On the continuum origin of Heisenberg’s indeterminacy relations

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If space is indistinguishable from the extension of a physical body, as is Descartes’s conception, then transformations of space become transformations of physical bodies. Every point of space then has properties of physical bodies in some suitable non-singular sense of average over the space. Every point of space is then thinkable as a non-singular point particle possessing such (averaged) physical properties. Then, the location of such a point particle is, relative to another (similar) point particle, indeterminate over the extension of the physical body. Further, transformations of the space may “move” such a point particle in relation to another such point particle. These notions then provide a non-probabilistic explanation of Heisenberg’s indeterminacy relations.

I. INTRODUCTION

Newtonian mechanics ascribes independent and real existence to space, time and matter. In Newton’s theory, space and time also play a dual role. Firstly, they play the role of a “background” for things happening physically to matter. Secondly, they also provide us the inertial systems which happen to be advantageous to describe Newton’s law of inertia. Then, if matter were to be removed completely, the space and time of the newtonian framework would “remain” behind.

Descartes opposed the concept of space as being independent of physical objects. Essentially, he stated: the space is identical with extension, but extension is connected with physical bodies; thus there should be no space without physical bodies and hence no empty space. There certainly are (philosophical) weaknesses of such an argument. However, let “the space be indistinguishable from physical bodies.”

The question is then of some suitable theoretical (mathematical) formulation which describes space (and physical bodies) as per Descartes’s conception. Such a description can be expected to possess the following characteristics.

In this description, Cartan’s volume-form should be well-defined at every spatial location. A point of space could also be prescribed, in some suitable non-singular sense, the inertia, electric charge etc. Then, we could also look at such a point as a point particle in the newtonian sense. We may also look at a physical body in the newtonian, non-singular, sense of a point particle possessing various properties of a physical body. Thus, physical bodies should everywhere in space be describable as singularity-free.

In this description, there cannot therefore be any spatial location without a physical property of a material object. Hence, any local motion of a physical body will, clearly, be a change in the (suitable) structure of the space.

The issue is of incorporating time in this framework. Now, the temporal evolution of “points of space” is a mathematically well-definable concept - as a dynamical system.

Then, Descartes’s conceptions could be realizable in some mathematically well-defined formalism that deals with dynamical systems defined on continuum as the underlying set.

Such a description then also follows the principle of general covariance: the laws regarding physical objects in it are based on the arbitrary coordinate transformations of the underlying space and also on time as an essentially arbitrary parameter of the dynamical system.

The question now is of suitable mathematical structure on the space that allows us the association of physical properties of material objects to the points of the space. Furthermore, the question is also of defining in a natural manner the boundary of any physical object.

A physical object has associated with it various (fundamental) physical properties, eg, (rest) energy. Then, the adjacent objects clearly separate by boundaries at which the spatial derivative(s) of that property under consideration, (eg, rest energy), change(s) the sign.

*This article is dedicated to Prof. Malcolm MacCallum on his 60th birthday. His suave and helpful nature, his easily accessible personality and his important scientific contributions to general relativity have indeed been inspiring for all the researchers of general relativity.
But, a physical object is a region of space as per Descartes’s conception. Therefore, some suitable structure definable on the space must, as per Descartes’s conception, then possess a similar property of its derivative changing its sign at a boundary of a physical object. It then also follows that a physical property can, essentially, be specified independently for each spatial direction since these directions are to be treated as independent of each other.

Now, the space (continuum) is characterizable by “distance” separating its points. Suitable “distance” function can then be expected to possess the property of its derivative(s) changing sign across boundaries separating regions of space corresponding to separated objects. This suitable distance function then, mathematically, becomes a pseudo-metric function on the space, remaining a metric function within a region.

Consider therefore a three-dimensional pseudo-Riemannian manifold, denoted as $\mathbb{B}$, admitting a pseudo-metric $d^2$:

$$d^2 = P^2Q^2R^2 \, dx^2 + P^2Q^2R^2 \, dy^2 + P^2Q^2R^2 \, dz^2$$

where we have $P \equiv P(x)$, $Q \equiv Q(y)$, $R \equiv R(z)$ and $P' = dP/dx$, $Q' = dQ/dy$, $R' = dR/dz$. The vanishing of any of these spatial functions is a curvature singularity, and constancy (over a range) is a degeneracy of $d^2$.

