Localization and adiabatic pumping in a generalized Aubry-André-Harper model

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A generalization of the Aubry-André-Harper (AAH) model \cite{1, 2} is a workhorse for the study of localization and topological states in one dimension. It is described by the tight-binding equation \cite{2}

\[ t(\psi_{n+1} + \psi_{n-1}) + V_1 \cos(Qn + k)\psi_n = E\psi_n, \quad (1) \]

where \(\psi_n\) is the wavefunction amplitude at site \(n\), \(t\) is a nearest-neighbor coupling, and \(V_1\), \(Q\), and \(k\) are the amplitude, frequency, and phase of the on-site potential. This model emerges naturally from the dimensional reduction of a two-dimensional (2D) Quantum Hall (QH) system to a one-dimensional (1D) chain, where the phase \(k\) is the quasi-momentum transverse to the chain \cite{1, 3}. When the potential is made quasiperiodic by setting \(Q = 2\pi \beta\), with \(\beta\) sufficiently irrational \cite{4, 5}, the model exhibits a localization transition: all bulk eigenstates are extended for \(0 < V < 2t\) and localized for \(V > 2t\), while at \(V = 2t\) the system is self-dual under the transformation \(\psi_n = \sum_m f_m \exp(i mnQ) \) \cite{2}. The relationship between quasiperiodicity and localization has been explored in many subsequent variants of the AAH model. Typically, altering the form of the potential modulation results in starkly different behaviors; some models exhibit co-existing localized states and extended states separated by a mobility edge \cite{2, 6, 8}, while others lack any localization transition \cite{9}. One particularly interesting variant, which preserves the critical properties of the original AAH model, involves incommensurate modulations in the off-diagonal couplings \cite{10, 13}. In this case, all states are localized for \(V_1 > 2 \max(t, V_2)\), while \(V_2\) is the amplitude of the off-diagonal modulation \cite{13}.

The off-diagonal AAH model has recently been subject to renewed attention \cite{14, 17}. Kraus et al. have demonstrated theoretically and experimentally that a purely off-diagonal AAH model can be realized as a lattice of coupled optical waveguides, and that topological boundary states can be pumped across the chain by adiabatically winding the phase of the modulation \cite{14}. In subsequent works, the generalized AAH model (including both diagonal and off-diagonal modulation) was shown to be topologically equivalent to Fibonacci quasicrystals of the same quasiperiodicity \cite{15, 16}, and the commensurate model was found to exhibit topological Majorana boundary states \cite{17}.

This paper discusses the effects of the relative phase difference \(\phi\) between the on-site and off-diagonal modulations in a quasiperiodic AAH model. Earlier studies of off-diagonal AAH models have set \(\phi = 0\), motivated by the derivation from a 2D QH system with uniform magnetic field and next-nearest-neighbor couplings \cite{10, 13}. However, this is an artificial restriction when considering fabricated 1D quasicrystals, such as waveguide arrays \cite{17}. We show that an AAH model with non-zero \(\phi\) retains many of the original AAH model’s compelling features, such as analytic tractability, while introducing several new features. For instance, by varying \(\phi\) with all other parameters fixed (including the modulation amplitudes and quasiperiodicity), one can induce a transition between purely extended and purely localized bulk states. By noting that 1D systems with non-zero \(\phi\) can be generated from 2D QH systems with uneven magnetic flux distribution in each unit cell, we derive a generalization of Aubry-André duality, as well as the \(\phi\)-dependent localization phase diagram. Moreover, introducing \(\phi\) as a tunable parameter allows us to group generalized AAH models into topologically distinct families. As shown in Refs. \cite{15, 16}, when AAH models are grouped by \(k\) (the phase common to both modulations), the \(E\) versus \(k\) bandstructures have topologically non-trivial and equivalent bandgaps. At face value, this seems to rule out using 1D quasicrystals to observe topological transitions. But if AAH models are grouped by the relative phase \(\phi\) instead of \(k\), the \(E\) versus \(\phi\) bandstructures can exhibit either topologically trivial or non-trivial bandgaps. Thus, topological transitions between models of the same quasiperiodicity can be observed. Finally, we show that because \(\phi\) affects both localization and the adiabatic pumping of topological boundary states, it is possible to observe interesting interactions between the two processes. All these phenomena should be observ-
and off-diagonal modulations. The on-site and off-diagonal modulations the same phase) \[10–16\]. This is because the diagonal and off-diagonal modulations have the same (to be derived below). The heat maps in Fig. 1(b)-(c) show the ground state’s inverse participation ratio (IPR) \(\sum_n |\psi_n|^4\), which vanishes for extended states \[25\]. Fig. 2 plots the IPR of all eigenstates, at three points in the phase diagram. In the extended phase [Fig. 2(a)], we verified by direct inspection that all high-IPR states are boundary states. In Fig. 2(b), \(V_1\) and \(V_2\) are chosen so the system is in the extended phase for \(\phi = 0\) and in the localized phase for \(\phi = \pi/2\), which agrees with the computed IPR values. In Fig. 2(c), all states are localized.

