Leptoquark and vector-like quark extended model for simultaneous explanation of $W$ boson mass and muon $g-2$ anomalies

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Abstract: The CDF collaboration recently announced a new measurement result for the $W$ boson mass, and it is in tension with the standard model prediction. In this paper, we explain this anomaly in the vector-like quark (VLQ) $(X,T,B)_{LR}$ and leptoquark (LQ) $S_3$ extended model. In this model, both the VLQ and LQ have positive corrections to the $W$ boson mass. Moreover, it may be a solution to the $(g-2)_\mu$ anomaly because of the chiral enhancements from top, $T$, and $B$ quarks.

Keywords: $W$ mass, muon $g-2$, leptoquark, vector-like quark

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I. INTRODUCTION

The standard model (SM) has provided powerful descriptions of elementary particle physics and can explain most experiments with high precision [1]. However, it has been challenged by several experiments in recent years, for example, by $(g-2)_\mu$ [2, 3] and $B$ decay anomalies [4]. Recently, the CDF collaboration announced a measurement of $W$ boson mass [5], which was seven standard deviations heavier than the SM prediction. On the one hand, we are yet to scrutinize the theoretical and experimental uncertainties [6]. On the other hand, this may be a signature of new physics [7]. Currently, there are many studies dedicated to explaining this CDF anomaly [8–74].

To explain the $W$ mass anomaly, we can introduce new fermions and scalars. Vector-like quarks (VLQs) are well motivated in many new physics models, such as composite Higgs models [75, 76], little Higgs models [77, 78], grand unified theories [79], and extra dimension models [80]. From the viewpoint of model building, one attractive reason is that VLQs can avoid the problem of the quantum anomaly. In contrast, the leptoquarks (LQs) are well motivated in the grand unified theories [81–83]. LQs can be the solution to the $(g-2)_\mu$, $B$ physics, and other flavor anomalies [4, 84]. If we consider the VLQ and scalar LQ simultaneously, there can be two sources of $W$ mass corrections. Furthermore, it is also possible to explain $(g-2)_\mu$ at the same time. Here, we study the triplet LQ and triplet VLQ extended model.

In this paper, we first construct the model in Sec. II. In Sec. III, we calculate the new physics contributions to the $W$ boson mass and $(g-2)_\mu$. Then, we perform the numerical analysis in Sec. IV. Finally, we present our summary and conclusions in Sec. V.

II. MODEL SETUP

The SM gauge group is $SU_C(3)\otimes SU_L(2)\otimes U_Y(1)$. New particles can then carry different representations under this group. There are typically six types of scalar LQs [84, 85] and seven types of VLQs [86]. In our previous paper [87], we considered the $S_3$ LQ and $(X,T,B)_{LR}$ VLQ extended model to explain $(g-2)_\mu$ ahead of the CDF $W$ mass anomaly. Here, we investigate this model again, which is referred to as $S_3 + (X,T,B)_{LR}$ for convenience.

The representation of $S_3$ is $(3,3,1/3)$, which is $(3,3,2/3)$ for $(X,T,B)_{LR}$ [1]. Then, the relevant Lagrangian can be decomposed as $\mathcal{L}_H + \mathcal{L}_S + \mathcal{L}_{YT} + \mathcal{L}_{S_{gauge}} + \mathcal{L}_{scalar}$. Here, $\mathcal{L}_H$, $\mathcal{L}_S$, $\mathcal{L}_{YT}$, $\mathcal{L}_{S_{gauge}}$, and $\mathcal{L}_{scalar}$ mark the VLQ Yukawa interactions with Higgs, Yukawa VLQ gauge interactions with LQs, VLQ gauge interactions, LQ gauge interactions, and LQ scalar sector interactions, respectively. Below, we study these interactions.
carefully.

A. VLQ Yukawa interactions with Higgs

First, let us express the Yukawa interactions with Higgs.

\[ \mathcal{L}_{\alpha} = -MT(X,T,B)L \left( \begin{array}{c} X \\ T \\ B \end{array} \right) - \frac{y_{\alpha}^{T}}{M_{T}} \bar{Q}_{L} u_{R}^{\prime} \phi \]

\[ = -\frac{y_{ij}^{\alpha}}{M_{T}} \bar{Q}_{L} d_{R}^{\prime} \phi - \frac{y_{ij}^{\alpha}}{M_{T}} \bar{Q}_{L} \Psi_{R} \phi + \text{h.c.} \]  

(1)

Here, we define \( \phi \equiv i \sigma^{2} \phi \), and \( \alpha^{a}(a = 1, 2, 3) \) are the Pauli matrices. The SM Higgs doublet \( \phi \) is parameterized as \( \phi = [0, (v + h)/\sqrt{2}] \) in the unitary gauge. \( Q_{L}, u_{R}^{\prime}, d_{R} \) represent the SM quark fields, and the triplet \( (X,T,B)L;R \) can be parameterized in the following form:

\[ \Psi_{L,R} \equiv \left( \begin{array}{c} T_{L,R} \\ \sqrt{2}X_{L,R} \\ -T_{L,R} \end{array} \right). \]

(2)

For simplicity, we only consider the mixing between the third generation and VLQ [86, 91, 92]. After electroweak symmetry breaking (EWSB), we obtain the following mass terms:

\[ \mathcal{L}_{\text{mass}} = -\left[ \begin{array}{cc} i_{L} & \bar{T}_{L} \end{array} \right] \left( \begin{array}{cc} \frac{1}{\sqrt{2}} & \sqrt{2} \bar{Y}_{\gamma} V \\\n & M_{T} \end{array} \right) \left[ \begin{array}{c} i_{R} \\ T_{R} \end{array} \right] \]

Then, we can rotate the quark fields into mass eigenstates through the following transformations:

\[ \begin{array}{c} T_{L} \\ T_{R} \end{array} \rightarrow \left( \begin{array}{cc} \cos \theta_{L} & \sin \theta_{L} \\ -\sin \theta_{L} & \cos \theta_{L} \end{array} \right) \left[ \begin{array}{c} i_{L} \\ T_{L} \end{array} \right], \]

\[ \begin{array}{c} T_{R} \\ T_{R} \end{array} \rightarrow \left( \begin{array}{cc} \cos \theta_{R} & \sin \theta_{R} \\ -\sin \theta_{R} & \cos \theta_{R} \end{array} \right) \left[ \begin{array}{c} i_{R} \\ T_{R} \end{array} \right]. \]

(3)

and

\[ \begin{array}{c} b_{L} \\ B_{L} \end{array} \rightarrow \left( \begin{array}{cc} \cos \theta_{b} & \sin \theta_{b} \\ -\sin \theta_{b} & \cos \theta_{b} \end{array} \right) \left[ \begin{array}{c} b_{L} \\ B_{L} \end{array} \right], \]

\[ \begin{array}{c} B_{R} \\ B_{R} \end{array} \rightarrow \left( \begin{array}{cc} \cos \theta_{b} & \sin \theta_{b} \\ -\sin \theta_{b} & \cos \theta_{b} \end{array} \right) \left[ \begin{array}{c} b_{R} \\ B_{R} \end{array} \right]. \]

(4)

After the above quark transformations, we have the following mass eigenstate Yukawa interactions with Higgs:

\[ \mathcal{L}_{\text{Yukawa}} = -\frac{m_{\alpha}}{v} c_{\alpha}^{\prime} h \bar{u}_{T} - \frac{m_{\beta}}{v} c_{\beta}^{\prime} h \bar{t}_{T} T - \frac{m_{B}}{v} c_{B}^{\prime} h \bar{b}_{T} B - \frac{m_{\bar{B}}}{v} c_{\bar{B}}^{\prime} h \bar{\bar{B}} \bar{B} \]

\[ - \frac{m_{\alpha}}{v} s_{\alpha}^{\prime} c_{\beta}^{\prime} h \bar{t}_{T} L_{R} - \frac{m_{B}}{v} s_{B}^{\prime} c_{\bar{B}}^{\prime} h \bar{\bar{B}} L_{R} - \frac{m_{\bar{B}}}{v} s_{\bar{B}}^{\prime} c_{\bar{B}}^{\prime} h \bar{L}_{R} B_{R}. \]

(5)

In the above, the physical masses are labeled as \( m_{L,T,B}, s_{L}^{\prime}, c_{L}^{\prime}, s_{R}^{\prime}, c_{R}^{\prime} \) represent \( \sin \theta_{L}^{(b)}, \cos \theta_{L}^{(b)}, \sin \theta_{R}^{(b)}, \cos \theta_{R}^{(b)} \), respectively. Moreover, we have the following relations:

\[ \tan \theta_{R}^{c} = \frac{m_{\beta}}{m_{\bar{B}}} \tan \theta_{L}, \quad M_{T}^{2} = m_{b}^{2}(c_{L}^{\prime})^{2} + m_{b}^{2}(s_{L}^{\prime})^{2}, \]

\[ \tan \theta_{R}^{b} = \frac{m_{b}}{m_{\bar{B}}} \tan \theta_{L}, \quad M_{T}^{2} = m_{b}^{2}(c_{\bar{B}})^{2} + m_{b}^{2}(s_{\bar{B}})^{2}, \]

\[ \sin 2\theta_{L}^{c} = \frac{\sqrt{2}(m_{\beta}^{2} - m_{\bar{B}}^{2})}{m_{b}^{2} - m_{\bar{B}}^{2}} \sin 2\theta_{L}^{b}. \]

(6)

Hence, there are two new independent input parameters \( m_{\beta} \) and \( \theta_{L}^{c} \) (also denoted as \( \theta_{L} \) in the following), and the parameters \( M_{T}, m_{b}, \theta_{R}^{c}, \theta_{L}^{c}, \theta_{L}^{b} \) can be determined from the above equations (see App. A). One interesting point is that the mass of the \( X \) quark is \( M_{T} \), which is less than \( m_{T} \) and \( m_{B} \).

