**SO(N) Superpotential, Seiberg-Witten Curves and Loop Equations**

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**Abstract**

We consider the exact superpotential of $\mathcal{N} = 1$ super Yang-Mills theory with gauge group $SO(N)$ and arbitrary tree-level polynomial superpotential of one adjoint Higgs field. A field-theoretic derivation of the glueball superpotential is given, based on factorization of the $\mathcal{N} = 2$ Seiberg-Witten curve. Following the conjecture of Dijkgraaf and Vafa, the result is matched with the corresponding $SO(N)$ matrix model prediction. The verification involves an explicit solution of the first non-trivial loop equation, relating the spherical free energy to that of the non-orientable surfaces with topology $\mathbb{RP}^2$. 
1 Introduction

Recently there has been a spectacular progress in the understanding the low-energy dynamics of a large class of $\mathcal{N} = 1$ supersymmetric gauge theories. Although the proposal of Dijkgraaf and Vafa (DV) linking effective superpotentials to random matrix quantities arose from string theoretic reasonings [1, 2, 3], its main part has subsequently been proven by purely field theoretic methods in [4, 5].

There has been much research in extending the DV framework to accommodate matter in the fundamental representation [6, 7, 8, 9, 10, 11, 12, 13], exhibit Seiberg-Witten curves [14, 15, 16, 17, 18, 19, 20, 21, 22, 23], calculate gravitational couplings [24, 25] and various other related developments [26, 27, 28, 29, 30, 31].

As pointed out in [32, 33] much of the physics of $\mathcal{N} = 2$ theories deformed by tree-level potentials can be effectively obtained from the knowledge that the appropriate Seiberg-Witten (SW) curve [34, 35] for the undeformed theory factorizes. This approach has been used in [16] (see also [4, 5]), together with the Intriligator-Leigh-Seiberg (ILS) linearity principle [36] and ‘integrating-in’ techniques to derive the random matrix DV superpotential directly from properties of SW curves for unitary groups.

The object of this paper is to i) calculate the effective glueball superpotentials for orthogonal groups from the SW perspective and then ii) subsequently verify the DV conjecture by deriving the same result from the random matrix perspective using loop equation techniques. This is interesting as these involve nonorientable graphs on the random matrix side and especially as the original proposal of DV [3] was somewhat ambiguous (and indeed has to be slightly modified). Very recently, as this work was in progress, there appeared papers which addressed the orthogonal groups from different perspectives, namely perturbative methods [37] and CY/diagrammatic methods [38]. The interesting paper [38] has some overlap in that the loop equation is also used there. We differ, however, on various aspects of the derivation of the equation as well as the solution of it.

The plan of this paper is as follows. In section 2 we present the factorization solution for SW curves for orthogonal groups and use this information to construct the effective glueball superpotential by integrating-in $S$. Then we discuss in section 3 how this result should be reproduced from a random matrix model following the DV proposal. This implies in particular a non-trivial relation between the spherical and $\mathbb{R}P^2$ contribution to the free energy, as was also noted in [37, 38]. Finally, in section 4 we prove for arbitrary tree level potentials the required random matrix identity using loop equation techniques. We close the paper with a discussion.

2 Exact $SO(N)$ superpotential from factorized SW curve

In this section we give a field-theoretic derivation of the exact superpotential in $\mathcal{N} = 1$ super Yang-Mills theory with gauge group $SO(N)$ and arbitrary polynomial superpotential
of one adjoint Higgs field.

We thus consider $\mathcal{N} = 2$ $SO(N)$ gauge theory broken to $\mathcal{N} = 1$ by a tree-level superpotential

$$W_{\text{tree}} = \sum_{p \geq 1} \frac{g_{2p}}{2p} \text{Tr} \Phi^{2p}$$  \hspace{1cm} (1)$$

where the field $\Phi$ is in the adjoint. Note that, due to the antisymmetry of $\Phi$ only even terms in the potential contribute. Following the computation for $SU(N)$ in Ref [16], we compute here the exact superpotential for the $SO(N)$ case, in the confining vacuum where $\langle \Phi \rangle = 0$ classically.

