Abstract

Distributionally Robust Optimization (DRO) has enabled to prove the equivalence between robustness and regularization in classification and regression, thus providing an analytical reason why regularization generalizes well in statistical learning. Although DRO’s extension to sequential decision-making overcomes external uncertainty through the robust Markov Decision Process (MDP) setting, the resulting formulation is hard to solve, especially on large domains. On the other hand, existing regularization methods in reinforcement learning only address internal uncertainty due to stochasticity. Our study aims to facilitate robust reinforcement learning by establishing a dual relation between robust MDPs and regularization. We introduce Wasserstein distributionally robust MDPs and prove that they hold out-of-sample performance guarantees. Then, we introduce a new regularizer for empirical value functions and show that it lower bounds the Wasserstein distributionally robust value function. We extend the result to linear value function approximation for large state spaces. Our approach provides an alternative formulation of robustness with guaranteed finite-sample performance. Moreover, it suggests to use regularization as a practical tool for dealing with external uncertainty in reinforcement learning methods.

1. Introduction

Markov Decision Processes (MDPs) originated from the seminal works of Bellman (1957) and Howard (1960) to model sequential decision-making problems and provide a theoretical basis for reinforcement learning (RL) methods. Real-world applications which include healthcare and marketing, for example, give rise to several challenging issues. Firstly, the model parameters are generally unknown but rather estimated through historical data. This may lead the performance of a learned strategy to significantly degrade when deployed (Mannor et al., 2007). Secondly, experimentation can be expensive or time-consuming which constrains policy evaluation and improvement to perform with limited data (Lange et al., 2012). Lastly, when the state-space is large, the value function is commonly approximated by a parametric function, which results in additional uncertainty regarding the efficiency of a learned policy (Farahmand et al., 2009; Farahmand, 2011).

This phenomenon is reminiscent of over-fitting in the statistical learning framework that can be interpreted as the following single-stage decision-making problem (Zhang et al., 2018). Consider a training set of random input-output vectors \((\hat{x}_i, \hat{y}_i)_{i=1}^n\) generated by a fixed distribution and assume one wants to find a parameter \(\theta \in \Theta\) that minimizes the expected loss function \(\ell_\theta\) with respect to \((w.r.t.)\) the generating distribution. Unfortunately, in most cases, the true distribution is unknown besides being hard to estimate accurately. A classical method to overcome this issue is to minimize the empirical risk:

\[
\min_{\theta \in \Theta} \frac{1}{n} \sum_{i=1}^{n} \ell_\theta(\hat{x}_i, \hat{y}_i),
\]

but this often yields solutions that perform poorly on out-of-sample data (Friedman et al., 2001).

Several methods ensure better generalization to new, unseen data (test set) while performing well on available data (training set). These may be categorized into two main approaches. The first one consists of regularizing the empirical risk and optimizing the resulting objective (Vapnik, 2013). Another approach is to robustify the objective function by introducing ambiguity w.r.t. the empirical distribution (Kuhn et al., 2019). The resulting problem is a distributionally robust stochastic optimization (DRSO) which can be formulated as

\[
\min_{\theta \in \Theta} \max_{Q \in \mathcal{M}(\hat{P}_n)} \mathbb{E}_{(x,y) \sim Q}[\ell_\theta(x, y)] \quad \text{(DRSO)}
\]

where \(\hat{P}_n\) is the empirical distribution w.r.t. the sample set and \(\mathcal{M}(\hat{P}_n)\) is an ambiguity set of probability distributions that must be consistent with the dataset. Such ambiguity sets can be based on specified properties such as moment
constraints (Delage & Ye, 2010; Bertsimas et al., 2018; Wiesemann et al., 2014), or on a given divergence from the empirical distribution (Hu & Hong, 2013; Ben-Tal et al., 2013; Erdoğan & Iyengar, 2006; Esfahani & Kuhn, 2017). The resulting problem (DRSO) can then be solved using Distributionally Robust Optimization (DRO).

Wasserstein distance-based ambiguity sets are of particular interest in DRO theory, as they display interesting ramifications in classical problems encountered in statistical learning. More precisely, the specific problem as defined in (DRSO) is equivalent to regularization for fundamental learning tasks such as classification (Xu et al., 2009; Shafieezadeh-Abadeh et al., 2015; Blanchet et al., 2016), regression (Shafieezadeh-Abadeh et al., 2017) and maximum likelihood estimation (Kuhn et al., 2019). However, to the best of our knowledge, equivalence between robustness and regularization has only been studied on single-stage decision problems.

Regularization techniques are widely used in RL to mitigate uncertainty in value function approximation (Farahmand et al., 2009) or to derive improved versions of policy optimization methods (Shani et al., 2019). Although regularized policy learning helps to derive risk-sensitive strategies that satisfy safety criteria (Ruszczynski, 2010; Tamar et al., 2015), existing connections between regularization in RL and robustness are still weak. Specifically, prior regularization methods address the internal uncertainty i.e., the inherent stochasticity of the dynamical system, without accounting for the external uncertainty of the MDP parameters i.e., transition and reward functions.

Robust MDPs provide a convenient framework for dealing with external uncertainty in sequential decision-making and enable to construct robust strategies with provable worst-case guarantees and better generalization to unseen data (Iyengar, 2005; Nilim & El Ghaoui, 2005; Xu & Mannor, 2010; Yu & Xu, 2015). While standard formulations focus on a tabular setting, previous work has enabled to scale up learning and planning for robust MDPs (Tamar et al., 2014; Roy et al., 2017; Lim & Autef, 2019; Derman et al., 2018; 2019). However, solving robust MDPs remains challenging even on small domains (Petrik & Russell, 2019), specifically because of the difficulty to construct an uncertainty set of models that yields a robust policy without being too conservative.

This study aims to facilitate robust RL by addressing a new regularization perspective on sequential decision-making settings. In Section 2, we recall the MDP framework and describe its robust and distributionally robust formulations. Then, in Section 3, we introduce Wasserstein distributionally robust MDPs as an analytical tool to establish a connection between robustness and regularization. We address our main result in Section 4: In Theorem 4.1, we devise the first dual relation between robustness to model uncertainty and regularized value functions. An extension to linear function approximation is addressed and formally stated in Theorem 4.2. Finally, we establish out-of-sample guarantees for Wasserstein distributionally robust MDPs, thus demonstrating the fact that our regularization method enables better generalization to unseen data. Detailed proofs can be found in the Appendix.

Main Contributions. To summarize, our specific contributions are: (1) A dual relation between regularized value functions and Wasserstein distributionally robust MDPs which suggests to use regularization as a practical tool for ensuring robustness to model uncertainty; (2) An extension of this dual relation to linear function approximation; (3) Out-of-sample performance guarantees for Wasserstein distributionally robust MDPs and as a result, for our regularization approach.

Notation. We denote by $\mathbb{R} = [-\infty, +\infty]$ the extended reals. The set of distributions over any Borel set $\mathcal{E}$ is defined as $\mathcal{M}(\mathcal{E})$. Given a norm $\|\cdot\|$ over an Euclidean space, the dual norm is defined through $\|\cdot\|_*= \sup_{\|x\|\leq 1} \langle \cdot, x \rangle$. For all integer $n \geq 1$, we denote by $[n]$ the set of integers $\{1, \cdots, n\}$.

2. Preliminaries

In this section, we provide the theoretical background used throughout this study. We first recall some definitions and fundamental results of convex analysis. This preliminary study is further applied in Section 4, where we define the conjugate robust value function, a tool borrowed from the conjugate transformation, which in turn plays a crucial role in our regularization method. Then, we describe the MDP setting and its generalization to robust and distributionally robust MDPs.

2.1. Convex Analysis

Consider a convex Euclidean space $X$ equipped with a scalar product $(\cdot, \cdot)$. Further denote by $U : X \to \mathbb{R}$ an extended real-valued function over $X$. We then define the following.

**Definition 2.1.** (a) Proper Function. We say that $U$ is proper if $U > -\infty$ and there exists $x \in X$ such that $U(x) \leq +\infty$.

(b) Closed Function. We say that $U$ is a closed function if its epigraph $\text{epi}(U) := \{(x, c) \in X \times \mathbb{R} | U(x) \leq c\}$ is a closed subset of $X \times \mathbb{R}$.

**Convex Closure.** The convex closure $\bar{c}l(U)$ of $U$ is the greatest closed and convex function upper-bounded by $U$ i.e., if $U_c$ is a closed and convex function that satisfies $U_c \leq U$, then $U_c \leq \bar{c}l(U)$.
In other words, a function is proper if and only if its epigraph is nonempty and does not contain a vertical line. Moreover, when dealing with a non-convex function, we may work with its convex closure instead, in order to apply standard results from convex analysis. In particular, if the convex closure of a function is proper, then it coincides with its double conjugate, as we detail below.

**Definition 2.2 (Conjugate Function).** The Legendre-Fenchel transform (or conjugate function) of $U$ is the mapping $U^* : X \to \mathbb{R}$ defined by

$$U^*(y) := \sup_{x \in X} \langle y, x \rangle - U(x),$$

We further define the conjugate function of $U^*$ as

$$U^{**}(x) := \sup_{y \in X^*} \langle y, x \rangle - U^*(y),$$

which is also the double conjugate of $U$.

Regardless of the initial function $U$, its conjugate $U^*$ is convex and closed but not necessarily proper. In fact, if $U$ is convex, then $U^*$ is proper if and only if $U$ is, as stated in the fundamental theorem below (see Bertsekas (2009); Barbu & Precupanu (2012)).

**Theorem 2.1 (Conjugacy Theorem).** The following holds:

(a) $U \geq U^{**}$
(b) If $U$ is convex and closed, then $U$ is proper if and only if $U^*$ is.
(c) If $U$ is closed, proper and convex, then $U = U^{**}$.
(d) The conjugates of $U$ and $\text{cl}(U)$ are equal.

### 2.2. From MDPs to distributionally robust MDPs

#### 2.2.1. Markov Decision Process

A Markov Decision Process (MDP) is a tuple $\langle S, A, r, P \rangle$ with finite state and action spaces $S$ and $A$ respectively, such that $r : S \times A \to \mathbb{R}$ is a deterministic reward function bounded by $R_{\text{max}}$ and $P : S \to \mathcal{M}(S)^{|A|}$ denotes the transition model i.e., for all $s \in S$, the elements of $p_s := (p(|s, a_1), \ldots, p(|s, a_A|)) \in \mathcal{M}(S)^{|A|} \subset \mathbb{R}^{|S| \times |A|}$ are listed in such a way that transition probabilities of the same action are arranged in the same block. At step $t$, the agent is in state $s_t$, chooses action $a_t$ according to a policy $\pi : S \to A$ and gets a reward $r(s, a)$. Then, it is brought to state $s_{t+1}$ with probability $p(s_{t+1}|s_t, a_t)$. The agent’s goal is to maximize the following value function over the set of policies $\Pi$:

$$v^\pi_p(s) = \mathbb{E}^\pi_p \left[ \sum_{t=0}^{\infty} \gamma^t r(s_t, a_t) \mid s_0 = s \right] \quad \forall s \in S, \quad (1)$$

where $\gamma \in [0, 1)$ is a discount factor determining the degree of myopia to upcoming rewards and the expectation is conditioned on transition model $p$, policy $\pi$ and initial state $s$. The value function (1) can be efficiently computed thanks to the Bellman operator:

$$T^\pi_p v_p(s) = \sum_{a \in A} \pi_s(a) \left( r(s, a) + \gamma \sum_{s' \in S} p(s'|s, a) v_p(s') \right) \quad \forall s \in S.$$

Besides being non-decreasing, $T^\pi_p$ is a $\gamma$-contracting operator w.r.t. the sup-norm so it admits $v^\pi_p$ as a unique fixed point (Puterman, 2014).

#### 2.2.2. Robust Markov Decision Process

A robust MDP $\langle S, A, r, P \rangle$ is an MDP with uncertain transition model $p \in \mathcal{P}$. The domain $\mathcal{P}$ is called the uncertainty set which we assume to be structured as a product set of transition models that are independent for each state. More formally, $\mathcal{P} \subset \mathbb{R}^{|S| \times |A| \times |S|}$ satisfies the rectangularity assumption $\mathcal{P} = \bigotimes_{s \in S} \mathcal{P}_s$ where for all $s \in S$, $\mathcal{P}_s$ is a set of transition matrices $p_s \in \mathcal{P}_s$ (Wieselmann et al., 2013). Accordingly, for all $s, s' \in S$ and $a \in A$, the probability of getting from state $s$ to state $s'$ after applying action $a$ is given by any $p_s \in \mathcal{P}_s$. Moreover, we assume that $\mathcal{P}$ is closed, convex and compact.

