Higher spin entanglement entropy at finite temperature with chemical potential

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Abstract

It is generally believed that the semiclassical AdS\textsubscript{3} higher spin gravity could be described by a two dimensional conformal field theory with $\mathcal{W}$-algebra symmetry in the large central charge limit. In this paper, we study the single interval entanglement entropy on the torus in the CFT with a $\mathcal{W}_3$ deformation. More generally we develop the monodromy analysis to compute the two-point function of the light operators under a thermal density matrix with a $\mathcal{W}_3$ chemical potential to the leading order. Holographically we compute the probe action of the Wilson line in the background of the spin-3 black hole with a chemical potential. We find exact agreement.

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1 Introduction

The AdS/CFT correspondence provides a new tool to study the entanglement entropy. It was proposed by Ryu and Takayanagi that the entanglement entropy in the conformal field theory (CFT) with a gravity dual can be evaluated by the area of a minimal surface in the bulk \(1\)

\[
S_{HEE} = \frac{A}{4\pi G_N}.
\]  

(1.1)

The Ryu-Takayanagi (RT) formula (1.1) defines the holographic entanglement entropy (HEE), which implies a deep and intriguing relation between the entanglement and the quantum gravity. The HEE could be understood as a generalized gravitational entropy \(2\), as suggested by the similarity of (1.1) with the Bekenstein-Hawking entropy of the black hole.

On the other hand, the holographic entanglement entropy opens a new window to study the AdS/CFT correspondence. Especially in the AdS\(_3\)/CFT\(_2\) correspondence, the semiclassical AdS\(_3\) gravity is dual to the large \(c\) limit of the two-dimensional conformal field theory. In this context, under reasonable assumptions the Ryu-Takayanagi formula has been derived in both the bulk \(3\) and the CFT \(4\) in AdS\(_3\)/CFT\(_2\). On the CFT side, the partition function of the \(n\)-sheeted Riemann surface could be simplified in the large \(c\) limit. In fact, under this limit, the conformal block of multi-point functions could be dominated by the vacuum block, which allows one to solve the conformal block in the leading order using the monodromy techniques. On the dual bulk side, loosely speaking, the classical handle-body solution ending on the \(n\)-sheeted Riemann surface could be constructed, and its on-shell action reproduces the leading order CFT partition function \(5\). Moreover, it has been shown that the 1-loop correction to the RT formula in the bulk is captured exactly by the next-leading order contribution in the CFT partition function \(6\)-\(14\). This is due to the fact that the 1-loop partition function of any handle-body configuration \(15\)-\(16\) could be reproduced by the CFT partition function \(17\).

Furthermore the study of the entanglement entropy sheds light on the correspondence between the higher spin (HS) gravity and the CFT with \(W\) symmetry. In the first order formulation of the AdS\(_3\) gravity, the theory could be rewritten in terms of the Chern-Simons (CS) theory with the gauge group \(SL(2, R) \times SL(2, R)\) \(18\). By generalizing the gauge group from \(SL(2, R)\) to \(SL(N, R)\), the higher spin theory up to spin \(N\) in AdS\(_3\) could be constructed in the Chern-Simons formulation. The construction could be extended to the full higher spin algebra \(hs[\lambda]\). More interestingly, by imposing the generalized Brown-Henneaux asymptotic boundary condition, the asymptotic symmetry group of the higher spin theory turns out to be generated by the \(W_N\) algebra. This suggests that the higher spin AdS\(_3\) gravity could be dual to a 2D CFT with \(W\)-algebra \(19\)-\(20\). One typical feature of the higher spin gravity is the loss of the diffeomorphism invariance. As a result, the usual geometrical notion like the horizon,
the singularity and the area make no much sense. As a result, the RT formula (1.1) may not be able to compute the HEE in a higher spin theory. More precisely, if one focus on the vacuum of the dual CFT, then the dual configuration is still gravitational and the higher spin fields appear only as the fluctuations around the classical configurations. In this case, one can still applies the RT formula and the higher spin fluctuations contribute only at the next-leading order \cite{9,10,13}. However, if one considers the highly excited states with $W$ charge, then the dual configuration could be a higher spin black hole. Now the RT formula does not apply and one has to find a new way to compute HEE.

One promising proposal for HEE in the higher spin $\text{AdS}_3$ gravity is to use the Wilson line \cite{21,22}. As the theory is defined in the framework of Chern-Simons theory, it is natural to consider the Wilson line operator defined in terms of the gauge potential. By considering the Wilson line which ends on the branch points of the interval, it was proposed that the probe action of the Wilson line captures the entanglement entropy. As a consistency check, the entropy of the HS black hole has been reproduced. Furthermore, it was shown in \cite{23} that the WL evaluated in a general asymptotically AdS background captures correctly the correlation function in the dual CFT. This puts the WL proposal on a firmer footing.

Let us review the work in \cite{23} in more details. On the dual field theory, the four-point correlation function, involving two heavy and two light operators, was considered. The heavy operator corresponds to the higher spin black hole or the conical defect with the higher spin hair. The light operator with conformal dimension $\Delta \ll c$ could be taken to be the twist operator in the $n \to 1$ limit. Therefore this four-point correlation function encodes the single-interval entanglement entropy of a highly excited states. From the operator product expansion (OPE), this four-point function can be decomposed into the contributions from the propagating states in different modules. The contribution from each module is called the conformal block. For a CFT with $W$ symmetry, the states in the theory are classified by the representations of $W$ algebra. Considering the large $c$ limit of the CFT with a sparse light spectrum, only vacuum conformal block dominates the four-point function. By studying the monodromy problem of a differential equation, the classical order of the conformal block can be computed. Remarkably, as shown in \cite{23} explicitly this $W_3$ vacuum block can be computed even more efficiently by the bulk WL in the $\text{AdS}_3$ background corresponding to the heavy operator.

However, there are two subtleties in the study in \cite{23}. First of all, the higher spin black hole solution usually contains two terms:

\begin{equation}
a = a_z dz + a_{\bar{z}} d\bar{z},
\end{equation}

for the holomorphic boundary condition \cite{35}, and

\begin{equation}
a = a_z dz + a_t dt,
\end{equation}

for the antiholomorphic boundary condition. \cite{50,51}
for the canonical boundary condition [24]. The first term $a_z$ contains the charge, while the second term $a_\bar{z}$ or $a_t$ contains the chemical potential [25]. In the higher spin black hole solution, the asymptotic condition is different from the one for pure AdS. The boundary condition corresponds to the higher spin deformation in the field theory. As shown in [24], there are two kinds of deformations to the CFT: the canonical deformation and the holomorphic deformation, corresponding to different asymptotic boundary conditions. For the holographic entanglement entropy we should evaluate the probe action of the WL in terms of the gauge potential, whose boundary condition should be in accord to the deformation in the field theory. However, in [23], the chemical potential has been turned off in the holographic calculation. In other words, only the gauge potential including only $a_z$ was discussed. Correspondingly, there is no deformation in CFT side such that the monodromy analysis is easy to do. Generically speaking, when there is a deformation in CFT, the entanglement entropy is hard to compute [26–28].

The second subtle point in [23] is that the higher spin black hole microstate was regarded to be created by a heavy operator in the CFT. On the other hand, it is quite often to use CFT at a finite temperature to represent a black hole. At the leading order, both pictures could be indistinguishable, but not at the quantum level [29]. It would be interesting to study the HEE in a higher spin black hole background in the finite temperature picture. This is the issue we want to address in this paper.

In this paper, we study the single-interval entanglement entropy at a finite temperature and with a higher spin chemical potential. We use a thermal density matrix with a finite chemical potential to describe a higher spin black hole. There is moreover a canonical deformation term in the Hamiltonian, corresponding to the canonical boundary condition [24] in the higher spin black hole solution. Our approach is different from the one in [26]. Instead of expanding the density matrix perturbatively in terms of the chemical potential, we treat the density matrix in a more exact way. This is feasible because we are only interested in the leading order result and we focus on the entanglement entropy rather than the Rényi entropy. Therefore we can use the saddle point approximation without worrying about the backreaction. In our case, the entanglement entropy is encoded in the two-point function of the twist operators under the density matrix. More generically we may consider the two-point function of two light primary operators with both conformal dimension and the spin-3 charge. Instead of studying the deformed theory directly, we take a picture-changing transformation and set the theory to a non-deformed theory. Under this picture transformation the two primary operators are transformed into two descendent operators, with the density matrix being invariant. As

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1 This defect is not important for the entanglement entropy. When we evaluate the entanglement entropy in canonical deformation the two twist operator are at the same time such that the chemical potential does not make effect. But for a general correlation function of two operators at different time the chemical potential make a difference. We will go back to this problem later.
the spatial direction is compact, the correlation function is defined on a torus, we need to study the conformal block on the torus. We may insert a complete state bases at the thermal cycle. Basically the 2-point correlation on the torus reduces to a sum of four-point functions which could be decomposed into the contributions from different propagating modules. We call the contribution from each module as a generalized conformal block. We can still use the monodromy analysis to study the leading order of generalized conformal block. Due to the presence of the chemical potential, we have one more differential equation, which correspond to evolving the operator by the higher spin charge. By solving the monodromy problem, we determine the leading-order correlation function of two general light operators under the thermal density matrix with the chemical potential.

Furthermore, we discuss the HEE by computing the probe action of the Wilson line in the background of the higher spin black hole with the chemical potential in the canonical boundary condition. We find complete agreement with field theory correlator. The agreement between the 2-point function on torus and its holographic computation via Wilson line not only holds for the twist operators, but also for more general operators at different time. On the bulk side, the picture-changing transformation could be understood as the gauge transformation between different boundary conditions.

