Waveless Approximation Theories of Gravity

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Note from the Author

During the 1970’s, there was a lot of interest in developing good numerical simulations of the production of gravitational radiation by strongly interacting astrophysical systems (the archetype being a binary pair of black holes). As a graduate student at the University of Maryland during that time, with strong encouragement from my advisor, Charles Misner, I developed a collection of approximation schemes which were designed to try to avoid the fatal numerical instabilities which arose during the course of all of the contemporary attempts to evolve the full system of Einstein’s equations. The idea of these schemes was to mathematically decouple the “gravitational wave” production from the evolution of the matter and the slowly changing “induced gravitational fields” of the astrophysical systems. The equations governing the evolution of the matter and induced gravitational field system were designed to be a form of enhanced Newton-Euler system: The matter fields would evolve via Euler-type equations, while the induced gravitational fields would be determined by a system of elliptic equations to be solved for certain pieces of the metric. These equations were set up to ignore gravitational radiation, except for the possible inclusion of terms accounting for the loss of energy due to wave propagation. To determine the gravitational radiation production, the idea was to incorporate the motion of the matter and induced gravitational fields into source terms for a system of linear wave equations.
In 1978, I wrote a paper which describes some of the equation systems I developed for the matter and induced gravitational field evolution. I called these systems "Waveless Approximation Theories of Gravity", and I submitted a paper with this title to Physical Review D. It was rejected by the referees, based on the reasonable contention that I hadn’t tried to numerically implement the ideas. Not being any good at numerical work, I lay the paper and the ideas aside, and I went on to other things. The only record of it which I left was a brief mention towards the end of an article I wrote with Jim Nester for the Einstein Centenary volume "General Relativity and Gravitation" [edited by A. Held, Plenum, 1980].

A number of years later, John Friedman noticed that one of the popular methods developed by J. R. Matthews and James Wilson to study gravitational radiation production (It is sometimes called the CFC or “Conformal Flatness Condition” approach) is very similar to one of my Waveless Approximation Theories. (John happened to have read my article with Nester.) He has been kind enough to suggest that people call this the Isenberg-Mathe\[\text{...}]

The version appearing here is unchanged from what I wrote in 1978. In reading it over, I have found no serious errors, except for the claim that the equations constituting “WAT-II” are elliptic, and consequently always admit a unique solution. For general fields, they are not. I hope people find this article useful.
Abstract

The analysis of a general multibody physical system governed by Einstein’s equations in quite difficult, even if numerical methods (on a computer) are used. Some of the difficulties – many coupled degrees of freedom, dynamic instability – are associated with the presence of gravitational waves. We have developed a number of “waveless approximation theories” (WAT) which repress the gravitational radiation and thereby simplify the analysis. The matter, according to these theories, evolves dynamically. The gravitational field, however, is determined at each time step by a set of elliptic equations with matter sources. There is reason to believe that for many physical systems, the WAT-generated system evolution is a very accurate approximation to that generated by the full Einstein theory.
According to Einstein’s theory, most nonspherical, nonstationary multibody systems must radiate gravitationally [1]. Whether or not one is interested in the gravitational waves themselves, they make the analysis of the evolution of such a system quite difficult to carry out. Numerical computers can overcome some of these difficulties, such as those associated with many coupled degrees of freedom, nonlinearity, and gauge coordinate freedom; but the hyperbolic nature of Einstein’s equations (including gravity waves) leads to problems of numerical stability.

While the gravitational waves make it hard to analyze the motion of a multibody system, the studies of Thorne [2], Smarr [3] and others indicate that in some cases, the real physical effects of the waves upon the motion of the bodies (“radiation back reaction”) may be quite small. We are thus motivated to develop an approximation to Einstein’s theory (coupled to matter) which neglects radiation in calculating the motion of a gravitating system of bodies. Such a computational tool could be useful not only in studies which ignore gravity waves, but also in studies which seek to find the radiation emitted by a system: A waveless approximation theory (“WAT”) could be used first, to determine the motion of the bodies; then one could compute what waves are emitted by bodies so moving. A WAT would effectively decouple and thereby simplify the analysis of gravity wave production.

