THE DIVERGENCE OF MOCK FOURIER SERIES FOR SPECTRAL MEASURES

WU-YI PAN AND WEN-HUI AI

Abstract. In this paper, we study divergence properties of Fourier series on Cantor-type fractal measure, also called Mock Fourier series. We give a sufficient condition under which the Mock Fourier series for doubling spectral measure is divergent on non-zero set. In particular, there exists an example of the quarter Cantor measure whose Mock Fourier sums is not almost everywhere convergent.

1. Introduction

Let \( \mu \) be a Borel probability measure on \( \mathbb{R}^d \) with compact support. We say that \( \mu \) is a spectral measure if there exists a discrete set \( \Lambda \subset \mathbb{R}^d \) such that \( E(\Lambda) := \{ e^{-2\pi i \lambda \cdot x} : \lambda \in \Lambda \} \) is an orthonormal basis for \( L^2(\mu) \). The first singular, nonatomic, spectral measure was constructed by Jorgensen and Pedersen [JP98]. They proved the surprising result that the quarter Cantor measure \( \mu_4 \) is a spectral. Over twenty years, many other interesting singular spectral measures on self-affine and Moran fractal sets have been constructed (see [AH14, DHL14, DHL19] and so on).

Given a spectral measure \( \mu \) with spectrum \( \Lambda \), for \( L^1(\mu) \) function \( f \), we define coefficients \( c_\lambda(f) = \int f(y) e^{-2\pi i \lambda \cdot y} d\mu(y) \) and the Mock Fourier series \( \sum_{\lambda \in \Lambda} c_\lambda(f) e^{2\pi i \lambda \cdot x} \). There is a natural sequence of finite subsets \( \Lambda_n \) increasing to \( \Lambda \) as \( n \to \infty \), and we define the partial sums of the Mock Fourier series by

\[
S_n(f)(x) = \sum_{\lambda \in \Lambda_n} c_\lambda(f) e^{2\pi i \lambda \cdot x}.
\]

We will use \( (S_n, \Lambda_n) \) to denote the Mock Dirichlet summation operator \( S_n \) with \( \Lambda_n \).

As analogue to classical Fourier analysis, an extremely natural question is whether \( S_n(f) \) converges to \( f \) as \( n \to \infty \). The answer has an added piquancy since: not only does it depend on the determining what the function space \( f \) is belonged to, but it also depends critically on how one defines “convergence”.

As we all know, for any continuous function, the assertion of uniform convergence of classical Fourier series is wrong [Zy68]. By contrast, Strichartz [St00] showed that it is true for some singular continuous spectral measure with standard spectrum. Unfortunately, for given spectral measure, different spectrum may have different convergence. Dutkay et al. [DHS14] proved there is a continuous function \( f \) whose \( (S_n(f), \Lambda_n) \) does not even converge pointwise to \( f \) for standard spectrum \( \Lambda = \bigcup_{n=0}^{\infty} \Lambda_n \).

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In this paper, we study divergence properties of doubling spectral measures. Let \((M, \rho)\) be a metric space and suppose that \(\mu\) is a positive locally finite Borel measure on \(M\). We call \(\mu\) a doubling measure if \(\mu\) satisfies the doubling condition

\[
\mu(B(x, 2r)) \leq A_1 \mu(B(x, r)) < \infty
\]

for all \(x \in M\) and \(r > 0\), where \(A_1\) is constant and independent of \(x, r\). Here \(B(x, r)\) denotes the closed ball \(B(x, r) = \{y \in M : \rho(y, x) \leq r\}\).

To conveniently state our main results, we just briefly introduce the related concepts, and their well-posedness will be given in Section 2. Recall the finite discrete measure defined on a measure space \((X, A, \mu)\) has the form

\[
\nu = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}
\]

for every finite collection \(x_1, x_2, \ldots, x_N \in X\) not necessarily pairwise different points, where \(\delta_{x_i}\) is the Dirac measure concentrated at the point \(x \in X\). Given a Mock Dirichlet summation operator \(S_n\) with \(\Lambda_n\), we formally write

\[
S_n(\nu)(x) = \frac{1}{N} \sum_{x \in \Lambda_n} \sum_{\lambda \in \Lambda_n} e^{2\pi i \lambda \cdot (x - x_n)}.
\]

Now we state our main result.

**Theorem 1.1.** Assume \(\mu\) is a doubling spectral measure. Let \((S_n, \Lambda_n)\) be the Mock Dirichlet summation operator and if

\[
\lim_{\alpha \to \infty} \sup_{\nu = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}} \mu(\{x \in X : \sup_{n} |S_n(\nu)(x)| > \alpha\}) > 0,
\]

then there exists an integrable function such that the Mock Fourier series diverges on \(\mu\)-non-zero set.

As a corollary, we find the example in [DHS14] diverges on non-zero set.

**Corollary 1.2.** Let \(\mu_4\) be quarter Cantor measure and suppose \((S_n, 17\Lambda_n)\) is the Mock Dirichlet summation operator with

\[
17\Lambda_n = \{17 \sum_{j=0}^{n} 4^j l_j : l_j \in \{0, 1\}, \ n \in \mathbb{N}\}.
\]

Then there exists an integrable function \(f \in L^1(\mu_4)\) whose Mock Fourier series \(S_n(f)(x)\) diverges on a non-zero set.

