FEEDBACK STABILIZATION
OF A LINEAR HYDRO-ELASTIC SYSTEM

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Abstract. It is known that the linear Stokes-Lamé system can be stabilized by a boundary feedback in the form of a dissipative velocity matching on the common interface [5]. Here we consider feedback stabilization for a generalized linear fluid-elasticity interaction, where the matching conditions on the interface incorporate the curvature of the common boundary and thus take into account the geometry of the problem. Such a coupled system is semigroup well-posed on the natural finite energy space [13], however, the system is not dissipative to begin with, which represents a key departure from the feedback control analysis in [5]. We prove that a damped version of the general linear hydro-elasticity model is exponentially stable. First, such a result is given for boundary dissipation of the form used in [5]. This proof resolves a more complex version, compared to the classical case, of the weighted energy methods, and addresses the lack of over-determination in the associated unique continuation result. The second theorem demonstrates how assumptions can be relaxed if a viscous damping is added in the interior of the solid.

1. Introduction. Fluid-structure interactions arise in a variety of physical, biological and engineering problems. Modeling of this phenomenon is crucial in the design of aircraft wings, turbine blades and bridges, and the study of aneurysms in large arteries, stents, and artificial heart valves. Mathematical analysis of the interface dynamics between a fluid and an elastic deformable body stands among the most challenging topics of the contemporary applied mathematics research. A quantification of this phenomenon entails in the general case both the nonlinear elasticity system and the Navier-Stokes equations, each subject in itself being an area of intense research. The challenge of parabolic-hyperbolic coupling is amplified by the fact that the interface between the two media is itself an unknown, thus creating a free and moving boundary problem.

Various useful simplifications of this description have been considered including, first and foremost, linearized models. To our knowledge, the first such system

2010 Mathematics Subject Classification. Primary: 74F10, 93D15; Secondary: 35Q74, 35Q35, 35R35, 93B18.

Key words and phrases. Fluid-structure, free boundary, moving boundary, linearization, boundary damping, stability.

The first author is partially supported by NSF Grant DMS-1312801.
The third author is partially supported by NSF Grant DMS-1616425.
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appeared in print as early as 1965 in [23] and was prompted by biomedical applications. The dynamics was linearized at rest, which effectively amounted to a fixed/cylindrical geometry on which Stokes flow and linear isotropic elasticity systems were coupled through suitable stress and velocity matching conditions on the common boundary. This setting and variations thereof have since been examined in many theoretical and numerical studies [12, 15, 26, 28, 29, 31, 36, 37, 38, 39, 40, 64, 69, 66, 24, 32, 33, 35, 30, 34, 43, 68, 67, 60, 61].

Here, we concern ourselves with feedback stabilization of linear hydro-elasticity models, the problem of stability being central to control studies. The classical coupling between linear elasticity and the Stokes system has been investigated in a recent series of papers by Avalos and Triggiani (in [2] the authors studied spectral properties of the semigroup generator; [3], by Avalos and Dvorak, provides an alternative proof of the maximality argument; [6] addresses semigroup well-posedness of a damped model; and [7] treats the backward uniqueness problem). The least amount of damping required for stabilization of the Stokes-elasticity model is the boundary interface feedback. The uniform exponential stability was demonstrated in [5]. Such a feedback represents a refinement accounting for the slip condition on the interface that dissipates energy through friction [65, p. 240]. From an operator theory viewpoint, the corresponding boundary term shifts the generator spectrum to the left, away from the imaginary axis.

In comparison, our goal is to provide a feedback stabilization law for a more general linear model for hydro-elasticity, which was recently proposed in [14]. The work in [14] was motivated by an optimal control problem associated with a free/moving boundary interaction, where the linearization of the system represents the sensitivity equations needed in the process of finding necessary optimality conditions. In this setting, the equations of fluid and solid dynamics were linearized simultaneously with the free and moving interface between them. This refinement led to a markedly different and more complex model (which agrees with the classical one when the steady regime is the zero equilibrium), where the matching conditions on the common interface account for the geometry of the problem through the presence of boundary curvatures. The well-posedness analysis of the above mentioned linearization was considered in [13], where it was demonstrated that the new model generates a $C_0$-semigroup when linearized around a steady regime (i.e., constant in time, but possibly nonzero, fluid velocity field), provided that the energy of the steady-state regime is not too high.

In the present work we demonstrate that a damped fluid-structure interaction model linearized around a steady regime is exponentially stable. Unlike the Stokes-elasticity scenario, the stability in this case is contingent on the presence of viscous dissipation in the elastic solid itself, similar to the analysis provided in [44]. Moreover, the dissipation through the interface naturally competes with the energy of the steady regime, hence the stability property may be lost if the steady regime’s energy is too high. This interplay of forces leads to interesting correlations between the stability of the system and the relative values of the physical parameters as presented in the discussion below.

The rest of the paper is organized as follows. In Section 2, we summarize the linear fluid-elasticity model obtained in [14] and present its dissipative counterpart. Section 3 addresses the well-posedness of the linear problem (primarily referencing [13]). Section 4 lists the main results of the paper: Theorems 4.1 and 4.2, for the cases without and with the interior viscous damping in the solid respectively.
Section 5 serves as a reference for various weighted energy identities to be used in the proofs. Sections 6 and 7 show how the multiplier identities lead to stabilization estimates which imply the two main results.

2. PDE model. We start by introducing the general linear model of hydro-elasticity obtained in [14]. Let \( D \subset \mathbb{R}^3 \) be a bounded open domain comprised of two sub-domains \( D = \Omega \cup \Omega_f \) and having smooth boundary \( \partial D = \Gamma_f \). The elastic body occupies open sub-domain \( \Omega \) with boundary \( \Gamma \). This elastic solid is surrounded by a fluid, which occupies open sub-domain \( \Omega_f = D \setminus \Omega \), with smooth boundary \( \Gamma \cup \Gamma_f \), assuming \( \Gamma \cap \Gamma_f = \emptyset \). Vector field \( n \) will denote the exterior unit normal field to the elastic domain \( \Omega \). A sample 2D domain is depicted in Figure 1.

The structure component is modeled by linear elasticity, while the fluid is described by the Stokes equation. On the common interface \( \Gamma \) we have suitable matchings of velocities and normal stress tensors, while on the outer boundary \( \Gamma_f \), the fluid is subject to a no-slip condition. The state variable is represented by \((u, w, p)\), where \( u \) denotes the elastic displacement, \( w \) the fluid velocity, and \( p \) the fluid pressure. The PDE model is given by:

\[
\begin{align*}
\left\{ \\
\begin{array}{ll}
\dot{w}_t - \nu \Delta w + (Dw)w_s + (Dw_s)w + \nabla p &= 0 \quad \text{in} \quad (0, T) \times \Omega_f \\
\text{div}(w) &= 0 \quad \text{in} \quad (0, T) \times \Omega_f \\
u_{tt} - \det(\Theta_s) \text{Div}[T(u)] &= 0 \quad \text{in} \quad (0, T) \times \Omega \\
w + (Dw_s)u = u_t \quad \text{on} \quad (0, T) \times \Gamma \\
T(u)n = \sigma(w, p)n + B(u) \quad \text{on} \quad (0, T) \times \Gamma \\
w &= 0 \quad \text{on} \quad (0, T) \times \Gamma_f \\
u(0) = u^0; \quad u_t(0) = u^1; \quad w(0) &= w^0
\end{array}
\right.
\end{align*}
\]

where the terms \( T, B \) and \( \sigma \) are introduced below.

This coupled system was obtained in [14], by performing linearization on a full nonlinear, moving boundary Navier-Stokes-St. Venant-Kirchhoff elasticity interaction, around an arbitrary steady regime. Therefore all the variables with subindex “s” present in the linearization considered here represent the states of the following steady state nonlinear fluid-structure coupling:

\[
\begin{align*}
\left\{ \\
\begin{array}{ll}
-\nu \Delta w_s + (Dw_s)w_s + \nabla p_s &= v_1 \quad \text{on} \quad \Omega_f \\
\text{div} \ w_s &= 0 \quad \text{on} \quad \Omega_f \\
w_s &= 0 \quad \text{on} \quad \Gamma_f \\
\text{Div} \ T_s &= v_2 \quad \text{on} \quad \Omega = \phi(\Omega^0) \\
T_sn &= \sigma(p_s, w_s)n \quad \text{on} \quad \Gamma = \phi(\Gamma^0) \\
\varphi_s &= I_{\Gamma_f} \quad \text{on} \quad \Gamma_f.
\end{array}
\right.
\end{align*}
\]
We recall here that the actual elasticity domain $\Omega$ is the image of some fixed reference domain $\Omega^0$ according to a deformation map (Lagrangian formulation). Consequently, the shape of the fluid domain $\Omega_f$ is induced by this structural deformation through the common interface $\Gamma$. More precisely, given a material reference configuration for the solid $\Omega^0 \subset \mathcal{D}$ with boundary $\Gamma^0$, let $\Omega^0_f = \mathcal{D} \setminus \overline{\Omega^0}$ be the reference fluid configuration. Then the entire volume $\mathcal{D}$ is described by a smooth, injective, orientation-preserving map:

$$\varphi_s : \mathcal{D} \rightarrow \overline{\mathcal{D}}, \ x \mapsto \varphi_s(x).$$

For any material point $x \in \Omega^0$, $\varphi_s(x)$ represents its physical position. On $\Omega^0_f$, $\varphi_s(x)$ is defined as an arbitrary extension of the restriction of $\varphi_s$ to $\Gamma^0$, which preserves the outside boundary $\Gamma_f$, i.e., $\varphi_s = I|_{\Gamma_f}$ on $\Gamma_f$. As usual, we define $J(\varphi_s) = \det(D\varphi_s) > 0$ as the Jacobian of the deformation $\varphi_s$.

