Eigenvalues and energy functionals with monotonicity formulae under Ricci flow

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Abstract. In this note, we construct families of functionals of the type of $F$-functional and $W$-functional of Perelman. We prove that these new functionals are nondecreasing under the Ricci flow. As applications, we give a proof of the theorem that compact steady Ricci breathers must be Ricci-flat. Using these new functionals, we also give a new proof of Perelman’s no non-trivial expanding breather theorem. Furthermore, we prove that compact expanding Ricci breathers must be Einstein by a direct method. In this note, we also extend Cao’s methods of eigenvalues[1] and improve their results.

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1. Introduction

Let $(M, g)$ be a closed Riemannian manifold. In [15], Perelman introduced a functional
\[ F(g, f) = \int_M (R + |\nabla f|^2)e^{-f}d\mu. \] (1)

If $(M, g)$ is a solution to the Ricci flow equation, Perelman proved that the $F$-functional is nondecreasing under the Ricci flow and the Ricci flow can be viewed as the gradient flow of this functional. He proved that under the following coupled system:
\[
\begin{aligned}
\frac{\partial}{\partial t}g_{ij} &= -2R_{ij}, \\
\frac{\partial}{\partial t}f &= -\Delta f - R + |\nabla f|^2,
\end{aligned} \] (2)

the $F$-functional is nondecreasing. More precisely,
\[ \frac{d}{dt}F = 2 \int_M |R_{ij} + \nabla_i \nabla_j f|^2 e^{-f}d\mu \geq 0. \] (3)

If one defines
\[ \lambda(g) = \inf F(g, f), \]

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where the infimum is taken over all the smooth $f$ which satisfies
\[ \int_M e^{-f} d\mu = 1, \]
then $\lambda(g)$ is the lowest eigenvalue of the operator
\[ -4\Delta + R, \]
and the nondecreasing of the $F$ functional implies the nondecreasing of
$\lambda(g)$. As an application, Perelman was able to show that there is no non-trivial steady or expanding Ricci breathers on closed manifolds.

In [1], X.D. Cao considered the eigenvalues of the operator
\[ -\Delta + \frac{R}{2}, \]
on manifolds with nonnegative curvature operator. They showed that the eigenvalues of these manifolds are nondecreasing along the Ricci flow. Using the monotonicity of the eigenvalues, they proved that the only steady Ricci breather with nonnegative curvature operator is the trivial one which is Ricci-flat.

In this note, we study monotonicity formulae of various energy functionals and the eigenvalues of the operator $-4\Delta + kR$. In section 2,\footnote{The results in section 2 are relatively independent of the rest of the paper and can be treated separately.} we study a monotonicity formula of eigenvalues of $-\Delta + \frac{R}{2}$, which improves the result in [1] based on the same technique Cao has used. The following is one of the main theorems in this paper:

**Theorem 11** Let $g(t), t \in [0,T)$, be a solution to the Ricci flow on a closed Riemannian manifold $M^n$. Assume that there is a $C^1$-family of smooth functions $f(t) > 0$, which satisfy
\[ \lambda(t)f(t) = -\Delta g(t)f(t) + \frac{1}{2}R g(t)f(t) \] (4)
\[ \int_M f^2(t)d\mu_{g(t)} = 1 \] (5)
where $\lambda(t)$ is a function of $t$ only. Then
\[ \frac{d}{dt}\lambda(t) = 4 \int R_{ij}\nabla^i f\nabla^j f d\mu + 2 \int |Rc|^2 f^2 d\mu \]
\[ = \int |R_{ij} + \nabla_i \nabla_j \varphi|^2 e^{-\varphi} d\mu + \int |Rc|^2 e^{-\varphi} d\mu \geq 0. \] (6)
where $\varphi$ satisfies $e^{-\varphi} = f^2$.

In section 3, we introduce a functional $E$ and derive its first variation along the Ricci flow. In section 4, we define new functionals $F_k$ and we prove that although the Ricci flow is not a gradient flow of $F_k$ in the sense of Perelman’s, $F_k$ are still nondecreasing under the Ricci flow. The monotonicity is strict unless we are on a Ricci-flat solution.

In section 5, by using the monotonicity of the eigenvalues, we rule out compact steady Ricci breathers.

In section 6, we introduce a family of new functionals $W_e$ and $W_{ek}$. These functionals are not scale invariant in contrast to Perelman’s $W$-entropy. We obtain their first variations:

**Theorem 12** Under the following coupled system

\[
\begin{align*}
\frac{\partial}{\partial t} g_{ij} &= -2R_{ij} \\
\frac{\partial}{\partial t} f &= -\Delta f + |\nabla f|^2 - R \\
\frac{d}{dt} \tau &= 1,
\end{align*}
\]

we have the following monotonicity:

\[
\begin{align*}
\frac{d}{dt} W_{ek} &= 2\tau^2 \int_M \left| R_{ij} + \nabla_i \nabla_j f + \frac{1}{2\tau} g_{ij} \right|^2 e^{-f} d\mu \\
&\quad + 2(k - 1)\tau^2 \int_M \left| R_{ij} + \frac{1}{2\tau} g_{ij} \right|^2 e^{-f} d\mu,
\end{align*}
\]

where $k \geq 1$. The monotonicity is strict unless we are on an expanding gradient Ricci soliton ($k = 1$) or an Einstein manifold ($k > 1$).

In section 7, we use the non-scale-invariant functional $W_e$ to prove Perelman’s no non-trivial expanding breather theorem.

In section 8, we use $W_{ek}$ functionals to rule out non-Einstein compact expanding Ricci breathers.

**Remark 13** The no non-Einstein steady or expanding Ricci breather theorem was first proved by T. Ivey in [11]. We give a different proof here which is based on a global functional method and does not use maximal principles.

