The parastatistics Fock space and explicit Lie superalgebra representations

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Abstract

It is known that the defining triple relations of \( m \) pairs of parafermion operators \( f^{\pm}_j \) and \( n \) pairs of paraboson operators \( b^{\pm}_j \) with relative parafermion relations can be considered as defining relations for the Lie superalgebra \( \mathfrak{osp}(2m+1|2n) \) in terms of \( 2(2m+n) \) generators. With the common hermiticity conditions, this means that the parastatistics Fock space of order \( p \) corresponds to an infinite-dimensional unitary irreducible representation \( V(p) \) of \( \mathfrak{osp}(2m+1|2n) \), with lowest weight \( (\frac{-p^2}{2}, \ldots, \frac{-p^2}{2} | \frac{p^2}{2}, \ldots, \frac{p^2}{2}) \). These representations (also in the simplest case \( m = n = 1 \)) had never been constructed due to computational difficulties, despite their importance. In the present paper we partially solve the problem in the general case using group theoretical techniques, in which the \( \mathfrak{u}(m|n) \) subalgebra of \( \mathfrak{osp}(2m+1|2n) \) plays a crucial role: a set of Gelfand–Zetlin patterns of \( \mathfrak{u}(m|n) \) can be used to label the basis vectors of \( V(p) \). An explicit and elegant construction of these representations \( V(p) \) for \( m = n = 1 \), and the actions or matrix elements of the \( \mathfrak{osp}(3|2) \) generators are given.

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1. Introduction

In 1953 Green [1] generalized the ordinary Fermi–Dirac and Bose–Einstein statistics introducing the parafermion and paraboson statistics. Parastatistics were also formulated algebraically in terms of generators and relations. The operators of parafermion statistics \( f^{\pm}_j \), \( j = 1, 2, \ldots, m \), satisfying

\[
[[f^+_j, f^+_k], f^-_l] = \frac{1}{2}(\epsilon - \eta)^2 \delta_{kl} f^+_j - \frac{1}{2}(\epsilon - \xi)^2 \delta_{jl} f^+_k,
\]

(1.1)

where \( j, k, l \in \{1, 2, \ldots, m\} \) and \( \eta, \epsilon, \xi \in \{+, -\} \) (to be interpreted as +1 and −1 in the algebraic expressions \( \epsilon - \xi \) and \( \epsilon - \eta \)), are generating elements of the orthogonal Lie algebra \( \mathfrak{so}(2m+1) \) [2, 3]. In a similar way \( n \) paraboson operators \( b^{\pm}_j \), satisfying

\[
[[b^+_j, b^+_k], b^-_l] = (\epsilon - \xi) \delta_{kl} b^+_j + (\epsilon - \eta) \delta_{jl} b^+_k,
\]

(1.2)
are generating elements of the orthosymplectic Lie superalgebra \( \mathfrak{osp}(1|2n) \) [4]. The important objects to construct are the generalizations of the fermion and boson Fock spaces. Parafermion and paraboson Fock spaces are characterized by a parameter \( p \), the order of the parastatistics. Although Green already proposed a general approach in 1953, known nowadays as the Green ansatz [1], for the construction of the parafermion and paraboson Fock spaces their structure was not known until a few years ago. The difficulties surrounding the Green ansatz are connected to the problem of finding proper bases of irreducible constituents of \( p \)-fold tensor products [5, 6], and did not lead to a solution of the problem. Recently, for the case of parafermions, this explicit construction of the Fock space of order \( p \) was given in [7], and for parabosons in [8]. The solutions use the algebraic formulations of parafermion and paraboson statistics. As a consequence, the parafermion Fock space of order \( p \) is the finite-dimensional unitary irreducible representation (unirrep) of \( \mathfrak{so}(2m + 1) \) with lowest weight \( (-\frac{p}{2}, -\frac{p}{2}, \ldots, -\frac{p}{2}) \), and the paraboson Fock space is the infinite-dimensional unirrep of \( \mathfrak{osp}(1|2n) \) also with lowest weight \( (\frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2}) \). The constructions use the branchings \( \mathfrak{so}(2m + 1) \supset \mathfrak{u}(m) \) for the parafermion Fock space and \( \mathfrak{osp}(1|2n) \supset \mathfrak{sp}(2n) \supset \mathfrak{u}(n) \) for the paraboson Fock space. The results are complete descriptions of proper bases and the explicit action of the parafermion and paraboson operators in the corresponding basis [7, 8].

As a next step, it is natural to extend these results to a system consisting of parafermions \( f^+_j \) and parabosons \( b^+_j \). The commutation relations among paraoperators were studied by Greenberg and Messiah [9]. As a consequence of some natural assumptions they came to the result that for each pair of paraoperators there can exist at most four types of relative commutation relations: straight commutation, straight anticommutation, relative paraboson, and relative parafermion relations. The case with relative paraboson relations and the corresponding Fock representations has been investigated in [10–13]. In the present paper we consider relative parafermion relations. It was proved by Palev [14] that \( m \) parafermions \( f^+_j \equiv c^+_j \) (1.1) and \( n \) parabosons \( b^+_j \equiv c^+_m \) (1.2) with relative parafermion relations lead to the result that they generate the orthosymplectic Lie superalgebra \( \mathfrak{osp}(2m + 1|2n) \). Then the parastatistics Fock space of order \( p \) corresponds to an infinite-dimensional unitary representation of \( \mathfrak{osp}(2m + 1|2n) \) and can be constructed explicitly using similar techniques as in [7, 8], namely using the branching \( \mathfrak{osp}(2m + 1|2n) \supset \mathfrak{gl}(m|n) \), an induced representation construction, a basis description for the covariant tensor representations of \( \mathfrak{gl}(m|n) \), Clebsch–Gordan coefficients of \( \mathfrak{gl}(m|n) \), and the method of reduced matrix elements. The covariant tensor representations of the Lie superalgebra \( \mathfrak{gl}(m|n) \) in an explicit form and the relevant Clebsch–Gordan coefficients (namely those corresponding to the tensor product \( V(|\mu\nu\rangle) \otimes V(|1, 0, \ldots, 0\rangle) \), where \( V(|\mu\nu\rangle) \) is any \( \mathfrak{gl}(m|n) \) irreducible covariant tensor representation and \( V(|1, 0, \ldots, 0\rangle) \) is the representation of \( \mathfrak{gl}(m|n) \) with highest weight \( (1, 0, \ldots, 0) \) were constructed and found in [15].

The structure of the paper is as follows. In section 2, we define the parastatistics Fock space \( V(p) \). In section 3, we consider the important relation between parastatistics operators and the Lie superalgebra \( \mathfrak{osp}(2m + 1|2n) \), and give a description of \( V(p) \) in terms of the representations of \( \mathfrak{osp}(2m + 1|2n) \). Section 4 is devoted to the analysis of the representations \( V(p) \) for \( \mathfrak{osp}(2m + 1|2n) \) and to finding the matrix elements for \( m = n = 1 \), where the main computational result is given in theorem 5. We conclude the paper with some final remarks.