To this end we use the ILS linearity principle [36] which implies that under the addition of (1), the exact effective superpotential is given by

$$W_q = \sqrt{2} \sum_{m=1}^{r} M_m M_m a_m^D(v_p, \Lambda) + \sum_{p \geq 1} g_p v_p$$  \hspace{1cm} (2)$$

Here, $\Lambda$ is the scale governing the running of the gauge coupling constant, $M_m$, $\tilde{M}_m$ are the monopole fields, $a_m^D$ are the dual $\mathcal{N} = 2$ $U(1)$ vector multiplet scalars, $r = \lfloor N/2 \rfloor$ is the rank of $SO(N)$, and we have defined

$$v_p \equiv \frac{1}{2p} \text{Tr} \Phi^{2p}$$  \hspace{1cm} (3)$$

Our aim is to obtain the universal superpotential $W(S, \Lambda)$ which we achieve by first integrating out the monopole fields and subsequently integrating in the $S$ field. Turning to the first step, the equation of motion reads

$$a_m^D(v_p, \Lambda) = 0 \hspace{0.5cm}, \hspace{0.5cm} m = 1 \ldots r$$  \hspace{1cm} (4)$$

where we assume that all species of monopoles condense. The $a_m^D$ are given by integrals of a meromorphic form over cycles of hyperelliptic curves [34, 35]. In particular, for the case of $SO(N)$ these were obtained in [39] and [40] for $N$ odd and even respectively, and may be summarized according to

$$g^2 = P(x)^2 - 4x^2q \lambda^{2\tilde{h}} \hspace{1cm}, \hspace{1cm} P(x) = \prod_{k=1}^{r} (x^2 - e_k^2)$$  \hspace{1cm} (5)$$

Here $\tilde{h} = N - 2$ is the dual Coxeter number of $SO(N)$ and $q = 2r - \tilde{h}$ (hence $q = 2$ for $SO(2N)$ and $q = 1$ for $SO(2N + 1)$). The relation between the $v_p$ in (3) and the moduli $e_k$ in (5) is

$$v_p = \frac{1}{p} \sum_{k=1}^{r} e_k^{2p}$$  \hspace{1cm} (6)$$

The vanishing of the $a_m^D$ implies a factorization constraint on $P(x)$. For $SU(N)$ this was solved in [41], while in the case of $SO(N)$ we find a similar solution

$$P(x) = 2x^q \lambda^{\tilde{h}} T_{\tilde{h}} \left( \frac{x}{2\Lambda} \right)$$  \hspace{1cm} (7)$$

\[\text{See also Refs. [42, 43, 23].}\]
where \( T_l(z) = \cos(l \arccos(z)) \) is a Chebyshev polynomial. Indeed, it is easy to check that for this choice the SW curve factorizes

\[
y^2 = 4^{q+1} \Lambda^{4r} z^{2q}(z^2 - 1)[U_{h-1}(z)]^2, \quad z = \frac{x}{2\Lambda}
\]  

(8)

For \( SO(2N) \) this shows one six-fold, two single and \( 2N - 4 \) double zeroes, while for \( SO(2N + 1) \) there are two single and \( 2N - 1 \) double zeroes. Moreover, it is not difficult to check that the corresponding meromorphic one-form is

\[
\lambda = \frac{1}{2\pi i} (qP(x) - xP'(x)) \frac{dx}{y} = -\frac{x^{q+1} \tilde{h}}{2\pi} \frac{dz}{\sqrt{1 - z^2}}
\]  

(9)

exhibiting singularities at the single zeroes \( z = \pm 1 \) of \( y \) only.

