A Method of Generating Knotted and Linked Gaussian Dots from Bessel Like Beams

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Abstract: We experimentally demonstrate that the second harmonic intensity profile generated by Bessel like beams, is composed of Gaussian spots of various geometries surrounded by concentric rings; one of which is two central spots of similar radius knotted by ellipsoidal concentric rings. We show that the spatial profile is invariant against propagation. We observe that their behavior is similar to that of screw dislocation in wave trains: they collide and rebound as they propagate. In this way, we have generated linked frequency doubled Bessel-type vector beam with a spatial polarization, that are knotted as they oscillate along the optical axis, when propagating in the laboratory environment.

Nye & Berry, in their seminal 1974 paper, observed that wavefronts can contain dislocation lines closely resembling those found in crystals [1]. The wavefront dislocations can be created as a loop or in pairs. The morphology of the dislocations was analyzed; they showed that the dislocations can be curved, they may intersect, collide, and rebound. Experimental studies in optics of waves with screw dislocations were later reported [2, 3]; where a double screw dislocation or phase singularity was created in the process of diffraction of a plane wave or a gaussian beam by computer-synthesized gratings.

The mathematical expression for a wave with phase singularities are exact solutions of the Helmholtz equation. These wavefront dislocations are ‘cylindrical tubes’ in which, the wave intensity vanishes and around which the phase changes by multiples of $2\pi$. It is proposed that a superposition of Bessel beams are ideal for the production of such singularities [4].

The ideal Bessel beams are also solutions to the Helmholtz equation. They are immune to diffractive spreading and can heal themselves after being disrupted by an obstacle. Bessel beams as proposed by Durnin, have a well defined bright central spot radius surrounded by concentric rings [5, 6] and the beam’s intensity distribution is invariant along its propagation direction. Such idealized beam are rigorously exact solutions, in infinite free space, of the scalar Helmholtz monochromatic wave equation. Any realizations of such beams in the laboratory will require an infinite amount of energy. However, over a limited spatial range, approximations to the ideal Bessel beam can be obtained. One such light field is termed Bessel–Gauss (BG) beams [7].

BG beams have an additional Gaussian factor that decreases the light amplitude away from the origin. They can be realized experimentally using conical lenses such as axicons [8], holographic masks [9], or by the use of spatial light modulators (SLM) [10]. BG beams still exhibit the self-healing property [11] in addition to the relatively diffraction-free propagation of ideal Bessel beams. These qualities have led to their adoption in a wide-range of fields which include, but are not limited to, optical communication [12, 13], and various other applications as listed in [14].

Nonlinear wave mixing is a method of interest for the superposition of BG beams. It was proposed that the structure of the screw dislocations of different order Bessel like beam can be obtained by nonlinear processes of Second Harmonic Generation (SHG) [2–4].

SHG with laboratory-generated Bessel beams, is an area of great interest [15, 16]. These experimental studies initially took place because it was thought that the non-diffracting region of high
intensity would enable high conversion efficiencies [17]. At the same time, it was demonstrated that SHG excited by a Bessel like distribution is gradually changed into a high-brightness axial frequency doubled beam by adjustment of the phase-matching conditions in the crystal [16].

In this letter, we investigate second-harmonic generation for Bessel-Gauss beams that are constructed by means of virtual axicons encoded as a hologram on an SLM. We closely follow the methods detailed in Rosales–Guzmán and Forbes [10]. We demonstrate experimentally that by increasing the refractive index of the virtual axicon, that produces the BG beam input, we can generate a frequency doubled light with spatial properties that closely resemble that of screw dislocations [18]. However, our experimental observations show that at the point where phase singularities are observed, the intensity is maximum contrary to that discussed by Berry and Nye where at the phase singularities the light intensities completely vanishes. We observe for the first time as far as we know, that the second harmonic (SH) light can have a bright ellipsoidal central spot that can be split into two distinct central spots of similar radius. We’ve dubbed them Gaussian–dots (GD). Both central spot geometries are surrounded by concentric rings with decaying brightness away from the central spot; similar to that of a BG beam. These vortex beams retain the self–healing properties of the BG beam by rotating and adopting a spring-like oscillation as they propagate through space. The ellipsoidal structure of the generated beams determines the global topology of the field with which the loop could be threaded. The superposition of the BG beams allows for the SH light to form links that are threaded and knotted, enabling the Gaussian–dots to intersect or collide, rotate, and rebound.

