Comprehensive survey on convex analysis in robust optimization

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Abstract. This paper presents a comprehensive survey on Convex Analysis (CA) in Robust Optimization (RO). Since RO is a class of continuous optimization problem which involved uncertain parameter thus convex analysis is needed to guarantee that the uncertain set is compact set, i.e., closed and bounded. Furthermore, in RO, to handle the uncertainty, the Robust Counterpart Methodology is employed to guarantee that the Robust Counterpart (RC) of the uncertain optimization problem will be ended up in one of the computationally tractable problems, whether it is a linear optimization, conic quadratic or semidefinite optimization problems. The importance of CA in RO lies in a fact that the robustness can be represented by a convex hull, i.e., the set which is the smallest convex set such that the uncertainty set is included in. This convex hull can replaces the uncertainty set. This can be done since determining the solution feasibility with respect to an uncertainty set is equivalent to taking the constraint left hand side supremum over the uncertainty set. If the uncertainty set can be replaced by a convex hull, this yields that the optimal solution remains the same. Thus, this paper discuss the recent researches, the most cited paper, the top journal and publishers also the bibliometric analysis on the topic of CA in RO.

1. Introduction

One area of optimization that is able to solve various problems related to optimization under uncertainty is called Robust Optimization (see Ben-Tal & Nemirovskii [1], Gorissen et al [2]). In Robust Optimization (RO), there is a suitable framework to formulate an Optimization problem model where the feasibility of optimal solution must be achieved for several parameter variations in an uncertainty set. The uncertainties in the Robust Counterpart model can produce viable solutions for all possibilities using box, ellipsoidal and polyhedral uncertainty conditions. The general formulation for Robust Optimization is given as follows

\[
\min_{x_0,x}\{x_0 : f_0(x, \zeta) \leq 0, f_i(x, \zeta) \leq 0, i = 1, ..., m \ \forall (\zeta \in \mathcal{U})\}
\]  

where \(x\) is a the decision vector, \(f_0\) is the objective function, \(f_i\) is constraint functions, \(\zeta\) stands for the uncertain data, and \(\mathcal{U}\) is the uncertain set. According to Ben-Tal & Nemirovskii in [1] the basic RO paradigm is based on the following assumptions. First, all decisions variable \(x \in \mathcal{R}^n\) represent here-and-now decisions means that the decisions are numeral value that is obtained by solving the problem. Second, when the data belongs to an uncertainty set \(\mathcal{U}\), thus the decision must be taken. Third, violations is not tolerated when the uncertainty set \(\mathcal{U}\) is
contained the data. This means that the constraints of the uncertain problem in question are hard. In addition to the basic assumptions, we can assume without loss of generality that there is a certainty in the objective function and constraint right-hand side, the set \( \mathcal{U} \) is compact and convex; and a constraint-wise term of the uncertainty is presented.

The main challenge in optimization with uncertainty data is determining how and when a problem. The indefinite optimization can be reformulated so that the Robust Counterpart (RC) can be obtained from the indeterminate problem. The optimization problem is computationally tractable and determines the RC estimate with problems that have been proven computationally tractable. Referring to Gabrel, Murat and Thiele in [3], on a convex set of parameter uncertainties, then the worst case of constraints is calculated. This is the focus of Robust Optimization on worst case optimization. Referring to Söüzüer, S. and Thiele, A.C., (2016) [4], the first step towards achieving the RO method was carried out by A.L. Soyster, follow by Mulvey, Vanderbei and Zenios in [5] also Ben-Tal and Nemirovskii in [1]. Robust optimization can be categorized into two, namely single stage and two stage models. In single stage RO, all decision variables with the "here and now" decision are considered to be resolved immediately. Meanwhile, in two-stage RO with the "wait and see" decision, the decision variables in the second stage are adjusted to the realization of parameter uncertainty. This two-stage RO approach was first introduced by Ben-Tal, Goryashko, Guslitzer and Nemirovski (2004) in [6], by considering two sets of variables, namely the first set must be determined before solving the uncertainty and the other can be calculated after the uncertainty is resolved. This multi stage Robust optimization is known as the Adjustable Robust Counterpart (ARC).

As discussed in [1], Convex Analysis (CA) plays important roles in the research of RO because the main aim of the research is to proof that the uncertain optimization problem that is indefinite problem can be replaced by a computionally tractable problem which belongs to a convex optimization problem. Convexity of the objective or constraint functions guarantees the achievement of global optimal solutions.

The problem of minimizing a linear objective function over the intersection of an affine set and a convex cone is called as Conic optimization (CO). The general form of a conic optimization problem is as follows:

$$\min_{x \in \mathbb{R}^n} \left\{ c^T x : Ax - b \in \mathcal{K} \right\}. \quad (2)$$

where \( c^T x \) is the objective function, with objective vector \( c \in \mathbb{R}^n \). An affine function from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) is represented as \( Ax - b \), a convex cone in \( \mathbb{R}^m \) is denoted by \( \mathcal{K} \) and the size of constraint matrix \( A \) is \( m \times n \).

