alpha, \quad \alpha_i = \alpha(H_i), \\
[E_\alpha, E_\beta] &= n_{\alpha, \beta} E_{\alpha + \beta}, \quad \alpha + \beta \in \Delta \setminus \{0\}, \\
[E_\alpha, \bar{E}_\beta] &= 0, \quad \alpha + \beta \notin \Delta \cup \{0\}, \\
[E_\alpha, E_{-\alpha}] &= H_\alpha = \sum \alpha_i H_i.
\end{align*}
\] (6.1)

As a consequence, we have also the commutation relations:

\[
\begin{align*}
[E_{-\gamma}, E_\gamma] &= -\gamma H, \quad \gamma H := (\gamma, H) = \sum_{j=1}^r \gamma_j H_j; \\
[H, E_\alpha] &= \alpha(H) E_\alpha.
\end{align*}
\] (6.2)

### 6.2 Perelomov’s coherent vectors for semisimple Lie groups

All representations of compact Lie groups are CS-representations because these representations are highest weight representations. Kostant and Sternberg \[18\] showed that for any representation of a compact group \(G\) the orbit to a projectivized highest weight vector is the only Kähler coherent state orbit. Harish-Chandra \[16\] has defined highest weight representations for non-compact semisimple (or even reductive) Lie groups. He has classified square integrable highest weight representations. This classification has been fully realized by Enright, Howe and Wallach, and independently by Jakobsen \[14\]. Lisiecki has emphasized (cf. \[21\] and Theorem 6.1 in \[23\]) that: a non-compact semisimple Lie group is a CS-group if and only if it is hermitian. If this is the case, the CS-representations of \(G\) are precisely the highest weight representations. Each of them has a unique CS-orbit, which is the orbit through the highest line. The starting point of the proof of Lisiecki is the paper of Borel \[11\], where it is proved: a non-compact semisimple Lie group \(G\) admits a homogeneous Kähler orbit if and only if it is hermitian, and such a manifold is of the form \(G/Z_G(S)\), where \(Z_G(S)\) is the centralizer of a torus \(S \subset G\); moreover, it is a holomorphic fiber bundle over the Hermitian symmetric space \(G/K\), where \(K\) is a maximal compact subgroup of \(G\), with (compact) flag manifolds \(K/Z_G(S)\) as fibers.

Let us consider again the triplet \((G, \pi, \mathfrak{g})\) where \((G, \pi)\) is a CS-representation. Then this representation can be realized as an extreme weight representation. For linear connected reductive groups with \(Z_K(\mathfrak{z}) = \mathfrak{t}\), where \(\mathfrak{z}\) denotes the center of the Lie algebra \(\mathfrak{k}\) of \(K\), the effective representation is furnished by the Harish-Chandra theorem (cf. e.g. \[19\], p. 158). The theorem furnishes the holomorphic discrete series for the non-compact case, and for the compact case it is equivalent with the Borel-Weil theorem (\[31\]; also cf. \[19\], p. 143).

In accord with the procedure of \[13\] for getting Perelomov’s CS-vectors, we start with \(e_{g,0}\). Then we consider for a everywhere dense subset \(G^0 \subset G\) the Gauss decomposition

\[
g = g_+ g_0 g_-, \quad g \in G^0.
\] (6.3)
If we restrict ourself to the complex semisimple Lie groups, then, in the notation of the § 6.1, eq. (6.3) reads
\[ g = \exp \sum_{\alpha \in \Delta} z_{\alpha} E_{\alpha} \exp \sum_{i=1}^{r} v_{i} H_{i} \exp \sum_{\alpha \in \Delta_{0}} \xi_{\alpha} E_{\alpha} \exp \sum_{\alpha \in \Delta_{-}} y_{\alpha} E_{\alpha}, \] (6.4)
where \((z_{\alpha}, v_{i}, \xi_{\alpha}, y_{\alpha})\) are local coordinates for \(G\).

