HOPF BIFURCATION OF A FRACTIONAL-ORDER OCTONION-VALUED NEURAL NETWORKS WITH TIME DELAYS

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Abstract. In this paper, the hopf bifurcation of a fractional-order octonion-valued neural networks with time delays is investigated. With this constructed model all the parameters would belong to the normed division algebra of octonians. Because of the non-commutativity of the octonians, the fractional-order octonion-valued neural networks can be decomposed into four-dimensional real-valued neural networks. Furthermore, the conditions for the occurrence of Hopf bifurcation for the considered model are firstly given by taking time delay as a bifurcation parameter. Also we investigate their bifurcation when the system loses its stability. Finally, we give one numerical simulation to verify the effectiveness of the our proposed method.

1. Introduction. Fractional-order neural networks has gained interest over the last few years due to their wide range of applications such as electro magnetic waves, visco elastic systems, dielectric polarization and biological systems [4, 14, 23, 27, 30]. Fractional calculus is the generalization of ordinary differentiation and integration to arbitrary fractional(non-integer) order. In recent years, fractional calculus has been extensively used to model the system behavior and physical system exhibiting memory and hereditary properties [28] and hence it gains more dominance than the classical integer-order systems [11, 19]. Taking all these facts into account, the incorporation of the memory term in the network model is an important improvement. Recently the dynamic behaviors of fractional-order neural networks have gained research interest and there were several literatures discussing the stability of the fractional-order neural networks which can be found in [3, 12, 16, 23].

In recent decades, the multidimensional neural network model becomes more popular, because of their increasing applications in the field of radar imaging, antenna design, quantum waves, filtering, communications signal processing, speech synthesis and so on [17, 21, 22]. The complex-valued neural networks(CVNNs) which is an extension of real-valued neural networks(RVNNs) [25] has received a rapid growth in these few years and has produced significant results in real life problems. Meanwhile, the quaternion-valued neural networks(QVNNs) are another multidimensional networks, which were discussed in [26] and both of theses type of neural networks are special cases of Clifford-valued neural networks. Octonion-valued neural networks(OVNNs) are generalization of both CVNNs and QVNNs. Moreover, the OVNNs have an important property of being a normed division
algebra, which means that a norm and a multiplicative inverse can be defined on it [5]. Based on these facts, it is imperative that the octonions are introduced in the neural networks and it should be noted that results on feed-forward octonions were reported in [15] and they have comprehensive applications in signal and high-dimensional data processing.

The dynamic properties of OVNNs were intensively discussed in recent days in which the global exponential stability of neutral-type OVNNs with time varying delays were studied in [21] and in [26] using Cayley-Dickson construction, the octonion numbers were decomposed into their complex components to establish the global exponential stability of OVNNs with delay. which gave sufficient criteria in terms of linear matrix inequalities. In [22], notable delay-dependent criteria in terms of complex-valued linear matrix inequality for global exponential stability of OVNNs with leakage and mixed time delays were investigated. In this case, the activation functions are separated into real and imaginary parts for discussing the problem of Hopf bifurcation, in which the discrete time delay is taken as the bifurcation parameter in [20] and in [7, 10] sum of the time delay is taken as the bifurcation parameter. Stability and bifurcation direction are discussed by using the central manifold theorem addressed in [6]. To the best of our knowledge so far, the problem of Hopf bifurcation is mostly concentrated on RVNNs with time delays [1, 2, 18, 29, 31] and CVNNs with time delays [8, 12, 20, 24]. Taking all the considerations, in this paper our main aim is to deal with the problem of bifurcation for a two-dimensional OVNNs with time delays. Inspired by the above mentioned reasons, this paper is formulated as follows:

1. Firstly, the fractional-order OVNNs model is transformed into an four n-dimensional fractional-order CVNNs for the activation functions considered here, the origin is taken as the equilibrium point and thus the decomposed fractional-order CVNNs model has been linearized with zero as the equilibrium point. The linearized system undergoes Laplace transformation through which we can be able to formulate the characteristic matrix of the fractional-order OVNNs.

2. Using, certain results and conditions available, the bifurcation analysis of the eigenvalues of the obtained characteristic equation. That is, the global asymptotically stability of the proposed model could be guaranteed when all the roots of the characteristic equation have negative real parts.

3. When the system has unstable position with respect to zero equilibrium, the Hopf bifurcation will occurs, and in this case the considered time-delay is taken to be a bifurcation parameter along with the corresponding critical frequency of the time-delay.

The remaining part of this paper is arranged as follows. In Section 2, some definitions and properties of Caputo-derivative are provided. In Section 3, the model of the octonion-valued network was given with some assumptions, and using these assumptions, the characteristic matrix was found. In Section 4, the conditions of Hopf bifurcation are established. In Section 5, numerical simulations are given to show the effectiveness of our theoretical results. Finally, conclusion is given in Section 6.

2. Preliminaries.

**Definition 2.1.** [13] The fractional-order integral of non-integer order \( \nu \) for an integral function \( w(t) \) is defined as follows:
\[ I^\nu w(t) = \frac{1}{\Gamma(\nu)} \int_{t_0}^{t} (t-\zeta)^{\nu-1} w(\zeta) d\zeta, \]

where, \( t \geq t_0, \nu > 0, \Gamma(.) \) is the Gamma function and is defined as \( \Gamma(u) = \int_{0}^{\infty} t^{u-1} e^{-t} dt \).

\textbf{Definition 2.2.} [13] The Caputo-fractional derivative of order \( \nu \) for a function \( w(t) \) is defined by

\[ C^\nu D^\nu_{t_0} w(t) = \frac{1}{\Gamma(n-\nu)} \int_{t_0}^{t} \frac{w^{(n)}(\zeta)}{(t-\zeta)^{\nu-n+1}} d\zeta, \]

where, \( n-1 \leq \nu < n, n \in \mathbb{Z}^+, t = t_0 \) is the initial time.

Especially, when \( 0 < \nu < 1 \)

\[ C^\nu D^\nu_{t_0} w(t) = \frac{1}{\Gamma(1-\nu)} \int_{t_0}^{t} \frac{w'(\zeta)}{(t-\zeta)^{\nu}} d\zeta. \]

It follows from (3) that taking the Laplace transform:

\[ \mathcal{L} \left[ C^\nu D^\nu_{t_0} w(t) : s \right] = s^\nu W(s) - \sum_{k=0}^{n-1} s^{\nu-k-1} w^{(k)}(0) \]

where, \( W(s) \) is the Laplace transform of \( w(t) \) i.e. \( W(s) = \mathcal{L} \{ w(t) \} \) if \( w^{(k)}(0) = 0, k = 1, 2, \ldots, n \), then \( \mathcal{L} \left[ C^\nu D^\nu_{t_0} w(t) : s \right] = s^\nu W(s) \).

\textbf{Property 2.3.} \( C^\nu D^\nu_{t_0} \mu = 0 \) holds, where \( \mu \) is any constant.

\textbf{Property 2.4.} For any constant \( \alpha \) and \( \beta \), the linearity of Caputo fractional-order derivative gives,

(i) \( C^\nu D^\nu_{t_0} (\alpha w(t) + \beta u(t)) = \alpha C^\nu D^\nu_{t_0} w(t) + \beta C^\nu D^\nu_{t_0} u(t) \),
(ii) For \( \nu > 0, \theta > 0 \), then \( C^\nu D^\nu_{t_0} C^\theta D^\theta_{t_0} w(t) = C^{\nu+\theta} D^{\nu+\theta}_{t_0} w(t) \),
(iii) \( C^\nu D^\nu_{t_0} C^\nu D^\nu_{t_0} w(t) = w(t) \).

An octonion is a number defined by an 8-dimensional algebra, twice the number of dimensions of the quaternions, with basis

\[ \{ e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7 \} \]

where, \( e_0 \) is the scalar or real element (it may be identified with the real number 1), if \( e_0 = 0 \) then the octonion is said to be pure. An octonion number \( x \in \mathbb{O} \) (set of octonions) can be written in the form

\[ x = x_0 e_0 + x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 + x_5 e_5 + x_6 e_6 + x_7 e_7, \]

where, \( x_i (i = 0, 1, \ldots, 7) \) are real coefficients. The addition and subtraction of octonions is defined by \( x + y = \sum_{i=0}^{7} (x_i + y_i) e_i, x - y = \sum_{i=0}^{7} (x_i - y_i) e_i \) and the multiplication of the unit octonions is given in the following table, which describes the results of multiplying the elements in the \( i \)th row by the element in the \( j \)th column. It follows from table that the octonion multiplication is neither commutative (\( e_i e_j = -e_j e_i \neq e_j e_i \) for if \( i, j \) are distinct and non zero) nor associative.
\((e_i e_j) e_k = -e_i (e_j e_k) \neq e_i (e_j e_k)\) for if \(i, j, k\) are distinct, non zero or \(e_i e_j \neq \pm e_k\).

The octonion conjugate is denoted as \(x^*\) and is defined by \(x^* = x_0 e_0 - \sum_{l=1}^{7} x_l e_l\).