**Remark 14** The no non-trivial expanding Ricci breather theorem was proved by Perelman in [15] by discussing the monotonicity of a scale invariant eigenvalue $\bar{\lambda}(g) = \lambda(g) V_{-\frac{n}{2}}(g)$. We present a different proof in section 7.

\[2 \text{ See the definition of } W_{ek} \text{ in (34)}\]
Remark 15 For compact shrinking Ricci breathers, one can find references in [15], [4], [5], [10], [11] and others.

Throughout this paper, we use Einstein convention, i.e. repeated index implies summation.

2. Eigenvalues of \((-\Delta + \frac{R}{2})\) on manifolds

Let \((M, g(t))\) be a compact Riemannian manifold, where \(g(t)\) is a smooth solution to the Ricci flow equation on \(0 \leq t < T\). In [1], Cao considers the problem from a viewpoint different from the entropy functional method. They study the eigenvalues \(\lambda\) and eigenfunctions \(f\) of the operator \(-\Delta + \frac{R}{2}\) with the normalization \(\int_M f^2 dv = 1\). First, they define:

\[
\lambda(h, t) = \int_M (-\Delta h + \frac{R}{2} h) hd\mu,
\]

where \(h\) is a smooth function satisfying \(\frac{d}{dt}(\int_M h^2 d\mu) = 0\) and \(\int_M h^2 d\mu = 1\).

They then obtain the monotonicity formula under the non-negative curvature operator assumption as following:

Theorem 21 [1] On a compact Riemannian manifold with non-negative curvature operator, the eigenvalues of the operator \(-\Delta + \frac{R}{2}\) are nondecreasing under the Ricci flow, i.e.

\[
\frac{d}{dt}\lambda(f, t)|_{t=t_0} = 2 \int_R^f f_i f_j d\mu + \int |Rc|^2 f^2 d\mu \geq 0.
\]

In this theorem, \(f_i\) denotes the covariant derivative of \(f\) with respect to \(\frac{\partial}{\partial x^i}\), (also denoted as \(\partial_i\)) and in (8), \(\frac{d}{dt}\lambda(f, t)\) is evaluated at time \(t = t_0\) and \(f\) is the corresponding eigenvalue at time \(t_0\). As a direct consequence of Theorem 21, they prove the following

Corollary 22 [1] There is no compact steady Ricci breather with non-negative curvature operator, other than the one which is Ricci-flat.

Remark 23 Clearly, at time \(t\), if \(f\) is the eigenfunction of the eigenvalue \(\lambda(t)\), then \(\lambda(f, t) = \lambda(t)\).

In this section, based on (8), we will drop the curvature assumption on the manifold and prove Theorem 11.
Proof. (Theorem 11) Let \( \varphi \) be a function satisfying \( f^2(x) = e^{-\varphi(x)} \) and plug it into (8), we have

\[
2 \frac{d}{dt} \lambda(t) = 2 \frac{d}{dt} \lambda(f, t) = \int R_{ij} \nabla^i \varphi \nabla^j \varphi e^{-\varphi} d\mu + 2 \int |Rc|^2 e^{-\varphi} d\mu. \tag{9}
\]

Using the divergence theorem and rearrangements, we derive the first term of the last identity

\[
\int R_{ij} \nabla^i \varphi \nabla^j \varphi e^{-\varphi} d\mu = \int \nabla^i R_{ij} \nabla^j \varphi e^{-\varphi} d\mu + \int R_{ij} \nabla^i \nabla^j \varphi e^{-\varphi} d\mu. \tag{10}
\]

By the contracted second Bianchi identity \( \nabla^i R_{ij} = \frac{1}{2} \nabla_j R \) and integration by parts, we have

\[
(10) = \frac{1}{2} \int \nabla_j R \nabla^j \varphi e^{-\varphi} d\mu + \int R_{ij} \nabla^i \nabla^j \varphi e^{-\varphi} d\mu
= \frac{1}{2} \int R \Delta e^{-\varphi} d\mu + \int R_{ij} \nabla^i \nabla^j \varphi e^{-\varphi} d\mu. \tag{11}
\]

This also implies the following

\[
\int R_{ij} \nabla^i \nabla^j \varphi e^{-\varphi} d\mu = \int R_{ij} \nabla^i \nabla^j \varphi e^{-\varphi} d\mu - \frac{1}{2} \int R \Delta e^{-\varphi} d\mu. \tag{12}
\]

On the other hand, using integration by parts and symmetry of the hessian of functions, we have

\[
\int |\nabla \nabla \varphi|^2 e^{-\varphi} d\mu = -\int \nabla_j \varphi \nabla^i \nabla^i \nabla^j \varphi e^{-\varphi} d\mu - \int \nabla_j \varphi \nabla^i \nabla^j \varphi \nabla_i e^{-\varphi} d\mu = -\int \nabla_j \varphi \nabla^i \nabla^j \varphi e^{-\varphi} d\mu + \int \frac{1}{2} |\nabla \varphi|^2 \Delta e^{-\varphi} d\mu. \tag{13}
\]

