QUASI-MULTIPLICATIVITY OF TYPICAL COCYCLES

KIHO PARK

Abstract. We show that typical (in the sense of [BV04] and [AV07]) Hölder and fiber-bunched $\text{GL}_d(\mathbb{R})$-valued cocycles over a subshift of finite type are uniformly quasi-multiplicative with respect to all singular value potentials. We prove the continuity of the singular value pressure and its corresponding (necessarily unique) equilibrium state for such cocycles, and apply this result to repellers. Moreover, we show that the pointwise Lyapunov spectrum is closed and convex, and establish partial multifractal analysis on the level sets of pointwise Lyapunov exponents for such cocycles.

Contents

1. Introduction 2
2. Preliminaries and statement of results 5
  2.1. Symbolic dynamics 5
  2.2. Holonomies and fiber-bunched cocycles 7
  2.3. Typical cocycles 8
  2.4. Quasi-multiplicativity and the main theorem 9
3. Thermodynamic formalism 11
  3.1. Additive thermodynamic formalism 11
  3.2. Subadditive thermodynamic formalism 12
  3.3. Bowen’s theorem for subadditive potentials 13
  3.4. Subadditive potentials with bounded distortion 14
  3.5. Exterior algebra 16
4. Quasi-multiplicativity 17
  4.1. Preliminary linear algebra 17
  4.2. Proof of Theorem A 21
  4.3. Proof of Theorem 4.1 26
  4.4. Proof of Theorem E 30
5. Continuity of the subadditive pressure 30
  5.1. Proof of Theorem B 30
  5.2. Applications in dimension theory and proof of Theorem C 32
6. Other applications of Theorem E 38
  6.1. Pointwise Lyapunov spectrum and proof of Theorem D 38
  6.2. Multifractal analysis 42
References 43
1. Introduction

Given a finite set of $M_{d \times d}(\mathbb{R})$ matrices $A = \{A_1, \ldots, A_q\}$ and an infinite word $x^+ = x_0 x_1 x_2 \ldots$ where each $x_j \in \{1, 2, \ldots, q\}$, consider the products

$$A_{x_n} \ldots A_{x_0},$$

for $n = 1, 2, \ldots$. The study of such products naturally arises in many settings and has numerous applications. For instance, suppose each $A_i$ is contracting, and $T_i$ is an affine transformation of $\mathbb{R}^d$ whose linear part is $A_i$; that is, $T_i(x) = A_i x + r_i$ for some translation vector $r_i$. Then there exists a unique self-affine attractor $X \subset \mathbb{R}^d$ invariant under $\{T_1, \ldots, T_q\}$, in the sense that $X = \bigcup_{i=1}^q T_i X$; see [Hut81]. The local geometry of the attractor $X$ depends on properties of the composition of the linear contractions (1.1); for example, the Hausdorff dimension of $X$ is intimately related to the growth of the product (1.1) over all possible words $x^+$. See Remark 5.4.

Among many methods to analyze the product (1.1), one is to study the limit (if it exists) of the following expression

$$\lim_{n \to \infty} \frac{1}{n} \log \|A_{x_{n-1}} \ldots A_{x_0}\|.$$

If the limit exists at $x^+ = x_0 x_1 \ldots$, we call it the pointwise Lyapunov exponent of $x^+$, and it measures the asymptotic growth rate of the product (1.1). Once we put a standard metric on the space of all possible words (see Section 2; roughly, two words $x^+$ and $y^+$ are close if they agree along a long initial string), it is not hard to see that in general the pointwise Lyapunov exponent is a discontinuous function in $x^+$. Nonetheless, under mild assumptions on the matrices $A$, the structure of the Lyapunov spectrum (i.e., the values of the pointwise Lyapunov exponent) is quite regular. For example, under the assumption that the matrices in $A$ do not preserve a common proper subspace of $\mathbb{R}^d$ (i.e., $A$ is irreducible), Feng [Fen03, Fen09] showed that the spectrum is a closed interval.

The product of matrices (1.1) can be placed in a broader context. To any dynamical system $f: X \to X$ and map $A: X \to M_{d \times d}(\mathbb{R})$, we can associate a linear cocycle $F_A: X \times \mathbb{R}^d \to X \times \mathbb{R}^d$ given by

$$F_A(x, v) = (fx, A(x)v).$$

We say that $F_A$ is generated by $f$ and $A$. For $n \in \mathbb{N}$ and $x \in X$, we write $F^n_A(x, v) = (f^n x, A^n(x)v)$, where

$$A^n(x) := A(f^{n-1}x) \ldots A(fx)A(x).$$

The definition of linear cocycle $F$ also extends to (not necessarily trivializable) vector bundles $\mathcal{E}$ over $X$ as a family of linear maps $F_x: \mathcal{E}_x \to \mathcal{E}_{fx}$ covering a base system $(X, f)$.

When the base system is the left shift operator on a one-sided shift $\Sigma^+_q = \{1, 2, \ldots, q\}^{\mathbb{N}_0}$, then the map $A: \Sigma^+_q \to M_{d \times d}(\mathbb{R})$ defined by $x = (x_i)_{i \in \mathbb{N}_0} \mapsto A_{x_0}$ generates a linear cocycle $F_A$. The cocycle $F_A$ encodes the products (1.1) in the sense that $A^n(x) = A_{x_{n-1}} \ldots A_{x_0}$, and it is an example of a locally constant cocycle (See Definition 2.1 and Remark 2.2).

Another natural class of linear cocycles comes from smooth dynamics. When the base system $f: M \to M$ is a smooth map or diffeomorphism of a closed Riemannian manifold $M$, the derivative cocycle $Df$ is a cocycle generated by the map $A(x) = D_x f: T_x M \to T_{fx} M$. More generally, for any $Df$-invariant sub-bundle $E \subset TM$, the derivative map restricted to $E$ gives rise to a linear cocycle $Df|_E$. If $f$ is uniformly hyperbolic (i.e., expanding or Anosov), then there exists a symbolic coding of $f$ by a subshift of finite type [Sm68].
From such a coding, the derivative cocycle of a uniformly hyperbolic map can effectively be regarded as a linear cocycle over a subshift of finite type.

The main objects of interest in this paper are linear cocycles $F_A$ over a subshift of finite type $(\Sigma, f)$ generated by $\text{GL}_d(\mathbb{R})$-valued functions on $\Sigma$. In particular, we study the thermodynamic formalism of such cocycles. Any $A: \Sigma \to \text{GL}_d(\mathbb{R})$ defines a sequence of continuous functions $\{\varphi_{A,n}\}_{n \in \mathbb{N}}$ on $\Sigma$ given by

$$\varphi_{A,n}(x) = \|A^n(x)\|,$$

where $\|\cdot\|$ is the operator norm. The submultiplicativity of the norm $\|\cdot\|$ implies that this sequence is submultiplicative in the sense that for any $m, n \in \mathbb{N}$,

$$0 \leq \varphi_{A,n+m} \leq (\varphi_{A,n} \circ f^m) \cdot \varphi_{A,m}.$$

A submultiplicative sequence gives rise to a singular value potential $\Phi_A = \{\log \varphi_{A,n}\}_{n \in \mathbb{N}}$. The singular value potential $\Phi_A$ is an example of a subadditive potential $\Psi = \{\log \psi_n\}_{n \in \mathbb{N}}$ which can be thought of as a generalization of the Birkhoff sum $S_n \psi$ for a potential $\psi \in C(\Sigma, \mathbb{R})$. The usual thermodynamical notions of the pressure and the equilibrium states of a potential $\psi$ extend to subadditive potentials [CFH08].

In his fundamental work on Thermodynamic formalism, Bowen [Bow74] showed that for any Hölder potential $\psi$ on a mixing hyperbolic system such as $(\Sigma, f)$, there exists a unique equilibrium state for $\psi$, and that such equilibrium state has the Gibbs property.

It is natural to ask if Bowen’s theorem (with suitable generalizations) holds for subadditive potentials such as $\Phi_A$. Unfortunately, the analogue of Bowen’s theorem does not necessarily hold for general subadditive potentials [FK10]. On the other hand, Bowen’s theorem remains valid for singular value potentials of certain cocycles, including the cocycles generated by locally constant $\text{GL}_d(\mathbb{R})$-valued functions satisfying an extra assumption known as quasi-multiplicativity. Denoting the set of all admissible words of $\Sigma$ by $\mathcal{L}$, for any $A: \Sigma \to \text{GL}_d(\mathbb{R})$ and $I \in \mathcal{L}$, we define

$$\|A(I)\| := \max_{x \in [I]} \varphi_{A,I}(x) = \max_{x \in [I]} \|A^I(x)\|. \tag{1.2}$$

We say $A$ is quasi-multiplicative if there exist $c > 0$ and $k \in \mathbb{N}$ such that for any words $I, J \in \mathcal{L}$, there exists $K = K(I, J) \in \mathcal{L}$ with $|K| \leq k$ such that $IKJ \in \mathcal{L}$ and

$$\|A(IKJ)\| \geq c \|A(I)\| \|A(J)\|. \tag{1.3}$$

Notice that quasi-multiplicativity of $A$ resembles Bowen’s specification property [Bow74] in some respects.

For locally constant cocycles, there is a sufficient condition that guarantees quasi-multiplicativity. We say that a locally constant $\text{GL}_d(\mathbb{R})$-valued function $A$ is irreducible if the image of $A$ (which is necessarily a finite set of matrices) doesn’t preserve a common proper subspace of $\mathbb{R}^d$. It is well-known that an irreducible locally constant cocycle is quasi-multiplicative [FK10], [BM18]. Hence, for such cocycles $F_A$, there is a unique equilibrium state for the singular value potential $\Phi_A$.

In this paper, we address the question of whether quasi-multiplicativity holds for more general cocycles beyond locally constant cocycles. It is not entirely clear what the natural counterpart to irreducibility might be for general cocycles. On the other hand, since quasi-multiplicativity is a typical feature of locally constant cocycles, it is reasonable to expect that quasi-multiplicativity holds for a more general class of cocycles with suitable assumptions.
We restrict our attention to Hölder continuous and fiber-bunched (See Section 2 for precise definitions) cocycles, a class that contains the locally constant cocycles. The fiber-bunching assumption is an open condition which roughly says that the cocycle is nearly conformal. We denote the space of $\alpha$-Hölder and fiber-bunched functions by $C^\alpha_b(\Sigma, \text{GL}_d(\mathbb{R}))$, viewed as a subset of $C^\alpha(\Sigma, \text{GL}_d(\mathbb{R}))$.

Our main result establishes that quasi-multiplicativity holds generically among these cocycles. More precisely, Bonatti and Viana in [BV04] introduced the notion of typical cocycles among fiber-bunched cocycles (see Definition 2.6, 2.8, and 2.10 for precise formulations). The set

$$U := \{ A \in C^\alpha_b(\Sigma, \text{GL}_d(\mathbb{R})): A \text{ is typical} \}$$

is open in $C^\alpha_b(\Sigma, \text{GL}_d(\mathbb{R}))$, and Bonatti and Viana [BV04] also proved that $U$ is dense in $C^\alpha(\Sigma, \text{SL}_d(\mathbb{R}))$ and that its complement has infinite codimension.

**Theorem A.** Every $A \in U$ is quasi-multiplicative. Moreover, the constants $c, k$ in (1.3) can be chosen uniformly in a neighborhood of $A$ in $U$.

Theorem A follows from a more general result: for $A \in U$, Theorem E (see Section 2) gives simultaneous quasi-multiplicativity of the exterior product cocycles $A \wedge t$, $t \in \{1, \ldots, d - 1\}$ with uniform constants $c, k$.

As an application, we prove the continuity of the subadditive pressure on $U$. More precisely, there exists a natural generalization of the singular value potential $\Phi^s_A$ for all $s \in [0, \infty)$ by considering the exterior product cocycles $A \wedge t$ (See Section 2). Denoting the pressure of $\Phi^s_A$ by $P(\Phi^s_A)$, we establish the following continuity result using the uniform constants $c, k$ from Theorem E:

**Theorem B.**

1. The map $(A, s) \mapsto P(\Phi^s_A)$ is continuous on $U \times [0, \infty)$.
2. For each $A \in U$ and $s \in [0, \infty)$, the singular value potential $\Phi^s_A$ has a unique equilibrium state $\mu_{A,s}$, which also varies continuously on $U \times [0, \infty)$.

Cao, Pesin, and Zhao [CPZ18] recently proved a result that implies Theorem B (1). See Section 3 for further comments.

Theorem B has further applications in dimension theory of repellers. Given a repeller $\Lambda$ (see Definition 5.2), one can associate a number $s(\Lambda)$ obtained as the unique zero of Bowen’s equation. Such number $s(\Lambda)$ is an upper bound, and often a natural estimate, on the Hausdorff dimension of the repeller. In fact, there are many settings in which $s(\Lambda)$ is equal to the Hausdorff dimension. See Section 5 and a survey [CP10] for more details on the number $s(\Lambda)$. From its definition, it follows that $s(\Lambda)$ varies upper semi-continuously under small perturbations of the repeller $\Lambda$. Using Theorem E we prove a result on the continuity of $s(\Lambda)$:

**Theorem C.** Let $M$ be a Riemannian manifold, and let $h: M \to M$ be a $C^r$ map with $r > 1$. Suppose $\Lambda \subset M$ is a $\alpha$-bunched repeller defined by $h$ for some $\alpha \in (0,1)$ with $r - 1 > \alpha$. Then there exist a $C^1$-neighborhood $\mathcal{V}_1$ of $h$ in $C^r(M,M)$ and a $C^1$-open and $C^r$-dense subset $\mathcal{V}_2 \subset \mathcal{V}_1$ such that the map

$$g \mapsto s(\Lambda_g)$$

is continuous on $\mathcal{V}_2$.

As another application of Theorem E, we extend and generalize Feng’s result [Fen03, Fen09] which states that the pointwise Lyapunov spectrum of an irreducible locally constant
cocycle is a closed interval. By considering the exterior product cocycle $A^\otimes$, we define

$$
\lambda_t(x) := \lim_{n \to \infty} \frac{1}{n} \log \varphi'(A^n(x)),
$$

if the limit exists, and set

$$
\vec{\lambda}(x) := (\lambda_1(x), \ldots, \lambda_d(x)),
$$

if $\lambda_t(x)$ exists for each $1 \leq t \leq d$. We define the pointwise Lyapunov spectrum of $A$ as

$$
L_A := \{ \vec{\alpha} \in \mathbb{R}^d : \vec{\alpha} = \vec{\lambda}(x) \text{ for some } x \in \Sigma \}.
$$

**Theorem D.** Let $A \in \mathcal{U}$. Then $L_A$ is a closed and convex subset of $\mathbb{R}^d$.

Recall that a repeller $\Lambda$ is conformal if the derivative map $D_x h$ is a conformal transformation for every $x \in \Lambda$. Combining Theorem D with the proof of Theorem C, we obtain the following corollary whose proof appears in Section 6.

**Corollary 1.1.** Let $\Lambda \subset M$ be a conformal repeller defined by a $C^r$ map $h : M \to M$ with $r > 1$. Then there exists a $C^1$-neighborhood $\mathcal{V}_1$ of $h$ in $C^r(M, M)$ and a $C^1$-open and $C^r$-dense subset $\mathcal{V}_2$ of $\mathcal{V}_1$ such that for every $g \in \mathcal{V}_2$, the pointwise Lyapunov spectrum $L_g$ of $g|_\Lambda$ is a closed and convex subset of $\mathbb{R}^d$.

Finally, we also obtain partial multifractal results on the level sets of pointwise Lyapunov exponents (Corollary 6.5) by applying general results in [FH10].

The paper is organized as follows. In Section 2, we introduce the setting of our results and state the main theorem (Theorem E) of the paper. In Section 3, we survey relevant results in thermodynamic formalism for both additive and subadditive settings. In Section 4, we prove Theorem E in a more general setting. In Section 5, we prove Theorem B and C. In Section 6, we establish Theorem D and Corollary 1.1. Moreover, we discuss further applications of Theorem E including the structure of the pointwise Lyapunov spectrum as well as some of its level sets.

**Acknowledgements** The author is very grateful to his advisor Amie Wilkinson for her support and numerous helpful discussions. The author would also like to thank Clark Butler for sharing his insights and for pointing out an error in Section 3 of the original draft, and Aaron Brown for many helpful suggestions. Lastly, the author also thanks De-Jun Feng for his comments and Ping Ngai Chung for for improving the readability of the paper.

### 2. Preliminaries and statement of results

#### 2.1. Symbolic dynamics.** An adjacency matrix $T$ is a square $(0,1)$-matrix. A one-sided subshift of finite type defined by a $q \times q$ adjacency matrix $T$ is a dynamical system $(\Sigma_T^+, f)$ where

$$
\Sigma_T^+ := \{(x_i)_{i \in \mathbb{N}_0} : x_i \in \{1, 2, \ldots, q\} \text{ and } T_{x_i, x_{i+1}} = 1 \text{ for all } i \in \mathbb{N}_0\}
$$

and $f$ is the left shift operator. Similarly, we define a two-sided subshift of finite type $(\Sigma_T, f)$ where

$$
\Sigma_T := \{(x_i)_{i \in \mathbb{Z}} : x_i \in \{1, 2, \ldots, q\} \text{ and } T_{x_i, x_{i+1}} = 1 \text{ for all } i \in \mathbb{Z}\}.
$$

Then $(\Sigma_T, f)$ is the natural extension of $(\Sigma_T^+, f)$: denoting the projection from $\Sigma_T$ onto $\Sigma_T^+$ by

$$
\pi : \Sigma_T \to \Sigma_T^+,
$$

each $x \in \Sigma_T$ corresponds to one possible sequence of preimages of $\pi(x) \in \Sigma_T^+$ under $f$. 

We will always assume that the adjacency matrix $T$ is \textit{primitive}, meaning that there exists $N > 0$ such that all entries of $T^N$ are positive. The primitivity of $T$ is equivalent to the mixing property of the corresponding subshift of finite type $(\Sigma_T, f)$.

Fix $\theta \in (0, 1)$ and endow $\Sigma_T$ with the metric $d$ defined as follows: for $x = (x_i)_{i \in \mathbb{Z}}, y = (y_i)_{i \in \mathbb{Z}} \in \Sigma_T$, we have

$$d(x, y) = \theta^k,$$

where $k$ is the largest integer such that $x_i = y_i$ for all $|i| < k$. Equipped with such metric, the subshift of finite type $(\Sigma_T, f)$ becomes a hyperbolic homeomorphism.

An \textit{admissible word of length $n$} is a word $i_0 \ldots i_{n-1}$ with $i_j \in \{1, \ldots, q\}$ such that $T_{i_j,i_{j+1}} = 1$ for all $0 \leq j \leq n - 2$. Let $\mathcal{L}$ be the collection of all admissible words. For $I \in \mathcal{L}$, we denote its length by $|I|$. For each $n \in \mathbb{N}$, let $\mathcal{L}(n) \subset \mathcal{L}$ be the set of all admissible words of length $n$. For any $I = i_0 \ldots i_{n-1} \in \mathcal{L}(n)$, we define the associated \textit{cylinder} by

$$[I] = [i_0 \ldots i_{n-1}] := \{y \in \Sigma_T : y_j = i_j \text{ for all } 0 \leq j \leq n - 1\}.$$

For $x \in \Sigma_T$ and $n \in \mathbb{N}$, we similarly define

$$[x]_n := \{y \in \Sigma_T : y_i = x_i \text{ for all } 0 \leq i \leq n - 1\}.$$

Using the superscript $w$, for each $x \in \Sigma_T$, we denote the word $x_0 \ldots x_{n-1}$ by

$$[x]_n^w := x_0 \ldots x_{n-1} \in \mathcal{L}(n).$$

We define the \textit{local stable set} $\mathcal{W}^s_{\text{loc}}(x)$ of $x \in \Sigma_T$ by

$$\mathcal{W}^s_{\text{loc}}(x) := \{y \in \Sigma_T : x_i = y_i \text{ for all } i \geq 0\}.$$

In other words, $y \in \Sigma_T$ belongs to $\mathcal{W}^s_{\text{loc}}(x)$ if the forward orbit of $y$ exponentially shadows the forward orbit of $x$, meaning that $d(f^n x, f^n y) \leq \theta^n$ for all $n \geq 0$. We extend the definition to define the \textit{stable set} $\mathcal{W}^s(x)$ of $x \in \Sigma_T$ by

$$\mathcal{W}^s(x) := \{y \in \Sigma_T : f^n y \in \mathcal{W}^s_{\text{loc}}(f^n x) \text{ for some } n \geq 0\}.$$

The (local) stable set of $f^{-1}$ is called the \textit{(local) unstable set} $\mathcal{W}^u$ of $f$.

For any $x, y \in \Sigma_T$ with $x_0 = y_0$, we say $y$ is in the \textit{local neighborhood} of $x$. For such $x$ and $y$, the following bracket operation

$$[x, y] := \mathcal{W}^s_{\text{loc}}(x) \cap \mathcal{W}^u_{\text{loc}}(y) \in \Sigma_T \quad (2.1)$$

is well-defined. From the definition, $[x, y]$ is the unique point in the local neighborhood of $x$ and $y$ that exponentially shadows the orbit of $x$ in the future and the orbit of $y$ in the past.

Recall from the introduction that to any dynamical system $(X, f)$ and $M_{d \times d}(\mathbb{R})$-valued function $A$ on $X$, we associate a linear cocycle $F_A$. It is clear from the definition of $A^n(\cdot)$ that the following cocycle equation holds:

$$A^{n+m}(x) := A^n(f^m x) A^m(x) \text{ for all } n, m \in \mathbb{N}.$$

If the base system $(X, f)$ is invertible and the image of $A$ is a subset of $\text{GL}_d(\mathbb{R})$, then we extend the definition to define $A^0(\cdot) \equiv I$ and $A^{-n}(x) := (A^n(f^{-n} x))^{-1}$ for $n \in \mathbb{N}$ such that the cocycle equation holds for all $n, m \in \mathbb{Z}$.

**Definition 2.1.** We say $A : \Sigma_T \to M_{d \times d}(\mathbb{R})$ is \textit{locally constant} if there exists $k \in \mathbb{N}$ such that $A(x)$ depends only on the word $x_{-k} \ldots x_k \in \mathcal{L}(2k + 1)$ for every $x = (x_i)_{i \in \mathbb{Z}} \in \Sigma_T$. A \textit{locally constant cocycle} $F_A$ is a cocycle whose generator $A$ is locally constant.
Remark 2.2. For any locally constant function $\mathcal{A}$ on $\Sigma_T$, there exists a re-coding of $\Sigma_T$ to another subshift of finite type $\tilde{\Sigma}_T$ such that $\mathcal{A}$ is carried to a function on $\tilde{\Sigma}_T$ depending only on the 0-th entry $x_0$ of $x = (x_i)_{i \in \mathbb{Z}} \in \tilde{\Sigma}_T$.

For simplicity, we assume that all locally constant functions considered in this paper are functions that depend only on the 0-th entry.

2.2. Holonomies and fiber-bunched cocycles. Let $\mathcal{A}$ be an $\alpha$-Hölder $GL_d(\mathbb{R})$-valued function on $\Sigma_T$, meaning that there exists $C > 0$ such that for all $x, y \in \Sigma_T$,

$$\|\mathcal{A}(x) - \mathcal{A}(y)\| \leq Cd(x, y)^\alpha,$$

where $\| \cdot \|$ is the standard operator norm.

Definition 2.3. A local stable holonomy for $F_\mathcal{A}$ is a family of matrices $H^s_{x,y} \in GL_d(\mathbb{R})$ defined for any $x, y \in \Sigma_T$ with $y \in \mathcal{W}^s_{\text{loc}}(x)$ such that

1. $H^s_{x,x} = I$ and $H^s_{y,z} \circ H^s_{x,y} = H^s_{x,z}$ for any $y, z \in \mathcal{W}^s_{\text{loc}}(x)$,
2. $\mathcal{A}(x) = H^s_{f_y f_x} \circ \mathcal{A}(y) \circ H^s_{x,y}$,
3. $H^s: (x, y) \mapsto H^s_{x,y}$ is continuous.