We organize our paper as follows. In Section 2, we firstly present a brief overview of the the relationship between the continuity of maximal operators and convergence almost everywhere. Succeeded by, we introduce the main tool for our proof of Theorem 1.1 i.e., the dyadic cube analysis constructed by Christ [Ch90]. In Section 3, as an application of Theorem 1.1 we consider the self-affine measure generated by Hadamard triple. Under some technical condition about the spectrum, we give a criterion on there exists an integrable function whose Mock Fourier series diverges on non-zero set. The criterion can be applied to cover Corollary 1.2.

2. The proof of theorem 1.1

Let \((X, A, \mu)\) be a complete finite measure space with a \(\sigma\)-field \(A\). To recall some basic facts firstly, the space of (equivalence classes of) all measurable functions
on $\langle X, \mathcal{A}, \mu \rangle$ is denoted by $L^0(\mu)$. It is endowed with the topology of convergence in measure by the metric

$$d(f, g) = \int_X \frac{|f - g|}{1 + |f - g|} \, d\mu.$$  

It is not difficult to show that $(L^0(\mu), d)$ is a complete metric space.

A mapping $T : (M, d_1) \to (L^0(\mu), d)$ from a metric space $M$ to $L^0(\mu)$ is said to be continuous at $x \in M$, if for any sequence $\{x_n\} \subset M$, we have $d(Tx_n, Tx) \to 0$ whenever $d_1(x_n, x) \to 0$. We call mapping $T$ is continuous if it is continuous at every point of $M$.

We firstly recall the following theorem due to Guzman [Gu81] on the almost everywhere finiteness of maximal operator.

**Theorem 2.1.** [Gu81, p. 10] Assume $\langle X, \mathcal{A}, \mu \rangle$ is a complete measure space and $T_k : L^1(\mu) \to (L^0(\mu), d)$ is a sequence of sub-linear operators with $\mu(X) < \infty$. If each $T_k$ is continuous and that the maximal operator $T^*$ defined for $f \in L^1(\mu)$ and $x \in X$ as

$$T^* f(x) = \sup_k |T_k f(x)| < \infty \quad \mu - a.e.$$

Then $T^*$ is also continuous at 0, and therefore

$$\lim_{\alpha \to \infty} \phi(\alpha) := \lim_{\alpha \to \infty} \sup_{\|f\|_{L^1(\mu)} \leq 1} \mu(\{x \in X : T^* f(x) > \alpha\}) = 0.$$  

Notice that bounded linear operator is always continuous. By Theorem 2.1, if $\{T_k\}$ is a sequence of bounded linear operators, then $\lim_{\alpha \to \infty} \phi(\alpha) > 0$ implies there exists an integrable function $g$ such that $T_k(g)$ is not almost everywhere convergent.

**Corollary 2.2.** Let $\langle X, \mathcal{A}, \mu \rangle$ be a complete measure space and $T_k : L^1(\mu) \to (L^0(\mu), d)$ be a sequence of bounded linear operators with $\mu(X) < \infty$. If

$$\lim_{\alpha \to \infty} \sup_{\|f\|_{L^1(\mu)} \leq 1} \mu(\{x \in X : T^* f(x) > \alpha\}) > 0,$$

then a function $g$ existing in $L^1(\mu)$ can be obtained that its $T_k(g)$ diverges on $\mu$-non-zero set in $X$.

However, it is difficult to verify (1), but we can consider a sum of Dirac measure firstly, which is rather easy to handle in many cases. Concretely, let $\langle X, \mathcal{A}, \mu \rangle$ and $\langle X, \mathcal{A}, \nu \rangle$ be complete Borel measure spaces defined on a Hausdorff space $X$, and $\mathcal{B}(X)$ be the space of locally finite Borel space measure on $X$. Consider a sequence $k_j$ of kernels satisfying the following two properties:

(i) Each $k_j : X \times X \to X$ is a measurable function such that $k_j(\cdot, y) \in L^1(\mu)$.

(ii) For each $j$ there exists $O_j$ such that

$$\|k_j(\cdot, y)\|_{L^1(\mu)} \leq O_j \quad \text{for every } y \in X.$$  

We write

$$K_j f(x) = \int_X k_j(x, y) f(y) \, d\mu(y) \quad \text{for } f \in X.$$  

Using Fubini-Tonelli’s theorem, the second property of kernels makes the maximal operator make sense as following:

$$K^* f(x) = \sup_j |K_j f(x)|.$$
Moreover, if \( k_j(x, y) \) is a continuous function with compact support on \( X \times X \) for any \( j \in \mathbb{N} \), then each \( K_j \) is bounded linear operator from \( L^1(\mu) \) to \( L^\infty(\mu) \). Such an operator has a natural extension to a bounded linear operator from \( \mathcal{B}(X) \) to \( L^\infty(\mu) \), which we denote by \( K_j \) again, namely
\[
K_j(\nu)(x) = \int_X k_j(x, y) d \nu(y), \quad K^*\nu(x) = \sup_j |K_j\nu(x)|. 
\]

Especially, choose a sum of Dirac measure \( \nu = \sum_{h=1}^{H} \delta_{x_h} \) for \( x_1, \cdots, x_H \in X \), then
\[
K_j\nu(x) = \frac{1}{H} \sum_{h=1}^{H} k_j(x, x_h), \quad K^*\nu(x) = \frac{1}{H} \sup_j |\sum_{h=1}^{H} k_j(x, x_h)|.
\]

In what follows, our aim is to extend the "pointillist principle" of Carena [Ca09, Theorem 1], then the similar conclusion can be obtained under slightly different condition. To certificate the Theorem 1.1, recall the dyadic cubes constructed by Christ in [Ch90], which is very important for extending results from harmonic analysis to the metric space setting.

