Multi-scale polarisation phenomena

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Multi-scale methods that separate different time or spatial scales are among the most powerful techniques in physics, especially in applications that study nonlinear systems with noise. When the time scales (noise and perturbation) are of the same order, the scales separation becomes impossible. Thus, the multi-scale approach has to be modified to characterise a variety of noise-induced phenomena. Here, based on stochastic modelling and analytical study, we demonstrate in terms of the fluctuation-induced phenomena and Hurst R/S analysis metrics that the matching scales of random birefringence and pump–signal states of polarisation interaction in a fibre Raman amplifier results in a new random birefringence-mediated phenomenon, which is similar to stochastic anti-resonance. The observed phenomenon, apart from the fundamental interest, provides a base for advancing multi-scale methods with application to different coupled nonlinear systems ranging from lasers (multimode, mode-locked, random, etc.) to nanostructures (light-mediated conformation of molecules and chemical reactions, Brownian motors, etc.).

Keywords: fibre Raman amplifier, multi-scale methods, polarisation, stochastic calculations

INTRODUCTION

Many processes in nature have different temporal and spatial scales that lead to multi-scale complexity. To describe this complexity on different levels, multi-scale methods have been developed and explored for more than 100 years¹–³. For example, in nonlinear fibre optics, three groups of scales have 13 orders of magnitude separation between the smallest length scale of 1.55 μm and the Southern Cross Cable Network, whose length is approximately 32 500 km. The shortest micron scale is related to the wavelength of light and the core diameter. Thus, Maxwell’s equations have to be explored for characterisation of fibre in the dispersion relations context. The next metre scale corresponds to the fibre beat and correlation lengths, e.g., lengths at which a state of polarisation (SOP) reproduces itself and preserves deterministic evolution. The longest kilometre length scale is the fibre attenuation and gain scale, chromatic dispersion and the Kerr non-linearity. At this length scale, the Manakov equation is obtained by averaging the randomly varying birefringence⁴–⁶.

To describe the pump and signal SOPs evolution in a fibre Raman amplifier (FRA), different multi-scale averaging techniques have been used⁷–²⁰. Some of them account for the scale of birefringence fluctuations (SBF)⁷–¹¹, whereas others account for the SBF and the scale where the pump and signal SOPs interact¹²–²⁰. All of the averaging techniques, stochastic modelling, and experimental study demonstrated polarisation pulling (polarisation trapping) of the signal SOP to the pump SOP²⁷–¹¹,¹⁶–¹⁹,²¹–²⁷.

Along with polarisation pulling, our recent theoretical study¹¹,¹³,¹⁴,¹⁶,¹⁷,¹⁹ revealed an additional phenomenon similar to the fluctuation-induced escape (FIE)²⁸–³¹, which occurred with an increased polarisation mode dispersion (PMD) parameter $D_p$²⁴,³². The simplest manifestation of this effect, in the form of the resonance-like increase of the gain fluctuations as a function of the PMD parameter, has been first found theoretically by Lin and Agrawal²² and experimentally by Popov and co-workers²⁴. Additionally, it has been studied theoretically in detail by Sergeyev and co-workers¹³,¹⁴,¹⁶,¹⁷,¹⁹.

Modern fibre Raman-based unrepeatered transmission systems use bidirectional pumping schemes³³. The co-propagating pump and signal provide a major contribution to the pump-to-signal relative intensity noise (RIN) transfer, which also depends on the PMD value³⁴. To study the statistical properties of forward pumped FRA is the key to unlocking the RIN characterisation, which is based on the vector models of FRA, and, thus, to developing efficient vector RIN suppression techniques.

To justify application of previously explored multi-scale techniques for studying statistical properties of co-propagating pump and signal SOPs, we, for the first time, use computer simulation of stochastic differential equations with application of the Kloeden-Platen-Schurz algorithm, which provides the fastest convergence (see Supplementary Information). We reveal both the polarisation pulling and resonance-like escape from polarisation pulling in terms of fluctuation-induced phenomena metrics, such as Kramers and intrawell relaxation lengths, gain, root mean square (RMS) gain fluctuations, as well as the spectrum, correlation function, the Hurst parameter and probability distribution function for projecting the signal SOP to the pump SOP. The FRA pump–signal SOPs interaction is defined by the rate of relative rotation of the signal SOP with respect to the pump SOP. According to the results of our analytical study, the stochastic modelling demonstrates that for some PMD parameters, which are typical for the...
currently used single mode fibres, this rate is on the same scale as
the birefringence correlation length. Thus, the rate has to be included
in the fast scale group with further averaging to obtain a correct
description in the region of resonance-like escape from the polarisation
pulling.

