A new integrable system related to the Toda lattice

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Abstract. A new integrable lattice system is introduced, and its integrable discretizations are obtained. A Bäcklund transformation between this new system and the Toda lattice, as well as between their discretizations, is established.
1 Introduction

We want to introduce in this paper a new integrable lattice system:

\[ \dddot{x}_k = \dot{x}_k \left( \exp(x_{k+1} - x_k) - \exp(x_k - x_{k-1}) \right), \]  

(1.1)

along with two integrable discretizations thereof. In the difference equations below \( x_k(x) \) are supposed to be functions of the discrete time \( t \in h\mathbb{Z} \), and \( \dddot{x}_k = x_k(t + h) \), \( \dot{x}_k = x_k(t - h) \). The first of our integrable discretizations is implicit with respect to the updates \( \dddot{x}_k \)’s:

\[ \exp(\dddot{x}_k - x_k) - 1 = \frac{1 + h \exp(x_{k+1} - x_k)}{1 + h \exp(x_k - \dddot{x}_{k-1})}, \]  

(1.2)

and the other is explicit:

\[ \exp(\dddot{x}_k - x_k) - 1 = \frac{1 + h \exp(x_{k+1} - x_k)}{1 + h \exp(x_k - x_{k-1})}, \]  

(1.3)

The system (1.1) resembles much the usual Toda lattice:

\[ \dddot{x}_k = \exp(x_{k+1} - x_k) - \exp(x_k - x_{k-1}), \]  

(1.4)

and in fact turns out to be closely related to it my means of a sort of Bäcklund transformation. To the author’s knowledge, this system has not appeared in the literature, despite its beauty and possible physical applications.

It is by now well known that the Toda lattice admits several apparently different integrable discretizations (which are, in fact, closely connected with each other, but these connections are rather nontrivial). Two of the discretizations are explicit with respect to \( \dddot{x}_k \)’s, namely the Hirota’s one [1]:

\[ \exp(\dddot{x}_k - x_k) - \exp(x_k - \dddot{x}_k) = h^2 \left( \exp(x_{k+1} - x_k) - \exp(x_k - x_{k-1}) \right), \]  

(1.5)

and a standard–like one [2]:

\[ \exp(\dddot{x}_k - 2x_k + \dddot{x}_k) = \frac{1 + h^2 \exp(x_{k+1} - x_k)}{1 + h^2 \exp(x_k - x_{k-1})}, \]  

(1.6)

which can be also presented in the form

\[ \exp(\dddot{x}_k - x_k) - \exp(x_k - \dddot{x}_k) = h^2 \left( \exp(x_{k+1} - x_k) - \exp(\dddot{x}_k - x_{k-1}) \right). \]  

(1.7)
Another two discretizations are implicit with respect to $\tilde{x}_k$'s. They were introduced in \cite{5} and read:

$$\exp(\tilde{x}_k - x_k) - \exp(x_k - \tilde{x}_k) = h^2 \left( \exp(x_{k+1} - x_k) - \exp(x_k - \tilde{x}_{k-1}) \right) \quad (1.8)$$

and

$$\exp(\tilde{x}_k - x_k) - \exp(x_k - \tilde{x}_k) = h^2 \left( \exp(x_{k+1} - x_k) - \exp(\tilde{x}_k - \tilde{x}_{k-1}) \right). \quad (1.9)$$

We discuss their algebraic structure, as well as their relations to each other further on. We shall demonstrate that (1.2) is related to (1.1) just in the same way as (1.8) is related to (1.4), and shall also elaborate an algebraic structure of the discretization (1.3).

All the systems above (continuous and discrete time ones) may be considered either on an infinite lattice ($k \in \mathbb{Z}$), or on a finite one ($1 \leq k \leq N$). In the last case one of the two types of boundary conditions may be imposed: open–end ($x_0 = \infty$, $x_{N+1} = -\infty$) or periodic ($x_0 \equiv x_N$, $x_{N+1} \equiv x_1$). We shall be concerned only with the finite lattices here, consideration of the infinite ones being to a large extent similar.

## 2 Newtonian equations of motion: Lagrangian and Hamiltonian formulations

All the equations introduced in the previous section, both continuous– and discrete–time, are written in the Newtonian form:

$$\ddot{x}_k = \Phi_k(\dot{x}, x) \quad \text{or} \quad \Psi_k(\tilde{x}, x, \dot{x}_k) = 0,$$

respectively. They all turn out to admit a Lagrangian formulation.