If the extreme weight \(j\) (here minimal) of the representation has the components \(j = (j_{1}, \cdots, j_{r})\), where \(r\) is the rank of the Cartan algebra, then
\[
\begin{align*}
H_{k} e_{j} &= j_{k} e_{j}, \quad k = 1, \ldots, r; \\
E_{\alpha} e_{j} &= 0 : \alpha \in \Delta_{-} \cup \Delta_{0}.
\end{align*}
\] (6.5)

Perelomov’s CS-vectors are
\[ e_{z,j} = \exp(\sum_{\alpha \in \Delta_{+}} z_{\alpha} E_{\alpha}) e_{j}, \] (6.6)
where \(z_{\alpha}\) are local coordinates for the coordinate neighborhood \(V_{0} \subset M, V_{0} = M \setminus \Sigma_{0}\).
\(\Sigma_{0}\) is the polar divisor \(\Sigma_{0} = \lambda(S)\). For hermitian symmetric spaces it was proved in [5] that \(\Sigma_{0} = CL_{0}\), where \(CL_{0}\) is the cut locus relative to the origin \(0 \in M\).

### 6.3 Differential operators on semisimple Lie group orbits

We start introducing the notation
\[ Z = \sum_{\alpha \in \Delta_{+}} z_{\alpha} E_{\alpha}, \]
and then \(\partial_{\alpha}(Z) = E_{\alpha}\), where \(\partial_{\alpha} := \frac{\partial}{\partial z_{\alpha}}, \alpha \in \Delta_{+}\).

We remember the definition of the Bernoulli numbers \(B_{i}\) (see e.g. [1]):
\[
\frac{x}{1 - e^{-x}} = 1 + \frac{1}{2} x + \sum_{k \geq 1} (-1)^{k-1} \frac{B_{k} x^{2k}}{(2k)!} = \sum_{n \geq 0} c_{n} x^{n}, \tag{6.7}
\]
\[ c_{0} = 1; \quad c_{1} = \frac{1}{2}; \quad c_{2k+1} = 0; \quad c_{2k} = \frac{(-1)^{k-1}}{(2k)!} B_{k}; \tag{6.8}
\]
\[ B_{1} = \frac{1}{6}; \quad B_{2} = \frac{1}{30}; \quad B_{3} = \frac{1}{42}; \quad B_{4} = \frac{1}{300}; \cdots \tag{6.9}
\]

The following lemma is needed:

**Lemma 1.** Let the relation:
\[ \frac{1}{n!} = \sum_{k=0}^{n} \frac{c_{k}}{(n-k+1)!} \] (6.10)

Then the constants \(c_{k}\) of eq. (6.10) verifies the definition (6.7).
We need also another formula similar to (6.10).

**Lemma 2.** Let the constants $d_k$ be defined by the relation:

$$\frac{1}{(n+2)!} = \sum_{k=0}^{n} d_k \frac{1}{(n-k+1)!}. \quad (6.11)$$

Then the constants $c$ and $d$ are related by

$$d_k = (-1)^k c_{k+1}. \quad (6.12)$$

Now we recall the main results established in [6]:

**Theorem 1.** Let $G$ be a semisimple Lie group. If $G_\lambda$ is the generator of the group $G$, then $G_\lambda \in \mathfrak{g}_1 = \mathfrak{d}_0 \oplus \mathfrak{d}_1'$. More exactly, $G_\lambda \in \mathfrak{a}_1$, i.e.

$$G_\lambda = P_\lambda + \sum_{\beta \in \Delta_+} Q_{\lambda,\beta} \partial_\beta, \lambda \in \Delta, \quad (6.13)$$

where $P_\lambda$ and $Q_{\lambda,\beta}$ are polynomials in $z$.

