A theory of quantum differential equation solvers: limitations and fast-forwarding

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Abstract

We study the limitations and fast-forwarding of quantum algorithms for linear ordinary differential equation (ODE) systems with a particular focus on non-quantum dynamics, where the coefficient matrix in the ODE is not anti-Hermitian or the ODE is inhomogeneous. On the one hand, for generic homogeneous linear ODEs, by proving worst-case lower bounds, we show that quantum algorithms suffer from computational overheads due to two types of “non-quantumness”: real part gap and non-normality of the coefficient matrix. We then show that homogeneous ODEs in the absence of both types of “non-quantumness” are equivalent to quantum dynamics, and reach the conclusion that quantum algorithms for quantum dynamics work best. We generalize our results to the inhomogeneous case and find that existing generic quantum ODE solvers cannot be substantially improved. To obtain these lower bounds, we propose a general framework for proving lower bounds on quantum algorithms that are \textit{amplifiers}, meaning that they amplify the difference between a pair of input quantum states. On the other hand, we show how to fast-forward quantum algorithms for solving special classes of ODEs which leads to improved efficiency. More specifically, we obtain quadratic improvements in the evolution time $T$ for inhomogeneous ODEs with a negative semi-definite coefficient matrix, and exponential improvements in both $T$ and the spectral norm of the coefficient matrix for inhomogeneous ODEs with efficiently implementable eigensystems, including various spatially discretized linear evolutionary partial differential equations. We give fast-forwarding algorithms that are conceptually different from existing ones in the sense that they neither require time discretization nor solving high-dimensional linear systems.

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1 Introduction

Differential equations are widely used to model the evolution and dynamics of systems in many disciplines including mathematics, physics, chemistry, engineering, biology, and economics. In this work, we consider the linear ordinary differential equation (ODE) system

\[
\frac{d}{dt} u(t) = Au(t) + b(t), \quad t \in [0, T],
\]

\[
u(0) = u_\text{in},
\]

where \( u(t) \in \mathbb{C}^N \) is the solution of the ODE, \( A \in \mathbb{C}^{N \times N} \) is the coefficient matrix, and \( b(t) \in \mathbb{C}^N \) is the inhomogeneous term. We always consider a time-independent coefficient matrix \( A \) and sometimes allow for a time-dependent inhomogeneous term. We call Equation (1) a homogeneous system of ODEs if \( b(t) = 0 \) for all \( t \) and an inhomogeneous system otherwise. A quantum algorithm is said to solve\(^1\) Equation (1) if it prepares a quantum state approximately equal to the normalized final solution \( u(T)/\|u(T)\| \) up to 2-norm error \( \epsilon \), where \( \| \cdot \| \) denotes the 2-norm of a vector.

One notable example of Equation (1) is quantum dynamics, where \( b(t) = 0 \) for all \( t \) and \( A \) is anti-Hermitian. Simulating quantum dynamics is also known as time-independent Hamiltonian simulation, which traces its formulation to Feynman [7] and is widely viewed as one of the most important and promising applications of quantum computers. The first quantum algorithm for time-independent Hamiltonian simulation was proposed by Lloyd [8], and there has been rapid and significant theoretical progress in the past few decades [9–25]. The best asymptotic complexity scaling is achieved by quantum signal processing (QSP) and quantum singular value transformation (QSVT), and is given by \( O(T\|A\| + \log(1/\epsilon)/\loglog(1/\epsilon)) \). This scaling matches the worst-case lower bound for time-independent Hamiltonian simulation in time \( T \), error \( \epsilon \), and the norm of \( A \) [14].

This work focuses on quantum algorithms for ODE systems that are not necessarily quantum dynamics, \( i.e. \), \( A \) is not anti-Hermitian and/or the system is inhomogeneous. Our work has three motivations. First, real-world systems and applications of non-quantum dynamics typically have a huge degree of freedom, leading to the need to employ high-dimensional differential equations for accurate modeling. This makes the topic of quantum algorithms for solving general ODE systems intriguing since quantum computers can potentially solve high-dimensional problems exponentially faster than classical computers. Second, compared to simulating quantum dynamics, there are much fewer quantum algorithms available [1–3, 5]. Most of them first discretize the time variable according to different classical numerical methods and solve the resulting linear system of equations using quantum linear system solvers (such as the HHL algorithm [26] or advanced ones [27–32]). Though the linear-in-\( T \), linear-in-\( \|A\| \) and polylog-in-\( \epsilon \) scalings can still be achieved, existing quantum algorithms for general non-quantum dynamics are not as efficient as those for quantum dynamics, in the sense that their complexity scalings typically involve extra factors related to the “non-quantumness” of the evolution (\( e.g. \), the norm of \( e^{At} \), the condition number of the eigenbasis and the rate of norm decay). Therefore it is desirable to investigate whether there exist quantum algorithms for generic non-quantum dynamics with better asymptotic scaling or whether the “non-quantumness” of such systems inherently limits the efficiency of any quantum algorithm that solves it. Third, the best existing algorithms with linear scalings in \( T \) and \( \|A\| \) can still be expensive in specific applications. Some examples include the inhomogeneous parabolic and hyperbolic partial differential equations. These equations involve an unbounded spatial Laplacian operator, which has a huge spectral norm after spatial discretization. This makes linear scaling in \( \|A\| \) undesirable for efficient simulation. The linear scaling in \( T \) also means that simulation for long times is costly. Therefore, it is worthwhile to investigate whether there exist quantum algorithms for specific and practically applicable systems that are more computationally efficient compared to their generic counterparts.

In this work, we study quantum algorithms for solving non-quantum dynamics from two different angles: 1. their limitations when applied to general non-quantum dynamics and 2. their fast-forwarding when applied to specific non-quantum dynamics. On the one hand, we identify two types of “non-quantumness” of ODEs that cause computational overhead. We show that generic quantum algorithms for solving homogeneous ODEs suffer from exponential computational overhead if the coefficient matrix has at least one pair of

\(^1\)Here the meaning of solving an ODE quantuminly follows existing literature on quantum ODE solvers, see, \( e.g. \), [1–6].
eigenvalues with different real parts and linear computational overhead if the coefficient matrix is non-normal\(^2\). To avoid the computational overheads from these two sources of non-quantumness, the coefficient matrix \(A\) should be normal and all of its eigenvalues should have the same real parts. We then show that this class of homogeneous ODEs is equivalent to quantum dynamics due to what we call the shifting equivalence. Our hardness results can also be generalized to the inhomogeneous case, and imply that existing generic quantum algorithms for inhomogeneous ODEs cannot be substantially improved. On the other hand, we show that fast-forwarding and improved asymptotic complexity scalings are possible for special cases of ODEs. More specifically, we quadratically improve the scaling in time \(T\) if the coefficient matrix \(A\) is negative semi-definite, and exponentially improve the scaling in \(T\) and \(\|A\|\) if \(A\) has classically computable eigenvalues and quantumly implementable eigenstates. The latter case includes common (spatially discretized) linear evolutionary partial differential equations, such as transport equation, heat equation, advection-diffusion equation, wave equation, and so on. Our algorithms are “one-shot” in the sense that they neither require time discretization nor solve high-dimensional linear systems. Instead, they are based on the linear combination of quantum states, which makes them conceptually different from existing generic algorithms that rely on time discretization and solving high-dimensional linear systems.

1.1 Limitations of generic quantum ODE solvers

1.1.1 Results

We first focus on homogeneous ODE systems and show that the most efficient class of homogeneous ODE systems that generic quantum algorithms can solve is quantum dynamics. We identify two sources of non-quantumness that introduce computational overhead: real part gap (i.e., the maximal difference in the eigenvalues' real parts) and non-normality of the coefficient matrix, and prove two corresponding lower bounds. Specifically, when the real part gap \(\delta > 0\) (i.e., there exists at least a pair of eigenvalues with different real parts), generic quantum ODE solvers must have worst-case complexity \(\Omega(e^{\delta T})\), an exponential overhead in both the real part gap and the evolution time. This lower bound is related to the inefficiency of quantum post-selection (assuming BQP \(\neq\) PP)[33], except our result is in the query model and generalizes to the case when the coefficient matrix \(A\) has a possibly non-orthonormal eigenbasis because we do not assume \(A\) is unitarily diagonalizable. Second, when \(A\) is non-normal (even if all the eigenvalues have the same real parts), we show another worst-case lower bound \(\Omega(\mu(A))\) for generic quantum ODE solvers, where \(\mu(A) = \|AA^\dagger - A^\dagger A\|^{1/2}\) is a measure of non-normality [34] in the sense that \(\mu(A)\) becomes larger if \(A\) is more non-normal. Therefore, the most efficient class that generic quantum ODE solvers can solve is the one that sufficiently avoids both types of “non-quantumness”, implying that the coefficient matrix should be normal and all the eigenvalues have the same real parts. Systems with such features are quantumly equivalent to quantum dynamics due to an observation that we call shifting equivalence: the normalized solution of a homogeneous ODE system remains unchanged if the coefficient matrix is shifted by a real number. Therefore we can shift the coefficient matrix by the real part of its eigenvalues and the resulting

---

\(^2\)A matrix \(A\) is normal iff \(AA^\dagger = A^\dagger A\), or equivalently \(A\) is unitarily diagonalizable.
one becomes an anti-Hermitian matrix. Our results and their implications are summarized in Figure 1.

We then generalize our lower bound results to inhomogeneous ODE systems. We show that general quantum ODE solvers have worst-case complexity $\Omega(e^{At} + \mu(A))$, where $\mu$ is defined to be 0 if all the eigenvalues have non-positive real parts and the real part gap otherwise. This lower bound differs from that for homogeneous systems because the shifting equivalence does not hold in the inhomogeneous case. Our lower bounds complement existing work on the upper bound side. Indeed existing quantum solvers of generic inhomogeneous ODE systems typically assume all the eigenvalues of $A$ have non-positive real parts [1–3], have complexity scaling with $O(\|A^0\|)$ [5], and/or scales linearly in some other equivalent measures of non-normality (e.g., the condition number $\kappa_V$ of the eigenbasis of $A$) [1–3]. Our lower bounds imply that the complexities of these existing algorithms are likely asymptotically optimal in the worst case.

Moreover, we prove lower bounds in terms of the time $T$ and error $\epsilon$ for solving inhomogeneous ODEs even when $\mu = 0$. For generic Hamiltonian simulation where $A$ is anti-Hermitian and $b = 0$, Berry et al. [9] proved an $\Omega(T)$ query lower bound which indicates that no generic fast-forwarding is possible. This bound was later improved to $\Omega(T + \log(1/\epsilon)/\log\log(1/\epsilon))$ by Berry, Childs, and Kothari [14]. However, these results do not say anything about what the lower bound should be for ODEs where $A$ is not anti-Hermitian or $b \neq 0$. We extend these results to the setting where $A$ has negative logarithmic norm — i.e., the eigenvalues of $(A + A^T)$ are negative — and $b \neq 0$. This setting is disjoint from that of Hamiltonian simulation since $b \neq 0$ and the logarithmic norm of an anti-Hermitian coefficient matrix is 0. In this setting, we prove by reduction that generic quantum ODE solvers require $\min \{\Omega(\log^\alpha(1/\epsilon)), \Omega(T^\alpha)\}$ queries if solving the linear system of equations $AX = b$ requires $\Omega(\log^\alpha(1/\epsilon))$ queries. Notice that our lower bound depends on the lower bound for solving a particular class of linear system of equations since $A$ has negative logarithmic norm. It is proved in [35] that when $A$ is unrestricted, solving $AX = b$ requires at least $\Omega(\log(1/\epsilon))$ queries. We leave to future work the investigation of how this lower bound changes if $A$ is restricted to have negative logarithmic norm.

1.1.2 Methods

We prove our lower bounds on ODE solvers by first introducing a general framework for proving lower bounds on a class of quantum algorithms that we call amplifiers. Our framework generalizes techniques used by Somma and Subasi to lower bound the complexity of verifying the output of a quantum linear system solver [36] and relates to those used to lower bound the complexity of quantum state discrimination [4–37]. Suppose we have access to unitaries $O$ and $O^{-1}$ (and their controlled versions) such that $O$ is promised to (consistently) prepare one of the states $|\psi\rangle$ or $|\phi\rangle$, with $|\langle\psi|\phi\rangle| \geq 1 - \epsilon$ (i.e., the two states are close). If we have an algorithm $A$ using queries to $O$ and $O^{-1}$ that amplifies the distance between $|\psi\rangle$ and $|\phi\rangle$ to a constant, then we can use $A$ a constant number of times to readily determine which state $O$ prepares [42]. But we know that the quantum state discrimination problem is hard when $|\psi\rangle$ and $|\phi\rangle$ are close, so $A$ must make many queries to $O$ and $O^{-1}$. We call quantum algorithms like $A$ that amplify the distance between two quantum states (to constant) amplifiers, and we show that amplifiers must use $\Omega(1/\sqrt{\epsilon})$ queries to $O$ and $O^{-1}$.

