Mixture model fitting using conditional models and modal Gibbs sampling

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Abstract

Mixture models are a convenient way of modeling data using a convex combination of different parametric distributions. In this paper, we present a novel approach to fitting mixture models based on estimating first the posterior distribution of the auxiliary variables that assign each observation to a group in the mixture. The posterior distributions of the remainder of the parameters in the mixture is obtained by averaging over their conditional posterior marginals on the auxiliary variables using Bayesian model averaging.

A new algorithm based on Gibbs sampling is used to approximate the posterior distribution of the auxiliary variables without sampling any other parameter in the model. In particular, the modes of the full conditionals of the parameters of the densities in the mixture are computed and these are plugged-in to the full conditional of the auxiliary variables to draw samples. This approximation, that we have called ‘modal’ Gibbs sampling, reduces the computational burden in the Gibbs sampling algorithm and still provides very good estimates of the posterior distribution of the auxiliary variables. Conditional models on the auxiliary variables are fitted using the Integrated Nested Laplace Approximation (INLA) to obtain the conditional posterior distributions, including modes, of the distributional parameters in the mixtures.

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This approach is general enough to consider mixture models with discrete or continuous outcomes from a wide range of distributions and latent models as conditional model fitting is done with INLA. This presents several other advantages, such as fast fitting of the conditional models, not being restricted to the use of conjugate priors on the model parameters and being less prone to label switching. Within this framework, computing the marginal likelihood of the mixture model when the number of groups in the mixture is known is easy and it can be used to tackle selection of the number of components. Finally, a simulation study has been carried out to assess the validity of ‘modal’ Gibbs sampling and two examples on well-known datasets are discussed using a mixture of Gaussian and Poisson distributions, respectively.

Keywords: Bayesian model averaging; INLA; Marginal likelihood; Mixture models; Model selection
1 Introduction

Mixture models are a convenient way of describing data when these are thought to come from different groups or components which are represented by different distributions. Some well known examples of this type of data include the velocities of galaxies (Carlin and Chib 1995; Chib 1995), waiting and eruption times of the Old Faithful geyser in the Yellowstone National Park (Azzalini and Bowman 1990) and the number of fetal movements in lambs (Leroux and Puterman 1992; Chib 1996). In all these cases, the data generating process is made of different components with different parametric densities and the observed data does not include information about the component to which observation belongs. For this reason, mixture models are often represented using latent auxiliary variables to indicate to which component each observation belongs and the parametric distribution of each component.

In this paper a new approach to fitting mixture models is developed by focusing on estimating the posterior distribution of the auxiliary variables first. Once this posterior distribution has been obtained, the posterior marginal of the remainder of the parameters in the model is obtained by Bayesian model averaging over conditional posterior distributions given the values of the auxiliary variables. This is possible because, given the value of the auxiliary variables, a mixture model becomes a model with different independent components and the posterior marginals of their parameters can easily be obtained.

Hence, model fitting will be split in two steps. First of all, the posterior distribution of the latent auxiliary variables will be estimated. This will be done by means of what we have called 'modal' Gibbs sampling because some of the parameters will not be sampled but the modes of their full conditional distributions will be used instead. Secondly, conditional models on these auxiliary variables will be fitted by considering values of the auxiliary...
variables with non-negligible posterior probability. These conditional models will be fitted with INLA during ‘modal’ Gibbs sampling and the resulting conditional marginals will be combined using Bayesian model averaging (Hoeting et al., 1999, Bivand et al., 2014). As conditional models are fitted with INLA, it is possible to consider a wide range of distributions for the mixture components and the use of non-conjugate priors for their parameters. Furthermore, this approach seems to be less likely to suffer from label switching and it allows for the computation of measures for the selection of the optimal number of components, such as the marginal likelihood.

The analysis of mixture models using Bayesian inference has been considered by several authors. Including a set of auxiliary variables to describe to which component each observation belongs simplifies model fitting with Markov chain Monte Carlo (MCMC) methods, as described in Section 2. Gibbs sampling (see, for example, Chib, 1995) and Metropolis-Hastings (see, for example Cappé et al., 2002) algorithms are popular MCMC methods to fit mixture models. See, for example, Marin et al. (2005) for a discussion. More recently, Gómez-Rubio and Rue (2017) describe how to combine the Integrated Nested Laplace Approximation (INLA) within the Metropolis-Hastings algorithm to fit models that can be expressed as a conditional Gaussian Markov random field (GMRF). Conditional on the values of the auxiliary variables that assign each observation to a group, a mixture model becomes a model with several likelihoods that can be easily fitted within the INLA framework.

In this paper we follow a different approach from Gómez-Rubio and Rue (2017) to estimate the posterior distribution of the auxiliary variables. A variation of the Gibbs sampling algorithm is developed to use the modes of the full conditional distributions for some of the parameters of the model and these are plugged-in to the full conditional distribution of the auxiliary variables to draw samples. This provides a good approximation to their posterior
distribution, as confirmed by a simulation study and the examples developed later.

The paper is organized as follows. In Section 2 an introduction to mixture models and Bayesian inference is provided. Next, INLA is described in Section 3. Fitting mixture models with INLA is detailed in Section 4. The computation of the marginal likelihood of a model for the selection of the number of components in the mixture is tackled in Section 5. A simulation study is carried out in Section 6. Several examples are developed in Section 7. Finally, the developments and main results of this paper are discussed in Section 8.

## 2 Bayesian Inference on Mixture Models

### 2.1 Mixture models

Given a set of $n$ observations $\mathbf{y} = (y_1, \ldots, y_n)$, a mixture model with $K$ components or groups is usually defined as follows:

$$y_i \sim \sum_{j=1}^{K} w_j f_j(y_i|\theta_j), \ i = 1, \ldots, n$$

Here, $f_j(\cdot|\theta_j)$ is the distribution of group or component $j$ in the mixture, which is defined by a set of parameters $\theta_j$. Parameters $w_j$ indicate the weight of each component in the mixture and these are taken to sum up to 1, i.e., $\sum_{j=1}^{K} w_j = 1$. Weights depend on the number of observations that belong to each component and the vector of weights $\{w_j\}_{j=1}^{K}$ will be denoted by $\mathbf{w}$. The ensemble of parameters $\{\theta_j\}_{j=1}^{K}$ will be denoted by vector $\mathbf{\theta}$ and a single element of it by $\theta_\bullet \in \mathbf{\theta}$.

Typically, the form of distributions $f_j(\cdot|\theta_j)$, and their associated parameters $\theta_j$, will depend on the actual problem. For example, for continuous observations, $f_j(\cdot|\theta_j)$ can be a Normal distribution with parameters $\theta_j = (\mu_j, \tau_j)$, where $\mu_j$ is the mean and $\tau_j$ the
precision. When the observations are count data, distributions $f_j(\cdot|\theta_j)$ could well be Poisson with different means, i.e., $\theta_j = \lambda_j$.