The norm of the octonion number \(x\) is defined by \(||x|| = \sqrt{x x^*} = \sqrt{\sum_{l=0}^{7} x_l^2}\) and the inverse of the octonion is defined by \(x^{-1} = \frac{x^*}{||x||^2}\).

| \(\times\) | \(e_0\) | \(e_1\) | \(e_2\) | \(e_3\) | \(e_4\) | \(e_5\) | \(e_6\) | \(e_7\) |
|---|---|---|---|---|---|---|---|---|
| \(e_0\) | \(e_0\) | \(e_1\) | \(e_2\) | \(e_3\) | \(e_4\) | \(e_5\) | \(e_6\) | \(e_7\) |
| \(e_1\) | \(-e_0\) | \(e_3\) | \(-e_2\) | \(e_5\) | \(-e_4\) | \(-e_7\) | \(e_6\) | \(-e_5\) |
| \(e_2\) | \(-e_3\) | \(-e_0\) | \(e_1\) | \(e_6\) | \(e_7\) | \(-e_4\) | \(-e_5\) | \(e_3\) |
| \(e_3\) | \(-e_2\) | \(e_1\) | \(-e_0\) | \(e_7\) | \(-e_6\) | \(e_5\) | \(-e_4\) | \(e_4\) |
| \(e_4\) | \(-e_5\) | \(-e_6\) | \(-e_7\) | \(e_0\) | \(e_1\) | \(e_2\) | \(e_3\) | \(e_5\) |
| \(e_5\) | \(-e_4\) | \(-e_7\) | \(-e_6\) | \(-e_0\) | \(-e_3\) | \(e_2\) | \(e_6\) | \(-e_0\) |
| \(e_6\) | \(-e_7\) | \(-e_6\) | \(-e_5\) | \(-e_2\) | \(e_3\) | \(e_6\) | \(-e_0\) | \(-e_1\) |
| \(e_7\) | \(-e_6\) | \(-e_3\) | \(-e_2\) | \(-e_1\) | \(e_0\) | \(-e_1\) | \(-e_2\) | \(-e_3\) |

3. The model. Consider the following fractional-order octonion-valued neural networks with time delays, for which the states and weights are from \(\mathbb{O}\).

\[
\begin{align*}
\frac{d^\alpha}{dt^\alpha} w_1(t) &= -d_1 w_1(t) + a f_1(w_1(t)) + b_1 g_1(w_n(t - \tau_n)), \\
\frac{d^\alpha}{dt^\alpha} w_p(t) &= -d_p w_p(t) + a f_p(w_p(t)) + b_p g_p(w_{p-1}(t - \tau_{p-1})), \quad p = 2, 3, \ldots, n,
\end{align*}
\]

\(w_p(t) \in \mathbb{O}\) is the state of the \(p\)th neuron, \(d_p > 0\), \(a \in \mathbb{O}\) and \(b_p \in \mathbb{O}\) are connection weights for neuron \(p\) to \(p\) neuron \(p\) to \(p-1\) respectively; \(f_p : \mathbb{O} \to \mathbb{O}\) is the nonlinear octonion-valued activation of \(p\)th neuron without time delay; \(g_p : \mathbb{O} \to \mathbb{O}\) is the nonlinear octonion-valued activation of \(p\)th neuron with time delay; \(\tau\) is the time delay; \(\forall p = 2, 3, \ldots, n\).

Using the Cayley-Dickson construction, the octonion units are written in the following form

\[e_0 = 1, e_1 = i, e_2 = j, e_3 = i j, e_4 = k, e_5 = i k, e_6 = j k, e_7 = i j k.\]

Since \(\mathbb{C} = \{z = z_1 + i z_2|z_1, z_2 \in \mathbb{R}, i^2 = -1\}\) is the set of complex number. Now we can write any octonion number \(x\) as

\[x = \hat{x} q_1 + j \hat{x} q_2 + k \hat{x} q_3 + j k \hat{x} q_4,
\]

where, \(\hat{x} q_1 = x_0 + i x_1, \hat{x} q_2 = x_2 + i x_3, \hat{x} q_3 = x_4 + i x_5, \hat{x} q_4 = x_6 + i x_7\). In this case the product of two octonions is defined as:

\[
\begin{align*}
x y &= \hat{x} \hat{y} q_1 - \hat{x} q_2 \hat{y} q_2 - \hat{x} q_3 \hat{y} q_3 - \hat{x} q_4 \hat{y} q_4 + j (\hat{x} q_1 \hat{y} q_2 + \hat{x} q_2 \hat{y} q_3 + \hat{x} q_3 \hat{y} q_4 + \hat{x} q_4 \hat{y} q_1) \\
&+ k (\hat{x} q_1 \hat{y} q_3 + \hat{x} q_3 \hat{y} q_1 + \hat{x} q_4 \hat{y} q_2 + \hat{x} q_2 \hat{y} q_4 + \hat{x} q_4 \hat{y} q_3 + \hat{x} q_3 \hat{y} q_2).\end{align*}
\]

\((\mathcal{M}_1): \) The neuron activations \(f\) and \(g\) can be written in the form

\[
\begin{align*}
f(x) &= \hat{f} q_1(x) + j \hat{f} q_2(x) + k \hat{f} q_3(x) + j k \hat{f} q_4(x), \\
g(x(t - \tau)) &= \hat{g} q_1(x(t - \tau)) + j \hat{g} q_2(x(t - \tau)) + k \hat{g} q_3(x(t - \tau)) + j k \hat{g} q_4(x(t - \tau)).
\end{align*}
\]

\((\mathcal{M}_2): \) Let \(w(t) = \hat{w} q_1(t) + j \hat{w} q_2(t) + k \hat{w} q_3(t) + j k \hat{w} q_4(t)\), the activation function \(f(w)\) can be separated into \(f(w) = \hat{f} q_1(w) + j \hat{f} q_2(w) + k \hat{f} q_3(w) + j k \hat{f} q_4(w)\). The derivatives of \(\hat{f} q_1(x_0, x_1), \hat{f} q_2(x_0, x_1), \hat{f} q_2(x_2, x_3), \hat{f} q_2(x_2, x_3), \hat{f} q_2(x_4, x_5), \hat{f} q_2(x_4, x_5), \hat{f} q_2(x_4, x_5), \hat{f} q_2(x_4, x_5).\)
(x_6, x_7), \hat{f}_p^q(x_6, x_7)$ with respect to \(x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7\) respectively, exist and continuous and \(\hat{f}_p^q(0, 0) = 0, \hat{f}_p^q(0, 0) = 0, \hat{f}_p^q(0, 0) = 0, \hat{f}_p^q(0, 0) = 0, \hat{f}_p^q(0, 0) = 0\).

(M₃) : The activation functions \(f_p^q(\cdot), g_p^q(\cdot), p = 1, 2, \ldots, n\) satisfies, there exist positive constants \(F_p^q, G_p^q\) such that

\[
|f_p^q(x^\kappa) - g_p^q(y^\kappa)| \leq F_p^q|x^\kappa - y^\kappa|, |g_p^q(x^\kappa) - g_p^q(y^\kappa)| \leq G_p^q|x^\kappa - y^\kappa|, \]

\(x^\kappa, y^\kappa \in \mathbb{R}, \kappa = R, I, J, K.\)

(M₄) : There exists constants \(\gamma_p, p = 1, 2, \ldots, n\), such that the following conditions hold:

\[
\gamma_p d_p > \sum_{q=1}^{n} \gamma_q (|F_p^q| |a_{pq}| + |G_p^q| |b_{pq}|), \kappa = R, I, J, K, p = 1, 2, \ldots, n.
\]

By the Cayley-Dickson construction, system (5) can be decomposed into the 4-dimensional complex valued systems:

\[
\begin{align*}
C_{t_0}D_t^\nu \hat{w}_1^1(t) &= -d_1 \hat{w}_1^1(t) + \hat{a}_1 \hat{f}_1^1(w_1(t)) - \hat{a}_2 \hat{f}_1^2(w_1(t)) - \hat{a}_3 \hat{f}_1^3(w_1(t)) \\
&- \hat{a}_4 \hat{f}_1^4(w_1(t)) + \hat{b}_1^1 \hat{g}_1^1(w_n(t - \tau_n)) - \hat{b}_1^2 \hat{g}_1^2(w_n(t - \tau_n)) - \hat{b}_1^3 \hat{g}_1^3(w_n(t - \tau_n)) - \hat{b}_1^4 \hat{g}_1^4(w_n(t - \tau_n)), \\
C_{t_0}D_t^\nu \hat{w}_1^2(t) &= -d_1 \hat{w}_1^2(t) + \hat{a}_1 \hat{f}_1^2(w_1(t)) + \hat{a}_2 \hat{f}_1^1(w_1(t)) + \hat{a}_3 \hat{f}_1^4(w_1(t)) \\
&- \hat{a}_4 \hat{f}_1^3(w_1(t)) + \hat{b}_1^1 \hat{g}_1^2(w_n(t - \tau_n)) + \hat{b}_1^2 \hat{g}_1^1(w_n(t - \tau_n)) + \hat{b}_1^3 \hat{g}_1^3(w_n(t - \tau_n)) + \hat{b}_1^4 \hat{g}_1^4(w_n(t - \tau_n)), \\
C_{t_0}D_t^\nu \hat{w}_1^3(t) &= -d_1 \hat{w}_1^3(t) + \hat{a}_1 \hat{f}_1^3(w_1(t)) - \hat{a}_2 \hat{f}_1^4(w_1(t)) + \hat{a}_3 \hat{f}_1^1(w_1(t)) \\
&+ \hat{a}_4 \hat{f}_1^2(w_1(t)) + \hat{b}_1^1 \hat{g}_1^3(w_n(t - \tau_n)) - \hat{b}_1^2 \hat{g}_1^4(w_n(t - \tau_n)) - \hat{b}_1^3 \hat{g}_1^1(w_n(t - \tau_n)) - \hat{b}_1^4 \hat{g}_1^2(w_n(t - \tau_n)), \\
C_{t_0}D_t^\nu \hat{w}_1^4(t) &= -d_1 \hat{w}_1^4(t) + \hat{a}_1 \hat{f}_1^4(w_1(t)) + \hat{a}_2 \hat{f}_1^3(w_1(t)) - \hat{a}_3 \hat{f}_1^2(w_1(t)) \\
&+ \hat{a}_4 \hat{f}_1^1(w_1(t)) + \hat{b}_1^1 \hat{g}_1^4(w_n(t - \tau_n)) + \hat{b}_1^2 \hat{g}_1^3(w_n(t - \tau_n)) + \hat{b}_1^3 \hat{g}_1^2(w_n(t - \tau_n)) + \hat{b}_1^4 \hat{g}_1^1(w_n(t - \tau_n)).
\end{align*}
\]

\[
\begin{align*}
C_{t_0}D_t^\nu \hat{w}_p^1(t) &= -d_p \hat{w}_p^1(t) + \hat{a}_1 \hat{f}_p^1(w_p(t)) - \hat{a}_2 \hat{f}_p^2(w_p(t)) - \hat{a}_3 \hat{f}_p^3(w_p(t)) \\
&- \hat{a}_4 \hat{f}_p^4(w_p(t)) + \hat{b}_p^1 \hat{g}_p^1(w_p(t - \tau_p)) - \hat{b}_p^2 \hat{g}_p^2(w_p(t - \tau_p)) \\
&- \hat{b}_p^3 \hat{g}_p^3(w_p(t - \tau_p)) - \hat{b}_p^4 \hat{g}_p^4(w_p(t - \tau_p)), \\
C_{t_0}D_t^\nu \hat{w}_p^2(t) &= -d_p \hat{w}_p^2(t) + \hat{a}_1 \hat{f}_p^2(w_p(t)) + \hat{a}_2 \hat{f}_p^1(w_p(t)) + \hat{a}_3 \hat{f}_p^4(w_p(t)) \\
&- \hat{a}_4 \hat{f}_p^3(w_p(t)) + \hat{b}_p^1 \hat{g}_p^2(w_p(t - \tau_p)) + \hat{b}_p^2 \hat{g}_p^1(w_p(t - \tau_p)) + \hat{b}_p^3 \hat{g}_p^3(w_p(t - \tau_p)) + \hat{b}_p^4 \hat{g}_p^4(w_p(t - \tau_p)), \\
C_{t_0}D_t^\nu \hat{w}_p^3(t) &= -d_p \hat{w}_p^3(t) + \hat{a}_1 \hat{f}_p^3(w_p(t)) - \hat{a}_2 \hat{f}_p^4(w_p(t)) + \hat{a}_3 \hat{f}_p^1(w_p(t)) \\
&+ \hat{a}_4 \hat{f}_p^2(w_p(t)) + \hat{b}_p^1 \hat{g}_p^3(w_p(t - \tau_p)) - \hat{b}_p^2 \hat{g}_p^4(w_p(t - \tau_p)) - \hat{b}_p^3 \hat{g}_p^1(w_p(t - \tau_p)) - \hat{b}_p^4 \hat{g}_p^2(w_p(t - \tau_p)), \\
C_{t_0}D_t^\nu \hat{w}_p^4(t) &= -d_p \hat{w}_p^4(t) + \hat{a}_1 \hat{f}_p^4(w_p(t)) + \hat{a}_2 \hat{f}_p^3(w_p(t)) - \hat{a}_3 \hat{f}_p^2(w_p(t)) \\
&+ \hat{a}_4 \hat{f}_p^1(w_p(t)) + \hat{b}_p^1 \hat{g}_p^4(w_p(t - \tau_p)) + \hat{b}_p^2 \hat{g}_p^3(w_p(t - \tau_p)) + \hat{b}_p^3 \hat{g}_p^2(w_p(t - \tau_p)) + \hat{b}_p^4 \hat{g}_p^1(w_p(t - \tau_p)), \quad p = 2, 3, \ldots, n.
\end{align*}
\]

Under the hypothesis (M₂) the origin is an equilibrium point of the above equations. By linearizing the systems (6) and (7) at the equilibrium point, we get the
following:

\begin{align}
\begin{aligned}
\tau_0 D_\tau^\rho \hat{w}_1^q(t) &= -d_1 \hat{w}_1^q(t) + \hat{a}_1 \hat{w}_1^q(t) - \hat{a}_2 \hat{w}_2^q(t) - \hat{a}_3 \hat{w}_3^q(t) - \hat{a}_4 \hat{w}_4^q(t) \\
&\quad + \hat{b}_1 \hat{w}_1^q(t - \tau_n) - \hat{b}_2 \hat{w}_2^q(t - \tau_n) - \hat{b}_3 \hat{w}_3^q(t - \tau_n) \\
&\quad + \hat{b}_4 \hat{w}_4^q(t - \tau_n), \\
\tau_0 D_\tau^\rho \hat{w}_1^{q2}(t) &= -d_1 \hat{w}_1^{q2}(t) + \hat{a}_1 \hat{w}_1^{q2}(t) + \hat{a}_2 \hat{w}_2^{q2}(t) + \hat{a}_3 \hat{w}_3^{q2}(t) - \hat{a}_4 \hat{w}_4^{q2}(t) \\
&\quad + \hat{b}_1 \hat{w}_1^{q2}(t - \tau_n) + \hat{b}_2 \hat{w}_2^{q2}(t - \tau_n) + \hat{b}_3 \hat{w}_3^{q2}(t - \tau_n) \\
&\quad + \hat{b}_4 \hat{w}_4^{q2}(t - \tau_n), \\
\tau_0 D_\tau^\rho \hat{w}_1^{q3}(t) &= -d_1 \hat{w}_1^{q3}(t) + \hat{a}_1 \hat{w}_1^{q3}(t) + \hat{a}_2 \hat{w}_2^{q3}(t) + \hat{a}_3 \hat{w}_3^{q3}(t) - \hat{a}_4 \hat{w}_4^{q3}(t) \\
&\quad + \hat{b}_1 \hat{w}_1^{q3}(t - \tau_n) - \hat{b}_2 \hat{w}_2^{q3}(t - \tau_n) + \hat{b}_3 \hat{w}_3^{q3}(t - \tau_n) \\
&\quad + \hat{b}_4 \hat{w}_4^{q3}(t - \tau_n), \\
\tau_0 D_\tau^\rho \hat{w}_1^{q4}(t) &= -d_1 \hat{w}_1^{q4}(t) + \hat{a}_1 \hat{w}_1^{q4}(t) + \hat{a}_2 \hat{w}_2^{q4}(t) + \hat{a}_3 \hat{w}_3^{q4}(t) - \hat{a}_4 \hat{w}_4^{q4}(t) \\
&\quad + \hat{b}_1 \hat{w}_1^{q4}(t - \tau_n) + \hat{b}_2 \hat{w}_2^{q4}(t - \tau_n) + \hat{b}_3 \hat{w}_3^{q4}(t - \tau_n) \\
&\quad + \hat{b}_4 \hat{w}_4^{q4}(t - \tau_n).
\end{aligned}
\end{align}

Then by taking Laplace transform on both sides of the above equations, the characteristic matrix can be obtained as

\begin{align}
\Delta(s) = \begin{pmatrix}
U & -V_1 & -V_2 & -V_3 \\
V_1 & U & -V_3 & V_2 \\
V_2 & V_3 & U & -V_1 \\
V_3 & -V_2 & V_1 & U
\end{pmatrix},
\end{align}

where,

\begin{align}
U &= \begin{pmatrix}
s^\rho + d_1 - \hat{a}_1 & 0 & \ldots & -\hat{b}_1 e^{-s\tau_n} \\
-\hat{b}_2 e^{-s\tau_1} & s^\rho + d_2 - \hat{a}_1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \ldots & -\hat{b}_n e^{-s\tau_{n-1}} & s^\rho + d_n - \hat{a}_1
\end{pmatrix},
\end{align}
occurrence of Hopf bifurcation of the system (5).

