A New Phase-Type Distribution to Break Kleinrock’s Independence Assumption
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Abstract—In this paper, we first introduce a new phase-type $PH(p,\lambda)$ distribution. The basic properties of the $PH(p,\lambda)$ distribution are investigated, including its Markov chain representation and the non-denseness property. Based on this new distribution, a model to predict message delay distribution in message-switched communication networks (MSCNs) is presented. Specifically, we consider that message lengths are kept unchanged when they traverse from node to node in a network. This scenario breaks Kleinrock’s independence assumption but corresponds to actual scenarios. Finally, simulation shows that our model provides close and much better prediction results than that under Kleinrock’s assumption.

Index Terms—Phase-type distribution, queueing networks, dependent service times, message-switched communication networks, message delay distribution, Kleinrock’s independence assumption.

I. INTRODUCTION

The phase-type distribution was first introduced by Neuts in 1975 [1]. It is defined as the time to absorption in a continuous-time Markov chain. Because it forms a broad class of distributions, it has been widely used in stochastic modeling in different areas (see the review of Bladt [2] and the references therein).

Many distributions that we are familiar with belong to the class of this distribution, e.g., 1) the degenerate, 2) exponential, 3) hypoexponential, 4) Erlang, 5) mixture of Erlang, 6) hyperexponential and 7) Coxian distributions. The Coxian distribution has an explicit form and generalizes the preceding (2)-(5) distributions. More importantly, the Coxian distribution has a nice property called denseness. That means it can arbitrarily closely represent any nonnegative random variables. Having an explicit form and the denseness property make the Coxian distribution a very useful modeling tool in practice [3].

Let us continue with another story: a new distribution function was published by me and my former M.Sc. and Ph.D. supervisors in 2010 for modeling end-to-end delay in wireless multi-hop communications [4]. The distribution was actually derived in 2008 by me with some help from my father (a retired math professor on functional analysis). I did not pay attention to this distribution in the first place but soon after I realized that this distribution had never been created before. Two basic questions about this distribution arose naturally to me:

1) What category does this distribution belong to?

2) What are the applications of this distribution?

It literally took me twelve years to answer them properly.

My quick answer to the first question is as follows: The distribution published in [4] is a new class of the phase-type distribution (the associated distribution is called the $PH(p,\lambda)$ distribution in this paper). It has three distinct properties: 1) it generalizes (1)-(4) distributions; 2) it isn’t dense and 2) it isn’t a special case of the Coxian distribution and vice versa.

For my second question, an interesting application of this $PH(p,\lambda)$ distribution is that it suggests a new way of modeling the message delay distribution in message-switched communication networks (MSCNs) [1]. A theoretical framework that uses queueing theory to analyze message flows was first setup up by Kleinrock in his Ph.D. proposal in 1961 [5], followed by his Ph.D. thesis [6] and his book on communication nets [7]. In the 50th anniversary issue of operations research in 2002, Kleinrock described his path to packet networks and presented some of his major results on nets [8]. To continue Kleinrock’s work and to break his independence assumption, I find it is more convenient to use the same terms and notations from his work [6], [8], including the term “message switching”.

Back in 1961, Kleinrock listed seven questions about communication nets on the first page of his Ph.D. proposal [5]; we here replicate his first question below (the question has two parts):

Question 1A. What is the probability density distribution for the total time lapse between the initiation and reception of a message between any two nodes?

Question 1B. In particular, what is the expected value of this distribution?

For Question 1B, Kleinrock gave an explicit formula [8]:

$$T_i = \frac{1}{\mu C_i - \lambda_i},$$  \hspace{1cm} (1)

where $\lambda_i$ is the average traffic on channel $i$, $T_i$ is the average delay and $\mu C_i$ is the capacity of channel $i$ in messages per sec. Moreover, (1) is derived based on the classic Kleinrock’s independence assumption (KIA):

Kleinrock’s Independence Assumption (KIA). Each time that a message is received at a node within the network, a

\[1\]It is unnecessary to differentiate the terms message-switched networks and packet-switched networks from the analysis point of view. This is because those networks are both store-and-forward networks and the theoretical basis for them is the same. However, I emphasize that the message switching and the packet switching are different technologies, while the latter dominates data networks like the Internet.
new length is chosen for this packet independently from an exponential distribution.

With KIA, an MSCN can be modeled as a Jackson network, which has been extensively studied since 1957 [9]. KIA has been widely used in the computer network analysis although this assumption does not correspond to actual situations. In reality, messages maintain their lengths as they pass through the networks. However, without KIA, the network modeling problem is considered to be analytically intractable [8].

To break KIA, an MSCN can’t be modeled as a Jackson network again but has to be modeled as a queuing network with dependent service times. Most work on this subject is limited to the two-node tandem case. In 1977, Mitchell et al. [10] investigated the influence of different degrees of correlation between the service times of a given customer on the performance of the two-node tandem system (with and without blocking). Their results were primarily based on simulation experiments. Boxma [11], [12] analyzed the $M/G/1 \rightarrow G/1$ system and derived an exact solution for the stationary distributions of the joint stationary distributions of waiting times at both queues, and sojourn times at the second queue (but end-to-end message delay wasn’t investigated). Rubin [13] and Calo [14], [15] studied the average end-to-end message delay estimation under different message distributions and transmission rates at different queues. The two-node case with equal service times at both nodes is also considered [16] but detailed results for average waiting times are only provided for exponential service times at the first node.

A few studies have reported tandem systems with more than two queues and dependent service times. Light traffic asymptotics for expected waiting times in a series of queues with correlated service times are investigated [17]. Sandmann [18] studied the average end-to-end message delay in a tandem network (for up to ten queueing nodes) by the use of simulation. In 2011, Popescu and Constantinescu [19] reported results on network latency distribution of an actual chain of IP routers experiments. By comparing the experimental results with predictions based on KIA, they concluded that KIA isn’t valid to model end-to-end delay distribution in these experiments. However, the above work only considered tandem networks. For the past five decades, only a two-node tandem network with equal capacity channels is known to have a non-explicit solution [11], [12], but this solution isn’t for the end-to-end message delay distribution. Question 1A without KIA is still unanswered.

In this paper, we apply the $\text{PH}(\mathbf{p}, \lambda)$ distribution to answer Kleinrock’s Question 1A. Considering an MSCN without KIA and under a fixed routing strategy, we propose a new model to predict the end-to-end message delay distribution for any traffic flows within the net. Moreover, the message delay distribution function based our model has a simple and explicit form.

In the simulation part, we consider two topologically different nets: the first net is a two-node tandem network, in which the service times of an arbitrary message at both queues are identical; the second net is a six-node hypothetical network with 30 (= $6 \times 5$) traffic flows. By analyzing 24 simulation data sets from the above two topologies, we show the superiority of our model in terms of some goodness-of-fit statistics in comparison to the delay model with KIA. In other words, our model provides close and better predictions than the others.

The rest of this paper is organized as follows: Section II introduces the distribution in (4), followed by its Markov chain and phase-type related properties. In Section II, we describe the procedure to model an MSCN without KIA, and then introduce some important symbols and their definitions used in this paper as well as three key matrices that are sufficient to model an MSCN. Section III is about modeling message delay distribution without and with KIA. In Section IV, two different networks are first simulated. The above two models to predict end-to-end delay distribution in topologically different networks are performed. Section V summarizes this paper’s work.