The remaining of the paper is organized as follows. In Section 2, we review the computation of the Rényi entropy at finite temperature. In particular, we give a derivation of the differential equation proposed in [7] to study the conformal block on the torus. In Section 3, we study the two-point function of light operators on the torus with a chemical potential. We discuss the picture-changing transformation and introduce an auxiliary periodic coordinate with which the two-point function is defined with respect to a theory without deformation. We furthermore establish the differential equations of the wavefunction and the monodromy condition. In Section 4, we solve the monodromy problem to read the correlation function. In Section 5, we compute the correlation function holographically by using the Wilson line proposal. In Section 6, we ends with conclusion and discussion.

2 Rényi entropy at finite temperature

Entanglement entropy measures the entanglement between the subsystem and its environment [30]. Assuming the whole system can be described by a density matrix $\rho$, we can define a reduced density matrix for the sub-system $A$ by tracing out the degrees of freedom in its environment $A^C$

$$\rho_A = \text{tr}_{A^C} \rho. \tag{2.1}$$
The entanglement entropy of subsystem $A$ is defined to be the Von Neumann entropy of the reduced density matrix

$$S_{EE}(A) = -\log \rho_A \log \rho_A.$$ (2.2)

Moreover we can define the Rényi entanglement entropy

$$S_n(A) = -\frac{1}{n-1} \log \text{tr} \rho_A^n,$$ (2.3)

which allows us to read the entanglement entropy

$$S_{EE} = \lim_{n \to 1} S_n,$$ (2.4)

if $n$ can be analytic extended to non-integer and the limit $n \to 1$ can be well taken.

The entanglement entropy and Rényi entropy can be computed by using the replica trick \[31\]. The $n$-th Rényi entropy is given by

$$S_n = -\frac{1}{n-1} \log \text{tr} \rho_A^n = -\frac{1}{n-1} \log \frac{Z_n}{Z_1^n},$$ (2.5)

where $Z_n$ is the partition function on an $n$-sheeted space-time connected with each other at the boundary of sub-region $A$. In the path integral formalism, the partition function $Z_n$ can be taken in another way: the field theory on the $n$-sheeted space-time is replaced by $n$-copies of the original theory on one-sheet spacetime with appropriate twisted boundary condition on the fields at the entangling surface. The entangling surface is at the boundary of sub-region $A$ at a fixed time, so is a surface of co-dimension 2. Circling around the entangling surface, the $i$-th copy of the field is connected with the $(i+1)$-th one. Specifically, in two dimensional case, the entangling surface shrinks to some branch points, and the boundary condition on the fields at the branch points requires the introduction of the twist fields. The partition function can be computed by inserting the twist and anti-twist operators at the branch points in a orbifold CFT\[2\]

$$\frac{Z_n}{Z_1^n} = \langle T(z_1, \bar{z}_1)T(z_2, \bar{z}_2)\ldots T(z_{2N}, \bar{z}_{2N})\rangle,$$ (2.6)

where $N$ is the number of the intervals. In Eq. (2.6), the correlation can be defined not only at the zero temperature but also at a finite temperature and even with a chemical potential as well.

In our case, we consider a two-point function on a torus, even with a higher spin current deformation. The correlation function can be decomposed into the generalised conformal block, whose leading order could be computed by using the monodromy analysis. Before going into the details, we would like to give some general comments on the accuracy of the

\[2\]The twist operator description can also be extended to higher dimensions, see \[32\].
calculation. For a multi-point correlation function, we can use the operator product expansion (OPE) recursively and the correlation function can be decomposed into the contributions from different propagating modules. If the correlation function is defined on a higher genus Riemann surface, we need to cut the Riemann surface open at some cycles and inserting the states in different modules. We may just consider one module propagating in each OPE channel or one module at each cycle. This allows us to define generalized conformal block, or conformal block on the torus. The multi-point function is a summation of the generalised conformal blocks. We have different ways to take the operator product expansion and cutting the cycles. For a theory with crossing symmetry and modular invariance, all of these expansions equal to each other, but with different convergent rates. It is believed that for a large $c$ theory, there is one kind of expansion in which the contribution from the vacuum module dominates and the contribution from other modules is non-perturbatively suppressed in the large $c$ limit. Therefore even if we do not know the exact construction of the CFT dual to the AdS$_3$ gravity, we can still compute the correlation function reliably from the vacuum block as long as we find the proper channel. For different locations of the inserted operators and different Riemann surfaces, we may need to use different channels to expand the correlation function such that the vacuum module dominates in the expansion. There could be a phases transition at the parameter space when the expansion channel change. This effect is already known for Hawking-Page transition and in holographic entanglement entropy calculation [4,12].

In the next subsections, we first review the conformal block of the four-point function on a complex plane as in [23]. Then we turn to the finite temperature case and show how to derive the partition function on a torus. Our discussion clarifies the proposal in [7].

2.1 Conformal block from the monodromy

Let us first consider the four-point function in full complex plane to show the general idea of the conformal block. For simplicity, we assume the symmetry of the theory is only generated by the Virasoro algebra. Our discussion follows [4,33]. For the higher spin case, see Ref. [24].

For a general four-point function, we can evaluate it by inserting an identity operator

$$\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\phi_4(z_4)\rangle = \sum_{\chi} \langle \phi_1(z_1)\phi_2(z_2) | \chi \rangle \langle \chi | \phi_3(z_3)\phi_4(z_4) \rangle,$$

(2.7)

where the states $\chi$ are normalized and orthogonal to each other. For a conformal field theory, the states can be classified by the representations of the conformal symmetry. The four-point function can be written as

$$\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\phi_4(z_4)\rangle = \sum_{\alpha} \sum_{\chi_\alpha} \langle \phi_1(z_1)\phi_2(z_2) | \chi_\alpha \rangle \langle \chi_\alpha | \phi_3(z_3)\phi_4(z_4) \rangle = \sum_{\alpha} F_\alpha(x_i).$$

(2.8)
The function $F^\alpha(x_i)$ is called the conformal block, which is the conformal partial wave related to the representation $\alpha$. The semi-classical limit is defined by taking $\Delta_c \to \infty$ with the ratio $\Delta_c$ being fixed. It is believed that under this limit the conformal block can be approximated to be

$$F^\alpha(x_i) \approx e^{-\frac{x}{6}f(x_i)}, \quad (2.9)$$

where $f(x_i)$ depends only on $\frac{\Delta_c}{c}$.

To determine the function $f(x_i)$, the standard way is to solve the monodromy problem.

We first introduce a null state

$$|\xi\rangle = \left( L^2 - \frac{3}{2(2\Delta_\psi + 1)} L_{-1}^2 \right) |\hat{\psi}\rangle, \quad (2.10)$$

where

$$\Delta_\psi = \frac{1}{16} (5 - c + \sqrt{(c-1)(c-25)}). \quad (2.11)$$

In the large $c$ limit, $\Delta_\psi \to -\frac{1}{2} - \frac{9}{2c}$ and the null states goes to

$$|\xi\rangle = \left( L^2 - \frac{c}{6} L_{-1}^2 \right) |\hat{\psi}\rangle. \quad (2.12)$$

Inserting the null state into the correlation function and defining

$$\psi(z) = \sum_{\alpha} \langle \phi_1(z_1) \phi_2(z_2) | \chi_\alpha \rangle \langle \chi_\alpha | \hat{\psi}(z) \phi_3(z_3) \phi_4(z_4) \rangle,$$

$$T(z) = \sum_{\alpha} \langle \phi_1(z_1) \phi_2(z_2) | \chi_\alpha \rangle \langle \chi_\alpha | \hat{T}(z) \phi_3(z_3) \phi_4(z_4) \rangle,$$

we find that the decoupling of the null state leads to

$$\psi''(z) + \frac{6}{c} T(z) \psi(z) = 0, \quad (2.15)$$

in the large $c$ limit, where by using the Ward identity, the stress tensor is of the form

$$T(z) = \sum_i \frac{h_i}{(z - z_i)^2} + \frac{1}{z - z_i} \frac{\partial}{\partial z_i} \log F. \quad (2.16)$$

In Eq. (2.13), there is a term $\langle \chi_\alpha | \hat{\psi}(z) \phi_3(z_3) \phi_4(z_4) \rangle$. Because that $\hat{\psi}$ is a null state, it leads to a differential equation. Solving the differential equation, we can get a monodromy when $z$ moves around $z_3$ and $z_4$. As the monodromy is the same for all of the $\chi_\alpha$, this indicates that $\psi(z)$ in (2.13) has such a monodromy as well. With this monodromy condition and (2.15), we can solve all the coefficients in the stress tensor (2.16) and fix the conformal block up to a constant. For the entanglement entropy, we need to consider vacuum conformal block. In this case the monodromy around $z_3$ and $z_4$ is trivial.
2.2 Conformal block at finite temperature

In this subsection, we discuss the conformal block at a finite temperature. The thermal density matrix is

$$\rho_{\text{thermal}} = e^{2\pi i \tau \mathcal{L}_0 - 2\pi i \bar{\tau} \bar{\mathcal{L}}_0}, \quad (2.17)$$

where

$$\mathcal{L}_0 = -\frac{1}{2\pi} \int_0^{2\pi} \hat{T}(w) dw,$$

$$\bar{\mathcal{L}}_0 = -\frac{1}{2\pi} \int_0^{2\pi} \hat{\bar{T}}(\bar{w}) d\bar{w}. \quad (2.18)$$