Remark 1. The dynamical behavior of the fractional-order neural networks with time delays were studied in [3]. The occurrence of Hopf bifurcation results with respect to the octonion parameters have not been discussed yet for fractional-order systems.

Remark 2. If the characteristic matrix (10) can be written as \( \Delta(s) \), the system (11) can be transformed equivalently as

\[
V_1 = \begin{pmatrix}
-a^q_2 & 0 & \ldots & -b^q_1 e^{-s\tau_n}
\end{pmatrix},
\]

\[
V_2 = \begin{pmatrix}
-a^q_3 & 0 & \ldots & -b^q_1 e^{-s\tau_n}
\end{pmatrix},
\]

\[
V_3 = \begin{pmatrix}
-a^q_4 & 0 & \ldots & -b^q_1 e^{-s\tau_n}
\end{pmatrix}
\]

Therefore the bifurcation analysis of the system (5) can be absolutely determined by the distribution of the eigenvalues of \( \det(\Delta(s)) \).

**Lemma 3.1.** [9] If all the roots of the characteristic equation \( \det(\Delta(s)) \) have negative real parts, then the zero solution of system (5) is Lyapunov globally asymptotically stable.

**Lemma 3.2.** [9] If \( 0 < \nu < 1 \), all the eigenvalue of the \( \Delta(s) \) satisfy \( |\arg(\lambda)| > \frac{\pi}{2} \) and the characteristic equation \( \det(\Delta(s)) \) has no purely imaginary roots for any \( \tau_p > 0, p = 1, 2, \ldots, n \), then the zero solution of system (5) is Lyapunov globally asymptotically stable.

**Remark 1.** The dynamical behavior of the fractional-order neural networks with time delays were studied in [3]. The occurrence of Hopf bifurcation results with respect to the octonion parameters have not been discussed yet for fractional-order systems.

**Remark 2.** If the characteristic matrix (10) can be written as \( \Delta(s) = \begin{pmatrix} \mathbb{A} & -\mathbb{B} \\ \mathbb{B} & \mathbb{A} \end{pmatrix} \), where \( \mathbb{A} = \begin{pmatrix} U & -V_1 \\ V_1 & U \end{pmatrix}, \mathbb{B} = \begin{pmatrix} V_2 & V_3 \\ V_3 & -V_2 \end{pmatrix} \). The characteristic roots of the characteristic equation \( P(s) = (\Delta(s) - sI_n) = 0 \) can be found either from the characteristic value of \( \mathbb{A} + i\mathbb{B} \) or from the characteristic value of \( \mathbb{A} - i\mathbb{B} \).

4. **Main results.** In this section, we perform the stability of model (5) by taking time delay as a bifurcation parameter and then we establish the conditions for the occurrence of Hopf bifurcation of the system (5).

Consider the following fractional-order OVNNs with two neurons

\[
\begin{align*}
\frac{C}{\tau_1} D_t^\nu w_1(t) &= -d_1 w_1(t) + a f_1(w_1(t)) + g_1(w_2(t - \tau_1)), \\
\frac{C}{\tau_2} D_t^\nu w_2(t) &= -d_p w_p(t) + a f_p(w_p(t)) + g_p(w_{p-1}(t - \tau_{p-1})),
\end{align*}
\]

(11)

where, \( \nu \in (0, 1] \), \( w_1(t), w_2(t) \) are state variables, \( a, b_1, b_2 \) are the connection weights, \( f(\cdot), g(\cdot) \) denote the nonlinear activations, \( \tau_1, \tau_2 \) are constant time delays. By Assumptions \((\mathcal{M}_1) - (\mathcal{M}_2)\), the system (11) can be transformed equivalently as
\[
\begin{align*}
C_t D_t^\nu \omega_1^{q_1}(t) &= -d_1 \omega_1^{q_1}(t) + \bar{a}_{q_1} f_1^{q_1}(w_1(t)) - \bar{a}_{q_2} f_2^{q_2}(w_1(t)) - \bar{a}_{q_3} f_3^{q_3}(w_1(t)) \\
&\quad - \bar{a}_{q_4} f_4^{q_4}(w_1(t)) + \bar{b}_1^1 g_1^{q_1}(w_2(t) - \tau_1) - \bar{b}_2^q g_2^{q_2}(w_2(t) - \tau_2) \\
&\quad + \bar{b}_1^q g_1^{q_1}(w_2(t) - \tau_2) - \bar{b}_1^q g_1^{q_1}(w_2(t) - \tau_2), \\
C_t D_t^\nu \omega_1^{q_2}(t) &= -d_1 \omega_1^{q_2}(t) + \bar{a}_{q_1} f_1^{q_1}(w_2(t)) - \bar{a}_{q_2} f_2^{q_2}(w_2(t)) - \bar{a}_{q_3} f_3^{q_3}(w_2(t)) \\
&\quad - \bar{a}_{q_4} f_4^{q_4}(w_2(t)) + \bar{b}_1^1 g_1^{q_1}(w_2(t) - \tau_1) - \bar{b}_2^q g_2^{q_2}(w_2(t) - \tau_2) \\
&\quad + \bar{b}_1^q g_1^{q_1}(w_2(t) - \tau_2) - \bar{b}_1^q g_1^{q_1}(w_2(t) - \tau_2), \\
C_t D_t^\nu \omega_1^{q_3}(t) &= -d_1 \omega_1^{q_3}(t) + \bar{a}_{q_1} f_1^{q_1}(w_1(t)) + \bar{b}_1^1 g_1^{q_1}(w_2(t) - \tau_1) - \bar{b}_2^q g_2^{q_2}(w_2(t) - \tau_2) \\
&\quad + \bar{b}_1^q g_1^{q_1}(w_2(t) - \tau_2) - \bar{b}_1^q g_1^{q_1}(w_2(t) - \tau_2), \\
C_t D_t^\nu \omega_1^{q_4}(t) &= -d_1 \omega_1^{q_4}(t) + \bar{a}_{q_1} f_1^{q_1}(w_1(t)) + \bar{b}_1^1 g_1^{q_1}(w_2(t) - \tau_1) - \bar{b}_2^q g_2^{q_2}(w_2(t) - \tau_2) \\
&\quad + \bar{b}_1^q g_1^{q_1}(w_2(t) - \tau_2) - \bar{b}_1^q g_1^{q_1}(w_2(t) - \tau_2), \\
C_t D_t^\nu \omega_1^{q_5}(t) &= -d_1 \omega_1^{q_5}(t) + \bar{a}_{q_1} f_1^{q_1}(w_2(t)) + \bar{b}_1^1 g_1^{q_1}(w_2(t) - \tau_1) - \bar{b}_2^q g_2^{q_2}(w_2(t) - \tau_2) \\
&\quad + \bar{b}_1^q g_1^{q_1}(w_2(t) - \tau_2) - \bar{b}_1^q g_1^{q_1}(w_2(t) - \tau_2), \\
&\quad + \bar{a}_{q_4} f_4^{q_4}(w_2(t)) - \bar{a}_{q_4} f_4^{q_4}(w_2(t)) - \bar{a}_{q_4} f_4^{q_4}(w_2(t)) - \bar{a}_{q_4} f_4^{q_4}(w_2(t)) - \bar{a}_{q_4} f_4^{q_4}(w_2(t)) - \bar{a}_{q_4} f_4^{q_4}(w_2(t)).
\end{align*}
\]

(12)

From Assumption \((M_2)\), the origin is the equilibrium point of the system (11), then the characteristic equation can be obtained as:

\[
\det \begin{pmatrix}
\mathcal{U} & -\mathcal{V} \\
\mathcal{V} & \mathcal{U}
\end{pmatrix} = 0,
\]

where,

\[
\mathcal{U} = \begin{pmatrix}
s' + d & \bar{b}_1^q e^{-s \tau_2} & \bar{a}_{q_3} & \bar{b}_1^q e^{-s \tau_2} \\
-\bar{a}_{q_2} & s' + d & \bar{b}_1^q e^{-s \tau_1} & \bar{a}_{q_4} \\
-\bar{b}_2^q e^{-s \tau_1} & -\bar{a}_{q_2} & s' + d & \bar{b}_1^q e^{-s \tau_2} \\
-\bar{a}_{q_3} & \bar{b}_1^q e^{-s \tau_2} & \bar{a}_{q_4} & s' + d
\end{pmatrix},
\]

\[
\mathcal{V} = \begin{pmatrix}
s' + d & \bar{b}_2^q e^{-s \tau_1} & \bar{a}_{q_3} & \bar{b}_2^q e^{-s \tau_2} \\
-\bar{a}_{q_2} & s' + d & \bar{b}_2^q e^{-s \tau_1} & \bar{a}_{q_4} \\
-\bar{b}_1^q e^{-s \tau_1} & -\bar{a}_{q_2} & s' + d & \bar{b}_2^q e^{-s \tau_2} \\
-\bar{a}_{q_3} & \bar{b}_2^q e^{-s \tau_2} & \bar{a}_{q_4} & s' + d
\end{pmatrix}.
\]

