New classes of quadratic vector fields admitting integral-preserving Kahan–Hirota–Kimura discretizations

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Abstract
We present some new families of quadratic vector fields, not necessarily integrable, for which their Kahan–Hirota–Kimura discretization exhibits the preservation of some of the characterizing features of the underlying continuous systems (conserved quantities and invariant measures).

Keywords: discrete integrable systems, geometric numerical integration, quadratic vector fields

1. Introduction

The purpose of this paper is to present some new classes of quadratic vector fields for which the Kahan–Hirota–Kimura discretization [10] produces birational maps which preserve some of the characterizing features of the underlying continuous systems, as for example conserved quantities and invariant measures.

It is a well-known fact in numerical integration that if a dynamical system has conserved quantities, or is volume-preserving, or has some other important geometrical feature (such as being invariant under the action of a Lie group of symmetries) a generic discretization scheme of the underlying differential equations does not share the same properties. On the contrary, Kahan–Hirota–Kimura discretizations, which are applicable whenever the continuous vector field is quadratic, seem to inherit several good properties of the continuous systems they are discretizing. In particular, for the special case of completely integrable systems, ideally one would like to obtain discretizations which are themselves completely integrable. As a matter of fact the Kahan–Hirota–Kimura discretization of many known integrable quadratic systems of differential equations possesses this remarkable integrability-preserving feature [2–4, 9, 12, 15–21].
The Kahan–Hirota–Kimura discretization scheme has been introduced in 1993 by Kahan in the unpublished notes [10]. It can be applied to any system of ordinary differential equations \( \dot{x} = f(x) \) for \( x : \mathbb{R} \to \mathbb{R}^n \) with

\[
f(x) = Q(x) + Bx + c, \quad x \in \mathbb{R}^n.
\]

Here each component of \( Q : \mathbb{R}^n \to \mathbb{R}^n \) is a quadratic form, while \( B \in \text{Mat}_{n \times n} (\mathbb{R}) \) and \( c \in \mathbb{R}^n \). Then the Kahan–Hirota–Kimura discretization is given by

\[
\frac{\tilde{x} - x}{2\varepsilon} = Q(x, \tilde{x}) + \frac{1}{2} B(x + \tilde{x}) + c,
\]

where

\[
Q(x, \tilde{x}) = \frac{1}{2} (Q(x + \tilde{x}) - Q(x) - Q(\tilde{x})),
\]

is the symmetric bilinear form corresponding to the quadratic form \( Q \). Here and below we use the following notational convention which will allow us to omit a lot of indices: for a sequence \( x : \mathbb{Z} \to \mathbb{R} \) we write \( x \) for \( x_k \) and \( \tilde{x} \) for \( x_{k+1} \). Equation (2) is linear with respect to \( \tilde{x} \) and therefore defines a rational map \( \tilde{x} = \Phi(x, \varepsilon) \). Clearly, this map approximates the time-(2\varepsilon)-shift along the solutions of the original differential system. (We have chosen a slightly unusual notation \( 2\varepsilon \) for the time step, in order to avoid appearance of powers of 2 in numerous formulas; a more standard choice would lead to changing \( \varepsilon \mapsto \varepsilon/2 \) everywhere.) Since equation (2) remains invariant under the interchange \( x \leftrightarrow \tilde{x} \) with the simultaneous sign inversion \( \varepsilon \mapsto -\varepsilon \), one has the reversibility property \( \Phi^{-1}(x, \varepsilon) = \Phi(x, -\varepsilon) \). In particular, the map \( \Phi \) is birational.

The explicit form of the map \( \Phi \) defined by (2) is

\[
\tilde{x} = \Phi(x, \varepsilon) = x + 2\varepsilon (I - \varepsilon f'(x))^{-1} f(x),
\]

where \( f'(x) \) denotes the Jacobi matrix of \( f(x) \). Moreover, if the vector field \( f(x) \) is homogeneous (of degree 2), then (3) can be equivalently rewritten as

\[
\tilde{x} = \Phi(x, \varepsilon) = (I - \varepsilon f'(x))^{-1} x.
\]

Due to the reversibility of the map, in the latter case we also have:

\[
x = \Phi(\tilde{x}, -\varepsilon) = (I + \varepsilon f'(\tilde{x}))^{-1} \tilde{x} \iff \tilde{x} = (I + \varepsilon f'(\tilde{x})) x.
\]

Kahan applied this discretization scheme to the famous Lotka–Volterra system and showed that in this case it possesses a very remarkable non-spiralling property.

The next, definitely more intriguing, appearance of this discretization was in the two papers by Hirota and Kimura who (being apparently unaware of the work by Kahan) applied it to two famous integrable systems of classical mechanics, the Euler top and the Lagrange top [7, 11]. Surprisingly, the Kahan–Hirota–Kimura discretization scheme produced integrable maps in both the Euler and the Lagrange cases of rigid body motion. Even more surprisingly, the mechanism which assures integrability in these two cases seems to be rather different from the majority of examples known in the area of integrable discretizations, and, more generally, integrable maps, see [23].

In the last years many efforts in understanding the integrability mechanism of the Kahan–Hirota–Kimura scheme have been spent. We refer to the papers [2–4, 9, 12, 15–21] for a rich collection of results, including many examples of integrability-preserving Kahan–Hirota–Kimura maps.

The aim of this paper is to enrich the already remarkable list of dynamical systems for which the Kahan–Hirota–Kimura scheme produces birational discretizations preserving features of the continuous systems that they discretize.
The paper is organized as follows. In section 2, we consider the application of the Kahan–Hirota–Kimura method to generalizations of reduced Nahm systems. In particular, we show the existence of conserved quantities for the discretization for instances with quartic and sextic Hamiltonian and the preservation of an invariant measure in the general case. In section 3, we consider higher-dimensional Nambu systems. We provide results on the existence of conserved quantities and the preservation of an invariant measure for the discretization. Finally, in section 4, we present some classes of systems where rational integrals are exactly preserved by the Kahan–Hirota–Kimura discretization scheme.

