Loosely displaced formation-keeping control for satellite swarm with continuous low-thrust

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Abstract
Aiming at a long-term formation-keeping problem for the satellite swarm, the concept of a loosely displaced formation is proposed in this article. On this basis, a continuous low-thrust control strategy for maintaining the loosely displaced formation is designed. The control objective is to reduce more fuel consumption during the formation-keeping. For achieving that, we proposed a forward-feedback control strategy by using pseudo-spectral method and sliding mode theory. To be specific, the control strategy includes two parts: a forward control and a feedback control. For the forward control, a numerical optimization with the Legendre pseudo-spectral method is attempted to convert the optimal control problem into a nonlinear programming problem and fuel consumption is selected as the optimization index. For stability issue, the feedback control via adaptive finite-time sliding mode theory is introduced as an additional control component. Finally, the numerical results demonstrate that propellant mass is effectively saved as well as the formation can be tracked accurately with this control strategy proposed in this article.

Keywords
Satellite formation-keeping, loosely displaced formation, pseudo-spectra method, finite-time stable, sliding mode control

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Introduction
Satellite swarm, as an emerging concept of distributed space system, is composed of several or more physically independent, “free-flying” satellites that collaborate on orbit. As compared to a monolithic satellite, the swarm can collectively achieve a certain level of system-wide functionality. Formation-keeping is one of the most key technologies for the satellite swarm, which requires that the satellites could adjust their relative motion to shape a desired configuration and maintain it steadily.

For keeping the formation, a relative motion equation is required for describing the relative motion between satellites. So far, varieties of relative motion equations were built by the researchers and the linearized C-W equation is the most commonly used form. By utilizing the analytical solution of the C-W equation, Schaub et al.,6 Lane and Axelrad,7 and Inalhan et al.8 designed quantities of particular formation configuration, such as the in-plane formation, the in-track formation, the circular formation, and the projected circular formation. On this basis, Schaub and
Alfriend, Schweighart, and Li et al. took the effect of $J_2$ perturbation into account and designed the formations under $J_2$ perturbation. However, in reality, C-W equation is merely suitable for close formations as well as there exist varieties of uncertain perturbations in space, hence, the formation configuration designed in previous studies cannot be maintained enduringly without providing external thrust.

With the wide application of the continuous low-thrust propulsion system, a variety of control methods for the satellite formation flying have been proposed by using the modern control theory, such as the adaptive control, the model predictive control, the sliding model techniques-based nonlinear tracking control, and the robust control. In addition, some theoretical methods in other areas are also helpful to this control problem.

The fuel consumption is a major concern for the formation-keeping mission, especially for a long-term one. All the methods mentioned above didn’t take the fuel consumption of the satellite into account. For saving the fuel consumption, a linear quadratic regulator (LQR) was used in the studies of Xu and Hu, Palacios et al., and Ke et al., however, LQR merely applies to the linear system and its convergence rate is slow. Wu et al. proposed a formation-keeping control scheme via real-time optimal control and iterative learning control, in which the optimal control via pseudo-spectral method aims to minimize the fuel consumption and the iterative learning control is used to improve formation-keeping accuracy, however, they didn’t consider the uncertain perturbations. In addition, all the studies were oriented toward the precisely displaced formation. In fact, for some space missions, keeping the formation precisely may be unnecessary and a loosely displaced formation can be tolerated. Compared to the precisely displaced formation, the loosely displaced formation does not require the satellites to maintain their positions accurately, which means it can reduce the propellant consumption by loosening the displacement constraint of the satellites. Based on the above, this article aims to find a loosely displaced formation oriented to the optimal fuel consumption, which is obviously an optimal control problem. By using the optimal control theory, it is difficult to obtain the optimal analytical solution. The pseudo-spectral method, as a classical direct numerical optimization method, can transform the continuous optimal control problem into a finite-dimensional nonlinear programming problem. It was proved that the pseudo-spectral method has good convergence and low initial sensitivity, thus, it is widely used for solving the optimal control problem. However, while solving the optimal solution by using the pseudo-spectral method, some unknown perturbations and unmolded characteristics can’t be dealt with, which will affect the formation-keeping accuracy. In addition, the pseudo-spectral method is a numerical optimization method and the optimal solution obtained is discrete, thus, a systematic residual error will be generated.

