CONCISENESS OF COPRIME COMMUTATORS IN FINITE GROUPS

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Abstract. Let $G$ be a finite group. We show that the order of the subgroup generated by coprime $\gamma_k$-commutators (respectively $\delta_k$-commutators) is bounded in terms of the size of the set of coprime $\gamma_k$-commutators (respectively $\delta_k$-commutators). This is in parallel with the classical theorem due to Turner-Smith that the words $\gamma_k$ and $\delta_k$ are concise.

1. Introduction

Let $w$ be a group-word in $n$ variables, and let $G$ be a group. The verbal subgroup $w(G)$ of $G$ determined by $w$ is the subgroup generated by the set $G_w$ consisting of all values $w(g_1, \ldots, g_n)$, where $g_1, \ldots, g_n$ are elements of $G$. A word $w$ is said to be concise if whenever $G_w$ is finite for a group $G$, it always follows that $w(G)$ is finite. More generally, a word $w$ is said to be concise in a class of groups $\mathcal{X}$ if whenever $G_w$ is finite for a group $G \in \mathcal{X}$, it always follows that $w(G)$ is finite. In the sixties P. Hall asked whether every word is concise but later Ivanov proved that this problem has a negative solution in its general form [6] (see also [9, p. 439]). On the other hand, many relevant words are known to be concise. For instance, Turner-Smith [15] showed that the lower central words $\gamma_k$ and the derived words $\delta_k$ are concise; here the words $\gamma_k$ and $\delta_k$ are defined by the positions $\gamma_1 = \delta_0 = x_1$, $\gamma_{k+1} = [\gamma_k, x_{k+1}]$ and $\delta_{k+1} = [\delta_k, \delta_k]$. Wilson showed in [16] that the multilinear commutator words (outer commutator words) are concise. It has been proved by Merzlyakov [8] that every word is concise in the class of linear groups.

In [3] a word $w$ was called boundedly concise in a class of groups $\mathcal{X}$ if for every integer $m$ there exists a number $\nu = \nu(\mathcal{X}, w, m)$ such that whenever $|G_w| \leq m$ for a group $G \in \mathcal{X}$ it always follows that $|w(G)| \leq \nu$. Fernández-Alcober and Morigi [4] showed that every word which is concise in the class of all groups is actually boundedly concise. Moreover they showed that whenever $w$ is a multilinear commutator word having at most $m$ values in a group $G$, one has $|w(G)| \leq (m-1)^{(m-1)}$. Questions on conciseness of words in the class of residually finite groups have been tackled in [1]. It was shown

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that if $w$ is a multilinear commutator word and $q$ a prime-power, then the word $w^q$ is concise in the class of residually finite groups; and if $w = \gamma_k$ is the $k$th lower central word and $q$ a prime-power, then the word $w^q$ is boundedly concise in the class of residually finite groups.

The concept of (bounded) conciseness can actually be applied in a much wider context. Suppose $\mathcal{X}$ is a class of groups and $\phi(G)$ is a subset of $G$ for every group $G \in \mathcal{X}$. One can ask whether the subgroup generated by $\phi(G)$ is finite whenever $\phi(G)$ is finite. In the present paper we show bounded conciseness of coprime commutators in finite groups.

