On the Laistrygonian Nichols algebras that are domains

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Abstract
We consider a class of Nichols algebras $\mathcal{B}(\mathcal{L}_q(1, G))$ introduced in Andruskiewitsch et al. which are domains and have many favourable properties like AS-regular and strongly noetherian. We classify their finite-dimensional simple modules and their point modules.

Keywords Hopf algebras · Nichols algebras · Simple modules · Point modules

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1 Introduction

1.1 The context

The classification of Nichols algebras with finite Gelfand–Kirillov dimension (GKdim) over abelian groups, although not yet complete, has recently made significant progress; see [3, 4, 8] and references therein. In particular, those that are domains are completely classified, see [3, Theorem 1.4.1] and [2]. Beyond those coming from quantum groups with generic parameter and the Jordan plane, the next examples are the Laistrygonian Nichols algebras $B(L_q(1, G))$, where $G \in \mathbb{N}$. The precise definition is recalled below but notice that there are other Laistrygonian Nichols algebras that are not domains. The purpose of this paper is to study the Laistrygonian Nichols algebras $B(L_q(1, G))$ as algebras, rather than as braided Hopf algebras.

For the importance of Nichols algebras over abelian groups towards the classification of pointed Hopf algebras with finite GKdim, see [1].

1.2 Main results

In Sect. 2, we recall the definition and basic properties of the algebras $B(L_q(1, G))$ from [3]. Since they have a PBW basis, they are iterated Ore extensions and therefore AS-regular and Cohen–Macaulay, see Proposition 2.3. Our first main result is the classification of the finite-dimensional simple modules of $B(L_q(1, G))$, see Theorem 3.5. Here is the basic idea of the proof: there is a surjective algebra map $\nu_G$ from $B(L_q(1, G))$ to the quantum plane $k_q[X, Y]$ (or the usual polynomial ring since $q = 1$ is allowed). We show that any finite-dimensional irreducible representation of $B(L_q(1, G))$ factorizes through $\nu_G$, and thus is known. This curious phenomenon appeared in other examples, see, for instance, [18, Lemma 2.1] and [5, Theorem 3.11].

In Sect. 4, we discuss relations between different Laistrygonian Nichols algebras. Our second main result is the classification of the point modules of $B(L_q(1, G))$, see Theorem 5.2.
Notations and conventions

We denote the natural numbers by \( \mathbb{N} \) and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). If \( k < t \in \mathbb{N}_0 \), then we denote \( \mathbb{I}_{k,t} = \{ n \in \mathbb{N}_0 : k \leq n \leq t \} \), and \( \mathbb{I}_t := \mathbb{I}_{1,t} \). We work over an algebraically closed field \( k \) of characteristic 0.

All modules are left. As usual, \( \text{rep} \ A \) is the category of finite-dimensional representations of an algebra \( A \); the set of isomorphism classes of simple objects in \( \text{rep} \ A \) is denoted \( \text{irrep} \ A \). As usual, we talk without distinctions of an element of \( \text{irrep} \ A \) or its representative. We use indistinctly the languages of representations and modules.

The braided tensor category of left Yetter–Drinfeld modules over a Hopf algebra \( H \) is denoted by \( H \mathcal{YD} \).

Our reference for Hopf algebras is [20]. We use the expression braided Hopf algebra as in [22]; that is a (rigid) braided vector space with compatible multiplication and comultiplication. As explained in loc. cit., this means that it can be realized as a Hopf algebra in the braided tensor category \( H \mathcal{YD} \) over some Hopf algebra \( H \).

2 Preliminaries

2.1 The Nichols algebra \( \mathcal{B}(\mathcal{L}_{q}(1, \mathcal{D})) \)

We introduce the algebra of our interest; see [3, §4.3.1] for details. Let \( \mathcal{D} \in \mathbb{N} \) and \( q \in k^\times \). The algebra \( \mathcal{B}(\mathcal{L}_{q}(1, \mathcal{D})) \) is presented by generators \( x_1, x_2, (z_n)_{n \in \mathbb{I}_{0,\mathcal{D}}} \) with defining relations

\[
\begin{align*}
    x_2 x_1 - x_1 x_2 + \frac{1}{2} x_1^2, \\
    x_1 z_0 - q^{-1} z_0 x_1, \\
    z_n z_{n+1} - q^{-1} z_{n+1} z_n, & \quad n \in \mathbb{I}_{0,\mathcal{D}-1}, \\
    x_2 z_n - q z_n x_2 - z_{n+1}, & \quad n \in \mathbb{I}_{0,\mathcal{D}-1}, \\
    x_2 z_{\mathcal{D}} - q z_{\mathcal{D}} x_2.
\end{align*}
\]

\( \mathcal{B}(\mathcal{L}_{q}(1, \mathcal{D})) \) is a domain and has a PBW basis

\[
B_{\mathcal{D}} = \{ x_1^{m_1} x_2^{m_2} z_{\mathcal{D}}^{n_\mathcal{D}} \cdots z_{1}^{n_1} z_0^{n_0} : m_i, n_j \in \mathbb{N}_0 \};
\]

hence, \( \text{GKdim} \mathcal{B}(\mathcal{L}_{q}(1, \mathcal{D})) = 3 + \mathcal{D} \). The algebra \( \mathcal{B}(\mathcal{L}_{q}(1, \mathcal{D})) \) is graded, with

\[
\text{deg} x_1 = \text{deg} x_2 = 1, \quad \text{deg} z_n = n + 1, \quad n \in \mathbb{I}_{0,\mathcal{D}}.
\]

Actually, \( \mathcal{B}(\mathcal{L}_{q}(1, \mathcal{D})) \) is the Nichols algebra of the braided vector space \( \mathcal{L}_{q}(1, \mathcal{D}) \) that has a basis \( x_1, x_2, x_3 := z_0 \), cf. [3, §4.1.1] and Sect. 4. Indeed (2.4) is just the recursive definition of \( z_n \) in terms of \( x_1, x_2, z_0 \). Notice that \( q = q_{12} = q_{21}^{-1} \) in the notation of [3]. In loc. cit., the parameter \( q \) was somehow neglected as the main focus.
was on the classification problem, see also Proposition 4.4. But for the sake of the algebraic properties, the role of $q$ is central, as we see in this paper.

**Remark 2.1** Notice that the subalgebra of $B(L_q(1, G))$ generated by $x_1$ and $x_2$ is isomorphic to the Jordan plane and has defining relation (2.1).

It follows from (2.1) by a standard argument that

$$x_1 x_2^j = (x_2 + \frac{1}{2} x_1)^j x_1, \quad j \in \mathbb{N}. \quad (2.7)$$

Also, one derives from [3, Lemmas 4.3.3, 4.3.4] that

$$x_1 z_n = q z_n x_1, \quad n \in \mathbb{Z}_0, G \quad (2.8)$$

$$z_m z_n = q^{m-n} z_n z_m, \quad m < n \in \mathbb{Z}_0, G. \quad (2.9)$$

### 2.2 Ring-theoretical properties

We show that $B(L_q(1, G))$ is an iterated Ore extension. Hence, it is strongly noetherian by [10, Proposition 4.10]; AS-regular by [9, Proposition 2] and Cohen–Macaulay by [23, Lemma 5.3].

We start by an auxiliary result. Consider the following subalgebras of $B(L_q(1, G))$:

- $R_1 = \mathbb{K}[x_1]$, $R_2 = \mathbb{K}(x_1, x_2)$, $R_3 = \mathbb{K}(x_1, x_2, z_G)$

and in general

$$R_{G+3-j} = \mathbb{K}(x_1, x_2, z_G, z_G^{-1}, \ldots, z_j), \quad j \in \mathbb{Z}_0, G.$$

Let $j \in \mathbb{Z}_0, G$. Because of the defining relations, (2.8) and (2.9), we have that

$$B_{G+3-j} = \{ x_1^{m_1} x_2^{m_2} z_G^{n_1} \ldots z_j^{n_j} : m_i, n_h \in \mathbb{Z}_0 \}$$

is a PBW basis of $R_{G+3-j}$.

Let now $j \in \mathbb{Z}_1, G$. We denote by $x_1, x_2, (z_n)_{n \in \mathbb{Z}_0, G-j}$ the generators of $B(L_q(1, G-j))$ and by $B_{G-j}$ the corresponding PBW basis. Then there is an algebra map $\psi : B(L_q(1, G-j)) \rightarrow R_{G+3-j}$ given by

$$x_1 \mapsto x_1, \quad x_2 \mapsto x_2, \quad z_n \mapsto z_{j+n}, \quad n \in \mathbb{Z}_0, G-j.$$

Indeed it is easy to see that this assignment preserves the defining relations.

**Lemma 2.2** The map $\psi : B(L_q(1, G-j)) \rightarrow R_{G+3-j}$ is an isomorphism.

**Proof** Clearly $\psi$ sends the PBW basis $B_{G-j}$ to the PBW basis $B_{G+3-j}$. \hfill $\Box$

To fix the notation, we recall that given a ring $R$, $\sigma \in \text{Aut}(R)$ and a $(\sigma, 1)$-derivation $\delta$ of $R$, i.e. $\delta(rr') = \sigma(r)\delta(r') + \delta(r)r'$, the Ore extension $R[X; \sigma, \delta]$ (or simply $R[X; \sigma]$ if $\delta = 0$) is the ring of polynomials $R[X]$ with the multiplication determined by $Xr = \sigma(r)X + \delta(r), r \in R$. 