To understand the above behavior, we derive a generalization of Aubry-Andr é duality. For \(\phi = 0\), the AAH model (including off-diagonal modulation) is spectrally invariant under the exchange \(t \leftrightarrow V_1/2\), with localized states mapped to extended states and vice versa \[12, 13\]. For the \(\phi \neq 0\) case, we take the QH system with couplings given by Eqs. (3)-(6), and apply the following gauge transformation:

\[
\hat{A}_{mn} \rightarrow \hat{A}_{mn} - nQ \hat{x} - mQ \hat{y}.
\]  

Fourier transforming the 2D tight-binding Hamiltonian in the \(+\hat{x}\) coordinate yields Eq. (2). Since \(\phi\) does not affect the total flux through a unit cell of the QH system, it conserves the quasiperiodicity of the 1D chains. The redistribution of magnetic flux is reminiscent of Haldane’s “zero field QH” model \[20\], which demonstrated that the band-topological properties of a QH system can be altered without changing the net flux through a unit cell. In the present model, the flux redistribution described by \(\phi\) will be shown to affect both localization and topological properties of the derived 1D quasicrystals. Although it may be difficult to realize a 2D QH system with the required nonuniform flux, the 1D quasicrystals themselves should be readily realizable, as discussed below.

**Localization transition.**—For \(\phi = 0\), the localization phase diagram was derived in Refs. \[11, 13\], and is shown in Fig. 1(b). For \(V_1 > 2\max(t, V_2)\), all bulk eigenstates are localized; for \(V_1,2V_2 < 2t\), all bulk eigenstates are extended; and in the remainder of the phase space, the eigenstates are critical \[13, 27\]. As in the original AAH model, the localization transition is driven by the modulation amplitudes. In particular, when the off-diagonal modulation amplitude \(V_2\) is less than \(t\), the critical value of \(V_1\) is exactly the same as in the original AAH model.

Varying \(\phi\) changes the phase diagram. As shown in Fig. 1(c), for \(\phi = \pi/2\) the critical phase shrinks onto the line of the \(V_2\) axis, while the boundary between the extended and localized phases becomes an elliptical arc (to be derived below). The heat maps in Fig. 1(b)-(c) show the ground state’s inverse participation ratio (IPR) \(\sum_n |\psi_n|^4\), which vanishes for extended states \[25\]. Fig. 2 plots the IPR of all eigenstates, at three points in the phase diagram. In the extended phase [Fig. 2(a)], we verified by direct inspection that all high-IPR states are boundary states. In Fig. 2(b), \(V_1\) and \(V_2\) are chosen so the system is in the extended phase for \(\phi = 0\) and in the localized phase for \(\phi = \pi/2\), which agrees with the computed IPR values. In Fig. 2(c), all states are localized.

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\[
\hat{A}_{mn} \rightarrow \hat{A}_{mn} - nQ \hat{x} - mQ \hat{y}.
\]
The system thus exhibits duality under a combination of exchanging $t \leftrightarrow V_1/2$, and moving the relative phase $\phi$ into the complex argument of the off-diagonal modulation. Note that this reduces to the usual Aubry-André duality for $\phi = 0$. Furthermore, only the magnitude of each off-diagonal term affects the spectrum; its complex argument can be gauged away.

Using Eq. (8), we can derive the spectral measure by following the argument of Thouless [11]. The result is that the generalized AAH model of Eq. (2) has purely extended bulk states when

$$
\sum_{\pm} \sqrt{(V_1/2)^2 \pm V_1V_2 \cos \phi + V_2^2} < 2t.
$$

For $\phi = 0$, this inequality states that the states are extended in a rectangular region of the phase diagram, $V_1 < 2t$, $V_2/t < 1$; and via Aubry-André duality, they are localized for $V_1 > 2 \max(t, V_2)$, in agreement with Fig. 1(b) and Ref. [13]. For $\phi = \pi/2$, the extended phase is the semi-elliptical region

$$
(V_1/2)^2 + V_2^2 < t^2,
$$

which agrees well with the numerical results shown in Fig. 1(c). For intermediate values of $\phi$, we find that Eq. (9) also agrees with numerical results. For $\phi \neq 0$, the critical phase in the $\phi = 0$ phase diagram appears to be unstable; with even small nonzero values of $\phi$, the entire region outside the extended phase is found numerically to consist of purely localized states.

