HYBRID BOUNDS ON TWO-PARAMETRIC FAMILY WEYL SUMS ALONG SMOOTH CURVES

CHANGHAO CHEN AND IGOR E. SHPARLINSKI

Abstract. We obtain a new bound on Weyl sums with degree \( k \geq 2 \) polynomials of the form \( (\tau x + c)\omega(n) + xn \), \( n = 1, 2, \ldots \), with fixed \( \omega(T) \in \mathbb{Z}[T] \) and \( \tau \in \mathbb{R} \), which holds for almost all \( c \in [0, 1) \) and all \( x \in [0, 1) \). We improve and generalise some recent results of M. B. Erdoğan and G. Shakan (2019), whose work also shows links between this question and some classical partial differential equations. We extend this to more general settings of families of polynomials \( xn + y\omega(n) \) for all \( (x, y) \in [0, 1)^2 \) with \( f(x, y) = z \) for a set of \( z \in [0, 1) \) of full Lebesgue measure, provided that \( f \) is some Hölder function.

Contents

1. Introduction 2
   1.1. Background 2
   1.2. Previous results 3
   1.3. Set-up 4
   1.4. New results 6
2. Preparations 7
   2.1. Notation and conventions 7
   2.2. Mean value theorems 7
   2.3. The completion technique 10
   2.4. Continuity of exponential sums 10
   2.5. Large values of Weyls sums 12
3. Proofs of main results 12
   3.1. Proof of Theorem 1.1 12
   3.2. Proof of Theorem 1.2 15
   3.3. Proof of Theorem 1.3 16
4. Comments 16
Acknowledgement 17
References 17

2010 Mathematics Subject Classification. 11L15, 35Q35.
Key words and phrases. Weyl sums, mean values theorem, slice of diagonal surface, partial differential equation.
1. Introduction

1.1. Background. For a natural number \( d \) let \( T_d = (\mathbb{R}/\mathbb{Z})^d \) be the \( d \)-dimensional unit torus. We also write \( T = \mathbb{R}/\mathbb{Z} \) instead of \( T_1 \).

Given a family \( \varphi = (\varphi_1(T), \ldots, \varphi_d(T)) \in \mathbb{Z}[T]^d \) of \( d \) distinct non-constant polynomials and a vector \( u = (u_1, \ldots, u_d) \in T_d \), we consider the Weyl \footnote{We refer to Section 1.1 for the definition of \( \varphi \).} sums

\[
S_{\varphi}(u; N) = \sum_{n=1}^N e \left( u_1 \varphi_1(n) + \ldots + u_d \varphi_d(n) \right),
\]

where throughout the paper we denote 

\[ e(x) = \exp(2\pi i x). \]

Recently, Wooley \footnote{See also Flaminio and Forni \cite{FlaminioForni}.} (see also Flaminio and Forni \cite{FlaminioForni}) has introduced a scenario which interpolates between individual bounds and bounds involving averaging over all \( u \in T_d \). In the setting of \cite{Wooley} the sums \( S_{\varphi}(u; N) \) are estimated for almost all (with respect to the Lebesgue measure) coordinates \( u_i, i \in I \), and for all coordinates \( u_j, j \in J \), where the sets \( I \) and \( J \) form a partition of the set \( \{1, \ldots, d\} \). The results of Wooley \cite{Wooley} have been recently improved and generalised in \cite{ChenShparlinski}. To be precise, we outline a special version of Wooley \cite{Wooley}, Flaminio and Forni \cite{FlaminioForni} and the authors \cite{ChenShparlinski} with \( d = 2 \).

Let \( \varphi_1(T), \varphi_2(T) \in \mathbb{Z}(T) \) then there exists a positive constant \( \rho < 1 \) depending only on \( \varphi_1 \) and \( \varphi_2 \) such that for almost all \( u \in T \) we have

\[
\sup_{v \in T} \left| \sum_{n=1}^N e(u \varphi_1(n) + v \varphi_2(n)) \right| \leq N^{\rho + o(1)}, \quad N \to \infty.
\]

Independently, motivated by applications to some families of partial differential equations, Erdoğan and Shakan \cite{ErdoğanShakan} (see also \cite{ChenShparlinski}) have considered the following more special case of dimension \( d = 2 \) with \( \varphi = (\omega(T), \tau \omega(T) + T) \) for some \( \tau \in \mathbb{Q} \) and function \( \omega : \mathbb{Z} \to \mathbb{R} \) (not necessary a polynomial). We now present some details for the motivation of \cite{ChenShparlinski, ErdoğanShakan}. Exponential series of the type

\[
q(t, x) = \sum_{n \in \mathbb{Z}} a_n e(t \omega(n) + xn)
\]

are solutions of various partial differential equations with respect to different function \( \omega : \mathbb{Z} \to \mathbb{R} \). The key examples are the linear Schrödinger equation:

\[ iq_t + q_{xx} = 0, \quad \text{with } \omega(n) = -n^2, \]
and the *Airy equation*:

\[ q_t + q_{xxx} = 0, \quad \text{with } \omega(n) = n^3. \]

Hence, it is important to investigate the properties of the exponential series (1.3); we refer to Erdoğan and Shakan [10] for more details. Among other things, Erdoğan and Shakan [10] have obtained bounds on the Minkowski, or box, dimension of the graphs of real and imaginary parts of the function

\[ q(t, x) \big|_{t=\tau x + c} = q(\tau x + c, x), \]

for almost all \( c \in \mathbb{R} \) and any fixed rational number \( \tau \), where, as usual, \( q(t, x) \big|_{t=\tau x + c} \) means that we consider (or restrict) the function \( q(t, x) \) on the line \( t = \tau x + c \). For this purpose, Erdoğan and Shakan [10] obtain exponential sum estimates of the following type: there exist a positive constant \( \vartheta < 1 \) depending only on \( \omega \) such that for any \( \tau \in \mathbb{Q} \) and almost all \( c \in \mathbb{R} \) the following estimates

\[ (1.4) \quad \sup_{x \in T} \left| \sum_{n=1}^{N} e((\tau x + c)\omega(n) + xn) \right| \leq N^{\vartheta + o(1)}, \]

holds as \( N \to \infty \). Note that this corresponds to sums (1.2) with \( \varphi = (\omega(T), \tau \omega(T) + T) \). It is important to remark that the uniformity with respect to \( c \) and \( \tau \) is *not required* in (1.4).

