Isotropic turbulence correlation functions modelling on the basis of Karman-Howarth equation

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Abstract A closure model for the von Karman-Howarth equation is considered. The offered algebraic closure model is based on a single spatial-point third-order correlation function, unlike the widely used models based on a distance between two separated points. The model holds for homogeneous isotropic turbulence theory. Numerical solution of the von Karman-Howarth equation for the flow behind a bi-plane grid of circular cylinders in comparison to experimental data is presented.

1. Introduction

It is known that the statistical theory of isotropic turbulence allows for a complete kinematic analysis of turbulent transfer. Most theoretical analyses are constrained by the use of additional hypotheses [1]. Isotropic turbulence is defined by the condition that the average values of velocity components and their derivatives are statistically uniform in all directions. This allows considering the average values to be slowly varying functions of time.

In view of the above, one can define the second correlation tensor between two arbitrary velocity components at two arbitrary flow points. The correlation between the three components is called the triple correlation. Averaged values of the derivatives of the velocity fluctuations are expressed in terms of the derivatives of the double correlation tensor. Eventually, this process results in the partial differential Karman-Howarth equation [2] connecting second and third correlations. Formally the original Karman-Howarth equation is not closed as it contains two unknown variables: the second- and third-order correlation functions.

Several closures of the Karman-Howarth equation have been developed including the direct interaction approximation model [3] and the eddy-damped quasi-normal Markovian (EDQNM) closure [4].

There are numerous publications related to the closure of the original Karman-Howarth equation. Different closure models are offered in references [5–11] in the form of functional relations between the third- and the second-order correlations.

Most of the closure models presented in references [5–11] consider the third-order correlation as a functional relation, comprising the second-order correlation function derivative with respect to the distance between two separated points. In the present work, an alternative closure model is considered in the form of algebraic equation using the second-order correlation function. Numerical simulation of the turbulent flow behind a bi-plane grid of circular cylinders and comparison to experimental data are also shown.
2. Closure of the Karman-Howarth equation

The original von Karman-Howarth equation for the two-point velocity correlation can be written as

$$\frac{\partial (\Delta u)^2}{\partial t} = \left( \frac{\partial}{\partial r} + \frac{4}{r} \right) \left[ (\Delta u)^3 + 2 \nu \frac{\partial (\Delta u)^2}{\partial r} \right],$$

(1)

where $(\Delta u)^2$ and $(\Delta u)^3$ are the second- and triple-correlation functions, $\nu$ is the viscosity, and $r$ and $t$ are the spatial and time coordinates.

The equation (1) cannot become a closed equation unless a modeled functional relation is employed. Most of the closure models in references [5] – [11] use a gradient type hypothesis in the form

$$\bar{(\Delta u)^3} = r^\beta A(t,r) \frac{\delta (\Delta u)^2}{\delta r},$$

(2)

where $\beta = \text{const}$, and some function $A(t,r)$ is determined by the closure models, presented in references [5] – [11].

From [11] for small scales, it follows a relation

$$\bar{(\Delta u)^3} = -S \bar{(\Delta u)^{23/2}}.$$

(3)

Here $S$ is the skewness of the longitudinal velocity derivative with respect to the longitudinal distance $x$ (in [11] it was accepted $S = \text{const}$).

In the present research and for the triple-correlation function an algebraic generalized expression (3) is used:

$$\bar{(\Delta u)^3} = -\bar{r} \alpha(\bar{r}) \bar{(\Delta u)^{23/2}},$$

(4)

where $\bar{r} = r/r_0$, $r_0$ is the reference size and $\alpha(\bar{r})$ is a smooth arbitrary function.

One can note that the gradient type closure models in [5–11] in some sense has a non-local character since the function $\bar{(\Delta u)^3}(r,t)$ is determined by two spatial points values $(\Delta u)^2(0,t)$ and $(\Delta u)^2(r,t)$. Thus, unlike the gradient type closure model the algebraic model is an one spatial point model.

Substitution (4) for the Karman-Howarth equation (1) results in equation for the second-correlation function

$$\frac{\partial (\Delta u)^2}{\partial t} = \left( \frac{\partial}{\partial r} + \frac{4}{r} \right) \left[ -\bar{r} \alpha(\bar{r}) \bar{(\Delta u)^{23/2}} + 2 \nu \frac{\partial (\Delta u)^2}{\partial r} \right].$$

(5)

To write equation (5) in dimensionless form, one can introduce the following variables:

$$U = \frac{(\Delta u)^2}{V_\infty^2}, \quad \bar{t} = t \frac{v}{r_0^2},$$

(6)

where $V_\infty$ is the reference velocity. The expression for the dimensionless time can be also written in the form

$$\bar{t} = t \frac{v}{r_0^2} = \frac{x}{V_\infty r_0^2} = \frac{x}{\text{Re}}.$$

(7)

Here $x = x/r_0$ is the dimensionless longitudinal distance and

$$\text{Re} = \frac{V_\infty r_0}{\nu}$$

is the reference Reynolds number. Taking into account (6) equation (5) can be written as
\[ \frac{\partial U}{\partial \tilde{x}} = \left( \frac{\partial}{\partial \tilde{r}} + \frac{4}{\tilde{r}} \right) \left[ -\tilde{r} \alpha(\tilde{r}) U \sqrt{\tilde{U}} + \frac{2}{\text{Re}} \frac{\partial U}{\partial \tilde{r}} \right]. \] (8)