An alternative formulation of the model, very useful for Bayesian inference with MCMC, is to express the mixture by including auxiliary variables $\{z_i\}_{i=1}^n$ to indicate to which component observation $i$ belongs. These auxiliary variables are defined so that the probability of observation $i$ belonging to component $j$, conditional on $w$, is:

$$\pi(z_i = j|w) = w_j, \ j = 1, \ldots, K$$

This means that, given $z_i = j$, the distribution of $y_i$ is $f_{z_i}(\cdot|\theta_{z_i})$. The ensemble of auxiliary variables $(z_1, \ldots, z_n)$ will be denoted by $z$.

### 2.2 Bayesian inference

We have followed [Marin et al. (2005)](#) in order to provide a brief summary of Bayesian inference for mixture models below and the description of the Gibbs sampling algorithm for mixture models described in Section 2.3. First of all, the graphical representation of a mixture model in Figure 1 shows the relationship between the different parameters in the mixture model.

Observations $y_i$ are independent given $z_i$ and $\theta_{z_i}$, hence:

$$\pi(y|z, \theta) = \prod_{i=1}^n f_{z_i}(y_i|\theta_{z_i})$$

Note also that the prior distribution on $z$ is $\pi(z|w)$, which is the distribution of $z$ given weights $w$ and it is defined as:

$$\pi(z|w) = \prod_{j=1}^K w_j^{n_j}$$
where \( n_j \) is the number of observations assigned to component \( j \).

Finally, in order to complete our Bayesian formulation of the model, priors on \( \theta \) and \( w \) must be defined. For convenience, conjugate priors are often used as this will make inference using MCMC easier (see, Section 2.3 below for details). However, as we will see in Section 4, sampling on \( \theta \) and \( w \) will not be done in practice. The prior on \( \theta \) will depend on the distributions in the mixture. For example, for a mixture of Normal distributions, means \( \{ \mu_j \}_{j=1}^K \) can be assigned a vague Normal prior (e.g., centered at zero with a small precision) whilst precisions \( \{ \tau_j \}_{i=1}^K \) can be assigned vague Gamma priors. For a mixture of Poisson distributions, means \( \{ \mu_j \}_{j=1}^K \) can be assigned a vague Gamma prior. However, given that model fitting will be done with INLA there is a wider choice of priors available.

Weights \( w \) can be assigned a Dirichlet prior with parameters \( (\alpha_1, \ldots, \alpha_K) \). A convenient vague prior on \( w \) is a Dirichlet with parameters \( \alpha_1 = \ldots = \alpha_K = 2 \), which also ensures that the mode of this prior distribution exists.

It is worth noting that the prior on \( \theta \) can be used to include some identifiability constraints in the model (see, e.g., [Carlin and Chib, 1995]). For example, in a mixture model with two components defined by Normal distributions with means \( \mu_1 \) and \( \mu_2 \), respectively, the prior may be informative so that \( \mu_1 < \mu_2 \), or include a constraint to make sure that
\( \mu_1 < \mu_2 \). This will make the two components fully identifiable. Hence, the prior in this case could be proportional to \( \pi(\mu_1)\pi(\mu_2)I(\mu_1 < \mu_2) \), where \( I(\cdot) \) is the indicator function.

### 2.3 Gibbs sampling

Gibbs sampling has been a convenient approach to obtain samples from the posterior distribution of the model parameters in mixture models (Chib (1995). Marin et al. (2005) have derived the full conditionals of the mixture model described in Section 2.2.

For weight parameters \( w \) the full conditional is a Dirichlet distribution with parameters \((\alpha_1 + n_1, \ldots, \alpha_K + n_K)\), where \( n_j \) represents the number of observations allocated to component \( j \), with \( j = 1, \ldots, K \). Note that this full conditional only depends on \( z \) and \( \pi(w) \). It may be worth setting \( \alpha_1 = \ldots = \alpha_K \) to 2 (instead of 1) so that the mode of the full conditional exists when some \( n_j \) is set to zero (i.e., if there are no observations assigned to group \( j \) at some point during Gibbs sampling).

The full conditional distribution for grouping variable \( z_i \) is a discrete distribution that will provide the probability of observation \( i \) being assigned to component \( j \). This can be stated as

\[
\pi(z_i = j|w, \theta, y) \propto w_j f_j(y_i|\theta_j), \quad j = 1, \ldots, K
\]

Actual probabilities are computed by obtaining the terms in the right hand side first and then re-scaling the probabilities to sum up to one:

\[
\pi(z_i = j|w, \theta, y) = \frac{w_j f_j(y_i|\theta_j)}{\sum_{j=1}^{K} w_j f_j(y_i|\theta_j)}
\]  

(3)

Finally, the full conditionals of \( \theta_* \in \theta \) will depend on the actual distribution being used but they are not difficult to obtain for many typical distributions.
For example, consider that \( f_j(\cdot | \theta_j) \) is a Gaussian distribution with mean \( \mu_j \) and precision \( \tau_j \). If the prior on \( \tau_j \) is a Gamma distribution with parameters \( a \) and \( b \), and the prior on \( \mu_j | \tau_j \) is a Gaussian distribution with mean \( \mu \) and precision \( \tau_j \), then the full conditionals are given by

\[
\pi(\mu_j | \tau_j, z, y) \propto N \left( \frac{\mu + s_j}{1 + n_j}, \frac{1}{\tau_j (1 + n_j)} \right)
\]

\[
\pi(\tau_j | \mu_j, z, y) \propto Ga \left( a + \frac{n_j + 1}{2}, b + 0.5(\mu_j - \mu)^2 + 0.5s s_j \right)
\]

In the previous expressions, \( n_j \) is the number of observations assigned to component \( j \), \( s_j \) the sum of their values and \( s s_j \) is the sum of squares of the values around mean \( \mu_j \).

For a mixture of Poisson with \( f_j(\cdot | \theta_j) \) a Poisson with mean \( \mu_j \) and a Gamma prior (i.e, \( \pi(\mu_j) \sim Ga(a, b) \)), the full conditional distribution is

\[
\pi(\mu_j | z, y) \propto Ga(a + s_j, b + n_j)
\]

Here, \( n_j \) is the number of observations assigned to component \( j \) and \( s_j \) the sum of their values.

