Effective field theory approach to many-body localization

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We construct an analytic theory of many-body localization (MBL) in random spin chains. The approach is based on a first quantized perspective in which MBL is understood as a localization phenomenon on the high dimensional lattice defined by the discrete Hilbert space of the clean system. We construct a field theory on that lattice and apply it to discuss the stability of a weak disorder (‘Wigner-Dyson’) and a strong disorder (‘Poisson’) phase.

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When shielded against external environments (‘baths’) interacting and disordered many particle quantum systems may enter a state of ‘many-body localization’ (MBL), a phase distinguished by the absence of ergodicity and the vanishing of transport coefficients. First observed in the low temperature regime of an interacting Fermi system \cite{bloch2008}, MBL is now recognized as a widespread phenomenon shown by fermionic \cite{Chattaraj_hong}, bosonic \cite{imambekov2010, balatsky2010}, and spin systems \cite{altland2010, altland2011} over extended parameter ranges; the first experimental observation of a many-body localized phase has been reported recently \cite{mirrahimi2015}.

We owe much of our understanding of MBL to theoretical work that puts the focus on \textit{fixed realizations} of a disorder configuration. MBL is then diagnosed from the analysis of phase space decay rates inspired by early work on single particle localization on random lattices \cite{gallavotti1998}, the real space RG approach to random spin systems \cite{fisher1998, ALTLAND1999}, or phenomenological modeling \cite{altland2014}. A recent proof of MBL \cite{altland2015} proceeds in terms of a sequence of local unitary transformations whose generator is a functional of the disorder configuration. While these approaches provide convincing evidence for the formation of localized regimes and phase transitions, they do not provide us with \textit{effective} theories of the MBL phase, i.e. descriptions similar in spirit to the powerful Ginzburg-Landau type field theories \cite{ginzburg1950, ginzburg1951} of single particle localized phases. Judging from experience with single particle localization the construction of such theories will be a powerful aid in the identification of universality classes, the description of critical phenomena, and that of observables. In this paper we introduce the foundations of such a theory.

Our approach differs from previous work in two important respects. First, we discuss our model system – a random spin 1/2 chain with local interactions – from a \textit{first quantized} perspective in which its Hamiltonian is considered as a matrix in Hilbert space. This formulation brings us in a position to apply established methods of transport on random lattices to the problem. Second averaging over disorder is performed at an early stage of the construction. From that point on the system is described by an interplay of the \textit{clean} spin chain Hamiltonian with effective field degrees of freedom introduced by the disorder average. Our main task will be the identification of ‘soft modes’, i.e. field fluctuations of lowest action which describe the physics at large distance scales. We will discuss how these fluctuations describe an Anderson localized and delocalized phase depending on whether we are in a regime of strong or weak disorder, respectively. However, the description of the transition between the two phases, which we argue is \textit{not} in the Anderson universality class, is beyond the scope of the present analysis.

\textbf{Model and qualitative considerations:} We consider a system of $L$ spins governed by a Hamiltonian $\hat{H} \equiv \hat{H}_1 + \hat{H}_2$, with random one-body $\hat{H}_1$ and deterministic two-body $\hat{H}_2$. Here, $\hat{H}_1 \equiv \sum_{l=1}^{L} \epsilon_l \sigma_{z,l}$ describes a random magnetic field in $z$-direction with Gaussian distributed field strengths $\epsilon_l$ of variance $\gamma^2$. The (non-random) two-body interaction $\hat{H}_2 \equiv \sum_{l} \hat{V}_l + \ldots$ contains a spin-exchange term $\hat{V}_l = \gamma \sigma_{l}^{+} \otimes \sigma_{l+1}^{-} + \text{h.c.}$, where $\sigma^{\pm} = \frac{1}{2}(\sigma_{x} \pm i \sigma_{y})$ in the standard Pauli basis, $v$ sets the interaction strength, and the ellipses denote integrability breaking contributions – such as rotational symmetry breaking magnetic fields or next nearest neighbor interactions – not specified explicitly. The assumed non-integrability of the \textit{clean} system will facilitate our later discussion of the strongly interacting regimes.

