PROJECTIVE FREENESS OF ALGEBRAS OF REAL
SYMMETRIC FUNCTIONS

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Abstract. Let $\mathbb{D}^n := \{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_j| < 1, \ j = 1, \ldots, n \}$, and let $\overline{\mathbb{D}}^n$ denote its closure in $\mathbb{C}^n$. Consider the ring $C_r(\overline{\mathbb{D}}^n; \mathbb{C}) = \{ f : \overline{\mathbb{D}}^n \to \mathbb{C} : f \text{ is continuous and } f(z) = \overline{f(\overline{z})} (z \in \overline{\mathbb{D}}^n) \}$ with pointwise operations, where $\overline{\cdot}$ is used appropriately to denote both (component-wise) complex conjugation and closure. It is shown that $C_r(\overline{\mathbb{D}}^n; \mathbb{C})$ is projective free, that is, finitely generated projective modules over $C_r(\overline{\mathbb{D}}^n; \mathbb{C})$ are free. Let $A$ denote the polydisc algebra, that is, the set of all continuous functions defined on $\mathbb{D}^n$ that are holomorphic in $\mathbb{D}^n$. For $N$ a positive integer, let $\partial^{-N}A$ denote the algebra of functions $f \in A$ whose complex partial derivatives of all orders up to $N$ belong to $A$. We show that each of the real symmetric algebras $\partial^{-N}A_r = \{ f \in \partial^{-N}A : f(z) = \overline{f(\overline{z})} (z \in \overline{\mathbb{D}}^n) \}$ is projective free.

1. Introduction

In this article we will show the projective freeness of some real Banach algebras of real symmetric functions on the polydisc $\overline{\mathbb{D}}^n := \{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_j| \leq 1, \ j = 1, \ldots, n \}$. We begin with the definition of a projective free ring.

Definition 1.1. Let $R$ be a commutative ring with identity. The ring $R$ is said to be projective free if every finitely generated projective $R$-module is free. Recall that if $M$ is a finitely generated $R$-module, then

1. $M$ is called free if $M \simeq R^k$ for some integer $k \geq 0$;
2. $M$ is called projective if there exists an $R$-module $N$ and an integer $m \geq 0$ such that $M \oplus N = R^m$.

In terms of matrices (see [3, Proposition 2.6] or [1, Lemma 2.2]), the ring $R$ is projective free if and only if every idempotent matrix $P$ is conjugate (by an invertible matrix $S$) to a diagonal matrix with 1s and 0s on the diagonal.
that is, for every $m \in \mathbb{N}$ and every $P \in R^{m \times m}$ satisfying $P^2 = P$, there exists an $S \in R^{m \times m}$ such that $S$ is invertible as an element of $R^{m \times m}$ and

$$S^{-1}PS = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}$$

for some $k \geq 0$.

The motivation for considering projective freeness of rings also comes from Control Theory in the context of stabilization of control systems over rings, see for instance [4].

It was shown in 1976 by Quillen and Suslin, independently, that the polynomial ring $\mathbb{F}[x_1, \ldots, x_n]$ with coefficients in a field $\mathbb{F}$ is projective free, settling Serre’s conjecture from 1955 (see [5]).

In the context of a commutative semisimple unital complex Banach algebra $R$, [2, Corollary 1.4] says that the contractability of the maximal ideal space $M(R)$ of $R$, equipped with the weak-$*$ topology, suffices for $R$ to be projective free. In particular, the complex Banach algebra $C(\overline{D^n}; \mathbb{C})$ of complex-valued continuous functions on the polydisc $\overline{D^n}$ is projective free, since its maximal ideal space can be identified with $\overline{D^n}$, which is clearly contractible. Similarly, the polydisc algebra $A$ (the set of all continuous functions defined on $\overline{D^n}$ that are holomorphic in $D^n$, with the usual point-wise operations and the supremum norm) is also projective free.

