The integral double Burnside ring of the symmetric group $S_3$

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The double Burnside $R$-algebra $B_R(G,G)$ of a finite group $G$ with coefficients in a commutative ring $R$ has been introduced by S. Bouc. It is $R$-linearly generated by finite $(G,G)$-bisets, modulo a relation identifying disjoint union and sum. Its multiplication is induced by the tensor product. B. Masterson described $B_Q(S_3,S_3)$ as a subalgebra of $Q^{8\times 8}$. We give a variant of this description and continue to describe $B_R(S_3,S_3)$ for $R \in \{\mathbb{Z}, \mathbb{Z}_2, \mathbb{F}_2, \mathbb{Z}_3, \mathbb{F}_3\}$ via congruences as suborders of certain $R$-orders respectively via path algebras over $R$.

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0 Introduction

Suppose given a commutative ring $R$. In [5] §3.1, S. Bouc introduced the \textit{biset category} $\text{Biset}_R$, see also [5] §1.4. As objects, the category $\text{Biset}_R$ has finite groups. Morphisms between two finite groups $G$ and $H$ are given by the double Burnside $R$-module $B_R(H, G)$, which is $R$-linearly generated by finite $(H, G)$-bisets, modulo a relation identifying disjoint union and sum. Composition of morphisms in $\text{Biset}_R$ is given by an operation on bisets that is similar to the tensor product of bimodules.

Let $\mathcal{X}$ and $\mathcal{Y}$ be classes of finite groups closed under forming subgroups, factor groups and extensions. Following Bouc [3] §3.4.1, we say that an $(H, G)$-biset $M$ is an $(\mathcal{X}, \mathcal{Y})$-\textit{free} $(H, G)$-biset if for any $m \in M$ the left stabilizer of $m$ in $H$ is in $\mathcal{X}$ and the right stabilizer of $m$ in $G$ is in $\mathcal{Y}$. We have the subcategory $\text{Biset}_{\mathcal{X}, \mathcal{Y}}^R$ of $\text{Biset}_R$, consisting of finite groups as objects and $R$-linear combinations of classes of $(\mathcal{X}, \mathcal{Y})$-free bisets as morphisms, see [3] Lemma 4.

There is an equivalence of categories between the category of functors from $\text{Biset}_{\mathcal{X}, \mathcal{Y}}^R$ to $R$-Mod and the category of \textit{globally-defined Mackey functors} $\text{Mack}_{\mathcal{X}, \mathcal{Y}}^R$ [6] §8. Here, a globally-defined Mackey functor, with respect to $\mathcal{X}$ and $\mathcal{Y}$, maps groups to $R$-modules and each group morphism $\alpha$ covariantly to an $R$-module morphism $\alpha_*$, provided $\text{ker} \alpha \in \mathcal{Y}$ and contravariantly to $\alpha^*$, provided $\text{ker} \alpha \in \mathcal{X}$. These morphisms should satisfy a list of compatibilities, amongst which a Mackey formula, see e.g. [6] §8. By the equivalence, these requirements are now subsumed as particular cases of operations coming from $\text{Biset}_{\mathcal{X}, \mathcal{Y}}^R$. The only remaining necessity, replacing the list of compatibilities, is that it actually is a functor from $\text{Biset}_{\mathcal{X}, \mathcal{Y}}^R$ to $R$-Mod.

We list two examples of functors from $\text{Biset}_{\mathcal{X}, \mathcal{Y}}^R$ to $R$-Mod.

- Suppose that $R$ is a field of characteristic 0. Let $\mathcal{X}$ and $\mathcal{Y}$ consist of all finite groups. Consider the functor $\text{Biset}_{\mathcal{X}, \mathcal{Y}}^R \to R$-Mod that maps a finite group $G$ to the representation group of $G$ over $R$, i.e. the Grothendieck group of the category of finitely generated $RG$-modules.

- Let $\mathcal{X}$ consist of all finite groups and $\mathcal{Y} = \{1\}$. Consider the functor $\text{Biset}_{\mathcal{X}, \mathcal{Y}}^R \to R$-Mod that maps a finite group $G$ to the cohomology $H^n(G, R)$ of $G$ in some degree $n$ with trivial coefficients.

For some more examples, see [6] §8. The example of the Burnside ring is also explained in [4] §6.1

Each finite group $G$ is an object in $\text{Biset}_R$. Its endomorphism ring is called \textit{double Burnside algebra} $B_R(G, G) := \text{Hom}_{\text{Biset}_R}(G, G)$.

The isomorphism classes of finite transitive $(G, G)$-bisets form an $R$-linear basis of $B_R(G, G)$. In particular, if we choose a system $L_{G\times G}$ of representatives for the conjugacy classes of subgroups of $G \times G$, we have the $R$-linear basis $\{(G \times G)/U : U \in L_{G\times G}\}$.

If $R$ is a field in which $|G|$ and $\varphi(|G|)$ are invertible, where $\varphi$ denotes Euler’s totient function, and if $G$ is cyclic, then the double Burnside algebra $B_R(G, G)$ is semisimple. This is treated in [7] Theorem 8.11, Remark 8.12(a)].
In case of $G = S_3$, we have 22 conjugacy classes of subgroups of $S_3 \times S_3$ and thus $\text{rk}_R(B_R(S_3, S_3)) = 22$. The double Burnside $\mathbb{Q}$-algebra $B_{\mathbb{Q}}(S_3, S_3)$ has been described by B. Masterson [1, §8] and then by B. Masterson and G. Pfeiffer [2, §7]. We describe $B_{\mathbb{Q}}(S_3, S_3)$ independently, using a direct Magma-supported calculation [9], with the aim of being able to pass from $B_{\mathbb{Q}}(S_3, S_3)$ to $B_{\mathbb{Z}}(S_3, S_3)$ in the sequel. The double Burnside $\mathbb{Q}$-algebra $B_{\mathbb{Q}}(S_3, S_3)$ is not semisimple [3, Proposition 6.1.5], thus not isomorphic to a direct product of matrix rings. As a substitute, we use a suitable isomorphic copy $A$ of $B_{\mathbb{Q}}(S_3, S_3)$ to be able to find a $\mathbb{Z}$-order $A_{\mathbb{Z}}$ inside $A$ that contains an isomorphic copy of $B_{\mathbb{Z}}(S_3, S_3)$, which we describe via congruences, cf. Proposition 5.

We calculate a path algebra for $B_{\mathbb{Z}}(S_3, S_3)$, cf. Proposition 11. We deduce that $B_{\mathbb{F}_2}(S_3, S_3)$ is Morita equivalent to the path algebra

$$B_{\mathbb{Q}}(S_3, S_3) \xrightarrow{\tau^{-1}} A \xleftarrow{\delta} B_{\mathbb{Z}}(S_3, S_3)$$

We calculate a path algebra for $B_{\mathbb{Z}}(S_3, S_3)$, cf. Proposition 15. We deduce that $B_{\mathbb{F}_3}(S_3, S_3)$ is Morita equivalent to the path algebra

We freely use this identification.

1 Preliminaries on bisets and the double Burnside algebra

Bisets. Recall that an $(G, G)$-biset $X$ is a finite set $X$ together with a left $G$ and a right $G$-action that commute with each other, i.e. $(h \cdot x) \cdot g = h \cdot (x \cdot g) =: h \cdot x \cdot g$ for $h, g \in G$ and $x \in X$.

Every $(G, G)$-biset $X$ can be regarded as a left $(G \times G)$-set by setting $(h, g)x := hxg^{-1}$ for $(h, g) \in G \times G$ and $x \in X$. Likewise, every left $(G \times G)$-set $Y$ can be regarded as an $(G, G)$-biset by setting $h \cdot y \cdot g := (h, g^{-1})y$ for $h, g \in G$ and $y \in Y$. We freely use this identification.
Tensor product. Let $M$ be an $(G, G)$-biset and let $N$ be a $(G, G)$-biset. The cartesian product $M \times N$ is a $(G, G)$-biset via $h(m, n)p = (hm, np)$ for $h, p \in G$ and $(m, n) \in M \times N$. It becomes a left $G$-set via $g(m, n) = (mg^{-1}, gn)$ for $g \in G$ and $(m, n) \in M \times N$. We call the set of $G$-orbits on $M \times N$ the tensor product $M \times N^G$ of $M$ and $N$. This also is an $(G, G)$-biset. The $G$-orbit of the element $(m, n) \in M \times N$ is denoted by $m \times n \in M \times N^G$. Moreover, let $L$ be a $(G, G)$-biset. Then $M \times (N \times L) \sim (M \times N) \times L$, $m \times (n \times \ell) \rightarrow (m \times n) \times \ell$ as $(G, G)$-bisets.

Double Burnside $R$-algebra. We denote by $B_R(G, G)$ the double Burnside $R$-algebra of $G$. Recall that $B_R(G, G)$ is the $R$-module freely generated by the isomorphism classes of finite $(G, G)$-bisets, modulo the relations $[M \sqcup N] = [M] + [N]$ for $(G, G)$-bisets $M, N$. Multiplication is defined by $[M] \cdot [N] = [M \times N]$ for $(G, G)$-bisets $M, N$. An $R$-linear basis of $B_R(G, G)$ is given by $((G \times G)/U) : U \in \mathcal{L}_{G \times G}$, where we choose a system $\mathcal{L}_{G \times G}$ of representatives for the conjugacy classes of subgroups of $G \times G$. Moreover, $1_{B_Z(G, G)} = [G].$

Abbreviation. In case of $G = S_3$, we often abbreviate $B_R := B_R(S_3, S_3)$.