The robust value function under any policy $\pi$ is the worst-case performance:

$$v^\pi_p(s) = \inf_{p \in \mathcal{P}} v^\pi_p(s) \quad \forall s \in S.$$  

Thanks to the rectangularity assumption, one can show that $v^\pi_p$ is the unique fixed point of the following robust Bellman operator (Iyengar, 2005; Nilim & El Ghaoui, 2005):

$$T^\pi_p v_p(s) = \sum_{a \in A} \pi_s(a) \left( r(s, a) + \gamma \inf_{p \in \mathcal{P}} \sum_{s' \in S} p(s'|s, a) v_p(s') \right) \quad \forall s \in S.$$  

Moreover, a robust policy is optimal whenever it solves the max-min problem given by $\max_{\pi \in \Pi} \min_{p \in \mathcal{P}} v^\pi_p$, and it can be computed efficiently using robust dynamic programming (Iyengar, 2005).

#### 2.2.3. Distributionally Robust Markov Decision Process

In a Distributionally Robust Markov Decision Process (DRMDP) $\langle S, A, r, P, \mathcal{M} \rangle$, the transition model is also unknown but instead, it is a random variable supported on $\mathcal{P}$ and obeying a distribution $\mu \in \mathcal{M} \subseteq \mathcal{M}(\mathcal{P})$ (Xu & Mannor, 2010; Yu & Xu, 2015). Here, the class of probability distributions $\mathcal{M}$ is the ambiguity set and each of them is supported on the uncertainty set $\mathcal{P}$. Note
that DRMDPs generalize robust MDPs: if $\mathcal{M}$ only contains Dirac distributions $\delta_p$ i.e., with full mass on transition $p$, then we recover a robust MDP of uncertainty set $\mathcal{P} := \{ p : \delta_p \in \mathcal{M} \}$. We further assume both $\mathcal{P}$ and $\mathcal{M}$ to be rectangular so that any $\mu \in \mathcal{M}$ is structured as a product of independent measures $\mu_s$ over $\mathcal{P}_s$.

Given any policy $\pi$, the distributionally robust value function is the worst-case expectation over the ambiguity set:

$$v^\pi_{\mathcal{M}}(s) = \inf_{\mu \in \mathcal{M}} \mathbb{E}_{p \sim \mu} \left[ v^\pi_p(s) \right] \forall s \in \mathcal{S}$$

and a distributionally robust optimal policy $\pi^*_{\mathcal{M}}$ satisfies $\pi^*_{\mathcal{M}} \in \mathop{\arg\max}_{\pi \in \Pi} v^\pi_{\mathcal{M}}$. Although the DRMDP setting can be very general, the ambiguity set must satisfy specific properties for the solution to be tractable (Xu & Mannor, 2010; Yu & Xu, 2015; Chen et al., 2019).

In the next section, we shall introduce DRMDPs with Wasserstein-distance-based ambiguity sets that yield a solvable reformulation. This setting will further serve us as an analytical tool for deriving a connection between robustness to model uncertainty and regularization in Section 4. Wasserstein DRMDPs will also be of further use in Section 5, as they will enable us to derive out-of-sample guarantees.

### 3. Wasserstein DRMDPs

Statistical-distance-based ambiguity sets have attracted a great deal of interest in data-driven DRO. In particular, Wasserstein-distance-based ambiguity sets have been shown to offer powerful out-of-sample guarantees while exhibiting strong connections with regularization in statistical learning (Blanchet et al., 2016; Kuhn et al., 2019).

In this section, we consider Wasserstein DRMDPs, an instance of DRMDP with a Wasserstein ball as ambiguity set. These will be further used for establishing a dual relation between DRMDPs and regularization (Section 4) and for deriving out-of-sample guarantees in sequential decision-making (Section 5).

#### 3.1. Framework

The Wasserstein metric arises from the theory of optimal transport and measures the optimal transport cost between two probability measures (Villani, 2008). It can intuitively be viewed as the minimal cost required for turning a pile of sand into another, where the cost function corresponds to the amount of sand times its distance to destination. More formally, for any $s \in \mathcal{S}$, let $\| \cdot \|$ be a norm on the uncertainty set $\mathcal{P}_s \subseteq \mathcal{M}(\mathcal{S})^{\mathcal{A}}$. We define the Wasserstein distance over distributions supported on $\mathcal{P}_s$ as follows.

**Definition 3.1 (Wasserstein metric).** Given the norm $\| \cdot \|$, the 1-Wasserstein distance between two probability distributions $\mu_s, \nu_s \in \mathcal{M}(\mathcal{P}_s)$ is defined as:

$$d(\mu_s, \nu_s) := \min_{\gamma \in \Gamma(\mu_s, \nu_s)} \left\{ \int_{\mathcal{P}_s \times \mathcal{P}_s} \| p_s - p'_s \| \gamma(dp_s, dp'_s) \right\}$$

where $\Gamma(\mu_s, \nu_s)$ is the set of distributions over $\mathcal{P}_s \times \mathcal{P}_s$ with marginals $\mu_s$ and $\nu_s$.

Given $\hat{\mu}_s \in \mathcal{M}(\mathcal{P}_s)$, the Wasserstein metric enables us to define the Wasserstein ball of radius $\alpha_s$ centered at $\hat{\mu}_s$ as:

$$\mathcal{M}_{\alpha_s}(\hat{\mu}_s) := \{ \nu_s \in \mathcal{M}(\mathcal{P}_s) : d(\hat{\mu}_s, \nu_s) \leq \alpha_s \},$$

which leads us to introduce the following DRMDP setting.

**Definition 3.2 (Wasserstein DRMDP).** A Wasserstein DRMDP is a tuple $(\mathcal{S}, \mathcal{A}, r, \mathcal{P}, \mathcal{M}_{\alpha}(\hat{\mu}))$ such that $\mathcal{M}_{\alpha}(\hat{\mu}) = \bigotimes_{s \in \mathcal{S}} \mathcal{M}_{\alpha_s}(\hat{\mu}_s)$ and for all $s \in \mathcal{S}$, the transition model $p_s \in \mathcal{P}_s$ is unknown. Indeed, it is a random variable that follows a distribution $\mu_s \in \mathcal{M}_{\alpha_s}(\hat{\mu}_s)$.

#### 3.2. Solving Wasserstein DRMDPs

Define the optimal Wasserstein distributionally robust Bellman operator as:

$$T_{\mathcal{M}_{\alpha}(\hat{\mu})} v(s) := \sup_{\pi_s \in \mathcal{M}(\mathcal{A})} \inf_{\mu_s \in \mathcal{M}_{\alpha_s}(\hat{\mu}_s)} T^\pi_{\hat{\mu}_s} v(s) \forall s \in \mathcal{S},$$

where

$$T^\pi_{\hat{\mu}_s} v(s) := \sum_{a \in \mathcal{A}} \pi_s(a) \left( r(s, a) + \gamma \int_{p_s \in \mathcal{P}_s} \sum_{s' \in \mathcal{S}} v(s') p_s(s'|s, a) d\mu_s(p_s) \right).$$

While we easily see that $T_{\mathcal{M}_{\alpha}(\hat{\mu})}$ is a non-decreasing contraction w.r.t. the sup-norm, the following result ensures that standard planning algorithms can be used for solving a Wasserstein DRMDP. It is a reformulation of Chen et al. (2019)[Theorem 4.5].

**Theorem 3.1.** There exists a policy $\pi^* \in \Pi$ and a unique function $v^*$ such that $v^* = T_{\mathcal{M}_{\alpha}(\hat{\mu})} v^*$. Furthermore, $\pi^*$ is a distributionally robust optimal policy that satisfies for all $s \in \mathcal{S}$

$$v^*(s) = \sup_{\pi_s \in \mathcal{M}(\mathcal{A})} \inf_{\mu_s \in \mathcal{M}_{\alpha_s}(\hat{\mu}_s)} \mathbb{E}_{p \sim \mu} \left[ v^\pi_p(s) \right] = \inf_{\mu_s \in \mathcal{M}_{\alpha_s}(\hat{\mu}_s)} \mathbb{E}_{p \sim \mu} \left[ v^\pi_p(s) \right] = v^\pi_{\mathcal{M}_{\alpha}(\hat{\mu})}(s).$$

Consequently, given an arbitrary $v_0 \in \mathbb{R}^{\mathcal{S}}$, the sequence $(v_k)_k$ defined through $v_{k+1} := T_{\mathcal{M}_{\alpha}(\hat{\mu})} v_k$ converges exponentially fast to the optimal distributionally robust value function (Puterman, 2014). Therefore, one can sequentially construct an optimal strategy using the following procedure:

$$\pi^k_s \in \mathop{\arg\max}_{\pi_s \in \mathcal{M}(\mathcal{A})} \inf_{\mu_s \in \mathcal{M}_{\alpha_s}(\hat{\mu}_s)} T^\pi_{\hat{\mu}_s} v_k(s).$$
Indeed, since the action space $\mathcal{A}$ is finite, the set of distributions $\mathcal{M}(\mathcal{A})$ is compact and $(\pi^k)_{k}$ admits a convergent subsequence $(\pi^k_{s,a})_{k}$. Moreover, any of its limit points yields an optimal random action so that an optimal distributionally robust policy is given by $\hat{\pi}_{\mathcal{M}_{\alpha}}(\cdot|s,a) := \lim_{k \to \infty} \pi^k_{s,a} \in \mathcal{S}$ (Chen et al., 2019)[Lemma 4.3.-Theorem 4.5.]

4. Regularization and Wasserstein DRMDPs

This section addresses the core contributions of our study. We first define the empirical value function as an MDP counterpart of the empirical risk encountered in supervised learning and defined in Section 1. Then, we introduce conjugate robust value functions, which will play a crucial role in our regularization method. The connection between our regularizer and robustness is detailed in Section 4.3.

4.1. Empirical Value Function and Distribution

Given a Wasserstein DRMDP $(\mathcal{S}, \mathcal{A}, r, \mathcal{M}_{\alpha}(\cdot))$, the nominal distribution $\bar{\mu}$ represents the center of a Wasserstein ball. In our setting, $\bar{\mu}$ is taken to be the empirical distribution over transition models and it is estimated based on the history of several episodes.

Tabular State-Space. In the tabular case, $n$ episodes of respective lengths $(T_i)_{i \in [n]}$ enable to estimate the transition model through visit counts as:

$$\hat{p}_i(s'|s,a) := \frac{n_i(s,a,s')}{\sum_{s'' \in \mathcal{S}} n_i(s,a,s'')}$$

where $s, s' \in \mathcal{S}$, $a \in \mathcal{A}$, $i \in [n]$ and $n_i(s,a,s')$ is the number of transitions $(s,a,s')$ occurred during episode $i$.

Large State-Space. When the state space is too large to be stored in a table, the transition model cannot be estimated according to the previous method. Similarly to (Lim & Autef, 2019), kernel averages may be used to approximate the empirical transition function. For all action $a \in \mathcal{A}$, define a kernel $\psi_a : \mathcal{S} \times \mathcal{S} \to \mathbb{R}_+$. Then, for all $s, s' \in \mathcal{S}$, the empirical transition function can be estimated through the following:

$$\hat{p}_i(s'|s,a) := \frac{\psi_a(s,a,s') n_i(s,a,s')}{\sum_{s'' \in \mathcal{S}} \psi_a(s,a,s'') n_i(s,a,s'')} \quad \forall i \in [n]$$

For both tabular and large state-spaces, an estimated transition model $\hat{p}_i$ can be deduced from each episode history, which yields an empirical value function $v^\pi_{\hat{p}_i}$ for any policy $\pi \in \Pi$.

The empirical distribution over transition functions can be defined as the cross product $\hat{\mu}_n := \bigotimes_{(s,a) \in \mathcal{S} \times \mathcal{A}} \hat{p}_n^{s,a}$ where for all $s \in \mathcal{S}, a \in \mathcal{A}, \hat{\mu}^{s,a}_n := \frac{1}{n} \sum_{i=1}^n \delta_{\hat{p}_i(\cdot|s,a)}$, and $\delta_{\hat{p}_i(\cdot|s,a)}$ is a Dirac distribution with full mass on model $\hat{p}_i(\cdot|s,a)$. Setting $\delta_i := \bigotimes_{(s,a) \in \mathcal{S} \times \mathcal{A}} \delta_{\hat{p}_i(\cdot|s,a)}$, the empirical distribution $\hat{\mu}_n$ can further be written as

$$\hat{\mu}_n = \frac{1}{n} \sum_{i=1}^n \delta_i$$

which defines our nominal. Based on the history of $n$ independent episode runs, we thus construct a Wasserstein DRMDP $(\mathcal{S}, \mathcal{A}, r, \mathcal{M}_{\alpha}(\cdot))$ where the ambiguity set $\mathcal{M}_{\alpha}(\hat{\mu}_n)$ is a Wasserstein ball centered at the empirical distribution $\hat{\mu}_n$.

4.2. Conjugate Robust Value Function

We introduce conjugate robust value functions, which provide an analytical tool for establishing a dual relation between regularized empirical value functions and Wasserstein DRMDPs. In that respect, we use technical results from convex analysis that play a crucial role in optimization, as they enable to derive duality results for minimax problems.