In summary, a loosely displaced formation-keeping control strategy oriented to the optimal fuel consumption is proposed in this article. More specifically, the controller is composed of two parts: a fuel-optimal controller and a feedback controller. For the first part, the control objective is to minimize the fuel consumption while keeping loosely displaced formation, and the Legendre pseudo-spectral method (LPM) is introduced for solving the optimal trajectory and control force. Considering that the pseudospectral method provides a numerical discrete solution and the optimal control is an open-loop control, a feedback controller should be introduced to make sure that the fuel-optimal trajectory can be tracked accurately. In this article, inspired by Huang et al., an adaptive finite-time sliding mode (AFSM) feedback controller is proposed, which can effectively compensate the uncertain perturbations and the linearization error of the system thereby improving the tracking accuracy. Finally, the simulation result has illustrated the fuel-optimality and the tracking accuracy of the control strategy in this article.

### Modeling and descriptions

#### Nonlinear relative orbital dynamic equation for the formation satellite

Usually, a local-vertical local-horizontal (LVLH) frame is served as the reference orbit for describing the swarm satellites’ relative motion, which can be established by using the following principle: The origin is located at the center of the reference satellite and the coordinate axes are defined as

$$
\hat{x}_c = \frac{r_c}{\|r_c\|}, \quad \hat{z}_c = \frac{r_c \times v_c}{\|r_c \times v_c\|}, \quad \hat{y}_c = \hat{z}_c \times \hat{x}_c
$$

(1)

where $r_c$ and $v_c$ are, respectively, the position and velocity coordinates of the reference satellite.

In space, the nonlinear dynamics of the satellite satisfies the following equation

$$\ddot{r} = -\frac{\mu}{r^3} r + u + d
$$

(2)

where $r \in \mathbb{R}^{3 \times 1}$ denotes the satellite position coordinates, $r = \|r\|$, $\mu = 3.9860044 \times 10^5$ km$^3$/s$^2$, $d \in \mathbb{R}^{3 \times 1}$ is the perturbation acceleration, and $u \in \mathbb{R}^{3 \times 1}$ is the control acceleration provided by the thruster.

We define $p = r - r_c$, and the relative orbital dynamic equation can be deduced as

$$\ddot{p} + 2\omega_c \times \dot{p} + [\omega_c \times (\omega_c \times p) + g(p) = u + d
$$

(3)

with

$$g(p) = \frac{\mu}{\|p + r_c\|^3} (p + r_c) - \frac{\mu}{r_c^3} r_c.$$
where \( \omega \) denotes the reference orbital angular rate and \( \omega^s \) denotes the skew symmetry matrix with the following expression:

\[
\omega^s = \begin{bmatrix}
0 & -\omega_{c,3} & \omega_{c,2} \\
\omega_{c,3} & 0 & -\omega_{c,1} \\
-\omega_{c,2} & \omega_{c,1} & 0
\end{bmatrix}
\]

In this article, we assume that the reference orbit is a two-body elliptic orbit and it is worth to note that equation (3) is established in the LVLH frame, thus, we have

\[
r_c = \begin{bmatrix} r_c \ 0 \ 0 \end{bmatrix}^T, \quad \omega_c = \begin{bmatrix} 0 & 0 & -\dot{\theta}_c \end{bmatrix}^T
\]

with satisfying

\[
r_c = \frac{a(1-e^2)}{(1+e\cos\theta_c)}, \quad \dot{\theta}_c = \frac{1}{r_c} \sqrt{\frac{\mu}{a(1-e^2)}}(1+e\cos\theta_c)
\]

where \( a \) is the semi-major axis of the reference orbit, \( e \) is the eccentricity, \( i \) denotes the inclination of the reference orbit, and \( \theta_c \) denotes the reference orbital angular rate.

Continuous low-thrust propulsion system is widely used in recent years because of its high specific impulse and precise thrust output. In this article, we adopt continuous low-thrust as the control input and the propellant mass flow under continuous thrust is as follows

\[
\dot{m} = -\frac{m}{I_{sp\delta_0}} \|u\| \quad \text{(5)}
\]

where \( m \) is the satellite mass, \( I_{sp} \) is the specific impulse of the engine, and \( g_\delta \) is the seal acceleration.

