Representations of the orthosymplectic Yangian

A. I. Molev

Abstract

We give a complete description of the finite-dimensional irreducible representations of the Yangian associated with the orthosymplectic Lie superalgebra $\text{osp}_{1|2}$. The representations are parameterized by monic polynomials in one variable, they are classified in terms of highest weights. We give explicit constructions of a family of elementary modules of the Yangian and show that a wide class of irreducible representations can be produced by taking tensor products of the elementary modules.

1 Introduction

It is well-known that the classification of finite-dimensional irreducible representations of the orthosymplectic Lie superalgebras $\text{osp}_{M|2n}$ is more complicated than that of its general linear counterpart $\text{gl}_{M|N}$; see e.g. books [6] and [15]. It is therefore not surprising that such a disparity extends to the respective Yangians.

The Yangian for $\text{gl}_{M|N}$ was introduced by Nazarov [16] and its finite-dimensional irreducible representations were classified by Zhang [19]. The orthosymplectic Yangians $Y(\text{osp}_{M|2n})$ were introduced by Arnaudon et al. [1], and the general classification problem still remains open. Our goal in this paper is to describe finite-dimensional irreducible representations of the Yangian $Y(\text{osp}_{1|2})$. One can expect that this simplest case will be instrumental in solving the general classification problem. In particular, an extension to $\text{osp}_{1|2n}$ appears to be straightforward.

According to [1], the $R$-matrix originated in [18] leads to the $RTT$-type definition of the extended Yangian $X(\text{osp}_{M|2n})$. The Yangian $Y(\text{osp}_{M|2n})$ can be regarded as the quotient of the extended Yangian by the ideal generated by the center, implying that the finite-dimensional irreducible representations of these two algebras are essentially the same. An explicit description of the center and the Hopf algebra structure were given in [1]. In the subsequent work [2], a Drinfeld-type presentation of the Yangian $Y(\text{osp}_{1|2})$ was produced, the double Yangian was constructed and its universal $R$-matrix was calculated in an explicit form. Applications of the orthosymplectic Yangians to spin chain models were discussed in [3]. More recently, linear and quadratic evaluations in the Yangian $Y(\text{osp}_{M|2n})$ were investigated in [8] and [10].

By analogy with the $R$-matrix form of its orthogonal and symplectic counterparts, every finite-dimensional irreducible representation of the orthosymplectic Yangian is a highest weight representation; cf. [4]. For the extended Yangian $X(\text{osp}_{1|2})$, this means that the representations
are determined by the triples \((\lambda_1(u), \lambda_2(u), \lambda_3(u))\) of formal series \(\lambda_i(u) \in 1 + u^{-1} \mathbb{C}[[u^{-1}]]\) which should satisfy the consistency condition

\[
\lambda_1(u)\lambda_3(u + 1/2) = \lambda_2(u)\lambda_2(u + 1/2). \tag{1.1}
\]

The key step in the classification is to find the conditions on such triples for the corresponding representation to be finite-dimensional. Here we need a super-extension of the approach originally used by Tarasov [17] and which had already been adapted and applied to the twisted Yangians associated with the classical Lie algebras; see [14, Ch. 3 & 4]. This leads to the following result.

**Main Theorem.** The finite-dimensional irreducible representation of the algebra \(X(\mathfrak{osp}_{1|2})\) associated with the highest weight \((\lambda_1(u), \lambda_2(u), \lambda_3(u))\) is finite-dimensional if and only if

\[
\frac{\lambda_2(u)}{\lambda_1(u)} = \frac{P(u + 1)}{P(u)} \tag{1.2}
\]

for some monic polynomial \(P(u)\) in \(u\). Hence, the finite-dimensional irreducible representations of the Yangian \(Y(\mathfrak{osp}_{1|2})\) are parameterized by monic polynomials \(P(u)\).

The parametrization turns out to be the same as for the representations of the Yangian \(Y(\mathfrak{sl}_2)\) given in [7], and we will call \(P(u)\) the Drinfeld polynomial of the representation. The appearance of the consistency conditions (1.1) is best explained by the Gauss decomposition of the generator matrix for \(X(\mathfrak{osp}_{1|2})\) and can be compared with the similar conditions for the Yangian \(Y(\mathfrak{o}_3)\); see [12, 13, Sec. 5.3], which were derived in a different way in [4]. However, unlike the case of the Yangian \(Y(\mathfrak{sl}_2)\) (or \(Y(\mathfrak{o}_3)\)), there is no epimorphism \(X(\mathfrak{osp}_{1|2}) \to \mathfrak{U}(\mathfrak{osp}_{1|2})\), so that in general, representations of \(\mathfrak{osp}_{1|2}\) do not extend to the Yangian.

An essential step in the proof of the Main Theorem is the analysis of the elementary modules \(L(\alpha, \beta)\) over \(X(\mathfrak{osp}_{1|2})\) associated with the highest weights of the form

\[
\lambda_1(u) = \frac{u + \alpha}{u + \beta}, \quad \lambda_2(u) = 1, \quad \lambda_3(u) = \frac{u + \beta - 1/2}{u + \alpha - 1/2}. \tag{1.3}
\]

The corresponding small Verma module \(M(\alpha, \beta)\) turns out to be irreducible if and only if \(\beta - \alpha\) and \(\beta - \alpha + 1/2\) are not nonnegative integers. The elementary modules \(L(\alpha, \beta)\) are the irreducible quotients of \(M(\alpha, \beta)\) and so they split into three families, according to these conditions. Such a module is finite-dimensional if and only if \(\beta - \alpha \in \mathbb{Z}_+\). In this case, when regarded as an \(\mathfrak{osp}_{1|2}\)-module, \(L(\alpha, \beta)\) has the character which decomposes as

\[
\text{ch} L(\alpha, \beta) = \sum_{k=0}^{[\frac{\beta - \alpha}{2}]} \text{ch} V(\beta - \alpha - 2k),
\]

where \(V(l)\) denotes the \(2l + 1\)-dimensional \(\mathfrak{osp}_{1|2}\)-module with the highest weight \(l \in \mathbb{Z}_+\). In particular,

\[
\dim L(\alpha, \beta) = \binom{\beta - \alpha + 2}{2}.
\]
We construct a basis of each small Verma module $M(\alpha, \beta)$ and give explicit formulas for the action of the generators of $X(\mathfrak{osp}_{1|2})$. This leads to a corresponding description of all elementary modules. We then show that, up to twisting by a multiplication automorphism of $X(\mathfrak{osp}_{1|2})$, every finite-dimensional irreducible representation of this algebra is isomorphic to the quotient of the submodule of the tensor product module of the form

$$L(\alpha_1, \beta_1) \otimes \ldots \otimes L(\alpha_k, \beta_k),$$

(1.4)

generated by the tensor product of the highest weight vectors. The final step is to investigate irreducibility conditions for such tensor products.

In the case of the Yangian $Y(sl_2)$, an irreducibility criterion for tensor products of evaluation modules was given by Chari and Pressley [5]; see also [14, Ch. 3]. Such tensor products exhaust all irreducible $Y(sl_2)$-modules. This property turns out not to extend to representations of the Yangian for $\mathfrak{osp}_{1|2}$; see Example 4.19 below, but a wide class of irreducible modules can still be constructed explicitly via tensor products of the form (1.4).

2 Sign conventions, definitions and preliminaries

Consider the three-dimensional space $\mathbb{C}^{1|2}$ over $\mathbb{C}$ with basis elements $e_1, e_2, e_3$ where we assume a $\mathbb{Z}_2$-gradation defined by setting that the vector $e_2$ is even and the vectors $e_1$ and $e_3$ are odd. It will be convenient to use the involution on the set $\{1, 2, 3\}$ defined by $i \mapsto j' = 4 - i$. The endomorphism algebra $\text{End} \mathbb{C}^{1|2}$ then gets a $\mathbb{Z}_2$-gradation with the parity of the matrix unit $e_{ij}$ found by $i + j \mod 2$.

We will consider $3 \times 3$ matrices with entries in superalgebras. All such matrices will be even so that the $(i, j)$ entry will have the parity $i + j \mod 2$. We want to posit that the product of two matrices with entries in a superalgebra $A$ is calculated in the standard way (without any additional signs). Accordingly, the algebra of even matrices over $A$ will be identified with the tensor product algebra $\text{End} \mathbb{C}^{1|2} \otimes A$. A matrix $A = [a_{ij}]$ will be regarded as the element

$$A = \sum_{i,j=1}^{3} e_{ij} \otimes a_{ij}(-1)^{i+j} \in \text{End} \mathbb{C}^{1|2} \otimes A.$$

We will use the involutive matrix super-transposition $t$ defined by $(A^t)_{ij} = A_{j'i'}(-1)^{i+j} \theta_i \theta_j$, where we set

$$\theta_1 = \theta_2 = 1 \quad \text{and} \quad \theta_3 = -1.$$

This super-transposition is associated with the bilinear form on the space $\mathbb{C}^{1|2}$ defined by the anti-diagonal matrix $G = [\delta_{i',j}]$. We will also regard $t$ as the linear map

$$t : \text{End} \mathbb{C}^{1|2} \to \text{End} \mathbb{C}^{1|2}, \quad e_{ij} \mapsto e_{j'i'}(-1)^{i+j} \theta_i \theta_j.$$

(2.1)

In the case of multiple tensor products of the endomorphism algebras, we will indicate by $t_a$ the map (2.1) acting on the $a$-th copy of $\text{End} \mathbb{C}^{1|2}$.
A standard basis of the general linear Lie superalgebra $\mathfrak{gl}_{1|2}$ is formed by elements $E_{ij}$ of the parity $i+j \mod 2$ for $1 \leq i, j \leq 3$ with the commutation relations
\[ [E_{ij}, E_{kl}] = \delta_{kj} E_{il} - \delta_{il} E_{kj}(-1)^{(i+j)(k+l)} . \]
We will regard the orthosymplectic Lie superalgebra $\mathfrak{osp}_{1|2}$ associated with the bilinear form defined by $G$ as the subalgebra of $\mathfrak{gl}_{1|2}$ spanned by the elements
\[ F_{ij} = E_{ij} - E_{j'i'}(-1)^{ij+i'} \theta_i \theta_j . \]
For any given $\mu \in \mathbb{C}$ we will denote by $V(\mu)$ the irreducible highest weight module over $\mathfrak{osp}_{1|2}$ generated by a nonzero vector $\xi$ such that
\[ F_{11}\xi = \mu \xi \quad \text{and} \quad F_{12}\xi = 0. \]
The module $V(\mu)$ is finite-dimensional if and only if $\mu \in \mathbb{Z}_+$. In that case, $\dim V(\mu) = 2\mu + 1$.