MATERIALS AND METHODS

Signal and pump states of polarisation evolution in terms of biased
Brownian motion

To provide an insight into the FIE phenomena for the fibre Raman
amplification, we, first, outline similarities between the SOP evolu-
tions in Raman and biased Brownian motion. The FIE from a meta-
stable state of an excitable system with probability controlled by an
external force is a fundamental phenomenon that is inherent in
many physical processes, such as, diffusion in crystals, protein folding,
activated chemical reactions and many others26–31. Figure 1a demon-
strates the escape of the Brownian particle from the bottom of
the potential well due to fluctuations and barrier height modulation.

The potential well $\Delta U$ is a source of polarisation pulling (i.e., ‘polari-
isation trapping’10). For the adiabatic forcing case, specifically, when
the period of barrier modulation $T$ is much longer than the intrawell
relaxation time $\tau_b$, FIE takes the form of stochastic resonance (SR),
as such, a synchronisation between the activated escape events from the
potential minimum with a periodic forcing, which results in the max-
imal signal-to-noise ratio at $\tau_b$ (the Kramers time, which characterises the average residence time with respect to the FIE28–31).

The evolution along the fibre length for signal and pump SOP is
similar to the evolution of the Brownian particle in the potential well
(Figure 1a). As follows from Figure 1b, the pump $\mathbf{p}=(p_1, p_2, p_3)$
and signal $s=(s_1, s_2, s_3)$ SOPs evolution comprises: (i) signal-to-pump
SOP pulling (i.e., polarisation trapping caused by potential well build-
up)7–11,16–19,21–27 caused by the stimulated Raman scattering aniso-
tropy; (ii) barrier modulation caused by the relative rotation of signal
SOP with respect to the pump SOP at the rate $\dot{\theta}=\tau_0/\tau_b p$, where $\tau_b$ is the beat length, $\tau_0$ is the intrawell relaxation time,
and the orientation angle $\theta$ is driven by a white-noise process
(fixed-modulus model10).

$$\frac{\partial \theta}{\partial t} = g(z), \quad \langle g(z) \rangle = 0, \quad \langle g(z)g(z') \rangle = \sigma^2 \delta(z-z').$$

(1)

where $\langle \ldots \rangle$ represents averaging of the birefringence fluctuations
along the fibre, $\delta(z)$ is the Dirac delta function, and $\sigma^2=1/L_c$ ($L_c$ is
the birefringence correlation length). As a result of evolution, the
signal wave is amplified and changes its direction as follows:

$$S(z) = s_0(z) \exp(\int_0^z gP(z)\,dz/2 \pm z)$$

Here, $s_0$ is part of the signal amplitude that is related to the pump
and signal SOPs interaction, $G_{ave} = \exp\left(\int_0^z gP(z)\,dz/2 \pm z\right)$ is the averaged Raman gain, $g$ is the Raman gain coefficient, $P_0$ is the pump
power at distance $z$, $P_0(z) = P_0 \exp(-\gamma z)$; $P_0$ is the input pump
power, $\gamma_s$ and $\gamma_p$ are the signal and pump losses, respectively; $L$ is the
fibre length.