B. VLQ Yukawa interactions with LQ

Now, let us consider the Yukawa interactions with LQs. In this \( S_{3} + (X,T,B)L;R \) model, gauge eigenstate interactions can be written as

\[ \mathcal{L} = \sum_{ij} (Q_{L})^{ij} (i \sigma^{2})^{ab}(S_{3})^{bc} L_{L}^{i} + x_{T} \text{Tr}[(\Psi_{R})^{2} S_{3}] e_{R}^{i} + \text{h.c.}. \]

(7)

Here, \( L_{L}^{i}, e_{R}^{i} \) denote the SM lepton fields. Similarly, we also parameterize the \( S_{3} \) triplet in the following form:
Here, $\omega_\pm$ are the chirality operators $(1 \pm \gamma^5)/2$. The new parameters $S_3^{\mu T}$ and $S_R^{\mu T}$ can be determined from the original $x_{ij}$ and $x_{Tj}$. We adopt the new parameters for convenience. When performing the transformations in Eqs. (4) and (5), we have the following mass eigenstate Yukawa interactions with LQs:

$$L_{\text{Yukawa}} \supset \mu (y_L^{S,\mu T} s_{3L} \omega_+ + y_R^{S,\mu T} c_{3L} \omega_+ + y_R^{3,\mu T} s_{3L} \omega_- + y_R^{3,\mu T} c_{3L} \omega_-) (S_3^{1/3})^* + \mu' (y_L^{S,\mu T} s_{3L} \omega_+ + y_R^{S,\mu T} c_{3L} \omega_+ + y_R^{3,\mu T} s_{3L} \omega_- + y_R^{3,\mu T} c_{3L} \omega_-) (S_3^{1/3})^*$$

$$+ \mu (y_L^{S,\mu T} s_{3L} \omega_+ + y_R^{S,\mu T} c_{3L} \omega_+ + y_R^{3,\mu T} s_{3L} \omega_- + y_R^{3,\mu T} c_{3L} \omega_-) (S_3^{1/3})^*$$

$$+ y_s^{3,\mu T} \mu \omega_- X (S_3^{2/3})^* + \text{h.c..}$$

(11)

C. VLQ gauge interactions

For the triplet VLQ, the covariant derivative of the electroweak part is defined as $D_\mu \Psi = \partial_\mu \Psi - ig[W_\mu^{a T}, \Psi] - ig^\prime Y B_\mu \Psi$, with $Y$ as the $U_Y(1)$ charge. $W_\mu^a$ and $B_\mu$ label the $SU(2)_L$ and $U_Y(1)$ gauge fields, respectively. The gauge interactions are written as Tr$(\Psi \not D \Psi)/2$, in which the factor 1/2 is to normalize the kinetic terms. After EWSB, the charged current interactions can be written as

$$L \supset \frac{g}{c_W} (\frac{2}{3} s_W^2) Z_\mu (\overline{T_L} Y^\mu T_L + \overline{T_R} Y^\mu T_R) + \frac{g}{c_W} (\frac{1}{3} + s_W^2) Z_\mu (\overline{B_L} Y^\mu B_L + \overline{B_R} Y^\mu B_R)$$

$$+ \frac{g}{c_W} (1 - s_W^2)(\overline{X_L} Y^\mu X_L + \overline{X_R} Y^\mu X_R).$$

(13)

As we know, third generation quarks interact with $W$ and $Z$ bosons in the following form:

$$L \supset \frac{g}{\sqrt{2}} W_\mu^a (\overline{T_L} Y^\mu T_L + \overline{T_R} Y^\mu T_R) + \frac{g}{c_W} Z_\mu [\frac{1}{2} - \frac{2}{3} s_W^2] \overline{B_L} Y^\mu B_L + \frac{1}{3} s_W^2 \overline{B_R} Y^\mu B_R.$$ 

(14)

After rotating the quark fields with Eqs. (4) and (5), we have following mass eigenstate charged current interactions $^1$:

$$L_{\text{gauge}} \supset \frac{g}{\sqrt{2}} W_\mu^a [\overline{\psi} Y^\mu (s_L^T c_R^T s_R^T c_L^T + \sqrt{2} s_L^T s_R^T c_L^T s_R^T + \sqrt{2} s_L^T s_R^T c_L^T s_R^T + \sqrt{2} c_L^T c_R^T c_L^T c_R^T - \sqrt{2} c_L^T c_R^T c_L^T c_R^T) + \text{h.c..}$$

$$+ \overline{T_Y} Y^\mu (s_L^T c_R^T s_R^T c_L^T + \sqrt{2} s_L^T s_R^T c_L^T s_R^T + \sqrt{2} s_L^T s_R^T c_L^T s_R^T + \sqrt{2} c_L^T c_R^T c_L^T c_R^T - \sqrt{2} c_L^T c_R^T c_L^T c_R^T) + \text{h.c..}$$

$$+ \frac{g}{c_W} [\overline{X_L} Y^\mu (s_L^T c_R^T c_L^T - \frac{1}{3} s_W^2) + \overline{X_R} Y^\mu (s_L^T c_R^T c_L^T - \frac{1}{3} s_W^2)] + \text{h.c..}$$

(15)

$^1$ The $WXt$ and $WXT$ interactions show sign difference from those in Ref. [86], while it can be absorbed through the redefinition of $X$ field and has no physical effects.
Similarly, we have the following mass eigenstate neutral current interactions:

\[
\mathcal{L}_{\text{gauge}}^{\text{EN}} \supset \frac{g}{2c_W} Z_\mu \left[ \bar{f}^\mu \{(c_L^T)^2 - \frac{4}{3} s_W^2\} \omega - \frac{4}{3} \bar{s}^2 W \omega + \bar{T} \gamma^\mu \{(s_L^T)^2 - \frac{4}{3} s_W^2\} \omega - \frac{4}{3} \bar{s}^2 W \omega \right] T \\
+ s_T \left[ (c_L^T) \left[ \bar{T} \gamma^\mu T_L + \bar{T} L \gamma^\mu L_L \right] + \bar{f}^\mu \left[ - (c_L^T)^2 + \frac{2}{3} s_W^2 \right] \omega \right] + \bar{l} \left[ (c_L^T) \right] \right] T \\
+ \tilde{b} y^\mu \left[ (s_L^T)^2 + \frac{2}{3} s_W^2 \right] \omega + \bar{s} \left[ (s_L^T)^2 + \frac{2}{3} s_W^2 \right] \omega \right] B + s_T \left[ (c_L^T) \left[ \bar{b} \gamma^\mu b_L + \bar{b} \gamma^\mu b_L \right] \\
+ b_T \left[ s_T \left[ \bar{b} \gamma^\mu b_L + \bar{b} \gamma^\mu b_L \right] + 2 \left[ 1 - \frac{5}{3} \bar{s} W \right] (\bar{X} \gamma^\mu X + \bar{X} \gamma^\mu X) \right]. 
\]

(16)

\[ D. \quad \text{LQ gauge interactions} \]

For the LQ $S_3$, the covariant derivative of the electroweak part is defined as $D_\mu S_3 = \partial_\mu S_3 - ig[W^\mu_\nu \tau^\nu, S_3] - ig' Y B S_3$. Then, the gauge interactions $\mathcal{L}_{S_3}^{\text{gauge}} \supset \frac{1}{2} \text{Tr}(D_\mu S_3) (D^\nu S_3)$ can be expanded as shown below.

- $S_3 S_3 W$ interaction:

\[
ig W^\mu_\nu \left[ (\partial^\nu S_3^{1/3}) S_3^{1/3} - (\partial^\nu S_3^{1/3})(S_3^{1/3})^* + (\partial^\nu S_3^{-2/3})(S_3^{1/3})^* - (\partial^\nu S_3^{1/3})^* S_3^{-2/3} \right] + \text{h.c.} 
\]

(17)

- $S_3 S_3 Z$ interaction:

\[
\frac{igZ_\mu}{c_W} \left[ 1 - \frac{2}{3} s_W^2 \left[ (\partial^\nu S_3^{1/3})(S_3^{1/3}) - (\partial^\nu S_3^{1/3})(S_3^{1/3})^* \right] + \left( 2 \frac{2}{3} s_W^2 - 1 \right) \left[ (\partial^\nu S_3^{-2/3})(S_3^{-2/3})^* - (\partial^\nu S_3^{-2/3}) S_3^{-2/3} \right] \\
+ \left( 1 - \frac{4}{3} s_W^2 \right) \left[ (\partial^\nu S_3^{1/3})(S_3^{1/3})^* - (\partial^\nu S_3^{1/3}) S_3^{1/3} \right] \right]. 
\]

(18)

- $S_3 S_3 Y$ interaction:

\[
ie A_\mu \left[ \frac{1}{3} \left[ (\partial^\nu S_3^{1/3})(S_3^{1/3}) - (\partial^\nu S_3^{1/3})(S_3^{1/3})^* \right] + \frac{2}{3} \left[ (\partial^\nu S_3^{-2/3})(S_3^{-2/3})^* - (\partial^\nu S_3^{-2/3}) S_3^{-2/3} \right] \\
+ \frac{4}{3} \left[ (\partial^\nu S_3^{1/3})(S_3^{1/3})^* - (\partial^\nu S_3^{1/3}) S_3^{1/3} \right] \right]. 
\]

(19)

- $S_3 S_3 WW$ interaction:

\[
g^2 \left[ W^\mu_\nu W^-^\mu \left[ 2 S_3^{1/3}(S_3^{1/3})^* + S_3^{-2/3}(S_3^{-2/3})^* + S_3^{4/3}(S_3^{4/3})^* \right] - S_3^{4/3}(S_3^{-2/3})^* W^-^\mu - S_3^{-2/3}(S_3^{4/3})^* W^\mu_\nu W^+^\mu \right] \right]. 
\]

(20)

- $S_3 S_3 WZ$ interaction:

\[
\frac{g^2}{c_W} W^\mu_\nu Z^\mu \left[ -1 + \frac{1}{3} s_W \right] S_3^{-2/3}(S_3^{1/3})^* + \left( -1 + \frac{5}{3} s_W \right) S_3^{1/3}(S_3^{4/3})^* \right] + \text{h.c.} 
\]

(21)

- $S_3 S_3 WY$ interaction:

\[
eg W^\mu_\nu A^\mu \left[ - \frac{1}{3} S_3^{-2/3}(S_3^{1/3})^* - \frac{5}{3} S_3^{1/3}(S_3^{4/3})^* \right] + \text{h.c.} 
\]