Introduce the permutation operator $P$ as the element
\[ P = \sum_{i,j=1}^{3} e_{ij} \otimes e_{ji}(-1)^{ij} \in \text{End} \mathbb{C}^{1|2} \otimes \text{End} \mathbb{C}^{1|2} \]
and set
\[ Q = P^{t_1} = P^{t_2} = \sum_{i,j=1}^{3} e_{ij} \otimes e_{j'i'}(-1)^{ij} \theta_i \theta_j \in \text{End} \mathbb{C}^{1|2} \otimes \text{End} \mathbb{C}^{1|2} . \]
The rational function in $u$ given by
\[ R(u) = 1 - \frac{P}{u} + \frac{Q}{u - \kappa}, \quad \kappa = -3/2, \]
is an $R$-matrix, it satisfies the Yang–Baxter equation as originally found in [18]. The $R$-matrices produced in that paper are known to extend to the Brauer algebra so that the Yang–Baxter equation can be verified by taking a suitable Brauer algebra representation in tensor products of the $\mathbb{Z}_2$-graded spaces; cf. [8], [11].

Following [1], define the extended Yangian $X(\mathfrak{osp}_{1|2})$ as an associative superalgebra with generators $t^{(r)}_{ij}$ of parity $i+j \mod 2$, where $1 \leq i, j \leq 3$ and $r = 1, 2, \ldots$, satisfying certain quadratic relations. In order to write them down, introduce the formal series
\[ t_{ij}(u) = \delta_{ij} + \sum_{r=1}^{\infty} t^{(r)}_{ij} u^{-r} \in X(\mathfrak{osp}_{1|2})[[u^{-1}]] \quad (2.2) \]
and combine them into the $3 \times 3$ matrix $T(u) = [t_{ij}(u)]$ so that
\[ T(u) = \sum_{i,j=1}^{3} e_{ij} \otimes t_{ij}(u)(-1)^{ij+j} \in \text{End} \mathbb{C}^{1|2} \otimes X(\mathfrak{osp}_{1|2})[[u^{-1}]]. \]
Consider the algebra \( \text{End} \mathbb{C}^{1|2} \otimes \text{End} \mathbb{C}^{1|2} \otimes X(\mathfrak{osp}_{1|2})[[u^{-1}]] \) and introduce its elements \( T_1(u) \) and \( T_2(u) \) by

\[
T_1(u) = \sum_{i,j=1}^{3} e_{ij} \otimes 1 \otimes t_{ij}(u)(-1)^{ij+j}, \quad T_2(u) = \sum_{i,j=1}^{3} 1 \otimes e_{ij} \otimes t_{ij}(u)(-1)^{ij+j}.
\]

The defining relations for the superalgebra \( X(\mathfrak{osp}_{1|2}) \) can then be written in the form of the *RTT*-relation

\[
R(u-v) T_1(u) T_2(v) = T_2(v) T_1(u) R(u-v).
\]

As shown in [1], the product \( T(u) T^t(u-\kappa) \) is a scalar matrix with

\[
T(u-\kappa) T^t(u) = c(u) 1,
\]

where \( c(u) \) is a series in \( u^{-1} \). All its coefficients belong to the center \( ZX(\mathfrak{osp}_{1|2}) \) of \( X(\mathfrak{osp}_{1|2}) \) and generate the center.

The *Yangian* \( Y(\mathfrak{osp}_{1|2}) \) is defined as the subalgebra of \( X(\mathfrak{osp}_{1|2}) \) which consists of the elements stable under the automorphisms

\[
t_{ij}(u) \mapsto f(u) t_{ij}(u)
\]

for all series \( f(u) \in 1 + u^{-1} \mathbb{C}[[u^{-1}]] \). We have the tensor product decomposition

\[
X(\mathfrak{osp}_{1|2}) = ZX(\mathfrak{osp}_{1|2}) \otimes Y(\mathfrak{osp}_{1|2}).
\]

The Yangian \( Y(\mathfrak{osp}_{1|2}) \) can be equivalently defined as the quotient of \( X(\mathfrak{osp}_{1|2}) \) by the relation

\[
T(u-\kappa) T^t(u) = 1.
\]

We will also use a more explicit form of the defining relations (2.3) written in terms of the series (2.2) as follows:

\[
[t_{ij}(u), t_{kl}(v)] = \frac{1}{u-v} \left( \sum_{p=1}^{3} t_{pj}(u) t_{p'l}(v)(-1)^{i+j+p+ip+jp} \theta_{i} \theta_{p} \right)
\]

(2.7)

The coefficients of the series \( t_{11}(u), t_{12}(u), t_{21}(u) \) and \( c(u) \) generate the algebra \( X(\mathfrak{osp}_{1|2}) \). The mapping \( t_{ij}(u) \mapsto t_{ij}(-u) \) defines an anti-automorphism of \( X(\mathfrak{osp}_{1|2}) \), while each of the mappings

\[
t_{ij}(u) \mapsto t_{ij}(u + a), \quad a \in \mathbb{C},
\]

(2.8)

and \( t_{ij}(u) \mapsto t_{ij'}(u) \theta_{i} \theta_{j} \) defines an automorphism. Consider their composition to define the anti-automorphism

\[
\omega : t_{ij}(u) \mapsto t_{ij'}(-u + 1/2) \theta_{i} \theta_{j}.
\]

(2.9)
The universal enveloping algebra $U(\mathfrak{osp}_{1|2})$ can be regarded as a subalgebra of $X(\mathfrak{osp}_{1|2})$ via the embedding 

$$F_{ij} \mapsto \frac{1}{2}\left(t_{ij}(1) - t_{ji}(1)(-1)^{i+j}\theta_i\theta_j\right)(-1)^i.$$  

(2.10)

This fact relies on the Poincaré–Birkhoff–Witt theorem for the orthosymplectic Yangian which was essentially pointed out in [1] and [2], as the associated graded algebra for $Y(\mathfrak{osp}_{1|2})$ is isomorphic to $U(\mathfrak{osp}_{1|2}[u])$. It states that given any total ordering on the set of generators $t_{ij}^{(r)}$ with $i + j \leq 4$ and $r \geq 1$, the ordered monomials in the generators with the powers of odd generators not exceeding 1, form a basis of $X(\mathfrak{osp}_{1|2})$. A detailed proof can be carried over by adapting the arguments of [4, Sec. 3] to the super case in a straightforward way with the use of the vector representation recalled below in (4.19).

The extended Yangian $X(\mathfrak{osp}_{1|2})$ is a Hopf algebra with the coproduct defined by

$$\Delta : t_{ij}(u) \mapsto \sum_{k=1}^{3} t_{ik}(u) \otimes t_{kj}(u).$$  

(2.11)

For the image of the series $c(u)$ we have $\Delta : c(u) \mapsto c(u) \otimes c(u)$ and so the Yangian $Y(\mathfrak{osp}_{1|2})$ inherits the Hopf algebra structure from $X(\mathfrak{osp}_{1|2})$.

## 3 Gaussian generators

A Drinfeld-type presentation of the Yangian for $\mathfrak{osp}_{1|2}$ was given in [2] with the use of the Gauss decomposition of the matrix $T(u)$. We will be using some calculations produced therein and derive consistency relations for the Gaussian generators.

Apply the Gauss decomposition to the generator matrix $T(u)$ for $X(\mathfrak{osp}_{1|2})$,

$$T(u) = F(u) \ H(u) \ E(u),$$  

(3.1)

where $F(u)$, $H(u)$ and $E(u)$ are uniquely determined matrices of the form

$$F(u) = \begin{bmatrix} 1 & 0 & 0 \\ f_{21}(u) & 1 & 0 \\ f_{31}(u) & f_{32}(u) & 1 \end{bmatrix}, \quad E(u) = \begin{bmatrix} 1 & e_{12}(u) & e_{13}(u) \\ 0 & 1 & e_{23}(u) \\ 0 & 0 & 1 \end{bmatrix},$$

and $H(u) = \text{diag}[h_1(u), h_2(u), h_3(u)]$. Explicit formulas for the entries of the matrices $F(u)$, $H(u)$ and $E(u)$ can be written with the use of the Gelfand–Retakh quasideterminants [9]; cf. [13, Sec. 4]. In particular, we have

$$h_1(u) = t_{11}(u), \quad h_2(u) = \begin{bmatrix} t_{11}(u) & t_{12}(u) \\ t_{21}(u) & t_{22}(u) \end{bmatrix}, \quad h_3(u) = \begin{bmatrix} t_{11}(u) & t_{12}(u) & t_{13}(u) \\ t_{21}(u) & t_{22}(u) & t_{23}(u) \\ t_{31}(u) & t_{32}(u) & t_{33}(u) \end{bmatrix},$$
whereas
\[ e_{12}(u) = h_1(u)^{-1} t_{12}(u), \quad e_{23}(u) = h_2(u)^{-1} \begin{bmatrix} t_{11}(u) & t_{13}(u) \\ t_{21}(u) & t_{23}(u) \end{bmatrix}, \]
and
\[ f_{21}(u) = t_{21}(u) h_1(u)^{-1}, \quad f_{32}(u) = \begin{bmatrix} t_{11}(u) \\ t_{31}(u) \end{bmatrix} t_{32}(u) h_2(u)^{-1}. \]

**Proposition 3.1.** The following relations for the Gaussian generators hold:

\[ e_{12}(u) = -e_{23}(u + 1/2), \quad f_{21}(u) = f_{32}(u + 1/2), \quad (3.2) \]

and

\[ h_1(u) h_3(u + 1/2) = h_2(u) h_2(u + 1/2). \quad (3.3) \]

Moreover,

\[ c(u) = h_1(u) h_1(u + 1)^{-1} h_2(u + 1) h_2(u + 3/2). \quad (3.4) \]

**Proof.** The argument is quite similar to the proof of the corresponding relations for the Gaussian generators of \( Y(\rho_3) \) given in [12]; see also [13, Sec. 5.3]. We will outline a few key steps.

By inverting the matrices on both sides of (3.1), we get

\[ T(u)^{-1} = E(u)^{-1} H(u)^{-1} F(u)^{-1}. \]

On the other hand, relation (2.4) implies \( T^t(u) = c(u) T(u - \kappa)^{-1} \). Hence, by equating the \((i, j)\) entries with \( i, j = 2, 3 \) in this matrix relation, we derive

\[ h_1(u) = c(u) h_3(u - \kappa)^{-1}, \]
\[ h_1(u) e_{12}(u) = -c(u) e_{23}(u - \kappa) h_3(u - \kappa)^{-1}, \quad (3.5) \]
\[ f_{21}(u) h_1(u) = c(u) h_3(u - \kappa)^{-1} f_{32}(u - \kappa), \]

and

\[ h_2(u) + f_{21}(u) h_1(u) e_{12}(u) = c(u) \left( h_2(u - \kappa)^{-1} + e_{23}(u - \kappa) h_3(u - \kappa)^{-1} f_{32}(u - \kappa) \right). \quad (3.6) \]

Calculating as in [2] and [12], we verify that the coefficients of the series \( h_1(u), h_2(u) \) and \( h_3(u) \) pairwise commute. Furthermore, we get

\[ h_1(u) e_{12}(u) = e_{12}(u + 1) h_1(u) \quad \text{and} \quad h_1(u) f_{21}(u + 1) = f_{21}(u) h_1(u) \]

which together with relations (3.5) imply the first two desired identities, where we replaced \( \kappa \) by its value \(-3/2\). They imply that relation (3.6) can be written in the form

\[ h_2(u) - c(u) h_2(u - \kappa)^{-1} = - \begin{bmatrix} e_{12}(u + 1), f_{21}(u) \end{bmatrix} h_1(u). \quad (3.7) \]
As a final step, use one more relation between the Gaussian generators,
\[
[e_{12}(u), f_{21}(v)] = \frac{h_1(u)^{-1}h_2(u) - h_1(v)^{-1}h_2(v)}{u - v},
\]
so that eliminating \( c(u) \) from (3.7) we come to (3.3). Relation (3.4) follows by eliminating \( h_3(u) \) from the first relation in (3.5) with the use of (3.3).