2. Higher-dimensional generalizations of reduced Nahm systems

The first class of quadratic differential equations we want to consider is a higher-dimensional generalization of the two-dimensional Nahm systems introduced in [8],

\[
\begin{align*}
\ell_1 &= x_1^2 - x_2^2, \\
\ell_2 &= -2x_1x_2, \\
\ell_3 &= -6x_1^2 - 4x_2^2, \\
\ell_4 &= -10x_1x_2 + x_2^2.
\end{align*}
\]

(5)

Such systems can be explicitly integrated in terms of elliptic functions and they admit integrals of motion given respectively by

\[
\begin{align*}
H_1 &= x_2(3x_1^2 - x_2^2), \\
H_2 &= x_2(2x_1 + x_3)(x_1 - x_3)^2, \\
H_3 &= x_2(3x_1 - x_2)^2(4x_1 + x_3)^3.
\end{align*}
\]

(6–8)

Systems (5) have been discussed in [8] and discretized by means of the Kahan–Hirota–Kimura scheme in [16] (see also [1]). There it is shown that the corresponding Kahan–Hirota–Kimura maps admit conserved quantities which are \(\varepsilon\)-perturbations of the continuous ones given by (6)–(8).

We now observe that systems (5) are special two-dimensional instances of the following parametric \(n\)-dimensional family of quadratic differential equations:

\[
\dot{x} = \ell_1^{-a}\ell_2^{-b}\ell_3^{-c} J \nabla (\ell_1^a\ell_2^b\ell_3^c),
\]

(9)

where \(\ell_1, \ell_2, \ell_3 : \mathbb{R}^n \rightarrow \mathbb{R}\) are linear forms, \(J \in \text{Mat}_{n \times n}(\mathbb{R})\) is skew-symmetric and \(a, b, c \in \mathbb{R}\). System (9) has

\[
H = \ell_1^a\ell_2^b\ell_3^c
\]
as integral of motion and

\[
\Omega = \frac{dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n}{\ell_1\ell_2\ell_3}
\]

(10)
as invariant measure form. Note that if \((a, b, c) = (1, 1, 1)\) and \(n\) is an even number then (9) is a canonical Hamiltonian system on \(\mathbb{R}^n\) with a homogeneous cubic Hamiltonian.

It is immediate to verify that, if \(n = 2\) and \(J \in \text{Mat}_{2 \times 2}(\mathbb{R})\) is the canonical symplectic matrix, the first system in (5) is obtained if \((a, b, c) = (1, 1, 1)\) and \((\ell_1, \ell_2, \ell_3) = (x_2, \sqrt{3}x_1 - x_2, \sqrt{3}x_1 + x_2)\), the second system in (5) is obtained if \((a, b, c) = (1, 1, 2)\) and \((\ell_1, \ell_2, \ell_3) = (x_2, 2x_1 + 3x_2, x_1 - x_2)\) and the third system in (5) is obtained if \((a, b, c) = (1, 2, 3)\) and \((\ell_1, \ell_2, \ell_3) = (x_2, 3x_1 - x_2, 4x_1 + x_2)\).
We now turn our attention to the Kahan–Hirota–Kimura discretization of the quadratic vector field (9). Here and below we avoid to write explicitly the discrete equations of motion discretizing the continuous vector fields we are going to consider. Such equations are constructed by using (2) and the explicit form of the Kahan–Hirota–Kimura map is given by (3) (or (4) if \( f \) is homogeneous).

We give the following statement (note that \((a, b, c)\) denotes any permutation of \(\{a, b, c\}\)).

**Proposition 2.1.** The following claims are true.

(i) The Kahan–Hirota–Kimura map discretizing system (9) admits (10) as invariant measure form.

(ii) Let \((a, b, c) = (1, 1, 2)\) and assume that \(\ell_3 = \alpha \ell_1 + \beta \ell_2\) for \(\alpha, \beta \in \mathbb{R}\). Then the corresponding Kahan–Hirota–Kimura map admits the conserved quantity

\[
H_\varepsilon = \frac{H}{1 + \varepsilon^2 P_2 + \varepsilon^4 P_4},
\]

where

\[
P_2 = -2\lambda_{12}^2 (5\alpha^2 \ell_1^2 + 6\alpha \beta \ell_1 \ell_2 + 5\beta^2 \ell_2^2),
\]

\[
P_4 = \lambda_{12}^4 (9\alpha^4 \ell_1^4 - 2\alpha^2 \beta^2 \ell_1^2 \ell_2^2 + 9\beta^4 \ell_2^4),
\]

with \(\lambda_{12} = \nabla \ell_1^T J \nabla \ell_2\).

(iii) Let \((a, b, c) = (1, 2, 3)\) and assume that \(\ell_3 = \alpha \ell_1 + \beta \ell_2\) for \(\alpha, \beta \in \mathbb{R}\). Then the corresponding Kahan–Hirota–Kimura map admits the conserved quantity

\[
H_\varepsilon = \frac{H}{1 + \varepsilon^2 P_2 + \varepsilon^4 P_4 + \varepsilon^6 P_6},
\]

where

\[
P_2 = -\lambda_{12}^4 (32\alpha^2 \ell_1^2 + 40\alpha \beta \ell_1 \ell_2 + 35\beta^2 \ell_2^2),
\]

\[
P_4 = \lambda_{12}^4 (256\alpha^4 \ell_1^4 + 640\alpha^3 \beta \ell_1^3 \ell_2 + 960\alpha^2 \beta^2 \ell_1^2 \ell_2^2 + 592\alpha \beta^3 \ell_1 \ell_2^3 + 259\beta^4 \ell_2^4),
\]

\[
P_6 = -\lambda_{12}^6 (-1152\alpha^3 \beta^3 \ell_1^2 \ell_2^2 - 864\alpha^2 \beta^4 \ell_1^2 \ell_2^3 - 216\alpha \beta^6 \ell_1 \ell_2^5 + 225\beta^8 \ell_2^6),
\]

with \(\lambda_{12} = \nabla \ell_1^T J \nabla \ell_2\).