2 Z-linear basis of $B_Z(S_3, S_3)$

The following calculations were done using the computer algebra system Magma \cite{ref}. The group $S_3$ has the subgroups $V_0 := \{\text{id}\}$, $V_1 := \{(1, 2)\}$, $V_2 := \{(1, 3)\}$, $V_3 := \{(2, 3)\}$, $V_4 := \{(1, 2, 3)\}$, $V_5 := S_3$. The set $\{V_0, V_1, V_4, V_5\}$ is a system of representatives for the conjugacy classes of subgroups of $S_3$. In $S_3$, we write $a := (1, 2)$, $b := (1, 2, 3)$ and $1 := \text{id}$. So $V_1 = \langle a \rangle$, $V_4 = \langle b \rangle$ and $V_5 = \langle a, b \rangle$.

A system of representatives for the conjugacy classes of subgroups of $S_3 \times S_3$ is given by

$$
\begin{align*}
U_{0,0} &:= V_0 \times V_0 = \{(1, 1)\}, & U_{4,1} &:= V_4 \times V_1 = \langle (b, 1), (1, a) \rangle, \\
U_{1,0} &:= V_1 \times V_0 = \langle (a, 1) \rangle, & U_{1,4} &:= V_1 \times V_4 = \langle (a, 1), (1, b) \rangle, \\
U_{0,1} &:= V_0 \times V_1 = \langle (1, a) \rangle, & U_7 &:= \langle (a, a), (b, 1) \rangle, \\
\Delta(V_1) &= \langle (a, a) \rangle, & \Delta(V_5) &= \langle (a, a), (b, b) \rangle, \\
U_{4,0} &:= V_4 \times V_0 = \langle (b, 1) \rangle, & U_{4,4} &:= V_4 \times V_4 = \langle (b, 1), (1, b) \rangle, \\
U_{0,4} &:= V_0 \times V_4 = \langle (1, b) \rangle, & U_{1,5} &:= V_1 \times V_5 = \langle (a, 1), (1, a), (1, b) \rangle, \\
\Delta(V_4) &= \langle (b, b) \rangle, & U_{5,1} &:= V_5 \times V_1 = \langle (a, 1), (b, 1), (1, a) \rangle, \\
U_{1,1} &:= V_1 \times V_1 = \langle (a, 1), (1, a) \rangle, & U_{4,5} &:= V_4 \times V_5 = \langle (b, 1), (a, 1), (1, b) \rangle, \\
U_{5,0} &:= V_5 \times V_0 = \langle (a, 1), (b, 1) \rangle, & U_{5,4} &:= V_5 \times V_4 = \langle (a, 1), (b, 1), (1, b) \rangle, \\
U_{0,5} &:= V_0 \times V_5 = \langle (1, a), (b, 1) \rangle, & U_8 &:= \langle (a, a), (b, 1), (1, b) \rangle, \\
U_6 &:= \langle (a, a), (1, b) \rangle, & U_{5,5} &:= V_5 \times V_5 = \langle (a, 1), (1, a), (b, 1) \rangle.
\end{align*}
$$

Let $H_{i,j} := [(S_3 \times S_3)/U_{i,j}]$ for $i, j \in \{0, 1, 4, 5\}$, $H_s := [(S_3 \times S_3)/U_s]$ for $s \in \{6, 8\}$ and $H_{t}^\Delta := [(S_3 \times S_3)/\Delta(U_t)]$ for $t \in \{1, 4, 5\}$.

So we obtain the Z-linear basis

$$
\mathcal{H} := \quad (H_{0,0}, H_{1,0}, H_{0,1}, H_{1,1}, H_{4,0}, H_{4,1}, H_{0,4}, H_{4,4}, H_{1,1}, H_{5,0}, H_{0,5}, H_{5,5}, H_{0,6}, H_{6,1}, H_{1,4}, H_{4,1}, H_{7,5}, H_{1,5}, H_{4,5}, H_{5,1}, H_{4,5}, H_{5,4}, H_{5,5}, H_{6,5})
$$

of $B_Z(S_3, S_3)$. Of course, $\mathcal{H}$ is also a $\mathbb{Q}$-linear basis of $B_{\mathbb{Q}}(S_3, S_3)$.
### 3 \( B_Q(S_3, S_3) \)

#### 3.1 Peirce decomposition of \( B_Q(S_3, S_3) \)

Using Magma [9] we obtain an orthogonal decomposition of \( 1_{B_Q} \) into the following idempotents of \( B_Q = B_Q(S_3, S_3) \).

\[
\begin{align*}
  e & := -\frac{1}{2}H_{0,0} + H_{1,0} + \frac{1}{2}H_{4,0} \\
  g & := \frac{1}{3}H_{0,0} - \frac{1}{2}H_{1,0} - \frac{1}{3}H_{0,1} - H_{4,0} + 2H_{1,1} + H_{4,1} \\
  h & := -\frac{1}{2}H_{0,0} + \frac{1}{3}H_{0,1} + \frac{1}{2}H_{4,0} - \frac{1}{2}H_{0,4} + \frac{2}{3}H_{4,4} - H_{4,1} \\
  \varepsilon_1 & := e + g + h.
\end{align*}
\]

Write \( \varepsilon_1 := e + g + h \). In [Remark 1] and [Remark 3] we shall see that these idempotents are primitive.

In a next step, we fix \( Q \)-linear bases of the Peirce components.

| Peirce component | \( Q \)-linear basis |
|------------------|----------------------|
| \( e B_Q e \)    | \( e = -\frac{1}{2}H_{0,0} + H_{1,0} + \frac{1}{2}H_{4,0} \) |
| \( e B_Q g \)    | \( b_{e,g} := -\frac{1}{3}H_{0,0} - \frac{1}{2}H_{1,0} - \frac{1}{3}H_{0,1} - \frac{1}{2}H_{4,0} + H_{1,1} + \frac{1}{3}H_{4,1} \) |
| \( e B_Q h \)    | \( b_{e,h} := -\frac{1}{3}H_{0,0} + \frac{1}{2}H_{1,0} + \frac{1}{3}H_{0,1} + \frac{1}{2}H_{4,0} - \frac{1}{2}H_{0,4} - H_{1,1} - \frac{1}{2}H_{4,1} + \frac{1}{3}H_{1,4} + \frac{1}{3}H_{4,4} \) |
| \( g B_Q e \)    | \( b_{g,e} := -\frac{1}{3}H_{0,0} + 2H_{1,0} + H_{4,0} \) |
| \( g B_Q g \)    | \( g = \frac{1}{3}H_{0,0} - 2H_{1,0} - \frac{1}{3}H_{0,1} - H_{4,0} + 2H_{1,1} + H_{4,1} \) |
| \( g B_Q h \)    | \( b_{g,h} := -\frac{1}{3}H_{0,0} + \frac{1}{2}H_{1,0} + \frac{1}{3}H_{0,1} + \frac{1}{2}H_{4,0} - H_{0,4} - 2H_{1,1} - H_{4,1} + \frac{2}{3}H_{1,4} + \frac{2}{3}H_{4,4} \) |
| \( h B_Q e \)    | \( b_{h,e} := -\frac{1}{3}H_{0,0} + H_{4,0} \) |
| \( h B_Q g \)    | \( b_{h,g} := \frac{1}{3}H_{0,0} - \frac{1}{2}H_{0,1} - H_{4,0} + H_{4,1} \) |
| \( h B_Q h \)    | \( h = -\frac{1}{2}H_{0,0} + \frac{1}{3}H_{0,1} + \frac{1}{2}H_{4,0} - \frac{1}{2}H_{0,4} + \frac{2}{3}H_{4,4} - H_{4,1} \) |
| \( e B_Q \varepsilon_4 \) | \( b_{e,\varepsilon_4} := \frac{1}{3}H_{0,0} + \frac{1}{2}H_{1,0} + \frac{1}{2}H_{0,1} + \frac{1}{2}H_{4,0} + \frac{1}{2}H_{0,4} - \frac{1}{2}H_{1,1} - \frac{1}{2}H_{0,5} - \frac{1}{2}H_{4,1} - \frac{1}{2}H_{4,4} - \frac{1}{2}H_{4,5} \) |
| \( g B_Q \varepsilon_4 \) | \( b_{g,\varepsilon_4} := -\frac{1}{3}H_{0,0} + \frac{1}{2}H_{1,0} + \frac{1}{2}H_{0,1} + \frac{1}{2}H_{4,0} - \frac{1}{2}H_{0,4} - \frac{1}{2}H_{1,1} - \frac{1}{2}H_{0,5} - \frac{1}{2}H_{4,1} - \frac{1}{2}H_{4,4} + \frac{1}{2}H_{4,5} \) |
| \( h B_Q \varepsilon_4 \) | \( b_{h,\varepsilon_4} := -\frac{1}{3}H_{0,0} + \frac{1}{2}H_{1,0} + \frac{1}{2}H_{0,1} + \frac{1}{2}H_{4,0} - \frac{1}{2}H_{0,4} - \frac{1}{2}H_{1,1} - \frac{1}{2}H_{0,5} - \frac{1}{2}H_{4,1} + \frac{1}{2}H_{4,5} \) |
| \( \varepsilon_2 B_Q \varepsilon_2 \) | \( \varepsilon_2 = -\frac{1}{2}H_{0,0} + H_{1,0} + \frac{1}{2}H_{4,0} + 2H_{1,1} \) |
| \( \varepsilon_2 B_Q \varepsilon_4 \) | \( b_{e,\varepsilon_2,\varepsilon_4} := \frac{1}{3}H_{0,0} + \frac{1}{2}H_{1,0} + \frac{1}{2}H_{0,1} + \frac{1}{2}H_{4,0} + \frac{1}{2}H_{0,4} - \frac{1}{2}H_{1,1} - \frac{1}{2}H_{0,5} + \frac{1}{2}H_{4,4} \) |
| \( \varepsilon_2 B_Q \varepsilon_3 \) | \( \varepsilon_3 = -\frac{1}{2}H_{0,0} + \frac{1}{2}H_{4,0} + \frac{1}{2}H_{4,4} - \frac{1}{2}H_{1,0} + \frac{1}{2}H_{4,5} - \frac{1}{2}H_{4,4} + \frac{1}{2}H_{4,5} \) |
| \( \varepsilon_3 B_Q \varepsilon_3 \) | \( b_{e,\varepsilon_3} := -\frac{1}{3}H_{0,0} - \frac{1}{2}H_{1,0} - \frac{1}{3}H_{4,0} + \frac{1}{2}H_{5,0} \) |
| \( \varepsilon_4 B_Q \varepsilon_4 \) | \( b_{e,\varepsilon_4} := -\frac{1}{2}H_{0,0} + \frac{1}{3}H_{1,0} + \frac{1}{2}H_{0,1} + \frac{1}{2}H_{1,0} - \frac{1}{2}H_{1,1} - \frac{1}{2}H_{0,5} - \frac{1}{2}H_{4,1} + \frac{1}{2}H_{5,1} \) |
| \( \varepsilon_4 B_Q \varepsilon_2 \) | \( b_{e,\varepsilon_4,\varepsilon_2} := \frac{1}{3}H_{0,0} - \frac{1}{2}H_{1,0} - \frac{1}{3}H_{0,1} + \frac{1}{2}H_{4,0} + \frac{1}{2}H_{0,4} + \frac{1}{2}H_{1,1} + \frac{1}{2}H_{5,0} + \frac{1}{2}H_{4,4} + \frac{1}{2}H_{4,5} \) |
| \( \varepsilon_4 B_Q \varepsilon_3 \) | \( b_{e,\varepsilon_4,\varepsilon_3} := \frac{1}{3}H_{0,0} - \frac{1}{3}H_{1,0} - \frac{1}{3}H_{0,1} - \frac{1}{3}H_{4,0} - \frac{1}{3}H_{4,4} - \frac{1}{3}H_{1,1} - \frac{1}{3}H_{0,5} + \frac{1}{3}H_{5,0} + \frac{1}{3}H_{4,5} + \frac{1}{3}H_{5,5} \) |

