Automatic Debiased Machine Learning via Neural Nets for Generalized Linear Regression

Victor Chernozhukov  Whitney K. Newey  Victor Quintas-Martinez
MIT  MIT  MIT

Vasilis Syrgkanis
Microsoft Research

April 2021

Abstract

We give debiased machine learners of parameters of interest that depend on generalized linear regressions, which regressions make a residual orthogonal to regressors. The parameters of interest include many causal and policy effects. We give neural net learners of the bias correction that are automatic in only depending on the object of interest and the regression residual. Convergence rates are given for these neural nets and for more general learners of the bias correction. We also give conditions for asymptotic normality and consistent asymptotic variance estimation of the learner of the object of interest.

*This research was supported by NSF grant 1757140.
1 Introduction

We give debiased machine learners of parameters of interest that depend on generalized linear regressions. Generalized linear regressions are functions of regressors in a linear set that make a residual orthogonal in the population to that linear set, where the residual may be nonlinear in the regression. These regressions generalize linear projections to nonlinear residuals. An important special case is a function of regressors that solves the first order conditions for a generalized linear model, Nelder and Wedderburn (1972).

Many causal and structural parameters of economic interest depend on generalized regressions. Examples include policy effects, average derivatives, regression decompositions, average treatment effects, causal mediation, and parameters of economic structural models. Often, a generalized regression may be high dimensional, depending on many variables. There may be many covariates for policy effects, average derivatives, and treatment effects. There may be many prices and covariates in the economic demand for some commodity or state variables in a dynamic model. This paper is about estimating economic and causal parameters that depend on high dimensional generalized regressions. We innovate in allowing for high dimensional generalized regressions to estimate objects of interest.

Machine learners provide remarkably good predictions in a variety of settings but are inherently biased. The bias arises from using regularization and/or model selection to control the variance of the prediction. To obtain small mean squared prediction errors, machine learners regularize and/or select among models so that variance and squared bias are balanced. Although such balance is good for prediction, it is not good for inference. Confidence intervals based on estimators with approximately balanced variance and squared bias will tend to have poor coverage. This inference problem can be even worse when machine learners are plugged into a formula for an effect of interest, which often has smaller variance but similar bias to the predictor.

To reduce regularization and model selection bias we use a Neyman orthogonal moment function where there is no first-order effect of the generalized regression on the expected moment function. The orthogonal moment function is constructed by adding to an identifying moment an adjustment term that is the product of the residual and a function $\alpha_0$ in the linear space of possible regressions. This construction is model free and nonparametric, so that the regression learners have no first order effect on the moments for unrestricted, possibly misspecified, nonparametric distributions. Consequently the standard errors are robust to misspecification because they constructed from moment functions that are orthogonal under misspecification.

We give learners of $\alpha_0$ that are automatic, in the sense that they depend only on the object of interest and not on the form of $\alpha_0$. The object of interest and the regression residual are used to form an objective function that is minimized in the population at the true $\alpha_0$. We estimate this objective function in way that also depends only on the object of interest and the

2
regression residual and minimize that estimate to learn $\alpha_0$. We give neural net learners of $\alpha_0$ based on this objective function with convergence rates for in terms of the width and depth of the network and the dimension and smoothness of the regression. We obtain the neural net rates from general critical radius and approximation rate conditions for the estimated objective function that could be used to obtain convergence rates for other learners of $\alpha_0$.

We allow any regression learner, including neural nets, random forests, Lasso, and other machine learners to be used in the orthogonal moment function. A primary requirement of the regression learner is that the product of mean-square convergence rates for the learner of $\alpha_0$ and the regression learner converges faster than $n^{-1/2}$. When the parameter of interest or the residual is nonlinear in the regression it is also required that the regression learner converge in mean square faster than $n^{-1/4}$. We combine these rate conditions with those we give for the learner of $\alpha_0$ to specify conditions for asymptotic normality and consistent variance estimation for parameters of interest.

We use cross-fitting as in Chernozhukov et al. (2018), where orthogonal moment functions are averaged over groups of observations, the regression and $\alpha_0$ learners use observations not in the group, and each observation is included in the average over one group. Cross-fitting removes a source of bias and eliminates any need for Donsker conditions for the regression learner. Early work by Bickel (1982), Schick (1986), and Klaassen (1987) used similar sample splitting ideas.

This paper builds on recent work on Neyman orthogonal moment functions and debiased machine learning. We use model free orthogonal moment functions like those of Chernozhukov et al. (2016) with adjustment term for generalized regression from Ichimura and Newey (2021). The learner of $\alpha_0$ we give generalizes those of Chernozhukov, Newey, and Robins (2018) and Chernozhukov, Newey, and Singh (2018) for approximately sparse linear regression and Singh, Xu, and Gretton (2020) for reproducing kernel Hilbert spaces. The learner of $\alpha_0$ is like these previous results in not requiring an explicit formula for $\alpha_0$. A convergence rate for a general learner of $\alpha_0$ is obtained by applying results of Foster and Syrgkanis (2019) that characterize convergence rates in terms of critical radius and approximation. The primitive rate conditions given for neural net learners of $\alpha_0$ use critical radius and approximation rate results given in Farrell, Liang, and Misra (2021). Additional neural net rate conditions could be obtained using the Schmidt-Heber (2020) and Yarotsky (2017, 2018). The learner of $\alpha_0$ differs from those of Farrell, Liang, and Misra (2020, 2021) in being automatic rather than using a known form for $\alpha_0$.

This work builds upon ideas in classical semi- and nonparametric learning theory with low-dimensional regressions using traditional smoothing methods (Van Der Vaart, 1991; Bickel et al., 1993; Newey 1994; Robins and Rotnitzky, 1995; Van der Vaart, 1998), that do not apply to the current high-dimensional setting. The orthogonal moment functions developed in Chernozhukov et al. (2016) and used here build on previous work on model free orthogonal moment
functions. Hasminskii and Ibragimov (1979) and Bickel and Ritov (1988) suggest such estimators for functionals of a density. Newey (1994) develops such scores for densities and regressions from computation of the semiparametric efficiency bound for regular functionals. Doubly robust estimating equations for treatment effects as in Robins, Rotnitzky, and Zhao (1995) and Robins and Rotnitzky (1995) constitute model based orthogonal moment functions and have motivated much subsequent work. Newey, Hsieh, and Robins (1998, 2004) extend model free orthogonal moment functions to any functional of a density or distribution in a low dimensional setting. Model free, orthogonal moments for any learner are given and their general properties derived in Chernozhukov et al. (2016, 2020). We use those model free, orthogonal moment functions for regressions.

This paper also builds upon and contributes to the literature on modern orthogonal/debiased estimation and inference, including Zhang and Zhang (2014), Belloni et al. (2012, 2014a,b), Robins et al. (2013), van der Laan and Rose (2011), Javanmard and Montanari (2014a,b, 2015), Van de Geer et al. (2014), Farrell (2015), Ning and Liu (2017), Chernozhukov et al. (2015), Neykov et al. (2018), Ren et al. (2015), Jankova and Van De Geer (2015, 2016a, 2016b), Bradic and Kolar (2017), Zhu and Bradic (2017a,b). This prior work is about regression coefficients, treatment effects, and semiparametric likelihood models. The objects of interest we consider are different than those analyzed in Cai and Guo (2017). The continuity properties of functionals we consider provide additional structure that we exploit, namely the $\alpha_0$, an object that is not considered in Cai and Guo (2017). Targeted maximum likelihood, Van Der Laan and Rubin (2006), based on machine learners has been considered by Van der Laan and Rose (2011) and large sample theory given by Luedtke and Van Der Laan (2016), Toth and van der Laan (2016), and Zheng et al. (2016). We use moment function methods with automatic debiasing rather than the likelihood based method with a known form for orthogonal moments as in van der Laan and Rose (2011).

In Section 2 we describe the objects of interest we consider and associated orthogonal moment functions. In Section 3 we give the learner of $\alpha_0$, the automatic debiased machine learner (Auto-DML) of the parameter of interest, and a consistent estimator of its asymptotic variance. Section 4 derives mean square convergence rates for the Lasso learner of $\alpha_0$ and conditions for root-n consistency and asymptotic normality of Auto-DML. Section 5 gives Auto-DML for nonlinear functionals of multiple regressions.
2 Average Linear Effects for Generalized Linear Regression

We consider data that consist of i.i.d. observations $W_1, \ldots, W_n$ with $W_i$ having CDF $F_0$. For expositional purposes, in this Section we first consider parameters that depend linearly on an unknown function. We take that function to depend on a vector of regressors $X$ that may be infinite dimensional. We will denote a possible such function by $\gamma$ with $\gamma(x)$ being its value at $X = x$. We will impose the restriction that $\gamma$ is in a set of functions $\Gamma$ that is linear and closed in mean square, meaning that every $\gamma$ in $\Gamma$ has finite second moment and that if $\gamma_k \in \Gamma$ for each positive integer $k$ and $E[\{\gamma_k(X) - \gamma(X)\}^2] \to 0$ then $\gamma \in \Gamma$. We give examples of $\Gamma$ in what follows.

We specify that the estimator $\hat{\gamma}$ is an element of $\Gamma$ with probability one and has a probability limit (plim) $\gamma(F)$ when $F$ is the distribution of a single observation $W_i$. We denote the plim of $\hat{\gamma}$ to be $\gamma_0 := \gamma(F_0)$ when the distribution of $W_i$ is the true distribution $F_0$. We suppose that $\gamma(F)$ satisfies an orthogonality condition where a residual $\rho(W, \gamma)$ with finite second moment is orthogonal in the population to all $b \in \Gamma$. That is we specify that $\gamma(F)$ satisfies

$E_F[b(X)\rho(W, \gamma(F))] = 0$ for all $b \in \Gamma$ and $\gamma(F) \in \Gamma$.  

(2.1)

This is an orthogonality condition where the function $\gamma(F)$ is required to be in the same space $\Gamma$ containing all $b(X)$. We refer to such a $\gamma(F)$ as a generalized linear regression where $X$ are regressors and equation (2.1) is the orthogonality condition that defines this regression. For example, if $\rho(W, \gamma) = Y - \gamma(X)$ this orthogonality condition is necessary and sufficient for $\gamma(F)$ to be the least squares projection of $Y$ on $\Gamma$, i.e. $\gamma(F) = \arg\min_{\gamma \in \Gamma} E_F[\{Y - \gamma(X)\}^2]$. Other important examples include quantile conditions where $\rho(W, \gamma) = p - 1(Y < \gamma(X))$ for some $0 < p < 1$ and first order conditions for generalized linear models where $\rho(W, \gamma) = \lambda(\gamma(X))[Y - \mu(\gamma(X))]$ for a link function $\mu(a)$ and another function $\lambda(a)$, Nelder and Wedderburn (1972). We refer to the solution $\gamma(F)$ to equation (2.1) as a generalized linear regression because the residual $\rho(W, \gamma)$ is allowed to be more general than $Y - \gamma(X)$. We assume existence and uniqueness of $\gamma(F)$ and that $\Gamma$ does not vary with $F$.

A parameter of interest is defined in terms of a function $m(w, \gamma)$ of a realized data observation $w$ and of the function $\gamma$, i.e. $m(w, \gamma)$ is a functional of $\gamma$. The parameter or object of interest has the form

$\theta_0 = E[m(W, \gamma_0)]$.  

(2.2)

This parameter $\theta_0$ is an expectation of some known formula $m(W, \gamma)$ of a data observation $W$ and a generalized regression $\gamma$. In this section we consider linear effects where $m(w, \gamma)$ is linear in $\gamma$. We also give estimators in Section 5 for important parameters that are nonlinear in multiple functions $\gamma$. 

5
Several important examples where $m(w, \gamma)$ is linear in $\gamma$ are:

**Example 1:** (Weighted Average Derivative). Here $\rho(W, \gamma)$ in equation (2.1) can be any function satisfying the conditions we will impose, $X = (D, Z)$ for a continuously distributed random variable $D$, $\gamma_0(x) = \gamma_0(d, z)$, $\omega(d)$ is a pdf, and

$$\theta_0 = E \left[ \int \omega(u) \frac{\partial \gamma_0(u, Z)}{\partial d} du \right] = E \left[ \int S(u) \gamma_0(u, Z) \omega(u) du \right] = E[S(U) \gamma_0(U, Z)],$$

where $S(u) = -\omega(u)^{-1} \partial \omega(u)/\partial u$ is the negative score for the pdf $\omega(u)$, the second equality follows by integration by parts, and $U$ is a random variable that is independent of $Z$ with pdf $\omega(u)$. This $U$ could be thought of as one simulation draw from the pdf $\omega(u)$. Here $m(w, \gamma) = S(u)\gamma(u, x)$ where $W$ includes $U$. As discussed in Chernozhukov, Newey, and Singh (2020), when $\rho(W, \gamma) = Y - \gamma(D, Z)$ this $\theta_0$ can be interpreted as an average treatment effect on $Y$ of a continuous treatment $D$. More generally $\theta_0$ will be the average derivative with respect to $D$ of a generalized regression averaged over the marginal distributions of $U$ and $Z$.