(22)
\[ S_3S_3ZZ \text{ interaction:} \]
\[
\frac{g^2}{c_w^2} Z \mu Z' \left[ \frac{1}{9} g^4 m_3^2 \left( S_3^{1/3} \right)^* + \left( 1 - \frac{4}{3} s_w^2 \right) S_3^{4/3} \left( S_3^{4/3} \right)^* + \left( \frac{2}{3} s_w^2 \right) S_3^{-2/3} \left( S_3^{-2/3} \right)^* \right].
\]

\[ S_3S_3Z\gamma \text{ interaction:} \]
\[
\frac{2e g}{c_w} Z \mu \phi \left[ -\frac{1}{9} g^2 m_3^2 S_3^{1/3} \left( S_3^{1/3} \right)^* + \frac{4}{3} \left( 1 - \frac{4}{3} s_w^2 \right) S_3^{4/3} \left( S_3^{4/3} \right)^* - \frac{2}{3} \left( 1 + \frac{2}{3} s_w^2 \right) S_3^{-2/3} \left( S_3^{-2/3} \right)^* \right].
\]

\[ S_3S_3\gamma\gamma \text{ interaction:} \]
\[
e^2 A_\mu A^\mu \left[ \frac{1}{9} g^2 S_3^{1/3} \left( S_3^{1/3} \right)^* + \frac{16}{9} g^2 S_3^{4/3} \left( S_3^{4/3} \right)^* + \frac{4}{9} g^2 S_3^{-2/3} \left( S_3^{-2/3} \right)^* \right].
\]

E. LQ scalar sector interactions

Here, we consider the scalar sector interactions. The mass related terms can be written as
\[
\mathcal{L}_{\text{scalar}}^{\text{mass}} = -\frac{1}{2} m_{S_3^{\psi}}^2 \text{Tr}(S_3^{\psi} S_3) - \lambda_{\phi S_3} \phi \text{Tr}(S_3^{\psi} S_3) - \lambda_{S_3} (\phi^* \phi) \text{Tr}(S_3^{\psi} S_3). \tag{26}
\]

After EWSB, we have the following mass equations \[93\]:
\[
m_{S_3^{\psi \psi}}^2 = m_{S_3^2}^2 + \lambda_{\phi S_3^2} v^2 + \lambda_{S_3^2} v^2,
\]
\[
m_{S_3^{\psi \psi}}^2 = m_{S_3^2}^2 + \lambda_{\phi S_3^2} v^2 + \frac{1}{2} \lambda_{S_3^2} v^2, \quad m_{S_3^{\psi \psi}}^2 = m_{S_3^2}^2 + \lambda_{S_3^2} v^2. \tag{27}
\]

Obviously, the \( \phi^* \phi \text{Tr}(S_3^{\psi} S_3) \) term does not contribute to the mass splittings. Meanwhile, there are tree level generated mass splittings, which are controlled by the coupling \( \lambda_{S_3^2} \). In fact, this is similar to the traditional colorless electroweak triplet. The difference is that the mass splittings for the traditional electroweak triplet can also be caused by the non-zero triplet vacuum expectation value \[94, 95\]. In Ref. \[96\], the authors studied the mass splittings originating from the \( S_1 \) and \( S_3 \) LQ mixing.

Although there are three mass parameters, \( m_{S_3^{\psi \psi}}^2 \), \( m_{S_3^{\psi \psi}}^2 \), and \( m_{S_3^{\psi \psi}}^2 \), only two of them are relevant because we can redefine the mass. For convenience, let us define the following mass splitting quantity:

\[ \Delta m^2(\approx 2m_{S_3^{\psi \psi}} \cdot \Delta m) \equiv m_{S_3^{\psi \psi}}^2 - m_{S_3^{\psi \psi}}^2 = \frac{1}{2} \lambda_{AS_3} v^2. \tag{28} \]

Thus, we can choose the input parameters to be \( (m_{S_3^{\psi \psi}}, \Delta m) \) or \( (m_{S_3^{\psi \psi}}, \Delta m) \).

III. EXPLANATION OF THE W BOSON MASS
AND \( (g - 2)_\mu \) ANOMALIES

A. Contributions to the W boson mass

If we choose the \( (\alpha, G_F, m_Z) \) scheme, the \( W \) boson mass can be determined from the formula \[97, 98\]:
\[
m_W^2 = \frac{m_Z^2}{2} \left( 1 + \sqrt{1 - \frac{\sqrt{8 \pi \alpha (1 + \Delta r)}}{G_F m_Z^2}} \right), \tag{29}
\]
and then we have the following approximation:
\[
\frac{\Delta m_W^2}{m_W^2} = -\frac{s_w^2 - c_w^2}{s_w^2 - c_w^2} \Delta r. \tag{30}
\]

In the above, we define the quantities \( \Delta m_W^2 \equiv (m_W^{\text{NP}})^2 - (m_W^{\text{SM}})^2 \) and \( \Delta r \equiv \Delta r_{\text{NP}} - \Delta r_{\text{SM}} \) to isolate the new physics contributions. If we neglect the new physics contributions from the wave function renormalization constants, vertex, and box diagrams in \( \mu \) decay\[9\], the \( W \) mass correction can be correlated with the \( S, T, U \) oblique parameters as \[99–104\].

1) There is another interaction term \( \phi^* \text{Tr}(S_3(S_3^3)^*) \phi \), in which the \( \text{Tr}(S_3) \) only acts on the \( SU(3) \) color space. However, it can be removed since we have the relation \( \phi^* \phi \text{Tr}(S_3(S_3^3)^*) = \phi^* (S_3^3)^* S_3 \phi + \phi^* \text{Tr}(S_3(S_3^3)^*) \phi \).

2) Strictly speaking, we need to analyse the complete new physics corrections to the \( \mu \) decay. Here, we will not study that hard work.
where we define the deviations $\Delta S \equiv S^{NP} - S^{SM}, \Delta T \equiv T^{NP} - T^{SM},$ and $\Delta U \equiv U^{NP} - U^{SM}.$ In most cases, the $T$ parameter dominates. There are mainly two contributions to the oblique parameters in the $S_3 + (X, T, B)_{L,R}$ model. One is from the $LQ$ contributions, and the other is from the VLQs.

Now, let us turn to the oblique contributions from the $LQ$ loops, which are denoted as $\Delta S^{S_1}, \Delta T^{S_1},$ and $\Delta U^{S_1}.$ Unlike the traditional electroweak Higgs triplet model, $S_3$ does not modify the $T$ parameter at tree level because of the exact color symmetry. The complete one-loop results can be calculated through the interactions given in Sec. II.D, and the details are given in App. B. The $U$ parameter formula is lengthy, and the $S$ and $T$ parameters have the following compact expressions:

$$\Delta S^{S_1} = \frac{\mathcal{N}_C}{9\pi} \log \frac{m_{S_{1,0}}^2}{m_{S_{2,0}}^2},$$

$$\Delta T^{S_1} = \frac{\mathcal{N}_C}{8\pi m_{W}^2 m_{W}^2} \left[ \theta_0 (m_{S_{1,0}}^2, m_{S_{2,0}}^2) + \theta_0 (m_{S_{2,0}}^2, m_{S_{1,0}}^2) \right].$$

(32)

Here, $\mathcal{N}_C = 3$ is a color factor, and the function $\theta_0$ is defined as

$$\theta_0 (y_1, y_2) \equiv y_1 + y_2 - \frac{2y_1 y_2}{y_1 - y_2} \log \frac{y_1}{y_2}.$$  

(33)

Obviously, we have $\Delta T^{S_1} \geq 0$ because of the inequality $\theta_0 (x, y) \geq 0,$ in which the equality applies if and only if $x = y.$ For $\Delta S^{S_1},$ it is negative if $m_{S_{1,0}} > m_{S_{2,0}}$ (i.e., $\Delta S^{S_1} > 0$) and positive if $m_{S_{1,0}} < m_{S_{2,0}}$ (i.e., $\Delta S^{S_1} < 0$). The mass expressions are shown in Eqs. (27) and (28). In the approximation of $\Delta \lambda_{S_1} v^2 \ll m_{S_{1,0}}^2$ (or $\Delta m \ll m_{S_{1,0}}$), the $S, T,$ and $U$ parameters can be expanded as

$$\Delta S^{S_1} \approx -\frac{\Delta \lambda_{S_1} v^2}{3 m_{S_{1,0}}^2} \approx \frac{4\Delta m}{3m_{S_{1,0}}^2},$$

$$\Delta T^{S_1} \approx \frac{(\Delta \lambda_{S_1})^2 v^4}{16\pi^2 m_{W}^2 m_{W}^2 m_{S_{1,0}}^2} \approx \frac{(\Delta m)^2}{\pi^2 m_{W}^2 m_{W}^2},$$

$$\Delta U^{S_1} \approx \frac{7(\Delta \lambda_{S_1})^2 v^4}{40\pi m_{W}^2 m_{W}^2 m_{S_{1,0}}^2} \approx \frac{14(\Delta m)^2}{5m_{W}^2 m_{W}^2}.$$  

(34)

In the limit $\Delta \lambda_{S_1} \to 0,$ they will vanish. These results completely agree with those in Refs. [105, 106].

For the VLQ part, the oblique corrections are caused by both the modification of SM quark gauge couplings and new VLQ loops, which are denoted as $\Delta S^{XTB}, \Delta T^{XTB},$ and $\Delta U^{XTB}.$ Their analytic expressions are lengthy; hence, the details are given in App. C. Considering $m_b \ll m_t \ll m_T$ and $s_{L} \ll 1,$ the formulae can be approximated as

$$\Delta S^{XTB} \approx \frac{N_C (s_L^2)^2}{18\pi} (-12\log \frac{m_T}{m_t} - 16\log \frac{m_T}{m_b} + 29),$$

$$\Delta T^{XTB} \approx \frac{N_C m_t^2 (s_L^2)^2}{8\pi m_{W}^2 m_{W}^2} (6\log \frac{m_T}{m_t} - 5),$$

$$\Delta U^{XTB} \approx \frac{N_C (s_L^2)^2}{18\pi} (24\log \frac{m_T}{m_b} - 5).$$  

(35)

If $s_L^2$ goes to zero, $\Delta S^{XTB}, \Delta T^{XTB},$ and $\Delta U^{XTB}$ will vanish. Our expansion results agree with those in Cao’s paper [90], whereas the expansion of the $S$ parameter differs from the result in Ref. [107]. According to Eq. (35), $\Delta T^{XTB}$ and $\Delta U^{XTB}$ are always positive, whereas $\Delta S^{XTB}$ is always negative. Typically, $\Delta T^{XTB}$ is several times larger than $\Delta S^{XTB}$ and $\Delta U^{XTB}$ because of the $m_T^2/(m_{W}^2 m_{W}^2)$ factor.