Recall that in the continuous time case Lagrangian equations are given by

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}_k} - \frac{\partial \mathcal{L}}{\partial x_k} = 0, \quad (2.1)$$

while their discrete time analog is given by

$$\partial \left( \Lambda(\tilde{x}, x) + \Lambda(x, \dot{x}) \right)/\partial x_k = 0. \quad (2.2)$$

Finally, recall that Lagrangian formulation implies also a possibility of introducing a Hamiltonian one. Namely, in the continuous time case one defines momenta $p_k$ canonically conjugated to the coordinates $x_k$ by

$$p_k = \partial \mathcal{L}/\partial \dot{x}_k. \quad (2.3)$$
Then the flow defined by (2.1), being expressed in terms of \((x,p)\), preserves the standard symplectic form \(\sum dx_k \wedge dp_k\) on the phase space \(\mathbb{R}^{2N}(x,p)\). Moreover, this flow may be written in a canonical form

\[
\dot{x}_k = \partial H/\partial p_k, \quad \dot{p}_k = -\partial H/\partial x_k,
\]

the Hamiltonian function \(H(x,p)\) being given by

\[
H = \sum_{k=1}^{N} \dot{x}_k p_k - \mathcal{L}. \tag{2.5}
\]

Analogously, in the discrete time case the momenta \(p_k\) canonically conjugated to \(x_k\) are given by

\[
p_k = \partial \Lambda(\bar{x},x)/\partial x_k. \tag{2.6}
\]

Then the map \((x,x) \mapsto (\bar{x},x)\) induces a symplectic map \((x,p) \mapsto (\bar{x},\bar{p})\) of the phase space \(\mathbb{R}^{2N}(x,p)\), i.e. a map preserving the standard symplectic form \(\sum dx_k \wedge dp_k\). Note that (2.6) implies that the equations (2.2) may be presented as

\[
p_k = -\partial \Lambda(\bar{x},x)/\partial x_k, \tag{2.7}
\]
\[
\bar{p}_k = \partial \Lambda(\bar{x},x)/\partial \bar{x}_k. \tag{2.8}
\]

3 Simplest flow of the Toda hierarchy and its bi–Hamiltonian structure

The both lattices (1.1) and (1.4) arise from the simplest flow of the Toda hierarchy under two different parametrizations of the relevant variables \((a,b)\) (called Flaschka variables) by the canonically conjugated variables \((x,p)\).

The simplest flow of the Toda hierarchy is:

\[
\dot{a}_k = a_k(b_{k+1} - b_k), \quad \dot{b}_k = a_k - a_{k-1}. \tag{3.1}
\]

It may be considered either under open–end boundary conditions \((a_0 = a_N = 0)\), or under periodic ones (all the subscripts are taken (mod \(N\)), so that \(a_0 \equiv a_N, b_{N+1} \equiv b_1\)).

It is easy to see that the flow (3.1) is Hamiltonian with respect to two different compatible Poisson brackets. The first of them is linear:

\[
\{a_k, b_k\}_1 = -\{a_k, b_{k+1}\}_1 = a_k \tag{3.2}
\]
(only the non–vanishing brackets are written down), and a Hamiltonian function generating the flow (3.1) in this bracket is equal to

\[ H^{(1)} = \frac{1}{2} \sum_{k=1}^{N} b_k^2 + \sum_{k=1}^{N} a_k. \]  

(3.3)

The second Poisson bracket is given by:

\[
\{b_{k+1}, b_k\} = a_k, \quad \{a_{k+1}, a_k\} = a_{k+1}a_k, \quad \{b_k, a_k\} = -b_k a_k, \quad \{b_{k+1}, a_k\} = b_{k+1}a_k,
\]

the corresponding Hamiltonian function being

\[ H^{(2)} = \sum_{k=1}^{N} b_k. \]  

(3.4)

An integrable discretization of the flow (3.1) is given by the difference equations [4], [5]

\[
\tilde{a}_k = a_k \frac{\beta_{k+1}}{\beta_k}, \quad \tilde{b}_k = b_k + h \left( \frac{a_k}{\beta_k} - \frac{a_{k-1}}{\beta_{k-1}} \right),
\]

