ON NON-EXISTENCE OF SOLUTIONS OF THE DIRICHLET PROBLEM FOR THE MINIMAL SURFACE SYSTEM

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Abstract. Recently we made systematic developments [25] regarding Lawson-Osserman constructions in their 1977 Acta Math paper “Non-existence, non-uniqueness and irregularity of solutions to the minimal surface system” in the aspects of non-uniqueness and irregularity. In this note we generalize Lawson-Osserman’s result on non-existence.

1. Introduction

Let \( \Omega \subset \mathbb{R}^{n+1} \) be an open bounded and strictly convex domain with boundary of class \( C^r \) for \( r \geq 2 \). The Dirichlet problem for minimal surfaces (cf. [9, 2, 4, 14, 10]) asks, for a given function \( f : \partial \Omega \to \mathbb{R}^{m+1} \) of class \( C^s \) with \( 0 \leq s \leq r \), what kind of and how many functions \( F : x = (x^1, \ldots, x^{n+1}) \mapsto (F^1, \ldots, F^{m+1}) \) is a weak solution to the minimal surface system

\[
\begin{cases}
\sum_{i=1}^{n+1} \frac{\partial}{\partial x^i} (\sqrt{g} g^{ij}) = 0, & j = 1, \ldots, n+1, \\
\sum_{i,j=1}^{n+1} \frac{\partial}{\partial x^i} (\sqrt{g} g^{ij} \frac{\partial F^\alpha}{\partial x^j}) = 0, & \alpha = 1, \ldots, m+1,
\end{cases}
\]

(1.1)

where \( g_{ij} = \delta_{ij} + \sum_{\alpha=1}^{m+1} \frac{\partial F^\alpha}{\partial x^i} \frac{\partial F^\alpha}{\partial x^j}, \quad (g^{ij}) = (g_{ij})^{-1} \) and \( g = \det(g_{ij}) \), satisfying the Dirichlet condition \( F|_{\partial \Omega} = f \). That means the graph of \( F \) is minimal with that of \( f \) being its boundary. When \( m = 0 \), by J. Douglas [6], T. Radó [17, 18], Jenkins-Serrin [9] and Bombieri-de Giorgi-Maranda [2], for any continuous boundary data there exists a unique Lipschitz solution; further by E. de Giorgi [4] and J. Moser [14], the solution is in fact analytic; and according to [7] the solution has an area-minimizing graph.

Dramatically different behaviors occur when \( m \geq 1 \). Considering \( \Omega = \mathbb{D}^{n+1} \) (the unit disk), Lawson-Osserman [10] found that, by T. Radó [18], Morse-Tompkins [13] and M. Shiffman [19], real analytic boundary data can be constructed for \( n, m \geq 1 \) so that there exist at least three analytic solutions; that, by density monotonicity of minimal surfaces, the Dirichlet problem is generally not solvable for \( n \geq m + 1 \geq 3 \); and that, in fact by group action, a Lipschitz but non-\( C^1 \) solution exists for certain boundary data.

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We made further systematic developments in [25] and showed that there are boundary data for which infinitely many analytic solutions and at least one singular solution can exist simultaneously. Such phenomena closely relates to both geometry (minimal surface theory) and analysis (PDE system [11] over the entire \( \mathbb{R}^{n+1} \)). As far as we know, this behavior is being observed for the first time and hard to be foreseen from the viewpoint of classic PDE system. Actually, all these phenomena trace back to the following non-existence result.

**Theorem 1.1** (Lawson-Osserman [10]). Let \( \eta : S^n \to S^n \subset \mathbb{R}^{m+1} \) be of \( C^2 \). Suppose \( n > m \) and \( [\eta] \neq 0 \in \pi_n(S^n) \). Then there exists \( R_\eta \in \mathbb{R}_+ \) s.t. for any number \( R \geq R_\eta \) there is no solution for the boundary function \( f = \eta_R := R \cdot \eta \).