Our lower bounds on quantum ODE solvers follow directly from our framework since quantum ODE solvers are amplifiers: the time evolution operator $e^{AT}$ of non-quantum dynamics is no longer unitary and thus small differences in the initial states can lead to large differences in the final states.

We highlight the fact that our lower bound on amplifiers differs from the standard lower bound on quantum state discrimination because the two settings differ. In the standard state discrimination setup [1, 37, 40], an information-theoretic perspective is taken that only assumes access to copies of the quantum states that are to be discriminated. In contrast, our setup is motivated from an algorithmic perspective that allows access to quantum state preparation oracles $O$ and their inverses $O^{-1}$. Allowing access to $O^{-1}$ is more appropriate when proving lower bounds on quantum algorithms (like quantum ODE solvers) because we typically know the circuit of $O$ and can therefore easily implement $O^{-1}$ by inverting each gate in the circuit of $O$ and reversing their order. As a result of such stronger access to the quantum states, our lower bound is $\Omega(1/\sqrt{\epsilon})$ which is quadratically smaller than the state discrimination lower bound of $\Omega(1/\epsilon)$.

Our lower bound framework may be of independent interest since it can be used to prove lower bounds on any quantum algorithm that can be viewed as an amplifier. For example, it can be easily used to recover lower bounds on quantum linear system solvers. We have used our framework to understand the power of quantum algorithms for solving “non-quantum” problems. More generally, we expect our framework to help
Table 1: Summary of our fast-forwarding results. Here the top result (on the third line) is the scaling of the best generic quantum ODE solver [5] and serves as the baseline. The parameters \( g, g', \) and \( g'' \) are defined by

\[
g := \max_{t \in [0,T]} \frac{\|u(t)\|}{\|u(T)\|},
\]

\[
g' := \frac{(\|u(0)\| + \|b\|)/\|u(T)\|}{\|u(T)\|},
\]

and \( g'' := \left(\|u(0)\| + T^{-1} \int_0^T \|b(t)\|dt\right)/\|u(T)\|). \)

A dash means the corresponding parameter does not exist in that case. The complexities are measured by the number of queries to the input oracles of \( A, b(t) \) and \( u(0). \)

1.2 Fast-forwarding of tailored quantum ODE solvers

The worst-case lower bounds discussed above apply to quantum solvers of generic ODEs. However, these lower bounds may be bypassed by quantum solvers tailored to solve particular classes of ODEs (see Figure 2). Indeed, in the case of Hamiltonian simulation, we know for example that fast-forwarding is possible when the Hamiltonian is 1-sparse [43], can be diagonalized via an efficient quantum circuit [44], or corresponds to real-space dynamics [18, 45, 46] or quadratic fermionic and bosonic systems [44, 47].

In contrast to previous work, we study the fast-forwarding of particular classes of non-quantum dynamics. Our results and comparisons with the best generic algorithms are summarized in Table 1.
1.2.1 Homogeneous ODE

In the homogeneous case, we find two types of possible fast-forwarding, one giving a quadratic speedup in $T$ and another giving an exponential speedup in both $T$ and $\|A\|$.

We obtain quadratic speedups assuming $A$ is a negative semi-definite Hermitian matrix with bounded spectral norm or $A = -H^2$, for some accessible Hermitian matrix $H$. These results are similar to existing work that show how to quadratically speed up the preparation of quantum Gibbs states [48, 49]. They are also related to the quadratic fast-forwarding of quantum walks on graphs [50–52]. Implementing quantum walks can be seen as a special case of simulating homogenous ODEs where the coefficient matrix $A$ is associated with a Markov chain on a graph. At the heart of these results is the surprising fact from approximation theory that the monomial $x^t$ can be well-approximated by a polynomial of degree $O(\sqrt{t})$ over certain domains, see, e.g. [53, Theorem 3.3].

We obtain exponential speedups assuming $A$ is a normal matrix with an eigenbasis that can be efficiently prepared on a quantum computer and eigenvalues that can be efficiently computed classically. This assumption is similar to the “QC-solvable” condition of Atia and Aharonov [44], which they show implies a violation of the so-called computational time-energy uncertainty principle. We sometimes refer to the assumption as knowing the eigenvalues and eigenvectors of $A$. Under this assumption, we show how to implement a block-encoding of $e^{AT}$ with query complexity independent of $T$ and $\|A\|$. Our technique is similar to that used to fast-forward 1-sparse Hamiltonian simulation [43]. Suppose $A = UAU^{-1}$ is the eigen-decomposition of $A$. Then $e^{AT} = U e^{AT} U^{-1}$ can be efficiently computed, since $U$ is assumed to be efficiently implementable, and the diagonal transformation $e^{AT}$ can be implemented using controlled rotations with $O(1)$ query complexity. We remark that, although in both scenarios we can achieve speedup explicitly in the parameters $T$ and/or $\|A\|$, the overall complexities still suffer from the norm decay rate due to a post-selection step. As a result, the overall complexity of preparing the final solution might still have exponential overhead in the evolution time (when $A$ is strictly negative definite), and a genuine fast-forwarding can only be achieved by placing further assumptions on the input states (e.g., the input state has non-trivial overlap with the zero-energy eigenstate).

1.2.2 Inhomogeneous ODE

We generalize our fast-forwarding results to inhomogeneous ODEs. We first consider the time-independent inhomogeneous term $b(t) \equiv b$, and show that, once the homogeneous evolution operator $e^{AT}$ can be fast-forwarded, the corresponding inhomogeneous ODE can also be fast-forwarded with the same speedup. Our algorithm first separately computes the homogeneous and the inhomogeneous parts of the solution and then combines them together using the technique of linear combination of quantum states. More specifically, notice that the solution of Equation (1) with constant $b$ can be written as

$$u(T) = e^{AT}u(0) + \int_0^T e^{A(T-s)}b \, ds,$$

where the first part is exactly the solution of the corresponding homogeneous equation. For the second part, when $A$ is negative semi-definite, it can also be approximated by polynomial with quadratically reduced degree and can thus be efficiently prepared using the QSVT technique. When the eigenvalues and eigenstates of $A$ are known, the operator $\int_0^T e^{A(T-s)} ds$ also has known eigensystem and can be exponentially fast-forwarded as in the homogeneous case. Finally, the two parts of Equation (2) can be linearly combined via a circuit similar to that arising in the linear combination of unitaries (LCU) technique [27].

In the time-independent case, our algorithm for solving inhomogeneous ODE systems is conceptually different from existing quantum ODE solvers. To the best of our knowledge, all existing algorithms for inhomogeneous ODE systems require two extra steps: 1. dividing the entire time interval using a very small time step size, and 2. transforming the problem to that of solving a linear system of equations in much higher dimension. However, our algorithm takes a one-shot strategy in the sense that our algorithm directly maps the input state to the final solution. It neither requires time discretization nor solving a high-dimensional linear system of equations. This is a more natural and straightforward way of designing quantum ODE solvers and facilitates fast-forwarding. We also expect our one-shot strategy to be applicable to the design of quantum solvers of generic ODEs with generic coefficient matrices $A$, since $e^{AT}$ can still be block-encoded (though the complexity might be super linear in $T$ or $\|A\|$) using the contour integral method [6, 54, 55].
We further generalize the exponential speedup in $T$ and $\|A\|$ to the case where the inhomogeneous term $b(t)$ is time-dependent and its norm is known at all times. In this case, the contribution of the inhomogeneous term to the final solution, $\int_0^T e^{A(T-s)}b(s)ds$, becomes a genuine integral over time rather than a matrix-vector multiplication. Nevertheless, we can still efficiently compute such an integral as follows. We first discretize the integral using first-order numerical quadrature (e.g., the Riemann sum formula), which results in a linear combination of vectors which again can be computed using the linear combination of quantum states. Although the number of the quadrature nodes needs to be very large to control the approximation error, the computational overhead is only logarithmic thanks to the efficiency of the LCU technique. Overall, the complexity of computing the final solution remains independent of the time $T$ and the norm $\|A\|$. Our generalization to time-dependent $b(t)$ crucially relies on the fact that the integral can be efficiently computed with only logarithmic scaling in the number of quadrature nodes. This fact is also the key to the efficiency of the truncated Dyson series method for interaction-picture Hamiltonian simulation [18] and the qHOP method for time-dependent Hamiltonian simulation [56]. In the time-dependent case, although we use time discretization to approximate the integral due to $b(t)$'s time dependence, our algorithm still does not require solving a high-dimensional linear system of equations.

1.2.3 Application to evolutionary PDEs

We identify various instances of high-dimensional systems of linear time-evolutionary partial differential equations (PDEs) that can be fast-forwarded exponentially in $T$ and $\|A\|$. Generally speaking, evolutionary PDEs can be classified into three types: parabolic (with degeneration), hyperbolic, and higher-order. All of these equations involve spatial differential operators such as the divergence and Laplacian operators. A key observation is that these differential operators (after spatial discretization) have known eigenvalues and eigenstates, since they can be diagonalized by the quantum Fourier transform circuit and the eigenvalues are explicit expressions in frequencies. Therefore we can apply our fast-forwarded algorithm for ODEs with known eigensystem.

In Section 4.6, we study parabolic PDEs and show how to simultaneously compute the eigenvalues and prepare the eigenstates of the (spatially discretized) divergence and Laplacian operators. We discuss typical instances of parabolic PDEs that can be fast-forwarded, including the transport equation from continuum and statistical physics, the inhomogeneous heat equation from thermodynamics, and the advection-diffusion equation that models chemical processes as well as biological and ecological networks.

In Section 4.7, we study hyperbolic PDEs. After spatial discretization, such a PDE involves a second-order time derivative. Our first step is to transform the PDE into a first-order ODE system by lifting. We then solve the first-order ODE system using our fast-forwarding techniques. Typical instances of hyperbolic PDEs that can be fast-forwarded include the wave equation that models fluid mechanics and electromagnetism and the Klein-Gordon equation from quantum field theory.

Our techniques are not limited to second-order PDEs. In Section 4.8, we show how two examples of higher-order PDEs can also be fast-forwarded, namely the Airy equation that models fluid and plasma dynamics and the beam equation from civil, mechanical, and aeronautical engineering.

1.3 Organization

The rest of the paper is organized as follows. We first introduce preliminary results in Section 2, including the notation, basic linear algebra results, the concept and operation of block encodings, and existing generic quantum ODE solvers. Then we discuss and prove our lower bound results for generic ODE solvers in Section 3 and fast-forwarding algorithms and results for specific applications in Section 4. We conclude with a brief summary and several open questions in Section 5.

2 Preliminaries

In this section, we introduce our notation and preliminary lemmas used in later analysis and discuss existing generic algorithms for solving ODEs.
2.1 Notation

Let \( u = (u_0, u_1, \cdots, u_{N-1})^T \) be an \( N \)-dimensional (possibly unnormalized) column vector. We use \( u^* \) to denote its conjugate transpose, \( \|u\| \) to denote its vector 2-norm, and \( \|u\|_1 \) to denote its vector 1-norm. The notation \( |u\rangle \) represents a (pure) quantum state that is the normalized vector under 2-norm

\[
|u\rangle = u/\|u\|.
\]

For quantum states, we follow the standard bra-ket notation where \( \langle u | \) denotes the conjugate transpose of \( |u\rangle \) and \( \langle u | v \rangle \) denotes the inner product between \( |u\rangle \) and \( |v\rangle \).

For two quantum states \( |u\rangle \) and \( |v\rangle \), we write \( |u\rangle_p |v\rangle_u \), as the multiple-register quantum state constructed by the tensor product of \( |u\rangle \) and \( |v\rangle \). Here the subscript “a” means “ancilla” and “w” means “working register.” We may use other letters (to be specified later) to represent the name of the corresponding register in the subscripts. The notation |⊥⟩ generally represents the parts in a quantum state that are orthogonal to those of interest.

Let \( A \) be an \( N \)-by-\( N \) matrix. We use \( A^\dagger \) to denote its conjugate transpose, i.e., the adjoint operator of \( A \). The norm \( \|A\| \) without subscript represents the spectral norm, i.e., the matrix 2-norm

\[
\|A\| = \sup_{u \neq 0} \|Au\|/\|u\|.
\]

The norm \( \|A\|_1 \) with subscript 1 is the Schatten 1-norm

\[
\|A\|_1 = \text{Tr} \left( \sqrt{A^\dagger A} \right).
\]

The trace distance between two matrices \( A \) and \( B \) is defined to be \( 1/2 \|A - B\|_1 \), and the trace distance between two pure states \( |u\rangle \) and \( |v\rangle \) is the trace distance between the corresponding density matrices \( |u\rangle \langle u| \) and \( |v\rangle \langle v| \).

Let two functions \( f, g : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \) represent some complexity scalings. We write \( f = O(g) \) if there exists a constant \( C > 0 \), independent of the arguments of \( f \) and \( g \), such that \( f(x) \leq Cg(x) \) for all sufficiently large \( x \). \( f = \Omega(g) \) if \( g = O(f) \). \( f = \Theta(g) \) if \( f = O(g) \) and \( f = \Omega(g) \). \( f = \tilde{O}(g) \) if \( f = O(g \text{ poly log}(g)) \).

