Interpolating statistics realized as Deformed Harmonic Oscillators

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Abstract

The idea that a system obeying interpolating statistics can be described by a deformed oscillator algebra has been an outstanding issue. This original concept introduced long ago by Greenberg is the motivation for this investigation. We establish that a q-deformed algebra can be used to describe the statistics of particles (anyons) interpolating continuously between Bose and Fermi statistics, i.e., fractional statistics. We show that the generalized intermediate statistics splits into the Boson-like and Fermion-like regimes, each described by a unique oscillator algebra. The B-anyon thermostatistics is described by employing the q-calculus based on the Jackson derivative but the F-anyons are described by ordinary derivatives of thermodynamics. Thermodynamic functions of both B-anyons and F-anyons are determined and examined.

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I. INTRODUCTION

The statistical mechanics of anyons existing in 2+1 space-time dimensions was investigated in detail sometime ago [1], where many of the thermodynamic properties of these particles, such as the partition function, entropy, pressure, internal energy, virial expansion and specific heat were determined. The analysis of thermostatistics in that investigation was performed in two space dimensions based on a simple ansatz for the anyon distribution function. In particular it was found that a natural distinction arises between boson-like and fermion-like anyons. Virial coefficients of the two dimensional anyon gas were determined and compared with earlier investigations on the subject [2]. More recently, the subject of generalized statistics has been studied in other investigations [3,4], where oscillator algebras and field theory in low space dimensions are employed. There has also been great interest in a comparative study of different theories of interpolation [5] in the context of Haldane and Gentile statistics.

Physical systems in 2+1 space-time dimensions display many unusual quantum properties, especially relating to rotation and spin. These features lead to interesting results for the quantum mechanics of angular momentum in two space dimensions giving rise to fractional spin, and hence not quantized in familiar half integer multiples of the Planck constant. These objects are named in the literature as anyons, representing fractional statistics which accordingly interpolate between Bosons and Fermions determined by the relation defining permutation symmetry for the many body wave function

$$
\psi(x_1, \cdots x_j \cdots x_i \cdots x_n) = e^{i\pi \alpha} \psi(x_1, \cdots x_i \cdots x_j \cdots x_n),
$$

(1)

where the statistics determining parameter $\alpha$ in this relation is a real number and the limits $\alpha = 0, 1$ correspond respectively to Bosons and Fermions. Since the permutation symmetry is related to rotations in two space dimensions, anyons may be described by the braid group, described by the infinite group of $N$ strands represented by products of Pauli matrices [6]. This connection between fractional spin and fractional statistics, with Chern-Simons Field theory providing a representation for the anyons, has been the reason why anyons have been thought to exist only in 2+1 dimensions. It must, however, be stressed that permutation symmetry or the exchange symmetry is more general than what arises from rotations.

Indeed, a theory of interpolating or intermediate statistics has recently been formulated which provides a definite clarification of this issue. In this investigation [7], a theory of the exchange symmetry of many particles is developed so as to allow a continuous interpolation between Bose and Fermi statistics. This theory has many interesting features, one of them being that the system is not restricted to 2+1 dimensions. The other features are: interpolating statistics arises directly from principles of thermodynamics, specifically the principle of detailed balance; basic numbers arise naturally and automatically in this theory; the theory reproduces standard expressions in the Bose and Fermi limits; the general expression for the distribution function is the solution of a transcendental equation which can be expressed as a power series and also in the form of a continued fraction; the first approximant of this solution agrees with an approximate distribution function introduced as an ansatz in the thermostatistics of anyons investigated earlier [1].
There is another generalization of the standard thermostatistics that has been studied extensively in the literature which has to do with the theory of $q$-deformed quantum oscillators i.e., quantum groups [8]. These are objects described by a deformed algebra of the harmonic oscillators signified by a parameter $q$ so that $q = 1$ corresponds to the boson oscillators. It has been shown that the $q$-calculus based on the Jackson derivative (JD) can be employed rather than the ordinary partial derivatives in order to describe the thermostatistics of $q$-bosons [9]. Indeed it has been shown that the complete theory of generalized thermodynamics of $q$-bosons and $q$-fermions can be developed using basic numbers with the base $q$ and employing the $q$-calculus based on JD. The thermodynamic functions such as entropy, pressure, internal energy, specific heat etc. of such systems have been studied [10] and compared with ordinary bosons and fermions.