Assume that for some \( \vartheta < 1 \), for any \( \tau \in \mathbb{Q} \), for a set of \( c \in T \) of full Lebesgue measure we have (1.4) (again, the uniformity with respect to \( c \) and \( \tau \) is not required). Then, for polynomials \( \omega \) with \( \deg \omega = k \geq 2 \), the argument of the proof of [10, Corollary 3.5] under the assumption (1.4) with any \( \vartheta < 1 \) gives a nontrivial bound

\[ \delta \leq 2 - \frac{(1 - \vartheta)}{k} \]

on the *fractal dimension* \( \delta \) of the graph of the Fourier coefficients of solutions to some linear dispersive partial differential equation, see [10, Equation (1)]. We refer for further details to [10], see also [4, 6–8, 14] for some related questions and further references.

Here we concentrate on obtaining new bounds of the form (1.4) for polynomials \( \omega(T) \in \mathbb{Z}[T] \) and in particular, using several ideas from [9] we improve and generalise some bounds from [10].

1.2. Previous results. We say that a certain property holds for *almost all* \( u \in T \) if it holds for a set \( U \subseteq T \) of Lebesgue measure \( \lambda(U) = 1 \).

We define \( \Omega_k^\mathbb{Q} \) as the smallest possible value (infimum) of \( \vartheta \) such that for any polynomials \( \omega(T) \in \mathbb{Z}[T] \) of degree \( k \) and any \( \tau \in \mathbb{Q} \) there is a set of \( u \in T \) of full Lebesgue measure satisfying (1.4).
We also define $\Theta_k$ as the smallest possible value (infimum) of $\vartheta$ such that for the polynomial $\omega(T) = T^k$ and $\tau = 1$ there is a set of $u \in \mathbb{T}$ of full Lebesgue measure satisfying (1.4).

The goal is to improve the trivial bounds

$$\Omega_k^\mathbb{Q} \leq 1 \quad \text{and} \quad \Theta_k \leq 1.$$ 

We remarks that in [10] (as well as in [4]) the Weyl sums in (1.4) are over dyadic intervals, but both formulations are certainly equivalent.

The first nontrivial bound

$$\Omega_k^\mathbb{Q} \leq 1 - \frac{1}{2^k + 1}$$

has been given by Erdős and Shakan [10, Proposition 3.3], which is based on the classical Weyl differencing method, see [15, Lemma 2.4]. It is also noticed in [10, Footnote 6] that for large $k$ the bound (1.5) can be improved if one uses the Vinogradov method to bound Weyl sums combined with the modern form of the the Vinogradov due to Bourgain, Demeter and Guth [3] (for $d \geq 4$) and Wooley [17] (for $d = 3$), see (2.1) below. Namely one easily verifies that using this bound, see, for example, [2, Theorem 5] or [10, Equation (16)], one derives

$$\Omega_k^\mathbb{Q} \leq 1 - \frac{1}{2(s_0(k) + 1)}.$$ 

Here we improve this bound and extend it to $\Omega_k^\mathbb{Q}$ as follows following,

$$\Omega_k^\mathbb{R} \leq 1 - \frac{1}{2s_0(k) + 1},$$

where $s_0(k)$ is given at (1.11) and (1.12) below. In fact the bound (1.7) is a very special case of a much more general result given in Theorem 1.1 below.

In the monomial case, the truly remarkable result of [4, Theorem 1.4] gives exact values

$$\Theta_2 = \Theta_3 = \frac{3}{4}.$$ 

It is very interesting that the exact values of $\Theta_2$ and $\Theta_3$ in (1.8) differ from the naively expected $1/2$.

1.3. Set-up. For $\omega(T) \in \mathbb{Z}[T]$ and $(x, y) \in \mathbb{T}_2$ we consider the two-parametric family of Weyl sums

$$S_\omega(x, y; N) = \sum_{n=1}^{N} e(xn + y\omega(n)).$$
For \( u = (x, y) \in T_2 \) we also use the notation
\[
S_\omega(u; N) = S_\omega(x, y; N)
\]
as well, and we also apply this convention to other similar sums.

We note that informally (1.4) means the existence of a nontrivial bound of Weyl sums \( S_\omega(x, y; N) \) along the all points bundle of lines \( y = \tau x + c \) which holds for any \( \tau \in \mathbb{Q} \) and almost all \( c \in \mathbb{R} \).

Certainly besides relaxing the condition \( \tau \in \mathbb{Q} \) to \( \tau \in \mathbb{R} \) it is also interesting to extend the above family of straight lines to more general curves, satisfying some smoothness conditions.

Let \( 0 < \rho \leq 1 \). We recall that a function \( f : T_2 \to \mathbb{R} \) is called a \( \rho \)-H"older function, if there is some constant \( C(f) \) depending only on \( f \) such that
\[
\|f(u) - f(v)\| \leq C(f)\|u - v\|^{\rho}, \quad u, v \in T_2,
\]
where \( \|z\| \) is the Euclidean norm of \( z \) (note that the left side of this inequality is the Euclidean norm in \( \mathbb{R} \), while the right side is the Euclidean norm in \( \mathbb{R}^2 \)).

In particular, in the case \( \rho = 1 \) the function \( f \) is often called a Lipschitz function. Moreover note that if \( f \) is a differentiable function on \( T_2 \) and the partial differentials \( f_x, f_y \) are uniformly bounded then the function is a Lipschitz function.