The part of the Raman gain $\langle G \rangle$, which is related to the pump–
signal SOPs coupling is

$$\langle G \rangle = 10 \log \left(\frac{\langle s_t(L) \rangle}{s_t(0)} / G_{ave} \right) = 10 \log \left(\frac{\langle s_t(L) \rangle}{s_t(0)} \right).$$

(3)

We excluded the averaged Raman gain $G_{ave}$ in Equation (3). This
allows us to concentrate on the vector nature of the processes under
consideration. If the input pump and signal SOPs are parallel, the
Raman gain adopts the maximum value, and, if the SOPs are ortho-
gonal, then the Raman gain adopts the minimum value7–27. The gain
difference is referred to as the polarisation-dependent gain (PDG) and
is defined as follows7–27:

$$PDG = 10 \log(\langle s_{t,\text{max}}(L) \rangle / \langle s_{t,\text{min}}(L) \rangle).$$

(4)

To quantify de-correlation of the pump and signal SOPs and polar-
sation pulling in terms of the fluctuation-induced phenomena, we
introduce the RMS gain fluctuations as follows12,14:

Figure 1  (a) Fluctuation-induced phenomena, where the escape probability is controlled by an external periodic force. $\Delta U$ – potential well, $T$ – period of barrier
modulation, $\tau_r$ – escape rate, $\tau_0$ – intrawell relaxation time, $\tau_4$ – residence time (the Kramers time); (b) Evolution of the pump $\mathbf{p}$ and signal $s$ states of polarisation
(SOPs) and the local birefringence vector (BV) $\mathbf{W}_i = (2b_i \cos\theta, 2b_i \sin\theta, 0)^T$ on the Poincaré sphere. Vectors $s$ and $p$ rotate around the local axis $\mathbf{W}$ at rates $b_s$ and $b_p$, vector $\mathbf{W}$ rotates randomly in the equatorial plane at the rate $\sigma = \tau_b^{-1/2}$ ($\tau_b$ is the correlation length). Anisotropy of fibre Raman amplification, which results in
the signal-to-pump SOP polarisation pulling, i.e., builds up a potential well, while relative rotation of the signal SOP, with the rate $b_s - b_p$, plays a barrier modulation role, and
the random fluctuation of BV defines the noise.
Thus, we introduced the \( <G> \), PDG, and \( \sigma_G^2 \) metrics to further justify the different multi-scale techniques\(^7\)–\(^{20}\) using stochastic modelling.

**Vector models of the fibre Raman amplifier and multi-scale techniques**

Here, we present two analytical models (where different averaging techniques have been used) and stochastic equations to validate these models. In the first model, the generic multi-scale technique has been applied, where only the randomly varying birefringence scale has been considered as the fastest scale\(^4\)–\(^6\). Next, we average the fast birefringence fluctuations (details are found in the Supplementary Information) and neglect the pump depletion, cross-phase and self-phase modulations (XPM and SPM) and time dependence, i.e. group velocity dispersion (GVD). This approximation is valid for pump powers \( P_m < 1 \text{ W} \), signal powers \( s < 10 \text{ mW} \),\(^{11,21}\) \( D_p > 0.01 \text{ ps km}^{-1} \).\(^{12}\)

It has been estimated\(^{12}\) that the GVD can be neglected when the fibre length \( L \) is much smaller than the dispersion length \( L_D = \frac{\tau_p^2}{\beta_2} \).

For pulse duration \( T_p = 2.5 \text{ ps} \), \( |\beta_2| = 5 \text{ ps}^2 \text{ km}^{-1} \), we have \( L_D > 100 \text{ km} \). Thus, GVD can be neglected for \( L < 20 \text{ km} \).

Taking into consideration Equation (2), we obtain the following equations, which describe the pump–signal SOPs coupling:

\[
\frac{d(s)}{dz} = \frac{g_s}{2} P_0(z) (s) (\hat{p}) + \left( \beta_s - \beta_p \right) \exp(-2\sigma_G^2 z) \left[ \begin{array}{c} 0 \\ -\langle s \rangle \\ \langle s \rangle \end{array} \right], \quad \frac{d(p)}{dz} = 0,
\]

\[
S = sG_{ave}, s = \langle s \rangle, |s| = 1, \quad s_0 = |s| = \sqrt{s_1^2 + s_2^2 + s_3^2}, \quad P = P_0(z), |p| = 1.
\]

