Connectivity Keeping Trees in 2-Connected Graphs with Girth Conditions

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Abstract

Mader conjectured in 2010 that for any tree $T$ of order $m$, every $k$-connected graph $G$ with minimum degree at least $\left\lfloor \frac{3k}{2} \right\rfloor + m - 1$ contains a subtree $T' \cong T$ such that $G - V(T')$ is $k$-connected. This conjecture has been proved for $k = 1$; however, it remains open for general $k \geq 2$; for $k = 2$, partially affirmative answers have been shown, all of which restrict the class of trees to special subclasses such as trees with at most 5 internal vertices, trees of order at most 8, trees with diameter at most 4, caterpillars, and spiders. We first extend the previously known subclass of trees for which Mader’s conjecture for $k = 2$ holds; namely, we show that Mader’s conjecture for $k = 2$ is true for the class of bifurcate quasi-unimodal caterpillars which includes every caterpillar and every tree of order $m$ with diameter at least $m - 4$. Instead of restricting the class of trees, we next consider 2-connected graphs with girth conditions. We then show that Mader’s conjecture is true for every 2-connected graph $G$ with $g(G) \geq \delta(G) - 8$, where $g(G)$ and $\delta(G)$ denote the girth of $G$ and the minimum degree of a vertex in $G$, respectively. Besides, we show that for every 2-connected graph $G$ with $g(G) \geq \delta(G) - 7$, the lower bound of $m + 2$ on $\delta(G)$ in Mader’s conjecture can be improved to $m + 1$ if $m \geq 10$. Moreover, the lower bound of $\delta(G) - 8$ (respectively, $\delta(G) - 7$) on $g(G)$ in these results can be improved to $\delta(G) - 9$ (respectively, $\delta(G) - 8$ with $m \geq 11$) if no six (respectively, four) cycles of length $g(G)$ have a common path of length $\left\lfloor \frac{g(G)}{2} \right\rfloor - 1$ in $G$. We also show that Mader’s conjecture holds for every 2-connected graph $G$ with $g^\circ(G) \geq \delta(G) - 8$, where $g^\circ(G)$ is the overlapping girth of $G$. Mader’s conjecture is interesting not only from a theoretical point of view but also from a practical point of view, since it may be applied to fault-tolerant problems in

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communication networks. Our proofs lead to $O(|V(G)|^4)$ time algorithms for finding a desired subtree in a given 2-connected graph $G$ satisfying the assumptions.

Keywords 2-connected graphs · Connectivity · Girth · Trees

1 Introduction

Throughout this paper, a graph $G = (V, E)$ means a simple undirected graph unless stated otherwise. For adjacent vertices $u$ and $v$, the edge joining them is denoted by $uv$ or $vu$. The minimum degree of a vertex in $G$ is denoted by $\delta(G)$. For a proper subset $S \subseteq V(G)$, we denote by $G - S$ the graph obtained from $G$ by deleting every vertex in $S$, where $G - \{v\}$ is abbreviated to $G - v$. For two sets $A$ and $B$, we denote by $A \setminus B$ the set difference \{ $x$ | $x \in A$, $x \notin B$ \}. For a nonempty subset $S \subseteq V(G)$, the subgraph of $G$ induced by $S$ is denoted by $\langle S \rangle_G$, i.e., $\langle S \rangle_G = G - V(G) \setminus S$. When a graph $G_1$ is isomorphic to a graph $G_2$, we write $G_1 \cong G_2$. For any $S \subseteq V(G)$ with $|S| < k < |V(G)|$, if $G - S$ is connected, then $G$ is $k$-connected.

In 1972, Chartrand, Kaugars, and Lick proved the following result on the existence of a vertex whose removal does not influence $k$-connectedness of a graph.

**Theorem 1** (Chartrand et al. [1]) Every $k$-connected graph $G$ with $\delta(G) \geq \lfloor \frac{3k}{2} \rfloor$ contains a vertex $v$ such that $G - v$ is $k$-connected.

After more than 30 years, Fujita and Kawarabayashi considered a similar problem for an edge of a graph and showed the following.

**Theorem 2** (Fujita and Kawarabayashi [3]) Every $k$-connected graph $G$ with $\delta(G) \geq \lfloor \frac{3k}{2} \rfloor + 2$ contains an edge $uv$ such that $G - \{u, v\}$ is $k$-connected.

In the same paper, they conjectured the next statement.

**Conjecture 1** There is a function $f(m)$ such that every $k$-connected graph $G$ with $\delta(G) \geq \lfloor \frac{3k}{2} \rfloor + f(m)$ contains a connected subgraph $W$ of order $m$ such that $G - V(W)$ is $k$-connected.

Note that the condition that $W$ is connected is essential, since by iteratively applying Theorem 1, we can see that every $k$-connected graph $G$ with $\delta(G) \geq \lfloor \frac{3k}{2} \rfloor + m - 1$ contains a subgraph $X$ of order $m$ such that $G - V(X)$ is $k$-connected. In 2010, Mader [8] settled Conjecture 1 by showing the following result. Mader’s result in fact improves the lower bound on $\delta(G)$ in Theorem 2 and generalizes Theorem 1.

**Theorem 3** (Mader [8]) Every $k$-connected graph $G$ with $\delta(G) \geq \lfloor \frac{3k}{2} \rfloor + m - 1$ contains a path $P$ of order $m$ such that $G - V(P)$ is $k$-connected.

Based on this result, Mader conjectured the following, i.e., a path in Theorem 3 can be generalized to any tree of the same order.

**Conjecture 2** (Mader [8]) For any tree $T$ of order $m$, every $k$-connected graph $G$ with $\delta(G) \geq \lfloor \frac{3k}{2} \rfloor + m - 1$ contains a subtree $T' \cong T$ such that $G - V(T')$ is $k$-connected.

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Mader’s conjecture is a generalization not only for Theorem 3 but also for the next well-known result on the existence of a subtree isomorphic to any given tree.

**Proposition 1**  For any tree $T$ of order $m$, every graph $G$ with $\delta(G) \geq m - 1$ contains a subtree $T' \cong T$.

Note that in Proposition 1, for any $v \in V(G)$ and any $w \in V(T)$, $G$ has a subtree $T' \cong T$ with an isomorphism $\phi$ from $V(T)$ to $V(T')$ such that $\phi^{-1}(v) = w$. Thus, in particular, $T'$ can be chosen to be a path ending on any vertex of $G$.

Apart from Mader’s conjecture, Locke’s conjecture concerning nonseparating trees in connected graphs is known. A $k$-cohesive graph is a non-trivial connected graph in which for any two distinct vertices $u$ and $v$, the sum of the degrees of $u$, $v$ and the distance between $u$ and $v$ is at least $k$.

**Conjecture 3** (Locke [6]) For any tree $T$ of order $m \geq 3$, every $2m$-cohesive graph $G$ has a subtree $T' \cong T$ such that $G - V(T')$ is connected.

Motivated by Locke’s conjecture, Diwan and Tholiya proved a theorem which is weaker than the conjecture, but it is the same as Mader’s conjecture for $k = 1$ (Mader in fact mentioned their result in the paper [8]). Note that if $G$ is connected and $\delta(G) \geq m$, then $G$ is $2m$-cohesive.

**Theorem 4** (Diwan and Tholiya [2]) For any tree $T$ of order $m$, every connected graph $G$ with $\delta(G) \geq m$ contains a subtree $T' \cong T$ such that $G - V(T')$ is connected.

For general $k \geq 2$, Mader’s conjecture remains open; however for $k = 2$, partially affirmative answers have been shown. Tian et al. [9] proved that Mader’s conjecture for $k = 2$ is true when $T$ is a star or a double-star, and they [10] extended their results to a path-star or a path-double-star. Hasunuma and Ono [4] showed that for any tree $T$ of order $m$, every 2-connected graph $G$ with $\delta(G) \geq \max\{m + n(T) - 3, m + 2\}$ contains a subtree $T' \cong T$ such that $G - V(T')$ is 2-connected, where $n(T)$ is the number of internal vertices of $T$. As a corollary, it follows that Mader’s conjecture for $k = 2$ holds for any tree $T$ with $n(T) \leq 5$ and for any tree of order $m \leq 8$. Lu and Zhang [7] also proved that Mader’s conjecture for $k = 2$ is true for any tree with diameter at most 4. Hong et al. [5] recently showed that Mader’s conjecture for $k = 2$ holds for any caterpillars or any spiders. Note that every known result which is a partially affirmative answer to Mader’s conjecture for $k = 2$ restricts the class of trees to special subclasses.

In this paper, we first extend the previously known subclass of trees for which Mader’s conjecture for $k = 2$ holds; namely, we show that Mader’s conjecture for $k = 2$ is true for the class of bifurcate quasi-unimodal caterpillars which includes every caterpillar and every tree of order $m$ with diameter at least $m - 4$. Next, we employ another approach to Mader’s conjecture for $k = 2$; that is, we add girth conditions to 2-connected graphs. The girth of a 2-connected graph $G$ denoted by $g(G)$ is the length of a smallest cycle in $G$. We then show that Mader’s conjecture is true for every 2-connected graph $G$ with girth at least $\delta(G) - 8$. Note that for any given integers $r \geq 2$ and $g \geq 3$, there exists an $r$-regular graph with girth $g$, which has been shown in [11].
Theorem 5 For any tree $T$ of order $m$, every 2-connected graph $G$ with $\delta(G) \geq m + 2$ and $g(G) \geq \delta(G) - 8$ contains a subtree $T' \cong T$ such that $G - V(T')$ is 2-connected.

Since $g(G) \geq 3$ for any 2-connected graph $G$, we have the following corollary which improves the previously known upper bound of 8 on the order of a tree for which Mader’s conjecture holds for $k = 2$.

Corollary 1 For any tree of order $m \leq 9$, every 2-connected graph $G$ with $\delta(G) \geq m + 2$ contains $T' \cong T$ such that $G - V(T')$ is 2-connected.

By increasing the lower bound of $\delta(G) - 8$ on $g(G)$ in Theorem 5 by one, we can improve the lower bound of $m + 2$ on $\delta(G)$ to $m + 1$ if $m \geq 10$. Namely, a stronger statement holds in such a case.

Theorem 6 For any tree $T$ of order $m \geq 10$, every 2-connected graph $G$ with $\delta(G) \geq m + 1$ and $g(G) \geq \delta(G) - 7$ contains a subtree $T' \cong T$ such that $G - V(T')$ is 2-connected.

Besides, by adding structural conditions, we can improve the girth conditions in Theorems 5 and 6 as follows.

Theorem 7 For any tree $T$ of order $m$, every 2-connected graph $G$ with $\delta(G) \geq m + 2$ and $g(G) \geq \delta(G) - 9$ in which no six cycles of length $g(G)$ have a common path of length $\left\lceil \frac{g(G)}{2} \right\rceil - 1$ contains a subtree $T' \cong T$ such that $G - V(T')$ is 2-connected.

