ISOMORPHIC BUSEMANN-PETTY PROBLEM FOR SECTIONS OF PROPORTIONAL DIMENSIONS

ALEXANDER KOLDOBSKY

Abstract. The main result of this note is a solution to the isomorphic Busemann-Petty problem for sections of proportional dimensions, as follows. Suppose that $0 < \lambda < 1$, $k > \lambda n$, and $K, L$ are origin-symmetric convex bodies in $\mathbb{R}^n$ satisfying the inequalities

$$|K \cap H| \leq |L \cap H|, \quad \forall H \in \text{Gr}_{n-k},$$

where $\text{Gr}_{n-k}$ is the Grassmanian of $(n-k)$-dimensional subspaces of $\mathbb{R}^n$, and $|K|$ stands for volume of proper dimension. Then

$$|K|^{\frac{n-k}{n}} \leq C^{k} \left( \frac{(1 - \log \lambda)^{3}}{\lambda} \right)^{k} |L|^{\frac{n-k}{n}},$$

where $C$ is an absolute constant.

1. Introduction

The Busemann-Petty problem, raised in 1956 in [BP], asks the following question. Suppose that $K, L$ are origin-symmetric convex bodies in $\mathbb{R}^n$ so that the $(n-1)$-dimensional volume of every central hyperplane section of $K$ is smaller than the same for $L$, i.e.

$$|K \cap \xi^\perp| \leq |L \cap \xi^\perp|, \quad \forall \xi \in S^{n-1}. \quad (1)$$

Does it follow that the $n$-dimensional volume of $K$ is smaller than that of $L$, i.e.

$$|K| \leq |L| ?$$

Here $\xi^\perp = \{ x \in \mathbb{R}^n : (x, \xi) = 0 \}$ is the central hyperplane perpendicular to $\xi$, and $|K|$ stands for volume of proper dimension. The problem was solved in the end of the 1990’s as the result of a sequence of papers [LR], [Ba2], [Gi], [Bo4], [L], [Pa], [G1], [G2], [Z1], [Z2], [K1], [K2], [Z3], [GKS] ; see [K4, p. 3] or [G3, p. 343] for details. The answer is affirmative if $n \leq 4$, and it is negative if $n \geq 5$.

The lower dimensional Busemann-Petty problem asks the same question for sections of lower dimensions. Suppose $K, L$ are origin-symmetric convex bodies in $\mathbb{R}^n$, and $1 \leq k \leq n-1$. Let $\text{Gr}_{n-k}$ be the Grassmanian...
of \((n - k)\)-dimensional subspaces of \(\mathbb{R}^n\), and suppose that
\[
|K \cap H| \leq |L \cap H|, \quad \forall H \in Gr_{n-k}.
\] (2)

Does it follow that \(|K| \leq |L|\)? It was proved in [BZ] (see also [K3], [K4, p.112], [RZ] and [M] for different proofs) that the answer is negative if the dimension of sections \(n - k > 3\). The problem is still open for two- and three-dimensional sections \((n - k = 2, 3, n \geq 5)\).

Since the answer to the Busemann-Petty problem is negative in most dimensions, it makes sense to ask the isomorphic Busemann-Petty problem, namely, does there exist an absolute constant \(C\) such that inequalities (1) imply
\[
|K| \leq C |L|.
\]

If the answer to the isomorphic Busemann-Petty problem was affirmative, then by iteration there would exist an absolute constant \(C\) such that for every \(1 \leq k \leq n - 1\)
\[
|K|^\frac{n-k}{n} \leq C^k |L|^\frac{n-k}{n}.
\] (3)

However, the isomorphic Busemann-Petty problem is still open and equivalent to the slicing problem [Bo1, Bo2, Ba1, MP], another major open problem in convex geometry. The slicing problem asks whether there exists an absolute constant \(C\) so that for any origin-symmetric convex body \(K\) in \(\mathbb{R}^n\) of volume 1 there is a hyperplane section of \(K\) whose \((n-1)\)-dimensional volume is greater than \(1/C\). In other words, does there exist an absolute constant \(C\) so that for any \(n \in \mathbb{N}\) and any origin-symmetric convex body \(K\) in \(\mathbb{R}^n\)
\[
|K|^{\frac{n-1}{n}} \leq C \max_{\xi \in S^{n-1}} |K \cap \xi^\perp|.
\] (4)

The best current result \(C \leq O(n^{1/4})\) is due to Klartag [Kl], who removed the logarithmic term from an earlier estimate of Bourgain [Bo3]. We refer the reader to [BGVV] for the history and partial results.

Iterating (4) one gets the lower dimensional slicing problem asking whether the inequality
\[
|K|^{\frac{n-k}{n}} \leq C^k \max_{H \in Gr_{n-k}} |K \cap H|
\] (5)
holds with an absolute constant \(C\), where \(1 \leq k \leq n - 1\). Inequality (5) was recently proved in [K5] in the case where \(k \geq \lambda n\), \(0 < \lambda < 1\), with the constant \(C = C(\lambda)\) dependent only on \(\lambda\).

