Distance-regular graphs obtained from the Mathieu groups $M_{11}$, $M_{12}$, $M_{22}$, $M_{23}$ and $M_{24}$

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Abstract

We construct distance-regular graphs (including strongly regular graphs) admitting a transitive action of the five sporadic simple groups discovered by E. Mathieu, the Mathieu groups $M_{11}$, $M_{12}$, $M_{22}$, $M_{23}$ and $M_{24}$. We discuss a possibility of permutation decoding of the codes spanned by the adjacency matrices of these graphs and find PD-sets for some of the codes.

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1 Introduction

The main motivation for this paper is to give further contribution to the classification of transitive DRGs, especially those admitting a transitive action of a simple group. The research presented in the paper can be seen as a continuation of the work given in [9], in which
transitive structures constructed from the Mathieu group $M_{11}$ were described. Moreover, a construction of distance-regular graphs (DRGs), and especially strongly regular graphs (SRGs), from finite groups gave an important contribution to the graph theory and the design theory (see [4, 6, 25]). There are further examples of the research whose main topic was to give classifications of particular DRGs under the actions of some simple groups, for example [11, 12].

We assume that the reader is familiar with the basic facts of the group theory, the theory of strongly regular graphs and the theory of distance-regular graphs. We refer the reader to [8, 26] for relevant background reading in the group theory, to [4, 28] for the theory of strongly regular graphs, and to [6, 13] for the theory of distance-regular graphs.

In this paper we study the Mathieu groups $M_{11}, M_{12}, M_{22}, M_{23}$ and $M_{24}$ i.e. the sporadic simple groups, which are the simple groups of orders 7920, 95040, 443520, 10200960 and 244823040, respectively. The Mathieu groups $M_{11}, M_{12}, M_{22}, M_{23}$ and $M_{24}$ are belonging to the class of simple sporadic groups, introduced by Emile Mathieu in [21, 22, 23]. The Mathieu groups have been studied so far in various literature. For example, $t$-designs arising from Mathieu groups $M_{22}, M_{23}$ and $M_{24}$ have been studied in a work by Kramer, Magliveras and Mesner (see [17]) and those arising from $M_{11}$ in [9]. Codes connected with Mathieu group $M_{11}$ have been studied in [24] and those connected with $M_{12}$ in [1]. More about small representations (up to rank 5) of these five sporadic simple groups can be found in [25]. We refer the reader to [8, 30] for more details about these groups.

Using the method outlined in Section 3 we constructed and classified SRGs and DRGs of diameter $d \geq 3$ from above mentioned simple groups as follows:

- up to 2000 vertices and for which the rank of the permutation representation of the group is at most 25 (i.e. the number of orbits of the stabiliser acting on the cosets is at most 25) admitting a transitive action of the group $M_{11}$,
- up to 2000 vertices and for which the rank of the permutation representation of the group is at most 20 admitting a transitive action of the group $M_{12}$,
• up to 2000 vertices and for which the rank of the permutation representation of the group is at most 30 admitting a transitive action of the group $M_{22}$,

• up to 10000 vertices and for which the rank of the permutation representation of the group is at most 20 admitting a transitive action of the group $M_{23}$,

• up to 10000 vertices and for which the rank of the permutation representation of the group is at most 20 admitting a transitive action of the group $M_{24}$.

We also study codes spanned by the adjacency matrices of the constructed DRGs. Codes with large automorphism groups are suitable for permutation decoding (see [16, 19]), the decoding method developed by Jessie MacWilliams in the early 60's that can be used when a linear code has a sufficiently large automorphism group to ensure the existence of a set of automorphisms, called a PD-set, that has some specific properties. Therefore, the codes constructed in this paper are suitable for permutation decoding, and we were searching for PD-sets for some of the constructed codes.

To find the graphs and compute their full automorphism groups, and to obtain PD-sets and the corresponding information sets of the codes, we used programmes written for Magma [3] and GAP [14]. The constructed DRGs, including the SRGs, and the obtained PD-sets and the corresponding information sets of the codes can be found at the link: http://www.math.uniri.hr/~asvob/DRGs_Mathieu_PD.7z.

2 Preliminaries

In this section we define coherent configurations and association schemes, which are the tools for the construction of graphs presented in this paper. We also give basic definitions and properties of DRGs and SRGs.

**Definition 1** A coherent configuration on a finite non-empty set $\Omega$ is an ordered pair $(\Omega, \mathcal{R})$ with $\mathcal{R} = \{R_0, R_1, \ldots, R_d\}$ a set of non-empty relations on $\Omega$, such that the following axioms hold.
(i) \( \sum_{i=0}^{t} R_i \) is the identity relation, where \( \{ R_0, R_1, \ldots, R_t \} \subseteq \{ R_0, R_1, \ldots, R_d \} \).