**Example 2:** (Average Treatment Effect). In this example $\rho(W, \gamma) = Y - \gamma(X), X = (D, Z)$ and $\gamma_0(x) = \gamma_0(d, z)$, where $D \in \{0, 1\}$ is the treatment indicator and $Z$ are covariates. The object of interest is

$$\theta_0 = E[\gamma_0(1, Z) - \gamma_0(0, Z)].$$

If potential outcomes are mean independent of treatment $D$ conditional on covariates $Z$, then $\theta_0$ is the average treatment effect (Rosenbaum and Rubin, 1983). Here $m(w, \gamma) = \gamma(1, z) - \gamma(0, z)$.

**Example 3:** (Linear Inverse Probability Weighting) In this example $\rho(W, \gamma) = 1 - \gamma(X)D$ where $D \in \{0, 1\}$ is a complete data indicator. Let $P(X) = \Pr(D = 1|X)$. The orthogonality condition of equation (2.1) is

$$0 = E[\{1 - D\gamma_0(X)\}b(X)] = E[P(X)\{P(X)^{-1} - \gamma_0(X)\}b(X)] = 0 \text{ for all } b(X) \in \Gamma.$$

If $\Gamma$ is all functions of $X$ with finite second moment then $\gamma_0(X) = P(X)^{-1}$. If $\Gamma$ is smaller then $\gamma_0$ is the linear projection of the inverse probability $P(X)^{-1}$ on $\Gamma$ for the expectation $E[P(X)(\cdot)]/E[D]$ that is weighted by the probability $P(X)$. The orthogonality condition is the first order conditions for minimization of $E[-2\gamma(X) + D\gamma(X)^2]$ so that

$$\gamma(F) = \arg \min_{\gamma \in \Gamma} E_F[-2\gamma(X) + D\gamma(X)^2].$$

The object of interest has $m(W, \gamma) = DU\gamma(X)$ where $U$ is a random variable that is only observed when $D = 1$. Here the parameter of interest is

$$\theta_0 = E[DU\gamma_0(X)] = E[DE[U|D, X]\gamma_0(X)] = E[DE[U|D = 1, X]\gamma_0(X)]$$

$$= E[P(X)\bar{U}(X)\gamma_0(X)], \quad \bar{U}(X) = E[U|D = 1, X].$$
This \( \theta_0 \) is like an inverse probability weighted expectation that is useful for estimating expectations of data that are only observed when \( D = 1 \). If \( U \) is missing at random in the sense that \( E[U|D = 1, X] = E[U|X] \) then \( \theta_0 = E[P(X)\bar{U}(X)\gamma_0(X)] = E[\bar{U}(X)] = E[U] \) when \( \gamma_0(X) = P(X)^{-1} \). In this case \( \theta_0 \) is a weighted expectation of \( U \) that equals \( E[U] \) when \( P(X)^{-1} \in \Gamma \).

**Example 4:** (Regression Effect for Generalized Linear Regression) In this example \( \rho(W, \gamma) = \lambda(\gamma(X))[Y - \mu(\gamma(X))] \) so that the orthogonality condition (2.1) corresponds to a generalized linear model. For any \( m(W, \gamma) \) the parameter \( \theta_0 \) in equation will be a linear function of the generalized linear regression \( \gamma_0 \), such as the weighted average derivative in Example 1. This example generalizes the setting of Chernozhukov, Newey, and Robins (2018) and Hirshberg and Wager (2018) to allow \( \rho(Y, \gamma) \) to be nonlinear in \( \gamma \).

We consider \( m(w, \gamma) \) where there exists a function \( v_m(X) \) with \( E[v_m(X)^2] < \infty \) and

\[
E[m(W, \gamma)] = E[v_m(X)\gamma(X)] \quad \text{for all } \gamma \text{ such that } E[\gamma(X)^2] < \infty.
\] (2.3)

By the Riesz representation theorem, existence of such a \( v_m(X) \) is equivalent to \( E[m(W, \gamma)] \) being a mean-square continuous functional of \( \gamma \), i.e. \( E[m(W, \gamma)] \leq C \| \gamma \| \) for all \( \gamma \), where \( \| \gamma \| = \sqrt{E[\gamma(X)^2]} \) and \( C > 0 \). We will refer to this \( v_m(X) \) as the Riesz representer (Rr).

Existence of the Rr is equivalent to the semiparametric variance bound for \( \theta_0 \) being finite when \( \rho(W, \gamma) = Y - \gamma(X) \), as in Newey (1994), Hirshberg and Wager (2018), and Chernozhukov, Newey, and Singh (2019).

Each of Examples 1-4 has such a Rr. Let \( f(d|z) \) be the pdf of \( D \) conditional on \( Z \) in Example 1 and \( \pi_0(z) = \Pr(D = 1|Z = z) \) the propensity score in Example 2. Table 1 summarizes the functional \( m(w, \gamma) \) and the Rr in each of the examples:

| Effect                                      | \( m(W, \gamma) \)                     | \( v_m(X) \)                                         |
|---------------------------------------------|----------------------------------------|------------------------------------------------------|
| Weighted Average Derivative                 | \( S(U)\gamma(U, Z) \)                 | \( f(D|Z)^{-1}\omega(D)S(D) \)                        |
| Average Treatment Effect                    | \( \gamma(1, Z) - \gamma(0, Z) \)     | \( \pi_0(Z)^{-1}D - (1 - \pi_0(Z))^{-1}(1 - D) \)   |
| Linear Inverse Probability Weight           | \( DU\gamma(X) \)                     | \( P(X)\bar{U}(X) \)                                |
| Generalized Linear Model                    | \( m(W, \gamma) \)                    | \( v_m(W) \)                                        |

Equation (2.3) follows in Example 1 by integration and multiplying and dividing by \( f(d|z) \) and in Example 2 in a standard way for average treatment effects. In Example 3 the Rr follows by iterated expectations and in Example 4 is general. For \( E[v_m(X)^2] < \infty \) to hold in Examples 1 and 2 the denominator must not be too small relative to the numerator in each \( v_m(X) \), on average. For instance Example 2 must have \( E[\{\pi_0(Z)(1 - \pi_0(Z))\}^{-1}] < \infty \).
We assume that the learner \( \hat{\gamma} \) is an element of \( \Gamma \) with \( \text{plim} \, \gamma(F) \) satisfying equation (2.1). For example, \( \Gamma \) may be the set of all functions \( \gamma(X) \) with finite second moment in which case \( \hat{\gamma} \) is unrestricted in functional form. Here the orthogonality condition (2.1) is a conditional moment restriction \( E_F[\rho(W, \gamma(F))|X] = 0 \). In another case \( X = (X_1, X_2, \ldots) \) and \( \Gamma \) is the mean square closure of finite linear combinations of \( X \). This corresponds to a high dimensional, approximately sparse, generalized linear regression \( \hat{\gamma} \) having a limit \( \gamma(F) \) satisfying \( E_F[X_j \rho(W, \gamma(F))] = 0 \) for all \( j \). In a third case, considered by Hirshberg and Wager (2018) and Farrell, Liang, and Misra (2020), \( \Gamma \) is the mean square closure of \( \{a(X_1) + X_1'b(X_1)\} \) where \( a(X_1) \) is a scalar function and \( b(X_1) \) a vector of functions each having unrestricted functional form. Another special case is the mean square closure of additive functions \( a(X_1) + a(X_2) \) where \( X_1 \) and \( X_2 \) are distinct components of \( X \).

The generalized regression learner \( \hat{\gamma} \) can be any of a variety of machine learners including neural nets, random forests, Lasso, and other high dimensional methods. All we require is that \( \hat{\gamma} \) converge in mean square at a sufficiently fast rate, as specified in Section 4. Whatever the choice of \( \hat{\gamma} \), estimating \( \theta_0 \) by plugging \( \hat{\gamma} \) into \( m(W, \gamma) \) and averaging over observations on \( W \) can lead to large biases when \( \hat{\gamma} \) involves regularization and/or model selection, as discussed in the Introduction. For that reason we use a Neyman orthogonal moment function for \( \theta_0 \), where the regression learner \( \hat{\gamma} \) has no first-order effect on the moments.

We follow Chernozhukov et al. (2016, 2020) in basing the orthogonal moment function on the plim \( \gamma(F) \) of \( \hat{\gamma} \) when one observation \( W \) has CDF \( F \), where \( F \) is unrestricted except for regularity conditions, and \( \gamma(F) \) solves (2.1). Here \( \gamma(F) \) can be thought of as the plim of \( \hat{\gamma} \) under general misspecification. We construct a Neyman orthogonal moment function as in Chernozhukov et al. (2016, 2020) by adding to the identifying moment function \( m(W, \gamma) - \theta \) the nonparametric influence function of \( E[m(W, \gamma(F))] \) which is the adjustment term of Newey (1994). The resulting moment function will be the efficient influence function for the nonparametric model where \( F \) is unrestricted and \( E_F[m(W, \gamma(F))] \) is the parameter of interest.

To construct the orthogonal moment function we use the formula for the adjustment term for generalized regression \( \gamma(F) \) given in Ichimura and Newey (2021). We suppose that there is \( v_\rho(W) \) such that

\[
\frac{\partial}{\partial \tau} E[b(X)\rho(W, \gamma_0 + \tau \delta)] = E[b(X)v_\rho(W)\delta(X)] \quad \text{for all } b, \delta \in \Gamma.
\] (2.4)

It will often be the case that this equation is satisfied for \( v_\rho(W) = \partial \rho(W, \gamma_0 + a)/\partial a \) at \( a = 0 \) for a constant \( a \). More generally it will be the case that \( v_\rho(X) = \partial E[\rho(W, \gamma_0 + a)|X]/\partial a \). For example when \( \rho(W, \gamma) = Y - \gamma(X) \) we have \( v_\rho(w) = -1 \). We note that by iterated expectations if equation (2.4) is satisfied then it is also satisfied for \( v_\rho(X) = E[v_\rho(W)|X] \). We allow for \( v_\rho(W) \) to depend on \( W \) rather than \( X \) because that makes estimation simpler as we will explain in Section 3. We will assume that \( v_\rho(X) < 0 \) is bounded and bounded away from zero.
By Proposition 1 of Ichimura and Newey (2021) the adjustment term is
\[ \alpha_0(X)\rho(W, \gamma_0), \quad \alpha_0(X) = \arg \min_{\alpha \in \Gamma} E[-v_{\rho}(X)\{v_m(X) - v_{\rho}(X)\} - \alpha(X)]^2, \]
where \( \alpha_0 \in \Gamma \) by construction. Evaluating this adjustment term at possible values \( \gamma \) and \( \alpha \) of \( \gamma_0 \) and \( \alpha_0 \) and adding it to the identifying moment function gives the orthogonal moment function
\[ \psi(w, \gamma, \alpha, \theta) = m(w, \gamma) - \theta + \alpha(x)\rho(w, \gamma), \quad \gamma \in \Gamma, \quad \alpha \in \Gamma. \]

The moment function \( \psi(w, \gamma, \alpha, \theta) \) depends on a possible value \( \alpha \) of the unknown function \( \alpha_0 \) as well as a possible value \( \gamma \) of the plim \( \gamma_0 \) of \( \hat{\gamma} \). A learner \( \hat{\alpha} \) of \( \alpha_0 \) is needed to use this orthogonal moment function to estimate \( \theta_0 \). In Section 3 we will describe how to construct \( \hat{\alpha} \).

The general results of Chernozhukov et al. (2020) imply that \( \psi(w, \gamma, \alpha, \theta) \) is robust to the value of \( \alpha \), in the sense that \( E[\psi(W, \gamma_0, \alpha, \theta_0)] = 0 \) for all \( \alpha \in \Gamma \), and that the first order effect of \( \gamma \) on \( E[\psi(W, \gamma, \alpha_0, \theta_0)] \) is zero. We can also verify these properties directly. Robustness in \( \alpha \) follows from equation (2.1) that gives
\[ E[\psi(W, \gamma_0, \alpha, \theta_0)] = E[\alpha(X)\rho(W, \gamma_0)] = 0, \]
for all \( \alpha \in \Gamma \). To show that the first order effect of \( \gamma \in \Gamma \) is zero consider \( \gamma_{\tau} = \gamma_0 + \tau \delta \) for any \( \delta \in \Gamma \), representing local variation in \( \gamma \) away from \( \gamma_0 \) in \( \Gamma \). They by equations (2.3) - (2.5),
\[ \frac{\partial}{\partial \tau} E[\psi(W, \theta_0, \gamma_{\tau}, \alpha_0)] = \frac{\partial}{\partial \tau} E[m(W, \gamma_{\tau})] + \frac{\partial}{\partial \tau} E[\alpha_0(X)\rho(W, \gamma_{\tau})] \]
\[ = E[v_m(X)\delta(X)] + E[\alpha_0(X)v_{\rho}(X)\delta(X)] \]
\[ = E[\{-v_{\rho}(X)\{v_m(X) - v_{\rho}(X)\} - \alpha_0(X)\}\delta(X)] = 0, \]
where the second equality follows by equations (2.3) and (2.4) and the third equality by the necessary and sufficient condition for equation (2.5). Equation (2.7) gives an explicit calculation showing that \( \gamma \) has zero first order effect on \( E[\psi(W, \theta, \gamma, \alpha_0)] \) so that \( \psi(W, \theta, \gamma, \alpha) \) is orthogonal. One can interpret this result as meaning that the addition of \( \alpha(x)\rho(w, \gamma) \) to \( m(w, \gamma) \) ”partials out” the effect of \( \gamma \) on \( m(w, \gamma) \).

