1 Introduction

The Abel-Jacobi theorem is an important result of algebraic geometry. The theory of divisors and the Riemann bilinear relations are fundamental to the development of this result: if a point $O$ is fixed in a Riemann compact surface $X$ of genus $g$, the Abel-Jacobi map identifies the Picard group $Pic_O (X)$ the quotient of divisors of a group of degree zero on the sub-group of divisors associated to meromorphic functions. The Riemann surface of genus $g \geq 1$ can be embedded in the Jacobian variety $Jac (X)$ via the Abel-Jacobi. In fact we generally have a map:

$$X^{(g)} = X^g / \mathcal{S}_g \longrightarrow Jac (X)$$

such that $X^{(g)}$ may be provided with an analytical structure. Indeed the two sets $X^{(g)} = X^g / \mathcal{S}_g$, $Jac (X)$ are algebraic varieties and the map

$$X^{(g)} \longrightarrow Jac (X)$$

is surjective. For reasons of dimension we can verify that is finite fibers. In fact this is a birational map.

2 Riemann bilinear relations

Let $X$ be a compact Riemannian surface. Recalling that,

$$H_1 (X, \mathbb{Z}) \cong \mathbb{Z}^{2g} \quad \text{and} \quad H^1_{dR} (X, \mathbb{R}) \cong \mathbb{R}^{2g}$$

where $g$ is the genus of $S$. The following map

$$H_1 (X, \mathbb{Z}) \times H^1_{dR} (X, \mathbb{R}) \longrightarrow \mathbb{R}$$

$$(\gamma, \omega) \longrightarrow \int_\gamma \omega$$

makes these two spaces in duality: for a basis $(\gamma_1, ..., \gamma_2g)$ in $H_1 (X, \mathbb{Z})$ there exist a dual basis

$$(\omega_1, ..., \omega_{2g}) \in H^1_{dR} (X, \mathbb{R})$$

such that for $i,j = 1, ..., 2g$

$$\int_{\gamma_i} \omega_j = \delta_{ij}$$
The intersection product

\[ H_1(X, \mathbb{Z}) \times H_1(X, \mathbb{Z}) \rightarrow \mathbb{Z} \]

\[ (\gamma_1, \gamma_2) \rightarrow \gamma_1 \# \gamma_2 \]

defines an antisymmetric bilinear form on \( H_1(X, \mathbb{Z}) \), which has a corresponding symplectic bases

**Proposition 1** For any symplectic basis \((a_1, \ldots, a_g, b_1, \ldots, b_g)\) of \( H_1(X, \mathbb{Z}) \) and for any closed 1-forms \( \eta \) and \( \eta' \) on the surface \( X \) we have

\[
\int_X \eta \wedge \eta' = \sum_{k=1}^g \left( \int_{a_k} \eta \int_{b_k} \eta' - \int_{a_k} \eta' \int_{b_k} \eta \right)
\]

**Preuve.** Let \((a_1, \ldots, a_g, b_1, \ldots, b_g)\) be a symplectic basis of \( H_1(X, \mathbb{Z}) \) associated with a cutting \( S \) into a \( 4g \)-Gones quotes \( \Delta \): \( A_1B_1A'_1B'_1, \ldots, A_gB_gA'_gB'_g \), where \( A_i \) and \( A'_i \) are identified by the map \( \varphi_i \) and \( B_i, B'_i \) are identified by the map \( \psi_i \) as in the following figure. Differential forms can be seen as differential forms on \( \Delta \). Since this last is simply connected, so there exist a function \( f \) such that \( df = \eta \). So for each \( x \in A \) and for each \( y \in B \) we have:

1. \[
\int_{b_i(x)} df = \int_{b_i} \eta = f \circ \varphi_i(x) - f(x)
\]
2. \[
\int_{a_i(x)} df = \int_{a_i} \eta = f(x) - f \circ \psi_i(x)
\]
Stokes formula implies
\[
\int_S \eta \wedge \eta' = \int_{\Delta} \eta \wedge \eta' = \int_D d\left(f \eta'\right) = \int_{\Delta} f \eta' = \sum_{k=1}^g \int_{A_i + B_i - A_i' - B_i'} f \eta'
\]
And it follows from the formulas (1) and (2):
\[
\int_{A_i - A_i'} f \eta' = \int_{A_i} (f - f \circ \varphi_i (x)) \eta' = -\int_{b_i} \eta \int_{a_i} \eta'
\]
\[
\int_{B_i - B_i'} f \eta' = \int_{B_i} (f - f \circ \psi_i (x)) \eta' = \int_{a_i} \eta \int_{b_i} \eta'
\]
which proves equality ■