From the solution (7) we may now read off that at the factorization point, the zeroes of \( P(x) \) are \( x = 0 \) and

\[
e_k = 2\Lambda \cos \left( \frac{\pi k}{\tilde{h}} - \frac{1}{2} \right), \quad k = 1 \ldots \tilde{h}
\]  

(10)

Because of the symmetry \( e_{\tilde{h}+1-k} = -e_k \) this means that the \( r \) zeroes \( e_k^2 \) in (5) are given by

\[
e_k^2 = (2\Lambda)^2 \cos^2 \left( \frac{\pi k}{\tilde{h}} - \frac{1}{2} \right), \quad k = 1, \ldots, r - 1 \, , \quad e_r^2 = 0
\]  

(11)

Substituting this in (6) then yields after some algebra

\[
v_p(\Lambda^2) = \frac{\tilde{h}}{2p} \binom{2p}{p} \Lambda^{2p}
\]  

(12)

which thus specifies the point in the moduli space where the monopoles coupling to each \( U(1) \) in the Cartan subalgebra of \( SO(N) \) have become massless. It then follows from (2) that the exact superpotential is

\[
W(\Lambda^2, g_{2p}) = \sum_{p \geq 1} g_{2p} v_p(\Lambda^2)
\]  

(13)

We can now integrate in \( S \) by Legendre transforming (13) using

\[
\frac{\partial W}{\partial \ln \Lambda^{N-2}} = S
\]  

(14)

which fixes the normalization\(^2\) of \( S \).

The final result for the exact \( SO(N) \) superpotential is then

\[
W_{\text{eff}}(S, \Lambda^2, g_{2p}) = \frac{N - 2}{2} \left[ -S \ln(\tilde{\Lambda}(S)/\Lambda)^2 + \sum_{p \geq 1} \frac{g_{2p}}{p} \binom{2p}{p} [\tilde{\Lambda}(S)]^{2p} \right]
\]  

(15)

\(^2\) This choice was also adopted in ref. [44].
where the function $\hat{\Lambda}(S)$ is determined by the solution of the equation

$$S = \sum_{p \geq 1} g_{2p} \binom{2p}{p} \hat{\Lambda}^{2p}$$

which follows from (14), using (13), (12).

At this point, it is useful to recall the corresponding result for $SU(N)$ (or $U(N)$ with even potential) [16]

$$W_{\text{eff}}^{SU(N)}(S, \Lambda^2, f_{2p}) = N \left[ -S \ln(\hat{\Lambda}(S)/\Lambda)^2 + \sum_{p \geq 1} \frac{f_{2p}}{2p} \binom{2p}{p} [\hat{\Lambda}(S)]^{2p} \right]$$

$$S = \frac{1}{2} \sum_{p \geq 1} f_{2p} \binom{2p}{p} \hat{\Lambda}^{2p}$$

where the tree-level potential is as in (1) with coupling constants $f_{2p}$. We thus note the simple relation between the two cases

$$W_{\text{eff}}^{SO(N)}(g_{2p}) = \frac{N - 2}{2N} W_{\text{eff}}^{SU(N)}(f_{2p} = 2g_{2p})$$

which will be relevant below.

### 3 The Dijkgraaf-Vafa proposal

Following the conjecture of Dijkgraaf and Vafa [3], we expect to reproduce the exact $SO(N)$ superpotential [15] from an appropriate matrix model. In this case we need to consider the partition function of a one-matrix model with $\Phi$ in the adjoint representation of $SO(N)$, i.e. real antisymmetric matrices$^3$. We thus consider

$$Z = \int d\Phi \exp \left( -\frac{1}{g_s} \text{Tr} W_{\text{tree}}(\Phi) \right) , \quad S = g_s M$$

where $W_{\text{tree}}(\Phi)$ is the tree-level superpotential in (14) and $M$ is the size of the matrices. Eliminating $g_s$, one may rewrite the partition function in the more standard random matrix model form

$$Z = \int \prod_{a>b} D\Phi_{ab} e^{-M \text{Tr} V(\Phi)}$$

where $V(\Phi) = \sum_p \frac{1}{2p} \tilde{g}_{2p} \Phi^{2p}$ and $\tilde{g}_{2p}$ is related to the tree level potential coefficients through $\tilde{g}_{2p} = g_{2p}/S$. The corresponding free energy in $Z = \exp(M^2F)$ of this random matrix model has a $1/M$ expansion