The orthogonality of \( \psi(W, \theta, \gamma, \alpha) \) only depends on \( \gamma, \alpha \in \Gamma \) so that it is model free, i.e. nonparametric. Consequently the estimator of \( \theta \) will be asymptotically normal and standard errors will be consistent under general misspecification. This robustness of the standard errors results from the orthogonality of the moments only depending on the plim of \( \hat{\gamma} \) and not on any model assumptions. The orthogonal moment function could also be viewed as the efficient influence function of \( \theta(F) = E_F[m(W, \gamma(F))] \) as discussed in Chernozhukov et al. (2020). This interpretation clarifies that the estimator of \( \theta \) is an efficient semiparametric estimator of \( \theta(F) \).
The orthogonal moment function will be doubly robust, in the sense that $E[\psi(W, \theta_0, \gamma, \alpha_0)] = 0$ for all $\gamma \in \Gamma$, if and only if $E[\alpha_0(X)\rho(W, \gamma)]$ is linear (affine) in $\gamma$. This follows from linearity of $E[m(W, \gamma)]$ in $\gamma$ and from Chernozhukov et al. (2020). There are many interesting cases where double robustness does not hold, such as $\rho(W, \gamma) = \lambda(\gamma(X))|Y - \mu(\gamma(X))|$ for some nonlinear link function $\mu(a)$. Although the moment function in equation (2.6) is not doubly robust in this case it will still be orthogonal, enabling the use of machine learning in the estimation of $\gamma$ and $\alpha$.

**Example 3:** This example illustrates the orthogonal moment function. Here $\rho(w, \gamma) = 1 - d \cdot \gamma(x)$ so that $v_p(W) = -D$ and $v_p(X) = -E[D|X] = -P(X)$. Recall that $v_m(X) = P(X)\bar{U}(X)$ so that $-v_m(X)/v_p(X) = \bar{U}(X)$. The adjustment term is then

$$\alpha_0(X)[1 - D\gamma_0(X)], \quad \alpha_0(X) = \arg\min_{\alpha \in \Gamma} E[P(X)\{\bar{U}(X) - \alpha(X)\}^2],$$

where $\alpha_0(X)$ is the projection of $\bar{U}(X) = E[U|D = 1, X]$ on $\Gamma$ weighted by $P(X)$. The orthogonal moment function is

$$\psi(W, \gamma, \alpha, \theta) = DU\gamma(X) - \theta + \alpha(X)\{1 - D\gamma(X)\}.$$

Here $\rho(W, \gamma)$ is linear in $\gamma$ so that this orthogonal moment function is doubly robust by Chernozhukov et al. (2016). In particular, for any $\gamma \in \Gamma$,

$$E[DU\gamma(X) + \alpha_0(X)\{1 - \gamma(X)D\}] = E[P(X)\bar{U}(X)\gamma(X)] + E[\alpha_0(X)] - E[\alpha_0(X)\gamma(X)P(X)]$$

$$= E[\alpha_0(X)] + E[P(X)\{\bar{U}(X) - \alpha_0(X)\}\gamma(X)] = E[\alpha_0(X)].$$

If $\bar{U}(X) \in \Gamma$ it follows that $\alpha_0(X) = \bar{U}(X)$ so that $E[\alpha_0(X)] = E[\bar{U}(X)]$. Also if $P(X)^{-1} \in \Gamma$ then

$$E[P(X)\{\bar{U}(X) - \alpha_0(X)\}P(X)^{-1}] = 0,$$

so that that $E[\alpha_0(X)] = E[\bar{U}(X)]$. Therefore if either $\bar{U}(X) \in \Gamma$ or $P(X)^{-1} \in \Gamma$ we have $\theta_0 = E[\bar{U}(X)]$.

## 3 Estimation

To estimate (learn) $\theta_0$ we use cross-fitting where the orthogonal moment function $\psi(W_i, \gamma, \alpha, \theta)$ is averaged over observations $W_i$ different than used to estimate $\gamma_0$ and $\alpha_0$. Let $I_\ell$, ($\ell = 1, ..., L$), be a partition of the observation index set $\{1, ..., n\}$ into $L$ distinct subsets of about equal size. In practice $L = 5$ (5-fold) or $L = 10$ (10-fold) cross-fitting is often used. Let $\hat{\gamma}_\ell$ and $\hat{\alpha}_\ell$ be estimators constructed from the observations that are *not* in $I_\ell$. We construct the estimator
\( \hat{\theta} \) by setting the sample average of \( \psi(W_i, \theta, \gamma_\ell, \alpha_\ell) \) to zero and solving for \( \theta \). This \( \hat{\theta} \) and an associated asymptotic variance estimator \( \hat{V} \) have explicit forms

\[
\hat{\theta} = \frac{1}{n} \sum_{\ell=1}^{L} \sum_{i \in I_\ell} \{ m(W_i, \gamma_\ell) + \hat{\alpha}_\ell(X_i) \rho(W_i, \gamma_\ell) \},
\]

\[
\hat{V} = \frac{1}{n} \sum_{\ell=1}^{L} \sum_{i \in I_\ell} \hat{\psi}_{i\ell}^2, \quad \hat{\psi}_{i\ell} = m(W_i, \gamma_\ell) - \hat{\theta} + \hat{\alpha}_\ell(X_i) \rho(W_i, \gamma_\ell).
\]

Any regression learner \( \hat{\gamma}_\ell \) can be used here as long as its mean-square convergence rate is a large enough power of \( 1/n \), as assumed in Section 4. Such a convergence rate is available for neural nets (Chen and White, 1999, Yarotsky, 2017, 2018, Schmidt-Heiber, 2020, Farrell, Liang, and Misra, 2021), random forests (Syrgkanis and Zampetakis, 2020), Lasso (Bickel, Ritov, and Tsybakov, 2009), boosting (Luo and Spindler, 2016), and other high dimensional methods. As a result any of these regression learners can be used to construct an Auto-DML \( \hat{\theta} \) from equation (3.1), in conjunction with a learner \( \hat{\alpha}_\ell \) of \( \alpha_0 \).

The cross fitting notation here allows for double cross-fitting where \( \hat{\gamma}_\ell \) and \( \hat{\alpha}_\ell \) use distinct subsamples of observations that are not in \( I_\ell \). Such double cross-fitting weakens conditions for asymptotic normality of \( \hat{\theta} \) as discussed in Section 4.

We use orthogonality of the moment function \( \psi(w, \gamma, \alpha, \theta) = m(w, \gamma) - \theta + \alpha(x) \rho(w, \gamma) \) to construct an estimator \( \hat{\alpha}_\ell \) similarly to Chernozhukov et al. (2020). By equation (2.7) for any \( \delta \in \Gamma \) we have

\[
0 = \frac{\partial}{\partial \tau} E[\psi(W, \gamma_0 + \tau b, \alpha_0, \theta_0)] = E[m(W, b)] + E[\alpha_0(X) v_\rho(X) b(X)]
\]

\[
= E[\{v_m(X) + v_\rho(X) \alpha_0(X)\} b(X)].
\]

for all \( b \in \Gamma \). These are first order conditions for minimization of the function

\[
R(\alpha) := -2E[v_m(X) \alpha(X)] - E[v_\rho(X) \alpha(X)^2] = E[-2m(W, \alpha) - v_\rho(X) \alpha(X)^2]
\]

at \( \alpha_0 \) over all \( \alpha \in \Gamma \). Adding a constant to the objective function does not change the minimum so that equation (3.2) is also the first order conditions for minimization of

\[
\bar{R}(\alpha) = E[\frac{v_m(X)^2}{-v_\rho(X)}] + R(\alpha)
\]

\[
= E[\frac{v_m(X)^2}{-v_\rho(X)}] - 2E[v_m(X) \alpha(X)] - E[v_\rho(X) \alpha(X)^2]
\]

\[
= E[-v_\rho(X)\{\frac{v_m(X)}{-v_\rho(X)} - \alpha(X)\}^2]
\]

over all \( \alpha \in \Gamma \). This objective function is minimized at \( \alpha_0 \) and thus so is \( R(\alpha) \). We use this result to construct an estimator of \( \alpha_0(X) \) by minimizing a sample version of \( R(\alpha) \).
A sample version of $R(\alpha)$ based only on observations not in $I_\ell$ is

$$
\hat{R}_\ell(\alpha) = \frac{1}{n - n_\ell} \sum_{i \notin I_\ell} \{-2m(W_i, \alpha) - \hat{v}_\rho(W_i)\alpha(X_i)^2\},
$$

(3.3)

where $\hat{v}_\rho(w)$ is an estimator of $v_\rho(w)$. We use $\hat{v}_\rho(w)$ as a function of $w$ here to avoid the need to estimate $E[v_\rho(W)|X]$ in constructing the objective function. When $\hat{v}_\rho(w) = v_\rho(w)$ the expectation of this objective function is $R(\alpha)$ and more generally $\hat{R}_\ell(\alpha)$ will estimate $R(\alpha)$.

We construct $\hat{\alpha}_\ell$ as

$$
\hat{\alpha}_\ell = \arg \min_{\alpha \in A} \hat{R}_\ell(\alpha),
$$

where $A \subset \Gamma$ is a set of functions that can approximate $\alpha_0$. For $\rho(w, \gamma) = y - \gamma(w)$ the $\hat{\alpha}_\ell$ in Chernozhukov, Newey, and Singh (2018) does this over the class of functions that are linear combinations of a dictionary of functions $(b_1(x), ..., b_p(x))'$ with $b_j \in \Gamma$, $j = 1, 2, ...$ and $p$ large with an $L_1$ penalty added to $\hat{R}_\ell(\alpha)$. Chernozhukov et al. (2020) gives an analogous construction for general $\rho(w, \gamma)$ including primitive conditions for quantiles. For $\rho(w, \gamma) = y - \gamma(w)$ Singh (2020) does this over a class of functions that are included in a reproducing kernel Hilbert space.

Any learner $\hat{\alpha}_\ell$ obtained from minimizing $\hat{R}_\ell(\alpha)$ is automatic in being based only on $m(w, \gamma)$, that determines the object of interest, and does not require knowing the form of $\alpha_0$. In particular, $\hat{\alpha}_\ell$ does not depend on plugging in nonparametric estimates of components of $\alpha_0$. Instead, just the known functional $m(w, \gamma)$ is used in the construction of $\hat{\alpha}_\ell$. This automatic nature of $\hat{\alpha}_\ell$ is especially useful when $\alpha_0$ does not have a simple form.

For causal parameters such as the average treatment effect this type of learner $\hat{\alpha}_\ell$ avoids inverting a learner of a conditional probability or pdf. The finite sample properties of methods that rely on inverses of high dimensional learners can be poor; see Singh and Sun (2019) for recent examples. Instead, $\hat{\alpha}_\ell$ approximates and learns $\alpha_0$ directly and so avoids instability from inverting a high dimensional estimator. This advantage of obtaining $\hat{\alpha}_\ell$ from minimizing $\hat{R}_\ell(\alpha)$ arises for many causal parameters because they involve shifts in regressors so that the Riesz representer $v_m(x)$ must include the pdf of $X$ in its denominator.

A neural net learner $\hat{\alpha}_\ell$ can be constructed by choosing the set $A$ to correspond to a neural net. A general neural network, nonlinear regression takes the form

$$
x \xrightarrow{f_1} H^{(1)} \xrightarrow{f_2} \ldots \xrightarrow{f_m} H^{(m)}
$$

where

$$
H^{(\ell)} = \{H^{(\ell)}_k\}_{k=1}^{K_\ell},
$$

are called neurons, $x$ is the original finite dimensional input, and the function $f_\ell$ maps one layer of neurons to the next as in

$$
f_\ell : v \xrightarrow{f_1} \{H^{(\ell)}_k(v)\}_{k=1}^{K_\ell} := (1, \{\sigma(v'\alpha_{k,\ell})\}_{k=2}^{K_\ell}),
$$

12
where each $\alpha_{k,\ell}$ is a $K_{\ell-1}$ vector of parameters and where $\sigma(u)$ is a nonlinear activation function. We will focus on the case where $\sigma(u)$ is the RELU function

$$\sigma(u) = \max\{0, u\}.$$  

An important special case is a multilayer perceptron (MLP) network where the number of neurons $K_\ell = K$ is the same for each layer, for which results were recently given by Farrell, Liang, and Misra (2021). Sparse versions of this specification, where many of the elements of the coefficient vectors $\alpha_{k,\ell}$ may be zero, have also been considered recently by Schmidt-Hieber (2020). Yarotsky (2017, 2018) gave other neural network specifications with good approximation properties.