2. The parastatistics Fock space \( V(p) \)

Before introducing the parastatistics Fock space, we will consider the Fock space \( V(1) \) corresponding to a system of \( m \) pairs of Fermi operators \( F^+_i, F^-_i \), \( i = 1, 2, \ldots, m \) \((a, b) = ab + ba\)

\[
\{F^+_i, F^-_i\} = \delta_{ik}, \quad \{F^-_i, F^+_i\} = \{F^-_i, F^-_i\} = 0 \quad (2.1)
\]
and \( n \) pairs of Bose operators \( B_j^\pm, j = 1, 2, \ldots, n \) \([a, b] = ab - ba\)

\[
[B_j^+, B_l^+] = \delta_{jl}, \quad [B_j^-, B_l^-] = [B_j^+, B_l^+] = 0,
\]

which mutually anticommute

\[
\{F_j^\pm, B_l^\mp\} = 0, \quad \xi, \eta = \pm, \quad i = 1, \ldots, m; \quad j = 1, \ldots, n.
\]

The Hilbert space is irreducible under the action of the algebra spanned by the elements \( B_j \) lowest weight among them. The defining triple relations for such a system are given by \(14\)

\[
\text{The Fock space } V(1) \text{ is defined as a Hilbert space with vacuum vector } |0\rangle, \text{ with }
\]

\[
|0\rangle = 1, \quad F_j^-|0\rangle = B_j^-|0\rangle = 0, \quad (F_j^\pm)^\dagger = F_j^\mp, \quad (B_j^\pm)^\dagger = B_j^\mp.
\]

The Hilbert space is irreducible under the action of the algebra spanned by the elements \( 1, F_j^\pm, B_j^\pm \). A set of (orthogonal and normalized) basis vectors of this space is given by

\[
|\theta_1, \ldots, \theta_m, k_1, \ldots, k_n\rangle = \frac{(F_1^+)^{\theta_1} \cdots (F_m^+)^{\theta_m} (B_1^+)^{k_1} \cdots (B_n^+)^{k_n}}{\sqrt{k_1! \cdots k_n!}} |0\rangle, \quad \theta_i = 0, 1; \quad k_j \in \mathbb{Z}_+.
\]

A straightforward calculation gives

\[
F_j^+|\theta_1, \ldots, \theta_m, k_1, \ldots, k_n\rangle = (-1)^{\theta_j+\theta_m+1} \sqrt{1-\theta_j}|\theta_1, \ldots, \theta_i+1, \ldots, k_j, \ldots\rangle,
\]

\[
F_j^-|\theta_1, \ldots, \theta_m, k_1, \ldots, k_n\rangle = (-1)^{\theta_j+\theta_m+1} \sqrt{\theta_j}|\theta_1, \ldots, \theta_i-1, \ldots, k_j, \ldots\rangle,
\]

\[
B_j^+|\theta_1, \ldots, \theta_m, k_1, \ldots, k_n\rangle = (-1)^{\theta_j+\theta_m} \sqrt{1+k_j}|\theta_1, \ldots, \theta_i, k_j+1, \ldots\rangle,
\]

\[
B_j^-|\theta_1, \ldots, \theta_m, k_1, \ldots, k_n\rangle = (-1)^{\theta_j+\theta_m} \sqrt{k_j}|\theta_1, \ldots, \theta_i, k_j-1, \ldots\rangle.
\]

This Fock space is a certain unirrep of the Lie superalgebra \( \mathfrak{osp}(2m+1|2n) \) \([14]\), with lowest weight \((-\frac{1}{2}, \ldots, -\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\)).

We are interested in a system of \( m \) pairs of parafermion operators \( f_i^\pm \equiv c_i^\pm \), \( i = 1, \ldots, m \) and \( n \) pairs of paraboson operators \( b_j^\pm \equiv c_{m+j}^\pm \), \( j = 1, \ldots, n \) with relative parafermion relations among them. The defining triple relations for such a system are given by \(14\)

\[
\{[c_j^\xi, c_l^\eta], c_i^\epsilon\} = -2\delta_{ij}\delta_{\xi\epsilon}\delta_{l-\eta} \epsilon^{(l)}(-1)^{\delta(l)\delta(\eta)} c_j^\eta + 2\epsilon^{(l)} \delta_{l\eta} \delta_{\xi-\epsilon} c_j^\xi,
\]

\[
\xi, \eta, \epsilon = \pm \text{ or } \pm 1; \quad j, k, l = 1, \ldots, n+m.
\]

where \( \{a, b\} = ab - (-1)^{\deg(a)\deg(b)} ba \) and \( \deg(c_i^\pm) \equiv \langle i \rangle = 0 \text{ if } j = 1, \ldots, m \)

\[
1 \text{ if } j = m+1, \ldots, n+m.
\]

(2.11)

In the case \( j, k, l = 1, \ldots, m \) (2.10) reduces to (1.1) and in the case \( j, k, l = m+1, \ldots, m+n \) (2.10) reduces to (1.2).

The parastatistics Fock space \( V(p) \) is the Hilbert space with vacuum vector \(|0\rangle\), defined by means of \((j, k, l = 1, 2, \ldots, m+n)\)

\[
|0\rangle = 1, \quad c_j^\dagger |0\rangle = 0, \quad (c_j^\dagger)^p = c_j^p, \quad [c_j^\dagger, c_k^\dagger]|0\rangle = p\delta_{jk}|0\rangle,
\]

(2.12)

and by irreducibility under the action of the algebra spanned by the elements \( c_j^\dagger, c_j, j = 1, \ldots, m+n \), subject to (2.10). The parameter \( p \) is referred to as the order of the parastatistics system and for \( p = 1 \) the parastatistics Fock space \( V(p) \) coincides with the Fock space \( V(1) \) (2.3)–(2.9) of \( n \) bosons and \( m \) fermions with unusual grading which anticommute.
Constructing a basis for the parastatistics Fock space $V(p)$ for general (integer) $p$-values turns out to be a difficult problem, unsolved so far not only in general but also for a single parafermion and paraboson. Even the simpler question of finding the structure of $V(p)$ (weight structure) is not solved. In the present paper we shall partially solve the last problem for any $m$ pairs of parafermions and $n$ pairs of parabosons and we shall construct an orthogonal (normalized) basis for $V(p)$, and give the actions of the generators $c_j^+$ on the basis vectors for $m = n = 1$.

3. The Lie superalgebras $B(m|n)$

The Lie superalgebra $B(m|n) \equiv \mathfrak{osp}(2m + 1|2n)$ [16] consists of matrices of the form

$$
\begin{pmatrix}
a & b & u & x \\
c & -d' & v & y \\
-v' & -u' & 0 & z \\
y_1' & x_1' & z_1' & d & e \\
-y_1' & -x_1' & -z_1' & f & -d'
\end{pmatrix},
$$

where $a$ is any $(m \times m)$-matrix, $b$ and $c$ are antisymmetric $(m \times m)$-matrices, $u$ and $v$ are $(m \times 1)$-matrices, $x, y, x_1, y_1$ are $(m \times n)$-matrices, $z$ and $z_1$ are $(1 \times n)$-matrices, $d$ is any $(n \times n)$-matrix, and $e$ and $f$ are symmetric $(n \times n)$-matrices. The even elements have $x = y = x_1 = y_1 = 0$, $z = z_1 = 0$ and the odd elements are those with $a = b = c = 0$, $u = v = 0$, $d = e = f = 0$. Denote the row and column indices running from 1 to $2m + 2n + 1$ and by $e_{ij}$ the matrix with zeros everywhere except for a 1 on position $(i, j)$. The Cartan subalgebra $H$ of $\mathfrak{osp}(2m + 1|2n)$ is the subspace of diagonal matrices with basis

$$h_i = e_{i+i,m,i+m}, \quad i = 1, \ldots, m; \quad h_{m+i} = e_{2m+1+j,2m+1+j} - e_{2m+1+n+j,2m+1+n+j}, \quad j = 1, \ldots, n.$