**Description for the loosely displaced formation**

In this section, the concept of the loosely displaced formation will be explained. As shown in Figure 2, if a desired formation has been given already, a loosely displaced formation can be described as: given a small loose region \( A \), it satisfies the following constraint

\[
\|p(t) - p_d(t)\| \leq \Lambda \quad \text{(6)}
\]

where \( p_d \) denotes the desired trajectory of the given precisely displaced formation.

**The objective of the formation-keeping control**

In this paper, the fuel consumption is mostly concerned, thus, the objective function can be selected as follows:

\[
J_1 = \Delta m = m(t_0) - m(t_f) = -m(t_0)\exp\left(-\frac{1}{I_{sp\delta_0}} \int_{t_0}^{t_f} \|u\| \, dt\right) \quad \text{(7)}
\]

**The design of the control scheme via LPM-AFSM**

For solving the formation-keeping problem mentioned above, a control scheme via LPM-AFSM is proposed in this section. To be specific, with LPM, we can solve the fuel-optimal control input and trajectory. In view of the optimal solution being discrete and the uncertain perturbations, a feedback controller via adaptive sliding mode theory is designed for keeping the tracking accuracy.

**Definitions, lemmas, and assumptions**

**Definition 1.** For the following system

\[
\dot{x} = f(t,x), f(t,0) = 0 \quad \text{(8)}
\]

if there exists a small region \( \varepsilon > 0 \) and \( T(\varepsilon,x_0) < \infty \) with satisfying \( \|x(t)\| < \varepsilon \) for all \( t > t_0 + T \), it can be concluded that the system is practical finite-time stable.

**Lemma 1.** For system (14), if there exists a Lyapunov function \( V(x) \) with satisfying

\[
\dot{V}(x) \leq \rho V^\theta(x) + \omega
\]

with \( \rho > 0, 0 < \theta < 1, \) and \( 0 < \omega < \infty \), it can be concluded that system (8) is practical finite-time stable.
Definition 2. For any \( x \in \mathbb{R}^{n \times 1}, \tan(x) = [\tan(x_1), \ldots, \tan(x_n)]^T \), where \( \tan(x_i) \) is the hyperbolic tangent function with the following expression

\[
\tan(x_i) = \frac{e^{2x_i} - 1}{e^{2x_i} + 1}
\]

The derivative of \( \tan(x_i) \) can be expressed as

\[
\tan'(x_i) = \text{sech}^2(x_i) = \left( \frac{2}{e^{x_i} + e^{-x_i}} \right)^2
\]

Lemma 2. For any \( x \in \mathbb{R}, \varepsilon > 0 \) and \( \kappa = 0.2785 \), the following inequality always holds

\[
0 \leq |x| - x \cdot \tanh(\varepsilon x) \leq \frac{\kappa}{\varepsilon}
\]

Lemma 3. For the vector \( x \in \mathbb{R}^{n \times 1} \) and \( 0 < \rho < 2 \), the following inequality always holds

\[
\left( \sum_{i=1}^{n} x_i^2 \right)^{\rho} \leq \left( \sum_{i=1}^{n} x_i^\rho \right)^{2}
\]

Definition 3. For any \( x \in \mathbb{R}^{n \times 1} \), \( \text{sign}(x) = [\text{sign}(x_1), \ldots, \text{sign}(x_n)]^T \), where \( \text{sign}(x_i) \) is the sign function with the following expression

\[
\text{sign}(x) = \begin{cases} 
1, x > 0 \\
0, x = 0 \\
-1, x < 0 
\end{cases}
\]

Assumption 1. In this article, we suppose the disturbance torque \( d \) is bounded with unknown upper bound.

Assumption 2. In this article, we assume that the reference satellite and the formation satellite have the same orbital period \( T \), which can be obtained with

\[
T = 2\pi \sqrt{\frac{a^3}{\mu}}
\]

Solving the optimal trajectory via LPM

Commonly, a Bolza optimal control problem (Bolza-OCP) can be described as: given the following index function

\[
J = \Phi(x(t_0), x(t_f), t_0, t_f) + \int_{t_0}^{t_f} L(t, x(t), u(t)) dt
\]

find the optimal state variables \( x^* \) and control variables \( u^* \) to minimize \( J \) with the following constraints

\[
\begin{align*}
\dot{x} &= f(t, x, u) \\
x(t) &\leq x(t) \leq u(t) \leq u_u \\
\phi(x(t_0), x(t_f)) &\leq 0
\end{align*}
\]

We know that it is difficult to obtain the analytic solution for the above optimal control problem and a numerical method is suggested. LPM has been widely used over the recent years to solve the optimal control problems, which can transform the continuous optimal control problem into a discrete nonlinear programming problem. The detailed algorithm steps of the LPM are as follows.