A specification of $A$ consisting of functions of $x$ having this neural net form with parameter vectors $\alpha_{k,\ell}$, $(k = 1, \ldots, K_\ell, \ell = 1, \ldots, m)$, is an unrestricted, fully nonparametric specification where elements of $A$ can approximate any function of $x$ for large enough $K_\ell$ and $m$. We can also use neural nets to specify $A$ that satisfies linear restrictions where $A \subset \Gamma$. For example consider

$$\Gamma = \{b_1(X_1) + X_2'b_2(X_1)\}$$

and let $A_1$ be a neural net specification like that just described. We can specify $A \subset \{b_1(X_1) + X_2'b_2(X_1)\}$ as

$$A = A_1 + \sum_{j=1}^J X_2jA_{1j}$$  \hspace{1cm} (3.4)

where each $A_{1j}$ is a copy of $A_1$. Here any $f \in A$ has the form $f(x) = f_1(x_1) + \sum_{j=1}^J x_2j f_{1j}(x_1)$, for $(f_1, f_{11}, \ldots, f_{1J}) \in A_1^{J+1}$, which is included in $\Gamma$. One could construct neural net functions contained in $\Gamma$ for other possible $\Gamma$ in similar ways.

Specifying $A$ to be a neural net as we have described and minimizing $\hat{R}(\alpha)$ over $A$ gives a neural net learner of $\alpha_0$ that is automatic in depending only on the residual through $\hat{v}_\rho(w)$ and on the object of interest $m(w, \alpha)$. The resulting estimated orthogonal moment function differs from those of Farrell, Liang, and Misra (2020, 2021) in not requiring an explicit formula for $\alpha_0$. By specifying $A$ to be other classes of functions one could obtain other learners of $\alpha_0$. We leave the development of these to future work.

**Example 3:** The orthogonal moment function for this example and the neural net learner of $\alpha_0$ can be used to construct Auto-DML for an inverse probability weighted expectation. Recall that $m(W, \gamma) = DU\gamma(X)$ and $v_\rho(W) = -D$. Then the learner of $\alpha_0$ is given by

$$\hat{\alpha}_\ell = \arg\min_{\alpha \in A} \left\{ \frac{1}{n - n_\ell} \sum_{i \notin I_\ell} [-2D_i U_i \alpha(X_i) + D_i \alpha(X_i)^2] \right\}$$  \hspace{1cm} (3.5)

$$= \arg\min_{\alpha \in A} \left\{ \frac{1}{n - n_\ell} \sum_{i \notin I_\ell} D_i [U_i - \alpha(X_i)]^2 \right\},$$
where the second equality holds by adding $D_i U_i^2$ inside the summation (which does not change the minimizer). This $\hat{\alpha}_\ell$ is simply a nonparametric regression of $U_i$ on $\alpha(X) \in \Gamma$ for the data where $U_i$ is observed. The estimator of $\theta_0$ and an asymptotic variance estimator are given by

$$\hat{\theta} = \frac{1}{n} \sum_{\ell=1}^{L} \sum_{i \in I_\ell} \{D_i U_i \hat{\gamma}_\ell(X_i) + \hat{\alpha}_\ell(X_i) [1 - D_i \hat{\gamma}_\ell(X_i)] \},$$  

(3.6)

$$\hat{V} = \frac{1}{n} \sum_{\ell=1}^{L} \sum_{i \in I_\ell} \hat{\psi}_{i\ell}^2, \quad \hat{\psi}_{i\ell} = D_i U_i \hat{\gamma}_\ell(X_i) - \hat{\theta} + \hat{\alpha}_\ell(X_i) [1 - D_i \hat{\gamma}_\ell(X_i)].$$

The estimator $\hat{\theta}$ will be consistent and asymptotically normal and $\hat{V}$ consistent for any $\hat{\gamma} \in \Gamma$ that converges to $\gamma_0$ at a fast enough rate. As shown in Section 2, $\hat{\theta}$ is doubly robust in the sense that $\theta_0 = E[E[U|D = 1, X]]$ whenever either $P(X)^{-1} \in \Gamma$ or $E[U|D = 1, X] \in \Gamma$.

4 Large Sample Inference

In this Section, we give mean square convergence rates for the neural net and other learners $\hat{\alpha}_\ell$ and root-n consistency and asymptotic normality results for the learner $\hat{\theta}$ of the object of interest and its asymptotic variance estimator $\hat{V}$. We first derive convergence rates for $\hat{\alpha}_\ell$.

4.1 Convergence Rates for $\hat{\alpha}_\ell$

In this subsection we suppress the $\ell$ subscript for notation convenience. We consider the problem of estimating

$$\alpha_0 = \arg \min_{\alpha \in A} \{E[-2m(W, \alpha) - v_\rho(W)\alpha(X)]^2]\},$$

where $A$ is a set of functions having certain properties we detail here. For any random variable $a(W)$ let $\|a\| = \sqrt{E[a(W)^2]}$ and $\|a\|_\infty = \sup_{w \in W} |a(w)|$. We assume that $m(W, \alpha)$ is mean square continuous in the following sense:

Assumption 1: For some $M > 0$ it is the case that $E[m(W, \alpha)^2] \leq M \|\alpha\|^2, v_\rho(W)$ is bounded, and $v_\rho(X) = E[v_\rho(W)|X] < 0$ is bounded away from zero.

Define

$$\text{star}(A - \alpha_0) = \{z \rightarrow \xi(\alpha(x) - \alpha_0(x)) : \alpha \in A, \xi \in [0, 1]\}$$

$$\text{star}(m \circ A - m \circ \alpha_0) = \{z \rightarrow \xi(m(w; \alpha) - m(w; \alpha_0)) : \alpha \in A, \xi \in [0, 1]\}$$

Assumption 2: $\|f\|_\infty \leq 1$ for all $f \in \text{star}(A - \alpha_0)$ and $f \in \text{star}(m \circ A - m \circ \alpha_0).$
Define
\[ \alpha^* = \arg\min_{\alpha \in \mathcal{A}} \{ E[-2m(W, \alpha) - v_\rho(W)\alpha(X)^2] \} \]
to be the best approximation of \( \alpha_0 \) by an element of \( \mathcal{A} \). We first give a general result on mean-square convergence of \( \hat{\alpha} \) to \( \alpha_0 \) when \( v_\rho(W) \) is known. The \( v_\rho(W) \) known case includes the important special case of a linear projection where \( v_\rho(W) = -1 \). See e.g. Foster and Srygkanis (2019) for the definition of the critical radius used in the statement of the following result.

**Theorem 1:** Let \( \delta_n \) be an upper bound on the critical radius of \( \text{star}(A - \alpha_0) \) and \( \text{star}(m \circ A - m \circ \alpha_0) \). If Assumptions 1 and 2 are satisfied then for some universal constant \( C \) it follows that with probability \( 1 - \zeta \)
\[ \| \hat{\alpha} - \alpha_0 \|^2 \leq C(M\delta_n^2 + \| \alpha^* - \alpha_0 \|^2 + \frac{M \ln(1/\zeta)}{n}) \]

To use Theorem 1 to obtain a mean square convergence rate for \( \hat{\alpha} \) it is important to know the critical radius and the rate at which \( \| \alpha_* - \alpha_0 \| \) shrinks as the approximating set \( \mathcal{A} \) becomes richer. Farrell, Liang, and Misra (2021) have recently obtained such results for deep, RELU neural nets. We can apply their results to obtain a mean square rate for such a learner of \( \alpha_0 \) when \( x \) is a \( d \) dimensional input for the multilayer perceptron (MLP) network with \( m \) layers and width \( K \).

The convergence rate depends on the smoothness of the function \( \alpha_0(x) \), as specified in the following result. Specifically we assume that the support of \( X \) is contained in a Cartesian product \( \mathcal{X} \) of compact intervals and \( \alpha_0(X) \) can be extended to a function that is continuously differentiable on \( \mathcal{X} \) and has \( \beta \) continuous derivatives.

**Corollary 2:** If i) the support of \( X \) is contained in a Cartesian product of compact intervals and \( \alpha_0(X) \) can be extended to a function that is continuously differentiable on \( X \) and has \( \beta \) continuous derivatives; ii) \( \mathcal{A} \) is a MLP network with \( d \) inputs, width \( K \), and depth \( m \) with \( K \rightarrow \infty \) and \( m \rightarrow \infty \); iii) \( m \circ \mathcal{A} \) is representable as such a network; then there is \( C > 0 \) such that for any \( \varepsilon > 0 \) with probability approaching one
\[ \| \hat{\alpha} - \alpha_0 \|^2 = O_p(K^2m^2 \ln(K^2m) \ln(n)/n) + [Km\sqrt{\ln(K^2m)}]^{-2(\beta/d)+\varepsilon}. \]

Condition iii) that \( m \circ \mathcal{A} \) is representable as a neural network is satisfied in Example 1 when \( S(U) \) is bounded, Example 2 evidently, and Example 3 when \( U \) is bounded because a bounded random variable times a neural network is a neural network. When \( \alpha_0 \) is smooth enough, in that \( \beta \) is large enough, the upper bound on \( \| \hat{\alpha} - \alpha_0 \| \) in Corollary 2 gives a mean square convergence that can be close to, but less than \( n^{-1/2} \). Such can be obtained by choosing \( K \) and \( m \) to approximately balance the two terms in Corollary 2. As in Farrell, Liang, and Misra (2021) this
rate is not optimal for a given smoothness. An optimal rate could be obtained using the neural networks of Yarotsky (2017, 2018) or the sparse networks of Schmidt-Heiber (2020). We focus on Corollary 2 for an MLP neural network because that is readily applicable and because the rates obtained are fast enough for the estimators of the parameter of interest to be asymptotically normal.

It is straightforward to extend this result to cover specifications of $\Gamma$ other than the unrestricted, nonparametric $\alpha(x)$ case. For example when $\Gamma = \{a(X_1) + X_2b(X_1)\}$ for bounded $X_2$ the same rate given in Corollary 2 will hold for the neural net specified in equation (3.4).

There are many generalized regression learners where the $v_\rho(W)$ is unknown. The next result extends Theorem 1 to allow for estimated $v_\rho(W)$.

**Theorem 3:** If the conditions of Theorem 1 are satisfied and $\sum_{i=1}^n |\hat{v}_\rho(W_i) - v_\rho(W_i)| / n = O_p(\epsilon_{mn})$ then
\[
\|\hat{\alpha} - \alpha_0\|^2 = O_p(\delta_n^2 + \|\alpha^* - \alpha_0\|^2 + \frac{\ln(n)}{n} + \epsilon_{pn}).
\]

The $\epsilon_{pn}$ is a slow rate but is convenient in applying to a wide range of estimators $\hat{v}_\rho(w)$ of $v_\rho(w)$. The slow rate could be improved by using additional cross-fitting where $\hat{v}_\rho(w)$ is obtained using observations different than those used in the rest of $\hat{R}_\ell(\alpha)$ in equation (3.3). This additional sample splitting makes the estimator much more complicated so we reserve it to future work. For $\hat{v}_\rho(w)$ that requires nonparametric estimation $\epsilon_{pn}$ will converge slower than $n^{-1/2}$ making the convergence rate for $\hat{\alpha}$ slower than $n^{-1/4}$. Asymptotic normality of the parameter of interest will still be possible in this case but does require that $\hat{\gamma}$ converge faster than $n^{-1/4}$.

### 4.2 Large Sample Inference for $\theta_0$

We use additional regularity conditions to show asymptotic normality of $\hat{\theta}$ and consistency of the asymptotic variance estimator $\hat{V}$. We will first give a general result for $\hat{\theta}$ that applies to any $\hat{\alpha}_\ell$ and does not rely on Theorem 1 or Corollary 2 for a convergence rate for $\hat{\alpha}_\ell$. The next condition imposes a few continuity and regularity conditions on the residual $\rho(W, \gamma)$ and its conditional expectation. Let $\bar{\rho}(X, \gamma) = E[\rho(W, \gamma)|X]$.

**Assumption 3:** i) $\alpha_0(X)$ and $E[\rho(W, \gamma_0)^2|X]$ are bounded and $E[m(W, \gamma_0)^2] < \infty$; ii) $E[\{\rho(W, \gamma) - \rho(W, \gamma_0)^2\}] \to 0$ if $\|\gamma - \gamma_0\| \to 0$; iii) there is $C > 0$ such that for all $\|\gamma - \gamma_0\|$ small enough $E[\{\bar{\rho}(X, \gamma) - \bar{\rho}(X, \gamma_0)\}^2] \leq C \|\gamma - \gamma_0\|^2$.

