The isotropy group of the matrix multiplication tensor

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Abstract

By an isotropy group of a tensor $t \in V_1 \otimes V_2 \otimes V_3 = \tilde{V}$ we mean the group of all invertible linear transformations of $\tilde{V}$ that leave $t$ invariant and are compatible (in an obvious sense) with the structure of tensor product on $\tilde{V}$. We consider the case where $t$ is the structure tensor of multiplication map of rectangular matrices. The isotropy group of this tensor was studied in 1970s by de Groote, Strassen, and Brockett-Dobkin. In the present work we enlarge, make more precise, expose in the language of group actions on tensor spaces, and endow with proofs the results previously known. This is necessary for studying the algorithms of fast matrix multiplication admitting symmetries. The latter seems to be a promising new way for constructing fast algorithms. 

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1. Introduction. The present article is related to the problem of fast matrix multiplication. In [1], [2] the author put forward an idea that good (i.e. fast) algorithms must have nontrivial symmetries, and this may be a fruitful way of searching for new algorithms. To confirm this idea, in [1] and [2] it was shown that some well-known algorithms (Strassen’s, Hopcroft’s and Laderman’s [3], [4], [5]) have rather large automorphism groups. (The definition of automorphism group of an algorithm was also given in [1], [2]).

The automorphism group of an algorithm is a subgroup of a certain ambient group, namely the isotropy group of the tensor associated with operation of matrix multiplication. Studying the latter group is a necessary part of studying symmetries of algorithms. However, this is a purely algebraic problem; it is not difficult, but its solution is not short. This is the subject of this article.

The algorithms themselves are not considered in the article. They will be considered in the future article [6] (and were considered in [1], [2]). The present article is a preparation for [6], quite similarly to the way how [7] was a preparation article for [8]. Thus, the aim of the present article is to liberate the reader of [6] of some necessary, but standard algebra.

It should be said from the very beginning that the present article is not especially original. The results themselves are not neither new, nor difficult. They were mainly known 40 years ago. See [7, Sect.3], and [9, Sect.4]. We have an intention to make them more precise, to enlarge them to necessary extent, to expose them in an appropriate language (group actions on tensor spaces), which is necessary for further applications, and to endow them with proofs.

J.M.Landsberg communicated to the author that he (with co-authors) independently came to the idea of using symmetry for analysis of constructing matrix multiplication algorithms, approximately at the same time as the author of the present article. See preprint [10] for more details. Also, it should be mentioned that symmetry groups were used in [15] to find some new algorithms. However, we want to warn the reader on the following. In the
listed works, where group action on algorithms, or symmetry of algorithms, were considered
(1],[2],[7],[9],[10],[15]), every author develops his own system of concepts, and uses his own
language (so that, it is possible that in the present work the author lays foundations only
for his own future work).

Finally, inform the reader that this article is a revised version of Sections 3 and 4 of [2].
However, the differences with [2] are significant. Some points are exposed in [2] with more
details and explanations. On the other hand, some places in [2] were too complicated and
difficult, and in this article the author tried to simplify them.

2. The isotropy group of a tensor. We assume the reader is familiar with the basics of
multilinear algebra and group representation theory, including the concept of tensor product
of representations. See [11, Ch.4], [12, Ch.8], [13, Ch.1]. (This remark is made for possible
readers who are not pure matematicians, but, for example, computer scientists).

Let $K$ be a field, $\tilde{V} = V_1 \otimes \ldots \otimes V_l$ a tensor product of several spaces. By a decomposable
automorphism we mean any invertible transformation $\varphi \in GL(\tilde{V})$, which is compatible, in
an obvious sense, with the structure of a tensor product on $\tilde{V}$. For example, let $\tilde{V} = V_1 \otimes V_2 \otimes V_3,$
and let $\alpha : V_1 \rightarrow V_2,$ $\beta : V_2 \rightarrow V_1,$ and $\gamma : V_3 \rightarrow V_3$ be some isomorphisms (so that
necessary $\dim V_1 = \dim V_2$). Then the transformation of $\tilde{V}$, defined by

$$v_1 \otimes v_2 \otimes v_3 \mapsto \beta(v_2) \otimes \alpha(v_1) \otimes \gamma(v_3),$$

is a decomposable automorphism.

The group of all decomposable automorphisms of $\tilde{V}$ will be denoted by $S(V_1, \ldots, V_l)$. Those that
preserve all factors $V_i$ form a normal subgroup, denoted by $S^0(V_1, \ldots, V_l)$. In
other words, $S^0(V_1, \ldots, V_l)$ is the group of all transformations of the form $g_1 \otimes \ldots \otimes g_l$, where
$g_i \in GL(V_i)$. The following statement holds.

Proposition 1. The group $S^0 = S^0(V_1, \ldots, V_l)$ is a central product of the groups $GL(V_i)$,
i = 1, \ldots, l. More precisely, $S^0 \cong A/B$, where $A = GL(V_1) \times \ldots \times GL(V_l)$ and $B$ is the
subgroup of all elements of the form $(\lambda_1 \text{id}_{V_1}, \ldots, \lambda_l \text{id}_{V_l})$, where $\lambda_i \in K^*$ and $\lambda_1 \ldots \lambda_l = 1$. The
quotient group $S(V_1, \ldots, V_l)/S^0$ may be naturally identified with the group of all permutations
of the set $\{V_i | \dim V_i > 1\}$, preserving dimensions.

The proof of this proposition is not difficult, but a bit tedious. It is left to the reader, or
can be found in [2, Sect.3].

Definition. Let $\tilde{V} = V_1 \otimes \ldots \otimes V_l$, and let $t \in \tilde{V}$ be an arbitrary non-zero tensor. The
groups

$$\Gamma(t) = \{ g \in S(V_1, \ldots, V_l) | g(t) = t \}$$

and

$$\Gamma^0(t) = \Gamma(t) \cap S^0(V_1, \ldots, V_l)$$

are called the isotropy group and the small isotropy group of $t$, respectively.

Let $M_{a,b}(K)$ be the space of all $a \times b$ matrices over $K$. Its usual basis is $\{e_{ij} | 1 \leq i \leq a,$ $1 \leq j \leq b\}$, where $e_{ij}$ are the usual matrix units. We will briefly denote $M_{a,b}(K)$ by $M_{ab}$,
and the group $GL_{ij}(K)$ by $GL_n$. Also we will use the notation $\mathbf{I} = \{1, \ldots, n\}$.

Let $m, n, p \in \mathbb{N}$. Consider the product $L = M_{mn} \otimes M_{np} \otimes M_{pm}$. In the theory of fast
matrix multiplication a very important role is played by the following tensor, often denoted by $(m, n, p)$:

$$\langle m, n, p \rangle = \sum_{i,j,k} e_{ij} \otimes e_{jk} \otimes e_{ki} \in L,$$
where the sum is over all \( i \in \overline{m}, \ j \in \overline{n}, \ k \in \overline{p} \).

To describe the group \( \Gamma(\langle m, n, p \rangle) \) is the main goal of the present work.