**Theorem 2.3.** [Ch90, Theorem 11] Let \((X, \rho)\) be a metric space and suppose that \( \mu \) is a regular doubling measure on \( X \). Then there exists a collection of open subsets \( \{Q^k_\alpha \subset X : k \in \mathbb{Z}, \alpha \in I_k\} \) satisfying the following properties.

(i) For each integer \( k \),
\[
\mu \left( X \setminus \bigcup_\alpha Q^k_\alpha \right) = 0.
\]

(ii) Each \( Q^k_\alpha \) has a center \( z_{Q^k_\alpha} \) such that
\[
B(z_{Q^k_\alpha}, C_1 \delta^k) \subseteq Q^k_\alpha \subseteq B(z_{Q^k_\alpha}, C_2 \delta^k),
\]
where \( C_1, C_2 \) and \( \delta \) are positive constants depending only on the doubling constant \( A_1 \) of the measure \( \mu \) and independent of \( Q^k_\alpha \).

(iii) For each \((k, \alpha)\) and each \( l < k \), there is a unique \( \beta \) such that \( Q^k_\alpha \subset Q^l_\beta \).

(iv) For any \( k, \alpha \) and \( t > 0 \), there exist constants \( \delta \in (0, 1), C_3, \eta > 0 \) depending only on \( \mu \) such that
\[
\mu \{x \in Q^k_\alpha : \rho(x, X \setminus Q^k_\alpha) \leq t\delta^k\} \leq C_3 t^\eta \mu(Q^k_\alpha).
\]

\( I_k \) denotes some index set, depending on \( k \). Dyadic cubes is constructed by
\[
\bigcup_{k \in \mathbb{Z}, \alpha \in I_k} \{Q^k_\alpha\}.
\]

It should be noted that the center of dyadic cubes satisfies maximal \( \delta^k \)-distance disperse condition, that is
\[
\rho(z_{Q^k_\alpha}, z_{Q^k_\beta}) \geq \delta^k \quad \text{for any } \alpha \neq \beta.
\]

In this context, maximality means that no new points of the space \( X \) can be added to the set \( \{z_{Q^k_\alpha}\} \) such that (3) remains valid. One other factor indeed, the last condition says that the area near the boundary of a "cube" \( Q^k_\alpha \) is small.

The main result is as follow.
Lemma 2.4. Let \((X, \rho)\) be a metric space and let \(\mu\) be a positive regular Borel measure satisfying the doubling condition on \(X\). Let \(\nu\) be a measure such that \(d\nu = g d\mu\) with \(g \in L_{\text{loc}}^1(X, \rho, \mu)\). Denote
\[
\psi_\alpha(f) : = \nu\{x \in X : |f(x)| > \alpha\}
\]
for a measurable function \(f\) defined on \((X, \rho, \nu)\). If each kernel \(k_j(x, y)\) is a continuous function with compact support on \(X \times X\) and \(K^*\) is defined in \((2)\), then
\[
\lim_{\alpha \to \infty} \sup_{\|f\|_{L^1(\nu)} \leq 1} \psi_\alpha(K^*f) = 0 \iff \lim_{\alpha \to \infty} \sup_{\omega} \psi_\alpha(K^*\omega) = 0
\]
for every finite collection \(a_1, a_2, \ldots, a_H \in X\) not necessarily pairwise different points.

Proof. For pairwise different points \(a_1, a_2, \ldots, a_H \in X\), we firstly denote following sets:
\[
\mathcal{M}_\mathbb{N} = \{\omega = \frac{\sum_{h=1}^H c_h \delta_{a_h}}{\sum_{h=1}^H c_h} : c_h \in \mathbb{N}^+\}, \\
\mathcal{M}_\mathbb{Q} = \{\omega = \frac{\sum_{h=1}^H c_h \delta_{a_h}}{\sum_{h=1}^H c_h} : c_h \in \mathbb{Q}^+\}, \\
\mathcal{M}_\mathbb{R} = \{\omega = \frac{\sum_{h=1}^H c_h \delta_{a_h}}{\sum_{h=1}^H c_h} : c_h \in \mathbb{R}^+\},
\]
and
\[
\mathcal{F}_\mathbb{R} = \{f = \frac{\sum_{h=1}^H c_h \chi_{Q_h}}{\sum_{h=1}^H c_h} : c_h \in \mathbb{R}^+, \ Q_i \cap Q_j = \emptyset\},
\]
where \(Q_h\) is dyadic cube constructed in Theorem 2.3 and \(\chi_{Q_h}\) is characteristic function.

**Necessity.** We divide the proof of
\[
\lim_{\alpha \to \infty} \sup_{\omega} \psi_\alpha(K^*\omega) = 0 \implies \lim_{\alpha \to \infty} \sup_{\|f\|_{L^1(\nu)} \leq 1} \psi_\alpha(K^*f) = 0
\]
into four steps.

**Step 1.** By \(\lim_{\alpha \to \infty} \sup_{\omega} \psi_\alpha(K^*\omega) = 0\), it is easy to see \(\lim_{\alpha \to \infty} \sup_{\omega \in \mathcal{M}_\mathbb{N}} \psi_\alpha(K^*\omega) = 0\). In this step, we verify that \(\lim_{\alpha \to \infty} \sup_{\omega \in \mathcal{M}_\mathbb{Q}} \psi_\alpha(K^*\omega) = 0\).