After summing the fermion and boson contributions, we can obtain the total oblique parameter deviations as

$$\Delta S \equiv \Delta S^{XTB} + \Delta S^{S_1}, \quad \Delta T \equiv \Delta T^{XTB} + \Delta T^{S_1},$$

and $\Delta U \equiv \Delta U^{XTB} + \Delta U^{S_1}.$ To explain the $W$ boson mass anomaly, a positive $\Delta T$ is required, which is satisfied for both the VLQ and $LQ$ contributions. There are four independent parameters involved in the oblique corrections: $m_T$ and $s_L$ for the VLQ, and $m_{S_{1,0}}$ and $\Delta \lambda_{S_1}$ for the $LQ$.

B. Contributions to $(g - 2)_\mu$

According to the BNL and FNAL experiments [2, 3], the most recent muon anomalous magnetic dipole moment was measured as $a_{\mu}^{\text{exp}} = 116592061(41) \times 10^{-11}.$ In the SM, it is predicted as $a_{\mu}^{\text{SM}} = 116591810(43) \times 10^{-11}$ [109]. Thus, the deviation is $\Delta a_{\mu} = a_{\mu}^{\text{exp}} - a_{\mu}^{\text{SM}} = (251 \pm 59) \times 10^{-11},$ which corresponds to a $4.2\sigma$ discrepancy. LQ models can be the solution to this $(g - 2)_\mu$ anomaly [84, 105, 110–115]. In our previous paper [87], we studied $(g - 2)_\mu$ in the $S_3 + (X, T, B)_{L,R}$ model, in which the contributions are mainly from $T$ and $B$ quarks. Considering all contributions from $\mu, T, b, B, X$ quarks, the complete expression is calculated as

1) In Ref. [107], the authors adopt the $S, T,$ and $U$ parameter formulae given in Ref. [108] directly. For the $(X, T, B)_{L,R}$ triplet, the $T$ parameter formula still holds, while the $S$ and $U$ parameter formulae are no longer applicable. The detailed clarification is given in App. C.

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\[ \Delta a_\mu = \frac{m^2_{\mu \tau}}{8\pi^2} \left( \left| \frac{c^S_\mu}{m^2_{S^{(1)}}} \right|^2 \frac{2m_{\mu}^2 c^L_\mu c^L_\tau}{m^2_{S^{(1)}} m^2_{S^{(2)}}} \mathcal{R}(y_{\mu S^{(1)}} y_{\mu S^{(2)}}) f_{L L}(m^2_{S^{(5)}}) \right) + \frac{m^2_{\mu \tau}}{8\pi^2} \left( \left| \frac{c^S_\tau}{m^2_{S^{(3)}}} \right|^2 \frac{2m_{\mu}^2 c^L_\mu c^L_\tau}{m^2_{S^{(1)}} m^2_{S^{(2)}}} \mathcal{R}(y_{\tau S^{(1)}} y_{\tau S^{(2)}}) f_{L L}(m^2_{S^{(5)}}) \right) \]

In the above, the subscripts "LL" and "LR" denote the contributions with and without chiral enhancements. The related functions are defined as

\[ f_{L L}(x) = \frac{1 + 4x - 5x^2 + 2x(2 + x)\log x}{4(1 - x)^2}, \]
\[ f_{L R}(x) = \frac{-7 - 8x + x^2 + (4 + 2x)\log x}{4(1 - x)^3}, \]
\[ \tilde{f}_{L L}(x) = \frac{-2 - 7x + 2x^2 + 3x^3 + 2x(1 - 4x)\log x}{4(1 - x)^2}, \]
\[ \tilde{f}_{L R}(x) = \frac{1 + 4x - 5x^2 - 2(8 - 8x)\log x}{4(1 - x)^3}, \]
\[ \tilde{f}_{L L}(x) = \frac{4 + x - 8x^2 + 3x^3 + 2x(5 - 2x)\log x}{4(1 - x)^4}. \] (37)

Considering \( m_\mu \ll m_\tau \approx m_B \), \( \Theta \ll 1 \), and \( m_{S^{(i)}} \approx m_{S^{(i)}} \approx m_{S^{(i)}}, \) it can be approximated as

\[ \Delta a_\mu = \frac{m_{\mu} m_{\mu\tau}}{4\pi^2 m^2_{S^{(5)}}} \left[ f_{L L}(m^2_{S^{(5)}}) m^2_{S^{(5)}} \right] + \frac{m^2_{\mu \tau}}{8\pi^2} \left( \left| \frac{c^S_\mu}{m^2_{S^{(1)}}} \right|^2 \frac{2m_{\mu}^2 c^L_\mu c^L_\tau}{m^2_{S^{(1)}} m^2_{S^{(2)}}} \mathcal{R}(y_{\mu S^{(1)}} y_{\mu S^{(2)}}) \right) \tilde{f}_{L L}(m^2_{S^{(5)}}). \] (38)

As we can see, the contributions to \((g-2)_\mu\) are mainly determined by the parameters \( m_\tau, S_{L}, \) and \( m_{S^{(i)}} \). Although the parameter \( \Delta a_{S_{L}} \) can also alter the correction, it is subdominated through the LQ mass differences.

**IV. NUMERICAL ANALYSIS**

In this paper, we choose the SM input parameters to be \( m_Z = 91.1876 \text{ GeV}, m_{W} = 80.379 \text{ GeV}, m_{\mu} = 105.66 \text{ MeV}, m_{t} = 172.5 \text{ GeV}, m_{b} = 4.2 \text{ GeV}, \alpha = 1/128, \) and \( c_{W} = m_{W}/m_{Z} \) [1].

We define the \( W \) mass deviation quantity \( \Delta m^\text{exp}_{W} = m^\text{exp}_{W} - m^\text{SM}_{W} \). Here, \( m^\text{exp}_{W} \) is the CDF result 80.4335 ± 9.4 MeV [5], and \( m^\text{SM}_{W} \) is the SM prediction 80.357 ± 6 MeV [1]. Then, \( \Delta m^\text{exp}_{W} \) is calculated to be 76.5 ± 11.2 MeV. For the VLF mass, the direct search requires it to be above 1.4 TeV [116, 117]. For the LQ mass, the direct search also requires it to be above 1.5 TeV [118, 119]. We also consider the constraints from electroweak precision observables. Because it is small for the correlation between the oblique corrections and Zbb couplings [120, 121], we can treat them separately for simplicity. There is a \(
\begin{align*}
\Delta S_{\text{fit}} &= 0.005, \quad \sigma_{S} = 0.096, \\
\Delta T_{\text{fit}} &= 0.040, \quad \sigma_{T} = 0.120, \\
\Delta U_{\text{fit}} &= 0.134, \quad \sigma_{U} = 0.087,
\end{align*}
\] (39)

with the correlation matrix

\[ \rho = \begin{pmatrix}
0.91 & 1.00 & -0.65 \\
-0.65 & -0.88 & 1.00
\end{pmatrix}. \] (40)

Then, we can define the \( \chi^2 \) quantity as

\[ \chi^2 = \sum_{i,j=1,2,3} \frac{O_{ij} - O_{ij}^{\text{fit}}}{\sigma_i \sigma_j} \] (41)

where the indices 1, 2, 3 label the \( S, T, U \) parameters. Next, we perform the \( \chi^2 \) fit of the oblique parameters.

First, let us roughly compare the contributions from \((X,T,B)_{LR}\) and \( S_{3} \). In Fig. 1, we show their individual contributions to the \( W \) boson mass. For the pure \((X,T,B)_{LR}\) case, the behavior is as expected because we have the approximation \( \Delta m_{W} \propto (s_{f}^{i})^{2} \log(m_{f}/m_{t}) \). Thus, we need larger \( s_{f}^{i} \) and \( m_{f} \) to produce a sizable \( W \) mass correction. For the pure \( S_{j} \) case, the behaviour is also as expected because we have the approximation...
$\Delta m_W \propto (\tilde{\lambda}_{\Delta S})^2/m_{S_1}^2$, if $\tilde{\lambda}_{\Delta S} \sim O(1)$. Thus, we need larger $\tilde{\lambda}_{\Delta S}$, and small $m_{S_1}$ to produce a sizable $W$ mass correction. In Fig. 2, we show the parameter space allowed by the CDF $W$ mass measurement and the oblique parameters. For the pure $(X,T,B)_{LR}$ case, we find that $m_T$ should be at least 3.8 TeV when $s_L = 0.05$. For the pure $S_3$ case, we find that $|\tilde{\lambda}_{\Delta S}|$ should be at least 2.3 when $m_{S_1} = 1.5$ TeV. $|\Delta m|$ lies at approximately 25 GeV, which is almost independent of $m_{S_1}$.

In Fig. 3, we consider the contributions from $(X,T,B)_{LR}$ and $S_3$ at the same time. In the two plots above, we show the $W$ mass allowed regions in the plane of $m_T - s_L$ with fixed $m_{S_1}$ and $\tilde{\lambda}_{\Delta S}$. For the scenarios $m_{S_1} = 1.5$ TeV and $\tilde{\lambda}_{\Delta S} = 0.8, 1$, we find that the lower limit of $m_T$ can be decreased to 3.1 TeV and 2.7 TeV when $s_L = 0.05$. In the two plots below, we show the $W$ mass allowed parameter space is shown at the $1\sigma$ (green) and $2\sigma$ (yellow) confidence levels (CLs), respectively. The blue line enclosed area is bounded by the $S,T,U$ parameters at the $2\sigma$ CL.

Fig. 1. (color online) Pure $(X,T,B)_{LR}$ contributions to $\Delta m_W$ as a function of $m_T$ for different $s_L$ (left). The pure $S_3$ contributions to $\Delta m_W$ as a function of $m_{S_1}$ for different $\tilde{\lambda}_{\Delta S}$ (right).

Fig. 2. (color online) Pure $(X,T,B)_{LR}$ case in the plane of $m_T - s_L$ (upper), the pure $S_3$ case in the plane of $m_{S_1} - |\Delta m|$ (lower left), and the pure $S_1$ case in the plane of $m_{S_1} - |\Delta m|$ (lower right). The CDF $W$ mass allowed parameter space is shown at the $1\sigma$ (green) and $2\sigma$ (yellow) confidence levels (CLs), respectively. The blue line enclosed area is bounded by the $S,T,U$ parameters at the $2\sigma$ CL.