Furthermore, for a function \( f : T_2 \to \mathbb{R} \) and \( z \in \mathbb{R} \) we denote the level set
\[
f^{-1}(z) = \{(x, y) : f(x, y) = z\}.
\]

Now for an integer \( k \geq 2 \) and \( 0 < \rho \leq 1 \), we denote \( \Omega_{k,\rho} \) as the smallest possible value (infimum) of \( \vartheta \) such that for any \( \omega(T) \in \mathbb{Z}[T] \) of degree \( k \), any \( \rho \)-H"older function \( f : T_2 \to \mathbb{R} \) and for almost all \( z \in R(f) \) we have
\[
\sup_{(x, y) \in f^{-1}(z)} |S_\omega(x, y; N)| \leq N^{\vartheta + o(1)}.
\]

We again note that the uniformity in \( f \) is not required.

Since we obviously have
\[
\Omega_{k,\rho}^\mathbb{R} \leq \Omega_{k,1},
\]
taking \( \rho = 1 \) in Theorem 1.1 below we obtain the aforementioned improvement (1.7) of (1.5) and (1.6) from [10].
1.4. **New results.** Following the definition of \( \vartheta(k) \) of Wooley [19, Equation (14.22)], it is convenient to introduce the following quantity

\[
\eta(k) = \begin{cases} 
0, & 2k + 2 \geq \lfloor \sqrt{2k + 2} \rfloor^2 + \lfloor \sqrt{2k + 2} \rfloor, \\
1, & \text{otherwise.}
\end{cases}
\]

We now define \( s_0 \) as follows. For \( k \in \{2, \ldots, 10\} \) we set

\[
s_0(2) = 3, \quad s_0(3) = 5, \quad s_0(4) = 8, \\
s_0(5) = 12, \quad s_0(6) = 18, \quad s_0(7) = 24, \\
s_0(8) = 31, \quad s_0(9) = 40, \quad s_0(10) = 49,
\]

while for \( k \geq 11 \) we define

\[
s_0(k) = k(k - 1)/2 + \lfloor \sqrt{2k + 2} \rfloor - \eta(k).
\]

**Theorem 1.1.** For any integer \( k \geq 2 \) and \( 0 < \rho \leq 1 \), we have

\[
\Omega_{k, \rho} \leq 1 - \frac{\rho}{2s_0(k) + 2 - \rho}.
\]

Taking \( \rho = 1 \), we see that Theorem 1.1 yields (1.7) and shows that the same bound holds along almost curves defined by level sets of a Lipschitz function.

Moreover, let \( f(x, y) = x^2 + y^2 \) then \( f \) is a Lipschitz function, and hence Theorem 1.1 implies that for almost all \( 0 < z < 1 \) we have

\[
\sup_{(x, y) \in f^{-1}(z)} |S_\omega(x, y; N)| \leq N^{1 - 1/(2s_0(k) + 1) + o(1)}, \quad N \to \infty.
\]

Note that the supremum is taken over a family of circles \( x^2 + y^2 = z \).

We now show some possible improvements and variants of Theorem 1.1 for some special functions \( f \). We start with the projection \( f(x, y) = y \) (also corresponding to \( \tau = 0 \) in (1.4)). We denote by \( \Pi_k \) the analogue of \( \Omega_{k, \rho} \) to only one function \( f(x, y) = y \).

**Theorem 1.2.** For any integer \( k \geq 2 \) we have

\[
\Pi_k \leq 1 - \frac{k}{2s_0(k) + 1}.
\]

Our methods also yield the result on restricting Weyl sums on a larger family of circles. More precisely, for \( 0 < r < 1 \) and \( \mathbf{z} \in \mathbb{T}_2 \), denote

\[
C(z, r) = \{ \mathbf{u} \in \mathbb{R}^2 : \| \mathbf{u} - \mathbf{z} \| = r \}
\]

the circle with center \( \mathbf{z} \) and radius \( r \).
We now define $\Gamma_k$ as the smallest possible value (infimum) of $\vartheta$ such that for any polynomials $\omega(T) \in \mathbb{Z}[T]$ of degree $k$ and any $r \in (0, 1)$ there is a set of $(x, y) \in \mathbb{T}_2$ of full Lebesgue such that

$$\sup_{(x,y) \in C(z,r)} |S_\omega(x, y; N)| \leq N^{1-\vartheta+o(1)}$$

as $N \to \infty$.

**Theorem 1.3.** For any integer $k \geq 2$ we have

$$\Gamma_k \leq 1 - \frac{1}{2s_0(k) + 1}.$$

Note that the bound in Theorem 1.3 is the same bound as Theorem 1.1 for the case $\rho = 1$. We may expect that Theorem 1.3 follows from Theorem 1.1. However, it seems that the argument is not immediately obvious. As for Theorem 1.1, the claim holds for almost all $z \in \mathbb{T}$ with respect to one dimensional Lebesgue measure. While in Theorem 1.3 the claim holds for almost all $\mathbf{z} = (z_1, z_2) \in \mathbb{T}_2$ with respect to the two dimensional Lebesgue measure. While this can be handled via the *Fubini theorem* there are still some issues with the uniformity of constants in the Lipschitz condition on the relevant functions.

## 2. Preparations

### 2.1. Notation and conventions.

Throughout the paper, the notation $U = O(V)$, $U \ll V$ and $V \gg U$ are equivalent to $|U| \leq cV$ for some positive constant $c$, which throughout the paper may depend, where obvious, on the polynomial $\omega$ (or sometimes only on $k = \deg \omega$) and the real $\tau$ and are absolute otherwise.