The ideal Bessel beams have electric field amplitude described mathematically by:

$$E(r, \varphi, z, t) = \mathcal{E} e^{i k z} J_\ell(k_r r) e^{i \ell \varphi}$$

where $\mathcal{E}$ is the amplitude, $J_\ell(k_r r)$ are the Bessel functions of order $\ell$, while $k_r$ and $k_z$ are the radial and longitudinal components of the wave vector $\mathbf{k}$, respectively, obeying the relation $|k|^2 = k_r^2 + k_z^2$.

This beam has an infinite extent and its intensity profile is a bright central spot surrounded by concentric rings. Each ring carries approximately the same power as the central spot. The beam’s intensity profile is also propagation invariant. This combination makes it impossible to realize experimentally. There are, however, a number of ways to generate good approximations. Here, we use the Bessel-Gauss approximation where an additional Gaussian factor is introduced that decreases the amplitude away from the origin, as follow [10]:

$$E(r, \varphi, z = 0) = \mathcal{E} e^{i k_z z} J_\ell(k_r r) e^{i \ell \varphi} \exp \left[ -\left( \frac{r}{w_o} \right)^2 \right]$$

where $w_o$ is a measure of the input Gaussian beam waist and is related to the BG finite propagation distance $z_{\text{max}}$ as: $\frac{z_{\text{max}}}{2} = \frac{k_r}{k_z} w_o$.

We generate our BG beams with an SLM by encoding an azimuthal variation (of modulus $2\pi$) and a blazed grating to separate the first order from the others [10]. The mathematical expression will take the form:

$$\Phi_{\text{SLM}} = \text{mod} \left[ k \alpha (n - 1) r + \ell \phi + 2 \pi (G_x x + G_y y), 2 \pi \right]$$

where ‘mod’ is the modulus function, and $G_x$ and $G_y$ are the grating frequencies along the $x$ and $y$ directions, $n$ is the refractive index of the axicon, $\alpha$ is the angle of the axicon: the apex angle of the conical prism, $\ell$ is the so called topological charge or azimuthal order of the BG beam. An illustration of the experimental setup is shown in Fig. 1. The laser source is a mode locked regenerative Ti:sapphire laser with repetition rate of 3kHz, pulse energy of 1.67 mJ. The laser system can produce a 38 fs pulse duration at 808 nm center wavelength. The linearly
polarized output of the laser is expanded to cover the face of the light modulator. The wavefront manipulator is a phase-only Santec reflective liquid crystal on silicon (LCOS) based Spatial Light Modulator (SLM-200). The nonlinear medium we use for frequency doubling, is a 500 μm thick type-I $\beta$–Barium Borate (BBO) crystal, with optical axis $\theta = 28^\circ$. The crystal is affixed to a kinematic mount which allows translation along the optical axis, and to be tilted and rotated for optimal phase matching. We generate the desired phase mask or holographic grating on a computer to display on the SLM. All the diffracted orders generated from the mask are collected by a lens $L_1$ and focused through the BBO by $L_2$. We’ve designed a homemade aperture (FB) to block the remaining gaussian beam. The sCMOS camera is used to characterize the shape and intensity of the incident beam on the BBO. The sCMOS and BBO crystal are equidistant from the beamsplitter (BS) to ensure that the desired beam mode is incident on the crystal. A second HWP is placed before the BBO to further optimize the phase matching conditions. The generated second harmonic signal is imaged using an EMCCD placed in the far-field. We use two interference filters centered at 405 ± 5 nm to ensure that only the frequency doubled signals are imaged.

In our experiment, we make full use of the flexibility that Eq. (3) allows. We are able to control the generated hologram by:

(i) keeping the refractive index of the axicon constant at $n = 1.5$ and varying the topological
charge $\ell$. The top row of Fig. 2 shows three generated holograms and their corresponding intensity profile of the BG beam; the zeroth order $\ell = 0$ and higher orders $\ell = 3, 10$.

(ii) We can also control the encoded holograms by keeping $\ell$ constant and set to zero such that only the zeroth order BG beam is excited. We vary the refractive index, which changes the depth-of-field (DOF) of the virtual axicon. The DOF is a function of the radius ($w_0$) of the beam ‘entering’ the axicon (reflected off the SLM), the axicon’s index of refraction ($n$), and its opening angle ($\alpha$); $w_0$ and $\alpha$ are kept constant throughout.