1) This is similar to the estimation in Ref. [96], in which the authors studied the singlet LQ $S_1$ and triplet LQ $S_3$ extended model. The $\Delta m$ in their work denotes the mass difference between $S_1$ and $S_1^{1/3}$. 
mass allowed regions in the plane of $m_{S_{1/3}} - \Delta_{\delta S_3}$, with fixed $m_T$ and $s_L$. For the scenario $s_L = 0.05$ and $m_T = 3$ TeV, we find that the lower limit of $|\Delta_{\delta S_3}|$ can be decreased to 0.9 when $m_{S_{1/3}} = 1.5$ TeV. For the scenario $s_L = 0.05$ and $m_T = 5$ TeV, $\Delta_{\delta S_3}$ can be zero because $(X, T, B)_{L,R}$ is sufficient to produce the $W$ mass correction.

Moreover, the $S_3 + (X, T, B)_{L,R}$ model can also explain the $(g-2)_{\mu}$ anomaly. In our previous paper [87], we took the LQs to have the same mass ($\Delta_{\delta S_3} = 0$). Here, we consider the LQ mass differences, which only lead to small effects. Based on the previous $W$ mass numerical analysis, we choose two benchmark points $m_T = 3$ TeV, $s_L = 0.05$, $m_{S_{1/3}} = 1.5$ TeV, $\Delta_{\delta S_3} = 1$ and $m_T = 5$ TeV, $s_L = 0.05$, $m_{S_{1/3}} = 1.5$ TeV, $\Delta_{\delta S_3} = 0$. Under the first benchmark point, the leading order numerical result of $\Delta_{\mu T}$ is $-0.5914 \times 10^{-7} \text{Re} [y_L^{S,\mu T} (y_R^{S,\mu T})^*]$, which constrains $\text{Re} [y_L^{S,\mu T} (y_R^{S,\mu T})^*]$ to be roughly in the ranges $(-0.052, -0.032)$ and $(-0.062, -0.022)$ at the $1\sigma$ and $2\sigma$ CLs, respectively. Under the second benchmark point, the leading order numerical result of $\Delta_{\mu T}$ is $-0.4542 \times 10^{-7} \times \text{Re} [y_L^{S,\mu T} (y_R^{S,\mu T})^*]$, which constrains $\text{Re} [y_L^{S,\mu T} (y_R^{S,\mu T})^*]$ to be roughly in the ranges $(-0.068, -0.042)$ and $(-0.081, -0.029)$ at the $1\sigma$ and $2\sigma$ CLs, respectively. In Fig. 4, we show the regions allowed by $(g-2)_{\mu}$ in the plane of $m_{S_{1/3}} - \Delta_{\delta S_3}$.

V. SUMMARY AND CONCLUSIONS

We consider the $(X, T, B)_{L,R}$ and $S_3$ extended model to explain the $W$ boson mass anomaly. The mass splittings of VLQs originate from mixing with SM quarks, and the mass splittings of LQs can be generated through interaction with SM Higgs. For VLQ oblique parameter corrections, some papers adopt the existing formulae directly without any examination, which are based on the singlet and doublet properties. In this paper, we obtain the complete VLQ and LQ contributions to the $S, T, U$ parameters. As we know, direct search experiments push the VLQ and LQ mass lower limits to approximately TeV. We also consider the constraints from electroweak precision measurements, of which $Zhb$ coupling imposes the strong bound $s_L \leq 0.05$. For the pure $(X, T, B)_{L,R}$ model, $m_T$ should be as heavy as 4 TeV for $s_L = 0.05$. For the pure $S_3$ model, $|\Delta_{\delta S_3}| \sim 2$ is required for $m_{S_{1/3}} = 1.5$ TeV. For the $S_3 + (X, T, B)_{L,R}$ model, we find that the $W$ boson mass and $(g-2)_{\mu}$ anomalies can be explained simultaneously. Because $W$ mass corrections can be shared by the VLQ and LQ, they allow for lower $m_T$ and smaller $|\Delta_{\delta S_3}|$. Depending on the choice of $m_T, s_L, m_{S_{1/3}}$,
the \((g-2)_{\mu}\) anomaly can also be explained when \(\text{Re} [\delta_{\mu}^{S \mu T} / (\delta_{\mu}^{S \mu T})^*]\) ranges from \(-O(-0.1)\) to \(O(-0.01)\).

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**APPENDIX A: PROPERTIES AND EXPANSIONS OF THE \((X;T;B)_{LR}\) RELATED PARAMETERS**

Relations between the VLQ parameters:

\[
\tan \theta_R = \frac{m_t}{m_T} \tan \theta_L, \\
\tan \theta_R = \frac{m_b}{m_B} \tan \theta_L, \\
\sin 2\theta_R = \frac{\sqrt{2}(m_T^2 - m_T^2)}{m_B^2 - m_B^2} \sin 2\theta_L, \\
M_T = m_T c_L^2 c_R + m_T s_L^2 s_R = m_B c_L c_R + m_B s_L s_R = \frac{m_{s_L}^2 s_R}{s_R}, \\
M_T = \frac{m_T c^2 + m_T s_L^2 s_R}{s_R} = \frac{m_B c^2 + m_B s_L s_R}{s_R}, \\
\sqrt{2}(m_T c_L^2 s_R - m_T c_L^2 s_R) = \sqrt{2}(m_B c_L^2 s_R - m_b c_L^2 s_R).
\]

From the approximations \(m_b \ll m_t \ll m_T\) and \(s_L \ll 1\), we obtain the following results:

\[
\theta_R^p = \frac{m_t}{m_T} \theta_L^p, \\
m_X = M_T \approx m_T \left[1 - \frac{1}{2} \left(1 - \frac{m_t^2}{m_T^2}\right) (\theta_L^p)^2 \right], \\
\theta_L^p = \sqrt{2} \frac{m_T^2 - m^2_T}{m_T^2 - m_B^2} \theta_L \approx \sqrt{2} \left(1 - \frac{m_t^2}{m_T^2}\right) \theta_L, \\
\theta_R^p = \sqrt{2} m_b \frac{m_T^2 - m^2_T}{m_T^2 - m_B^2} \theta_L^p \approx \sqrt{2} m_b \frac{m_T^2 - m^2_T}{m_T^2 - m_B^2} \left(1 - \frac{m_t^2}{m_T^2}\right) \theta_L^p, \\
m_B \approx m_T \left[1 + \frac{(m_T^2 - m_T^2)(m_T^2 - 2m_T^2 + m_B^2)}{2m_T^2 - m_B^2} (\theta_L^p)^2 \right] \\
= m_T \left[1 + \frac{1}{2} \left(1 - \frac{m_t^2}{m_T^2}\right) \left(1 - \frac{2m_t^2}{m_T^2}\right) (\theta_L^p)^2 \right]. (A2)
\]

**APPENDIX B: LQ CONTRIBUTIONS TO THE OB-LIQUE PARAMETERS**

First, let us define the \(B_0\) function as

\[
B_0(p^2, m_T^2, m_B^2) \equiv \Delta \epsilon - \int_0^1 \frac{dx \log \left[ x m_T^2 + (1-x) m_B^2 - x(1-x)p^2 \right]}{\mu^2}.
\]

In the above, \(\Delta \epsilon\) is defined as \(1/\epsilon - \gamma_E + \log 4\pi\). Here, we adopt the dimensional regularization, and \(D = 4 - 2\epsilon\) is the space-time dimension. \(\gamma_E\) is Euler’s constant, and \(\mu\) is the renormalization scale.

According to the LQ gauge interactions derived in Sec. II.D, the self energies of neutral gauge bosons are...
calculated as

\[
\Pi_{\gamma\gamma}(p^2) = \frac{e^2 N_C}{16\pi^2} \sum_{S_i} Q_{S_i}^2 \left[ \frac{1}{3} (4m_{S_i}^2 - p^2) B_0(p^2, m_{S_i}^2, m_{S_i}^2) - \frac{4}{3} m_{S_i}^2 B_0(0, m_{S_i}^2, m_{S_i}^2) - \frac{2}{9} p^2 \right],
\]

\[
\Pi_{\gamma Z}(p^2) = \frac{eg N_C}{16\pi^2 c_w^2} \sum_{S_i} Q_{S_i}^2 (t_3^S - Q_{S_i} s_w^2) \left[ \frac{1}{3} (4m_{S_i}^2 - p^2) B_0(p^2, m_{S_i}^2, m_{S_i}^2) - \frac{4}{3} m_{S_i}^2 B_0(0, m_{S_i}^2, m_{S_i}^2) - \frac{2}{9} p^2 \right],
\]

\[
\Pi_{ZZ}(p^2) = \frac{g^2 N_C}{16\pi^2 c_w^2} \sum_{S_i} (t_3^S - Q_{S_i} s_w^2)^2 \left[ \frac{1}{3} (4m_{S_i}^2 - p^2) B_0(p^2, m_{S_i}^2, m_{S_i}^2) - \frac{4}{3} m_{S_i}^2 B_0(0, m_{S_i}^2, m_{S_i}^2) - \frac{2}{9} p^2 \right],
\]

(B2)

where \( S_i^s = S_i^{s/3}, S_i^{1/3}, S_i^{s-2/3} \). \( Q_{S_i} \) and \( t_3^S \) denote their electric charge and the third component of weak isospin, which means \( Q_{S_i^{s/3}} = 4/3, Q_{S_i^{1/3}} = 1/3, Q_{S_i^{s-2/3}} = -2/3 \) and \( t_3^{S_i^{s/3}} = 1, t_3^{S_i^{1/3}} = 0, t_3^{S_i^{s-2/3}} = -1 \).