**Proof.** We prove all claims.

(i) Equations (9) can be written as

\[
\dot{x} = J \left( a \ell_2 \ell_3 \nabla \ell_1 + b \ell_1 \ell_3 \nabla \ell_2 + c \ell_1 \ell_2 \nabla \ell_3 \right). \tag{13}
\]

Their Kahan–Hirota–Kimura discretization reads

\[
\tilde{x} - x = \varepsilon J \left( a (\ell_2 \tilde{\ell}_3 + \tilde{\ell}_2 \ell_3) \nabla \ell_1 + b (\ell_1 \tilde{\ell}_3 + \tilde{\ell}_1 \ell_3) \nabla \ell_2 + c (\ell_1 \tilde{\ell}_2 + \tilde{\ell}_1 \ell_2) \nabla \ell_3 \right). \tag{14}
\]

Here and below we adopt the shortcut notation \(\tilde{F} = F(\tilde{x})\) for any map \(F : \mathbb{R}^n \rightarrow V\), where \(V = \mathbb{R}, \mathbb{R}^n\) or \(\text{Mat}_{n \times n}(\mathbb{R})\). Now, multiplying (14) from the left by the vectors \(\nabla \ell_i^T\), \(i = 1, 2, 3\), we obtain

\[
\tilde{\ell}_1 - \ell_1 = \varepsilon \lambda_{12} b (\ell_1 \tilde{\ell}_3 + \tilde{\ell}_1 \ell_3) + \varepsilon \lambda_{13} c (\ell_1 \tilde{\ell}_2 + \tilde{\ell}_1 \ell_2), \tag{15}
\]
\[
\bar{\ell}_2 - \ell_2 = -\varepsilon \lambda_{12} a(\bar{\ell}_1 \bar{\ell}_3 + \bar{\ell}_3 \bar{\ell}_1) + \varepsilon \lambda_{23} c(\bar{\ell}_1 \bar{\ell}_3 + \bar{\ell}_3 \bar{\ell}_1),
\]
\[
\bar{\ell}_3 - \ell_3 = -\varepsilon \lambda_{13} a(\bar{\ell}_1 \bar{\ell}_3 + \bar{\ell}_3 \bar{\ell}_1) - \varepsilon \lambda_{23} b(\bar{\ell}_1 \bar{\ell}_3 + \bar{\ell}_3 \bar{\ell}_1),
\]

where \( \lambda_j = \nabla \ell_i^T J \nabla \ell_j \), for \( 1 \leq i < j \leq 3 \).

Now, the Jacobian of the vector field (13) is
\[
f' = J(A_1 \nabla \ell_1^T + A_2 \nabla \ell_2^T + A_3 \nabla \ell_3^T),
\]
where
\[
A_1 = b \ell_3 \nabla \ell_2 + c \ell_2 \nabla \ell_3, \quad A_2 = a \ell_3 \nabla \ell_1 + c \ell_1 \nabla \ell_3, \quad A_3 = a \ell_2 \nabla \ell_1 + b \ell_1 \nabla \ell_2.
\]

As for any Kahan–Hirota–Kimura discretization we have
\[
\det \left( \frac{\partial \tilde{x}}{\partial x} \right) = \frac{\det(I + \varepsilon f')}{\det(I - \varepsilon f')}.
\]

Using Sylvester’s determinant formula (see for instance formula B.1.16 in [22]) we obtain
\[
\det(I - \varepsilon f') = (1 - \varepsilon \nabla \ell_1^T BA_1)(1 - \varepsilon \nabla \ell_2^T B_2 A_2)(1 - \varepsilon \nabla \ell_3^T B_1 A_3),
\]
where
\[
B_1 = (I - \varepsilon J(A_1 \nabla \ell_1^T + A_2 \nabla \ell_2^T))^{-1}, \quad B_2 = (I - \varepsilon J A_1 \nabla \ell_1^T)^{-1},
\]

or, more explicitly (use the Sherman–Morrison formula (see for instance formula B.2.2 in [22])),
\[
B_1 = I + \varepsilon J(\eta_{11} A_1 \nabla \ell_1^T + \eta_{12} A_1 \nabla \ell_2^T + \eta_{21} A_2 \nabla \ell_1^T + \eta_{22} A_2 \nabla \ell_2^T),
\]
\[
B_2 = I + \frac{\varepsilon JA_1 \nabla \ell_1^T}{1 - \varepsilon \nabla \ell_1^T A_1},
\]

where
\[
\eta_{11} = \frac{1 - \varepsilon \nabla \ell_1^T A_2}{\Delta}, \quad \eta_{12} = \frac{\varepsilon \nabla \ell_1^T A_2}{\Delta},
\]
\[
\eta_{21} = \frac{\varepsilon \nabla \ell_1^T A_1}{\Delta}, \quad \eta_{22} = \frac{1 - \varepsilon \nabla \ell_1^T A_1}{\Delta},
\]

with
\[
\Delta = 1 - \varepsilon (\nabla \ell_1^T A_1 + \nabla \ell_2^T A_2) + \varepsilon^2 (\nabla \ell_1^T A_1 \nabla \ell_2^T A_2 - \nabla \ell_1^T A_2 \nabla \ell_2^T A_1).
\]

