Omni-Lie Superalgebras and Lie 2-superalgebras

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Abstract We introduce the notion of omni-Lie superalgebra as a super version of the omni-Lie algebra introduced by Weinstein. This algebraic structure gives a nontrivial example of Leibniz superalgebra and Lie 2-superalgebra. We prove that there is a one-to-one correspondence between Dirac structures of the omni-Lie superalgebra and Lie superalgebra structures on subspaces of a super vector space.

1 Introduction

In [17], Weinstein introduced the notion of omni-Lie algebra, which can be regarded as the linearization of the Courant bracket. An omni-Lie algebra associated to a vector space $V$ is the direct sum space $\mathfrak{gl}(V) \oplus V$ together with the nondegenerate symmetric pairing $\langle \cdot, \cdot \rangle$ and the skew-symmetric bracket $\llbracket \cdot, \cdot \rrbracket$ given by

$$\langle A + u, B + v \rangle = \frac{1}{2}(Av + Bu),$$

and

$$\llbracket A + u, B + v \rrbracket = [A, B] + \frac{1}{2}(Av - Bu).$$

The bracket $\llbracket \cdot, \cdot \rrbracket$ does not satisfy the Jacobi identity so that an omni-Lie algebra is not a Lie algebra. An omni-Lie algebra is actually a Lie 2-algebra since Roytenberg and Weinstein proved that every Courant algebroid gives rise to a Lie 2-algebra ([13]). Recently, omni-Lie algebras are studied from several aspects ([4], [9], [16]) and are generalized to omni-Lie algebroids and omni-Lie 2-algebras in [5, 6, 15]. The corresponding Dirac structures are also studied therein.

In this paper, we introduce the notion of omni-Lie superalgebra, which is the super analogue of omni-Lie algebra. We also study Dirac structures of omni-Lie superalgebra in order to characterize all possible Lie superalgebra structures on a super vector space. We prove that omni-Lie superalgebra is a Leibniz superalgebra as well as a Lie 2-superalgebra, which is a super version of Lie 2-algebra or a 2-term $L_\infty$-algebra ([2, 8, 10]).
The paper is organized as follows. In Section 2, we recall some basic facts for Lie superalgebras. In Section 3, we define omni-Lie superalgebra on \( \mathcal{E} = \mathfrak{gl}(V) \oplus V \) for a super vector space \( V \) and study Dirac structures. In Section 4, we prove an omni-Lie superalgebra is a Lie 2-superalgebra.

2 Lie Superalgebras and Leibniz Superalgebras

We first recall some facts and definitions about Lie superalgebras, basic reference is Kac [7]. We work on a fixed field \( \mathbb{K} \) of characteristic 0.

A super vector space \( V \) is a \( \mathbb{Z}_2 \)-graded vector space with a direct sum decomposition \( V = V_0 \oplus V_1 \). An element \( x \in V_0 \cup V_1 \) is called homogeneous. The degree of a homogeneous element \( x \in V_\alpha, \; \alpha \in \mathbb{Z}_2 \) is defined by \(|x| = \alpha \).

A morphism between two super vector spaces, \( V \) and \( W \), is a grade-preserving linear map:

\[
f : V \rightarrow W, \quad f(V_\alpha) \subseteq W_\alpha, \quad \forall \alpha \in \mathbb{Z}_2.
\]

The direct sum \( V \oplus W \) is graded by

\[
(V \oplus W)_\alpha = V_\alpha \oplus W_\alpha, \quad (V \oplus W)_1 = V_1 \oplus W_1,
\]

and the tensor product \( V \otimes W \) is graded by

\[
(V \otimes W)_\alpha = (V_0 \otimes W_\alpha) \oplus (V_1 \otimes W_1), \quad (V \otimes W)_1 = (V_0 \otimes W_1) \oplus (V_1 \otimes W_0).
\]

**Definition 2.1.** A Lie superalgebra is a super vector space (i.e. \( \mathbb{Z}_2 \)-graded vector space) \( L = L_0 \oplus L_1 \) together with a bracket \( [\cdot, \cdot] : L \otimes L \rightarrow L \) satisfies,

(i) graded condition: \([L_\alpha, L_\beta] \subseteq L_{\alpha + \beta} \; \forall \alpha, \beta \in \mathbb{Z}_2,
(ii) super skew-symmetry:

\[
[x, y] + (-1)^{|x||y|}[y, x] = 0, \quad (1)
\]

(iii) super Jacobi identity:

\[
J_1 := (-1)^{|y||x|}[x, [y, z]] + (-1)^{|x||y|}[y, [x, z]] + (-1)^{|y||z|}[z, [x, y]] = 0, \quad (2)
\]

where \( x, y, z \in L \) are homogeneous elements of degree \(|x|, |y|, |z| \) respectively.

One can rewrite the super Jacobi identity in another form:

\[
J_2 := -(-1)^{|x||z|}J_1 = [x, [y, z]] - [x, [y, z]] - (-1)^{|y||z|}[y, [x, z]] = 0, \quad (3)
\]

which will be convenient for our use below.