(ii) \( \mathcal{R} \) is a partition of \( \Omega^2 \).

(iii) For every relation \( R_i \in \mathcal{R} \), its converse \( R_i^T = \{ (y, x) : (x, y) \in R_i \} \) is in \( \mathcal{R} \).

(iv) There are constants \( p_{ij}^k \) known as the intersection numbers of the coherent configuration \( \mathcal{R} \), such that for \( (x, y) \in R_k \), the number of elements \( z \in \Omega \) for which \( (x, z) \in R_i \) and \( (z, y) \in R_j \) equals \( p_{ij}^k \).

We say that a coherent configuration is homogeneous if it contains the identity relation, i.e., if \( R_0 = I \). If \( \mathcal{R} \) is a set of symmetric relations on \( \Omega \), then a coherent configuration is called symmetric. A symmetric coherent configuration is homogeneous (see [7]). Symmetric coherent configurations are introduced by Bose and Shimamoto in [2] and called association schemes. An association scheme with relations \( \{ R_0, R_1, \ldots, R_d \} \) is called a \( d \)-class association scheme.

Let \( \Gamma \) be a graph with diameter \( d \), and let \( \delta(u, v) \) denote the distance between vertices \( u \) and \( v \) of \( \Gamma \). The \( i \)th-neighborhood of a vertex \( v \) is the set \( \Gamma_i(v) = \{ w : \delta(v, w) = i \} \). Similarly, we define \( \Gamma_i \) to be the \( i \)th-distance graph of \( \Gamma \), that is, the vertex set of \( \Gamma_i \) is the same as for \( \Gamma \), with adjacency in \( \Gamma_i \) defined by the \( i \)th distance relation in \( \Gamma \). We say that \( \Gamma \) is distance-regular if the distance relations of \( \Gamma \) give the relations of a \( d \)-class association scheme, that is, for every choice of \( 0 \leq i, j, k \leq d \), all vertices \( v \) and \( w \) with \( \delta(v, w) = k \) satisfy \( |\Gamma_i(v) \cap \Gamma_j(w)| = p_{ij}^k \) for some constant \( p_{ij}^k \). In a distance-regular graph, we have that \( p_{ij}^k = 0 \) whenever \( i + j < k \) or \( k < |i - j| \). A distance-regular graph \( \Gamma \) is necessarily regular with degree \( p_{i1}^0 \); more generally, each distance graph \( \Gamma_i \) is regular with degree \( k_i = p_{ii}^0 \).

An equivalent definition of distance-regular graphs is the existence of the constants \( b_i = p_{i+1,1}^i \) and \( c_i = p_{i-1,1}^i \) for \( 0 \leq i \leq d \) (notice that \( b_d = c_0 = 0 \)). The sequence \( \{ b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d \} \), where \( d \) is the diameter of \( \Gamma \) is called the intersection array of \( \Gamma \). Clearly, \( b_0 = k \), \( b_d = c_0 = 0 \), \( c_1 = 0 \).
A regular graph is strongly regular with parameters \((v, k, \lambda, \mu)\) if it has \(v\) vertices, degree \(k\), and if any two adjacent vertices are together adjacent to \(\lambda\) vertices, while any two non-adjacent vertices are together adjacent to \(\mu\) vertices. A strongly regular graph with parameters \((v, k, \lambda, \mu)\) is usually denoted by \(\text{SRG}(v, k, \lambda, \mu)\). A strongly regular graph is a distance-regular graph with diameter 2 whenever \(\mu \neq 0\). The intersection array of an SRG is given by \(\{k, k - 1 - \lambda; 1, \mu\}\).

3 DRGs constructed from the Mathieu groups

Let \(G\) be a finite permutation group acting on the finite set \(\Omega\). This action induce the action of the group \(G\) on the set \(\Omega \times \Omega\). For more information see [29]. The orbits of this action are the sets of the form \(\{(\alpha g, \beta g) : g \in G\}\). If \(G\) is transitive, then \(\{(\alpha, \alpha) : \alpha \in \Omega\}\) is one such orbit. If the rank of \(G\) is \(r\), then it has \(r\) orbits on \(\Omega \times \Omega\). Let \(|\Omega| = n\) and \(\Delta_i\) is one of these orbits. We say that the \(n \times n\) matrix \(A_i\), with rows and columns indexed by \(\Omega\) and entries

\[
A_i(\alpha, \beta) = \begin{cases} 
1, & \text{if } (\alpha, \beta) \in \Delta_i \\
0, & \text{otherwise.}
\end{cases}
\]

is called the adjacency matrix for the orbit \(\Delta_i\).