A local unstable holonomy $H^u_{x,y}$ is likewise defined for $y \in \mathcal{W}^u_{\text{loc}}(x)$ satisfying the analogous properties above.

We say that a stable/unstable holonomy $H^{s/u}$ is uniformly continuous if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $y \in \mathcal{W}^{s/u}(x)$, we have

$$d(x, y) \leq \delta \implies \|H^{s/u}_{x,y} - I\| \leq \varepsilon.$$

Definition 2.4. An $\alpha$-Hölder function $\mathcal{A}$ is fiber-bunched if for all $x \in \Sigma_T$, we have

$$\|\mathcal{A}(x)\|\|\mathcal{A}(x)^{-1}\|\theta^\alpha < 1,$$

where $\theta$ is the hyperbolicity constant defining the metric on the base $\Sigma_T$.

By projectivizing the action on the fibers, $F_\mathcal{A}$ gives rise to the projective cocycle $\mathbb{P}(F_\mathcal{A}): \Sigma_T \times \mathbb{P}(\mathbb{R}^d) \to \Sigma_T \times \mathbb{P}(\mathbb{R}^d)$. Then the fiber-bunching condition is equivalent to the condition that the rate of expansion (respectively, contraction) of the projective cocycle $\mathbb{P}(F_\mathcal{A})$ at every point $x \in \Sigma_T$ is bounded above by $1/\theta^\alpha$ (respectively, below by $\theta^\alpha$). In particular, the Hölder continuity and the fiber-bunching assumption on $\mathcal{A} \in C^\alpha_b(\Sigma_T, GL_d(\mathbb{R}))$ together ensure the convergence of the canonical stable/unstable holonomy $H^{s/u}_{x,y}$: for any $y \in \mathcal{W}^{s/u}_{\text{loc}}(x)$,

$$H^s_{x,y} := \lim_{n \to \infty} \mathcal{A}^n(y)^{-1}\mathcal{A}^n(x) \quad \text{and} \quad H^u_{x,y} := \lim_{n \to -\infty} \mathcal{A}^n(y)^{-1}\mathcal{A}^n(x). \quad (2.2)$$

See [KS13] or [BGMV03] for details.

A cocycle may admit multiple holonomies. However, when the cocycle is fiber-bunched, the canonical holonomies are unique in the sense that they are the only holonomies varying Hölder continuously in the base points $\Sigma_T$ with the same Hölder exponent $\alpha$: there exists $C > 0$ such that

$$\|H^{s/u}_{x,y} - I\| < Cd(x, y)^\alpha, \quad (2.3)$$

for any $y \in \mathcal{W}^{s/u}_{\text{loc}}(x)$. In particular, the canonical holonomies are uniformly continuous. We will always work with the canonical holonomies for fiber-bunched cocycles.
Remark 2.5. It is worth noting a special family of cocycles trivially admitting the canonical holonomies. For any locally constant \( GL_d(\mathbb{R}) \)-valued function \( A \), the canonical holonomies of \( F_A \) from (2.2) converge to the identity and satisfy the properties listed in Definition 2.3.

The canonical holonomies of a fiber-bunched cocycle identify the fibers over points on the same (local) stable or unstable set, similar to how fibers over two nearby points can be trivially identified for locally constant cocycles.

Using (2) of Definition 2.3, we can extend the definition of the local stable holonomy to the global stable holonomy \( H^s_{x,y} \) for \( y \in W^s(x) \) not necessarily in the local stable set of \( x \):

\[
H^s_{x,y} := A(y)^{-1} H^s_{f^nx,f^ny} A^n(x),
\]

for some large enough \( n \in \mathbb{N} \) so that \( f^n y \in W^s_{loc}(f^n x) \). We can likewise define the global unstable holonomy.

A point \( z \in \Sigma_T \) is a homoclinic point of a periodic point \( p \) if \( z \in W^s(p) \cap W^u(p) \setminus \{ p \} \). The homoclinic points of \( p \) are characterized as the points other than \( p \) whose orbits synchronously approach the orbit of \( p \) in both forward and backward time. For a hyperbolic system such as \( (\Sigma_T, f) \), the set of homoclinic points of any periodic point is dense in \( \Sigma_T \).

2.3. Typical cocycles. We now formulate the assumptions building up to the main theorem. Consider any periodic point \( p \) and a homoclinic point \( z \in W^s(p) \cap W^u(p) \setminus p \). We define the holonomy loop \( \psi^s_p \) as the composition of the unstable holonomy from \( p \) to \( z \) and the stable holonomy from \( z \) to \( p \):

\[
\psi^s_p := H^s_{z,p} \circ H^u_{p,z}. \tag{2.4}
\]

The following definition is a slight weakening of the assumptions of Theorem 1 in [BV04]. See Remark 2.11.

Definition 2.6. Let \( A \in C^0_b(\Sigma_T, GL_d(\mathbb{R})) \) and \( H^{s/u} \) be the canonical holonomies for \( F_A \). We say that \( A \) is 1-typical if it satisfies the following two extra conditions:

(A0) there exists a periodic point \( p \) such that \( P := A^{per(p)}(p) \) has simple real eigenvalues of distinct norms. Let \( \{v_i\}_{1 \leq i \leq d} \) be the eigenvectors of \( P \).

(B0) there exists a homoclinic point \( z \) of \( p \) such that \( \psi^s_p \) twists the eigenvectors of \( P \) into general position: for any \( 1 \leq i, j \leq d \), the image \( \psi^s_p(v_i) \) does not lie in any hyperplane \( W_j \) spanned by all eigenvectors of \( P \) other than \( v_j \). Equivalently, the coefficients \( c_{i,j} \) in

\[
\psi^s_p(v_i) = \sum_{1 \leq j \leq d} c_{i,j} v_j,
\]

are nonzero for all \( 1 \leq i, j \leq d \).

Remark 2.7. We will often refer (A0) and (B0) by pinching and twisting conditions, respectively.

For each \( 1 \leq t \leq d \), we denote by \( A^t \) the action of \( A \) on the exterior product \( (\mathbb{R}^d)^\wedge t \). See subsection 3.5 for basic properties of the exterior product. From the canonical holonomies \( H^{s/u} \) for \( F_A \), the cocycles generated by \( A^t \), \( 1 \leq t \leq d \) also admit stable and unstable holonomies, namely \( (H^{s/u})^\wedge t \). So, for a 1-typical function \( A \), we consider similar conditions appearing in Definition 2.6 on \( A^t \).

Definition 2.8. Let \( A \) be 1-typical. For \( 2 \leq t \leq d - 1 \), we say \( A \) is \( t \)-typical if the same points \( p, z \in \Sigma_T \) from Definition 2.6 satisfy

(A1) all the products of \( t \) distinct eigenvalues of \( P \) are distinct;
Remark 2.9. Notice that the definition of $t$-typicality only asks for (A1) and (B1); the definition does not require that $A^t$ also be fiber-bunched.

On the other hand, we will use the fact that the stable and unstable holonomies $(H^{s/u})^t$ are uniformly continuous. This follows from the Hölder continuity $2.3$ of the canonical holonomies $H^{s/u}$ for $F_A$.

Definition 2.10. We say $A$ is typical if $A$ is $t$-typical for all $1 \leq t \leq d - 1$. We denote $\mathcal{U} \subset C^0_b(\Sigma_T, \text{GL}_d(\mathbb{R}))$ to be the set of all typical functions.

Remark 2.11. A few comments regarding the assumptions are in order. First, similar (but slightly stronger) assumptions are introduced in Bonatti and Viana [BV04] as a sufficient condition to establish the simplicity of the Lyapunov exponents of $F_A$ for any ergodic $f$-invariant measure with continuous local product structure. Their setting is $\text{SL}_d(\mathbb{R})$-valued cocycles, and they also show that $\mathcal{U}$ is open and dense in $C^0_b(\Sigma_T, \text{SL}_d(\mathbb{R}))$. We remark that the difference in the settings ($\text{SL}_d(\mathbb{R})$ for [BV04] and $\text{GL}_d(\mathbb{R})$ for this paper) does not cause any issues in translating the relevant statements and results from [BV04] to this paper.

Avila and Viana in [AV07] improved the result by removing the assumption on the exterior powers and allowing the number of symbols of $\Sigma_T$ to be countably infinite. Under many different settings, such assumptions serve as checkable conditions to establish the simplicity of the Lyapunov exponents; see [BPVL16] for instance. Our twisting condition (B1) on $\psi_p^z$ is weaker than both [BV04] and [AV07], but we still require the assumption on the simplicity of the eigenvalues of $P^t$ for all $1 \leq t \leq d - 1$. In all cases, such assumptions are satisfied by an open and dense subset $\mathcal{U}$ of maps in $C^0_b(\Sigma_T, \text{GL}_d(\mathbb{R}))$, and the complement of $\mathcal{U}$ has infinite codimension.

Remark 2.12. If $z$ is a homoclinic point of $p$, then $f^r z$ for any $r \in \mathbb{Z}$ is also a homoclinic point of $p$. Their holonomy loops are conjugated by $P^r$:

$$P^r \psi_p^z = \psi_p^{f^r z} P^r.$$ 

It then follows that if the twisting condition (B0) holds at $z$, then it also holds at $f^r z$.

Suppose $z$ is a homoclinic point of $p$ on $W^s_{loc}(p)$ and $f^\ell z \in W^s_{loc}(p)$ for some $\ell \in \mathbb{N}$. From Proposition $2.3$, $\psi_p^z$ satisfies the relation

$$\psi_p^z = P^{-\ell} \circ H^s_{f^\ell z,p} \circ A^\ell(z) \circ H^u_{p,z}.$$ 

(2.5)

2.4. **Quasi-multiplicativity and the main theorem.** In order to state the main theorem, we need to introduce the notion of quasi-multiplicativity. Recalling that $\mathcal{L}$ is the set of all admissible words, a function $\psi: \mathcal{L} \to \mathbb{R}_{\geq 0}$ is submultiplicative if

$$\psi(I) \psi(J) \geq \psi(IJ)$$

for all $I, J \in \mathcal{L}$ with $IJ \in \mathcal{L}$. Let $\mathcal{D}$ be the set of non-negative and submultiplicative functions on $\mathcal{L}$:

$$\mathcal{D} = \{ \psi: \mathcal{L} \to \mathbb{R}_{\geq 0}: \psi \text{ is submultiplicative} \}.$$ 

Definition 2.13. A non-negative and submultiplicative function $\psi \in \mathcal{D}$ is quasi-multiplicative if there exist constants $c > 0$ and $k \in \mathbb{N}$ such that for any words $I, J \in \mathcal{L}$, there exists $K = K(I,J) \in \mathcal{L}$ with $|K| \leq k$ such that $IKJ \in \mathcal{L}$ and

$$\psi(IKJ) \geq c \psi(I) \psi(J).$$ 

(2.6)
Remark 2.14. Suppose $\psi: \mathcal{L} \to \mathbb{R}_{\geq 0}$ is not submultiplicative, but still satisfies the following weaker property: there exists $C \geq 1$ such that for all $I, J \in \mathcal{L}$, we have

$$C \psi(I) \psi(J) \geq \psi(IJ).$$

(2.7)

Then, $C \psi$ is submultiplicative, and we can consider quasi-multiplicativity of the function $C \psi$. For such $\psi$, (2.6) and (2.7) together resemble the usual notion of a quasimorphism for the function $\log \psi$.

However, we are mainly interested in the singular value potentials (see Section 3 for the definition), which are automatically submultiplicative. Hence, we have stated the definition of quasi-multiplicativity for submultiplicative functions.

Consider a family of quasi-multiplicative functions on $\mathcal{L}$. If they all admit uniform constants $c > 0$ and $k \in \mathbb{N}$ as well as the common connecting word $K = K(I, J)$ for any $I, J \in \mathcal{L}$, then it would make sense to consider such functions as being simultaneously quasi-multiplicative.

**Definition 2.15.** Let $\mathcal{I}$ be an index set. A family of non-negative and submultiplicative functions $\psi^{(i)} \in \mathcal{D}$, $i \in \mathcal{I}$ are simultaneously quasi-multiplicative if there exist constants $c > 0$ and $k \in \mathbb{N}$ such that for any words $I, J \in \mathcal{L}$, there exists a word $K = K(I, J) \in \mathcal{L}$ with $|K| \leq k$ such that $IKJ \in \mathcal{L}$ and

$$\psi^{(i)}(IKJ) \geq c \psi^{(i)}(I) \psi^{(i)}(J),$$

for all $i \in \mathcal{I}$.

We are most interested in quasi-multiplicativity of the singular value functions related to a cocycle $F_A$. The singular values of $A \in M_{d \times d}(\mathbb{R})$ are eigenvalues of $\sqrt{A^t A}$. We define the singular value function $\varphi^s: M_{d \times d}(\mathbb{R}) \to \mathbb{R}$ with parameter $s \geq 0$ as follows:

$$\varphi^s(A) = \begin{cases} \alpha_1(A) \ldots \alpha_{\lfloor s \rfloor}(A) \alpha_{\lfloor s \rfloor + 1}(A)^{\lfloor s \rfloor} & 0 \leq s \leq d, \\ \|\det(A)\|^s & s > d, \end{cases}$$

where $\alpha_1(A) \geq \ldots \geq \alpha_d(A) \geq 0$ are the singular values of $A$. It is well-known that $\varphi^s$ is submultiplicative for all $s$: for any $A, B \in M_{d \times d}(\mathbb{R})$ and $s \in [0, \infty)$,

$$\varphi^s(A) \varphi^s(B) \geq \varphi^s(AB).$$

Moreover, the function $(A, s) \mapsto \varphi^s(A)$ is upper semi-continuous, and has a discontinuity at $s = k \in \mathbb{N}$ only when there is a jump in the singular values of the form $\alpha_{k-1}(A) > \alpha_k(A) = 0$. In particular, if $A$ takes value in $\text{GL}_d(\mathbb{R})$, then $\varphi^s(A)$ is continuous in both $A$ and $s$.

For any $A: \Sigma_T \to \text{GL}_d(\mathbb{R})$ and $s \geq 0$, we can associate them to a non-negative function (which we also call the singular value function) on $\mathcal{L}$ denoted by $\varphi^s_A \in \mathcal{D}$ as follows: for any $n \in \mathbb{N}$ and $I \in \mathcal{L}(n)$, we define

$$\varphi^s_A(I) := \max_{x \in [0,1]} \varphi^s(A^n(x)).$$

(2.9)

Notice that this definition is similar to how we define $\|A(I)\|$ in the introduction (1.2). From the submultiplicativity of $\varphi^s$, it follows that $\varphi^s_A$ is also submultiplicative. We are now ready to state the main theorem of the paper.

**Theorem E.** Let $A \in \mathcal{U}$ be typical. Then the singular value functions $\varphi^s_A$ with $s \in [0, d]$ are simultaneously quasi-multiplicative. Moreover, the constants $c, k$ can be chosen uniformly in a neighborhood of $A$ in $\mathcal{U}$.
Remark 2.16. We make a few remarks on Theorem E. In the statement of Theorem E we note that the parameter \( s \) of the singular value function \( \tilde{\varphi}_A^s \) varies only within the range \([0, d]\). This is mainly due to two reasons: first, the singular value function \( \varphi^s \) takes a particularly simple form when \( s > d \), and second, it suffices to consider \( s \in [0, d] \) in the applications appearing in Section 5. If the parameter \( s \) were to vary within \([0, s_0]\) for some \( s_0 \in \mathbb{R}^+_0 \), then the theorem still remains true except that the constant \( c \) will have to change depending on \( s_0 \). We also note that the theorem is not necessarily true if we consider simultaneous quasi-multic和平性 of \( \tilde{\varphi}_A^s \) over the range \([0, \infty)\) for the parameter \( s \). See Remark 4.11 for further comments.

Lastly, note that Theorem A is an immediate corollary of Theorem E. The proof of Theorem E appears in Section 4.

3. Thermodynamic formalism

3.1. Additive thermodynamic formalism. Let \( f \) be a continuous map on a compact metric space \( X \). A potential on \( X \) is a continuous function \( \psi: X \to \mathbb{R} \).

A subset \( E \subset X \) is \((n, \varepsilon)\)-separated if every pair of distinct \( x, y \in E \) satisfies
\[
d_n(x, y) := \max_{0 \leq i \leq n-1} d(f^i x, f^i y) \geq \varepsilon.
\]

Using \((n, \varepsilon)\)-separated subsets, we can define a thermodynamical object called the pressure \( P(\psi) \) of \( \psi \) as follows:
\[
P(\psi) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \sup \left\{ \sum_{x \in E} e^{S_n \psi(x)} : E \text{ is an } (n, \varepsilon)\text{-separated subset of } X \right\},
\]
where \( S_n \psi = \psi + \psi \circ f + \ldots + \psi \circ f^{n-1} \).

When \( \psi \equiv 0 \), the pressure \( P(0) \) is equal to the topological entropy \( h(f) \), which measures the complexity of the system \((X, f)\).

Denoting the space of \( f \)-invariant probability measures on \( X \) by \( \mathcal{M}(f) \), the pressure satisfies the variational principle:
\[
P(\psi) = \sup \left\{ h_\mu(f) + \int \psi d\mu : \mu \in \mathcal{M}(f) \right\},
\]
where \( h_\mu(f) \) is the measure-theoretic entropy. See [Wal00].

Any invariant measure \( \mu \in \mathcal{M}(f) \) achieving the supremum in the variational principle is called an equilibrium state of \( \psi \). If the entropy map \( \mu \mapsto h_\mu(f) \) is upper semi-continuous, then any potential has an equilibrium state. However, the existence, the finiteness, or the uniqueness of the equilibrium state for a given potential is a subtle question that depends on the system \((X, f)\) as well as the potential \( \psi \).

On the other hand, there are specific settings where such questions have an affirmative answer. When \((X, f)\) is a mixing hyperbolic system such as \((\Sigma_T, f)\), and the potential \( \psi \) is Hölder, then the result of Bowen [Bow74] states that there exists a unique equilibrium state \( \mu_\psi \), which has the Gibbs property.

Proposition 3.1. Let \((\Sigma_T, f)\) be a mixing subshift of finite type, and \( \psi \) be Hölder continuous. Then there exists a unique equilibrium state \( \mu_\psi \) of \( \psi \), characterized as the unique \( f \)-invariant measure satisfying the Gibbs property: there exists \( C \geq 1 \) such that for any \( n \in \mathbb{N} \) and \( I \in \mathcal{L}(n) \), we have
\[
C^{-1} \leq \frac{\mu_\psi(I)}{e^{-n \psi(x)} + S_n \psi(x)} \leq C
\] (3.1)
Remark 3.2. One necessary condition for the Gibbs property (3.1) to hold is that the variation within each cylinder should be uniformly bounded: there exists a constant $C \geq 0$ such that for every $n \in \mathbb{N}$ and $I \in \mathcal{L}(n)$,

$$|S_n \psi(x) - S_n \psi(y)| \leq C$$

for every $x, y \in I$. We denote this property by bounded distortion.

In the setting of Bowen’s theorem, the hyperbolicity of the system and the Hölder regularity of the potential guarantee the bounded distortion property.

3.2. Subadditive thermodynamic formalism. The additive theory of thermodynamic formalism extends to the subadditive theory with suitable generalizations. A sequence of continuous functions $\{\psi_n\}_{n \in \mathbb{N}}$ on $\Sigma_T$ is submultiplicative if each $\psi_n$ is a non-negative function on $\Sigma_T$ satisfying

$$0 \leq \psi_{m+n} \leq \psi_m \cdot \psi_n \circ f^n,$$

for all $m, n \in \mathbb{N}$. If we set $\Psi = \{\log \psi_n\}_{n \in \mathbb{N}}$, then $\Psi$ becomes a subadditive sequence of functions on $\Sigma_T$. We will consider such $\Psi$ obtained from a submultiplicative sequence $\{\psi_n\}_{n \in \mathbb{N}}$ as a subadditive potential on $\Sigma_T$. A natural example of a subadditive potential is a singular value potential of a continuous $GL_d(\mathbb{R})$-valued function $A$ on $\Sigma_T$: for $s \geq 0$, we define

$$\Phi_s^A := \{\log \varphi^s(A^n(\cdot))\}_{n \in \mathbb{N}}.$$

We define the subadditive pressure of a subadditive potential $\Psi = \{\log \psi_n\}_{n \in \mathbb{N}}$ as

$$P(\Psi) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \sup \left\{ \sum_{x \in E} \psi_n(x) : E \text{ is an (n, } \varepsilon \text{-separated subset of } \Sigma_T \right\},$$

(3.3)

where the existence of the limit is guaranteed from the subadditivity of $\Psi$.

There are a few different generalizations of the additive notion of the pressure to the subadditive setting: Barreira [Bar96] defines the subadditive pressure by open covers while Cao, Feng, and Huang [CFH08] define it using Bowen balls. Our definition of the subadditive pressure (3.3) is based on [CFH08]. See also [Fal88a]. It is not known whether two definitions of the subadditive pressure are equal in general, but there are known settings in which two definitions agree. In particular, it is shown in [CFH08] that two notions are equivalent when the entropy map $\mu \mapsto h_\mu(f)$ is upper semi-continuous, which includes our setting of mixing subshifts of finite type $(\Sigma_T, f)$.

Cao, Feng, and Huang [CFH08] also establish the subadditive variational principle:

$$P(\Psi) = \sup \left\{ h_\mu(f) + \mathcal{F}(\Psi, \mu) : \mu \in \mathcal{M}(f), \mathcal{F}(\Psi, \mu) \neq -\infty \right\},$$

(3.4)

where

$$\mathcal{F}(\Psi, \mu) := \lim_{n \to \infty} \frac{1}{n} \int \log \psi_n \, d\mu = \inf_{n \to \infty} \frac{1}{n} \int \log \psi_n \, d\mu,$$

whose limit is again guaranteed from the subadditivity of $\Psi$.

Similar to the additive setting, any invariant measure $\mu \in \mathcal{M}(f)$ achieving the supremum in (3.4) is called an equilibrium state of $\Psi$. Also, at least one equilibrium state necessarily exists for any subadditive potential $\Psi$ if the entropy map $\mu \mapsto h_\mu(f)$ is upper semi-continuous [Fen11]. See also [Rae03].
Recall that $D$ is the set of non-negative and submultiplicative functions on $L$. For any submultiplicative sequence $\{\psi_n\}_{n \in \mathbb{N}}$ on $\Sigma_T$, we associate a function $\psi \in D$ similar to (1.2) and (2.9): for any $n \in \mathbb{N}$ and $I \in \mathcal{L}(n)$, let

$$\psi(I) := \max_{x \in [I]} \psi_n(x).$$

(3.5)

Hence, we can extend the notion of quasi-multiplicativity to submultiplicative sequences as follows.

**Definition 3.3.** We say that a submultiplicative sequence of continuous functions $\{\psi_n\}_{n \in \mathbb{N}}$ on $\Sigma_T$ (or its associated subadditive potential $\Psi = \{\log \psi_n\}_{n \in \mathbb{N}}$) is quasi-multiplicative if the function $\psi \in D$ obtained from $\{\psi_n\}_{n \in \mathbb{N}}$ by (3.5) is quasi-multiplicative in the sense of Definition 2.13.

We say that $A : \Sigma_T \to \text{GL}_d(\mathbb{R})$ is quasi-multiplicative if its singular value potential $\Phi_A^1$ is quasi-multiplicative. This agrees with the definition of quasi-multiplicativity (1.3) of $A$ from the introduction.

Conversely, for any $\psi \in D$, we can associate a subadditive potential $\Psi = \{\log \psi_n\}_{n \in \mathbb{N}}$ in an obvious way:

$$\psi_n(x) := \psi([x]^w_n),$$

(3.6)

Hence, we can consider the pressure and the equilibrium states of functions in $D$.

