The Monotony Criterion for a Finite Size Scaling Analysis of Phase Transitions

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Abstract

We propose a new criterion to analyse the order of phase transitions within a finite size scaling analysis. It refers to response functions like order parameter susceptibilities and the specific heat and states different monotony behaviour in volume for first and second order transitions close to the transition point. The criterion applies to analytical and numerical studies of phase diagrams including tricritical behaviour.

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1 Introduction

Finite size scaling analysis is particularly useful in numerical simulations to distinguish first from second order phase transitions by their different onset in a large but finite volume. Usually it is the specific scaling behaviour of the peak in a susceptibility, the shift in the critical coupling/temperature and the width of the critical region that anticipates the nature of the transition in the thermodynamic limit [1]. In this letter we formulate a new criterion in a finite volume to distinguish first from second order phase transitions. For a certain interval of the scaling region response functions with a nonanalytic behaviour in the infinite volume limit show different monotony behaviour for 1st and 2nd order transitions. Examples for such functions are the specific heat and order parameter susceptibilities. They are increasing in volume in a certain neighbourhood of $T_c$ for 2nd order transitions, and decreasing for 1st order transitions for some range in the scaling region, which will be specified below.

Figure 1: $(t, v)$-plane in the vicinity of a phase transition at $t = v = 0$. $t$ denotes the scaling field $t = (T - T_c(L))/T_c(L)$, $v$ is inverse to some power of the volume $L^x$ with $x > 0$. Let us consider the susceptibility as a function of $t$ and $v$. Then, for a 1st order transition, $\chi(t, v_1) > \chi(t, v_2)$, whereas for a 2nd order transition $\chi(t, v_1) < \chi(t, v_2)$. The shaded part is the crossover region between different asymptotic behaviour as $(t, v) \to 0$.

For definiteness we fix the notation in terms of an order parameter susceptibility $\chi(T, L)$, considered as a function of the temperature $T$ and the spatial size parameter $L$, so that a symmetric $D$-dimensional volume is of size $L^D$. By $T_c(L)$ we denote the location of the maximum of $\chi(T, L)$ in volume $L^D$ associated with the phase transition at $T_c \equiv T_c(\infty)$. In the infinite volume

$$\chi(t + T_c(\infty), L = \infty) < \infty \quad \text{as } t \to 0$$  \hspace{1cm} (1)

for a 1st order transition at $T_c$, with a possible discontinuity at $t = 0$ in the associated order parameter, $t$ denoting the scaling field $t = (T - T_c(L))/T_c(L)$, whereas

$$\chi(t + T_c(\infty), L = \infty) \simeq A|t|^{-\gamma} \quad \text{as } t \to 0$$  \hspace{1cm} (2)

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{image.png}
\caption{(t, v)-plane in the vicinity of a phase transition at t = v = 0. t denotes the scaling field t = (T - T_c(L))/T_c(L), v is inverse to some power of the volume L^x with x > 0. Let us consider the susceptibility as a function of t and v. Then, for a 1st order transition, \(\chi(t, v_1) > \chi(t, v_2)\), whereas for a 2nd order transition \(\chi(t, v_1) < \chi(t, v_2)\). The shaded part is the crossover region between different asymptotic behaviour as \((t, v) \to 0\).}
\end{figure}

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for a 2nd order transition with critical exponent $\gamma > 0$. On the other hand, in the opposite order of limits, $\chi(T_c(L), L)$ diverges in both cases as $L$ approaches infinity. Stated differently, at $T_c(L)$, in the limit $L \to \infty$ $\chi$ has a $\delta$-function singularity or a power law type of singularity for a 1st or 2nd order transition, respectively. It is this difference that is responsible for the different monotony properties for $t \neq 0$ in the finite volume.

