RICCI FLOW WITH SURGERY ON MANIFOLDS WITH
POSITIVE ISOTROPIC CURVATURE

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Abstract. We study the Ricci flow for initial metrics with positive isotropic curvature (strictly PIC for short).

In the first part of this paper, we prove new curvature pinching estimates which ensure that blow-up limits are uniformly PIC in all dimensions. Moreover, in dimension $n \geq 12$, we show that blow-up limits are weakly PIC. This can be viewed as a higher-dimensional version of the fundamental Hamilton-Ivey pinching estimate in dimension 3.

In the second part, we develop a theory of ancient solutions which have bounded curvature; are $\kappa$-noncollapsed; are weakly PIC; and are uniformly PIC. This is an adaptation of Perelman’s work; the additional ingredients needed in the higher dimensional setting are the differential Harnack inequality for solutions to the Ricci flow satisfying the PIC condition, and a rigidity result due to Brendle-Huisken-Sinestrari for ancient solutions that are uniformly PIC.

In the third part of this paper, we prove a Canonical Neighborhood Theorem for the Ricci flow with initial data with positive isotropic curvature, which holds in dimension $n \geq 12$. This relies on the curvature pinching estimates together with the structure theory for ancient solutions. This allows us to adapt Perelman’s surgery procedure to this situation. As a corollary, we obtain a topological classification of all compact manifolds with positive isotropic curvature which do not contain non-trivial incompressible space forms.

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1. Introduction

Recall the following curvature conditions which were introduced in \[28\] and \[8\]:

**Definition 1.1.** (i) We denote by PIC the set of all algebraic curvature tensors which have nonnegative isotropic curvature in the sense that \( R(\varphi, \bar{\varphi}) \geq 0 \) for all complex two-forms of the form \( \varphi = (e_1 + ie_2) \wedge (e_3 + ie_4) \), where \( \{e_1, e_2, e_3, e_4\} \) is an orthonormal four-frame.

(ii) We denote by PIC1 the set of all algebraic curvature tensors satisfying \( R(\varphi, \bar{\varphi}) \geq 0 \) for all complex two-forms of the form \( \varphi = (e_1 + i\mu e_2) \wedge (e_3 + i\lambda e_4) \), where \( \{e_1, e_2, e_3, e_4\} \) is an orthonormal four-frame and \( \lambda \in [0, 1] \).

(iii) We denote by PIC2 the set of all algebraic curvature tensors satisfying \( R(\varphi, \bar{\varphi}) \geq 0 \) for all complex two-forms of the form \( \varphi = (e_1 + i\mu e_2) \wedge (e_3 + i\lambda e_4) \), where \( \{e_1, e_2, e_3, e_4\} \) is an orthonormal four-frame and \( \lambda, \mu \in [0, 1] \).

Note that PIC2 \( \subset \) PIC1 \( \subset \) PIC. The curvature tensor of a Riemannian manifold \( M \) lies in the PIC1 cone if and only if the curvature tensor of \( M \times \mathbb{R} \) lies in the PIC cone. Similarly, the curvature tensor of \( M \) lies in the PIC2 cone if and only if the curvature tensor of \( M \times \mathbb{R}^2 \) lies in the PIC cone.

The significance of the curvature conditions above stems from the fact that they are all preserved by the Ricci flow. For an initial metric that is weakly PIC2, the subsequent solution of the Ricci flow satisfies a differential Harnack inequality (cf. \[18\], \[3\]). For an initial metric that is strictly PIC1, it was shown in \[2\] that the Ricci flow will converge to a metric of constant curvature after rescaling (see \[11\], \[12\], \[21\], \[25\], \[26\], \[27\], \[30\] for earlier work on the subject). For an initial metric that is strictly PIC, it has been conjectured that the Ricci flow should only form so-called neck-pinching singularities.

For \( n = 4 \), this was proved in a landmark paper by Hamilton \[20\] (see also \[11\], \[12\]). Our goal in this paper is to confirm the conjecture for \( n \geq 12 \).

A key step in our analysis is a new curvature pinching estimate in higher dimensions. By work of Hamilton \[16\], \[17\], the curvature tensor satisfies the evolution equation

\[
D_t R = \Delta R + Q(R).
\]

Here, \( Q(R) \) is a quadratic expression in the curvature tensor. More precisely, \( Q(R) = R^2 + R^\# \), where \( R^2 \) and \( R^\# \) are defined by

\[
(R^2)_{ijkl} = \sum_{p, q = 1}^{n} R_{ijpq} R_{klpq}.
\]
Note that the definitions of $R^2$, $R^\#$, and $Q(R)$ make sense for any algebraic curvature tensor $R$. In fact, the definitions even make sense if $R$ does not satisfy the first Bianchi identity. It is sometimes convenient to consider curvature-type tensors that do not satisfy the first Bianchi identity (see Section 4 below); however, unless stated otherwise, we will assume that the first Bianchi identity is satisfied.

In order to prove pinching estimates for the Ricci flow, we need to analyze the Hamilton ODE $\frac{d}{dt} R = Q(R)$ on the space of algebraic curvature tensors.

Our first main result is a pinching estimate for the Hamilton ODE:

**Theorem 1.2.** Assume that $n \geq 5$ is an integer such that Assumptions 4.1, 5.1, and 5.4 are satisfied. Let $K$ be a compact set of algebraic curvature tensors in dimension $n$ which is contained in the interior of the PIC cone, and let $T > 0$ be given. Then there exist a small positive real number $\theta$, a large positive real number $N$, an increasing concave function $f$ satisfying $\lim_{s \to \infty} f(s) = 0$, and a continuous family of closed, convex, O($n$)-invariant sets $\{F_t: t \in [0, T]\}$ such that the family $\{F_t: t \in [0, T]\}$ is invariant under the Hamilton ODE $\frac{d}{dt} R = Q(R)$; $K \subset F_0$; and

\[
F_t \subset \{ R : R - \theta \text{ scal id} \otimes \text{id} \in \text{PIC} \}
\cap \{ R : \text{Ric}_{11} + \text{Ric}_{22} - \theta \text{ scal} + N \geq 0 \}
\cap \{ R : R + f(\text{scal}) \text{id} \otimes \text{id} \in \text{PIC}_2 \}
\]

for all $t \in [0, T]$.

Here, $\otimes$ denotes the Kulkarni-Nomizu product. More precisely, if $A$ and $B$ are symmetric bilinear forms, then $(A \otimes B)_{ijkl} = A_{ik}B_{jl} - A_{il}B_{jk} - A_{jk}B_{il} + A_{jl}B_{ik}$.

Assumptions 4.1, 5.1, and 5.4 require that certain systems of inequalities for a parameter $b_{\text{max}}$ are satisfied. It is easy to see that Assumptions 4.1, 5.1, and 5.4 are satisfied when $n$ is sufficiently large. A closer examination shows that Assumptions 4.1, 5.1, and 5.4 are satisfied for $n \geq 12$ and $b_{\text{max}} = \frac{1}{12}$.

Via Hamilton’s PDE-ODE principle (cf. [13], Theorem 3, and [17]), Theorem 1.2 immediately gives curvature pinching estimates for solutions to the Ricci flow starting from initial metrics with positive isotropic curvature:

**Corollary 1.3.** Let $(M, g_0)$ be a compact manifold of dimension $n \geq 12$ with positive isotropic curvature. Then there exist a small positive real number $\theta$, a large positive real number $N$, and an increasing concave function $f$ satisfying $\lim_{s \to \infty} f(s) = 0$ such that the curvature tensor of $(M, g(t))$ satisfies $R - \theta \text{ scal id} \otimes \text{id} \in \text{PIC}$, $\text{Ric}_{11} + \text{Ric}_{22} - \theta \text{ scal} + N \geq 0$, and $R + f(\text{scal}) \text{id} \otimes \text{id} \in \text{PIC}_2$ for all $t \geq 0$. 

and

\[
(R^\#)_{ijkl} = 2 \sum_{p,q=1}^n (R_{ipkq}R_{jplq} - R_{iplq}R_{jpkq}).
\]
Corollary 1.3 can be viewed as a higher dimensional version of the fundamental Hamilton-Ivey pinching estimate in dimension 3 (cf. [19], [24]).

The proof of Theorem 1.2 will occupy Sections 2–6. In Section 7 we develop a theory of ancient solutions to the Ricci flow which have bounded curvature; are \(\kappa\)-noncollapsed; are weakly PIC2; and are uniformly PIC. In Section 8 we establish an analogue of Perelman’s Canonical Neighborhood Theorem. This result gives a precise description of the high curvature regions. This makes it possible to extend the flow beyond singularities by a surgery procedure as in Perelman’s work (cf. [31],[32],[33]). Moreover, the surgically modified flow must become extinct in finite time. This is discussed in Sections 9–11. One simplification compared to Perelman’s work is that we have an upper bound for the extinction time; this allows us to choose the surgery parameters independent of time \(t\).

As a corollary, we obtain a topological classification of all compact manifolds of dimension \(n \geq 12\) which admit metrics of positive isotropic curvature and do not contain non-trivial incompressible space forms. More precisely, we prove that such a manifold must be diffeomorphic to a connected sum of finitely many pieces, each of which is diffeomorphic to a quotient of \(S^n\) or \(S^{n-1} \times \mathbb{R}\). Conversely, it follows from work of Micallef and Wang [29] that every manifold which is diffeomorphic to a connected sum of quotients of \(S^n\) and \(S^{n-1} \times \mathbb{R}\) admits a metric with positive isotropic curvature. Earlier results on the topology of manifolds with positive isotropic curvature (which are based on minimal surface techniques) are discussed in [14],[15],[28].

Finally, we mention an interesting connection between Ricci flow on manifolds with positive isotropic curvature and mean curvature flow for two-convex hypersurfaces. A compact hypersurface in \(\mathbb{R}^{n+1}\) \((n \geq 4)\) is two-convex if and only if the induced metric has positive isotropic curvature. If we evolve a two-convex hypersurface by mean curvature flow, then results of Huisken and Sinestrari [22] imply that singularities are asymptotically convex. Moreover, in the two-convex setting, Huisken and Sinestrari [23] showed that mean curvature flow can be extended beyond singularities by a surgery procedure.

2. Auxiliary results

In this section, we collect various auxiliary results that will be needed in the later sections. Throughout this paper, we assume that \(n \geq 5\).

**Lemma 2.1.** Suppose that \(S \in \text{PIC}\). Then the largest eigenvalue of the Ricci tensor of \(S\) is bounded from above by \(\frac{1}{2}\) scal\((S)\).

**Proof.** Since \(S\) has nonnegative isotropic curvature, we have

\[
\text{scal}(S) - 2 \text{Ric}(S)_{nn} = \sum_{k,l=1}^{n-1} S_{klll} \geq 0,
\]
as claimed.

**Lemma 2.2.** Assume $S \in \text{PIC}$. Then $\text{Ric}(S)_{11} + \text{Ric}(S)_{22} + \text{Ric}(S)_{33} + \text{Ric}(S)_{44} \geq 0$ for every orthonormal four-frame $\{e_1, e_2, e_3, e_4\}$.

**Proof.** Note that $\text{Ric}(S)_{11} + \text{Ric}(S)_{33} \geq 2S_{1313}$, $\text{Ric}(S)_{11} + \text{Ric}(S)_{44} \geq 2S_{1414}$, $\text{Ric}(S)_{22} + \text{Ric}(S)_{33} \geq 2S_{2323}$, $\text{Ric}(S)_{22} + \text{Ric}(S)_{44} \geq 2S_{2424}$. Taking the sum of all four inequalities yields

$$\text{Ric}(S)_{11} + \text{Ric}(S)_{22} + \text{Ric}(S)_{33} + \text{Ric}(S)_{44} \geq S_{1313} + S_{1414} + S_{2323} + S_{2424} \geq 0,$$

as claimed.

**Lemma 2.3.** Let $0 \leq \zeta \leq 1$ and $0 < \rho \leq 1$. Assume that $S \in \text{PIC}$ and $\text{Ric}(S)_{11} + \text{Ric}(S)_{22} \geq \frac{2(1-\zeta)}{n} \text{scal}(S)$ for every pair of orthonormal vectors $\{e_1, e_2\}$. Then

$$\frac{n-2}{n} \text{scal}(S) \left( \text{Ric}(S)_{11} + \text{Ric}(S)_{22} \right) - \rho \left( \frac{\text{oRic}(S)^2}{\text{Ric}(S)_{11}} + \frac{\text{oRic}(S)^2}{\text{Ric}(S)_{22}} \right) \geq \frac{2(n-2)(1-\zeta) - 2\zeta^2 \rho \frac{n^2 - 2n + 2}{(n-2)^2}}{n^2} \text{scal}(S)^2$$

for every pair of orthonormal vectors $\{e_1, e_2\}$.

**Proof.** By scaling, we may assume that $\text{scal}(S) = n$. Moreover, we may assume that $\text{Ric}(S)$ is diagonal with diagonal entries $\lambda_1 \leq \ldots \leq \lambda_n$. By assumption, $\lambda_1 + \lambda_2 \geq 2(1-\zeta)$. It suffices to prove that

$$(n-2)(\lambda_i + \lambda_j) - \rho ((\lambda_i - 1)^2 + (\lambda_j - 1)^2) \geq 2(n-2)(1-\zeta) - 4\zeta^2 \rho \frac{n^2 - 2n + 2}{(n-2)^2}$$

for $i < j$. This inequality can be rewritten as

$$\lambda_i \left( \frac{n-2}{\rho} + 2 - \lambda_i \right) + \lambda_j \left( \frac{n-2}{\rho} + 2 - \lambda_j \right) \geq 2(1-\zeta) \left( \frac{n-2}{\rho} + 1 + \zeta \right) - \frac{2n^2\zeta^2}{(n-2)^2}.$$ 

for $i < j$. Note that $\frac{n-2}{\rho} + 2 \geq n$. Moreover, $\lambda_n \leq \frac{n}{2}$ by Lemma 2.1. We distinguish two cases:

**Case 1:** Suppose first that $\lambda_i \geq 1 - \zeta$. Since $i < j$, we have $1 - \zeta \leq \lambda_i \leq \lambda_j \leq \frac{n}{2}$. This implies

$$\lambda_i \left( \frac{n-2}{\rho} + 2 - \lambda_i \right) + \lambda_j \left( \frac{n-2}{\rho} + 2 - \lambda_j \right) \geq 2(1-\zeta) \left( \frac{n-2}{\rho} + 1 + \zeta \right),$$

which implies the claim.
Putting these facts together, we conclude that

$$\lambda_i \left( \frac{n-2}{\rho} + 2 - \lambda_i \right) + \lambda_j \left( \frac{n-2}{\rho} + 2 - \lambda_j \right)$$

$$\geq \lambda_i \left( \frac{n-2}{\rho} + 2 - \lambda_i \right) + (2 - 2\zeta - \lambda_i) \left( \frac{n-2}{\rho} + 2\zeta + \lambda_i \right)$$

$$= 2(1 - \zeta) \left( \frac{n-2}{\rho} + 1 + \zeta \right) - 2(1 - \zeta - \lambda_i)^2.$$  

The inequality \(\lambda_1 + \lambda_2 \geq 2(1 - \zeta)\) implies \(\lambda_i \geq \lambda_1 \geq 1 - \frac{2(n-1)\zeta}{n-2}\). Consequently, \(0 \leq 1 - \zeta - \lambda_i \leq \frac{n\zeta}{n-2}\). Thus, we conclude that

$$\lambda_i \left( \frac{n-2}{\rho} + 2 - \lambda_i \right) + \lambda_j \left( \frac{n-2}{\rho} + 2 - \lambda_j \right)$$

$$\geq 2(1 - \zeta) \left( \frac{n-2}{\rho} + 1 + \zeta \right) - \frac{2n^2\zeta^2}{(n-2)^2},$$

as claimed.

**Proposition 2.4.** Suppose \(R(t)\) is a solution of the Hamilton ODE \(\frac{d}{dt} R = Q(R)\). For each \(t\), let us denote the smallest isotropic curvature by \(4\kappa\). If \(\kappa \geq 0\), then \(\frac{d}{dt} \kappa \geq 2(n-1)\kappa^2\).

**Proof.** By definition of \(\kappa\), the curvature tensor \(S_{ijkl} = R_{ijkl} - \kappa (g_{ik}g_{jl} - g_{il}g_{jk})\) has nonnegative isotropic curvature. We compute

$$Q(S)_{ijkl} = Q(R)_{ijkl} + 2(n-1)\kappa^2 (g_{ik}g_{jl} - g_{il}g_{jk})$$

$$- 2\kappa (\text{Ric}_{ik}g_{jl} - \text{Ric}_{il}g_{jk} - \text{Ric}_{jk}g_{il} + \text{Ric}_{jl}g_{ik})$$

(cf. [3]). Suppose that \(\{e_1, e_2, e_3, e_4\}\) is an orthonormal four-frame satisfying \(R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} = 4\kappa\). Then \(S_{1313} + S_{1414} + S_{2323} + S_{2424} - 2S_{1234} = 0\). Using Proposition 7.5 in [3], we obtain

$$0 \leq Q(S)_{1313} + Q(S)_{1414} + Q(S)_{2323} + Q(S)_{2424} - 2Q(S)_{1234}$$

$$= Q(R)_{1313} + Q(R)_{1414} + Q(R)_{2323} + Q(R)_{2424} - 2Q(R)_{1234}$$

$$+ 8(n-1)\kappa^2 - 4\kappa (\text{Ric}_{11} + \text{Ric}_{22} + \text{Ric}_{33} + \text{Ric}_{44}).$$

Lemma [2] gives

$$0 \leq \text{Ric}(S)_{11} + \text{Ric}(S)_{22} + \text{Ric}(S)_{33} + \text{Ric}(S)_{44}$$

$$= \text{Ric}_{11} + \text{Ric}_{22} + \text{Ric}_{33} + \text{Ric}_{44} - 4(n-1)\kappa.$$

Putting these facts together, we conclude that

$$\frac{d}{dt} (R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234})$$

$$= Q(R)_{1313} + Q(R)_{1414} + Q(R)_{2323} + Q(R)_{2424} - 2Q(R)_{1234} \geq 8(n-1)\kappa^2,$$
as claimed.

In the following, we say that $T \geq 0$ if $T(\varphi, \bar{\varphi}) \geq 0$ for all complex two-forms $\varphi \in \text{so}(n, \mathbb{C})$.

**Proposition 2.5.** Assume that $S \in \text{PIC}$ and $T \geq 0$. Then $S^\#, S^#T \in T_5 \text{PIC}$. Note that we do not require that $S$ and $T$ satisfy the Bianchi identity.

**Proof.** Consider a two-form $\varphi$ of the form $\varphi = (e_1 + ie_2) \wedge (e_3 + ie_4)$. We define a linear transformation $L : \text{so}(n, \mathbb{C}) \to \text{so}(n, \mathbb{C})$ by $L\sigma = [\varphi, \sigma]$. If $S(\varphi, \bar{\varphi}) = 0$, then the second variation identity gives $L^*SL \geq 0$ and $LSL^* \geq 0$ (cf. [34]). Let $P : \text{so}(n, \mathbb{C}) \to \text{so}(n, \mathbb{C})$ be a linear transformation such that $LPL = L$. Then

$$S^\#(\varphi, \bar{\varphi}) = \text{tr}(SL^*SL) = \text{tr}((L^*SL)P(LSL^*)P^*) \geq 0$$

and

$$(S^#T)(\varphi, \bar{\varphi}) = \text{tr}(TL^*SL) \geq 0.$$

Since this holds for every $\varphi = (e_1 + ie_2) \wedge (e_3 + ie_4)$ satisfying $S(\varphi, \bar{\varphi}) = 0$, we conclude that $S^\#, S^#T \in T_5 \text{PIC}$.

**Lemma 2.6.** Assume that $S \in \text{PIC}$. Then

$$S_{1313} + \lambda^2 S_{1414} + S_{2323} + \lambda^2 S_{2424} - 2\lambda S_{1234} + \frac{1 - \lambda^2}{n - 4} (\text{Ric}(S)_{11} + \text{Ric}(S)_{22} - 2S_{1212}) \geq 0$$

for every orthonormal four-frame $\{e_1, e_2, e_3, e_4\}$ and every $\lambda \in [0, 1]$.

**Proof.** Since $S \in \text{PIC}$, we have

$$S_{1313} + S_{1414} + S_{2323} + S_{2424} - 2S_{1234} \geq 0$$

for every orthonormal four-frame $\{e_1, e_2, e_3, e_4\}$. We now replace $e_4$ by $\lambda e_4 \pm \sqrt{1 - \lambda^2} e_p$, where $p \in \{5, \ldots, n\}$, and take the arithmetic mean of the resulting inequalities. This gives

$$S_{1313} + \lambda^2 S_{1414} + S_{2323} + \lambda^2 S_{2424} - 2\lambda S_{1234} + (1 - \lambda^2)(S_{1p1p} + S_{2p2p}) \geq 0$$

for all $p \in \{5, \ldots, n\}$. In the next step, we take the mean value over all $p \in \{5, \ldots, n\}$. Using the inequality

$$\sum_{p=5}^{n} (S_{1p1p} + S_{2p2p}) \leq \sum_{p=3}^{n} (S_{1p1p} + S_{2p2p}) = \text{Ric}(S)_{11} + \text{Ric}(S)_{22} - 2S_{1212},$$

we obtain

$$S_{1313} + \lambda^2 S_{1414} + S_{2323} + \lambda^2 S_{2424} - 2\lambda S_{1234} + \frac{1 - \lambda^2}{n - 4} (\text{Ric}(S)_{11} + \text{Ric}(S)_{22} - 2S_{1212}) \geq 0.$$

This proves the assertion.
Proposition 2.7. Let $S$ be an algebraic curvature tensor and let $H$ be a symmetric matrix such that

$$Z := S_{1313} + \lambda^2 S_{1414} + S_{2323} + \lambda^2 S_{2424} - 2\lambda S_{1234} + (1 - \lambda^2)(H_{11} + H_{22}) \geq 0$$

for every orthonormal four-frame $\{e_1, e_2, e_3, e_4\}$ and every $\lambda \in [0, 1]$. Moreover, suppose that $Z = 0$ for one particular orthonormal four-frame $\{e_1, e_2, e_3, e_4\}$ and one particular $\lambda \in [0, 1]$. Then

$$Q(S)_{1313} + \lambda^2 Q(S)_{1414} + Q(S)_{2323} + \lambda^2 Q(S)_{2424} - 2\lambda Q(S)_{1234}$$
$$+ (H \otimes H)_{1313} + \lambda^2 (H \otimes H)_{1414}$$
$$+ (H \otimes H)_{2323} + \lambda^2 (H \otimes H)_{2424}$$
$$- 2\lambda (H \otimes H)_{1234}$$
$$+ 2(1 - \lambda^2)((S * H)_{11} + (S * H)_{22})$$
$$\geq 2(1 + \lambda^2)(H_{11} + H_{22})^2.$$

Here, $S \ast H$ is defined by $(S \ast H)_{ik} := \sum_{p,q=1}^{n} S_{ipkq}H_{pq}$.

Proof. Let us define an extended curvature tensor $T$ on $\mathbb{R}^{n+1}$ by

$$T_{ijkl} = S_{ijkl}, \quad T_{ijk0} = 0, \quad T_{i0k0} = H_{ik}.$$ 

Since $Z \geq 0$ for every orthonormal four-frame $\{e_1, e_2, e_3, e_4\}$ and every $\lambda \in [0, 1]$, it follows that $T \in \text{PIC}$. Moreover, if $Z = 0$ for one orthonormal four-frame $\{e_1, e_2, e_3, e_4\}$ and one particular $\lambda \in [0, 1]$, then $\{e_1, e_2, e_3, \lambda e_4 + \sqrt{1 - \lambda^2}e_0\}$ is an orthonormal four-frame in $\mathbb{R}^{n+1}$ which has zero isotropic
curvature for $T$. Using Proposition 7.4 in [5], we obtain

\[
0 \leq \sum_{p,q=0}^{n} (T_{1p1q} + T_{2p2q}) \left( T_{3p3q} + \lambda^2 T_{4p4q} + 2\lambda \sqrt{1 - \lambda^2 T_{0p4q}} + (1 - \lambda^2) T_{0p0q} \right) \\
- \sum_{p,q=0}^{n} T_{12pq} \left( \lambda T_{34pq} + \sqrt{1 - \lambda^2 T_{30pq}} \right) \\
- \sum_{p,q=0}^{n} (T_{1p3q} + \lambda T_{2p4q} + \sqrt{1 - \lambda^2 T_{2p0q}}) \left( T_{3p1q} + \lambda T_{4p2q} + \sqrt{1 - \lambda^2 T_{0p2q}} \right) \\
- \sum_{p,q=0}^{n} (\lambda T_{1p4q} + \sqrt{1 - \lambda^2 T_{1p0q}} - T_{2p3q}) \left( \lambda T_{4p1q} + \sqrt{1 - \lambda^2 T_{0p1q}} - T_{3p2q} \right) \\
= \sum_{p,q=0}^{n} (T_{1p1q} + T_{2p2q}) \left( T_{3p3q} + \lambda^2 T_{4p4q} + (1 - \lambda^2) T_{0p0q} \right) \\
- \sum_{p,q=0}^{n} T_{12pq} \lambda T_{34pq} \\
- \sum_{p,q=0}^{n} (T_{1p3q} + \lambda T_{2p4q}) \left( T_{3p1q} + \lambda T_{4p2q} \right) \\
- \sum_{p,q=0}^{n} (\lambda T_{1p4q} - T_{2p3q}) \left( \lambda T_{4p1q} - T_{3p2q} \right).
\]
This implies

\[ 0 \leq \sum_{p,q=1}^{n} (T_{1p1q} + T_{2p2q}) (T_{3p3q} + \lambda^2 T_{4p4q} + (1 - \lambda^2) T_{0p0q}) \]

\[ - \sum_{p,q=1}^{n} T_{12pq} \lambda T_{34pq} \]

\[ - \sum_{p,q=1}^{n} (T_{1p3q} + \lambda T_{2p4q}) (T_{3p1q} + \lambda T_{4p2q}) \]

\[ - \sum_{p,q=1}^{n} (\lambda T_{1p4q} - T_{2p3q}) (\lambda T_{3p1q} - T_{3p2q}) \]

\[ + (H_{11} + H_{22})(H_{33} + \lambda^2 H_{44}) \]

\[ - (H_{13} + \lambda H_{24})^2 - (\lambda H_{14} - H_{23})^2 \]

\[ = \sum_{p,q=1}^{n} (S_{1p1q} + S_{2p2q}) (S_{3p3q} + \lambda^2 S_{4p4q} + (1 - \lambda^2) H_{pq}) \]

\[ - \sum_{p,q=1}^{n} S_{12pq} \lambda S_{34pq} \]

\[ - \sum_{p,q=1}^{n} (S_{1p3q} + \lambda S_{2p4q}) (S_{3p1q} + \lambda S_{4p2q}) \]

\[ - \sum_{p,q=1}^{n} (\lambda S_{1p4q} - S_{2p3q}) (\lambda S_{4p1q} - S_{3p2q}) \]

\[ + (H_{11} + H_{22})(H_{33} + \lambda^2 H_{44}) \]

\[ - (H_{13} + \lambda H_{24})^2 - (\lambda H_{14} - H_{23})^2 \]
Consequently,

$$Q(S)_{1313} + \lambda^2 Q(S)_{1414} + Q(S)_{2323} + \lambda^2 Q(S)_{2424} - 2\lambda Q(S)_{1234}$$

$$= \sum_{p,q=1}^{n} (S_{13pq} - \lambda S_{24pq})^2 + \sum_{p,q=1}^{n} (\lambda S_{14pq} + S_{23pq})^2$$

$$+ S^\#(e_1, e_3, e_1, e_3) + \lambda^2 S^\#(e_1, e_4, e_1, e_4)$$

$$+ S^\#(e_2, e_3, e_2, e_3) + \lambda^2 S^\#(e_2, e_4, e_2, e_4)$$

$$+ 2\lambda S^\#(e_1, e_3, e_4, e_2) + 2\lambda S^\#(e_1, e_4, e_2, e_3)$$

$$\geq \sum_{p,q=1}^{n} (S_{13pq} - \lambda S_{24pq})^2 + \sum_{p,q=1}^{n} (\lambda S_{14pq} + S_{23pq})^2$$

$$- 2 (1 - \lambda^2) \sum_{p,q=1}^{n} (S_{1p1q} + S_{2p2q}) H_{pq}$$

$$- 2 (H_{11} + H_{22}) (H_{33} + \lambda^2 H_{44})$$

$$+ 2 (H_{13} + \lambda H_{24})^2 + 2 (\lambda H_{14} - H_{23})^2.$$

Note that

$$(S \ast H)_{11} + (S \ast H)_{22} = \sum_{p,q=1}^{n} (S_{1p1q} + S_{2p2q}) H_{pq}$$

and

$$(H \otimes H)_{1313} + \lambda^2 (H \otimes H)_{1414}$$

$$+ (H \otimes H)_{2323} + \lambda^2 (H \otimes H)_{2424}$$

$$- 2\lambda (H \otimes H)_{1234}$$

$$= 2 (H_{11} + H_{22}) (H_{33} + \lambda^2 H_{44})$$

$$- 2 (H_{13} + \lambda H_{24})^2 - 2 (\lambda H_{14} - H_{23})^2.$$

Moreover, since $Z = \frac{\partial Z}{\partial \lambda} = 0$, we obtain

$$\lambda S_{1414} + \lambda S_{2424} - S_{1234} = \lambda (H_{11} + H_{22})$$

and

$$S_{1313} + S_{2323} - \lambda S_{1234} = -(H_{11} + H_{22}).$$
This allows us to bound
\[(1 + \lambda^2)(H_{11} + H_{22})^2 \leq \left[ (\lambda S_{1414} + S_{1423}) + (\lambda S_{2424} - S_{1324}) \right]^2 \]
\[+ \left[ (S_{1313} - \lambda S_{1324}) + (S_{2323} + \lambda S_{1423}) \right]^2 \]
\[\leq 2(\lambda S_{1414} + S_{1423})^2 + 2(\lambda S_{2424} - S_{1324})^2 \]
\[+ 2(\lambda S_{1313} - \lambda S_{1324})^2 + 2(\lambda S_{2323} + \lambda S_{1423})^2 \]
\[\leq \sum_{p,q=1}^{n} (S_{13pq} - \lambda S_{24pq})^2 + \sum_{p,q=1}^{n} (\lambda S_{14pq} + S_{23pq})^2. \]

Putting these facts together, the assertion follows.

3. The initial pinching set

In this section, we describe a set of inequalities (which depends on time \(t\)) which is preserved under the Hamilton ODE. Throughout this section, we assume that \(n \geq 5\).