In this framework, we write the St. Venant-Kirchhoff equations of elasticity on the deformed configuration $\Omega$, using the symmetric Cauchy stress tensor $T_s : \overline{\Omega} \to \mathfrak{S}^3$,

$$T_s = [J(\varphi_s)^{-1} \mathcal{P}(D\varphi_s)^*] \circ \varphi_s^{-1}$$

given in terms of the Green-St. Venant nonlinear strain tensor

$$\sigma(\varphi_s) = \frac{1}{2}(D\varphi_s)^* D\varphi_s - I,$$

and the Piola transform of the Cauchy stress tensor

$$\mathcal{P} = (D\varphi_s)(\lambda \text{Tr}[\sigma(\varphi_s)] I + 2\mu \sigma(\varphi_s))$$

Here $\lambda, \mu > 0$ are the Lamé parameters of the material [21]. The boundary dynamics prescribes the matching of the normal elastic stress $T_sn$ and the fluid stress $\sigma(w_s, p_e)n$ given by

$$\sigma(w_s, p_e) = -p_e I + 2\nu \varepsilon(w_s).$$

In particular, $\sigma(w, p)$ in (1) is given by

$$\sigma(w, p) := -p I + 2\nu \varepsilon(w). \quad (3)$$

Following [13], in order to exhibit the particular structures of the linearized elastic stress tensor $T(u)$ and boundary form $\mathcal{B}(u)$ in (1), we introduce the following notation:

- $\Theta_s := D\varphi_s \circ \varphi_s^{-1} = [D(\varphi_s^{-1})]^{-1}$.
- $D\Theta := \Theta_s^*(D\varphi_s)\Theta_s$.
- $\Sigma(D\varphi) := \lambda \text{Tr}(D\varphi_s) I + \mu (D\varphi_s + \Theta_s^* D\varphi_s)$

Then the elasticity stress tensor $T(u)$ takes the following form:

$$T(u) = \frac{1}{\det \Theta_s} \Theta_s \Sigma(D\varphi) \Theta_s^* + (D\varphi) T_s, \quad (4)$$

while the boundary operator $\mathcal{B}(u)$ is given by

$$\mathcal{B}(u) = (T_s - \sigma(w_s, p_e))([(D\varphi_s)^* n + (D^2 b_{ij}) u_{ij}] + [(D\varphi_s) u] n + \text{div}(u) T_s n - T_s (D\varphi_s)^* n$$

$$- \langle u, n \rangle \left( - \text{Div}_f(T_s) + \left[ \frac{\partial p_e}{\partial n} I - 2\nu \frac{\partial \varepsilon(w_s)}{\partial n} \right] n \right), \quad (5)$$

where $b_{ij}(x) = d_{ij}(x) - d_{ij}(x)$ is the oriented distance function from $x$ to the elastic domain $\Omega$, and $D^2 b_{ij}$ is the matrix of curvatures, whose trace equals the additive curvature $b$ of $\Gamma$. 

In [13] it was shown that the boundary form $\mathcal{B}$ is a linear operator involving only zero-order terms and tangential derivatives on the boundary of order one, and therefore $\mathcal{B}$ is continuous $\mathbf{H}^1(\Omega) \to \mathbf{H}^{-1/2}(\Gamma)$. This represented a key feature in the proof of well-posedness of finite energy solution for the linearized system (1).

Note that the linear coupled system for $u$, $w$ and $p$ is quite different from usual couplings of linear models. In particular, extra terms are present on the common interface in the normal stress tensors matching, some of them involving boundary curvatures (i.e., $D^2b_\Omega$), proving that the connection on the interface between elasticity and fluid is tightly linked to the geometry of the boundary.

2.1. Dissipative model. Semigroup well-posedness of (1) was considered in [13]. The system shares many features with the classical Stokes-elasticity model, but at the same time has a number of crucial differences from the latter.

From (5), one can see that the operator $\mathcal{B}$ incorporates tangential derivatives of the elastic displacement $u$, with the coefficients dependent, among other things, on the curvature of the interface $\Gamma$. The presence of $\mathcal{B}$ renders the stress matching on the interface as an oblique boundary condition. In context of elliptic problems, a small tangential perturbation can typically be accommodated as a small perturbation of the associated bilinear form. In contrast, a hyperbolic problem with an oblique derivative condition is not well-posed in the standard finite energy space [45, 17, 63] regardless of the magnitude of the coefficients. Because of the sharp regularity requirements on the well-posedness of a hyperbolic-parabolic coupling, no loss of smoothness is admissible; so the problem (1) appears ill-posed to begin with. The well-posedness stems from the diffusive effect of the fluid which regularizes the elastic dynamics and preserves the finite energy regularity. The fluid viscosity plays an essential role and if the classical linear model admitted any positive viscosity, then in (1) the value of $\nu$ has to be large enough relatively to the energy of the steady regime $(w_s, p_s, u_s)$.

Another new aspect of (1) is that the ensuing dynamical system is inherently non-dissipative (and not conservative) unless the linearization is considered around rest. The non-dissipativity stems, among other things, from energy-level tangential derivatives on the interface incorporated by $\mathcal{B}$ (5).

Remark 2.1 (Non-expansive property). Let us further elaborate on the non-expansive property of system (1). In general, even if the linearization is considered about a conservative system, the linear solution need not preserve energy. For instance, let $A, B$ denote symmetric matrix fields that are uniformly (strictly) positive definite. Consider the scalar wave equation

$$y_{tt} - \text{div}(A\nabla y) = 0$$

on a smooth domain $X$ with the boundary condition $y|_{\partial X} = 0$. Let $y^s$ be the solution of the perturbed system

$$y_{tt}^s - \text{div}((A + sB)\nabla y^s) = 0.$$

By $y^s$ we indicate the derivative of $y^s$ with respect to $s$ at $s = 0$. Then $y^s$ solves

$$y_{tt}^s - \text{div}(A\nabla y^s) = \text{div}(B\nabla y).$$

So the system for $y^s$ is conservative, while the one associated to $y^s$ is not, except when the linearization trajectory $y$ is zero to begin with.

Classical Stokes-elasticity couplings, like the one present in [5], could be stabilized solely by a boundary feedback control where the size of the positive dissipation
coefficient was unrestricted. In addition, the zero eigenvalue of the semigroup generator could be eliminated by requiring the boundary to incorporate a small flat segment [6]. This same stabilization argument cannot be cited for (1) due to the extra terms in both the conditions on the common boundary. Instead, the presented results entail more complex multiplier estimates and suitable restrictions on the interplay between viscosity \( \nu \), boundary damping \( K \) and the energy of the steady regime \((w_s, p_s, u_s)\) about which the system is linearized.

The general damped model can be formulated as follows: for \( \gamma, K > 0 \) and \( \alpha \geq 0 \) (both \( \alpha = 0 \) and \( \alpha > 0 \) will be considered):

\[
\begin{aligned}
&w_t - \nu \Delta w + (Dw)w_s + (Dw_s)w + \nabla p = 0 \quad \text{in } (0, T) \times \Omega_f \\
&\text{div}(w) = 0 \quad \text{in } (0, T) \times \Omega_f \\
&u_{tt} - \det(\Theta_s) \text{Div}[\mathcal{T}(u)] + \gamma \mathcal{T}(u) = 0 \quad \text{in } (0, T) \times \Omega \\
&w + (Dw_s)u = u_t + K\mathcal{T}(u)n \quad \text{on } (0, T) \times \Gamma \\
&\mathcal{T}(u)n = \sigma(w, p)n + B(u) \quad \text{on } (0, T) \times \Gamma \\
&w = 0 \quad \text{on } (0, T) \times \Gamma_f \\
&u(0) = u^0; \quad u_t(0) = u^1; \quad w(0) = w^0
\end{aligned}
\] (6)

The term \( \alpha \) (if nonzero) quantifies viscous dissipation in the solid. We also introduce additional stiffness via \( \gamma > 0 \) to avoid constant eigenfunctions. In the Stokes-elasticity scenario \( \gamma > 0 \) could be replaced with a geometric condition [2], however, the same argument doesn’t carry over due to the presence of the perturbation term \((Dw_s)u\) in the velocity matching condition. The term \( K\mathcal{T}(u)n \) is the natural generalization of the Stokes-elasticity feedback [5] and describes a slip condition on the interface analogously to [65, p. 240]. The friction forces provide a dissipative effect, which, from the functional-analytic perspective, is reflected in the shift of the generator spectrum to the left complex half-plane.

3. Well-posedness analysis. Following [13] we appeal to the fact that \( \det \Theta_s > 0 \) on \( \bar{\Omega} \) and denote by \( L^2_{\Theta_s}(\Omega) \) the space \( L^2(\Omega) \) with an equivalent inner product

\[
(u, z)_{L^2_{\Theta_s}(\Omega)} := \int_{\Omega} \frac{1}{\det \Theta_s}(u, z).
\] (7)

From the properties of the Cauchy stress tensor \( \mathcal{T} \) for the linearized problem, the following bilinear form defines a semi-norm on \( H^1(\Omega) \) (see [13, p. 1978]):

\[
(u, z)_{1, \Omega} := \int_{\Omega} \mathcal{T}(u) \cdot Dz \quad \text{and} \quad |u|_{1, \Omega}^2 := (u, u)_{1, \Omega}.
\] (8)

The state of the system (6) will be described by the vector

\[
(u, u_t, w) \in \mathcal{H} := H^1(\Omega) \times L^2_{\Theta_s}(\Omega) \times \mathcal{H}_f,
\] (9)

where the space \( \mathcal{H}_f \) is the closure in the \( L^2(\Omega_f) \)-topology of divergence-free vector fields vanishing on the boundary \( \Gamma_f \), thus obtaining

\[
\mathcal{H}_f := \left\{ z \in L^2(\Omega_f) : \text{div } z = 0, \quad (z, n)|_{\Gamma_f} = 0 \right\}.
\]