The essential statement of the monotony criterion is the following (details will be given below): For a sufficiently large size $L_s < \infty$, there is always a size $L_l$ with $L_l > L_s$ and such that (cf. Fig. 1)

\[ \chi(\delta + T_c(L), L \geq L_l) < \chi(\delta + T_c(L_s), L_s) \] for 1st order,

\[ \chi(\delta + T_c(L), L \geq L_l) > \chi(\delta + T_c(L_s), L_s) \] for 2nd order,

that is, $\chi(\delta + T_c(L), L)$ is decreasing or increasing in volume for a 1st or 2nd order phase transition, respectively. The susceptibility is measured at fixed distance $\delta$ to the volume dependent maximum, constraint by

\[ c_1 \sigma(L)^{1+\varepsilon} < |\delta| < c_2 \sigma(L_s). \] (5)

Here $\sigma(L) = L^{-D}$ or $\sigma(L) = L^{-1/\nu}$ for 1st or 2nd order transition, respectively, with $\nu$ being the critical exponent of the correlation length, $\xi \sim |T - T_c|^{-\nu}$ as $T \to T_c$. Hence $\sigma(L)$ is the width of the critical region in the volume $L^D$, $c_1$, $c_2$ and $\varepsilon$ are positive constants, typically $\varepsilon = 1$. Beyond the general constraint that both linear sizes $L_s$ and $L_l$ have to be sufficiently large, in addition $L_l$ has to be sufficiently larger than $L_s$, so that $\sigma(L_l)$ is considerably smaller than $\sigma(L_s)$. The monotony behaviour of $\chi$ does not refer to values of $L_l$ close to $L_s$. In terms of Fig. 1, $L_s$ and $L_l$ are separated by the shaded area, for which we do not make any predictions on the behaviour of $\chi$.

In section 2 we make these statements more precise. We specify the conditions under which the criterion holds and give the proof. As it turns out, the criterion applies whenever the standard conjectures on finite size scaling behaviour hold. Rigorous proofs of these standard conjectures are missing in general, neither do we attempt to give such proofs in the treatment below. In section 3 we conclude with some remarks on testing the criterion in analytical and numerical calculations.

2 Statements of the criterion

Let $t$ denote the scaling field, i.e. $t = (T - T_c(L))/T_c(L)$. $T_c(L)$ locates the maximum of the susceptibility in the volume $L^D$. Furthermore we set $v = L^{-m}$ with some $m > 0$. The infinite volume limit is obtained as $v \to 0$ from above.

The transition region is given by small $t$ and $v$. Let

\[ H^2 := \{(t, v) \in \mathbb{R}^2 \mid v \geq 0\} \] (6)
denote a half plane, $\mathcal{U} \subseteq H^2$ the intersection of an open neighbourhood of $0 \in \mathbb{R}^2$ with $H^2$ and $\mathcal{U}^* = \mathcal{U} \setminus \{0\}$.
2.1 First order transitions

The typical behaviour of a susceptibility close to a first order transition is summarized by the following definition (with $v = L^{-D}$). As usual, $C(U^*)$ denotes the set of real valued continuous functions in $U^*$.

**Definition 1** Let $\omega > 0$. $\Psi_1^\omega(U)$ is the set of functions $\chi \in C(U^*)$ with the following properties.

1. $\chi \in C^1(U^* \setminus \{(R, 0)\})$, that is, $\chi$ is continuously differentiable for $v \neq 0$.
2. There exists $\nu(t) > 0$ such that for all $v < |t|/\nu(t)$
   \[ |\chi(t, v)| \leq \omega. \]
3. With appropriate positive constants $c, K_1, K_2$ and $\epsilon$ we have in $U^*$ for $v \neq 0$
   \[
   |\chi(0, v) - \frac{c}{v}| < \frac{K_1}{v^{1-\epsilon}}, \\
   \left| \frac{\partial}{\partial t} \chi(t, v) \right| < \frac{K_2}{v^2}.
   \]

It follows already from this definition that $\nu(t)$ is bounded from below by some positive constant. This is part of the following proposition. The main statement is the monotony behaviour of a function that obeys Definition 1.