Observe that the coefficients of the series \( e_{12}(u) \) and \( f_{21}(u) \) are stable under all automorphisms (2.5) and so belong to the subalgebra \( \mathcal{Y}(osp_{1|2}) \) of \( X(osp_{1|2}) \). Together with the coefficients of the series \( h(u) = h_1(u)^{-1}h_2(u) \) they generate the Yangian \( \mathcal{Y}(osp_{1|2}) \), and the defining relations for these generators are given in [2] in a slightly different setting.

4 Yangian representations

4.1 Highest weight representations

A representation \( V \) of the algebra \( X(osp_{1|2}) \) is called a highest weight representation if there exists a nonzero vector \( \xi \in V \) such that \( V \) is generated by \( \xi \),
\[
\begin{align*}
t_{ij}(u) \xi &= 0 \quad \text{for} \quad 1 \leq i < j \leq 3, \\
t_{ii}(u) \xi &= \lambda_i(u) \xi \quad \text{for} \quad i = 1, 2, 3,
\end{align*}
\]
for some formal series
\[
\lambda_i(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]].
\]
The vector \( \xi \) is called the highest vector of \( V \) and the triple \( \lambda(u) = (\lambda_1(u), \lambda_2(u), \lambda_3(u)) \) is the highest weight of \( V \).

The quasideterminant formulas for the Gaussian generators \( h_i(u) \) given in Sec. 3 imply that the conditions (4.1) in the above definition can be replaced with
\[
h_i(u) \xi = \lambda_i(u) \xi \quad \text{for} \quad i = 1, 2, 3.
\]
Hence, Proposition 3.1 implies the consistency condition (1.1) for the components \( \lambda_i(u) \) of the highest weight.

Given any triple \( \lambda(u) = (\lambda_1(u), \lambda_2(u), \lambda_3(u)) \) of formal series of the form (4.2) satisfying the consistency condition (1.1), define the Verma module \( M(\lambda(u)) \) as the quotient of \( X(osp_{1|2}) \) by the left ideal generated by all coefficients of the series \( t_{ij}(u) \) with \( 1 \leq i < j \leq 3 \), and \( t_{ii}(u) - \lambda_i(u) \) for \( i = 1, 2, 3 \). The Poincaré–Birkhoff–Witt theorem implies that the Verma module \( M(\lambda(u)) \) is nonzero and we denote by \( L(\lambda(u)) \) its irreducible quotient.

Remark 4.1. In terms of the Drinfeld presentation of the Yangian \( Y(osp_{1|2}) \) given in [2], the highest vector conditions take the form
\[
e_{12}(u) \xi = 0 \quad \text{and} \quad h(u) \xi = \mu(u) \xi
\]
for a series \( \mu(u) \in 1 + u^{-1} \mathbb{C}[[u^{-1}]] \). The Main Theorem implies that the irreducible highest weight representation of \( \mathcal{Y}(\mathfrak{osp}_{1|2}) \) associated with \( \mu(u) \) is finite-dimensional if and only if

\[
\mu(u) = \frac{P(u + 1)}{P(u)}
\]

for some monic polynomial \( P(u) \) in \( u \). \qed

**Proposition 4.2.** Every finite-dimensional irreducible representation of the algebra \( \mathcal{X}(\mathfrak{osp}_{1|2}) \) is isomorphic to \( L(\lambda(u)) \) for a certain highest weight \( \lambda(u) \) satisfying (1.1).

**Proof.** The argument is a straightforward adaptation of the proof of [4, Thm 5.1] to the super case, which amounts to taking care of additional signs in the calculations. \qed

To prove the Main Theorem, we only need to determine which of the modules \( L(\lambda(u)) \) are finite-dimensional. The first step is to establish some necessary conditions.

**Proposition 4.3.** If the module \( L(\lambda(u)) \) is finite-dimensional, then

\[
\frac{\lambda_1(u)}{\lambda_2(u)} = \frac{(u + \alpha_1) \ldots (u + \alpha_k)}{(u + \beta_1) \ldots (u + \beta_k)}
\]

for \( k \in \mathbb{Z}_+ \) and certain complex numbers \( \alpha_i, \beta_i \).

**Proof.** We follow the proof of a similar property for the Yangian \( \mathcal{Y}(\mathfrak{gl}_2) \); see [14, Prop. 3.3.1]. Note that due to (1.1), the series \( \lambda_3(u) \) is uniquely determined by \( \lambda_1(u) \) and \( \lambda_2(u) \), and so we can parameterize the highest weights by arbitrary pairs of series \( \lambda(u) = (\lambda_1(u), \lambda_2(u)) \), omitting \( \lambda_3(u) \). By twisting the action of the extended Yangian \( \mathcal{X}(\mathfrak{osp}_{1|2}) \) on \( L(\lambda(u)) \) by the automorphism (2.5) with \( f(u) = \lambda_2(u)^{-1} \), we get a \( \mathcal{X}(\mathfrak{osp}_{1|2}) \)-module isomorphic to \( L(\mu(u), 1) \) with \( \mu(u) = \lambda_1(u)/\lambda_2(u) \). Let \( \xi \) denote the highest vector of \( L(\mu(u), 1) \). Since this representation is finite-dimensional, the vectors \( t_{21}^{(i)} \xi \in L(\mu(u), 1) \) with \( i \geq 1 \) are linearly dependent,

\[
\sum_{i=1}^{m} c_i t_{21}^{(i)} \xi = 0
\]

with \( c_i \in \mathbb{C} \), assuming \( c_m \neq 0 \). Apply the operators \( t_{12}^{(r)} \) for all \( r \geq 1 \) to the linear combination on the left hand side and take the coefficient of \( \xi \). Since \( t_{12}(u) \xi = 0 \), we get from the defining relations (2.7) that

\[
t_{12}(u) t_{21}(v) \xi = \frac{1}{u - v} \left( t_{22}(u) t_{11}(v) - t_{22}(v) t_{11}(u) \right) \xi = -\frac{\mu(u) - \mu(v)}{u - v} \xi.
\]

Hence, writing

\[
\mu(u) = 1 + \mu^{(1)} u^{-1} + \mu^{(2)} u^{-2} + \ldots, \quad \mu^{(i)} \in \mathbb{C},
\]

we derive

\[
t_{12}^{(r)} t_{21}^{(i)} \xi = \mu^{(r+i-1)} \xi.
\]
Therefore, for all $r \geq 1$ we have the relations

$$
\sum_{i=1}^{m} c_i \mu^{(r+i-1)} = 0.
$$

They imply

$$
\mu(u)(c_1 + c_2 u + \cdots + c_m u^{m-1}) = (b_1 + b_2 u + \cdots + b_m u^{m-1})
$$

so that $\mu(u)$ can be written as a rational function in $u$, as required.

We will use the name elementary module for the module $L(\lambda(u))$ with

$$
\lambda_1(u) = \frac{u + \alpha}{u + \beta} \quad \text{and} \quad \lambda_2(u) = 1
$$

(4.3)

and denote it by $L(\alpha, \beta)$.

The Hopf algebra structure on the extended Yangian $X(\mathfrak{osp}_{1|2})$ allows us to regard tensor products of the form

$$
L = L(\alpha_1, \beta_1) \otimes \cdots \otimes L(\alpha_k, \beta_k)
$$

(4.4)

as $X(\mathfrak{osp}_{1|2})$-modules. Let $\xi^{(i)}$ denote the highest vector of $L(\alpha_i, \beta_i)$.

**Proposition 4.4.** The $X(\mathfrak{osp}_{1|2})$-module $L(\lambda(u))$ with

$$
\lambda_1(u) = \frac{(u + \alpha_1) \cdots (u + \alpha_k)}{(u + \beta_1) \cdots (u + \beta_k)} \quad \text{and} \quad \lambda_2(u) = 1
$$

(4.5)

is isomorphic to the irreducible quotient of the submodule of $L$, generated by the tensor product of the highest vectors $\xi^{(1)} \otimes \cdots \otimes \xi^{(k)}$.

**Proof.** The coproduct formula (2.11) implies that the cyclic span $X(\mathfrak{osp}_{1|2})(\xi^{(1)} \otimes \cdots \otimes \xi^{(k)})$ is a highest weight module with the highest weight $(\lambda_1(u), \lambda_2(u))$ which implies the claim.

We will need to find the conditions for the elementary modules to be finite-dimensional and establish some sufficient conditions for the module $L$ in (4.4) to be irreducible.

### 4.2 Small Verma modules

Note that by twisting the action of the extended Yangian on a highest weight module with the highest weight (4.3) by the shift automorphism (2.8) with $a = -\beta$, we get the corresponding module whose highest weight is found by shifting $\alpha \mapsto \alpha - \beta$ and $\beta \mapsto 0$. We will now assume that $\beta = 0$. Let $\alpha \in \mathbb{C}$ and consider the Verma module $M(\lambda(u))$ with

$$
\lambda_1(u) = \frac{u + \alpha}{u}, \quad \lambda_2(u) = 1, \quad \lambda_3(u) = \frac{u - 1/2}{u + \alpha - 1/2}
$$

(4.6)
Let $K$ be the submodule of $M(\lambda(u))$ generated by all vectors of the form

$$t_{21}^{(r)} \xi \quad \text{for} \quad r \geq 2 \quad \text{and} \quad \left(t_{31}^{(r)} + (\alpha - 1/2) t_{31}^{(r-1)}\right) \xi \quad \text{for} \quad r \geq 3,$$

where $\xi$ denotes the highest vector of the Verma module. Introduce the small Verma module $M(\alpha)$ as the quotient $M(\lambda(u))/K$. We will keep the notation $\xi$ for the image of the highest vector of the Verma module in the quotient. More general small Verma modules of the form $M(\alpha, \beta)$ corresponding to the highest weights (4.3) are then obtained by twisting the modules $M(\alpha)$ by suitable automorphisms (2.8).

