HAMILTONIAN MECHANICS
ON DUALS OF GENERALIZED LIE ALGEBROIDS

by
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Abstract

A new description, different by the classical theory of Hamiltonian Mechanics, in the general framework of generalized Lie algebroids is presented. In the particular case of Lie algebroids, new and important results are obtained. We present the dual mechanical systems called by use, dual mechanical \((\rho, \eta)\)-systems, Hamilton mechanical \((\rho, \eta)\)-systems or Cartan mechanical \((\rho, \eta)\)-systems. We obtain the canonical \((\rho, \eta)\)-semi(spray) associated to a dual mechanical \((\rho, \eta)\)-system. The Hamilton mechanical \((\rho, \eta)\)-systems are the spaces necessary to develop a Hamiltonian formalism. We obtain the \((\rho, \eta)\)-semispray associated to a regular Hamiltonian \(H\) and external force \(F_e\) and we derive the equations of Hamilton-Jacobi type.

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Contents

1 Introduction 2
2 Preliminaries 3
3 Natural and adapted basis 6
4 The lift of a differentiable curve 11
5 Remarkable Mod-endomorphisms 13
  5.1 Projectors 14
  5.2 The almost product structure 16
  5.3 The almost tangent structure 17
6 Tensor \(d\)-fields. Distinguished linear \((\rho, \eta)\)-connections 17
7 Dual mechanical systems 22
8 \((\rho, \eta)\)-semisprays and \((\rho, \eta)\)-sprays for dual mechanical \((\rho, \eta)\)-systems 24
9 A Hamiltonian formalism for Hamilton mechanical \((\rho, \eta)\)-systems 32
References 35
1 Introduction

The concept of Hamilton space, introduced in [15], was intensively studied in [6, 7, 8, 11, 14], and it has been successful, as a geometric theory of the Hamiltonian function. In the classical sense, a regular Hamiltonian on $T^*M$ is a smooth function $H: T^*M \rightarrow \mathbb{R}$ such that the Hessian matrix with entries

$$g^{ij}(x, p) = \frac{1}{2} \frac{\partial^2 H(x, p)}{\partial p_i \partial p_j}$$

is everywhere nondegenerate on $T^*M$ (or on a domain of $T^*M$) and a Hamilton space is a pair $H_n = (M, H)$, where $H$ is a regular Hamiltonian. (see [16]) The case when $H$ is square of a function on $T^*M$, positively, 1-homogeneous with respect to the momentum $p_i$, provides an important class of Hamilton spaces called Cartan spaces. The modern formulation of the geometry of Cartan spaces was given by R. Miron [12, 13] although some results were obtained by É. Cartan [5] and A. Kawaguchi [9].

The geometry of $T^*M$ is from one point of view different from that of $TM$, because not exists a natural tangent structure and a semispray can not be introduced as usual for the tangent bundle. Two geometrical ingredients are of great importance on $T^*M$: the canonical 1-form $p_idx^i$ and its exterior derivative $dp_i \wedge dx^i$ (the canonical symplectic structure of $T^*M$). They are systematically used to defined new useful tools in the classical theory.

A Hamiltonian description of Mechanics on duals of Lie algebroids was presented in [10]. (see also [17, 18, 19, 20, 21]) The role of cotangent bundle of the configuration manifold was played by the prolongation $L^*E$ of $E$ along the projection $E^* \twoheadrightarrow M$. The Lie algebroid version of the classical results concerning the universality of the standard Liouville 1-form on cotangent bundles is presented in Theorem 3.4 and Corollary 3.6. Given a Hamiltonian function $E^* \rightarrow \mathbb{R}$ and the symplectic form $\Omega_E$ on $E^*$, the dynamics are obtained solving the equation

$$i_{\xi_H} \Omega_E = dL^*E H$$

with the usual notations. The solutions of $\xi_H$ (curves in $E^*$) are the ones of the Hamilton equations for $H$.

The purpose of the present paper is to find the answer to the following question:

- **Could we to give a Hamiltonian description of Mechanics on duals of generalized Lie algebroids (see [1, 2, 3]) similar with the Lagrangian description of Mechanics on generalized Lie algebroids presented in the paper [4] without the symplectic form?**

In Sections 3, 4, 5 and 6 we set up the basic notions and terminology. In Section 7 we present for the first time the dual mechanical systems called by use, dual mechanical $(\rho, \eta)$-systems, Hamilton mechanical $(\rho, \eta)$-systems or Cartan mechanical $(\rho, \eta)$-systems.

In Section 8 we obtain the canonical $(\rho, \eta)$-semispray associated to the dual mechanical $(\rho, \eta)$-system $\left(\hat{E}, \hat{\pi}, M, \hat{F}_e, (\rho, \eta) \Gamma\right)$ and from locally invertible $B^Y$-morphism $(g, h)$. Also, we present the canonical $(\rho, \eta)$-spray associated to mechanical system $\left(\hat{E}, \hat{\pi}, M, \hat{F}_e, (\rho, \eta) \Gamma\right)$ and from locally invertible $B^Y$-morphism $(g, h)$.
The Section 9 is dedicated to study the geometry of Hamilton mechanical \((\rho, \eta)\)-systems. These mechanical systems are the spaces necessary to obtain a Hamiltonian formalism in the general framework of generalized Lie algebroids. We determine and we study the \((\rho, \eta)\)-semispray associated to a regular Hamiltonian \(H\) and external force \(\mathbf{F}_e\) which are applied on the dual of the total space of a generalized Lie algebroid and we derive the equations of Hamilton-Jacobi type.

Finally, we obtain that the integral curves of the canonical \((\rho, \eta)\)-semispray associated to Hamilton mechanical \((\rho, \eta)\)-system \(((\mathcal{E}, \mathbf{F}_e), (\rho, \eta)\) \(\Gamma\)) and from locally invertible \(\mathbf{B}^p\)-morphism \((g, h)\) are the \((g, h)\)-lifts solutions for the equations of Hamilton-Jacobi type (9.10).

Our researches are very important because, if \(h = Id_M = \eta\), then all results presented in this paper become new results in the framework of Lie algebroids.

2 Preliminaries

Let \(\text{Vect}, \text{Liealg}, \text{Mod}, \text{Man}\) and \(\mathbf{B}^p\) be the category of real vector spaces, Lie algebras, modules, manifolds and vector bundles respectively.

We know that if \((E, \pi, M) \in |\mathbf{B}^p|\) so that \(M\) is paracompact and if \(A \subseteq M\) is closed, then for any section \(u\) over \(A\) it exists \(\tilde{u} \in \Gamma (E, \pi, M)\) so that \(\tilde{u}|_A = u\). In the following, we consider only vector bundles with paracompact base.

Additionally, if \((E, \pi, M) \in |\mathbf{B}^p|, \Gamma (E, \pi, M) = \{u \in \text{Man} (M, E) : u \circ \pi = Id_M\}\) and \(\mathcal{F} (M) = \text{Man} (M, \mathbb{R})\), then \((\Gamma (E, \pi, M), +, \cdot)\) is a \(\mathcal{F} (M)\)-module. If \((\varphi, \varphi_0) \in \mathbf{B}^p ((E, \pi, M), (E', \pi', M'))\) such that \(\varphi_0 \in \text{Iso}_{\text{Man}} (M, M')\), then, using the operation \(\mathcal{F} (M) \times \Gamma (E', \pi', M') \rightarrow \Gamma (E', \pi', M')\)

\[
(f, u') \mapsto f \circ \varphi_0^{-1} \cdot u'
\]

it results that \((\Gamma (E', \pi', M'), +, \cdot)\) is a \(\mathcal{F} (M)\)-module and we obtain the \(\text{Mod}\)-morphism

\[
\Gamma (E, \pi, M) \xrightarrow{\Gamma (\varphi, \varphi_0)} \Gamma (E', \pi', M')
\]

\[
u \mapsto \Gamma (\varphi, \varphi_0) \nu
\]

defined by

\[
\Gamma (\varphi, \varphi_0) \nu (y) = \varphi \left( u_{\varphi_0^{-1}} (y) \right),
\]

for any \(y \in M'\).

Let \(M, N \in |\text{Man}|, h \in \text{Iso}_{\text{Man}} (M, N)\) and \(\eta \in \text{Iso}_{\text{Man}} (N, M)\).

We know (see [2,3]) that if \((F, \nu, N) \in |\mathbf{B}^p|\) so that there exists

\((\rho, \eta) \in \mathbf{B}^p ((F, \nu, N), (TM, \tau_M, M))\)

and an operation

\[
\Gamma (F, \nu, N) \times \Gamma (F, \nu, N) \xrightarrow{[\cdot]_{F,h}} \Gamma (F, \nu, N)
\]

\[
(u, v) \mapsto [u, v]_{F,h}
\]

with the following properties:
GLA_1. the equality holds good

\[ [u, f \cdot v]_{F,h} = f [u, v]_{F,h} + \Gamma (Th \circ \rho, h \circ \eta) (u) f \cdot v, \]

for all \( u, v \in \Gamma (F, \nu, N) \) and \( f \in \mathcal{F} (N) \).

GLA_2. the 4-tuple \( \left( \Gamma (F, \nu, N), +, \cdot, [~, \cdot]_{F,h} \right) \) is a Lie \( \mathcal{F} (N) \)-algebra,

GLA_3. the \textbf{Mod}-morphism \( \Gamma (Th \circ \rho, h \circ \eta) \) is a \textbf{LieAlg}-morphism of

\[ \left( \Gamma (F, \nu, N), +, \cdot, [~, \cdot]_{F,h} \right) \]

source and

\[ \left( \Gamma (TN, \tau_N, N), +, \cdot, [~, \cdot]_{TN} \right) \]

target, then the triple \( \left( (F, \nu, N), [~, \cdot]_{F,h}, (\rho, \eta) \right) \) is called generalized Lie algebroid.

In particular, if \( h = Id_M = \eta \), then we obtain the definition of the Lie algebroid.

We can discuss about the category \( \textbf{GLA} \) of generalized Lie algebroids. (see [3])

Examples of objects of this category are presented in the paper [2].

Let \( \left( (F, \nu, N), [~, \cdot]_{F,h}, (\rho, \eta) \right) \) be an object of the category \( \textbf{GLA} \).

- Locally, for any \( \alpha, \beta \in \Gamma, p \), we set \( [t_\alpha, t_\beta]_{F,h} = L^\gamma_{\alpha\beta} t_\gamma \). We easily obtain that \( L^\gamma_{\alpha\beta} = -L^\gamma_{\beta\alpha} \), for any \( \alpha, \beta, \gamma \in \Gamma, p \).

The real local functions \( L^\gamma_{\alpha\beta} \), \( \alpha, \beta, \gamma \in \Gamma, p \) will be called the \textit{structure functions of the generalized Lie algebroid} \( \left( (F, \nu, N), [~, \cdot]_{F,h}, (\rho, \eta) \right) \).

- We assume the following diagrams:

\[ \begin{array}{cccc}
F & \xrightarrow{\rho} & TM & \xrightarrow{Th} & TN \\
\downarrow \nu & & \downarrow \tau_M & & \downarrow \tau_N \\
N & \xrightarrow{\eta} & M & \xrightarrow{h} & N \\
(\chi^i, z^\alpha) & & (x^i, y^i) & & (\chi^i, z^\tilde{i})
\end{array} \]

where \( i, \tilde{i} \in \Gamma, m \) and \( \alpha \in \Gamma, p \).

If

\[ (\chi^i, z^\alpha) \longrightarrow (\chi^\tilde{i} (\chi^i), z^\alpha (\chi^i, z^\alpha)) , \]

\[ (x^i, y^i) \longrightarrow (x^\tilde{i} (x^i), y^i (x^i, y^i)) \]

and

\[ (\chi^i, z^\tilde{i}) \longrightarrow (\chi^\tilde{i} (\chi^i), z^\tilde{i} (\chi^i, z^\tilde{i})) , \]

then

\[ z^\alpha' = \Lambda^\alpha_{\alpha} z^\alpha , \]

\[ y^i = \frac{\partial z^i}{\partial x^\alpha} y^\alpha \]

and

\[ z^\tilde{i} = \frac{\partial z^\tilde{i}}{\partial x^\alpha} z^\alpha . \]
We assume that \((\theta, \mu) = (Th \circ \rho, h \circ \eta)\). If \(z^\alpha t_\alpha \in \Gamma (F, \nu, N)\) is arbitrary, then

\[
(2.1) \quad \Gamma (Th \circ \rho, h \circ \eta) (z^\alpha t_\alpha) f (h \circ \eta (\varkappa)) = \left( \theta^i \cdot z^\alpha \frac{\partial f}{\partial \varkappa^i} \right) (h \circ \eta (\varkappa)),
\]

for any \(f \in \mathcal{F} (N)\) and \(\varkappa \in N\).

The coefficients \(\rho^i\) respectively \(\theta^i\) change to \(\rho^{\prime i}\) respectively \(\theta^{\prime i}\) according to the rule:

\[
(2.2) \quad \rho^{\prime i} = \Lambda^i_\alpha \rho^i \frac{\partial x^{\prime i}}{\partial x^i},
\]

respectively

\[
(2.3) \quad \theta^{\prime i} = \Lambda^i_\alpha \theta^i \frac{\partial x^{\prime i}}{\partial x^i},
\]

where

\[
\|\Lambda^\alpha_i\| = \|\Lambda^\alpha_i\|^{-1}.
\]

Remark 2.1 The following equalities hold good:

\[
(2.4) \quad \rho^i \circ h \frac{\partial f \circ h}{\partial x^i} = \left( \theta^i \cdot \frac{\partial f}{\partial \varkappa^i} \right) \circ h, \forall f \in \mathcal{F} (N).
\]

and

\[
(2.5) \quad \left( L^\gamma_\alpha \circ h \right) \left( \rho^k \circ h \right) = \left( \rho^i \circ h \right) \frac{\partial \left( \rho^k \circ h \right)}{\partial x^i} - \left( \rho^j \circ h \right) \frac{\partial \left( \rho^k \circ h \right)}{\partial x^j}.
\]

Let \((E, \pi, M) \in |B|\) and \(\left( \hat{E}, \hat{\pi}, \hat{M} \right)\) its dual. We have the \(B\)-morphism

\[
(2.6) \quad \begin{array}{c}
\pi^* (h^* F) \quad \pi^* (h^* \nu) \\
\downarrow \quad \downarrow
\end{array} \quad \begin{array}{c}
F \\
\nu
\end{array} \quad \begin{array}{c}
M \quad \xrightarrow{h \circ \pi^*} \quad \hat{N}
\end{array}
\]

Let \(\left( \hat{\pi}^* (h^* F), Id_{\hat{E}} \right)\) be the \(B\)-morphism of \(\left( \pi^* (h^* F), \pi^* (h^* \nu), \hat{E} \right)\) source and \(\left( \hat{E}, \tau_{\hat{E}}, \hat{E} \right)\) target, where

\[
(2.7) \quad \begin{array}{c}
\pi^* (h^* F) \quad \pi^* (h^* \nu) \\
\downarrow \quad \downarrow
\end{array} \quad \begin{array}{c}
\hat{T} \hat{E} \\
\hat{Z}^\alpha T \left( \frac{\partial \pi^* (h^* \nu)}{\partial x^i} \right)
\end{array} \quad \begin{array}{c}
Z^\alpha \cdot \rho^i \circ h \circ \pi^* \frac{\partial \pi^* (h^* \nu)}{\partial x^i} \\
\left( \hat{u}_x \right)
\end{array}
\]

Using the operation

\[
\Gamma \left( \pi^* (h^* F), \pi^* (h^* \nu), \hat{E} \right)^2 \quad \xrightarrow{[\cdot]_{\hat{\pi}^* (h^* F)}} \quad \Gamma \left( \pi^* (h^* F), \pi^* (h^* \nu), \hat{E} \right)
\]

5
defined by

\[
[T_\alpha, T_\beta]_{\pi^*(h^*F)} = \left( L_{\alpha\beta}^\gamma \circ h \circ \pi^* \right) T_\gamma,
\]

(2.8)

\[
[T_\alpha, fT_\beta]_{\pi^*(h^*F)} = f \left( L_{\alpha\beta}^\gamma \circ h \circ \pi^* \right) T_\gamma + \left( \rho_\alpha^\gamma \circ h \circ \pi^* \right) \frac{\partial f}{\partial x^\gamma} T_\beta,
\]

\[
[fT_\alpha, T_\beta]_{\pi^*(h^*F)} = -\left[ T_\beta, fT_\alpha \right]_{\pi^*(h^*F)},
\]

for any \( f \in \mathcal{F}(\tilde{E}) \), it results that

\[
\left( \left( \pi^* (h^*F), \pi^* (h^*\nu), \tilde{E} \right), \left[ \cdot, \cdot \right]_{\pi^*(h^*F)} \right) \left( \tilde{E}, \tilde{\nu}, \tilde{E} \right) \left( \rho^* \right), \left( \tilde{E}, \tilde{\pi}, \tilde{E} \right) \left( \rho, \eta \right)
\]

is a Lie algebroid.

