A completeness result for implicit justification stit logic

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Abstract. We present a completeness result for the implicit fragment of justification stit logic introduced in [5]. Although this fragment allows for no strongly complete axiomatization, we show that a restricted form of strong completeness (subsuming weak completeness) is available, as well as deduce a version of restricted compactness property.

1 Introduction

Basic justification stit (or jstit, for short) logic was introduced in [5] as an environment for analysis of doxastic actions related to proving activity within a somewhat idealized community of agents, combining expressive means of stit logic by N. Belnap et al. [3] with those of justification logic by S. Artemov et al. [2]. This logic, therefore, retains the full set of expressive means of the two above-mentioned logics and introduces some new expressive means on top of them. These new expressive means were called in [5] proving modalities and they capture different modes in which one can speak about proving activity of an agent. The general idea behind jstit logic is that one gets a right classification of such modes if one intersects the distinction between agentive and factual (aka moment-determinate) events developed in stit logic with the distinction between explicit and implicit modes of knowledge which is central to justification logic. The first distinction, when applied to proofs, corresponds to a well-known philosophical discussion of proofs-as-objects vs proofs-as-acts. One refers to a proof-as-act when one says that agent \(j\) proves some proposition \(A\), but one refers to a proof-as-object when saying that \(A\) was proved. While doing that, one can either simply say that \(A\) was proved, or add that \(A\) was proved by some proof \(t\); and the difference between these two modes of speaking is exactly the difference between implicit and explicit reference to proofs. All in all this gives us the following classification of proving modalities:

|            | Agentive          | Moment-determinate |
|------------|-------------------|--------------------|
| Explicit   | \(j\) proves \(A\) by \(t\) | \(A\) has been proven by \(t\) |
|            | \textit{Prove}(\(j, t, A\)) | \textit{Proven}(\(t, A\)) |
| Implicit   | \(j\) proves \(A\) | \(A\) has been proven |
|            | \textit{Prove}(\(j, A\)) | \textit{Proven}(\(A\)) |

In [5] the semantics of these modalities was presented and informally motivated in some detail. However, in the present paper, we are going to look into one fragment of
basic jstit logic rather than the full system. The reason for this is the relatively high level of complexity of the full basic jstit logic. The fragment in question is, in fact, the basic jstit logic without the two explicit proving modalities given in the first row of table above. The resulting restricted system, therefore, features the full set of expressive means inherited from justification logic and stit logic plus the two implicit modalities, \( \text{Prove}(j, A) \) and \( \text{Proven}(A) \). For the same reason (i.e. keeping the complexity down), we also use a slightly simplified version of the semantics introduced in [5] to interpret this logic.

The resulting system, which we will call the implicit jstit logic, still allows for an analysis of the interplay between proofs-as-acts and proofs-as-objects, although it limits the format of such an analysis to some extent and also zeros out the interplay between implicit and explicit modes of speech. But even this restricted logic, has, as will be shown below, a challenging degree of complexity, which makes the problem of axiomatizing it both interesting and non-trivial.

The present paper is devoted to solving this exact problem. Its layout is as follows. In Section 2 we define the language and the semantics of the logic at hand. We also show some features of implicit jstit logic, which limit the power and the scope of possible completeness results, namely, the failure of compactness and finite model properties. The latter fails in a rather strong form; as a result, one cannot impose any finite bound not only on the overall size of a model satisfying a given formula, but also on the length of histories in such a model. The failure of compactness also means that one cannot have a strongly complete axiomatization for this logic while retaining a finitary notion of proof.

Despite all these challenges, however, it turns out that with implicit jstit logic one can do much better than just weak completeness; in fact, our main result is much closer to the strong completeness and only differs from the latter in that some restrictions are imposed on proof variables occurring in a given set of formulas. The exact formulation of this result is given in Section 3, where we also define the axiom system which displays this exact degree of completeness w.r.t. implicit jstit logic. We immediately show this system to be sound w.r.t. the semantics introduced in Section 2 and we end the section by proving a number of theorems in the system.

Section 4 then contains the bulk of technical work necessary for the completeness theorem. It gives a stepwise construction and adequacy check for all the numerous components of the canonical model and ends with a proof of a truth lemma. Section 5 then reaps the fruits of the hard work done in Section 4 giving a concise proof of the completeness result and drawing some quick corollaries including the weak completeness theorem and a restricted form of compactness property. Then follows Section 6 giving some conclusions and drafting directions for future work.

In what follows we will be assuming, due to space limitations, a basic acquaintance with both stit logic and justification logic. We recommend to peruse [1] for a quick introduction into the basics of stit logic, and [4, Ch. 2] for the same w.r.t. justification logic.

## 2 Basic definitions and notation

We fix some preliminaries. First we choose a finite set \( A_g \) disjoint from all the other sets to be defined below. Individual agents from this set will be denoted by
letters $i$ and $j$. Then we fix countably infinite sets $PVar$ of proof variables (denoted by $x, y, z, w, u$) and $PConst$ of proof constants (denoted by $a, b, c, d$). When needed, subscripts and superscripts will be used with the above notations or any other notations to be introduced in this paper. Set $Pol$ of proof polynomials is then defined by the following BNF:

$$t := x \mid c \mid s + t \mid s \times t \mid !t,$$

with $x \in PVar$, $c \in PConst$, and $s, t$ ranging over elements of $Pol$. In the above definition $+$ stands for the sum of proofs, $\times$ denotes application of its left argument to the right one, and $!$ denotes the so-called proof-checker, so that $!t$ checks the correctness of proof $t$.

In order to define the set $Form$ of formulas we fix a countably infinite set $Var$ of propositional variables to be denoted by letters $p, q, r, s$. Formulas themselves will be denoted by letters $A, B, C, D$, and the definition of $Form$ is supplied by the following BNF:

$$A := p \mid A \land B \mid \neg A \mid [j]A \mid \Box A \mid t: A \mid KA \mid Prove(j, A) \mid Proven(A),$$

with $p \in Var$, $j \in Ag$ and $t \in Pol$.

It is clear from the above definition of $Form$ that we are considering a version of modal propositional language. As for the informal interpretations of modalities, $[j]A$ is the so-called cstit action modality and $\Box$ is historical necessity modality, both modalities are borrowed from stit logic. The next two modalities, $KA$ and $t: A$, come from justification logic and the latter is interpreted as “$t$ proves $A$”, whereas the former is the strong epistemic modality “$A$ is known”. The two remaining modalities, $Prove(j, A)$ and $Proven(A)$ are implicit modalities related to the proving activity of agents and their informal interpretation was considered in Section 1.

We assume $\Diamond$, $(K)$, and $(j)$ for a $j \in Ag$ as notations for the dual modalities of $\Box$, $K$ and $[j]$, respectively.

For the language at hand, we assume the following semantics. A jstit model is a structure

$$M = \langle Tree, \leq, Choice, Act, R, E, V \rangle$$

such that:

- $Tree$ is a non-empty set. Elements of $Tree$ are called moments.
- $\leq$ is a partial order on $Tree$ for which a temporal interpretation is assumed.
- $Hist$ is the set of maximal chains in $Tree$ w.r.t. $\leq$. Since $Hist$ is completely determined by $Tree$ and $\leq$, it is not included into the structure of a model as a separate component. Elements of $Hist$ are called histories. The set of histories containing a given moment $m$ will be denoted $H_m$. The following set:

$$MH(M) = \{(m, h) \mid m \in Tree, h \in H_m\},$$

called the set of moment-history pairs, will be used to evaluate formulas of the above language.

1 Perhaps, “$A$ is provable” will be an even better reading.
• \( \text{Choice} \) is a function mapping \( \text{Tree} \times \text{Agent} \) into \( 2^{2^{H_m}} \) in such a way that for any given \( j \in \text{Agent} \) and \( m \in \text{Tree} \) we have as \( \text{Choice}(m, j) \) (to be denoted as \( \text{Choice}^m_j \) below) a partition of \( H_m \). For a given \( h \in H_m \) we will denote by \( \text{Choice}^m_j(h) \) the element of partition \( \text{Choice}^m_j \) containing \( h \).

• \( \text{Act} \) is a function mapping \( MH(\mathcal{M}) \) into \( 2^{\mathit{Pol}} \).

• \( R \) is a pre-order on \( \text{Tree} \) called epistemic accessibility.

• \( \mathcal{E} \) is a function mapping \( \text{Tree} \times \mathit{Pol} \) into \( 2^{\mathit{Form}} \).

• \( V \) is an evaluation function, mapping the set \( \mathit{Var} \) into \( 2^{MH(\mathcal{M})} \).

However, not all structures of the above described type are admitted as jstit models. A number of additional restrictions needs to be satisfied. More precisely, we assume satisfaction of the following constraints:

1. Historical connection:
   \[ (\forall m, m_1 \in \text{Tree})(\exists m_2 \in \text{Tree})(m_2 \leq m \land m_2 \leq m_1). \]

2. No backward branching:
   \[ (\forall m, m_1, m_2 \in \text{Tree})(m_1 \leq m \land m_2 \leq m) \Rightarrow (m_1 \leq m_2 \lor m_2 \leq m_1). \]

3. No choice between undivided histories:
   \[ (\forall m, m' \in \text{Tree})(\forall h, h' \in H_m)(m < m' \land m' \in h \land h' \rightarrow \text{Choice}^m_j(h) = \text{Choice}^m_j(h')) \]
   for every \( j \in \text{Agent} \).

4. Independence of agents:
   \[ (\forall m \in \text{Tree})(\forall f : \text{Agent} \rightarrow 2^{H_m})(\forall j \in \text{Agent})(f(j) \in \text{Choice}^m_j) \Rightarrow \bigcap_{j \in \text{Agent}} f(j) \neq \emptyset. \]

5. Monotonicity of evidence:
   \[ (\forall t \in \mathit{Pol})(\forall m, m' \in \text{Tree})(R(m, m') \Rightarrow \mathcal{E}(m, t) \subseteq \mathcal{E}(m', t)). \]

6. Evidence closure properties. For arbitrary \( m \in \text{Tree}, s, t \in \mathit{Pol} \) and \( A, B \in \mathit{Form} \) it is assumed that:
   
   \[ \begin{align*}
   (a) & \quad A \Rightarrow B \in \mathcal{E}(m, s) \land A \in \mathcal{E}(m, t) \Rightarrow B \in \mathcal{E}(m, s \times t); \\
   (b) & \quad \mathcal{E}(m, s) \cup \mathcal{E}(m, t) \subseteq \mathcal{E}(m, s + t). \\
   (c) & \quad A \in \mathcal{E}(m, t) \Rightarrow t : A \in \mathcal{E}(m, t!); \\
   \end{align*} \]

7. Expansion of presented proofs:
   \[ (\forall m, m' \in \text{Tree})(m' < m \Rightarrow \forall h \in H_m(\text{Act}(m', h) \subseteq \text{Act}(m, h))). \]
8. No new proofs guaranteed:

\[(\forall m \in \text{Tree})(\bigcap_{h \in H_m} (\text{Act}(m, h)) \subseteq \bigcup_{m' < m, h \in H_{m'}} (\text{Act}(m', h))).\]

9. Presenting a new proof makes histories divide:

\[(\forall m \in \text{Tree})(\forall h, h' \in H_m)(\exists m' > m(m' \in h \cap h') \Rightarrow (\text{Act}(m, h) = \text{Act}(m, h'))).\]

10. Future always matters:

\[\leq \subseteq R.\]

11. Presented proofs are epistemically transparent:

\[(\forall m, m' \in \text{Tree})(R(m, m') \Rightarrow (\bigcap_{h \in H_m} (\text{Act}(m, h)) \subseteq \bigcap_{h' \in H_{m'}} (\text{Act}(m', h'))))).\]

We offer some intuitive explanation for the above defined notion of jstit model. Due to space limitations, we only explain the intuitions behind jstit models very briefly, and we urge the reader to consult [5, Section 3] for a more comprehensive explanations, whenever needed.

The components like Tree, ≤, Choice and V are inherited from stit logic, whereas R and E come from justification logic. The only new component is Act. The intuition behind the semantics is that Ag, our community of agents, is engaged in proving activity and this proving activity consists in making proof polynomials public within the community. One can think of a group of researchers, assembled before a whiteboard in a conference room and putting the proofs they discover on this whiteboard. Function Act gives out the current state of this whiteboard at any given moment under any given history. The whole situation is somewhat idealized in that we assume that nothing ever gets erased from the whiteboard, that there is always enough free space on it, and that the agents do not send one another any private messages.

The numbered list of semantical constraints above then just builds on these intuitions. Constraints 1–4 are borrowed from stit logic, constraints 5 and 6 are inherited from justification logic. Constraint 7 just says that nothing gets erased from the whiteboard, constraint 8 says a new proof cannot spring into existence as a static (i.e. moment-determinate) feature of the environment out of nothing, but rather has to come as a result (or a by-product) of a previous activity. Constraint 9 is just a corollary to constraint 3 in the richer environment of jstit models, constraint 10 says that the possible future of the given moment is always epistemically relevant in this moment, and constraint 11 says that the community knows everything that has firmly made its way onto the whiteboard.

For the members of Form, we will assume the following inductively defined satisfaction relation. For every jstit model \(\mathcal{M} = \langle \text{Tree}, \leq, \text{Choice}, \text{Act}, R, E, V \rangle\) and for every \((m, h) \in \text{Pair}_\mathcal{M}\) we stipulate that:
\[ M, m, h \models p \iff (m, h) \in V(p); \]
\[ M, m, h \models [j]A \iff (\forall h' \in \text{Choice}^m_j(h))(M, m, h' \models A); \]
\[ M, m, h \models \Box A \iff (\forall h' \in H_m)(M, m, h' \models A); \]
\[ M, m, h \models KA \iff \forall m'\forall h'(R(m, m') \& h' \in H_m' \Rightarrow M, m', h' \models A); \]
\[ M, m, h \models t:A \iff A \in E(m, t) \& M, m, h \models KA; \]
\[ M, m, h \models \text{Prove}(j, A) \iff (\forall h' \in \text{Choice}^m_j(h))(\exists t \in \text{Act}(m, h'))(M, m, h' \models t:A) \&
(\forall s \in \text{Pol})(\exists h'' \in H_m)(M, m, h \models s:A \Rightarrow s \notin \text{Act}(m, h'')); \]
\[ M, m, h \models \text{Proven}(A) \iff (\exists t \in \text{Pol})(\forall h' \in H_m)(t \in \text{Act}(m, h') \& M, m, h \models t:A) \]

In the above clauses we assume that \( p \in \text{Var}; \) we also assume standard clauses for Boolean connectives. Note that the satisfaction clause for \( \text{Prove}(j, A) \) consists of two conjuncts, one stating that some proof of \( A \) must be presented at every history in a given choice cell, and the other saying that no proof of \( A \) is presented in all histories through the given moment. These conjuncts show some similarity to the conjuncts in the satisfaction clause for the \textit{dstit} operator, which are known in the existing literature under the names of positive and negative condition, respectively. Following this usage, we will name the first conjunct in the satisfaction clause for \( \text{Prove}(j, A) \) the positive condition for \( \text{Prove}(j, A) \), and the second one the negative condition for \( \text{Prove}(j, A) \). The intuitive motivation for the satisfaction clauses of \( \text{Prove}(j, A) \) and \( \text{Proven}(A) \) was worked out in detail in [5] and we do not dwell on it here.

We further assume standard definitions for satisfiability and validity of formulas and sets of formulas in the presented semantics.

Before we proceed to proving things about the defined system, we want to briefly comment on how the above semantics relates to the semantics introduced in [5]. The main difference is that the latter semantics uses two epistemic accessibility relations \( R \) and \( R_e \) with the constraint that \( R \subseteq R_e \), whereas in the jstit models as defined above one only finds one such relation \( R \), and this relation serves the functions of both \( R \) and \( R_e \). Thus the semantics defined above arises from the more general semantics presented in [5] as a particular case with \( R \) and \( R_e \) being identified with one another.

The exact import of this additional restriction on the semantics presented in [5] is not yet clear. It is known that on the level of pure justification logic identifying \( R \) and \( R_e \) does not change the set of validities (see, e.g. [2, Comment 6.5]). Our tentative hypothesis would be, then, that imposing \( R = R_e \) in the richer context of jstit logic might be just as irrelevant as it is in justification logic. However, we have no proof of this hypothesis at the moment, so it stands as an open problem.

The semantics just defined admits of no finitary strongly complete system since it is not compact. Indeed, the set
\[ \{ \text{Proven}(p) \} \cup \{ \neg t:p \mid t \in \text{Pol} \} \]
is unsatisfiable, even though every finite subset of it can be satisfied. Still, the main result of this paper shows that we can do better than just weak completeness; in fact we can show that also infinite consistent sets of formulas can be satisfied provided that there is an infinite set of proof variables that do not occur in those formulas. Thus
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we get something considerably stronger than just weak completeness including also a restricted form of the compactness theorem.

It is also worth noting that under the presented semantics some satisfiable formulas cannot be satisfied over finite models. As an example of this phenomenon, consider $K(\Diamond p \land \neg \Diamond p)$. If $\mathcal{M}, m_1, h \models K(\Diamond p \land \neg \Diamond p)$, then, by reflexivity of $R$, also $\mathcal{M}, m_1, h \models \Diamond p \land \Diamond \neg p$, which means that at least two different histories are running through $m_1$ in $\mathcal{M}$. Therefore, $m_1$ cannot be a $\leq$-maximal moment in $\mathcal{M}$, so that there is at least one moment $m_2 \in h$ such that $m_1 < m_2$. By the future always matters constraint we get then that $R(m_1, m_2)$, which, by transitivity of $R$, means that we also have $\mathcal{M}, m_2, h \models K(\Diamond p \land \Diamond \neg p)$. Iterating this construction $\omega$ times, we get a countably infinite sequence of moments along $h$:

$$m_1 < m_2 < \ldots < m_n < \ldots,$$

showing that the moments in these sequence are pairwise different (by antisymmetry of $\leq$) and that $\mathcal{M}$ is consequently an infinite model. Since $\mathcal{M}$ was chosen arbitrarily, this shows that $K(\Diamond p \land \Diamond \neg p)$ cannot be satisfied over finite jstit models. On the other hand, $K(\Diamond p \land \Diamond \neg p)$ is clearly satisfiable when one allows for infinite models. One can consider, for example, a jstit model $\mathcal{M} = \langle \text{Tree}, \leq, \text{Choice}, \text{Act}, R, E, V \rangle$ for a community $\{j\}$ consisting of a single agent, setting:

$$\text{Tree} := \{(a_1, \ldots, a_n) \mid a_i \in \{0, 1\} \text{ for } i \leq n\} \cup \{\Lambda\},$$

where $\Lambda$ is the empty sequence;

$$(a_1, \ldots, a_n) \leq (b_1, \ldots, b_k) \iff (n \leq kk \land \forall i \leq n)(a_i = b_i), \text{ Choice}^m_j := H_m, \text{ Act}(m, h) = \emptyset$$

for every $m \in \text{Tree}$ and $h \in H_m$;

$$R := \leq, E(m, t) := \text{Form},$$

for every $m \in \text{Tree}$ and $t \in \text{Pol}$;

$$V(p) := \{(m, h) \mid (m, 1) \in h\}, V(q) = \emptyset$$

provided $q \in \text{Var} \setminus \{p\}$. It is straightforward to check then that with these settings we get that $\mathcal{M}, \Lambda, h \models K(\Diamond p \land \Diamond \neg p)$ for an arbitrary history $h$ over $\mathcal{M}$.