Thus, the multi-scale method includes averaging of the fast birefringence fluctuations and results in the averaged gain value and in the absence of pump and signal SOPs correlation. The method neglects gain fluctuations. Thus, one condition for the validity of the method is the low gain fluctuations. Equation (6) has been developed using unitary transformation to exclude the pump SOP fluctuations due to the random birefringence. The applied transformation preserves the length of the pump and signal SOP vectors as well as the scalar and vector products. As a result, evolution of the signal SOP includes a term (the second one), which accounts for the relative orientation of the signal SOP with respect to the pump SOP. However, Kozlov and co-workers\(^7\) have applied unitary transformations to the pump and signal SOPs to exclude both the pump and signal SOP fluctuations due to random birefringence and, as a result, have obtained equations that differ from Equation (6) and those derived by Sergeyev and co-workers\(^13\). The stimulated Raman scattering and XPM introduce a coupling between the pump and signal SOPs. Thus, the adopted transformations\(^7\) do not preserve either the vector and scalar lengths or the vector products.

To justify the multi-scale method that results in Equation (6), we use stochastic equations derived from the coupled Manakov–PMD equations to calculate the part of the gain, which is related to the pump–signal SOPs coupling, gain fluctuations and correlation properties of signal and pump SOPs (details are found in Supplementary Information)\(^12\):

\[
\frac{ds}{dz} = \frac{g_s}{2} P_0(z) s + \left( \begin{array}{c} s_2 \\ -s_1 \\ 0 \end{array} \right) g_s + \beta_s \left( \begin{array}{c} 0 \\ -s_1 \\ s_2 \end{array} \right), \quad \frac{dp}{dz} = \left( \begin{array}{c} p_2 \\ -p_1 \\ 0 \end{array} \right) g_p + \beta_p \left( \begin{array}{c} 0 \\ -p_1 \\ p_2 \end{array} \right).
\]
Here, $\Delta = 2L_c V_1 \exp(-\gamma_z L_c) / L_c$, $\Delta_c = 2L_c V_3 / L_c$. Using the linear stability analysis of Equation (10) near $\langle \hat{x}_0 \rangle, \langle \hat{y}_0 \rangle$, we find eigenvalues:

$$A_{1,2} = -\frac{1}{4L_c} (3\Delta < \hat{x}_0 > + 1) + \frac{1}{4L_c} \sqrt{1 + \Delta^2 < \hat{x}_0 >^2 - 2\Delta < \hat{x}_0 > - 4\Delta_1^2 + \Delta_1^2 \hat{x}_0^2}. \quad (12)$$

We introduce the intrawell relaxation length $L_R = 1 / |\text{Re}(A_{1,2})|$. If $\text{Im}(A_{1,2}) < 0$, the system escapes by oscillating around the states $\langle \hat{x}_0 \rangle, \langle \hat{y}_0 \rangle$. Thus, we define the Kramers length as $L_K = 2\pi L_c / |\text{Im}(A_{1,2})|$. To study the long-range memory effects for the Raman-induced polarisation pulling and escape, we provide the Hurst rescale range $G$.

$$G \equiv \frac{1}{4L_c} \sqrt{1 + \Delta^2 < \hat{x}_0 >^2 - 2\Delta < \hat{x}_0 > - 4\Delta_1^2 + \Delta_1^2 \hat{x}_0^2}. \quad (13)$$

Next, the range $R_N$ and the standard deviation $S_N$ are calculated as

$$R_N = \max \Gamma_{N,k} - \min \Gamma_{N,k} S_N = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (X_i - \mu_N)^2}. \quad (14)$$

The rescale range is found as $R_N / S_N$. Then, the time series of $N$ points are divided into two $N/2$-points time series, and the rescale range $R_{N/2} / S_{N/2}$ is calculated for both time series and is, then, averaged. This process is repeated for partial series which comprise $n = N/4, N/8, \ldots$ points. The Hurst parameter $H$ is estimated by fitting the power law of averaged $R_n / S_n$ for $n \to \infty$, e.g.

$$\langle R_n / S_n \rangle_{n \to \infty} = C n^H. \quad (15)$$

It has been determined by many authors that the Hurst parameter varies as $0 < H < 1$. The parameter $0.5 < H < 1$ is related to the persistent statistics. Thus, a positive increment in the past makes it more probable to have a positive trend in the future. This conclusion