**3. A subgroup of** \( \Gamma^0 \). Let \( C_l \) and \( R_l \) denote the spaces of all columns of height \( l \), resp.
the rows of length \( l \), over \( K \), and let \((e_i \mid i \in \overline{l})\) and \((e^i \mid i \in \overline{l})\) be the usual bases in \( C_l \) and \( R_l \), respectively. If \( c \in C_l \) and \( r \in R_l \), then \( rc \) is a \( 1 \times 1 \) matrix, i.e. a scalar. The map \((r, c) \mapsto rc\) is a pairing (that is, a nondegenerate bilinear map), and \((e_i)\) and \((e^i)\) are dual bases. Thus, we can consider \( C_l \) and \( R_l \) as dual spaces.

The group \( G = GL_l \) acts on \( C_l \) as usually: \((g, v) \mapsto gv\), where \( gv \) is the usual product of a matrix by a column. Also, there is a left action of \( G \) on \( R_l \) by

\[
(g, v') \mapsto g \circ v' := v'g^{-1}.
\]

(This is a left action indeed, that is, \((gh) \circ v' = g \circ (h \circ v')\) for all \( g, h \in G \) and \( v' \in V'\). Indeed, \( g \circ (h \circ v') = g \circ (v'h^{-1}) = (v'h^{-1})g^{-1} = v'h^{-1}g^{-1} = v'(gh)^{-1} = (gh) \circ v'\). So there is a left action of \( G \) on \( C_l \otimes R_l \) such that

\[
g(v \otimes v') = gv \otimes v'g^{-1}, \quad \forall g \in G, \ v \in C_l, \ v' \in R_l.
\]

Consider the tensor

\[
\delta = \delta(l) = \sum_{i=1}^{l} e_i \otimes e^i \in C_l \otimes R_l
\]

(so-called identity tensor). The next lemma is standard; nevertheless we give a proof.

**Lemma 2.** We have \( g\delta = \delta \), for all \( g \in G \).

**Proof.** Let \( a_{ij} \) and \( b_{ij} \) be the coefficients of the matrices \( g \) and \( g^{-1} \), i.e.,

\[
g = \sum_{i,j=1}^{l} a_{ij} e_{ij} \quad \text{and} \quad g^{-1} = \sum_{i,j=1}^{l} b_{ij} e_{ij}.
\]

Then \( ge_i = \sum_{j=1}^{l} a_{ji} e_j \) and \( e^i g^{-1} = \sum_{j=1}^{l} b_{ij} e^j \). Hence

\[
ge\delta = g(\sum_{i=1}^{l} e_i \otimes e^i) = \sum_{i=1}^{l} ge_i \otimes e^i g^{-1} = \sum_{i=1}^{l} (\sum_{j=1}^{l} a_{ji} e_j) \otimes (\sum_{k=1}^{l} b_{ik} e^k)
\]

\[
= \sum_{j,k=1}^{l} (\sum_{i=1}^{l} a_{ji} b_{ik}) e_j \otimes e^k = \sum_{j,k=1}^{l} (\delta_{jk}) e_j \otimes e^k = \sum_{j=1}^{l} e_j \otimes e^j = \delta,
\]

as \( \sum_{i=1}^{l} a_{ji} b_{ik} = \delta_{jk} \) for all \( 1 \leq j, k \leq l \) (because matrices \( g \) and \( g^{-1} \) are inverse). \( \square \)

Let \( m, n, p \in \mathbb{N} \), and \( L = L_1 \otimes L_2 \otimes L_3 \), where \( L_1 = M_{mn}, \ L_2 = M_{np}, \ L_3 = M_{pm} \). Define certain action of \( G = GL_m \times GL_n \times GL_p \) on \( L \). For \((a, b, c) \in G \) define the transformation \( T(a, b, c) \) of \( L \) by the formula

\[
T(a, b, c)(x \otimes y \otimes z) = axb^{-1} \otimes byc^{-1} \otimes cz^{-1}a^{-1}.
\]

It is easy to see that the rule \( g \mapsto T(g) \) is a homomorphism from \( G \) to \( GL(L) \), that is always

\[
T(a_1, b_1, c_1)T(a_2, b_2, c_2) = T(a_1a_2, b_1b_2, c_1c_2), \quad \text{and} \quad T(E_m, E_n, E_p) = \text{id}_L.
\]
Indeed, for any \( x \otimes y \otimes z \in L \) we have

\[
T(a_1, b_1, c_1)(T(a_2, b_2, c_2)(x \otimes y \otimes z)) = T(a_1, b_1, c_1)(a_2b_2^{-1} \otimes b_2y c_2^{-1} \otimes c_2z a_2^{-1})
\]

\[
= a_1a_2b_2^{-1}b_1^{-1} \otimes b_1b_2y c_2^{-1}c_1^{-1} \otimes c_1c_2z a_2^{-1}a_1^{-1}
\]

\[
= (a_1a_2)x(b_1b_2)^{-1} \otimes (b_1b_2)y(c_1c_2)^{-1} \otimes (c_1c_2)z(a_1a_2)^{-1}
\]

\[
= T(a_1a_2, b_1b_2, c_1c_2)(x \otimes y \otimes z).
\]

**Proposition 3.** The transformations \( T(a, b, c) \) preserve \( t = (m, n, p) \).

**Proof.** Since \( g \mapsto T(g) \) is a homomorphism, and \( G = GL_m \times GL_n \times GL_p \) is a direct product, it suffices to prove that \( t \) is invariant under \( T(g) \) if \( g \) is in one of the factors \( GL_m \), \( GL_n \), or \( GL_p \). For instance, let \( g \in GL_m \), that is, more precisely, \( g = (a, E_n, E_p) \), where \( a \in GL_m \). Let \( a' = (a'_{ij}) = a^{-1} \) be the matrix inverse to \( a \). Then

\[
T(g)t = T(a, E_n, E_p) \sum_{(i, j, k) \in \pi \times \pi \times \pi} e_{ij} \otimes e_{jk} \otimes e_{ki} = \sum_{(i, j, k) \in \pi \times \pi \times \pi} ae_{ij} \otimes e_{jk} \otimes e_{ki} a'.
\]

It is sufficient to prove that for all \((j, k) \in \pi \times \pi \) the sum of all summands in \( T(g)t \) having \( e_{jk} \) in the middle coincides with the similar sum in \( t \), that is,

\[
\sum_{i=1}^{m} ae_{ij} \otimes e_{jk} \otimes e_{ki} a' = \sum_{i=1}^{m} e_{ij} \otimes e_{jk} \otimes e_{ki}.
\]

We have

\[
a = \sum_{r, s=1}^{m} a_{rs} e_{rs}, \quad a' = \sum_{r, s=1}^{m} a'_{rs} e_{rs},
\]

whence

\[
a e_{ij} = \sum_{r=1}^{m} a_{ri} e_{rj}, \quad e_{ki} a' = \sum_{s=1}^{m} a'_{is} e_{ks}.
\]

So

\[
\sum_{i=1}^{m} ae_{ij} \otimes e_{jk} \otimes e_{ki} a' = \sum_{i, r, s=1}^{m} a_{ri} a'_{is} e_{rj} \otimes e_{jk} \otimes e_{ks} = \sum_{r, s=1}^{m} \delta_{rs} e_{rj} \otimes e_{jk} \otimes e_{ks}
\]

\[
= \sum_{r=1}^{m} e_{rj} \otimes e_{jk} \otimes e_{kr} = \sum_{i=1}^{m} e_{ij} \otimes e_{jk} \otimes e_{ki},
\]

as required. Here we have used the equality \( \sum_{i=1}^{m} a_{ri} a'_{is} = \delta_{rs} \), as \( a \) and \( a' \) are inverse matrices.