By the commutator formulae for covariant derivatives which are known as Ricci identities, see page 286 in [3], we have

\[
\nabla_i \nabla_j \nabla^i \varphi = \nabla_j \nabla_i \nabla^i \varphi - R^k_{ijkj} \nabla^k \varphi = \nabla_j \nabla_i \nabla^i \varphi + R_{kj} \nabla^k \varphi, \tag{14}
\]

where \( R^l_{ijkj} \) represents the Riemann curvature (3,1)-tensor and \( R_{kj} \) denotes the Ricci curvature tensor. We use the following convention of Riemann curvature tensor throughout this paper:

\[
R_m(X, Y)Z \equiv [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,
\]

where \( \{X, Y, Z\} \) are vector fields on the manifold.
Combing (13), (14), and the contracted second Bianchi identity, we have

\[ \int |\nabla \nabla \varphi|^2 e^{-\varphi} d\mu = -\int \nabla_j \varphi (\nabla_j \nabla^i \varphi - R_{jk} \nabla^k \varphi) e^{-\varphi} d\mu + \int \frac{1}{2} |\nabla \varphi|^2 \Delta e^{-\varphi} d\mu \]

\[ = -\int \nabla_j \varphi \nabla^j \Delta \varphi e^{-\varphi} d\mu + \int R_{jk} \nabla^j \varphi \nabla^k \varphi e^{-\varphi} d\mu + \int \frac{1}{2} |\nabla \varphi|^2 \Delta e^{-\varphi} d\mu \]

\[ = -\int \Delta e^{-\varphi} \Delta \varphi d\mu + \int (\nabla_j R_{jk} \nabla^k \varphi + R_{jk} \nabla^j \nabla^k \varphi) e^{-\varphi} d\mu \]

\[ + \int \frac{1}{2} |\nabla \varphi|^2 \Delta e^{-\varphi} d\mu \]

\[ = -\int \Delta e^{-\varphi} \Delta \varphi d\mu + \int \frac{1}{2} \nabla_k R \nabla^k \varphi e^{-\varphi} d\mu - \int R_{jk} \nabla^j \nabla^k \varphi e^{-\varphi} d\mu \]

\[ + \int \frac{1}{2} |\nabla \varphi|^2 \Delta e^{-\varphi} d\mu \]

\[ = -\int \Delta e^{-\varphi} (\Delta \varphi + \frac{1}{2} R - \frac{1}{2} |\nabla \varphi|^2) d\mu - \int R_{jk} \nabla^j \nabla^k \varphi e^{-\varphi} d\mu. \]

(15)

We notice that we are free to change the dummy index from \(\{i, j, k, l\}\) to other index or exchange among them whenever necessary.

Combing (12) and (15), we have the following

\[ \int 2 R_{ij} \nabla^i \nabla^j \varphi e^{-\varphi} d\mu + \int |\nabla \nabla \varphi|^2 e^{-\varphi} d\mu \]

\[ = \int R_{jk} \nabla^j \nabla^k \varphi e^{-\varphi} d\mu - \int \Delta e^{-\varphi} (\Delta \varphi + \frac{1}{2} R - \frac{1}{2} |\nabla \varphi|^2) d\mu \]

\[ = \int R_{ij} \nabla^i \varphi \nabla^j \varphi e^{-\varphi} d\mu - \int \Delta e^{-\varphi} (\Delta \varphi + R - \frac{1}{2} |\nabla \varphi|^2) d\mu \]

(16)

Recall (4), then a simple calculation yields

\[ 2 \lambda(t) = \Delta_{g(t)} \varphi + R_{g(t)} - \frac{1}{2} |\nabla \varphi|^2_{g(t)} \]

(17)

Plugging (17) into (16), by divergence theorem on closed manifolds, we have

\[ \int 2 R_{ij} \nabla^i \nabla^j \varphi e^{-\varphi} d\mu + \int |\nabla \nabla \varphi|^2 e^{-\varphi} d\mu \]

\[ = \int R_{ij} \nabla^i \varphi \nabla^j \varphi e^{-\varphi} d\mu - \int 2 \lambda(t) \Delta e^{-\varphi} d\mu \]

(18)

\[ = \int R_{ij} \nabla^i \varphi \nabla^j \varphi e^{-\varphi} d\mu. \]
In the end, we plug (18) into (9) and have the following

\[
2 \frac{d}{dt} \lambda(t) = \int R_{ij} \nabla^i \varphi \nabla^j \varphi e^{-\varphi} d\mu + \int |Rc|^2 e^{-\varphi} d\mu + \int |Rc|^2 e^{-\varphi} d\mu
\]

\[
= \int |R_{ij} + \nabla^i \nabla^j \varphi|^2 e^{-\varphi} d\mu + \int |Rc|^2 e^{-\varphi} d\mu \geq 0
\]

(19)

**Remark 24** One can find applications of Theorem 11 in [1] (e.g. Theorem 3). When the lowest eigenvalues are concerned, the eigenfunctions are always smooth and positive, see in [4]. For the monotonicity of eigenvalues of ordinary Laplace operator under Ricci flow, one can see a result in [13].

**Remark 25** For evolution of Yamabe constant under Ricci flow, see a recent preprint [2]. We thank Xiao-Dong Cao for pointing out the reference to us.

3. Construct entropy functional from first variation formula

In this section, we will construct a functional $E$ and show that it is monotone along the Ricci flow.

Under the following coupled system:

\[
\begin{align*}
\frac{\partial}{\partial t} g_{ij} &= -2(R_{ij} + \nabla_i \nabla_j f) \\
\frac{\partial}{\partial t} f &= -\Delta f - R
\end{align*}
\]

(20)

one has the following first variation formula, (see [15])

\[
\frac{d}{dt} \mathcal{F} = 2 \int_M |R_{ij} + \nabla_i \nabla_j f|^2 e^{-f} d\mu \geq 0,
\]

(21)

where $\mathcal{F}$ is defined as in (1). Hence, the modified Ricci flow can be viewed as an $L^2$ gradient Ricci flow of Perelman’s $\mathcal{F}$-functional.

