GENERAL MODULAR QUANTUM DILOGARITHM
AND BETA INTEGRALS

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ABSTRACT. We consider a univariate beta integral composed from general modular quantum
dilogarithm functions and prove its exact evaluation formula. It represents the partition function
of a particular 3d supersymmetric field theory on the general squashed lens space. Its possible
applications to 2d conformal field theory are briefly discussed as well.

CONTENTS

1. Introduction 1
2. The modular quantum dilogarithm 3
3. The rarefied hyperbolic beta integral 8
4. Limiting relations 13
Appendix A. The Dedekind sum 16
Appendix B. Dimofte’s notations 16
References 18

To A. A. Slavnov on the occasion of his 80th birthday

1. INTRODUCTION

For many generations of students in Russia, including the authors, the basic sources of knowl-
edge on quantum field theory were the textbook by Bogoliubov and Shirkov [7] and the mono-
graph of Slavnov and Faddeev [41], complementing the fundamental treatise [7] by a considera-
tion of the gauge fields theory. However, nowadays they do not form a complete background for
entering a modern research. One of the most important missing ingredients in these books is
supersymmetry, which became vital for understanding perturbative and non-perturbative theo-
retical mechanisms behind some physical phenomena. In particular, the last three decades saw
the rise of the highly powerful localization technique allowing for exact computation of various
partition functions (including the superconformal indices) of supersymmetric field theories on
curved space-times [35].

These partition functions are expressed in terms of the complicated special functions of hy-
pergeometric type. Trying to understand their properties, it is natural to ask – whether they
are new or not? Clearly they are beyond the existing textbooks like [8] and handbooks like [34]
and require searching in the modern mathematical literature. Raising such a question, Dolan
and Osborn have found in 2008 [15] a direct connection of superconformal indices of 4d $\mathcal{N} = 1$
theories to the elliptic hypergeometric integrals constructed eight years earlier [12]. The discovery of elliptic hypergeometric functions at the turn of millenium became a big surprise to mathematicians, because the theory of special functions of hypergeometric type has been developed since the times of Euler only in two instances – the plain hypergeometric functions, like the Euler-Gauss $\text{}_2\text{F}_1$-function, and their $q$-analogues [3], and there were no indications on the existence of the third level for such functions. Moreover, these new functions unified two previously separately considered families of classical special functions (elliptic and hypergeometric ones) and generalized all previously found hypergeometric objects [44].

The connection with quantum field theory appeared to be very fruitful, since it resulted in new understanding of the structure of these functions and brought many new both mathematical and physical results (see, e.g. [47] or [35]). The present paper represents another step in the development of such relations. Namely, we describe an extension of $q$-hypergeometric functions constructed from the modular quantum dilogarithm. Modular analogue of the quantum dilogarithm function was suggested by Faddeev in connection to the lattice Virasoro algebra [18,19], and it has found many applications, in particular, in the hyperbolic Ruijsenaars model [38] and relativistic Toda chain [32], Yang-Baxter equation [4,10,11,28,49], topological invariants [14,29], usual 2d conformal field theory [20] and its discretizations [5]. In the relatively recent time it was found to play a key role in the computations of the supersymmetric partition functions of 3d models [27] on the squashed three-sphere $S^3_\tau$ (see [23]).

In the theory of special functions, a slight modification of Faddeev’s function was called the hyperbolic gamma function [38], and the integrals composed out of them were called the hyperbolic hypergeometric integrals. The top univariate hyperbolic beta integral was described in [38] and it represented a special degeneration of the elliptic beta integral [42], or, more generally, of the elliptic analogue of the Euler-Gauss hypergeometric function [44], as described in [9]. This integral has been directly connected to the functional star-triangle relation [45] and the Yang-Baxter equation [12], supersymmetric partition function of 3d theories [16], and a topological field theory [31].

Faddeev’s function used the simplest $\tau \to -1/\tau$ transformation from the modular group $SL(2, \mathbb{Z})$ applied to the infinite product $(z; q)_\infty = \prod_{j=0}^{\infty} (1 - zq^j)$, $q = e^{2\pi i \tau}$. Its generalization to a function based on an arbitrary modular transformation from $SL(2, \mathbb{Z})$ was suggested by Dimofte in [13]. The key motivation was an interest in the partition function of the Chern-Simons theory on the general squashed lens space $L(c, a)_\tau$. The $a = -1$ case of this function was considered earlier in [25] and around the same time in [1,2,33], $q$-Hypergeometric functions associated with $L(c, -1)_\tau$ were investigated in [21,46]. A manifestation of the corresponding hyperbolic integrals in 2d conformal field theory was discussed in [39].

Dimofte proved a pentagon relation for the general modular quantum dilogarithm, which can be considered as a special identity for the corresponding generalized hyperbolic hypergeometric integral. Its special case associated with the manifold $S^3_\tau/\mathbb{Z}_k$ was independently established in [2]. Following the terminology suggested in [46], we will be calling also the function introduced in [13] as the rarefied hyperbolic gamma function. Taking it as a building block, we construct the univariate hyperbolic beta integral associated with $L(c, a)_\tau$ and prove its exact evaluation
formula. This integral represents the partition function of a particular 3d supersymmetric field theory and defines a new class of solutions of the star-triangle relation. In particular, this formula and its degenerations extend the considerations of [21, 39]. Also, the derived formula should be applicable to the general lens space extension of 2d conformal field theory considered in [8], similarly to the parafermionic case [39]. The main result of the present work was announced in a short note [40].

2. The modular quantum dilogarithm

Let us take \( SL(2, \mathbb{Z}) \) group of modular transformations described by the matrices

\[
M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1, \quad a, b, c, d \in \mathbb{Z}.
\]

Its projective realization allowing simultaneous change of the signs of all integer parameters,

\[
\tau \rightarrow \frac{a\tau + b}{c\tau + d},
\]

plays an important role in the theory of automorphic forms. In the following we use only transformation (2) with the fixed sign of \( c \), \( c \geq 0 \),

In order to match with the notations of [13], we also set

\[
a = -p, \quad b = -s, \quad c = k, \quad d = -r, \quad k > 0, \quad pr + ks = 1.
\]

Since \( k \) will play a special role, we write also

\[
pr = 1 - ks = 1 \mod k,
\]

so that \((p|s) = (p|k) = (r|s) = 1\) (here \((p|k)\) denotes the greatest common divisor of \( p \) and \( k \)).

The three-dimensional sphere \( S^3 \) can be described by two complex variables \( z_1 \) and \( z_2 \) satisfying the equation \(|z_1|^2 + |z_2|^2 = 1\). Its squashed form \( S^3_\tau \) is defined by the relation \(|\kappa z_1|^2 + |\kappa^{-1} z_2|^2 = 1\), where \( \kappa = \kappa(\tau) \) is some deformation parameter. The squashed lens space \( L(c, a)_\tau \) is defined by the same relation after the following identification of points [50]

\[
(z_1, z_2) \sim (e^{\frac{2\pi i}{c} z_1}, e^{\frac{2\pi i}{c} z_2}), \quad a \in \mathbb{Z}_{>0}, \quad (a|c) = 1.
\]

In this context, because of the mod \( c \) restrictions, one has an additional reduction of the modular group parameters – there remain only two integer variables \( a \) and \( c \).

Following Dimofte [13], we define the general unnormalized modular quantum dilogarithm, or the rarefied hyperbolic gamma function, as

\[
\gamma_M(\mu, m) = \gamma_M(\mu, m; \omega_1, \omega_2) := \frac{(\tilde{q} e^{2\pi i u(\mu, m)}; \tilde{q})_\infty}{(e^{2\pi i u(\mu, m)}; q)_\infty}, \quad |q| < 1,
\]

where \((a; q)_\infty = \prod_{j=0}^{\infty} (1 - aq^j)\),

\[
u(\mu, m) := \frac{\mu + m\omega_2}{k\omega_2}, \quad q := e^{2\pi i \tau}, \quad \tau := \frac{\omega_1 + r\omega_2}{k\omega_2},
\]

and

\[
\tilde{q} := e^{2\pi i \tilde{\tau}}, \quad \tilde{\tau} := \frac{a\tau + b}{c\tau + d} = -\frac{p\omega_1 + \omega_2}{k\omega_1},
\]

together with

\[
e^{2\pi i u(\mu, m)} = e^{2\pi i \frac{\mu - pm\omega_1}{k\omega_1}} = \tilde{q}^m e^{2\pi i u(\mu, m)} \quad c\tau + d = \frac{\omega_1}{\omega_2}.
\]
Because of the evident periodicity $\gamma_M(\mu, m+k) = \gamma_M(\mu, m)$ we shall assume that $m \in \mathbb{Z}_k, \mathbb{Z}_k = \{0, 1, \ldots, k-1\}$. As argued in [13], $\gamma_M(\mu, m)$ is a general lens space analogue of Faddeev’s quantum modular dilogarithm [18]. We shall call it also the rarefied hyperbolic gamma function. This function, after appropriate normalization, gives rise to a compact form of the general rarefied hyperbolic beta integral evaluation formula, which is the main desired goal of this work.

Let us denote $2\pi i\omega_1/\omega_2 = -\delta$ and take the limit $\delta \to 0^+$. Then $q \to \epsilon$, $\epsilon = e^{2\pi i r/k}$, $\epsilon^k = 1$ and $\tilde{q} \to 0$. As a result we have the asymptotics

$$
\gamma_M(\mu, m) = \exp \sum_{n=1}^{\infty} \left( \frac{e^{2\pi i u/n}}{1-q^n} - \frac{\tilde{q}^n e^{2\pi i u/\tilde{q}^n}}{1-\tilde{q}^n} \right) \sim \exp \left( \frac{1}{k\delta} \text{Li}_2(e^{2\pi i u/2}) \right),
$$

(9)

where $\text{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ is the dilogarithm function. This relation justifies the name modular quantum dilogarithm for $\gamma_M(\mu, m)$, as suggested by Faddeev for the $k = p = r = 1$ case in [19].