II. The New $\text{PH}(\mathbf{p}, \lambda)$ Distribution

A. On the Distribution in (4)

We now briefly present the distribution in (4). Let $k \geq 1$ be a positive integer, $p_1, p_2, \ldots, p_k$ be $k$ real numbers between 0 and 1, and $X_1, X_2, \ldots, X_k$ be $k$ independent Bernoulli distributed random variables (rvs) with parameter $p_1, p_2, \ldots, p_k$; the probability mass function (pmf) of $X_i$ is given by

$$f_{X_i}(x) = \begin{cases} p_i & \text{if } x = 1, \\ q_i = 1 - p_i & \text{if } x = 0. \end{cases}$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be $k$ positive real numbers and $Y_1, Y_2, \ldots, Y_k$ be $k$ independent rvs, with $Y_i$ following an exponential distribution of parameter $\lambda_i$; the probability density function (pdf) of $Y_i$ is given by

$$f_{Y_i}(x) = \lambda_i e^{-\lambda_i x}.$$  

If $V_i$ is the product of $X_i$ and $Y_i$ (i.e., $V_i = X_i Y_i$), then $V_1, V_2, \ldots, V_k$ are $k$ independent rvs; the pdf of $V_i$ is [4]

$$f_{V_i}(x) = p_i \lambda_i e^{-\lambda_i x} + q_i \delta(x).$$

The cumulative density function (cdf) of $V_i$ is

$$F_{V_i}(x) = 1 - p_i e^{-\lambda_i x}.$$  

The random variable (rv) $Z = \sum_{i=1}^{k} V_i$ follows a distribution in (4) with two parameter vectors $\mathbf{p} = (p_1, p_2, \ldots, p_k)$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$. Under the condition of $\lambda_i \neq \lambda_j (i \neq j, \forall i, j \leq k)$, the pdf of $Z$ is given by

$$f_Z(x) = \sum_{i=1}^{k} \frac{\lambda_i}{P_i} e^{-\lambda_i x} + Q \delta(x),$$

where

$$\frac{1}{P_i} = p_i \prod_{j=1, j \neq i}^{k} \left(1 + \frac{p_j \lambda_j}{\lambda_j - \lambda_i}\right)$$

and

$$Q = 1 - F(0) = \prod_{i=1}^{k} q_i.$$
The complementary cumulative density function (ccdf) of $Z$ is given by

$$P(Z > x) = S_Z(x) = \sum_{i=1}^{k} \frac{1}{P_i} e^{-\lambda_i x},$$

(9)

Note that $P_i$ in (7) is completely determined by $p$ and $\lambda$ and can be positive or negative according to the sign of

$$\prod_{j=1, j \neq i}^{k} \left(1 + \frac{p_{j} \lambda_{j}}{\lambda_{i} - \lambda_{j}}\right).$$

B. Properties of the Distribution with a PDF of (6)

Proposition II.1 shows what the distribution with a pdf of (6) is:

**Proposition II.1.** The distribution with a pdf of (6) is a phase-type distribution.

In this paper, we call the distribution with a pdf of (6) the following properties:

1. State $(k + 1)$ is an absorbing state.
2. The process always starts at state 1 and go through every state in order.
3. The time in state $i (i \leq k)$ is either exponential with mean $1/\lambda_i$ (with probability $p_i$) or zero (with probability $q_i = 1 - p_i$).

Let $Z$ denote the time until the process reaches the absorbing state $(k + 1)$. Then $Z$ is a phase-type distributed rv with a pdf of (6).

**Proposition II.2.** The PH($p, \lambda$) distribution generalizes the 1) degenerate, 2) exponential, 3) hypoexponential and 4) Erlang distributions.

The PH($p, \lambda$) distribution will be in the form of a degenerate distribution if $k = 1$ and $p_1 = 0$; it will be the exponential distribution if $k = 1$ and $p_1 = 1$; it will be the hypoexponential distribution if $k \geq 2$, $\forall p_i = 1 (i \leq k)$ and $\lambda_i \neq \lambda_j (i \neq j, i, j \leq k)$; it will be the Erlang distribution if $k \geq 2$, $\forall p_i = 1 (i \leq k)$ and $\lambda_i = \lambda_j (i \neq j, i, j \leq k)$.

**Lemma II.3.** Under the condition of $p_i \in [0,1]$, $(i \leq k)$ and $\lambda_i = \lambda_j (i \neq j, i, j \leq k)$, the cdf of the PH($p, \lambda$) distribution can be expressed as

$$P(Z \leq x) = F_Z(x) = \sum_{i=1}^{k} \left(1 - \frac{1}{K_i} \sum_{j=0}^{i-1} \frac{1}{j!} e^{-\lambda x j}\right),$$

(10)

where

$$\frac{1}{K_i} = \sum_{m_1 + m_2 + \cdots + m_k = i} \frac{k!}{m_1! m_2! \cdots m_k!} q_1^{1-m_1}.$$

(11)

It can be observed that the PH($p, \lambda$) with a cdf of (10) is a special case of the mixture of Erlang distribution:

$$F_Z(x) = \begin{cases} u(x), & P(Z = 0) = 1/K_0, \\ \text{Erl}(1, \lambda), & P(Z = V_1) = 1/K_1, \\ \text{Erl}(2, \lambda), & P(Z = V_1 + V_2) = 1/K_2, \\ \vdots & \\ \text{Erl}(k, \lambda), & P(Z = V_1 + \cdots + V_k) = 1/K_k, \end{cases}$$

(12)

where $u(x)$ is a step function, $\text{Erl}(k, \lambda)$ is the Erlang distribution and $1/K_0 = 1 - \sum_{i=1}^{k} 1/K_i$.

**Lemma II.4.** The PH($p, \lambda$) distribution is not dense.

Schassberger proved that any non-negative distribution function can be approximated by the mixture of Erlang distribution that is arbitrarily close to the original function as $k \to \infty$ [20]. Such a property is called the denseness property. Because the mixture of Erlang is a special case of the Coxian distribution, the Coxian distribution is dense as well.

We now show that the PH($p, \lambda$) distribution doesn’t generalize the mixture of Erlang when $\lambda_i = \lambda_j (i \neq j, i, j \leq k)$. This can be proved by contradiction. Consider a PH($p, \lambda$) distribution with $k = 3$. According to (12), the cdf is

$$F_Z(x) = \begin{cases} u(x), & P(Z = 0) = 1 + p_1 p_2 - p_1 - p_2, \\ \text{Erl}(1, \lambda), & P(Z = V_1) = p_1 - p_2 + p_2(1 - p_1), \\ \text{Erl}(2, \lambda), & P(Z = V_1 + V_2) = p_1 p_2. \end{cases}$$

Next, consider a mixture of Erlang distribution with $k = 3$ with a cdf being

$$F_Z(x) = \begin{cases} u(x), & P(Z = 0) = a_1, \\ \text{Erl}(1, \lambda), & P(Z = V_1) = a_2, \\ \text{Erl}(2, \lambda), & P(Z = V_1 + V_2) = a_3, \end{cases}$$

where $a_1$, $a_2$ and $a_3$ can be any values that are subject to the condition: $a_1 + a_2 + a_3 = 1$. If the PH($p, \lambda$) distribution generalizes the mixture of Erlang, then the following equations always have a solution such that $p_1, p_2, p_3 \in [0,1]$:

$$\begin{cases} 1 + p_1 p_2 - p_1 - p_2 = a_1 \\ p_1 + p_2 - 2p_1 p_2 = a_2 \\ p_1 p_2 = a_3 \end{cases}$$

(13)

However, this is not always true. For example, let $a_1 = 0.4, a_2 = 0.2, a_3 = 0.4$. It is not possible to find $p_1, p_2, p_3 \in [0,1]$ that satisfy (13).

Now we have the last proposition:

**Proposition II.5.** The PH($p, \lambda$) distribution isn’t a special case of the Coxian distribution, and vice verse.

**Proof.** Let us first show that the PH($p, \lambda$) distribution isn’t a special case of the Coxian distribution. Consider a 4-state Markov chain for the PH($p, \lambda$) distribution. State 4 is an absorbing state. The probability of the process going through states 1 and 3 (except state 2) is $p_1 (1 - p_2) p_3$ However, such a process can not be constructed by the Coxian distribution.