Explicitly:

a) For $\alpha \in \Delta_+$,

$$E_\alpha = \sum_{k \geq 0} c_k \sum_{\beta \in \Delta_+} p_{k \alpha \beta}(z) \partial_{\alpha+\beta}, \quad (6.14)$$

where the coefficients $c_k$, related to the Bernoulli numbers by eq. (6.8), are introduced by eq. (6.10). The polynomials $p_{k \alpha \beta}, k \in \mathbb{N}, \alpha \in \Delta_+$ are given by the equation:

$$p_{k \alpha \beta}(z) = \sum_{\alpha_1 \cdots \alpha_k = \beta \atop \alpha_1 + \cdots + \alpha_k = \alpha} n_{\alpha_1 \cdots \alpha_k} z_{\alpha_1} \cdots z_{\alpha_k}, \quad k \geq 1 \quad (6.15)$$

where

$$n_{\alpha_1 \cdots \alpha_k} = n_{\alpha_1,\alpha} n_{\alpha_2,\alpha+\alpha_1} \cdots n_{\alpha_k,\alpha+\alpha_1+\cdots+\alpha_{k-1}}, \quad (k \geq 1, \alpha_0 = 0) \quad (6.16)$$

and $n_{\alpha,\beta,\gamma} \in \Delta_+$ are the structure constants of eq. (6.1), and for $k = 0$ the sum (6.14) is just $\partial_\alpha$.

The expression (6.14) can be put also into a form in which the Bernoulli numbers are explicit:

$$E_\alpha = \partial_\alpha + \frac{1}{2} \sum_{\beta \in \Delta_+} z_{\beta} n_{\beta,\alpha} \partial_{\alpha+\beta} + \sum_{k \geq 1} \frac{(-1)^{k-1}}{(2k)!} B_k \sum_{\beta \in \Delta_+} p_{k \alpha \beta} \partial_{\alpha+\beta}. \quad (6.17)$$

The degree of the polynomial $p$ has the property: degree $p_{k \alpha \beta} \leq \nu$; $p_{k \alpha \beta}$ as a function of $z$ contains only even powers. The table below contains the values of $\nu$. 

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Degree $\nu$ for simple Lie algebras

| $A_l$ : $\nu = l - 1$ | $l \geq 1$ | $E_6$ : $\nu = 10$ | $G_2$ : $\nu = 4$ |
| $B_l$ : $\nu = 2l - 2$ | $l \geq 2$ | $E_7$ : $\nu = 16$ |
| $C_l$ : $\nu = 2l - 2$ | $l \geq 2$ | $E_8$ : $\nu = 28$ |
| $D_l$ : $\nu = 2l - 4$ | $l \geq 3$ | $F_4$ : $\nu = 10$ |

b) The differential action of the generators of the Cartan algebra is:

$$H = j + \sum_{\beta \in \Delta_+} \beta z_{\beta} \partial_{\beta}$$

(6.18)

c) If $(\alpha, j) = 0$, then

$$E_{\alpha} = - \sum_{\beta \in \Delta_+} n_{\beta, -\alpha} z_{\beta - \alpha} \partial_{\beta}.$$  

(6.19)

d) If $\gamma \in \Delta_-$ is simple root, then

$$E_{\gamma} = j \gamma z_{-\gamma} + \sum_{k \geq 0} d_k \sum_{\delta, \beta \in \Delta_+} q_{\gamma \delta}(z)p_{k\delta \beta}(z)\partial_{\beta + \delta},$$

(6.20)

where the coefficients $d$ are expressed through the coefficients $c$ by eq. (6.12).

The expression of the polynomials $q_{\gamma \delta}, \gamma \in \Delta_-, \delta \in \Delta_+$ is

$$q_{\gamma \delta} = -\gamma z_{-\gamma} \delta z_{\delta} + \sum_{\mu \in \Delta_+} z_{\delta - \mu - \gamma} n_{\delta - \mu - \gamma, \gamma} z_{\mu} n_{\mu, \delta - \mu}.$$  

(6.21)

In the case of Hermitian symmetric cases eq. (6.14) becomes simply:

$$E_{\alpha} = \partial_{\alpha}$$

(6.22)

while eq. (6.21) becomes

$$-E_{\alpha}^- = K_{\alpha}^- = (\alpha, j)z_{\alpha} + \frac{1}{2} z_{\alpha} \sum_{\beta \in \Delta_+} (\alpha, \beta)z_{\beta} \partial_{\beta}$$

$$-\frac{1}{2} z_{\alpha} \sum_{\gamma - \alpha \in \Delta_k \setminus \{0\}} n_{\gamma - \alpha} n_{\gamma - \alpha, -\beta} z_{\beta + \alpha - \gamma} \partial_{\beta}.$$  