**Example 2.2.** Let \( A = A_0 \oplus A_1 \) be an associative superalgebra with multiplication \( A_\alpha A_\beta \subseteq A_{\alpha + \beta} \) for all \( \alpha, \beta \in \mathbb{Z}_2 \). Define the bracket:

\[
[x, y] := xy - (-1)^{|x||y|}yx, \quad \forall x, y \in A.
\]

Then \((A, [\cdot, \cdot])\) is a Lie superalgebra which is denoted by \( A_L \).
Example 2.3. Let \( V = V_0 \oplus V_1 \) be a super vector space, then we have the general linear Lie superalgebra

\[
\mathfrak{gl}(V) = \mathfrak{gl}(V)_0 \oplus \mathfrak{gl}(V)_1,
\]

such that

\[
\mathfrak{gl}(V)_0 = \text{Hom}(V_0, V_0) \oplus \text{Hom}(V_1, V_1),
\]

\[
\mathfrak{gl}(V)_1 = \text{Hom}(V_0, V_1) \oplus \text{Hom}(V_1, V_0),
\]

and the bracket is given by (4). When \( \dim V_0 = m \), \( \dim V_1 = n \), \( \mathfrak{gl}(V) \) is usually denoted by \( \mathfrak{gl}(m|n) \).

A homomorphism between two Lie superalgebras \( (L, [\cdot, \cdot]) \) and \( (L', [\cdot, \cdot])' \) is linear map \( \varphi : L \to L' \) such that

\[
\varphi(L_\alpha) \subseteq L'_\alpha, \quad \varphi([x, y]) = [\varphi(x), \varphi(y)]', \quad \forall x, y \in L, \forall \alpha \in \mathbb{Z}_2.
\]

A super vector space \( V = V_0 \oplus V_1 \) is called a module of a Lie superalgebra \( L \) or, equivalently, say that \( L \) acts on \( V \) if there is a homomorphism \( \rho : L \to \mathfrak{gl}(V) \), i.e.,

\[
\rho([x, y])v = \rho(x)\rho(y)v - (-1)^{|x||y|} \rho(y)\rho(x)v.
\]

(5)

For simplicity, one often writes \( xv = \rho(x)v \) to denote such an action. A new Lie superalgebra can constructed as follows.

Proposition 2.4. [14] Let \( L \) be a Lie superalgebra with an action on \( V \). Define a bracket on \( L \oplus V \) by

\[
[x + u, y + v] := [x, y]_L + xv - (-1)^{|x||y|} yu.
\]

(6)

Then \( (L \oplus V, [\cdot, \cdot]) \) becomes a Lie superalgebra, denoted by \( L \ltimes V \) and called semidirect product of \( L \) and \( V \).

In [12], Loday introduced a new algebraic structure, which is usually called Leibniz algebra. Its super version is as follows.

Definition 2.5. [1] A Leibniz superalgebra is a super vector space \( L = L_0 \oplus L_1 \) together with a morphism \( \circ : L \otimes L \to L \) satisfying \( L_\alpha \circ L_\beta \subseteq L_{\alpha + \beta} \) for all \( \alpha, \beta \in \mathbb{Z}_2 \), and the super Leibniz rule:

\[
x \circ (y \circ z) = (x \circ y) \circ z + (-1)^{|x||y|} y \circ (x \circ z),
\]

(7)

for all homogeneous elements \( x, y, z \in L \).

By definition, it is easy to see that a Leibniz superalgebra is just a Lie superalgebra if the operation "\( \circ \)" is super skew-symmetric. In this case, the super Leibniz rule above is actually the super Jacobi identity for \( J_2 \) given by (3).
3 Omni-Lie Superalgebras and Dirac Structures

Let $V$ be a super vector space, recall that the space $\mathfrak{gl}(V) \oplus V$ has a $\mathbb{Z}_2$-grading

$$\mathcal{E} := \mathfrak{gl}(V) \oplus V = (\mathfrak{gl}(V)_0 \oplus V_0) \oplus (\mathfrak{gl}(V)_1 \oplus V_1).$$

Like in [9], we define an operation $\circ$ on $\mathfrak{gl}(V) \oplus V$ as follows:

$$(A + x) \circ (B + y) = [A, B] + Ay.$$  \hspace{1cm} (8)

Then we have

**Proposition 3.1.** $(\mathcal{E}, \circ)$ is a Leibniz superalgebra.

**Proof.** To check the super Leibniz rule (7) holds on $\mathcal{E}$ under the operation $\circ$, let $e_1 = A + x, e_2 = B + y, e_3 = C + z$ be homogenous elements of degree $|A| = |x|, |B| = |y|$ and $|C| = |z|$. By definition, we have

$$\begin{align*}
\{e_1 \circ e_2\} \circ e_3 &= e_1 \circ \{e_2 \circ e_3\} - (-1)^{|y||e_2|} e_2 \circ \{e_1 \circ e_3\} \\
&= ([A, B] + Ay) \circ (C + z) - (A + x) \circ ([B, C] + Bz) \\
&\quad - (-1)^{|y||e_2|} (B + y) \circ ([A, C] + Az) \\
&= [[A, B], C] - [A, [B, C]] - (-1)^{|y||e_2|}[B, [A, C]] \\
&\quad + [A, B]z - ABz - (-1)^{|y||e_2|} BAz \\
&= 0,
\end{align*}$$

where the equality holds because $\mathfrak{gl}(V)$ is a Lie superalgebra acting on $V$. \hfill \Box