Replacing \((x, \varepsilon)\) by \((\bar{x}, -\varepsilon)\) in (18) we obtain \(\det(I + \varepsilon f')\). Note that the expressions for \(\det(I - \varepsilon f')\) and \(\det(I + \varepsilon f')\) are rational functions in the variables \(\varepsilon, \ell_1, \ell_2, \ell_3\) and \(\varepsilon, \bar{\ell}_1, \bar{\ell}_2, \bar{\ell}_3\) respectively, i.e. \(\det(I + \varepsilon f') = \bar{P}/Q\) and \(\det(I - \varepsilon f') = P/Q\) for polynomials \(\bar{P}, \bar{Q} \in \mathbb{Q}[\varepsilon, \bar{\ell}_1, \bar{\ell}_2, \bar{\ell}_3]\) and \(P, Q \in \mathbb{Q}[\varepsilon, \ell_1, \ell_2, \ell_3]\). Now, using a computer algebra program like Maple it can be checked that the polynomial \(\bar{P}Q\ell_1 \ell_2 \ell_3 - \bar{Q}P \ell_1 \ell_2 \ell_3\) is contained in the ideal \(I \subset \mathbb{Q}[\varepsilon, \ell_1, \ell_2, \ell_3, \bar{\ell}_1, \bar{\ell}_2, \bar{\ell}_3]\) generated by (15)–(17), i.e.
\[
\frac{\det(I + \varepsilon f')}{\det(I - \varepsilon f')} = \frac{\bar{P}_1 \bar{Q}_1}{Q_1} \frac{\ell_1 \ell_2 \ell_3}{\bar{\ell}_1 \bar{\ell}_2 \bar{\ell}_3}
\]
is an algebraic identity. This proves the claim.

(ii) In this case we find that the discrete equations of motion can be written as
\[ \tilde{x} - x = \varepsilon J \left( (3\alpha(\tilde{\ell}_1 \tilde{\ell}_2 + \tilde{\ell}_1 \tilde{\ell}_2) + 2\beta(\tilde{\ell}_2 \tilde{\ell}_2) \right) \nabla \ell_1 + (3\beta(\tilde{\ell}_1 \tilde{\ell}_2 + \tilde{\ell}_1 \tilde{\ell}_2) + 2\alpha(\tilde{\ell}_1 \tilde{\ell}_1) \nabla \ell_2). \]  \hfill (19)

Then multiplying (19) from the left by \( \nabla \ell_i^T, i = 1, 2 \), we obtain
\[ \tilde{\ell}_1 - \ell_1 = \varepsilon \lambda_{12} (3\beta(\tilde{\ell}_1 \tilde{\ell}_2 + \tilde{\ell}_1 \tilde{\ell}_2) + 2\alpha(\tilde{\ell}_1 \tilde{\ell}_1)), \]  \hfill (20)
\[ \tilde{\ell}_2 - \ell_2 = -\varepsilon \lambda_{12} (3\alpha(\tilde{\ell}_1 \tilde{\ell}_2 + \tilde{\ell}_1 \tilde{\ell}_2) + 2\beta(\tilde{\ell}_1 \tilde{\ell}_1)). \]  \hfill (21)

Now, using (20) and (21) it can be verified that (11) is indeed a conserved quantity of the Kahan–Hirota–Kimura map.

(iii) In this case we find that the discrete equations of motion can be written as
\[ \tilde{x} - x = \varepsilon J \left( (4\alpha(\tilde{\ell}_1 \tilde{\ell}_2 + \tilde{\ell}_1 \tilde{\ell}_2) + 2\beta(\tilde{\ell}_2 \tilde{\ell}_2) \right) \nabla \ell_1 + (5\beta(\tilde{\ell}_1 \tilde{\ell}_2 + \tilde{\ell}_1 \tilde{\ell}_2) + 4\alpha(\tilde{\ell}_1 \tilde{\ell}_1) \nabla \ell_2). \]  \hfill (22)

Then multiplying (22) from the left by \( \nabla \ell_i^T, i = 1, 2 \), we obtain
\[ \tilde{\ell}_1 - \ell_1 = \varepsilon \lambda_{12} (5\beta(\tilde{\ell}_1 \tilde{\ell}_2 + \tilde{\ell}_1 \tilde{\ell}_2) + 4\alpha(\tilde{\ell}_1 \tilde{\ell}_1)), \]  \hfill (23)
\[ \tilde{\ell}_2 - \ell_2 = -\varepsilon \lambda_{12} (4\alpha(\tilde{\ell}_1 \tilde{\ell}_2 + \tilde{\ell}_1 \tilde{\ell}_2) + 2\beta(\tilde{\ell}_1 \tilde{\ell}_1)). \]  \hfill (24)

Now, using (23) and (24) it can be verified that (12) is indeed a conserved quantity of the Kahan–Hirota–Kimura map.

We remark that there exist at least other two remarkable choices for the parameters \( a, b, c \). The first one is given by \( (a, b, c) = (1, 1, 1) \). The Kahan–Hirota–Kimura map in this case admits a conserved quantity as well, but such a case is covered by Proposition 3 in [2]. The second one is \( (a, b, c) = (1, 1, -1) \). We will see that such a case is contained in proposition 4.2.

### 3. Higher-dimensional Nambu systems

An important class of three-dimensional vector fields is given by Nambu systems [14]:
\[ \dot{x} = \nabla H \times \nabla K, \]  \hfill (25)
where \( H, K : \mathbb{R}^n \rightarrow \mathbb{R} \). In these coordinates one has a divergenceless vector field, that means that the canonical measure is preserved by the flow. The famous Euler top is an example of Nambu mechanics. The integrability properties of the Kahan–Hirota–Kimura discretization of Nambu systems in \( \mathbb{R}^3 \), with \( H, K \) being quadratic polynomials, has been studied in [3] and [9].

In this paper we consider the following superintegrable \( n \)-dimensional generalization of (25). Let \( H, K : \mathbb{R}^n \rightarrow \mathbb{R} \) be two functionally independent quadratic homogeneous polynomials and \( v_1, \ldots, v_{n-3} \in \mathbb{R}^n \) be \( n - 3 \) linearly independent vectors in \( \mathbb{R}^n \). Then define the divergenceless system of quadratic differential equations given by
\[ \dot{x} = \det(\varepsilon, v_1, \ldots, v_{n-3}) \nabla H \times \nabla K, \]  \hfill (26)
where \( \varepsilon = (e_1, \ldots, e_n)^T \), \( e_i \) being the \( i \)-th unit basis vector in \( \mathbb{R}^n \). Equivalently, one has
\[ \dot{x} = \left( \det A_1, \ldots, (-1)^{i+1} \det A_i, \ldots, (-1)^{n+1} \det A_n \right), \]
where $A_i$ denotes the matrix obtained from $(v_1, \ldots, v_{n-3}, \nabla K, \nabla K)$ by deleting the $i$th row. It is easy to see that (26) is a superintegrable system. In the generic case (i.e. $\Delta \not\equiv 0$), it admits $H, K$ and

$$V_i = \sum_{k=1}^{n} (v_j)_k x_k, \quad i = 1, \ldots, n - 3. \quad (27)$$

as $n - 1$ functionally independent integrals of motion.