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Proposition 2.3 The algebra $\mathcal{B}(\mathcal{L}_q(1, \mathcal{G}))$ is an iterated Ore extension.

**Proof** It is well known that $R_2$ is an Ore extension of $R_1$ and it follows easily that $R_3 \cong R_2[X; \sigma_g]$ where $\sigma_g(x_1) = q^{-1}x_1$, $\sigma_g(x_2) = q^{-1}x_2$. Let $j \in \mathbb{I}_{0,\mathcal{G}-1}$. By the preceding discussion, $R_{\mathcal{G}+3-j}$ is a free $R_{\mathcal{G}+2-j}$-module with basis $(z_i^n)_{n \in \mathbb{N}_0}$. Using Lemma 2.2, we check that there are an algebra automorphism $\sigma_j$ and a $(\sigma_j, 1)$-derivation $\delta_j$ of $R_{\mathcal{G}+2-j}$ determined by

$$
\begin{align*}
\sigma_j(x_1) &= q^{-1}x_1, & \sigma_j(x_2) &= q^{-1}x_2, & \sigma_j(z_i) &= q^{j-i}z_i, \\
\delta_j(x_1) &= 0, & \delta_j(x_2) &= -q^{-1}z_{j+1}, & \delta_j(z_i) &= 0,
\end{align*}
$$

$i \in \mathbb{I}_{j+1,\mathcal{G}}$. Therefore, $R_{\mathcal{G}+3-j} \cong R_{\mathcal{G}+2-j}[X; \sigma_j, \delta_j]$, for all $j \in \mathbb{I}_{0,\mathcal{G}-1}$. \hfill \Box$

2.3 Quotients of the Laistrygon

The notion of exact sequence of Hopf algebras in braided tensor categories was first discussed in [15]. In the particular setting of braided Hopf algebras as in [22], the first reference we are aware of is [6]. We recall from loc. cit. that the sequence of braided Hopf algebras and braided Hopf algebra morphisms

$$
0 \longrightarrow S \xrightarrow{\iota} R \xrightarrow{\pi} T \longrightarrow 0
$$

is exact if $\iota$ is injective, $\pi$ is surjective, $\ker \pi \cong RS^+$ and $R^{\text{co} \pi} = S$. A braided Hopf algebra $R$ fitting into the previous exact sequence is called an extension of $T$ by $S$. Clearly $*$ implies the equality $RS^+ = S^+R$; when this last equality holds, we say that $S$ is normal in $R$. Notice that there are exact sequences where either $S$ or $T$, or both, are usual Hopf algebras but $R$ is braided in a strict sense.

We now present $\mathcal{B}(\mathcal{L}_q(1, \mathcal{G}))$ as an extension of braided Hopf algebras. By (2.7) and (2.8), we have that

$$
\mathcal{B}(\mathcal{L}_q(1, \mathcal{G}))(x_1) \mathcal{B}(\mathcal{L}_q(1, \mathcal{G})) = \mathcal{B}(\mathcal{L}_q(1, \mathcal{G}))x_1.
$$

Hence, $k[x_1]$ is a normal Hopf subalgebra of $\mathcal{B}(\mathcal{L}_q(1, \mathcal{G}))$ and

$$
\mathcal{D}_q(\mathcal{G}) := \mathcal{B}(\mathcal{L}_q(1, \mathcal{G}))/\mathcal{B}(\mathcal{L}_q(1, \mathcal{G}))x_1 \mathcal{B}(\mathcal{L}_q(1, \mathcal{G}))
$$

is a braided Hopf algebra quotient that fits into an exact sequence

$$
0 \rightarrow k[x_1] \rightarrow \mathcal{B}(\mathcal{L}_q(1, \mathcal{G})) \xrightarrow{\sigma} \mathcal{D}_q(\mathcal{G}) \rightarrow 0
$$

of braided Hopf algebras. Using the PBW basis, we see that $\mathcal{D}_q(\mathcal{G})$ is generated by $x_2, (z_n)_{n \in \mathbb{I}_{0,\mathcal{G}}}$ with defining relations (2.3), (2.4) and (2.5). Here and below we use the same notation for $x_2, z_n$ and their images in $\mathcal{D}_q(\mathcal{G})$.

The above projection $\sigma$ induces a map irrep $\mathcal{D}_q(\mathcal{G}) \rightarrow \text{irrep } \mathcal{B}(\mathcal{L}_q(1, \mathcal{G}))$. 

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Lemma 2.4 The above map is bijective: irrep \( \mathcal{B}(\mathfrak{I}_q(1, \mathcal{I})) \cong \text{irrep } \mathcal{I}_q(\mathcal{I}) \).

**Proof** By Remark 2.1, \( x_1 \) and \( x_2 \) generate a subalgebra of \( \mathcal{B}(\mathfrak{I}_q(1, \mathcal{I})) \) isomorphic to the Jordan plane. Thus, by [18, Lemma 2.1] we have that \( x_1 \) acts nilpotently on every finite-dimensional \( \mathcal{B}(\mathfrak{I}_q(1, \mathcal{I})) \)-module; but \( \ker x_1 \) is a submodule by the preceding; hence, \( x_1 \) acts by 0 on every finite-dimensional simple \( \mathcal{B}(\mathfrak{I}_q(1, \mathcal{I})) \)-module. \( \square \)

Lemma 2.5 Let \( \mathcal{I} > 1 \). The map \( \pi_{\mathcal{I}} : \mathcal{B}(\mathfrak{I}_q(1, \mathcal{I})) \to \mathcal{B}(\mathfrak{I}_q(1, \mathcal{I} - 1)) \) given by

\[
\pi_{\mathcal{I}}(x_i) = x_i, \quad \pi_{\mathcal{I}}(z_j) = z_j, \quad \pi_{\mathcal{I}}(z_{\mathcal{I}}) = 0, \quad i \in \mathbb{I}_2, \quad j \in \mathbb{I}_{0, \mathcal{I} - 1}.
\]

is an algebra epimorphism.

Clearly \( \ker \pi_{\mathcal{I}} = z_{\mathcal{I}} \mathcal{B}(\mathfrak{I}_q(1, \mathcal{I})) \); thus, we have an isomorphism of algebras

\[
\mathcal{B}(\mathfrak{I}_q(1, \mathcal{I}))/z_{\mathcal{I}} \mathcal{B}(\mathfrak{I}_q(1, \mathcal{I})) \cong \mathcal{B}(\mathfrak{I}_q(1, \mathcal{I} - 1)). \quad (2.10)
\]

### 3 Simple modules of \( \mathcal{B}(\mathfrak{I}_q(1, \mathcal{I})) \)

The purpose of this section is to give the classification of the finite-dimensional simple \( \mathcal{B}(\mathfrak{I}_q(1, \mathcal{I})) \)-modules. We reduce this computation to those of the quantum plane, see Proposition 3.1.

#### 3.1 Simple modules of the quantum plane

Let \( k_q[X, Y] \) denote the algebra generated by \( X \) and \( Y \) with defining relation \( XY = qYX \). Then there is a surjective algebra map \( \nu_{\mathcal{I}} : \mathcal{B}(\mathfrak{I}_q(1, \mathcal{I})) \to k_q[X, Y] \) given by

\[
\nu_{\mathcal{I}}(x_1) = \nu_{\mathcal{I}}(z_j) = 0, \quad j \in \mathbb{I}_{\mathcal{I}}, \quad \nu_{\mathcal{I}}(x_2) = X, \quad \nu_{\mathcal{I}}(z_0) = Y
\]

for any \( \mathcal{I} \in \mathbb{N} \). Clearly \( \nu_{\mathcal{I}} = \nu_1 \pi_2 \ldots \pi_{\mathcal{I}} \), cf. Lemma 2.5.

If \( q = 1 \), then \( k_1[X, Y] = k[X, Y] \) is the polynomial ring in 2 variables; by Hilbert’s Nullstellensatz its finite-dimensional simple modules are all one-dimensional and parametrized by the points of the plane. Assume that \( q \neq 1 \); then \( k_q[X, Y] \) is called the quantum plane of parameter \( q \). We recall the well-known classification of its finite-dimensional simple modules. First, there are the one-dimensional \( k_q[X, Y] \)-modules \( k_q^X = k \) with action \( X \cdot 1 = a, Y \cdot 1 = 0 \) and \( k_q^Y = k \) with action \( X \cdot 1 = 0, Y \cdot 1 = a \), for every \( a \in k^\times \). Second, suppose that ord \( q =: N < \infty \) and let \( (e_i)_{i \in \mathbb{I}_N} \) be the canonical basis of \( k_N \). Given \( a, b \in k^\times \), the \( k_q[X, Y] \)-module \( \mathcal{U}_{a, b} \) is \( k_N \) with the action defined by

\[
Xe_i = aq^{i-1}e_i, \quad Ye_j = e_{j+1}, \quad Ye_N = be_1, \quad i \in \mathbb{I}_N, \quad j \in \mathbb{I}_{N-1}.
\]

It is easy to see that \( \mathcal{U}_{a, b} \) is simple.
Proposition 3.1 Assume that $q \neq 1$. Let $V \in \text{irrep } k_q[X, Y]$.

(a) If $\dim V = 1$, then $V$ is isomorphic to $k_q^X$ or to $k_q^Y$ for a unique $a \in k^\times$.

(b) If $\dim V > 1$, then $\text{ord } q =: N < \infty$ and $V \simeq U_{a, b}$ for unique $a, b \in k^\times$.