For any quantity $V > 1$ we write $U = V^{o(1)}$ (as $V \to \infty$) to indicate a function of $V$ which satisfies $|U| \leq V^\varepsilon$ for any $\varepsilon > 0$, provided $V$ is large enough. The advantage of using $V^{o(1)}$ is that it absorbs $\log V$ and other similar quantities without changing the whole expression and the need to re-define $\varepsilon$.

### 2.2. Mean value theorems.

We start with recalling the the Vinogradov mean value theorem for the Weyl sums has recently been established by Bourgain, Demeter and Guth [3] (for $d \geq 4$) and Wooley [17] (for $d = 3$) (see also [19]) in the best possible form

$$(2.1) \quad \int_{\mathbb{T}_k} \left| \sum_{n=1}^{N} e \left( x_1 n + \ldots + x_k n^k \right) \right|^{2s} dx \leq N^{s+o(1)} + N^{2s-s(k)+o(1)}$$
as $N \to \infty$ and the integration is over $x = (x_1, \ldots, x_k) \in T_k$ and

$$s(k) = \frac{k(k + 1)}{2}.$$

However, for our application we use a special case of a result of Wooley [19, Corollary 14.8] and combined with the table at the end of [19, Section 14] for $k \geq 4$, and the classical method of Hua [12], see also [5, Lemma 5] or [19, Equation (14.27)] for $k = 2, 3$, see also [1, Table 1].

We define $\sigma_0(k)$ as follows. For $k \in \{2, \ldots, 10\}$ we set

$$\sigma_0(2) = 6, \quad \sigma_0(3) = 10, \quad \sigma_0(4) = 15,$$

(2.2) $$\sigma_0(5) = 70/3, \quad \sigma_0(6) = 34, \quad \sigma_0(7) = 93/2,$$

$$\sigma_0(8) = 306/5, \quad \sigma_0(9) = 78, \quad \sigma_0(10) = 678/7,$$

while for $k \geq 11$ we define

(2.3) $$\sigma_0(k) = k(k - 1) + 2 \left\lfloor \sqrt{2k + 2} \right\rfloor - 1 - \eta(k),$$

where $\eta(k)$ is given by (1.10).

**Remark 2.1.** We note that for small values of $k$, the underlying result

$$\int_0^1 \int_0^1 \left| \sum_{n=1}^N e\left(xn + y\omega(n)\right) \right|^{2k+2} \leq N^{2k-k+1+o(1)}$$

within the method of Hua [12] is traditionally formulated for monomials $\omega(T) = T^k$. However one verifies that this bound holds for any polynomial $\omega(T) \in \mathbb{Z}[T]$ with $\deg \omega = k$.

**Lemma 2.2.** For any polynomial $\omega(T) \in \mathbb{Z}[T]$ of degree $\deg \omega = k \geq 2$, and any fixed real $\sigma$ with

$$\sigma \geq \sigma_0(k),$$

for $k = 2, 3$ or $k \geq 11$ and

$$\sigma > \sigma_0(k),$$

for integer $k \in [4, 10]$, where $\sigma_0(k)$ is given by (2.2) and (2.3), we have

$$\int_0^1 \int_0^1 \left| \sum_{n=1}^N e\left(xn + y\omega(n)\right) \right|^{\sigma} dx dy \leq N^{\sigma-k-1+o(1)}$$

as $N \to \infty$.

**Remark 2.3.** The term $o(1)$ appears in the exponent of $N$ only in the purely classical cases $k = 2, 3$ and in fact can be replaced with a power of $\log N$, however this causes no effect on the final result.
For a polynomial \( \omega(T) \in \mathbb{Z}[T] \) and an integer \( s \geq 1 \) we now define the mean value of the exponential polynomials, which are more general than the sums (1.9)

\[
I_{\omega,s}(a; N) = \left| \int_0^1 \int_0^1 \left( \sum_{n=1}^N a_n e(xn+y\omega(n)) \right)^{2s} \, dx \, dy \right|
\]

where \( a = (a_n)_{n=1}^{\infty} \) is some sequence of complex weights.

**Lemma 2.4.** For any polynomial \( \omega(T) \in \mathbb{Z}[T] \) of degree \( \deg \omega = k \geq 2 \), weights \( a \) with \( a_n = n^{o(1)} \) and fixed integer \( s \geq s_0(k) \) where \( s_0(k) \) is given by (1.11) and (1.12), we have

\[
I_{\omega,s}(N) \leq N^{2s-k-1+o(1)}
\]

as \( N \to \infty \).

**Proof.** Using that \( |z|^2 = z \bar{z} \) for \( z \in \mathbb{C} \) to compute the \( 2s \)-th power of the inner sum, after changing the order of summations and integration, we derive

\[
\int_0^1 \int_0^1 \left| \sum_{n=1}^N a_n e(xn+y\omega(n)) \right|^{2s} \, dx \, dy
\]

\[
= \sum_{n_1,\ldots,n_{2s}=1}^N \bar{a}_{n_1}a_{n_2} \ldots \bar{a}_{n_{2s-1}}a_{n_{2s}}
\]

\[
\times \int_0^1 e(xf(n_1,\ldots,n_{2s}) + yg(n_1,\ldots,n_{2s})) \, dx \, dy,
\]

where

\[
f(n_1,\ldots,n_{2s}) = \sum_{j=1}^{2s} (-1)^j n_j \quad \text{and} \quad g(n_1,\ldots,n_{2s}) = \sum_{j=1}^{2s} (-1)^j w(n_j).
\]

Hence, recalling the condition \( a_n = n^{o(1)} \) we conclude that

(2.4)

\[
I_{\omega,s}(N) \leq JN^{o(1)},
\]

where \( J \) is the number of solutions to the system of equations

\[
f(n_1,\ldots,n_{2s}) = g(n_1,\ldots,n_{2s}) = 0,
\]

\[
1 \leq n_1,\ldots,n_{2s} \leq N.
\]
Again, using the orthogonality of exponential functions, we write

$$J = \int_0^1 \int_0^1 \left| \sum_{n=1}^N e(xn + y\omega(n)) \right|^{2s} dx dy.$$

One verifies that $\sigma = 2s_0(k)$ is an admissible value of $\sigma$ in Lemma 2.2, which together with (2.4) concludes the proof.