In terms of the dual basis $e_i, i = 1, \ldots, m; \delta_j, j = 1, \ldots, n$ of $H^*$, the even root vectors and corresponding roots of $\mathfrak{osp}(2m + 1|2n)$ are given by

$$e_{jk} - e_{k+m,j+m} \leftrightarrow \delta_j - \delta_k, \quad j \neq k = 1, \ldots, m,$$

$$e_{j,k+m} - e_{k,j+m} \leftrightarrow \delta_j + \delta_k, \quad j < k = 1, \ldots, m,$$

$$e_{j+k,m} - e_{k+m,j} \leftrightarrow -\delta_j - \delta_k, \quad j < k = 1, \ldots, m,$$

$$e_{j,2m+1} - e_{2m+1,j} \leftrightarrow \delta_j, \quad j = 1, \ldots, m,$$

$$e_{j+m,2m+1} - e_{2m+1,j} \leftrightarrow -\delta_j, \quad j = 1, \ldots, m,$$

and the odd ones by

$$e_{j,2m+1+k} - e_{2m+1+n+k,j+m} \leftrightarrow \delta_j - \delta_k, \quad j = 1, \ldots, m; \ k = 1, \ldots, n,$$

$$e_{m+j,2m+1+k} - e_{2m+1+n+k,j} \leftrightarrow -\delta_j - \delta_k, \quad j = 1, \ldots, m; \ k = 1, \ldots, n,$$

$$e_{2m+1+k,2m+1+k} - e_{2m+1+k,j} \leftrightarrow -\delta_k, \quad k = 1, \ldots, n,$$

$$e_{j,2m+1+n+k} + e_{2m+1+k,j} \leftrightarrow \delta_j + \delta_k, \quad j = 1, \ldots, m; \ k = 1, \ldots, n,$$

$$e_{m+j,2m+1+n+k} + e_{2m+1+k,j} \leftrightarrow -\delta_j + \delta_k, \quad j = 1, \ldots, m; \ k = 1, \ldots, n,$$

$$e_{2m+1+k,2m+1+n+k} + e_{2m+1+k,2m+1+k} \leftrightarrow \delta_k, \quad k = 1, \ldots, n.$$

If we introduce the following multiples of the even vectors with roots $\pm\epsilon_j, \ j = 1, \ldots, m$

$$c_j^+ = f_j^+ = \sqrt{2}(e_{j,2m+1} - e_{2m+1,j+m}),$$

$$c_j^- = f_j^- = \sqrt{2}(e_{2m+1,j} - e_{j+m,2m+1}),$$

(3.2)
and of the odd vectors with roots $\pm \delta_j$, $j = 1, \ldots, n$
\[
c^{+}_{m+j} = b^j_+ = \sqrt{2}(e_{2m+1, 2m+1+n+j} + e_{2m+1+j, 2m+1}),
\]
\[
c^{-}_{m+j} = b^-_j = \sqrt{2}(e_{2m+1, 2m+1+n+j} - e_{2m+1+j, 2m+1}),
\]
(3.3)

it is easy to verify that these operators satisfy the triple relations (2.10).

Moreover, the following holds [14].

**Theorem 1** ([14]). As a Lie superalgebra defined by generators and relations, $\mathfrak{osp}(2m + 1|2n)$ is generated by $2m + 2n$ elements $c^\pm_j$ subject to the parastatistics relations
\[
\left[\left[\left[ c^\xi_j, c^\eta_k \right], c^\zeta_l \right], c^\nu_m \right] = -2\delta_j^l\delta_k^m\delta_{\xi+\zeta,\eta+\nu} - (\zeta n - \xi n)(\eta n - \nu n)\delta_j^l\delta_k^m, \\
\xi, \eta, \zeta, \nu = \pm; \quad j, k, l = 1, \ldots, n + m.
\]
(3.4)

The paraoperators $c^+_j$ are positive root vectors, and the $c^-_j$ are negative root vectors.

We are interested in the construction of the parastatistics Fock space $V(p)$ defined by (2.12):
\[
\langle 0|0\rangle = 1, \quad c^-_j\langle 0|0\rangle = 0, \quad (c^+_j)^\dagger = c^+_j, \\
\|c^-_j, c^+_j\|\langle 0|0\rangle = p\delta_{jk}\langle 0|0\rangle
\]

and by irreducibility under the action of the algebra spanned by the elements $c^+_j$, $c^-_j$, $j = 1, \ldots, m + n$, subject to (2.10). It is straightforward to see that
\[
[c^-_i, c^+_j] = -2h_i, \quad i = 1, \ldots, m, \quad \text{and} \quad [c^-_{m+j}, c^+_m] = 2h_{m+j}, \quad j = 1, \ldots, n.
\]
(3.5)

Therefore we have the following.

**Corollary 2.** The parastatistics Fock space $V(p)$ is the unitary irreducible representation of $\mathfrak{osp}(2m + 1|2n)$ with lowest weight $(-\frac{p}{2}, \ldots, -\frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2})$.

In order to construct the representations $V(p)$ in general one can use an induced module procedure. The relevant subalgebras of $\mathfrak{osp}(2m + 1|2n)$ are easy to describe by means of the generators $c^\pm_j$.

**Proposition 3.** A basis for the even subalgebra $so(2m + 1) \oplus sp(2n)$ of $\mathfrak{osp}(2m + 1|2n)$ is given by the elements
\[
[c^\xi_j, c^\eta_k], \quad c^\xi_j, \quad i, k, l = 1, \ldots, m; \quad \{c^\xi_{m+j}, c^\eta_{m+s}\}, \quad j, s = 1, \ldots, n, \quad \xi, \eta = \pm.
\]
(3.6)

The $(m + n)^2$ elements
\[
\|c^+_j, c^-_k\| \quad (j, k = 1, \ldots, m + n)
\]
(3.7)

are a basis for the subalgebra $u(m|n)$.

Note that with $\frac{1}{2}\|c^+_j, c^-_k\| = E_{jk}$, the triple relations (3.4) imply the relations $[E_{ij}, E_{kl}] = \delta_{jk}E_{ij} - (\xi n - \zeta n)\delta_{\xi+\zeta,\eta+\nu}E_{ij}$. Therefore, the elements $\|c^+_j, c^-_k\|$ form, up to a factor 2, the standard $u(m|n)$ or $gl(m|n)$ basis elements.

The superalgebra $u(m|n)$ is, algebraically, the same as the general linear Lie superalgebra $gl(m|n)$. However the condition $(c^+_k)^\dagger = c^-_k$ implies that we are dealing here with the ‘compact form’ $u(m|n)$.