A choice of functions, say, $P_o$, $Q_o$, $R_o$, is a specific distribution of “physical properties” in the space of $d^2$. As some “region” of physical properties “moves” in the space, we have the original set of functions changing to the “new” set of corresponding functions, say, $P_1$, $Q_1$, $R_1$.

Clearly, we are considering the isometries of $d^2$ while considering “motion” of this kind. Then, we will remain within the group of the isometries of $d^2$ by restricting to the triplets of nowhere-vanishing functions $P$, $Q$, $R$. We also do not consider any degenerate situations for $d^2$.

If we denote by $\ell$ the pseudo-metric function corresponding to $d^2$, then $(\mathbb{B}, \ell)$ is an uncountable, separable, complete pseudo-metric space. If we denote by $d$, a metric function canonically obtainable from the pseudo-metric $d^2$, then the space $(\mathbb{B}, d)$ is an uncountable, separable, complete metric space. If $\Gamma$ denotes the metric topology induced by $d$ on $\mathbb{B}$, then $(\mathbb{B}, \Gamma)$ is a Polish topological space. Further, we also obtain a Standard Borel Space $(\mathbb{B}, \mathcal{B})$ where $\mathcal{B}$ denotes the Borel $\sigma$-algebra of the subsets of $\mathbb{B}$, the smallest one containing all the open subsets of $(\mathbb{B}, \Gamma)$.

A measurable, one-one map of $\mathbb{B}$ onto itself is a Borel automorphism. Now, the Borel automorphisms of $(\mathbb{B}, \mathcal{B})$, forming a group, are natural for us to consider here.

But, the pseudo-metric $d^2$ is a metric function on certain “open” sets, to be called the P-sets, of its Polish topology $\Gamma$. A P-set of $(\mathbb{B}, d)$ is therefore never a singleton subset, $\{ \{x\} : x \in \mathbb{B}\}$, of the space $\mathbb{B}$. Note also that every open set of $(\mathbb{B}, \Gamma)$ is not a P-set of $(\mathbb{B}, d)$.

Now, the differential of the volume-measure on $\mathbb{B}$, defined by $d\mu = P^2Q^2R^2 \left( \frac{dP}{dx} \frac{dQ}{dy} \frac{dR}{dz} \right) \, dx \, dy \, dz$ (2)

This differential of the volume-measure vanishes when any of the derivatives, of $P$, $Q$, $R$ with respect to their arguments, vanishes. (Functions $P$, $Q$, $R$ are non-vanishing over $\mathbb{B}$.)

A P-set of the space $\mathbb{B}$ is then also thinkable as the interior of a region of $\mathbb{B}$ for which the differential of the volume-measure, $d\mu$, is vanishing on its boundary while it being non-vanishing at any of its interior points.

Any two P-sets, $P_i$ and $P_j$, $i, j \in \mathbb{N}$, $i \neq j$, are, consequently, pairwise disjoint sets of $\mathbb{B}$. Also, each P-set is, in its own right, an uncountable, complete, separable, metric space.

Evidently, a P-set is the mathematically simplest form of “localized” physical properties in the space $\mathbb{B}$ and we call it a physical particle. This suggests that suitable mathematical properties of a P-set can represent physical properties.

We then recall that the Galilean concept of the (inertial) mass of a physical body is that of the measure of its inertia. Therefore, some appropriate measure definable for a P-set is the property of inertia of a physical body, a P-set in question. So also should be the case with the gravitational mass of a physical body. Such should also be the case with other relevant properties of physical bodies, for example, its electric charge.

Also, signed measures are definable on a P-set as well. Signed measures then provide us the notion of the “polarity” of certain properties. For example, a signed measure can provide the polarity of an electric charge.

Thus, we associate with every attribute of a physical body, a suitable class of (Lebesgue) measures on such P-sets. Therefore, a P-set is a physical particle, always an extended body, since a P-set cannot be a singleton set of $(\mathbb{B}, d)$.

Therefore, various physical properties (measures) change only when the region of space (P-set) changes. Thus, a region of space (P-set) and physical properties (the measures on P-sets) are,
then, are amalgamated into one thing here. This union of the space and the physical properties is then clearly perceptible here.