Note that the above operation is not super skew-symmetric, we can define a super skew-symmetric bracket on $\mathcal{E} = \mathfrak{gl}(V) \oplus V$ as its skew symmetrization:

$$[A + x, B + y] \triangleq \frac{1}{2}((A + x) \circ (B + y) - (B + y) \circ (A + x))$$

$$= [A, B] + \frac{1}{2} (Ay - (-1)^{|y||B|} Bx), \hspace{1cm} (9)$$

and define a $V$-valued inner product, i.e., a non-degenerated super symmetric bilinear form:

$$\langle A + x, B + y \rangle \triangleq \frac{1}{2}(Ay + (-1)^{|y||B|} Bx). \hspace{1cm} (10)$$

We call the triple $(\mathcal{E}, [\cdot, \cdot], \langle \cdot, \cdot \rangle)$ an **omni-Lie superalgebra**. Without the factor 1/2 in bracket $[\cdot, \cdot]$, this would be the semidirect product Lie superalgebra for the action of $\mathfrak{gl}(V)$ on $V$ described in Proposition 2.4. With the factor 1/2, the bracket does not satisfy the super Jacobi identity, which leads to the concept of Lie 2-superalgebra defined in the next section. Next we compute the Jacobiator for this bracket.

**Proposition 3.2.** For $e_1 = A + x, e_2 = B + y, e_3 = C + z \in \mathcal{E}$, define

$$T(e_1, e_2, e_3) := \frac{3}{4}(1)^{|z||x|} \langle [e_1, e_2], e_3 \rangle + (-1)^{|z||y|} \langle [e_2, e_3], e_1 \rangle + (-1)^{|y||z|} \langle [e_3, e_1], e_2 \rangle.$$
Let $J_1$ denote the Jacobiator given in (2) for the bracket $[\cdot, \cdot]$ on $\mathcal{E}$, then we have

$$J_1(e_1, e_2, e_3) = T(e_1, e_2, e_3).$$

**Proof.** We compute both the sides as follows:

$$J_1(e_1, e_2, e_3) = (-1)^{|x||y|}[[[A + x, B + y], C + z] + c.p.] = [[(-1)^{|x||y|}[A, B] + \frac{1}{2}(-1)^{|x||y|} A y - (-1)^{|x||y|} B x , C + z] + c.p.] = (-1)^{|x||y|}[[A, B], C] + c.p. + \frac{1}{2} ((-1)^{|x||y|}[A, B] z - \frac{1}{2}(-1)^{|x||y|} (-1)^{|x|+|y|+|z|} C (A y - (-1)^{|x||y|} B x ) + \frac{1}{2} ((-1)^{|x||y|}[B, C] x - \frac{1}{2}(-1)^{|x||y|} (-1)^{|x|+|y|+|z|} A (B z - (-1)^{|y||z|} C y ) + \frac{1}{2} ((-1)^{|y||z|}[C, A] y - \frac{1}{2}(-1)^{|y||z|} (-1)^{|x|+|y|+|z|} B (C x - (-1)^{|z||x|} A z )) = \frac{1}{4}(-1)^{|x||y|}|x| [AB] z - \frac{1}{4}(-1)^{|x||y|} (-1)^{|x||y|} B A z + \frac{1}{4}(-1)^{|y||z|} |z| C A y - \frac{1}{4}(-1)^{|y||z|} (-1)^{|x||y|} C B x + \frac{1}{4}(-1)^{|x||y|} |y| B C x - \frac{1}{4}(-1)^{|x||y|} (-1)^{|y||z|} C B x + \frac{1}{4}(-1)^{|x||y|} [AB] z - \frac{1}{4}(-1)^{|x||y|} (-1)^{|x||y|} B A z + \frac{1}{4}(-1)^{|y||z|} |z| C A y - \frac{1}{4}(-1)^{|y||z|} (-1)^{|x||y|} C B x - \frac{1}{4}(-1)^{|x||y|} (-1)^{|y||z|} |y| B C x + \frac{1}{4}(-1)^{|y||z|} [AC] y - \frac{1}{4}(-1)^{|y||z|} (-1)^{|x||y|} C B x + \frac{1}{4}(-1)^{|x||y|} |y| B C x - \frac{1}{4}(-1)^{|x||y|} (-1)^{|y||z|} |z| C A y.