One natural question to ask is: can we find a functional $E$ such that the ‘honest’ Ricci flow is the $L^2$ gradient Ricci flow of it in a certain sense? Namely, one expects that under the following coupled system :

\[
\begin{align*}
\frac{\partial}{\partial t} g_{ij} &= -2R_{ij} \\
\frac{\partial}{\partial t} f &= -\Delta f - R + |\nabla f|^2,
\end{align*}
\]

(22)

there is a functional $E$ such that the following is true,

\[
\frac{d}{dt} E = 2 \int_M |Rc|^2 e^{-f} d\mu \geq 0.
\]

(23)
It turns out that this can be done by integrating the above first variation formula over a time interval \([0, t]\). We carry out the computations as the following

\[
\mathcal{E}(t) - \mathcal{E}(0) = \int_0^t \frac{d\mathcal{E}}{ds} ds = 2 \int_0^t \int_M |Rc|^2 e^{-f} d\mu ds
\]

\[
= 2 \int_M \int_0^t g^{ij} R_{ik} R_{jl} e^{-f} \sqrt{\det g} ds \, dx^n
\]

\[
= \int_M \int_0^t \frac{\partial g^{ij}}{\partial t} R_{jl} e^{-f} \sqrt{\det g} ds \, dx^n
\]

The last step follows from a lemma in [3], see page 67, Lemma 3.1.

Integration by parts yields

\[
\mathcal{E}(t) - \mathcal{E}(0) = \int_M \left[ g^{ij} R_{ij} e^{-f} \sqrt{\det g} \right]^t_0 - \int_0^t g^{ij} \frac{\partial R_{ij}}{\partial t} e^{-f} \sqrt{\det g} ds \right] \, dx^n
\]

\[
= \int_M Re^{-f} d\mu - \int_0^t \left[ \int_M g^{ij} \frac{\partial R_{ij}}{\partial t} e^{-f} \sqrt{\det g} ds \right] \, dx^n
\]

Using Lemma 3.5 in [3] and the fact that metric tensor \(g\) is a covariant constant, by letting \(h = -2Rc\), we have the following

\[
g^{ij} \frac{\partial R_{ij}}{\partial t} = -g^{ij} g^{pq} (\nabla_q \nabla_j R_{tp} + \nabla_q \nabla_t R_{jp} - \nabla_q \nabla_p R_{ji} - \nabla_j \nabla_i R_{pq})
\]

In the above computations, we have used the contracted second Bianchi identity. With the help of Lemma 3.9 in [3], We obtain

\[
\frac{\partial e^{-f} \sqrt{\det g}}{\partial t} = \left( -\frac{\partial}{\partial t} f - R \right) e^{-f} \sqrt{\det g}
\]

As suggested by the referee, a simpler way to obtain equation (26) is:

\[
\frac{\partial R}{\partial t} = \Delta R + 2|Rc|^2 = g^{ij} \frac{\partial R_{ij}}{\partial t} + R_{ij} \frac{\partial g^{ij}}{\partial t}
\]

\[
= g^{ij} \frac{\partial R_{ij}}{\partial t} + 2|Rc|^2.
\]
Plug (26) and (27) into (25), we have

\[
\mathcal{E}(t) - \mathcal{E}(0) = \int_M Re^{-f} d\mu \bigg|_0^s - \int_0^t \int_M \left[ \Delta R e^{-f} + R(-\frac{\partial}{\partial t} f - R)e^{-f} \right] d\mu \, ds \\
= \int_M Re^{-f} d\mu \bigg|_0^s - \int_0^t \int_M \left[ -\Delta f + |\nabla f|^2 - \frac{\partial}{\partial t} f - R \right] e^{-f} d\mu \, ds \\
= \int_M Re^{-f} d\mu \bigg|_{s=t} - \int_M Re^{-f} d\mu \bigg|_{s=0}.
\]

The last step follows from the third equation in the coupled system (22).

The computation above suggests that we could define \( \mathcal{E} \)-functional as

\[
\mathcal{E} = \int_M Re^{-f} d\mu.
\]  

(28)

Next, we prove the monotonicity property for \( \mathcal{E} \)-functional directly.

**Proposition 31** Assume \( g(t) \) satisfies the Ricci flow equation over the time interval \([0,T)\), and also function \( f \) satisfies the evolution equation (2) (also (22)), then

\[
\frac{d\mathcal{E}}{dt} = 2 \int_M |Rc|^2 e^{-f} d\mu.
\]

Proof. (First proof of Proposition 31) Using \( \frac{\partial R}{\partial t} = \Delta R + 2|Rc|^2 \), we derive the following

\[
\frac{d}{dt} \int_M Re^{-f} d\mu = \int_M \frac{dR}{dt} e^{-f} d\mu + R \frac{df}{dt} (e^{-f} d\mu) \\
= \int_M (\Delta R + 2|Rc|^2)e^{-f} d\mu + \int_M R(-f_t - R)e^{-f} d\mu \\
= 2 \int_M |Rc|^2 e^{-f} d\mu + \int_M R[-\Delta f + |\nabla f|^2 - f_t - R]e^{-f} d\mu \\
= 2 \int_M |Rc|^2 e^{-f} d\mu.
\]

The last equality comes from the second equation of the coupled system (2).

(Second proof) Under the following coupled system with modified Ricci flow (see also similar system in (35)),

\[
\begin{align*}
\frac{\partial}{\partial t} g_{ij} &= -2 \left( R_{ij} + \nabla_i \nabla_j f \right) \\
\frac{\partial}{\partial t} f &= -\Delta f - R,
\end{align*}
\]

(30)
we have the following
\[
\frac{d}{dt} \int_M Re^{-f} d\mu = \int_M (\Delta R - \nabla R \nabla f + 2|Rc|^2)e^{-f} d\mu \\
= 2 \int_M |Rc|^2 e^{-f} d\mu.
\]
By using the diffeomorphism invariance, see e.g. Proposition 1.2 in [15], we prove that under the original coupled system we still have
\[
\frac{d}{dt} \int_M Re^{-f} d\mu = 2 \int_M |Rc|^2 e^{-f} d\mu.
\]

Remark 32 We will apply the diffeomorphism invariance principle generally in this paper. See details in the proof of Corollary 66 and Corollary 67.