The next condition imposes mean square consistency of $\hat{\gamma}_\ell$ and $\hat{\alpha}_\ell$, that the product of their mean-square convergence rates is smaller than $1/\sqrt{n}$, and a boundedness condition for $\hat{\alpha}_\ell$.

**Assumption 4:** i) $\|\hat{\gamma}_\ell - \gamma_0\| \overset{p}{\to} 0$ and $\|\hat{\alpha}_\ell - \alpha_0\| \overset{p}{\to} 0$; ii) $\sqrt{n} \|\hat{\gamma}_\ell - \gamma_0\| \|\hat{\alpha}_\ell - \alpha_0\| \overset{p}{\to} 0$; iii) $\hat{\alpha}_\ell(X)$ is bounded.
The results we have obtained for the neural net learner \( \hat{\alpha}_\ell \) can be used to verify these conditions and we do so in Corollary 5 to follow. The mean square convergence of \( \hat{\gamma}_\ell \) is a primitive condition for this paper and allows use of a wide variety of \( \hat{\gamma}_\ell \) in the construction of the estimator. The next condition allows for \( \rho(W, \gamma) \) to be nonlinear in \( \gamma \).

**Assumption 5:** Either \( \rho(W, \gamma) \) is affine in \( \gamma \) or \( n^{1/4} \| \hat{\gamma}_\ell - \gamma_0 \| \overset{p}{\to} 0 \) and there is \( C > 0 \) such that

\[
|E[m(W, \gamma) - \theta_0 + \alpha_0(X)\rho(W, \gamma)]| \leq C \| \gamma - \gamma_0 \|^2.
\]

This Assumption imposes the usual faster than \( n^{-1/4} \) convergence rate for \( \hat{\gamma}_\ell \) when \( \rho(w, \gamma) \) is nonlinear in \( \gamma \) but does not require that rate when \( \rho(W, \gamma) \) is linear in \( \gamma \). This condition combines a small remainder condition on \( E[\alpha_0(X)\rho(W, \gamma)] \) as a function of \( \gamma \) with existence of a Riesz representer \( v_m(X) \) satisfying \( E[m(W, \gamma)] = E[v_m(X)\gamma(X)] \). This feature of this Assumption makes it a little involved to verify but simple to state.

We have the following large sample inference result under these conditions.

**Theorem 4:** If Assumptions 1 and 3-5 are satisfied then for \( V = E[\{m(W, \gamma_0) - \theta_0 + \alpha_0(X)\rho(W, \gamma_0)\}]^2 \),

\[
\sqrt{n}(\hat{\theta} - \theta_0) \overset{p}{\to} N(0, V), \quad \hat{V} \overset{p}{\to} V.
\]

Next we use Corollary 2 and Theorem 3 to formulate regularity conditions when \( \hat{\alpha}_\ell \) is the neural net learner of \( \alpha_0 \) of Section 3. Let

\[
\epsilon_{an} = K^2 m^2 \ln(K^2 m) \ln(n)/n + [Km \sqrt{\ln(K^2 m)}/n]^{-2(\beta/d)+\varepsilon}.
\]

This \( \epsilon_{an} \) is taken from the upper bound for \( \| \hat{\alpha}_\ell - \alpha_0 \|^2 \) in Corollary 2 and so characterizes the mean square convergence rate of the automatic neural net learner \( \hat{\alpha}_\ell \) when \( v_\rho(w) \) is known. We also want to allow for estimated \( \hat{v}_\rho(w) \). The next condition does so, as well as giving a sufficient condition for Assumption 2 and rate conditions for asymptotic normality.

**Assumption 6:** i) \( |m(W, \alpha)| \leq C \| \alpha \|_\infty \); ii) \( \sum_{i=1}^{n} |\hat{v}_\rho(W_i) - v_\rho(W_i)|/n = o_p(\epsilon_{an}); \ iii) \| \hat{\gamma}_\ell - \gamma_0 \| \overset{p}{\to} 0, \epsilon_{an} \to 0, \epsilon_{\rho n} \to 0, \text{ and } \sqrt{n} \| \hat{\gamma}_\ell - \gamma_0 \| (\sqrt{\epsilon_{an} + \epsilon_{\rho n}}) \overset{p}{\to} 0.

This condition does allow for \( v_\rho(w) \) to be known in which case we can take \( \epsilon_{\rho n} = 0 \). Under this condition and others the learner \( \hat{\theta} \) will be asymptotically normal and \( \hat{V} \) will be a consistent estimator of the asymptotic variance.

**Corollary 5:** If Assumptions 1, 3, 5, and 6 and the hypotheses of Corollary 2 are satisfied then for the neural net \( \hat{\alpha}_\ell \) of Section 3 and \( V = E[\{m(W, \gamma_0) - \theta_0 + \alpha_0(X)\rho(W, \gamma_0)\}]^2 \),

\[
\sqrt{n} (\hat{\theta} - \theta_0) \overset{p}{\to} N(0, V), \quad \hat{V} \overset{p}{\to} V.
\]
It is straightforward to specify primitive conditions that are sufficient for the conditions of Corollary 5. We will illustrate with the inverse probability weighting in Example 3.

**Example 3:** The following result specifies conditions that are sufficient for those of Corollary 5. Since \( v_\rho(w) \) is known in this case we can take \( \epsilon_{\rho n} = 0 \).

**Corollary 6:** If i) \( U \) is bounded; ii) \( P(X) = \Pr(D = 1|X) \) is bounded away from zero; iii) \( \gamma_0 \) is bounded; iv) \( \alpha_0 \) satisfies the hypotheses of Corollary 2; v) \( \|\hat{\gamma}_\ell - \gamma_0\| \xrightarrow{p} 0, \epsilon_{an} \longrightarrow 0 \), and \( \sqrt{n}\|\hat{\gamma}_\ell - \gamma_0\| \xrightarrow{\epsilon_{an}} 0 \) then for \( V = \text{Var}(DU\gamma_0(X) + \alpha_0(X)\{1 - D\gamma_0(X)\}) \),

\[
\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{p} N(0, V), \quad \hat{V} \xrightarrow{p} V.
\]

## 5 Nonlinear Effects of Multiple Regressions

Some important objects of interest are expectations of nonlinear functionals of multiple regressions. In this Section we give Auto-DML for such effects. Such effects have the form \( \theta_0 = E[m(W, \gamma)] \) where \( m(w, \gamma) \) is nonlinear in a possible value \( \gamma \) of multiple generalized regressions \( (\gamma_1(X_1), ..., \gamma_J(X_J))' \) with regressors \( X_j \), residual \( \rho_j(W, \gamma_j) \), and \( \Gamma_j \) specific to each regression \( \gamma_j(X_j) \). The corresponding orthogonal moment functions are like those discussed in Section 3 except that the bias correction is a sum of \( J \) terms with the \( j \)-th term being the bias correction for the learner of \( \gamma_j \), as in Newey (1994, p. 1357). The estimated bias corrections are like those of Section 4 with the \( j \)-th term being the product of a learner \( \hat{\alpha}_j(\ell)(X_j) \) and the residual \( \rho_j(W, \hat{\gamma}_j) \).

Each \( \hat{\alpha}_{j\ell}(X_j) \) differs from Section 3 in way needed to account correctly for nonlinearity of \( m(W, \gamma) \) in \( \gamma \). The difference is that in the objective function for \( \hat{\alpha}_{j\ell}(X_j) \) the functional of interest \( m(w, \alpha) \) is replaced by an estimated Gateaux derivative with respect to the \( j \)-th component of \( \gamma \). Let

\[
\hat{D}_{j}(W_i, \alpha_j) = \frac{d}{d\tau} m(W_i, \gamma_{j\ell} + \tau e_j\alpha_j) \bigg|_{\tau=0}
\]

be such a Gateaux derivative where \( e_j \) denotes the \( j \)-th column of the identity matrix. This derivative will often be straightforward to calculate as an analytic derivative with respect to the scalar \( \tau \). When \( m(w, \gamma) \) is linear in a single \( \gamma \) this derivative just evaluates \( m(W_i, \gamma) \) at \( \gamma = \alpha \) giving the \( m(W_i, \alpha) \) of Section 3.

To obtain \( \hat{\alpha}_{j\ell}(X_j) \) we also make use of an estimated derivative \( \hat{v}_{\rho_j}(W_i) \) of \( \rho_j(W, \gamma_j) \) with respect to \( \gamma_j \) at \( \hat{\gamma}_j \). Then \( \hat{\alpha}_{j\ell} \) is given by

\[
\hat{\alpha}_{j\ell} = \arg\min_{\alpha_j \in \mathcal{A}} \frac{1}{n - n_\ell} \left\{ \sum_{\ell \neq \ell} \left[ -2\hat{D}_{j}(W_i, \alpha_j) - \hat{v}_{\rho_j}(W_i)\alpha_j(X_{ji})^2 \right] \right\},
\]
where $\mathcal{A}^j$ is the set of approximating functions for $\alpha_j$. As with linear $m(w, \gamma)$ this $\hat{\alpha}_{j\ell}$ depends just on $m(w, \gamma)$ and the first step. Thus $\hat{\alpha}_{j\ell}$ is automatic, in the same way as in Section 3, in only requiring $m(w, \gamma)$ and the regression residual $\rho_j(W_i, \gamma_j)$ for its construction.

It is straightforward to obtain a convergence rate for $\hat{\gamma}_j$ analogous to Theorem 3. The following result does so while accounting for the presence of $\hat{\gamma}_i$ in $\hat{D}_j(W_i, \alpha_j)$. For notational convenience we suppress the $j$ subscripts.

**Theorem 7:** If the conditions of Theorem 1 are satisfied, $\sum_{i=1}^n |\hat{\epsilon}_p(W_i) - v_p(W_i)|/n = o_p(\epsilon_{pm})$, and $\sup_\|x\| \leq 1 \sum_{i=1}^n |\hat{D}(W_i, \alpha) - D(W_i, \alpha)|/n = o_p(\epsilon_{mn})$ then

$$\|\hat{\alpha} - \alpha_0\|^2 = O_p(\delta_n^2 + \|\alpha^* - \alpha_0\|^2 + \frac{\ln(n)}{n} + \epsilon_{pm} + \epsilon_{pn}).$$

Similar to Theorem 3 the rate here is slower than could be obtained with additional cross fitting in the construction of $\hat{\epsilon}_p(W_i)$ and $\hat{D}(W_i, \alpha)$. For brevity we leave that to future work.

The construction of $\hat{\theta}$ is analogous to that in Section 3 with the bias correction term being the sum of terms for each $\gamma_j$ in $\gamma$. This construction is

$$\hat{\theta} = \frac{1}{n} \sum_{\ell=1}^L \sum_{i \in I_\ell} \left\{ m(W_i, \hat{\gamma}_\ell) + \sum_{j=1}^J \hat{\alpha}_{j\ell}(X_{ji}) \rho_j(W_i, \hat{\gamma}_{j\ell}) \right\},$$

$$\hat{V} = \frac{1}{n} \sum_{\ell=1}^L \sum_{i \in I_\ell} \hat{\psi}_{i\ell}^2, \ \hat{\psi}_{i\ell} = m(W_i, \hat{\gamma}_\ell) - \hat{\theta} + \sum_{j=1}^J \hat{\alpha}_{j\ell}(X_{ji}) \rho_j(W_i, \hat{\gamma}_{j\ell}).$$

It is straightforward to specify conditions for asymptotic normality of $\hat{\theta}$ and consistency of $\hat{V}$ by combining the conditions of Section 4 with the convergence rate result of Theorem 7. For relative simplicity we give a result only for neural net learners. We also assume for simplicity that each $X_j$ has the same dimension $d$. Let $\rho_j(X_j, \gamma_j) = E[\rho_j(W, \gamma_j)|X_j]$.

**Assumption 7:** $E[\{m(W_i, \gamma_0)\}^2] < \infty$ and for each $j$, i) $E[\rho_j(W_i, \gamma_j)|X_j]$ is bounded ii) $E[\{\rho_j(W_i, \gamma_j) - \rho_j(W_i, \gamma_0)\}^2] \rightarrow 0$ if $\|\gamma_j - \gamma_{j0}\| \rightarrow 0$; iii) there is $C > 0$ such that for all $\|\gamma_j - \gamma_{j0}\|$ small enough $E[\{\bar{\rho}_j(X_j, \gamma_j) - \bar{\rho}_j(X, \gamma_{j0})\}^2] \leq C \|\gamma_j - \gamma_{j0}\|^2$.

This condition is analogous to Assumption 3. For the next condition let $\epsilon_{an}$ be as specified in Section 4 where $d$ is the dimension of each $X_j$ and we assume that the smoothness index $\beta$ of $\alpha_{j0}(x)$ is the same for each $j$.