Step 1: For dealing with the optimal control problem by using LPM, the time interval should be mapped to \( t \in [-1, 1] \)

\[
t = \frac{(t_f - t_0) \tau + (t_f + t_0)}{2}
\]

Step 2: The continuous state and control variables can be approximated by the following discretized variables

\[
x(t) \approx x^N(t) = \sum_{l=0}^{N} x_l(t) L_l(t) \\
u(t) \approx u^N(t) = \sum_{l=0}^{N} u_l(t) L_l(t)
\]

with

\[
L_l(t) = \frac{1}{N(N+1)P_N(t)} \frac{(\tau^2 - 1)P_N(t)}{\tau - \tau_l}
\]

where \( \tau_l (l = 0, 1, \ldots, N) \) are the Legendre-Gauss-Lobatto (LGL) points and they are the roots of the \( N \)-order polynomial \( P_N(t) \) with

\[
P_N(t) = \frac{1}{2^N N!} d^N \left[ (\tau^2 - 1)^N \right]
\]

Step 3: Take the derivative of \( x(t) \) at the point \( \tau_k \), we have

\[
\dot{x}(\tau_k) \approx \dot{x}^N(\tau_k) = \sum_{l=0}^{N} x_l(\tau_k) \dot{L}_l(\tau_k) = \sum_{l=0}^{N} D_{kl} x_l(\tau_k)
\]

with

\[
[D_{kl}] = \begin{cases} 
P_N(\tau_k) \frac{1}{P_N(\tau_l)} \frac{1}{\tau_k - \tau_l}, & k, l \neq 0, l \neq k \\
-\frac{N(N+1)}{4}, & k = l = 0 \\
\frac{N(N+1)}{4}, & k = l = N \\
0, & \text{otherwise}
\end{cases}
\]
Finally, equations (15) and (19) can be replaced by

$$J = \phi(x_0, x_N, \tau_0, \tau_N) + \frac{t_f - t_0}{2} \sum_{k=1}^{N} \omega_k L(\tau_k, x_k, u_k) \tag{20}$$

$$\sum_{i=0}^{N} D_{kj} x_i = \frac{t_f - t_0}{2} f(x_k, u_k, \tau_k) \tag{21}$$

where $x_k = x(\tau_k), u_k = u(\tau_k)$, and $\omega_k$ is the weight coefficient with

$$\omega_k = \frac{2}{N(N + 1)} \frac{1}{\beta_N(\tau_k)}$$

Thus, the Bolza-OCP can be transformed to a standard nonlinear programming (NLP) problem and solved by using the LPM. Lots of literatures introduced the theory of the LPM. \(^{34-36}\) Ross and Fahroo \(^{34}\) proved that the NLP problem obtained by the LPM converges to the continuous Bolza problem at a spectral rate. Fahroo and Ross \(^{36}\) pointed that the Karush–Kuhn–Tucker multipliers of the NLP problem map linearly to the spectrally discretized covectors with the optimality conditions of the Bolza problem.

However, formation-keeping usually has a long-term duration, it is unpractical as well as unnecessary to make a global optimization for that. In view of the fact that the propellant mass flow rate is slow and the satellite mass can be regarded as a constant in a finite duration, the index function $J_1$ can be equivalent to the following

$$J_2 = -\int_{t_0}^{t_f} \|u\| \, dt \tag{22}$$

If we ignore the tiny uncertain perturbations $d$ and introduce the following constraints

$$p(kT - T) = \hat{p}(kT), \quad \hat{p}(kT - T) = \hat{p}(kT), \quad k \in \mathbb{Z}^+ \tag{23}$$

we can get a periodic system as

$$\ddot{p} + 2\omega_c \dot{p} + \left[\omega_c^2 + (\omega_c^2)^2\right] p + g(p) = u, \quad t \in [0, T] \tag{24}$$