4. Entropy functionals \( \mathcal{F}_k \) and their monotonicity

Definition 41 We define the following variations of \( \mathcal{F} \)-functional,
\[
\mathcal{F}_k(g, f) = \int_M (kR + |\nabla f|^2)e^{-f} d\mu, \tag{31}
\]
where \( k \geq 1 \). When \( k = 1 \), this is the \( \mathcal{F} \)-functional.

Next we derive the monotonicity formula for these functionals \( \mathcal{F}_k(g, f) \).

Theorem 42 Suppose the Ricci flow of \( g(t) \) exists for \([0, T)\), then all the functionals \( \mathcal{F}_k(g, f) \) will be monotone under the coupled system (22), i.e.
\[
\frac{d}{dt} \mathcal{F}_k(g_{ij}, f) = 2(k - 1) \int_M |Rc|^2 e^{-f} d\mu + 2 \int_M |R_{ij} + \nabla_i \nabla_j f|^2 e^{-f} d\mu \geq 0 \tag{32}
\]
Furthermore, the monotonicity is strict unless the Ricci flow is a trivial Ricci soliton and \( f \) is a constant function. Namely, the metric is Ricci-flat.

Proof. (Theorem 42) Under the coupled system (2)
\[
\begin{align*}
\frac{\partial}{\partial t} g_{ij} &= -2R_{ij} \\
\frac{\partial}{\partial t} f &= -\Delta f - R + |\nabla f|^2,
\end{align*}
\]
we have shown in Proposition 31 that,
\[
\frac{d}{dt} \int_M Re^{-f} d\mu = 2 \int_M |Rc|^2 e^{-f} d\mu.
\]
On the other hand, in [15], it was shown that under the same system (2),
\[
\frac{d}{dt} \int_M (R + |\nabla f|^2) e^{-f} d\mu = 2 \int_M |R_{ij} + \nabla_i \nabla_j f|^2 e^{-f} d\mu.
\]

Put them together we get the following formula:
\[
\frac{d}{dt} \mathcal{F}_k(g, f) = \frac{d}{dt} \int_M (kR + |\nabla f|^2) e^{-f} d\mu \\
= 2(k - 1) \int_M |Rc|^2 e^{-f} d\mu + 2 \int_M |R_{ij} + \nabla_i \nabla_j f|^2 e^{-f} d\mu \\
\geq 0.
\]

For \( k = 1 \), the above is consistent with the monotonicity formula of \( \mathcal{F} \)-functional. The second statement is obvious. This finishes the proof of Theorem 42.

**Remark 43** We notice that under the coupled system (2), the Ricci flow can be viewed as a \( L^2 \) gradient flow of Perelman’s \( \mathcal{F} \) functional up to a diffeomorphism where our functionals are not. But the monotonicity is still retained.

**Remark 44** Our functionals yield more information about the Ricci tensor itself.

5. Eigenvalues and compact steady Ricci breathers

In this section, we discuss the applications of the monotonicity formula (32) we derived in Theorem 42.

First, we recall the definition of Ricci breathers, see original definition in [11] and [15].

**Definition 51** A metric \( g(t) \) evolving by the Ricci flow is called a breather, if for some \( t_1 < t_2 \) and \( \alpha > 0 \) the metrics \( \alpha g(t_1) \) and \( g(t_2) \) differ only by a diffeomorphism; the cases \( \alpha = 1 \), \( \alpha < 1 \), \( \alpha > 1 \) correspond to steady, shrinking or expanding breathers, respectively.

Trivial breathers are called Ricci solitons for which the above properties are true for each pair of \( t_1 \) and \( t_2 \).

Define \( \lambda_k(g) = \inf \mathcal{F}_k(g, f) \), where infimum is taken over all smooth \( f \), satisfying \( \int_M e^{-f} d\mu = 1 \). \( \lambda_k \) is the lowest eigenvalue of the corresponding operators \( -4\Delta + kR \) for \( k > 1 \). By applying direct methods and elliptic regularity theory (see [7],§8.12), one can see that the infimum is always attained.

Using the monotonicity in Theorem 42, we have
**Theorem 52** On a compact Riemannian manifold \((M, g(t))\), where \(g(t)\) satisfies the Ricci flow equation for \(t \in [0, T)\), the lowest eigenvalue \(\lambda_k\) of the operator \(-4\Delta + kR\) is nondecreasing under the Ricci flow. The monotonicity is strict unless the metric is Ricci-flat.

**Proof.** (Theorem 52) For any time \(t_1, t_2\) in \([0, T)\), suppose that at time \(t_2\), the lowest eigenvalue \(\lambda_k(g(t_2))\) is attained by a function \(f_k(x)\). Evolving under the backward Ricci flow, we get a solution \(f_k(x, t)\) to the coupled system (2) which satisfies the initial condition \(f_k(x, t_2) = f_k(x)\).

Using the monotonicity formula of (32), we have

\[
\lambda_k(g(t_2)) = \mathcal{F}_k(g(t_2), f_k(t_2)) \\
\geq \mathcal{F}_k(g(t_1), f_k(t_1)) \\
\geq \inf \mathcal{F}_k(g(t_1), f) \\
= \lambda_k(g(t_1)).
\]

This proves that \(\lambda_k\) is nondecreasing under the Ricci flow. Since the monotonicity of \(\mathcal{F}_k\) is strict unless the metric is Ricci-flat, this finishes the proof of Theorem 52.

As a corollary, we generalize the theorem of Cao in the case of the lowest eigenvalue.

**Corollary 53** On a compact Riemannian manifold, the lowest eigenvalues of the operator \(-\Delta + \frac{R}{2}\) are nondecreasing under the Ricci flow.