As the generalized thermodynamics of generalized bosons and fermions can be successfully described by a theory based on the $q$-deformed algebra, we can accordingly pose the following question: What kind of deformed algebra would describe interpolating statistics, the statistics of anyons? This question is even more interesting due to the fact that the recent derivation of interpolating statistics from the principle of detailed balance [7] introduces the basic numbers naturally but without the benefit of an oscillator algebra which leads to these basic numbers. Attempts have been made by Greenberg and others [11] to investigate $q$-deformed algebras as possible foundations of interpolating statistics.

At one time it was regarded as conventional wisdom that generalized or interpolating statistics can have no relation to the algebra of deformed harmonic oscillators since oscillators exist in any dimensions [6]. The connection with quantum groups was originally discovered in [12] and subsequent investigations have employed quantum groups in lower dimensions [3,4]. It has been pointed out that the issues of quantum statistics are very different from commutation and anticommutation relations for local fields [6]. It has been conjectured that quantum groups might be the characteristic symmetry structures of anyon systems even though explicit realizations of this fact are still missing and thus a direct relation between the two should exist [4]. If anyons are described in two dimensions, and since the real world is of three space dimensions, anyons may not be real particles. All of this might be an indication that there ought to be an approach to interpolating statistics other than from spin and rotation and this is primarily the goal of the present investigation.

Anyons might well be composites from charged particles and magnetic flux tubes. In what follows we shall use the term *anyons*, to refer to particles obeying interpolating statistics. We shall use this term in its generic sense and not as particles described by any particular model investigated in the literature and not restricted to 2+1 space dimensions. This has indeed been an outstanding problem and never resolved until the present investigation. The fundamental question that comes to mind is: Can a deformed oscillator algebra describe interpolating statistics? If so, what kind of algebra would be most appropriate to describe a statistical theory which would interpolate between Bose and Fermi statistics? It is indeed the spirit of current wisdom that interpolation might be considered in some sense as a deformation and thus we expect the two aspects to be closely related. The investigation of several possible algebras leads one to conclude that many of these are not satisfactory:
some do not have the correct Bose or Fermi limits; some are inconsistent; others do not lead to suitable real distribution functions etc. Recently a successful formulation of the theory of $q$-deformed bosons and fermions was based on the premise that while $q$-bosons and $q$-fermions arise from different oscillator algebras, they can be considered as special cases of one fundamental algebra. We shall accordingly begin with this algebra which may be defined by

$$aa^\dagger - \kappa q a^\dagger a = q^{-N}, \quad 0 \leq q \leq 1,$$

where $\kappa = 1$ will describe boson-like anyons, and $\kappa = -1$ will be used to describe fermion-like anyons. Here $N$ is the number operator which acts on the Fock space of oscillator states. We shall introduce the abbreviations B-anyons and F-anyons for simplicity. This algebra was investigated for arbitrary parameter $q$ in [10] and $\kappa$ was taken to be either 1 or -1 to describe $q$-bosons and $q$-fermions. In the present work, we shall investigate the B-anyons and F-anyons, corresponding to $\kappa = 1$ and $\kappa = -1$ separately arising from the corresponding algebra and treat $q$ as a parameter which will provide the interpolation between the Bose and Fermi cases. It must be stressed that the interpretation in the present work is different and the consequences are also quite different from [10].

II. THE ALGEBRA OF B-ANYONS

We shall examine the algebra defined by

$$aa^\dagger - qa^\dagger a = q^{-N}, \quad 0 \leq q \leq 1,$$

where $N$ is the number operator and $q \to 1$ corresponds to the Bose limit. We may also add the relations

$$[N, a] = -a, \quad [N, a^\dagger] = a^\dagger,$$

which can be established by examining the action on Fock states. It is well-known that this algebra leads to the introduction of basic numbers. Nevertheless, it is instructive to briefly review the analysis.