(There is a less computational and more conceptual proof, which is, in brief, as follows. If \( c \in C_r \) and \( r \in R_s \), then \( cr \in M_{rs} \) is an \( r \times s \) matrix. Now we consider the product

\[
N = C_m \otimes R_m \otimes C_n \otimes R_n \otimes C_p \otimes R_p
\]

and the linear map \( \tau : N \rightarrow L \) defined by

\[
c_1 \otimes r_1 \otimes c_2 \otimes r_2 \otimes c_3 \otimes r_3 \mapsto c_1r_2 \otimes c_2r_3 \otimes c_3r_1
\]
(as the expression $c_1r_2 \otimes c_2r_3 \otimes c_3r_1$ is linear in each of the arguments $c_1, \ldots, r_3$, this is a well-defined map indeed). It is easy to see that this is an isomorphism of vector spaces.

We can consider the “identity tensors” $\delta_{(m)} \in C_m \otimes R_m$, $\delta_{(n)}$, and $\delta_{(p)}$). Then $\delta_{(m)} \otimes \delta_{(n)} \otimes \delta_{(p)}$ is an element of $N$. It is easy to see that $T(\delta_{(m)} \otimes \delta_{(n)} \otimes \delta_{(p)})$ is nothing else but $\langle m, n, p \rangle$.

There is an action of $G = GL_m \times GL_n \times GL_p$ on $N$ by the rule

$$(a, b, c)(c_1 \otimes r_1 \otimes c_2 \otimes r_2 \otimes c_3 \otimes r_3) = ac_1 \otimes r_1 a^{-1} \otimes bc_2 \otimes r_2 b^{-1} \otimes cc_3 \otimes r_3 c^{-1}.$$ 

It is possible to check that $\tau$ is a $G$-homomorphism, with respect to this action. Next, it can be deduced from Lemma 2 that $\delta_{(m)} \otimes \delta_{(n)} \otimes \delta_{(p)}$ is an invariant element. So its image $\langle m, n, p \rangle$ is an invariant element too.

Thus, we have the following statement.

**Proposition 4.** The group $H$ of all transformations of the form $T(a, b, c)$ is a subgroup of $\Gamma^0(t)$.

The inverse inclusion is also true, but to prove it is more difficult. We prove it later in this article.

**Proposition 5.** Any element of $\Gamma^0(t)$ has the form $T(a, b, c)$, for some $(a, b, c) \in GL_m \times GL_n \times GL_p$, and therefore $\Gamma^0(t) = H$.

The representation of an element of $H$ in the form $T(g)$, $g = (a, b, c)$, is not unique. Evidently, $T(g_1) = T(g_2)$ if and only if $T(g_2 g_1^{-1}) = 1$ ($= id_L$; we will sometimes write 1 for identity map, or the identity element of a group).

Let us find out when $T(h) = 1$. We need two auxiliary statements, whose proofs are left to the reader.

1. If $\tilde{V} = V_1 \otimes \ldots \otimes V_l$ is any tensor product, then two nonzero decomposable tensors $u_1 \otimes \ldots \otimes u_l$ and $v_1 \otimes \ldots \otimes v_l$ coincide if and only if $v_i = \lambda_i u_i$, $\lambda_1 \ldots \lambda_l = 1$.

2. Let $a \in GL_m$, $b \in GL_n$, and suppose that $axb$ is proportional to $x$ for all $x \in M_{mn}$. Then both $a$ and $b$ are scalar matrices.

**Proposition 6.** $T(h) = 1$ if and only if $h = (\lambda E_m, \mu E_n, \nu E_p)$, where $\lambda, \mu, \nu \in K^*$.

**Proof.** If $h$ is of this form, then

$$T(h)(x \otimes y \otimes z) = \lambda x \mu^{-1} \otimes \mu y \nu^{-1} \otimes \nu z \lambda^{-1} = x \otimes y \otimes z,$$

for all $x$, $y$, and $z$, whence $T(h) = 1$.

Conversely, let $h = (a, b, c)$ and $T(h) = 1$. Then $axb^{-1} \otimes byc^{-1} \otimes cza^{-1} = x \otimes y \otimes z$, for all $x \in L_1, \ y \in L_2, \ z \in L_3$. It follows from (1) that $axb^{-1} \sim x$, for all $x$. Now (2) implies that both $a$ and $b$ are scalar matrices. Similarly $c$ is a scalar matrix also.

**Corollary 7.** $T(a, b, c_1) = T(a, b, c_2)$ if and only if $a_2 = \lambda a_1$, $b_2 = \mu b_1$, $c_2 = \nu c_1$, for some $\lambda, \mu, \nu \in K^*$.

Now we can describe the structure of $\Gamma^0(t)$ as an abstract group. Recall that the projective general linear group is $PGL_n(K) = GL_n(K)/Z_n(K)$, where $Z_n(K) = \{ \lambda E_n \mid \lambda \in K^* \}$ is the group of all scalar matrices of size $n$.

**Proposition 8.** $\Gamma^0(t) \cong PGL_m(K) \times PGL_n(K) \times PGL_p(K)$.

**Proof.** We have $\Gamma^0(t) = H$ by Proposition 5. The map $g \mapsto T(g)$ is a surjective homomorphism of the group $G = GL_m \times GL_n \times GL_p$ onto $H$. So $H$ is isomorphic to the quotient group $H/B$, where $B = \{ h \mid T(h) = 1 \}$. It follows from Proposition 6 that $B = Z_m(K) \times Z_n(K) \times Z_p(K)$. Finally, it is easy to see that the quotient group of $G$ by the latter subgroup is isomorphic to $PGL_m(K) \times PGL_n(K) \times PGL_p(K)$. □
4. Structure of $\Gamma(t)$. The full isotropy group $\Gamma(t)$, where $t = \langle m, n, p \rangle$, may be larger than $\Gamma^0(t)$. However, the relations between $\Gamma(t)$ and $\Gamma^0(t)$ can be easily described.

In this section we assume that at most one of the three numbers $m$, $n$, and $p$ is equal to 1. Then $mn, np, pm > 1$.