**Proposition 1** Let $\omega > 0, \chi \in \Psi_1^\omega(U)$. There exist positive $\delta, \epsilon$ such that the following statements hold.

1. For all $t$ with $|t| < \epsilon\delta$,
   \[
   \nu(t) > \epsilon. \tag{7}
   \]
2. For all $(t, v), (t, w) \in U^*$ with $\nu(t)w < |t| < \epsilon v$ and $v < \delta$, we have $w < v$ and
   \[
   \chi(t, v) > |\chi(t, w)|. \tag{8}
   \]

The proposition includes the case $w = 0$ of infinite volume. In this case, in the critical region $|t|/v < \epsilon$, the susceptibility $\chi$ is always larger in the finite volume than in the infinite one.

**Proof:** Let $\chi \in \Psi_1^\omega(U), \omega > 0$.

\alpha. There are $\epsilon, \delta > 0$ such that for all $t, v$ with $v < \delta, |t| < \epsilon v$

\[
\chi(t, v) \geq 2\omega. \tag{9}
\]
For,
\[ \chi(t, v) = \chi(0, v) + t \int_0^1 ds \frac{\partial}{\partial \eta} \chi(\eta, v) \bigg|_{\eta=st}, \]
and there are \( c_0, K > 0 \) so that
\[ \chi(0, v) > \frac{c_0}{v}, \]
\[ \left| \frac{\partial}{\partial \eta} \chi(\eta, v) \right| < \frac{K}{v^2} \]
in \( U^* \). Hence, with \( \epsilon = \frac{c_0}{2K} \),
\[ \chi(t, v) \geq \frac{c_0}{v} - \frac{|t|K}{v^2} > \frac{c_0 - \epsilon K}{v} = \frac{c_0}{2v}. \]
Choosing \( \delta = \frac{c_0}{4\omega} \), we get for \( v < \delta \) the lower bound on \( \chi(t, v) \).

\( \beta \). By definition of \( \nu(t) \), we have whenever \( \nu(t)w < |t| \) the upper bound
\[ |\chi(t, w)| \leq \omega. \tag{10} \]

\( \gamma \). Summarizing \( \alpha \) and \( \beta \), it follows that for all \( w, v \) with \( \nu(t)w < |t| < \epsilon v \), \( v < \delta \), the strict bound
\[ |\chi(t, w)| < \chi(t, v) \]
holds. Furthermore, (10) implies that \( |\chi(t, w')| \leq \omega \) for all \( 0 \leq w' \leq \omega \). Hence it must hold that \( w < v \). It follows that
\[ \frac{|t|}{\nu(t)} \leq \frac{|t|}{\epsilon} \]
and hence \( \nu(t) > \epsilon \) for \( |t| < \epsilon \delta \). \( \square \)

The remainder of this section is devoted to the function class \( \Psi_1^\omega(U) \). As an example of such a function we consider the following typical representation of the magnetic susceptibility in the volume \( L^D \) close to a first order phase transition,
\[ \chi_2(T, L) = c_2 L^D \exp \left( -f_2 L^2(T - T_c(L))^2 \right) + \eta_2(T, L). \tag{11} \]
Here \( c_2, f_2 \) are positive real constants and \( \eta_2(\cdot, L) \) is analytic for \( L < \infty \), locally uniformly convergent as \( L \nearrow \infty \) (so that \( \eta_2(\cdot, \infty) \) is analytic). Furthermore, \( T_c(L) = T_c - d_2 L^{-m} + o(L^{-m}) \), with \( m \geq 0 \).

We can view \( \chi_2 \) as a function of some \( \Psi_1^\omega \) in essentially two ways. The first one corresponds to the point of view given in the introduction, in which the scaling field \( t \) measures the distance of the temperature \( T \) to the volume dependent location of the peak \( T_c(L) \). It does not refer to a particular volume dependence of \( T_c(L) \) itself. That is, \( m \) does not need further specification except that \( m \geq 0 \).