Let $\mathcal{P}$ be an uncertainty set of transition models. Additionally, fix a behavior policy $\pi \in \Pi$ and a state $s \in \mathcal{S}$.

**Definition 4.1 (Conjugate Robust Value).** The conjugate robust value function at state $s$ under policy $\pi$ is defined for all $z \in \mathbb{R}^{|\mathcal{S}| \times |\mathcal{A}| \times |\mathcal{S}|}$ as:

$$v^\pi_{s,\pi}(z) := \inf_{p \in \mathcal{P}} v^\pi_p(s) - \langle z, p \rangle.$$ 

Furthermore, we define the effective domain of $v^\pi_{s,\pi}$ as

$$\mathcal{Z} := \{ z \in \mathbb{R}^{|\mathcal{S}| \times |\mathcal{A}| \times |\mathcal{S}|} : v^\pi_{s,\pi}(z) > -\infty \}.$$ 

In words, the conjugate robust value is minus the Legendre-Fenchel transformation of the value function over uncertainty set $\mathcal{P}$. Since RL generally focuses on maximizing the value, we reformulated the conjugate as an infimum instead of the classical transformation $\sup_{p \in \mathcal{P}} (z, p) - v^\pi_p(s)$. This formulation suggests a new representation of the value as a function of the transition law in addition to the initial state.

4.3. Robustification by Regularization

We now hold the tools for establishing our main result, which we address for both tabular and large state-spaces in this section.

4.3.1. The Tabular Case

**Theorem 4.1 (Robustification by Regularization).** Let $(\mathcal{S}, \mathcal{A}, r, \mathcal{M}_{\alpha}(\cdot))$ be a finite Wasserstein DRMDP with a radius-$\alpha$-ball ambiguity set. Then, for any policy $\pi \in \Pi$, it holds that

$$v^\pi_{\mathcal{M}_{\alpha}(\cdot)}(s) \geq \frac{1}{n} \sum_{i=1}^n v^\pi_{\hat{p}_i}(s) - \kappa \alpha$$
where \( \kappa := \sup \{ \| z \|_\alpha : z \in \mathcal{D} \} \) and \( \mathcal{D} \) is the effective domain of \( v^{\ast, \pi} \).

Theorem 4.1 can be formulated as follows: The regularized empirical value function is a lower bound of a Wasserstein DRMMDP. Moreover, our regularization term can easily be expressed as the product between the Wasserstein ball radius and the diameter of the conjugate robust value’s effective domain.

**Sketch of proof.** We briefly describe the proof here. Its full version is provided in the Appendix. First, we explicit the constraints involved in the worst-case expectation \( v^{\ast, \pi}_{\mathcal{M}, \mu}(\hat{\mu}_n)(s) = \inf_{\mu \in \mathcal{M}, \mu} \mathbb{E}_{\pi \sim \mu}[v^{\ast, \pi}_p(s)] \). Then, the minimax inequality and duality arguments enable to derive the following bound:

\[
v^{\ast, \pi}_{\mathcal{M}, \mu}(\hat{\mu}_n)(s) \geq \max_{\lambda, x_1, \ldots, x_n} \frac{1}{n} \sum_{i=1}^n x_i - \lambda \alpha \text{ subject to }
\begin{cases}
\min_{\mu \in \mathcal{M}(S) | |S| \times |A|} & \max_{\|u\|_\pi \leq \lambda} u^{\ast, \pi}_p(s) + \langle u_i, p_p - \hat{p}_i \rangle \geq x_i, \ i \in [n] \\
\lambda & \geq 0
\end{cases}
\]

By definition of the convex closure and Proposition 1.3.13. of (Bertsekas, 2009) we have

\[
\begin{align*}
\max_{\mu \in \mathcal{M}(S) | |S| \times |A|} & \max_{\|u\|_\pi \leq \lambda} u^{\ast, \pi}_p(s) + \langle u_i, p_p - \hat{p}_i \rangle \geq x_i, \ i \in [n] \\
\lambda & \geq 0 \\
\end{align*}
\]

where \( u^{\ast, \pi}_p : \mathcal{P} \rightarrow \mathbb{R} \) denotes the mapping \( p \mapsto v^{\ast, \pi}_p(s) \). We then introduce the conjugate of \( \tilde{\mathcal{c}}(u^{\ast, \pi}_p) \) w.r.t. \( \mathcal{M}(S) | |S| \times |A| \) as:

\[
\tilde{\mathcal{c}}(u^{\ast, \pi}_p)(z) := \max_{\mu \in \mathcal{M}(S) | |S| \times |A|} \left( \langle z, p \rangle - \tilde{\mathcal{c}}(u^{\ast, \pi}_p)(p) \right)
\]

for \( z \in \mathbb{R}^{|S| \times |A| \times |S|} \). Based on the fact that \( \tilde{\mathcal{c}}(u^{\ast, \pi}_p) \) is proper, we can use Theorem 2.1(c) to conclude that \( \tilde{\mathcal{c}}(u^{\ast, \pi}_p) \) coincides with its bi-conjugate function so

\[
\tilde{\mathcal{c}}(u^{\ast, \pi}_p)(p) = \max_{z \in \mathbb{R}^{|S| \times |A| \times |S|}} \left( \langle z, p \rangle - \tilde{\mathcal{c}}(u^{\ast, \pi}_p)^*(z) \right)
= \max_{z \in \mathbb{R}^{|S| \times |A| \times |S|}} \left( \langle z, p \rangle - \tilde{\mathcal{c}}(u^{\ast, \pi}_p)^*(z) \right)
= \max_{z \in \mathbb{R}^{|S| \times |A| \times |S|}} \left( \langle z, p \rangle + v^{\ast, \pi}(z) \right)
\]

where \( \mathcal{Z} = \{ z : \tilde{\mathcal{c}}(u^{\ast, \pi}_p)^*(z) < +\infty \} = \{ z : v^{\ast, \pi}(z) > -\infty \} \) is the effective domain of the conjugate robust value \( v^{\ast, \pi} \). Thus, if we use the reformulation of the convex closure and apply the minimax theorem 1 we obtain

\[
\min_{\mu \in \mathcal{M}(S) | |S| \times |A|} \max_{\|u\|_\pi \leq \lambda} \left( \tilde{\mathcal{c}}(u^{\ast, \pi}_p)(p) + \langle u_i, p_p - \hat{p}_i \rangle \right)
= \max_{\|u\|_\pi \leq \lambda} \max_{z_i \in \mathcal{Z}} \left( v^{\ast, \pi}(z_i) - \sigma_{\mathcal{M}(S) | |S| \times |A|}(z_i - u_i) \right)
= \max_{\|u\|_\pi \leq \lambda} \max_{z_i \in \mathcal{Z}} \left( v^{\ast, \pi}(z_i) - \sigma_{\mathcal{M}(S) | |S| \times |A|}(z_i - u_i) \right)
\]

where \( \sigma_{\mathcal{P}}(u) := \sup_{p \in \mathcal{P}} \langle p, u \rangle \) denotes the support function of a general set \( \mathcal{P} \). We use the conservative bound \( \sigma_{\mathcal{M}(S) | |S| \times |A|} \leq \sigma_{\mathcal{R}[|S| \times |A|]} \) to deduce

\[
\min_{\mu \in \mathcal{M}(S) | |S| \times |A|} \max_{\|u\|_\pi \leq \lambda} \left( \tilde{\mathcal{c}}(u^{\ast, \pi}_p)(p) + \langle u_i, p_p - \hat{p}_i \rangle \right)
= \max_{\|u\|_\pi \leq \lambda} \max_{z_i \in \mathcal{Z}} \left( v^{\ast, \pi}(z_i) - \sigma_{\mathcal{R}[|S| \times |A|]}(z_i - u_i) \right)
= \max_{\|u\|_\pi \leq \lambda} \max_{z_i \in \mathcal{Z}} \left( v^{\ast, \pi}(z_i) - \sigma_{\mathcal{R}[|S| \times |A|]}(z_i - u_i) \right)
\]

Recalling the notation \( \kappa := \sup \{ \| z \|_\alpha : z \in \mathcal{D} \} \), we obtain

\[
\max_{\lambda, x_1, \ldots, x_n} \frac{1}{n} \sum_{i=1}^n x_i - \lambda \alpha \text{ s.t. }
\begin{cases}
\min_{\mu \in \mathcal{M}(S) | |S| \times |A|} & \max_{\|u\|_\pi \leq \lambda} \left( \tilde{\mathcal{c}}(v^{\ast, \pi}_p)(p) + \langle u_i, p_p - \hat{p}_i \rangle \right) \geq x_i \\
\lambda & \geq 0 \\
\geq \max_{\|u\|_\pi \leq \lambda} \frac{1}{n} \sum_{i=1}^n x_i - \lambda \alpha \text{ s.t. } \left( v^{\ast, \pi}_p(s) \right) \geq x_i \\
\lambda & \geq \kappa \\
= \frac{1}{n} \sum_{i=1}^n v^{\ast, \pi}_p(s) - \kappa \alpha
\end{cases}
\]

Putting this altogether yields the desired result.

It is clear from the proof of Theorem 4.1 and it has already been mentioned in (Kuhn et al., 2019)[Theorem 6.3] that restricting the support of distributions to transition functions is what prevents from establishing an exact equivalence. In practice, we could extend the uncertainty set \( \mathcal{M}(S) | |S| \times |A| \) to the whole space \( \mathbb{R}^{|S| \times |A| \times |S|} \) so the ambiguity set would include distributions supported on \( \mathbb{R}^{|S| \times |A| \times |S|} \) and the worst-case expectation would reach the lower-bound. However, this expectation would no longer reflect the value function of an MDP, since \( p \in \mathbb{R}^{|S| \times |A| \times |S|} \) would then not necessarily represent a transition function.

Theorem 4.1 suggests a regularization term structured as the product between the Wasserstein ball radius and the diameter of the effective domain \( \mathcal{D} \). However, since the value function does not necessarily have a closed form and the

\footnote{Proposition 5.5.4. of (Bertsekas, 2009)}
effective domain of its robust conjugate may be hard to assess, this regularized value function cannot be used in practice. The following result provides an upper bound of the diameter, thus enabling to calibrate the regularization term.

**Proposition 4.1.** Define as $L := \frac{\beta \gamma R_{\max}}{(1 - \gamma)^2}$ where $\beta$ is such that $\|\cdot\|_1 \leq \beta \|\cdot\|$ and consider $\hat{u}_\pi^\ast : \mathbb{R}^{|S| \times |A| \times |S|} \rightarrow \mathbb{R}$ the continuation function of $u_\pi^\ast$ which is defined as

$$\hat{u}_\pi^\ast(\hat{p}) := \inf_{\hat{p} \in \mathcal{M}_\alpha(\hat{\mu}_n)} u_\pi^\ast(p) + L \|p - \hat{p}\|$$

Further denote by $\tilde{\delta} := \{\tilde{z} : (\hat{u}_\pi^\ast)^+ (\tilde{z}) < +\infty\}$. If $\tilde{\delta} \subseteq \delta$, then $\kappa \leq L$.

Based on the above proposition, if $\tilde{\delta} \subseteq \tilde{\delta}$, then

$$v_{\mathcal{M}_\alpha(\hat{\mu}_n)}(s) \geq \frac{1}{n} \sum_{i=1}^{n} v_{\hat{p}_i}^\pi(s) - \frac{\beta \gamma R_{\max}}{(1 - \gamma)^2} \alpha$$

and since $v_{\mathcal{M}_\alpha(\hat{\mu}_n)}^\pi(s) = \inf_{\mu \in \mathcal{M}_\alpha(\hat{\mu}_n)} \mathbb{E}_{p \sim \mu}[v_{\pi}^\ast(s)]$ we obtain the following bound:

$$\frac{1}{n} \sum_{i=1}^{n} v_{\hat{p}_i}^\pi(s) \geq v_{\mathcal{M}_\alpha(\hat{\mu}_n)}^\pi(s) \geq \frac{1}{n} \sum_{i=1}^{n} v_{\hat{p}_i}^\pi(s) - \frac{\beta \gamma R_{\max}}{(1 - \gamma)^2} \alpha.$$

This inequality suggests to use the product $L \alpha$ as a practical regularizer that yields a lower bound of the distributionally robust value function. In particular, the regularized value function

$$\frac{1}{n} \sum_{i=1}^{n} v_{\hat{p}_i}^\pi(s) - \frac{\beta \gamma R_{\max}}{(1 - \gamma)^2} \alpha$$

is guaranteed to be distributionally robust w.r.t. the Wasserstein ball $\mathcal{M}_\alpha(\hat{\mu}_n)$ centered at the empirical distribution. This is of particular use for RL methods, as it enables to ensure better generalization without resorting the additional computations that DRMDP requires. As a matter of fact, standard value iteration can be performed to learn $v_{\hat{p}_i}^\pi(s)$ for all $i \in [n]$ and subtracting $L \alpha$ to the resulting average ensures distributional robustness w.r.t. the ambiguity set $\mathcal{M}_\alpha(\hat{\mu}_n)$.