The full conditionals of \( \theta_\bullet \) presented before for Gaussian and Poisson mixtures can be derived in a closed form because of the conjugate priors on the elements in \( \theta_j \). As we shall see in Section 4, it is not always necessary to use conjugate priors as the full conditionals can be obtained using numerical approximations with INLA. Before describing how to fit mixture models with INLA, we describe the INLA methodology in the next Section. Furthermore, collapsed Gibbs sampling (Liu 1994) can be used if the remainder of the parameters in \( \theta \), denoted by \( \theta_{\bullet \bullet} \), are integrated out in the full conditional of \( \theta_\bullet \).

Figure 2 shows the steps to run the collapsed Gibbs sampling algorithm using the full
conditionals derived before. After a suitable burn-in time, Gibbs sampling will provide
(correlated) draws from the joint posterior distribution of \((w, z, \theta)\). Posterior inference will
be based on these samples, which can be used to obtain estimates of the posterior marginal
distributions, summary statistics, etc.

1. Assign initial values to \(\theta\) and \(z\): \(\theta^{(0)}, z^{(0)}\).

2. For \(l = 1, 2, \ldots\), repeat:
   
   (a) Sample \(w^{(l)}\) from \(\pi(w|y, z^{(l-1)}, \theta^{(l-1)})\):

   \[
   \pi(w|y, z^{(l-1)}) = \text{Dirichlet}(\alpha_1 + n_1^{(l-1)}, \ldots, \alpha_K + n_K^{(l-1)})
   \]

   (b) Sample \(z_i^{(l)}\) from \(\pi(z_i|y, w^{(l-1)}, \theta^{(l-1)})\):

   \[
   \pi(z_i = j|w, \theta) \propto w_j f_j(y_i | \theta_j); \ i = 1, \ldots, n; \ j = 1, \ldots, K
   \]

   (c) Sample \(\theta^{(l)}\) from \(\pi(\theta|y, z^{(l)})\).

Figure 2: Collapsed Gibbs sampling algorithm to fit a mixture model.

3 Integrated Nested Laplace Approximation

Rue et al. (2009) describe a novel approach for Bayesian model fitting that focuses on
obtaining good approximations to the posterior marginals of the model parameters. In
addition, they only consider models that can be expressed as a latent Gaussian Markov
random field (GMRF) because of their important computational properties (for details, see,
The approximation to the posterior marginals is based on repeated use of the Laplace approximation and, hence, it has been termed Integrated Nested Laplace Approximation (INLA).

This method can be summarized as follows. We will consider a vector of observations \( y = (y_1, \ldots, y_n) \) whose elements have distributions family which depend on a vector of parameters \( \theta_1 \). The means \( \mu_i \) of these distributions are conveniently linked to a linear predictor that depends on a number of latent effects \( x \), which is a GMRF that depends on hyperparameters \( \theta_2 \). The ensemble of hyperparameters will be denoted by \( \theta = (\theta_1, \theta_2) \). Elements in \( x \) include the linear predictor that is linked to the mean \( \mu_i \), as well as the coefficients of the fixed effects and other types of latent effects.

INLA assumes that the observations are independent given the latent effects and the hyperparameters:

\[
\pi(y|x, \theta) = \prod_{i \in I} \pi_{l(i)}(y_i|x_i, \theta)
\]

In the previous equation, \( I \) is a index of observed responses and \( x_i \) are the elements of \( x \) that represent the linear predictor. The likelihood associated to observation \( i \) is defined by index \( l(i) \). This allows for several observations to have the same likelihood. Hence, the joint posterior distribution can be written as follows:

\[
\pi(x, \theta|y) \propto \pi(y|x, \theta) \pi(x, \theta) = \pi(y|x, \theta) \pi(x|\theta) \pi(\theta) = \pi(x|\theta) \pi(\theta) \prod_{i \in I} \pi_{l(i)}(y_i|x_i, \theta).
\]

Given that \( x \) is a GMRF, equation (5) can be rewritten as
\[ \pi(x, \theta | y) \propto \pi(\theta) | Q(\theta) |^{n/2} \exp \left\{ -\frac{1}{2} x^T Q(\theta) x + \sum_{i \in I} \log \left( \pi_{I(i)}(y_i | x_i, \theta) \right) \right\}. \]  (6) \{eq2\}

Here, \( Q(\theta) \) is the precision matrix of \( x \).

INLA starts by finding a good approximation to \( \pi(\theta | y) \), \( \tilde{\pi}(\theta | y) \), that can be used to approximate \( \pi(x_j | y) \) because it can be written down as:

\[ \pi(x_j | y) = \int \pi(x_j | \theta, y) \pi(\theta | y) d\theta. \]  (7)

The approximation provided by INLA is

\[ \tilde{\pi}(x_j | y) = \sum_g \tilde{\pi}(x_j | \theta_g, y) \times \tilde{\pi}(\theta_g | y) \times \Delta_g. \]  (8)

In the previous equation, \( \tilde{\pi}(x_j | \theta_g, y) \) is an approximation to \( \pi(x_j | \theta_g, y) \). Rue et al. (2009) describe different ways in which this approximation can be obtained. \( \theta_g \) is a vector with values of \( \theta \) over a grid, with associated weights \( \Delta_g \).

INLA is implemented as a package for the R statistical software (R Core Team, 2016) called R-INLA. This package implements a number of latent effects and allows for an easy model fitting and visualization of the output. A recent review on INLA and the R-INLA package can be found in Rue et al. (2017).

As stated before, INLA (and R-INLA) can handle different likelihoods. We will be using this feature to define and fit mixture models with INLA (see Section 4). Furthermore, an approximation to the marginal likelihood is also provided, so that model selection (Gómez-Rubio et al., 2017) and Bayesian model averaging (Bivand et al., 2014) can be carried out. This approximation is based on

\[ \tilde{\pi}(y) = \int \frac{\pi(\theta, x, y)}{\tilde{\pi}_G(x | \theta, y)} \bigg|_{x=x^*(\theta)} d\theta. \]
Hubin and Storvik (2016) and Gómez-Rubio and Rue (2017) show that this approximation is very accurate on a wide range of models.

4 Fitting Mixture Models with INLA

We will start by noting that, given \( z \), INLA can be used to fit the resulting model because all observations are assigned to a particular group in the mixture. Hence, given \( z \), the model can be expressed as a model with several likelihoods. In this case, the approximations provided by INLA are for (conditional) posterior marginals \( \pi(\theta_\bullet | y, z) \), with \( \theta_\bullet \in \Theta \). In addition, the (conditional) marginal likelihood obtained is \( \pi(y | z) \).

Hence, the posterior marginals of the parameters in the mixture model can be obtained as follows:

\[
\pi(\theta_\bullet | y) = \sum_{z \in Z} \pi(\theta_\bullet | y, z = z) \pi(z = z | y), \quad \theta_\bullet \in \Theta
\]

Here, \( Z \) is the parameter space of the of the auxiliary variables, which is the \( n \)-dimensional Cartesian product of \( \{1, \ldots, K\} \).