To prove this, we write \(c_h = \frac{m_h}{n_h}\) with \(n_h, m_h \in \mathbb{N}^+\) for \(c_h \in \mathbb{Q}^+\). Since
\[
\frac{\sum_{h=1}^H c_h \delta_{a_h}}{\sum_{h=1}^H c_h} = \left(\sum_{h=1}^H \frac{\tau_{a_h}}{\sum_{h=1}^H \tau_{a_h}}\right), \text{ where } \tau_{a_h} = n_h \prod_{j=1, j \neq h}^H m_j \in \mathbb{N},
\]
we know \(\mathcal{M}_\mathbb{N} = \mathcal{M}_\mathbb{Q}\). Hence \(\lim_{\alpha \to \infty} \sup_{\omega \in \mathcal{M}_\mathbb{Q}} \psi_\alpha(K^*\omega) = 0\).

**Step 2.** In this step, we want to prove that \(\lim_{\alpha \to \infty} \sup_{\omega \in \mathcal{M}_\mathbb{R}} \psi_\alpha(K^*\omega) = 0\). Let us start by defining maximal truncated operator \(K^*_Nf = \max_{1 \leq j \leq N} |K_j f(x)|\), then it is clear that
\[
\lim_{N \to \infty} \psi_\alpha(K^*_Nf) = \psi_\alpha(K^*f)
\]
for each \(\alpha > 0\). Similarly, we define \(K^*_N\omega\) for finite discrete measure \(\omega\).
We next claim that, for each integer $N$, real numbers $\alpha, \varepsilon > 0$, $0 < \beta < \alpha$ and $\omega \in \mathcal{M}_R$, we can find a finite discrete measure $\varpi \in \mathcal{M}_Q$ satisfying the inequality
\[
\psi_\alpha(K_N^* \omega) \leq \psi_{\alpha-\beta}(K_N^* \varpi) + 2\varepsilon.
\]

In fact, for $\omega \in \mathcal{M}_R$, take $d_h \in \mathbb{Q}^+$ such that $c_h = d_h + r_h$, where $r_h > 0$ will be determined later. Since
\[
\frac{\sum_{h=1}^H c_h k_j(x, a_h)}{\sum_{h=1}^H c_h} = \frac{\sum_{h=1}^H d_h k_j(x, a_h) + \sum_{h=1}^H r_h k_j(x, a_h)}{\sum_{h=1}^H c_h} = \frac{\sum_{h=1}^H r_h k_j(x, a_h)}{\sum_{h=1}^H c_h} + \frac{\sum_{h=1}^H c_h k_j(x, a_h)}{\sum_{h=1}^H c_h} - \frac{\sum_{h=1}^H r_h}{\sum_{h=1}^H c_h},
\]
we simplify above equality as
\[
K_j(c) = K_j(d) + K_j(rd) - K_j(crd).
\]
Then, for $0 \leq \beta \leq \alpha$, we have
\[
\psi_\alpha(K_N^* \omega) = \psi_\alpha \left( \sup_{1 \leq j \leq N} |K_j(c)| \right)
\leq \psi_{\alpha-\beta} \left( \sup_{1 \leq j \leq N} |K_j(d)| \right) + \psi_{\frac{\beta}{2}} \left( \sup_{1 \leq j \leq N} |K_j(rd)| \right) + \psi_{\frac{\beta}{2}} \left( \sup_{1 \leq j \leq N} |K_j(crd)| \right).
\]
Furthermore,
\[
\psi_{\frac{\beta}{2}} \left( \sup_{1 \leq j \leq N} |K_j(rd)| \right) \leq \psi_{\frac{\beta}{2}} \left( \frac{\sum_{j=1}^N \sum_{h=1}^H r_h |k_j(x, a_h)|}{\sum_{h=1}^H d_h} \right)
\leq \frac{2}{\beta} \sum_{j=1}^N \sum_{h=1}^H r_h \int_X |k_j(x, a_h)| |d\nu| \cdot \frac{1}{\sum_{h=1}^H d_h},
\]
and $\int_X |k_j(x, a_h)| |d\nu| < \infty$ for all $j$, we can choose small $r_h$ such that the right side hand of the above inequalities are all less than arbitrary $\varepsilon > 0$. If we write $\varpi = \frac{\sum_{h=1}^H d_h \delta_{\gamma_h}}{\sum_{h=1}^H d_h}$, our claim is proved.

Hence we use $K_N^* \leq K^*$ to get $\psi_\alpha(K_N^* \omega) \leq \sup_{\varpi \in \mathcal{M}_Q} \psi_{\alpha-\beta}(K^* \varpi) + 2\varepsilon$. Moreover, taking the maximum in the measure family $\mathcal{M}_R$ and letting $\beta \to 0$, $\varepsilon \to 0$, we have
\[
\sup_{\varpi \in \mathcal{M}_Q} \psi_\alpha(K^* \varpi) = \sup_{\omega \in \mathcal{M}_R} \psi_\alpha(K_N^* \omega) \leq \lim_{\alpha \to 0} \sup_{\omega \in \mathcal{M}_R} \psi_\alpha(K^* \varpi).
\]
Observing that $\sup_{\varpi \in \mathcal{M}_Q} \psi_\alpha(K^* \varpi)$ monotonically decreases with $\alpha$ and $\sup_{\varpi \in \mathcal{M}_Q} \psi_\alpha(K^* \varpi)$ monotonically decreases with $\alpha$, we have
\[
\lim_{\alpha \to 0} \sup_{\varpi \in \mathcal{M}_Q} \psi_\alpha(K^* \varpi) = 0.
\]
By sandwich theorem, we obtain
\[
\lim_{\alpha \to 0} \sup_{\varpi \in \mathcal{M}_Q} \psi_\alpha(K^* \varpi) = 0.
\]
By the way, this trick will be used repeatedly. To facilitate the writing, we will not describe this process in detail.

