On weighted spectral radius of unraveled balls and normalized Laplacian eigenvalues

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Abstract
For a graph $G$, the unraveled ball of radius $r$ centered at a vertex $v$ is the ball of radius $r$ centered at $v$ in the universal cover of $G$. We obtain a lower bound on the weighted spectral radius of unraveled balls of fixed radius in a graph with positive weights on edges, which is used to present an upper bound on the $s$-th (where $s \geq 2$) smallest normalized Laplacian eigenvalue of irregular graphs under minor assumptions. Moreover, when $s = 2$, the result may be regarded as an Alon–Boppana type bound for a class of irregular graphs.

Key words: Weighted spectral radius; unraveled ball; Alon-Boppana bound; normalized Laplacian eigenvalue; weighted graph.

AMS Classification: 05C50

1 Introduction
For a simple graph $G$ of order $n$, denote the eigenvalues of its adjacency matrix by $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. The degree of a vertex $u$ in $G$, denoted by $d_G(u)$, is defined as the number of edges incident to $u$, and we write $d(u)$ when $G$ is clear.

The well-known result of Alon–Boppana bound may be stated as follows:

**Theorem 1.1.** [11] For any $d$-regular graph $G$ containing two edges with distance at least $2k + 2$,

$$\lambda_2(G) \geq 2\sqrt{d-1} \left(1 - \frac{1}{k+1}\right) + \frac{1}{k+1}. \quad (1)$$

It is natural to generalize the Alon–Boppana bound to graphs that are not necessarily regular. One plausible extension is to verify whether the inequality $\liminf_{i \to \infty} \lambda_2(G_i) \geq 2\sqrt{d-1}$ still holds for any sequence of graphs $\{G_i\}$ with average degree at least $d$ and growing diameter. However,
Hoory [6] constructed a counterexample to disprove the problem. Furthermore, he proposed a new idea of r-robust and provided an Alon–Boppana type bound for a class of irregular graphs as follows. For a graph $G$, the ball of radius $r$ centered at $v$, denoted by $G(v, r)$, is the induced subgraph of $G$ on the vertices with distance at most $r$ from $v$. A graph $G$ has an r-robust average degree $\geq d$ if for the induced subgraph obtained by deleting any ball of radius $r$, its average degree is at least $d$.

**Theorem 1.2.** [6] Given a real number $d \geq 2$ and a natural number $r \geq 2$, if a graph $G$ has an $r$-robust average degree $\geq d$, then

$$\max\{\lambda_2(G), |\lambda_n(G)|\} \geq 2\sqrt{d-1} \left(1 - c \cdot \frac{\log r}{r}\right),$$  
where $c$ is an absolute constant.

Recently, Jiang [7] presented a method of unraveled balls to improve the bound above.

**Theorem 1.3.** [7] Given a real number $d \geq 1$ and a natural number $r \geq 1$, if a graph $G$ has an $r$-robust average degree $\geq d$, then

$$\lambda_2(G) \geq 2\sqrt{d-1} \cos \left(\frac{\pi r}{r+1}\right).$$

It is benefit to consider normalized Laplacian eigenvalues, since they can reveal many fundamental properties of a graph (see [4]). Especially, the second smallest one is tightly related with expansion and algorithmic properties of a graph (see [4] Chapter 2). The normalized Laplacian matrix $\mathcal{L}(G)$ of a graph $G$ is defined to be $I - D^{-\frac{1}{2}}AD^{-\frac{1}{2}}$, where $D$ is the diagonal degree matrix with diagonal element $D(v, v) = d(v)$ for $v \in V(G)$, and $A$ is the adjacency matrix of $G$. Denote the eigenvalues of $\mathcal{L}(G)$ by $\mu_1(G) = \mu_2(G) \leq \cdots \leq \mu_n(G) \leq 2$. In terms of the normalized Laplacian, the well-known Alon–Boppana bound theorem says that for any $d$-regular graph $G$,

$$\mu_2(G) \leq 1 - 2\sqrt{d-1} + o(1),$$

as the diameter of $G$ going to infinity. One may ask whether the assertion that $\limsup_{\epsilon \rightarrow \infty} \mu_2(G_i) \leq 1 - 2\sqrt{\frac{d-1}{d}} + o(1)$ for any sequence of graphs $G_i$ with average degree at least $d$ and growing diameter still holds. Indeed, Young [13] proved that there exists some fixed $\epsilon \geq 0$ and a sequence of graphs $\{G_i\}$ with the common average degree $d$ and common maximum degree (and hence growing diameter) such that for every $i$,

$$\mu_2(G_i) \geq 1 - 2\sqrt{\frac{d-1}{d}} + \epsilon.$$  
(5)

Furthermore, Young [13] generalized the Alon–Boppana bound on the second smallest normalized Laplacian eigenvalue to graphs that may be irregular by adapting the idea of Hoory [6]. Young indicated that [13, Theorem 7] can be phrased in the r-robust average degree framework of Hoory.