First assume that $m$, $n$, and $p$ are pairwise distinct. Then $\dim L_1 = mn$, $\dim L_2 = np$, and $\dim L_3 = mp$ are pairwise distinct also. So $S(L_1, L_2, L_3) = S^0(L_1, L_2, L_3)$, whence $\Gamma(t) = \Gamma^0(t)$.

Next assume that $|\{m, n, p\}| = 2$. We consider the case $m = n \neq p$ only; the remaining two cases $m = p \neq n$ and $m \neq n = p$ can be obtained from this case in an obvious way. Define $\rho_{(23)} : L \to L$ for the tensor

$$\rho_{(23)}(x \otimes y \otimes z) = x^t \otimes z^t \otimes y^t$$

we use the same symbol $t$ for the tensor $t = \langle m, n, p \rangle$ and the transpose map, but hope this will not lead to a confusion). Note that $\rho_{(23)}$ is well-defined, because operation of taking the transpose matrix maps the spaces $L_2 = M_{np} = M_{mp}$ and $L_3 = M_{pm}$ each onto the other, and $L_1 = M_{mn}$ onto itself. Observe next that $\rho_{(23)}^2 = 1 = (\text{id}_L)$, as

$$\rho_{(23)}^2(x \otimes y \otimes z) = \rho_{(23)}(\rho_{(23)}(x \otimes y \otimes z)) = \rho_{(23)}(x^t \otimes z^t \otimes y^t) = (x^t)^t \otimes (y^t)^t \otimes (z^t)^t = x \otimes y \otimes z.$$ 

Finally, we have $\rho_{(23)} \in \Gamma(t)$, because

$$\rho_{(23)}(t) = \rho_{(23)} \left( \sum_{1 \leq i, j \leq m} e_{ij} \otimes e_{jk} \otimes e_{ki} \right) = \sum_{1 \leq i, j \leq m} e_{ji} \otimes e_{ik} \otimes e_{kj} = t.$$ 

To formulate the statement on the structure of $\Gamma(t)$, it will be convenient to use the notion of semidirect product. Recall that a group $G$ is the product of its subgroups $A$ and $B$, which is denoted by $G = AB$, if for each $g \in G$ there exist $a \in A$ and $b \in B$ such that $g = ab$. If in addition $A \cap B = 1$, then it is easy to see that the representation of $g$ in the form $g = ab$ is unique. Finally, a group $G$ is said to be a semidirect product of $A$ by $B$, which is denoted by $G = A \rtimes B$, if $G = AB$, $A$ is normal in $G$, and $A \cap B = 1$.

Let $Q = \{1, \rho_{(23)}\}$ be the subgroup of $\Gamma(t)$ of order 2 generated by $\rho_{(23)}$. Show that $\Gamma(t) = \Gamma^0(t) \ltimes Q$. We have $\Gamma^0(t) \leq \Gamma(t)$ from the definition of $\Gamma^0(t)$, because $S^0(V_1, \ldots, V_l) \leq S(V_1, \ldots, V_l)$. Next, $Q \cap \Gamma^0(t) = 1$, because $\rho_{(23)}$ induces a nontrivial permutation of factors. Finally $\Gamma(t) = \Gamma^0(t)Q$. Indeed, let $x \in \Gamma(t)$. Then the permutation $\pi$, induced by $x$ on the factors $\{L_1, L_2, L_3\}$, preserves the dimensions, whence $\pi = 1$ or $\pi = (23)$. If $\pi = 1$, then $x \in \Gamma^0(t)$. If $\pi = (23)$, then the element $x' = x\rho_{(23)}$ is in $\Gamma(t)$ and induces the trivial permutation, whence $x' \in \Gamma^0(t)$ and $x = x'\rho_{(23)} \in \Gamma^0(t)Q$.

It remains to consider the case $m = n = p$. Define

$$\rho_{(12)} : x \otimes y \otimes z \mapsto y^t \otimes x^t \otimes z^t.$$ 

Then $\rho_{(12)}^2 = 1$ and $\rho_{(12)} \in \Gamma(t)$ similarly to $\rho_{(23)}$. Consider the group $Q = \langle \rho_{(12)}, \rho_{(23)} \rangle$. A direct checking, left to the reader, shows that $Q \cong S_3$ and that any permutation of the factors $L_1, L_2$, and $L_3$ is induced by a unique element of $Q$. It follows, quite similarly to the case $m = n \neq p$, that $\Gamma(t) = \Gamma^0(t) \ltimes Q$. 

It may be useful to have explicit formulae for conjugation of an element of $H$ by an element of $Q$. For a matrix $x \in GL_l(K)$ we denote by $x^\vee$ the matrix $x^\vee = (x^t)^{-1} = (x^{-1})^t$ (which is usually called the matrix cotragradient to $x$).

**Proposition 9.** If $m = n$, then

$$\rho_{(23)} T(a, b, c) \rho_{(23)} = T(b^\vee, a^\vee, c^\vee).$$

If $m = n = p$, and for a permutation $\pi \in S_3$, $\rho_\pi$ is the element of $Q = \langle \rho_{(23)}, \rho_{(12)} \rangle$ inducing this permutation on $\{L_1, L_2, L_3\}$, then in addition the following relations hold:

$$\rho_{(12)} T(a, b, c) \rho_{(12)} = T(c^\vee, b^\vee, a^\vee),$$

$$\rho_{(13)} T(a, b, c) \rho_{(13)} = T(a^\vee, c^\vee, b^\vee),$$

$$\rho_{(123)} T(a, b, c) \rho_{(123)}^{-1} = T(c, a, b),$$

$$\rho_{(132)} T(a, b, c) \rho_{(132)}^{-1} = T(b, c, a).$$

**Proof.** We prove the relation for $\rho_{(12)}$ as an example. Note that $\rho_{(12)}^{-1} = \rho_{(12)}$, as $\rho_{(12)}^2 = 1$. For $x \in L_1$, $y \in L_2$, and $z \in L_3$ we have $\rho_{(12)} (x \otimes y \otimes z) = y^t \otimes x^t \otimes z^t$, whence

$$x \otimes y \otimes z \xrightarrow{\rho_{(12)}} y^t \otimes x^t \otimes z^t \xrightarrow{T(a,b,c)} ay^t b^{-1} \otimes bx^t c^{-1} \otimes cz^{t}a^{-1}$$

$$\xrightarrow{\rho_{(12)}} (bx^t c^{-1} b^{-1} y^t) \otimes (cz^{t}a^{-1} c^{-1} z^{t}a^{-1} c^{-1} z^{t}a^{-1}) = (c^{-1})^t x b^t \otimes (b^{-1})^t y a^t \otimes (a^{-1})^t z c^t$$

$$= c^\vee x(b^\vee)^{-1} \otimes b^\vee y(a^\vee)^{-1} \otimes a^\vee z(c^\vee)^{-1} = T(c^\vee, b^\vee, a^\vee)(x \otimes y \otimes z),$$

whence

$$\rho_{(12)} T(a, b, c) \rho_{(12)} = T(c^\vee, b^\vee, a^\vee).$$

$\square$

We summarize the statements obtained so far in the next theorem, which is the main result of the present work.