Theorem 8 For any tree $T$ of order $m \geq 11$, every 2-connected graph $G$ with $\delta(G) \geq m + 1$ and $g(G) \geq \delta(G) - 8$ in which no four cycles of length $g(G)$ have a common path of length $\left\lceil \frac{g(G)}{2} \right\rceil - 1$ contains a subtree $T' \cong T$ such that $G - V(T')$ is 2-connected.

Moreover, by introducing a variant of the girth of a 2-connected graph, we can generalize Theorem 5. For a 2-connected graph $G$ which is not a cycle, let $g^\circ(G)$ denote the largest integer $\ell$ such that any two cycles of length at most $\ell - 1$ in $G$ are edge-disjoint. We call $g^\circ(G)$ the overlapping girth of $G$. Since there is no cycle of length $g(G) - 1$, it holds that $g^\circ(G) \geq g(G)$. Note that $g^\circ(G) = g(G)$ if there exist two cycles $C_1, C_2$ of length $g(G)$ such that $E(C_1) \cap E(C_2) \neq \emptyset$. Using the overlapping girth of a 2-connected graph, we can show the following. Namely, $g(G)$ in Theorem 5 can be replaced with $g^\circ(G)$.

Theorem 9 For any tree $T$ of order $m$, every 2-connected graph $G$ with $\delta(G) \geq m + 2$ and $g^\circ(G) \geq \delta(G) - 8$ contains a subtree $T' \cong T$ such that $G - V(T')$ is 2-connected.

Mader’s conjecture is interesting not only from a theoretical point of view but also from a practical point of view, since it has a potential application to fault-tolerant problems in communication networks. Mader’s conjecture concerns the reliability of a communication network for a faulty subtree structure rather than a set of faulty vertices. Our proofs are constructive and lead to $O(|V(G)|^4)$ time algorithms for finding a desired subtree in a given 2-connected graph $G$ in Theorems 5, 6, 7, 8, and 9.
This paper is organized as follows. Section 2 presents notations, terminology, and known results used in this paper. Section 3 gives an outline of our proofs. Section 4 extends the class of caterpillars to a larger class for which Mader’s conjecture for \( k = 2 \) holds. Detailed proofs of Theorems 5 and 6 (respectively, Theorems 7, 8, and 9) are given in Sect. 5 (respectively, Sects. 6 and 7). Section 8 concludes the paper with several remarks.

2 Preliminaries

For a nonempty subset \( E' \subseteq E(G) \), we denote by \( G - E' \) and \( \langle E' \rangle \) the graph obtained from \( G \) by deleting every edge in \( E' \) and the edge-induced subgraph of \( G \) by \( E' \), respectively. For \( v \in V(G) \), we denote by \( N_G(v) \) the set of neighbors of \( v \) in \( G \), i.e., vertices adjacent to \( v \) in \( G \). The cardinality of \( N_G(v) \) may be written by \( \text{deg}_G(v) \). Let \( \Delta(G) = \max_{v \in V(G)} \text{deg}_G(v) \). For \( S \subseteq V(G) \), \( N_G(S) \) is defined as \( \{ \cup_{v \in S} N_G(v) \} \setminus S \). For \( G' \subseteq G \), let \( N_G(G') = N_G(V(G')) \).

A component of a graph \( G \) is a maximal connected subgraph of \( G \). A cut vertex of \( G \) is a vertex whose deletion increases the number of components of \( G \). A block of \( G \) is a maximal connected subgraph of \( G \) without a cut vertex. A cyclic block is a block with order at least 3. For a tree \( T \), the set of internal vertices, i.e., vertices with degree at least two, is denoted by \( V_I(T) \), while the set of leaves, i.e., vertices with degree one, is denoted by \( V_L(T) \). For a vertex \( v \) of a tree \( T \), if \( v \) is adjacent to at least \( \text{deg}_T(v) - 1 \) leaves, then \( v \) is called a pseudo-leaf of \( T \). A path is a tree \( T \) with \( \Delta(T) \leq 2 \), and it may be denoted by a sequence \( (v_1, v_2, \ldots, v_m) \) of vertices if \( V(T) = \{v_1, v_2, \ldots, v_m\} \) and \( E(T) = \{v_i v_{i+1} \mid 1 \leq i < m\} \). A caterpillar is a tree \( T \) such that \( \langle V_I(T) \rangle_T \) is a path if \( V_I(T) \neq \emptyset \). For a subtree \( T' \) of \( G \) and a subgraph \( H \) of \( G \) such that \( V(T') \cap V(H) = \emptyset \), let \( |V_H(T')| = \{u \in V(T') \mid |N_G(u) \cap V(H)| \geq 2 \} \).

The length of a path or a cycle is the number of edges in it. The distance \( d_G(u, v) \) between two vertices \( u \) and \( v \) in a connected graph \( G \) is the length of a shortest path between them. Let \( d_G(u, u) = 0 \) for any \( u \in V(G) \). The eccentricity \( \text{ecc}_G(v) \) of \( v \) in \( G \) is defined as \( \max_{w \in V(G)} d_G(v, w) \). A central vertex of \( G \) is a vertex \( u \) with \( \text{ecc}_G(u) = \min_{v \in V(G)} \text{ecc}_G(v) \), while a peripheral vertex is a vertex \( u \) with \( \text{ecc}_G(u) = \max_{v \in V(G)} \text{ecc}_G(v) \). The diameter \( \text{diam}(G) \) of a connected graph \( G \) is the maximum distance for every pair of vertices in \( G \), i.e., \( \text{diam}(G) = \max_{u, v \in V(G)} d_G(u, v) \). Let \( \text{diam}(G) = 0 \) if \( |V(G)| = 1 \).

On the existence of a subtree isomorphic to a given tree \( T \) in a graph \( G \), by assuming the existence of a subtree \( S' \) isomorphic to a subtree \( S \) of \( T \), we do not need to consider the degree condition for a vertex of \( S' \) whose degree is the same as that of the corresponding vertex in \( T \). Namely, Proposition 1 can be stated in a more general form as follows.

Lemma 1 [4] Let \( T \) be a tree of order \( m \) and \( S \) a subtree of \( T \). If a graph \( G \) contains a subtree \( S' \cong S \) such that \( \text{deg}_G(v) \geq m - 1 \) for any \( v \in V(G) \) except for \( v \in V(S') \) with \( \text{deg}_S(\phi^{-1}(v)) = \text{deg}_T(\phi^{-1}(v)) \) where \( \phi \) is an isomorphism from \( V(S) \) to \( V(S') \), then \( G \) contains a subtree \( T' \cong T \) such that \( S' \subseteq T' \).
In particular, the remark after Proposition 1 corresponds to the case that $S$ is a trivial tree, i.e., $|V(S)| = 1$. By considering the case that $S$ is a subtree obtained from $T$ by deleting leaves, the following lemma is obtained.

**Lemma 2** [4] *Let $T$ be a tree of order $m$ and $S$ a subtree obtained from $T$ by deleting leaves adjacent to a vertex in $V_S \subseteq V_1(T)$. If a graph $G$ contains a subtree $S' \cong S$ such that $\deg_G(u) \geq m - 1$ for any $u \in \{\phi(v) \mid v \in V_S\}$ where $\phi$ is an isomorphism from $V(S)$ to $V(S')$, then $G$ contains a subtree $T' \cong T$ such that $S' \subseteq T'$.*

Note that in Lemmas 1 and 2, $T'$ can be greedily obtained from $S'$, i.e., given a subtree $S'$ of a graph $G$ satisfying the assumptions, $T'$ can be computed in $O(|E(G)|)$ time.

An orientation $D$ of a graph $G$ is a directed graph obtained from $G$ by replacing each edge by an arc (directed edge) with the same end-vertices. The outdegree $\deg_D^+(v)$ (respectively, indegree $\deg_D^-(v)$) of a vertex $v$ in $D$ is the number of arcs from (respectively, to) $v$ in $D$. If for any $v \in V(G)$, $\deg_D^+(v)$ is even, then any component of $G$ is Eulerian and $G$ has an orientation $D$ in which for any $v \in V(D)$, $\deg_D^+(v) = \deg_D^-(v)$. If $G$ has a vertex with odd degree, we can find a directed path $W$ connecting two vertices with odd degree, and by inductively applying a similar discussion for $G - E(W)$, we can see the following lemma holds. We here remark that Lemma 3 holds for multigraphs.

**Lemma 3** *Every graph $G$ has an orientation $D$ such that $|\deg_D^+(v) - \deg_D^-(v)| \leq 1$ for any $v \in V(D)$.***

By applying the depth-first search (for a directed cycle) or the breadth-first search (for a directed path connecting two vertices with odd degree) once, we can replace at least one edge with an arc. Thus, an orientation in Lemma 3 can be found in $O(|E(G)|^2)$ time.

### 3 Outline of Proofs

In this section, we explain the outline of our constructive proofs and the time complexity for the algorithms based on the proofs.

Let $T$ be a tree of order $m$. Let $G$ be a 2-connected graph with $\delta(G) \geq m + 2$. From Proposition 1, $G$ contains a subtree $T' \cong T$. Let $B$ be a maximum block in $G - V(T')$, i.e., a block with the maximum order among all the blocks in $G - V(T')$. Note that $B$ is a cyclic block since $\delta(G - V(T')) \geq 2$. If $B = G - V(T')$, then $T'$ is a desired subtree. Suppose that $B \neq G - V(T')$. Then there is a vertex in $G - V(T') \cup V(B)$. For any vertex $w \in V(G) \setminus (V(T') \cup V(B))$, $|N_G(w) \cap V(B)| \leq 1$. Now let $P = (v_1, v_2, \ldots, v_t)$, where $v_1, v_t \in V(B)$ and $v_i \notin V(B)$ for $1 < i < t$, be a shortest path among all the paths of order at least 3 of $G$ connecting two vertices in $B$ such that every internal vertex is not in $B$. Let $w$ be a vertex in $G - V(B)$. Since $G$ is 2-connected, there are paths $Q_1$ and $Q_2$ such that $Q_1$ is a path in $G$ from $w$ to a vertex $v \in V(B)$ where $V(Q_1) \cap V(B) = \{v\}$, and $Q_2$ is a path in $G - v$ from $w$ to a vertex $v' \in V(B) \setminus \{v\}$ where $V(Q_2) \cap V(B) = \{v'\}$. Let $u$ be the last vertex in $Q_1$.
(from \(w\) to \(v\)) that is also in \(Q_2\). Note that it is possible that \(u = w\). Now let \(Q'_1\) be the subpath of \(Q_1\) from \(u\) to \(v\), and \(Q'_2\) be the subpath of \(Q_2\) from \(u\) to \(v'\). Then the path \((E(Q'_1) \cup E(Q'_2))\) from \(v\) to \(v'\) has order at least 3 such that every internal vertex is not in \(B\). Thus, \(P\) is well-defined.