The fact that, for a sufficiently small number \( R \), Dirichlet problem is solvable due to the Implicit Functional Theorem, e.g. see [15], together with the above theorem led Lawson and Osserman to the philosophy that there should exist \( R_0 \) such that the boundary condition \( \eta_{R_0} \), supports a singular solution. For first examples of such kind, they employed the three Hopf maps between unit spheres. In complex coordinates \( \eta(z_1, z_2) = (|z_1|^2 - |z_2|^2, 2z_1z_2) \) is the first. Note that for \( R \in \mathbb{R} \) the cone \( C = C(\text{graph of } \eta_R) \) over graph of \( \eta_R \) is an entire graph on \( \mathbb{R}^{n+1} \). Moreover, \( C \) is minimal if and only if its link of “spherical graph” type

\[
L := C \cap S^6 = \left\{ \left( \frac{x}{\sqrt{1 + R^2}}, \frac{R \cdot \eta(x)}{\sqrt{1 + R^2}} \right) : x \in S^3 \right\}
\]

is minimal in \( S^6 \). If one uses quaternions, then, isometrically up to a sign, \( \eta(q) = \overline{q}iq \) for \( q \) of unit length into \( \mathbb{H} \) into \( \text{Im } \mathbb{H} \), and \( L \) can be viewed as an orbit through \( ((\alpha, 0, 0, 0), (\sqrt{1 - \alpha^2})i) \) under action \( Sp(1) \cong S^3 \) on \( \mathbb{H}^7 \) where \( q \cdot (x, y) = (qx, qyq) \) for \( x \in \mathbb{H} \) and \( y \in \text{Im } \mathbb{H} \). As a result, the orbit of maximal volume, corresponding to \( \alpha = \frac{2}{3} \), is minimal in \( S^6 \). Hence slope \( R_0 \) can take value \( \sqrt{\frac{1 - \alpha^2}{\alpha}} = \sqrt{\frac{5}{2}} \). Similar procedures can be done for the other two Hopf maps.

In [25] we employed the theory of isometrically minimal immersions from projective spaces into spheres (see [3, 16, 22, 20, 21], using eigenvalues and eigenfunctions of Laplacian operators on projective spaces) for a constellation of uncountably many boundary functions that support irregular solutions. The boundary data are suitable multiple of LOMSE map \( \eta \). Besides the original three in [10], \( \eta \) maps the source sphere into a sphere of higher dimension and thus represents a zero homotopy class of the target sphere. In order to figure out the situation of Dirichlet problem to these \( \eta_R \), an important step is to understand whether or not it has no solution for large \( R \).

Our main result is a generalization of Theorem 1.1

**Theorem 1.2.** If a \( C^2 \) map \( \eta : S^n \to \mathbb{R}^{m+1} \) has its image \( N = \eta(S^n) \) being an \( l \)-dimensional embedded submanifold with \( l < n \) and \( [\eta] \neq 0 \in \pi_n(N) \), then there exists \( R_\eta \in \mathbb{R}_+ \) s.t. for constant \( R \geq R_\eta \) there is no solution for the boundary function \( f = \eta_R \).

A direct application to LOMSEs in [25] leads to
Corollary 1.3. For most $LOMSE \eta : S^n \to S^m$, there exists $R_\eta \in \mathbb{R}$ s.t. for constant $R \geq R_\eta$ there is no solution for the boundary function $f = \eta_R$.

In [I] we strengthen the corollary to

**Theorem 1.4.** For every $LOMSE \eta$, there exists $R_\eta \in \mathbb{R}$ s.t. for constant $R \geq R_\eta$ there is no solution for the boundary function $f = \eta_R$.

On the other hand, if $n = 1$ or $\eta(S^n)$ is contained in a line of $\mathbb{R}^{m+1}$, then the problem can be solvable for $f = \eta_R$ for any $R \in \mathbb{R}$. Boundary data induced by holomorphic maps between complex spaces also have such property. Consequently, we have

**Corollary 1.5.** Let $F : \mathbb{C}^n \to \mathbb{C}^m$ be holomorphic and $f = F|_{S^{2n-1}}$. If the image $N$ of $f$ is an embedded submanifold of lower dimension, then $[f] = 0 \in \pi_{2n-1}(N)$.