In Control Theory, rings of real symmetric functions in play a particularly important role. A function $f : D^n \to \mathbb{C}$ is said to be real symmetric if

$$f(z) = \overline{f(\overline{z})} \quad (z \in D^n).$$

(For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $\overline{z} := (\overline{z_1}, \ldots, \overline{z_n})$.) Algebraic-analytic properties (some of which are relevant in control theory) of various real symmetric algebras have also been studied recently for example in [9], [10], [11], [16].

It is known that the projective freeness of the complexification $R \otimes \mathbb{C}$ of an algebra $R$ over the reals does not in general imply the projective freeness of $R$. For example, the ring $C(S^2; \mathbb{C})$ of complex-valued continuous functions on the unit sphere $S^2$ (in $\mathbb{R}^3$) can be shown to be projective free (see for example [5 p. 38]). On the other hand, a consequence of the Hairy Ball Theorem (see for example [7]) is that the ring $C(S^2; \mathbb{R})$ of real-valued continuous functions on $S^2$ is not projective free (see for example [5 p. 30-33]).

In this article, our first main result is that the ring $C_r(D^n; \mathbb{C})$ of real symmetric functions from $C(D^n; \mathbb{C})$, that is,

$$C_r(D^n; \mathbb{C}) = \{ f \in C(D^n; \mathbb{C}) : f(z) = \overline{f(\overline{z})} \ (z \in D^n) \}$$

is projective free. We note that $C_r(D^n; \mathbb{C})$ is a real Banach algebra, and not a complex Banach algebra with the usual operations (for example we have $1 \in C_r(D^n; \mathbb{C})$ but $i \cdot 1$).

**Theorem 1.2.** $C_r(D^n; \mathbb{C})$ is a projective free ring.
We will also show the projective freeness of certain subalgebras of the real symmetric polydisc algebra. In order to state this result, we will first introduce some notation. For \(N\) a positive integer, let \(\partial^{-N}A\) denote the Banach algebra of all functions \(f \in A\) whose complex partial derivatives of all orders up to \(N\) belong to \(A\), with the norm
\[
\|f\|_{\partial^{-N}A} := \sum_{\alpha_1 + \cdots + \alpha_n \leq N} \frac{1}{\alpha_1! \cdots \alpha_n!} \sup_{z \in \mathbb{D}^n} \left| \frac{\partial^{\alpha_1 + \cdots + \alpha_n} f}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}} (z) \right|.
\]
We show that each of the real symmetric algebras
\[
\partial^{-N}A_r = \{ f \in \partial^{-N}A : f(z) = f(\overline{z}) \ (z \in \mathbb{D}^n) \}
\]
is projective free.

**Theorem 1.3.** \(\partial^{-N}A_r\) is a projective free ring.

We give the proofs of these two main results, namely Theorem 1.2 and Theorem 1.3, in Section 2 and Section 3, respectively.

### 2. Proof of Theorem 1.2

We will first recall the following well-known result from vector bundle theory.

**Proposition 2.1.** \(C([-1, 1]^n; \mathbb{R})\) is a projective free ring.

**Proof.** If \(Q\) is a finitely generated projective module over \(C([-1, 1]^n; \mathbb{R})\), then it can be shown that there exists a vector bundle \(\xi\) such that \(Q \simeq S(\xi)\), where \(S(\xi)\) denotes the \(C([-1, 1]^n; \mathbb{R})\)-module consisting of all cross sections of \(\xi\); see for example [8, p.36, 3-F.(b)]. Moreover, \(S(\xi)\) is free if and only if \(\xi\) is trivial; see [8, p.36, 3-F.(a)]. But as \([-1, 1]^n\) is contractible, \(\xi\) is trivial (see for example [12, Proposition 3.2.1, p.40]).

**Proof of Theorem 1.2.** We will prove by induction on \(k\) (0 \(\leq\) \(k\) \(\leq\) \(n\)) that each of the rings \(C_r([-1, 1]^{n-k} \times \mathbb{D}^k; \mathbb{C})\), that is,
\[
\left\{ f : [-1, 1]^{n-k} \times \mathbb{D}^k \to \mathbb{C} : f \text{ is continuous and } f(z) = \overline{f(\overline{z})} \ (z \in [-1, 1]^{n-k} \times \mathbb{D}^k) \right\},
\]
is projective free.