Let \(A_0, \ldots, A_{r-1}\) be the adjacency matrices for the orbits of \(G\) on \(\Omega \times \Omega\). These satisfy the following conditions.

(i) \(A_0 = I\), if \(G\) is transitive on \(\Omega\). If \(G\) has \(s\) orbits on \(\Omega\), then \(I\) is a sum of \(s\) adjacency matrices.

(ii) \(\sum_i A_i = J\), where \(J\) is the all-one matrix.

(iii) If \(A_i\) is an adjacency matrix, then so is its transpose \(A_i^T\).

(iv) If \(A_i\) and \(A_j\) are adjacency matrices, then their product is an integer-linear-combination of adjacency matrices.
If $A_i$ is symmetric, then the corresponding orbit is called self-paired. Further, if $A_i = A_j^T$, then the corresponding orbits are called mutually paired.

The graphs obtained in this paper are constructed using the method described in [10] which can be rewritten in terms of coherent configurations in the following way.

**Theorem 1** Let $G$ be a finite permutation group acting transitively on the set $\Omega$ and $A_0, \ldots, A_d$ be the adjacency matrices for orbits of $G$ on $\Omega \times \Omega$. Let $\{B_1, \ldots, B_t\} \subseteq \{A_1, \ldots, A_d\}$ be a set of adjacency matrices for the self-paired or mutually paired orbits. Then $M = \sum_{i=1}^t B_i$ is the adjacency matrix of a regular graph $\Gamma$. The group $G$ acts transitively on the set of vertices of the graph $\Gamma$.

Using this method one can construct all regular graphs admitting a transitive action of the group $G$. We will be interested only in those regular graphs that are distance-regular, and especially strongly regular.

**Remark 1** Because of the large number of possibilities for building the first row of the adjacency matrix of a DRG, the only way to obtain the classification of DRGs given in this paper was with the use of computers. The running time complexity of the algorithm used for the construction of graphs depends on a number of parameters, such as the size of the used subgroup, the number of orbits of a vertex stabilizer, the number of vertices of the graphs and the number of self-paired and mutually paired orbits in a particular case.

### 3.1 DRGs from the group $M_{11}$

The Mathieu group $M_{11}$ has the order 7920 and up to conjugation has 39 subgroups. In Table[1] we give the list of all the subgroups $H_i^1 \leq M_{11}$ which lead to the construction of SRGs or DRGs of diameter $d \geq 3$. 

Using the method described in Theorem 1 we obtained all DRGs with at most 2000 vertices and for which the rank of the permutation representation of the group is at most 25, i.e. we gave the classification of such DRGs.

**Theorem 2** Up to isomorphism there are exactly five strongly regular graphs and exactly three distance-regular graphs of diameter $d \geq 3$ with at most 2000 vertices and for which the rank of the permutation representation of the group is at most 25, admitting a transitive action of the group $M_{11}$. The SRGs have parameters $(55, 18, 9, 4)$, $(66, 20, 10, 4)$, $(144, 55, 22, 20)$, $(144, 66, 30, 30)$ and $(330, 63, 24, 9)$, and the DRGs have 165, 220 and 330 vertices, respectively. Details about the obtained strongly regular graphs are given in Table 2 and details about the obtained DRGs with $d \geq 3$ are given in Table 3.

| Subgroup | Structure | Order | Index | Rank | Primitive |
|----------|-----------|-------|-------|------|-----------|
| $H_1^1$  | $S_5$     | 144   | 55    | 3    | yes       |
| $H_2^1$  | $Z_9 : QD16$ | 120   | 66    | 4    | yes       |
| $H_3^1$  | $Z_{11} : Z_3$ | 55    | 144   | 6    | no        |
| $H_4^1$  | $GL(2, 3)$ | 48    | 165   | 8    | yes       |
| $H_5^1$  | $S_3 \times S_3$ | 36    | 220   | 16   | no        |
| $H_6^1$  | $S_4$     | 24    | 330   | 23   | no        |

Table 1: Subgroups of the group $M_{11}$

| Graph $\Gamma$ | Parameters | $Aut(\Gamma)$ |
|----------------|------------|---------------|
| $\Gamma_1^1 = \Gamma(M_{11}, H_1^1)$ | (55, 18, 9, 4) | $S_{11}$ |
| $\Gamma_2^1 = \Gamma(M_{11}, H_2^1)$ | (66, 20, 10, 4) | $S_{12}$ |
| $\Gamma_3^1 = \Gamma(M_{11}, H_3^1)$ | (144, 55, 22, 20) | $M_{11}$ |
| $\Gamma_4^1 = \Gamma(M_{11}, H_4^1)$ | (144, 66, 30, 30) | $M_{12} : Z_2$ |
| $\Gamma_5^1 = \Gamma(M_{11}, H_5^1)$ | (330, 63, 24, 9) | $S_{11}$ |