For the sake of simplicity, we assume that \(\tau_1 = \tau_2 = \tau\) and \(d_1 = d_2 = d\). Hence, (14) is equivalent to

\[
\Lambda_1(s) + \Lambda_2(s)e^{-2s\tau} + \Lambda_3(s)e^{-4s\tau} + \Lambda_4(s)e^{-6s\tau} + \Lambda(s)e^{-8s\tau} = 0
\]

(15)
Multiplying both sides of (15) by $e^{4\pi \tau}$ it can be found that

$$A_1(s)e^{4\pi \tau} + A_2(s)e^{2\pi \tau} + A_3(s) + A_4(s)e^{-2\pi \tau} + \Lambda(s)e^{-4\pi \tau} = 0 \quad (16)$$

where,

$$A_1(s) = \phi_1 + \phi_2 s^\nu + \phi_3 s^{2\nu} + \phi_4 s^{3\nu} + \phi_5 s^{4\nu} + \phi_6 s^{5\nu} + \phi_7 s^{6\nu} + \phi_8 s^{7\nu} + s^{8\nu},$$

$$A_2(s) = \phi_9 + \phi_{10} s^\nu + \phi_{11} s^{2\nu} + \phi_{12} s^{3\nu} + \phi_{13} s^{4\nu} + \phi_{14} s^{5\nu} + \phi_{15} s^{6\nu},$$

$$A_3(s) = \phi_{16} + \phi_{17} s^\nu + \phi_{18} s^{2\nu} + \phi_{19} s^{3\nu} + \phi_{20} s^{4\nu},$$

$$A_4(s) = \phi_{21} + \phi_{22} s^\nu + \phi_{23} s^{2\nu},$$

$$A_5(s) = \phi_{24}.$$

Let $s = i\omega = \omega \left( \cos \frac{\pi}{2} + \sin \frac{\pi}{2} \right), \omega > 0$ into the equation (16), then we can get

$$(A_1 + iB_1)(\cos 4\omega \tau + \sin 4\omega \tau) + (A_2 + iB_2)(\cos 2\omega \tau + \sin 2\omega \tau) + (A_3 + iB_3)$$

$$+ (A_4 + iB_4)(\cos 2\omega \tau - \sin 2\omega \tau) + (A_5 + iB_5)(\cos 4\omega \tau - \sin 4\omega \tau) = 0 \quad (17)$$

where, $A_p, B_p$ are the real and imaginary parts of $\Lambda_p(s)(p = 1, 2, 3, 4, 5)$, respectively and are represented as follows

$$A_1 = \omega^{8\nu}\cos 4\nu \pi + \phi_8^R s^{7\nu}\cos \frac{7\nu \pi}{2} - \phi_8^I s^{7\nu}\sin \frac{7\nu \pi}{2} + \phi_9^R s^{6\nu}\cos 3\nu \pi - \phi_9^I s^{6\nu}\sin 3\nu \pi$$

$$+ \phi_6^R s^{5\nu}\cos \frac{5\nu \pi}{2} - \phi_6^I s^{5\nu}\sin \frac{5\nu \pi}{2} + \phi_5^R s^{4\nu}\cos 2\nu \pi - \phi_5^I s^{4\nu}\sin 2\nu \pi$$

$$+ \phi_4^R s^{3\nu}\cos \frac{3\nu \pi}{2} - \phi_4^I s^{3\nu}\sin \frac{3\nu \pi}{2} + \phi_3^R s^{2\nu}\cos \nu \pi - \phi_3^I s^{2\nu}\sin \nu \pi + \phi_2^R s^{\nu}\cos \frac{\nu \pi}{2}$$

$$- \phi_2^I s^{\nu}\sin \frac{\nu \pi}{2} + \phi_1^R,$$

$$B_1 = \omega^{8\nu}\sin 4\nu \pi + \phi_8^I s^{7\nu}\sin \frac{7\nu \pi}{2} + \phi_8^R s^{7\nu}\sin \frac{7\nu \pi}{2} + \phi_9^I s^{6\nu}\cos 3\nu \pi + \phi_9^R s^{6\nu}\sin 3\nu \pi$$

$$+ \phi_6^I s^{5\nu}\cos \frac{5\nu \pi}{2} + \phi_6^R s^{5\nu}\sin \frac{5\nu \pi}{2} + \phi_5^I s^{4\nu}\cos 2\nu \pi + \phi_5^R s^{4\nu}\sin 2\nu \pi$$

$$+ \phi_4^I s^{3\nu}\cos \frac{3\nu \pi}{2} + \phi_4^R s^{3\nu}\sin \frac{3\nu \pi}{2} + \phi_3^I s^{2\nu}\cos \nu \pi + \phi_3^R s^{2\nu}\sin \nu \pi + \phi_2^I s^{\nu}\cos \frac{\nu \pi}{2}$$

$$+ \phi_2^R s^{\nu}\sin \frac{\nu \pi}{2} + \phi_1^I,$$

$$A_2 = \phi_{10}^R s^{6\nu}\cos 3\nu \pi - \phi_{12}^I s^{5\nu}\sin 3\nu \pi + \phi_{14}^R s^{5\nu}\cos \frac{5\nu \pi}{2} - \phi_{14}^I s^{5\nu}\sin \frac{5\nu \pi}{2}$$

$$+ \phi_{13}^R s^{4\nu}\cos 2\nu \pi - \phi_{13}^I s^{4\nu}\sin 2\nu \pi + \phi_{12}^R s^{3\nu}\cos \frac{3\nu \pi}{2} - \phi_{12}^I s^{3\nu}\sin \frac{3\nu \pi}{2}$$

$$+ \phi_{11}^R s^{2\nu}\cos \nu \pi - \phi_{11}^I s^{2\nu}\sin \nu \pi + \phi_{10}^R s^{\nu}\cos \frac{\nu \pi}{2} - \phi_{10}^I s^{\nu}\sin \frac{\nu \pi}{2} + \phi_9^R,$$

$$B_2 = \phi_{10}^I s^{6\nu}\cos 3\nu \pi + \phi_{12}^R s^{5\nu}\sin 3\nu \pi + \phi_{14}^I s^{5\nu}\cos \frac{5\nu \pi}{2} + \phi_{14}^R s^{5\nu}\sin \frac{5\nu \pi}{2}$$

$$+ \phi_{13}^I s^{4\nu}\cos 2\nu \pi + \phi_{13}^R s^{4\nu}\sin 2\nu \pi + \phi_{12}^I s^{3\nu}\cos \frac{3\nu \pi}{2} + \phi_{12}^R s^{3\nu}\sin \frac{3\nu \pi}{2}$$

$$+ \phi_{11}^I s^{2\nu}\cos \nu \pi + \phi_{11}^R s^{2\nu}\sin \nu \pi + \phi_{10}^I s^{\nu}\cos \frac{\nu \pi}{2} + \phi_{10}^R s^{\nu}\sin \frac{\nu \pi}{2} + \phi_9^I,
the real and imaginary parts of (17) we can obtain

\[ A_3 = \phi_2^R \omega^4 \cos^2 \nu \pi - \phi_2^I \omega^4 \sin 2 \nu \pi + \Phi^R_{19} \omega^3 \nu \cos \frac{3 \nu \pi}{2} - \Phi^I_{19} \omega^3 \nu \sin \frac{3 \nu \pi}{2} \\
+ \phi^R_{18} \omega^2 \nu \cos \nu \pi - \phi^I_{18} \omega^2 \nu \sin \nu \pi + \phi^R_{17} \omega \nu \cos \frac{\nu \pi}{2} - \phi^I_{17} \omega \nu \sin \frac{\nu \pi}{2} + \phi^R_{16}, \]

\[ B_3 = \phi^R_{20} \omega^4 \cos^2 \nu \pi + \phi^R_{20} \omega^4 \sin 2 \nu \pi + \Phi^R_{19} \omega^3 \nu \cos \frac{3 \nu \pi}{2} + \Phi^I_{19} \omega^3 \nu \sin \frac{3 \nu \pi}{2} \\
+ \phi^R_{18} \omega^2 \nu \cos \nu \pi + \phi^R_{17} \omega \nu \cos \frac{\nu \pi}{2} + \phi^R_{16} \nu \sin \frac{\nu \pi}{2} + \phi^I_{16}, \]

\[ A_4 = \phi^R_{23} \omega^4 \cos^2 \nu \pi - \phi^I_{23} \omega^2 \nu \cos \nu \pi + \phi^R_{22} \omega \nu \cos \frac{\nu \pi}{2} - \phi^I_{22} \omega \nu \sin \frac{\nu \pi}{2} + \phi^R_{21}, \]

\[ B_4 = \phi^I_{23} \omega^2 \nu \sin \nu \pi + \phi^I_{22} \omega \nu \cos \frac{\nu \pi}{2} + \phi^I_{22} \omega \nu \sin \frac{\nu \pi}{2} + \phi^I_{21}, \]

\[ A_5 = \phi^R_{24}, B_5 = \phi^I_{24}. \]

where, \( \phi^R_p, \phi^I_p \) are the real and imaginary parts of \( \phi_p(p = 1, 2, \ldots, 24) \). Separating the real and imaginary parts of (17) we can obtain

\[ (A_1 + A_5) \cos 4 \omega \tau + (-B_1 + B_5) \sin 4 \omega \tau + (A_2 + A_4) \cos 2 \omega \tau \\
+ (-B_2 + B_4) \sin 2 \omega \tau + A_3 = 0, \]