Before turning to the field theoretical description of the system let us formulate a brief synopsis. The key players of our analysis will be fluctuation amplitudes $Q_{nm} \sim \bar{\psi}_n \psi_m$ describing the phase coherent propagation of pairs of many-body wave function amplitudes $\psi_n$, where $n$ are spin-$z$ states $n = (n_1, \ldots, n_L)$, $n_l \in \{0,1\}$. In a single particle problem with independently distributed site diagonal disorder (and likewise in the many-body ‘random energy model’ considered in Ref. \cite{anderson1958}) such fluctuations would be strictly confined in Hilbert space, $Q_n \equiv Q_{nn}$. However, the scarcity of independent disorder amplitudes in the MBL problem gives rise to off-diagonal fluctuations subject to damping that grows continuously in $|n-m|$, where $|n-m| = \frac{1}{2} \sum_{l} |n_l - m_l|$ is the Hamming distance between sites. These non-local fluctuations are responsible for the main differences between conventional and many body localization.

In the absence of interactions, sites are strictly uncorrelated and the independent fluctuations of $Q_{nn}$ describe...
a many-body Poisson phase with maximal localization. Interactions play the role of a ‘hopping operator’ in the lattice of sites. If we treat the problem ignoring the off-diagonal modes, it reduces to a conventional Anderson localization problem in a lattice whose coordination number $\sim L$ is set by the number of states $m$ connected to a given $n$ by nearest neighbor interactions. The extensive growth of the effective dimensionality of the lattice then leads to the incorrect prediction of delocalization, no matter how weak the interaction. (This phenomenon has indeed been observed numerically in the random energy model [23].)

Below we will discuss how the inclusion of off-diagonal fluctuations is the key to rectifying the picture both in the weak and the strong interaction phase. For weak interactions they describe an essentially important parametric renormalization of hopping parameters, and a perturbative elimination of ‘hopping operators’ similar in spirit to that employed in Imbrie’s proof of MBL in Ising chains [14]. We will show how an analogous weak rotation of basis in Hilbert space can be applied to perturbatively eliminate the nearest neighbor hopping operators in the field theory describing the disorder averaged system. This establishes the stability of the localized phase relative to interactions. In the complementary limit of strong interactions off-diagonal modes are required to correctly describe the translational invariance of the clean system and the formation of an ergodic Wigner-Dyson regime when weak disorder is turned on.

Field theory construction: Localization properties of the system may be probed by products of advanced and retarded Green functions $G^\pm(n,m,\epsilon \pm \frac{\Omega}{2}) \equiv \langle n|\langle \epsilon \pm \frac{\Omega}{2} - \hat{H}\rangle^{-1}|m\rangle$ at weakly different energies $\epsilon \pm \Omega/2$, respectively. Such products can be obtained from the generating functional $Z \equiv \int \mathcal{D}(\hat{\psi},\hat{\psi}) \exp(i\hat{\psi}(\hat{\epsilon} - \hat{H})\hat{\psi})$, where $\psi = \{\psi_n\}$ is a $4 \times 2^L$ dimensional supervector, and the four component index $\alpha = (\lambda,s)$ comprises an index $s = \pm$ discriminating between retarded and advanced Green functions and between complex commuting ($\lambda = 0 \equiv b$) and Grassmann components ($\lambda = 1 \equiv f$), respectively. The Green function energy arguments are contained in $\hat{\epsilon} = \epsilon + \frac{i}{2}\omega^T\tau_3$ where $\tau_3$ is a Pauli matrix acting in advanced/retarded space. For simplicity we will focus on the band center throughout, $\epsilon = \omega = 0$, in which case $\hat{\epsilon} = i\delta\tau_3$ merely contains an infinitesimal imaginary part. Green functions are obtained from $Z$ by differentiation w.r.t. suitably introduced sources, however, we suppress these for the sake of clarity throughout. Following standard procedures [1] we average the functional over the Gaussian fluctuations of $\epsilon$ to generate a quartic term in $\psi$ which in a second step is decoupled by means of a supermatrix Hubbard Stratonovich field $A = A^a_{nm}$ comprising commuting and anti-commuting elements. After integration over the then Gaussian $\psi$-fields, the effective $A$-action reads $S[A] = \frac{1}{2\gamma^2} \text{str}(A_{nm}A_{nm})f_{nm} + \text{str} \ln(\hat{\epsilon} - \hat{H}_2 + A)$, where the supertrace [1] ‘str’ extends over all indices not shown explicitly and the weight function $f_{nm} = 1/(L - |n - m|)$.