Figure 2 The gain $<G>$ (part of the Raman gain, which is related to the pump–signal SOPs coupling), which is averaged for 100 stochastic trajectories (solid black curves), and the corresponding RMS gain fluctuations $\sigma_G$ (solid red curve) in comparison with the $<G>$ (black dashed curves) and RMS gain fluctuations (red dashed curves) as a function of the PMD parameter $D_p$. (a) and (b), (c) and (d) plots correspond to the Stokes parameters of the pump and the input signal fields as follows: $\hat{p} = (1,0,0)$, $\hat{s} = (1,0,0)$ (a); $\hat{p} = (1,0,0)$, $\hat{s} = (0,0,1)$ (c); $\hat{p} = (0,0,1)$, $\hat{s} = (0,0,1)$ (b); $\hat{p} = (0,0,1)$, $\hat{s} = (0,0,0)$ (d). Points A, B and C correspond to the signal beat lengths $L_{95}$ of 200, 20 and 10 m, respectively. Inset: ten stochastic trajectories of the signal power $s_0$. The orange dashed curve shows the average values of $s_0$. The dashed domains demonstrate ranges of the $s_0$ standard deviation. The parameters correspond to point B and $\hat{p} = (1,0,0)$, $\hat{s} = (1,0,0)$. The gain $<G>$ is normalised to $G_{\text{max}}$ in agreement with Equation (3).
works in reverse for the anti-persistent statistics. Specifically, a positive increment in the past will result in a more probable negative trend in the future. Hurst first suggested an application of the R/S analysis to study water storage in the Nile River\textsuperscript{36,57}. Since then, the Hurts parameter has been proven as a feasible metric for analysing long-range dependence in network traffic\textsuperscript{39,40}, turbulence\textsuperscript{41}, heartbeat\textsuperscript{35}, coalition of neurons dynamics\textsuperscript{42}, detection of low observable targets within sea clutter\textsuperscript{43}, identification and prediction of epileptic seizures, earthquakes, and crashes in financial market\textsuperscript{3,44,45}. 

RESULTS AND DISCUSSION

Equation (7) have been solved using the Wolfram Mathematica 9.0 computer algebra system using the built-in Klöden–Platen–Schurz method, which provided the fastest convergence compared with the Runge–Kutta and Milstein algorithms (see the Supplementary Information). The averaging procedure was performed for an ensemble of $N = 100$ stochastic trajectories. This provides the precision of $1/N^{1/2} \sim 10\%$, which is sufficient to justify the analytical results obtained by Sergeevy and co-workers\textsuperscript{13,14}. We used the following parameters: the Raman gain coefficient $g$ was $0.8$ W$^{-1}$ km$^{-1}$; the input signal $s_0$ and the pump $P_{in}$ powers were $10$ mW and $1$ W, respectively; the fibre length $L$ was $5$ km; the correlation length of birefringence vector $L_s$ was $100$ m. The Stokes parameters for the pump and the input signal fields, which correspond to the maximum and minimum PDG values, were: $\hat{p} = (1,0,0)$, $\hat{s} = (1,0,0)$ for the maximum gain and maximum PDG; $\hat{p} = (1,0,0)$, $\hat{s} = (-1,0,0)$ for the minimum gain and maximum PDG; $\hat{p} = (0,0,1)$, $\hat{s} = (0,0,1)$ for the maximum gain and minimum PDG; and $\hat{p} = (0,0,1)$, $\hat{s} = (0,0,1)$ for the minimum gain and minimum PDG. Based on our previous publications and on Equations (7), the Poincaré sphere reference frame is chosen to have a local birefringence as $W_{i\text{tot}} = (2\tilde{h},0,0)$ for all stochastic realisations. Thus, all trajectories for the Stokes parameter $s_0$ have the same starting point with respect to the chosen reference frame with further divergence being caused by the random birefringence fluctuations, as shown in the Figure 2 inset.