CO problem is classified as an important optimization problem due to two facts: first a conic can models many nonlinear problems, and, secondly, interior-point methods (IPM), see [7], can be used to solved a wide class of conic optimization problems can be solved efficiently. An algorithm with polynomial complexity is provided by IPM, which is developed by Nesterov and Nemirovski in [8]. This means that a self-concordant barrier that is computationally tractable is belong to the cone \( \mathcal{K} \). When the cone \( \mathcal{K} \) is the nonnegative orthant of \( \mathbb{R}^m \), i.e., when \( \mathcal{K} = \mathbb{R}^m_+ \) then the occurrence of the easiest and most well known case is provided by this cone.

$$\min_{x \in \mathbb{R}^n} \left\{ c^T x : Ax - b \in \mathbb{R}^m_+ \right\}. \quad (3)$$

This is the well known Linear Optimization (LO) problem. This explain that one of the special case of CO is LO.

This paper presents a comprehensive survey on convex analysis (CA) in robust optimization (RO). A discussion on methodology of literature search is presented in Section 2. In Section 3, the bibliometric analysis on convex analysis in robust ptimizationare is presented. Section 4 stands for bibliographic mapping analysis and discuss the CA and RO topics. The conclusions of this literature review is presented in Section 5.
2. Method
Google Scholar (GS) databases is used for the literature search. The search is restricted to articles published from 2010-2020 for English written international peer-reviewed journals. Keyword convex analysis and RO is used in literature search and the results returned with 995 papers, for keyword convex analysis, it yields 830 papers. The keyword search was applied to title and abstract only.

This study uses Publish or Perish and VOSViewer software to obtain and analyze the data. Refers to Harzing in [9], Publish or Perish is a program that can be used to retrieve and analyze academic citation using a variety of data source, including Google Scholar and Microsoft Academic Search. Publish or Perish is used to obtain raw citations then analyze them. A file that is resulted by Publish or Perish is a bibliographic citation file saved in a format developed by Research Information Systems (RIS). It contains a series of lines delimited by two-character codes and a corresponding value. Information such as title, author, publication date, keywords, publisher, issue number, and start and end page is provided by a RIS file. Furthermore, VOSViewer is used to visualise and analyse trends in the form of a bibliometric map (see Van Eck & Waltman in [10]). All articles used in this study sourced from Google Scholar, because the data source can be obtained free of charge and has a maximum number of 1000 documents.

3. Bibliometric Analysis on Convex Analysis in Robust Optimization
3.1. Research Summary by Year, Journals and Citations
A summary of research into the importance of Convex Analysis in Robust Optimization is presented in this section. The articles number on the topic of Convex Analysis in Robust Optimization has been increasing, see Figure 1. As can be seen in Table 1, these articles are published in journals in the topics of optimization, mathematics analysis, mathematical programming, control optimal, operational research. The top publisher that publish paper on the topic of CA in RO also can be seen in Table 1.

The most cited paper is presented in Table 2. The first most cited is Vardakas et al with 632 citations, followed by Goh and Sim [11] with topic on distributionally robust optimization with 501 citations, and the third one is A Ben-tal et al [12] on Robust optimization problems affected by uncertain probabilities with total 432 citations.

3.2. Main Research Themes
In this section, topics within CA for RO is highlighted. Using VOSviewer software (Van Eck & Waltman, 2009 in [10]) a co-occurrence network of the terms is created by obtaining data.
Table 1. Top Journals and Top Publisher that publish CA in RO

| Journal Name                        | Number of Paper | Publisher            | Number of papers |
|-------------------------------------|-----------------|----------------------|------------------|
| arXiv preprint                      | 142             | Springer             | 206              |
| Mathematical Programming            | 33              | arxiv.org            | 130              |
| J. of Opt Theory and Applications   | 20              | ieeexplore.ieee.org  | 104              |
| EJOR                                | 19              | Elsevier             | 73               |
| Operations Research                 | 18              | pubsonline.informs.org | 40             |
| Optimization                        | 16              | Taylor & Francis     | 29               |
| SIAM Journal on Optimization        | 15              | SIAM                 | 27               |
| Annals of Operations Research       | 14              | books.google.com     | 20               |
| IEEE Trans. on Automatic Control    | 12              | researchgate.net     | 19               |
| J. of Global Optimization          | 12              | aimsciences.org      | 13               |
| J. of Industrial & Management Opt.  | 11              | optimization-online.org | 13          |
| Optimization Online                 | 11              | Willey Online Library | 7             |
| Operations Research Letters         | 10              | JSTOR                | 3                |
| Optimization Letters               | 10              | Hindawi              | 7                |
| Others                              | 652             | Others               | 300              |

Table 2. The most cited papers on Convex Analysis in Robust Optimization

| Authors                              | Topic                                           | Cites |
|--------------------------------------|-------------------------------------------------|-------|
| J.S Vardakas, N. Zorba 2014 [13]    | A survey on demand response in smart grids      | 634   |
| J. Goh, M. Sim 2010 [11]            | Distributionally robust optimization             | 501   |
| A. Ben-tal, D. Den Hertog 2013 [12] | Robust solutions affected by uncertain probabilities | 432   |

from the titles, abstract and keywords. If two terms both occur on the same line then the two terms are said to co-occur. VOSviewer thesaurus is used to group terms with similar meaning. In VOSviewer network, o-occurrences of those terms. is indicated by the distance between two terms. A number of clusters is identified based on this VosViewer network.