### 3 Natural and adapted basis

In the following we consider the following diagram:

\[
\begin{array}{c}
\tilde{E} \\
\pi \downarrow \\
M \\
\downarrow h \\
N
\end{array}
\quad \left( F, [\cdot, \cdot]_{F,h}, (\rho, \eta) \right)
\]

(3.1)

where \((E, \pi, M) \in \mathcal{B}^\mathbb{V}\) and \((F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta)\) is a generalized Lie algebroid.

Let \((\rho, \eta) \Gamma\) be a \((\rho, \eta)\)-connection for the vector bundle \((\tilde{E}, \tilde{\pi}, M)\).

We take \((x^i, p_a)\) as canonical local coordinates on \((\tilde{E}, \tilde{\pi}, M)\), where \(i \in \mathbb{1, m}\) and \(a \in \mathbb{1, r}\). Let

\[
(x^i, p_a) \rightarrow (x^i(x^i), p_a(a(x^i, p_a)))
\]

be a change of coordinates on \((\tilde{E}, \tilde{\pi}, M)\). Then the coordinates \(p_a\) change to \(p_a^\prime\) by the rule:

\[
p_a^\prime = M_a^\alpha p_a.
\]

Let

\[
\frac{\partial}{\partial x^i} = \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial p_a} \right) = \left( \partial_i, \partial_a \right)
\]

(3.2)

be the natural base of the dual tangent Lie algebroid \( \left( T\tilde{E}^* \tau_{\tilde{E}}^* \tilde{E} \right), [\cdot, \cdot]_{T\tilde{E}^*}, \left( Id, \tilde{E} \right) \left( \tilde{E}, \tilde{\pi}, \tilde{E} \right) \left( \rho, \eta \right) \).

For any sections

\[
Z^\alpha T_\alpha \in \Gamma \left( \pi^* (h^*F), \pi^* (h^*F), \tilde{E} \right)
\]
and
\[ Y_α \hat{\partial} \in \Gamma (V^* T^* E, \tau^*, E) \]
we obtain the section
\[ Z^α \tilde{\partial}_α + Y_α \hat{\partial} = Z^α \left( T_α \oplus \left( \rho_α^* \circ h \circ \pi^* \right) \hat{\partial}_i \right) + Y_α \left( 0_{\pi^* (h^* F)} \oplus \hat{\partial} \right) \]
\[ = Z^α T_α \oplus \left( Z^α \left( \rho_α^* \circ h \circ \pi^* \right) \hat{\partial}_i + Y_α \hat{\partial} \right) \in \Gamma \left( \pi^* (h^* F) \oplus T^* E, \pi, E \right). \]

Since we have
\[ Z^α \tilde{\partial}_α + Y_α \hat{\partial} = 0 \]
\[ Z^α T_α = 0 \land Z^α \left( \rho_α^* \circ h \circ \pi^* \right) \hat{\partial}_i + Y_α \hat{\partial} = 0, \]
it implies \( Z^α = 0, \) \( α \in \overline{1, \rho} \) and \( Y_α = 0, \) \( α \in \overline{1, \tau}. \)

Therefore, the sections \( \tilde{\partial}_1, ..., \tilde{\partial}_p, \hat{\partial}, ..., \hat{\partial} \) are linearly independent.

We consider the vector subbundle \( \left( \rho, \eta \right) T^* E, \left( \rho, \eta \right) \tau^*, E \) of the vector bundle \( \left( \pi^* (h^* F) \oplus T^* E, \pi, E \right), \) for which the \( \mathcal{F} E \)-module of sections is the \( \mathcal{F} E \)-submodule of \( \left( \Gamma \left( \pi^* (h^* F) \oplus T^* E, \pi, E \right), +, \cdot \right), \) generated by the set of sections \( \left( \tilde{\partial}_α, \hat{\partial} \right) \) which is called the natural \( \left( \rho, \eta \right) \)-base.

The matrix of coordinate transformation on \( \left( \rho, \eta \right) T^* E, \left( \rho, \eta \right) \tau^*, E \) at a change of fibred charts is

\[
\begin{pmatrix}
\Lambda^α_0 \circ h \circ \pi^* & 0 \\
\left( \rho_α^* \circ h \circ \pi^* \right) \frac{\partial M^a_α \circ \pi^*}{\partial x_i} & y^b M^a_α \circ \pi^*
\end{pmatrix}
\]

We have the following

**Theorem 3.1** Let \( \left( \hat{\rho}, \text{Id}_E^* \right) \) be the \( \mathbf{B}^\gamma \)-morphism of \( \left( \rho, \eta \right) T^* E, \left( \rho, \eta \right) \tau^*, E \) source and \( \left( T^* E, \tau^*, E \right) \) target, where

\[
\left( \rho, \eta \right) T^* E \xrightarrow{\hat{\rho}} T^* E
\]

\[ \left( Z^α \tilde{\partial}_α + Y_α \hat{\partial} \right) (u_x) \mapsto \left( Z^α \left( \rho_α^* \circ h \circ \pi^* \right) \tilde{\partial}_i + Y_α \hat{\partial} \right) (u_x) \]

Using the operation

\[ \Gamma \left( \rho, \eta \right) T^* E, \left( \rho, \eta \right) \tau^*, E \]

\[ \xrightarrow{\left[ \right]_{\rho, \eta} T^* E} \Gamma \left( \rho, \eta \right) T^* E, \left( \rho, \eta \right) \tau^*, E \]
defined by

\[
\left[ \left( Z_1^\alpha \partial^\alpha_a + Y_1^a \partial_a \right), \left( Z_2^\beta \partial^\beta_b + Y_2^b \partial_b \right) \right]_{(\rho, \eta) T^* E} = \left[ Z_1^\alpha T_a, Z_2^\beta T_b \right]_{(h^* F)} \oplus \left[ \left( \rho^\gamma_a \circ h \circ \pi^* \right) Z_1^\gamma \partial^\gamma_a + Y_1^a \partial^a, \left( \rho^\beta_b \circ h \circ \pi^* \right) Z_2^\beta \partial^\beta_b + Y_2^b \partial_b \right]_{\pi^* T^* E},
\]

for any \( Z_1^\alpha \partial^\alpha_a + Y_1^a \partial_a \) and \( Z_2^\beta \partial^\beta_b + Y_2^b \partial_b \), we obtain that the couple

\[
\left( \left[ \cdot, \cdot \right]_{(\rho, \eta) T^* E}, \left( \cdot, \pi^* \right) \right)
\]

is a Lie algebroid structure for the vector bundle \( \left( (\rho, \eta) T^* E, (\rho, \eta) \tau^* \pi, \pi^* \right) \).

The Lie algebroid

\[
\left( \left( (\rho, \eta) T^* E, (\rho, \eta) \tau^* \pi, \pi^* \right), \left[ \cdot, \cdot \right]_{(\rho, \eta) T^* E}, \left( \cdot, \pi^* \right) \right)
\]

is called the Lie algebroid generalized tangent bundle of dual vector bundle \( \left( E, \pi^* \right) \).

**Remark 3.1** The following equalities hold good:

\[
\begin{align*}
\left[ \partial^\alpha_a, \partial^\beta_b \right]_{(\rho, \eta) T^* E} & = \left( L^\gamma_{\alpha\beta} \circ h \circ \pi^* \right) \partial^\gamma_a, \\
\left[ \partial^\alpha_a, \partial_b \right]_{(\rho, \eta) T^* E} & = 0_{(\rho, \eta) T^* E}, \\
\left[ \partial^\alpha_a, \partial^\beta_b \right]_{(\rho, \eta) T^* E} & = 0_{(\rho, \eta) T^* E}.
\end{align*}
\]

We consider the \( B^v \)-morphism \( (\rho, \eta) \pi^* !, \pi^* (h^* F) \) given by the commutative diagram

\[
\begin{array}{ccc}
(\rho, \eta) T^* E & \xrightarrow{(\rho, \eta) \pi^* !} & \pi^* (h^* F) \\
(\rho, \eta) \tau^* \pi & \xrightarrow{id_*} & \pi^* \tau^* E \\
\pi^* & \xrightarrow{pr_1} & E
\end{array}
\]

Using the components, this is defined as:

\[
(\rho, \eta) \pi^* ! \left( \tilde{Z}^\alpha \partial^\alpha_a + Y_a \partial_a \right) \left( u_x \right) = \left( \tilde{Z}^\alpha \partial^\alpha_a \right) \left( u_x \right),
\]

for any \( \tilde{Z}^\alpha \partial^\alpha_a + Y_a \partial_a \in \left( (\rho, \eta) T^* E, (\rho, \eta) \tau^* \pi, \pi^* \right) \).
Using the $\mathcal{B}^\vee$-morphism $\left((\rho, \eta) \bar{\pi}^!, Id_{\bar{E}}\right)$ and the $\mathcal{B}^\vee$-morphism (2.7) we obtain the tangent $(\rho, \eta)$-application $\left(\left((\rho, \eta) T\bar{\pi}, h \circ \bar{\pi}\right)\right)$ of $\left(\left((\rho, \eta) \tau^*_{\bar{E}}, \bar{E}\right)\right)$ source and $(F, \nu, N)$ target.

Using the $\mathcal{B}^\vee$-morphisms (2.6) and (3.7) we obtain the tangent $(\rho, \eta)$-application $\left(\left((\rho, \eta) T\bar{\pi}^*, h \circ \bar{\pi}^*\right)\right)$ of $\left(\left((\rho, \eta) \tau^*_{\bar{E}}, \bar{E}\right)\right)$ source and $(F, \nu, N)$ target.

**Definition 3.1** The kernel of the tangent $(\rho, \eta)$-application is written

$$\left(\left(V(\rho, \eta) T\bar{E}, (\rho, \eta) \tau^*_{\bar{E}}, \bar{E}\right)\right)$$

and is called the **vertical subbundle**.

We remark that the set $\left\{\bar{\partial}^a, a \in \Gamma, r\right\}$ is a base of the $\mathcal{F}(\bar{E})$-module

$$\left(\left(\Gamma\left(\left(V(\rho, \eta) T\bar{E}, (\rho, \eta) \tau^*_{\bar{E}}, \bar{E}\right)\right), +, \cdot\right)\right).$$

**Proposition 3.1** The short sequence of vector bundles

\begin{equation}
\begin{array}{cccccc}
0 & \rightarrow & V(\rho, \eta) T\bar{E} & \rightarrow & (\rho, \eta) \pi^! & \rightarrow & (\rho, \eta) \pi^* (h^*F) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & Id_{\bar{E}} & \rightarrow & Id_{\bar{E}} & \rightarrow & Id_{\bar{E}} & \rightarrow & 0 \\
E & & E & & E & & E & & E
\end{array}
\end{equation}

is exact.

Let $(\rho, \eta) \Gamma$ be a $(\rho, \eta)$-connection for the vector bundle $\left(\left(\bar{E}, \pi^*, M\right)\right)$, i. e. a **Man**-morphism of $\left((\rho, \eta) T\bar{E}\right)$ source and $\left(\left(V(\rho, \eta) T\bar{E}\right)\right)$ target defined by

\begin{equation}
(\rho, \eta) \Gamma \left(\left((\bar{Z}^a \partial_{\bar{a}} + Y_b \partial)^b \bar{u}_x\right)\right) = \left((Y_b - (\rho, \eta) \Gamma_{bc} \bar{Z}^a) \partial \right)^b \bar{u}_x,
\end{equation}

such that the $\mathcal{B}^\vee$-morphism $\left((\rho, \eta) \Gamma, Id_{\bar{E}}\right)$ is a split to the left in the previous exact sequence. Its components satisfy the law of transformation

\begin{equation}
(\rho, \eta) \Gamma_{bc} = M_{\bar{a}}^{\bar{b}} p_\alpha^{\bar{a}} \left(- (\rho^\alpha \circ h \circ \pi^*) \frac{\partial M^{\alpha \bar{a}}_{bc}}{\partial \bar{x}} + (\rho, \eta) \Gamma_{bc} \right) \Lambda^r_{\gamma} \circ h \circ \pi^*.
\end{equation}

The kernel of the $\mathcal{B}^\vee$-morphism $\left((\rho, \eta) \Gamma, Id_{\bar{E}}\right)$ is written $\left(\left(H(\rho, \eta) T\bar{E}, (\rho, \eta) \tau^*_{\bar{E}}, \bar{E}\right)\right)$ and is called the **horizontal vector subbundle**.

We remark that the horizontal and the vertical vector subbundles are interior differential systems of the Lie algebroid generalized tangent bundle

$$\left(\left((\rho, \eta) T\bar{E}, (\rho, \eta) \tau^*_{\bar{E}}, \bar{E}\right), \left[\cdot, \cdot\right]_{(\rho, \eta) T\bar{E}}, \left(\rho, Id_{\bar{E}}\right)\right).$$
We put the problem of finding a base for the \( F(\tilde{E}) \)-module 
\[
\left( \Gamma \left( H(\rho, \eta) T\tilde{E}, (\rho, \eta) \tau_{\tilde{E}}^*, E \right), \cdot, \cdot \right)
\]
of the type 
\[
\tilde{\delta}_\alpha = Z_\alpha^\beta \tilde{\delta}_\beta + Y_{\alpha\alpha} \tilde{\partial}^a , \alpha \in \Gamma, r
\]
which satisfies the following conditions:
\[
\begin{align}
\Gamma \left( (\rho, \eta) \pi! , Id_{\tilde{E}} \right) \left( \tilde{\delta}_\alpha \right) &= T_\alpha , \\
\Gamma \left( (\rho, \eta) \Gamma , Id_{\tilde{E}} \right) \left( \tilde{\delta}_\alpha \right) &= 0 .
\end{align}
\]
Then we obtain the sections 
\[
(3.14) \quad \tilde{\delta}_\alpha = \tilde{\delta}_\alpha + (\rho, \eta) \Gamma_{ba} \tilde{\partial}^b = T_\alpha \oplus \left( (\rho^j_\alpha \circ h \circ \pi^* ) \tilde{\partial}_i^* - (\rho, \eta) \Gamma_{ba} \tilde{\partial}^b \right) ,
\]
such that their law of change is a tensorial law under a change of vector fiber charts.

The base \( \left( \tilde{\delta}_\alpha , \tilde{\partial}^a \right) \) will be called the *adapted* \((\rho, \eta)\)-base.