Moreover, a given measure can be integrated over the underlying P-set in question. The integration procedure is always a well-defined one for obvious reasons.

The value of the integral provides then an “averaged quantity characteristic of a P-set” under question. Of course, this is a property of the entire P-set under consideration.

For example, let us define an almost-everywhere finite-valued positive-definite measurable function, \( \rho \), on \((\mathcal{B}, \mathcal{B})\). Let us call its class the energy density. Integrating it over the volume of a P-set, the resultant quantity can be called a total mass, \( m_r \), of that P-set under consideration. The total mass, \( m_r \), is a property of that entire P-set and, hence, of every point of that P-set.

Clearly, the “location” of the mass \( m_r \), will be indeterminate over the size of that P-set because the averaged property is also the property of every point of the set under consideration.

Now, in a precise mathematical sense \( \mathcal{B} \), sets can be touching and that describes our intuitive notion of touching physical bodies. Of course, the corresponding point particles are then “touching” within the limits defined by the sizes, boundaries, of the corresponding P-sets.

A Borel automorphism of \((\mathcal{B}, \mathcal{B})\) then induces an associated transformation of \((\mathcal{B}, \mathcal{B})\), say, to \(\mathcal{B}, d'\), and that “moves” P-sets about in \(\mathcal{B}\), since (suitably defined) distance between the P-sets can change under that Borel automorphism.

The individuality of a point particle is clearly that of the corresponding P-set. As noted before, in the present formalism, a point particle is a point of the P-set of the space \(\mathcal{B}\) with associated integrated measures defined on that P-set. As a Borel automorphism of the space \(\mathcal{B}\), changes that P-set, the integrated properties also change and, hence, the initial particle changes into another particle(s), since integrated measures change.

Here, the notion of energy of a point particle is then that of some suitably defined integrated measure defined on a P-set of the space \(\mathcal{B}\). The notion of the momentum of a point particle (as a point of the P-set of \(\mathcal{B}\) with associated integrated measures on that P-set) is then that of the appropriately defined notion of the rate of change of, some suitably defined, “physical distance” under the action of a Borel automorphism of \(\mathcal{B}\), including evidently any changes that may occur to measures definable on that P-set. Therefore, the notions of energy and momentum of a particle are certainly (well-) definable in the present formalism.

Of course, we then need to discover various laws of such transformations of point particles into one another in the present formalism. But, it is clear at the outset that these will crucially depend on the structure of the group of Borel automorphisms of the space \(\mathcal{B}\).

Further, if a P-set splits into two or more P-sets, we have the process of creation of particles since the measures are now definable individually over the split parts, two or more P-sets. On the other hand, if two or more P-sets unite to become a single P-set, we have the process of annihilation of particles since the measures are now definable over a single P-set.

Clearly, the laws of creation and annihilation of particles will require of us to specify the corresponding transformations causing the splitting and the merger of the P-sets.

Now, we call as an object a region of \(\mathcal{B}\) bounded by the vanishing of \(\mathcal{B}\) but containing interior points for which it vanishes (so such a region is not a P-set). Such a region of \(\mathcal{B}\) is then a collection of P-sets. But, a P-set is a particle. Therefore, an object is a collection of particles.

Objects may also unite to become a single object or an object may also split into two or more than two objects under transformations of P-sets. We may then also think of the corresponding laws for these processes involving objects.

Moreover, the metric of \((\mathcal{B}, d)\) allows us the precise definition of the sizes of P-sets and objects. Then, given an object of specific size, we may use it as a measuring rod to measure “distance” between two other objects.

We call this the “physical” distance separating P-sets (as extended bodies). We also (naturally) define distance separating objects.

Now, the Borel automorphisms of the space \(\mathcal{B}\) can be classified as

1. those which preserve and,
2. those which do not preserve

measures defined on a specific P-set.
Note that we are restricting our attention to only a specific P-set/Object and not every P-set/Object is under consideration here.

Measure-preserving Borel automorphisms of the space $\mathbb{B}$ then “transform” a P-set maintaining its characteristic classes of (Lebesgue) measures, that is, its physical properties.

