SYMMETRIES AND REDUCTIONS ON THE NONCOMMUTATIVE KADOMTSEV-PETVIASHVILI AND GELFAND-DICKEY HIERARCHIES

CHUANZHONG LI

Department of Mathematics, Ningbo University, Ningbo, 315211 Zhejiang, P. R. China
Email:lichuanzhong@nbu.edu.cn

ABSTRACT. In this paper, we construct the additional flows of the noncommutative Kadomtsev-Petviashvili(KP) hierarchy and the additional symmetry flows constitute an infinite dimensional Lie algebra $W_{1+\infty}$. In addition, the generating function of the additional symmetries can also be proved to have a nice form in terms of wave functions and this generating symmetry is used to construct the noncommutative KP hierarchy with self-consistent sources and the constrained noncommutative KP hierarchy. The above results will be further generalized to the noncommutative Gelfand-Dickey hierarchies which contains many interesting noncommutative integrable systems such as the noncommutative KdV hierarchy and noncommutative Boussinesq hierarchy. Meanwhile, we construct two new noncommutative systems including odd noncommutative C type Gelfand-Dickey and even noncommutative C type Gelfand-Dickey hierarchies. Also using the symmetry, we can construct a new noncommutative Gelfand-Dickey hierarchy with self-consistent sources. Basing on the natural differential Lax operator of the noncommutative Gelfand-Dickey hierarchy, the string equations of the noncommutative Gelfand-Dickey hierarchy are also derived.

Mathematics Subject Classifications (2010): 37K05, 37K10, 35Q53. Keywords: noncommutative KP hierarchy, noncommutative Gelfand-Dickey hierarchy, noncommutative Gelfand-Dickey hierarchy with self-consistent sources, odd noncommutative C type Gelfand-Dickey hierarchy, even noncommutative C type Gelfand-Dickey hierarchy, additional symmetry, $W_{1+\infty}$ Lie algebra, String equation.

1. INTRODUCTION

The Kadomtsev-Petviashvili(KP) hierarchy([1],[2]) is one of the most important integrable hierarchy. It arises in many different fields of mathematics and physics such as enumerative algebraic geometry, topological field theory, string theory and so on. The KP hierarchy has the well-known Virasoro symmetry which was extensively studied in literature([2],[3]).
The noncommutative field theory is a fruitful subject in both mathematics and physics particularly in noncommutative integrable systems \[^{[4, 5, 6, 7, 8]}\]. The noncommutative theory gives rise to various new physical objects in quantum mechanics such as the canonical commutation relation \([q, p] = i\hbar\). As said in \[^{[9]}\], the noncommutative parameter is closely related to the existence of a background flux in the effective theory of D-branes. With the presence of background magnetic fields the noncommutative gauge theories were found to be equivalent to ordinary gauge theories and noncommutative solitons play important roles in the study of D-brane dynamics\[^{[10]}\].

Similar to the additional symmetry flows of the KP hierarchy which was given by Orlov and Shulman\[^{[3]}\], the additional symmetry flows can constitute a centerless \(W_{1+\infty}\) algebra. Therefore motivated by the results on the classical KP hierarchy\[^{[11, 12]}\], we will give generating functions of the additional symmetries of the noncommutative KP hierarchy in terms of wave functions which can lead to the noncommutative KP hierarchy with self-consistent sources and the constrained noncommutative KP hierarchy.

The Gelfand-Dickey hierarchy was introduced by I. M. Gelfand and L. A. Dickey \[^{[13]}\] and has attracted a lot of attention in the research of integrable systems. The Hamiltonian theory of Gelfand-Dickey hierarchy as developed in terms of Lax pair which can be seen from the Dicky’s book\[^{[2]}\] for detail. As we know, the string equation formally as \([P, Q] = 1\) which connects the Lax operator and Orlov-Shulman operator is very useful in application on the partition function of the string theory \[^{[14]}\]. The possible physical interest in a non-commutative generalization of Orlov’s work \[^{[15, 16]}\] might be an exciting and interesting subject and this becomes one important motivation for us to consider additional symmetries of noncommutative Gelfand-Dickey hierarchy of which the Bäcklund transformation was constructed in \[^{[17]}\].

The non-commutative KP and Gelfand-Dickey hierarchies have shown to be Hamiltonian systems for example in \[^{[4, 6, 18]}\]. Non-commutative Hamiltonian systems in finite dimension \[^{[19]}\] and infinite dimension \[^{[20]}\] have a nice algebraic formulation and can be described for any associative algebra. In this paper, we will restrict the associative algebra to the case under the Moyal product which will reviewed later in detail in the next section.

The organization of this paper is as follows. We firstly review the Lax equation of the noncommutative KP hierarchy in Section 2. In Section 3, under the basic Sato theory, we construct the additional symmetry of the noncommutative KP hierarchy and later the noncommutative KP hierarchy with self-consistent sources and the constrained noncommutative KP hierarchy will be constructed. In Section 4. We further do a reduction on the noncommutative KP hierarchy to get the noncommutative Gelfand-Dickey hierarchy and we also construct two new noncommutative systems including odd noncommutative C
type Gelfand-Dickey and even noncommutative C type Gelfand-Dickey hierar-
chies. In Section 5, we construct the additional symmetry of the noncommutative
Gelfand-Dickey hierarchy and meanwhile we define the noncommutative
Gelfand-Dickey hierarchy with self-consistent sources. The String equations of
the noncommutative Gelfand-Dickey hierarchy will be studied in Section 6.