**Remark 3.2** The following equality holds good
\[
(3.15) \quad \Gamma \left( \tilde{\rho} , Id_{\tilde{E}} \right) \left( \tilde{\delta}_\alpha \right) = (\rho^j_\alpha \circ h \circ \pi^* ) \tilde{\partial}_i^* - (\rho, \eta) \Gamma_{ba} \tilde{\partial}^b .
\]

Moreover, if \((\rho, \eta) \Gamma\) is the \((\rho, \eta)\)-connection associated to a connection \( \Gamma \) (see [1]), then we obtain
\[
(3.16) \quad \Gamma \left( \tilde{\rho} , Id_{\tilde{E}} \right) \left( \tilde{\delta}_\alpha \right) = (\rho^j_\alpha \circ h \circ \pi^* ) \tilde{\partial}_i^* ,
\]
where \( \left( \tilde{\delta}_i^* , \tilde{\partial}^a \right) \) is the adapted base for the \( F(\tilde{E}) \)-module \( \left( \Gamma \left( T\tilde{E}, \tau_{\tilde{E}}^*, E \right), \cdot, \cdot \right) \).

**Theorem 3.2** The following equality holds good
\[
(3.17) \quad \left[ \tilde{\delta}_\alpha^* , \delta_\beta^* \right]_{(\rho, \eta) T\tilde{E}} = \left( L_{\alpha\beta}^\gamma \circ h \circ \pi^* \right) \tilde{\delta}_\gamma^* + (\rho, \eta, h) \mathbb{R}_{b_{\alpha\beta}} \tilde{\partial}^b ,
\]
where
\[
(3.18) \quad (\rho, \eta, h) \mathbb{R}_{b_{\alpha\beta}} = \Gamma \left( \tilde{\rho} , Id_{\tilde{E}} \right) \left( \tilde{\delta}_\beta^* \right) ((\rho, \eta) \Gamma_{ba} ) \]
\[
+ \Gamma \left( \tilde{\rho} , Id_{\tilde{E}} \right) \left( \tilde{\delta}_\alpha^* \right) ((\rho, \eta) \Gamma_{b\alpha} ) - \left( L_{\alpha\beta}^\gamma \circ h \circ \pi^* \right) (\rho, \eta) \Gamma_{b\gamma} ,
\]

Moreover, we have:
\[
(3.19) \quad \left[ \tilde{\delta}_\alpha^* , \tilde{\partial}^a \right]_{(\rho, \eta) T\tilde{E}} = -\Gamma \left( \tilde{\rho} , Id_{\tilde{E}} \right) \left( \tilde{\partial}^a \right) ((\rho, \eta) \Gamma_{ba} ) \tilde{\partial}^b ,
\]
and

\[(3.20) \quad \Gamma \left( \tilde{\rho}, \text{Id}_E \right) \left[ \delta_\alpha, \delta_\beta \right]_{(\rho, \eta) T_E} = \left[ \Gamma \left( \tilde{\rho}, \text{Id}_E \right) \left( \delta_\alpha \right), \Gamma \left( \tilde{\rho}, \text{Id}_E \right) \left( \delta_\beta \right) \right]_{T_E}.
\]

Let \((d\tilde{z}^\alpha, d\tilde{p}_a)\) be the natural dual \((\rho, \eta)\)-base of natural \((\rho, \eta)\)-base \(\left( \partial_\alpha, \tilde{\partial} \right)\).

This is determined by the equations

\[
\begin{cases}
\langle d\tilde{z}^\alpha, \partial_\beta \rangle = \delta_\alpha^\beta, \\
\langle d\tilde{p}_a, \partial_\beta \rangle = 0,
\end{cases}
\]

\[
\begin{cases}
\langle d\tilde{z}^\alpha, \tilde{\partial} \rangle = 0, \\
\langle d\tilde{p}_a, \tilde{\partial} \rangle = \delta_a^b.
\end{cases}
\]

We consider the problem of finding a base for the \(F \left( ^*E \right)\)-module

\[
\left( \Gamma \left( \left( V (\rho, \eta) T^* \right), (\rho, \eta) \tau \right), (\rho, \eta), +, \cdot \right)
\]

of the type

\[
\delta \tilde{p}_a = \theta_a d\tilde{z}^a + \omega_a^b d\tilde{p}_b, \quad a \in \mathbb{1}, r
\]

which satisfies the following conditions:

\[(3.21) \quad \langle \delta \tilde{p}_a, \tilde{\partial} \rangle = \delta_a^b \wedge \langle \delta \tilde{p}_a, \tilde{\partial} \rangle = 0,
\]

We obtain the sections

\[(3.22) \quad \delta \tilde{p}_a = -(\rho, \eta) \Gamma_a d\tilde{z}^a + d\tilde{p}_a, \quad a \in \mathbb{1}, r.
\]

such that their changing rule is tensorial under a change of vector fiber charts. The base \((d\tilde{z}^\alpha, \delta \tilde{p}_a)\) will be called the adapted dual \((\rho, \eta)\)-base.

4 The lift of a differentiable curve

We consider the following diagram:

\[(4.1) \quad \begin{array}{c}
\begin{array}{c}
E
F, [.,]_{F,h}, (\rho, \eta)
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
M
h \quad N
\end{array}
\end{array}
\]

where \((E, \pi, M) \in |\mathbf{B}^\gamma|\) and \((\left( F, \nu, N \right), [.,]_{F,h}, (\rho, \eta)) \in |\mathbf{GLA}|.

We admit that \((\rho, \eta) \Gamma\) is a \((\rho, \eta)\)-connection for the vector bundle \(\left( ^*E, \pi, M \right)\) and \(I \xrightarrow{\cdot} M\) is a differentiable curve. We know that

\[
\left( \tilde{E} | \text{Im}(\eta \circ h \circ c), \tilde{\pi} | \text{Im}(\eta \circ h \circ c), \text{Im} (\eta \circ h \circ c) \right)
\]

11
is a vector subbundle of the vector bundle \( (\dot{E}, \pi, M) \).

**Definition 4.1** If

\[
I \xrightarrow{\dot{c}} \dot{E}_{|\text{Im}(\eta \circ h)\circ c(t)} \quad t \mapsto p_a(t) s^\alpha(\eta \circ h \circ c(t))
\]

is a differentiable curve such that there exists \( g \in \text{Man}(\dot{E}, F) \) such that the following conditions are satisfied:

1. \( (g, h) \in B^v \left( (\dot{E}, \dot{\pi}, M), (F, \nu, N) \right) \) and

2. \( \rho \circ g \circ \dot{c}(t) = \frac{d(\eta \circ h \circ c)(t)}{dt} \frac{\partial}{\partial x^i} ((\eta \circ h \circ c)(t)) \), for any \( t \in I \), then we will say that \( \dot{c} \) is the \((g, h)\)-lift of the differentiable curve \( c \).

**Remark 4.1** The condition 2 is equivalent with the following affirmation:

\[
\rho^a_i (\eta \circ h \circ c(t)) g^{\alpha \alpha} (h \circ c(t)) p_a(t) = \frac{d(\eta \circ h \circ c)(t)}{dt}, \; i \in \overline{1, m}.
\]

**Definition 4.2** If \( I \xrightarrow{\dot{c}} \dot{E}_{|\text{Im}(\eta \circ h)\circ c(t)} \) is a differentiable \((g, h)\)-lift of the differentiable curve \( c \), then the section

\[
\text{Im} (\eta \circ h \circ c) \xrightarrow{\tilde{u}(c, \dot{c})} \dot{E}_{|\text{Im}(\eta \circ h)\circ c(t)} \quad \eta \circ h \circ c(t) \mapsto \dot{c}(t)
\]

will be called the **canonical section associated to the couple** \((c, \dot{c})\).

**Definition 4.3** If \((g, h) \in B^v \left( (\dot{E}, \dot{\pi}, M), (F, \nu, N) \right) \) has the components \( g^{\alpha \alpha}; a \in \overline{1, r}, \; \alpha \in \overline{1, p} \) such that for any vector local \((n + p)\)-chart \((V, t_V)\) of \((F, \nu, N)\) there exists the real functions

\[
V \xrightarrow{\tilde{g}_{aa}} \mathbb{R} ; \; a \in \overline{1, r}, \; \alpha \in \overline{1, p}
\]

such that

\[
\tilde{g}_{aa}(\varkappa) \cdot g^{ab}(\varkappa) = \delta^b_a, \; \forall \varkappa \in V,
\]

then we will say that the \(B^v\)-morphism \((g, h)\) is **locally invertible**.

**Remark 4.2** In particular, if \((\text{Id}_{TM}, \text{Id}_M, \text{Id}_M) = (\rho, \eta, h)\) and the \(B^v\) morphism \((g, \text{Id}_M)\) is locally invertible, then we have the differentiable \((g, \text{Id}_M)\)-lift

\[
I \xrightarrow{\dot{c}} TM \\
(t \mapsto \tilde{g}_{ji}(c(t)) \frac{dc^j(t)}{dt} dx^i(c(t)))
\]
Definition 4.4 If \( \tilde{c} : E \to \mathbb{R} \) is a differentiable \((g, h)\)-lift for the curve 
\( c \) such that its components functions \( (p_b, b \in \Gamma, \tau) \) are solutions for the differentiable 
system of equations:

\[
\frac{du_a}{dt} + (\rho, \eta)_{\Gamma_{ba}} \circ \tilde{u}(c, \tilde{c}) \circ (\eta \circ h \circ c) \cdot g^{ha} \circ h \circ c \cdot u_a = 0,
\]

then we will say that the \((g, h)\)-lift \( \tilde{c} \) is parallel with respect to the \((\rho, \eta)\)-connection 
\( (\rho, \eta)_{\Gamma} \).

Remark 4.3 \( \tilde{g}_{ji} \circ c \cdot \frac{dc^i}{dt}, j \in \Gamma, m \) are solutions for the differentiable system of equations

\[
\frac{du_j}{dt} + \Gamma_{jk} \circ \tilde{u}(c, \tilde{c}) \circ g^{kh} \circ c \cdot u_h = 0,
\]

namely

\[
\frac{d}{dt} \left( \tilde{g}_{ji} \circ c(t) \cdot \frac{dc^i(t)}{dt} \right) + \Gamma_{jk} \left( c(t), \left( \tilde{g}_{ji} \circ c(t) \cdot \frac{dc^i(t)}{dt} \right) \cdot dx^i(c(t)) \right) \cdot \frac{dc^k(t)}{dt} = 0.
\]

5 Remarkable Mod-endomorphisms

Now, let us consider the following diagram:

\[
\begin{array}{ccc}
E & \rightarrow & (F, [,]_{F,h}, (\rho, \eta)) \\
\downarrow \tilde{z} & \downarrow \nu & \downarrow \pi \\
M & \rightarrow & N
\end{array}
\]

where \((E, \pi, M) \in |B^\nu| \) and \((F, \nu, N), [,]_{F,h}, (\rho, \eta) \) is a generalized Lie algebroid.

Let \((\rho, \eta)_{\Gamma} \) be a \((\rho, \eta)\)-connection for the vector bundle \((E, \pi, M)\).

Definition 5.1 For any Mod-endomorphism \( e \) of

\[
\begin{pmatrix}
\Gamma \left( (\rho, \eta)_{\tau_E}, (\rho, \eta)_{\tau_E} \right), +, \cdot
\end{pmatrix}
\]

we define the application of Nijenhuis type

\[
\Gamma \left( (\rho, \eta)_{\tau_E}, (\rho, \eta)_{\tau_E} \right)^2 \xrightarrow{N_e} \Gamma \left( (\rho, \eta)_{\tau_E}, (\rho, \eta)_{\tau_E} \right)
\]

defined by

\[
N_e(X, Y) = [eX, eY]_{(\rho, \eta)_{\tau_E}} + e^2 [X, Y]_{(\rho, \eta)_{\tau_E}} - e [eX, Y]_{(\rho, \eta)_{\tau_E}} - e [X, eY]_{(\rho, \eta)_{\tau_E}},
\]

for any \( X, Y \in \Gamma \left( (\rho, \eta)_{\tau_E}, (\rho, \eta)_{\tau_E} \right) \).
5.1 Projectors

**Definition 5.1.1** Any Mod-endomorphism $e$ of $\Gamma \left( (\rho, \eta)^* \tau^*_E, \tau^*_E \right)$ with the property

\[(5.1.1) \quad e^2 = e\]

will be called *projector*.

**Example 5.1.1** The Mod-endomorphism

\[
\Gamma \left( (\rho, \eta)^* \tau^*_E, \tau^*_E \right) \xrightarrow{\mathcal{V}} \Gamma \left( (\rho, \eta)^* \tau^*_E, \tau^*_E \right)
\]

\[
\tilde{Z}^\alpha \partial^a + Y_a \overline{\partial} \quad \rightarrow \quad Y_a \overline{\partial}
\]

is a projector which will be called the *vertical projector*.

**Remark 5.1.1** We have $\mathcal{V} \left( \tilde{\delta}^\alpha \right) = 0$ and $\mathcal{V} \left( \tilde{\partial}^a \right) = \tilde{\partial}^a$. Therefore, it follows

\[
\mathcal{V} \left( \tilde{\partial}^a \right) = -(\rho, \eta) \Gamma^b_{\alpha a} \tilde{\partial}^b.
\]

In addition, we obtain the equality

\[(5.1.2) \quad \Gamma ((\rho, \eta) \Gamma, Id_E) \left( Z^\alpha \tilde{\partial}^a + Y_a \overline{\partial} \right) = \mathcal{V} \left( Z^\alpha \tilde{\partial}^a + Y_a \overline{\partial} \right),
\]

for any $Z^\alpha \tilde{\partial}^a + Y_a \overline{\partial} \in \Gamma \left( (\rho, \eta)^* \tau^*_E, \tau^*_E \right)$.

**Theorem 5.1.1** A $(\rho, \eta)$-connection for the vector bundle $(E, \pi, M)$ is characterized by the existence of a Mod-endomorphism $\mathcal{V}$ of $\left( \Gamma \left( (\rho, \eta)^* \tau^*_E, \tau^*_E \right), +, \cdot \right)$ with the properties:

\[(5.1.3) \quad \mathcal{V}(X) = X \iff X \in \Gamma \left( (\rho, \eta)^* \tau^*_E, \tau^*_E \right)
\]

**Example 5.1.2** The Mod-endomorphism

\[
\Gamma \left( (\rho, \eta)^* \tau^*_E, \tau^*_E \right) \xrightarrow{\mathcal{H}} \Gamma \left( (\rho, \eta)^* \tau^*_E, \tau^*_E \right)
\]

\[
\tilde{Z}^\alpha \partial^a + Y_a \overline{\partial} \quad \rightarrow \quad \tilde{Z}^\alpha \partial^a
\]

is a projector which will be called the *horizontal projector*.