We set \( v = L^{-D} \), \( t = (T - T_c(L))/T_c(L) \) and
\[ \chi(t, v) = \chi_2(T, L) = cv^{-1} \exp \left( -f(v)v^{-2}t^2 \right) + \eta(t, v), \tag{12} \]
where \( \eta(t, v) \equiv \eta_2(T, L) \) and \( f(v) = f_2 T_c(L)^2 \), \( c = c_2 \).
Lemma 1 \( \chi \in \Psi_1^\omega(U) \) for some \( \omega > 0 \) and sufficiently small \( U \). \( \nu(t) \) of Def. 4 satisfies \( \nu(t)^{-1} \to 0 \) as \( |t| \to 0 \).

Proof: For \( \alpha, \beta > 0 \) and \( x \in \mathbb{R} \) the function
\[
f_{\alpha, \beta}(x) = x^\alpha \exp(-\beta x^2)
\]
attains its maximum at \( x^2 = \alpha/(2 \beta) \), is positive and monotone decreasing for \( x > (\alpha/(2 \beta))^{1/2} \), and satisfies on \( \mathbb{R} \) the bound
\[
|f_{\alpha, \beta}(x)| \leq \left( \frac{\alpha}{2 \beta e} \right)^{\frac{1}{2}}.
\]
We prove that \( \chi(t, v) \) match the criteria of Definition 4 to belong to some \( \Psi_1^\omega(U) \). There is \( \delta > 0 \) such that for all \( v < \delta \) we have \( \hat{f} \leq f(v) \leq \overline{f} \) for some \( \hat{f}, \overline{f} > 0 \). We choose \( U \) such that \( (t, v) \in U \) implies that \( v < \delta \) and \( |t| < \hat{f}^{1/2} \) and put
\[
K_\eta = \max_{(t,v) \in U} (|\eta(t,v)|, |\partial_t \eta(t,v)|).
\]
Condition 1. is obviously satisfied. To show 2. we notice that for
\[
\frac{|t|}{v} > \nu(t) > \left( \frac{1}{2 \hat{f}} \right)^{\frac{1}{2}}
\]
on one obtains
\[
|\chi(t,v)| < \left| \frac{c}{|t|} \frac{|t|}{v} \exp\left(-f(v) \left( \frac{|t|}{v} \right)^2 \right) + \eta \right| \leq c \frac{\nu(t)}{|t|} \exp(-f(v)\nu(t)^2) + |\eta|.
\]
We choose \( \nu(t) = |t|^{-1} \) and obtain with \( \omega = c/(\hat{f} e) + K_\eta \)
\[
|\chi(t, v)| \leq \omega.
\]
To verify the 3rd condition, we compute
\[
|\chi(0, v) - cv^{-1}| = |\eta(t, v)| \leq K_\eta,
\]
and notice that there is \( K > 0 \) such that
\[
\left| \frac{\partial}{\partial t} \chi(t, v) \right| \leq \frac{2|t|}{v^2} f|\chi| + |\partial_t \eta(t, v)|
\]
\[
\leq \frac{2\overline{f}}{v^2} \left[ c \left( \frac{1}{2 \hat{f} e} \right)^{\frac{1}{2}} + \hat{f}^{1/2} K_\eta \right] + K_\eta
\]
\[
\leq \frac{K}{v^2}.
\]
Hence, \( \chi \in \Psi_1^\omega (U) \). \( \square \)

The next lemma states that the susceptibility \( \chi_2 \) matches the criteria of Definition 1 also if the scaling field \( t \) measures the distance of \( T \) from the phase transition point, i.e. to \( T_c(L = \infty) \). The only additional constraint then is that the peak \( T_c(L) \) of \( \chi_2 \) converges sufficiently fast to \( T_c(\infty) \), namely that

\[
|T_c(L) - T_c(\infty)| \leq \frac{c}{L^m} \quad \text{with } m \geq D.
\]

We set \( v = L^{-D} \), \( t = (T - T_c(\infty))/T_c(\infty) \) and

\[
\chi(t, v) = \chi_2(T, L) = cv^{-1} \exp (-f v^{-2} (t + dv^n)^2) + \eta(t, v),
\]

where now \( \eta(t, v) \equiv \eta_2(T, L) \) and \( f = f_2 T_c(\infty)^2 \), \( d = d_2/T_c(\infty) \), and \( n = m/D \).