Accordingly, we define the energy of the system proportional to the square of an equivalent norm in \( \mathcal{H} \):

\[
E(w, u, u_t) := \frac{1}{2} ||w||^2_{L^2(\Omega_f)} + \frac{1}{2} ||u||^2_{L^2_{\Theta_s}(\Omega)} + \frac{1}{2} |u|^2_{1, \Omega} + \frac{1}{2} \gamma ||u||^2_{L^2_{\Theta_s}(\Omega)}.
\] (10)
with the usual abbreviation
\[ E(t) := E(w(t), u(t), u_t(t)). \]
By \( E_w \) we will indicate the fluid component of the energy:
\[ E_w(t) = E_w(w(t), 0, 0) = \frac{1}{2} \| w(t) \|^2_{L^2(\Omega_f)}, \quad (11) \]
and \( E_u \) will denote respectively the elastic component:
\[ E_u(t) := E(0, u(t), u_t(t)) = \frac{1}{2} \| u_t \|^2_{L^2(\Omega_s)} + \frac{1}{2} \| u \|^2_{L^2(\Omega_s)} \quad (12) \]
Let’s also define
\[ \mathbf{H}^1_{\gamma_f}(\Omega_f) := \{ \varphi \in \mathbf{H}^1(\Omega_f) : \varphi|_{\Gamma_f} = 0 \}. \quad (13) \]
From Korn’s inequality ([22, Thm. 6.15-4, p. 409]) it follows that we may define an equivalent inner product on the space \( \mathbf{H}^1_{\gamma_f}(\Omega_f) \) by
\[(\hat{w}, \hat{v}) \mapsto \int_{\Omega_f} \varepsilon(\hat{w}) \cdot \varepsilon(\hat{v}). \]
The entire system (6) can then be written as an evolution problem on \( \mathcal{H} \)
\[ y' = \mathbb{A}y \]
with the generator \( \mathbb{A} \) given by
\[ \mathbb{A}y = \mathbb{A} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} v \\ \det(\Theta_s) \operatorname{Div}(\mathcal{T}(u)) - \alpha v - \gamma u \\ \nu \Delta w - (Dw)\nu_s - (Dw)\nu - \nabla p \end{bmatrix} \quad (14) \]
The function \( \pi \) represents the elliptic solver which reconstructs the pressure of the fluid. In particular, by taking the divergence of the fluid equation and invoking the solenoidal condition on \( w \) we obtain an elliptic boundary value problem for variable \( p \). The pressure can then be expressed in terms of the fluid velocity \( w \) and elastic deformation \( u \) via a combination of appropriate Neumann and Dirichlet harmonic extension maps \( p = \pi(w, u) \). More precisely, the domain can be described, following [13, 6], as follows: \((u_0, w_0, v_0) \in \mathcal{H} \) belong to \( \mathcal{D}(\mathbb{A}) \) if and only if (restrictions denote trace maps)
- \( u_0 \in \mathbf{H}^1(\Omega) \) and \( \operatorname{Div}(\mathcal{T}(u)) \in L^2_{\partial_\Omega_s}(\Omega) \) in the sense of distributions.
- \( w_0 \in \mathbf{H}^1(\Omega) \cap \mathcal{H}_f \) \[ \mathbf{Remark} \]. Consequently \( \mathcal{T}(u)n|_\Gamma \) is well-defined in \( \mathbf{H}^{-1/2}(\Gamma) \) [13, Prop. 6.4].
- \( v_0 \in \mathbf{H}^1(\Omega) \) \[ \mathbf{Remark} \]. The conditions so far yield that the pressure \( p_0 = \pi(w_0, u_0) \), as a solution of an appropriate elliptic boundary value problem, and the fluid variable \( w_0 \) possess the following trace regularity on the full fluid boundary \( \partial\Omega_f = \Gamma \cup \Gamma_f \):
\[ p_0|_{\partial\Omega_f} \in H^{-1/2}(\partial\Omega_f), \quad \frac{\partial p_0}{\partial n} \in H^{-3/2}(\partial\Omega_f) \]
\[ [Dw]n|_{\partial\Omega_f}, \varepsilon(w)n|_{\partial\Omega_f} \in H^{-1/2}(\partial\Omega_f), \quad \langle \Delta w \rangle|_{\partial\Omega_f}, n \in H^{-3/2}(\partial\Omega_f). \]
Therefore, the stress matching condition can be consistently embedded as a part of the domain definition in the next bullet.
- \( \mathcal{T}(u_0)n|_\Gamma = -p_0 n + 2\varepsilon(w_0) + \mathcal{B}(u_0) \) as elements of \( \mathbf{H}^{-1/2}(\Gamma) \)
- \( v_0 \in \mathbf{H}^1(\Omega) \) (implying \( v_0|_\Gamma \in \mathbf{H}^{1/2}(\Gamma) \))
• If $K = 0$ (no viscous damping) then the velocity matching is incorporated into the domain via $w_0 + (Dw_s)u_0 = v_0$, the equality satisfied in $H^{1/2}(\Gamma)$. If $K > 0$, then an implicit requirement is $T(u_0)n \in H^{1/2}(\Gamma)$ and the condition reads $w_0 + (Dw_s)u_0 = v_0 + K T(u)n$, again as elements of $H^{1/2}(\Gamma)$.

The semigroup well-posedness result for (6) can be formulated as follows:

**Theorem 3.1.** Let $R_\varepsilon := W^{15/4+\varepsilon,4}(\Omega) \times W^{11/4+\varepsilon,4} (\Omega_f) \times W^{7/4+\varepsilon,4}(\Omega_f)$. There exists $r_0 > 0$ such that if the steady regime solution satisfies (for some, arbitrarily small, $\varepsilon > 0$)

$$||(\varphi_s - \text{Id}), w_s, p_s||_{\mathcal{R}_\varepsilon} = r < r_0$$

and if $w_s$ admits a $C^1$ extension to $\mathcal{D}$, then there exists $v_0(r) \geq 0$, such that for any viscosity $\nu > v_0$, system (6) generates a $C_0$ semigroup on the state space $\mathcal{H}$. Moreover, any weak solution $(u, w, \eta)$ possesses additional regularity

$$w \in L^2(0,T; H^1(\Omega_f)), w|_{\Gamma} \in L^2(0,T; H^{1/2}(\Gamma)) \quad \text{and} \quad T(u)n|_{\Gamma} \in L^2(0,T; L^2(\Gamma)).$$

This well-posedness argument follows as in [13] taking into account the modified domain of the generator as mentioned above. As an additional bonus, the dissipative formulation ($K > 0$) yields the indicated stronger regularity of the co-normal elastic trace. Now weak solutions have $T(u)n|_{\Gamma} \in L^2(\Gamma)$.

4. Main results.

**Theorem 4.1** (Stability without interior damping). Suppose $\alpha = 0$ and $\gamma > 0$. Let $\nu_0$, $r_0$ and $r$ be as in Theorem 3.1. Then there exist $\nu_1 \geq \nu_0$, $0 < r_1 \leq r_0$ and $K_1 > 0$ (dependent on $(w_s, p_s, u_s)$) such that for $r \leq r_1$ and $\nu > \nu_1$, $K > K_1$ the dynamical system generated by (6) is exponentially stable. In particular, there exist $C > 0$ and $\omega > 0$ such that the energy $E(t)$ defined by (10) satisfies

$$E(t) \leq CE(0)e^{-\omega t}, \quad \text{for } t \geq 0.$$

**Theorem 4.2** (Stability with interior damping). Suppose $\alpha > 0$ and $\gamma > 0$, assume that in addition to the condition $\nu > \nu_0$ furnished by the well-posedness theorem (see Thm. 3.1), the viscosity value also admits some $\eta \in (0,1)$ such that

$$\int_{\Omega_f} \langle w, (Dw)w_s + (Dw_s)w \rangle \leq 2\eta \nu ||w||^2_{H^1(\Omega_f)}.$$ 

Then there exists $K_1 > 0$ dependent on $\alpha$, $\gamma$, $\nu$ (and the norm of $(w_s, p_s, u_s)$) such that for $K > K_1$ the dynamical system generated by (6) is exponentially stable.

Naturally, when $\alpha > 0$, the assumptions beyond those asserted by the well-posedness theorem are relatively precise, and the only “large” magnitude condition is on the coefficient $K$ of the boundary damping. The stability result for $\alpha = 0$ requires a more general hypothesis.

**Remark 4.1** (Working around general equilibria). The results here rely on the low energy of the steady regime about which the linearization takes place. A potential extension would be to relax this condition and instead enforce the smallness on the proximity (in the state space topology) between the steady regime and the initial data. The closeness to an equilibrium condition for Navier-Stokes and fluid-structure coupled models has been a topic of active research [10, 8, 9, 52, 57, 58]. The stabilization in this case would be to the nearby steady regime rather than to the zero state. Such a generalization would naturally require a further revision.
of the linearized model and the feedback, but the analysis would benefit from the techniques used in the present case.

5. **Toolbox of multiplier identities.** To obtain the desired energy estimates and infer stability we will need a “toolbox” of several multiplier identities. The (weighted) multiplier method is somewhat coarser than spectral estimates in the context of linear semigroup problems, but is more tractable when dealing with boundary controls, variable coefficients, energy-level perturbations and nonlinearities. Multiplier identities together with unique continuation results form a powerful (and more streamlined) alternative to Lyapunov function techniques. Multiplier approaches to boundary stabilization were pioneered in [59, 62, 18, 47, 50, 53]; for a more general discussion see the classical texts [48, 46, 51]. These techniques could also be viewed through the lens of Carleman estimates whose applications to boundary control problems were explored in [70, 71, 54, 1, 72]. The complete account of pertinent publications of course extends far beyond what is listed above. Among the later works we would like to mention long-time behavior analysis via multipliers in the papers by Chueshov et al. [16, 19, 20], as well as the actively cited here study of the Stokes-Lamé system [5].