**Theorem.** Let $m, n, p \in \mathbb{N}$, $(m, n, p) \neq (1, 1, 1)$, let $L_1 = M_{mn} = M_{mn}(K)$, $L_2 = M_{np}$, $L_3 = M_{pm}$, let $L = L_1 \otimes L_2 \otimes L_3$, and let

$$t = \sum_{1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq p} e_{ij} \otimes e_{jk} \otimes e_{ki} \in L.$$

For elements $a \in GL_m(K)$, $b \in GL_n(K)$, $c \in GL_p(K)$ define the transformation $T(a, b, c): L \longrightarrow L$ by the formula

$$T(a, b, c)(x \otimes y \otimes z) = axb^{-1} \otimes byc^{-1} \otimes cz^{-1}a^{-1}.$$

Put

$$H = \{ T(a, b, c) \mid (a, b, c) \in GL_m(K) \times GL_n(K) \times GL_p(K) \}.$$

Then $\Gamma(t) = H$. The transformations $T(a, b, c)$ and $T(a_1, b_1, c_1)$ are equal if and only if $a_1 = \lambda a$, $b_1 = \mu b$, and $c_1 = \nu c$, for some $\lambda, \mu, \nu \in K^*$. The group $H$ is isomorphic to $PGL_m(K) \times PGL_n(K) \times PGL_p(K)$.

If $m$, $n$, and $p$ are pairwise distinct, then $\Gamma(t) = \Gamma^0(t)$. 
If \( m = n \) or \( m = p \), then we define the transformations \( \rho_{(23)}, \rho_{(12)} : L \rightarrow L \) by
\[
\rho_{(23)}(x \otimes y \otimes z) = x^t \otimes z^t \otimes y^t,
\rho_{(12)}(x \otimes y \otimes z) = y^t \otimes x^t \otimes z^t,
\]
respectively. In the case \( m = n \neq p \) put \( Q = \langle \rho_{(23)} \rangle \), and in the case \( m = n = p \) put \( Q = \langle \rho_{(12)}, \rho_{(23)} \rangle \). Then \( Q \cong \mathbb{Z}_2 \) or \( Q \cong S_3 \) in the first and second case, respectively. \( Q \) is a subgroup of \( \Gamma(t) \), and \( \Gamma(t) = \Gamma^0(t) \setminus Q \). Any permutation of the factors \( L_1, L_2, L_3 \), preserving the dimensions, is induced by a unique element of \( Q \).

In the case \( m = n \) the relation
\[
\rho_{(23)}T(a, b, c)\rho_{(23)} = T(b^\vee, a^\vee, c^\vee),
\]
holds, and in the case \( m = n = p \) the relations
\[
\rho_{(12)}T(a, b, c)\rho_{(12)} = T(c^\vee, b^\vee, a^\vee),
\rho_{(13)}T(a, b, c)\rho_{(13)} = T(a^\vee, c^\vee, b^\vee),
\rho_{(123)}T(a, b, c)\rho_{(123)}^{-1} = T(c, a, b),
\rho_{(132)}T(a, b, c)\rho_{(132)}^{-1} = T(b, c, a)
\]
holds also. Here \( \rho_\pi \) is the element of \( Q \) inducing the permutation \( \pi \in S_3 \) on the factors.

In the cases \( m = p \neq n \) and \( n = p \neq m \) the statements similar to those for the case \( m = n \neq p \) are true.

The rest of the article is devoted to the proof of Proposition 5.

5. Transformations of matrix spaces. There is a well-known theorem stating that any automorphism of the algebra of square matrices over a field is induced by the conjugation by a nondegenerate matrix. In this section we prove an analogue of this theorem for the multiplication of rectangular matrices.

By \( R_n \) and \( C_n \) we denote the spaces of all rows of length \( n \), respectively the columns of height \( n \), over a given field \( K \); i.e. \( R_n(K) = M_{1n}(K) \) and \( C_n = M_{n1}(K) \). Note that if \( c \in C_m \) and \( r \in R_n \), then \( cr \in M_{mn} \). Moreover, the rule \( c \otimes r \mapsto cr \) defines an isomorphism \( C_m \otimes R_n \rightarrow M_{mn} \). Also, observe that \( \operatorname{rk}(X) = 1 \) if and only if \( X = cr \) for some \( C \in C_m \) and \( r \in R_n \). Here \( \operatorname{rk} \) is the rank of a matrix. Finally note that for any \( c \in C_l \) and \( r \in R_l \) the product \( rc \) is a \( 1 \times 1 \) matrix, that is, a scalar.

Recall that a (non-zero) element of a tensor product \( \bar{V} = V_1 \otimes \ldots \otimes V_l \) of the form \( v_1 \otimes \ldots \otimes v_l \) is called a decomposable tensor. It is clear that if \( \bar{U} = U_1 \otimes \ldots \otimes U_l, \bar{V} = V_1 \otimes \ldots \otimes V_l \), and \( \varphi : \bar{U} \rightarrow \bar{V} \) is a decomposable isomorphism, then \( \varphi \) takes decomposable tensors to decomposable ones. The converse is also true:

**Proposition 10.** Let \( \bar{U} = U_1 \otimes \ldots \otimes U_l, \bar{V} = V_1 \otimes \ldots \otimes V_l \), \( X \) and \( Y \) be the sets of all decomposable tensors in \( \bar{U} \) and \( \bar{V} \) respectively, and \( \varphi : \bar{U} \rightarrow \bar{V} \) be an isomorphism of linear spaces that bijectively maps \( X \) onto \( Y \). Then \( \varphi \) is a decomposable isomorphism.

A proof of this proposition is contained in [1]. It is not difficult. Another proof can be found in [7]. Anyway, this statement, no doubt, is an old result of the classical projective algebraic geometry (“automorphisms of Segre embeddings”). (However, the author could
not find it, in an explicit form, in the available textbooks. In [7] the book [14] is mentioned, which is not available to the author.) So, we will not prove this statement here.

Proposition 11. Let $A$ be an invertible linear transformation of the space $M_{mn}$, having the property that $\text{rk}(Ax) = 1$ for all $x$ such that $\text{rk}(x) = 1$. Then either there exist $a \in GL_m$ and $b \in GL_n$ such that $Ax = axb$ for all $x \in M_{mn}$, or $m = n$ and there exist $a, b \in GL_n$ such that $Ax = axb$ for all $x$.

Proof. Let $\varphi : C_m \otimes R_n \rightarrow M_{mn}$, $\varphi(c \otimes r) = cr$ be the isomorphism described above. Then $\varphi$ maps bijectively the set of decomposable tensors in $C_m \otimes R_n$ onto the set of all rank 1 matrices. Consider $A' = \varphi^{-1}A\varphi$. Then $A'$ is an automorphism of the linear space $C_m \otimes R_n$, taking decomposable tensors to decomposable ones. So $A'$ is a decomposable automorphism by Proposition 10.