Suppose that \(t \geq 4\). Then, we have that \(N_G(v_2) \cap V(B) = \{v_1\}\) and \(N_G(v_2) \cap V(P) = \{v_1, v_3\}\). Therefore, \(|N_G(v_2) \setminus (V(P) \cup V(B))| \geq m + 2 - 2 \geq m\), which implies that \(V(G) \setminus (V(P) \cup V(B)) \neq \emptyset\). Let \(w\) be any vertex in \(G - V(P) \cup V(B)\). By the definition of \(P\), \(w\) can be adjacent to at most three vertices in \(V(B) \cup V(P)\). Thus, \(\delta(G - V(P) \cup V(B)) \geq m + 2 - 3 = m - 1\). An example of this situation is illustrated in Fig. 1. Hence, by Proposition 1, \(G - V(P) \cup V(B)\) contains a subtree \(T'' \cong T\) such that \(G - V(T'')\) has a block \(B' \supseteq (V(B) \cup V(P))_G\). Thus, we can find a block of order at least \(|V(B)| + 2\). This observation can be stated as the following lemma. Note that \(t \geq 4\) if and only if \(V_{[B]}(T') = \emptyset\), where \(V_{[B]}(T') = \{u \in V(T') \mid |N_G(u) \cap V(B)| \geq 2\}\).

**Lemma 4** Let \(T\) be a tree of order \(m\) and \(G\) a 2-connected graph with \(\delta(G) \geq m + 2\). For any subtree \(T' \cong T\) in \(G\) and a maximum block \(B\) in \(G - V(T')\), if \(B \neq G - V(T')\) and \(V_{[B]}(T') = \emptyset\), then there exists a subtree \(T'' \cong T\) in \(G\) such that \(G - V(T'')\) contains a larger block \(B'\) than \(B\).

Suppose that \(t = 3\). Then \(v_2 \in V_{[B]}(T')\). If there exists a subtree \(T'' \subset G - V(B) \cup \{v_2\}\) such that \(T'' \cong T\), then \(G - V(T'')\) has a block \(B' \supseteq (V(B) \cup \{v_2\})_G\), i.e., we have a block of order at least \(|V(B)| + 1\). If we have a manipulation to find such a subtree \(T''\), where we use the term “manipulation” to mean a part of a constructive proof concerning graphs, then by applying the manipulations for \(t \geq 4\) or \(t = 3\) iteratively, we finally obtain a desired subtree \(T'' \cong T\) such that \(G - V(T'')\) is 2-connected.

From the above discussion, Mader’s conjecture for \(k = 2\) can be reduced to the following statement. Namely, if we can show Statement 1, then it is concluded that Mader’s conjecture for \(k = 2\) is true.

**Statement 1** Let \(T\) be a tree of order \(m\) and \(G\) a 2-connected graph with \(\delta(G) \geq m + 2\). For any subtree \(T' \cong T\) in \(G\) and a maximum block \(B\) in \(G - V(T')\), if \(B \neq G - V(T')\) and \(V_{[B]}(T') \neq \emptyset\), then there exist a vertex \(v \in V_{[B]}(T')\) and a subtree \(T'' \cong T\) in \(G - V(B) \cup \{v\}\), i.e., \(G - V(T'')\) contains a larger block \(B'\) than \(B\).

The above manipulations can be algorithmically described as follows.

1. Compute a subtree \(T' \cong T\) in \(G\).
2. Compute a maximum block $B$ in $G - V(T')$.
3. If $B = G - V(T')$ then output $T'$ as a desired subtree of $G$ and stop.
4. If $B \neq G - V(T')$ then compute a shortest path $P$ of order at least 3 connecting two vertices in $B$ such that every internal vertex is not in $B$.
5. Compute a subtree $T'' \cong T$ in $G - V(B) \cup V(P)$, let $T" = T''$, and return to Step 2.

We here check the complexity of the above algorithm under the assumption that there exists a constructive proof of Statement 1. A subtree $T' \cong T$ in $G$ can be computed in $O(|E(G)|)$ time at Step 1, and a maximum block $B$ can also be found in $O(|E(G)|)$ time at Step 2. At Step 4, a shortest path $P$ can be found by computing all shortest paths for vertices of $V(B)$ in $G - E(B)$. Thus, it takes $O(|V(G)|^3)$ time. Since the number of iterations is $O(|V(G)|)$, if Statement 1 can be shown by a constructive proof from which a procedure within $O(|V(G)|^3)$ time is obtained, we have an $O(|V(G)|^4)$ time algorithm. These observations and the above discussions are summarized as follows.

**Lemma 5** If Statement 1 holds, then by iteratively applying Lemma 4 or Statement 1, it follows that $G$ contains a subtree $T' \cong T$ such that $G - V(T')$ is 2-connected. Besides, if there is a procedure for Statement 1 within $O(|V(G)|^3)$ time, then we have an $O(|V(G)|^4)$ time algorithm for finding a desired subtree.

Next, we consider the case that a 2-connected graph $G$ has no triangle, i.e., $g(G) \geq 4$ and show similar lemmas with a similar statement. Note that the minimum degree condition $\delta(G) \geq m + 2$ is replaced with $\delta(G) \geq m + 1 \geq 3$.

**Lemma 6** Let $T$ be a tree of order $m \geq 2$ and $G$ a 2-connected graph with $\delta(G) \geq m + 1$ and $g(G) \geq 4$. For any subtree $T' \cong T$ in $G$ and a maximum block $B$ in $G - V(T')$, if $B \neq G - V(T')$ and $V[B](T') = \emptyset$, then there exists a subtree $T'' \cong T$ in $G$ such that $G - V(T'')$ contains a larger block $B'$ than $B$.

**Proof** Let $P = (v_1, v_2, \ldots, v_t)$ be a shortest path of order at least 3 between two vertices in $B$ such that every internal vertex is not in $B$. Then, $t \geq 4$ since $V[B](T') = \emptyset$. By the definition of $P$ and the girth condition $g(G) \geq 4$, any vertex $w$ in $G - V(P) \cup V(B)$ can be adjacent to at most two vertices in $V(B) \cup V(P)$. Thus, $\delta(G - V(P) \cup V(B)) \geq m - 1$. Therefore, $G - V(P) \cup V(B)$ contains a subtree $T'' \cong T$ such that $G - V(T'')$ has a block $B' \cong (V(B) \cup V(P))_G$. Note that the condition $m \geq 2$ is necessary to guarantee that $V(G) \setminus (V(P) \cup V(B)) \neq \emptyset$. \hfill $\square$

**Statement 2** Let $T$ be a tree of order $m \geq 2$ and $G$ a 2-connected graph with $\delta(G) \geq m + 1$ and $g(G) \geq 4$. For any subtree $T' \cong T$ in $G$ and a maximum block $B$ in $G - V(T')$, if $B \neq G - V(T')$ and $V[B](T') \neq \emptyset$, then there exist a vertex $v \in V[B](T')$ and a subtree $T'' \cong T$ in $G - V(B) \cup \{v\}$, i.e., $G - V(T'')$ contains a larger block $B'$ than $B$.

**Lemma 7** If Statement 2 holds, then by iteratively applying Lemma 6 or Statement 2, it follows that $G$ contains a subtree $T' \cong T$ such that $G - V(T')$ is 2-connected. Besides, if there is a procedure for Statement 2 within $O(|V(G)|^3)$ time, then we have an $O(|V(G)|^4)$ time algorithm for finding a desired subtree.
Let $T' \subset G$ such that $T' \cong T$. Let $B$ be a maximum block in $G - V(T')$. Since $\delta(G - V(T')) \geq 1$, it may happen that $B$ is not a cyclic block, i.e., $B$ is a block with two vertices. Note that if $B$ is not a cyclic block, then $B$ is not 2-connected. Suppose that $B$ is not a cyclic block. Assume that $B = G - V(T')$. Then, $|V(G)| = m + 2$. Since $\delta(G) \geq m + 1$, $G$ must be a complete graph with at least four vertices, which contradicts the girth condition $g(G) \geq 4$. Therefore, if $B$ is not a cyclic block, then $B \neq G - V(T')$. Hence, the algorithm for Lemma 5 also works well under the assumption that Statement 2 holds; namely, in the case that $G - V(T')$ has no cyclic block, the algorithm does not incorrectly output a subtree at Step 3.

In Statement 2, if $B$ is not a cyclic block, then by the girth condition $g(G) \geq 4$, we have that $V[B](T') = \emptyset$. Thus, we may assume that a maximum block $B$ is a cyclic block in Statement 2.

### 4 Bifurcate Quasi-Unimodal Caterpillars

In this section, we extend a class of special trees for which Mader’s conjecture for $k = 2$ holds. A **quasi-unimodal caterpillar** is a caterpillar $T$ such that if $V_1(T) \neq \emptyset$ and $(V_1(T))_T = (v_1, v_2, \ldots, v_n)$ then there exists a positive integer $1 \leq k \leq n$ such that $\deg_T(v_i) \leq \deg_T(v_{i+1})$ for each $i < k - 1$ and $\deg_T(v_i) \geq \deg_T(v_{i+1})$ for each $i \geq k + 1$, where $v_k$ is called a **free vertex** of $T$. A **bifurcate quasi-unimodal caterpillar** is a tree $T$ obtained from a quasi-unimodal caterpillar $T_1$ and a caterpillar $T_2$ by identifying a free vertex of $T_1$ (or any vertex of $T_1$ if $|V(T_1)| \leq 2$) and a peripheral vertex of $T_2$. Note that the class of bifurcate quasi-unimodal caterpillars properly contains the class of caterpillars, since $T_1$ may be trivial, i.e., $|V(T_1)| = 1$. Figure 2 illustrates an example of a bifurcate quasi-unimodal caterpillar.

In order to show that Mader’s conjecture is true for the class of bifurcate quasi-unimodal caterpillars, we prove the following lemma.

**Lemma 8** Let $T$ be a bifurcate quasi-unimodal caterpillar of order $m$. Let $G$ be a 2-connected graph with $\delta(G) \geq m + 1$. For any subtree $T' \cong T$ in $G$ and a maximum...
block $B$ in $G - V(T')$, if $B \neq G - V(T')$ and $V_{[B]}(T') \neq \emptyset$, then there exists a vertex $v \in V_{[B]}(T')$ and a subtree $T'' \cong T$ in $G - (V(B) \cup \{v\})$ such that $v$ and $T''$ can be found in $O(|E(G)|)$ time.