The coprime commutators $\gamma_k^*$ and $\delta_k^*$ have been introduced in [13] as a tool to study properties of finite groups that can be expressed in terms of commutators of elements of coprime orders. Let $G$ be a finite group. Every element of $G$ is a $\gamma_k^*$-commutator as well as a $\delta_k^*$-commutator. Now let $k \geq 2$ and let $X$ be the set of all elements of $G$ that are powers of $\gamma_k^*$-commutators. An element $g$ is a $\gamma_k^*$-commutator if there exist $a \in X$ and $b \in G$ such that $g = [a, b]$ and $([a], [b]) = 1$. For $k \geq 1$ let $Y$ be the set of all elements of $G$ that are powers of $\delta_k^*$-commutators. The element $g$ is a $\delta_k^*$-commutator if there exist $a, b \in Y$ such that $g = [a, b]$ and $([a], [b]) = 1$. The subgroups of $G$ generated by all $\gamma_k^*$-commutators and all $\delta_k^*$-commutators will be denoted by $\gamma_k^*(G)$ and $\delta_k^*(G)$, respectively. One can easily see that if $N$ is a normal subgroup of $G$ and $x$ an element whose image in $G/N$ is a $\gamma_k^*$-commutator (respectively a $\delta_k^*$-commutator), then there exists a $\gamma_k^*$-commutator $y \in G$ (respectively a $\delta_k^*$-commutator) such that $x \in yN$. It was shown in [13] that $\gamma_k^*(G) = 1$ if and only if $G$ is nilpotent and $\delta_k^*(G) = 1$ if and only if the Fitting height of $G$ is at most $k$. It follows that for any $k \geq 2$ the subgroup $\gamma_k^*(G)$ is precisely the last term of the lower central series of $G$ (which is sometimes denoted by $\gamma_\infty(G)$) while for any $k \geq 1$ the subgroup $\delta_k^*(G)$ is precisely the last term of the lower central series of $\delta_k^*(G)$. In the present paper we prove the following results.

**Theorem 1.1.** Let $k \geq 1$ and $G$ a finite group in which the set of $\gamma_k^*$-commutators has size $m$. Then $|\gamma_k^*(G)|$ is $m$-bounded.

**Theorem 1.2.** Let $k \geq 0$ and $G$ a finite group in which the set of $\delta_k^*$-commutators has size $m$. Then $|\delta_k^*(G)|$ is $m$-bounded.

We remark that the bounds for $|\gamma_k^*(G)|$ and $|\delta_k^*(G)|$ in the above results do not depend on $k$. Thus, we observe here the phenomenon that in [4] was dubbed “uniform conciseness”. We make no attempts to provide explicit bounds for $|\gamma_k^*(G)|$ and $|\delta_k^*(G)|$ in Theorems 1.1 and 1.2. Throughout the paper we use the term $m$-bounded to mean that the bound is a function of $m$.

2. Preliminaries

We begin with a well-known result about coprime actions on finite groups. Recall that $[[K, H]]$ is the subgroup generated by $\{[k, h] : k \in K, h \in H\}$, and $[K, iH] = [[K, iH], H]$ for $i \geq 2$. 
Lemma 2.1 ([5], Lemma 4.29). Let $A$ act via automorphisms on $G$, where $A$ and $G$ are finite groups, and suppose that $(|G|, |A|) = 1$. Then $[G, A, A] = [G, A]$.

For the following result from [14], recall that a subset $B$ of a group $A$ is normal if $B$ is a union of conjugacy classes of $A$.

Lemma 2.2. Let $A$ be a group of automorphisms of a finite group $G$ with $(|A|, |G|) = 1$. Suppose that $B$ is a normal subset of $A$ such that $A = \langle B \rangle$. Let $k \geq 1$ be an integer. Then $[G, A]$ is generated by the subgroups of the form $[G, b_1, \ldots, b_k]$, where $b_1, \ldots, b_k \in B$.

The following is an elementary property of $\delta^*_k$-commutators.

Lemma 2.3. Let $G$ be a finite group. For $k$ a non-negative integer,

$$\delta^*_k(\delta^*_1(G)) = \delta^*_k(G).$$

Proof. We argue by induction. For $k = 0$, the result is obvious by the definition of $\delta^*_0$-commutators.