2. NONCOMMUTATIVE KP HIERARCHY

The KP hierarchy is one of the most important topics in the area of classical
integrable systems. In the noncommutative system, Moyal produc-
t ⋆ is defined
by a skew symmetric matrix θuv as

\[
\exp \left( \frac{i}{2} \theta_{uv} \partial_u \partial_v \right) f(t) g(t) |_{t = \bar{t} = t} = f(t) g(t) + \frac{i}{2} \theta_{uv} \partial_u f(t) \partial_v g(t) + \theta(\partial^2)
\]

where θ(θ2) means the higher order terms. The matrix θ = (θuv) can contain
functions in the variables tu. We can get that [tu, tv]⋆ = tu ⋆ tv − tv ⋆ tu = iθuv, and when θuv → 0, the noncommutative system can be reduced to the
commutative ones. The noncommutative KP hierarchy is constructed by the
pseudo-differential operator \(L = \partial + u_2 \partial^{-1} + u_3 \partial^{-2} + \ldots\) like this:

\[
L_{tn} = [B_n, L]_* := B_n \ast L - L \ast B_n,
\]

where \(B_n = (L^n)_+\) and “+” means the projection on nonnegative powers of \(\partial\). In order to define the noncommutative KP hierarchy, we need a formal adjoint
operation \(\ast\) for an arbitrary pseudo-differential operator \(P = \sum p_i \ast \partial^i\), we
have \(P^* = \sum (-1)^i \partial^i \ast p_i\). Meanwhile, we have \(\partial^* = -\partial, (\partial^{-1})^* = -\partial^{-1}\), and
\((A \ast B)^* = B^* \ast A^*\) for two operators. The noncommutative KP hierarchy in
the Lax equation has the form

\[
\frac{\partial L}{\partial t_n} = [B_n, L]_*, \quad n = 1, 2, \ldots
\]

(2.2)

where \(u_i = u_i(t_1, t_2, t_3, \ldots)\).

From the first equation of the noncommutative KP hierarchy we have that
\(\partial_x = \partial_{t_1}\), hence a solution depends only on \(x + t_1; t_2; \ldots\). The combination
\(x + t_1\) will be relabeled by \(x\). This hierarchy contains the \((2 + 1)\) dimensional
noncommutative KP equation:

\[
v_{t_3} - \frac{1}{4} v_{xxx} - \frac{3}{4} v_x \ast v - \frac{3}{4} v \ast v_x - \frac{3}{4} \int v_{t_2} v dx + \frac{3}{4} \int v v_{t_2} dx = 0,
\]

(2.3)

where \(v = \frac{1}{2} u_2\).
Meanwhile, we can give the noncommutative KP hierarchy by the consistent conditions of the following set of linear partial differential equations

\[
L \ast \omega(t, \lambda) = \lambda \ast \omega(t, \lambda), \quad \frac{\partial \omega(t, \lambda)}{\partial t_n} = B_n \ast \omega(t, \lambda), \quad t = (t_1, t_2, \ldots). \quad (2.4)
\]

Here, \( \omega(t, \lambda) \) is defined as a wave function, and let \( \phi = 1 + \sum_{i=1}^{\infty} \omega_i \partial^{-i} \) be the wave operator of the noncommutative KP hierarchy. The Lax operator and the wave function can be represented as

\[
L = \phi \ast \partial \ast \phi^{-1}, \quad \omega(t, \lambda) = \phi(t) \ast e^{\xi(t, \lambda)},
\]

in which \( \xi(t, \lambda) = \lambda \ast t_1 + \lambda^2 \ast t_2 + \ldots \). The Lax equation is equivalent to the following Sato equation

\[
\frac{\partial \phi}{\partial t_n} = -L_n \ast \phi. \quad (2.6)
\]

3. ADDITIONAL SYMMETRY OF NONCOMMUTATIVE KP HIERARCHY

Firstly, we define the operator \( \Gamma \) and the Orlov-Shulman’s operator \( M \) as

\[
M = \phi \ast \Gamma \ast \phi^{-1}, \quad \Gamma = \sum_{i=0}^{\infty} i t_i \partial^i. \quad (3.1)
\]

Meanwhile, the Orlov-Shulman’s operator \( M \) satisfies the following identities

\[
[L, M]_\ast = 1, \quad \partial_n M = [B_n, M]_\ast, \quad M \ast \omega(t, z) = \partial_z \omega(t, z). \quad (3.2)
\]

Further, we can get

\[
\frac{\partial M}{\partial t_n} = [B_n, M], \quad \frac{\partial M L^l}{\partial t_n} = [B_n, M^l]. \quad (3.3)
\]

Moreover, to the wave function \( \omega(t, z) \), \((L, M)\) is anti-isomorphic to \((z, \partial_z)\) with \([z, \partial_z]_\ast = -1\). We get

\[
M^m \ast L^l \ast \omega(t, z) = z^l \ast (\partial_z^m \omega(t, z)), \quad L^l \ast M^m \ast \omega(t, z) = \partial_z^m (z^l \ast \omega(t, z)). \quad (3.4)
\]

Next, we should consider the adjoint wave function \( \omega^\ast \) and the adjoint Orlov-Shulman’s operator \( M^\ast \) that are useful for constructing the additional symmetry of the noncommutative KP hierarchy, we have

\[
\omega^\ast(t, z) = (\phi^\ast)^{-1} \ast e^{-\xi(t, z)}, \quad \xi(t, z) = \sum_{i=0}^{\infty} t_i \lambda_i. \quad (3.5)
\]

However, \( L^\ast \) and \( M^\ast \) satisfy \([L^\ast, M^\ast]_\ast = [M, L]_\ast = -1\). Furthermore, we have

\[
L^\ast \ast \omega^\ast = z \ast \omega^\ast, \partial_n \omega^\ast = -B_n \ast \omega^\ast. \quad (3.6)
\]
For $B^*_m \star \omega^*$, if the operator $A$ is a differential operator and has form $A := \sum_{n=0}^{\infty} \partial^n a_n$, then we define $A^* \star g(x) = \sum_{m=0}^{\infty} (-1)^m (\partial^m g(x)) \star a_m$. Next, we give the additional symmetries of the noncommutative KP hierarchy. Firstly, we introduce additional independent variables $t^*_m,l$ and define the action of the additional flows on the wave operator as

$$\frac{\partial \phi}{\partial t^*_m,l} = -(C_{m,l} - \star \phi), \quad (3.7)$$

in which

$$C_{m,l} = M^m \star L^l, \quad m, l \in \mathbb{Z}. \quad (3.8)$$

In addition, we give some useful identities in the following proposition.