We proceed to subject the action to a stationary phase analysis and seek for solutions of the equation $\delta S[A]/\delta A^a_{nm} = 0$, or

$$f_{nm}A^a_{nm} = -\gamma^2 \left( \frac{1}{\hat{\epsilon} - \hat{H}_2 + A} \right)^a_{nm}.$$  \hspace{1cm} (1)

The structure of the equation tells us that the mean field configuration plays the role of a ‘self energy’ describing the influence of the disorder. Indeed, it is straightforward to verify that the equation is solved by the fully diagonal configuration $A = i\gamma\tau_3 \otimes \mathbb{I}$ where $\mathbb{I}$ is the unit-matrix in Hilbert space, the value of the energy scale $\kappa$ depends on the regime we are in and the sign of $\pm i\kappa$ is determined by the causality $\pm i\delta$ of the Green function. The physics of (de)localization is encoded in soft fluctuations around the diagonal configuration. Depending on the relative strength of disorder and interactions, these fluctuations are determined by the condition of approximate commutativity with the interaction operator $\hat{H}_2$, or the quadratic weight governed by the correlation function $f_{nm}$, respectively. We first discuss the more involved latter regime, $\gamma \gg v$.

**Strong disorder:** Referring to the supplementary material for details we note that for $\gamma \gg v$ the strength of the impurity self energy is set by $\kappa \simeq \gamma\sqrt{L}$, i.e. the sum of $L$ random numbers $\pm \epsilon_l$. Due to the assumed weakness of the clean Hamiltonian, $\hat{H}_2$, the mean field solutions possess a large family of approximate solutions $A = i\gamma\sqrt{L}Q$, where $Q_{nm} \equiv Q_n\delta_{nm}$ are matrices site-diagonal in Hilbert space, $Q_n = T_n\tau_3T_n^{-1}$, and $T_n = \{T_n^{ab}\} = 4 \times 4$-supermatrices describing fluctuations away from the diagonal $\tau_3$. Substitution of these configurations into the action shows that in the non-interacting limit, $v = 0$, the fluctuations $T_n$ fully cancel out. A straightforward expansion to leading (quadratic) order in the interaction leads to the soft fluctuation action

$$S[Q] = \frac{v^2}{2\gamma^2 L} \sum_{n,m} \text{str}(Q_nQ_m)X_{nm}$$ \hspace{1cm} (2)

where $X_{nm} = v^{-2} \langle |n| \hat{V} |m| \rangle^2$ is a connectivity matrix assuming the value 1/0 if two states $n,m$ are coupled/not coupled by exchange interactions. This action is equivalent to that of an a-periodic Anderson lattice with sites, $n$, and bond connectivities $X_{nm}$. For each $n$, we have $O(L)$ non-vanishing elements $X_{nm} = 1$ meaning that the lattice has characteristic coordination number $Z = L$ and hopping strength $\alpha \equiv v^2/\gamma^2 L$. Lattices of this type have an Anderson metal-insulator transition at $\alpha_c$ determined by the equation $Z\sqrt{\alpha_c/2\pi} \ln(\alpha_c/2) \sim (v/\gamma)L^{1/2} = 1$, i.e. the action [2] predicts Anderson delocalization in the thermodynamic limit, $L \to \infty$, no matter how weak the interaction. This result is in conflict with our understanding of MBL, and it means that the restriction to diagonal fluctuations commutative with the disorder weight must have been premature.

The key to resolving the situation lies in the observation that the model supports a large number of nearly soft
that the vertex fluctuations get weighed with an action $S[Q, v] = \frac{v^2}{2\pi L^2} \text{str}(T_d^{-1}XT_dQ_v)^2$ and to the generator of the similarity transformation $S[Q, G] = \frac{1}{4} \text{str}(T_d^{-1}GT_d\tau_3Q_v)^2$. The vanishingly small weight $L^{-1}$ multiplying the vertex fluctuations means that the integral over the non-linear super-manifold spanned by the matrix $Q_v$ has to be done rigorously. Referring for details to the supplementary material, we here merely state the remarkably simple result of the integration, $S_{\text{eff}}[Q, G] = \sqrt{\frac{2\pi}{\sqrt{L}\gamma}} \text{str}(Q_nQ_m)X_{nm} - \sqrt{\frac{2\pi}{\sqrt{L}\gamma}} \text{str}(QGQG)$, i.e. the action (2) multiplied by a factor $\sqrt{L}$ plus a term coupling to the generator of the similarity transform. This effective action affords a straightforward physical interpretation: the dimensionless coupling constants, $c$, of site-to-site hopping terms in the field theories of disordered systems are products, $c = \rho_0\Gamma$ of the local (i.e. on-site) density of states of the disordered non-interacting system, $\rho_0$, and a characteristic hopping rate $\Gamma$. Presently, $\rho_0 \sim 1/\sqrt{L}\gamma$ is given by the ratio of one over the characteristic band-width. The hopping rate due to interactions equals $\nu^2/\gamma$, where $\nu \sim |\epsilon_n - \epsilon_m|$ is the characteristic energy difference between two levels of Hamming distance $O(1)$. Technically, the full integration over the vertex fluctuations was required to resolve this small energy difference with accuracy.