**Proposition 1.** ([K5, Corollary 3]) There exists an absolute constant \(C\) such that for every \(n \in \mathbb{N}\), every \(0 < \lambda < 1\), every \(k > \lambda n\), and every
origin-symmetric convex body $K$ in $\mathbb{R}^n$

$$|K|^{\frac{n-k}{n}} \leq C^k \left( \frac{(1 - \log \lambda)^3}{\lambda} \right)^{\frac{k}{n}} \max_{H \in \text{Gr}_{n-k}} |K \cap H|.$$  

In this note we prove an isomorphic Busemann-Petty problem for sections of proportional dimensions.

**Theorem 1.** Suppose that $0 < \lambda < 1$, $k > \lambda n$, and $K, L$ are origin-symmetric convex bodies in $\mathbb{R}^n$ satisfying the inequalities

$$|K \cap H| \leq |L \cap H|, \quad \forall H \in \text{Gr}_{n-k}.$$  

Then

$$|K|^{\frac{n-k}{n}} \leq C^k \left( \frac{(1 - \log \lambda)^3}{\lambda} \right)^{\frac{k}{n}} |L|^{\frac{n-k}{n}},$$

where $C$ is an absolute constant.

It is easy to see that Theorem 1 implies Proposition 1; see Remark 2. It is not known whether Theorem 1 can be deduced from Proposition 1, we provide an independent proof here.

Proposition 1 was proved in [K5] for arbitrary measures in place of volume. The arguments of this paper do not allow for an extension of Theorem 1 to arbitrary measures, and, therefore, the possibility of such an extension remains open. Note that a version of the isomorphic Busemann-Petty problem for arbitrary measures was established in [KZ], but with the constant $C = \sqrt{n}$ depending on the dimension.

**2. Proof of Theorem 1**

We need several definitions and facts. A closed bounded set $K$ in $\mathbb{R}^n$ is called a **star body** if every straight line passing through the origin crosses the boundary of $K$ at exactly two points different from the origin, the origin is an interior point of $K$, and the **Minkowski functional** of $K$ defined by

$$\|x\|_K = \min\{a \geq 0 : x \in aK\}$$

is a continuous function on $\mathbb{R}^n$.

We use the polar formula for volume of a star body

$$|K| = \frac{1}{n} \int_{S^{n-1}} \|\theta\|_K^{-n} d\theta.$$  

(6)

The solution of the original Busemann-Petty problem was based on a connection with intersection bodies found by Lutwak [L]. In this paper we use a more general class of generalized intersection bodies
introduced by Zhang [Z4] in connection with the lower dimensional Busemann-Petty problem.

For $1 \leq k \leq n-1$, the $(n-k)$-dimensional spherical Radon transform $R_{n-k} : C(S^{n-1}) \to C(Gr_{n-k})$ is a linear operator defined by

$$R_{n-k}g(H) = \int_{S^{n-1} \cap H} g(x) \, dx, \quad \forall H \in Gr_{n-k}$$

for every function $g \in C(S^{n-1})$. By the polar formula for volume, for every $H \in Gr_{n-k}$, we have

$$|K \cap H| = \frac{1}{n-k} \int_{S^{n-1} \cap H} \|x\|_K^{n+k} \, dx$$

$$= \frac{1}{n-k} R_{n-k}(\| \cdot \|_K^{n+k})(H).$$

We say that an origin symmetric star body $K$ in $\mathbb{R}^n$ is a generalized $k$-intersection body, and write $K \in \mathcal{BP}_k^n$, if there exists a finite Borel non-negative measure $\mu$ on $Gr_{n-k}$ so that for every $g \in C(S^{n-1})$

$$\int_{S^{n-1}} \|x\|_K^{-k} g(x) \, dx = \int_{Gr_{n-k}} R_{n-k}g(H) \, d\mu(H).$$

When $k = 1$ we get the original class of intersection bodies introduced by Lutwak in [L].

For a star body $K$ in $\mathbb{R}^n$ and $1 \leq k < n$, denote by

$$\text{o.v.r.} (K, \mathcal{BP}_k^n) = \inf \left\{ \left( \frac{|D|}{|K|} \right)^{1/n} : K \subset D, \ D \in \mathcal{BP}_k^n \right\}$$

the outer volume ratio distance from $K$ to the class $\mathcal{BP}_k^n$. This quantity is directly related to the isomorphic Busemann-Petty problem.

**Theorem 2.** Suppose that $1 \leq k \leq n-1$, and $K, L$ are origin-symmetric star bodies in $\mathbb{R}^n$ such that

$$|K \cap H| \leq |L \cap H|, \quad \forall H \in Gr_{n-k}.$$

Then

$$|K|^{\frac{n-k}{n}} \leq (\text{o.v.r.} (K, \mathcal{BP}_k^n))^k |L|^{\frac{n-k}{n}}.$$

**Proof:** Let $s > \text{o.v.r.} (K, \mathcal{BP}_k^n)$, then there exists a star body $D \in \mathcal{BP}_k^n$ such that $K \subset D$ and

$$|D|^{\frac{1}{n}} \leq s |K|^{\frac{1}{n}}.$$

Let $\mu$ be the measure on $Gr_{n-k}$ corresponding to $D$ by definition (8).