**Proof** Suppose that $T'$ consists of a quasi-unimodal caterpillar $T'_1$ and a caterpillar $T'_2$. We may assume that $V_I(T'_1) \neq \emptyset$ and $V_I(T'_2) \neq \emptyset$, since the other case can be treated more easily. Let $(V_I(T'_1))' = (u_1, u_2, \ldots, u_n_1)$ and $(V_I(T'_2))' = (v_1, v_2, \ldots, v_n_2)$ such that $V(T'_1) \cap V(T'_2) = \{u_p\}, v_{n_2+1} \in V_L(T'_2), v_{n_2} v_{n_2+1} \in E(T'_2),$ and $v_{n_2+1} = u_p$. Note that $u_p$ is a free vertex of $T'_1$. Let $H = G - V(T') \cup \{V(B)\}$.

Case 1: $V(I(T'_1) \cap V_{[B]}(T')) \neq \emptyset$.

Let $v_q \in V_{[B]}(T')$ where $1 \leq q \leq n_2$ such that for any $j < q$, $v_j \notin V_{[B]}(T')$. Let $H' = (V(H) \cup \{v_j \mid j < q\})_G$. Since any vertex in $H'$ has at most one neighbor in $B$ and $\delta(G) \geq m + 1$, $\delta(H') \geq m + 1 - (m - (q - 1) + 1) = q - 1$. If there exists a vertex $v_{q+k} \in V_I(T'_1) \cup \{v_{n_2+1}\}$ where $k \geq 1$ such that $N_G(v_{q+k}) \cap V(H') \neq \emptyset$ and for any $1 \leq k' < k, N_G(v_{q+k'}) \cap V(H') = \emptyset$, then $\delta(H') \geq (q - 1) + (k - 1) = q + k - 2$ and there exists a path $(w_1, \ldots, w_{q+k-1})$ in $H'$ such that $w_{q+k} - v_{q+k} \in E(G)$. Note that from Proposition 1 or Lemma 1, such a path can be greedily constructed by starting the vertex $w_{q+k-1}$. Thus, by letting $w_i$ correspond to $v_i$ for each $1 \leq i < q + k$, we have a subtree $S'$ in $G - (V(B) \cup \{v_q\}$ which is isomorphic to the subtree $S'$ obtained from $T'$ by deleting every leaf of $T'_2$ adjacent to a vertex in $\{v_1, \ldots, v_{q+k-1}\}$. Since $\text{deg}_{G - V(B) \cup \{v_q\}}(v_i) \geq m - 1$ for each $w_i \in V(H')$, by applying Lemma 2, we have a desired subtree $T'' \cong T$ in $G - (V(B) \cup \{v_q\})$. Therefore, we may assume that for any $q < j \leq n_2 + 1$, $N_G(v_j) \cap V(H') = \emptyset$ which implies that $|N_G(v_j) \cap V(B)| \geq 2$ since $\text{deg}_G(v_j) - (m - q) \geq q + 1 \geq 2$, i.e., $v_j \in V_{[B]}(T')$.

Besides, $\delta(H') \geq q - 1 + (n_2 + 1 - q) \geq n_2$.

Case 1.1: $\{j \mid p + 1 \leq j \leq n_1, N_G(u_j) \cap V(H') \neq \emptyset\} \neq \emptyset$.

Let $b = \min\{j \mid p + 1 \leq j \leq n_1, N_G(u_j) \cap V(H') \neq \emptyset\}$. Let $x_{b-1} \in N_G(u_b) \cap V(H')$. Since $\delta(H') \geq n_2 + (b - 1) - p = n_2 + b - p - 1$, we can construct a path $(y_1, y_2, \ldots, y_{n_2}, x_p, x_{p+1}, \ldots, x_{b-1})$ of order $n_2 + b - p$ in $H'$. If either $p = 0$ or $p \geq 2$ and $x_p u_{p-1} \notin E(G)$, then by letting $y_i$ (respectively, $x_i$) correspond to $v_i$ (respectively, $u_i$) for each $1 \leq i \leq n_2$ (respectively, $p \leq i < b$), we have a subtree $S''$ in $G - (V(B) \cup \{u_p\})$ which is isomorphic to the subtree $S'$ obtained from $T'$ by deleting every leaf of $T'_2$ and every leaf of $T'_1$ adjacent to a vertex in $\{u_p, u_{p+1}, \ldots, u_{b-1}\}$. Since $\text{deg}_{G - V(B) \cup \{u_p\}}(w) \geq m - 1$ for each $w \in V(H')$, it follows from Lemma 2 that there exists a subtree $T'' \cong T$ in $G - (V(B) \cup \{u_p\})$. If $p \geq 2$ and $x_p u_{p-1} \notin E(G)$, then $\text{deg}_{H'}(x_p) \geq n_2 + b - p$ which implies that we can select a vertex $x_p e_{p-1} \in N_G(x_p) \cap (V(H') \setminus \{y_1, y_2, \ldots, y_{n_2}, x_{p+1}, \ldots, x_{b-1}\})$. If either $p = 2$ or $p \geq 3$ and $\{x_{p-1} u_{p-1}, x_{p-1} u_{p-2}\} \cap E(G) \neq \emptyset$, then we can construct a subtree $S''$ in $G - (V(B) \cup \{u_p\})$ which is isomorphic to the subtree $S'$ obtained from $T'$ by deleting every leaf of $T'_2$ and every leaf of $T'_1$ adjacent to a vertex in $\{u_{p-1}, u_p, u_{p+1}, \ldots, u_{b-1}\}$. Note that when $x_{p-1} u_{p-1} \in E(G)$ and $x_{p-1} u_{p-2} \notin E(G)$, this construction is possible since the monotone property that $\text{deg}_{T'_1}(u_i) \leq \text{deg}_{T'_1}(u_{i+1})$ for each $i < p - 1$ holds. Otherwise, since $\text{deg}_{H'}(x_p) \geq n_2 + b - p + 1$, we can also select a vertex $x_{p-2} \in N_G(x_{p-1}) \cap (V(H') \setminus \{y_1, y_2, \ldots, y_{n_2}, x_p, x_{p+1}, \ldots, x_{b-1}\})$. Repeating a similar process, we can finally obtain a subtree $S''$ in $G - (V(B) \cup \{u_p\})$ which is isomorphic to the subtree $S'$ obtained from $T'$ by deleting every leaf of $T'_2$ and every
leaf of $T'$ adjacent to a vertex in \{${u_p-i, u_p-i+1, \ldots, u_b-1}$\} for some $i < p$. An example of a situation in Case 1.1 is illustrated in Fig. 3. Since $\deg_{G-V(B)\cup \{u_p\}}(w) \geq m-1$ for each $w \in V(H')$, by Lemma 2, we have a desired subtree $T'' \cong T$ in $G-V(B)\cup \{u_p\}$.

Case 2: \{${j | p+1 \leq j \leq n_1, N_G(u_j) \cap V(H') \neq \emptyset}$\} $\neq \emptyset$.

Since $\delta(H') \geq n_2+n_1-p$, we can construct a path $(y_1, y_2, \ldots, y_{n_2}, x_p, x_{p+1}, \ldots, x_{n_1})$ of order $n_2+n_1-p+1$ in $H'$. Similarly to Case 1.1, we can obtain a subtree $S''$ and by Lemma 2, we have a desired subtree $T'' \cong T$ in $G-V(B)\cup \{u_p\}$.

Case 2: $V_I(T'_2) \cap V_B(T') = \emptyset$.

Let $H'' = \langle V(H) \cup V_I(T'_2) \rangle_G$. Then, we have $\delta(H'') \geq n_2$. If $u_p \in V_B(T')$, then we can apply a similar discussion in Case 1 to obtain the desired result. Thus, we may assume that $u_p \notin V_B(T')$. Suppose that $z \in V_L(T') \cap V_B(T')$ and $zz' \in E(T')$. If $z' \notin V_B(T')$, then $N_G(z') \cap V(H) \neq \emptyset$ and for $z'' = N_G(zz') \cap V(H)$, the subtree $T'' = \langle E(T-z) \cup \{z'z''\} \rangle$ is a desired subtree in $G-V(B)\cup \{z\}$. Thus, we may also assume that $V_I(T') \cap V_B(T') \neq \emptyset$. Therefore, it is sufficient to consider the following two subcases.

Case 2.1: \{${i | u_i \in V_B(T'), 1 \leq i < p}$\} $\neq \emptyset$.

Let $a = \max\{i | u_i \in V_B(T'), 1 \leq i < p\}$. Then $u_{a+1} \notin V_B(T')$ and $N_G(u_{a+1}) \cap V(H) \neq \emptyset$. Let $x_a \in N_G(u_{a+1}) \cap V(H)$. If either $a = 1$ or $a \geq 2$ and $x_a u_{a-1} \in E(G)$, then we have a desired subtree $S''$ isomorphic to the subtree $S'$ obtained from $T'$ by deleting every leaf of $T'_1$ adjacent to $u_a$. If $a \geq 2$ and $x_a u_{a-1} \notin E(G)$, then we can select a vertex $x_{a-1} \in N_G(x_a) \cap V(H)$. If either $a = 2$ or $a \geq 3$ and $\{x_{a-1} u_{a-1}, x_{a-1} u_{a-2}\} \cap E(G) \neq \emptyset$, then we can construct a desired subtree $S''$ isomorphic to the subtree $S'$ obtained from $T'$ by deleting every leaf of $T'_1$ adjacent to a vertex in \{${u_{a-1}, u_a}$\}. Note that the monotone property that $\deg_{T'_1}(u_i) \leq \deg_{T'_1}(u_{i+1})$ for each $i < p - 1$ holds. Thus, similarly to Case 1.1, we can finally obtain a desired subtree $T'' \cong T$ in $G-V(B)\cup \{u_a\}$.
Case 2.2: \( \{ i \mid u_i \in V_{[B]}(T'), \ p < i \leq n_1 \} \neq \emptyset \).

Let \( b' = \min \{ i \mid u_i \in V_{[B]}(T'), \ p < i \leq n_1 \} \). From the monotone property that 
\( \deg_{T_1'}(u_i) \geq \deg_{T_1'}(u_{i+1}) \) for each \( i \geq p + 1 \), similarly to Case 2.1, we can construct 
a desired subtree \( T'' \cong T \) in \( G - V(B) \cup \{ u_{b'} \} \).

For determining the appropriate case and finding a specified vertex such as \( v_q \), \( v_{q+k} \), 
\( x_{b-1} \), and \( u_{a+1} \), it takes \( O(|E(G)|) \) time. In Case 1, a path of a certain length in \( H' \) can be found in \( O(|E(G)|) \) time and it can be extended to obtain a subtree \( S'' \) in 
\( O(|E(G)|) \) time. By Lemma 2, \( S'' \) can further be extended to \( T'' \) in \( O(|E(G)|) \) time. Therefore, a desired subtree \( T'' \) 
with a vertex in \( V_{[B]}(T') \) can be found in \( O(|E(G)|) \) time. \( \Box \)

From Lemma 5 (respectively, Lemma 7) and Lemma 8, we have Theorem 10 
(respectively, Theorem 11) such that a desired subtree \( T' \) in each theorem can be 
found in \( O(|V(G)|^4) \) time.