We may now choose a perturbative basis rotation $\mathbb{I} + G \equiv 1 + \frac{X}{\gamma}X_{nm}$ to effect a mutual cancellation of the two terms in the action. This should be compared to the construction of Imbrie in which a similarity transformation generated by $\nu^2 \sum_{nm} \Sigma_{nm}$ was applied to eliminate a spin-flip operator in the unaveraged model of an Ising chain [14]. In effect, our approach achieves a construction of similar nature within the framework of an effective averaged theory. A perturbative extension to higher orders in $\nu/\gamma$ may be applied [24] to eliminate site hopping over larger distances at higher order in the $X$-expansion. Evidently, this scheme breaks down at values $\nu > \gamma$ when a transition outside the standard Anderson universality class is expected to take place. (Within the field theoretical framework, an Anderson transition would be described by local hopping operators as in [2] which the above construction shows are not present in this form.) Before discussing the situation in the complementary limit of strong interactions, it is worth comparing to the random energy model which has been introduced as a phenomenological model of many-body Hilbert space localization in Ref. [23]. In that model, the on-site energies $\epsilon_n$ are chosen as $2L$ independent random variables and the ensuing field theory is described by strictly local fluctuation modes, $T_n$. The absence of nonlocal modes means that Eq. (2) describes the theory at weak interactions by a Schrieffer-Wolff type transformation after the vertex fluctuations are integrated out. The vanishingly small weight $L^{-1}$ multiplying the vertex fluctuations means that the integral over the non-linear super-manifold spanned by the matrix $Q_v$ has to be done rigorously. Referring for details to the supplementary material, we here merely state the remarkably simple result of the integration, $S_{\text{eff}}[Q, G] = \sqrt{\frac{2\pi}{\sqrt{L}\gamma}} \text{str}(Q_nQ_m)X_{nm} - \sqrt{\frac{2\pi}{\sqrt{L}\gamma}} \text{str}(QGQG)$, i.e. the action (2) multiplied by a factor $\sqrt{L}$ plus a term coupling to the generator of the similarity transform. This effective action affords a straightforward physical interpretation: the dimensionless coupling constants, $c$, of site-to-site hopping terms in the field theories of disordered systems are products, $c = \rho_0\Gamma$ of the local (i.e. on-site) density of states of the disordered non-interacting system, $\rho_0$, and a characteristic hopping rate $\Gamma$. Presently, $\rho_0 \sim 1/\sqrt{L}\gamma$ is given by the ratio of one over the characteristic band-width. The hopping rate due to interactions equals $\nu^2/\gamma$, where $\nu \sim |\epsilon_n - \epsilon_m|$ is the characteristic energy difference between two levels of Hamming distance $O(1)$. Technically, the full integration over the vertex fluctuations was required to resolve this small energy difference with accuracy.

