On the algebraic structures connected with the linear Poisson brackets of hydrodynamics type

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Abstract

The generalized form of the Kac formula for Verma modules associated with linear brackets of hydrodynamics type is proposed. Second cohomology groups of the generalized Virasoro algebras are calculated. Connection of the central extensions with the problem of quantization of hydrodynamics brackets is demonstrated.

1. Poisson brackets of hydrodynamics type (PBHT)

\[ \{ u^i(x), u^j(y) \} = g^{ij}(u(x))\delta'(x - y) + u^k b^{ij}_k(u(x))\delta(x - y) \quad (1) \]

(here and below we assumed a summation on repeat indexes) were introduced and studied in [1, 2] to construct a theory of conservative systems of hydrodynamics type and to develop a Bogolubov-Whitham method of averaging Hamiltonian field-theoretic systems. We refer to the recent expository article [3] and the extensive bibliography therein. In [4] S P Novikov and first author considered and gave a classification of these Poisson brackets depending linearly on the fields \( u^j \) relative to linear change \( u^k = A^k_j u^j \). Some examples were discussed in [5, 6].

For the reader’s convenience we recall some construction from [4]. The simplest local Lie algebras arising from the brackets of hydrodynamics type are especially interesting, where, according to [1], in the case when all metrics are linear in \( u \) we have

\[
\begin{align*}
g^{ij} &= C^{ij}_k u^k + g_0^{ij} \\
b^{ij}_k &= \text{const}, \quad g_0^{ij} = \text{const} \\
C^{ij}_k &= b^{ij}_k + b^{ji}_k.
\end{align*} \quad (2)
\]

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Linear (homogeneous) part of such PBHT determines some very interesting class of the infinite-dimensional Lie algebras ("hydrodynamic algebras"): for two vector-functions \( f(x) \) and \( g(x) \) with \( N \) components \( f_p, g_q \) we may define the commutator in the "local translation-invariant first-order Lie algebra" or the hydrodynamic algebra

\[
[f, g]_k(x) = b_{k}^{pq}(f'_p(x)g_q(x) - g'_q(x)f_p(x)).
\] (3)

A bracket (1) or Lie algebra (3), linear in the fields, is called symmetric if \( b_{k}^{ij} = b_{k}^{ji} \). Here \( f_p \) are adjoint variables for \( u^i \).

It is useful to introduce a new algebra \( B \) as a multiplication in \( N \)-space \( M \) with basis \( e^1, e^2, \ldots, e^N \)

\[
e^i e^j = b_{k}^{ij} e^k. \] (4)

For the functions \( f(x) = f_p(x)e^p \) and \( g(x) = g_q(x)e^q \) we write (3) in the form \( f' g - g' f \) using multiplication (4) in the algebra \( B \). The tensor \( b_{k}^{ij} \) defines by (3) a local translationally invariant Lie algebra of first order if and only if the multiplication law (4) defines a finite-dimensional algebra \( B \) in which the following identities hold:

\[
a, b, c \in B, \quad (ab)c = (ac)b, \quad (ab)c - a(bc) = (ba)c - b(ac). \] (5)

In the symmetric case \( 2b_{k}^{ij} = 2b_{k}^{ji} = C_{k}^{ij} \) this algebra is commutative and associative.

**Remark** After introducing an operation \( [a, b] = ab - ba \) on \( B \) it is Lie algebra and as was first proved in [2] this is a solvable Lie algebra. If a finite-dimensional algebra \( B \) (4),(3) is commutative then it is automatically associative, and if it has the right unit then it’s commutative. The theory of extensions for the algebras \( B \) (4), (3) was constructed in [4, 8]. The method of construction the wide class of such algebras was proposed by S I Gelfand: if algebra \( A \) is commutative and associative, \( \partial \) - differentiation of \( A \) then multiplication \( a \circ b = a(\partial b) \) satisfied (3).

The formula (4) define Poisson bracket if and only if an algebra (4) satisfied (3) and for the following symmetric bilinear product

\[
(e^i, e^j)_0 = g_{0}^{ij}
\]

we have

\[
(ab, c)_0 = (a, cb)_0, \] (6)

for all \( a, b, c \in B \). In this case the "quasifrobenius property" (3) hold true for all symmetric bilinear products with matrix \( g^{ij}(u) = C_{k}^{ij} u^k + g_{0}^{ij} \) for any \( u^i \). If the algebra \( B \) is commutative and has the unit then we have the classical frobenius algebra.

Poisson bracket (4) is called nondegenerate if the pseudo-Riemannian metric \( g^{ij} = C_{k}^{ij} u^k + g_{0}^{ij} \) is nondegenerate at a "generic point":

\[
det(g^{ij}) := P_N(u^1, u^2, \ldots u^N) \neq 0. \]
Lie algebra (3) and a finite-dimensional algebra $B$ (5) are called nondegenerate if the pseudo-Riemannian metric $g^{ij} = C_{ij}^k u^k$ is nondegenerate at a "generic point", where $b_k^{ij} + b_k^{ji} = C_k^{ij}$. From the main theorem of [1] we know that in a nondegenerate case these pseudo-Riemannian metrics have a vanishing curvature.