No one knows exactly how to identify the “radiation terms” in Einstein’s theory. Thus there is no clearcut, unique way to neglect radiation in Einstein’s theory. We have therefore developed a few waveless approximation theories. Some of them are formally superior, some are easier to use in doing calculations, and some appear to give good approximations for a wider class of systems. Much testing of the various versions of WAT remains to be done before we can say which of the WAT’s holds the most practical promise. It may happen that more than one version can be used profitably.

The original (and in some ways the best) “waveless approximation to Einstein’s theory” is the Newtonian theory of gravity. When supplemented by the Euler equations for fluid motion [4], Newton’s theory determines the evolution of a multibody (continuous fluid) self-gravitating system as follows: The matter is described by the mass-density \( \rho(x,t) \) the 3-velocity \( v^\alpha(x,t) \), and the pressure \( p(x,t) \). [The pressure is assumed to be some function of \( \rho \), according to a prescribed equation of state.] The gravity is described by a single potential \( \Phi(x,t) \). Only the matter variables \( \rho \) and \( v^\alpha \) are dynamic.
They change in time according to the equations [5].

\[
\dot{\rho} = -\partial_a (pv^a) \tag{1}
\]

and

\[
\dot{v}^a = -v^m \partial_m v^a - \frac{1}{p} \partial^a p + \partial^a \Phi. \tag{2}
\]

As the matter moves about, the gravitational potential \( \Phi \) slavishly follows, as determined by the elliptic equation

\[
\partial^2 \Phi = -4\pi \rho. \tag{3}
\]

Gravity has no dynamics of its own, in the Newtonian theory.

For many systems, (most of those we may encounter in the solar system) the Newtonian theory provides a very accurate description of the motion. And it is very easy to work with. [Indeed, the gravitational potential equation (3) can always be solved, in integral form, thereby eliminating \( \Phi \) explicitly from the matter evolution equations (1) and (2).] However at relativistic speeds \((v^2 \rightarrow 1)\) and relativistic densities \((p \rightarrow 10^{15} gm/cc)\) the Newtonian description becomes progressively inaccurate.

Einstein’s theory, which agrees with present-day experiments to a high degree of precision, replaces the single gravitational potential by the ten components of the metric. And it replaces the single linear elliptic equation (3) by the nonlinear, mixed elliptic-hyperbolic set of Einstein’s equations. Broken down into 3 + 1 dynamic form (convenient for later discussion), the Einstein system with fluid source takes the following form: The field variables are the metric

\[
g = -N^2 dt^2 + \gamma_{ab}(dx^a + M^a dt)(dx^b + M^b dt), \tag{4}
\]

the extrinsic curvature

\[
K_{ab} = -2N L_+(\gamma_{ab}) \tag{5}
\]

and the fluid stress energy tensor

\[
T^{\alpha,\beta} = \rho U^{\alpha} U^{\beta} + pg^{\alpha\beta}. \tag{6}
\]

The lapse \(N\) and shift \(M^\alpha\) are completely arbitrary throughout time. Four of Einstein’s equations,

\[
0 = R + (tr K)^2 - K^m_n K^n_m - 16\pi [(U^\perp)^2 \rho - p] \tag{7}
\]
and

\[ 0 = \nabla_b K^c_b - \nabla_b (tr K) - 8\pi\rho U^a U^\perp \]  

are constraint equations for the initial data. These constraints can be made explicitly elliptic, using the conformal-transverse-trace decomposition scheme of York [6]. The rest of the system consists of time evolution equations:

\[ \dot{\gamma}_{ab} = -2NK_{ab} + \mathcal{L}_M \gamma_{ab}, \]  

\[ \dot{K}^a_n = N\left\{ R^a_n + trKK^a_n - [\rho\delta^a_n + \rho U^a U_n] \right\} - \frac{1}{2} \delta^a_n [\rho(U^\perp)^2 - 3\rho] \]  

\[ - \nabla^a \nabla_n N + \frac{1}{N} \mathcal{L}_M K^a, \]  

\[ \dot{\rho} = \mathcal{F}[N, M, \gamma, K; \rho, U, p], \]  

\[ \dot{U}^m = \mathcal{F}^m[N, M, \gamma, K; \rho, U, p]. \]