Then, the self energy of the \( W \) boson is calculated as

\[
\Pi_{WW}(p^2) = \frac{g^2 N_C}{16\pi^2} \left[ -\frac{p^2}{9} \left[ 3B_0(p^2, m_{S_i^{s/3}}^2, m_{S_i^{s/3}}^2) + 3B_0(p^2, m_{S_i^{1/3}}^2, m_{S_i^{1/3}}^2) + 4 \right]
\]

\[
+ \frac{2}{3} [(m_{S_i^{s/3}}^2 + m_{S_i^{1/3}}^2) B_0(p^2, m_{S_i^{s/3}}^2, m_{S_i^{s/3}}^2) + (m_{S_i^{s/3}}^2 + m_{S_i^{1/3}}^2) B_0(p^2, m_{S_i^{s/3}}^2, m_{S_i^{s/3}}^2) - m_{S_i^{s/3}}^2 B_0(0, m_{S_i^{s/3}}^2, m_{S_i^{s/3}}^2) - 2m_{S_i^{s/3}}^2 B_0(0, m_{S_i^{s/3}}^2, m_{S_i^{s/3}}^2)]
\]

\[
- \frac{1}{3p^2} [(m_{S_i^{1/3}}^2 - m_{S_i^{s/3}}^2) B_0(p^2, m_{S_i^{s/3}}^2, m_{S_i^{s/3}}^2) + (m_{S_i^{1/3}}^2 - m_{S_i^{s/3}}^2) B_0(p^2, m_{S_i^{s/3}}^2, m_{S_i^{s/3}}^2) - m_{S_i^{1/3}}^2 B_0(0, m_{S_i^{s/3}}^2, m_{S_i^{s/3}}^2) - 2m_{S_i^{1/3}}^2 B_0(0, m_{S_i^{s/3}}^2, m_{S_i^{s/3}}^2)]
\]

\[
+ m_{S_i^{s/3}}^2 (m_{S_i^{1/3}}^2 + m_{S_i^{s/3}}^2) B_0(0, m_{S_i^{s/3}}^2, m_{S_i^{s/3}}^2) - (m_{S_i^{1/3}}^2 - m_{S_i^{s/3}}^2)^2 B_0(0, m_{S_i^{s/3}}^2, m_{S_i^{s/3}}^2) - (m_{S_i^{1/3}}^2 - m_{S_i^{s/3}}^2)^2 B_0(0, m_{S_i^{s/3}}^2, m_{S_i^{s/3}}^2)],
\]

(B3)

Based on the exact expressions of \( \Pi_{VV}(p^2) \), we can derive \( \Pi_{VV}(0) \) and \( \Pi_{VV}(0) \equiv \frac{d\Pi_{VV}(p^2)}{dp^2}\big|_{p=0} \) as follows:

\[
\Pi_{\gamma\gamma}(0) = \Pi_{\gamma Z}(0) = \Pi_{ZZ}(0) = 0,
\]

\[
\Pi_{\gamma\gamma}'(0) = -\frac{e^2 N_C}{48\pi^2} \sum_{S_i} Q_{S_i}^2 (\Delta_\alpha - \log \frac{m_{S_i}^2}{\mu^2}),
\]

\[
\Pi_{\gamma Z}'(0) = -\frac{eg N_C}{48\pi^2 c_w^2} \sum_{S_i} Q_{S_i}^2 (t_3^S - Q_{S_i} s_w^2) (\Delta_\alpha - \log \frac{m_{S_i}^2}{\mu^2}),
\]

\[
\Pi_{ZZ}'(0) = -\frac{g^2 N_C}{48\pi^2 c_w^2} \sum_{S_i} (t_3^S - Q_{S_i} s_w^2)^2 (\Delta_\alpha - \log \frac{m_{S_i}^2}{\mu^2}),
\]

\[
\Pi_{WW}(0) = \frac{g^2 N_C}{16\pi^2} \left[ \frac{1}{2} (m_{S_i^{s/3}}^2 + m_{S_i^{1/3}}^2 - 2m_{S_i^{s/3}}^2) - \frac{m_{S_i^{s/3}}^2 m_{S_i^{1/3}}^2}{m_{S_i^{s/3}}^2 - m_{S_i^{s/3}}^2} \log \frac{m_{S_i^{s/3}}^2}{m_{S_i^{s/3}}^2} - \frac{m_{S_i^{s/3}}^2 m_{S_i^{s/3}}^2}{m_{S_i^{s/3}}^2 - m_{S_i^{s/3}}^2} \log \frac{m_{S_i^{s/3}}^2}{m_{S_i^{s/3}}^2} \right],
\]

\[
\Pi_{WW}'(0) = \frac{g^2 N_C}{16\pi^2} \left[ -\frac{2}{3} (\Delta_\alpha - \log \frac{m_{S_i^{s/3}}^2}{\mu^2}) - \frac{4}{9} - \frac{m_{S_i^{s/3}}^2 + m_{S_i^{s/3}}^2 - 14m_{S_i^{s/3}}^2 m_{S_i^{s/3}}^2}{18(m_{S_i^{s/3}}^2 - m_{S_i^{s/3}}^2)^2} - \frac{m_{S_i^{s/3}}^2 + m_{S_i^{s/3}}^2 - 14m_{S_i^{s/3}}^2 m_{S_i^{s/3}}^2}{18(m_{S_i^{s/3}}^2 - m_{S_i^{s/3}}^2)^2}
\]

\[
+ \frac{m_{S_i^{s/3}}^2 (m_{S_i^{s/3}}^2 - 3m_{S_i^{s/3}}^2) \log \frac{m_{S_i^{s/3}}^2}{m_{S_i^{s/3}}^2}}{3(m_{S_i^{s/3}}^2 - m_{S_i^{s/3}}^2)^3} + \frac{m_{S_i^{s/3}}^2 (m_{S_i^{s/3}}^2 - 3m_{S_i^{s/3}}^2) \log \frac{m_{S_i^{s/3}}^2}{m_{S_i^{s/3}}^2}}{3(m_{S_i^{s/3}}^2 - m_{S_i^{s/3}}^2)^3} \right].
\]

(B4)

The \( S, T, \) and \( U \) parameters are defined as [99–101]
\[ \frac{\alpha S}{4s_w c_w} = \frac{\Pi_{ZZ}(m_Z^2) - \Pi_{ZZ}(0)}{m_Z^2} = \frac{c_w^2 - s_w^2}{s_w c_w} \Pi_{\gamma^*}(0) - \Pi_{\gamma^*}(0) = \Pi_{\gamma^*}(0) - \frac{c_w^2 - s_w^2}{s_w c_w} \Pi_{\gamma^*}(0) - \Pi_{\gamma^*}(0), \]
\[ \alpha T = \frac{\Pi_{WW}(m_W^2) - \Pi_{WW}(0)}{m_W^2} - \frac{\Pi_{ZZ}(m_Z^2) - \Pi_{ZZ}(0)}{m_Z^2} = 2s_w c_w \Pi_{\gamma^*}(0) - s_w^2 \Pi_{\gamma^*}(0) = \Pi_{WW}(0) - s_w^2 \Pi_{\gamma^*}(0) - 2s_w c_w \Pi_{\gamma^*}(0) - s_w^2 \Pi_{\gamma^*}(0). \] (B5)

When we adopt Eq. (B4) in the above definitions, the LQ contributions to the $S$, $T$, and $U$ parameters are calculated in the following explicit forms:

\[ \Delta S_{\gamma} = -\frac{N_c}{9\pi} \log \frac{m_{S_{\gamma}}^2}{m_{S_{\gamma}}^2}, \quad \Delta T_{\gamma} = -\frac{N_c}{8\pi m_{W}^2 s_{W}^2} \left[ \theta_{\epsilon}(m_{S_{\gamma}}^2, m_{S_{\gamma}}^2) + \theta_{\epsilon}(m_{S_{\gamma}}^2, m_{S_{\gamma}}^2) \right], \]
\[ \Delta U_{\gamma} = \frac{N_c}{\pi} \left[ -\frac{m_{S_{\gamma}}^4 + m_{S_{\gamma}}^4 - 14m_{S_{\gamma}}^2 m_{S_{\gamma}}^2}{18(m_{S_{\gamma}}^2 - m_{S_{\gamma}}^2)^2} \log \frac{m_{S_{\gamma}}^2}{m_{S_{\gamma}}^2} + \frac{m_{S_{\gamma}}^4 - 3m_{S_{\gamma}}^2 m_{S_{\gamma}}^2 - 3m_{S_{\gamma}}^2 m_{S_{\gamma}}^2}{3(m_{S_{\gamma}}^2 - m_{S_{\gamma}}^2)^3} \log \frac{m_{S_{\gamma}}^2}{m_{S_{\gamma}}^2} \right]. \] (B6)

As we can see, the divergence and scale $\mu$ are exactly canceled in the oblique parameters.

**APPENDIX C: VLQ CONTRIBUTIONS TO THE OBLIQUE PARAMETERS**

Because the triplet VLQ is involved, we cannot simply adopt the formulae of the $S$ and $T$ parameters in Ref. [108], in which some calculations are based on singlet and doublet properties. In this section, we present a detailed deduction.

Let us generally denote the quark interactions with gauge bosons $V_1$ and $V_2$ as $\tilde{f}_j \gamma_{\mu}(g_{ij}^{V_1} + g_{ij}^{V_2} \gamma^5)f_j V_1^\mu + \tilde{f}_j \gamma_{\mu}(g_{ij}^{V_1} + g_{ij}^{V_2} \gamma^5)f_j V_2^\mu$. Here, the masses of $f_i$ and $f_j$ are labeled as $m_i$ and $m_j$. Thus, the self energy of $V_1 - V_2$ is calculated as [108]

\[ \Pi_{V_1 V_2}(0) = \frac{-N_c}{16\pi^2} \left[ (g_{ij}^{V_1} + g_{ij}^{V_2} \gamma^5)(2m_i^2 + m_j^2) \Delta\epsilon - 2m_i^2 \log \frac{m_i^2}{\mu^2} + m_j^2 \log \frac{m_j^2}{\mu^2} + \theta_{\epsilon}(m_i^2, m_j^2) \right] + (g_{ij}^{V_1} - g_{ij}^{V_2} \gamma^5)[-4m_i m_j \Delta\epsilon + 2m_i m_j \log \frac{m_i^2 m_j^2}{\mu^2} + \theta_{\epsilon}(m_i^2, m_j^2)]. \] (C1)

Moreover, the first derivative of the $V_1 - V_2$ self energy is calculated as [108]

\[ \frac{d\Pi_{V_1 V_2}(p^2)}{dp^2} |_{p^2 = 0} = \frac{-N_c}{4\pi^2} \left[ (g_{ij}^{V_1} + g_{ij}^{V_2} \gamma^5)[-\frac{1}{3} \Delta\epsilon + \frac{1}{6} \log \frac{m_i^2 m_j^2}{\mu^4} - \frac{1}{2} \theta_{\epsilon}(m_i^2, m_j^2)] + (g_{ij}^{V_1} - g_{ij}^{V_2} \gamma^5)[-\frac{m_i^2 + m_j^2}{12m_i m_j} - \frac{1}{2} \theta_{\epsilon}(m_i^2, m_j^2)]. \] (C2)

In Ref. [108], the derivation of the $S$ and $T$ parameters depends on the following relations:

\[ (U^a)^2 = U^a, \quad (D^a)^2 = D^a, \quad D^a M_a D^a = (V^a)^* M_a V^a, \quad U^a M_a U^a = V^a M_a (V^a)^*. \] (C3)
They are only valid for singlet and doublet VLQs. For the \((X,T,B)\) case, they no longer hold. In fact, the cancelation of divergence is guaranteed by the relations in Eq. (7).