The **second order average degree** of a graph $G$ is defined to be

$$\bar{d}_G = \frac{\sum_{u \in V(G)} d(u)^2}{\sum_{u \in V(G)} d(u)}.$$  

A graph $G$ is $(r, d, \bar{d})$-robust if for the induced subgraph of $G$ obtained by deleting any ball of radius $r$, its average degree is at least $d$, and its second order average degree is at most $\bar{d}$.
Theorem 1.4. [13] Given real numbers $\tilde{d} \geq d \geq 2$ and a natural number $r \geq 2$, if a graph $G$ is $(r, d, \tilde{d})$-robust, then

$$\mu_2(G) \leq 1 - \frac{2\sqrt{\tilde{d}^2 - 1}}{d} \left(1 - c \cdot \frac{\log r}{r}\right).$$

(6)

In addition, Chung [3] used a different approach to obtain, under some technical assumptions on a graph, another analogous upper bound $\mu_2(G) \leq 1 - \sigma \left(1 - \frac{c}{r}\right)$, where $\sigma = \frac{2\sum_{u \in V(G)} d(u)\sqrt{d(u) - 1}}{\sum_{u \in V(G)} d(u)^2}$, $k$ is the diameter of $G$, and $c$ is a constant.

Let $G$ be a simple graph. The matrix $D^{-\frac{1}{2}}AD^{-\frac{1}{2}}$ associated with $G$ may be regarded as the adjacency matrix of the weighted graph $(G, w_0)$ with edge weight $w_0(uv) = (d(u)d(v))^{-\frac{1}{2}}$. Moreover, if we denote the second largest eigenvalue of $D^{-\frac{1}{2}}AD^{-\frac{1}{2}}$ by $\lambda_2(G, w_0)$, then the second smallest normalized Laplacian eigenvalue $\mu_2(G)$ is equal to $1 - \lambda_2(G, w_0)$. Based on the observations above, it is believable that by considering weighted graphs one could provide a tighter upper bound on $\mu_2(G)$. The related results are referred to [2, 4, 12, 9]. The motivation of this paper is to solve the problem above by combining the weighting idea of Young [13] and the idea of unraveled ball of Jiang [7]. Indeed, we present an upper bound on $\mu_s(G)$ for $s \geq 2$. Hereafter, denote the set of real numbers and positive real numbers by $\mathbb{R}$ and $\mathbb{R}^+$ respectively.

Recall that the degree of a vertex $u$ in $G$, denoted by $d(u)$, is the number of edges incident to $u$.

One of the main result in this paper is as follows:

Theorem 1.5. If a connected (positively) weighted graph $(G, w)$ has minimum degree at least 2, then for any natural number $r$ with $r \geq 1$ and any function $g : V(G) \rightarrow \mathbb{R}^+$, there exists a vertex $v$ of $G$ such that the weighted spectral radius of the unraveled ball $\tilde{G}(v, r)$ of $G$ satisfies

$$\lambda_1(\tilde{G}(v, r), w) \geq \frac{2\sum_{v_1 \in V(G)} \sqrt{d(v_1)}}{\sum_{v \in V(G)} g(v)d(v)} \left(\sum_{v_2 \in N(v_1)} w(v_1v_2) \sqrt{g(v_1)g(v_2)} \cos \left(\frac{\pi}{r+2}\right)\right),$$

(7)

where the neighborhood $N(v_1)$ of $v_1$ is $\{v \in V(G) : v_1v \in E(G)\}$.

Remark 1. (1) The function $g(x)$ in Theorem 1.5 is just a technical function which may be appropriately chosen to get simpler bounds for different weight graphs. For instance, if $w \equiv 1$, then let $g \equiv 1$ in Theorem 1.5, and we obtain [7, Theorem 1]. As a result, Theorem 1.5 extends the result of Jiang to weighted graphs.

(2) A graph is positively weighted if the graph has a positive weight on every edge.

The other main result in this paper is to derive an upper bound on the $s$th smallest normalized Laplacian eigenvalue. A graph $G$ is $(r, d, \tilde{d}, s)$-robust if for the induced subgraph of $G$ obtained by sequentially deleting any $s$ ball of radius $r$, its average degree is at least $\tilde{d}$, and its second order average degree is at most $d$. When $s = 1$, $(r, d, \tilde{d}, 1)$-robust is just $(r, d, \tilde{d})$-robust. We prove the following:

Theorem 1.6. Given real numbers $\tilde{d} \geq d \geq 2$, and natural numbers, $r \geq 1$ and $s \geq 2$, if a graph $G$ is $(r, d, \tilde{d}, s-1)$-robust, then the $s$-th smallest normalized Laplacian eigenvalue satisfies

$$\mu_s(G) \leq 1 - \frac{2\sqrt{\tilde{d}^2 - 1}}{d} \cos \left(\frac{\pi}{r+1}\right).$$

(8)
Remark 2. Note that
\[ \cos \left( \frac{\pi}{r+1} \right) = 1 - \frac{\pi^2}{2(r+1)^2} + o \left( \frac{1}{r^3} \right). \]
It is easy to see that (8) is a slight improvement of (6) for large \( r \) and \( s = 2 \).

The rest of the paper is organized as follows. In Section 2, some related concepts and symbols are introduced. In Section 3, we prove Theorem 1.5 and include some corollaries. In Section 4, we present a lower bound on the weighted spectral radius of a ball, which is used to prove Theorem 1.6 in Section 5.

2 Preliminary

Definition 2.1. A graph \( G' \) (possibly infinite) is a covering of another graph \( G \) via a covering map \( \varphi: V(G') \to V(G) \) if \( \varphi \) is a surjective map and is a local isomorphism: for every vertex \( v \) of \( G' \), the map \( \varphi \) induces a bijection from the edges incident to \( v \) in \( G' \) to the edges incident to \( \varphi(v) \) in \( G \).