(3.6)

where \( \beta_k = \beta_k(a, b) \) is defined as a unique set of functions satisfying the recurrent relation

\[ \beta_k = 1 + \frac{h b_k - h^2 a_{k-1}}{\beta_{k-1}} \]  

(3.7)

together with an asymptotic relation

\[ \beta_k = 1 + h b_k + O(h^2). \]  

(3.8)

In the open–end case, due to \( a_0 = 0 \), we obtain from (3.7) the following finite continued fractions expressions for \( \beta_k \):

\[
\beta_1 = 1 + h b_1; \quad \beta_2 = 1 + h b_2 - \frac{h^2 a_1}{1 + h b_1}; \quad \ldots \quad \beta_N = 1 + h b_N - \frac{h^2 a_{N-1}}{1 + h b_{N-1} - \frac{h^2 a_{N-2}}{1 + h b_{N-2} - \ldots - \frac{h^2 a_1}{1 + h b_1}}}. \]
In the periodic case (3.7), (3.8) uniquely define $\beta_k$’s as $N$-periodic infinite continued fractions. It can be proved that for $h$ small enough these continued fractions converge and their values satisfy (3.8).

It can be proved [5] that the map (3.6) is Poisson with respect to the both brackets (3.2) and (3.4), and hence with respect to their arbitrary linear combination.

Let us recall also the Lax representations of the flow (3.1) and of the map (3.6). They are given in terms of the $N \times N$ Lax matrix $T$ depending on the phase space coordinates $a_k, b_k$ and (in the periodic case) on the additional parameter $\lambda$:

$$ T(a,b,\lambda) = \sum_{k=1}^{N} b_k E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} + \lambda^{-1} \sum_{k=1}^{N} a_k E_{k,k+1}. \quad (3.9) $$

Here $E_{jk}$ stands for the matrix whose only nonzero entry on the intersection of the $j$th row and the $k$th column is equal to 1. In the periodic case we have $E_{N+1,N} = E_{1,N}, E_{N,N+1} = E_{N,1}$; in the open–end case we set $\lambda = 1$, and $E_{N+1,N} = E_{N,N+1} = 0$.

The flow (3.1) is equivalent to the following matrix differential equation:

$$ \dot{T} = [T, B], \quad (3.10) $$

where

$$ B(a,b,\lambda) = \sum_{k=1}^{N} b_k E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k}, \quad (3.11) $$

and the map (3.6) is equivalent to the following matrix difference equation:

$$ \tilde{T} = B^{-1} T B, \quad (3.12) $$

where

$$ B(a,b,\lambda) = \sum_{k=1}^{N} \beta_k E_{kk} + h \lambda \sum_{k=1}^{N} E_{k+1,k}. \quad (3.13) $$

The spectral invariants of the matrix $T(a,b,\lambda)$ serve as integrals of motion for the flow (3.1), as well as for the map (3.6).

In particular, it is easy to see that the Hamiltonian functions (3.3), (3.5) are spectral invariants of the Lax matrix:

$$ H^{(1)} = \frac{1}{2} \text{tr}(T^2), \quad H^{(2)} = \text{tr}(T). $$
4 Reminding the Toda lattice case

The Toda lattice (1.4) admits a Lagrangian formulation with a Lagrange function

\[ L^{(1)}(x, \dot{x}) = \frac{1}{2} \sum_{k=1}^{N} \dot{x}_k^2 - \sum_{k=1}^{N} \exp(x_k - x_{k-1}). \] (4.1)

A general procedure implies that the momenta \( p_k \) are given by

\[ p_k = \partial L^{(1)}/\partial \dot{x}_k = \dot{x}_k, \]

so that the corresponding Hamiltonian function is

\[ H^{(1)} = \frac{1}{2} \sum_{k=1}^{N} p_k^2 + \sum_{k=1}^{N} \exp(x_k - x_{k-1}), \] (4.2)

and the flow (3.1) takes the form of canonical equations of motion:

\[ \dot{x}_k = \partial H^{(1)}/\partial p_k = p_k, \]
\[ \dot{p}_k = -\partial H^{(1)}/\partial x_k = \exp(x_{k+1} - x_k) - \exp(x_k - x_{k-1}). \]

One sees immediately that this coincides with the flow (3.1), if the Flaschka variables \((a, b)\) are introduced according to the formulas

\[ a_k = \exp(x_{k+1} - x_k), \quad b_k = p_k. \] (4.3)

Obviously, this leads immediately to the linear Poisson brackets (3.2).