Remarque 2 If the surface X is provided with a riemann structure, and if \( \eta, \eta' \) are holomorphic 1-forms, then \( \int_X \eta \wedge \eta' = 0 \)

Proposition 3 Let X be a compact Riemannian of which is fixed 2g simple closed curves \((a_1, \ldots, a_g, b_1, \ldots, b_g)\), forming a symplectic basis of the space \( H_1 (X, \mathbb{Z}) \) and let \( \omega_1 \) be a holomorphic 1-form on X and \( \omega_2 \) non-sigular 1-meromorphic form along all the curves \( a_i, b_i \). Given a point \( z_0 \in X - \{a_i, b_i\} \) such that, \( u (z) = \int_{z_0}^z \omega_1 \), then
\[
2i\pi \sum \text{Res} (u, \omega_2) = \sum_{i=1}^g \left( \int_{a_i} \omega_1 \int_{b_i} \omega_2 - \int_{a_i} \omega_2 \int_{b_i} \omega_1 \right)
\]

Preuve. The proposal follows from the Residue formula and equations (1) and (2): \( 2i\pi \sum \text{Res}(u, \omega_2) = \int_{\partial \Delta} u, \omega_2 \) ■

Whether now \((a_1, \ldots, a_g, b_1, \ldots, b_g)\) is a 2g simple closed curves on a compact Riemann surface X which form basis of the space \( H_1 (X, \mathbb{Z}) \) and \((\omega_1, \ldots, \omega_g)\) is a fixed basis of the space of 1-holomorphic forms on X.
Definition 4 Let’s call the period matrices $A, B \in \mathcal{M}_g(\mathbb{C})$ defined by

$$A_{ij} = \int_{a_i} \omega_j$$

$$B_{ij} = \int_{b_i} \omega_j$$

Theorme 5 (Riemann bilinear relations)

1. The matrix $A$ is invertible
2. The matrix $\Omega = A^{-1}B$ is symmetrical and its imaginary part

$$\text{Im} \, \Omega = (\text{Im} \, \Omega_{ij})_{i,j \leq g}$$

is positive definite

Preuve. Whether $\lambda = (\lambda_1, ..., \lambda_g) \in \mathbb{C}^g$ such that $\sum_{i=1}^{g} \lambda_i A_{ij} : j = 1, ..., g$. Consider the holomorphic 1-form

$$\omega = \sum_{i=1}^{g} \lambda_i \omega_i$$

By definition of the matrix $A$, we have:

$$\int_{a_i} \omega = 0 = \sum_{i=1}^{g} \lambda_i A_{ij}$$

so is

$$\int_{a_i} \overline{\omega} = 0$$

Then it follows from the Proposition1,

$$\int_{a_i} \omega \wedge \overline{\omega} = 0 : \omega = 0$$

so $\lambda_i = 0$, $i = 1, ..., g$. For the other one, we easily verify that $\Omega$ is independent of the basis $(\omega_1, ..., \omega_g)$. Since the matrix $A$ is invertible, so a base change we can consider $A = I$: $A_{ij} = \delta_{ij}$. Hence $\Omega_{ij} = B_{ij}$, and it still follows from the Proposition1:

$$0 = \int_{X} \omega_i \wedge \omega_j = \sum_{k=1}^{g} \left( \int_{a_k} \omega_i \int_{b_k} \omega_j - \int_{a_k} \omega_j \int_{b_k} \omega_i \right)$$

$$= \int_{b_i} \omega_j - \int_{b_j} \omega_i$$
Finally, if \( v = (v_1, \ldots, v_g) \in \mathbb{R}^g - \{0\} \), then we have:

\[
\iota_v \operatorname{Im} \Omega.v = \frac{i}{2} \int_X \eta \wedge \eta > 0, \quad \text{when} \quad \eta = \sum_{k=1}^g v_k \omega_k
\]

\[\blacksquare\]

3  Lattice of periods

Let \( X \) be a compact Riemannian surface with two \( 2g \) fixed simply closed curves which form a basis of the space \( H_1(X, \mathbb{Z}) \), \((\omega_1, \ldots, \omega_g)\) a basis of the space \( \Omega^1(X) \) of holomorphic 1-forms is fixed. The image of the following map

\[
p : H_1(X, \mathbb{Z}) \longrightarrow \Omega^1(X)^* \\
\gamma \longmapsto p(\gamma)
\]

is a lattice \( \Lambda \) in \( \Omega^1(X)^* \), where \( p(\gamma)(\omega) = \int_\gamma \omega \).