Definition 2.2. The universal cover \( \tilde{G} \) of a connected graph \( G \) is a covering of \( G \) that is a (possibly infinite) tree.

The universal cover of a connected graph is unique up to isomorphism. If \( G \) is a finite tree, then the universal cover of \( G \) is itself. Otherwise, the universal cover of \( G \) is an infinite graph. For instance, the universal cover of a \( d \)-regular graph is the infinite \( d \)-regular tree.

A non-backtracking walk of \( G \) is defined as a walk \((v_0, v_1, \ldots)\) on \( G \) satisfying \( v_i \neq v_{i+2} \) for every \( i \) with \( i \geq 0 \). Specifically, a walk of length at most 1 is just non-backtracking. From the view of random walks, the universal cover can be defined in an equivalent way:

Definition 2.3. The universal cover \( \tilde{G} \) of a connected graph \( G \) is defined as follows: the vertex set consists of all non-backtracking walks on \( G \) starting at a fixed vertex \( v_0 \), and two vertices are adjacent if and only if one is a simple extension of the other. The covering map \( \varphi: V(\tilde{G}) \to V(G) \) is defined by \( \varphi((v_0, \ldots, v_i)) = v_i \) for \((v_0, \ldots, v_i) \in V(\tilde{G}) \).

In fact, the universal cover is independent of the choice of the fixed vertex \( v_0 \). Given a graph \( G \), the unraveled ball of \( G \) is the ball of radius \( r \) centered at \( v \) in the universal cover \( \tilde{G} \) of \( G \), and thus it has an equivalent definition:

Definition 2.4. Given a graph \( G \) and a vertex \( v \) of \( G \), the unraveled ball of \( G \), denoted by \( \tilde{G}(v, r) \), is defined as follows: the vertex set contains all non-backtracking walks on \( G \) of length at most \( r \) starting at \( v \), and two vertices are adjacent if and only if one is a simple extension of the other.

Next we introduce weighted graphs.

Definition 2.5. A weighted graph \((G, w)\) is a graph \( G \) along with a weight function on edges, \( w: E(G) \to \mathbb{R}^+ \). A weighted graph \((G_1, w_1)\) is called the weighted subgraph of \((G, w)\) if \( G_1 \) is a subgraph of \( G \) and \( w_1 = w|_{E(G_1)} \). For simplicity of notations, we denote the weighted subgraph by \((G_1, w)\) instead of \((G_1, w_1)\).
Note that unweighed graphs are just the special case where all the edge weights are equal to 1. Let \((G, w)\) be a weighted graph, and \(\tilde{G}\) the universal cover of \(G\). By the covering map \(\varphi: V(\tilde{G}) \to V(G)\): \(\varphi((v_0, \ldots, v_i)) = v_i\), the weighted function \(w\) can lift in a natural way to a weighted function \(\tilde{w}: E(\tilde{G}) \to \mathbb{R}^+\), which is defined by \(\tilde{w}((v_0, \ldots, v_{i-1})(v_0, \ldots, v_{i-1}, v_i)) = w(v_{i-1}v_i)\). Thus we naturally get a weighted universal cover \((\tilde{G}, \tilde{w})\) from the weighted graph \((G, w)\). For simplicity of notation, we write \((\tilde{G}, w)\) instead of \((\tilde{G}, \tilde{w})\).

**Definition 2.6.** For a weighted graph \((G, w)\) of order \(n\). The adjacency matrix \(A(G, w)\) of \((G, w)\) is defined by

\[
(A(G, w))_{u,v} = \begin{cases} w(uv), & \text{if } uv \in E(G); \\ 0, & \text{otherwise.} \end{cases}
\]

The weighted spectral radius of \(G\) is the spectral radius of \(A(G, w)\), and denoted by \(\lambda_1(G, w)\). Order the eigenvalues of \(A(G, w)\) as \(\lambda_1(G, w) \geq \lambda_2(G, w) \geq \cdots \geq \lambda_n(G, w)\).

### 3 Proof of Theorem 1.5 and Corollaries

The proof of Theorem 1.5 uses an old idea of constructing a weighted test function via non-backtracking walks (see e.g. [7] [5] [22]), and also via the eigenvector of a path (see e.g. [7]).

**Proof of Theorem 1.5.** For \(e = (v_0, v_1) \in W_1\), let \(T_e\) be the component of \(\tilde{G}(v_0, r+1) - (v_0)\) containing the vertex \(e\), and it is a tree. Let \(T\) be the disjoint union of a class of graphs \(\{T_e\}_{e \in W_1}\), and thus \(T\) is a forest. The vertex set of \(T\) is \(\bigcup_{i=1}^{r+1} W_i\), where \(W_i\) is defined as the set of all non-backtracking walks of length \(i\) on \(G\). By regarding every vertex \((v_0, v_1, \ldots, v_i)\) in \(T_e\), \(1 \leq i \leq r+1\), as the vertex \((v_1, \ldots, v_i)\) in \(\tilde{G}(v_1, r)\), we observe that \((T_e, w)\) is a weighted subgraph of \((\tilde{G}(v_1, r), w)\). By monotonicity of weighted spectral radius, \(\lambda_1(\tilde{G}(v_1, r), w) \geq \lambda_1(T_e, w)\) for \(e = (v_0, v_1) \in W_1\). Since \(\lambda_1(T, w) = \max\{\lambda_1(T_e, w): e \in W_1\}\), there exists a vertex \(e^*\) such that \(\lambda_1(T, w) = \lambda_1(T_{e^*}, w)\). It derives that there exists a vertex \(v_i^*\), the terminal vertex of \(e^*\), such that \(\lambda_1(\tilde{G}(v_i^*, r), w) \geq \lambda_1(T, w)\). It suffices to prove that for any function \(g: V \to \mathbb{R}^+\),

\[
\lambda_1(T, w) \geq 2 \cos\left(\frac{\pi}{r+2}\right) \frac{\sum_{(v_1, v_2) \in W_1} \sqrt{d(v_1) - 1} w(v_1v_2) \sqrt{g(v_1)g(v_2)}}{\sum_{v \in V(G)} g(v) d(v)}.
\]