Thus, the two sides are equal. \(\square\)

The bracket $[\cdot, \cdot]$ does not satisfy the super Jacobi identity so that an omni-Lie superalgebra is not a Lie superalgebra. However, all possible Lie superalgebra structures on $V$ can be characterized by means of the omni-Lie superalgebra.

For a bilinear operation $\omega$ on $V$ such that $\omega : V_\alpha \times V_\beta \rightarrow V_{\alpha + \beta}$, we define the adjoint operator

$$\text{ad}_\omega : V_\alpha \rightarrow \mathfrak{gl}(V)_\alpha, \quad \text{ad}_\omega(x)(y) = \omega(x, y) \in V_{\alpha + \beta}$$

where $x \in V_\alpha, y \in V_\beta$. Then the graph of the adjoint operator:

$$\mathcal{F}_\omega = \{ \text{ad}_\omega x + x : \forall x \in V \} \subset \mathcal{E} = \mathfrak{gl}(V) \oplus V$$

is a super subspace of $\mathcal{E}$. Denote $\mathcal{F}_\omega^\perp$ the orthogonal complement of $\mathcal{F}_\omega$ in $\mathcal{E}$ with respect to the super symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathcal{E}$ given in (10).
Proposition 3.3. With the above notations, \((V, \omega)\) is a Lie superalgebra if and only if its graph \(F_{\omega}\) is maximal isotropic, i.e. \(F_{\omega} = F_{\omega}^+\), and is closed with respect to the bracket \([\cdot, \cdot]\).

Proof. First we see that
\[
\langle \text{ad}_\omega(x) + x, \text{ad}_\omega(y) + y \rangle = \frac{1}{2}(\text{ad}_\omega(x)y + (-1)^{|x||y|}\text{ad}_\omega(y)x) = \frac{1}{2}(\omega(x, y) + (-1)^{|x||y|}\omega(y, x)).
\]
This means that \(\omega\) is super skew-symmetric if and only if its graph is isotropic, i.e. \(F_{\omega} \subseteq F_{\omega}^+\). Moreover, by dimension analysis, we have \(F_{\omega}\) is maximal isotropic.

Next let \([x, y] := \omega(x, y)\), we shall check that the super Jacobi identity on \(V\) is satisfied if and only if \(F_{\omega}\) is closed under bracket (9) on \(E\). In fact,
\[
[\text{ad}_\omega(x) + x, \text{ad}_\omega(x) + y] = [\text{ad}_\omega(x) + x, \text{ad}_\omega(y)] = \frac{1}{2}(\omega(x, y) - (-1)^{|x||y|}\omega(y, x)) + \omega(x, y).
\]
Thus this bracket is closed if and only if
\[
[\text{ad}_\omega(x), \text{ad}_\omega(y)] = \text{ad}_\omega(\omega(x, y)).
\]
In this case, for \(\forall z \in V\), we have
\[
[\text{ad}_\omega(x), \text{ad}_\omega(y)](z) = \text{ad}_\omega(\omega(x, y))(z)
\]
\[
= \text{ad}_\omega(x)\text{ad}_\omega(y)(z) - (-1)^{|x||y|}\text{ad}_\omega(y)\text{ad}_\omega(x)(z) - \text{ad}_\omega(\omega(x, y))(z)
\]
\[
= \omega(x, \omega(y, z)) - (-1)^{|x||y|}\omega(y, \omega(x, z)) - \omega(\omega(x, y), z)
\]
\[
= [x, [y, z]] - (-1)^{|x||y|}[y, [x, z]] - [[x, y], z]
\]
\[
= 0.
\]
This is exactly the super Jacobi identity on \(V\). Therefore, the conclusion follows from Definition 2.1.

In [3], quadratic Lie superalgebras are studied for a given inner product \(B\) on \(V\). In this case, one has the orthogonal Lie superalgebra \(\mathfrak{o}(V) \subseteq \mathfrak{gl}(V)\) and it is easy to see that \((V, \omega, B)\) is a quadratic Lie superalgebra if and only if \(\omega\) satisfies the two conditions in Proposition 3.3 above as well as \(\text{ad}_\omega x \in \mathfrak{o}(V), \forall x \in V\).

Definition 3.4. A Dirac structure \(L\) of the omni-Lie superalgebra \((\mathfrak{gl}(V) \oplus V, [\cdot, \cdot], \langle \cdot, \cdot \rangle)\) is a maximal isotropic subspace \((L = L^+)\) and closed under the bracket \([\cdot, \cdot]\).

Remark 3.5. According to Proposition 3.2, for a Dirac structure \(L\), we have
\[
J_1(e_1, e_2, e_3) = T(e_1, e_2, e_3) = 0, \quad \forall e_i \in L.
\]
Thus a Dirac structure is a Lie superalgebra, though omni-Lie superalgebra is not for itself. In fact, a Dirac structure is also a Leibniz subalgebra under the operation $\circ$.

By Proposition 3.3, $(V, \omega)$ is a Lie superalgebras if and only if $\mathcal{F}_\omega$ is a Dirac structure of the omni-Lie superalgebra $\mathfrak{gl}(V) \oplus V$. In order to give a general characterization for all Dirac structures of $\mathcal{E}$, we adapt the theory of characteristic pairs developed in [11] (see also [15]).