Consider a multi-correlation function of the primary fields $\phi_j$ with the conformal dimension $h_j$ on a torus. The torus is characterized by the moduli $\tau$, and is doubly periodic

$$z \sim z + 2\pi, \quad z \sim z + 2\pi \tau. \quad (2.19)$$

By inserting a complete set of states we change the correlation function on the torus to a summation of the correlator on a cylinder

$$\langle \prod_j \phi_j(z_j) \rangle_{\tau} = \text{Tr} \left( e^{2\pi i \tau \mathcal{L}_0 - 2\pi i \bar{\tau} \bar{\mathcal{L}}_0} \prod_j \phi_j(z_j) \right) = \sum_{h,k,\bar{k}} \langle h,k,\bar{k} | \prod_j \phi_j(z_j) | h,k,\bar{k} \rangle e^{2\pi i (h+k-\frac{c}{24}) - 2\pi i (\bar{h}+\bar{k}-\frac{c}{24})}, \quad (2.20)$$

where the index $h$ denotes the different primary modules propagating on the torus and $k$ and $\bar{k}$ denote the descendants in that module. Under the conformal transformation the different modules do not mix with each other so we define a finite conformal block which only sum over the states in one module

$$\mathcal{F}(\tau, h; z_j, h_j; h_{p,r}) = \sum_k \langle h, k | \prod_j \phi_j(z_j) | h, k \rangle_{h_{p,r}} e^{2\pi i (h+k-\frac{c}{24})}, \quad (2.21)$$

where we only consider holomorphic part. It is a multi-point conformal block on the cylinder, with two descendants operators at the past infinity and the future infinity respectively, and $h_{p,r}$ denote the conformal dimension of the propagators.

Let us focus on the holomorphic sector and derive the Ward identity on the conformal block following the paper [34]. Consider the correlation function

$$\sum_k \langle h, k | \hat{T}(z) \prod_j \phi_j(z_j) | h, k \rangle_{h_{p,r}} e^{2\pi i (h+k-\frac{c}{24})}. \quad (2.22)$$
Even though we only sum over one module on the torus, the above function should be periodic in both direction. The periodic condition \( z \to z + 2\pi \) is trivial. For \( z \to z + 2\pi \tau \), we have

\[
\hat{T}(z) \sum_k |h, k\rangle \langle h, k| e^{2\pi i r (h + k - \frac{\tau}{2\pi})} = \sum_k |h, k\rangle \langle h, k| T(z + 2\pi \tau) e^{2\pi i r (h + k - \frac{\tau}{2\pi})}. \tag{2.23}
\]

Therefore we find that

\[
\sum_k \langle h, k| \hat{T}(z) \prod_j \phi_j(z_j) | h, k\rangle_{h,p, r} e^{2\pi i r (h + k - \frac{\tau}{2\pi})} \]

\[
= \sum_j \sum_n \frac{1}{4\sin^2 \frac{\pi}{2}(z - z_j + 2\pi n \tau)} \sum_k \langle h, k\rangle \langle h, k| \phi_j(z_j) \prod_j \phi_j(z) | h, k\rangle_{h, p, r} e^{2\pi i r (h + k - \frac{\tau}{2\pi})} \]

\[
+ \sum_j \sum_n \frac{1}{2} \cot \frac{1}{2}(z - z_j + 2\pi n \tau) \frac{\partial}{\partial z_j} \sum_k \langle h, k\rangle \prod_j \phi_j(z_j) | h, k\rangle_{h, p, r} e^{2\pi i r (h + k - \frac{\tau}{2\pi})} + f(\tau), \tag{2.24}
\]

where

\[
\sum_n \frac{1}{4\sin^2 \frac{\pi}{2}(z - z_j + 2\pi n \tau)} = \frac{1}{4\pi^2} \psi(\frac{z - z_j}{2\pi} | \tau) + \frac{1}{2\pi^2} \eta_1(\tau),
\]

\[
\sum_n \frac{1}{2} \cot \frac{1}{2}(z - z_j + 2\pi n \tau) = \frac{1}{2\pi} \zeta(z - z_j | \tau) - \frac{1}{2\pi^2} \eta_1(\tau)(z - z_j). \tag{2.25}
\]

To fix the \( f(\tau) \), we need to take an integral along the spatial direction. Considering that

\[
\int_0^{2\pi} dz \hat{T}(z) = -4\pi^2 L_0 + \frac{c}{6\pi^2}. \tag{2.26}
\]

we get

\[
f(\tau) = 2\pi i \frac{\partial}{\partial \tau} F(\tau). \tag{2.27}
\]

Similarly, by inserting the null state \( |\xi\rangle \) in the correlation function as in (2.10), we have a differential equation

\[
\left\{-\frac{3}{2(2\Delta + 1)} \frac{\partial}{\partial z^2} + \frac{2}{2\pi} \eta_1(\tau) \Delta + \sum_j \left( \frac{1}{4\pi^2} \psi(\frac{z - z_j}{2\pi} | \tau) + \frac{1}{2\pi^2} \eta_1(\tau) \right) h_j \right.
\]

\[
+ \sum_j \left( \frac{1}{2\pi} \zeta(\frac{z - z_j}{2\pi} | \tau) - \frac{1}{2\pi^2} \eta_1(\tau)(z - z_j) \right) \frac{\partial}{\partial z_j}
\]

\[
+ 2\pi \frac{\partial}{\partial \tau} \right) \sum_k \langle h, k| \psi(z) \prod_j \phi_j(z_j) | h, k\rangle_{h, p, r} e^{2\pi i r (h + k - \frac{\tau}{2\pi})} = 0, \tag{2.28}
\]

where \( \psi(z) \) is the vertex operator for the state \( |\psi\rangle \). Defining the function

\[
\Psi \equiv \frac{\sum_k \langle h, k| \psi(z) \prod_j \phi_j(z_j) | h, k\rangle_{h, p, r} e^{2\pi i r (h + k - \frac{\tau}{2\pi})}}{\sum_k \langle h, k| \prod_j \phi_j(z_j) | h, k\rangle_{h, p, r} e^{2\pi i r (h + k - \frac{\tau}{2\pi})}}. \tag{2.29}
\]
which is assumed to be order $c^0$, and taking the large $c$ limit, we have the equation

$$
\frac{-c}{6} \frac{\partial}{\partial z^2} \Psi + \sum_j \left( \frac{1}{4\pi^2} \phi \left( \frac{z - z_j}{2\pi} \right) + \frac{1}{2\pi^2} \eta_1(\tau) h_j \right) + \left( \frac{1}{2\pi^2} \eta_1(\tau) (z - z_j) \frac{\partial}{\partial z_j} \log F \right) \Psi 
+ 2\pi i \frac{\partial}{\partial \tau} \log F \Psi = 0
$$

(2.30)

This is the equation (14) given in [7]. Here we give a field theory derivation for that equation.

We need to fix the monodromy condition around the propagator with

$$M = \lim_{c \to \infty} \begin{pmatrix} e^{\pi i (1 + (1 - 24h_{p, r})^{\frac{1}{2}})} & 0 \\ 0 & e^{\pi i (1 - (1 - 24h_{p, r})^{\frac{1}{2}})} \end{pmatrix},$$

(2.31)

and the monodromy around the spatial cycle

$$M = \lim_{c \to \infty} \begin{pmatrix} e^{\pi i (1 - 24h)}^{\frac{1}{2}} & 0 \\ 0 & e^{-\pi i (1 - 24h)^{\frac{1}{2}}} \end{pmatrix}.$$

(2.32)

The matrix denote the transformation for two independent solutions of the equation (2.30) up to conjugate, when the argument moves around the two cycles.

### 3 Correlation function at finite temperature with higher spin deformation

We now turn to compute the entanglement entropy at a finite temperature with a finite chemical deformation. The entanglement entropy of a single interval could be read from the two-point function of two primary twist operators in this system. The spatial direction of the torus is $-L/2 \leq \sigma \leq L/2$. We define

$$
\mathcal{L}_0 = -\frac{1}{2\pi} \int_{-L/2}^{L/2} \hat{T}(z) d\sigma,
$$

$$
\mathcal{W}_0 = \frac{1}{2\pi} \int_{-L/2}^{L/2} \hat{W}(z) d\sigma,
$$

$$
\bar{\mathcal{L}}_0 = -\frac{1}{2\pi} \int_{-L/2}^{L/2} \hat{T}(\bar{z}) d\sigma,
$$

$$
\bar{\mathcal{W}}_0 = \frac{1}{2\pi} \int_{-L/2}^{L/2} \hat{W}(\bar{z}) d\sigma,
$$

(3.1)

where the integral is over the real axis $-L/2 \leq \sigma \leq L/2$, and the Hamiltonian for the non-deformed theory is

$$
H_0 = \mathcal{L}_0 + \bar{\mathcal{L}}_0,
$$

$$
P_0 = \mathcal{L}_0 - \bar{\mathcal{L}}_0.
$$

(3.2)
For a theory with a higher spin current deformation, the modified Hamiltonian is

\[ H = H_0 - \frac{2\pi i\alpha}{\beta} \mathcal{W}_0 + \frac{2\pi i\bar{\alpha}}{\beta} \bar{\mathcal{W}}_0. \] (3.3)

In a deformed theory the system is evolved by this Hamiltonian.