In the following subsection, we will discuss a sufficient condition for quasi-multiplicativity of locally constant cocycles as well as some of its consequences.

### 3.3. Bowen’s theorem for subadditive potentials

In this subsection, we show that Bowen’s theorem (Proposition 3.1) remains to hold (with suitable generalizations) for subadditive potentials with quasi-multiplicativity.

For subadditive potentials, equilibrium states are often not unique, and such examples can be found where the subadditive potential is given by the singular value potential of some $M_{d \times d}(\mathbb{R})$-valued function. See [FK10], for instance.

More specifically, consider a subadditive potential $\Psi$ obtained from $\psi \in D$ by (3.6). Alternatively, we can characterize such $\Psi = \{\log \psi_n\}_{n \in \mathbb{N}}$ by the condition that for any $n \in \mathbb{N}$ and $I \in \mathcal{L}(n)$,

$$\psi_n(x) = \psi_n(y) \text{ for all } x, y \in [I].$$

(3.7)

Such $\Psi$ can be thought of as a subadditive potential with zero variation within cylinders. An example of such $\Psi$ is the singular value potential $\Phi_A^1$ for locally constant $\text{GL}_d(\mathbb{R})$-valued functions $A$.

The main consequence of quasi-multiplicativity of $\psi \in D$ is the uniqueness of the equilibrium state for the corresponding subadditive potential $\Psi$.

**Proposition 3.4.** [Fen11, Theorem 5.5] Let $\psi \in D$ be quasi-multiplicative. Then the associated subadditive potential $\Psi = \{\log \psi_n\}_{n \in \mathbb{N}}$ obtained from $\psi$ as in (3.6) has a unique equilibrium state $\mu_\psi \in \mathcal{M}(f)$. Such $\mu$ is ergodic and has the following Gibbs property: there exists $C \geq 1$ such that

$$C^{-1} \leq \frac{\mu_\psi(I)}{e^{-nP(\Psi_\psi)(I)}} \leq C$$

(3.8)

for any $n \in \mathbb{N}$ and $I \in \mathcal{L}(n)$.

**Remark 3.5.** In Feng [Fen11, Theorem 5.5], this result is proved for one-sided subshifts of finite type. This generalizes easily to two-sided subshifts of finite type. We briefly summarize the proof, which is similar to Bowen’s original proof. Define a sequence of probability
measures $\nu_n$ on the $\sigma$-algebra generated by $n$-cylinders by $\nu_n(I) = \psi(I)/ \sum_{J \in \mathcal{L}(n)} \psi(J)$, and considers any subsequential weak-$*$ limit $\mu \in \mathcal{M}(f)$ of the new sequence of probability measures $\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} f_i \nu_n$. Quasi-multiplicativity then gives the Gibbs property as well as the ergodicity on $\mu$. In fact, the sequence $\mu_n$ actually converges to $\mu$ (i.e., a subsequential limit is an actual limit), and $\mu$ is the unique equilibrium state of $\Psi$. The same proof readily extends to our setting of two-sided subshifts of finite type.

The following remark provides a criterion to establish quasi-multiplicativity for a locally constant $\text{GL}_d(\mathbb{R})$-valued function. 

**Remark 3.6.** Recall that an $\text{GL}_d(\mathbb{R})$-valued function $A$ on $\Sigma_T$ is irreducible if there does not exist a proper subspace of $\mathbb{R}^d$ preserved under the image of $A$ (which is necessarily a finite set of matrices). It is well-known that irreducibility of a locally constant function implies quasi-multiplicativity. See [Pen09].

The typicality assumption in Theorem [E] is related to irreducibility of locally constant cocycles because a locally constant and typical cocycle is necessarily irreducible. This follows because any $A$-invariant subspace has to be a span of some eigendirections of $A(p)$; if $A$ is not irreducible, then then $A$ would not satisfy the twisting condition (B0) and consequently fail to be typical.

### 3.4. Subadditive potentials with bounded distortion

In the previous subsection, we saw that quasi-multiplicativity of $\psi \in \mathcal{D}$ is a sufficient condition for Bowen’s theorem (Proposition [3.4]) to hold for a subadditive potential $\Psi$ with zero variation within cylinders (i.e., satisfying (3.7)).

In this subsection, we show that Bowen’s theorem in the subadditive setting (Proposition [3.4]) can be considered on a bigger class of subadditive potentials than those satisfying (3.7). Such class consists of subadditive potentials $\Psi = \{\log \psi_n\}_{n \in \mathbb{N}}$ with bounded distortion: there exists $C \geq 1$ such that for any $n \in \mathbb{N}$ and $I \in \mathcal{L}(n)$, we have

$$C^{-1} \leq \frac{\psi_n(x)}{\psi_n(y)} \leq C$$

for any $x, y \in [I]$.

As noted in Remark [3.2] in order to generalize the Gibbs property (3.1) to the general subadditive setting, one necessary condition on the subadditive potential $\Psi = \{\log \psi_n\}_{n \in \mathbb{N}}$ is that $\Psi$ satisfies the bounded distortion (3.9). It is clear that subadditive potentials $\Psi$ considered in the previous subsection (i.e., $\Psi$ obtained from $\psi \in \mathcal{D}$ by (3.6), or equivalently, $\Psi$ satisfying (3.7)) has the bounded distortion property with $C = 1$.

**Remark 3.7.** For a subadditive potential $\Psi = \{\log \psi_n\}_{n \in \mathbb{N}}$ with bounded distortion, we can restate the Gibbs property (3.8) and quasi-multiplicativity from Definition (3.3) by replacing $\psi(I)$ to $\psi_n(x)$ for any $x \in [I]$.

More precisely, an $f$-invariant measure $\mu \in \mathcal{M}(f)$ has the Gibbs property with respect to $\Psi = \{\log \psi_n\}_{n \in \mathbb{N}}$ if there exists $C \geq 1$ such that for any $n \in \mathbb{N}$ and $I \in \mathcal{L}(n)$,

$$C^{-1} \leq \frac{\mu(I)}{e^{-n \mathcal{P}(\Psi) \psi_n(x)}} \leq C$$

for any $x \in [I]$. This formulation resembles the Gibbs property of the additive setting (3.1) more closely.
Quasi-multiplicativity of such sequence \( \{ \psi_n \}_{n \in \mathbb{N}} \) (or, equivalently, of the subadditive potential \( \Psi \)) is equivalent to the existence of \( c > 0 \) and \( k \in \mathbb{N} \) such that for any words \( I, J \in \mathcal{L} \), there exists \( K = K(I, J) \in \mathcal{L} \) with \( |K| \leq k \) such that

\[
\psi_{|IKJ|}(x) \geq c \psi_{|I|}(y) \psi_{|J|}(z)
\]

for any \( x \in [IKJ], y \in [I], \) and \( z \in [J] \).

The following proposition states that Proposition 3.4 remains valid for subadditive potentials with bounded distortion.

**Proposition 3.8.** Let \( \Psi = \{ \log \psi_n \}_{n \in \mathbb{N}} \) be a subadditive potential with bounded distortion (3.9). If \( \{ \psi_n \}_{n \in \mathbb{N}} \) is quasi-multiplicative, then \( \Psi \) has a unique equilibrium state. Such equilibrium state is ergodic and has the Gibbs property with respect to \( \Psi \).

**Proof.** Let \( \psi \in \mathcal{D} \) be the submultiplicative function on \( \mathcal{L} \) obtained from \( \Psi \) as in (3.5):

\[
\psi(I) := \max_{x \in [I]} \psi_n(x)
\]

Then, \( \psi \) is quasi-multiplicative. Let \( \tilde{\Psi} = \{ \log \tilde{\psi}_n \}_{n \in \mathbb{N}} \) be the subadditive potential obtained from \( \psi \) by (3.6). Note that \( \tilde{\psi}_n \) satisfies (3.7), and \( \tilde{\psi}_n \) and \( \psi_n \) are related by the identity

\[
\tilde{\psi}_n(x) = \max_{y \in [x]_n} \psi_n(y).
\]

The proposition will follow from the following claim relating the thermodynamical objects of \( \Psi \) and \( \tilde{\Psi} \).

**Claim:** \( P(\Psi) = P(\tilde{\Psi}) \). Moreover, the set of equilibrium states of \( \Psi \) is equal to the set of equilibrium states of \( \tilde{\Psi} \).

**Proof of the claim.** Both statements made in the claim easily follow from the bounded distortion property on \( \Psi \).

For any \( (n, \varepsilon) \)-separated set \( E \), we have from the bounded distortion and the definition of \( \tilde{\psi}_n \) that

\[
1 \leq \frac{\sum_{x \in E} \tilde{\psi}_n(x)}{\sum_{x \in E} \psi_n(x)} \leq C.
\]

Then, it follows from the definition of the subadditive pressure (3.3) that \( P(\Psi) = P(\tilde{\Psi}) \).

For the second statement in the claim, again from the bounded distortion, we have \( \mathcal{F}(\Psi, \mu) = \mathcal{F}(\tilde{\Psi}, \mu) \) for any \( f \)-invariant measure \( \mu \). Since the measure-theoretic entropy \( h_\mu(f) \) does not depend on the potential, the second claim follows from the subadditive variational principle (3.4). □

Since \( \tilde{\Psi} \) satisfies (3.7), we obtain the unique equilibrium state \( \mu \) of \( \tilde{\Psi} \) from Proposition 3.4. From the claim, we conclude that \( \mu \) is the unique equilibrium state of \( \Psi \). To conclude the proof, we note from the bounded distortion property that the Gibbs property of \( \mu \) with respect to \( \tilde{\Psi} \) is equivalent to the Gibbs property of \( \mu \) with respect to \( \Psi \). □

Recalling that the singular value potential of a \( \text{GL}_d(\mathbb{R}) \)-valued function \( A \) is defined by

\[
\Phi^s_A := \{ \log \varphi^s(A^n(\cdot)) \}_{n \in \mathbb{N}},
\]

the following lemma shows that the singular value potentials \( \Phi^s_A, s \in [0, \infty) \) of a fiber-bunched \( A \in C_b^\alpha(\Sigma_T, \text{GL}_d(\mathbb{R})) \) have bounded distortion.
**Lemma 3.9** (bounded distortion). Let $A$ be a Hölder and fiber-bunched $GL_d(\mathbb{R})$-valued function on $\Sigma_T$. Then $\Phi_A^s$ has bounded distortion for any $s \in [0, \infty)$.

**Proof.** From Hölder continuity of the canonical holonomies (2.3), we can fix $c > 1$ such that $\|H_{s,u}^x\|$ is bounded above by $c$ whenever $d(x, y) \leq \theta$. Hence, for any $x, y \in \Sigma_T$ with $d(x, y) \leq \theta$, we have that $\varphi^s(H_{s,u}^x)$ is uniformly bounded above by $c^s$.

Consider any $n \in \mathbb{N}$, $I \in \mathcal{L}(n)$, and $x, y \in [I]$. Then, setting $z := [x, y]$ and using (2) of Definition 2.3 as well as the submultiplicativity (2.8) of $\varphi^s$, we have

$$c^{-2s} \varphi^s(A^n(x)) \leq \varphi^s(A^n(z)) = \varphi^s(H_{f^n,x,f^n,z}^\alpha \circ A^n(x) \circ H_{z,x}^s) \leq c^{2s} \varphi^s(A^n(x)).$$

Using the canonical unstable holonomy instead, we have $c^{-2s} \varphi^s(A^n(y))/\varphi^s(A^n(z)) \leq c^{2s}$. Then, the statement follows by setting the constant $C$ equal to $c^{4s}$. □

**Remark 3.10.** Note that Lemma 3.9 also holds for any $A : \Sigma_T \to GL_d(\mathbb{R})$ admitting uniformly continuous holonomies $H_{s,u}^x$. Moreover, the canonical holonomies $H_{s,u}^x$ vary continuously in $A$. Hence, for a fixed $s$, by increasing $C$ from Lemma 3.9 if necessary, the bounded distortion holds on $\Phi_A^s$ for all $B \in C_0^0(\Sigma_T, GL_d(\mathbb{R}))$ sufficiently close to $A$ with the uniform constant $C$.

Recall that the subset $U$ of $C_0^0(\Sigma_T, GL_d(\mathbb{R}))$ consists of typical $GL_d(\mathbb{R})$-valued functions. Using the uniform constant $c$ from Theorem E, we show that the subadditive pressure $P(\Phi_A^s)$ is continuous on $U \times [0, \infty)$ by adapting the proof of Fekete’s lemma. Since the equilibrium state of $\Phi_A^s$ for a typical $A \in U$ is unique from quasi-multiplicativity, it follows that the unique equilibrium state also varies continuously on $U \times [0, \infty)$.

**Theorem (Theorem B).**

1. The map $(A, s) \mapsto P(\Phi_A^s)$ is continuous on $U \times [0, \infty)$.
2. For each $A \in U$ and $s \in [0, \infty)$, the singular value potential $\Phi_A^s$ has a unique equilibrium state $\mu_{A,s}$, which also varies continuously on $U \times [0, \infty)$.

The proof of Theorem B appears in Section 5. From the definition and the subadditivity, the map $(A, s) \mapsto P(\Phi_A^s)$ is upper semi-continuous, and hence is generically continuous on its domain $C_0^0(\Sigma_T, GL_d(\mathbb{R}))$. Theorem B establishes that, when restricted to $C_0^0(\Sigma_T, GL_d(\mathbb{R}))$, the subadditive pressure varies continuously on an open and dense subset $U$.

Cao, Pesin, and Zhao [CPZ18] recently showed that the map $(A, s) \mapsto P(\Phi_A^s)$ is jointly continuous on $C_0^0(\Sigma_T, GL_d(\mathbb{R})) \times [0, \infty)$, and Theorem B (1) is implied by their result. However, the methods of proof are different. Cao, Pesin, and Zhao construct a horseshoe with dominated splitting which carries most of the pressure. Using the structural stability of horseshoes, they establish the lower semi-continuity of the pressure. See [CPZ18] for details. On the other hand, we compare $P(\Phi_A^s)$ to $P(\Phi_B^s)$ for $B$ sufficiently close to $A$ using uniform constants from simultaneous quasi-multiplicativity of Theorem E.

For similar results in this direction, Feng and Shmerkin [FS14] showed that locally constant functions are continuity points of $P(\Phi_A^s)$ in $L^\infty(\Sigma_T, M_{d \times d}(\mathbb{R}))$.

### 3.5 Exterior algebra

We will make use of the exterior algebra in studying the singular value potential $\Phi_A^s$. For $1 \leq k \leq d$, we denote the $k$-th exterior power of $\mathbb{R}^d$ by $(\mathbb{R}^d)^{\wedge k}$. It is a $(\binom{d}{k})$-dimensional $\mathbb{R}$-vector space spanned by decomposable vectors $v_1 \wedge \ldots \wedge v_k$ with the usual identifications.
Any linear transformation \( A \) and the standard inner product \( \langle \cdot, \cdot \rangle \) on \( \mathbb{R}^d \) naturally extend to \( (\mathbb{R}^d)^\wedge k \) for any two decomposable vectors \( v_1 \wedge \ldots \wedge v_k, u_1 \wedge \ldots \wedge u_k \in (\mathbb{R}^d)^\wedge k \), we have
\[
A^\wedge k(v_1 \wedge \ldots \wedge v_k) := Av_1 \wedge \ldots \wedge Av_k,
\]
\[
\langle v_1 \wedge \ldots \wedge v_k, u_1 \wedge \ldots \wedge u_k \rangle := \det(\langle v_i, u_j \rangle)_{1 \leq i,j \leq k},
\]
and we extend it to the entire \( (\mathbb{R}^d)^\wedge k \) by linearity. The exterior algebra satisfies the following properties: for any linear transformations \( A, B \) of \( \mathbb{R}^d \),
\[
(AB)^\wedge k = A^\wedge k B^\wedge k, \quad (A^\wedge k)^T = (A^T)^\wedge k,
\]
\[
\|A^\wedge k\| = \alpha_1(A) \ldots \alpha_k(A) = \varphi^k(A).
\]
Under the induced inner product on \( (\mathbb{R}^d)^\wedge k \), it follows that \( \Phi^k_A = \Phi^1_{A^\wedge k} \).

4. Quasi-multiplicativity

In this section, we prove Theorem \[A\]. We will first illustrate the ideas by proving a simpler result, Theorem \[A\]. Building on the proof Theorem \[A\] and using an inductive argument, we will prove a more general result which we describe now.

In what follows, we let \( V_t, t = 1,2,\ldots,\kappa \) be normed \( \mathbb{R} \)-vector spaces of dimension \( d_t \). For any \( \mathcal{A}_t : \Sigma_T \to \text{GL}(V_t) \), we define \( \tilde{\varphi}^1_{\mathcal{A}_t} : \mathcal{L} \to \mathbb{R}^+_0 \) analogously to (1.2) and (2.9):
\[
\tilde{\varphi}^1_{\mathcal{A}_t}(I) = \|\mathcal{A}_t(I)\| := \max_{x \in [I]} \|\mathcal{A}_t(x)\|.
\]

**Theorem 4.1.** Let \( \mathcal{A}_t : \Sigma_T \to \text{GL}(V_t) \), \( t = 1,2,\ldots,\kappa \) be Hölder functions admitting uniformly continuous stable and unstable holonomies. Suppose there exist a fixed point \( p \in \Sigma_T \) and a homoclinic point \( z \in W^s(p) \cap W^u(p) \setminus \{p\} \) such that each \( \mathcal{A}_t \) satisfies the pinching (A0) and the twisting (B0) conditions of Definition \[2.6\] at \( p \) and \( z \). Then the singular value functions \( \tilde{\varphi}^1_{\mathcal{A}_t} \), \( t = 1,2,\ldots,\kappa \) are simultaneously quasi-multiplicative: there exist \( c > 0, k \in \mathbb{N} \) such that for any words \( I, J \in \mathcal{L} \), there exists \( K = K(I,J) \in \mathcal{L} \) with \( |K| \leq k \) such that \( IKJ \in \mathcal{L} \) and that for each \( 1 \leq t \leq \kappa \), we have
\[
\|\mathcal{A}_t(IKJ)\| \geq c \|\mathcal{A}_t(I)\| \cdot \|\mathcal{A}_t(J)\|.
\]
Moreover, the constants \( c, k \) can be chosen uniformly in a small neighborhood of each \( \mathcal{A}_t \).

**Remark 4.2.** The first statement is the main content of Theorem \[4.1\] the uniform choice of the constants \( c \) and \( k \) follows from the fact that all parameters vary continuously on the data \( \mathcal{A}_t \).

Although the constants \( c, k \) can be chosen uniformly in a small neighborhood of each \( \mathcal{A}_t \), we cannot necessarily choose the connecting word \( K \) uniformly. See Remark \[4.10\].

We will prove Theorem \[4.1\] in subsection \[4.3\]. In subsection \[4.4\] we will then show that Theorem \[4.1\] follows as a corollary of Theorem \[4.1\].

4.1. Preliminary linear algebra. We first collect preliminary lemmas and relevant constants needed in the proof of Theorem \[A\] and Theorem \[4.1\]. Throughout the section, \( V \) is a finite dimensional \( \mathbb{R} \)-vector space equipped with a norm \( \| \cdot \| \).

**Definition 4.3.** For \( A \in \text{End}(V) \), we choose a singular value decomposition (SVD)
\[
A = U\Lambda V^\dagger,
\]
where the singular values in \( \Lambda \) are listed in a non-increasing order. We define \( u(A) \) and \( v(A) \) as the first column of \( U \) and \( V \), respectively.
If the singular values of \( A \) are distinct, then the SVD of \( A \) is unique (up to signs), and hence so are \( u(A) \) and \( v(A) \). If there are repeated singular values, then the singular value decomposition of \( A \) is not necessarily unique. In this case, we simply choose a singular value decomposition of \( A \), and set \( u(A) \) and \( v(A) \) accordingly.

Roughly speaking, \( u(A) \) and \( v(A) \) can be thought of as the most expanding direction of \( A^* = A^\top \) and \( A \), respectively. From the definition, we have

\[
\|A\| u(A) = A v(A). \tag{4.1}
\]

Throughout the section, when we measure an angle between nonzero vectors, we mean the angle between the lines spanned by the vectors. Similarly, when we measure an angle between a nonzero vector \( v \) and a hyperplane \( W \), we mean the minimum angle \( \angle(v, W) \) over all \( w \in W \setminus \{0\} \). Also, we will not distinguish between a vector in \( V \setminus \{0\} \) and its corresponding point in the projective space \( \mathbb{P}(V) \) when there is no confusion. We have an easy lemma from linear algebra.

**Lemma 4.4.** Given any \( A \in \text{Aut}(V) \) and any \( w \in V \), we have

\[
\|A w\| \geq \cos \angle(w, v(A)) \|A\| \cdot \|w\|.
\]

**Proof.** Let \( v = v(A) \), and write \( w = av + v' \) where \( |a| = \|w\| \cos \angle(v, w) \) and \( v' \in v^\perp \).

Letting \( u = u(A) \), we have from (4.1) that

\[
A w = a\|A\| u + Av'.
\]

Since the singular vectors are pairwise orthogonal (i.e., columns of \( U \) are pairwise orthogonal), we have \( Av' \in u^\perp \) and the lemma follows. \( \square \)

Recall that the co-norm \( m(A) \) of \( A \in \text{GL}(V) \) is defined by

\[
m(A) = \|A^{-1}\|^{-1}.
\]

The following lemma will be useful in proving Theorem 4.1.

**Lemma 4.5.** Let \( \theta > 0 \) be given and \( A, B, C, D \in \text{Aut}(V) \) such that

\[
\angle(B^* v(A), (Cu(D))^\perp) > \theta.
\]

Then,

\[
\|ABCD\| \geq \|A\| \cdot \|D\| \cdot \sin(\theta) \frac{m(B)^2 m(C)^2}{\|B\| \|C\|}.
\]

**Proof.** We have

\[
\|BCu(D)\| \cos \angle(BCu(D), v(A)) = \langle v(A), BCu(D) \rangle = \langle B^* v(A), Cu(D) \rangle, \geq \|B^* v(A)\| \|Cu(D)\| \sin(\theta).
\]

Hence,

\[
\cos \angle(BCu(D), v(A)) \geq \sin(\theta) \frac{m(B)m(C)}{\|B\| \|C\|}.
\]

It then follows from (4.1) and Lemma 4.4 that

\[
\|ABCD\| \geq \|ABCDv(D)\| = \|D\| \cdot \|ABCu(D)\|, \geq \|D\| \cdot \|ABCu(D)\| \|A\| \cdot \|BCu(D)\|, \geq \|A\| \cdot \|D\| \cdot \sin(\theta) \frac{m(B)m(C)}{\|B\| \|C\|} \cdot m(BC).
\]
This completes the proof. 

We will also make use of the adjoint cocycle. For a cocycle $F_A$ generated by a $\text{GL}(V)$-valued function $A$ over $f$, we define the adjoint cocycle $F_A^*$ over $f^{-1}$ generated by $A$, where $A^*$ is defined by the relation

$$\langle A^*(x)u, v \rangle = \langle u, A(f^{-1}x)v \rangle \quad \text{for all } x \in \Sigma_T \text{ and } u, v \in V.$$  

(4.2)

Suppose $F_A$ admits holonomies $H^{s/u}$. Then the adjoint cocycle $F_A^*$ also admits holonomies given by

$$H_{x,y}^{s,u} = (H_{x,y}^{s,u})^* \quad \text{and} \quad H_{x,y}^{u,s} = (H_{x,y}^{u,s})^*.$$  

This can be easily seen by plugging $u = (H_{x,y}^{s,u})^*\bar{w}$ and $v = H_{f^{-1}y, f^{-1}x}^{s,u}\bar{v}$ into (4.2) for some $y$ in the stable set of $x$ with respect to $f$. The following lemma shows that many properties of $A$ carry over to $A^*$.