In order to avoid cumbersome notation, we set
\[
\Delta_k := [-1, 1]^{n-k} \times \mathbb{D}^k.
\]
Then \([-1, 1]^n = \Delta_0 \subset \Delta_1 \subset \cdots \subset \Delta_{n-1} \subset \Delta_n = \mathbb{D}^n\).

For \(k = 0\), the ring \(C_r(\Delta_0; \mathbb{C})\) is precisely \(C([-1, 1]^n; \mathbb{R})\), which is indeed projective free by Proposition 2.1.

Suppose that for some \(k\) such that \(0 \leq k \leq n\), the ring \(C_r(\Delta_k; \mathbb{C})\) is projective free. We want to show that \(C_r(\Delta_{k+1}; \mathbb{C})\) is then also projective free.
To this end, let us consider an idempotent $P$ with entries from $C_r(\Delta_{k+1}; \mathbb{C})$. In particular, the restriction $P|_{\Delta_k}$ is an idempotent with all its entries belonging to $C_r(\Delta_k; \mathbb{C})$. By the induction hypothesis, since $C_r(\Delta_k; \mathbb{C})$ is projective free, there is a $v \in (C_r(\Delta_k; \mathbb{C}))^{m\times m}$ such that $v$ is invertible in $(C_r(\Delta_k; \mathbb{C}))^{m\times m}$, and

$$v^{-1}(z)P(z)v(z) = \begin{bmatrix} I_{\ell} & 0 \\ 0 & 0 \end{bmatrix} \quad (z \in \Delta_k)$$

for some $\ell \geq 0$. Consider the matrix $\tilde{P}$ given as follows: for elements $z$ of

$$\Delta_{k+1}^+ := \left\{ (r, a + ib, \zeta) \in \mathbb{C}^n : r \in [-1, 1]^{n-k-1} \text{ and } a \in \mathbb{R}, b \geq 0, a + ib \in \mathbb{R}, \right\},$$

we set

$$\tilde{P}(z) = (v(r, a, \zeta))^{-1}P(z)v(r, a, \zeta) \quad (z = (r, a + ib, \zeta) \in \Delta_{k+1}^+). \quad (2.1)$$

Then for $z = (r, a + ib, \zeta) \in \Delta_{k+1}^+$, we have

$$(\tilde{P}(z))^2 = (v(r, a, \zeta))^{-1}P(z)v(r, a, \zeta)(v(r, a, \zeta))^{-1}P(z)v(r, a, \zeta)
= (v(r, a, \zeta))^{-1}P(z)v(r, a, \zeta) = \tilde{P}(z).$$

Also, it is clear that $\tilde{P} \in (C(\Delta_{k+1}^+; \mathbb{C}))^{m\times m}$. As $\Delta_{k+1}^+$ is contractible, $C(\Delta_{k+1}^+; \mathbb{C})$ is projective free. Hence we can find a $V \in (C(\Delta_{k+1}^+; \mathbb{C}))^{m\times m}$ which is invertible as an element of $(C(\Delta_{k+1}^+; \mathbb{C}))^{m\times m}$ such that

$$(V(z))^{-1}\tilde{P}(z)V(z) = \begin{bmatrix} I_{\tilde{\ell}} & 0 \\ 0 & 0 \end{bmatrix} \quad (z \in \Delta_{k+1}^+) \quad (2.2)$$

for some $\tilde{\ell} \geq 0$. As the pointwise rank of $P$ matches with that of $\tilde{P}$ on $\Delta_{k+1}^+$, it follows from (2.1) and (2.2) that $\ell = \tilde{\ell}$. Thus (2.2) becomes

$$(V(z))^{-1}\tilde{P}(z)V(z) = \begin{bmatrix} I_{\ell} & 0 \\ 0 & 0 \end{bmatrix} \quad (z \in \Delta_{k+1}^+).$$

In particular, we have for $z = (r, a, \zeta) \in \Delta_k$ that

$$(V(r, a, \zeta))^{-1}\tilde{P}(r, a, \zeta)V(r, a, \zeta) = \begin{bmatrix} I_{\ell} & 0 \\ 0 & 0 \end{bmatrix}. \quad (2.3)$$