**Remark 1.** The idempotents \( e, g, h, \varepsilon_2, \varepsilon_3 \) are primitive, as \( e B_Q e \cong Q, g B_Q g \cong Q, h B_Q h \cong Q, \varepsilon_2 B_Q \varepsilon_2 \cong Q \) and \( \varepsilon_3 B_Q \varepsilon_3 \cong Q \).
We have the following multiplication table for the basis elements of $B_Q = B_Q(S_3, S_3)$.

|   | $e$ | $b_{e,g}$ | $b_{e,h}$ | $b_{g,e}$ | $b_{g,h}$ | $h$ | $b_{e,c}$ | $b_{h,c}$ | $e_2$ | $b_{e,c},e_{4}$ | $b_{h,c},e_{4}$ | $e_{4}$ | $b_{e,c},e_{4}^\prime$ | $b_{h,c},e_{4}^\prime$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $e$ | $e$ | $b_{e,g}$ | $b_{e,h}$ | $b_{g,e}$ | $b_{g,h}$ | $h$ | $b_{e,c}$ | $b_{h,c}$ | $e_2$ | $b_{e,c},e_{4}$ | $b_{h,c},e_{4}$ | $e_{4}$ | $b_{e,c},e_{4}^\prime$ | $b_{h,c},e_{4}^\prime$ |
| $b_{e,g}$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $b_{e,h}$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $b_{g,e}$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $b_{g,h}$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $h$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $e_2$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $b_{e,c}$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $b_{h,c}$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $b_{e,c},e_{4}$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $b_{h,c},e_{4}$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $e_{4}$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |

We see that $e_3$ is even central.

**Lemma 2.** Consider $Q[\eta, \xi]/(\eta^2, \eta \xi, \xi^2) = Q[\overline{\eta}, \overline{\xi}]$, where $\overline{\xi} := \xi + (\eta^2, \eta \xi, \xi^2)$ and $\overline{\eta} := \eta + (\eta^2, \eta \xi, \xi^2)$. We have the $Q$-algebra isomorphism

$$
\mu : Q[\overline{\eta}, \overline{\xi}] \rightarrow e_{4} B_Q e_{4}
$$

$$
\overline{\eta} \mapsto b'_{e_{4},e_{4}}, \overline{\xi} \mapsto b''_{e_{4},e_{4}}.
$$

**Proof.** Since $e_{4} B_Q e_{4} = Q(e_{4}, b'_{e_{4},e_{4}}, b''_{e_{4},e_{4}})$ is commutative and $(b'_{e_{4},e_{4}})^2 = 0$, $(b''_{e_{4},e_{4}})^2 = 0$ and $b'_{e_{4},e_{4}} b''_{e_{4},e_{4}} = 0$, the map $\mu$ is a well-defined $Q$-algebra isomorphism.

As the $Q$-linear basis $(1, \overline{\eta}, \overline{\xi})$ is mapped to the $Q$-linear basis $(e_{4}, b'_{e_{4},e_{4}}, b''_{e_{4},e_{4}})$, it is bijective. \qed

**Remark 3.** The ring $Q[\overline{\eta}, \overline{\xi}]$ is local. In particular, $e_{4}$ is a primitive idempotent of $B_Q$.

**Proof.** We have $U(Q[\overline{\eta}, \overline{\xi}]) = Q[\overline{\eta}, \overline{\xi}] \setminus \langle \overline{\eta}, \overline{\xi} \rangle$, as for $u := a + b \overline{\eta} + c \overline{\xi}$ the inverse is given by $u^{-1} = a^{-1} - a^{-2} b \overline{\eta} - a^{-2} c \overline{\xi}$ for $a, b, c \in Q$, with $a \neq 0$. Thus the nonunits of $Q[\overline{\eta}, \overline{\xi}]$ form an ideal and so $Q[\overline{\eta}, \overline{\xi}]$ is a local ring. \qed

To standardize notation, we aim to construct a $Q$-algebra $A := \bigoplus_{i,j} A_{i,j}$ such that $A \cong B_Q(S_3, S_3)$. 

In a first step to do so, we choose $\mathbb{Q}$-vector spaces $A_{i,j}$ and $\mathbb{Q}$-linear isomorphisms $\gamma_{i,j} : A_{i,j} \xrightarrow{\sim} \varepsilon_i B_{Q} \varepsilon_j$ for $i, j \in [1, 4]$. We define the tuple of $\mathbb{Q}$-vector spaces

\[
\begin{align*}
(A_{1,1}, & A_{1,2}, A_{1,3}, A_{1,4}, (Q^{3 \times 3}, 0, 0, Q^{3 \times 1}) , \\
A_{2,1}, & A_{2,2}, A_{2,3}, A_{2,4}, := 0, Q , 0 , Q , \\
A_{3,1}, & A_{3,2}, A_{3,3}, A_{3,4}, 0, 0, Q , 0 , \\
A_{4,1}, & A_{4,2}, A_{4,3}, A_{4,4}, Q^{1 \times 3}, Q , 0 , Q^{[\eta, \xi]} \), cf. \text{Lemma 2}
\end{align*}
\]

We have $\gamma_{s,t} = 0$ for $(s, t) \in \{(1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2), (3, 4), (4, 3)\}$.

Let \( \beta : B_{Q} \times B_{Q} \to B_{Q} \) be the multiplication map on $B_{Q}$. Write

\[
\beta_{i,j,k} := \beta|\varepsilon_i B_{Q} \varepsilon_j \times \varepsilon_j B_{Q} \varepsilon_k : \varepsilon_i B_{Q} \varepsilon_j \times \varepsilon_j B_{Q} \varepsilon_k \to \varepsilon_i B_{Q} \varepsilon_k .
\]

Now, we construct $\mathbb{Q}$-bilinear multiplication maps $\alpha_{i,j,k}$ for $i, j, k \in [1, 4]$ such that the following quadrangle of maps commutes.

\[
\begin{array}{c}
A_{i,j} \times A_{j,k} \\
\gamma_{i,j} \times \gamma_{j,k} \downarrow \alpha_{i,j,k} \downarrow \gamma_{i,k} \\
\varepsilon_i B_{Q} \varepsilon_j \times \varepsilon_j B_{Q} \varepsilon_k \\
\beta_{i,j,k} \downarrow \varepsilon_i B_{Q} \varepsilon_k
\end{array}
\]

I.e. we set $\alpha_{i,j,k} := \gamma_{i,k}^{-1} \circ \beta_{i,j,k} \circ (\gamma_{i,j} \times \gamma_{j,k})$. This leads to

- $\alpha_{i,j,k} = 0$
  if $(i, j)$, $(j, k)$ or $(i, k)$ is contained in $\{(1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2), (3, 4), (4, 3)\}$
- $\alpha_{1,1,1} : A_{1,1} \times A_{1,1} \to A_{1,1}, (X, Y) \mapsto XY$
- $\alpha_{1,1,4} : A_{1,1} \times A_{1,4} \to A_{1,4}, (X, u) \mapsto Xu$
- $\alpha_{1,4,1} = 0$
- $\alpha_{1,4,4} : A_{1,4} \times A_{4,4} \to A_{1,4}, (u, a + b \eta + c \xi) \mapsto ua$
- $\alpha_{2,1,2} : A_{2,1} \times A_{2,2} \to A_{2,2}, (u, v) \mapsto uv$
Accordingly, elements of this direct sum are written as matrices with entries in the respective summands, i.e. in the form $[m_{i,j}]_{i,j}$ with $m_{i,j} \in M_{i,j}$ for $i, j \in [1, r]$.