Table 2: SRGs constructed from the group $M_{11}$
Graph $\Gamma$

| Number of vertices | Diameter | Intersection array | $Aut(\Gamma)$ |
|--------------------|----------|--------------------|---------------|
| $\Gamma_6 = \Gamma(M_{11}, H_1^1)$ | 165 | 3 | $\{24, 14, 6, 1, 4, 9\}$ | $A_{11} : Z_2$ |
| $\Gamma_7 = \Gamma(M_{11}, H_2^1)$ | 220 | 3 | $\{27, 16, 7, 1, 4, 9\}$ | $S_{12}$ |
| $\Gamma_8 = \Gamma(M_{11}, H_3^1)$ | 330 | 4 | $\{28, 18, 10, 4, 1, 4, 9, 16\}$ | $A_{11} : Z_2$ |

Table 3: DRGs constructed from the group $M_{11}$, $d \geq 3$

Proof. There are 39 conjugacy classes of subgroups of $M_{11}$, but only 19 of them lead to a permutation representation of rank at most 25 and of index at most 2000. Applying the method described in Theorem 1 to the permutation representations on cosets of these 19 subgroups we obtain the results. □

Remark 2 All SRGs given in Table 2 are isomorphic to the ones constructed in [9].

Remark 3 The graphs $\Gamma_6^1$, $\Gamma_7^1$ and $\Gamma_8^1$ are unique graphs with the given intersection arrays, known as Johnson graphs, $J(11, 3)$, $J(12, 3)$ and $J(11, 4)$, respectively (see [6]).

3.2 DRGs from the group $M_{12}$

The Mathieu group $M_{12}$ has the order 95040 and up to conjugation has 147 subgroups. In Table 4 we give the list of all the subgroups $H_i^2 \leq M_{12}$ which lead to the construction of SRGs or DRGs of diameter $d \geq 3$.

| Subgroup | Structure | Order | Index | Rank | Primitive |
|----------|-----------|-------|-------|------|-----------|
| $H_1^2$ | $A_6 : Z_2$ | 1440 | 66 | 3 | yes |
| $H_2^2$ | $L(2, 11)$ | 660 | 144 | 5 | no |
| $H_3^2$ | $(E_9 : Q_8) : Z_2$ | 432 | 220 | 5 | yes |
| $H_4^2$ | $(E_8 : E_3) : Z_2$ | 192 | 495 | 11 | yes |
| $H_5^2$ | $S_5$ | 120 | 792 | 15 | no |

Table 4: Subgroups of the group $M_{12}$

Using the method described in Theorem 1 we obtained all DRGs with at most 2000 vertices and for which the rank of the permutation representation of the group is at most 20, i.e. we gave the classification of such DRGs.
**Theorem 3** Up to isomorphism there are exactly seven strongly regular graphs and exactly three distance-regular graphs of diameter \(d \geq 3\) with at most 2000 vertices and for which the rank of the permutation representation of the group is at most 20, admitting a transitive action of the group \(M_{12}\). The SRGs have parameters \((66, 20, 10, 4)\), \((144, 66, 30, 30)\), \((144, 55, 22, 20)\), \((144, 22, 10, 2)\) and \((495, 238, 109, 119)\), and the DRGs have 220, 495 and 792 vertices, respectively. Details about the obtained strongly regular graphs are given in Table 5 and details about the obtained DRGs with \(d \geq 3\) are given in Table 6.

| Graph \(\Gamma\) | Parameters                        | \(\text{Aut}(\Gamma)\) |
|-----------------|-----------------------------------|-------------------------|
| \(\Gamma^2_1\)  | \((66, 20, 10, 4)\)              | \(S_{12}\)               |
| \(\Gamma^2_2\)  | \((144, 66, 30, 30)\)            | \(M_{12} : Z_2\)         |
| \(\Gamma^2_3\)  | \((144, 66, 30, 30)\)            | \(M_{12} : Z_2\)         |
| \(\Gamma^2_4\)  | \((144, 55, 22, 20)\)            | \(M_{12}\)               |
| \(\Gamma^2_5\)  | \((144, 22, 10, 2)\)             | \(S_{12} \wr S_2\)       |
| \(\Gamma^2_6\)  | \((495, 238, 109, 119)\)         | \(O^-((10, 2) : Z_2)\)   |