Now we consider a Markov chain on \(W_1\) as follows: the initial state \(E_1\) is chosen from \(W_1\) uniformly at random, and if the current state \(E_i = (v_{i-1}, v_i)\) is given, the next state \(E_{i+1}\) will be chosen from \(\{(v_i, v_{i+1}) \in W_1: v_{i+1} \neq v_{i-1}\}\) uniformly at random. The transition matrix \(P\) is

\[
P_{(u,v),(w,z)} = \begin{cases} \frac{1}{d(w)-1}, & \text{if } v = w \text{ and } z \neq u; \\ 0, & \text{otherwise.} \end{cases}
\]

We can attach \(E_1, \ldots, E_i\) one by one to form a non-backtracking walk on \(G\) of length \(i\), which is denoted by the random variables \(Y_i = (X_0, X_1, \ldots, X_i)\).

It is known that \(\lambda = 2 \cos\left(\frac{\pi}{r+2}\right)\) is the spectral radius of the path \(P_{r+1}\) on \(r+1\) vertices. Let \((x_1, \ldots, x_{r+1}) \in \mathbb{R}^{r+1}\) be a positive eigenvector of \(P_{r+1}\) associated with \(\lambda\). By the Rayleigh principle, it follows that

\[
\sum_{i=2}^{r+1} 2x_{i-1}x_i = \lambda \cdot \sum_{i=1}^{r+1} x_i^2. \tag{9}
\]
Define the vector

\[ f: \bigcup_{i=1}^{r+1} W_i \to \mathcal{R}; \quad f(\omega) = x_i \sqrt{g(v_i) \Pr(Y_i = \omega)} \]

for \( \omega = (v_0, v_1, \ldots, v_i) \in W_i \), where \( g: V(G) \to \mathcal{R}^+ \) is a fixed vertex weight function. Let \( A(w) \) be the adjacency matrix of the weighted forest \((T, w)\). For \( \omega = (v_0, \ldots, v_{i-1}, v_i) \), let \( \omega^- = (v_0, \ldots, v_{i-1}) \).

By simple calculations, we have

\[ \langle f, A(w)f \rangle = \sum_{i=2}^{r+1} \sum_{\omega \in W_i} 2f(\omega) \cdot f(\omega) \cdot w(\omega^-) \]

\[ = \sum_{i=2}^{r+1} 2x_{i-1}x_i \sum_{\omega \in W_i} w(\omega^-) \sqrt{g(v_{i-1})g(v_i)} \Pr(Y_{i-1} = \omega^-) \cdot \Pr(Y_i = \omega). \]

(12)

For simplicity of notation, let

\[ I_i = \sum_{\omega \in W_i} g(v_i) \Pr(Y_i = \omega), \]

\[ J_i = \sum_{\omega \in W_i} w(\omega^-) \sqrt{g(v_{i-1})g(v_i)} \Pr(Y_{i-1} = \omega^-) \cdot \Pr(Y_i = \omega). \]

In order to complete the proof, it suffices to simplify \( I_i \) and \( J_i \) for every \( i \). Firstly, we have

\[ I_i = \mathbb{E}[g(X_i)] = \sum_{v \in V(G)} g(v) \cdot \Pr(X_i = v). \]

(13)

for \( i \geq 1 \). Secondly, by the Markov property, for \( i \geq 2 \) and \( \omega = (v_0, v_1, \ldots, v_i) \in W_i \),

\[ \Pr(Y_{i-1} = \omega^-) = \frac{\Pr(Y_i = \omega)}{\Pr(E_i = (v_{i-1}, v_i)|E_{i-1} = (v_{i-2}, v_{i-1}))} = (d(v_i) - 1) \Pr(Y_i = \omega). \]

Note that \( w(\omega^-) \) is defined to be \( w(v_{i-1}v_i) \). Thus it follows that for \( i \geq 2 \),

\[ J_i = \sum_{\omega = (v_0, v_1, \ldots, v_i) \in W_i} \sqrt{d(v_{i-1}) - 1} w(v_{i-1}v_i) \sqrt{g(v_{i-1})g(v_i)} \Pr(Y_i = \omega) \]

\[ = \mathbb{E} \left[ \sqrt{d(X_{i-1}) - 1} w(X_{i-1}X_i) \sqrt{g(X_{i-1})g(X_i)} \right] \]

\[ = \sum_{(v_1, v_2) \in W_1} \sqrt{d(v_1) - 1} w(v_1v_2) \sqrt{g(v_1)g(v_2)} \Pr(X_{i-1} = v_1, X_{i-2} = v_2). \]

(14)