Note also, that the same example shows that one cannot put a finite bound on the length of histories in the models satisfying a given formula, so that what one might have called a “finite history property” which is satisfied, e.g., by the canonical model of the logic of dstit operator (see [3, Section 17C] for the definition) also fails for the implicit jstit logic.

3 Axiomatic system and soundness

We consider the following set of axiomatic schemes:
A full set of axioms for classical propositional logic (A0)

\(S5\) axioms for \(\square\) and \([j]\) for every \(j \in \text{Agent}\) (A1)

\(\square A \rightarrow [j]A\) for every \(j \in \text{Agent}\) (A2)

\((\Diamond [j]A_1 \land \ldots \land [j]A_n) \rightarrow \Diamond ([j]A_1 \land \ldots \land [j]A_n)\) (A3)

\((s: (A \rightarrow B)) \rightarrow (t: A \rightarrow (s \times t): B)\) (A4)

\(t: A \rightarrow (\langle t: A \rangle \land KA)\) (A5)

\((s: A \lor t: A) \rightarrow (s + t): A\) (A6)

\(S4\) axioms for \(K\) (A7)

\(KA \rightarrow \square KA\) (A8)

\(\text{Prove}(j, A) \rightarrow (\neg \text{Proven}(A) \land [j] \text{Prove}(j, A) \land KA)\) (A9)

\(\square \text{Prove}(j, A) \rightarrow \square \text{Prove}(i, A)\) (A10)

\(\text{Proven}(A) \rightarrow (K \text{Proven}(A) \land KA)\) (A11)

\(\neg K(\bigvee_{i=1}^{n} \langle K \rangle \Diamond \text{Prove}(j_i, A_i))\) (A12)

\(\neg \text{Prove}(j, A) \rightarrow (\langle j \rangle (\bigwedge_{i \in \text{Ag}} \neg \text{Prove}(i, A))\) (A13)

The assumption is that in (A3) \(j_1, \ldots, j_n\) are pairwise different.

To this set of axiom schemes we add the following rules of inference:

\(A, A \rightarrow B \Rightarrow B;\) (R1)

\(A \Rightarrow KA;\) (R2)

If \(A\) is an instance of (A0)–(A13) and \(c \in \text{Const}\), then infer \(c: A;\) (R3)

\(KA \rightarrow (\neg \text{Proven}(B_1) \lor \ldots \lor \neg \text{Proven}(B_n)) \Rightarrow \)

\(\Rightarrow KA \rightarrow (\bigwedge_{j \in \text{Ag}} \neg \text{Prove}(j, B_1) \lor \ldots \lor \bigwedge_{j \in \text{Ag}} \neg \text{Prove}(j, B_n)).\) (R4)

We call a jstit model \(\mathcal{M} = \langle \text{Tree}, \leq, \text{Choice}, \text{Act}, R, \mathcal{E}, V \rangle\) normal iff the following condition is satisfied:

\((\forall c \in \text{Const}) (\forall m \in \text{Tree}) (\{A \mid A \text{ is a substitution instance}\)

of one of the schemes among \([A1] - [A13]\}\subseteq \mathcal{E}(m, c)).\)

Our goal is now a restricted completeness theorem w.r.t. the class of normal models. We start by establishing soundness, and we precede the soundness theorem with the following rather straightforward technical claim:

**Lemma 1.** For every \(A \in \text{Form}\) and every \(t \in \text{Pol}\), all of the formulas \(\square A, KA, t: A\) and \(\text{Proven}(A)\) are moment-determinate, that is to say, if \(\alpha \in \{\square A, KA, t: A, \text{Proven}(A)\}\), then for an arbitrary normal jstit model \(\mathcal{M} = \langle \text{Tree}, \leq, \text{Choice}, \text{Act}, R, \mathcal{E}, V \rangle\) and \(m \in \text{Tree}, \) if \(h, h' \in H_m,\) then:

\(\mathcal{M}, m, h \models \alpha \iff \mathcal{M}, m, h' \models \alpha.\)
Also, Boolean combinations of these formulas are moment-determinate.

Proof. For \( \alpha = \Box A \) and \( \alpha = KA \) it suffices to note that the semantical conditions for satisfaction of \( KA \) and \( \Box A \) at a given \( (m, h) \in MH(M) \) in a given \( M \) have no free occurrences of \( h \). When we turn, further, to the corresponding condition for \( t:A \), the only free occurrence of \( h \) will be within the context \( \mathcal{M}, m, h \models KA \) which was shown to be moment-determinate. Similarly, in the satisfaction condition for \( \text{Proven}(A) \) the only free occurrence of \( h \) is within a moment determinate context \( \mathcal{M}, m, h \models t:A \).

Of course, Boolean combinations of moment-determinate formulas must be moment-determinate, too.

It follows from Lemma\(^1\) that the truth of moment-determinate formulas at a given moment-history pair only depends on the moment, so that we might as well omit the histories when discussing satisfaction of such formulas and write \( \mathcal{M}, m \models KA \) instead of \( \mathcal{M}, m, h \models KA \), etc.

Establishing soundness mostly reduces to a routine check that every axiom is valid and that rules preserve validity. We treat the less obvious cases in some detail:

**Theorem 1.** Every instance of \( (A1) – (A13) \) is valid over the class of normal jstit models. Every application of rules \( (R1) – (R4) \) to formulas which are valid over the class of normal jstit models yields a formula which is valid over the class of normal jstit models.

Proof. First, note that if \( \mathcal{M} = \langle \text{Tree}, \leq, \text{Choice}, \text{Act}, R, \mathcal{E}, V \rangle \) is a normal jstit model, then \( \langle \text{Tree}, \leq, \text{Choice}, V \rangle \) is a model of stit logic. Therefore, axioms \( (A0) – (A3) \), which were copy-pasted from the standard axiomatization of dstit logic (see, e.g. \( 3 \) Ch. 17) must be valid. Second, note that if \( \mathcal{M} = \langle \text{Tree}, \leq, \text{Choice}, \text{Act}, R, \mathcal{E}, V \rangle \) is a normal jstit model, then \( \langle \text{Tree}, \leq, \text{Act}, R \rangle \) is what is called in \( 2 \) p. 1067 a frame for a Fitting justification model with the form of constant specification defined by \( (R0) \). This means that all of the \( (A1) – (A7) \) must be valid, whereas \( (R1) – (R3) \) must preserve validity. The validity of other elements of the above-presented axiomatic system will be motivated below in some detail. In what follows, \( \mathcal{M} = \langle \text{Tree}, \leq, \text{Choice}, \text{Act}, R, \mathcal{E}, V \rangle \) will always stand for an arbitrary normal jstit model, and \( (m, h) \) for an arbitrary element of \( MH(M) \).

As for \( (A8) \), assume for reductio that \( \mathcal{M}, m \models KA \land \neg \Box KA \). Then \( \mathcal{M}, m, h \models KA \) and also \( \mathcal{M}, m \not\models \Box KA \). The latter means that for some \( h' \in H_m \) we have \( \mathcal{M}, m \not\models K \Box A \). Therefore, there must be some \( n' \in \text{Tree} \) such that \( R(m, n') \) and some \( g \in H_{m'} \) such that \( \mathcal{M}, n', g \not\models \Box A \). Whence for some \( g' \in H_{m'} \) we will have \( \mathcal{M}, n', g' \not\models A \). Since \( R(m, n') \), this means that \( KA \) must fail at \( (m, h) \) in \( \mathcal{M} \), a contradiction.

We consider next \( (A9) \). Assume that \( \text{Prove}(j, A) \) is true at \( (m, h) \) in \( \mathcal{M} \). Note that the negative condition for \( \text{Prove}(j, A) \) at \( (m, h) \) is logically equivalent to the negation of the satisfaction condition for \( \text{Proven}(A) \), which means that \( \neg \text{Proven}(A) \) must be true at \( (m, h) \) in \( \mathcal{M} \). Further, since clearly \( h \in \text{Choice}^m_j(h) \) and thus \( \text{Choice}^m_j(h) \) cannot be empty, it follows from the positive condition for \( \text{Prove}(j, A) \) that for some \( t \in \text{Pol} \) we will have \( \mathcal{M}, m, h \models t:A \), and therefore, by validity of \( (A5) \), \( \mathcal{M}, m, h \models KA \). Finally, note that since \( \text{Choice}^m_j \) is a partition of \( H_m \), then for any \( h' \in \text{Choice}^m_j(h) \),

\(^{2}\)But note, that in \( 2 \) they do not include \( \mathcal{E} \) in justification frames; however, this is of no consequence for the present setting.
if $h'' \in \text{Choice}^m(h')$, then $h'' \in \text{Choice}^m(h)$. Therefore, since the positive condition for $\text{Prove}(j, A)$ is satisfied at $(m, h)$, there must be some $t \in \text{Pol}$ such that both $t \in \text{Act}(m, h'')$ and $\mathcal{M}, m \models t : A$. Therefore, the positive condition for $\text{Prove}(j, A)$ will be satisfied at $(m, h')$ for every $h' \in \text{Choice}^m(h)$. As for the negative condition, recall that it is equivalent to the negation of the satisfaction condition for $\text{Proven}(j, A)$ and the latter is, by Lemma 1, moment-determinate. Therefore, the negative condition for $\text{Prove}(j, A)$ must be moment-determinate as well, and, once satisfied at a given $(m, h)$, it will be satisfied at every history through $m$. Therefore, once we have $\text{Prove}(j, A)$ true at $(m, h)$ in $\mathcal{M}$, we must also have $\mathcal{M}, m, h \models [j]\text{Prove}(j, A)$.

The next axiom is (A10). If $\Box \text{Prove}(j, A)$ is true at $(m, h)$ in $\mathcal{M}$, this means that $\text{Prove}(j, A)$ is true at $(m, h')$ in $\mathcal{M}$ for every $h' \in H_m$. Now, take an arbitrary such $h'$. We know that the negative condition for $\text{Prove}(i, A)$ is the same as for $\text{Prove}(j, A)$, and is therefore satisfied at $(m, h')$. As for the positive condition, assume that $h'' \in \text{Choice}^m(h')$. We know that $\text{Prove}(j, A)$ is true at $(m, h'')$, therefore, since $h''$ is obviously in $\text{Choice}^m(h')$, for some $t \in \text{Pol}$ we must have both $t \in \text{Act}(m, h'')$ and $\mathcal{M}, m \models t : A$. Thus the positive condition for $\text{Prove}(i, A)$ at $(m, h')$ is satisfied as well. Since $h'$ was chosen as an arbitrary history through $m$, this means that $\Box \text{Prove}(i, A)$ must be satisfied at $(m, h)$ in $\mathcal{M}$.

We now take up (A11). If $\text{Proven}(A)$ is true at $m$ in $\mathcal{M}$, then there is a $t \in \text{Pol}$ such that $t \in \bigcap_{h \in H_m} \text{Act}(m, h)$ and $t : A$ is true at $m$. By validity of (A7), we immediately get that $\mathcal{M}, m \models KA$. Further, the fact that $t : A$ is true at $m$ means that $A \in \mathcal{E}(m, t)$. Now, assume that $m' \in \text{Tree}$ is such that $R(m, m')$. By the epistemic transparency of presented proofs constraint we know that $t \in \bigcap_{h' \in H_{m'}} \text{Act}(m', h')$. By monotonicity of evidence, we know that $A \in \mathcal{E}(m', t)$. By the S4 reasoning for $K$ we know that $\mathcal{M}, m' \models KA$. Summing up, we must have $\text{Proven}(A)$ true at $m'$, and since $m'$ was chosen as an arbitrary $R$-successor of $m$, this means that we also have $\mathcal{M}, m \models K\text{Proven}(A)$.

To prove the validity of (A12) over the class of normal jist models, we proceed by induction on $n \geq 1$.

Basis. $n = 1$. Assume, for reductio, that $\mathcal{M}, m \models K(K) \Box \text{Prove}(j_1, A_1)$. Then, by validity of (A7), $\mathcal{M}, m \models (K) \Box \text{Prove}(j_1, A_1)$. Therefore, for some $m' \in \text{Tree}$ such that $R(m, m')$, we must have $\mathcal{M}, m' \models \Box \text{Prove}(j_1, A_1)$. The latter, in turn, means that for some $h' \in H_{m'}$ we will have $\mathcal{M}, m', h' \models \text{Prove}(j_1, A_1)$. We know then that $m'$ must have some $<\text{-successors}$, where $<$ is the irreflexive companion of $\leq$ in $\mathcal{M}$. Indeed, if $m'$ were a $\leq$-maximal moment, then we would have $H_{m'} = \{h'\}$, that is to say, $h'$ would be the only history passing through $m'$. But then, of course $h' \in \text{Choice}^m(j_1)(h')$, therefore, for some $t \in \text{Pol}$ we would have then both $t \in \text{Act}(m', h')$ and $\mathcal{M}, m' \models t : A$ by the positive condition for $\text{Prove}(j_1, A_1)$ at $(m', h')$. But then, given that $H_{m'} = \{h'\}$, this would mean that $t \in \bigcap_{g \in H_{m'}} \text{Act}(m', g)$ so that the negative condition for $\text{Prove}(j_1, A_1)$ at $(m', h')$ would be violated, contradicting our assumption that $\mathcal{M}, m', h' \models \text{Prove}(j_1, A_1)$.

Therefore, we can choose a moment $m''$ such that both $m'' > m'$ and $h'$ passes through $m''$; consider then $H_{m''}$. All the histories passing through $m''$ are pairwise undivided at $m'$, therefore, by the presenting a new proof makes histories divide constraint we must have $\text{Act}(m', g) = \text{Act}(m', g')$ for any $g, g' \in H_{m''}$. We also know that, since $\mathcal{M}, m', h' \models \text{Prove}(j_1, A_1)$, there must be a $t \in \text{Pol}$ such that $t \in \text{Act}(m', h')$ and $\mathcal{M}, m' \models t : A$. Since $h' \in H_{m''}$, this further means that $t \in \bigcap_{g \in H_{m''}} \text{Act}(m', g)$.
By the expansion of presented proofs constraint, we may infer from the latter that $t \in \bigcap_{g \in H_{m''}} \text{Act}(m'',g)$. By the future always matters constraint, we know that, since $m' < m''$, then we must have $R(m', m'')$, whence, given that $\mathcal{M}, m' \models t : A$, we must also have $\mathcal{M}, m'' \models t : A$. Summing this up with $t \in \bigcap_{g \in H_{m''}} \text{Act}(m'',g)$, we get that $\mathcal{M}, m'' \models \text{Proven}(A)$, which, by (A11), means that $\mathcal{M}, m'' \models K \text{Proven}(A)$, whence further, by (A5), $\mathcal{M}, m'' \models \Box K \Box \text{Proven}(A)$. Validity of (A11) yields then $\mathcal{M}, m'' \models K \Box \text{Proven}(A)$. Note, further, that $\text{Proven}(A) \rightarrow \neg \text{Prove}(j, A)$ must be valid as a consequence of (A9), and by S5 reasoning for $\Box$ and S4 reasoning for $K$ we get from this the validity of:

$$K \Box \text{Proven}(A) \rightarrow K \Box \neg \text{Prove}(j, A).$$

The latter means that $\mathcal{M}, m'' \models K \Box \neg \text{Prove}(j, A)$, and, pushing the negation outside, $\mathcal{M}, m'' \models \neg (K) \Diamond \text{Prove}(j, A)$. It remains then to note we already established both $R(m, m')$ and $R(m, m'')$ so that by transitivity of $R$ we get that $R(m, m'')$. Therefore, the consequence that $\mathcal{M}, m'' \models \neg (K) \Diamond \text{Prove}(j, A)$ turns out to be in contradiction with our initial hypothesis that $\mathcal{M}, m \models K (K) \Diamond \text{Prove}(j_1, A_1)$. The obtained contradiction shows that we must have $\neg K (K) \Diamond \text{Prove}(j_1, A_1)$ true throughout any given normal jstit model for any $A_1 \in \text{Form}$ and $j_1 \in Ag$.

**Induction step.** Assume that for a $k \geq 1$ the validity of all instances of the scheme $\neg K (\bigvee_{l=1}^{k+1} (K) \Diamond \text{Prove}(j_l, A_l))$ has been successfully shown and assume that $n = k + 1$. Assume, further, that:

$$\mathcal{M}, m \models \bigvee_{l=1}^{k+1} (K) \Diamond \text{Prove}(j_l, A_l).$$

Then, by S4 reasoning for $K$, we know that

$$\mathcal{M}, m \models \bigvee_{l=1}^{k+1} (K) \Diamond \text{Prove}(j_l, A_l),$$

so that at least one of $(K) \Diamond \text{Prove}(j_l, A_l)$ must be true at $m$; suppose, wlog, that $l = 1$. Then, arguing as in the base case, we find a moment $m''$ such that $R(m, m'')$ and $\mathcal{M}, m'' \models K \Box \neg \text{Prove}(j_1, A_1)$. Applying to this S4 reasoning for $K$, we get further that $\mathcal{M}, m'' \models KK \Box \neg \text{Prove}(j_1, A_1)$, and, pushing out the negation, that $\mathcal{M}, m'' \models K \neg (K) \Diamond \text{Prove}(j_1, A_1)$. Since we have $R(m, m'')$, it follows that we also have:

$$\mathcal{M}, m'' \models \bigvee_{l=1}^{k+1} (K) \Diamond \text{Prove}(j_l, A_l).$$

From the latter two facts, S4 reasoning for $K$ yields that:

$$\mathcal{M}, m'' \models \bigvee_{l=2}^{k+1} (K) \Diamond \text{Prove}(j_l, A_l),$$

contradicting the induction hypothesis. The obtained contradiction shows the validity of (A12) for $n = k + 1$. 
The last axiom is (A13). So, assume that $\mathcal{M}, m, h \models \neg \text{Prove}(j, A)$. We have to consider then two cases.

Case 1. The negative condition for $\text{Prove}(j, A)$ fails at $(m, h)$. Then we must have $\mathcal{M}, m, h \models \text{Proven}(A)$, and by (A9) we know that $\mathcal{M}, m, h \models \bigwedge_{i \in Ag} \neg \text{Prove}(i, A)$, thus also $\mathcal{M}, m, h \models (j) \bigwedge_{i \in Ag} \neg \text{Prove}(i, A)$ by S5 reasoning for $[j]$.