But

$$\tilde{P}(r, a, \zeta) = (v(r, a, \zeta))^{-1}P(r, a, \zeta)v(r, a, \zeta) = \begin{bmatrix} I_{\ell} & 0 \\ 0 & 0 \end{bmatrix}.$$ 

Hence from (2.3) and the above, we have

$$(V(r, a, \zeta))^{-1} \begin{bmatrix} I_{\ell} & 0 \\ 0 & 0 \end{bmatrix} V(r, a, \zeta) = \begin{bmatrix} I_{\ell} & 0 \\ 0 & 0 \end{bmatrix} \quad ((r, a, \zeta) \in \Delta_k).$$
By pre- and post- multiplying this with $V(r, a, \zeta)$ and $(V(r, a, \zeta))^{-1}$, respectively, we obtain
\[
\begin{bmatrix}
I_\ell & 0 \\
0 & 0
\end{bmatrix}
= V(r, a, \zeta)
\begin{bmatrix}
I_\ell & 0 \\
0 & 0
\end{bmatrix}
(V(r, a, \zeta))^{-1} \quad ((r, a, \zeta) \in \Delta_k).
\]
Now define $U$ by
\[
U(z) = v(r, a, \zeta)V(z)(V(r, a, \zeta))^{-1} \quad (z = (r, a + ib, \zeta) \in \Delta^+_{k+1}).
\]
First of all, $U \in (C(\Delta^+_{k+1}; \mathbb{C}))^{m \times m}$ and moreover, it is invertible as an element of $(C(\Delta^+_{k+1}; \mathbb{C}))^{m \times m}$. We also have for $z = (r, a + ib, \zeta) \in \Delta^+_{k+1}$ that
\[
(U(z))^{-1}P(z)U(z) = V(r, a, \zeta)(V(z))^{-1}(v(r, a, \zeta))^{-1}P(z)v(r, a, \zeta)V(z)(V(r, a, \zeta))^{-1}
= V(r, a, \zeta)(V(z))^{-1}\tilde{P}(z)V(z)(V(r, a, \zeta))^{-1}
= V(r, a, \zeta)
\begin{bmatrix}
I_\ell & 0 \\
0 & 0
\end{bmatrix}
(V(r, a, \zeta))^{-1} =
\begin{bmatrix}
I_\ell & 0 \\
0 & 0
\end{bmatrix}.
\]
(2.4)

Finally, for $(r, a, \zeta) \in \Delta_k,$
\[
U(r, a, \zeta) = v(r, a, \zeta)V(r, a, \zeta)(V(r, a, \zeta))^{-1} = v(r, a, \zeta),
\]
and so $U|_{\Delta_k}$ belongs to $(C(\Delta_k; \mathbb{C}))^{m \times m}$. Finally for $z = (r, a + ib, \zeta) \in \Delta^+_{k+1},$ we set
\[
S(z) = S(r, a + ib, \zeta) = \begin{cases} 
U(r, a + ib, \zeta) & \text{if } b \geq 0 \\
\overline{U(r, a - ib, \zeta)} & \text{if } b \leq 0.
\end{cases}
\]
(Here the notation $\overline{U(\xi)}$ is used to denote the matrix obtained from $U(\xi)$ by taking complex conjugates entry-wise.) Owing to the fact that $U|_{\Delta_k}$ belongs to $(C(\Delta_k; \mathbb{C}))^{m \times m}$, the above gives us a well-defined element $S$ in $(C(\Delta^+_{k+1}; \mathbb{C}))^{m \times m}$. Moreover, $S$ is invertible in $(C(\Delta^+_{k+1}; \mathbb{C}))^{m \times m}$. Finally, using the real symmetry of $P$ and (2.4), we obtain that
\[
(S(z))^{-1}P(z)S(z) =
\begin{bmatrix}
I_\ell & 0 \\
0 & 0
\end{bmatrix} \quad (z \in \Delta^+_{k+1}).
\]
Thus $C(\Delta^+_{k+1}; \mathbb{C})$ is projective free. By induction, it follows that $C(\Delta_k; \mathbb{C})$ is projective free for all $0 \leq k \leq n$. In particular, for $k = n$, the ring $C(\Delta_n; \mathbb{C}) = C(\mathbb{D}^n; \mathbb{C})$ is projective free. This completes the proof. \(\square\)

