THE LOCAL SHAPE OF THE UNIVERSE
IN THE INFLATIONARY LIMIT

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Recent high precision data by WMAP and SDSS have provided strong evidence to suggest that the universe is nearly flat. They are also making it possible to probe the topology of the universe. Motivated by these results, we have recently studied the consequences of taking the inflationary limit, i.e. \(|\Omega_0 - 1| \ll 1\). We have shown that in this limit a generic detectable spherical or hyperbolic topology is locally indistinguishable from either \(\mathbb{R}^2 \times S^1\) or \(\mathbb{R} \times T^2\), irrespective of its global shape. Here we briefly present these results and further discuss their observational implications.

Keywords: Cosmic topology; observational cosmology; cosmic microwave background; inflation.

1. Introduction

Among the most fundamental questions in cosmology are those relating to the natures of the geometry and topology of the universe. The geometry can be obtained from the matter-energy content of the universe, through Einstein’s field equations. In this general relativistic context, the universe seems to be well described by a locally homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) metric, with a constant spatial curvature with sign \(k = 0, \pm 1\).

General relativity is however a local metrical theory, and as such does not determine the topology of spacetime. In general, the spatial section of a FLRW spacetime can be a multiply connected manifold (which we assume compact and orientable) \(M = \tilde{M}/\Gamma\), where the covering space \(\tilde{M}\) is either \(\mathbb{E}^3\), \(S^3\) or \(H^3\), depending upon \(k\); and \(\Gamma\) is a discrete and fixed-point free group of isometries of \(\tilde{M}\). Thus, unless specified by a fundamental theory, the topology of the universe is expected to be inferred from observations. An immediate consequence of such multiple-
connectedness, is that there will be multiple images of each radiating source, one for each isometry in $\Gamma$. Therefore, searching for a non-trivial topology amounts to a search for multiple images or pattern repetitions. Due to the existence of a cosmological horizon, however, only the subset of images within the observable universe (i.e. a sphere of radius $\chi_{\text{obs}}$, where $\chi_{\text{obs}}$ is the redshift–distance relation evaluated at $z = z_{\text{max}}$ of the survey used, centered at the observer’s position) is detectable.²

Most studies of cosmic topology so far have concentrated on particular manifolds. Given that there are an infinite number of possible topologies for each curvature sign, a more reasonable search strategy is to ask what is the set of all detectable topologies. In general, each candidate manifold has a distinct isometry group $\Gamma$. Since not all isometries give rise to detectable multiple images, it is reasonable to restrict our focus only to the subgroup generated by detectable isometries, which we refer to as the local shape of the universe. As we shall show, this greatly simplifies the search for image repetitions, while making it possible to obtain some very general results. In the following we briefly outline our recent results for the spherical spaces,³ which show that in an appropriate limit typical manifolds have approximately the same local shape, and further discuss their observational implications.

2. Main Results

Recent high precision data by WMAP⁴ are making it possible to comprehensively probe the topology of the universe for the first time, using pattern repetition. In addition they have provided strong evidence suggesting that the universe is nearly flat, with $\Omega_0 \simeq 1$. According to the most recent estimates of the density parameters, $\Omega_0 \in [0.99, 1.03]$ with a 2$\sigma$ confidence⁵ which still allows the 3-space to be spherical, hyperbolic or flat. Furthermore, in a non-flat inflationary universe one would typically expect $|1 - \Omega_0| \ll 1$, implying $\chi_{\text{obs}} \ll 1$ in units of the curvature radius. In addition, non-flat manifolds are rigid, in the sense that all metrical quantities, such as the length of closed geodesics, are fixed in units of the curvature radius. Hence, the inflationary limit imposes important constraints on the set of detectable isometries for non-flat manifolds, and hence on their local shape.

Our aim here is to take a general point of view and ask what are the set of topologies that would be detectable in very nearly flat universes. Thus to proceed we shall, in addition to assuming $|\Omega_0 - 1| \ll 1$, make two further physically motivated assumptions: (i) the observer is at a position $x$ where the topology is detectable, i.e. $r_{\text{inj}}(x) < \chi_{\text{obs}}$, where $r_{\text{inj}}(x)$ is the injectivity radius at $x$ defined as half the length of the smallest closed geodesic passing through $x$ (see Ref.⁶ for details), and (ii) the topology is not excludable, i.e. it does not produce too many images so as to be ruled out by present observations. Thus, our main physical assumption can be summarized as

$$r_{\text{inj}}(x) \lesssim \chi_{\text{obs}} \ll 1.$$  (1)

These assumptions severely restricts the set of detectable non-flat manifolds. Thus in the case of spherical manifolds, only lens spaces (with $r_{\text{inj}} = \frac{\pi}{p}$) and binary
dihedral spaces (with $r_{inj} = \frac{\pi}{m}$) of sufficiently high order of $p$ or $m$ are detectable. In the hyperbolic case, the only detectable manifolds are the so-called nearly cusped manifolds, which are sufficiently similar to the cusped manifolds (cusped manifolds are not compact, and possess regions with arbitrarily small $r_{inj}(x)$.)

In a recent study, we considered both classes of manifolds and showed that a generic detectable spherical or hyperbolic manifold is locally indistinguishable from either a cylindrical ($\mathbb{R}^2 \times S^1$) or toroidal ($\mathbb{R} \times T^2$) manifold, irrespective of its global shape. Here, we shall briefly review our results, and further discuss their observational implications.