**Remark 2.5.** Our main ingredient, Lemma 2.2 holds for any real $\sigma \geq \sigma_0(k)$. However, in Lemma 2.4, to pass from $I_{\omega,s}(N)$ to the bound of Lemma 2.2 we need the integrality of $s$ (which is not needed for the rest of our argument). It is interesting to avoid this and find a more efficient way of linking Lemmas 2.2 and 2.4, perhaps via an efficient use of the Hölder inequality, and thus obtain numerically stronger results (which improve with decreasing $s$).

2.3. **The completion technique.** We now recall a result from [9] obtained via the standard completion technique (see [13, Section 12.2]) for the Weyl sums $S_\omega(u;N)$ as in (1.1) which we adjust to our setting of the sums $S_\omega(x,y;N)$ given by (1.9). Namely, by a special case of [9, Lemma 3.2] we have

**Lemma 2.6.** For $(x,y) \in T_2$ and $1 \leq M \leq N$ we have

$$S_\omega(x,y;M) \ll W_\omega(x,y;N),$$

where

$$W_\omega(x,y;N) = \sum_{h=-N}^N \frac{1}{|h|+1} \left| \sum_{n=1}^N e(hn/N + xn + y\omega(n)) \right|.$$

2.4. **Continuity of exponential sums.** For $u = (x,y) \in T_2$ and $\zeta = (\zeta_1,\zeta_2)$, we define the square centred at $u$ with “side length” $\zeta$ by

$$\mathcal{R}(u;\zeta) = [x-\zeta_1, x+\zeta_1] \times [y-\zeta_2, y+\zeta_2]$$

Here we present a close analogue of [9, Lemma 3.2], see also [18, Lemma 2.1]. However, we use the following version of summation by parts which is slightly different from the proof in [9, Lemma 3.2] and [18, Lemma 2.1].

Let $a_n$ be a sequence and for each $t \geq 1$ denote

$$A(t) = \sum_{1 \leq n \leq t} a_n.$$

Let $\psi : [1, N] \to \mathbb{R}$ be a differential function. Then

$$\sum_{n=1}^N a_n \psi(n) = A(N)\psi(N) - \int_1^N A(t)\psi'(t)dt.$$
Lemma 2.7. Suppose that $\omega(T) \in \mathbb{Z}[T]$ is of degree $\deg \omega = k$. Let $0 < \alpha < 1$. Let $\varepsilon > 0$ and

$$0 < \zeta_1 \leq N^{\alpha - 2 - \varepsilon}, \quad 0 < \zeta_2 \leq N^{\alpha - k - 1 - \varepsilon}.$$  

If $W_\omega(x, y; N) \geq N^\alpha$ for some $(x, y) \in R(u, v, \zeta)$, then for any $(a, b) \in R(u, v, \zeta)$ we obtain

$$W_\omega(a, b; N) \geq N^\alpha/2,$$

provided that $N$ is large enough.

Proof. As in [9, Section 2.3] we observe that for any $N$ there exists a sequence of complex numbers $b_N(n)$ such that

$$b_N(n) \ll \log N, \quad n = 1, \ldots, N,$$

and $W_\omega(x, y; N)$ can be written as

$$(2.5) \quad W_\omega(x, y; N) = \sum_{n=1}^{N} b_N(n) e(xn + y\omega(n)).$$

For $\delta_1, \delta_2 \in \mathbb{R}$ applying partial summation, we obtain

$$W_\omega(x + \delta_1, y + \delta_2; N) - W_\omega(x, y; N)$$

$$= \sum_{n=1}^{N} b_N(n) e(xn + y\omega(n))(e(n\delta_1 + \omega(n)\delta_2) - 1)$$

$$= A(N)\psi(N) - \int_{1}^{N} A(t)\psi'(t)dt,$$

where

$$A(t) = \sum_{n \leq t} b_N(n) e(xn + y\omega(n)) \quad \text{and} \quad \psi(t) = e(t\delta_1 + \omega(t)\delta_2) - 1.$$  

Observe that $A(N) \ll N^{1+o(1)}$ as $N \to \infty$. Since $|e(t) - 1| \ll |t|$ for all $t \in \mathbb{R}$ and $\omega(N) \ll N^k$ for all large enough $N$, we obtain

$$\psi(N) \ll N|\delta_1| + N^k|\delta_2|.$$  

Thus we derive

$$(2.7) \quad A(N)\psi(N) \ll N^{2+o(1)}|\delta_1| + N^{k+1+o(1)}|\delta_2|.$$  

Furthermore, since $\psi'(t) \ll \delta_1 + t^{k-1}\delta_2$, we have

$$(2.8) \quad \int_{1}^{N} A(t)\psi'(t)dt \ll N^{1+o(1)} \int_{1}^{N} |\psi'(t)|dt$$

$$\leq N^{2+o(1)}|\delta_1| + N^{k+1+o(1)}|\delta_2|.$$
Combining (2.6) with (2.7) and (2.8) we obtain
\[
W_\omega(x + \delta_1, y + \delta_2; N) - W_\omega(x, y; N) 
\ll N^{2+o(1)}|\delta_1| + N^{k+1+o(1)}|\delta_2|.
\]

Therefore, we conclude that for any fixed $\varepsilon > 0$ and the choice of $\zeta_1$ and $\zeta_2$, the claim holds for all large enough $N$. □