3. Proof of Theorem 1.3

We begin by showing that the maximal ideal space of $\partial^{-N}A$ can be identified with the polydisc $\mathbb{D}^n$. This is a generalization of an analogous one variable result; see [13 Theorem 1.3].
Proposition 3.1. The maximal ideal space of $\partial^{-N}A$ is homeomorphic to $\mathbb{D}^n$.

Proof. For each $\lambda \in \mathbb{D}^n$, point evaluation at $\lambda$ is a complex homomorphism $\varphi_\lambda$. Moreover $\partial^{-N}A$ contains the functions $z_1, \ldots, z_n$ and so we see that if $\lambda \neq \mu$, the corresponding point evaluations $\varphi_\lambda$ and $\varphi_\mu$ are distinct. Thus the map $\lambda \mapsto \varphi_\lambda$ embeds $\mathbb{D}^n$ in the maximal ideal space of $\partial^{-N}A$. Also it is easy to see that this inclusion is continuous, since if $\lambda$ is convergent to $\lambda' \in \mathbb{D}^n$, then for each function $f \in \partial^{-N}A$, we have that

$$\varphi_{\lambda_n}(f) = f(\lambda_n) \longrightarrow f(\lambda) = \varphi_\lambda(f),$$

by the continuity of $f$, and so $(\varphi_{\lambda_n})_{n \in \mathbb{N}}$ converges to $\varphi_\lambda$ in the weak-$*$ topology in the maximal ideal space of $\partial^{-N}A$. Now we will show that every complex homomorphism arises in this manner.

Let $\varphi$ be a nontrivial multiplicative linear functional on $\partial^{-N}A$, and let $\lambda := (\varphi(z_1), \ldots, \varphi(z_n))$. Then clearly for every polynomial $p$, $\varphi(p) = p(\lambda)$. We show below that for all $f \in \partial^{-N}A$,

$$|\varphi(f)| \leq \|f\|_\infty. \quad (3.1)$$

On applying this estimate to the coordinate functions $z \mapsto z_\ell$, we see that $\lambda \in \mathbb{D}^n$. Since $\partial^{-N}A \subset A$, any function $f \in \partial^{-N}A$ can be approximated by polynomials in the $L^\infty$-norm. But (3.1) implies that $\varphi$ is continuous in the $L^\infty$-norm, and so $\varphi(f) = f(\lambda)$ holds for all $f \in \partial^{-N}A$. (Note that in this argument, we do not need the density of polynomials in the norm of $\partial^{-N}A$, which doesn’t hold for $N \geq 1$)

Now we will prove (3.1). If $f \in \partial^{-N}A$ is such that

$$\inf_{z \in \mathbb{D}^n} |f(z)| > 0,$$

then $f$ is invertible in $\partial^{-N}A$. Indeed, since $\partial^{-N}A \subset A$, the condition

$$\inf_{z \in \mathbb{D}^n} |f(z)| > 0$$

implies that $f$ is invertible in $A$. Differentiating $1/f$ $N$ times we get that all its complex partial derivatives up to the order $N$ are in $A$. Therefore, if $0 \not\in \text{clos.range}(f) = \text{range}(f)$, then $f$ is invertible in $\partial^{-N}A$, and so $f$ does not belong to any proper ideal of $\partial^{-N}A$. Thus $\varphi(f) \neq 0$ for any maximal ideal (multiplicative linear functional) $\varphi$. Replacing $f$ by $f - a$, $a \in \mathbb{C}$, we get that if $a \not\in \text{range}(f)$, then for any multiplicative linear functional $\varphi$, $\varphi(f) \neq a$, that is, $\varphi(f) \subset \text{range}(f)$. Thus $|\varphi(f)| \leq \|f\|_\infty$, and (3.1) is proved.