Table 5: SRGs constructed from the group \(M_{12}\)

| Graph \(\Gamma\) | Number of vertices | Diameter | Intersection array | \(\text{Aut}(\Gamma)\) |
|-----------------|--------------------|----------|--------------------|-------------------------|
| \(\Gamma^2_8\)  | 220                | 3        | \{27, 16, 7; 1, 4, 9\} | \(S_{12}\)               |
| \(\Gamma^2_9\)  | 495                | 4        | \{32, 21, 12, 5; 1, 4, 9, 16\} | \(S_{12}\)               |
| \(\Gamma^2_{10}\) | 792               | 5        | \{35, 24, 15, 8, 3; 1, 4, 9, 16, 25\} | \(S_{12}\)               |

Table 6: DRGs constructed from the group \(M_{12}\), \(d \geq 3\)

**Proof.** There are 147 conjugacy classes of subgroups of \(M_{12}\), but only 31 of them lead to a permutation representation of rank at most 20 and of index at most 2000. Applying the method described in Theorem 1 to the permutation representations on cosets of these 31 subgroups we obtain the results. \(\Box\)

**Remark 4** The strongly regular graph \(\Gamma^2_1\) is isomorphic to the triangular graph \(T(12)\). The adjacency matrices of non-isomorphic SRGs \(\Gamma^2_2\), \(\Gamma^2_3\) and \(\Gamma^2_4\) are the incidence matrices of symmetric designs with parameters \((144, 66, 30)\), designs with Menon parameters (related to
a regular Hadamard matrix of order 144). These symmetric designs have been described in [18, 31]. According to Brouwer’s table (see [5]), known graphs with the parameters equal to the parameters of the graph $\Gamma_9^2$ (not isomorphic to $\Gamma_3^1$) are obtainable from orthogonal arrays $OA(12,5)$. Since our method does not use orthogonal arrays, it is likely that our graph is new. The graph $\Gamma_6^3$ is unique graph with the given parameters and the graph $\Gamma_7^2$ is isomorphic to the $O^-(10,2)$ polar graph. Strongly regular graphs with parameters $(144,66,30,30)$ have been known before (see [4, 5]).

**Remark 5** The graphs $\Gamma_8^2$, $\Gamma_9^2$, and $\Gamma_{10}^2$ are unique graphs with the given intersection arrays, known as Johnson graphs, $J(12,3)$, $J(12,4)$ and $J(12,5)$, respectively (see [6]).

### 3.3 DRGs from the group $M_{22}$

The Mathieu group $M_{22}$ has the order 443520 and up to conjugation 156 subgroups. In Table 7 we give the list of all the subgroups $H_i^3 \leq M_{22}$ which lead to the construction of SRGs or DRGs of diameter $d \geq 3$.

| Subgroup | Structure | Order | Index | Rank | Primitive |
|----------|-----------|-------|-------|------|-----------|
| $H_1^3$  | $E_{16} : A_6$ | 5760  | 77    | 3    | yes       |
| $H_2^3$  | $A_7$     | 2520  | 176   | 3    | yes       |
| $H_3^3$  | $E_{16} : S_5$ | 1920  | 231   | 4    | yes       |
| $H_4^3$  | $E_8 : L(3,2)$ | 1344  | 330   | 5    | yes       |
| $H_5^3$  | $L(2,11)$  | 660   | 672   | 6    | yes       |
| $H_6^3$  | $(A_4 \times A_4) : Z_2$ | 288   | 1540  | 22   | no        |

Table 7: Subgroups of the group $M_{22}$

Using the method described in Theorem 1 we obtained all DRGs with at most 2000 vertices and for which the rank of the permutation representation of the group is at most 30, i.e. we gave the classification of such DRGs.

**Theorem 4** Up to isomorphism there are exactly five strongly regular graphs and exactly three distance-regular graphs of diameter $d \geq 3$ with at most 2000 vertices and for which
the rank of the permutation representation of the group is at most 30, admitting a transitive action of the group $M_{22}$. The SRGs have parameters $(77,16,0,4)$, $(176,70,18,34)$, $(231,30,9,3)$, $(231,40,20,4)$ and $(672,176,40,48)$, and the DRGs have 330, 672 and 1540 vertices, respectively. Details about the obtained strongly regular graphs are given in Table 8 and details about the obtained DRGs with $d \geq 3$ are given in Table 9.