2.2 Linear algebra results

We discuss several linear algebra results that will be frequently used in our analysis. The first one is useful for bounding the error and the success probability related to quantum states from those related to unnormalized vectors. The second one is the relations of the overlaps between quantum states or the approximate versions. The third one is about the relations among trace distance, fidelity and 2-norm for pure states. Here we only state the results, and the proofs can be found in Appendix A.

**Lemma 1.** For any two non-zero vectors \( a, \bar{a} \) such that \( \|a - \bar{a}\| \leq \epsilon \), we have

1. \( \|\bar{a}\| \geq \|a\| - \epsilon \),
2. \( \|a/\|a\| - \bar{a}/\|ar{a}\|\| \leq 2\epsilon/\|a\| \).

**Lemma 2.** Let \( |\psi\rangle, |\phi\rangle, |\overline{\psi}\rangle \) and \( |\overline{\phi}\rangle \) be quantum states. Then

\[
|\langle \psi | \overline{\psi}\rangle| \leq |\langle \psi | \phi \rangle| + \| |\overline{\psi}\rangle - |\psi\rangle \| + \| |\overline{\phi}\rangle - |\phi\rangle \|. \tag{6}
\]

**Lemma 3.** For two pure states \( |\psi\rangle \) and \( |\phi\rangle \), we have

\[
\frac{1}{2} \| |\psi\rangle \langle \psi| - |\phi\rangle \langle \phi| \|_1 = \sqrt{1 - |\langle \psi | \phi \rangle|^2} \leq \| |\psi\rangle - |\phi\rangle \|. \tag{7}
\]
2.3 Block-encoding

Block-encoding is a powerful technique to represent arbitrary matrices with unitary matrices. For a possibly non-unitary matrix $A$ and a parameter $\alpha \geq \|A\|$, the intuitive idea of block-encoding is to construct a unitary $U$ with higher dimension such that $A$ appears in its upper-left block,

$$U = \begin{pmatrix} A/\alpha & * \\ * & * \end{pmatrix}.$$  \hfill (8)

A formal definition with the presence of errors is as follows [48].

**Definition 4** (Block-encoding). Suppose $A$ is a $2^n$-dimensional matrix such that $\|A\| \leq \alpha$. Then a $2^{n+n_a}$-dimensional unitary matrix $U$ is an $(\alpha, n_a, \epsilon)$-block-encoding of $A$, if

$$\|A - \alpha(0|_a \otimes I)U(0|_a \otimes I)\| \leq \epsilon.$$  \hfill (9)

Block-encoding allows us to encode a general matrix on quantum computers and then perform matrix operations, including summation, multiplication, inverse, and polynomial transformation. Here we mostly follow [48] again and briefly summarize several properties that will be used in our work.

**Definition 5** (State preparation pair). Let $y$ be a $2^n$-dimensional vector and $\|y\|_1 \leq \beta$. Then a pair of unitaries $(P_L, P_R)$ is a $(\beta, n, 0)$-state-preparation-pair, if $P_L |0\rangle = \sum_{j=0}^{2^n-1} c_j |j\rangle$, $P_R |0\rangle = \sum_{j=0}^{2^n-1} d_j |j\rangle$ and $\beta c_j^* d_j = y_j$ for any $0 \leq j \leq 2^n - 1$.

**Lemma 6** (Linear combination of block-encodings [48, Lemma 52]). Let $A = \sum_{j=0}^{2^n-1} y_j A_j$ be a $2^n$-dimensional matrix. Suppose that $(P_L, P_R)$ is a $(\beta, k, 0)$-state-preparation-pair of $y$, and $W = \sum_{j=0}^{2^n-1} |j\rangle \langle j| \otimes U_j$ is a $2^{n+n_a+k}$-dimensional unitary matrix such that $U_j$ is an $(\alpha, n_a, \epsilon)$-block-encoding of $A_j$. Then $(P_L^\dagger \otimes I_{n_a} \otimes I_n) W (P_R \otimes I_{n_a} \otimes I_n)$ is an $(\alpha \beta, n_a + k, \alpha \beta \epsilon)$-block-encoding of $A$, with a single use of $W, P_L^\dagger$ and $P_R$.

**Lemma 7** (Multiplication of block-encodings [48, Lemma 53]). Let $A, B$ be $2^n$-dimensional matrices, $U_A$ be an $(\alpha, n_a, \delta)$-block-encoding of $A$, and $U_B$ be a $(\beta, n_b, \epsilon)$-block-encoding of $B$. Then $(I_{n_b} \otimes U_A)(I_{n_a} \otimes U_B)$ is an $(\alpha \beta, n_a + n_b, \alpha \epsilon + \beta \delta)$-block-encoding of $AB$, with a single use of $U_A$ and $U_B$.

**Lemma 8** (Inverse of a block-encoding [55, Appendix B]). Suppose $A$ is a $2^n$-dimensional invertible Hermitian matrix such that all the eigenvalues are within $[-1, -\delta] \cup [\delta, 1]$, and $U_A$ is a $(1, n_A, 0)$-block-encoding of $A$. Then a $(4/(3\delta), n_A + 1, \epsilon)$-block-encoding of $A^{-1}$ can be constructed, using $O((1/\delta) \log(1/(\delta \epsilon)))$ queries to $U_A$ and its inverse.

**Lemma 9** (Polynomial of a block-encoding [48, Theorem 56]). Let $A$ be a $2^n$-dimensional Hermitian matrix, and $U_A$ is an $(\alpha, n_A, \epsilon)$-block-encoding of $A$. If $P(x)$ is a degree-$d$ real polynomial such that $|P(x)| \leq 1/2$ for all $x \in [-1, 1]$, then a $(1, n_A + 2, 4d \sqrt{\epsilon/\alpha})$-block-encoding of $P(A/\alpha)$ can be constructed using $d$ applications of $U_A$ and $U_A^{-1}$, a single application of controlled $U$ and $O((n_A + 1)d)$ additional one- and two-qubit gates.

2.4 Existing generic algorithms for solving ODEs

The best generic algorithm for solving ODEs with time-independent $A$ and $b$ is [5] based on Taylor’s expansion, and that for ODE with time-dependent $b(t)$ is the quantum spectral method [3].\(^3\) We summarize the main results of these two algorithms for comparison with our lower bounds and fast-forwarding results.

**Lemma 10** ([5]). Consider solving Equation (1) with time-independent $b$. Suppose $A$ is an $N$-by-$N$ $s$-sparse matrix with sparse input oracles. For $u(0)$ and $b$ we assume that their norms are known and their prepare oracles are given. Then there exists a quantum algorithm which produces an $\epsilon$-approximation of $u(T)/\|u(T)\|$ in the 2-norm sense, succeeding with probability $\Omega(1)$ with a flag indicating success, with query complexity

$$\tilde{O}\left( gT\|A\|C(A)\text{poly}(s, \log(N), \log(1 + T e^2\|b\|/\|u(T)\|), \log(1/\epsilon)) \right),$$  \hfill (10)

\(^3\)The quantum spectral method can also deal with the time-dependent coefficient matrix $A(t)$, but in our work, we constrain ourselves with time-independent $A$.\(^1\)
where
\[
g = \sup_{t \in [0, T]} \|u(t)\| / \|u(T)\|, \quad C(A) = \sup_{t \in [0, T]} \|e^{At}\|.
\]

Lemma 11 ([3]). Consider solving Equation (1) with time-dependent \(b(t)\). Suppose \(A\) is an \(N\)-by-\(N\) \(s\)-sparse matrix with sparse input oracles, and can be diagonalized as \(A = V \Lambda V^{-1}\), where \(\Lambda = \text{diag}(\lambda_0, \ldots, \lambda_{N-1})\) with \(\text{Re}(\lambda_j) \leq 0\) for each \(j\). For \(u(0)\) and \(b(t)\) we assume that their norms are known and their prepare oracles are given. Then there exists a quantum algorithm which produces an \(\epsilon\)-approximation of \(u(T)/\|u(T)\|\) in the 2-norm sense, succeeding with probability \(\Omega(1)\) with a flag indicating success, with query complexity
\[
\tilde{O}\left(\frac{gT\|A\|s\kappa_V}{\epsilon\|u(T)\|} \right) \text{ polylog} \left(\frac{\max_{t \in [0,T]} \|u^{(k)}(t)\|}{\epsilon\|u(T)\|}\right),
\]
where
\[
g = \sup_{t \in [0, T]} \frac{\|u(t)\|}{\|u(T)\|}, \quad \kappa_V = \|V\|\|V^{-1}\|.
\]

In most existing algorithms based on solving a linear system of equations, the scaling in the number of queries to \(A\) and to the prepare oracle of \(u(0)\) are asymptotically the same. However, a recent work [6] proposes a time-marching strategy for solving homogeneous ODE where the number of queries to the prepare oracle is asymptotically smaller. We summarize the result as follows.\(^4\)

Lemma 12 ([6]). Consider solving the homogeneous case of Equation (1) (i.e., \(b = 0\)). Suppose \(\|u(0)\| = 1\), and we are given an \((\alpha, n_A, 0)\)-block-encoding of the matrix \(A\) and the prepare oracle of \(u(0)\). Then there exists a quantum algorithm which produces an \(\epsilon\)-approximation of \(u(T)/\|u(T)\|\) in the 2-norm sense, succeeding with probability \(\Omega(1)\) with a flag indicating success, using
\[
\tilde{O}\left(\alpha^2T^2Q \log(1/\epsilon)\right)
\]
queries to the block-encoding of \(A\), and
\[
\mathcal{O}(Q)
\]
queries to the prepare oracle of \(u(0)\). Here
\[
Q = \frac{\|e^{AT}\|}{\|u(T)\|}.
\]

3 Limitations

In this section, we consider ODEs with time-independent coefficient matrix \(A\) and inhomogeneous term \(b\)
\[
\frac{d}{dt}u(t) = Au(t) + b, \quad t \in [0, T],
\]
\[
u(0) = u_{\text{in}}.
\]

We discuss the limitations of generic quantum algorithms for solving ODEs by proving worst-case lower bounds. Motivated by the quantum state discrimination problem, we first discuss and prove a lower bound for quantum algorithms as amplifiers with the access of the prepare oracle and its inverse. Then we introduce the idea of the lower bounds for quantum ODE solvers using the lower bound of amplifiers and prove them in the homogeneous case. After that, we discuss the quantum shifting equivalence for homogeneous ODEs and explain the implications of our lower bounds. Finally, we present a generalization to the inhomogeneous case, implying that existing quantum algorithms cannot be significantly improved.
3.1 Quantum state discrimination

The goal of quantum state discrimination is to distinguish two quantum states $|\psi\rangle$ and $|\phi\rangle$ with large overlap. Standard results from the information-theoretic perspective show that, in order to distinguish two states with a constant average probability of error, the copies of required states scale at least linearly in terms of the infidelity between two states [37, 40].

Here we study a different scenario to perform state discrimination with stronger assumptions. Let $O_\psi$ and $O_\phi$ be two oracles such that $O_\psi|0\rangle = |\psi\rangle$ and $O_\phi|0\rangle = |\phi\rangle$. Suppose that we are consistently given one of the two oracles, and we aim at determining which one is given. Then a general algorithm with query access can be illustrated as in Figure 3. Here a generic circuit is decomposed into layers and segments based on whether the operation depends on the oracles to be determined or not. Specifically, in Figure 3, $U_1, \cdots, U_{q+1}$ are unitaries which are independent of the choice of $|\psi\rangle$ and $|\phi\rangle$, and can be a composition of several unitary evolutions and gates. $O$ is the given oracle, either $O_\psi$ or $O_\phi$, and its inverse and controlled versions are also accessible. The algorithm alternatively applies $U_j$ and $O$, its inverse or controlled versions. After all the unitary evolutions, some projection measurements are performed over the ancillas to get a desired output quantum state in the working registers. Here $O$ can be applied to or controlled by any register, $U_j$ are arbitrary unitaries independent of the states to be determined, and the measurements are postponed without loss of generality thanks to the principle of deferred measurement. Therefore Figure 3 represents a generic quantum algorithm using the oracle $O$ to output another quantum state.

Let $\rho_\psi$ be the output state with $O_\psi$. For two quantum states $|\psi\rangle$ and $|\phi\rangle$ such that $|\langle \psi | \phi \rangle| \geq 1 - \epsilon$, we call a quantum algorithm an amplifier if the outputs $\rho_\psi$ and $\rho_\phi$ can be obtained with constant probability and have $\Omega(1)$ trace distance. We claim that the query complexity of an amplifier must have some lower bound, since the quantum state discrimination can be solved with the help of an amplifier as illustrated in Figure 4. Specifically, with the given oracle $O$, we perform the amplifier to get the output state $\rho$ and perform the standard state discrimination technique over the output states. This is an alternative approach to distinguish $|\psi\rangle$ and $|\phi\rangle$, and therefore should be more expensive if the overlap between $|\psi\rangle$ and $|\phi\rangle$ is larger. Meanwhile, the cost of determining the output states $\rho_\psi$ and $\rho_\phi$ takes $O(1)$ time since they have at least $\Omega(1)$ trace distance. This implies that the complexity of the amplifier must be high.