Utilizing Assumption 2, the formation-keeping duration can be divided into an integral number of circles with $T$, thus, we let

$$t_f - t_0 = nT \tag{25}$$

where $n \in \mathbb{Z}^+$. On this basis, we merely optimize for the first circle of the formation-keeping duration, that is

$$J = -\int_{t_0}^{T} \|u\| \, dt \tag{26}$$

Hence, the loosely displaced formation-keeping control problem oriented to the optimal fuel consumption in this paper can be transformed to the following NLP problem:

Find the optimal state variables and control variables

$$X^* = (p_0, \ldots, p_N, \dot{p}_0, \ldots, \dot{p}_N, u_0, \ldots, u_N)$$

for minimizing

$$J = -\frac{T}{2} \sum_{k=1}^{N} \omega_k \|u_k\| \tag{27}$$

subject to

$$\left\{ \begin{array}{l}
\sum_{i=0}^{N} D_{kj} x_i = \frac{T}{2} \left[ -2\omega_c \dot{p}_k + \left[\omega_c^2 + (\omega_c^2)^2\right] p_k - g(p_k) + u_k \right] \\
\|p_k - p_{k,d}\| \leq \Delta, \|\dot{p}_k\| \leq v_{\text{max}}, 0 \leq \|u_k\| \leq u_{\text{max}}, p_0 = p_N, \dot{p}_0 = \dot{p}_N
\end{array} \right. \tag{28}$$

The above NLP problem can be solved by using the PSOPT software,\(^{37}\) which is an open source optimal control package written in C++.

The feedback control design via AFSM

In the section “Solving the optimal trajectory via LPM,” we have solved the fuel-optimal trajectory of the satellite by using LPM, however, there still exist two issues that cannot be dealt with:

i) From equation (3), we can see it contains the unknown perturbation term $d$, which cannot be expressed exactly, however, the optimal trajectory and control input are solved without considering this.

ii) We have already known that the optimal trajectory and control input are discrete with the LPM, however, the control system is continuous obviously.

Conclusively, for keeping the control accuracy, a feedback controller is quite essential to be taken into for improving the tracking accuracy. In this article, we proposed an AFSM tracking controller.

Firstly, by using the linear interpolation method, the optimal solution can be serialized as $p^*, \dot{p}^*, \text{ and } u^*$, however, the linearization errors also will be introduced, thus, the following equation will be obtained
\[ \ddot{\rho} + 2\omega_c^\infty \dot{\rho} + \left[ \omega_c^\infty + (\omega_c^\infty)^2 \right] \rho' + g(\rho') = u' + \epsilon_1 \quad (29) \]

where \( \epsilon_1 \) denotes the linearization errors.

Define \( \rho = p - \rho^* \), we have

\[ \ddot{\rho} + 2\omega_c^\infty \dot{\rho} + \left[ \omega_c^\infty + (\omega_c^\infty)^2 \right] \rho + g(\rho + \rho^*) - g(\rho') = u + d - u' - \epsilon_1 \quad (30) \]

where the term \( g(\rho + \rho^*) \) can be linearized at \( \rho^* + r_c \) and we can obtain that

\[ \ddot{\rho} = \frac{\mu}{\| \rho^* + r_c \|^3} \left[ I_{3\times3} - \frac{3}{\| \rho^* + r_c \|^2} (\rho^* + r_c)(\rho^* + r_c)^T \right] \rho + \epsilon_2 \quad (31) \]

Define \( u = u_{fb} + u^* \) and \( u_{fb} \) denotes the feedback control scheme, by using equations (30) and (31), the following equation can be deduced

\[ \dot{\rho} + 2\omega_c^\infty \dot{\rho} + H(\rho^*) = u_{fb} + \ddot{d} \quad (32) \]

with

\[ \ddot{d} = -\epsilon_1 - \epsilon_2 + d \]

\[ H(\rho^*) = \omega_c^\infty + (\omega_c^\infty)^2 + \frac{\mu}{\| \rho^* + r_c \|^3} \left[ I_{3\times3} - \frac{3}{\| \rho^* + r_c \|^2} (\rho^* + r_c)(\rho^* + r_c)^T \right] \]

For designing the feedback controller, a sliding mode surface is firstly given as

\[ s = \dot{\rho} + k_1 \tanh \left( \frac{\rho}{k_2} \right) \quad (33) \]

where \( k_1, k_2 > 0 \).