**Remark 5.1.2** We have $\mathcal{H} \left( \tilde{\delta}^\alpha \right) = \tilde{\delta}^\alpha$ and $\mathcal{H} \left( \tilde{\partial}^a \right) = 0$. Therefore, we obtain $\mathcal{H} \left( \tilde{\partial}^a \right) = \tilde{\partial}^a$.
Theorem 5.1.2 A \((\rho, \eta)\)-connection for the vector bundle \(\left( \overset{\ast}{E}, \overset{\ast}{\pi}, M \right)\) is characterized by the existence of a \(\text{Mod}\)-endomorphism \(\overset{\ast}{H}\) of

\[
\left( \Gamma \left( (\rho, \eta) T\overset{\ast}{E}, (\rho, \eta) \tau_{\overset{\ast}{E}} \overset{\ast}{E} \right), +, \cdot \right)
\]

with the properties:

\[
\overset{\ast}{H} \left( (\rho, \eta) T\overset{\ast}{E}, (\rho, \eta) \tau_{\overset{\ast}{E}} \overset{\ast}{E} \right) \subset \Gamma \left( H (\rho, \eta) T\overset{\ast}{E}, (\rho, \eta) \tau_{\overset{\ast}{E}} \overset{\ast}{E} \right)
\]

\[
(5.1.4)
\]

\[
\overset{\ast}{H} X = X \iff X \in \Gamma \left( H (\rho, \eta) T\overset{\ast}{E}, (\rho, \eta) \tau_{\overset{\ast}{E}} \overset{\ast}{E} \right).
\]

Corollary 5.1.1 A \((\rho, \eta)\)-connection for the vector bundle \(\left( \overset{\ast}{E}, \overset{\ast}{\pi}, M \right)\) is characterized by the existence of a \(\text{Mod}\)-endomorphism \(\overset{\ast}{H}\) of

\[
\left( \Gamma \left( (\rho, \eta) T\overset{\ast}{E}, (\rho, \eta) \tau_{\overset{\ast}{E}} \overset{\ast}{E} \right), +, \cdot \right)
\]

with the properties:

\[
\overset{\ast}{2} \overset{\ast}{H} = \overset{\ast}{H}
\]

\[
(5.1.5)
\]

\[
\text{Ker} \left( \overset{\ast}{H} \right) = \left( \Gamma \left( V (\rho, \eta) T\overset{\ast}{E}, (\rho, \eta) \tau_{\overset{\ast}{E}} \overset{\ast}{E} \right), +, \cdot \right).
\]

Remark 5.1.3 For any

\[
X \in \Gamma \left( (\rho, \eta) T\overset{\ast}{E}, (\rho, \eta) \tau_{\overset{\ast}{E}} \overset{\ast}{E} \right)
\]

we obtain the following unique decomposition

\[
X = \overset{\ast}{H} X + \overset{\ast}{V} X.
\]

Proposition 5.1.1 After some calculations we obtain

\[
N_\overset{\ast}{V} (X, Y) = \overset{\ast}{V} \left[ \overset{\ast}{H} X, \overset{\ast}{H} Y \right]_{(\rho, \eta) T\overset{\ast}{E}} = N_\overset{\ast}{H} (X, Y),
\]

for any \(X, Y \in \Gamma \left( (\rho, \eta) T\overset{\ast}{E}, (\rho, \eta) \tau_{\overset{\ast}{E}} \overset{\ast}{E} \right)\).

Corollary 5.1.2 The horizontal interior differential system

\[
\left( H (\rho, \eta) T\overset{\ast}{E}, (\rho, \eta) \tau_{\overset{\ast}{E}} \overset{\ast}{E} \right)
\]

is involutive if and only if \(N_\overset{\ast}{V} = 0\) or \(N_\overset{\ast}{H} = 0\).
5.2 The almost product structure

Definition 5.2.1 Any $\text{Mod}$-endomorphism $e$ of

$$
\left( \Gamma \left( (\rho, \eta) T^*E, (\rho, \eta) \tau^*_{E^*}, E^* \right), +, \cdot \right)
$$

with the property

$$(5.2.1) \quad e^2 = \text{Id}$$

will be called the almost product structure.

Example 5.2.1 The $\text{Mod}$-endomorphism

$$
\left( \Gamma \left( (\rho, \eta) T^*E, (\rho, \eta) \tau^*_{E^*}, E^* \right), +, \cdot \right)
$$

$\tilde{Z}^\alpha \delta^\alpha + Y^a \partial^a \mapsto \tilde{Z}^\alpha \delta^\alpha - Y^a \partial^a$

is an almost product structure.

Remark 5.2.1 The previous almost product structure has the properties:

$$(5.2.2) \quad \mathcal{P} = 2\mathcal{H} - \text{Id};$$

$$\mathcal{P} = \text{Id} - 2\mathcal{V};$$

$$\mathcal{P} = \mathcal{H} - \mathcal{V}.$$

Remark 5.2.2 We obtain that $\mathcal{P} \left( \delta^\alpha \right) = \delta^\alpha$ and $\mathcal{P} \left( \partial^a \right) = -\partial^a$. Therefore, it follows

$$\mathcal{P} \left( \delta^\alpha \right) = \delta^\alpha - \rho \Gamma_{b\alpha} \partial^b.$$

Theorem 5.2.1 A $(\rho, \eta)$-connection for the vector bundle $\left( E^*, \pi, M \right)$ is characterized by the existence of a $\text{Mod}$-endomorphism $\mathcal{P}$ of

$$
\left( \Gamma \left( (\rho, \eta) T^*E, (\rho, \eta) \tau^*_{E^*}, E^* \right), +, \cdot \right)
$$

with the following property:

$$(5.2.3) \quad \mathcal{P}(X) = -X \iff X \in \Gamma \left( V (\rho, \eta) T^*E, (\rho, \eta) \tau^*_{E^*}, E^* \right).$$

Proposition 5.2.1 After some calculations, we obtain

$$N_{\mathcal{P}}(X, Y) = 4\mathcal{V} \left[ \mathcal{H}X, \mathcal{H}Y \right],$$

for any $X, Y \in \Gamma \left( (\rho, \eta) T^*E, (\rho, \eta) \tau^*_{E^*}, E^* \right)$.

Corollary 5.2.1 The horizontal interior differential system $\left( \mathcal{H} (\rho, \eta) T^*E, (\rho, \eta) \tau^*_{E^*}, E^* \right)$ is involutive if and only if $N_{\mathcal{P}} = 0$. 

16
5.3 The almost tangent structure

**Definition 5.3.1** Any $\text{Mod}$-endomorphism $e$ of $\left(\Gamma((\rho, \eta)T^*E, (\rho, \eta)\tau^*_E, E)^*\right)$ with the property

\[(5.3.1)\quad e^2 = 0\]

will be called the *almost tangent structure*.

**Example 5.3.1** If $(E, \pi, M) = (F, \nu, N)$, $g \in \text{Man}(E, E)^*$ such that $(g, h)$ is a locally invertible $B^\nu$-morphism, then the $\text{Mod}$-endomorphism

\[\Gamma\left(\Gamma((\rho, \eta)T^*E, (\rho, \eta)\tau^*_E, E)^*, E\right) \xrightarrow{J_{(g,h)}} \Gamma\left(\Gamma((\rho, \eta)T^*E, (\rho, \eta)\tau^*_E, E)^*, E\right)\]

\[\tilde{Z}^a\partial_a + Y_b\tilde{\partial} \mapsto \left(\tilde{g}_{ba} \circ h \circ \pi\right)\tilde{Z}^b\]

is an almost tangent structure which will be called the *almost tangent structure associated to the $B^\nu$-morphism $(g, h)$*. (See: Definition 4.3)

**Remark 5.3.1** We obtain that

\[J_{(g,h)}(\delta_a) = J_{(g,h)}(\partial_a) = \left(\tilde{g}_{ba} \circ h \circ \pi\right)\tilde{\partial}\]

and

\[J_{(g,h)}(\tilde{\partial}) = 0.\]

and we have the following properties:

\[(5.3.2)\quad J_{(g,h)} \circ \mathcal{P} = J_{(g,h)};\]

\[\mathcal{P} \circ J_{(g,h)} = -J_{(g,h)};\]

\[J_{(g,h)} \circ \mathcal{H} = J_{(g,h)};\]

\[\mathcal{H} \circ J_{(g,h)} = 0;\]

\[J_{(g,h)} \circ \mathcal{V} = 0;\]

\[\mathcal{V} \circ J_{(g,h)} = J_{(g,h)};\]

\[N \ast J_{(g,h)} = 0.\]

6 Tensor $d$-fields. Distinguished linear $(\rho, \eta)$-connections

We consider the following diagram:
where \( (E, \pi, M) \in |\mathcal{B}^v| \) and \( \left( (F, \nu, N), [\cdot]_{F,b}, (\rho, \eta) \right) \) is a generalized Lie algebroid.

Let \( (\rho, \eta) \Gamma \) be a \((\rho, \eta)\)-connection for the vector bundle \( \left( \ast E, \pi, M \right) \).

Let
\[
\mathcal{T}^{p,r}_{q,s} \left( (\rho, \eta) T \ast E, (\rho, \eta) \tau_{\ast E}, \ast E \right), +, \cdot
\]
be the \( \mathcal{F} \left( \ast E \right) \)-module of tensor fields by \((p,r)\)-type from the generalized tangent bundle
\[
H (\rho, \eta) T \ast E \oplus V (\rho, \eta) T \ast E, (\rho, \eta) \tau_{\ast E}, \ast E \right).
\]

An arbitrarily tensor field \( T \) is written by the form:
\[
T = T_{\rho_1 \ldots \rho_p a_1 \ldots a_r} \delta_{\alpha_1} \otimes \ldots \otimes \delta_{\alpha_p} \otimes d \tilde{z}^{a_1} \otimes \ldots \otimes d \tilde{z}^{a_r} \otimes \partial \otimes \ldots \otimes \partial \otimes \delta \tilde{p}_{\alpha_1} \otimes \ldots \otimes \delta \tilde{p}_{\alpha_r}.
\]

Let
\[
\left( \mathcal{T} \left( (\rho, \eta) T \ast E, (\rho, \eta) \tau_{\ast E}, \ast E \right), +, \cdot, \otimes \right)
\]
be the tensor fields algebra of generalized tangent bundle \( \left( (\rho, \eta) T \ast E, (\rho, \eta) \tau_{\ast E}, \ast E \right) \).

If \( T_1 \in \mathcal{T}^{p_1,r_1}_{q_1,s_1} \left( (\rho, \eta) T \ast E, (\rho, \eta) \tau_{\ast E}, \ast E \right) \) and \( T_2 \in \mathcal{T}^{p_2,r_2}_{q_2,s_2} \left( (\rho, \eta) T \ast E, (\rho, \eta) \tau_{\ast E}, \ast E \right) \), then the components of product tensor field \( T_1 \otimes T_2 \) are the products of local components of \( T_1 \) and \( T_2 \).

Therefore, we obtain \( T_1 \otimes T_2 \in \mathcal{T}^{p_1+p_2,r_1+r_2}_{q_1+q_2,s_1+s_2} \left( (\rho, \eta) T \ast E, (\rho, \eta) \tau_{\ast E}, \ast E \right) \).

Let \( \mathcal{D T} \left( (\rho, \eta) T \ast E, (\rho, \eta) \tau_{\ast E}, \ast E \right) \) be the family of tensor fields
\[
T \in \mathcal{T} \left( (\rho, \eta) T \ast E, (\rho, \eta) \tau_{\ast E}, \ast E \right)
\]
for which there exists
\[
T_1 \in \mathcal{T}^{p,0}_{q,0} \left( (\rho, \eta) T \ast E, (\rho, \eta) \tau_{\ast E}, \ast E \right)
\]
and
\[
T_2 \in \mathcal{T}^{0,r}_{0,s} \left( (\rho, \eta) T \ast E, (\rho, \eta) \tau_{\ast E}, \ast E \right)
\]
such that \( T = T_1 + T_2 \).

The \( \mathcal{F} \left( \ast E \right) \)-module \( \mathcal{D T} \left( (\rho, \eta) T \ast E, (\rho, \eta) \tau_{\ast E}, \ast E \right), +, \cdot \) will be called the module of distinguished tensor fields or the module of tensor \( d \)-fields.

Remark 5.1 The elements of
\[
\Gamma \left( (\rho, \eta) T \ast E, (\rho, \eta) \tau_{\ast E}, \ast E \right)
\]
respectively
\[
\Gamma \left( ((\rho, \eta) T \ast E), ((\rho, \eta) \tau_{\ast E}), \ast E \right)
\]
are tensor $d$-fields.

**Definition 6.1** Let $(\rho, \eta) \Gamma$ be a $(\rho, \eta)$-connection for the vector bundle $\left( \overset{\star}{E}, \pi, M \right)$ and let

\[(X, T) \xrightarrow{(\rho, \eta) \overset{\star}{D}} (\rho, \eta) \overset{\star}{D} X T\]

be a covariant $(\rho, \eta)$-derivative for the tensor algebra of generalized tangent bundle

\[\left( (\rho, \eta) T \overset{\star}{E}, (\rho, \eta) \tau _{E}, \overset{\star}{E} \right)\]

which preserves the horizontal and vertical distributions by parallelism.

If $(U, s_U)$ is a vector local $(m + r)$-chart for $\left( \overset{\star}{E}, \pi, M \right)$, then the real local functions

\[\left( (\rho, \eta) \overset{\star}{H}^\alpha _{\beta \gamma}, (\rho, \eta) \overset{\star}{H}^a _{b \gamma}, (\rho, \eta) \overset{\star}{V}^\alpha _{c \beta}, (\rho, \eta) \overset{\star}{V}^b _{c a} \right)\]

defined on $\pi^{-1}(U)$ and determined by the following equalities:

\[\begin{align*}
(\rho, \eta) \overset{\star}{D}_x \overset{\star}{\delta}_\beta &= (\rho, \eta) \overset{\star}{H}^\alpha _{\beta \gamma} \overset{\star}{\delta}_\alpha, & (\rho, \eta) \overset{\star}{D}_x \overset{\star}{\partial}^a &= (\rho, \eta) \overset{\star}{H}^a _{b \gamma} \overset{\star}{\partial}^b \\
(\rho, \eta) \overset{\star}{D}_x \overset{\star}{\partial}^\beta &= (\rho, \eta) \overset{\star}{V}^\alpha _{b \beta} \overset{\star}{\delta}_\alpha, & (\rho, \eta) \overset{\star}{D}_x \overset{\star}{\partial}^\gamma &= (\rho, \eta) \overset{\star}{V}^b _{c \alpha} \overset{\star}{\partial}^a
\end{align*}\]

are the components of a linear $(\rho, \eta)$-connection

\[\left( (\rho, \eta) \overset{\star}{H}, (\rho, \eta) \overset{\star}{V} \right)\]

for the generalized tangent bundle $\left( (\rho, \eta) T \overset{\star}{E}, (\rho, \eta) \tau _{E}, \overset{\star}{E} \right)$ which will be called the **distinguished linear $(\rho, \eta)$-connection**.

If $h = Id_M$, then the distinguished linear $(Id_{TM}, Id_M)$-connection will be called the **distinguished linear connection**.

The components of a distinguished linear connection $\left( \overset{\star}{H}, \overset{\star}{V} \right)$ will be denoted

\[\left( \overset{\star}{H}^j_k, \overset{\star}{H}^b_k, \overset{\star}{V}^j_a, \overset{\star}{V}^b_a \right)\].