**Assumption 8:** For each $j = 1, \ldots, J$, i) $|D_j(W_i, \alpha_j)| \leq C \|\alpha_j\|_\infty$; ii) $\sum_{i=1}^n |\hat{\epsilon}_p(W_i) - v_p(W_i)|/n = o_p(\epsilon_{pm})$; iii) $\sup_\|\alpha_j\|_\infty \leq 1 \sum_{i=1}^n |\hat{D}_j(W_i, \alpha_j) - D_j(W_i, \alpha_j)|/n = o_p(\epsilon_{mn})$; iv) $\|\hat{\gamma}_\ell - \gamma_0\|_\infty \rightarrow 0$, $\epsilon_{an} \rightarrow 0$, $\epsilon_{pm} \rightarrow 0$, $\epsilon_{mn}$ and $\sqrt{n} \|\hat{\gamma}_\ell - \gamma_0\| (\sqrt{\epsilon_{an}} + \epsilon_{pm} + \epsilon_{mn}) \rightarrow 0$.  

19
This condition is analogous to Assumption 6.

**Assumption 9:** $n^{1/4} \| \hat{\gamma}_{j\ell} - \gamma_{j0} \| \to 0$ for each $j$ and there are $C, \varepsilon > 0$ such that

$$
\left| E[m(W, \gamma) - \theta_0 + \sum_{j=1}^J \alpha_{j0}(X_j)\rho(W, \gamma_j)] \right| \leq C \sum_{j=1}^J \| \gamma_j - \gamma_{j0} \|^2.
$$

This condition is analogous to Assumption 5. These conditions imply asymptotic normality of $\hat{\theta}$ and consistency of the asymptotic variance matrix $\hat{V}$.

**Theorem 8:** If Assumptions 7-9 are satisfied and the hypotheses of Corollary 2 are satisfied for each $\alpha_j$ then for $V = E[\{m(W, \gamma_0) - \theta_0 + \sum_{j=1}^J \alpha_{j0}(X_j)\rho_j(W, \gamma_j)\}^2]$, 

$$
\sqrt{n}(\hat{\theta} - \theta_0) \overset{p}{\to} N(0, V), \quad \hat{V} \overset{p}{\to} V.
$$

## 6 Proofs

**Proof of Theorem 1:** Let $\mathbb{E}_n[\cdot]$ denote the empirical expectation over a sample of size $n$, i.e.

$$
\mathbb{E}_n[Z] = \frac{1}{n} \sum_{i=1}^n Z_i,
$$

$$
L_n(\alpha) = \mathbb{E}_n[-2m(W; \alpha) - v_\rho(W)\alpha(X)^2],
$$

$$
L(\alpha) = E[-2m(W; \alpha) - v_\rho(W)\alpha(X)^2] = E[-2v_m(X)\alpha(X) - v_\rho(W)\alpha(X)^2].
$$

Note that

$$
\hat{\alpha} = \arg\min_{\alpha \in \mathcal{A}} L_n(\alpha).
$$

By $E[v_m(X)\alpha(X)] = E[-v_\rho(X)\alpha_0(X)\alpha(X)]$ for all $\alpha \in \mathcal{A} \subset \Gamma$ we have,

$$
L(\alpha) - L(\alpha_0) = E[-2v_m(X)\{\alpha(X) - \alpha_0(X)\} - v_\rho(X)\{\alpha(X)^2 - \alpha_0(X)^2\}]
$$

$$
= E[2v_\rho(X)\alpha_0(X)\{\alpha(X) - \alpha_0(X)\} - v_\rho(X)\{\alpha(X)^2 - \alpha_0(X)^2\}]
$$

$$
= E[-v_\rho(X)\{\alpha(X) - \alpha_0(X)\} \{\alpha(X) + \alpha(X) + \alpha(X)\}]
$$

$$
= E[-v_\rho(X)\{\alpha(X) - \alpha_0(X)\}^2] \leq C \| \alpha - \alpha_0 \|^2,
$$

where the last inequality follows by $-v_\rho(X) \leq C$.

Next, by Lemma 11 of Foster and Srygkanis (2019), the fact that $-2m(W, \alpha) - v_\rho(W)\alpha(X)^2$ is $4$-Lipschitz with respect to the vector $(m(W, \alpha), \alpha(X))$ and by our choice of $\delta := \delta_n + \varepsilon_0 \sqrt{\log(c_1/\zeta)/n}$, where $\delta_n$ is an upper bound on the critical radius of $\text{star}(\mathcal{A} - \alpha_0)$ and $\text{star}(m \circ \mathcal{A} - m \circ \alpha_0)$, with probability $1 - \zeta$,

$$
\forall \alpha \in \mathcal{A} : |L_n(\alpha) - L_n(\alpha_0) - (L(\alpha) - L(\alpha_0))| \leq O \left( \delta \left( \| \alpha - \alpha_0 \| + \sqrt{\mathbb{E}[m(W; \alpha) - m(W; \alpha_0)]^2} \right) + \delta^2 \right)
$$

$$
= O \left( \delta M \| \alpha - \alpha_0 \| + \delta^2 \right) := \epsilon_1.
$$
Also for $\alpha^* = \arg\min_{\alpha \in \mathcal{A}} L(\alpha)$ and by Bennet’s inequality and MSE-continuity we have that w.p. $1 - \zeta$:

$$
|L_n(\alpha^*) - L_n(\alpha_0) - (L(\alpha^*) - L(\alpha_0))| \leq O\left(\|\alpha^* - \alpha_0\| \sqrt{\frac{M \log (1/\zeta)}{n}} + \frac{\log (1/\zeta)}{n}\right) =: \epsilon_2
$$

In addition since $\hat{\alpha} = \arg\min_{\alpha \in \mathcal{A}} L_n(\alpha)$, we have that:

$$
L_n(\hat{\alpha}) - L_n(\alpha^*_0) \leq 0 \quad (6.1)
$$

Combining the previous inequalities it follows that

$$
\|\hat{\alpha} - \alpha_0\|^2 \leq C\{L(\hat{\alpha}) - L(\alpha_0)\} \leq C\{L_n(\hat{\alpha}) - L_n(\alpha_0) + \epsilon_1\}
\leq C\{L_n(\alpha^*) - L_n(\alpha_0) + \epsilon_1\} \leq C\{L(\alpha^*) - L(\alpha_0) + \epsilon_2 + \epsilon_1\}
= C\|\alpha^* - \alpha_0\|^2 + \epsilon_2 + \epsilon_1
$$

We can thus conclude that for some universal constant $C$:

$$
\|\hat{\alpha} - \alpha_0\|^2 \leq C\left(\delta M \|\alpha - \alpha_0\| + \delta^2 + \|\alpha^* - \alpha_0\|^2 + \|\alpha^* - \alpha_0\| \sqrt{\frac{M \log (1/\zeta)}{n}} + \frac{\log (1/\zeta)}{n}\right)
\leq \frac{1}{2}\|\alpha - \alpha_0\|^2 + 2C\left(\delta^2 M + \delta^2 + \|\alpha^* - \alpha_0\|^2 + \|\alpha^* - \alpha_0\| \sqrt{\frac{M \log (1/\zeta)}{n}} + \frac{\log (1/\zeta)}{n}\right)
$$

where we invoked the AM-GM inequality. Re-arranging, yields the proof. Q.E.D.

**Proof of Corollary 2:** An upper bound for the critical radius of a MLP neural net is given in equation (A.10) of Farrell, Liang, and Misra (2021, FLM). Using the fact that the number of parameters given there is bounded by $CK^2m$ it follows that

$$
\delta_n \leq C \sqrt{\frac{K^2m^2 \ln (K^2m) \ln (n)}{n}}, \quad (6.3)
$$

where $C$ denotes a generic positive constant. Let $\epsilon_n = \inf_{\alpha \in \mathcal{A}, x \in X} |\alpha(x) - \alpha_0(x)|$. It follows by the uniform approximating bounds given in FLM, in particular in the first inequality on the top of p. 206, that

$$
K^2m^2 \ln (K^2m) \leq C \epsilon_n^{-2d/\beta}(\ln (1/\epsilon_n) + 1)^7.
$$

It follows that for any $\epsilon > 0$ and $n$ large enough,

$$
\epsilon_n \leq C\{Km \sqrt{\ln (K^2m)}\}^{-\beta/d+\epsilon},
$$

where the presence of $\epsilon$ allows us to ignore the $(\ln (1/\epsilon_n) + 1)^7$ term. It follows that

$$
\|\alpha^* - \alpha_0\| \leq C\epsilon_n \leq C\{Km \sqrt{\ln (K^2m)}\}^{-\beta/d+\epsilon}. \quad (6.4)
$$
The conclusion then follows from Theorem 1 and squaring and plugging in the inequalities from equations (6.3) and (6.4). Q.E.D.

**Proof of Theorem 3**: Let

\[ \hat{L}_n(\alpha) = E_n[-2m(W; \alpha) - \tilde{v}_p(W)\alpha(X)^2]. \]

Then by the triangle inequality and Assumption 2,

\[
\sup_{\alpha \in A} \left| \hat{L}_n(\alpha) - L_n(\alpha) \right| = \sup_{\alpha \in A} \left| E_n[\left\{ \tilde{v}_p(W) - v_p(W) \right\}\alpha(X)^2] \right| \leq \sup_{\alpha \in A} E_n[|\tilde{v}_p(W) - v_p(W)|\alpha(X)^2],
\]

\[
\leq C E_n[|\tilde{v}_p(W) - v_p(W)|] = O_p(\epsilon_p).
\]

The proof then follows exactly as in the proof of Theorem 1 with \( \hat{L}_n(\hat{\alpha}) - \hat{L}_n(\alpha_0) \), \( \epsilon_1 + \epsilon_p \), \( \hat{L}_n(\alpha_s) - \hat{L}_n(\alpha_0) \), and \( \epsilon_2 + \epsilon_p \), replacing \( \hat{L}_n(\hat{\alpha}) - \hat{L}_n(\alpha_0) \), \( \epsilon_1 \), \( L_n(\alpha_s) - L_n(\alpha_0) \), and \( \epsilon_2 \) respectively in the proof and by letting \( \zeta \) shrink slowly enough that \( \ln(1/\zeta) \leq C \ln(n) \). Q.E.D.

**Proof of Theorem 4**: To show the first conclusion we verify Assumptions 1 - 3 of Chernozhukov, Escanciano, Ichimura, Newey, and Robins (2020, CEINR), with \( g(w, \gamma, \theta) \) and \( \phi(w, \gamma, \alpha, \theta) \) there given by \( m(w, \gamma) - \theta \) and \( \alpha(x)\rho(w, \gamma) \) respectively. By Assumption 1 and \( ||\hat{\gamma}_\ell - \gamma_0|| \xrightarrow{P} 0 \),

\[
\int \|g(w, \hat{\gamma}_\ell, \theta_0) - g(w, \gamma_0, \theta_0)\|^2 F_0(dw) = \int [m(w, \hat{\gamma}_\ell) - m(w, \gamma_0)]^2 F_0(dw) \leq C ||\hat{\gamma}_\ell - \gamma_0||^2 \xrightarrow{P} 0.
\]

Also by Assumption 3 i) and ii), \( \alpha_0(X) \) bounded and \( \rho(W, \gamma) \) mean square continuous in \( \gamma \) and \( ||\hat{\gamma}_\ell - \gamma_0|| \xrightarrow{P} 0 \),

\[
\int \|\phi(w, \hat{\gamma}_\ell, \alpha_0, \theta_0) - \phi(w, \gamma_0, \alpha_0, \theta_0)\|^2 F_0(dw)
\]

\[
\leq \int \alpha_0(x)^2[\rho(w, \hat{\gamma}_\ell) - \rho(w, \gamma_0)]^2 F_0(dw) \leq C \int [\rho(w, \hat{\gamma}_\ell) - \rho(w, \gamma_0)]^2 F_0(dw)
\]

\[
\leq C ||\hat{\gamma}_\ell - \gamma_0||^2 \xrightarrow{P} 0.
\]

Also by \( E[\rho(W, \gamma_0)^2|X] \) bounded and \( ||\hat{\alpha}_\ell - \alpha_0|| \xrightarrow{P} 0 \), iterated expectations gives

\[
\int \|\phi(w, \gamma_0, \hat{\alpha}_\ell, \tilde{\theta}_\ell) - \phi(w, \gamma_0, \alpha_0, \theta_0)\|^2 F_0(dw)
\]

\[
= \int [\hat{\alpha}_\ell(x) - \alpha_0(x)]^2 \rho(w, \gamma)^2 F_0(dw) \leq C ||\hat{\alpha}_\ell - \alpha_0||^2 \xrightarrow{P} 0.
\]

Therefore Assumption 1 parts i), ii), and iii) of CEINR is satisfied.