**Proposition 5.** Let

$$A := \bigoplus_{i, j \in [1, 4]} A_{i,j} = \begin{bmatrix}
A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\
A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\
A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\
A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4}
\end{bmatrix} = \begin{bmatrix}
\mathbb{Q}^{3 \times 3} & 0 & 0 & \mathbb{Q}^{3 \times 1} \\
0 & \mathbb{Q} & 0 & \mathbb{Q} \\
0 & 0 & \mathbb{Q} & 0 \\
\mathbb{Q}^{1 \times 3} & \mathbb{Q} & 0 & \mathbb{Q}[\overline{\eta}, \overline{\xi}]
\end{bmatrix}.$$  

Define the multiplication

$$A \times A \rightarrow A$$

$$([a_{i,j}]_{i,j} \ , \ [a'_{s,t}]_{s,t}) \rightarrow \left[ \sum_{r \in [1,4]} \alpha_{i,r,j}(a_{i,r} a'_{r,j}) \right]_{i,j}.$$  

We obtain a $\mathbb{Q}$-algebra isomorphism

$$A \xrightarrow{\sim} B_\mathbb{Q}(S_3, S_3)$$

$$[a_{i,j}]_{i,j \in [1,4]} \rightarrow \sum_{i,j \in [1,4]} \gamma_{i,j}(a_{i,j}).$$
3.2 $B_Q(S_3,S_3)$ as path algebra modulo relations

We aim to write $B_Q = B_Q(S_3,S_3) \cong A$, up to Morita equivalence, as a path algebra modulo relations.

We denote by $e_{i,j} \in A_{1,1} = Q^{3 \times 3}$ the elements that have a single non-zero entry at position $(i,j)$. We have $a_{1,1} := \gamma^{-1}(e) = e_{1,1} \in Q^{3 \times 3} \subseteq A$, $\gamma^{-1}(g) = e_{2,2} \in Q^{3 \times 3} \subseteq A$, $\gamma^{-1}(e_{3,3}) \in Q^{3 \times 3} \subseteq A$ and $a_{k,k} := \gamma^{-1}(\varepsilon_k)$ for $k \in [2,4]$, cf.

We have $AA_{1,1} \cong A e_{2,2}$ as $A$-modules, using multiplication with $e_{1,2}$ from the right from $AA_{1,1}$ to $A e_{2,2}$ and multiplication with $e_{2,1}$ from the right from $A e_{2,2}$ to $AA_{1,1}$. Note that $e_{1,2} e_{2,1} = a_{1,1}$ and $e_{2,1} e_{1,2} = e_{2,2}$. Similarly $AA_{1,1} \cong A e_{3,3}$.

Therefore, $A$ is Morita equivalent to

$$A' := \left( \sum_{i \in [1,4]} a_{i,i} \right) A \left( \sum_{i \in [1,4]} a_{i,i} \right) = \bigoplus_{i,j \in [1,4]} a_{i,i} a_{j,j} = \bigoplus_{i,j \in [1,4]} a_{i,i} A_{i,j} a_{j,j} .$$

Write $A'_{i,j} := a_{i,i} A_{i,j} a_{j,j} = A_{i,j}$ for $i,j \in [2,4]$.

Identify $A'_{1,1} := Q = \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) = a_{1,1} A_{1,1} a_{1,1} \subseteq A_{1,1} = Q^{3 \times 3}$.

Identify $A'_{1,4} := Q = \left( \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \right) = a_{1,1} A_{1,4} a_{4,4} \subseteq A_{1,4} = Q^{2 \times 1}$.

Identify $A'_{1,4} := Q = \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) = a_{4,4} A_{4,1} a_{1,1} \subseteq A_{4,1} = Q^{1 \times 3}$. Let $A'_{i,j} := 0$ and $A'_{j,1} := 0$ for $j \in [2,3]$.

We have the $Q$-linear basis of $A'$

\[
\begin{align*}
a_{1,1} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & a_{2,2} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
a_{3,3} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & a_{4,4} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
a_{1,4} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & a_{4,1} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \\
a_{2,4} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} & a_{4,2} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\
a'_{4,4} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta \end{bmatrix} & a''_{4,4} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi \end{bmatrix}
\end{align*}
\]
We have the following multiplication table for the basis elements.

| (,) | $a_{1,1}$ | $a_{1,4}$ | $a_{2,2}$ | $a_{2,4}$ | $a_{3,3}$ | $a_{4,1}$ | $a_{4,2}$ | $a_{4,4}$ | $a_{4}''$ | $a_{4}'''$ |
|-----|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| $a_{1,1}$ | $a_{1,1}$ | $a_{1,4}$ | $0$       | $0$       | $0$       | $0$       | $0$       | $0$       | $0$       | $0$       |
| $a_{1,4}$ | $0$       | $0$       | $0$       | $0$       | $0$       | $0$       | $0$       | $0$       | $0$       | $0$       |
| $a_{2,2}$ | $0$       | $a_{2,2}$ | $a_{2,4}$ | $0$       | $0$       | $0$       | $0$       | $0$       | $0$       | $0$       |
| $a_{2,4}$ | $0$       | $0$       | $0$       | $0$       | $0$       | $a_{2,4}$ | $0$       | $0$       | $0$       | $0$       |
| $a_{3,3}$ | $0$       | $0$       | $0$       | $0$       | $a_{3,3}$ | $0$       | $0$       | $0$       | $0$       | $0$       |
| $a_{4,1}$ | $a_{4,1}$ | $a_{4,4}$ | $0$       | $0$       | $0$       | $0$       | $0$       | $0$       | $0$       | $0$       |
| $a_{4,2}$ | $0$       | $a_{4,2}$ | $a_{4,4}'' - 12a_{4,4}'$ | $0$       | $0$       | $0$       | $0$       | $0$       | $0$       | $0$       |
| $a_{4,4}$ | $0$       | $0$       | $0$       | $0$       | $0$       | $a_{4,4}$ | $0$       | $0$       | $0$       | $0$       |
| $a_{4}''$ | $0$       | $0$       | $0$       | $0$       | $0$       | $0$       | $0$       | $0$       | $a_{4}''$ | $0$       |
| $a_{4}'''$ | $0$       | $0$       | $0$       | $0$       | $0$       | $0$       | $0$       | $0$       | $0$       | $a_{4}'''$ |

We have $a_{4,4}' = a_{4,1} \cdot a_{4,4}$ and $a_{4,4}'' = a_{4,2} \cdot a_{2,4} + 12a_{4,1} \cdot a_{1,4}$. Hence, as a $\mathbb{Q}$-algebra $A'$ is generated by $a_{1,1}, a_{1,2}, a_{3,3}, a_{4,4}, a_{1,4}, a_{4,1}, a_{2,4}, a_{4,2}$.

Consider the quiver $\Psi := \begin{bmatrix} \tilde{a}_{3,3} & \tilde{a}_{2,2} \\ \tilde{a}_{4,4} & \tilde{a}_{1,1} \end{bmatrix}$.

We have a surjective $\mathbb{Q}$-algebra morphism $\varphi : \mathbb{Q}\Psi \to A'$ by sending

$\tilde{a}_{1,1} \mapsto a_{1,1}$, $\tilde{a}_{2,2} \mapsto a_{2,2}$, $\tilde{a}_{3,3} \mapsto a_{3,3}$, $\tilde{a}_{4,4} \mapsto a_{4,4}$,

$\rho \mapsto a_{4,1}$, $\pi \mapsto a_{1,4}$, $\vartheta \mapsto a_{4,2}$, $\sigma \mapsto a_{2,4}$.

We establish the following multiplication trees, where we underline the elements that are not in a $\mathbb{Q}$-linear relation with previously underlined elements.

\[
\begin{array}{c}
\xymatrix{
\text{Diagram 1:} \\
\end{array}
\]

\[
\begin{array}{c}
\xymatrix{
\text{Diagram 2:} \\
\end{array}
\]

The multiplication tree of the idempotent $a_{3,3}$ consists only of the element $a_{3,3}$.

So the kernel of $\varphi$ contains the elements:

$\pi \rho$, $\sigma \vartheta$, $\rho \pi \rho$, $\vartheta \sigma \rho$, $\pi \vartheta$, $\sigma \rho$, $\rho \pi \vartheta$, $\vartheta \sigma \vartheta$. 