**Theorem 10** For any bifurcate quasi-unimodal caterpillar \( T \) of order \( m \), every 2-connected 
graph \( G \) with \( \delta(G) \geq m+2 \) contains a subtree \( T' \cong T \) such that \( G - V(T') \) 
is 2-connected.

**Theorem 11** For any bifurcate quasi-unimodal caterpillar \( T \) of order \( m \geq 2 \), every 
2-connected graph \( G \) with \( \delta(G) \geq m+1 \) and \( g(G) \geq 4 \) contains a subtree \( T' \cong T \) 
such that \( G - V(T') \) is 2-connected.

We here remark that Theorem 10 can be shown, even if the lower bound of \( m+1 \) 
on \( \delta(G) \) is replaced with \( m+2 \) in Lemma 8. Under such a stronger degree condition 
in Lemma 8, we can slightly weaken the definition of a quasi-unimodal caterpillar as 
follows: a quasi-unimodal caterpillar is a caterpillar \( T \) such that if \( V_1(T) \neq \emptyset \) and 
\( V_1(T) = (v_1, v_2, \ldots, v_n) \) then there exists a positive integer \( 1 \leq k \leq n \) such that 
\( \deg_T(v_i) \leq \deg_T(v_{i+1}) \) for each \( i < k - 2 \) and \( \deg_T(v_i) \geq \deg_T(v_{i+1}) \) for each 
\( i \geq k + 2 \). Thus, we can employ the corresponding weaker version of a bifurcate 
 quasi-unimodal caterpillar in Theorem 10.

As shown in the next proposition, the class of bifurcate quasi-unimodal caterpillars 
contains trees with large diameter.

**Proposition 2** Every tree \( T \) of order \( m \) with \( \text{diam}(T) \geq m - 4 \) is a bifurcate 
 quasi-unimodal caterpillar.

**Proof** Let \( T \) be a tree \( T \) of order \( m \) with \( \text{diam}(T) \geq m - 4 \). Let \( P = 
(v_1, v_2, \ldots, v_{\text{diam}(T)+1}) \) be a longest path in \( T \). If \( \text{diam}(T) = m - 1 \), then \( T \) is a 
path. If \( \text{diam}(T) = m - 2 \) or \( T - V(P) \) has no edge, then \( T \) is a caterpillar since any 
vertex in \( V(T) \setminus V(P) \) is a leaf of \( T \). If \( \text{diam}(T) = m - 3 \) and \( T - V(P) \) has an edge 
\( xy \), then either \( x \) or \( y \) is adjacent to a vertex of \( P \) which implies that \( T \) is a bifurcate 
 quasi-unimodal caterpillar. Let \( \text{diam}(T) = m - 4 \) and \( V(T) \setminus V(P) = \{ x, y, z \} \).
Suppose that \( T - V(P) \) has exactly one edge \( xy \) such that \( yv_i, zv_j \in E(T) \). If \( i = j \), 
then \( T \) is a bifurcate quasi-unimodal caterpillar consisting of a quasi-unimodal caterpillar 
\( \langle V(P) \cup \{ z \} \rangle_T \) and a path \( (x, y, v_i) \). If \( i < j \) (respectively, \( j < i \)), then \( T \) is a bifurcate quasi-unimodal caterpillar consisting of a path \( (v_1, \ldots, v_i, y, x) \) and
a caterpillar \(\langle \{v_1, \ldots, v_{\text{diam}(T) + 1}, z\} \rangle_T\) (respectively, a path \((x, y, v_1, \ldots, v_{\text{diam}(T) + 1})\) and a caterpillar \(\langle \{v_1, \ldots, v, z\} \rangle_T\)). Suppose that \(T - V(P)\) has two edges \(xy\) and \(yz\). Let \(v_k\) be the vertex of \(P\) adjacent to one of \(\{x, y, z\}\) in \(T\). Then, \(T\) is a bifurcate quasi-unimodal caterpillar consisting of a path \(P\) and a caterpillar \(\langle \{x, y, z, v_k\} \rangle_T\). \(\square\)

Note that there exists a tree of order \(m\) with \(\text{diam}(T) = m - 5\) which is not a bifurcate quasi-unimodal caterpillar, e.g., a tree of order \(m \geq 10\) obtained from a path \((v_1, v_2, \ldots, v_{m-4})\) by adding four vertices \(x_1, x_2, y_1, y_2\) with four edges \(x_1x_2, x_2v_1, y_1y_2, y_2v_j\), where \(3 \leq i < j \leq m - 6\).

From Theorems 10 and 11 and Proposition 2, we have the following corollaries which are used to show our main results.

**Corollary 2** For any tree of order \(m\) with \(\text{diam}(T) \geq m - 4\), every 2-connected graph \(G\) with \(\delta(G) \geq m + 2\) contains a subtree \(T' \cong T\) such that \(G - V(T')\) is 2-connected.

**Corollary 3** For any tree of order \(m \geq 2\) with \(\text{diam}(T) \geq m - 4\), every 2-connected graph \(G\) with \(\delta(G) \geq m + 1\) and \(g(G) \geq 4\) contains a subtree \(T' \cong T\) such that \(G - V(T')\) is 2-connected.

### 5 Proofs of Theorems 5 and 6

In order to show our main results, we prove the following lemma.

**Lemma 9** Let \(T\) be a tree of order \(m\) and \(G\) a 2-connected graph with \(\delta(G) \geq m + 1\) and \(g(G) \geq \text{diam}(T) - 1\). For any subtree \(T' \cong T\) in \(G\) and a maximum block \(B\) in \(G - V(T')\), if \(B \neq G - V(T')\) and \(V_B(T') \neq \emptyset\), then there exist a vertex \(v \in V_B(T')\) and a subtree \(T'' \cong T\) in \(G - V(B) \cup \{v\}\) such that \(v\) and \(T''\) can be found in \(O(|V(G)|^2)\) time.

**Proof** Let \(T' \subset G\) such that \(T' \cong T\). Let \(B\) be a maximum block in \(G - V(T')\) such that \(B \neq G - V(T')\). Also, let \(v \in V_B(T')\) and \(H = G - V(T') \cup V(B)\). When \(m \leq 2\), the lemma can be easily checked. Let \(m \geq 3\). Suppose that \(v\) is a leaf of \(T'\) and for the neighbor \(v'\) of \(v\) in \(T'\), \(v' \notin V_B(T')\), i.e., \(|N_G(v') \cap V(B)| \leq 1\). Then, \(|N_G(v') \cap V(H)| \geq 1\). For any \(v'' \in N_G(v') \cap V(H)\), \(T'' = (E(T' - v) \cup \{v'v''\}) \cong T\) such that \(T'' \subset G - V(B) \cup \{v\}\). Thus, w.l.o.g., we may assume that \(v\) is not a leaf of \(T'\). Let \(S' = \langle V_1(T') \rangle_{T'}\). Then \(v \in V(S')\). Since \(\text{diam}(S') = \text{diam}(T') - 2\), \(g(G) \geq \text{diam}(S') + 1\). We regard \(S'\) as a rooted tree at \(v\) and denote by \(C(u)\) the set of children of a vertex \(u\) in \(S'\). Besides, we denote by \(h(S')\) the height of \(S'\), i.e., \(h(S') = \text{ecc}_{S'}(v)\).

Since \(\delta(G) \geq m + 1\), it holds that for any vertex \(w \in V(H)\), \(\deg_{G - V(B) \cup \{v\}}(w) \geq m - 1\). If there exists a subtree \(W \subset \langle V(H) \cup (V(T') - v) \rangle_G\) such that \(W\) is isomorphic to a subtree \(U\) obtained from \(T'\) by deleting leaves adjacent to a vertex in \(V' \subset V(S')\) and \(\phi(V') = \{\phi(u) \mid u \in V'\} \subseteq V(H)\), where \(\phi\) is an isomorphism from \(V(U)\) to \(V(W)\), then by Lemma 2, there exists a subtree \(T'' \subset G - (V(B) \cup \{v\})\) such that \(T'' \cong T\). In particular, if there exists a vertex \(w\) in \(H\) such that \(C(w) \subseteq N_G(w)\), then we can employ the subtree \((E(T' - v) \cup \{wu \mid u \in C(v)\})\) as a desired subtree \(W\), where \(V' = \{v\}\).
and $\phi(V') = \{w\}$. Note that $C(v) = \emptyset$ when $\text{diam}(S') = 0$, i.e., $|V(S')| = 1$. Suppose that $v$ is a leaf of $S'$. Let $C(v) = \{v'\}$. If there is no vertex in $H$ adjacent to $v'$, i.e., $C(v) \not\subseteq N_G(w)$ for any $w \in V(H)$, then $\delta(H) \geq 1$ and $v' \in V_{[B]}(T')$. In such a case, if $\text{diam}(S') = 1$, then for $xy \in E(H)$, we can employ $\langle \{xy\} \rangle$ as a desired subtree $W$, where $V' = \{v, v'\}$ and $\phi(V') = \{x, y\}$. From these observations, we may assume that there is no vertex $w$ in $H$ with $N_G(w) \supseteq C(v)$, $\text{diam}(S') \geq 2$, and $v$ is not a leaf of $S'$ (since we can employ $v'$ instead of $v$ if $v' \in V_{[B]}(T')$).