For a maximal isotropic subspace $L \subset \mathfrak{gl}(V) \oplus V$, set the subspace $D = L \cap \mathfrak{gl}(V)$. Define $D^0 \subset V$ to be the null space of $D$:

$$D^0 = \{ x \in V | X(x) = 0, \forall X \in D \}.$$

It is easy to see that $D = (D^0)^0$.

**Lemma 3.6.** With notations above, a subspace $L$ is maximal isotropic if and only if $L$ is of the form

$$L = D \oplus \mathcal{F}_{\pi|_{D^0}} = \{ X + \pi(x) + x | X \in D, x \in D^0 \}, \quad (11)$$

where $\pi : V \to \mathfrak{gl}(V)$ is a super skew-symmetric map.

**Proof.** In the following, we also denote $\pi(x, y) = \pi(x)(y) \in V$ for convenience. First suppose that $L$ is given by (11), then

$$\langle X + \pi(x) + x, Y + \pi(y) + y \rangle = \frac{1}{2} \{ X(y) + \pi(x, y) + (-1)^{|x||y|} Y(x) + (-1)^{|x||y|} \pi(y, x) \}$$

$$= \frac{1}{2} \{ \pi(x, y) + (-1)^{|x||y|} \pi(y, x) \}$$

$$= 0, \quad \forall X + \pi(x) + x, Y + \pi(y) + y \in L,$$

since $\pi : V \to \mathfrak{gl}(V)$ is super skew-symmetric so that $L$ is isotropic. Next we prove that $L$ is maximal isotropic. For $\forall Z + z \in L^\perp$,

$$\langle X, Z + z \rangle = X(z) = 0, \quad \forall X \in D \Rightarrow z \in D^0.$$

Moreover, $\forall x \in D^0$, the equality below

$$\langle X + \pi(x) + x, C + z \rangle = X(z) + \pi(x)(z) + (-1)^{|x||z|} Cx$$

$$= (-1)^{|x||z|}(C - \pi(z))(x) = 0,$$

implies that $C - \pi(z) \triangleq Z \in D$. Thus

$$C + z = Z + \pi(z) + z \in L = D \oplus \mathcal{F}_{\pi|_{D^0}} \Rightarrow L = L^\perp.$$

The converse part is straightforward so we omit the details. \qed

The proof of the following Lemma is skipped since it is straightforward and similar to that in [15].
Lemma 3.7. Let \((D, \pi)\) be given above. Then \(L\) is a Dirac structure if and only if the following conditions are satisfied:

1. \(D\) is a subalgebra of \(\mathfrak{gl}(V)\);
2. \(\pi(\pi(x, y)) - [\pi(x), \pi(y)] \in D, \quad \forall \ x, y \in D^0;\)
3. \(\pi(x, y) \in D^0, \quad \forall \ x, y \in D^0.\)

Such a pair \((D, \pi)\) is called a characteristic pair of a Dirac structure \(L\).

By means of the two lemmas above, we can mention the main result in this section.

Theorem 3.8. There is a one-to-one correspondence between Dirac structures of the omni-Lie superalgebra \((\mathfrak{gl}(V) \oplus V, \langle \cdot, \cdot \rangle, [\cdot, \cdot])\) and Lie superalgebra structures on subspaces of \(V\).

Proof. For any Dirac structure \(L = D \oplus F\pi|_{D^0}\), a Lie superalgebra structure on \(D^0\) is as follows:

\[
[x, y]_{D^0} \triangleq \pi(x, y) \in D^0, \quad \forall \ x, y \in D^0.
\]

It easy to see that this is a super skew-symmetric map. For super Jacobi identity, we have for all \(x, y, z \in D^0\),

\[
[[x, y]_{D^0}, z]_{D^0} = \pi([x, y]_{D^0})(z) = \pi((\pi(x)\pi(y))(z) = [\pi(x), \pi(y)](z) = \pi(x)(\pi(y)(z)) - (-1)^{|x||y|}\pi(y)(\pi(x)(z)) = [x, [y, z]_{D^0}]_{D^0} - (-1)^{|x||y|}|y, [x, z]_{D^0}]_{D^0}.
\]

Thus we get a Lie superalgebra \((D^0, [\cdot, \cdot]_{D^0}).\)

Conversely, for any Lie superalgebra \((W, [\cdot, \cdot]_{W})\) on a subspace \(W\) of \(V\). Define \(D\) by

\[
D = W^0 \triangleq \{X \in \mathfrak{gl}(V) | X(x) = 0, \forall x \in W\}.
\]

Then \(D^0 = (W^0)^0 = W\). Since Lie superalgebra structure \([\cdot, \cdot]_{W}\) gives a super skew symmetric morphism:

\[
ad : W \to \mathfrak{gl}(W), \quad \text{ad}_x(y) = [x, y]_{W},
\]

we take a super skew symmetric morphism \(\pi : V \to \mathfrak{gl}(V)\), as an extension of \(\text{ad}\) from \(W = D^0\) to \(V\). Thus we get a maximal isotropic subspace \(L = D \oplus F\pi|_{W}\) from the pair \((D, \pi)\) as in Lemma 3.6.