The quantitative lower bound on the query complexity of an amplifier is given as follows. The proof

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4Similar as the quantum spectral method, the time marching method can also deal with the time-dependent coefficient matrix $A(t)$, but in our work we constrain ourselves with time-independent $A$. 

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is inspired by [36], and we remark that, although our lower bound for amplifiers is related to and can be
intuitively understood via the quantum state discrimination problem, the following theorem is standalone
and does not use any results from the quantum state discrimination problem.

**Theorem 13.** Let $|\psi\rangle$ and $|\phi\rangle$ be two quantum states such that $\langle\psi|\phi\rangle$ is real and $\langle\psi|\phi\rangle \geq 1 - \epsilon$. Suppose
that we are given an oracle $O$ and promised that it is consistently either $O_{\psi}$ or $O_{\phi}$, such that $O_{\psi}|0\rangle = |\psi\rangle$
and $O_{\phi}|0\rangle = |\phi\rangle$. Let an amplifier be a quantum algorithm in Figure 3 that outputs either $\rho_{\psi}$ or $\rho_{\phi}$ with
$\Omega(1)$ success probability such that $\|\rho_{\psi} - \rho_{\phi}\| = \Omega(1)$. Then the amplifier must use $\Omega(1/\sqrt{\epsilon})$
queries to the given oracle, its inverse or controlled versions in the worst case.

**Proof.** Let the quantum circuit before measurement be $U_{\alpha} = U_{q+1}V_{q,u}U_{q} \cdots V_{2,u}U_{2}V_{1,u}U_{1}$, where $U_{j}$’s are
unitaries independent of the state $u$, and $V_{j,u}$ denotes a single application of $O_{u}$, its inverse or controlled version. Then

\[
\left\|U_{\psi}|0\rangle \langle 0|U_{\psi}^{\dagger} - U_{\phi}|0\rangle \langle 0|U_{\phi}^{\dagger}\right\|_{1} \\
\leq 2 \left\|U_{\psi}|0\rangle - U_{\phi}|0\rangle\right\| \\
\leq 2 \|U_{\psi} - U_{\phi}\| \\
= 2 \|U_{q+1}V_{q,u}U_{q} \cdots V_{2,u}U_{2}V_{1,u}U_{1} - U_{q+1}V_{q,\phi}U_{q} \cdots V_{2,\phi}U_{2}V_{1,\phi}U_{1}\| \\
= 2 \sum_{j=1}^{q} \|V_{j,\psi}U_{j} - V_{j,\phi}U_{j}\| \\
= 2 \sum_{j=1}^{q} \|V_{j,\psi} - V_{j,\phi}\|.
\]

We bound $\|V_{j,\psi} - V_{j,\phi}\|$ by first analyzing $\|O_{\psi} - O_{\phi}\|$ in the worst case. Here the worst case means that,
besides preparing the desired state from $|0\rangle$, the prepare oracle $O$ is required to satisfy further assumptions
on the orthogonal subspace. Specifically, for two quantum states $|\psi\rangle$ and $|\phi\rangle$, there exist two prepare oracles
$O_{\psi}$ and $O_{\phi}$ such that

\[
\|O_{\psi} - O_{\phi}\| = \| |\psi\rangle - |\phi\rangle \|.
\]

For example, an explicit construction, given in [36, Eqs. (A26-A29)], is that $O_{\phi}$ is a rotation of $O_{\psi}$ in the
two-dimensional subspace orthogonal to $|\phi\rangle$ and $|\psi\rangle$:

\[
O_{\phi} = e^{i\theta M}O_{\psi}, \\
\theta = \arccos(\langle\phi|\psi\rangle), \\
M = i|\psi\rangle \langle \psi^{\perp}| - i|\psi^{\perp}\rangle \langle \psi|, \\
|\psi^{\perp}\rangle = (I - |\psi\rangle \langle \psi|)|\phi\rangle / \| (I - |\psi\rangle \langle \psi|) |\phi\rangle \|.
\]

Then, noting that $\langle\psi|\phi\rangle$ is real,

\[
\|O_{\psi} - O_{\phi}\| = \| |\psi\rangle - |\phi\rangle \| = \sqrt{2(1 - \langle\psi|\phi\rangle)} \leq \sqrt{2\epsilon},
\]

and

\[
\|O_{\psi}^{-1} - O_{\phi}^{-1}\| = \|O_{\phi}(O_{\psi}^{-1} - O_{\phi}^{-1})O_{\psi}\| = \|O_{\psi} - O_{\phi}\| \leq \sqrt{2\epsilon}.
\]

The controlled versions of $O$ and $O^{-1}$ satisfy the same bound. Therefore, in the worst case, we have

\[
\|V_{j,\psi} - V_{j,\phi}\| \leq \sqrt{2\epsilon},
\]

and thus

\[
\left\|U_{\psi}|0\rangle \langle 0|U_{\psi}^{\dagger} - U_{\phi}|0\rangle \langle 0|U_{\phi}^{\dagger}\right\|_{1} \leq 2q\sqrt{2\epsilon}.
\]
Measuring \( U_\psi |0\rangle \) or \( U_\phi |0\rangle \) and succeeding give the states \( \rho_\psi \) or \( \rho_\phi \). Since the success probability is \( \Omega(1) \), the normalization factor due to the measurement is \( \Omega(1) \), and we have
\[
\| \rho_\psi - \rho_\phi \|_1 \leq O(q \sqrt{\epsilon}).
\] (25)
Together with \( \| \rho_\psi - \rho_\phi \|_1 = \Omega(1) \) according to the definition of an amplifier, we must have
\[
q = \Omega(1/\sqrt{\epsilon}).
\] (26)

We remark that standard lower bounds for quantum state discrimination problem typically only assume access to multiple copies of the state, and show that the number of the copies required is \( \Omega(1/\epsilon) \) \cite{4, 37, 40}. Our result in Theorem 13 assumes a stronger query access, i.e., the prepare oracle, its inverse and controlled versions, and the query lower bound becomes quadratically worse. Another remark is that, though in this paper we focus on the problem of solving ODEs, Theorem 13 holds for general quantum algorithms that can be viewed as amplifiers for at least one pair of quantum states and thus can potentially be applied to study other problems with certain “non-quantumness”. For example, \cite{4} applies a similar idea to prove a lower bound for solving generic non-linear ODEs in terms of non-linearity. In Appendix B, we show how to use Theorem 13 to recover existing lower bounds for solving linear systems of equations.

### 3.2 Proving lower bounds on generic quantum ODE solvers

The central idea of our lower bound proofs is to regard an ODE solver as an amplifier. Non-quantum ODE with specific features may amplify the difference between two quantum states through the time evolution due to its non-unitary evolution operator \( e^{AT} \), and thus satisfies the definition of an amplifier and has a worst-case query lower bound according to Theorem 13.

More specifically, consider the system in Equation (17). The solution can be explicitly given as
\[
u(T) = e^{AT}u(0) + \int_0^T e^{A(T-s)}bds.
\] (27)
Let \( v(T) \) denote the solution to Equation (17) with a differential initial condition \( v(0) \), then
\[
u(T) - v(T) = e^{AT}(u(0) - v(0)).
\] (28)

Equation (28) indicates that the difference between \( u(T) \) and \( v(T) \) may be amplified through the operator \( e^{AT} \). For example, let us assume that the spectral norm of \( e^{AT} \) is large and \( w = \operatorname{argmax}\| e^{AT}x \|/\|x \|. \) If we choose \( u(0) - v(0) \) to be proportional to \( w \), then \( \| u(T) - v(T) \| = \| e^{AT}\|\| u(0) - v(0) \| \), which indicates that a quantum ODE solver may serve as an amplifier for \( |u(0)\rangle \) and \( |v(0)\rangle \) and the cost has a lower bound according to Theorem 13.

We remark that even if \( \|e^{AT}\| \) is large, Equation (28) does not necessarily imply that \( |u(T)\rangle \) and \( |v(T)\rangle \) have large distance. One reason is that the difference between \( u(T) \) and \( v(T) \) still depends on the overlap between \( u(0) - v(0) \) and the “worst” vector that \( e^{AT} \) can amplify. Another important reason is that a large difference between \( u(T) \) and \( v(T) \) does not necessarily imply a large difference between \( |u(T)\rangle \) and \( |v(T)\rangle \) due to possible normalization. Consider a trivial example with \( A = aI \) and \( b = 0 \) for some real number \( a > 0 \). The solution of the ODE becomes \( u(T) = e^{aT}u(0) \) and thus \( \| u(T) \| = \| u(0) \|. \) As a result, \( \| u(T) - v(T) \| = \| u(0) - v(0) \| \), and the distance between quantum states is preserved, although \( \| e^{aT} \| \) or even \( \| e^{aT}(u(0) - v(0))\| \) may be very large.\footnote{This is also related to the shifting equivalence for homogeneous ODEs. See Section 3.3.3.}

Therefore more careful computations are needed to identify the intrinsic sources of any computational overhead and prove the lower bounds.

### 3.3 Homogeneous case

We first study homogeneous ODEs, i.e., \( b = 0 \) in Equation (17). Typical examples of homogeneous ODEs include quantum dynamics, where \( A \) is an anti-Hermitian matrix, and imaginary time evolution, where \( A \)
is a Hermitian matrix. For technical simplicity, we consider a diagonalizable matrix $A$, i.e., there exist an invertible matrix $V$ and a diagonal matrix $D = \text{diag}(\lambda_1, \cdots, \lambda_N)$, such that $A = VDV^{-1}$. Note that here we do not assume $V$ to be unitary. We will first show that, once there exists a pair of eigenvalues with different real parts, then the scaling of generic quantum algorithms for this differential equation must be at least exponential in evolution time and real part gap. Our second result is that generic quantum ODE solvers will become less efficient if the coefficient matrix $A$ is more “non-normal”. These two hardness results together imply that “quantum algorithms for quantum dynamics” is the most efficient. Finally, we apply our results to quantum algorithms for imaginary time evolution and show that the cost scales at least linearly in the decay of the solution norm.

3.3.1 Eigenvalues with different real parts

**Proposition 14.** Given a diagonalizable matrix $A = VDV^{-1}$ where $D = \text{diag}(\lambda_1, \cdots, \lambda_N)$ is a diagonal matrix and $V$ is an invertible matrix. Suppose that $b = 0$. Then there exists an instance of the initial condition $u(0)$ of Equation (17) such that any quantum algorithm outputting $u(T)/\|u(T)\|$ with bounded error and failure probability must have worst-case query complexity $\Omega\left(e^{T\max_{i \neq j} |\Re(\lambda_i - \lambda_j)|}\right)$ to the prepare oracle of $|u(0)\rangle$, its inverse or controlled versions.

*Proof.* Let $\lambda_j = \alpha_j + i\beta_j$ where $\alpha_j$ and $\beta_j$ are the real part and the imaginary part of $\lambda_j$, respectively. Without loss of generality, assume that

$$\max_{i \neq j} |\Re(\lambda_i - \lambda_j)| = \Re(\lambda_1 - \lambda_2) = \alpha_1 - \alpha_2 > 0. \quad (29)$$

Let $V = (v_1, \cdots, v_N)$ where each $v_j$ is the eigenvector of $A$ corresponding to the eigenvalue $\lambda_j$ and are normalized such that $\|v_j\| = 1$. Notice that $V$ is not necessarily unitary so $\langle v_i | v_j \rangle$’s are not necessarily 0, but we always have

$$0 \leq |\langle v_i | v_j \rangle| < 1. \quad (30)$$

For any $0 < \epsilon < 1$, we consider solving Equation (17) with two possible initial conditions

$$u(0) = e^{i\theta} v_2, \quad w(0) = \sqrt{\epsilon} v_1 + \xi v_2. \quad (31)$$

Here $\theta$ is chosen such that $\langle u(0) | w(0) \rangle$ is real, and $\xi$ is chosen such that $\|w(0)\| = 1$, i.e.,

$$1 = w(0)^* w(0) = \epsilon + |\xi|^2 + 2\Re(\sqrt{\epsilon} \langle v_1 | v_2 \rangle). \quad (32)$$

Let the evolution time be

$$T = \frac{1}{2(\alpha_1 - \alpha_2)} \log(1/\epsilon). \quad (33)$$

The solutions $u(T)$ and $w(T)$ can be solved as follows. We start with the equation $V^{-1}V = I$, which implies that

$$V^{-1} v_j = e_j, \quad (34)$$

where $e_j$ is the vector with the only non-zero entry to be 1 at the $j$-th position. As a result,

$$e^{AT} v_j = V e^{DT} V^{-1} v_j = V e^{DT} e_j = e^{\lambda_j T} v_j. \quad (35)$$

Therefore,

$$u(T) = e^{i\theta} e^{\lambda_2 T} v_2 = e^{i\theta} e^{\alpha_2 T} e^{i\beta_2 T} v_2, \quad (36)$$

and

$$w(T) = \sqrt{\epsilon} e^{\lambda_1 T} v_1 + \xi e^{\lambda_2 T} v_2 = \sqrt{\epsilon} e^{\alpha_1 T} e^{i\beta_1 T} v_1 + \xi e^{\alpha_2 T} e^{i\beta_2 T} v_2. \quad (37)$$