Proof Since $\ker X$ is a $k_q[X, Y]$-submodule of $V$, then $\ker X = V = 0$. If $\ker X = V$, then $V = \langle v \rangle$ is one-dimensional, $V = av$ and $V \simeq k_q^X$ for a unique $a \in k^\times$. If $\ker X = 0$, then from $XY = qYX$ we see that $(1 - q^{\dim V}) \det Y = 0$. If $\det Y = 0$, then $V \simeq k_q^X$ for a unique $a \in k^\times$ as before. If $\det Y \neq 0$, then $q^{\dim V} = 1$, so that $\text{ord } q < \infty$. Since $XY^N = Y^NX$, there exist $v \in V - 0$ and $a, b \in k^\times$ such that $Xv = av$ and $Y^N v = bv$. Therefore, $V = \langle Y^iv : i \in \mathbb{Z}_{0, N - 1} \rangle$ and consequently $V \simeq U_{a, b}$. \hfill $\square$

Remark 3.2 The infinite-dimensional simple $k_q[X, Y]$-modules were computed in [13] using results from [14].

3.2 Finite-dimensional simple modules

We proceed now with the classification of the finite-dimensional simple $B(\mathbb{L}_q(1, G))$-modules.

Recall that $B_q(G)$ is generated by $x_2, (z_n)_{n \in \mathbb{Z}_q}$ with defining relations (2.3), (2.4) and (2.5). The relations (2.4) and (2.5) imply that

$$z_{q - 1} x_2^j = q^{-j} x_2^j z_{q - 1} - j q^{-j} x_2^j z_q, \quad j \in \mathbb{N}. \quad (3.1)$$

We start with an auxiliary result.

Lemma 3.3 Let $V \in \text{rep } B_q(G), n = \dim V$. If the action of $z_{q^j}$ is invertible, then the actions of $z_{q^j - 1}, x_2$ are invertible and $q^n = 1$.

Proof We prove that $z_{q^j - 1}$ is invertible; the proof for $x_2$ is similar. Suppose that $\ker z_{q^j - 1} \neq 0$. Note that $z_{q^j}$ ker $z_{q^j - 1} \subseteq \ker z_{q^j - 1}$ by (2.3). Hence, there exist $\lambda \in k^\times$ and $0 \neq v_0 \in \ker z_{q^j - 1}$ such that $z_{q^j} v_0 = \lambda v_0$. Let $v_j : = x_2^j v_0, j \in \mathbb{N}_0$. By (2.5), $z_{q^j} v_j = \lambda q^{-j} v_j$. By (3.1),

$$z_{q^j} v_j = -j \lambda q^{-j} v_{j - 1}, \quad j \in \mathbb{N}. \quad (3.2)$$

(This is also valid for $j = 0$ if we agree that $v_{j - 1} = 0$). Since $\dim V < \infty$, there exists $m \in \mathbb{N}$ such that $v_m \in \langle v_j : j \in \mathbb{Z}_{0, m - 1} \rangle$. Pick $m$ minimal (here we use that $v_0 \neq 0$) and write $v_m = \sum_{j \in \mathbb{Z}_{0, m - 1}} a_j v_j$. Applying $-z_{q^j - 1}$, we see from (3.2) that

$$m \lambda q^{-m} v_{m - 1} = \sum_{j \in \mathbb{Z}_{0, m - 1}} j \lambda q^{-j} a_j v_{j - 1}.$$

Since $\lambda \neq 0$, we conclude that $v_{m - 1} \in \langle v_j : j \in \mathbb{Z}_{0, m - 2} \rangle$, a contradiction to the minimality of $m$. Hence, $z_{q^j - 1}$ is invertible. From (2.3) follows that $(1 - q^m) \det z_q \det z_{q^j - 1} = 0$ and consequently $q^n = 1$. \hfill $\square$
Lemma 3.4 Let $V \in \text{irrep } D_q(\mathcal{G})$. Then $z_g = 0$ on $V$.

**Proof** Let $V \in \text{irrep } D_q(\mathcal{G})$, $n = \dim V$. Then $z_g$ is a submodule of $V$ by (2.5) and (2.9); consequently $\ker z_g = 0$ or $\ker z_g = V$. Suppose that $\ker z_g = 0$. Then $q^n = 1$ and $\ker z_{g-1} = \ker x_2 = 0$ by Lemma 3.3. Hence, $n = lN$, where $N = \text{ord } q$ and $l \in \mathbb{N}$. Let $\lambda \in \mathbb{k}^\times$ an eigenvalue of $z_g$ and $\lambda_j := \lambda q^j$, $j \in \mathbb{Z}$. Let $V^\lambda = \ker(z_g - \lambda)$ denote the eigenspace of eigenvalue $\lambda$. By (2.3), $z_{g-1}V^{\lambda_j} \subseteq V^{\lambda_{j+1}}$, $j \in \mathbb{N}_0$. Since $z_{g-1}$ is invertible, $z_{g-1}V^{\lambda_j} = V^{\lambda_{j+1}}$. Similarly, $x_2V^{\lambda_j} = V^{\lambda_{j-1}}$. Thus, $V = V^{\lambda_0} \oplus \ldots \oplus V^{\lambda_{N-1}}$ and $\dim V^{\lambda_j} = l$. Let $B_{\lambda} = \{v_j : j \in \mathbb{I}_l\}$ be a basis of $V^\lambda$. Then $B = \bigcup_{j \in \mathbb{I}_{2,N-1}} z_g^{-1} B_{\lambda}$ is a basis of $V$ and the actions of $z_g^{-1}, z_g$ and $x_2$ in this basis are given, respectively, by the following matrices

\[
\begin{pmatrix}
0 & \ldots & 0 & A \\
\text{id}_l & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \ldots & \text{id}_l & 0
\end{pmatrix},
\begin{pmatrix}
\lambda \text{id}_l & \ldots & 0 \\
0 & \ldots & 0 \\
\vdots & \ddots & \ddots \\
0 & \ldots & \lambda_{N-1} \text{id}_l
\end{pmatrix},
\begin{pmatrix}
0 & B_2 & \ldots & 0 \\
0 & \ldots & 0 \\
\vdots & \ddots & \ddots \\
0 & \ldots & B_N & B_1
\end{pmatrix},
\]

with $A, B_l \in \text{GL}_l(\mathbb{k})$, $l \in \mathbb{I}_N$. By (2.4), we have for all $j \in \mathbb{I}_{2,N-1}$,

$$B_2 = qAB_1 + \lambda \text{id}_l, \quad B_{j+1} = qB_j + \lambda_{j-1} \text{id}_l, \quad B_1 = qB_N + \lambda_{N-1} \text{id}_l.$$  

Arguing inductively we see that $AB_1 + N\lambda_{N-1} \text{id}_l = B_1 A$. Applying the trace map to this identity we get that $n\lambda_{N-1} = 0$; thus, $\lambda = 0$, a contradiction. \qed

**Theorem 3.5** irrep $D(\mathcal{L}_q(1, \mathcal{G})) \simeq \text{irrep } k_q[X, Y]$.

**Proof** Let $V \in \text{irrep } D(\mathcal{L}_q(1, \mathcal{G}))$. By Lemma 2.4, $V \in \text{irrep } D_q(\mathcal{G})$, and thus, by Lemma 3.4, $z_g = 0$ on $V$. Notice that $D_q(\mathcal{G})z_g D_q(\mathcal{G}) = z_g D_q(\mathcal{G})$. Since $D_q(\mathcal{G} - 1) = D_q(\mathcal{G})/D_q(\mathcal{G})z_g D_q(\mathcal{G})$, using Lemma 3.4 again, we conclude that $z_{g-1} = 0$ on $V$. Repeating this $\mathcal{G}$-times, we see that $V \in \text{irrep } k_q[X, Y]$. \qed

### 4 Twisting and isomorphisms

#### 4.1 Twisting

In this subsection, following [7], we use the term twisting to refer to the twisting of the multiplication introduced in [16] which is dual to the twisting of the comultiplication in an appropriate sense. Precisely, let $H$ be a Hopf algebra and $\sigma : H \otimes H \to \mathbb{k}$ be an invertible 2-cocycle. Consider the Hopf algebra $H_\sigma$ which has the same coalgebra structure of $H$ and multiplication given by

$$x \cdot_\sigma y = \sigma(x_1, y_1)x_2 y_2 \sigma^{-1}(x_3, y_3), \quad x, y \in H; \quad (4.1)$$

$H_\sigma$ is obtained by twisting the multiplication of $H$.

We start by a definition implicit in [7, §2.4]. Let $R$ be a Hopf algebra in $\mathcal{YD}$, $A := R#H$, $\pi : A \to H$ and $\iota : H \to A$ be the canonical projection and injection.
Define $\sigma^\pi : A \otimes A \to k$ by $\sigma^\pi := \sigma(\pi \otimes \pi)$. Since the maps $\pi : A_{\sigma^\pi} \to H_\sigma$ and $\iota : H_\sigma \to A_{\sigma^\pi}$ are still Hopf algebra maps and the comultiplication is not changed, $A_{\sigma^\pi} \simeq R_\sigma \# H_\sigma$ where $R_\sigma$ is a Hopf algebra in $H_\sigma \mathcal{YD}$ that coincides with $R$ as vector subspace of $A$, with multiplication

$$x \cdot_\sigma y = \sigma(x_0, y_0)x_1y_1,$$

$x, y \in R_\sigma$. (4.2)

**Definition 4.1** Let $R$ and $S$ be braided Hopf algebras in the sense of [22]. We say that $R$ and $S$ are **twist-equivalent** if there exist a Hopf algebra $H$ and an invertible 2-cocycle $\sigma : H \otimes H \to k$ such that

- $R$ is realizable in $H_{\sigma} \mathcal{YD}$;
- $S$ is isomorphic to $R_{\sigma}$ as a braided Hopf algebra.