**Theorem 3.1.** Let \(K\) be a compact set of algebraic curvature tensors in dimension \(n\) which is contained in the interior of the PIC cone, and let \(T > 0\) be given. Then there exist a small positive real number \(\theta\), a large positive real number \(N\) and a continuous family of closed, convex, \(O(n)\)-invariant sets \(\{G_t^{(0)} : t \in [0, T]\}\), such that the family \(\{G_t^{(0)} : t \in [0, T]\}\) is invariant under the Hamilton ODE \(\frac{d}{dt} R = Q(R); K \subset G_t^{(0)}\); and

\[G_t^{(0)} \subset \{ R : R - \theta \text{scal} \otimes \text{id} \in \text{PIC} \} \cap \{ R : \text{Ric}_{11} + \text{Ric}_{22} - \theta \text{scal} + N \geq 0 \}\]

for all \(t \in [0, T]\).

Without loss of generality, we may assume that \(\text{Ric}_{11} + \text{Ric}_{22} \geq -1\) for all \(R \in K\). One ingredient in our construction is the family of maps \(\ell_{a,b}\) introduced in [1]. Following [1], we define

\[\ell_{a,b}(S) = S + b\text{Ric}(S) \otimes \text{id} + \frac{1}{n}(a - b)\text{scal}(S) \otimes \text{id}\]

for every algebraic curvature tensor \(S\). It is shown in [1] that

\[\ell_{a,b}^{-1}(Q(\ell_{a,b}(S))) = Q(S) + D_{a,b}(S),\]

where \(D_{a,b}(S)\) is defined by

\[D_{a,b}(S) = (2b + (n - 2)b^2 - 2a)^\circ\text{Ric}(S) \otimes \text{Ric}(S) \]
\[+ 2a\text{Ric}(S) \otimes \text{Ric}(S) + 2b^2 \text{Ric}(S)^2 \otimes \text{id} \]
\[+ \frac{nb^2(1 - 2b) - 2(a - b)(1 - 2b + nb^2)}{n(1 + 2(n - 1)a)} |\text{Ric}(S)|^2 \otimes \text{id}.\]
In particular, if $R$ evolves by the Hamilton ODE $\frac{d}{dt} R = Q(R)$, then $S = \ell_{a,b}^{-1}(R)$ evolves by the ODE $\frac{d}{dt} S = Q(S) + D_{a,b}(S)$. We now give the definition of the sets $G_t^{(0)}$. The definition depends on a parameter $\delta > 0$, which we choose small enough.

**Definition 3.2.** For $\delta > 0$ small, we denote by $G_t^{(0)}$ the set of all algebraic curvature tensors $R$ satisfying the following conditions:

(i) If $0 \leq b \leq \delta e^{-8t}$, $2a = 2b + (n-2)b^2$, and $S = \ell_{a,b}^{-1}(R)$, then $S \in \text{PIC}$.

(ii) If $0 \leq b \leq \delta e^{-8t}$, $2a = 2b + (n-2)b^2$, and $S = \ell_{a,b}^{-1}(R)$, then $\text{Ric}(S)_{11} + \text{Ric}(S)_{22} + 2b^2 \text{scal}(S) \geq -2$.

(iii) For every orthonormal four-frame $\{e_1, e_2, e_3, e_4\}$, the inequality $R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \geq 32\delta$.

(iv) For every orthonormal frame $\{e_1, \ldots, e_n\}$, the inequality

$$\text{Ric}(S)_{11} + \text{Ric}(S)_{22} + 2b^2 \text{scal}(S) \geq -2,$$

holds.

Clearly, $G_t^{(0)}$ is convex for each $t$. Moreover, $\mathcal{K} \subset G_t^{(0)}$ if $\delta > 0$ is sufficiently small. We claim that the family $\{G_t^{(0)} : t \in [0, T]\}$ is invariant under the Hamilton ODE $\frac{d}{dt} R = Q(R)$ if $\delta > 0$ is sufficiently small.

We begin with two technical lemmata:

**Lemma 3.3.** Assume that $R \in G_t^{(0)}$, $0 < b \leq \delta e^{-8t}$, and $2a = 2b + (n-2)^2$. Let $S = \ell_{a,b}^{-1}(R) \in \text{PIC}$, and let

$$U = \frac{1}{1 + (n-2)b} \text{Ric}(S) \otimes \text{id} - \frac{1}{n} \left( \frac{1}{1 + (n-2)b} - \frac{1 + (n-2)b}{1 + 2(n-1)a} \right) \text{scal}(S) \otimes \text{id}.$$

Then $U \in T_S \text{PIC}$.

**Proof.** By assumption,

$$\ell_{b+(n-2)b^2/2,b}^{-1}(R) \in \text{PIC}$$

for all $0 < b \leq \delta e^{-8t}$. Differentiating with respect to $b$ gives

$$U = -\frac{d}{db} \ell_{b+(n-2)b^2/2,b}^{-1}(R) \in T_{\ell_{b+(n-2)b^2/2,b}^{-1}(R)} \text{PIC}$$

for all $0 < b \leq \delta e^{-8t}$. 
Lemma 3.4. Assume that $R \in G_t(0)$, $0 < b \leq \delta e^{-\delta t}$, and $2a = 2b + (n-2)^2$. Let $S = \ell^{-1}_{a,b}(R) \in \text{PIC}$, and let

$$U = \frac{1}{1 + (n-2)b} \text{Ric}(S) \otimes \text{id} - \frac{1}{n} \left( \frac{1}{1 + (n-2)b} - \frac{1}{1 + 2(n-1)a} \right) \text{scal}(S) \text{id} \otimes \text{id}.$$  

Then

$$8bU - 4a \text{Ric}(S) \otimes \text{id} - 4a \text{id} \otimes \text{id} + b^{17} \text{scal}(S)^2 \text{id} \otimes \text{id} \in T_S \text{PIC}$$

if $\delta > 0$ is sufficiently small.

Proof. Suppose that $\{e_1, e_2, e_3, e_4\}$ is an orthonormal four-frame satisfying

$$S_{1313} + S_{1414} + S_{2323} + S_{2424} - 2S_{1234} = 0.$$  

We need to show that

$$b(U_{1313} + U_{1414} + U_{2323} + U_{2424} - 2U_{1234})$$

$$- a(Ric(S)_{11} + Ric(S)_{22} + Ric(S)_{33} + Ric(S)_{44}) - 4a + b^{17} \text{scal}(S)^2 \geq 0$$

for this particular orthonormal four-frame.

Case 1: Suppose first that $Ric(S)_{11} + Ric(S)_{22} + Ric(S)_{33} + Ric(S)_{44} \leq 8$. Using condition (iii), we obtain

$$32b \leq R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234}$$

$$- S_{1313} - S_{1414} - S_{2323} - S_{2424} + 2S_{1234}$$

$$= 2b(Ric(S)_{11} + Ric(S)_{22} + Ric(S)_{33} + Ric(S)_{44}) + \frac{4(n-2)b^2}{n} \text{scal}(S)$$

$$\leq 16b + \frac{4(n-2)b^2}{n} \text{scal}(S),$$

hence $b \text{scal}(S) \geq \frac{4n}{n-2}$. Since $U \in T_S \text{PIC}$ by Lemma 3.3, we conclude that

$$b(U_{1313} + U_{1414} + U_{2323} + U_{2424} - 2U_{1234})$$

$$- a(Ric(S)_{11} + Ric(S)_{22} + Ric(S)_{33} + Ric(S)_{44}) - 4a + b^{17} \text{scal}(S)^2$$

$$\geq -a(Ric(S)_{11} + Ric(S)_{22} + Ric(S)_{33} + Ric(S)_{44}) - 4a + b^{17} \text{scal}(S)^2$$

$$\geq -12a + \frac{16n^2}{(n-2)^2} b^\frac{1}{8}$$

$$> 0$$

if $b > 0$ is sufficiently small.
Case 2: Suppose finally that $\text{Ric}(S)_{11} + \text{Ric}(S)_{22} + \text{Ric}(S)_{33} + \text{Ric}(S)_{44} \geq 8$. By definition of $U$, we obtain

$$b \left( U_{1313} + U_{1414} + U_{2323} + U_{2424} - 2U_{1234} \right) - a \left( \text{Ric}(S)_{11} + \text{Ric}(S)_{22} + \text{Ric}(S)_{33} + \text{Ric}(S)_{44} \right) - 4a + b \frac{17}{8} \text{scal}^2(S)$$

$$= \left( \frac{2b}{1 + (n - 2)b} - a \right) \left( \text{Ric}(S)_{11} + \text{Ric}(S)_{22} + \text{Ric}(S)_{33} + \text{Ric}(S)_{44} \right)$$

$$\geq 8 \left( \frac{2b}{1 + (n - 2)b} - a \right) - 4a + b \frac{17}{8} \text{scal}^2(S)$$

$$\geq b + b \frac{17}{8} \text{scal}^2(S)^2 - C(n) b^2 \text{scal}(S)$$

$$> 0$$

if $b > 0$ is sufficiently small.

**Lemma 3.5.** Let $R(t)$ be a solution of the Hamilton ODE, and let $S(t) = \ell^{-1}_{a(t),b(t)}(R(t))$, where $0 < b(t) \leq \delta e^{-8t}$, $2a(t) = 2b(t) + (n - 2)b(t)^2$, and $b'(t) = -8b(t)$. If $\delta > 0$ is sufficiently small, then the condition $S \in \text{PIC}$ is preserved.

**Proof.** The evolution of $S$ is given by

$$\frac{d}{dt} S = Q(S) + D_{a,b}(S) - b' U = Q(S) + D_{a,b}(S) + 8b U,$$

where

$$U = \frac{1}{1 + (n - 2)b} \text{Ric}(S) \otimes \text{id}$$

$$- \frac{1}{n} \left( \frac{1}{1 + (n - 2)b} - \frac{1 + (n - 2)b}{1 + 2(n - 1)a} \right) \text{scal}(S) \text{id} \otimes \text{id}.$$ 

Hence, it suffices to show that $Q(S) + D_{a,b}(S) + 8b U \in T_S \text{PIC}$. Since $Q(S) \in T_S \text{PIC}$ by Proposition 7.5 in [5], it is enough to show that $D_{a,b}(S) + 8b U \in T_S \text{PIC}$. To verify this, we distinguish two cases:

**Case 1:** Suppose first that $\text{Ric}(S)$ is two-positive. In this case, $D_{a,b}(S) \in \text{PIC}$. Since $U \in T_S \text{PIC}$ by Lemma 3.3, it follows that $D_{a,b}(S) + 8b U \in T_S \text{PIC}$, as claimed.

**Case 2:** Suppose next that $\text{Ric}(S)$ is not two-positive. In this case, $\text{scal}(S) \leq C(n) |\text{Ric}(S)|$. Since $2a = 2b + (n - 2)b^2$, it follows that

$$D_{a,b}(S) - 2a \text{Ric}(S) \otimes \text{Ric}(S) - 2b \frac{17}{8} \text{scal}(S)^2 \text{id} \otimes \text{id} \in \text{PIC}.$$
if $b > 0$ is sufficiently small. The condition (ii) implies that $\text{Ric}(S) + (1 + b\frac{8}{5} \text{scal}(S)) \text{id}$ is weakly two-positive. Consequently,

$$2a \text{Ric}(S) \otimes \text{Ric}(S) + 4a(1 + b\frac{8}{5} \text{scal}(S)) \text{Ric}(S) \otimes \text{id} + 2a(1 + b\frac{8}{5} \text{scal}(S))^2 \text{id} \otimes \text{id}$$

$$= 2a [\text{Ric}(S) + (1 + b\frac{8}{5} \text{scal}(S)) \text{id}] \otimes [\text{Ric}(S) + (1 + b\frac{8}{5} \text{scal}(S)) \text{id}] \in \text{PIC}.$$ Using Lemma 3.3 we easily obtain

$$8bU - 4a(1 + b\frac{8}{5} \text{scal}(S)) \text{Ric}(S) \otimes \text{id} - 2a(1 + b\frac{8}{5} \text{scal}(S))^2 \text{id} \otimes \text{id} + 2b\frac{12}{5} \text{scal}(S)^2 \text{id} \otimes \text{id} \in T_5 \text{PIC}$$

if $b > 0$ is sufficiently small. Adding these formulas, we conclude that $D_{a,b}(S) + 8bU \in T_5 \text{PIC}$. This completes the proof.

**Lemma 3.6.** Let $R(t)$ be a solution of the Hamilton ODE, and let $S(t) = \ell_{a,b}^{-1}(R(t))$, where $0 < b \leq \delta e^{-st}$, $2a = 2b + (n - 2)b^2$, and $b$ does not depend on $t$. If $\delta > 0$ is sufficiently small, then the condition $\text{Ric}(S)_{11} + \text{Ric}(S)_{22} + 2b\frac{8}{5} \text{scal}(S) \geq -2$ is preserved.

**Proof.** The evolution of $S$ is given by

$$\frac{d}{dt}S = Q(S) + D_{a,b}(S).$$

Suppose that $\text{Ric}(S)_{11} + \text{Ric}(S)_{22} + 2b\frac{8}{5} \text{scal}(S) = -2$ and

$$\frac{d}{dt} \left( \text{Ric}(S)_{11} + \text{Ric}(S)_{22} + 2b\frac{8}{5} \text{scal}(S) \right) \leq 0.$$ We may assume that $\text{Ric}(S)$ is diagonal. We compute

$$\frac{1}{2} \frac{d}{dt} \left( \text{Ric}(S)_{11} + \text{Ric}(S)_{22} + 2b\frac{8}{5} \text{scal}(S) \right)$$

$$\geq \sum_{p=1}^{n} (S_{1p1p} + S_{2p2p}) \text{Ric}(S)_{pp}$$

$$+ 2b\frac{8}{5} |\text{Ric}(S)|^2 - 2b (\delta \text{Ric}(S)_{11} + \delta \text{Ric}(S)_{22}) - 4b\frac{9}{5} |\text{Ric}(S)|^2$$

$$\geq \sum_{p=3}^{n} (S_{1p1p} + S_{2p2p}) \text{Ric}(S)_{pp} - \frac{1}{2} (\text{Ric}(S)_{11} + \text{Ric}(S)_{22})$$

$$+ 2b\frac{8}{5} |\text{Ric}(S)|^2 - b |\text{Ric}(S)_{11} - \text{Ric}(S)_{22}|^2 - 4b\frac{9}{5} |\text{Ric}(S)|^2.$$ We distinguish two cases:
Case 1: Suppose first that \( S_{1p1p} + S_{2p2p} \geq 0 \) for all \( p \in \{3, \ldots, n\} \). In this case,

\[
\sum_{p=3}^{n} (S_{1p1p} + S_{2p2p})(\text{Ric}(S)_{pp} - \frac{1}{2}(\text{Ric}(S)_{11} + \text{Ric}(S)_{22}))
\geq \frac{1}{2} \sum_{p=3}^{n} (S_{1p1p} + S_{2p2p})|\text{Ric}(S)_{11} - \text{Ric}(S)_{22}|,
\]

hence

\[
\frac{1}{2} \frac{d}{dt}(\text{Ric}(S)_{11} + \text{Ric}(S)_{22} + 2b^\frac{4}{7} \text{scal}(S))
\geq \frac{1}{2} \sum_{p=3}^{n} (S_{1p1p} + S_{2p2p})|\text{Ric}(S)_{11} - \text{Ric}(S)_{22}|
+ 2b^\frac{4}{7} |\text{Ric}(S)|^2 - b |\text{Ric}(S)_{11} - \text{Ric}(S)_{22}|^2 - 4b^\frac{4}{7} |\text{Ric}(S)|^2.
\]

If \( |\text{Ric}(S)_{11} - \text{Ric}(S)_{22}| \leq b^\frac{4}{7} |\text{Ric}(S)| \) or \( \sum_{p=3}^{n} (S_{1p1p} + S_{2p2p}) \geq 2b |\text{Ric}(S)_{11} - \text{Ric}(S)_{22}| \), then the right hand side is positive, contrary to our assumption. Therefore, we must have \( |\text{Ric}(S)_{11} - \text{Ric}(S)_{22}| \geq b^\frac{4}{7} |\text{Ric}(S)| \) and \( \sum_{p=3}^{n} (S_{1p1p} + S_{2p2p}) \leq 2b |\text{Ric}(S)_{11} - \text{Ric}(S)_{22}| \). This implies \( |\text{Ric}(11) - \text{Ric}(22)| \geq c(n) b^\frac{4}{7} \text{scal} \) and \( \sum_{p=3}^{n} (R_{1p1p} + R_{2p2p}) \leq C(n) b \text{scal} \). On the other hand, using condition (iv) we obtain

\[
\sum_{p=3}^{n} (R_{1p1p} + R_{2p2p}) \geq c(n) b^\frac{4}{7} \text{scal}^{-3} \left( \sum_{p=3}^{n} (|R_{1p1p}| + |R_{2p2p}|) \right)^4
\geq c(n) b^\frac{4}{7} \text{scal}^{-3} |\text{Ric}_{11} - \text{Ric}_{22}|^4
\geq c(n) b^\frac{4}{7} \text{scal}.
\]

This leads to a contradiction if \( b > 0 \) is sufficiently small.

Case 2: Suppose next that \( S_{1m1m} + S_{2m2m} < 0 \) for some \( m \in \{3, \ldots, n\} \). Since \( S \in \text{PIC} \), Lemma \( \text{[2.1]} \) gives

\[
\sum_{p \in \{1, \ldots, n\} \setminus \{m\}} \text{Ric}(S)_{pp} \geq \text{Ric}(S)_{mm},
\]

hence

\[
\sum_{p \in \{3, \ldots, n\} \setminus \{m\}} (\text{Ric}(S)_{pp} - \frac{1}{2}(\text{Ric}(S)_{11} + \text{Ric}(S)_{22}))
\geq \text{Ric}(S)_{mm} - \frac{1}{2}(\text{Ric}(S)_{11} + \text{Ric}(S)_{22}).
\]
This implies
\[ \sum_{p=3}^{n} (S_{1p1p} + S_{2p2p})(\text{Ric}(S)_{pp} - \frac{1}{2}(\text{Ric}(S)_{11} + \text{Ric}(S)_{22})) \]
\[ \geq \sum_{p \in \{3,...,n\}\setminus\{m\}} (S_{1m1m} + S_{2m2m} + S_{1p1p} + S_{2p2p}) \cdot (\text{Ric}(S)_{pp} - \frac{1}{2}(\text{Ric}(S)_{11} + \text{Ric}(S)_{22})) \]
\[ \geq \frac{1}{2} \sum_{p \in \{3,...,n\}\setminus\{m\}} (S_{1m1m} + S_{2m2m} + S_{1p1p} + S_{2p2p}) |\text{Ric}(S)_{11} - \text{Ric}(S)_{22}|, \]

hence
\[ \frac{1}{2} \frac{d}{dt} (\text{Ric}(S)_{11} + \text{Ric}(S)_{22} + 2b^\frac{5}{2} \text{scal}(S)) \geq \frac{1}{2} \sum_{p \in \{3,...,n\}\setminus\{m\}} (S_{1m1m} + S_{2m2m} + S_{1p1p} + S_{2p2p}) |\text{Ric}(S)_{11} - \text{Ric}(S)_{22}| \]
\[ + 2b^\frac{5}{2} |\text{Ric}(S)|^2 - b |\text{Ric}(S)_{11} - \text{Ric}(S)_{22}|^2 - 4b^\frac{9}{2} |\text{Ric}(S)|^2. \]

If $|\text{Ric}(S)_{11} - \text{Ric}(S)_{22}| \leq b^\frac{9}{2} |\text{Ric}(S)|$ or $\sum_{p \in \{3,...,n\}\setminus\{m\}} (S_{1m1m} + S_{2m2m} + S_{1p1p} + S_{2p2p}) \geq 2b |\text{Ric}(S)_{11} - \text{Ric}(S)_{22}|$, then the right hand side is positive, contrary to our assumption. Therefore, we must have $|\text{Ric}(S)_{11} - \text{Ric}(S)_{22}| \geq b^\frac{9}{2} |\text{Ric}(S)|$ and $\sum_{p \in \{3,...,n\}\setminus\{m\}} (S_{1m1m} + S_{2m2m} + S_{1p1p} + S_{2p2p}) \leq 2b |\text{Ric}(S)_{11} - \text{Ric}(S)_{22}|$. This implies $|\text{Ric}_{11} - \text{Ric}_{22}| \geq c(n) b^\frac{9}{2} \text{scal}$ and $\sum_{p \in \{3,...,n\}\setminus\{m\}} (R_{1m1m} + R_{2m2m} + R_{1p1p} + R_{2p2p}) \leq C(n)b \text{scal}$. On the other hand, using condition (iv) we obtain
\[ \sum_{p \in \{3,...,n\}\setminus\{m\}} (R_{1m1m} + R_{2m2m} + R_{1p1p} + R_{2p2p}) \]
\[ \geq c(n) b^\frac{9}{2} \text{scal}^{-3} \left( \sum_{p \in \{3,...,n\}\setminus\{m\}} (|R_{1m1m}| + |R_{2m2m}| + |R_{1p1p}| + |R_{2p2p}|) \right)^4 \]
\[ \geq c(n) b^\frac{9}{2} \text{scal}^{-3} |\text{Ric}_{11} - \text{Ric}_{22}|^4 \]
\[ \geq c(n) b^\frac{9}{2} \text{scal}. \]

This again leads to a contradiction if $b > 0$ is sufficiently small.

**Lemma 3.7.** The inequality $R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \geq 32\delta$ is preserved.

**Proof.** This follows immediately from Proposition 2.4.
**Lemma 3.8.** If $\delta > 0$ is sufficiently small, then the condition

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \geq \delta^{\frac{1}{4}} \text{scal}^{-3} \left( \sum_{p,q=1}^{n} (R_{13pq} - R_{24pq})^2 + \sum_{p,q=1}^{n} (R_{14pq} + R_{23pq})^2 \right)^2$$

is preserved.

**Proof.** Consider the borderline situation when

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} = \delta^{\frac{1}{4}} \text{scal}^{-3} \left( \sum_{p,q=1}^{n} (R_{13pq} - R_{24pq})^2 + \sum_{p,q=1}^{n} (R_{14pq} + R_{23pq})^2 \right)^2.$$

The proof that PIC is preserved leads to the following inequality:

$$R^{\#}_{1313} + R^{\#}_{1414} + R^{\#}_{2323} + R^{\#}_{2424} + 2R^{\#}_{1342} + 2R^{\#}_{1423} \geq -C(n) \delta^{\frac{1}{4}} \left( \sum_{p,q=1}^{n} (R_{13pq} - R_{24pq})^2 + \sum_{p,q=1}^{n} (R_{14pq} + R_{23pq})^2 \right).$$

This implies

$$\frac{d}{dt} \left( R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \right) \geq (1 - C(n) \delta^{\frac{1}{4}}) \left( \sum_{p,q=1}^{n} (R_{13pq} - R_{24pq})^2 + \sum_{p,q=1}^{n} (R_{14pq} + R_{23pq})^2 \right).$$

On the other hand,

$$\frac{d}{dt} \left[ \text{scal}^{-3} \left( \sum_{p,q=1}^{n} (R_{13pq} - R_{24pq})^2 + \sum_{p,q=1}^{n} (R_{14pq} + R_{23pq})^2 \right)^2 \right] \leq C(n) \left( \sum_{p,q=1}^{n} (R_{13pq} - R_{24pq})^2 + \sum_{p,q=1}^{n} (R_{14pq} + R_{23pq})^2 \right).$$

Thus, we conclude that

$$\frac{d}{dt} \left[ R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \right] - \delta^{\frac{1}{4}} \text{scal}^{-3} \left( \sum_{p,q=1}^{n} (R_{13pq} - R_{24pq})^2 + \sum_{p,q=1}^{n} (R_{14pq} + R_{23pq})^2 \right)^2 \geq (1 - C(n) \delta^{\frac{1}{4}}) \left( \sum_{p,q=1}^{n} (R_{13pq} - R_{24pq})^2 + \sum_{p,q=1}^{n} (R_{14pq} + R_{23pq})^2 \right),$$

and the right hand side is nonnegative if $\delta > 0$ is sufficiently small.
4. A ONE-PARAMETER FAMILY OF INVARIANT CONES $\mathcal{C}(b)$

In this section, we construct a one-parameter family of cones that are preserved under the Hamilton ODE $\frac{d}{dt} R = Q(R)$. Throughout this section, we assume that $n \geq 5$. Moreover, we make the following assumption:

**Assumption 4.1.** Let $b_{\text{max}}$ be a positive real number. To each $0 < b \leq b_{\text{max}}$, we associate real numbers $a$, $\gamma$, and $\omega$ by

\[
\begin{align*}
a &= \frac{(2 + (n - 2)b)^2}{2(2 + (n - 3)b)}b, \\
\gamma &= \frac{b}{2 + (n - 3)b}, \\
\omega^2 &= \frac{27(2 + (n - 2)b)(1 - 4(n - 2)b^2)^{\frac{3}{2}}(1 + (n - 2)b)^2}{8n^2b^2(2 + (n - 3)b)^2}.
\end{align*}
\]

We assume that

\[
2 + (n - 8)b - 2(n + 2)(n - 2)b^2 - n(n - 2)^2b^3 > 0
\]

and

\[
\left(\frac{n - 2}{n - 3} + \frac{2(n - 2)}{n}\right)\left(2b + (n - 2)a + 8(n - 3)b^2\right)\left(\frac{n - 2}{n - 3} + 2(2b + (n - 2)a)\right) < \frac{\omega}{2}
\]

for $0 < b \leq b_{\text{max}}$.

**Definition 4.2.** For $0 < b \leq b_{\text{max}}$ as above, let $\mathcal{E}(b)$ denote the set of all curvature tensors $S$ for which there exists a curvature tensor $T$ satisfying the following conditions:

(i) $T \geq 0$.

(ii) $S - T \in \text{PIC}$.

(iii) $\text{Ric}(S)_{11} + \text{Ric}(S)_{22} + \frac{2}{n} \text{scal}(S) \geq 0$.

(iv) For every orthonormal frame $\{e_1, \ldots, e_n\}$, the inequality

\[
\text{Ric}(S)_{22} - \text{Ric}(S)_{11} \leq \omega^\frac{1}{2} \text{scal}(S)^{\frac{1}{2}} \left(\sum_{p=3}^{n}(T_{1p1p} + T_{2p2p})\right)^{\frac{1}{2}}
\]

\[
+ 2(n - 2)b \sum_{p=3}^{n}(T_{1p1p} + T_{2p2p})
\]

holds.

Note that we do not require $T$ to satisfy the first Bianchi identity. Finally, we define $\mathcal{C}(b) = \ell_{a,b}(\mathcal{E}(b))$.

Clearly, $\mathcal{C}(b)$ is convex for each $b$.

**Theorem 4.3.** Suppose that Assumption 4.1 is satisfied. Then, for each $0 < b \leq b_{\text{max}}$, the cone $\mathcal{C}(b)$ is transversally invariant under the Hamilton ODE $\frac{d}{dt} R = Q(R)$ away from 0.
It suffices to show that $E(b)$ is transversally invariant under the ODE $\frac{d}{dt} S = Q(S) + D_{a,b}(S)$ for $0 < b \leq b_{\text{max}}$. We begin with an elementary lemma:

**Lemma 4.4.** We have

$$(2b + (n - 2)b^2 - 2a)xy + 2a(x + 2)(y + 2) + b^2(x^2 + y^2) \geq 0$$

whenever $x, y \geq -\frac{2(2+(n-2)b)}{2+(n-3)b}$.

**Proof.** Let $D = \{(x, y) \in \mathbb{R}^2 : x, y \geq -\frac{2(2+(n-2)b)}{2+(n-3)b}\}$, and define $\psi : D \to \mathbb{R}$ by $\psi(x, y) = (2b + (n - 2)b^2 - 2a)xy + 2a(x + 2)(y + 2) + b^2(x^2 + y^2)$. Clearly, $\psi(x, y) \to \infty$ as $(x, y) \in D$ approaches infinity. Hence, there exists a point in $D$ where $\psi$ attains its minimum. The Hessian of $\psi$ is given by

$$\begin{bmatrix}
2b^2 & (2b + (n - 2)b^2) \\
(2b + (n - 2)b^2) & 2b^2
\end{bmatrix}.$$ 

Since the Hessian of $\psi$ has two eigenvalues of opposite signs, the function $\psi$ attains its minimum on the boundary of $D$. On the other hand, a straightforward calculation gives

$$\psi\left(-\frac{2(2+(n-2)b)}{2+(n-3)b}, y\right) = b^2y^2 \geq 0.$$ 

Thus, $\psi \geq 0$ on $\partial D$. This implies $\psi \geq 0$ on $D$.

**Lemma 4.5.** Suppose that $S \in E(b)$. Then $(2b + (n - 2)b^2 - 2a) \overset{\circ}{\text{Ric}}(S) \otimes \overset{\circ}{\text{Ric}}(S) + 2a \overset{\circ}{\text{Ric}}(S) \otimes \overset{\circ}{\text{Ric}}(S) + 2b^2 \overset{\circ}{\text{Ric}}(S)^2 \otimes \text{id} \in \text{PIC}.$

**Proof.** Let

$$U := (2b + (n - 2)b^2 - 2a) \overset{\circ}{\text{Ric}}(S) \otimes \overset{\circ}{\text{Ric}}(S) + 2a \overset{\circ}{\text{Ric}}(S) \otimes \overset{\circ}{\text{Ric}}(S) + 2b^2 \overset{\circ}{\text{Ric}}(S)^2 \otimes \text{id}.$$ 

Moreover, let $\varphi$ be a complex two-form of the form $\varphi = \zeta \wedge \eta$ such that $g(\zeta, \zeta) = g(\zeta, \eta) = g(\eta, \eta) = 0$. We claim that $U(\varphi, \varphi) \geq 0$. We can find vectors $z, w \in \text{span}(\zeta, \eta)$ such that $g(z, z) = g(w, w) = 2$, $g(z, w) = 0$, and $\text{Ric}(S)(z, w) = 0$. The identities $g(\zeta, \zeta) = g(\zeta, \eta) = g(\eta, \eta) = 0$ give $g(z, z) = g(z, w) = g(w, w) = 0$. Consequently, we may write $z = e_1 + ie_2$ and $w = e_3 + ie_4$ for some orthonormal four-frame $\{e_1, e_2, e_3, e_4\}$. Using the
identity \( \text{Ric}(S)(z, \bar{w}) = 0 \), we obtain

\[
U_{1313} + U_{1414} + U_{2323} + U_{2424} - 2U_{1234} = 2(2b + (n - 2)b^2 - 2a) (\text{Ric}(S))_{11} + \text{Ric}(S)_{22}) (\text{Ric}(S))_{33} + \text{Ric}(S)_{44}) + 4a (\text{Ric}(S))_{11} + \text{Ric}(S)_{22}) (\text{Ric}(S))_{33} + \text{Ric}(S)_{44}) + 4b^2 ((\text{Ric}(S))^2)_{11} + (\text{Ric}(S))^2)_{22} + (\text{Ric}(S))^2)_{33} + (\text{Ric}(S))^2)_{44}) \geq 2(2b + (n - 2)b^2 - 2a) (\text{Ric}(S))_{11} + \text{Ric}(S)_{22}) (\text{Ric}(S))_{33} + \text{Ric}(S)_{44}) + 4a (\text{Ric}(S))_{11} + \text{Ric}(S)_{22}) (\text{Ric}(S))_{33} + \text{Ric}(S)_{44}) + 2b^2 ((\text{Ric}(S))_{11} + \text{Ric}(S))_{22})^2 + (\text{Ric}(S))_{33} + \text{Ric}(S))_{44})^2) \]

\[
= \frac{2\text{scal}(S)^2}{n^2} [(2b + (n - 2)b^2 - 2a)xy + 2a(x + 2y + 2) + b^2(x^2 + y^2)],
\]

where \( x \) and \( y \) are defined by \( x := \frac{n}{\text{scal}(S)} (\text{Ric}(S))_{11} + \text{Ric}(S))_{22}) \) and \( y := \frac{n}{\text{scal}(S)} (\text{Ric}(S))_{33} + \text{Ric}(S))_{44}) \). The condition (iii) implies \( x, y \geq -2\gamma - 2 = -\frac{2(2b + (n - 2)b^2)}{2 + (n - 3)b} \). Using Lemma 4.4, we obtain

\[
U_{1313} + U_{1414} + U_{2323} + U_{2424} - 2U_{1234} \geq 0,
\]

hence \( U(\varphi, \bar{\varphi}) \geq 0 \). Thus, \( U \in \text{PIC} \), as claimed.