The dependences of gain ($G$), which, according to Equation (3), is part of the Raman gain, are related to the pump–signal SOPs coupling (black solid and dashed curves), and RMS gain fluctuations $\sigma G$ (red solid and dashed curves) on the PMD parameter are shown in Figure 2. The solid and dashed curves correspond to the numerical solution of Equations (7)–(9). Based on Figure 2, the stochastic calculations of the gain and RMS gain fluctuations, which are derived using Equation (7), perfectly fit in the range of the PMD parameters using results from our previously developed model, which is based on Equations (8) and (9)\textsuperscript{13–20}. This is a significant result because it provides analysis tools for long fibre communication systems without using time-consuming calculations, which are based on the solution of the underlying stochastic equations.

Based on Equation (6), we conclude that in view of the exponential decay of the term related to escape from polarisation pulling, an application of the multi-scale technique with averaging, excluding scale of the pump–signal SOPs interactions, results in polarisation pulling in all ranges of the PMD parameters. As a result, the gain values coincide with gain values obtained from Equations (7)–(9) in the limit of $D_p \to 0$ and are close to the values for an ideal Raman polarizer\textsuperscript{11}, by taking into account normalisation of $\langle G \rangle$ to $G_{\text{ave}}$ in Equation (3). However, the averaging technique, which accounts for the signal and pump SOPs interaction\textsuperscript{13–20} scale, better agrees with the stochastic modelling results (Figure 2). In addition, these analytical techniques, which resulted in Equations (8) and (9), predict a resonant enhancement of the RMS gain fluctuations within the range of PMD parameters of $10^{-2}$ to $10^{-1}$ ps km$^{-1/2}$, which are typical for the modern single mode fibres (the red dashed curves in Figure 2 in comparison with the red solid curves, which are obtained numerically from Equation (7)).

Though the analytic theory predicts a constant asymptotic of $0.34$ dB for the PDG parameter (red curve in Figure 3), the numerical PDG disappears approximately monotonically with PMD (solid curves in Figure 3), which is in agreement with the Ref. 10 results. Nevertheless, the averaging of the $N = 100$ trajectories provides a precision $\Delta \sim 1/N^{1/2}$ of $0.4$ dB. Thus, the asymptotic cannot be validated for the parameters used here.

To characterise the transition from polarisation pulling to escape from pulling, we determined Kramers and the intrawell relaxation lengths from Equations (11) and (12) and the Hurst parameter for the pump-to-signal SOP projection $(x) \equiv \langle \hat{p} \cdot \hat{s} \rangle / \langle |s| \rangle$ (insets 1 and 2 in Figure 3). As follows from Figure 3 (inset 1), the transition from polarisation pulling to escape has the threshold at $D_p \sim 0.02$ ps km$^{-1/2}$, which, according to Equation (12), corresponds to the escape rate $|\Delta x_{1,2}| \gtrsim 0$ for $x < e$ evolving along the fibre length. In contrast to our previous results on SR in the fibre Raman amplification\textsuperscript{13}, here the escape from polarisation pulling happens in many uncorrelated steps rather than in one step, as for SR. Therefore, we have increased gain fluctuations instead of increased the signal-to-noise ratio, e.g., the stochastic anti-resonance\textsuperscript{17}.

To gain insight into statistical properties of the pump and signal SOPs interaction, we studied the stochastic evolution of the signal-to-pump SOP projection $(x) \equiv \langle \hat{p} \cdot \hat{s} \rangle / \langle |s| \rangle$ along the fibre instead of the $s_0$ evolution (Figures 4–6). By comparing Figures 3–6 with Figure 2 (inset) and Equation (8) (first row), we determine that the
evolution of $\langle x \rangle$ reflects the statistics of SOP interactions by including small scales, whereas for $x_0$ the small-scale statistics disappears due to the propagation distance averaging. The asymptotic behaviour $(D_p \to 0)$ of $\langle x \rangle$ demonstrates the Raman-induced polarisation pulling effect$^{5, 11, 16-19, 21-27}$, where the Raman amplification plays an effective fibre polariser role because $\langle x \rangle \equiv \langle \hat{p} \cdot \hat{s} \rangle / \langle |s| \rangle \to 1$, i.e., the signal SOP is attracted to the pump SOP (see the top row of Figure 4 and the black solid curve in Figure 2a). For $D_p \to 0$, the fibre becomes effectively ‘isotropic’$^{10}$. Thus, the Raman amplification anisotropy results in the strong amplification of the co-polarised to pump signal SOP and the attenuation of the cross-polarised signal SOP.