### 3.1 Picture-changing transformation

In the modified system, we can define the Euclidian version of the two-point function at a finite temperature with a non-zero potential \( \Phi \) conjugate to the momentum

\[
\frac{1}{Z} \text{Tr} \left( \rho \phi^r_1(t_E, 1, \sigma_1) \phi^r_2(t_E, 2, \sigma_2) \right),
\]

where

\[
\rho = e^{-\beta H + i\Phi P},
\]

\[
\phi^r(t_E, \sigma) = e^{iE_H \phi^r(0, \sigma)e^{-t_E H}} = e^{iE_H e^{-i\sigma P_0} \phi^r(0, 0)e^{i\sigma P_0} e^{-t_E H}}.
\] (3.5)

The superscript \( r \) denote that the operator is evolved by the modified Hamiltonian. In the finite temperature system, the operator is doubly periodic

\[
\phi^r(t_E + \beta, \sigma + \Phi) = \phi^r(t_E, \sigma + L) = \phi^r(t_E, \sigma).
\] (3.6)

For the discussion we can also define an operator evolved by the original Hamiltonian without deformation, as

\[
\phi(z, \bar{z}) = e^{-izL_0}e^{i\bar{z}\bar{L}_0}\phi(0, 0)e^{izL_0}e^{-i\bar{z}\bar{L}_0},
\] (3.7)

where we have introduced the complex coordinate

\[
z = \sigma + it_E, \quad \bar{z} = \sigma - it_E.
\] (3.8)

If we regard the \( \mathcal{W}_0 \mathcal{W}_0 \) term in Eq. (3.3) as an interaction, then the operator \( \phi^r \) in (3.5) is the operator in the Hamiltonian picture and the operator \( \phi \) in (3.7) is the operator in the interaction picture. We can recombine the chemical potential \( \Phi \) and the inverse temperature as a parameter in the complex coordinate

\[
2\pi \tau = \Phi + i\beta, \quad 2\pi \bar{\tau} = \Phi - i\beta.
\] (3.9)

Note that here we do not normalize the spatial direction so that the complex quantity \( \tau \) is not the moduli of the torus.

From (3.2) and (3.3), we can rewrite the density matrix as

\[
\rho = e^{2\pi i\tau L_0 + 2\pi i\alpha \mathcal{W}_0} e^{-2\pi i\bar{\tau}\bar{L}_0 - 2\pi i\bar{\alpha}\bar{\mathcal{W}}_0}.
\] (3.10)
The operators in the Hamiltonian picture and interaction picture are related to each other as

$$
\phi^r(\tau, \sigma) = e^{-\frac{2\pi i}{\beta} t_E \mathcal{V}_0} e^{\frac{2\pi i}{\beta} t_E \mathcal{V}_0} e^{-iz \mathcal{L}_0 e^{iz \mathcal{L}_0} \phi^r(0,0) e^{iz \mathcal{L}_0} e^{-\frac{2\pi i}{\beta} t_E \mathcal{V}_0}} e^{-\frac{2\pi i}{\beta} t_E \mathcal{V}_0} e^{\frac{2\pi i}{\beta} t_E \mathcal{V}_0} e^{-iz \mathcal{L}_0 e^{iz \mathcal{L}_0} \phi^r(z, \bar{z}) e^{\frac{2\pi i}{\beta} t_E \mathcal{V}_0} e^{-\frac{2\pi i}{\beta} t_E \mathcal{V}_0}},
$$

(3.11)

where we have used the relation

$$
[i \mathcal{L}_0, \phi(z, \bar{z})] = -\frac{\partial}{\partial z} \phi(z, \bar{z}),
$$

$$
[i \bar{\mathcal{L}}_0, \phi(z, \bar{z})] = -\frac{\partial}{\partial \bar{z}} \phi(z, \bar{z}).
$$

(3.12)

The relation (3.12) can be proved as follows. By the path integral we have

$$
[i \mathcal{L}_0, \phi(z)] = \left[-\frac{i}{2\pi} \int_{-\frac{L}{2}}^{\frac{L}{2}} T(z') dz', \phi(z)\right] = \left(\frac{i}{2\pi} \oint dz' \hat{T}(z') \phi(z, \bar{z})\right)
$$

$$
= \frac{i}{2\pi} \oint dz' \sum_{m=-\infty}^{\infty} \frac{\hat{L}_m}{(z' - z)^{m+2}} \phi(z)
$$

$$
= -(\hat{L}_1 \phi)(z) = -\hat{\partial} \phi(z).
$$

(3.13)

Thus, we see that the translation along \( z \) is induced by the conserved charge \( \mathcal{L}_0 \). Similarly we may consider the evolution with respect to the charge \( \mathcal{W}_0 \) as well

$$
[i \mathcal{W}_0, \phi(z)] = \left[-\frac{i}{2\pi} \int_{-\frac{L}{2}}^{\frac{L}{2}} \hat{W}(z') dz', \phi(z)\right] = \left(-\frac{i}{2\pi} \oint dz' \hat{W}(z') \phi(z, \bar{z})\right)
$$

$$
= -\frac{i}{2\pi} \oint dz' \sum_{m=-\infty}^{\infty} \frac{\hat{W}_m}{(z' - z)^{m+3}} \phi(z)
$$

$$
= (\hat{W}_{-2} \phi)(z).
$$

(3.14)

By introducing two other auxiliary coordinates \( y, \bar{y} \), we can define

$$
\phi(z, y; \bar{z}, \bar{y}) \equiv e^{-i\mathcal{W}_0 y} e^{i\mathcal{W}_0 \bar{y}} \phi(z, \bar{z}) e^{i\mathcal{W}_0 y} e^{i\mathcal{W}_0 \bar{y}},
$$

(3.15)

then we have

$$
\phi^r(t_E, \sigma) = \phi(z, y; \bar{z}, \bar{y})
$$

(3.16)

with

$$
y = \frac{2\pi \alpha}{\beta} t_E, \quad \bar{y} = \frac{2\pi \bar{\alpha}}{\beta} t_E.
$$

(3.17)

To compute the single-interval entanglement entropy, \( \phi_1 \) and \( \phi_2 \) are taken to be the twist and anti-twist operators at the branch points respectively. Both operators are primary. With (3.16), we can write (3.4) as

$$
\text{Tr} \left( \rho \phi(z_1, y_1; \bar{z}_1, \bar{y}_1) \phi(z_2, y_2; \bar{z}_2, \bar{y}_2) \right).
$$

(3.18)
The operator \( \phi(z_i, y_i; \bar{z}_i, \bar{y}_i) \) can be regarded to be a descendant operator inserted at \((z_i, \bar{z}_i)\), which is evolved by the non-deformed Hamiltonian \( \mathcal{H}_0 \).

Up to now, we have transformed the correlation function of two primary operators at a finite temperature in a deformed theory to the correlation function of two descendant operators in a non-deformed theory under a density matrix at a finite temperature and with a finite chemical potential. We can regard this transformation to be a picture-changing transformation in quantum theory. In Eq. (3.3), \(-\frac{2\pi i\alpha}{D} \mathcal{W}_0 + \frac{2\pi i\bar{\alpha}}{D} \bar{\mathcal{W}}_0\) can be regarded as an interaction term. In Eq. (3.15) \( \phi(z_i, y_i; \bar{z}_i, \bar{y}_i) \) on the left hand side can be regarded to be in the Heisenberg picture and its evolution is respect to the Hamiltonian with interaction, and the operator \( \phi(z, \bar{z}) \) on the the right hand side can be regarded to be in the interaction picture. In the Heisenberg picture we compute the correlation function of two primary operators in the deformed theory, while in the interaction picture we compute the correlation function of two descendant operators in non-deformed theory. In Section 5, we will show that the picture transformation here corresponds to the gauge transformation in the bulk theory. Different pictures here correspond to different boundary conditions in the bulk solutions.

With the relations (3.12) and (3.15) it is easy to prove
\[
e^{-iL_0z_1}e^{-i\mathcal{W}_0y_1}\phi(z,y)e^{iL_0z_1}e^{i\mathcal{W}_0y_1} = \phi(z + z_1, y + y_1).
\]

Furthermore using (3.6) or (3.10) and (3.18), we have
\[
\langle \phi(z + 2\pi i\tau, y + 2\pi i\alpha) \rangle|_{\tau, \alpha} = \langle \phi(z, y) \rangle|_{\tau, \alpha},
\]
which is a generalized version of cyclic boundary condition in the thermal direction. However one should be aware that this periodicity is only true for a complete theory. That means we need to sum over all contributions from different channels with proper combination between the holomorphic and anti-holomorphic part. If we consider only one conformal block, the periodicity may break down. For example in [3,7] the second order differential equation has been defined for the wavefunction of a multi-point function as Eq. (2.15) and Eq. (2.30). However, it was shown that the solution is not single-valued along the non-trivial cycle. Furthermore because the conformal block only contain holomorphic part, even in trivial cycle, the conformal block may have an extra phase, as shown in [7].

### 3.2 Monodromy problem

In this subsection, we will show how to expand the correlation function (3.18) in terms of the generalized conformal block and set up the monodromy condition to compute the generalized conformal block from propagating vacuum module states. In the semi-classical limit, we assume that the propagating vacuum module dominates the contribution. In a theory
with \( W_3 \) symmetry, the vacuum module include the excitations of Virasoro generators and \( W_3 \) generators acting on the vacuum.