Case 2. The negative condition for $\text{Prove}(j, A)$ holds at $(m, h)$. Then, since we have $\mathcal{M}, m, h \not\models \neg \text{Prove}(j, A)$, the positive condition for $\text{Prove}(j, A)$ at $(m, h)$ must fail. Therefore, we can choose a $g \in \text{Choice}^m_j(h)$ such that for no $t \in \text{Pol}$ do we have both $t \in \text{Act}(m, g)$ and $\mathcal{M}, m \models t : A$. Note, further, that $g \in \text{Choice}^m_j(h)$ for every $i \in Ag$, and therefore the positive condition for every formula of the form $\text{Prove}(i, A)$ fails at $(m, g)$. Therefore, we must have $\mathcal{M}, m, g \models \bigwedge_{i \in Ag} \neg \text{Prove}(i, A)$, and, since $g \in \text{Choice}^m_j(h)$, also $\mathcal{M}, m, h \models (j) \bigwedge_{i \in Ag} \neg \text{Prove}(i, A)$ as desired.

It only remains to show that (R4) preserves validity over normal jat models. Assume that $KA \rightarrow (\neg \text{Proven}(B_1) \lor \ldots \lor \neg \text{Proven}(B_n))$ is valid over normal jat models, and assume also that we have:

$$\mathcal{M}, m, h \models KA \land (\bigvee_{j \in Ag} \text{Prove}(j, B_1) \land \ldots \land \bigvee_{j \in Ag} \text{Prove}(j, B_n)).$$

This means that we can choose $j_{B_1}, \ldots, j_{B_n} \in Ag$ in such a way that we end up having:

$$\mathcal{M}, m, h \models KA \land \text{Prove}(j_{B_1}, B_1) \land \ldots \land \text{Prove}(j_{B_n}, B_n).$$

We can now re-use the manner of reasoning employed above for the base case of (A12). More precisely, since $\mathcal{M}, m, h \models \text{Prove}(j_{B_1}, B_1)$ then $m$ must have some $<\text{-successors}$, otherwise $h$ would be the unique history through $m$. Then, if there existed $t \in \text{Pol}$ such that both $t \in \text{Act}(m, h)$ and $\mathcal{M}, m \models t : B_1$, the negative condition for $\text{Prove}(j_{B_1}, B_1)$ at $(m, h)$ would be violated. On the other hand, if there were no such $t$, then the positive condition for $\text{Prove}(j_{B_1}, B_1)$ at $(m, h)$ would be violated.

Since $m$ is not a $\leq\text{-maximal}$ moment in $\text{Tree}$, then we can choose an $m' \in \text{Tree}$ such that both $m' > m$ and $h \in H_{m'}$. All the histories passing through $m'$ are pairwise undivided at $m$, therefore, by the presenting a new proof makes histories divide constraint we must have $\text{Act}(m, g) = \text{Act}(m, g')$ for any $g, g' \in H_{m'}$. We also know that, since

$$\mathcal{M}, m, h \models \text{Prove}(j_{B_1}, B_1) \land \ldots \land \text{Prove}(j_{B_n}, B_n),$$

there must be $t_1, \ldots, t_n \in \text{Pol}$ such that $t_1, \ldots, t_n \in \text{Act}(m, h)$ and $\mathcal{M}, m \models t_i : B_i$ for all $i$ such that $1 \leq i \leq n$. Since $h \in H_{m'}$, this further means that $t_1, \ldots, t_n \in \bigcap_{g \in H_{m'}} \text{Act}(m, g)$. By the expansion of presented proofs constraint, we may infer from the latter that $t_1, \ldots, t_n \in \bigcap_{g \in H_{m'}} \text{Act}(m', g)$. By the future always matters constraint, we know that, since $m < m'$, then we must have $R(m, m')$, whence, given that $\mathcal{M}, m \models t_i : B_i$ for all $i$ such that $1 \leq i \leq n$, we must also have $\mathcal{M}, m' \models t_i : B_i$ for all such $i$. Summing this up with $t_1, \ldots, t_n \in \bigcap_{g \in H_{m'}} \text{Act}(m', g)$, we get that

$$\mathcal{M}, m' \models \text{Proven}(B_1) \land \ldots \land \text{Proven}(B_n).$$

Further, we know that $\mathcal{M}, m' \models KA$, so that by $R(m, m')$ and S4 properties of $K$ we must also have $\mathcal{M}, m' \models KA$. Thus we get that $KA \land \text{Proven}(B_1) \land \ldots \land \text{Prove}(B_n)$ is satisfied at $m'$ which is in contradiction with the assumed validity of $KA \rightarrow (\neg \text{Proven}(B_1) \lor \ldots \lor \neg \text{Proven}(B_n))$. \hfill \qed
We then define a proof in the above-presented axiomatic system as a finite sequence of formulas such that every formula in it is either an axiom or is obtained from earlier elements of the sequence by one of the inference rules. A proof is a proof of its last formula. If an $A \in \text{Form}$ is provable in our system, we will write $\vdash A$.

The presence in our system of the rules like $\text{R2}$ and especially $\text{R4}$ complicates the issue of finding the right notion of an inference from premises and the right format for Deduction Theorem. Given that these problems lie beyond the scope of the present paper, we will take a little detour and will base our definition of consistency of a set of formulas upon the notion of provable formula, rather than just saying that a set $\Gamma \subseteq \text{Form}$ is inconsistent iff $\bot$ is derivable from $\Gamma$. Moreover, due to the form of our main result we need to relativize our notions to sets of proof variables occurring in a given set of formulas.

More precisely, assume that $Z \subseteq \text{PVar}$. Then we can define $\text{Pol}_Z$ and $\text{Form}_Z$ as the sets of proof polynomials (resp. formulas) containing proof variables from $Z$ only. Note that this imposes no restrictions on proof constants, so that the set of closed proof polynomials is contained in $\text{Pol}_Z$ for every $Z \subseteq \text{PVar}$. Now, for a given $Z \subseteq \text{PVar}$ we say that $\Gamma \subseteq \text{Form}_Z$ is a set of formulas in $Z$. We say that $\Gamma$ is inconsistent iff for some $A_1, \ldots, A_n \in \Gamma$ we have $\vdash (A_1 \land \ldots \land A_n) \rightarrow \bot$, and we say that $\Gamma$ is consistent iff it is not inconsistent. $\Gamma$ is maxiconsistent in $Z$ iff $\Gamma \subseteq \text{Form}_Z$ and no consistent subset of $\text{Form}_Z$ properly extends $\Gamma$.

Even with this slightly non-standard definition of inconsistency, we can still do many familiar things, e.g. extend consistent sets with new formulas and eventually make them maxiconsistent. More precisely, the following lemma holds:

**Lemma 2.** Let $Z \subseteq \text{PVar}$, let $\Gamma \subseteq \text{Form}_Z$ be consistent, and let $A, B \in \text{Form}_Z$. Then:

1. There exists a $\Delta \subseteq \text{Form}_Z$ such that $\Delta$ is maxiconsistent in $Z$ and $\Gamma \subseteq \Delta$.
2. If $\Gamma$ is maxiconsistent in $Z$, then exactly one element of $\{A, \neg A\}$ is in $\Gamma$.
3. If $\Gamma$ is maxiconsistent in $Z$, then $A \lor B \in \Gamma$ iff $(A \in \Gamma$ or $B \in \Gamma)$.
4. If $\Gamma$ is maxiconsistent in $Z$ and $A, (A \rightarrow B) \in \Gamma$, then $B \in \Gamma$.
5. If $\Gamma$ is maxiconsistent in $Z$, then $A \land B \in \Gamma$ iff $(A \in \Gamma$ and $B \in \Gamma)$.

**Proof.** (Part 1) Just as in the standard case, we enumerate the elements of $\text{Form}_Z$ as $A_1, \ldots, A_n, \ldots$ and form the sequence of sets $\Gamma_1, \ldots, \Gamma_n, \ldots$, such that $\Gamma_1 := \Gamma$ and for every natural $i \geq 1$:

$$
\Gamma_{i+1} := \left\{ \begin{array}{ll} 
\Gamma_i, & \text{if } \Gamma_i \cup \{A_i\} \text{ is inconsistent;} \\
\Gamma_i \cup \{A_i\}, & \text{otherwise.}
\end{array} \right.
$$

We now define $\Delta := \bigcup_{i \geq 1} \Gamma_i$. Of course, we have $\Gamma \subseteq \Delta$, and, moreover, $\Delta$ is maxiconsistent in $Z$. To see this, note that for every $i \geq 1$ the set $\Gamma_i$ is consistent by construction. Now, if $\Delta$ is inconsistent, then there must be a valid implication from a finite conjunction of formulas in $\Delta$ to $\bot$. These formulas must be mentioned in our numeration of $\text{Form}_Z$ so that the valid implication in question can presented as $\vdash (A_{i_1} \land \ldots \land A_{i_n}) \rightarrow \bot$ for appropriate natural $i_1, \ldots, i_n$. Since all of $A_{i_1}, \ldots, A_{i_n}$ are in $\Delta$, we must have, by
the construction of \( \Gamma_1, \ldots, \Gamma_n, \ldots \), that \( A_{i_1}, \ldots, A_{i_n} \in \Gamma_{\max(i_1, \ldots, i_n)} \). But then this latter set must be inconsistent which contradicts our construction.

Further, if some consistent \( \Xi \subseteq \text{Form}_Z \) is such that \( \Delta \subseteq \Xi \), then let \( A_n \in \Xi \setminus \Delta \). We must have then \( \Gamma_n \cup \{A_n\} \) inconsistent, but we also have \( \Gamma_n \cup \{A_n\} \subseteq \Xi \), which implies inconsistency of \( \Xi \), in contradiction to our assumptions. Therefore, \( \Delta \) is not only consistent, but also maxiconsistent in \( Z \).

(Part 2) We cannot have both \( A \) and \( \neg A \) in \( \Gamma \), since we have, of course, \( \vdash (A \land \neg A) \rightarrow \bot \).

If, on the other hand, neither \( A \), nor \( \neg A \) is in \( \Gamma \), then both \( \Gamma \cup \{A\} \) and \( \Gamma \cup \{\neg A\} \) must be inconsistent, so that for some \( B_1, \ldots, B_n \in \Gamma \) we will have:
\[
\vdash (B_1 \land \ldots \land B_n \land A) \rightarrow \bot,
\]
whereas for some \( C_1, \ldots, C_k \in \Gamma \) we will have:
\[
\vdash (C_1 \land \ldots \land C_k \land \neg A) \rightarrow \bot,
\]
whence we get, using \( \text{(A0)} \) and \( \text{(R1)} \):
\[
\vdash (C_1 \land \ldots \land C_k) \rightarrow A,
\]
and further:
\[
\vdash (B_1 \land \ldots \land B_n \land C_1 \land \ldots \land C_k) \rightarrow \bot,
\]
so that \( \Gamma \) turns out to be inconsistent, contrary to our assumptions.

(Part 3) Assume \( (A \lor B) \in \Gamma \). If neither \( A \) nor \( B \) are in \( \Gamma \), then, by Part 2, both \( \neg A \) and \( \neg B \) are in \( \Gamma \). Using \( \text{(A0)} \) and \( \text{(R1)} \) we get that:
\[
\vdash ((A \lor B) \land \neg A \land \neg B) \rightarrow \bot,
\]
since \( (A \lor B) \neg \) is inconsistent, contrary to our assumptions. In the other direction, if, say \( A \in \Gamma \) and \( (A \lor B) \notin \Gamma \), then, by Part 2, we must have \( \neg (A \lor B) \in \Gamma \). Using \( \text{(A0)} \) and \( \text{(R1)} \) we get that:
\[
\vdash (\neg (A \lor B) \land A) \rightarrow \bot,
\]
showing, again, that \( \Gamma \) is inconsistent, contrary to our assumptions. The case when \( B \in \Gamma \) is similar.

Parts 4 and 5 are similar to Part 3.

Remark. Note that one can recover the notion of non-relativized maxiconsistent set and its properties just by setting \( Z := PVar \). But this will not be needed in the present paper.

We are now prepared to formulate our main result:

**Theorem 2.** Let \( X \subseteq PVar \) be such that \( PVar \setminus X \) is countably infinite. Then an arbitrary \( \Gamma \subseteq \text{Form}_X \) is consistent iff it is satisfiable in a normal jstit model.

The rest of the paper is mainly concerned with proving Theorem 2. One part of it we have, of course, right away, as a consequence of Theorem 1.

**Corollary 1.** Let \( X \subseteq PVar \) be such that \( PVar \setminus X \) is countably infinite. If \( \Gamma \subseteq \text{Form}_X \) is satisfiable in a normal jstit model, then \( \Gamma \) is consistent.
Proof. Let $\Gamma \subseteq Form_X$ be satisfiable in a normal jstit model so that we have, say $\mathcal{M}, m, h \models \Gamma$ for some $(m, h) \in MH(\mathcal{M})$. If $\Gamma$ were inconsistent this would mean that for some $A_1, \ldots, A_n \in \Gamma$ we would have $\vdash (A_1 \land \ldots \land A_n) \rightarrow \bot$. By Theorem 1 this would mean that:

$$\mathcal{M}, m, h \models (A_1 \land \ldots \land A_n) \rightarrow \bot,$$

whence clearly $\mathcal{M}, m, h \models \bot$, which is impossible. Therefore, $\Gamma$ must be consistent.

Before we move further, we mention some theorems in the above axiom system to be used later in the proof of the main result:

**Lemma 3.** The following holds for every $A \in Form$, $t \in Pol$, $x \in PVar$, and $j \in Ag$:

1. $\vdash t:A \rightarrow \Box t:A$;
2. $\vdash Proven(A) \rightarrow \Box Proven(A)$;
3. $\not\vdash x:A$;
4. $\vdash (Prove(j, A) \land \neg \Box Prove(j, A)) \rightarrow [j](Prove(j, A) \land \neg \Box Prove(j, A))$;
5. $\vdash KA \rightarrow \Box KA$.

**Proof.** (Part 1) We have:

$$
\begin{align*}
t:A & \rightarrow !t:A & \text{(by (A5))} \\
& \rightarrow Kt:A & \text{(by (A2))} \\
& \rightarrow \Box K \Box t:A & \text{(by (A8))} \\
& \rightarrow K \Box t:A & \text{(by (A1))} \\
& \rightarrow \Box t:A & \text{(by (A7))}
\end{align*}
$$

Our theorem follows then by transitivity of implication.

(Part 2) Again, we proceed by building a chain of implications:

$$
\begin{align*}
Proven(A) & \rightarrow KP proven(A) & \text{(by (A11))} \\
& \rightarrow \Box K \Box Proven(A) & \text{(by (A8))} \\
& \rightarrow K \Box Proven(A) & \text{(by (A1))} \\
& \rightarrow \Box Proven(A) & \text{(by (A7))}
\end{align*}
$$

(Part 3). Take an arbitrary normal jstit model $\mathcal{M} = \langle Tree, \leq, Choice, Act, R, \mathcal{E}, V \rangle$ and consider another model $\mathcal{M}' = \langle Tree, \leq, Choice, Act, R, \mathcal{E}', V \rangle$ such that:

$$\mathcal{E}'(m, t) = \begin{cases} 
\mathcal{E}(m, t), & \text{if } t \neq x; \\
\emptyset, & \text{if } t = x.
\end{cases}$$

It is straightforward to verify that $\mathcal{M}'$ is again a normal jstit model, and we obviously have $\mathcal{M}', m \not\models x:A$ for every $m \in Tree$. Therefore, $x:A$ is not valid, and, by Theorem 1, cannot be provable in our system.
(Part 4). We chain the implications as follows:

\[(\text{Prove}(j, A) \land \neg \Box \text{Prove}(j, A)) \rightarrow (\square [\text{Prove}(j, A) \land \neg \Box \text{Prove}(j, A)) \quad (\text{by A}_1 \text{ and A}_9)\\
\rightarrow (\square [\text{Prove}(j, A) \land [\neg \Box \text{Prove}(j, A)]) \quad (\text{by A}_2)\\
\rightarrow [\text{Prove}(j, A) \land \neg \Box \text{Prove}(j, A)] \quad (\text{by A}_1)\]

\[\square\]

(Part 5). By S5 properties of $\Box$ and S4 properties of $K$, we clearly have $\vdash \Box K \Box A \rightarrow \Box KA$.

Part 5 follows then by (A8) and transitivity of implication.

4 The canonical model

The main aim of the present section is to prove the inverse of Corollary 1. The method used is a variant of the canonical model technique, but, due to the complexity of the case, we do not define our model in one full sweep. Rather, we proceed piecewise, defining elements of the model one by one, and checking the relevant constraints as soon, as we have got enough parts of the model in place. The last subsection proves the truth lemma for the defined model.

Throughout this section we fix an $X \subseteq PVar$ such that $PVar \setminus X$ is countably infinite. We then present the set of proof variables in the following form:

\[PVar = X \cup Y \cup W \cup U \cup \{z\},\]

where:

\[Y := \{y_{(i,A)} \mid i \in Ag, A \in Form_X\},\]
\[W := \{w_A \mid A \in Form_X\},\]
\[U := \{u_A \mid A \in Form_X\}.\]

Since $Form$ is countably infinite and $Ag$ is finite, this presentation of $PVar$ is well-defined. Also throughout this section we will use $M = \langle Tree, \leq, Choice, Act, R, E, V \rangle$ as a fixed notation for our canonical model.

The ultimate building blocks of $M$ we will call elements. Before going on with the definition of $M$, we define what these elements are and explore some of their properties.

**Definition 1.** An element is a sequence of the form $(\Gamma_1, \ldots, \Gamma_n)\alpha$ for some natural $n \geq 1$ such that:

- $\alpha \in \{\uparrow, \downarrow\};$
- For every $i \leq n$, $\Gamma_i$ is maxiconsistent in $X$;
- For every $i < n$, for all $A \in Form_X$, if $KA \in \Gamma_i$, then $KA \in \Gamma_{i+1}$;
- For every $i$ such that $1 < i \leq n$, for all $j \in Ag$ and $A \in Form_X$, if $\text{Prove}(j, A) \in \Gamma_1$, then $\text{Proven}(A) \in \Gamma_i$;

---

3More precisely, we divide $PVar \setminus X$ into three countably infinite subsets plus a single proof variable which we will denote $z$. For the first of these three subsets (denoted $Y$) we fix a bijection onto the Cartesian product of $Ag$ and $Form$, for the other two (denoted $W$ and $U$) we fix bijections onto $Form$. 

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\begin{itemize}
  \item For every $i$ such that $1 < i \leq n$, for all $A \in \text{Form}_X$, it is true that $K \square \neg \text{Prove}(j, A) \in \Gamma_i$.
\end{itemize}

In other words, elements are sequences of subsets of $\text{Form}_X$ of a rather special kind, which are signed by either $\downarrow$ or $\uparrow$. The (purely technical) reason for including these arrows in the structure of elements is that one normally needs at least two copies of one element in order to get the truth conditions for formulas like $\square \text{Prove}(j, A)$ right. Both $\downarrow$ or $\uparrow$ mainly become relevant after we define $\text{Act}$ and for most other purposes they can be more or less overlooked.

We prove the following lemma:

\textbf{Lemma 4.} Whenever $(\Gamma_1, \ldots, \Gamma_n)\alpha$ is an element, then, for some $\Gamma_{n+1} \subseteq \text{Form}_X$, the sequence $(\Gamma_1, \ldots, \Gamma_{n+1})\alpha$ is also an element.