Let us define the operator $\tilde{N} = a^\dagger a$ and its action on the Fock state by $\tilde{N}|n\rangle = \alpha_n|n\rangle$, assuming that the eigenvalue depends on $n$. First we derive the algebra $\tilde{N}a^\dagger - qa^\dagger \tilde{N} = a^\dagger q^{-N}$ as a consequence of the algebra in Eq.(3). This leads to the recurrence relation

$$\alpha_{n+1} = q^{-n} + q\alpha_n,$$

the solution of which is immediately seen to be

$$\alpha_n = q^{n-1} + q^{n-3} + \cdots + q^{-n+3} + q^{-n+1}.$$ 

This is recognized as the familiar basic number

$$\alpha_n = [n] = \frac{q^n - q^{-n}}{q - q^{-1}},$$

with the corresponding operator form defined by
\[ [N] = \frac{q^N - q^{-N}}{q - q^{-1}}. \quad (7) \]

In this manner we see that the basic number \( a^\dagger a = [N] \), which is different from the number operator, arises as a consequence of the algebra in Eq.(3) and is uniquely related to this algebra.

In order to build the Fock space, we first observe the results \( a|n\rangle = \sqrt{\alpha_n} |n - 1\rangle \) and \( a^\dagger|n\rangle = \sqrt{\alpha_{n+1}} |n + 1\rangle \). We may then proceed to construct the Fock states in the familiar straightforward manner, thus obtaining

\[ |n\rangle = \frac{(a^\dagger)^n}{[n]!} |0\rangle, \quad (8) \]

where\[
[n]! = [n][n - 1] \cdots [2] \cdot 1. \quad (9)\]

All of these reduce to the standard results of boson oscillators in the Bose limit. We note that there are no restrictions on the Fock states, with \( n \) corresponding to any integer.

To proceed further, we introduce the Hamiltonian

\[ H = \sum_i N_i (E_i - \mu). \quad (10) \]

We note that contrary to appearances, this Hamiltonian does indeed incorporate interpolation (or deformation), since the occupation number depends on the parameter \( q \) as we shall see. From the definition of the mean value

\[ q^{\pm n_k} = \frac{1}{\mathcal{Z}} \text{Tr}(e^{-\beta H} q^\pm N_k), \quad (11) \]

we derive

\[ [n_k] = \frac{1}{\mathcal{Z}} \text{Tr}(e^{-\beta H} [N_k]). \quad (12) \]

From the cyclic property of the trace and the property \( f(N) a^\dagger = a^\dagger f(N + 1) \), valid for polynomial functions, we obtain the result

\[ e^{\beta(E_k - \mu)} = \frac{q^{-n_k} + q^{[n_k]}}{[n_k]}, \quad (13) \]

where the right hand side can be expressed in terms of \([n_k + 1]\). This leads to the solution for the mean occupation number:

\[ n_i = \frac{1}{\ln q} \ln \left( \frac{e^{\beta(E_i - \mu)} - q^{-1}}{e^{\beta(E_i - \mu)} - q} \right). \quad (14) \]
This is a familiar result for q-boson oscillators, signifying the effects of \( q \)-deformation [10]. Here, however, we are interpreting \( q \) as the parameter of interpolating statistics. This is seen to reduce to the ordinary Boson distribution function in the Bose limit.

The theory of \( q \)-deformed bosons is also characterized by the Bargman-Wigner holomorphic representation of the annihilation and creation operators, namely

\[
a \iff D^q(x), \quad x \iff a^\dagger,
\]

where \( D^q(x) \) denotes the Jackson derivative (JD) defined by

\[
D^q(x)f(x) = \frac{f(qx) - f(q^{-1}x)}{x(q - q^{-1})}.
\]

The JD arises naturally in \( q \)-deformed algebras [13] and is intimately linked with basic numbers. It has also been shown that the connection of the basic number with JD may be attributed to deformed Heisenberg algebra [14]. It has been further established [10] that the structure of thermostatistics is preserved if ordinary derivatives of thermodynamics are replaced by JD [9,10]. Accordingly, the various thermodynamic quantities of interest can be obtained by employing the JD. The results obtained in [10] are therefore readily seen to apply to the interpolating statistics for the case of B-anyons. Specifically we have to set \( \kappa = 1 \) in the results of [10]. We shall now summarize these results for the various thermodynamic functions.