Let us extend the subalgebra $u(m|n)$ to a parabolic subalgebra $\mathcal{P}$ of $\mathfrak{osp}(2m + 1|2n)$
\[
\mathcal{P} = \text{span}\{c^\xi_j, \|c^+_j, c^-_k\|, \|c^+_j, c^-_k\| | j, k = 1, \ldots, m + n\}.
\]
(3.8)

Since $\|c^+_j, c^-_k\|\langle 0|0\rangle = p\delta_{jk}\langle 0|0\rangle$, with $[c^+_j, c^+_k] = -2h_i, \quad i = 1, \ldots, m, \quad \text{and} \quad [c^-_{m+j}, c^+_m] = 2h_{m+j}, \quad j = 1, \ldots, n$, the space spanned by $\langle 0|0\rangle$ is a trivial one-dimensional $u(m|n)$ module $\mathbb{C}\langle 0\rangle$. 

\[5\]
of weight \((-\frac{p}{2}, \ldots, -\frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2}\)). As \(c_j^+|0\rangle = 0\), the \(u(m)\) module \(C|0\rangle\) can be extended to a one-dimensional \(P\) module. The induced \(osp(2m + 1|2n)\) module \(\overline{V}(p)\) is defined by

\[
\overline{V}(p) = \text{Ind}_{\mathfrak{osp}(2m + 1|2n)}^{\mathfrak{osp}(2m + 1|2n)} C|0\rangle.
\]

(3.9)

This is an \(osp(2m + 1|2n)\) representation with lowest weight \((-\frac{p}{2}, \ldots, -\frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2}\)). By the Poincaré–Birkhoff–Witt theorem \([17]\), it is easy to write a basis for \(\overline{V}(p)\)

\[
(c +)^k_1 \cdots (c +)^k_{m+n} (\mathfrak{c}_1^+ c_2^+ \cdots)^{k_2} (\mathfrak{c}_1^+ c_3^+ \cdots)^{k_3} \cdots (\mathfrak{c}_m^+ c_{m+n}^+ \cdots)^{k_{m+n}} |0\rangle,
\]

\(k_1, \ldots, k_{m+n}, k_1, k_2, k_1, k_2, \ldots, k_{m-1,m}, k_{m+1,m+2}, k_{m+1,m+3}, \ldots, k_{m+n-1,m+n} \in \mathbb{Z}_+\),

\(k_1, k_2, \ldots, k_{m+n}, k_2, k_2+1, \ldots, k_{m+n}, 0, 1, 1, 1\).

(3.10)

The problem and difficulty come from the fact that in general \(\overline{V}(p)\) is not a simple module (i.e. not an irreducible representation) of \(osp(2m + 1|2n)\). Let \(M(p)\) be the maximal nontrivial submodule of \(\overline{V}(p)\). Then the simple module (irreducible module), corresponding to the parastatistics Fock space, is

\[
V(p) = \overline{V}(p)/M(p).
\]

(3.11)

The problem is to determine the vectors belonging to \(M(p)\), and therefore to find the structure of \(V(p)\).

### 4. The parastatistics Fock space of \(osp(2m + 1|2n)\) and matrix elements for \(m = n = 1\)

Consider the induced module \(\overline{V}(p)\) in the case \(m = n = 1\), with basis vectors

\[
|k, l, \theta\rangle = (c^+_1)^k (c^+_2)^l (|c^+_1, c^+_2\rangle)^0 |0\rangle, \quad k, l \in \mathbb{Z}_+; \ \theta = 0, 1.
\]

(4.1)

The weight of this vector is

\[
\left(-\frac{p}{2}, \frac{p}{2}\right) + k \epsilon_1 + l \delta_1 + \theta (\epsilon_1 + \delta_1).
\]

(4.2)

The level of such a vector is defined as \(k + l + 2\theta\). The actions of the generators \(c^+_1, c^+_2\) on the basis vectors \(|k, l, \theta\rangle\) can be computed, using the triple relations. For the positive root vectors the computations are easy

\[
c^+_1|k, l, \theta\rangle = |k + 1, l, \theta\rangle,
\]

\[
c^+_2|k, l, 0\rangle = |k, l + 1, 0\rangle - (-1)^k |k - 1, l, 1\rangle,
\]

\[
c^+_2|k, l, 1\rangle = |k, l + 1, 1\rangle.
\]

(4.3)

However, for the negative root vectors this requires some tough computations

\[
c^+_1|k, l, 0\rangle = k (p - k + 1) |k - 1, l, 0\rangle,
\]

\[
c^+_1|k, l, 1\rangle = k (p - k - 1) |k - 1, l, 1\rangle + (-1)^k |k - 1, l, 0\rangle,
\]

\[
c^+_2|k, 2l, 0\rangle = -2k |k - 1, 2l - 2, 1\rangle + 2l |k, 2l - 1, 0\rangle,
\]

\[
c^+_2|k, 2l + 1, 0\rangle = 2k |k - 1, 2l - 1, 1\rangle - (p + 2l - 2k) |k, 2l, 0\rangle,
\]

\[
c^+_2|k, 2l, 1\rangle = 2l |k, 2l - 1, 1\rangle - 2 |k + 1, 2l, 0\rangle,
\]

\[
c^+_2|k, 2l + 1, 1\rangle = (p + 2l - 2k) |k, 2l, 1\rangle + 2 |k + 1, 2l + 1, 0\rangle.
\]

(4.4)

Using \(|0\rangle = 1\) and \((c^+_1)^k = c^+_1\) we can compute ‘inner products’ of the vectors \(|k, l, \theta\rangle\)

\[
\langle k, l, \theta|k, l, \theta\rangle = ((c^+_1)^k (c^+_2)^l (|c^+_1, c^+_2\rangle)^0 |0\rangle, (c^+_1)^k (c^+_2)^l (|c^+_1, c^+_2\rangle)^0 |0\rangle)
\]

\[
= ((|c^+_1, c^+_1\rangle)^0 (c^+_2)^k (c^+_1)^k (|c^+_1, c^+_2\rangle)^0 |0\rangle, |0\rangle).
\]

(4.5)
Straightforward long computations give:

\[(k, 2l, 0)k, 2l, 0) = k!(p - k + 1)_{k}^{2l}t_{i}^{(p/2)}\]  
(4.6)

\[(k, 2l + 1, 0)k, 2l + 1, 0) = k!(p - k + 1)_{2l+1}^{2l+1}t_{i}^{(p/2)}\]  
(4.7)

\[(k, 2l, 1)k, 2l, 1) = 4k!(p - k + 1)_{k}^{2l+1}t_{i}^{(p/2 + 1)}\]  
(4.8)

\[(k, 2l + 1, 1)k, 2l + 1, 1) = 4k!(p - k + 1)_{k+1}^{2l+1}t_{i}^{(p/2 + 1)}\]  
(4.9)

where the symbol \((a)_{k} = a(a + 1)\cdots(a + k - 1)\) is the common Pochhammer symbol.

From (4.6) \((l = 0, k = 1)\) it follows that \(p\) should be a positive number, otherwise the inner product is not positive definite. In a similar way, whenever \(p\) is fixed then equations (4.6)–(4.9) give that \(k = 0, 1, \ldots, p - \theta\). Now, bearing in mind that vectors of different weight have an inner product of zero, we must find the inner product of vectors with one and the same weight. At level 0 there is one vector of weight \((-\frac{p}{2} | \frac{p}{2}\) only, \(|0, 0, 0) = |0); at level 1 there is one vector of weight \((-\frac{p}{2} + 1 | \frac{p}{2}\) and one of weight \((-\frac{p}{2} | \frac{p}{2} + 1); at level 2 there is one vector of weight \((-\frac{p}{2} + 2 | \frac{p}{2}\), one of weight \((-\frac{p}{2} | \frac{p}{2} + 2), but two vectors of weight \((-\frac{p}{2} + 1 | \frac{p}{2} + 1)\.