**Lemma 4.6.** Let $A \in C^\alpha(\Sigma_T, \text{GL}_d(\mathbb{R}))$. Then,

1. $F_A^*$ is fiber-bunched if and only if $F_A$ is fiber-bunched.
2. $A^*$ is 1-typical if and only if $A$ is 1-typical.
3. $A^*$ is typical if and only if $A$ is typical.

**Proof.** See Lemma 7.2 of [BV04] for the proof of (1). The setting in [BV04] is $\text{SL}_d(\mathbb{R})$-valued cocycles, but the proof readily extends to $\text{GL}_d(\mathbb{R})$-valued cocycles. For (2), we note that the eigenvalues of the adjoint matrix $P^*$ are equal to the eigenvalues of $P$; in particular, they are simple and distinct in modulus. Indeed, if we define $W_j$ to be the hyperplane spanned by all but the $j$-th eigenvector $v_j$ of $P$, then the $j$-th eigendirection of $P^*$ is given by $w_j := (W_j)^\perp$: for any $1 \leq i \neq j \leq d$, we have

$$\langle v_i, P^*w_j \rangle = \langle Pv_i, w_j \rangle = \lambda_i \langle v_i, w_j \rangle = 0.$$  

The twisting condition (B0) from Definition 2.6 is then equivalent to

$$\langle \psi_p^* (v_i), w_j \rangle \neq 0 \quad \text{for all } 1 \leq i, j \leq d.$$  

Hence, $\langle v_i, (\psi_p^*)^*w_j \rangle \neq 0$ for all $1 \leq i, j \leq d$; this is equivalent to $A^*$ being 1-typical because $\psi_p^* = (\psi_p^*)^*$. (3) then trivially follows from (2). 

For $v \in \mathbb{P}(V)$, let the cone around $v$ of size $\varepsilon$ be

$$C(v, \varepsilon) := \{ w \in \mathbb{P}(V) : \angle(v, w) < \varepsilon \}.$$  

If $P \in \text{GL}(V)$ has simple eigenvalues of distinct norms, then any $v \in \mathbb{P}(V)$ can be mapped close to one of the eigendirections of $P$ under iterations of $P$. Even though the number of iterations needed depends on the given direction $v$, the following lemma shows that such number of iterations can be uniformly bounded above, independent of $v$. A quick illustration of ideas in $\mathbb{P}(\mathbb{R}^3)$ is as follows: suppose $\{v_i\}_{1 \leq i \leq 3}$ are eigendirections of $P$ with $|\lambda_1| > |\lambda_2| > |\lambda_3|$. If given $v$ is already close to some $v_i$, then no iteration of $P$ is necessary. If not, then a large but bounded number of iterations of $P$ will either map $v$ close to one of the $v_i$'s or map it out of the $\varepsilon$-neighborhood of span$\{v_2, v_3\}$ for some fixed $\varepsilon > 0$, in which case further bounded number of iterations of $P$ will map it close to $v_1$.

**Lemma 4.7.** Suppose $V$ is $d$-dimensional, and $P \in \text{GL}(V)$ has simple eigenvalues of distinct norms with corresponding eigenvectors $\{v_i\}_{1 \leq i \leq d}$. Given $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that for any $v \in \mathbb{P}(V)$, there exists $n = n(v) \leq N$ such that

$$P_\alpha v \in \bigcup_{i=1}^{d} C(v_i, \varepsilon).$$
Proof. Without loss of generality, suppose that the eigenvalues \( \{\lambda_i\}_{1 \leq i \leq d} \) of \( P \) corresponding to \( \{v_i\}_{1 \leq i \leq d} \) are decreasing in modulus. We adopt the same notation as in Lemma 4.6 and define \( W_j \) to be the hyperplane spanned by all but the \( j \)-th eigenvector \( v_j \). We make a few observations:

1. If \( v \) is close to each hyperplane \( W_i \) for every \( i \neq j \), then \( v \) has to be close to \( v_j \). We fix \( \eta > 0 \) depending on the given \( \varepsilon > 0 \) such that the following holds: if \( v \in \mathbb{P}(V) \) with \( \angle(v, W_i) < \eta \) for all \( i \in \{1, \ldots, j-1, j+1, \ldots, d\} \), then \( v \in C(v_j, \varepsilon) \).

2. Let \( v = \sum_{i=1}^{d} c_i v_i \). If the angle \( \angle(v, W_j) \) is not too small, then the ratio \( |c_j/c_i| \) is not too small (if \( c_i = 0 \), the ratio is \( \infty \)) for every \( i \). Since \( |\lambda_j| > |\lambda_i| \) for all \( i \geq j+1 \), we choose some large \( m \in \mathbb{N} \) such that \( |\lambda_j^m c_j/\lambda_i^m c_i| \) is sufficiently large for all \( i \geq j+1 \); this implies that \( P^m v \) makes a small angle with each \( W_i \) for \( i \geq j+1 \).

Formally, for \( \eta > 0 \) chosen in (1), we choose \( m \in \mathbb{N} \) such that for any \( 1 \leq j \leq d \), if \( \angle(v, W_j) > \eta \), then the angle \( P^m v \) makes with each hyperplane \( W_i \) with \( i \geq j+1 \) is at most \( \eta \). The existence of such \( m \in \mathbb{N} \) follows from the simplicity of the eigenvalues of \( P \).

We claim that for any \( v \in \mathbb{P}(V) \), there exists \( 0 \leq k \leq d-1 \) with \( P^{km} v \in \bigcup_{i=1}^{d} C(v_i, \varepsilon) \).

If \( w_0 := v \in \bigcup_{i=1}^{d} C(v_i, \varepsilon) \), then there is nothing to be done; we set \( k = 0 \).

If \( w_0 \not\in \bigcup_{i=1}^{d} C(v_i, \varepsilon) \), then we find the smallest \( j_0 \in \mathbb{N} \) such that \( \angle(w_0, W_{j_0}) > \eta \). From the choice of \( m \), the angle \( w_1 := P^m w_0 \) makes with each hyperplane \( W_i \), \( i = j_0 + 1, \ldots, d \) is smaller than \( \eta \).

If \( w_1 \in \bigcup_{i=1}^{d} C(v_i, \varepsilon) \), then we set \( k = m \). If not, from \( w_1 \not\in C(v_{j_0}, \varepsilon) \) and (1), there exists some \( i \in \{1, \ldots, j_0 - 1, j_0 + 1, \ldots, d\} \) such that \( \angle(w_1, W_i) > \eta \). Since we already know \( w_1 \) makes an angle less than \( \eta \) with each \( W_i \) with \( i \geq j_0 + 1 \), such \( i \) is necessarily smaller than \( j_0 \). We then set \( j_1 \) to be the smallest number (necessarily smaller than \( j_0 \)) among such \( i \). Again from the choice of \( m \), the angle \( w_2 := P^m w_1 \) makes with each \( W_i \), \( i = j_1 + 1, \ldots, d \) is smaller than \( \eta \).

We repeat the process inductively as follows: given \( w_n := P^m w_{n-1} \) from the previous step, we set \( k = nm \) if \( w_n \in \bigcup_{i=1}^{d} C(v_i, \varepsilon) \). If not, from \( w_n \not\in C(v_{j_{n-1}}, \varepsilon) \) and (1), we can necessarily find some \( i \) smaller than \( j_{n-1} \) such that \( \angle(w_n, W_i) > \eta \). We set \( j_n \) to be the smallest such \( i \). Then, \( w_{n+1} := P^m w_n \) makes an angle less than \( \eta \) with each \( W_i \), \( i = j_n + 1, \ldots, d \).

We continue this process until \( j_n = 1 \). From the construction, \( \angle(w_{n+1}, W_i) < \eta \) for all \( i = 2, \ldots, d \), which implies that \( w_{n+1} \in C(v_1, \varepsilon) \). Note that \( j_0 \leq d - 1 \), because if \( j_0 \) were equal to \( d \), then \( w_0 \) must have been in \( C(v_d, \varepsilon) \) contradicting \( w_0 \not\in \bigcup_{i=1}^{d} C(v_i, \varepsilon) \). Since \( \{j_n\} \) is a strictly decreasing sequence bounded below by \( 1 \), the inductive process necessarily terminates in at most \( d - 1 \) steps. We complete the proof by setting \( N := (d - 1)m \). \[ \square \]

Remark 4.8. Since the eigenvalues of \( P \) from Lemma 4.7 vary continuously in \( P \), we can choose \( N \) to work uniformly near \( P \): given \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) such that for any
\[ \tilde{P} \in \text{GL}(V) \] sufficiently close to \( P \) and any \( v \in \mathbb{P}(V) \), there exists \( n = n(v, \tilde{P}) \leq N \) such that \( \tilde{P}^n v \in \bigcup_{i=1}^{d} C(\tilde{v}_i, \varepsilon) \) where \( \{\tilde{v}_i\}_{1 \leq i \leq d} \) are distinct eigendirections of \( \tilde{P} \).

In the following lemma, we also adopt the same notations from Lemma 4.6.

**Lemma 4.9.** Let \( \varepsilon > 0 \) be given, and suppose \( P, \psi, R \in \text{GL}(V) \) and \( \ell \in \mathbb{N} \) satisfy the following properties:

- \( P \) has simple real eigenvalues of distinct norms,
- For any \( v \in \bigcup_{i=1}^{d} C(v_i, \varepsilon) \), we have \( \angle(\psi(v), \mathbb{W}_i) > \varepsilon \) for each \( i \),
- \( \angle(R(v), v) < \varepsilon/2 \) for any \( v \in \mathbb{P}(V) \),
- For any \( v \in \mathbb{P}(V) \) with \( \angle(v, \mathbb{W}_i) > \varepsilon \) for each \( i \), we have \( P^\ell v \in \mathcal{C}(v_1, \varepsilon/2) \).

Then for any \( v \in \bigcup_{i=1}^{d} C(v_i, \varepsilon/2) \), we have

\[ P^\ell \psi R(v) \in C(v_1, \varepsilon/2). \]

**Proof.** The proof is immediate from the properties of \( P, \psi, R \) and \( \ell \).

**4.2. Proof of Theorem [A]** In this subsection, we prove Theorem [A]

**Theorem** (Theorem [A]). Every \( A \in \mathcal{U} \) is quasi-multiplicative. Moreover, the constants \( c, k \) in (1.3) can be chosen uniformly in a neighborhood of \( A \) in \( \mathcal{U} \).

**Proof of Theorem [A]** Given \( A \in \mathcal{U} \), we will set uniform constants \( c \) and \( k \) such that for any given \( I, J \in \mathcal{L} \), there exists \( K \in \mathcal{L} \) with \( |K| \leq k \) such that quasi-multiplicativity (1.3) holds.

Let \( p \) and \( z \) be the periodic and homoclinic point given by the hypothesis. For simplicity, we assume that \( p \) is a fixed point of \( f \). In the case where the reference point \( p \) is a periodic point, we replace \( f \) by its suitable power so that \( p \) becomes a fixed point and the proof readily extends with relevant modifications. From Remark 2.12, we also assume that \( z \) is on \( \mathcal{W}^u_{\text{loc}}(p) \).

**Step 1.** We begin by setting up the notations and constants to be used in the proof.

- For any \( (\omega, n) \in \Sigma_T \times \mathbb{N} \), we identify it with the orbit segment starting at \( \omega \) of length \( n \).
- Let \( \{v_i\}_{1 \leq i \leq d} \) be eigendirections of \( P = A(p) \) listed in the order of decreasing modulus of eigenvalues. Similarly, we denote the eigendirections of \( P_* := A_*(p) \) by \( \{w_j\}_{1 \leq j \leq d} \). We define \( \mathbb{W}_i \) be the hyperplane in \( \mathbb{R}^d \) spanned by all \( v_i \)'s except \( v_j \).
- As in the proof of Lemma 4.6, we have \( w_j = (\mathbb{W}_j)^\perp \) for each \( 1 \leq j \leq d \).
- The angle formed by the top eigendirections \( v_1 \) and \( w_1 \) of \( P \) and \( P_* \) is necessarily bounded away from \( \pi/2 \). Let \( \beta := \angle(v_1, w_1^\perp) = \angle(v_1, \mathbb{W}_1) > 0 \).
- The twisting condition (B0) implies that there exists \( \varepsilon_0 > 0 \) such that

\[ \angle(\psi_p^\ell(v), \mathbb{W}_j) > \varepsilon_0, \quad (4.3) \]

for all \( 1 \leq j \leq d \) whenever \( v \in \bigcup_{i=1}^{d} C(v_i, \varepsilon_0) \). Fix such an \( \varepsilon_0 \in (0, \beta/8) \).
Suppose \( a, b, c, d \in \Sigma_T \) are related by
\[
[a, c] = b \quad \text{and} \quad [c, a] = d,
\]
where the bracket operation is defined in (2.1). Then we say such points form a rectangle with vertices \( a, b, c, \) and \( d \), and denote it by \([a, b, c, d]_r\).

Note that a rectangle is defined by prescribing two opposite vertices. All rectangles appearing in the proof will have one of its vertices at \( p \).

For \( q \in \Sigma_T \) in the local neighborhood of \( p \), but not on \( W^u_{loc}(p) \cup W^s_{loc}(p) \), consider the rectangle \([p, x, q, y]_r\) having \( p \) and \( q \) as opposite vertices. We define “the holonomy of the rectangle \([p, x, q, y]_r\)” by
\[
R_q := H^u_{y,p} \circ H^s_{q,y} \circ H^u_{x,q} \circ H^s_{p,x}.
\] (4.4)
Since canonical holonomies are uniformly continuous, the holonomy composition \( R_q \) uniformly approaches the identity as the rectangle degenerates (i.e., as a pair of opposite sides degenerates to a point) to a line or a point.

Recall \( \theta \in (0, 1) \) is the hyperbolicity constant defining the metric on the base \( \Sigma_T \). We fix \( m \in \mathbb{N} \) such that the following conditions hold: suppose \([p, x, q, y]_r\) is a rectangle.

(i) If \([p, x, q, y]_r\) has an edge whose length is at most \( \theta^m \), then
\[
\angle(R_q(v), v) < \frac{\varepsilon_0}{2} \quad \text{for any} \quad v \in \mathbb{P}(\mathbb{R}^d).
\]
(ii) If all edges of \([p, x, q, y]_r\) are no longer than \( \theta^m \), then
\[
\angle(H^u_{x,q} \circ H^s_{p,x}(v), v) < \varepsilon_0/2 \quad \text{and} \quad \angle(H^u_{y,q} \circ H^s_{p,y}(v), v) < \varepsilon_0/2,
\]
for any \( v \in \mathbb{P}(\mathbb{R}^d) \).

The existence of such \( m \in \mathbb{N} \) is guaranteed from the uniform continuity of the canonical holonomies \( H^{s/u} \).

Recall that we assumed \( z \in W^u_{loc}(p) \). Fix \( \ell \in \mathbb{N} \) such that \( f^\ell z \in W^s_{loc}(p) \). Increase \( \ell \) if necessary such that for any \( v \in \bigcup_{i=1}^d C(v_i, \varepsilon_0) \), we have
\[
P^\ell \psi^*_y(v) \in C(v_1, \varepsilon_0/2).
\]
The existence of such \( \ell \) is guaranteed from (4.3) and pinching condition (A0) on \( P \).

Notice that further increasing \( \ell \) does not disturb the defining properties of \( \ell \). So, we further increase \( \ell \) if necessary so that \( d(f^\ell z, p) \leq \theta^m \).

Set \( \Upsilon := \max \left( \max_{x \in \Sigma_T} \|A(x)\|, 1 \right) \) and \( \varrho := \min \left( \min_{x \in \Sigma_T} m(A(x)), 1 \right) \).
Using the uniform continuity of the canonical holonomies, we fix \( C_0 > 1 \) so that 
\[ \|H_{x,y}^s/u\| \leq C_0 \]
for any \( x, y \in \Sigma_T \) with \( d(x, y) \leq \theta \). Increase \( C_0 \) if necessary so that it also serves as a constant for the bounded distortion property \([1.9]\) of the singular value potential \( \Phi_A^1 \); for any \( n \in \mathbb{N} \) and \( I \in \mathcal{L}(n) \), we have
\[
C_0^{-1} \leq \frac{\|A^n(x)\|}{\|A^n(y)\|} \leq C_0,
\]
for any \( x, y \in I \).

- Let \( N \in \mathbb{N} \) be given by applying Lemma 4.7 to \( P \) and \( \varepsilon_0/2 \). Then for any \( v \in \mathcal{P}(\mathbb{R}^d) \), there exists \( n = n(v) \leq N \) such that \( P^nv \in \bigcup_{i=1}^d \mathcal{C}(v_i, \varepsilon_0/2) \).
- Let \( k_1 := N + \ell \).

By adjusting the constants \( \beta, \varepsilon_0, m, \ell, \Upsilon, \varrho, C_0, N \) and \( k_1 \) in the order they are defined, we may assume that they work for the adjoint cocycle as well. For the adjoint cocycle, we interchange the role of \( z \) and \( f^r z \), and denote the corresponding points (on the orbit of \( z \)) by \( z, f^r z, \ldots, f^{r\ell} z \in \mathcal{W}^s_\text{loc}(p) \).

The constants \( \beta, \varepsilon_0, m, r, \ell, \Upsilon, \varrho, C_0, N \) and \( k_1 \) also work uniformly in a small neighborhood of \( \mathcal{A} \). We will comment regarding the uniform choice of the constants \( c, k \) at the end of the proof.

**Step 2.** Since the adjacency matrix \( T \) is primitive, there exists \( \bar{\tau} \in \mathbb{N} \) such that \( T^\bar{\tau} > 0 \). Such \( \bar{\tau} \) is the mixing rate of the system \((\Sigma_T, f)\). Then for any given \( I \in \mathcal{L} \), there exists \( \bar{\omega}_0 \in \left[ I \right] \cap \mathcal{W}^s(p) \) such that \( f^{[1+\bar{\tau}]} \bar{\omega}_0 \in \mathcal{W}^s_\text{loc}(p) \).

We set
\[
\bar{\omega}_0 := f^{\bar{\tau}} \bar{\omega}_0 \text{ where } \tau = \tau(I) := |I| + \bar{\tau} + m.
\]

Since \( f^{[1+\bar{\tau}]} \bar{\omega}_0 \) is already on the local stable set \( \mathcal{W}^s_\text{loc}(p) \) of \( p \), we have \( d(\omega_0, p) \leq \theta^m \).

Let
\[
u_{\bar{\omega}_0} := H_{\bar{\omega}_0, p}^s u(\mathcal{A}^\tau(\bar{\omega}_0)).
\]

Lemma 4.7 implies that there exists \( n = n(u_{\bar{\omega}_0}) \leq N \) such that \( P^n u_{\bar{\omega}_0} \in \mathcal{C}(v_i, \varepsilon_0/2) \), for some \( 1 \leq i \leq d \). From \( u_{\bar{\omega}_0} \) and \( n \), we construct a new point
\[
\bar{\omega}_1 := f^{r-\tau-n} [z, f^n \omega_0];
\]

note that \( \bar{\omega}_1 \in \mathcal{W}^u_\text{loc}(\bar{\omega}_0) \cap \left[ I \right] \). We set
\[
\omega_1 := f^{\bar{\tau}} \bar{\omega}_1, \text{ and } \bar{\omega}_1 := f^{n+\ell} \omega_1.
\]

The forward orbit segment starting at \( \bar{\omega}_1 \in \left[ I \right] \) first comes close to \( p \), arriving at \( \omega_1 \), then dwells near \( p \) for \( n \) iterates, and then shadows the orbit segment from \( z \) to \( f^{\ell} z \) to finally land on \( \mathcal{W}^s_\text{loc}(p) \) at the point \( \bar{\omega}_1 \). Since \( n \) is bounded above by \( N \), the length of the orbit segment \( (\omega_1, n + \ell) \) is bounded above by \( k_1 \).

The holonomy of the rectangle with opposite vertices at \( p \) and \( f^n \omega_1 = f^{-\ell} \bar{\omega}_1 \) is given by
\[
R_f f^{-\ell} \bar{\omega}_1 = H_{z, p}^u H^s_{f^{-\ell} \bar{\omega}_1, z} H^u_f f^n \omega_0 f^{-\ell} \bar{\omega}_1 H^s_f p, f^n \omega_0.
\]
Combining this with the relation $H_{\omega_1,\tau}^s A^\ell(f^{-\ell} \omega_1) = A^\ell(z)H_{f^{-\ell} \omega_1, z}^s$ and (2.5), we obtain

$$H_{\omega_1,\tau}^s A^{n+\ell}(\omega_1)H_{\omega_0,\omega_1}^u = H_{\omega_1,\tau}^s A^\ell(f^{-\ell} \omega_1)A^n(\omega_1)H_{\omega_0,\omega_1}^u$$

$$= H_{\omega_1,\tau}^s H_{f^{-\ell} \omega_1, z}^s A^\ell(z)H_{f^{-\ell} \omega_1, z}^s H_{f^{-\ell} \omega_0, f^{-\ell} \omega_1}^s A^n(\omega_0)$$

$$= H_{f^{-\ell} \omega_1, z}^s A^\ell(z)H_{f^{-\ell} \omega_1, z}^s H_{f^{-\ell} \omega_0, f^{-\ell} \omega_1}^s A^n(\omega_0)$$

$$= P^\ell \psi^\tau_z R_{f^{-\ell} \omega_1} H_{f^{-\ell} \omega_0, f^{-\ell} \omega_1}^s A^n(\omega_0)$$

$$= P^\ell \psi^\tau_z R_{f^{-\ell} \omega_1} P^n H_{f^{-\ell} \omega_0, \cdot}^s.$$  

Then $u_{\omega_1} := H_{\omega_1,\tau}^s A^{n+\ell}(\omega_1)H_{\omega_0,\omega_1}^u u(A^\tau(\omega_0))$ is related to $u_{\omega_0}$ as follows:

$$u_{\omega_1} = H_{\omega_1,\tau}^s A^{n+\ell}(\omega_1)H_{\omega_0,\omega_1}^u u(A^\tau(\omega_0))$$

$$= P^\ell \psi^\tau_z R_{f^{-\ell} \omega_1} P^n H_{f^{-\ell} \omega_0, \cdot}^s u_{\omega_0}.$$  

From (4.6), it follows that

$$u_{\omega_1} \in C(v_1, \epsilon_0/2).$$  

Indeed, the choice of $n = n(u_{\omega_0})$ gives $P^n u_{\omega_0} \in C(v_1, \epsilon_0/2)$ for some $1 \leq i \leq d$. Since the edge between $p$ and $f^n \omega_0$ is no longer than $\theta^n$, $R_{f^{-\ell} \omega_1}$ doesn’t move any line off itself more than $\epsilon_0/2$ in angle. Lemma 4.9 then gives (4.7). Note from the choice of $\ell$, we have $d(\omega_1, p) \leq \theta^n$. This fact will be used in Step 4.

Let us briefly summarize what we have done so far. From a given word $I \in \mathcal{L}$, we construct an orbit segment $(\omega_0, \tau)$ starting at $\omega_0 \in [I]$ and ending at $\omega_0 \in \mathcal{W}_{\text{loc}}(p)$ using the mixing property of the base system $(\Sigma_T, f)$. We do not however have any control of the singular direction $u_{\omega_0}$; it could be anywhere in $P(\mathbb{R}^d)$. So we construct a new orbit segment $(\tilde{\omega}_0, \tau + n + \ell)$ which first shadows the orbit of $\omega_0$ for time $\tau + n$ and then shadows the orbit of $z$ for time $\ell$. By choosing $n$ in such a way that $P^n u_{\omega_0}$ is close to one of the eigendirections of $P$, we ensure that $u_{\tilde{\omega}_0}$ is close enough to the top eigendirection $v_1$ of $P$.