In the first row of Fig. 3 we show the intensity distribution of the BG beam that is incident on the BBO crystal. One can observe that as the refractive index of the axicon is changed, the DOF of the virtual lens also varies. These changes in turn adjust the distance at which the BG beam forms prior to interacting with the nonlinear crystal. Fig. 3–(a) shows the zeroth order BG beam for $n = 1.5$ as the fundamental input to the frequency doubled beam in Fig. 3–(d). The resulting beam in Fig. 3–(d) shows that a considerable part of the SH signal is generated in the axial direction, along the axis of the BG cone as previously demonstrated for a non-collinear phase matching, where wave vectors with the opposite radial components generate second-harmonic output along the beam axis and give a single on-axis output in the far field [15, 16]. In addition to our experimental verification, we also observe that the single on axis beam is contained or bounded by circular concentric rings.

Fig. 3. The top row shows intensity profile of the BG beam (input beam) incident on the BBO, imaged using the sCMOS. The bottom row displays the intensity distribution of the SH light generated; imaged using the EMCCD: for $n = 1.5$ the input beam is shown in (a) and its output is displayed in (d). For $n = 2.0$, its input beam is (b) and its output is (e). For $n = 2.5$ its input beam is (c) and its corresponding output is (f).

We then increase the refractive index to $n = 2.0$, Fig. 3–(b) shows that the fundamental beam still maintains its zeroth order BG beam but the interference pattern at the DOF of the virtual axicon is quite noticeable. The resulting SH of such an input beam is shown in Fig. 3–(e). The frequency doubled beam still displays a considerable part of its intensity in the axial direction, along the axis of the BG cone. This single on axis beam has an ellipsoidal shape contained or bounded by ellipsoidal rings or vortices.

When the refractive index of the axicon is set to $n = 2.5$, the incident light beam in Fig. 3–(c) has intensity distribution similar to that found within the DOF of the axicon. In this way, we allow the zeroth order beam generated by this virtual axicon to form within the bulk of the nonlinear crystal. Fig. 3–(f) shows the resulting SH light generated. It is composed of two central spots (Gaussian–dots) of similar radius that are linked and knotted by ellipsoidal concentric rings.

The GD in our experimental finding have properties similar to the “knotted nothings” observed in [2, 3] and discussed in Berry et al. [4]. Knotted nothings are wavefront dislocations or phase singularities where the wave function vanishes. In our findings in the position of the singularities we observe bright center spots or GD, where the light intensity is maximum. These results can also be interpreted as a superposition of two zeroth order BG beams, that
through the nonlinear mixing process, have their bright central spots linked and knotted together by two indistinguishable vortex lines.

![Fig. 4. Gaussian-dot's geometrical rotation/evolution during free-space propagation along $z$: the top row shows the ellipsoidal GD intensity distribution as the refractive index is varied from $n = 2.0$ to $3.0$. The middle row are images of the various bright central point of the beams collide and rebound before flipping its polarization shown in the bottom row and recorded here.](image)

We then smoothly vary the refractive index of the virtual axicon and observe their behaviors as they propagate. In the first row of Fig. 4 we present the evolution of the GD separation as we increase $n$. We can observe that as the refractive index increases from $n = 2$ to $n = 3$, the GD separation widens, which in turn modifies the shape of the vortex lines from ellipsoidal to shapes similar to a parallelogram. In Fig. 4, we also capture the GD as they collide, rebound, and flip from horizontal to vertical as they propagate along $z$.

In conclusion, we have observed for the first time that the second harmonic signal from zeroth order Bessel-Gauss beams produced by virtual axicons with varying refractive indices, have intensity distributions composed of Gaussian spots of various geometries surrounded by concentric bright rings. When the axicon’s refractive index is set to $n = 2$ the central spot is cylindrical, which splits into two central spots of similar radius as we increase the refractive index of the axicon. The central spots are knotted by ellipsoidal concentric rings but as $n$ increases to 3 or greater, the vortex line adopt shapes similar to parallelograms. We further show that these GD collide, rebound, and flip as they propagate while maintaining their non-diffractive properties of the BG beams used to generate them. Further studies of our results, specifically the two-dot configuration, could be used to describe stationary configurations of electrons in free space where Schrödinger’s equation reduces to the Helmholtz equation. One can also note that as the GD collide and flip direction, the two become indistinguishable. For this reason we believe that our experimental findings could spur new methods of generating entangled photons.

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