Now we focus on the probabilities in (13) and (14). Since the minimum degree of \( G \) is at least 2, the Markov chain has no absorbing states. And it is easy to see that the uniform distribution \( x = (\frac{1}{|W_1|}, \ldots, \frac{1}{|W_1|}) \) on \( W_1 \) is a stationery distribution of the Markov chain, that is \( x = xP^{i-1} \) for \( i \geq 1 \), where \( P \) is the transition matrix. Thus we have \( \Pr(E_i = e) = 1/|W_1| \) for \( i \geq 1 \) and \( e \in W_1 \), which derives that for \( i \geq 1 \) and \( v \in V(G) \),

\[ \Pr(X_i = v) = \sum_{e \sim v} \Pr(E_i = e) = \frac{d(v)}{|W_1|}, \]

(15)
where $e \sim v$ denotes ranging over all the edges incident to $v$. Hence for $i \geq 2$ and $(v_1, v_2) \in W_1$,

\[
\Pr(X_{i-1} = v_1, X_i = v_2) = \sum_{\{u: (u, v_1) \in W_1, u \neq v_2\}} \Pr(E_{i-1} = (u, v_1), E_i = (v_1, v_2))
\]

\[
= \sum_{\{u: (u, v_1) \in W_1, u \neq v_2\}} \Pr(E_i = (v_1, v_2)|E_{i-1} = (u, v_1))\Pr(E_{i-1} = (u, v_1))
\]

\[
= \sum_{\{u: (u, v_1) \in W_1, u \neq v_2\}} \frac{1}{(d(v_1) - 1)|W_1|} \frac{1}{|W_1|}.
\]

By rewriting $\Pr(X_i = v)$ in (13) and $\Pr(X_{i-1} = v_1, X_i = v_2)$ in (14), we can simplify $I_i$ and $J_i$, and substitute them to (11) and (12) to obtain

\[
\langle f, f \rangle = \sum_{i=1}^{r+1} x_i^2 \sum_{v \in V(G)} g(v) \frac{d(v)}{|W_1|},
\]

\[
\langle f, A(w) f \rangle = \sum_{i=2}^{r+1} 2x_{i-1}x_i \sum_{(v_1, v_2) \in W_1} \sqrt{d(v_1) - 1} w(v_1 v_2) \sqrt{g(v_1)g(v_2)} \frac{1}{|W_1|}.
\]

Finally, combining (11), the equalities above, and the Rayleigh principle, we obtain

\[
\lambda_1(T, w) \geq \frac{\langle f, A(w) f \rangle}{\langle f, f \rangle} = 2\cos\left(\frac{\pi}{r + 2}\right) \frac{\sum_{(v_1, v_2) \in W_1} \sqrt{d(v_1) - 1} w(v_1 v_2) \sqrt{g(v_1)g(v_2)}}{\sum_{v \in V(G)} g(v) d(v)},
\]

for any vertex weight function $g$, and complete the proof.

If the weight function $w_0$ is defined by $w_0(uv) = (d(u)d(v))^{-\frac{1}{2}}$ for every edge $uv$ of a graph $G$, then let $g(v) = d(v)$ for every vertex $v$ of $G$ in Theorem 1, and we derive the following corollary.

**Corollary 3.1.** If a connected weighted graph $(G, w_0)$ with edge weight $w_0(uv) = (d(u)d(v))^{-\frac{1}{2}}$ has minimum degree at least 2, then for any natural number $r$ with $r \geq 1$, there exists a vertex $v$ of $G$ such that

\[
\lambda_1(\tilde{G}(v, r), w_0) \geq 2\cos\left(\frac{\pi}{r + 2}\right) \frac{\sum_{u \in V(G)} d(u) \sqrt{d(u) - 1}}{\sum_{u \in V(G)} d(u)^2}.
\]

Since the weighted unraveled ball $(\tilde{G}(v, r), w_0)$ is a weighted induced subgraph of the weighted universal cover $(\tilde{G}, w_0)$, it follows that $\lambda_1(\tilde{G}, w_0) \geq \lambda_1(\tilde{G}(v, r), w_0)$ by the monotonicity of weighted spectral radius. Thus we can obtain a lower bound on $\lambda_1(\tilde{G}, w_0)$ by letting $r$ go to infinity in Corollary 3.1.

**Corollary 3.2.** If a connected weighted graph $(G, w_0)$ with edge weight $w_0(uv) = (d(u)d(v))^{-\frac{1}{2}}$ has minimum degree at least 2, then the weighted spectral radius of its universal cover satisfies

\[
\lambda_1(\tilde{G}, w_0) \geq 2\cos\left(\frac{\pi}{r + 2}\right) \frac{\sum_{u \in V(G)} d(u) \sqrt{d(u) - 1}}{\sum_{u \in V(G)} d(u)^2}.
\]

### 4 Weighted spectral radius of a ball

For a weighted graph, the weight of a closed walk is the product of weights of all edges on the closed walk. The following result is well-known.
Lemma 4.1. [10] For any connected weighted graph \((G, w)\) (possibly infinite) and every vertex \(v\) of \(G\), the weighted spectral radius of \((G, w)\) is

\[
\lambda_1(G, w) = \limsup_{k \to \infty} \sqrt[k]{t_{2k}^{(w)}(v)},
\]

where \(t_{2k}^{(w)}(v)\) is the total weight of all closed walks of length \(2k\) from \(v\) to itself in \(G\).

The following lemma establishes connections between the weighted spectral radius of a ball and its corresponding unraveled ball, and is an extension of [9] Theorem 2.2).