$$F = \sum_{n=0}^{\infty} \frac{1}{M^n} F_n$$

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$^3$This is quite different from the standard orthogonal ensemble in random matrix theory where the matrices are real symmetric.
For our purposes we will only be concerned with the contributions $F_0, F_1$ arising from the sphere $S^2 (\chi = 2)$ and the projective plane $\mathbb{RP}^2 (\chi = 1)$ respectively. In terms of these quantities we extract the free energies from (20) according to

$$F_{\chi=2} = -S^2 F_0 , \quad F_{\chi=1} = -SF_1$$

(23)

where we have expanded $Z \simeq \exp(-M^2/S^2 F_{\chi=2} - (M/S) F_{\chi=1})$. According to the conjecture of [3] (in the form given in Refs. [37, 38] for $SO/Sp$), the perturbative part of the superpotential is then given by

$$W_{\text{pert}} = N \partial_s F_{\chi=2} + 4F_{\chi=1}$$

(24)

in terms of the free energy contributions $F_{\chi=1,2}$ defined in (23). Comparing with the exact result (15) obtained from factorization of the SW curve, we thus see that in order for the conjecture to hold for all $N$ one needs the relation

$$\partial_s F_{\chi=2} = -2F_{\chi=1}$$

(25)

Moreover, given this relation, one should have that

$$\partial_s F_{\chi=2}(g) = \frac{1}{N-2} W_{\text{pert}}(g) = \frac{1}{2N} W^\text{SU}(N)(f = 2g) = \frac{1}{2} \partial_s F^\text{SU}(N)(f = 2g)$$

(26)

where we used the relation (19) in the second step and the last step follows from the DV conjecture for $SU(N)$. Here, the arguments $g$ and $f = 2g$ indicate the coupling constant dependence.

In the next section we will use the loop equation to prove the non-trivial identity (25) relating the $\mathbb{RP}^2$ free energy to the spherical contribution. We will also derive the relation (26), after which the proof of the $SO(N)$ conjecture (24) immediately follows from the one for $SU(N)$, which was proven in [16, 4, 5].

### 4 Loop equation

In this section we will derive the result (24) obtained by factorization of the SW curve for the orthogonal groups$^5$ using the loop equation for the relevant random matrix model. We also prove the relation (26) by comparing the zeroth order loop equation for $SO(N)$ and $SU(N)$.

The loop equation allows to find recursive relations among the contributions in the $1/M$ expansion of the free energy (see (22)) of a random matrix model. In practice the

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$^4$All quantities $W$, $F$, $F$ in this paper that do not carry an explicit superscript referring to the group are for $SO(N)$.

$^5$The same result has been recently obtained using other methods [37, 38]. In Ref. [38] the loop equation was used as well, but we differ considerably on various aspects in the derivation.
loop equations do not involve the free energy directly but are rather expressed in terms of its derivatives – the resolvents

\[
W(z) \equiv \left( \frac{1}{M} \text{Tr} \frac{1}{z - \Phi} \right) = \frac{1}{z} + \frac{d}{dV(z)} F
\]  

(27)

\[
W(z, z) \equiv M^2 \left( \left( \frac{1}{M^2} \text{Tr} \frac{1}{z - \Phi} \text{Tr} \frac{1}{z - \Phi} \right) - \left( \frac{1}{M} \text{Tr} \frac{1}{z - \Phi} \right)^2 \right)
\]  

(28)

where \( \frac{d}{dV(z)} \) is the loop insertion operator

\[
\frac{d}{dV(z)} = -\sum_{p=1}^{\infty} \frac{2p}{z^{2p+1}} \frac{\partial}{\partial g_{2p}}
\]  

(29)

The \( 1/M \) expansion of the free energy [22] induces the appropriate expansion of the resolvent \( W(z) = W_0(z) + (1/M)W_1(z) + \ldots \). The form of the loop insertion operator then determines the asymptotic large \( z \) behavior of \( W_i(z) \). In particular we have \( W_0(z) \sim 1/z \) and \( W_i(z) \sim O(1/z^2) \) for \( i > 0 \). These conditions in general ensure the uniqueness of the solution of the loop equations.