Any decomposable automorphism of $C_m \otimes R_n$ is either of the form $c \otimes r \mapsto ac \otimes rb$, for some $a \in GL_m$ and $b \in GL_n$; or $m = n$, and the automorphism has the form $c \otimes r \mapsto ar^t \otimes c^t b$, where $a, b \in GL_m$. In the first case we have for matrices of the form $cr$

$$A(cr) = (\varphi A' \varphi^{-1})(cr) = (\varphi A')(c \otimes r) = \varphi(A'(c \otimes r)) = \varphi(ac \otimes rb) = acrb,$$

and therefore $Ax = axb$ for all $x$, because $x$ is a linear combination of rank 1 matrices. In the second case

$$A(cr) = \varphi(A'(c \otimes r)) = \varphi(ar^t \otimes c^t b) = ar^t c^t b = a(c r^t) b,$$

whence again $Ax = ax^t b$ for all $x$. \qed

Let $m, n, p \in \mathbb{N}$. Consider matrix multiplication $M_{mn} \times M_{np} \rightarrow M_{mp}$. In particular, for any $x \in M_{mn}$ we can consider the subspace

$$xM_{np} = \{xy \mid y \in M_{np}\} \subseteq M_{mp}.$$

Lemma 12. $\dim xM_{np} = p \cdot \text{rk}(x)$ and $\dim M_{mn}x' = m \cdot \text{rk}(x')$ for all $x \in M_{mn}$ and $x' \in M_{np}$.

Proof. Prove the first equality; the second can be considered similarly. Let $r = \text{rk}(x)$. There exist nondegenerate matrices $a \in GL_m$ and $b \in GL_n$ such that $x = aE_r b$, where $E_r = \sum_{i=1}^{r} e_{ii}$. Now $xM_{np} = aE_r b \cdot M_{np}$. As the left multiplication by $a$ is an invertible linear transfromation on $M_{mp}$, we have

$$\dim aE_r b \cdot M_{np} = \dim E_r b \cdot M_{np}.$$

Moreover, $bM_{np} = M_{np}$, so $\dim xM_{np} = \dim E_r M_{np}$. But the space $E_r M_{np}$ is the space of all $m \times p$ matrices that have zero $j$-th rows for all $j \geq r + 1$. Hence $\dim E_r M_{np} = rp = p \cdot \text{rk}(x)$. \qed

Proposition 13. Let $A, B,$ and $C$ be linear transformations of matrix spaces $M_{mn}$, $M_{np}$, and $M_{mp}$, such that $A(x)B(y) = C(xy)$ for all $x \in M_{mn}$ and $y \in M_{np}$. Then there exist $a \in GL_m$, $b \in GL_n$, and $c \in GL_p$ such that $A(x) = axb$, $B(y) = b^{-1}yc$, $C(z) = azc$.

Proof. For an arbitrary element $x \in M_{mn}$ and a subspace $Y \subseteq M_{np}$ put

$$xy = \{xy \mid y \in Y\}.$$
Also, for any two subspaces \( X \subseteq M_{mn} \) and \( Y \subseteq M_{np} \) define

\[
XY = \{ xy \mid x \in X, \ y \in Y \}.
\]

It is easy to deduce from the hypothesis that always

\[
A(x)B(Y) = C(xy), \quad A(X)B(Y) = C(XY).
\]

In particular, let \( x \in M_{mn} \) be an arbitrary element, and \( Y = M_{np} \). Then \( A(x)B(Y) = A(x)M_{np} \), and therefore \( A(x)M_{np} = C(xM_{np}) \). Hence \( \dim A(x)M_{np} = \dim xM_{np} \), whence \( p \cdot \text{rk}(A(x)) = p \cdot \text{rk}(x) \) by Lemma 12 and therefore \( \text{rk}(A(x)) = \text{rk}(x) \). Thus, \( A \) preserves the rank.

By Proposition 11, either there exist \( a \in GL_m \) and \( b \in GL_n \) such that \( A(x) = axb \), or \( m = n \) and \( A(x) = ax^t b \), for all \( x \in M_{mn} \).

Admit the second possibility for \( A \), and get a contradiction. We may assume that \( m = n \geq 2 \). Take \( X = e_1R_n \) and \( Y = M_{np} \). Then \( XY = e_1R_nM_{np} = e_1R_p \), whence \( \dim XY = p \). On the other hand, \( A(X) = a(e_1R_n)^t b = aC_n e^1 b \), and further

\[
A(X)B(Y) = A(X)M_{np} = (aC_n e^1 b)(M_{np}) = (aC_n e^1)(bM_{np}).
\]

As \( b \in GL_n \), we have \( bM_{np} = M_{np} \). It is also clear that \( e^1 M_{np} = R_p \). So \( A(X)B(Y) = aC_n R_p = a \cdot M_{np} = M_{np} \); in particular \( \dim A(X)B(Y) = np \). So \( \dim A(X)B(Y) \neq \dim XY = \dim C(XY) \), a contradiction.

Thus, \( A(x) = axb \). The reader can prove similarly that \( B(y) = b_1yc_1 \), for some \( b_1 \in GL_n \) and \( c_1 \in GL_p \).

Show that \( b_1 \) is proportional to \( b^{-1} \). For any \( x \in M_{mn} \), \( y \in M_{np} \) we have

\[
C(xy) = A(x)B(y) = axb \cdot b_1 yc_1,
\]

whence for any \( d \in GL_n \)

\[
C(xy) = C(xd \cdot d^{-1} y) = axdb \cdot b_1 d^{-1} yc_1.
\]

So \( axbb_1 yc_1 = axdbb_1 d^{-1} yc_1 \). As both \( a \) and \( c_1 \) are invertible, the latter equality implies \( xb_1 y = xdbb_1 d^{-1} y \). Since this equality holds for all \( x \in M_{mn} \) and \( y \in M_{np} \), it follows that \( bb_1 = dbb_1 d^{-1} \), for all \( d \in GL_p \). That is, \( bb_1 \) commutes with all elements of \( GL_p \) and so is a scalar matrix, \( bb_1 = \lambda E_p \), \( \lambda \in K^* \). That is, \( b_1 = \lambda b^{-1} \).

Hence \( B(y) = b_1 yc_1 = b^{-1} yc, \) where \( c = \lambda^{-1} c_1 \).

Thus, \( A \) and \( B \) can be defined by formulae \( A(x) = axb \), \( B(x) = b^{-1} yc \). So \( C(xy) = A(x)B(y) = axyc, \) for all \( x \in M_{mn}, \ y \in M_{np} \). As \( M_{mn} M_{np} = M_{mp} \), we see that \( C(z) = azc \), for all \( z \in M_{np} \).

6. Structure tensors and contragradient maps. In this section we recall, briefly and without proofs, some well-known concepts.

By \( V^* \) we denote dual space of \( V \), as usually. As only finite-dimensional spaces are considered, we identify \( (V^*)^* \) with \( V \).