Here \( \mathcal{F} \) and \( \mathcal{F}^m \) are easily calculated (from the action) functionals of the indicated variables. Despite the elliptic initial value constraints (7) and (8), this set of equations is predominantly a dynamic (hyperbolic) set. The gravitational field does not slavishly follow the matter; it evolves on its own. This contributes to computational instability. This instability, together with strong coupling and nonlinearities evident in (7)- (12), make the full Einstein theory very difficult to use for computation, even on a large computer.

The waveless approximation theories are designed to compromise between the Newtonian and Einsteinian theories. We would like as much of the gravitational fields as possible to be nondynamic - to be determined elliptically at each time by the evolving matter. We would like the nonlinearities and coupling in the equations to be cut down. At the same time, we want the WAT-generated motion of a given system to be as close as possible to that computed using Einstein’s theory.

WAT-I, which leans slightly towards the Newtonian side of the compromise, is the best motivated among all the WAT’s. It is founded on the notion that the conformal 3-geometry on a maximal \((tr K = 0)\) spacelike slice is closely associated to gravitational radiation [7]. Thus WAT-I produces maximal spacelike slices only, and it represses the intrinsic conformal curvature on these slices.
We derive the WAT-I field equations by building intrinsic conformal flatness and maximal slicing directly into the Lagrangian action of Einstein:

\[ S = \int [4R + L_M] \eta \]  

(13)

[Here \( 4R \) is the spacetime scalar curvature, \( L_M \) is the matter Lagrangian, and \( \eta \) is the volume element]. Thus we compute \( 4R \) out of the metric

\[ g = -N^2 dt^2 + e^{2\psi} \delta_{ab}(dx^a + M^a dt)(dx^b + M^b dt) \]  

(14)

[where \( \delta_{ab} \) is the flat metric] and we require that the momentum conjugate to \( \psi \) must vanish. The matter Lagrangian is chosen as in Einstein’s theory, but with the metric taking the specialized form (14). Varying the action (13), with perfect fluid source fields [8], we obtain the following set of field equations for the five gravitational potentials \( (N, M^0, \psi) \) and for the matter fields \( \rho, U^\alpha, p \):

\[ \partial^2 \psi + (\partial_m \psi)(\partial^m \psi) = \frac{e^{2\psi}}{8Na} \left[ (\ell M)^{b}_a (\ell M)^b_a \right] + 2\pi [\rho (U^\alpha)^2 - p] \]  

(15)

\[ \partial^2 N + (\partial_m N)(\partial^m \psi) = \frac{1}{2N} e^{2\psi} \left[ (\ell M)^{b}_a (\ell M)^b_a \right] + N \left( \frac{3}{2} p - \rho \right), \]  

(16)

\[ \partial_b \left[ \frac{e^{2\psi}}{N} (\ell M)^{b}_a \right] = 8\pi U_f U^\perp \rho, \]  

(17)

\[ \dot{\rho} = \mathcal{F}[N, M, \psi; \rho, U, p] \]  

(18)

\[ \dot{u}^m = \mathcal{F}^m[N, M, \psi; p, U, p] \]  

(19)

Here \( (\ell M)^{a}_b = \partial^a M_b + \partial_b M^0 - \frac{2}{3} \delta^{a}_b \partial_m M^m \) (conformal killing operator in flat space).