**Proposition 3.1.** The following identities hold true

$$\frac{\partial L^n}{\partial t^*_m,l} = -[(C_{m,l})_-, L^n_\star], \quad \frac{\partial M^m}{\partial t^*_m,l} = -[(C_{m,l})_-, M^m_\star], \quad (3.9)$$

$$\frac{\partial C_{n,k}}{\partial t^*_m,l} = -[(C_{m,l})_-, C_{n,k}]_\star, \quad \frac{\partial C_{n,k}}{\partial t_n} = [B_n, C_{n,k}]_\star. \quad (3.10)$$

**Proof.** The proof is very classical. We only need to consider the corresponding dressing structures

$$L^n = \phi \star \partial^n \star \phi^{-1}, \quad M^m = \phi \star \Gamma^m \star \phi^{-1}, \quad (3.11)$$

and a direct calculation will lead to the results (3.9) using the additional Sato equation (3.7). Similarly we can also consider the dressing structure

$$C_{n,k} = \phi \star \Gamma^n \star \phi^k \star \phi^{-1}, \quad (3.12)$$

a direct calculation will lead to the results (3.10) using the additional Sato equation (3.7). We need to note that $\Gamma$ depends on $t_n$ and does not depend on $t^*_m,l$. \qed

**Proposition 3.2.** The additional flows $\partial^*_m,l$ commute with the flows $\partial_n$ of the noncommutative KP hierarchy, which can be shown as

$$[\partial^*_m,l, \partial_n]_\star = 0. \quad (3.13)$$

**Proof.** Let the additional flows $\partial^*_m,l$ and the flows $\partial_{2n+1}$ act on $\phi$, we can get

$$[\partial^*_m,l, \partial_n]_\star \phi = \partial^*_m,l (\partial_n \phi) - \partial_n (\partial^*_m,l \phi) = \partial^*_m,l (L^n_\star \star \phi) + \partial_n ((C_{m,l})_- \star \phi) = - (\partial^*_m,l L^n)_- \star \phi - (L^n)_- \star (\partial^*_m,l \phi) + (\partial_n C_{m,l})_- \star \phi + (C_{m,l})_- \star (\partial_n \phi) = ([C_{m,l}^-, L^n]_\star)_- \star \phi + (L^n)_- \star (C_{m,l})_- \star \phi$$
\[
+([L^n_+, C_{m,l}]_*) - \phi - (C_{m,l})_+ \ast (L^n)_- \ast \phi = ([C_{m,l}^-]_-, L^n_+) \ast \phi - ([C_{m,l}^-]_-, L^n) \ast \phi + [L^n_-, (C_{m,l})_-]_+ \ast \phi = ([C_{m,l}^-]_-, L^n) \ast \phi + [L^n_-, (C_{m,l})_-]_+ \ast \phi = 0.
\]

In the above proof, \(([L^n_+, (C_{m,l})_+])_+ = 0\) and \(([L^n_+, C_{m,l}]_+ = ([L^n_+, (C_{m,l})_-])_-\) have been used. \(\square\)

Therefore, the additional flows \(\partial_{\ast_{m,l}}\) are symmetries of the noncommutative KP hierarchy.

**Proposition 3.3.** The additional symmetry flows \(\partial_{\ast_{m,l}}\) of the noncommutative KP hierarchy form a centerless \(W_{1+\infty}\) algebra.

**Proof.** By a direct calculation, we can easily derive

\[
[C_{m,l}, C_{n,k}]_+ = \sum_{p,q} C_{nk,ml}^p C_{p,q},
\]

because the classical structure

\[
[z^l \partial_m, z^k \partial_n]_+ = \sum_{p,q} C_{nk,ml}^p z^q \partial_p.
\]

This further implies that

\[
([C_{m,l}, C_{n,k}]_+)_- = - \sum_{p,q} C_{nk,ml}^p (C_{p,q})_-,
\]

where the coefficient \(C_{nk,ml}^p\) is the standard coefficient of the \(W\) algebra \(3\).

Using eq. (3.16), we can get

\[
[\partial_{\ast_{m,l}}, \partial_{\ast_{n,k}}]_+ \phi = - \sum_{p,q} C_{nk,ml}^p (C_{p,q})_- \ast \phi = \sum_{p,q} C_{nk,ml}^p \partial_{\ast_{p,q}} \phi,
\]

which is equal to

\[
[\partial_{\ast_{m,l}}, \partial_{\ast_{n,k}}]_+ = \sum_{p,q} C_{nk,ml}^p \partial_{\ast_{p,q}}.
\]

\(\square\)

To have a better understanding of the additional symmetry flows in noncommutative case, we give a typical example of the noncommutative KP hierarchy. Let \(m = 1\), the corresponding flow on \(L\) is

\[
\frac{\partial L}{\partial t_{1,1}^\ast} = -([M \ast L]_-, L)_+ = L + ([M \ast L]_+, L)_+.
\]

6
Using eq.(3.1), \( M \star L \) is expressed by
\[
M \star L = \phi \star x \partial \star \phi^{-1} + \sum_i i \phi \star t_i \star \phi^{-1} \star \phi \partial^i \star \phi^{-1}.
\] (3.18)

Furthermore, using \( \partial x = x \partial + 1 \) and \( \partial^{-i} x = x \partial^{-i} - i \partial^{-i-1} \), we get
\[
(\phi \star x \partial \star \phi^{-1})_+ = x \partial + \omega_1 \star x - x \star \omega_1,
\] (3.19)
with the \( \phi^{-1} = 1 - \omega_1 \partial^{-1} + ... \) being used. Taking eq.(3.19) into eq.(3.18), we have
\[
(M \star L)_+ = x \partial + \omega_1 \star x - x \star \omega_1 + \sum_i i (\phi \star t_i \star \phi^{-1} \star L^i)_+.
\] (3.20)