The infinite product standing in the denominator of (5) has zeros at the points

$$
\mu = -j\omega_1 - (kn + m + jr)\omega_2, \quad n \in \mathbb{Z}, \quad j \in \mathbb{Z}_{\geq 0}.
$$

The numerator function has zeros at the points

$$
\mu = (kn' + p(m + j' + 1))\omega_1 + (j' + 1)\omega_2, \quad n' \in \mathbb{Z}, \quad j' \in \mathbb{Z}_{\geq 0}.
$$

There are coinciding points in these two sets, namely, the zeros with the coordinates

$$
kn + m + j' + 1 + rj = 0, \quad kn' + p(m + j' + 1) + j = 0
$$

cancel each other. Multiplying the first equation by $p$ and subtracting the second one we find

$$
n' = pm - sj, \quad kn + rj + m = -1, -2, \ldots.
$$

Therefore true poles of function (5) are located at the points

$$
\mu = -j\omega_1 - (kn + rj + m)\omega_2, \quad n \in \mathbb{Z}, \quad j \in \mathbb{Z}_{\geq 0}, \quad kn + rj + m \in \mathbb{Z}_{\geq 0},
$$

(10)

and true zeros lie at the points

$$
\mu = (p(m + j + 1) + kn)\omega_1 + (j + 1)\omega_2, \quad n \in \mathbb{Z}, \quad j \in \mathbb{Z}_{\geq 0}, \quad p(m + j + 1) + kn \in \mathbb{Z}_{>0}.
$$

(11)

Relations

$$
e^{2\pi i u(\mu + \omega_1, m+r)} = q e^{2\pi i u(\mu, m)}, \quad e^{2\pi i u(\mu + \omega_2, m-1)} = e^{2\pi i u(\mu, m)},
$$

$$
e^{2\pi i \tilde{u}(\mu + \omega_1, m+r)} = e^{2\pi i \tilde{u}(\mu, m)}, \quad e^{2\pi i \tilde{u}(\mu + \omega_2, m-1)} = \tilde{q}^{-1} e^{2\pi i \tilde{u}(\mu, m)}
$$

lead to the equations

$$
\gamma_M(\mu + \omega_1, m + r) = (1 - e^{2\pi i \frac{\mu + m + \omega_2}{k\omega_2}}) \gamma_M(\mu, m),
$$

(12)

$$
\gamma_M(\mu + \omega_2, m - 1) = (1 - e^{2\pi i \frac{\mu - pm + \omega_1}{k\omega_1}}) \gamma_M(\mu, m).
$$

(13)

Making back shifts of the variables, we obtain

$$
\gamma_M(\mu - \omega_1, m - r) = \frac{\gamma_M(\mu, m)}{1-q^{-1} e^{2\pi i \frac{\mu + m + \omega_2}{k\omega_2}}}, \quad \gamma_M(\mu - \omega_2, m + 1) = \frac{\gamma_M(\mu, m)}{1-\tilde{q} \tilde{e}^{2\pi i \frac{\mu - pm + \omega_1}{k\omega_1}}}.
$$
For $k = p = r = 1, s = 0$, i.e. $M = M_{st} := \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$, the dependence on $m$ disappears and one gets the standard Faddeev’s modular quantum dilogarithm (hyperbolic gamma function)

$$
\gamma(\mu) = \gamma(\mu; \omega_1, \omega_2) := \gamma_{M_{st}}(\mu, m) = \frac{(qe^{2\pi i \frac{\mu}{\omega_1}}; \tilde{q})_{\infty}}{(e^{2\pi i \frac{\mu}{\omega_2}}; q)_{\infty}}, \quad q = e^{2\pi i \frac{\mu}{\omega_1}}, \quad \tilde{q} = e^{-2\pi i \frac{\mu}{\omega_1}}. \tag{14}
$$

Using the identity $(a; q)_{\infty} = \prod_{n=0}^{k-1} (aq^n; q^k)_{\infty}$, one can write the function $\gamma_M(\mu, m)$ as the following product $\left[13\right]$

$$
\gamma_M(\mu, m) = \prod_{\gamma, \delta \in \Delta(k, p, m)} \gamma\left(\frac{1}{k}(\mu + \omega_1 \delta + \omega_2 \gamma); \omega_1, \omega_2\right), \tag{15}
$$

where $\Delta(k, p, m) = \{\gamma, \delta \in \mathbb{Z}, 0 \leq \gamma, \delta < k, p\gamma - \delta \equiv pm \text{ mod } k\}$. Multiplying the constraint $p\gamma - \delta \equiv pm \text{ mod } k$ by $r$, we can write it also as $\gamma - r\delta \equiv m \text{ mod } k$.

Using expressions [5] and [14], for general $M_1 = \begin{pmatrix} -p & -s \\ k & -r \end{pmatrix}$, $M_2 = \begin{pmatrix} r & k \\ -s & p \end{pmatrix}$, $M_3 = \begin{pmatrix} p & -k \\ s & r \end{pmatrix}$ it is straightforward to derive the identities

$$
\gamma_{M_1}(\mu, m; \omega_1, \omega_2) = \gamma_{M_2}(-s(\mu + m\omega_2), 0; p\omega_1 + \omega_2, r\omega_2 + \omega_1) \times \gamma\left(\frac{\mu + (k + m)\omega_2}{k}; r\omega_2 + \omega_1, \omega_2\right) \gamma\left(\frac{\mu + (k - pm)\omega_1}{k}; \omega_1, \omega_2 + p\omega_1\right), \tag{16}
$$

$$
\gamma_{M_1}(\mu, m; \omega_1, \omega_2) \gamma_{M_3}(s(\mu + m\omega_2), 0; r\omega_2 + \omega_1, p\omega_1 + \omega_2) = \gamma\left(\frac{\mu + m\omega_2}{k}; r\omega_2 + \omega_1, \omega_2\right) \gamma\left(\frac{\mu + (pm)\omega_1}{k}; \omega_1, \omega_2 + p\omega_1\right). \tag{17}
$$

It follows from (16) that for $s = -1$, i.e. for $rp = 1 + k$, the function $\gamma_M(\mu, m)$ can be written as a product of three $\gamma(\mu)$ functions:

$$
\gamma\left(\begin{pmatrix} -p & 1 \\ k & -r \end{pmatrix}\right)(\mu, m; \omega_1, \omega_2) = \gamma\left(\frac{\mu + (k + m)\omega_2}{k}; r\omega_2 + \omega_1, \omega_2\right) \times \gamma\left(\frac{\mu + (k - pm)\omega_1}{k}; \omega_1, \omega_2 + p\omega_1\right). \tag{18}
$$

Similarly, for the case $s = 1$, i.e. for $rp = 1 - k$, $\gamma_M(\mu, m)$ can be expressed via three $\gamma(\mu)$ functions as well,

$$
\gamma\left(\begin{pmatrix} -p & -1 \\ k & -r \end{pmatrix}\right)(\mu, m; \omega_1, \omega_2) = \frac{\gamma\left(\frac{\mu + m\omega_2}{k}; r\omega_2 + \omega_1, \omega_2\right) \gamma\left(\frac{\mu + (pm)\omega_1}{k}; \omega_1, \omega_2 + p\omega_1\right)}{\gamma(\mu + m\omega_2; r\omega_2 + \omega_1, p\omega_1 + \omega_2)}. \tag{19}
$$

It is instructive to compare the derived formulae with the one corresponding to parafermionic functions emerging for $p = r = 1$ and $s = 0$ $\left[21,25,33,39\right]$

$$
\gamma\left(\begin{pmatrix} -1 & 0 \\ k & -1 \end{pmatrix}\right)(\mu, m; \omega_1, \omega_2) = \gamma\left(\frac{\mu + m\omega_2}{k}; \omega_2 + \omega_1, \omega_2\right) \gamma\left(\frac{\mu + (k - m)\omega_1}{k}; \omega_1, \omega_2 + \omega_1\right). \tag{20}
$$

Let us take definition of the Dedekind $\eta$-function and Jacobi $\theta_1$-function

$$
\eta(\tau) = e^{\frac{\pi i \tau}{12}}(e^{2\pi i \tau}; e^{2\pi i \tau})_{\infty}, \tag{21}
$$

$$
\theta_1(u | \tau) = -\theta_{11}(u) = -\sum_{\ell \in \mathbb{Z} + 1/2} e^{\pi i \ell^2} e^{2\pi i (u + 1/2)} = i q^{1/8} e^{-\pi i u} (q; q)_{\infty} \theta(e^{2\pi i u}; q), \tag{22}
$$
with $q = e^{2\pi ir}$ and $\theta(z; q) = (z; q)_{\infty}(qz^{-1}; q)_{\infty}$. Modular transformation laws for $c > 0$ have the form

$$\eta \left( \frac{a\tau + b}{c\tau + d} \right) = \varepsilon(a, b, c, d) \sqrt{-i(c\tau + d)} \eta(\tau),$$

(23)

and

$$\theta_1 \left( \frac{u}{c\tau + d} \right) \frac{a\tau + b}{c\tau + d} = -i\varepsilon(a, b, c, d)^3 \sqrt{-i(c\tau + d)} e^{\frac{i\pi u^2}{c\tau + d}} \theta_1(u\mid \tau),$$

(24)

where the character (24-th root of unity)

$$\varepsilon = \varepsilon(a, b, c, d) := \left\{ \begin{array}{ll} (\frac{d}{c}) & \text{for odd } c, \\ (\frac{2}{d}) & \text{for odd } d. \end{array} \right.$$ 

(25)

Here $(\frac{d}{c})$ is the Legendre-Jacobi symbol. For odd $c > 0$

$$\left( \frac{d}{c} \right) = (-1)^{ge(d)}, \quad g_e(d) = \sum_{\nu=1}^{\frac{(c-1)/2}{c}} \left\lfloor \frac{2d\nu}{c} \right\rfloor,$$

where $[x]$ is the integer part of $x \in \mathbb{R}$.