Definition 6 We call \( \Lambda \) the lattice of periods. The dual basis \((\omega_1, \ldots, \omega_g)\) identifies the space \( \Omega^1(X)^* \) to \( \mathbb{C}^g \). As a lattice in the space \( \mathbb{C}^g \), \( \Lambda = A\mathbb{Z} + B\mathbb{Z} \).

Remarque 7 Note that the set \( \Lambda \) is a lattice since it comes from the Riemann bilinear relations and the real range of \((A, B)\) is equal \( 2g \). The Riemann bilinear relations even show that \( \Lambda \) is a particular lattice.

Definition 8 A divisor on a Riemannian surface is the data of a finite set the points \((P_i, n_i)\), weighted by nonzero integers. The set of divisors is naturally equipped with a commutative group structure. It is a \( \mathbb{Z} \)-module generated by \( X \). A divisor is called effective if its degree \( \sum_i n_i = 0 \), and the divisor \( D \) is principal if \( D = \text{div}(f) \) is given by the poles and zeros of a meromorphic function \( f \).

Notation 9 \( D = \sum_i n_i P_i, \quad \deg D = \sum_i n_i \)

4  Abel-Jacobi map

Wether \( O \) and \( P \) are two points of a Riemann compact surface \( X \). Two paths \( \gamma \) and \( \gamma' \) link \( O \) to \( P \) in \( X \) differ only by a factor of \( H_1(X, \mathbb{Z}) \). In another word: \( p(\gamma) = p(\gamma') \mod \Lambda \). For any path \( \gamma \) the following map

\[
u_O : X \longrightarrow \mathbb{C}^g/\Lambda \\
P \longmapsto (\int_\gamma \omega_1, \ldots, \int_\gamma \omega_g)
\]

5
is well defined, but depending on the point $O$. Moreover, for each point $P \in X$ we can associate the divisor $P - O$ of degree zero. A divisor $\text{div}(f)$ associated to a meromorphic function $f$ is also of degree zero.

**Definition 10** The set of divisors of degree zero is naturally an Abelian group. We call group of Picard $\text{Pic}_O(X)$ the quotient of divisor group of degree zero by the sub-group of divisors associated to meromorphic functions.

**Proposition 11** The map $u_O$ extends naturally into a group morphism:

$$u : \text{Pic}_O(X) \longrightarrow \mathbb{C}^g/\Lambda$$

$$\sum_P n_p P \longrightarrow \sum_P n_p u_O(P)$$

which does not depend on the point $O$.

**Preuve.** Let’s show first the map $u$ is well defined. Wether $\text{div}(f) = \sum_P n_p$ where $f$ is a meromorphic function and we set

$$\omega = \frac{df}{2i\pi f}$$

We note $F_k(z) = \int_O^z \omega_k$ for $k = 1, \ldots, g$. So Proposition 3 implies

$$\sum \text{Res} \left( F_k \frac{df}{f} \right) = \sum_{j=1}^g \left( \int_{a_j}^{b_j} \omega \int_{a_j}^{b_j} \omega_k - \int_{a_j}^{b_j} \omega_k \int_{a_j}^{b_j} \omega \right)$$

The right side is a linear combination in integers of periods $\int_{a_j}^{b_j} \omega_k$, $\int_{b_j}^{a_j} \omega_k$ as integer, because the periods of the 1-form $\omega$ are integers (Residue formula). The left side is equal to

$$\sum_P n_p F_k(P)$$

Finally the $k^{th}$ coordinate of the image $u_O(P)$ equals $F_k(P)$. Whether we change the point $O$ in another one $O' \in X$ in another one, then

$$(u_O - u_{O'}) \left( \sum_P n_p P \right) = -\sum_P n_p \left( \int_{O'}^{O} \omega_1, \ldots, \int_{O'}^{O} \omega_g \right)$$

But the sum of the right hand is zero, because the degree $\sum_P n_p = 0$
Definition 12 The map $u$ defined as above is called the Abel-Jacobi map.

Theorem 13 (Abel) The Abel-Jacobi map is injective.