Let us turn now to the discrete time case. Consider first the equations of motion (1.8). It is easy to see that they admit a Lagrangian formulation with the Lagrange function

\[ \Lambda_1(\bar{x}, x) = \sum_{k=1}^{N} \phi_1(\bar{x}_k - x_k) - h \sum_{k=1}^{N} \exp(x_k - \bar{x}_{k-1}), \] (4.4)

where \( \phi_1(\xi) = (\exp(\xi) - 1 - \xi)/h \). Hence they are equivalent to the symplectic map \((x, p) \mapsto (\bar{x}, \bar{p})\) with

\[ h p_k = \exp(\bar{x}_k - x_k) - 1 + h^2 \exp(x_k - \bar{x}_{k-1}), \] (4.5)
\[ h \bar{p}_k = \exp(\bar{x}_k - x_k) - 1 + h^2 \exp(x_{k+1} - \bar{x}_k). \] (4.6)

We demonstrate now that they may be put in the form (3.6).
Proposition 1. If the variables \(a_k, b_k\) are defined by (4.3), and
\[
\beta_k = \exp(\bar{x}_k - x_k),
\] (4.7)
then (4.5), (4.6) imply (3.6), (3.7).

Proof. The first equation of motion in (3.6) follows immediately from the definitions of
\(a_k = \exp(x_{k+1} - x_k)\), \(\beta_k = \exp(\bar{x}_k - x_k)\). The recurrent relation (3.7) is just a reformulation
of (4.5) in the variables \(a_k, b_k, \beta_k\). Finally, the second equation in (3.6) follows immediately
from (4.6) and (4.5).

Note that this proposition implies immediately that the map (3.6) is Poisson with
respect to the linear bracket (3.2).

A very remarkable circumstance was found in [5]: an apparently different discretization
(1.9) is in fact only another parametrization of the same map (3.6). It is easy to see that
(1.9) admits a Lagrangian formulation with a Lagrange function
\[
\Lambda_2(\bar{x}, x) = \sum_{k=1}^{N} \frac{1}{2h}(\bar{x}_k - x_k)^2 - \sum_{k=1}^{N} \phi_2(x_k - \bar{x}_{k-1}),
\] (4.8)
where \(\phi_2(\xi) = h^{-1} \int_0^\xi \log(1 + h^2 \exp(\eta))d\eta\). (It is easy to see that the corresponding
equations (2.2) read:
\[
\bar{x}_k - 2x_k + x_k = \log \left(1 + h^2 \exp(x_{k+1} - x_k)\right) - \log \left(1 + h^2 \exp(x_k - \bar{x}_{k-1})\right),
\]
which is equivalent to (1.9)). Hence an equivalent form of writing (1.9) in canonically
conjugated variables \((x, p)\) is:
\[
\exp(hp_k) = \exp(\bar{x}_k - x_k) \left(1 + h^2 \exp(x_{k+1} - x_k)\right) = \exp(\bar{x}_k - x_k) + h^2 \exp(\bar{x}_k - \bar{x}_{k-1}),
\] (4.9)
\[
\exp(h\bar{p}_k) = \exp(\bar{x}_k - x_k) \left(1 + h^2 \exp(x_{k+1} - \bar{x}_k)\right) = \exp(\bar{x}_k - x_k) + h^2 \exp(x_{k+1} - x_k).
\] (4.10)

Proposition 2. If the variables \(a_k, b_k\) are defined by
\[
a_k = \exp(x_{k+1} - x_k + hp_k), \quad 1 + hb_k = \exp(hp_k) + h^2 \exp(x_k - x_{k-1}),
\] (4.11)
and
\[
\beta_k = \exp(hp_k),
\] (4.12)
then (4.9), (4.10) imply (3.6), (3.7).

Proof. From (1.9), (4.10), and the first equation in (4.11) it follows that the first
equation of motion in (3.6) is satisfied, if
\[
\beta_k = \exp(\bar{x}_k - x_k) \left(1 + h^2 \exp(x_k - \bar{x}_{k-1})\right),
\]
which is just (4.12). Now (3.7) is a reformulation of the second equation in (4.11), if one takes into account that
\[ a_k / \beta_k = \exp(x_{k+1} - x_k). \]
The second equation of motion in (3.6) follows directly from the second equation in (4.11), (4.10), and (4.9).