For compactness and simplicity, let us reformulate the VLQ gauge interactions in Sec. II.C with the matrix form. The gauge interactions with the \(W\) boson can be written as

\[
\frac{g}{\sqrt{2}} \left\{ X_L \gamma^\mu V_{L}^X \begin{pmatrix} t_L \\ T_L \end{pmatrix} + X_R \gamma^\mu V_{R}^X \begin{pmatrix} t_R \\ T_R \end{pmatrix} + (\overline{T_L}, T_L)\gamma^\mu V_{L}^b \begin{pmatrix} b_L \\ B_L \end{pmatrix} + (\overline{T_R}, T_R)\gamma^\mu V_{R}^b \begin{pmatrix} b_R \\ B_R \end{pmatrix} \right\} + \text{h.c.} \tag{C4}
\]

Similarly, the gauge interactions with the \(Z\) boson can be written as

\[
\frac{g}{2\cos\theta_W} Z_\mu \left( X_L \gamma^\mu (U_L^X - 2Q_X s_W^2) X_L + X_R \gamma^\mu (U_R^X - 2Q_X s_W^2) X_R + (\overline{U_L}, T_L)\gamma^\mu (U_L^b - 2Q_b s_W^2) \begin{pmatrix} b_L \\ B_L \end{pmatrix} + (\overline{U_R}, T_R)\gamma^\mu (U_R^b - 2Q_b s_W^2) \begin{pmatrix} b_R \\ B_R \end{pmatrix} \right). \tag{C5}
\]

As for the gauge interactions with the photon, it has the trivial form \(e Q_f \gamma^\mu f A_\mu\). In the above, the \(V\) and \(U\) matrices are given as

\[
V_{L}^{X} = \sqrt{2}(s_{L} - c_{L}), \quad V_{R}^{X} = \sqrt{2}(s_{R} - c_{R}),
\]

\[
V_{L}^{b} = \begin{pmatrix} c_{L}c_{L} + \sqrt{2}s_{L}s_{L} & c_{L}s_{L} - \sqrt{2}s_{L}c_{L} \\ s_{L}c_{L} - \sqrt{2}s_{L}s_{L} & s_{L}s_{L} + \sqrt{2}c_{L}c_{L} \end{pmatrix}, \quad V_{R}^{b} = \begin{pmatrix} \sqrt{2}s_{R}c_{R} - \sqrt{2}s_{R}c_{R} \\ -\sqrt{2}c_{R}s_{R} \sqrt{2}c_{R}s_{R} \end{pmatrix}, \tag{C6}
\]

and

\[
U_{L}^{X} = U_{L}^{X} = 2, \quad U_{L}^{b} = \begin{pmatrix} (c_{L}c_{L})^2 & s_{L}c_{L} \\ s_{L}c_{L} & (s_{L}c_{L})^2 \end{pmatrix}, \quad U_{R}^{b} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

\[
U_{L}^{b} = \begin{pmatrix} 1 + (s_{L}c_{L})^2 & -s_{L}c_{L} \\ -s_{L}c_{L} & 1 + (c_{L}c_{L})^2 \end{pmatrix}, \quad U_{R}^{b} = \begin{pmatrix} 2(s_{b}c_{b})^2 & -2s_{b}c_{b} \\ -2s_{b}c_{b} & 2(c_{b}c_{b})^2 \end{pmatrix}. \tag{C7}
\]

The \(U\) and \(V\) matrices can be correlated through the following identities:

\[
U_{L/R}^{X} = V_{L/R}^{X}(V_{L/R}^{X})^\dagger, \quad U_{L/R}^{b} = V_{L/R}^{b}(V_{L/R}^{b})^\dagger - (V_{L/R}^{X})^\dagger V_{L/R}^{X}, \quad U_{L/R}^{b} = (V_{L/R}^{b})^\dagger V_{L/R}^{b}. \tag{C8}
\]

### A. Derivation of the \(T\) parameter

According to Eq. (C1), the self energy consists of \(\theta_z\) and non-\(\theta_z\) parts. Here, let us consider the non-\(\theta_z\) part. Based on the definition in Eq. (B5), it can be calculated as

\[
\alpha T_{\text{non-}\theta_z} = \frac{N e \lambda^2}{32 \pi^2 m_W^2} \left\{ V_{L/R}^{X} (V_{L/R}^{X})^\dagger U_{L/R}^{X} (V_{L/R}^{X})^\dagger + V_{L/R}^{X} (V_{L/R}^{X})^\dagger U_{L/R}^{X} (V_{L/R}^{X})^\dagger \right\} \Delta \log \frac{m_X^2}{\mu^2}
\]

\[
+ \left\{ V_{L/R}^{X} (V_{L/R}^{X})^\dagger U_{L/R}^{X} (V_{L/R}^{X})^\dagger + V_{L/R}^{X} (V_{L/R}^{X})^\dagger U_{L/R}^{X} (V_{L/R}^{X})^\dagger \right\} \Delta \log \frac{m_X^2}{\mu^2}
\]

\[
\alpha T_{\text{non-}\theta_z} = \frac{N e \lambda^2}{32 \pi^2 m_W^2} \left\{ V_{L/R}^{X} (V_{L/R}^{X})^\dagger U_{L/R}^{X} (V_{L/R}^{X})^\dagger + V_{L/R}^{X} (V_{L/R}^{X})^\dagger U_{L/R}^{X} (V_{L/R}^{X})^\dagger \right\} \Delta \log \frac{m_X^2}{\mu^2}
\]

\[
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\]
\begin{equation}
\Delta T^{XTB} = \frac{N_C}{16\pi^2 m_W^2} \left[ 2[(s_{L}^2)^2 + (s_{R}^2)^2] \theta_+(m_{L}^2, m_{T}^2) + 4s_{L}^2 s_{R}^2 \theta_-(m_{L}^2, m_{T}^2) \\
+ 2[(c_{L}^2)^2 + (c_{R}^2)^2] \theta_+ (m_{L}^2, m_{T}^2) + 4c_{L}^2 c_{R}^2 \theta_-(m_{L}^2, m_{T}^2) \\
+ [(c_{L}^2 - s_{L}^2) \theta_+ + (c_{R}^2 - s_{R}^2) \theta_-] \left( m_{L}^2, m_{T}^2, m_{B}^2 \right) \\
+ [(c_{L}^2 + s_{L}^2) \theta_+ + (c_{R}^2 + s_{R}^2) \theta_-] \left( m_{L}^2, m_{T}^2, m_{B}^2 \right) \\
+ [(s_{L}^2 - c_{L}^2) \theta_+ + (s_{R}^2 - c_{R}^2) \theta_-] \left( m_{L}^2, m_{T}^2, m_{B}^2 \right) \\
+ [(s_{L}^2 + c_{L}^2) \theta_+ + (s_{R}^2 + c_{R}^2) \theta_-] \left( m_{L}^2, m_{T}^2, m_{B}^2 \right) \\
- (s_{L}^2)^2 \theta_+(m_{L}^2, m_{T}^2) - [(s_{L}^2)^2 + 4(s_{R}^2)^2 \theta_+(m_{L}^2, m_{T}^2) - 4(s_{L}^2 s_{R}^2) \theta_-(m_{L}^2, m_{B}^2)] \right].
\end{equation}

Here, the \( \theta_+ \) function has been previously defined, and the \( \theta_- \) function is defined as
\begin{equation}
\theta_-(y_1, y_2) \equiv 2 \frac{\text{log}(y_1)}{y_1 \text{log}(y_1) - y_2 \text{log}(y_2) - 2}.
\end{equation}

This is consistent with the result of the \( T \) parameter in Ref. [123].

\section{Derivation of the \( S \) parameter}

If we adopt the \( S \) parameter formula in Ref. [108], it will give the following result:
\begin{equation}
\Delta S_{\text{wrong}} = \frac{N_C}{2\pi} \left[ 2[(s_{L}^2)^2 + (s_{R}^2)^2] \psi_+(m_{L}^2, m_{T}^2) + 4s_{L}^2 s_{R}^2 \psi_-(m_{L}^2, m_{T}^2) \\
+ 2[(c_{L}^2)^2 + (c_{R}^2)^2] \psi_+ (m_{L}^2, m_{T}^2) + 4c_{L}^2 c_{R}^2 \psi_- (m_{L}^2, m_{T}^2) \\
+ [(c_{L}^2 - s_{L}^2) \psi_+ + (c_{R}^2 - s_{R}^2) \psi_-] \left( m_{L}^2, m_{T}^2, m_{B}^2 \right) \\
+ [(c_{L}^2 + s_{L}^2) \psi_+ + (c_{R}^2 + s_{R}^2) \psi_-] \left( m_{L}^2, m_{T}^2, m_{B}^2 \right) \\
+ [(s_{L}^2 - c_{L}^2) \psi_+ + (s_{R}^2 - c_{R}^2) \psi_-] \left( m_{L}^2, m_{T}^2, m_{B}^2 \right) \\
+ [(s_{L}^2 + c_{L}^2) \psi_+ + (s_{R}^2 + c_{R}^2) \psi_-] \left( m_{L}^2, m_{T}^2, m_{B}^2 \right) \\
- (s_{L}^2)^2 \psi_+(m_{L}^2, m_{T}^2) - [(s_{L}^2)^2 + 4(s_{R}^2)^2 \psi_+(m_{L}^2, m_{T}^2) - 4(s_{L}^2 s_{R}^2) \psi_-(m_{L}^2, m_{B}^2)] \right].
\end{equation}
\[
+[(c_L^c)^2 - \sqrt{s_L^c c_L^c})^2 + 2(s_L^c c_L^c)^2] \psi_+ (m_T^2, m_B^2) - 2 \sqrt{s_L^c c_L^c}(c_L^c s_L^c - \sqrt{s_L^c c_L^c}) \psi_- (m_T^2, m_B^2)
\]
\[
+[(s_L^c)^2 + \sqrt{s_L^c s_L^c})^2 + 2(c_L^c c_L^c)^2] \psi_+(m_T^2, m_B^2) - 2 \sqrt{s_L^c s_L^c}(s_L^c s_R^c + \sqrt{s_L^c s_L^c}) \psi_- (m_T^2, m_B^2)
\]
\[
+[(c_R^c)^2 + \sqrt{s_R^c s_R^c})^2 + 2(c_R^c c_R^c)^2] \psi_+(m_T^2, m_B^2) + 2 \sqrt{s_R^c s_R^c}(c_R^c c_R^c + \sqrt{s_R^c s_R^c}) \psi_- (m_T^2, m_B^2)
\]
\[
-[(s_R^c)^2 + 4(s_R^c c_R^c)^2] [\chi+(m_T^2, m_B^2) - 4(s_R^c c_R^c)(s_R^c c_R^c)] \chi- (m_T^2, m_B^2)].
\]