Because of the fact that the PH($p, \lambda$) distribution isn’t dense (see Lemma II.4) and the fact that the Coxian distribution is
dense, it is clear that the Coxian distribution isn’t a special case of the $PH(p, \lambda)$ distribution.

\[ \mu_Z = E[Z] = \sum_{i=1}^{k} \frac{p_i}{\lambda_i}, \quad (14) \]

and

\[ \sigma^2_Z = Var[Z] = \sum_{i=1}^{k} \left( \frac{2p_i}{\lambda_i} - \left( \frac{p_i}{\lambda_i} \right)^2 \right). \quad (15) \]

The mean and the variance of $Z$ have been derived in [4].

**Remark 1.** The mean and the variance of $Z$ are

\[ \mu_Z = E[Z] = \sum_{i=1}^{k} \frac{p_i}{\lambda_i}, \quad (14) \]

and

\[ \sigma^2_Z = Var[Z] = \sum_{i=1}^{k} \left( \frac{2p_i}{\lambda_i} - \left( \frac{p_i}{\lambda_i} \right)^2 \right). \quad (15) \]

Let $\Delta = \lambda_2 - \lambda_1$. If the following condition holds:

\[ \lambda_2 - \lambda_1 = p_1 \lambda_2, \quad (17) \]

then $Z$ is another $PH(p, \lambda)$ distributed rv with $k = 1$, $\lambda = \lambda_1$ and

\[ p = \frac{\Delta p_1 + p_1 p_2 \lambda_1}{\Delta}. \quad (18) \]

For a proof of Remark 2 see Appendix A. This property means that under a condition of (17), the sum of two independent $PH(p, \lambda)$ distributed rvs with $k = 1$ is another $PH(p, \lambda)$ distributed rv with $k = 1$. This property will be used in Section IV.C.

**III. Message-Switched Communication Network (MSCN)**

**A. Modeling an MSCN without KIA**

An MSCN has a number of nodes (representing switches and routers in packet-switched networks) and communication channels. In the language of graph theory, the topology of an MSCN can be described as a directed graph $G = (V, E)$, where $V$ is a set of nodes $\{1, 2, \cdots, v\}$ and $E$ is a set of channels that connect pairs of distinct nodes. If node $m$ and node $n$ are connected (or adjacent), then the pair $(m, n)$ denotes a one-way communication link from node $m$ to node $n$. Moreover, the channel capacity of the communication link $(m, n)$ is denoted by $C_{mn}$. An example of an MSCN is illustrated in Fig. 2 which shows an interconnection of four nodes. Each Node has two full-duplex channels represented by the symbol "$\bullet \bullet \bullet $"; the symbol "$\bullet \bullet $" was used in [2] for the same purpose.

Besides the topological graph $G = (V, E)$, a functioning MSCN must carry a number of traffic flows and implement a routing strategy. A traffic flow is a stream of messages with the same source-destination pair (i.e., from the same node $i$ to the same node $j$), and it has the following two properties:

1) Message arrivals follow a Poisson process with the mean arrival rate $\gamma_{ij}$.

2) Message lengths are independent exponentially distributed random variables with the mean message length $1/\mu$.

Specifically, let $A_n$ and $V_n$ be the time between the arrivals of the $n^{th}$ and the $(n+1)^{th}$ messages, and the $n^{th}$ message length, respectively. The pdf of $A_n$ is given by

\[ f_{A_n}(x) = \gamma_i e^{-\gamma_i x}. \quad (19) \]

The pdf of message length $V_n$ is given by

\[ f_{V_n}(x) = \mu e^{-\mu x}. \quad (20) \]

Furthermore, a fixed routing strategy is considered, i.e., inside each node, the next node that messages visit with origin $i$ and destination $j$ is fixed.

An MSCN is a store-and-forward communication net. When messages are passing through a node, they are stored in a queue, if necessary, and then forwarded (transmitted) to the next node on the way to their destination. Such a store-and-forward mechanism is demonstrated by the reference design architecture of a generic router as shown in Fig. 2 [21].

Finally, the MSCN model considered in this paper does not adopt KIA. In this respect, messages maintain their lengths as they pass through the network and the service time (i.e., transmission time) at each link is directly proportional to the message length.

**B. Notations and Definitions**

For convenience, we here list some of the important symbols and their definitions (similar symbols and definitions can also be found in [7]). Note that certain relations among the definitions are given as well.

- $ij$ symbols with the subscript or the superscript $ij$ is referred to as the quantities for messages with origin $i$ and destination $j$.
C. Three Matrices to Define an MSCN Model

According to the discussion in Section III-A and notations in III-B we are now able to define an MSCN by three matrices, namely, the 1) adjacency matrix $I$, 2) traffic matrix $\tau$ and 3) routing matrix $R$.

An adjacency matrix is a $v \times v$ matrix, in which an entry of unity in the $mn$ position indicates the presence of a one-way communication link from node $m$ to node $n$ ($m, n = 1, 2, \cdots, v$), and the absence of an entry indicates the absence of a channel:

$$ I = \begin{bmatrix} 0 & c_{12} & \cdots & c_{1v} \\ c_{21} & 0 & \cdots & c_{2v} \\ \vdots & \vdots & \ddots & \vdots \\ c_{v1} & c_{v2} & \cdots & 0 \end{bmatrix}. \quad (21) $$

The diagonal elements of $I$ should be all zero. It worth noting that the matrix $I$ and the graph $G = [V, A]$ are equivalent when they are defining the same topological structure of a net.

When the channel capacity assigned to each link is specified, $G = [V, A]$ is a weighted graph and the adjacency matrix $I$ of a weighted graph, i.e., the $mn$-th entry of $I$ is the channel capacity $c_{mn}$ instead of the value of 1.

A traffic matrix $\tau$ is the second $v \times v$ matrix, where an entry in the $ij$ position specifies the external arrival rate of messages with origin $i$ and destination $j$:

$$ \tau = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1v} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2v} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{v1} & \gamma_{v2} & \cdots & \gamma_{vv} \end{bmatrix}. \quad (22) $$

Under a fixed routing strategy, a routing matrix $R$ is the third $v \times v$ matrix such that the $ij$-th entry is the next node to visit:

$$ R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1v} \\ r_{21} & r_{22} & \cdots & r_{2v} \\ \vdots & \vdots & \ddots & \vdots \\ r_{v1} & r_{v2} & \cdots & r_{vv} \end{bmatrix}. \quad (23) $$

3In this paper, symbols with two dots hovered refer to quantities when KIA is made; symbols without these two dots refer to quantities without KIA.
In practice, the routing matrix $R$ can be constructed by shortest path routing algorithms based on the adjacency matrix $I$ (such as Dijkstra’s and Bellman-Ford algorithms) or by some dynamic traffic-aware routing algorithms based on the $I$ and $\tau$ matrices.

When a routing matrix $R$ is determined, a path for messages with any origin-destination pair is determined accordingly. By using the traffic matrix $\tau$, it is simple to obtain the total arrival rate at any link $(m, n)$:

$$\lambda_{mn} = \sum_{p_{i,j} = (m, n), \exists j \neq p} \tau_{ij},$$

resulting the fourth $v \times v$ matrix called the total traffic matrix:

$$\Lambda = \begin{bmatrix} 0 & \lambda_{12} & \cdots & \lambda_{1v} \\ \lambda_{21} & 0 & \cdots & \lambda_{2v} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{v1} & \lambda_{v2} & \cdots & 0 \end{bmatrix}.$$  

(25)

The diagonal elements of the matrix $\Lambda$ should be all zero. Here, we emphasize that the fourth matrix $\Lambda$ is derived from the matrices $\tau$ and $R$ but it plays a central role in message delay prediction.

Finally, we impose that the traffic load of every channel in the network should always be less than one (otherwise, the queue inside a link will be unstable). This means that if we let the maximum traffic load of all links be

$$\rho_{\max} = \max(\rho_{11}, \rho_{12}, \cdots, \rho_{mn}, \cdots, \rho_{vv}) \ (m, n = 1, 2, \cdots, v),$$

then $\rho_{\max}$ should always be less than one:

$$\rho_{\max} < 1.$$  

(26)

IV. MESSAGE DELAY DISTRIBUTION MODELS IN MSCNs

In an MSCN without or with KIA, let us first 1) consider a traffic flow with origin $i$ and destination $j$ and 2) the $n^{th}$ message of this flow with its length being $V_n$. Under fixed routing, this message shall follow a path that is a sequence of nodes $\{s_1, s_2, \cdots, s_{k+1}\}$.