Let $I$ be the ideal in $\mathbb{Q}\Psi$ generated by those elements. So $I \subseteq \ker(\varphi)$. Therefore, $\varphi$ induces a surjective $\mathbb{Q}$-algebra morphism from $\mathbb{Q}\Psi/I$ to $A'$. Note that $\mathbb{Q}\Psi/I$ is $\mathbb{Q}$-linearly generated by

$$\mathcal{N} := \{\tilde{a}_{3,3} + I, \tilde{a}_{2,2} + I, \tilde{a}_{4,1} + I, \sigma + I, \pi + I, \vartheta + I, \rho + I, \vartheta\sigma + I, \rho\pi + I\},$$

cf. the underlined elements above. To see that, note that a product $\xi$ of $k$ generators may be written as a product in $\mathcal{N}$ of $k'$ generators and a product of $k''$ generators, where $k = k' + k''$ and where $k'$ is chosen maximal. We call $k''$ the excess of $\xi$. If $k'' \geq 1$ then, using the trees above, we may write $\xi$ as an $\mathbb{Q}$-linear combination of products of generators that have excess $\leq k'' - 1$. In the present case, we even have $\xi = 0$.

Moreover, note that $|\mathcal{N}| = 10 = \dim_{\mathbb{Q}}(A')$.

Since we have a surjective $\mathbb{Q}$-algebra morphism from $\mathbb{Q}\Psi/I$ to $A'$, this dimension argument shows this morphism to be bijective. In particular, $I = \ker(\varphi)$.

We may reduce this list to obtain $\ker(\varphi) = (\pi\rho, \sigma\vartheta, \pi\vartheta, \sigma\rho)$. So we obtain the

**Proposition 6.** Recall that $I = (\pi\rho, \sigma\vartheta, \pi\vartheta, \sigma\rho)$. We have the isomorphism of $\mathbb{Q}$-algebras

$$A' \cong \mathbb{Q}\left\llbracket \tilde{a}_{3,3}, \tilde{a}_{2,2}, \tilde{a}_{4,4}, \tilde{a}_{1,1} \right\rrbracket / I = \mathbb{Q}\Psi/I$$

$$a_{1,1} \mapsto \tilde{a}_{1,1} + I$$
$$a_{2,2} \mapsto \tilde{a}_{2,2} + I$$
$$a_{3,3} \mapsto \tilde{a}_{3,3} + I$$
$$a_{4,4} \mapsto \tilde{a}_{4,4} + I$$
$$a_{4,1} \mapsto \rho + I$$
$$a_{1,4} \mapsto \pi + I$$
$$a_{4,2} \mapsto \vartheta + I$$
$$a_{2,4} \mapsto \sigma + I.$$

In particular, $\mathbb{Q}\Psi/I$ is Morita equivalent to $A \cong B_{\mathbb{Q}}(S_3, S_3)$.

4 The double Burnside $R$-algebra $B_R(S_3, S_3)$ for $R \in \{\mathbb{Z}, \mathbb{Z}(2), \mathbb{F}_2, \mathbb{Z}(3), \mathbb{F}_3\}$

4.1 $B\mathbb{Z}(S_3, S_3)$ via congruences

Recall that $A = \bigoplus_{i,j \in [1,4]} A_{i,j} \cong B_{\mathbb{Q}}$, cf. **Proposition 5**. In the $\mathbb{Q}$-algebra $A$, we define the $\mathbb{Z}$-order

$$A_{\mathbb{Z}} := \begin{bmatrix} A_{\mathbb{Z},1,1} & A_{\mathbb{Z},1,2} & A_{\mathbb{Z},1,3} & A_{\mathbb{Z},1,4} \\ A_{\mathbb{Z},2,1} & A_{\mathbb{Z},2,2} & A_{\mathbb{Z},2,3} & A_{\mathbb{Z},2,4} \\ A_{\mathbb{Z},3,1} & A_{\mathbb{Z},3,2} & A_{\mathbb{Z},3,3} & A_{\mathbb{Z},3,4} \\ A_{\mathbb{Z},4,1} & A_{\mathbb{Z},4,2} & A_{\mathbb{Z},4,3} & A_{\mathbb{Z},4,4} \end{bmatrix} := \begin{bmatrix} \mathbb{Z}^{3 \times 3} & 0 & 0 & \mathbb{Z}^{3 \times 1} \\ 0 & \mathbb{Z} & 0 & \mathbb{Z} \\ 0 & 0 & \mathbb{Z} & 0 \\ \mathbb{Z}^{1 \times 3} & \mathbb{Z} & 0 & \mathbb{Z}[\varpi, \overline{\xi}] \end{bmatrix} \subseteq A.$$
In fact, \( A_Z \) is a subring of \( A \), as \( \alpha_{i,j,k}(A_Z, A_Z \times A_Z, A_Z) \subseteq A_Z, k \) for \( i, j, k \in [1, 4] \).

**Remark 7.** As \( A \cong B_Q \) is not semisimple, there are no maximal \( Z \)-orders in \( A \), \([6, \S 10]\). So \( A_Z \) is not a canonical choice of a \( Z \)-order in \( A \), but it nonetheless enables us to describe \( \Lambda \) inside \( A_Z \) via congruences.

Consider the following elements of \( U(A) \).

\[
x_1 := \begin{bmatrix} 0 & -2 & 0 & 0 & 0 & 0 \\ 6 & 6 & -4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad x_2 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 7 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad x_3 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.
\]

We define the injective ring morphism \( \delta : B_Z \to A \), \( y \mapsto x_3^{-1} \cdot x_2^{-1} \cdot x_1^{-1} \cdot \gamma^{-1}(y) \cdot x_1 \cdot x_2 \cdot x_3 \). The conjugating element \( x_1 \) was constructed such that its image lies in \( A_Z \). The elements \( x_2, x_3 \) serve the purpose of simplifying the congruences of \( \delta(B_Z) \).

**Theorem 8.** The image \( \delta(B_Z) \) in \( A_Z \) is given by

\[
\Lambda := \delta(B_Z) = \left\{ \begin{bmatrix} s_{1,1} & s_{1,2} & s_{1,3} & 0 & 0 & t_1 \\ s_{2,1} & s_{2,2} & s_{2,3} & 0 & 0 & t_2 \\ s_{3,1} & s_{3,2} & s_{3,3} & 0 & 0 & t_3 \\ 0 & 0 & 0 & u & 0 & v \\ 0 & 0 & 0 & 0 & w & 0 \\ x_1 & x_2 & x_3 & y & 0 & z_1 + z_2\nu + z_3\xi \end{bmatrix} \mid \begin{bmatrix} 2w - 2z_1 \equiv 8z_2 \equiv 4z_3 \equiv 40 \\ x_1 \equiv 40 \\ x_2 \equiv 40 \\ x_3 \equiv 40 \\ y \equiv 20 \\ t_1 \equiv 20 \\ t_2 \equiv 20 \\ t_3 \equiv 20 \\ v \equiv 20 \\ x_1 \equiv 30 \\ x_2 \equiv 30 \\ x_3 \equiv 30 \\ z_2 \equiv 30 \end{bmatrix} \right\} \subseteq A_Z.
\]

In particular, we have \( B_Z = B_Z(S_3, S_3) \cong \Lambda \) as rings.

More symbolically written, we have

\[
\Lambda = \begin{bmatrix} Z & Z & Z & 0 & 0 & (2) \\ Z & Z & Z & 0 & 0 & (2) \\ Z & Z & Z & 0 & 0 & (2) \\ 0 & 0 & 0 & Z & 0 & (2) \\ 0 & 0 & 0 & 0 & Z & 0 \end{bmatrix}_{-2}. \]

\[
(12) (12) (12) (2) 0 Z \quad + (12)\nu + (4)\xi.
\]
Proof. We identify $Z^{22 \times 4}$ and $A_Z$ along the isomorphism

$$
\left( s_{1,1}, s_{2,1}, s_{3,1}, s_{1,2}, s_{2,2}, s_{3,2}, s_{1,3}, s_{2,3}, s_{3,3},
\begin{array}{c}
x_1, x_2, x_3, u, y, t_1, t_2, t_3, v, z_1, z_2, z_3
\end{array}
\right) ^t
\to
\begin{bmatrix}
s_{1,1} & s_{1,2} & s_{1,3} & 0 & 0 & t_1 \\
s_{2,1} & s_{2,2} & s_{2,3} & 0 & 0 & t_2 \\
s_{3,1} & s_{3,2} & s_{3,3} & 0 & 0 & t_3 \\
0 & 0 & 0 & u & 0 & v \\
0 & 0 & 0 & 0 & w & 0 \\
x_1 & x_2 & x_3 & y & 0 & z_1 + z_2 + \eta + z_3 \bar{\eta}
\end{bmatrix}.
$$