The notion of twist equivalence is useful for classification purposes.

**Lemma 4.2** [7, Lemma 2.13] Let $H$ and $\sigma$ be as above. If $R = \bigoplus_{n \in \mathbb{N}_0} R(n)$ is a graded Hopf algebra in $H_{\sigma} \mathcal{YD}$, then $R_{\sigma}$ is a graded Hopf algebra in $H_{\sigma} \mathcal{YD}$ with $R(n) = R_{\sigma}(n)$ as vector spaces, for all $n \geq 0$. Moreover, $R$ is a Nichols algebra if and only if $R_{\sigma}$ is a Nichols algebra.

We recall that two matrices $q = (q_{ij})_{i,j \in \mathbb{Z}_0}$ and $q' = (q'_{ij})_{i,j \in \mathbb{Z}_0}$ with entries in $k^\times$ are twist-equivalent if

$$q_{ii} = q'_{ii} \quad \text{and} \quad q_{ij}q_{ji} = q'_{ij}q'_{ji}, \quad \text{for all } i \neq j \in \mathbb{Z}_0.$$  

See [7, Definition 3.8]. Suppose that this is the case. Let $V$ and $V'$ be braided vector spaces of diagonal type with braiding matrices $q$ and $q'$, respectively. Then [7, Proposition 3.9] essentially says that the Nichols algebras $\mathcal{B}(V)$ and $\mathcal{B}(V')$ are twist-equivalent. Our first goal in this subsection is to extend this result to a class of braided vector spaces of dimension 3.

More precisely, let $(q_{ij})_{i,j \in \mathbb{Z}_2}$ be a matrix of nonzero scalars and $a \in k$. Let $V'$ be the braided vector space with basis $(x_i)_{i \in \mathbb{Z}_3}$ and braiding

$$c(x_i \otimes x_j))_{i,j \in \mathbb{Z}_3} = \begin{pmatrix} q_{11}x_1 \otimes x_1 & q_{11}(x_2 + x_1) \otimes x_1 & q_{12}x_3 \otimes x_1 \\ q_{11}x_1 \otimes x_2 & q_{11}(x_2 + x_1) \otimes x_2 & q_{12}x_3 \otimes x_2 \\ q_{21}x_1 \otimes x_3 & q_{21}(x_2 + ax_1) \otimes x_3 & q_{22}x_3 \otimes x_3 \end{pmatrix}. \quad (4.3)$$

We realize $V'$ in $k^2 \mathcal{YD}$ as follows. If $\alpha_1, \alpha_2$ is the canonical basis of $\mathbb{Z}^2$, then the action of $\mathbb{Z}^2$ on $V'$ and the $\mathbb{Z}^2$-grading are given by

$$\alpha_1 \rightarrow x_1 = q_{11}x_1, \quad \alpha_1 \rightarrow x_2 = q_{11}(x_2 + x_1), \quad \alpha_1 \rightarrow x_3 = q_{12}x_3;$$

$$\alpha_2 \rightarrow x_1 = q_{21}x_1, \quad \alpha_2 \rightarrow x_2 = q_{21}(x_2 + ax_1), \quad \alpha_2 \rightarrow x_3 = q_{22}x_3; \quad (4.4)$$

$$\deg x_1 = \alpha_1, \quad \deg x_2 = \alpha_1, \quad \deg x_3 = \alpha_2.$$
Then the Nichols algebra \( B \) is a Hopf algebra in \( Z^2 \mathbb{Z} \mathcal{YD} \) and we may consider the bosonization \( A = B(Y) \triangleright \mathbb{Z}^2 \), used in the proof below.

Let \( Y \) be the braided vector space with basis \( (x_i)_{i \in \mathbb{Z}} \) and braiding \((4.3)\) but with respect to \((q_{ij})_{i,j \in \mathbb{Z}}\) and the same \( a \in \mathbb{Z} \). Assume that \((q_{ij})\) and \((q_{ij}')\) are twist-equivalent, i.e. \( q_{11} = q_{11}' \), \( q_{22} = q_{22}' \) and \( q_{12}q_{21} = q_{12}'q_{21}' \).

**Lemma 4.3** The Nichols algebras \( B(Y) \) and \( B(Y') \) are twist-equivalent.

**Proof** We argue as in \([7, \text{Lemma 2.12}]\). Let \((p_{ij})_{i,j \in \mathbb{Z}} \in (k^\times)^{2 \times 2}\). Let \( \sigma : \mathbb{Z}^2 \times \mathbb{Z}^2 \to k^\times \) be the bilinear form, hence a 2-cocycle, given by \( \sigma(\alpha_i, \alpha_j) = p_{ij} \), that we extend to an invertible 2-cocycle \( \sigma : k\mathbb{Z}^2 \otimes k\mathbb{Z}^2 \to k \) with the same name. Let us twist the multiplication of \( A \) by the cocycle \( \sigma^\pi := \sigma(\pi \otimes \pi) \) where \( \pi : A \to k\mathbb{Z}^2 \) is the natural projection. Then \( A_{\sigma^\pi} = B(Y)_\sigma \# k\mathbb{Z}^2 \) where \( B(Y)_\sigma \in \mathbb{Z}^2 \mathcal{YD} \) has the same \( k\mathbb{Z} \) grading as \( B(Y) \). As object of \( A_{\sigma^\pi} \), the coaction on \( A_{\sigma^\pi} \) coincides with the coaction on \( B(Y) \), while the action of \( k\mathbb{Z}^2 \) on \( B(Y)_\sigma \) is determined by

\[
\alpha_1 \mapsto x_1 = q_{11} x_1, \quad \alpha_2 \mapsto x_1 = p_{21}^{-1} q_{21} x_1.
\]

\[
\alpha_1 \mapsto x_2 = q_{11}(x_2 + x_1), \quad \alpha_2 \mapsto x_2 = p_{21}^{-1} q_{21} (x_2 + ax_1), \quad (4.5)
\]

\[
\alpha_1 \mapsto x_3 = p_{12}^{-1} q_{12} x_3, \quad \alpha_2 \mapsto x_3 = q_{22} x_3.
\]

Indeed, observe that

\[
\Delta^2(x_i) = x_i \otimes 1 \otimes 1 + \alpha_1 \otimes x_i \otimes 1 + \alpha_1 \otimes \alpha_1 \otimes x_i, \quad i \in \mathbb{Z};
\]

\[
\Delta^2(x_3) = x_3 \otimes 1 \otimes 1 + \alpha_2 \otimes x_3 \otimes 1 + \alpha_2 \otimes \alpha_2 \otimes x_3.
\]

Let \( j \in \mathbb{Z} \) and \( g \in \mathbb{Z}^2 \). Since \( \pi(x_j) = 0 \), we have by \((4.1)\) that

\[
g \cdot x_j = \sigma(g, \alpha_j) g x_j, \quad x_j \cdot g = \sigma(\alpha_j, g) x_j g.
\]

Given \( i \in \mathbb{Z} \), we compute

\[
\alpha_i \cdot x_1 = \sigma(\alpha_i, \alpha_1) \alpha_i x_1 = p_{i1}^{-1} q_{i1} x_1 \alpha_i = p_{i1}^{-1} q_{i1} \cdot x_1 \cdot x_i \cdot \sigma^\pi \alpha_i.
\]

Hence, \( \alpha_i \cdot x_1 = p_{i1}^{-1} q_{i1} x_1 \). Similarly, \( \alpha_i \cdot x_3 = p_{i2}^{-1} q_{i2} x_3 \). For the action on \( x_2 \), we set \( a_1 = 1, a_2 = a \). Then

\[
\alpha_i \cdot x_2 = \sigma(\alpha_i, \alpha_1) \alpha_i x_2 = p_{i1}^{-1} q_{i1} (x_2 + a_1 x_1) \alpha_i = p_{i1}^{-1} q_{i1} (x_2 + a_1 x_1) \cdot x_2 \cdot \sigma^\pi \alpha_i
\]

and the verification of \((4.5)\) is complete. Therefore, the braiding \( c^\sigma \) of \( \mathcal{Y}_\sigma = B(Y)_\sigma \) is determined by \((c^\sigma(x_i \otimes x_j))_{i,j \in \mathbb{Z}} =\)
If we choose \( p_{11} = p_{21} = p_{22} = 1 \) and \( p_{12} = q'_{12}q_{12}^{-1} \), then clearly \( \mathcal{V}_\sigma \simeq \mathcal{V}' \) as braided vector spaces. Now \( \mathcal{B}(\mathcal{V}_\sigma) \simeq \mathcal{B}(\mathcal{V}_\sigma') \) by Lemma 4.2.

If \( q_{11} = q_{22} = 1 \), \( q = q_{12} = q_{21}^{-1} \) and \( \mathcal{G} = -2a \), then \( \mathcal{V} =: \Sigma_q(1, \mathcal{G}) \).

**Proposition 4.4** Let \( q, q' \in \mathbb{k}^\times \). Then \( \mathcal{B}(\Sigma_q(1, \mathcal{G})) \) and \( \mathcal{B}(\Sigma_{q'}(1, \mathcal{G})) \) are twist-equivalent.