We have seen that the Gelfand topology of the maximal ideal space of $\partial^{-N}A$ is weaker than the usual Euclidean topology of $\mathbb{D}^n$, and moreover it is Hausdorff. Also with the Euclidean topology, $\mathbb{D}^n$ is compact. Consequently, the two topologies coincide; see for example [13 §3.8.(a)]. □
We now prove Theorem 1.3 by following the same method as in the proof of the main result in [15].

**Proof of Theorem 1.3.** If $M$ is a finitely generated projective $\partial^{-N}A_1$-module, then $M \otimes \mathbb{C}$ is a finitely generated projective $\partial^{-N}A_1 \approx \partial^{-N}A_r \otimes \mathbb{C}$-module. But by Proposition 3.1, the maximal ideal space of $\partial^{-N}A_r$ is $\mathbb{D}^n$, which is contractible. So by [2], it follows that $\partial^{-N}A_1$ is projective free, and so $M \otimes \mathbb{C}$ is free as a $\partial^{-N}A_r$-module. Using this, we will show that $M$ is free as a $\partial^{-N}A_1$-module.

Let $e_1, \ldots, e_k$ be a basis for $M \otimes \mathbb{C}$ over $\partial^{-N}A_r$. The involution $\cdot^*$ on $\partial^{-N}A_r$ defined by $f^*(z) = \overline{f(z)}$ ($z \in \mathbb{D}^n$) induces an involution on $M \otimes \mathbb{C}$, which by slight abuse of notation, we will also denote by $\cdot^*$: thus if $u + iv \in M \otimes \mathbb{C}$, where $u, v \in M$, then $(u + iv)^* = u - iv$. It is clear that $e_1^*, \ldots, e_k^*$ also form a basis for $M \otimes \mathbb{C}$ over $\partial^{-N}A_r$.

Let $U = [u_{ij}] \in (\partial^{-N}A_r)^{k \times k}$ be the change of basis matrix from the basis \{e_1, \ldots, e_k\} to the basis \{e_1^*, \ldots, e_k^*\}. Then since

$$e_m^* = \sum_{\ell=1}^k u_{m\ell} e_{\ell},$$

we have

$$e_m = e_m^{**} = \sum_{\ell=1}^k u_{m\ell}^{*} e_{\ell}^*,$$

and so

$$e_m^* = \sum_{\ell=1}^k u_{m\ell} \sum_{j=1}^k u_{ij}^* e_j^* = \sum_{\ell=1}^k \sum_{j=1}^k u_{m\ell} u_{ij}^* e_j^*.$$ 

Consequently,

$$U(z)\overline{U(z)} = I \quad (z \in \mathbb{D}^n). \quad (3.2)$$

Suppose we are able to find a matrix $V \in (\partial^{-N}A_r)^{k \times k}$ satisfying

$$U(z) = V(z)(\overline{V(z)})^{-1} \quad (z \in \mathbb{D}^n). \quad (3.3)$$

Let us first observe that this would give the desired freeness of the $\partial^{-N}A_1$-module $M$. To this end, let us define