Taking eq.(3.20) into eq.(3.17), we have
\[
\frac{\partial L}{\partial t_{1,1}} = L + [x \partial, L]_s + \left[ \sum_i i (\phi \star t_i \star \phi^{-1} \star L^i)_+, L \right]_s
+ [\omega_1 \star x, L]_s - [x \star \omega_1, L]_s.
\]

Next, we define one generating function of the additional symmetries of the noncommutative KP hierarchy. Firstly, we define a generating operator \( Y(\lambda, \mu) \) of the additional symmetries as
\[
Y(\lambda, \mu) = \sum_{m=0}^{\infty} (\mu - \lambda)^m m! \sum_{l=-\infty}^{\infty} \lambda^{-l-m-1}(C_{m,m+l})_-,
\]
which can be expressed by a simple form in the following proposition.

**Proposition 3.4.** The following identity holds true
\[
Y(\lambda, \mu) = \frac{1}{\lambda} \omega(t,-\lambda) \partial^{-1} \ast \omega^*(t,\mu).
\] (3.21)

**Proof.** Basing on eq.(3.7), and the above lemma, we get
\[
(M^m \star L^{m+l})_- = \sum_{i=1}^{\infty} \partial^{-i} \text{res}_z[z^{-1} \ast (\partial^{-1}(M^m \star W \partial^{m+l+1} \ast e^\xi)) \ast (W^{-1} \ast e^{-\xi})^*],
\]
\[
= \sum_{i=1}^{\infty} \text{res}_z[z^m \partial^{-i} (M^m \star \omega) \partial^{-1} \ast \omega^*(t,z)],
\]
\[
= \text{res}_z[z^m \ast (\partial_z^m \omega) \partial^{-1} \ast \omega^*(t,z)],
\]
with the help of the identity \( f \partial^{-1} = \partial^{-1} f + \partial^{-1} f_z \partial^{-1} \).

Thus, we have one generating function of the noncommutative KP hierarchy,
\[
Y(\lambda, \mu) = \sum_{m=0}^{\infty} (\mu - \lambda)^m m! \sum_{l=-\infty}^{\infty} \lambda^{-l-m-1} \ast \text{res}_z[z^{l+m} \ast (\partial_z^m \omega(t,z)) \partial^{-1} \omega^*(t,z)].
\]
\[ res_z \left[ \sum_{n=-\infty}^{+\infty} \frac{z^n}{\lambda^{n+1}} \star \omega(t, z + \mu - \lambda) \star \partial^{-1} \omega^*(t, z) \right] \]
\[ = \frac{1}{\lambda} [\omega(t, -\lambda) \partial^{-1} \star \omega^*(t, \mu)]. \]

3.1. **Noncommutative KP hierarchy with self-consistent sources.** The noncommutative KP hierarchy with self-consistent sources can be constructed by taking derivatives with respect to a new time variable \( y_n \)

\[ \quad L_{y_n} = [B_n + P(t)\partial^{-1} \star Q(t), L]_*, \quad \text{(3.22)} \]

where

\[ \partial_t P = B_n \star P, \quad \partial_t Q = -B^*_n \star Q. \quad \text{(3.23)} \]

Here \( P, Q \) take specific values of \( \omega(t, -\lambda), \omega^*(t, \mu) \) respectively. And the corresponding Sato equation becomes

\[ \phi_{y_n} = (-L^n - P(t)\partial^{-1} \star Q(t)) \star \phi. \quad \text{(3.24)} \]

3.2. **Constrained noncommutative KP hierarchy.** The constrained noncommutative KP hierarchy [21] is constructed by the following pseudo-differential Lax operator \( \bar{L} = \partial + q(t)\partial^{-1} \star r(t) \)

\[ \bar{L}_{t_n} = [\bar{B}_n, \bar{L}]_*, \quad \bar{B}_n = \bar{L}_n, \quad \text{(3.25)} \]

where

\[ \partial_t q = \bar{B}_n \star q, \quad \partial_t r = -\bar{B}^*_n \star r. \quad \text{(3.26)} \]

In order to get the explicit form of the flow equations, we need the operator \( \bar{B}_n \),

\[ \bar{B}_1 = \partial, \]
\[ \bar{B}_2 = \partial^2 + 2(q \star r), \]
\[ \bar{B}_3 = \partial^3 + 3q \star r \star \partial + 3q_x \star r, \]

\[ \ldots \quad \ldots \quad \ldots \]

Then we can get the first few flows of the noncommutative cKP hierarchy

\[ \begin{cases} q_{t_1} = q_x \\ r_{t_1} = r_x, \end{cases} \quad \text{(3.27)} \]

\[ \begin{cases} q_{t_2} = q_{xx} + 2(q \star r \star q) \\ r_{t_2} = -r_{xx} - 2(r \star q \star r), \end{cases} \quad \text{(3.28)} \]
\[ q_{t_3} = q_{xxx} + 3q_x \ast r \ast q + 3q \ast r \ast q_x \\
+ q \ast r_x \ast q + r \ast q_x \ast q \\
r_{t_3} = r_{xxx} + 3r_x \ast q \ast r + 3r \ast q \ast r_x, \quad \cdots \cdots \cdots \] (3.29)

4. Noncommutative Gelfand-Dickey hierarchies

The Gelfand-Dickey hierarchy is one of the most important topics in the area of classical integrable systems. The noncommutative Gelfand-Dickey hierarchy can be constructed by the pseudo-differential operator \( \mathcal{L} = \partial^N + u_2 \partial^{N-2} + u_3 \partial^{N-3} + \ldots u_0 \) like this:

\[ \mathcal{L}_{t_n} = [\mathcal{B}_n, \mathcal{L}] := \mathcal{B}_n \ast \mathcal{L} - \mathcal{L} \ast \mathcal{B}_n, \quad n \neq 0 \mod N, \quad (4.1) \]

where \( \mathcal{B}_n = (\mathcal{L}^\dagger)^+ \) and “+” means the projection on nonnegative powers of \( \partial \).