Using the triple product Jacobi identity, we deduce the modular transformation rule for the $\theta(z; q)$-function:

$$\theta(e^{-\frac{2\pi i m}{c\tau + d}}; e^{2\pi i (\frac{a+c}{c\tau + d})}) = i\varepsilon^2 e^{\frac{2\pi i}{c} \left( \tau - \frac{a+c}{c\tau + d} \right)} e^{-\pi i \frac{m+1}{c\tau + d}} e^{\frac{\pi i m^2}{c\tau + d}} \theta(e^{2\pi i u}; e^{2\pi i r}).$$

(26)

Now we can deduce the reflection formula for our rarefied hyperbolic gamma function

$$\gamma_M(\omega_1 + \omega_2 - \mu, r - 1 - m) = \frac{(\bar{q} - m e^{\frac{2\pi i u}{c\tau + d}}; \bar{q})_{\infty}(m+1 e^{2\pi i u}; \bar{q})_{\infty}(q e^{-2\pi i u}; q)_{\infty}(e^{2\pi i u}; q)_{\infty}}{(q e^{2\pi i u}; q)_{\infty}},$$

$$= (-1)^m q^{-m+1/2} e^{-\frac{2\pi i m}{c\tau + d}} \theta(e^{-\frac{2\pi i u}{c\tau + d}}; \bar{q}) \theta(e^{2\pi i u}; e^{2\pi i r})$$

$$= i\varepsilon^2 e^{\frac{2\pi i (r-3-1)}{6k}} e^{\pi i (1-s)m e^{2\pi i k \pi} m(m-r+1)} e^{\frac{\pi i}{k} B_{2,2}(\mu \omega_1, \omega_2)},$$

(27)

where

$$B_{2,2}(\mu; \omega_1, \omega_2) = \frac{1}{\omega_1 \omega_2} \left( (\mu - \frac{\omega_1 + \omega_2}{2})^2 - \frac{\omega_1^2 + \omega_2^2}{12} \right)$$

is the second order multiple Bernoulli polynomial. Note that the right-hand side expression is invariant with respect to the shift $m \to m + k$.

For $k = p = r = 1$ one has $\varepsilon^2 = e^{-\pi i / 3}$, i.e. $i\varepsilon^2 e^{\frac{2\pi i (r+p-3)}{6k}} = 1$, and $(-1)^m e^{\pi i \frac{2\pi i k}{m(m-r+1)} = 1}$. This yields the standard result

$$\gamma(\omega_1 + \omega_2 - \mu) \gamma(\mu) = e^{\pi i B_{2,2}(\mu; \omega_1, \omega_2)}.$$ 

(28)

Define now the normalized rarefied hyperbolic gamma function

$$\Gamma_M(\mu, m) := Z(m) e^{-\frac{\pi i}{2k} B_{2,2}(\mu; \omega_1, \omega_2)} \gamma_M(\mu, m),$$

(29)

where

$$Z(m) = \frac{e^{-\frac{\pi i}{2k} (1 + \frac{r+p-3}{6k})}}{\varepsilon(-p, -s, k, -r)} e^{\pi i \frac{(1-s)k-p}{2k} (m-r+1)}.$$ 

(30)
The relation to the Dedekind sum $S(-d,c)$ (see formula (67.6) in [36] and Appendix A)

$$\varepsilon(a,b,c,d) = e^{\pi i S(-d,c)} e^{\frac{\pi i}{4m}(a+d)}$$

(31)

implies that we can write $Z(m)$ also in the form

$$Z(m) = e^{-\frac{\pi i}{4}(1-\frac{1}{2})} e^{-\pi i S(r,k)} e^{\pi i \frac{(1-s)k-p}{2k} m(m-r+1)}.$$  

(32)

The multiplier $Z(m)$ guarantees the following simple reflection equation

$$\Gamma_M(\omega_1 + \omega_2 - \mu, r - 1 - m) \Gamma_M(\mu, m) = 1.$$  

(33)

During its derivation one should use the identity $e^{\pi i (1-s)m(m-r+1)} = (-1)^{(1-s)m}$ following from the property $(r|s) = 1$. Equality (33) is equivalent to the relation

$$\Gamma_M(\pm \mu, \pm m) := \nu \Gamma_M(\mu, m) \Gamma_M(-\mu, -m) = \frac{(-1)^{sm}}{4 \sin \frac{\pi (\mu + m \omega_2)}{k \omega_2} \sin \frac{\pi (\mu + pm \omega_1)}{k \omega_1}}.$$  

(34)

Note that $\Gamma_M(\mu, m)$ is an appropriate generalization of the function

$$\gamma^{(2)}(\mu; \omega) := e^{-\frac{1}{4} \pi i B_{2,2}(\mu \omega_1, \omega_2)} \gamma(\mu; \omega),$$

(35)

used in the previous considerations [35] and [39]. This gamma function is not periodic

$$\Gamma_M(\mu, m + k) = \xi \Gamma_M(\mu, m), \quad \xi = e^{\pi i \frac{(1-s)k-p}{2} (k+2m-r+1)}.$$  

(36)

One can check that $\xi^2 = 1$, i.e. the quasiperiodicity factor is a pure sign, $\xi = \pm 1$.

Let us consider in more detail the rightmost exponential multiplier in (30). From the expression (32) it follows that it is only the choice of this root of unity that $\Gamma_M$ depends on the full set of discrete variables $k, p, r, s$, and the rest depends only on $p$ and $k$. Indeed, under the first transformation $r \rightarrow r + nk$ and $s \rightarrow s - np$ keeping the condition $pr + ks = 1$ intact, one has

$$e^{\pi i \frac{(1-s)k-p}{2k} m(m-r+1)} \rightarrow e^{\pi i \frac{(1-s)k-p}{2k} m(m-r+1)} e^{\frac{\pi i mn}{2} (2k+2m)}$$

Under the second transformation $p \rightarrow p + nk$ and $s \rightarrow s - nr$ preserving $pr + ks = 1$, one has

$$e^{\pi i \frac{(1-s)k-p}{2k} m(m-r+1)} \rightarrow e^{\pi i \frac{(1-s)k-p}{2k} m(m-r+1)} e^{\frac{\pi i n}{2} (r-1)m(m-r+1)}.$$  

One can check that both quasiperiodicity multipliers in these two relations are pure signs, i.e. the $\Gamma_M$-dependence on other integer parameters in $M$ beyond $c$ and $d$ is minimal, reduced to the sign choice.

To compute the residues in our integrals we will need the following limit

$$\lim_{\mu \to 0} \mu \Gamma_M(\mu, 0) = \frac{\gamma(\omega_1 \omega_2)}{2\pi k}. $$

(37)

Indeed,

$$\lim_{\mu \to 0} \mu \Gamma_M(\mu, 0) = \frac{e^{-\frac{\pi i}{4} (1+\frac{r+p-3}{8})} e^{-\frac{\pi i}{4} B_{2,2}(0, \omega_1, \omega_2)}}{\varepsilon(-p, -s, k, -r)} \lim_{\mu \to 0} \mu \gamma_M(\mu, 0).$$

Since

$$\lim_{\mu \to 0} \mu \gamma_M(\mu, 0) = \frac{k \omega_2 (\bar{q}; q)_\infty}{2\pi i (q; q)_\infty} = \frac{k \sqrt{-\omega_1 \omega_2}}{2\pi i} \varepsilon(-p, -s, k, -r) \left(\frac{q}{\bar{q}}\right)^{\frac{1}{24}},$$

in combination we obtain (37).
This normalized function satisfies the equations
\[
\frac{\Gamma_M(\mu + \omega_1, m + r)}{\Gamma_M(\mu, m)} = (-1)^m e^{\pi i \frac{(r-1)(s-1)}{2}} 2\sin \frac{\pi (\mu + m\omega_2)}{k\omega_2},
\] (38)
where the sign factor in front of the sine-function represents some quadratic character (note that \(e^{\pi i (r-1)(s-1)} = 1\)), and
\[
\frac{\Gamma_M(\mu + \omega_2, m - 1)}{\Gamma_M(\mu, m)} = (-1)^{(s-1)m} e^{\pi i \frac{(s-1)(r-1)}{2}} 2\sin \frac{\pi (\mu - pm\omega_1)}{k\omega_1}.\] (39)

Analogously,
\[
\frac{\Gamma_M(\mu - \omega_1, m - r)}{\Gamma_M(\mu, m)} = \frac{(-1)^{m-r} e^{\pi i \frac{(r-1)(s-1)}{2}}}{2\sin \frac{\pi (\mu + m\omega_2 - (\omega_1 + r\omega_2))}{k\omega_2}}
\] and
\[
\frac{\Gamma_M(\mu - \omega_2, m + 1)}{\Gamma_M(\mu, m)} = \frac{(-1)^{(s-1)(m+1)} e^{\pi i \frac{(s-1)(r-1)}{2}}}{2\sin \frac{\pi (\mu - pm\omega_1 - (\omega_2 + p\omega_1))}{k\omega_1}}.
\]

For \(k = p = r = 1\) one obtains the standard equations for \(\gamma^{(2)}(u; \omega_1, \omega_2)\).

For further applications we need to know also the asymptotics of \(\Gamma_M(\mu, m)\)-function. Assume that \(\text{Im}(\omega_1/\omega_2) > 0\). With this condition the asymptotics of \(\gamma(y; \omega_1, \omega_2)\) has the form
\[
\lim_{y \to \infty} \gamma(y; \omega_1, \omega_2) = 1, \quad \text{for arg } \omega_1 < \arg y < \arg \omega_2 + \pi,
\]
\[
\lim_{y \to \infty} e^{-\pi \frac{B_2(y; \omega_1, \omega_2)}{2}} \gamma(y; \omega_1, \omega_2) = \gamma(y; \omega_1, \omega_2), \quad \text{for arg } \omega_1 - \pi < \arg y < \arg \omega_2.
\]
Recalling relations (15), (29), and (33), we obtain
\[
\lim_{\mu \to \infty} \Gamma_M(\mu, m) \sim Z(m) e^{-\frac{\pi}{2} B_2(y; \omega_1, \omega_2)}, \quad \text{if arg } \omega_1 < \arg \mu < \arg \omega_2 + \pi,
\] (40)
\[
\lim_{\mu \to \infty} \Gamma_M(\mu, m) \sim Z^{-1}(m) e^{\frac{\pi}{2} B_2(y; \omega_1, \omega_2)}, \quad \text{if arg } \omega_1 - \pi < \arg \mu < \arg \omega_2.
\] (41)

3. The rarefied hyperbolic beta integral

Now we are going to construct a general univariate rarefied hyperbolic beta integral composed out of the function \(\Gamma(\mu, m)\), which will be a hyperbolic analogue of the elliptic beta integral [42] for the general lens space. For proving its exact evaluation formula we use an appropriate modification of the method suggested in [43].