By (7), the condition $|K \cap H| \leq |L \cap H|$ can be written as

$$R_{n-k}(\| \cdot \|_K^{n+k})(H) \leq R_{n-k}(\| \cdot \|_L^{n+k})(H), \quad \forall H \in Gr_{n-k}.$$
Integrating this inequality over $Gr_{n-k}$ with respect to the measure $\mu$ and using (8) we get
\[
\int_{S^{n-1}} \|x\|^{-k}_{D} \|x\|^{-n+k}_{K} dx \leq \int_{S^{n-1}} \|x\|^{-k}_{D} \|x\|^{-n+k}_{L} dx.
\] (10)
Since $K \subset D$, we have $\|x\|^{-k}_{K} \leq \|x\|^{-k}_{D}$, so the left-hand side of (10) can be estimated from below by
\[
\int_{S^{n-1}} \|x\|^{-n}_{K} dx = n|K|.
\]
By Hölder’s inequality and (9), the right-hand side of (10) can be estimated from above by
\[
\left(\int_{S^{n-1}} \|x\|^{-n}_{D} dx\right)^{\frac{k}{n}} \left(\int_{S^{n-1}} \|x\|^{-n}_{L} dx\right)^{\frac{n-k}{n}} = n|D|^\frac{k}{n}|L|^{\frac{n-k}{n}}
\]
\[
\leq ns^k |K|^\frac{k}{n} |L|^{\frac{n-k}{n}}.
\]
Combining these estimates and sending $s$ to o.v.r. $(K, \mathcal{BP}^n_k)$, we get the result.

The outer volume ratio distance from a general convex body to the class of generalized $k$-intersection bodies was estimated in [KPZ].

**Proposition 2.** ([KPZ, Theorem 1.1]) Let $K$ be an origin-symmetric convex body in $\mathbb{R}^n$, and let $1 \leq k \leq n - 1$. Then
\[
o.v.r.(K, \mathcal{BP}^n_k) \leq C \sqrt{n \left(\frac{en}{k}\right)^{3/2}},
\]
where $C$ is an absolute constant.

**Remark 1.** In [KPZ, Theorem 1.1], the result was formulated with the logarithmic term raised to the power 1/2 instead of 3/2, due to a mistake. The correction was made in [K5].

Our main result immediately follows.

**Proof of Theorem 1.** Since $n/k < 1/\lambda$, by Proposition 2,
\[
o.v.r.(K, \mathcal{BP}^n_k) < C \sqrt{\frac{(1 - \log \lambda)^3}{\lambda}}.
\]
The result follows from Theorem 2.
Remark 2. Theorem 1 implies Proposition 1 via a simple argument, similar to the one employed in [MP] for hyperplane sections. Indeed, assume that $\lambda$ and $k$ are as in Theorem 1, and suppose that
\[ |K|^{\frac{n-k}{n}} = C(\lambda)^k |B_2^n|^{\frac{n-k}{n}}, \]  
where
\[ C(\lambda) = C \sqrt{\frac{(1-\log \lambda)^3}{\lambda}}, \]
$B_2^n$ is the unit Euclidean ball in $\mathbb{R}^n$, and $K$ is an origin-symmetric convex body in $\mathbb{R}^n$. Then, by Theorem 1, it is not possible that
\[ |K \cap H| < |B_2^n \cap H| = |B_2^{n-k}| \]
for all $H \in Gr_{n-k}$, so
\[ \max_{H \in Gr_{n-k}} |K \cap H| \geq |B_2^{n-k}|. \]
Dividing both sides by equal numbers from (11), we get
\[ \frac{\max_{H \in Gr_{n-k}} |K \cap H|}{|K|^{\frac{n-k}{n}}} \geq \frac{|B_2^{n-k}|}{C(\lambda)^k |B_2^n|^{\frac{n-k}{n}}}. \]
By homogeneity, the condition (11) can be dropped, and the latter inequality holds for arbitrary origin-symmetric convex $K$. Now Proposition 1 follows from
\[ \frac{|B_2^n|^{\frac{n-k}{n}}}{|B_2^{n-k}|} \in (e^{-k/2}, 1); \]
see for example [KL, Lemma 2.1].

Finally, we mention the following fact which can be proved by applying Theorem 1 twice.

Corollary 1. Suppose that $0 < \lambda < 1$, $k > \lambda n$, $0 < c_1 < c_2$, and $K, L$ are origin-symmetric convex bodies in $\mathbb{R}^n$ such that
\[ c_1 \leq \frac{|L \cap H|}{|K \cap H|} \leq c_2, \quad \forall H \in Gr_{n-k}. \]
Then
\[ \frac{c_1}{C^k \left( \frac{(1-\log \lambda)^3}{\lambda} \right)^k} \leq \frac{|L|^{\frac{n-k}{n}}}{|K|^{\frac{n-k}{n}}} \leq c_2 C^k \left( \frac{(1-\log \lambda)^3}{\lambda} \right)^k, \]
where $C$ is an absolute constant.
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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MO 65211

E-mail address: koldobskiya@missouri.edu