After these preparations, we now show that the conditions (i) and (ii) are preserved. To that end, we will evolve \( T \) by the ODE

\[
\frac{d}{dt} T = S^2 + T^\# + \frac{nb^2(1 - 2b) - 2(a - b)(1 - 2b + nb^2)}{n(1 + 2(n - 1)a)} |\text{Ric}(S)|^2 \text{id} \otimes \text{id}.
\]

Note that

\[
\frac{nb^2(1 - 2b) - 2(a - b)(1 - 2b + nb^2)}{2 + (n - 3)b} (2 + (n - 8)b - 2(n + 2)(n - 2)b^2 - n(n - 2)^2b^3) > 0
\]

for \( 0 < b \leq b_{\text{max}} \).

**Lemma 4.6.** The condition \( T \geq 0 \) is preserved.

**Proof.** This follows directly from the evolution equation of \( T \).

**Lemma 4.7.** The condition \( S - T \in \text{PIC} \) is preserved.
**Proof.** We compute

\[
\frac{d}{dt}(S - T) = (S - T) + 2(S - T)T \\
+ (2b + (n - 2)b^2 - 2a) \mathring{\o}(S) \otimes \mathring{\o}(S) \\
+ 2a \mathring{\o}(S) \otimes \mathring{\o}(S) + 2b^2 \mathring{\o}(S)^2 \otimes \text{id}.
\]

By Proposition 2.5 and Lemma 4.5, we have \( \frac{d}{dt}(S - T) \in T_{S - T}\text{PIC} \). Therefore, the condition \( S - T \in \text{PIC} \) is preserved.

**Lemma 4.8.** The condition \( \text{Ric}(S)_{11} + \text{Ric}(S)_{22} + \frac{2\gamma}{n} \text{scal}(S) \geq 0 \) is preserved.

**Proof.** Suppose that \( \text{Ric}(S)_{11} + \text{Ric}(S)_{22} + \frac{2\gamma}{n} \text{scal}(S) = 0 \). We may assume that \( \text{Ric}(S) \) is diagonal. We compute

\[
\frac{1}{2} \frac{d}{dt}(\text{Ric}(S)_{11} + \text{Ric}(S)_{22} + \frac{2\gamma}{n} \text{scal}(S)) \\
= \sum_{p=1}^{n}(S_{1p1p} + S_{2p2p}) \text{Ric}(S)_{pp} \\
+ \frac{4(1 + \gamma)}{n^2} (a - b) \text{scal}(S)^2 + \frac{2\gamma}{n} (1 - 2b) |\text{Ric}(S)|^2 - 2b (\text{Ric}(S)_{11} + \text{Ric}(S)_{22}) \\
+ 2(1 + \gamma) \frac{n^2b^2 - 2(n - 1)(a - b)(1 - 2b)}{n(1 + 2(n - 1)a)} |\text{Ric}(S)|^2 \\
= \sum_{p=3}^{n}(S_{1p1p} + S_{2p2p})(\text{Ric}(S)_{pp} - \frac{1}{2} (\text{Ric}(S)_{11} + \text{Ric}(S)_{22})) \\
+ \frac{4(1 + \gamma)}{n^2} (a - b) \text{scal}(S)^2 + \frac{2\gamma(1 + \gamma)}{n^2} (1 - 2b) \text{scal}(S)^2 - b |\text{Ric}(S)_{11} - \text{Ric}(S)_{22}|^2 \\
+ \frac{2\gamma}{n} (1 - 2b) |\text{Ric}(S)|^2 + 2(1 + \gamma) \frac{n^2b^2 - 2(n - 1)(a - b)(1 - 2b)}{n(1 + 2(n - 1)a)} |\text{Ric}(S)|^2.
\]

Using the identity

\[
2(a - b) + \gamma(1 - 2b) = \frac{b(1 + (n - 2)b)^2}{2 + (n - 3)b}
\]

and the inequality

\[
n^2b^2 - 2(n - 1)(a - b)(1 - 2b) \\
= \frac{2b^2}{2 + (n - 3)b} ((2n - 1) + (3n - 2)(n - 2)b + (n - 1)(n - 2)^2b^2) > 0,
\]
we obtain
\[ \frac{1}{2} \frac{d}{dt} \left( \text{Ric}(S)_{11} + \text{Ric}(S)_{22} + \frac{2\gamma}{n} \text{scal}(S) \right) \]
> \sum_{p=3}^{n} \left( S_{1p1p} + S_{2p2p} \right) \left( \text{Ric}(S)_{pp} - \frac{1}{2} (\text{Ric}(S)_{11} + \text{Ric}(S)_{22}) \right)
+ \frac{2(1 + \gamma)}{n^2} \frac{b(1 + (n - 2)b)^2}{2 + (n - 3)b} \text{scal}(S)^2 - b |\text{Ric}(S)_{11} - \text{Ric}(S)_{22}|^2.

By Lemma 2.1,
\[ \sum_{p \in \{1, \ldots, n\} \backslash \{m\}} \text{Ric}(S)_{pp} \geq \text{Ric}(S)_{mm}, \]
hence
\[ \sum_{p \in \{3, \ldots, n\} \backslash \{m\}} \left( \text{Ric}(S)_{pp} - \frac{1}{2} \text{Ric}(S)_{11} + \text{Ric}(S)_{22} \right) \geq \text{Ric}(S)_{mm} - \frac{1}{2} \text{Ric}(S)_{11} + \text{Ric}(S)_{22} \]
for each \( m \in \{3, \ldots, n\} \). Since \( S - T \in \text{PIC} \), it follows that
\[ \sum_{p=3}^{n} \left( S_{1p1p} + S_{2p2p} - T_{1p1p} - T_{2p2p} \right) \left( \text{Ric}(S)_{pp} - \frac{1}{2} \text{Ric}(S)_{11} + \text{Ric}(S)_{22} \right) \geq 0. \]
This gives
\[ \sum_{p=3}^{n} \left( S_{1p1p} + S_{2p2p} \right) \left( \text{Ric}(S)_{pp} - \frac{1}{2} \text{Ric}(S)_{11} + \text{Ric}(S)_{22} \right) \]
\[ \geq \frac{1}{2} \sum_{p=3}^{n} \left( T_{1p1p} + T_{2p2p} \right) |\text{Ric}(S)_{11} - \text{Ric}(S)_{22}|. \]
Therefore, we obtain
\[ \frac{1}{2} \frac{d}{dt} \left( \text{Ric}(S)_{11} + \text{Ric}(S)_{22} + \frac{2\gamma}{n} \text{scal}(S) \right) \]
> \frac{1}{2} \sum_{p=3}^{n} \left( T_{1p1p} + T_{2p2p} \right) |\text{Ric}(S)_{11} - \text{Ric}(S)_{22}| \]
\[ + \frac{2(1 + \gamma)}{n^2} \frac{b(1 + (n - 2)b)^2}{2 + (n - 3)b} \text{scal}(S)^2 - b |\text{Ric}(S)_{11} - \text{Ric}(S)_{22}|^2. \]
At this point, we distinguish two cases:

Case 1: Suppose that \( \sum_{p=3}^{n} (T_{1p1p} + T_{2p2p}) \geq 2b |\text{Ric}(S)_{11} - \text{Ric}(S)_{22}|. \) In this case, we clearly have
\[ \frac{1}{2} \frac{d}{dt} \left( \text{Ric}(S)_{11} + \text{Ric}(S)_{22} + \frac{2\gamma}{n} \text{scal}(S) \right) > 0. \]
Case 2: Suppose next that \( \sum_{p=3}^{n} (T_{1p1p} + T_{2p2p}) \leq 2b |\text{Ric}(S)_{11} - \text{Ric}(S)_{22}| \).

In this case, the condition (iv) gives

\[
|\text{Ric}(S)_{11} - \text{Ric}(S)_{22}| \leq \omega \frac{1}{2} \text{scal}(S)^{\frac{1}{2}} \left( \sum_{p=3}^{n} (T_{1p1p} + T_{2p2p}) \right)^{\frac{1}{2}} \\
+ 2(n-2)b \sum_{p=3}^{n} (T_{1p1p} + T_{2p2p}) \\
\leq \omega \frac{1}{2} \text{scal}(S)^{\frac{1}{2}} \left( \sum_{p=3}^{n} (T_{1p1p} + T_{2p2p}) \right)^{\frac{1}{2}} \\
+ 4(n-2)b^2 |\text{Ric}(S)_{11} - \text{Ric}(S)_{22}|.
\]

Consequently,

\[
\sum_{p=3}^{n} (T_{1p1p} + T_{2p2p}) \geq \frac{(1 - 4(n-2)b^2)^{\frac{1}{2}}}{\omega} \frac{|\text{Ric}(S)_{11} - \text{Ric}(S)_{22}|}{\text{scal}(S)}.
\]

In view of our choice of \( \omega \), we have

\[
\frac{27(1 - 4(n-2)b^2)^4}{8\omega^2} \frac{1 + \gamma}{n^2} \frac{b(1 + (n-2)b)^2}{2 + (n-3)b} = b^3,
\]

hence

\[
\frac{(1 - 4(n-2)b^2)^{\frac{1}{2}}}{\omega} \frac{|\text{Ric}(S)_{11} - \text{Ric}(S)_{22}|}{\text{scal}(S)} \\
+ \frac{2(1 + \gamma)}{n^2} \frac{b(1 + (n-2)b)^2}{2 + (n-3)b} \frac{\text{scal}(S)^2}{|\text{Ric}(S)_{11} - \text{Ric}(S)_{22}|^2} \geq b.
\]

Thus, we conclude that

\[
\frac{1}{2} \frac{d}{dt} (|\text{Ric}(S)_{11} + \text{Ric}(S)_{22} + \frac{2\gamma}{n} \text{scal}(S)) \\
> \frac{(1 - 4(n-2)b^2)^{\frac{1}{2}}}{\omega} \frac{|\text{Ric}(S)_{11} - \text{Ric}(S)_{22}|^3}{\text{scal}(S)} \\
+ \frac{2(1 + \gamma)}{n^2} \frac{b(1 + (n-2)b)^2}{2 + (n-3)b} \frac{\text{scal}(S)^2}{|\text{Ric}(S)_{11} - \text{Ric}(S)_{22}|^2} - b|\text{Ric}(S)_{11} - \text{Ric}(S)_{22}|^2 \\
\geq 0.
\]

This proves the assertion.

**Lemma 4.9.** The condition

\[
\text{Ric}(S)_{22} - \text{Ric}(S)_{11} \leq \omega \frac{1}{2} \text{scal}(S)^{\frac{1}{2}} \left( \sum_{p=3}^{n} (T_{1p1p} + T_{2p2p}) \right)^{\frac{1}{2}} \\
+ 2(n-2)b \sum_{p=3}^{n} (T_{1p1p} + T_{2p2p}) \\
\]


is preserved.

**Proof.** The condition is equivalent to the inequality

$$\text{Ric}(S)_{22} - \text{Ric}(S)_{11} \leq (\sigma \omega + 2(n - 2)b) \sum_{p=3}^{n} (T_{1p1p} + T_{2p2p}) + \frac{1}{4\sigma} \text{scal}(S)$$

for all $\sigma > 0$. In order to show that this inequality is preserved, we assume that equality holds. Clearly, $\text{Ric}(S)_{22} - \text{Ric}(S)_{11} \geq 0$ and $\text{Ric}(S)_{12} = 0$. Moreover, we can arrange that $\text{Ric}(S)_{pq} = 0$ for $3 \leq p < q \leq n$. Note that

$$\frac{d}{dt} \sum_{p=3}^{n} (T_{1p1p} + T_{2p2p}) \geq \sum_{p=3}^{n} ((S^2)_{1p1p} + (S^2)_{2p2p})$$

and

$$\frac{d}{dt} \text{scal}(S) \geq 2 |\text{Ric}(S)|^2.$$

Moreover, we may write

$$\frac{d}{dt} (\text{Ric}(S)_{22} - \text{Ric}(S)_{11}) = J_1 + J_2 + J_3 + J_4 + J_5 + J_6,$$

where

$$J_1 = 2 \sum_{p=3}^{n} (S_{2p2p} - S_{1p1p}) \text{Ric}(S)_{pp},$$

$$J_2 = -4 \sum_{p=3}^{n} S_{121p} \text{Ric}(S)_{2p},$$

$$J_3 = 4 \sum_{p=3}^{n} S_{212p} \text{Ric}(S)_{1p},$$

$$J_4 = -2S_{1212} (\text{Ric}(S)_{22} - \text{Ric}(S)_{11}),$$

$$J_5 = 4b ((\text{Ric}(S)^2)_{11} - (\text{Ric}(S)^2)_{22}),$$

$$J_6 = \frac{4}{n} (2b + (n - 2)a) \text{scal}(S) (\text{Ric}(S)_{22} - \text{Ric}(S)_{11}).$$

The terms $J_1, J_2, J_3$ can be estimated as follows:

$$J_1 \leq \tau \sum_{p=3}^{n} (S_{2p2p} - S_{1p1p})^2 + \frac{1}{\tau} \sum_{p=3}^{n} (\text{Ric}(S)_{pp})^2,$$

$$J_2 \leq \frac{2(n - 2)\tau}{n - 3} \sum_{p=3}^{n} (S_{121p})^2 + \frac{2(n - 3)}{(n - 2)\tau} \sum_{p=3}^{n} (\text{Ric}(S)_{2p})^2,$$

$$J_3 \leq \frac{2(n - 2)\tau}{n - 3} \sum_{p=3}^{n} (S_{212p})^2 + \frac{2(n - 3)}{(n - 2)\tau} \sum_{p=3}^{n} (\text{Ric}(S)_{1p})^2.$$
Here, $\tau > 0$ is arbitrary. To estimate $J_4$, we recall that $S$ has nonnegative isotropic curvature. This gives

$$0 \leq \sum_{3 \leq p,q \leq n, p \neq q} (S_{1212} + S_{1p1p} + S_{2q2q} + S_{pqpq})$$

$$\leq (n - 2)(n - 3)S_{1212} + (n - 4) \sum_{p=3}^{n} (S_{1p1p} + S_{2p2p}) + \sum_{p=3}^{n} \text{Ric}(S)_{pp},$$

hence

$$J_4 \leq \frac{2(n - 4)}{(n - 2)(n - 3)} \sum_{p=3}^{n} (S_{1p1p} + S_{2p2p}) (\text{Ric}(S)_{22} - \text{Ric}(S)_{11})$$

$$+ \frac{2}{(n - 2)(n - 3)} \left( \sum_{p=3}^{n} (S_{2p2p} - S_{1p1p}) \right) \left( \sum_{p=3}^{n} \text{Ric}(S)_{pp} \right)$$

$$\leq \frac{(n - 2)^2}{n - 3} \sum_{p=3}^{n} (S_{1p1p} + S_{2p2p})^2 + \frac{\tau}{n - 3} \sum_{p=3}^{n} (S_{2p2p} - S_{1p1p})^2$$

$$+ \frac{(n - 4)^2}{(n - 2)^2(n - 3)\tau} (\text{Ric}(S)_{22} - \text{Ric}(S)_{11})^2 + \frac{1}{(n - 3)\tau} \sum_{p=3}^{n} (\text{Ric}(S)_{pp})^2.$$

We next observe

$$J_5 \leq -4b (\text{Ric}(S)_{22} - \text{Ric}(S)_{11}) (\text{Ric}(S)_{11} + \text{Ric}(S)_{22}) + 4b \sum_{p=3}^{n} (\text{Ric}(S)_{1p})^2$$

$$\leq 8(n - 3)b^2 \tau \sum_{p=3}^{n} (S_{2p2p} - S_{1p1p})^2 + \frac{n - 2}{2(n - 3)\tau} (\text{Ric}(S)_{11} + \text{Ric}(S)_{22})^2$$

$$+ 4b \sum_{p=3}^{n} (S_{12p2} + \sum_{q=3}^{n} S_{1pq})^2$$

$$\leq 8(n - 3)b^2 \tau \sum_{p=3}^{n} (S_{2p2p} - S_{1p1p})^2 + \frac{n - 2}{2(n - 3)\tau} (\text{Ric}(S)_{11} + \text{Ric}(S)_{22})^2$$

$$+ 4(n - 2)b \sum_{p=3}^{n} \left( (S_{12p2})^2 + \sum_{q=3}^{n} (S_{1pq})^2 \right).$$

Therefore,

$$J_1 + J_2 + J_3 + J_4 + J_5 \leq \left( \frac{(n - 2)\tau}{n - 3} + 8(n - 3)b^2 \tau + 2(n - 2)b \right)$$

$$\cdot \sum_{p=3}^{n} ((S^2)_{1p1p} + (S^2)_{2p2p})$$

$$+ \frac{n - 2}{(n - 3)\tau} |\text{Ric}(S)|^2.$$
Finally,
\[
J_6 \leq \frac{2(n-2)}{n} \left(2b + (n-2)a\right) \sum_{p=3}^{n} (S_{2p} - S_{1p})^2 \\
+ \frac{2(2b + (n-2)a)}{\tau} |\text{Ric}(S)|^2.
\]

Putting these facts together, we obtain
\[
\frac{d}{dt}(\text{Ric}(S)_{22} - \text{Ric}(S)_{11}) \\
\leq \left[\left(\frac{n-2}{n-3} + \frac{2(n-2)}{n}\right)(2b + (n-2)a) + 8(n-3)b^2\right] \tau + 2(n-2)b \\
\cdot \sum_{p=3}^{n} ((S^2)_{1p} + (S^2)_{2p}) \\
+ \left(\frac{n-2}{n-3} + 2(2b + (n-2)a)\right) \frac{1}{\tau} |\text{Ric}(S)|^2.
\]

At this point, we are still free to choose \(\tau\). Putting \(\tau = 2\sigma \left(\frac{n-2}{n-3} + 2(2b + (n-2)a)\right)\) and using the inequality
\[
\left(\frac{n-2}{n-3} + \frac{2(n-2)}{n}\right)(2b + (n-2)a) + 8(n-3)b^2 \leq \frac{\omega}{2},
\]
we conclude that
\[
\frac{d}{dt}(\text{Ric}(S)_{22} - \text{Ric}(S)_{11}) \\
\leq (\omega \sigma + 2(n-2)b) \sum_{p=3}^{n} ((S^2)_{1p} + (S^2)_{2p}) + \frac{1}{2\sigma} |\text{Ric}(S)|^2 \\
\leq \frac{d}{dt}\left[(\omega \sigma + 2(n-2)b) \sum_{p=3}^{n} (T_{1p} + T_{2p}) + \frac{1}{4\sigma} \text{scal}(S)\right].
\]

This proves the assertion.

5. A ONE-PARAMETER FAMILY OF INVARIANT CONES \(\tilde{C}(b)\) WHICH PINCH TOWARD PIC1

In this section, we construct a second family of invariant cones. Let \(b_{\text{max}}\) be defined as in the previous section, and let
\[
a_{\text{max}} = \frac{(2 + (n-2)b_{\text{max}})^2}{2(2 + (n-3)b_{\text{max}})} b_{\text{max}}
\]
and
\[
\gamma_{\text{max}} = \frac{b_{\text{max}}}{2 + (n-3)b_{\text{max}}}.
\]
We impose the following assumption:
Assumption 5.1. Assume that $\tilde{b}_{\text{max}}$ is a positive real number such that
\[
\frac{1 + (n - 2)b_{\text{max}}}{1 + (n - 2)b} \sqrt{2b + (n - 2)b^2} = \frac{n^2 - 5n + 4}{n^2 - 7n + 14} \frac{1}{n - 4},
\]
and
\[
\frac{a_{\text{max}} - a}{1 + 2(n - 1)a} - \frac{b_{\text{max}} - b}{1 + (n - 2)b} \geq 0,
\]
for $b = \tilde{b}_{\text{max}}$ and $2a = 2b + (n - 2)b^2$.

Definition 5.2. Assume that $0 \leq b \leq \tilde{b}_{\text{max}}$ and $2a = 2b + (n - 2)b^2$. We denote by $\tilde{E}(b)$ the set of all algebraic curvature tensors $S$ such that $\ell_{a,b}(S) \in C(b_{\text{max}})$ and
\[
Z := S_{1313} + \lambda^2 S_{1414} + S_{2323} + \lambda^2 S_{2424} - 2\lambda S_{1234}
\]
\[
+ \sqrt{2a} (1 - \lambda^2) (\text{Ric}(S)_{11} + \text{Ric}(S)_{22}) \geq 0
\]
for every orthonormal four-frame $\{e_1, e_2, e_3, e_4\}$ and every $\lambda \in [0,1]$. Moreover, we define $\tilde{C}(b) = \ell_{a,b}(\tilde{E}(b))$.

Clearly, $\tilde{C}(b)$ is convex for each $b$.

Proposition 5.3. Suppose that Assumption 5.1 is satisfied. Then $\tilde{C}(\tilde{b}_{\text{max}}) = C(b_{\text{max}})$.

Proof. The inclusion $\tilde{C}(\tilde{b}_{\text{max}}) \subset C(b_{\text{max}})$ follows immediately from the definition. We now prove the reverse inclusion $C(b_{\text{max}}) \subset \tilde{C}(\tilde{b}_{\text{max}})$. For abbreviation, we put $b = \tilde{b}_{\text{max}}$ and $2a = 2b + (n - 2)b^2$. Moreover, suppose that $S$ is an algebraic curvature tensor such that $\ell_{a,b}(S) \in C(b_{\text{max}})$. We claim that $S \in \tilde{E}(b)$. To verify this claim, we need to show that
\[
Z := S_{1313} + \lambda^2 S_{1414} + S_{2323} + \lambda^2 S_{2424} - 2\lambda S_{1234}
\]
\[
+ \sqrt{2a} (1 - \lambda^2) (\text{Ric}(S)_{11} + \text{Ric}(S)_{22}) \geq 0
\]
for every orthonormal four-frame $\{e_1, e_2, e_3, e_4\}$ and every $\lambda \in [0,1]$. To prove this, we consider the algebraic curvature tensor $T = \ell_{a_{\text{max}},b_{\text{max}}}^{-1}(\ell_{a,b}(S)) \in E(b_{\text{max}})$. Clearly, $T \in \text{PIC}$ and
\[
\text{Ric}(T)_{11} + \text{Ric}(T)_{22} + \frac{2\gamma_{\text{max}}}{n} \text{scal}(T) \geq 0.
\]
Note that $S$ and $T$ have the same Weyl tensor. Moreover,

$$\text{Ric}^o(S) = \frac{1 + (n-2)b_{\max}}{1 + (n-2)b} \text{Ric}^o(T)$$

and

$$\text{scal}(S) = \frac{1 + 2(n-1)a_{\max}}{1 + 2(n-1)a} \text{scal}(T),$$

hence

$$\text{Ric}(S) = \frac{1 + (n-2)b_{\max}}{1 + (n-2)b} \text{Ric}(T)$$

$$+ \frac{1}{n} \left( \frac{1 + 2(n-1)a_{\max}}{1 + 2(n-1)a} - \frac{1 + (n-2)b_{\max}}{1 + (n-2)b} \right) \text{scal}(T) \text{id}.$$ 

Moreover,

$$S = T + \frac{b_{\max} - b}{1 + (n-2)b} \text{Ric}(T) \otimes \text{id}$$

$$+ \frac{1}{n} \left( \frac{a_{\max} - a}{1 + 2(n-1)a} - \frac{b_{\max} - b}{1 + (n-2)b} \right) \text{scal}(T) \text{id} \otimes \text{id}.$$ 

Consequently,

$$Z = T_{1313} + \lambda^2 T_{1414} + T_{2323} + \lambda^2 T_{2424} - 2\lambda T_{1234}$$

$$+ \frac{b_{\max} - b}{1 + (n-2)b} \left[ (1 + \lambda^2) \text{Ric}(T)_{11} + (1 + \lambda^2) \text{Ric}(T)_{22} \right.$$ 

$$+ 2 \text{Ric}(T)_{33} + 2\lambda^2 \text{Ric}(T)_{44} \right]$$

$$+ \frac{4}{n} \left( \frac{a_{\max} - a}{1 + 2(n-1)a} - \frac{b_{\max} - b}{1 + (n-2)b} \right) (1 + \lambda^2) \text{scal}(T)$$

$$+ \frac{1 + (n-2)b_{\max}}{1 + (n-2)b} \sqrt{2a} (1 - \lambda^2) (\text{Ric}(T)_{11} + \text{Ric}(T)_{22})$$

$$+ \frac{2}{n} \left( \frac{1 + 2(n-1)a_{\max}}{1 + 2(n-1)a} - \frac{1 + (n-2)b_{\max}}{1 + (n-2)b} \right) \sqrt{2a} (1 - \lambda^2) \text{scal}(T).$$
Using Lemma 2.6, we obtain

\[
Z \geq -\frac{1 - \lambda^2}{n - 4} (\text{Ric}(T)_{11} + \text{Ric}(T)_{22} - 2T_{1212}) \\
+ \frac{b_{\text{max}} - b}{1 + (n - 2)b} \left[ (1 + \lambda^2) \text{Ric}(T)_{11} + (1 + \lambda^2) \text{Ric}(T)_{22} \\
+ 2 \text{Ric}(T)_{33} + 2\lambda^2 \text{Ric}(T)_{44} \right] \\
+ \frac{4}{n} \left( \frac{a_{\text{max}} - a}{1 + 2(n - 1)a} - \frac{b_{\text{max}} - b}{1 + (n - 2)b} \right) (1 + \lambda^2) \text{scal}(T) \\
+ \frac{1 + (n - 2)b_{\text{max}}}{1 + (n - 2)b} \sqrt{2a} (1 - \lambda^2) (\text{Ric}(T)_{11} + \text{Ric}(T)_{22}) \\
+ \frac{2}{n} \left( \frac{1 + 2(n - 1)a_{\text{max}}}{1 + 2(n - 1)a} - \frac{1 + (n - 2)b_{\text{max}}}{1 + (n - 2)b} \right) \sqrt{2a} (1 - \lambda^2) \text{scal}(T) \\
=: \text{RHS}.
\]

We claim that \( \text{RHS} \geq 0 \) for all \( \lambda \in [0, 1] \). Since \( \text{RHS} \) is a linear function of \( \lambda^2 \), it suffices to examine the endpoints of the interval:

**Case 1:** Suppose first that \( \lambda = 1 \). In this case, the quantity

\[
\text{RHS} = 2 \frac{b_{\text{max}} - b}{1 + (n - 2)b} [\text{Ric}(T)_{11} + \text{Ric}(T)_{22} + \text{Ric}(T)_{33} + \text{Ric}(T)_{44}] \\
+ \frac{8}{n} \left( \frac{a_{\text{max}} - a}{1 + 2(n - 1)a} - \frac{b_{\text{max}} - b}{1 + (n - 2)b} \right) \text{scal}(T)
\]

is nonnegative by Lemma 2.2 and Assumption 5.1.

**Case 2:** Suppose next that \( \lambda = 0 \). Using the estimate

\[
\text{Ric}(T)_{11} + \text{Ric}(T)_{22} + 2\text{Ric}(T)_{33} \geq -\frac{4\gamma_{\text{max}}}{n} \text{scal}(T)
\]

and the identity

\[
\frac{1 + (n - 2)b_{\text{max}}}{1 + (n - 2)b} \sqrt{2a} = \frac{n^2 - 5n + 4}{n^2 - 7n + 14} \frac{1}{n - 4},
\]

we obtain

\[
\text{RHS} \geq -\frac{1}{n - 4} (\text{Ric}(T)_{11} + \text{Ric}(T)_{22} - 2T_{1212}) - \frac{4\gamma_{\text{max}}}{n} \frac{b_{\text{max}} - b}{1 + (n - 2)b} \text{scal}(T) \\
+ \frac{4}{n} \left( \frac{a_{\text{max}} - a}{1 + 2(n - 1)a} - \frac{b_{\text{max}} - b}{1 + (n - 2)b} \right) \text{scal}(T) \\
+ \frac{n^2 - 5n + 4}{n^2 - 7n + 14} \frac{1}{n - 4} (\text{Ric}(T)_{11} + \text{Ric}(T)_{22}) \\
+ \frac{2}{n} \left( \frac{1 + 2(n - 1)a_{\text{max}}}{1 + 2(n - 1)a} - \frac{1 + (n - 2)b_{\text{max}}}{1 + (n - 2)b} \right) \sqrt{2a} \text{scal}(T).
\]
In the next step, we estimate the term $T_{1212}$ using the inequality
\[
0 \leq \sum_{3 \leq p, q \leq n, p \neq q} (T_{1212} + T_{1p1p} + T_{2q2q} + T_{pqpq})
= (n^2 - 7n + 14) T_{1212} + (n - 5)(\text{Ric}(T)_{11} + \text{Ric}(T)_{22}) + \text{scal}(T).
\]
This finally gives
\[
\text{RHS} \geq -\frac{2}{n^2 - 7n + 14} \frac{1}{n - 4} \text{scal}(T) - \frac{4 \gamma_{\max}}{n} \frac{b_{\max} - b}{1 + (n - 2)b} \text{scal}(T)
+ \frac{4}{n} \left( \frac{a_{\max} - a}{1 + 2(n - 1)a} - \frac{b_{\max} - b}{1 + (n - 2)b} \right) \text{scal}(T)
+ \frac{2}{n} \left( \frac{1 + 2(n - 1)a_{\max}}{1 + 2(n - 1)a} - \frac{1 + (n - 2)b_{\max}}{1 + (n - 2)b} \right) \sqrt{2a} \text{scal}(T),
\]
and the right hand side is nonnegative by Assumption 5.1. This completes the proof of Proposition 5.3.