Let $M$ be the representation matrix of $\delta$, with respect to the bases $\mathcal{H} = (H_{0,0}, H_{0,1}, H_{1,0},$ $H_{1,1}, H_{0,5}, H_{5,0}, H_{5,1}, H_{1,4}, H_{4,1}, H_{6,4}, H_{4,4}, H_{5,1}, H_{1,5}, H_{5,4}, H_{4,5}, H_{8,5})$ of $B_Z$ and the standard basis of $A_Z$. We obtain

$$
M = \begin{bmatrix}
0 & 0 & 15 & -3 & 0 & 20 & 8 & 6 & 0 & 25 & 7 & 9 & 8 & -3 & 1 & 12 & 10 & 3 & 15 & 4 & 3 & 5 \\
0 & 0 & -18 & 0 & 0 & -24 & 0 & -9 & 0 & -30 & -12 & -6 & -12 & 0 & 0 & -8 & -15 & -3 & -10 & -4 & -4 & -5 \\
0 & 0 & 126 & -6 & 0 & 168 & 12 & 60 & 0 & 210 & 78 & 48 & 80 & 0 & 64 & 100 & 21 & 80 & 28 & 26 & 35 \\
-5 & -2 & -60 & 9 & -3 & -55 & -23 & -24 & -1 & -85 & -16 & -36 & -22 & 10 & 0 & -33 & -34 & -12 & -51 & -11 & -5 & -17 \\
6 & 3 & 72 & 3 & 2 & 66 & 2 & 36 & 1 & 102 & 33 & 24 & 33 & 1 & 1 & 22 & 51 & 12 & 34 & 11 & 11 & 17 \\
-42 & -20 & -504 & 2 & -16 & -462 & -46 & -240 & -7 & -714 & -208 & -192 & -220 & 15 & 0 & -176 & -340 & -84 & -272 & -77 & -65 & -119 \\
0 & 0 & -10 & 2 & 0 & -10 & -4 & -4 & 0 & -15 & -3 & -6 & -4 & 2 & 0 & -6 & -6 & -2 & -9 & -2 & -1 & -3 \\
0 & 0 & 12 & 0 & 0 & 12 & 0 & 6 & 0 & 18 & 6 & 4 & 6 & 0 & 0 & 0 & 4 & 9 & 2 & 6 & 2 & 3 \\
0 & 0 & -84 & 0 & -84 & -6 & -40 & 0 & -126 & -38 & -32 & -40 & 4 & 1 & -32 & -60 & -14 & -48 & -14 & -12 & -21 & -3 \\
0 & 0 & -756 & 36 & 0 & -1008 & -72 & -360 & 0 & -1260 & -468 & -288 & -480 & 72 & 0 & -384 & -600 & -108 & -480 & -100 & -120 & -180 \\
252 & 120 & 3024 & -12 & 96 & 2772 & 276 & 1440 & 36 & 4284 & 1248 & 1152 & 1320 & -228 & 0 & 1056 & 2040 & 432 & 1632 & 396 & 252 & 612 \\
0 & 0 & 504 & -24 & 0 & 504 & 36 & 240 & 0 & 756 & 228 & 192 & 240 & -48 & 0 & 192 & 360 & 72 & 288 & 72 & 48 & 108 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -10 & 2 & 0 & -10 & -4 & -4 & 0 & -10 & -4 & -6 & -4 & 2 & 0 & -6 & -4 & -2 & -6 & -2 & -2 & -2 \\
0 & 0 & 12 & 0 & 0 & 12 & 0 & 6 & 0 & 12 & 6 & 4 & 6 & 0 & 0 & 4 & 6 & 2 & 4 & 2 & 2 \\
0 & 0 & -84 & 0 & -84 & -6 & -40 & 0 & -84 & -40 & -32 & -40 & 4 & 0 & -32 & -40 & -14 & -32 & -14 & -14 & -14 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 504 & -24 & 0 & 504 & 36 & 240 & 0 & 504 & 240 & 192 & 240 & -48 & 0 & 192 & 240 & 72 & 192 & 72 & 24 & 72 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\
\end{bmatrix}
$$

Let $\lambda :=
\begin{bmatrix}
s_{1,1} & s_{1,2} & s_{1,3} & 0 & 0 & t_1 \\
s_{2,1} & s_{2,2} & s_{2,3} & 0 & 0 & t_2 \\
s_{3,1} & s_{3,2} & s_{3,3} & 0 & 0 & t_3 \\
0 & 0 & 0 & u & 0 & v \\
0 & 0 & 0 & 0 & w & 0 \\
x_1 & x_2 & x_3 & y & 0 & z_1 + z_2 + \eta + z_3 \bar{\eta}
\end{bmatrix}
\in A_Z$, identified with $\lambda \in Z^{22 \times 4}$.

We have
\( \lambda \in A \iff \exists \ q \in \mathbb{Z}^{2 \times 1} \text{ such that } \lambda = Mq \)
\( \iff \exists \ q \in \mathbb{Z}^{2 \times 1} \text{ such that } M^{-1} \cdot \lambda = q \)
\( \iff \ 24M^{-1} \cdot \lambda \in 24\mathbb{Z}^{2 \times 1} \)

\[
\left( \begin{array}{cccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array} \right) \begin{pmatrix}
\mathbf{s}_{1,1} \\
\mathbf{s}_{2,1} \\
\mathbf{s}_{3,1} \\
\vdots \\
\mathbf{s}_{3,3} \\
\mathbf{x}_1 \\
\mathbf{x}_2 \\
\mathbf{x}_3 \\
\mathbf{y} \\
\mathbf{y} \\
\mathbf{y} \\
\mathbf{t}_1 \\
\mathbf{t}_2 \\
\mathbf{t}_3 \\
\mathbf{u} \\
\mathbf{u} \\
\mathbf{u} \\
\mathbf{z}_1 \\
\mathbf{z}_2 \\
\mathbf{z}_3 \\
\end{pmatrix} \in 24\mathbb{Z}^{11 \times 1}
\]

\( \iff \begin{pmatrix}
2w - 2z_1 \equiv_8 z_2 \equiv_4 z_3 \equiv_4 0 \\
x_1 \equiv_4 0 \\
x_2 \equiv_4 0 \\
x_3 \equiv_4 0 \\
y \equiv_2 0 \\
t_1 \equiv_2 0 \\
t_2 \equiv_2 0 \\
t_3 \equiv_2 0 \\
v \equiv_2 0 \\
x_1 \equiv_3 0 \\
x_2 \equiv_3 0 \\
x_3 \equiv_3 0 \\
z_2 \equiv_3 0 \\
\end{pmatrix} \)

\[\text{4.2 Localisation at } 2: \mathbb{B}_{\mathbb{Z}(2)}(S_3, S_3) \text{ via congruences}\]

Write \( R := \mathbb{Z}(2) \). In the \( \mathbb{Q} \)-algebra \( A \), cf. \textbf{Proposition 5}, we have the \( R \)-order

\[
A_R := \left[ \begin{array}{cccc}
A_{R,1,1} & A_{R,1,2} & A_{R,1,3} & A_{R,1,4} \\
A_{R,2,1} & A_{R,2,2} & A_{R,2,3} & A_{R,2,4} \\
A_{R,3,1} & A_{R,3,2} & A_{R,3,3} & A_{R,3,4} \\
A_{R,4,1} & A_{R,4,2} & A_{R,4,3} & A_{R,4,4} \\
\end{array} \right] := \left[ \begin{array}{cccc}
R^{3 \times 3} & 0 & 0 & R^{3 \times 1} \\
0 & R & 0 & R \\
0 & 0 & R & 0 \\
R^{1 \times 3} & R & 0 & R[\eta, \xi] \\
\end{array} \right] \subseteq A.
\]

\textbf{Corollary 9.} We have

\[
\Lambda_{(2)} = \left\{ \begin{pmatrix}
s_{1,1} & s_{1,2} & s_{1,3} & 0 & 0 & t_1 \\
s_{2,1} & s_{2,2} & s_{2,3} & 0 & 0 & t_2 \\
s_{3,1} & s_{3,2} & s_{3,3} & 0 & 0 & t_3 \\
0 & 0 & 0 & u & 0 & v \\
x_1 & x_2 & x_3 & y & 0 & z_1 + z_2 \eta + z_3 \xi \\
\end{pmatrix} \in A_R : \begin{pmatrix}
2w - 2z_1 \equiv_8 z_2 \equiv_4 z_3 \equiv_4 0 \\
x_1 \equiv_4 0 \\
x_2 \equiv_4 0 \\
x_3 \equiv_4 0 \\
y \equiv_2 0 \\
t_1 \equiv_2 0 \\
t_2 \equiv_2 0 \\
t_3 \equiv_2 0 \\
v \equiv_2 0 \\
\end{pmatrix} \subseteq A_R.\right\}
\]
In particular, we have $B_R = B_R(S_3, S_3) \cong \Lambda_{(2)}$ as $R$-algebras.

More symbolically written, we have

$$
\Lambda_{(2)} = \begin{bmatrix}
R & R & R & 0 & 0 & (2) \\
R & R & R & 0 & 0 & (2) \\
R & R & R & 0 & 0 & (2) \\
0 & 0 & 0 & R & 0 & (2) \\
0 & 0 & 0 & 0 & R & 0 \\
(4) & (4) & (4) & (2) & 0 & R + (4)\bar{\eta} + (4)\bar{\xi}
\end{bmatrix}.
$$

Remark 10. We claim that $1_{\Lambda_{(2)}} = e_1 + e_2 + e_3 + e_4 + e_5$ is an orthogonal decomposition into primitive idempotents, where

$$
e_1 := \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\quad e_2 := \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\quad e_3 := \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\quad e_4 := \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\quad e_5 := \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

Proof. We have $e_1 \Lambda_{(2)} e_1 \cong R$, $e_2 \Lambda_{(2)} e_2 \cong R$, $e_3 \Lambda_{(2)} e_3 \cong R$ and $e_4 \Lambda_{(2)} e_4 \cong R$. So, it follows that $e_1, e_2, e_3, e_4$ are primitive.

As $R$-algebras, we have

$$
e_5 \Lambda_{(2)} e_5 \cong \left\{ \left( w, \ z_1 + z_2 \bar{\eta} + z_3 \bar{\xi} \right) \in R \times R[\bar{\eta}, \bar{\xi}] : 2w - 2z_1 \equiv_8 z_2 \equiv_4 z_3 \equiv_4 0 \right\} =: \Gamma
\subseteq R \times R[\bar{\eta}, \bar{\xi}] .
$$

To show that $e_5$ is primitive, we show that $\Gamma$ is local.