The main ingredient in the proof [6] of Theorem 1 is the formula:

$$\text{Ad}(\exp Z) = \exp \text{ad}_Z, \ Z \in \mathfrak{g},$$

(6.23)

i.e.

$$e^Z X e^{-Z} = \sum_{n \geq 0} \frac{1}{n!} \text{ad}_Z^n X, \ X, Z \in \mathfrak{g}.$$  

(6.24)
where

\[ \text{ad}_Y X = [Y, X], \quad \text{ad}_Y^m X = [Y, \text{ad}_Y^{m-1} X], \quad m > 1, \quad \text{ad}_Y^0 X = X. \]

We also have used the relation

\[ e^Z \partial_\alpha (e^{-Z}) = - \left[ \partial_\alpha(Z) + \sum_{n \geq 1} \frac{1}{(n+1)!} \text{ad}_Z^n \partial_\alpha(Z) \right], \quad (6.25) \]

\[ \partial_\alpha(Y) = \partial_\alpha Y - Y \partial_\alpha = -\text{ad}_Y(\partial_\alpha). \]

Due to the correspondence (5.8), the operator of the left hand side of eq. (6.24) corresponds to the differential action on \( \mathcal{F}_{3\epsilon} \), and \( e^Z X e^{-Z} \sim -X \).

### 6.4 An example: \( SU(3)/S(U(1) \times U(1) \times U(1)) \)

In this section we follow closely [20] for the example of the compact non-symmetric space \( M = SU(3)/S(U(1) \times U(1) \times U(1)) \).

The commutation relations of the generators are:

\[ [C_{ij}, C_{kl}] = \delta_{jk}C_{il} - \delta_{il}C_{kj}, \quad 1 \leq i, j \leq 3. \quad (6.26) \]

Let us consider the following parametrizations useful for the Gauss decomposition and also in the definition of the coherent states for the manifold \( M \):

\[ V_+ (\zeta) = \exp(\zeta_{12} C_{12} + \zeta_{13} C_{13} + \zeta_{23} C_{23}), \quad (6.27) \]

\[ V'_+ (z) = \exp(z_{23} C_{23}) \exp(z_{12} C_{12} + z_{13} C_{13}). \quad (6.28) \]

Let us denote by the same letter \( C_{ij} \) the \( n \times n \)-matrix having all elements 0 except at the intersection of the line \( i \) with the column \( j \), that is \( C_{ij} = (\delta_{ai} \delta_{bj})_{1 \leq a, b \leq n} \); here \( n = 3 \). Then:

\[ V_+ (\zeta) = \begin{pmatrix} 1 & \zeta_{12} & \zeta_{13} + \frac{1}{2} \zeta_{12} \zeta_{23} \\ 0 & 1 & \zeta_{23} \\ 0 & 0 & 1 \end{pmatrix}, \quad (6.29) \]

\[ V'_+ (z) = \begin{pmatrix} 1 & z_{12} & z_{13} \\ 0 & 1 & z_{23} \\ 0 & 0 & 1 \end{pmatrix}. \quad (6.30) \]

Now observing that for

\[ z_{12} = \zeta_{12}; \quad z_{13} = \zeta_{13} + \frac{1}{2} \zeta_{12} \zeta_{23}; \quad z_{23} = \zeta_{23}, \quad (6.31) \]

we get

\[ V_+ (\zeta) = V'_+ (z). \quad (6.32) \]
So, we have two parametrizations of the compact non-symmetric flag manifold $M = SU(3)/S(U(1) \times U(1) \times U(1))$: one in $\zeta$, given by eq. (6.29) and the other one in $z$, given by (6.30), which are identified with the relations (6.31).