In order to show that the cones $\tilde{C}(b)$ are preserved by the Hamilton ODE, we need to impose one additional assumption:

**Assumption 5.4.** For each $0 < b < \tilde{b}_{\max}$, we define numbers $a$ and $\zeta$ by
\[
2a = 2b + (n - 2)b^2 \quad \text{and} \quad \zeta := \frac{1 + 2(n - 1)a_{\max}}{1 + 2(n - 1)a_{\max}} \frac{1 + (n - 2)b_{\max}}{1 + (n - 2)b} (1 + \gamma_{\max}).
\]
We then assume that
\[
nb^2(1 - 2b) - 2(a - b)(1 - 2b + nb^2) \geq 0,
\]
\[
n^2b^2 - 2(n - 1)(a - b)(1 - 2b) \geq 0,
\]
and
\[
\sqrt{2a} (1 - \zeta)^2 + a (1 + (n - 2)(1 - \zeta)) - 2b\zeta^2 \frac{n^2 - 2n + 2}{(n - 2)^2} > 0
\]
for all $0 < b < \tilde{b}_{\max}$.

After these preparations, we now prove the main result of this section:

**Theorem 5.5.** Suppose that Assumptions 4.1, 5.1, and 5.4 are satisfied. If $0 < b < \tilde{b}_{\max}$ and $2a = 2b + (n - 2)b^2$, the cone $\tilde{C}(b)$ is transversally invariant under the Hamilton ODE $\frac{d}{dt} R = Q(R)$ away from 0.

**Proof.** It suffices to show that $\tilde{E}(b)$ is transversally invariant under the ODE $\frac{d}{dt} S = Q(S) + D_{a,b}(S)$. Suppose that $S \in \tilde{E}(b)$, so that
\[
Z := S_{1313} + \lambda^2 S_{1414} + S_{2323} + \lambda^2 S_{2424} - 2\lambda S_{1234}
+ \sqrt{2a} (1 - \lambda^2) (\text{Ric}(S)_{11} + \text{Ric}(S)_{22}) \geq 0
\]
for every orthonormal four-frame \( \{e_1, e_2, e_3, e_4\} \) and every \( \lambda \in [0, 1] \). Moreover, we assume that \( Z = 0 \) for one particular orthonormal four-frame \( \{e_1, e_2, e_3, e_4\} \) and one particular \( \lambda \in [0, 1] \). We will show that \( \frac{d}{dt} Z > 0 \). Using Proposition 2.7, we obtain

\[
Q(S)_{1313} + \lambda^2 Q(S)_{1414} + Q(S)_{2323} + \lambda^2 Q(S)_{2424} - 2\lambda Q(S)_{1234}
\]
\[
+ 2a (\text{Ric}(S) \otimes \text{Ric}(S))_{1313} + 2a\lambda^2 (\text{Ric}(S) \otimes \text{Ric}(S))_{1414}
\]
\[
+ 2a (\text{Ric}(S) \otimes \text{Ric}(S))_{2323} + 2a\lambda^2 (\text{Ric}(S) \otimes \text{Ric}(S))_{2424}
\]
\[
- 4a\lambda (\text{Ric}(S) \otimes \text{Ric}(S))_{1234}
\]
\[
+ 2\sqrt{2a} (1 - \lambda^2) ((S \ast \text{Ric}(S))_{11} + (S \ast \text{Ric}(S))_{22})
\]
\[
\geq 4a(1 + \lambda^2) (\text{Ric}(S)_{11} + \text{Ric}(S)_{22})^2.
\]

Using the identities

\[
\frac{d}{dt} S = Q(S) + 2a \text{Ric}(S) \otimes \text{Ric}(S) + 2b^2 \text{Ric}(S)^2 \otimes \text{id}
\]
\[
+ \frac{nb^2(1 - 2b) - 2(a - b)(1 - 2b + nb^2)}{n(1 + 2(n - 1)a)} \text{Ric}(S)^2 \otimes \text{id}
\]

and

\[
\frac{d}{dt} \text{Ric}(S) = 2 S \ast \text{Ric}(S) - 4b \ast \text{Ric}(S)^2
\]
\[
+ \frac{4(n - 2)a}{n} \text{scal}(S) \text{Ric}(S) + \frac{4a}{n^2} \text{scal}(S)^2 \text{id}
\]
\[
+ 2 \frac{n^2b^2 - 2(n - 1)(a - b)(1 - 2b)}{n(1 + 2(n - 1)a)} \ast \text{Ric}(S)^2 \text{id}
\]

we obtain

\[
\frac{d}{dt} Z \geq 4a(1 + \lambda^2) (\text{Ric}(S)_{11} + \text{Ric}(S)_{22})^2
\]
\[
+ 2b^2 [(1 + \lambda^2) (\text{Ric}(S)^2)_{11} + (1 + \lambda^2) (\text{Ric}(S)^2)_{22}
\]
\[
+ 2 (\text{Ric}(S)^2)_{33} + 2\lambda^2 (\text{Ric}(S)^2)_{44}
\]
\[
+ 4 \frac{nb^2(1 - 2b) - 2(a - b)(1 - 2b + nb^2)}{n(1 + 2(n - 1)a)} (1 + \lambda^2) |\text{Ric}(S)|^2
\]
\[
- 4b \sqrt{2a} (1 - \lambda^2) ((\text{Ric}(S)_{11})^2 + (\text{Ric}(S)_{22})^2)
\]
\[
+ \frac{4(n - 2)a}{n} \sqrt{2a} (1 - \lambda^2) \text{scal}(S) (\text{Ric}(S)_{11} + \text{Ric}(S)_{22})
\]
\[
+ \frac{8a}{n^2} \sqrt{2a} (1 - \lambda^2) \text{scal}(S)^2
\]
\[
+ 4 \frac{n^2b^2 - 2(n - 1)(a - b)(1 - 2b)}{n(1 + 2(n - 1)a)} \sqrt{2a} (1 - \lambda^2) |\text{Ric}(S)|^2.
\]
All except one term on the right hand side have a favorable sign. In the next step, we estimate
\[
\frac{d}{dt} Z \geq 4a(1 - \lambda^2) \left( \text{Ric}(S)_{11} + \text{Ric}(S)_{22} \right)^2 \\
- 4b \sqrt{2a} (1 - \lambda^2) \left( (\text{Ric}(S)_{11})^2 + (\text{Ric}(S)_{22})^2 \right) \\
+ \frac{4(n-2)a}{n} \sqrt{2a} (1 - \lambda^2) \text{scal}(S) \left( \text{Ric}(S)_{11} + \text{Ric}(S)_{22} \right) \\
+ \frac{8a}{n^2} \sqrt{2a} (1 - \lambda^2) \text{scal}(S)^2.
\]

The condition \( \ell_{a,b}(S) \in C(b_{\text{max}}) \) gives
\[
\text{Ric}(S)_{11} + \text{Ric}(S)_{22} \geq \frac{2(1 - \zeta)}{n} \text{scal}(S),
\]
where
\[
\zeta := \frac{1 + 2(n-1)a}{1 + 2(n-1)a_{\text{max}}} \frac{1 + (n-2)b_{\text{max}}}{1 + (n-2)b} (1 + \gamma_{\text{max}}).
\]
Applying Lemma 2.3 with \( \rho = b a \) gives
\[
\frac{(n-2)a}{n} \text{scal}(S) \left( \text{Ric}(S)_{11} + \text{Ric}(S)_{22} \right) - b \left( (\text{Ric}(S)_{11})^2 + (\text{Ric}(S)_{22})^2 \right)
\geq \frac{2}{n^2} \left( a(n-2)(1 - \zeta) - 2b\kappa^2 \frac{n^2 - 2n + 2}{(n-2)^2} \right) \text{scal}(S)^2,
\]

hence
\[
\frac{d}{dt} Z \geq \frac{16a}{n^2} (1 - \zeta)^2 (1 - \lambda^2) \text{scal}(S)^2 \\
+ \frac{8}{n^2} \sqrt{2a} \left( a(1 + (n-2)(1 - \zeta)) - 2b\kappa^2 \frac{n^2 - 2n + 2}{(n-2)^2} \right) (1 - \lambda^2) \text{scal}(S)^2.
\]

Using Assumption 5.4, we conclude that \( \frac{d}{dt} Z > 0 \). This completes the proof of Theorem 5.5.

**Theorem 5.6.** Suppose that Assumptions 4.1, 5.1, and 5.4 are satisfied. Let \( \mathcal{K} \) be a compact set of algebraic curvature tensors in dimension \( n \) which is contained in the interior of the PIC cone, and let \( T > 0 \) be given. Then there exist a small positive real number \( \theta \), a large positive real number \( N \), an increasing concave function \( g \) satisfying \( \lim_{s \to \infty} \frac{g(s)}{s} = 0 \), and a continuous family of closed, convex, \( O(n) \)-invariant sets \( \{ \mathcal{G}_t : t \in [0, T] \} \) such that the family \( \{ \mathcal{G}_t : t \in [0, T] \} \) is invariant under the Hamilton ODE \( \frac{d}{dt} R = Q(R) \); \( \mathcal{K} \subset \mathcal{G}_0 \); and
\[
\mathcal{G}_t \subset \left\{ R : R - \theta \text{scal} \otimes \text{id} \in \text{PIC} \right\} \\
\cap \left\{ R : \text{Ric}_{11} + \text{Ric}_{22} - \theta \text{scal} + N \geq 0 \right\} \\
\cap \left\{ R : R + g(\text{scal}) \text{id} \otimes \text{id} \in \text{PIC}_1 \right\}
\]
for all $t \in [0, T]$.

Proof. By Proposition 5.3 we have $\hat{C}(b_{\text{max}}) = C(b_{\text{max}})$. For each $b \in [b_{\text{max}}, b_{\text{max}} + \hat{b}_{\text{max}}]$, we define $C(b) := \hat{C}(b_{\text{max}} + \hat{b}_{\text{max}} - b)$. Therefore, $C(b)$ is a family of closed, convex, $O(n)$-invariant cones which depends continuously on the parameter $b \in (0, b_{\text{max}} + \hat{b}_{\text{max}}]$. Moreover, for each $b \in (0, b_{\text{max}} + \hat{b}_{\text{max}}]$, the cone $C(b)$ is transversally invariant under the Hamilton ODE. Finally, we conclude that $\hat{C}(0) = C(b_{\text{max}}) \cap \text{PIC1}$.

We now describe the construction of the family $\{G_t : t \in [0, T]\}$. By Theorem 3.1 we can find a family of closed, convex, $O(n)$-invariant sets $\{G_{t}^{(0)} : t \in [0, T]\}$ such that the family $\{G_{t}^{(0)} : t \in [0, T]\}$ is invariant under the Hamilton ODE $\frac{d}{dt} R = Q(R); K \subset G_{t}^{(0)}$ and
\[
G_{t}^{(0)} \subset \{ R : R - \theta \text{scal id} \otimes \text{id} \in \text{PIC} \}
\cap \{ R : \text{Ric}_{11} + \text{Ric}_{22} - \theta \text{scal} + N \geq 0 \}
\]
for all $t \in [0, T]$. Given $\theta$ and $N$, we can find a small positive number $\beta_0$ such that
\[
\{ R : R - \theta \text{scal id} \otimes \text{id} \in \text{PIC} \}
\cap \{ R : \text{Ric}_{11} + \text{Ric}_{22} - \theta \text{scal} + N \geq 0 \}
\subset \{ R : R + N \text{id} \otimes \text{id} \in C(\beta_0) \}.
\]
Therefore, $G_{t}^{(0)} \subset \{ R : R + N \text{id} \otimes \text{id} \in C(\beta_0) \}$ for all $t \in [0, T]$.

Arguing as in Theorem 4.1 in [1] (see also Proposition 16 in [8]), we can construct an increasing sequence $\beta_0 < \beta_1 < \beta_2 < \ldots$ with $\lim_{j \to \infty} \beta_j = b_{\text{max}} + \hat{b}_{\text{max}}$ and a sequence of families of sets $\{G_{t}^{(j)} : t \in [0, T]\}$ with the following properties:

- For $j = 0$, the family of sets $\{G_{t}^{(0)} : t \in [0, T]\}$ coincides with the one constructed in Theorem 3.1.
- For each $j \geq 1$, the family of sets $\{G_{t}^{(j)} : t \in [0, T]\}$ is defined by
  \[
  G_{t}^{(j)} = \{ R : R + 2^j N \text{id} \otimes \text{id} \in C(\beta_j) \}
  \]
  for $t \in [0, T]$.
- For each $j \geq 1$, the family $\{G_{t}^{(j)} : t \in [0, T]\}$ is invariant under the Hamilton ODE.
- For each $j \geq 1$, we have $K \subset G_{t}^{(j)}$.

We now define $G_t = \bigcap_{j \in \mathbb{N}} G_{t}^{(j)}$ for each $t \in [0, T]$. Then $\{G_t : t \in [0, T]\}$ is a family of closed, convex, $O(n)$-invariant sets satisfying $K \subset G_0$. Moreover, the family of sets $\{G_t : t \in [0, T]\}$ is invariant under the Hamilton ODE. Finally, since $G_t \subset \{ R \in G_{t}^{(0)} : R + 2^j N \text{id} \otimes \text{id} \in C(\beta_j) \}$ and $\lim_{j \to \infty} \beta_j = b_{\text{max}} + \hat{b}_{\text{max}}$, we conclude that
\[
G_t \subset \{ R \in G_{t}^{(0)} : R + g(\text{scal}) \text{id} \otimes \text{id} \in \text{PIC1} \}
\]
for all \( t \in [0, T] \), where \( g \) is an increasing concave function satisfying \( \lim_{s \to \infty} \frac{g(s)}{s} = 0 \).

6. Pinching towards PIC2 and proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2. Let \( \{G_t : t \in [0, T]\} \) denote the family of sets constructed in Theorem 5.6. We can find a positive number \( \theta \) and an increasing concave function \( g \) such that \( \lim_{s \to \infty} \frac{g(s)}{s} = 0 \) and

\[
G_t \subset \{ R : R - \theta \text{ scal id } \oplus \text{id} \in \text{PIC} \} 
\cap \{ R : R + g(\text{scal}) \text{id } \oplus \text{id} \in \text{PIC1} \}
\]

for all \( t \in [0, T] \).

**Lemma 6.1.** There exists a small positive constant \( \kappa > 0 \) with the following property. If \( R \in G_t \), then

\[
R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2 \mu^2 R_{2424} - 2\lambda \mu R_{1234} \geq 0
\]

for every orthonormal four-frame \( \{e_1, e_2, e_3, e_4\} \) and all \( \lambda, \mu \in [0, 1] \) satisfying \( \lambda^2(1 - \mu^2)^2 + \mu^2(1 - \lambda^2)^2 + (1 - \lambda^2)^2(1 - \mu^2)^2 \leq \kappa \).

**Proof.** This follows immediately from the fact that \( R - \theta \text{ scal id } \oplus \text{id} \in \text{PIC} \) for all \( R \in G_t \).

To any curvature tensor \( R \in G_t \), we associate a curvature tensor \( T \in \text{PIC1} \) by

\[
T = R + 2 g(\text{scal}) \text{id } \oplus \text{id}.
\]

If \( R \) evolves by the Hamilton ODE \( \frac{d}{dt} R = Q(R) \), then \( T \) satisfies

\[
\frac{d}{dt} T - Q(T) + L(n) g(\text{scal}) \text{id } \oplus \text{id} \in \text{PIC2},
\]

where \( L(n) \) is a large constant that depends only on \( n \).

**Definition 6.2.** Suppose that \( f \) is an increasing concave function satisfying \( f(s)^2 \geq \frac{128 L(n)}{\kappa} s g(s) \) and \( \lim_{s \to \infty} \frac{f(s)}{s} = 0 \). Let \( F_t \) denote the set of all algebraic curvature tensors \( R \in G_t \) such that

\[
T_{1313} + \lambda^2 T_{1414} + \mu^2 T_{2323} + \lambda^2 \mu^2 T_{2424} - 2\lambda \mu T_{1234}
+ (1 - \lambda^2)(1 - \mu^2) f(\text{scal}) \geq 0
\]

for \( \lambda, \mu \in [0, 1] \), where \( T \) is defined by \( T = R + 2 g(\text{scal}) \text{id } \oplus \text{id} \).

**Theorem 6.3.** The family \( \{F_t : t \in [0, T]\} \) defined above is invariant under the Hamilton ODE \( \frac{d}{dt} R = Q(R) \). Moreover, by a suitable choice of \( f \) we can arrange that \( K \subset F_0 \).
Proof. We first show that the family of sets \( \mathcal{F}_t : t \in [0, T] \) is preserved by the Hamilton ODE \( \frac{d}{dt} R = Q(R) \). To prove this, we assume that \( R \in \mathcal{F}_t \) and

\[
T_{1313} + \lambda^2 T_{1414} + \mu^2 T_{2323} + \lambda^2 \mu^2 T_{2424} - 2\lambda\mu T_{1234} + (1 - \lambda^2)(1 - \mu^2) f(\text{scal}) = 0
\]

for one particular orthonormal four-frame \( \{e_1, e_2, e_3, e_4\} \) and one particular pair \( \lambda, \mu \in [0, 1] \). Clearly,

\[
R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2 \mu^2 R_{2424} - 2\lambda\mu R_{1234} < T_{1313} + \lambda^2 T_{1414} + \mu^2 T_{2323} + \lambda^2 \mu^2 T_{2424} - 2\lambda\mu T_{1234} \leq 0.
\]

Hence, Lemma 6.1 implies that

\[
\lambda^2 (1 - \mu^2)^2 + \mu^2 (1 - \lambda^2)^2 + (1 - \lambda^2)^2 (1 - \mu^2)^2 > \kappa.
\]

Using Proposition 7.26 in [5], we obtain

\[
T_{1313} + \lambda^2 T_{1414} + \mu^2 T_{2323} + \lambda^2 \mu^2 T_{2424} - 2\lambda\mu T_{1234} + 2\lambda\mu T_{1342} + 2\lambda\mu T_{1423} \geq 0,
\]

hence

\[
Q(T)_{1313} + \lambda^2 Q(T)_{1414} + \mu^2 Q(T)_{2323} + \lambda^2 \mu^2 Q(T)_{2424} - 2\lambda\mu Q(T)_{1234} \geq 0.
\]

This implies

\[
\frac{d}{dt} (T_{1313} + \lambda^2 T_{1414} + \mu^2 T_{2323} + \lambda^2 \mu^2 T_{2424} - 2\lambda\mu T_{1234} + (1 - \lambda^2)(1 - \mu^2) f(\text{scal})) \\
\geq Q(T)_{1313} + \lambda^2 Q(T)_{1414} + \mu^2 Q(T)_{2323} + \lambda^2 \mu^2 Q(T)_{2424} - 2\lambda\mu Q(T)_{1234} \\
- 2L(n) (1 + \lambda^2)(1 + \mu^2) \text{scal} g(\text{scal}) \\
\geq \sum_{p,q=1}^{n} (T_{13pq} - \lambda\mu T_{24pq})^2 + \sum_{p,q=1}^{n} (\lambda T_{14pq} + \mu T_{23pq})^2 \\
- 2L(n) (1 + \lambda^2)(1 + \mu^2) \text{scal} g(\text{scal}).
\]

The first order conditions corresponding to variations of \( \lambda \) and \( \mu \) give

\[
\lambda (1 - \mu^2) f(\text{scal}) = \lambda T_{1414} + \lambda \mu^2 T_{2424} - \mu T_{1234} \\
= (\lambda T_{1414} + \mu T_{1423}) - \mu (T_{1324} - \lambda T_{2424})
\]

and

\[
(1 - \lambda^2) \mu f(\text{scal}) = \mu T_{2323} + \lambda^2 \mu T_{2424} - \lambda T_{1234} \\
= (\lambda T_{1423} + \mu T_{2323}) - \lambda (T_{1324} - \lambda T_{2424})
\]
Moreover,
\[(1 - \lambda^2)(1 - \mu^2) f(\text{scal}) = -((T_{1313} - \lambda \mu T_{1324}) - \lambda (\lambda T_{1414} + \mu T_{1423}))
- \mu (\lambda T_{1423} + \mu T_{2323}) + \lambda \mu (T_{1324} - \lambda \mu T_{2424}).\]

Consequently,
\[
[\lambda^2 (1 - \mu^2)^2 + \mu^2 (1 - \lambda^2)^2 + (1 - \lambda^2)^2 (1 - \mu^2)^2] f(\text{scal})^2 \\
\leq 2 (\lambda T_{1414} + \mu T_{1423})^2 + 2 \mu^2 (T_{1324} - \lambda \mu T_{2424})^2 \\
+ 2 (\lambda T_{1423} + \mu T_{2323})^2 + 2 \lambda^2 (T_{1324} - \lambda \mu T_{2424})^2 \\
+ 4 (T_{1313} - \lambda \mu T_{1324})^2 + 4 \lambda^2 (\lambda T_{1414} + \mu T_{1423})^2 \\
+ 4 \mu^2 (\lambda T_{1423} + \mu T_{2323})^2 + 4 \lambda^2 \mu^2 (T_{1324} - \lambda \mu T_{2424})^2 \\
\leq 16 \sum_{p,q=1}^{n} (T_{13pq} - \lambda \mu T_{24pq})^2 + 16 \sum_{p,q=1}^{n} (\lambda T_{14pq} + \mu T_{23pq})^2.
\]

Putting these facts together, we obtain
\[
\frac{d}{dt} \left( T_{1313} + \lambda^2 T_{1414} + \mu^2 T_{2323} + \lambda^2 \mu^2 T_{2424} - 2 \lambda \mu T_{1234} \\
+ (1 - \lambda^2)(1 - \mu^2) f(\text{scal}) \right) \\
\geq \frac{1}{16} \left[ \lambda^2 (1 - \mu^2)^2 + \mu^2 (1 - \lambda^2)^2 + (1 - \lambda^2)^2 (1 - \mu^2)^2 \right] f(\text{scal})^2 \\
- 2L(n) (1 + \lambda^2)(1 + \mu^2) \text{scal} g(\text{scal}) \\
\geq \frac{\kappa}{16} f(\text{scal})^2 - 8L(n) \text{scal} g(\text{scal}),
\]
and the right hand side is nonnegative in view of our choice of $f(s)$. This shows that the family \( \{F_t : t \in [0, T]\} \) is invariant under the Hamilton ODE.

Finally, if we choose $f$ sufficiently large, we can arrange that the following holds: if $R \in \mathcal{K}$ and $T = R + g(\text{scal}) \text{id} \otimes \text{id}$, then
\[
T_{1313} + \lambda^2 T_{1414} + \mu^2 T_{2323} + \lambda^2 \mu^2 T_{2424} - 2 \lambda \mu T_{1234} \\
+ (1 - \lambda^2)(1 - \mu^2) f(\text{scal}) \geq 0
\]
for $\lambda, \mu \in [0, 1]$. This ensures that $\mathcal{K} \subset \mathcal{F}_0$.

7. Ancient solutions which are weakly PIC2 and uniformly PIC

In this section, we study ancient solutions to the Ricci flow of dimension $n \geq 5$ which are weakly PIC2 and uniformly PIC. A fundamental ingredient is the differential Harnack inequality for the curvature tensor. This inequality was proved in a fundamental paper by Hamilton [18] for solutions to the Ricci flow with nonnegative curvature operator. In [3], we showed that the differential Harnack inequality holds on any solution to the Ricci flow which is weakly PIC2.
Theorem 7.1 (cf. R. Hamilton [18]; S. Brendle [3]). Assume that \((M, g(t))\), \(t \in (0, T)\), is a solution to the Ricci flow which is complete with bounded curvature, and is weakly PIC2. Then
\[
\frac{\partial}{\partial t} \text{scal} + 2 \left\langle \nabla \text{scal}, v \right\rangle + 2 \text{Ric}(v, v) + \frac{1}{t} \text{scal} \geq 0
\]
for every tangent vector \(v\). In particular, the product \(t \cdot \text{scal}\) is monotone increasing at each point in space.

On an ancient solution, the Harnack inequality gives the following estimate:

Corollary 7.2. Assume that \((M, g(t))\) is an ancient solution to the Ricci flow which is complete with bounded curvature, and is weakly PIC2. Then
\[
\frac{\partial}{\partial t} \text{scal} + 2 \left\langle \nabla \text{scal}, v \right\rangle + 2 \text{Ric}(v, v) \geq 0
\]
for every tangent vector \(v\). In particular, the scalar curvature is monotone increasing at each point in space.

The following inequality is obtained by integrating the differential Harnack inequality along paths in space-time:

Corollary 7.3. Assume that \((M, g(t))\) is an ancient solution to the Ricci flow which is complete with bounded curvature, and is weakly PIC2. Then
\[
\text{scal}(x_1, t_1) \leq \exp \left( \frac{d_{g(t_1)}(x_1, x_2)^2}{2(t_2 - t_1)} \right) \text{scal}(x_2, t_2)
\]
whenever \(t_1 < t_2\).

Moreover, the following result from [7] plays a central role in our argument:

Theorem 7.4 (S. Brendle, G. Huisken, C. Sinestrari [7]). Assume that \((M, g(t))\) is a complete, non-flat ancient solution to the Ricci flow with bounded curvature. Suppose that \((M, g(t))\) is uniformly PIC1, so that \(R - \theta \text{scal} \otimes \text{id} \in \text{PIC}1\) for some uniform constant \(\theta > 0\). Then \((M, g(t))\) has constant curvature for each \(t\).

Proof. If \(M\) is compact, Theorem 7.4 is equivalent to Theorem 2 in [7]. However, the proof of Theorem 2 in [7] relies exclusively on the maximum principle (see Corollary 7 in that paper). Hence, the argument also works for complete manifolds with bounded curvature.

We next state several splitting theorems, which are based on the strict maximum principle (see [17] and [5]).

Proposition 7.5. Let \((M, g(t))\), \(t \in (0, T]\), be a (possibly incomplete) solution to the Ricci flow which is weakly PIC2 and strictly PIC. Moreover, suppose that there exists a point \((p_0, t_0)\) in space-time and a unit vector \(v \in T_{p_0}M\) with the property that \(\text{Ric}(v, v) = 0\). Then, for each \(t \leq t_0\), the
flow \((M,g(t))\) locally splits as a product of an \((n-1)\)-dimensional manifold with an interval.

**Proof.** Suppose that the assertion is false. Then there exists a real number \(\tau \in (0, t_0)\) with the property that \((M,g(\tau))\) does not locally split as a product. The Ricci tensor of \((M,g(t))\) satisfies the evolution equation

\[
\frac{\partial}{\partial t} \text{Ric} = \Delta \text{Ric} + 2R \ast \text{Ric}.
\]

Since \(R\) is weakly PIC2, \(R \ast \text{Ric}\) is weakly positive definite. Using the strict maximum principle (see [5], Section 9, or [17]), we conclude that the nullspace of \(\text{Ric}_{g(\tau)}\) defines a parallel subbundle of the tangent bundle of \((M,g(\tau))\). Since \((M,g(\tau))\) is strictly PIC, this subbundle has rank 0 or 1. Since \((M,g(\tau))\) does not split as a product, this subbundle must have rank 0. In other words, the Ricci curvature of \((M,g(\tau))\) is strictly positive.

Let \(\Omega\) be a bounded open neighborhood of the point \(p_0\) with smooth boundary. Let us choose a smooth function \(f: \Omega \to \mathbb{R}\) with the property that \(f > 0\) in \(\Omega\), \(f = 0\) on \(\partial \Omega\), and \(\text{Ric}_{g(\tau)} - f\text{id}\) is weakly positive definite. Let \(F: \bar{\Omega} \times [\tau, t_0] \to \mathbb{R}\) denote the solution of the linear heat equation with initial data \(F(\cdot, \tau) = f\) and Dirichlet boundary conditions. Using the maximum principle, we conclude that \(\text{Ric}_{g(t)}(\cdot, t)\text{id}\) is weakly positive definite. In particular, the Ricci curvature at \((p_0, t_0)\) is strictly positive, contrary to our assumption.

**Proposition 7.6.** Let \((M,g(t)), t \in (0,T],\) be a complete solution to the Ricci flow (possibly with unbounded curvature). Moreover, we assume that \((M,g(t))\) is weakly PIC2 and strictly PIC. Suppose that there exists a point \((p_0,t_0)\) in space-time with the property that the curvature tensor at \((p_0,t_0)\) lies on the boundary of the PIC2 cone. Then, for each \(t \leq t_0\), the universal cover of \((M,g(t))\) splits off a line.

**Proof.** By Berger’s holonomy classification theorem, there are five possible cases:

**Case 1:** Suppose that \(\text{Hol}^0(M,g(\tau)) = SO(n)\) for some \(\tau \in (0,t_0)\). If we fix an arbitrary pair of real numbers \(\lambda, \mu \in [0,1]\), then Theorem 9.13 in [2] implies that the set of all orthonormal four-frames \(\{e_1,e_2,e_3,e_4\}\) satisfying

\[
\begin{align*}
R_{g(\tau)}(e_1,e_3,e_1,e_3) + \lambda^2 R_{g(\tau)}(e_1,e_4,e_1,e_4) \\
+ \mu^2 R_{g(\tau)}(e_2,e_3,e_2,e_3) + \lambda^2 \mu^2 R_{g(\tau)}(e_2,e_4,e_2,e_4) \\
- 2\lambda\mu R_{g(\tau)}(e_1,e_2,e_3,e_4) = 0
\end{align*}
\]

is invariant under parallel transport. Since \(\text{Hol}^0(M,g(\tau)) = SO(n)\), we conclude that

\[
\begin{align*}
R_{g(\tau)}(e_1,e_3,e_1,e_3) + \lambda^2 R_{g(\tau)}(e_1,e_4,e_1,e_4) \\
+ \mu^2 R_{g(\tau)}(e_2,e_3,e_2,e_3) + \lambda^2 \mu^2 R_{g(\tau)}(e_2,e_4,e_2,e_4) \\
- 2\lambda\mu R_{g(\tau)}(e_1,e_2,e_3,e_4) > 0
\end{align*}
\]
for all orthonormal four-frames \( \{e_1, e_2, e_3, e_4\} \) and all \( \lambda, \mu \in [0, 1] \). In other words, the curvature tensor of \((M, g(\tau))\) lies in the interior of the PIC2 cone.