We have the $R$-linear basis $(b_1, b_2, b_3, b_4)$ of $\Gamma$, where

$$
b_1 = \begin{pmatrix} 1, & 1 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0, & 2 + 4\bar{\eta} \end{pmatrix},
\quad b_3 = \begin{pmatrix} 0, & 8\bar{\eta} \end{pmatrix}, \quad b_4 = \begin{pmatrix} 0, & 4\bar{\xi} \end{pmatrix} .
$$

We claim that the Jacobson radical of $\Gamma$ is given by $J := R\langle 2b_1, b_2, b_3, b_4 \rangle$, that $\Gamma/J \cong F_2$ and that $\Gamma$ is local.

In fact, the multiplication table for the basis elements is given by
\[
\begin{array}{c|cc|ccc}
(\cdot) & b_1 & b_2 & b_3 & b_4 \\
\hline 
b_1 & b_1 & b_2 & b_3 & b_4 \\
b_2 & b_2 & 2b_2 + b_3 & 2b_3 & 2b_4 \\
b_3 & b_3 & 2b_3 & 0 & 0 \\
b_4 & b_4 & 2b_4 & 0 & 0 \\
\end{array}
\]

This shows that \( J \) is an ideal. Moreover, \( J \) is topologically nilpotent as
\[
J^3 = R(8b_1, 4b_2, 2b_3, 4b_4) \subseteq 2e_5 \Lambda(2) e_5.
\]
Since \( \Gamma/J \cong \mathbb{F}_2 \), the claim follows.

### 4.3 \( B_{\mathbb{Z}(2)}(S_3, S_3) \) and \( B_{\mathbb{F}_2}(S_3, S_3) \) as path algebras modulo relations

Write \( R := \mathbb{Z}(2) \). We aim to write \( \Lambda(2) \), up to Morita equivalence, as path algebra modulo relations. The \( R \)-algebra \( \Lambda(2) \) is Morita equivalent to \( \Lambda'(2) := (e_3 + e_4 + e_5)\Lambda(2)(e_3 + e_4 + e_5) \) since \( \Lambda(2)e_1 \cong \Lambda(2)e_2 \cong \Lambda(2)e_3 \) using multiplication with elements of \( \Lambda(2) \) with a single nonzero entry 1 in the upper \( (3 \times 3) \)-corner.

We have the \( R \)-linear basis of \( \Lambda'(2) \) consisting of

\[
e_3 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad e_4 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad e_5 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},
\]

\[
\tau_1 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \tau_2 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \end{bmatrix}, \quad \tau_3 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},
\]

\[
\tau_4 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \tau_5 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 87 \end{bmatrix}, \quad \tau_6 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 47 \end{bmatrix},
\]

\[
\tau_7 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 + 47 \end{bmatrix}
\]

We have \( \tau_5 = \tau_1\tau_2 \) and \( \tau_6 = \tau_3\tau_4 + 6\tau_1\tau_2 \). Hence, as an \( R \)-algebra \( \Lambda'(2) \) is generated by \( e_3, e_4, e_5, \tau_1, \tau_2, \tau_3, \tau_4, \tau_7 \).
Consider the quiver \( \Psi := \begin{pmatrix} \tilde{e}_3 & \tilde{e}_2 & \tilde{e}_4 & \tilde{e}_5 & \tilde{e}_7 \\ \tau_1 & \tau_2 & \tau_3 & \tau_7 & \tau_1 \end{pmatrix} \).

We have a surjective \( R \)-algebra morphism \( \varphi : R\Psi \to \Lambda'_{(2)} \) by sending

\[
\begin{align*}
\tilde{e}_3 & \mapsto e_3 \\
\tilde{e}_4 & \mapsto e_4 \\
\tilde{e}_5 & \mapsto e_5 \\
\tilde{\tau}_1 & \mapsto \tau_1 \\
\tilde{\tau}_2 & \mapsto \tau_2 \\
\tilde{\tau}_3 & \mapsto \tau_3 \\
\tilde{\tau}_4 & \mapsto \tau_4 \\
\tilde{\tau}_7 & \mapsto \tau_7 \\
\end{align*}
\]

We establish the following multiplication trees, where we underline the elements that are not in an \( R \)-linear relation with previous elements.

We may reduce the list of generators \( I \subseteq \ker(\varphi) \). Therefore, \( \varphi \) induces a surjective \( R \)-algebra morphism from \( R\Psi/I \) to \( \Lambda'_{(2)} \). We may reduce the list of generators to obtain

\[
I = \left( \tilde{\tau}_2 \tilde{\tau}_1, \tilde{\tau}_4 \tilde{\tau}_1, \tilde{\tau}_7 \tilde{\tau}_1 - 2\tilde{\tau}_1, \tilde{\tau}_2 \tilde{\tau}_3, \tilde{\tau}_4 \tilde{\tau}_3, \tilde{\tau}_7 \tilde{\tau}_3 - 2\tilde{\tau}_3, \tilde{\tau}_2 \tilde{\tau}_7 - 2\tilde{\tau}_2, \tilde{\tau}_4 \tilde{\tau}_7 - 2\tilde{\tau}_4, \tilde{\tau}_7^2 - 2\tilde{\tau}_7 - \tilde{\tau}_1 \tilde{\tau}_2 \right).
\]

Let \( I \) be the ideal generated by these elements. So \( I \subseteq \ker(\varphi) \). Therefore, \( \varphi \) induces a surjective \( R \)-algebra morphism from \( R\Psi/I \) to \( \Lambda'_{(2)} \). We may reduce the list of generators to obtain

\[
I = \left( \tilde{\tau}_2 \tilde{\tau}_1, \tilde{\tau}_4 \tilde{\tau}_1, \tilde{\tau}_7 \tilde{\tau}_1 - 2\tilde{\tau}_1, \tilde{\tau}_2 \tilde{\tau}_3, \tilde{\tau}_4 \tilde{\tau}_3, \tilde{\tau}_7 \tilde{\tau}_3 - 2\tilde{\tau}_3, \tilde{\tau}_2 \tilde{\tau}_7 - 2\tilde{\tau}_2, \tilde{\tau}_4 \tilde{\tau}_7 - 2\tilde{\tau}_4, \tilde{\tau}_7^2 - 2\tilde{\tau}_7 - \tilde{\tau}_1 \tilde{\tau}_2 \right).
\]

Note that \( R\Psi/I \) is \( R \)-linearly generated by

\[
\mathcal{N} := \{ \tilde{e}_3 + I, \tilde{e}_4 + I, \tilde{e}_5 + I, \tilde{\tau}_1 + I, \tilde{\tau}_2 + I, \tilde{\tau}_3 + I, \tilde{\tau}_4 + I, \tilde{\tau}_7 + I, \tilde{\tau}_3 \tilde{\tau}_4 + I, \tilde{\tau}_1 \tilde{\tau}_2 + I \},
\]
4.4 Localisation at 3: $B_{\mathbb{Z}(3)}(S_3, S_3)$ via congruences

Write $R = \mathbb{Z}(3)$. In the $\mathbb{Q}$-algebra $A$, cf. Proposition 5, we have the $R$-order

$$A_R := \begin{bmatrix}
A_{R,1,1} & A_{R,1,2} & A_{R,1,3} & A_{R,1,4} \\
A_{R,2,1} & A_{R,2,2} & A_{R,2,3} & A_{R,2,4} \\
A_{R,3,1} & A_{R,3,2} & A_{R,3,3} & A_{R,3,4} \\
A_{R,4,1} & A_{R,4,2} & A_{R,4,3} & A_{R,4,4}
\end{bmatrix} := \begin{bmatrix}
R^{3 \times 3} & 0 & 0 & R^{3 \times 1} \\
0 & R & 0 & R \\
0 & 0 & R & 0 \\
R^{1 \times 3} & R & 0 & R(\overline{7}, \overline{\xi})
\end{bmatrix} \subseteq A.$$
Corollary 13. We have

\[
\Lambda(3) = \begin{bmatrix}
    1 & s_{1,1} & s_{1,2} & s_{1,3} & 0 & 0 & t_1 \\
    0 & s_{2,1} & s_{2,2} & s_{2,3} & 0 & 0 & t_2 \\
    0 & 0 & 0 & u & 0 & v & t_3 \\
    0 & 0 & 0 & w & 0 & x_1 x_2 x_3 & y
\end{bmatrix}
\in A_R : \begin{bmatrix}
    x_1 & x_2 & x_3 & z_1 + z_2 \eta + z_3 \xi
\end{bmatrix}
\subseteq A_R.
\]

In particular, we have \( B_R = B_R(S_3, S_4) \cong \Lambda(3) \) as \( R \)-algebras.