Next note that

\[
\hat{\Delta}_\ell(w) := \phi(w, \hat{\gamma}_\ell, \hat{\alpha}_\ell, \tilde{\theta}_\ell) - \phi(w, \gamma_0, \hat{\alpha}_\ell, \tilde{\theta}_\ell) - \phi(w, \hat{\gamma}_\ell, \alpha_0, \theta_0) + \phi(w, \gamma_0, \alpha_0, \theta_0)
\]

\[
= [\hat{\alpha}_\ell(x) - \alpha_0(x)]|\rho(w, \hat{\gamma}_\ell) - \rho(w, \gamma_0)|.
\]
Let \( \bar{\rho}(X, \gamma) = E[\rho(W, \gamma)|X] \). Then by iterated expectations, the Cauchy-Schwartz inequality, and Assumptions 3 and 4,

\[
\int \hat{\Delta}_\ell(w) F_0(dw) = \int [\hat{\alpha}_\ell(x) - \alpha_0(x)] [\bar{\rho}(x, \hat{\gamma}_\ell) - \bar{\rho}(x, \gamma_0)] F_0(dx)
\]

\[
\leq \|\hat{\alpha}_\ell - \alpha_0\| \|\bar{\rho}(\cdot, \hat{\gamma}_\ell) - \bar{\rho}(\cdot, \gamma_0)\| \leq C \|\hat{\alpha}_\ell - \alpha_0\| \|\hat{\gamma}_\ell - \gamma_0\| = o_p(\frac{1}{\sqrt{n}}).
\]

Also by \( \hat{\alpha}(x) \) and \( \alpha_0(x) \) bounded,

\[
\int \|\Delta_\ell(w)\|^2 F_0(dw) = \int [\hat{\alpha}_\ell(x) - \alpha_0(x)]^2 [\bar{\rho}(w, \hat{\gamma}_\ell) - \bar{\rho}(w, \gamma_0)]^2 F_0(dw)
\]

\[
\leq C \int [\bar{\rho}(w, \hat{\gamma}_\ell) - \bar{\rho}(w, \gamma_0)]^2 F_0(dw) \overset{p}{\to} 0,
\]

as in equation (6.6). By equations (6.8) and (6.9) it follows that Assumption 2 i) of CEINR is satisfied.

Next, Assumption 3 of CEINR follows by Assumption 5. Therefore each of Assumptions 1-3 of CEINR are satisfied, so the first conclusion follows by Lemma 15 of CEINR and the Lindberg-Levy central limit theorem.

Finally by the first conclusion \( \hat{\theta} \overset{p}{\to} \theta_0 \) so that \( \int \{m(w, \hat{\gamma}_\ell) - \hat{\theta} - m(w, \gamma_0) + \hat{\theta}\}^2 F_0(dw) \overset{p}{\to} 0 \) by Assumptions 1 and 3 so that the hypotheses of Lemma 16 of CEINR are satisfied, giving the second conclusion. Q.E.D.

**Proof of Corollary 5:** Note that Assumption 2 is satisfied by Assumption 6 i). Also, by Assumption 6 ii), Theorem 3, and Corollary 2 it follows that \( \|\hat{\alpha}_\ell - \alpha_0\| = O_p(\sqrt{\epsilon_{\alpha n} + \epsilon_{\rho n}}) \overset{p}{\to} 0 \). Then Assumption 4 is satisfied by Assumption 6 iii) and by \( \hat{\alpha}_\ell \) uniformly bounded. The conclusion then follows by Theorem 4. Q.E.D.

**Proof of Corollary 6:** Assumption 1 is satisfied by \( U \) bounded, by \( v_\rho(W) = -D \), and by \( E[v_\rho(W)|X] = -P(X) \) bounded away from zero. Assumption 2 is satisfied by \( U \) bounded. Assumption 3 i) is satisfied by \( U \) and \( \gamma_0(X) \) bounded. Also

\[
E[\{\rho(W, \gamma) - \rho(W, \gamma_0)\}^2] = E[D^2\{\gamma(X) - \gamma_0(X)\}^2] \leq \|\gamma - \gamma_0\|^2
\]

by \( |D| \leq 1 \), so Assumption 3 ii) is satisfied. Assumption 3 iii) also follows by this inequality. Assumption 5 is satisfied by \( \rho(W, \gamma) \) linear (affine) in \( \gamma \). Assumption 6 is satisfied by condition vi) of this Corollary and \( v_\rho(W) \) known so that \( \epsilon_{\rho n} = 0 \). The conclusion then follows by Corollary 5. Q.E.D.

**Proof of Theorem 7:** The proof follows analogously to the proof of Theorem 3 for

\[
\hat{L}_n(\alpha) = \mathbb{E}_n[-2\hat{D}(W; \alpha) - \hat{v}_\rho(W)\alpha(X)^2], \quad L_n(\alpha) = \mathbb{E}_n[-2D(W; \alpha) - v_\rho(W)\alpha(X)^2],
\]

23
Then by the triangle inequality and Assumption 2,
\[
\sup_{\alpha \in A} \left| \hat{L}_n(\alpha) - L_n(\alpha) \right| \leq \sup_{\alpha \in A} \left| \mathbb{E}_n[\{\hat{D}(W, \alpha) - D(W, \alpha)\}] \right| + \sup_{\alpha \in A} \left| \mathbb{E}_n[\{\hat{e}_\rho(W) - e_\rho(W)\} \alpha(X)^2] \right|
\leq \sup_{\alpha \in A} \mathbb{E}_n[\{\hat{D}(W, \alpha) - D(W, \alpha)\}] + O_p(\epsilon_m) = O_p(\epsilon_m + \epsilon_n).
\]
The remainder of the proof follows exactly as in Theorem 3 with \( \epsilon_m + \epsilon_n \) replacing \( \epsilon_n \). Q.E.D.

**Proof of Theorem 8:** It follows exactly as in the proof of Lemma 15 of CEINR that for each \( j \)
\[
\frac{1}{\sqrt{n}} \sum_{i \in I_i} [\hat{\alpha}_{j\ell}(X_{ji}) \rho_j(W_i, \hat{\gamma}_{j\ell}) - \alpha_{j0}(X_{ji}) \rho_j(W_i, \gamma_{j0})]
= \frac{1}{\sqrt{n}} \sum_{i \in I_i} [\hat{\alpha}_{j\ell}(X_{ji}) - \alpha_{j0}(X_{ji})] \rho(W_i, \gamma_{j0}) + \frac{1}{\sqrt{n}} \sum_{i \in I_i} \alpha_{j0}(X_{ji}) [\rho(W_i, \hat{\gamma}_{j\ell}) - \rho(W_i, \gamma_{j0})] + o_p(1)
= \frac{n_{\ell}}{\sqrt{n}} \int \alpha_{j0}(x_j)[\rho(w, \hat{\gamma}_{j\ell}) - \rho(w, \gamma_{j0})] F_0(dw) + o_p(1),
\]
\[
\frac{1}{\sqrt{n}} \sum_{i \in I_i} [m(W_i, \hat{\gamma}_{\ell}) - m(W_i, \gamma_{0})] = \frac{n_{\ell}}{\sqrt{n}} \int [m(w, \hat{\gamma}_{\ell}) - \theta_0] F_0(dw) + o_p(1).
\]
Also by Assumption 9 it is the case that \( \|\hat{\gamma}_{j\ell} - \gamma_{j0}\| < \varepsilon \) for all \( j \) with probability approaching one, so that by the triangle inequality and Assumption 9 iii) we have
\[
\left| \frac{1}{\sqrt{n}} \sum_{i \in I_i} [m(W_i, \hat{\gamma}_{\ell}) - \theta_0 + \sum_{j=1}^J \hat{\alpha}_{j\ell}(X_{ji}) \rho_j(W_i, \hat{\gamma}_{j\ell}) - \psi(W_i, \gamma_0, \alpha_0, \theta_0)] \right|
\leq \frac{n_{\ell}}{\sqrt{n}} \left| \int [m(w, \hat{\gamma}_{\ell}) - \theta_0 + \sum_{j=1}^J \alpha_{j0}(x_j) \rho_j(w, \hat{\gamma}_{j\ell})] F_0(dw) \right| + o_p(1)
\leq \sqrt{nC} \sum_{j=1}^J \|\hat{\gamma}_j - \gamma_{j0}\|^2 = \sqrt{n} \alpha_p((n^{-1/4})^2) = o_p(1),
\]
where
\[
\psi(w, \gamma_0, \alpha_0, \theta_0) := m(w, \gamma_0) - \theta_0 + \sum_{j=1}^J \alpha_{j0}(x_j) \rho_j(w, \gamma_{j0}).
\]
The first conclusion then follows by the triangle inequality and the central limit theorem. The second conclusion follows in analogous way, treating each \( j \) separately, using the arguments in Lemma 16 of CEINR. Q.E.D.
References

Ahn, H. and C.F. Manski (1993): “Distribution Theory for the Analysis of Binary Choice under Uncertainty with Nonparametric Estimation of Expectations,” *Journal of Econometrics* 56, 291–321.

Athey, S., G. Imbens, and S. Wager (2018): “Approximate Residual Balancing: Debias Inference of Average Treatment Effects in High Dimensions,” *Journal of the Royal Statistical Society, Series B* 80, 597–623.

Avagyan, V. and S. Vansteelandt (2017): ”Honest data-adaptive inference for the average treatment effect under model misspecification using penalised bias-reduced double-robust estimation,” https://arxiv.org/abs/1708.03787.

Belloni, A., D. Chen, and V. Chernozhukov (2012): “Sparse Models and Methods for Optimal Instruments with an Application to Eminent Domain,” *Econometrica* 80, 2369–429.

Belloni, A. and V. Chernozhukov (2013): ”Least Squares After Model Selection in High-dimensional Sparse Models,” *Bernoulli* 19, 521–547.

Belloni, A., V. Chernozhukov, and C. Hansen (2014a): “Inference on Treatment Effects after Selection among High-Dimensional Controls,” *Review of Economic Studies* 81, 608–650.

Belloni, A., V. Chernozhukov, L. Wang (2014b): “Pivotal Estimation via Square-Root Lasso in Nonparametric Regression,” *Annals of Statistics* 42, 757–788.

Belloni, A., V. Chernozhukov, K. Kato (2015): ”Uniform Post-selection Inference for Least Absolute Deviation Regression and Other Z-estimation Problems,” *Biometrika* 102, 77–94.

Bickel, P.J. (1982): “On Adaptive Estimation,” *Annals of Statistics* 10, 647–671.

Bickel, P.J. and Y. Ritov (1988): “Estimating Integrated Squared Density Derivatives: Sharp Best Order of Convergence Estimates,” *Sankhyā: The Indian Journal of Statistics, Series A* 238, 381–393.

Bickel, P.J., C.A.J. Klaassen, Y. Ritov and J.A. Wellner (1993): *Efficient and Adaptive Estimation for Semiparametric Models*, Baltimore: Johns Hopkins University Press.

Bickel, P.J., Y. Ritov, and A. Tsybakov (2009): “Simultaneous Analysis of Lasso and Dantzig Selector,” *Annals of Statistics* 37, 1705–1732.

Blundell, R.W. and J.L. Powell (2004): ”Endogeneity in Binary Response Models,” *Review of Economic Studies* 71, 655-679.

Bradic, J. and M. Kolar (2017): “Uniform Inference for High-Dimensional Quantile Regression: Linear Functionals and Regression Rank Scores,” arXiv:1702.06209.

Bradic, J., S. Wager, and Y. Zhu (2019): ”Sparsity Double Robust Inference of Average Treatment Effects,” https://arxiv.org/pdf/1905.00744.pdf.

Bradic, J., V. Chernozhukov, W. Newey, and Y. Zhu (2019): ”Minimax Semiparametric Learning with Approximate Sparsity,” arXiv.
Cai, T.T. and Z. Guo (2017): “Confidence Intervals for High-Dimensional Linear Regression: Minimax Rates and Adaptivity,” *Annals of Statistics* 45, 615-646.

Candes, E. and T. Tao (2007): “The Dantzig Selector: Statistical Estimation when $p$ is much Larger than $n,” *Annals of Statistics* 35, 2313–2351.

Chatterjee, S. and J. Jafarov (2015): “Prediction Error of Cross-Validated Lasso,” arXiv:1502.06291.

Chen, X. and H. White (1999): ”Improved Rates and Asymptotic Normality for Nonparametric Neural Network Estimators,” *IEEE Transactions on Information Theory* 45, 682-691.

Chernozhukov, V., D. Chetverikov, and K. Kato (2013a): “Gaussian Approximations and Multiplier Bootstrap for Maxima of Sums of High-Dimensional Random Vectors,” *Annals of Statistics* 41, 2786–2819.

Chernozhkov, V., C. Hansen, and M. Spindler (2015): “Valid Post-Selection and Post-Regularization Inference: An Elementary, General Approach,” *Annual Review of Economics* 7, 649–688.

Chernozhukov, V., J. C. Escanciano, H. Ichimura, W.K. Newey, and J. Robins (2016): “Locally Robust Semiparametric Estimation,” https://arxiv.org/abs/1608.00033v1.