Similarly as above, we get

**Lemma 2** Let \( n \geq 1 \). Then \( \chi \in \Psi_1^\omega (U) \) for some \( \omega > 0 \) and sufficiently small \( U \). \( \nu(t) \) of Def. 1 satisfies \( \nu(t) - 1 \to 0 \) as \( |t| \to 0 \).

**Proof:** For \( a, b, \epsilon \in \mathbb{R} \), \( \epsilon \neq 0 \),

\[
(a + b)^2 \geq a^2 \left( 1 - \frac{1}{\epsilon^2} \right) + b^2 \left( 1 - \epsilon^2 \right).
\]

We choose \( \delta, \eta > 0 \) sufficiently small such that the following estimates on \( \chi \) are true for all \( t, v \) with \( |t| < \eta \), \( 0 < v < \delta \), identifying \( U \).

\( \chi \) obviously satisfies condition 1. of Def. 1. To show 2. we estimate

\[
|\chi(t, v)| \leq cv^{-1} \exp (-f v^{-2} \left( \frac{t^2}{2} - (dv^n)^2 \right) ) + |\eta|
\]

\[
\leq 2cv^{-1} \exp (-\frac{f}{2} v^{-2} t^2) + |\eta|.
\]

Proceeding as in the proof of the previous lemma we obtain condition 2, with appropriate \( \omega \). To realize the 3rd condition, we first notice that

\[
|\chi(0, v) - cv^{-1}| \leq cv^{-1} \left| e^{-f d^2 v^{2(n-1)}} - 1 \right| + |\eta| \leq \frac{K}{v^{1-\epsilon}}
\]

for some positive constant \( K \), with \( \epsilon = \min (1, 2(n-1)) \geq 0 \). Finally there is \( K' > 0 \) such that

\[
\left| \frac{\partial}{\partial t} \chi(t, v) \right| = 2fv^{-2} |t + dv^n||\chi| + |\partial_t \eta(t, v)|
\]

\[
\leq 2fcv^{-2} \left( \frac{1}{2fe} \right)^{\frac{1}{2}} \cdot 2 = \frac{K'}{v^2}.
\]

Hence \( \chi \in \Psi_1^\omega (U) \). \( \square \)
Finally, we state without proof that every function $\chi: U^* \rightarrow \mathbb{R}$ of the form

$$\chi(t, v) = \frac{1}{v} f(\frac{t}{v}) + \tilde{\chi}(t, v)$$

(17)

belongs to $\Psi^\omega(U)$ for some $U$ and with appropriate $\omega > 0$, if the following conditions are satisfied.

1a. $\tilde{\chi}(t, v) \in C^1(U^*)$,

1b. $\tilde{\chi}(t, v)$ together with its (first) partial derivatives are uniformly bounded in $U^*$.

2a. $f \in C^1(\mathbb{R})$ is a nonnegative function with $f(0) > 0$,

2b. $\lim_{x \to \pm \infty} |x|^{1+\epsilon} f(x) = 0$ for some $\epsilon > 0$,

2c. $(d/dx)f(x)$ is uniformly bounded on $\mathbb{R}$.