The logarithm of the partition function is given by

\[
\ln Z = -\sum_i \ln(1 - ze^{-\beta E_i})
\]

where \( z = e^{\beta \mu} \) is the fugacity, as inferred from

\[
N = \sum_i n_i = zD^{(q)}(z) \ln Z.
\]

The \( q \)-dependence arises after employing the JD. To obtain the various thermodynamic functions, it is convenient to replace the sums over states by integrals in the usual manner

\[
\sum_i \implies V(2\pi)^{-3} \int d^3k,
\]

where \( V \) is the volume, and introduce [15] the thermal wavelength \( \lambda = h/\sqrt{2\pi mkT} \). In this manner, we obtain the expression for the internal energy

\[
U = \frac{3}{2\lambda^3} VT g_{5/2}(q, z),
\]

where

\[
g_n(q, z) = \frac{1}{q - q^{-1}} \left( \sum_{r=1}^{\infty} \frac{(q z)^r}{r^{n+1}} - \sum_{r=1}^{\infty} \frac{(q^{-1} z)^r}{r^{n+1}} \right),
\]

\[6\]
corresponds to the familiar [1,10] generalized Riemann Zeta functions involving both $q$ and $q^{-1}$, which reduces to the standard $g_n$ function in the Bose limit. The entropy of the B-anyons is given by

$$\frac{S}{V} = \frac{1}{\lambda^3} \left\{ \frac{5}{2} g_{5/2}(q, z) - g_{3/2}(q, z) \ln z \right\}.$$  \hspace{1cm} (22)

Recalling the range $0 \leq q \leq 1$, we observe that the thermodynamic quantities such as the internal energy, entropy etc. for the B-anyons are smaller than for ordinary Bosons. This follows from the known behavior of these functions [15] as a function of the argument. The pressure of the B-anyon gas is given by

$$\frac{P}{T} = \frac{1}{\lambda^3} g_{3/2}(q, z),$$  \hspace{1cm} (23)

which is discussed in [10].

Finally, considering that the effects due to the statistics parameter $q$ are rather subtle, we would like to consider, as an illustration, the virial expansion in the case of B-anyons in some detail. We begin with the result

$$\frac{\lambda^3}{v} = g_{3/2}(z, q),$$  \hspace{1cm} (24)

where $v = V/N$. The generalized function here can be conveniently expressed in the form of the series

$$g_{3/2}(z, q) = z + \frac{[2]}{2^{5/2}} z^2 + \frac{[3]}{3^{5/2}} z^3 + \cdots,$$  \hspace{1cm} (25)

and it is easy to see that it reduces to the ordinary function $g_{3/2}(z)$ in the limit $q \to 1$. Now for $q \neq 1$, we have the result in Eq.(24) given by the above series, which can be reverted to obtain

$$z = \left( \frac{\lambda^3}{v} \right) - \frac{[2]}{2^{5/2}} \left( \frac{\lambda^3}{v} \right)^2 + \left( \frac{[2]}{2^2} - \frac{[3]}{3^{5/2}} \right) \left( \frac{\lambda^3}{v} \right)^3 + \cdots.$$  \hspace{1cm} (26)

Since

$$\frac{P v}{kT} = \frac{v}{\lambda^3} g_{5/2}(q, z),$$  \hspace{1cm} (27)

upon substitution, we immediately obtain the virial expansion given by

$$\frac{P v}{kT} = 1 - \frac{[2]}{2^{7/2}} \frac{\lambda^3}{v} + \cdots.$$  \hspace{1cm} (28)

In the Bose limit, this reduces to the standard virial expansion. However, the above result shows that for B-anyons, for $q \neq 1$, the virial coefficients generally involve the basic numbers and hence $q$ dependent. On the basis of the second term, we might state that the second
and other virial coefficients for the B-anyons are $q$-dependent and larger than for standard Bosons.

In conclusion, the algebra in Eq.(3) describes the interpolating statistics of Boson-like particles, B-anyons. The system is described in terms of JD and the basic numbers. All the thermodynamic functions can be determined in terms of the statistics determining parameter $q$ and the properties of the gas, such as the equation of state etc., can be described as in [10].