If an approximation \( \hat{\pi}(z | y) \) to \( \pi(z | y) \) is available, the posterior marginal of \( \theta_\bullet \) can be approximated as

\[
\pi(\theta_\bullet | y) \approx \sum_{z \in Z} \hat{\pi}(\theta_\bullet | y, z = z) \hat{\pi}(z = z | y)
\]

Approximation \( \hat{\pi}(z | y) \) can be obtained in a number of ways. In the next Section, it will be described how Gibbs sampling can be used together with INLA to obtain the posterior of \( z \).
4.1 Gibbs sampling with INLA

Gibbs sampling can be implemented using the approximations to the conditional marginals of $\theta_*$ provided by INLA. This algorithm is described below and it will follow the lines described in Section 2.3 with the difference that some of the steps will be done with some of the output provided by INLA.

First of all, it is worth noting that instead of providing an approximation to full conditional of $\theta_*$, $\pi(\theta_*|y, z, w, \theta_{-*})$, INLA provides an approximation to $\pi(\theta_0|y, z, w)$. However, using the approximation to the former distribution can be regarded as using a collapsed Gibbs sampling (Liu, 1994) because $\pi(\theta_0|y, z, w)$ is obtained by integrating $\theta_{-*}$ out:

$$\pi(\theta_0|y, z, w) = \int \pi(\theta|y, z, w) d\theta_{-*}$$

This simplifies model fitting as the approximation is computed by INLA, which also allows for a wide range of distributions in the mixture components and non-conjugate priors when defining the models. This is summarized in Figure 3.

4.2 ’Modal’ Gibbs sampling with INLA

As stated previously, in order to fit a mixture model with INLA only the posterior of $z$ is required. We now propose a new sampling algorithm which can be regarded as a collapsed Gibbs sampling (Liu, 1994) in the sense that some of the parameters are integrated out in the full conditionals and not all parameters in the model are sampled. During Gibbs sampling only new values of $z$ will be sampled given the (conditional) modes of $w$ and $\theta$, i.e., samples from $w$ and $\theta$ are replaced by the modes of their respective full conditional distributions.

The full conditional distribution of $w$ is a Dirichlet distribution with parameters $(\alpha_1 +$
1. Assign initial values to $\theta$ and $z$: $\theta^{(0)}$, $z^{(0)}$.

2. For $l = 1, 2, \ldots$, repeat:
   
   (a) Sample $w^{(l)}$ from $\pi(w|y, z^{(l-1)}, \theta^{(l-1)})$:
   
   $$\pi(w|y, z^{(l-1)}) = \text{Dirichlet}(\alpha_1 + n_1^{(l-1)}, \ldots, \alpha_K + n_K^{(l-1)})$$

   (b) Sample $z_i^{(l)}$ from $\pi(z_i|y, w^{(l)}, \theta^{(l-1)})$:
   
   $$\pi(z_i = j|w_j, \theta) \propto w_j f_j(y_i|\theta_j); \ i = 1, \ldots, n; \ j = 1, \ldots, K$$

   (c) Fit model with INLA to approximate conditional marginals $\pi(\theta_\bullet|y, z^{(l)})$, $\theta_\bullet \in \theta$.

   (d) Sample $\theta^{(l)}$ from $\pi(\theta_\bullet|y, z^{(l)})$ using the approximated conditional marginal obtained with INLA.

Figure 3: Gibbs sampling to fit mixture models with INLA.

$n_1, \ldots, \alpha_K + n_K$, which has a mode at $((\alpha_1 + n_1 - 1)/n^*, \ldots, (\alpha_K + n_K)/n^*)$, with $n^* = n + \sum_{i=1}^{K} \alpha_i - K$. For this reason, to make sure that the model is well defined it is better to take $\alpha_1 = \ldots = \alpha_K$ equal to 2 (for example) instead of 1 in order to propose a vague prior. The modes of the conditional distributions of $\theta$ used in the Gibbs sampling algorithm are provided by numerical approximation of INLA and these can be directly used.

Hence, the sampling process can be simplified as in Figure 4.
1. Assign initial values to $z$: $z^{(0)}$.

2. For $l = 1, 2, \ldots$, repeat:

   (a) Fit model (conditional on $z^{(l-1)}$) with INLA to approximate conditional marginals $\pi(\theta_\bullet | y, z^{(l-1)}), \theta_\bullet \in \theta$.

   (b) Obtain (conditional) modes of $\mathbf{w}$ and $\theta$: $\hat{\mathbf{w}}^{(l)}$ and $\hat{\theta}^{(l)}$.

   (c) Sample $z_i^{(l)}$ from $\pi(\cdot | y, \hat{\mathbf{w}}^{(l)}, \hat{\theta}^{(l)})$:

   $$\pi(z_i = j | y, \hat{\mathbf{w}}^{(l)}, \hat{\theta}^{(l)}) \propto \hat{w}_j f_j(y_i | \hat{\theta}_j); \quad i = 1, \ldots, n; \quad j = 1, \ldots, K$$

Figure 4: ‘Modal’ Gibbs sampling algorithm to fit mixture models with INLA.

4.3 Accuracy of ‘model’ Gibbs sampling

The critical point in this approach is the reliability of ‘modal’ Gibbs sampling to obtain a good approximation to $\pi(z | y)$. This essentially means that all assignments with non-negligible probability are explored and that their probabilities are estimated with accuracy.

Let us denote by $Z^* \subseteq Z$ the subset of all possible assignments explored by ‘modal’ Gibbs sampling. Because of the structure of ‘modal’ Gibbs sampling in which different assignments are simulated by conditioning on the modes of the parameters of the distributions of the components, we believe that the parameter space $Z$ is conveniently explored as this mimics the data generating process. All the assignments left out are likely to have a very small probability of occurring under the data generating process (see, Porteous et al. 2008, for a similar discussion).

INLA can be used to assess that parameter space $Z$ has been conveniently explored.
Assuming that only assignments in $\mathcal{Z}^*$ have a non-negligible probability, the posterior probability of $\pi(z|y)$ can be computed as

$$\pi(z|y) \propto \pi(y|z)\pi(z); \ z \in \mathcal{Z}^*$$

Here, $\pi(y|z)$ is the marginal likelihood, for which INLA provides a good approximation. Hence, the posterior probabilities can be computed as

$$\pi(z|y) \simeq \tilde{\pi}_I(z|y) = \frac{\tilde{\pi}(y|z)\pi(z)}{\sum_{z \in \mathcal{Z}} \tilde{\pi}(y|z = z)\pi(z = z)} \quad (9)$$

\{eq:postz\}

Note that this approximation is easy and fast to compute from the conditional models fitted during ‘modal’ Gibbs sampling. Hence, this can be computed and compared to the probabilities obtained with the samples from ‘modal’ Gibbs sampling, $\hat{\pi}_G(z|y)$. If both estimates of the posterior probabilities do not match then there is reason to suspect that ‘modal’ Gibbs sampling has not explored the parameter space $\mathcal{Z}$ conveniently. In this case, the posterior probabilities computed as in equation (9) can be used to approximate the posterior distribution of $z$.