(18)

\[ (A_1 - A_5) \sin 4 \omega \tau + (B_1 + B_5) \cos 4 \omega \tau + (A_2 - A_4) \sin 2 \omega \tau \\
+ (B_2 + B_4) \cos 2 \omega \tau + B_3 = 0. \]

(19)

Since \( \sin^2 \theta + \cos^2 \theta = 1 \), directly we can get two cases as follows:

**Case (1).** If \( \sin 2 \omega \tau = \sqrt{1 - \cos^2 2 \omega \tau} \), then (18) can be changed into

\[ (r_1 \cos 2 \omega \tau + r_2 + r_3 \cos 2 \omega \tau) = \sqrt{1 - \cos^2 2 \omega \tau} (r_4 \cos 2 \omega \tau + r_5) \]

(20)

It follows from (20), that

\[ K_1 \cos 4 \omega \tau + K_2 \cos 3 \omega \tau + K_3 \cos 2 \omega \tau + K_4 \cos 2 \omega \tau + K_5 = 0 \]

(21)

where,

\[ K_1 = r_1^2 + r_4^2, \quad r_1 = 2(A_1 + A_5), \]

\[ K_2 = 2(r_1 + r_3 + r_4 r_5), \quad r_2 = (A_3 - (A_1 + A_5)), \]

\[ K_3 = (r_4^2 - r_2^2 + r_5^2 + 2r_1 r_2), \quad r_3 = (A_2 + A_4), \]

\[ K_4 = 2(r_2 r_3 - r_4 r_5), \quad r_4 = 2(B_1 - B_5), \]

\[ K_5 = r_2^2 - r_5^2, \quad r_5 = (B_2 - B_4). \]

Then (21) is equivalent to

\[ h^4 + \beta_1 h^3 + \beta_2 h^2 + \beta_3 h + \beta = 0 \]

(22)

where, \( h = \cos 2 \omega \tau, \beta_1 = \frac{K_2}{K_1}, \beta_2 = \frac{K_3}{K_1}, \beta_3 = \frac{K_4}{K_1}, \beta_4 = \frac{K_5}{K_1} \). Then doing theoretical calculation with the help of Mathematica, the expression of \( \cos 2 \omega \tau \) can be obtained and it is defined as

\[ \cos 2 \omega \tau = \Xi_1(\omega). \]

(23)

Similarly, by utilizing (19) the expression of \( \sin 2 \omega \tau \) can be derived and it is denoted by

\[ \sin 2 \omega \tau = \Xi_2(\omega) \]

(24)
From (23) and (24), it is not hard to deduce
\[ \Xi_1^2(\omega) + \Xi_2^2(\omega) = 1 \] (25)
By using the numerical software Maple 18, we can easily find the roots \( w \) of the polynomial (25). From (23), it can be found that
\[ \tau^\sigma = \frac{1}{2\omega} \{ \arccos \Xi_1(\omega) + 2\sigma\pi \}, \sigma = 0, 1, 2, \ldots \] (26)

**Case(2).** If \( \sin 2\omega \tau = -\sqrt{1 - \cos^2 2\omega \tau} \), do the same calculations as Case(1), we conclude that
\[ \cos 2\omega \tau = \Upsilon_1(\omega) \quad \text{and} \quad \sin 2\omega \tau = \Upsilon_2(\omega). \] (27)
Then it can be found from (27) that
\[ \Upsilon_1^2(\omega) + \Upsilon_2^2(\omega) = 1 \] (28)
From (28), we can find the value of $\omega$, then it follows from (27) that

$$\tau_2^\sigma = \frac{1}{2\omega} \left\{ \arccos \Upsilon_1(\omega) + 2\sigma \pi \right\}, \sigma = 0, 1, 2, \ldots.$$  \hspace{1cm} (29)

Assume that (25) and (28) admits at least one positive real root. Then the bifurcation parameter of the system (11) is defined as

$$\tau_0 = \min \left\{ \tau_1^\sigma, \tau_2^\sigma \right\}, \sigma = 0, 1, 2, \ldots.$$ \hspace{1cm} (30)

where, $\tau_1^\sigma$ and $\tau_2^\sigma$ are defined in (26) and (29) respectively. When $\tau = 0$, the characteristic equation (15) becomes

$$\chi^8 + \Delta_1 \chi^7 + \Delta_2 \chi^6 + \Delta_3 \chi^5 + \Delta_4 \chi^4 + \Delta_5 \chi^3 + \Delta_6 \chi^2 + \Delta_7 \chi + \Delta_8 = 0$$ \hspace{1cm} (31)

where,

$$\Delta_1 = \phi_8, \quad \Delta_5 = \phi_4 + \phi_{12} + \phi_{19},$$
Figure 5. The waveform and the phase diagram of real and imaginary parts of $v_1(t)$ when $\tau = 0.14 < \tau_0 = 0.15$.

Figure 6. The waveform and the phase diagram of real and imaginary parts of $v_1(t)$ when $\tau = 0.39 > \tau_0 = 0.15$.

\[ \Delta_2 = \phi_7 + \phi_{15}, \quad \Delta_6 = \phi_3 + \phi_{17} + \phi_{18} + \phi_{23}, \]
\[ \Delta_3 = \phi_6 + \phi_{1014}, \quad \Delta_7 = \phi_2 + \phi_{10} + \phi_{17} + \phi_{22}, \]
\[ \Delta_4 = \phi_5 + \phi_{13} + \phi_{20}, \quad \Delta_8 = \phi_1 + \phi_9 + \phi_{16} + \phi_{21} + \phi_{24}. \]

Define $\Theta_1, \Theta_2, \ldots, \Theta_8$ as follows:

\[ \Theta_1 = \Delta_1, \]
\[ \Theta_2 = \begin{bmatrix} \Delta_1 & 1 \\ \Delta_3 & \Delta_2 \end{bmatrix}, \]
\[ \Theta_3 = \begin{bmatrix} \Delta_1 & 1 & 0 \\ \Delta_3 & \Delta_2 & \Delta_1 \\ \Delta_5 & \Delta_4 & \Delta_3 \end{bmatrix}, \ldots, \Theta_7 = \begin{bmatrix} \Delta_1 & 1 & \cdots & 0 \\ \Delta_3 & \Delta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Delta_7 \end{bmatrix}, \]
\[ \Theta_8 = \Theta_7 \Delta_8. \]
Assumption:
(M₅) : Θₚ > 0 (p = 1, 2, . . . , 8),
(M₆) : Re $\frac{ds}{d\tau}$ |(τ = τ₀, ω = ω₀) ≠ 0,
where τ₀ and ω₀ represents the bifurcation point and the critical frequency respectively.

By implicit theorem and differentiating (15) with respect to τ we have

$$\frac{ds}{d\tau} = \frac{\Sigma_1(s)}{\Sigma_2(s)}$$  \hspace{1cm} (32)

where,

$$\Sigma_1 = s \left[ 2(\phi_9 + \phi_{10}s^\nu + \phi_{11}s^{2\nu} + \phi_{12}s^{3\nu} + \phi_{13}s^{4\nu} + \phi_{14}s^{5\nu} + \phi_{15}s^{6\nu})e^{-2s\tau} 
+ 4(\phi_6 + \phi_{16}s^\nu + \phi_{17}s^{2\nu} + \phi_{18}s^{3\nu} + \phi_{19}s^{4\nu} + \phi_{20}s^{5\nu})e^{-4s\tau} + 6(\phi_2 + \phi_{21}s^\nu 
+ \phi_{22}s^{2\nu} + \phi_{23}s^{3\nu})e^{-6s\tau} + 8\phi_{24}e^{-8s\tau} \right]$$,
Figure 9. The waveform and the phase diagram of real and imaginary parts of $v_2(t)$ when $\tau = 0.14 < \tau_0 = 0.15$.