Let us consider also the CS-vectors
\[ \phi_z = [V'_+(z)]^+ \phi_w = \exp(\bar{z}_{12}C_{21} + \bar{z}_{13}C_{31}) \exp(\bar{z}_{23}C_{32}) \phi_w. \] (6.33)

Now $\phi_w$ is chosen as maximal weight vector corresponding to the weight $w = (w_1, w_2, w_3)$ such that $j_1 = \omega_1 - \omega_2 \geq 0$, $j_2 = \omega_2 - \omega_3 \geq 0$ and the lowering operators are $C_{ij}, i > j$, and $C_{ii}$ generates the Cartan algebra, i.e.
\[ \left\{ \begin{array}{l} C_{ij} \phi_w \neq 0, \quad i > j, \\ C_{ij} \phi_w = 0, \quad i < j, \\ C_{ii} \phi_w = w_i \phi_w. \end{array} \right. \] (6.34)

The coherent state vectors corresponding to the representation $\pi_w$ determined by eqs. (6.34) are introduced as
\[ e_z = \pi_w(\{V'_+(z)\}) \phi_w. \] (6.35)

Denoting by $Z$ the matrix
\[ Z = \begin{pmatrix} 1 & z_{12} & z_{13} \\ 0 & 1 & z_{23} \\ 0 & 0 & 1 \end{pmatrix}, \] (6.36)
the reproducing kernel which determines the scalar product $(e_z, e_z)$ has the expression:
\[ K(ZZ^+) = \Delta_1^j (ZZ^+) \Delta_2^j (ZZ^+); \]
\[ \Delta_1 = 1 + |z_{12}|^2 + |z_{13}|^2; \]
\[ \Delta_2 = (1 + |z_{12}|^2 + |z_{13}|^2)(1 + |z_{23}|^2) - |z_{12} + z_{13}z_{23}|^2. \]

In particular, it is observed that for $z_{23} = 0$, $M$ becomes $SU(3)/S(U(2) \times U(1)) = G_1(C^3) = \mathbb{C}P^2$. In order to compare with the scalar product for coherent states on $M \approx \mathbb{C}P^2 \approx G_1(C^3)$ we remember that in the case of the Grassmannian we have used in [3, 4] a weight which here corresponds to $w_1 = 1, w_2 = w_3 = 0$ and then on $\mathbb{C}P^2$ the reproducing kernel is just $K(ZZ^+) = \Delta_1$.

The calculation which has as result Lemma 3 (cf. [6]) does not use the value of the reproducing kernel and is an algebraic one.

**Lemma 3.** The differential operators $C_{ij}$ associated to the generators $C_{ij}$ are given by
the formulas:

\[
\begin{align*}
C_{11} &= -z_{12}\partial_{12} - z_{13}\partial_{13} + w_1, \\
C_{12} &= \partial_{12}, \\
C_{13} &= \partial_{13}, \\
C_{21} &= -z_{12}^2\partial_{12} - z_{12}z_{13}\partial_{13} + (z_{12}z_{23} - z_{13})\partial_{23} + (w_1 - w_2)z_{12}, \\
C_{22} &= z_{12}\partial_{12} - z_{23}\partial_{23} + w_2, \\
C_{23} &= z_{12}\partial_{13} + \partial_{23}, \\
C_{31} &= -z_{12}z_{13}\partial_{12} - z_{13}^2\partial_{13} + (z_{12}z_{23} - z_{13})z_{23}\partial_{23} + \\
&\quad (w_1 - w_3)z_{13} - (w_2 - w_3)z_{12}z_{23}, \\
C_{32} &= z_{13}\partial_{12} - z_{23}^2\partial_{23} + (w_2 - w_3)z_{23}, \\
C_{33} &= z_{13}\partial_{13} + z_{23}\partial_{23} + w_3.
\end{align*}
\]

We have underlined the apparition of a third-degree polynomial multiplying the partial derivative of \(C_{31}\). Note also the relation \(C_{11} + C_{22} + C_{33} = w_1 + w_2 + w_3\).

