Tunable planar Josephson junctions driven by time-dependent spin-orbit coupling

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The integration of conventional superconductors with common III–V semiconductors provides a versatile platform to implement tunable Josephson junctions (JJs) and their applications. We propose that with gate-controlled time-dependent spin-orbit coupling, it is possible to strongly modify the current-phase relations and Josephson energy and provide a mechanism to drive the JJ dynamics, even in the absence of any bias current. We show that the transition between stable phases is realized with a simple linear change in the strength of the spin-orbit coupling, while the transition rate can exceed the gate-induced electric field gigahertz changes by an order of magnitude. The resulting interplay between the constant effective magnetic field and changing spin-orbit coupling has direct implications for superconducting spintronics, the control of Majorana bound states, and emerging qubits. We argue that topological superconductivity, sought for fault-tolerant quantum computing, offers simpler applications in superconducting electronics and spintronics.

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In the push to implement beyond-CMOS applications, Josephson junctions (JJs) have found their broad use due to their high-speed switching, low-power dissipation, and intrinsic nonlinearities.[1, 2] In addition to the well-established role of JJs as the key elements for superconducting electronics and superconducting qubits,[1–6] there is a growing interest in tailoring their spin-dependent properties to enable dissipationless spin currents, cryogenic memory,[7–14] and fault-tolerant quantum computing.[15–22] The role of spin-orbit coupling (SOC) has been extensively studied in the normal-state properties and recognized for its importance in spintronics.[23–25]. However, the superconducting analogs of the SOC-related effects remain to be understood. They might even be important when their normal-state counterparts are negligibly small.[26–32] Motivated by the recent progress in gate-controlled SOC in planar JJs based on a two-dimensional electron gas (2DEG),[33–35], we reveal how time-dependent SOC tunes many of their key properties and offers an unexplored mechanism to drive JJs.

A common description of a JJ circuit, is given by a Josephson element, resistor, and capacitor connected in parallel, using the resistively and capacitively shunted junction (RSCJ) model. The bias current through the junction, $i$, is the sum of the supercurrent and the quasiparticle current flowing in the resistor and capacitor. The supercurrent is usually assumed as $I_c\sin(\varphi + \varphi_0)$, where $I_c$ is the maximum supercurrent, $\varphi$ is the phase difference between the superconducting regions, and the anomalous phase, $\varphi_0 \neq 0, \pi$, arises from the broken time-reversal and inversion symmetries.[37–43]

For a ballistic JJ depicted in Fig. 1(a), the interplay between SOC and the effective Zeeman field $h$, yields a more complex current-phase relation (CPR) than $I_c(\varphi)$ given above, such that for a generalized RSCJ model

$$d^2\varphi/d\tau^2 + (d\varphi/d\tau)/\sqrt{\beta_c} + I(\varphi, \mu, h, \alpha)/I_c = -\omega_p t + I_c,$$  

where $\tau = \omega_p t$ is a dimensionless time, expressed using the JJ plasma frequency, $\omega_p = \sqrt{2\pi I_c / \Phi_0 C}$, $\Phi_0 = h/2e$ is the magnetic flux quantum, and $C$ is the capacitance. The damping of this nonlinear oscillator is characterized by the Stewart-McCumber parameter, $\beta_c = 2\pi I_c CR^2 / \Phi_0$, where $R$ is the resistance $[44, 45]$ and $Q = \sqrt{\beta_c}$ is the quality factor. The generalized CPR can be modified by the chemical potential $\mu$, and $h$, arising from the applied magnetic field or magnetic proximity effect.[46] Since $h_z$ does not induce $\varphi_0$,[47, 48] and only produces CPR reversals, we focus on $h_z = 0$ [Fig. 1(a)]. The CPR can also be tuned by the Rashba SOC, illustrated in Fig. 1(a), which is parametrized by its strength $\alpha$, in the Hamiltonian, $H_{so} = \alpha(\sigma \times \mathbf{p}) \cdot \hat{z}$. Here, $\sigma$ is the Pauli matrix vector, and $\mathbf{p}$ is the in-plane momentum, for 2DEG with the inversion symmetry broken along the $z$-direction.[49]