Let $x \in V(H)$ and $C(v) \setminus N_G(x) = \{v_1, v_2, \ldots, v_p\}$. Since $|N_G(x) \cap V(B)| \leq 1$ and $|N_G(x) \cap V(T')| \leq m - p$, $|N_H(x)| \geq p$. Let $\{x_1, x_2, \ldots, x_p\} \subseteq N_H(x)$. If $h(S') = 1$, then we can employ $\langle E(T' - V') \cup \{xu \mid u \in C(v) \cap N_G(x) \cup \{x_i \mid 1 \leq i \leq p\} \rangle$ as a desired subtree $W$, where $V' = \{v, v_1, v_2, \ldots, v_p\}$ and $\phi(V') = \{x, x_1, x_2, \ldots, x_p\}$. Suppose that $h(S') \geq 2$. Let $\{C(v_i) \setminus N_G(x_i)\} = q_i$ for each $i$. Since $|C(v) \cap N_G(x_i)| \geq 1$, $|N_H(x_i)| \geq q_i + 1$. Thus, we can select $q_i$ vertices $y_{i,1}, y_{i,2}, \ldots, y_{i,q_i}$ as children of $x_i$ in the subtree $\langle \{x_i \mid 1 \leq i \leq p\} \rangle$ rooted at $x$. By letting these children correspond to the $q_i$ children of $v_i$ in $C(v_i) \setminus N_G(x_i)$ for each $i$ with $q_i > 0$, we can obtain a desired subtree $W$ if $h(S') = 2$. Note that when $\text{diam}(S') = 3$, there is at most one $i$ such that $C(v_i) \neq \emptyset$, and if $q_i > 0$, then $\{x_1, x_2, \ldots, x_p\} \cap \{y_{i,1}, y_{i,2}, \ldots, y_{i,q_i}\} = \emptyset$, $\{y_{i,1}, y_{i,2}, \ldots, y_{i,q_i}\} = \emptyset$ for any pair of $i$ and $i'$ with $q_i > 0$ and $q_{i'} > 0$. Thus, the subtree defined by $\langle E(T' - V') \cup \{xu \mid u \in C(v) \cap N_G(x) \cup \{x_i \mid 1 \leq i \leq p\} \cup \bigcup_{1 \leq i \leq p} \{x_i - u \mid u \in C(v_i) \cap N_G(x_i)\} \cup \{x_i - y_{i,j} \mid 1 \leq j \leq q_i\}\rangle$ can be employed as a desired subtree $W$, where $V' = \{v, v_1, v_2, \ldots, v_p\} \cup \bigcup_{1 \leq i \leq p} (C(v_i) \setminus N_G(x_i))$ and $\phi(V') = \{x, x_1, x_2, \ldots, x_p\} \cup \bigcup_{1 \leq i \leq p} q_i \cdot \{y_{i,1}, y_{i,2}, \ldots, y_{i,q_i}\}$. If $h(S') \geq 3$, by inductively applying similar manipulations to descendants of $x$, we can finally obtain a desired subtree $W$. Figure 4 illustrates a construction of a desired subtree $W$. Note that in each extension process, disjointness of the sets of new children for descendants of $x$ is guaranteed by the girth condition $g(G) \geq \text{diam}(S') + 1$.  

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{diagram.png}
\caption{A construction of a desired subtree $W$ in the proof of Lemma 9}
\end{figure}
The assumption that \( v \) is neither a leaf of \( T' \) nor a leaf of \( S' \) can be realized by preferentially selecting a vertex in \( V_{[B]}(T') \setminus (V_L(T') \cup V_L(S')) \) if \( V_{[B]}(T') \setminus (V_L(T') \cup V_L(S')) \neq \emptyset \). For \( v \in V_{[B]}(T') \setminus (V_L(T') \cup V_L(S')) \), we can check whether there exists a vertex \( w \in V(H) \) such that \( C(v) \subseteq N_G(w) \) in \( O(|E(G)|) \) time. If we find such a vertex, then we can immediately obtain a desired subtree \( W \). Otherwise, we apply the manipulations for constructing \( W \) in a breadth-first search order for \( S' \). When \( \text{diam}(S') \geq 4 \), the selection process for new children of a descendant of \( x \) with a correspondence to descendants of \( v \) can be done in \( O(|V(G)|^2) \) time. Thus the desired subtree \( W \) can be computed in \( O(|V(G)|^2) \) time. (If we prepare the adjacency matrix at the outset, then the construction of \( W \) itself can be executed in \( O(|E(G)|) \) time.)

The extension process from \( W \) to \( T'' \) can be done in \( O(|E(G)|) \) time. If \( V_{[B]}(T') \setminus (V_L(T') \cup V_L(S')) = \emptyset \), then we can directly obtain either \( W \) or \( T'' \). Hence, a desired subtree \( T'' \) with a vertex \( v \in V_{[B]}(T') \) can finally be found in \( O(|V(G)|^2) \) time. \( \Box \)

Lemma 9 is indeed stronger than Statement 1 under the assumption that \( g(G) \geq \text{diam}(T) - 1 \). Therefore, by Lemmas 5 and 9, we have the following.

**Theorem 12** For any tree \( T \) of order \( m \), every 2-connected graph \( G \) with \( \delta(G) \geq m + 2 \) and \( g(G) \geq \text{diam}(T) - 1 \) contains a subtree \( T' \cong T \) such that \( G - V(T') \) is 2-connected.

For any 2-connected graph \( G \), it holds that \( g(G) \geq 3 \). Thus, the following result by Lu and Zhang [7] is obtained as a corollary from Theorem 12.

**Corollary 4** [7] For any tree \( T \) of order \( m \) with \( \text{diam}(T) \leq 4 \), every 2-connected graph \( G \) with \( \delta(G) \geq m + 2 \) contains a subtree \( T' \cong T \) such that \( G - V(T') \) is 2-connected.

Besides, by combining Lemmas 7 and 9, we have the following.

**Theorem 13** For any tree \( T \) of order \( m \geq 2 \), every 2-connected graph \( G \) with \( \delta(G) \geq m + 1 \) and \( g(G) \geq \max\{\text{diam}(T) - 1, 4\} \) contains a subtree \( T' \cong T \) such that \( G - V(T') \) is 2-connected.

From Theorem 13, the following result for 2-connected graphs without a triangle is obtained as a corollary.

**Corollary 5** For any tree \( T \) of order \( m \geq 2 \) with \( \text{diam}(T) \leq 5 \), every 2-connected graph \( G \) with \( \delta(G) \geq m + 1 \) and \( g(G) \geq 4 \) contains a subtree \( T' \cong T \) such that \( G - V(T') \) is 2-connected.

Now, we are ready to show our main two results stated in the introduction. Let \( T \) be a tree of order \( m \). Suppose that \( G \) is a 2-connected graph with \( \delta(G) \geq m + 2 \) and \( g(G) \geq \delta(G) - 8 \). Then, \( g(G) \geq m - 6 \). From Corollary 2, it is sufficient to consider a tree \( T \) with \( \text{diam}(T) \leq m - 5 \). That is, we have \( g(G) \geq \text{diam}(T) - 1 \). Therefore, Theorem 5 follows from Corollary 2 and Theorem 12. Next, suppose that \( m \geq 10 \) and \( G \) is a 2-connected graph with \( \delta(G) \geq m + 1 \) and \( g(G) \geq \delta(G) - 7 \). Then, \( g(G) \geq m - 6 \geq 4 \). From Corollary 3, we may assume that \( \text{diam}(T) \leq m - 5 \). Thus, we have \( g(G) \geq \max\{\text{diam}(T) - 1, 4\} \). Hence, Theorem 6 follows from Corollary 3 and Theorem 13.

From Lemmas 5, 7, 8, and 9, we can see that a desired subtree \( T' \) in Theorems 5 and 6 can be found in \( O(|V(G)|^4) \) time.
6 Proofs of Theorems 7 and 8

In this section, we show that the lower bounds on $g(G)$ in Theorems 5 and 6 can be improved if a 2-connected graph $G$ satisfies a structural property on the smallest cycles.

We first prove the following lemma. Note that for any two cycles $C_1$ and $C_2$ of length $g(G)$, it holds that $|E(C_1) \cap E(C_2)| \leq \left\lfloor \frac{g(G)}{2} \right\rfloor$.

**Lemma 10** Let $T$ be a tree of order $m$. Let $G$ be a 2-connected graph with $\delta(G) \geq m+2$ and $g(G) \geq \text{diam}(T) - 2$ in which no six cycles of length $g(G)$ have a common path of length $\left\lfloor \frac{g(G)}{2} \right\rfloor - 1$ in $G$. For any subtree $T' \cong T$ in $G$ and a maximum block $B$ in $G - V(T')$, if $B \neq G - V(T')$ and $V[B](T') \neq \emptyset$, then there exist a vertex $v \in V[B](T')$ and a subtree $T'' \cong T$ in $G - V(B) \cup \{v\}$ such that $v$ and $T''$ can be found in $O(|V(G)|^2)$ time.

**Proof** We use the notations such as $T'$, $B$, $v$, $H$, $S'$, $C(u)$, $W$, and $x$ with the same meaning as in the proof of Lemma 9. It is sufficient to consider the case that $g(G) = \text{diam}(T) - 2 = \text{diam}(S')$. Thus, suppose that $\text{diam}(S') \geq 3$. By the discussion in the proof of Lemma 9, we suppose that $v$ is not a leaf of $S'$, i.e., $|C(v)| \geq 2$, and there is no vertex $w$ in $H$ such that $C(v) \subseteq N_{G}(w)$. For $u \in C(v)$, we denote by $S'_{u}$ the subtree rooted at $u$ in $S'$. Let $F'$ be a component of $H$ containing $x$. Note that $|N_{G}(F') \cap V(B)| \leq 1$. In the following discussion, w.l.o.g., we may assume that $N_{G}(F') \cap V(B) \neq \emptyset$. Then, let $N_{G}(F') \cap V(B) = \{v_{B}\}$ and $F' = (V(F') \cup \{v_{B}\})_{G}$.

Suppose that $v$ is a pseudo-leaf of $S'$ and $v'$ is the non-leaf vertex adjacent to $v$ in $S'$. If there exists a vertex $y$ in $H$ such that $v' \in N_{G}(y)$, then by letting the vertex $y$ correspond to $v$, we can easily obtain a desired subtree $W$. If there is no vertex in $H$ which is adjacent to $v'$, then $v' \in V[B](T')$. Thus, we may assume that if $\text{diam}(S') \geq 4$ then $v$ is not a pseudo-leaf of $S'$, and if $\text{diam}(S') = 3$ then the central vertices $v, v'$ are in $V[B](T')$ such that $\{v, v'\} \cap N_{G}(w) = \emptyset$ for any $w \in V(H)$. Suppose that $\text{diam}(S') = 3$. Let $xy \in E(F)$. By the minimum degree condition $\delta(G) \geq m+2$, we have $|N_{F'}(x)| \geq |C(v) \setminus N_{G}(x)| + 3$ and $|N_{F'}(y) \setminus \{x\}| \geq |C(v') \setminus N_{G}(y)| + 3$. Since no six triangles have a common edge, $|N_{F'}(x) \cap N_{F'}(y)| \leq 5$. Therefore, we can easily find $y \in C'(x) \subseteq N_{F}(x)$ and $C'(y) \subseteq N_{F}(y) \setminus \{x\}$ so that $C'(x) \cap C'(y) = \emptyset$, $|C'(x)| = |C(v) \setminus N_{G}(x)|$, and $|C'(y)| = |C(v') \setminus N_{G}(y)|$. Thus, a desired subtree $W$ can be constructed. In what follows, we suppose that $\text{diam}(S') \geq 4$.