### Table 8: SRGs constructed from the group $M_{22}$

| Graph $\Gamma$ | Parameters   | Aut($\Gamma$) |
|----------------|--------------|---------------|
| $\Gamma_1^3 = \Gamma(M_{22}, H_1^3)$ | (77,16,0,4) | $M_{22} : Z_2$ |
| $\Gamma_2^3 = \Gamma(M_{22}, H_2^3)$ | (176,70,18,34) | $M_{22}$ |
| $\Gamma_3^3 = \Gamma(M_{22}, H_3^3)$ | (231,30,9,3) | $M_{22} : Z_2$ |
| $\Gamma_4^3 = \Gamma(M_{22}, H_4^3)$ | (231,40,20,4) | $S_{22}$ |
| $\Gamma_5^3 = \Gamma(M_{22}, H_5^3)$ | (672,176,40,48) | $(U(6,2) : Z_2) : Z_2$ |

### Table 9: DRGs constructed from the group $M_{22}$, $d \geq 3$

| Graph $\Gamma$ | Number of vertices | Diameter | Intersection array | Aut($\Gamma$) |
|----------------|--------------------|----------|-------------------|---------------|
| $\Gamma_6^3 = \Gamma(M_{22}, H_6^3)$ | 330 | 4 | $\{7,6,4,4;1,1,1,6\}$ | $M_{22} : Z_2$ |
| $\Gamma_7^3 = \Gamma(M_{22}, H_7^3)$ | 672 | 3 | $\{110,81,12;1,18,90\}$ | $M_{22} : Z_2$ |
| $\Gamma_8^3 = \Gamma(M_{22}, H_8^3)$ | 1540 | 3 | $\{57,36,17;1,4,9\}$ | $S_{22}$ |

**Proof.** There are 156 conjugacy classes of subgroups of $M_{22}$, but only 21 of them lead to a permutation representation of rank at most 30 and of index at most 2000. Applying the method described in Theorem 1 to the permutation representations on cosets of these 21 subgroups we obtain the results. ☐

**Remark 6** The strongly regular graphs $\Gamma_1^3$ and $\Gamma_2^3$ are unique graphs with these parameters. The graph $\Gamma_3^3$ is isomorphic to the SRG known as the Cameron graph. The SRG $\Gamma_4^3$ is isomorphic to the triangular graph $T(22)$ and $\Gamma_5^3$ is isomorphic to the graph known as $U(6,2)$-graph. For more information we refer the reader to [4, 5].

**Remark 7** The graph $\Gamma_6^3$ is isomorphic to the graph known as $M_{22}$-graph or doubly truncated Witt graph. The graph $\Gamma_7^3$ is isomorphic to the one constructed by Soicher in [27]. So far, it
is the only known example of DRG with this intersection array. The graph $\Gamma^3_8$ is known as Johnson graph $J(22,3)$. (see \cite{9})

### 3.4 DRGs from the group $M_{23}$

The Mathieu group $M_{23}$ has order 10200960 and up to conjugation 204 subgroups. In Table 10 we give the list of all the subgroups $H_i^4 \leq M_{23}$ which lead to the construction of SRGs or DRGs of diameter $d \geq 3$.

| Subgroup | Structure | Order | Index | Rank | Primitive |
|----------|-----------|-------|-------|------|-----------|
| $H_1^4$  | $L(3,4):Z_2$ | 40320 | 253   | 3    | yes       |
| $H_2^4$  | $E_{16}:A_7$  | 40320 | 253   | 3    | yes       |
| $H_3^4$  | $A_8$       | 20160 | 506   | 4    | yes       |
| $H_4^4$  | $M_{11}$    | 7920  | 1288  | 4    | yes       |
| $H_5^4$  | $E_{16}:(A_5:S_3)$ | 5760 | 1771  | 8    | yes       |

Table 10: Subgroups of the group $M_{23}$

Using the method described in Theorem 5 we obtained all DRGs with at most 10000 vertices and for which the rank of the permutation representation of the group is at most 20, i.e. we gave the classification of such DRGs.

**Theorem 5** Up to isomorphism there are exactly three strongly regular graphs and exactly two distance-regular graphs of diameter $d \geq 3$ with at most 10000 vertices and for which the rank of the permutation representation of the group is at most 20, admitting a transitive action of the group $M_{23}$. The SRGs have parameters $(253, 42, 21, 4)$, $(253, 112, 36, 60)$ and $(1288, 495, 206, 180)$, and the DRGs have 506 and 1771 vertices, respectively. Details about the obtained strongly regular graphs are given in Table 11 and details about the obtained DRGs with $d \geq 3$ are given in Table 12.
Table 11: SRGs constructed from the group $M_{23}$

| Graph $\Gamma$ | Parameters | $\text{Aut}(\Gamma)$ |
|----------------|------------|----------------------|
| $\Gamma_4^1 = \Gamma(M_{23}, H_{4}^1)$ | $(253,42,21,4)$ | $S_{23}$ |
| $\Gamma_4^2 = \Gamma(M_{23}, H_{4}^2)$ | $(253,112,36,60)$ | $M_{23}$ |
| $\Gamma_4^3 = \Gamma(M_{23}, H_{4}^3)$ | $(1288,495,206,180)$ | $M_{24}$ |