While quasistatic gate-tunable changes in SOC and $\varphi_0$ have been demonstrated in 2DEG-based JJs,[33, 34], the implications of dynamically tuned SOC on the CPR remain unexplored. For a conventional CPR without any $\varphi_0$, Eq. (1) has a mechanical analog with a driven and damped pendulum, in which $\varphi$ becomes the displacement angle$[44, 45]$. A JJ driven by $i$ is equivalent to the pendulum displaced by an external torque from its stable equilibrium, determined by the gravitational acceleration $g$, while $\omega_p$ determines the oscillation frequency around a stable equilibrium point.[1]

Instead of using $i$, Fig. 1(b) suggests an entirely different way to drive the pendulum: By changing the orientation of the effective $g'$ and the new equilibrium, resulting from the interplay of the static $h$ and time-dependent $\alpha$. 

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With JJ advances and gate changes exceeding the gigahertz range[3], there is a tantalizing prospect for dynamically controlled CPR by time-dependent SOC. Unlike assuming a specific relation, \( I(\varphi) = I_c \sin(\varphi + \varphi_0) \), the CPR can have a more general and anharmonic form which should be obtained microscopically. To this end, a single-particle Hamiltonian, \( \hat{H}(\mathbf{p}) = p^2/2m^* + \sigma \cdot \mathbf{h} + H_{\text{so}}(\mathbf{p}) \), where \( m^* \) is the effective mass, can be used to solve a BCS model of superconductivity, given by the effective Hamiltonian

\[
\hat{H}(\mathbf{p}) = \begin{pmatrix}
\hat{H}(\mathbf{p}) - \mu \hat{1} & \frac{\Delta}{\Delta^*} \\
\frac{\Delta^*}{\Delta} & -\hat{H}^\dagger(-\mathbf{p}) + \mu \hat{1}
\end{pmatrix}, \tag{2}
\]

where \( \Delta \) is a 2 \times 2 superconducting gap in spin space[47].

After diagonalizing the resulting Bogoliubov-de Gennes equations, \( \hat{H}\hat{\psi} = E\hat{\psi} \), where \( \hat{\psi} \) is the four-component wave function for quasiparticle states with energy \( E \), we match the wave functions and generalized velocities at interfaces \( (x = 0, d) \), shown in Fig. 1(a). This allows us to obtain the ground-state JJ energy \( E_{\text{GS}} \), together with the corresponding CPR, using charge conservation and the quantum definition of current[47]. The CPR is related to the JJ energy: \( I(\varphi) \propto \partial E_{\text{GS}}/\partial \varphi \)[50].

Our numerical findings are illustrated for the JJ depicted in Fig. 1(a). The normal region \( (N) \) has a length \( L = 0.3\xi_S \) and a width \( W = 10L \), such that lengths are normalized by \( \xi_S = \hbar/\sqrt{2m^*\Delta} \), where \( \Delta \) is the superconducting gap in \( S \). The energies are normalized by \( \Delta \) and the supercurrent \( I_0 = 2|e\Delta|/h \), where \( e \) is the electron charge and \( |e\Delta|/h \) is the maximum supercurrent in a single-channel short \( S-N-S \) JJ[50].

To explore the tunability of CPRs and JJ energies with SOC, we focus on the parameters for high-quality epitaxial InAs-Al based JJs, \( \Delta_{\text{Al}} = 0.2 \text{ meV} \), with a 2-dimensional factor of 10 for InAs, while its \( m^* \) is 0.03 times the electron mass[33, 34]. In these JJs the gate control of Rashba SOC and thus its magnitude in the range \( \alpha \in (0, 180 \text{ meV}A) \) has been demonstrated[33, 34]. In Fig. 2, at \( h_x = (2/3)\Delta \approx 450 \text{ mT} \), we assume gate control process that primarily changes \( \alpha \), not \( \mu \). Experimentally, this could be realized with dual-gate schemes[51] to independently tune the carrier density and the electric field, \( E \). However, for a continuous change of \( \alpha \), we are unaware that even in a static case the calculated CPR and \( E_{\text{GS}} \) are given.