\textbf{Proof.} Assume $(\Gamma_1, \ldots, \Gamma_n)\alpha$ is an element. We have two cases to consider:

\begin{enumerate}
  \item \textbf{Case 1.} $n = 1$. Then consider the set:

  \[ \Delta := \{ KA \mid KA \in \Gamma_1 \} \cup \{ \text{Proven}(A) \mid (\exists j \in \text{Ag})(\text{Prove}(j, A) \in \Gamma_1) \} \cup \{ K \square \neg \text{Prove}(j, A) \mid A \in \text{Form} \}. \]

  We show that $\Delta$ is consistent. Of course, the set $\{ KA \mid KA \in \Gamma_1 \}$ is consistent since it is a subset of $\Gamma_1$ and the latter is assumed to be consistent.

  Further, if the set

  \[ \Delta' := \{ KA \mid KA \in \Gamma_1 \} \cup \{ \text{Proven}(A) \mid (\exists j \in \text{Ag})(\text{Prove}(j, A) \in \Gamma_1) \} \]

  is inconsistent, this would mean, wlog, that for some $B_1, \ldots, B_k, C_1, \ldots, C_l \in \text{Form}$ and $j_1, \ldots, j_l \in \text{Ag}$ such that $KB_1, \ldots, KB_k$ and $\text{Prove}(j_1, C_1), \ldots, \text{Prove}(j_l, C_l)$ are in $\Gamma_1$, we have that:

  \[ \vdash (KB_1 \land \ldots \land KB_k) \rightarrow (\neg \text{Proven}(C_1) \lor \ldots \lor \neg \text{Proven}(C_l)), \]

  whence, by \[A7]\):

  \[ \vdash K(B_1 \land \ldots \land B_k) \rightarrow (\neg \text{Proven}(C_1) \lor \ldots \lor \neg \text{Proven}(C_l)), \]

  and further, by \[K4]\):

  \[ \vdash K(B_1 \land \ldots \land B_k) \rightarrow (\bigwedge_{j \in \text{Ag}} \neg \text{Prove}(j, C_1) \lor \ldots \lor \bigwedge_{j \in \text{Ag}} \neg \text{Prove}(j, C_l)). \]

  Since the latter formula is in $X$ it is, of course, in $\Gamma_1$ by its maxiconsistency in $X$. Also, given Lemma \[2\] $K(B_1 \land \ldots \land B_k)$ is in $\Gamma_1$ by the fact that $KB_1, \ldots, KB_k \in \Gamma_1$, \[A7]\), and the fact that $\Gamma_1$ is maxiconsistent in $X$. Therefore, we get:

  \[ \bigwedge_{j \in \text{Ag}} \neg \text{Prove}(j, C_1) \lor \ldots \lor \bigwedge_{j \in \text{Ag}} \neg \text{Prove}(j, C_l) \in \Gamma_1 \]

  by Lemma \[2\]4. By Lemma \[2\]3, we further get that for some $r$ such that $1 \leq r \leq l$ all of the formulas $\neg \text{Prove}(j, C_j)$, where $j \in \text{Ag}$ are in $\Gamma_1$. By the choice of $C_1, \ldots, C_l$ this makes $\Gamma_1$ inconsistent and we get a contradiction, which shows that $\Delta'$ is consistent.
Assume, further, that $\Delta$ is inconsistent. In view of consistency of $\Delta'$ this will mean that for some $KB_1, \ldots, KB_k$ in $\Gamma_1$, and some $Proven(C_1), \ldots, Proven(C_l)$ from $\Delta' \setminus \Gamma_1$ and some $Prove(j_1, D_1), \ldots, Prove(j_r, D_r) \in Form$, we will have:

$$\vdash (KB_1 \land \ldots \land KB_k \land Proven(C_1) \land \ldots \land Proven(C_l)) \rightarrow (\langle K \rangle \Diamond Prove(j_1, D_1) \lor \ldots \lor \langle K \rangle \Diamond Prove(j_r, D_r)).$$

From the latter validity, by (R2) and (A7) we get that:

$$\vdash K(KB_1 \land \ldots \land KB_k \land Proven(C_1) \land \ldots \land Proven(C_l)) \rightarrow K(\langle K \rangle \Diamond Prove(j_1, D_1) \lor \ldots \lor \langle K \rangle \Diamond Prove(j_r, D_r)),$$

whence, by (A12), we obtain:

$$\vdash K(KB_1 \land \ldots \land KB_k \land Proven(C_1) \land \ldots \land Proven(C_l)) \rightarrow \bot,$$

and, by (A7), and (A11) we further obtain:

$$\vdash (KB_1 \land \ldots \land KB_k \land Proven(C_1) \land \ldots \land Proven(C_l)) \rightarrow \bot,$$

showing that $\Delta'$ must be consistent. This makes up a contradiction showing that $\Delta$ must be consistent.

Since $\Delta$ is shown to be inconsistent, then, by Lemma 2.1, it is also extendable to a set $\Gamma_2$ which is maxiconsistent in $X$. By the choice of $\Delta$, this means that $(\Gamma_1, \Gamma_2)\alpha$ must be an element.

**Case 2.** $n > 1$. Then it is easy to see that $(\Gamma_1, \ldots, \Gamma_n, \Gamma_n)\alpha$ is an element. 

The structure of elements will be important in what follows. If $\xi = (\Gamma_1, \ldots, \Gamma_n)\alpha$ is an element, then its initial segment is any element $\tau$ of the form $(\Gamma_1, \ldots, \Gamma_k)\alpha$ with $k \leq n$. If, moreover, $k < n$, then $\tau$ is a proper initial segment of $\xi$, and if $k = n - 1$, then $\tau$ is the greatest proper initial segment of $\xi$. Moreover, we define $n$ to be the length of $\xi$. Thus, any element of length 1 has no proper initial segments. Furthermore, we define that $\Gamma_n$ is the end element of $\xi$ and write $\Gamma_n = end(\xi)$.

We now define the following relation $\equiv$ between elements of equal length. For the elements of length 1 we set:

$$(\Gamma)\alpha \equiv (\Delta)\beta \iff (\forall A \in Form_X)(\Box A \in \Gamma \Rightarrow A \in \Delta);$$

and for the elements of greater length we set:

$$(\Gamma_1, \ldots, \Gamma_n, \Gamma_{n+1})\alpha \equiv (\Delta_1, \ldots, \Delta_n, \Delta_{n+1})\beta \iff$$

$$\iff (\Gamma_1 = \Delta_1 \land \ldots \land \Gamma_n = \Delta_n \land \alpha = \beta \land (\Gamma_{n+1})\alpha \equiv (\Delta_{n+1})\beta).$$

It is routine to check that $\equiv$ is an equivalence relation given that $\Box$ is an S5 modality. We will denote the equivalence class of element $(\Gamma_1, \ldots, \Gamma_n)\alpha$ generated by $\equiv$ by $[(\Gamma_1, \ldots, \Gamma_n)\alpha]_{\equiv}$. Since all the elements inside a given $\equiv$-equivalence class are of the same length, we may extend the notion of length to these classes setting that the length of $[(\Gamma_1, \ldots, \Gamma_n)\alpha]_{\equiv}$ also equals $n$.

We now proceed to definitions of components for the canonical model.
4.1 *Tree*, ≤, and Hist

The first two components of the canonical model $M$ are as follows:

- *Tree* is the set of $\equiv$-equivalence classes of elements plus $\dagger$ and $\ddagger$ as additional moments;

- We set that both $\dagger < m$ and $m \not< \dagger$ for every $m \in \text{Tree} \setminus \{\dagger\}$. We further set that $\ddagger$ is only $<$-comparable to $\dagger$ (in which case we already have $\dagger < \ddagger$), and for any two $\equiv$-equivalence classes of elements $m$ and $m'$, we have that $m < m'$ iff there is an element $\xi \in m$ such that $\xi$ is a proper initial segment of every element $\tau \in m'$.

The relation $\leq$ is then defined as the reflexive companion to $<$. Before we move on to the choice- and justifications-related components, let us pause to check that the restraints imposed by our semantics on *Tree* and $\leq$ are satisfied:

**Lemma 5.** The relation $\leq$, as defined above, is a partial order on *Tree*, which satisfies both historical connection and no backward branching constraints. Moreover, every element in *Tree*, except for $\dagger$, has at least one immediate $<$-successor.

**Proof.** Transitivity and reflexivity of $\leq$ are obvious. As for antisymmetry, assume that we have both $m < m'$ and $m' < m$. Then $m$ and $m'$ must be equivalence classes of elements. Let $\xi \in m$ be a proper initial segment of every element in $m'$ and let $\tau \in m'$ be a proper initial segment of every element in $m$. It follows that $\xi$ is a proper initial segment of $\tau$ and also $\tau$ is a proper initial segment of $\xi$, a contradiction.

Historical connection is satisfied since $\dagger$ is the $\leq$-least element of *Tree*. Let us prove the absence of backward branching. Assume that we have both $m \leq m''$ and $m' \leq m''$ but neither $m \leq m'$ nor $m' \leq m$ holds. This means that all the three moments are pairwise different and none of them is either $\dagger$ or $\ddagger$, otherwise our assumptions about them would be immediately falsified. Therefore, all the three moments are some equivalence classes of elements and we also have $m \neq m'$, $m < m''$ and $m' < m''$. So let $\xi \in m$ and $\tau \in m'$ be such that both $\xi$ and $\tau$ are proper initial segments of every element in $m''$. Then, since $m \neq m'$, $\xi$ and $\tau$ must be different, hence either $\xi$ must be a proper initial segment of $\tau$ or $\tau$ must be a proper initial segment of $\xi$. Assume, wlog, that $\xi$ is a proper initial segment of $\tau$. Then $\xi$ is included into the greatest proper initial segment of $\tau$. Let $\tau'$ be any element in $m'$. It follows from the definition of $\equiv$ that all the elements within $m'$ share the same greatest proper initial segment, therefore $\xi$ must be a proper initial segment of $\tau'$ as well. It follows that $m < m'$, contrary to our assumptions.

Consider the $<$-successors of a given $m \in \text{Tree}$. If $m \neq \dagger$, then either $m = \dagger$, or $m$ is an equivalence class of elements. If $m = \dagger$, then take any $\Gamma \subseteq \text{Form}_X$ which is maxiconsistent in $X$ and any $\alpha \in \{\dagger, \ddagger\}$. Then $(\Gamma)\alpha$ is an element and we have $\dagger < [(\Gamma)\alpha]_{\equiv}$. Moreover, no other moment can be in between them: this cannot be either $\ddagger$, or $\dagger$, or an equivalence class of elements (since the greatest proper initial segment of $(\Gamma)\alpha$ is empty). If, on the other hand, $m$ is an equivalence class of elements, then assume that $m = [(\Gamma_1, \ldots, \Gamma_n)\alpha]_{\equiv}$. By Lemma A we know that for some $\Delta \subseteq \text{Form}$, the tuple $(\Gamma_1, \ldots, \Gamma_n, \Delta)\alpha$ must be an element. But then we must have

$$[(\Gamma_1, \ldots, \Gamma_n)\alpha]_{\equiv} < [(\Gamma_1, \ldots, \Gamma_n, \Delta)\alpha]_{\equiv},$$

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and again, no moments are strictly in between them since \((\Gamma_1, \ldots, \Gamma_n)\alpha\) is the greatest proper initial segment of \((\Gamma_1, \ldots, \Gamma_n, \Delta)\alpha\).

Moreover, it is easy to see that if \(m, m' \in \text{Tree}\) are two equivalence classes of elements, and \(m < m'\), then the length of \(m\) is less than the length of \(m'\), and if \(m'\) is an immediate \(<\)-successor of \(m\), then length of \(m\) is the length of \(m'\) minus one.

Before we move on, let us have a quick look into the structure of histories induced by \(\text{Tree}\) and \(\leq\). Lemma \[\text{Lemma 5}\] shows that we must have the history \((†, ‡)\) plus a bunch of infinite histories of the form \((†, m_1, \ldots, m_n, \ldots)\), ordered in the type of \(ω\), where, for \(n \geq 1\), \(m_n\) is an equivalence class of elements of length \(n\) and every next element is the immediate \(<\)-successor of the previous one. Every such infinite history we can also represent in the form \((†, ξ_1, \ldots, ξ_n, \ldots)\) such that for every \(n \geq 1\):

- \(ξ_n \in m_n\);
- \(ξ_n\) is the greatest proper initial segment of \(ξ_{n+1}\).

Moreover, one can show that such a representation, for a given history of the form \((†, m_1, \ldots, m_n, \ldots)\), is unique. Indeed, suppose that \((†, ξ_1, \ldots, ξ_n, \ldots)\) and \((†, ξ'_1, \ldots, ξ'_n, \ldots)\) are two different representations for \((†, m_1, \ldots, m_n, \ldots)\). Then let \(i \geq 1\) be the first natural number such that \(ξ_i \neq ξ'_i\). Consider \(m_{i+1}\). We have \(ξ_{i+1}, ξ'_{i+1} \in m_{i+1}\) so that \(ξ_{i+1} ≡ ξ'_{i+1}\). Since \(ξ_i\) and \(ξ'_i\) are the greatest proper initial segments of \(ξ_{i+1}\) and \(ξ'_{i+1}\), respectively, the greatest proper initial segments of \(ξ_{i+1}\) and \(ξ'_{i+1}\) are non-empty and, by \(ξ_{i+1} ≡ ξ'_{i+1}\), must coincide, which cannot be the case since \(ξ_i \neq ξ'_i\).

Therefore, if \(h = (†, m_1, \ldots, m_n, \ldots)\) is a history in \(\text{Tree}\) and \((†, ξ_1, \ldots, ξ_n, \ldots)\) is the unique representation of \(h\) as a sequence of elements, we define \(m_n \cap h\) to be \(ξ_n\).

All the above statements admit of an inversion. Not only can every history be uniquely represented as a sequence of elements, but also every sequence of elements of an appropriate form represents a unique history in \(M\). Not only is every intersection of an equivalence class of elements and a history an element in this class, but also, conversely, every element defines the intersection of at least one history with the equivalence class induced by this element. More precisely:

Lemma 6. The following statements are true:

1. Fix a sequence \((†, ξ_1, \ldots, ξ_n, \ldots)\) where all of \(ξ_1, \ldots, ξ_n, \ldots\) are elements and, for every natural \(n\), \(ξ_n\) is the greatest proper initial segment of \(ξ_{n+1}\). Then there is a unique history \(h = (†, m_1, \ldots, m_n, \ldots)\) in \(M\) such that for all natural \(n\) it is true that \(ξ_n \in m_n\) (thus \(ξ_n = m_n \cap h\)).

2. Let \(ξ\) be an element. Then there is at least one history \(h \in H_{[\xi]}=\) such that \([ξ]≡ \cap h = ξ\).

Proof. As for Part 1, consider \((†, [ξ_1]≡, \ldots, [ξ_n]≡, \ldots)\); it is obviously a history in \(M\) and we also have \(ξ_n \in [ξ_n]≡\) for all natural \(n\).

As for Part 2, we have to consider two cases.
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Case 1. The length of $\xi$ equals 1, so that $\xi = (\Gamma)\alpha$ for appropriate $\Gamma$ and $\alpha$. Then we know, by the proof of Lemma 4 above, that for some $\Delta \subseteq Form_X$ the sequence:

\[
\begin{align*}
\xi_1 := & (\Gamma)\alpha; \\
\xi_2 := & (\Gamma, \Delta)\alpha; \\
\vdots \\
\xi_{n+1} := & (\Gamma, \Delta, \ldots, \Delta)\alpha; \\
\vdots 
\end{align*}
\]

is a sequence of elements in which every element is the greatest proper initial segment of the next one. Therefore, by Part 1, there must be a history $h$ in $M$ such that $h = (\hat{1}, [\xi_1] \equiv, \ldots, [\xi_n] \equiv, \ldots)$ and also $\xi = \xi_1 = [\xi_1] \equiv \cap h$.

Case 2. The length of $\xi$ is greater than 1, so that $\xi = (\Gamma_1, \ldots, \Gamma_n)\alpha$ for appropriate $n > 1$, $\Gamma_1, \ldots, \Gamma_n$ and $\alpha$. Then we define the following sequence of elements:

\[
\begin{align*}
\xi_1 := & (\Gamma_1)\alpha; \\
\xi_2 := & (\Gamma_1, \Gamma_2)\alpha; \\
\vdots \\
\xi_n := & (\Gamma_1, \ldots, \Gamma_n)\alpha; \\
\xi_{n+1} := & (\Gamma_1, \ldots, \Gamma_n, \Gamma_n)\alpha; \\
\vdots \\
\xi_{n+k} := & (\Gamma_1, \ldots, \Gamma_n, \Gamma_n, \ldots, \Gamma_n)\alpha; \\
\vdots 
\end{align*}
\]

Again, it is easy to see that every element in this sequence is the greatest proper initial segment of the next one, so that, arguing as in the previous case, we get that $h = (\hat{1}, [\xi_1] \equiv, \ldots, [\xi_n] \equiv, \ldots, [\xi_{n+k}] \equiv, \ldots)$ is a history in $M$ and $\xi = \xi_n = [\xi_n] \equiv \cap h$. □

4.2 Choice

We now define the choice structures of our canonical model:

- $Choice^m_j = H_m$, if $m \in \{\hat{t}, \hat{\emptyset}\}$,

- $Choice^m_j(h) = \{h' \mid h' \in H_m, (\forall A \in Form)([j]A \in end(h \cap m) \Rightarrow A \in end(h' \cap m))\}$, if $m$ is an equivalence class of elements.

Since for every $j \in Ag$, $[j]$ is an S5-modality, $Choice$ induces a partition on $H_m$ for every given $m \in Tree$. We check that the choice function verifies the relevant semantic constraints:

Lemma 7. The tuple $(Tree, \leq, Choice)$, as defined above, verifies both the independence of agents and the no choice between undivided histories constraints.
Proof. We first tackle no choice between undivided histories. Consider a moment \( m \) and two histories \( h, h' \in H_m \) such that \( h \) and \( h' \) are undivided at \( m \). Since the agents’ choices are only non-vacuous at moments represented by equivalence classes of elements, we may safely assume that \( m \) is such a class. Since \( h \) and \( h' \) are undivided at \( m \), this means that there is a moment \( m' \) such that \( m < m' \) and \( m' \) is shared by \( h \) and \( h' \).

Hence we know that also \( m' \) is some equivalence class of elements. Suppose the length of \( m \) is \( n \) and the length of \( m' \) is \( n' \). Then \( n < n' \), also \( h \cap m \) is the initial segment of length \( n \) of \( h \cap m' \), and similarly, \( h' \cap m \) is the initial segment of length \( n \) of \( h' \cap m' \). But both \( h \cap m' \) and \( h' \cap m' \) are, by definition, in \( m' \), therefore, they must share the greatest proper initial segment. Hence, their initial segments of length \( n \) must coincide as well, and we must have \( h \cap m = h' \cap m \), whence \( \text{end}(h \cap m) = \text{end}(h' \cap m) \). Now, if \( j \in Ag \) and \( [j]A \in \text{end}(h \cap m) \), then, by \( \text{(A3)} \) and maxiformcompleteness of \( \text{end}(h \cap m) \) in \( X \), we will have also \( A \in \text{end}(h \cap m) = \text{end}(h' \cap m) \), and thus \( h' \in \text{Choice}^m_j(h) \), so that \( \text{Choice}^m_j(h) = \text{Choice}_j^m(h') \) since \( \text{Choice} \) is a partition of \( H_m \).