The inner products of the latter are given by

\[(1, 1, 0)[0, 0, 1) = p^{2}. \quad (1, 1, 0)[0, 0, 1) = 2p. \quad (0, 0, 1)[0, 0, 1) = 4p. \quad (4.10)

The matrix of inner products of the vectors of weight \((-\frac{p}{2} + 1 | \frac{p}{2} + 1)\) has the determinant

\[\det(\frac{p^{2}}{2p} - \frac{2p}{4p}) = 4p^{2}(p - 1). \quad (4.11)

This matrix is positive definite only if \(p > 1\). Therefore, for \(p > 1\) both vectors of weight \((-\frac{p}{2} + 1 | \frac{p}{2} + 1)\) belong to \(V(p)\); but for \(p = 1\) one vector \((2[1, 1, 0) \sim |0, 0, 1)\) belongs to \(M(p)\) and the subspace of \(V(p)\) of weight \((-\frac{p}{2} + 1 | \frac{p}{2} + 1)\) is one-dimensional.

We can continue this analysis level by level, but the computations become complicated and in order to find a technique that works for arbitrary \(m\) and \(n\) one should find a better way of analyzing \(\overline{V}(p)\). For this purpose, we will construct a different basis for \(\overline{V}(p)\). It is indicated by the character \(\overline{V}(p)\); this is a formal infinite series of terms \(\overline{V}(p) = \sum_{j_{1}, \ldots, j_{n}}[\overline{V}(p)]\) a weight of \(\overline{V}(p)\) and \(n\) the dimension of this weight space. So the vacuum vector \(|0\) of \(\overline{V}(p)\), of weight \((-\frac{p}{2}, \ldots, -\frac{p}{2} | \frac{p}{2}, \ldots, \frac{p}{2}\), yields a term \(x_{1}^{-\frac{p}{2}} \cdots x_{m}^{-\frac{p}{2}} y_{1}^{\frac{p}{2}} \cdots y_{n}^{\frac{p}{2}}\) in the character \(\overline{V}(p)\). Since the basis vectors are given by

\[(c_{m+1}^{(k)})^{(k)}(c_{m+1}^{(k)})^{(k)}(c_{m+1}^{(k)})^{(k)}(c_{m+1}^{(k)})^{(k)}(c_{m+1}^{(k)})^{(k)}\]

where \(k_{1}, \ldots, k_{m+n}, k_{12}, k_{13}, \ldots, k_{m-1,m}, k_{m+1,m+2}, k_{m+1,m+3}, \ldots, k_{m+n-1,m+n} \in \mathbb{Z}

k_{1,m+1}, k_{1,m+2}, \ldots, k_{1,m+n}, k_{2,m+1}, \ldots, k_{m,m+n} = 0, 1, it follows that

\[\det(\frac{p^{2}}{2p} - \frac{2p}{4p}) = 4p^{2}(p - 1). \quad (4.11)

Such expressions have an interesting expansion in terms of supersymmetric Schur polynomials, valid for general \(m\) and \(n\).

**Proposition 4** ([18]). Let \((x) = (x_{1}, x_{2}, \ldots, x_{m})\) and \((y) = (y_{1}, y_{2}, \ldots, y_{n})\) be sets of \(m\) and \(n\) variables, respectively. Then [18]

\[\prod_{j}(1 + x_{j}y_{j}) \prod_{j}(1 - x_{j}x_{k}) \prod_{j}(1 - y_{j}) \prod_{j}(1 - y_{j}) = \prod_{k \in \mathcal{H}} s_{k}(x_{1}, \ldots, x_{m}, y_{1}, \ldots, y_{n}) = \sum_{k \in \mathcal{H}} s_{k}(x)(y). \quad (4.13)\]
On the right-hand side, the sum is over all partitions \( \lambda \), satisfying the so-called hook condition \( \lambda_{m+1} \leq n \ (\lambda \in \mathcal{H}) \) and \( s_\lambda(x|y) \) is the supersymmetric Schur function \([19]\) defined by

\[
s_\lambda(x|y) = \sum_{\tau} \sigma_{\lambda\tau} s_\tau(x)s_\tau(y),
\]

with \( l(\sigma) \leq m \), \( l(\tau') \leq n \), \( \tau' \) the conjugate partition to \( \tau \); \( c_{\sigma\tau}^{\lambda} \) the famous Littlewood–Richardson coefficients, \( |\lambda| = |\sigma| + |\tau| \) and \( s_\lambda(x) \) the ordinary Schur function.

The characters of the irreducible covariant \( u(m|n) \) tensor representations \( \text{V}([\Lambda^\lambda]) \), which are necessarily finite-dimensional, are given by the supersymmetric Schur functions \( s_\lambda(x|y) \), \( \lambda \in \mathcal{H} \). The relation between the partitions \( \lambda = (\lambda_1, \lambda_2, \ldots) \), \( \lambda_{m+1} \leq n \) and the highest weights \( \Lambda^\lambda = [\mu]| = [\mu_1, \ldots, \mu_m, |\mu_{m+1}, \ldots, \mu_r]; \ r = m + n \), of the irreducible covariant \( u(m|n) \) tensor representations is given by \([20]\):

\[
\mu_{ir} = \lambda_i, \quad 1 \leq i \leq m, \quad (4.14)
\]

\[
\mu_{m+i,r} = \max\{0, \lambda'_i - m\}, \quad 1 \leq i \leq n, \quad (4.15)
\]

where \( \lambda' \) is the partition conjugate \([19]\) to \( \lambda \). Therefore expansion \((4.13)\) yields the branching to \( u(m|n) \) of the \( \text{osp}(2m + 1|2n) \) representation \( \text{V}(p) \). This gives the possibility of labeling the basis vectors of \( \text{V}(p) \). For each irreducible covariant \( u(m|n) \) tensor representations one can use the corresponding Gelfand–Zetlin basis \([15]\). The union of all these GZ bases is then the basis for \( \text{V}(p) \). Thus the new basis of \( \text{V}(p) \) consists of vectors of the form \( p \) is dropped from the notation of the vectors.

\[
|\mu| \equiv |\mu|' = \begin{pmatrix}
|\mu_1| & \cdots & |\mu_m| & |\mu_{m+1}| & \cdots & |\mu_{r-1}| & |\mu_r| \\
|\mu_{1,r}| & \cdots & |\mu_{m,r}| & |\mu_{m+1,r}| & \cdots & |\mu_{r-1,r}| & |\mu_{rr}|
\end{pmatrix}
\]

(4.16)

which satisfy the conditions

1. \( \mu_{ir} \in \mathbb{Z}_+ \) are fixed and \( \mu_{jr} - \mu_{j+1,r} \in \mathbb{Z}_+, \ j \neq m, \ 1 \leq j \leq r - 1, \)

\[
\mu_{mr} \geq \#\{i : \mu_{ir} > 0, \ m + 1 \leq i \leq r\};
\]

2. \( \mu_{ip} - \mu_{i,p-1} = \theta_{i,p-1} \in \{0, 1\}, \ 1 \leq i \leq m; \ m + 1 \leq p \leq r; \)

3. \( \mu_{mp} \geq \#\{i : \mu_{ip} > 0, \ m + 1 \leq i \leq p\}, \ m + 1 \leq p \leq r; \)

4. if \( \mu_{m,m+1} = 0 \), then \( \theta_{mn} = 0; \)

5. \( \mu_{ip} - \mu_{i+1,p} \in \mathbb{Z}_+, \ 1 \leq i \leq m - 1; \ m + 1 \leq p \leq r - 1; \)

6. \( \mu_{i,j+1} - \mu_{ij} \in \mathbb{Z}_+ \) and \( \mu_{i,j} - \mu_{i+1,j+1} \in \mathbb{Z}_+, \)

\[
1 \leq i \leq j \leq m - 1 \text{ or } m + 1 \leq i \leq j \leq r - 1.
\]

(4.17)

For \( m = n = 1 \), the new basis vectors are given by

\[
|\mu| = \begin{pmatrix}
|\mu_{12}| & |\mu_{13}|
\end{pmatrix},
\]

(4.18)
where
\[ \mu_{12} \in \mathbb{N} \text{ if } \mu_{22} = 0; \mu_{12} \in \mathbb{Z}_+ \text{ if } \mu_{22} \in \mathbb{Z}_+ \]
and \( \mu_{11} = \mu_{12} - \mu_{12} - 1 \) (if \( \mu_{12} = 0 \), then \( \mu_{11} = 0 \)).