7 Equations of motion

Let \((M \approx G/H, \omega)\) be a quantizable homogeneous Kähler manifold. Let us consider the triplet \((L, h, \nabla)\), where \(L\) is a \(G\)-homogeneous holomorphic positive line bundle, \(h\) the hermitian metric and \(\nabla\) the connection compatible with the complex structure. The quantization condition reads:

\[
\omega = i\Theta_L = \pi c_1(L) = -i\partial\bar{\partial}\log(h),
\]

where \(\Theta_L\) is the curvature matrix of \(L\) and \(c_1\) is first Chern class. Here the hermitian metric is given by

\[
h(z, \bar{z}) = (e_{\bar{z}}, e_{\bar{z}})^{-1},
\]

while the Kähler potential is \(K(z) = -\log h(z) = \log(e_z, e_{\bar{z}})\). The Kähler two-form is

\[
\omega(z) = i \sum_{\alpha, \beta \in \Delta_+} g_{\alpha, \beta} dz_\alpha \wedge d\bar{z}_\beta,
\]

where

\[
g_{\alpha, \beta}(z) = \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} \log(e_z, e_{\bar{z}}).
\]

If \(H \in \mathfrak{g}\) and the corresponding element in \(A_M\) is \(H\), then the energy function (covariant symbol) attached to \(H\) is

\[
H(z, \bar{z}) := (e_{\sigma(z)}, H e_{\sigma(z)}) = N^2(z)(e_z, H e_{\bar{z}}) = \frac{(e_{\bar{z}}, H e_{\bar{z}})}{(e_z, e_{\bar{z}})}.
\]
The homogeneous quantizable Kähler manifolds $M \cong G/H$ for which the group $G$ verify some obstructions which for semisimple Lie groups are $H^1(g) = H^2(g) = \{0\}$ can be organized as elementary Hamiltonian $G$-spaces [13]. Passing on from the dynamical system problem in the Hilbert space $\mathcal{H}$ to the corresponding one on $M$ is called sometimes dequantization, and the system on $M$ is a classical one. Following Berezin [10], the motion on the classical phase space can be described by the local equation of motion
\[
\dot{z}_\gamma = i\{H, z_\gamma\}, \gamma \in \Delta_+.
\] (7.6)
In eq. (7.6) $\{\cdot, \cdot\}$ denotes the Poisson bracket:
\[
\{f, g\} = \sum_{\alpha, \beta \in \Delta_+} g^{-1}_{\alpha, \beta} \left\{ \frac{\partial f}{\partial z_\alpha} \frac{\partial g}{\partial \bar{z}_\beta} - \frac{\partial f}{\partial \bar{z}_\alpha} \frac{\partial g}{\partial z_\beta} \right\}, f, g \in C^\infty(M).
\]
The equations of motion (7.6) can be written down as
\[
\begin{pmatrix}
0 & \mathcal{G} \\
-\bar{\mathcal{G}} & 0
\end{pmatrix}
\begin{pmatrix}
\dot{z} \\
\dot{\bar{z}}
\end{pmatrix}
= -i \begin{pmatrix}
\frac{\partial}{\partial z} \\
\frac{\partial}{\partial \bar{z}}
\end{pmatrix} H.
\] (7.7)
We consider algebraic Hamiltonian linear in the generators of the group
\[
H = \sum_{\lambda \in \Delta} \epsilon_\lambda G_\lambda.
\] (7.8)
If we take into account eq. (5.10), then the equations of motion (7.6) are (cf. [3, 4])
\[
i \dot{z}_\alpha = \sum_{\lambda \in \Delta} \epsilon_\lambda Q_{\lambda, \alpha}, \alpha \in \Delta_+.
\] (7.9)
For linear Hamiltonians (7.8), we look for the solution of the Schrödinger equation
\[
H \psi = i \dot{\psi}
\] (7.10)
via Perelomov’s coherent state vectors:
\[
\psi = \psi(z) = e^{i\varphi} e^z.
\] (7.11)
It can be proved (cf. [3, 4]) that: If the initial state is a coherent state and the Hamiltonian is linear in the generators of the group, then the state will evolve into a coherent state. More exactly,

**Proposition 3.** On the manifold $M$ of coherent states for which formulas (5.10) are true, the classical motion and the quantum evolution generated by the linear Hamiltonian (7.8) are given both by the same equation of motion (7.9). For semisimple Lie groups the expression of the polynomials $Q$-s is that given in Theorem 1. The phase in eq. (7.11) is given by the sum
\[
\varphi = -\int_0^t H dt - \Im \int_0^t \frac{(e_z, d\bar{e}_z)}{(e_z, e_z)}
\] (7.12)
of the dynamical and Berry phase.
8 Explicit examples of equations of motions

The first two examples presented below are pedagogical. The case of the Grassmann manifold and its non-compact dual are taken from [3, 4]. The last example is derived as application of the expressions of the differential operators presented in §6.4.