Table 12: DRG constructed from the group $M_{23}$, $d \geq 3$

| Graph $\Gamma$ | Number of vertices | Diameter | Intersection array | $\text{Aut}(\Gamma)$ |
|----------------|--------------------|----------|-------------------|----------------------|
| $\Gamma_4^4 = \Gamma(M_{23}, H_{5}^4)$ | 506 | 3 | $\{15, 14, 12; 1, 1, 9\}$ | $M_{23}$ |
| $\Gamma_4^5 = \Gamma(M_{23}, H_{4}^5)$ | 1771 | 3 | $\{60, 38, 18; 1, 4, 9\}$ | $S_{23}$ |

Proof. There are 204 conjugacy classes of subgroups of $M_{23}$, but only 14 of them lead to a permutation representation of rank at most 20 and of index at most 10000. Applying the method described in Theorem 1 to the permutation representations on cosets of these 14 subgroups we obtain the results. □

Remark 8 The graph $\Gamma_4^1$ is isomorphic to the triangular graph $T(23)$. The graph $\Gamma_4^1$ can be constructed from the group $M_{23}$ as a rank 3 graph, and $\Gamma_4^4$ (isomorphic to the graph $\Gamma_5^2$) can be constructed from the group $M_{24}$ as a rank 3 graph.

Remark 9 The graph $\Gamma_4^4$ is isomorphic to the distance-regular graph that can be obtained from residual design of Steiner system $S(5, 8, 24)$. The graph $\Gamma_4^5$ is known as Johnson graph $J(23, 3)$. For more information we refer the reader to [6].

3.5 DRGs from the Mathieu group $M_{24}$

The Mathieu group $M_{24}$ has order 244823040 and up to conjugation 1529 subgroups. In Table 13 we give the list of all the subgroups $H_{5}^i \leq M_{24}$ which lead to the construction of SRGs or DRGs of diameter $d \geq 3$. 

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Using the method described in Theorem 1 we obtained all DRGs with at most 10000 vertices and for which the rank of the permutation representation of the group is at most 20, i.e. we gave the classification of such DRGs.

**Theorem 6** Up to isomorphism there are exactly two strongly regular graphs and exactly two distance-regular graphs of diameter \( d \geq 3 \) with at most 10000 vertices and for which the rank of the permutation representation of the group is at most 20, admitting a transitive action of the group \( M_{24} \). The SRGs have parameters \((276, 44, 22, 4)\) and \((1288, 495, 206, 180)\), and the DRGs have 759 and 2024 vertices, respectively. Details about the obtained strongly regular graphs are given in Table 14 and details about the obtained DRGs with \( d \geq 3 \) are given in Table 15.

### Table 13: Subgroups of the group \( M_{24} \)

| Subgroup | Structure | Order | Index | Rank | Primitive |
|----------|-----------|-------|-------|------|-----------|
| \( H_1^5 \) | \( M_22 : Z_2 \) | 887040 | 276 | 3 | yes |
| \( H_2^5 \) | \( E_{16} : A_8 \) | 322560 | 759 | 4 | yes |
| \( H_3^5 \) | \( M_{12} : Z_2 \) | 190080 | 1288 | 3 | yes |
| \( H_1^4 \) | \((L(3,4) : Z_3) : Z_2 \) | 120960 | 2024 | 5 | yes |

### Table 14: SRGs constructed from the group \( M_{24} \)

| Graph \( \Gamma \) | Parameters | Aut(\( \Gamma \)) |
|------------------|------------|----------------|
| \( \Gamma_5^3 = \Gamma(M_{24}, H_1^5) \) | \((276, 44, 22, 4)\) | \( S_{24} \) |
| \( \Gamma_5^4 = \Gamma(M_{24}, H_2^5) \) | \((1288, 495, 206, 180)\) | \( M_{24} \) |

### Table 15: DRGs constructed from the group \( M_{24} \), \( d \geq 3 \)

| Graph \( \Gamma \) | Number of vertices | Diameter | Intersection array | Aut(\( \Gamma \)) |
|------------------|-------------------|----------|-------------------|----------------|
| \( \Gamma_3^3 = \Gamma(M_{24}, H_2^5) \) | 759 | 3 | \( \{30, 28, 24; 1, 3, 15\} \) | \( M_{24} \) |
| \( \Gamma_4^4 = \Gamma(M_{24}, H_4^5) \) | 2024 | 3 | \( \{63, 40, 19; 1, 4, 9\} \) | \( S_{24} \) |

**Proof.** There are 1529 conjugacy classes of subgroups of \( M_{24} \), but only 15 of them lead to a permutation representation of rank at most 20 and of index at most 10000. Applying
the method described in Theorem 1 to the permutation representations on cosets of these 15 subgroups we obtain the results. □

Remark 10 The graph $\Gamma_1^2$ is isomorphic to the triangular graph $T(24)$. The graph $\Gamma_2^3$ (isomorphic to the graph $\Gamma_3^4$) can be constructed from the group $M_{24}$ as a rank 3 graph.