Suppose the result holds for $k - 1$. So

$$\delta^*_{k-1}(\delta^*_1(G)) = \delta^*_k(G).$$

It was mentioned in the introduction that $\delta^*_k(G) = \gamma_\infty(\delta^*_k(G))$. By induction,

$$\delta^*_{k+1}(G) = \gamma_\infty(\delta^*_k(\delta^*_1(G))),$$

and viewing $\delta^*_1(G)$ as the group in consideration, we have

$$\gamma_\infty(\delta^*_k(\delta^*_1(G))) = \delta^*_k(\delta^*_1(G))$$

as required. □

Here is a helpful observation that we will use in both of our main results. Recall that a Hall subgroup of a finite group is a subgroup whose order is coprime to its index. Also, a finite group $G$ is metanilpotent if and only if $\gamma_\infty(G)$ is nilpotent.

Lemma 2.4. Let $G$ be a finite metanilpotent group and $P$ a Sylow $p$-subgroup of $\gamma_\infty(G)$, and let $H$ be a Hall $p'$-subgroup of $G$. Then $P = [P, H]$.

Proof. For simplicity, we write $K$ for $\gamma_\infty(G)$. By passing to the quotient $G/O_{p'}(G)$, we may assume that $P = K$.

Let $P_1$ be a Sylow $p$-subgroup of $G$. So $G = P_1 H$. Now $P_1/P$ is normal in $G/P$ as $G/P$ is nilpotent, but also $P \leq P_1$; hence $P_1$ is normal in $G$. It follows that $K = [P_1, H]$, since in a nilpotent group all coprime elements commute. By Lemma 2.2, $[P_1, H, H] = [P_1, H] = P$, and so $P = [P_1, H] = [P, H]$. □

As it turns out, in the proofs of our main results we often reduce to the following case.
Lemma 2.5. Let $i$ and $m$ be positive integers. Let $P$ be an abelian $p$-group acted on by a $p'$-group $A$ such that
\[ |\{[x, a_1, \ldots, a_i] : x \in P, a_1, \ldots, a_i \in A\}| = m. \]
Then $|[P, A]| = 2^m$, so is $m$-bounded.

Proof. We enumerate the set $\{[x, a_1, \ldots, a_i] : x \in P, a_1, \ldots, a_i \in A\}$ as $\{c_1, \ldots, c_m\}$. As $P$ is abelian, we have that
\[ [x, a_1, \ldots, a_i]^l = [x^l, a_1, \ldots, a_i] \] for all $x \in P, a_1, \ldots, a_i \in A$, and $l$ a positive integer.

Consider $g \in [P, A]$, which can be expressed as some product $c_1^{l_1} \cdots c_m^{l_m}$ for non-negative integers $l_1, \ldots, l_m$. We claim that $l_1, \ldots, l_m \in \{0, 1\}$. For, if $l_j > 1$ with $j \in \{1, \ldots, m\}$, we know from (i) that $c_j^{l_j} \in \{c_1, \ldots, c_m\}$. We replace all such $c_j^{l_j}$ accordingly, so that $g$ is now expressed as $c_1^{k_1} \cdots c_m^{k_m}$ with $k_1, \ldots, k_m \in \{0, 1\}$. Hence $|[P, A]| = 2^m$. \qed

The well-known Focal Subgroup Theorem \cite{12} states that if $G$ is a finite group and $P$ a Sylow $p$-subgroup of $G$, then $P \cap G'$ is generated by the set of commutators $\{[g, z] \mid g \in G, z \in P, [g, z] \in P\}$. In particular, it follows that $P \cap G'$ can be generated by commutators lying in $P$. This observation led to the question on generation of Sylow subgroups of verbal subgroups of finite groups. More specifically, the following problem was addressed in \cite{2}.

Given a multilinear commutator word $w$ and a Sylow $p$-subgroup $P$ of a finite group $G$, is it true that $P \cap w(G)$ can be generated by $w$-values lying in $P$?

The answer to this is still unknown. The main result of \cite{2} is that if $G$ has order $p^n n$, where $n$ is not divisible by $p$, then $P \cap w(G)$ is generated by $n$th powers of $w$-values. In the present paper we will require a result on generation of Sylow subgroups of $\delta_k^*(G)$.