### 4.3.2. The Linear Approximation Case

When the state-space is large, one generally approximates the value function using feature vectors. Specifically, define as $\Phi(\cdot) \in \mathbb{R}^m$ a feature vector function such that for all $p \in \mathcal{P}$ we have $v_p^\pi(s) \approx \Phi(s)^\top w_p$ and assume all feature vectors are linearly independent. Based on this approximation, we define an approximate conjugate robust value as:

$$w_{\pi^\ast}(z) := \inf_{p \in \mathcal{P}} \Phi(s)^\top w_p - \langle z, p \rangle \quad \forall z \in \mathbb{R}^{|S| \times |A| \times |S|}.$$  

We also denote by $\mathcal{M} := \{z : w_{\pi^\ast}(z) > -\infty\}$ the effective domain of the approximate conjugate robust value.

Under standard conditions, Theorem 4.1 generalizes to large scale MDPs using this linear value function approximation.

**Theorem 4.2.** Let $(\mathcal{S}, \mathcal{A}, r, \mathcal{M}_\alpha(\hat{\mu}_n))$ be a Wasserstein DRMDP. Then, for any policy $\pi \in \Pi$ it holds that

$$\inf_{\mu \in \mathcal{M}_\alpha(\hat{\mu}_n)} \mathbb{E}_{p \sim \mu}[\pi^\ast] w_p - \eta \alpha \geq \frac{1}{n} \sum_{i=1}^{n} \Phi(s)^\top w_{\hat{p}_i} - \eta \alpha$$

where $\eta := \sup\{\|z\|_\infty : z \in \mathcal{M}\}$.

### 5. Out-of-Sample Performance Guarantees

Assume that for each episode, a transition model $p$ has been generated according to a true (but unknown) distribution $\mu$. A potential defect of non-robust MDP formulations is that optimal policies may perform poorly once deployed on new, unseen data. In this section, we provide a guaranteed out-of-sample performance in Wasserstein DRMDPs for carefully determined Wasserstein-ball radii.

Given the Wasserstein DRMDP $(\mathcal{S}, \mathcal{A}, r, \mathcal{P}, \mathcal{M}_\alpha(\hat{\mu}_n))$, the nominal distribution $\hat{\mu}_n$ represents the empirical distribution over transition models and it is estimated based on the history of several episodes, as depicted in Section 4.1. From a statistical learning viewpoint, the transition models estimates can be interpreted as a training set $\hat{\mathcal{P}}_n := \{(\hat{p}_i)_{1 \leq i \leq n}\}$ which in turn represents a random vector that follows a distribution $\mu^\pi$ supported on $\mathcal{P}$. To avoid cluttered notation, we shall denote by $\hat{\pi}^\ast := \pi_{\mathcal{M}_\alpha(\hat{\mu}_n)}^\pi$ an optimal policy for the Wasserstein DRMDP $(\mathcal{S}, \mathcal{A}, r, \mathcal{P}, \mathcal{M}_\alpha(\hat{\mu}_n))$ induced by the training set and $\hat{\pi}^\ast := v_{\mathcal{M}_\alpha(\hat{\mu}_n)}^\pi$ the optimal distributionally robust value function. Then, the out-of-sample performance of $\hat{\pi}^\ast$ is given by $\mathbb{E}_{p \sim \mu}[v_p^\pi(s)]$. The following result establishes that $\hat{\pi}^\ast$ satisfies a performance guarantee of the type

$$\mu^n(A) \geq 1 - \epsilon$$

where $\epsilon \in (0, 1)$ determines the confidence level, $A$ denotes the event

$$A := \{\hat{p} : \mathbb{E}_{p \sim \mu}[v_p^\pi(s)] \geq v_{\mathcal{M}_\alpha(\hat{\mu}_n)}^\pi(s) \quad \forall s \in \mathcal{S}\}$$

and $v_{\mathcal{M}_\alpha(\hat{\mu}_n)}^\pi(s)$ is a certificate for the out-of-sample performance. This bound is the best we can hope for, as the true generating distribution $\mu$ is unknown. The proof is based on (Fournier & Guillin, 2015)[Theorem 2] and uses the contracting property of $T_{\mathcal{M}_\alpha(\hat{\mu}_n)}$. While similar result has been suggested by Yang (2018) in a stochastic control setting, Theorem 5.1 generalizes this statement to MDPs. It establishes the fact that with high probability, the distributionally robust optimal policy cannot yield lower value than the certificate performance $v_{\mathcal{M}_\alpha(\hat{\mu}_n)}^\pi(s)$ that resulted
from the training set $\hat{P}_n$. As a corollary result, the regularized value function a fortiori satisfies this performance guarantee.

**Theorem 5.1 (Finite-sample Guarantee).** Let $\epsilon \in (0,1)$, $m := |S| \times |A|$. Denote by $\hat{\pi}^*$ an optimal policy of the Wasserstein DRMDP $(S, A, r, \mathcal{M}_{\alpha(n,\epsilon)}(\hat{\mu}_n))$ and $\tilde{v}^*$ its optimal value. If for all $s \in S$ the radius of the Wasserstein ball at $s$ satisfies

$$\alpha_s(n_s, \epsilon) := \left\{ \begin{array}{ll} c_0 \left( \frac{1}{n_s c_2} \log \left( \frac{c_1}{\epsilon} \right) \right)^{1/(m^2)} & \text{if } n_s \geq C_m^c, \\ c_0 & \text{o.w.} \end{array} \right.$$  

with $C_m^c := \frac{1}{c_2} \log \left( \frac{\hat{\alpha}}{c_1} \right)$ and

$$n_s := \sum_{i \in [n], a \in A, s' \in S} n_s(s, a, s'),$$

then it holds that

$$\mu^n \left( \left\{ \tilde{p} \mid \mathbb{E}_{p \sim \mu}[v_p^{\hat{\pi}}(s)] \geq v_{\mathcal{M}_{\alpha(n,\epsilon)}(\hat{\mu}_n)}(\hat{\pi}_n)(s) \quad \forall s \in S \right\} \right) \geq 1 - \epsilon$$

where $c_0, c_1, c_2$ are positive constants that only depend on $m \neq 2^2$.

The positive constant $c_0$ corresponds to the diameter of the Wasserstein space $\mathcal{M}(\mathcal{P}_s)$. Therefore, if the sample size is smaller than $C_m^c$, then the Wasserstein DRMDP as defined above recovers a robust MDP with uncertainty set $\mathcal{P}_s$. Moreover, for any fixed $\epsilon > 0$, the radius $\alpha_s(n_s, \epsilon)$ tends to 0 when $n_s$ goes to infinity, which ensures the solution to become less conservative as the sample size increases.

**6. Related Work**

Our work is at the crossroads of DRO theory, statistical learning and robust RL. More precisely, it uses analytical tools from DRO theory to derive out-of-sample performance guarantees for Wasserstein DRMDPs. Moreover, it enables to establish a relation with regularization, thus building the bridge with statistical learning.

Regularization in statistical learning precedes its robust formulations. Indeed, a robust optimization interpretation has first been suggested by Xu et al. (2009) for support vector machines, long after the regularization methods of Vapnik (2013). Then, advancing research on data-driven DRO has enabled to establish an equivalence between robustness and regularization in a wider range of statistical learning problems (Shafieezadeh-Abadeh et al., 2015; 2017; Blanchet et al., 2016; Kuhn et al., 2019). Differently, in RL, RMDPs date back to 2005 with the concurrent works of Iyengar (2005) and Nilim & El Ghaoui (2005) while to our knowledge, our study suggests the first connection between regularization and robustness in RL.

DRMDPs have been introduced in (Xu & Mannor, 2010; Yu & Xu, 2015) for specifically structured ambiguity sets motivated by confidence interval estimates. Then, advances in distributionally robust optimization have inspired new models on parameter uncertainty, suggesting various types of ambiguity sets. These can be based on specified properties such as moment constraints (Delage & Ye, 2010; Bertsimas et al., 2018; Wiesemann et al., 2014), or on a given divergence from the empirical distribution. Popular choices of divergence include Kullback-Leibler divergence (Hu & Hong, 2013), φ-divergences (Ben-Tal et al., 2013) as well as the Prohorov metric (Erdogan & Iyengar, 2006) and Wasserstein distances (Esfahani & Kuhn, 2017). Yang (2017) introduced Wasserstein DRMDPs for the finite horizon case and proposed a tractable variant of dynamic programming to solve them. An extension to the infinite horizon case has been proposed in (Chen et al., 2019) while a similar stochastic control setting has been studied in (Yang, 2018).

Finally, the emerging field of distributional RL (Bellemare et al., 2017) differs from our approach. There, an optimal policy is learned through the internal distribution of the cumulative reward while in our setting, we study its worst-case expectation to account for the external uncertainty of the MDP parameters. Moreover, differently than our work, the metric used in (Bellemare et al., 2017) for measuring the distance between two distributional value functions is a minimum of Wasserstein distances over the state-action space. Such a metric may be problematic in the Wasserstein DRMDP setting, as the rectangularity assumption would falter.

**7. Discussion**

Our study highlighted the possibility of rewriting the distributionally robust value function as a regularized problem for sequential decision-making. Interestingly, unlike most of the frameworks for solving MDPs, e.g., value iteration, policy iteration, most variants of policy optimization and others that rely on the contracting properties of the Bellman operators without being convex, our analysis requires solving convex optimization problems exclusively. This may open up new algorithmic avenues, where recent advances in convex optimization are harnessed to solve planning and learning problems.

Our regularization approach simplifies robust RL. Since it yields a lower-bound of the distributionally robust value, further study should analyze that bound’s tightness. Other compelling directions for future work include the exten-
sion of our results to non-linear function approximation and deep architectures (Levine et al., 2017). It would be interesting to experimentally test if this additional regularization component ensures better generalization. One can consider extensions to policy optimization w.r.t. our regularized formulation and build a connection with regularized policy search. Another natural extension would be to analyze the asymptotic consistency of our approach for increasing sample size, in the same spirit of (Esfahani & Kuhn, 2017)[Theorem 3.6.] for the statistical learning setup.

Acknowledgements

The authors would like to thank Shimrit Shtern for pointers to the relevant literature. Thanks also to Shirli Di Castro Shashua, Guy Tennenholtz and Nadav Merlis for their comprehensive review of an earlier draft.

References

Barbu, V. and Precupanu, T. Convexity and optimization in Banach spaces. Springer Science & Business Media, 2012.

Bellemare, M. G., Dabney, W., and Munos, R. A distributional perspective on reinforcement learning. In Proceedings of the 34th International Conference on Machine Learning-Volume 70, pp. 449–458. JMLR. org, 2017.

Bellman, R. A Markovian decision process. Journal of mathematics and mechanics, pp. 679–684, 1957.

Ben-Tal, A., den Hertog, D., De Waegenaere, A., Melenberg, B., and Rennen, G. Robust solutions of optimization problems affected by uncertain probabilities. Management Science, 59(2):341–357, 2013.

Bertsekas, D. P. Convex optimization theory. Athena Scientific Belmont, 2009.

Bertsekas, D. P. and Tsitsiklis, J. N. Neuro-Dynamic Programming. Athena Scientific, 1st edition, 1996. ISBN 1886529108.

Bertsimas, D., Sim, M., and Zhang, M. Adaptive distributionally robust optimization. Management Science, 65 (2):604–618, 2018.

Blanchet, J., Kang, Y., and Murthy, K. Robust Wasserstein profile inference and applications to machine learning. arXiv:1610.05627, 2016.

Chen, Z., Yu, P., and Haskell, W. B. Distributionally robust optimization for sequential decision-making. Optimization, 68(12):2397–2426, 2019.

Delage, E. and Ye, Y. Distributionally robust optimization under moment uncertainty with application to data-driven problems. Operations Research, 58(3):595–612, 2010.

Derman, E., Mankowitz, D., Mann, T., and Mannor, S. Soft-robust actor-critic policy-gradient. AUA press for Association for Uncertainty in Artificial Intelligence, pp. 208–218, 2018.

Esfahani, P. and Kuhn, D. Data-driven distributionally robust optimization using the Wasserstein metric: performance guarantees and tractable reformulations. Mathematical Programming, 2017.

Esfahani, P. and Kuhn, D. Robust Wasserstein distributionally robust optimization: Performance guarantees and tractable reformulations. Mathematical Programming, 2017.

Farahmand, A. M. Regularization in reinforcement learning. 2011.

Farahmand, A. M., Ghavamzadeh, M., Mannor, S., and Szepesvári, C. Regularized policy iteration. In Advances in Neural Information Processing Systems, pp. 441–448, 2009.

Fournier, N. and Guillin, A. On the rate of convergence in Wasserstein distance of the empirical measure. Probability Theory and Related Fields, 162(3-4):707–738, 2015.