We shall prove that \(L\) is a Dirac structure. Firstly, \(\forall X, Y \in D\) and \(x \in W\), we have

\[
[X, Y](x) = XY(x) - (-1)^{|X||Y|}YX(x) = 0,
\]

which implies that \(D\) is a subalgebra of \(\mathfrak{gl}(V)\).
Next step is to prove that \( L \) is closed under the bracket \([\cdot, \cdot]\). Remember that \( \pi|_W = \text{ad} \) and \([\cdot, \cdot]|_W \) satisfies the super Jacobi identity, we obtain
\[
[\text{ad}_x, \text{ad}_y] = \text{ad}_{\text{ad}_x y} = \text{ad}_{x \text{ad}_y}, \quad \forall \ x, y \in W.
\]
For any \( X \in D \) and \( x, y \in W \), we have
\[
[X, \text{ad}_x](y) = X([x, y]|_W) - (-1)^{|x||y|}[x, X(y)] = 0,
\]
thus \([X, \text{ad}_x] \in D\). On the other hand, we have
\[
[[X + \text{ad}_x + x, Y + \text{ad}_y + y]
= [X, Y] + [X, \text{ad}_y] + [\text{ad}_x, Y] + [\text{ad}_x, \text{ad}_y] + \frac{1}{2}(\text{ad}_x(y) - (-1)^{|x||y|}\text{ad}_y(x))
= [X, Y] + [X, \text{ad}_y] + [\text{ad}_x, Y] + \text{ad}[x, y]|_W + [x, y]|_W
\in D \oplus \mathcal{F}_{\pi|W},
\]
Thus, we conclude that \( L \) is a Dirac structure. Finally, it is easy to see that the Dirac structure \( L \) is independent of the choice of extension \( \pi \). This completes the proof. \( \square \)

4 Lie 2-superalgebras

The concept of Lie \( n \)-superalgebras is introduced in [8]. In particular, the axiom of a Lie 2-superalgebra can be expressed explicitly as follows:

**Definition 4.1.** A Lie 2-superalgebra \( \mathcal{V} = (\mathcal{V}^1 \xrightarrow{d} \mathcal{V}^0, l_2, l_3) \) consists of the following data:

- two super vector spaces \( \mathcal{V}^0 \) and \( \mathcal{V}^1 \) together with a morphism \( d: \mathcal{V}^1 \to \mathcal{V}^0 \);
- a morphism \( l_2 = [\cdot, \cdot]: \mathcal{V}^i \otimes \mathcal{V}^j \to \mathcal{V}^{i+j} \);
- a morphism \( l_3: \mathcal{V}^0 \otimes \mathcal{V}^0 \otimes \mathcal{V}^0 \to \mathcal{V}^1 \);

such that, \( \forall \ x, y, z, w \in \mathcal{V}^0, \forall \ h, k \in \mathcal{V}^1 \);

(a) \([x, y] + (-1)^{|x||y|}[y, x] = 0\);

(b) \([x, h] + (-1)^{|x||h|}[h, x] = 0\);

(c) \([h, k] = 0\);

(d) \( l_3(x, y, z) \) is totally super skew-symmetric;

(e) \( d([x, h]) = [x, dh] \);

(f) \( [dh, k] = [h, dk] \);

(g) \( d(l_3(x, y, z)) = -[[x, y], z] + [x, [y, z]] + (-1)^{|y||z|}[[x, z], y] \);
(h) \( l_3(x, y, dh) = -[[x, y], h] + [x, [y, h]] + (-1)^{|y|h}|[x, z], h]; \)

(i) \( \delta l_3(x, y, z, w) := [x, l_3(y, z, w)] - (-1)^{|x||y|}[y, l_3(x, z, w)] 
+ (-1)^{|y||z||w||x|} l_3([x, y], z, w) - l_3([x, y], z, w) 
+ (-1)^{|y||z||w||x|} l_3([x, z], y, w) - (-1)^{|y||z||w||x|} l_3([x, w], y, z) 
- l_3(x, [y, z], w) + (-1)^{|z||w||x|} l_3(x, [y, w], z) - l_3(x, y, [z, w]) = 0. \)

This is the super analogue of a 2-term \( L_\infty \)-algebra which is equivalent to a Lie 2-algebra (see [2] for more details). Here we use the terminology Lie 2-superalgebra instead of 2-term \( L_\infty \)-superalgebra.