The loop equation is derived by requiring the invariance of the partition function [21] under the coordinate reparameterization

\[
\Phi = \Phi' - \varepsilon \sum_{k=0}^{\infty} \frac{\Phi'^{2k+1}}{z^{2k+2}}
\]  

(30)

The transformation properties of the measure follow from

\[
d\Phi_{ab} = d\Phi'_{ab} - \varepsilon \sum_{k=0}^{\infty} \sum_{l=0}^{2k} \frac{\Phi'^{2k-l}}{z^{2k+2}} \Phi'_{ac} d\Phi'_{cd} \Phi'_{db}
\]  

(31)

This is the initial starting point of ref. [38] but from now on our treatment differs considerably.

The Jacobian matrix for the reparameterization (30) is

\[
\frac{\partial \Phi_{ab}}{\partial \Phi'_{ij}} = \delta_{ai} \delta_{bj} - \varepsilon \sum_{k=0}^{\infty} \sum_{l=0}^{2k} \frac{1}{z^{2k+2}} \left[ \Phi'^{2k-l}_{ai} \Phi'^{l}_{jb} - \Phi'^{2k-l}_{aj} \Phi'^{l}_{ib} \right]
\]  

(32)

where \( a > b \) and \( i > j \). The resulting Jacobian is then obtained from

\[
J = 1 - \varepsilon \sum_{a>b} \sum_{k=0}^{\infty} \sum_{l=0}^{2k} \frac{1}{z^{2k+2}} \left[ \Phi'^{2k-l}_{aa} \Phi'^{l}_{bb} - \Phi'^{2k-l}_{ab} \Phi'^{l}_{ba} \right]
\]  

(33)

Using the symmetry with respect to the interchange of \( a \) and \( b \) we have \( \sum_{a>b} = \frac{1}{2} \sum_{a,b} \). Furthermore since \( \Phi'^{l}_{ab} = (-1)^l \Phi'^{l}_{ba} \) we can rewrite \( J \) in a compact form

\[
J = 1 - \varepsilon \left( \frac{\text{Tr} \frac{1}{z - \Phi'}}{2} \right)^2 + \varepsilon \frac{\text{Tr} \frac{1}{z^2 - \Phi'^2}}{2} \equiv 1 - \varepsilon \left( \frac{\text{Tr} \frac{1}{z - \Phi'}}{2} \right)^2 + \varepsilon \frac{\text{Tr} \frac{1}{z^2 - \Phi'}}{2}
\]  

(34)
Combining this with the transformation property of the potential $\text{Tr} V(\Phi) = \text{Tr} V(\Phi') - \varepsilon \text{Tr} \frac{V'(\Phi')}{z-\Phi'}$ the loop equation follows from the invariance property

$$\int D\Phi e^{-M\text{Tr} V(\Phi)} \equiv \int D\Phi' J \cdot \left( 1 + M \varepsilon \text{Tr} \frac{V'(\Phi')}{z-\Phi'} \right) e^{-M\text{Tr} V(\Phi')} = \int D\Phi' e^{-M\text{Tr} V(\Phi')}$$

(35)

Standard manipulations then yield

$$\frac{1}{2} \left( W(z)^2 + \frac{1}{M^2} W(z, z) \right) - \frac{1}{M} \frac{1}{2z} W(z) - \int_C \frac{dw}{2\pi i} \frac{V'(w)}{z-w} W(w) = 0$$

(36)

Let us denote by $\hat{K}$ the integral operator

$$\hat{K} f(z) = \int_C \frac{dw}{2\pi i} \frac{V'(w)}{z-w} f(w)$$

(37)

Then it follows from (36) that the equations for the $S^2$ and $\mathbb{R}P^2$ contributions to the resolvent are respectively

$$\hat{K} W_0(z) = \frac{1}{2} W_0^2(z)$$

(38)

$$\hat{K} W_1(z) = W_0(z) \left( W_1(z) - \frac{1}{2z} \right)$$

(39)

These equations should be solved subject to the asymptotic conditions $W_0(z) \sim 1/z$ for $z \to \infty$ and $W_1(z) \sim O(1/z^2)$.