Consider, next, the independence of agents. Let \( m \in \text{Tree} \) and let \( f \) be a function on \( Ag \) such that \( \forall j \in Ag(f(j) \in \text{Choice}^m_j) \). We are going to show that in this case \( \bigcap_{j \in Ag} f(j) \neq \emptyset \). If \( m \in \{ \sharp, \sharp' \} \), then this is obvious, since every agent will have a vacuous choice. We treat the case when \( m \) is an equivalence class of elements. Assume that \( m = [\Gamma_1, \ldots, \Gamma_{n+1}]_\equiv \). We have two cases to consider:

Case 1. \( n = 0 \). By \( \text{(A1)} \) we know that there is a set \( \Delta \) of formulas of the form \( \Box A \) which is shared by all sets of the form \( \text{end}(\xi) \) with \( \xi \in m \) in the sense that if \( \xi \in m \), then \( \Box A \in \text{end}(\xi) \) iff \( \Box A \in \Delta \). By the same axiom scheme and Lemma \( \text{(A2)} \), we also know that for every \( j \in Ag \) there is set \( \Delta_j \) of formulas of the form \( [j]A \) which is shared by all sets of the form \( \text{end}(\xi) \) such that \( \exists h \in f(j) \land \xi = m \cap h \). More precisely:

\[
\xi \in m \Rightarrow (\exists h \in f(j) \land \xi = m \cap h) \Leftrightarrow (\forall A \in \text{Form})([j]A \in \text{end}(\xi) \Leftrightarrow [j]A \in \Delta_j)).
\]

We now consider the set \( \Delta \cup \bigcup \{ \Delta_j \mid j \in Ag \} \) and show its consistency. Indeed, if this set is inconsistent, then, wlog, we would have a provable formula of the following form:

\[
\vdash (\Box A \land \bigwedge_{j \in Ag} [j]A_j) \rightarrow \bot.
\] (1)

But then, for every \( j \in Ag \) an element \( \xi_j \in m \) such that \( (\forall A \in \text{Form})([j]A \in \text{end}(\xi_j) \Leftrightarrow [j]A \in \Delta_j) \). This is possible, since we may simply choose an arbitrary \( h_j \in f(j) \) and set \( \xi_j := m \cap h_j \). Then we will have \( [j]A_j \in \xi_j \) for every \( j \in Ag \). Next, consider \( \Gamma_1 \). Since \( m = [\Gamma_1]_\equiv \) and \( \Box \) is an \( S5 \)-modality, we must have:

\[
\{ \Diamond [j]A_j \in Ag \} \subseteq \Gamma_1,
\]

whence, by Lemma \( \text{(A5)} \):

\[
\bigwedge_{j \in Ag} \Diamond [j]A_j \in \Gamma_1,
\]

and further, by \( \text{(A3)} \) and Lemma \( \text{(A2)} \):

\[
\Diamond \bigwedge_{j \in Ag} [j]A_j \in \Gamma_1.
\]
Also, by definition of $\Delta$ and the fact that $m = \lbrack (\Gamma_1)\alpha \rbrack_\Xi$, we get successively:

$$\Box A \in \Gamma_1,$$

then, by Lemma 2.5:

$$\Box A \land \bigwedge_{j \in Ag} [j]A_j \in \Gamma_1,$$

and finally, by the fact that $\Box$ is an S5-modality:

$$\bigwedge_{j \in Ag} [j]A_j \in \Gamma_1. \quad (2)$$

From (1), together with (2), it follows by S5 reasoning for $\Box$ that $\Box \bot \in \Gamma_1$, so that, again by S5 properties of $\Box$ and Lemma 2.4, it follows that $\bot \in \Gamma_1$, which is in contradiction with maxiconsistency of $\Gamma_1$ in $X$.

Hence $\Delta \cup \bigcup \{\Delta_j \mid j \in Ag\}$ is consistent, and since it is in $X$, we can extend it to a set $\Xi$ which is maxiconsistent in $X$. We now consider $(\Xi)\alpha$ which is obviously an element, and since, moreover $\Delta \subseteq \Xi$, then also $(\Xi)\alpha \in m$. By Lemma 0.2, we can choose a history $g$ such that $(\Xi)\alpha = g \cap n$. We also know that for every $j \in Ag$, there is a history $h_j \in f(j)$ such that $h_j \cap m = \xi_j$ by the choice of $\xi_j$. Therefore, for every $j \in Ag$,

$\text{Choice}_m^n(h_j) = f(j)$. Also, if $[j]A \in \text{end}(\xi_j) = \text{end}(h_j \cap m)$, then $[j]A \in \Delta_j$, hence $[j]A \in \Xi = \text{end}(g \cap m)$, therefore, by (A1), $A \in \text{end}(g \cap m)$. Thus we get that $g \cap \bigcap_{j \in Ag} \text{Choice}_m^n(h_j) = \bigcap_{j \in Ag} f_j$ so that the independence of agents is verified.

Case 2. $n > 0$. For the most part, we can re-use our reasoning from Case 1. We again form the sets $\Delta_j, \{\Delta_j \mid j \in Ag\}$ and $\Xi$, and consider element $(\Gamma_1, \ldots, \Gamma_n, \Xi)\alpha \in m$. We then choose a history $g \in H_m$ for which we have $(\Gamma_1, \ldots, \Gamma_n, \Xi)\alpha = m \cap g$ and show that $g \cap \bigcap_{j \in Ag} \text{Choice}_m^n(h_j) = \bigcap_{j \in Ag} f_j$.

The only new ingredient is that now seeing that $(\Gamma_1, \ldots, \Gamma_n, \Xi)\alpha$ is in fact an element is much less trivial and has to be argued separately. We show this as follows. If $KA \in \Gamma_n$, then $KA \in \Gamma_{n+1}$ by definition of an element. But then $\Box KA \in \Gamma_{n+1}$ by Lemma 3.5 and maxiconsistency of $\Gamma_{n+1}$ in $X$, whence $\Box KA \in \Delta$ and, therefore, $\Box KA \in \Xi$. By (A1) and maxiconsistency of $\Xi$ we get then $KA \in \Xi$. Similarly, if $\text{Prove}(j, A) \in \Gamma_1$, then $\text{Proven}(A) \in \Gamma_{n+1}$ by definition of an element. But then $\Box \text{Proven}(A) \in \Gamma_{n+1}$, by Lemma 3.2 and maxiconsistency of $\Gamma_{n+1}$ in $X$, whence $\Box \text{Proven}(A) \in \Delta$ and, therefore, $\Box \text{Proven}(A) \in \Xi$. By (A1) and maxiconsistency of $\Xi$ in $X$, we get then $\text{Proven}(A) \in \Xi$. Finally, if $A \in \text{Form}_X$ and $j \in Ag$, then, since $n > 0$, we must have $K \Box \neg \text{Prove}(j, A) \in \Gamma_{n+1}$ by definition of an element, whence $\Box K \Box \neg \text{Prove}(j, A) \in \Gamma_{n+1}$ by (A3) and maxiconsistency of $\Gamma_{n+1}$ in $X$, so that $\Box K \Box \neg \text{Prove}(j, A) \in \Delta$ and, further, $K \Box \neg \text{Prove}(j, A) \in \Xi$. By (A1) and maxiconsistency of $\Xi$ in $X$, we get then that $K \Box \neg \text{Prove}(j, A) \in \Xi$. Thus $(\Gamma_1, \ldots, \Gamma_n, \Xi)\alpha$ is an element, and the rest is shown exactly as in Case 1.

4.3 $R$ and $E$

We now define the justifications-related components of our canonical model. We first define $R$ as follows:

- $R(((\Gamma)\alpha)_{\Xi, m'}) \iff (m' \in \text{Tree} \setminus \{\perp, \top\}) \&$
  - $\& (\forall \tau \in m')(\forall A \in \text{Form}_X)(KA \in \Gamma \Rightarrow KA \in \text{end}(\tau));$
If \( n > 1 \), then
\[
R(([\Gamma_1, \ldots, \Gamma_n]a)\equiv, m') \iff \\
(\exists \Delta_1, \ldots, \Delta_k \subseteq Form_X)(k > 0 \& m' = ([\Gamma_1, \Delta_1, \ldots, \Delta_k]a)\equiv \& \forall \xi \in Form_X)(KA \in \Gamma_n \Rightarrow KA \in \Delta_k)\);
\]
• \( R(\xi, m) \), for all \( m \in Tree \);
• \( R(\xi, m) \iff m = \xi \).

Now, for the definition of \( \mathcal{E} \):
• For all \( t \in Pol \): \( \mathcal{E}(\xi, t) = \mathcal{E}(\xi, t) = \{ A \in Form \mid \vdash t : A \} \);
• For all \( t \in Pol_x \) and \( m \in Tree \setminus \{ \xi, \xi \} \):
\[
(\forall A \in Form)(A \in \mathcal{E}(m, t) \iff \\
(\exists t_1 : B_1 \ldots t_n : B_n)(\forall \xi \in m)(t_1 : B_1, \ldots, t_n : B_n \in end(\xi)) \& \\
\& \vdash (t_1 : B_1 \land \ldots \land t_n : B_n) \rightarrow t : A)\);
\]
• \( (\forall A \in Form)(A \in \mathcal{E}(m, z) \iff (\forall \xi \in m)(Proven(A) \in end(\xi))) \), for all \( m \in Tree \setminus \{ \xi, \xi \} \);
• \( \mathcal{E}(m, t) = Form \), if \( m \in Tree \setminus \{ \xi, \xi \} \) and \( t \in Pol \setminus (Pol_x \cup Y \cup W \cup U \cup \{ z \}) \).

We start by mentioning a straightforward corollary to the above definition:

**Lemma 8.** For all \( m \in Tree \) and \( t \in Pol \) it is true that \( \{ A \in Form \mid \vdash t : A \} \subseteq \mathcal{E}(m, t) \).

**Proof.** This holds simply by the definition of \( \mathcal{E} \) when \( m \in \{ \xi, \xi \} \). If \( m \in Tree \setminus \{ \xi, \xi \} \), then we have another obvious case for \( t \in Pol \setminus (Pol_x \cup Y \cup W \cup U \cup \{ z \}) \).

If \( t \in Pol_x \), and \( \vdash t : A \), then we have just a border case in the definition of \( \mathcal{E}(m, t) \), with \( t : A \) following from the empty conjunction of elements present in \( end(\xi) \) for every \( \xi \in m \).

Finally, if \( t \in Y \cup W \cup U \cup \{ z \} \), then \( t \in PVar \), therefore, by Lemma 3, we must have:
\[
\{ A \in Form \mid \vdash t : A \} = 0 \subseteq \mathcal{E}(m, t).
\]

Note that since we know that for every \( c \in PConst \) and every instance \( A \) of one of the axiom schemes in the list \( [A10, A13] \), it is true that \( \vdash c : A \) (by \( [K3] \)), it follows, among other things, that the above-defined function \( \mathcal{E} \) satisfies the additional normality condition on jst models.

It is straightforward to check that \( R \), as defined above, is a preorder on \( Tree \), using \( [A7] \) and \( [AN] \). Let us briefly look into why the future always matters constraint is verified as well. Assume \( m \in Tree \). If \( m = \xi \), then it is connected to all the elements in \( Tree \) by both \( \leq \) and \( R \), and if \( m = \xi \), then it is connected only to itself by both \( \leq \) and \( R \), so these moments cannot falsify the constraint. So let us assume that \( m \) is a class of
equivalence generated by some element, say \( m = [(\Gamma_1, \ldots, \Gamma_n)\alpha]_\equiv \). If \( m \leq m' \), then \( m' \) must be an equivalence class as well, and \((\Gamma_1, \ldots, \Gamma_n)\alpha \) must be an initial segment of every element in \( m' \), so that we may assume, wlog, that \( m' = [(\Gamma_1, \ldots, \Gamma_k)\alpha]_\equiv \) for some \( k \geq n \). In particular, if \( n > 1 \), then \( k - 1 > 0 \). But then take an arbitrary \( A \in \text{Form}_X \).

If \( KA \in \Gamma_n \), then, since \((\Gamma_1, \ldots, \Gamma_k)\alpha \) is an element, \( KA \in \Gamma_k \). By Lemma 5 and maxiconsistency of \( \Gamma_k \) in X we must have then \( \Box KA \in \Gamma_k \). Now, by definition of \( \equiv \), we get \( KA \in \text{end}(\tau) \) for any given \( \tau \in m' \). It follows then that \( R(m, m') \) as desired.

We further check that the semantical constraints for \( \mathcal{E} \) are verified:

**Lemma 9.** The function \( \mathcal{E} \), as defined above, satisfies both the monotonicity of evidence and the evidence closure properties.

**Proof.** We start with the monotonicity of evidence. Assume \( R(m, m') \) and \( t \in \text{Pol} \). If \( m \in \{1, \sharp\} \) then, by Lemma 3 \( \mathcal{E}(m, t) = \{ A \in \text{Form} \mid \vdash t \vdash A \} \subseteq \mathcal{E}(m', t) \) for any \( m' \in \text{Tree} \).

Assume, further, that \( m \) is an equivalence class of elements. Then, since we have \( R(m, m') \), \( m' \) must be an equivalence class of elements as well. Also, we are done if \( m = m' \). On the other hand, if \( m \neq m' \), then consider \( t \). If \( t \in \text{Pol} \setminus (Pol_X \cup \{z\}) \), then we must have \( \mathcal{E}(m, t) = \mathcal{E}(m', t) \) by definition, since \( m, m' \in \text{Tree} \setminus \{1, \sharp\} \).

If \( t = z \), then take an arbitrary \( A \in \mathcal{E}(m, z) \). By the above definition of \( \mathcal{E} \), this means that \( \text{Proven}(A) \in \text{end}(\xi) \) for every element \( \xi \) of \( m \). By maxiconsistency of \( \text{end}(\xi) \) in \( X \) and \([A1]\), this further means that \( \text{Proven}(A) \in \text{end}(\xi) \) for every element \( \xi \) of \( m \). Therefore, by \( R(m, m') \), and the fact that \( m, m' \in \text{Tree} \setminus \{1, \sharp\} \), we get that \( \text{Proven}(A) \in \text{end}(\sigma) \) for every element \( \sigma \) of \( m' \), whence, by \([A7]\), it follows that \( \text{Proven}(A) \in \text{end}(\sigma) \) for every element \( \sigma \) of \( m' \). Therefore \( A \in \mathcal{E}(m', z) \). Since \( A \) was arbitrary, this means that \( \mathcal{E}(m, z) \subseteq \mathcal{E}(m', z) \), as desired.

Finally, assume that \( t \in \text{Pol}_X \) and take an arbitrary \( A \in \mathcal{E}(m, t) \). Then we can choose \( t_1: B_1, \ldots, t_n: B_n \) in such a way that for all \( \xi \in m \) we have \( t_1: B_1, \ldots, t_n: B_n \in \text{end}(\xi) \), and, moreover, \( \vdash (t_1: B_1 \wedge \ldots \wedge t_n: B_n) \rightarrow \vdash t: A \). Since \( t_1: B_1, \ldots, t_n: B_n \in \text{end}(\xi) \), we know that \( (t_1: B_1, \ldots, t_n: B_n) \in \text{Form}_X \). We also know that, for every \( \xi \in m \), \( \text{end}(\xi) \) is maxiconsistent in \( X \). Therefore, using Lemma 2 we obtain, successively:

\[
(\forall \xi \in m)(Kt_1: B_1, \ldots, Kt_n: B_n \in \text{end}(\xi)) \quad \text{by Lemma 3 1}
\]
\[
(\forall \xi \in m)((Kt_1: B_1 \wedge \ldots \wedge Kt_n: B_n \in \text{end}(\xi)) \quad \text{by Lemma 2 5}
\]
\[
(\forall \xi \in m)(K(t_1: B_1 \wedge \ldots \wedge t_n: B_n) \in \text{end}(\xi)) \quad \text{by \([A7]\)}
\]

From the latter it follows by \( R(m, m') \) that \( K(t_1: B_1 \wedge \ldots \wedge t_n: B_n) \in \text{end}(\tau) \) for all \( \tau \in m' \). We also know that for every \( \tau \in m' \), \( \text{end}(\tau) \) is maxiconsistent in \( X \) so that, applying Lemma 2 and \([A7]\), we get that \( t_1: B_1, \ldots, t_n: B_n \in \text{end}(\tau) \) for all \( \tau \in m' \). Adding this to our initial assumption that \( \vdash (t_1: B_1 \wedge \ldots \wedge t_n: B_n) \rightarrow \vdash t: A \), we obtain that \( A \in \mathcal{E}(m', t) \).

We turn now to the closure conditions. We verify the first two conditions, and the third one can be verified in a similar way, restricting attention to \( t \) rather than considering both \( s \) and \( t \). Let \( s, t \in \text{Pol} \). We need to consider two cases:

**Case 1.** \( m \in \{1, \sharp\} \). If \( A \in \mathcal{E}(m, s) \), then \( \vdash s: A \). Therefore, by \([A6]\), we must also have \( \vdash (s + t): A \) so that \( A \in \mathcal{E}(m, s + t) \). Similarly, if \( A \in \mathcal{E}(m, t) \), then also \( A \in \mathcal{E}(m, s + t) \) and the closure constraint (b) is verified. If, on the other hand, it is true that for some \( A, B \in \text{Form} \) we have both \( A \rightarrow B \in \mathcal{E}(m, s) \) and \( A \in \mathcal{E}(m, t) \),
Lemma 10. The following statements are true:

Case 2. \( m \in Tree \setminus \{\|, \|\} \). If \( s + t, s \times t \not\in Pol_X \), then we have:

\[
E(m, s + t) = E(m, s \times t) = Form,
\]

so that all the closure conditions are verified trivially. Therefore, assume that \( s + t, s \times t \in Pol_X \). If \( A \in Form \) and \( A \in E(m, s) \), then we can choose \( t_1 : B_1, \ldots, t_n : B_n \) such that for all \( \xi \in m \) we have both \( t_1 : B_1, \ldots, t_n : B_n \in end(\xi) \) and

\[
\vdash (t_1 : B_1 \land \ldots \land t_n : B_n) \rightarrow s : A.
\]

By \([A0]\) and \([A6]\) we get then that \( \vdash (t_1 : B_1 \land \ldots \land t_n : B_n) \rightarrow s + t : A \), which means that \( A \in E(m, s + t) \). Similarly, if \( A \in E(m, t) \), then \( A \in E(m, s + t) \) as well, and closure condition (b) is verified.