**Proof.** (Corollary 53) Let \(k = 2\), then \(\frac{1}{4} \lambda_2\) is the lowest eigenvalue of \(-\Delta + \frac{R}{2}\) and the result will follow.

As an application of Theorem 21, Cao proved the following

**Theorem 54** [1] There is no compact steady Ricci breather with nonnegative curvature operator, other than the one which is Ricci-flat.

As a corollary of Theorem 52, we drop the nonnegative curvature operator condition and have the following

**Corollary 55** There is no compact steady Ricci breather other than the one which is Ricci-flat.

**Proof.** (Theorem 55) For a Ricci breather, let \(t_1\) and \(t_2\) be as in the definition, then \(\lambda_k(t_1) = \lambda_k(t_2)\) for a steady breather due to the diffeomorphism invariance of the eigenvalues. The fact that \(\lambda_k(t)\) fails to be strictly increasing yields that the manifold must be Ricci-flat.

**Remark 56** This result was first proved by T. Ivey in [11] with a different approach. See also details in the book [4].
6. New formulae over expanders

In this section, we define the following functionals and discuss their first variation formulae under modified Ricci flow and Ricci flow respectively

\[ \mathcal{W}_e(g, \tau(t), f) = \tau^2 \int_M \left[ R + \frac{n}{2\tau} + \Delta f \right] e^{-f} d\mu, \]
\[ \mathcal{W}_{ek}(g, \tau(t), f) = \tau^2 \int_M \left[ k(R + \frac{n}{2\tau}) + \Delta f \right] e^{-f} d\mu, \]

where \( k > 1 \).

**Remark 61** The functionals we obtain in this section are different from Perelman’s \( \mathcal{W} \)-functional. They are not scale invariant.

**Remark 62** After this paper was submitted, we found out that in [15], Perelman has defined a functional similar to our functional \( \mathcal{W}_e \) which he called \(- < E > \). Perelman’s \(< E > \) is modeled on shrinking Ricci solitons and ours are the corresponding version on expanding solitons. Furthermore, the functionals \( \mathcal{W}_{ek} \) are new to our knowledge. A different motivation which made us discover these functionals, \( \mathcal{W}_e \) and \( \mathcal{W}_{ek} \), will appear somewhere else.

**Remark 63** In [6], M. Feldman, T. Ilmanen, and L. Ni constructed a scale invariant \( \mathcal{W} \)-entropy which is an analogue of Perelman’s \( \mathcal{W} \)-entropy but has vanishing first variation over expanders. There is also a very good unified treatment about entropy formulae over steady, expanding, and shrinking Ricci breathers in [4]. See a related generalization of Perelman’s formula also in a recent preprint [12].

We start with deriving the first variation formulae of \( \mathcal{W}_e \) and \( \mathcal{W}_{ek} \).

**Theorem 64** Under the following coupled system

\[
\begin{aligned}
\frac{\partial}{\partial t} g_{ij} &= -2 \left( R_{ij} + \nabla_i \nabla_j f \right) \\
\frac{\partial}{\partial t} f &= -\Delta f - R \\
\frac{d}{dt} \tau &= 1,
\end{aligned}
\]

the first variation formula for \( \mathcal{W}_e(g, \tau(t), f) \) is

\[
\frac{d}{dt} \mathcal{W}_e = 2\tau^2 \int_M \left| R_{ij} + \nabla_i \nabla_j f + \frac{1}{2\tau} g_{ij} \right|^2 e^{-f} d\mu.
\]
Theorem 65 Under the coupled system (35), the first variation formula for $W_{ek}(g, \tau(t), f)$ is
\[
\frac{d}{dt}W_{ek} = 2\tau^2 \int_M |R_{ij} + \nabla_i \nabla_j f + \frac{1}{2\tau} g_{ij}|^2 e^{-f} d\mu \\
+ 2(k-1)\tau^2 \int_M |R_{ij} + \frac{1}{2\tau} g_{ij}|^2 e^{-f} d\mu.
\]
(37)

where $k > 1$.

Proof. (Theorem 64) The proof is by direct computations. We notice that $W_e = \tau^2 F + \frac{n}{2\tau}$. Under the coupled system (35), it is known that
\[
\frac{d}{dt}F = 2\int_M |R_{ij} + \nabla_i \nabla_j f|^2 e^{-f} d\mu.
\]
This implies
\[
\frac{d}{dt}W_e = 2\tau^2 \int_M |R_{ij} + \nabla_i \nabla_j f|^2 e^{-f} d\mu + 2\tau F + \frac{n}{2}
= 2\tau^2 \int_M |R_{ij} + \nabla_i \nabla_j f + \frac{1}{2\tau} g_{ij}|^2 e^{-f} d\mu.
\]
(38)

Proof. (Theorem 65) By definition, we have $W_{ek} = W_e + (k-1)\tau^2 \int_M (R + \frac{n}{2\tau}) e^{-f} d\mu$. Recall we defined $E = \int_M R e^{-f} d\mu$ in (29). This yields $W_{ek} = W_e + (k-1)(\tau^2 E + \frac{n}{2\tau})$.

As in the second proof of Proposition 31, we have, under the coupled system (35), $\frac{d}{dt}E = 2\int_M |Rc|^2 e^{-f} d\mu$. Using Theorem 64, the rest of the proof is direct computations as the following
\[
\frac{d}{dt}W_{ek} = 2\tau^2 \int_M |R_{ij} + \nabla_i \nabla_j f + \frac{1}{2\tau} g_{ij}|^2 e^{-f} d\mu \\
+ (k-1) \left[ 2\tau^2 \int_M |Rc|^2 e^{-f} d\mu + 2\tau E + \frac{n}{2} \right]
\]
(39)

As direct corollaries of Theorem 64 and Theorem 65, we also obtain first variation formulae under the ‘honest’ Ricci flow.