2.5. Large values of Weyl's sums. Let $0 < \alpha < 1$ and let $\varepsilon$ be sufficiently small. We set
\[
\zeta_1 = 1 / \left[ N^{2+\varepsilon-\alpha} \right], \quad \zeta_2 = 1 / \left[ N^{k+1+\varepsilon-\alpha} \right],
\]
and divide $T_2$ into $(\zeta_1 \zeta_2)^{-2}$ squares of the form
\[
[\ell \zeta_1, (\ell + 1) \zeta_1] \times [m \zeta_2, (m + 1) \zeta_2],
\]
where $\ell = 0, \ldots, \zeta_1^{-1} - 1$ and $m = 0, \ldots, \zeta_2^{-1} - 1$. Let $\mathcal{R}$ be the collection of these squares. We now consider the subset of $\mathcal{R}$ that consists of squares which contain a large sum $W_\omega(u; N)$ for some $u = (x, y) \in T_2$. More precisely, we denote
\[
\tilde{\mathcal{R}} = \{ R \in \mathcal{R} : \exists u \in R \text{ with } W_\omega(u; N) \geq N^\alpha \}.
\]

To present our results in full generality we assume that there are positive $s$ and $t$ such that
\[
\int_0^1 \int_0^1 W_\omega(u, v; N)^{2s} \, du \, dv \leq N^{2s-t+o(1)}
\]
for $N \to \infty$. Then we specialise $s$ and $t$ to get concrete estimates.

**Lemma 2.8.** Suppose (2.11) holds. Then
\[
\#\tilde{\mathcal{R}} \leq (\zeta_1 \zeta_2)^{-1} N^{2s(1-\alpha)-t+o(1)}.
\]

**Proof.** For each $R \in \tilde{\mathcal{R}}$, by Lemma 2.7 we have $W_\omega(x, y; N) \geq N^\alpha/2$ for all $(x, y) \in R$. Combining this with (2.11) we obtain
\[
N^{2s \alpha} \zeta_1 \zeta_2 \#\tilde{\mathcal{R}} \ll \int_0^1 \int_0^1 W_\omega(u, v; N)^{2s} \, du \, dv \leq N^{2s-t+o(1)},
\]
which yields the desired bound. □

3. Proofs of main results

3.1. Proof of Theorem 1.1. We start with the following statement which could be of independent interest.
3.1.1. Lebesgue measure of large weighted Weyl sums. We continue to use \( \lambda \) to denote the Lebesgue measure.

**Lemma 3.1.** Suppose (2.11) holds. Let \( f : T^2 \to \mathbb{R} \) be a \( \rho \)-Hölder function. For \( 0 < \alpha < 1 \) we have

\[
\lambda \left( \{ z \in \mathbb{R} : \sup_{u \in f^{-1}(z)} |W_\omega(u; N)| \geq N^\alpha \} \right) \leq N^{(2-\alpha)(1-\rho)+(k+1-\alpha)+2s(1-\alpha)-t+o(1)}.
\]

**Proof.** We fix some sufficiently small \( \varepsilon > 0 \) and define the set

\[
\Omega = \bigcup_{R \in \tilde{R}} R.
\]

For \( A \subseteq T^2 \) denote \( f(A) = \{ f(u) : u \in A \} \). Observe that

\[
\left\{ z \in \mathbb{R} : \sup_{u \in f^{-1}(z)} |W_\omega(u; N)| \geq N^\alpha \right\} \subseteq f(\Omega) \subseteq \bigcup_{R \in \tilde{R}} f(R),
\]

where \( \tilde{R} \) is as in (2.10).

Since \( f \) is \( \rho \)-Hölder, for any \( A \subseteq T^2 \) we obtain

\[
\lambda(f(A)) \ll (\text{diam} A)^\rho,
\]

where \( \text{diam} A = \sup\{ \|a - b\| : a, b \in A \} \). For each \( R \in \tilde{R} \), by (2.9) we have

\[
\text{diam} R \ll \zeta_1 \ll N^{\alpha-2-\varepsilon}.
\]

Combining with Lemma 2.8 and the estimate (3.1), we derive

\[
\lambda \left( \{ z \in \mathbb{R} : \sup_{u \in f^{-1}(z)} |W_\omega(u; N)| \geq N^\alpha \} \right) \ll N^{(\alpha-2-\varepsilon)\rho} \#\tilde{R} \ll N^{(2-\alpha+\varepsilon)(1-\rho)} N^{k+1-\alpha+\varepsilon} N^{2s(1-\alpha)-t}.
\]

Since \( \varepsilon > 0 \) is arbitrary, this finishes the proof. \qed

3.1.2. Conditional estimate. We process a similarly to as in [9, Section 4.1].

**Lemma 3.2.** Suppose (2.11) holds. Let \( f : T^2 \to \mathbb{R} \) be a \( \rho \)-Hölder function. Then

\[
\Omega_{k,\rho} \leq 1 - \frac{t - k - 1 + \rho}{2s + 2 - \rho}.
\]
Proof. We fix some $\alpha > 1/2$ and set

$$N_i = 2^i, \quad i = 1, 2, \ldots.$$  

We now consider the set

$$B_i = \{ z \in \mathbb{R} : \exists u \in f^{-1}(z) \text{ with } W_{\omega}(u; N_i) \geq N_i^\alpha \}.$$ 

By Lemma 3.1 we have

$$\lambda(B_i) \leq N_i^{(2-\alpha)(1-\rho) + 2s(1-\alpha) + k + 1 - t - \alpha + 2\varepsilon + o(1)}.$$ 

We ask that the parameters satisfy the following convergency condition

$$\sum_{i=1}^{\infty} N_i^{(2-\alpha)(1-\rho) + 2s(1-\alpha) + k + 1 - t - \alpha + 2\varepsilon + o(1)} < \infty,$$

which, due to the exponential growth of $N_i$ and the arbitrary small choice of $\varepsilon > 0$, is equivalent to the inequality

$$(3.3) \quad (2 - \alpha)(1 - \rho) + 2s(1 - \alpha) + k + 1 - t - \alpha < 0.$$ 