**Step 3.** In this step, we show that \( \lim_{\alpha \to \infty} \sup_{f \in \mathcal{F}_R} \psi_\alpha(K^*f) = 0 \). Repeat the trick in **Step 2**, it is enough to prove that for each integer \( N \), real numbers \( \alpha, \varepsilon > 0 \), \( 0 < \beta < \alpha \) and \( f \in \mathcal{F}_R \), there exists a finite discrete measure \( \omega \in \mathcal{M}_R \) satisfying the inequality

\[
\psi_\alpha(K^*_N f) \leq \psi_{\alpha - \beta}(K^*_N \omega) + \varepsilon.
\]

The desired limiting behavior will follow by letting \( \beta \to 0 \), \( \varepsilon \to 0 \) and \( N \to \infty \).

Let \( f = \frac{\sum_{h=1}^H c_h \chi_{Q_h}}{\sum_{h=1}^H c_h \nu(Q_h)} \in \mathcal{F}_R \) and \( \omega = \frac{\sum_{h=1}^H c_h \nu(Q_h) \delta_{z_Q}}{\sum_{h=1}^H c_h \nu(Q_h)} \in \mathcal{M}_R \), where \( z_Q \) denotes the center of the dyadic cube \( Q \). If \( 0 \leq \beta \leq \alpha \), for fixed \( N \), we obtain that

\[
\psi_\alpha(K^*_N f) \leq \psi_{\alpha - \beta}(K^*_N \omega) + \psi_{\beta}(K^*_N (f - \omega)).
\]

From

\[
\left| \left( \sum_{h=1}^H c_h \nu(Q_h) \right) K_j(f - \omega) \right| = \left| \sum_{h=1}^H c_h \left( \int_{Q_h} k_j(x, y) d\nu(y) - \int_{Q_h} k_j(x, z_Q) d\nu(y) \right) \right|
\]

\[
\leq \sum_{h=1}^H c_h \int_{Q_h} |k_j(x, y) - k_j(x, z_Q)| d\nu(y),
\]

we have

\[
\psi_\beta(K^*_N (f - \omega))
\]

\[
\leq \sum_{j=1}^N \frac{1}{\beta} \int_X |K_j(f - \omega)(x)| d\nu(x)
\]

\[
\leq \frac{1}{\beta} \left( \sum_{h=1}^H c_h \nu(Q_h) \right) \sum_{j=1}^N \int_X \left( \sum_{h=1}^H c_h \int_{Q_h} |k_j(x, y) - k_j(x, z_Q)| d\nu(y) \right) d\nu(x)
\]

\[
\leq \frac{1}{\beta} \left( \sum_{h=1}^H c_h \nu(Q_h) \right) \sum_{j=1}^N \sum_{h=1}^H \int_{Q_h} \left( \int_{F_j} |k_j(x, y) - k_j(x, z_Q)| d\nu(x) \right) d\nu(y),
\]

where \( F_j \) denotes the projection of the support of \( k_j(x, y) \). Clearly, \( F_j \) is a bounded set with finite measure. Since every \( k_j(x, y) \) is a uniformly continuous function with compact support in \( X \times X \), we can take small \( \text{diam}(Q_j) \) such that the term in above inequalities is small enough. Similar to the previous proof in **Step 2**, we conclude that \( \lim_{\alpha \to \infty} \sup_{f \in \mathcal{F}_R} \psi_\alpha(K^*f) = 0 \).

**Step 4.** Finally, from the fact that the set of all real coefficients linear combinations of characteristic functions of dyadic sets is dense in \( L^1(\mu) \), by a standard argument, one obtains

\[
\lim_{\alpha \to \infty} \sup_{\|f\|_1 \leq 1} \psi_\alpha(K^*f) = \lim_{\alpha \to \infty} \sup_{f \in \mathcal{F}_R} \psi_\alpha(K^*f) = 0.
\]

The proof of Lemma 2.4 in one direction is complete.
2.3. We claim the set in 

\[ \{ \text{large integer such that } h \text{ for } \omega \text{ then } Q \text{ and (ii) in Theorem 2.3, then there exists} \} \]

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Hence, \[ \omega \text{ Clearly we have } \rho \text{ the metric by } \}

\[ \text{This is a contradiction to } \rho(x_h, x_m) \leq \rho(z_{Q^n_{im}}) + \rho(x_{Q^n_{im}}, x) = \rho(z_{Q^n_{im}}, x_m) < d. \]