Clearly the gravitational fields of WAT-I are of the desired “slave” type. That is, while the matter evolves according to (18) and (19), the potentials \( N, M^a, \psi \) follow it around, as determined by the five elliptic equations (15)-(17) and the accompanying boundary value problem. The equations (17)-(19) are coupled and non-linear. However, the differential operators appearing in these equations are all strongly elliptic and self adjoint [9]. So while the WAT-I boundary value problem is nowhere as simple as that of Newton’s theory, it seems to admit unique solutions for any chosen matter fields \( \rho, U^\alpha, p(p) \) [10]. Moreover, the nonlinearities and coupling in these
equations are relatively mild. We see this especially if we consider matter fields and boundary conditions with planar symmetry, in which case we can always obtain explicit analytic solutions for \((N, M^a, \psi)\) [11]. More generally, while we can’t always find analytic solutions, the boundary value problem for WAT-I is amenable to numerical solution, without any particular inherent danger of numerical instability.

How accurately does the WAT-I treatment of a given multibody system approximate the Einstein treatment of the same system (as specified by free initial data)? We first note that, unlike Newtonian solutions, WAT-I solutions can satisfy the full set of Einstein’s equations. A prime example is the Schwarzschild solution. This can happen because the five WAT-I gravitational equations are a subset of the Einstein equations (7)-(10) for a maximal, intrinsically conformally flat, slicing. [Such is not the case for the Newtonian equation (3).] A means of gauging the inaccuracy of the WAT-I prescribed motion for a given system now suggests itself: Consider the collection of terms in the Einstein’s equations which are not included in the WAT system. They are primarily the matter and curvature terms appearing in the tracefree right hand side of eq. (10) which prevent a set of intrinsically conformally flat, maximal initial data from retaining these properties as it evolves. So it is these terms which cause the WAT-I evolution and the Einstein evolution to diverge. Thus if we monitor these terms as we carry out a WAT-I analysis of a given multibody system, we get some idea of how different an Einstein analysis of the same system would be.

Another means for comparing WAT-I and Einstein treatments of multibody systems, particularly those on an astrophysical scale, is provided by the PPN scheme [12], along with its higher order extensions [13]. Since the standard PPN spacetime metric takes exactly the form (14) [i.e. it admits an intrinsically conformally flat slicing], we see that to PPN order, WAT-I and the Einstein theory are identical. The two theories diverge at higher (“PPPN”) orders, but this is not surprising, since gravitational radiation is important in these higher order approximations. We can use the PPPN analyses to parametrize the inaccuracy of WAT-I, as applied to systems of interest.

It is not only gravitational radiation that is ruled out when we require (as in WAT-I) that a spacetime admit a maximal, intrinsically conformally flat metric. The Kerr “rotating black hole” spacetime is stationary [and therefore contains no radiation], and yet it admits no such metric foliation [14]. We would like a waveless approximation theory which could handle Kerr, and
other systems of rotating bodies. Thus we look for a WAT which permits intrinsically conformal curvature, and in fact determines the intrinsic conformal geometry on every (maximal?) slice elliptically.

No way has been found to directly build into the action principle an elliptic determination of the intrinsic conformal geometry on each (maximal) spacelike slice. If, however, we are willing to sacrifice the advantages of an action [e.g., the canonical formalism, and the Noether conservation laws], then we can set up new waveless approximation theories which do include non-dynamic conformal curvature. We describe a few of them here.