In this case, by the Borel–Cantelli lemma, we obtain that

$$\lambda\left(\bigcap_{q=1}^{\infty} \bigcup_{i=q}^{\infty} B_i\right) = 0.$$ 

Since

$$\{ z \in \mathbb{R} : \exists u \in f^{-1}(z) \text{ with } W_{\omega}(u; N_i) \geq N_i^\alpha \}$$

for infinite many $i \in \mathbb{N}$ is a subset of $\bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} B_i$, we conclude that for almost all $z \in \mathbb{R}$ there exists $i_z$ such that for any $i \geq i_z$ one has

$$(3.4) \quad \sup_{u \in f^{-1}(z)} W_{\omega}(u; N_i) \leq N_i^\alpha.$$ 

We fix one of such $z \in \mathbb{R}$ in the following argument. For any $N \geq N_{i_z}$ we find $i > i_z$ such that

$$N_{i-1} \leq N < N_i.$$ 

By Lemma 2.6 and (3.4) we have

$$\sup_{u \in T_2} |S_{\omega}(u; N)| \ll \sup_{u \in T_2} W_{\omega}(u; N_i) \ll N_i^\alpha \ll N^\alpha.$$ 

Note that the condition (3.3) can be written as

$$\alpha > \frac{2(1 - \rho) + 2s + k + 1 - t}{2s + 2 + 1 - \rho} = 1 - \frac{t - k - 1 + \rho}{2s + 2 - \rho},$$

which finishes the proof. \qed
3.1.3. **Concluding the proof.** Similar to the proof of Lemma 2.7, we write \( W_\omega(u, v; N) \) as in (2.5). Note that \( b_N(n) \ll \log N \) for all \( n = 1, \ldots, N \). Combining (2.5) with Lemma 2.4 we see that (2.11) holds with
\[
(3.5) \quad s = s_0(k) \quad \text{and} \quad t = k + 1
\]
where \( s_0(k) \) is given by (1.11) and (1.12), which after substitution in (3.2) implies Theorem 1.1.

3.2. **Proof of Theorem 1.2.** It is sufficient to obtain the following analogue of the estimate of Lemma 3.1, and then use the similar argument as in the proof of Theorem 1.1.

**Lemma 3.3.** Suppose (2.11) holds. Let \( f : T_2 \to \mathbb{T} \) with \( f(x, y) = y \). For \( 0 < \alpha < 1 \) we have
\[
\lambda\{z \in \mathbb{R} : \sup_{u \in f^{-1}(z)} |W_\omega(u; N)| \geq N^\alpha\} \leq N^{2s(1-\alpha) - t + 2 - \alpha + o(1)}.
\]

**Proof.** Let \( \pi_y : T_2 \to \mathbb{T} \) be the projection that \( \pi(x, y) = y \). Then
\[
(3.6) \quad \{z \in \mathbb{T} : \sup_{u \in f^{-1}(z)} |W_\omega(u; N)| \geq N^\alpha\} \subseteq \pi_y\left( \bigcup_{\mathcal{R} \in \tilde{\mathcal{R}}} \mathcal{R} \right).
\]

Observe that each square of \( \mathcal{R} \) is of the following form
\[
[\ell \zeta_1, (\ell + 1) \zeta_1] \times [m \zeta_2, (m + 1) \zeta_2)
\]
for some \( \ell \) and \( m \), see (2.9). Combining with (3.1) and (3.6), we derive
\[
\lambda\left( \left\{ z \in \mathbb{T} : \sup_{u \in f^{-1}(z)} |W_\omega(u; N)| \geq N^\alpha \right\} \right) \\
\ll \lambda\left( \pi_y \left( \bigcup_{\mathcal{R} \in \tilde{\mathcal{R}}} \mathcal{R} \right) \right) \\
\ll \zeta_2 \# \tilde{\mathcal{R}} \ll N^{2s(1-\alpha) - t + 2 - \alpha + o(1)}.
\]

Applying Lemma 2.8 we finish the proof.

Applying Lemma 3.3 with \( N = 2^i \), \( i = 0, 1, \ldots \), we obtain the desired results provided
\[
2s(1 - \alpha) - t + 2 - \alpha < 0
\]
or, equivalently
\[
\alpha > \frac{1 - t}{2s + 1}.
\]
With the choice (3.5) we conclude the proof.
3.3. **Proof of Theorem 1.3.** Recall that we fix $0 < r < 1$. It is sufficient to prove the following estimate.

**Lemma 3.4.** Suppose (2.11) holds. Fix $0 < r < 1$. The for $0 < \alpha < 1$ we have

$$\lambda\left(\{z \in T_2 : \sup_{u \in C(z, r)} |W_\omega(u; N)| \geq N^\alpha\}\right) \leq N^{k+1-\alpha+2s(1-\alpha)-t+o(1)}$$

**Proof.** For each square $R$ as in Section 2.5, denote $C(R) = \bigcup_{z \in R} C(z, r)$. Observe that

$$\{z \in T_2 : \sup_{u \in C(z, r)} |W_\omega(u; N)| \geq N^\alpha\} \subseteq \bigcup_{R \in \tilde{R}} C(R).$$

Moreover, for each $R \in \tilde{R}$ we have

$$\lambda(C(R)) \ll \zeta_1.$$

Applying Lemma 2.8 we derive

$$\lambda\left(\{z \in T_2 : \sup_{u \in C(z, r)} |W_\omega(u; N)| \geq N^\alpha\}\right) \leq \zeta_1 \#\tilde{R} \ll N^{k+1-\alpha+2s(1-\alpha)-t},$$

which gives the desired bound. \(\square\)

Applying Lemma 3.4 with $N = 2^i$, $i = 0, 1, \ldots$, we obtain the desired results provided

$$k + 1 - \alpha + 2s(1 - \alpha) - t < 0$$

or, equivalently

$$\alpha > 1 - \frac{t - k}{2s + 1}.$$  

With the choice (3.5) we conclude the proof.