**Proposition 4.5.** The module $M(\alpha)$ is spanned by the vectors

$$t_{31}^{(2)} t_{21}^{(1)} s \xi, \quad r, s \in \mathbb{Z}_+.$$  

**Proof.** By the Poincaré–Birkhoff–Witt theorem for the extended Yangian, the Verma module $M(\lambda(u))$ has the basis

$$t_{31}^{(k_1)} \cdots t_{31}^{(k_p)} t_{21}^{(l_1)} \cdots t_{21}^{(l_q)} \xi,$$

where $k_1 \geq \cdots \geq k_p \geq 1$ and $l_1 > \cdots > l_q \geq 1$. Hence, the induction on the length of the monomial in (4.9) reduces the argument to the verification of the property that the span of the vectors (4.8) is stable under the action of the generators $t_{31}^{(k)}$ and $t_{21}^{(l)}$.

The defining relations (2.7) imply that $[t_{31}^{(k)}, t_{31}^{(m)}] = 0$ and $[t_{31}^{(k)}, t_{21}^{(l)}] = 0$ for all $k, m$. Therefore, for $k \geq 2$ in $M(\lambda(u))$ we have

$$t_{31}^{(k)} t_{31}^{(2)} t_{21}^{(1)} s \xi \equiv \left(-\alpha + 1/2\right)^{k-2} t_{31}^{(2)} t_{21}^{(1)} s \xi \mod K.$$

The property is also clear for $k = 1$ because $t_{31}^{(1)} = 2 t_{21}^{(1)}$. Furthermore, since

$$[t_{21}^{(l)}, t_{31}^{(2)}] = t_{21}^{(l)} t_{31}^{(2)} - t_{31}^{(1)} t_{21}^{(l)},$$

and $[t_{21}^{(l)}, t_{21}^{(1)}] = t_{21}^{(l)}$, the property for the generators $t_{21}^{(l)}$ easily follows too.

We will regard $M(\alpha)$ as an $\mathfrak{osp}_{1|2}$-module via the embedding (2.10). We get the weight space decomposition

$$M(\alpha) = \bigoplus_{k=0}^{\infty} M(\alpha)_{-\alpha-k},$$

where we define the weight subspaces of an arbitrary $\mathfrak{osp}_{1|2}$-module $V$ by

$$V_\gamma = \{ v \in V \mid F_{11} v = \gamma v \}.$$  

Proposition 4.5 implies that

$$\dim M(\alpha)_{-\alpha-k} \leq \lfloor k/2 \rfloor + 1.$$  

For all values $i, j \in \{1, 2, 3\}$ set $T_{ij}(u) = u(u + \alpha - 1/2) t_{ij}(u)$. We will regard the coefficients of these Laurent series in $u$ as operators in $M(\alpha)$. 

11
**Proposition 4.6.** All operators $T_{ij}(u)$ on the small Verma module $M(\alpha)$ are polynomials in $u$.

**Proof.** Calculating modulo $K$, we get

$$t_{21}(u)\xi = u^{-1}t_{21}^{(1)}\xi \quad \text{and} \quad t_{31}(u)\xi = \left(u^{-1}t_{31}^{(1)} + \frac{1}{u(u+\alpha-1/2)}t_{31}^{(2)}\right)\xi$$

so that the claim holds for the action of the operators $T_{21}(u)$ and $T_{31}(u)$ on $\xi$. By acting on the vectors (4.8) of the spanning set, we note that the operator $T_{31}(u)$ commutes with $t_{31}^{(2)}$ and $t_{21}^{(1)}$, while for the operator $T_{21}(u)$ we have the relations

$$[T_{21}(u), t_{31}^{(2)}] = t_{21}^{(1)}T_{21}(u) - t_{31}^{(1)}T_{31}(u) \quad \text{and} \quad [T_{21}(u), t_{21}^{(1)}] = T_{31}(u).$$

Hence the property for the operators $T_{21}(u)$ and $T_{31}(u)$ follows by an obvious induction.

As a next step, consider the relations for the series $T_{11}(u)$ implied by (2.7):

$$[T_{11}(u), t_{21}^{(1)}] = T_{21}(u), \quad [T_{11}(u), t_{31}^{(1)}] = 2T_{31}(u)$$

and

$$[T_{11}(u), t_{31}^{(2)}] = T_{31}(u)\left(2u + 1/2 + t_{11}^{(1)}\right) - 2t_{31}^{(1)}T_{11}(u) - t_{21}^{(1)}T_{21}(u).$$

Together with the relation

$$T_{11}(u)\xi = (u + \alpha - 1/2)(u + \alpha)\xi \quad (4.12)$$

they imply the claim for the operator $T_{11}(u)$. For the remaining operators the property follows from the relations

$$[t_{12}^{(1)}, T_{21}(u)] = T_{11}(u) - T_{22}(u), \quad [t_{21}^{(1)}, T_{22}(u)] = T_{32}(u) - T_{21}(u)$$

and

$$[t_{12}^{(1)}, T_{11}(u)] = T_{12}(u), \quad [t_{21}^{(1)}, T_{32}(u)] = T_{33}(u) - T_{22}(u), \quad [t_{23}^{(1)}, T_{33}(u)] = -T_{23}(u),$$

which are consequences of (2.7). \qed

For any $r, s \in \mathbb{Z}_+$ introduce vectors of the small Verma module $M(\alpha)$ by setting

$$\xi_{rs} = T_{21}(-\alpha - r + 3/2) \cdots T_{21}(-\alpha - 1/2)T_{21}(-\alpha + 1/2) \times T_{21}(-\alpha - s + 1) \cdots T_{21}(-\alpha - 1)T_{21}(-\alpha)\xi.$$

We would like to show that under certain additional conditions the vectors $\xi_{rs}$ form a basis of $M(\alpha)$; see Theorem 4.10 and Corollary 4.11 below. This will require a few lemmas where the action of the operators $T_{ij}(u)$ on these vectors is calculated.

**Lemma 4.7.** In the module $M(\alpha)$ we have

$$T_{11}(u)\xi_{rs} = (u + \alpha + r - 1/2)(u + \alpha + s)\xi_{rs}.$$
Proof. The formula holds for $\xi_0 = \xi$ by (4.12). The defining relations (2.7) give

$$T_{11}(u)T_{21}(v) = \frac{u-v+1}{u-v} T_{21}(v)T_{11}(u) - \frac{1}{u-v} T_{21}(u)T_{11}(v),$$

which implies the desired formula by an obvious induction. \hfill \square

**Lemma 4.8.** In the module $M(\alpha)$ for all $r \leq s + 1$ we have

$$T_{21}(u) \xi_{rs} = \frac{(-1)^{r+1}(s-r+1)(2u+2\alpha + 2r - 1)}{(s+1)(2s - 2r + 1)} \xi_{r,s+1} + \frac{2(u+\alpha + s)}{2s - 2r + 1} \xi_{r+1,s}. $$

**Proof.** By the definition of the vectors $\xi_{rs}$ we have $T_{21}(-\alpha - r + 1/2) \xi_{rs} = \xi_{r+1,s}$. Next we point out the following relation for generators of $\text{X} (\text{osp}_{1|2})$:

$$(u-v-1/2) t_{21}(u) t_{21}(v) + (u-v+1/2) t_{21}(v) t_{21}(u) = t_{31}(v) t_{11}(u) - t_{31}(u) t_{11}(v).$$

It is derived by calculating the commutators $[t_{21}(u), t_{21}(v)]$ and $[t_{11}(u), t_{31}(v)]$ by (2.7) and eliminating the term $t_{11}(u) t_{31}(v)$. By Lemma 4.7 we have $T_{11}(u) \xi_{rs} = 0$ for $u = -\alpha - r + 1/2$ and $u = -\alpha - s$. Hence, we come to the relation

$$(r-s-1) T_{21}(-\alpha - s) T_{21}(-\alpha - r + 1/2) \xi_{rs} = -(r-s) T_{21}(-\alpha - r + 1/2) T_{21}(-\alpha - s) \xi_{rs}. $$

Since $T_{21}(-\alpha - s) \xi_{0,s} = \xi_{0,s+1}$, applying the relation repeatedly, we get the formula

$$T_{21}(-\alpha - s) \xi_{rs} = \frac{(-1)^{r}(s-r+1)}{s+1} \xi_{r,s+1}$$

(4.13)

which is valid for all $r \leq s + 1$. Finally, using the Lagrange interpolation formula

$$T_{21}(u) = \frac{u+\alpha + r - 1/2}{r-s-1/2} T_{21}(-\alpha - s) - \frac{u+\alpha + s}{r-s-1/2} T_{21}(-\alpha - r + 1/2).$$

we get the relation in the lemma. \hfill \square

**Lemma 4.9.** In the module $M(\alpha)$ for all $r \leq s$ we have

$$T_{12}(u) \xi_{rs} = -\frac{r(s-r+1)(2\alpha + 2r - 3)(u+\alpha + s)}{2(2s - 2r + 1)} \xi_{r-1,s}$$

$$+ \frac{(-1)^{r+1}s(2s + 1)(\alpha + s - 1)(2u + 2\alpha + 2r - 1)}{4(2s - 2r + 1)} \xi_{r,s-1}. $$

**Proof.** By Proposition 4.6, the operator $T_{12}(u)$ is a polynomial in $u$ of degree one. As in the proof of Lemma 4.8, it will be sufficient to calculate the action of the operator for two different values $u = -\alpha - r + 1/2$ and $u = -\alpha - s$, and then apply the Lagrange interpolation formula.

Recall from Sec. 3 that the coefficients of the series $h_1(u)$ and $h_2(u)$ pairwise commute. Set $d(u) = h_1(u) h_2(u + 1)$. Using the defining relations (2.7), we can also write this series in the form

$$d(u) = t_{22}(u) t_{11}(u + 1) + t_{12}(u) t_{21}(u + 1).$$
The coefficient of the series \( c(u) \) act by scalar multiplication in the small Verma module. The scalars are found from (3.4) and given by
\[
c(u) \mapsto \frac{(u + 1)(u + \alpha)}{u(u + \alpha + 1)}.
\] (4.14)

On the other hand, by Lemma 4.7, the coefficients of the series \( h_1(u) = t_{11}(u) \) act on each vector \( \xi_{rs} \) as multiplications by scalars depending on \( r \) and \( s \). Hence the same property holds for the coefficients of \( d(u) \) whose action is uniquely determined by the relation
\[
d(u) d(u + 1/2) = c(u) h_1(u + 1/2) h_1(u + 1)
\] implied by (3.4). Therefore, the action is found by
\[
d(u) \mapsto \frac{(u + 1/2)(u + \alpha)}{u(u + \alpha + 1/2)} h_1(u + 1/2).
\]

For the corresponding polynomial operator
\[
D(u) = T_{22}(u) T_{11}(u + 1) + T_{12}(u) T_{21}(u + 1)
\] (4.15)
we then have
\[
D(u) = (u + 1)(u + \alpha - 1/2) T_{11}(u + 1/2).
\] (4.16)

For any \( r, s \in \mathbb{Z}_+ \) we find from (4.15) by applying Lemma 4.7 that
\[
D(-\alpha - r - 1/2) \xi_{rs} = T_{12}(-\alpha - r - 1/2) T_{21}(-\alpha - r - 1/2) \xi_{rs}
= T_{12}(-\alpha - r - 1/2) \xi_{r+1,s}.
\]

Hence using (4.16) and replacing \( r \) by \( r - 1 \) we find
\[
T_{12}(-\alpha - r + 1/2) \xi_{rs} = -\frac{1}{4} r(s - r + 1)(2\alpha + 2r - 3) \xi_{r-1,s},
\]
which holds for \( r \geq 1 \). To extend this formula to the case \( r = 0 \) use Lemma 4.7 and relations
\[
[T_{12}(u) T_{21}(v)] = \frac{1}{u - v} \left( T_{22}(u) T_{11}(v) - T_{22}(v) T_{11}(u) \right)
\] (4.17)
implied by (2.7) to derive by induction on \( s \) that \( T_{12}(-\alpha + 1/2) \xi_{0s} = 0 \).