All the previous algorithms will provide an approximation to the posterior of $z$, and a set of conditional posterior marginal distributions $\pi(\theta_\bullet|z,y)$ for the model parameters $\theta_\bullet \in \theta$. The actual posterior marginals of the model parameters can be approximated as

$$\pi(\theta_\bullet|y) \simeq \sum_{z \in \mathcal{Z}} \tilde{\pi}(\theta_\bullet|z = z,y)\tilde{\pi}(z = z|y) \quad (10)$$

where $\hat{\pi}(z = z|y)$ is the approximation to $\pi(z = z|y)$ provided by Gibbs sampling, modal Gibbs sampling or the one computed using the approximations to the conditional marginal likelihoods provided by INLA, as in equation (9) above.
The full conditional distribution of $\mathbf{w}$ is the same as in Section 2, i.e., a Dirichlet distribution. Similarly as before, the posterior distribution of $\mathbf{w}$ can be expressed as

$$\pi(\mathbf{w}|\mathbf{y}) = \sum_{z \in \mathcal{Z}} \pi(\mathbf{w}|z, \mathbf{y}) \pi(z = z|\mathbf{y}),$$

which can be approximated as

$$\pi(\mathbf{w}|\mathbf{y}) \simeq \sum_{z \in \mathcal{Z}} \pi(\mathbf{w}|z, \mathbf{y}) \hat{\pi}(z = z|\mathbf{y})$$

Note that the previous distribution is a mixture of Dirichlet distributions, so it can be computed very efficiently.

Griffiths and Steyvers (2004) have proposed a similar algorithm to assign documents to a mixture of topics. However, in their case they are able to derive the full conditional of $z_i$ on $z_{-i}$ and $\mathbf{y}$, so there is no need to deal with parameters $\mathbf{\theta}$ during Gibbs sampling. Once they have obtained the posterior distribution on the auxiliary variables $\mathbf{z}$ they are able to compute the posterior distribution on the remainder of the model parameters by averaging over $\mathbf{z}$ using its posterior distribution.

Porteous et al. (2008) follow a similar approach by they develop a fast algorithm by arguing that, for a given document, posterior probabilities will be concentrated on a number of topics. Hence, they first explore the set of topics fast and then refine the estimated posterior probabilities.

Finally, the marginal likelihood of the model can also be expressed in a similar way:

$$\pi(\mathbf{y}) = \sum_{z \in \mathcal{Z}} \pi(\mathbf{y}|z = z) \pi(z = z).$$

Given that $\pi(\mathbf{y}|\mathbf{z})$ is the conditional marginal likelihood, which can be approximated with INLA, and $\pi(\mathbf{z})$ is the prior distribution of $\mathbf{z}$, so the marginal likelihood is easy to compute.
Hence, it can be approximated as

$$\pi(y) \simeq \tilde{\pi}_I(y) = \sum_{z \in Z} \tilde{\pi}(y|z = z) \pi(z = z). \tag{11}$$

Note that $\pi(z)$ is (see, for example Griffiths and Steyvers, 2004):

$$\pi(z) = \int_{\mathcal{W}} \pi(z|w)\pi(w)dw = \frac{\Gamma(\sum_{i=1}^{K} \alpha_i) \prod_{i=1}^{K} \Gamma(n_i + \alpha_i)}{\prod_{i=1}^{K} \Gamma(\alpha_i) \Gamma(n + \sum_{i=1}^{K} \alpha_i)}$$

Here, $\Gamma(\cdot)$ is the Gamma function and $\mathcal{W}$ is the parametric space of the weights vector $w$.

5 Selecting the number of components

Finding the number of components is an important problem in mixture models. From a Bayesian perspective, this implies computing the posterior probabilities of the number of components over a range of values. The problem of choosing the number of components in the mixture can be regarded as a model selection problem as well, as discussed below.

Given models $\mathcal{M}_1, \ldots, \mathcal{M}_p$, we can identify each model by its number of components (i.e., $\mathcal{M}_1$ has one component, $\mathcal{M}_2$ has two components, and so on) and assign them prior probabilities $\pi(\mathcal{M}_k), \ k = 1, \ldots, p$. Posterior probabilities can be derived as:

$$\pi(\mathcal{M}_k|y) \propto \pi(y|\mathcal{M}_k)\pi(\mathcal{M}_k)$$

Here, $\pi(y|\mathcal{M}_k)$ is the marginal likelihood of model $\mathcal{M}_k$ and the actual posterior probabilities can be computed as follows:

$$\pi(\mathcal{M}_k|y) = \frac{\pi(y|\mathcal{M}_k)\pi(\mathcal{M}_k)}{\sum_{k=1}^{p} \pi(y|\mathcal{M}_k)\pi(\mathcal{M}_k)} \tag{12}$$
The marginal likelihood \( \pi(y|M_k) \) of a given model \( M_k \) can be approximated with INLA when the model is completely fitted with it. However, in the case of mixture models, this is not possible because INLA is combined with MCMC. For this reason, we need to resort to other methods to compute the marginal likelihood.

Chib (1995) and Chib and Jeliazkov (2001) describe several approaches to compute the marginal likelihood of a model using the MCMC output. They note that the marginal likelihood is the scaling constant when writing the posterior distribution using Bayes’ rule. For mixture models, this can be written by conditioning on \( z \) as

\[
\pi(y) = \frac{\pi(y|z)\pi(z)}{\pi(z|y)} \tag{13} \]

Note that this holds for any \( z \) but, as pointed out by Chib (1995) and Chib and Jeliazkov (2001), values with a high posterior probability (or density, if \( z \) is a continuous random variable) are preferred since \( \pi(z|y) \) will be away from zero. In this case, posterior mode \( z^m \) will be used to compute the log-marginal likelihood as:

\[
\log(\pi(y)) = \log(\pi(y|z^m)) + \log(\pi(z^m)) - \log(\pi(z^m|y)) \tag{14} \]

Note that the first and second terms in the right hand side are the conditional marginal likelihood (for which an accurate approximation is provided by INLA) and the prior evaluated at the posterior mode of \( z \), respectively. The last term needs to be obtained from the MCMC output. In practice, the following approximation will be used:

\[
\log(\pi(y)) \simeq \log(\tilde{\pi}(y|z^m)) + \log(\pi(z^m)) - \log(\hat{\pi}(z^m|y)) \tag{15} \]

Note that the last term in equation (15) can be approximated in different ways, that will lead to different approximations to \( \pi(y) \). If it is the estimate provided by modal Gibbs
sampling then the approximation to the marginal likelihood will be denoted by \( \hat{\pi}_G(y) \). If the last term is computed using the conditional marginal likelihoods provided by INLA, as in equation (9), the approximation will be denoted by \( \tilde{\pi}_M(y) \).