Lemma 2.6. Let $k \geq 0$ and let $G$ be a finite soluble group of order $p^n n$, where $p$ is a prime and $n$ is not divisible by $p$, and let $P$ be a Sylow $p$-subgroup of $G$. Then $P \cap \delta_k^*(G)$ is generated by $n$th powers of $\delta_k^*$-commutators lying in $P$.

It seems likely that Lemma 2.6 actually holds for all finite groups. In particular, the result in \cite{2} was proved without the assumption that $G$ is soluble. It seems though that proving Lemma 2.6 for arbitrary groups is a complicated task. Indeed, one of the tools used in \cite{2} is the proof of the Ore Conjecture by M. W. Liebeck, E. A. O’Brien, A. Shalev, and P. H. Tiep \cite{7} that every element of any finite simple group is a commutator. Recently it was conjectured in \cite{13} that every element of a finite simple group is a commutator of elements of coprime orders. If this is confirmed, then extending Lemma 2.6 to arbitrary groups would be easy. However the conjecture that every element of a finite simple group is a commutator of elements of coprime orders at present is known to be true only for the alternating
groups [13] and the groups PSL(2, q) [10]. Thus, we prove Lemma 2.6 only for soluble groups, which is quite adequate for the purposes of the present paper.

Before we embark on the proof of Lemma 2.6 we note a key result from [2] that we will need.

Lemma 2.7. Let \( G \) be a finite group, and let \( P \) be a Sylow \( p \)-subgroup of \( G \). Assume that \( N \leq L \) are two normal subgroups of \( G \), and use the bar notation in the quotient group \( G/N \). Let \( X \) be a normal subset of \( G \) consisting of \( p \)-elements such that \( P \cap L = \langle P \cap X \rangle \). Then \( P \cap L = \langle P \cap X, P \cap N \rangle \).

We are now ready to prove Lemma 2.6.

Proof. Let \( G \) be a counter-example of minimal order. Then \( k \geq 1 \).

By induction on the order of \( G \), the lemma holds for any proper subgroup and any proper quotient of \( G \). We observe that \( \delta_1^*(G) \neq 1 \) since \( G \) is not perfect, and by Lemma 2.3 we have \( \delta_{k+1}^*(G) = \delta_k^*(\delta_k^*(G)) \). Since the result holds for \( \delta_k^*(G) \), it follows that \( P \cap \delta_{k+1}^*(G) \) is generated by \( n \)-th powers of \( \delta_k^* \)-commutators in \( G \). Note that we made use of Remark 3.2 of [2].

If \( \delta_{k+1}^*(G) \neq 1 \), by induction the result holds for \( G/\delta_{k+1}^*(G) \). Combining this with the fact that \( P \cap \delta_{k+1}^*(G) \) can be generated by \( n \)-th powers of \( \delta_k^* \)-commutators, we get a contradiction by Lemma 2.7. Hence \( \delta_{k+1}^*(G) = 1 \). Further \( O_{p'}(G) = 1 \) since \( G \) is a minimal counter-example. Therefore \( \delta_{k}^*(G) \subseteq P \), and it is now obvious that \( P \cap \delta_{k}^*(G) \) is generated by \( n \)-th powers of \( \delta_k^* \)-commutators lying in \( P \). So we have our required contradiction.

\( \square \)

3. Proofs of the main results

We mention here a needed result of Schur and Wiegold. The much celebrated Schur Theorem states that if \( G \) is a group with \( |G/Z(G)| \) finite, then \( |G'| \) is finite. It is implicit in the work of Schur that if \( |G/Z(G)| = m \), then \( |G'| \) is \( m \)-bounded. However, Wiegold produced a shorter proof of this second statement, which also gives the best possible bound. The reader is directed to Robinson ([11], pages 102-103) for details.

Additionally, for the proof of Theorem 1.2 we require the following result from [13].