Let us take variables \(\mu, a_j \in \mathbb{C}\) and \(m, n_j \in \mathbb{Z} + \nu\) with \(j = 1, \ldots, 6\) and \(\nu = 0, \frac{1}{2}\) and impose the balancing constraints
\[
\sum_{j=1}^{6} a_j = \omega_1 + \omega_2, \quad \sum_{j=1}^{6} n_j = r - 1.
\] (42)
Define the function
\[
\rho(\mu, m; a, n) = \frac{\prod_{j=1}^{6} \Gamma_M(a_j \pm \mu, n_j \pm m)}{\Gamma_M(\pm 2\mu, \pm 2m) \prod_{1 \leq \ell < j \leq 6} \Gamma_M(a_\ell + a_j, n_\ell + n_j)},
\] (43)
where we assume the notation
\[
\Gamma_M(a \pm \mu, n \pm m) := \Gamma_M(a + \mu, n + m) \Gamma_M(a - \mu, n - m).
\]
Let us resolve the balancing conditions by setting \( a_0 = \omega_1 + \omega_2 - A \) and \( n_0 = r - 1 - N \), where \( A = \sum_{j=1}^{5} a_j \) and \( N = \sum_{j=1}^{5} n_j \). Using the inversion relation for \( \Gamma_M \)-function we can write now

\[
\rho(\mu, m; a, n) = \frac{\prod_{j=1}^{5} \Gamma_M(a_j \pm \mu, n_j \pm m)}{\Gamma_M(A \pm \mu, N \pm m) \prod_{1 \leq \ell < j \leq 5} \Gamma_M(a_\ell + a_j, n_\ell + n_j)}.
\]

Recurrence relations for the rarefied hyperbolic gamma function guarantee validity of the following equation for the \( \rho \)-function

\[
\rho(\mu, m; a_0 + \omega_1, a_2, \ldots, n_1 + r, n_2, \ldots) - \rho(\mu, m; a_0, n) = g_1(\mu - \omega_1, m - r; a, n) - g_1(\mu, m; a, n),
\]

where

\[
\frac{g_1(\mu, m; a, n)}{\rho(\mu, m; a, n)} = \frac{\prod_{j=1}^{5} \sin \frac{\pi(a_j + \mu + (n_j + m)\omega_2)}{k\omega_2}}{\prod_{j=2}^{5} \sin \frac{\pi(a_1 + a_j + (n_1 + n_j)\omega_2)}{k\omega_2} \cdot \sin \frac{\pi(2\mu + 2m\omega_2)}{k\omega_2} \cdot \sin \frac{\pi(A + \mu + (m + N)\omega_2)}{k\omega_2}}.
\]

One has the ratios

\[
\frac{\rho(\mu, m; a_0 + \omega_1, \ldots, n_1 + r, \ldots)}{\rho(\mu, m; a_0, n)} = \frac{\sin \frac{\pi(a_0 \pm \mu + (n_0 + m)\omega_2)}{k\omega_2}}{\sin \frac{\pi(A \pm \mu + (N + m)\omega_2)}{k\omega_2}} \prod_{j=2}^{5} \sin \frac{\pi(a_j - \mu + (n_j - m)\omega_2)}{k\omega_2} \cdot \sin \frac{\pi(A \pm \mu + (N + m)\omega_2 - (\omega_1 + r\omega_2))}{k\omega_2} \cdot \sin \frac{2\pi(\mu + m\omega_2 - \omega_1 - r\omega_2)}{k\omega_2} \cdot \sin \frac{2\pi(\mu + m\omega_2)}{k\omega_2}.
\]

Therefore, dividing equation (44) by \( \rho(\mu, m; a, n) \) we obtain the following trigonometric identity

\[
\frac{\sin \frac{\pi(a_1 \pm \mu + (n_1 + m)\omega_2)}{k\omega_2}}{\sin \frac{\pi(A \pm \mu + (N + m)\omega_2)}{k\omega_2}} \prod_{j=2}^{5} \sin \frac{\pi(a_j - \mu + (n_j + m)\omega_2)}{k\omega_2} - 1
\]

\[
= \frac{\sin \frac{\pi(A \pm \mu + (N + n_1)\omega_2)}{k\omega_2}}{\sin \frac{2\pi(\mu + m\omega_2)}{k\omega_2}} \prod_{j=1}^{5} \sin \frac{\pi(a_j - \mu + (n_j - m)\omega_2)}{k\omega_2} \cdot \sin \frac{\pi(A \pm \mu + (N - m)\omega_2)}{k\omega_2} - \prod_{j=1}^{5} \sin \frac{\pi(a_j + \mu + (n_1 + m)\omega_2)}{k\omega_2}.
\]

or

\[
\frac{1 - t_1 z^{\pm 1}}{1 - T z^{\pm 1}} \prod_{j=2}^{5} \frac{1 - T t_j^{-1}}{1 - t_1 t_j} - 1
\]

\[
= \frac{t_1(1 - t_1 T)}{z(1 - z^2)} \prod_{j=2}^{5} \frac{1 - t_1 t_j}{1 - t_1 t_j} \left( \frac{\prod_{j=1}^{5} (1 - t_1 t_j^{-1})}{1 - T z^{-1}} - \prod_{j=1}^{5} (1 - t_j z) \right),
\]

where

\[
z = e^{2\pi i \frac{\mu + m\omega_2}{k\omega_2}}, \quad t_j = e^{2\pi i \frac{n_j + \omega_2}{k\omega_2}}, \quad T = e^{2\pi i \frac{A + N\omega_2}{k\omega_2}}.
\]
This relation is precisely the $e^{2\pi i \tau} \to 0$ case of the elliptic function identity established in \cite{43} (see also \cite{46}).

Analogously, one proves the equation

$$
\rho(\mu, m; a_1 + \omega_2, a_2, \ldots, n_1 - 1, n_2, \ldots) - \rho(\mu, m; a, n) = g_2(\mu - \omega_2, m + 1, \ldots; a, n) - g_2(\mu, m; a, n),
$$

where $g_2/\rho$-function is obtained from $g_1/\rho$ in \cite{45} by the replacement of $\omega_2 \to -p\omega_1$ in the numerators of the arguments of sin-functions and $\omega_2 \to \omega_1$ in the denominators. This results again in the equation \cite{17} with

$$
z = e^{2\pi i \frac{\mu - m\omega_1}{k\omega_1}}, \quad t_j = e^{2\pi i \frac{a_j - n_j\omega_1}{k\omega_1}}, \quad T = e^{2\pi i \frac{A - N\omega_1}{k\omega_1}}.
$$

It can be checked that the functions $\rho$ and $g_{1,2}$ are $k$-periodic

$$
f(\mu, m + k; a, n) = f(\mu, m; a, \ldots, n_j + k, \ldots) = f(\mu, m; a, n), \quad j = 1, 2, \ldots, 5.
$$

Suppose that $\text{Re}(\omega_j) > 0$, $\text{Im}(\omega_j) > 0$. Then the poles of $\Gamma_M(\mu, m)$ are located in the left lower quarter of the complex plane, and zeros lie in the right upper quarter. Denote now

$$
I(a, n) := \sum_{m \in \mathbb{Z}_k + \nu} \int_{-i\infty}^{i\infty} \rho(\mu, m; a, n) d\mu,
$$

(50)

where $\mathbb{Z}_k = \{0, 1, \ldots, k - 1\}$ and $\nu = 0, \frac{1}{2}$. We impose also the restrictions $\text{Re}(a_t) > 0$, which ensure that all pole arrays of the integrand lie either to the right or to the left of the imaginary axis.

To prove the convergence of this integral, recall the asymptotics of the $\Gamma_M$ function \cite{40, 41}. Substitute these relations into the kernel $\rho$ for the limit $\mu = +i\lambda$, $\lambda \to +\infty$. This yields the asymptotics

$$
\lim_{\lambda \to +\infty} \rho(\mu, m; a, n) \propto e^{-\frac{6\pi \lambda}{k} \left( \frac{1}{\omega_1} + \frac{1}{\omega_2} \right)}
$$

(51)

Since we took $\text{Re}(\omega_{1,2}) > 0$ we see that the kernel vanishes exponentially fast and the integral converges. Similar exponential fallout takes place in the limit $\lambda \to -\infty$, i.e. the kernel vanishes sufficiently fast even after the shift of $\mu$ by any constant, which will be necessary below.

Actually the contour of integration can be chosen in a substantially more general form – it is only necessary to demand convergence of the integral in the indicated domain of period values $\omega_{1,2}$. In particular, choosing appropriately this contour one can relax taken restrictions on the $\omega_{1,2}$-variables.

Using equalities \cite{14} and \cite{18}, for the function \cite{50} we obtain the equations

$$
I(a_1 + \omega_1, a_2, \ldots, n_1 + r, n_2 \ldots) - I(a, n) = \sum_{m \in \mathbb{Z}_k + \nu} \int_{-i\infty}^{i\infty} (g_1(\mu - \omega_1, m - r; a, n) - g_1(\mu, m; a, n)) d\mu,
$$

$$
I(a_1 + \omega_2, a_2, \ldots, n_1 - 1, n_2 \ldots) - I(a, n) = \sum_{m \in \mathbb{Z}_k + \nu} \int_{-i\infty}^{i\infty} (g_2(\mu - \omega_2, m + 1; a, n) - g_2(\mu, m; a, n)) d\mu.
$$
Because of the $k$-periodicity of the functions $g_{1,2}$ in the variable $m$, we can replace $m - r$ by $m$ in the first equation, and $m + 1$ by $m$ in the second.