It is easy to calculate that a Poisson bracket for the variables \( a_k, b_k \) resulting from the parametrization (4.11) reads:

\[
\{ b_{k+1}, b_k \} = h a_{k+1}, \quad \{ a_{k+1}, a_k \} = h a_{k+1} a_k, \quad \{ b_k, a_k \} = -a_k - h b_k a_k,
\]

which is exactly a linear combination \( \{ \cdot, \cdot \}_1 + h\{ \cdot, \cdot \}_2 \).

We conclude this section by noting that the both Lagrange functions (4.4) and (4.8) serve as difference approximations to the continuous time one (4.1).

5 A new lattice

We turn now to the system (1.1). First of all, one sees readily that it admits a Lagrangian formulation with

\[
L^{(2)}(x, \dot{x}) = \sum_{k=1}^{N} [\dot{x}_k \log(\dot{x}_k) - \dot{x}_k] - \sum_{k=1}^{N} \exp(x_k - x_{k-1}).
\]

(5.1)

Hence the momenta \( p_k \) are introduced by

\[ p_k = \partial L^{(2)}/\partial \dot{x}_k = \log(\dot{x}_k), \]

hence the corresponding Hamiltonian function is equal to

\[
H^{(2)} = \sum_{k=1}^{N} \exp(p_k) + \sum_{k=1}^{N} \exp(x_k - x_{k-1}),
\]

(5.2)

and the canonical form of the equations of motion is:

\[
\dot{x}_k = \partial H^{(2)}/\partial p_k = \exp(p_k), \quad \dot{p}_k = -\partial H^{(2)}/\partial x_k = \exp(x_{k+1} - x_k) - \exp(x_k - x_{k-1}).
\]

It is now easy to see that if one introduces variables \( a_k, b_k \) according to

\[
a_k = \exp(x_{k+1} - x_k + p_k), \quad b_k = \exp(p_k) + \exp(x_k - x_{k-1}),
\]

(5.3)

then their evolution induced by the flow above just coincides with (3.1).
It can be readily checked that (5.3) leads to Poisson brackets (3.4), and that the notation $H^{(2)}$ for the function (5.2) is consistent with (3.5).

So, the equations (1.1) admit a Lax representation (3.10) with the matrices (3.9), (3.11), for the entries of which one has the formulas (5.3), which is equivalent also to

$$a_k = \dot{x}_k \exp(x_{k+1} - x_k), \quad b_k = \dot{x}_k + \exp(x_k - x_{k-1}).$$

Turning to the discrete time system (3.6), we find the following results. It admits a Lagrangian formulation with

$$\Lambda_3(\tilde{x}, x) = \sum_{k=1}^{N} \phi(\tilde{x}_k - x_k) - \sum_{k=1}^{N} \psi(x_k - \tilde{x}_{k-1}),$$

(5.4)

where the two functions $\phi(\xi), \psi(\xi)$ are defined by

$$\phi(\xi) = \int_{0}^{\xi} \log \left| \frac{\exp(\eta) - 1}{h} \right| d\eta, \quad \psi(\xi) = \int_{0}^{\xi} \log(1 + h \exp(\eta)) d\eta.$$ (5.5)

Hence a symplectic map $(x, p) \mapsto (\tilde{x}, \tilde{p})$ generated by (1.2) may be defined by the following relations:

$$h \exp(p_k) = \left( \exp(\tilde{x}_k - x_k) - 1 \right) \left( 1 + h \exp(x_k - \tilde{x}_{k-1}) \right),$$ (5.6)

$$h \exp(\tilde{p}_k) = \left( \exp(\tilde{x}_k - x_k) - 1 \right) \left( 1 + h \exp(x_{k+1} - \tilde{x}_k) \right).$$ (5.7)

It is very remarkable that this map can be again reduced to (3.6)!

**Proposition 3.** If the variables $a_k, b_k$ are defined by (5.3), and

$$\beta_k = \exp(\tilde{x}_k - x_k) \left( 1 + h \exp(x_k - \tilde{x}_{k-1}) \right),$$ (5.8)

then (5.6), (5.7) imply (3.6), (3.7).