Proof. Whether $D = \sum n_P P$ is a divisor of degree zero such that $u(D) = 0$, we will find a meromorphic function $f$ such that $D = \text{div}(f)$. Indeed we will construct a 1-form

$$\omega = \frac{df}{2i\pi f}$$

Let $\omega$ be a 1-meromorphic form on the surface $S$ with simples poles in the points $P$ of divisor $D$ with residues $n_P$. Hence once again by Proposition 1:

$$u(D) = \sum n_P u_o(P) = \sum \text{Res}(u_o \omega) = \sum_{j=1}^{g} \left(\int_{a_j}^{b_j} \omega \int_{b_j}^{a_j} \omega_k - \int_{a_j}^{b_j} \omega \int_{b_j}^{a_j} \omega_k \right)_{k=1,\ldots,g}$$

We will modify $\omega$ so that all its periods will become integers. 

Lemma 14 Whether $x_1, \ldots, x_g, y_1, \ldots, y_g$ are complex numbers, then there exists a holomorphic 1-form $\eta$ such that

$$\int_{a_i} \eta = x_i \quad \text{and} \quad \int_{b_i} \eta = y_i$$

if and only if

$$\sum_{k=1}^{g} \left(y_k \int_{a_k} \omega_i - x_k \int_{b_k} \omega_i \right) = 0 \quad i = 1, \ldots, g$$

Proof. As the matrix $A$ is invertible, then the vectors

$$(\int_{a_1} \omega_1, \ldots, \int_{a_g} \omega_g) \quad i = 1, \ldots, g$$

are linearly independent. Now the following linear map is surjective

$$\Phi : \mathbb{C}^{2g} \longrightarrow \mathbb{C}^g$$

$$(x_1, \ldots, x_g, y_1, \ldots, y_g) \longrightarrow \left( \sum_{k=1}^{g} \left(y_k \int_{a_k} \omega_k - x_k \int_{b_k} \omega_i \right)_{i=1,\ldots,g} \right)$$
So $\dim \ker \Phi = g$. But if $\eta$ is a holomorphic 1-form, $\eta \wedge \omega_i = 0 : i = 1, \ldots, g$, and then Proposition 1 implies

$$\left( \int_{a_1} \eta, \ldots, \int_{a_g} \eta, \int_{b_1} \eta, \ldots, \int_{b_g} \eta \right) \in \ker \Phi$$

The lemma follows from that the dimension of the space of the holomorphic 1-forms is equal to the genus $g$. Since $u(D) = 0$ in the quotient $\mathbb{C}^g/\Lambda$, then there exists integers $(A_1, \ldots, A_g, B_1, \ldots, B_g)$ such that

$$\sum_{k=1}^g \left( \int_{a_k} \omega - B_k \int_{a_k} \omega_i - \int_{a_k} \omega - A_k \int_{a_k} \omega_i \right) i = 1, \ldots, g$$

So by the lemma above, there exists a holomorphic 1-form $\eta$ such that all the periods of the 1-form $\eta - \omega$ are integers. Hence we can consider that $\omega$ has integer periods. A primitive of the form between $O$ and $z$ gives the meromorphic function

$$f(z) = \exp \left( 2i\pi \int_{O}^{z} \omega \right)$$

which is well defined, satisfying $\text{div} (f) = D$.

**Theorem 15 (Jacobi)** The Abel-Jacobi map is injective

**Preuve.** The map $u$ is a group morphism. So it suffices to show that the image of the map $u$ contains a neighborhood of the point $O$. This will follow from the inverse function theorem.

**Lemma 16** There exists $g$ distinct points $P_1, \ldots, P_g \in X$ such that any holomorphic 1-form which vanishes in each $P_k$ is identically zero

**Preuve.** For any point $P \in X$ the sub-space

$$H_P = \{ \omega \in \Omega^1(X)^* : \omega(P) = 0 \}$$

is of codimension $\leq 1$ in $\Omega^1(X)$. But the intersection

$$\bigcap_{P \in S} H_P$$

is trivial and $\dim \Omega^1(X) = g$. Then there exists points $P_1, \ldots, P_g \in S$ such that

$$H_{P_1} \cap \ldots \cap H_{P_2} \cap H_{P_g} = 0$$
Let $P_1, ..., P_g \in X$ be fixed points as in the lemma with simply connected disjoint local coordinates $(U_i, z_i)$ around these points and $z_i (P_i) = 0$ $i \leq g$. In fact each 1-form $\omega_i$ is written as:

$$\omega_i = \varphi_{ij} dz_j$$
onumber

on $U_j$. The matrix $(\varphi_{ij})_{1 \leq i,j \leq g}$ is invertible by lemma above.