**Step 3.** We apply the argument in Step 2 to the adjoint cocycle $A_*$ with $\hat{z}$ and $f^{-\ell} \hat{z}$ playing the role of $z$ and $f^\ell z$.

Similar to $\tilde{\omega}_0$, we obtain $\tilde{t}_0 \in f^{[1]}J$ from the mixing property of $(\Sigma_T, f)$ such that

$$t_0 := f^{-\tau(J)} \tilde{t}_0 \in \mathcal{W}_{\text{loc}}^u(p) \text{ where } \tau(J) = |J| + \tau + m.$$  

Applying Lemma 4.7 to $P_*$ and the direction $P_{f^{-\ell} \omega_0}^\tau u(A_*^{\tau(J)}(\tilde{t}_0))$ gives $\hat{n} \leq N$ such that $P_\hat{n} H_{f^{-\ell} \omega_0}^{\tau(J)} u(A_*^{\tau(J)}(\tilde{t}_0))$ belongs to the cone $C(w_1, \epsilon_0/2)$ for some $1 \leq i \leq d$. Define

$$\hat{t}_1 := f^{\tau(J)+\hat{n}}[f^{-\tilde{n}} \tilde{t}_0, \hat{z}],$$

and set

$$t_1 = f^{-\tau(J)} \hat{t}_1 \text{ and } \tilde{t}_1 := f^{-\hat{n}-\ell} \tilde{t}_1.$$  

Then the analogue of (4.7) holds:

$$H_{f^{-\ell} \omega_1, \tau}^s A_*^{n+\ell}(t_1)H_{f^{-\ell} \omega_1, t_1}^s u(A_*^{\tau(J)}(\tilde{t}_0)) \in C(w_1, \epsilon_0/2).$$  

The length of the $f^{-1}$-orbit from $t_1$ to $\tilde{t}_1$ is bounded above by $k_1$.

Having two points $\tilde{\omega}_1 \in \mathcal{W}^u(p)$ and $t_1 \in \mathcal{W}_{\text{loc}}^u(p)$ with the desired control on the singular directions (4.7) and (4.8), we connect their orbits near $p$ by

$$\chi := [\tilde{t}_1, \tilde{\omega}_1].$$
and set \( \tilde{\chi} := f^{-\tau(I)-n-\ell} \chi \in I \) and \( \hat{\chi} := f^{r(J)+\hat{n}+\ell} \chi \in f^{[J]}[J] \).

From the construction, every edge of the rectangle \([p, \tilde{\omega}_1, \chi, \hat{\omega}_1]_r\) is no longer than \( \theta^m \).

From the choice of \( m \), \( H^u_{\tilde{\omega}_1, \chi} \circ H^s_{\hat{\omega}_1} \) is sufficiently close to the identity in that it does not move any line off itself more than \( \varepsilon_0/2 \) in angle. Then from (4.7),

\[
u_{\hat{\chi}} := A^{n+\ell}(f^{r(I)} \hat{\chi}) H^u_{\omega_0, f^{r(I)} \hat{\chi}} u(A^T(\omega_0)) = H^u_{\omega_1, \chi} H^s_{\hat{\omega}_1} u_{\omega_1}
\]

belongs to \( \mathcal{C}(v_1, \varepsilon_0) \).

Similarly, \( H^u_{\tilde{\omega}_1, \chi} \circ H^s_{\hat{\omega}_1} \) doesn’t move any line off itself more than \( \varepsilon_0/2 \) in angle. Notice that

\[
u_{\hat{\chi}} := A^{n+\ell}(f^{\hat{n}+\ell} \chi) H^u_{\omega_0, f^{\hat{n}+\ell} \chi} v(A^T(J)(t_0))
\]

belongs to \( \mathcal{C}(w_1, \varepsilon_0) \).

Then \( u_{\tilde{\chi}} \in \mathcal{C}(v_1, \varepsilon_0) \) and \( v_{\hat{\chi}} \in \mathcal{C}(w_1, \varepsilon_0) \) together give the uniform angle gap (using the choice of \( \varepsilon_0 \in (0, \beta/8) \)):

\[
\angle \left( u_{\tilde{\chi}}, u_{\hat{\chi}} \right) > \frac{3\beta}{4}.
\]  

(4.9)

**Step 4.** We use the orbit of \( \chi \) to construct a connecting word \( K \). Let \( k := 2m + 2\tilde{\tau} + 2k_1 \), and note that \( k \) is independent of \( I \) and \( J \). Then we define the connecting word

\[
K := [f^{[I]} \tilde{\chi}]^w_k,
\]

where \( \tilde{k} = 2m + 2\tilde{\tau} + n + \hat{n} + 2\ell \). The length of \( K \) is at most \( k \). We apply Lemma 4.5 with \( A = A^T(I)(t_0) \), \( B = H^s_{\hat{\omega}_1, \chi} A^{n+\ell}(\chi) \), \( C = A^{n+\ell}(f^{r(I)} \hat{\chi}) H^u_{\omega_0, f^{r(I)} \hat{\chi}} \), and \( D = A^T(I)(\omega_0) \):

recalling that \( H^s_{u,x} = (H^s_{u,x})^* \), from (4.9), such choice of \( A, B, C \) and \( D \) satisfies the assumption of Lemma 4.5 with \( \theta = 3\beta/4 \). Since \( C_0 \) is the constant from the bounded distortion as well as the upper bound on \( \|H^s_{u,x}\| \) whenever \( d(x, y) \leq \theta \), we have

\[
\|A(IKJ)\| \geq \|A^{[k] + [I] + [J]}(\hat{\chi})\|,
\]

\[
\geq \|A^{r(I)}(f^{\hat{n}+\ell} \chi) A^{n+\ell}(\chi) A^{r(I)}(\hat{\chi})\|,
\]

\[
\geq C_0^{-2}\|H^s_{\hat{\omega}_1, \chi} A^{r(I)}(f^{\hat{n}+\ell} \chi) A^{n+\ell}(\chi) A^{r(I)}(\hat{\chi})\|,
\]

\[
\geq C_0^{-2}\|A^{r(I)}(t_0) H^s_{\hat{\omega}_1, \chi} A^{n+\ell}(\chi) A^{r(I)}(\hat{\chi}) H^u_{\omega_0, \hat{\chi}}\|,
\]

\[
\geq C_0^{-2}\|ABCD\|,
\]

\[
\geq C_0^{-2}\|ABCD\| \geq \frac{m(B) m(C)^2}{\|B\| \|C\|},
\]

\[
\geq \frac{C_0^{-8}\|A^{T(I)}(t_0)\| \cdot \|A^{r(I)}(\omega_0)\|}{\|B\| \|C\|},
\]

\[
\geq \frac{C_0^{-8}\|A^{T(I)}(t_0)\| \cdot \|A^{[I]}(\hat{\chi})\| \cdot \|A^{r(I)}(\hat{\chi})\|}{\|B\| \|C\|},
\]

\[
\geq C_0^{-10} \sin(3\beta/4) \frac{d_{2k_1} + 2(\tau + m)}{\mathcal{Y}^{2k_1}} \|A^{[I]}(f^{-[I]} t_0)\| \cdot \|A^{[I]}(\omega_0)\|,
\]

\[
\geq c \|A(1)\| \|A(J)\|,
\]

where \( c := C_0^{-10} \sin(3\beta/4) \frac{d_{2k_1} + 2(\tau + m)}{\mathcal{Y}^{2k_1}} \).
4.3. Proof of Theorem 4.1. The proof of Theorem 4.1 closely follows the proof of Theorem A. We will use the same notations as in the proof of Theorem A whenever applicable.

**Proof of Theorem 4.1.**

**Step 1.** Let $\mathcal{A}_t : \Sigma_T \to \text{GL}(\mathcal{V}_t)$, $1 \leq t \leq \kappa$ be Hölder functions with uniformly continuous holonomies $H^{s/u, \{t\}}$ such that the pinching (A0) and the twisting (B0) conditions from Definition 2.6 hold at the common fixed point $p$ and its homoclinic point $z$. Let $P_t := \mathcal{A}_t(p)$.

First, we fix the constants $\beta, \varepsilon_0, m, \ell, \Upsilon, \rho, C_0$ and $N$ from the proof of Theorem A such that their properties work uniformly for all $\mathcal{A}_t$, $t \in \{1, 2, \ldots, \kappa\}$. For instance, denoting

$$\beta_t := \angle(v_1^{(t)}), (w_1^{(t)}) = \angle(v_1^{(t)}, \mathcal{W}_1^{(t)}) > 0,$$

let $\beta$ be the minimum of all $\beta_t$:

$$\beta := \min_{1 \leq t \leq \kappa} \beta_t,$$

which is necessarily bounded away from 0. We define $N \in \mathbb{N}$ by taking the maximum among the $N$’s obtained by applying Lemma 4.7 to $P_t$ and $\varepsilon_0/2$ for each $1 \leq t \leq \kappa$. Similarly, other constants are chosen to work uniformly for all $\mathcal{A}_t$, $1 \leq t \leq \kappa$.

For $k_1$, we re-define it as

$$k_1 := \kappa(N + \ell).$$

By further relaxing these constants, they work uniformly in a small neighborhood of each $\mathcal{A}_t$.

In order to avoid overloading the super/subscripts, for the rest of the proof, we will often write $\mathcal{A}$ to denote $\mathcal{A}_t$ for some $1 \leq t \leq \kappa$ when the context is clear. Similarly, we will suppress the index $t$ from related expressions (especially from the holonomies $H^{s/u, \{t\}}_x(y)$) when there is no confusion.

**Step 2.** Following Step 2 of the proof of Theorem A, we obtain $\bar{\tau} \in \mathbb{N}$ from the mixing property of $(\Sigma_T, f)$ such that given any $I \in \mathcal{L}$, there exists $\bar{\omega}_0 \in [I] \cap \mathcal{W}^s(p)$ such that $f^{[I]+\bar{\tau}}(\bar{\omega}_0) \in \mathcal{W}^s_{\text{loc}}(p)$. We set

$$\bar{\omega}_0 := \omega_0 = f^{\tau} \bar{\omega}_0,$$  

where $\tau = \tau(I) := |I| + \bar{\tau} + m$.

Since $f^{[I]+\tau} \bar{\omega}_0$ is already on the local stable set $\mathcal{W}^s_{\text{loc}}(p)$ of $p$, we have $d(\bar{\omega}_0, p) \leq \theta^m$. For each $1 \leq t \leq \kappa$, let

$$u^{(t)}(\omega_0) := H^{s}_{\omega_0, p} u(\mathcal{A}_t^t(\bar{\omega}_0)) \in \mathbb{P}(\mathcal{V}_t).$$

With $(\bar{\omega}_0, \tau)$ as the base case, we will inductively construct orbit segments $\{(\bar{\omega}_j, \tau + n_j)\}_{1 \leq j \leq \kappa}$ with $\bar{\omega}_j \in [I]$ such that the $j$-th orbit segment $(\bar{\omega}_j, \tau + n_j)$ satisfies the following properties: setting

$$\omega_j := f^{\tau} \bar{\omega}_j \quad \bar{\omega}_j := f^{n_j} \omega_j,$$

we have $\omega_j \in \mathcal{W}^s_{\text{loc}}(\bar{\omega}_0)$ and $\bar{\omega}_j \in \mathcal{W}^s_{\text{loc}}(p)$ with $d(\bar{\omega}, p) \leq \theta^m$. Moreover, setting

$$u^{(t)}(\omega_j) := H^{s}_{\omega_j, \bar{\omega}_j} A^{n_j}(\omega_j) H^{u}_{\bar{\omega}_j, \omega_j} u(\mathcal{A}_t^t(\bar{\omega}_0)),$$

we have

$$u^{(t)}(\omega_j) \in \mathcal{C}(v_1^{(t)}, \varepsilon_0/2) \quad \text{for } 1 \leq t \leq j.$$
First, we construct \( \tilde{\omega}_1 \) similarly how we constructed \( \bar{\omega}_1 \) in Step 2 of the proof of Theorem A by applying Lemma 4.7 to \( u^{(1)}(\omega_0) \), we obtain \( \bar{n}_0 \leq N \) such that \( P^{\bar{n}_0}_1 u^{(1)}(\omega_0) \) belongs to \( C(v_1^{(1)}, \varepsilon_0/2) \). We then set

\[
\bar{\omega}_1 = f^{-\tau - \bar{n}_0} \omega_1 \quad \text{and} \quad n_1 = \bar{n}_0 + \ell,
\]

and define \( \omega_1, \tilde{\omega}_1 \) according to (4.10). Following the same argument that established \( u_{\omega_1} \in C(v_1, \varepsilon_0/2) \) from Step 2 in the proof of Theorem A, we see that \( u^{(1)}(\omega_1) \) defined in (4.11) belongs to \( C(v_1^{(1)}, \varepsilon_0/2) \). This establishes (4.12) for \( j = 1 \).

For the inductive step, suppose we have \((\omega_j, \tau + n_j)\) such that (4.12) holds. Applying Lemma 4.7 to \( u^{(j+1)}(\omega_j) \) gives \( \bar{n}_j \leq N \) such that \( P^{\bar{n}_j}_{j+1} u^{(j+1)}(\omega_j) \) belongs to \( C(v_{j+1}^{(j+1)}, \varepsilon_0/2) \) for some \( 1 \leq i \leq d_{j+1} \). Setting

\[
\bar{\omega}_{j+1} = f^{-\tau - n_j - \bar{n}_j} \omega_{j+1} \quad \text{and} \quad n_{j+1} = n_j + \bar{n}_j + \ell,
\]

we obtain \( \omega_{j+1} \) and \( \tilde{\omega}_{j+1} \) according to (4.10). From the choice of \( \ell \), we have \( d(\tilde{\omega}_{j+1}, p) \leq \theta^m \).

We need to show that for such \( \omega_{j+1} \), \( u^{(j+1)}(\omega_{j+1}) \) belongs to \( C(v_{j+1}^{(j+1)}, \varepsilon_0/2) \) for each \( 1 \leq t \leq j+1 \).

The analogous calculations to (4.5) and (4.6) show that \( u^{(j+1)}(\omega_{j+1}) \) and \( u^{(j)}(\omega_j) \) are related by

\[
u^{(t)}(\omega_{j+1}) = P^t \nu^{(t)}_{\omega_j} P^t_{j+1} \omega_{j+1} \quad \text{for each} \quad 1 \leq t \leq \kappa.
\]

From the inductive hypothesis as well as the choice of \( n_j \), it follows that \( P^t \nu^{j+1} / u^{(j)}(\omega_j) \) belongs to \( C(v_{j+1}^{(j+1)}, \varepsilon_0/2) \) for each \( 1 \leq t \leq j+1 \). Indeed for \( 1 \leq t \leq j \), we already have \( u^{(j)}(\omega_{j+1}) \in C(v_{j+1}^{(j+1)}, \varepsilon_0/2) \) from the hypothesis, and since \( v_{j+1}^{(j+1)} \) is the eigendirection of \( P_j \) corresponding to the largest eigenvalue in modulus, \( P^t_{j+1} \) maps it even closer toward \( v_{j+1}^{(j+1)} \).

For \( t = j+1 \), the number \( \bar{n}_j \leq N \) is chosen so that \( P^{\bar{n}_j}_{j} u^{(j+1)}(\omega_j) \) belongs to \( C(v_{j+1}^{(j+1)}, \varepsilon_0/2) \) for some \( 1 \leq i \leq d_{j+1} \).

Since \( P^t_{j+1} \) does not move any line off itself more than \( \varepsilon_0/2 \) in angle, it follows from Lemma 4.9 that \( u^{(j+1)}(\omega_{j+1}) \) belongs to \( C(v_{j+1}^{(j+1)}, \varepsilon_0/2) \) for each \( 1 \leq t \leq j+1 \), completing the inductive step.

\textbf{Figure 4.2.}

\[ p = \ldots \]
See Figure 4.3 below for first two iterations of the inductive step. Also see Figure 4.2 for the symbolic description of the inductive step; the checks and arrows indicate the 0-th entry of the indicated points.

The induction ends at $\omega_\kappa$ where $\kappa$ is the number of cocycles $A_t$. Setting $\bar{\omega}_1 := \bar{\omega}_\kappa$ and $n_1 = n_\kappa$, we summarize the properties of the orbit segment $(\bar{\omega}_1, \tau + n_1)$ as follows:

$$\bar{\omega}_1 \in [I], \; \omega_1 := \omega_\kappa \in W^u_{loc}(\omega_0), \; \bar{\omega}_1 := \bar{\omega}_\kappa \in W^s_{loc}(p)$$

with $d(\bar{\omega}_1, p) \leq \theta^m$, and

$$u^{(t)}(\omega_1) := u^{(t)}(\omega_\kappa) \in C(v_1^{(t)}, \varepsilon_0/2) \text{ for every } 1 \leq t \leq \kappa.$$  

Recalling that

$$k_1 := \kappa(N + \ell),$$

we have $n_1 \leq k_1$. This follows because $n_{j+1} - n_j = \tilde{n}_j + \ell \leq N + \ell$ for each $j$ and the inductive process terminates in $\kappa$ iterations.

**Step 3.** We apply the same argument above to the adjoint cocycle $A_\ast$ similar to Step 3 in the proof of Theorem $A$.

For any $J \in L$, there exist $\hat{t}_0 \in f| |J|$ such that

$$t_0 := f^{- \tau(J)} \hat{t}_0 \in W^u_{loc}(p) \text{ with } \tau(J) = |J| + \bar{\tau} + m.$$
Similar to how we constructed \( \tilde{\omega}_1 \) inductively from \( \tilde{\omega}_0 \), we construct \( \tilde{\omega}_j \in f^{[J]}(I) \) from \( \tilde{\omega}_0 \) such that properties analogous to (4.13) and (4.14) hold: denoting \( \tilde{\omega}_j := f^{-\tau(j)} \tilde{\omega}_j \) and \( \tilde{\omega}_j := f^{-\hat{n}_j} \tilde{\omega}_j \in W_{\text{loc}}^u(p) \), we have

\[
i_j \in W_{\text{loc}}^s(i_0) \quad \text{and} \quad \tilde{\omega}_j \in W_{\text{loc}}^u(p)
\]

with \( d(\tilde{\omega}_j, p) \leq \theta^m \). Also, \( \hat{n}_j \) is bounded above by \( k_1 \). Moreover,

\[
H_{\tilde{\omega}_j, p}^s, A^s_{\bar{\omega}_j} (\tilde{\omega}_j) H_{\omega_0, i_0, \tilde{\omega}_j}^u (A^u_{\bar{\omega}_j}(i_0)) \in C(w_1^1, \varepsilon_0/2) \quad \text{for every} \quad 1 \leq t \leq \kappa.
\]

(4.15)

Having constructed two points \( \tilde{\omega}_1 \in W_{\text{loc}}^u(p) \) and \( \tilde{\omega}_j \in W_{\text{loc}}^u(p) \) with the desired control on the singular directions (4.12) and (4.15), we connect them near \( p \) by

\[
\chi := [\tau_1, \tilde{\omega}_1],
\]

and set \( \tilde{\chi} := f^{-\tau(I) - n_1} \chi \in [I] \) and \( \hat{\chi} := f^{\tau(J) + \hat{n}_j} \chi \in f^{[J]}(J) \).

From the choice of \( m \), following the same argument as in Step 3 in the proof of Theorem \( \Box \) we obtain the uniform angle gap:

\[
\angle \left( v^{(t)}(\tilde{\chi}), u^{(t)}(\tilde{\chi}) \right) > \frac{3\beta}{4} \quad \text{for every} \quad 1 \leq t \leq \kappa,
\]

where

\[
u^{(t)}(\tilde{\chi}) = A^{n_1}(f^{\tau(I)} \chi) H_{\omega_0, f^{\tau(I)} \chi}^u u(A^u(\tilde{\omega}_0))
\]

and

\[
u^{(t)}(\hat{\chi}) = A^{n_1}(f^{\hat{n}_j} \chi) H_{\omega_0, f^{\hat{n}_j} \chi}^u u(A^u(J)(i_0)).
\]

**Step 4.** This step also follows Step 4 in the proof of Theorem \( \Box \) Setting

\[
K := [f^{[I]} \chi]_k
\]

where \( k = 2m + 2\tau + n_1 + \hat{n}_j \), the length of \( K \) is bounded above by \( k := 2m + 2\tau + 2k_1 \), a number defined independent of \( I \) and \( J \). We then apply Lemma 4.5 to each \( A_t, t \in \{1, 2, \ldots, \kappa\} \). This gives

\[
\| A_t(IKJ) \| \geq c\| A_t(I) \|\| A_t(J) \|,
\]

where \( c := C_0^{-10} \sin(3\beta/4) \frac{2^{6k_1 + 2(\tau + m)}}{Y^{2k_1}} \). Here we have used the fact that all constants from Step 1 have been chosen to work uniformly over all \( A_t \). Lastly, \( c \) and \( k \) can be slightly relaxed to work uniformly in a small neighborhood of each \( A_t \).

**Remark 4.10.** Unlike constants \( c \) and \( k \), it is clear from the proof of Theorem 4.11 that the connecting word \( K = K(I, J) \in \mathcal{L} \) cannot be chosen uniformly in a small neighborhood of \( A \). This is because although \( B \) may be arbitrarily close to \( A \), the singular direction \( u(B^I(\tilde{\omega}_0)) \) from Step 2 could be drastically different from \( u(A_I(\tilde{\omega}_0)) \) if the length of \( I \) (and, hence, \( \tau = m + \tau + |I| \)) is arbitrarily large. Then the number of iterates \( n \) of \( P \) needed to turn \( H_{\omega_0, P}^s A^s_I(\tilde{\omega}_0) \) close to one of the eigendirections of \( P \) would be different from that of \( H_{\omega_0, P}^s A^s_J(\tilde{\omega}_0) \). Hence we cannot expect \( K \) to be chosen uniformly near \( A \).

**Remark 4.11.** From the proof of Theorem 4.11 it is clear that given any \( s_0 \in \mathbb{R}^+_0 \) (i.e., not necessarily belonging to the range \( [0, d] \)) and \( A \in \mathcal{U} \), the singular value functions \( \tilde{\varphi}_A^s, s \in [0, s_0] \) are simultaneously quasi-multiplicative. Moreover, the constants \( c, k \) can be chosen uniformly in a small neighborhood of \( A \).
Suppose $s_0 > d$. Let $c_1, k > 0$ be constants from Theorem 4.1 and $c_2 > 0$ be a small constant such that
\[
\min_{x \in \Sigma_t} \det(B(x))^{s/d} \geq c_2
\]
for all $s \in [d, s_0]$ and $B$ sufficiently close to $A$. Let $C > 1$ be the maximum among the bounded distortion constant (see Remark 3.10) of $\Phi^s_{B}$ for all $s \in (d, s_0)$ and $B$ sufficiently close to $A$. From the definition of the pressure ($3.3$), it follows from the proof of Fekete’s lemma. For any sequence $\{\alpha_n\}$, we have
\[
\sum_{|I| = n} \alpha_n(I) = \sum_{|I| = n} \max_{x \in [I]} \varphi^s(A^n(x)).
\]

4.4. Proof of Theorem E. From Theorem 4.1, Theorem E easily follows.

Proof of Theorem E. Given $A \in \mathcal{U}$, we set $\kappa = d - 1$, $\forall_t = \mathbb{R}^{(t)}$ and $A_t = A^{\kappa t}$ for each $1 \leq t \leq d - 1$. Then each $A_t$ admits holonomies $(H^s)^{\kappa t}$ where $H^s$ are the canonical holonomies of $A$ given by the fiber-bunching assumption on $A$. Also, $(H^s)^{\kappa t}$ varies Hölder continuously because $H^s$ does from (2.3). Hence, it follows from $A$ being typical that $A_t$, $1 \leq t \leq d - 1$ satisfy the assumptions of Theorem 4.1.