**Theorem 6.1** If $\left( (\rho, \eta) \overset{\star}{H}, (\rho, \eta) \overset{\star}{V} \right)$ is a distinguished linear $(\rho, \eta)$-connection for the generalized tangent bundle $\left( (\rho, \eta) T \overset{\star}{E}, (\rho, \eta) \tau _{E}, \overset{\star}{E} \right)$, then its components satisfy the change relations:
and for any

\[ \{ \gamma \} H_{\beta \gamma} = \Lambda_{\alpha}^{\beta} \circ h \circ \pi \left[ \Gamma \left( \frac{\partial}{\partial \rho}, \text{Id}_E \right) \left( \frac{\partial}{\partial \eta} \right) \left( \Lambda_{\beta}^{\alpha} \circ h \circ \pi \right) \right] + \]

\[ + (\rho, \eta) H_{\beta \gamma} \cdot \Lambda_{\beta}^{\alpha} \circ h \circ \pi \cdot \Lambda_{\gamma}^{\beta} \circ h \circ \pi, \]

(6.3)

\[ (\rho, \eta) H_{\beta \gamma} = M_{\alpha}^{\beta} \circ \pi \left[ \Gamma \left( \frac{\partial}{\partial \rho}, \text{Id}_E \right) \left( \frac{\partial}{\partial \eta} \right) \left( M_{\beta}^{\alpha} \circ \pi \right) \right] + \]

\[ + (\rho, \eta) H_{\beta \gamma} \cdot M_{\alpha}^{\beta} \circ \pi \cdot \Lambda_{\gamma}^{\beta} \circ h \circ \pi, \]

\[ (\rho, \eta) V_{\beta}^{\alpha} = \Lambda_{\alpha}^{\beta} \circ h \circ \pi \cdot (\rho, \eta) V_{\beta} \cdot M_{\alpha}^{\beta} \circ \pi \cdot M_{\alpha}^{\beta} \circ \pi, \]

(6.3')

\[ (\rho, \eta) V_{\beta}^{\alpha} = M_{\alpha}^{\beta} \circ \pi \cdot (\rho, \eta) V_{\beta} \cdot M_{\alpha}^{\beta} \circ \pi \cdot M_{\alpha}^{\beta} \circ \pi. \]

The components of a distinguished linear connection \( \left( H, V \right) \) verify the change relations:

\[
H_{jk}^{i} = \frac{\partial x^{i}}{\partial x^{j}} \circ \pi \cdot \frac{\delta}{\delta x^{k}} \left( \frac{\partial x^{i}}{\partial x^{j}} \circ \pi \right) + H_{jk} \cdot \frac{\partial x^{i}}{\partial x^{j}} \circ \pi \cdot \frac{\partial x^{k}}{\partial x^{k}} \circ \pi,
\]

(6.4)

\[
H_{bk}^{i} = M_{a}^{b} \circ \pi \cdot \frac{\delta}{\delta x^{k}} \left( M_{a}^{b} \circ \pi \right) + H_{bk} \cdot M_{a}^{b} \circ \pi \cdot \frac{\partial x^{k}}{\partial x^{k}} \circ \pi,
\]

\[
V_{j}^{i} = \frac{\partial x^{i}}{\partial x^{j}} \circ \pi \cdot V_{j}^{i} \cdot \frac{\partial x^{j}}{\partial x^{j}} \circ \pi \cdot M_{c}^{b} \circ \pi,
\]

\[
V_{bk}^{i} = M_{a}^{b} \circ \pi \cdot V_{bk}^{i} \cdot M_{a}^{b} \circ \pi M_{c}^{b} \circ \pi.
\]

Example 6.1 If \( \left( \pi^{*}, \pi, M \right) \) is endowed with the \((\rho, \eta)\)-connection \((\rho, \eta)\) \( \Gamma \), then the local real functions

\[
\left( \frac{\partial (\rho, \eta) \Gamma_{bk}}{\partial \rho a}, \frac{\partial (\rho, \eta) \Gamma_{bk}}{\partial \rho a}, 0, 0 \right)
\]

are the components of a distinguished linear \((\rho, \eta)\)-connection for the generalized tangent bundle

\[
\left( (\rho, \eta) T \pi^{*}, (\rho, \eta) \tau_{E}^{*} \pi^{*} E \right)
\]

which will by called the Berwald linear \((\rho, \eta)\)-connection.

Theorem 6.2 If the generalized tangent bundle \( \left( (\rho, \eta) T \pi^{*}, (\rho, \eta) \tau_{E}^{*} \pi^{*} E \right) \) is endowed with a distinguished linear \((\rho, \eta)\)-connection \( ((\rho, \eta) H, (\rho, \eta) V) \), then, for any

\[
X = \tilde{Z}^{\gamma} \delta_{\gamma} + Y_{a} \theta_{a}^{\gamma} \in \Gamma \left( (\rho, \eta) T \pi^{*}, (\rho, \eta) \tau_{E}^{*} \pi^{*} E \right)
\]

and for any

\[
T \in T_{qs} \pi^{*} \left( (\rho, \eta) T \pi^{*}, (\rho, \eta) \tau_{E}^{*} \pi^{*} E \right),
\]

20
we obtain the formula:

\[
(\rho, \eta) \, D_X \left( T^\alpha_{\beta_1...\beta_q b_1...b_s} \delta^*_{\alpha_1} \otimes ... \otimes \delta^*_{\alpha_p} \otimes d\bar{z}^\beta_1 \otimes ... \otimes d\bar{z}^\beta_q \otimes \hat{\partial} \otimes ... \otimes \hat{\partial} \otimes \delta \bar{p}_{\alpha_1} \otimes ... \otimes \delta \bar{p}_{\alpha_r} \right)
\]

\[
= \tilde{Z}^\gamma T^\alpha_{\beta_1...\beta_q b_1...b_s} \gamma^* \delta^*_{\alpha_1} \otimes ... \otimes \delta^*_{\alpha_p} \otimes d\bar{z}^\beta_1 \otimes ... \otimes d\bar{z}^\beta_q \otimes \hat{\partial} \otimes ... \otimes \hat{\partial} \otimes \delta \bar{p}_{\alpha_1} \otimes ... \otimes \delta \bar{p}_{\alpha_r} + Y_c T^\alpha_{\beta_1...\beta_q b_1...b_s} \mid c \delta^*_{\alpha_1} \otimes ...
\]

\[
\otimes \delta^*_{\alpha_p} \otimes d\bar{z}^\beta_1 \otimes ... \otimes d\bar{z}^\beta_q \otimes \hat{\partial} \otimes ... \otimes \hat{\partial} \otimes \delta \bar{p}_{\alpha_1} \otimes ... \otimes \delta \bar{p}_{\alpha_r},
\]

where

\[
T^\alpha_{\beta_1...\beta_q b_1...b_s} = \Gamma \left( \tilde{\rho}, Id_\delta \right) T^\alpha_{\beta_1...\beta_q b_1...b_s} + (\rho, \eta) H_{\alpha\gamma} T^\alpha_{\beta_1...\beta_q b_1...b_s} + ... + (\rho, \eta) H_{\alpha\gamma} T^\alpha_{\beta_1...\beta_q b_1...b_s}
\]

and

\[
(\rho, \eta) V^\alpha_{\beta_1...\beta_q b_1...b_s} = \Gamma \left( \tilde{\rho}, Id_\delta \right) T^\alpha_{\beta_1...\beta_q b_1...b_s} + ...
\]

\[
+ (\rho, \eta) V^\alpha_{\beta_1...\beta_q b_1...b_s} + ... + (\rho, \eta) V^\alpha_{\beta_1...\beta_q b_1...b_s} + ...
\]

\[
- (\rho, \eta) V^\alpha_{\beta_1...\beta_q b_1...b_s} + ...
\]

\[
+ (\rho, \eta) V^\alpha_{\beta_1...\beta_q b_1...b_s} + ...
\]

\[
+ (\rho, \eta) V^\alpha_{\beta_1...\beta_q b_1...b_s} + ...
\]

\[
\text{Definition 6.2} \text{ We assume that } (E, \pi, M) = (F, \nu, N). \]

If \((\rho, \eta) \Gamma\) is a \((\rho, \eta)\)-connection for the vector bundle \((E, \pi, M)\) and

\[
\left( \begin{array}{c}
\tilde{H}_{bc}, (\rho, \eta) H_{bc}, (\rho, \eta) V^\alpha_b, (\rho, \eta) V^\alpha_b
\end{array} \right)
\]

are the components of a distinguished linear \((\rho, \eta)\)-connection for the generalized tangent bundle \(\left( (\rho, \eta) T_E, (\rho, \eta) \tau^*_E, E \right)\) such that

\[
(\rho, \eta) H_{bc} = (\rho, \eta) \tilde{H}_{bc} \text{ and } (\rho, \eta) V^\alpha_b = (\rho, \eta) V^\alpha_b,
\]

on pages 21.
then we will say that the generalized tangent bundle \( (\rho, \eta) T^* E, (\rho, \eta) \tau^* E, \hat{E} \) is endowed with a normal distinguished linear \((\rho, \eta)\)-connection on components

\[
(\rho, \eta) \tilde{H}_{bc}, (\rho, \eta) \tilde{V}_b.
\]

The components of a normal distinguished linear \((Id_{TM}, Id_{M})\)-connection \((\tilde{H}, \tilde{V})\) will be denoted \((\tilde{H}_{jk}, \tilde{V}_{jk})\).

7 Dual mechanical systems

Using the diagram:

\[
\begin{array}{c}
E^* \\
\tilde{E}^* \\
\tilde{\pi}
\end{array}
\begin{array}{c}
\downarrow \pi \\
M \\
\downarrow h \\
M
\end{array}
\]

where \(((E, \pi, M), [[\cdot], (\rho, \eta))]\) is a generalized Lie algebroid, we build the generalized tangent bundle

\[
((\rho, \eta) T^* E, (\rho, \eta) \tau^* E, [\cdot], (\rho, \eta) \tau^* E, (\rho, \eta) T^* E, (\rho, \eta) \tau^* E).
\]

**Definition 7.1** A triple

\[
(7.2) \quad \left( \left( E^*, \tilde{\pi}, M \right), F_e, (\rho, \eta) \tilde{\Gamma} \right),
\]

where

\[
(7.3) \quad F_e = F_{\alpha} \tilde{a} \partial \in \Gamma \left( V (\rho, \eta) T^* E, (\rho, \eta) \tau^* E, \hat{E} \right)
\]

is an external force and \((\rho, \eta) \tilde{\Gamma}\) is a \((\rho, \eta)\)-connection for \( (E^*, \tilde{\pi}, M)\), will be called dual mechanical \((\rho, \eta)\)-system.

**Definition 7.2** A smooth Hamilton fundamental function on the dual vector bundle \( (E^*, \tilde{\pi}, M)\) is a mapping \( E \overset{H}{\rightarrow} \mathbb{R} \) which satisfies the following conditions:

1. \( H \circ \hat{u} \in C^\infty (M) \), for any \( \hat{u} \in \Gamma \left( E^*, \tilde{\pi}, M \right) \setminus \{0\}; \)

2. \( H \circ 0 \in C^0 (M) \), where 0 means the null section of \( (E^*, \tilde{\pi}, M) \).

Let \( H \) be a differentiable Hamiltonian on the total space of the dual vector bundle \( (E^*, \tilde{\pi}, M) \).
If \((U, s_U)\) is a local vector \((m + r)\)-chart for \((\dot{E}, \dot{\pi}, M)\), then we obtain the following real functions defined on \(\pi^{-1}(U)\):

\[
\begin{align*}
H_i & \overset{\text{put}}{=} \frac{\partial H}{\partial x^i} & \quad & H^b_i & \overset{\text{put}}{=} \frac{\partial^2 H}{\partial x^i \partial y^b} \\
H^a_i & \overset{\text{put}}{=} \frac{\partial H}{\partial p^a_i} & \quad & H^{ab}_{i} & \overset{\text{put}}{=} \frac{\partial^2 H}{\partial p^a_i \partial p^b}
\end{align*}
\]

\[(7.4)\]

**Definition 7.3** If for any local vector \((m + r)\)-chart \((U, s_U)\) of \((\dot{E}, \dot{\pi}, M)\), we have:

\[
\text{rank } \left\| H^{ab}_{x} \right\| = r,
\]

then we say that the Hamiltonian \(H\) is regular.

**Proposition 7.1** If the Hamiltonian \(H\) is regular, then for any local vector \((m + r)\)-chart \((U, s_U)\) of \((\dot{E}, \dot{\pi}, M)\), we obtain the real functions \(\tilde{H}_{ba}\) locally defined by

\[
\pi^{-1}(U) \quad \overset{\text{H}_{ba}}{\longrightarrow} \quad \mathbb{R}
\]

\[
\begin{array}{c}
\pi^{-1}(U) \\
\{0\}
\end{array}
\]

\[
\begin{array}{c}
\tilde{H}_{ba}(\pi^{-1}(U)) \\
\{0\}
\end{array}
\]

\[(7.6)\]

where \(\left\| \tilde{H}_{ba}(\pi^{-1}(U)) \right\| = \left\| H^{ab}_{x} \right\|^{-1}\), for any \(\pi^{-1}(U)\) of \((\dot{E}, \dot{\pi}, M)\).

**Definition 7.4** A smooth Cartan fundamental function on the vector bundle \((\dot{E}, \dot{\pi}, M)\)

is a mapping \(\dot{E} \overset{K}{\longrightarrow} \mathbb{R}_+\) which satisfies the following conditions:

1. \(K \circ u \in C^\infty(M)\), for any \(u \in \Gamma \left(\dot{E}, \dot{\pi}, M\right) \setminus \{0\};\)

2. \(K \circ 0 \in C^0(M)\), where 0 means the null section of \((\dot{E}, \dot{\pi}, M)\);

3. \(K\) is positively 1-homogenous on the fibres of vector bundle \((\dot{E}, \dot{\pi}, M)\);

4. For any local vector \((m + r)\)-chart \((U, s_U)\) of \((\dot{E}, \dot{\pi}, M)\), the hessian:

\[
\left\| K^{ab}_{ba}(\pi^{-1}(U)) \right\|
\]

is positively define for any \(\pi^{-1}(U)\).

**Definition 7.5** If \(H\) respectively \(K\) is a smooth Hamilton respectively Cartan function, then we put the triple

\[
\left( \left(\dot{E}, \dot{\pi}, M\right), F, H \right),
\]

respectively

\[
\left( \left(\dot{E}, \dot{\pi}, M\right), F, K \right),
\]

23
where
\[ F_c = F_a \partial_a \in \Gamma \left( V(\rho, \eta) T E, (\rho, \eta) \tau_E^*, E \right) \]
is an external force. These are called Hamilton mechanical \((\rho, \eta)\)-system and Cartan mechanical \((\rho, \eta)\)-system respectively.

Any Hamilton mechanical \((\text{Id}_{TM}, \text{Id}_M)\)-system and any Cartan mechanical \((\text{Id}_{TM}, \text{Id}_M)\)-system will be called Hamilton mechanical system and Cartan mechanical system, respectively.

8 \((\rho, \eta)\)-semisprays and \((\rho, \eta)\)-sprays for dual mechanical \((\rho, \eta)\)-systems

Let \( \left( \hat{E}, \pi, M \right), \hat{F}_c, (\rho, \eta) \Gamma \) be an arbitrary dual mechanical \((\rho, \eta)\)-system.

**Definition 8.1** The vertical section \( \hat{G} = p_o \hat{\partial} \) will be called the Liouville section.

A section \( \hat{S} \in \Gamma \left( (\rho, \eta) T \hat{E}, (\rho, \eta) \tau_{\hat{E}}^*, \hat{E} \right) \) will be called \((\rho, \eta)\)-semispray if there exists an almost tangent structure \( e \) such that \( e \left( \hat{S} \right) = \hat{C} \).

Let \( g \in \text{Man} \left( \hat{E}, E \right) \) be such that \((g, h)\) is a locally invertible \( B^v \)-morphism of \( \left( \hat{E}, \pi, M \right) \) source and \((E, \pi, M)\) target.