Similarly, taking \( u = -\alpha - s - 1 \) in (4.15) and (4.16), we get by using (4.13) that
\[
T_{12}(-\alpha - s) \xi_{rs} = \frac{1}{4} (-1)^s (2s + 1)(\alpha + s - 1) \xi_{r,s-1},
\]
which holds for \( r < s \). This formula extends to the case \( r = s \) by applying relation (4.17) and taking into account Lemma 4.7.

**Theorem 4.10.** Suppose that \( -\alpha \notin \mathbb{Z}_+ \) and \( -\alpha + 1/2 \notin \mathbb{Z}_+ \). Then the \( X(\mathfrak{osp}_{1\mid2}) \)-module \( M(\alpha) \) is irreducible. Moreover, the vectors \( \xi_{rs} \) with \( r \leq s \) form a basis of \( M(\alpha) \) and \( \xi_{rs} = 0 \) for \( r > s \).
Proof. We start by showing that all vectors \( \xi_{rs} \) with \( 0 \leq r \leq s \) are nonzero in \( M(\alpha) \). The conditions on \( \alpha \) and Lemma 4.9 imply that it is sufficient to verify that \( \xi \neq 0 \); the vector \( \xi_{rs} \) will then also have to be nonzero, because the application of suitable operators \( T_{12}(v) \) to \( \xi_{rs} \) gives the vector \( \xi \) with a nonzero coefficient.

The relation \( \xi = 0 \) in \( M(\alpha) \) would mean that \( \xi \), as an element of the Verma module \( M(\lambda(u)) \) with the highest weight given in (4.6), belongs to the submodule \( K \). That is, \( \xi \) is a linear combination of vectors of the form

\[ x_r t_{21}^{(r)} \xi \quad \text{for} \quad r \geq 2 \quad \text{and} \quad y_r \left( t_{31}^{(r)} + (\alpha - 1/2) t_{31}^{(r-1)} \right) \xi \quad \text{for} \quad r \geq 3, \]

with \( x_r, y_r \in X(osp_{1|2}) \). The elements \( x_r \) and \( y_r \) must have the respective \( osp_{1|2} \)-weights 1 and 2 as eigenvectors of the operator \( F_{11} \). Write these elements as linear combinations of the vectors of the Poincaré–Birkhoff–Witt basis of \( X(osp_{1|2}) \) by using any ordering on the generators consistent with the increasing \( osp_{1|2} \)-weights. The right-most generators occurring in each basis monomial will have positive \( osp_{1|2} \)-weights. On the other hand, calculating in the Verma module \( M(\lambda(u)) \) we find

\[ t_{12}(u) \left( t_{21}(v) - v^{-1} t_{21}^{(1)} \right) \xi = \frac{1}{u-v} \left( t_{22}(u) t_{11}(v) - t_{22}(v) t_{11}(u) \right) \xi \]

\[ - v^{-1} \left( t_{11}(u) - t_{22}(u) \right) \xi = 0, \]

as the coefficient of \( \xi \) equals

\[ \frac{1}{u-v} \left( \frac{v+\alpha}{v} - \frac{u+\alpha}{u} \right) - \alpha u^{-1} v^{-1} = 0. \]

Now combine the second family of generators of the submodule \( K \) given in (4.7) into the generating series

\[ t_{31}(v) - v^{-1} t_{31}^{(1)} \]

\[ - \frac{1}{v(v+\alpha - 1/2)} t_{31}^{(2)} \]

which can be written as the anti-commutator of \( t_{21}^{(1)} \) with the series

\[ t_{21}(v) - v^{-1} t_{21}^{(1)} \]

\[ - \frac{1}{v(v+\alpha - 1/2)} t_{21}^{(2)} \]

whose coefficients are also generators of \( K \). Working first with one part of the anti-commutator and using the previous calculation we get

\[ t_{12}(u) t_{21}^{(1)} \left( t_{21}(v) - v^{-1} t_{21}^{(1)} \right) \xi = \left( t_{11}(u) - t_{22}(u) \right) \left( t_{21}(v) - v^{-1} t_{21}^{(1)} \right) \xi. \]

By the previous argument, the coefficients of this series vanish under the action of the coefficients of the series \( t_{12}(v) \).

Turning to the second part of the anti-commutator, we find that the expression

\[ t_{12}(u) \left( t_{21}(v) - v^{-1} t_{21}^{(1)} \right) - \frac{1}{v(v+\alpha - 1/2)} t_{21}^{(2)} \]

\[ t_{21}^{(1)} \xi \]

15
equals
\[ -\left(t_{21}(v) - v^{-1}t_{21}^{(1)} - \frac{1}{v(v + \alpha - 1/2)}t_{21}^{(2)}\right) t_{12}(u) t_{21}(\xi) \]  
(4.18)
plus
\[ \frac{1}{u-v} \left(t_{22}(u)t_{11}(v) - t_{22}(v)t_{11}(u)\right) t_{21}(\xi) - v^{-1}\left(t_{11}(u) - t_{22}(u)\right) t_{21}(\xi) \]
\[ - \frac{1}{v(v + \alpha - 1/2)} \left((u + t_{21}^{(1)}t_{11}(u) - t_{22}(u)(u + t_{11}^{(1)})) t_{21}(\xi).\right] \]

The expression (4.18) vanishes under the action of the coefficients of the series \( t_{12}(w) \), so we only need to transform the second expression. We will do this modulo terms of the form \( x_{r}t_{21}(r)\xi \) with \( r \geq 2 \) which were already considered above. Note the commutators
\[ [t_{11}(u), t_{21}^{(1)}] = t_{21}(u), \quad [t_{22}(u), t_{21}^{(1)}] = t_{21}(u) - t_{32}(u). \]

Using the second relation in (3.2) and writing the Gaussian generators in terms of the \( t_{ij}(u) \), we find
\[ t_{21}(u)t_{22}(u + 1/2)\xi = t_{32}(u)t_{11}(u + 1/2)\xi. \]

Since \( t_{21}(u)\xi \equiv u^{-1}t_{21}^{(1)}\xi \), we derive that \( t_{32}(u)\xi \equiv (u + \alpha - 1/2)^{-1}t_{21}^{(1)}\xi \). Therefore, the expression in question is then simplified by using relations
\[ t_{11}(u)t_{21}^{(1)}\xi \equiv u^{-1}t_{21}^{(1)}\xi \quad \text{and} \quad t_{22}(u)t_{21}^{(1)}\xi \equiv \frac{u^2 + (\alpha - 1/2)(u + 1)}{u(u + \alpha - 1/2)}t_{21}^{(1)}\xi \]
and thus verifying that it reduces to zero. This completes the proof that \( \xi \not\equiv 0 \mod K \).

As a next step, observe that since the vectors \( \xi_{rs} \) with \( 0 \leq r \leq s \) are nonzero in \( M(\alpha) \), they are eigenvectors for the operator \( T_{11}(u) \), whose eigenvalues are distinct as polynomials in \( u \). Hence they are linearly independent. The number of those vectors of the \( \mathfrak{osp}_{1|2} \)-weight \(-\alpha - k\) equals \( |k/2| + 1 \), which together with the inequality (4.11) proves that they form a basis of the weight space \( M(\alpha)_{-\alpha-k} \). Thus, all vectors \( \xi_{rs} \) with \( 0 \leq r \leq s \) form a basis of \( M(\alpha) \). Any vector \( \xi_{rs} \) with \( r > s \) cannot be nonzero, because otherwise it would be an eigenvector for the operator \( T_{11}(u) \) whose eigenvalue does not occur among those of the vectors in \( M(\alpha) \).

Finally, we prove the irreducibility of \( M(\alpha) \). As we noted in the beginning of the proof, the application of suitable operators \( T_{12}(v) \) to an arbitrary basis vector \( \xi_{rs} \) yields the highest vector \( \xi \) with a nonzero coefficient. This implies that any nonzero submodule of \( M(\alpha) \) must contain \( \xi \) and so coincide with \( M(\alpha) \).

**Corollary 4.11.** For any \( \alpha \in \mathbb{C} \) the vectors \( \xi_{rs} \) with \( 0 \leq r \leq s \) form a basis of \( M(\alpha) \).

**Proof.** Consider the vector space \( \tilde{M}(\alpha) \) with basis elements \( \tilde{\xi}_{rs} \) labelled by \( r, s \in \mathbb{Z}_{+} \) with \( 0 \leq r \leq s \). Define the action of the generators \( t_{11}^{(r)}, t_{21}^{(r)} \) and \( t_{12}^{(r)} \) of \( \mathfrak{osp}_{1|2} \) in \( \tilde{M}(\alpha) \) by using the formulas of Lemmas 4.7, 4.8 and 4.9, where the vectors \( \xi_{rs} \) with \( r \leq s \) are respectively replaced with \( \tilde{\xi}_{rs} \), while all vectors \( \xi_{rs} \) with \( r > s \) are replaced by 0. Also, let the coefficients
of the series $c(u)$ act in $\tilde{M}(\alpha)$ by scalar multiplication defined by (4.14). By Theorem 4.10, this assignment endows the space $\tilde{M}(\alpha)$ with a $X(\mathfrak{osp}_{1|2})$-module structure for all $-\alpha \not\in \mathbb{Z}_+$ and $-\alpha + 1/2 \not\in \mathbb{Z}_+$. Since the matrix elements of the generators in the basis depend polynomially on $\alpha$, the same formulas define a representation of $X(\mathfrak{osp}_{1|2})$ in $\tilde{M}(\alpha)$ for all values of $\alpha$ by continuity.

The formulas for the action of the generators in the basis $\tilde{\xi}_{rs}$ show that for any $\alpha \in \mathbb{C}$ there is an $X(\mathfrak{osp}_{1|2})$-module epimorphism $\pi : M(\lambda(u)) \to \tilde{M}(\alpha)$ defined by $\xi \mapsto \tilde{\xi}_{00}$, where the highest weight $\lambda(u)$ of the Verma module is given by (4.6). Moreover, the submodule $K$ of $M(\lambda(u))$ is contained in the kernel of $\pi$ which gives rise to an epimorphism $\bar{\pi} : M(\alpha) \to \tilde{M}(\alpha)$ with $\xi_{rs} \mapsto \tilde{\xi}_{rs}$. By taking into account the dimensions of the respective $\mathfrak{osp}_{1|2}$-weight components, we conclude from (4.11) that $\bar{\pi}$ is an isomorphism.