More symbolically written, we have

\[
\Lambda(3) = \begin{bmatrix}
    R & R & R & 0 & 0 & R \\
    R & R & R & 0 & 0 & R \\
    R & R & R & 0 & 0 & R \\
    0 & 0 & 0 & R & 0 & R \\
    0 & 0 & 0 & 0 & R & 0 \\
    (3) & (3) & (3) & R & 0 & R + (3) \eta + R \xi
\end{bmatrix}.
\]

Remark 14. We claim that \( 1_{\Lambda(3)} = e_1 + e_2 + e_3 + e_4 + e_5 + e_6 \) is an orthogonal decomposition into primitive idempotents, where

\[
e_1 := \begin{bmatrix}
    1 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad e_2 := \begin{bmatrix}
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad e_3 := \begin{bmatrix}
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
e_4 := \begin{bmatrix}
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad e_5 := \begin{bmatrix}
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad e_6 := \begin{bmatrix}
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

Proof. We have \( e_s \Lambda(3) e_s \cong R \) for \( s \in [1, 5] \). Therefore it follows that \( e_1, e_2, e_3, e_4, e_5 \) are primitive.

To show that \( e_6 \) is primitive, we claim that the ring \( e_6 \Lambda(3) e_6 \cong R[\eta, \xi] \) is local.

We have \( U(R[\eta, \xi]) = R[\eta, \xi] \setminus (3, \eta, \xi) \). In fact, for \( u := a + b \eta + c \xi \) with \( a \in R \setminus (3) \) and \( b, c \in R \), the inverse is given by \( u^{-1} = a^{-1} - a^{-2}b \eta - a^{-2}c \xi \) as

\[
uu^{-1} = aa^{-1} + (-a^{-1}b + a^{-1}b) \eta + (-a^{-1}c + a^{-1}c) \xi = 1.
\]

Thus the nonunits of \( R[\eta, \xi] \) form an ideal and so \( R[\eta, \xi] \) is a local ring. This proves the claim. \( \square \)
4.5 \( B_{Z(3)}(S_3, S_3) \) and \( B_{F_3}(S_3, S_3) \) as path algebras modulo relations

Write \( R := Z_{(3)} \). We aim to write \( \Lambda_{(3)} \), up to Morita equivalence, as path algebra modulo relations.

The \( R \)-algebra \( \Lambda_{(3)} \) is Morita equivalent to \( \Lambda'_{(3)} := (e_3 + e_4 + e_5 + e_6)\Lambda_{(3)}(e_3 + e_4 + e_5 + e_6) \) since \( \Lambda_{(3)} e_1 \cong \Lambda_{(3)} e_2 \cong \Lambda_{(3)} e_3 \) using multiplication with elements of \( \Lambda_{(3)} \) with a single nonzero entry 1 in the upper \((3 \times 3)\)-corner. We have the \( R \)-linear basis of \( \Lambda'_{(3)} \) consisting of

\[
\begin{align*}
e_3 & := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, & e_4 & := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, & e_5 & := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
\end{align*}
\]

\[
\begin{align*}
e_6 & := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, & \tau_1 & := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, & \tau_2 & := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
\end{align*}
\]

\[
\begin{align*}
\tau_3 & := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, & \tau_4 & := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, & \tau_5 & := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}.
\end{align*}
\]

We have \( \tau_5 = \tau_1 \tau_2 \) and \( \tau_6 = \tau_3 \tau_4 + 4 \tau_1 \tau_2 \). Hence, as an \( R \)-algebra \( \Lambda'_{(3)} \) is generated by \( e_3, e_4, e_5, e_6, \tau_1, \tau_2, \tau_3, \tau_4 \).

Consider the quiver \( \Psi := \begin{tikzpicture}
\node (e5) at (0,0) {$\tilde{e}_5$};
\node (e3) at (0.5,0) {$\tilde{e}_3$};
\node (e4) at (1,0) {$\tilde{e}_4$};
\node (e6) at (1.5,0) {$\tilde{e}_6$};
\node (e1) at (2,0) {$\tilde{e}_1$};
\node (e2) at (2.5,0) {$\tilde{e}_2$};
\node (e5) at (3,0) {$\tilde{e}_5$};
\node (e6) at (3.5,0) {$\tilde{e}_6$};
\node (e1) at (4,0) {$\tilde{e}_1$};
\node (e2) at (4.5,0) {$\tilde{e}_2$};
\node (e3) at (5,0) {$\tilde{e}_3$};
\node (e4) at (5.5,0) {$\tilde{e}_4$};
\path
(e3) edge[->] (e1)
(e3) edge[<->] (e5)
(e3) edge[->] (e2)
(e4) edge[<->] (e6)
(e4) edge[->] (e5)
(e4) edge[->] (e2)
(e5) edge[<->] (e6)
(e5) edge[<->] (e1)
(e5) edge[<->] (e2)
node at (2.5,0) {$\tilde{\tau}_1$};
node at (3,0) {$\tilde{\tau}_2$};
node at (3.5,0) {$\tilde{\tau}_3$};
node at (4,0) {$\tilde{\tau}_4$};
\end{tikzpicture} \). We have a surjective \( R \)-algebra morphism \( \varphi : R\Psi \to \Lambda'_{(3)} \) by sending

\[
\begin{align*}
\tilde{e}_3 & \mapsto e_3, & \tilde{e}_4 & \mapsto e_4, & \tilde{e}_5 & \mapsto e_5, & \tilde{e}_6 & \mapsto e_6,
\end{align*}
\]
\[
\begin{align*}
\tilde{\tau}_1 & \mapsto \tau_1, & \tilde{\tau}_2 & \mapsto \tau_2, & \tilde{\tau}_3 & \mapsto \tau_3, & \tilde{\tau}_4 & \mapsto \tau_4.
\end{align*}
\]

We establish the following multiplication trees, where we underline the elements that are not in an \( R \)-linear relation with previous elements.

The multiplication tree of the idempotent \( e_5 \) consists only of the element \( e_5 \).
Proposition 15. Recall that $I = (\tilde{\tau}_4\tilde{\tau}_3, \tilde{\tau}_2\tilde{\tau}_1, \tilde{\tau}_2\tilde{\tau}_3)$. We have the isomorphisms of $R$-algebras

$$\Lambda'_{(3)} \xrightarrow{\sim} R \left[ \begin{array}{ccc} \tilde{e}_3 & \tilde{e}_3 & \tilde{e}_6 \\ \tilde{\tau}_2 & \tilde{\tau}_4 & \tilde{\tau}_3 \end{array} \right] /I$$

$e_i \mapsto \tilde{e}_i + I$ for $i \in [3, 6]$

$\tau_i \mapsto \tilde{\tau}_i + I$ for $i \in [1, 4]$

Recall that $Bz_{(3)}(S_3, S_3)$ is Morita equivalent to $\Lambda'_{(3)}$. 

So the kernel of $\varphi$ contains the elements: $\tilde{\tau}_4\tilde{\tau}_3, \tilde{\tau}_4\tilde{\tau}_1, \tilde{\tau}_2\tilde{\tau}_1, \tilde{\tau}_2\tilde{\tau}_3, \tilde{\tau}_3\tilde{\tau}_4\tilde{\tau}_3, \tilde{\tau}_2\tilde{\tau}_3, \tilde{\tau}_3\tilde{\tau}_4\tilde{\tau}_1, \tilde{\tau}_1\tilde{\tau}_2\tilde{\tau}_3, \tilde{\tau}_1\tilde{\tau}_2\tilde{\tau}_1$. Let $I$ be the ideal generated by these elements. So, $I \subseteq \ker(\varphi)$. Therefore, $\varphi$ induces a surjective $R$-algebra morphism from $R\Psi/I$ to $\Lambda'_{(3)}$. We may reduce the list of generators to obtain $I = (\tilde{\tau}_4\tilde{\tau}_3, \tilde{\tau}_4\tilde{\tau}_1, \tilde{\tau}_2\tilde{\tau}_1, \tilde{\tau}_2\tilde{\tau}_3)$. Note that $R\Psi/I$ is $R$-linearly generated by

$$\mathcal{N} := \{ \tilde{e}_3 + I, \tilde{e}_4 + I, \tilde{e}_5 + I, \tilde{e}_6 + I, \tilde{\tau}_1 + I, \tilde{\tau}_2 + I, \tilde{\tau}_3 + I, \tilde{\tau}_4 + I, \tilde{\tau}_3\tilde{\tau}_4 + I, \tilde{\tau}_1\tilde{\tau}_2 + I \},$$

cf. the underlined elements above.

To see that, note that a product $\xi$ of $k$ generators may be written as a product in $\mathcal{N}$ of $k'$ generators and a product of $k''$ generators, where $k = k' + k''$ and where $k'$ is choosen maximal. If $k'' \geq 1$ then, using the trees above, we have $\xi = 0$. Moreover, note that $|\mathcal{N}| = 10 = \text{rk}_R(\Lambda'_{(3)})$.

Since we have an surjective algebra morphism from $R\Psi/I$ to $\Lambda'_{(3)}$, this rank argument shows this morphism to be bijective. In particular, $I = \ker(\varphi)$.

So, we obtain the
Corollary 16. As $F_3$-algebras, we have

$$\Lambda'_3/3\Lambda'_3 \cong F_3 \left[ \begin{array}{ccc} \tilde{e}_5 & \tilde{e}_3 & \tilde{e}_6 \\ \tilde{\tau}_2 & \tilde{\tau}_4 & \tilde{\tau}_3 \\ \downarrow & \downarrow & \downarrow \\ \tilde{\tau}_1 & \tilde{\tau}_4 \tilde{\tau}_3, & \tilde{\tau}_2 \tilde{\tau}_1, & \tilde{\tau}_2 \tilde{\tau}_3 \end{array} \right].$$

Recall that $B_{F_3}(S_3, S_3)$ is Morita equivalent to $\Lambda'_3/3\Lambda'_3$.

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