Denote by $T_n^{s_k s_{k+1}}$ the delay of the $n^{th}$ message experienced when it passes through the channel $\{s_k, s_{k+1}\}$ along the path $p_{ij}$. $T_n^{s_k s_{k+1}}$ is the sum of the queuing delay $W_n^{s_k s_{k+1}}$ and the service time $R_n^{s_k s_{k+1}}$ (the processing and propagation delays are neglected):

$$T_n^{s_k s_{k+1}} = W_n^{s_k s_{k+1}} + R_n^{s_k s_{k+1}}.$$

(27)

Moreover, the service time of the $n^{th}$ message at the link $\{s_k, s_{k+1}\}$ is

$$R_n^{s_k s_{k+1}} = \frac{V_n}{C_{s_k s_{k+1}}}.$$  

(28)

The message delay $Z_n^{ij}$ along the path $p_{ij}$ is the sum of delays $T_n^{s_k s_{k+1}} (k = 1, 2, \cdots, h + 1)$:

$$Z_n^{ij} = \sum_{k=1}^{h} T_n^{s_k s_{k+1}} = \sum_{k=1}^{h} (W_n^{s_k s_{k+1}} + R_n^{s_k s_{k+1}}).$$

(29)

where the 2\textsuperscript{nd} equality holds because of (27) and the 3\textsuperscript{rd} equality holds because of (28).

Eq. (29) breaks down an end-to-end message delay into smaller delays, and it serves as a starting point for modeling message delay without KIA in Section [IV-B] and with KIA in Section [IV-C]. But before we start modeling message delay, we need to first introduce three assumptions in the following section.

A. Three Assumptions for MSCNs without KIA

In an MSCN without KIA, messages maintain their lengths when they enter every node along the path. The message delay distribution is derived based upon the following three assumptions:

A1: The flow into each queue of the network is Poisson;

A2: The queueing delays $W_n^{s_k s_{k+1}} (k = 1, 2, \cdots, h + 1)$ at each link are mutually independent;

A3: The service times and queueing delays are mutually independent.

These assumptions must be made because they allow us to analyze queues in isolation, making the end-to-end message delay prediction possible. However, we here show that none of these assumptions can actually exist in an MSCN without KIA mainly because the network preserves the service time dependence.

A1 is not true. Consider a two-node tandem network. If the service times at two nodes are dependent, then the interarrival times at the second node and service times will become dependent. For example, if the service time of the $n^{th}$ message at the first queue is relatively long, then the interarrival time between the $n^{th}$ and $(n + 1)^{st}$ messages at the second queue will become relatively long. Because the Poisson process has the memoryless property, the flow into the second queue is not Poisson.

A2 is not true as well. By extending the above discussion on a two-node tandem net to a three-node tandem net, we understand that the interarrival times and the service times at the second and the third queue are dependent. Based on the Lindley equation (29),

$$W_{n+1} = \begin{cases} W_n + S_n - A_n \leq 0 & \text{if } W_n + S_n - A_n > 0 \\ 0 & \text{if } W_n + S_n - A_n \leq 0 \end{cases},$$

(30)

we conclude that the waiting times at the second and the third queues are dependent.

A3 again is not true. The influence of different degrees of correlation between the server utilization and the waiting time has been investigated in a tandem queueing system with dependent service times [10], [16].

B. Message Delay Distribution Model without KIA

Eq. (29) can be expanded as

$$Z_n^{ij} = \sum_{k=1}^{h} W_n^{s_k s_{k+1}} + \sum_{k=1}^{h} \frac{V_n}{C_{s_k s_{k+1}}} = W_n^{ij} + R_n^{ij},$$

(31)

where $W_n^{ij}$ and $R_n^{ij}$ are called the network queuing delay and network service time in this paper.
Because messages keep their lengths unchanged, the network service time in (31) is

$$R^{ij}_n = \sum_{k=1}^{h} \frac{V_n}{C_{s_k s_{k+1}}} = V_n \sum_{k=1}^{h} \frac{1}{C_{s_k s_{k+1}}} = \frac{V_n}{C_{ij}}, \quad (32)$$

where

$$\frac{1}{C_{ij}} = \sum_{k=1}^{h} \frac{1}{C_{s_k s_{k+1}}} \quad (33)$$

Because the message length is exponential with a pdf of (20), (32) indicates that $R^{ij}_n$ is another exponentially distributed rv with the mean network service time being

$$E[R^{ij}_n] = E[V_n] \frac{1}{C_{ij}} = \frac{1}{\mu C_{ij}}. \quad (34)$$

So the pdf of the network service time $R^{ij}_n$ is

$$f_{R^{ij}_n}(x) = \mu C_{ij} e^{-\mu C_{ij} x} \quad (35)$$

If the assumption A1 is made, then a path $p_{ij}$ can be decomposed into $h$ individual $M/M/1$’s (for each communication link $\{s_k, s_{k+1}\}$ along the path). The flow into each link is a Poisson process with mean rate $\lambda_{s_k s_{k+1}}$. Moreover, it is a property of an $M/M/1$ queue that the stationary distribution function of the queuing delay $W_{s_k s_{k+1}}$ at the $\{s_k, s_{k+1}\}$ is

$$P(W_{s_k s_{k+1}} \leq t) = 1 - e^{-\lambda_{s_k s_{k+1}} t}, \quad (36)$$

where

$$\lambda_{s_k s_{k+1}} = \frac{\lambda_{s_k s_{k+1}}}{\mu C_{s_k s_{k+1}}} \quad (37)$$

and

$$\theta_{s_k s_{k+1}} = \mu C_{s_k s_{k+1}} - \lambda_{s_k s_{k+1}}. \quad (38)$$

By comparing the cdf of (3) with the cdf of (36), we can confirm that the rv $W_{s_k s_{k+1}}$ is in fact a $PH(p, \theta)$ distributed rv with $k = 1, p_1 = \rho_{s_1 s_{h+1}}$ and $\lambda_1 = \theta_{s_1 s_{h+1}}$.

Based on the assumption A2, the network queuing delay $W^{ij}$ is the sum of $W_{s_k s_{k+1}}$ along the path so $W^{ij}_n$ follows the $PH(p, \theta)$ distribution with a cdf of (9). Therefore, the stationary cdf of the network waiting time $W_{ij}$ is given by

$$S_{W_{ij}}(x) = \sum_{f=1}^{h} \prod_{g=1}^{h} \left(1 + \frac{p_f \theta_f}{\theta_g - \theta_f}\right) p_f e^{-\theta_f x}, \quad (39)$$

where

$$p_f = \rho_{s_1 s_{h+1}}, \quad \theta_f = \theta_{s_1 s_{h+1}}, (f = 1, 2, \cdots, h). \quad (40)$$

Speaking intuitively, consider a patient who must perform a number of tests in a hospital and each test has a queue. There is a chance that the patient does not have to wait in a queue if he/she finds the queue is empty. The total time the patient spent in all the queues is a $PH(p, \theta)$ distributed rv with $k = h$. The Markov chain for such a process describing the waiting time in queues is shown in Fig. 1.