(C12)

In the above, the functions \(\psi_+\) and \(\chi\) are defined as

\[
\psi_+(y_1, y_2) \equiv \frac{1}{3} - \frac{1}{9} \log \frac{y_1}{y_2}, \quad \psi_-(y_1, y_2) \equiv -\frac{y_1 + y_2}{6 \sqrt{y_1 y_2}}.
\]

\[
\chi_+(y_1, y_2) \equiv \frac{(y_1^2 + y_2^2) - 2y_1 y_2}{9(y_1 - y_2)^2} + \frac{3y_1 y_2 (y_1 + y_2) - y_1 - y_2}{3(y_1 - y_2)^4} \log \frac{y_1}{y_2},
\]

\[
\chi_-(y_1, y_2) \equiv -\frac{\sqrt{y_1 y_2}}{6(y_1 - y_2)^2} \left[\frac{y_1 + y_2}{y_1 - y_2} + \frac{2y_1 y_2}{(y_1 - y_2)^3} \log \frac{y_1}{y_2}\right].
\]

(C13)

This is not solid because the \(S\) parameter formula in Ref. [108] relies on the singlet and doublet representations, which should be considered for the \((X, T, B)_{L,R}\) triplet.

According to Eq. (C2), the first derivative of self energy consists of \(\chi\) and non-\(\chi\) parts. Here, let us consider the non-\(\chi\) part. Based on the definition in Eq. (B5), it can be calculated as

\[
\frac{a S^{XTB}}{4 s_w^2 c_w} = \frac{N c g^2}{96 m^2 c_w} \left[ U^X_L U^X_R + \frac{U^X_L U^X_R}{2} \right] - \frac{U^X_L U^X_R}{2}
\]

\[
+ \text{Tr}[U^X_L (U^X_L + U^X_R)] \left( \Delta_\epsilon + \log \frac{m_T^2}{m_B^2} \right) + \frac{1}{2} \left[ \text{Tr}[U^X_L U^X_R + U^X_R U^X_L] \right] - \text{Tr}[U^X_L U^X_R]
\]

\[
+ \text{Tr}[U^X_{L,R} U^X_{L,R}] - 2\text{Tr}[U^X_{L,R}] \left( \Delta_\epsilon + \log \frac{m_T^2}{m_B^2} \right) + \frac{1}{2} \left[ \text{Tr}[U^X_{L,R} U^X_{L,R} + U^X_{R,L} U^X_{R,L}] \right] - \text{Tr}[U^X_{L,R} U^X_{L,R}]
\]

\[
= \frac{N c g^2}{32 m^2 c_w} \left[ \frac{2}{3} - \frac{1}{3} \cos(2\theta^c) \cos(2\theta^b) \frac{(m_T^2 + m_B^2) \sin(2\theta^c) \sin(2\theta^b)}{6m_B m_B} \right] - \frac{16}{9} \left[ (s_L^c)^2 \log \frac{m_T^2}{m_B^2} + (c_L^c)^2 \log \frac{m_T^2}{m_B^2} \right]
\]

\[
- \frac{5}{3} (s_R^c)^2 \log \frac{m_T^2}{m_B} + (c_R^c)^2 \log \frac{m_T^2}{m_B^2} \right] - \frac{1}{9} \log \frac{m_T^2}{m_B^2} + \frac{7\cos(2\theta^b) + 8\cos(2\theta^b)}{18} \left[ \frac{m_B^2}{m_B^2} \right].
\]

(C14)

As we can see, the contributions from the non-\(\chi\) part cannot simply be described by the \(\psi_+\) functions, which depend on the singlet and doublet properties. The correct expression for \(\Delta S^{XTB}\) can be calculated as follows:

\[
\Delta S^{XTB} = \frac{a S^{XTB}}{a \rho} + \frac{N c g^2}{2\pi} \left[ \psi_+(m_T^2, m_B^2) - (s_L^c)^2 \chi-(m_T^2, m_B^2)
\]

\[
- (s_R^c)^2 + 4(s_R^c c_R^c)^2 \chi-(m_T^2, m_B^2) - 4(s_R^c c_R^c)(s_R^c c_R^c) \chi-(m_T^2, m_B^2) \right].
\]

(C15)

C. Derivation of the \(U\) parameter

According to Eq. (C2), the \(U\) parameter also consists of \(\chi\) and non-\(\chi\) parts. For the \(\chi\) part, it can be calculated as

---

1) I would like to thank Haiying Cai for talking about this.
\[
\Delta U^{XTB}_{\chi} = -\frac{N_C}{2} \left[ \frac{1}{2} \left( s'_L \right)^2 + (s'_R)^2 \right] \chi + (m^2_X, m^2_b) + 4 s'_L s'_R \bar{\chi} \chi - (m^2_X, m^2_b)
+ 2 (c'_L)^2 + (c'_R)^2 \chi + (m^2_X, m^2_b) + 4 c'_L c'_R \bar{\chi} \chi - (m^2_X, m^2_b)
+ (c'_L - c'_R)^2 \chi - (m^2_X, m^2_b) + 2 \sqrt{2} G_F s'_L (c'_L + \sqrt{G_F} s'_L) \bar{\chi} \chi - (m^2_X, m^2_b)
+ (c'_L - c'_R)^2 \chi - (m^2_X, m^2_b) + 2 \sqrt{2} G_F s'_R (c'_R - \sqrt{G_F} s'_R) \bar{\chi} \chi - (m^2_X, m^2_b)
- (c'_L c'_R)^2 \chi - (m^2_X, m^2_b) - [2 (s'_L s'_R)^2 \chi + (m^2_X, m^2_b)]
- 2 (s'_L s'_R)^2 \chi - (m^2_X, m^2_b) - 2 (s'_L s'_R)^2 \chi - (m^2_X, m^2_b)
\]
\]  
(C16)

For the non-\( \chi \) part, it can be calculated as

\[
\frac{\alpha U^{XTB}_{\text{non-} \chi}}{4 \pi} = \frac{N_C G_F}{96\pi^2} \left[ \frac{1}{2} \left( V^{X_L}_L (V^{X_R}_L)^* - U^{X_L}_L U^{X_R}_L + V^{X_L}_R (V^{X_R}_R)^* - U^{X_L}_R U^{X_R}_R \right) \right] (-\Delta_\pi + \log \frac{m^2_X}{\mu^2})
+ 2 \left[ V^{X_L}_L U^{X_R}_L \right]^{-1} + \left[ V^{X_L}_R U^{X_R}_R \right]^{-1} + \left[ U^{X_L}_L U^{X_R}_L \right]^{-1} + \left[ U^{X_L}_R U^{X_R}_R \right]^{-1}
- \frac{1}{2} \left[ U^{X_L}_L U^{X_R}_L \right]^{-1} + \left[ V^{X_L}_L U^{X_R}_L \right]^{-1} + \left[ V^{X_L}_R U^{X_R}_R \right]^{-1} + \left[ U^{X_L}_L U^{X_R}_L \right]^{-1} + \left[ U^{X_L}_R U^{X_R}_R \right]^{-1}
+ \frac{1}{2} \left[ U^{X_L}_L U^{X_R}_L \right]^{-1} + \left[ V^{X_L}_L U^{X_R}_L \right]^{-1} + \left[ V^{X_L}_R U^{X_R}_R \right]^{-1} + \left[ U^{X_L}_L U^{X_R}_L \right]^{-1} + \left[ U^{X_L}_R U^{X_R}_R \right]^{-1}
+ \frac{1}{2} \left[ U^{X_L}_L U^{X_R}_L \right]^{-1} + \left[ V^{X_L}_L U^{X_R}_L \right]^{-1} + \left[ V^{X_L}_R U^{X_R}_R \right]^{-1} + \left[ U^{X_L}_L U^{X_R}_L \right]^{-1} + \left[ U^{X_L}_R U^{X_R}_R \right]^{-1}
\]

Note that the non-\( \chi \) contributions vanish for the singlet and doublet VLQs [108], whereas they are non-zero for the \((X,T,B)_{L,R}\) triplet. Thus, the total contributions of the \( U \) parameter should be

\[
\Delta U^{XTB} = \Delta U^{XTB}_{\chi} + \Delta U^{XTB}_{\text{non-} \chi}.
\]

References

[1] P. A. Zyla et al. (Particle Data Group), PTEP 2020, 083C01 (2020)
[2] G. W. Bennett et al., Phys. Rev. D 73, 072003 (2006), arXiv:hep-ex/0602035
[3] B. Abi et al., Phys. Rev. Lett. 126, 141801 (2021), arXiv:2104.03281[hep-ex]
[4] D. London and J. Mattias, Annu. Rev. Nucl. Part. Sci. 2, 37 (2022), arXiv:2110.13270[hep-ph]
[5] T. Aaltonen et al. (CDF Collaboration), Science 376, 170 (2022)
[6] A. Bacchetta, G. Bozzi, M. Radici et al., Phys. Lett. B 788, 542 (2019), arXiv:1807.02101[hep-ph]
[7] C. Campagnari and M. Mulders, Science 376, 136 (2022)
[8] Y.-Z. Fan, T.-P. Tang, Y.-L. S. Tsai et al., Phys. Rev. Lett. 129, 091802 (2022), arXiv:2204.03693[hep-ph]
[9] C.-T. Lu, L. Wu, Y. Wu et al., Phys. Rev. D 106, 035034 (2022), arXiv:2204.03796[hep-ph]
[10] X. K. Du, Z. Li, F. Wang et al., arXiv:2204.04286
[11] T.-P. Tang, M. Abdughani, L. Feng et al., arXiv:2204.04356
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