**Proof.** The first equation of motion in (3.6) follows immediately from $a_k = \exp(x_{k+1} - x_k + p_k)$ and (5.6), (5.7), (5.8). The recurrent relation (3.7) follows from (5.8), (5.6), if one takes into account that

$$h^2 a_k / \beta_k = h \left( \exp(x_{k+1} - x_k) - \exp(x_{k+1} - \tilde{x}_k) \right),$$ (5.9)

and hence

$$1 + h \exp(x_k - x_{k-1}) - \frac{h^2 a_k}{\beta_k} = 1 + h \exp(x_k - \tilde{x}_{k-1}).$$

The second equation of motion follows from (5.3), (5.6), and (5.7) with the help of (5.9).
6 Discretizations related to relativistic Toda lattice

We now turn to the discretizations (1.5), (1.7). A simple observation shows that these models are equivalent to (1.8), (1.9), respectively, when considered as equations on the lattice with the coordinates \((t, k)\). More precisely, the equations of motion (1.8), (1.9) are recovered from (1.5), (1.7) after renaming \(x_k(t)\) to \(x_k(t - kh)\). However, such renaming mixes the ”spatial” and ”temporal” variables, and this changes the properties of the initial value problem, which we are concerned with, dramatically.

First of all, from a practical point of view we must remark that the Hirota’s and the standard–like models are explicit with respect to \(\tilde{x}_k\), while the models (1.8), (1.9) require to solve certain nonlinear algebraic equations (or, equivalently, to evaluate continued fractions) in order to obtain the \(\tilde{x}_k\).

Another important difference between our new models and the old ones lies in their algebraic, \(r\)–matrix structure. According to the observation in [4], the Hirota’s and the standard–like models are in essence equivalent. More precisely, they both arise from the following system of difference equations:

\[
\tilde{d}_k + h^2 \tilde{c}_{k-1} = d_k + h^2 c_k, \quad \tilde{d}_{k+1} c_k = d_k \tilde{c}_k, \quad \quad (6.1)
\]

if the variables \((c, d)\) are parametrized by canonically conjugated variables \((x, p)\) in two different ways. An equivalent form of equations (6.1) may be obtained, if one resolves for \((\tilde{c}_k, d_k)\):

\[
\tilde{d}_k = d_{k-1} \frac{d_k + h^2 c_k}{d_{k-1} + h^2 c_{k-1}}, \quad \tilde{c}_k = c_k \frac{d_{k+1} + h^2 c_{k+1}}{d_k + h^2 c_k}. \quad (6.2)
\]

The map defined by these difference equations is Poisson with respect to two different compatible Poisson brackets: a linear one,

\[
\{c_k, d_{k+1}\}_1 = -c_k, \quad \{c_k, d_k\}_1 = c_k, \quad \{d_k, d_{k+1}\}_1 = h^2 c_k, \quad (6.3)
\]

and a quadratic one,

\[
\{c_k, c_{k+1}\}_2 = -c_k c_{k+1}, \quad \{c_k, d_{k+1}\}_2 = -c_k d_{k+1}, \quad \{c_k, d_k\}_2 = c_k d_k. \quad (6.4)
\]

The Lax representation for the map (6.1) may be given in terms of the \(N \times N\) matrices depending on the dynamical variables \((c, d)\) and an additional parameter \(\lambda\):

\[
L(c, d, \lambda) = \sum_{k=1}^{N} d_k E_{kk} + h \lambda \sum_{k=1}^{N} E_{k+1,k}, \quad (6.5)
\]

\[
U(c, d, \lambda) = \sum_{k=1}^{N} E_{kk} - h \lambda^{-1} \sum_{k=1}^{N} c_k E_{k,k+1}. \quad (6.6)
\]
It is easy to check that the difference equations \((6.1)\) are equivalent to the matrix equation
\[
U L = L U, \quad \text{or} \quad L U^{-1} = U L^{-1}. \tag{6.7}
\]

In terms of the Lax matrix
\[
T(c, d, \lambda) = L(c, d, \lambda) U^{-1}(c, d, \lambda), \tag{6.8}
\]
the equation \((6.7)\) takes the form
\[
\tilde{T} = U^{-1} T U = L^{-1} T L, \tag{6.9}
\]
which implies, in particular, that the spectral invariants of the matrix \(T\) are integrals of motion for the map \((6.1)\).

As observed in \([2], [3]\), the matrix \(T\) from \((6.8)\) just coincides with the Lax matrix of the relativistic Toda hierarchy (which is also bi–Hamiltonian with respect to both brackets \((6.3), (6.4)\)).