Recalling that $\varphi_A = \varphi_A^{\kappa t}$ for $1 \leq t \leq d - 1$, Theorem 4.1 gives simultaneous quasi-multiplicativity of $\varphi_A$ when $t$ is restricted to $[1, d - 1] \cap \mathbb{N}$. Moreover, $\varphi_A^0 \equiv 1$ is trivially quasi-multiplicative, and by decreasing $c$ if necessary (depending on $k$ and $g$ from the proof of Theorem 4.1), $\varphi_A^d$ is also simultaneously quasi-multiplicative with the same constants $c$ and $k$.

Simultaneous quasi-multiplicativity easily extends to include all $t \in [0, d]$ as follows: for any $t \in (n, n + 1)$ with $n \in \{0, 1, \ldots, d - 1\}$, we write $t = n\gamma + (n + 1)(1 - \gamma)$ for some $\gamma \in (0, 1)$. We raise the inequality from simultaneous quasi-multiplicativity of $\varphi_A^n$ by power $\gamma$:
\[
(\varphi_A^n(IKJ))^{\gamma} \geq (c\varphi_A^n(I)\varphi_A^n(J))^{\gamma}.
\]

Similarly, we raise the inequality from simultaneous quasi-multiplicativity of $\varphi_A^{n+1}$ by power $1 - \gamma$:
\[
(\varphi_A^{n+1}(IKJ))^{1-\gamma} \geq (c\varphi_A^{n+1}(I)\varphi_A^{n+1}(J))^{1-\gamma}.
\]

Noticing that $\varphi_A^t$ is uniformly comparable to $\varphi_A^n(\varphi_A^{n+1})^{1-\gamma}$, multiplying the two inequalities gives simultaneous quasi-multiplicativity of $\varphi_A^t$; there exists $c_0 > 0$ such that
\[
\varphi_A^t(IKJ) \geq c_0 \varphi_A^t(I) \varphi_A^t(J).
\]

5. Continuity of the subadditive pressure

5.1. Proof of Theorem B. In this subsection, we prove Theorem B based on the proof of Fekete’s lemma. For any $A : \Sigma_T \to GL_d(\mathbb{R})$ and $s \in [0, \infty)$, we obtain a subadditive sequence $\{\log \alpha^s_n(A)\}_{n \in \mathbb{N}}$ where
\[
\alpha^s_n(A) := \sum_{|I| = n} \varphi_A^n(I) = \sum_{|I| = n} \max_{x \in [I]} \varphi^s(A^n(x)).
\]

Since the base system is a subshift of finite type $(\Sigma_T, f)$, we have
\[
P(\Phi_A) = \lim_{n \to \infty} \frac{1}{n} \log \alpha^s_n(A);
\]
that is, we can compute the pressure by looking $(n, 1)$-separated sets, and drop the limit in $\varepsilon$ from the definition of the pressure ($3.3$). See Section 4 of [Kel98].
We say that a sequence \( \{a_n\}_{n \in \mathbb{N}} \) is almost superadditive with constant \( C > 0 \) if for all \( n, m \in \mathbb{N} \), we have
\[
a_{n+m} \geq a_n + a_m - C.
\]

In the following lemma, we use quasi-multiplicativity from Theorem \([E]\) to show that given any \( \mathcal{A} \in \mathcal{U} \) and \( s \in [0, \infty) \), the sequence \( \{\log \alpha^s_n(\mathcal{B})\} \) is almost superadditive with the uniform constant \( C > 0 \) for all \( \mathcal{B} \) sufficiently close to \( \mathcal{A} \).

**Lemma 5.1.** Let \( \mathcal{A} \in \mathcal{U} \) and \( s \in [0, \infty) \). Then there exists \( C = C_s > 0 \) such that the following holds: there exists a small neighborhood of \( \mathcal{A} \) in \( \mathcal{U} \) such that for all \( \mathcal{B} \) in the neighborhood, the sequence \( \{\log \alpha^s_n(\mathcal{B})\}_{n \in \mathbb{N}} \) is almost superadditive with constant \( C \).

**Proof.** There exists \( C_1 > 0 \) such that for any \( n \in \mathbb{N} \),
\[
\alpha^s_{n+1}(\mathcal{A}) \leq C_1 \alpha^s_n(\mathcal{A}).
\]
In fact, denoting the number of alphabets in \( \Sigma_T \) by \( q \), set \( C_1 = \Upsilon^s \cdot q \) where \( \Upsilon = \max_{x \in \Sigma_T} \|A(x)\| \).

Increase \( C_1 \) slightly to ensure that \((5.1)\) also holds for all \( \mathcal{B} \) in a small neighborhood of \( \mathcal{A} \).

After shrinking the neighborhood if necessary, we have
\[
\alpha^s_n(\mathcal{B}) \alpha^s_m(\mathcal{B}) \leq \sum_{i=0}^{k} \alpha^s_{n+m+i}(\mathcal{B}) \leq \left( \sum_{i=0}^{k} C_1^i \right) \alpha^s_{m+n}(\mathcal{B})
\]
where \( c \) and \( k \) are the uniform constants from simultaneous quasi-multiplicativity in Theorem \([E]\) and Remark 4.11. The lemma follows by setting \( C = \log \left( c^{-1} \cdot \sum_{i=0}^{k} C_1^i \right) \). \( \square \)

We are now ready to prove Theorem \([B]\) (1).

**Proof of Theorem \([B]\) (1).** Let \( \mathcal{A} \in \mathcal{U} \), \( s \in [0, \infty) \), and \( \varepsilon > 0 \) be given.

First, we show that there exists \( \delta > 0 \) such that for any \( \mathcal{B} \) sufficiently close to \( \mathcal{A} \) and \( t \in [0, \infty) \) with \( |s - t| < \delta \), we have
\[
\left| \mathcal{P}(\Phi^s_\mathcal{B}) - \mathcal{P}(\Phi^t_\mathcal{B}) \right| < \varepsilon/2.
\]

For any \( \mathcal{B} \) near \( \mathcal{A} \), consider the ratio
\[
\frac{\varphi^s_\mathcal{B}(I)}{\varphi^t_\mathcal{B}(I)},
\]
for some \( n \in \mathbb{N} \) and \( I \in \mathcal{L}(n) \). Suppose \( x, y \in [I] \) such that \( \varphi^s_\mathcal{B}(I) = \varphi^s(\mathcal{B}^n(x)) \) and \( \varphi^t_\mathcal{B}(I) = \varphi^t(\mathcal{B}^n(y)) \). We then write
\[
\frac{\varphi^s_\mathcal{B}(I)}{\varphi^t_\mathcal{B}(I)} = \frac{\varphi^s(\mathcal{B}^n(x))}{\varphi^t(\mathcal{B}^n(y))} = \frac{\varphi^s(\mathcal{B}^n(y))}{\varphi^t(\mathcal{B}^n(y))} \cdot \frac{\varphi^s(\mathcal{B}^n(x))}{\varphi^t(\mathcal{B}^n(y))}.
\]
Using the bounded distortion property (Lemma 3.9 and Remark 3.10) of \( \mathcal{B} \), the first term in the ratio \( \varphi^s(\mathcal{B}^n(x))/\varphi^t(\mathcal{B}^n(y)) \) can be bounded above and below by \( C_1 \) and \( C_1^{-1} \) for some uniform constant \( C_1 \) independent of \( \mathcal{B} \) and \( n \).

To bound the second term \( \varphi^s(\mathcal{B}^n(y))/\varphi^t(\mathcal{B}^n(y)) \) in the ratio, choose \( \Upsilon \) so that it serves as an upper bound on \( \max_{x \in \Sigma_T} \|\mathcal{B}(x)\| \) for any \( \mathcal{B} \) sufficiently close to \( \mathcal{A} \). If \( |s - t| < \delta \), then \( \varphi^s(\mathcal{B}^n(y))/\varphi^t(\mathcal{B}^n(y)) \) can be bounded above and below by \( \Upsilon^{n\delta} \) and \( \Upsilon^{-n\delta} \). Then it follows from the definition of \( \alpha^s_n(\mathcal{B}) \) that
\[
\left| \frac{1}{n} \log \alpha^s_n(\mathcal{B}) - \frac{1}{n} \log \alpha^t_n(\mathcal{B}) \right| \leq \delta \log \Upsilon + \frac{1}{n} \log C_1.
\]
Sending \( n \) to infinity, (5.2) follows by setting \( \delta = \varepsilon/(2 \log \Upsilon) \).

We then show that there exists a neighborhood of \( \mathcal{A} \) in \( \mathcal{U} \) such that for any \( \mathcal{B} \) in the neighborhood,

\[
|\mathcal{P}(\Phi^s_{\mathcal{B}}) - \mathcal{P}(\Phi^s_{\mathcal{A}})| < \varepsilon/2.
\]  

(5.3)

For any \( t, n \in \mathbb{N} \), we write \( n = qt + r \) with \( 0 \leq r < t \). For all \( \mathcal{B} \) in a small neighborhood of \( \mathcal{A} \), Lemma 5.1 gives

\[
-C(q+1)/n + \frac{q}{n} \log \alpha^s_n(\mathcal{B}) + \frac{1}{n} \log \alpha^s_n(\mathcal{B}) \leq \frac{1}{n} \log \alpha^s_n(\mathcal{B}) \leq \frac{q}{n} \log \alpha^s_n(\mathcal{B}) + \frac{1}{n} \log \alpha^s_n(\mathcal{B}).
\]

Notice that as \( n \to \infty \), we have \( q/n \to 1/t \) and \( \frac{1}{n} \log \alpha^s_n(\mathcal{B}) \to 0 \) because there are only \( t \) possible values of \( \alpha^s_n(\mathcal{B}) \). Sending \( n \to \infty \),

\[
\left| \mathcal{P}(\Phi^s_{\mathcal{B}}) - \frac{1}{t} \log \alpha^s_n(\mathcal{B}) \right| \leq C/t.
\]

We choose \( t \in \mathbb{N} \) large so that \( C/t < \varepsilon/8 \). Then we shrink the neighborhood of \( \mathcal{A} \) if necessary such that for any \( \mathcal{B} \) in the neighborhood,

\[
\left| \frac{1}{t} \log \alpha^s_n(\mathcal{A}) - \frac{1}{t} \log \alpha^s_n(\mathcal{B}) \right| < \varepsilon/4.
\]

Then for all \( \mathcal{B} \) in such neighborhood of \( \mathcal{A} \), (5.3) follows.

Combining (5.2) and (5.3), we have

\[
|\mathcal{P}(\Phi^s_{\mathcal{A}}) - \mathcal{P}(\Phi^s_{\mathcal{B}})| < \varepsilon
\]

for any \( \mathcal{B} \) sufficiently close to \( \mathcal{A} \) and any \( t \in [0, \infty) \) with \( |s - t| < \delta = \varepsilon/(2 \log \Upsilon) \).

**Proof of Theorem B (2).** From Proposition 3.4, the equilibrium state \( \mu_{\mathcal{A}, s} \) of \( \Phi^s_{\mathcal{A}} \) is unique due to quasi-multiplicativity of \( \Phi^s_{\mathcal{A}} \). Together with the continuity the map \( (\mathcal{A}, s) \mapsto \mathcal{P}(\Phi^s_{\mathcal{A}}) \) on \( \mathcal{U} \times [0, \infty) \), it follows that \( \mu_{\mathcal{A}, s} \) also varies continuously on \( \mathcal{U} \times [0, \infty) \).

Indeed, suppose \((\mathcal{A}_n, s_n) \in \mathcal{U} \times [0, \infty)\) converges to \((\mathcal{A}, s) \in \mathcal{U} \times [0, \infty)\). By passing to a subsequence, let \( \nu \) be any weak-* limit of \( \mu_{\mathcal{A}_n, s_n} \). We recall that two maps \( \mu \mapsto h_\mu(f) \) and \( (\Phi, \mu) \mapsto \mathcal{F}(\Phi, \mu) \) are upper semi-continuous; the entropy map is upper semi-continuous from the expansivity of the base system \((\Sigma_T, f)\), and \( \mathcal{F} \) is upper semi-continuous from being an infimum of continuous functions. From Theorem B (1), \( \nu \) must be an equilibrium state of \( \Phi^s_{\mathcal{A}} \):

\[
\mathcal{P}(\Phi^s_{\mathcal{A}}) = \lim_{n \to \infty} \mathcal{P}(\Phi^s_{\mathcal{A}_n}) = \lim_{n \to \infty} h_{\mu_{\mathcal{A}_n, s_n}}(f) + \mathcal{F}(\Phi^s_{\mathcal{A}_n}, \mu_{\mathcal{A}_n, s_n}),
\]

\[
\leq h_\nu(f) + \mathcal{F}(\Phi^s_{\mathcal{A}}, \nu).
\]

Since \( \mathcal{A} \in \mathcal{U} \), the equilibrium state \( \mu_{\mathcal{A}, s} \) of \( \Phi^s_{\mathcal{A}} \) is unique. Hence \( \nu = \mu_{\mathcal{A}, s} \), as desired.

**5.2. Applications in dimension theory and proof of Theorem C.** Theorem B has applications in the dimension theory of fractals. More specifically, we consider repellers of expanding maps. Let \( M \) be a \( d \)-dimensional Riemannian manifold, and \( h: M \to M \) be a \( C^1 \) map.

**Definition 5.2.** A compact \( h \)-invariant subset \( \Lambda \subset M \) is a repeller if

1. \( h \) is expanding on \( \Lambda \): there exists \( \lambda > 1 \) such that
   \[
   \|D_x h(v)\| \geq \lambda \|v\|
   \]
   for all \( x \in \Lambda \) and \( v \in T_x M \);
2. there exists a bounded open neighborhood \( V \) of \( \Lambda \) such that
   \[
   \Lambda = \{ x \in V : h^n x \in V \text{ for all } n \geq 0 \}.
   \]
For any repeller $\Lambda$ and $s \in [0, d]$, we associate a subadditive sequence $\Phi^s_\Lambda = \{\log \varphi^s_{\Lambda,n}\}_{n \in \mathbb{N}}$ on $\Lambda$ where
\[
\varphi^s_{\Lambda,n}(x) := \varphi^s((D_x h^n)^{-1}).
\]
Then the function $s \mapsto P(\Phi^s_\Lambda)$ is strictly decreasing, and the equation
\[
P(\Phi^s_\Lambda) = 0
\]
has a unique solution (see [Bar03], [Fal94], and [BCH10]) which we denote by $s(\Lambda)$. Such equation is a variation of so-called Bowen’s equation first introduced in [Bow79], and its unique solution often carries geometric information of the underlying object. In our case, $s(\Lambda)$ is an upper bound for the upper box dimension of $\Lambda$:

**Proposition 5.3.** [BCH10] Let $\Lambda$ be a repeller. Then $s(\Lambda)$ is an upper bound on the upper box dimension of $\Lambda$; that is,
\[
\dim_B \Lambda \leq s(\Lambda).
\]

Such $s(\Lambda)$ is a good candidate for estimating the Hausdorff dimension of $\Lambda$, and there are many settings in which $s(\Lambda)$ is equal to the Hausdorff dimension. See [Bar03], [Fal94], and [BCH10].

**Remark 5.4.** The idea of relating the Hausdorff dimension of dynamically defined fractals to the unique zero of the pressure function applies to other settings as well, including self-affine sets described in the introduction.

Let $T = \{T_i\}_{i=1}^q$ be a set of affine transformations of $\mathbb{R}^d$ with $T_i(x) = A_i x + r_i$ where $A_i \in \text{GL}_d(\mathbb{R})$ is a contraction (i.e., $\|A_i\| < 1$) and $r_i \in \mathbb{R}^d$ is a translation vector. Then there exists a unique self-affine attractor $X \subset \mathbb{R}^d$ invariant under $T$ in the sense that
\[
X = \bigcup_{i=1}^q T_i X;
\]
see [Hut81]. Let $F_A$ be the locally constant cocycle over the one-sided shift $(\Sigma_\mathbb{N}^+, f)$ on $q$ alphabets generated by $A(x) = A_{x_0}$. Then the unique zero $s(X)$ of the function
\[
s \mapsto P(\Phi^s_{A})
\]
is an upper bound of, and often equal to, the Hausdorff dimension of $X$.

Indeed, Falconer [Fal88a] showed that when $\|A_i\| < 1/3$ for each $i$, then for Lebesgue almost all translation vectors $(r_1, \ldots, r_q)$, the upper bound $s(X)$ is in fact equal to the Hausdorff dimension of $X$. Solomyak [Sol98] then relaxed the assumption $\|A_i\| < 1/3$ to $\|A_i\| < 1/2$.

Moreover, Bárány, Hochman, and Rapaport [BHR17] recently considered self-affine sets $X \subset \mathbb{R}^d$ satisfying the strong open set condition. They showed that under mild and checkable conditions, $s(X)$ is equal to the Hausdorff dimension of $X$.

From the structural stability of hyperbolic sets, for any $C^1$-small perturbation $g$ of $h$, there exists a continuation $\Lambda_g$ of $\Lambda$ such that $h|_{\Lambda}$ is conjugate to $g|_{\Lambda_g}$. In particular, $\Lambda_g$ is also a repeller (with respect to $g$). Notice from its definition (i.e., from the subadditivity of $\Phi^s_\Lambda$) that $s(\Lambda_g)$ varies upper semi-continuously in $g$.

We will now prove Theorem [C] by applying Theorem [B]. First, we introduce the analogue of the fiber-bunching condition on $\Lambda$.

**Definition 5.5.** Suppose $\Lambda$ is a repeller defined by $h$. For $\alpha \in (0, 1)$, we say $h|_{\Lambda}$ is $\alpha$-bunched if
\[
\|(D_x h)^{-1}\|^{1+\alpha} \cdot \|D_x h\| < 1,
\]
for all $x \in \Lambda$.

**Remark 5.6.** A natural class of $\alpha$-bunched repellers are small perturbations of conformal repellers.

**Theorem (Theorem C).** Let $M$ be a Riemannian manifold, and let $h: M \to M$ be a $C^r$ map with $r > 1$. Suppose $\Lambda \subset M$ is a $\alpha$-bunched repeller defined by $h$ for some $\alpha \in (0, 1)$ with $r - 1 > \alpha$. Then there exist a $C^1$-neighborhood $\mathcal{V}_1$ of $h$ in $C^r(M, M)$ and a $C^1$-open and $C^r$-dense subset $\mathcal{V}_2 \subset \mathcal{V}_1$ such that the map

$$g \mapsto s(\Lambda_g)$$

is continuous on $\mathcal{V}_2$.

**Remark 5.7.** The neighborhood $\mathcal{V}_1$ is chosen such that $\Lambda$ has a continuation $\Lambda_g$ for every $g \in \mathcal{V}_1$.

We begin by relating the setting of Theorem C to the setting in Theorem B. It is well-known that the dynamics on any repeller can be coded by a one-sided subshift of finite type $(\Sigma_T, f)$ via a Markov partition $\mathcal{R}$ of arbitrarily small diameter. See [Bow79], [Rue82], [Rue89] for discussions on Markov partitions and the coding of repellers into subshifts of finite type.

Once we fix such a Markov partition $\mathcal{R}$ for $\Lambda$, there exists a continuous and surjective coding map

$$\chi: \Sigma_T^+ \to \Lambda$$

such that $\chi \circ f = h \circ \chi$. The number $\text{card}(\chi^{-1}x)$ is bounded on $\Lambda$, and there exists a residual set $\tilde{\Lambda} \subset \Lambda$ such that every $x \in \tilde{\Lambda}$ has a unique pre-image under $\chi$.

We now take the natural extension $(\Sigma_T, f)$ of $(\Sigma_T^+, f)$, and consider its inverse $(\Sigma_T, f^{-1})$. Recalling that $\pi: \Sigma_T \to \Sigma_T^+$ is the projection map, we define a cocycle $F_B$ over $(\Sigma_T, f^{-1})$ generated by

$$B(x) = (D\chi(\pi x)h)^{-1}. \quad (5.4)$$

The reason why we consider $F_B$ other than the usual derivative cocycle is because $B^n: \Sigma_T \to \text{GL}_d(\mathbb{R})$ is related to $\varphi_{\lambda, n}: \Lambda \to \text{GL}_d(\mathbb{R})$ in a following way: for any $x \in \Sigma_T$ and $n \in \mathbb{N}$, we have

$$B^n(f^{-n}x) = (D\chi(\pi x)h)^{-1} \cdots (D\chi(\pi(f^{-n-1}x))h)^{-1} = \varphi_{\lambda, n}(\chi(x)). \quad (5.5)$$

Let $g$ be a $C^1$-small perturbation of $h$ in $C^r(M, M)$. If the perturbation is sufficiently small, then we may use the same Markov partition $\mathcal{R}$ of $\Lambda$ to code the dynamics of $g$ on $\Lambda_g$ via $\chi_g$, and take its natural extension. Then we realize the perturbation $h|_\Lambda$ to $g|_{\Lambda_g}$ as the perturbation of the cocycle $F_B$ to $F_{B_g}$ over the same subshift of finite type $(\Sigma_T, f^{-1})$ where $B_g(x) = (D\chi_g(\pi x)g)^{-1}$.

Now consider the typicality assumption on the cocycle $F_B$ over $(\Sigma_T, f^{-1})$. We fix $\theta$, the constant defining the metric on the base $\Sigma_T$, such that $\theta < \|Dx(h)^{-1}\|$ for all $x \in \Lambda$. If $h$ is $C^r$ and $\alpha$-bunched for some $r > 1$ and $\alpha \in (0, 1)$ satisfying $r - 1 > \alpha$, then the corresponding cocycle $F_B$ over $(\Sigma_T, f^{-1})$ is also fiber-bunched. Denoting the canonical holonomies of $F_B$ by $H^u$ (the minus sign in the superscript indicates that the cocycle is over $(\Sigma_T, f^{-1})$), the local unstable holonomy $H^u_z$ is trivial from the definition of $B$: $H^u_{z, y} \equiv I$ for any $y$ in the local unstable set of $x$ with respect to $f^{-1}$.

A homoclinic point $y$ of a fixed point $p$ in $\Sigma_T$ corresponds to a sequence of points $\{z_n\}_{n \in \mathbb{N}} \in \Lambda$ such that $z_0 = \chi(\pi z)$, $h^\ell z_0 = \chi(\pi p)$ for some $\ell \in \mathbb{N}$ and

$$h z_n = z_{n-1}, \quad \text{and} \quad z_n \xrightarrow{n \to \infty} \chi(\pi p). \quad (5.6)$$
Symbolically, if \( p = [\ldots a|a|a\ldots] \in \Sigma_T \) and \( z = [\ldots a|a|b_1\ldots b_{t-1}|a|a\ldots] \in \Sigma_T \), then for each \( n \in \mathbb{N} \), (from now on, we will drop the notation for the coding map \( \chi \) between \( \Sigma_T^+ \) and \( \Lambda \)) we have
\[
z_n = [a\ldots a|a|b_1\ldots b_{t-1}|a|a\ldots] \in \Sigma_T^+.
\]
Moreover, \( H_{z,p}^\pm \) is given by
\[
H_{z,p}^\pm = \lim_{n \to \infty} [(D_{\pi p}h)^n(D_{z_{n-1}}h)^{-1}\ldots(D_{z_0}h)^{-1}].
\]
Using the fact that \( h^\ell z_0 = \pi p \), we have \( H_{p,f^\ell z}^{u,-} = I \), and
\[
H_{p,z}^{u,-} = (D_{h^{\ell-1}z_0}h)^{-1}\ldots(D_{h^{\ell-1}2_0}h)^{-1}(D_{\pi p}h)^{\ell-1}.
\]
Via \( H^{s/u,-} \), the holonomy loop \( \psi_p^{z,-} \) with respect to \( F_B \) over \((\Sigma_T, f^{-1})\) is given by
\[
\psi_p^{z,-} = H_{z,p}^{\pm} \circ H_{p,z}^{u,-},
\]
where \( H_{z,p}^{\pm} \) and \( H_{p,z}^{u,-} \) are given as in the paragraph above. We say that an \( \alpha \)-bunched repeller \( \Lambda \) defined by \( h \) (or simply \( h \)) is \textit{typical} if the corresponding cocycle \( F_B \) is typical over \((\Sigma_T, f^{-1})\).