2. For the vector-valued functions periodic in $x$, by passing to an expansion in Fourier series, we obtain a basis $(L_n^i)$ for the algebra (3) with the relations

$$[L_n^i, L_m^j] = (nb_k^{ij} - mb_k^{ji}) L_n^{k} L_{m+n}^k,$$

which we call the generalized Witt algebra. For this algebra we have the following generalization of the Gelfand–Fuks theorem [9] on the central extension of the algebra of vector fields on the circle:

**Theorem 1** If the algebra (4) is commutative and has the unit then

$$H^2(V) = B^*,$$

where $V$ - algebra (4) and $H^2$ - second cohomology group of Lie algebra and $B^*$ is the dual space for the algebra (4). The all central extensions of (4) have the following form

$$[L_n^i, L_m^j] = (n - m)b_k^{ij} L_n^{k} L_{m+n}^k + b_k^{ij} l^k (n^3 - n) \frac{12}{\delta_{n+m,0}} Z,$$

where $Z$ - central element and $l = (l_i) \in B^*$.

In the case when $B = \mathbb{C}$ (algebra of complex numbers) we have Gelfand–Fuks theorem.

For the algebra (8) we may consider Verma module $V_{h,l}$, $h, l \in B^*$, over this algebra: $V_{h,l}$ free generate by $| v >$ over $L_n^i$ with $n > 0$, $i = 1, \ldots, N$ and

$$L_n^i | v > = 0, \quad n < 0;$$

$$L_0^i | v > = h(e_i) | v >;$$

$$Z | v > = | v >.$$

An element of the Verma module is singular if it is generates the Verma submodule, i.e. it is eigenvector for all $L_0^i$, and annihilated by the all $L_n^i$, $n < 0$.

**Theorem 2** For the algebra (8) with the unital $B$ Verma module is reducible if and only if it has the singular vector.

When algebra (4) is commutative and has the unit then for (8) from the root decomposition of $B$ we have the following analog of the famous Kac–Feigin–Fuks criteria:
Theorem 3  Verma module $V_{h,l}$ is reducible if and only if in the algebra $B$ (4) exist one-dimensional ideal $< a >$ such that if $a^2 = 0$ then $h(a) = l(a)(a^2 - 1)/2$ for some $\alpha \in N_+$, or if $aa = \mu a$ with $\mu \in C$, $\mu \neq 0$ then $\mathcal{R} := h(a)/\mu$, $\mathcal{C} := l(a)/\mu$ satisfied to Kac condition [10] for the ordinary Virasoro algebra.

3. In this part we consider the problem of quantization of the PBHT (1). Its decision may be find by means of the change of variables $u = u(v)$ such that in the new variables $v$ bracket (4) is constant. After canonical quantizations of the constant bracket we can come back to the old variables $u$. But we run up against the problem of ordering. Thus we need the simplest as it is possible change of the variables. The linear change is not suitable. Since according to [1] in the case of nondegenerate algebras (3),(4) the metric $g^{ij} = C^{ij}_k u^k$ must have zero curvature, we appeal to changes $u(v)$, which are now nonlinear, where metric in the new coordinates $(v^1, \ldots, v^N)$ is constant

$$g^{ij}(u(v)) = g^{\alpha\beta}_0 (\partial u^i/\partial v^\alpha)(\partial u^j/\partial v^\beta), \quad g^{\alpha\beta}_0 = \text{const}.$$ 

We consider the purely quadratic changes

$$u^i = \frac{1}{2} F^{i\alpha\beta} v^\alpha v^\beta. \quad (9)$$

Then for a change (4) to reduce the nondegenerate metric of zero curvature $g^{ij} = C^{ij}_k u^k$ ( from PBHT ) to constant form it is necessary and sufficient that the following conditions hold

- $b^{ij}_k = b^{ji}_k$;
- $F$ and $g^{ij}_0$ determine a Frobenius representation of the algebra (3), where the $F^{i\alpha\gamma}_0$ give a representation of the basis $e^i$ of the algebra in the form of linear operator in $v$-space which are selfadjoint in this inner product, so that

$$e^i \rightarrow (F^i)^\alpha_\beta = g^{\alpha\gamma}_0 F^{i\gamma}_\beta, \quad F^i F^j = C^{ij}_k F^k / 2 \quad \det(F^{i\alpha\beta}) \neq 0).$$

Thus if the algebra $B$ (4) is commutative and nondegenerate (in this case it has the unit) by the quadratic changes we obtained constant bracket. After the linear change of variables from $v$ to $(\phi)$ we have

$$\{(\phi^i(x))', (\phi^j(y))'\} = \delta^{ij}(x - y).$$

Dirac quantization lead to the theory of free fields in 2D quantum field theory, and the Fourier components of $(\phi)'$ form the following famous algebra($i = 1, \ldots, N, \ s \in Z$):

$$[a^i(s), a^j(k)] = k \delta^{ij} \delta_{s+k,0}.$$
After choice the ordering procedure for $a^i(s)$ we obtain for (7) not algebra (7), but the algebra (8), and the ordering procedures are in the one to one correspondence with central extensions of (7). Thus algebra (8) is the quantization of PBHT (1) in the case when algebra (4) is symmetric and nondegenerate.

**Remark** It will be very interesting investigate the algebras (4) and PBHT with another type of the change of variables $u = u(v)$ such that in the new variables $v$ bracket (1) is constant. Some examples of such noncommutative algebras (for the Poisson brackets of one-dimensional hydrodynamics) were investigated in [4].

We are very grateful to Professor S P Novikov for numerous discussion and suggestion.
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