Remark 11 The graph $\Gamma_3^5$ is unique distance-regular graph known as near hexagon which can be obtained from Steiner system $S(5, 8, 24)$. The graph $\Gamma_1^5$ is known as Johnson graph $J(24, 3)$. For more information we refer the reader to [6].

4 Permutation decoding

Let $C \subseteq \mathbb{F}_p^n$ be a linear $[n, k, d]$ code. For $I \subseteq \{1, ..., n\}$, let $p_I : \mathbb{F}_p^n \to \mathbb{F}_p^{|I|}$, $x \mapsto x|_I$, be the $I$-projection of $\mathbb{F}_p^n$. Then $I$ is called an information set for $C$ if $|I| = k$ and $p_I(C) = \mathbb{F}_p^{|I|}$.

The set of the first $k$ coordinates for a code with a generator matrix in the standard form is an information set. The first $k$ coordinates are then called information symbols and the last $n-k$ coordinates are the check symbols and they form the corresponding check set.

Let $C \subseteq \mathbb{F}_p^n$ be a linear $[n, k, d]$ code that can correct at most $t$ errors (i.e. $t$-error-correcting code) and let $I$ be an information set for $C$. A subset $S \subseteq \text{Aut}C$ is a PD-set for $C$ if every $t$-set of coordinate positions can be moved by at least one element of $S$ out of the information set $I$. The property of having a PD-set for a code is not invariant under isomorphism of codes, it depends on the choice of the information set.

The algorithm of permutation decoding (see [20]) uses PD-sets and it is more efficient the smaller the size of a PD-set is. A lower bound on the size of a PD-set is given in the following theorem and it is due to Gordon [15].

Theorem 7 If $S$ is a PD-set for an $[n, k, d]$ code $C$ that can correct $t$ errors, $r = n - k$, then

$$|S| \geq \left\lceil \frac{n}{r} \left\lceil \frac{n - 1}{r - 1} \left[ \cdots \left\lceil \frac{n - t + 1}{r - t + 1} \right\rceil \cdots \right] \right\rceil \right\rceil.$$ 

PD-sets for codes do not always exist. Even if they exist, PD-sets are not easy to find, since they depend on the chosen information set of the code.
Let $A$ be the adjacency matrix of a graph $\Gamma$. Then the full automorphism group of $\Gamma$ is a subgroup of the full automorphism group of the linear code spanned by $A$ over $\mathbb{F}_p$. Codes with large automorphism groups are likely to have PD-sets, therefore, we were looking for PD-sets for the codes spanned by adjacency matrices of the DRGs constructed in this paper.

For any of the constructed DRG $\Gamma^i_j$ from the previous section, let $C^i_j$ denote the linear code spanned by the adjacency matrix of the graph $\Gamma^i_j$. Sizes of the obtained PD-sets (for specific information sets) for some of these codes are given in Table 16. For the other codes computation of PD-sets was not feasible. We denote by $t$ the error correcting capacity of the code, and by $g$ the Gordon bound for the size of the PD-set of a code, from Theorem 7.

| Code $C$ | Parameters $[n, k, d]$ | Aut($C$) | $t$ | $g$ | Size of PD-set |
|---------|------------------------|----------|-----|----|----------------|
| $C^1_1$ | [55,10,10]             | $S_{11}$ | 4   | 5  | 5              |
| $C^1_2$ | [165,120,4]            | $S_{11}$ | 1   | 4  | 5              |
| $C^1_3$ | [330,286,6]            | $S_{11}$ | 2   | 60 | 420            |
| $C^1_4$ | [330,120,8]            | $S_{11}$ | 3   | 7  | 22             |
| $C^2_1$ | [66,10,20]             | $S_{12}$ | 9   | 15 | 660            |
| $C^2_2$ | [77,20,16]             | $M_{22} : Z_2$ | 7 | 19 | 110 |
| $C^2_3$ | [1771,1540,4]          | $S_{23}$ | 1   | 8  | 23             |

Table 16: PD-sets for codes from constructed DRGs from Mathieu groups

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