We assume that the action of the \( u(1|1) \) generators is as given in [15] and since the weight of \( |0\rangle \) is \( (-\frac{p}{2}, \frac{p}{2}) \), the weight of the vector \( |\mu\rangle \) is determined by
\[
\left( -\frac{p}{2} \frac{p}{2} \right) + (\mu_{11}|\mu_{12} + \mu_{22} - \mu_{11}).
\]

Our aim is to compute the action of \( c_i^+ \) and \( c_i^+ \) on the basis vectors \( |\mu\rangle \). From the triple relations (3.4), it follows that under the \( u(1|1) \) basis (3.7), the set \( (c_i^+, c_i^+) \) forms a tensor of rank \( (1,0) \). Therefore one can attach a unique GZ pattern with top line 10 to \( c_i^+ \) and \( c_i^+ \), corresponding to the weight \( \epsilon_1 \) and \( \delta_1 \), respectively. Explicitly
\[
c_i^+ \sim \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad c_i^+ \sim \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.
\]

In general, for the action of \( c_i^+ \), \( j = 1, 2 \) on the basis vectors \( |\mu\rangle \) one can write
\[
(c_j^+ |\mu\rangle = \sum_{\mu'} (\mu' |c_j^+ |\mu\rangle |\mu\rangle),
\]
where the matrix elements can be written as follows:
\[
(\mu' |c_j^+ |\mu\rangle = \begin{pmatrix} \mu_{12} & \mu_{22} & 1 & 0 \\ \mu_{11} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mu_{12}' & \mu_{22}' \end{pmatrix} \times (\mu_{12}' \mu_{22}' |c_j^+ |\mu_{12} \mu_{22}))
\]
\[
(\mu' |c_j^+ |\mu\rangle = \begin{pmatrix} \mu_{12} & \mu_{22} & 1 & 0 \\ \mu_{11} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mu_{12}' & \mu_{22}' \end{pmatrix} \times (\mu_{12}' \mu_{22}' |c_j^+ |\mu_{12} \mu_{22})).
\]

The first factor on the right-hand side of (4.22)–(4.23) is a \( u(1|1) \) Clebsch–Gordan coefficient (CGC) [15], and the second factor is a reduced matrix element. The possible values of the patterns \( \mu' \) are determined by the \( u(1|1) \) tensor product \( (1,0) \otimes (\mu_{12}, \mu_{22}) = (\mu_{12} + 1, \mu_{22}) \oplus (\mu_{12}, \mu_{22} + 1) \), and by the additivity property of the internal labels \( (\mu_{11}' = \mu_{11} + 1 \text{ in the above expression}) \). The only \( gl(1|1) \) CGCs of relevance are given below, their values taken from [15]:
\[
\begin{pmatrix} 1 \mu_{12} & 0 \mu_{12} + 1 \mu_{12} + 1 \end{pmatrix} = 1,
\]
\[
\begin{pmatrix} 1 \mu_{12} & 0 \mu_{12} + 1 \mu_{12} + 1 \end{pmatrix} = \sqrt{\frac{\mu_{12} + \mu_{22}}{\mu_{12} + \mu_{22} + 1}},
\]
\[
\begin{pmatrix} 1 \mu_{12} & 0 \mu_{12} + 1 \mu_{12} + 1 \end{pmatrix} = \sqrt{\frac{1}{\mu_{12} + \mu_{22} + 1}},
\]
\[
\begin{pmatrix} 1 \mu_{12} & 0 \mu_{12} + 1 \mu_{12} + 1 \end{pmatrix} = -\sqrt{\frac{1}{\mu_{12} + \mu_{22} + 1}},
\]
\[
\begin{pmatrix} 1 \mu_{12} & 0 \mu_{12} + 1 \mu_{12} + 1 \end{pmatrix} = \sqrt{\frac{\mu_{12} + \mu_{22}}{\mu_{12} + \mu_{22} + 1}}.
\]
Thus the problem now is finding explicit expressions for the functions $\tilde{G}_i$ and $G_i$, $i = 1, 2$, where

$$G_1(\mu) = (\mu_{12} + 1, \mu_{22} || e_{12}^+ || \mu_{12}, \mu_{22}), \quad G_2(\mu) = (\mu_{12}, \mu_{22} + 1 || e_{12}^+ || \mu_{12}, \mu_{22}),$$

We can write

$$c_1^1 \left| \begin{array}{c} \mu_{12}, \mu_{22} \\ \mu_{12} + 1, \mu_{22} \\ \mu_{12} + 1 \\ \mu_{12} - 1 \end{array} \right| = \tilde{G}_1(\mu) \left| \begin{array}{c} \mu_{12} + 1, \mu_{22} \\ \mu_{12} + 1 \\ \mu_{12} \end{array} \right|,$$

$$c_1^2 \left| \begin{array}{c} \mu_{12}, \mu_{22} \\ \mu_{12} + 1, \mu_{22} \\ \mu_{12} \\ \mu_{12} - 1 \end{array} \right| = \tilde{G}_2(\mu) \left| \begin{array}{c} \mu_{12}, \mu_{22} + 1 \\ \mu_{12} + 1 \\ \mu_{12} \end{array} \right|,$$

and the action of $c_2^1$ follows from $(\mu' | c_2^1 | \mu) = (\mu | c_2^1 | \mu')$.

Now it remains to determine the functions $G_i$ and $\tilde{G}_i$, $i = 1, 2$. From the action

$$(c_2^1, c_2^2) | \mu) = 2\hbar_2 | \mu) = (p + 2(\mu_{12} + \mu_{22} - \mu_{11})) | \mu),$$

one deduces the following recurrence relations for $G_1$ and $G_2$:

$$G_1(\mu_{12}, \mu_{22}) G_2(\mu_{12} + 1, \mu_{22} - 1) \sqrt{\mu_{12} + \mu_{22} + 1} + G_1(\mu_{12}, \mu_{22} - 1) G_2(\mu_{12}, \mu_{22} - 1) = 0,$$

$$G_1(\mu_{12}, \mu_{22})^2 + \frac{\mu_{12} + \mu_{22} - 1}{\mu_{12} + \mu_{22}} G_2(\mu_{12}, \mu_{22} - 1)^2 + \frac{\mu_{12} + \mu_{22}}{\mu_{12} + \mu_{22} + 1} G_2(\mu_{12}, \mu_{22})^2 = p + 2\mu_{22},$$

$$G_1(\mu_{12} - 1, \mu_{22})^2 + \frac{\mu_{12} - 1}{\mu_{12} + \mu_{22}} G_2(\mu_{12}, \mu_{22} - 1)^2 + \frac{\mu_{12} + \mu_{22}}{\mu_{12} + \mu_{22} + 1} G_2(\mu_{12}, \mu_{22})^2 = p + 2\mu_{22} + 2.$$
\[ G_1(\mu_{12}, \mu_{22}) = -\sqrt{\mu_{12}(p - \mu_{12})}, \quad \text{if } \mu_{22} \text{ is odd} \] (4.40)

\[ G_2(\mu_{12}, \mu_{22}) = \sqrt{\mu_{12} + \mu_{22} + 1}, \quad \text{if } \mu_{22} \text{ is even} \] (4.41)

\[ G_2(\mu_{12}, \mu_{22}) = -\sqrt{\frac{(\mu_{22} + 1)(p + \mu_{22} + 1)}{\mu_{12} + \mu_{22}}}, \quad \text{if } \mu_{22} \text{ is odd.} \] (4.42)