Let \( \Omega \) be a bounded open neighborhood of the point \( p_0 \) with smooth boundary. Let us choose a smooth function \( f : \Omega \to \mathbb{R} \) with the property that \( f > 0 \) in \( \Omega \), \( f = 0 \) on \( \partial \Omega \), and \( R_{g(\tau)} - f \, \text{id} \otimes \text{id} \in \text{PIC2} \). Let \( F : \bar{\Omega} \times [\tau, t_0] \to \mathbb{R} \) denote the solution of the linear heat equation with initial data \( F(\cdot, \tau) = f \) and Dirichlet boundary conditions. Using the maximum principle, we conclude that \( R_{g(t)} - e^{-L(t-\tau)} \, F(\cdot, t) \, \text{id} \otimes \text{id} \in \text{PIC2} \) for some large constant \( L \). In particular, the curvature tensor at \((p_0, t_0)\) lies in the interior of the PIC2 cone, contrary to our assumption.

**Case 2:** Suppose that \( n = 2m \) and \( \text{Hol}^0(M, g(\tau)) = U(m) \) for some \( \tau \in (0, t_0) \). In this case, the universal cover of \((M, g(\tau))\) is a Kähler manifold. This contradicts the fact that \((M, g(\tau))\) is strictly PIC.

**Case 3:** Suppose next that \( n = 4m \geq 8 \) and \( \text{Hol}^0(M, g(\tau)) = \text{Sp}(m) \cdot \text{Sp}(1) \) for some \( \tau \in (0, t_0) \). In this case, the universal cover of \((M, g(\tau))\) is a quaternionic Kähler manifold. This contradicts the fact that \((M, g(\tau))\) is strictly PIC.

**Case 4:** Suppose that \((M, g(\tau))\) is locally symmetric for some \( \tau \in (0, t_0) \). In this case, the universal cover of \((M, g(\tau))\) is isometric to a symmetric space. It is easy to check that the symmetric spaces of rank 1 are not strictly PIC. Moreover, the symmetric spaces of rank 2 or higher admit flat subspaces of dimension 2; hence, these spaces are not strictly PIC. This contradicts our assumption that \((M, g(\tau))\) is strictly PIC.

**Case 5:** Suppose finally that \((M, g(t))\) is locally reducible for each \( t \in (0, t_0) \). In this case, the universal cover of \((M, g(t))\) is isometric to a Riemannian product \( X \times Y \) for each \( t \in (0, t_0) \). Since \((M, g(t))\) is strictly PIC, we conclude that either \( \dim X = 1 \) or \( \dim Y = 1 \). Therefore, the universal cover of \((M, g(t))\) splits off a line for each \( t \in (0, t_0) \). This completes the proof of Theorem 7.6.

By combining Proposition 7.6 and Theorem 7.4 we can draw the following conclusion:

**Corollary 7.7.** Let \((M, g(t)), t \in (-\infty, T], \) be a complete, non-flat ancient solution to the Ricci flow with bounded curvature. Moreover, we assume that \((M, g(t))\) is weakly PIC2 and uniformly PIC, so that \( R - \theta \cdot \text{scal} \, \text{id} \otimes \text{id} \in \text{PIC} \) for some uniform constant \( \theta > 0 \). Suppose that there exists a point \((p_0, t_0)\) in space-time with the property that the curvature tensor at \((p_0, t_0)\) lies on the boundary of the PIC2 cone. Then, for each \( t \leq t_0 \), the universal cover of \((M, g(t))\) is isometric to a round cylinder \( S^{n-1} \times \mathbb{R} \).

**Proof.** By Proposition 7.6, the universal cover of \((M, g(t))\) is isometric to a product \((X, g_X(t)) \times \mathbb{R} \) for each \( t \leq t_0 \). Clearly, \((X, g_X(t)), t \leq t_0, \) is a complete, non-flat ancient solution to the Ricci flow of dimension \( n - 1 \), which is weakly PIC2 and uniformly PIC1. Since \((X, g_X(t))\) has bounded curvature, Theorem 7.4 implies that \((X, g_X(t))\) is isometric to \( S^{n-1} / \mathbb{T} \). This
completes the proof of Corollary 7.7.

We next recall two results due to Perelman, which play a fundamental role in the argument:

**Proposition 7.8** (G. Perelman). Assume that \((M, g)\) is a complete non-compact manifold which is weakly PIC2. Fix a point \(p \in M\) and let \(p_j\) be a sequence of points such that \(d(p, p_j) \to \infty\). Moreover, suppose that \(\lambda_j\) is a sequence of positive real numbers satisfying \(\lambda_j d(p, p_j)^2 \to \infty\). If the rescaled manifolds \((M, \lambda_j g, p_j)\) converge in the Cheeger-Gromov sense to a smooth limit, then the limit splits off a line.

**Proposition 7.9** (G. Perelman). Let \((M, g)\) be a complete noncompact Riemannian manifold which is weakly PIC2. Then \((M, g)\) does not contain a sequence of necks with radii converging to 0.

The proofs of Proposition 7.8 and Proposition 7.9 are based on Toponogov’s theorem. For a detailed exposition of these results of Perelman we refer to [11], Lemma 2.2 and Proposition 2.3.

We now define the class of ancient solutions that we will study.

**Definition 7.10.** An ancient \(\kappa\)-solution is a non-flat ancient solution to the Ricci flow of dimension \(n\) which is complete; has bounded curvature; is weakly PIC2; and is \(\kappa\)-noncollapsed on all scales.

The following result is a consequence of Proposition 7.9 and Theorem 7.4.

**Proposition 7.11.** Suppose that \((M, g(t)), t \in (-\infty, 0]\), is a complete ancient solution to the Ricci flow which is \(\kappa\)-noncollapsed on all scales; is weakly PIC2; and is uniformly PIC. Moreover, suppose that \((M, g(t))\) satisfies the Harnack inequality

\[
\frac{\partial}{\partial t} \text{scal} + 2 \langle \nabla \text{scal}, v \rangle + 2 \text{Ric}(v, v) \geq 0
\]

for every tangent vector \(v\). Then \((M, g(t))\) has bounded curvature.

**Proof.** Since \((M, g(t))\) satisfies the Harnack inequality, it suffices to show that \((M, g(0))\) has bounded curvature. The proof is by contradiction. Suppose that \((M, g(0))\) has unbounded curvature. We distinguish two cases:

**Case 1:** Suppose first that \((M, g(0))\) is strictly PIC2. In this case, \(M\) is diffeomorphic to \(\mathbb{R}^n\) by the soul theorem (cf. [10]). By a standard point-picking argument, there exists a sequence of points \(x_j \in M\) such that \(Q_j := \text{scal}(x_j, 0) \geq j\) and

\[
\sup_{x \in B_{g(0)}(x_j, jQ_j^{-1})} \text{scal}(x, 0) \leq 4Q_j.
\]

Since \((M, g(t))\) satisfies the Harnack inequality, we obtain

\[
\sup_{(x, t) \in B_{g(0)}(x_j, jQ_j^{-1}) \times (-\infty, 0]} \text{scal}(x, t) \leq 4Q_j.
\]
Using Shi’s interior derivative estimate, we obtain bounds for all the derivatives of curvature on \( B_{g(0)}(x_j, \frac{1}{2} Q_j^{-\frac{j}{2}}) \times (-Q_j^{-1}, 0] \). We now dilate the flow \((M, g(t))\) around the point \((x_j, 0)\) by the factor \(Q_j\). Using the noncollapsing assumption and the curvature derivative estimates, we conclude that, after passing to a subsequence, the rescaled flows converge in the Cheeger-Gromov sense to a smooth non-flat ancient solution \((M^\infty, g^\infty(t)), t \in (-\infty, 0]\). The limit \((M^\infty, g^\infty(t))\) is complete; has bounded curvature; is weakly PIC2; and is uniformly PIC. By Proposition [7,8] the limit \((M^\infty, g^\infty(0))\) splits off a line. Consequently, the flow \((M^\infty, g^\infty(t))\) is isometric to a product \((X, g_X(t)) \times \mathbb{R}\), where \((X, g_X(t))\) is a smooth non-flat ancient solution to the Ricci flow in dimension \(n - 1\) which is complete; has bounded curvature; is weakly PIC2; and is uniformly PIC1. By Theorem [7,4], \((X, g_X(0))\) is isometric to \(S^{n-1}/\Gamma\) and \((M^\infty, g^\infty(0))\) is isometric to \((S^{n-1}/\Gamma) \times \mathbb{R}\). If \(\Gamma\) is non-trivial, then a result of Hamilton implies that \(M\) contains a non-trivial incompressible space form \(S^{n-1}/\Gamma\) (cf. [6], Theorem A.2), but this is impossible since \(M\) is diffeomorphic to \(\mathbb{R}^n\). Thus, \(\Gamma\) is trivial, and \((M^\infty, g^\infty(0))\) is isometric to a round cylinder \(S^{n-1} \times \mathbb{R}\). Consequently, \((M, g(0))\) contains a sequence of points \(x_j \in X\) such that \(Q_j := \text{scal}(x_j, 0) \geq j\) and

\[
\sup_{x \in B_{g(0)}(x_j, j^{-1} Q_j^{-\frac{j}{2}})} \text{scal}(x, 0) \leq 4Q_j.
\]

As in Case 1, the Harnack inequality and Shi’s interior derivative estimate give bounds for all the derivatives of curvature. We now dilate the manifold \((X, g_X(0))\) around the point \(x_j\) by the factor \(Q_j\). Passing to the limit as \(j \to \infty\), we obtain a smooth non-flat limit which is uniformly PIC1 and which must split off a line by Proposition [7,8]. This is a contradiction.

We next recall a key result from Perelman’s first paper:

**Theorem 7.12** (cf. G. Perelman [31], Corollary 11.6). Given a positive real number \(w > 0\), we can find positive constants \(B\) and \(C\) such that the following holds: Let \((M, g(t)), t \in [0, T]\), be a solution to the Ricci flow which is weakly PIC2. Suppose that the ball \(B_{g(t)}(x_0, r_0)\) is compactly contained in \(M\), and \(\text{vol}_{g(t)}(B_{g(t)}(x_0, r_0))) \geq w r_0^n\) for each \(t \in [0, T]\). Then \(\text{scal}(x, t) \leq C r_0^{-2} + B t^{-1} \) for all \(t \in (0, T]\) and all \(x \in B_{g(t)}(x_0, \frac{1}{4} r_0)\).
Note that Perelman imposes the stronger assumption \((M, g(t))\) has non-negative curvature operator. However, his proof works under the weaker assumption that \((M, g(t))\) is weakly PIC2.

One of the main tools in Perelman’s theory is the longrange curvature estimate for ancient \(\kappa\)-solutions in dimension 3. In the next step, we verify that this estimate holds in our situation.

**Theorem 7.13** (cf. G. Perelman [31], Section 11.7). Given \(\kappa > 0\), we can find a large positive constant \(\eta\) and a positive function \(\omega : [0, \infty) \to (0, \infty)\) (depending on \(\kappa\)) with the following property: Let \((M, g(t))\) be an ancient \(\kappa\)-solution. Then

\[
\text{scal}(x, t) \leq \text{scal}(y, t) \omega(\text{scal}(y, t) d_{g(t)}(x, y)^2)
\]

for all points \(x, y \in M\) and all \(t\). Moreover, the covariant derivatives of the Riemann curvature tensor satisfy the pointwise estimates \(|\nabla R| \leq \eta \text{scal}^2\) and \(|\nabla^2 R| \leq \eta \text{scal}^2\).

**Proof.** The proof is essentially the same as in Section 11.7 in Perelman’s paper [31] (see also [11]). We sketch the argument for the convenience of the reader. We only need to prove the first statement, as the second one follows from the first together with Shi’s interior derivative estimate. Let us fix a point \(y \in M\). By a rescaling, we can arrange that \(\text{scal}(y, 0) = 1\). For abbreviation, let \(A = \{x \in M : \text{scal}(x, 0) d_{g(0)}(y, x)^2 \geq 1\}\). We distinguish two cases:

**Case 1:** Suppose that \(A = \emptyset\). In this case, we can find a point \(z \in M\) such that \(\text{scal}(z, 0)\) is maximal. Using the Harnack inequality, we obtain

\[
\sup_{x \in M} \text{scal}(x, t) \leq \text{scal}(z, 0)
\]

for all \(t \in (-\infty, 0]\). The fact that \(A = \emptyset\) implies that \(d_{g(0)}(y, z) \leq \text{scal}(z, 0)^{-\frac{1}{2}}\). Hence, we can find a small positive constant \(\beta\), depending only on \(n\), such that \(d_{g(t)}(y, z) \leq 2 \text{scal}(z, 0)^{-\frac{1}{2}}\) for all \(t \in [-\beta \text{scal}(z, 0)^{-1}, 0]\). Moreover, if we choose \(\beta\) small enough, then Shi’s interior derivative estimate guarantees that \(\text{scal}(z, t) \geq \frac{1}{2} \text{scal}(z, 0)\) for all \(t \in [-\beta \text{scal}(z, 0)^{-1}, 0]\). If we apply the Harnack inequality (cf. Corollary 7.3) with \(t = -\beta \text{scal}(z, 0)^{-1}\), then we obtain

\[
\frac{1}{2} \text{scal}(z, 0) \leq \text{scal}(z, t) \leq \exp \left(-\frac{d_{g(t)}(y, z)^2}{2t}\right) \text{scal}(y, 0)
\]

\[
\leq \exp \left(-\frac{2}{t \text{scal}(z, 0)}\right) \text{scal}(y, 0)
\]

\[
= \exp \left(\frac{2}{\beta}\right).
\]
Therefore, the curvature is uniformly bounded from above by a constant that depends only on $n$.

**Case 2:** Suppose now that $A \neq \emptyset$. In this case, we choose a point $z \in A$ which has minimal distance from $y$ with respect to the metric $g(0)$. Note that $\text{scal}(z,0) = d_{g(0)}(y,z)^{-2}$ since $z$ lies on the boundary of $A$. Let $p$ be the mid-point of the minimizing geodesic in $(M,g(0))$ joining $y$ and $z$. Clearly, $B_{g(0)}(p,\frac{1}{4} d_{g(0)}(y,z)) \cap A = \emptyset$. This implies

$$\sup_{x \in B_{g(0)}(p,\frac{1}{4} d_{g(0)}(y,z))} \text{scal}(x,0) \leq 16 d_{g(0)}(y,z)^{-2}.$$ 

By the Harnack inequality,

$$\sup_{x \in B_{g(t)}(p,\frac{1}{4} d_{g(t)}(y,z))} \text{scal}(x,t) \leq 16 d_{g(t)}(y,z)^{-2}$$

for all $t \in (-\infty,0]$. Using the noncollapsing assumption, we obtain

$$\text{vol}_{g(t)}(B_{g(t)}(p,\frac{1}{4} d_{g(t)}(y,z))) \geq \kappa \left(\frac{1}{4} d_{g(t)}(y,z)\right)^n$$

for all $t \in (-\infty,0]$. Once we have a lower bound for the volume of geodesic balls, Theorem 7.12 gives an upper bound on the curvature. Consequently,

$$\sup_{x \in B_{g(0)}(p,r)} \text{scal}(x,0) \leq d_{g(0)}(y,z)^{-2} \omega(d_{g(0)}(y,z)^{-1} r)$$

for all $r \geq 0$, where $\omega : [0,\infty) \to [0,\infty)$ is a positive and increasing function that may depend on $n$ and $\kappa$. If we put $r = d_{g(0)}(y,z)$ and apply the Harnack inequality, we obtain

$$\sup_{x \in B_{g(0)}(p,r)} \text{scal}(x,t) \leq d_{g(0)}(y,z)^{-2} \omega(1),$$

for all $t \in (-\infty,0]$. Therefore, we can find a small positive constant $\beta$, depending only on $n$ and $\kappa$, such that $d_{g(t)}(y,z) \leq 2 d_{g(0)}(y,z)$ for all $t \in [-\beta d_{g(0)}(y,z)^2,0]$. Moreover, $\text{scal}(z,0) = d_{g(0)}(y,z)^{-2}$. Hence, if we choose $\beta$ small enough, then Shi’s interior derivative estimate guarantees that $\text{scal}(z,t) \geq \frac{1}{2} d_{g(t)}(y,z)^{-2}$ for all $t \in [-\beta d_{g(0)}(y,z)^2,0]$. If we apply the Harnack inequality (cf. Corollary 7.13) with $t = -\beta d_{g(0)}(y,z)^2$, then we obtain

$$\frac{1}{2} d_{g(0)}(y,z)^{-2} \leq \text{scal}(z,t)$$

$$\leq \exp \left(-\frac{d_{g(t)}(y,z)^2}{2t}\right) \text{scal}(y,0)$$

$$\leq \exp \left(-\frac{2 d_{g(0)}(y,z)^2}{t}\right) \text{scal}(y,0)$$

$$= \exp \left(\frac{2}{\beta}\right).$$
This finally implies
\[
\sup_{x \in B_g(0)(y,r)} \text{scal}(x,0) \leq \sup_{x \in B_g(0)(y,r+d_g(0)(y,z))} \text{scal}(x,0) \\
\leq d_g(0)(y,z)^{-2} \omega(d_g(0)(y,z)^{-1} r + 1) \\
\leq 2 e^{\frac{\beta}{\theta}} \omega(\sqrt{2} e^{\frac{\beta}{\theta}} r + 1)
\]
for all \( r \geq 0 \). This completes the proof of Theorem 7.13.

We next establish an analogue of Perelman’s compactness theorem for ancient \( \kappa \)-solutions:

**Corollary 7.14** (cf. G. Perelman [31], Section 11.7). Fix \( \kappa > 0 \) and \( \theta > 0 \). Assume that \((M^{(j)}, g^{(j)}(t))\) is a sequence of ancient \( \kappa \)-solutions satisfying \( R - \theta \text{scal} id \otimes id \in \text{PIC} \). Suppose that \( x_j \) is a point on \( M^{(j)} \) satisfying \( \text{scal}(x_j,0) = 1 \). Then, after passing to a subsequence if necessary, the sequence \((M^{(j)}, g^{(j)}(t), x_j)\) converges in the Cheeger-Gromov sense to an ancient \( \kappa \)-solution satisfying \( R - \theta \text{scal} id \otimes id \in \text{PIC} \).

**Proof.** It follows from the noncollapsing assumption and the longrange curvature estimate in Theorem 7.13 that, after passing to a subsequence if necessary, the sequence \((M^{(j)}, g^{(j)}(t), x_j)\) converges in the Cheeger-Gromov sense to a smooth non-flat ancient solution \((M^\infty, g^\infty(t))\). Clearly, \((M^\infty, g^\infty(t))\) is \( \kappa \)-noncollapsed on all scales; is weakly PIC2; and satisfies \( R - \theta \text{scal} id \otimes id \in \text{PIC} \). Since the Harnack inequality
\[
\frac{\partial}{\partial t} \text{scal} + 2(\nabla \text{scal}, v) + 2 \text{Ric}(v, v) \geq 0
\]
holds on \((M^{(j)}, g^{(j)}(t))\) for each \( j \), it also holds on the limit \((M^\infty, g^\infty(t))\). Consequently, \((M^\infty, g^\infty(t))\) has bounded curvature by Proposition 7.11.

Finally, we establish a structure theorem and a universal noncollapsing theorem for ancient \( \kappa \)-solutions. This was first established by Perelman [32] in the three-dimensional case, and adapted to dimension 4 in [11]. We first consider the noncompact case:

**Theorem 7.15** (cf. G. Perelman [31], Corollary 11.8; Chen-Zhu [11], Proposition 3.4). Given \( \varepsilon > 0 \) and \( \theta > 0 \), we can find large positive constants \( C_1 = C_1(n, \theta, \varepsilon) \) and \( C_2 = C_2(n, \theta, \varepsilon) \) with the following property: Suppose that \((M, g(t))\) is a noncompact ancient \( \kappa \)-solution satisfying \( R - \theta \text{scal} id \otimes id \in \text{PIC} \) which is not locally isometric to a round cylinder. Then, for each point \((x_0, t_0)\) in space-time, we can find an open neighborhood \( B \) of \( x_0 \) satisfying \( B_{g(t_0)}(x_0, C_1^{-1} \text{scal}(x_0, t_0)^{-\frac{2}{n}}) \subset B \subset B_{g(t_0)}(x_0, C_1 \text{scal}(x_0, t_0)^{-\frac{2}{n}}) \) and \( C_2^{-1} \text{scal}(x_0, t_0) \leq \text{scal}(x, t_0) \leq C_2 \text{scal}(x_0, t_0) \) for all \( x \in B \). Moreover, \( B \) satisfies at least one of the following conditions:

- \( B \) is an \( \varepsilon \)-neck.
• $B$ is an $\varepsilon$-cap; that is, $B$ is diffeomorphic to a ball and the boundary $\partial B$ is a cross-sectional sphere of an $\varepsilon$-neck.

In particular, $(M, g(t_0))$ is $\kappa_0$-noncollapsed for some universal constant $\kappa_0 = \kappa_0(n, \theta)$.

A key point is that the constants $C_1$ and $C_2$ in Theorem 7.15 do not depend on $\kappa$.

**Proof.** It suffices to prove the assertion for $t_0 = 0$. Suppose that $x_0$ does not lie at the center of an $\varepsilon$-neck in $(M, g(0))$. Since $(M, g(t))$ is not locally isometric to a round cylinder, Corollary 7.4 implies that $(M, g(t))$ is strictly PIC2. By the soul theorem (cf. [10]), $M$ is diffeomorphic to $\mathbb{R}^n$. We denote by $M_{\varepsilon}$ the set of all points in $(M, g(0))$ which do not lie at the center of an $\varepsilon$-neck. Note that $x_0 \in M_{\varepsilon}$; in particular, $M_{\varepsilon} \neq \emptyset$.

**Step 1:** We claim that the closure of $M_{\varepsilon}$ is compact. If this is false, we can find a sequence of points $x_j \in M_{\varepsilon}$ such that $d_{g(0)}(x_0, x_j) \to \infty$. The long-range curvature estimate in Theorem 7.13 implies that $\lim_{x_j \to \infty} \mathrm{scal}(x_j, 0) d_{g(0)}(x_0, x_j)^2 = \infty$. We now dilate the flow $(M, g(t))$ around the point $(x_j, 0)$ by the factor $\mathrm{scal}(x_j, 0)$. By Corollary 7.14 the rescaled flows converge in the Cheeger-Gromov sense to an ancient $\kappa$-solution satisfying $R - \theta \mathrm{scal} \circ \mathrm{id} \in \mathrm{PIC}$. By Proposition 7.8 the limiting ancient $\kappa$-solution splits off a line. Using Theorem 7.4 we conclude that the limiting ancient $\kappa$-solution is isometric to $(S^{n-1}/\Gamma) \times \mathbb{R}$. If $\Gamma$ is non-trivial, then Theorem A.2 in [5] implies that $M$ contains a non-trivial incompressible space form $S^{n-1}/\Gamma$, but this is impossible since $M$ is diffeomorphic to $\mathbb{R}^n$. Thus, $\Gamma$ is trivial, and the limit is isometric to a round cylinder $S^{n-1} \times \mathbb{R}$. Consequently, $x_j$ lies on an $\varepsilon$-neck if $j$ is sufficiently large. This contradicts the assumption that $x_j \in M_{\varepsilon}$.

Therefore, $M_{\varepsilon}$ has compact closure. In particular, $M_{\varepsilon} \neq M$ since $M$ is noncompact. Since $M_{\varepsilon} \neq \emptyset$, it follows that $\partial M_{\varepsilon} \neq \emptyset$.

**Step 2:** We now consider an arbitrary point $y \in \partial M_{\varepsilon}$. By definition, $y$ lies on an $\varepsilon$-neck in $(M, g(0))$. The Harnack inequality gives $\mathrm{scal}(x, t) \leq \mathrm{scal}(x, 0) \leq 2 \mathrm{scal}(y, 0)$ for all $x \in B_{g(0)}(y, \mathrm{scal}(y, 0)^{-\frac{1}{2}})$ and all $t \leq 0$. Hence, we can find a small constant $\beta = \beta(n) > 0$ such that

\[
\mathrm{vol}_{g(t)}(B_{g(t)}(y, \mathrm{scal}(y, 0)^{-\frac{1}{2}})) \geq \beta \mathrm{scal}(y, 0)^{-\frac{n}{2}}
\]

for all $t \in [-\beta \mathrm{scal}(y, 0)^{-1}, 0]$. Applying Theorem 7.12 we obtain

\[
\mathrm{scal}(x, 0) \leq \mathrm{scal}(y, 0) \omega(\mathrm{scal}(y, 0) d_{g(0)}(x, y)^2)
\]

for all $x \in M$, where $\omega : [0, \infty) \to [0, \infty)$ denotes a positive function that does not depend on $\kappa$. Using the Harnack inequality, we conclude that

\[
\mathrm{scal}(x, t) \leq \mathrm{scal}(y, 0) \omega(\mathrm{scal}(y, 0) d_{g(0)}(x, y)^2)
\]

for all $x \in M$ and all $t \leq 0$.

**Step 3:** We again consider an arbitrary point $y \in \partial M_{\varepsilon}$. By definition of $M_{\varepsilon}$, the point $y$ lies on an $\varepsilon$-neck in $(M, g(0))$. By work of Hamilton [20], a
neck admits a unique foliation by CMC spheres. There is a unique leaf in Hamilton’s CMC foliation which passes through the point \( y \); let us denote this leaf by \( \Sigma_y \). Since \( M \) is diffeomorphic to \( \mathbb{R}^n \), the complement \( M \setminus \Sigma_y \) has exactly one bounded connected component, which must be diffeomorphic to a ball by the solution of the Schoenflies conjecture in dimension \( n \neq 4 \). We will denote this connected component by \( \Omega_y \).

We next show that \( \text{scal}(y,0) \text{diam}_{g(0)}(\Omega_y)^2 \leq C \), where \( C \) depends on \( n, \theta, \) and \( \varepsilon \), but not on \( \kappa \). The proof is by contradiction. Suppose that \( (M^{(j)}, g^{(j)}(t)) \) is a sequence of noncompact ancient \( \kappa \)-solutions which satisfy \( R - \theta \text{scal} \geq 0 \) and which are not locally isometric to a round cylinder. Suppose further that \( y_j \in \partial M^{(j)}_\varepsilon \) is a sequence of points such that \( y_j \in \partial M^{(j)}_\varepsilon \) and \( \text{scal}(y_j,0) \text{diam}_{g^{(j)}(0)}(\Omega_{y_j})^2 \rightarrow \infty \), where \( \Omega_{y_j} \) denotes the region in \( (M^{(j)}, g^{(j)}(0)) \) which is bounded by the CMC sphere passing through \( y_j \). We dilate the flow \( (M^{(j)}, g^{(j)}(t)) \) around the point \( (y_j,0) \) by the factor \( \text{scal}(y_j,0) \). Using the longrange curvature estimate established in Step 2, we conclude that, after passing to a subsequence if necessary, the rescaled manifolds converge to a complete, smooth, non-flat ancient solution \( (M^\infty, g^\infty(t)) \) which is weakly PIC2 and satisfies \( R - \theta \text{scal} \geq 0 \) and \( \left[\frac{1}{2}\text{scal}(y,0)^{-1},0\right] \). Note that the cross-section is isometric to a round cylinder \( S^{n-1} \times \mathbb{R} \). Therefore, if \( j \) is sufficiently large, then \( y_j \) lies on an \( n \)-neck. This contradicts the fact that \( y_j \) lies on the boundary of \( M^{(j)}_\varepsilon \).

**Step 4**: A consequence of the longrange curvature estimate in Step 2 and the diameter estimate in Step 3 is that \( \text{scal}(x,t) \leq C \text{scal}(y,0) \) for all \( y \in \partial M_\varepsilon \), all \( x \in \Omega_y \), and all \( t \leq 0 \). Here, \( C \) is a positive constant that depends only on \( n, \theta, \) and \( \varepsilon \), but not on \( \kappa \). Hence, we can find a small positive number \( \alpha \) and a large positive number \( C \) such that \( \text{scal}(y,0) \text{diam}_{g(t)}(\Omega_y)^2 \leq C \) and \( \text{scal}(y,t) \geq \frac{1}{2} \text{scal}(y,0) \) for all \( y \in \partial M_\varepsilon \) and all \( t \in [-\alpha \text{scal}(y,0)^{-1},0] \). Note that \( \alpha \) and \( C \) depend only on \( n, \theta, \) and \( \varepsilon \), but not on \( \kappa \). Using the Harnack inequality (cf. Corollary 7.3), we obtain

\[
\text{scal}(x,0) \geq \frac{1}{C} \text{scal}(y, -\alpha \text{scal}(y,0)^{-1}) \geq \frac{1}{2C} \text{scal}(y,0)
\]

for all \( y \in \partial M_\varepsilon \) and all \( x \in \Omega_y \), where \( C \) depends only on \( n, \theta, \) and \( \varepsilon \), but not on \( \kappa \).

**Step 5**: Finally, we note that, given two points \( y, y' \in \partial M_\varepsilon \), the associated CMC spheres \( \Sigma_y \) and \( \Sigma_{y'} \) are either disjoint or identical. Consequently, the sets \( \Omega_y \) are nested. In other words, given two points \( y, y' \in \partial M_\varepsilon \), we
either have \( \Omega_y \subset \Omega_{y'} \) or \( \Omega_{y'} \subset \Omega_y \). Since \( \partial M_\varepsilon \) is compact, we can find a point \( y_0 \in \partial M_\varepsilon \) such that \( \Omega_y \subset \Omega_{y_0} \) for all \( y \in \partial M_\varepsilon \). Consequently, the set \( \partial M_\varepsilon \) is contained in the closure of \( \Omega_{y_0} \). From this, we deduce that \( M_\varepsilon \) is contained in the closure of \( \Omega_{y_0} \). In particular, our original point \( x_0 \) lies in \( \Omega_{y_0} \). Since \( x_0 \) does not lie at the center of an \( \varepsilon \)-neck, the distance of \( x_0 \) to the boundary \( \partial \Omega_{y_0} = \Sigma_{y_0} \) is bounded from below by \( \text{scal}(y_0, 0)^{-\frac{n}{2}} \geq C^{-1} \text{scal}(x_0, 0)^{-\frac{n}{2}} \). This shows that \( B_{g(0)}(x_0, C \text{scal}(x_0, 0)^{-\frac{n}{2}}) \subset \Omega_{y_0} \). On the other hand, using the diameter estimate in Step 3, we obtain \( \Omega_{y_0} \subset B_{g(0)}(x_0, C \text{scal}(x_0, 0)^{-\frac{n}{2}}) \). To summarize, the set \( B := \Omega_{y_0} \) is a neighborhood of the point \( x_0 \) which has all the required properties.

**Theorem 7.16** (cf. G. Perelman [31]; Chen-Zhu [11]). Fix \( \theta > 0 \). We can find a constant \( \kappa_0 = \kappa_0(n, \theta) \) such that the following holds: Suppose that \((M, g(t))\) is an ancient \( \kappa \)-solution for some \( \kappa > 0 \), which in addition satisfies \( R - \theta \text{scal} \) \( \odot \) \( \text{id} \in \text{PIC} \). Then either \((M, g(t))\) is \( \kappa_0 \)-noncollapsed for all \( t \); or \((M, g(t))\) is a metric quotient of the round sphere \( S^n \); or \((M, g(t))\) is a noncompact quotient of the round cylinder \( S^{n-1} \times \mathbb{R} \).