We now determine the braided Hopf algebra structure of \( \mathcal{D}_q(\mathcal{G}) \).

**Proposition 4.5** As braided Hopf algebra, \( \mathcal{D}_q(\mathcal{G}) \) is twist-equivalent to the enveloping algebra of a graded nilpotent Lie algebra.

**Proof** We realize \( \mathcal{B}(\Sigma_q(1, \mathcal{G})) \) in \( \mathbb{Z}_2^2 \mathcal{V}\mathcal{D} \) by (4.4); since \( \mathbb{k}x_1 \) is a Yetter–Drinfeld submodule of \( \mathcal{B}(\Sigma_q(1, \mathcal{G})) \), \( \mathcal{D}_q(\mathcal{G}) \) is an object in \( \mathbb{Z}_2^2 \mathcal{V}\mathcal{D} \). Let \( x_2 \) and \( z_n \) be the images of \( x_2 \) and \( z_n \) in \( \mathcal{D}_q(\mathcal{G}) \), \( n \in \mathbb{I}_{0,\mathcal{G}} \). We claim the vector subspace \( n_q \) of \( \mathcal{D}_q(\mathcal{G}) \) spanned by \( x_2 \) and \( z_n \), \( n \in \mathbb{I}_{0,\mathcal{G}} \), is a subobject in \( \mathbb{Z}_2^2 \mathcal{V}\mathcal{D} \). Indeed, by (4.4) we have that

\[
\alpha_1 \mapsto x_2 = x_2, \quad \alpha_2 \mapsto x_2 = q^{-1}x_2, \quad \delta(x_2) = \alpha_1 \otimes x_2.
\]

On the other hand, we know by [3, Lemma 4.2.1] that

\[
\alpha_1 \mapsto z_i = qz_i, \quad \alpha_2 \mapsto z_i = q^{-i}z_i, \quad \delta(z_i) = \alpha_i^i \alpha_2 \otimes z_i, \quad i \in \mathbb{I}_{0,\mathcal{G}}.
\]

Hence, the analogous formulas for \( z_n \) hold in \( \mathcal{D}_q(\mathcal{G}) \) and \( n_q \) is of diagonal type with braiding given by

\[
c(x_2 \otimes x_2) = x_2 \otimes x_2, \quad c(x_2 \otimes z_i) = qz_i \otimes x_2,
\]

\[
c(z_i \otimes x_2) = q^{-1}x_2 \otimes z_i, \quad c(z_i \otimes z_j) = q^{ij}z_j \otimes z_i, \quad i, j \in \mathbb{I}_{0,\mathcal{G}}.
\]

Let \( v_1, v_2 \) be primitive elements of a braided Hopf algebra whose braiding satisfies \( c(v_i \otimes v_j) = q_{ij}v_j \otimes v_i \), where \( q_{ij} \in \mathbb{k}^\times \) and \( q_{12}q_{21} = 1 \). A well-known argument shows that \( v_1v_2 - q_{12}v_2v_1 \) is again primitive. Hence, \( \mathcal{Z}_n \in \mathcal{D}_q(\mathcal{G}) \) is primitive, \( n \in \mathbb{I}_{0,\mathcal{G}} \). When \( q = 1 \), the braiding of \( n := n_1 \) is the usual flip so that \( n \) is a nilpotent Lie algebra and \( \mathcal{D}_1(\mathcal{G}) \simeq \mathcal{U}(n) \).

Let \( (p_{ij})_{i,j \in \mathbb{Z}^2} = \begin{pmatrix} 1 & q^{-1} \\ 1 & 1 \end{pmatrix} \) and \( \sigma : \mathbb{k}\mathbb{Z}^2 \otimes \mathbb{k}\mathbb{Z}^2 \to \mathbb{k} \) be the invertible 2-cocycle determined by \( \sigma(\alpha_i, \alpha_j) = p_{ij} \) as in Lemma 4.3. Consider the bosonization \( \mathcal{K} = \mathcal{D}_q(\mathcal{G}) \# \mathbb{k}\mathbb{Z}^2 \); as explained above, \( \mathcal{K}_{\sigma,\pi} \simeq \mathcal{D}_q(\mathcal{G})_{\sigma} \# \mathbb{k}\mathbb{Z}^2 \). Arguing as in the verification of (4.5), we conclude that

\[
\alpha_1 \mapsto_\sigma x_2 = \alpha_2 \mapsto_\sigma x_2 = x_2, \quad \alpha_1 \mapsto_\sigma z_0 = \alpha_2 \mapsto_\sigma z = z_0.
\]

Thus, the action on \( \mathcal{D}_q(\mathcal{G})_{\sigma} \) is trivial. Now we appeal to (4.2):

\[
x_2 \cdot_\sigma z_n = q^{-1}x_2z_n, \quad z_n \cdot_\sigma x_2 = z_nx_2, \quad z_n \cdot_\sigma z_m = q^{-n}z_nz_m.
\]
We claim that \( \tilde{x}_2 \) and \( \tilde{z}_n = q^{-n} \tilde{z}_n \) in \( \mathcal{D}_q(\mathcal{G})_\sigma \) satisfy the defining relations of \( \mathcal{D}_1(\mathcal{G}) \). Indeed for \( n \in \mathbb{I}_{0,\mathcal{G}} \), we have

\[
\tilde{z}_n \cdot \tilde{z}_{n+1} = q^{-2n-1} \tilde{z}_n \cdot \tilde{z}_{n+1} = q^{-3n-2} \tilde{z}_n \tilde{z}_{n+1} = q^{-3n-1} \tilde{z}_n \tilde{z}_{n+1} = q^{-3n-1} \tilde{z}_n \tilde{z}_{n+1} \tag{2.3}
\]

\[
\tilde{x}_2 \cdot \tilde{z}_n = q^{-n} \tilde{x}_2 \cdot \tilde{z}_n = q^{-n} \tilde{x}_2 \tilde{z}_n = q^{-n-1} (q \tilde{x}_2 \tilde{z}_n + \tilde{z}_n) = q^{-n-1} (q \tilde{x}_2 \tilde{z}_n + \tilde{z}_n + 1) \tag{2.4}
\]

\[
\tilde{x}_2 \cdot \tilde{z}_\mathcal{G} = q^{-\mathcal{G}} \tilde{x}_2 \cdot \tilde{z}_\mathcal{G} = q^{-\mathcal{G}} \tilde{x}_2 \tilde{z}_\mathcal{G} = q^{-\mathcal{G}} \tilde{x}_2 \tilde{z}_\mathcal{G} = q^{-\mathcal{G}} \tilde{x}_2 \tilde{z}_\mathcal{G} = q^{-\mathcal{G}} \tilde{z}_\mathcal{G} \tilde{x}_2 = \tilde{z}_\mathcal{G} \cdot \tilde{x}_2. \tag{2.5}
\]

It follows now easily that \( \mathcal{D}_q(\mathcal{G})_\sigma \) is isomorphic to \( \mathcal{D}_1(\mathcal{G}) \) as braided Hopf algebras, i.e. \( \mathcal{D}_q(\mathcal{G}) \) and \( \mathcal{D}_1(\mathcal{G}) \) are twist-equivalent.

Summarizing, \( \mathcal{B}(\mathcal{L}_q(1, \mathcal{G})) \) and \( \mathcal{B}(\mathcal{L}_1(1, \mathcal{G})) \) are twist-equivalent and there is an extension of braided Hopf algebras

\[
0 \to \mathbb{k}[x_1] \to \mathcal{B}(\mathcal{L}_1(1, \mathcal{G})) \to U(\mathfrak{n}) \to 0
\]

where \( \mathfrak{n} \) is the Lie algebra with basis \( \{x, z_n : n \in \mathbb{I}_{0,\mathcal{G}}\} \) and bracket

\[
[x, z_n] = z_{n+1}, \quad n \in \mathbb{I}_{0,\mathcal{G}-1}, \quad [x, z_\mathcal{G}] = 0, \quad [z_n, z_m] = 0, \quad n, m \in \mathbb{I}_{0,\mathcal{G}}.
\]

### 4.2 Isomorphisms

Let \( q, q' \in \mathbb{k}^\times \) and \( \mathcal{G}, \mathcal{G}' \in \mathbb{N} \).

(a) Assume that \( \mathcal{B}(\mathcal{L}_q(1, \mathcal{G})) \simeq \mathcal{B}(\mathcal{L}_{q'}(1, \mathcal{G}')) \) as braided Hopf algebras. Then \( \mathcal{G} = \mathcal{G}' \) and \( q = q' \). Indeed the isomorphism should preserve the space of primitive elements and the braiding by [21].

(b) Assume that \( \mathcal{B}(\mathcal{L}_q(1, \mathcal{G})) \simeq \mathcal{B}(\mathcal{L}_{q'}(1, \mathcal{G}')) \) as algebras. Then \( \mathcal{G} = \mathcal{G}' \), since \( \mathcal{G} + 3 = \text{GKdim} \mathcal{B}(\mathcal{L}_q(1, \mathcal{G})) = \text{GKdim} \mathcal{B}(\mathcal{L}_{q'}(1, \mathcal{G}')) = \mathcal{G}' + 3 \). Furthermore, if \( 1 < \text{ord} q = N < \infty \), then \( \text{ord} q' = N < \infty \) since \( \mathcal{B}(\mathcal{L}_q(1, \mathcal{G})) \) has a simple module of dimension \( N \).