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**Strong interaction:** We next explore what happens in the complementary limit \( v / \gamma \gg 1 \) of the strongly interacting system. In this case, a spectral decomposition of the r.h.s. of Eq. (1) in the eigenfunctions of \( \hat{H}_2 \) shows that the impurity self energy \( \kappa = \pi \gamma^2 \rho \) multiplying the solutions \( A \) of Eq. (1) is given by a golden rule product of the on-site band center density of states of the clean interacting system, \( \rho \sim 1/Lv \), and the scattering probability \( \sim \gamma^2 \) (see supplementary material). Low action fluctuations \( T \) around the diagonal solution must be commutative with the interaction Hamiltonian \( [T, \hat{H}_2] = 0 \). For a generic (non-integrable) \( \hat{H}_2 \) a set of modes satisfying this condition can be constructed by switching to a basis \( |n, \bar{n}, s \rangle \) in which each site \( n \equiv T^{s} \bar{n} \) is represented as an \( s \)-fold translation of a site \( \bar{n} \) of a \( 2^L / L \)-dimensional unit cell, \( n_t = (T^{s} \bar{n})_{t} \equiv (\bar{n})_{t-s} \). It is then straightforward to check that modes \( T_{n,m} \equiv T_{(r,\bar{n}), (s, \bar{m})} \equiv T_{r-s, \bar{n}, \bar{m}} \) depending only on the translation sector commute with the translationally invariant Hamiltonian. The substitution of \( A = i\kappa T \gamma T^{-1} \) into the disorder weight \( S_{\text{dis}}[A] = (1/2\pi^2) \sum_{n,m} \text{str}(A_{n,m} A_{m,n}) \) leads to a term coupling these modes, \( S_{\text{dis}}[Q] = \text{const.} \times \rho^2 \sum_{r} \text{str}(Q_{r} Q_{-r}) f(r) \), where the sum is over translation sectors, and the effective weight function \( f(r) = \frac{1}{L} \sum_{n} f(\bar{n}, r), (\bar{n}, 0) \) measures the characteristic Hemming distance between translated states. The coupling constant can be interpreted as the product of the on-site density of states on the lattice of translational modes, \( \rho = \rho 2^L / L \), and the golden rule scattering rate scattering rate between clean eigenstates of the system. Due to the large density of states \( \rho \sim 2^L \) disorder strengths exponentially small in system size suffice to effect a freezing of the translation modes to a single effective zero mode \( T_{n,m} \equiv T_{0, \bar{n}, \bar{m}} = T_{0, \bar{n}, \bar{m}} T_{0, r-s, \bar{n}, \bar{m}} \) fully diagonal in Hilbert space. This mode has vanishing action and describes fully ergodic behavior at large interactions \( v > \gamma \).

**Phenomenological consequences:** The differences between the regimes discussed above show in the behavior of system observables, for example in its many body spectral correlations. In the clean limit, \( \gamma = 0 \), the \( L \) translation sectors of fluctuations \( T \) describe an equal number of Hilbert space sectors irreducibly transforming under the translationally invariant Bloch Hamiltonian. Each sector individually shows Wigner-Dyson correlations (as described by the fluctuations of the corresponding translation mode), however, the lack of statistical correlations between them implies that the many-body spectrum of the nearly clean system does not show level repulsion. For disorder strong enough to couple the translation sectors, and up to values \( \gamma \lesssim v \) a Wigner-Dyson regime described by the fluctuations of a fully ergodic mode \( T_0 \) ensues. The transition region \( \gamma \approx v \) is beyond the control of the present theory. However, for strong disorder \( \gamma \gg v \), the perturbative decoupling of the fluctuations \( T_d \), which describes the statistical independence of fluctuations in the locally diagonalized Hilbert space basis, implies Poissonian statistics. Both, the Wigner-Dyson and the Poissonian forms of the spectral statistics can be explicitly derived by integration over the corresponding field modes using the techniques described in Ref. [1].

**Discussion:** We have derived an effective field theory describing the physics of XXZ-chains in the presence of local disorder. Formulated in a first quantized language the theory effectively describes quantum transport on a translationally non-invariant lattice subject to site-diagonal disorder. Striking differences to standard single particle Anderson localization lie in the extensively high dimensionality \( \sim L \) of the lattice, and in the scarcity of only \( O(L) \) independent disorder parameters on a lattice with an exponentially large number \( 2^L \) of sites. Where the high configuration number favors delocalization, the statistical dependence of the site disorder leads to the formation of off-diagonal fluctuation modes describing statistical correlations between different sites. These fluctuations support localization via an effective decoupling of sites which we described to leading order in perturbation theory in \( v / \gamma \), and which we argued puts the system outside the Anderson class. As a sanity check we compared to the phenomenological random energy model which lacks this mechanism and in the consequence shows a delocalization instability at arbitrarily weak interactions. It is also tempting to compare the fluctuations of the theory in the effectively decoupled frame to those of the phenomenological theory of l-bits [20], however further work is required to substantiate this picture.