**Proof.** In the noncompact case, the universal noncollapsing property follows from Theorem 7.15. Hence, we may assume that \( M \) is compact. The noncollapsing assumption implies that \((M, g(t))\) cannot be a compact quotient of a round cylinder. Using Corollary 7.14, we conclude that \((M, g(t))\) is strictly PIC2. Let us consider the asymptotic shrinking soliton \((\bar{M}, \bar{g}(t))\) associated with \((M, g(t))\) (cf. [31], Section 11.2). Perelman proved that \((\bar{M}, \bar{g}(t))\) is a non-flat shrinking gradient soliton. Moreover, Corollary 7.14 implies that \((\bar{M}, \bar{g}(t))\) is an ancient \( \kappa \)-solution satisfying \( R - \theta \text{scal} \) \( \odot \) \( \text{id} \in \text{PIC} \). We distinguish two cases:

**Case 1:** We first consider the case that the asymptotic shrinking soliton \( \bar{M} \) is compact. A shrinking soliton \((\bar{M}, \bar{g}(t))\) cannot be isometric to a compact quotient of a round cylinder. By Corollary 7.14, \((\bar{M}, \bar{g}(t))\) is strictly PIC2. Since \((\bar{M}, \bar{g}(t))\) is a shrinking soliton, results in [8] imply that \((\bar{M}, \bar{g}(t))\) must be isometric to a metric quotient of the round sphere \( S^n \). Consequently, the original ancient solution \((M, g(t))\) is a metric quotient of \( S^n \).

**Case 2:** Suppose next that the asymptotic shrinking soliton \( \bar{M} \) is noncompact. We will show that \((\bar{M}, \bar{g}(t))\) is noncollapsed with a universal constant that depends only on \( n \) and \( \theta \). In view of Theorem 7.15, there are two possibilities: either the asymptotic shrinking soliton \((\bar{M}, \bar{g}(t))\) is \( \kappa_0 \)-noncollapsed for some universal constant \( \kappa_0 \), or else \((\bar{M}, \bar{g}(t))\) is isometric to a noncompact quotient of the round cylinder. Let us focus on the second case, and divide it into two subcases:

- If the dimension \( n \) is odd and \( \bar{M} \) is a noncompact quotient \( (S^{n-1} \times \mathbb{R})/\Gamma \), then there are only finitely many possibilities for the group \( \Gamma \), and the resulting quotients are all noncollapsed with a universal constant.
• If the dimension $n$ is even and $\bar{M}$ is a noncompact quotient $(S^{n-1} \times \mathbb{R})/\Gamma$, the center slice $(S^{n-1} \times \{0\})/\Gamma$ is incompressible in $M$ by Theorem A.1 in [3]. However, the fundamental group of an even-dimension manifold which is strictly PIC2 is at most 2 by Synge’s theorem. Thus, $\pi_1(M)$ has order at most 2. Hence, there are only finitely many possibilities for the group $\Gamma$, and the resulting quotients $(S^{n-1} \times \mathbb{R})/\Gamma$ are noncollapsed with a universal constant.

Therefore, the asymptotic shrinking soliton $(\bar{M}, \bar{g}(t))$ is noncollapsed with a universal constant that depends only on $n$ and $\theta$. Once we know that the asymptotic shrinking soliton is noncollapsed with a universal constant, it follows from Perelman’s monotonicity formula for the reduced volume that the original ancient solution $(M, g(t))$ is noncollapsed with a universal constant that depends only on $n$ and $\theta$. This is a consequence of work of Perelman [31], Section 7.3. Additional details can be found in [11], pp. 205–208.

**Corollary 7.17** (cf. G. Perelman [32], Section 1.5). *Given $\varepsilon > 0$ and $\theta > 0$, there exist positive constants $C_1 = C_1(n, \theta, \varepsilon)$ and $C_2 = C_2(n, \theta, \varepsilon)$ such that the following holds: Assume that $(M, g(t))$ is an ancient $\kappa$-solution satisfying $R - \theta \text{scal id} \cap \text{id} \in \text{PIC}$. Then, for each point $(x_0, t_0)$ in space-time, there exists a neighborhood $B$ of $x_0$ such that $B_{g(t_0)}(x_0, C_1^{-1} \text{scal}(x_0, t_0)^{-\frac{1}{2}}) \subset B \subset B_{g(t_0)}(x_0, C_1 \text{scal}(x_0, t_0)^{-\frac{1}{2}})$ and $C_2^{-1} \text{scal}(x_0, t_0) \leq \text{scal}(x, t_0) \leq C_2 \text{scal}(x_0, t_0)$ for all $x \in B$. Finally, $B$ satisfies at least one of the following conditions:

- $B$ is an $\varepsilon$-neck.
- $B$ is an $\varepsilon$-cap; that is, $B$ is diffeomorphic to a ball and the boundary $\partial B$ is a cross-sectional sphere of an $\varepsilon$-neck.
- $B$ is a closed manifold diffeomorphic to $S^n/\Gamma$.
- $B$ is an $\varepsilon$-quotient neck of the form $(S^{n-1} \times [-L, L])/\Gamma$.

**Proof.** As usual, it is enough to consider the case $t_0 = 0$. Suppose that the assertion is false. Then we can find a sequence of ancient $\kappa_j$-solutions $(M^{(j)}, g^{(j)}(t))$ satisfying $R - \theta \text{scal id} \cap \text{id} \in \text{PIC}$ and a sequence of points $x_j \in M^{(j)}$ such that the following holds: if $B$ is a neighborhood of the point $x_j$ such that $B_{g^{(j)}(0)}(x_j, j^{-1} \text{scal}(x_j, 0)^{-\frac{1}{2}}) \subset B \subset B_{g^{(j)}(0)}(x_j, j \text{scal}(x_j, 0)^{-\frac{1}{2}})$ and $j^{-1} \text{scal}(x_j, 0) \leq \text{scal}(x, 0) \leq j \text{scal}(x_j, 0)$ for all $x \in B$, then $B$ is neither an $\varepsilon$-neck; nor an $\varepsilon$-cap; nor a closed manifold diffeomorphic to $S^n/\Gamma$; nor an $\varepsilon$-quotient neck. Clearly, $(M^{(j)}, g^{(j)}(t))$ cannot be isometric to a quotient of a round sphere or a round cylinder. It follows from Theorem 7.16 that $(M^{(j)}, g^{(j)}(t))$ is $\kappa_0$-noncollapsed for some uniform constant $\kappa_0$ which is independent of $j$.

By a rescaling, we can arrange that $\text{scal}(x_j, 0) = 1$ for each $j$. We now apply the compactness theorem for ancient $\kappa_0$-solution (cf. Corollary 7.14) to the sequence $(M^{(j)}, g^{(j)}(t))$. Consequently, after passing to a subsequence if necessary, the sequence $(M^{(j)}, g^{(j)}(t), x_j)$ will converge in the Cheeger-Gromov sense to an ancient $\kappa_0$-solution satisfying $R - \theta \text{scal id} \cap \text{id} \in \text{PIC}$.
Let us denote this limiting ancient $\kappa_0$-solution by $(M^\infty, g^\infty(t), x_\infty)$. There are two possibilities:

Case 1: We first consider the case that $M^\infty$ is compact. In this case, the diameter of $(M^{(j)}, g^{(j)}(0))$ has a uniform upper bound independent of $j$. Therefore, if $j$ is sufficiently large, then $B^{(j)} := B^{(j)}(x_j/j^{-1}) \subset B^{(j)}(x_j/j) \subset B^{(j)}(x_j/j)$ and $j^{-1} \leq \text{scal}(x,0) \leq j$ for all $x \in B^{(j)}$. Since $(M^{(j)}, g^{(j)}(t))$ is strictly PIC2, results in [8] imply that $B^{(j)} = M^{(j)}$ is diffeomorphic to a space form. This contradicts the assumption on $x_j$.

Case 2: Finally, we consider the case that $M^\infty$ is noncompact. If $(M^\infty, g^\infty(t))$ is isometric to a noncompact quotient of the round cylinder, then, for $j$ large enough, the point $x_j$ lies at the center of an $\varepsilon$-neck or an $\varepsilon$-quotient neck, and this contradicts the assumption on $x_j$. Consequently, $(M^\infty, g^\infty(t))$ is not isometric to a quotient of the round cylinder. At this point, we apply Theorem 7.4 to $(M^\infty, g^\infty(t))$ (and with $\varepsilon$ replaced by $\frac{\varepsilon}{2}$). Therefore, we can find a neighborhood $B^\infty \subset M^\infty$ of the point $x_\infty$ satisfying $B_{g^\infty(0)}(x_\infty, C_1^{-1}) \subset B^\infty \subset B_{g^\infty(0)}(x_\infty, C_1)$ and $C_1^{-1} \leq \text{scal}(x,0) \leq C_1$ for all $x \in B^\infty$. Furthermore, $B^\infty$ is either an $\frac{\varepsilon}{2}$-neck or an $\frac{\varepsilon}{2}$-cap. Hence, if we choose $j$ sufficiently large, then we can find a neighborhood $B^{(j)} \subset M^{(j)}$ of the point $x_j$ satisfying $B_{g^{(j)}(0)}(x_j, (2C_1)^{-1}) \subset B^{(j)} \subset B_{g^{(j)}(0)}(x_j, 2C_1)$ and $(2C_2)^{-1} \leq \text{scal}(x,0) \leq 2C_2$ for all $x \in B^{(j)}$. Finally, $B^{(j)}$ is either an $\varepsilon$-neck or an $\varepsilon$-cap. This contradicts our assumption on $x_j$.

8. A Canonical Neighborhood Theorem in higher dimensions

In this section, we consider a solution of the Ricci flow starting from a compact manifold of dimension $n \geq 12$ with positive isotropic curvature. Our goal is to establish an analogue of Perelman’s Canonical Neighborhood Theorem. We begin with a definition:

Definition 8.1. Assume that $f : [0, \infty) \to [0, \infty)$ is a concave and increasing function satisfying $\lim_{s \to \infty} \frac{f(s)}{s} = 0$, and $\theta$ is a positive real number. A Riemannian manifold is said to have $(f, \theta)$-pinched curvature if $R + f(\text{scal}) \text{id} \oplus \text{id} \in \text{PIC}$ and $R - \theta \text{scal} \text{id} \oplus \text{id} \in \text{PIC}$.

If $(M, g_0)$ is a compact manifold of dimension $n \geq 12$ with positive isotropic curvature, then Corollary 7.3 implies that the subsequent solution to the Ricci flow has $(f, \theta)$-pinched curvature for some suitable $f$ and $\theta$.

Theorem 8.2 (cf. G. Perelman [31], Theorem 12.1). Let $(M, g_0)$ be a compact manifold with positive isotropic curvature of dimension $n \geq 12$, which does not contain any non-trivial incompressible space forms. Let $g(t)$, $t \in [0, T]$, denote the solution to the Ricci flow with initial metric $g_0$. Given a small number $\varepsilon > 0$ and a large number $C_0$, we can find a positive number
\( \hat{r} \) with the following property. If \((x_0, t_0)\) is a point in space-time with \( Q := \text{scal}(x_0, t_0) \geq \hat{r}^{-2} \), then the parabolic neighborhood \( \{(x, t) : d_{g(t_0)}(x_0, x) \leq C_0Q_j^{-\frac{1}{2}}, 0 \leq t_0 - t \leq C_0Q_j^{-1}\} \) is, after scaling by the factor \( Q \), \( \varepsilon \)-close to the corresponding subset of an ancient \( \kappa_0 \)-solution satisfying \( R - \theta \text{scal} \odot \text{id} \in \text{PIC} \).

The proof of Theorem 8.2 is an adaptation of Perelman’s work [31] (see also [11], where the four-dimensional case is treated). We sketch the argument for the convenience of the reader. The proof of Theorem 8.2 is by contradiction. If the assertion is false, then we can find a sequence of points \((x_j, t_j)\) in space-time with the following properties:

(i) \( Q_j := \text{scal}(x_j, t_j) \geq j^2 \).

(ii) After dilating by the factor \( Q_j \), the parabolic neighborhood \( \{(x, t) : d_{g(t)}(x, x) \leq C_0Q_j^{-\frac{1}{2}}, 0 \leq t_j - t \leq C_0Q_j^{-1}\} \) is not \( \varepsilon \)-close to the corresponding subset of any ancient \( \kappa_0 \)-solution satisfying \( R - \theta \text{scal} \odot \text{id} \in \text{PIC} \).

By a point-picking argument, we can arrange that \((x_j, t_j)\) satisfies the following condition:

(iii) If \((\tilde{x}, \tilde{t})\) is a point in space-time such that \( \tilde{t} \leq t_j \) and \( \tilde{Q} := \text{scal}(\tilde{x}, \tilde{t}) \geq 4Q_j \), then the parabolic neighborhood \( \{(x, t) : d_{g(t)}(\tilde{x}, x) \leq C_0Q_j^{-\frac{1}{2}}, 0 \leq \tilde{t} - t \leq C_0\tilde{Q}^{-1}\} \) is, after dilating by the factor \( \tilde{Q} \), \( \varepsilon \)-close to the corresponding subset of an ancient \( \kappa_0 \)-solution satisfying \( R - \theta \text{scal} \odot \text{id} \in \text{PIC} \).

The strategy is to rescale the flow \((M, g(t))\) around the point \((x_j, t_j)\) by the factor \( Q_j \). We will show that the rescaled flows converge to an ancient \( \kappa_0 \)-solution satisfying \( R - \theta \text{scal} \odot \text{id} \in \text{PIC} \).

**Step 1:** We first establish a pointwise curvature derivative estimate. Using (iii), we conclude that \( |D \text{scal}| \leq \eta \text{scal}^{\frac{3}{2}} \) and \( |\frac{\partial}{\partial t} \text{scal}| \leq \eta \text{scal}^2 \) for each point \((x, t)\) in space-time satisfying \( t \leq t_j \) and \( \text{scal}(x, t) \geq 4Q_j \).

**Step 2:** We next prove a longrange curvature estimate. Given any \( \rho > 0 \), we define

\[
\mathcal{M}(\rho) = \limsup_{j \to \infty} \sup_{x \in B_{g(t)}(x_j, \rho Q_j^{-\frac{1}{2}})} Q_j^{-1} \text{scal}(x, t_j).
\]

Here, we allow the possibility that \( \mathcal{M}(\rho) = \infty \). The pointwise curvature derivative estimate in Step 1 implies that \( \mathcal{M}(\rho) \leq 8 \) for \( 0 < \rho < \frac{1}{1000} \).

We claim that \( \mathcal{M}(\rho) < \infty \) for all \( \rho > 0 \). Suppose this is false. Let \( \rho^* = \sup\{\rho \geq 0 : \mathcal{M}(\rho) < \infty\} < \infty \).

By definition of \( \rho^* \), we have a uniform curvature bound on the geodesic ball \( B_{g(t)}(x_j, \rho Q_j^{-\frac{1}{2}}) \) for each \( \rho < \rho^* \). Moreover, Perelman’s noncollapsing estimate gives a lower bound for the volume. We rescale around \((x_j, t_j)\) by the factor \( Q_j \) and pass to the limit as \( j \to \infty \). In the limit, we obtain a
smooth, but incomplete manifold \((B^\infty, g^\infty)\). Let \(x_\infty = \lim_{j \to \infty} x_j\). Then \((B^\infty, g^\infty)\) contains a minimizing geodesic \(\gamma_\infty\), parametrized by the interval \([0, \rho^*]\) such that \(\gamma_\infty(0) = x_\infty\) and \(\text{scal}_{g^\infty}(\gamma_\infty(s)) \to \infty\) as \(s \to \rho^*\).

Using property (iii) and Corollary 7.17, we conclude that, for \(s\) sufficiently close to \(\rho^*\), the point \(\gamma_\infty(s)\) has a neighborhood which is either a \(2\varepsilon\)-neck; or a \(2\varepsilon\)-cap; or a \(2\varepsilon\)-quotient neck. The second and third possibility can be ruled out as follows. If \(\gamma_\infty(s)\) lies on a \(2\varepsilon\)-cap for some \(s\) close to \(\rho^*\), then the geodesic \(\gamma_\infty\) must enter and exit the cap, but this is impossible since \(\gamma_\infty\) minimizes length. Moreover, if \(\gamma_\infty(s)\) lies on a \(2\varepsilon\)-quotient neck for some \(s\) close to \(\rho^*\), then \((M, g(t_j))\) contains a quotient neck for \(j\) sufficiently large, and Theorem A.1 in [6] then implies that \(M\) contains a non-trivial incompressible space form, contrary to our assumption. Thus, if \(s\) is sufficiently close to \(\rho^*\), then \(\gamma_\infty(s)\) lies on a \(2\varepsilon\)-neck. Moreover, since the curvature blows up as \(s \to \rho^*\), the radius of the neck converges to 0 as \(s \to \rho^*\).

As in [31], Section 12.1, there is a sequence of rescalings which converges to a piece of a non-flat metric cone in the limit. Using the pointwise curvature derivative estimate established in Step 1, we can extend the metric backwards in time. This gives a locally defined solution to the Ricci flow which is weakly PIC2 and which, at the final time, is a piece of non-flat metric cone. This contradicts Proposition 7.5.

**Step 3:** We now rescale the manifold \((M, g(t_j))\) around the point \(x_j\) by the factor \(Q_j\). By Step 2, we have uniform bounds for the curvature at bounded distance. Using the curvature derivative estimate in Step 1 together with Shi’s interior derivative estimates, we conclude that the covariant derivatives of curvature are bounded at bounded distance. Combining this with Perelman’s noncollapsing estimate, we conclude that (after passing to a subsequence) the rescaled manifolds converge in the Cheeger-Gromov sense to a complete smooth limit manifold \((M^\infty, g^\infty)\). Since \((M, g(t_j))\) has \((f, \theta)\)-pinched curvature, the curvature tensor of \((M^\infty, g^\infty)\) is weakly PIC2 and satisfies \(R - \theta \text{scal} \otimes \text{id} \in \text{PIC}\). Using property (iii) and Corollary 7.17, we conclude that every point in \((M^\infty, g^\infty)\) with scalar curvature greater than 4 has a neighborhood which is either a \(2\varepsilon\)-neck; or a \(2\varepsilon\)-cap; or a \(2\varepsilon\)-quotient neck. Note that the last possibility cannot occur; indeed, if \((M^\infty, g^\infty)\) contains a quotient neck, then \((M, g(t_j))\) contains a quotient neck for \(j\) sufficiently large, and Theorem A.1 in [6] then implies that \(M\) contains a non-trivial incompressible space form, contrary to our assumption.

After these preparations, we now prove that \((M^\infty, g^\infty)\) has bounded curvature. Indeed, if there is a sequence of points in \((M^\infty, g^\infty)\) with curvature going to infinity, then \((M^\infty, g^\infty)\) contains a sequence of necks with radii converging to 0, contradiction Proposition 7.9. Thus, \((M^\infty, g^\infty)\) has bounded curvature.

**Step 4:** We now extend the limit \((M^\infty, g^\infty)\) backwards in time. By Step 3, the scalar curvature of \((M^\infty, g^\infty)\) is bounded from above by a constant \(\Lambda > 4\). Let \(\tau_i := -\frac{1}{100\Lambda}\). Using the pointwise curvature derivative estimate in Step 1, we can extend \((M^\infty, g^\infty)\) backwards in time to a complete solution
\((M^\infty, g^\infty(t)), t \in [\tau_1, 0]\). Moreover, \(\Lambda_1 := \sup_{t \in [\tau_1, 0]} \sup_{M^\infty} \text{scal}_{g^\infty(t)} \leq 2\Lambda\).

In the next step, we put \(\tau_2 := \tau_1 - \frac{1}{100\eta \Lambda_1}\). Using the pointwise curvature derivative estimate in Step 1, we can extend the solution \((M^\infty, g^\infty)\), \(t \in [\tau_1, 0]\), backwards in time to a solution \((M^\infty, g^\infty(t)), t \in [\tau_2, 0]\). Moreover, \(\Lambda_2 := \sup_{t \in [\tau_2, 0]} \sup_{M^\infty} \text{scal}_{g^\infty(t)} \leq 2\Lambda_1\). Continuing this process, we can extend the solution backwards in time to the interval \([\tau_m, 0]\), where \(\tau_m + 1 := \tau_m - \frac{1}{100\eta \Lambda_m}\) and \(\Lambda_{m+1} := \sup_{t \in [\tau_m + 1, 0]} \sup_{M^\infty} \text{scal}_{g^\infty(t)} \leq 2\Lambda_m\).

Let \(\tau^* = \lim_{m \to \infty} \tau_m \leq -\frac{1}{100\eta \Lambda}\). Using a standard diagonal sequence argument, we obtain a complete, smooth limit flow \((M^\infty, g^\infty(t))\) which is defined on the interval \((\tau^*, 0]\) and which has bounded curvature for each \(t \in (\tau^*, 0]\). Since \((M, g(t))\) has \((f, \theta)\)-pinched curvature, the curvature tensor of the limit flow \((M^\infty, g^\infty(t))\) is weakly PIC2 and satisfies \(R - \theta \text{id} \leq 0\).

**Step 5**: We claim that \(\tau^* = -\infty\). To prove this, we argue by contradiction. Suppose that \(\tau^* > -\infty\). Clearly, \(\lim_{m \to \infty} (\tau_m - \tau_{m+1}) = 0\), hence \(\lim_{m \to \infty} \Lambda_m = \infty\). Consequently, the limit flow \((M^\infty, g^\infty(t)), t \in (\tau^*, 0]\), has unbounded curvature.

By the Harnack inequality (cf. Theorem 7.1 above), the function \(t \mapsto (t - \tau^*) \text{scal}_{g^\infty(t)}(x)\) is monotone increasing at each point \(x \in M^\infty\). In particular,

\[
\text{scal}_{g^\infty(t)}(x) \leq \frac{\tau^* \Lambda}{t - \tau^*}
\]

for all \(x \in M^\infty\) and all \(t \in (\tau^*, 0]\). Using Lemma 8.3(b) in [31], we conclude that

\[
-C(n) \sqrt{\frac{\tau^* \Lambda}{t - \tau^*}} \leq \frac{d}{dt} g^\infty(t)(x, y) \leq 0
\]

for all \(x, y \in M^\infty\) and all \(t \in (\tau^*, 0]\). This finally implies

\[
\frac{d}{dt} g^\infty(t)(x, y) \leq \frac{d}{dt} g^\infty(0)(x, y) \leq \frac{d}{dt} g^\infty(0)(x, y) - C(n) \tau^* \sqrt{\Lambda}
\]

for all \(x, y \in M^\infty\) and all \(t \in (\tau^*, 0]\).

By the maximum principle,

\[
\inf_{M^\infty} \text{scal}_{g^\infty(t)} \leq \inf_{M^\infty} \text{scal}_{g^\infty(0)} \leq 1
\]

for all \(t \in (\tau^*, 0]\). Hence, we can find a point \(y_\infty \in M^\infty\) such that \(\text{scal}_{g^\infty(t)}(y_\infty) \leq 4\) for \(t = \tau^* + \frac{1}{100\eta \Lambda}\). Using the pointwise curvature derivative estimate in Step 1, we obtain \(\text{scal}_{g^\infty(t)}(y_\infty) \leq 8\) for all \(t \in (\tau^*, \tau^* + \frac{1}{100\eta \Lambda}]\). Arguing as in Step 2, we can show that

\[
\lim_{m \to \infty} \sup_{B_{g^\infty(\tau_m)}(y_\infty, A)} \text{scal}_{g(\tau_m)} < \infty
\]

for every \(\Lambda > 0\). We claim that there exists a constant \(\Lambda^* > \Lambda\) (independent of \(\Lambda\)) such that

\[
\lim_{m \to \infty} \sup_{B_{g^\infty(\tau_m)}(y_\infty, A)} \text{scal}_{g(\tau_m)} \leq \Lambda^*
\]
for every $A > 0$. Indeed, if no such $\Lambda^*$ exists, we can extract a complete smooth limit which has unbounded curvature, and hence (by property (iii) above) contains a sequence of necks with radii converging to 0, contradiction Proposition 7.9. This shows that we can find a constant $\Lambda^*$ with the required property.

Using the distance estimate, we obtain $B_{g_\infty(0)}(y_\infty, A) \subset B_{g_\infty(\tau_m)}(y_\infty, A - C(n)\tau^*\sqrt{\Lambda})$. Putting these facts together, we conclude that

$$\limsup_{m \to \infty} \sup_{B_{g_\infty(0)}(y_\infty, A)} \text{scal}_{g_\infty(\tau_m)} \leq \Lambda^*$$

for every $A > 0$. Using the pointwise derivative estimate in Step 1, we obtain

$$\sup_{t \in (\tau^*, \tau^* + \frac{1}{1000\Lambda^*}]} \sup_{B_{g_\infty(0)}(y_\infty, A)} \text{scal}_{g_\infty(t)} \leq 2\Lambda^*$$

for every $A > 0$. Since $\Lambda^*$ is independent of $A$, we conclude that $(M_\infty, g_\infty(t))$, $t \in (-\infty, 0]$, has bounded curvature. This is a contradiction. Therefore, $\tau^* = -\infty$.

To summarize, if we dilate the flow $(M, g(t))$ around the point $(x_j, t_j)$ by the factor $Q_j$, then (after passing to a subsequence), the rescaled flows converge in the Cheeger-Gromov sense to an ancient $\kappa$-solution $(M_\infty, g_\infty(t))$, $t \in (-\infty, 0]$, satisfying $R - \theta \text{scal} \otimes \text{id} \in \text{PIC}$. Here, $\kappa$ depends on the initial data. The universal noncollapsing property in Theorem 7.16 implies that the limit is an ancient $\kappa_0$-solution. This contradicts statement (ii). This completes the proof of Theorem 8.2.

Finally, by combining Theorem 8.2 with Theorem 7.17 we can draw the following conclusion:

**Corollary 8.3** (cf. G. Perelman [31], Theorem 12.1). Let $(M, g_0)$ be a compact manifold with positive isotropic curvature of dimension $n \geq 12$, which does not contain any non-trivial incompressible space forms. Let $g(t)$, $t \in [0, T)$, denote the solution to the Ricci flow with initial metric $g_0$. Given any $\varepsilon > 0$, there exists a positive number $\tilde{r}$ with the following property. If $(x_0, t_0)$ is a point in space-time with $Q := \text{scal}(x_0, t_0) \geq \tilde{r}^{-2}$, then we can find a neighborhood $B$ of $x_0$ such that $B_{g(t_0)}(x_0, (2C_1)^{-1}\text{scal}(x_0, t_0)^{-\frac{1}{2}}) \subset B \subset B_{g(t_0)}(x_0, 2C_1\text{scal}(x_0, t_0)^{-\frac{1}{2}})$ and $(2C_2)^{-1}\text{scal}(x_0, t_0) \leq \text{scal}(x, t_0) \leq 2C_2\text{scal}(x_0, t_0)$ for all $x \in B$. Furthermore, $B$ satisfies one of the following conditions:

- $B$ is a $2\varepsilon$-neck.
- $B$ is a $2\varepsilon$-cap; that is, $B$ is diffeomorphic to a ball and the boundary $\partial B$ is the cross-sectional sphere of a $2\varepsilon$-neck.
- $B$ is a closed manifold diffeomorphic to $S^n/\Gamma$.

Here, $C_1 = C_1(n, \theta, \varepsilon)$ and $C_2 = C_2(n, \theta, \varepsilon)$ are the constants appearing in Corollary 7.17. Finally, we have $|D\text{scal}| \leq \eta \text{scal}^{\frac{3}{2}}$ and $|\partial_{\nu}\text{scal}| \leq \eta \text{scal}^2$ at the point $(x_0, t_0)$, where $\eta$ is a constant that depends only on $n$ and $\theta$. 
9. The behavior of the flow at the first singular time

Throughout this section, we fix a compact initial manifold \((M, g_0)\) of dimension \(n \geq 12\) which has positive isotropic curvature and does not contain any non-trivial incompressible space forms. Let \((M, g(t))\) be the solution of the Ricci flow with initial metric \(g_0\), and let \([0, T)\) denote the maximal time interval on which the solution is defined. Note that \(T \leq \frac{2}{n} \inf_{x \in M} \text{scal}(x, 0)\).

By Theorem \ref{thm:1.2}, we can find a continuous family of closed, convex, \(O(n)\)-invariant sets \(\{F_t : t \in [0, T]\}\) such that the family \(\{F_t : t \in [0, T]\}\) is invariant under the Hamilton ODE \(\frac{d}{dt}R = Q(R)\); the curvature tensor of \((M, g_0)\) lies in the set \(F_0\); and \(F_t \subset \{R : R - \theta \text{scal}\text{id} \otimes \text{id} \in \text{PIC}\} \cap \{R : R + f(\text{scal}) \text{id} \otimes \text{id} \in \text{PIC}^2\}\) for all \(t \in [0, T]\). Here, \(f\) is a concave and increasing function satisfying \(\lim_{s \to \infty} \frac{f(s)}{s} = 0\), and \(\theta\) and \(N\) are positive numbers. Note that \(f\), \(\theta\), and \(N\) depend only on the initial data. By Hamilton’s PDE-ODE principle, the curvature tensor of \((M, g(t))\) lies in the set \(F_t\) for each \(t \in [0, T]\).

By Corollary \ref{cor:8.3} every point in space-time where the scalar curvature is sufficiently large admits a Canonical Neighborhood which is either a 2\(\varepsilon\)-neck; or a 2\(\varepsilon\)-cap; or a closed manifold diffeomorphic to \(S^n/\Gamma\). Let \(\Omega := \{x \in M : \limsup_{t \to T} \text{scal}(x, t) < \infty\}\).

The pointwise curvature derivative estimate implies that \(\Omega\) is an open subset of \(M\). We distinguish two cases:

Case 1: Suppose that \(\Omega = \emptyset\). Using the inequality \(\frac{\partial}{\partial t} \text{scal} \leq \eta \text{scal}^2\), we obtain \(\inf_{x \in M} \text{scal}(x, t) \to \infty\) as \(t \to T\). Hence, if \(t\) is sufficiently close to \(T\), then every point on \((M, g(t))\) admits a Canonical Neighborhood as described above. This implies that \(M\) is diffeomorphic to a quotient of \(S^n\) or a quotient of \(S^{n-1} \times \mathbb{R}\).

Case 2: Suppose now that \(\Omega \neq \emptyset\). Using the pointwise curvature estimates, we conclude the metrics \(g(t)\) converge to a smooth metric \(g(T)\) on \(\Omega\). Let \(\rho\) be a small positive number such that the Canonical Neighborhood Theorem holds at all points with \(\text{scal} \leq 4\rho^{-2}\). Following \cite{31}, we consider the set \(\Omega_{\rho} := \{x \in M : \limsup_{t \to T} \text{scal}(x, t) \leq \rho^{-2}\}\).