**Remark 1.** The sliding mode surface proposed here can ensure a finite-time stability for the system states. Compared with the conventional terminal sliding mode surface, the singularity problem can be effectively avoided. Furthermore, it possesses more clarity than the existing nonsingular sliding mode surface.30

Because the boundary of \( \ddot{d} \) is unknown, we define \( \ddot{D} = D - \ddot{D} \), where \( \ddot{D} \) is the estimation of \( D \). With equation (36), the feedback control scheme is given as equations (37) and (38).

\[ u_{fb} = - \left[ \frac{k_1}{k_2} \sum_{i=3}^{3} \rho_i \text{sech}^2 \left( \frac{\rho_i}{k_2} \right) + k_3 s + k_4 \text{sign}(s) + \dot{D} \text{sign}(s) \right] + 2\omega_c^\infty \dot{\rho} + \left[ \omega_c^\infty + (\omega_c^\infty)^2 - 2I_{3\times3} \right] \rho \quad (34) \]

\[ \dot{D} = \gamma (\| s \| - k_5 \dot{D}) \quad (35) \]

where \( k_3, k_4, k_5 > 0, \gamma > 0 \).

**Theorem 1.** For the system equation (32), if we suppose \( \| \ddot{d} \| \leq D \) and the value of \( D \) can only be obtained exactly, given the feedback control scheme as equations (33) to (35), the system states will converge to a small residual set around the origin in finite time.

The proof process for Theorem 1 is given as follows.

**Proof.**

(i) Firstly, to illustrate the stability of \( s \), a Lyapunov function \( V_1 \) is chosen as

\[ V_1 = \frac{1}{2} s^T s + \frac{1}{2\gamma} \dot{D}^2 \quad (36) \]

Utilizing equations (32) to (35), the derivation of \( V_1 \) is obtained as
\[
\dot{V}_1 = s^T \dot{s} + \frac{1}{\gamma} \dot{D}^2 \\
= s^T \left[ \frac{\rho}{k_2} + \frac{k_1}{k_2} \sum_{i=1}^{3} \dot{r}_i \tanh^2 \left( \frac{\rho_i}{\kappa_2} \right) \right] + \frac{1}{\gamma} \dot{D} \left( \dot{D} - \dot{\hat{D}} \right) \\
= s^T \left[ -2\omega^e \dot{\rho} - H(p^e) \rho + u_f + \dot{d} + \frac{k_1}{k_2} \sum_{i=1}^{3} \dot{r}_i \tanh^2 \left( \frac{\rho_i}{\kappa_2} \right) \right] + \frac{1}{\gamma} \dot{D} \left( -\gamma ||s|| + \gamma k_s \dot{D} \right) \\
= -s^T \left[ k_3 s + k_4 \text{sign}(s) + \dot{D} \text{sign}(s) \right] + s^T \dot{d} - \dot{D} \left( ||s|| - k_3 \dot{D} \right) \\
\leq -k_3 ||s||^2 - k_4 \sum_{i=1}^{3} |s_i| - \dot{D} \sum_{i=1}^{3} |s_i| + D ||s|| - \dot{D} ||s|| + k_3 \dot{D} \dot{D} \\
= -k_4 ||s|| + k_5 \dot{D} \dot{D} - k_3 ||s||^2 + (k_4 + \dot{D}) \left( ||s|| - \sum_{i=1}^{3} |s_i| \right)
\]

In equation (37), the term \( k_5 \dot{D} \dot{D} \) can be further written as

\[
k_5 \dot{D} \dot{D} = k_5 \left( \dot{D}^2 + \ddot{D} \right) \leq k_5 \left( \dot{D}^2 + \frac{1}{2 \lambda_1} \dot{D}^2 + \frac{\lambda_1}{2} D^2 \right)
\]

\[
= -k_5 \frac{2 \lambda_1 - 1}{2 \lambda_1} \dot{D}^2 + k_5 \frac{\lambda_1}{2} D^2 
\]

(38)