III. B-ANYON DISTRIBUTION FUNCTION

We may further investigate the distribution function of the B-anyons as follows. We notice that the logarithmic form in Eq.(14) enables us to express the distribution function in the form

$$n_i = -\frac{1}{2 \ln q^{-1}} \ln \left( 1 - \frac{q^{-1} - q}{e^{\eta_i} - q} \right),$$

where $\eta_i = \beta(E_i - \mu)$. We may then express this in the form of a continued fraction,

$$n_i = \left( \frac{1}{2 \ln q^{-1}} \right) \frac{y_i}{1 - \frac{1^2 y_i}{2 - \frac{2^2 y_i}{3 - \frac{3^2 y_i}{4 - \cdots}}}},$$

where $y_i = (q^{-1} - q)/(e^{\beta(E_i - \mu)} - q)$. We note that $q^{-1} > q$. The infinite continued fraction form [16], besides being a convenient tool to express an infinite series in a compact form, has many desirable properties. For instance we observe that the first two convergents (approximants) are given by

$$n_i^{(1)} = \left( \frac{q^{-1} - q}{2 \ln q^{-1}} \right) \frac{1}{e^{\eta_i} - q},$$

and

$$n_i^{(2)} = \left( \frac{q^{-1} - q}{2 \ln q^{-1}} \right) \frac{1}{e^{\eta_i} - \left( \frac{q + q^{-1}}{2} \right)}.$$

The convergents play important roles in the theory of continued fractions. For instance, there exists the theorem of inequalities,

$$n^{(1)} < n^{(3)} < \cdots < n < \cdots n^{(4)} < n^{(2)},$$

(33)
which provides a convenient algorithm to address the question of how best to approximate
the continued fraction. In particular, we observe that the exact distribution function of the
B-anyons is bounded between the first two convergents, thus:

\[ n^{(1)} < n < n^{(2)}. \]  

(34)

If we choose a value \( q = \frac{1}{2} \) for the purpose of illustration, we obtain the result

\[ \left( \frac{3}{4 \ln 2} \right) \frac{1}{e^{n} - 0.5} < n_i < \left( \frac{3}{4 \ln 2} \right) \frac{1}{e^{n} - 2.5}, \]

(35)

valid when \( q = \frac{1}{2} \). It is important to stress that each of the quantities on the left, on the
right and in the middle of the above inequality relation are exact. Moreover, Eqs.(31,32)
are exact results for arbitrary values of \( q \). This is undoubtedly a remarkable result defined
by the exact upper and lower bounds. Moreover, the form of the distribution function is,
other than the numerical values, quite similar to the form used in the ansatz of [1].

**IV. F-ANYONS**

In order to describe the Fermion-like anyons, we shall now investigate the algebra defined
by

\[ aa^\dagger + q^{-1}a^\dagger a = q^{-N}, \]

(36)
together with the relations

\[ [N, a] = -a, \quad [N, a^\dagger] = a^\dagger, \]

(37)

where \( N \) is the number operator, \( 0 \leq q \leq 1 \) and the Fermi limit is defined by \( q \to 1 \). This
corresponds to the case when we set \( \kappa = -1 \) in Eq.(2). This algebra has been investigated in
the literature [10] but here we use this algebra to describe interpolating statistics of Fermion-
like particles (F-anyons) with \( q \) as the statistics determining parameter. Let us introduce
the operator \( a^\dagger a = \hat{N} \) and assume that the action on the Fock state can be described by
\( \hat{N}|n\rangle = \beta_n|n\rangle \) where the eigenvalue depends on \( n \). We can determine \( \beta_n \) as before. First we
observe that the relation \( \hat{N}a^\dagger + q^{-1}a^\dagger\hat{N} = a^\dagger q^{-N} \) is true. We also know that the annihilation
and creation operators \( a, a^\dagger \) upon acting on state \( |n\rangle \) lower and raise the number of quanta,
with appropriate constants to be determined. As a consequence of this relation, we obtain
the result

\[ \beta_{n+1} = q^{-n} - q^{-1}\beta_n. \]  

(38)

We can choose \( \beta_0 = 0 \), thus defining the ground state as vacuum and may then determine
\( \beta_n \) by solving the above recurrence relation. We accordingly obtain

\[ \beta_n = 0, 1, 0, q^{-2}, 0, q^{-4}, \ldots. \]  

(39)

Thus
\[ \beta_n = \frac{1 - (-1)^n}{2} q^{-n+1}, \]  

which reduces to 0 and \( q^{-n+1} \) respectively when \( n \) is an even or odd number. This behavior is totally in contrast with the case of B-anyons.

From the action of the creation and annihilation operators on the Fock states, we further obtain the results

\[ a^\dagger |0\rangle = \sqrt{\beta_1} |1\rangle = |1\rangle; \quad a^\dagger a^\dagger |0\rangle = \sqrt{\beta_1} \sqrt{\beta_2} |2\rangle = 0. \]  

Consequently the sequence of states terminates and hence the Fock states are \( |0\rangle, |1\rangle \) only. Therefore in spite of the interpolating statistics, the F-anyons obey Pauli exclusion principle, just as ordinary Fermions do. This fact makes the interpolating statistics in this investigation very different from other theories.