The co-occurrence network of the terms used in CA and RO literature in shown in Figure 2. The clusters is identified by different colours, each colour represents one cluster. The most occur terms among those 115 terms are robust optimization, convex analysis, optimization, application, approach, algorithm, robust optimization problem. The popular keywords in 8 clusters in presented in Table 3.

3.3. Most Productive Authors
In addition to mapping the field of study, by using VOSViewer we can also create co-authorship mapping from the bibliographic data obtained. This co-authorships is built based on bibliographic map in keyword co-occurence, citation, bibliographic coupling or co-citation map. The RIS (Research Information System) file with 995 papers from Publish or Perish searching results is used in this co-authorship analysis.

The analysis type is co-authorships with full counting method which consider the maximum number of authors per paper is 25 authors. The threshold for this analysis is the minimum
number of document of an author is 5. The results shows among the GS databases 995 papers, there are 418 authors. There is only 53 authors meet the threshold. The total strength of co-authorship links with other author is calculated, the greatest totally link strength is selected. In this case, authorship map can be seen in Figure 3. The largest set of connected authors consists of 17 authors is shown in Figure 4. The most linked productive authors can be seen in Table 4.
Table 4. Total Link Strength of Authorships

| Authors     | Documents | Link Strength |
|-------------|-----------|---------------|
| Jeyakumar, V| 28        | 33            |
| Goberna     | 20        | 32            |
| Li, G       | 19        | 22            |
| Lee, GM     | 18        | 16            |
| Thi, HL     | 16        | 16            |
| Lopez, MA   | 14        | 22            |
| Chuong, TD  | 12        | 9             |
| Hertog, d Den| 11     | 12            |
| Ackooij, TD | 11        | 9             |
| Dinh, N     | 10        | 17            |
| Lee, JH     | 10        | 11            |
| Kobis, E    | 10        | 7             |
| Ben-Tal, A  | 10        | 6             |
| Zhang, X    | 9         | 3             |
| Bertsimas   | 9         | 1             |
| Volle,M     | 7         | 14            |
| Co, XT      | 6         | 9             |

Figure 3. Network visualization of co-authorship

Figure 4. The largest connected co-authorship

4. Discussions
4.1. The recent results
Several recent papers on CA in RO is highlighted in this paper. A discussion on convex minimization on adjustable robust counterpart can be found in Selvi et al [14]. Burak Kocuk in [15] discuss the Kullback-Leibler divergence constrained distributionally robust optimization and its conic formulation also its applications. Li et al in [16] presents Robust Farkas-Minkowski constraint qualification for convex inequality System under data uncertainty. Robustness in nonsmooth nonconvex optimization problems is discussed by Mashkoorzadeh et al in [17]. Papers on literature review on CA in RO are discussed as follows. Rahimian and Mehrotra reviews the distributionally robust optimization in [18], Leyffer et al in [19] discussed a survey of nonlinear robust optimization, Yanikoglu, Gorissen and den Hertog in [20] discussed
a survey and tutorial on adjustable robust optimization. In Peykani et al [21], a discussion is performed for topic on data envelopment analysis and its review with robust optimization. In Rudnick-Cohen, Herrmann and Azarm [22] discuss the non-convex feasibility robust optimization via scenario generation and local refinement.

4.2. The state of research on Convex Analysis in Robust Optimization

Refers to Table 3 where the keyword clusters presented, it can be seen that application of CA in RO has a very important role. Nevertheless, the chance to do more research on the topics still widely open. The keyword can be chosen from one of the clusters presented in Table 3.

![Figure 5. Hibert Space bibliometric map.](image)

As future research agenda, another bibliometric map is generate with keyword "Convex Analysis" only. One of interesting results shows that researches on Hilbert Space for Robust Optimization is can be one option to do. See Figure 5. In this figure the keyword Hilbert Space is strongly connected to Convex Optimization, but there is no line connected to keyword Robust Optimization. The same result is obtained also for the keyword Functional Analysis, see Figure 6.

![Figure 6. Functional Analysis bibliometric map.](image)
5. Conclusion and Recommendations

A literature review of research in convex analysis (CA) for Robust Optimization (RO) is presented. Through this bibliometric analysis, it can be shown that there are still many studies on convex analysis and robust optimization that can be studied further. For a more in-depth and accurate research, use more specific keywords in searching the data or use other sources such as Scopus or Web of Science.

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