**Topological properties.**—AAH models can be topologically characterized by noting that a family of AAH chains with different $k$’s is essentially a 2D system, with $k$ acting as an additional compact dimension. When $Q$ is given by a rational approximant $2\pi q/p$, the 2D system has well-defined topological invariants in the form of the Chern numbers of the $p$ bands [29]. For $p \gg 1$, the Chern flux is independent of $k$ and the modulation amplitudes $V_1$ and $V_2$. Based on this, Kraus et al. concluded that every individual AAH chain of the same $Q$ and $\phi$ is topologically equivalent and non-trivial [13] [15]. However, as Madsen et al. have explained, this is misleading: the association of a 1D chain with a Chern number makes sense in the context of how the family of chains was constructed (e.g. grouping by $k$), but is meaningless outside that context; in fact, any individual AAH chain can be continuously deformed into a conventional 1D insulator [20]. In this light, it is desirable to look for alternative families of AAH chains, which may exhibit different topological properties than those derived in Refs. [14] [15].

One such interesting alternative is to use the parameter $\phi$ in place of $k$ as a compact dimension for grouping AAH chains. The bandstructures of $E$ versus $\phi$ reveal the existence of topologically distinct regimes, as shown in Fig. 3. For $V_1 \lesssim 2V_2$, the bandgaps are topologically trivial, and for $V_1 \gtrsim 2V_2$ the bandgaps are topologically non-trivial and spanned by boundary states. Hence, by varying $V_1$ and/or $V_2$, a topological transition can be induced between families of AAH chains with the same
FIG. 4: (color online) Bandstructures of $E$ versus $\phi$ for (a) $k = 0$ and (b) $k = 0.3\pi$. The other parameters are $V_1 = 1.9$, $V_2 = 0.5$, $t = 1$, $Q = (1 + \sqrt{5})\pi$, and $N = 101$. The areas bounded by vertical dashes show the $\phi$ intervals over which the bulk states are localized, based on Eq. (9). The red and purple circles show the expectation value $\langle \psi(t)|E|\psi(t)\rangle$, starting from a boundary state and taking $\phi(t) = 10^{-5}t$. The wavefunctions are plotted in Fig. 5.

Due to the choice of parameters in this figure, all bulk states are extended, and the boundary states in Fig. 3(c) can be pumped from one end of the chain to the other by adiabatically winding $\phi$, similar to the pumping by winding $k$ demonstrated in Ref. [14]. In these bandstructures, varying $k$ shifts the dispersion relation of the boundary states without altering the bulk bands, similar to tuning boundary conditions on a two-dimensional strip [31, 32].

The $\phi$ dependence of bulk state localization also affects the adiabatic pumping of topological boundary states. As described in Ref. [14], the pumping process involves a boundary state adiabatically merging into a bulk band and becoming an extended state, which then evolves into a boundary state at the opposite end. However, when the bulk states are localized, pumping fails due to a breakdown of adiabaticity [33]. Fig. 4 shows a particularly interesting situation, in which $V_1, V_2$ are chosen so that extended and localized bulk states exist at different values of $\phi$. Since $k$ changes the dispersion of the boundary states, it can control whether a boundary state merges into the band in the extended or localized regime. We simulate the pumping process by numerically solving the Schrödinger equation $i\hbar \frac{d}{dt} |\psi(t)\rangle = H[\phi(t)] |\psi(t)\rangle$, starting from a boundary state and increasing $\phi$ slowly as $\phi(t) = 10^{-5}t$. For fixed $d\phi/dt = 10^{-5}$, Fig. 4(a) and (b) show successful pumping of a boundary state. For another value of $k$, with $d\phi/dt$ and all other parameters unchanged, the adiabatic pump breaks down as shown in Fig. 4(b) and Fig. 5(b), due to the fact that the boundary state now merges into a localized bulk.

**Discussion.**—The most feasible experimental platform for studying these phenomena is an array of evanescently-coupled optical waveguides [18–20]. Such systems have previously been used to demonstrate Anderson localization in disordered lattices [21, 22], localization in AAH chains with purely on-site modulation [23], and adiabatic pumping of boundary states in AAH chains with purely off-diagonal modulation [14]. Our results motivate the development and study of quasiperiodic waveguide arrays with simultaneous on-site and off-diagonal modulations. The on-site potential can be modulated via the waveguide’s width [23] or refractive index [21, 25], while the off-diagonal hopping term can be modulated via the inter-waveguide separation [14, 22, 24]. Since the two types of modulation are independently designed, there should be no difficulty setting the relative phase $\phi$ to any desired value [17, 25]. Realizations with cold-atom lattices may also be possible [34].

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