**Lemma 4.2.** For every vertex \(v\) of a weighted graph \((G, w)\) and any natural number \(r \geq 1\),

\[
\lambda_1(G(v, r), w) \geq \lambda_1(G(v, r), w).
\]

**Proof.** Recall that the vertex set of \(\tilde{G}(v, r)\) consists of all non-backtracking walks of length at most \(r\) starting at \(v\). By the covering map \(\varphi: V(\tilde{G}) \to V(G); \ \varphi((v_0, \ldots, v_1)) = v_i,\) we can naturally construct a map \(\sigma\), mapping a closed walk \((\omega = \omega_0, \ldots, \omega_{2k} = \omega)\) of length \(2k\) in \(\tilde{G}(v, r)\) to a closed walk \((v = v_0, \ldots, v_{2k} = v)\) of length \(2k\) in \(G(v, r)\) for \(k \geq 0\), where \(v_j\) is the terminal vertex of \(\omega_j\) for every \(j\).

It is obvious that \(\sigma\) is an injective map. In fact, since the covering map \(\varphi\) is a local isomorphism, there exists an inverse map \(\tau\) such that \(\tau \sigma = \text{id}\). In addition, by naturally lifting the weight function \(w\) to a weight function of \(\tilde{G}\), and the weights of walks are invariant under the map \(\sigma\). Hence, the sum of the weights of closed walks of length \(2k\) on \(\tilde{G}(v, r)\) is no more than the sum of the weights of closed walks of length \(2k\) on \(G(v, r)\). Therefore, Lemma 4.1 gives that \(\lambda_1(G(v, r), w) \geq \lambda_1(G(v, r), w)\).

Combining Theorem 1.5 and Lemma 4.2, we prove a lower bound on the weighted spectral radius of a ball of a graph with edge weight \(w_0(uv) = (d(u)d(v))^{-\frac{1}{2}}\), which also has its own interest.

**Theorem 4.3.** Let \((G, w_0)\) be a weighted graph with edge weight \(w_0(uv) = (d(u)d(v))^{-\frac{1}{2}}\). If the graph \(G\) has average degree \(d\) with \(d \geq 2\) and second order average degree \(\tilde{d}\), then for any natural number \(r \geq 1\), there exists a vertex \(v\) of \(G\) such that

\[
\lambda_1(G(v, r), w_0) \geq \frac{2\sqrt{d - 1}}{d} \cos \left(\frac{\pi}{r + 2}\right).
\]

**Proof.** Since \(G\) may have vertices of degree 1, we cannot use Corollary 3.1 directly. Instead, we consider the 2-core \(H\) in \(G\), the largest induced subgraph of \(G\) with minimum degree at least 2. Observe that the 2-core \(H\) can be obtained from \(G\) by deleting vertices of degree 1 sequentially. Since removing vertices of degree 1 from a graph of average degree at least cannot decrease its average degree, the 2-core \(H\) is non-empty. We decompose the proof into two parts.

**Case 1:** The 2-core \(H\) is connected. Similarly with Corollary 3.1 by setting \(w_0(uv) = (d_G(u)d_G(v))^{-\frac{1}{2}}\) and \(g(v) = d_G(v)\) in Theorem 1.5 we can derive that there exists a vertex \(v\) of \(H\) such that

\[
\lambda_1(\tilde{H}(v, r), w_0) \geq \frac{2\sum_{u \in V(H)} d_H(u) \sqrt{d_H(u) - 1}}{\sum_{u \in V(H)} d_H(u)d_G(u)} \cos \left(\frac{\pi}{r + 2}\right).
\]
Since \((H(v, r), w_0)\) is a weighted subgraph of \((G(v, r), w_0)\), the monotonicity of weighted spectral radius and Lemma 4.2 derive that
\[
\lambda_1(G(v, r), w_0) \geq \lambda_1(H(v, r), w_0) \geq \lambda_1(\overline{H}(v, r), w_0).
\]

Combing the inequalities above, we only need to prove that
\[
\frac{\sum_{u \in V(H)} d_H(u)\sqrt{d_H(u)} - 1}{\sum_{u \in V(H)} d_H(u)d_G(u)} \geq \frac{\sqrt{d-1}}{d}.
\]  \hspace{1cm} (16)

Recall that the average degree of \(H\) is at least \(d\). Note that \(h(x) = x\sqrt{x-1}\) is a convex function for \(x \geq 2\). It follows from Jensen’s inequality that
\[
\sum_{u \in V(H)} d_H(u)\sqrt{d_H(u)} - 1 \geq \sum_{u \in V(H)} d_H(u)\sqrt{d-1}.
\]  \hspace{1cm} (17)

Then it suffices to prove
\[
\frac{\sum_{u \in V(H)} d_H(u)d_G(u)}{\sum_{u \in V(H)} d_H(u)} \leq \overline{d} = \frac{\sum_{u \in V(G)} d_G(u)^2}{\sum_{u \in V(G)} d_G(u)},
\]  \hspace{1cm} (18)

since (17) and (18) imply (16). Let \(H'\) be the spanning subgraph of \(G\) composed of \(H\) plus all isolated vertices in \(V \setminus V(H)\). It is obvious that
\[
\frac{\sum_{u \in V(H)} d_H(u)d_G(u)}{\sum_{u \in V(H)} d_H(u)} = \frac{\sum_{u \in V(G)} d_H(u)d_G(u)}{\sum_{u \in V(G)} d_H(u)}.
\]  \hspace{1cm} (19)