More explorations will be made in §5 and §6 for various situations.

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2. **Proof of Theorem 1.2**

For the sake of completeness, we recall the original idea of the proof of Theorem 1.1 by Lawson-Osserman. It’s been done through a contradiction argument and interesting combinations of geometry, topology and analysis.

Note that the $C^2$ assumption allows us to apply Allard’s Boundary Regularity Theorem, which guarantees many computations in this paper valid.

**Theorem 2.1** (Allard [I]). Suppose the boundary map is of $C^{s,\alpha}$ for $2 \leq s \leq \infty$ or $s = \omega$ and $F$ be any solution to the Dirichlet problem in $\Omega$. Then there exists a neighborhood $U$ of $\partial \Omega$ such that $F \in C^{s,\alpha}(U \cap \overline{\Omega})$.

Assume the opposite of the conclusion, that is there exist a sequence $\{R_i\} \to +\infty$ such that the Dirichlet problem can be solved for each $\eta_{R_i}$. Let $\{F_i\}$ be the solutions and $\{G_i\}$ be their images. Then two steps are the followings.

**Step 1.** For minimal graph $G_i$ in Euclidean space, a special formula of volume is given by boundary terms

$$M(G_i) = \frac{1}{n+1} \int_{\partial G_i} <\nu, p> \ast 1_{\partial G_i},$$

(2.1)
where \( p \) is the position vector and \( \nu \) is the unit exterior normal field to \( \partial G_i \). So

\[
M(G_i) \leq \frac{1}{n+1} \int_{\partial G_i} \|p\| \, * 1_{\partial G_i}
\]

(2.2)

\[
= \frac{\sqrt{1 + R_i^2}}{n+1} \int_{S^n} \|(e_1, d(\eta R_i)(e_1)) \wedge \cdots \wedge (e_n, d(\eta R_i)(e_n))\| \, * 1_{S^n}
\]

\[
\leq \sqrt{1 + R_i^2} R_i^{m} \cdot (\text{Volume of graph of } \eta)
\]

for \( R_i \geq 1 \), where \( \{e_1, \cdots, e_n\} \) is a pointwise orthogonal frame on \( S^n \). Therefore, one can get a uniform upper bound

(2.3)

\[
M(G_i) \leq C \cdot R_i^{m+1},
\]

for some positive constant \( C \), e.g. \( C \) can be \( \frac{\sqrt{2}}{n+1} \cdot (\text{Volume of graph of } \eta) \).

**Step 2.** For a lower bound of \( M(G_i) \), the density monotonicity for minimal varieties in Euclidean space plays a crucial role.

**Theorem 2.2** (e.g. see \([7, 8, 11]\)). Let \( G \) be an \((n+1)\)-dimensional minimal variety of \( \mathbb{R}^N \) and \( p \in \mathbb{R}^N \). Then the density function, defined as

\[
\Theta(G, p, d) := \frac{M(G \cap B_d(p))}{\omega_{n+1} r^{n+1}}
\]

where \( B_d(p) \) is the solid ball of radius \( d \) centered at \( p \) and \( \omega_{n+1} \) the volume of unit solid ball in \( \mathbb{R}^{n+1} \), is increasing in the Euclidean distance \( d \) for \( 0 < d \leq \text{dist}(p, \partial G) \).

Note that \( \inf_{d>0} \Theta(G, p, d) \geq 1 \) for a Lipschitz graph \( G \) and \( p \in G \). Therefore, if there exists some \( p \in G \) with large \( \text{dist}(p, \partial G) \), then one can have a big lower bound of \( M(G) \). In \([10]\) Lawson and Osserman observed that, under the topological assumption of Theorem 1.1, there must exist point \( p_i \) of form \((x_i, 0)\) lying in \( G_i \) for each \( i \). Otherwise,

\[
H(x, r) := \frac{F_i(rx)}{\|F_i(rx)\|} \quad \text{for } x \in S^n \text{ and } r \in [0, 1]
\]

is a well-defined map from \( S^n \times [0, 1] \) to \( S^m \) and leads to the null homotopy of \( \eta \). Hence \( \text{dist}(p_i, \partial G_i) \geq R_i \) and consequently by Theorem 2.2