Chernozhukov, V., D. Chetverikov, M. Demirer, E. Duflo, C. Hansen, W.K. Newey (2017): ”Double/Debiased/Neyman Machine Learning of Treatment Effects,” American Economic Review 107, 261-65.

Chernozhukov, V., D. Chetverikov, M. Demirer, E. Duflo, C. Hansen, W.K. Newey, J.M. Robins (2018): ”Double/debiased machine learning for treatment and structural parameters,” Econometrics Journal 21, C1-C68.

Chernozhukov, V., W.K. Newey, and J. Robins (2018): “Double/De-Biased Machine Learning Using Regularized Riesz Representers,” https://arxiv.org/pdf/1802.08667v1.pdf.

Chernozhukov, V., W.K. Newey, and R. Singh (2018): ”Learning L2-Continuous Regression Functionals via Regularized Riesz Representers,” https://arxiv.org/pdf/1809.05224v1.pdf.

Chernozhukov, V., W.K. Newey, and R. Singh (2019): ”Double/De-Biased Machine Learning of Global and Local Parameters Using Regularized Riesz Representers,” https://arxiv.org/abs/1802.08667v1.

Chernozhukov, V., J.A. Hausman, W.K. Newey (2019): ”Demand Analysis with Many Prices,” NBER Working Paper 26424.

Chernozhukov, V., J. C. Escanciano, H. Ichimura, W.K. Newey, and J. Robins (2020): “Locally Robust Semiparametric Estimation,” https://arxiv.org/abs/1608.00033v4.

Daubechies, I., M Defrise, and C. De Mol (2004): “An Iterative Thresholding Algorithm for Linear Inverse Problems with a Sparsity Constraint,” *Communications on Pure and Applied Mathematics* 57, 1413–57.

Farbmacher, M., M. Huber, L. Lafférs, H. Langen, M. Spindler (2020): ”Causal Mediation Analysis with Double Machine Learning,” https://arxiv.org/abs/2002.12710.

Farrell, M. (2015): “Robust Inference on Average Treatment Effects with Possibly More
Covariates than Observations,” *Journal of Econometrics* 189, 1–23.

Farrell, M., T. Liang, S. Misra (2020): ”Deep Learning for Individual Heterogeneity,” https://arxiv.org/abs/2010.14694.

Farrell, M., T. Liang, S. Misra (2021): ”Deep Neural Networks for Estimation and Inference,” *Econometrica* 89, 181–213.

Foster, D.J. and V. Srygkanis (2019): ”Orthogonal Learning,” https://arxiv.org/abs/1901.09036

Hasminskii, R.Z. and I.A. Ibragimov (1979): “On the Nonparametric Estimation of Functionals,” in P. Mandl and M. Huskova (eds.), *Proceedings of the 2nd Prague Symposium on Asymptotic Statistics, 21-25 August 1978*, Amsterdam: North-Holland, pp. 41-51.

Hirshberg, D.A. and S. Wager (2017): “Balancing Out Regression Error: Efficient Treatment Effect Estimation without Smooth Propensities,” arXiv:1712.00038v1.

Hirshberg, D.A. and S. Wager (2020): “Debiased Inference of Average Partial Effects in Single-Index Models,” *Journal of Business and Economic Statistics* 38, 19-24.

Hirshberg, D.A. and S. Wager (2018): “Augmented minimax linear estimation,” arXiv:1712.00038v5.

Ichimura and Newey (2021): ”The Influence Function of Semiparametric Estimators,” working paper.

Imai, K, L. Keele, and D. Tingley (2010): ”A General Approach to Causal Mediation Analysis,” *Psychological Methods* 15, 309 –334.

Imbens, G.W. and W.K. Newey (2009): ”Identification and Estimation of Triangular Simultaneous Equations Models Without Additivity,” *Econometrica* 77, 1481-1512.

Jankova, J. and S. Van De Geer (2015): “Confidence Intervals for High-Dimensional Inverse Covariance Estimation,” *Electronic Journal of Statistics* 90, 1205–1229.

Jankova, J. and S. Van De Geer (2016a): “Semi-Parametric Efficiency Bounds and Efficient Estimation for High-Dimensional Models,” arXiv:1601.00815.

Jankova, J. and S. Van De Geer (2016b): “Confidence Regions for High-Dimensional Generalized Linear Models under Sparsity,” arXiv:1610.01353.

Javanmard, A. and A. Montanari (2014a): “Hypothesis Testing in High-Dimensional Regression under the Gaussian Random Design Model: Asymptotic Theory,” *IEEE Transactions on Information Theory* 60, 6522–6554.

Javanmard, A. and A. Montanari (2014b): “Confidence Intervals and Hypothesis Testing for High-Dimensional Regression,” *Journal of Machine Learning Research* 15: 2869–2909.

Javanmard, A. and A. Montanari (2015): “De-Biasing the Lasso: Optimal Sample Size for Gaussian Designs,” arXiv:1508.02757.

Jing, B.Y., Q.M. Shao, and Q. Wang (2003): “Self-Normalized Cramér-Type Large Deviations for Independent Random Variables,” *Annals of Probability* 31, 2167–2215.

Kennedy, E.H. (2020): ”Optimal Doubly Robust Estimation of Heterogeneous Causal Effects,” https://arxiv.org/pdf/2004.14497.pdf.
Klaassen, C.A.J. (1987): "Consistent Estimation of the Influence Function of Locally Asymptotically Linear Estimators," *Annals of Statistics* 15, 1548-1562.

Leeb, H., and B.M. Pötscher (2008a): “Recent Developments in Model Selection and Related Areas,” *Econometric Theory* 24, 319–22.

Leeb H., and B.M. Pötscher (2008b): “Sparse Estimators and the Oracle Property, or the Return of Hodges’ Estimator,” *Journal of Econometrics* 142, 201–211.

Luo, Ye and M. Spindler (2016): "High-Dimensional L2 Boosting: Rate of Convergence," https://arxiv.org/pdf/1602.08927.pdf.

Luedtke, A. R. and M. J. van der Laan (2016): “Optimal Individualized Treatments in Resource-limited Settings,” *The International Journal of Biostatistics* 12, 283-303.

Nelder, J. and R. Wedderburn (1972): "Generalized Linear Models,” *Journal of the Royal Statistical Society. Series A* 135, 370–384.

Newey, W.K. (1994): “The Asymptotic Variance of Semiparametric Estimators,” *Econometrica* 62, 1349–1382.

Newey, W.K., F. Hsieh, and J.M. Robins (1998): “Undersmoothing and Bias Corrected Functional Estimation,” MIT Dept. of Economics working paper 98-17.

Newey, W.K., F. Hsieh, and J.M. Robins (2004): “Twicing Kernels and a Small Bias Property of Semiparametric Estimators,” *Econometrica* 72, 947–962.

Newey, W.K. and J.M. Robins (2017): “Cross Fitting and Fast Remainder Rates for Semiparametric Estimation,” arXiv:1801.09138.

Neykov, M., Y. Ning, J.S. Liu, and H. Liu (2015): “A Unified Theory of Confidence Regions and Testing for High Dimensional Estimating Equations,” arXiv:1510.08986.

Ning, Y. and H. Liu (2017): “A General Theory of Hypothesis Tests and Confidence Regions for Sparse High Dimensional Models,” *Annals of Statistics* 45, 158-195.

Ren, Z., T. Sun, C.H. Zhang, and H. Zhou (2015): “Asymptotic Normality and Optimalities in Estimation of Large Gaussian Graphical Models,” *Annals of Statistics* 43, 991–1026.

Robins, J.M. and A. Rotnitzky (1995): “Semiparametric Efficiency in Multivariate Regression Models with Missing Data,” *Journal of the American Statistical Association* 90 (429): 122–129.

Robins, J.M., A. Rotnitzky, and L.P. Zhao (1995): “Analysis of Semiparametric Regression Models for Repeated Outcomes in the Presence of Missing Data,” *Journal of the American Statistical Association* 90, 106–121.

Robins, J., P. Zhang, R. Ayyagari, R. Logan, E. Tchetgen, L. Li, A. Lumley, and A. van der Vaart (2013): “New Statistical Approaches to Semiparametric Regression with Application to Air Pollution Research,” Research Report Health E Inst.

Rosenbaum, P.R. and D. B. Rubin (1983): “The Central Role of the Propensity Score in Observational Studies for Causal Effects,” *Biometrika* 70: 41–55.
Rothenhäusler, D. and B. Yu (2019): “Incremental Causal Effects,” arXiv:1907.13258.
Rudelson, M. and S. Zhou (2013): ”Reconstruction From Anisotropic Random Measurements,” IEEE Transactions on Informating Theory 59, 3434–3447.
Schick, A. (1986): “On Asymptotically Efficient Estimation in Semiparametric Models,” Annals of Statistics 14, 1139–1151.
Schmidt-Hieber, J. (2020): ”Nonparametric Regression Using Deep Neural Networks with RELU Activation Function,” The Annals of Statistics 48, 1875–1897.
Singh, R. and L. Sun (2019): “De-biased Machine Learning for Compliers,” arXiv:1909.05244.
Singh, R., L. Xu, and A. Gretton (2020): ”Kernel Methods for Nonparametric Treatment Effects,” draft.
Stock, J.H. (1989): “Nonparametric Policy Analysis,” Journal of the American Statistical Association 84, 567–575.
Syrgkanis, V., and M. Zampetakis (2020): ”Estimation and Inference with Trees and Forests in High Dimensions,” https://arxiv.org/abs/2007.03210.
Tchetgen Tchetgen, E.J. and I. Shipster (2012): ”Semiparametric Theory for Causal Mediation Analysis: Efficiency Bounds, Multiple Robustness and Sensitivity Analysis,” The Annals of Statistics 40, 1816-1845.
Toth, B. and M. J. van der Laan (2016), “TMLE for Marginal Structural Models Based On An Instrument,” U.C. Berkeley Division of Biostatistics Working Paper Series, Working Paper 350.
Tseng, P. (2001): “Convergence of a Block Coordinate Descent Method for Nondifferentiable Minimization,” Journal of Optimization Theory and Applications 109, 475–94.
Van De Geer, S., P. Bühlmann, Y. Ritov, and R. Dezeure (2014): “On Asymptotically Optimal Confidence Regions and Tests for High-Dimensional Models,” Annals of Statistics, 42: 1166–1202.
Van der Laan, M. and D. Rubin (2006): “Targeted Maximum Likelihood Learning,” International Journal of Biostatistics 2.
Van der Laan, M. J. and S. Rose (2011): Targeted Learning: Causal Inference for Observational and Experimental Data, Springer.
Van der Vaart, A.W. (1991): “On Differentiable Functionals,” Annals of Statistics, 19: 178–204.
Van der Vaart, A.W. (1998): Asymptotic Statistics. New York: Cambridge University Press.
Vermeulen, K. and S. Vansteelandt (2015): ”Bias-Reduced Doubly Robust Estimation,” Journal of the American Statistical Association 110, 1024-1036.
Vershynin, R. (2018): High-Dimensional Probability, New York: Cambridge University Press.
White, H. (1982): ”Maximum Likelihood Estimation of Misspecified Models,” Econometrica 50, 1-25.
Wooldridge, J.M. (2002): *Econometric Analysis of Cross-Section and Panel Data*, Cambridge, MIT Press.

Wooldridge, J.M. (2019): “Correlated Random Effects Models with Unbalanced Panels,” *Journal of Econometrics* 211, 137–50.

Wooldridge, J.M. and Y. Zhu (2020): “Inference in Approximately Sparse Correlated Random Effects Probit Models With Panel Data,” *Journal of Business and Economic Statistics* 38, 1-18.

Yarotsky, D. (2017): “Error Bounds for Approximations With Deep ReLU Networks,” *Neural Networks* 94, 103–114. [184,189,191,206,208]

Yarotsky, D. (2018): “Optimal approximation of continuous functions by very deep ReLU networks,” in 31st Annual Conference on Learning Theory 639–649. [184,189,192,206]

Zhang, C. and S. Zhang (2014): “Confidence Intervals for Low-Dimensional Parameters in High-Dimensional Linear Models,” *Journal of the Royal Statistical Society, Series B* 76, 217–242.

Zheng, W., Z. Luo, and M. J. van der Laan (2016), “Marginal Structural Models with Counterfactual Effect Modifiers,” U.C. Berkeley Division of Biostatistics Working Paper Series, Working Paper 348.

Zhu, Y. and J. Bradic (2017a): “Linear Hypothesis Testing in Dense High-Dimensional Linear Models,” *Journal of the American Statistical Association* 112.

Zhu, Y. and J. Bradic (2017b): “Breaking the Curse of Dimensionality in Regression,” arXiv: 1708.00430.

Zubizarreta, J.R. (2015): “Stable Weights that Balance Covariates for Estimation with Incomplete Outcome Data,” *Journal of the American Statistical Association* 90 (429): 122–129.