Any such function has the property that it ”approaches $\delta$” locally, i.e. there is $c > 0$ so that for any test function $g(t)$ supported in a small neighbourhood of $t = 0$

$$\lim_{v \to 0^+} [\chi(t, v)]g = \lim_{v \to 0^+} \int dt \; \chi(t, v) g(t) = cg(0).$$

The example discussed above corresponds to

$$g(x) = \left(\frac{C}{\pi}\right)^{1/2} \exp\left(-cx^2\right), \; c > 0.$$

### 2.2 Second order transitions

In contrast to the first order case, at a second order transition the susceptibility $\chi$ can be divergent in the infinite volume limit as the critical temperature is approached. (It need not be divergent, since another second derivative of a suitable thermodynamic potential may be divergent instead.) If $\chi$ is divergent with a critical exponent, it is described by the following definition (with $\nu = L^{-1/\nu}$, $\nu > 0$ being the critical exponent of the correlation length $\xi$).

**Definition 2** Let $\gamma > 0$. $\Psi^\gamma_2(U)$ is the set of functions $\chi \in C(U^*)$ that satisfy the following conditions.

1. There are constants $A, K, \epsilon > 0$ such that in $U^*$

$$|\chi(t, 0) - A|t|^{-\gamma}| \leq K |t|^{-\gamma+\epsilon}.$$

Furthermore, there exist $\nu, \mathcal{C} > 0$ so that for $|t| > \nu \nu$,

$$\chi(t, v) \geq \mathcal{C} \chi(t, 0).$$
2. There are constants $\eta, B > 0$ such that for $|t| < \eta v$

$$|\chi(t, v)| < B v^{-\gamma}.$$  

Functions that obey Definition 2 are always increasing in volume close to the transition.

Proposition 2 Let $\gamma > 0$ and $\chi \in \Psi^2_\gamma(\mathcal{U})$. There are constants $v, \epsilon > 0$ such that for all $t, v, w$ with $\nu w < |t| < \epsilon v$ we have $w < v$ and

$$|\chi(t, v)| < \chi(t, w).$$

The inequality includes the case $w = 0$, corresponding to infinite volume. That is, the susceptibility is always smaller in finite volume than in the infinite volume as long as we are in the critical region $|t|/v < \epsilon$.

**Proof:** Let $\chi \in \Psi^2_\gamma(\mathcal{U}), \gamma > 0$. There are numbers $C, D, \nu > 0$ such that in $\mathcal{U}^*$ for $|t| > \nu w$

$$\chi(t, w) > C \chi(t, 0) > D |t|^{-\gamma}.$$  

Furthermore, there are $\eta, B > 0$ such that for $|t| < \eta v$

$$|\chi(t, v)| < B v^{-\gamma}.$$  

We choose $\epsilon = \min (\eta, (D/B)^{1/\gamma})$ and get for $\nu w < |t| < \epsilon v$

$$|\chi(t, v)| < B v^{-\gamma} < B \left(\frac{\epsilon}{|t|}\right)^\gamma < \frac{B\epsilon^\gamma}{D} \chi(t, w) < \chi(t, w).$$  

This proves the lemma. □

The specific property for a 2nd order transition that the singular part of the free energy density behaves as a generalized homogeneous function implies for the susceptibility in a volume $L^D$ a typical form like

$$\chi_2(T, L) = |T - T_c|^{-\gamma} Q((T - T_c) L^{1/\nu}) + \eta(T, L), \quad (18)$$

with some $\gamma > 0$. Here $\eta(\cdot, L)$ has similar analyticity properties as in (11) above, $\nu > 0$ is the critical exponent of the correlation length $\xi \sim |T - T_c|^{-\nu}$. $Q$ is continuous and behaves as

$$\lim_{x \to 0} |x|^{-\gamma} Q(x) = K > 0,$$

$$\lim_{x \to \pm \infty} Q(x) = C > 0.$$  

The first equation expresses the absence of a divergence of $\chi_2$ for finite $L$, the second one its presence in the infinite volume case. With $t = (T - T_c)/T_c, \nu = L^{-1/\nu}$ and

$$\chi(t, v) := \chi_2(T, L)$$
it is straightforward to verify that $\chi$ belongs to $\Psi_2^\gamma(U)$.