As was pointed out in the proof of Corollary 4.11, for any $\alpha \in \mathbb{C}$ the vectors (4.8) form a basis of $M(\alpha)$, and (4.11) is in fact an equality:

$$\dim M(\alpha)_{-\alpha-k} = \lfloor k/2 \rfloor + 1.$$

### 4.3 Elementary modules

The elementary modules $L(\alpha)$ are defined as the irreducible quotients of $M(\alpha)$. We would like to describe the structure of $L(\alpha)$ for the values of $\alpha$ which do not satisfy the assumptions of Theorem 4.10; that is, $-\alpha \in \mathbb{Z}_+$ or $-\alpha + 1/2 \in \mathbb{Z}_+$.

**Proposition 4.12.** Suppose that $-\alpha = k \in \mathbb{Z}_+$. The linear span $J$ of all basis vectors $\xi_{rs}$ of $M(-k)$ with $s > k$ is an $X(\mathfrak{osp}_{1|2})$-submodule. The module $L(-k)$ is isomorphic to the quotient $M(-k)/J$ and has the basis formed by the vectors $\xi_{rs}$ mod $J$ with $0 \leq r \leq s \leq k$.

**Proof.** The formula of Lemma 4.9 gives

$$T_{12}(u) \xi_{r,k+1} = \frac{1}{2} r(k-r+2)(u+1) \xi_{r-1,k+1}$$

for all $r \leq k+1$. This implies that the subspace $J$ of $M(-k)$ is invariant under the action of $X(\mathfrak{osp}_{1|2})$. Furthermore, the formula of Lemma 4.9 also shows that the quotient $M(-k)/J$ is irreducible and hence isomorphic to $L(-k)$.

**Proposition 4.13.** Suppose that $-\alpha + 1/2 = k \in \mathbb{Z}_+$. The linear span $N$ of all basis vectors $\xi_{rs}$ of $M(-k+1/2)$ with $r > k$ is an $X(\mathfrak{osp}_{1|2})$-submodule. The module $L(-k+1/2)$ is isomorphic to the quotient $M(-k+1/2)/N$ and has the basis formed by the vectors $\xi_{rs}$ mod $N$ with $0 \leq r \leq k$.

**Proof.** The formula of Lemma 4.9 now gives

$$T_{12}(u) \xi_{k+1,s} = \frac{(-1)^k}{4} s(2s+1)(u+1) \xi_{k+1,s-1}$$

for all $k+1 \leq s$. Recalling that $\xi_{rs} = 0$ for $r > s$ we conclude that the subspace $N$ of $M(-k+1/2)$ is invariant under the action of $X(\mathfrak{osp}_{1|2})$. Furthermore, Lemma 4.9 implies that the quotient $M(-k+1/2)/N$ is irreducible and hence isomorphic to $L(-k+1/2)$.
Corollary 4.14. We have the following criteria.

1. The $X(\mathfrak{osp}_{1|2})$-module $M(\alpha)$ is irreducible if and only if $-\alpha \notin \mathbb{Z}_+$ and $-\alpha + 1/2 \notin \mathbb{Z}_+$.
2. The $X(\mathfrak{osp}_{1|2})$-module $L(\alpha)$ is finite-dimensional if and only if $-\alpha = k \in \mathbb{Z}_+$. Moreover,
   \[ \dim L(-k) = \binom{k + 2}{2}. \]

Proof. All parts are immediate from Theorem 4.10 and Propositions 4.12 and 4.13.

As the above description of the elementary modules shows, they admit bases formed by $\mathfrak{osp}_{1|2}$-weight vectors. Accordingly, we can define their characters by using formal exponents of a variable $q$ and using the definition (4.10) of $\mathfrak{osp}_{1|2}$-weight subspaces. Namely, we set

\[ \text{ch } V = \sum_{\gamma} \dim V_{-\gamma} q^{\gamma}. \]

In particular, the character of the irreducible highest weight module $V(\mu)$ over $\mathfrak{osp}_{1|2}$ is found by

\[ \text{ch } V(\mu) = \frac{q^{-\mu}}{1-q}, \]

if $\mu \notin \mathbb{Z}_+$, and by

\[ \text{ch } V(\mu) = \frac{q^{-\mu} - q^{\mu+1}}{1-q}, \]

if $\mu \in \mathbb{Z}_+$.

Corollary 4.15. 1. The character of $M(\alpha)$ is given by

\[ \text{ch } M(\alpha) = \frac{q^\alpha}{(1-q)(1-q^2)}. \]

2. For $-\alpha = k \in \mathbb{Z}_+$ we have

\[ \text{ch } L(-k) = q^{-k} \frac{(1-q^{k+1})(1-q^{k+2})}{(1-q)(1-q^2)}. \]

3. For $-\alpha + 1/2 = k \in \mathbb{Z}_+$ we have

\[ \text{ch } L(-k + 1/2) = q^{-k+1/2} \frac{1 - q^{2k+2}}{(1-q)(1-q^2)}. \]

Proof. The formulas follow by evaluating the dimensions of the weight subspaces.
In terms of the characters of the \(\mathfrak{osp}_{1\mid 2}\)-modules, we can write the above formulas as

\[
\text{ch } L(-k) = \sum_{p=0}^{\lfloor k/2 \rfloor} \text{ch } V(k - 2p)
\]

and

\[
\text{ch } L(-k + 1/2) = \sum_{p=0}^{k} \text{ch } V(k - 1/2 - 2p).
\]

This implies the following.

**Corollary 4.16.** The restriction of the module \(L(\alpha)\) to the Lie superalgebra \(\mathfrak{osp}_{1\mid 2}\) is irreducible if and only if \(\alpha = 0, -1\) or \(1/2\). \(\square\)

Corollary 4.16 shows that the \(\mathfrak{osp}_{1\mid 2}\)-modules \(V(0), V(1)\) and \(V(-1/2)\) can be extended to \(X(\mathfrak{osp}_{1\mid 2})\). The Yangian action on the three-dimensional vector representation \(V(1) = \mathbb{C}^{1\mid 2}\) arises from the replacement of \(T(u)\) in the \(RTT\)-relation (2.3) by a transposed \(R\)-matrix \(R(u)\) which satisfies the Yang–Baxter equation; cf. [2]. It is given explicitly by setting

\[
t_{ij}(u) \mapsto \delta_{ij} + u^{-1}e_{ij}(-1)^i - (u + \kappa)^{-1}e_{j'i'}(-1)^{ij}\theta_i\theta_j
\]

and is isomorphic to \(L(-1)\).

### 4.4 Tensor product modules

We will now use the results of the previous sections to complete the proof of the Main Theorem.

Recall that the elementary modules of the form \(L(\alpha, \beta)\) and small Verma modules \(M(\alpha, \beta)\) are associated with the highest weights of the form (1.3). They can be obtained by twisting the respective modules \(L(\alpha)\) and \(M(\alpha)\) with the shift automorphisms (2.8). Corollary 4.14(2) implies that the module \(L(\alpha, \beta)\) is finite-dimensional if and only if \(\beta - \alpha \in \mathbb{Z}_+\).

For the highest weight of the form (4.5), the existence of a monic polynomial \(P(u)\) satisfying (1.2) is equivalent to the condition that the parameters \(\beta_1, \ldots, \beta_k\) can be renumbered in such a way that all differences \(\beta_i - \alpha \in \mathbb{Z}_+\). If this condition holds, then the tensor product module (4.4) is finite-dimensional and so is its irreducible subquotient \(L(\lambda(u))\). This thus proves that the conditions of the Main Theorem are sufficient for the irreducible highest weight module to be finite-dimensional. In the rest of this section, we will show that the conditions are also necessary.

By the results of Sec. 4.2, each small Verma module \(M(\alpha, \beta)\) has the basis \(\xi_{rs}\) parameterized by \(r, s \in \mathbb{Z}_+\) with \(r \leq s\) and the generators of the extended Yangian \(X(\mathfrak{osp}_{1\mid 2})\) act by the rules implied by Lemmas 4.7, 4.8 and 4.9. Namely, for all \(i, j \in \{1, 2, 3\}\) we now introduce the operators \(T_{ij}(u) = (u + \alpha - 1/2)(u + \beta) t_{ij}(u)\), and the formulas take the following form, where the vectors \(\xi_{rs}\) with \(r > s\) are equal to zero:

\[
T_{11}(u) \xi_{rs} = (u + \alpha + r - 1/2)(u + \alpha + s) \xi_{rs}
\]
implies that all operators
\[ T_{21}(u) \xi_{rs} = \frac{(-1)^{r+1}(s-r+1)(2u+2\alpha+2r-1)}{(s+1)(2s-2r+1)} \xi_{r,s+1} + \frac{2(u+\alpha+s)}{2s-2r+1} \xi_{r+1,s} \]
and
\[ T_{12}(u) \xi_{rs} = -\frac{r(s-r+1)(2\alpha-2\beta+2r-3)(u+\alpha+s)}{2(2s-2r+1)} \xi_{r-1,s} + \frac{(-1)^{r+1}s(2s+1)(\alpha-\beta+s-1)(2u+2\alpha+2r-1)}{4(2s-2r+1)} \xi_{r,s-1}. \]
The coefficients of the series \( c(u) \) act on \( M(\alpha, \beta) \) by scalar multiplication, with the scalars found from (3.4) and given by
\[ c(u) \mapsto \frac{(u+\alpha)(u+\beta+1)}{(u+\alpha+1)(u+\beta)}. \]

By Corollary 4.14, the \( \mathfrak{osp}_{1|2} \)-module \( M(\alpha, \beta) \) is irreducible if and only if \( \beta - \alpha \notin \mathbb{Z}_+ \) and \( \beta - \alpha + 1/2 \notin \mathbb{Z}_+ \). In the cases where \( M(\alpha, \beta) \) is reducible, the above formulas for the action of \( T_{ij}(u) \) extend to the irreducible quotients \( L(\alpha, \beta) \) with the assumption that the vectors \( \xi_{rs} \) belonging to the maximal nontrivial submodule of \( M(\alpha, \beta) \) are understood as equal to zero.

Our argument will rely on certain sufficient conditions for the tensor product of the form (4.4) to be irreducible as an \( \mathfrak{osp}_{1|2} \)-module. To state the conditions we will use a notation involving multisets of complex numbers \( \{z_1, \ldots, z_l\} \). For such a multiset we will write \( \{z_1, \ldots, z_l\}_+ \) to denote the multiset formed by all elements \( z_i \) which belong to \( \mathbb{Z}_+ \).