4. **Comments**

We see from (3.2) that any reduction in the value of $s_0(k)$ in the condition on $s$ in Lemma 2.4 immediately leads to an improvement of Theorems 1.1, 1.2 and Theorem 1.3, see also Remarks 2.5 for one of the possible ways to achieve this.

Certainly, the case of non-polynomial functions $\omega(T)$, such as, for example, $\omega(T) = T^\kappa$ with some $\kappa \in \mathbb{R}$, which has also been considered in [10], are of interest as well. Our method can be applied to such
functions as well, provided appropriate mean value theorems become available. It is easy to see that one can have analogues of Lemmas 2.7 and 3.1 for any function $\omega : \mathbb{N} \to \mathbb{R}$ with some smoothness conditions such as

$$\omega'(x) \ll x^{\kappa-1+o(1)}, \quad \text{as } x \to \infty.$$  \hfill (4.1)

We can also define natural analogues of the sums $S_\omega(x, y; N)$ and $W_\omega(x, y; N)$. One then easily checks that our method produces nontrivial results for the sums $S_\omega(x, y; N)$ for any function $\omega$ satisfying (2.11) and (4.1) with $t > \kappa$.

Our methods can also be used to address the following general scenario. Let $(\Gamma, \mu)$ be a measure space and suppose that for each $\gamma \in \Gamma$ there is a corresponding set $A_\gamma \subseteq T_2$ satisfying some “regular” conditions. Then the goal is there may exist some positive $\vartheta < 1$, depending only on $\mu$ and $\Gamma$, (and the properties of the sets $A_\gamma$) such that for $\mu$-almost all $\gamma \in \Gamma$ we have

$$\sup_{u \in A_\gamma} |S_\omega(u; N)| \leq N^{\vartheta+o(1)}.$$

For example, the sets $A_\gamma$, $\gamma \in \Gamma$, in [10] is a family of lines with rational direction, while the sets $A_\gamma$, $\gamma \in \Gamma$, in Theorem 1.1 is the level sets of some Hölder function $f$. Furthermore, the sets $A_\gamma$, $\gamma \in \Gamma$, of Theorem 1.3 is a family of circles of fixed radius $r$. Certainly more general sets are also of interest and can be investigated via our approach.

Acknowledgement

The authors would like to thank Julia Brandes, Burak Erdoğan, George Shakan and Trevor Wooley for helpful discussions and patient answering their questions. In particular, the authors are very grateful to Trevor Wooley for directing them to the results of [19, Section 14] which have led to improved bounds.

This work was supported by ARC Grant DP170100786.

References

[1] T. C. Anderson, B. Cook, K. Hughes and A. Kumchev, ‘Improved $\ell^p$-boundedness for integral $k$-spherical maximal functions’, Discrete Anal., 2018, available at https://arxiv.org/abs/1707.08667.

[2] J. Bourgain, ‘On the Vinogradov mean value’, Proc. Steklov Math. Inst., 296 (2017), 30–40.

[3] J. Bourgain, C. Demeter and L. Guth, ‘Proof of the main conjecture in Vinogradov’s mean value theorem for degrees higher than three’, Ann. Math., 184 (2016), 633–682.
[4] J. Brandes, S. T. Parsell, C. Poulias, G. Shakan and R. C. Vaughan, ‘On generating functions in additive number theory, II: Lower-order terms and applications to PDEs’, Preprint, 2019, available at https://arxiv.org/abs/2001.05629.

[5] J. Brüdern and O. Robert, ‘Rational points on linear slices of diagonal hypersurfaces’, Nagoya Math. J., 218 (2015), 51–100.

[6] F. Chamizo and A. Córdoba, ‘Differentiability and dimension of some fractal Fourier series’, Adv. Math. 142 (1999), 335–354.

[7] F. Chamizo, A. Córdoba and A. Ubis, ‘Fourier series in BMO with number theoretical implications’, Math. Ann. 376 (2020), 457–473.

[8] F. Chamizo and A. Ubis, ‘Multifractal behavior of polynomial Fourier series’, Adv. Math. 250 (2014), 1–34.

[9] C. Chen and I. E. Shparlinski, ‘New bounds of Weyl sums’, Int. Math. Research Notices (to appear).

[10] M. B. Erdoğan and G. Shakan, ‘Fractal solutions of dispersive partial differential equations on the torus’, Selecta Math. 25 (2019), Art. 11, 1–26.

[11] L. Flaminio and G. Forni, ‘On effective equidistribution for higher step nilflows’, Preprint, 2014, available at https://arxiv.org/abs/1407.3640.

[12] L.-K. Hua, Additive theory of prime numbers, Amer. Math. Soc., Providence, RI, 1965.

[13] H. Iwaniec and E. Kowalski, Analytic number theory, Amer. Math. Soc., Providence, RI, 2004.

[14] L. B. Pierce, ‘On discrete fractional integral operators and mean values of Weyl sums’, Bull. London Math. Soc., 43 (2011), 597–612.

[15] R. C. Vaughan, The Hardy-Littlewood method, Cambridge Tracts in Math. vol. 25, Cambridge Univ. Press, 1997.

[16] H. Weyl, ‘Über die Gleichverteilung von Zahlen mod Eins’, Math. Ann., 77 (1916), 313–352.

[17] T. D. Wooley, ‘The cubic case of the main conjecture in Vinogradov’s mean value theorem’, Adv. in Math., 294 (2016), 532–561.

[18] T. D. Wooley, ‘Perturbations of Weyl sums’, Internat. Math. Res. Notices, 2016 (2016), 2632–2646.

[19] T. D. Wooley, ‘Nested efficient congruencing and relatives of Vinogradov’s mean value theorem’, Proc. London Math. Soc., 118 (2019), 942–1016.