(c) However, we do not know whether \( \mathcal{B}(\mathcal{L}_q(1, \mathcal{G})) \simeq \mathcal{B}(\mathcal{L}_{q'}(1, \mathcal{G})) \) as algebras implies \( q = q' \). In particular, it is natural to guess that \( \mathcal{B}(\mathcal{L}_q(1, \mathcal{G})) \) is isomorphic to \( \mathcal{B}(\mathcal{L}_1(1, \mathcal{G})) \) as algebras only when \( q = 1 \). We show that this is indeed the case by an argument based on the determination of the finite-dimensional simple modules.

Let \( \mathcal{R} \) be a ring. For brevity, we say ideal for two-sided ideal. The set of isomorphism classes of simple \( \mathcal{R} \)-modules is denoted \( \text{Irrep} \mathcal{R} \). For each \( p \in \text{Irrep} \mathcal{R} \), we fix a representative \( N_p \). By definition, see [17], the closed sets of the Zariski topology on \( \text{Irrep} \mathcal{R} \) are the sets

\[
\mathcal{V}(I) = \{ p \in \text{Irrep} \mathcal{R} : I \cdot N_p = 0 \}, \quad I \text{ ideal of } \mathcal{R}.
\]
When $R$ is commutative, $\text{Irrep } R = \text{irrep } R$ with this topology is naturally equivalent to the maximal spectrum of $R$ with the classical Zariski topology. In general, $\text{irrep } R$ is a topological space with the induced topology.

Let $\varphi : R \to S$ be a ring homomorphism and let $\varphi' : \text{Irrep } S \to \text{Irrep } R$ denote the natural map given by induction along $\varphi$.

**Lemma 4.6** If $\varphi$ is surjective, then $\varphi'$ is a closed continuous map.

**Proof** It suffices to show that for any ideals $I$ of $R$ and $J$ of $S$ we have that

$$(\varphi')^{-1}(\mathcal{V}(I)) = \mathcal{V}(\varphi(I)), \quad \varphi'(\mathcal{V}(J)) = \mathcal{V}(\varphi^{-1}(J)).$$

Here $\varphi(I)$ is an ideal because $\varphi$ is surjective. Since $I \cdot \varphi'(N_p) = \varphi(I) \cdot N_p$, we have

$$(\varphi')^{-1}(\mathcal{V}(I)) = \{ p \in \text{Irrep } S : I \cdot \varphi'(N_p) = 0 \} = \mathcal{V}(\varphi(I)).$$

Given $p \in \text{Irrep } S$, we have $J \cdot N_p = \varphi^{-1}(J) \cdot \varphi(N_p)$ as $\varphi$ is surjective; thus, $\varphi'(\mathcal{V}(J)) \subseteq \mathcal{V}(\varphi^{-1}(J))$. Also if $q \in \mathcal{V}(\varphi^{-1}(J))$, then $\ker \varphi \cdot N_q = 0$, i.e. $N_q \in \text{Im } \varphi'$ and the other contention holds. $\square$

**Proposition 4.7** If $B(\mathbb{L}_q(1, \mathcal{I})) \simeq B(\mathbb{L}_1(1, \mathcal{I}))$ as algebras, then $q = 1$.

**Proof** If $q = 1$, then $\text{irrep } B(\mathbb{L}_1(1, \mathcal{I}))$ is homeomorphic to the plane with the Zariski topology, by Theorem 3.5 and Lemma 4.6. Let $q \neq 1$; we may assume that $q$ is not a root of 1. By Theorem 3.5 and Lemma 4.6, $\text{irrep } B(\mathbb{L}_q(1, \mathcal{I}))$ is homeomorphic to $\text{irrep } k_q[X, Y] = U_1 \cup U_2$ where $U_1$ and $U_2$ are homeomorphic to $k \times 0$ and $0 \times k$, respectively; just apply Lemma 4.6 to the projections $k_q[X, Y] \to k[X]$ and $k_q[X, Y] \to k[Y]$ and Proposition 3.1. Thus, $\text{irrep } B(\mathbb{L}_1(1, \mathcal{I}))$ is not homeomorphic to $\text{irrep } B(\mathbb{L}_q(1, \mathcal{I}))$. $\square$

5 Point modules over $B(\mathbb{L}_q(1, \mathcal{I}))$

5.1 Point modules

Let $A = \bigoplus_{n \in \mathbb{N}_0} A^n$, $A^0 \simeq k$, be a graded $k$-algebra with $\dim_k A^n$ finite, $n \in \mathbb{N}$, generated in degree 1. A point module over $A$ is a (left) graded module $V = \bigoplus_{n \in \mathbb{N}_0} V^n$ over $A$ such that $V$ is cyclic, generated in degree 0, and has Hilbert series $h_V(t) = 1/(1 - t)$, in other words $\dim_k V_n = 1, n \in \mathbb{N}_0$. Point modules, introduced in [11], allow the introduce projective geometry in graded ring theory. See the survey [19]. If $A$ is strongly noetherian, then the point modules for $A$ are parametrized by a projective scheme [12, Corollary E4.12], [19, Theorem 3.10]. Our goal in this section is to compute the projective scheme parametrizing the point modules over $B(\mathbb{L}_q(1, \mathcal{I}))$ which we have shown in Sect. 2 that is strongly noetherian. We do this by essentially elementary calculations.

We first recall the parametrization of point modules over a free associative algebra given in [19, Proposition 3.5]. As usual, $(a_0 : a_1 : \cdots : a_n)$ with $a_i \in k$ denotes a point of the projective space $\mathbb{P}^n = \mathbb{P}^n(k)$.
Theorem 5.1 Let $A = k\langle x_i : i \in \mathbb{N}_{0,n} \rangle$ be the free associative algebra. The isomorphism classes of point modules over $A$ are in bijective correspondence with $\mathbb{N}_0$-indexed sequences of points in $\mathbb{P}^n$, in other words, points in the infinite product $\prod_{i=0}^{\infty} \mathbb{P}^n$. The correspondence is given by:

$$V = \bigoplus_{i \in \mathbb{N}_0} \langle v_i \rangle \mapsto (P_0, P_1, \ldots) \in \prod_{i=0}^{\infty} \mathbb{P}^n, \quad P_i := (a_{0,i} : \cdots : a_{n,i}),$$

where $x_j v_i = a_{j,i} v_{i+1}$.

Given an homogeneous element $F$ of the polynomial ring $k[X_0, X_1, X_2]$, $\mathcal{V}(F)$ denotes the projective subvariety of $\mathbb{P}^2$ of zeros of $F$.

Theorem 5.2 The isomorphism classes of point modules over $\mathcal{B}(\mathcal{L}_q(1, \mathcal{G}))$ are parametrized by $\mathcal{V}(X_0X_2)$.

The parametrization is given by $V \mapsto P_0$ in the notation of Theorem 5.1. To prove Theorem 5.2, observe that $\mathcal{V}(X_0X_2) = B \cup C \cup \{ (0 : 0 : 1) \}$ where

$$B := \{(1 : b : 0) : b \in k\}, \quad C := \{(0 : 1 : c) : c \in k\}.$$ 

We deal with the point modules parametrized by $B$ and $C$ in Lemmas 5.4 and 5.5 while we show in Lemma 5.6 that the rest corresponds to $(0 : 0 : 1)$.

Recall from Lemma 2.5 and Sect. 3.1 the algebra surjections

$$\mathcal{B}(\mathcal{L}_q(1, \mathcal{G})) \xrightarrow{\pi_q} \mathcal{B}(\mathcal{L}_q(1, \mathcal{G} - 1)) \cdots \xrightarrow{\pi_2} \mathcal{B}(\mathcal{L}_q(1, 1)) \xrightarrow{\nu_1} k_q [X, Y].$$

The associated maps $\pi_q^t, \pi_{\mathcal{G} - 1}^t, \ldots$ between the varieties of point modules are all isomorphisms, while $\nu_q^t, \nu_{\mathcal{G} - 1}^t, \ldots$ identify the variety corresponding to the quantum plane, which is $\mathbb{P}^1$ by [19, Example 3.2], with $C \cup \{ (0 : 0 : 1) \}$.

5.2 Proof of Theorem 5.2

In the rest of the section, $V = \bigoplus_{i \in \mathbb{N}_0} V_i$ denotes a point module over $\mathcal{B}(\mathcal{L}_q(1, \mathcal{G}))$ with $V_i = \langle v_i \rangle, i \in \mathbb{N}_0$. Since $x_1, x_2, z_0$ have degree one and $V$ is cyclic, there exists $P_i = (a_i : b_i : c_i) \in \mathbb{P}^2$ such that

$$x_1 v_i = a_i v_{i+1}, \quad x_2 v_i = b_i v_{i+1}, \quad z_0 v_i = c_i v_{i+1}, \quad i \in \mathbb{N}_0. \quad (5.1)$$
By Theorem 5.1, \( V \) is completely determined by \( P := (P_0, P_1, \ldots) \in \prod_{i=0}^{\infty} \mathbb{P}^2 \). We start by the following identity in \( \mathcal{B}(\mathcal{L}_q(1, \mathcal{G})) \):

\[
\sum_{i \in \mathbb{L}_{0,\mathcal{G}+1}} \binom{\mathcal{G}+1}{i} (-q)^i x_2^{\mathcal{G}+1-i} z_0 x_2^i = 0. \tag{5.2}
\]

**Proof** One proves recursively on \( n \in \mathbb{G}+1 \) that \( z_n = \sum_{i \in \mathbb{L}_{0,n}} \binom{n}{i} (-q)^i x_2^{n-i} z_0 x_2^i \). The claim follows because of the defining relation (2.5). \( \square \)

**Remark 5.3** The following are equivalent: (i) \( a_0 = 0 \), (ii) \( a_i = 0 \) for some \( i \in \mathbb{N}_0 \), (iii) \( a_i = 0 \) for all \( i \in \mathbb{N}_0 \).