In Fig. 2(a), for \( \mu = \Delta \), the anharmonic CPR changes significantly with \( \alpha \). There is a competition between \( \sin \varphi \) and the next harmonic, \( \sin 2\varphi \), resulting in \( I(\varphi) = -I(\varphi) \). However, there is no spontaneous current, \( I(\varphi = 0) \equiv 0 \), only \( I_c \) reversal with \( \alpha \). Such a continuous and symmetric \( 0-\pi \) transition is well studied without SOC in \( S/\text{ferromagnet}/S \) JJs due to the changes in the effective magnetization, temperature, or the thickness of the magnetic region[52–60]. The corresponding JJ energy landscape in Fig. 2(b), shifted such that its overall minimum value is 0, corroborates this SOC evolution. By increasing \( \alpha \) from 0 to 200 meV, the minimum in \( E_{\text{GS}} \) changes from \( \varphi = 0 \) to \( \pi \), and then goes back to 0. A gray trace indicates that by increasing \( \alpha \) in a smaller range, the JJ minimum can transition from \( \varphi = 0 \) to approximately \( \pi/2 \).

While we use an exact (complete) CPR with its anharmonicities, their prior descriptions have often relied on an approximate simple harmonic expansion \( \sin n\varphi, \cos n\varphi \)[61, 62]. However, this approach is not very efficient with SOC. Instead, it is better to use a compact form where only a small number of terms gives a more accurate description[47]

\[
I(\varphi, \mu, \alpha) \approx \sum_{\sigma=\pm} \sum_{n=1}^{N} \frac{I_n^{\sigma}(n\varphi + \varphi_0^{\sigma})}{\sqrt{1 - \tau_n^{\sigma} \sin^2(n\varphi/2 + \varphi_0^{\sigma}/2)}}. \tag{3}
\]
where $\tau_0^\sigma$ is the JJ transparency for spin channel $\sigma$ and the phase shifts $\varphi_{0n}$ are additional fitting parameters. This description includes the anomalous Josephson effect $I(\varphi = 0) \neq 0$, revisited in JJ diode effects[63–67]. For a simple picture of a single anomalous phase[47, 48],

$$\varphi_0 \propto h_y \alpha^3,$$

(4)

therefore vanishing in Fig. 2, where $h = h_x \hat{x}$.

A quasistatic gate-controlled SOC suggests that more important opportunities are available using fast gate changes, compatible with the advances in JJ circuits[3]. However, the implications of gigahertz changes in SOC and a different mechanism to drive JJ, as sketched in Fig. 1(b), remain unexplored. To obtain the resulting JJ dynamics we use Eq. (1) with $i = 0$, where the driving arises from $\alpha = \alpha(t)$, viewed as a time-dependent effective $g'$. Some guidance as to what to expect for JJ dynamics can be given from the InAs-Al samples, where, in addition to the previous range of $\alpha$, $I_e \sim 4 \mu A$, $R \sim 100 \Omega$, and $C \sim 15 \text{fF}$, leading to $\omega_p \sim 900 \text{GHz}$ and the damping $\beta_c \sim 1$, which is also suitable for the rapid single-flux quantum (RSFQ) applications[1, 4]. We keep $h_x = (2/3)\Delta$.

The JJ dynamics are driven by a simple linear variation of $\alpha(t)$ from the gate-controlled $E$, as shown in Fig. 3(a). We first consider in Fig. 3(b) the reduction of $\omega_p$, from 1000 GHz (similar to InAs-Al JJs[33]) to 10 GHz (much faster than the $\alpha(t)$ variation), at $\beta_c = 1$. The results reveal a strong delay in the onset in the $\varphi = 0$ to approximately $\pi/2$ transition, which is indicated from the static picture in Fig. 2(b). Simultaneously, the time for the $\varphi = 0$ to approximately $\pi/2$ transition is increased by an order of magnitude.