This is a contradiction to \( \rho(x_h, x_m) \geq d \). Apparently, let \( c_h \in \mathbb{N}^+ \) and

\[ \omega = \sum_{h=1}^{H} c_h \delta_{x_h}, \quad f = \sum_{h=1}^{H} c_h \nu(Q^n_{ih}) \chi_{Q^n_{ih}}, \]

then \( \omega \in M_N \) and \( \| f \|_1 = 1 \). Fix \( N \in \mathbb{N}^+ \) and \( \beta > 0 \), through simple calculation, we have

\[ \psi_\beta(K^*_N(f - \omega)) \leq \sum_{j=1}^{N} \psi_\beta(K_j(f - \omega)) \leq \sum_{j=1}^{N} \frac{1}{\beta} \left| \int_X k_j(x, y)(f(y)d\nu - d\omega) \right| \]

and

\[ \left| \int_X k_j(x, y)(f(y)d\nu - d\omega) \right| \leq \frac{1}{\sum_{h=1}^{H} c_h} \left| \int_{Q^n_{ih}} k_j(x, x_h) - k_j(x, x_h)d\nu(y) \right| . \]

Hence,

\[ \psi_\beta(K^*_N(f - \omega)) \leq \frac{1}{\sum_{h=1}^{H} \beta c_h} \left( \int_{Q^n_{ih}} \int_X |k_j(x, y) - k_j(x, x_h)|d\nu(y) \right) d\nu(x) \]

\[ \leq \frac{1}{\sum_{h=1}^{H} \beta c_h} \left( \int_{Q^n_{ih}} \int_{F_j} |k_j(x, y) - k_j(x, x_h)|d\nu(x) \right) d\nu(y), \]

where \( F_j \) is the projection of the support of \( k_j(x, y) \). Similar to Step 3, we can choose the diameter of the dyadic sets such that \( \psi_\beta(K^*_N(f - \omega)) \leq \varepsilon \) for any \( \varepsilon > 0 \). Repeat the same step like before, we obtain that \( \lim_{\alpha \to \infty} \sup_{\omega \in M_N} \psi_\alpha(K^*\omega) = 0 \).

This completes the proof of Lemma 2.3.

Combining with Corollary 2.2 and Lemma 2.4 we obviously have the following corollary.
Corollary 2.5. Let $(X, \rho)$ be a metric space. Assume every kernel $k_j(x, y)$ is a continuous function with compact support on $X \times X$ and $K^*$ is defined in (2). If $\mu$ is a totally finite complete measure satisfying the doubling condition and
\[
\lim_{\alpha \to \infty} \sup_{\omega} \mu(\{x \in X : K^* \omega(x) > \alpha\}) > 0,
\]
then there exists a function $g \in L^1(\mu)$ such that $T_k(g)$ diverges on $\mu$-non-zero set in $X$.

Based on above lemmas, we can prove Theorem 1.1.

Proof of Theorem 1.1. Let $\mu$ denote a spectral measure supported on a compact subset of $X \subset (\mathbb{R}^d, \rho)$, where $(\mathbb{R}^d, \rho)$ is the Euclidean space. Let $\{e^{-2\pi i \lambda \cdot x} : \lambda \in \Lambda\}$ be an exponential orthonormal basis of $L^2(\mu)$. Define the Mock Dirichlet kernel as
\[
k_n(x, y) = \sum_{\lambda \in \Lambda_n} e^{2\pi i \lambda \cdot (x - y)}.
\]
Then the Mock Dirichlet summation operator $S_n$ with $\Lambda_n$ can be written as
\[
S_n(f)(x) = \sum_{\lambda \in \Lambda_n} c_\lambda(f) e^{2\pi i \lambda \cdot x} = \int_X k_n(x, y) f(y) d\mu(y).
\]
From Corollary 2.5, the theorem follows immediately.

3. Application to self-affine spectral measures

In this section, we apply our results to self-affine spectral measures. Recall that the self-affine measure is defined by iterated function system (IFS).

Definition 3.1 (Self-affine measure). Let $R$ be a $d \times d$ expansive matrix (all its eigenvalues have modulus strictly bigger than one). Let $B = \{b_0, b_1, \cdots, b_{N-1}\}$ be a finite subset of $\mathbb{R}^d$. We define the affine iterated function system
\[
\varphi_b(x) = R^{-1}(x + b) \quad \text{for} \ x \in \mathbb{R}^d \text{ and } b \in B.
\]
The self-affine measure (with equal weights) is the unique probability measure satisfying
\[
\mu(E) = \frac{1}{N} \sum_{j=1}^{N} \mu(\varphi_b^{-j}(E)) \quad \text{for all Borel subsets } E \text{ of } \mathbb{R}^d.
\]
This measure is supported on the attractor $T(R, B)$ which is the unique compact set that satisfies
\[
T(R, B) = \bigcup_{b \in B} \varphi_b(T(R, B)).
\]
The set $T(R, B)$ is also called the self-affine set associated with the IFS. It can also be described as
\[
T(R, B) = \left\{ \sum_{k=1}^{\infty} R^{-k} b_k : b_k \in B \right\}.
\]
One can refer to [Hu81] for a detailed exposition of the theory of iterated function systems. In this section, we will use $\mu_{R, \{\pm 1\}}$ to denote Cantor measure which is the special case when $d = 1$ and $B = \{ \frac{-1}{2}, \frac{1}{2} \}$.

To the best of our knowledge, most of self-affine spectral measures are constructed by Hadamard triples.
Definition 3.2 (Hadamard triple). For a given expansive $d \times d$ matrix $R$ with integer entries. Let $B, L \subset \mathbb{Z}^d$ be finite sets of integer vectors with the same cardinality $N \geq 2$. We say that the triple $(R, B, L)$ forms a Hadamard triple if the matrix

$$H = \frac{1}{\sqrt{N}} \left[ \sum_{l \in L} e^{2\pi i R^{-1} b \cdot l} \right]_{b \in B}$$

is unitary, i.e., $H^* H = I$, where $H^*$ denotes conjugate transpose of $H$.