Lemma 3.1. Let \( G \) be a finite group and let \( y_1, \ldots, y_k \) be \( \delta_k^* \)-commutators in \( G \). Suppose the elements \( y_1, \ldots, y_k \) normalize a subgroup \( N \) such that \( (|y_i|, |N|) = 1 \) for every \( i = 1, \ldots, k \). Then for every \( x \in N \) the element \( [x, y_1, \ldots, y_k] \) is a \( \delta_{k+1}^* \)-commutator.

Now we are ready to begin.

Proof of Theorem 1.2. Let \( X \) be the set of all \( \gamma_k^* \)-commutators. We wish to show that if \( |X| = m \), then \( |\gamma_k^*(G)| \) is \( m \)-bounded. For convenience we write \( K \) for \( \langle X \rangle \). Of course, \( K = \gamma_\infty(G) \).
The subgroup $C_G(X)$ has index $\leq m!$, so $|K/Z(K)| \leq m!$ too. By Schur, $K'$ has $m$-bounded order. Therefore, by passing to the quotient, we may assume $K' = 1$, and so $K$ is abelian with $G$ metanilpotent.

It is enough to bound the order of each Sylow subgroup of $K$. We choose a Sylow $p'$-subgroup $P$. By passing to the quotient $G/O_{p'}(G)$, we may assume $K = P$.

By Lemma 2.4, a Hall $p'$-subgroup $H$ of $G$ satisfies $P = [P, k_1 H]$. We know that $P$ is abelian and $P$ is normal in $PH$.

We denote the set $\{[x, h_1, \ldots, h_{k_1}] : x \in P, h_1, \ldots, h_{k_1} \in H\}$ by $\hat{X}$.

For $x \in P, h_1, \ldots, h_{i-1} \in H$, where $i \geq 2$, we note that $[x, h_1, \ldots, h_{i-1}]$ is a $\gamma_i^*$-commutator. Therefore $\hat{X} \subseteq X$, and $|\hat{X}| \leq m$.

By Lemma 2.5, it follows that $|[P, k_1 H]|$ is $m$-bounded. Appealing to Lemma 2.3, we conclude that $|P|$ is $m$-bounded.

Proof of Theorem 1.2. Let $X$ be the set of $\delta_k^*$-commutators in $G$. We wish to show here that if $|X| = m$, then $|\delta_k^*(G)|$ is $m$-bounded. We recall that $\delta_k^*(G) = \gamma_\infty(\delta_{k-1}^*(G))$. For ease of notation we define $Q := \delta_{k-1}^*(G)$, and we write $K$ for $\delta_k^*(G)$.

The subgroup $C_G(X)$ has index $\leq m!$ in $G$, so $|K/Z(K)| \leq m!$ and as in the proof of Theorem 1.1, we may assume $K' = 1$. Hence $K$ is assumed to be abelian with $Q$ metanilpotent. In what follows, we now restrict to the group $Q$.

It is sufficient to show that the order of each Sylow subgroup of $K$ is $m$-bounded. We choose $P$ a Sylow $p$-subgroup of $K$. By passing to the quotient $G/O_{p'}(G)$, we may assume $K = P$.

By Lemma 2.4, a Hall $p'$-subgroup $H$ of $Q$ satisfies $P = [P, H]$. By Lemma 2.6, since $H$ is generated by its Sylow subgroups, we have $H$ is generated by a normal subset $B$ of powers of $\delta_{k-1}^*$-commutators that are of $p'$ order.

Lemma 2.2 now implies that $[P, H]$ is generated by subgroups $[P, b_1, \ldots, b_k]$ for $b_1, \ldots, b_k \in B$. By Lemma 3.1, for $x \in P$ we have $[x, b_1, \ldots, b_k]$ is a $\delta_k^*$-commutator, and we deduce that $|[P, b_1, \ldots, b_k]|$ is $m$-bounded.

It follows that the number of generators of $[P, H]$ is at most $m$, and furthermore the exponent of $[P, H]$ is $m$-bounded. Hence, the finite abelian $p$-group $P = [P, H]$ has $m$-bounded order.

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