According to Huang et al., the following inequality always holds

\[
\dot{V}_1 \leq -k_4 ||s|| - \left( \frac{k_5 \left( 2 \lambda_1 - 1 \right)}{2 \lambda_1} \dot{D}^2 \right)^{\frac{\alpha+1}{\alpha}} + k_5 \frac{\lambda_1}{2} D^2 \\
\leq -\sqrt{2} k_4 \left( \frac{1}{2} s^T s \right)^{\frac{1}{2}} - \left( \frac{\gamma k_5 \left( 2 \lambda_1 - 1 \right)}{\lambda_1} \right) \left( \frac{1}{2 \gamma} \dot{D}^2 \right)^{\frac{1}{2}} + k_5 \frac{\lambda_1}{2} D^2 \\
\leq -\eta_1 \left( \frac{1}{2} s^T s + \frac{1}{2 \gamma} \dot{D}^2 \right)^{\frac{1}{2}} + k_5 \frac{\lambda_1}{2} D^2 \\
= -\eta_1 V_{\dot{\rho}} + \eta_1
\]

(40)

where the expressions of \( \eta_1 \) and \( \eta_1 \) as follows

\[
o_1 = \frac{k_5 \lambda_1}{2} D^2. \eta_1 = \min \left( \sqrt{2} k_4, \left( \frac{\gamma k_5 \left( 2 \lambda_1 - 1 \right)}{\lambda_1} \right) \right)
\]

Thus, according to Lemma 1, the state variables \( s \) and \( \dot{D} \) are practical finite-time stable.

(ii) To illustrate the stability of \( \dot{\rho} \) and \( \dot{\hat{\rho}} \), a Lyapunov function \( V_2 \) is chosen as follows

\[
V_2 = \frac{1}{2} \dot{\rho}^T \dot{\rho}
\]

(41)

Take the derivation of \( V_2 \), we have
\[ \dot{V}_2 = \rho^T \ddot{\rho} = \rho^T s - k_1 \rho^T \tanh \left( \frac{\rho}{k_2} \right) \] (42)

From Definition 1, we can conclude that there exists a small region \( \varepsilon > 0 \) and \( T_1 < T_{\text{max}} < \infty \) satisfying \( \|s\| \leq \varepsilon \) for all \( t > T_1 \). In addition, according to Lemma 2, we have

\[ -k_1 \rho^T \tanh \left( \frac{\rho}{k_2} \right) = -k_1 \sum_{i=1}^{3} |\rho_i| \tanh \left( \frac{\rho_i}{k_2} \right) \leq -k_1 \sum_{i=1}^{3} |\rho_i| + 3k_1k_2 \varepsilon \] (43)

Utilizing equations (18) and (43), \( \dot{V}_2 \) can be further written to the following

\[ \dot{V}_2 \leq \varepsilon \sum_{i=1}^{3} |\rho_i| - k_1 \sum_{i=1}^{3} |\rho_i| + 3k_1k_2 \varepsilon \]

\[ \leq -(k_1 - \varepsilon) \sqrt{\sum_{i=1}^{3} |\rho_i|^2} + 3k_1k_2 \varepsilon \] (44)

\[ = -\eta_2 V_2 + \sigma_2 \]

where \( \eta_2 = k_1 - \varepsilon > 0 \), \( \sigma_2 = 3k_1k_2 \varepsilon \).

Hence, the state variable \( \rho \) is practical finite-time stable, that is, there exists a small region \( \Delta > 0 \) and \( \|\rho\| \leq \Delta \) can be reached in finite time.

According to equation (20), the following inequality can be obtained

\[ \|\dot{\rho}\| = \left\| \mathbf{s} - k_1 \tanh \left( \frac{\rho}{k_2} \right) \right\| \leq \|\mathbf{s}\| + k_1 \left\| \tanh \left( \frac{\rho}{k_2} \right) \right\| \]

\[ \leq \|\mathbf{s}\| + k_1 \|\rho\| \leq \varepsilon + k_1 \frac{\Delta}{k_2^2} \] (45)

Hence, there exists a small region \( \varepsilon + k_1 \Delta > 0 \) and \( \|\dot{\rho}\| \leq \varepsilon + k_1 k_2^{-1} \Delta \) can be reached in finite time.