**Example 3.1.** The periodic Volterra chain with $N = 4$ is governed by the following system of differential equations:

$$\begin{align*}
\dot{x}_1 &= x_1(x_2 - x_4), \\
\dot{x}_2 &= x_2(x_3 - x_1), \\
\dot{x}_3 &= x_3(x_4 - x_2), \\
\dot{x}_4 &= x_4(x_1 - x_3).
\end{align*} \quad (28)$$

This system is completely integrable and possesses three functionally independent integrals of motion: $H = x_1 x_3$ and $K = x_2 x_4$ and $V_1 = x_1 + x_2 + x_3 + x_4$. The vector field (28) can be expressed as

$$\dot{x} = \det(\varepsilon, v_1, \nabla H, \nabla K),$$

where $v_1 = \nabla V_1$ and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_4)^T$.

We now turn our attention to the Kahan–Hirota–Kimura discretization of system (26).

**Proposition 3.2.** The following claims are true.

(i) Assume that $H = \ell_1 \ell_2$, where $\ell_1, \ell_2: \mathbb{R}^n \rightarrow \mathbb{R}$ are linear forms. Then the Kahan–Hirota–Kimura map discretizing system (26) admits

$$H_\varepsilon = \frac{\ell_1 \ell_2}{1 - \varepsilon^2 \Delta T},$$

where $\Delta = \det(\nabla \ell_1, v_1, \ldots, v_{n-3}, \nabla \ell_2, \nabla K)$ and $V_{\ell_i} i = 1, \ldots, n - 3$, given in (27), as $n - 2$ functionally independent conserved quantities.

(ii) Assume that $H = \ell_1 \ell_2, K = \ell_3 \ell_4$, where $\ell_1, \ell_2, \ell_3, \ell_4: \mathbb{R}^n \rightarrow \mathbb{R}$ are linear forms. Then the Kahan–Hirota–Kimura map discretizing system (26) admits

$$H_\varepsilon = \frac{\ell_1 \ell_2}{1 - \varepsilon^2 (\ell_1 \Delta_{124} + \ell_2 \Delta_{132})^2}, \quad K_\varepsilon = \frac{\ell_3 \ell_4}{1 - \varepsilon^2 (\ell_3 \Delta_{134} + \ell_4 \Delta_{123})^2},$$

where $\Delta_{jk} = \det(\nabla \ell_j, v_1, \ldots, v_{n-3}, \nabla \ell_k, \nabla K)$ and $V_{\ell_i} i = 1, \ldots, n - 3$, given in (27), as $n - 1$ functionally independent conserved quantities. In such case the Kahan–Hirota–Kimura map admits the invariant measure form

$$\Omega = \frac{dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n}{\ell_1 \ell_2 \ell_3 \ell_4}.$$

**Proof.** We prove both claims.

(i) The Kahan–Hirota–Kimura map is given by

$$\tilde{x} - x = \varepsilon \left( \det(\varepsilon, v_1, \ldots, v_{n-3}, \ell_2 \nabla \ell_1 + \ell_1 \nabla \ell_2, \nabla K) \right. \left. + \det(\varepsilon, v_1, \ldots, v_{n-3}, \ell_3 \nabla \ell_1 + \ell_1 \nabla \ell_3, \nabla K) \right). \quad (29)$$
Now, multiplying (29) from the left by $\nabla \ell_i^T$, $i = 1, 2$, yields

$$
\tilde{\ell}_1(1 - \varepsilon \Delta) = \ell_1(1 + \varepsilon \tilde{\Delta}),
$$

$$
\tilde{\ell}_2(1 + \varepsilon \Delta) = \ell_2(1 - \varepsilon \tilde{\Delta}),
$$

which justifies the claim. Concerning functions (27), we note that the Kahan–Hirota–Kimura discretization preserves all linear integrals.

(ii) In this case we find that the discrete equations of motion can be written as

$$
\frac{x - x}{\varepsilon} = (\ell_1 \tilde{\ell}_3 + \ell_3 \tilde{\ell}_1) \Delta_{24} + (\ell_1 \tilde{\ell}_4 + \ell_4 \tilde{\ell}_1) \Delta_{23} + (\ell_2 \tilde{\ell}_3 + \ell_3 \tilde{\ell}_2) \Delta_{14} + (\ell_2 \tilde{\ell}_4 + \ell_4 \tilde{\ell}_2) \Delta_{13},
$$

where $\Delta_{ij} = \det(\varepsilon, v_1, \ldots, v_n, \nabla \ell_i, \nabla \ell_j)$. Then multiplying (30) from the left by the vectors $\nabla \ell_i^T$, $i = 1, \ldots, 4$, yields

$$
\tilde{\ell}_1(1 - \varepsilon (\ell_3 \Delta_{124} + \ell_4 \Delta_{123})) = \ell_1(1 + \varepsilon (\tilde{\ell}_3 \Delta_{124} + \tilde{\ell}_4 \Delta_{123})),
$$

$$
\tilde{\ell}_2(1 + \varepsilon (\ell_3 \Delta_{124} + \ell_4 \Delta_{123})) = \ell_2(1 - \varepsilon (\tilde{\ell}_3 \Delta_{124} + \tilde{\ell}_4 \Delta_{123})),
$$

$$
\tilde{\ell}_3(1 + \varepsilon (\ell_1 \Delta_{234} + \ell_2 \Delta_{134})) = \ell_3(1 - \varepsilon (\tilde{\ell}_1 \Delta_{234} + \tilde{\ell}_2 \Delta_{134})),
$$

$$
\tilde{\ell}_4(1 - \varepsilon (\ell_1 \Delta_{234} + \ell_2 \Delta_{134})) = \ell_4(1 + \varepsilon (\tilde{\ell}_1 \Delta_{234} + \tilde{\ell}_2 \Delta_{134})).
$$