Let $P(S')$ and $Q(S')$ be the set of peripheral vertices in $S'$ and the set of parents of a peripheral vertex in $S'$, respectively. Let $S'' = S' - P(S')$. Note that $\text{diam}(S'') = \text{diam}(S') - 2$, and $v \notin P(S') \cup Q(S')$ since any vertex in $P(S')$ is a leaf of $S'$ and any vertex in $Q(S')$ is a pseudo-leaf of $S'$. For the subtree $S''$ rooted at $v$, we apply the manipulations in the proof of Lemma 9. Let $W'$ be the subtree obtained after such manipulations and let $W'_F = (V(W') \cap V(F))_{G}$. Suppose that $\{z_1, z_2, \ldots, z_q\}$ is the set of vertices in $W'_F$ which are corresponding to vertices in $Q(S')$. Note that $q$ may be less than $|Q(S')|$. Let $\{u_1, u_2, \ldots, u_q\} \subseteq Q(S')$ such that $u_i$ is corresponding to $z_i$ for $1 \leq i \leq q$. For each $1 \leq i \leq q$, let $D(z_i) = N_{F}(z_i) \setminus \{p(z_i)\}$ where $p(z_i)$ is the parent of $z_i$ in $W'_F$ rooted at $x$. Also let $r_i = \{C(u_1) \setminus N_{G}(z_i)\}$ for each $1 \leq i \leq q$, where $C(u_i)$ is the set of children of $u_i$ in $S'$. Since $g(G) = \text{diam}(S'), V(W'_F) \cap D(z_i) = \emptyset$
for each $1 \leq i \leq q$ but it may happen that $D(z_i) \cap D(z_j) \neq \emptyset$ for $1 \leq i < j \leq q$. It follows from $\delta(G) \geq m + 2$ and $|C(v) \setminus N_G(z_i)| \geq 1$ that $|D(z_i)| \geq r_i + 1$ for each $i$.

Suppose that $|D(z_k)| = r_k + 1$ for some $k \in \{1, 2, \ldots, q\}$. Then $|C(v) \setminus N_G(z_k)| = 1$ and $z_k$ is adjacent to every vertex in $T'$ except for ones in $(C(v) \cup C(u_k)) \setminus N_G(z_k)$. Thus, we may assume that $V_{[B]}(S') \subseteq N_S'(C(v) \setminus N_G(z_k)) \cup N_S'(C(u_k) \setminus N_G(z_k))$, since otherwise there exists $v' \in V_{[B]}(S')$ such that $N_S'(v') \subseteq N_G(z_k)$ and we can obtain the desired result with respect to $v'$ instead of $v$. Note that $N_S'(v')$ corresponds to $C(v')$ if the root of $S'$ is $v'$. Let $C(v) \setminus N_G(z_k) = \{w_k\}$. Instead of $x$, we let $z_k$ correspond to $v$ and apply the manipulations in the proof of Lemma 9 for $S''$. Let $W''$ be the resultant subtree and let $W''' = (V(W'') \cap V(F))_{W''}$. If $w_k$ is either a leaf or a pseudo-leaf of $S'$, then we can immediately obtain a desired subtree $W$ in this setting. Otherwise, there is no pseudo-leaf adjacent to $v$ in $S'$ which corresponds to a vertex in the subtree $W''$. Thus, w.l.o.g., we may assume that $\text{diam}(S') \geq 5, u_k \notin C(v)$, and $u_k \neq w_k$. Since $g(G) = \text{diam}(S') \geq 5$, it does not hold that $|C(v) \cap N_G(z_k)| \geq 2$, which implies that $|C(v)| = 2$. Then let $C(v) = \{u, w_k\}$. Since $N_S'(w_k) \subseteq N_G(z_k)$, we may assume that $w_k \notin V_{[B]}(S')$. Since $N_S'(C(u_k) \setminus N_G(z_k)) = \{u_k\}$, if $u_k \notin V_{[B]}(S')$ then $V_{[B]}(S') \subseteq N_S'(w_k)$ which implies that $V_{[B]}(S') = \emptyset$. Suppose that $u_k \in V_{[B]}(S')$. Since $u_k$ is a pseudo-leaf of $S'$, by the previous discussion, we may assume that $p(u_k) \in V_{[B]}(S')$ where $p(u_k)$ is the parent of $u_k$ in $S'$. Here $p(u_k) \notin N_{S'}(C(u_k) \setminus N_G(z_k))$. Thus, $p(u_k) \in N_{S'}(w_k)$. This means that $w_k = p(p(u_k))$. Hence, it is concluded that $V_{[B]}(S'_u) = \emptyset$. Let $H' = (V(H) \cup V(S'_u))_G$. For every vertex $u' \in V(S'_u), |N_G(u') \cap V(B)| \leq 1$. Thus, it holds that $\delta(H') \geq 1 + |V(S'_u)|$. Since $w_k \notin V_{[B]}(S')$, $N_G(w_k) \cap V(H) \neq \emptyset$. Let $w_k' \in N_G(w_k) \cap V(H)$. Then, we can construct a subtree $U''_{H'}$ in $H'$ which is isomorphic to $S' \setminus V(S'_w)$ such that $w_k'$ corresponds to $v$ in an isomorphism from $V(S') \setminus V(S'_w)$ to $V(U''_{H'})$. Thus, $(E(T - V(S'_u) \cup \{v\}) \cup \{w_kw_k'\} \cup E(U''_{H'}))$ can be employed as a desired subtree $W$. Constructions of $U''_{H'}$ and $W$ are illustrated in Fig. 5. Consequently, we may assume that any vertex $z_i$ in $\{z_1, z_2, \ldots, z_q\}$ satisfies that $|D(z_i)| \geq r_i + 2$.

Let $D'(z_i) = N_F(z_i) \setminus \{p(z_i)\}$ for $1 \leq i \leq q$. Then, $|D'(z_i)| \geq r_i + 3$ for each $i$. Note that either $D'(z_i) = D(z_i)$ or $D'(z_i) = D(z_i) \cup \{v_B\}$. Define $I_G$ as
the (multi)graph with vertex set \{z_1, z_2, \ldots, z_q\} in which \(z_i\) and \(z_j\) are joined by \(|D'(z_i) \cap D'(z_j)|\) edges. Note that \(I_G\) may be a multigraph if \(\text{diam}(S') = 4.\) Since no six cycles of length \(g(G)\) have a common path of length \(\left\lfloor \frac{g(G)}{2} \right\rfloor - 1 = \left\lfloor \frac{\text{diam}(S')}{2} \right\rfloor\) in \(G,\) each vertex in \(I_G\) is incident to at most five edges, i.e., \(\Delta(I_G) \leq 5.\) Let \(D'' = \{v'' \in D'(z_i) \cap D'(z_j) \cap D'(z_k) \mid 1 \leq i_1 < i_2 < i_3 \leq q\}.\) Note that \(D''\) may be nonempty if \(\text{diam}(S')\) is even. By the definition of \(I_G,\) for each edge \(e\) of \(I_G,\) there exists a vertex corresponding to \(e.\) The correspondence is not bijective when \(D'' \neq \emptyset.\) If \(v'' \in D'',\) then \(v''\) generates a complete subgraph \(K''\) of order at least three in \(I_G,\) i.e., every edge in \(K''\) corresponds to \(v''.\) We modify the graph \(I_G\) as follows, and let \(J_G\) be the resultant (multi)graph.

1. Delete every edge in a complete subgraph generated by a vertex in \(D''.\)
2. Delete the edge generated by \(v_B\) if \(v_B\) is contained in exactly two sets \(D'(z_{i_1})\) and \(D'(z_{i_2}).\)

Note that in \(J_G,\) each edge \(z_i z_j\) bijectively corresponds to a vertex in \((D'(z_i) \cap D'(z_j)) \setminus (D'' \cup \{v_B\})\). The degree of each vertex in a complete subgraph generated by a vertex in \(D''\) in \(I_G\) decreases by at least two in \(J_G.\) Thus, for any \(z_i\) in such a complete subgraph in \(I_G,\) if \(2\ell \leq \deg_{I_G}(z_i) \leq 2\ell + 1\) then \(|D'(z_i) \setminus D''| \geq r_i + 1 + \ell\) for \(\ell = 0, 1, 2.\) Besides, for \(z_i\) such that \(v_B\) is contained in exactly two sets \(D'(z_i)\) and \(D'(z'_i),\) if \(2\ell - 1 \leq \deg_{I_G}(z_i) \leq 2\ell\) then \(|D'(z_i) \setminus (D'' \cup \{v_B\})| \geq r_i + \ell\) for \(\ell = 0, 1, 2.\) Note that if \(v_B\) is contained in exactly one set \(D'(z'_i),\) then \(|D(z'_i)| \geq r_i + 2\) and \(|D(D(z'))| \geq r_j + 3\) for any \(j \neq i.\) By Lemma 3, \(J_G\) has an orientation \(D_G\) such that \(|\deg_{D_G}^+(z) - \deg_{D_G}^-(z)| \leq 1\) for any \(z \in V(D_G)\) and if \(v_B\) is contained in exactly one set \(D'(z'_i)\) then \(\deg_{D_G}^-(z'_i) \leq 2.\) Note that if an orientation of \(J_G\) satisfying the first condition does not satisfy the second condition, the reverse orientation satisfies both the conditions since \(\Delta(J_G) \leq 5.\) For each arc from \(z_i\) to \(z_j\) in \(I_G,\) we select the vertex in \(D(z_i) \cap D(z_j)\) corresponding to \(z_i\) as a child of \(z_j\) in \(D_G.\) Thus, for each \(i, \deg_{D_G}(z_i)\) vertices in \(D(z_i)\) are not selected as children of \(z_i.\) Note that any vertex in \(D'' \cup \{v_B\}\) is not selected as a child of any \(z_i.\) An example of the selection process is illustrated in Fig. 6. By adding vertices in \(D(z_i)\) which are not in \(D(z_j)\) for any \(j \neq i\) as children of \(z_i,\) we can disjointly obtain \(r_i\) children of \(z_i\) for \(1 \leq i \leq q.\) This process works well from the conditions on \(\deg_{I_G}(z_i), \deg_{D_G}^+(z_i), \deg_{D_G}^-(z_i),\) and \(|D'(z'_i) \setminus (D'' \cup \{v_B\})|\) for \(1 \leq i \leq q.\) Consequently, we can appropriately extend \(W'\) for a desired subtree \(W\) and finally obtain a subtree \(T'' \cong T\) in \(G - V(B) \cup \{v\}.\)

The central vertex or the two central vertices of \(S'\) can be computed in \(O(|V(S')|)\) time by iteratively deleting leaves. The sets \(P(S')\) and \(Q(S')\) can be found in \(O(|V(S')|)\) time by a breadth-first search from a central vertex of \(S'.\) Manipulations for obtaining a desired subtree \(T''\) except for those concerning \(I_G, J_G,\) and \(D_G\) can be done in \(O(|V(G)|^2)\), since we apply either manipulations similar to those of the proof of Lemma 9 or manipulations which can be done in \(O(|E(G)|)\) time, within a constant number of times. The graphs \(I_G\) and \(J_G\) can be constructed in \(O(|V(G)|^2)\) time. We can obtain \(D_G\) from \(J_G\) by applying Lemma 3 in \(O(|E(J_G)|)\) time. Since \(|E(J_G)| \leq \frac{3}{2}|V(I_G)| \leq \frac{3}{2}|V(G)|,\) we have the desired result in \(O(|V(G)|^2)\) time. □

Next, we consider the case \(\delta(G) \geq m + 1.\) In this case, we need to strengthen the structural condition on the smallest cycles in Lemma 10.
Lemma 11  Let $T$ be a tree of order $m$. Let $G$ be a 2-connected graph with $\delta(G) \geq m+1$ and $g(G) \geq \text{diam}(T) - 2$ in which no four cycles of length $g(G)$ have a common path of length $\left\lceil \frac{g(G)}{2} \right\rceil - 1$ in $G$. For any subtree $T' \cong T$ in $G$ and a maximum block $B$ in $G - V(T')$, if $B \neq G - V(T')$ and $V[B](T') \neq \emptyset$, then there exist a vertex $v \in V[B](T')$ and a subtree $T'' \cong T$ in $G - V(B) \cup \{v\}$ such that $v$ and $T''$ can be found in $O(|V(G)|^2)$ time.