On the other hand, if \( A, B \in Form \) and we have both both \( A \rightarrow B \in E(m, s) \) and \( A \in E(m, t) \), then we can choose \( t_1 : B_1, \ldots, t_n : B_n \) and also \( s_1 : C_1, \ldots, s_k : C_k \) such that for every \( \xi \in m \) we have all of the following:

\[
\begin{align*}
t_1 : B_1, \ldots, t_n : B_n, s_1 : C_1, \ldots, s_k : C_k & \in end(\xi); \\
\vdash (t_1 : B_1 \land \ldots \land t_n : B_n) \rightarrow t : A; \\
\vdash (s_1 : C_1 \land \ldots \land s_k : C_k) \rightarrow s : (A \rightarrow B);
\end{align*}
\]

It follows then by \([A0]\) and \([A4]\) that:

\[
(t_1 : B_1 \land \ldots \land t_n : B_n \land s_1 : C_1 \land \ldots \land s_k : C_k) \rightarrow s \times t : B,
\]

so that \( B \in E(m, s \times t) \) follows and closure condition (a) is verified.

\[\square\]

4.4 Act and V

It only remains to define \( Act \) and \( V \) for our canonical model, and we define them as follows:

- \((m, h) \in V(p) \iff p \in end(m \cap h)\), for all \( p \in Var \);
- \(Act(\|, (\|, \|)) = Act(\|, (\|, \|)) = \emptyset\);
- \(Act(\|, h) = \{z\}, \text{ if } h \neq (\|, \|)\);
- \(Act(m, h) = \{z\} \cup \{y(j, A) \mid Prove(j, A) \land \neg \Box Prove(j, A) \in \Gamma_1\} \cup \{u_A \mid \Box Prove(j, A) \in \Gamma_1\}, \text{ if } m \cap h = (\Gamma_1, \ldots, \Gamma_n) \uparrow;\)
- \(Act(m, h) = \{z\} \cup \{y(j, A) \mid Prove(j, A) \land \neg \Box Prove(j, A) \in \Gamma_1\} \cup \{w_A \mid \Box Prove(j, A) \in \Gamma_1\}, \text{ if } m \cap h = (\Gamma_1, \ldots, \Gamma_n) \downarrow.\)

We begin by establishing some consequences of the above definition:

**Lemma 10.** The following statements are true:
1. If $(\Gamma)\alpha$ is an element, then:

$$\bigcap_{h \in H_{[(\Gamma)\alpha]}} (Act(\{(\Gamma)\alpha\}_{\equiv}, h) = \{z\}).$$

2. If $n > 1$ and $(\Gamma_1, \ldots, \Gamma_n)\alpha$ is an element and $g \in H_{[(\Gamma_1, \ldots, \Gamma_n)\alpha]_{\equiv}}$ is arbitrary, then:

$$\bigcap_{h \in H_{[(\Gamma_1, \ldots, \Gamma_n)\alpha]_{\equiv}}} (Act(\{(\Gamma_1, \ldots, \Gamma_n)\alpha\}_{\equiv}, h) = Act(\{(\Gamma_1, \ldots, \Gamma_n)\alpha\}_{\equiv}, g).$$

Proof. (Part 1). Set $m := |(\Gamma)\alpha|_{\equiv}$. It is clear from the definition of $Act$ that $z \in \bigcap_{h \in H_m} (Act(m, h))$, so that we only need to show that $z$ is the only member in this intersection. The other elements of $Act$, according to the definition, can have one of the following forms: either $y_{(j,A)}$, or $u_A$, or $w_A$, for some $A \in Form_X$ and $j \in Ag$. We know, further, that both $(\Gamma)\alpha \equiv (\Gamma) \uparrow$ and $(\Gamma)\alpha \equiv (\Gamma) \downarrow$ then, using Lemma 2, take any $h, h' \in H_m$ for which $h \cap m = (\Gamma) \uparrow$ and $h' \cap m = (\Gamma) \downarrow$. By definition, $Act(m, h)$ is disjoint from $\{w_A \mid A \in Form_X\}$ whereas $Act(m, h')$ is disjoint from $\{u_A \mid A \in Form_X\}$; therefore, $\bigcap_{h \in H_m} (Act(m, h))$ must be disjoint from $\{w_A \mid A \in Form_X\} \cup \{u_A \mid A \in Form_X\}$. Finally, consider a variable of the form $y_{(j,A)}$ for arbitrary $A \in Form_X$ and $j \in Ag$. If $y_{(j,A)} \in \bigcap_{h \in H_m} (Act(m, h))$, then recall that for every $(\Delta)\alpha \equiv m$ there exists, by Lemma 2, a history $h_{\Delta} \in H_m$ such that $(\Delta)\alpha = m \cap h_{\Delta}$. This means that $Prove(j, A) \land \neg \Box Prove(j, A) \in \Delta$ for every $(\Delta)\alpha \equiv m$, and thus, by maximum consistency of $\Delta$ in $X$ and Lemma 5, that $Prove(j, A), \neg \Box Prove(j, A) \in \Delta$ for every $(\Delta)\alpha \equiv m$. In particular, we will have $Prove(j, A), \neg \Box Prove(j, A) \in \Gamma$. Consider then the following set of formulas in $X$:

$$\Xi = \{B \mid \Box B \in \Gamma\} \cup \{\neg Prove(j, A)\}.$$

$\Xi$ is consistent, for otherwise we would have:

$$\vdash (B_1 \land \ldots \land B_k) \rightarrow Prove(j, A),$$

for some $B_1, \ldots, B_k$ such that $\Box B_1, \ldots, \Box B_k$ are all in $\Gamma$. Since $\Box$ is an S5-modality, we would obtain that

$$\vdash (\Box B_1 \land \ldots \land \Box B_k) \rightarrow \Box Prove(j, A),$$

whence, by maximum consistency of $\Gamma$ in $X$, it would follow that $\Box Prove(j, A) \in \Gamma$, and the latter, given that also $\neg \Box Prove(j, A) \in \Gamma$, would contradict $\Gamma$’s maximum consistency. Therefore, $\Xi$ is consistent and we can extend $\Xi$ to a set $\Theta \subseteq Form_X$, which is maximum consistent in $X$. By definition, we will have $(\Gamma)\alpha \equiv (\Theta)\alpha$, and thus $(\Theta)\alpha \in m$. But we will also have $\neg Prove(j, A) \in \Theta$ which contradicts our assumption that $Prove(j, A) \in \Delta$ for every $(\Delta)\alpha \equiv m$. This contradiction shows that no proof variable of the form $y_{(j,A)}$ is in $\bigcap_{h \in H_m} (Act(m, h))$. Therefore, finally, we get our claim that $\bigcap_{h \in H_m} (Act(m, h)) = \{z\}$ verified.

\footnote{One of these two elements even coincides with $(\Gamma)\alpha$, but we cannot tell, which one.}
(Part 2). We set \( m := [\Gamma_1, \ldots, \Gamma_n] \). It will suffice to show that, for all \( h, h' \in H_m \), we have \( \text{Act}(m, h) = \text{Act}(m, h') \). We know that for some appropriate \( \Delta_1, \ldots, \Delta_n, \Theta_1, \ldots, \Theta_n \) we will have:

\[
m \cap h = (\Delta_1, \ldots, \Delta_n) \alpha,
\]

and:

\[
m \cap h' = (\Theta_1, \ldots, \Theta_n) \alpha.
\]

Since the length of \( m \) is greater than 1, we know that all elements in \( m \) share the same greatest proper initial segment, so that we have:

\[
\Gamma_i = \Delta_i = \Theta_i
\]

for all \( i < n \), and, in particular:

\[
\Gamma_1 = \Delta_1 = \Theta_1.
\]

Now it is clear from the definition of \( \text{Act} \), that \( \text{Act}(m, h) \) and \( \text{Act}(m, h') \) are completely determined by \( \alpha, \Delta_1 \) and \( \Theta_1 \), respectively, therefore, it follows that \( \text{Act}(m, h) = \text{Act}(m, h') \).

\[\square\]

We now check that the remaining semantic constraints on normal jsit models:

**Lemma 11.** The canonical model, as defined above, satisfies the constraints as to the expansion of presented proofs, no new proofs guaranteed, presenting a new proof makes histories divide, and epistemic transparency of presented proofs.

**Proof.** We consider the expansion of presented proofs first. Let \( m' < m \) and let \( h \in H_m \).

Then \( m' \neq \dagger \), since \( \dagger \) has no \( \prec \)-successors. If \( m' = \dagger \) and \( m = \dagger \), then \( h \) must be \( (\dagger, \dagger) \) and we have \( \text{Act}(\dagger, (\dagger, \dagger)) = \text{Act}(\dagger, (\dagger, \dagger)) = \emptyset \), so that the expansion of presented proofs holds. If \( m' = \dagger \) and \( m \) is an equivalence class of elements, then \( h \neq (\dagger, \dagger) \), and we have \( \text{Act}(\dagger, h) = \{z\} \) and \( z \in \text{Act}(m, h) \). Finally, if \( m' \) is an equivalence class of elements, then \( m \) is also an equivalence class of elements. In this case, \( m \cap h \) must be of the form \( (\Gamma_1, \ldots, \Gamma_n) \alpha \) for the respective \( \Gamma_1, \ldots, \Gamma_n \subseteq \text{Form}_X \) and \( \alpha \in \{\uparrow, \downarrow\} \). But then, for some \( k \leq n \), \( m' \cap h \) must be of the form \( (\Gamma_1, \ldots, \Gamma_k) \alpha \). Since the extension of both \( \text{Act}(m, h) \) and \( \text{Act}(m', h) \) is determined by \( \alpha \) and \( \Gamma_1 \), and these are shared by both \( m \cap h \) and \( m' \cap h \), it follows that \( \text{Act}(m, h) = \text{Act}(m', h) \) and thus \( \text{Act}(m, h) \subseteq \text{Act}(m', h) \).

We consider next the no new proofs guaranteed constraint. Let \( m \in \text{Tree} \). If \( m \in \{\dagger, \dagger\} \), then \( \bigcap_{h \in H_m} (\text{Act}(m, h)) = \bigcup_{m' < m, h \in H_m} \text{Act}(m', h) = \emptyset \) and the constraint is trivially satisfied. If \( m \in \text{Tree} \setminus \{\dagger, \dagger\} \), then we need to distinguish between two cases:

**Case 1.** The length of \( m \) equals 1. Then \( m \) is of the form \( [[\Gamma] \alpha] \) for the respective \( \Gamma \subseteq \text{Form}_X \) and \( \alpha \in \{\uparrow, \downarrow\} \). By Lemma 10.1, we have then that \( \bigcap_{h \in H_m} \text{Act}(m, h) = \{z\} \). On the other hand, note that the only \( \prec \)-predecessor of \( [[\Gamma] \alpha] \) is \( m \) must be \( \dagger \) and therefore, by definition of \( \text{Act} \), we get that \( \bigcup_{h \in H_m} \text{Act}(\dagger, h) = \{z\} \) so that the no new proofs guaranteed constraint is verified for \( m \).

**Case 2.** The length of \( m \) is greater than 1. Then \( m \) must be of the form \( [[\Gamma_1, \ldots, \Gamma_n, \Gamma_{n+1}] \alpha] \) for the respective \( \Gamma_1, \ldots, \Gamma_n, \Gamma_{n+1} \subseteq \text{Form}_X, n > 0 \), and \( \alpha \in \{\uparrow, \downarrow\} \). Then we choose,
by Lemma 6.2, an arbitrary \( g \in H_m \) such that \( m \cap g = (\Gamma_1, \ldots, \Gamma_n, \Gamma_{n+1}) \alpha \). For this \( g \) we get, using Lemma 10.2:

\[
\bigcap_{h \in H_m} (Act([\Gamma_1, \ldots, \Gamma_n, \Gamma_{n+1}]_{\alpha} \vDash, h)) = Act([\Gamma_1, \ldots, \Gamma_n, \Gamma_{n+1}]_{\alpha} \vDash, g)
\]

\[
= \{ z \} \cup \{ y_{(j,A)} \mid \text{Prove}(j, A) \land \neg \Box \text{Prove}(j, A) \in \Gamma_1 \} \cup \{ w_A \mid \Box \text{Prove}(j, A) \in \Gamma_1 \}
\]

\[
= Act([\Gamma_1]_{\alpha} \vDash, g)
\]

\[
\subseteq \bigcup_{m' < m, h \in H_m} (Act(m', h)),
\]

since \( [\Gamma_1]_{\alpha} \vDash < [\Gamma_1, \ldots, \Gamma_n, \Gamma_{n+1}]_{\alpha} \vDash \).

We turn next to the presenting a new proof makes histories divide constraint. Consider an \( m, m' \in Tree \) such that \( m < m' \) and arbitrary \( h, h' \in H_{m'} \). If \( m = \hat{\downarrow} \), then the constraint is verified trivially since \( \hat{\downarrow} \) has no \(<\)-successors. If \( m = \hat{\uparrow} \) and \( m' = \hat{\uparrow} \), then we must have \( h = h' = (\hat{\downarrow}, \hat{\downarrow}) \) and the constraint is verified trivially. If \( m = \hat{\downarrow} \) and \( m' \neq \hat{\downarrow} \), then both \( h \) and \( h' \) are different from \( (\hat{\downarrow}, \hat{\downarrow}) \), which means that \( Act(h, h) = Act(\hat{\downarrow}, h') = \{ z \} \), and the constraint is again verified. Finally, if \( m \in Tree \setminus \{ \hat{\downarrow}, \hat{\uparrow} \} \), then we must have \( m = [(\Gamma_1, \ldots, \Gamma_n)_{\alpha}]_{\vDash} \) for some appropriate \( \Gamma_1, \ldots, \Gamma_n, \alpha \). But then, since \( m' > m \), it must be that \( m' = [(\Gamma_1, \ldots, \Gamma_k)_{\alpha}]_{\vDash} \) for some \( k > n \) (so that, among other things, we know that \( k > 1 \)). Now, given that \( h, h' \in H_{m'} \), this means that for appropriate \( \Delta, \Delta' \subseteq Form_X \) we must have \( h \cap m' = (\Gamma_1, \ldots, \Gamma_{k-1}, \Delta)_{\alpha} \) and \( h' \cap m' = (\Gamma_1, \ldots, \Gamma_{k-1}, \Delta')_{\alpha} \), which, in turn, means that:

\[ h \cap m = h' \cap m = (\Gamma_1, \ldots, \Gamma_n)_{\alpha}. \]

It follows, by definition of \( Act \), that in this case \( Act(m, h) = Act(m, h') \), and the constraint is verified.

It remains to check the epistemic transparency of presented proofs constraint. Assume that \( m, m' \in Tree \) are such that \( R(m, m') \). If we have \( m \in \{ \hat{\downarrow}, \hat{\uparrow} \} \), then, by definition, we must have \( \bigcap_{h \in H_m} (Act(m, h)) = \emptyset \), and the constraint is verified in a trivial way. If, on the other hand, \( m \in Tree \setminus \{ \hat{\downarrow}, \hat{\uparrow} \} \), then, by \( R(m, m') \), we must also have \( m' \in Tree \setminus \{ \hat{\downarrow}, \hat{\uparrow} \} \). We have then two cases to consider:

**Case 1.** The length of \( m \) equals 1. Then, by Lemma 10.1, we know that

\[ \bigcap_{h \in H_m} (Act(m, h)) = \{ z \}. \]

It is also obvious, by the fact that \( m' \in Tree \setminus \{ \hat{\downarrow}, \hat{\uparrow} \} \), that \( z \in \bigcap_{h \in H_m} (Act(m', h)) \) and thus the constraint is satisfied.

**Case 2.** The length of \( m \) is greater than 1. Then \( m = [(\Gamma_1, \ldots, \Gamma_n)_{\alpha}]_{\vDash} \) for appropriate \( \Gamma_1, \ldots, \Gamma_n, \alpha \), and, since we have \( R(m, m') \), we must also have \( m' = [(\Gamma_1, \Delta_1, \ldots, \Delta_k)_{\alpha}]_{\vDash} \) for appropriate \( \Delta_1, \ldots, \Delta_k \). We assume that in fact \( \alpha = \hat{\downarrow} \), the other subcase is similar. Using Lemma 6.2, we choose \( g \in H_m \) and \( g' \in H_{m'} \) in such a way that \( m \cap g = (\Gamma_1, \ldots, \Gamma_n) \downarrow \) and \( m' \cap g' = (\Gamma_1, \Delta_1, \ldots, \Delta_k) \downarrow \). We get then, by Lemma
4.5 The truth lemma

It follows from Lemmas 5–11 that our above-defined canonical model is in fact a normal jstit model. Now we need to supply a truth lemma:

**Lemma 12.** Let $A \in \text{Form}_X$, let $m \in \text{Tree} \setminus \{\top, \bot\}$, and let $h \in H_m$. Then:

$$\mathcal{M}, m, h \models A \iff A \in \text{end}(m \cap h).$$

**Proof.** As is usual, we prove the lemma by induction on the construction of $A$. The basis of induction with $A = p \in \text{Var}$ we have by definition of $\text{V}$, whereas Boolean cases for the induction step are trivial. We treat the modal cases:

**Case 1.** $A = \Box B$. If $\Box B \in \text{end}(m \cap h)$, then note that for every $h' \in H_m$ we must have $m \cap h' \equiv m \cap h$. By definition of $\equiv$ and the fact that $m \in \text{Tree} \setminus \{\top, \bot\}$, we must have then $B \in (m \cap h')$ for all $h' \in H_m$ and thus, by induction hypothesis, we obtain that $\mathcal{M}, m, h \models \Box B$. If, on the other hand, $\Box B /\in \text{end}(m \cap h)$, we need to consider then two subcases:

**Case 1.1.** The length of $m$ equals 1. We must have then $m \cap h = (\Gamma)\alpha$ for some appropriate $\Gamma$ and $\alpha$ so that $\Gamma = \text{end}(m \cap h)$. Then the set

$$\Xi = \{\Box C \mid \Box C \in \Gamma\} \cup \{\neg B\}$$

must be consistent, since otherwise we would have

$$\vdash (\Box C_1 \wedge \ldots \wedge \Box C_n) \rightarrow B$$

for some $\Box C_1, \ldots, \Box C_n \in \Gamma$, whence, since $\Box$ is an S5-modality, we would get

$$\vdash (\Box C_1 \wedge \ldots \wedge \Box C_n) \rightarrow \Box B,$$

which would mean that $\Box B \in \Gamma$, contrary to our assumption. Therefore, $\Xi$ is consistent and we can extend $\Xi$ to a set $\Delta \in \text{Form}_X$ which is maxiconsistent in $X$. Of course, in this case $B \notin \Delta$. We will have then that $(\Delta)\alpha$ is an element, and, by definition of $\equiv$, that $(\Gamma)\alpha \equiv (\Delta)\alpha$. By Lemma 6.2, for some $h' \in H_m$ we will have $(\Delta)\alpha = m \cap h'$ and, therefore, $\Delta = \text{end}(m \cap h')$. Since $B \notin \Delta$, it follows, by induction hypothesis, that $\mathcal{M}, m, h' \not\models B$, hence $\mathcal{M}, m, h \not\models \Box B$ as desired.