We now compute the fidelity of the input states and the output states. For the input states, we have

$$|\langle u(0) | w(0) \rangle| = \sqrt{|\langle (\sqrt{\epsilon} v_1 + \xi v_2) | (\sqrt{\epsilon} v_1 + \xi v_2) \rangle|}$$

$$= \sqrt{\epsilon |\langle v_1 | v_2 \rangle|^2 + |\xi|^2 + 2\Re(\sqrt{\epsilon} \langle v_1 | v_2 \rangle)}. \quad (38)$$

$$\sqrt{\epsilon |\langle v_1 | v_2 \rangle|^2 + |\xi|^2 + 2\Re(\sqrt{\epsilon} \langle v_1 | v_2 \rangle)}.$$
Using Equation (32), we have

$$| \langle u(0)|w(0) \rangle | = \sqrt{\epsilon} | \langle v_1|v_2 \rangle |^2 + 1 - \epsilon$$

$$\geq \sqrt{1 - \epsilon}$$

$$> 1 - \epsilon.$$  \hfill (39)

For the output states, we have

$$| \langle u(T)|w(T) \rangle |^2 = \frac{w(T)^*u(T)u(T)^*w(T)}{\|u(T)\|^2\|w(T)\|^2}$$

$$= \frac{(\sqrt{\epsilon}e^{\alpha_1 T}u_{\beta_1}v_1 + \xi e^{\alpha_2 T}u_{\beta_2}v_2)^*v_2v_2^*(\sqrt{\epsilon}e^{\alpha_1 T}e^{i\beta_1}v_1 + \xi e^{\alpha_2 T}e^{i\beta_2}v_2)}{\|\sqrt{\epsilon}e^{\alpha_1 T}e^{i\beta_1}v_1 + \xi e^{\alpha_2 T}e^{i\beta_2}v_2\|^2}$$

$$= \frac{(\sqrt{\epsilon}e^{(\alpha_1-\alpha_2) T}e^{i(\beta_1-\beta_2)}v_1 + \xi v_2)^*v_2v_2^*(\sqrt{\epsilon}e^{(\alpha_1-\alpha_2) T}e^{i(\beta_1-\beta_2)}v_1 + \xi v_2)}{\|\sqrt{\epsilon}e^{(\alpha_1-\alpha_2) T}e^{i(\beta_1-\beta_2)}v_1 + \xi v_2\|^2}. \hfill (40)$$

The choice of $T$ in Equation (33) ensures that $\sqrt{\epsilon}e^{(\alpha_1-\alpha_2) T} = 1$, and thus

$$| \langle u(T)|w(T) \rangle |^2 = \frac{(e^{i(\beta_1-\beta_2) T}v_1 + \xi v_2)^*v_2v_2^*(e^{i(\beta_1-\beta_2) T}v_1 + \xi v_2)}{e^{i(\beta_1-\beta_2) T}v_1 + \xi v_2}$$

$$= \frac{|\langle v_1|v_2 \rangle |^2 + |\xi|^2 + 2\text{Re}(e^{-i(\beta_1-\beta_2) T} \xi \langle v_1|v_2 \rangle)}{1 + |\xi|^2 + 2\text{Re}(e^{-i(\beta_1-\beta_2) T} \xi \langle v_1|v_2 \rangle)}. \hfill (41)$$

Notice that

$$|2\text{Re}(e^{-i(\beta_1-\beta_2) T} \xi \langle v_1|v_2 \rangle)| \leq 2|e^{-i(\beta_1-\beta_2) T} \xi \langle v_1|v_2 \rangle| \leq |\xi|^2 + |\langle v_1|v_2 \rangle |^2,$$  \hfill (42)

and that function $f(x) = \frac{\alpha + x}{b + x}$ with $b > a > 0$ is monotonically increasing for $x \geq -a$, then we can further bound the final fidelity as

$$| \langle u(T)|w(T) \rangle |^2 \leq \frac{2|\langle v_1|v_2 \rangle |^2 + 2|\xi|^2}{1 + |\langle v_1|v_2 \rangle |^2 + 2|\xi|^2}. \hfill (43)$$

Using Equation (32), we can bound $|\xi|^2$ as

$$|\xi|^2 = 1 - \epsilon - 2\text{Re}(\sqrt{\epsilon} \xi \langle v_1|v_2 \rangle)$$

$$\leq 1 + 2|\sqrt{\epsilon} \xi \langle v_1|v_2 \rangle|$$

$$\leq 1 + 2|\xi|,$$  \hfill (44)

which implies that

$$|\xi| \leq 1 + \sqrt{2}. \hfill (45)$$

Therefore,

$$| \langle u(T)|w(T) \rangle | \leq \sqrt{\frac{2|\langle v_1|v_2 \rangle |^2 + 2(3 + 2\sqrt{2})}{1 + |\langle v_1|v_2 \rangle |^2 + 2(3 + 2\sqrt{2})}} =: C. \hfill (46)$$

Here $C < 1$ and only depends on $V$, and thus we regard $C$ as a constant and will absorb it into the notation $O$ and $\Omega$.

Given an oracle to consistently prepare either $|u(0)\rangle$ or $|w(0)\rangle$, we denote $|\bar{u}(T)\rangle$ and $|\bar{w}(T)\rangle$ as the corresponding outputs of a quantum ODE solver with 2-norm distance at most $(1 - C)/4$ of $|u(T)\rangle$ and $|w(T)\rangle$, respectively. According to Lemma 3, Lemma 2, Equation (39) and Equation (46), we have

$$\| |\bar{u}(T)\rangle \langle u(T)\rangle - |\bar{w}(T)\rangle \langle w(T)\rangle \|_1 = 2\sqrt{1 - |\langle \bar{u}(T)|\bar{w}(T) \rangle |^2} \geq 2\sqrt{1 - (C + (1 - C)/4 + (1 - C)/4)^2}$$

$$= \sqrt{(3 + C)(1 - C)} = \Omega(1). \hfill (47)$$

Therefore, Equation (39) and Equation (47), together with Theorem 13, imply that solving the ODE requires $\Omega(1/\sqrt{\epsilon})$ queries to the given prepare oracle. According to the choice of $T$ in Equation (33), the complexity with the worst-case input state is bounded from below by $\Omega(1/\sqrt{\epsilon}) = \Omega(e^{(\alpha_1-\alpha_2) T}).$
We remark that Proposition 14 only requires the input state to be the worst case. More specifically, for any given diagonalizable coefficient matrix \( A \), we can always find an input state such that the complexity is bounded from below as claimed. Meanwhile, our lower bound is only in terms of the prepare oracle of \( u(0) \), and no result is obtained in the input oracle of \( A \). This is different from the counterparts in the Hamiltonian simulation problem (e.g., [14]), which require both \( A \) and \( u(0) \) to be the worst case but can yield lower bounds in terms of the input oracles of both \( A \) and \( u(0) \).

### 3.3.2 Non-normal matrices

It has been demonstrated that quantum algorithms can efficiently solve Hamiltonian simulation problems. With the linear overhead of solution decay, quantum devices can also efficiently solve imaginary time evolution problems. In both cases the coefficient matrix \( A \) is normal, i.e., \( A^\dagger A = AA^\dagger \) or, equivalently, \( A \) can be diagonalized by a unitary matrix.

In this section, we wish to understand the capability of quantum algorithms for solving ODEs with a coefficient matrix beyond a normal matrix. We will show that generic quantum ODE solvers will become less efficient if the coefficient matrix is more “non-normal”. To establish an explicit hardness result, we need to quantify the non-normality of a matrix. In this work, we use the following function

\[
\mu(A) = \|A^\dagger A - AA^\dagger\|^{1/2}.
\]  

(48)

This is a quite natural measure because of \( \mu(A) = 0 \) for any normal matrix according to the definition. We refer to [34] for alternative ways to measure the non-normality of a matrix.

We are now ready to prove the following result, which shows that generic quantum ODE solvers must take \( \Omega(\mu(A)) \) cost in the worst case.

**Proposition 15.** Consider the linear ODE problem in Equation (17). Let \( \mu(A) = \|A^\dagger A - AA^\dagger\|^{1/2} \). Then there exists an instance of Equation (17), satisfying \( b = 0 \), \( A \) is diagonalizable and all the eigenvalues have equal real parts, such that any quantum algorithm outputting \( u(T) / \|u(T)\| \) with bounded error and failure probability must have worst-case query complexity \( \Omega(\mu(A)) \) to the prepare oracle of \( |u(0)\rangle \), its inverse or controlled versions.

**Proof.** Consider the example with \( N = 3 \), \( u = (u_1, u_2, u_3)^T \),

\[
A = \begin{pmatrix}
i & i/\delta & 0 \\
0 & 2i & 0 \\
0 & 0 & 3i
\end{pmatrix}.
\]  

(49)

Here \( \delta \) is a real positive parameter in \((0, 1)\). Notice that the matrix \( A \) can be diagonalized such that \( A = VDV^{-1} \) where

\[
V = \begin{pmatrix}1 & 1 & 0 \\
0 & \delta & 0 \\
0 & 0 & 1\end{pmatrix}, \quad D = \begin{pmatrix}i & 0 & 0 \\
0 & 2i & 0 \\
0 & 0 & 3i\end{pmatrix},
\]  

(50)

and that

\[
A^\dagger A - AA^\dagger = \begin{pmatrix}-1/\delta^2 & -1/\delta & 0 \\
-1/\delta & 1/\delta^2 & 0 \\
0 & 0 & 0\end{pmatrix},
\]  

(51)

\[
\mu(A) = \frac{\sqrt{1 + \delta^2}}{\delta} = \Theta \left( \frac{1}{\delta} \right).
\]  

(52)

We choose two initial conditions

\[
u(0) = (0, 0, 1)^T, \\
w(0) = (0, \delta, \sqrt{1 - \delta^2})^T.
\]  

(53)

According to Equation (27) and noting that \( e^A \) can be computed as \( Ve^DV^{-1} \), we obtain up to \( T = 1 \)

\[
u(1) = (0, 0, e^{3i})^T, \\
w(1) = (e^{2i} - e^{i}, e^{2i} \delta, e^{3i} \sqrt{1 - \delta^2})^T.
\]  

(54)
Now, suppose that we are given an oracle to prepare either $|u(0)\rangle$ or $|w(0)\rangle$. Let $|\tilde{u}(1)\rangle$ and $|\tilde{w}(1)\rangle$ be the corresponding outputs of a quantum ODE solver with 2-norm distance at most $1/10$ of $|u(1)\rangle$ and $|w(1)\rangle$, respectively. On the one hand,

$$\langle u(0)|w(0)\rangle = \sqrt{1 - \delta^2} \geq 1 - \delta^2. \quad (55)$$

On the other hand,

$$|\langle u(1)|w(1)\rangle| = \frac{|e^{-3i}e^{3i}\sqrt{1 - \delta^2}|}{\|w(1)\|} = \frac{\sqrt{1 - \delta^2}}{\sqrt{|e^{2i} - e^i|^2 + \delta^2 + 1 - \delta^2}} \leq \frac{1}{\sqrt{|e^{2i} - e^i|^2 + 1}}. \quad (56)$$

According to Lemma 3 and Lemma 2, we have

$$\| |\tilde{u}(1)\rangle\langle u(1)| - |\tilde{w}(1)\rangle\langle w(1)| \|_1 = 2 \sqrt{1 - |\langle u(1)|w(1)\rangle|^2} \geq 2 \left[ 1 - \left( \frac{1}{\sqrt{|e^{2i} - e^i|^2 + 1} + 1/10 + 1/10} \right)^2 \right] > 0.77 = \Omega(1). \quad (57)$$

Therefore, Theorem 13, Equation (55) and Equation (52) imply that preparing $|\tilde{u}(1)\rangle$ or $|\tilde{w}(1)\rangle$ requires query complexity $\Omega(1/\delta) = \Omega(\mu(A))$.

### 3.3.3 Implications and shifting equivalence

Besides identifying two types of “non-quantumness” of ODEs, Proposition 14 and Proposition 15 together also imply that the class of ODEs that generic quantum algorithms can solve most efficiently is equivalent to quantum dynamics.

To illustrate this, let us consider a linear homogeneous ODE $du/dt = Au$ where $A$ is a general matrix. According to Proposition 15, there exist worst-case examples such that the non-normality of the matrix $A$ introduces extra computational cost, and thus a sufficient condition to avoid such overhead is to restrict the matrix $A$ to be normal, i.e., unitarily diagonalizable. According to Proposition 14, to avoid any possible exponential overhead in time and real part gap, all the eigenvalues of $A$ must have equal real parts. These two observations suggest that the structure of the matrix $A$ is

$$A = U(aI + J)U^\dagger, \quad (58)$$

where $U$ is a unitary matrix, $a$ is a real number, $I$ is the identity matrix and $J$ is a diagonal matrix with purely imaginary diagonal elements.