Corollary 66 Under the following coupled system
\[
\begin{align*}
\frac{\partial}{\partial t}g_{ij} &= -2R_{ij} \\
\frac{\partial}{\partial t}f &= -\Delta f + |\nabla f|^2 - R \\
\frac{d}{dt} \tau &= 1,
\end{align*}
\]
(40)
the first variation formula for $W_e(g, \tau(t), f)$

$$\frac{d}{dt} W_e = 2\tau^2 \int_M |R_{ij} + \nabla_i \nabla_j f + \frac{1}{2\tau} g_{ij}|^2 e^{-f} d\mu.$$  \hspace{1cm} (41)

**Corollary 67**: Under the coupled system (40), the first variation formula for $W_{ek}(g, \tau(t), f)$ is

$$\frac{d}{dt} W_{ek} = 2\tau^2 \int_M |R_{ij} + \nabla_i \nabla_j f + \frac{1}{2\tau} g_{ij}|^2 e^{-f} d\mu + 2(k-1)\tau^2 \int_M |R_{ij} + \frac{1}{2\tau} g_{ij}|^2 e^{-f} d\mu,$$  \hspace{1cm} (42)

where $k > 1$.

**Proof.** (Corollary 66) We adapt a proof similar to the one of Lemma 5.15 in [4] (see also in Proposition 1.2 of [15]). We observe that the modified Ricci flow and Ricci flow differ only by a diffeomorphism. Since $(g(t), f(t))$ is a solution to (40), the pair $(\tilde{g}(t), \tilde{f}(t))$ defined by $\tilde{g}(t) = \Phi^*(t)g(t)$ and $\tilde{f}(t) = f(t) \circ \Phi(t)$, is a solution to (35), where $\Phi(t): M \to M$ is a one parameter family of diffeomorphisms defined by

$$\frac{d}{dt}\Phi(t) = \nabla_{\tilde{g}(t)} \tilde{f}(t)$$
$$\Phi(0) = id_M.$$ \hspace{1cm} (43)

Now $W_e(g, f) = W_e(\tilde{g}, \tilde{f})$, so that by (36) we have

$$\frac{d}{dt} W_e(g(t), f(t)) = \frac{d}{dt} W_e(\tilde{g}(t), \tilde{f}(t)) = 2\tau^2 \int_M |R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau} g_{ij}|^2 e^{-\tilde{f}} d\tilde{\mu}.$$ \hspace{1cm} (44)

**Proof.** (Corollary 67) The proof is very similar to the proof of Corollary 66 and shall be skipped here.

If we unify Corollary 66 and Corollary 67, we obtain Theorem 12 given in the introduction section.

**Remark 68**: One can define exactly similar new formulae $W_s$ and $W_{sk}$ over shrinkers which differ only by replacing $\frac{d}{dt}\tau$ by $-\frac{d}{dt}\tau$ and letting $\frac{d}{dt}\tau = -1$. Similar monotonicity formulae still hold.
7. No expanding breathers theorem

We will focus on compact expanding Ricci breathers. First, we define 
\[ \mu_e(g, \tau) = \inf \mathcal{W}_e(g, \tau, f) = \inf \tau^2 \int_M [R + \frac{n}{2\tau} + \Delta f] e^{-f} d\mu, \]
where the infimum is taken over all the smooth functions satisfying \( \int_M e^{-f} d\mu = 1. \)

We prove the following result of no non-trivial expanding Ricci breathers which was obtained by Perelman in [15] with a different method.

**Corollary 71** There is no expanding Ricci breather on compact Riemannian manifolds other than expanding gradient Ricci solitons.

**Proof.** (Corollary 71) By the definition of Ricci breathers, there exist a pair of time moments \( t_1 \) and \( t_2 \), such that the Ricci flow solution \( g(t) \) at these two moments differ only by a diffeomorphism and a scaling \( \alpha \), i.e., 
\[ g(t_2) = \alpha \Phi^* g(t_1), \]
where \( \Phi \) is a diffeomorphism and the scalar \( \alpha > 1 \).

By the standard argument about the existence of a minimizer of \( \mu_e \) at a fixed time moment, there exists a smooth function \( f \) which attains the infimum. By solving the backward heat equation in Corollary 66, we get a smooth function \( f(x, t) : M \times [t_1, t_2] \rightarrow \mathbb{R} \) with initial condition \( f(\cdot, t_2) = f(\cdot). \) We define a linear function \( \tau : [t_1, t_2] \rightarrow (0, +\infty) \) with \( \tau(t_2) = T + t_2 \), where \( T \) is a real number. Under the coupled system (40), by the monotonicity formula, we obtain the following

\[
\mu_e(g(t_1), \tau(t_1)) = \inf_{t=t_1} \mathcal{W}_e(g, \tau, f) \\
\leq \mathcal{W}_e(g, \tau, f)_{t=t_1} \\
\leq \mathcal{W}_e(g, \tau, f)_{t=t_2} \\
= \mu_e(g(t_2), \tau(t_2)).
\]

On the other hand, it is easy to see that, if we simultaneously scale \( \tau \) and \( g \) by a scalar \( \alpha \), we have the following scaling property of \( \mu_e \),

\[
\mu_e(\alpha g(t), \alpha \tau) = \alpha \mu_e(g(t), \tau).
\]

Combining (45), (46), and the diffeomorphic invariant property of the functionals, we have

\[
\mu_e(g(t_1), \tau(t_1)) \leq \mu_e(g(t_2), \tau(t_2)) = \alpha \mu_e(g(t_1), \tau(t_1)).
\]

This yields \( 0 \leq (\alpha - 1)\mu_e(g(t_1), \tau(t_1)), \) i.e.