By the path integral the function \( (3.18) \) can be normalized to be

\[
C_2 = \frac{\text{Tr} e^{2\pi i\sigma L_0 + 2\pi i\alpha W_0 - 2\pi i\bar{\sigma} L_0 - 2\pi i\bar{\alpha} W_0} \phi(z_1, y_1; \bar{z}_1, \bar{y}_1) \phi(z_2, y_2; \bar{z}_2, \bar{y}_2)}{\text{Tr} e^{2\pi i\sigma L_0 + 2\pi i\alpha W_0 - 2\pi i\bar{\sigma} L_0 - 2\pi i\bar{\alpha} W_0} \left( e^{2\pi i\alpha W_0 - 2\pi i\bar{\alpha} W_0} \right)^{2\pi \tau, 2\pi \bar{\tau}}}.
\]

(3.21)

The correlation functions in the second line are defined on a torus with \((L, 2\pi \tau)\) being its periods. The \( e^{2\pi i\alpha W_0 - 2\pi i\bar{\alpha} W_0} \) is a non-local operator inserting at a time slice. In this correlation function, the local operator can be continuously deformed in any contour away from the locations of the other operators, and the expectation value is continuously changed along this contour. When the contour crosses a non-local operator, the situation becomes subtle. In the case at hand, the non-local operators induce a jump of the operators. More precisely, we have the relation

\[
e^{2\pi i z_1 L_0} \phi(z, y) = \phi(z - z_1, y) e^{2\pi i y_1 L_0},
e^{2\pi i y_1 W_0} \phi(z, y) = \phi(z, y - y_1) e^{2\pi i y_1 W_0}.
\]

(3.22)

This means that crossing an operator \( e^{2\pi i z_1 L_0} \) has the effect of evolving \( -z_1 \) along the \( z \) direction and crossing an operator \( e^{2\pi i y_1 W_0} \) has the effect of evolving \( -y_1 \) in the \( y \) direction.

These equation can be written in the path integral formalism as

\[
\langle e^{2\pi i z_1 L_0} \phi_{\text{lower}}(z, y) \cdots \rangle = \langle \phi_{\text{upper}}(z - z_1, y) e^{2\pi i y_1 L_0} \cdots \rangle,
\]

\[
\langle e^{2\pi i y_1 W_0} \phi_{\text{lower}}(z, y) \cdots \rangle = \langle \phi_{\text{upper}}(z, y - y_1) e^{2\pi i y_1 W_0} \cdots \rangle.
\]

(3.23)

where the subscript ”lower” or ”upper” denotes the operator \( \phi \) is below or above the non-local operators. Because the operator \( W_0 \) commutes with the Hamiltonian, the inserted non-local operator \( \alpha W_0 \) can be moved to any imaginary time slice if the movement don’t touch other operators. Therefore, the correlation functions in the numerator and the denominator in Eq. (3.21) are represented respectively as in (1a) and (1b) in Fig.1.

As in previous section, both the numerator and denominator of (3.21) are the correlation functions on the torus. By inserting a complete basis in thermal cycle, they can be decomposed into contributions from the states in different modules. Furthermore for the two operators in the numerator we can take an OPE and the expansion can be decomposed into the contribution from different modules. For each choice of modules in the OPE and in thermal cycle, the contribution defines a generalised conformal block. The numerator and denominator can be written as a summation of generalised conformal blocks from different modules. In the
large central charge limit, we assume that the generalised conformal block from the vacuum module dominates the contribution and the ones with other modules are non-perturbatively suppressed. We note that the jump from the crossing a non-local operator (3.22, 3.23) remains in the conformal block.

To determine this conformal block, we may use the monodromy analysis as before. However, due to the presence of $W_3$, there is one more differential equation to consider. Introduce a primary state $|\hat{\psi}\rangle$ such that

\[ L_0 |\hat{\psi}\rangle = - |\psi\rangle, \]
\[ W_0 |\hat{\psi}\rangle = \frac{1}{3} |\psi\rangle. \] (3.24)

In its descendants there are null states at level 1, 2, 3. In the large $c$ limit, they are

\[ (L_{-1} + 2W_{-1}) |\hat{\psi}\rangle = 0, \]
\[ (L_{-1}^2 - L_{-2} + \frac{16}{c} L_{-2}) |\hat{\psi}\rangle = 0, \]
\[ (L_{-1}^3 + \frac{24}{c} L_{-2} L_{-1} + \frac{12}{c} L_{-3} + \frac{24}{c} W_{-3}) |\hat{\psi}\rangle = 0. \] (3.25)

Inserting the null states into the correlation function, we get three differential equations on the correlation function involving the operator $\hat{\psi}$ corresponding to the state $|\hat{\psi}\rangle$. In particular the third equation can be transformed to

\[ \psi'''(z, y) + \frac{24}{c} T(z, y) \psi'(z, y) + \frac{12}{c} T'(z, y) \psi(z, y) + \frac{24}{c} W(z, y) \psi(z, y) = 0, \] (3.26)

where the prime denotes the derivative with respect to $z$ and

\[ \psi(z, y) = \frac{\langle e^{2\pi i\alpha W_0 - 2\pi i\bar{\alpha} W_0} \hat{\psi}(z, y) \rangle}{\langle e^{2\pi i\alpha W_0 - 2\pi i\bar{\alpha} W_0} \rangle}, \] (3.27)

and similarly

\[ T(z, y) = \frac{\langle e^{2\pi i\alpha W_0 - 2\pi i\bar{\alpha} W_0} \hat{T}(z, y) \rangle}{\langle e^{2\pi i\alpha W_0 - 2\pi i\bar{\alpha} W_0} \rangle}, \]
\[ W(z, y) = \frac{\langle e^{2\pi i\alpha W_0 - 2\pi i\bar{\alpha} W_0} \hat{W}_{-3}(z, y) \rangle}{\langle e^{2\pi i\alpha W_0 - 2\pi i\bar{\alpha} W_0} \rangle}. \] (3.28)
The ellipsis in (3.27) denotes other local operators at \((z_i, y_i)\). Unlike the Virasoro case without deformation, the functions \(T(z, y)\) and \(W(z, y)\) can not be determined simply by imposing the doubly periodic condition.

With the \(sl(3, R)\) algebra the equation (3.26) can be rewritten in a compact form

\[
\frac{\partial \Psi(z, y)}{\partial z} = a(z, y) \Psi(z, y),
\]

where

\[
a(z, y) = \left( L_1 + \frac{6}{c} T(z, y) L_{-1} - \frac{6}{c} W(z, y) W_{-2} \right) = \begin{pmatrix}
0 & -\frac{12}{c} T(z, y) & -\frac{24}{c} W(z, y) \\
1 & 0 & -\frac{12}{c} T(z, y) \\
0 & 1 & 0
\end{pmatrix},
\]

\[
\Psi(z, y) = \begin{pmatrix}
\psi''(z, y) + \frac{2}{c} T(z, y) \psi(z, y) \\
\psi'(z, y) \\
\psi(z, y)
\end{pmatrix}.
\]

From the definition of \(\hat{\psi}(z, y)\), we have

\[
\frac{\partial \hat{\psi}(y, z)}{\partial y} = e^{-iyW_0} [-iW_0, \hat{\psi}(z)] e^{iyW_0} = e^{-iyW_0} (-\hat{W}_{-2} \hat{\psi}(z)) e^{iyW_0},
\]

where \((\hat{W}_{-2}\psi)(z)\) is the corresponding vertex operator for the state \(\hat{W}_{-2} | \psi \rangle\). Using the relation (3.25), we have

\[
\frac{\partial \Psi(z, y)}{\partial y} = b(z, y) \Psi(z, y),
\]

where

\[
b = - \begin{pmatrix}
\frac{4}{c} T(z, y) & -\frac{4}{c} T'(z, y) - \frac{24}{c} W(z, y) & \frac{4}{c} T''(z, y) + \frac{144}{c} T(z, y)^2 \\
0 & -\frac{8}{c} T(z, y) & \frac{4}{c} T'(z, y) - \frac{8}{c} W(z, y) \\
1 & 0 & \frac{4}{c} T(z, y)
\end{pmatrix}.
\]

If the insertion of the operator \(\hat{\psi}\) is away from the position of the non-local operator, the equations (3.29) and (3.33) can be solved formally by introducing an evolution operator,

\[
\Psi(z, y) = U(z, y; z_0, y_0) \Psi(z_0, y_0),
\]

where

\[
U(z, y; z_0, y_0) = P \exp \left[ \int_{(z_0, y_0)}^{(z, y)} a(z, y) dz + b(z, y) dy \right],
\]

is a path-ordered integral on a contour in the two-dimensional complex plane. The consistency condition for the path-ordered integral is

\[
- \frac{\partial}{\partial y} a(z, y) + \frac{\partial}{\partial z} b(z, y) - [a(z, y), b(z, y)] = 0,
\]
or explicitly as
\[
\frac{\partial T}{\partial y} + 2 \frac{\partial W}{\partial z} = 0, \\
\frac{\partial^3 T}{\partial z^3} - 6 \frac{\partial W}{\partial y} + \frac{96}{c} T \frac{\partial T}{\partial z} = 0,
\]
which can be derived directly by \( \mathcal{W} \) algebra in the large \( c \) limit. In Eq. (3.35), \( z \) and \( y \) have to be regarded as two independent complex coordinates, representing the evolution by \( L_0 \) and \( \mathcal{W}_0 \) respectively.