Furthermore we observe that the algebra in Eq.(36) does not admit of a basic number as a solution of Eq.(38), i.e., there is no basic number corresponding to the algebra in Eq.(36). The result \( a^\dagger a = [N], aa^\dagger = [1 - N] \) is not true in the present case when the creation and annihilation operators satisfy the algebra, Eq.(36) in contrast to the case in [10]. Instead, our analysis reveals the operator relations

\[ a^\dagger a = \hat{N} = \frac{1 - (-1)^N}{2} q^{-N+1}, \quad aa^\dagger = q^{-N} - q^{-1} \hat{N}, \]  

Moreover, since the algebra is not related to basic numbers, the theory of F-anyons does not require the use of JD and accordingly we would employ the ordinary derivatives of thermodynamics.

From the definition of the mean value,

\[ \hat{n} = \frac{1}{Z} \text{Tr}(e^{-\beta H} a^\dagger a), \]  

proceeding as before, we obtain the result

\[ \frac{1}{2}(1 - (-1)^n) = \frac{q^{-1}}{e^{\beta(E - \mu)} + q^{-1}}. \]  

We may re-express this to obtain the distribution function in the form

\[ n_i = \frac{2}{\pi} \arcsin \left( \sqrt{\frac{q^{-1}}{e^{\eta_i} + q^{-1}}} \right), \]  

where \( \eta_i = \beta(E_i - \mu) \). It is possible to express the distribution function in the form of a series containing powers of \( g \), where

\[ g = \frac{q^{-1}}{e^{\eta} + q^{-1}}, \]
is a familiar form which occurs in the theory of anyons [1] and hence this form is useful. Thus, using Mathematica, we obtain the result as a power series,

\[
n = \frac{1}{\sqrt{g}} + \frac{7\sqrt{g}}{6} + \frac{149g^{3/2}}{120} + \frac{2161g^{5/2}}{1680} + \cdots .
\]  

(47)

This is the form from which we may obtain all the thermodynamic functions for the F-anyons. For instance, the logarithm of the partition function can also be expressed in the form of a power series. However, a simpler approach prevails as we shall see below.

V. THERMOSTATISTICS OF F-ANYONS

The consequences of Eq.(47) may be rather intractable. However, we can take advantage of the considerable simplification that exists in the case of the F-anyons as follows, and thus there is no need to deal with the series form. We recall that the Fock states reduce to \( n = 0, 1 \) only. In this case, referring to Eqs.(44,45), \( \sin^2 \frac{n\pi}{2} \) takes the values 0,1 only and hence can be replaced by \( n \). Thus the distribution function of Eq.(44) simplifies to the form

\[
n_i = \frac{q^{-1}}{e^{\beta(E_i-\mu)} + q^{-1}} .
\]  

(48)

Other than the factor in the numerator, we recognize this to be of the same form as an ansatz used in a previous investigation [1]. It has the correct Fermi limit. We shall now use this to investigate the thermostatistics. It is important to stress here that in the case of F-anyons, we should not use the JD but must employ ordinary derivatives.

We begin by observing that the logarithm of the partition function is given by

\[
\ln Z = \sum_i \ln(1 + q^{-1}ze^{-\beta E_i}) ,
\]  

(49)

considerably different from the expression in Eq.(17). We also note that partition function is explicitly dependent on the parameter \( q \). Since

\[
n_i = z \frac{\partial}{\partial z} \ln Z ,
\]  

(50)

this produces the distribution function given by Eq.(48). Upon converting the sum over states by an integration in the standard manner, and introducing the thermal wavelength, we obtain for the thermodynamic potential the expression

\[
\Omega = -\frac{1}{\beta} \ln Z = -\frac{1}{\beta\lambda^3} \ln(1 + q^{-1}z) - \frac{1}{\beta\lambda^3} f_{5/2}(q, z) ,
\]  

(51)

where the function \( f_n \) defined by

\[
f_n(q^{-1}z) = \sum_{r=1}^{\infty} (-1)^{r+1} \frac{(q^{-1}z)^r}{r^n} ,
\]  