Computing the Jacobian of the vector field (26) we obtain

$$
f' = A_1 \nabla \ell_1^T + A_2 \nabla \ell_2^T + A_3 \nabla \ell_3^T + A_4 \nabla \ell_4^T,
$$

where the vectors $A_i$, $i = 1, \ldots, 4$ are defined by

$$
A_1 = \ell_3 \Delta_{24} + \ell_4 \Delta_{23}, \quad A_2 = \ell_3 \Delta_{14} + \ell_4 \Delta_{13},
$$

$$
A_3 = \ell_1 \Delta_{24} + \ell_2 \Delta_{14}, \quad A_4 = \ell_1 \Delta_{23} + \ell_2 \Delta_{13}.
$$

As for any Kahan–Hirota–Kimura discretization we have

$$
\det \left( \frac{\partial \tilde{x}}{\partial x} \right) = \frac{\det(I + \varepsilon f')}{\det(I - \varepsilon f')}.
$$

Using Sylvester’s determinant formula we obtain

$$
det(I - \varepsilon f') = (1 - \varepsilon \nabla \ell_1^T A_1)(1 - \varepsilon \nabla \ell_2^T B_2 A_2)(1 - \varepsilon \nabla \ell_3^T B_3 A_3)(1 - \varepsilon \nabla \ell_4^T B_4 A_4),
$$

where

$$
B_1 = (I - \varepsilon (A_1 \nabla \ell_1^T + A_2 \nabla \ell_2^T + A_3 \nabla \ell_3^T))^{-1},
$$

$$
B_2 = (I - \varepsilon (A_1 \nabla \ell_1^T + A_2 \nabla \ell_2^T))^{-1}, \quad B_3 = (I - \varepsilon A_1 \nabla \ell_1^T)^{-1}.
$$

As in the proof of proposition 2.1 the matrices $B_i$, $i = 1, 2, 3$, can be computed using the Sherman–Morrison formula. Replacing $(x, \varepsilon)$ by $(\tilde{x}, -\varepsilon)$ we obtain $det(I + \varepsilon f')$. Note that the expressions for $det(I - \varepsilon f')$ and $det(I + \varepsilon f')$ are rational functions in the variables $\varepsilon, \ell_1, \ldots, \ell_4$ and $\varepsilon, \tilde{\ell}_1, \ldots, \tilde{\ell}_4$ respectively. Finally, using the equations (31)–(34) it can be verified that
\[
\frac{\det(I + \varepsilon f^\prime)}{\det(I - \varepsilon f^\prime)} = \tilde{\ell}_1 \tilde{\ell}_2 \tilde{\ell}_3 \tilde{\ell}_4
\]
is an algebraic identity. This proves the claim. □

4. Classes with exact preservation of integrals

This section is devoted to two families of systems of quadratic ordinary differential equations in \(\mathbb{R}^n\) which admit a rational conserved quantity. We will show that the corresponding Kahan–Hirota–Kimura discretizations preserve exactly such integrals of motion.

The first family of differential equations is given by

\[
\dot{x} = \ell^2 J \nabla \left( \frac{H \ell}{\ell} \right),
\]

where \(\ell : \mathbb{R}^n \to \mathbb{R}\) is a linear form, \(H : \mathbb{R}^n \to \mathbb{R}\) is a quadratic homogeneous polynomial and \(J \in \text{Mat}_{n \times n}(\mathbb{R})\) is skew-symmetric. System (35) admits the rational function

\[
F = \frac{H \ell}{\ell}
\]
as integral of motion.

The second family of differential equations we want to consider is given by

\[
\dot{x} = \det \left( \varepsilon, v_1, \ldots, v_{n-3}, \ell_2^2 \nabla \left( \frac{\ell_1}{\ell_2} \right), \nabla H \right) - \alpha H \det(\nabla \ell_1, \ldots, v_{n-3}, \nabla \ell_2, \varepsilon),
\]

with parameter \(\alpha \in \mathbb{R}, \ell_1, \ell_2 : \mathbb{R}^n \to \mathbb{R}\) linear forms, \(H : \mathbb{R}^n \to \mathbb{R}\) quadratic homogeneous polynomial and \(v_1, \ldots, v_{n-3} \in \mathbb{R}^n\) are constant vectors such that \(\nabla \ell_1, \nabla \ell_2, v_1, \ldots, v_{n-3}\) are linearly independent. System (37) admits the rational function

\[
F = \frac{\ell_1}{\ell_2}
\]
as integral of motion. Moreover the linear forms

\[
V_i = \sum_{k=1}^{n} (v_i)_k x_k, \quad i = 1, \ldots, n - 3,
\]

provide \(n - 3\) additional functionally independent integrals of motion. Therefore system (37) is superintegrable. Additionally, if \(\alpha = 1\), system (37) admits the rational function

\[
G = \frac{H \ell_1}{\ell_2}
\]
as integral of motion.

Example 4.1. A remarkable three-dimensional example belonging to class (37) is given by

\[
\begin{align*}
\dot{x}_1 &= \frac{1}{2}(a_1 + a_2 + a_3)x_1^2 + \frac{1}{2}(a_1 - a_2 - a_3)(x_1 - x_2)(x_1 - x_3), \\
\dot{x}_2 &= \frac{1}{2}(a_1 + a_2 + a_3)x_2^2 + \frac{1}{2}(a_2 - a_3 - a_1)(x_2 - x_3)(x_2 - x_1), \\
\dot{x}_3 &= \frac{1}{2}(a_1 + a_2 + a_3)x_3^2 + \frac{1}{2}(a_3 - a_1 - a_2)(x_3 - x_1)(x_3 - x_2),
\end{align*}
\]

where \(a_1, a_2, a_3\) are parameters. Indeed, system (41) is a degenerate subcase of the famous Halphen system (1881) [6]:
\[
\begin{aligned}
\dot{x}_1 &= a_1 x_1^2 + (\lambda - a_1) (x_1 x_2 - x_2 x_3 + x_3 x_1), \\
\dot{x}_2 &= a_2 x_2^2 + (\lambda - a_2) (x_2 x_3 - x_3 x_1 + x_1 x_2), \\
\dot{x}_3 &= a_3 x_3^2 + (\lambda - a_3) (x_3 x_1 - x_1 x_2 + x_2 x_3),
\end{aligned}
\]  