For two elements \( v \in V \) and \( l \in V^* \) it will be convenient to denote \( l(v) \) either by \( \langle l, v \rangle \) or by \( \langle v, l \rangle \). Thus, the element \( \langle u_1, u_2 \rangle \) is defined, if one of the elements \( u_1 \) and \( u_2 \) is in \( V \).
the other is in \( V^* \); and we always have \( \langle u_1, u_2 \rangle = \langle u_2, u_1 \rangle \). The symbol \( \langle u_1, u_2 \rangle \) is called the pairing of \( u_1 \) and \( u_2 \).

For any linear map \( f : X \longrightarrow Y \) there exists a unique linear map \( f^* : Y^* \longrightarrow X^* \), called the dual map, such that \( (l, f(x)) = (f^*(l), x) \) for all \( x \in X \) and \( l \in Y^* \).

If \( f \) is an isomorphism, then \( f^* \) is an isomorphism also, and \( f^\dual = (f^*)^{-1} : X^* \longrightarrow Y^* \) is called the map, contragradient to \( f \). This is the unique map \( X^* \longrightarrow Y^* \) satisfying the condition \( (x, l) = \langle f(x), f^*(l) \rangle \) for all \( x \in X, l \in X^* \).

If \( f : X \longrightarrow Y \) and \( g : Y \longrightarrow Z \) are linear maps, then \( (gf)^\dual = f^\dual g^\dual \). If \( f \) and \( g \) are isomorphisms, then \( (gf)^\dual = g^\dual f^\dual \). Also, \( (f^\dual)^\dual = f \).

In particular, suppose that \( \varphi : G \longrightarrow GL(X) \) is a representation of a group \( G \) on a space \( X \). Then the map \( \varphi^* : G \longrightarrow GL(X^*) \), defined by \( \varphi^*(g) = \varphi(g)^\dual \), is a representation also, called a representation contragradient (or more often dual) to \( \varphi \).

Let \( X, Y, Z \) be spaces. By \( \mathcal{L}(X, Y) \) we denote the space of all linear maps from \( X \) to \( Y \), and by \( \mathcal{L}_2(X, Y; Z) \) the space of all bilinear maps \( f : X \times Y \longrightarrow Z \). The spaces \( \mathcal{L}(X, Y) \) and \( \mathcal{L}_2(X, Y; Z) \) may be identified, in a canonical way, with \( X^* \otimes Y \) and \( X^* \otimes Y^* \otimes Z \), respectively (see [11], §4.2). Describe this identification. Let \( l \in X^* \) and \( y \in Y \). Consider the map \( \varphi_{l,y} : X \longrightarrow Y \), defined by

\[
\varphi_{l,y}(x) = l(x)y.
\]

Clearly, \( \varphi_{l,y} \) is a linear map. Furthermore, the expression \( l(x)y \) is linear in all three arguments \( l, x, \) and \( y \), and therefore the rule \( (l, y) \mapsto \varphi_{l,y} \) defines a bilinear map from \( X^* \times Y \) to \( \mathcal{L}(X, Y) \). By the universal property of tensor product there exists a unique linear map \( \varphi : X^* \otimes Y \longrightarrow \mathcal{L}(X, Y) \) such that \( \varphi(l \otimes y) = \varphi_{l,y} \) for all \( l \) and \( y \). It can be shown (see [11]) that \( \varphi \) is an isomorphism.

We can define the isomorphism \( \varphi : X^* \otimes Y^* \otimes Z \longrightarrow \mathcal{L}_2(X, Y; Z) \) in a similar way. Namely, \( \varphi \) is the unique linear map such that

\[
(\varphi(l \otimes m \otimes z))(x, y) = l(x)m(y)z \quad \forall x \in X, \ y \in Y, \ z \in Z, \ l \in X^*, \ m \in Y^*
\]

(the details are left to the reader).

Let \( f \in \mathcal{L}(X, Y) \) (resp., \( f \in \mathcal{L}_2(X, Y; Z) \)), and let \( h \in X^* \otimes Y \) (resp., \( h \in X^* \otimes Y^* \otimes Z \)) be the tensor such that \( \varphi(h) = f \). This \( h \) is called the structure tensor of \( f \), and will be denoted by \( \widetilde{f} \).

Consider the group \( G = GL(X) \times GL(Y) \). It acts on the spaces \( X^* \otimes Y \) and \( \mathcal{L}(X, Y) \) as usual. That is, an element \( g = (g_1, g_2) \in G \) acts on \( X^* \otimes Y \) as \( g_1^\dual \otimes g_2 \), and the action of \( g \) on \( \mathcal{L}(X, Y) \) is defined by \( g(f) = g_2 f g_1^{-1} \) (we leave to the reader to show that this is indeed a left action). Similarly, the group \( G = GL(X) \times GL(Y) \times GL(Z) \) acts on \( X^* \otimes Y^* \otimes Z \) and on \( \mathcal{L}_2(X, Y; Z) \). The element \( g = (g_1, g_2, g_3) \in G \) acts on \( X^* \otimes Y^* \otimes Z \) as \( g_1^\dual \otimes g_2^\dual \otimes g_3 \), and the action on \( \mathcal{L}_2(X, Y; Z) \) is described by the rule

\[
(g(f))(x, y) = g_3(f(g_1^{-1} x, g_2^{-1} y))
\]

(i.e., \( g \) takes \( f \) to the map \( f_1 \) defined by \( f_1(x, y) = g_3(f(g_1^{-1} x, g_2^{-1} y)) \); we may also write this as \( g(f) = g_3 \circ f \circ (g_1^{-1} \times g_2^{-1}) \)).

The following proposition is well known.

**Proposition 14.** Let \( G = GL(X) \times GL(Y) \) (resp. \( G = GL(X) \times GL(Y) \times GL(Z) \)), and let \( \varphi : X^* \otimes Y \longrightarrow \mathcal{L}(X, Y) \) (resp. \( \varphi : X^* \otimes Y^* \otimes Z \longrightarrow \mathcal{L}_2(X, Y; Z) \)) be the canonical isomorphism. Then \( \varphi \) is an isomorphism of \( KG \)-modules.
7. The isotropy group of a bilinear map. Let \( X, Y, \) and \( Z \) be vector spaces and let \( f \in \mathcal{L}_2(X,Y;Z) \) be a bilinear map. The group \( G = GL(X) \times GL(Y) \times GL(Z) \) acts on \( \mathcal{L}_2(X,Y;Z) \) in the way described in the end of the previous section. The stabilizer of \( f \) in \( G \) with respect to this action will be called the *isotropy group* of \( f \), and will be denoted by \( \Delta(f) \). The reader can easily check that this definition is equivalent to the following: \( \Delta(f) \) is the set of all triples \( (A,B,C) \in G \) such that \( f(Ax,By) = Cf(x,y) \) for all \( x \in X \) and \( y \in Y \). In other words, the diagram

\[
\begin{array}{ccc}
X \times Y & \xrightarrow{f} & Z \\
A \times B \downarrow & & \downarrow C \\
X \times Y & \xrightarrow{f} & Z 
\end{array}
\]

must commute.