(2.4)

\[
M(G_i) \geq \omega_{n+1} R_i^{m+1}.
\]

Now (2.3) and (2.4) give

\[
CR_i^{m+1} \geq M(G_i) \geq \omega_{n+1} R_i^{m+1},
\]

a contradiction for large \( R_i \) under the assumption \( n > m \). This completes the proof of Theorem 1.1.

**Proof of Theorem 1.2** We take similar strategy. Assume the opposite of the conclusion. Then there exist a sequence \( \{R_i\} \to +\infty \) with \( \{F_i\} \) being solutions of
the Dirichlet problem to \( \{ \eta_{R_i} \} \). Denote the image of \( F_i \) by \( G_i \). Step 1 for upper bound of \( M(G_i) \) is the same. Since \( N = \eta(S^n) \) has dimension \( l(< n) \), we have

\[
M(G_i) \leq C \cdot R_i^{l+1}.
\]

**Step 2.** Now let us focus on lower bound of \( M(G_i) \). Let \( \epsilon_0(N) \) be the normal injectivity radius of the embedded submanifold \( N = \eta(S^n) \) in \( \mathbb{R}^{m+n+2} \). Here the normal injectivity radius \( \epsilon_0(N) \) means the largest number of which radius the disk normal bundle of \( N \) is differomorphic to a neighborhood of \( N \) through the exponential map restricted to normals. Then it easily follows that \( R \cdot \epsilon_0(N) \) would be the normal injectivity radius of \( R \cdot N \) for \( R \in \mathbb{R}_+ \).

**Claim (⋆).** For each \( i \in \mathbb{N} \), there exists point \( p_i \) in \( G_i \) and \( \epsilon_0 \cdot R_i \) away from \( \partial G_i \).

Note that \( G_i \) is a Lipschitz graph and the density function is upper semicontinuous. With above claim and Theorem 2.2 we establish

\[
M(G_i) \geq \omega_{n+1} \epsilon_0^{n+1} R_i^{n+1}.
\]

Therefore, (2.5) and (2.6) give rise to a contradiction and Theorem 1.2 gets proved.

Now we show Claim (⋆) by contradiction. Assume that there exists some \( i \in \mathbb{N} \) for which \( G_i \) is entirely contained in the \( \epsilon_0 \)-neighborhood \( \tilde{N} \) of \( \partial G_i \). Then

\[
F(x, r) := F_i(rx) \text{ for } x \in S^n \text{ and } r \in [0, 1]
\]
gives a null homotopy of \( R_i \eta \) as a map from \( S^n \) into \( \tilde{N} \). Let \( \pi^\perp \) be the projection of \( \tilde{N} \) to \( N \) along normal exponential leaves. We can define

\[
H(x, r) := \pi^\perp \circ F(x, r) = \pi^\perp(F_i(rx)) \text{ for } x \in S^n \text{ and } r \in [0, 1],
\]

which provides a null homotopy of \( R_i \eta \) as a map from \( S^n \) to \( N \). This contradiction to the topological assumption \([\eta] \neq 0 \in \pi_n(N)\) completes the proof of Theorem 1.2.

**Remark 2.3.** Generally, \( p_i \) can be a point of \( \partial \tilde{N} \). For the special case of Lawson-Osserman, \( p_i \) can always take point of form \((y_i, 0)\) for some \( y_i \in \mathbb{D}^{n+1} \).