We claim that the ODE with coefficient matrix Equation (58) is equivalent to quantum dynamics. Let

$$H = iUJU^\dagger. \quad (59)$$

Here $H$ is a Hermitian matrix because (noticing that $J^\dagger = -J$)

$$H^\dagger = -iUJ^\dagger U^\dagger = iUJU^\dagger = H. \quad (60)$$

Then the matrix $A$ becomes

$$A = U(aI + J)U^\dagger = aI - iH. \quad (61)$$

The solution of the ODE can be written as

$$u(t) = e^{At}u(0) = e^{(aI-iH)t}u(0) = e^{at}e^{-iHt}u(0), \quad (62)$$

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and thus the normalized solution $|u(t)| = e^{-iHt} |u(0)|$ exactly solves the quantum dynamics with Hamiltonian $H$.

The above calculations can be generally summarized as the shifting equivalence: if two matrices $A$ and $B$ satisfy $A = \alpha I + B$ for some real number $\alpha$, then the ODEs with coefficient matrices $A$ and $B$, respectively, are quantumly equivalent in the sense that their normalized solutions are the same. We remark that the quantities we use to describe “non-quantumness” in our lower bounds Proposition 14 and Proposition 15, namely the real part gap $\delta$ and the measure of non-normality $\mu(A)$, are both invariant under shifting by a real number.

### 3.3.4 Norm decay and imaginary time evolution

Now we restrict ourselves to the imaginary time evolution

$$\frac{d}{dt} u(t) = -Hu(t), \quad t \in [0,T],$$

where $H$ is a positive semi-definite Hermitian matrix. The solution of Equation (63) is given as

$$u(T) = e^{-HT} u(0),$$

and the imaginary time evolution problem typically aims at preparing an approximation of the quantum state $e^{-HT} |u(0)\rangle / \xi$ where $\xi = \|e^{-HT} |u(0)\rangle \|$.

Existing approaches for imaginary time evolution, including those based on quantum phase estimation (QPE) [57] and LCU [58], scale linearly in terms of $1/\xi$, which indicates that the quantum algorithms might be expensive if the unnormalized solution $u(T)$ decays fast during the evolution. Similar dependence on the solution decay is also observed in quantum solvers for more general inhomogeneous linear ODE, for example the linear-in-$g$ dependence shown in Lemma 11 and Lemma 10. It is noted in [2, 33] that dramatically improved dependence on the norm decay (e.g., exponentially improved) is unlikely since such improvement implies $\text{BQP} = \text{PP}$, which is considered unlikely. We establish an explicit hardness result in the query model, showing that the linear dependence on the norm decay is asymptotically tight and further ruling out possible polynomial improvement in the next corollary, which is a direct consequence of Proposition 14.

**Corollary 16.** Consider the imaginary time evolution problem Equation (63) where $H$ is a positive semi-definite Hermitian matrix, and let $\xi = \|e^{-HT} |u(0)\rangle \|$. Then there exists an instance of Equation (63) such that any quantum algorithm outputting $u(T)/\|u(T)\|$ with bounded error and failure probability must have worst-case query complexity $\Omega(1/\xi)$ to the prepare oracle of $|u(0)\rangle$, its inverse or controlled versions.

**Proof.** This is a direct consequence of Proposition 14, by considering a positive semi-definite Hermitian matrix $H$ with at least one eigenvalue equal to 0 and noticing that

$$e^T \max_{i,j} |\text{Re}(\lambda_i - \lambda_j)| = \|H\|^T = \|e^{-HT} |u(0)\rangle\|^{-1} = 1/\xi,$$

where $|u(0)\rangle$ is chosen to be an eigenstate of $H$ with eigenvalue $\|H\|$.

### 3.4 Inhomogeneous case

Both Proposition 14 and Proposition 15 can be generalized to the inhomogeneous case using similar analysis and constructions. The results are given as follows, and we refer to Appendix C for detailed proofs.

**Proposition 17.** Given a diagonalizable matrix $A = VDV^{-1}$ where $D = \text{diag}(\lambda_1, \cdots, \lambda_N)$ is a diagonal matrix and $V$ is an invertible matrix. Then there exists an instance of the vector $b$ and the initial condition $u(0)$ of Equation (17) such that any quantum algorithm outputting $u(T)/\|u(T)\|$ with bounded error and failure probability must have worst-case query complexity $\Omega(e^{\gamma T} / (T + 1 + \sqrt{2}))$ to the prepare oracle of $|u(0)\rangle$, its inverse or controlled versions, where $\gamma = \min \{ \max_j \text{Re}(\lambda_j), \max_{i,j} \text{Re}(\lambda_i - \lambda_j) \}$.

**Proposition 18.** Consider the linear ODE problem in Equation (17). Let $\mu(A) = \|A^T A - AA^T\|^{1/2}$. Then there exists an instance of Equation (17) such that any quantum algorithm outputting $u(T)/\|u(T)\|$ with bounded error and failure probability must have worst-case query complexity $\Omega(\mu(A))$ to the prepare oracle of $|u(0)\rangle$, its inverse or controlled versions.
Compared to the homogeneous case, the lower bound due to non-normality (Proposition 18) is quite similar, yet the condition for exponential overhead due to the eigenvalues has changed. In the homogeneous case (Proposition 14), exponential computational overhead occurs once there exists any difference in eigenvalues’ real parts, but in the inhomogeneous case (Proposition 18), that happens under an additional requirement that at least one eigenvalue has positive real part. One intuitive reason that such a difference is reasonable to occur is that the shifting equivalence does not hold anymore for inhomogeneous ODE due to the presence of the inhomogeneous term \( b \).

Our lower bounds indicate that existing generic quantum ODE solvers cannot be significantly improved in the sense that the assumptions cannot be extensively relaxed and the scalings due to the non-normality cannot be exponentially improved. We consider a diagonalizable matrix \( A \). An important assumption in the existing algorithms is that all the eigenvalues of \( A \) have a non-positive real part. This is assumed explicitly in [1–3] (see, e.g., Lemma 11) and implicitly in [5] through the dependence on the parameter \( \max_t \| e^{At} \| \) (see Lemma 10). Our Proposition 17 indicates a likely exponential computational cost when an eigenvalue has a positive real part, except a constrained scenario that all the eigenvalues of \( A \) have the same real part.

Besides, there are some computational overheads due to the non-normality in the existing algorithms, for example, the parameter \( \kappa_V \) in Lemma 11 and \( \max_t \| e^{At} \| \) in Lemma 10. Our Proposition 18 suggests that those dependence is unlikely to be exponentially improved. We remark that all of our lower bounds require or construct the worst case via diagonalizable matrix \( A \), while Lemma 10 also applies to non-diagonalizable matrices.

### 3.5 Lower bound in time and precision

We now discuss lower bounds in time \( T \) and tolerated error \( \epsilon \) for solving ODE. It is known that the lower bound for Hamiltonian simulation is \( \Omega(T + \log(1/\epsilon) / \log \log(1/\epsilon)) \) [14]. Here we establish a different lower bound for inhomogeneous ODE beyond quantum dynamics. The difference between our lower bound and the existing one for Hamiltonian simulation is that our result considers the coefficient matrix to have a negative logarithmic norm (i.e., the eigenvalue of \( A + A^\dagger \) is negative), while the coefficient matrix in Hamiltonian simulation is always anti-Hermitian (so the logarithmic norm is 0).

The lower bound can be established by linking a quantum ODE solver with solving a linear system of equations and using the lower bound for quantum linear system solvers. The key observation is that the solution of a dissipative system of differential equations exponentially converges to its equilibrium, which is the solution of a system of linear equations. We prove the following proposition, which implies a lower bound \( \Omega(T^\alpha) \) for sufficiently accurate simulation and a lower bound \( \Omega(\log^\alpha(1/\epsilon)) \) for sufficiently long time simulation.

**Proposition 19.** Consider the linear ODE problem in Equation (17). Let \( l(A) \) denote the logarithmic norm of \( A \) defined to be the maximum eigenvalue of \( (A + A^\dagger)/2 \). Suppose that the worst-case quantum query lower bound for solving a linear system of equations \( Ax = b \) with \( l(A) < 0 \) up to error \( \epsilon \) is \( \Omega(\log^\alpha(1/\epsilon)) \), then any quantum algorithm approximating \( u(T)/\|u(T)\| \) up to error \( \epsilon \) must have worst-case query complexity \( \Omega(\min \{ \Omega(\log^\alpha(1/\epsilon)), \Omega(T^\alpha) \}) \) to the same oracles.

**Proof.** The definition of the logarithmic norm implies that it bounds the decay rate of the exponential operator in the sense that for any \( T > 0 \),

\[
\|e^{AT}\| \leq e^{l(A)T}.
\]  

(66)

For now, we do not impose further structure on the coefficient matrix \( A \). \( b \) and \( u(0) \) can also be arbitrary vectors with no constraint other than that \( \|b\| = \|u(0)\| = 1 \) for technical simplicity. Using Lemma 1, the
solution of Equation (17) is an approximation of the solution of the linear system $Ax = b$, as

\[
\left\| \tilde{u}(T) - A^{-1}b \right\| \leq \frac{2}{\|A^{-1}b\|} \left\| u(T) + A^{-1}b \right\| \\
= \frac{2}{\|A^{-1}b\|} \left\| e^{AT}u(0) + (e^{AT} - I)A^{-1}b + A^{-1}b \right\| \\
\leq \frac{2}{\|A^{-1}b\|} \left( \|e^{AT}\| + \|e^{AT}\| \|A^{-1}\| \right) \\
\leq \frac{2}{\|A^{-1}b\|} \left( 1 + \|A^{-1}\| \right) e^{l(A)T} \\
\leq 2 \left( \|A\| + \kappa(A) \right) e^{-|l(A)|T},
\]

where the last inequality is because $1 = \|b\| = \|AA^{-1}b\| \leq \|A\|\|A^{-1}b\|$, and $\kappa(A) = \|A\|\|A^{-1}\|$ is the condition number of $A$.

Let $|\tilde{u}(T)\rangle$ be the numerical state obtained by a generic quantum ODE solver up to time $T$ with tolerated error $\epsilon$. Then we have

\[
\left\| \tilde{u}(T) - A^{-1}b \right\| \leq \epsilon + 2 \left( \|A\| + \kappa(A) \right) e^{-|l(A)|T}.
\]

In other words, a quantum ODE solver up to time $T$ and precision $\epsilon$ can be regarded as a quantum linear system solver with precision specified in the above estimate.

Suppose that a generic quantum linear system solver for matrices with negative logarithmic norm has worst-case query complexity $\Omega((\log(1/\epsilon'))^{\alpha})$ where $\epsilon'$ is the tolerated error. Then a generic quantum ODE solver for solving ODE with the corresponding worst-case complexity $\Omega((\log(1/\epsilon))^{\alpha})$.

Let $|\tilde{u}(T)\rangle$ be the numerical state obtained by a generic quantum ODE solver up to time $T$ with tolerated error $\epsilon$. Then we have

\[
\left\| \tilde{u}(T) - A^{-1}b \right\| \leq \epsilon + 2 \left( \|A\| + \kappa(A) \right) e^{-|l(A)|T}.
\]

In other words, a quantum ODE solver up to time $T$ and precision $\epsilon$ can be regarded as a quantum linear system solver with precision specified in the above estimate.

Suppose that a generic quantum linear system solver for matrices with negative logarithmic norm has worst-case query complexity $\Omega((\log(1/\epsilon'))^{\alpha})$ where $\epsilon'$ is the tolerated error. Then a generic quantum ODE solver for solving ODE with the corresponding worst-case complexity $\Omega((\log(1/\epsilon))^{\alpha})$.

\[
\Omega \left( \log^\alpha \left( \frac{1}{\epsilon + 2 \|A\| + \kappa(A) e^{-|l(A)|T}} \right) \right) \geq \Omega \left( \log^\alpha \left( \frac{1}{2 \max \left\{ \epsilon, \frac{1}{2} \|A\| + \kappa(A) e^{-|l(A)|T} \right\}} \right) \right) \\
= \Omega \left( \log^\alpha \left( \frac{1}{2} \min \left\{ \frac{1}{\epsilon}, \frac{1}{2} \|A\| + \kappa(A) e^{-|l(A)|T} \right\} \right) \right) \\
= \Omega \left( \min \left\{ \log^\alpha \left( \frac{1}{2\epsilon} \right), \log^\alpha \left( \frac{e^{-|l(A)|T}}{4 (\|A\| + \kappa(A))} \right) \right\} \right) \\
= \min \{ \Omega (\log^\alpha (1/\epsilon)) , \Omega (T^\alpha) \}.
\]

4 Fast-forwarding

In this section, we investigate several specific classes of ODEs and design quantum algorithms which scale better than existing generic ones (Lemma 11 and Lemma 10). We obtain quadratic improvement in $T$ if the coefficient matrix is bounded and negative definite or negative semi-definite with square-root access, and exponential improvement in $T$ and $\|A\|$ if the eigenvalues and eigenstates of $A$ are known. The exponential improvement can also be obtained in solving common linear evolutionary PDEs.