Consider now the following map

$$F : U_1 \times ... \times U_g \rightarrow \mathbb{C}^g$$

such that

$$F_i (z) = \sum_{j=1}^g \int_{P_j}^{z_i} \omega_i : i = 1, ..., g$$

The integral

$$\int_{P_j}^{z_i} \omega_i$$

is well defined since each $U_i$ is simply connected. Hence the map $F$ is differentiable in complexe coordonates $z_1, ..., z_g$ and the expression of the jacobian matrix is

$$\left( \frac{\partial F_i}{\partial x_j} \right)_{1 \leq i,j \leq g} = (\varphi_{ij} (P))_{1 \leq i,j \leq g}$$

This matrix is invertible in the point $P = (P_1, ..., P_g)$. So by the local inverse theorem we have a neighborhood of $F(P) = 0$:

$$W = F(U_1 \times ... \times U_g) \subset \mathbb{C}^g$$

Finally if $\xi \in W$ then there exists points $Q_1, ..., Q_g \in \mathbb{C}^g$ such that

$$\left( \sum_{j=1}^g \int_{P_j}^{Q_j} \omega_1, ..., \sum_{j=1}^g \int_{P_j}^{Q_j} \omega_g \right) = \xi$$

In another words

$$u \left( \sum_{j=1}^g (Q_j - P_j) \right) = \xi$$

Summarizing the theorem of Abel-Jacobi:
**Thorme 17** (Abel-Jacobi) The Abel-Jacobi map $u : Pic(X) \rightarrow Jac(X) = \mathbb{C}^g/\Lambda$ is bijective

Furthermore whether a point $O \in X$ is fixed, we have the following map

$$
\begin{align*}
    u_O : X & \rightarrow Jac(X) \\
    P & \rightarrow u(P - O)
\end{align*}
$$

When $g = 1$ this map is an isomorphisme. In general it is still:

**Proposition 18** If the genus $g \geq 1$, the map $u_O : X \rightarrow Jac(X)$ is an embedding

**Preuve.** Since $S$ is compact, it suffices to show that $u_O$ is an injective immersion map. Let’s prove firstable $u_O$ is injective. Suppose by contradiction that $u_O(P) = u_O(P')$. So the map $u$ concels on the divisor of degree zero, $P - P'$. This last is the divisor of a meromorphic function $f$. This one has a single pole and a single zero; so it is a map:

$$X \rightarrow \mathbb{CP}^1$$

of degree one. Thus is absurde since $g \geq 1$. Let’s prove that $u_O$ is an immersion map. As in the proof the Abel-Jacobi theorem:

$$d_P u_O(\xi) = (\omega_1(P)(\xi), ..., \omega_g(P)(\xi))$$

The proposition follows again from the local inverse theorem and the next lemma ■

**Lemme 19** The holomorphic 1-forms $(\omega_1, ..., \omega_g)$ have no common zero

**Preuve.** Once again by contradiction: if a point $P$ is a common zero. According to Riemann-Roch theorem: the dimension of the space of holomorphic functions having more then one simple pole in $P$ equals:

$$
\begin{align*}
    \text{deg } u_O - g + 1 + \dim \{ \omega \in \Omega^1(X) : \omega(P) = 0 \} \\
    & = 1 - g + 1 + \dim \{ \omega \in \Omega^1(X) : \omega(P) = 0 \} = 2
\end{align*}
$$

Then there exists a function $f \in X$, which has a unique simple pole in $P$. So it is a map $f : X \rightarrow \mathbb{CP}^1$ of one degree, when even an absurdity since $g \geq 1$ ■
Remarque 20  Once a point $O \in X$ is fixed we have more generally a map

$$X^{(g)} = X^g / \mathfrak{S}_g \rightarrow \text{Jac}(X)$$

$$(P_1, \ldots, P_g) \rightarrow u \left( \sum_{j=1}^g (P_j - O) \right)$$

and $X^{(g)}$ can be provided with an analytical structure. We showed that the map $X^{(g)} \rightarrow \text{Jac}(X)$ is surjective. For reasons of dimensions we can verify that is finite fibers. We can show:

- $X^{(g)}$ and $\text{Jac}(X)$ are algebraic variety
- The map $X^{(g)} \rightarrow \text{Jac}(X)$ is birational
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