Let us impose such constraints on the parameters $a_\ell$ that there will not be poles in the vertical stripes of $\mu$ bounded by the points $-\text{Re}(\omega_1)$ and 0, as well as by the points $-\text{Re}(\omega_2)$ and 0. The $\mu$-dependent part of the $g_{1}(\mu, m; a, n)$-function has the form

$$
\frac{\prod_{\ell=1}^{5} \Gamma_M(a_\ell - \mu, n_\ell - m) \Gamma_M(a_\ell + \mu + \omega_1, n_\ell + m + r)}{\Gamma_M(A - \mu, N - m) \Gamma_M(A + \mu + \omega_1, N + m + r)} \sin \frac{\pi(-2\mu + 2pm\omega_1)}{k\omega_1}.
$$

The poles of this function in $\mu$ are located at the points

$$
a_\ell - \mu, a_\ell + \mu + \omega_1 = -j_\omega_1 - (kn + n_\ell - m + jr)\omega_2, \quad \ell = 1, \ldots, 5,
$$

where $j, kn + n_\ell - m + jr \geq 0$, and

$$
A - \mu, A + \mu + \omega_1 = (p(N + m + r + j + 1) + kn)\omega_1 + (j + 1)\omega_2,
$$

where $j + 1, p(N + m + r + j + 1) + kn > 0$. One can see that the first set of points does not have representatives in the vertical strip bounded by the points $-\text{Re}(\omega_1)$ and 0. The second set of points do not enter this region, if $\text{Re}(\omega_2 - A) > 0$. Imposing the latter condition we find

$$
I(a_1 + \omega_1, a_2, \ldots, n_1 + r, n_2, \ldots) = I(a, n).
$$

(52)

After imposing the constraint $\text{Re}(\omega_1 - A) > 0$, the second equation for the $\rho$-function yields the equality

$$
I(a_1 + \omega_2, a_2, \ldots, n_1 - 1, n_2, \ldots) = I(a, n).
$$

(53)

Repeating these relations $k$ times and using the $k$-periodicity we find

$$
I(a_1 + k\omega_1, a_2, \ldots, n) = I(a_1 + k\omega_2, a_2, \ldots, n) = I(a, n).
$$

(54)

Note that the function $\Gamma_M(\mu, m)$ is well defined for $\omega_{1,2} > 0$ (i.e., when $|q| = 1$). But for incommensurate real periods $\omega_1$ and $\omega_2$ the derived relations show that $I(a, n)$ is a constant independent of $a_\ell$ and $n_\ell$. Let us compute this constant using the residue calculus.

Let us take the limit when both pairs of poles of a particular integral entering $I(a, n)$ start to pinch the contour of integration. To find it, we indicate the poles of the integrand coming from the $\Gamma_M$-function poles:

$$
a_\ell \pm \mu = -j_\omega_1 - j'_\omega_2, \quad j' := n_\ell \pm m + rj + kn \geq 0, \quad j \in \mathbb{Z}_{\geq 0}, \ n \in \mathbb{Z}, \ \ell = 1, \ldots, 6.
$$

Now we fix the values of $n_\ell$ as follows

$$
n_1 = n_2 = n_3 = \nu, \quad n_4 = n_5 = -\nu, \quad N = \sum_{\ell=1}^{5} n_\ell = \nu, \quad \nu = 0, \frac{1}{2},
$$

(55)

and take the limit $a_1 + a_4 \to 0$. As a result, two pairs of poles pinch the integration contour; the first pair being $\mu = -a_1, a_4$, which satisfies necessary conditions for $m = \nu$, and the second one $\mu = a_1, -a_4$ emerging for $m = 0$, if $\nu = 0$, and $m = k - \nu$, if $\nu = 1/2$.

Deform now the contour of integration to the left, pick up residues of the poles at $\mu = -a_1, -a_4$ and take the limit $a_1 + a_4 \to 0$. Applying the Cauchy theorem we find

$$
I(a, n) = 2\pi i(\lim_{\mu \to -a_4} (\mu + a_4)\rho(\mu, \nu; a, n) + \lim_{\mu \to -a_1} (\mu + a_1)\rho(\mu, k - \nu; a, n)) = 2i\sqrt{\omega_1\omega_2}k. \quad (56)
$$
E.g.,
\[
\lim_{\mu \to a_1} (\mu + a_1) \rho(\mu, k - \nu; a, n) = \lim_{\mu \to a_1} (\mu + a_1) \Gamma_M(a_1 + \mu, 0)
\]
\[
\times \prod_{\ell=2}^{6} \Gamma_M(a_\ell - a_1, 0) \prod_{\ell=4}^{5} \Gamma_M(a_\ell - a_1, -2\nu) \Gamma_M(a_\ell + a_1, 0)
\]
\[
\prod_{\ell=1}^{6} \Gamma_M(A - a_\ell, 0) \prod_{\ell=1}^{5} \Gamma_M(A - a_\ell, 2\nu)
\]
\[
\times \prod_{\ell=1}^{6} \Gamma_M(A + a_\ell, 2\nu)
\]
\[
\prod_{\ell=1}^{\nu} \Gamma(a_\ell + a_k, -2\nu) \Gamma(a_\ell + a_k, -2\nu)
\]
\[
\prod_{\ell=1}^{\nu} \Gamma(a_\ell + a_k, 0) \Gamma(a_\ell + a_k, -2\nu)
\]
where \( A = a_2 + a_3 + a_5 \). Thus we have proved the following theorem.

**Theorem.** Let \( a_j \in \mathbb{C}, \) \( \text{Re}(a_j) > 0, j = 1, \ldots, 6, \) and \( \text{Re}(\omega_{1,2}) > 0. \) Also take \( n_j \in \mathbb{Z} + \nu, j = 1, \ldots, 6, \nu = 0, \frac{1}{2} \) and impose the following balancing condition
\[
\sum_{j=1}^{6} a_j = \omega_1 + \omega_2, \quad \sum_{j=1}^{6} n_j = r - 1.
\]
Then the following identity holds:
\[
\sum_{m \in \mathbb{Z}_k + \nu} \int_{-\infty}^{\infty} \frac{\prod_{j=1}^{6} \Gamma_M(a_j \pm \mu, n_j \pm m)}{\Gamma_M(\pm 2\mu, \pm 2m)} \frac{d\mu}{2ik\sqrt{\omega_1 \omega_2}} = \prod_{1 \leq \ell < j \leq 6} \Gamma_M(a_\ell + a_j, n_\ell + n_j),
\]
where \( \mathbb{Z}_k = \{0, 1, \ldots, k - 1\}. \)

The constraints \( \text{Re}(\omega_{1,2} - A) > 0 \) (or \( \text{Re}(a_6 - \omega_{1,2}) > 0) \) used in the proof of the theorem, as well as restrictions on \( \omega_{1,2}, \) are lifted by the analytical continuation. Equivalently, relation (58) can be rewritten in terms of the \( \gamma_M \)-function
\[
\sum_{m \in \mathbb{Z}_k + \nu} \int_{-\infty}^{\infty} e^{-\frac{2\pi i (m^2 - \nu^2)}{k}} \prod_{j=1}^{6} \frac{\gamma_M(a_j \pm \mu, n_j \pm m)}{\gamma_M(\pm 2\mu, \pm 2m)} \frac{d\mu}{2ik\sqrt{\omega_1 \omega_2}}
\]
\[
eq e^{-\frac{2\pi i (1 - \frac{1}{k}) (N + \nu^2)}{k}} \prod_{1 \leq \ell < j \leq 6} \gamma_M(a_\ell + a_j, n_\ell + n_j),
\]
where \( N = \sum_{1 \leq \ell < j \leq 6} n_\ell n_j. \) Under the shift \( n_j \to n_j + k \) for some fixed \( j \) one has \( N \to N + k \sum_{\ell=1, \neq j}^{6} n_\ell \). Therefore, for \( \nu = 0 \) both sides of the equality (59) are invariant with respect to such shifts. Correspondingly, in this case we can set \( n_j \in \mathbb{Z}_k \) and take a slightly more general discrete balancing condition \( \sum_{j=1}^{6} n_j = r - 1 \mod k. \) However, if \( \nu = 1/2, \) then \( N \) is shifted by the half-integer and an analogous statement will not be true. In this case we can replace the multiplier \( Z(m) \) in (29) by the exponential of a cubic polynomial of \( m \) which, in difference from (56), will guarantee the periodicity \( \Gamma_M(\mu, m + k) = \Gamma_M(\mu, m). \) It will produce in the right-hand side of (59) a different \( a_j \)-independent multiplier, which will be invariant under the shifts \( n_\ell \to n_\ell + k \) for all \( \nu \) and which will coincide with the one given above for the reduced balancing condition (57).

Since we use the general modular transformation for the Dedekind \( \eta \)-function and Jacobi theta-function, the theory of rarefied hyperbolic beta integrals can be considered as a complement to the theory of Jacobi forms (17), because the kernels of integrals are composed of “one-halves” of the meromorphic Jacobi forms (in the sense of the number of divisor forms).
The key identity (58) can be rewritten in the form of the star-triangle relation, which leads to a new solvable model of 2$d$ lattice spin system, and as a consequence to new solutions of the Yang-Baxter equation similar to [4,10–12,28,30,49]. These and some other applications, as well as the problem of elliptic generalization of the equality (58) will be considered in a separate work.