Now let us discuss the monodromy condition. In the case at hand, there are two types of expansions: one of them is the operator product expansion of two operators, and the other one is for inserting a complete bases along the thermal circle. Correspondingly, we need to impose the monodromy condition on two circles, the thermal circle and the circle enclosing the two operators. At each of the circle there is a monodromy condition. Because we only keep the vacuum module in the OPE of two operators, the monodromy around the the circle enclosing the two operators is trivial. The monodromy along the thermal cycle is more subtle. By inserting a complete set of state basis the torus is cut open and becomes a cylinder, we can take a conformal transformation
\[
w = e^{\frac{2\pi}{\beta} z},
\]
which maps the cylinder to the full complex plane. The operators are related by
\[
\hat{\psi}(z) = \left( \frac{\partial w}{\partial z} \right)^{2h} \hat{\psi}(w) = \left( \frac{2\pi}{\beta} \right)^{2h} e^{\frac{2\pi}{\beta} h z} \hat{\psi}(w).
\]
The monodromy in the \( w \) coordinate is trivial. While in the \( z \) coordinate, there is an extra phase \( e^{\frac{2\pi}{\beta} h z} \) from the conformal transformation. When the conformal dimension of \( \hat{\psi} \) is a half-integer, the monodromy around the thermal circle is -1, as in [7]. When the conformal dimension of \( \hat{\psi} \) is an integer, as in our case, the monodromy around the thermal circle must be trivial
\[
\Psi(z + 2\pi \tau)_{\text{upper}} = \Psi(z)_{\text{lower}}.
\]
Considering the relation (3.23), we have the monodromy condition around the thermal cycle
\[
\Psi(z + 2\pi \tau, y + 2\pi \alpha)_{\text{lower}} = \Psi(z, y)_{\text{lower}}.
\]

### 4 Monodromy analysis

In this section, we use the monodromy condition to compute the correlation function (3.21) on the torus. Firstly we study the function in the denominator \( \langle e^{2\pi i \alpha \mathcal{W}_0 - 2\pi i \bar{\alpha} \bar{\mathcal{W}}_0} \rangle_{2\pi \tau} \), which is just the partition function of a higher spin black hole. We discuss the expectation values of
the stress tensor and the higher spin charge. Then we compute the two-point function in the numerator. In the discussion, we assume the conformal dimension of the operator is of order \( c \) but still light compared to the charge of the higher spin black hole so that we can ignore the backreaction to the background.

### 4.1 Thermodynamics of the ensemble

In this subsection, we show that the monodromy condition can determine the thermodynamics of the ensemble with the higher spin deformation. The thermodynamics of higher spin black hole was studied holographically in [35] and in [24] from the point of view of canonical deformation. A field theory derivation was presented in [36] by using the perturbation expansion. Here we give another field theory derivation for the thermodynamics. In our derivation the relation with the holographic study becomes more clear.

In this case, because of the translation invariance, the matrices \( a \) and \( b \) are constant-valued

\[
a_0 = \begin{pmatrix} 0 & -\frac{12}{c}\langle T \rangle_0 & -\frac{24}{c}\langle W \rangle_0 \\ 1 & 0 & -\frac{12}{c}\langle T \rangle_0 \\ 0 & 1 & 0 \end{pmatrix},
\]

\[
b_0 = -\begin{pmatrix} \frac{4}{c}\langle T \rangle_0 & -\frac{24}{c}\langle W \rangle_0 & \frac{144}{c^2}\langle T \rangle_0^2 \\ 0 & -\frac{8}{c}\langle T \rangle_0 & -\frac{24}{c^2}\langle W \rangle_0 \\ 1 & 0 & \frac{4}{c}\langle T \rangle_0 \end{pmatrix}.
\]

Here the subscript 0 denotes the expectation value with no operator insertion. Because the operators \( L \) and \( W \) commute, the matrices \( a_0 \) and \( b_0 \) also commute with each other. Then the monodromy condition (3.42) can be written as

\[
\exp[2\pi \alpha b_0 + 2\pi \tau a_0] = 1.
\]

This is exactly the monodromy condition suggested in [35]. Now we derive it from the field theory. With this monodromy condition we can easily solve the expectation value and derive the thermodynamics law as in [35]. Here we omit the details.

For our later study, we diagonalize the matrices \( a_0 \) and \( b_0 \) by

\[
a_0 = M \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} M^{-1},
\]

\[
b_0 = -M \begin{pmatrix} \frac{1}{3}(\lambda_2 + \lambda_3)^2 + \frac{2}{3}\lambda_2\lambda_3 & 0 & 0 \\ 0 & \frac{1}{3}(\lambda_1 + \lambda_3)^2 + \frac{2}{3}\lambda_1\lambda_3 & 0 \\ 0 & 0 & \frac{1}{3}(\lambda_1 + \lambda_2)^2 + \frac{2}{3}\lambda_1\lambda_2 \end{pmatrix} M^{-1},
\]

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where
\[
M = \begin{pmatrix}
\frac{3}{4} \lambda_1^2 & -\frac{1}{4} \lambda_2^2 & \frac{3}{4} \lambda_3^2 \\
\lambda_1 & \lambda_2 & \lambda_3 \\
1 & 1 & 1
\end{pmatrix}.
\] (4.6)

Here \(\lambda_1, \lambda_2, \lambda_3\) are three roots of the following cubic equation
\[
\lambda^3 + \frac{24}{c} \langle T \rangle_0 \lambda + \frac{24}{c} \langle W \rangle_0 = 0,
\] (4.7)
with \(\lambda_1 > \lambda_2 > \lambda_3\).

### 4.2 Two-point function

In this subsection, we evaluate the correlation function \((3.21)\) by imposing the monodromy condition. The conformal dimensions and the higher spin charges of two operators are respectively \((h_1, q_1)\) \((h_2, q_2)\). As we are interested in the entanglement entropy, we set
\[
h_1 = h_2, \\
q_1 = -q_2,
\] (4.8)

such that the operators can fuse to the vacuum module. We only consider light operators with \(1 << h, q << c\), so that we can use the saddle point approximation and ignore their back reaction to the background. When we calculate the entanglement entropy, in \(n \to 1\) limit, the twist operator satisfy the light operator condition.

Because \(h\) and \(q\) are much smaller than \(a_0\) and \(b_0\) which are the charges with no operator inserting, we can take a linear perturbation about the solution \(a_0, b_0\) as
\[
a = a_0 + a_1, \\
b = b_0 + b_1,
\] (4.9)

where
\[
a_1 = \begin{pmatrix}
0 & -\frac{12}{c} (T - \langle T \rangle_0) & -\frac{24}{c} (W - \langle W \rangle_0) \\
0 & 0 & -\frac{12}{c} (T - \langle T \rangle_0) \\
0 & 0 & 0
\end{pmatrix},
\]
\[
b_1 = -\begin{pmatrix}
\frac{4}{c} (T - \langle T \rangle_0) & -\frac{4}{c} T' - \frac{24}{c} (W - \langle W \rangle_0) & \frac{4}{c} T'' + \frac{288}{c^2} \langle T \rangle_0 (T - \langle T \rangle_0) \\
0 & -\frac{8}{c} (T - \langle T \rangle_0) & \frac{4}{c} T' - \frac{24}{c} (W - \langle W \rangle_0) \\
0 & 0 & \frac{4}{c} (T - \langle T \rangle_0)
\end{pmatrix}.
\]

Define
\[
U_0(z, y) = \exp[a_0(z - z_0) + b_0(y - y_0)],
\]
\[
U(z, y) = U_0(z, y) U_1(z, y).
\] (4.10)
The differential equations (3.29) and (3.33) can be rewritten as
\[
\frac{\partial}{\partial z} U_1(z, y) = U_0^{-1}(z, y) a_1(z, y) U_0(z, y), \\
\frac{\partial}{\partial y} U_1(z, y) = U_0^{-1}(z, y) b_1(z, y) U_0(z, y).
\] (4.11)

The equations can be solved by
\[
U_1(z, y) = P \exp \int_{(z_0, y_0)}^{(z, y)} [U_0^{-1}(z, y) a_1(z, y) U_0(z, y) dz + U_0^{-1}(z, y) b_1(z, y) U_0(z, y) dy].
\] (4.12)

Because of the consistency relation (3.37) and (3.38) we can continuously deform the contour as long as the contour is away from the singular points. The singular points of \( a(z, y) \) and \( b(z, y) \) can only appear at the locations of the operators, \( z_1 \) and \( z_2 \). By OPE we also have
\[
T(z, y_1) \sim \frac{h_1}{(z - z_1)^2} + \frac{r_1}{z - z_1}, \\
T(z, y_2) \sim \frac{h_2}{(z - z_2)^2} + \frac{r_2}{z - z_2}, \\
W(z, y_1) \sim \frac{q_1}{(z - z_1)^3} + \frac{p_1}{(z - z_1)^2} + \frac{s_1}{z - z_1}, \\
W(z, y_2) \sim \frac{q_2}{(z - z_2)^3} + \frac{p_2}{(z - z_2)^2} + \frac{s_2}{z - z_2}.
\] (4.13)

To the linear order the monodromy condition can be written as
\[
\oint U_0^{-1}(z, y) a_1(z, y) U_0(z, y) dz + U_0^{-1}(z, y) b_1(z, y) U_0(z, y) dy = 0.
\] (4.14)

This condition should satisfy for both the contour around the two operators and the contour around the thermal circle. Now we choose a special contour as follows
\[
(z, y) = \begin{cases} 
(z_2 - \epsilon, y_2 + t_1(y_1 - y_2)) & 0 < t_1 < 1 \\
(z_2 - \epsilon + (z_1 - z_2 + 2\epsilon) t_2, y_1) & 0 < t_2 < 1 \\
(z_1 + \epsilon * e^{2\pi i t_3}, y_1) & 0 < t_3 < 1 \\
(z_1 + \epsilon + (z_2 - z_1 - 2\epsilon) t_4, y_1) & 0 < t_4 < 1 \\
(z_2 - \epsilon, y_1 + t_5(y_2 - y_1)) & 0 < t_5 < 1 \\
(z_2 - \epsilon * e^{2\pi i t_6}, y_2) & 0 < t_6 < 1.
\end{cases}
\] (4.15)