We can give the noncommutative Gelfand-Dickey hierarchies by the consistent conditions of the following set of linear partial differential equations

\[ \mathcal{L} \ast \hat{\omega}(t, \lambda) = \lambda \ast \hat{\omega}(t, \lambda), \quad \frac{\partial \hat{\omega}(t, \lambda)}{\partial t_n} = \mathcal{B}_n \ast \hat{\omega}(t, \lambda), \quad t = (t_1, t_2, \ldots, t_{iN-1}, t_{iN+1}, \ldots). \quad (4.2) \]

Here, \( W(t, \lambda) \) is defined as a wave operator, and let \( W = 1 + \sum_{i=1}^{\infty} \alpha_i \ast \partial^{-i} \) be the wave operator of the noncommutative Gelfand-Dickey hierarchies. The Lax operator and the wave function can be represented as

\[ \mathcal{L} = W \ast \partial \ast W^{-1}, \quad \hat{\omega}(t, \lambda) = W(t) \ast e^{\xi_\mathcal{L}(t, \lambda)}, \quad (4.3) \]

in which \( \xi_\mathcal{L}(t, \lambda) = \lambda \ast t_1 + \lambda^2 \ast t_2 + \ldots + \lambda^{iN-1} \ast t_{iN-1} + \lambda^{iN+1} \ast t_{iN+1} + \ldots. \)

The Lax equation is equivalent to the following Sato equation

\[ \frac{\partial W}{\partial t_n} = -\mathcal{L}^\dagger_n \ast W. \quad (4.4) \]

When \( N = 2 \), one can derive the noncommutative KdV hierarchy which contains the following noncommutative KdV equation

\[ u_{t_3} = \frac{1}{4} u_{xxx} + \frac{3}{4} (u_x \ast u + u \ast u_x), \quad (4.5) \]

and the following noncommutative fifth order KdV equation

\[ u_{t_5} = \frac{1}{16} u_{xxxxx} + \frac{5}{16} (u_{xxx} \ast u + u \ast u_{xxx}) + \frac{5}{8} (u_x \ast u_x + u \ast u \ast u). \quad (4.6) \]
When \( N = 3 \), one can derive the noncommutative Boussinesq hierarchy which contains the following noncommutative Boussinesq equation
\[
\frac{1}{3}u_{xxx} + (u \ast u)_{xx} + ([u, \partial^{-1}u_{t2}]_*)_x.
\] (4.7)

4.1. Odd noncommutative C type Gelfand-Dickey hierarchies. The Odd noncommutative C type Gelfand-Dickey hierarchy can be constructed by the differential operator
\[
\mathcal{L} = \partial^{2N+1} + (\partial^N u_1 \partial^{N-1} + \partial^{N-1} u_1 \partial^N) + (\partial^{N-1} u_2 \partial^{N-2} + \partial^{N-2} u_2 \partial^{N-1}) + ... + (\partial u_N + u_N \partial),
\] (4.8)
which satisfies
\[
\mathcal{L}^* = -\mathcal{L}.
\] (4.9)

The Lax equation of the Odd noncommutative C type Gelfand-Dickey hierarchy is as
\[
\mathcal{L}_n = [\mathcal{B}_n, \mathcal{L}]_* := \mathcal{B}_n \ast \mathcal{L} - \mathcal{L} \ast \mathcal{B}_n, \quad n \neq 0 \ mod \ 2N+1,
\] (4.10)
where \( \mathcal{B}_n = (\mathcal{L}^{2N+1+})_+ \) and “+” means the nonnegative projection on powers of \( \partial \). While, we can give the Odd noncommutative C type Gelfand-Dickey hierarchies by the consistent conditions of the following set of linear partial differential equations
\[
\mathcal{L} \ast \omega(t, \lambda) = \lambda \ast \omega(t, \lambda), \quad \frac{\partial \omega(t, \lambda)}{\partial t_n} = \mathcal{B}_n \ast \omega(t, \lambda),
\] (4.11)
where \( t = (t_1, t_2, ..., t_i, ...) \), \( i \neq 0 \ mod \ 2N+1 \). Here, \( W(t, \lambda) \) is defined as a wave function, and let \( W = 1 + \sum_{i=1}^{\infty} \alpha_i \ast \partial^{-i} \) be the wave operator of the noncommutative Gelfand-Dickey hierarchies. The Lax operator and the wave function can be represented as
\[
\mathcal{L} = W \ast \partial \ast W^{-1}, \quad \omega(t, \lambda) = W(t) \ast e^{\xi(t, \lambda)},
\] (4.12)
in which \( \xi(t, \lambda) = \lambda \ast t_1 + \lambda^2 \ast t_2 + ... + \lambda^{i(2N+1)-1} \ast t_{i(2N+1)-1} + \lambda^{i(2N+1)+1} \ast t_{i(2N+1)+1} + ... \). The Lax equation is equivalent to the following Sato equation
\[
\frac{\partial W}{\partial t_n} = -\mathcal{L}^\frac{n}{2} \ast W.
\] (4.13)
4.2. **Even noncommutative C type Gelfand-Dickey hierarchies.** The noncommutative Gelfand-Dickey hierarchy can be constructed by the differential operator

\[ \mathcal{L} = \partial^{2N} + \partial^{N-1}u_1\partial^{N-1} + \cdots + \partial u_{N-1}\partial + u_N, \]

which satisfies

\[ \mathcal{L}^* = \mathcal{L}. \quad (4.14) \]

The Lax equation of the Odd noncommutative C type Gelfand-Dickey hierarchy is as

\[ \mathcal{L}_{t_n} = [B_n, \mathcal{L}]_* := B_n \star \mathcal{L} - \mathcal{L} \star B_n, \quad n \neq 0 \mod 2N, \quad (4.15) \]

where \( B_n = (\mathcal{L} \hat{\pi}_n)_+ \) and “+” means the nonnegative projection on powers of \( \partial \).