The Canonical Neighborhood Theorem guarantees that every point in \(\Omega \setminus \Omega_{\rho}\) lies either on a 2\(\varepsilon\)-tube with boundary components in \(\Omega_{\rho}\); or on a 2\(\varepsilon\)-cap with boundary in \(\Omega_{\rho}\); or on a 2\(\varepsilon\)-horn with boundary in \(\Omega_{\rho}\); or on a double 2\(\varepsilon\)-horn; or on a capped 2\(\varepsilon\)-horn; or on a closed manifold diffeomorphic to \(S^n/\Gamma\). (Here, we use the definitions from Perelman’s paper \cite{32}). Following Perelman \cite{32}, we perform surgery on each 2\(\varepsilon\)-horn with boundary in \(\Omega_{\rho}\). We discard all double 2\(\varepsilon\)-horns, all capped 2\(\varepsilon\)-horns, and all closed manifolds.
diffeomorphic to $S^n/\Gamma$. We leave unchanged all the $2\varepsilon$-tubes with boundary in $\Omega_\rho$, and all $2\varepsilon$-caps with boundary in $\Omega_\rho$.

In the remainder of this section, we show that the surgery procedure preserves our curvature pinching estimates, provided that the surgery parameters are sufficiently fine.

**Proposition 9.1.** Suppose that the curvature tensor of a $\delta$-neck lies in the set $F_t$ prior to surgery. If $\delta$ is sufficiently small and the curvature of the neck is sufficiently large, then the curvature tensor of the surgically modified manifold lies in the set $F_t$. Moreover, the scalar curvature is pointwise increasing under surgery.

**Proof.** Suppose that the scalar curvature of the neck is close to $h^{-2}$, where $h$ is small. Let us rescale by the factor $h^{-1}$ so that the scalar curvature of the neck is close to 1 after rescaling. Let us, therefore, assume that $g$ is a Riemannian metric on $S^{n-1} \times [-10, 10]$ which is close to the round metric with scalar curvature 1, and which has curvature in the set $h^2 F_t$. Let $z$ denote the height function on $S^{n-1} \times [-10, 10]$. The metric after surgery is given by $\tilde{g} = e^{-2\varphi} g$, where $\varphi = e^{-\frac{1}{z}}$ for $z > 0$ and $\varphi = 0$ for $z \leq 0$.

Let $\{e_1, \ldots, e_n\}$ denote a local orthonormal frame with respect to the metric $g$. If we put $\tilde{e}_i = e^{\varphi} e_i$, then $\{\tilde{e}_1, \ldots, \tilde{e}_n\}$ is an orthonormal frame with respect to the metric $\tilde{g}$. We will express geometric quantities associated with the metric $g$ relative to the frame $\{e_1, \ldots, e_n\}$, while geometric quantities associated with $\tilde{g}$ will be expressed in terms of $\{\tilde{e}_1, \ldots, \tilde{e}_n\}$. With this understood, the curvature tensor after surgery is related to the curvature tensor before surgery by the formula

$$\tilde{R} = e^{2\varphi} R + e^{2\varphi} \left( D^2 \varphi + d\varphi \otimes d\varphi - \frac{1}{2} |d\varphi|^2 \text{id} \right) \otimes \text{id}.$$ 

This implies

$$|\tilde{R} - R - z^{-4} e^{-\frac{1}{2}} (dz \otimes dz) \otimes \text{id}| \ll z^{-4} e^{-\frac{1}{2}}$$

for $z > 0$ sufficiently small. Consequently, $\text{scal}(\tilde{R}) > \text{scal}(R)$ if $z > 0$ is sufficiently small. Since the metric $g$ is close to the cylindrical metric, we obtain

$$|R - \frac{1}{2} \left( \text{id} - 2 z \otimes z \right) \otimes \text{id}| \ll 1,$$

hence

$$|\tilde{R} - (1 - z^{-4} e^{-\frac{1}{2}}) R - \frac{1}{2} z^{-4} e^{-\frac{1}{2}} \text{id} \otimes \text{id}|$$

$$\leq |\tilde{R} - R - z^{-4} e^{-\frac{1}{2}} (dz \otimes dz) \otimes \text{id}| + z^{-4} e^{-\frac{1}{2}} \left| R - \frac{1}{2} \left( \text{id} - 2 z \otimes z \right) \otimes \text{id} \right|$$

$$\ll z^{-4} e^{-\frac{1}{2}}$$

for $z > 0$ sufficiently small. Therefore, we may write

$$\tilde{R} = (1 - z^{-4} e^{-\frac{1}{2}}) R + z^{-4} e^{-\frac{1}{2}} S,$$
where $|S - \frac{1}{2} \text{id} \otimes \text{id}| \ll 1$ for $z > 0$ sufficiently small. Consequently, $S \in h^2 F_t$ if $z > 0$ is sufficiently small. Moreover, $R \in h^2 F_t$ in view of our assumption. Since $F_t$ is a convex set, we conclude that $R \in h^2 F_t$ if $z > 0$ is sufficiently small. This easily implies that $R \in h^2 F_t$ for all $z \in (0, 10)$.

### 10. A priori estimates for Ricci flow with surgery

In this section, we define Ricci flow with surgery. Moreover, we discuss how Perelman’s noncollapsing estimate and the Canonical Neighborhood Theorem can be extended to Ricci flow with surgery.

We first recall some basic facts concerning the so-called standard solution. The standard solution is a noncompact rotationally symmetric solution to the Ricci flow which is asymptotic to a cylinder at infinity. It is used to model the evolution of a cap that is glued in during a surgery procedure. The following results were proved by Perelman [32] in dimension 3 and were extended to higher dimensions in [11].

**Theorem 10.1** (G. Perelman [32], Section 2; B.L. Chen, X.P. Zhu [11], Theorem A.1). There exists a complete solution $(\mathbb{R}^n, g(t))$, $t \in [0, \frac{n-1}{2})$, to the Ricci flow with the following properties:

1. For each $t \in [0, \frac{n-1}{2})$, the manifold $(\mathbb{R}^n, g(t))$ is rotationally symmetric.
2. The initial manifold $(\mathbb{R}^n, g(0))$ is isometric to a round cylinder with scalar curvature $1$ outside of a compact set, and this compact set is isometric to the cap used in the surgery procedure.
3. For each $t \in [0, \frac{n-1}{2})$, the manifold $(\mathbb{R}^n, g(t))$ is asymptotic to a cylinder with scalar curvature $\frac{1}{1 - \frac{2}{n-1}}$ at infinity.
4. The scalar curvature is bounded from below by $K_{\text{stand}} \frac{1}{(1 - \frac{2}{n-1})}$, where $K_{\text{stand}}$ is a positive constant that depends only on $n$.
5. For each $t \in [0, \frac{n-1}{2})$, the manifold $(\mathbb{R}^n, g(t))$ is weakly PIC2 and satisfies $R - \theta \text{scal} \text{id} \otimes \text{id} \in \text{PIC}$.
6. The flow $(\mathbb{R}^n, g(t))$ is $\kappa$-noncollapsed for some constant $\kappa > 0$ which depends only on $n$.
7. There exists a function $\omega : [0, \infty) \to (0, \infty)$ such that $\text{scal}(x, t) \leq \text{scal}(y, t) \omega(\text{scal}(y, t) d(g(t), (x, y))^2)$ for all points $x, y$ and all $t \in [0, \frac{n-1}{2})$.

**Proof.** The statements (i), (ii), (iii), (iv), (vi), (vii) are established in [11], Appendix A. Moreover, it is shown in [11] that $(\mathbb{R}^n, g(t))$ has nonnegative curvature operator. Hence, it remains to show that $R - \theta \text{scal} \text{id} \otimes \text{id} \in \text{PIC}$. To see this, we observe that the initial manifold $(\mathbb{R}^n, g(0))$ is uniformly PIC. Moreover, on the initial manifold $(\mathbb{R}^n, g(0))$, the sum of the two smallest eigenvalues of the Ricci tensor is bounded from below by a small multiple of the scalar curvature. Hence, we can find a small constant $b \in (0, b_{\text{max}})$ such that the curvature tensor of $(\mathbb{R}^n, g(0))$ lies in $\mathcal{C}(b)$. By the maximum
principle, the curvature tensor of \((\mathbb{R}^{n}, g(0))\) lies in \(\mathcal{C}(b)\) for each \(t \geq 0\). Consequently, the curvature tensor of \((\mathbb{R}^{n}, g(t))\) satisfies \(R - \theta \, \text{scal} \, \text{id} \in \text{PIC}\) for each \(t \geq 0\).

Following Perelman [32], we will refer to the solution in Theorem 10.1 as the standard solution. It turns out that the standard solution satisfies a Canonical Neighborhood Property:

**Theorem 10.2** (cf. G. Perelman [32]; B.L. Chen, X.P. Zhu [11], Corollary A.2). Given a small number \(\varepsilon > 0\) and a large number \(C_0 > 0\), we can find a number \(\alpha \in [0, \frac{n-1}{2}]\) with the following property. If \((x_0, t_0)\) is a point on the standard solution such that \(t_0 \in [\alpha, \frac{n-1}{2}]\), then the parabolic neighborhood \(P(x_0, t_0, C_0 \, \text{scal}(x_0, t_0)^{-\frac{1}{2}}, -C_0 \, \text{scal}(x_0, t_0)^{-1})\) is, after scaling by the factor \(\text{scal}(x_0, t_0)^{-\frac{1}{2}}\), \(\varepsilon\)-close to the corresponding subset of a noncompact ancient \(\kappa_0\)-solution satisfying \(R - \theta \, \text{scal} \, \text{id} \in \text{PIC}\).

**Proof.** Suppose that the assertion is false. Then we can find a sequence of points \((x_j, t_j)\) on the standard solution such that \(t_j \to \frac{n-1}{2}\) and the parabolic neighborhood \(P(x_j, t_j, C_0 \, \text{scal}(x_j, t_j)^{-\frac{1}{2}}, -C_0 \, \text{scal}(x_j, t_j)^{-1})\) is not \(\varepsilon\)-close to the corresponding subset of a noncompact ancient \(\kappa_0\)-solution satisfying \(R - \theta \, \text{scal} \, \text{id} \in \text{PIC}\). We dilate the solution around the point \((x_j, t_j)\) by the factor \(\text{scal}(x_j, t_j)^{-\frac{1}{2}}\). Using statement (vii) in Theorem 10.1 together with the Harnack inequality (cf. Theorem 7.11), we conclude that the rescaled flows converge to a complete, noncompact ancient solution \((M^\infty, g^\infty(t))\). The limiting ancient solution \((M^\infty, g^\infty(t))\) is weakly PIC2 and satisfies \(R - \theta \, \text{scal} \, \text{id} \in \text{PIC}\). Moreover, the limiting ancient solution is \(\kappa\)-noncollapsed.

By Theorem 7.11 the standard solution satisfies the Harnack inequality

\[
\frac{\partial}{\partial t} \text{scal} + 2 \langle \nabla \text{scal}, v \rangle + 2 \text{Ric}(v, v) + \frac{1}{t} \text{scal} \geq 0
\]

for \(t \in (0, \frac{n-1}{2})\). Consequently, the limiting ancient solution \((M^\infty, g^\infty(t))\) satisfies

\[
\frac{\partial}{\partial t} \text{scal} + 2 \langle \nabla \text{scal}, v \rangle + 2 \text{Ric}(v, v) \geq 0.
\]

Using Proposition 7.11 we conclude that \((M^\infty, g^\infty(t))\) has bounded curvature. Thus, \((M^\infty, g^\infty(t))\) is a noncompact ancient \(\kappa_0\)-solution satisfying \(R - \theta \, \text{scal} \, \text{id} \in \text{PIC}\). This is a contradiction.

**Corollary 10.3** (cf. G. Perelman [32]; B.L. Chen, X.P. Zhu [11], Corollary A.2). Given \(\varepsilon > 0\), there exist positive constants \(C_1 = C_1(n, \varepsilon)\) and \(C_2 = C_2(n, \varepsilon)\) such that the following holds: For each point \((x_0, t_0)\) on the standard solution, there exists a neighborhood \(B\) of \(x_0\) such that \(B_{g(t_0)}(x_0, C_1^{-1} \, \text{scal}(x_0, t_0)^{-\frac{1}{2}}) \subset B \subset B_{g(t_0)}(x_0, C_1 \, \text{scal}(x_0, t_0)^{-\frac{1}{2}})\) and \(C_2^{-1} \, \text{scal}(x_0, t_0) \leq \text{scal}(x, t_0) \leq C_2 \, \text{scal}(x_0, t_0)\).
for all $x \in B$. Furthermore, $B$ satisfies at least one of the following conditions:

- $B$ is an $\varepsilon$-neck.
- $B$ is an $\varepsilon$-cap; that is, $B$ is diffeomorphic to a ball and the boundary $\partial B$ is a cross-sectional sphere of an $\varepsilon$-neck.

Finally, we have $|D_{scal}| \leq \eta_{scal}^{\frac{3}{2}}$ and $|\frac{\partial}{\partial t}_{scal}| \leq \eta_{scal}^{\frac{3}{2}}$.

**Proof.** If $t_0$ is sufficiently close to $\frac{n-1}{2}$ (depending on $\varepsilon$), this follows from Theorem [10.2] and Theorem [7.15]. If $t_0$ is bounded away from 1, this follows from the fact that the standard solution is asymptotic to a cylinder at infinity.

Finally, we state a lemma which will be needed later.

**Lemma 10.4.** Given $\alpha \in [0, \frac{n-1}{2})$ and $l > 0$, we can find a large number $A$ (depending on $\alpha$ and $l$) with the following property. Suppose that $t_1 \in [0, \alpha]$ and $\gamma$ is a space-time curve on the standard solution (parametrized by the interval $[0, t_1]$) such that $\gamma(0)$ lies on the cap at time 0, and $\int_{t_0}^{t_1} |\gamma'(t)|^2_{g(t)} dt \leq l$. Then the curve $\gamma$ is contained in the parabolic neighborhood $P(\gamma(0), 0, \frac{A}{2}, t_1)$.

**Proof.** Using the inequality $\int_{t_0}^{t_1} |\gamma'(t)|^2_{g(t)} dt \leq l$ and Hölder’s inequality, we obtain $\int_{t_0}^{t_1} |\gamma'(t)|_{g(t)} dt \leq \alpha^{\frac{1}{2}} l^{\frac{1}{2}}$. From this, the assertion follows easily.

After these preparations, we now give the definition of Ricci flow with surgery. For the remainder of this section, we fix a compact initial manifold $(M, g_0)$ of dimension $n \geq 12$ which has positive isotropic curvature and does not contain any non-trivial incompressible space forms. As above, let $\{\mathcal{F}_t : t \in [0, T]\}$ be a family of closed, convex, $O(n)$-invariant sets such that the family $\{\mathcal{F}_t : t \in [0, T]\}$ is invariant under the Hamilton ODE $\frac{d}{dt} R = Q(R)$; the curvature tensor of $(M, g_0)$ lies in the set $\mathcal{F}_0$; and

$$\mathcal{F}_t \subset \{ R : R - \theta \text{ scal id} \otimes \text{id} \in \text{PIC} \} \cap \{ R : R + f(\text{scal}) \text{id} \otimes \text{id} \in \text{PIC2} \}$$

for all $t \in [0, T]$. Here, $f$ is a concave and increasing function satisfying $\lim_{s \to \infty} f(s) = 0$, and $\theta$ and $N$ are positive numbers.

In the following, we fix a small positive number $\varepsilon > 0$. Moreover, we fix constants $C_1 = C_1(n, \theta, \varepsilon)$ and $C_2 = C_2(n, \theta, \varepsilon)$ such that the conclusions of Corollary [7.17] and Corollary [10.3] hold.

**Definition 10.5.** A Ricci flow with surgery on the interval $[0, T)$ consists of the following data:

- A decomposition of $[0, T)$ into a disjoint union of finitely many subintervals $[t_k^-, t_k^+]$, $k \in \{0, 1, \ldots, l\}$. In other words, $t_0^- = 0$, $t_l^+ = T$, and $t_{k}^- = t_{k-1}^+$ for $k \in \{1, \ldots, l\}$.
• A collection of smooth Ricci flows \((M^{(k)}, g^{(k)}(t))\), defined for \(t \in [t_k^-, t_k^+]\) and going singular as \(t \to t_k^+\) when \(k \in \{0, 1, \ldots, l - 1\}\).

• Positive numbers \(\varepsilon, r, \delta, h\), where \(\delta \leq \varepsilon\) and \(h \leq \delta r\).

For each \(k\), we put \(\Omega^{(k)} = \{x \in M^{(k)} : \limsup_{t \to t_k^+} \text{scal}(x, t) < \infty\}\). We assume that the following conditions are satisfied:

• The manifold \((M^{(0)}, g^{(0)}(0))\) is isometric to the given initial manifold \((M, g_0)\).

• The manifold \((M^{(k)}, g^{(k)}(t_k^-))\) is obtained from \((\Omega^{(k-1)}, g^{(k-1)}(t_{k-1}^+))\) by performing surgery on finitely many \(\delta\)-necks. For each neck on which we perform surgery, we can find a point \((x_0, t_0)\) at the center of that neck such that \(\text{scal}(x_0, t_0) = h^{-2}\) and the parabolic neighborhood \(P(x_0, t_0, \delta^{-1} h, -\delta^{-1} h^2)\) is free of surgeries.

• Each flow \((M^{(k)}, g^{(k)}(t))\) satisfies the Canonical Neighborhood Property with accuracy \(4\varepsilon\) on all scales less than \(r\). That is to say, if \((x_0, t_0)\) is an arbitrary point in space-time satisfying \(\text{scal}(x_0, t_0) \geq r^{-2}\), then there exists a neighborhood \(B\) of \(x_0\) with the property that \(B_x(x_0, (8C_1)^{\frac{1}{32}} \text{scal}(x_0, t_0)^{-\frac{1}{2}}) \subset B \subset B_y(x_0, 8C_1 \text{scal}(x_0, t_0)^{-\frac{1}{2}})\) and \((8C_2)^{-1} \text{scal}(x_0, t_0) \leq \text{scal}(x, t_0) \leq 8C_2 \text{scal}(x_0, t_0)\) for all \(x \in B\). Moreover, \(B\) is either a strong \(4\varepsilon\)-neck (in the sense of [32]) or a \(4\varepsilon\)-cap.

• The manifold \((M^{(k)})\) may have multiple connected components. However, we assume that no connected component of \((M^{(k)})\) is diffeomorphic to \(S^n / \Gamma\) (otherwise we discard it immediately).

In the following, we will write the surgically modified solution simply as \(g(t)\). However, it is important to remember that the underlying manifold changes across surgery times.

In the first step, we prove an upper bound for the length of the time interval on which the solution is defined.

**Proposition 10.6.** Suppose that we have a Ricci flow with surgery starting from \((M, g_0)\) which is defined on \([0, T)\). Then \(T \leq \frac{n}{2 \inf_{x \in M} \text{scal}(x, 0)}\).

**Proof.** By the maximum principle, the function
\[
t \mapsto \frac{n}{2 \inf_{x \in M} \text{scal}(x, t)} + t
\]
is monotone decreasing under smooth Ricci flow. By Proposition 9.1, this function is monotone decreasing across surgery times. From this, the assertion follows.

**Proposition 10.7.** Let \(f\) and \(\theta\) be as above. Moreover, let \((M, g(t))\) be a Ricci flow with surgery starting from \((M, g_0)\). Then \((M, g(t))\) has \((f, \theta)\)-pinched curvature.

**Proof.** By Theorem 1.2 and Hamilton’s PDE-ODE principle, the property that the curvature tensor lies in \(\mathcal{F}_t\) is preserved by the Ricci flow. By
Proposition 9.1, the property that the curvature tensor lies in $F_t$ is preserved under surgery. Thus, we conclude that the curvature tensor of the surgically modified flow lies in the set $F_t$ at time $t$. From this, the assertion follows.

**Proposition 10.8.** Suppose that we have a Ricci flow with surgery with parameters $\varepsilon, r, \delta, h$. Then $|D\text{scal}| \leq \eta \text{scal}^2$ and $|\frac{\partial}{\partial t}\text{scal}| \leq \eta \text{scal}$ for all points $(x, t)$ in space-time satisfying $\text{scal}(x, t) \geq r$. Here, $\eta$ is a universal constant that depends only on $n$ and $\theta$.

**Proof.** This follows immediately from the Canonical Neighborhood Assumption.

**Proposition 10.9** (cf. G. Perelman [32], Lemma 4.5). Fix $\varepsilon > 0$ small, $\alpha \in [0, \frac{n-1}{2})$, and $A > 1$. Then there exists $\bar{\delta} > 0$ (depending on $\alpha$ and $A$) with the following property. Suppose that we have a Ricci flow with surgery with parameters $\varepsilon, r, \delta, h$, where $\delta \leq \bar{\delta}$. Suppose that $T_0 \in [0, T)$ is a surgery time, and let $x_0$ be a point which lies on a gluing cap at time $T_0$. Let $T_1 = \min\{T, T_0 + \alpha h^2\}$. Then one of the following statements holds:

(i) The solution is defined on $P(x_0, T_0, Ah, T_1 - T_0)$. Moreover, after dilating the solution by $h^{-2}$ and shifting time $T_0$ to 0, the parabolic neighborhood $P(x_0, T_0, Ah, T_1 - T_0)$ is $A^{-1}$-close to the corresponding subset of the standard solution.

(ii) There exists a surgery time $t^+ \in (T_0, T_1)$ such that the solution is defined on $P(x_0, T_0, Ah, t^+ - T_0)$. Moreover, the parabolic neighborhood $P(x_0, T_0, Ah, t^+ - T_0)$ is, after dilation by the factor $h^{-2}$, $A^{-1}$-close to the corresponding subset of the standard solution. Finally, for each point $x \in B_g(T_0) (x_0, Ah)$, the flow exists exactly until time $t^+$.

The proof is the same as the proof of Lemma 4.5 in Perelman’s paper [32]. We omit the details.

As in Perelman’s work [32], it is crucial to establish a noncollapsing estimate in the presence of surgeries.

**Definition 10.10.** Suppose we are given a Ricci flow with surgery. We say that the flow is $\kappa$-noncollapsed on scales less than $\rho$ if the following holds. If $(x_0, t_0)$ is a point in space-time and $r_0$ is a positive number such that $r_0 \leq \rho$ and $\text{scal}(x, t) \leq r_0^{-2}$ for all points $(x, t) \in P(x_0, t_0, r_0, -r_0^2)$ for which the flow is defined, then $\text{vol}_{g(t_0)}(B_g(t_0)(x_0, r_0)) \geq \kappa r_0^n$.

As in Perelman [32], the noncollapsing estimate for Ricci flow with surgery will follow from the monotonicity formula for the reduced volume.

**Definition 10.11.** Suppose we are given a Ricci flow with surgery. A curve in space-time is said to be admissible if it stays in the region unaffected by surgery. A curve in space-time is called barely admissible if it is on the boundary of the set of admissible curves.
Lemma 10.12 (cf. G. Perelman [32], Lemma 5.3). Fix \(\varepsilon, r, L\). Then we can find a real number \(\delta > 0\) (depending on \(r\) and \(L\)) with the following property. Suppose that we have Ricci flow with surgery with parameters \(\varepsilon, r, \delta, h\), where \(\delta \leq \bar{\delta}\). Let \((x_0, t_0)\) be a point in space-time such that \(\text{scal}(x_0, t_0) \leq r^{-2}\), and let \(T_0 < t_0\) be a surgery time. Finally, let \(\gamma\) be a barely admissible curve (parametrized by the interval \([T_0, t_0]\)) such that \(\gamma(T_0)\) lies on the boundary of a surgical cap at time \(T_0\), and \(\gamma(t_0) = x_0\). Then

\[
\int_{T_0}^{t_0} \sqrt{t_0 - t} \left(\text{scal}(\gamma(t), t) + |\gamma'(t)|^2_{g(t)}\right) dt \geq L.
\]

Proof. Since \(\text{scal}(x_0, t_0) \leq r^{-2}\), Proposition [10.8] implies that \(\text{scal} \leq 4r^{-2}\) in \(P(x_0, t_0, \frac{r}{100}, \frac{r^2}{10000})\). Let \(\gamma\) be a barely admissible curve in space-time satisfying the assumptions of Lemma 10.12 and suppose that

\[
\int_{T_0}^{t_0} \sqrt{t_0 - t} \left(\text{scal}(\gamma(t), t) + |\gamma'(t)|^2_{g(t)}\right) dt < L.
\]

Using Hölder’s inequality, we obtain

\[
\int_{t_0 - \tau}^{t_0} |\gamma'(t)|_{g(t)} dt < (2L)^{\frac{1}{2}} \tau^{\frac{1}{2}}
\]

for \(\tau > 0\). Hence, we can find a real number \(\tau \in (0, \frac{r^2}{1000})\), depending only on \(r\) and \(L\), such that \(\gamma|_{[t_0 - \tau, t_0]}\) is contained in the parabolic neighborhood \(P(x_0, t_0, \frac{r}{100}, \frac{r^2}{10000})\). This implies

\[
\text{scal}(\gamma(t), t) \leq 4r^{-2}
\]

for all \(t \in [t_0 - \tau, t_0]\).

Having chosen \(\tau\), we define real numbers \(\alpha \in [0, \frac{n-1}{2})\) and \(l > 0\) by the relations

\[
\frac{(n-1)\sqrt{\tau}}{4K_{\text{std}}} \left|\log \left(1 - \frac{2\alpha}{n-1}\right)\right| = L
\]

and

\[
\frac{l}{2} \sqrt{\tau} = L.
\]

Having fixed \(\alpha\) and \(l\), we choose a large constant \(A\) so that the conclusion of Lemma 10.3 holds. Having chosen \(\alpha\) and \(A\), we choose \(\delta\) so that the conclusion of Proposition 10.9 holds. Moreover, by choosing \(\delta\) small enough, we can arrange that \(K_{\text{std}} \delta^2 \leq \frac{1}{16}\).

In the following, we assume that \(\delta \leq \bar{\delta}\). Let \(T_1 = [T_0, T_0 + \alpha h^2]\) denote the largest number with the property that \(\gamma|_{[T_0, T_1]}\) is contained in the parabolic neighborhood \(P(\gamma(T_0), T_0, Ah, \alpha h^2)\). By Proposition 10.9 the parabolic neighborhood \(P(\gamma(T_0), T_0, Ah, T_1 - T_0)\) is close to the corresponding subset of the standard solution. This implies

\[
\text{scal}(\gamma(t), t) \geq \frac{1}{2K_{\text{std}} \left(h^2 - \frac{2(t - T_0)}{n-1}\right)} \geq \frac{1}{2K_{\text{std}} \delta^2 r^2} \geq 8r^{-2}
\]

for all \(t \in [T_0, T_1]\). Consequently, the intervals \([T_0, T_1]\) and \([t_0 - \tau, t_0]\) are disjoint; that is, \(T_1 \leq t_0 - \tau\). We distinguish two cases:
Case 1: Suppose that $T_1 < T_0 + \alpha h^2$. In this case, the curve $\gamma|_{[T_0, T_1]}$ exits the parabolic neighborhood $P(\gamma(T_0), T_0, Ah, \alpha h^2)$ at time $T_1$. Since the parabolic neighborhood $P(\gamma(T_0), T_0, T_1 - T_0)$ is close to the corresponding subset of the standard solution, Lemma 11.4 implies that $\int_{T_0}^{T_1} |\gamma'(t)|_{g(t)}^2 dt \geq \frac{l}{2}$. (Here, we have used the fact that $\int |\gamma'(t)|_{g(t)}^2 dt$ is invariant under scaling.) Consequently, 

$$L > \int_{T_0}^{T_1} \sqrt{T_0 - t} \left( \text{scal}(\gamma(t), t) + |\gamma'(t)|_{g(t)}^2 \right) dt$$

$$\geq \sqrt{T} \int_{T_0}^{T_1} |\gamma'(t)|_{g(t)}^2 dt$$

$$\geq \frac{l}{2} \sqrt{T},$$

which contradicts our choice of $l$.

Case 2: Suppose that $T_1 = T_0 + \alpha h^2$. In this case,

$$L > \int_{T_0}^{T_1} \sqrt{T_0 - t} \left( \text{scal}(\gamma(t), t) + |\gamma'(t)|_{g(t)}^2 \right) dt$$

$$\geq \sqrt{T} \int_{T_0}^{T_1} \text{scal}(\gamma(t), t) dt$$

$$\geq \sqrt{T} \int_{T_0}^{T_1} 2K_{\text{stand}} \frac{1}{2(t - T_0)} dt$$

$$= \frac{(n - 1) \sqrt{T}}{4K_{\text{stand}}} \left| \log \left(1 - \frac{2\alpha}{n - 1}\right) \right|,$$

which contradicts our choice of $\alpha$.

**Proposition 10.13** (cf. G. Perelman [32], Lemma 5.2). Fix a small number $\varepsilon > 0$. Then we can find a positive number $\kappa$ and a positive function $\delta(\cdot)$ with the following property. Suppose that we have a Ricci flow with surgery with parameters $\varepsilon, r, \delta, h$, where $\delta \leq \delta(r)$. Then the flow is $\kappa$-noncollapsed on all scales less than $\varepsilon$.

Note that the constant $\kappa$ in the noncollapsing estimate is independent of the surgery parameters.

**Proof.** Consider a point $(x_0, t_0)$ in space-time and a positive number $r_0 \leq \varepsilon$ such that scal$(x, t) \leq r_0^{-2}$ for all points $(x, t) \in P(x_0, t_0, r_0, -r_0^2)$ for which the flow is defined. We need to show that vol$_{g(t_0)}(B_{g(t_0)}(x_0, r_0)) \geq \kappa r_0^n$ for some uniform constant $\kappa > 0$. We distinguish three cases:

Case 1: Suppose first that scal$(x_0, t_0) \geq r^{-2}$. In this case, the Canonical Neighborhood Assumption implies that vol$_{g(t_0)}(B_{g(t_0)}(x_0, r_0)) \geq \kappa r_0^n$ for some uniform constant $\kappa > 0$. 


Case 2: Suppose next that the parabolic neighborhood \( P(x_0, t_0, \frac{r_0}{2}, -\frac{r_0^2}{4}) \) contains surgeries. Let \((x, t)\) be a point in \( P(x_0, t_0, \frac{r_0}{2}, -\frac{r_0^2}{4}) \) which lies on a surgical cap. Clearly, \( \frac{1}{4} h^{-2} \leq \text{scal}(x, t) \leq r_0^{-2} \), hence \( r_0 \leq 2h \). This implies \( \text{vol}_{g(t)}(B_g(t)(x, \frac{r_0}{100})) \geq \kappa r_0^n \) for some uniform constant \( \kappa > 0 \). From this, we deduce that \( \text{vol}_{g(t)}(B_g(t)(x_0, r_0)) \geq \text{vol}_{g(t)}(B_g(t)(x, \frac{r_0}{4})) \geq \kappa r_0^n \).