4. Limiting relations

For simplicity we consider further only the $\nu = 0$ case. Let us reparametrize $a_j, n_j$ in the identity (58) in the following asymmetric way

$$a_j = f_j + i\xi, \quad a_{j+3} = g_j - i\xi, \quad l_j := n_{j+3}, \quad j = 1, 2, 3.$$  \hspace{1cm} (60)

Then the balancing condition takes the form

$$\sum_{j=1}^{3} (f_j + g_j) = \omega_1 + \omega_2, \quad \sum_{j=1}^{3} (n_j + l_j) = r - 1.$$  \hspace{1cm} (61)

Now we shift in (58) the integration variable $\mu \to \mu - i\xi$ and take the limit $\xi \to -\infty$ using the asymptotics of $\Gamma_M(\mu, m)$. Since the integrand is an even function (in fact the parity transformation reschedules the separate terms keeping the sum intact), one can write

$$2 \int_0^{i\infty} \sum_{m=0}^{r-1} \left[ \prod_{j=1}^{3} \frac{\Gamma_M(\mu + f_j + i\xi, n_j + m)\Gamma_M(\mu + g_j - i\xi, l_j + m)}{\Gamma_M(2\mu, 2m)\Gamma_M(-2\mu, -2m)} \right] \frac{d\mu}{2ik\sqrt{\omega_1\omega_2}}$$

$$= 2 \int_{i\xi}^{i\infty} \sum_{m=0}^{3} \prod_{j=1}^{3} \Gamma_M(\mu + f_j, n_j + m)\Gamma_M(-\mu + g_j, l_j - m) e^{-\frac{8\pi i}{3}\sigma_1} \frac{d\mu}{2ik\sqrt{\omega_1\omega_2}}$$ \hspace{1cm} (62)

where in the limit $\xi \to -\infty$

$$\sigma_1 = \sum_{j=1}^{3} \left[ B_{2,2}(\mu + g_j - 2i\xi) - B_{2,2}(\mu + f_j + 2i\xi) \right] - B_{2,2}(2\mu - 2i\xi) + B_{2,2}(-2\mu + 2i\xi)$$

$$- ((1-s)k-p) \left( \sum_{j=1}^{3} [(l_j + m)^2 - (r-1)(l_j + m) - (n_j - m)^2 + (r-1)(n_j - m)] + 4m(r-1) \right).$$

On the right-hand side of equality (58) we have

$$\prod_{\ell,j=1}^{3} \Gamma_M(f_{\ell} + g_j, n_{\ell} + l_j) e^{-\frac{4\pi i}{3}\sigma_2}, \quad \sigma_2 = \sum_{1 \leq i < j \leq 3} \left[ B_{2,2}(g_i + g_j - 2i\xi) - B_{2,2}(f_i + f_j + 2i\xi) \right]$$

$$- ((1-s)k-p) \sum_{1 \leq \ell < j \leq 3} \left[ (l_{\ell} + l_j)^2 - (r-1)(l_{\ell} + l_j) - (n_{\ell} + n_j)^2 + (r-1)(n_{\ell} + n_j) \right].$$
Similar to the considerations of [35] for $k = 1$ case, it can be checked that all $B_{2,2}$-terms appearing on the left- and right-hand sides cancel each other. Taking care about the rest yields:

$$
\int_{-i\infty}^{i\infty} \sum_{m=0}^{r-1} \prod_{j=1}^{3} \Gamma_{M}(\mu + f_j, n_j + m) \Gamma_{M}(\mu + g_j, l_j - m) \frac{d\mu}{ik\sqrt{\omega_{1}\omega_{2}}} = \prod_{\ell,j=1}^{3} \Gamma_{M}(f_{\ell} + g_j, n_{\ell} + l_j). \tag{63}
$$

Let us compare this result with the analogous formula in paper [39] for the parafermionic hyperbolic gamma function corresponding to the choice $p = r = 1$ and $s = 0$. That formula in [39] contains an additional sign factor in the integral. To see how it appears we should compare the definition of $\Gamma_{M}$ (29) for $p = 1$ with the definition in [39],

$$
\Lambda(y, m; \omega_{1}, \omega_{2}) = \prod_{k=0}^{m-1} \gamma^{(2)} \left( \frac{y}{r} + \omega_{2} \left( 1 - \frac{m}{r} \right) + (\omega_{1} + \omega_{2}) \frac{k}{r}; \omega_{1}, \omega_{2} \right) \times \prod_{k=0}^{r-m-1} \gamma^{(2)} \left( \frac{y}{r} + \frac{m}{r}\omega_{1} + (\omega_{1} + \omega_{2}) \frac{k}{r}; \omega_{1}, \omega_{2} \right),
$$

Remembering that [37] $S(1, k) = -\frac{1}{4} + \frac{1}{6k} + \frac{k}{12}$ and using (32) one can write

$$
\Gamma\left( \begin{array}{cc} -1 & 0 \\ k & -1 \end{array} \right) (\mu, m) = e^{\frac{\pi i}{2} \left( \frac{1}{\pi} - k \right)} e^{\frac{\pi i k}{2r} m^2} e^{-\frac{\pi i}{2r} B_{2,2}(\mu; \omega_{1}, \omega_{2})} \gamma\left( \begin{array}{cc} -1 & 0 \\ k & -1 \end{array} \right) (\mu, m).
$$

On the other hand, using the definition (35) we obtain

$$
\Lambda(\mu, m; \omega_{1}, \omega_{2}) = e^{\frac{\pi i}{2} \left( \frac{1}{\pi} - k \right)} e^{\frac{\pi i k}{2r} m^2} e^{-\frac{\pi i}{2r} B_{2,2}(\mu; \omega_{1}, \omega_{2})} \gamma\left( \begin{array}{cc} -1 & 0 \\ k & -1 \end{array} \right) (\mu, m).
$$

Finally, we come to the relation

$$
\Gamma\left( \begin{array}{cc} -1 & 0 \\ k & -1 \end{array} \right) (\mu, m) = e^{\frac{\pi i}{2} (m^2 - m)} \Lambda(\mu, m; \omega_{1}, \omega_{2}) \tag{64}
$$

and it is this sign difference between $\Gamma_{M}$ and $\Lambda$ that gives rise to the sign factor in the integral considered in [39].

Now we can obtain three more integral relations taking various limits of the parameters $f_{\ell}$ and $g_{\ell}$. Resolving the balancing condition (61) for $g_{3}$ and taking the limit $f_{3} \to i\infty$ in the (63) we obtain:

$$
\int_{-i\infty}^{i\infty} \sum_{m=0}^{k-1} e^{\frac{\pi i m}{2r}(p-k(1-s))(n_{1}+n_{2}+l_{1}+l_{2})} e^{\left( \frac{\pi i}{2r} \omega_{1}\omega_{2}(y f_{1} f_{2} g_{1} g_{2} + f_{1} f_{2} - g_{1} g_{2}) \right)} \Gamma_{M}(y + f_{1}, n_{1} + m) \times \Gamma_{M}(y + f_{2}, n_{2} + m) \Gamma_{M}(-y + g_{1}, l_{1} - m) \Gamma_{M}(-y + g_{2}, l_{2} - m) \frac{dy}{i\sqrt{\omega_{1}\omega_{2}}} \tag{65}
$$

$$
= ke^{\frac{\pi i}{2r}(p-k(1-s))(l_{1} l_{2} - n_{1} n_{2})} \Gamma_{M}(\omega_{1} + \omega_{2} - f_{1} - f_{2} - g_{1} - g_{2}, r - 1 - n_{1} - l_{1} - n_{2} - l_{2}) \times \Gamma_{M}(f_{1} + g_{1}, n_{1} + l_{1}) \Gamma_{M}(f_{1} + g_{2}, n_{2} + l_{2}) \Gamma_{M}(f_{2} + g_{1}, n_{1} + l_{1}) \Gamma_{M}(f_{2} + g_{2}, n_{2} + l_{2}).
$$
Further on, taking in (65a) the limit \( f_2 \to -i\infty \) and \( g_2 \to i\infty \) with \( f_2 + g_2 = \alpha \) kept fixed, we obtain

\[
\int_{-i\infty}^{i\infty} \sum_{m=0}^{k-1} e^{\frac{\pi i m}{\omega}} \left[ (p-k(1-s))(2N+l+1-r) \right] e^{\frac{\pi i m}{\omega}} \left[ g\left(\frac{Q}{2} - \alpha\right) - \frac{y^2}{4} + y(2\alpha + g - Q) \right] \times \Gamma_M(y, m) \Gamma_M(-y + g, l - m) \frac{dy}{i\sqrt{\omega_1 \omega_2}}
\]

\[
= k e^{\frac{\pi i m}{\omega}} \left[ (p-k(1-s))(l(1-r + (l+2N)) \right] \Gamma_M(Q - \alpha - g, r - 1 - N - l) \Gamma_M(\alpha, N) \Gamma_M(g, l).
\]

Here \( N := n_2 + l_2, Q := \omega_1 + \omega_2 \). We also denoted \( g_1 = g, l_1 = l \) and set \( f_1 = 0, n_1 = 0 \) since these variables can be restored by the shifts \( y \to y + f_1, g \to g + f_1 \) and \( m \to m + n_1, l \to l + n_1 \).

Finally, taking in equality (66) the limit \( g \to -i\infty \), we obtain

\[
e^{-\frac{\pi i}{4}} e^{\pi i S(r, k)} e^{-\frac{\pi i}{\omega_1} + \frac{\pi i}{2\omega_2}} \int_{-i\infty}^{i\infty} \sum_{m=0}^{k-1} e^{\frac{\pi i m}{\omega}} \left[ (p-k(1-s))(2N+l+1-r) \right] e^{\frac{\pi i m}{\omega}} (p-k(1-s)) \times \Gamma_M(y, m) \frac{dy}{i\sqrt{\omega_1 \omega_2}} = k e^{\frac{\pi i m}{\omega}} \left[ (p-k(1-s))(N(r-1)+N^2) - \frac{\pi i \omega}{2} \left( \frac{Q}{2} - \alpha \right) ^2 \right] \Gamma_M(\alpha, N).
\]

To compare these integrals with those computed by Dimofte in [13], note that the function \( Z_b^{(k,p)}(y, m) \) used in [13] is related to \( \Gamma_M(y, m) \) after setting \( \omega_1 = b^{-1}, \omega_2 = b \), by the relation

\[
\Gamma_M(y, m) = Z(m)^{-1} e^{\frac{\pi i}{2} B z(y, b, b^{-1})} Z_b^{(k,p)}(iy, m),
\]

which can be obtained from formula (83) in Appendix B. Note that here \( b \) is the \( q \)-deformation parameter and it should not be mixed with the integer \( b \)-variable entering the description of modular group transformation (2).