In addition to these two ways of approximating the marginal likelihood, equation (11) provides another way to estimate the marginal likelihood, denoted by \( \tilde{\pi}_I(y) \). Computing these three different estimates can be a way to perform an assessment of whether the parametric space \( Z \) has been conveniently explored and the posterior probabilities of its elements estimated with accuracy. In particular, \( \hat{\pi}_G(y) \) will be very sensitive to this issue and large differences with \( \tilde{\pi}_M(y) \) and \( \tilde{\pi}_I(y) \) will indicate that the posterior probabilities provided by modal Gibbs sampling may not be accurate.

Estimates of the posterior probability of a given model, as defined in equation (13), can be obtained with any of the three estimates of the marginal likelihood and they will be denoted similarly. For example, \( \hat{\pi}_G(M_i|y), \ i = 1, \ldots, p \) are the estimates of the posterior probabilities of the different models based on estimates \( \{\hat{\pi}_G(y|M_k]\}_{k=1}^p \) of their marginal likelihoods.

## 6 Simulation Study

In order to assess the performance of the methods presented to fit mixture models with INLA we have conducted a simulation study. The aim of this study is twofold, as we are interested in estimating both the actual number of components and the posterior distributions of the model parameters. Furthermore, in this simulation study we have considered mixtures of Gaussian and Poisson distributions.
6.1 Gaussian data

The first dataset comprises simulated observations from three Gaussian distributions centered at 0, 5 and 10, respectively, with precision one. We have generated 50 observations from each Gaussian distribution to obtain the final dataset. Note that this will produce a mixture with overlapping distributions between groups and, hence, provides a good testing framework. Mixture models with up to 5 components will be fitted to the data to assess model choice using the marginal likelihood and the estimation of the model parameters. A density estimate of the simulated data can be found in Figure 5 (left plot).

In principle, we will not assume that all \( K \) components have the same variances and the components will be defined by \( K \) Gaussian distributions with different means and variances. Furthermore, we will compare the results obtained with our current approach to those obtained with MCMC with the JAGS software (Plummer, 2016). For this, we have used a burn-in of 200 iterations plus other 10000 iterations, of which we have only have kept one every ten, so that a final 1000 samples have been used for inference. The means of the Gaussian distributions in the mixture have been ordered in increasing order at every step of the sampling process with JAGS in order to reclass the assignment to the mixture components to reduce label switching.

In order to define the full Bayesian model, we have used a Gaussian prior on the means with zero mean and precision 1/1000. For the precisions, we have used a Gamma distribution with parameters 0.5 and 0.5. Also, all models have been assigned the same probability a priori.

Table 1 shows the results of the simulation study for the Gaussian data. The marginal likelihoods favor the model with 3 components, which is the actual number of components in the mixture. All three methods to approximate the marginal likelihood give very similar values and their respective posterior probabilities agree to choose the model with 3
Figure 5: Simulated data from a mixture of three Gaussian distributions (left) and three Poisson distributions (right).

Table 1: Results of simulation study using a mixture of Gaussian distributions. Symbol ∗ means that model $\mathcal{M}_1$ has entirely been computed with INLA as it has a single component. $\hat{\pi}(\mathcal{M}_j|y)$ is the estimate of the posterior probability of the models, which is the same regardless of the approximation to the marginal likelihood used. Parameter estimates are summarized using posterior mean and standard deviation (between parentheses).

| Model | $\log(\tilde{\pi}_I(y))$ | $\log(\hat{\pi}_G(y))$ | $\log(\tilde{\pi}_M(y))$ | $\hat{\pi}(\mathcal{M}_j|y)$ | Parameter estimates |
|-------|----------------|----------------|----------------|----------------|-------------------|
| $\mathcal{M}_1^*$ | -431.87 | -431.87 | -431.87 | 0.00 | $\mu$ | 5.02 (0.34) |
| | | | | | $\tau$ | 0.06 (0.01) |
| $\mathcal{M}_2$ | -415.73 | -417.75 | -417.87 | 0.00 | $\mu$ | 2.88 (0.33) |
| | | | | | $\tau$ | 1.34 (0.33) |
| $\mathcal{M}_3$ | -385.28 | -386.74 | -386.72 | 1.00 | $\mu$ | 1.43 (0.29) |
| | | | | | $\tau$ | 1.13 (0.25) |
| $\mathcal{M}_4$ | -392.69 | -395.38 | -395.06 | 0.00 | $\mu$ | -0.41 (0.61) |
| | | | | | $\tau$ | 7.2 (114.57) |
| $\mathcal{M}_5$ | -406.82 | -464.62 | -408.11 | 0.00 | $\mu$ | -0.82 (0.78) |
| | | | | | $\tau$ | 102.88 (108.72) |

components. Also, this model provides very good estimates of the means and precisions of the Gaussian distributions in the mixture. Figure 6 shows the posterior marginals of the
model parameters obtained with MCMC and INLA (by Bayesian model averaging on the conditional posterior marginals). In all cases the agreement is quite high, which confirms that our approach provides similar estimates to MCMC.

![Density plots](image)

Figure 6: Estimates of the model parameters for the mixture with three components (Gaussian mixture).
6.2 Poisson data

A similar simulation study based on a mixture of Poisson distributions has been conducted.
A mixture of three Poisson distributions with means 1, 15 and 45, respectively, has been used and 50 observations have been simulated from each distribution. A histogram of the simulated data can be found in Figure 5 (right plot). Similarly as in the Gaussian case, mixture models up to 5 components have been fitted to the data. Model fitting has also been done using MCMC with the JAGS software, for which we have used 200 burn-in iterations, plus 1000 iterations for inference (after keeping one in 10 iterations from the total 10000 iterations).