Recall that the 2-core \(H\) can be obtained from \(G\) by deleting vertices of degree 1 sequentially. Thus we can recover \(G\) by \(H'\), by sequentially adding some edges of \(G\) in the opposite order, and each of the edges joins a non-isolated vertex and an isolated vertex in the current state. Assume that there are \(m\) edges to be added. For simplicity of notation, let \(G^{(0)} = H'\) and \(G^{(m)} = G\). In the \(i\)th step, assume that some edge \(v_1v_2\) of \(G\) is added to \(G^{(i-1)}\), and \(G^{(i)}\) is the resulting graph. Note that \(\frac{d}{y} \leq \frac{d}{y+2}\) for \(y \geq 2x\). It follows that for all \(i\),
\[
\frac{d_{G^{(i)}}(v_j)}{\sum_{u \in V(G)} d_{G^{(i)}}(u)} \leq \frac{d_{G^{(i)}}(v_j) + 1}{\sum_{u \in V(G)} d_{G^{(i)}}(u) + 2} = \frac{d_{G^{(i+1)}}(v_j)}{\sum_{u \in V(G)} d_{G^{(i+1)}}(u)}, \forall j = 1, 2,
\]
\[
\frac{d_{G^{(i)}}(z)}{\sum_{u \in V(G)} d_{G^{(i)}}(u)} \leq \frac{d_{G^{(i)}(z)}}{\sum_{u \in V(G)} d_{G^{(i)}}(u) + 2} = \frac{d_{G^{(i+1)}(z)}}{\sum_{u \in V(G)} d_{G^{(i+1)}}(u)}, \forall z \in V(G) \setminus \{v_1, v_2\}.
\]

Consequently, we have
\[
\frac{d_{H^{(i)}}(z)}{\sum_{u \in V(G)} d_{H^{(i)}}(u)} \leq \frac{d_{G^{(i)}}(z)}{\sum_{u \in V(G)} d_{G^{(i)}}(u)} \leq \cdots \leq \frac{d_{G}(z)}{\sum_{u \in V(G)} d_{G}(u)}, \forall z \in V(G).
\]  \hspace{1cm} (20)

Therefore, (18) follows by (19) and (20).

Case 2: The 2-core \(H\) is disconnected. Now \(G\) is also disconnected. Assume that the disconnected graph \(G\) is composed of \(G_1, \ldots, G_t\). Let \(H_i\) be the connected 2-core of \(G_i\) with vertex set \(V_i\) for \(1 \leq i \leq t\). By the same argument with Case 1, there exists a vertex \(u_i\) of \(H_i\) such that
\[
\lambda_1(G(u_i, r), w_0) \geq \frac{2}{\pi} \frac{d_H(u)\sqrt{d_H(u)} - 1}{\sum_{u \in V(G)} d_H(u)d_G(u)} \cos \left( \frac{\pi}{r + 2} \right) = \frac{M_i}{N_i} \cos \left( \frac{\pi}{r + 2} \right).
\]
One can simply verify that there exists an \(i_0 \in [1, t]\) such that
\[
\frac{M_1 + \cdots + M_t}{N_1 + \cdots + N_t} \leq \max \left\{ \frac{M_1}{N_1}, \ldots, \frac{M_t}{N_t} \right\} = \frac{M_{i_0}}{N_{i_0}}.
\]
Thus there exists a vertex \(u_{i_0}\) of \(H_{i_0}\) such that
\[
\lambda_1(G(u_{i_0}, r), w_0) \geq \frac{2\sum_{u \in V(H)} d_H(u)\sqrt{d_H(u) - 1}}{\sum_{u \in V(H)} d_H(u)d_G(u)} \cos \left( \frac{\pi}{r + 2} \right).
\]
Using the same argument with Case 1, we can prove that there exists a vertex \(u_{i_0}\) of \(G\) such that
\[
\lambda_1(G(u_{i_0}, r), w_0) \geq \frac{2\sqrt{d^2 - 1}}{d} \cos \left( \frac{\pi}{r + 2} \right).
\]
\[\square\]

5 Proof of Theorem 1.6

Recall that a graph \(G\) is \((r, d, \tilde{d}, s, r, d, s - 1)\)-robust if for the induced subgraph of \(G\) obtained by sequentially deleting any \(s\) ball of radius \(r\), its average degree is at least \(d\), and its second order average degree is at most \(\tilde{d}\). Before proving Theorem 1.6, we provide a lower bound on the \(s\)-th largest weighted eigenvalue of a graph.

Lemma 5.1. Let \((G, w_0)\) be a weighted graph with edge weight \(w_0(\{u, v\}) = (d(u)d(v))^{-\frac{1}{2}}\). Given real numbers, \(\tilde{d} \geq d \geq 2\), and natural numbers, \(r \geq 1\) and \(s \geq 2\), if \(G\) is \((r, d, \tilde{d}, s - 1)\)-robust, then
\[
\lambda_s(G, w_0) \geq \frac{2\sqrt{d^2 - 1}}{d} \cos \left( \frac{\pi}{r + 1} \right).
\]

Proof. We show how to sequentially construct \(G_1, \ldots, G_s\), a collection of pairwise disjoint weighted induced subgraphs of \(G\) such that no vertex in \(V(G_i)\) is adjacent to a vertex of \(V(G_j)\) for \(i \neq j\), and \(\lambda_1(G_i, w_0) \geq \frac{2\sqrt{d^2 - 1}}{d} \cos \left( \frac{\pi}{r + 1} \right)\) holds for \(1 \leq i \leq s\).