5. **Additional symmetry of noncommutative Gelfand-Dickey hierarchies**

Firstly, we define the operator \( \Gamma_g \) and the Orlov-Shulman’s operator \( \mathcal{M} \) as

\[ \mathcal{M} = W \star \Gamma_g \star W^{-1}, \quad \Gamma_g = \sum_{j=0}^{N-1} \sum_{i=0}^{\infty} (Ni + j)t_{Ni+j}\partial^{Ni+j}. \quad (5.1) \]

Meanwhile, the Orlov-Shulman’s operator \( \mathcal{M} \) satisfy the following identities

\[ [\mathcal{L}, \mathcal{M}]_* = 1, \quad \partial_{t_n} \mathcal{M} = [\mathcal{B}_n, \mathcal{M}]_*, \quad \mathcal{M} \star \hat{\omega}(t, z) = \partial_z \hat{\omega}(t, z). \quad (5.2) \]

Further, we can get

\[ \frac{\partial \mathcal{M}^m}{\partial t_{n_m}} = [\mathcal{B}_n, \mathcal{M}^m]_*, \quad \frac{\partial \mathcal{M}^m \mathcal{L}^l}{\partial t_{n_l}} = [\mathcal{B}_n, \mathcal{M}^m \mathcal{L}^l]_* \quad (5.3) \]

Moreover, to the wave function \( \hat{\omega}(t, z) \), \( (\mathcal{L}, \mathcal{M}) \) is anti-isomorphic to \( (z, \partial_z) \) with \([z, \partial_z]_* = -1\). We get

\[ \mathcal{M}^m \star \mathcal{L}^l \star \hat{\omega}(t, z) = z^l \star (\partial_z^m \hat{\omega}(t, z)), \quad \mathcal{L}^l \star \mathcal{M}^m \star \hat{\omega}(t, z) = \partial_z^m (z^l \star \hat{\omega}(t, z)). \quad (5.4) \]

Next, we should consider the adjoint wave function \( \hat{\omega}^* \) and the adjoint Orlov-Shulman’s operator \( \mathcal{M}^* \) that are useful for constructing the additional symmetry of the noncommutative Gelfand-Dickey hierarchies, we have

\[ \hat{\omega}^*(t, z) = (W^*)^{-1} \star e^{-\xi(t, z)}. \quad (5.5) \]

However, \( \mathcal{L}^* \) and \( \mathcal{M}^* \) satisfy \( [\mathcal{L}^*, \mathcal{M}^*]_* = [\mathcal{M}, \mathcal{L}]_* = -1 \). Furthermore, we have

\[ \mathcal{L}^* \star \hat{\omega}^* = z \star \hat{\omega}^*, \partial_{t_n} \hat{\omega}^* = -\mathcal{B}_n^* \star \hat{\omega}^*. \quad (5.6) \]
Next, we give the additional symmetries of the noncommutative Gelfand-Dickey hierarchies. Firstly, we introduce additional independent variables $t^*_{m,l}$ and define the action of the additional flows on the wave operator as

$$\frac{\partial W}{\partial t^*_{m,l}} = -(D_{m,l})_- \ast W;$$

(5.7)

in which

$$D_{m,l} = \mathcal{M}^m \ast L^l.$$  

(5.8)

Remark 5.1. For the C type noncommutative Gelfand-Dickey hierarchy, to keep the C type condition the operator $D_{m,l}$ needs to take the following form

$$D_{m,l} = \mathcal{M}^m \ast L^l - (-1)^l L^l \ast \mathcal{M}^m.$$  

(5.9)

In addition, we give some useful identities in the following proposition.

**Proposition 5.2.** The following identities hold true

$$\frac{\partial \mathcal{L}^n}{\partial t^*_{m,l}} = -[(D_{m,l})_-, \mathcal{L}^n]_\ast, \quad \frac{\partial \mathcal{M}^m}{\partial t^*_{m,l}} = -[(D_{m,l})_-, \mathcal{M}^m]_\ast,$$

(5.10)

$$\frac{\partial D_{n,k}}{\partial t^*_{m,l}} = -[(D_{m,l})_-, D_{n,k}]_\ast, \quad \frac{\partial D_{n,k}}{\partial t_n} = [\mathcal{B}_n, D_{n,k}]_\ast.$$  

(5.11)

**Proof.** The proof is very classical. We only need to consider the corresponding dressing structures

$$\mathcal{L}^n = W \ast \partial^n \ast W^{-1}, \quad \mathcal{M}^m = W \ast \Gamma^m_g \ast W^{-1},$$

(5.12)

and

$$D_{n,k} = W \ast (\Gamma^k_g \ast \partial^k - (-1)^k \partial^k \ast \Gamma^k_g) \ast W^{-1},$$

(5.13)

a direct calculation will lead to the proposition using the additional Sato equation. We need to note that $\Gamma_g$ depends on $t_n$ and does not depend on $t^*_{m,l}$. □

**Proposition 5.3.** The additional flows $\partial_{t^*_{m,l}}$ commute with the flows $\partial_{t_n}$ of the noncommutative Gelfand-Dickey hierarchies, which can be shown as

$$[\partial_{t^*_{m,l}}, \partial_{t_n}]_\ast = 0, \quad n \neq 0 \mod N.$$  

(5.14)