**$S^2$ topology**

It is convenient to relate the solution of the genus 0 loop equation (38) to the result for matrix models relevant for the unitary case. In that case, the relevant equation is

$$\hat{K}^\text{SU} W_0^\text{SU}(z) = (W_0^\text{SU}(z))^2$$

(40)

where $\hat{K}^\text{SU}$ is given by the same formula as (37) but with the coupling constants $\tilde{g}_{2p}$ substituted by coupling constants of the complex matrix model $\tilde{f}_{2p}$. Comparing the two we see that the orthogonal resolvent $W_0 \equiv W_0^\text{SO}$ is equal to the ‘unitary’ resolvent calculated with the couplings $\tilde{f}_{2p} = 2\tilde{g}_{2p}$. Using the defining relation (27) we see that this implies that

$$F_0^\text{SO}(\tilde{g}) = \frac{1}{2} F_0^\text{SU}(\tilde{f} = 2\tilde{g})$$

(41)

**$\mathbb{R}P^2$ topology**

In order to find the solution for $W_1(z)$ it is convenient to introduce the function $\hat{D} W_0(z)$ where $\hat{D}$ is the differential operator

$$\hat{D} = \sum_p \tilde{g}_{2p} \frac{\partial}{\partial \tilde{g}_{2p}} = \sum_p g_{2p} \frac{\partial}{\partial g_{2p}}$$

(42)
\[ [\hat{\mathcal{D}}, \hat{\mathcal{K}}] = \hat{\mathcal{K}} \quad [\hat{\mathcal{D}}, \frac{d}{dV}] = -\frac{d}{dV} \]  

(43)

Note that this differential operator does not depend on whether the couplings \( \{ g_{2p} \} \) are rescaled by \( S \) or not. Acting with \( \hat{\mathcal{D}} \) on the genus 0 equation (38) one obtains

\[ \hat{\mathcal{K}}(\hat{\mathcal{D}}W_0) = W_0\hat{\mathcal{D}}W_0 - \frac{1}{2}W_0^2 \]  

(44)

where we used the first commutator in (43). In terms of these quantities we can find a solution of (39) in the form

\[ W_1(z) = \alpha \left( W_0 - \frac{1}{z} \right) + \beta \hat{\mathcal{D}}W_0 \]  

(45)

where we recall the condition \( W_1(z) \sim O(\frac{1}{z^2}) \). Substituting this back into (39) yields uniquely \( \alpha = -\beta = -\frac{1}{2} \) thus

\[ W_1 = -\frac{1}{2} \left( W_0 - \frac{1}{z} - \hat{\mathcal{D}}W_0 \right) \equiv -\frac{1}{2} \frac{d}{dV(z)} \left( 2F_0 - \sum_p \tilde{g}_{2p} \frac{\partial}{\partial \tilde{g}_{2p}} F_0 \right) \]  

(46)

where we used that \( \hat{\mathcal{K}} \frac{1}{z} = 0 \) and the second commutator in (43). Since \( W_1 = dF_1/dV(z) \) we can uniquely reconstruct \( F_1 \) from the above equation up to an inessential coupling constant independent additive constant. This yields finally

\[ F_1 = -\frac{1}{2} \left( 2 - \sum_p \tilde{g}_{2p} \frac{\partial}{\partial \tilde{g}_{2p}} \right) F_0 \]  

(47)

**Link with the DV proposal**

Let us now reinterpret the matrix model identities obtained in the previous section within the gauge theoretical framework.

