RAPID DATA SEARCH USING ADIABATIC QUANTUM COMPUTATION

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We show that by a suitable choice of time-dependent Hamiltonian, the search for a marked item in an unstructured database can be achieved in unit time, using Adiabatic Quantum Computation. This is a considerable improvement over the $O(\sqrt{N})$ time required in previous algorithms. The trade-off is that in the intermediate stages of the computation process, the ground state energy of the computer increases to a maximum of $O(\sqrt{N})$, before returning to zero at the end of the process.

1 Adiabatic Quantum Computation

Adiabatic Quantum Computation (AQC) is a new paradigm in quantum computation, in which an initial state $|\Psi_0\rangle$ is adiabatically transformed into a final state $|\Psi_1\rangle$ by means of a time-dependent Hamiltonian:

$$H(s) = f(s)H_0 + g(s)H_1$$  \hspace{1cm} (1)

where

$$H_0 = I - |\Psi_0\rangle\langle\Psi_0|$$  \hspace{1cm} (2)

$$H_1 = I - |\Psi_1\rangle\langle\Psi_1|$$  \hspace{1cm} (3)

$s(t)$ is a time-like parameter, which monotonically increases with time $t$, such that $s(0) = 0$ and $s(T) = 1$, where $T$ is the required running time of the AQC). The boundary conditions on $f(s)$ and $g(s)$ are as follows:

$$f(0) = g(1) = 1 , \hspace{0.5cm} f(1) = g(0) = 0 .$$  \hspace{1cm} (4)

Note that $|\Psi_0\rangle$ and $|\Psi_1\rangle$ are ground states of $H_0$ and $H_1$, respectively. The adiabaticity condition, which must be preserved at all times, is given by:

$$\frac{|(-|dH/dt)|+|}{(E_+ - E_-)^2} \leq \epsilon ,$$  \hspace{1cm} (5)

where $|\pm\rangle$ = ground state & first excited state : $H|\pm\rangle = E_{\pm}|\pm\rangle$.

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The problem is to estimate the running time $T$ for a given computational problem implemented via an AQC.

2 Data Search Problem

For a completely unstructured database of $N$ items, one would like to find a marked item (say “$m$”) in the shortest possible time. Schematically:

Marked item

$\begin{array}{cccccccc}
| & | & | & | & | & | & |
| \cdots & | \cdots & | \cdots & | \cdots & | \cdots & | \cdots & | \cdots & | \cdots & | m | \cdots & | \cdots & | \cdots & | \cdots & | \cdots & | \cdots & | \cdots & | \cdots & | \cdots & | \cdots & | \cdots & | \cdots & | \cdots & | \cdots & | \cdots & |
\end{array}$

→ Find $m$ in shortest possible time

unstructured database of $N$ items

Now, it is well-known that classically, $\mathcal{O}(N)$ steps are required on an average to find $m$. On the other hand, it was shown by Grover, that quantum mechanics can be used to quadratically speed-up the process. That is, by associating each item in the database with an eigenvector in a $N$-dimensional Hilbert space (such that $m$ corresponds to the ket $|m\rangle$), then starting with the symmetric superposition of all states, and applying a series of unitary operations, one can evolve to the marked state in $\mathcal{O}(\sqrt{N})$ steps.

3 Data Search Using AQC

In this case, the adiabatic Hamiltonian replaces the set of unitary transformations mentioned in the previous section. The final state $|\Psi_1\rangle$ is again the marked state $|m\rangle$. Now, it can be proved, by closely following the arguments of Grover’s algorithm, that the following inequality holds (see Appendix):

$$\int_0^T g(s(t)) \geq \frac{k\sqrt{N}}{4},$$

(6)

(where $k$ is a constant of order unity). Note that previously it had been assumed that $g(s(t)) = 1 - s = 1 - f(s(t)) \leq 1 \ \forall s$, for which the above theorem implies that the running time $T \geq k\sqrt{N}/4$ which is at par with Grover’s algorithm. Result (6) can be thought of as AQC generalization of the lower bound on the number of steps of any search algorithm that has been proved previously.

The ‘gap’ between the ground state and first excited state of the Hamiltonian is given by

$$\Delta(s) = \sqrt{(f-g)^2 + \frac{4}{N}fg}$$

(7)

For the choice $g = 1 - f = 1 - s$, Min($\Delta$) = $\Delta(1/2)$ = $1/\sqrt{N}$, which is in conformity with the general idea that the running time and minimum gap are inverses of each other. However, (6) and (7) suggest that by suitably changing $g(s(t))$, so as to increase the gap considerably, the running time can
be significantly reduced. We make such a choice, which satisfies the boundary conditions (4):

\[ f = 1 - s + \sqrt{Ns}(1 - s), \quad g = s + \sqrt{Ns}(1 - s). \] (8)

It is easy to see that for the above choice, \( \text{Min}(\Delta) = \mathcal{O}(1) \) and the running time is

\[ T = \frac{1}{\epsilon} \left( 1 + \frac{\pi}{2} \right) \]

which is a constant. The original gap and the modified gap are plotted in figure 1 for \( N = 10,000 \).

4 Discussion

(1) Note that although the ground state energy rises to \( \mathcal{O}(\sqrt{N}) \) for intermediate times, no energy is actually being used in the process. Thus, energy need not be regarded as a resource in this context. Furthermore, by subtracting \( E_- \) from Hamiltonian (1) at all times, the resultant ground state energy can be made to vanish, without modifying the gap or the running time.

Error correcting processes would presumably entail heat losses, commensurate with the second law of thermodynamics.

(3) A larger gap in our case may signal greater fault-tolerance than previously considered Hamiltonians.

(4) Work is in progress towards similar considerations for structured data searches, to see whether similar speed-ups are possible.

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Appendix

To prove (6), first write (1) as

\[ H(s) = H_1(s) + H_2m(s) \]  

where

\[ H_1(s) = (f(s) + g(s)) - f(s)|\psi_0\rangle\langle\psi_0| \]  
\[ H_2m(s) = -g(s)|m\rangle\langle m| \]  

Consider two computers \(|\psi_m,t\rangle\) and \(|\psi_{m'},t\rangle\) respectively at any instant \(t\), evolving to states \(|m\rangle\) and \(|m'\rangle\). The Schrödinger equations are:

\[ i\frac{\partial}{\partial t}|\psi_{m,m',t}\rangle = (H_1 + H_{2m,m'})|\psi_{m,m',t}\rangle \]  

subject to the boundary conditions:

\[ |\psi_m,0\rangle = |\psi_{m'},0\rangle = |\psi_0\rangle \quad |\psi_m,T\rangle = |m\rangle, \quad |\psi_{m'},T\rangle = |m'\rangle \]  

From (12), it follows that:

\[ \frac{\partial}{\partial t} \sum_{m,m'} [1 - |\langle\psi_m,t|\psi_{m',t}\rangle|^2] \leq 4N^{3/2}g(s) \]  

Integrating (14) from \(t = 0\) to \(t = T\), and using the boundary conditions (13), we get:

\[ \sum_{m,m'} [1 - |\langle\psi_m,T|\psi_{m',T}\rangle|^2] \leq 4N^{3/2} \int_0^T g(s(t))dt \]  

Finally, using the fact:

\[ 1 - |\langle\psi_m,T|\psi_{m',T}\rangle|^2 \geq k \quad \forall m \neq m' \]  

which simply means that different computers evolve the same initial state to sufficiently different final states. This yields (6), for \(N \gg 1\).

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