The system $(R, B, L)$ forms a Hadamard triple if and only if the Dirac measure $\delta_{R^{-1}D} = \frac{1}{#B} \sum_{b \in B} \delta_{R^{-1}b}$ is a spectral measure with spectrum $L$. Moreover, this property is a key property in producing spectrum of self-affine spectral measures. Laba and Wang [LW02], Dutkay, Hausermann and Lai [DHL19] eventually proved that Hadamard triple generates self-affine spectral measure in all dimension.

If $(R, B, L)$ forms a Hadamard triple, we let

$$\Lambda_n = L + R^1 L + (R^1)^2 L + \cdots + (R^1)^{n-1} L = \sum_{k=0}^{n-1} (R^1)^k L,$$

and

$$\Lambda = \bigcup_{n=0}^{\infty} \Lambda_n = \sum_{k=0}^{\infty} (R^1)^k L,$$

where $R^t$ denotes transpose of $R$.

The set $\Lambda$ forms an orthonormal set for the self-affine spectral measure $\mu := \mu_{R, B}$. But the set $\Lambda$ can be incomplete (see [DLW17, p. 4]). In this paper, we assume that the self-affine spectral measure $\mu$ generated by Hadamard triple $(R, B, L)$ always has a spectrum like $\Lambda$.

We say that the self-affine measure $\mu$ in Definition 3.1 satisfies the no-overlap condition or measure disjoint condition if

$$\mu(\varphi_b(T(R, B)) \cap \varphi_{b'} T(R, B)) = 0 \text{ for all } b \neq b' \in B.$$

Dutkay, Hausermann and Lai [DHL19] proved that if the self-affine spectral measure is generated by Hadamard triple, then the no-overlap condition is satisfied.

Following the work of Dutkay et al. [DHS14], encoding map plays the key role in linking no-overlap self-affine spectral measure and code space. Let $\mathbb{N}^*$ denote the positive integer numbers. For symbolic space $B^{\mathbb{N}^*}$ with the product probability measure $dP$ where each digit in $B$ has probability $1/N$, Dutkay et al. [DHS14] proved following proposition.

Proposition 3.3. [DHS14, Proposition 1.11] Define the encoding map $h : B^{\mathbb{N}^*} \mapsto T(R, B)$ by

$$h(b_1 b_2 b_3 \ldots) = \sum_{i=1}^{\infty} R^{-i} b_i,$$

then $h$ is onto, it is one to one on a set of full measure, and it is measure preserving.

Define the right shift $S : B^{\mathbb{N}^*} \mapsto B^{\mathbb{N}^*}$,

$$S(b_1 b_2 b_3 \cdots) = b_2 b_3 \cdots,$$

and the map $R : T(R, B) \mapsto T(R, B)$,

$$R \left( \sum_{i=1}^{\infty} R^{-i} b_i \right) = \sum_{i=1}^{\infty} R^{-i} b_{i+1}.$$
Then 
\[ hS = R h. \]

By Proposition 3.3, we immediately obtain the following corollary.

**Corollary 3.4.** The dynamical systems of \((T(R, B), F, \mu, R)\) is ergodic.

For a self-affine spectral measure \(\mu := \mu_{R, B}\) generated by Hadamard triple \((R, B, L)\), let \(\tau\) be an integer such that

\[ \tau \Lambda = \tau \bigcup_{n=0}^{\infty} \Lambda_n = \tau \sum_{k=0}^{\infty} (R^t)^k L \]

is a spectrum of \(\mu\). Define the Dirichlet kernel

\[ D_n(x) := \sum_{\lambda \in \tau \Lambda_n} e^{2\pi i \lambda \cdot x} \ (x \in \mathbb{R}). \]

For \(f \in L^1(\mu)\), the Mock Dirichlet summation operator

\[ S_n(f)(x) = \sum_{\lambda \in \tau \Lambda_n} \left( \int_{T(R, B)} f(y) e^{-2\pi i \lambda \cdot y} d\mu(y) \right) e^{2\pi i \lambda \cdot x} \]

can be written as

\[ S_n(f)(x) = \int_{T(R, B)} f(y) D_n(x - y) d\mu(y). \]

Similar to [DHS14, Proposition 2.2], we have following result.

**Proposition 3.5.** Define trigonometric polynomials

\[ m_\tau(x) = \sum_{l \in L} e^{2\pi i (\tau l) \cdot x} \ (x \in \mathbb{R}). \]

Then the Dirichlet kernel satisfies the formula

(6) \[ D_n(x) = \prod_{k=0}^{n} m_\tau((R^t)^k x). \]

**Proof.** The proof is by induction on \(n\). It is clear that

\[ D_0(x) = \sum_{\lambda \in \tau \Lambda_0} e^{2\pi i \lambda \cdot x} = \sum_{l \in L} e^{2\pi i (\tau l) \cdot x} = m_\tau(x), \]

Suppose \(D_n(x) = \prod_{k=0}^{n} m_\tau((R^t)^k x)\), we need to prove

(7) \[ D_{n+1}(x) = m_\tau(x) D_n(R^t x). \]

Since \(\Lambda_{n+1} = R^t \Lambda_n + L\), we see that every point \(\lambda_{n+1}\) in \(\Lambda_{n+1}\) will have a unique representation of the form \(\lambda_{n+1} = R^t \lambda_n + l\) with \(\lambda_n \in \Lambda_n\) and \(l \in L\). This yields

\[ D_{n+1}(x) = \sum_{\lambda_n \in \Lambda_n} \sum_{l \in L} e^{2\pi i (R^t \lambda_n + l) \cdot x} \]
\[ = \sum_{l \in L} e^{2\pi i \tau l \cdot x} \sum_{\lambda_n \in \Lambda_n} e^{2\pi i R^t \lambda_n \cdot x} \]
\[ = m_\tau(x) D_n(R^t x). \]

Then equation (6) follows by induction from equation (7). \qed
Using above propositions and Theorem 1.1, we can obtain following lemma about doubling spectral measure generated by Hadamard triple.