For the initially cross-polarised pump and signal SOPs, this attraction occurs (the top row of Figure 5) with a lower rate and is initiated by the birefringence fluctuations due to the escape from the metastable state with $\langle x \rangle \equiv \langle \hat{p} \cdot \hat{s} \rangle / \langle |s| \rangle \to -1$. As a result, the average gain $\langle G \rangle$ remains minimal for the considered fibre length (Figure 2b and 2d). An important polarisation pulling property for both considered initial signal SOPs is minimisation of RMS fluctuations of the average gain (points A on the red curves in Figure 2a and 2b) and a regular structure of the spectral energy density (Figure 6a and 6b). This is due to ‘fine graining’ of the birefringence fluctuations, which play a role of white noise perturbations around a stable polarisation state (Figure 2a and 2c and the top row in Figure 4), or perturbations pulling out a metastable polarisation state (Figure 2b and 2d and the top row in Figure 5). The corresponding correlation functions demonstrate a damped oscillation behaviour (insets in Figure 6a and 6b).

The opposite extreme case is $L_c \gg L_b$ (the large PMD parameters, i.e., the case of a ‘standard Raman amplifier’$^{10, 11}$), when the deterministic evolution, which is induced by the fibre birefringence,

![Figure 4](image-url)

**Figure 4** Left column: Evolution of the averaged projection $\langle x \rangle \equiv \langle \hat{p} \cdot \hat{s} \rangle / \langle |s| \rangle$ (red dashed curve) and its standard deviation (filled area) for the input gain and the signal $\rho = (1,0,0), \hat{s} = (1,0,0)$ (a “maximum gain”, (a) in Figure 2) with the A-, B-, C-PMD parameters of Figure 2. Right column: the corresponding (A, B and C) histograms.
prevails over the stochastics. In this case, a single mode fibre is similar to the polarization-maintaining (PM) fibre, which has comparatively rare stochastic switches of the birefringence axis fluctuations. Thus, the RMS gain fluctuations decrease (e.g., the points C on the red curves in Figure 2a and 2b), and \( \langle G \rangle \) approaches a constant small but non-zero value (black curves in Figure 2a and 2c). Because the evolution is driven by the fast pump–signal decorrelation, the average gain is minimal (but non-zero) for the initially co-polarised pump and signal. This means that there is a weak correlation between the pump and signal SOPs (bottom row in Figure 4).

The decrease of SOP correlation manifests itself in the Hurst parameter reduction \( H < 1 \) (inset 2 in Figure 3). For the initially cross-polarised pump and signal SOPs, the residual correlation (bottom row in Figure 5) maximises the gain \( \langle G \rangle \) (black curves in Figure 2b and 2d and the bottom row in Figure 5). An oscillatory evolution, which underlies localisation, along the fibre reveals itself in the modulated power spectrum densities and the corresponding correlation functions (Figure 6a and 6f).

The intermediate case of \( L_b \approx L_c = \frac{4}{2} (D_p < 10^{-2} \text{ km}^{-1/2}) \) demonstrates a resonant enhancement of polarisation stochastic evolution, where the RMS gain fluctuations have a set of spikes (in the vicinity of points B in Figure 2). Such spikes correspond to enhanced 'wandering' of the trajectories for the signal-to-pump SOP projections \( \langle x \rangle \) (Figures 4 and 5). In view of the increased rather than decreased gain fluctuations for point B, this phenomenon is contrary to the SR and, thus, is referred to as stochastic anti-resonance. The stochasticity intensification is demonstrated by the threshold-like dropping of the Hurst parameter to \( H < 0.7 \) and the corresponding collapse of the Kramers length (insets in Figure 3). This switching between the statistical scenarios is the distinguishing characteristic of the 'stochastic anti-resonance' under consideration. The average polarisation state remains 'localised' (the middle rows of