### 3. Application to interior LOMSEs

Recall that in [25] a mapping \( \eta \) between unit spheres is called an LOMSE if and only if, as a map, it is a composition of a Hopf fibration \( \pi \) to projective spaces and a minimal isometric immersion \( \iota \) into Euclidean sphere of certain radius, i.e.,

\[
\eta = \iota \circ \pi
\]

with commutative diagram

\[
\begin{array}{ccc}
S^n & \xrightarrow{\eta} & S^m \\
\downarrow{\pi} & \downarrow{\iota} & \\
P^l & \rightarrow & S^m
\end{array}
\]
Note that there are three families $S^{2s+1} \rightarrow \mathbb{C}P^s$, $S^{4s+3} \rightarrow \mathbb{H}P^s$, and $S^{15} \rightarrow S^8$ for the choice of $\pi$, and usually uncountably many for $\iota$ (using eigenfunctions of $k$-th eigenvalue of Laplacian operator, see [3, 16, 22, 20, 21]). Hence in such way we got a constellation of uncountably many LOMSEs (of $(n, l, 2k)$-type), to each $\eta$ of them there is a boundary function $\tan \theta \cdot \eta$ for some acute angle $\theta$ (see [25] for details) such that the cone over its graph gives an irregular solution of the Dirichlet problem. Therefore, we systematically generalized the irregular part about solutions of the Dirichlet problem in [10].

As detected in [3, 16, 22, 20, 21], the moduli space of $\iota$ is quite beautiful. Its interior part contains a standard isometric immersion $\eta_0$. Since its image $N_0$ is an orbit of corresponding group action under certain equivalent property, $N_0$ is an embedded submanifold in the Euclidean space. All other interior point $\eta$ is simply $A\eta_0$ for a full rank square matrix $A$ of size of dimension of target Euclidean space, where

$$A = \sqrt{C + I}$$

for symmetric nonnegative matrix $C + I$ with $C \in W_2$. Here $C \in W_2$ is an equivalent algebraic requirement for

$$\|A\eta_0(x)\| = \|\eta_0(x)\|,$$

$$\begin{align*}
\angle(A\eta_0(x), A\eta_0(v_x)) &= \angle(\eta_0(x), d\eta_0(v_x)) = 0, \\
\angle(A\eta_0(v_x), A\eta_0(w_x)) &= \angle(d\eta_0(v_x), d\eta_0(w_x)),
\end{align*}$$

(3.1)

where $x \in S^n$ and $v_x, w_x \in T_xS^n$ are arbitrary. The moduli space is in fact described by this compact convex body $W_2$ in vector space. See [3, 23] for details. As a result, each interior point $\eta = A\eta_0$ of the moduli space has an embedded submanifold image $N = A(N_0)$ in an Euclidean sphere.

Moreover, according to [12] and [24], $\iota$ is a finite covering on $N$, so the diagram

$$\begin{array}{ccc}
S^n & \xrightarrow{\eta} & N^l \\
\downarrow & \searrow \pi & \\
N^l & \xrightarrow{\iota} & P^l
\end{array}$$

has the homotopy lifting property. Therefore, $\eta$ is null-homotopic if and only if so is $\pi$. However, it is well known that Hopf fibration $\pi$ is not homotopic to a constant map. For example, one can use long exact sequence for homotopy groups and apply the celebrated finiteness result due to Serre which says homotopy groups of spheres are all finite except for those of $\pi_1(S^1)$ or $\pi_{4s-1}(S^{2s})$. In particular $\pi_{2s+1}(S^1)$, $\pi_{4s+3}(S^3)$ and $\pi_{15}(S^7)$ are finite.

Since $[\eta] \neq 0 \in \pi_n(N)$, Corollary 1.3 is indeed an immediate corollary of Theorem 1.2. We shall deal with the boundary point of the moduli space in the next section.
4. Application to boundary LOMSEs

In general, the moduli space of isometric minimal immersions of projective space into Euclidean space (corresponding to $k$-th eigenvalue for $k \geq 2$) could be quite subtle in its boundaries, e.g. see [3] for some exploration.

It seems still unknown whether the image of a boundary point of the moduli space is always an embedded submanifold or not. If a boundary point comes from a subcover of the background group action, then its image is an embedded submanifold. However, there might be a possibility that after smashing certain directions (when $A$ is not of full rank) the image $N_0$ of the standard minimal isometric immersion is then immersed into Euclidean spheres of lower dimensions with contact points however not coinciding in local patches.