In some sense, the dynamics of the intrinsic conformal geometry is contained in the transverse traceless part of the extrinsic curvature tensor $K_{(TT)}$. We are thus motivated to kill both $K_{(TT)}$ and its time evolution. The vanishing of $K_{(TT)}$, together with the maximal slide condition, allows us to replace $K^a_b$ by the conformal Killing form:

$$K^a_b = (LW)^a_b = \nabla^a W_b + \nabla_b W^a - \frac{2}{3} \delta^a_b (\nabla \cdot W)$$

(20)

The requirement that $K_{TT}$ vanish in the Einstein equations is a complicated condition, involving inverse operators. But if we instead ask that the time derivative of the full traceless part of $K^a_b$ vanish, then we get a manageable equation:

$$0 = R^c_b - \frac{1}{3} \delta^c_b R - \frac{1}{N^2} [\nabla^c N \nabla_b N - \frac{1}{3} \delta^c_b \nabla^c N]$$

$$- \rho [U^c U_b - \frac{1}{3} \delta^c_b U^m U_m] + \frac{1}{N} \partial_M (LW)^c_b$$

(21)

Examination of $R_{ab}$ as a differential operator on the conformal metric $\tilde{\gamma}_{ab}$ indicates that (21) determines $\tilde{\gamma}_{ab}$ elliptically. We build WAT-II by combining (21) with some means of determining the conformal scale factor $\psi$, the lapse $N$, and the shift $M^a$. The method for doing this which is most faithful to Einstein’s equations (and therefore most accurate) is to combine (20) and (21) with the Einstein constraint equations, the matter evolution equations (11)-(12), and finally the maximal slice and minimal distortion conditions [15]:

$$0 = (tr \dot{K}) \Rightarrow 0 = R - \frac{\nabla^2 N}{N} - \frac{3}{2} \rho (U_2)^2 - \frac{3}{2} p - \rho U^m U_m$$

(22)

$$0 = (\tilde{\gamma})^m_n \Rightarrow 0 = \nabla_m [L M]^m_n - \nabla_m [N (LW)^m_n]$$

(23)
Then WAT-II proceeds by (a) evolving the matter fields according to (11) and (12), and (b) solving for the thirteen gravity potentials $\tilde{\gamma}_{ab}, \psi, N, M^a, W^a$ using (9), (10), (21), (22), and (23). The determination of the gravitational field is apparently quite complicated, but since it is elliptic in nature, WAT-II is still a simpler system than Einstein’s equations for numerical computation.

A simpler version of WAT-II is obtained if we decouple the determination of $\tilde{\gamma}_{ab}$ from that of the other gravitational potentials. We can do this by using the equations of WAT-I to find $N, M^a$ and $\psi$, setting $W^a = M^a$, and then solving equation (21) for $\tilde{\gamma}_{ab}$. This decoupled version of WAT-II involves much simpler equations, but is likely to be much less accurate than the coupled version.

WAT-III is obtained by forgetting all about WAT-I, [i.e. forgetting conformal decomposition and maximal slicing] and carrying the idea of WAT-II [requirement that the traceless part of (10) vanish] one step further. For WAT-III, we replace $\dot{K}^a_b$ and $\dot{\gamma}_{ab}$ by zero everywhere in the Einstein equations (7)-(10) and then attempt to solve what remains. Setting $\dot{\gamma}_{ab} = 0$ in (9) allows us to replace $K_{ab}$ by $\frac{1}{2N} [\nabla_a M_b + \nabla_b M_a]$ in the rest of the equations, so we obtain

$$ R + \frac{1}{N^2} (\nabla \cdot M)^2 - \frac{1}{N^2} \nabla^a M_b \nabla^b M^a = 16\pi [((U^\perp)^2 \rho - p] $$

$$ \nabla_c \left[ \frac{1}{2N} (\nabla^c M_b + \nabla_b M^c) \right] - \nabla_b (\nabla \cdot M) = 8\pi \rho U_b U^\perp $$

and

$$ R^a_b + \frac{1}{2N} (\nabla M) [\nabla^a M_b + \nabla_b M^a] - \frac{\nabla^a \nabla_b N}{N} + \frac{1}{N} \mathcal{L}_M [\nabla_b M^a + \nabla^a M_b] $$

$$ = \frac{1}{2} \rho \delta^a_b - \rho [U^a U_b + \frac{1}{2} \delta^a_b (U^\perp)^2] $$

which are to be solved for $N, M^a$ and $\gamma_{ab}$. The matter evolution for WAT-III is governed by (11)-(12) and then on each time slice we solve the ten equations (24)-(26) for the ten gravitational variables.