**Case 1.2.** The length of $m$ is greater than 1. We must have then $m \cap h = (\Gamma_1, \ldots, \Gamma_n, \Gamma)\alpha$ for some appropriate $n > 0$, $\Gamma_1, \ldots, \Gamma_n, \Gamma$ and $\alpha$ so that $\Gamma = \text{end}(m \cap h)$. We then define
\( \Delta \) as in Case 1.1 so that we have \((\Gamma_1, \ldots, \Gamma_n, \Gamma)\alpha \equiv (\Gamma_1, \ldots, \Gamma_n, \Delta)\alpha \) and show that for any \( h' \in H_m \) such that \((\Gamma_1, \ldots, \Gamma_n, \Delta)\alpha = m \land h' \) and, \( \text{io ipso} \), \( \Delta = \text{end}(m \land h') \), we will have \( \mathcal{M}, m, h' \not\models B \), whence \( \mathcal{M}, m, h \not\models B \) as desired. The only new ingredient is that we need to supply a proof that \((\Gamma_1, \ldots, \Gamma_n, \Delta)\alpha \) is actually an element. Well, if for any \( C \in \text{Form}_X \) we have that \( KC \in \Gamma_n \), then, since \((\Gamma_1, \ldots, \Gamma_n, \Gamma)\alpha \) is an element, we will have \( KC \in \Gamma \), whence, by maxiconsistency of \( \Gamma \) in \( X \) and Lemma 3.5, \( \Box KC \in \Gamma \), and since every boxed formula from \( \Gamma \) is also in \( \Delta \), we get that \( \Box \Delta \in \Delta \), whence \( \Box KC \in \Delta \) by maxiconsistency of \( \Delta \) in \( X \) and \( \text{S5} \) reasoning for \( \Box \). Further, if we have \( \text{Proven}(j, C) \in \Gamma_1 \) for \( j \in Ag \), then, since \((\Gamma_1, \ldots, \Gamma_n, \Gamma)\alpha \) is an element, we will have \( \text{Proven}(C) \in \Gamma \), whence, by maxiconsistency of \( \Gamma \) in \( X \) and Lemma 3.2, \( \Box \text{Proven}(C) \in \Gamma \), and since every boxed formula from \( \Gamma \) is also in \( \Delta \), we get that \( \Box \text{Proven}(C) \in \Delta \) by maxiconsistency of \( \Delta \) in \( X \) and \( \text{S5} \) reasoning for \( \Box \). Finally, if \( C \in \text{Form}_X \) and \( j \in Ag \), then \( \Box \neg \text{Proven}(j, C) \in \Gamma \), because \((\Gamma_1, \ldots, \Gamma_n, \Gamma)\alpha \) is an element, whence \( \Box \neg \text{Proven}(j, C) \in \Gamma \) by \[ \{A2\} \] and maxiconsistency of \( \Gamma \) in \( X \). And since every boxed formula from \( \Gamma \) is also in \( \Delta \), we get that \( \Box \neg \text{Proven}(j, C) \in \Delta \) as well, hence \( \Box \neg \text{Proven}(j, C) \in \Delta \) by \[ \{A1\} \] and maxiconsistency of \( \Delta \) in \( X \).

**Case 2.** \( A = [j]B \) for some \( j \in Ag \). Then, if \( [j]B \in \text{end}(m \land h) \), by definition of \( \text{Choice} \) and the fact that \( m \in \text{Tree} \setminus \{\dagger, \ddagger\} \) we must have:

\[
\text{Choice}_m^j(h) = \{ h' \mid h' \in H_m, \ (\forall C \in \text{Form}_X) ([j]C \in \text{end}(h \land m) \Rightarrow C \in \text{end}(h' \land m)) \}.
\]

Therefore, if \( h' \in \text{Choice}_m^j(h) \), then we must have \( B \in \text{end}(h' \land m) \), and further, by induction hypothesis, that \( \mathcal{M}, m, h' \models B \), so that we get \( \mathcal{M}, m, h \models [j]B \). On the other hand, if \( [j]B \not\in \text{end}(m \land h) \), we need to consider then two subcases:

**Case 2.1.** The length of \( m \) equals 1. We must have then \( m \land h = (\Gamma)\alpha \) for some appropriate \( \Gamma \) and \( \alpha \) so that \( \Gamma = \text{end}(m \land h) \). Then the set

\[
\Xi = \{ [j]C \mid [j]C \in \Gamma \} \cup \{ \neg B \}
\]

must be consistent, since otherwise we would have

\[
\vdash ([j]C_1 \land \ldots \land [j]C_n) \rightarrow B
\]

for some \( [j]C_1, \ldots, [j]C_n \in \Gamma \), whence, since \( [j] \) is an \( \text{S5} \)-modality, we would get

\[
\vdash ([j]C_1 \land \ldots \land [j]C_n) \rightarrow [j]B,
\]

which would mean that \( [j]B \in \Gamma \), contrary to our assumption. Therefore, \( \Xi \) is consistent and we can extend \( \Xi \) to a set \( \Delta \subseteq \text{Form}_X \) which is maxiconsistent in \( X \). Of course, in this case \( B \not\in \Delta \). We will have then that \( (\Delta)\alpha \) is an element.

Now, if \( D \in \text{Form}_X \) is such that \( \Box D \in \Gamma \), then, by \[ \{A2\} \] and maxiconsistency of \( \Gamma \) in \( X \), we know that \( [j]D \in \Gamma \), so that also \( [j]D \in \Delta \), and hence, by \[ \{A1\} \] and maxiconsistency of \( \Delta \) in \( X \), \( D \in \Delta \). We have thus shown that:

\[
(\forall D \in \text{Form}_X)(\Box D \in \Gamma \Rightarrow D \in \Delta),
\]

and it follows that \((\Gamma)\alpha \equiv (\Delta)\alpha \) by definition of \( \equiv \). By Lemma 3.2, for some \( h' \in H_m \) we will have \((\Delta)\alpha = m \land h' \) and, therefore, \( \Delta = \text{end}(m \land h') \). Also, since \( \Delta \) contains
all the \([j]\)-modalized formulas from \(\Gamma\), we know that for any such \(h'\) we will have \(h' \in \text{Choice}_j^m(h)\). Since \(B \notin \Delta\), it follows, by induction hypothesis, that \(\mathcal{M}, m, h' \not \models B\), hence \(\mathcal{M}, m, h \not \models [j]B\) as desired.

**Case 2.2.** The length of \(m\) is greater than 1. We must have then \(m \cap h = (\Gamma_1, \ldots, \Gamma_n, \Gamma)\alpha\) for some appropriate \(n > 0, \Gamma_1, \ldots, \Gamma_n, \Gamma\) and \(\alpha\) so that \(\Gamma = \text{end}(m \cap h)\). We then define \(\Delta\) as in Case 2.1 so that \((\Gamma_1, \ldots, \Gamma_n, \Gamma)\alpha \equiv (\Gamma_1, \ldots, \Gamma_n, \Delta)\alpha\) and show that for any \(h' \in H_m\) such that \((\Gamma_1, \ldots, \Gamma_n, \Delta)\alpha = m \cap h'\) and, eo ipso, \(\Delta = \text{end}(m \cap h')\), we will have both \(h' \in \text{Choice}_j^m(h)\) and \(\mathcal{M}, m, h' \not \models B\), whence \(\mathcal{M}, m, h \not \models [j]B\) as desired. Again, a separate argument for \((\Gamma_1, \ldots, \Gamma_n, \Delta)\alpha\) being an element needs to be supplied, and it can be done in the same way as in Case 1.2, given that by \((3)\) and S5 properties of \(\Box\) we know that every boxed formula from \(\Gamma\) is also in \(\Delta\).

**Case 3.** \(A = KB\). Assume that \(KB \in \text{end}(m \cap h)\). We clearly have then \(m = [(m \cap h)\Xi]\). Hence, by definition of \(R\) and the fact that \(m \in \text{Tree} \setminus \{\dagger, \ddagger\}\) we must have for every \(m' \in \text{Tree}\):

\[
R(m, m') \Rightarrow (\forall \tau \in m')((\forall C \in \text{Form}_X)(KC \in \text{end}(m \cap h) \Rightarrow KC \in \text{end}(\tau)).
\]

Therefore, if \(R(m, m')\) and \(h' \in H_{m'}\) is arbitrary, then, of course, \((h' \cap m') \in m'\) so that \(KB \in \text{end}(h' \cap m')\), and, further, \(B \in \text{end}(h' \cap m')\) by S4 reasoning for \(K\). Therefore, by induction hypothesis, we get that \(\mathcal{M}, m', h' \models B\), whence \(\mathcal{M}, m, h \models KB\). On the other hand, if \(KB \notin \text{end}(m \cap h)\), we need to consider then two subcases:

**Case 3.1.** The length of \(m\) equals 1. We must have then \(m \cap h = (\Gamma)\alpha\) for some appropriate \(\Gamma\) and \(\alpha\) so that \(\Gamma = \text{end}(m \cap h)\). Then the set

\[
\Xi = \{KC | KC \in \Gamma\} \cup \{\neg \Box B\}
\]

must be consistent, since otherwise we would have

\[
\vdash (KC_1 \land \ldots \land KC_n) \rightarrow \Box B
\]

for some \(KC_1, \ldots, KC_n \in \Gamma\), whence, since \(K\) is an S4-modality, we would get

\[
\vdash (KC_1 \land \ldots \land KC_n) \rightarrow K\Box B,
\]

which would mean that \(K\Box B \in \Gamma\), hence, by \((\text{A1}), (\text{A7})\) and maxiconsistency of \(\Gamma\) in \(X\), that \(KB \in \Gamma\), contrary to our assumption. Therefore, \(\Xi\) is consistent and we can extend \(\Xi\) to a set \(\Delta \subseteq \text{Form}_X\) which is maxiconsistent in \(X\). Of course, in this case \(\Box B \not \in \Delta\). We will have then that \((\Delta)\alpha\) is an element. So we set \(m' = [(\Delta)\alpha]_{\Xi}\). Assume that \((\Delta')\alpha' \equiv (\Delta)\alpha\). Then every boxed formula from \(\Delta\) will be in \(\Delta'\). In particular, whenever \(KC \in \Delta\), then also \(\Box KC \in \Delta\) and thus \(KC \in \Delta'\), by Lemma \((\text{B}5)\) and maxiconsistency of \(\Delta\) in \(X\). Therefore, whenever \(KC \in \Gamma\) and \(\tau \in m' = [(\Delta)\alpha]_{\Xi}\), we have that \(KC \in \text{end}(\tau)\) so that we must have \(R(m, m')\). On the other hand, since \(\Box B \not \in \Delta\), then, by Case 1, there must be a \(\sigma \in m'\) such that \(B \notin \text{end}(\tau)\). But then, by Lemma \((\text{B}2)\), we can choose \(h' \in H_{m'}\) in such a way that \(\tau = m' \cap h'\), and we get that \(B \not \in \text{end}(m' \cap h')\). Therefore, by induction hypothesis, we get \(\mathcal{M}, m', h' \not \models B\). In view of the fact that also \(R(m, m')\), this means that \(\mathcal{M}, m, h \not \models KB\) as desired.

**Case 3.2.** The length of \(m\) is greater than 1. We must have then \(m \cap h = (\Gamma_1, \ldots, \Gamma_n, \Gamma)\alpha\) for some appropriate \(n > 0, \Gamma_1, \ldots, \Gamma_n, \Gamma\) and \(\alpha\) so that \(\Gamma = \text{end}(m \cap h)\). We then define \(\Delta\) as in Case 3.1 and consider \(m' = [(\Gamma_1, \Delta)\alpha]_{\Xi}\). We get then \(R(m, m')\) immediately by definition of \(R\). Just as in Case 3.1, we will use the fact that \(\Box B \not \in \Delta\) to find
\( \tau \in m' \) and \( h' \in H_{m'} \), so that \( \tau = m' \cap h' \) and \( B \notin \text{end}(\tau) \). It will follow by induction hypothesis that \( M, m', h' \not\models B \). Therefore, given that \( R(m, m') \), that \( M, m, h \not\models KB \).

The only new ingredient is that now we need to supply a proof that \( (\Gamma_1, \Delta)\alpha \) is actually an element. Well, if for any \( C \in \text{Form}_X \) \( M \) have \( M \) also have \( 3.1 \) and maxiconsistency of \( \bigcap \), then we have to consider two subcases:

\( \xi_\alpha \n > 0 \), \( \alpha \in X \) for in this case we would also have \( M \) and maxiconsistency of \( \bigcap \). On the other hand, if \( t \not\in \text{end}(m \cap h) \), then for no \( t_1:B_1, \ldots, t_n:B_n \in \text{end}(m \cap h) \) can it be that:

\[
\vdash (t_1:B_1 \land \ldots \land t_n:B_n) \rightarrow t:B,
\]

for in this case we would also have \( t:B \in \text{end}(m \cap h) \) by maxiconsistency of \( \text{end}(m \cap h) \) in \( X \). Therefore, we must have \( A \not\in \mathcal{E}(m, t) \) so that \( M, m, h \not\models t:B \).

**Case 4.** \( A = \text{Proven}(B) \). Assume that \( \text{Proven}(B) \in \text{end}(m \cap h) \). Then, by Lemma 10.1 and maxiconsistency of \( \text{end}(m \cap h) \), we will also have \( (\Gamma_1, \Delta)\alpha \not\models \text{Proven}(B) \). Now, choose an arbitrary \( \xi \in m \). We know that \( \text{end}(\xi) \equiv \text{end}(m \cap h) \), therefore, we must have \( \text{Proven}(B) \in \text{end}(\xi) \) by definition of \( \equiv \), which means that \( B \in \mathcal{E}(m, z) \). We also have \( z \in \text{Act}(m, h') \) for all \( h' \in H_m \) and we will have \( KB \in \text{end}(m \cap h) \) by (A11) so that we have \( M, m, h \models z:B \) by Case 3 and induction hypothesis. It follows then that \( M, m, h \models \text{Proven}(B) \). On the other hand, assume that \( \text{Proven}(B) \notin \text{end}(m \cap h) \). Then we have to consider two subcases:

**Case 5.1.** The length of \( m \) equals 1. Then, since \( \text{Proven}(B) \notin \text{end}(m \cap h) \), we will have \( B \not\in \mathcal{E}(m, z) \) by definition. Also, by Lemma 10.1, we know that \( \cap_{h' \in H_m} \text{Act}(m, h') = \{ z \} \). It follows that \( M, m, h \models \neg \text{Proven}(B) \).

**Case 5.2.** The length of \( m \) is greater than 1. We have to consider two subcases:

\[
\cap_{h' \in H_m} \text{Act}(m, h') = \{ z \} \cup \{ y_{(j, C)} | \text{Proven}(j, C) \land \neg \Box \text{Prove}(j, C) \in \Gamma_1 \} \cup \{ u_A | \Box \text{Prove}(j, C) \in \Gamma_1 \}.
\]

We know that \( B \not\in \mathcal{E}(m, z) \) since \( \text{Proven}(B) \not\in \Gamma \). If, for some \( j \in Ag \), we would have \( \text{Prove}(j, B) \in \Gamma_1 \), it would follow that \( \text{Proven}(B) \in \Gamma \), since \( (\Gamma_1, \ldots, \Gamma_n, \Gamma)\alpha \) is...
an element. Therefore, if \( u_C, y_{(j,C)} \in \bigcap_{h' \in \mathcal{H}_m} \text{Act}(m, h') \) for any \( j \in \mathcal{A}_g \), then \( C \neq B \) and therefore \( B \notin \mathcal{E}(m, u_C) = \mathcal{E}(m, y_{(j,C)}) = \{C\} \). It follows then that for no proof which is presented under all histories through \( m \), this proof will be acceptable for \( B \), hence we get \( \mathcal{M}, m, h \nvDash \text{Proven}(B) \).

**Case 6.** \( \Box \text{Prove}(j, B) \) for some \( j \in \mathcal{A}_g \). Assume that \( \text{Prove}(j, B) \in \text{end}(m \cap h) \). Then we know that the length of \( m \) must be \( 1 \). Indeed, if length of \( m \) were greater than \( 1 \), then we would have \( K \Box \neg \text{Prove}(j, B) \in \text{end}(m \cap h) \), whence, by S4 reasoning for \( K, S5 \) reasoning for \( \square \), and maxiconsistency of \( \text{end}(m \cap h) \) in \( X \) we would have \( \neg \text{Prove}(j, B) \in \text{end}(m \cap h) \) would be impossible.

So, for some appropriate \( \Gamma \) and \( \alpha \) we will have both \( m = [(\Gamma)\alpha] \) and \( m \cap h = (\Gamma)\alpha \). We need to consider two subcases:

**Case 6.1.** \( \Box \text{Prove}(j, B) \in \Gamma \). Then, for all \( h' \in \text{Choice}^m_j(h) \), we will have, of course, \( m \cap h' = m \cap h \) which means, by maxiconsistency and S5 reasoning for \( \Box \), that we will also have \( \Box \text{Prove}(j, B) \in \text{end}(m \cap h') \). This will mean that for all \( h' \in \text{Choice}^m_j(h) \) we will have either \( u_B \) or \( w_B \) in \( \text{Act}(m, h') \) and we will have, of course \( B \in \mathcal{E}(m, u_B) = \mathcal{E}(m, w_B) \). Further, by \([\mathcal{A}9]\) and maxiconsistency in \( X \) of every \( \text{end}(m \cap h) \) with \( h' \in \text{Choice}^m_j(h) \) we know that also \( KB \in m \cap h' \) for every such \( h' \). Therefore, we know by Case 3 above that either \( \mathcal{M}, m, h' \vdash u_B : B \), or \( \mathcal{M}, m, h' \vdash w_B : B \) for every \( h' \in \text{Choice}^m_j(h) \).