Instead of an attempt on a direct application of Theorem 1.2 we alter the argument to fit boundary LOMSEs. Even if $\iota$ is not a covering, it is always an immersion. Locally $\iota$ is an embedding and one can pull back the normal bundle of the immersed submanifold to $P$. Let $\mathcal{N}$ be the pull-back bundle and $\epsilon_0$ the normal radius of first focal point. Set $(\mathcal{N}, P, \pi^\perp)$ to be the pull-back normal bundle via $R_i\iota$. Then $R_i\epsilon_0$ would be the corresponding normal radius of first focal point.

**Proof of Theorem 1.4.** Follow the proof of Theorem 1.2. Assume the opposite of the conclusion. Then there exist a sequence $\{R_i\} \to +\infty$ with $\{F_i\}$ being solutions of the Dirichlet problem to $\{\eta_{R_i}\}$. Denote the image of $F_i$ by $G_i$. Step 1 is unchanged for upper bound of $M(G_i)$.

For Step 2, we only need to show

**Claim (★).** For each $i \in \mathbb{N}$, there exists point $p_i$ in $G_i$ and $\epsilon_0R_i$ away from $\partial G_i$.

If (★) is not the case, then $G_i$ is entirely contained in $(\epsilon_0R_i)$-neighborhood $\tilde{N}$ of $\partial G_i$. In particular, $F_i(\mathbb{D}^{n+1})$ is contained in the $(\epsilon_0R_i)$-neighborhood $\tilde{N}$ of $R_i\eta(S^n)$. Note that $\tilde{N}$ is the exponential image set of $(\epsilon_0R_i)$-disk bundle of $\mathcal{N}$. Although $\eta(S^n)$ may not be an embedded submanifold, we have “local lifting” property for (segments of) curves within $\tilde{N}$ to $\mathcal{N}$.

For each fixed $x \in S^n$, consider curve $F_i(rx)$ where $r \in [0,1]$. When $r = 1$, $F_i(x) = R_i\eta(x) \in \tilde{N}$. Hence for this fixed ending curve map there is a unique lifting $\hat{F}_i(rx)$ to $\mathcal{N}$ by applying the “local lifting” property backward from $r = 1$ to $r = 0$.
with \( \hat{F}_i(x) = (\pi(x), 0) \).

\[
\begin{array}{c}
\hat{N} \\
\cup \\
(x, r) \xrightarrow{F} F_i(rx) \xrightarrow{r=1} R_i \eta(x) \\
P \ni H(x, r) \xleftarrow{\pi^+} \hat{F}_i(rx) \subset \hat{N} \ni (\pi(x), 0)
\end{array}
\]

As illustrated above, we end up with a map \( H(x, r) := \pi^+(\hat{F}_i(rx)) \) from \( S^n \times [0, 1] \) to \( P \). Clearly, \( H \) is continuous in both slots and gives a null homotopy of \( \pi \). Contradiction with \( \pi \neq 0 \in \pi_n(P) \) where \( \pi \) is a Hopf fibration submersion to projective space! Hence (\( \star \)) is true.

Similarly, the statement of the theorem follows due to the same contradiction argument for theorem \( \text{[1.2]} \).

**Question 1.** Although for all LOMSE we obtain this non-existence result for large \( R \), it still remains mysterious to us at this moment for middle part \( R \).

**Question 2.** For a boundary data \( \phi \) which support singular solutions to the Dirichlet problem, is it always the case that similar non-existence result holds when stretching \( \phi \) severely?