Inserting (68) in (66) we obtain

\[
e^{-\frac{\pi i}{4}} e^{\pi i S(r, k)} e^{-\frac{\pi i}{\omega_1} + \frac{\pi i}{2\omega_2}} \int_{-\infty}^{\infty} \sum_{m=0}^{k-1} (-1)^{(1-s)m} e^{\frac{\pi i m}{\omega}} (m+2Nm+1-r)m \times e^{\frac{\pi i}{\omega}} \left[ -y^2 + y(-2\alpha + Q) + \frac{Q}{2} \left( \frac{Q}{2} - \alpha \right) \right] Z_b^{(k,p)}(y, m) \times \frac{dy}{i\sqrt{\omega_1 \omega_2}} = k e^{\frac{\pi i m}{\omega}} (p-k(1-s)) \left[ \frac{1}{2} N + l - 1 - N - l \right] \times Z_b^{(k,p)}(iQ - \alpha - g, r - 1 - N - l) Z_b^{(k,p)}(\alpha, N) Z_b^{(k,p)}(g, l).
\]

This integral relation has been suggested first and verified numerically for \( p = r = 1 \) and \( s = 0 \) in [25, 26]. Then it has been proved in [13] for the case of even \( s \) and odd \( p \) and \( r \), and our result confirms the corresponding computations. Note that in [13] this relation is written in terms of the function \( \kappa(k, r) = 3(k-1)^2 - \frac{12}{k} \sum_{j=0}^{k-1} \frac{1}{2} r_j + \frac{r-1}{2} \mod k \). To see that both formulae indeed coincide one should take into account that for odd \( r \) the function \( \kappa(k, r) \) satisfies the relation

\[
e^{-\frac{\pi i}{2} \kappa(k, r)} = e^{\pi i F(\frac{r-1}{2})} = e^{\frac{\pi i}{2} (1-k)} e^{2\pi i S(r, k)} e^{\frac{\pi i m}{\omega} \frac{r-1}{2} (-1)^{\frac{r-1}{2}} e^{-\frac{\pi i m}{\omega} \frac{r-1}{2}},}
\]

where the function \( F(m) \) is described in Appendix B.
Now inserting (68) in (67) we obtain
\[
e^{\frac{\pi i}{4}(1-\delta k)} e^{\frac{\pi i S(r,k)}{k}} \int_{-\infty}^{\infty} \left( \sum_{m=0}^{k-1} (-1)^{(1-s)m} e^{\frac{\pi i m}{k}} (m^2 + 2Nm + m(1-r)) \right) (70)
\times e^{\frac{\pi i}{k}[-y^2 + 2y(iQ^2 - \alpha)]} Z_b(k,p)(y,m) dy = kZ_b^{(k,p)}(\alpha, N).
\]

For \( p = r = 1 \) and \( s = 0 \) this relation reduces to the equality considered in [13]. The identity (70) expresses a 3d mirror symmetry between the theory of free chiral field and the \( U(1) \) gauge theory with a chiral field with the 1/2 Chern-Simons coupling on the lens space \( L(k, k-p) \) [14]. The term \( e^{\frac{\pi i m^2}{k}} \) gives a contribution of the flat connection with the holonomy \( m \) to the Chern-Simons action in agreement with [22, 24].

As mentioned above, general modular quantum dilogarithm can be written as a product of a number of hyperbolic gamma functions with arguments lying on a specific lattice of points [13]. It is interesting to note that precisely the same special lattice of points emerged first in [8] in the products of so-called \( \Upsilon \)-functions that were used for writing structure constants of an 2d conformal field theory related through the AGT correspondence to \( \mathcal{N} = 2 \) four-dimensional gauge theory on \( \mathbb{C}^2/\Gamma_{c,d} \), where \( \Gamma_{c,d} \subset U(2) \) is a finite group acting on local coordinates according to formula (4). The lattice appearing in a particular case \( d = -1 \) was introduced earlier in the paper [6] for description of structure constants in the quantum Liouville field theory interacting with parafermions (para Liouville field theory). It is expected that the general rarefied hyperbolic gamma function will play a similar important role in the mentioned 2d conformal field theory for description of the fusion matrix and boundary correlation functions.

The authors are indebted to A. B. Kalmynin and R. M. Kashaev for a discussion of obtained results. This work is partially supported by Laboratory of Mirror Symmetry NRU HSE, RF government grant, ag. no. 14.641.31.0001.

**APPENDIX A. THE DEDEKIND SUM**

The Dedekind function is defined as the following sum [37]
\[
S(r, k) = \sum_{\delta=1}^{k-1} \delta \left( \frac{r\delta}{k} - \left\lfloor \frac{r\delta}{k} \right\rfloor - \frac{1}{2} \right),
\]
where \( \lfloor x \rfloor \) is the integer part of \( x \in \mathbb{R} \). Its key properties are
\[
S(-r, k) = -S(r, k), \quad S(r, k) = S(p, k),
S(r, k) = \frac{1}{4k} \sum_{m=1}^{k-1} \cot \frac{\pi m}{k} \cot \frac{\pi rm}{k}, \quad S(r + k, k) = S(r, k).
\]

**APPENDIX B. DIMOFTE’S NOTATIONS**

The original modular quantum dilogarithm [18,19] can be written in the form [4]
\[
\phi(z) = \exp \left( \frac{1}{4} \int_{\mathbb{R}+i\epsilon} dw \frac{e^{-2\pi i z w}}{w \sinh(\pi bw) \sinh(\pi b^{-1} w)} \right).
\]
In [13] the function $Z_b^{(1,1)}(z,0)$ is defined as

$$Z_b^{(1,1)}(z,0) = \phi \left( -z + i \frac{Q}{2} \right), \quad Q = \omega_1 + \omega_2. \quad (74)$$

Then the function $Z_b^{(k,p)}(z,m)$ can be rewritten as

$$Z_b^{(k,p)}(z,m) = \prod_{\gamma, \delta \in \Delta(k,p,m)} Z_b^{(1,1)} \left( \frac{1}{k} (z + ib^{-1} \delta + i b \gamma); \omega_1, \omega_2 \right), \quad (75)$$

where $\Delta(k,p,m) = \{ \gamma, \delta \in \mathbb{Z}, 0 \leq \gamma, \delta < k, \gamma - r \delta \equiv m \mod k \}$. Now we find relation between $\gamma_M(z,m)$ for $\omega_1 = b^{-1}, \omega_2 = b$, and $Z_b^{(k,p)}(z,m)$. First, reminding the connection with $\gamma$-function [14],

$$\phi(z) = \gamma \left( \frac{Q}{2} - i z \right)^{-1},$$

we see that

$$Z_b^{(1,1)}(z,0) = \gamma \left( Q + i z \right)^{-1}. \quad (76)$$

Recalling the reflection formula (28), we can write

$$\gamma(z) = e^{\pi i B_2,2(z;b^{-1})} Z_b^{(1,1)}(iz,0). \quad (76)$$

Taking the product of both sides of this relation over the lattice points $\Delta(k,p,m)$ and reminding [15], we obtain

$$\gamma_M(z,m) = e^{\pi i A(z,m)} Z_b^{(k,p)}(z,m), \quad (77)$$

where

$$A(z,m) = \sum_{\delta = 0}^{k-1} B_{2,2} \left( \frac{1}{k} (z + \omega_1 \delta + \omega_2 [m + r \delta]); \omega_1, \omega_2 \right), \quad (78)$$

$[x] \equiv x \mod k \in \mathbb{Z}_k$. In fact,

$$[m + r \delta] = m + r \delta - N(\delta)k, \quad \text{if} \quad N(\delta)k \leq m + r \delta < (N(\delta) + 1)k, \quad (79)$$

where $N(\delta) = \left[ \frac{m + r \delta}{k} \right]$. Note that since $k$ and $p$ are relatively prime, when $\delta$ runs the values $0, \ldots, k - 1$, the function $[m + r \delta]$ also runs all these values, but in a different order.

Using equalities

$$\sum_{\delta = 0}^{k-1} \delta = \sum_{\delta = 0}^{k-1} [m + r \delta] = \frac{k(k - 1)}{2}, \quad \sum_{\delta = 0}^{k-1} \delta^2 = \sum_{\delta = 0}^{k-1} [m + r \delta]^2 = \frac{k(k - 1)(2k - 1)}{6}, \quad (80)$$

one can show that

$$\sum_{\delta = 0}^{k-1} B_{2,2} \left( \frac{1}{k} (z + \omega_1 \delta + \omega_2 [m + r \delta]); \omega_1, \omega_2 \right) = \frac{1}{k} B_{2,2}(z; \omega_1, \omega_2) + F(m), \quad (81)$$

where

$$F(m) = -\frac{1}{2k} - \frac{k}{2} + 1 + \frac{2}{k^2} \sum_{\delta = 0}^{k-1} \delta [m + r \delta]. \quad (82)$$

Equalities (80) follow from the remark given after the statement (79). Thus we have

$$\gamma_M(z,m) = e^{\frac{\pi i}{k} B_{2,2}(z; \omega_1, \omega_2)} e^{\pi i F(m)} Z_b^{(k,p)}(iz,m). \quad (83)$$
Let us now show that the function $F(m)$ is connected with the Dedekind sum as follows

$$e^{\pi i F(m)} = e^{\frac{\pi i}{2} (1 - \frac{1}{k})} e^{2\pi i S(r,k)} e^{\frac{\pi i}{k} m(m+1)} (-1)^m e^{-\frac{\pi i}{k} m}.$$  \hfill (84)

First, write $F(m)$ in the form

$$F(m) = \frac{1}{2} - \frac{1}{2k} + 2 \sum_{\delta=0}^{k-1} \frac{\delta}{k} \left( \left\lfloor \frac{m + r\delta}{k} \right\rfloor - \left\lfloor \frac{m + \frac{r\delta}{k}}{k} \right\rfloor - \frac{1}{2} \right).$$

We saw already that equation (84) is satisfied for $m = 0$. To establish it for generic $m$, we should find $m$-dependence of the sum in the second line of (85). Namely, we should show that

$$e^{2\pi i \sum_{\delta=0}^{k-1} \frac{\delta}{k} \left( \frac{m + \frac{r\delta}{k}}{k} - \left\lfloor \frac{m + \frac{r\delta}{k}}{k} \right\rfloor - \frac{1}{2} \right)} = e^{2\pi i S(r,k)} e^{\frac{2\pi i}{k} m(m+1)}.$$  \hfill (86)

To see validity of this equality, we analyze for which $\delta$ the integer $\left\lfloor \frac{m + r\delta}{k} \right\rfloor$ is different from $\left\lfloor \frac{\delta}{k} \right\rfloor$. Assume that $r\delta = kN - n$, for some positive integers $N$ and $n = 0, 1, \ldots, k - 1$. It is clear that if $m < n$, then $\left\lfloor \frac{m + r\delta}{k} \right\rfloor = \left\lfloor \frac{\delta}{k} \right\rfloor$, and if $n \leq m$, then $\left\lfloor \frac{m + r\delta}{k} \right\rfloor - \left\lfloor \frac{\delta}{k} \right\rfloor = 1$. Therefore we should find $\delta$ satisfying the relation $r\delta = kN - n$ with $n \leq m$ and sum over them. From the condition $pr = 1 - ks$, we obtain $-rnp = ksn - n$, i.e. the required $\delta$'s satisfy the condition $\delta \equiv -np \mod k$ with $n \leq m$. Since we want to compute only the phase, this condition is sufficient for our purpose. Collecting all terms, we see that the additional phase factor created by $m \neq 0$ is $e^{\pi ip(m+1)/k}$.