Now, for a super vector space \( V \), let \( V_0 = gl(V) \oplus V, V_1 = V, d : V \rightarrow gl(V) \oplus V, \) where \( i \) is the inclusion map and define operations:

\[ l_2 = [\cdot, \cdot], \quad l_3 = -(-1)^{|\cdot|\cdot|\cdot|} T. \]

**Theorem 4.2.** With notations above, the omni-Lie superalgebra \( (\mathcal{E}, [\cdot, \cdot], \langle \cdot, \cdot \rangle) \) defines a Lie 2-superalgebra \( (V \rightarrow gl(V) \oplus V, l_2, l_3). \)

**Proof.** For Condition (a), by the grading in \( gl(V) \oplus V \) we have \( \deg(A + x) = \deg(A) = \deg(x) \), then

\[
\begin{align*}
[A + x, B + y] + (-1)^{|x||y|}[B + y, A + x] &= [A, B] + \frac{1}{2}(Ay - (-1)^{|x||y|} Bx) + (-1)^{|A||B|}[B, A] \\
&\quad + (-1)^{|x||y|} \frac{1}{2} (Bx - (-1)^{|y||x|}Ay) \\
&= [A, B] + (-1)^{|x||y|}[B, A] + \frac{1}{2}(Ay - (-1)^{|x||y|} Bx) \\
&\quad + \frac{1}{2}((-1)^{|y||x|} Bx - Ay) \\
&= 0.
\end{align*}
\]

Conditions (b), (c), (e) and (f) are easy to be checked. By Proposition 3.2, we have \( l_3 = J_2 \), thus Conditions (g)–(h) hold. For Condition (i), we first verify a special case by taking \( e_1 = A, e_2 = B, e_3 = C, e_4 = w \). In fact, by the definition
of \( l_3 \) and Proposition 3.2, we get
\[
\begin{align*}
[A, l_3(B, C, w)] &- (-1)^{[x][y]} [B, l_3(A, C, w)] \\
&+ (-1)^{([x]+[y])z} l_3([C, l_3(A, B, w)]) - [l_3(A, B, C, w)]
\end{align*}
\]

\(-l_3([A, B], C, w) + (-1)^{[y][z]} l_3([A, C], B, w) - (-1)^{([y]+[z])w} l_3([A, w], B, C)
\]

\(-l_3(A, [B, C], w) + (-1)^{[z][w]} l_3(A, [B, w], C) - l_3(A, [B, C], w))
\]

\[
\begin{align*}
&= -\frac{1}{8} A[B, C]w + \frac{1}{8} (-1)^{[y][z]} B[A, C]w - \frac{1}{8} (-1)^{([x]+[y])z} C[A, B]w + 0 \\
&+ \frac{1}{8} [[A, B], C]w - \frac{1}{8} (-1)^{[y][z]} [[A, C], B]w + \frac{1}{8} (-1)^{([x]+[y])z} [[B, C], A]w \\
&+ \frac{1}{8} (-1)^{([y][z][w])} [B, C]A w - \frac{1}{8} (-1)^{[y][z][w]} [A, C]B w + \frac{1}{8} [A, B]C w
\end{align*}
\]

\[
\begin{align*}
&= \frac{1}{4} \{[[A, B], C] + (-1)^{([y][z][w])} [[B, C], A] + (-1)^{([x]+[y])z} [[C, A], B] \} w \\
&- \frac{1}{8} \{A[B, C] - (-1)^{[y][z]} B[A, C] + (-1)^{([x]+[y])z} C[A, B] \} w \\
&- (-1)^{([y][z][w])} B[A, C]B - [A, B]C w
\end{align*}
\]

\[
\begin{align*}
&= \frac{1}{4} \{[[A, B], C] - [A, [B, C]] + (-1)^{[y][z]} [B, [A, C]] \} w \\
&- \frac{1}{8} \{[A, [B, C]] - (-1)^{[y][z]} [B, [A, C]] - [[A, B], C] \} w
\end{align*}
\]

\[
= 0.
\]

The general case can be checked similarly. \( \square \)

A Lie 2-superalgebra is called strict if \( l_3 = 0 \). This kind of Lie 2-superalgebras can be described in terms of crossed module.

**Definition 4.3.** A crossed module of Lie superalgebras consists of a pair of Lie superalgebras \((g, [\cdot, \cdot]_g)\) and \((h, [\cdot, \cdot]_h)\) together with an action of \( g \) on \( h \) and a homomorphism \( \varphi : h \to g \) such that
\[
\varphi(xh) = [x, \varphi(h)]_g, \quad \varphi(h)k = [h, k]_h, \quad \forall h, k \in h, \forall x \in g.
\]

**Proposition 4.4.** Strict Lie 2-superalgebras are in one-to-one correspondence with crossed modules of Lie superalgebras.