Non-measure-preserving Borel automorphisms change the characteristic classes of Lebesgue measures (physical properties) of a P-set while “transforming” it. Evidently, such considerations also apply to objects.

It is therefore permissible that a particular periodic Borel automorphism leads to an oscillatory motion of a P-set or an object while preserving its class of characteristic measures.

We can then think of an object undergoing periodic motion as a (physically realizable) time-measuring clock. Such an object undergoing oscillatory motion then “measures” the time-parameter of the corresponding (periodic) Borel automorphism since the period of the motion of such an object is precisely the period of the corresponding Borel automorphism.

Then, within the present formalism, a measuring clock is therefore any P-set or an object undergoing periodic motion.

Then, crucially, the present formalism represents measuring apparatuses, measuring rods and measuring clocks, on par with every other thing that the formalism intends to treat.

Such considerations then suggest an appropriate distance function, physical distance, on the family of all P-sets/objects of the space $(\mathbb{B}, d)$. More than one such distance function will be definable, depending obviously on the collection of P-sets or objects that we may be considering in the form of a measuring rod or measuring clock.

This above is permissible since we are dealing here with a continuum which is a standard Borel space with Polish topology. Relevant mathematical results can be found in [2].

A Borel automorphism of $(\mathbb{B}, \mathcal{B})$ may change the physical distance resulting into “relative motion” of objects. We also note here that the sets invariant under the specific Borel automorphism are characteristic of that automorphism. Hence, such sets will then have their distance “fixed” under that Borel automorphism and will be stationary relative to each other.

On a different note, an automorphism, keeping invariant a chain of objects separating two other objects, can describe the situation of two or more relatively stationary objects.

Effects of the Borel automorphisms of $(\mathbb{B}, \mathcal{B})$ on the (mathematically well defined) physical distance are then motions of physical bodies.

Obviously, various concepts such as the density of point particles, a flux of point particles across some surface etc. are then well definable in terms of the transformations of P-sets and the effects of these transformations on the measures definable over the P-sets under consideration.

Then, we note that such “averaging procedures” are well-defined over any collection of P-sets and, also, of objects. Thus, we may, in a mathematically meaningful way, define a suitable “energy-momentum tensor” $\mathcal{T}$ and some relation between the averaged quantities, an “equation of state” defining appropriately the “state of the fluid matter” under consideration.

(Such conceptions require however the notion of transformations of P-sets and objects. Moreover, this averaging is a “sum total” of the effects of various such transformations of P-sets and objects and, hence, will require corresponding mathematical machinery. This is, then, the premise of the ergodic theory. Recall that $(\mathbb{B}, \mathcal{B})$ is a Standard Borel Space.

Einstein’s field equations are then definable in the sense (only) of these averages. Therefore, Einstein’s equations are “obtainable” on the basis of the temporal evolution of point particles, points of the 3-space $\mathbb{B}$. Descartes’ conceptions are then also realizable in the present formalism.)

Clearly, a joint manifestation of Borel automorphisms of the space $(\mathbb{B}, \mathcal{B})$ and the association, as a point particle, of integrated measures definable on a P-set with the points of a P-set is a candidate reason behind Heisenberg’s indeterminacy relations in the present continuum formulation $\mathbb{B}$ since intrinsic indeterminacy exists here.

Intuitively, as well as in a mathematically precise sense, it can be seen that as the size of the P-set gets smaller and smaller the position of the point-particle (of integrated characteristics of a P-set) is determinable more and more accurately. (But, recall that a P-set is never a singleton subset of $\mathbb{B}$. So, complete positional localization of a point particle is not permissible.)

Now, any experimental arrangement to determine a physical property of a P-set is based on some specific “arrangement” of P-sets and involves corresponding Borel automorphisms of $\mathbb{B}$ affecting those P-sets. For example, Heisenberg’s microscope attempting the determination of the location of an electron involves the collision of a photon with an electron. It therefore has an associated Borel automorphism producing the motion of a specific P-set, a photon.

Although we have not specified the sense $\mathbb{B}$ in which a P-set can be a photon, it is clear that the Borel automorphism causing its motion will also affect an electron as a P-set.
Thus, a P-set “transforms” as a result of our efforts to “determine any of its characteristic measures” since these “efforts or experimental arrangements” are also Borel automorphisms, not necessarily the members of the class of Borel automorphisms keeping invariant that P-set (as well as the class of its characteristic measures).