**Theorem 4.17.** Suppose that for each \( h = 1, \ldots, k-1 \) the following holds:

1. If the multiset \( \{\beta_h - \alpha_i, \beta_i - \alpha_h \mid i = h, \ldots, k\}_+ \) is not empty, then \( \beta_h - \alpha_h \) is a minimal element of the multiset \( \{\beta_h - \alpha_i, \beta_i - \alpha_h, \beta_h - \alpha_i + 1/2, \beta_i - \alpha_h + 1/2 \mid i = h, \ldots, k\}_+ \).

2. If the multiset \( \{\beta_h - \alpha_i, \beta_i - \alpha_h \mid i = h, \ldots, k\}_+ \) is empty and the multiset \( \{\beta_h - \alpha_i + 1/2, \beta_i - \alpha_h + 1/2 \mid i = h, \ldots, k\}_+ \) is not empty, then \( \beta_h - \alpha_h + 1/2 \) is a minimal element of this multiset.

Then the \( \mathfrak{osp}_{1|2} \)-module \( L \) defined in (4.4) is irreducible.

**Proof.** We let \( \xi^{(l)}_{ij} \) denote the basis vectors of the module \( L(\alpha_l, \beta_l) \) with the highest vector \( \xi^{(l)} \). Proposition 4.6 implies that all operators
\[ T_{ij}(u) = \prod_{l=1}^{k} (u + \alpha_l - 1/2)(u + \beta_l) t_{ij}(u) \]
acting in the module \( L \) are polynomials in \( u \).

As a first step, we will show by induction on \( k \) that any vector \( \zeta \in L \) satisfying the condition \( T_{12}(u) \zeta = 0 \) is proportional to \( \xi^{(1)} \otimes \ldots \otimes \xi^{(k)} \). The case \( k = 1 \) is clear so we suppose that \( k \geq 2 \). We may assume that such a vector \( \zeta \) is an \( \mathfrak{osp}_{1|2} \)-weight vector and write
\[ \zeta = \sum_{r,s} \xi^{(1)}_{rs} \otimes \zeta_{rs}, \quad \zeta_{rs} \in L(\alpha_2, \beta_2) \otimes \ldots \otimes L(\alpha_k, \beta_k). \]
The sum is finite and taken over the pairs \( r \leq s \) with the condition that the \( \xi_{rs}^{(1)} \) are basis vectors of \( L(\alpha_1, \beta_1) \). Let \( p \) be the maximal sum \( r + s \) for which there are nonzero elements \( \zeta_{rs} \) in the expression. By taking the coefficient of \( \xi_{rs} \) with \( r + s = p \) in the relation \( T_{12}(u) \zeta = 0 \), we get \( T_{12}(u) \zeta_{rs} = 0 \). By the induction hypothesis, \( \zeta_{rs} \) is proportional to the vector \( \xi' = \xi^{(2)} \otimes \cdots \otimes \xi^{(k)} \). Furthermore, the defining relations (2.7) give

\[
T_{12}(u)T_{11}(v) = \frac{u - v}{u - v} T_{11}(v)T_{12}(u) + \frac{1}{u - v} T_{11}(u)T_{12}(v).
\]

Hence, for any value of \( v \), the vector \( T_{11}(v) \zeta \) is also annihilated by the operator \( T_{12}(u) \). Note that the basis vectors \( \xi_{rs}^{(1)} \) are eigenvectors for the operator \( T_{11}(v) \) with distinct eigenvalues as polynomials in \( v \). This implies that by taking a common eigenvector for the commuting family of operators \( T_{11}(v) \), we may conclude that there is an \( \mathfrak{osp}_{1|2} \)-weight vector \( \zeta \) of the form

\[
\zeta = \xi_{r_0,s_0}^{(1)} \otimes \xi' + \sum_{r+s<p} \xi_{rs}^{(1)} \otimes \zeta_{rs}, \tag{4.20}
\]

with \( r_0 + s_0 = p \) such that \( T_{12}(u) \zeta = 0 \).

Next we will show that the condition \( r_0 < s_0 \) is impossible in such a vector. Indeed, if this condition holds, consider the coefficient of the vector \( \xi_{r_0,s_0-1}^{(1)} \otimes \xi' \) in the relation \( T_{12}(u) \zeta = 0 \). This coefficient can only arise from the terms

\[
T_{12}(u) \xi_{r_0,s_0}^{(1)} \otimes T_{22}(u) \xi' \pm T_{11}(u) \xi_{r_0,s_0-1}^{(1)} \otimes T_{12}(u) \xi_{r_0,s_0-1}
\]

with the sign depending on the parity of the vector \( \xi_{r_0,s_0-1}^{(1)} \). The \( \mathfrak{osp}_{1|2} \)-weight condition implies that

\[
\zeta_{r_0,s_0-1} = \sum_{l=2}^{k} c_l \xi^{(2)} \otimes \cdots \otimes \xi_{01}^{(l)} \otimes \cdots \otimes \xi^{(k)}
\]

for some constants \( c_l \in \mathbb{C} \). We have

\[
T_{12}(u) \zeta_{r_0,s_0-1} = \sum_{l=2}^{k} \pm c_l T_{11}(u) \xi^{(2)} \otimes \cdots \otimes T_{12}(u) \xi_{01}^{(l)} \otimes \cdots \otimes T_{22}(u) \xi^{(k)}.
\]

By using the formulas for the action of the operators \( T_{ij}(u) \) and equating the coefficient in question to zero, we get

\[
b(u + \alpha_1 + r_0 - 1/2) \prod_{i=2}^{k} (u + \alpha_i - 1/2)(u + \beta_i)
\]

\[
+ (u + \alpha_1 + r_0 - 1/2)(u + \alpha_1 + s_0 - 1) \sum_{l=2}^{k} b_l \prod_{i=2}^{l-1} (u + \alpha_i - 1/2)(u + \alpha_i)
\]

\[
\times (u + \alpha_l - 1/2) \prod_{i=l+1}^{k} (u + \alpha_i - 1/2)(u + \beta_i) = 0,
\]

21
where \( b_l \) are some constants, while \( b \) is a nonzero constant. By cancelling the common factors and setting \( u = -\alpha_1 - s_0 + 1 \) we get

\[
\prod_{i=2}^{k}(\beta_i - \alpha_1 - s_0 + 1) = 0.
\]

It follows from this relation that the multiset \( \{\beta_i - \alpha_1 \mid i = 1, \ldots, k\} \) is not empty, because \( \beta_i - \alpha_1 = s_0 - 1 \in \mathbb{Z}_+ \) for some \( i \in \{2, \ldots, k\} \). By assumption (1) of the theorem, we have \( \beta_1 - \alpha_1 \in \mathbb{Z}_+ \) and \( \beta_1 - \alpha_1 \leq \beta_i - \alpha_1 \). However, this makes a contradiction, as by Proposition 4.12 we must have \( s_0 \leq \beta_1 - \alpha_1 \).

Excluding the condition \( r_0 < s_0 \) in (4.20), we show next that the condition \( r_0 = s_0 \geq 1 \) is impossible either. If this condition holds, consider the coefficient of the vector \( \xi_{r_0-1,r_0}^{(1)} \otimes \xi' \) in the relation \( T_{12}(u) \xi = 0 \). This coefficient can only arise from the terms

\[
T_{12}(u) \xi_{r_0-1,r_0}^{(1)} \otimes T_{22}(u) \xi' \pm T_{11}(u) \xi_{r_0-1,r_0}^{(1)} \otimes T_{12}(u) \xi_{r_0-1,r_0}.
\]

By the \( \text{osp}_{1|2} \)-weight condition,

\[
\xi_{r_0-1,r_0} = \sum_{l=2}^{k} c_l \xi^{(2)} \otimes \ldots \otimes \xi_{01}^{(l)} \otimes \ldots \otimes \xi^{(k)}
\]

for some constants \( c_l \in \mathbb{C} \). Calculating as in the previous case, we now come to the relation

\[
b(u + \alpha_1 + r_0) \prod_{i=2}^{k}(u + \alpha_i - 1/2)(u + \beta_i)
\]

\[
+ (u + \alpha_1 + r_0 - 3/2)(u + \alpha_1 + r_0) \sum_{l=2}^{k} b_l \prod_{i=2}^{l-1}(u + \alpha_i - 1/2)(u + \alpha_i)
\]

\[
\times (u + \alpha_l - 1/2) \prod_{i=l+1}^{k} (u + \alpha_i - 1/2)(u + \beta_i) = 0,
\]

where \( b_l \) are some constants, while \( b \) is a nonzero constant. Cancel the common factors and set \( u = -\alpha_1 - r_0 + 3/2 \) to get

\[
\prod_{i=2}^{k}(\beta_i - \alpha_1 - r_0 + 3/2) = 0.
\]

This means that for some \( i \in \{2, \ldots, k\} \) we have \( \beta_i - \alpha_1 + 1/2 = r_0 - 1 \in \mathbb{Z}_+ \). If the multiset \( \{\beta_j - \alpha_1 \mid j = 1, \ldots, k\} \) is not empty, then by assumption (1) of the theorem, we have \( \beta_1 - \alpha_1 \in \mathbb{Z}_+ \) and \( \beta_1 - \alpha_1 \leq \beta_j - \alpha_1 + 1/2 \). This is impossible because by Proposition 4.12 we must have \( r_0 \leq \beta_1 - \alpha_1 \). Hence assumption (2) of the theorem for \( h = 1 \) should apply, and we have \( \beta_1 - \alpha_1 + 1/2 \in \mathbb{Z}_+ \) together with the inequality

\[
\beta_1 - \alpha_1 + 1/2 \leq \beta_i - \alpha_1 + 1/2.
\]

This makes a contradiction, as by Proposition 4.13 we must have \( r_0 \leq \beta_1 - \alpha_1 + 1/2 \).
We have thus showed that any vector \( \zeta \in L \) with \( T_{12}(u) \zeta = 0 \) is proportional to \( \xi(1) \otimes \xi' \). By looking at the set of \( \mathfrak{osp}_{1|2} \)-weights of any nonzero submodule of \( L \) we derive that such a submodule must contain a nonzero vector \( \zeta \) with \( T_{12}(u) \zeta = 0 \), and so contain the vector \( \xi(1) \otimes \xi' \). It remains to prove this vector is cyclic in \( L \).