Table 2: Results of simulation study using a mixture of Gaussian distributions. Symbol * means that model $\mathcal{M}_1$ has entirely been computed with INLA as it has a single component. $\hat{\pi}(\mathcal{M}_j|y)$ is the estimate of the posterior probability of the models, which is the same regardless of the approximation to the marginal likelihood used. Parameter estimates are summarized using posterior mean and standard deviation (between parentheses).

| Model | $\log(\hat{\pi}_I(y))$ | $\log(\hat{\pi}_G(y))$ | $\log(\hat{\pi}_M(y))$ | $\hat{\pi}(\mathcal{M}_j|y)$ | Parameter estimates |
|-------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $\mathcal{M}_1$ | -1756.54 | -1756.54 | -1756.54 | 0.00 | $\lambda$ 3.01 (0.02) |
| $\mathcal{M}_2$ | -813.59 | -815.49 | -815.47 | 0.00 | $\lambda$ 7.22 (0.31) 42.70 (0.99) |
| $\mathcal{M}_3$ | -548.09 | -549.11 | -549.10 | 1.00 | $\lambda$ 1.09 (0.15) 14.79 (0.54) 44.71 (0.94) |
| $\mathcal{M}_4$ | -553.60 | -554.60 | -554.95 | 0.00 | $\lambda$ 1.17 (52.09) 1.23 (0.55) 14.79 (0.54) 44.71 (0.94) |
| $\mathcal{M}_5$ | -563.22 | -610.21 | -564.26 | 0.00 | $\lambda$ 0.00 (0.00) 1.10 (0.57) 1.31 (0.49) 14.79 (0.54) 44.71 (0.94) |

Table 2 summarizes the results obtained when fitting the different mixture models to the data. Similarly as in the previous example, all three methods to compute the marginal likelihoods provide very similar values, which lead to very similar posterior probabilities of the models. The preferred mixture model is clearly the one with three components. Figure 7 shows the posterior marginal distributions of the means of the different components for the mixture with three components. Again, there is a very good agreement between the
estimates provided by INLA and MCMC.

![Graph showing density estimates for λ₁, λ₂, and λ₃](image)

Figure 7: Estimates of the model parameters for the mixture with three components (Poisson mixture).

### 6.3 Final remarks

Although the simulation study only considers Gaussian and Poisson examples, it shows that the proposed approach based on modal Gibbs sampling to fit mixture models with INLA is feasible and works well in practice. The estimates of the posterior marginals of the model parameters obtained with modal Gibbs sampling match those obtained with MCMC. The selection of the number of components in the mixture has also been tackled by providing a new way to estimate the marginal likelihood of a given mixture model. In the simulation study this criterion selects the model with the same number of components used to simulate the data.

However, fitting mixture models is hard in practice [Celeux et al., 2000]. In particular, if the number of components is not specified correctly most algorithms will struggle. For example, when the number of components is higher than three in the simulation study it can be seen how the components with the highest means are accurately estimated, whilst the components with lower means are not properly estimated. This can happen because
the algorithm converges to a local optimum. When we fit a model with more than the actual number of components (three, in our examples) some components have a mean close to zero. We believe that this is because there is not enough data allocated to components with the lowest means and that the prior (centered at zero) has a stronger effect on the posterior marginal.

When the fitted model has less components than the actual model, then some groups will be merged together in one component. Usually, this will make observations coming from ‘neighbor’ components being grouped together. Hence, there is a potential identifiability problem because it is not clear which components will be merged together. When the number of components in the model is larger than its actual number, then some of the components may not have enough observations allocated as to obtain good estimates of the parameters of that component.

Identifiability problems are often observed as label switching during model fitting. However, we have not observed this problem when fitting the models in the simulation study. We also believe that using conditional modes for the model parameters during modal Gibbs sampling protects from label switching because the components are better identified. Although this sampling framework may also lead to a loss of variability in the samples obtained of the auxiliary variables, the results in the simulation study suggest that it is not the case and the parametric space of the auxiliary variables is conveniently explored.

7 Examples

7.1 Gaussian mixture models: galaxy data

Carlin and Chib (1995) and Chib (1995) have studied the velocities of 82 galaxies in the
Corona Borealis region using mixture models. They have fitted mixture models with different components to try to estimate the optimal number of components in the mixture using different methods. Carlin and Chib (1995) compute the posterior probability of each model using MCMC methods, whilst Chib (1995) focuses on computing the marginal likelihood to derive Bayes factors for model choice. We will fit mixture models with a different number of components in order to obtain the parameter estimates and the marginal likelihood of each model, so that model choice can be performed.

Figure 8 shows a kernel density estimate of the data. Three groups seem to appear in the data. Hence, it makes sense to fit a mixture model to the data. In order to consider several models, mixture models with up to 4 components have been fitted to the data.

In this case, all models are a mixture of Gaussian distributions. We have considered a vague Gaussian prior (with zero mean and precision 0.001) for the means, and a Gamma (with parameters 0.5 and 0.5) for the precisions. Furthermore, all precisions have been considered to be equal, i.e., all components have the same precision (as in Carlin and Chib, 1995). For all models, simulations included 200 burn-in iterations followed by other 1000 iterations for inference, obtained after thinning (by keeping one in ten iterations).

Table 3 summarizes the results obtained after fitting the different mixture models. The values of the marginal likelihood with all three methods are very similar, as well as the posterior probabilities. In addition, Chib (1995) report the marginal likelihoods of the model with two and three components and these values are very similar to the ones shown in Table 3. The small discrepancies may be related to the choice of the priors. Furthermore, Carlin and Chib (1995) report point estimates of the model parameters for the models with 3 and 4 components, which are very similar to the ones in Table 3 as well. In this case, the results favor the mixture model with three components. Hence, modal Gibbs sampling seems to be a valid approach to fit Gaussian mixture models given that similar results to
other published bibliography have been obtained.

7.2 Non-Gaussian mixture models: earthquake data

The next example will consider distributions of non-Gaussian mixtures. In particular, the yearly number of major earthquakes (magnitude 7 or greater) from 1990 to 2006 has been considered. Figure 9 shows the actual dataset and a histogram of the yearly counts. This dataset has been analyzed in Zucchini et al. (2016) where it is suggested that this dataset could be analyzed using a mixture model given its overdispersion. Given that now the response variable represents counts, it makes sense to use a mixture of Poisson
Table 3: Summary of results for the different mixture models fitted on the Galaxies dataset. Symbol * means that model $\mathcal{M}_1$ has entirely been computed with INLA as it has a single component. $\hat{\pi}(\mathcal{M}_j|y)$ is the estimate of the posterior probability of the models, which is the same regardless of the approximation to the marginal likelihood used. Parameter estimates are summarized using posterior mean and standard deviation (between parentheses).