**Theorem 8.1** The section

\[ \hat{S} = \left( g^{ab} \circ h \circ \pi^a \right) p_o \hat{\partial}_a - 2 \left( G_a - \frac{1}{2} F_a \right) \hat{\partial}_a \]
is a \((\rho, \eta)\)-semispray such that the real local functions \( G_a, a \in 1, n \), satisfy the following conditions

\[ (\rho, \eta) \Gamma_{bc} = \left( g_{ca} \circ h \circ \pi^a \right) \frac{\partial (G_a - \frac{1}{2} F_a)}{\partial p_a} - \frac{1}{2} \left( g^{de} \circ h \circ \pi^e \right) p_e \left( L_{de}^c \circ h \circ \pi^c \right) \left( \tilde{g}_{fb} \circ h \circ \pi^f \right) + \frac{1}{2} \left( \rho^c_i \circ h \circ \pi^c \right) \frac{\partial (g^{ac} \circ h \circ \pi^c)}{\partial x^c} p_e \left( \tilde{g}_{ab} \circ h \circ \pi^a \right) - \frac{1}{2} \left( g^{ae} \circ h \circ \pi^e \right) p_e \left( \rho^c_i \circ h \circ \pi^c \right) \frac{\partial (g_{ac} \circ h \circ \pi^c)}{\partial x^c} \]

\[ \hat{\Gamma}_{bc} = \left( g_{ca} \circ h \circ \pi^a \right) \frac{\partial G_a}{\partial p_a} - \frac{1}{2} \left( g^{de} \circ h \circ \pi^e \right) p_e \left( L_{de}^c \circ h \circ \pi^c \right) \left( \tilde{g}_{fb} \circ h \circ \pi^f \right) + \frac{1}{2} \left( \rho^c_i \circ h \circ \pi^c \right) \frac{\partial (g^{ac} \circ h \circ \pi^c)}{\partial x^c} p_e \left( \tilde{g}_{ab} \circ h \circ \pi^a \right) - \frac{1}{2} \left( g^{ae} \circ h \circ \pi^e \right) p_e \left( \rho^c_i \circ h \circ \pi^c \right) \frac{\partial (g_{ac} \circ h \circ \pi^c)}{\partial x^c} \]

In addition, we remark that the local real functions

\[ (\rho, \eta) \hat{\Gamma}_{bc} = \left( g_{ca} \circ h \circ \pi^a \right) \frac{\partial G_a}{\partial p_a} - \frac{1}{2} \left( g^{de} \circ h \circ \pi^e \right) p_e \left( L_{de}^c \circ h \circ \pi^c \right) \left( \tilde{g}_{fb} \circ h \circ \pi^f \right) + \frac{1}{2} \left( \rho^c_i \circ h \circ \pi^c \right) \frac{\partial (g^{ac} \circ h \circ \pi^c)}{\partial x^c} p_e \left( \tilde{g}_{ab} \circ h \circ \pi^a \right) - \frac{1}{2} \left( g^{ae} \circ h \circ \pi^e \right) p_e \left( \rho^c_i \circ h \circ \pi^c \right) \frac{\partial (g_{ac} \circ h \circ \pi^c)}{\partial x^c} \]
are the components of a $(\rho, \eta)$-connection $(\rho, \eta)\hat{\Gamma}$ for the vector bundle $\left(\tau_{E}^{*}, \pi, M\right)$.

The $(\rho, \eta)$-semispray $S$ will be called the canonical $(\rho, \eta)$-semispray associated to mechanical $(\rho, \eta)$-system $\left(\left(\tau_{E}^{*}, \pi, M\right), F_e, (\rho, \eta)\Gamma\right)$ and from locally invertible $B^y$-morphism $(g, h)$.

Proof. We consider the $\text{Mod}$-endomorphism

$$\Gamma\left((\rho, \eta)TE, (\rho, \eta)\tau_{E}^{*}, \pi, \pi\right) \to \Gamma\left((\rho, \eta)TE, (\rho, \eta)\tau_{E}^{*}, \pi, \pi\right)$$

$$X \mapsto \mathbf{J}_{(g, h)} \left[\pi, M\right]_{(\rho, \eta)TE}^{\pi, Y} - \left[\pi, \mathbf{J}_{(g, h)} X\right]_{(\rho, \eta)TE}^{\pi, Y}.$$

Let $X = Z^a \partial_a + Y_a\partial^a$ be an arbitrary section. Since

$$\left[\pi, Z_{(\rho, \eta)TE}^{a}\partial_a \partial^a\right] = \left[(g^{ae} \circ h \circ \pi)^{a}_{e} p_c \partial_a, Z^b \partial_b\right]_{(\rho, \eta)TE}^{\pi, Y} + \left[(g^{ae} \circ h \circ \pi)^{a}_{e} p_c \partial_a, Y_b\partial^b\right]_{(\rho, \eta)TE}^{\pi, Y} - \left[2\left(G_a - \frac{1}{4}F_a\right)\partial^a, Z^b \partial_b\right]_{(\rho, \eta)TE}^{\pi, Y} - \left[2\left(G_a - \frac{1}{4}F_a\right)\partial^a, Y_b\partial^b\right]_{(\rho, \eta)TE}^{\pi, Y}$$

and

$$\left[(g^{ae} \circ h \circ \pi)^{a}_{e} p_c \partial_a, Z^b \partial_b\right]_{(\rho, \eta)TE}^{\pi, Y} = \left[(g^{ae} \circ h \circ \pi)^{a}_{e} p_c \partial_a \partial^a\right]_{(\rho, \eta)TE}^{\pi, Y}$$

$$- Z^b \left(\rho_c^j \circ h \circ \pi\right) \partial_j \left((g^{ce} \circ h \circ \pi)^{e}_{c} p_c \partial_c\right)$$

$$+ \left(g^{ae} \circ h \circ \pi\right) p_c Z^b \left(L^c_{ab} \circ h \circ \pi\right) \partial_c,$$

$$\left[(g^{ae} \circ h \circ \pi)^{a}_{e} p_c \partial_a, Y_b\partial^b\right]_{(\rho, \eta)TE}^{\pi, Y} = \left[(g^{ae} \circ h \circ \pi)^{a}_{e} p_c \partial_a \partial^a\right]_{(\rho, \eta)TE}^{\pi, Y}$$

$$- Y_b \left(g^{bc} \circ h \circ \pi\right) \partial_c,$$

$$\left[2\left(G_a - \frac{1}{4}F_a\right)\partial^a, Z^b \partial_b\right]_{(\rho, \eta)TE}^{\pi, Y} = 2\left(G_a - \frac{1}{4}F_a\right) \partial^a \partial^c$$

$$- 2Z^b \left(\rho_c^j \circ h \circ \pi\right) \partial_j \left(G_c - \frac{1}{4}F_c\right) \partial^c,$$

$$\left[2\left(G_a - \frac{1}{4}F_a\right)\partial^a, Y_b\partial^b\right]_{(\rho, \eta)TE}^{\pi, Y} = 2\left(G_a - \frac{1}{4}F_a\right) \partial^c \partial^c$$

$$- 2Y_b \left(\partial_b \left(G_c - \frac{1}{4}F_c\right) \partial^c\right),$$

$$25$$
it results that

\[ \mathcal{J}_{(g,h)} \left[ \mathcal{S}, X \right]_{(\rho,\eta)TE}^* = \left( g^{ae} \circ h \circ \pi \right) p_e \left( \rho_a^i \circ h \circ \pi \right) \frac{\partial Z^c}{\partial x^i} \left( \tilde{g}_{cd} \circ h \circ \pi \right) \\hat{\partial} \]

\[ \begin{aligned}
&- Z^b \left( \rho_b^i \circ h \circ \pi \right) \frac{\partial}{\partial x^j} \left( g^{ae} \circ h \circ \pi \right) p_e \left( \tilde{g}_{cd} \circ h \circ \pi \right) \\hat{\partial}^d \\
&+ \left( g^{ae} \circ h \circ \pi \right) p_e Z^b \left( F_{ab} \circ h \circ \pi \right) \\hat{\partial}^d - Y_d \\hat{\partial}^d \\
&- 2 \left( G_a - \frac{1}{4} F_a \right) \frac{\partial Z^c}{\partial \rho_a} \left( \tilde{g}_{cd} \circ h \circ \pi \right) \\hat{\partial}^d .
\end{aligned} \]

(P1)

Since

\[ \begin{aligned}
\left[ S, \mathcal{J}_{(g,h)} X \right]_{(\rho,\eta)TE}^* &= \left[ \left( g^{ae} \circ h \circ \pi \right) p_e \hat{\partial}_a, Z^b \left( \tilde{g}_{bc} \circ h \circ \pi \right) \hat{\partial}^c \right]_{(\rho,\eta)TE}^* \\
&= \left[ 2 \left( G_a - \frac{1}{4} F_a \right) \hat{\partial}^a, Z^b \left( \tilde{g}_{bc} \circ h \circ \pi \right) \hat{\partial}^c \right]_{(\rho,\eta)TE}^* \\
&= -2 \left( G_a - \frac{1}{4} F_a \right) \frac{\partial Z^c}{\partial \rho_a} \left( \tilde{g}_{cd} \circ h \circ \pi \right) \hat{\partial}^d \\
&- Z^b \left( \tilde{g}_{bc} \circ h \circ \pi \right) \frac{\partial^2 (G_d - \frac{1}{4} F_d)}{\partial \rho_c} \hat{\partial}^d.
\end{aligned} \]

and

\[ \left[ \left( g^{ae} \circ h \circ \pi \right) p_e \hat{\partial}_a, Z^b \left( \tilde{g}_{bc} \circ h \circ \pi \right) \hat{\partial}^c \right]_{(\rho,\eta)TE}^* = -Z^d \hat{\partial}_d \]

\[ \begin{aligned}
&+ \left( g^{ae} \circ h \circ \pi \right) p_e \left( \rho_a^i \circ h \circ \pi \right) \frac{\partial Z^b}{\partial x^i} \left( \tilde{g}_{bd} \circ h \circ \pi \right) \\hat{\partial}^d \\
&- \left( g^{ae} \circ h \circ \pi \right) p_e \left( \rho_a^i \circ h \circ \pi \right) Z^b \frac{\partial \left( \tilde{g}_{bd} \circ h \circ \pi \right)}{\partial x^i} \\hat{\partial}^d
\end{aligned} \]

it results that

(P2)

\[ \begin{aligned}
\left[ S, \mathcal{J}_{(g,h)} X \right]_{(\rho,\eta)TE}^* &= -Z^d \hat{\partial}_d + \left( g^{ae} \circ h \circ \pi \right) p_e \left( \rho_a^i \circ h \circ \pi \right) \frac{\partial Z^b}{\partial x^i} \left( \tilde{g}_{bd} \circ h \circ \pi \right) \\hat{\partial}^d \\
&- \left( g^{ae} \circ h \circ \pi \right) p_e \left( \rho_a^i \circ h \circ \pi \right) Z^b \frac{\partial \left( \tilde{g}_{bd} \circ h \circ \pi \right)}{\partial x^i} \\hat{\partial}^d \\
&- 2 \left( G_a - \frac{1}{4} F_a \right) \frac{\partial Z^b}{\partial \rho_a} \left( \tilde{g}_{bd} \circ h \circ \pi \right) \\hat{\partial}^d \\
&+ Z^b \left( \tilde{g}_{bc} \circ h \circ \pi \right) \frac{\partial^2 (G_d - \frac{1}{4} F_d)}{\partial \rho_c} \hat{\partial}^d.
\end{aligned} \]
Using equalities \((P_1)\) and \((P_2)\), we obtain:

\[
P \left( Z^a \tilde{\partial}_a + Y \tilde{\partial} \right) = Z^a \tilde{\partial}_a + Y \tilde{\partial} + \left( g^{ae} h \circ \pi \right) p_e Z^b \left( L_{ab}^c h \circ \pi \right) \left( \tilde{g}_{cd} h \circ \pi \right) \tilde{\partial}
\]

\[
- Z^b \left( \rho^i_b h \circ \pi \right) \frac{\partial (g^{ae} h \circ \pi)}{\partial x^i} p_e \left( \tilde{g}_{cd} h \circ \pi \right) \tilde{\partial}
\]

\[
+ \left( g^{be} h \circ \pi \right) p_e \left( \rho^i_a h \circ \pi \right) Z^b \frac{\partial \left( \tilde{g}_{cd} h \circ \pi \right)}{\partial x^i} \tilde{\partial}
\]

\[
- Z^b \left( \tilde{g}_{bc} h \circ \pi \right) \frac{\partial (G_d \nabla F_a)}{\partial p_c} \tilde{\partial}
\]

After some calculations, it results that \(P\) is an almost product structure.

Using the equalities \((5.1.2)\) and \((5.2.2)\) it results that

\[
P \left( Z^a \tilde{\partial}_a + Y \tilde{\partial} \right) = (I - 2 (\rho, \eta) \Gamma) \left( Z^a \tilde{\partial}_a + Y \tilde{\partial} \right),
\]

for any \( Z^a \tilde{\partial}_a + Y \tilde{\partial} \in \Gamma \left( (\rho, \eta) T E, (\rho, \eta) T \pi_E, E \right) \) and we obtain

\[
(\rho, \eta) \Gamma \left( Z^a \tilde{\partial}_a + Y \tilde{\partial} \right) = Y \tilde{\partial} - \frac{1}{2} \left( g^{ae} h \circ \pi \right) p_e Z^b \left( L_{ab}^c h \circ \pi \right) \left( \tilde{g}_{cd} h \circ \pi \right) \tilde{\partial}
\]

\[
+ \frac{1}{2} Z^b \left( \rho^i_b h \circ \pi \right) \frac{\partial (g^{ae} h \circ \pi)}{\partial x^i} p_e \left( \tilde{g}_{cd} h \circ \pi \right) \tilde{\partial}
\]

\[
- \frac{1}{2} \left( g^{ae} h \circ \pi \right) p_e \left( \rho^i_a h \circ \pi \right) Z^b \frac{\partial \left( \tilde{g}_{cd} h \circ \pi \right)}{\partial x^i} \tilde{\partial}
\]

\[
+ Z^b \left( \tilde{g}_{bc} h \circ \pi \right) \frac{\partial (G_d \nabla F_a)}{\partial p_c} \tilde{\partial}
\]

Since

\[
(\rho, \eta) \Gamma \left( Z^a \tilde{\partial}_a + Y \tilde{\partial} \right) = (Y + (\rho, \eta) \Gamma_{\partial a} Z^b) \tilde{\partial}
\]

it results the relations \((8.3)\). In addition, since

\[
(\rho, \eta) \Gamma_{bc} = (\rho, \eta) \Gamma_{bc} + \frac{1}{4} \left( \tilde{g}_{cd} h \circ \pi \right) \frac{\partial F_c}{\partial p_c}
\]

and

\[
(\rho, \eta) \Gamma_{bc} = (\rho, \eta) \Gamma_{bc} - \frac{1}{4} \left( \tilde{g}_{cd} h \circ \pi \right) \frac{\partial F_c}{\partial p_c}
\]

\[
= M^b_{\partial} h \circ \pi \left( - \left( \rho^i_c h \circ \pi \right) \frac{\partial M^a_{\partial}}{\partial x^i} p_a + (\rho, \eta) \Gamma_{bc} \right) M^c h \circ \pi
\]

\[
+ M^b_{\partial} h \circ \pi \left( \frac{1}{4} \left( \tilde{g}_{ce} h \circ \pi \right) \frac{\partial F_c}{\partial p_c} \right) M^c h \circ \pi
\]

\[
= M^b_{\partial} h \circ \pi \left( - \left( \rho^i_c h \circ \pi \right) \frac{\partial M^a_{\partial}}{\partial x^i} p_a + (\rho, \eta) \Gamma_{bc} - \frac{1}{4} \left( \tilde{g}_{cd} h \circ \pi \right) \frac{\partial F_c}{\partial p_c} \right) M^c h \circ \pi
\]

\[
= M^b_{\partial} h \circ \pi \left( - \left( \rho^i_c h \circ \pi \right) \frac{\partial M^a_{\partial}}{\partial x^i} p_a + (\rho, \eta) \Gamma_{bc} M^c h \circ \pi
\]

it results the conclusion of the theorem.

\[q.e.d.\]
Remark 8.1 In particular, if \((\rho, \eta) = (Id_{TM}, Id_M)\), \((g, h) = (Id_E, Id_M)\), and \(F_e = 0\), then we obtain the classical canonical semispray associated to connection \(\Gamma\).