Figure 10. The waveform and the phase diagram of real and imaginary parts of $v_2(t)$ when $\tau = 0.39 > \tau_0 = 0.15$.

$$\Sigma_2 = \left[ (\phi_2 \nu s^{\nu-1} + 2\phi_3 \nu s^{2\nu-1} + 3\phi_4 \nu s^{3\nu-1} + 4\phi_5 \nu s^{4\nu-1} + 5\phi_6 \nu s^{5\nu-1} \\
+ 6\phi_7 \nu s^{6\nu-1} + 7\phi_8 \nu s^{7\nu-1} + 8\nu s^{8\nu-1}) + [-2\nu (\phi_9 + \phi_{10} s^\nu + \phi_{11} s^{2\nu} \\
+ \phi_{12} s^{3\nu} + \phi_{13} s^{4\nu} + \phi_{14} s^{5\nu} + \phi_{15} s^{6\nu}) + (\phi_{10} \nu s^{\nu-1} + 2\phi_{11} \nu s^{2\nu-1} \\
+ 3\phi_{12} \nu s^{3\nu-1} + 4\phi_{13} \nu s^{4\nu-1} + 5\phi_{14} \nu s^{5\nu-1} + 6\phi_{15} \nu s^{6\nu-1})] e^{-2s\tau} \\
+ [-4s^\tau (\phi_1 + \phi_{17} s^\nu + \phi_{18} s^{2\nu} + \phi_{19} s^{3\nu} + \phi_{20} s^{4\nu}) + (\phi_{17} \nu s^{\nu-1} \\
+ 2\phi_{18} s^{2\nu-1} + 3\phi_{19} \nu s^{3\nu-1} + 4\phi_{20} \nu s^{4\nu-1})] e^{-4s\tau} + [-6s^\tau (\phi_{21} \\
+ \phi_{22} s^\nu + \phi_{23} s^{2\nu}) + (\phi_{22} \nu s^{\nu-1} + 2\phi_{23} \nu s^{2\nu-1})] e^{-6s\tau} - 8s^\tau \phi_{24} e^{-8s\tau} \right]$$

It can be deduced from (32),

$$\text{Re} \left[ \frac{ds}{d\tau} \right] |_{(\tau=\tau_0, \omega=\omega_0)} = \frac{\dot{\Sigma}_1 \Sigma + \dot{\Sigma}_2 \Sigma_2}{\Sigma_1^2 + \Sigma_2^2}$$

(33)
where, $\hat{\Sigma}_1, \check{\Sigma}_1$ are the real and imaginary parts of $\Sigma_1$ and $\hat{\Sigma}_2, \check{\Sigma}_2$ are the real and imaginary parts of $\Sigma_2$ and are defined as follows:

$$
\hat{\Sigma}_1 = \omega_0 \left[ 2 \sum_{p=9}^{15} \left( \phi_p R_{p-9} \omega_0 (p-9) \nu \cos \left( \frac{p-9}{2} \nu \pi \right) - \phi_p I_{p-9} \omega_0 (p-9) \nu \sin \left( \frac{p-9}{2} \nu \pi \right) \right) \sin 2\omega_0 \tau_0 
- 2 \sum_{p=9}^{15} \left( \phi_p R_{p-9} \omega_0 (p-9) \nu \sin \left( \frac{p-9}{2} \nu \pi \right) + \phi_p I_{p-9} \omega_0 (p-9) \nu \cos \left( \frac{p-9}{2} \nu \pi \right) \right) \cos 2\omega_0 \tau_0 
+ 4 \sum_{p=16}^{20} \left( \phi_p R_{p-16} \omega_0 (p-16) \nu \cos \left( \frac{p-16}{2} \nu \pi \right) - \phi_p I_{p-16} \omega_0 (p-16) \nu \sin \left( \frac{p-16}{2} \nu \pi \right) \right) \sin 4\omega_0 \tau_0 
- 4 \sum_{p=16}^{20} \left( \phi_p R_{p-16} \omega_0 (p-16) \nu \sin \left( \frac{p-16}{2} \nu \pi \right) + \phi_p I_{p-16} \omega_0 (p-16) \nu \cos \left( \frac{p-16}{2} \nu \pi \right) \right) \cos 4\omega_0 \tau_0 \right].
$$

**Figure 11.** The waveform and the phase diagram of real and imaginary parts of $w_2(t)$ when $\tau = 0.14 < \tau_0 = 0.15$.

**Figure 12.** The waveform and the phase diagram of real and imaginary parts of $w_2(t)$ when $\tau = 0.39 > \tau_0 = 0.15$. 
Figure 13. The waveform and the phase diagram of real and imaginary parts of $v_3(t)$ when $\tau = 0.14 < \tau_0 = 0.15$.

Figure 14. The waveform and the phase diagram of real and imaginary parts of $v_3(t)$ when $\tau = 0.39 > \tau_0 = 0.15$.

\[ + 6 \sum_{p=22}^{23} \left( \phi^R_{p,\nu_0} \frac{(p-21)\nu\pi}{2} \cos \left( \frac{(p-21)\nu\pi}{2} \right) \right) - \phi^I_{p,\nu_0} \frac{(p-21)\nu\pi}{2} \sin \left( \frac{(p-21)\nu\pi}{2} \right) \right) \sin 6\omega_0\tau_0 \]

\[ - 6 \sum_{p=22}^{21} \left( \phi^R_{p,\nu_0} \frac{(p-21)\nu\pi}{2} \sin \left( \frac{(p-21)\nu\pi}{2} \right) \right) + \phi^I_{p,\nu_0} \frac{(p-21)\nu\pi}{2} \cos \left( \frac{(p-21)\nu\pi}{2} \right) \right) \cos 6\omega_0\tau_0 \]

\[ + 8 \left( \phi^R_{24,\nu_0} \cos 8\omega_0\tau_0 + \phi^I_{24,\nu_0} \sin 8\omega_0\tau_0 \right) \]
Figure 15. The waveform and the phase digram of real and imaginary parts of $w_3(t)$ when $\tau = 0.14 < \tau_0 = 0.15$.

Figure 16. The waveform and the phase digram of real and imaginary parts of $w_3(t)$ when $\tau = 0.39 > \tau_0 = 0.15$.

\[
+ 4 \sum_{p=16}^{20} \left( \phi_p R_0 \omega_0^{(p-16)} \cos \frac{(p-16) \nu \pi}{2} - \phi_p I_0 \omega_0^{(p-16)} \sin \frac{(p-16) \nu \pi}{2} \right) \cos^{4} \omega_0 \tau_0
\]

\[
+ 4 \sum_{p=16}^{20} \left( \phi_p R_0 \omega_0^{(p-16)} \sin \frac{(p-16) \nu \pi}{2} + \phi_p I_0 \omega_0^{(p-16)} \cos \frac{(p-16) \nu \pi}{2} \right) \sin^{4} \omega_0 \tau_0
\]

\[
+ 6 \sum_{p=22}^{23} \left( \phi_p R_0 \omega_0^{(p-21)} \cos \frac{(p-21) \nu \pi}{2} - \phi_p I_0 \omega_0^{(p-21)} \sin \frac{(p-21) \nu \pi}{2} \right) \cos^{6} \omega_0 \tau_0
\]

\[
+ 6 \sum_{p=22}^{23} \left( \phi_p R_0 \omega_0^{(p-21)} \sin \frac{(p-21) \nu \pi}{2} + \phi_p I_0 \omega_0^{(p-21)} \cos \frac{(p-21) \nu \pi}{2} \right) \sin^{6} \omega_0 \tau_0
\]

\[
+ 8(\phi_{24} R \omega_0 \tau_0 - \phi_{24} I \omega_0 \tau_0)
\]
\[ \Sigma_2 = \sum_{p=2}^{8} \left( \phi_p^R (p-1) \nu \omega_0^{(p-1) \nu - 1} \cos \left( \frac{(p-1) \nu - 1}{2} \right) - \phi_p^I (p-1) \nu \omega_0^{(p-1) \nu - 1} \cos \left( \frac{(p-1) \nu - 1}{2} \right) \right) \times \sin \left( \frac{(p-1) \nu - 1}{2} \right) + 8 \nu \omega_0^8 \cos \left( \frac{(8 \nu - 1) \pi}{2} \right) \]

\[
+ \left[ -2 \tau \sum_{p=9}^{15} \left( \phi_p^R (p-9) \nu \cos \left( \frac{(p-9) \nu \pi}{2} \right) - \phi_p^I (p-9) \nu \sin \left( \frac{(p-9) \nu \pi}{2} \right) \right) \right] \cos 2 \omega_0 \tau_0
\]

\[
+ \left[ -2 \tau \sum_{p=9}^{15} \left( \phi_p^R (p-9) \nu \cos \left( \frac{(p-9) \nu \pi}{2} \right) - \phi_p^I (p-9) \nu \sin \left( \frac{(p-9) \nu \pi}{2} \right) \right) \right] \cos 2 \omega_0 \tau_0
\]

\[
+ \phi_p^I (p-9) \nu \omega_0^{(p-9) \nu - 1} \cos \left( \frac{(p-9) \nu \pi}{2} \right) + \sum_{p=9}^{15} \left( \phi_p^R (p-9) \nu \omega_0^{(p-9) \nu - 1} \cos \left( \frac{(p-9) \nu \pi}{2} \right) - \phi_p^I (p-9) \nu \omega_0^{(p-9) \nu - 1} \sin \left( \frac{(p-9) \nu \pi}{2} \right) \right) \sin 2 \omega_0 \tau_0
\]

\[
+ \phi_p^I (p-9) \nu \omega_0^{(p-9) \nu - 1} \cos \left( \frac{(p-9) \nu \pi}{2} \right) + \sum_{p=9}^{15} \left( \phi_p^R (p-9) \nu \omega_0^{(p-9) \nu - 1} \sin \left( \frac{(p-9) \nu - 1}{2} \right) \right) \sin 2 \omega_0 \tau_0
\]