As usual, all derivations and estimates below are performed on strong semigroup solutions and extended to weak ones by density of the generator domain and the fact that the concluding inequalities are continuous with respect to the finite energy space $\mathcal{H}$.

5.1. **Energy identity.** First, apply the multiplier $w$ to the fluid equation to get the energy identity for the fluid component. Recall that on the common interface $n$ and $n_f = -n$ denote the outward normal fields with respect to the solid and fluid domains respectively.

\[
E_w(T) + 2\nu \int_0^T \|w\|^2_{H^1(\Omega_f)} = E_w(0) - \int_0^T \int_{\Omega_f} \langle w, (Dw)w_s + \sigma(w,p)n \rangle.
\]  

In turn, the multiplier $u_t$ for the solid equation yields the energy relation for the elastic solid:

\[
E_u(T) + \alpha \int_0^T \|u_t\|^2_{L^2_{\sigma}(\Omega)} = E_u(0) + \int_0^T \int_{\Gamma} \langle u_t, \sigma(u)n \rangle.
\]  

From (16) we derive a couple of useful modifications to be used later. First, if we replace the lower limit of integration 0 by $t$ in (16) and apply the boundary condition $u_t = w + (Dw)s - KT(u)n$, then we get after a rearrangement:

\[
E_u(t) = E_u(T) + \alpha \int_t^T \|u_t\|^2_{L^2_{\sigma}(\Omega)} + K \int_t^T \int_{\Gamma} \|T(u)n\|^2 - \int_t^T \int_{\Gamma} \langle w + (Dw)s, T(u)n \rangle.
\]  

And if we integrate (17) over $t \in [\delta, T - \delta]$ (for $0 < \delta < T/2$ to be specified later), we obtain:

\[
(T - 2\delta)E_u(T) + \alpha \int_\delta^{T-\delta} \int_t^T \|u_t\|^2_{L^2_{\sigma}(\Omega)} + \int_{\Gamma} \int_{\Gamma} K |T(u)n|^2.
\]
\[
T^{-\delta} \int_\delta^T E_u(t) dt + \int_\delta^T \int_{\Gamma} \langle w + (Dw_x)u, T(u)n \rangle \quad (18)
\]

Adding (15) and (16) and applying the two boundary conditions of (6) yields the main energy identity for the system:

\[
E(T) + 2\nu \int_0^T \|w\|^2_{H^1(\Omega_T)} + \alpha \int_0^T \|u_t\|^2_{L^2(\Omega_T)} + K \int_0^T \int_\Gamma |T(u)n|^2
=E(0) - \int_0^T \int_{\Omega_T} \langle w, (Dw)w_x + (Dw_x)w \rangle + \int_0^T \int_{\Gamma} \langle (Dw_x)_u, T(u)n \rangle
+ \int_0^T \int_{\Gamma} \langle w, B(u) \rangle
\quad (19)
\]

A notable difference from the Stokes-elasticity scenario is that the very last term, containing \(B(u)\), cannot be directly absorbed into the left-hand side, and thus renders this energy law non-dissipative. So one requires some kind of an estimate on \(\int_0^T \|u\|^2_{H^1(\Omega_T)}\) not only to obtain a suitable stabilization inequality, but also to make the energy identity “usable” to begin with.

5.2. “Equipartition” of energy. Next, the “equipartition” multiplier \(u\) for the solid equation leads to:

\[
(u, u_t)_{L^2(\Omega_T)} - \int_0^T \|u_t\|^2_{L^2(\Omega)} + \int_0^T |u|^2_{1,\Omega} - \int_0^T \int_{\Gamma} \langle u, T(u)n \rangle
+ \gamma \int_0^T \|u\|^2_{L^2(\Omega_T)} + \frac{\alpha}{2} \|u\|^2_{L^2(\Omega_T)} = 0.
\quad (20)
\]

Consider the following (to be used later) rearrangement of this identity. Relabel the integration interval into \([\delta, T - \delta]\) and pick some positive parameter \(\rho > 0\) (both \(\rho\) and \(\delta\) are small, to be specified later). Then consider a \((1 - \rho)\) multiple of the preceding identity and rearrange it as follows:

\[
\int_\delta^{T-\delta} \left(\|u_t\|^2_{L^2(\Omega_T)} - |u|^2_{1,\Omega} - \gamma \|u\|^2_{L^2(\Omega_T)} \right)
= \rho \int_\delta^{T-\delta} \|u_t\|^2_{L^2(\Omega_T)} - \rho \int_\delta^{T-\delta} \|u\|^2_{L^2(\Omega_T)} - \gamma \|u\|^2_{L^2(\Omega_T)}
+ (1 - \rho)(u, u_t)_{L^2(\Omega_T)} \left|_{T-\delta}^{T} \right. + \frac{(1 - \rho)\alpha}{2} \|u\|^2_{L^2(\Omega_T)} \left|_{\delta}^{T-\delta} \right.
- (1 - \rho) \int_\delta^{T-\delta} \int_{\Gamma} \langle u, T(u)n \rangle
\quad (21)
\]

5.3. “Flux” multiplier \([Du]/h\). The following derivations will be used in the absence of damping in the solid interior, i.e., for \(\alpha = 0\). Then we need an estimate on the elastic energy via boundary terms only. Such a result is often derived with the help of the classical “flux” multiplier of the form \([Du]/h\), for a smooth vector field \(h\), applied to the elasticity equation. This step, however, becomes substantially more involved and delicate than in classical treatments [5, pp. 350-53] due to the complex structure of the stress tensor \(T(u)\).
Some of the terms in the ensuing identities will be bounded via integrals on a larger time interval (in particular Proposition 6.1). So for convenience of subsequent notation, the integration in time for this multiplier will start out not over $[0, T]$ but its proper subset interval $[\delta, T - \delta]$ for some $0 < \delta < T/2$. Take the $L^2_{\Theta_s}(\Omega)$ inner product of the elasticity equation with $[Du|h)$, and proceed term by term. First, $u_{tt}$ gives us

$$
\int_\delta^{T-\delta} (u_{tt}, [Du|h) L^2_{\Theta_s}(\Omega)
$$

$$= (u_t, [Du|h) L^2_{\Theta_s}(\Omega) \bigg|_{\delta}^{T-\delta} - \frac{1}{2} \int_\delta^{T-\delta} (\nabla|u_t|^2, h) L^2_{\Theta_s}(\Omega)
$$

$$= (u_t, [Du|h) L^2_{\Theta_s}(\Omega) \bigg|_{\delta}^{T-\delta} + \frac{1}{2} \int_\delta^{T-\delta} (|u_t|^2, \text{div} h) L^2_{\Theta_s}(\Omega)
$$

$$+ \frac{1}{2} \int_\delta^{T-\delta} \int_\Omega |u_t|^2 \left\langle \nabla \left( \frac{1}{\text{det} \Theta_s} \right), h \right\rangle - \frac{1}{2} \int_\delta^{T-\delta} \int_\Gamma \frac{1}{\text{det} \Theta_s} |u_t|^2 \langle h, n \rangle
$$

Here we remark that $|u_t|^2 \left\langle \nabla \left( \frac{1}{\text{det} \Theta_s} \right), h \right\rangle$ is a quadratic form in $u_t$ and each of the coefficients is proportional for some partial derivative of a component of $\Theta_s$. The well-posedness conditions in Theorem 3.1 (in particular, the bound on $\varphi_s - \text{Id}$ in $W^{15/4+\varepsilon, 4}(\Omega)$) assert smallness of these partial derivatives which will be invoked later. For the time being, label

$$C_0[D\Theta_s]|u_t, u_t| := |u_t|^2 \left\langle \nabla \left( \frac{1}{\text{det} \Theta_s} \right), h \right\rangle
$$

Next, the product of $[Du|h$ with the stress tensor:

$$- (\det(\Theta_s) \text{Div}([T(u)]), [Du|h) L^2_{\Theta_s}(\Omega) = - \int_\Omega \langle \text{Div} T(u), (Du)h \rangle =
$$

$$= \int_\Omega T(u)[Du][Dh] + \int_\Omega T(u)[(D^2u)h] - \int_\Omega [(T(u)n), (Du)h]
$$

Note that $D^2u$ is a 3-tensor. In general, for a matrix $M$ define the tensor action of $DM$ on a vector $v$ to be the $i-k$ matrix (using the summation convention)

$$[(DM)v]_{ik} := (\partial_k M_{ij})v_j
$$

(22)

So, accordingly, for $(D^2u)h$ we compute $(D[Du])h$, which gives:

$$[(D^2u)h]_{ik} := (\partial_k^2 u_i)h_j
$$

In particular, the $i$-th row of $(D^2u)h$ could also be written as $[(D^2u_i)h]^*$. The bulk of the discussion will be on how to rewrite the product $([Tu]..[(D^2u)h])$ From the definition (4) of $T$ we have

$$T(u)\Rightarrow [(D^2u)h] = Du[T_s..[(D^2u)h] + \frac{1}{\text{det} \Theta_s} \Theta_s \Sigma(Du)\Theta_s^*..[(D^2u)h]
$$

We will analyze the two summands in this order.