The integrals from \( t_1, t_5 \) and \( t_2, t_4 \) are canceled with each other. The integrals from \( t_3 \) and \( t_6 \) can be evaluated by using the residue theorem. Then the monodromy condition (4.14) leads to
\[
\sum_{i=1,2} \text{Res}_{z_i} (D(z, y_i)^{-1} M^{-1} a_0 M D(z, y_i)) = 0.
\] (4.16)
where
\[
D(z, y) = \text{Diag} \left( e^{\lambda_1 z - (\frac{1}{3} (\lambda_2 + \lambda_3)^2 + \frac{2}{3} \lambda_2 \lambda_3) y}, e^{\lambda_2 z - (\frac{1}{3} (\lambda_1 + \lambda_3)^2 + \frac{2}{3} \lambda_1 \lambda_3) y}, e^{\lambda_3 z - (\frac{1}{3} (\lambda_1 + \lambda_2)^2 + \frac{2}{3} \lambda_1 \lambda_2) y} \right).
\]

It is easy to solve these equations. The solution is
\[
r_1 = -r_2 = m_1 h_1 + m_2 q_1, \quad s_1 = -s_2 = n_1 h_1 + n_2 s_2, \quad (4.17)
\]
where
\[
m_1 = \frac{1}{2} (K_1 - K_2), \quad m_2 = \frac{3}{2} (K_1 + K_2), \quad n_1 = \frac{1}{2} (K_3 - K_4), \quad n_2 = \frac{3}{2} (K_3 + K_4)
\]
with
\[
K_1 = \frac{a_3 (\lambda_1 - \lambda_2) \lambda_3 + a_1 (\lambda_2 - \lambda_3) \lambda_1 + a_2 (\lambda_3 - \lambda_1) \lambda_2}{a_3 (\lambda_1 - \lambda_2) + a_1 (\lambda_2 - \lambda_3) + a_2 (\lambda_3 - \lambda_1)}
\]
\[
K_2 = \frac{a_2 a_3 (\lambda_2 - \lambda_3) \lambda_1 + a_1 a_2 (\lambda_1 - \lambda_2) \lambda_3 + a_3 a_1 (\lambda_3 - \lambda_1) \lambda_2}{a_2 a_3 (\lambda_2 - \lambda_3) + a_1 a_2 (\lambda_1 - \lambda_2) + a_3 a_1 (\lambda_3 - \lambda_1)}
\]
\[
K_3 = \frac{a_3 (\lambda_1 - \lambda_2) (\frac{2}{3} \lambda_3^2 - \frac{1}{3} \lambda_1^2 - \frac{1}{3} \lambda_2^2) + a_1 (\lambda_2 - \lambda_3) (\frac{2}{3} \lambda_1^2 - \frac{1}{3} \lambda_2^2 - \frac{1}{3} \lambda_3^2) + a_2 (\lambda_3 - \lambda_1) (\frac{2}{3} \lambda_2^2 - \frac{1}{3} \lambda_1^2 - \frac{1}{3} \lambda_3^2)}{a_3 (\lambda_1 - \lambda_2) + a_1 (\lambda_2 - \lambda_3) + a_2 (\lambda_3 - \lambda_1)}
\]
\[
K_4 = \frac{a_2 a_3 (\lambda_2 - \lambda_3) (\frac{2}{3} \lambda_3^2 - \frac{1}{3} \lambda_1^2 - \frac{1}{3} \lambda_2^2) + a_1 a_2 (\lambda_1 - \lambda_2) (\frac{2}{3} \lambda_1^2 - \frac{1}{3} \lambda_2^2 - \frac{1}{3} \lambda_3^2) + a_3 a_1 (\lambda_3 - \lambda_1) (\frac{2}{3} \lambda_2^2 - \frac{1}{3} \lambda_1^2 - \frac{1}{3} \lambda_3^2)}{a_2 a_3 (\lambda_2 - \lambda_3) + a_1 a_2 (\lambda_1 - \lambda_2) + a_3 a_1 (\lambda_3 - \lambda_1)}
\]
\[
a_1 = \exp(- (z_1 - z_2) \lambda_1 + (y_1 - y_2) (\frac{2}{3} \lambda_1^2 - \frac{1}{3} \lambda_2^2 - \frac{1}{3} \lambda_3^2))
\]
\[
a_2 = \exp(- (z_1 - z_2) \lambda_2 + (y_1 - y_2) (\frac{2}{3} \lambda_2^2 - \frac{1}{3} \lambda_1^2 - \frac{1}{3} \lambda_3^2))
\]
\[
a_3 = \exp(- (z_1 - z_2) \lambda_3 + (y_1 - y_2) (\frac{2}{3} \lambda_3^2 - \frac{1}{3} \lambda_1^2 - \frac{1}{3} \lambda_2^2)).
\]

Using Eq. (3.12) we find that the holomorphic part of the correlator (3.21) obeys the equation
\[
\frac{\partial}{\partial z_1} \log C_2(z_1, y_1; z_2, y_2) = r_1
\]
\[
\frac{\partial}{\partial y_1} \log C_2(z_1, y_1; z_2, y_2) = -s_1. \quad (4.18)
\]

Finally, we obtain
\[
\log C_2(z_1, y_1; z_2, y_2) = -\frac{1}{2} \log[a_3 (\lambda_1 - \lambda_2) + a_1 (\lambda_2 - \lambda_3) + a_2 (\lambda_3 - \lambda_1)] [a_1^{-1} (\lambda_2 - \lambda_2) + a_2^{-1} (\lambda_3 - \lambda_1) + a_3^{-1} (\lambda_1 - \lambda_2)] h_1
\]
\[
- \frac{3}{2} \log[a_3 (\lambda_1 - \lambda_2) + a_1 (\lambda_2 - \lambda_3) + a_2 (\lambda_3 - \lambda_1)] [a_1^{-1} (\lambda_2 - \lambda_2) + a_2^{-1} (\lambda_3 - \lambda_1) + a_3^{-1} (\lambda_1 - \lambda_2)]^{-1} q_1. \quad (4.19)
\]

We have similar result for the anti-holomorphic part.
5 Holographic computation

5.1 Wilson line probe action

The holographic computation of the correlation function \( C_2 \) is to use the Wilson line proposal. The action of the Wilson line probe should give the function \( C_2 \). The general framework for defining and computing the probe action can be found in [21].

To calculate the two-point function holographically we need the flat connection for the spin-3 black hole

\[
A = e^{-\rho L_0} (a + d) e^{\rho L_0}
\]

(5.1)

\[
\bar{A} = e^{\rho L_0} (a + d) e^{-\rho L_0}
\]

(5.2)

with

\[
\begin{align*}
    a & = (L_1 + \frac{6}{c} \langle T \rangle_0 L_{-1} - \frac{6}{c} \langle W \rangle_0 W_{-2}) dz \\
    & - \langle W_2 + \frac{12}{c} \langle T \rangle_0 W_0 + \frac{36}{c^2} \langle T \rangle_0^2 W_{-2} + \frac{12}{c} \langle W \rangle_0 L_{-1}) dy \\
    \bar{a} & = (L_{-1} + \frac{6}{c} \langle \bar{T} \rangle_0 L_1 - \frac{6}{c} \langle \bar{W} \rangle_0 W_2) d\bar{z} \\
    & - \langle W_{-2} + \frac{12}{c} \langle \bar{T} \rangle_0 W_0 + \frac{36}{c^2} \langle \bar{T} \rangle_0^2 W_2 + \frac{12}{c} \langle \bar{W} \rangle_0 L_1) d\bar{y},
\end{align*}
\]

(5.3)

where \( y = \frac{2\pi \alpha'}{\beta t E} \). The terms proportional to \( dy \) in \( a, \bar{a} \) show that we are actually considering the black hole solution with a chemical potential in the canonical boundary condition. Correspondingly, the dual CFT is canonically deformed by the spin-3 current. Here we use \( y \) instead of \( t \) just to show the relation with the field theory analysis more clearly.

To calculate the action of the Wilson line probe, we introduce

\[
L = e^{-\rho L_0} e^{-(a + d) y},
\]

(5.5)

\[
R = e^{\bar{a} + \bar{d} y} e^{-\rho L_0}
\]

(5.6)

such that

\[
A = L d L^{-1}, \quad \bar{A} = R^{-1} d R.
\]

(5.7)

Then the probe action is defined by the diagonalized matrix

\[
H \simeq (L_i L_f^{-1} R_f^{-1} R_i)
\]

\[
= \text{diag}(t_1 e^{4\rho_0}, t_2, \frac{1}{t_1 t_2} e^{-4\rho_0}),
\]

(5.8)

where the subscripts \( i, f \) denote the endpoints of the Wilson line at the boundary; \( t_1 e^{4\rho_0}, t_2, e^{-4\rho_0}/t_1 t_2 \) are the eigenvalues of the matrix \( L_i L_f^{-1} R_f^{-1} R_i \) and \( \rho_0 \) is the IR cut-off for the boundary. By
direct calculation we get

\[ t_1 = 4\left( e^{\lambda_1(z_f-z_i)} + \left( -\frac{2}{3} \lambda_2 \lambda_3 - \frac{1}{3} (\lambda_2 + \lambda_3)^2 \right) (y_f - y_i) \right) \frac{1}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} + e^{\lambda_2(z_f-z_i)} + \left( -\frac{2}{3} \lambda_1 \lambda_3 - \frac{1}{3} (\lambda_1 + \lambda_3)^2 \right) (y_f - y_i) \frac{1}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} \]

\[ + e^{\lambda_3(z_f-z_i)} + \left( -\frac{2}{3} \lambda_1 \lambda_2 - \frac{1}{3} (\lambda_1 + \lambda_2)^2 \right) (y_f - y_i) \frac{1}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \cdot \left[ e^{-\lambda_3(z_f-z_i)} + \left( -\frac{2}{3} \lambda_1 \lambda_3 + \frac{1}{3} (\lambda_1 + \lambda_3)^2 \right) (y_f - y_i) \frac{1}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \right]. \]

\[ t_1 t_2 = 4\left( e^{-\lambda_1(z_f-z_i)} + \left( -\frac{2}{3} \lambda_2 \lambda_3 + \frac{1}{3} (\lambda_2 + \lambda_3)^2 \right) (y_f - y_i) \right) \frac{1}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} + e^{-\lambda_2(z_f-z_i)} + \left( -\frac{2}{3} \lambda_1 \lambda_3 + \frac{1}{3} (\lambda_1 + \lambda_3)^2 \right) (y_f - y_i) \frac{1}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} \]

\[ + e^{-\lambda_3(z_f-z_i)} - \left( -\frac{2}{3} \lambda_1 \lambda_2 + \frac{1}{3} (\lambda_1 + \lambda_2)^2 \right) (y_f - y_i) \frac{1}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \cdot \left[ e^{\lambda_3(z_f-z_i)} + \left( -\frac{2}{3} \lambda_1 \lambda_3 - \frac{1}{3} (\lambda_1 + \lambda_3)^2 \right) (y_f - y_i) \frac{1}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \right]. \]