**Proof.** Let \( \mathcal{V}^1 \to \mathcal{V}^0 \) be a strict Lie 2-superalgebra. Define \( g = \mathcal{V}^0, h = \mathcal{V}^1 \), and the following two brackets on \( g \) and \( h \):
\[
\begin{align*}
[h, k]_h &= l_2(dh, k) = [dh, k], \quad \forall h, k \in h = \mathcal{V}^1; \\
[x, y]_g &= l_2(x, y) = [x, y], \quad \forall x, y \in g = \mathcal{V}^0.
\end{align*}
\]

Obviously, \((g, [\cdot, \cdot]_g)\) is a Lie superalgebra by (a) and (g) in Definition 4.1. By Condition (h), we have
\[
\begin{align*}
-[[h, k]_h, l]_h + (-1)^{[k][l]} [[h, l]_h k]_h + [h, [k, l]_h]_h \\
-[[d(dh, k), l] + (-1)^{[k][l]} [d(dh, l), k] + [dh, [dk, l]] \\
-[[dh, dk], l] + (-1)^{[k][l]} [[dh, dl], k] + [dh, [dk, l]]
\end{align*}
\]

\[
= 0.
\]
This means that \((\mathfrak{h}, [, , ]_\mathfrak{h})\) is also Lie superalgebra. By Condition (e) and taking \(\varphi = d\), we have
\[
\varphi([h, k]_\mathfrak{h}) = d([dh, k]) = [dh, dk] = [\varphi(h), \varphi(k)]_\mathfrak{g},
\]
which implies that \(\varphi\) is a homomorphism of Lie superalgebras. Next we define an action of \(\mathfrak{g}\) on \(\mathfrak{h}\) by
\[
xh \triangleq l_2(x, h) = [x, h] \in \mathfrak{h},
\]
which is an action because the equality,
\[
[x, y]h - x(yh) + (-1)^{|x||y|}y(xh) = \begin{cases} [x, y], & \text{if } x = h, y \in \mathfrak{g}; \\ xh, & \text{if } x \in \mathfrak{h}; \\ 0, & \text{if } x, y \in \mathfrak{h}. \end{cases}
\]
holds by Condition \((h)\). Finally, it is easy to check that
\[
\varphi(xh) = d([x, h]) = [x, dh] = [x, \varphi(h)]_\mathfrak{g}
\]
\[
\varphi(h)k = [dh, k] = [h, k]_\mathfrak{g}.
\]
Therefore, we obtain a crossed module of Lie superalgebras.

Conversely, a crossed module of Lie superalgebras gives rise to a Lie 2-superalgebra with \(d = \varphi\), \(\mathcal{V}^0 = \mathfrak{g}\), \(\mathcal{V}^1 = \mathfrak{h}\), \(l_3 = 0\) and the following operations:
\[
l_2(x, y) \triangleq [x, y]_\mathfrak{g}, \quad \forall \ x, y \in \mathfrak{g}; \\
l_2(x, h) \triangleq xh, \quad \forall \ x \in \mathfrak{h}; \\
l_2(h, k) \triangleq 0.
\]
All of the conditions for a Lie 2-superalgebra can be verified directly from the definition of a crossed module. 

Another kind of Lie 2-superalgebras is called skeletal if \(d = 0\). As pointed in [8], Skeletal Lie 2-superalgebras are in one-to-one correspondence with quadruples \((\mathfrak{g}, V, \rho, l_3)\) where \(\mathfrak{g}\) is a Lie superalgebra, \(V\) is a super vector space, \(\rho\) is a representation of \(\mathfrak{g}\) on \(V\) and \(l_3\) is a 3-cocycle on \(\mathfrak{g}\) with values in \(V\). See [14] for more details of the cohomology of Lie superalgebras.

**Example 4.5.** Given a quadratic Lie superalgebra \((\mathfrak{g}, [, , ]_\mathfrak{g}, B)\) over \(K\), where \(B\) is the supertrace \(B(x, y) = \text{str}(xy)\) by Kac [7], a skeletal Lie 2-superalgebra can be constructed as follows: \(\mathcal{V}^1 = K\), \(\mathcal{V}^0 = \mathfrak{g}\), \(d = 0\) and
\[
l_2(x, y) = [x, y], \quad l_2(x, h) = 0, \quad l_3(x, y, z) = B([x, y], z) \quad (12)
\]
where \(x, y, z \in \mathfrak{g}, h \in K\). Condition \((i)\) is from the fact that Cartan 3-form \(l_3\) is closed. That is, by the invariance of \(B\), we have
\[
\delta l_3(w, x, y, z) = B(w, [x, [y, z]]) + B(w, [x, [y, z]]) + (-1)^{|y||z|}B(w, [[x, z], y]) + (-1)^{|x||y|}B(w, [z, [x, y]]) - B(w, [[x, y], z]) - (-1)^{|x||y|}B(w, [[y, x], z]) - 2B(w, [x, [y, z]]) + \begin{cases} 0, & \text{if } x, y, z \in \mathfrak{g}; \\ 2B(w, [x, [y, z]]) + (-1)^{|y||z|}B(w, [x, z], y) + (-1)^{|x||y|}B(w, [y, [x, z]]) = 0, & \text{if } x, y, z \in \mathfrak{g}. \end{cases}
\]
which holds from the super Jacobi identity on $\mathfrak{g}$ . Therefore, $(\mathbb{K} \to \mathfrak{g}, l_2, l_3)$ is a Lie 2-superalgebra as a super version of the string Lie algebra.

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