Friedman, J., Hastie, T., and Tibshirani, R. The elements of statistical learning, volume 1. Springer series in statistics New York, 2001.

Howard, R. A. Dynamic programming and Markov processes. 1960.

Hu, Z. and Hong, L. J. Kullback-Leibler divergence constrained distributionally robust optimization. Available at Optimization Online, 2013.

Iyengar, G. N. Robust dynamic programming. Mathematics of Operations Research, 30(2):257–280, 2005.

Kuhn, D., Esfahani, P. M., Nguyen, V. A., and Shafieezadeh-Abadeh, S. Wasserstein distributionally robust optimization: Theory and applications in machine learning. In Operations Research & Management Science in the Age of Analytics, pp. 130–166. INFORMS, 2019.

Lange, S., Gabel, T., and Riedmiller, M. Batch reinforcement learning. In Reinforcement learning, pp. 45–73. Springer, 2012.
Levine, N., Zahavy, T., Mankowitz, D. J., Tamar, A., and Mannor, S. Shallow updates for deep reinforcement learning. In Advances in Neural Information Processing Systems, pp. 3135–3145, 2017.

Lim, S. H. and Autef, A. Kernel-based reinforcement learning in robust Markov decision processes. In International Conference on Machine Learning, pp. 3973–3981, 2019.

Mannor, S., Simester, D., Sun, P., and Tsitsiklis, J. N. Bias and variance approximation in value function estimates. Management Science, 53(2):308–322, 2007.

Nilim, A. and El Ghaoui, L. Robust control of Markov decision processes with uncertain transition matrices. Operations Research, 53(5):780–798, 2005.

Petrik, M. and Russell, R. H. Beyond confidence regions: Tight Bayesian ambiguity sets for robust MDPs. arXiv preprint arXiv:1902.07605, 2019.

Puterman, M. L. Markov Decision Processes: Discrete Stochastic Dynamic Programming. John Wiley & Sons, 2014.

Roy, A., Xu, H., and Pokutta, S. Reinforcement learning under model mismatch. In Advances in neural information processing systems, pp. 3043–3052, 2017.

Ruszczynski, A. Risk-averse dynamic programming for markov decision processes. Mathematical programming, 125(2):235–261, 2010.

Shafieezadeh-Abadeh, S., Esfahani, P., and Kuhn, D. Distributionally robust logistic regression. Advances in Neural Information Processing Systems, pp. 1576–1584, 2015.

Shafieezadeh-Abadeh, S., Kuhn, D., and Esfahani, P. Regularization via mass transportation. arXiv:1710.10016, 2017.

Shani, L., Efroni, Y., and Mannor, S. Adaptive trust region policy optimization: Global convergence and faster rates for regularized mdp’s. arXiv preprint arXiv:1909.02769, 2019.

Strehl, A. L. and Littman, M. L. An analysis of model-based interval estimation for markov decision processes. Journal of Computer and System Sciences, 74(8):1309–1331, 2008.

Tamar, A., Mannor, S., and Xu, H. Scaling up robust MDPs using function approximation. In International Conference on Machine Learning, pp. 181–189, 2014.

Tamar, A., Chow, Y., Ghavamzadeh, M., and Mannor, S. Policy gradient for coherent risk measures. In Advances in Neural Information Processing Systems, pp. 1468–1476, 2015.
A. Wasserstein DRMDPs

A.1. Proof of Theorem 3.1

Although the proof of this result can be found in (Chen et al., 2019), we briefly recall its main steps here.

**Theorem.** There exists a policy \( \pi^* \in \Pi \) and a unique function \( v^* \) such that \( v^* = T_{\mathcal{M}(\mu)} \pi^* \). Furthermore, \( \pi^* \) is a distributionally robust optimal policy that satisfies for all \( s \in S \)

\[
v^*(s) = \sup_{\pi \in \Pi} \inf_{\mu \in \mathcal{M}(\mu)} \mathbb{E}_{p \sim \mu}[v^*_p(s)] = \inf_{\mu \in \mathcal{M}(\mu)} \mathbb{E}_{p \sim \mu}[v^*_p(s)] = v^*_\mathcal{M}(\mu)(s).
\]

**Proof.** Step 1 (Chen et al., 2019)[Lemma 4.2.]. We show that \( T_{\mathcal{M}(\mu)} \) is a non-decreasing \( \gamma \)-contraction w.r.t. the sup-norm, using standard RL tools (Puterman, 2014). By the Banach fixed-point theorem, the operator \( T_{\mathcal{M}(\mu)} \) admits a unique fixed point \( v^* \) and for any initial value \( v_0 \), the sequence \( (v_k)_{k \geq 0} \) defined as \( v_k := (T_{\mathcal{M}(\mu)})^k v_0 \) for all \( k \geq 0 \) converges exponentially fast to \( v^* \).

Now given \( (v_k) \), we build a sequence of policies \( (\pi^*_k) \) such that for all \( k \geq 0 \):

\[
\pi^*_k(s) \in \arg \max_{\pi \in \Pi} \inf_{\mu \in \mathcal{M}(\mu)} T_{\mu} \pi k(s).
\]

Since \( \mathcal{M}(A) \) is compact, the sequence \( (\pi^*_k) \) admits a subsequence \( (\pi^*_k) \) converging to some limit point \( \pi^*_\infty \).

**Step 2** (Chen et al., 2019)[Lemma 4.3.]. We show that for all \( s \in S \),

\[
\pi^*_\infty(s) \in \arg \max_{\pi \in \Pi} \inf_{\mu \in \mathcal{M}(\mu)} T_{\mu} \pi^*_\infty(s).
\]

**Step 3** (Chen et al., 2019)[Theorem 4.5.]. We finally show that the previous condition \( \pi^*_\infty(s) \in \arg \max_{\pi \in \Pi} \inf_{\mu \in \mathcal{M}(\mu)} T_{\mu} \pi^*_\infty(s) \) is sufficient for \( \pi^*_\infty \) to be a distributionally robust optimal policy, so \( \pi^*_\infty = \pi^* \) and the conclusion follows.

B. Regularization and Wasserstein DRMDPs

B.1. Proof of Theorem 4.1

We will use the following result throughout the proof.

**Lemma B.1.** For all policy \( \pi \in \Pi \) and state \( s \in S \) define the mapping \( u^*_\pi : \mathcal{P} \rightarrow \mathbb{R} \) as \( p \mapsto v^*_p(s) \). Then, the following holds:

(i) \( u^*_\pi \) is proper.

(ii) If \( \mathcal{P} \) is closed, then \( u^*_\pi \) is continuous and thus, it is a closed function.

(iii) For \( \mathcal{P} = \mathcal{M}(S) | S \times |A| \), the convex closure \( \bar{c}(u^*_\pi) \) of \( u^*_\pi \) is proper.

**Proof.** Claim (i). By assumption, the reward function is bounded by \( R_{\max} \), so we have

\[
|v^*_p(s)| = \left| \mathbb{E}_p \left[ \sum_{t=0}^{\infty} \gamma^t r(s_t, a_t) \right] s_0 = s \right| \leq \sum_{t=0}^{\infty} \gamma^t R_{\max} = \frac{R_{\max}}{1 - \gamma}
\]

and \( u^*_\pi \) is proper.

Claim (ii). Denote by \( (p_n)_{n \geq 1} \in \mathcal{P}^N \) a sequence that converges to \( p \). Since \( \mathcal{P} \) is closed, \( p \in \mathcal{P} \) and \( u^*_\pi(p) \) is well defined. Moreover, \( u^*_\pi(p) = v^*_p(s) \). For all \( n \geq 1 \) we introduce the Bellman operator \( T_{p_n}^\pi \) w.r.t. transition \( p_n \):

\[
T_{p_n}^\pi v(s) = \sum_{a \in A} \pi_n(a) r(s, a) + \gamma \sum_{a \in A} \pi_n(a) \sum_{s' \in S} p_n(s, a, s') v(s')
\]

and \( T_{p}^\pi \) the Bellman operator w.r.t. transition \( p \):

\[
T_{p}^\pi v(s) = \sum_{a \in A} \pi(a) r(s, a) + \gamma \sum_{a \in A} \pi(a) \sum_{s' \in S} p(s, a, s') v(s')
\]
Then, we have
\[
\lim_{n \to \infty} T_p^n v(s) = \sum_{a \in A} \pi_s(a) r(s, a) + \gamma T_p \sum_{a \in A} \pi_s(a) \sum_{s' \in S} p_n(s, a, s') v(s')
\]
where equality (a) holds since $S$ and $A$ are finite sets. Remark that here, we established the continuity of the Bellman operator with respect to the transition function.

As a result, for all $\epsilon > 0$, there exists $n_\epsilon$ such that for all $n \geq n_\epsilon$ we have $|T_p^n v_p^n(s) - T_p^n v_p^n(s)| \leq (1 - \gamma) \epsilon$. Using the fact that $v_p^n$ (resp. $v_p^n$) is the unique fixed point of $T_p^n$ (resp. $T_p^n$) and that $T_p^n$ is a $\gamma$-contraction, for all $n \geq n_\epsilon$ we can write:
\[
|v_p^n(s) - v_p^n(s)| = |T_p^n v_p^n(s) - T_p^n v_p^n(s)|
\]
\[
\leq |T_p^n v_p^n(s) - T_p^n v_p^n(s)| + |T_p^n v_p^n(s) - T_p^n v_p^n(s)|
\]
\[
\leq (1 - \gamma) \epsilon + \gamma |v_p^n(s) - v_p^n(s)|
\]
so $(1 - \gamma)|v_p^n(s) - v_p^n(s)| \leq (1 - \gamma) \epsilon$. Since $\gamma \in (0, 1)$, $(1 - \gamma)$ is positive and dividing both sides by $(1 - \gamma)$ yields $|v_p^n(s) - v_p^n(s)| \leq \epsilon$. Based on the fact that $\epsilon > 0$ is arbitrary, we have shown that $v_p^n(s) \to_{n \to \infty} v_p^n(s)$, which concludes the proof.

**Claim (iii).** By definition, $\overline{\text{cl}}(u_p^n) \leq u_p^n$ and $u_p^n$ is proper by Claim (i). Therefore, there exists $p \in M(S)^{|S| \times |A|}$ such that $\overline{\text{cl}}(u_p^n)(p) < +\infty$. Moreover, based on (Bertsekas, 2009)[Proposition 1.3.13.], we have
\[
\inf_{p \in M(S)^{|S| \times |A|}} u_p^n(s) = \inf_{p \in M(S)^{|S| \times |A|}} \overline{\text{cl}}(u_p^n)(p).
\]
Since $M(S)^{|S| \times |A|}$ is closed and compact, Claim (ii) ensures that $u_p^n$ is continuous and thus, the infimum is a minimum besides being finite i.e.,
\[
\min_{p \in M(S)^{|S| \times |A|}} u_p^n(s) = \inf_{p \in M(S)^{|S| \times |A|}} \overline{\text{cl}}(u_p^n)(p) > -\infty.
\]
In particular, $\overline{\text{cl}}(u_p^n)(p) > -\infty$ for all $p \in M(S)^{|S| \times |A|}$, which concludes the proof.

We now prove Theorem 4.1, whose statement is recalled below.

**Theorem (Robustification by Regularization).** Let $(S, A, r, M_\alpha(\mu_n))$ be a finite Wasserstein DRMDP with a radius-$\alpha$-ball ambiguity set. Then, for any policy $\pi \in \Pi$, it holds that
\[
u_{M_\alpha(\mu_n)}^\pi(s) \geq \frac{1}{n} \sum_{i=1}^n v_{\mu_i}^\pi(s) - \kappa \alpha
\]
where $\kappa := \sup\{\|z\|_\infty : z \in \mathcal{Z}\}$ and $\mathcal{Z}$ is the effective domain of $v_p^n$.

**Proof.** Without loss of generality, we consider the stationary model formulation where the distribution over transitions is initially chosen by Nature and remains fixed thereafter. Indeed, dynamic and stationary models are equivalent when the horizon is infinite, as depicted in (Nilim & El Ghaoui, 2005; Xu & Mannor, 2010; Chen et al., 2019).