Figure 5 As in Figure 4, but for \( \rho = (1,0,0), \delta = (-1,0,0) \) (a “minimum gain”, (b) in Figure 2).
technique leads to the results that are close to those obtained using aging over the correlation length scale using the generic multi-scale much longer than the birefringence correlation length. Thus, averaging relative rotation of the pump SOP with respect to the signal SOP) is the length of the pump-to-signal SOP interaction (beat length of the the signal SOP is attracted to the pump SOP leads to polarisation pulling when the low PMD values the fibre become almost isotropic. Thus, the birefringence properties (PMD parameter). We demonstrated that for multi-scale polarisation phenomena for the FRA as a function of its CONCLUSIONS In summary, using stochastic modelling, we provided insights into multi-scale polarisation phenomena for the FRA as a function of its birefringence properties (PMD parameter). We demonstrated that for the low PMD values the fibre become almost isotropic. Thus, the Raman gain (Figure 2). Only by including the scale of the signal-to-pump SOPs interactions, we demonstrate that it is possible to obtain the correct results (Figures 2 and 3). A further decrease of the interaction length corresponds to an almost deterministic birefringence case, where the pump and signal SOPs rotate without interaction (Figure 2). Detailed statistical analysis of the pump-to-signal SOP projection evolution along the fibre unveiled different types of fractional Brownian motions as a function of PMD values in terms of the Hurst parameter $H$. For the low PMD values, the polarisation pulling leads to $H\rightarrow1$, which corresponds to the persistent statistics. For the PMD values that correspond to the gain fluctuations maximum, the Hurst parameter decreases to $H=0.7$ and, therefore, approaches the Brownian motion with $H=0.5$. Further increase in the PMD parameter corresponds to the almost deterministic SOPs evolution and, thus, the persistent statistics with $H\rightarrow0.8$.

The obtained results are further generalised by accounting for the pump depletion, XPM and SPM, and time dependence (GVD and

Figures 4–5) but its sensitivity to the input SOP disappears with the PMD parameter growth (Figure 3). This means that the PDG decreases with $D_p$ (solid lines in Figure 3). Therefore, the Raman gain in the vicinity of the standard deviation peak behaves as an ‘effective depolariser’, which diminishes the PDG.

**CONCLUSIONS**

In summary, using stochastic modelling, we provided insights into multi-scale polarisation phenomena for the FRA as a function of its birefringence properties (PMD parameter). We demonstrated that for the low PMD values the fibre become almost isotropic. Thus, the Raman amplification anisotropy leads to polarisation pulling when the signal SOP is attracted to the pump SOP due to the decreased interaction length (increase in PMD), deterministic rotation of the signal SOP with respect to the pump SOP is intensified and results in escape phenomena, which is similar to stochastic anti-resonance, in view of the increased RMS gain fluctuations (Figure 2). When the rotation rate approaches the correlation length, the scale averaging of the correlation length is no longer valid and cannot provide correct results for the gain (Figure 2). Only by including the scale of the signal-to-pump SOPs interactions, we demonstrate that it is possible to obtain the correct results (Figures 2 and 3). A further decrease of the interaction length corresponds to an almost deterministic birefringence case, where the pump and signal SOPs rotate without interaction (Figure 2). Detailed statistical analysis of the pump-to-signal SOP projection evolution along the fibre unveiled different types of fractional Brownian motions as a function of PMD values in terms of the Hurst parameter $H$. For the low PMD values, the polarisation pulling leads to $H\rightarrow1$, which corresponds to the persistent statistics. For the PMD values that correspond to the gain fluctuations maximum, the Hurst parameter decreases to $H=0.7$ and, therefore, approaches the Brownian motion with $H=0.5$. Further increase in the PMD parameter corresponds to the almost deterministic SOPs evolution and, thus, the persistent statistics with $H\rightarrow0.8$.

The obtained results are further generalised by accounting for the pump depletion, XPM and SPM, and time dependence (GVD and
walk-off between the pump and signal waves). This manipulation provides an opportunity to gain insight into the RIN as a function of the FMD parameters and to adapt the developed methods to characterise the parametric amplifiers and Brillouin amplifiers. Additionally, these results can be applied, in the context of new multi-scale methods development, to study the complex nonlinear coupled systems, such as lasers (multimode, mode-locked, random), nanostructures (light-mediated conformation of molecules and chemical reactions, Brownian motors), and other systems.

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