\[
\mu_e(g(t_1), \tau(t_1)) \geq 0.
\]
make \( \tau(t_2) = \alpha \tau(t_1) \). This yields that \( T + t_2 = \alpha(T + t_1) \). Equivalently, we can choose \( T = \frac{t_2 - t_1}{\alpha - 1} \). \(^4\)

Under the above choice of \( T \), one can easily prove that

\[
\frac{\tau(t_2)}{\sqrt{g(t_2)}} = \frac{\tau(t_1)}{\sqrt{g(t_1)}}.
\]

Hence, there exists a \( t \in [t_1, t_2] \), such that \( 0 = \frac{d}{dt} \log \frac{\tau}{\sqrt{V}} = \frac{n}{2\tau} + \int \frac{R d\mu}{V} \geq \inf \int (R + \frac{n}{2\tau} + \Delta f) e^{-f} d\mu \). Equivalently, \( \mu_e(g(t), \tau(t)) \leq 0 \). Recall (48) and (45), we obtain \( 0 \leq \mu_e(g(t_1), \tau(t_1)) \leq \mu_e(g(t_1), \tau(t_1)) \leq 0 \), i.e. \( \mu_e(g(t_1), \tau(t_1)) = 0 \). Hence all the inequalities in (45) must be equalities and the first variation of \( W_e \) must vanish. Therefore, the Ricci breather must be a gradient Ricci soliton, namely,

\[
R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau} g_{ij} = 0.
\]

**Remark 72** One cannot get similar results for shrinking breathers by using the same method because of the “wrong” sign of \((\alpha - 1)\) in the shrinking case.

### 8. Expanding Ricci breathers are Einstein

In this section, we prove the following result by using the functional method. This result was first proved by T. Ivey in [11], see also in [4].

**Corollary 81** Expanding Ricci breathers on compact Riemannian manifolds must be Einstein.

**Proof.** (Corollary 81) The proof is very similar to the proof of Corollary 71. We define \( \mu_e(g, \tau) = \inf \mathcal{W}_{ek}(g, \tau, f) = \inf \tau^2 \int_M \left( R + \frac{n}{2\tau} \right) + \Delta f - f d\mu \), where the infimum is taken over all the smooth functions satisfying \( \int_M e^{-f} d\mu = 1 \). By solving the backward heat equation in Corollary 67, we get a smooth function \( f : M \times [t_1, t_2] \rightarrow \mathbb{R} \) with initial condition \( f(t_2) = f \). We define a linear function \( \tau : [t_1, t_2] \rightarrow (0, +\infty) \) with \( \tau(t_2) = T + t_2 \), where \( T \) is a real number. Under the coupled system (40), by the monotonicity formula, we obtain the following

\[
\mu_e(g(t_1), \tau(t_1)) = \inf \mathcal{W}_{ek}(g, \tau, f) \\
\leq \mathcal{W}_{ek}(g, \tau(t_1)) |_{t=t_1} \\
\leq \mathcal{W}_{ek}(g, \tau, f) |_{t=t_2} \\
= \mu_e(g(t_2), \tau(t_2)).
\]

\(^4\) By the choice of \( T \), we can show that \( \tau(t_1) = T + t_1 = \frac{\alpha - t_1}{\alpha - 1} > 0 \) and \( \tau(t_2) = T + t_2 = \frac{\alpha t_2 - t_1}{\alpha - 1} > 0 \).
On the other hand, it is easy to see that, if we simultaneously scale $\tau$ and $g$ by a scalar $\alpha > 1$, we have the following scaling property of $\mu_{ek}$,

$$
\mu_{ek}(\alpha g(t), \alpha \tau) = \alpha \mu_{ek}(g(t), \tau).
$$

(51)

Combining (50), (51), and the diffeomorphic invariant property of the functionals, we have

$$
\mu_{ek}(g(t_1), \tau(t_1)) \leq \mu_{ek}(g(t_2), \tau(t_2)) = \alpha \mu_{ek}(g(t_1), \tau(t_1)).
$$

(52)

This yields $0 \leq (\alpha - 1) \mu_{ek}(g(t_1), \tau(t_1))$, i.e.

$$
\mu_{ek}(g(t_1), \tau(t_1)) \geq 0.
$$

(53)

As in the previous section, we need to choose $T = \frac{t_2 - t_1}{\alpha - 1}$ in order to simultaneously change $g$ and $\tau$. Again, one can show that $\frac{\tau(t_2)}{V(g(t_2))} = \frac{\tau(t_1)}{V(g(t_1))}$. There exists a $t \in [t_1, t_2]$, such that $0 = \frac{d}{dt} \frac{\tau}{V}$. Hence, at this $t$, $0 = \frac{d}{dt} k \log \frac{\tau}{V} = k \frac{n}{2\tau} + \frac{\int R d\mu}{V} = \frac{\int k(R + \frac{\Delta f}{2\tau}) d\mu}{V} \geq \inf \int k(R + \frac{n}{2\tau}) + \Delta f e^{-f} d\mu$. Equivalently, $\mu_{ek}(g(t), \tau(t)) \leq 0$. Recall (53) and (50), we obtain $0 \leq \mu_{ek}(g(t_1), \tau(t_1)) \leq \mu_{ek}(g(t), \tau(t)) \leq 0$, i.e. $\mu_{ek}(g(t_1), \tau(t_1)) = 0$. Hence all the inequalities in (50) must be equalities and the first variation of $W_{ek}$ must vanish. Therefore, the Ricci breather must be a gradient Ricci soliton, and furthermore, it must be Einstein, namely

$$
R_{ij} + \nabla_i \nabla_j f - \frac{1}{\tau} g_{ij} = 0
$$

and

$$
R_{ij} - \frac{1}{\tau} g_{ij} = 0.
$$

(54)

**Remark 82** One cannot get similar results for shrinking breathers by using the same method because $\alpha - 1 < 0$ in the shrinking case.

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