**Proof** The relations (2.1) and (2.2) imply that

\[
a_i b_{i+1} - a_{i+1} b_i + \frac{1}{2} a_i a_{i+1} = 0, \quad a_{i+1} c_i - q a_i c_{i+1} = 0, \quad i \in \mathbb{N}_0. \tag{5.3}
\]

Assume that \( a_i \neq 0 \). We claim that \( a_{i+1} \neq 0 \). Indeed, if \( a_{i+1} = 0 \), then (5.3) implies that \( b_{i+1} = c_{i+1} = 0 \), that is \( V \) is not cyclic, a contradiction. Similarly if \( a_{i+1} \neq 0 \) and \( a_i = 0 \), then \( b_i = c_i = 0 \), again a contradiction. Hence, \( a_i = 0 \) if and only if \( a_{i+1} = 0 \) and the remark follows. \( \square \)

**Lemma 5.4** If \( a_0 \neq 0 \), then \( P_i = (1 : b_0 - i/2 : 0) \) for all \( i \in \mathbb{N}_0 \).

**Proof** Given \( i \in \mathbb{N}_0 \), by Remark 5.3 \( a_i \neq 0 \); hence, we can assume that \( a_i = 1 \). By (5.3), \( b_{i+1} = b_i - \frac{1}{2} \) and \( c_{i+1} = q^{-1} c_i \). Therefore,

\[
P_i = (1 : b_0 - i/2 : q^{-i} c_0), \quad i \in \mathbb{N}_0.
\]

It remains to prove that \( c_0 = 0 \). Evaluating both sides of (5.2) on \( v_0 \) and reordering, we have that

\[
\sum_{i \in \mathbb{L}_{0,\mathcal{G}+1}} \binom{\mathcal{G}+1}{i} (-q)^i b_0 \cdots b_{i-1} b_{i+1} \cdots b_{\mathcal{G}+1} c_i = 0. \tag{5.4}
\]

Suppose first that \( b_j = 0 \) for some \( j \in \mathbb{L}_{0,\mathcal{G}+1} \), that is \( b_0 = j/2 \). Then \( b_i \neq 0 \) for all \( i \neq j \). By (5.4), \( b_0 \cdots b_{j-1} b_{j+1} \cdots b_{\mathcal{G}+1} c_j = 0 \); thus, \( c_j = 0 \) and \( c_0 = 0 \).

Hence, we may assume that \( b_i \neq 0 \), \( i \in \mathbb{L}_{0,\mathcal{G}+1} \). Set \( b := b_0 b_1 \cdots b_{\mathcal{G}+1} \neq 0 \) and \( \hat{b}_i := b/b_i \). By (5.4), we have that

\[
0 = \sum_{i \in \mathbb{L}_{0,\mathcal{G}+1}} \binom{\mathcal{G}+1}{i} (-q)^i \hat{b}_i c_i = \sum_{i \in \mathbb{L}_{0,\mathcal{G}+1}} (-1)^i \binom{\mathcal{G}+1}{i} \hat{b}_i c_0
\]

\[
= b c_0 \sum_{i \in \mathbb{L}_{0,\mathcal{G}+1}} (-1)^i \binom{\mathcal{G}+1}{i} \frac{1}{b_0 - i/2}
\]

\[
= 2 b c_0 \sum_{i \in \mathbb{L}_{0,\mathcal{G}+1}} (-1)^i \binom{\mathcal{G}+1}{i} \frac{1}{2b_0 - i}. \tag{5.5}
\]
It is easy to prove by induction on $n$ that
\[
\sum_{i \in \{0, n\}} (-1)^i \binom{n}{i} \frac{1}{t-i} = \frac{(-1)^n n!}{t(t-1) \ldots (t-n)}, \quad n \in \mathbb{N}, \ t \in \mathbb{R}\setminus\mathbb{N}_0. \tag{5.6}
\]

Applying (5.6) to $t = 2b_0$, we obtain from (5.5) that
\[
\frac{2b_0(-1)^{\varphi+1} (\varphi + 1)!}{2b_0(2b_0 - 1) \ldots (2b_0 - (\varphi + 1))} = 0.
\]

Hence, $c_0 = 0$, consequently $c_i = 0$ for $i \in \mathbb{N}_0$ and $P_i = (1 : b_0 - i/2 : 0)$. \qed

\textbf{Lemma 5.5} Assume that $a_0 = 0$ and $b_j \neq 0$ for all $j \in \mathbb{N}_0$. Then
\[
P_j = \left(0 : 1 : q^{-j} \frac{c_0}{b_0}\right), \quad j \in \mathbb{N}_0.
\]

\textbf{Proof} By Remark 5.3, $a_j = 0$; hence, $P_j = (0 : 1 : c_j/b_j)$, $j \in \mathbb{N}_0$. Set
\[
\lambda_j^{(0)} := \frac{c_j}{b_j}, \quad \lambda_j^{(n+1)} := \lambda_j^{(n)} - q \lambda_j^{(n+1)}, \quad \beta_{j,n} := b_j b_{j+1} \ldots b_{j+n}, \tag{5.7}
\]
for $j, n \in \mathbb{N}_0$. Applying repeatedly (2.4), we have
\[
z_1 v_j = (b_{j+1} c_j - q c_{j+1} b_j) v_{j+2} = \beta_{j,1} (\frac{c_j}{b_j} - q \frac{c_{j+1}}{b_{j+1}}) v_{j+2} = \beta_{j,1} \lambda_j^{(1)} v_{j+2},
\]
\[
z_n v_j = \beta_{j,n} \lambda_j^{(n)} v_{j+n+1}, \quad j \in \mathbb{N}_0, \ n \in \mathbb{N}. \tag{5.8}
\]

From (2.5) and (5.8), it follows that
\[
\beta_{j,\varphi+1} \left(\lambda_j^{(\varphi)} - q \lambda_j^{(\varphi+1)}\right) = 0.
\]

By (2.3), $z_{n+1} z_n - q z_n z_{n+1} = 0$. Thus, (5.8) implies also that
\[
\beta_{j,2n+1} \left(\lambda_j^{(n)} \lambda_{j+n+1}^{(n+1)} - q \lambda_j^{(n+1)} \lambda_{j+n+2}^{(n)}\right) = 0, \quad n \in \mathbb{N}_0, \varphi-1.
\]

Since all $\beta_{j,i}$’s are $\neq 0$, we are led to deal with the following systems of polynomial equations on the variables $L_j^{(0)}$, $j \in \mathbb{N}_0$. Define recursively
\[
L_j^{(n+1)} = L_j^{(n)} - q L_{j+1}^{(n)} \tag{5.9}
\]

We consider for each $M \in \mathbb{N}$ the infinite system
\[
\left\{
\begin{array}{ll}
L_j^{(M)} - q L_{j+1}^{(M)} = 0, \\
L_j^{(n)} L_{j+n+1}^{(n+1)} - q L_j^{(n+1)} L_{j+n+2}^{(n)} = 0,
\end{array}
\right. \quad j \in \mathbb{N}_0, \ n \in \mathbb{N}_0, M-1. \tag{\mathcal{S}_M}
\]
Claim The system \( (\mathcal{S}_M) \) has a unique solution \( (\ell_j^{(0)})_{j \in \mathbb{N}_0} \) for each \( x \in \mathbb{K} \), namely
\[
\ell_j^{(0)} = q^{-j} x, \quad j \in \mathbb{N}_0. \tag{5.10}
\]

It is easy to see that (5.10) is a solution of \( (\mathcal{S}_M) \). For the converse, we proceed by induction on \( M \). Let \( (\ell_j^{(0)})_{j \in \mathbb{N}_0} \) be a solution of \( (\mathcal{S}_1) \). Then

\[
\begin{cases}
\ell_j^{(0)} - 2q \ell_{j+1}^{(0)} + q^2 \ell_{j+2}^{(0)} = 0, \\
\ell_j^{(0)} - 2q \ell_{j+1}^{(0)} \ell_{j+2}^{(0)} + q^2 \ell_{j+1}^{(0)} \ell_{j+2}^{(0)} = 0,
\end{cases} \quad j \in \mathbb{N}_0, \tag{5.11}
\]

by (5.9). The second equation of (5.11) minus the first multiplied by \( \ell_{j+1}^{(0)} \) gives \( (\ell_{j+1}^{(0)})^2 - \ell_j^{(0)} \ell_{j+2}^{(0)} = 0 \); replacing \( \ell_{j+1}^{(0)} \) by \( \frac{1}{q^2} \left( \ell_j^{(0)} - 2q \ell_{j+1}^{(0)} \right) \), we get

\[
(\ell_{j+1}^{(0)})^2 + q^{-2}(\ell_j^{(0)})^2 - 2q^{-1}\ell_j^{(0)} \ell_{j+1}^{(0)} = \left( \ell_{j+1}^{(0)} - q^{-1}\ell_j^{(0)} \right)^2 = 0.
\]

That is, \( \ell_{j+1}^{(0)} = q^{-1} \ell_j^{(0)} \) for all \( j \in \mathbb{N}_0 \); this implies (5.10).