(52)
is the generalization of the Riemann Zeta function for the Fermion-like particles. All of
these functions reduce to the standard Fermion case in the limit when \( q \to 1 \). The first term
corresponds to the fact that we have isolated the zero momentum state in the standard
manner. The pressure is obtained in the thermodynamic limit as

\[
P = \lim_{V \to \infty, N \to \infty} \left( -\frac{\Omega}{V} \right).
\]  

(53)

We thus determine the pressure of the F-anyons to be

\[
P = \frac{1}{\beta \lambda^3} f_{5/2}(q^{-1}z)
\]  

(54)

which agrees with the standard expression [15] in the Fermi limit. In a similar manner we
determine the mean density to be given in the thermodynamic limit by

\[
\frac{n}{V} = \frac{1}{\lambda^3} f_{3/2}(q^{-1}z).
\]  

(55)

We can learn how these thermodynamic quantities for F-anyons compare with the corre-
sponding ones for ordinary fermions by studying the graphs [15] for the functions
\( f_{3/2}, f_{5/2} \). Observing that \( f_n(q^{-1}z) > f_n(z) \), it thus follows for instance that the pressure of the F-
anyon gas is greater than that of ordinary fermions at the same temperature and for the
same fugacity. We observe that this conclusion is qualitatively the same as the conclusions
for the q-fermions discussed in [10]. However, it must be stressed that \( q \) in [10] refers to
arbitrary values whereas here the parameter \( q \) is restricted to be in the range \( 0 \leq q \leq 1 \).
The actual expressions are different and the results follow from a different algebra.

Next we may investigate the virial expansion. In the standard notation, we obtain the
result

\[
\frac{Pv}{kT} = 1 + \frac{1}{2^{5/2}} \left( \frac{\lambda^3}{v} \right) + \left( \frac{1}{8} - \frac{2}{3^{5/2}} \right) \left( \frac{\lambda^3}{v} \right)^2 + \cdots.
\]  

(56)

It is interesting to note that the virial coefficients are independent of \( q \). Indeed it is the same
as for ordinary fermions [15]. This situation contrasts with the conclusions of [1] where the
ansatz for the distribution function was introduced differently and where the calculations
were done in two dimensional space and it is also different from the conclusion in [10].

We can obtain the internal energy in the form

\[
U = \frac{3kTV}{2\lambda^3} f_{5/2}(q^{-1}z),
\]  

(57)

which has the correct Fermi limit. For the entropy of the F-anyons we obtain the expression

\[
\frac{S}{Nk} = \frac{5}{2} \frac{f_{5/2}(q^{-1}z)}{f_{3/2}(q^{-1}z)} - \ln z,
\]  

(58)
which agrees with the result for standard fermions in the fermi limit. We also observe that for \( q \neq 1 \) the entropy is larger than that of ordinary fermions.

We may conclude with some general results for the F-anyons. In the limit of large energy, the F-anyon distribution function reduces to

\[
n_i \to q^{-1} e^{-\beta E_i},
\]

which, other than the normalization factor, is the same as in classical Boltzmann case. In the limit when \( E = \mu \), the distribution reduces to

\[
n_i = \frac{q^{-1}}{1 + q^{-1}} \geq \frac{1}{2},
\]

which takes the value \( \frac{1}{2} \) only in the Bose limit when \( q = 1 \). In the high temperature limit, when \( T \to 0 \), it is clear that the distribution function reduces to the standard unmodified step form for all values of \( q \). Hence the interpolation statistics may be interpreted as solely a finite temperature effect. The modification at higher temperatures is similar to the standard Fermions except that the parameter \( q \) also plays a role.

The dependence on the parameter \( q \) is somewhat subtle for many of the thermodynamic functions. As an illustration, we may consider the dependence of the Fermi-energy in some detail. The number density of F-anyons is given by the distribution function

\[
\frac{N}{V} = \frac{g}{\lambda^3} f_{3/2}(q^{-1}z),
\]

where \( g \) is the multiplicity factor. This can be expressed by the series

\[
\frac{N}{V} = 4\pi g \left( \frac{2mkT}{\hbar^2} \right)^{3/2} (\ln(q^{-1}z)^{3/2}) \left( 1 + \frac{\pi^2}{8} (\ln(q^{-1}z)^{-2}) + \cdots \right).
\]