8.1 The oscillator Group

The canonical commutation relations of the creation and annihilation operators are

\[ [a_\lambda, a_\lambda^+] = \delta_{\lambda, \lambda'} 1; \lambda, \lambda' = 1, \ldots, n. \]

The Perelomov’s coherent state vectors (Glauber’s coherent states) are

\[ e_z = e^{\sum z_\lambda a_\lambda^+} e_0, \]

where

\[ a_\lambda e_0 = 0, \]
\[ a_\lambda^+ e_z = \partial_\lambda e_z, \]
\[ a_\lambda e_z = z_\lambda e_z. \]

The differential operators are \( G_{\lambda, \mu} = P_{\lambda, \mu} + \sum Q_{\lambda, \mu; \beta} \partial_\beta, \) where \( P_{\lambda, \mu} = 0; Q_{\lambda, \mu; \beta} = z_\mu \delta_{\beta, \lambda}. \)

The linear Hamiltonian

\[ H = \sum \omega_{\lambda, \mu} a_\lambda^+ a_\mu + f_\lambda a_\lambda^+ + \bar{f}_\lambda a_\lambda \]

implies the equation of motion

\[ i\dot{z}_\alpha = \omega_{\alpha, \mu} z_\mu + f_\alpha . \]

8.2 \( SU(2)/U(1) \& SU(1, 1)/U(1) \)

The generators verify the commutation relations

\[ [J_0, J_\pm] = \pm J_\pm; [J_-, J_+] = -2J_0, \]

and

\[ \begin{cases} J_+ e_{j, -j} \neq 0, \\ J_- e_{j, -j} = 0, \\ J_0 e_{j, -j} = -j e_{j, -j}. \end{cases} \]

We have

\[ \begin{cases} J_0 e_z = (-j + z \partial) e_z, \\ J_+ e_z = \partial e_z, \\ J_- e_z = (2j z - z^2 \partial) e_z. \end{cases} \]
The linear Hamiltonian

\[ H = \epsilon_+ J_+ + \epsilon_- J_- + \epsilon_0 J_0, \]

where \( \epsilon^+_0 = \epsilon_0; \; \epsilon^+_- = \epsilon_-; \; \epsilon^+ = \epsilon_+ \), implies the Riccati equation of motion

\[ i \dot{z} = \epsilon_0 z - \epsilon_- z^2 + \epsilon_+. \] (8.1)

In the non-compact case, the generators verifies the commutation relations

\[ [K_0, K_\pm] = \pm K_\pm; \; [K_-, K_+] = 2K_0 \]

and finally we have the equation of motion with a minus sign in the front of the term \( z^2 \) comparatively with the equation (8.1) corresponding to the compact case:

\[ i \dot{z} = \epsilon_0 z + \epsilon_- z^2 + \epsilon_+. \] (8.2)

The change of sign for the Riemann sphere and his non-compact dual as in equations (8.1) and (8.2) take place also in the case of the Grassmann manifold.

### 8.3 The complex Grassmannian \( G_n(\mathbb{C}^{m+n}) \)

The complex Grassmann manifold is \( G_n(\mathbb{C}^{m+n}) \approx SU(n+m)/S((U(n) \times U(m)). \) The non-compact dual of the complex Grassmann manifold is \( SU(n,m)/S((U(n) \times U(m)). \)

The differential operators on \( G_n(\mathbb{C}^{m+n}) \) ("Slater determinant manifold") are (cf. [3, 4]):

\[
\begin{align*}
E^+_{im} &= K^+_{im} = \partial_{im}, \\
E^-_{mi} &= K^-_{mi} = (j_i - j_m)z_{im} + \sum z_{jm}z_{in}\partial_{jn}, \\
H_{\mu\nu} &= \delta_{\mu\nu}j_{\nu} - z_{im}(\delta_{m\mu}\partial_{\nu} - \delta_{\nu\mu}\partial_{m}).
\end{align*}
\]