**Lemma 5.8.** Let \( h : M \to M \) be a \( C^r \) map defining an \( \alpha \)-bunched repeller \( \Lambda \). Then there exist a \( C^1 \)-neighborhood \( \mathcal{V}_1 \) of \( h \) in \( C^r(M, M) \) and a \( C^1 \)-open and \( C^r \)-dense subset \( \mathcal{V}_2 \) of \( \mathcal{V}_1 \) such that any \( g \in \mathcal{V}_2 \) is typical.

**Proof.** As mentioned in Remark 5.7, we begin by choosing \( \mathcal{V}_1 \) sufficiently small so that \( \Lambda \) has a continuation \( \Lambda_g \) for every \( g \in \mathcal{V}_1 \).

We code \( h|_\Lambda \) using a Markov partition to a one-sided subshift \((\Sigma_T^+, f)\) and take its natural extension \((\Sigma_T, f)\). Then consider \( F_B \) over \((\Sigma_T, f^{-1})\) defined as in (5.4). By choosing \( \mathcal{V}_1 \) sufficiently small, we ensure that \( \Lambda_g \) for every \( g \in \mathcal{V}_1 \) can be coded by the same Markov partition. For simplicity, we will continue to suppress the notation for the coding map \( \chi_g : \Sigma_T^+ \to \Lambda_g \) and write \( \mathcal{B}_g(x) = (D_{\pi x}g)^{-1} \) where \( \pi x \) refers to \( \chi_g(\pi x) \in \Lambda_g \).

Following Section 9 of [BV04], we will show that the pinching condition (A0) is \( C^r \)-dense via the claim below and briefly sketch the proof here.

**Claim:** given any \( C^r \)-neighborhood \( \mathcal{W} \) of \( \mathcal{V}_1 \), there exists \( g \in \mathcal{W} \) and a periodic point \( p_g \in \Lambda_g \) such that \( D_{p_g}g^{per(p_g)} \) has simple real eigenvalues of distinct norms.

First, notice that the lemma follows from the claim. Indeed, suppose there exists a fixed (or periodic) point \( p \in \Lambda_{g_0} \) of some \( g_0 \in \mathcal{V}_1 \) such that \( D_{p_0}g_0 \) has simple real eigenvalues of distinct norms. For \( g \) sufficiently \( C^1 \)-close to \( g_0 \), the property of having simple real eigenvalues of distinct norms persists at \( D_{p_g}g \) where \( p_g \) is the continuation of \( p \) with respect to \( g \). Denoting the corresponding fixed point in \( \Sigma_T \) by \( \tilde{p}_g \), the property of having simple real eigenvalues of distinct norms is equivalent on \( D_{\tilde{p}_g}g \) and its inverse \( B_g(\tilde{p}_g) = (D_{p_g}g)^{-1} \). Hence, the pinching condition (A0) on \( F_B \) is \( C^1 \)-open. Moreover, it is \( C^r \)-dense in \( \mathcal{V}_1 \) assuming that the claim holds.

It is clear that the twisting condition (B0) on \( F_{B_g} \) is \( C^1 \)-open because the canonical holonomies \( H^{s/u,-} \) vary continuously in \( g \). The twisting condition is also \( C^r \)-dense; given any \( \{z_n\}_{n \in \mathbb{N}} \) homoclinic (as in (5.6)) to a periodic point whose derivative of the return map has simple real eigenvalues of distinct norms, the twisting assumption (B0) on \( F_{B_g} \) can be obtained with an arbitrarily small \( C^r \)-perturbation of \( g \) near \( z_0 \). This is because an
arbitrarily small $C^r$-perturbation of $g$ near $z_0$ only changes $(D_{z_0}g)^{-1}$ without affecting other terms in $\psi_p^{z_0}$, and the perturbation can be chosen to destroy any configuration preventing the twisting condition (B0) on $\psi_p^{z_0}$. Hence, in order to prove the lemma, it suffices to prove the claim.

Proof of claim. Let $g_0$ be any map in $\mathcal{W}$. Given any fixed (or periodic) point $p \in \Lambda_{g_0}$, upon a small $C^r$-perturbation of $g_0$ near $p$, we assume that $P := D_p g_0$ has simple real eigenvalues of distinct norms except for some pairs of complex conjugate eigenvalues. Fix any sequence $\{z_n\}_{n \in \mathbb{N}_0}$ homoclinic to $p$ as in [5,6], and let $z$ be the corresponding homoclinic point in $\Sigma_T$. Upon another small perturbation of $g_0$ near $z_0$, we assume stronger twisting condition (i.e., original formulation in [BV04]) holds for $\psi_p^{z_0}$. From such twisting condition, it follows that there exists a small neighborhood $\mathcal{N}$ around $(\text{orbit of } z_0) \cup p$ such that any $g_0$-invariant set in $\mathcal{N}$ admits a $Dg_0$-invariant dominated splitting $E^1 \oplus \ldots \oplus E^k$ which agrees with the eigenspace splitting of $P$ at $p$.

Denoting $p = [aa \ldots] \in \Sigma_T^+$, consider a periodic point $x_m \in \Sigma_T^+$ which repeats the word $ab_1 \ldots b_{t-1}a \ldots a \in \mathcal{L}(\ell + m)$. We denote the corresponding periodic point in $\Sigma_T$ by $\bar{x}_m$.

When $m$ is sufficiently large, the orbit of $x_m$ belongs to $\mathcal{N}$. Since the dominated splitting is robust, there exists a dominated splitting over the orbit of $x_m$ (for all sufficiently large $m$) with respect to any sufficiently small $C^r$-perturbation $g$ of $g_0$. Moreover, such splitting has the same index as the eigenspace splitting of $P$ at $p$.

Assuming $E^1 \oplus \ldots \oplus E^k$ is ordered in the decreasing norm of the eigenvalues of $P$, let $j$ be the largest index such that $E^j$ is 2-dimensional (i.e., corresponds to a pair of complex conjugate eigenvalues). Then consider a 1-parameter family of perturbations $g_t, t \in [0,1]$ near $p$ such that $Dg_t$ near $p$ is given by the post-composition of $Dg_0$ with a rotation $R_{t\varepsilon}$ by angle $t\varepsilon$ along the $E^j$-plane. Here $\varepsilon > 0$ is chosen sufficiently small so that $g_t$ remains in $\mathcal{W}$ for all $t \in [0,1]$.

We can then show that given any small $\delta > 0$, there exists $t_0 \in [0,\delta]$ and a sufficiently large $m$ such that the rotation number of $B_{g_0}^{\ell+m}(\bar{x}_m)|_{E^j}$ is an integer. By an arbitrarily small $C^r$-perturbation of $g_0$ near $x_m$ preserving $E^j$, we can ensure that $B_{g_0}^{\ell+m}(\bar{x}_m)|_{E^j}$ has two real and distinct eigenvalues. Repeating this process on $g_0$ and $x_m$, we inductively resolve all complex conjugate pairs of eigenvalues into real eigenvalues of distinct norms by arbitrary small $C^r$-perturbations. See Section 9 in [BV04] for more details. \hfill \Box

This completes the proof of the lemma. \hfill \Box

Remark 5.9. The main content in the proof of Lemma 5.8 shows that the pinching condition (A0) is $C^r$-dense in $\mathcal{V}_1$. Then we concluded that there exists a $C^1$-open and $C^r$-dense subset $\mathcal{V}_2$ of $\mathcal{V}_1$ such that the cocycle $\mathcal{B}_g(x) = (D_{x_0}g)^{-1}$ over $(\Sigma_T, f^{-1})$ is typical for every $g \in \mathcal{V}_2$. From the same result, we can also conclude that there exists another $C^1$-open and $C^r$-dense subset $\bar{V}_2$ of $\mathcal{V}_1$ such that the derivative cocycle $Dg$ is typical (in the sense of Definition 2.6) for every $g \in \bar{V}_2$. This remark will be useful in proving Corollary 1.1 in Section 6.

Combining Theorem B and Lemma 5.8, the map

$$(g,s) \mapsto P(\Phi^s_{B_g})$$

is continuous on $\mathcal{V}_2 \times [0, \infty)$. Since $s(\Lambda)$ is the unique zero of the pressure function $s \mapsto P(\Phi^s_\Lambda)$, the following lemma relates the pressure $P(\Phi^s_B)$ defined over $(\Sigma_T, f^{-1})$ to the pressure $P(\Phi^s_\Lambda)$ defined over $(\Lambda, h|_\Lambda)$.

Lemma 5.10. $P(\Phi^s_B) = P(\Phi^s_\Lambda)$. 

Proof. Let $\Phi_A^{s,+} = \{\log \varphi_{\Lambda,n}^{s,+}\}_{n \in \mathbb{N}}$ be a subadditive sequence on $\Sigma^+_T$ defined by

$$\varphi_{\Lambda,n}^{s,+}(x) := \varphi_{\Lambda,n}(\chi(x)) = \varphi^s((D_{\chi(x)}h)^n)^{-1}.$$ 

Then $P(\Phi_A^{s,+})$ and $P(\Phi_A^{s,-})$ are equal, as described below.

Consider any $\mu \in \mathcal{M}(f)$ and $\nu \in \mathcal{M}(h)$ related by $\chi_*\mu = \nu$. From the fact that $\chi$ is the coding map, we have

$$h_\mu(f) = h_\nu(h).$$

Indeed, we have $h_\mu(f) \geq h_\nu(h)$ since $(\Lambda, h|_\Lambda)$ is a factor of $(\Sigma^+_T, f)$. For the reverse inequality, Ledrappier-Walters’ relativised variational principle [LW77] states that for any $\nu \in \mathcal{M}(h)$, we have

$$\sup_{\tilde{\mu}: \chi_*\tilde{\mu} = \nu} h_{\tilde{\mu}}(f) = h_\nu(h) + \int h(f, \chi^{-1}(x))d\nu(x).$$

Since the pre-image $\chi^{-1}(x)$ is finite, $h(f, \chi^{-1}(x)) = 0$ for every $x \in \Lambda$, and we have $h_\mu(f) \leq h_\nu(h)$.

For $\chi_*\mu = \nu$, we also have $\mathcal{F}(\Phi_A^{s,+}, \mu) = \mathcal{F}(\Phi_A^{s}, \nu)$ from the definition of $\Phi_A^{s,+}$. Then it follows that $P(\Phi_A^{s,+}) = P(\Phi_A^{s})$ from the subadditive variational principle [3.4]. Hence, it suffices to show that $P(\Phi_A^{s,+})$ is equal to $P(\Phi_B^{s})$.

From the expansivity of $(\Sigma_T, f)$, it suffices to consider $(n, 1)$-separated sets in the definition of the pressure (see [Kel98]). Notice that on $(\Sigma^+_T, f)$, a subset $E \subset \Sigma^+_T$ is $(n, 1)$-separated if any two distinct $x, y \in E$ satisfy $x_i \neq y_i$ for some $0 \leq i \leq n - 1$ (i.e., $y \not\in [x]_T$).

For every $x \in \Sigma^+_T$, we choose a point $\tilde{x} \in \Sigma_T$ such that $\pi\tilde{x} = x$. Then [5.4] and [5.5] gives

$$\varphi_{\Lambda}(B^n(f^{n-1}\tilde{x})) = \varphi_{\Lambda,n}^{s,+}(x).$$

(5.7)

We observe a simple relationship between $(n, 1)$-separated sets in $(\Sigma^+_T, f)$ and $(n, 1)$-separated sets in $(\Sigma_T, f^{-1})$. Given any $(n, 1)$-separated set $E$ in $(\Sigma^+_T, f)$, for each $x \in E$ we choose any point $\tilde{x} \in \Sigma_T$ from $\pi^{-1}(x)$, and call the corresponding set $\tilde{E} \subset \Sigma_T$. Then $f^{n-1}\tilde{E}$ is a $(n, 1)$-separated set in $(\Sigma_T, f^{-1})$. Conversely, given any $(n, 1)$-separated set $\tilde{E}$ of $(\Sigma_T, f^{-1})$, the projection $\pi(f^{-n-1}\tilde{E})$ is a $(n, 1)$-separated set in $(\Sigma^+_T, f)$.

From [5.7],

$$\sup \{ \sum_{x \in \tilde{E}} \varphi_{\Lambda,n}^{s,+}(x) : \tilde{E} \text{ is (n, 1)-separated in } (\Sigma_T, f^{-1}) \}$$

is equal to

$$\sup \{ \sum_{x \in E} \varphi_{\Lambda,n}^{s,+}(x) : E \text{ is (n, 1)-separated in } (\Sigma^+_T, f) \}$$

for each $n \in \mathbb{N}$. Hence, the definition of the subadditive pressure [3.3] gives $P(\Phi_B^s) = P(\Phi_A^{s,+})$. 

□

Proof of Theorem C. From Lemma 5.8 there exist a $C^1$-neighborhood $\mathcal{V}_1$ of $h$ in $C^r(M, M)$ and a $C^1$-open and $C^r$-dense subset $\mathcal{V}_2$ of $\mathcal{V}_1$ such that every $g \in \mathcal{V}_2$ is typical. Theorem B and Lemma 5.10 give us that the map

$$(g, s) \mapsto P(\Phi_B^{s,g}) = P(\Phi_A^{s})$$

is continuous on $\mathcal{V}_2 \times [0, \infty)$. Hence the map $g \mapsto s(\Lambda_g)$ is continuous on $\mathcal{V}_2$. 

□
6. OTHER APPLICATIONS OF THEOREM \[E\]

6.1. Pointwise Lyapunov spectrum and proof of Theorem \[D\]. We prove Theorem \[D\] in this subsection.

Recall from the introduction that
\[
\lambda_t(x) := \lim_{n \to \infty} \frac{1}{n} \log \varphi(f^n(x)),
\]
if the limit exists (See [BP02] for a general discussion on the pointwise Lyapunov exponent).

We may think of \(\lambda_t(x)\), if it exists, as the sum of top \(t\) Lyapunov exponents of \(x\). Let
\[
\vec{\lambda}(x) = (\lambda_1(x), \ldots, \lambda_d(x)),
\]
if \(\lambda_t(x)\) exists for each \(1 \leq t \leq d\). Let
\[
L_A := \{ \vec{\alpha} \in \mathbb{R}^d : \vec{\alpha} = \vec{\lambda}(x) \text{ for some } x \in \Sigma_T \}.
\]

Theorem (Theorem \[D\]). Let \(A \in \mathcal{U}\). Then \(L_A\) is a closed and convex subset of \(\mathbb{R}^d\).

Remark 6.1. Theorem \[D\] is a generalization of earlier works on the structure of various spectrums. For instance, the pointwise Lyapunov exponent \(\lambda_t(x)\) may be considered as a subadditive generalization of the Birkhoff average of a continuous function \(\varphi\) defined as
\[
\overline{\varphi}(x) := \lim_{n \to \infty} \frac{1}{n}(\varphi(x) + \ldots + \varphi(f^{n-1}x)),
\]
if the limit exists. For any Hölder continuous potential \(\varphi\) over a mixing subshift of finite type, Pesin and Weiss [PW01] showed that the spectrum of the Birkhoff average \(\overline{\varphi}\) is a closed interval.

For a class of subadditive potentials, Feng [Fen03, Fen09] considered the pointwise top Lyapunov spectrum for locally constant cocycles over a subshift of finite type. Under the irreducibility assumption, he obtained a similar result to [PW01] that the spectrum is a closed interval.

We prove Theorem \[D\] using Theorem \[E\] and ideas in [Fen03, Fen09]. Theorem \[D\] extends the result of Feng in two ways: we consider more general class of cocycles (i.e., fiber-bunched) and we consider the spectrum of all pointwise exponents \(\lambda_t\) for \(1 \leq t \leq d\) simultaneously as opposed to the top exponent \(\lambda_1\) only.

Proof of Theorem \[D\]. The idea is to carefully concatenate (using quasi-multiplicativity) a sequence of words such that the pointwise Lyapunov exponents exist and behave as controlled. Although this idea applies in showing both convexity and closedness of \(L_A\), the constructions are slightly different, and hence we divide the proof into two parts.

For any \(x \in \Sigma_T\), the pointwise Lyapunov exponent \(\vec{\lambda}(x)\) depends only on the forward trajectory \(\pi x\) of \(x\). For instance, any two points on the same stable set have the same pointwise Lyapunov exponents (if they exist). This can be seen from the bounded distortion on \(\Phi_A^t\) coming the existence of the canonical stable holonomy. Hence, we will focus on constructing a one-sided word \(\omega^+ \in \Sigma_T^+\) so that any \(\omega \in \Sigma_T\) with \(\pi \omega = \omega^+\) has the desired pointwise Lyapunov exponents.

Throughout the proof, we denote (over all \(1 \leq t \leq d\)) the uniform constant from bounded distortion on \(\Phi_A^t\) by \(C\), \(\max_{x \in \Sigma_T} \| A^t(x) \|\) by \(\Upsilon\), \(\min_{x \in \Sigma_T} m(A^t(x))\) by \(\varrho\), and simultaneous quasi-multiplicativity constant by \(c \in (0, 1)\). Also, similar to the proof of Theorem \[4.1\] we always consider all \(1 \leq t \leq d\) simultaneously even when it is not explicitly stated.

(1) \(L_A\) is closed.
Let \( \{x_i\}_{i \in \mathbb{N}} \) be a sequence of points in \( \Sigma_T \) such that their Lyapunov exponents exist and limit to some \( \bar{\lambda} \):

\[
\bar{\lambda}(x_i) \xrightarrow{i \to \infty} \bar{\lambda} = (\lambda_1, \ldots, \lambda_d).
\]

Replacing \( x_i \) by a subsequence if necessary, fix a strictly decreasing sequence \( \{\varepsilon_i\}_{i \geq 1} \) with \( \varepsilon_i \to 0 \) and assume that

\[
|\lambda_t(x_i) - \lambda_i| < \varepsilon_{i+1}, \tag{6.1}\]

for each \( i \in \mathbb{N} \) and \( 1 \leq t \leq d \). We then fix a strictly increasing sequence \( N_i \to \infty \) such that for any \( i \in \mathbb{N} \) (serving as a common index for both \( x \) and \( \varepsilon \)) and \( 1 \leq t \leq d \),

\[
\left| \frac{1}{N} \log \varphi^t(A^N(x_i)) - \lambda_t(x_i) \right| < \varepsilon_{i+1} \text{ for each } N \geq N_i. \tag{6.2}
\]

Suppose we have chosen another sequence \( m_i \to \infty \) with \( m_i \gg N_{i+1} \) for each \( i \in \mathbb{N} \) that satisfies a few extra properties to be determined below. Define

\[
\omega^+ := [x_1]^w_1K_1[x_2]^w_2K_2[x_3]^w_3K_3 \ldots \in \Sigma_T^w
\]

where \( K_i \in \mathcal{L} \) is the connecting word with \( |K_i| = k_i \leq k \) given by simultaneous quasi-multiplicativity of \( \Phi^t_A \), \( t = 1, 2, \ldots, d \). Let \( \omega \) be any point in \( \Sigma_T \) with \( \pi \omega = \omega^+ \).

We claim that with appropriate choices of \( m_i \)'s, the pointwise Lyapunov exponent \( \bar{\lambda}(\omega) \) exists and is equal to \( \bar{\lambda} \). Since \( \varepsilon_i \to 0 \), in order to establish the claim, it suffices to show that for each \( i \in \mathbb{N} \) and \( 1 \leq t \leq d \) that

\[
\left| \frac{1}{m} \log \varphi^t(A^m(\omega)) - \lambda_t \right| < 2\varepsilon_i \text{ for each } 1 \leq i \leq m.
\]

Consider any \( m_1 \in \mathbb{N} \) with \( m_1 \gg N_2 \). For any \( m = m_1 + a \) with \( 0 \leq a < k_1 + N_2 \), (6.1) and (6.2) give

\[
\frac{1}{m} \log \varphi^t(A^m(\omega)) < \frac{1}{m} \left( \log \varphi^t(A^{m_1}(x_1)) + \log C + a \log T \right),
\]

\[
\leq \frac{1}{m_1 + a} \left( m_1 \lambda_t + 2m_1 \varepsilon_2 + \log C + a \log T \right). \tag{6.4}
\]

For the lower bound, we similarly have

\[
\frac{1}{m_1 + a} \left( m_1 \lambda_t - 2m_1 \varepsilon_2 - \log C + a \log \rho \right) \leq \frac{1}{m} \log \varphi^t(A^m(\omega)). \tag{6.5}
\]

Since \( \varepsilon_2 < \varepsilon_1 \) and \( a \) is bounded above by \( k_1 + N_2 \), if we choose \( m_1 \) sufficiently large, the upper bound (6.4) is bounded above by \( \lambda_t + 2\varepsilon_1 \) for all \( 0 \leq a < k_1 + N_2 \). Likewise, the lower bound (6.5) is bounded below by \( \lambda_t - 2\varepsilon_1 \) for all \( 0 \leq a < k_1 + N_2 \). This establishes (6.3) for \( m = m_1, m_1 + k_1 + N_2 \).

Now consider \( m = m_1 + k_1 + a \) with \( 0 \leq a \leq N_2 \) and bounded above by \( m_2 \) to be chosen. We obtain different bounds on \( \frac{1}{m} \log \varphi^t(A^m(\omega)) \) by using (6.1) and (6.2) for \( i = 2 \) on the last \( a \) terms in the product \( A^m(\omega) \):

\[
\frac{1}{m} \log \varphi^t(A^m(\omega)) \leq \frac{1}{m} \left( \log \varphi^t(A^{m_1}(x_1)) + 2 \log C + k_1 \log T + \log \varphi^t(A^{k_1}(x_2)) \right),
\]

\[
\leq \frac{1}{m_1 + k_1 + a} \left( \lambda_t(m_1 + a) + 2(m_1 \varepsilon_2 + a\varepsilon_3) + 2 \log C + k_1 \log T \right). \tag{6.6}
\]
Similarly using quasi-multiplicativity of Theorem [6], we get
\[
\frac{1}{m_1 + k_1 + a} \left( \lambda_t(m_1 + a) - 2(m_1\varepsilon_2 + a\varepsilon_3) - 2\log C + \log c \right) \leq \frac{1}{m} \log \varphi^t(A^n(\omega)). \tag{6.7}
\]
We further increase \(m_1\) if necessary so that the upper and lower bounds (6.6) and (6.7) still belong to \((\lambda_t - 2\varepsilon_2, \lambda_t + 2\varepsilon_2)\) at \(m = m_1 + k_1 + N_2\). Since the upper (6.6) and lower (6.7) bounds limit to \(\lambda_t \pm 2\varepsilon_3\) as \(a \to \infty\), this gives (6.3) for \(m \in [m_1 + k_1 + N_2, m_1 + k_1 + m_2]\), once we choose \(m_2\) in the following paragraph.