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Supplementary Material to “Effective field theory approach to many-body localization”

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In this supplementary material we discuss solutions to the mean field equation in the limits of weak and strong disorder, and provide details on the renormalization of the second order interaction vertex.

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I. MEAN FIELD EQUATION

We here discuss solutions to the mean field equation (1) of the main text. We start from a matrix-diagonal ansatz \(A_{nm} = i\kappa\tau_3 \otimes 1^{b_f}\delta_{nm}, \kappa > 0\), homogeneous in boson-fermion and Hilbert-space, and where the sign change described by \(\tau_3 = \tau_3^f\) is dictated by the causality of the Green function. At the band center the equation then assumes the form

\[
A = -\gamma^2 L \left( \frac{1}{i\delta\tau_3 - H_2 + A} \right)_{nn},
\]

where we used that \(f_{nn} = 1/L\). Strong disorder: In the strong disorder limit one may neglect the interaction, \(H_2 = 0\), to obtain a quadratic equation which is solved by \(\kappa = \gamma\sqrt{L}\). Strong interaction: In the strongly interacting limit, we may expand the equation in eigenfunctions \(|\alpha\rangle\) of \(H_2\).

\[
A = -\gamma^2 L \sum_{\alpha} \frac{|\langle\alpha|\rangle|^2}{i\delta\tau_3 - E_{\alpha} + A},
\]

Ignoring a real energy shift contribution to \(A\) which may be absorbed by a redefinition of the energy argument we then find

\[
\kappa \simeq \gamma^2 \text{Im} \int dE \frac{\rho(E)}{E - i\kappa} \simeq \pi\gamma^2 \rho,
\]

where \(\rho \sim 1/Lv\) is the on-site density of states of the clean interacting system, and we made the self consistent assumption that the disorder generated smearing of states, \(\sim \kappa\), is weak enough that \(\rho(E)\) remains approximately constant for energies at the band center.

II. RENORMALIZATION OF THE INTERACTION-VERTEX

We present details on the derivation of the effective second order interaction-vertex by integration over the vertex mode. Starting out from the action \(S = S_w + S_X + S_X^2\), where \(S_w = -\frac{1}{2L} \sum_{m'n'} \text{str}(Q_{nm'}f_{n'n'}Q_{n'n})\) is the disorder weight and \(S_X^1 = \frac{\gamma^2}{2L} \text{str}(XQXQ)\), are the expansion of the tr ln to first and second order in the interaction, respectively, the goal is to integrate out fluctuations coupling nearest-neighbor sites to arrive at an effective interaction between Hilbert-space diagonal modes.

To this end we fix a pair \(n, m\) of nearest neighbors and parametrize fluctuations as discussed in the main text, \(Q = T_1 T_2 Q v^T T_2^{-1} T_1^{-1}\), where \(Q_v = e^{v\tau_3} e^{-w^T} W = B_{mn}^{\alpha} B_{nm}^{\beta} + (n \leftrightarrow m)\). The two contributions to the generator \(W\) commute which means that the matrix \(Q_v\) splits into two independent matrices with block elements

\[
\left( Q_v^{++} \right)_{nn}, \left( Q_v^{++} \right)_{nm}, \left( Q_v^{++} \right)_{mn}, \left( Q_v^{--} \right)_{mm}, \left( Q_v^{--} \right)_{mn}, \left( Q_v^{--} \right)_{nm},
\]

and a second set of blocks with \(n \leftrightarrow m\). The two \((n \leftrightarrow m)\) interchanged matrix sectors are independently generated and can be integrated separately. We focus on the first, and add the \((n \leftrightarrow m)\) contribution at the end of the calculation. A straightforward expansion of the action to second order in \(G \sim X\) then leads to

\[
S_w^0 = \frac{1}{4L} \text{str}(Q_v \tau_3 Q_v \tau_3),
\]

\[
S_w^2 = -\frac{1}{L} \text{str}(2\Phi_G P^- Q_v P^- \Phi_G P^+ Q_v P^+ - \Phi_G P^- Q_v P^- \Phi_G P^+ Q_v P^+ - \Phi_G P^+ Q_v P^- \Phi_G P^- Q_v P^-),
\]

\[
S_X^1 = \frac{iv}{\gamma \sqrt{L}} \text{str}(\Phi_X P^- Q_v P^+ + \Phi_X P^+ Q_v P^-),
\]