Consider the vector space \( L^* \) dual to \( L \) which is spanned by all linear maps \( \sigma : L \rightarrow \mathbb{C} \) satisfying the condition that the linear span of the vectors \( \eta \in L \) such that \( \sigma(\eta) \neq 0 \), is finite-dimensional. Equip \( L^* \) with a \( X(\mathfrak{osp}_{1|2}) \)-module structure by setting
\[
(x \sigma)(\eta) = \sigma(\omega(x) \eta) \quad \text{for} \quad x \in X(\mathfrak{osp}_{1|2}) \quad \text{and} \quad \sigma \in L^*, \ \eta \in L,
\]
where \( \omega \) is the anti-automorphism of the algebra \( X(\mathfrak{osp}_{1|2}) \) defined in (2.9). It is easy to verify that \( L^* \) is isomorphic to the tensor product module
\[
L(-\beta_1, -\alpha_1) \otimes \cdots \otimes L(-\beta_k, -\alpha_k).
\]
Moreover, the highest vector of the module \( L(-\beta_1, -\alpha_1) \) can be identified with the dual basis vector \( \xi^{(i)*} \). Suppose now that the submodule \( K = X(\mathfrak{osp}_{1|2}) \otimes \xi^{(1)} \otimes \cdots \otimes \xi^{(k)} \) of \( L \) is proper and consider its annihilator
\[
\text{Ann} \ K = \{ \rho \in L^* \mid \rho(\eta) = 0 \quad \text{for all} \quad \eta \in K \}.
\]
Then \( \text{Ann} \ K \) is a nonzero submodule of \( L^* \), which does not contain the vector \( \xi^{(1)*} \otimes \cdots \otimes \xi^{(k)*} \). However, this contradicts the claim verified in the first part of the proof, because the conditions on the parameters \( \alpha_i \) and \( \beta_i \) stated in the theorem will remain satisfied after we replace each \( \alpha_i \) by \( -\beta_i \) and each \( \beta_i \) by \( -\alpha_i \).

**Proposition 4.18.** Suppose that the \( X(\mathfrak{osp}_{1|2}) \)-module \( L(\lambda(u)) \) with the highest weight (4.5) is finite-dimensional. Then for any nonnegative integers \( l_1, \ldots, l_k \) and \( m_1, \ldots, m_k \) the module \( L(\lambda^+(u)) \) with the highest weight
\[
\lambda_1^+(u) = \frac{(u + \alpha_1 - l_1) \cdots (u + \alpha_k - l_k)}{(u + \beta_1 + m_1) \cdots (u + \beta_k + m_k)} \quad \text{and} \quad \lambda_2^+(u) = 1 \quad (4.21)
\]
is also finite-dimensional.

**Proof.** The highest weight module \( L(\lambda^+(u)) \) is isomorphic to an irreducible subquotient of the finite-dimensional module
\[
L(\lambda(u)) \otimes L(\alpha_1 - l_1, \alpha_1) \otimes \cdots \otimes L(\alpha_k - l_k, \alpha_k) \otimes L(\beta_1 + m_1) \otimes \cdots \otimes L(\beta_k, \beta_k + m_k)
\]
and hence is finite-dimensional. \( \square \)

We now return to proving the Main Theorem. Let the irreducible highest weight module \( L(\lambda(u)) \) with the highest weight (4.5) be finite-dimensional. To argue by contradiction, suppose that it is impossible to renumber the parameters \( \beta_1, \ldots, \beta_k \) in such a way that all differences \( \beta_i - \alpha_i \) with \( i = 1, \ldots, k \) belong to \( \mathbb{Z}_+ \). By Proposition 4.18, all modules \( L(\lambda^+(u)) \) with the
highest weight of the form \((4.21)\) are also finite-dimensional. Choose nonnegative integers \(l_i\) and \(m_i\) to ensure that the assumptions of Theorem 4.17 are satisfied by the shifted parameters \(\alpha_i - l_i\) and \(\beta_i + m_i\). This is done by induction, beginning with the multiset
\[
\{\beta_1 - \alpha_i, \beta_i - \alpha_1 \mid i = 1, \ldots, k\}
\]
and renumbering the parameters \(\alpha_i\) and \(\beta_i\), if necessary, to ensure that \(\beta_1 - \alpha_1\) is a minimal element of the multiset
\[
\{\beta_1 - \alpha_i, \beta_i - \alpha_1 \mid i = 1, \ldots, k\}_+ \tag{4.22}
\]
if it is nonempty. Then assumption (1) of the theorem for \(h = 1\) is achieved by suitable shifts \(\alpha_i \mapsto \alpha_i - l_i\) and \(\beta_i \mapsto \beta_i + m_i\) for \(i = 2, \ldots, k\). If the multiset (4.22) is empty, then assumption (2) for \(h = 1\) is achieved by a suitable renumbering the parameters \(\alpha_i\) and \(\beta_i\). Then we continue in the same way to consider the multisets for \(h = 2\), etc.

By Theorem 4.17, the module \(L(\lambda^+(u))\) is isomorphic to the tensor product of the corresponding elementary modules and is therefore infinite-dimensional, because the parameters of at least one of the elementary modules do not satisfy the conditions of Corollary 4.14(2). This contradiction completes the proof of the first part of the Main Theorem. The second part concerning representations of the Yangian \(Y(\mathfrak{osp}_{1|2})\) is immediate from the decomposition (2.6); cf. [4, Sec. 5.3].

Comparing the irreducibility conditions with those for the evaluation modules over the Yangian \(Y(\mathfrak{gl}_2)\) (see e.g. [14, Sec. 3.3]), note that unlike that case, it is not possible, in general, to renumber the parameters of the given highest weight (4.5) to satisfy the assumptions of Theorem 4.17. In fact, not every module \(L(\lambda(u))\) is isomorphic to a tensor product module of the form (4.4), as illustrated by the following example.

**Example 4.19.** The \(X(\mathfrak{osp}_{1|2})\)-module \(L(\lambda^+(u))\) with
\[
\lambda_1(u) = \frac{(u - 1)(u - 5/2)}{u(u - 3/2)}, \quad \lambda_2(u) = 1,
\]
is isomorphic to a subquotient of the tensor product \(L = L(-1, 0) \otimes L(-5/2, -3/2)\) of two three-dimensional modules. Note that the respective parameters do not satisfy the assumptions of Theorem 4.17. The module \(L\) turns out to have a proper submodule \(K\) which is generated by the vector
\[
\zeta = \xi_{11}^{(1)} \otimes \xi^{(2)} + 3 \xi_{01}^{(1)} \otimes \xi_{01}^{(2)} - \xi_{11}^{(1)} \otimes \xi_{11}^{(2)}.
\]

The submodule \(K\) is one-dimensional, isomorphic to a highest weight module \(L(\mu(u))\) with the components
\[
\mu_1(u) = \mu_2(u) = \frac{(u - 1/2)(u - 5/2)}{u(u - 3/2)}.
\]

The module \(L(\lambda(u))\) is isomorphic to the quotient \(L/K\) with \(\dim L(\lambda(u)) = 8\) and does not admit a tensor product decomposition of the form (4.4).
To conclude, we note that by analysing submodules of reducible modules $M(\alpha, \beta)$ we can obtain explicit constructions of some modules $L(\lambda(u))$ beyond the elementary modules. In particular, for any $k \in \mathbb{Z}_+$ the submodule of $M(-k)$ generated by the vector $\xi_{0,k+1}$ is isomorphic to the highest weight module $L(\lambda(u))$ with

$$\lambda_1(u) = \frac{u + 1}{u} \quad \text{and} \quad \lambda_2(u) = \frac{(u + 1/2)(u - k - 1)}{u(u - k - 1/2)}.$$ 

Its basis vectors are $\xi_{r,s}$ subject to the conditions $r \leq s$ and $s > k$ with the action of the generators described in Sec. 4.2. The character of $L(\lambda(u))$, as defined in Sec. 4.3, is found by

$$\text{ch } L(\lambda(u)) = \frac{q + q^2 - q^{k+3}}{(1 - q)(1 - q^2)}.$$

References

[1] D. Arnaudon, J. Avan, N. Crampé, L. Frappat, E. Ragoucy, R-matrix presentation for super-Yangians $Y(\mathfrak{osp}(m|2n))$, J. Math. Phys. 44 (2003), 302–308.

[2] D. Arnaudon, N. Crampé, L. Frappat, E. Ragoucy, Super Yangian $Y(\mathfrak{osp}(1|2))$ and the universal R-matrix of its quantum double, Commun. Math. Phys. 240 (2003), 31–51.

[3] D. Arnaudon, J. Avan, N. Crampé, A. Doikou, L. Frappat, E. Ragoucy, Bethe ansatz equations and exact S matrices for the $\mathfrak{osp}(M|2n)$ open super-spin chain, Nuclear Phys. B 687 (2004), 257–278.

[4] D. Arnaudon, A. Molev and E. Ragoucy, On the R-matrix realization of Yangians and their representations, Annales Henri Poincaré 7 (2006), 1269–1325.

[5] V. Chari and A. Pressley, Yangians and R-matrices, Enseign. Math. 36 (1990), 267–302.

[6] Sh.-J. Cheng and W. Wang, Dualities and representations of Lie superalgebras. Graduate Studies in Mathematics, 144. AMS, Providence, RI, 2012.

[7] V. G. Drinfeld, A new realization of Yangians and quantized affine algebras, Soviet Math. Dokl. 36 (1988), 212–216.

[8] J. Fuksa, A. P. Isaev, D. Karakhanyan, R. Kirschner, Yangians and Yang-Baxter R-operators for ortho-symplectic superalgebras, Nuclear Phys. B 917 (2017), 44–85.

[9] I. M. Gelfand and V. S. Retakh, Determinants of matrices over noncommutative rings, Funct. Anal. Appl. 25 (1991), 91–102.

[10] A. P. Isaev, D. Karakhanyan, R. Kirschner, Yang–Baxter R-operators for osp superalgebras, Nuclear Phys. B 965 (2021), Paper No. 115355, 28 pp.
[11] A. P. Isaev, A. I. Molev and O. V. Ogievetsky, *A new fusion procedure for the Brauer algebra and evaluation homomorphisms*, Int. Math. Res. Not. (2012), 2571–2606.

[12] N. Jing and M. Liu, *Isomorphism between two realizations of the Yangian $Y(\mathfrak{so}_3)$*, J. Phys. A 46 (2013), 075201, 12 pp.

[13] N. Jing, M. Liu and A. Molev, *Isomorphism between the $R$-matrix and Drinfeld presentations of Yangian in types $B$, $C$ and $D$*, Comm. Math. Phys. 361 (2018), 827–872.

[14] A. Molev, *Yangians and classical Lie algebras*, Mathematical Surveys and Monographs, 143. American Mathematical Society, Providence, RI, 2007.

[15] I. M. Musson, *Lie superalgebras and enveloping algebras*. Graduate Studies in Mathematics, 131. AMS, Providence, RI, 2012.

[16] M. L. Nazarov, *Quantum Berezinian and the classical Capelli identity*, Lett. Math. Phys. 21 (1991), 123–131.

[17] V. O. Tarasov, *Irreducible monodromy matrices for the $R$-matrix of the $XXZ$-model and lattice local quantum Hamiltonians*, Theor. Math. Phys. 63 (1985), 440–454.

[18] A. B. Zamolodchikov and Al. B. Zamolodchikov, *Factorized $S$-matrices in two dimensions as the exact solutions of certain relativistic quantum field models*, Ann. Phys. 120 (1979), 253–291.

[19] R. B. Zhang, *The $gl(M|N)$ super Yangian and its finite-dimensional representations*, Lett. Math. Phys. 37 (1996), 419–434.

School of Mathematics and Statistics
University of Sydney, NSW 2006, Australia
alexander.molev@sydney.edu.au