The stationary model is given by
\[
\mathcal{M}^\alpha(\alpha) = \left\{ \tilde{\mu} \in \mathbb{R}^{|S| \times |A|} : \mu_t = \mu \quad \forall t \geq 0; \mu \in M_\alpha(\mu_n) \right\}
\]
and we have

$$v_p^\pi(\mu)(s) = \inf_{\mu \in \mathcal{M}_a(\hat{\mu}_s)} \mathbb{E}_{p \sim \hat{\mu}}[v_p^\pi(s)] = \inf_{\mu \in \mathcal{M}_a(\alpha)} \mathbb{E}_{p \sim \mu}[v_p^\pi(s)].$$

The constraint $\hat{\mu} \in \mathcal{M}^\alpha(\alpha)$ corresponds to the following:

$$\begin{cases}
\hat{\mu} = \bigotimes_{t \geq 0} \mu_t; \\
\mu_t = \mu_s \quad \forall t \geq 0; \\
\mu = \bigotimes_{s \in S} \mu_s \\
\mu_s \in \mathcal{M}_a(\hat{\mu}_s^n) \quad \forall s \in S.
\end{cases}$$

Denote $\hat{p}_s^i := \bigotimes_{a \in A} \hat{p}_{s,a}^i$. By definition of the ambiguity set, $\mu_s \in \mathcal{M}_a(\hat{\mu}_s^n)$ if and only if $d(\mu_s, \hat{\mu}_s^n) \leq \alpha_s$. Recalling the definition of the Wasserstein metric

$$d(\mu_s, \hat{\mu}_s^n) := \min_{\gamma \in \mathcal{P}(\mu_s, \hat{\mu}_s^n)} \left\{ \int_{\mathcal{P}_s \times \mathcal{P}_s} \gamma(dp_s, dp_s') \right\},$$

and the empirical distribution $\hat{\mu}_s^n = \frac{1}{n} \sum_{i=1}^n \delta_{\hat{p}_s}(\cdot)$, the constraint $\mu_s \in \mathcal{M}_a(\hat{\mu}_s^n)$ is equivalent to the existence of $\mu_1^s, \ldots, \mu_n^s \in \mathcal{M}(\mathcal{P}_s)$ such that $\mu_s = \frac{1}{n} \sum_{i=1}^n \mu_i^s$ and $\frac{1}{n} \sum_{i=1}^n \mathbb{E}_{\hat{p}_s \sim \mu_i^s} ||p_s - \hat{p}_s|| \leq \alpha_s$.

Define $\mu_i := \bigotimes_{s \in S} \mu_i^s$ and $p_i := \bigotimes_{s \in S} \hat{p}_{s,a}^i$ and consider the product space $\mathcal{M}(S)^{|S| \times |A|}$ with the product norm corresponding to $\| \cdot \|$. e.g., if $\| \cdot \| = \| \cdot \|_2$ on $\mathcal{P}_s$, take the $\ell_2$-norm on $\mathcal{P}$ defined as $\|p\|_2 := \left( \sum_{s \in S} \|p_s\|_2^2 \right)^{1/2}$. Then, with the slight abuse of notation $\|p\| \equiv \|p_s\|$, the worst-case distributionally robust value function may be formulated as

$$\inf_{\mu \in \mathcal{M}(\alpha)} \mathbb{E}_{p \sim \mu}[v_p^\pi(s)] = \min_{\hat{\mu} \in \mathcal{M}(\alpha)} \mathbb{E}_{p \sim \hat{\mu}}[v_p^\pi(s)] \text{ s.t.}$$

$$\frac{1}{n} \sum_{i=1}^n \mathbb{E}_{p_i \sim \mu_i} \|p_i - \hat{p}_i\| \leq \alpha.$$
Here, we remark that since $S$ is finite, $\mathcal{M}(S)$ is the $(|S| - 1)$-dimensional simplex which is compact so the infima in the first-line constraints are minima. Moreover, by definition of the dual norm $\|\cdot\|_*$, the right hand side of the inequality is equivalent to the following:

$$
\max_{\lambda, x_1, \ldots, x_n} \frac{1}{n} \sum_{i=1}^{n} x_i - \lambda \alpha \text{ s.t. } \begin{cases} 
\min_{p \in \mathcal{M}(S) \times |A| \cup_u \|u_i\|_* \leq \lambda} (v^*_{p}(s) + \langle u_i, p - \tilde{p}_i \rangle) \geq x_i \quad \forall i \in [n] \\
\lambda \geq 0
\end{cases}
$$

Consequently, the worst-case expectation can be reformulated as

$$
\inf_{\tilde{\mu} \in \mathcal{M}^*(\alpha)} \mathbb{E}_{p \sim \tilde{\mu}}[v^*_{p}(s)] \geq \max_{\lambda, x_1, \ldots, x_n} \frac{1}{n} \sum_{i=1}^{n} x_i - \lambda \alpha \text{ s.t. } \begin{cases} 
\min_{p \in \mathcal{M}(S) \times |A| \cup_u \|u_i\|_* \leq \lambda} (v^*_{p}(s) + \langle u_i, p - \tilde{p}_i \rangle) \geq x_i \quad \forall i \in [n] \\
\lambda \geq 0
\end{cases}
$$

By definition of $u^*_s : p \mapsto v^*_{p}(s)$ and its convex closure $\tilde{c}(u^*_s)$, we have the following inclusion of feasible points

$$
\left\{ \lambda, x_i : \min_{p \in \mathcal{M}(S) \times |A| \cup_u \|u_i\|_* \leq \lambda} \left( \tilde{c}(u^*_s)(p) + \langle u_i, p - \tilde{p}_i \rangle \right) \geq x_i \right\} \subseteq \left\{ \lambda, x_i : \min_{p \in \mathcal{M}(S) \times |A| \cup_u \|u_i\|_* \leq \lambda} \left( v^*_{p}(s) + \langle u_i, p - \tilde{p}_i \rangle \right) \geq x_i \right\}
$$

which is in fact a double inclusion, by Proposition 1.3.13. of (Bertsekas, 2009). This implies that

$$
\max_{\lambda, x_1, \ldots, x_n} \frac{1}{n} \sum_{i=1}^{n} x_i - \lambda \alpha \text{ s.t. } \begin{cases} 
\min_{p \in \mathcal{M}(S) \times |A| \cup_u \|u_i\|_* \leq \lambda} (v^*_{p}(s) + \langle u_i, p - \tilde{p}_i \rangle) \geq x_i \\
\lambda \geq 0
\end{cases}
$$

Now introduce the conjugate transform of $\tilde{c}(u^*_s)$ w.r.t. uncertainty set $\mathcal{P} := \mathcal{M}(S) \times |A|$ as:

$$
\tilde{v}^*_{s, \pi}^*(z) := - \max_{p \in \mathcal{M}(S) \times |A|} \left( \langle z, p \rangle - \tilde{c}(u^*_s)(p) \right) = -\tilde{c}(u^*_s)^*(z)
$$

By Theorem 2.1(d) and by definition of the conjugate robust value (Def. 4.1), $\tilde{c}(u^*_s)^*(z) = (u^*_s)^*(z)$ so $\tilde{v}^*_{s, \pi}(z) = v^*_{s, \pi}(z) = -u^*_{s, \pi}(z)$. Moreover, by Lemma B.1, the function $\tilde{c}(u^*_s)$ is proper, so using Theorem 2.1(c), $\tilde{c}(u^*_s)$ coincides with its bi-conjugate function and

$$
\tilde{c}(u^*_s)(p) = \tilde{c}(u^*_s)^{**}(p) = \max_{\zeta \in \mathcal{Z}} \langle \zeta, p \rangle - \tilde{c}(u^*_s)^*(p) = \max_{\zeta \in \mathcal{Z}} \langle \zeta, p \rangle - (u^*_s)^*(p) = \max_{\zeta \in \mathcal{Z}} \langle \zeta, p \rangle + u^*_{s, \pi}(\zeta)
$$

where $\mathcal{Z} := \{ z : \tilde{v}^*_{s, \pi}(z) > -\infty \} = \{ z : v^*_{s, \pi}(z) > -\infty \}$ is the effective domain of $v^*_{s, \pi}$. Thus, if we use the reformulation of the convex closure and apply the minimax theorem \footnote{Proposition 5.5.4. of (Bertsekas, 2009)} we obtain

$$
\min_{p \in \mathcal{M}(S) \times |A| \cup_u \|u_i\|_* \leq \lambda} \max_{p \in \mathcal{M}(S) \times |A| \cup_u \|u_i\|_* \leq \lambda} \left( \tilde{c}(u^*_s)(p) + \langle u_i, p - \tilde{p}_i \rangle \right) = \min_{z_i \in \mathcal{Z}} \max_{\|u_i\|_* \leq \lambda} v^*_{s, \pi}(z_i) + \langle p, z_i \rangle + \langle u_i, p - \tilde{p}_i \rangle
$$

$$
= \max_{z_i \in \mathcal{Z}} \max_{\|u_i\|_* \leq \lambda} v^*_{s, \pi}(z_i) + \langle p, z_i \rangle + \langle u_i, p - \tilde{p}_i \rangle
$$

$$
= \max_{z_i \in \mathcal{Z}} \max_{\|u_i\|_* \leq \lambda} v^*_{s, \pi}(z_i) - \min_{\|u_i\|_* \leq \lambda} \langle p, z_i \rangle + \langle u_i, z_i \rangle
$$

$$
= \max_{z_i \in \mathcal{Z}} \max_{\|u_i\|_* \leq \lambda} v^*_{s, \pi}(z_i) - \sigma_{\mathcal{M}(S) \times |A| \cup_u \|u_i\|_* \leq \lambda}(-z_i - u_i) - \langle u_i, \tilde{p}_i \rangle
$$
where $\sigma_P(u) := \sup_{p \in P} \langle p, u \rangle$ denotes the support function of a general set $P$. We use the conservative bound $\sigma_{\mathcal{M}(S) | S \times |A|} \leq \sigma_{\mathcal{R}(S) | S \times |A|}$ to deduce

$$\min_{p \in \mathcal{M}(S) | S \times |A|} \max_{\|u\|_\infty \leq \lambda} \left( \tilde{\ell}(u^*_a)(p) + \langle u_i, p - \tilde{p}_i \rangle \right)$$

$$= \max_{z_1, \ldots, z_n} \max_{\|u\|_\infty \leq \lambda} \left( v_i^*(z_i) - \langle u_i, \tilde{p}_i \rangle - \sigma_{\mathcal{M}(S) | S \times |A|}(-z_i - u_i) \right)$$

$$\geq \max_{z_1, \ldots, z_n} \max_{\|u\|_\infty \leq \lambda} \left( v_i^*(z_i) + \langle z_i, \tilde{p}_i \rangle \right)$$

$$= \left\{ \begin{array}{ll}
v_i^*(s) & \text{if } \sup \{ \|z\|_\infty : z_i \in \mathcal{F} \} \leq \lambda \\
-\infty & \text{otherwise}
\end{array} \right.$$ 

Therefore, recalling the notation $\kappa := \sup \{ \|z\|_\infty : z \in \mathcal{F} \}$, we obtain

$$\sup_{x_1, \ldots, x_n} \frac{1}{n} \sum_{i=1}^n x_i - \lambda \alpha \text{ s.t. } \left\{ \begin{array}{l}
\min_{p \in \mathcal{M}(S) | S \times |A|} \max_{\|u\|_\infty \leq \lambda} \left( \tilde{\ell}(u^*_a)(p) + \langle u_i, p - \tilde{p}_i \rangle \right) \geq x_i \\
\lambda \geq 0
\end{array} \right.$$ 

$$\geq \sup_{x_1, \ldots, x_n} \frac{1}{n} \sum_{i=1}^n x_i - \lambda \alpha \text{ s.t. } \left\{ \begin{array}{l}
v_i^*(s) \geq x_i \\
\lambda \geq \kappa
\end{array} \right.$$ 

$$= \frac{1}{n} \sum_{i=1}^n v_i^*(s) - \kappa \alpha$$

Putting this altogether yields

$$v_{\mathcal{M} \alpha, \mu_\alpha}(s) \geq \frac{1}{n} \sum_{i=1}^n v_i^*(s) - \kappa \alpha,$$

which ends the proof.

**B.2. Proof of Proposition 4.1**

We first use the following result, which is a reformulation of (Strehl & Littman, 2008)[Lemma 1].

**Lemma B.2.** For all $p_1, p_2 \in \mathcal{M}(S) | S \times |A|$ we have

$$|v_{p_1}^* - v_{p_2}^*| \leq \frac{\beta \gamma R_{\max} \|p_1 - p_2\|}{(1 - \gamma)^2}$$

where $\beta$ is such that $\|\cdot\|_1 \leq \beta \|\cdot\|$.

**Proof.** Denote by $\Delta := \sup_{s \in S} |v_{p_1}^* - v_{p_2}^*|$. For all $s \in S$ we have

$$|v_{p_1}^* - v_{p_2}^*| = \sum_{a \in A} \pi_s(a) r(s, a) + \gamma \sum_{a \in A} \sum_{s' \in S} p_1(s, a, s') v_{p_1}^*(s') - \sum_{a \in A} \pi_s(a) r(s, a) - \gamma \sum_{a \in A} \sum_{s' \in S} p_2(s, a, s') v_{p_2}^*(s')$$

$$= \gamma \sum_{a \in A} \pi_s(a) \sum_{s' \in S} p_1(s, a, s') (v_{p_1}^*(s') - v_{p_2}^*(s')) + \gamma \sum_{a \in A} \sum_{s' \in S} (p_1(s, a, s') - p_2(s, a, s')) v_{p_2}^*(s')$$

$$\leq \gamma \sum_{a \in A} \pi_s(a) \sum_{s' \in S} |p_1(s, a, s') - p_2(s, a, s')| v_{p_2}^*(s')$$

$$\leq \gamma \Delta + \gamma \sum_{a \in A} \pi_s(a) \|p_1(s, a, \cdot) - p_2(s, a, \cdot)\|_1 R_{\max} \frac{1}{1 - \gamma}$$

$$\leq \gamma \Delta + \|p_1 - p_2\|_1 \frac{\gamma R_{\max}}{1 - \gamma}$$
Therefore, by definition of $\beta$,

$$(1 - \gamma)\Delta \leq \beta\|p_1 - p_2\| \frac{\gamma R_{\max}}{1 - \gamma}$$

and the result follows. \hfill \square

Now, we can prove Proposition 4.1 whose statement is recalled below.