\[
S_X^2 = \frac{\gamma^2}{2L} \text{str}(2\Phi_X P^- Q_v P^- \Phi_X P^+ Q_v P^+ + \Phi_X P^- Q_v P^- \Phi_X P^+ Q_v P^- + \Phi_X P^+ Q_v P^- \Phi_X P^+ Q_v P^-),
\]

where \(P^\pm = \frac{1}{2}(1 \pm \tau_3)\) are projection operators in advanced-retarded space, and we introduced \(\Phi_O = T^{-1}OT\). We note that \(\Phi_{X,G}\) have the same \((n,m)\) Hilbert space structure as \(Q_v\) in Eq. (6), suppressed for notational simplicity.

The effective interaction vertex between Hilbert-space diagonal modes at sites \(n, m\), is obtained by perturbative expansion in the interaction,

\[
S_{\text{eff}} = \langle S_w^0 + S_X^2 - \frac{1}{2} S_X^1 S_X^1/Q_v,\rangle,
\]
where $\langle...\rangle_Q = \int dQ_v e^{-S^0_v}$. We are left with the task of doing the integrals in (11). Following Efetov [1] we parametrize $Q_v = U_2 U_1 Q_0 U_1^{-1} U_2^{-1}$, where $(i = 1, 2)$

$$Q_0 = \begin{pmatrix} \cos \theta & i \sin \theta \\ -i \sin \theta & -\cos \theta \end{pmatrix}$$,

$$U_i = \begin{pmatrix} u_i^+ & 0 \\ 0 & u_i^- \end{pmatrix}$$, (12)

and the supermatrices $\hat{\theta} = \text{diag}(i\theta_b, \theta_f)$, $u_1^\pm = e^{-2\eta^\pm}$ with $\eta^\pm = (\eta^+ - \eta^-)$, $u_2^+ = (e^{i\phi} e^{i\chi})$, and $u_2^- = 1$. Here $0 \leq \theta_1 < \pi$ and $0 \leq \theta_b$ parametrize the compact fermionic and non-compact bosonic sectors, respectively, $0 \leq \phi, \chi < 2\pi$, and $\eta^\pm, \eta^\mp$ are independent Grassmann variables. The non-interacting action reads $S^0_v = \frac{1}{2} \left( \cosh^2 \theta_b - \cos^2 \theta_1 \right)$ and we notice that the main contribution to the integrals results from the large integration volume of the non-compact bosonic angle, $\theta_b \ll \ln L$. We may thus approximate $Q_0 = (\cosh \theta_b - \sinh \theta_b) \otimes P^b$, where $P^b = \frac{1}{2}(1 + \sigma_3^f)$ is a projector onto the bosonic sector.

Using the measure of the above polar representation [1] $dQ = \frac{1}{2\pi^2} \sinh \theta_b \sin \theta_1 d\phi d\theta_1 \sinh \eta^+ d\eta^+ d\eta^- d\eta^-$, and integrating over commuting and anti-commuting variables we find

$$\langle S^2_v \rangle_{Q_v} = 2 \sqrt{\frac{\pi}{L}} \text{str}(\Phi G P^- \Phi G P^+),$$ (13)

$$\langle S^2_{\bar{X}} \rangle_{Q_v} = - \sqrt{\frac{\pi \nu^2}{L \gamma^2}} \text{str}(\Phi X P^- \Phi X P^+),$$ (14)

$$\langle S^1_{\bar{X}} S^1_{X} \rangle_{Q_v} = \sqrt{\frac{\pi}{L}} \frac{2\nu^2}{\gamma^2} \text{str}(\Phi X P^- \Phi X P^+).$$ (15)

This can be rewritten as

$$S_{\text{eff}} = \frac{\sqrt{\pi}}{2 \sqrt{L}} \left( \frac{\nu^2}{\gamma^2} \text{str}(Q_n Q_m) X_{nm} - \text{str}(G_{nm} Q_m G_{mn} Q_n) \right),$$ (16)

where $Q_n = T_n \tau_3 T_n^{-1}$ are the Hilbert-space diagonal modes, and we re-introduced Hilbert-space indices. Adding the second contribution from generators with $n \leftrightarrow m$ exchanged increases the action [16] by a factor two.

[1] K. B. Efetov, *Supersymmetry in Disorder and Chaos* (Cambridge Univ. Press, 1999).