Briefly the key arguments are the following. At any given point $x$, an isometry $\gamma$ will generate a closed geodesic with length $\ell = d(x, \gamma x)$. But given the limit, we are only interested in geodesics short enough so that both the sources and their images lie within the cosmological horizon. The question then is what do these detectable closed geodesics look like? In particular how much do they deviate from the Clifford translations (i.e. isometries with constant distance function $d$) within the observable universe? We have found that for a lens space $L(p, q)$ with generator $g$, the following results hold:

i) The manifold $L(p, q)$ is equivalent to the manifold $L(p_{N-1}, q)$, where $p_{N-1}$ is the second to last convergent in the continued fractions expansion of $p/q$.

ii) The shortest geodesic at any point is generated by $g^{p_j}$, for some convergent $p_j$.

iii) $r_{inj}(x) \leq \frac{\pi}{\sqrt{p}}$. Thus by choosing $p$ to be sufficiently large, one can at each point of a lens space intersects a closed geodesic which is small in units of the curvature radius. We employ this bound to divide the $(q, p)$ plane into detectable, undetectable and observationally excluded regions, as shown in (Fig. 1a). Thus any systematic search for a lens space-shaped universe needs only to concentrate on manifolds that lie in the white central region of the diagram.

iv) Consider an observable sphere of radius $\chi_{obs}$, centered at an observer’s position $x_0$ (Fig. 1b). The isometry $g$ identifies points in two faces of the fundamental polyhedron (dashed lines, shown edgewise for simplicity). Let $d_0$ be the distance $d(x_0, gx_0)$ and let $d_{max}$ and $d_{min}$ denote the maximum and minimum values of $d(x, gx)$ for a point $x$ within the detectable sphere. We then find

$$\Delta d \leq 2 \frac{1}{|z_2||z_1|} \chi_{obs} \ll 1,$$

where $|z_1|$ and $|z_2|$ are position dependent, obeying $|z_1|^2 + |z_1|^2 = 1$. A similar result holds for the cusp-like parts of nearly cusped hyperbolic manifolds, namely

$$\Delta d \leq 2 \chi_{obs} \ll 1.$$

The above inequalities show clearly that subject to condition (1), the detectable isometries of lens spaces (as well as cusp-like hyperbolic manifolds) are very close to Clifford translations for most observers. We also note that all isometries in the isometry groups of the binary dihedral spaces are Clifford translations, and therefore trivially obey the inequality above. These results have important consequences
Fig. 1. Detectability of lens spaces \( L(p, q) \) in terms of the parameters \( p \) and \( q \). A pattern-repetition search should focus on manifolds on the white region (1a); and maximum and minimum lengths of closed geodesics generated by an isometry inside the observable universe of radius \( \chi_{\text{obs}} \). (1b)

for developing search strategies for cosmic topology. They show that for a typical observer in a very nearly flat universe, the 'detectable part' of the topology would be indistinguishable from either \( \mathbb{R}^2 \times S^1 \) or \( \mathbb{R} \times T^2 \) manifold. Note that the detectable isometries cannot have more than 2 independent generators because the fundamental domain has a far greater volume than the observable universe. We also emphasize that no matter how flat the universe turns out to be, there are always an infinite number of candidate manifolds, both spherical and hyperbolic, that have detectable isometries. Furthermore, in the case of (detectable) spherical manifolds, we have also proven that through any point in the manifold there is a closed geodesic of length much smaller than the curvature radius, and therefore the fraction of the total manifold's volume where there are detectable isometries is indeed large.

So far our discussion has assumed the inflationary limit, \(|\Omega_0 - 1| \ll 1\). However, the resolutions required to test such limits are not expected to be attainable in near future. Currently the best fit value for \( \Omega_0 \) is \( 1.02 \pm 0.02 \) to \( 1\sigma \). These bound are also compatible with other topologies not considered here, such as the recently proposed Poincaré dodecahedral space. It is therefore important to ask what the present results can tell us for less restrictive bounds. Results (i)-(iii) hold regardless of the value of \( \Omega_0 \). The bounds in (iv) however rely on approximations that remain valid only if \(|\Omega_0 - 1| \lesssim 10^{-4}\). For \( \Omega_0 \) of this order, the local shape becomes distinguishable from \( \mathbb{R}^2 \times S^1 \) or \( \mathbb{R} \times T^2 \). Nevertheless, the bounds shown here still allow us to severely constrain the magnitude of the deviation.

Currently one of the most promising methods of searching for cosmic topology is the so-called circles-in-the-sky method, where one looks for matching cir-
cles of anisotropies in CMB radiation due to topological identification. For Clifford translations in flat space these pairs must be strictly antipodal in the celestial sphere. Hence, in the inflationary limit, the circles due to isometries of spherical and hyperbolic manifolds should also be nearly antipodal. A preliminary search for nearly antipodal (i.e., with deviation $\theta \leq 10^\circ$) and with radius larger than 25$^\circ$ was undertaken in Ref. [8] with negative results. It can be shown, however, that for $|\Omega_0 - 1| \sim 10^{-4}$ the result (iv) still allows $\theta \geq 10^\circ$. Also, Ref. [9] claims to have found evidence of antipodal circles of radius $< 25^\circ$.

3. Final Remarks

Given the infinite number of possible candidate manifolds for cosmic topology, any realistic search strategy must be able to radically restrict the expected possibilities. This is particularly important in the inflationary limit, where the order of any detectable cyclic subgroup as well as the number of candidate isometries which generate small geodesics are extremely large.

The results presented here are quite general, imposing severe constraints on the set of detectable isometries for all detectable very nearly flat manifolds. By severely restricting the expected topological signatures of detectable isometries, we are able to provide an effective framework for searching for evidence of a non-trivial topology in cosmological observations. More specifically, these results can be used to confine the parameter space which realistic search strategies such as the ‘circles-in-the-sky’ method, need to concentrate on.

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