We next examine, in the inset of Fig. 3(b), the influence of reducing $\omega_p$, from 1000 GHz (similar to InAs-Al JJs[33]) to 10 GHz (much faster than the $\alpha(t)$ variation), at $\beta_c = 1$. The results reveal a strong delay in the onset in the $\varphi = 0$ to approximately $\pi/2$ transition, which is indicated from the static picture in Fig. 2(b). Simultaneously, the time for the $\varphi = 0$ to approximately $\pi/2$ transition is increased by an order of magnitude.

Finally, in Fig. 3(c), the $\varphi = 0$ to approximately $\pi/2$ transition occurs first for the slower $\alpha(t)$ variation, but takes approximately the same time as the faster gigahertz $\alpha(t)$ variation. This is encouraging for various applications, since (i) $E$ control of SOC allows tailoring of the onset of the transition between different states, (ii) a high-frequency switching between different equilibrium states and driving JJs is not limited by the characteristic times for the $E$ variation. $\alpha(t)$ changes at 0.2 GHz give an order-of-magnitude faster transition between the stable phases.

While the $E$ control of $\alpha$ and the evolution of the $E_{GS}$ minima in Fig. 2 largely determine the JJ dynamics in Fig. 3, it helps to identify other opportunities for SOC-driven JJs. In Fig. 4, we consider $\omega_p = 10 \text{GHz}$ and a triangular-like $\alpha(t)$ at $\mu = 10 \Delta$. For an underdamped regime, $\beta_c = 10$, the pendulum analogy from Fig. 1(b) explains the phase evolution of the gray trajectory from Fig. 4(a), also reproduced in Fig. 4(c). By increasing $\alpha$ to the maximum at $192 \text{meV/\AA}$, the pendulum is at an unstable position and will swing toward the $\varphi = 0$ minimum (equivalently shown as $\varphi = 2\pi$), implying that $g'$ points vertically down. With small damping (gray trajectory), the pendulum passes the equilibrium point, even when, with $\alpha < 80 \text{meV/\AA}$, the equilibrium and the overall minimum shift to $\varphi = \pi$, with $g'$ vertically up. Eventually, with damping it reaches the $\varphi = \pi$ minimum.

For critical damping, with the same starting point [see also Fig. 4(c)], the brown trajectory reveals a very different evolution with $\alpha$. Instead of the overall $E_{GS}$ minimum $\varphi = \pi$, for $\alpha = 0$, the phase is locked at the local minimum $\varphi = 0$. With a stronger damping, the $\varphi$ oscil-
FIG. 5. The evolution of (a) JJ CPR and (b) the JJ energy with $\varphi$ and $\alpha$, for $\mu = \Delta$ and $h_y = (2/3)\Delta$, rotated by $\pi/2$ from Fig. 2. An anharmonic CPR breaks the $I(-\varphi) = -I(\varphi)$ symmetry in (a) and the corresponding anomalous phase, $\varphi_0$, increases with $\alpha$ in (b). The inset in (a) shows $\varphi(t)$ for $\omega_p = 1000\text{GHz}, \beta_c = 1$ with a linearly increasing $\alpha$ from 0 to 160meVÅ over 1ns, which is then held at a maximum, with its JJ energy path in (b).

lations are insufficient to overcome the SOC-dependent barrier which, for $\alpha = 0$, separates the local minimum at $\varphi = \pi$ from the global one at $\varphi = 2\pi$. The tunability of the SOC-controlled energy landscape alone does not fully determine the generalized CPRs. The influence of the JJ circuit parameters can enable different $\varphi$ transitions.