4.1 Negative-definite coefficient matrix

4.1.1 Oracles

Let $A \in \mathbb{C}^{N \times N}$ be a negative definite Hermitian matrix. For technical simplicity, we assume $\|A\| \leq 1$ throughout this scenario. Suppose we have a $(1, n_A, 0)$-block-encoding of $A$ by a unitary $U_A$. Furthermore, suppose $O_u$ and $O_b$ are the oracles such that $O_u |0\rangle = \frac{1}{\|u(0)\|} \sum_{j=0}^{N-1} u_j(0) |j\rangle$ and $O_b |0\rangle = \frac{1}{\|b\|} \sum_{j=0}^{N-1} b_j |j\rangle$, and assume that $\|u(0)\|$, $\|b\|$ are known.
4.1.2 Homogeneous case

The approach largely follows [48], and the key component for quadratic speedup is that the exponential function $e^{-T(1-x)}$ can be approximated by a degree-$O(\sqrt{T})$ polynomial and can be implemented using QSVT (Lemma 9). Then, starting from a block-encoding of $A$, one can first construct a block-encoding of $(I + A)/2$ using the linear combination of unitaries technique (Lemma 6), then construct a block-encoding of $(I + A)$ by uniformly amplifying singular values [48, Theorem 30]. Notice that here we require the matrix $A$ to be negative definite to control the approximation error. Then, the operator $e^{AT} = e^{-T(I-(I+A))}$ can be block-encoded using the circuit for $e^{-T(1-x)}$ with $O(\sqrt{T})$ query complexity.

Lemma 20. Consider solving Equation (70) with $b = 0$ and $A$ is a negative definite Hermitian matrix such that all the eigenvalues of $A$ are within the interval $[-1, -\delta]$ for a $\delta > 0$. Suppose that we are given a $(1, n_A, 0)$-block-encoding of $A$, denoted by $U_A$. Then for any $T > 0$ and $\epsilon < 1/4$, a $(3, n_A+4, \epsilon)$-block-encoding of $e^{AT}$ can be constructed using

$$O\left(\frac{\sqrt{T}}{\delta} \log\left(\frac{1}{\epsilon}\right) \log\left(T \log\left(\frac{1}{\epsilon}\right)\right)\right)$$

queries to $U_A$, its inverse and controlled versions, and

$$O\left(\left(n_A + \frac{1}{\delta} \log\left(\frac{T \log\left(\frac{1}{\epsilon}\right)}{\epsilon}\right)\right) \sqrt{T} \log\left(\frac{1}{\epsilon}\right)\right)$$

additional one- or two-qubit gates.

Proof. Starting from the block-encoding of $A$, $V'' := (H \otimes I) \cdot U(H \otimes I)$ is a $(2, n_A+1, 0)$ block-encoding of $(I + A)$. Note that $V''$ uses one call to $c\cdot U$. Equivalently, $V''$ is a $(1, n_A+1, 0)$ block-encoding of $(I + A)/2$.

By assumption, $\|I + A\| \leq 1 - \delta$ for the parameter $\delta > 0$. Therefore, $\|(I + A)/2\| \leq (1 - \delta)/2$. Then, for $\epsilon_1 > 0$, by [48, Theorem 30], we can construct a $(1, n_A+2, \epsilon_1)$-block-encoding of $(I + A)$ using $O(\frac{1}{\epsilon_1} \log(1/\epsilon_1))$ calls to $V''$, $V'''$ and additional one- and two-qubit gates. Call this block-encoding $V'$.

Let $T, \epsilon_2 > 0$. By [48, Corollary 64], there exists an efficiently constructable real polynomial $P_{T, \epsilon_2}(x)$ of degree $d_{T, \epsilon_2} = O(\sqrt{\max(T, \log(1/\epsilon_2)) \log(1/\epsilon_2)})$ such that

$$|e^{-T(1-x)} - P_{T, \epsilon_2}(x)|_{[-1, 1]} \leq \epsilon_2.$$  \hfill (70)

Let $\hat{P}_{T, \epsilon_2} := \frac{1}{3} P_{T, \epsilon_2}(x)$. Then, for any $x \in [-1, 1]$ and $\epsilon_2 \leq 1/2$, we have

$$\left|\hat{P}_{T, \epsilon_2}(x)\right| = \left|\frac{P_{T, \epsilon_2}(x)}{3}\right| \leq \frac{1}{3} \max_{x \in [-1, 1]} e^{-T(1-x)} \leq \frac{1}{3} e^{-T(1-x)} + \frac{\epsilon_2}{3} \leq \frac{1}{2},$$ \hfill (71)

Then, by Lemma 9, we can construct a $(1, n_A+4, 4d_{T, \epsilon_2} \sqrt{\epsilon_1})$-block-encoding of $\hat{P}_{T, \epsilon_2}(I + A)$ using $d_{T, \epsilon_2}$ calls to $V'$ and $V''$, a single call to $c\cdot V'$ and $O(n_A d_{T, \epsilon_2})$ additional gates. Call this block-encoding $V$.

Note that $\|\hat{P}_{T, \epsilon_2}(I + A) - e^{AT}/3\| \leq \epsilon_2/3$, $V$ can also be regarded as a $(3, n_A+4, 12d_{T, \epsilon_2} \sqrt{\epsilon_1} + \epsilon_2)$-block-encoding of $e^{AT}$. To bound the overall error by $\epsilon > 0$, it suffices to choose $12d_{T, \epsilon_2} \sqrt{\epsilon_1} + \epsilon_2 \leq \epsilon$, and this can be achieved by choosing

$$\epsilon_2 = \Theta(\epsilon), \quad \epsilon_1 = \Theta\left(\frac{\epsilon^2}{\max(T, \log(1/\epsilon) \log(1/\epsilon))}\right).$$ \hfill (72)

The overall number of calls to $U_A$ (and its inverse and controlled versions) is of order

$$O\left(d_{T, \epsilon_2} \times \frac{1}{\delta} \log(1/\epsilon_1)\right) = O\left(\sqrt{T} \frac{\log\left(\frac{1}{\epsilon}\right)}{\delta} \log\left(T \log\left(\frac{1}{\epsilon}\right)\right)\right),$$ \hfill (73)

and the number of additional gates required is

$$O\left(n_A d_{T, \epsilon_2} + d_{T, \epsilon_2} \frac{1}{\delta} \log(1/\epsilon_1)\right) = O\left(\left(n_A + \frac{1}{\delta} \log\left(\frac{T \log(1/\epsilon)}{\epsilon}\right)\right) \sqrt{T} \log\left(\frac{1}{\epsilon}\right)\right).$$ \hfill (74)

\hfill $\square$
To solve the homogeneous ODE, we first apply \( O_a \) to prepare \(|0\rangle_a |u(0)\rangle_s \) (where “a” refers to “ancilla” and “s” refers to “state register”) and then apply the block-encoding of \( e^{AT} \) to obtain \(|0\rangle_a C|\psi\rangle_a + |1\rangle_a C'|\psi\rangle_a \) for some rescaling factor \( C \). Here \( C|\psi\rangle_a \) is an \( \mathcal{O}(\epsilon') \)-approximation of \( \frac{1}{\Vert u(T) \Vert} e^{AT}u(0) = \frac{1}{\Vert u(T) \Vert} u(T) \). According to Lemma 1, measuring the ancilla qubits and getting 0 yield an \( \mathcal{O}(\epsilon' \Vert u(0) \Vert/\|u(T)\|) \) approximation of \(|u(T)\rangle\). We choose \( \epsilon' = \epsilon \|u(T)\|/\|u(0)\| \) to bound the error by \( \epsilon \). The success probability can be boosted to \( \Omega(1) \) using amplitude amplification and repeating the procedure for \( \mathcal{O}(\|u(0)\|/\|u(T)\|) \) times. Therefore the overall query complexity is

\[
\tilde{O} \left( \frac{\|u(0)\|}{\|u(T)\|} \sqrt{\frac{T}{\delta}} \log \left( \frac{1}{\epsilon} \right) \right).
\]  

(75)

We remark that, although the explicit scaling in \( T \) seems quadratic, the actual overall scaling is always exponential in \( T \) since \( \|u(T)\| \) is always exponentially small for negative definite matrix \( A \). However, if we consider the inhomogeneous ODE instead, a truly quadratic speedup in \( T \) may be obtained because the solution of inhomogeneous ODE does not exhibit exponential decay, and the block-encoding of the homogeneous evolution operator \( e^{AT} \) is the first step to solve inhomogeneous ODE. We will discuss this in detail next.

### 4.1.3 Inhomogeneous case

Now we consider solving Equation (1) with a negative-definite Hermitian matrix \( A \) and a time-independent inhomogeneous term \( b \). The idea is directly based on the solution form given in Equation (2): we first construct block-encodings of the operators \( e^{AT} \) and \( \int_0^T e^{A(T-s)} ds \), then apply them to the corresponding vectors to obtain \( e^{AT} u_0 \) and \( \int_0^T e^{A(T-s)} b ds \), and finally use a technique similar to that of LCU to construct the linear combination of these two vectors.

We start with the block-encodings of the operators. The block-encoding for \( e^{AT} \) has been constructed in Lemma 20. We construct the block-encoding of \( \int_0^T e^{A(T-s)} ds \) as in Figure 5, which is based on the equation \( \int_0^T e^{A(T-s)} ds = (e^{AT} - I)A^{-1} \).

**Lemma 21.** Suppose that \( A \) is a negative-definite Hermitian matrix such that all the eigenvalues of \( A \) are within \([-1, -\delta]\), and \( U_A \) is a \((1, n_A, 0)\)-block-encoding of \( A \). Then for any \( T > 0 \) and \( \epsilon < 1/2 \), an \((16/(3\delta), 2n_A + 6, \epsilon)\)-block-encoding of \( \int_0^T e^{A(T-s)} ds \) can be constructed, using

\[
\mathcal{O} \left( \frac{\sqrt{T}}{\delta} \log \left( \frac{1}{\delta \epsilon} \right) \log \left( \frac{T \log(1/(\delta \epsilon))}{\delta \epsilon} \right) \right)
\]

queries to \( U_A \), its inverse and controlled version, and

\[
\mathcal{O} \left( \left( n_A + \frac{1}{\delta} \log \left( \frac{T \log(1/(\delta \epsilon))}{\delta \epsilon} \right) \right) \sqrt{T} \log \left( \frac{1}{\delta \epsilon} \right) \right)
\]

(76)

extra one or two-qubit gates.

**Proof.** Let \( n_1 = n_A + 4 \). We start with the \((3, n_1, \epsilon')\)-block-encoding of \( e^{AT} \) (or equivalently a \((1, n_1, \epsilon'/3)\)-block-encoding of \( e^{AT/3} \)), denoted by \( U'_{A} \), constructed in Lemma 20 with \( \epsilon' \) to be determined later. Let

---

Figure 5: Quantum circuit for constructing a block-encoding of \( \int_0^T e^{A(T-s)} ds \) for a negative definite Hermitian matrix \( A \). Here for a matrix \( M \), \( U_M \) represents its block-encoding.
assume that $U$ that block-encoding to $e$ extra one or two-qubit gates. The proof is completed by noticing that $(U - I)A^{-1}$ can be constructed using $O((1/\delta)(\log(1/\epsilon')(\log(1/\epsilon')))$ extra one or two-qubit gates according to Lemma 20.

Meanwhile, Lemma 8 tells that a $(4/(3\delta), n_A + 1, \epsilon'')$-block-encoding of $A^{-1}$ can be constructed using $O((1/\delta)/(\delta\epsilon''))$ queries to $U_A$ and its inverse.

Multiplying this two block-encodings together and using Lemma 7, we can construct a $(16/(3\delta), n_A + n_1 + 2, 4\epsilon'' + 16\epsilon'/(9\delta))$-block-encoding of $(e^{AT} - I)A^{-1}$, using $O \left( \left( \frac{1}{\delta} \sqrt{T} \log \left( \frac{1}{\epsilon'} \right) \log \left( \frac{T \log(1/\epsilon')}{\epsilon'} \right) + \log \left( \frac{1}{\delta\epsilon''} \right) \right) \right)$ \hfill (79)

queries to $U_A$, its inverse and controlled version, and

$O \left( \left( n_A + \frac{1}{\delta} \log \left( \frac{T \log(1/\epsilon')}{\epsilon'} \right) \right) \sqrt{T} \log \left( \frac{1}{\epsilon'} \right) \right)$ \hfill (80)

two-qubit gates. The proof is completed by noticing that $(e^{AT} - I)A^{-1} = \int_0^T e^{A(T-s)} ds$ and choosing $8\epsilon'/9\delta = 2\epsilon'' = \epsilon$.

Next, we show how to construct a linear combination of the homogeneous and the inhomogeneous parts. To make the result more general, here we estimate the complexity in terms of the number of queries to the block-encoding to $e^{AT}$ and $\int_0^T e^{A(T-s)} ds$. The quantum circuit is given in Figure 6.