For the subsequent calculations with this multiplier, gradients will be interpreted as row-vectors (to be consistent with matrix multiplication and interpreting them as elements of $R^d$-dual):
1. **Analysis of** $Du \mathcal{T}_s \cdot [(D^2 u) h]$. First, look at the scalar product of the $i$-th row of $Du \mathcal{T}_s$ and the corresponding row of $[(D^2 u) h]$. It has the form

$$\langle \nabla u_i \rangle_{\mathcal{T}_s} [(D^2 u)_i] h \quad (23)$$

Recall that matrix $\mathcal{T}_s$ is symmetric. Further note the following identity for any symmetric space-dependent matrix field $M$:

$$\nabla(v^t M v) = 2 v^t M Dv + v^t [(DM) v]$$

(where tensor contraction $(DM)v$ is defined as in (22)). Substituting $v^t = \nabla u_i$ and $M = \mathcal{T}_s$ we have $Dv = D^2 u_i$ and

$$\langle \nabla u_i \rangle_{\mathcal{T}_s} [(D^2 u)_i] = \frac{1}{2} \nabla \langle \nabla u_i \rangle_{\mathcal{T}_s} [(\nabla u_i)^*] - \frac{1}{2} \langle \nabla u_i \rangle [(D\mathcal{T}_s)(\nabla u_i)^*]$$

This identity leads to:

$$\int_{\Omega} Du \mathcal{T}_s \cdot [(D^2 u)_i] h = \frac{1}{2} \int_{\Omega} \nabla \text{Tr} [Du \mathcal{T}_s (Du)^*] h - \frac{1}{2} \sum_{i=1}^{3} \int_{\Omega} \langle \nabla u_i \rangle [(D\mathcal{T}_s)(\nabla u_i)^*] h$$

$$= - \frac{1}{2} \int_{\Omega} \text{Tr} [Du \mathcal{T}_s (Du)^*] \text{div} h - \frac{1}{2} \sum_{i=1}^{3} \int_{\Omega} \langle \nabla u_i \rangle [(D\mathcal{T}_s)(\nabla u_i)^*] h$$

$$+ \frac{1}{2} \int_{\Gamma} \text{Tr} [Du \mathcal{T}_s (Du)^*] \langle h, n \rangle$$

Writing $\text{Tr} [Du \mathcal{T}_s (Du)^*] = Du \mathcal{T}_s \cdot Du$, we can abbreviate this derivation as follows:

$$\int_{\Omega} Du \mathcal{T}_s \cdot [(D^2 u)_i] h = - \frac{1}{2} \int_{\Omega} (Du \mathcal{T}_s \cdot Du) \text{div} h + \frac{1}{2} \int_{\Omega} (Du \mathcal{T}_s \cdot Du) \langle h, n \rangle$$

$$+ \int_{\Gamma} C_1 [D\mathcal{T}_s] \{Du, Du \}$$

where $C_1 [D\mathcal{T}_s] \{Du, Du \}$ a quadratic form in $Du$ with the magnitude of the coefficients controlled by the $L^\infty(\Omega)$ norms of the entries of $D\mathcal{T}_s$ (for Cauchy stress tensor $\mathcal{T}_s$ of the state $(w_s, \rho_s, u_s)$).

2. **Analysis of** $\frac{1}{\det(\Theta_s)} \Theta_s \Sigma(D\overline{u}) \Theta_s^* \cdot [(D^2 u) h]$. First rewrite $D^2 u_i$ via the $i$-th row of the differential $Du$ as

$$D^2 u_i = D[(Du)_i]$$

Next, since $D\overline{u} = \Theta_s^* Du \Theta_s$ for an invertible matrix $\Theta_s$ (close to identity by well-posedness assumptions), we can write

$$Du = \Theta_s^{-*} (D\overline{u}) \Theta_s^{-1},$$

Then we have

$$D^2 u_i = D[(\Theta_s^{-*} D\overline{u} \Theta_s^{-1})_i]$$

which in component form reads

$$[D^2 u_i]_{jk} = \partial_h [\Theta_s^{-*}]_{ip} (D\overline{u})_{pq} [\Theta_s^{-1}]_{qj}$$

We can write it as

$$[D^2 u_i]_{jk} = [\Theta_s^{-*}]_{ip} (\partial_h D\overline{u})_{pq} [\Theta_s^{-1}]_{qj} + C_{jk} [D\Theta_s] \{Du\}$$
where $C_{jk}[D\Theta_s]\{Du\}$ is linear in $Du$ with the magnitude of the coefficients controlled by the $L^\infty(\Omega)$ norm of $D\Theta_s$. Likewise, in component form, using summation convention:

$$[\Theta_s \Sigma(D\vec{u})\Theta_s^*]_{ij} = [\Theta_s]_{ip}[\Sigma(D\vec{u})]_{pq}[\Theta_s^*]_{qj}$$

From this representation it follows that for each $k$:

$$[\Theta_s^{-1}]_{ip}(\partial_k D\vec{u})_{pq}[\Theta_s^*]_{qj} = \Sigma(D\vec{u})..\partial_k D\vec{u} = \frac{1}{2}(\partial_k (\Sigma(D\vec{u})..D\vec{u}))$$

Thus

$$\int_\Omega \frac{1}{\det \Theta_s} \Sigma(D\vec{u})\Theta_s^*..(D^2u)h$$

$$= \int_\Omega \frac{1}{2 \det \Theta_s} \nabla (\Sigma(D\vec{u})..D\vec{u}) h + \int_\Omega C_2[D\Theta_s]\{Du, Du\}$$

$$= -\frac{1}{2} \{\Sigma(D\vec{u})..D\vec{u}, \text{div} h\} \Omega \right)_s \right)\{Du, Du\} + \frac{1}{2} \int_\Gamma \frac{1}{\det \Theta_s} \Sigma(D\vec{u})..D\vec{u} (h, n)$$

$$+ \int_\Omega C_3[D\Theta_s]\{Du, Du\}$$

Here the terms $C_i[D\Theta_s]\{Du, Du\}$, as before, denote quadratic forms in $Du$ with coefficients proportional to $D\Theta_s$ entries.

Altogether, multiplier $(Du)h$ applied to $-\det \Theta_s \text{Div} (T(u))$ gives the following identity:

$$- \left( \text{det}(\Theta_s) \text{Div} (T(u)), [Du]h \right) \Omega \right)_s \Omega \right)\{Du, Du\}$$

$$= \int_\Omega (T(u)..[Du][Dh] - \frac{1}{2} \int_\Omega (T(u)..Du) \text{div} h$$

$$+ \int_\Omega C_3[D\Theta_s, D\Theta_s]\{Du, Du\} + \frac{1}{2} \int_\Gamma (T(u)..Du) (h, n) - \int_\Omega (T(u)n, [Du]h)$$

(24)

The two remaining terms in the elasticity equation are $\alpha u$ and $\gamma u$. The term $(\alpha u, [Du]h)$ will be left as is, primarily because we will be using these derivations when $\alpha = 0$. And when multiplier $[Du]h$ is applied to the $\gamma u$ we get:

$$(\gamma u, [Du]h) \Omega \right)_s \Omega \right)\{Du, Du\} = \gamma \int_\Omega \frac{1}{\det \Theta_s} \left( \frac{1}{2} \nabla |u|^2, h \right)$$

$$= -\frac{\gamma}{2} \left( |u|^2, \text{div} h \right) \Omega \right)_s \Omega \right)\{Du, Du\} + \gamma \int_\Gamma \frac{1}{\det \Theta_s} |u|^2 (h, n) - \frac{\gamma}{2} \int_\Omega \left( \nabla \frac{1}{\det \Theta_s}, h \right) |u|^2.$$ 

Now use the basic radial choice for the vector field $h$: fix any $x_0 \in \mathbb{R}^d$ and let $h(x) := x - x_0$

**Remark 5.1.** With no geometric condition imposed, the location of $x_0$ is irrelevant. See Remark 6.1 for additional comments.

Consequently,

$$\text{div} h = 3 \quad \text{and} \quad Dh = \text{Id}.$$ 

Integrate (24) over $t \in [\delta, T - \delta]$. Furthermore, for convenience and exposition, we introduce two generic labels:
$X_s$ to denote the combined “small” terms:

$$X_s = C_0[D\Theta_s]\{u_t, u_t\} + C_4[D\mathcal{T}_s, D\Theta_s]\{Du, Du\} + C_5[D\Theta_s]\{u, u\},$$

(25)

namely, a sum of quadratic forms in $Du, u$ and $u_t$ each with $L^\infty(\Omega)$ coefficients whose magnitudes are controlled by the $L^\infty(\Omega)$ norms of $D\mathcal{T}_s$ and $D\Theta_s$. These terms will eventually be asserted small in line with the underlying assumptions on the $R$-norm in Theorem 3.1.

- $X_{tr,0}$ to denote integrals on the boundary—the “trace terms”.

Combining the results of multiplication of $u_{tt}$ and of $-\det\Theta_s\text{Div}(\mathcal{T}(u))$ by $[Du]h$, for the above “radial” choice of $h$ yields:

$$\frac{3}{2} \int_\delta^{T-\delta} \left( \|u_t\|^2_{L^2_{\delta_\omega}(\Omega)} - \|u\|^2_{L^2_{\delta_\omega}(\Omega)} - \frac{\gamma}{2} \|u\|^2_{H^1_{\omega}(\Omega)} \right) + \int_\delta^{T-\delta} \int_\Omega \mathcal{T}(u)_t [Du]h$$

(26)

Now invoke a 3/2 multiple of the identity (21) to rewrite the first integral on the left-hand side of (26):

$$\frac{3\rho}{2} \int_\delta^{T-\delta} \|u_t\|^2_{L^2_{\delta_\omega}(\Omega)} + \int_\delta^{T-\delta} \int_\Omega \mathcal{T}(u)_t [Du]h$$

$$+ \frac{3(1-\rho)}{2} \gamma \int_\delta^{T-\delta} \|u\|^2_{L^2_{\delta_\omega}(\Omega)} + \frac{3(1-\rho)\alpha}{4} \|u\|^2_{H^1_{\omega}(\Omega)}$$

(27)

$$= \frac{3\rho}{2} \int_\delta^{T-\delta} \|u_t\|^2_{L^2_{\delta_\omega}(\Omega)} + \gamma \|u\|^2_{H^1_{\omega}(\Omega)} + X_s$$

$$- \gamma \int_\delta^{T-\delta} \int_\Omega \mathcal{T}(u)_t [Du]h$$

$$- \left( u_t, [Du]h + \frac{3(1-\rho)\alpha}{4} u \right)_{L^2_{\delta_\omega}(\Omega)} + X_{tr,1}$$

The term $X_{tr,1}$ introduced above consists of $X_{tr,0}$ plus the trace terms arising from (21) amounting to:

$$X_{tr,1} := \frac{1}{2} \int_\delta^{T-\delta} \int_\Omega \left( \frac{1}{\det\Theta_s} |u_t|^2 - \mathcal{T}(u)_t [Du]h - \frac{\gamma}{\det(\Theta_s)} |u|^2 \right) \langle h, n \rangle$$

(28)
6. **Proof of Theorem 4.1.** Throughout this section it will be assumed that $\alpha = 0$. Let us recall the now-standard in control literature concept of **lower order terms**. The generic label

$$l.o.t.(u)$$

will denote quadratic forms in $u$ and its first-order partial derivatives, such that the time-integral over $\Omega \times (0, T)$ of these forms can be estimated by an arbitrarily small multiple of the energy $E_{u}$ and norms of $u$ and $u_{t}$ in spaces that admit compact embeddings of $H^{1}(\Omega)$ and $L^{2}(\Omega)$ (or $L^{2}_{\alpha t}(\Omega)$) respectively.