In the above discussion, the tunability of CPRs and $E_{\text{GS}}$ does not exploit the anomalous Josephson effect[37–40, 70], which can be understood in analogy to $g'$ pointing sideways and therefore, breaking the symmetry from Figs. 2–4 and $I(-\varphi) \neq -I(\varphi)$. This situation can be simply realized by rotating $\mathbf{h}$ along the $y$ axis, while we retain all the other parameters from Fig. 2(a). The resulting CPR in Fig. 5(a) confirms that the JJ supercurrent is driven not only by $\varphi$ but also by $\varphi_0$, which is responsible for the stated symmetry breaking and, equivalently, the tilted $g'$. As for SOC cubic in $k$[47], there is a strong anharmonic behavior and the expected diode effect, where the sign and magnitude of the supercurrent depend on the polarity of the applied bias [34].

The implications of the combined broken time-reversal and inversion symmetries, responsible for the anomalous Josephson effect, are further illustrated in Fig. 5(b), which shows the SOC-tunable $E_{\text{GS}}$, single valued for the gray path, and leading to the time-dependent diode effect. This behavior is qualitatively different from the doubly degenerate $\varphi_0$ state in Fig. 2(b), which results from the second-harmonic generation in the CPR.

Even with a moderate $h_y \approx 450\text{mT}$ for InAs based JJs, with increasing $\alpha(t)$ we see an evolution of the single global minimum and thus the changes in the corresponding values of $\varphi_0$ from $\varphi = 0$ to approximately $3\pi/4$, in good agreement with the measured values[33]. This suggests that at a larger $\mathbf{h}$, for example, in In(As,Sb) with a much larger $g$ factor[35], it may be possible to fully control the tilt angle of $g'$ and simply swap between 0 and $\pi$ states in JJs, further controlling how the JJ dynamics are driven.

The same geometry in Al-InAs JJs at a larger $h_y$ has been experimentally shown to also support topological superconductivity[34]. This is important for several reasons, beyond hosting Majorana bound states[16]. The resulting topological superconductivity is associated with equal-spin $p$-wave superconductivity which could offer gate-controlled dissipationless spin currents, a key element for superconducting spintronics[7, 8]. Such spin-triplet supercurrents could be extended over a long range[71] and could overcome the usual competition between superconductivity and ferromagnetism. A transition to topological superconductivity is accompanied by an extra phase jump, of approximately $\pi[72, 73]$. Such a $\pi$ jump in Al-InAs JJs has been observed at $h_y \approx 600\text{mT}[34]$, an effective field about 25 times smaller, than expected for the $0-\pi$ transition

$$B_{0-\pi} = (\pi/2)h\nu_F/(g\mu_B L),$$

for a spin-polarized system in the absence of SOC[74], where $\nu_F$ is the Fermi velocity, $\mu_B$ the Bohr magneton, and $L$ the JJ length. Therefore, SOC plays a crucial role in understanding various transitions and, at larger $h_y$, the range of an effective $\varphi_0$ could exceed $2\pi[34]$ and support $2\pi$ pendulum rotation from Fig. 1(b), as used in RSFQ logic and memory[1, 4]. Therefore, in addition to the prospect of fault-tolerant quantum computing, the search for topological superconductivity also offers a promising platform for superconducting electronics and spintronics.

Without previous studies on SOC-driven JJ dynamics, we focus on a simple model and do not consider time-dependent magnetic fields[75] or noise[76]. A more general description could simultaneously include the role of changing $\mu$ and other SOC forms, linear and cubic in $k$, shown to give different routes to topological superconductivity and control of Majorana states[47, 77–79]. However, we expect that our focus only on linearized Rashba SOC, easily tunable by $\mathbf{E}$ field[33, 34], already clarifies its important role in JJ dynamics. With changing SOC, there are further opportunities for gate-controlled Majorana states and the probing of their non-Abelian statistics[80, 81] or an added tunability in the implementation of superconducting qubits[3, 82, 83]. This would extend the previously studied qubit tunability by voltage or flux[3, 84] as well as the use of $\pi$-phase states for an improved qubit operation[85, 86].

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