Eq. (31) shows that the message delay $Z^{ij}_n$ is the sum of the network queueing delay $W^{ij}_n$ (a $PH(p, \theta)$ distributed rv) and the network service time $R^{ij}_n$ (an exponentially distributed rv). By using the assumption A3 and the fact that the exponential distribution is a special case of the $PH(p, \theta)$ distribution (see Proposition 1), we have the following result:

**Proposition IV.1.** In an MSCN without KIA, consider a traffic flow with origin $i$ and destination $j$ and with a fixed path $p_{ij}$ under a fixed routing strategy. The stationary cdf of message delay is

$$S_{Z_{ij}}(t) = \sum_{f=1}^{h+1} \prod_{g=1}^{h+1} \left(1 + \frac{p_f \theta_f}{\theta_g - \theta_f}\right) p_f e^{-\theta_f t}, \quad (41)$$

where

$$p_f = \rho_{s_1 s_{h+1}}, \theta_f = \theta_{s_1 s_{h+1}}, (f = 1, 2, \cdots, h), \quad (42)$$

$$p_{h+1} = 1, \theta_{h+1} = \mu C_{ij}, (f = h + 1). \quad (43)$$

Clearly, the message delay $Z_{ij}$ follows a $PH(p, \theta)$ distribution with $k = h+1$. An $(h+3)$-state Markov chain representation for the message delay along an $h$-hop path is show in Fig. 3. State $h + 3$ is an absorbing state.

**Remark 3.** The average delay and the jitter for messages with origin $i$ and destination $j$ are

$$E[Z_{ij}] = \sum_{f=1}^{h} \frac{1}{\theta_f} = \sum_{f=1}^{h} \frac{1}{\mu C_{s_k s_{k+1}} - \lambda_{s_k s_{k+1}}} \quad (44)$$

and

$$\sigma_{Z_{ij}} = \sqrt{Var[Z_{ij}]} = \sqrt{\sum_{f=1}^{h+1} \left(\frac{2p_f \theta_f}{\theta_f^2} - \frac{p_f}{\theta_f^2}\right)} \quad (45)$$

where $p_f$ and $\theta_f$ are defined in (42) and (43).
Proof. According to (14) in Remark 1, we have

\[
E[Z_{ij}] = \sum_{k=1}^{h} \frac{P_f \left( \frac{1}{\mu_C} \right) + 1}{\mu_C + \frac{1}{C_{S_{k+1}S_{k+1}}} + \frac{1}{\mu_C + \frac{1}{C_{S_{k+1}S_{k+1}}}}}
\]

The 3rd equality holds because of (35). The jitter \( \sigma_{Z_{ij}} \) is a direct result of (15) in Remark 1.

\[\square\]

C. Message Delay Distribution Model with KIA

We can follow roughly the same steps in Section [IV-B] to derive the message delay distribution in an MSCN with KIA. Consider again a traffic flow with origin \( i \) and destination \( j \) and the \( n \)th message that goes through a sequence of nodes \( \{s_1, s_2, \ldots, s_{k+1}\} \). With KIA, each time the message is received at a node (say node \( k \)) within the net, a new length, \( \nu_k \), is chosen for this message from the pdf of \( \nu_k \). Therefore, \( \nu_{k,1}, \nu_{k,2}, \ldots, \nu_{k,k} \) are independent and identically distributed rvs identical to the rv \( \nu_k \) in Section [IV-B].

With KIA, (29) can be written as

\[
2_{ij}^n = \sum_{k=1}^{h} \left( \frac{\lambda_{s_k s_{k+1}}}{\mu_C + \mu_C + \frac{1}{C_{S_{k+1}S_{k+1}}}} + \frac{1}{\mu_C + \frac{1}{C_{S_{k+1}S_{k+1}}}} \right) \nu_{k}.
\]

If the assumption A1 is made, the stationary distribution function of the queueing delay \( \nu_{s_k s_{k+1}} \) at the \( \{s_k, s_{k+1}\} \) is

\[
P(\nu_{s_k s_{k+1}} \leq t) = 1 - \rho_{s_k s_{k+1}} e^{-\theta_{s_k s_{k+1}} t},
\]

where \( \rho_{s_k s_{k+1}} \) and \( \theta_{s_k s_{k+1}} \) are defined in (37) and (38), respectively. Note that \( \nu_{s_k s_{k+1}} \) is again a PH(p, \( \lambda \)) distributed rv with \( k = 1, p_1 = \rho_{s_k s_{k+1}}, \lambda_1 = \theta_{s_k s_{k+1}} \).

According to (47), \( \nu_{s_k s_{k+1}} \) is the sum of \( \nu_{s_k s_{k+1}} \) and \( \nu_{s_k s_{k+1}} \). Considering that the service time \( \nu_{s_k s_{k+1}} \) is exponential with the mean service time equal to \( 1/\mu_C + \mu_C + \frac{1}{C_{S_{k+1}S_{k+1}}} \), we have the following result:

Remark 4. \( \nu_{s_k s_{k+1}} \) is exponential with the mean delay equal to \( 1/\theta_{s_k s_{k+1}} \); the cdf of \( \nu_{s_k s_{k+1}} \) is given by

\[
P(\nu_{s_k s_{k+1}} \leq t) = 1 - e^{-t},
\]

Proof. This result can be obtained from Remark 2 in Section III because \( \nu_{s_k s_{k+1}} \) and \( \nu_{s_k s_{k+1}} \) satisfy the condition \( \lambda_2 - \lambda_1 = \theta_{s_k s_{k+1}} \) of Remark 2.

Let us temporarily let \( \lambda_1 = \mu_C + \frac{1}{C_{s_k s_{k+1}}} + \mu_C + \frac{1}{C_{s_k s_{k+1}}} \), \( \lambda_2 = \mu_C + \frac{1}{C_{s_k s_{k+1}}} \), \( p_1 = \rho_{s_k s_{k+1}} \), \( p_2 = \frac{1}{C_{s_k s_{k+1}}} \), and \( p_2 = 1 \). We can verify that the condition \( \lambda_2 - \lambda_1 = p_1 \theta_{s_k s_{k+1}} \) holds. Moreover, let \( \Delta = \lambda_2 - \lambda_1 = \lambda_{s_k s_{k+1}} \), we have

\[
p = \frac{\Delta p_1 + p_1 p_2 \Delta_1}{\Delta} = \frac{\Delta p_1 + \Delta p_1}{\Delta} = \frac{\Delta p_1}{\Delta} = 1.
\]

The message delay \( \nu_{ij}^n \) is the sum of the delay \( \nu_{s_k s_{k+1}} \) at each node. Because the stationary distribution of \( \nu_{s_k s_{k+1}} \) is exponential, the stationary distribution of \( \nu_{ij}^n \) is an hypoexponential distribution; the cdf of \( \nu_{ij}^n \) is

\[
F_{\nu_{ij}^n}(x) = \sum_{k=1}^{h} \prod_{k=1}^{k} \left( \frac{\theta_{s_k s_{k+1}} \theta_{s_k s_{k+1}}}{} \right) e^{-\theta_{s_k s_{k+1}} x},
\]

where \( \theta_{s_k s_{k+1}} \) is defined in (38). The distribution function of (51) with KIA may first be found by Wong in 1978 [23] and was also used in Popescu and Constantinescu’s work [19].

Remark 5. In an MSCN with KIA, the average delay and the jitter for messages with origin \( i \) and destination \( j \) are

\[
E[\nu_{ij}^n] = \sum_{k=1}^{h} \mu_{s_k s_{k+1}} C_{s_k s_{k+1}}
\]

and

\[
\sigma_{Z_{ij}} = \sqrt{\text{Var}[\nu_{ij}^n]} = \sum_{k=1}^{h} \frac{1}{\mu_{s_k s_{k+1}} C_{s_k s_{k+1}}},
\]

which is identical to (1).

Remark 6. When the path length is one (i.e., \( h = 1 \)), the delay distribution without KIA and the delay distribution with KIA are the same.

It is straightforward to show the above remark is true: If the path length is one, then both the MSCN without KIA and the MSCN with KIA can be modeled as an \( M/M/1 \) system with the same parameters. Therefore, the delay distributions for both networks are the same.

Remark 7. The average message delays without KIA and with KIA are the same, i.e.,

\[
E[\nu_{ij}^n] = E[\nu_{ij}^n].
\]

By comparing (44) in Remark 1 with (52) in Remark 5, it can be concluded that (55) is true.