Assume, further, that for some \( s \in \mathcal{P}ol \) we have \( \mathcal{M}, m, h \models s : B \). Then, in particular, we must have \( B \in \mathcal{E}(m, s) \). By definition of \( \mathcal{E} \), \( s \) cannot then be a proof variable of the form \( u_C, w_C \), or \( y_{(j,C)} \) for any \( j \in \mathcal{A}_g \) and any formula \( C \) different from \( B \). Moreover, \( s \) cannot be \( z \), since we have \( \text{Prove}(j, B) \in \text{end}(m \cap h) \) whence by maxiconsistency of \( \text{end}(m \cap h) \) in \( X \) and \([\mathcal{A}9]\), \( \neg \text{Proven}(B) \in \text{end}(m \cap h) \), so that, again by maxiconsistency, \( \text{Proven}(B) \notin \text{end}(m \cap h) \in m \), which means, by the above definition of \( \mathcal{E} \), that \( B \notin \mathcal{E}(m, z) \). Therefore, assuming that \( \mathcal{M}, m, h \models s : B \), \( s \) can be either in \( \mathcal{P}ol_X \) or in \( \mathcal{P}ol \setminus \{\mathcal{P}ol_X \cup Y \cup W \cup U \cup \{z\}\} \), or else in \( \{w_B, u_B, y_{(j,B)} \mid j \in \mathcal{A}_g\} \). Well, if \( s \) is either in \( \mathcal{P}ol_X \) or in \( \mathcal{P}ol \setminus \{\mathcal{P}ol_X \cup Y \cup W \cup U \cup \{z\}\} \), then it is immediate from the definition of \( \text{Act} \) that \( s \notin \text{Act}(m, h) \). On the other hand, if \( s \in \{y_{(j,B)} \mid j \in \mathcal{A}_g\} \), then that by maxiconsistency of \( \text{end}(m \cap h) \) in \( X \) we must have \( \text{Prove}(j, B) \land \neg \Box \text{Prove}(j, B) \notin \Gamma \) whence it immediately follows that, again \( s \notin \text{Act}(m, h) \). Finally, consider two elements \( (\Gamma) \uparrow \) and \( (\Gamma) \downarrow \). One of these elements is actually \( (\Gamma)\alpha \), both elements are in \( m \), and, by Lemma \([\mathcal{A}2]\), we can choose \( h', h'' \in \mathcal{H}_m \) in such a way that we have both \( (\Gamma) \uparrow = m \cap h' \) and \( (\Gamma) \downarrow = m \cap h'' \). It clearly follows then from the definition of \( \text{Act} \) that \( w_B \notin \text{Act}(m, h') \), whereas \( u_B \notin \text{Act}(m, h'') \).

**Case 6.2.** \( \Box \text{Prove}(j, B) \notin \Gamma \). Then, by maxiconsistency of \( \Gamma = \text{end}(m \cap h) \) in \( X \), we must have \( \text{Prove}(j, B) \land \neg \Box \text{Prove}(j, B) \in \Gamma \) as well as (again, by maxiconsistency of \( \Gamma \) in \( X \) and Lemma \([\mathcal{A}4]\)) \([\mathcal{A}9]\) \( \text{Prove}(j, B) \land \neg \Box \text{Prove}(j, B) \in \Gamma \). Therefore, for every \( h' \in \text{Choice}^m_j(h) \) we will have \( \text{Prove}(j, B) \land \neg \Box \text{Prove}(j, B) \in \text{end}(m \cap h') \) simply by definition of \( \text{Choice} \). This further means that for every such \( h' \), the proof variable \( y_{(j,B)} \) will be in \( \text{Act}(m, h') \). Besides, it is immediate from the definition of \( \mathcal{E} \) that \( B \in \mathcal{E}(m, y_{(j,B)}) \). Finally, note that by \([\mathcal{A}9]\) and maxiconsistency of the respective \( \text{end}(m \cap h') \) in \( X \), we will have \( KB \in \text{end}(m \cap h') \) for every \( h' \in \text{Choice}^m_j(h) \). Therefore, by Case 3 above, we will have \( \mathcal{M}, m, h' \vdash y_{(j,B)} : B \) for every \( h' \in \text{Choice}^m_j(h) \).

Assume, further, that for some \( s \in \mathcal{P}ol \) we have \( \mathcal{M}, m, h \models s : B \). Just as in Case 6.1, we can show that \( s \) cannot be of the form \( z, u_C, w_C \), or \( y_{(j,C)} \) for any \( j \in \mathcal{A}_g \) and any formula \( C \) different from \( B \). Then, again borrowing our reasoning from the
Case 6.1 above, we can show that if \( s \in Pol_X \) or \( s \in Pol \setminus (Pol_X \cup Y \cup W \cup U \cup \{z\}) \), then we must have \( s \notin Act(m, h) \). If \( s = u_B \) or \( w_B \) then we must have \( s \notin Act(m, h) \) since \( \Box \text{Prove}(i, B) \notin \Gamma = end(m \cap h) \), and therefore, by maxiconsistency of \( \Gamma \) in \( X \) and \((A10)\) we must have \( \Box \text{Prove}(i, B) \notin \Gamma = end(m \cap h) \) for all \( i \in Ag \). Assume then that \( s \) is \( y_{(i,B)} \) for some \( i \in Ag \). If \( y_{(i,B)} \notin Act(m, h) \), then we are done. If, on the other hand, \( y_{(i,B)} \in Act(m, h) \), then, by definition of \( Act \), we must have \( \text{Prove}(i, B) \land \neg \Box \text{Prove}(i, B) \in \Gamma = end(m \cap h) \), hence, by Lemma \( \Box \text{5} \), \( \neg \Box \text{Prove}(i, B) \in end(m \cap h) \). Then the set

\[
\Xi = \{ \Box C \mid \Box C \in \Gamma \} \cup \{ \neg \text{Prove}(i, B) \}
\]

must be consistent, since otherwise we would have

\[
\vdash (\Box C_1 \land \ldots \land \Box C_n) \to \text{Prove}(i, B)
\]

for some \( \Box C_1, \ldots, \Box C_n \in \Gamma \), whence, since \( \Box \) is an S5-modality, we would get

\[
\vdash (\Box C_1 \land \ldots \land \Box C_n) \to \Box \text{Prove}(i, B),
\]

which would mean that \( \Box \text{Prove}(i, B) \in \Gamma \), contrary to our assumption. Therefore, \( \Xi \) is consistent and we can extend \( \Xi \) to a set \( \Delta \subseteq Form_X \) which is maxiconsistent in \( X \). Of course, in this case \( \text{Prove}(i, B) \notin \Delta \). We will have then that \( (\Delta)\alpha \) is an element, and, by definition of \( \equiv \), that \( (\Gamma)\alpha \equiv (\Delta)\alpha \). By Lemma \( \Box \text{2} \), for some \( h' \in H_m \) we will have \( (\Delta)\alpha = m \cap h' \) and, therefore, \( \Delta = end(m \cap h') \). Since \( \text{Prove}(i, B) \notin \Delta \), it follows that \( y_{(i,B)} \notin Act(m, h') \).

Thus we have shown that if \( \text{Prove}(j, B) \in end(m \cap h) \), then \( \mathcal{M}, m, h \models \text{Prove}(j, B) \). For the inverse direction, assume that \( \text{Prove}(j, B) \notin end(m \cap h) \). Again, we have to consider two further subcases:

**Case 6.3.** The length of \( m \) equals 1 so that, for some appropriate \( \Gamma \) and \( \alpha \) we have both \( m = [(\Gamma)\alpha] \equiv \) and \( m \cap h = (\Gamma)\alpha \). If \( \mathcal{M}, m, h \models \text{Proven}(B) \), then by \((A9)\) we will have \( \mathcal{M}, m, h \not\models \text{Prove}(j, B) \), and thus we will be done. Therefore, assume that \( \mathcal{M}, m, h \not\models \text{Proven}(B) \). Moreover, if \( \mathcal{M}, m, h \not\models KB \) then we will again have, by \((A9)\), that \( \mathcal{M}, m, h \not\models \text{Prove}(j, B) \), so that we may also safely assume that \( \mathcal{M}, m, h \models KB \). Under these assumptions, in order to show that \( \mathcal{M}, m, h \not\models \text{Prove}(j, B) \) we have to show that the positive condition fails in that there is an \( h' \in Choice^p_m(h) \) such that no acceptable proof of \( B \) is present in \( Act(m, h') \). To this end, we consider the set

\[
\Xi = \{ [j]C \mid [j]C \in \Gamma \} \cup \bigwedge_{i \in Ag} \neg \text{Prove}(i, B).
\]

This set must be consistent, since otherwise we would have

\[
\vdash ([j]C_1 \land \ldots \land [j]C_n) \to \bigvee_{i \in Ag} \text{Prove}(i, B)
\]

for some \([j]C_1, \ldots, [j]C_n \in \Gamma \), whence, since \([j] \) is an S5 modality, we would get

\[
\vdash ([j]C_1 \land \ldots \land [j]C_n) \to [j](\bigvee_{i \in Ag} \text{Prove}(i, B)),
\]

which would mean that \( [j]\text{Prove}(i, B) \in \Gamma \), contrary to our assumption. Therefore, \( \Xi \) is maxiconsistent and we can extend \( \Xi \) to a set \( \Delta \subseteq Form_X \) which is maxiconsistent in \( X \). Of course, in this case \( \text{Prove}(i, B) \notin \Delta \). We will have then that \( (\Delta)\alpha \) is an element, and, by definition of \( \equiv \), that \( (\Gamma)\alpha \equiv (\Delta)\alpha \). By Lemma \( \Box \text{2} \), for some \( h' \in H_m \) we will have \( (\Delta)\alpha = m \cap h' \) and, therefore, \( \Delta = end(m \cap h') \). Since \( \text{Prove}(i, B) \notin \Delta \), it follows that \( y_{(i,B)} \notin Act(m, h') \).

Thus we have shown that if \( \text{Prove}(j, B) \in end(m \cap h) \), then \( \mathcal{M}, m, h \models \text{Prove}(j, B) \). For the inverse direction, assume that \( \text{Prove}(j, B) \notin end(m \cap h) \). Again, we have to consider two further subcases:

**Case 6.3.**
which would mean that \( [j](\bigvee_{i \in A_g} \text{Prove}(i, B)) \in \Gamma \). On the other hand, since \( \text{Prove}(j, B) \notin \Gamma \), this means, by maxiconsistency of \( \Gamma \) in \( X \), that \( \neg \text{Prove}(j, B) \in \Gamma \), whence, again by maxiconsistency and (A13), we obtain that \( [j](\bigwedge_{i \in A_g} \neg \text{Prove}(i, B)) \in \Gamma \). Therefore, by maxiconsistency of \( \Gamma \) in \( X \), we must have \( \neg [j](\bigvee_{i \in A_g} \text{Prove}(i, B)) \in \Gamma \), a contradiction.

Therefore, \( \Xi \) is consistent and we can extend \( \Xi \) to a set \( \Delta \subseteq \text{Form}_X \) which is maxiconsistent in \( X \). Of course, in this case we will have \( \text{Prove}(i, B) \notin \Delta \) for all \( i \in A_g \). We will have then that \( (\Delta)\alpha \) is an element of \( \Xi \), and, arguing as in Case 2.1 we can show (3) so that \( \Delta \) contains all boxed formulas from \( \Gamma \). Therefore, by definition of \( \Xi \), we know that \( (\Gamma)\alpha \equiv (\Delta)\alpha \). By Lemma 6.2, we know that for some \( h' \in H_m \) we will have \( (\Delta)\alpha = m \cap h' \) and, therefore, \( \Delta = \text{end}(m \cap h') \). Also, since \( \Delta \) contains all the \([j]\)-modalized formulas from \( \Gamma \), we know that for any such \( h' \) we will have \( h' \in \text{Choice}_m^\pi(h) \).

We also know that \( \text{Proven}(B) \notin \Delta \), for otherwise we would have, by maxiconsistency of \( \Delta \) and Lemma 5.2, that \( \Box \text{Proven}(B) \in \Delta \), whence, by the fact that \( (\Gamma)\alpha \equiv (\Delta)\alpha \) we would have that \( \text{Proven}(B) \in \Gamma \), contradicting our assumptions.

Consider then \( \text{Act}(m, h') \). We may assume that \( \alpha = \uparrow \), the reasoning for the case when \( \alpha = \downarrow \) is similar. We have, by definition of \( \text{Act} \) that:

\[
\text{Act}(m, h') = \{ z \} \cup \{ y_{i,C} \mid \text{Prove}(i, C) \wedge \neg \Box \text{Prove}(i, C) \in \Delta \} \cup \{ u_C \mid \Box \text{Prove}(i, C) \in \Delta \}.
\]

We know that \( B \notin \mathcal{E}(m, z) \), since we have established that \( \text{Proven}(B) \notin \Delta \); we also know that if \( u_C, y_{i,C} \in \text{Act}(m, h') \) for any \( i \in A_g \), then \( C \neq B \) since for all \( i \in A_g \) we have \( \text{Prove}(i, B) \notin \Delta \), and this means that if \( u_{C, y_{i,C}} \in \text{Act}(m, h') \) for any \( i \in A_g \), then both \( B \notin \mathcal{E}(m, u_C) \) and \( B \notin \mathcal{E}(m, y_{i,C}) \). Therefore, at \( (m, h') \) there exists no presented proof which would be acceptable for \( B \), and since \( h' \in \text{Choice}_m^\pi(h) \), this means that the positive condition for \( \text{Prove}(j, B) \) at \( (m, h) \) is violated, so that we get \( M, m, h \models \neg \text{Prove}(j, B) \) as desired.

Case 6.4. The length of \( m \) is greater than 1. Then, by Lemma 10.2, for all \( h' \in H_m \) we have that

\[
\text{Act}(m, h') = \bigcap_{h'' \in H_m} \text{Act}(m, h'').
\]

Assume then, that we have both \( s \in \text{Act}(m, h) \) and \( M, m, h \models s:B \) for some \( s \in \text{Pol} \). Then \( s \in \bigcap_{h'' \in H_m} \text{Act}(m, h'') \), which means that the negative condition for \( \text{Prove}(j, B) \) at \( (m, h) \) is violated and we must have \( M, m, h \models \neg \text{Prove}(j, B) \). Assume, on the contrary, that there is no \( s \in \text{Pol} \) for which both \( s \in \text{Act}(m, h) \) and \( M, m, h \models s:B \). Then, since \( h \) is of course in \( \text{Choice}_m^\pi(h) \), it turns out that the positive condition for \( \text{Prove}(j, B) \) at \( (m, h) \) is violated and again have \( M, m, h \models \neg \text{Prove}(j, B) \). So, in any case \( M, m, h \models \neg \text{Prove}(j, B) \), as desired.

This finishes the list of the modal induction cases at hand, and thus the proof of our truth lemma is complete.

\[\square\]

5 The main result

We are now in a position to prove Theorem \[2\]. The proof proceeds as follows. One direction of the theorem was proved as Corollary \[1\]. In the other direction, assume that \( \Gamma \subseteq \text{Form}_X \) is consistent. Then, by Lemma \[2\], \( \Gamma \) can be extended to a \( \Delta \) which is maxiconsistent in \( X \). But then choose an arbitrary \( \alpha \in \{ \uparrow, \downarrow \} \) and consider
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\[ M = \langle \text{Tree}, \leq, \text{Choice}, \text{Act}, R, \mathcal{E}, V \rangle \], the canonical model defined in Section 4. The structure \((\Delta)\alpha\) is an element, therefore \([(\Delta)\alpha]_\equiv \in \text{Tree}. By Lemma 6.2, there is a history \(h \in H_\alpha\) such that \((\Delta)\alpha = [(\Delta)\alpha]_\equiv \cap h\). For this \(h\), we will also have \(\Delta = \text{end}([(\Delta)\alpha]_\equiv \cap h)\). By Lemma 12, we therefore get that:

\[ M, [(\Delta)\alpha]_\equiv, h \models \Delta \supseteq \Gamma, \]

and thus \(\Gamma\) is shown to be satisfiable in a normal jstit model.

**Remark.** Note that the canonical model used in this proof is \(X\)-universal in the sense that it satisfies every consistent subset of \(\text{Form}_X\).

As an obvious corollary of Theorem 2 we get the following weak completeness result:

**Corollary 2.** For every \(A \in \text{Form}\), \(\vdash A\) iff \(A\) is valid over normal jstit models.

**Proof.** One direction follows from Theorem 1. In the other direction, if \(\nvdash A\), then \(\{\neg A\}\) is consistent. Setting \(X\) to be the set of proof variables occurring in \(A\), we see that \(\text{PVar} \setminus X\) must be countably infinite. Therefore, Theorem 2 applies, \(\{\neg A\}\) must be satisfied in some normal jstit model, and \(A\) cannot be valid.

As a further corollary, we deduce a restricted form of compactness property:

**Corollary 3.** Let \(X \subseteq \text{PVar}\) be such that \(\text{PVar} \setminus X\) is countably infinite. Then an arbitrary \(\Gamma \subseteq \text{Form}_X\) is satisfiable iff every finite \(\Gamma_0 \subseteq \Gamma\) is satisfiable.

**Proof.** If \(\Gamma\) is satisfiable, then clearly every finite \(\Gamma_0 \subseteq \Gamma\) is satisfiable. On the other hand, if every finite \(\Gamma_0 \subseteq \Gamma\) is satisfiable, then for no \(A_1, \ldots, A_n \in \Gamma\) can we have that \(\vdash (A_1 \land \ldots \land A_n) \rightarrow \bot\), for otherwise, by Theorem 1 the finite set \(\{A_1, \ldots, A_n\}\) would be unsatisfiable. Therefore, \(\Gamma\) must be consistent, and, by Theorem 2 also satisfiable.

6 Conclusions and future research

Theorem 2, the main result of this paper, proves what might be called a restricted strong completeness theorem for the implicit jstit logic. As we have shown in Section 5 this means, among other things, that this logic allows for a finitary proof system and enjoys a restricted form of compactness property. Taken together, these results show that, given the rich variety of expressive means present in the implicit jstit logic and non-trivial semantic constraints imposed on its models, this logic displays a surprising degree of regularity.

Of course, the results of the present paper give room to some generalization. One obvious observation would be that the rule (R3) gives but one variant out of the infinite family of the so-called constant specifications allowed for in justification logic; and it is straightforward to see that the above completeness proof can be easily adapted for the systems with other versions of constant specification. The other obvious direction of generalizing the results above would be to relieve the restriction that \(R = R_e\) and consider the semantics of [5] in its full generality, although, as we have already mentioned, it is not so clear whether this generalization will affect the set of validities.

In the broader perspective, Theorem 2 is a step towards axiomatization of the full basic justification stit logic in case such an axiomatization is possible. Viewing
Theorem 2 as a partial success in axiomatizing the full basic jstit logic, it is easy to see which steps shall come next. First, one needs to understand the mechanics behind the proving modalities omitted from the implicit jstit logic and axiomatize the logic of \( \text{Prove}(j, t, A) \) and \( \text{Proven}(t, A) \) placed on top of stit and justification modalities; then an axiomatization of a system combining both explicit and implicit proving modalities and their interplay may turn out to be possible. As a promising further step in this direction, one can consider, for example, the logic of the so-called \( E \)-notions, introduced in \([6]\). It allows one to define a combination of implicit and explicit proving modalities, even though this combination is but a subset of the variety of proving modalities definable within the full basic jstit logic, and can, therefore, provide a demo version of the problems to be encountered in an attempt to explore the properties of the full system.

7 Acknowledgements

To be inserted.

References

[1] S. Artemov and M. Fitting. Justification logic. In E. N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, winter 2016 edition, 2016.

[2] S. N. Artemov and E. Nogina. Introducing justification into epistemic logic. *J. Log. Comput.*, 15(6):1059–1073, 2005.

[3] N. Belnap, M. Perloff, and M. Xu. *Facing the Future: Agents and Choices in Our Indeterminist World*. Oxford University Press, 2001.

[4] J. Horty. *Agency and Deontic Logic*. Oxford University Press, USA, 2001.

[5] G. Olkhovikov and H. Wansing. Inference as doxastic agency. Part I: The basics of justification stit logic. (to appear), 2017.

[6] G. Olkhovikov and H. Wansing. Inference as doxastic agency. Part II: Ramifications and refinements. (submitted), 2017.