The linear Hamiltonian is

\[
H = \sum_{1 \leq \mu, \nu \leq m} (\epsilon^0_1)_{\mu\nu} H_{\mu\nu} + \sum_{m < \mu, \nu \leq m+n} (\epsilon^0_2)_{\mu\nu} H_{\mu\nu} + \\
\sum_{i=1}^{m+n} \sum_{p=m+1}^{m+n} \epsilon^+_{ip} F^+_{ip} + \epsilon^-_{ip} F^-_{ip},
\]

where \( F (K) \) are generators for compact (resp. non-compact) Grassmannian.

The equation of motion on the compact (resp., non-compact) Grassmann manifold is the Matrix Riccati equation (cf. [3, 4])

\[ i \dot{Z} = -Z\epsilon^0 Z + \epsilon^0 Z \pm Z \epsilon^- Z, \] (8.3)

\[ (\epsilon^0_{1,2})^+ = \epsilon^0_{1,2}, (\epsilon^+)^+ = \epsilon^- . \]

With the linear fractional change of variables \( Z = XY^{-1} \) we get a linearization of the matrix Riccati equation (8.3)

\[
\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = h_{n,c} \begin{pmatrix} X \\ Y \end{pmatrix}, \quad h_{n,c} = \begin{pmatrix} -i\epsilon^0_1 & -i\epsilon^+ \\ \pm\epsilon^- & i\epsilon^0_2 \end{pmatrix},
\] (8.4)
where the subindex $c$ ($n$) corresponds to the compact (resp., non-compact) case. The dimension of the matrices $X, Y, Z, \epsilon^0_0, \epsilon^0_2, \epsilon^+, \epsilon^-$ are, respectively: $m \times n, n \times n, m \times m, n \times n, m \times n$, and $n \times m$.

### 8.4 $SU(3)/S(U(1) \times U(1) \times U(1))$

Now we come back to the example of §6.4 and we consider the linear hermitian Hamiltonian:

$$H = \sum_{i,j=1}^{3} \epsilon_{ij} C_{ij}; \quad \epsilon_{ij}^+ = \epsilon_{ij}. \quad (8.5)$$

**Proposition 4.** In the parametrization (6.30) on $SU(3)/S(U(1) \times U(1) \times U(1))$, the equations of motion (7.9) associated to the Hamiltonian (8.5) are:

$$\begin{align*}
i \dot{z}_{12} &= -\epsilon_{11} z_{12} + \epsilon_{12} - \epsilon_{21} z_{12}^2 + \epsilon_{22} z_{12} - \epsilon_{31} z_{12} z_{13} + \epsilon_{32} z_{13} \\
i \dot{z}_{13} &= -\epsilon_{11} z_{13} + \epsilon_{13} - \epsilon_{21} z_{12} z_{13} + \epsilon_{23} z_{12} - \epsilon_{31} z_{12}^2 + \epsilon_{33} z_{13} \\
i \dot{z}_{23} &= \epsilon_{21} (z_{12} z_{23} - z_{13}) - \epsilon_{22} z_{23} + \epsilon_{23} + \epsilon_{31} (z_{12} z_{23} - z_{13}) z_{23} + \epsilon_{32} z_{23} + \epsilon_{33} z_{23} \quad (8.6)
\end{align*}$$

**Comment** In the $z$-parametrization given by eq. (6.30) the first two equations in (8.6) do not depend on $z_{23}$ and are in fact a matrix Riccati equation on $SU(3)/S(U(2) \times U(1)) = G_1(C^3) = \mathbb{CP}^2$, in accord with eq. (8.3). In the situation $z_{23} = 0$, the representations in $z$ and $\zeta$ coincides, and the terms in Proposition 4 corresponding to $\partial / \partial z_{12}$ are missing and we get in the equation of motion (8.6) only the first two equations. In the parametrization with $z$ the flag structure of $SU(3)/S(U(1)) \times U(1) \times U(1))$ is reflected in the decoupling of the first two equations of motion of the third.

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