\[
+ \phi_p^I (p-16) \nu \omega_0^{(p-16) \nu - 1} \sin \left( \frac{(p-16) \nu - 1}{2} \right) \cos 4 \omega_0 \tau_0
\]

\[
+ \phi_p^I (p-16) \nu \omega_0^{(p-16) \nu - 1} \sin \left( \frac{(p-16) \nu - 1}{2} \right) \cos 4 \omega_0 \tau_0
\]

\[
+ \phi_p^I (p-16) \nu \omega_0^{(p-16) \nu - 1} \sin \left( \frac{(p-16) \nu - 1}{2} \right) \sin 4 \omega_0 \tau_0
\]

\[
+ \phi_p^I (p-16) \nu \omega_0^{(p-16) \nu - 1} \sin \left( \frac{(p-16) \nu - 1}{2} \right) \sin 4 \omega_0 \tau_0
\]

\[
+ \phi_p^I (p-21) \nu \omega_0^{(p-21) \nu - 1} \cos \left( \frac{(p-21) \nu - 1}{2} \right) \cos 6 \omega_0 \tau_0
\]

\[
+ \phi_p^I (p-21) \nu \omega_0^{(p-21) \nu - 1} \cos \left( \frac{(p-21) \nu - 1}{2} \right) \cos 6 \omega_0 \tau_0
\]

\[
+ \phi_p^I (p-21) \nu \omega_0^{(p-21) \nu - 1} \sin \left( \frac{(p-21) \nu - 1}{2} \right) \sin 6 \omega_0 \tau_0
\]

\[
+ \phi_p^I (p-21) \nu \omega_0^{(p-21) \nu - 1} \sin \left( \frac{(p-21) \nu - 1}{2} \right) \sin 6 \omega_0 \tau_0
\]
\[\begin{align*}
&-\phi_p^t (p-21) \nu \omega_0^{(p-21)\nu-1} \sin \left(\frac{(p-21)\nu - 1}{2}\pi\right) \cos \nu \omega_0 \tau_0 \\
&+ \left[-6\tau \sum_{p=21}^{23} \phi_p^R \omega_0^{(p-21)\nu} \sin \left(\frac{(p-21)\nu}{2}\pi\right) + \phi_p^t \omega_0^{(p-21)\nu} \cos \left(\frac{(p-21)\nu}{2}\pi\right) \right] \sin \nu \omega_0 \tau_0 \\
&\times \cos \left(\frac{(p-21)\nu}{2}\pi\right) + \sum_{p=22}^{23} \left[ \phi_p^R (p-21) \nu \omega_0^{(p-21)\nu-1} \sin \left(\frac{(p-21)\nu - 1}{2}\pi\right) \right] \sin \nu \omega_0 \tau_0 \\
&+ \phi_p^t (p-21) \nu \omega_0^{(p-21)\nu-1} \cos \left(\frac{(p-21)\nu - 1}{2}\pi\right) \right] \sin 2\omega_0 \tau_0 \\
&- 8\tau \left[ \phi_p^R \cos 8\omega_0 \tau_0 + \phi_p^t \sin 8\omega_0 \tau_0 \right], \\
\Sigma_2 &= \sum_{p=2}^{8} \left[ \phi_p^R (p-1) \nu \omega_0^{(p-1)\nu-1} \sin \left(\frac{(p-1)\nu - 1}{2}\pi\right) + \phi_p^t (p-1) \nu \omega_0^{(p-1)\nu-1} \right] \\
&\times \cos \left(\frac{(p-1)\nu - 1}{2}\pi\right) + 8\nu \omega_0^{8\nu-1} \sin \left(\frac{8\nu - 1}{2}\pi\right) \\
&- \left[-2\tau \sum_{p=9}^{15} \left[ \phi_p^R \omega_0^{(p-9)\nu} \cos \left(\frac{(p-9)\nu}{2}\pi\right) + \phi_p^t \omega_0^{(p-9)\nu} \sin \left(\frac{(p-9)\nu - 1}{2}\pi\right) \right] \sin 2\omega_0 \tau_0 \\
&+ \left[-2\tau \sum_{p=9}^{15} \left[ \phi_p^R \omega_0^{(p-9)\nu} \sin \left(\frac{(p-9)\nu}{2}\pi\right) + \phi_p^t \omega_0^{(p-9)\nu} \cos \left(\frac{(p-9)\nu - 1}{2}\pi\right) \right] \sin 2\omega_0 \tau_0 \\
&+ \phi_p^t (p-9) \nu \omega_0^{(p-9)\nu-1} \cos \left(\frac{(p-9)\nu - 1}{2}\pi\right) \right] \cos 2\omega_0 \tau_0 \\
&- 4\tau \sum_{p=16}^{20} \left[ \phi_p^R \omega_0^{(p-16)\nu} \cos \left(\frac{(p-16)\nu}{2}\pi\right) + \phi_p^t \omega_0^{(p-16)\nu} \sin \left(\frac{(p-16)\nu - 1}{2}\pi\right) \right] \sin 4\omega_0 \tau_0 \\
&+ \left[-4\tau \sum_{p=16}^{20} \left[ \phi_p^R \omega_0^{(p-16)\nu} \sin \left(\frac{(p-16)\nu}{2}\pi\right) + \phi_p^t \omega_0^{(p-16)\nu} \cos \left(\frac{(p-16)\nu - 1}{2}\pi\right) \right] \sin 4\omega_0 \tau_0 \\
&\times \cos \left(\frac{(p-16)\nu}{2}\pi\right) + 20 \left[ \phi_p^R (p-16) \nu \omega_0^{(p-16)\nu-1} \sin \left(\frac{(p-16)\nu - 1}{2}\pi\right) \right] \sin 4\omega_0 \tau_0 \\
&+ \phi_p^t (p-16) \nu \omega_0^{(p-16)\nu-1} \cos \left(\frac{(p-16)\nu - 1}{2}\pi\right) \right] \cos 4\omega_0 \tau_0 \right] \\
\end{align*}\]
Example 5.1. Consider the following fractional-order OVNNs with two neurons:

\[\begin{align*}
\text{(i)} & \quad \text{At zero equilibrium, the neuron model (11) exhibits the occurrence of Hopf bifurcation at } \tau = \tau_0. \\
\text{(ii)} & \quad \text{The equilibrium point of the octonion network (11) is stable at origin for } \tau \in [0, \tau_0]. \\
\end{align*}\]

5. Numerical simulation. In this section, one numerical example is provided to affirm the effectiveness of our proposed results. The simulation results all are based on Adams-Bashforth-Moulton predictor-corrector scheme.

Example 5.1. Consider the following fractional-order OVNNs with two neurons:

\[
\begin{align*}
C \frac{d^{0.96} w(t)}{dt^{0.96}} &= -d_1 w(t) + a \tanh(w(t)) + b_1 \tanh(w(t - \tau_2)), \\
C \frac{d^{0.96} v(t)}{dt^{0.96}} &= -d_2 v(t) + a \tanh(v(t)) + b_2 \tanh(v(t - \tau_1)),
\end{align*}
\]

where, \(w(t) = w_0(t) + jw_1(t) + kw_2(t) + jk w_3(t), v(t) = v_0(t) + jv_1(t) + kv_2(t) + jk v_3(t)\) we introduce the parameters as follows, \(d_1 = d_2 = 2.6, a = (0.4 + 0.5i) + j(0.3 - 0.6i) + k(-0.4 + 0.2i) + jk(-1.2 + 0.7i), b_1 = (1 + 0.3i) + j(-1 + 0.9i) + k(-1 + 0.1i) + jk(-0.2 + 0.3i), b_2 = (0.2 + 0i) + j(-1 + 0.5i) + k(-1 + 0.2i) + jk(-0.9 + 0.2i).\)

The initial value is selected as \(w(0) = (0.1 + 0.6i) + j(0.2 + 0.4i) + k(0.5 + 0.5i) + jk(0.1 + 0.3i), v(0) = (0.1 + 0.6i) + j(0.2 + 0.4i) + k(0.5 + 0.5i) + jk(0.1 + 0.3i).\)

By some calculation, it is obtained that the critical frequency \(\omega_0 = 1.48\), and the bifurcation parameter \(\tau_0 = 0.15\). Hence, from Theorem 4.1., the zero equilibrium of the system (34) is locally asymptotically stable when \(\tau = 0.14 < \tau_0 = 0.15\). The system (34) is unstable with the zero equilibrium and the bifurcation occurs when \(\tau_0 = 0.15 < \tau = 0.39\).

6. Conclusions. In this paper, we investigate the problem of Hopf bifurcation of a fractional-order octonion-valued neural networks with time delay. It was shown that some sufficient criteria for the equilibrium solution of the model has loses its stability and a bifurcation occurs. In this case, the considered time delay is taken to be a bifurcating parameter along with the corresponding critical frequency of the
time delay, thus it guarantees the behaviors of the Hopf bifurcation. Finally, one numerical example is given to demonstrate the occurrence of the bifurcation for the considered model.

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