**Lemma 3.6.** Let $R$ be an integer symmetric matrix and let $\mu := \mu_{R,B}$ be a self-affine spectral measure generated by Hadamard triple $(R, B, L)$. Assume $\mu$ is a doubling measure with spectrum $\tau_\Lambda = \sum_{k=0}^{\infty} R^k \tau_L$. Let

$$\Delta(m_{\tau, b}) := \exp \left( \int_{T(R,B)} \log |m_\tau(x - (I - R^{-1})^{-1}b)| d\mu(y) \right), \quad b \in B,$$

where $m_\tau(x)$ is defined in Proposition 3.5. If $\Delta(m_{\tau, b}) > 1$ for some $b \in B$, then there exists an integrable function such that the Mock Fourier series diverges on non-zero set.

**Proof.** Recall that the points in $T(R, B)$ have the form $x = \sum_{i=1}^{\infty} R^{-i}b_i$ with $b_i \in B$, and the map $R$ is

$$R \left( \sum_{i=1}^{\infty} R^{-i}b_i \right) = \sum_{i=1}^{\infty} R^{-i}b_{i+1}.$$

Denote $R^k = R \cdots R$ and $y = \sum_{i=1}^{\infty} R^{-i}c_i$, we see that

$$(R^k x - R^k y) - R^k (x - y) = - \sum_{i=1}^{k} R^{k-i}(b_i - c_i) \in \mathbb{Z}.$$

Since $m_\tau(x)$ is $\mathbb{Z}$-periodic, we have

$$m_\tau(R^k x - R^k y) = m_\tau(R^k (x - y))$$

for all $x \in T(R, B)$ and $k \in \mathbb{N}$. By Proposition 3.5, we obtain

$$D_n(x - (I - R^{-1})^{-1}b) = \prod_{k=0}^{n} m_\tau(R^k(x - (I - R^{-1})^{-1}b))$$

$$= \prod_{k=0}^{n} m_\tau(R^k x - R^k((I - R^{-1})^{-1}b))$$

$$= \prod_{k=0}^{n} m_\tau(R^k x - R^k(\sum_{i=0}^{\infty} R^{-i}b))$$

$$= \prod_{k=0}^{n} m_\tau(R^k x - (I - R^{-1})^{-1}b).$$

Combining with Corollary 3.4 and Birkhoff’s ergodic theorem, one has that for $\mu$-a.e. $x$ in $T(R, B)$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \log |m_\tau(R^k x - (I - R^{-1})^{-1}b)| = \log \Delta(m_{\tau, b}),$$

i.e.,

$$\lim_{n \to \infty} \frac{1}{n} \log D_{n-1}(x - (I - R^{-1})^{-1}b) = \log \Delta(m_{\tau, b}).$$

Thus we can get a subset $A \subset T(R, B)$ with measure $\mu(A) > \frac{1}{2}$ such that the limit above is uniform on $A$. If $\Delta(m_{\tau, b}) > 1$ for some $b \in B$, for $x \in A$, taking
1 < \rho < \Delta(m_{\tau,b}), there exists \( n_\rho \) such that for \( n > n_\rho \),
\[
\frac{1}{n} \log D_{n-1}(x - (I - R^{-1})^{-1}b) > \log \rho.
\]

For \( x \in A \), it is easy to see
\[
\sup_n |D_{n-1}(x - (I - R^{-1})^{-1}b)| \geq \sup_{n>n_\rho} |D_{n-1}(x - (I - R^{-1})^{-1}b)| \\
\geq \sup_{n>n_\rho} \rho^n = +\infty.
\]

Hence for \( \alpha \geq 0 \), the Mock Dirichlet summation operator \( S_n(\delta_{(I-R^{-1})^{-1}b})(x) \) satisfies
\[
\mu(\{x \in T(R, B) : \sup_n |S_n(\delta_{(I-R^{-1})^{-1}b})(x)| > \alpha \}) \geq \mu(A) \geq \frac{1}{2}.
\]

By Theorem 1.1, the proof is complete.

As an example of Lemma 3.6, we can now prove Corollary 1.2.

Proof of Corollary 1.2. By [Yu07, Theorem 1.6], the quarter Cantor measure \( \mu_{4,\{-\frac{1}{4},\frac{1}{4}\}} \) is doubling spectral measure on \( T(4,\{-\frac{1}{4},\frac{1}{4}\}) \). Under a similarity transformation, \( \Delta_{2m_L} \) in [DHS14, Example 2.5] is equal to \( \Delta(m_{17, -1}) \). Hence, according to numerical results in [DHS14, Example 2.5] and Lemma 3.6 Corollary 1.2 is proved.

\[
\square
\]

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