**Proof.** Let the additional flows $\partial_{t^*_{m,l}}$ and the flows $\partial_{t_n}$ act on $W$, we can get

$$[\partial_{t^*_{m,l}}, \partial_{t_n}]_\ast W = \partial_{t^*_{m,l}} (\partial_{t_n} W) - \partial_{t_n} (\partial_{t^*_{m,l}} W)$$

$$= -\partial_{t^*_{m,l}} (\mathcal{L}^n \ast W) + \partial_{t_n} ((D_{m,l})_- \ast W)$$

$$= - (\partial_{t^*_{m,l}} \mathcal{L}^n)_- \ast W - (\mathcal{L}^n)_- \ast (\partial_{t^*_{m,l}} W)$$

$$+ (\partial_{t_n} (D_{m,l})_- \ast W + (D_{m,l})_- \ast (\partial_{t_n} W)$$

$$= ((D_{m,l})_- \mathcal{L}^n)_- \ast W + (\mathcal{L}^n)_- \ast (D_{m,l})_- \ast W$$

12
\begin{align*}
+([L_+^N, D_{m,l}]_*)_W - (D_{m,l})_- * ([L_+^N]_*)_W \\
= ([D_{m,l}]_-, [L_+^N]_*)_W - ([D_{m,l}]_-) * ([L_+^N]_*)_W + [L_+^N, (D_{m,l})_-]_W * W \\
= ([D_{m,l}]_-, [L_+^N]_*)_W * W + [L_+^N, (D_{m,l})_-]_W * W \\
= 0.
\end{align*}

In the above proof, \(([L_+^N, (D_{m,l})_+])_- = 0\) and \(([L_+^N, D_{m,l}]_+) = ([L_+^N, (D_{m,l})_-])_-\) have been used.

Therefore, the additional flows \(\partial_{m,l}^*\) are symmetries of the noncommutative Gelfand-Dickey hierarchies.

**Proposition 5.4.** The additional symmetry flows \(\partial_{m,l}^*\) of the noncommutative Gelfand-Dickey hierarchies form the centerless \(W_{1+\infty}\).

**Proof.** By a direct calculation, we can easily derive

\[ [D_{m,l}, D_{n,k}]_* = \sum_{p,q} C_{nk,ml}^{pq} D_{p,q}, \quad (5.15) \]

and it implies that

\[ ([D_{m,l}, D_{n,k}]_*)_W = -\sum_{p,q} C_{nk,ml}^{pq} (D_{p,q})_- * W \]

\[ = \sum_{p,q} C_{nk,ml}^{pq} \partial_{p,q}^* W, \]

where the coefficient \(C_{nk,ml}^{pq}\) is the standard coefficient of the W algebra. Using eq.(5.16), we can get

\[ [\partial_{m,l}^*, \partial_{n,k}^*]_W = -\sum_{p,q} C_{nk,ml}^{pq} (C_{p,q})_- * W = \sum_{p,q} C_{nk,ml}^{pq} \partial_{p,q}^* W, \]

which is equal to

\[ [\partial_{m,l}^*, \partial_{n,k}^*]_* = \sum_{p,q} C_{nk,ml}^{pq} \partial_{p,q}^* \].

To have a better understanding of the additional symmetry flows in noncommutative case, we give a typical example of the noncommutative Gelfand-Dickey hierarchies. Let \(m = 1\), the corresponding flow on \(\mathcal{L}\) is

\[ \frac{\partial \mathcal{L}}{\partial t_{1,1}^*} = -[(\mathcal{M} * \mathcal{L})_-]_* = \mathcal{L} + [(\mathcal{M} * \mathcal{L})_+]_*. \quad (5.17) \]

Using eq.(5.1), \(\mathcal{M} * \mathcal{L}\) is expressed by

\[ \mathcal{M} * \mathcal{L} = W x \partial W^{-1} + \sum_{i \neq 0 \text{ mod } N} i W \star t_i \star W^{-1} \star W \star \partial^i \star W^{-1}. \quad (5.18) \]
Furthermore, using $\partial x = x \partial + 1$ and $\partial^{-i} x = x \partial^{-i} - i \partial^{-i-1}$, we get
\[
(W \ast x \ast \partial W^{-1})_+ = x \partial + \hat{\omega}_1 \ast x - x \ast \hat{\omega}_1,
\] (5.19)
with the $W^{-1} = 1 - \hat{\omega}_1 \partial^{-1} + ...$ being used. Taking eq.(5.19) into eq.(5.18), we have
\[
(W \ast x \ast \partial W^{-1})_+ = x \partial + \hat{\omega}_1 \ast x - x \ast \hat{\omega}_1 + \sum_{i \neq 0 \mod N} i(W \ast t_i \ast W^{-1} \ast \mathcal{L}_i)_+.
\] (5.20)

Taking eq.(5.20) into eq.(5.17), we have
\[
\partial \mathcal{L} \partial t^*_{1,1} = \mathcal{L} + \{x \partial, \mathcal{L}\}_* + \sum_{i \neq 0 \mod N} i(W \ast t_i \ast W^{-1} \ast \mathcal{L}_i)_+, \mathcal{L}_* \}
+ \{\hat{\omega}_1 \ast x, \mathcal{L}_*\} - \{x \ast \hat{\omega}_1, \mathcal{L}_*\}.
\]

Next, we define one generating function of the additional symmetries of the noncommutative Gelfand-Dickey hierarchies. We define a generating operator $Y_g(\lambda, \mu)$ of the additional symmetries as
\[
Y_g(\lambda, \mu) = \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-l-m-1}(D_{m,m+l})_-
- \frac{1}{\lambda} \hat{\omega}(t, -\lambda) \partial^{-1} \ast \hat{\omega}^*(t, \mu).
\] (5.22)

5.1. Noncommutative Gelfand-Dickey hierarchy with self-consistent sources. The noncommutative Gelfand-Dickey hierarchy with self-consistent sources can be constructed by the differential operator $\mathcal{L} = \partial^N + u_2 \partial^{N-2} + u_3 \partial^{N-3} + ... u_0$ like this:
\[
\mathcal{L}_{yn} = \{B_n + \Phi(t) \partial^{-1} \ast \Psi(t), \mathcal{L}\}_*, \ n \neq 0 \mod N,
\] (5.21)
where
\[
\partial t_n \Phi = B_n \ast \Phi, \ \partial t_n \Psi = -B^*_n \ast \Psi.
\] (5.22)