**Proposition.** Define as $L := \frac{\beta R_{\max}}{(1 - \gamma)\Delta}$ where $\beta$ is such that $\|\cdot\|_1 \leq \beta\|\cdot\|$ and consider $\tilde{u}_s^\pi : \mathbb{R}^{|S| \times |A| \times |S|} \to \mathbb{R}$ the continuation function of $u_s^\pi$ which is defined as

$$\tilde{u}_s^\pi(\tilde{p}) := \inf_{p \in \mathcal{M}(S) \times |A|} u_s^\pi(p) + L\|p - \tilde{p}\|$$

Further denote by $\tilde{\mathcal{Z}} := \{\tilde{z} : (\tilde{u}_s^\pi)^* (\tilde{z}) < +\infty\}$. If $\tilde{\mathcal{Z}} \subseteq \tilde{\mathcal{Z}}$, then $\kappa \leq L$.

**Proof.** By Lemma B.2, for all $p_1, p_2 \in \mathcal{M}(S)^{|S| \times |A|}$ we have

$$u_s^\pi(p_2) - L\|p_1 - p_2\| \leq u_s^\pi(p_1) \leq u_s^\pi(p_2) + L\|p_1 - p_2\|$$

so $u_s^\pi$ is $L$-Lipschitz continuous over $\mathcal{M}(S)^{|S| \times |A|}$, and by construction, so is its continuation function $\tilde{u}_s^\pi$ over $\mathbb{R}^{|S| \times |A| \times |S|}$. Therefore, if we fix a transition function $p_1 \in \mathcal{M}(S)^{|S| \times |A|}$ it holds that

$$-(\tilde{u}_s^\pi)^*(z) := \min_{\tilde{p} \in \mathbb{R}^{|S| \times |A| \times |S|}} \tilde{u}_s^\pi(\tilde{p}) - \langle z, \tilde{p} \rangle$$

$$\leq \min_{\tilde{p} \in \mathbb{R}^{|S| \times |A| \times |S|}} v_{p_1}^\pi(s) + L\|p_1 - \tilde{p}\| - \langle z, \tilde{p} \rangle$$

$$= \min_{\mathbb{R}^{|S| \times |A| \times |S|}} v_{p_1}^\pi(s) + \max_{\|x\|_s \leq L} \langle x, p_1 - \tilde{p} \rangle - \langle z, \tilde{p} \rangle$$

$$= \min_{\mathbb{R}^{|S| \times |A| \times |S|}} \max_{\|x\|_s \leq L} v_{p_1}^\pi(s) + \langle x, p_1 - \tilde{p} \rangle - \langle z, \tilde{p} \rangle$$

using the minimax theorem on the right hand side of the inequality, we obtain

$$-(\tilde{u}_s^\pi)^*(z) \leq \max_{\|x\|_s \leq L} \min_{\tilde{p} \in \mathbb{R}^{|S| \times |A| \times |S|}} v_{p_1}^\pi(s) + \langle x, p_1 \rangle - \langle x + z, \tilde{p} \rangle$$

$$= v_{p_1}^\pi(s) + \max_{\|x\|_s \leq L} \langle x, p_1 \rangle + \min_{\tilde{p} \in \mathbb{R}^{|S| \times |A| \times |S|}} \langle -x - z, \tilde{p} \rangle$$

$$= \max_{\|x\|_s \leq L} -\sigma_{\mathbb{R}^{|S| \times |A| \times |S|}}(x + z) + v_{p_1}^\pi(s) + \langle x, p_1 \rangle$$

$$= \begin{cases} v_{p_1}^\pi(s) - \langle z, p_1 \rangle & \text{if } \|z\|_s \leq L \\ -\infty & \text{otherwise} \end{cases}$$

Therefore, the effective domain of $-(\tilde{u}_s^\pi)^*$ is included in the $\|\cdot\|_s$-ball of radius $L$, and by assumption on $\tilde{\mathcal{Z}}$, this implies that $\kappa \leq L$. \hfill \square

**B.3. Proof of Theorem 4.2**

We proceed similarly as we did for proving Theorem 4.1. We first establish the following result.

**Lemma B.3.** For all state $s \in S$ and all policy $\pi$, define the mapping $w_s^\pi : \mathcal{P} \to \mathbb{R}$ as $p \mapsto \Phi(s)^\top w_p$. If $w_s^\pi$ is proper and closed, then its convex closure $\tilde{c}(w_s^\pi)$ is proper.

**Proof.** We use Weierstrass theorem (Barbu & Precupanu, 2012)[Theorem 2.8.]: since $\mathcal{P}$ is compact and $w_s^\pi$ is closed, $w_s^\pi$ takes a minimum value on $\mathcal{P}$. Moreover, we have

$$\inf_{p \in \mathcal{P}} \tilde{c}(w_s^\pi)(p) = \inf_{p \in \mathcal{P}} w_s^\pi(p)$$

so since $w_s^\pi$ is proper, $\inf_{p \in \mathcal{P}} \tilde{c}(w_s^\pi)(p) > -\infty$ and $\tilde{c}(w_s^\pi)$ is proper. \hfill \square
We now prove Theorem 4.2, whose statement is recalled below.

**Theorem.** Let $(\mathcal{S}, \mathcal{A}, r, \mathbb{M}_n(\mu))$ be a Wasserstein DRMDP. Then, for any policy $\pi \in \Pi$ it holds that

$$\inf_{\mu \in \mathbb{M}_n(\bar{\mu}_n)} \mathbb{E}_{\mu \sim \mu_n} [\Phi(s)^\top w_p] \geq \frac{1}{n} \sum_{i=1}^n \Phi(s)^\top w_{\bar{p}_i} - \eta \alpha$$

where $\eta := \sup\{\|z\|_* : z \in \mathbb{M}\}$.

**Proof.** Similarly to the proof of Theorem 4.1, we consider the stationary model formulation so the worst-case distributionally robust value function can be expressed as

$$\inf_{\mu \in \mathbb{M}_n(\bar{\mu}_n)} \mathbb{E}_{\mu \sim \mu_n} [\Phi(s)^\top w_p] = \min_{\tilde{\mu}} \mathbb{E}_{\mu \sim \tilde{\mu}} [\Phi(s)^\top w_p] \text{ s.t. } \begin{cases} \tilde{\mu} = \bigotimes_{t \geq 0, s \in \mathcal{S}} \left( \frac{1}{n} \sum_{i=1}^n \mu^i_s \right) \\ \frac{1}{n} \sum_{i=1}^n \mathbb{E}_{\mu_i \sim \mu^i_s} [\|p_i - \tilde{p}_i\|] \leq \alpha \end{cases}$$

for an $\alpha$ determined by radii $\alpha_i$'s ($s \in \mathcal{S}$). For all $i \in [n]$, define $\bar{\mu}_i := \bigotimes_{t \geq 0, s \in \mathcal{S}} \mu^i_s$ and introduce the notation $\bar{\mu} := \frac{1}{n} \sum_{i=1}^n \bar{\mu}_i$. Thus, replacing distribution $\tilde{\mu}$ by its constrained law and using a duality argument, we obtain

$$\inf_{\mu \in \mathbb{M}_n(\bar{\mu}_n)} \mathbb{E}_{\mu \sim \mu_n} [\Phi(s)^\top w_p] = \sup_{\lambda \geq 1} \inf_{\mu_i = \bigotimes_{s \in \mathcal{S}} \mu^i_s} \left( \mathbb{E}_{\mu_i \sim \mu^i_s} [\Phi(s)^\top w_p] - \lambda \left( \alpha - \frac{1}{n} \sum_{i=1}^n \mathbb{E}_{p_i \sim \mu_i} [\|p_i - \tilde{p}_i\|] \right) \right)$$

Thanks to the maxmin inequality and setting all $\mu_i$-s to a Dirac distribution with full mass on the worst-case model, we can write

$$\inf_{\mu \in \mathbb{M}_n(\bar{\mu}_n)} \mathbb{E}_{\mu \sim \mu_n} [\Phi(s)^\top w_p] \geq \sup_{\lambda \geq 1} \inf_{\mu_i = \bigotimes_{s \in \mathcal{S}} \mu^i_s} \left( \mathbb{E}_{\mu_i \sim \mu^i_s} [\Phi(s)^\top w_p] - \lambda \left( \alpha - \frac{1}{n} \sum_{i=1}^n \mathbb{E}_{p_i \sim \mu_i} [\|p_i - \tilde{p}_i\|] \right) \right)$$

$$= \sup_{\lambda \geq 1} \frac{1}{n} \sum_{i=1}^n \inf_{\mu_i = \bigotimes_{s \in \mathcal{S}} \mu^i_s} \left( \mathbb{E}_{p_i \sim \mu_i} [\Phi(s)^\top w_p + \lambda \|p_i - \tilde{p}_i\|] - \lambda \alpha \right)$$

We introduce auxiliary variables $x_1, \ldots, x_n$ in order to reformulate the bound as

$$\inf_{\mu \in \mathbb{M}_n(\bar{\mu}_n)} \mathbb{E}_{\mu \sim \mu_n} [\Phi(s)^\top w_p] \geq \max_{\lambda, x_1, \ldots, x_n} \frac{1}{n} \sum_{i=1}^n x_i - \lambda \alpha \text{ s.t. } \begin{cases} \inf_{p \in \mathcal{M}(\mathcal{S})^{|S| \times |A|}} (\Phi(s)^\top w_p + \lambda \|p - \tilde{p}_i\|) \geq x_i & \forall i \in [n] \\ \lambda \geq 0 \end{cases}$$

Since $\mathcal{S}$ is finite, $\mathcal{M}(\mathcal{S})$ is compact and the infima in the first-line constraints are minima. Moreover, by definition of the dual norm $\|\cdot\|_*$, the right hand side of the inequality is equivalent to the following:

$$\max_{\lambda, x_1, \ldots, x_n} \frac{1}{n} \sum_{i=1}^n x_i - \lambda \alpha \text{ s.t. } \begin{cases} \min_{p \in \mathcal{M}(\mathcal{S})^{|S| \times |A|}} \max_{\|u_i\|_* \leq \lambda} (\Phi(s)^\top w_p + \langle u_i, p - \tilde{p}_i \rangle) \geq x_i & \forall i \in [n] \\ \lambda \geq 0 \end{cases}$$

so the worst-case expectation can be reformulated as

$$\inf_{\mu \in \mathbb{M}_n(\bar{\mu}_n)} \mathbb{E}_{\mu \sim \mu_n} [\Phi(s)^\top w_p] \geq \max_{\lambda, x_1, \ldots, x_n} \frac{1}{n} \sum_{i=1}^n x_i - \lambda \alpha \text{ s.t. } \begin{cases} \min_{p \in \mathcal{M}(\mathcal{S})^{|S| \times |A|}} \max_{\|u_i\|_* \leq \lambda} (\Phi(s)^\top w_p + \langle u_i, p - \tilde{p}_i \rangle) \geq x_i & \forall i \in [n] \\ \lambda \geq 0 \end{cases}$$
Similarly to the tabular case, consider the convex closure \( \tilde{\text{c}}l(w_\pi^*) \) of the mapping \( w_\pi^* : p \mapsto \Phi(s)^T w_p \). We then have

\[
\max_{\lambda, x_1, \ldots, x_n} \frac{1}{n} \sum_{i=1}^n x_i - \lambda \alpha \quad \text{s.t.} \quad \min_{p \in \mathcal{M}(S)^{\mathcal{A}}} \max_{i \leq \lambda} \left( \Phi(s)^T w_p + \langle u_i, p - \hat{p}_i \rangle \right) \geq x_i \quad \forall i \in [n]
\]

\[
= \max_{\lambda, x_1, \ldots, x_n} \frac{1}{n} \sum_{i=1}^n x_i - \lambda \alpha \quad \text{s.t.} \quad \min_{p \in \mathcal{M}(S)^{\mathcal{A}}} \max_{i \leq \lambda} \left( \tilde{\text{c}}l(w_\pi^*)(p) + \langle u_i, p - \hat{p}_i \rangle \right) \geq x_i \quad \forall i \in [n]
\]