We first recall how can the equations \((1.5), (1.7)\) be reduced to \((6.1)\), and then show that the same is true for \((1.3)\).

We start with \((1.5)\). It is easy to find a Lagrangian formulation of these equations with a Lagrange function
\[
\Lambda_4(\bar{x}, x) = \sum_{k=1}^{N} \phi_1(\bar{x}_k - x_k) - h \sum_{k=1}^{N} \exp(\bar{x}_k - \bar{x}_{k-1}), \tag{6.10}
\]
(where, as in the previous section, \(\phi_1(\xi) = (\exp(\xi) - 1 - \xi)/h\)). Hence the equations \((1.3)\) are equivalent to a symplectic map \((x, p) \mapsto (\bar{x}, \bar{p})\) with
\[
h p_k &= \exp(\bar{x}_k - x_k) - 1, \tag{6.11}
\]
\[
h \bar{p}_k &= \exp(\bar{x}_k - x_k) - 1 + h^2 \exp(\bar{x}_{k+1} - \bar{x}_k) - h^2 \exp(\bar{x}_k - \bar{x}_{k-1}). \tag{6.12}
\]

**Proposition 4.** Let the coordinates \((c, d)\) be parametrized by the canonically conjugated variables \((x, p)\) according to the formulas
\[
c_k = \exp(x_{k+1} - x_k), \quad d_k = 1 + hp_k - h^2 \exp(x_{k+1} - x_k). \tag{6.13}
\]

Then \((6.11), (6.12)\) imply \((6.4)\).

**Proof.** Obviously, we have from \((6.11), (6.12)\), and \((6.13)\):
\[
d_k + h^2 c_k = \exp(\bar{x}_k - x_k), \quad \bar{d}_k + h^2 \bar{c}_{k-1} = \exp(\bar{x}_k - x_k).
\]
Comparing these expressions, we get the first equation of motion in \((6.1)\), and the first of the expressions above together with \(c_k = \exp(x_{k+1} - x_k)\) implies the second equation in \((6.2)\). \(\blacksquare\)
It is important to notice that the parametrization (6.13) results in the linear Poisson bracket (6.3), which proves independently that the map (6.2) is Poisson with respect to this bracket.

Turning now to (1.7), we find a Lagrangian formulation of these equations (in the form (1.6)) with

\[ \Lambda_5(\tilde{x}, x) = \sum_{k=1}^{N} \frac{1}{2\hbar} (\tilde{x}_k - x_k)^2 - \sum_{k=1}^{N} \phi_2(x_k - x_{k-1}), \]

where, as in the previous section, \( \phi_2(\xi) = h^{-1} \int_{0}^{\xi} \log(1 + h^2 \exp(\eta)) d\eta \). Hence the expression for the momenta \( p_k \) and their updates, equivalent to (1.7), are:

\[ \exp(h\tilde{p}_k) = \exp(\tilde{x}_k - x_k), \]

(6.14)

\[ \exp(hp_k) = \exp(\tilde{x}_k - x_k) \left( 1 + \frac{h^2 \exp(x_{k+1} - x_k)}{1 + h^2 \exp(x_k - x_{k-1})} \right), \]

(6.15)

(6.16)

**Proposition 5.** Let the coordinates \((c, d)\) be parametrized by the canonically conjugated variables \((x, p)\) according to the formulas

\[ c_k = \exp(x_{k+1} - x_k + hp_k), \quad d_k = \exp(hp_k). \]

(6.17)

Then (6.13), (4.6) imply (6.1).

**Proof.** It is easy to see that (6.17) allows to rewrite the second equation in (6.1) as \( \exp(x_{k+1} - x_k + h\tilde{p}_k) = \exp(\tilde{x}_{k+1} - \tilde{x}_k + h\tilde{p}_k) \). This equality is an obvious consequence of (6.16). Further, (6.17) allows to rewrite the first equation in (6.2) as

\[ \exp(\tilde{p}_k) = \exp(p_k) \frac{1 + h^2 \exp(x_{k+1} - x_k)}{1 + h^2 \exp(x_k - x_{k-1})}, \]

(6.19)

which follows immediately from (6.15), (6.16).

This time we notice that (6.17) results (up to the factor \( h \)) in the quadratic Poisson bracket (6.4), which proves independently that the map (6.2) is Poisson with respect to this bracket.