Assume now that the claim holds for \( M > 0 \). Let \( (\ell_j^{(0)})_{j \in \mathbb{N}_0} \) be a solution of \( (\mathcal{S}_{M+1}) \). By (5.9), the first equation gives

\[
\ell_j^{(M)} = 2q^{-1}\ell_{j+1}^{(M)} - q^{-2}\ell_j^{(M)}, \quad j \in \mathbb{N}_0.
\]

Then it is easy to prove recursively that

\[
\ell_{j+h}^{(M)} = hq^{1-h}\ell_{j+1}^{(M)} - (h-1)q^{-h}\ell_j^{(M)}, \quad h \geq 2. \tag{5.12}
\]

When \( n = M \), the second equation of \( (\mathcal{S}_{M+1}) \) together with (5.9) says that

\[
\ell_j^{(M)} \ell_{j+1}^{(M)} - 2q \ell_j^{(M)} \ell_{j+2}^{(M)} + q^2 \ell_{j+1}^{(M)} \ell_{j+2}^{(M)} = 0, \quad j \in \mathbb{N}_0.
\]

Plugging (5.12) into the previous equality, we see that

\[
(M + 2) \left( q^{-M-1} \left( \ell_j^{(M)} \right)^2 - 2q^{-M} \ell_j^{(M)} \ell_{j+1}^{(M)} + q^{-M+1} \left( \ell_{j+1}^{(M)} \right)^2 \right) = 0.
\]

That is, \( (\ell_j^{(M)} - q \ell_{j+1}^{(M)})^2 = 0 \). Hence, we have that \( \ell_j^{(0)} \), \( j \in \mathbb{N}_0 \), is a solution of \( (\mathcal{S}_M) \). By the inductive hypothesis, \( \ell_j^{(0)} = q^{-j} \ell_0^{(0)} \) and the claim follows.

Since \( \frac{c_j}{b_j} \) is a solution of \( (\mathcal{S}_q) \) by the above discussion, the claim implies that \( \frac{c_j}{b_j} = q^{-j} \frac{c_0}{b_0} \), \( j \in \mathbb{N}_0 \). The lemma follows.

We next proceed with the remaining possibility.
Lemma 5.6 Assume that \( a_0 = 0 \) and \( b_i = 0 \) for some \( i \in \mathbb{N}_0 \). Then

\[
P_j = (0 : 0 : 1), \quad j \in \mathbb{N}_0.
\]

Proof We set \( z_n v_i = \zeta_i(n) v_{i+n+1}, i \in \mathbb{N}_0, n \in \mathbb{Z}_{0,\geq} \). Recall that \( a_i = 0 \) for all \( i \in \mathbb{N}_0 \) by Remark 5.3; thus, \( b_i \) and \( \zeta_i(0) = c_i \) could not be both 0, as \( V \) is cyclic. The proof of (5.13) is easy and follows a well-known pattern:

\[
z_n^{(2.4)} = x_2 z_{n-1} - q z_{n-1} x_2 = \sum_{k \in \mathbb{I}_{0,n}} \binom{n}{k} (-q)^{k} x_2^{n-k} z_0 x_2^k, \quad n \in \mathbb{Z}_{0,\geq}. \tag{5.13}
\]

Evaluating these identities at \( v_i, i \in \mathbb{N}_0 \), we get for \( n \in \mathbb{Z}_{0,\geq} \):

\[
\zeta_i(n) = \zeta_i(n-1) b_{i+n} - q b_i \zeta_i(n-1) = \sum_{k \in \mathbb{I}_{0,n}} \binom{n}{k} (-q)^{k} \zeta_i^{(0)} b_{i+n}^{(k)}
\]

where \( b_{i,n}^{(k)} = b_i b_{i+1} \cdots b_{i+k-1} b_{i+k+1} \cdots b_{i+n} = \prod_{h \in \mathbb{I}_{0,n}, h \neq k} b_{i+h} \). \tag{5.14}

Evaluating (2.3), respectively (2.5), at \( v_i \) and plugging in appropriate instances of (5.14), we get for \( i \in \mathbb{N}_0 \) and \( n \in \mathbb{Z}_{0,\geq-1} \):

\[
\zeta_i(n) \zeta_{i+n+1} b_{i+2n+2} - 2 q \zeta_i(n) \zeta_{i+n+2} b_{i+n+1} + q^2 \zeta_i(n) \zeta_{i+n+2} b_i = 0, \tag{5.15}
\]

\[
\zeta_i^{(\geq)} b_{i+\geq+1} - q b_i \zeta_i^{(\geq)} = 0. \tag{5.16}
\]

We fix for the remaining of the proof \( i \in \mathbb{N}_0 \) such that \( b_i = 0 \). \( \square \)

Step 1 We have \( b_{i+1} = 0 \) if and only if \( b_{i+2} = 0 \). If, in addition, \( b_{i+1} = 0 \), then \( b_j = 0, j \in \mathbb{N}_0 \).

Since \( c_i = \zeta_i(0) \neq 0 \), it follows from (5.15) that \( \zeta_i(0) b_{i+2} - 2 q \zeta_i(0) b_{i+1} = 0 \). Thus, \( b_{i+1} = 0 \) if and only if \( b_{i+2} = 0 \). Consequently, if \( b_{i+1} = 0 \), then \( b_{i+\ell} = 0, \ell \geq 2 \).

Assume that there exists \( i \in \mathbb{N}_0 \) such that \( b_i = b_{i+1} = 0 \). Let \( t \in \mathbb{N}_0 \) be the smallest one such that \( b_t = b_{i+1} = 0 \). If \( t > 0 \), then (5.15) implies that \( \zeta_t(0) \zeta_t(0) b_{t+1} - 2 q \zeta_t(0) \zeta_t(0) b_{t+1} = 0 \). Since \( b_t = b_{i+1} = 0 \), we get that \( b_{i+1} = 0 \), \( b_{t+1} = 0 \), \( b_{t+\ell} = 0, \ell \geq 2 \), a contradiction because \( \zeta_t(0) \neq 0 \neq \zeta_t(0) \). Hence, \( t = 0 \) and the second part of the claim follows from the first.

Step 2 Either \( b_j = 0 \) for all \( j \in \mathbb{N}_0 \) or else \( b_{i+m} \neq 0 \) for all \( m \in \mathbb{N} \).

Assume that the first possibility does not hold. We shall prove by induction that \( b_{i+2n+1} \neq 0 \) and \( b_{i+2n+2} \neq 0 \) for all \( n \in \mathbb{N}_0 \). When \( n = 0, b_{i+1} \neq 0 \) and \( b_{i+2} \neq 0 \) by Step 1. Let \( n \in \mathbb{N} \) and suppose that \( b_{i+1} \ldots b_{i+2n} \neq 0 \). We claim that \( b_{j+2n+1} \neq 0 \)
and $b_{j+2n+2} \neq 0$. Since $b_i = 0$, $\zeta_i^{(0)} \neq 0$, we have

$$\zeta_i^{(n)} (5.14) = \sum_{k \in I_{0,n}} (-q)^k \zeta_i^{(0)} b_{i,n}^{(k)} = \zeta_i^{(0)} b_{i,n} = \zeta_i^{(0)} b_{i+1} \ldots b_{i+n} \neq 0.$$  

Then (5.15) implies that

$$\zeta_i^{(n)} b_{i+2n+2} - 2q \zeta_i^{(n)} b_{i+n+1} = 0.$$  

(5.17)

If $b_{i+2n+2} = 0$, then $\zeta_i^{(0)} \neq 0$ and

$$0 (5.17) \zeta_i^{(n)} b_{i+2n+2} = \sum_{k \in I_{0,n}} (-q)^k \zeta_i^{(0)} b_{i,n}^{(k)} = (-q)^n \zeta_i^{(0)} b_{i+2n+2} b_{i+n+3} \ldots b_{i+2n+1} \implies b_{i+2n+1} = 0.$$

By Step 1, $b_j = 0$ for all $j \in \mathbb{N}_0$, contradicting the assumption. Similarly assume that $b_{i+2n+1} = 0$. Then $\zeta_i^{(0)} \neq 0$ and

$$\zeta_i^{(n)} = (-q)^n \zeta_i^{(0)} b_{i+2n+1} b_{i+n+1} \ldots b_{i+2n}$$

$$\zeta_i^{(n)} = n (-q)^{n-1} \zeta_i^{(0)} b_{i+2n+1} b_{i+n+2} \ldots b_{i+2n+2} b_{i+2n+2},$$

$$\implies 0 (5.17) (1 - 2n) (-q)^n \zeta_i^{(0)} b_{i+n+1} \ldots b_{i+2n+1} \implies 0 = b_{i+2n+2}.$$

Again $b_j = 0$ for all $j \in \mathbb{N}_0$ by Step 1, a contradiction. The step is proved. To finish the proof of the lemma, we just observe that

$$0 (5.16) \zeta_i^{(\mathcal{G})} b_{i+\mathcal{G}+1} = \sum_{k \in I_{0,\mathcal{G}}} (-q)^k \zeta_i^{(0)} b_{i,\mathcal{G}} b_{i+\mathcal{G}+1}$$

$$= \zeta_i^{(0)} b_{i+1} b_{i+2} b_{i+3} \ldots b_{i,\mathcal{G}} b_{i+\mathcal{G}+1}$$

Hence, $b_{i+3} \ldots b_{i+\mathcal{G}+1} = 0$. Step 2 implies that $b_j = 0$ for all $j \in \mathbb{N}_0$ and the lemma follows.

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