5. Application to region \( \mathbb{D}(2) - \mathbb{D}(1) \)

All previous discussions are devoted to the case of domain \( \Omega = \mathbb{D}^{n+1}(1) \subset \mathbb{R}^{n+1} \). It is also natural and interesting to study Dirichlet problem over region \( \mathbb{D}(2) - \mathbb{D}(1) \) between to spheres: given \( C^2 \) maps \( f_1 : S^n(1) \to \mathbb{R}^{m+1} \) and \( f_2 : S^n(2) \to \mathbb{R}^{m+1} \), whether there exists a Lipschitz function \( F : \mathbb{D}(2) - \mathbb{D}(1) \to \mathbb{R}^{m+1} \) such that the graph of \( F \) is minimal in \( \mathbb{R}^{m+n+2} \) with \( F|_{S^n(1)} = f_1 \) and \( F|_{S^n(2)} = f_2 \).

From (\[2\]) one would be able to establish

**Theorem 5.1.** If \( f_1(S^n(1)) \) and \( f_2(S^n(2)) \) are disjoint of Hausdorff dimensions lower than \( n \), then there exists \( R_{f_1, f_2} \in \mathbb{R}_+ \) such that for \( R \geq R_{f_1, f_2} \) the Dirichlet problem has no solution defined on \( \mathbb{D}(2) - \mathbb{D}(1) \) to boundary data \( R \cdot f_1 \) and \( R \cdot f_2 \).

**Proof.** Suppose the opposite: we have a sequence \( R_i \to +\infty \) with solutions \( F_i \) to boundary data \( R_i \cdot f_1, R_i \cdot f_2 \). Let \( d := \text{dist}(f_1(S^n(1)), f_2(S^n(2))) > 0 \). Then \( \text{dist}(R \cdot f_1(S^n(1)), R \cdot f_2(S^n(2))) = Rd \). By continuity, there must be some point \( p = (x_i, F_i(x_i)) \) where \( F_i(x_i) \) has distance \( \frac{Rd}{2} \) from both \( R \cdot f_1(S^n(1)) \) and \( R \cdot f_2(S^n(2)) \). Therefore, \( p \) is at least \( \frac{Rd}{2} \) away from boundary of the graph of \( F_i \).
It is clear that the conclusion follows from the arguments that we have already made before.

**Example 1.** Let \( f_1 : S^3(1) \to S^2(1) \) be the Hopf map and \( f_2 : S^3(2) \to 2 \cdot S^2(1) \) by \( x \mapsto 2 \cdot f_1(\frac{x}{\|x\|}) \). Then for large \( R \), the Dirichlet problem to rescaled boundary data \( R \cdot f_1, R \cdot f_2 \) has no solutions.

**Example 2.** Let \( f_1 : S^3(1) \to S^2(1) \) be the Hopf map and \( f_2 \) a zero map. Then for large \( R \), the Dirichlet problem to rescaled boundary data \( R \cdot f_1, 0 \) has no solutions.

How about \( f_1 \) being the first Hopf map and \( f_2(x) = T \circ f_1(\frac{x}{\|x\|}) \) where \( T(z, x_3) = (\frac{x_3^2}{\|z\|}, x_3) \)? Note that, as a self-map of \( S^2 \), \( T \) is smooth. The similar phenomena will occur.

**Theorem 5.2.** Suppose \( n > m \). As \( C^2 \) mappings, if \( f_1, f_2 : S^n(1) \to S^m(1) \) are not homotopic to each other, then there exists \( R_{f_1,f_2} \in \mathbb{R}_+ \) such that for \( R \geq R_{f_1,f_2} \) the Dirichlet problem has no solution defined on \( \mathbb{D}(2) - \mathbb{D}(1) \) to boundary data \( R \cdot f_1 \) and \( R \cdot f_2 \).

**Proof.** A punch line is that if there exists an solution \( F \) to rescaled boundary data \( R \cdot f_1 \) and \( R \cdot f_2 \). There must be some point \( x_R \in \mathbb{D}(2) - \mathbb{D}(1) \) with \( F(x_R) = 0 \). Then one can apply the density monotonicity theorem for a contradiction as in §2.

6. Some other connections

Note that a complex variety in a Kähler manifold is always minimal. If \( F : \mathbb{D}^{2n} \subset \mathbb{C}^n \to \mathbb{C}^m \) is a holomorphic map and \( \eta = F|_{\mathbb{S}^{2n-1}} \), then for any \( R \in \mathbb{R} \) the Dirichlet problem to \( \phi = \eta_R \) has solution \( R \cdot F \). Thus Corollary 1.3 is true due to Theorem 1.2.