**Proof** We use the notations in the proof of Lemma 10 with the same meaning and apply similar discussions under the condition $\delta(G) \geq m+1$.

Suppose that $\text{diam}(S') = 3$. Then $|N_{F'}(x)| \geq |C(v) \setminus N_G(x)| + 2$ and $|N_{F'}(y) \setminus \{x\}| \geq |C(v') \setminus N_G(y)| + 2$. Since no four triangles have a common edge, $|N_{F'}(x) \cap N_{F'}(y)| \leq 3$. Thus, we can easily find $y \in C'(x) \subset N_F(x)$ and $C'(y) \subset N_F(y) \setminus \{x\}$ so that $C'(x) \cap C'(y) = \emptyset$, $|C'(x)| = |C(v) \setminus N_G(x)|$, and $|C'(y)| = |C(v') \setminus N_G(y)|$.

Suppose that $\text{diam}(S') \geq 4$. We may assume that every vertex $z_i$ in $\{z_1, z_2, \ldots, z_q\}$ satisfies that $|D(z_i)| \geq r_i + 1$ and $|D'(z_i)| \geq r_i + 2$. Note that the degree condition $\delta(G) \geq m + 1$ implies that $\delta(H') \geq |V(S'_u)|$, where the lower bound on $\delta(H')$ is smaller than that in the proof of Lemma 10; however, the condition on $\delta(H')$ is sufficient for a construction of the subtree $U'_H$ in $H'$. The assumption that no four cycles of length $g(G)$ have a common path of length $\left\lceil \frac{g(G)}{2} \right\rceil - 1$ in $G$ implies that $\Delta(I_G) \leq 3$. By Lemma 3, $J_G$ has an orientation $D_G$ such that $|\text{deg}^+_G(z) - \text{deg}^-_G(z)| \leq 1$ for any $z \in V(D_G)$ and if $v_B$ is contained in exactly one set $D'(z_i)$ then $\text{deg}^-_G(z_i) \leq 1$. Based on $D_G$, we can disjointly obtain $r_i$ children of $z_i$ for $1 \leq i \leq q$ by the similar process as in the proof of Lemma 10. Hence, we can appropriately extend $W'_F$ to $W$ and finally obtain a desired subtree $T''$. Besides, $v$ and $T''$ can be found in $O(|V(G)|^2)$ time. \qed

From Lemmas 5, 7, 10, and 11, we have the following results.
Theorem 14  For any tree $T$ of order $m$, every 2-connected graph $G$ with $\delta(G) \geq m + 2$ and $g(G) \geq \text{diam}(T) - 2$ in which no six cycles of length $g(G)$ have a common path of length $\left\lceil \frac{g(G)}{2} \right\rceil - 1$ contains a subtree $T' \cong T$ such that $G - V(T')$ is 2-connected.

Theorem 15  For any tree $T$ of order $m \geq 2$, every 2-connected graph $G$ with $\delta(G) \geq m + 1$ and $g(G) \geq \max\{\text{diam}(T) - 2, 4\}$ in which no four cycles of length $g(G)$ have a common path of length $\left\lceil \frac{g(G)}{2} \right\rceil - 1$ contains a subtree $T' \cong T$ such that $G - V(T')$ is 2-connected.

Theorems 7 and 8 follow from Theorem 14 with Corollary 2 and Theorem 15 with Corollary 3, respectively. Manipulations in the proofs of Lemmas 10 and 11 can be done in $O(|V(G)|^2)$ time. Therefore, we can find a desired subtree $T'$ in Theorems 7 and 8 in $O(|V(G)|^3)$ time.

As special cases of Theorems 14 and 15, we have the following corollaries.

Corollary 6  For any tree $T$ of order $m$, every 2-connected graph $G$ with $\delta(G) \geq m + 2$ and $g(G) \geq \text{diam}(T) - 2$ in which any two smallest cycles are edge-disjoint contains a subtree $T' \cong T$ such that $G - V(T')$ is 2-connected.

Corollary 7  For any tree $T$ of order $m \geq 2$, every 2-connected graph $G$ with $\delta(G) \geq m + 1$ and $g(G) \geq \max\{\text{diam}(T) - 2, 4\}$ in which any two smallest cycles are edge-disjoint contains a subtree $T' \cong T$ such that $G - V(T')$ is 2-connected.

7 Proof of Theorem 9

In this section, we show that Theorem 5 can be generalized by using the overlapping girth of $G$ instead of the girth of $G$. Note that $g^\circ(G) \geq g(G)$ by definition, $g^\circ(G) \geq g(G) + 1$ if any two cycles of length $g(G)$ are edge-disjoint, and $g^\circ(G)$ may be much larger than $g(G)$.

We prove the following lemma which is a generalization for both Lemma 9 under the condition $\delta(G) \geq m + 2$ and the special case that any two cycles of length $g(G)$ are edge-disjoint in Lemma 10.

Lemma 12  Let $T$ be a tree of order $m$. Let $G$ be a 2-connected graph with $\delta(G) \geq m + 2$ and $g^\circ(G) \geq \text{diam}(T) - 1$. For any subtree $T' \cong T$ in $G$ and a maximum block $B$ in $G - V(T')$, if $B \neq G - V(T')$ and $V_{[B]}(T') \neq \emptyset$, then there exist a vertex $v \in V_{[B]}(T')$ and a subtree $T'' \cong T$ in $G - V(B) \cup \{v\}$ such that $v$ and $T''$ can be found in $O(|V(G)|^2)$ time.

Proof  We use the notations with the same meaning and apply similar manipulations as in the proof of Lemma 9 under the condition $\delta(G) \geq m + 2$. Note that $g^\circ(G) \geq \text{diam}(S') + 1$. If $\text{diam}(S') \leq 2$, then we can easily obtain a desired subtree $W$.

Suppose that $\text{diam}(S') \geq 3$. Consider a manipulation to a descendant $x'_i$ of $x$ for a construction of $W$ in the breadth-first search order for $S'$. Let $W_{H,i}$ be the subtree rooted at $x$ in $H$ which has been constructed just before the manipulation for $x'_i$. Suppose that we need to select $q'_i$ vertices as children of $x'_i$ in the subtree $W_{H,i}$. By
the condition $\delta(G) \geq m + 2$, we can select $q_i' + 1$ vertices $z_{i,1}, z_{i,2}, \ldots, z_{i,q_i'+1}$ as candidates for children of $x_i'$. Since we use the overlapping girth instead of the girth, there may be a case that $\{z_{i,1}, z_{i,2}, \ldots, z_{i,q_i'+1}\} \cap V(\mathcal{W}_H, i) \neq \emptyset$. However, in such a case, $|\{z_{i,1}, z_{i,2}, \ldots, z_{i,q_i'+1}\} \cap V(\mathcal{W}_H, i)| = 1$, since otherwise there are two cycles of length at most $\text{diam}(S')$ with a common edge $x_i' p(x_i')$ where $p(x_i')$ is the parent of $x_i'$ in $\mathcal{W}_H, i$, which contradicts the condition $g^o(G) \geq \text{diam}(S') + 1$. Therefore, we can correctly select $q_i'$ children of $x_i'$ and proceed the procedure to obtain a desired subtree $\mathcal{W}$. Hence, we have the desired result.

From Lemmas 5 and 12, we have the following from which Theorem 9 is obtained similarly to the discussion for Theorem 5 using Corollary 2 and Theorem 12.

**Theorem 16** For any tree $T$ of order $m$, every 2-connected graph $G$ with $\delta(G) \geq m + 2$ and $g(G) \geq \text{diam}(T) - 1$ contains a subtree $T' \cong T$ such that $G - V(T')$ is 2-connected.

## 8 Concluding Remarks

In this paper, we have proved that Mader’s conjecture for $k = 2$ holds for the class of bifurcate quasi-unimodal caterpillars which includes every caterpillar and every tree of order $m$ with diameter at least $m - 4$. We then have shown that Mader’s conjecture for $k = 2$ (with a weak degree condition $\delta(G) \geq m + 1$) is true for every 2-connected graphs with large girth. We have also shown that our constructive proofs lead to $O(|V(G)|^4)$ time algorithms.

Our lower bounds on the girth or the overlapping girth in Theorems 5, 7, and 9 can be improved if the lower bound of $m - 4$ on the diameter of a tree of order $m$ for which Mader’s conjecture for $k = 2$ holds is improved. For example, the following result follows from Theorem 12.

**Theorem 17** If Mader’s conjecture for $k = 2$ holds for any tree $T$ of order $m$ with $\text{diam}(T) \geq m - \ell$, then Mader’s conjecture for $k = 2$ holds for any 2-connected graph $G$ with $\text{girth}(G) \geq \delta(G) - \ell - 4$.

On the other hand, in order to improve the lower bounds on the girth or the overlapping girth in Theorems 12, 13, 14, 15, and 16 directly, we may need some other techniques.

Even though Mader’s conjecture for $k = 2$ still remains open, from Theorem 11 and Corollary 5, we may conjecture the following.

**Conjecture 4** For any tree $T$ of order $m \geq 2$, every 2-connected graph $G$ with $\delta(G) \geq m + 1$ and $\text{girth}(G) \geq 4$ contains a subtree $T' \cong T$ such that $G - V(T')$ is 2-connected.

Although we consider Mader’s conjecture only for $k = 2$, it would be interesting to approach Mader’s conjecture for general $k \geq 2$ by considering girth conditions.

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