$$\begin{bmatrix}
e_1 \\
\vdots \\
e_k \\
\end{bmatrix} = \overline{\begin{bmatrix}
e_1 \\
\vdots \\
e_k \\
\end{bmatrix}} \quad V(z) \begin{bmatrix}
e_1^* \\
\vdots \\
e_k^* \\
\end{bmatrix}.$$
We note first of all that \( \bar{e}_m = \bar{e}^*_m \) and so \( \bar{e}_m \in M \). Next we claim that \( \bar{e}_1, \ldots, \bar{e}_k \) are linearly independent over \( \partial^{-N}A_t \). Indeed, if there exist elements \( \alpha_1, \ldots, \alpha_k \in \partial^{-N}A_t \) such that
\[
\sum_{m=1}^k \alpha_m \bar{e}_m = 0,
\]
then we obtain
\[
0 = \left[ \begin{array}{c} \alpha_1 \\ \vdots \\ \alpha_k \end{array} \right] \left[ \begin{array}{c} \bar{e}_1 \\ \vdots \\ \bar{e}_k \end{array} \right] = 2 \left[ \begin{array}{c} \alpha_1 \\ \vdots \\ \alpha_k \end{array} \right] \left( \overline{V(z)} \right)^{-1} \left[ \begin{array}{c} e_1 \\ \vdots \\ e_k \end{array} \right],
\]
which implies, by the linear independence of \( e_1, \ldots, e_k \) over \( \partial^{-N}A \) that
\[
2 \left[ \begin{array}{c} \alpha_1 \\ \vdots \\ \alpha_k \end{array} \right] \left( \overline{V(z)} \right)^{-1} = 0,
\]
and thus \( \alpha_1 = \cdots = \alpha_k = 0 \). Also, as
\[
\left[ \begin{array}{c} e_1 \\ \vdots \\ e_k \end{array} \right] = \overline{V(z)} \left[ \begin{array}{c} \bar{e}_1 \\ \vdots \\ \bar{e}_k \end{array} \right],
\]
it follows that the span of \( \bar{e}_1, \ldots, \bar{e}_k \) over the ring \( \partial^{-N}A_t \) is \( M \). This completes the proof that once we have a \( V \in (\partial^{-N}A)^{k \times k} \) satisfying (3.3), then we have the desired freeness of \( M \). Now we will give the construction of the required \( V \). In order to do this, we will have to first deform \( U \). Set
\[
U_t(z) := U(tz) \quad (0 \leq t \leq 1, \ z \in \mathbb{D}^n).
\]
This defines a path from \( U(0) \) to \( U(z) \). We claim that we can find the needed \( V(z) \) along this path. Hence we want
\[
U_t(z) = V_t(z)(\overline{V_t(z)})^{-1}.
\]
Differentiating both sides with respect to \( t \) (with a fixed \( z \)) and post-multiplying by \( (U_t(z))^{-1} \) gives us
\[
\frac{\partial U_t}{\partial t} U_t^{-1} = \frac{\partial V_t}{\partial t} V_t^{-1} - U_t \frac{\partial V_t}{\partial t} V_t^{-1}(z) U_t^{-1}.
\]
So we need to solve this differential equation.

Consider instead the equation
\[
\frac{\partial V_t}{\partial t} V_t^{-1} = \frac{1}{2} \frac{\partial U_t}{\partial t} U_t^{-1}
\]
with the initial condition \( V_0(z) \) being any complex matrix such that
\[
U(0) = V_0(z)(\overline{V_0(z)})^{-1}.
\]
The existence of this matrix \( V_0(z) \) will be shown later. For each fixed \( z \), the equation (3.6) has a solution \( t \mapsto V_t(z) \), and by [6, §2.3, p.59], we have that for each fixed \( t \), the map \( z \mapsto V_t(z) : \mathbb{D}^n \to \mathbb{C}^{k \times k} \) is holomorphic. Also, since \( (t,z) \mapsto U(tz) : [0,1] \times \mathbb{D}^n \to \mathbb{C}^{k \times k} \) is \( C^N \), it follows that the right hand side
of the equation (3.6) is $CN^{-1}$, and so the map $(t, z) \mapsto V_t(z) : [0, 1] \times \mathbb{D}^n \to \mathbb{C}^{k \times k}$ is $CN$ as well. (In particular, from the above observations, it follows that $V := V_1(\cdot) \in (\partial^{-N} A)^{k \times k}$.)

Now we show that the solution $V_t$ to our auxiliary differential equation (3.6) in fact solves the original differential equation (3.5). Note first that from (3.2), we obtain

$$U_t^{-1} \frac{\partial U_t}{\partial t} + \frac{\partial U_t}{\partial t} U_t^{-1}(z) = 0$$