Remark 8. The jitter without KIA is higher than the jitter with KIA, i.e.,

\[
\sigma_{Z_{ij}} > \sigma_{Z_{ij}}.
\]

Proof. For the \( n \)th message, we have

\[
\sigma_{Z_{ij}}^2 - \sigma_{Z_{ij}}^2 = \text{Var} \left[ \sum_{k=1}^{h} R_{s_k s_{k+1}} \right] - \text{Var} \left[ \sum_{k=1}^{h} R_{s_k s_{k+1}} \right]
\]

\[
= \frac{1}{C_{ij}} \text{Var} [V_n] - \sum_{k=1}^{h} \left( \frac{C_{s_k s_{k+1}}}{C_{s_k s_{k+1}}} \right) \text{Var} [V_{s_k s_{k+1}}]
\]

\[
= \left( \sum_{k=1}^{h} \frac{1}{C_{s_k s_{k+1}}} \right)^2 \text{Var} [V_n] - \left( \sum_{k=1}^{h} \left( \frac{1}{C_{s_k s_{k+1}}} \right)^2 \right) \text{Var} [V_{s_k s_{k+1}}]
\]

\[
= \left( \sum_{k=1}^{h} \frac{1}{C_{s_k s_{k+1}}} \right)^2 - \left( \sum_{k=1}^{h} \frac{1}{C_{s_k s_{k+1}}} \right)^2 \text{Var} [V_{s_k s_{k+1}}]
\]

\[
= \left( \sum_{k=1}^{h} \frac{1}{C_{s_k s_{k+1}}} \right)^2 \text{Var} [V_n] - \left( \sum_{k=1}^{h} \frac{1}{C_{s_k s_{k+1}}} \right)^2 \text{Var} [V_{s_k s_{k+1}}]
\]

\[
= \left( \sum_{k=1}^{h} \frac{1}{C_{s_k s_{k+1}}} \right)^2 \text{Var} [V_n] - \left( \sum_{k=1}^{h} \frac{1}{C_{s_k s_{k+1}}} \right)^2 \text{Var} [V_{s_k s_{k+1}}]
\]

\[
= \left( \sum_{k=1}^{h} \frac{1}{C_{s_k s_{k+1}}} \right)^2 \text{Var} [V_n] - \left( \sum_{k=1}^{h} \frac{1}{C_{s_k s_{k+1}}} \right)^2 \text{Var} [V_{s_k s_{k+1}}]
\]
Based on the Cauchy-Schwarz inequality, we have the following inequality:

\[ \left( \sum_{k=1}^{h} \left( \frac{1}{C_{t, x_{k+1}}} \right)^2 \right)^{\frac{1}{2}} > \sum_{k=1}^{h} \left( \frac{1}{C_{t, x_{k+1}}} \right)^2 \]

Therefore, we have

\[ \sigma_{Z_i}^2 - \sigma_{Z_j}^2 > 0 \]

\[ \iff \sigma_{Z_i} > \sigma_{Z_j} \] \hspace{1cm} (59)

which is (56). The “⇔” in (59) is an equivalence sign.

Lastly, when an MSCN with KIA under fixed routing is modeled as a Jackson network and when all the paths are simple (this is assumed in Section V-B), it seems to me that all the assumptions A1-A3 are valid. For the assumption A1, Burke [24] proved that the departure process of an $M/M/c$ queue is Poisson in 1956. Moreover, because of the merging and splitting properties of the Poisson processes, A1 is valid. For the assumption A2, Walrand and Varaiya [25] proved that the sojourn times of a customer in an AMJN are all mutually independent for overtake-free paths in 1980. Because all paths are assumed to be simple, A2 is also valid. The assumption A3 is valid because message lengths are chosen independently based on KIA. Therefore, we may conclude that the distribution function of (51) is the solution to the end-to-end message distribution in an MSCN with KIA.

V. MESSAGE DELAY DISTRIBUTION RESULTS AND DISCUSSION

In this section, we put the message delay distribution models without and with KIA together and compare them with simulation results from two topologically different networks: a two-node tandem network (see Section V-A) and a six-node hypothetical network (see Section V-A). The effect of the service time independence is discussed in Section V-C.

A. Two-Node Tandem Network

The network topology of a two-node tandem network is shown in Fig. [5a]. The channel capacities of all links are the same, i.e., $C_{12} = C_{23} = C$. Only one traffic flow is considered with origin 1 and destination 3.

Such a network is essentially an $M/M/1 \rightarrow M/1$ system where any given message has equal service times at two queues. This simulation has been done by [10]–[12], [16] but is first reproduced for message delay distribution. The $\rho$ values we considered are 0.1, 0.2, 0.3, 0.36818, 0.5, 0.58, 0.6, 0.7, 0.75, 0.8, 0.9 and 0.99. The number of messages simulated in each of our simulation runs was 30,000.

Because we only have one traffic flow in the net, the path length is 2 in such a tandem net. The total arrival rates at both queues are the same, i.e., $\lambda_{12} = \lambda_{23} = \lambda$. For demonstration purpose, let the average service times at both queues equal to one [16], i.e., $\mu C = 1$. Now we can model the delay distribution for messages (with origin 1 and destination 3) without and with KIA:

1) Model without KIA: The pdf and the cccdf of message delays are given by

\[ f_{Z_1}(t) = \left( 1 + \frac{p_1 \theta_1}{\theta_2 - \theta_1} \right) \left( 1 + \frac{\theta_1}{\theta_3 - \theta_1} \right) p_1 e^{-\theta_1 t} \]

\[ S_{Z_1}(t) = \left( 1 + \frac{p_1 \theta_1}{\theta_2 - \theta_1} \right) \left( 1 + \frac{\theta_1}{\theta_3 - \theta_1} \right) p_1 e^{-\theta_1 t} \]

2) Model with KIA: The pdf and the cccdf of message delays are given by

\[ f_{Z_1}(x) = \frac{\theta_1}{\theta_2 - \theta_1} e^{-\theta_1 x} + \frac{\theta_1}{\theta_3 - \theta_1} e^{-\theta_3 x} \]

\[ S_{Z_1}(x) = \frac{\theta_1}{\theta_2 - \theta_1} e^{-\theta_1 x} + \frac{\theta_1}{\theta_3 - \theta_1} e^{-\theta_3 x} \]

Note that in (60)–(63), the case $\theta_1 = \theta_2$ will generate infinity. This problem can be mitigated by slightly changing either the value of $\theta_1$ or the value of $\theta_2$. Algorithm 1 is developed and it can be used for this purpose.

Algorithm 1

1: Initialize $\epsilon_i$ [e.g., $\epsilon_i = 10^{-4}$]
2: for $i = 1$ to the number of elements in $\theta$ do
3: \hspace{1cm} SET $\theta_i$ to ref_value
4: \hspace{2cm} for $j = i + 1$ to the number of elements in $\theta$ do
5: \hspace{3cm} if $\theta_i$ EQUAL ref_value then
6: \hspace{4cm} $\theta_j = \theta_j + \epsilon$
7: \hspace{3cm} end if
8: \hspace{2cm} end for
9: end for

There are twelve data sets generated for twelve different $\rho$ values. For each data set, we compare the fitted delay distributions without and with KIA using three criteria: $-2 \log(L)$, Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC). Let us precise that $\log(L)$ is the log-likelihood function using either the pdf of (60) or the pdf of (62). AIC = $-2 \log(L) + 2k$, BIC = $-2 \log(L) + k \log(n)$, where $k$ is the number of distribution-related parameters and $n$ is
between prediction and simulation results is distinct when
the distribution functions of the network queueing delay are
the same from both delay models. Moreover, the dis-

The adjacency matrix
these matrices to define the six-node network that we simu-

the sample size (that is 30,000 in this work). The best fitted
distribution corresponds to lower \(-2 \log(L)\), AIC and BIC. We
see in Table II that these criteria values for the delay model
without KIA are always smaller than that with KIA except the
case when \(\rho = 0.99\).