Using Theorem 8.1, we obtain the following:

**Theorem 8.2** The following properties hold good:

1° Since \(\delta_c = \partial_c + \rho (\eta) \tilde{\Gamma}_{bc}\), \(c \in \Gamma, r\), it results that

\[
\delta_c = \delta_c - \frac{1}{4} \left( \tilde{g}_{ce} \circ h \circ \pi \right) \frac{\partial F_b}{\partial p_c}, \quad c \in \Gamma, r.
\]

(8.4)

2° Since \(\delta \tilde{p}_b = -(\rho, \eta) \tilde{\Gamma}_{bc} d\tilde{z}^c + d\tilde{p}_b\), \(b \in \Gamma, r\), it results that

\[
\delta \tilde{p}_b = \delta \tilde{p}_b + \frac{1}{4} \left( \tilde{g}_{ce} \circ h \circ \pi \right) \frac{\partial F_b}{\partial p_c} d\tilde{z}^c, \quad b \in \Gamma, r.
\]

(8.5)

**Theorem 8.3** The real local functions

\[
\left( \frac{\partial (\rho, \eta) \tilde{\Gamma}_{bc}}{\partial p_a}, \frac{\partial (\rho, \eta) \tilde{\Gamma}_{bc}}{\partial p_a}, 0, 0 \right), \quad a, b, c \in \Gamma, r
\]

(8.6)

and

\[
\left( \frac{\partial (\rho, \eta) \tilde{\Gamma}_{bc}}{\partial p_a}, \frac{\partial (\rho, \eta) \tilde{\Gamma}_{bc}}{\partial p_a}, 0, 0 \right), \quad a, b, c \in \Gamma, r
\]

respectively, are the coefficients to a normal Berwald linear \((\rho, \eta)\)-connection for the generalized tangent bundle \(\left( (\rho, \eta) E^*, (\rho, \eta) \Gamma_{E}, \tilde{E} \right)\).

**Theorem 8.4** The tensor of integrability of the \((\rho, \eta)\)-connection \((\rho, \eta) \tilde{\Gamma}\) is as follows:

\[
(\rho, \eta, h) \tilde{R}_{b cd} = (\rho, \eta, h) R_{b cd} + \frac{1}{4} \left( \left( \tilde{g}_{de} \circ h \circ \pi \right) \frac{\partial F_b}{\partial p_c} \right)
\]

\[
+ \frac{1}{16} \left( \left( \tilde{g}_{cd} \circ h \circ \pi \right) \frac{\partial F_b}{\partial p_c} \right)
\]

\[
\cdot \left( \tilde{g}_{ef} \circ h \circ \pi \right) \frac{\partial^2 F_b}{\partial p_c \partial p_f} - \left( \tilde{g}_{ef} \circ h \circ \pi \right) \frac{\partial F_b}{\partial p_c} \left( \tilde{g}_{de} \circ h \circ \pi \right) \frac{\partial^2 F_b}{\partial p_c \partial p_f} +
\]

\[
\frac{1}{4} \left( L_{cd} \circ h \circ \pi \right) \left( \tilde{g}_{ef} \circ h \circ \pi \right) \frac{\partial F_b}{\partial p_c},
\]

where \(\mid_c\) is the h-covariant derivation with respect to the normal Berwald linear \((\rho, \eta)\)-connection (8.6).

Proof. Since

\[
(\rho, \eta, h) \tilde{R}_{b cd} = \Gamma \left( \tilde{\rho}, \tilde{I}_E \right) \left( \tilde{\delta}_c \right) \left( (\rho, \eta) \tilde{\Gamma}_{bd} \right) - \Gamma \left( \tilde{\rho}, \tilde{I}_E \right) \left( \tilde{\delta}_d \right) \left( (\rho, \eta) \tilde{\Gamma}_{bc} \right)
\]

\[
- \left( L_{cd} \circ h \circ \pi \right) (\rho, \eta) \tilde{\Gamma}_{be},
\]

and

\[
\Gamma \left( \tilde{\rho}, \tilde{I}_E \right) \left( \tilde{\delta}_c \right) \left( (\rho, \eta) \tilde{\Gamma}_{bd} \right) = \Gamma \left( \tilde{\rho}, \tilde{I}_E \right) \left( \tilde{\delta}_c \right) \left( (\rho, \eta) \Gamma_{bd} \right)
\]

\[
+ \frac{1}{4} \Gamma \left( \tilde{\rho}, \tilde{I}_E \right) \left( \tilde{\delta}_c \right) \left( \tilde{g}_{de} \circ h \circ \pi \right) \frac{\partial F_b}{\partial p_c}
\]

\[
- \frac{1}{4} \left( \tilde{g}_{ce} \circ h \circ \pi \right) \frac{\partial F_b}{\partial p_c} \left( (\rho, \eta) \Gamma_{bd} \right)
\]

\[
- \frac{1}{16} \left( \tilde{g}_{ce} \circ h \circ \pi \right) \frac{\partial F_b}{\partial p_c} \frac{\partial F_b}{\partial p_f} \left( (\rho, \eta) \Gamma_{bd} \right),
\]

\[
\end{equation}

\[
\end{array}
\]

\[
\begin{array}{l}
\end{equation}

\[
\end{array}
\]
\[ \Gamma \left( \tilde{\rho}, I_d E \right) \left( \begin{bmatrix} \delta^i_d \end{bmatrix} \right) \left( (\rho, \eta) \tilde{\Gamma}_{bc} \right) = \Gamma \left( \tilde{\rho}, I_d E \right) \left( \begin{bmatrix} \delta^i_d \end{bmatrix} \right) \left( (\rho, \eta) \Gamma_{bc} \right) \]

\[ + \frac{1}{4} \Gamma \left( \tilde{\rho}, I_d E \right) \left( \begin{bmatrix} \delta^i_d \end{bmatrix} \right) \left( (\tilde{g} \circ \rho \circ \eta)^* \frac{\partial F_b}{\partial \eta} \right) \]

\[ - \frac{1}{4} \left( \tilde{g} \circ \rho \circ \eta \right) \frac{\partial F_b}{\partial \eta} \frac{\partial F_b}{\partial \eta} \left( (\rho, \eta) \Gamma_{bc} \right) \]

\[ + \frac{1}{8} \left( \tilde{g} \circ \rho \circ \eta \right) \frac{\partial F_b}{\partial \eta} \frac{\partial F_b}{\partial \eta} \left( \left( \tilde{g} \circ \rho \circ \eta \right) \frac{\partial F_b}{\partial \eta} \right) , \]

the conclusion results.

**Proposition 8.1 If \( S \) is the canonical \((\rho, \eta)\)-semispray associated to the mechanical \((\rho, \eta)\)-system \( \left( \left( \tilde{E}, \pi, M \right), \tilde{F}_e, (\rho, \eta) \tilde{\Gamma} \right) \) and from locally invertible \( B^v \)-morphism \((g, h)\), then

\[ 2G_b = 2G_b \circ h \circ \pi - \left( g^{\alpha\epsilon} \circ h \circ \pi \right) p_e \rho^i_o \circ h \circ \pi \frac{\partial p_e}{\partial x^i} . \]

**Proof.** Since the Jacobian matrix of coordinates transformation is

\[ \begin{vmatrix} M^a_o & 0 \\ \rho^i_o & \frac{\partial M^a_o \circ \pi}{\partial x^i} \end{vmatrix} \begin{vmatrix} 0 & 0 \\ M^b_o \circ \pi & \frac{\partial p_e}{\partial x^i} \end{vmatrix} = \begin{vmatrix} M^a_o \circ \pi & 0 \\ \rho^i_o \circ h \circ \pi & \frac{\partial p_e}{\partial x^i} \end{vmatrix} \begin{vmatrix} M^b_o \circ \pi \end{vmatrix} \]

and

\[ \begin{vmatrix} M^a_o \circ h \circ \pi & 0 \\ \rho^i_o \circ h \circ \pi & \frac{\partial p_e}{\partial x^i} \end{vmatrix} \begin{vmatrix} 0 & 0 \\ M^b_o \circ \pi & \frac{\partial p_e}{\partial x^i} \end{vmatrix} \begin{vmatrix} \left( g^{\alpha\epsilon} \circ h \circ \pi \right) \rho^i_o \circ h \circ \pi \frac{\partial p_e}{\partial x^i} \end{vmatrix} = \begin{vmatrix} \left( g^{\alpha\epsilon} \circ h \circ \pi \right) \rho^i_o \circ h \circ \pi \frac{\partial p_e}{\partial x^i} \end{vmatrix} \]

the conclusion results.

In the following we consider a differentiable curve \( I \rightarrow M \) and its \((g, h)\)-lift \( \dot{c} \).

**Definition 8.3** If it is verifies the following equality:

\[ \frac{dc(t)}{dt} = \Gamma \left( \tilde{\rho}, I_d E \right) \left( \begin{bmatrix} \delta^i_d \end{bmatrix} \right) S (\dot{c}(t)) , \]

then we say that the curve \( \dot{c} \) is an integral curve of the \((\rho, \eta)\)-semispray \( S \) of the dual mechanical \((\rho, \eta)\)-system \( \left( \tilde{E}, \pi, M \right), \tilde{F}_e, (\rho, \eta) \tilde{\Gamma} \) .

**Theorem 8.5** The integral curves of the canonical \((\rho, \eta)\)-semispray associated to the dual mechanical \((\rho, \eta)\)-system \( \left( \tilde{E}, \pi, M \right), \tilde{F}_e, (\rho, \eta) \tilde{\Gamma} \) and from locally invertible \( B^v \)-morphism \((g, h)\), are the \((g, h)\)-lifts solutions of the equations:

\[ \frac{dp_e(t)}{dt} + 2G_b \circ \dot{u} (c, \dot{c}) (x(t)) = \frac{1}{2} F_b \circ \dot{u} (c, \dot{c}) (x(t)) , \]

where \( x(t) = (\eta \circ h \circ c) (t) . \)
Proof. Since the equality
\[
\frac{de(t)}{dt} = \Gamma \begin{pmatrix} \rho, \Id \end{pmatrix} S(\dot{c}(t))
\]
is equivalent with
\[
\frac{dp(t)}{dt}((\eta \circ h \circ c)^i(t), p(t)) = \rho_a^i \circ \eta \circ h \circ c(t) g^{ae} \circ h \circ c(t) p_e(t), \quad 2 \left( G_b - \frac{1}{2} F_b \right) \left((\eta \circ h \circ c(t), p(t))\right),
\]
it results
\[
\frac{dx^i(t)}{dt} = \rho_a^i \circ \eta \circ h \circ c(t) g^{ae} \circ h \circ c(t) p_e(t),
\]
where \( x^i(t) = (\eta \circ h \circ c)^i(t) \).

q.e.d.

**Definition 8.4** If \( S^* \) is a \((\rho, \eta)\)-semispray, then the vector field

\[
(8.11) \quad \left[ \mathcal{C}, S^* \right]_{(\rho, \eta) T E}^* - S^*
\]

will be called the *derivation of \((\rho, \eta)\)-semispray \( S^* \).

The \((\rho, \eta)\)-semispray \( S^* \) will be called \((\rho, \eta)\)-spray if there are verified the following conditions:

1. \( S^* \circ 0 \in C^1 \), where 0 is the null section;
2. Its derivation is the null vector field.

The \((\rho, \eta)\)-semispray \( S^* \) will be called quadratic \((\rho, \eta)\)-spray if there are verified the following conditions:

1. \( S^* \circ 0 \in C^2 \), where 0 is the null section;
2. Its derivation is the null vector field.

In particular, if \((\rho, \eta) = (\Id_T M, \Id_M)\) and \((g, h) = (\Id_E, \Id_M)\), then we obtain the spray and the quadratic spray which is similar with the classical spray and quadratic spray.

**Theorem 8.6** If \( S \) is the canonical \((\rho, \eta)\)-spray associated to the dual mechanical \((\rho, \eta)\)-system \( \left( \left( E, \pi, M \right), F_e, (\rho, \eta) \Gamma \right) \) and from locally invertible \( B^* \)-morphism \((g, h)\), then

\[
2 \left( G_b - \frac{1}{2} F_b \right) = (\rho, \eta) \Gamma_{bc} \left( \tilde{\rho}^f \circ h \circ \pi \right) p_f \\
+ \frac{1}{2} \left( g^{de} \circ h \circ \pi \right) p_e \left( \tilde{F}_{d_e} \circ h \circ \pi \right) \tilde{g}_{ab} \circ h \circ \pi \left( g^{cf} \circ h \circ \pi \right) p_f \\
- \frac{1}{2} \left( \tilde{g}_{bc} \circ h \circ \pi \right) \frac{\partial (g^{ae} \circ h \circ \pi)}{\partial x^i} \tilde{g}_{ab} \circ h \circ \pi \left( g^{cf} \circ h \circ \pi \right) p_f \\
+ \frac{1}{2} \left( g^{ae} \circ h \circ \pi \right) p_e \left( \tilde{\rho}^i \circ h \circ \pi \right) \frac{\partial (g^{cf} \circ h \circ \pi)}{\partial x^i} \left( g^{cf} \circ h \circ \pi \right) p_f
\]
We obtain the spray

\[ S = (g^a \circ h \circ \pi)^* p_e \partial_a - (\rho, \eta) \Gamma_{bc} (g^f \circ h \circ \pi) p_f \partial^b \]

(8.13)

\[ -\frac{1}{2} \left( g^{de} \circ h \circ \pi \right) p_e \left( L^a_{dc} \circ h \circ \pi \right) (\tilde{g}_{ab} \circ h \circ \pi) (g^f \circ h \circ \pi) p_f \partial^b \]

\[ + \frac{1}{2} \left( \rho^i \circ h \circ \pi \right) \partial (g^{ah \circ \pi}) p_e (\tilde{g}_{ab} \circ h \circ \pi) \left( g^f \circ h \circ \pi \right) p_f \partial^b \]

\[ - \frac{1}{2} \left( g^{ae} \circ h \circ \pi \right) p_e \left( \rho^i_a \circ h \circ \pi \right) \partial (\tilde{g}_{be} \circ \pi) \left( g^f \circ h \circ \pi \right) p_f \partial^b \]

This spray will be called the canonical \((\rho, \eta)\)-spray associated to the dual mechanical system \((\dot{\tilde{E}}, \dot{\tilde{T}}, M), \dot{\tilde{F}}, \dot{\tilde{\Gamma}}\) and from locally invertible \(B^y\)-morphism \((g, h)\).

In particular, if \((\rho, \eta) = (id_T, id_M)\) and \((g, h) = (id_E, id_M)\), then we get the canonical spray associated to connection \(\Gamma\) which is similar with the classical canonical spray associated to connection \(\Gamma\).

**Proof.** Since

\[ \left[ \dot{\tilde{C}}, \dot{\tilde{S}} \right]_{(\rho, \eta) \dot{\tilde{T}}} = \left[ p_a \partial^a, (g^{be} \circ h \circ \pi)^* p_e \partial^b \right]_{(\rho, \eta) \dot{\tilde{T}}} - 2 \left[ p_a \partial^a, (G_b - \frac{1}{4} F_b) \partial^b \right]_{(\rho, \eta) \dot{\tilde{T}}}, \]

and

\[ \left[ p_a \partial^a, (G_b - \frac{1}{4} F_b) \partial^b \right]_{(\rho, \eta) \dot{\tilde{T}}} = p_a \partial \left( G_b - \frac{1}{4} F_b \right) \partial^b - \left( G_b - \frac{1}{4} F_b \right) \partial^b \]

it results that

\[ (S_1) \quad \left[ \dot{\tilde{C}}, \dot{\tilde{S}} \right]_{(\rho, \eta) \dot{\tilde{T}}} - \dot{\tilde{S}} = 2 \left( \frac{\partial (G_b - \frac{1}{4} F_b)}{\partial p_f} + 2 \left( G_b - \frac{1}{4} F_b \right) \right) \partial^b \]

Using equality (8.3), it results that

\[ \frac{\partial (G_b - \frac{1}{4} F_b)}{\partial p_f} = (\rho, \eta) \Gamma_{bc} \left( g^c \circ h \circ \pi \right) \]

(8.2)

\[ + \frac{1}{2} \left( g^{de} \circ h \circ \pi \right) p_e \left( L^a_{dc} \circ h \circ \pi \right) (\tilde{g}_{ab} \circ h \circ \pi) \left( g^f \circ h \circ \pi \right) \]

\[ - \frac{1}{2} \left( \rho^i \circ h \circ \pi \right) \partial (g^{ae} \circ \pi) p_e (\tilde{g}_{ab} \circ h \circ \pi) \left( g^f \circ h \circ \pi \right) \]

\[ + \frac{1}{2} \left( g^{ae} \circ h \circ \pi \right) p_e \left( \rho^i_a \circ h \circ \pi \right) \partial (\tilde{g}_{be} \circ \pi) \left( g^f \circ h \circ \pi \right) \]

Using equalities \((S_1)\) and \((S_2)\), it results the conclusion of the theorem. \(q.e.d.\)
Theorem 8.7  All \((g, h)\)-lifts solutions of the following system of equations:

\[
\begin{align*}
\frac{dp_b}{dt} + (\rho, \eta) \Gamma_{bc} (g^{cf} \circ h \circ \pi^*) \, p_f \\
+ \frac{1}{2} \left( g^{de} \circ h \circ \pi^* \right) p_e \left( L^{b}_{dc} \circ h \circ \pi^* \right) \left( g_{ba} \circ h \circ \pi^* \right) \left( g^{cf} \circ h \circ \pi^* \right) \, p_f \\
- \frac{1}{2} \left( \rho_i \circ h \circ \pi^* \right) \frac{\partial \left( g^{be} \circ h \circ \pi^* \right)}{\partial x^j} p_e \left( g_{ba} \circ h \circ \pi^* \right) \left( g^{cf} \circ h \circ \pi^* \right) \, p_f \\
+ \frac{1}{2} \left( g^{ae} \circ h \circ \pi^* \right) p_e \left( \rho_i \circ h \circ \pi^* \right) \frac{\partial \left( g_{be} \circ h \circ \pi^* \right)}{\partial x^j} \left( g^{cf} \circ h \circ \pi^* \right) \, p_f = 0,
\end{align*}
\]

are the integral curves of canonical \((\rho, \eta)\)-spray associated to the dual mechanical \((\rho, \eta)\)-system \( \left( \left( \tilde{E}, \pi^*, M \right), \tilde{F}_e, (\rho, \eta) \right) \) and from locally invertible \( B^\nu \)-morphism \((g, h)\).