We have

$$\frac{\partial U_t}{\partial t} U_t^{-1} - U_t \frac{\partial U_t}{\partial t} V_t^{-1}(z) U_t^{-1} = \left( \frac{1}{2} \frac{\partial U_t}{\partial t} U_t^{-1} - \frac{1}{2} \frac{\partial U_t}{\partial t} U_t^{-1}(z) U_t^{-1} \right) U_t^{-1}$$

$$= \frac{1}{2} \frac{\partial U_t}{\partial t} U_t^{-1} - \frac{1}{2} U_t \left( - U_t^{-1} \frac{\partial U_t}{\partial t} \right) U_t^{-1}$$

$$= \frac{\partial U_t}{\partial t} U_t^{-1}.$$

The only remaining step is to show the existence of a matrix $V_0(z)$ such that (3.7) holds. Let $U(0) = P + iQ$, where $P, Q \in \mathbb{R}^{k \times k}$. Using (3.2), it follows that $P^2 + Q^2 = I$ and $PQ = QP$. Since $U(0)$ is invertible, it has a logarithm, given by the holomorphic function calculus:

$$\log U(0) = \frac{1}{2\pi i} \int_\gamma (\log \zeta)(\zeta I - (P + iQ))^{-1} d\zeta,$$

where $\gamma$ is any simple curve in the domain $\mathbb{C} \setminus e^{i\theta}(-\infty, 0]$ that contains the spectrum of $U(0)$ in its interior, and the $\theta$ is chosen so that the ray $e^{i\theta}(-\infty, 0]$ avoids all eigenvalues of $U(0)$. Also, since $PQ = QP$, it is clear that the real and imaginary parts of $(\zeta I - (P + iQ))^{-1}$ commute with each other. As $\log \zeta$ is just a scalar, it follows that the real and imaginary parts of $\log U(0)$ commute, that is, if $U(0) = X + iY$, with $X, Y \in \mathbb{R}^{k \times k}$, then $XY = YX$. In particular, $X + iY$ and $X - iY$ also commute. We have $U(0) = e^{X+iy}$, and thus (3.2) gives

$$e^{2X} = e^{X+iy} e^{X-iy} = U(0) \overline{U(0)} = I.$$

Now let $S$ be an invertible matrix taking $X$ to its Jordan canonical form:

$$X = SJS^{-1},$$

where $J$ is a block diagonal matrix with Jordan blocks. Since $e^{2X} = I$, we obtain $Se^{2J}S^{-1} = I$, and so $(e^J)^2 = I$. But for a Jordan sub-block of the type

$$J_t := \begin{bmatrix} \lambda & 1 \\ \vdots & \ddots & \ddots \\ & & 1 \\ \vdots & & \ddots & \ddots \\ 1 & & & \lambda \end{bmatrix} \in \mathbb{C}^{m \times m},$$

we have

$$e^{2J} = \begin{bmatrix} e^{2\lambda} & 2 & 0 & \cdots \\ 0 & e^{2\lambda} & \ddots & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & & 0 & e^{2\lambda} & 2 \\ 0 & 0 & \cdots & 0 & e^{2\lambda} \end{bmatrix}.$$
we have
\[ e^{J_{\ell}} = e^{\lambda} \begin{bmatrix} 1 & \frac{1}{1!} & \frac{1}{2!} & \ldots & \frac{1}{(m-1)!} \\ 1 & \frac{1}{1!} & \frac{1}{2!} & & \\ \vdots & \ddots & \ddots & \frac{1}{2!} & \\ \vdots & & \ddots & \frac{1}{1!} & \\ 1 & & & \frac{1}{1!} & \\ \end{bmatrix} \]

and so we conclude that all eigenvalues of $J$ must be 0, and moreover, each Jordan block $J_{\ell}$ is of size 1\times1. So $J = 0$ and so $X = 0$. Define $V_0(z) := e^{iY/2}$. Then $(V_0(z))^{-1} = (e^{-iY/2})^{-1} = e^{iY/2}$. Consequently,
\[ V_0(z)(V_0(z))^{-1} = e^{iY/2} \cdot e^{iY/2} = e^{iY} = U(0), \]
as required. \hfill \square

In light of our result above, it is natural to ask the following question: Is the real symmetric polydisc algebra $A_r := \{ f \in A : f(z) = f(\overline{z}) \ (z \in \mathbb{D}^n) \}$ projective free?

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