Fig. 6 shows the simulation and prediction results under
different traffic loads. The ccdfs of message delays from
simulation data, the ccdfs of (61) and the ccdfs of (63)
are plotted in blue, red and black dotted lines, respectively.
The y-axis uses logarithmic scale. When \(\rho \leq 0.6\), simulation
results show the accuracy of (61). On the other hand, the
ccdfs of (63) significantly deviate from the simulation results.
When \(\rho > 0.6\), it is hard to differentiate the ccdfs of (61)
and (63). This is because, first, the network queueing
delay dominates the end-to-end message delay, and second,
the distribution functions of the network queueing delay are
the same from both delay models. Moreover, the difference
between prediction and simulation results is distinct when
\(\rho > 0.75\), suggesting that both models are inaccurate in
predicting message delay distribution for high traffic loads.

B. Six-Node Hypothetical Network

The network topology of a six-node hypothetical network
is shown in Fig. 5. In Section III-C we listed three matrices
required to specify a communication net. Here, we first give
these matrices to define the six-node network that we simu-
lated. The adjacency matrix \(I\) and the relative traffic matrix \(\tau\)
are given by

\[
I = \begin{bmatrix}
0 & 1 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 0
\end{bmatrix}
\quad \text{and} \quad
\tau = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 0
\end{bmatrix}.
\]

The routing matrix \(R\) is constructed by Dijkstra’s algorithm
and is given by

\[
R = \begin{bmatrix}
0 & 2 & 2 & 4 & 5 & 2 \\
1 & 0 & 3 & 1 & 5 & 6 \\
2 & 2 & 0 & 2 & 2 & 6 \\
1 & 1 & 1 & 0 & 5 & 5 \\
1 & 2 & 6 & 4 & 0 & 6 \\
2 & 2 & 3 & 5 & 5 & 0
\end{bmatrix}
\]

According to the traffic matrix \(\tau\), the net is carrying 30 (= 6×5)
traffic flows. Based on the matrices \(R\) and \(\tau\), we can compute
1) the average path length, which is 1.4667, and 2) the total
traffic matrix, which is given by

\[
\begin{bmatrix}
0 & 0.9205 & 0 & 0.5523 & 0.1841 & 0 \\
0.9205 & 0 & 0.5523 & 0 & 0.3682 & 0.3682 \\
0 & 0.7364 & 0 & 0 & 0 & 0.1841 \\
0.5523 & 0 & 0 & 0 & 0.3682 & 0 \\
0.1841 & 0.1841 & 0 & 0.3682 & 0 & 0.5523 \\
0 & 0.3682 & 0.3682 & 0 & 0.3682 & 0
\end{bmatrix}
\]

The channel capacity assigned to each link is assumed to
be equal. Furthermore, the values of \(\gamma_{ij}, \mu\), and \(C_{mn}\) are left
unspecified and are adjusted to impose a particular network
load. In the previous section (Section V -A), we observe
that both delay models perform badly for high traffic loads.
Therefore, we simulated a highly loaded network with the
network load \(\rho\) equal to 0.45. The maximum traffic load of
all links is 0.92045.

The data set generated by simulation contains 30 traffic
flows, each of which has 30,000 message delays. Among the
30 flows, there are 18 flows with a path length \(h\) being 1, 10
flows with \(h = 2\) and 2 flows with \(h = 3\). Flows with \(h = 1\)
won’t be considered because the delay statistics predicted by
both models are the same (see Remark 6). For the rest of 12
flows with \(h = 2\) or 3, we list their path information in Table II,
including the nodes they visit and the traffic loads of the links
along their routing paths. We again compare the fitted delay
distributions using \(-2 \log(L)\), AIC and BIC criteria. We see
in Table II that the distribution using the delay model without
KIA has the smallest \(-2 \log(L)\), AIC and BIC for all the 12
traffic flows with no exceptions.

Fig. 7 shows the simulation and the prediction results for the
12 traffic loads with \(h = 2\) or 3. The ccdfs of message delay
from simulation data, the ccdfs of (41) (the model without
KIA) and the ccdfs of (51) (the model with KIA) are plotted
in blue, red and black dotted lines, respectively. The y-axis
uses log scale. Note that the figures display message delay \(t\)
multiplied by \(\mu C\) for the normalization purposes. At this time,
we see that the simulation and the prediction results using the
delay model without KIA are always in good agreement for
all traffic loads under traffic loads from medium (e.g., path
10 in Table II) to high values (e.g., path 4 in Table II). On
Fig. 6: Two-node tandem network: simulation results and predictions (w/o KIA model (Section IV-B) and w KIA model (Section IV-C)) of cccfs of message delay under different traffic loads.
TABLE I: Two-node tandem network under different traffic loads: $-2 \log(L)$, AIC and BIC of the fitted pdfs of message delays; sample mean of message delays, sample jitter; predictions of the mean and the jitter using the delay models w/o KIA (Section IV-B) and w KIA (Section IV-C).

| $\rho$ | Model | $-2 \log(L)$ | AIC | BIC | $Z_{13}$ | $\sigma_{Z_{13}}$ | $E[Z_{13}]$ | $\sigma_{Z_{13}}$ |
|-------|-------|--------------|-----|-----|---------|--------------|-------------|--------------|
| 0.1   | w/o KIA | 110506.64  | 10518.64 | 110586.50 | 120236.27 | 133086.27 | 156251.36 | 166094.94 |
|       | w KIA | 120237.65  | 120339.65 | 120236.27 | 133086.27 | 156251.36 | 166094.94 | 166094.94 |
| 0.2   | w/o KIA | 120681.42  | 118262.42 | 118764.37 | 1242.64 | 1342.85 | 1497.01 | 262.86 |
|       | w KIA | 126163.75  | 124653.75 | 126164.37 | 1342.85 | 1497.01 | 262.86 | 262.86 |
| 0.3   | w/o KIA | 131735.50  | 126747.50 | 126797.36 | 3.07 | 26.66 | 28.66 | 28.66 |
|       | w KIA | 133086.64  | 133092.64 | 133109.26 | 2.86 | 2.86 | 2.86 | 2.86 |
| >=0.3681 | w/o KIA | 132231.99  | 132243.99 | 132293.84 | 3.38 | 2.85 | 3.17 | 2.65 |
|       | w KIA | 137551.58  | 137553.58 | 137552.20 | 3.38 | 2.85 | 3.17 | 2.65 |
| 0.5   | w/o KIA | 141718.32  | 141718.32 | 141718.32 | 4.17 | 3.30 | 4.00 | 2.83 |
|       | w KIA | 151606.34  | 151618.34 | 151668.19 | 4.82 | 3.70 | 4.76 | 3.65 |
| 0.58  | w/o KIA | 150299.35  | 150305.35 | 150320.17 | 5.02 | 3.81 | 5.00 | 3.81 |
|       | w KIA | 153755.02  | 153767.02 | 153816.87 | 5.02 | 3.81 | 5.00 | 3.81 |
| 0.6   | w/o KIA | 156247.36  | 156251.36 | 156267.98 | 6.26 | 4.56 | 6.67 | 4.92 |
|       | w KIA | 160608.29  | 160694.94 | 160614.80 | 6.26 | 4.56 | 6.67 | 4.92 |
| 0.7   | w/o KIA | 173277.14  | 173283.14 | 173333.32 | 7.15 | 5.08 | 8.00 | 5.83 |
|       | w KIA | 178452.79  | 178458.79 | 178475.41 | 7.15 | 5.08 | 8.00 | 5.83 |
| 0.75  | w/o KIA | 184322.14  | 184334.14 | 184384.00 | 8.46 | 5.94 | 10.00 | 7.21 |
|       | w KIA | 185202.17  | 185206.17 | 185222.79 | 8.46 | 5.94 | 10.00 | 7.21 |
| 0.8   | w/o KIA | 222685.71  | 222699.71 | 222757.57 | 15.05 | 11.27 | 20.00 | 14.21 |
|       | w KIA | 222889.49  | 222899.49 | 222907.11 | 15.05 | 11.27 | 20.00 | 14.21 |
| 0.9   | w/o KIA | 234554.28  | 234566.28 | 234616.15 | 100.43 | 47.43 | 200.00 | 14.14 |
|       | w KIA | 234554.28  | 234566.28 | 234616.15 | 100.43 | 47.43 | 200.00 | 14.14 |

the other hand, the delay model with KIA only provides good predictions for high traffic loads. This is again because if the high traffic load is the case, then the network queuing delay dominates the end-to-end message delay. Moreover, even in the high traffic load scenarios, it is shown in Table II that the $-2 \log(L)$, AIC and BIC criteria still suggest the superiority of our delay model without KIA.