(42)

where \(\lambda\) is an additional parameter. The general solution of (42) can be written in terms of hypergeometric functions provided that \(\lambda \neq 0, a_1 + a_2 + a_3 \neq 2\lambda\) [6]. It turns out that (41) comes from (42) with the choice \(\lambda = (a_1 + a_2 + a_3)/2\) with \(a_1 + a_2 + a_3 \neq 0\), which evidently violates the previous condition. Obviously, such choice renders system (41) much simpler than the original Halphen system (42).

By straightforward inspection one can see that (41) admits two functionally independent rational integrals of motion of the form

\[
F_{ij} = \frac{a_1 (x_1 x_2 - x_2 x_3 + x_3 x_1) + a_2 (x_2 x_3 - x_3 x_1 + x_1 x_2) + a_3 (x_3 x_1 - x_1 x_2 + x_2 x_3)}{x_i - x_j},
\]

(43)

where \(i, j = 1, 2, 3, i \neq j\). Incidentally we observe that system (41) has been explicitly considered in [13]. Here the author claims that the parameter-independent quantities \((x_1 - x_2)/(x_2 - x_3)\) and \((x_2 - x_3)/(x_3 - x_1)\) are integrals of motion of (41). Although such a statement is correct, these integrals are functionally dependent, so that nothing can be argued about the complete integrability of the system. On the contrary, the existence of two independent integrals of motion (43) allows one to find a (rank 2) bi-Hamiltonian structure of (41), thus proving its complete integrability in the Arnold–Liouville sense. A representative polynomial map is the following one:

\[
\{x_1, x_2\} = (x_1 - x_2)(x_2 - x_3), \quad \{x_2, x_3\} = (x_2 - x_3)^2, \quad \{x_3, x_1\} = (x_2 - x_3)(x_3 - x_1).
\]

(44)

It turns out that (41) is Hamiltonian with respect to (44) with Hamiltonian \(F_{23}/2\) while \(F_{23}/F_{31} = (x_3 - x_1)/(x_2 - x_3)\) is a Casimir function.

We conclude by saying that system (41) belongs to the family of vector fields (37) by setting \(n = 3, \ell_1 = x_3 - x_1, \ell_2 = x_2 - x_3, H = (x_1 - x_2)F_{12} + \alpha = 1\).

The following statements show that the Kahan–Hirota–Kimura maps discretizing systems (35) and (37) preserve exactly the continuous integrals of motion.

**Proposition 4.2.** The Kahan–Hirota–Kimura discretization of system (35) admits (36) as conserved quantity.

**Proof.** Here and below we use the notation \(\tilde{S}_t = \tilde{S}^T (\nabla^2 H) x, \tilde{S} = \tilde{S}^T (\nabla^2 H) \tilde{x}\) and \(\tilde{S} = \tilde{x}^T (\nabla^2 H) \tilde{x}\), where \(\nabla^2 H\) denotes the Hessian of a function \(H : \mathbb{R}^n \to \mathbb{R}\).

The Kahan–Hirota–Kimura map is given by

\[
\tilde{x} - x = \varepsilon \left( \ell J \nabla H + \tilde{\ell} J \nabla H - \tilde{\ell} J \nabla \ell \right).
\]

(45)

Now, multiplying (45) from the left by \(\tilde{\ell} \nabla \tilde{H}^T, \ell \nabla H^T\) and \(\nabla \ell^T\) we obtain

\[
\tilde{\ell} \tilde{S}_t - \ell \tilde{S}_t = \varepsilon \left( \ell \tilde{\ell} \nabla H^T J \nabla H - \ell \tilde{\ell} \tilde{S} \nabla H^T J \nabla \ell \right),
\]

(46)

\[
\tilde{\ell} \tilde{S}_t - \ell \tilde{S}_t = \varepsilon \left( \ell \tilde{\ell} \nabla H^T J \nabla H - \tilde{\ell} \tilde{S} \nabla H^T J \nabla \ell \right),
\]

(47)
\begin{equation}
\tilde{\ell} - \ell = \varepsilon \left( \nabla \ell^T J \nabla H + \tilde{\ell} \nabla \ell^T J \nabla H \right). \tag{48}
\end{equation}

Finally, adding (47) to (46) and substituting (48) we obtain \( \tilde{\mathbf{S}} = \tilde{\mathbf{S}} \), which gives the claim. \( \square \)

**Proposition 4.3.** The following claims are true.

(i) The Kahan–Hirota–Kimura discretization of system (37) admits (38) and (39) as \( n - 2 \) functionally independent conserved quantities.

(ii) The Kahan–Hirota–Kimura discretization of system (37) with \( \alpha = 1 \) admits (40) as additional conserved quantity.

**Proof.** We prove all claims.

(i) The Kahan–Hirota–Kimura map is given by

\[
\frac{\bar{x} - x}{\varepsilon} = \det(\varepsilon, v_1, \ldots, v_{n-3}, \tilde{\ell}_2 \nabla \ell_1 - \tilde{\ell}_1 \nabla \ell_2, \nabla H) + \det(\varepsilon, v_1, \ldots, v_{n-3}, \ell_2 \nabla \ell_1 - \ell_1 \nabla \ell_2, \nabla H) - \alpha \varepsilon \det(\nabla \ell_1, v_1, \ldots, v_{n-3}, \nabla \ell_2, \varepsilon). \tag{49}
\]

Now, multiplying (49) from the left by \( \nabla \ell_i^T, i = 1, 2 \), we obtain

\[
\tilde{\ell}_1 - \ell_1 = -\varepsilon \left( \tilde{\ell}_1 \Delta + \ell_1 \tilde{\Delta} \right), \tag{50}
\]

\[
\tilde{\ell}_2 - \ell_2 = -\varepsilon \left( \tilde{\ell}_2 \Delta + \ell_2 \tilde{\Delta} \right). \tag{51}
\]

where \( \Delta = \det(\nabla \ell_1, v_1, \ldots, v_{n-3}, \nabla \ell_2, \nabla H) \). Thus, we see that \( \tilde{\ell}_1 \ell_2 = \tilde{\ell}_2 \ell_1 \).