As one particular example, we could, for instance, label

$$u_{t}w, Dw_{s}, Dw_{t}$$

relative to $\nu$ and $K$, in the energy identity (19) to obtain the following inequality:

$$E(T) + C_{0} \int_{0}^{T} \| w \|_{H^{1}(\Omega_{T})}^{2} + K_{0} \int_{0}^{T} \int_{\Gamma} |T(u)n|^{2} \leq E(0) + c_{B} \int_{0}^{T} \| w \|_{H^{1}(\Omega)}^{2}$$  \hspace{1cm} (29)

The constant $c_{B}$ above depends on the $L^{\infty}(\Gamma)$-norms of the coefficients in $\mathcal{B}$ which in turn are dictated by the energy of the steady regime $(w_{s}, p_{s}, u_{s})$ about which linearization is performed.

The desired exponential stability will follow if we can verify the following stabilization inequality:

**Lemma 6.1.** There exists $\bar{T} > 0$ and a a constant $C_{\bar{T}}$ such that every solution obeys the estimate:

$$E(\bar{T}) + \int_{0}^{\bar{T}} E_{u}(t) dt \leq C_{\bar{T}} \left( C_{0} \int_{0}^{\bar{T}} \| w \|_{H^{1}(\Omega_{T})}^{2} + K_{0} \int_{0}^{\bar{T}} \int_{\Gamma} |T(u)n|^{2} \right)$$  \hspace{1cm} (30)

Indeed, assuming that Lemma 6.1 is true, then we can use (30) back into (29). The norm of the steady regime, hence the constant $c_{B}$, needs to be small enough so that the term $c_{B} \int_{0}^{\bar{T}} \| w \|_{H^{1}(\Omega)}^{2}$ can be absorbed into the $1/C_{\bar{T}}$ multiple of the integral of $E_{u}$ on the left. Note that $c_{B}$ originates from estimating the product $\int_{T} \langle w, B(u) \rangle$, so $c_{B}$ can be made small either by requiring a low energy of the steady regime $(w_{s}, p_{s}, u_{s})$ or, via Young’s inequality, by asserting a larger viscosity (or a combination of both these factors). Then we arrive at the typical inequality

$$E((m + 1)T) \leq \rho_{T}E(mT) \quad \text{for all} \quad m \in \mathbb{N}, \quad \text{with} \quad \rho_{T} \in (0, 1),$$
which implies exponential decay of the sequence $E(mT) \leq C_0 E(0)e^{-m\omega_0}$. At the same time, from the energy inequality (29) (where $T > 0$ is arbitrary) we have, for every $t \in [mT,(m+1)T]$, the following Grönwall’s estimate

$$E(t) \leq E(mT)e^{c_1 T}, \quad mT \leq t \leq (m+1)T$$

(with $c_1$ proportional to $c_E$ and independent of the solution). This estimate and the exponential decay of the energy: with $t = mT + \tau$ for $\tau \in [0,T]$ we have

$$E(t) \leq E(mT)e^{c_1 \tau} \leq C_0 E(0)e^{(mT-\tau)\omega_0}e^{c_1 T} =: CE(0)e^{-t\omega}.$$  

The rest of the argument is devoted to the proof of Lemma 6.1.

### 6.2. Stabilization estimate for the elastic energy

From the fundamental identity (27) we obtain the following inequality:

$$\int_{T-\delta}^{T} E_u(t) dt \leq C_{DT,\delta,\Theta_s} \int_{T-\delta}^{T} \|u\|_{H^1(\Omega)}^2 dt + C_1 (E_u(\delta) + E_u(T-\delta)) + |X_{tr,1}| + \text{l.o.t.1}(u)$$

The constant $C_{DT,\delta,\Theta_s}$ stems from the term $X_s$ (see (25)) and can be estimated in terms of the $L^\infty(\Omega)$ norms of partial derivatives of entries in $T_s$ and $\Theta_s$. The energies $E_u(\delta)$ and $E_u(T-\delta)$ can be rewritten via (17) with $t = \delta, T-\delta$ respectively:

$$E_u(\delta) + E_u(T-\delta) = 2E_u(T) + K \left( \int_0^T + \int_{T-\delta}^T \right) |T(u)n|^2 - \left( \int_0^T + \int_{T-\delta}^T \right) \int_{\Gamma} \langle w + (Dw_s) u, T(u)n \rangle$$

which we can then bound by:

$$E_u(\delta) + E_u(T-\delta) \leq 2E_u(T) + C_2 \left( \int_0^T \int_{\Gamma} |T(u)n|^2 + \int_0^T \|w\|_{H^1(\Omega)}^2 \right) + \text{l.o.t.2}(u)$$

**Proposition 6.1** (Boundary terms). For trace products in $X_{tr,1}$ as defined by (28) pick any $0 < \delta < T/2$. Then there exists a constant $C_{3,T,\delta}$ dependent on $T$ and $\delta$, such that

$$|X_{tr,1}| \leq C_{3,T,\delta} \left( \int_0^T \int_{\Gamma} u_i^2 + \int_0^T \int_{\Gamma} |T(u)n|^2 + \text{l.o.t.3}(u) \right)$$

from which via the boundary condition it follows that

$$|X_{tr,1}| \leq C_{4,T,\delta} \left( \int_0^T \|w\|_{H^1(\Omega)}^2 + \int_0^T \int_{\Gamma} |T(u)n|^2 + \text{l.o.t.4}(u) \right)$$

**Proof.** At the first stage it is necessary to show that the traces of all the first-order derivatives of $u$ can be expressed in terms of the co-normal term $T(u)n$ and tangential derivatives of $u$ on $\Gamma$. To this end, note that if $\Theta_s = 1$ and $T_s = 0$, then $T(u)$ reduces to the classical isotropic elastic stress operator $\Sigma(Du)$. In that case, there is an invertible linear map (with the determinant dependent on the Lamé parameters only) that expresses $Du_{\Gamma}$ in terms of $\Sigma(Du)_{\Gamma}$ and $D_{\Gamma}u_{\Gamma}$, [5, Appendix] (also see [4, p. 7] for a related result for the Mindlin plate operator). The determinant of the transformation changes continuously with respect to the $L^\infty$ norms of the coefficients. Hence the smallness assumptions on $\Theta_s - Id$ and
$T_s$ can be sufficiently restrictive to ensure that the same result holds for the stress given by $T(u)$. The tangential estimate then directly follows the strategy of [41, Thm. 4.1] (also see [42]).

For the next proof ingredient, use (18) to get the inequality

$$ (T - 2\delta) E_u(T) \leq \int_\delta^T E_u(t)dt + C_5 \int_0^T \|w\|^2_{H^1(\Omega_f)} + C_5 \int_0^T \left|\mathcal{T}(n)\right|^2 + \text{l.o.t.}_6(u) $$

Henceforth the lower order terms l.o.t.($u$) will denote the combined l.o.t.($u$) from the above estimates. These obey the following inequality:

**Proposition 6.2** (Lower order terms). There exists $T_1 \geq 0$ such that for each $T \geq T_1$, there is $C_{5,T} > 0$ depending on $T$, but independent of the solution, such that for every $\varepsilon > 0$,

$$ \text{l.o.t.} (u) \leq \varepsilon \int_0^T E_u(t)dt + \frac{1}{\varepsilon} C_{5,T} \left( \int_0^T \|w\|^2_{H^1(\Omega_f)} + \int_0^T \left|\mathcal{T}(n)\right|^2 \right) $$

(37)

**Proof.** The argument is carried out using the standard compactness-uniqueness strategy [56, 55, 49, 53], or in fluid-structure context, see [2]. The heart of the argument is the unique continuation result for the problem where the right-hand side terms of (37) are zero. Thus one must consider an elastic system:

$$ z_{tt} - \det(\Theta_s) \text{Div} \mathcal{T}(z) + \gamma z = 0 \quad \text{in} \quad \Omega, \quad t \geq 0 $$

$$ \mathcal{T}(z)n = 0 \quad \text{and} \quad z_t = (Dw_s)z \quad \text{on} \quad \Gamma, \quad t \geq 0 $$

and verify that it has only the zero solution.

Since $z_t \neq 0$ on the boundary, then after differentiation in time one does not get an overdetermined hyperbolic problem for $v = z_t$, which is what typically happens in such proofs. Instead, we appeal to the fundamental identity (27) for the $z$-system.