And the corresponding Sato equation becomes
\[
W_{yn} = (-\mathcal{L}_{yn}^* + \Phi(t) \partial^{-1} \ast \Psi(t)) \ast W.
\] (5.23)

6. String equations of the noncommutative Gelfand-Dickey hierarchy

In this section, we will consider the string equation of the noncommutative Gelfand-Dickey hierarchy. Firstly, we get a special action of the additional flows on $\mathcal{L}_i$
\[
\partial t_{i_1,-(l-1)} \mathcal{L}_i = \{-(D_{1,-(l-1)})_-, \mathcal{L}_i\}_*
= \{(D_{1,-(l-1)})_+, \mathcal{L}_i\}_* + [-D_{1,-(l-1)}, \mathcal{L}_i]_*
\]
\[ = \left[(\mathcal{M} \star \mathcal{L}^{-(l-1)})_+, \mathcal{L}^l\right]_* + l. \quad (6.1) \]

Basing on the above knowledge, we can get the following proposition on the String equation.

**Proposition 6.1.** If \( \mathcal{L}^l \) is independent on the additional variable \( t^*_{1-(l-1)} \), then

\[ [\mathcal{L}^l, \frac{1}{l}(\mathcal{M} \star \mathcal{L}^{-(l-1)})_+]_* = 1, \quad l \in \mathbb{Z} \setminus \{0\}, \quad (6.2) \]

is a string equation similar as the standard form \([P, Q]_* = 1\) (\(P \) and \(Q\) are two differential operators) of the noncommutative Gelfand-Dickey hierarchy.

**Acknowledgements:** This work is supported by National Natural Science Foundation of China under Grant No. 11571192, and K. C. Wong Magna Fund in Ningbo University.

**REFERENCES**

[1] E. Date, M. Kashiwara, M. Jimbo, T. Miwa, in Nonlinear Integrable Systems-Classical and Quantum Theory, edited by M. Jimbo and T. Miwa (World Scientific, Singapore, 1983) p. 39-119.

[2] L. A. Dickey, Soliton Equations and Hamiltonian Systems(2nd Edition) (World Scientific, Singapore, 2003).

[3] A. Yu. Orlov, E. I.Schulman, Additional symmetries of integrable equations and conformal algebra representation, Lett. Math. Phys. 12(1986), 171-179.

[4] I. Y. Dorfman, A. S. Fokas, Hamiltonian theory over non-commutative rings and integrability in multidimensions, J. Math. Phys. 33 (1992), 2504-2514.

[5] M. Hamanaka, K. Toda, Towards noncommutative integrable systems, Physics Letters A 316 (2003), 77-83.

[6] B. A. Kupershmidt, KP or mKP. Noncommutative mathematics of Lagrangian, Hamiltonian, and integrable systems, Mathematical Surveys and Monographs 78. American Mathematical Society, Providence, RI, 2000.

[7] M. Sakakibara, Factorization methods for Noncommutative KP and Toda hierarchy, J. Phys. A 37(2004), L599-L604.

[8] C. Z. Li, T. Song, Bi-Hamiltonian structure of the extended noncommutative Toda hierarchy, Journal of Nonlinear Mathematical Physics, 23(2016), 368-382.

[9] M. Hamanaka, Noncommutative Integrable Systems and Quasideterminants, arXiv:1012.6043.

[10] M. R. Douglas and N. A. Nekrasov, Noncommutative Field Theory, Rev. Mod. Phys. 73 (2002), 977.

[11] M. Adler, T. Shiota, P. van Moerbeke, From the \( \omega_{\infty} \)-algebra to its central extension: a \( \tau \)-function approach, Phys. Lett. A 194(1994), 33-43.

[12] L. A. Dickey, On additional symmetries of the KP hierarchy and Sato’s Backlund transformation, Comm. Math. Phys. 167(1995), 227-233.

[13] I. M. Gelfand and L. A. Dickey, Fractional powers of operators and Hamiltonian systems, Funct. Anal. Appl. 10 (1976) 259.

[14] T. Nakatsu, On the string equation at \( c = 1 \), Mod. Phys. Lett. A 9 (1994), 3313.
[15] Y. A. Orlov and P. Winternitz. $P_\infty$ algebra of the KP equation, free fermions and a 2-cocycle in the Lie algebra of pseudodifferential operators, Int. J. Mod. Phys. B, 11(1997), 3159-3193.

[16] A. Y. Orlov and P. Winternitz. Algebra of pseudodifferential operators and symmetries of equations in the Kadomtsev-Petviashvili hierarchy, J. Math. Phys., 38(1997), 4644-4674.

[17] Z. Zheng, J. S. He and Y. Cheng, Bäcklund transformation of the noncommutative Gelfand-Dickey hierarchy, JHEP 02(2004), 069.

[18] A. De Sole, V. G. Kac, D. Valeri, Adler-Gelfand-Dickey approach to classical W-algebras within the theory of Poisson vertex algebras, Int. Math. Res. Not. 21 (2015), 11186-11235.

[19] M. Van den Bergh, Double Poisson algebras, Trans. Amer. Math. Soc. 360 (2008), 5711-5769.

[20] A. De Sole, V. G. Kac, D. Valeri, Double Poisson vertex algebras and noncommutative Hamiltonian equations, Adv. Math. 281 (2015), 1025-1099.

[21] A. De Sole, V. G. Kac, D. Valeri, Classical affine $\mathfrak{w}$-algebras for $\mathfrak{g}_l$ and associated integrable Hamiltonian hierarchies, Comm. Math. Phys. 348 (2016), 265-319.