(ii) Let \( \alpha = 1 \). Multiplying (49) from the left by the vectors \( \ell_1 \tilde{\nabla}^T \) and \( \ell_1 \nabla^T \) we obtain

\[
\ell_1 \tilde{\mathbf{S}} - \ell_1 \mathbf{S} = -\varepsilon \left( \ell_1 \det(\nabla \ell_1, v_1, \ldots, v_{n-3}, \tilde{\ell}_2 \nabla \ell_1 - \tilde{\ell}_1 \nabla \ell_2, \nabla H) - \ell_1 \mathbf{S} \tilde{\Delta} \right), \tag{52}
\]

\[
\tilde{\ell}_1 \mathbf{S} - \ell_1 \mathbf{S} = \varepsilon \left( \tilde{\ell}_1 \det(\nabla \ell_1, v_1, \ldots, v_{n-3}, \ell_2 \nabla \ell_1 - \ell_1 \nabla \ell_2, \nabla H) - \tilde{\ell}_1 \mathbf{S} \Delta \right). \tag{53}
\]

Adding (52), (53) and substituting (50), we conclude that \( \ell_1 \tilde{\mathbf{S}} = \tilde{\ell}_1 \mathbf{S} \).

The next claim follows.

**Corollary 4.1.** Assume that \( \ell_1, \ell_2, \ell_3, \ell_4 : \mathbb{R}^2 \to \mathbb{R} \) are linear forms. Then the Kahan–Hirota–Kimura map discretizing

\[
\dot{x} = \det \left( \varepsilon, v_1, \ldots, v_{n-3}, \ell_2^T \nabla \left( \frac{\ell_1}{\ell_2} \right), \ell_4^T \nabla \left( \frac{\ell_3}{\ell_4} \right) \right), \tag{54}
\]

where \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)^T \), admits \( F_1 = \ell_1/\ell_2 \), \( F_2 = \ell_3/\ell_4 \) and (39) as \( n - 1 \) functionally independent conserved quantities. Moreover the Kahan–Hirota–Kimura map has the invariant measure form

\[
\Omega = \frac{dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n}{\ell_1 \ell_2 \ell_3 \ell_4}. \tag{55}
\]
Proof. The conserved quantities follow from proposition 4.3. Indeed, we have
\[
\frac{\dot{x} - x}{\varepsilon} = (\ell_1 \tilde{\ell}_3 + \tilde{\ell}_1 \ell_3) \Delta_{24} - (\ell_1 \tilde{\ell}_4 + \tilde{\ell}_1 \ell_4) \Delta_{23} - (\ell_2 \tilde{\ell}_3 + \tilde{\ell}_2 \ell_3) \Delta_{14} + (\ell_2 \tilde{\ell}_4 + \tilde{\ell}_2 \ell_4) \Delta_{13},
\]
where \(\Delta_{ij} = \det(\ell_i, v_1, \ldots, v_{n-3}, \nabla \ell_i, \nabla \ell_j)\). Then multiplying (56) from the left by the vectors \(\nabla \ell_i^T, i = 1, \ldots, 4\), yields
\[
\tilde{\ell}_1 (1 + \varepsilon(\ell_4 \Delta_{123} - \ell_3 \Delta_{124})) = \ell_1 (1 + \varepsilon(\tilde{\ell}_4 \Delta_{124} - \tilde{\ell}_3 \Delta_{123})),
\]
\[
\tilde{\ell}_2 (1 + \varepsilon(\ell_4 \Delta_{123} - \ell_3 \Delta_{124})) = \ell_2 (1 + \varepsilon(\tilde{\ell}_4 \Delta_{124} - \tilde{\ell}_3 \Delta_{123})),
\]
\[
\tilde{\ell}_3 (1 + \varepsilon(\ell_1 \Delta_{234} - \ell_2 \Delta_{134})) = \ell_3 (1 + \varepsilon(\tilde{\ell}_1 \Delta_{234} - \tilde{\ell}_2 \Delta_{134})),
\]
\[
\tilde{\ell}_4 (1 + \varepsilon(\ell_1 \Delta_{234} - \ell_2 \Delta_{134})) = \ell_4 (1 + \varepsilon(\tilde{\ell}_1 \Delta_{234} - \tilde{\ell}_2 \Delta_{134})),
\]
where \(\Delta_{jk} = \det(\nabla \ell_i, v_1, \ldots, v_{n-3}, \nabla \ell_j, \nabla \ell_k)\). The existence of the invariant measure (55) can be verified by similar computations as in the proof of proposition 3.2.

5. Conclusions

In this paper we presented some new examples of Kahan–Hirota–Kimura discretizations. Some of the results in section 2 were found independently by Celledoni, McLachlan, McLaren, Owren, Quispel [5]. On the one hand we enriched the already existing long list of systems for which such discretization scheme preserves good features of the continuous systems (complete integrability in the optimal case), on the other one we generalized some of the results found in previous works on Kahan–Hirota–Kimura discretizations. In particular, we studied generalizations of the reduced Nahm systems considered in [8, 16] and explained the integrability properties of the discretization of the periodic Volterra chain with \(N = 4\) (see [16]) as an instance of a more general class of higher-dimensional Nambu systems. As a matter of fact, the integrability mechanism of the Kahan–Hirota–Kimura discretization is not unveiled yet. One of the major open problems is indeed to identify and characterize those structural properties of an integrable continuous vector field which ensure the preservation of integrability in the discrete setting.

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