| Model | $\log(\hat{\pi}_I(y))$ | $\log(\hat{\pi}_G(y))$ | $\log(\hat{\pi}_M(y))$ | $\hat{\pi}(\mathcal{M}_j|y)$ | $\mu_1$ | $\mu_2$ | $\mu_3$ | $\mu_4$ | $\tau$ |
|-------|----------------|----------------|----------------|----------------|--------|--------|--------|--------|--------|
| $\mathcal{M}_1$ | -248.09* | -248.09* | -248.09* | 0.00 | 20.82 (0.51) | – | – | – | 0.05 (0.007) |
| $\mathcal{M}_2$ | -244.62 | -245.70 | -245.65 | 0.00 | 9.85 (1.21) | 21.87 (0.36) | – | – | 0.11 (0.017) |
| $\mathcal{M}_3$ | -233.84 | -234.99 | -234.87 | 1.00 | 9.75 (0.82) | 21.40 (0.25) | 32.89 (1.28) | – | 0.23 (0.036) |
| $\mathcal{M}_4$ | -239.53 | -243.35 | -241.93 | 0.00 | 9.71 (0.53) | 20.06 (0.30) | 23.47 (0.38) | 33.02 (0.82) | 0.53 (0.118) |

Figure 9: Number of yearly major earthquakes from 1900 to 2006.

For this reason, different mixture models with up to four components have been fitted to the data. The prior on the mean of each group is a vague Gaussian distribution, with
zero mean and precision equal to 0.001. The results are summarized in Figure 4. Modal Gibbs sampling seems to provide a different estimate to the the marginal likelihood to the other two methods. This may indicate that posterior probabilities of the auxiliary variables may be unreliable in this case. However, all estimates seem to favor the model with two components and this is the one with the highest posterior probabilities in all cases.

*Zucchini et al.* (2016) report results where the chosen model is the one with three components. However, they use a parameterization of the model, as well as different priors on the mean of the groups. This may be the reason why model choice differs. Point estimates of the means seem to be very similar to those reported in *Zucchini et al.* (2016) though.

Table 4: Summary of results for the different mixture models fitted on the earthquakes dataset. Symbol * means that model $\mathcal{M}_1$ has entirely been computed with INLA as it has a single component. $\hat{\pi}(\mathcal{M}_j|y)$ is the estimate of the posterior probability of the models, which is the same regardless of the approximation to the marginal likelihood used. Parameter estimates are summarized using posterior mean and standard deviation (between parentheses).

| Model | $\log(\hat{\pi}_I(y))$ | $\log(\hat{\pi}_G(y))$ | $\log(\hat{\pi}_M(y))$ | $\hat{\pi}_I(\mathcal{M}_j|y)$ | $\hat{\pi}_G(\mathcal{M}_j|y)$ | Parameter estimates |
|-------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $\mathcal{M}_1^*$ | -399.18 | -399.18 | -399.18 | 0.00 | 1.00 | $\mu_1$, $\mu_2$, $\mu_3$, $\mu_4$ |
| $\mathcal{M}_2$ | -387.56 | -413.45 | -388.60 | 1.00 | 0.00 | 15.67 (0.73) 26.77 (1.44) |
| $\mathcal{M}_3$ | -402.51 | -415.91 | -403.56 | 0.00 | 0.00 | 12.78 (1.47) 19.68 (1.39) 30.97 (2.41) |
| $\mathcal{M}_4$ | -409.98 | -479.66 | -411.66 | 0.00 | 0.00 | 0.00 (0.00) 13.29 (2.26) 19.80 (1.60) 30.95 (2.47) |
8 Discussion

This paper introduces a novel approach to fit mixture models based on obtaining first the posterior distribution of the latent auxiliary variables using a variation of Gibbs sampling. Then, the conditional (on the auxiliary variables) posterior marginals of the remainder of the parameters in the mixture model are obtained with the Integrated Nested Laplace Approximation. Finally, the posterior marginals of these parameters are obtained by Bayesian model averaging of their conditional posterior marginals. Because INLA is used to fit the conditional models, the distributions of the components can be taken from a wide range of distributions, conjugate priors are not required and label switching does not seem to be a problem.

This novel approach also provides a simple way to compute the posterior probabilities of the number of components, by exploiting the marginal likelihood provided by INLA and the output from sampling. The simulation study carried out supports that this is a valid inferential framework for mixture models. The results in the simulation study and the comparison with MCMC included in the case studies suggest that the estimates provided are very accurate.

The Gibbs sampling algorithm employed in model fitting only draws samples from the auxiliary variables. For the other parameters, instead of sampling from their full conditional distributions their respective modes are used. The modes are available as part of the output provided by INLA at every step of the Gibbs sampling algorithm when a conditional model on the current value of the auxiliary variables is fitted. This simplifies the model fitting algorithm as well as reduces computing time as actual sampling is reduced. Given the results obtained in the simulation study, this simple approximation to the full conditional of the model parameters appears to be enough to obtain a good estimate of the posterior
distribution of the indicator variables. This algorithm resembles collapsed Gibbs sampling 
(Liu, 1994) because some of the model parameters are integrated out in the full conditionals.

Furthermore, if modal Gibbs sampling is thought to provide biased estimates of the posterior probabilities of the auxiliary variables, then the posterior probabilities can be recomputed with the estimates of the marginal likelihoods provided by INLA. This is similar to what Porteous et al. (2008) have done in the context of the classification of documents by topic. They develop a fast Gibbs sampling algorithm for latent Dirichlet allocation that explores the set of possible topics very quickly and then refine the posterior probabilities because they argue that the probability mass will be concentrated in a small number of topics.

The idea of obtaining estimates of the model parameter by averaging over the indicator variables has been considered by several authors. See, for example, Gelman, Carlin, Stern, Dunson, and Vehtari (Gelman et al., page 523), for a general overview, and Griffiths and Steyvers (2004), for an early application of this idea on the determination of topics covered in documents. However, instead of averaging over single values of the model parameters, this new approach performs Bayesian model averaging on the (conditional) posterior distributions of the model parameters obtained with INLA. Hence, no samples from the model parameters are required, which simplifies model fitting.

Finally, we believe that this approach can be applied to mixture models in a variety of areas. For example, it can be used to fit cure rate models (Berkson and Gage, 1952) in survival analysis, which are expressed as a mixture model with two components. One of the components is defined by a survival model, which can be modeled with INLA as a typical parametric survival distribution or a Cox model. Similarly, mixture models could be fitted to classify areas which follow a shared spatio-temporal risk pattern or which depart from this shared pattern in disease mapping (see, for example, Abellan et al., 2008). Here,
spatial and temporal patterns in a given area can be modeled as the mixture of a shared spatio-temporal pattern and an area-specific one. INLA provides several latent effects to model spatial and temporal effects, and a mixture model would fall within the current paradigm to fit mixture models with INLA.

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