Therein, choose $\rho$ small and absorb the quantity $\frac{3\rho}{2} \int_\delta^T ([z_{1,1}]^2_1 + \gamma \|z\|^2_{\mathbf{L}^2_{\Theta_s}(\Omega)}) + X_s$ into the left-hand side which leaves us with:

$$ \int_\delta^T E_z(t)dt \leq E_z(\delta) + E_z(T - \delta) + \frac{1}{2} \int_\delta^T \int_\Gamma \frac{1}{\det(\Theta_s)} |z_t|^2 (h, n) $$

$$ - \frac{1}{2} \int_\delta^T \int_\Gamma \mathcal{T}(z) .. D_z (h, n) $$

(39)

**Remark 6.1** (Geometry). It is interesting to note that one could not merely ask for a geometric condition $(h, n) \geq 0$ ($\Omega$ star-shaped with respect to $x_0$) to quickly dispense with the higher-order trace terms, by claiming $\mathcal{T}(z) .. D_z \geq 0$. Recall that

$$ \mathcal{T}(z) .. D_z = \frac{1}{\det(\Theta_s)} \Sigma(\partial D_z) .. \partial D_z + D_z .. \mathcal{T}_s D_z. $$

Matrix $T_s$ comes from the Cauchy stress tensor for the nonlinear steady regime about which we linearized our model; its entries and their derivatives are, by assumption, small, and when both the above terms are integrated over $\Omega$, the integral of $\frac{1}{\det(\Theta_s)} \Sigma(\partial D_z) .. \partial D_z$ dominates the total sum by virtue of Korn’s inequality. On the boundary, this argument no longer holds as the two quantities must be compared point-wise. The value of $\Sigma(\partial D_z) .. \partial D_z$ is always non-negative. However, the matrix $T_s$, though symmetric, is not necessarily positive definite. Thus $D_z .. \mathcal{T}_s D_z$ may take
negative values, possibly proportional to the derivatives that are not controlled by
\(\Sigma(Dz), Dz\).

Now invoke the trace estimate (34) to bound \(T(z), Dz\). Since \(T(z)|_r = 0\) and
\(z_t = (Dw_s)z\) then the right-hand side only contains lower order terms. Use the
interpolation inequality to estimate the resulting lower order terms via a small
multiple of the energy \(E_z\) and a large multiple of the \(L^2(\Omega)\) norm. Consequently,
for any \(\varepsilon > 0\) we have
\[
\int_\delta^{T-\delta} E_z(t)\,dt \leq E_z(\delta) + E_z(T - \delta) + \varepsilon \int_0^T E_z(t)\,dt + C_{\varepsilon,T} \int_0^T \|z\|_{L^2(\Omega)}^2.
\]

Now appeal to the energy identity (16), which for the \(z\)-system (38) implies conser-
vation:
\[E_z(t) = E_z(0) \quad \text{for } t \geq 0.\]

Hence the inequality we have is:
\[(T - 2\delta)E_z(0) \leq 2E_z(0) + \varepsilon T E_z(0) + C_{\varepsilon,T} \int_0^T \|z\|_{L^2(\Omega)}^2.\]

For small \(\delta, \varepsilon\) and large enough \(T = T_1\) we get
\[E_z(0) \leq C_{T_1} \int_0^{T_1} \|z\|_{L^2(\Omega)}^2.\]

Now following [11] we see that the set of such solutions \(z\) must be a finite-dimensional
subspace of \(H^1(\Omega \times (0, T_1))\), and time-derivatives of these solutions are also in
\(H^1(\Omega \times (0, T_1))\). Hence \(\partial_t\) is a linear operator on the solution space. Assuming this
space is non-trivial, \(\partial_t\) has a real eigenvalue \(\theta\) with eigenfunction \(\tilde{z}\) which satisfies
the elliptic problem
\[\theta^2\tilde{z} - \det(\Theta_s) \text{Div} \, T(\tilde{z}) + \gamma \tilde{z} = 0 \quad \text{with} \quad T(\tilde{z}) n|_r = 0.\]

This problem admits only zero solution (see, e.g., [13]) which contradicts \(\tilde{z}\) being
an eigenfunction of \(\partial_t\). Hence the solution space for the \(z\)-system (38) must be \(\{0\}\),
i.e.,
\[z \equiv 0.\]

This step verifies the unique continuation argument for our system. The rest of the
steps in the compactness-uniqueness proof follow the classical strategy: for instance,
see [2].

With Proposition 6.2 in place, we can put the estimates together:

- Start with (32), invoke the assumption on the smallness of \(DT_s\) and \(D\Theta_s\) to
  assert that \(c - C_{Dz,D\Theta_s} > 0\) (thus absorbing the first right-hand term of (32)
  into the left-hand side);
- Then bound \(E_u(\delta) + E_u(T - \delta)\) using (33);
- Use a fraction of the energy integral, e.g. \(\frac{1}{2} \int_\delta^{T-\delta} E(t)\,dt\), to estimate the corre-
sponding fraction of (36);
- Finally bound the lower order terms using (37).
After some rearrangement and relabeling of the constants we obtain that for some small $\eta_0 > 0$ and any $\varepsilon_0 > 0$,
\[
(T - 2\delta)E_u(T) + \eta_0 \int_0^T E_u(t) dt \\
\leq CE_u(T) + \varepsilon_0 \int_0^T E_u(t) dt + C_{T,\varepsilon_0,\delta} \left( \int_0^T \int_{\Gamma} |T(u)n|^2 + \int_0^T \|w\|^2_{H^1(\Omega_f)} \right)
\]
(40)
where $T \geq T_1$ from Proposition 6.2. The coefficient $C$ in front of $E_u(T)$ is itself independent of $T$. Thus further increasing, if needed, time $T$ to some $\bar{T} \geq T_1$ permits to state for an arbitrarily small $\varepsilon > 0$,
\[
E_u(\bar{T}) + \int_0^{\bar{T}} E_u(t) dt \leq \varepsilon \int_0^2 E_u(t) dt + C_{2T,\varepsilon,\delta} \left( \int_0^{\bar{T}} \int_{\Gamma} |T(u)n|^2 + \int_0^T \|w\|^2_{H^1(\Omega_f)} \right) -
\]
(41)
\[
\int_{I_\delta} E_u(t) dt \leq 2\delta E_u(\bar{T}) + C_{K,\eta} \int_0^{\bar{T}} \int_{\Gamma} |T(u)n|^2 + \int_0^{\bar{T}} \|w\|^2_{H^1(\Omega_f)} + \eta C_{D,\omega,\varepsilon} \int_0^{\bar{T}} \|u\|^2_{H^1(\Omega)}.
\]
Since $|I_\delta| = 2\delta$, then
\[
\int_{I_\delta} E_u(t) dt \leq 2\delta \left( E_u(\bar{T}) + C_{K,\eta} \int_0^{\bar{T}} \int_{\Gamma} |T(u)n|^2 + \int_0^{\bar{T}} \|w\|^2_{H^1(\Omega_f)} + \eta C_{D,\omega,\varepsilon} \int_0^{\bar{T}} \|u\|^2_{H^1(\Omega)} \right)
\]
Next, we set the parameter of our choice $\delta$ (for which the only restriction so far has been $(\bar{T} - 2\delta - C) > 0$ for $C$ from (40)) to be also no more than, for example, 1/4. Thus the term $2\delta E_u(\bar{T})$ when added to (41) can be absorbed into $E_u(\bar{T})$ on the left. Furthermore, choose $\eta$ small enough so that $2\delta\eta C_{D,\omega,\varepsilon} < 1/2$. Then
\[
\frac{1}{2} \int_{I_\delta} E_u(t) dt \leq 2\delta E_u(\bar{T}) + 2\delta \left( C_{K,\eta} \int_0^{\bar{T}} \int_{\Gamma} |T(u)n|^2 + \int_0^{\bar{T}} \|w\|^2_{H^1(\Omega_f)} \right)
\]
Adding this inequality to (41) yields
\[
\frac{1}{2} E_u(\bar{T}) + \frac{1}{2} \int_0^{\bar{T}} E_u(t) dt \leq \varepsilon \int_0^{\bar{T}} E_u(t) dt + C_{2,T,\varepsilon,\delta} \left( \int_0^{\bar{T}} \int_{\Gamma} |T(u)n|^2 + \int_0^{\bar{T}} \|w\|^2_{H^1(\Omega_f)} \right)
\]
For $\varepsilon < 1/2$ this step is equivalent to the result of Lemma 6.1, and consequently finishes the proof of Theorem 4.1.
7. Proof of Theorem 4.2. Let $c$ be a positive parameter, to be specified later. Multiply (20) by $c$ and add to (15) and (16). We get an analog of the “fundamental identity” (27), but now formulated to take advantage of $\alpha > 0$:

$$E(T) + 2\nu \int_0^T \| w \|_{H^1(\Omega_t)}^2 + \alpha \int_0^T \| u_t \|_{L^2_{\theta_\alpha}(\Omega)}^2 + c \int_0^T \| u \|_{L^2_{\gamma}(\Omega)}^2 + c\gamma \int_0^T \| u_t \|_{L^2_{\theta_\alpha}(\Omega)}^2$$

$$= E(0) + \int_0^T \int_\Gamma \langle w, \nu n - 2\nu c(w)n \rangle + \int_0^T \int_\Gamma \langle u_t + c\nu (u)n \rangle + c \int_0^T \| u_t \|_{L^2_{\theta_\alpha}(\Omega)}^2$$

$$- \int_0^T \int_{\Omega_f} \langle w, (Dw)w_s + (Dw)_s \rangle$$

$$- c(u, u_t)_{L^2_{\theta_\alpha}(\Omega)} \bigg|_0^T - \frac{c}{2} \| u \|_{L^2_{\theta_\alpha}(\Omega)}^2 \bigg|_0^T.$$

(42)

7.1. Outline ($\alpha > 0$). The goal is to turn (42) into an inequality of the form:

$$E(T) + C_1 \int_0^T E(t) dt \leq C_2 E(0).$$

To this end, we will proceed in two phases. First, we rewrite or estimate the unwanted terms on the right-hand side via any of the following:

- multiples of $E(0)$ (or $E_u(0)$ which is smaller),
- negative multiples of $E(T)$ or $E_u(T)$,
- trace products $\int_\Gamma \langle \mathcal{T}(u)n, u_t \rangle$ and $\int_\Gamma \langle \mathcal{T}(u)n, u \rangle$,
- integrals over $\Omega_f$.