From the action
\[ [c_1^+, c_1^+] |\mu\rangle = -2h_1|\mu\rangle = (p - 2\mu_{12})|\mu\rangle, \] (4.43)
one deduces the following recurrence relations for \( \tilde{G}_1 \) and \( \tilde{G}_2 \):

\[ \sqrt{\mu_{12} + \mu_{22} + 1} \]
\[ - \tilde{G}_1(\mu_{12}, \mu_{22} - 1)\tilde{G}_2(\mu_{12}, \mu_{22} - 1)\sqrt{\mu_{12} + \mu_{22} - 1} \]
\[ \tilde{G}_1(\mu_{12}, \mu_{22} - 1) = 0, \] (4.44)

\[ \frac{\mu_{12} + \mu_{22}}{\mu_{12} + \mu_{22} + 1} \tilde{G}_1(\mu_{12}, \mu_{22})^2 - \frac{\mu_{12} + \mu_{22} - 1}{\mu_{12} + \mu_{22}} \tilde{G}_1(\mu_{12} - 1, \mu_{22})^2 \]
\[ + \frac{\mu_{12} + \mu_{22}}{\mu_{12} + \mu_{22} + 1} \tilde{G}_2(\mu_{12}, \mu_{22})^2 = p - 2\mu_{12} + 2, \] (4.45)

\[ \tilde{G}_1(\mu_{12}, \mu_{22})^2 = \tilde{G}_1(\mu_{12} - 1, \mu_{22})^2 - \frac{\tilde{G}_2(\mu_{12}, \mu_{22} - 1)^2}{\mu_{12} + \mu_{22}} = p - 2\mu_{12}. \] (4.46)

The action \( c_1^+ |\mu\rangle \) leads to the boundary condition \( \tilde{G}_2(\mu_{12}, \mu_{22} - 1) = 0 \) if \( \mu_{22} = 0 \). The boundary condition together with the recurrence relations (4.44)–(4.46) lead to the following solution for the unknown functions \( \tilde{G}_1 \) and \( \tilde{G}_2 \):

\[ \tilde{G}_i(\mu_{12}, \mu_{22}) = G_i(\mu_{12}, \mu_{22}), \quad i = 1, 2, \quad \text{if } \mu_{22} \text{ is even}, \] (4.47)

\[ \tilde{G}_i(\mu_{12}, \mu_{22}) = -G_i(\mu_{12}, \mu_{22}), \quad i = 1, 2, \quad \text{if } \mu_{22} \text{ is odd}. \] (4.48)

The solution for \( \tilde{G}_i \) and \( G_i \), \( i = 1, 2 \) is unique up to a choice of the sign factor. At this point, only the actions of \([c_1^+, c_1^+] \) and \([c_2^+, c_2^+] \) have been used in the process. Now it remains to verify whether the actions of \( c_1^+ \) and \( c_2^+ \) thus determined do indeed yield a solution, i.e. one should verify that all triple relations are satisfied. This is a straightforward but tedious computation; the only result provided by this calculation is that the sign factors are restricted and their choice in (4.39)–(4.42), (4.47)–(4.48) is the simplest solution.

The explicit expressions for the reduced matrix elements (4.39)–(4.42) and (4.47)–(4.48) give the action of the generators in the basis of \( V(p) \), for arbitrary \( p \). The structure of the maximal submodule \( M(p) \) and hence of the irreducible factor module \( V(p) \) is revealed by examining when these matrix elements vanish. The only crucial factor is

\( (p - \mu_{12}) \).

So, starting from the vacuum vector, with a GZ pattern consisting of all zeros, one can raise the entries in the GZ pattern by applying the operators \( c_1^+ \). However, when \( \mu_{12} \) has reached the value \( p \) it can no longer be increased. As a consequence, all vectors \(|\mu\rangle \) with \( \mu_{12} > p \) belong to the submodule \( M(p) \). This uncovers the structure of \( V(p) \). We summarize.
Theorem 5. An orthonormal basis for the parastatistics Fock space $V(p)$ of one pair of parafermions and one pair of parabosons is given by the vectors $|\mu\rangle$ (see (4.18)–(4.19)), with $\mu_{12} \leq p$. The action of the Cartan algebra elements of $osp(3|2)$ is:

$$h_1|\mu\rangle = \left(-\frac{p}{2} + \mu_{11}\right)|\mu\rangle, \quad h_2|\mu\rangle = \left(\frac{p}{2} + \mu_{12} + \mu_{22} - \mu_{11}\right)|\mu\rangle.$$  (4.49)

For the action of the parastatistics operators $c_j^+, \ j = 1, 2$ we have:

$$c_j^+ \begin{vmatrix} \mu_{12}, \mu_{22} \\ \mu_{12} \end{vmatrix} = (-1)^{\mu_{22}}G_1(\mu_{12}, \mu_{22}) \begin{vmatrix} \mu_{12} + 1, \mu_{22} \\ \mu_{12} + 1 \end{vmatrix},$$  (4.50)

$$c_j^+ \begin{vmatrix} \mu_{12}, \mu_{22} \\ \mu_{12} - 1 \end{vmatrix} = \frac{\mu_{12} + \mu_{22}}{\mu_{12} + \mu_{22} + 1} (-1)^{\mu_{22}}G_1(\mu_{12}, \mu_{22}) \begin{vmatrix} \mu_{12} + 1, \mu_{22} \\ \mu_{12} \end{vmatrix}$$

$$- \frac{1}{\mu_{12} + \mu_{22} + 1} (-1)^{\mu_{22}}G_2(\mu_{12}, \mu_{22}) \begin{vmatrix} \mu_{12}, \mu_{22} + 1 \end{vmatrix}.\ (4.51)$$

$$c_j^+ \begin{vmatrix} \mu_{12}, \mu_{22} \\ \mu_{12} \end{vmatrix} = \frac{1}{\mu_{12} + \mu_{22} + 1} G_1(\mu_{12}, \mu_{22}) \begin{vmatrix} \mu_{12} + 1, \mu_{22} \\ \mu_{12} \end{vmatrix}$$

$$+ \frac{\mu_{12} + \mu_{22}}{\mu_{12} + \mu_{22} + 1} G_2(\mu_{12}, \mu_{22}) \begin{vmatrix} \mu_{12}, \mu_{22} + 1 \end{vmatrix},\ (4.52)$$

$$c_j^+ \begin{vmatrix} \mu_{12}, \mu_{22} \\ \mu_{12} - 1 \end{vmatrix} = G_2(\mu_{12}, \mu_{22}) \begin{vmatrix} \mu_{12}, \mu_{22} + 1 \end{vmatrix}.\ (4.53)$$

and the action of the parastatistics operators $c_j^-, \ j = 1, 2$ is given by:

$$c_j^- \begin{vmatrix} \mu_{12}, \mu_{22} \\ \mu_{12} \end{vmatrix} = (-1)^{\mu_{22}}G_1(\mu_{12} - 1, \mu_{22}) \begin{vmatrix} \mu_{12} - 1, \mu_{22} \\ \mu_{12} \end{vmatrix}$$

$$+ \frac{1}{\mu_{12} + \mu_{22}} (-1)^{\mu_{22}}G_2(\mu_{12} - 1, \mu_{22}) \begin{vmatrix} \mu_{12}, \mu_{22} - 1 \end{vmatrix}.\ (4.54)$$

$$c_j^- \begin{vmatrix} \mu_{12}, \mu_{22} \\ \mu_{12} - 1 \end{vmatrix} = \frac{\mu_{12} + \mu_{22} - 1}{\mu_{12} + \mu_{22}} G_1(\mu_{12} - 1, \mu_{22}) \begin{vmatrix} \mu_{12} - 1, \mu_{22} \end{vmatrix},\ (4.55)$$

$$c_j^- \begin{vmatrix} \mu_{12}, \mu_{22} \\ \mu_{12} \end{vmatrix} = \frac{\mu_{12} + \mu_{22} - 1}{\mu_{12} + \mu_{22}} G_2(\mu_{12}, \mu_{22} - 1) \begin{vmatrix} \mu_{12}, \mu_{22} - 1 \end{vmatrix}, (4.56)$$

$$c_j^- \begin{vmatrix} \mu_{12}, \mu_{22} \\ \mu_{12} - 1 \end{vmatrix} = \frac{1}{\mu_{12} + \mu_{22}} G_1(\mu_{12} - 1, \mu_{22}) \begin{vmatrix} \mu_{12} - 1, \mu_{22} \end{vmatrix}$$

$$+ G_2(\mu_{12}, \mu_{22} - 1) \begin{vmatrix} \mu_{12}, \mu_{22} - 1 \end{vmatrix},\ (4.57)$$

where $G_i, \ i = 1, 2$, are given by equations (4.39)–(4.42).
5. Summary and conclusion

In this paper we have investigated the Fock spaces $V(p)$ of $m$ parafermions and $n$ parabosons with relative parafermion relations among them, which are the unitary irreducible representations of $\mathfrak{osp}(2m+1|2n)$ with lowest weight $(-\frac{p}{2}, \ldots, -\frac{p}{2})$. We have used group theoretical methods and computational techniques. A crucial role in the analysis is played by the $u(m|n)$ subalgebra of $\mathfrak{osp}(2m+1|2n)$, generated by all supercommutators of the parafermions and parabosons. Taking a certain parabolic subalgebra $\mathcal{P}$ containing $u(m|n)$ and a trivial module of $\mathcal{P}$ generated from the vacuum, i.e. the lowest weight vector of weight $(-\frac{p}{2}, \ldots, -\frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2})$, an induced module $\overline{V}(p)$ of $\mathfrak{osp}(2m+1|2n)$ is constructed. The Fock space $V(p)$ is the quotient of this induced module by its maximal submodule $M(p)$. The character of the induced module is obtained and is rewritten as an infinite sum over certain partitions of supersymmetric Schur functions. This can be reinterpreted as a decomposition of the $\mathfrak{osp}(2m+1|2n)$ module into an infinite sum of finite-dimensional simple $u(m|n)$ modules labeled by partitions, namely the covariant $u(m|n)$ tensor modules. For each such representation of $u(m|n)$ one can use the corresponding GZ basis. The union of all these GZ basis vectors is the basis for the induced module $\overline{V}(p)$. The main calculation is then the action of one pair of parafermion and one pair of paraboson operators on this basis. In order to calculate the matrix elements, they are written as a product of a certain $u(1|1)$ CGC and a reduced matrix element. As the relevant $u(1|1)$ (in general $u(m|n)$) CGCs are known, the problem is in finding the reduced matrix elements. Solving a set of recurrence relations for these leads to their expressions. The last give not only the action of the generators in the basis of $\overline{V}(p)$, they also yield the structure of the maximal submodule $M(p)$ and hence of the irreducible factor module $V(p)$. This leads to an explicit basis of $V(p)$ (consisting of all possible GZ patterns with the $\mu_{12}$ integer at most $p$) and the explicit action of the generators in this basis.

We have found the matrix elements only in the case $m = n = 1$. The real interest lies in such quantum systems (parabosons and parafermions) with any degree of freedom, including an infinite degree of freedom. We hope to be able to find the matrix elements for any $m$ and $n$ and to report the result soon.

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References

[1] Green H S 1953 A generalized method of field quantization Phys. Rev. 90 270–3
[2] Kamefuchi S and Takahashi Y 1962 A generalization of field quantization and statistics Nucl. Phys. 36 177–206
[3] Ryan C and Sudarshan E C G 1963 Representations of para-Fermi rings Nucl. Phys. 47 207–11
[4] Ganchev A Ch and Palev T D 1980 A Lie superalgebraic interpretation of the para-Bose statistics J. Math. Phys. 21 797–9
[5] Quesne C 1999 Interpretation and extension of Green’s ansatz for paraparticles Phys. Lett. A 260 437–40
[6] Kanakoglou K and Daskaloyannis C 2007 A braided look at Green ansatz for parabosons J. Math. Phys. 48 113516
[7] Stoilova N I and Van der Jeugt J 2008 The parafermion Fock space and explicit $\mathfrak{so}(2n+1)$ representations J. Phys A: Math. Theor. 41 075202
[8] Lievens S, Stoilova N I and Van der Jeugt J 2008 The paraboson Fock space and unitary irreducible representations of the Lie superalgebra $\mathfrak{osp}(1|2n)$ Commun. Math. Phys. 281 805–26
[9] Greenberg O W and Messiah A M 1965 Selection rules for parafields and the absence of para particles in nature Phys. Rev. 138 B1155–67
[10] Yang W and Jing S 2001 A new kind of graded Lie algebra and parastatistical supersymmetry Sci. China A 44 1167–73
[11] Yang W and Jing S 2001 Fock space structure for the simplest parasupersymmetric system Mod. Phys. Lett. A 16 963–71
[12] Kanakoglou K and Herrera-Aguilar A 2011 Graded Fock-like representations for a system of algebraically interacting paraparticles J. Phys.: Conf. Ser. 287 011237
[13] Kanakoglou K 2011 Ladder operators, Fock-spaces, irreducibility and group gradings for the relative para-Bose set algebra Int. J. Algebra 5 413–28
[14] Palev T D 1982 Para-Bose and para-Fermi operators as generators of orthosymplectic Lie superalgebras J. Math. Phys. 23 1100–2
[15] Stoilova N I and Van der Jeugt J 2010 Gel’fand–Zetlin basis and Clebsch–Gordan coefficients for covariant representations of the Lie superalgebra gl(m|n) J. Math. Phys. 51 093523
[16] Kac V G 1977 Lie superalgebras Adv. Math. 26 8–96
[17] Kac V G 1978 Representations of classical Lie superalgebras Lect. Notes Math. 626 597–626
[18] Cummins C J and King R C 2013 Some noteworthy S-function identities, private communication
[19] Macdonald I G 1995 Symmetric Functions and Hall Polynomials 2nd edn (Oxford: Oxford University Press)
[20] Van der Jeugt J, Hughes J W B, King R C and Thierry-Mieg J 1990 Character formulas for irreducible modules of the Lie superalgebras sl(m|n) J. Math. Phys. 18 2278–304