We now describe the choice of \(m_2 \in \mathbb{N}\) satisfying the following properties. As pointed out in the previous paragraph, the upper (6.6) and lower (6.7) bounds limit to \(\lambda_t \pm 2\varepsilon_3\) as \(a \to \infty\). So, we choose \(m_2 \gg N_3\) sufficiently large such that the upper (6.6) and lower (6.7) bounds at \(m = m_1 + k_1 + m_2\) are close enough to \(\lambda_t \pm 2\varepsilon_3\) and \(\lambda_t - 2\varepsilon_3\), respectively. By doing so, we ensure that the upper bound
\[
\frac{1}{m} \left( \lambda_t(m_1 + m_2) + 2(m_1\varepsilon_2 + m_2\varepsilon_3) + 2\log C + (a + k_1) \log \Upsilon \right)
\]
and the lower bound
\[
\frac{1}{m} \left( \lambda_t(m_1 + m_2) - 2(m_1\varepsilon_2 + m_2\varepsilon_3) - 2\log C + \log c + a \log \varrho \right)
\]
of \(\frac{1}{m} \log \varphi^t(A^m(\omega))\) both belong to \((\lambda_t - 2\varepsilon_2, \lambda_t + 2\varepsilon_2)\) for \(m = m_1 + k_1 + m_2 + a\) with \(0 \leq a < k_2 + N_3\). From the construction, (6.3) now holds for \(m\) in the range \([m_1 + k_1 + m_2, m_1 + k_1 + m_2 + k_2 + N_3]\). Similar to (6.6) and (6.7), \(\frac{1}{m} \log \varphi^t(A^m(\omega))\) for \(m = m_1 + k_1 + m_2 + k_2 + a\) with \(a \geq N_3\) admits the following upper and lower bound by using (6.1) and (6.2) for \(i = 3\) on the last \(a\) terms:
\[
\frac{1}{m} \left( \lambda_t(m_1 + m_2 + a) + 2(m_1\varepsilon_2 + m_2\varepsilon_3 + a\varepsilon_4) + 3\log C + (k_1 + k_2) \log \Upsilon \right)
\]
and
\[
\frac{1}{m} \left( \lambda_t(m_1 + m_2 + a) - 2(m_1\varepsilon_2 + m_2\varepsilon_3 + a\varepsilon_4) - 3\log C + 2\log c \right).
\]
We further increasing \(m_2\) if necessary such that these bounds at \(m = m_1 + k_1 + m_2 + k_2 + N_3\) belong to \((\lambda_t - 2\varepsilon_2, \lambda_t + 2\varepsilon_2)\). Since these bounds limit to \(\lambda_t \pm 2\varepsilon_4\) as \(a \to \infty\), this gives (6.3) for \(m \in [m_1 + k_1 + m_2 + k_2 + N_3, m_1 + k_1 + m_2 + k_2 + k_3]\), once we choose \(m_3\).

We continue this inductive process of choosing \(m_i\) so that (6.3) holds. Similar to how we chose \(m_2\), we choose \(m_i \in \mathbb{N}\) sufficiently large such that the upper and lower bounds (obtained similar to (6.6) and (6.7)) of \(\frac{1}{m} \log \varphi^t(A^m(\omega))\) at \(m = \sum_{j=1}^{i} m_j + \sum_{j=1}^{i-1} k_j\) are close enough to \(\lambda_t \pm 2\varepsilon_{i+1}\). In estimating \(\frac{1}{m} \log \varphi^t(A^m(\omega))\), the large magnitude of \(m_i\) helps compensate for the next \(k_i + N_{i+1}\) terms following \(\sum_{j=1}^{i} m_j + \sum_{j=1}^{i-1} k_j\) which only admit crude bounds using \(\Upsilon\) and \(\varrho\). This ensures that \(\frac{1}{m} \log \varphi^t(A^m(\omega))\) remains in the range of \((\lambda_t - 2\varepsilon_i, \lambda_t + 2\varepsilon_i)\) for all \(m = \sum_{j=1}^{i} m_j + \sum_{j=1}^{i-1} k_j + a\) with \(0 \leq a < k_i + N_{i+1}\).

For \(m = \sum_{j=1}^{i-1} m_j + \sum_{j=1}^{i} k_j + a\) with \(a \geq N_{i+1}\), we use (6.1) and (6.2) on the last \(a\) terms with \(\varepsilon_{i+2}\), and further increase \(m_i\) if necessary such that (6.3) remains to hold up to \(m = \sum_{j=1}^{i+1} m_j + \sum_{j=1}^{i} k_j\) for some \(m_{i+1}\) to be chosen. Repeating this construction, we have
Consider any multiplicativity from Theorem E. that is, \( N(\omega) = \omega \alpha + (1 - \gamma) \beta \); the proof will construct \( \omega^+ \in \Sigma_T^+ \) by concatenating the words \([x]_n^w\) and \([y]_n^w\) with proportions \( \gamma \) and \( 1 - \gamma \), respectively.

We begin by defining a sequence \( \{N_i\}_{i \in \mathbb{N}} \) of integers given by \( N_i = \lfloor \gamma i \rfloor \) if \( i \) is odd and \( N_i = [(1 - \gamma) i] \) if \( i \) is even. Then such sequence \( \{N_i\}_{i \in \mathbb{N}} \) satisfies

\[
\lim_{i \to \infty} N_i = \infty, \quad \lim_{i \to \infty} \frac{(i + 1)N_{i+1}}{\sum_{j=1}^{i+1} jN_j} = 0, \quad \text{and} \quad \lim_{i \to \infty} \frac{\sum_{j=1}^{i} (2j - 1)N_{2j-1}}{\sum_{j=1}^{i} jN_j} = \gamma. \tag{6.8}
\]

In fact, the first limit is obvious from the definition of \( N_i \). Using \( a - 1 < |a| \leq a \) for any \( a \in \mathbb{R} \), the third limit follows because both the lower and upper bounds from

\[
\gamma \sum_{j=1}^{i} (2j - 1)^2 - \sum_{j=1}^{i} (2j - 1) \leq \frac{\sum_{j=1}^{i} (2j - 1)N_{2j-1}}{\sum_{j=1}^{i} jN_j} \leq \frac{\gamma \sum_{j=1}^{i} (2j - 1)^2}{\gamma \sum_{j=1}^{i} (2j - 1)^2 + (1 - \gamma) \sum_{j=1}^{i} (2j - 1) + (1 - \gamma) \sum_{j=1}^{i} (2j - 1)}
\]

converge to \( \gamma \). Similarly, the second limit also follows along the same reasoning.

Let \( \{\omega_n\}_{n \in \mathbb{N}} \) be a sequence of words defined as follows:

\[
[x]_1^w, [x]_2^w, \ldots, [y]_1^w, [y]_2^w, [x]_3^w, \ldots, [x]_4^w, [y]_4^w, \ldots;
\]

that is, \( \omega_i = [x]_1^w \) for \( 1 \leq i \leq N_1 \), \( \omega_i = [y]_2^w \) for \( N_1 + 1 \leq i \leq N_1 + N_2 \), and so on.

Consider

\[
\omega^+ := \omega_1 K_1 \omega_2 K_2 \omega_3 K_3 \ldots \in \Sigma_T^+
\]

where each connecting word \( K_i \in \mathcal{L} \) with \( |K_i| = k_i \leq k \) is given by simultaneous quasi-multiplicativity from Theorem E.

We will show that \( \lim_{m \to \infty} \frac{1}{m} \log \varphi^t(A^m(\omega)) = \gamma \alpha_t + (1 - \gamma) \beta_t \) for all \( 1 \leq t \leq d \). First choose \( \varepsilon_m \to 0 \) such that for each \( 1 \leq t \leq d \) and \( m \in \mathbb{N} \),

\[
\left| \frac{1}{m} \log \varphi^t(A^m(x)) - \alpha_t \right| < \varepsilon_m \quad \text{and} \quad \left| \frac{1}{m} \log \varphi^t(A^m(y)) - \beta_t \right| < \varepsilon_m.
\]

Consider any \( m \in \mathbb{N} \) with

\[
m = \sum_{j=1}^{i} jN_j + \sum_{j=1}^{N_1 + \ldots + N_i} k_j + a \quad \text{with} \quad 0 \leq a < (i + 1)N_{i+1} + k_{N_1 + \ldots + N_i + 1} + \ldots + k_{N_1 + \ldots + N_i + 1}.
\tag{6.9}
\]
Denoting \( r_j = \alpha_t \) for \( j \) odd and \( r_j = \beta_t \) for \( j \) even, we have
\[
\frac{1}{m} \log \varphi^t(A^m(\omega)) \leq \frac{1}{m} \left( \sum_{j=1}^{i} jN_j(r_j + \varepsilon_j) + \log T \left( \sum_{j=1}^{N_1+\cdots+N_i} k_j + a \right) + \log C \left( \sum_{j=1}^{i} N_j \right) \right),
\]
\[
\leq \frac{1}{m} \sum_{j=1}^{i} jN_j(r_j + \varepsilon_j) + \log \Upsilon \left( (i+1)N_{i+1} + \sum_{j=1}^{N_1+\cdots+N_{i+1}} k_j \right) + \log C \left( \sum_{j=1}^{i} N_j \right).}
\]

Sending \( m \) to \( \infty \), the last two terms both limit to 0 from the definition of \( N_j \) and \( (6.8) \).

The first term limits to \( \gamma \alpha_t + (1 - \gamma)\beta_t \) from the third property of \( (6.8) \) and the fact that \( \varepsilon_j \to 0 \). Hence,
\[
\limsup_{m \to \infty} \frac{1}{m} \log \varphi^t(A^m(\omega)) \leq \gamma \alpha_t + (1 - \gamma)\beta_t \text{ for each } 1 \leq t \leq d.
\]

Conversely, for \( m \) in the same range \( (6.9) \), we obtain from simultaneous quasi-multiplicativity that
\[
\frac{1}{m} \log \varphi^t(A^m(\omega)) \geq \frac{1}{m} \left( \sum_{j=1}^{i} jN_j(r_j + \varepsilon_j) + \log c \left( \sum_{j=1}^{i} N_j \right) + a \log g - \log C \left( \sum_{j=1}^{i} N_j \right) \right).
\]

It then follows from \( (6.8) \) that this lower bound also limits to \( \gamma \alpha_t + (1 - \gamma)\beta_t \) as \( m \) tends to \( \infty \). Hence we have constructed \( \omega^+ \in \Sigma_T^+ \) such that \( \tilde{\lambda}(\omega) \) exists and is equal to \( \gamma \tilde{\alpha} + (1 - \gamma)\tilde{\beta} \) for any \( \omega \in \Sigma_T \) with \( \pi\omega = \omega^+ \). This completes the proof.

**Remark 6.2.** For each \( 1 \leq t \leq d \), let
\[
\tilde{\lambda}_t(x) := (\lambda_1(x), \ldots, \lambda_t(x)),
\]
if each \( \lambda_i \) exists. Note \( \tilde{\lambda}_d(x) \) is equal to \( \tilde{\lambda}(x) \).

Then the same proof of Theorem [D] shows that the \( t \)-th pointwise Lyapunov spectrum \( L_{A,t} \) is also closed and convex for any \( A \in U \).

**Proof of Corollary [I].** Fix any \( \alpha \in (0,1) \) such that \( r - 1 > \alpha \). Since \( h|_A \) is conformal, by choosing \( \mathcal{V}_1 \) sufficiently small, we ensure that any \( g \in \mathcal{V}_1 \) is \( \alpha \)-bunched. From Lemma [5.8] and Remark [5.9] there exists a \( C^1 \)-open and \( C^r \)-dense subset \( \mathcal{V}_2 \) of \( \mathcal{V}_1 \) such that the derivative cocycle \( Dg \) of any \( g \in \mathcal{V}_2 \) is typical. Then Theorem [D] gives that \( L_g \) is closed and convex.

### 6.2. Multifractal analysis.

Using simultaneous quasi-multiplicativity of \( \Phi_A^s \) for \( A \in U \), we perform partial multifractal analysis of the \( \tilde{\alpha} \)-level set
\[
E(\tilde{\alpha}) := \{ x \in \Sigma_T : \tilde{\lambda}_t(x) = \tilde{\alpha} \}
\]
for certain \( \tilde{\alpha} \in \mathbb{R}^t \). For a general introduction on the multifractal analysis, see [BPS97], [PW01], [Cht10], [CHT14], and [PHT10].

For an arbitrary system \((X,f)\), arbitrary map \( A : X \to GL_d(\mathbb{R}) \), and arbitrary vector \( \tilde{\alpha} \), the \( \tilde{\alpha} \)-level set \( E(\tilde{\alpha}) \) may be empty. Even when \( E(\tilde{\alpha}) \) is non-empty, its structure may be irregular. With extra assumptions such as quasi-multiplicativity of the potential \( \Phi_A^s \), we can study such level set \( E(\tilde{\alpha}) \) for certain \( \tilde{\alpha} \in \mathbb{R}^n \).
We recall the general setting in which \cite{FH10} is applicable. Let \((X, f)\) be a compact metric space. For any \(\vec{q} = (q_1, \ldots, q_t) \in \mathbb{R}_+^t\) and \(\vec{\Phi} = (\Phi_1, \ldots, \Phi_t)\) where each \(\Phi_i = \{\log \varphi_{i,n}\}_{n \in \mathbb{N}}\) is a subadditive sequence of potential on \(X\), we define

\[
\vec{q} \cdot \vec{\Phi} := \sum_{i=1}^{m} q_i \Phi_i = \left\{ \sum_{i=1}^{m} q_i \log \varphi_{i,n} \right\}_{n \in \mathbb{N}}.
\]

In what follows, let

\[
P_{\vec{q}}(\vec{\Phi}) := P(\vec{q} \cdot \vec{\Phi}) \quad \text{and} \quad \mathcal{F}(\vec{\Phi}, \mu) := (\mathcal{F}(\Phi_1, \mu), \ldots, \mathcal{F}(\Phi_t, \mu)),
\]

where \(\mathcal{F}\) is defined as in (3.4).

Using Bowen's definition of entropy of non-compact sets \cite{Bow73}, Feng and Huang showed that

**Proposition 6.3.** \cite{FH10} \cite{FH10}. Theorem 4.8] Suppose the entropy map of the system \((X, f)\) is upper semi-continuous. If \(\vec{q}_0 \in \mathbb{R}_+^t\) such that \(\vec{q}_0 \cdot \vec{\Phi}\) has a unique equilibrium state \(\mu_{\vec{q}_0}\), then the subadditive pressure \(P_{\vec{q}_0}(\vec{\Phi})\) is differentiable at \(\vec{q}_0\) and the gradient \(\nabla P_{\vec{q}_0}\) at \(\vec{q}_0\) is equal to \(\mathcal{F}(\vec{\Phi}, \mu_{\vec{q}_0})\). Moreover, denoting \(\vec{\alpha} := \nabla P_{\vec{q}_0}(\vec{q}_0)\), the \(\vec{\alpha}\)-level set \(E(\vec{\alpha})\) is non-empty and satisfies

\[
h_{\text{top}}(E(\vec{\alpha})) = h_{\mu_{\vec{q}_0}}(f).
\]

**Remark 6.4.** We have only stated parts of \cite{FH10} Theorem 4.8 in order to keep the proposition simple. Indeed, under the same assumptions and notations \(\vec{\alpha} := \nabla P_{\vec{q}_0}(\vec{q}_0)\), the topological entropy of the \(\vec{\alpha}\)-level set \(E(\vec{\alpha})\) is also equal to other quantities:

\[
h_{\text{top}}(E(\vec{\alpha})) = \inf_{\vec{t} \in \mathbb{R}_+^t} \left( P_{\vec{q}_0} (\vec{t}) - \vec{\alpha} \cdot \vec{t} \right) = P_{\vec{q}_0}(\vec{q}_0) - \vec{\alpha} \cdot \vec{q}_0,
\]

\[
= \sup\{h_\mu(f) : \mu \in \mathcal{M}(f), \mathcal{F}(\vec{\Phi}, \mu) = \vec{\alpha} \}.
\]

Barreira-Gelfert \cite{BG06} first obtained similar results for repellers of \(C^{1+\alpha}\) maps satisfying the cone condition and bounded distortion. \cite{FH10} improved the result to the more general setting, described in Proposition 6.3. See also \cite{PW01} and \cite{FFW01} for related earlier works, establishing similar results for additive potentials.

We apply the proposition to \(\vec{\Phi}_A = (\Phi_A^1, \ldots, \Phi_A^t)\) for \(A \in \mathcal{U}\). From Theorem E, it follows that the subadditive potential \(\vec{q}_0 \cdot \vec{\Phi}_A\) is quasi-multiplicative for any \(\vec{q}_0 \in \mathbb{R}_+^t\). Then Proposition 3.4 gives the unique equilibrium state \(\mu_{\vec{q}_0}\) of \(\vec{q}_0 \cdot \vec{\Phi}_A\). Hence, we obtain the following corollary:

**Corollary 6.5.** For any \(A \in \mathcal{U}\) and any \(\vec{q}_0 \in \mathbb{R}_+^t\), the subadditive potential \(\vec{q}_0 \cdot \vec{\Phi}_A\) is quasi-multiplicative, and hence, has a unique equilibrium state \(\mu_{\vec{q}_0}\). Also, (6.11) and (6.12) hold with \(\vec{\alpha} := \nabla P_{\vec{q}_0}(\vec{q}_0)\).

**References**

- \cite{AV07} Artur Avila and Marcelo Viana, *Simplicity of lyapunov spectra: a sufficient criterion*, Portugal Math 64 (2007).
- \cite{Bar96} Luis M Barreira, *A non-additive thermodynamic formalism and applications to dimension theory of hyperbolic dynamical systems*, Ergodic Theory Dynamical Systems 16 (1996), no. 5, 871–927.
- \cite{Bar03} Luis Barreira, *Dimension estimates in nonconformal hyperbolic dynamics*, Nonlinearity 16 (2003), no. 5, 1657.
[BCH10] Jungchao Ban, Yongluo Cao, and Huiy Hu, *The dimensions of a non-conformal repeller and an average conformal repeller*, Transactions of the American Mathematical Society 362 (2010), no. 2, 727–751.

[BG06] Luis Barreira and Katrin Gelfert, *Multifractal analysis for lyapunov exponents on nonconformal repellers*, Communications in mathematical physics 267 (2006), no. 2, 393.

[BGMV03] Christian Bonatti, Xavier Gómez-Mont, and Marcelo Viana, *Généricité d’exposants de lyapunov non-nuls pour des produits déterministes de matrices*, Annales de l’Institut Henri Poincare (C) Non Linear Analysis, vol. 20, Elsevier, 2003, pp. 579–624.

[BHR17] Balázs Bárány, Michael Hochman, and Ariel Rapaport, *Hausdorff dimension of planar self-affine sets and measures*, arXiv preprint arXiv:1712.07353 (2017).

[BM18] Jairo Bochi and Ian D Morris, *Equilibrium states of generalised singular value potentials and applications to affine iterated function systems*, Geometric And Functional Analysis (2018), 1–34.

[Bow70] Rufus Bowen, *Markov partitions for axiom a diffeomorphisms*, American Journal of Mathematics 92 (1970), no. 3, 725–747.

[Bow73], *Topological entropy for noncompact sets*, Transactions of the American Mathematical Society 184 (1973), 125–136.

[Bow74], *Some systems with unique equilibrium states*, Mathematical Systems Theory 8 (1974), no. 3, 193–202.

[Bow79], *Hausdorff dimension of quasi-circles*, Publications Mathématiques de l’Institut des Hautes Études Scientifiques 50 (1979), no. 1, 11–25.

[BP02] Luis Barreira and Ya B Pesin, *Lyapunov exponents and smooth ergodic theory*, vol. 23, American Mathematical Soc., 2002.

[BPS97] Luis Barreira, Yakov Pesin, and Jörg Schmeling, *On a general concept of multifractality: multifractal spectra for dimensions, entropies, and lyapunov exponents. multifractal rigidity*, Chaos: an Interdisciplinary Journal of Nonlinear Science 7 (1997), no. 1, 27–38.

[BPVL16] Lucas Backes, Mauricio Poletti, Paulo Varandas, and Yuri Lima, *Simplicity of lyapunov spectrum for linear cocycles over non-uniformly hyperbolic systems*, arXiv preprint arXiv:1612.05056 (2016).

[BV04] Christian Bonatti and Marcelo Viana, *Lyapunov exponents with multiplicity 1 for deterministic products of matrices*, Ergodic Theory and Dynamical Systems 24 (2004), no. 5, 1295–1330.

[CFH08] Yongluo Cao, Dejun Feng, and Wen Huang, *The thermodynamic formalism for sub-additive potentials*, Discrete and Continuous Dynamical Systems 20 (2008), no. 3, 639.

[Cli10] Vaughn Climenhaga, *Multifractal formalism derived from thermodynamics*, arXiv preprint arXiv:1002.0789 (2010).

[Cli14] , *The thermodynamic approach to multifractal analysis*, Ergodic Theory and Dynamical Systems 34 (2014), no. 5, 1409–1450.

[CP10] Jianyu Chen and Yakov Pesin, *Dimension of non-conformal repellers: a survey*, Nonlinearity 23 (2010), no. 4, R93.

[CPZ18] YONGLUO CAO, YAKOV PESIN, and YUN ZHAO, *Dimension estimates for non-conformal repellers and continuity of sub-additive topological pressure*.

[Fal88a] Kenneth J Falconer, *The hausdorff dimension of self-affine fractals*, Mathematical Proceedings of the Cambridge Philosophical Society, vol. 103, Cambridge University Press, 1988, pp. 339–350.

[Fal88b] , *A subadditive thermodynamic formalism for mixing repellers*, Journal of Physics A: Mathematical and General 21 (1988), no. 14, L737.

[Fal94] , *Bounded distortion and dimension for non-conformal repellers*, Mathematical Proceedings of the Cambridge Philosophical Society, vol. 115, Cambridge University Press, 1994, pp. 315–334.

[Fen03] De-Jun Feng, *Lyapunov exponents for products of matrices and multifractal analysis. part i: Positive matrices*, Israel Journal of Mathematics 138 (2003), no. 1, 353–376.

[Fen09] , *Lyapunov exponents for products of matrices and multifractal analysis. part ii: General matrices*, Israel Journal of Mathematics 170 (2009), no. 1, 355.

[Fen11] , *Equilibrium states for factor maps between subshifts*, Advances in Mathematics 226 (2011), no. 3, 2470–2502.

[FFW01] Ai-Hua Fan, De-Jun Feng, and Jun Wu, *Recurrence, dimension and entropy*, Journal of the London Mathematical Society 64 (2001), no. 1, 229–244.
De-Jun Feng and Wen Huang, *Lyapunov spectrum of asymptotically sub-additive potentials*, Communications in Mathematical Physics 297 (2010), no. 1, 1–43.

De-Jun Feng and Antti Kaenmaki, *Equilibrium states of the pressure function for products of matrices*, arXiv preprint arXiv:1009.3129 (2010).

De-Jun Feng and Pablo Shmerkin, *Non-conformal repellers and the continuity of pressure for matrix cocycles*, Geometric and Functional Analysis 24 (2014), no. 4, 1101–1128.

John E Hutchinson, *Fractals and self similarity*, Indiana University Mathematics Journal 30 (1981), no. 5, 713–747.

Antti Käenmäki, *On natural invariant measures on generalised iterated function systems*, University of Jyväskylä, Department of Mathematics and Statistics, 2003.

Gerhard Keller, *Equilibrium states in ergodic theory*, vol. 42, Cambridge university press, 1998.

Boris Kalinin and Victoria Sadovskaya, *Cocycles with one exponent over partially hyperbolic systems*, Geometric and Functional Analysis 167 (2013), no. 1, 167–188.

François Ledrappier and Peter Walters, *A relativised variational principle for continuous transformations*, Journal of the London Mathematical Society 2 (1977), no. 3, 568–576.

Yakov Pesin and Howard Weiss, *The multifractal analysis of birkhoff averages and large deviations*, Global analysis of dynamical systems (2001), 419–431.

David Ruelle, *Repellers for real analytic maps*, Ergodic Theory and Dynamical Systems 2 (1982), no. 1, 99–107.

David Ruelle, *The thermodynamic formalism for expanding maps*, Communications in Mathematical Physics 125 (1989), no. 2, 239–262.

Ya G Sinai, *Markov partitions and c-diffeomorphisms*, Functional Analysis and its applications 2 (1968), no. 1, 61–82.

Boris Solomyak, *Measure and dimension for some fractal families*, Mathematical Proceedings of the Cambridge Philosophical Society, vol. 124, Cambridge University Press, 1998, pp. 531–546.

Peter Walters, *An introduction to ergodic theory*, vol. 79, Springer Science & Business Media, 2000.