**Example.** Let \( U, V, \) and \( W \) be three spaces, let \( X = \mathcal{L}(U,V), Y = \mathcal{L}(V,W), Z = \mathcal{L}(U,W) \), and let \( f : X \times Y \to Z \) be the usual composition of mappings, i.e., \( f(x,y) = xy \). Clearly \( f \) is bilinear. For \( g = (g_1,g_2,g_3) \in GL(U) \times GL(V) \times GL(W) \) put \( R(g) = (A,B,C) \), where \( A \in GL(X), B \in GL(Y) \) and \( C \in GL(Z) \) are defined by \( Ax = g_2xg_1^{-1}, By = g_3yg_2^{-1} \), and \( Cz = g_3zg_1^{-1} \), respectively. Then it is easy to see that \( R(g) \in \Delta(f) \) if and only if \( g \) fixes \( f \). But \( g(f) = (A^{-1} \otimes B^{-1} \otimes C) \overline{f} \) by the definition of the action of \( G \) on \( X^\ast \otimes Y^\ast \otimes Z \).

**Proposition 15.** Let \( f : X \times Y \to Z \) be a bilinear mapping and let \( \overline{f} \in X^\ast \otimes Y^\ast \otimes Z \) be its structure tensor. Let \( (A,B,C) \in G = GL(X) \times GL(Y) \times GL(Z) \). Then \( (A,B,C) \in \Delta(f) \) if and only if \( A^\ast \otimes B^\ast \otimes C \in \Gamma^0(\overline{f}) \).

**Proof.** By Proposition 14, the map \( h \mapsto \overline{h} \) is a \( G \)-isomorphism from \( \mathcal{L}_2(X,Y;Z) \) to \( X^\ast \otimes Y^\ast \otimes Z \). So \( g = (A,B,C) \in G \) is in \( \Delta(f) \) if and only if \( g \) fixes \( f \). But \( g(f) = (A^\ast \otimes B^\ast \otimes C) \overline{f} \) by the definition of the action of the group \( G \) on \( X^\ast \otimes Y^\ast \otimes Z \).

8. Proof of Proposition 5. We start with the following observation. Let \( x \) and \( y \) be \( a \times b \) and \( b \times a \) matrices, respectively. Then \( \text{Tr}(xy) = \text{Tr}(yx) \). Moreover,

\[
\langle x, y \rangle = \text{Tr}(xy) = \text{Tr}(yx)
\]

is a nondegenerate bilinear pairing between \( M_{ab} \) and \( M_{ba} \). Therefore we may identify \( M^\ast_{ab} \) with \( M_{ba} \), and \( M^\ast_{ab} \) with \( M_{ab} \).

Further, the group \( G = GL_a \times GL_b \) acts on both \( M_{ab} \) and \( M_{ba} \) in a usual way, that is, \( g = (g_1,g_2) \) takes \( x \in M_{ab} \) and \( y \in M_{ba} \) to \( g_1xg_2^{-1} \) and \( g_2yg_1^{-1} \), respectively. The pairing is invariant under this action. Indeed, if \( x \in M_{ab} \), \( y \in M_{ba} \), and \( g = (g_1,g_2) \in G \), then

\[
\langle gx, gy \rangle = \text{Tr}((g_1xg_2^{-1})(g_2yg_1^{-1})) = \text{Tr}(g_1xg_1^{-1}) = \text{Tr}(xy) = \langle x, y \rangle.
\]

Therefore the transformations, induced by \( g \) on \( M_{ab} \) and \( M_{ba} \), are contragradient each to the other.

Let \( L_1 = M_{mn}, L_2 = M_{np} \) and \( L_3 = M_{pm} \) be as in the hypothesis of the Proposition, and let \( N_1 = M_{nm} \) and \( N_2 = M_{pn} \). Then \( N_i \) is dual to \( L_i, i = 1, 2 \). Let \( \varphi : N_1 \times N_2 \to L_3 \) be the usual product map, that is, \( \varphi(x,y) = xy \). Its structure tensor \( \overline{\varphi} \in N^\ast_1 \otimes N^\ast_2 \otimes L_3 \) may be considered as an element of \( L_1 \otimes L_2 \otimes L_3 \). We show that \( \overline{\varphi} = t = \langle m, n, p \rangle \).
Indeed, we have
\[ t = \sum_{(i,j,k)\in \overline{m}\times \overline{n}\times \overline{p}} e_{ij} \otimes e_{jk} \otimes e_{ki}. \]

Let
\[ \psi : L_1 \otimes L_2 \otimes L_3 = N_1^* \otimes N_2^* \otimes L_3 \rightarrow \mathcal{L}_2(N_1, N_2; L_3) \]
be the canonical map, described in Section 6 (denoted by \( \varphi \) there). We must show that the bilinear map \( \rho = \psi(t) \) coincides with \( \varphi \). The bases of \( N_1 \) and \( N_2 \) are \( \{e_{uv} \mid u \in \overline{m}, \ v \in \overline{n}\} \) and \( \{e_{wq} \mid w \in \overline{p}, \ q \in \overline{p}\} \), respectively. It follows from the definition of \( \psi \) that the value of \( \rho \) on the pair \((e_{uv}, e_{wq})\) equals
\[ \sum_{i,j,k} \langle e_{ij}, e_{uv} \rangle \langle e_{jk}, e_{wq} \rangle e_{ki} = \sum_{i,j,k} \text{Tr}(e_{ij} e_{uv}) \text{Tr}(e_{jk} e_{wq}) e_{ki} = \sum_{i,j,k} \delta_{iv} \delta_{ju} \delta_{kw} \delta_{kJ} e_{ki} = \delta_{uv} \delta_{wq}, \]
where the sum is taken over all \((i,j,k)\in \overline{m}\times \overline{n}\times \overline{p}\).

On the other hand, \( \varphi(e_{uv}, e_{wq}) = e_{uv} e_{wq} = \delta_{uv} \delta_{wq}. \) Thus, \( \rho(e_{uv}, e_{wq}) = \varphi(e_{uv}, e_{wq}) \) for all \( u, v, w, q \), that is, \( \varphi = \rho \). Thus, \( t = \tilde{\varphi}. \)

It follows from the discussion in the beginning of the proof that the transformation on \( L_1 \), contragradient to the transformation \( x \mapsto b_1^{-1} x c_1 \) on \( N_1 \), may be described by the formula \( x' \mapsto c_1^{-1} x' b_1 \). Similarly \( A_2 \) acts by the rule \( y' \mapsto b_1^{-1} y' a_1^{-1} \). Therefore \( A \) acts by
\[ A(x \otimes y \otimes z) = c_1^{-1} x b_1 \otimes b_1^{-1} y a_1^{-1} \otimes a_1 z c_1. \]
That is,
\[ A(x \otimes y \otimes z) = axb^{-1} \otimes byc^{-1} \otimes cza^{-1}, \]
where \( a = c_1^{-1}, \ b = b_1^{-1}, \) and \( c = a_1. \) Thus, \( A = T(a, b, c). \)

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