Case 3: Suppose finally that the parabolic neighborhood \( P(x_0, t_0, \frac{r_0}{2}, -\frac{r_0^2}{4}) \) is free of surgeries. Note that \( t_0 \) is bounded from above by Proposition 10.6. By Lemma 10.12 we can find a positive function \( \tilde{\delta}(\cdot) \) such that the following holds: Suppose that the surgery parameters satisfy \( \delta \leq \tilde{\delta}(r) \), and suppose further that \( T_0 < t_0 \) is a surgery time and \( \gamma \) is a barely admissible curve (parametrized by the interval \([T_0, t_0]\)) such that \( \gamma(T_0) \) lies on the boundary of a surgical cap at time \( T_0 \) and \( \gamma(t_0) = x_0 \). Then

\[
\int_{t_0}^{t_0} \sqrt{t_0 - t} \left( \text{scal}(\gamma(t), t) + \left| \gamma'(t)_{g(t)} \right|^2 \right) dt \geq 8n \sqrt{t_0}.
\]

Thus, if \( \delta \leq \tilde{\delta}(r) \), then every barely admissible curve has reduced length greater than \( 2n \).

In the following, we assume that \( \delta \leq \tilde{\delta}(r) \). For \( t < t_0 \), we denote by \( \ell(x, t) \) the reduced distance from \((x_0, t_0)\), i.e. the infimum of the reduced length over all admissible curves joining \((x, t)\) and \((x_0, t_0)\). We claim that \( \inf_x \ell(x, t) \leq \frac{\delta}{4} \) for all \( t < t_0 \). This is clearly true if \( t \) is sufficiently close to \( t_0 \). Now, if \( \ell(x, t) < 2n \) for some point \((x, t)\) in space-time, then the reduced length is attained by a strictly admissible curve. Hence, we may apply results of Perelman (cf. [31], Section 7) to conclude that

\[
\frac{\partial}{\partial t} \ell \geq \Delta \ell + \frac{1}{t_0 - t} \left( \ell - \frac{n}{2} \right)
\]

whenever \( \ell < 2n \). From this, we deduce that \( \inf_x \ell(x, t) \leq \frac{\delta}{8} \) for all \( t < t_0 \).

In particular, there exists a point \( y \in M \) such that \( \ell(y, \varepsilon) \leq \frac{\delta}{8} \). Hence, we can find a radius \( \rho > 0 \) such that \( \ell(x, 0) \leq n \) for all points \( x \in B_{g(0)}(y, \rho) \). Note that \( \rho \) depends only on \( \varepsilon \) and the initial data \((M, g_0)\), but not on the surgery parameters. Hence, for each point \( x \in B_{g(0)}(y, \rho) \), the reduced distance is attained by a strictly admissible curve, and this curve must be an \( \mathcal{L} \)-geodesic.

Given a tangent vector \( v \) at \((x_0, t_0)\), we denote by \( \gamma_v(t) = \mathcal{L}_{t, t_0} \exp_{x_0}(v) \), \( t \in [0, t_0) \), the \( \mathcal{L} \)-geodesic satisfying \( \lim_{t \to t_0^-} \sqrt{t_0 - t} \gamma'_v(t) = v \). Let \( \mathcal{V} \) denote the set of all tangent vectors \( v \) at \((x_0, t_0)\) with the property that \( \gamma_v \) minimizes the \( \mathcal{L} \)-length and \( \gamma_v(0) \in B_{g(0)}(y, \rho) \). Finally, we put

\[
V(t) = \int_{\mathcal{V}} (t_0 - t)^{-\frac{n}{2}} e^{-\ell(\gamma_v(t), t)} J_v(t),
\]

where \( J_v(t) = \det(D\mathcal{L}_{t, t_0} \exp_{x_0})_v \) denotes the Jacobian determinant of the \( \mathcal{L} \)-exponential map, and the integration is with respect to the standard Lebesgue measure on the tangent space. For each tangent vector \( v \in \mathcal{V} \),
Perelman’s Jacobian comparison theorem (cf. [31], Section 7) implies that the function \( t \mapsto (t_0 - t)^{-\frac{3}{2}} e^{-\ell(\gamma_v(t), t)} J_v(t) \) is monotone increasing. Moreover, \( \lim_{t \to 0} (t_0 - t)^{-\frac{3}{2}} e^{-\ell(\gamma_v(t), t)} J_v(t) = 2^n e^{-|v|^2} \) for each \( v \in \mathcal{V} \). The monotonicity property for the Jacobian determinant implies that the function \( t \mapsto V(t) \) is monotone increasing. Since \( \ell(x, 0) \leq n \) for all points \( x \in B_{g(0)}(y, \rho) \), we obtain a uniform lower bound for \( V(0) \):

\[
V(0) = \int_{\mathcal{V}} (t_0 - \tilde{t})^{-\frac{3}{2}} e^{-\ell(\gamma_v(0), \tilde{t})} J_v(\tilde{t})
\]

By assumption, the parabolic neighborhood \( P(x_0, t_0, \frac{r_0}{t_0}, -\frac{r_0^2}{t_0^2}) \) is free of surgeries, and we have \( \text{scal} \leq r_0^{-2} \) in \( P(x_0, t_0, \frac{r_0}{t_0}, -\frac{r_0^2}{t_0^2}) \). Using Shi’s interior derivative estimates, we obtain \( |DR| \leq C(n) r_0^{-\frac{3}{2}} \) and \( |D^2 R| \leq C(n) r_0^{-2} \) in \( P(x_0, t_0, \frac{r_0}{t_0}, -\frac{r_0^2}{t_0^2}) \). Using the \( \mathcal{L} \)-geodesic equation, we conclude that there exists a small positive constant \( \mu(n) \) (depending only on \( n \)) with the following property: if \( \tilde{t} \in [t_0 - \mu(n) r_0^2, t_0] \) and \( |v| \leq \frac{r_0}{32 \sqrt{t_0^2 - t}} \), then \( \sqrt{t_0 - \tilde{t}} |\gamma'_v(t)|_{g(t)} \leq \frac{r_0}{16 \sqrt{t_0^2 - t}} \) and \( \gamma_v(t) \in B_{g(t_0)}(x_0, \frac{r_0 \sqrt{t_0^2 - t}}{4}) \) for all \( t \in [\tilde{t}, t_0] \). This implies

\[
V(0) \leq V(\tilde{t}) \leq \int_{\{ v \in \mathcal{V} : |v| \leq \frac{r_0}{32 \sqrt{t_0^2 - t}} \}} (t_0 - \tilde{t})^{-\frac{3}{2}} e^{-\ell(\gamma_v(\tilde{t}), \tilde{t})} J_v(\tilde{t})
\]

\[
+ \int_{\{ v \in \mathcal{V} : |v| \geq \frac{r_0}{32 \sqrt{t_0^2 - t}} \}} (t_0 - \tilde{t})^{-\frac{3}{2}} e^{-\ell(\gamma_v(\tilde{t}), \tilde{t})} J_v(\tilde{t})
\]

\[
\leq \int_{B_{g(t_0)}(x_0, \frac{r_0}{4})} (t_0 - \tilde{t})^{-\frac{3}{2}} e^{-\ell(x, \tilde{t})} d \text{vol}_{g(\tilde{t})}(x)
\]

\[
+ \int_{\{ |v| \geq \frac{r_0}{32 \sqrt{t_0^2 - t}} \}} 2^n e^{-|v|^2}
\]

\[
\leq (t_0 - \tilde{t})^{-\frac{3}{2}} \text{vol}_{g(\tilde{t})}(B_{g(t_0)}(x_0, \frac{r_0}{4}))
\]

\[
+ \int_{\{ |v| \geq \frac{r_0}{32 \sqrt{t_0^2 - t}} \}} 2^n e^{-|v|^2}
\]

for all \( \tilde{t} \in [t_0 - \mu(n) r_0^2, t_0] \). Putting these facts together gives

\[
(t_0 - \tilde{t})^{-\frac{3}{2}} \text{vol}_{g(\tilde{t})}(B_{g(t_0)}(x_0, \frac{r_0}{4}))
\]

\[
\geq t_0^{-\frac{3}{2}} e^{-n \text{vol}_{g(0)}(B_{g(0)}(y, \rho))} - \int_{\{ |v| \geq \frac{r_0}{32 \sqrt{t_0^2 - t}} \}} 2^n e^{-|v|^2}
\]

for all \( \tilde{t} \in [t_0 - \mu(n) r_0^2, t_0] \).
for all \( \ell \in [t_0 - \mu(n)r_0^2, t_0] \). Finally, we choose \( \ell \in [t_0 - \mu(n)r_0^2, t_0] \) so that \( t_0 - \ell \) is a small, but fixed, multiple of \( r_0^2 \), and the quantity

\[
\frac{1}{r_0^{'2}} e^{-n} \vol_{g(0)}(B_{g(0)}(y, \rho)) - \int_{\{|e|\geq \frac{\rho}{2\sqrt{n-1}}\}} 2^ne^{-|e|^2}
\]

is bounded from below by a positive constant. This gives a lower bound \( r_0^{-n} \vol_{g(i)}(B_{g(t_0)}(x_0, \frac{r_0}{4})) \), as desired.

**Theorem 10.14** (cf. G. Perelman [32], Section 5). Fix a small number \( \varepsilon > 0 \). Then we can find positive numbers \( \hat{r}, \hat{\delta}, h \) which is defined on some interval \([0, T]\). Then the flow satisfies the Canonical Neighborhood Property with accuracy \( 2\varepsilon \) on all scales less than \( 2\hat{r} \). More precisely, if \((x_0, t_0)\) is an arbitrary point in space-time satisfying \( \text{scal}(x_0, t_0) \geq (2\hat{r})^{-2} \), then there exists a neighborhood \( B \) of \( x_0 \) such that \( B_{g(t_0)}(x_0, (2C_1)^{-1} \text{scal}(x_0, t_0)^{-\frac{1}{2}}) \subset B \subset B_{g(t_0)}(x_0, 2C_1 \text{scal}(x_0, t_0)^{-\frac{1}{2}}) \) and \( (2C_2)^{-1} \text{scal}(x_0, t_0) \leq \text{scal}(x, t_0) \leq 2C_2 \text{scal}(x_0, t_0) \) for all \( x \in B \). Moreover, \( B \) is either a strong \( 2\varepsilon \)-neck (in the sense of [32]) or a \( 2\varepsilon \)-cap.

**Proof.** We argue by contradiction. Suppose that the assertion is false. Then we can find a sequence of Ricci flows with surgery \( \mathcal{M}^{(j)} \) and a sequence of points \((x_j, t_j)\) in space-time with the following properties:

(i) The flow \( \mathcal{M}^{(j)} \) is defined on the time interval \([0, T_j]\) and has parameters \( \varepsilon, \hat{r}_j, h_j, \hat{\delta}_j \), where \( \hat{r}_j \leq \frac{1}{3} \) and \( \hat{\delta}_j \leq \min\{\hat{\delta}(\hat{r}_j), \frac{1}{j}\} \). Here, \( \hat{\delta}(\cdot) \) is the function in Proposition 10.13.

(ii) \( Q_j := \text{scal}(x_j, t_j) \geq (2\hat{r}_j)^{-2} \).

(iii) There does not exist a neighborhood \( B \) of \( x_j \) with the property that \( B_{g(t_j)}(x_j, (2C_1)^{-1} \text{scal}(x_j, t_j)^{-\frac{1}{2}}) \subset B \subset B_{g(t_j)}(x_j, 2C_1 \text{scal}(x_j, t_j)^{-\frac{1}{2}}) \) and \( (2C_2)^{-1} \text{scal}(x_j, t_j) \leq \text{scal}(x, t_j) \leq 2C_2 \text{scal}(x_j, t_j) \) for all \( x \in B \), and such that \( B \) is a strong \( 2\varepsilon \)-neck (in the sense of [32]) or a \( 2\varepsilon \)-cap.

By Proposition 10.8 we have \( |D\text{scal}| \leq \eta \text{scal}^\frac{1}{2} \) and \( \frac{\partial}{\partial t}\text{scal} \leq \eta \text{scal}^2 \) for each point \((x, t)\) in space-time satisfying \( \text{scal}(x, t) \geq 4Q_j \). Moreover, by Proposition 10.13 the flow \( \mathcal{M}^{(j)} \) is \( \kappa \)-noncollapsed on scales less than \( \varepsilon \) for some uniform constant \( \kappa \) which is independent of \( j \). We proceed in several steps:

**Step 1:** We claim that, if \((x_0, t_0)\) is a point in space-time with \( \text{scal}(x_0, t_0) + Q_j \leq r_0^{-2} \), then \( \vol_{g(t_0)}(B_{g(t_0)}(x_0, r_0)) \geq \kappa r_0^n \) for some uniform constant \( \kappa \) which is independent of \( j \). Indeed, if \( \text{scal}(x_0, t_0) + Q_j \leq r_0^{-2} \), then the point-wise curvature derivative estimate implies that \( \text{scal} \leq 8r_0^{-2} \) in the parabolic neighborhood \( P(x_0, t_0, \frac{r_0}{400}, \frac{r_0^2}{10000}) \). Hence, the lower bound for the volume follows from Proposition 10.13.
Step 2: We next prove a longrange curvature estimate. Given any \( \rho > 0 \), we put
\[
M(\rho) = \limsup_{j \to \infty} \sup_{x \in B_{g(t_j)}(x_j, \rho Q_j^{-1/2})} Q_j^{-1} \text{scal}(x, t_j).
\]
The pointwise curvature derivative estimate implies that \( M(\rho) \leq 16 \) for \( 0 < \rho < 100 \). We claim that \( M(\rho) < \infty \) for all \( \rho > 0 \). Suppose this is false. Let \( \rho^* = \sup\{\rho \geq 0 : M(\rho) < \infty\} < \infty \).

By definition of \( \rho^* \), we have a curvature bound on the geodesic ball \( B_{g(t_j)}(x_j, \rho Q_j^{-1/2}) \) for each \( \rho < \rho^* \). Moreover, the noncollapsing estimate in Step 1 gives a lower bound for the volume. We rescale around \( (x_j, t_j) \) by the factor \( Q_j \) and pass to the limit as \( j \to \infty \). In the limit, we obtain an incomplete manifold \( (B^\infty, g^\infty) \). Let \( x_\infty = \lim_{j \to \infty} x_j \). The limit \( (B^\infty, g^\infty) \) contains a minimizing geodesic \( \gamma_\infty \), parametrized by the interval \( [0, \rho^*] \) such that \( \gamma_\infty(0) = x_\infty \) and \( \text{scal}_{g^\infty}(\gamma_\infty(s)) \to \infty \) as \( s \to \rho^* \).

If \( s \) is sufficiently close to \( \rho^* \), then the Canonical Neighborhood Assumption with accuracy \( 4\epsilon \) implies that the point \( \gamma_\infty(s) \) has a Canonical Neighborhood which is either a strong \( 4\epsilon \)-neck or a \( 4\epsilon \)-cap. The second possibility can be ruled out as follows. If \( \gamma_\infty(s) \) lies on a \( 4\epsilon \)-cap for some \( s \) close to \( \rho^* \), then the geodesic \( \gamma_\infty \) must enter and exit the cap, but this is impossible since \( \gamma_\infty \) is minimizing. Hence, if \( s \) is sufficiently close to \( \rho^* \), then \( \gamma_\infty(s) \) lies on a strong \( 4\epsilon \)-neck, and the radius of the neck converges to 0 as \( s \to \rho^* \).

As in [31], Section 12.1, there is a sequence of rescalings which converges to a piece of a non-flat metric cone in the limit. Let us fix a point on this metric cone. In view of the preceding discussion, this point has a Canonical Neighborhood which must be a strong \( 4\epsilon \)-neck. This gives a locally defined solution to the Ricci flow which is weakly PIC2 and which, at the final time, is a piece of non-flat metric cone. This contradicts Proposition 7.5.

Step 3: We now dilate the manifold \( (M, g(t_j)) \) around the point \( x_j \) by the factor \( Q_j \). By Step 2, we have uniform bounds for the curvature at bounded distance. Using these estimates together with the noncollapsing estimate in Step 1, we conclude that the rescaled manifolds converge in the Cheeger-Gromov sense to a complete limit manifold \( (M^\infty, g^\infty) \). Since \( (M, g(t_j)) \) has \( (f, \theta) \)-pinched curvature, the curvature tensor of \( (M^\infty, g^\infty) \) is weakly PIC2 and satisfies \( R - \theta \text{scal id} \in \text{PIC} \). Using the Canonical Neighborhood Assumption with accuracy \( 4\epsilon \), we conclude that every point in \( (M^\infty, g^\infty) \) with scalar curvature greater than 4 has a neighborhood which is either a strong \( 4\epsilon \)-neck or a \( 4\epsilon \)-cap.

We claim that \( (M^\infty, g^\infty) \) has bounded curvature. Indeed, if there is a sequence of points in \( (M^\infty, g^\infty) \) with curvature going to infinity, then \( (M^\infty, g^\infty) \) contains a sequence of necks with radii converging to 0, contradicting Proposition 7.9. This shows that \( (M^\infty, g^\infty) \) has bounded curvature.
Step 4: We now extend the limit \((M^\infty, g^\infty)\) backwards in time. By Step 3, the scalar curvature of \((M^\infty, g^\infty)\) is bounded from above by a constant \(\Lambda > 4\). We claim that, given any \(A > 0\), the parabolic neighborhood \(P(x_j, t_j, AQ_j^{-\frac{1}{2}}, -\frac{1}{100^A} Q_j^{-1})\) is free of surgeries if \(j\) is sufficiently large.

To prove this, suppose that \(P(x_j, t_j, AQ_j^{-\frac{1}{2}}, -\frac{1}{100^A} Q_j^{-1})\) contains surgeries. Let \(s_j \in [0, \frac{1}{100^A}]\) be the largest number such that \(P(x_j, t_j, AQ_j^{-\frac{1}{2}}, -s_j Q_j^{-1})\) is free of surgeries. The pointwise curvature derivative estimate gives

\[
\sup_{P(x_j, t_j, AQ_j^{-\frac{1}{2}}, -s_j Q_j^{-1})} \text{scal} \leq 2\Lambda Q_j
\]

if \(j\) is sufficiently large. Since the scalar curvature is greater than \(\frac{1}{2} h_j^{-2}\) in the surgery region, we deduce that \(Q_j^{-1} \leq 4\Lambda h_j^2\). Let us choose \(\alpha \in (0, \frac{n-1}{2})\) so that \(\frac{\alpha}{K_{\text{std}} (1 - \frac{2\alpha}{n-1})} \geq 1\). If \(s_j Q_j^{-1} \geq \alpha h_j^2\) for \(j\) sufficiently large, then Proposition 10.9 implies

\[
\sup_{P(x_j, t_j, AQ_j^{-\frac{1}{2}}, -s_j Q_j^{-1})} \text{scal} \geq \frac{1}{2 K_{\text{std}} (1 - \frac{2\alpha}{n-1})} h_j^{-2} s_j^{-1} Q_j \\
\geq 4\Lambda Q_j
\]

for \(j\) sufficiently large, which is impossible. Therefore, \(s_j Q_j^{-1} \leq \alpha h_j^2\) for \(j\) sufficiently large. Hence, Proposition 10.9 implies that the parabolic neighborhood \(P(x_j, t_j, AQ_j^{-\frac{1}{2}}, -s_j Q_j^{-1})\) is, after dilating by the factor \(h_j\), arbitrarily close to a piece of the standard solution when \(j\) is sufficiently large. This implies that \((x_j, t_j)\) lies on a \(2\varepsilon\)-neck or a \(2\varepsilon\)-cap when \(j\) is sufficiently large. Moreover, if \((x_j, t_j)\) lies on an \(2\varepsilon\)-neck, then this neck is actually a strong \(2\varepsilon\)-neck, as each \(\delta_j\)-neck on which we perform surgery has a large backward parabolic neighborhood that is free of surgeries. This contradicts statement (iii).

To summarize, we have shown that, for each \(A > 0\), the parabolic neighborhood \(P(x_j, t_j, AQ_j^{-\frac{1}{2}}, -\frac{1}{100^A} Q_j^{-1})\) is free of surgeries if \(j\) is sufficiently large. Consequently, we can extend the limit \((M^\infty, g^\infty)\) backward in time to a solution \((M^\infty, g^\infty(t))\), \(t \in [\tau_1, 0]\), where \(\tau_1 := -\frac{1}{100^A}\). Moreover, \(\Lambda_1 := \sup_{t \in [\tau_1, 0]} \sup_{M^\infty} \text{scal}_{g^\infty(t)} \leq 2\Lambda\) by the pointwise curvature derivative estimate.

In the next step, we put \(\tau_2 := \tau_1 - \frac{1}{100^A}\). Arguing as above, we can show that, for each \(A > 0\), the parabolic neighborhood \(P(x_j, t_j, AQ_j^{-\frac{1}{2}}, \tau_1 - \frac{1}{100^A} Q_j^{-1})\) is free of surgeries if \(j\) is sufficiently large. Hence, we can extend the solution \((M^\infty, g^\infty)\), \(t \in [\tau_1, 0]\), backwards in time to a solution \((M^\infty, g^\infty(t))\), \(t \in [\tau_2, 0]\). Moreover, \(\Lambda_2 := \sup_{t \in [\tau_2, 0]} \sup_{M^\infty} \text{scal}_{g^\infty(t)} \leq 2\Lambda_1\) by the pointwise
curvature derivative estimate. Continuing this process, we can extend the solution backwards in time to the interval $[\tau_m, 0]$, where $\tau_{m+1} := \tau_m - \frac{1}{100\eta\Lambda_m}$ and $\Lambda_{m+1} := \sup_{t \in [\tau_{m+1}, 0]} \sup_{M^\infty} \text{scal}_{g^\infty(t)} \leq 2\Lambda_m$.

Let $\tau^* = \lim_{m \to \infty} \tau_m \leq -\frac{1}{100\eta}$. Using a standard diagonal sequence argument, we obtain a complete, smooth limit flow $(M^\infty, g^\infty(t))$ which is defined on the interval $(\tau^*, 0]$ and which has bounded curvature for each $t \in (\tau^*, 0]$.

Step 5: Arguing as in Step 5 in the proof of Theorem 8.2, we can show that $\tau^* = -\infty$. Hence, if we dilate the flow $(M, g(t))$ around the point $(x_j, t_j)$ by the factor $Q_j$, then (after passing to a subsequence) the rescaled flows converge to an ancient solution which is complete; has bounded curvature; is weakly PIC; and satisfies $R - \theta \text{scal} \leq \text{id} \in \text{PIC}$. By Proposition 10.13, the limiting ancient solution is $\kappa$-noncollapsed for some $\kappa > 0$. The universal noncollapsing property in Theorem 7.16 implies that the limit is an ancient $\kappa_0$-solution. By Corollary 7.17, every point on the limiting ancient solution has a Canonical Neighborhood which is either a strong $2\epsilon$-neck; or a $2\epsilon$-cap; or a quotient neck. If the Canonical Neighborhood is a strong $2\epsilon$-neck or a $2\epsilon$-cap, we obtain a contradiction with statement (iii). If the Canonical Neighborhood is a quotient neck, then the manifold $(M, g(t_j))$ contains a quotient neck if $j$ is sufficiently large. Theorem A.1 in [6] then implies that $M$ contains a non-trivial incompressible space form, contrary to our assumption. This completes the proof of Theorem 10.14.

11. Global existence of surgically modified flows

As in the preceding sections, we fix a compact initial manifold $(M, g_0)$ of dimension $n \geq 12$ which has positive isotropic curvature and does not contain any non-trivial incompressible space forms. We begin by finalizing our choice of the surgery parameters $\epsilon, \hat{r}, \hat{\delta}$. As above, we fix a small number $\epsilon > 0$. Having chosen $\epsilon$, we choose numbers $\hat{r}, \hat{\delta}$ such that the conclusion of Theorem 10.14 holds. Having chosen $\epsilon, \hat{r}, \hat{\delta}$, we choose $h$ so that the following holds:

**Proposition 11.1** (cf. G. Perelman [32], Lemma 4.3). Given $\epsilon, \hat{r}, \hat{\delta}$, we can find a small number $h \in (0, \hat{\delta} \hat{r})$ with the following property. Suppose that we have a Ricci flow with surgery with parameters $\epsilon, \hat{r}, \hat{\delta}, h$ which is defined on the time interval $[0, T)$ and goes singular at time $T$. Let $x$ be a point which lies in an $4\epsilon$-horn in $(\Omega, g(T))$ and has curvature $\text{scal}(x, T) = h^{-2}$. Then the parabolic neighborhood $P(x, T, \hat{\delta}^{-1}h, -\hat{\delta}^{-1}h^2)$ is free of surgeries. Moreover, $P(x, T, \hat{\delta}^{-1}h, -\hat{\delta}^{-1}h^2)$ is a $\hat{\delta}$-neck.

**Proof.** Suppose that the assertion is false. Then there exists a sequence of positive numbers $h_j \to 0$, a sequence of Ricci flows with surgery $\mathcal{M}^{(j)}$ and a sequence of points $x_j$ with the following properties:
The second case can easily be ruled out, so
\( y \) is sufficiently large. By Proposition 10.8, we have a non-flat metric cone. This proves the claim. PIC2. Hence, Proposition 7.5 implies that the limit cannot be a piece of a horn which is free of surgeries. In view of Proposition 10.7, the limit is weakly neck. In particular, there exists a small parabolic neighborhood of \((y_j, T_j)\) which is free of surgeries. In view of Proposition 10.7, the limit is weakly PIC2. Hence, Proposition 7.5 implies that the limit cannot be a piece of a non-flat metric cone. This proves the claim.

Therefore, for every \( A > 1 \) we can find a constant \( Q(A) \) (depending on \( A \), but not on \( j \)) such that \( \sup_{x \in B_{g(T_j)}(x_j, Ah_j)} \text{scal}(x, T_j) \leq Q(A) h_j^{-2} \) if \( j \) is sufficiently large. By Proposition 10.8 we have \( |D\text{scal}| \leq \eta \text{scal}^2 \) whenever \( \text{scal} \geq \hat{r}^{-2} \). Since \( h_j \to 0 \), it follows that \( \inf_{x \in B_{g(T_j)}(x_j, Ah_j)} \text{scal}(x, T_j) \geq (1 + \eta A)^{-2} h_j^{-2} \geq 100 (\hat{r})^{-2} \) if \( j \) is sufficiently large. Consequently, if \( j \) is sufficiently large, then the distance of the point \( x_j \) from either end of the horn is at least \( Ah_j \).

We now continue with the proof of Proposition 11.1. Let us fix a number \( A > 1 \). Since the point \((x_j, T_j)\) lies on a \( 4\varepsilon \)-horn, no point in \( B_{g(T_j)}(x_j, Ah_j) \) can lie on a \( 4\varepsilon \)-cap. Hence, the Canonical Neighborhood Assumption implies that every point in \( B_{g(T_j)}(x_j, Ah_j) \) lies on a strong \( 4\varepsilon \)-neck. In particular, the parabolic neighborhood \( P(x_j, T_j, Ah_j, -\frac{h_j^2}{2Q(A)}) \) is free of surgeries. We dilate the flow around \((x_j, T_j)\) by the factor \( h_j^{-2} \). After rescaling, the curvature is uniformly bounded in this parabolic neighborhood. Hence, Shi’s estimate gives bounds for all the covariant derivatives of curvature. Note that these bounds are independent of \( j \), for every fixed \( A \). We now pass to the limit, sending \( j \to \infty \) first and \( A \to \infty \) second. In the limit, we obtain a complete manifold, which is uniformly PIC and weakly PIC2 by Proposition 10.7. By the Cheeger-Gromoll splitting theorem, the limit must split off a line; moreover, the cross section must be diffeomorphic to \( S^{n-1} \). Using the Canonical Neighborhood Assumption again, we conclude that, for each \( A > 1 \), the parabolic neighborhood \( P(x_j, T_j, Ah_j, -\frac{h_j^2}{2Q(A)}) \) is free of surgeries if \( j \) is sufficiently large. After rescaling and passing to the limit, we obtain
a solution to the Ricci flow which is defined on the time interval $[-\frac{1}{2}, 0]$ and which splits off a line. Moreover, every point on the limit solution has a Canonical Neighborhood which is a $4\varepsilon$-neck. Repeating this argument, we can extend the limit solution backwards in time, so that it is defined on $[-1, 0], [-\frac{3}{2}, 0], [-2, 0], \ldots$ etc. To summarize, we produce a limit solution which is ancient; uniformly PIC; weakly PIC2; and splits as a product of a line with a manifold diffeomorphic to $S^{n-1}$. By Theorem 7.4, the limiting solution is a family of round cylinders. Therefore, if $j$ is sufficiently large, the parabolic neighborhood $P(x_j, T_j, \delta^{-1}h, -\delta^{-1}h^2)$ is free of surgeries, and $P(x_j, T_j, \delta^{-1}h, -\delta^{-1}h^2)$ is a $\delta$-neck. This contradicts (iii).

We are now able to prove the main result of this section:

**Theorem 11.2.** Let us fix a small number $\varepsilon > 0$. Let $\tilde{r}, \tilde{\delta}$ be chosen as described at the beginning of this section, and let $h$ be chosen as in Proposition 11.1. Then there exists a Ricci flow with surgery with parameters $\varepsilon, \tilde{r}, \tilde{\delta}, h$, which is defined on some finite time interval $[0, T)$ and becomes extinct as $t \to T$.

**Proof.** We evolve the initial metric $g_0$ by smooth Ricci flow until the flow becomes singular for the first time. At the first singular time, we perform finitely many surgeries on $\delta$-necks which have curvature level $h$. After performing surgery, we restart the flow, and continue until the second singular time. Using Theorem 10.14 and a standard continuity argument, we conclude that the flow with surgery satisfies the Canonical Neighborhood Property with accuracy $2\varepsilon$ on all scales less than $2\tilde{r}$, up until the second singular time. Consequently, Proposition 11.1 ensures that, at the second singular time, we can again find $\delta$-necks on which to perform surgery. We then perform surgery again, and continue the flow until the third singular time. It follows from Theorem 10.14 and a standard continuity argument that the flow with surgery satisfies the Canonical Neighborhood Property with accuracy $2\varepsilon$ on all scales less than $2\tilde{r}$, up until the third singular time. We can now perform surgery again, and repeat the process.

Since each surgery reduces the volume by at least $h^n$, the surgery times cannot accumulate. By Proposition 10.6, the flow with surgery must become extinct by time $\frac{n}{2} \inf_{x \in M} \text{scal}(x, 0)$ at the latest.

**Corollary 11.3.** A compact manifold of dimension $n \geq 12$ which has positive isotropic curvature and does not contain any non-trivial incompressible space forms is diffeomorphic to a connected sum of quotients of $S^n$ and $S^{n-1} \times \mathbb{R}$.

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