**References**

[1] S. Alexandrov and B. Pioline, *Theta series, wall-crossing and quantum dilogarithm identities*, Lett. Math. Phys. 106 (2016), 1037–1066.

[2] J. E. Andersen and R. Kashaev, *Complex quantum Chern-Simons*, arXiv:1409.1208 [math.QA].

[3] G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, Encyclopedia of Math. Appl. 71, Cambridge Univ. Press, Cambridge, 1999.

[4] V. V. Bazhanov, V. V. Mangazeev, and S. M. Sergeev, *Exact solution of the Faddeev-Volkov model*, Phys. Lett. A 372 (2008), 1547–1550.

[5] V. V. Bazhanov, V. V. Mangazeev, and S. M. Sergeev, *Faddeev-Volkov solution of the Yang-Baxter equation and discrete conformal symmetry*, Nucl. Phys. B 784 (2007), 234–258.

[6] M. A. Bershtein, V. A. Fateev, and A. V. Litvinov, *Parafermionic polynomials, Selberg integrals and three-point correlation function in parafermionic Liouville field theory*, Nucl. Phys. B 847 (2011), 413–459.

[7] N. N. Bogolyubov, D. V. Shirkov, *Introduction to the Theory of Quantized Fields*, Moscow: Nauka, 1984, 4th edition.

[8] G. Bonelli, K. Maruyoshi, A. Tanzini, and F. Yagi, $\mathcal{N} = 2$ gauge theories on toric singularities, blow-up formulae and W-algebrae, J. High Energy Phys. 1301 (2013), 014.

[9] F. J. van de Bult, E. M. Rains, and J. V. Stokman, *Properties of generalized univariate hypergeometric functions*, Commun. Math. Phys. 275 (2007), 37–95.

[10] A. G. Bytsko and J. Teschner, *Quantization of models with non-compact quantum group symmetry: Modular XXZ magnet and lattice sinh-Gordon model*, J. Phys. A 39 (2006), 12927.

[11] D. Chicherin and S. Derkachov, *The R-operator for a modular double*, J. Phys. A 47 (2014), 115203.
[12] D. Chicherin and V. P. Spiridonov, *The hyperbolic modular double and the Yang-Baxter equation*, Advanced Studies in Pure Mathematics 76 (2018), 95–123.

[13] T. Dimofte, *Complex Chern-Simons theory at level k via the 3d-3d correspondence*, Commun. Math. Phys. 339 (2015), 619–662.

[14] T. Dimofte, D. Gaiotto, and S. Gukov, *Gauge theories labelled by three-manifolds*, Commun. Math. Phys. 325 (2014), 367–419.

[15] F. A. Dolan and H. Osborn, *Applications of the superconformal index for protected operators and q-hypergeometric identities to $\mathcal{N} = 1$ dual theories*, Nucl. Phys. B 818 (2009), 137–178.

[16] F. A. H. Dolan, V. P. Spiridonov, and G. S. Vartanov, *From 4d superconformal indices to 3d partition functions*, Phys. Lett. B 704 (2011), 234–241.

[17] M. Eichler and D. Zagier, *The Theory of Jacobi Forms*, Progress in Math. 55, Birkhäuser, Boston, 1985.

[18] L. Faddeev, *Current-like variables in massive and massless integrable models*, In: Quantum groups and their applications in physics (Varenna, 1994), Proc. Internat. School Phys. Enrico Fermi 127, IOS Press, Amsterdam, 1994, pp. 117–135.

[19] L. D. Faddeev, *Discrete Heisenberg-Weyl group and modular group*, Lett. Math. Phys. 34 (1995), 249–254.

[20] V. Fateev, A. B. Zamolodchikov, and A. B. Zamolodchikov, *Boundary Liouville field theory. I: Boundary state and boundary two-point function*, arXiv:hep-th/0001012.

[21] I. Gahramanov and A. P. Kels, *The star-triangle relation, lens partition function, and hypergeometric sum/integrals*, J. High Energy Phys. 1702 (2017), 040.

[22] L. Griguolo, D. Seminara, R. J. Szabo, and A. Tanzini, *Black holes, instanton counting on toric singularities and q-deformed two-dimensional Yang-Mills theory*, Nucl. Phys. B 772 (2007), 1–24.

[23] N. Hama, K. Hosomichi, and S. Lee, *SUSY gauge theories on squashed three-spheres*, J. High Energy Phys. 1105 (2011), 014.

[24] S. K. Hansen and T. Takata, *Reshetikhin-Turaev invariants of Seifert 3-manifolds for classical simple Lie algebras and their asymptotic expansions*, J. Knot Theory Ramifications 13:5 (2004), 617–668.

[25] Y. Imamura and D. Yokoyama, *$S^3/\mathbb{Z}_n$ partition function and dualities*, J. High Energy Phys. 1211 (2012), 122.

[26] Y. Imamura, H. Matsuno and D. Yokoyama, *Factorization of the $S^3/\mathbb{Z}_n$ partition function*, Phys. Rev. D 89:8 (2014), 085003.

[27] A. Kapustin, B. Willett, and I. Yaakov, *Exact results for Wilson loops in superconformal Chern-Simons theories with matter*, J. High Energy Phys. 1003 (2010), 089.

[28] R. Kashaev, *The quantum dilogarithm and Dehn twist in quantum Teichmüller theory*, in S. Pakuliak, G. von Gehlen (eds.), Integrable structures of exactly solvable two-dimensional Yang-Mills theory, pp. 211-221, NATO Sci. Ser. II, v. 35, Kluwer Acad. Publ., 2001.

[29] R. M. Kashaev, *Beta pentagon relations*, Theor. Math. Phys. 181:1 (2014), 1194–1205 (Teor. Mat. Fiz. 181 (1) (2014), 73–85).

[30] R. Kashaev, *The Yang-Baxter relation and gauge invariance*, J. Phys. A: Math. Theor. 49 (2016), 164001.

[31] R. M. Kashaev, F. Luo, and G. Vartanov, *A TQFT of Turaev-Viro type on shaped triangulations*, Ann. Henri Poincaré 17 (2016), 1109–1143.

[32] S. Kharchev, D. Lebedev, and M. Semenov-Tian-Shansky, *Unitary representations of $U_q(sl(2,\mathbb{R}))$, the modular double and the multiparticle q-deformed Toda chains*, Commun. Math. Phys. 225 (2002), 573–609.

[33] F. Nieri and S. Pasquetti, *Factorisation and holomorphic blocks in 4d*, J. High Energy Phys. 1511 (2015), 155.

[34] NIST Digital Library of Mathematical Functions. [http://dlmf.nist.gov/](http://dlmf.nist.gov/)

[35] V. Pestun et al., *Localization techniques in quantum field theories*, J. Phys. A: Math. Theor. 50 (2017), 440301.
[36] H. Rademacher, *Topics in Analytic Number Theory*, Springer, Berlin, 1973.
[37] H. Rademacher and E. Grosswald, *Dedekind Sums*, The Mathematical Association of America, 1972.
[38] S. N. M. Ruijsenaars, *A generalized hypergeometric function satisfying four analytic difference equations of Askey–Wilson type*, Commun. Math. Phys. 206 (1999), 639–690.
[39] G. Sarkissian and V. P. Spiridonov, *From rarefied elliptic beta integral to parafermionic star-triangle relation*, J. High Energy Phys. 1810 (2018), 097.
[40] G. A. Sarkissian and V. P. Spiridonov, *The modular group and the hyperbolic beta integral*, Uspekhi Mat. Nauk 75:3 (2020), 187–188 (Russian Math. Surveys 56:3 (2020), 557–577).
[41] A. A. Slavnov, L. D. Faddeev, *Introduction to Quantum Theory of Gauge Fields*, Moscow: Nauka, 1978.
[42] V. P. Spiridonov, *On the elliptic beta function*, Uspekhi Mat. Nauk 56:1 (2001), 181–182 (Russ. Math. Surveys 56:1 (2001), 185–186).
[43] V. P. Spiridonov, *Short proofs of the elliptic beta integrals*, Ramanujan J. 13 (2007), 265–283.
[44] V. P. Spiridonov, *Essays on the theory of elliptic hypergeometric functions*, Uspekhi Mat. Nauk 63:3 (2008), 3–72 (Russian Math. Surveys 63:3 (2008), 405–472).
[45] V. P. Spiridonov, *Elliptic beta integrals and solvable models of statistical mechanics*, Contemp. Math. 563 (2012), 181–211.
[46] V. P. Spiridonov, *Rarefied elliptic hypergeometric functions*, Adv. in Math. 331 (2018), 830–873.
[47] V. P. Spiridonov and G. S. Vartanov, *Elliptic hypergeometry of supersymmetric dualities*, Commun. Math. Phys. 304 (2011), 797–874.
[48] J. V. Stokman, *Hyperbolic beta integrals*, Adv. in Math. 190 (2004), 119–160.
[49] A. Yu. Volkov and L. D. Faddeev, *Yang-Baxterization of the quantum dilogarithm*, Zap. Nauchn. Semin. POMI 224 (1995), 146–154 (J. Math. Sciences 88:2 (1998), 202–207).
[50] Wikipedia contributors, *Lens space*, Wikipedia, https://en.wikipedia.org/wiki/Lens_space.

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