The gravitational equations for WAT-III and for WAT-II are just about equal in complexity. How is one to choose between these two approximation schemes? WAT-II, we have seen, is motivated by the relationship between gravitational radiation and the conformal geometry on maximal slices. WAT-III, on the other hand, is the natural generalization of the Bardeen-Wagoner approximation treatment of axially symmetric systems [16].
But the motivation for the various versions of WAT must be only a secondary consideration. These waveless theories are proposed as possible tools for the approximate calculation of the motion of multibody physical systems; they are not to be treated as possible candidates for the “true theory of gravity.” [Like the Newtonian theory of gravity, all versions of WAT involve action-at-a-distance and therefore violate any possible notion of causality.] So the various versions of WAT should be judged using more practical considerations primarily: ease of application, accuracy of approximation. Such practical considerations cannot yet be properly judged. More experience is needed in applying WAT-I, WAT-II (coupled or uncoupled), WAT-III, or any other version, to physical systems of interest.


References

[1] Szekeres has provided examples of nonspherical, nonstationary solutions
of Einstein’s equations which do not radiate. See Comm. Math. Phys. 
41, 55 (1975).

[2] Thorne and co-workers have looked at non-radial neutron star pulsations. See Ap. J. 149, 591 (1967) and Ap. J. 155, 163 (1969).

[3] Smarr and others have examined (numerically) the collision of two black
holes. See his article (p. 569) in the Annals of the N.Y. Academy of 
Sciences, vol. 302, “Eighth Texas Symposium” (1977).

[4] For definiteness, we consider our “multibody system” to be one which
is filled with perfect fluid, satisfying Euler’s equations of motion. Our
discussion here would change very little if we were to consider instead a 
system of discrete particles.

[5] Our conventions largely follow those of Gravitation (1973), by Misner,
Thorna and Wheeler. Note in particular the following:

Latin indices are spatial (1,3,3)
Greek indices are spatio-temporal (0,1,2,3)
refers to the “perp” or surface-orthogonal direction
∂α is the partial derivative
∂2 is the flat space Laplacian operator
∇α is the covariant derivative
∇2 is the curved space Laplacian
“·” is the Lie derivative in the time direction
L is the Lie derivative in spatial directions
G = 1
C = 1

[6] See N. O. Murchadha and J. W. York, Phys. Rev. D. 10 428 (1974) and 
Phys. Rev. D. 10 437 (1974).

[7] See J. W. York, Phys. Rev. Lett. 28, 1082 (1972).
[8] As in the Newtonian discussion, we choose perfect fluid sources here simply as an example. WAT-I works with any desired source field (as represented in $L_M$).

[9] The properties of the differential operator appearing in Eq. 17 are discussed by York in Ann. Inst. H. Poincare, XXI, 319 (1974).

[10] Existence and uniqueness proofs have so far been carried through only for systems with spherical or planar symmetry.

[11] Unpublished work of the author.

[12] See Will’s article in Bertotti, Experimental Gravitation, Academic Press, New York (1974).

[13] Chandrasekhar discusses the higher order PPN theory in a number of articles. See, for example, Ap. J. 160, 153 (1970), and Ap. J. 158, 55 (1969).

[14] The fact that the maximal slicing of Kerr is not intrinsically conformally flat is proven by Monroe in his 1976 Ph.D. dissertation (U. of N.C.). It is not known whether there exists any conformally flat slicing that is on-maximal.

[15] See Smarr and York, “The Radiation Gauge in General Relativity,” Prepring 847, Center for Astrophysics, Cambridge, MASS (1977).

[16] See Bardeen and Wagoner, Ap. J. 167, 359, (1971).