Finally, both \( \rho \) and \( \dot{\rho} \) are proved to be finite-time stable. From (i) and (ii), it can be concluded that the system in equation (32) is finite-time stable with the feedback control scheme as equations (33) to (35). Thus, Theorem 1 has been proved.

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**Figure 3.** The curves of the fuel-optimal trajectory for the satellite. (a) The desired and optimal position, (b) the desired and optimal velocity, and (c) the three-dimensional desired and optimal position and velocity vectors.
Test results

For proving the validity of the control strategy proposed in this article, we will make a test validation in this section. The test is simulated on a PC with Windows 10 OS, Core (TM) i5-8400 CPU and 8G RAM. The fuel-optimal solution is solved based on the PSOPT software version 3.9.3 and the Simulink toolbox is used to validate the convergence of the feedback controller.

In the test, a flying-around formation is selected as the desired precisely displaced formation, which can be expressed as

\[ p_d(t) = 200 \times \begin{bmatrix} \cos \left( \frac{2\pi}{T} t \right) \\ \cos \left( \frac{\pi}{6} \right) \\ \cos \left( \frac{2\pi}{T} t \right) \sin \left( \frac{\pi}{6} \right) \\ \sin \left( \frac{2\pi}{T} t \right) \end{bmatrix}^T \text{m} \]

The test parameters are selected as follows:

1. The reference orbital elements: \( a = 6652.56 \text{ km,} \) \( e = 0.001, \theta_c(0) = 30^\circ. \)
2. The constraints: \( v_{\text{max}} = 0.2 \text{ m/s,} \) \( a_{\text{c, max}} = 0.025 \text{ m/s}^2, \)
3. The initial satellite mass: \( m_0 = 100 \text{ kg.} \)
4. The specific impulse and the seal acceleration: \( I_{\text{sp}} = 1000 \text{ s,} \) \( g_0 = 9.81 \text{ m/s}^2. \)
5. The formation-keeping duration: \( t_f - t_0 = 1 \text{ day} = 16T. \)
6. The uncertain disturbing force is assumed as \( d(t) = \begin{bmatrix} 5 + \sin(0.002t) \\ 10 - \cos(0.002t) \sin(0.002t) \end{bmatrix}^T \times 10^{-7} \text{N}. \)

Figure 3 shows the desired and the optimal trajectory of the satellite in one orbital cycle. It is quite clear that the initial and final states are equal, that is, the satellite trajectory is varying periodically.

In this article, we set the control parameters as \( k_1 = 1, k_2 = 2, k_3 = 0.01, k_4 = k_5 = 1, \) and the tracking trajectories with feedback control and with no feedback control are shown in Figure 4. From the figure, it can be seen that the optimal trajectory can be tracked accurately with the feedback control as well as with no feedback control, the satellite trajectory will departure from the optimal gradually. Thus, we can conclude that the feedback controller proposed in this article is essential and effective.

Figure 5 shows the fuel consumption of the satellite. From the figure, it can be seen that for keeping the desired formation, about 2.63 kg propellant is required and correspondingly, merely 0.68 kg propellant is required for keeping the optimal trajectory. In addition, we also can see that if the uncertain disturbance is ignored, the fuel consumption needed will be decreased to 0.56 kg, which is fairly close to 0.68 kg, that is, for keeping the optimal formation stability, it requires very little propellant to eliminate the effect of the uncertain disturbance. Hence, we can conclude that under a tiny external interference, the formation keeping control strategy proposed in this article can effectively reduce the fuel consumption.

Conclusions

For reducing the propellant consumption of the satellite swarm while keeping the formation, this article proposes a forward-feedback control strategy oriented to the loosely displaced formation. In advance of designing the controller, the concept of a loosely displaced formation is firstly introduced for the purpose of saving the fuel consumption. On this basis, the control is composed of two parts: an optimal forward control via LPM and a feedback control via AFSM. For the optimal control, the LPM is used to transform the optimal control problem into a nonlinear programming one and the fuel-optimal trajectory and control will be finally obtained with this method. After that, a feedback control scheme via AFSM is designed to make sure that the optimal trajectory can be accurately tracked by the satellite under the unknown disturbing force. Theoretical proof process has proved that the sliding mode surface designed in this article is finite-time stable and the tracking error will also
converge to zero in a finite time. Finally, extensive simulation results demonstrate the effectiveness of the proposed control strategy.

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