The action of the probe is given by

\[ I_{\text{probe}} = \text{Tr} \left( \log(H) \left( \frac{L_0}{2} + \frac{3q_1}{2} W_0 \right) \right). \tag{5.9} \]

It is straightforward to check that

\[ C_2 = e^{-I_{\text{probe}}}. \tag{5.10} \]

The agreement between the correlation function in the field theory and its holographic computation is remarkable. The correlation function of two light operators is defined on the torus, and there is no restriction on the locations of the operators. When one considers the entanglement entropy, the operators are set to the same time slice. In this case, the correlation function is independent of the chemical potential and reduces to the one found in \cite{23}. In other words, the holographic computation of the entanglement entropy is not sensitive to the choice of the chemical potential with or without the chemical potential. This is not the case if one considers more general two-point functions on the torus.

### 5.2 Holographic correspondence for picture transformation

On the field side, we can transform two primary operators in the deformed theory into two descendant operators in a non-deformed theory by the picture-changing transformation. In this subsection, we would like to discuss the holographic correspondence of the picture-changing transformation. We suggest that the picture-changing transformation in the field theory correspond to a time-dependent gauge transformation on the gauge potential in the bulk.
Let us focus on a simple case. We assume that the state in the field theory is translational invariant. In a canonical deformed theory its holographic dual is just like the higher spin black hole as in [24,36].

\[
A = e^{-\rho L_0 (a + d)} e^{\rho L_0},
\]

(5.11)

where

\[
a = (L_1 + \frac{6}{c} \langle T \rangle_0 L_{-1} - \frac{6}{c} \langle W \rangle_0 W_{-2}) dz
\]

\[
+ \mu (W_2 + \frac{12}{c} \langle T \rangle_0 W_0 + \frac{36}{c^2} W_{-2} + \frac{12}{c} \langle W \rangle_0 L_{-1}) dt.
\]

(5.12)

where \( \langle T \rangle_0 \) and \( \langle W \rangle_0 \) are constants. However, the state we consider here can be any translation invariant state, not necessarily the thermal state, so \( \langle T \rangle_0 \) and \( \langle W \rangle_0 \) can take any values.

In the field theory we have \( \mathcal{W} \) symmetry so that we can take a symmetry transformation on the state

\[
| \tilde{O} \rangle = e^{-i\lambda W_0} | O \rangle.
\]

(5.13)

In the gravity side, the transformation can be written as

\[
\tilde{a} = U^{-1} (a + d) U,
\]

(5.14)

where

\[
U = \exp \left\{ -\lambda (W_2 + \frac{12}{c} \langle T \rangle_0 W_0 + \frac{36}{c^2} \langle T \rangle_0 W_{-2} + \frac{12}{c} \langle W \rangle_0 L_{-1}) \right\},
\]

(5.15)

which is an asymptotic symmetry in the bulk. This asymptotic symmetry was derived in [19] for asymptotic AdS boundary condition and was extended to the canonical deformed boundary condition in [39]. In this simple case we can give the transformation explicitly. Furthermore, taking (5.12) into (5.14), we see that the gauge transformation keeps \( a \) invariant. This corresponds to the fact that the state \( | O \rangle \) is an eigenstate of the symmetry generator \( W_0 \).

In the above discussion, we take the gauge transformation parameter \( \lambda \) to be constant. We can furthermore extend it to be time dependent such that it correspond to a picture-changing transformation in the field theory.\(^3\) We take \( \lambda = ts \), then the states transform as

\[
| \tilde{O} \rangle = e^{-it\lambda W_0} | O \rangle,
\]

(5.16)

and the corresponding operators transform as

\[
\tilde{\phi}(t) = e^{-it\lambda W_0} \phi(t) e^{it\lambda W_0},
\]

(5.17)

\(^3\)In [24], they regard the source term as a gauge field, and the time dependent transformation as the gauge transformation.
which is exactly the picture-changing transformation. Taking the parameter $\lambda = ts$ into (5.14), we get

$$\tilde{a} = a - s(W_2 + \frac{12}{c} (T)_0 W_0 + \frac{36}{c^2} (T)_0^2 W_{-2} + \frac{12}{c} (W)_0 L_{-1})dt.$$  (5.18)

For different $s$, it defines a different picture in the field theory, and it corresponds to different asymptotic boundary condition. Specifically, when $s = \mu$, the gauge transformation cancels the chemical potential term in (5.12), and we get the gauge connection used in [23].

As we shown before, the probe action of the WL reproduces exactly the correlation function of two light operators located on the torus. One interesting question is if it is possible to read the correlation function from the gauge potential without chemical potential. The question is related to the holographic computation of two-point function of descendent operators. If these two operators are at the same slice, the direct computation of the probe action gives the correct answer. But if the operators are at different time slices, then one has to develop the WL proposal to address this issue.

6 Conclusion and Discussion

In this article, we studied the entanglement entropy on a torus in the large $c$ CFT with $W$ current deformation. More generally, we discussed the two-point function of the light operator with $h, q << c$ under a thermal density matrix with a chemical potential. Due to the presence of the deformation, the correlation function seems to be hard to compute. However, in the large $c$ limit, if we accept that the vacuum module dominates the propagation, the problem is still tractable. First of all, under the limit, the $W_0$ charge commutes with the Hamiltonian so that we may apply a picture-changing transformation to turn off the deformation. Moreover, just as the Hamiltonian induce the translation of the time, the spin-3 operator induce the translation along an auxiliary coordinate. This leads us to find another differential equation on the wavefunction of the two-point function such that the monodromy problem could be well-defined. By imposing the monodromy condition, the two-point function could be determined at the linear order. Holographically we computed the probe action of the Wilson line in the background of spin-3 black hole with the chemical potential, and we found perfect agreement with field theory result.

Our treatment in the field theory could be applied to other deformation, as long as they are conserved. In the large $c$ limit, the other higher spin currents could be studied straightforwardly. This may help us to understand the $W$ conformal block on the torus [37]. Our study keeps to the leading order of $\frac{1}{c}$ expansion. It would be interesting to discuss the next leading order effect, especially considering the fact that the finite size correction to the entanglement entropy appears only at the next-leading order [11][38]. On the bulk this corresponds to the
1-loop quantum correction to the holographic entanglement entropy.

Our study supports the Wilson line proposal to compute the holographic entanglement entropy. With canonical boundary condition, the probe action of the Wilson line computes the HEE even with non-zero chemical potential. This suggests that in the probe limit, the Wilson line proposal can be applied to more general bulk configuration [40, 41]. On the other hand, how to determine the 1-loop quantum correction in the Wilson line proposal is an interesting question [42].

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Appendix A \( \mathcal{W}_3 \) algebras

For completeness we list the \( \mathcal{W}_3 \) algebras in this section as

\[
\begin{align*}
T(z_1)T(z_2) & \sim \frac{c/2}{(z_1 - z_2)^4} + \frac{2T(z_2)}{(z_1 - z_2)^2} + \frac{\partial T(z_2)}{T(z_1 - z_2)} + \ldots, \\
T(z_1)W(z_2) & \sim \frac{3W(z_2)}{(z_1 - z_2)^2} + \frac{\partial W(z_2)}{(z_1 - z_2)} + \ldots, \\
\frac{1}{N_3} W(z_1)W(z_2) & \sim \frac{1}{(z_1 - z_2)^6} + \frac{6T(z_2)}{(z_1 - z_2)^4} + \frac{3}{c} \frac{\partial T(z_2)}{\partial z} + \frac{\partial^2 T(z_2)}{(z_1 - z_2)^2} + \frac{96}{c(5c + 22)} \frac{\Lambda(z_2)}{(z_1 - z_2)^2} \\
& \quad + \frac{1}{5c} T(z_2) z_1 - z_2 + \frac{48}{c(5c + 22)} \frac{\partial \Lambda(z_2)}{z_1 - z_2} + \ldots, 
\end{align*}
\]

(A.1)

where \( N_3 = -\frac{5c}{6}, \) and

\[
\Lambda(z) = T(z)^2 - \frac{3}{10} \partial^2 T(z). 
\]

(A.2)
From the OPE coefficient, we can derive the commutators

\[
\begin{align*}
[L_m, L_n] &= (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0}, \\
[L_m, W_n] &= (2m - n)W_{m+n}, \\
[W_m, W_n] &= -\frac{1}{12}(m - n)(2m^2 - mn + 2n^2 - 8)L_{m+n} - \frac{40}{5c + 22}(m - n)\Lambda_{m+n} \\
&\quad - \frac{5c}{6} \frac{1}{5!} m(m^2 - 1)(m^2 - 4).
\end{align*}
\]

(A.3)

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