C. The Effect of the Service Time Dependence

For a two-node tandem net with identical service times at both queues, several papers already concluded the following surprising result [10]–[12], [16].

Pinedo and Wolff’s Claim. For heavy traffic, the expected delay in queue is lower when service times are dependent than when they are independent, but for light traffic, the reverse is true.

Pinedo and Wolff’s claim can also be confirmed in our simulation. Table I lists the sample mean of message delays and predicted average delay using (52) in the two-node tandem net. It can be observed that for high traffic loads ($\rho > 0.6$), the sample means are lower than the predicted average delays; for $\rho < 0.6$, the opposite is true. Their claim can also explain why the ccdfs of simulation data are always lower than the ccdfs of [61] for high traffic loads.

By investigating the average message delay via extensive simulation, Kleinrock argued that [7]:

Kleinrock’s Claim. If there is sufficient mixing of traffic, then the dependence effect may be small, resulting in an effect of restoring the independence of interarrival times and message lengths.

Kleinrock’s claim is partially justifiable. Table II lists sample means of message delays and predicted average delays using (52) in the six-node net for all the traffic flows. Consider the case of $\rho = 0.36818$ in the tandem net and Path 10 in the six-node net. They both have the same path length and the same traffic loads along their paths, but the sample mean in the latter case is much closer to the predicted average delay. This partially justifies Kleinrock’s claim because the predicted average delays using (44) and (52) are equal (see Remark 1). Table II also lists sample standard deviations of message delays (sample jitters) and predicted jitters using (45) and (53) in the six-node net for all the traffic flows. It is shown in the table that the sample jitters are 66.7% more likely to be closer to the predicted jitters using (45), which is based on the model without KIA. Therefore, it may be more appropriate to state that mixing considerable traffic streams on a channel has an effect of restoring the independence of interarrival times (not message lengths), which is the same assumption A1 we made in Section IV-A.

VI. Conclusion

In this paper, a new phase-type distribution is introduced. Some of its structural properties are studied. We apply this new distribution to model the message delay distribution in MSCNs without the classic Kleinrock’s independence assumption. The analysis of data sets from the simulation of two topologically different networks shows that our model with this new distribution is superior to the delay model based on Kleinrock’s independence assumption. Moreover, we show that the influence of the correlations due to dependent service times can be significantly reduced in a network with a larger number of nodes and complex traffic mixing. Although MSCNs with a small number of nodes and traffic flows are important in
Fig. 7: Six-node network: simulation results and predictions (w/o KIA model (Section IV-B) and w KIA model (Section IV-C)) of cdfs of message delay for messages pass through different paths when the network load is 0.45.
TABLE II: Six-node network when $\rho = 0.45$: $-2\log(L)$, AIC and BIC of the fitted pdfs of message delays; sample mean of message delays, sample jitter; predictions of the mean and the jitter using the delay models w/o KIA (Section IV-B) and w KIA (Section IV-C).

| Flow No. | Path $p_{ij}$ | $p_{13,12}$ | $p_{13,33}$ | $p_{33,34}$ | Model | $-2\log(L)$ | AIC | BIC |
|----------|---------------|-------------|-------------|-------------|-------|--------------|------|------|
| 1        | $p_{13}: 1 \rightarrow 2 \rightarrow 3$ | 0.92045 | 0.55227 | | w/o KIA | 219218.74 | 219230.74 | 219280.59 |
| 2        | $p_{16}: 1 \rightarrow 2 \rightarrow 6$ | 0.92045 | 0.36818 | | w KIA | 219899.16 | 219903.16 | 219919.78 |
| 3        | $p_{34}: 2 \rightarrow 1 \rightarrow 4$ | 0.92045 | 0.55227 | | w/o KIA | 217852.01 | 217856.01 | 217872.63 |
| 4        | $p_{31}: 3 \rightarrow 2 \rightarrow 1$ | 0.73636 | 0.92045 | | w KIA | 217852.01 | 217856.01 | 217872.63 |
| 5        | $p_{35}: 3 \rightarrow 2 \rightarrow 5$ | 0.73636 | 0.36818 | | w/o KIA | 217852.01 | 217856.01 | 217872.63 |
| 6        | $p_{42}: 4 \rightarrow 1 \rightarrow 2$ | 0.55227 | 0.92045 | | w/o KIA | 217852.01 | 217856.01 | 217872.63 |
| 7        | $p_{46}: 4 \rightarrow 5 \rightarrow 6$ | 0.36818 | 0.55227 | | w KIA | 217852.01 | 217856.01 | 217872.63 |
| 8        | $p_{53}: 5 \rightarrow 6 \rightarrow 3$ | 0.55227 | 0.36818 | | w/o KIA | 217852.01 | 217856.01 | 217872.63 |
| 9        | $p_{61}: 6 \rightarrow 2 \rightarrow 1$ | 0.36818 | 0.92045 | | w KIA | 217852.01 | 217856.01 | 217872.63 |
| 10       | $p_{64}: 6 \rightarrow 5 \rightarrow 4$ | 0.36818 | 0.36818 | | w/o KIA | 217852.01 | 217856.01 | 217872.63 |
| 11       | $p_{34}: 3 \rightarrow 2 \rightarrow 5 \rightarrow 4$ | 0.73636 | 0.92045 | 0.55227 | w KIA | 227920.30 | 227936.30 | 228002.77 |
| 12       | $p_{43}: 4 \rightarrow 1 \rightarrow 2 \rightarrow 3$ | 0.55227 | 0.92045 | 0.55227 | w KIA | 227936.30 | 228002.77 | 228027.72 |

some scenarios, in many practical situations they are of limited interest; and hence, it is possible and highly practical to use our delay model to make predictions of message delay in MSCNs as well as packet-switched networks.

APPENDIX A

PROOF OF REMARK 2

In the case $k = 2$, let $\Delta = \lambda_2 - \lambda_1$. The cdf of (67) can be expressed as

$$S_Z(x) = \left(1 + \frac{p_{21}A_1}{\lambda_2} \right) p_1 e^{-\lambda_1 x} + \left(1 + \frac{p_{12}A_2}{\lambda_1 - \lambda_2} \right) p_2 e^{-\lambda_2 x}$$

$$= \left(1 + \frac{p_{21}A_1}{\Delta} \right) p_1 e^{-\lambda_1 x} + \left(1 + \frac{p_{12}A_2}{\Delta} \right) p_2 e^{-\lambda_2 x}$$

$$= e^{-\lambda_1 x} \left(1 + \frac{p_{21}A_1}{\Delta} \right) p_1 + e^{-\lambda_2 x} \left(1 + \frac{p_{12}A_2}{\Delta} \right) p_2$$

(64)

If we arbitrarily let $1 + \frac{p_{12}A_2}{\Delta} = 0$, i.e.,

$$1 + \frac{p_{12}A_2}{\Delta} = 0$$

(65)

$$\iff \Delta = p_1 (A_1 + \Delta)$$

(66)

$$\iff \lambda_2 - \lambda_1 = p_1 \lambda_2,$$

then (64) can be written as

$$S_Z(x) = pe^{-\lambda_1 x},$$

(68)

where

$$p = \frac{\Delta p_1 + p_1 p_2 A_1}{\Delta}.$$  

(69)

Eq. (67) is the condition of (17) and eq. (68) is (18).
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