In the present work it is accepted that \( \alpha(\tilde{r}) = \alpha \tilde{r} \), where \( \alpha = \text{const} \). After substitution to (8) one can obtain

\[ \frac{\partial U}{\partial \tilde{x}} = \left( \frac{\partial}{\partial \tilde{r}} + \frac{4}{\tilde{r}} \right) \left[ -\alpha \tilde{r}^2 U \sqrt{\tilde{U}} + \frac{2}{\text{Re}} \frac{\partial U}{\partial \tilde{r}} \right]. \] (9)

3. Final equation and numerical simulation conditions

The point \( \tilde{r} = 0 \) is a singular point of equation (9). To avoid this singularity a new variable \( \tilde{Z} = \frac{\tilde{r}^2}{2} \) is introduced. Then, equation (9) can be transformed into

\[ \frac{\partial U}{\partial \tilde{x}} = -6\sqrt{2} \tilde{Z} U \sqrt{\tilde{U}} + \left( -3\alpha \tilde{Z} \sqrt{\tilde{U}} + \frac{10}{\text{Re}} \frac{\partial U}{\partial \tilde{Z}} + \frac{4}{\text{Re}} \tilde{Z} \frac{\partial^2 U}{\partial \tilde{Z}^2} \right). \] (10)

Equation (10) was discretized on a Cartesian grid, using the implicit finite difference method and the iteration process for nonlinear terms, \( \Delta \tilde{x}, \Delta \tilde{Z} \).

Numerical solution of equation (10) was carried out for the experimental conditions described in references [12], [13]. The measurements were made in the turbulent flow behind a bi-plane grid of the circular cylinders for a reference Reynolds number value of 8700. Boundary conditions for simulation were taken in the form:

\[ \frac{\partial U}{\partial \tilde{Z}}(\tilde{Z}, 0) = \frac{\partial U}{\partial \tilde{Z}}(\tilde{Z}, \tilde{Z}_2) = 0. \] (11)

Here \( \tilde{Z}_2 \) is the right boundary of the spatial integration area corresponding to the value \( \tilde{Z}_2 = \frac{\tilde{r}_2^2}{2} \), where \( \tilde{r}_2 = 4\sqrt{2} \). The cylinder diameter \( d = r_0 \) was used as the reference length. The parameter \( \alpha \) was set at 0.75.

To perform numerical simulations, it is necessary to determine the initial condition \( U(0, \tilde{Z}) \) for equation (10). The task of the function \( U(0, \tilde{Z}) \) description is not unique due to a complicated character of the flow around the cylinder. Figure 1(a) shows the flow structure past the isolated cylinder for \( \text{Re} = 10^4 \) obtained experimentally [14].

![Flow visualization past the circular cylinder obtained experimentally (a) and with numerical CFD simulation (b).](image)

Figure 1. Flow visualization past the circular cylinder obtained experimentally (a) and with numerical CFD simulation (b).

Numerical modeling was conducted in ANSYS Fluent using the incompressible Navier-Stokes equations. Computational hexa-grid contained about \( 6 \times 10^6 \) cells and was constructed for a 3D cylinder section with the cylinder diameter length. Boundary conditions of the “symmetry” type were assigned for the lateral surfaces of the computational domain. The “pressure outlet” boundary condition was determined at the distance of \( 15d \). The mesh had the first surface cells height of 0.001d.
The Karman vortex street and the two separating shear layers on both sides of the cylinder are clearly observed in the figure. The shear layer distance $r_{sh}$ is of about one cylinder diameter, and the area of isotropic turbulence must satisfy the condition $r > r_{sh} \approx r_0 = d$. So, in the current research, the initial condition for the second-correlation function $U(0, \tilde{z})$ was taken in the form

$$U(0, \tilde{z}) = U_0 \text{ for } \tilde{z} \leq \tilde{z}_1; U(0, \tilde{z}) = 0 \text{ for } \tilde{z} > \tilde{z}_1.$$  \hspace{1cm} (12)

Here $\tilde{z}_1 = \tilde{r}_1^2 / 2$, where the value $\tilde{r}_1 = 1.54$ corresponds to the condition $r_{sh} < \tilde{r}_1 < \tilde{r}_2$. $U_0$ of 0.00145 approximately corresponds to the data of reference [12].

4. Numerical simulation results

In Figure 2(a) the calculated and experimental [12] data for the second-correlation function $U(\tilde{x}, 0)$ are plotted against the dimensionless longitudinal distance $\tilde{x}$. The overall behavior of the predicted second-correlation function values correspond to the experimental data. Thus the values of $U(\tilde{x}, 0)$ are overpredicted at the middle part of the simulated range and are underpredicted for high $\tilde{x}$ values. Figure 2(b) shows the predicted and experimental values of $U(\tilde{x}, r) / U(\tilde{x}, 0)$ function against the spatial $\tilde{r}$ coordinate for $\tilde{x} = 240$.

**Figure 2.** Second-correlation function $U(\tilde{x}, 0)$ vs. the dimensionless longitudinal distance (a) and Second-correlation function $U(\tilde{x}, r) / U(\tilde{x}, 0)$ radial distribution for $\tilde{x} = 240$ (b).

Figure 2(b) demonstrates good agreement between the computed and experimental data for the second-correlation function.

The proposed point-wise third-order correlation function for the Karman-Howarth equation closure allows obtaining the second-correlation function simulation results in fair agreement with experimental data. Further improvements can be achieved by investigating the effect of the initial data distribution.

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