We first examine the perturbative expansion of \( F_0 \):

\[ F_0 = \sum_{\{n_{2p}\}} a_{\{n_{2p}\}} \prod_p \tilde{g}_{2p}^{n_{2p}} = \sum_{\{n_{2p}\}} a_{\{n_{2p}\}} \prod_p g_{2p}^{n_{2p}} S^{-\sum n_{2p}} \]  

(48)

where the sum runs over the various possible numbers of vertices of different types and \( a_{\{n_{2p}\}} \) are the relevant combinatorial factors. A crucial property is now that the differential operator \( \partial_S \) can be directly related to the operator \( \hat{\mathcal{D}} \) in the matrix model. In particular, using the fact that \( \mathcal{F}_{\chi=2} = -S^2 F_0 \) we easily get

\[ \frac{\partial \mathcal{F}_{\chi=2}(S)}{\partial S} = -\sum_{\{n_{2p}\}} \left( 2 - \sum n_{2p} \right) a_{\{n_{2p}\}} \prod_p g_{2p}^{n_{2p}} S^{1-\sum n_{2p}} \]  

(49)

The above can be rewritten in terms of random matrix quantities as

\[ \frac{\partial \mathcal{F}_{\chi=2}(S)}{\partial S} = -S \left( 2 - \sum_p g_{2p} \frac{\partial}{\partial g_{2p}} \right) F_0 \]  

(50)
Using (42) and the fact that $F_{\chi=1}(S) = -SF_1$, the solution of the loop equation (47) thus implies

$$\frac{\partial F_{\chi=2}(S)}{\partial S} = -2F_{\chi=1}(S)$$

(51)

which verifies the announced identity (25). Turning to the other relation (26), we note that the matrix model couplings for the unitary and orthogonal groups were related to the gauge theoretical couplings via $\tilde{f}_2 = 2\tilde{g}_2$ which before the rescaling is equivalent to $f_2 = 2g_2$. In particular it then follows from (41) that $F_{SO\chi=2} = \frac{1}{2}F_{SU\chi=2}(f = 2g)$ which proves (25).

The effective potential obtained in section 2 from the Seiberg-Witten curve for orthogonal groups was shown to be equal to

$$W_{\text{eff}} = \frac{N - 2}{2} \frac{\partial F_{\chi=2}(f = 2g)}{\partial S}$$

(52)

This can be rewritten in terms of the quantities related to the orthogonal matrix model as

$$W_{\text{eff}} = (N - 2) \frac{\partial F_{SO\chi=2}}{\partial S} = N \frac{\partial F_{SO\chi=2}}{\partial S} + 4F_{SO\chi=1}$$

(53)

The multiplicative factor 4 has been first identified in [37] where it was shown to arise from the fact that the field theoretical determinants gave 1 for graphs with topology of $S^2$ and 4 for graphs with the topology of $\mathbb{R}\mathbb{P}^2$.

5 Conclusions

We have performed a field-theoretic computation of the exact superpotential for $N = 1$ $SO(N)$ gauge theory with arbitrary tree-level potential of an adjoint field. Here we used the factorization properties of SW curves for the orthogonal groups together with the ILS linearity principle. Comparison of this result with the matrix model conjecture [3] (see also [37, 38]) implied the existence of a non-trivial identity relating the spherical free energy to that of the next contribution on $\mathbb{R}\mathbb{P}^2$. By explicitly solving the loop equation, we have been able to derive this identity for arbitrary potential. Moreover, the zeroth order loop equation enabled us to relate the spherical part in the $SO(N)$ theory to that of the $SU(N)$ theory, thereby showing that the validity of the $SU(N)$ conjecture directly implies the corresponding one for $SO(N)$.

The fact that certain quantities in supersymmetric gauge theories apparently know about information encoded in the loop equation, may be regarded as further evidence for the deep connection between these gauge theories and matrix models. It is also interesting to note that, in the end, the $SO(N)$ superpotential can be expressed in terms of the corresponding $SU(N)$ planar free energy. Our work also lends further support for the conjecture [16] that the ILS hypothesis is intimately related to the DV matrix theory proposal.
One obvious generalization is to apply the analysis of this note to the symplectic case. In particular, it would be interesting to use the methods of section 4 to derive from the loop equation the expected relation $\partial_S F_{\chi=2} = 2F_{\chi=1}$. Another area that has so far not received much attention is to consider the addition of matter in the fundamental representation to these $\mathcal{N} = 1$ $SO(N)$ gauge theories. Here, the appearance of different types of $\chi = 1$ contributions might give interesting results, while a possible connection to the case of $SU(N)$ with fundamental matter might generate further insights as well.

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