It remains to perform analogous considerations for an explicit discretization (1.3) of our new lattice (1.1). Remarkably, this system turns out to be still another realization of the same map (6.2)! To demonstrate this, note that (1.3) admits a Lagrangian formulation with

\[ \Lambda_6(\tilde{x}, x) = \sum_{k=1}^{N} \phi(\tilde{x}_k - x_k) - \sum_{k=1}^{N} \psi(x_k - x_{k-1}), \]

(6.18)

where \( \phi(\xi), \psi(\xi) \) are defined by (5.5). Hence a Hamiltonian formulation of this system is given by:

\[ h \exp(p_k) = \left( \exp(\tilde{x}_k - x_k) - 1 \right) \frac{1 + h \exp(x_k - x_{k-1})}{1 + h \exp(x_{k+1} - x_k)}, \]

(6.19)
\[ h \exp(\bar{p}_k) = \left( \exp(\bar{x}_k - x_k) - 1 \right), \quad (6.20) \]

**Proposition 6.** Let the coordinates \((c, d)\) be parametrized by the canonically conjugated variables \((x, p)\) according to the formulas

\[ c_k = \exp(x_{k+1} - x_k + p_k), \quad d_k = 1 + h \exp(p_k) + h \exp(x_k - x_{k-1}). \quad (6.21) \]

Then \((6.19), (6.20)\) imply \((6.1)\).

**Proof.** From \((6.21)\) and \((6.19)\) it follows:

\[ d_k + h^2 c_k = 1 + h \exp(x_k - x_{k-1}) + h \exp(p_k)(1 + h \exp(x_{k+1} - x_k)) = \exp(\bar{x}_k - x_k)(1 + h \exp(x_k - x_{k-1})). \quad (6.22) \]

Analogously, from \((6.21)\) and \((6.20)\) it follows:

\[ \ddot{d}_k + h^2 \bar{c}_{k-1} = 1 + h \exp(\bar{p}_k) + h \exp(\bar{x}_k - \bar{x}_{k-1})(1 + h \exp(\bar{p}_{k-1})) = \exp(\bar{x}_k - x_k)(1 + h \exp(x_k - x_{k-1})). \quad (6.23) \]

Comparing these expressions, we get the first equation of motion in \((6.1)\). The second equation in \((6.2)\) is a direct consequence of \((6.19), (6.20), (6.22)\). \(\square\)

It is easy to calculate that the parametrization \((6.21)\) generates the following Poisson bracket:

\[ \{c_{k+1}, c_k\} = c_{k+1} c_k, \quad \{d_{k+1}, d_k\} = h^2 c_k, \quad \{d_k, c_k\} = c_k - d_k c_k, \quad \{d_{k+1}, c_k\} = -c_k + d_{k+1} c_k. \quad (6.24) \]

This is, obviously, a linear combination of the brackets \((6.3)\) and \((6.4)\), namely \(\{\cdot, \cdot\}_2 - \{\cdot, \cdot\}_1\). Of course, the Poisson property of the map \((6.2)\) with respect to this bracket follows from the previous results, but the Proposition 6 gives an alternative way to prove this.

### 7 Conclusion

Identifying the variables \((a, b)\) in \((4.3)\) and in \((5.3)\), we get a transformation between two sets of variables \((x, p)\) (and, consequently, between two sets of variables \((\dot{x}, x)\)). This is exactly the Bäcklund transformation between the lattice \((1.1)\) and the Toda lattice \((1.4)\).

For each of these systems one has different integrable discretizations. Some of them share the Lax matrix with the continuous time prototype. These discretizations generate Newtonian equations implicit with respect to the updates \(\bar{x}_k\). Other discretizations have Lax representations with the Lax matrix defining the *relativistic* Toda hierarchy. These
discretizations turn out to be explicit. All three apparently different implicit discretizations turn out to be connected by Bäcklund transformations. An underlying fact is that all three appear from one and the same integrable map, if the relevant variables \((a,b)\) are parametrized by canonically conjugated ones \((x,p)\) in three different ways, generating three different Poisson brackets on the set of \((a,b)\) (and hence on the set of Lax matrices). Exactly the same holds true for the three explicit discretizations.

We would like to note here that all the Poisson brackets on the sets of Lax matrices were given an \(r\)-matrix interpretation in [3].

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