9  A Hamiltonian formalism for Hamilton mechanical \((\rho, \eta)\)-systems

Let \( \left( \left( \tilde{E}, \pi^*, M \right), \tilde{F}_e, H \right) \) be an arbitrarily Hamilton mechanical \((\rho, \eta)\)-system.

Let \((d\tilde{z}^a, d\tilde{\rho}_a)\) be the natural dual \((\rho, \eta)\)-base of the natural \((\rho, \eta)\)-base \( \left( \tilde{\partial}_a, \tilde{\partial} \right) \).

It is very important to remark that the 1-forms \( d\tilde{z}^a, d\tilde{\rho}_a, \, a \in \Gamma, \rho \) are not the differentials of coordinates functions as in the classical case, but we will use the same notations. In this case

\[
(d\tilde{z}^a) \neq \text{d}^{(\rho, \eta)T\tilde{E}}(\tilde{z}^a),
\]

where \( \text{d}^{(\rho, \eta)T\tilde{E}} \) is the exterior differentiation operator associated to exterior differential \( F(\tilde{E}) \)-algebra

\[
\left( \Lambda \left( (\rho, \eta)T\tilde{E}, (\rho, \eta) \tau^*_{\tilde{E}}, \tilde{E} \right), +, \cdot, \wedge \right).
\]

Let \( H \) be a regular Hamiltonian and let \((g, h)\) be a locally invertible \( B^\nu \)-morphism of \( \left( \tilde{E}, \pi^*, M \right) \) source and \((E, \pi, M)\) target.

**Definition 9.1** The 1-form

\[
\theta_H = \left( g_{ae} \circ h \circ \pi^* \right) H^e \, d\tilde{z}^a
\]

will be called the 1-form of Poincaré-Cartan type associated to the regular Hamiltonian \( H \) and from locally invertible \( B^\nu \)-morphism \((g, h)\).

We obtain easily:

\[
\theta_H \left( \tilde{\partial}_b \right) = \left( g_{be} \circ h \circ \pi^* \right) \cdot H^e, \quad \theta_H \left( \tilde{\partial} \right) = 0.
\]
Definition 9.2 The 2-form
\[ \omega_H = d^{(\rho, \eta)}_H \] will be called the 2-form of Poincaré-Cartan type associated to the Hamiltonian \( H \) and to the locally invertible \( B^v \)-morphism \( (g, h) \).

By the definition of \( d^{(\rho, \eta)}_H \), we obtain:
\[ \omega_H (U, V) = \Gamma \left( ^* \tilde{\rho}, \text{Id}^*_{E} \right) (U) (\theta_H (V)) - \Gamma \left( ^* \tilde{\rho}, \text{Id}^*_{E} \right) (V) (\theta_H (U)) - \theta_H \left( [U, V]^{(\rho, \eta)}_{\tau} \right), \]
for any \( U, V \in \Gamma \left( (\rho, \eta) T^*E, (\rho, \eta) T^*F \right) \).

**Definition 9.3** The real function
\[ \mathcal{E}_H = p_a H^a - H \]
will be called the energy of regular Hamiltonian \( H \).

**Theorem 9.1** The equation
\[ i_S (\omega_H) = -d^{(\rho, \eta)}_H (\mathcal{E}_H), \quad S \in \Gamma \left( (\rho, \eta) T^*E, (\rho, \eta) T^*F \right), \]
has an unique solution \( \dot{S}_H (g, h) \) of the type:
\[ \left( g^{ae} \circ h \circ \pi \right) p_e \partial_a - 2 \left( G_a - \frac{1}{4} F_a \right) \dot{\theta}^a, \]
where
\[ -2 \left( G_a - \frac{1}{4} F_a \right) = E_{b} (H, g, h) \dot{H}_ae \left( g^{eb} \circ h \circ \pi \right) \]
and
\[ E_{b} (H, g, h) = \left( \rho^*_b \circ h \circ \pi \right) H_i - \left( \rho^*_b \circ h \circ \pi \right) p_b H_i^a \]
\[ - \left( g^{df} \circ h \circ \pi \right) p_f \left( g^{de} \circ h \circ \pi \right) \frac{\partial \left( \tilde{g}_{ae} \circ \pi \right) H^*}{\partial x^*} \]
\[ + \left( g^{df} \circ h \circ \pi \right) p_f \left( g^{de} \circ h \circ \pi \right) \frac{\partial \left( \tilde{g}_{ae} \circ \pi \right) H^*}{\partial x^*} \]
\[ + \left( g^{df} \circ h \circ \pi \right) p_f \left( L_{ab} \circ h \circ \pi \right) \left( \tilde{g}_{ae} \circ h \circ \pi \right) H^e \]
\( \dot{S}_H (g, h) \) will be called the canonical \( (\rho, \eta) \)-semispray associated to the Hamilton mechanical \( (\rho, \eta) \)-system \( \left( E, \pi, M \right), \dot{F}_e, H \) and from locally invertible \( B^v \)-morphism \( (g, h) \).

**Proof.** We obtain that
\[ i_S (\omega_H) = -d^{(\rho, \eta)}_H (\mathcal{E}_H) \]
if and only if
\[ \omega_H \left( \mathring{S}, X \right) = -\Gamma \left( \mathring{\rho}, \text{Id}_E \right) (X) (E_H), \]
for any \( X \in \Gamma \left( (\rho, \eta) T \mathring{E}, (\rho, \eta) \tau_{\mathring{E}}, \mathring{E} \right). \)

Particularly, we obtain:
\[ \omega_L \left( \mathring{S}, \mathring{\partial}_b \right) = -\Gamma \left( \mathring{\rho}, \text{Id}_E \right) \left( \partial_b \right) (E_H). \]

If we expand this equality, we obtain
\[
\left( g^{df} \circ h \circ \pi \right) P_f \left( \rho^i \circ h \circ \pi \right) \frac{\partial (g_{b \circ h \circ \pi} \mathring{H}^e)}{\partial x^i} - \left( \rho^i \circ h \circ \pi \right) \frac{\partial (g_{b \circ h \circ \pi} \mathring{H}^e)}{\partial x^i}
\]
\[ - \left( L^c_{ab} \circ h \circ \pi \right) \left( g_{ec} \circ h \circ \pi \right) H^e - 2 \left( G_b - \frac{1}{4} F_b \right) \left( g_{ae} \circ h \circ \pi \right) \cdot H^e
\]
\[ = \left( \rho^i \circ h \circ \pi \right) L_i - \left( \rho^i \circ h \circ \pi \right) \frac{\partial (p_a H^a)}{\partial x^i}. \]

After some calculations, we obtain the conclusion of the theorem. \( q.e.d. \)

**Theorem 9.2** If \( \mathring{S}_H (g, h) \) is the canonical \((\rho, \eta)\)-semispray associated to the Hamilton mechanical \((\rho, \eta)\)-system \( \left( \mathring{E}, \mathring{\pi}, M, F_e, H \right) \) and from locally invertible \( B^V \)-morphism \( (g, h) \), then the real local functions
\[
(\rho, \eta) \Gamma_{bc} = -\frac{1}{2} \left( g_{cd} \circ h \circ \pi \right) \frac{\partial (E_b (H, g, h) H_{ac} (g^{eb} \circ h \circ \pi))}{\partial p_d}
\]
\[ - \frac{1}{2} \left( g^{df} \circ h \circ \pi \right) p_e \left( L^d_{bc} \circ h \circ \pi \right) \left( g_{fe} \circ h \circ \pi \right)
\]
\[ + \frac{1}{2} \left( \rho^i \circ h \circ \pi \right) \frac{\partial (g^{ec} \circ h \circ \pi)}{\partial x^i} \left( \rho^i \circ h \circ \pi \right) \frac{\partial (g_{dc} \circ h \circ \pi)}{\partial x^i}
\]
\[ - \frac{1}{2} \left( g^{de} \circ h \circ \pi \right) p_e \left( \rho^i \circ h \circ \pi \right) \frac{\partial (g_{bc} \circ h \circ \pi)}{\partial x^i} \]

are the components of a \((\rho, \eta)\)-connection \((\rho, \eta) \Gamma \) for the vector bundle \( \mathring{E}, \mathring{\pi}, M \) which will be called the \((\rho, \eta)\)-connection associated to the Hamilton mechanical \((\rho, \eta)\)-system \( \left( \mathring{E}, \mathring{\pi}, M, \mathring{F}_e, H \right) \) and from locally invertible \( B^V \)-morphism \( (g, h) \).

**Theorem 9.3** The parallel \((g, h)\)-lifts with respect to \((\rho, \eta)\)-connection \((\rho, \eta) \Gamma \) are the integral curves of the canonical \((\rho, \eta)\)-semispray associated to the Hamilton mechanical \((\rho, \eta)\)-system \( \left( \mathring{E}, \mathring{\pi}, M, \mathring{F}_e, H \right) \) and from locally invertible \( B^V \)-morphism \( (g, h) \).

**Definition 9.4** The equations
\[
\frac{dp_a (t)}{dt} - E_b (H, g, h) \mathring{H}_{ae} \left( g^{eb} \circ h \circ \pi \right) \circ u (c, \dot{c}) (x (t)) = 0,
\]
where \( x (t) = \eta \circ h \circ c (t) \), will be called the equations of Hamilton-Jacobi type associated to the Hamilton mechanical \((\rho, \eta)\)-system \( \left( \mathring{E}, \mathring{\pi}, M, \mathring{F}_e, H \right) \) and from locally invertible \( B^V \)-morphism \( (g, h) \).
Remark 9.1 The integral curves of the canonical \((\rho, \eta)\)-semispray associated to the Hamilton mechanical \((\rho, \eta)\)-system \(\left(\left(\overset{\circ}{E}, \overset{\circ}{\pi}, M\right), \overset{\circ}{F}_e, H\right)\) and from locally invertible \(B\)-morphism \((g, h)\) are the \((g, h)\)-lifts solutions for the equations of Hamilton-Jacobi type (9.10).

Using our theory, we obtain the following

**Theorem 9.4** If \(K\) is a Cartan fundamental function, then the geodesics on the manifold \(M\) are the curves such that the components of their \((g, h)\)-lifts are solutions for the equations of Hamilton-Jacobi type (9.10).

Therefore, it is natural to propose to extend the study of Cartan geometry from the dual of the Lie algebroid \(((\tau_M, M), [, ], (Id_{\tau_M}, Id_M))\), to the dual of an arbitrary (generalized) Lie algebroid \(((E, \pi, M), [, ]_{E, h}, (\rho, \eta))\).

References

[1] C. M. Arcu¸s, *Algebraic constructions in the category of Lie algebroids*, arXiv:math.DG/1101.0960v3, 20 Jul (2011).

[2] C. M. Arcu¸s, *Algebraic constructions in the category of vector bundles*, arXiv: math.DG/1101.0956 v3, 25 Jul (2011).

[3] C. M. Arcu¸s, *Interior and exterior differential systems for Lie algebroids*, Advanced in Pure Mathematics (accepted).

[4] C. M. Arcu¸s, *Lagrangian mechanics on generalized Lie algebroids*, arXiv:math-ph/1108.2844v2, 23 Aug (2011).

[5] É. Cartan, *Les espaces métriques fondés dur la notion d’aire*, Actual. Sci. Industr., No. 2, Paris, (1933).

[6] D. Hrimiuc, *On the differential geometry of an infinite dimensional Hamilton space*, Tensor N. S., 49, 238-249, (1990).

[7] D. Hrimiuc, *Hamilton Geometry*, Pergamon Press, Math. Comput. Modeling, 20, no. 415, 57-65, (1994).

[8] D. Hrimiuc, H. Shimada, *On the L-duality between Lagrange and Hamilton manifolds*, Nonlinear World, 3, 613-641, (1996).

[9] A. Kawaguchi, *Space of n-dimensions with a connection depending on n-dimensional plane elements*, Abh. Sem. Vektor und Tensor-Analysis, 5, 290-300, (1941).

[10] M. de Leon, J. Marrero, E. Martinez, *Lagrangian submanifolds and dynamics on Lie algebroids*, arXiv: math. DG/0407528 v1, (2004).

[11] R. Miron, *Hamilton geometry*, Ann. St. ale Univ. Al. I. Cuza, Iasi, s.I-a, Mat., 35, 33-67, (1989).

[12] R. Miron, *Cartan spaces in a new point of view by considering them as duals of Finsler spaces*, Tensor N. S., 46, 330-334, (1987).
[13] R. Miron, *The geometry of Cartan spaces*, Prog. of Math., India, (I, II), 22, 1-38, (1988).

[14] R. Miron, *Sur la géométrie des espaces Hamilton*, C.R Acad. Sci. Paris, Ser. I, 306, no. 4, 195-198, (1988).

[15] R. Miron, *Hamilton geometry*, Univ. Timișoara, Sem. Mecanică, 3, 54, (1987).

[16] R, Miron, Dragoș Hrimiuc, Hideo Shimada, Sorin V. Sabau, *The Geometry of Hamilton and Lagrange Spaces*, Kluver Academic Publishers, FTPH 118, 2001.

[17] L. Popescu, *Singular Hamilton Spaces*, Libertas Math., Texas University, 24, 111-114, (2004).

[18] L. Popescu, *Aspects of Lie algebroids geometry and Hamiltonian formalism*, Annals. Univ. Al. I. Cuza, Iasi, Series I, Math., LIII, 297-308, (2007).

[19] L. Popescu, *Hamiltonian formalism on Lie algebroids and its applications*, Differential Geometry and its Applications, Proc. Conf. Olomouc, 2007, World Scientific Publishing, 665-673, (2008).

[20] S. Vacaru, *Clifford-Finsler algebroids and nonholonomic Einstein-Dirac structures*, J. of Math. Phys. 47, 2093504,1-20, (2006).

[21] S. Vacaru, *Nonholonomic Algebroids, Finsler Geometry and Lagrange-Hamilton Spaces*, ArXiv: math-ph/0705.0032v1, (2007).