Combining this idea and Theorem 5.2 we gain the following

**Corollary 6.1.** Assume \( F : \mathbb{C}^n \to \mathbb{C}^m \) be a homomorphic map with \( F(S^{2n-1}(1)) \) and \( F(S^{2n-1}(2)) \) of dimensions strictly less than \( 2n - 1 \). Then their intersection \( F(S^{2n-1}(1)) \cap F(S^{2n-1}(2)) \neq \emptyset \).

**Example 3.** Let \( F : \mathbb{C}^2 \to \mathbb{C}^1 \) by \( (z_1, z_2) \mapsto z_1^3z_2 \). Then

\[
S^3(1) = \{ (a \cdot e^{i\alpha}, b \cdot e^{i\beta}) : a^2 + b^2 = 1, \text{ and } \alpha, \beta \in \mathbb{R} \}.
\]

Clearly, \( F(S^3(1)) = \{ a^3b \cdot e^{i(3\alpha + \beta)} : a^2 + b^2 = 1, \text{ and } \alpha, \beta \in \mathbb{R} \} \) contains the origin and so does \( F(S^3(2)) \).

We would like to point it out that actually all our discussions in this paper for non-existence extend automatically for many other bounded domains with \( C^2 \) boundaries as long as the required “corresponding” topological relations remain
true. For example, results for $\mathbb{D}(1)$ are valid for (high dimensional) solid ellipsoid regions; Corollary 6.1 also works for a $(2n-1)$-dimensional solid torus minus a small $(2n-1)$-dimensional solid torus inside. Regions with more complicated topological structures can be studied accordingly. Here we only mention a couple of relatively simple cases.

**Theorem 6.2.** Let $T^{n+1} = \mathbb{D}^{n}(\frac{1}{2}) \times S^1(1) \subset \mathbb{R}^{n+1}$. If $f : \partial T = S^{n-1}(\frac{1}{2}) \times S^1(1) \rightarrow \mathbb{R}^{m+1}$ has a simply connected embedded submanifold $N$ of lower dimension as its image and $f$ is not homotopic to zero as a map between $\partial T$ and $N$, then there exists $R_f \in \mathbb{R}_+$ such that for $R \geq R_f$ the Dirichlet problem on $T$ with boundary data $R \cdot f$ has no solutions.

**Proof.** A crucial point is that any solution $F$ for boundary data $R \cdot f$ would give a homotopy between $f : \partial T \rightarrow N$ and some $f_0 : \partial T \rightarrow F(\{0\} \times S^1(1)) \subset N$ if the image of $F$ is completely contained in certain open neighborhood of $N$ as in step 2 of the proof of Theorem 1.2. However the simply connectedness contradicts with the topological assumption on $f$. The property in the Claim (⋆) can be established and a proof for Theorem 6.2 follows in a similar way. $\square$

**Example 4.** Let $m = n - 1 \geq 2$ and $f : \partial T \rightarrow S^{n-1}(\frac{1}{2})$ by $(x, t) \mapsto x$. Then the identity map of composition $S^{n-1}(\frac{1}{2}) \hookrightarrow \partial T = S^{n-1}(\frac{1}{2}) \times S^1(1) \xrightarrow{f} S^{n-1}(\frac{1}{2})$ shows that $f$ is not null-homotopic. According to Theorem 6.2, there is no solution for boundary data $R \cdot f$ over $T$ for large $R$.

**Remark 6.3.** For $n - 2 \geq 1$ and domain $\mathbb{D}^{n-1}(\frac{1}{2}) \times S^2(1)$, if $f$ mapping the boundary into $\mathbb{R}^{m+1}$ has an embedded submanifold image $N$ with $\pi_2(N) = 0$ instead of $\pi_1(N) = 0$ in Theorem 6.2 then the same type conclusion holds.

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