Now introduce the conjugate transform of \( \tilde{\text{c}}l(w_\pi^*) \) as follows:

\[
\tilde{w}_\pi^{*, \pi}(z) := - \max_{z' \in \mathcal{W}} \left( \langle z', p \rangle - \tilde{\text{c}}l(w_\pi^*)(p) \right)
\]  
\[
= - \tilde{\text{c}}l(w_\pi^*)^*(z)
\]

By Theorem 2.1(d), \( \tilde{\text{c}}l(w_\pi^*)^* = (w_\pi^*)^* \) so \( \tilde{w}_\pi^{*, \pi} = -(w_\pi^*)^* \). Moreover, since the feature vectors \( \langle \Phi(s) \rangle_{s \in \mathcal{S}} \) are linearly independent, by (Bertsekas & Tsitsiklis, 1996)[Lemma 6.8.] \( w_\pi^* \) is proper and closed. As a result, thanks to Lemma B.3, the convex closure \( \tilde{\text{c}}l(w_\pi^*) \) is proper. Therefore, by Theorem 2.1(c), \( \tilde{\text{c}}l(w_\pi^*) \) coincides with its bi-conjugate function and

\[
\tilde{\text{c}}l(w_\pi^*)(p) = \tilde{\text{c}}l(w_\pi^*)^{**}(p)
\]

\[
= \max_{z \in \mathcal{W}} \left( \langle z, p \rangle - \tilde{\text{c}}l(w_\pi^*)^*(p) \right)
\]

\[
= \max_{z \in \mathcal{W}} \langle z, p \rangle - (w_\pi^*)^*(p)
\]

\[
= \max_{z \in \mathcal{W}} \langle z, p \rangle + w_\pi^{*, \pi}(z)
\]

where \( \mathcal{W} := \{ z : \tilde{w}_\pi^{*, \pi}(z) > -\infty \} \) is the effective domain of \( \tilde{w}_\pi^{*, \pi} \). Thus, if we use the reformulation of the convex closure and apply the minimax theorem (Bertsekas, 2009)[Proposition 5.5.4.] we obtain

\[
\max_{p \in \mathcal{M}(S)^{\mathcal{A}}} \max_{\|u\| \leq \lambda} \left( \tilde{\text{c}}l(w_\pi^*)(p) + \langle u_i, p - \hat{p}_i \rangle \right)
\]

\[
= \max_{z_i \in \mathcal{W}} \max_{\|u\| \leq \lambda} \left( w_\pi^{*, \pi}(z_i) + \langle p, z_i \rangle + \langle u_i, p - \hat{p}_i \rangle \right)
\]

\[
= \max_{z_i \in \mathcal{W}} \min_{p \in \mathcal{M}(S)^{\mathcal{A}}} \max_{\|u\| \leq \lambda} \left( w_\pi^{*, \pi}(z_i) + \langle p, z_i \rangle + \langle u_i, p - \hat{p}_i \rangle \right)
\]

\[
= \max_{z_i \in \mathcal{W}} \max_{\|u\| \leq \lambda} \left( w_\pi^{*, \pi}(z_i) - \langle u_i, \hat{p}_i \rangle + \min_{p \in \mathcal{M}(S)^{\mathcal{A}}} \langle p, z_i + u_i \rangle \right)
\]

where \( \sigma_p \) is the support function of \( \mathcal{P} \). We use the conservative bound \( \sigma_{\mathcal{M}(S)^{\mathcal{A} \times \{s\}}} \leq \sigma_{\mathcal{S}^{\mathcal{A} \times \{s\}}} \) to deduce

\[
\min_{p \in \mathcal{M}(S)^{\mathcal{A}}} \max_{\|u\| \leq \lambda} \left( \tilde{\text{c}}l(w_\pi^*)(p) + \langle u_i, p - \hat{p}_i \rangle \right)
\]

\[
= \max_{z_i \in \mathcal{W}} \max_{\|u\| \leq \lambda} \left( w_\pi^{*, \pi}(z_i) - \langle u_i, \hat{p}_i \rangle - \sigma_{\mathcal{M}(S)^{\mathcal{A} \times \{s\}}} (-z_i - u_i) \right)
\]

\[
\geq \max_{z_i \in \mathcal{W}} \max_{\|u\| \leq \lambda} \left( w_\pi^{*, \pi}(z_i) - \langle u_i, \hat{p}_i \rangle - \sigma_{\mathcal{S}^{\mathcal{A} \times \{s\}}} (-z_i - u_i) \right)
\]

\[
= \max_{z_i \in \mathcal{W}} \max_{\|u\| \leq \lambda} \left( w_\pi^{*, \pi}(z_i) + \langle z_i, \hat{p}_i \rangle \right)
\]

\[
= \left\{ \begin{array}{ll} 
\Phi(s)^T w_{\hat{p}_i} & \text{if } \sup \{ \|z_i\| : z_i \in \mathcal{W} \} \leq \lambda \\
-\infty & \text{otherwise} 
\end{array} \right.
\]
Therefore, recalling the notation \( \eta := \sup\{\|z\|_s : z \in \mathbb{W}\} \), we obtain

\[
\sup_{\lambda,x_1,\ldots,x_n} \frac{1}{n} \sum_{i=1}^{n} x_i - \lambda \alpha \text{ s.t. } \begin{cases} 
\min_{p \in \mathcal{M}(|S| \times |A|)} \max_{\|u_i\|_s \leq \lambda} \left( \tilde{c}(w_i^s)(p) + \langle u_i, p - \tilde{p}_i \rangle \right) \geq x_i & \forall i \in [n] \\
\lambda \geq 0 
\end{cases}
\geq \sup_{\lambda,x_1,\ldots,x_n} \frac{1}{n} \sum_{i=1}^{n} x_i - \lambda \alpha \text{ s.t. } \begin{cases} 
\Phi(s)^T w_i \geq x_i & \forall i \in [n] \\
\lambda \geq \eta 
\end{cases}
\]

Putting this altogether yields

\[
\inf_{\mu \in \mathfrak{M}_s(\tilde{p}_n)} \mathbb{E}_{p \sim \mu} [\Phi(s)^T w_i] \geq \frac{1}{n} \sum_{i=1}^{n} \Phi(s)^T w_i - \eta \alpha,
\]

which ends the proof. \( \square \)

C. Out-of-Sample Performance Guarantees

C.1. Proof of Theorem 5.1

Theorem 5.1 is based on the following result, which is a direct consequence of (Fournier & Guillin, 2015)[Lemma 5;Prop. 10].

**Lemma C.1.** Let \( \epsilon \in (0, 1) \), \( m := |S| \times |A| \) and \( \tilde{p}_n \in \mathcal{M}([0, 1]^m) \) be the empirical distribution at \( s \in S \). Then for all \( \alpha_s \in (0, \infty) \)

\[
\mu_s^* \left( \{\tilde{p}_s : d(\tilde{p}_s, \mu_s) \geq c_0 \beta_s \} \right) \leq c_1 b_1(n_s, \beta_s) \mathbb{I}_{\beta_s \leq 1}
\]

where \( b_1(n_s, \beta_s) := \exp(-c_2 n_s \cdot (\beta_s)^{m^2}) \) and \( c_0, c_1, c_2 \) are positive constants that only depend on \( m \neq 2 \).

The positive constant \( c_0 \) corresponds to \( c_0 := 3 \times 2^{1+m/2} \) that appears in (Fournier & Guillin, 2015)[Lemma 5]. Moreover, remark that in Lemma C.1, the probability vanishes when \( \beta_s > 1 \), since the Wasserstein diameter of \( \mathcal{M}([0, 1]^m) \) is bounded by \( c_0 \) (see Notation 4 and the proof of Proposition 10 in (Fournier & Guillin, 2015)).

We are now ready to prove Theorem 5.1 which we recall below.

**Theorem (Finite-sample Guarantee - Compact version).** Let \( \epsilon \in (0, 1) \), \( m := |S| \times |A| \). Denote by \( \tilde{\pi}^* \) an optimal policy of the Wasserstein DRMDP \( (S, A, r, \mathfrak{M}_{\alpha_s}(\tilde{p}_n)) \) and \( \tilde{v}^* \) its optimal value. If for all \( s \in S \) the radius of the Wasserstein ball at \( s \) satisfies

\[
\alpha_s(n_s, \epsilon) := \begin{cases} 
c_0 \cdot \left( \frac{1}{n_s c_2} \log \left( \frac{c_1}{\epsilon} \right) \right)^{1/(m^2/2)} & \text{if } n_s \geq C'_m^s \\
c_0 & \text{o.w.}
\end{cases}
\]

with \( C'_m^s := \frac{1}{c_2} \log \left( \frac{c_2}{\epsilon} \right) \) and

\[
n_s := \sum_{i \in [n], a \in A, s' \in S} n_i(s, a, s'),
\]

then it holds that

\[
\mu^n \left( \left\{ \tilde{p} \mid \mathbb{E}_{p \sim \mu} [v^*_{\tilde{p}}(s)] \geq \tilde{v}^*_{\mathfrak{M}_{\alpha_s}(\tilde{p}_n)}(s) \quad \forall s \in S \right\} \right) \geq 1 - \epsilon
\]

where \( c_0, c_1, c_2 \) are positive constants that only depend on \( m \neq 2^4 \).

\footnote{A comparable but more intricate conclusion can be established for \( m = 2 \) (Fournier & Guillin, 2015)[Proposition 10].}
Proof. Set
\[
\beta_s(n_s, \epsilon) := \begin{cases} 
\left( \frac{1}{n_s c_2} \log \left( \frac{c_1}{\epsilon} \right) \right)^{1/(m+2)} & \text{if } n_s \geq C_m^r \\
\text{o.w.} & 
\end{cases}
\]
so we have \(c_0 \cdot \beta_s(n_s, \epsilon) = \alpha_s(n_s, \epsilon)\) and Lemma C.1 ensures that the radius \(\alpha_s(n_s, \epsilon)\) provides the following guarantee:
\[
\mu_s^n(\{ \hat{\mu}_s : d(\hat{\mu}_s, \mu_s) \leq \alpha_s(n_s, \epsilon) \}) \geq 1 - \epsilon.
\]
Now, we introduce the operator \(T_{\tilde{\mu}_s}^s\) defined in Section 3.2, where by assumption, \(\hat{\pi}^* = (\hat{\pi}^*_s)_{s \in S}\) is the optimal policy of the Wasserstein DRMDP \(\langle S, A, r, \mathcal{M}_{\alpha(n, \epsilon)}(\hat{\mu}_n) \rangle\). By Equation (C.1), we have the following one-step guarantee:
\[
\mu_s^n(\{ \hat{\mu}_s : T_{\tilde{\mu}_s}^s v(s) \geq T_{\tilde{\mu}_n(s)} v(s) \}) \geq 1 - \epsilon
\]
so the induction assumption holds for all \(k \geq 1\), then \((T_{\tilde{\mu}_s}^s)^k v(s) \geq (T_{\tilde{\mu}_n(s)})^k v(s)\). Hence, supposing that the condition holds for an arbitrary \(k \geq 1\), we have
\[
(T_{\tilde{\mu}_s}^s)^{k+1} v(s) = T_{\tilde{\mu}_s}^s ((T_{\tilde{\mu}_s}^s)^k v(s)) \geq (T_{\tilde{\mu}_n(s)})^k v(s) \geq (T_{\tilde{\mu}_n(s)}) v(s)
\]
so the induction assumption holds for all \(k \geq 1\). Using the contracting property of \(T_{\tilde{\mu}_s}^s\), we have \(\lim_{k \to \infty} (T_{\tilde{\mu}_s}^s)^k v(s) = E_{p \sim u}[v_{p, n}^s(s)]\) and by Theorem 3.1 \(\lim_{k \to \infty} (T_{\tilde{\mu}_n(s)})^k v(s) = v_{\tilde{\mu}_n(s)}^s(s)\). Therefore, by setting \(k \to \infty\), if \(\mu_s \in \mathcal{M}_{\alpha(n, \epsilon)}(\hat{\mu}_n)\), then \(E_{p \sim u}[v_{p, n}^s(s)] \geq v_{\tilde{\mu}_n(s)}^s(s)\) and the following probabilistic guarantee holds for all \(s \in S\):
\[
\mu_s^n(\{ \hat{\mu}_s : E_{p \sim u}[v_{p, n}^s(s)] \geq v_{\tilde{\mu}_n(s)}^s(s) \}) \geq 1 - \epsilon
\]
which can be rewritten as
\[
\mu_s^n(\{ \hat{\mu}_s : E_{p \sim u}[v_{p, n}^s(s)] < v_{\tilde{\mu}_n(s)}^s(s) \}) \leq \epsilon.
\]
The independence structure \(\mu^n = \otimes_{s \in S} \mu^n_s\) enables to obtain
\[
\mu^n(\{ \hat{\mu} : E_{p \sim u}[v_{p, n}^s(s)] < v_{\tilde{\mu}_n(s)}^s(s) \ \forall s \in S\}) \leq \prod_{s \in S} \epsilon \leq \epsilon,
\]
which concludes the proof, by taking the complementary event. \(\square\)