More generally, every function $\chi: U^* \to R$ of the form

$$\chi(t, v) = \frac{1}{v^\gamma} f\left(\frac{t}{v}\right) + \tilde{\chi}(t, v)$$

with $\gamma > 0$ belongs to some $\Psi_2^\gamma(U)$, if the following conditions are satisfied:

1. $\tilde{\chi}(t, v) \in C^1(U^*)$.
2a. $f \in C^1(R)$, and $f(0) > 0$,
2b. $\lim_{x \to \pm\infty} |x|^\gamma f(x) = C$ for some finite $C > 0$.

Compared to the first order case (17) the essential difference comes from property (2b).

As an example,

$$\chi(t, v) = (t^2 + v^2)^{-\frac{m}{2}}, \quad m > 0,$$

belongs to the class $\Psi_2^m$.

3 Applications

The monotony criterion has been applied in the framework of convergent series expansions of $O(N)$ symmetric spin models on the lattice, performed to the 20th order in the hopping parameter (corresponding to the inverse temperature) both in the infinite and finite volume [2]. By means of the series representation it is hard to compute the scaling behaviour of a susceptibility $\chi(T, L)$ at the maximum $T_c(L)$, since $T_c(L)$ as well as the transition at $T_c(\infty)$ are extrapolated as the convergence radius of the series. The monotony criterion circumvents this difficulty in that it allows us to determine the order of the transition by the different $L$-dependence of $\chi(T, L)$ already close to $T_c(L)$, that is by an increase in volume for a second order transition, and a decrease for a first order transition.

In series representations the larger volume $L^D_l$ is conveniently chosen to be infinite so that (3)-(5), with $L = \infty$ and $\sigma(L) = 0$, give strong criteria in the whole scaling region. In Monte Carlo simulations both volumes $L^D_s$ and $L^D_l$ must be finite. The monotony criterion works equally well in this case. Both volumes have to be sufficiently large in order to match the conditions of the criterion (which reflect the standard assumptions of finite size scaling). Some care is needed to ensure that the susceptibility is actually measured at a temperature that satisfies (4), i.e. out of a region where the difference in the volume dependence between first and second order transitions appears. In particular this concerns the lower bound on $\delta$ in (4) because for $L_l < \infty$, also in case of a first order transition, a small neighbourhood around the peak at $T_c(L)$ exists, where $\chi(T, L)$ is increasing in volume because of the rounding of the $\delta$-singularity.
To summarize, let $\lambda$ denote a generic coupling constant. Application of the monotony criterion amounts to a choice of two sizes $L_s$ and $L_l$ with $L_s$, $L_l$ and $L_l/L_s$ sufficiently large. Let us define the ratio $r$

$$r(L_s, L_l; \lambda) := 1 - \frac{\chi(T_c(L_s) + \delta, L_s; \lambda)}{\chi(T_c(L_l) + \delta, L_l; \lambda)},$$

(20)

with $\delta$ satisfying the constraint (3). The monotony criterion then says that

$$r(L_s, L_l; \lambda) \begin{cases} > 0, & 2\text{nd order} \\ < 0, & 1\text{st order} \\ = 0, & \text{tricritical point for } r \text{ changing sign at } \lambda. \end{cases}$$

In (20) the susceptibility is measured at fixed distance to the volume-dependent location of its maximum. A reasonable alternative is to measure $\chi(T, L)$ for both volumes at the same temperature $T$ regardless of $L$, but $T$ chosen as $T = T_c(\infty) + \delta$. This requires that $T_c(L)$ approaches $T_c = T_c(\infty)$ according to

$$|T_c(L) - T_c(\infty)| \leq c \sigma(L),$$

(21)

cf. (3) and (13). The behaviour (21) meets the standard finite size scaling behaviour.

References

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