This can be employed to determine the chemical potential \( \mu \) of F-anyons in terms of the Fermi-energy of standard Fermions,

\[
E_F = \frac{3N}{4\pi gV} \frac{2^3}{2m} \frac{\hbar^2}{2}.
\]

In the lowest approximation, we obtain

\[
\mu = \frac{3N}{4\pi gV} \frac{2^3}{2m} \frac{\hbar^2}{2} - kT \ln q^{-1},
\]

which shows that the \( q \)-dependence appears only at finite temperatures. The expression beyond the zeroth approximation is given by

\[
\mu = -kT \ln q^{-1} + E_F \left( 1 - \frac{\pi^2}{12} \left( \frac{kT}{E_F} \right)^2 + \cdots \right).
\]

Thus the temperature dependence of the chemical potential of the F-anyons is different from that of standard Fermions due to dependence on the statistics determining factor, and it is relatively decreased.
VI. CONCLUDING REMARKS

In summary, we have addressed the following problem: what kind of oscillator algebra can adequately describe the statistics which interpolates continuously between Bose and Fermi statistics? This is of considerable interest due to the fact that any connection between particles obeying fractional statistics, anyons, and deformed Lie algebras (quantum groups) has been looked at in the literature with disfavor, other than in two space dimensions. Indeed, many investigations described the anyons by the braid group because of the connection between permutation symmetry and the rotation group in two space dimensions. We have thus established that interpolating statistics existing in ordinary 3+1 dimensions can arise from deformed quantum oscillator algebra.

Starting from the quantum group corresponding to $q$-deformed oscillators, described by one algebra containing two parameters $q$ and $\kappa$, we interpret the parameter $q$ as the statistics determining parameter, which then delineates the algebra into two distinct regimes for $\kappa = 1$ and $\kappa = -1$ corresponding to Boson-like and Fermion-like anyons. We have investigated the two distinct algebras in detail and derived the consequences.

We have investigated the thermostatistics of the B-anyons in order to ascertain how they differ from ordinary Bosons. Various thermodynamic functions have been determined. We have been able to express the distribution function in this case in terms of continued fractions. One advantage of the continued fraction form is that we can obtain convenient upper and lower bounds for the distribution function. As an example, we were able to obtain exact upper and lower bounds of the distribution function of the B-anyons for arbitrary values of $q$. We have obtained the virial expansion for B-anyons. We also obtain the second virial coefficient to establish its dependence on the statistics parameter via basic numbers.

We have examined in detail the algebra which gives rise to the interpolating statistics for the F-anyons. While the B-anyon algebra leads to basic numbers and the $q$-calculus based on Jackson derivatives, the features of the algebra of F-anyons are quite different: it does not lead to basic numbers; rules of ordinary derivatives prevail; there are only two Fock states, and hence F-anyons obey Pauli exclusion principle just as ordinary Fermions do. We have studied the thermostatistics of F-anyons as a consequence of this algebra. The thermodynamic properties of F-anyons are quite different from $q$-Fermions, although both of these depend on the parameter. For instance we are able to show that the virial coefficients of the F-anyons are no different from those of the standard Fermions, while that is not the case for $q$-deformed Fermions. We have obtained the temperature dependence of the Fermi energy of F-anyons in the form of a power series and show that the Fermi energy depends linearly on temperature in the first approximation which vanishes in Fermi limit, $q \to 1$.

Some investigations in the past [11] considered interpolation in order to accommodate a deviation from Pauli exclusion principle and tended to develop a theory to account for any experimental evidence along these lines. Here in the present investigation we have discussed different situations including the consequence that Pauli principle is strictly valid as for ordinary Fermions. This might indicate, that contrary to appearances, our algebra
is different from the original idea of Greenberg and others even at the domain of the single level system. It is rather surprising that the consequences of a similar algebra for B-anyons and F-anyons would be so strikingly different. It is interesting that in this theory, Pauli exclusion principle is valid for F-anyons while it has to be imposed by hand explicitly or implicitly for $q$-deformed Fermions [10]. It is also interesting that F-anyons would have a two dimensional Fock space, that Pauli principle would be valid for a range (dense set) of the statistics parameter.

We believe our investigation provides a confirmation of the notion that there is a close connection between quantum Lie algebra of deformed quantum harmonic oscillators and intermediate statistics as well as the concept that interpolation statistics is not related to 2+1 dimensions of space-time.

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