Hence, a Borel automorphism (experimental arrangement) “determining” a characteristic measure of a P-set changes, in effect, the very quantity that it is trying to determine. This peculiarity then leads to Heisenberg’s (corresponding) indeterminacy relation.

Then, in the present continuum description, it is indeed possible to explain the origin of Heisenberg’s indeterminacy relations. The present continuum description provides us therefore an origin of indeterminacy relations. This is in complete contrast to their probabilistic origin as advocated by the standard quantum theory.

Notice also that, in the present considerations, we began with none of the fundamental considerations of the concept of a quantum. But, one of the basic characteristics of the conception of a quantum, Heisenberg’s indeterminacy relation, emerges out of the present formalism.

Then, in the present framework, we have also done away with the “singular nature” of the particles and, hence, also with the unsatisfactory dualism of the field (space) and the source particle. Furthermore, we have, simultaneously, well-defined laws of motion (Borel automorphisms) for the field (space) and also for the well-defined conception of a point particle (of integrated measure characteristics). The present formalism is therefore a complete field theory.

Further, none of the two notions of location and momentum is any deficient for a description of the facts since Heisenberg’s indeterminacy relations are also “explainable” within the present formalism. This explanation crucially hinges on the fact that the points of the space $\mathbb{B}$, as singleton subsets of the space $\mathbb{B}$, are never the P-sets. It is only in the sense of associating the measures integrated over a P-set that the points of the space $\mathbb{B}$ are point particles.

The measurable location of a particle is essentially a different conception and that depends on the physical distance definable on the class of all P-sets of the space $\mathbb{B}$. The measurable momentum of a particle is also dependent on the notion of the physical distance changing under the action of a Borel automorphism of $\mathbb{B}$.

A comment on the mathematical methods would not be out of place here. Then, we note that the mathematical formalism of the ergodic theory is what is of immediate use for the present physical framework. This much is already clear from the above considerations.

However, it is not entirely satisfactory to use the present methods of ergodic theory. One of the primary reasons for this state of affairs is the inability of the present methods in ergodic theory to let us handle, in a physical sense, the P-sets. Some newer methods are then required here.

[1] Einstein A (1968) Relativity: The Special and the General Theory (Methuen & Co. Ltd, London) (See, in particular, Appendix V: Relativity and the Problem of Space.)

[2] Wagh S M (2004) Some fundamental issues in General Relativity and their resolution Database: gr-qc/0402003 and references therein

[3] Joshi K D (1983) Introduction to General Topology (Wiley Eastern, New Delhi) and references therein

[4] Nadkarni M G (1995) Basic Ergodic Theory (Texts and Readings in Mathematics - 6: Hindustan Book Agency, New Delhi) and references therein

[5] The concept of extension owes its origin to our experience of bringing into contact physical bodies. But, from this alone it cannot be concluded that the concept of an extension is unjustifiable in cases which themselves have not given rise to the formation of this concept.

[6] This is a “necessary” consequence of Descartes’s conception. We can, of course, choose the defining property (of a physical object) itself as a “distance” function.

[7] We define an equivalence relation “~” such that $x \sim y$ iff $\ell(x, y) = 0$ where $\ell$ is the pseudo-metric distance defined on the space $X$. Denote by $Y$ the set of all equivalence classes of $X$ under the equivalence relation ~. If $A, B \subseteq Y$ are two equivalence classes, then let $e(A, B) = \ell(x, y)$ where $x \in A$ and $y \in B$. The (metric) function $e$ on $Y$ is the canonical distance.

[8] This is a field-theoretic comprehension, in a definite sense, of the energy-momentum tensor.

[9] Recall that Einstein, in Albert Einstein: Philosopher - Scientist (Ed. P. A. Schlipp, La Salle: Open Court Publishing Company - The Library of Living Philosophers, Vol. VII: 1970), p. 666, regarded the correctness of Heisenberg’s indeterminacy relations as being “finally demonstrated”. But, he looked for some non-probabilistic explanation for them.

[10] It follows that this “sense” is that of measures definable on a P-set. We therefore need to “isolate” the classes of measures corresponding to a photon and an electron here.