In the second phase, we will invoke the boundary conditions and will be able to estimate all the terms, other than $E(0)$, appearing on the right-hand side after phase one in terms of either: multiples of $E(0)$, or fractional multiples of the terms on the left.

To start with, consider $c \int_0^T \| u_t \|_{L^2_{\theta_\alpha}(\Omega)}^2$ from the right-hand side of (42). If $c \leq \alpha$, say $c = \alpha/2$, then we could absorb the term into the corresponding integral on the left-hand side, but that, as we will see later, indirectly places additional restrictions on the viscosity. Instead, we rewrite this kinetic energy term using (16):

$$c \int_0^T \| u_t \|_{L^2_{\theta_\alpha}(\Omega)}^2 = \frac{c}{\alpha} (E_u(0) - E_u(T)) + \frac{c}{\alpha} \int_0^T \langle u_t, \mathcal{T}(u)n \rangle. \quad (43)$$

Since $\gamma > 0$ in the elasticity equation, then we have that

$$-c(u, u_t)_{L^2_{\theta_\alpha}(\Omega)} \bigg|_0^T - \frac{c}{2} \| u \|_{L^2_{\theta_\alpha}(\Omega)}^2 \bigg|_0^T \leq \frac{c(1 + \alpha + \gamma)}{\gamma} (E_u(0) + E_u(T)).$$

This last estimate, however, brings in the term $E_u(T)$, so we further invoke (16) to estimate $E_u(T)$ from above and obtain

$$-c(u, u_t)_{L^2_{\theta_\alpha}(\Omega)} \bigg|_0^T - \frac{c}{2} \| u \|_{L^2_{\theta_\alpha}(\Omega)}^2 \bigg|_0^T \leq \frac{c(1 + \alpha + \gamma)}{\gamma} \left( 2E_u(0) + \int_0^T \langle u_t, \mathcal{T}(u)n \rangle \right). \quad (44)$$

Applying (43) and (44) to (42), we obtain

$$E(T) + 2\nu \int_0^T \| w \|_{H^1(\Omega_t)}^2 + c \int_0^T \| u_t \|_{L^2_{\theta_\alpha}(\Omega)}^2 + c \int_0^T \| u \|_{L^2_{\gamma}(\Omega)}^2 + c\gamma \int_0^T \| u_t \|_{L^2_{\theta_\alpha}(\Omega)}^2$$

$$- \int_0^T \int_\Gamma \langle w, (Dw)w_s + (Dw)_s \rangle$$

$$- c(u, u_t)_{L^2_{\theta_\alpha}(\Omega)} \bigg|_0^T - \frac{c}{2} \| u \|_{L^2_{\theta_\alpha}(\Omega)}^2 \bigg|_0^T$$

$$\leq \frac{c(1 + \alpha + \gamma)}{\gamma} \left( 2E_u(0) + \int_0^T \langle u_t, \mathcal{T}(u)n \rangle \right).$$
\[ \leq E(0) + \int_0^T \int_\Gamma \langle w, p n - 2\nu \varepsilon(w)n \rangle \\
+ \int_0^T \int_\Gamma \left\langle c \left( \frac{1 + \alpha + \gamma}{\gamma} + \frac{1}{\alpha} \right) u_t + cu, T(u)n \right\rangle \\
- \int_0^T \int_{\Omega_j} \langle w, (Dw)w_s + (Dw_s)w \rangle \\
+ \left( \frac{2c(1 + \alpha + \gamma)}{\gamma} + \frac{c}{\alpha} \right) E_u(0) - \frac{c}{\alpha} E_u(T). \]

At this point, the first phase of the estimates has been completed and we are ready to invoke the boundary conditions. We label, for a shorthand

\[ C_{c,\alpha,\gamma} := c \left( \frac{1 + \alpha + \gamma}{\gamma} + \frac{1}{\alpha} \right) + 1, \]

and rewrite the integrals over \( \Gamma \) on the right-hand side of (45) using the boundary conditions of our system (6):

\[ \int_0^T \int_\Gamma \langle w, B(u) - T(u)n \rangle \\
+ C_{c,\alpha,\gamma} \int_0^T \int_\Gamma \langle w, (Dw)u - KT(u)n, T(u)n \rangle + c \int_0^T \int_\Gamma \langle u, T(u)n \rangle \\
= \int_0^T \int_\Gamma \langle (C_{c,\alpha,\gamma} - 1) w, T(u)n \rangle + \int_0^T \int_\Gamma \langle w, B(u) \rangle \\
+ C_{c,\alpha,\gamma} \int_0^T \int_\Gamma \langle (Dw)u, T(u)n \rangle \\
- C_{c,\alpha,\gamma} K \int_0^T \int_\Gamma |T(u)n|^2 + \int_0^T \int_\Gamma c(u, T(u)n). \]

In (45) we drop the term \(- (c/\alpha)E_u(T)\), replace \( E_u(0) \) by the larger total energy \( E(0) \) and plug in (46):

\[ E(T) + 2\nu \int_0^T \|w\|_{H^1(\Omega)}^2 + \alpha \int_0^T \|u_t\|_{L^2_{\alpha,\gamma}(\Omega)}^2 + c \int_0^T |u|_{L^1(\Omega)}^2 \]

\[ + c_\gamma \int_0^T \|u\|_{L^2_{\alpha,\gamma}(\Omega)}^2 + C_{c,\alpha,\gamma} K \int_0^T \int_\Gamma |T(u)n|^2 \]

\[ \leq \left( 1 + \frac{2c(1 + \alpha)}{\gamma} + \frac{c}{\alpha} \right) E(0) - \int_0^T \int_{\Omega_j} \langle w, (Dw)w_s + (Dw_s)w \rangle \\
+ \int_0^T \int_\Gamma \langle (C_{c,\alpha,\gamma} - 1) w, T(u)n \rangle + \int_0^T \int_\Gamma \langle w, B(u) \rangle \\
+ \int_0^T \int_\Gamma \langle (c + C_{c,\alpha,\gamma} Dw_s)u, T(u)n \rangle. \]

7.2. Concluding estimates with interior damping \((\alpha > 0)\). By assumption, viscosity \( \nu \) is large enough (or \( w_s \) is small enough) so that there exists \( \eta \in (0, 1) \) for
which
\[
\left| \int_0^T \int_\Omega \langle w, (Dw)w_s + (Dw_s)w \rangle \right| \leq 2\eta \nu \int_0^T \|w\|_{H^1(\Omega_f)}^2.
\]

Recall that \( c \) was a parameter of our choice. Also, operator \( B \) is a tangential boundary operator with variable coefficients (dependent on the steady regime \((w_s, p_s, u_s))\), which is continuous \( H^1(\Omega) \to H^{-1/2}(\Gamma) \). Hence we can choose
\[
c = c(\nu, \alpha, \gamma, w_s, p_s, u_s) > 0
\]
large enough so that the following inequality on the trace integrals from the right-hand side of (47) holds (by means of the Schwartz and Young and trace inequalities)
\[
\left| \int_0^T \int_\Gamma \langle (c + C_{c,\alpha,\gamma} Dw_s)u, T(u)n \rangle \right|
\]
\[
+ \left| \int_0^T \int_\Gamma \langle (C_{c,\alpha,\gamma} - 1) w, T(u)n \rangle \right|
\]
\[
+ \left| \int_0^T \int_\Gamma \langle w, B(u) \rangle \right|
\]
\[
\leq (1 - \eta) \nu \int_0^T \|w\|_{H^1(\Omega_f)}^2 + c \left( \int_0^T \|u\|_1^2_{L^2(\Omega)} + \gamma \int_0^T \|u\|_{L^2_{x_0}(\Omega)}^2 \right)
\]
\[
+ C_2(c, \alpha, \gamma) \int_0^T \int_\Gamma |T(u)n|^2.
\]

With this inequality in mind and looking back at (47), we see that we need the boundary feedback coefficient \( K \) large enough to satisfy
\[
K \geq \frac{C_2(c, \alpha, \gamma)}{C_{c,\alpha,\gamma}}.
\]

In this case, the right-most term of (48) can be absorbed into the damping term \( C_{c,\alpha,\gamma} K \int_0^T \int_\Gamma |T(u)n|^2 \) on the left of (47). Therefore (47) becomes
\[
E(T) + (1 - \eta) \nu \int_0^T \|w\|_{H^1(\Omega_f)}^2 + \alpha \int_0^T \|u_t\|^2_{L^2_{x_0}(\Omega)}
\]
\[
+ \frac{c}{2} \int_0^T |u|_1^2_{H^1(\Omega_f)} + C_2(c, \alpha, \gamma) \int_0^T \int_\Gamma |T(u)n|^2 \leq \left( 1 + \frac{\alpha(1 + \alpha)}{\gamma} + \frac{c}{\alpha} \right) E(0).
\]

The integral terms on the left-hand side of the latter dominate (a multiple of) the integral of the total energy \( E(t) \). So, we have arrived at the following inequality with parameters for \( C > 0, \delta > 0, \) independent of the solution itself,
\[
E(T) + \delta \int_0^T E(t)dt \leq CE(0) \quad \text{for any} \; T > 0.
\]

Since \( \delta \) is independent of \( T \), then \( \int_0^\infty E(t)dt \) converges, whence exponential stability follows [27].

Remark 7.1. In fact, since the system is autonomous and \( T \) represents only the “relative” time, meaning that we have for any \( t \geq T \) the estimate \( E(t) + \delta \int_{t-T}^t E(s)ds \leq CE(t - T) \), then the exponential stability would follow here even if \( \delta \) and \( C \) were dependent on \( T \) itself.
Acknowledgments. We thank the Referees for the many suggestions which made this work a substantially stronger paper. We would also like to thank Matthias Eller for his expert advice and fruitful discussions.

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Received March 2017; revised August 2017.

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