For simplicity of notation, let \(G^{(0)} = G\). For \(1 \leq i \leq s\), assume that we have obtained \(G^{(i-1)}\), an induced subgraph of \(G\), by sequentially deleting \(i - 1\) balls of radius \(r\) during the previous steps. In the \(i\)th step, let \(H_i\) be the graph obtained by sequentially deleting any \(s - i\) balls of radius \(r\) from \(G^{(i-1)}\). Since \(G\) is \((r, d, \tilde{d}, s - 1)\)-robust, the average degree of \(H_i\) is at least \(d\), and the second order average degree of \(H_i\) is at most \(\tilde{d}\). By Theorem 4.3 there exists a vertex \(v_i\) of \(H_i\) such that
\[
\lambda_1(H_i(v_i, r - 1), w_0) \geq \frac{2\sqrt{d^2 - 1}}{d} \cos \left( \frac{\pi}{r + 1} \right).
\]
Since \(H_i\) is a subgraph of \(G^{(i-1)}\), \(H_i(v_i, r - 1)\) is a subgraph of \(G^{(i-1)}(v_i, r - 1) =: G_i\). Thus as desired we obtain
\[
\lambda_1(G_i, w_0) \geq \lambda_1(H_i(v_i, r - 1), w_0) \geq \frac{2\sqrt{d^2 - 1}}{d} \cos \left( \frac{\pi}{r + 1} \right).
\]
Let \(G^{(i)}\) be the induced subgraph of \(G^{(i-1)}\) obtained by deleting a ball \(G^{(i-1)}(v_i, r)\). Turn to the next step, until we get \(G_1, \ldots, G_s\).
For $1 \leq i \leq s$, let $A_i(w_0)$ be the adjacency matrix of $(G_i, w_0)$. Let $A(w_0)$ be the adjacency matrix of $(G, w_0)$. Note that $A_i(w_0)$ is just the principal sub-matrix of $A(w_0)$ corresponding to $G_i$, but not equal to $I - \mathcal{L}(G_i)$. Additionally, let $f_i$ be a positive unit eigenvector of $A_i(w_0)$ associated with $\lambda_i(G_i, w_0)$. We can define a vector $g_i: V(G) \rightarrow \mathbb{R}$ by

$$g_i(u) = \begin{cases} f_i(u), & \text{if } u \in V(G_i); \\ 0, & \text{otherwise}, \end{cases}$$

for every $i$ with $1 \leq i \leq s$. It is obvious that $V(G_i) \cap V(G_j) = \emptyset$ for $i \neq j$, so $\{g_1, \ldots, g_s\}$ is a set of orthonormal vectors. Define $W_0$ as an $s$-dimension vector space spanned by $\{g_1, \ldots, g_s\}$. By the Rayleigh Principle, it follows that

$$\lambda_s(G, w_0) = \max_{\dim W = s} \min_{f \in W} \frac{\langle f, A(w_0)f \rangle}{\langle f, f \rangle} \geq \min_{f \in W_0} \frac{\langle f, A(w_0)f \rangle}{\langle f, f \rangle}.$$ 

It is obvious that $uv \notin E$ for $u \in V(G_i)$ and $v \in V(G_j)$ with $i \neq j$. Taking any non-zero element $f = c_1g_1 + \cdots + c_sg_s$ in $W_0$, we have

$$\frac{\langle f, A(w_0)f \rangle}{\langle f, f \rangle} = \frac{c_1^2\langle f_1, A_1(w_0)f_1 \rangle + \cdots + c_s^2\langle f_s, A_s(w_0)f_s \rangle}{c_1^2\langle f_1, f_1 \rangle + \cdots + c_s^2\langle f_s, f_s \rangle} = \frac{c_1^2\lambda_1(G_1, w_0)\langle f_1, f_1 \rangle + \cdots + c_s^2\lambda_s(G_s, w_0)\langle f_s, f_s \rangle}{c_1^2\langle f_1, f_1 \rangle + \cdots + c_s^2\langle f_s, f_s \rangle} \geq \frac{2\sqrt{d-1}}{d} \cos \left( \frac{\pi}{r+1} \right).$$

Finally, we complete the proof by

$$\lambda_s(G, w_0) \geq \min_{f \in W_0} \frac{\langle f, A(w_0)f \rangle}{\langle f, f \rangle} \geq \frac{2\sqrt{d-1}}{d} \cos \left( \frac{\pi}{r+1} \right).$$

\[\square\]

**Proof of Theorem 1.6** Recall that the normalized Laplacian matrix $\mathcal{L}$ of $G$ is defined to be $I - D^{-\frac{1}{2}}AD^{-\frac{1}{2}}$, where $D$ is the diagonal degree matrix of $G$ and $A$ is the adjacency matrix of $G$. If we consider a weighted graph $(G, w_0)$ with edge weight $w_0(uv) = (d(u)d(v))^{-\frac{1}{2}}$, then we have $\mathcal{L} = I - A(G, w_0)$. By applying Lemma 5.1, we can obtain the desired upper bound on $\mu_s(G) = 1 - \lambda_s(G, w_0)$. \[\square\]

**Acknowledgements**

The authors would be grateful to the referees for their valuable suggestions and comments which make a great improvement of the manuscript.

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