**Lemma 22.** Consider solving the ODE system Equation (1) with time-independent $b$ up to time $T$. Suppose that $U_0$ and $U_1$ are $(\alpha_0, n', \epsilon_0)$- and $(\alpha_1, n', \epsilon_1)$-block-encodings of $e^{AT}$ and $\int_0^T e^{A(T-s)} ds$, respectively, and assume that $\alpha_0$ and $\alpha_1$ are known. Then, for $0 \leq \epsilon < 1/2$, there exists a quantum algorithm that outputs an $\epsilon$-approximation of $|u(T)|$ with $\Omega(1)$ success probability, using

1. $O \left( \frac{\alpha_0 \|u(0)\| + \alpha_1 \|b\|}{\|u(T)\|} \right)$ \hfill (81)

queries to the controlled versions of $O_u, O_b, U_0$ and $U_1$, with the tolerated error in $U_0$ and $U_1$ to be $\epsilon_0 = \frac{\|u(T)\| \epsilon}{4\|u(0)\|}, \quad \epsilon_1 = \frac{\|u(T)\| \epsilon}{4\|b\|}, \hfill (82)$

2. $(n' + 1)$ ancilla qubits,
We start with the state $|0\rangle_c |0\rangle_u |0\rangle_v$, where the subscript "u" represents ancilla qubits in the block-encodings, "c" represents a single control qubit, and "v" represents the vector space encoding the solution. The first step is applying a rotation $R_y(-2\arcsin(\alpha_1 ||b||/\sqrt{\alpha_0^2 ||u(0)||^2 + \alpha_1^2 ||b||^2}))$ on the control register to obtain

$$\frac{1}{\sqrt{\alpha_0^2 ||u(0)||^2 + \alpha_1^2 ||b||^2}}(\alpha_0 ||u(0)|| |0\rangle_c + \alpha_1 ||b|| |1\rangle_c) |0\rangle_a |0\rangle_v.$$  \hspace{1cm} (84)

Applying $|0\rangle_c |0\rangle_c \otimes O_u + |1\rangle_c \otimes O_b$ on the control and vector registers gives

$$\frac{1}{\sqrt{\alpha_0^2 ||u(0)||^2 + \alpha_1^2 ||b||^2}}(\alpha_0 ||u(0)|| |0\rangle_c |u(0)\rangle_v + \alpha_1 ||b|| |1\rangle_c |0\rangle_a |b\rangle_v).$$ \hspace{1cm} (85)

Next, applying $|0\rangle_c |0\rangle_c \otimes U_0 + |1\rangle_c \otimes U_1$, which is the block-encoding of $e^{AT}$ and $\int_0^T e^{A(T-s)} ds$ controlled by the control register, gives

$$\frac{1}{\sqrt{\alpha_0^2 ||u(0)||^2 + \alpha_1^2 ||b||^2}}(|u(0)|| |0\rangle_c |e^{AT} |u(0)\rangle_v |e_0\rangle_v + ||b|| |1\rangle_c |\rho\rangle_a e^{AT} ds |b\rangle_v + |e_1\rangle_v) + |\perp\rangle,$$ \hspace{1cm} (86)

where $e_0$ and $e_1$ are two error terms with $||e_0|| \leq \epsilon_0$ and $||e_1|| \leq \epsilon_1$, and $|\perp\rangle$ represents terms where the ancilla qubit is not 0. Applying a Hadamard gate on the control register yields

$$\frac{1}{\sqrt{2\alpha_0^2 ||u(0)||^2 + \alpha_1^2 ||b||^2}}(|0\rangle_c |0\rangle_a (||e_0|| |e_0\rangle_v + ||b|| |e_1\rangle_v) + |\perp\rangle,$$ \hspace{1cm} (87)

and, according to Equation (27), this quantum state is the same as

$$\frac{1}{\sqrt{2\alpha_0^2 ||u(0)||^2 + \alpha_1^2 ||b||^2}}(|0\rangle_c |0\rangle_a (||u(T)|| |u(T)\rangle_v + ||u(0)|| |e_0\rangle_v + ||b|| |e_1\rangle_v) + |\perp\rangle.$$ \hspace{1cm} (88)

If we measure the control and ancilla registers and get 0 for both measurements, then the vector register encodes an approximation of $|u(T)\rangle$. The cost of a single run is a single use of the controlled versions of $O_u$, $O_b$, $U_0$, and $U_1$, and $O(1)$ extra one-qubit gates.

Now we analyze the approximation error and the success probability. Let $|\psi\rangle$ denote the quantum state in the vector register after a successful measurement. According to Lemma 1, the approximation error can be bounded as

$$||\psi - |u(T)\rangle|| \leq \frac{2\sqrt{2\alpha_0^2 ||u(0)||^2 + \alpha_1^2 ||b||^2}}{||u(T)||} \left( ||u(0)|| ||e_0|| ||e_0\rangle_v + ||b|| ||e_1|| ||e_1\rangle_v \right)$$ \hspace{1cm} (89)

and the success probability, after amplitude amplification, can be bounded as

$$\text{Prob} \geq \frac{||u(T)||}{\sqrt{2\alpha_0^2 ||u(0)||^2 + \alpha_1^2 ||b||^2}} - \frac{||u(0)|| ||e_0|| ||e_0\rangle_v + ||b|| ||e_1|| ||e_1\rangle_v}{\sqrt{2\alpha_0^2 ||u(0)||^2 + \alpha_1^2 ||b||^2}}.$$ \hspace{1cm} (90)
Choose $\epsilon_0 = \|u(T)\|\epsilon/(4\|u(0)\|)$ and $\epsilon_1 = \|u(T)\|\epsilon/(4\|b\|)$, then
\[ \|\psi - |u(T)\rangle\| \leq \epsilon \tag{91} \]
and
\[ \text{Prob} \geq \frac{\|u(T)\|^{(1 - \epsilon/2)}}{\sqrt{2}\alpha_0^2\|u(0)\|^2 + \alpha_1^2\|b\|^2} \geq \frac{\|u(T)\|}{2\sqrt{2}\alpha_0^2\|u(0)\|^2 + \alpha_1^2\|b\|^2} \tag{92} \]
Therefore, the success probability can be boosted to $\Omega(1)$ using
\[ \mathcal{O}\left(\frac{\alpha_0\|u(0)\| + \alpha_1\|b\|}{\|u(T)\|}\right) \tag{93} \]
repeats with amplitude amplification.

We are now ready to state our main result, which is a direct consequence of Lemma 20, Lemma 21 and Lemma 22.

**Theorem 23.** Consider solving the ODE system Equation (1) with time-independent $b$ up to time $T$, where $A$ is a bounded negative definite Hermitian matrix such that all the eigenvalues of $A$ are within $[-1, -\delta]$. Suppose the oracles described in Section 4.1.1. Then for any $T, \epsilon$ such that $0 < \epsilon < 1/2$, there exists a quantum algorithm that outputs an $\epsilon$-approximation of $|u(T)\rangle$ with $\Omega(1)$ success probability, using
1. $\mathcal{O}\left(\frac{\|u(0)\| + \|b\|/\delta}{\|u(T)\|} \sqrt{T}\log \left(\frac{1}{\delta\epsilon}\right) \log \left(\frac{T\log(1/(\delta\epsilon))}{\delta\epsilon}\right)\right)$ \tag{94}
2. $\mathcal{O}\left(\frac{\|u(0)\| + \|b\|/\delta}{\|u(T)\|}\right)$ \tag{95}
3. $(2n_A + 7)$ ancilla qubits,
4. $\mathcal{O}\left(\frac{\|u(0)\| + \|b\|/\delta}{\|u(T)\|} \left(n_A + \frac{1}{\delta} \log \left(\frac{T\log(1/(\delta\epsilon))}{\delta\epsilon}\right)\right) \sqrt{T}\log \left(\frac{1}{\delta\epsilon}\right)\right)$ \tag{96}
extra one or two-qubit gates.

Although there is still some $T$-dependence through the parameter $\|u(T)\|$ in the query complexity, **Theorem 23** indeed shows that quadratic fast-forwarding in $T$ can be achieved when solving an inhomogeneous ODE system with a negative definite coefficient matrix. This is because the norm of $\|u(T)\|$ can be asymptotically bounded independent of $T$ for a negative definite matrix $A$ (but might depend on $\delta$). To see this more clearly, let us consider a simplified case where we assume $\|u(0)\|, \|b\|$ and $\delta$ are $\Theta(1)$. Then, according to Equation (2),
\[ \|u(T)\| \geq \|e^{AT} - I\|A^{-1}|b| - \|e^{AT}u(0)\| \geq \|A^{-1}|b| - \|e^{AT}A^{-1}|b| - \|e^{AT}u(0)\| \geq \|b\| - e^{-\delta T}/\delta - e^{-\delta T} = \Omega(1). \tag{97} \]
Therefore the overall query complexity can be bounded by $\tilde{\mathcal{O}}\left(\sqrt{T}\log^2(1/\epsilon)\right)$, yielding a quadratic speedup in $T$ up to a logarithmic factor.
4.2 Negative semi-definite coefficient matrix with square-root access

4.2.1 Oracles

Let $A \in \mathbb{C}^{N \times N}$ be a negative semi-definite Hermitian matrix such that $A = -H^2$ for a Hermitian matrix $H$. Suppose we have an $(\alpha_H, n_H, 0)$-block-encoding of $H$ by a unitary $U_H$. Furthermore, suppose $O_u$ and $O_b$ are the oracles such that $O_u \{0\} = \frac{1}{\|u(0)\|} \sum_{j=0}^{N-1} u_j(0) |j\rangle$ and $O_b \{0\} = \frac{1}{\|b\|} \sum_{j=0}^{N-1} b_j |j\rangle$, and assume that $\|u(0)\|$, $\|b\|$ are known.

4.2.2 Homogeneous case

Fast-forwarding $e^{-TH^2}$ has been studied in several works including [48, 49, 52]. Here we follow the idea suggested in [48] and work out technical details. The idea is straightforward: the function $e^{-\beta x^2}$ can be approximated with a degree-$O(\beta)$ polynomial and then $e^{AT} = e^{-TH^2}$ can be implemented via Lemma 9.

**Lemma 24.** For any $\beta > 0$ and $0 < \epsilon < 1$, there exists an efficiently computable even polynomial $P(x)$ such that
\[
\sup_{x \in [-1,1]} |e^{-\beta x^2} - P(x)| \leq \epsilon
\]
and has degree $O(\sqrt{\max(\beta, \log(1/\epsilon)) \log(1/\epsilon)})$.

**Proof.** This is a direct consequence following [53, Theorem 4.1] that $e^{-y}$ on the interval $[0, \beta]$ can be approximated using a polynomial of degree $O(\sqrt{\max(\beta, \log(1/\epsilon)) \log(1/\epsilon)})$ and let $y = \beta x^2$.

**Lemma 25.** Consider solving Equation (1) with $b = 0$ and $A$ is a negative semi-definite Hermitian matrix such that $A = -H^2$ for a Hermitian $H$. Suppose that we are given a $(\alpha_H, n_H, 0)$-block-encoding of $H$, denoted by $U_H$. Then for any $T > 0$ and $\epsilon < 1/4$, a $(3, n_H + 2, \epsilon)$-block-encoding of $e^{AT}$ can be constructed using $O(\sqrt{\max(T \alpha_H^2, \log(1/\epsilon)) \log(1/\epsilon)})$ queries to $U_H$, its inverse and controlled versions, and $O(n_H \sqrt{\max(T \alpha_H^2, \log(1/\epsilon)) \log(1/\epsilon)})$ additional one- or two-qubit gates.

**Proof.** Let $\beta = T \alpha_H^2$, $P(x)$ be the polynomial in Lemma 24 with error $\epsilon$, and $d = O(\sqrt{\max(\beta, \log(1/\epsilon)) \log(1/\epsilon)})$. Then, according to Lemma 9, a $(1, n_H + 2, 0)$-block-encoding of $P(H/\alpha_H)/3$ can be constructed using $O(d)$ queries and $O(n_H d)$ additional gates. Equivalently, this is a $(3, n_H + 2, \epsilon)$-block-encoding of $e^{-TH^2} = e^{AT}$.

The parameter $\alpha_H$ depends on specific circuits that realize the oracle $U_H$, and it can be as small as $\sqrt{\|A\|}$. In this optimal case, the overall complexity becomes $\tilde{O}(\sqrt{T \|A\| \log(1/\epsilon)})$ and achieves quadratic speedup in both $T$ and $\|A\|$ for constructing block-encoding. When it comes to solving homogeneous ODE, similar to the discussion after Lemma 20, an extra post-selection step is required and the overall query complexity becomes
\[
\tilde{O} \left( \frac{\|u(0)\|}{\|u(T)\|} \sqrt{T \|A\| \log \left( \frac{1}{\epsilon} \right)} \right).
\]

However, unlike the negative definite case, a real quadratic improvement can hold once the matrix $A$ has an 0 eigenvalue and $\|u(0)\|$ has non-trivial overlap with the corresponding eigenstate. In this case, $\|u(T)\| = \|e^{AT} u(0)\| \geq \|u(0)\| \|u(0)\rangle \langle u(0)\| = \Omega(\|u(0)\|)$ where $|\psi_0\rangle$ is the eigenstate corresponding to 0 eigenvalue, and the overall complexity becomes $\tilde{O}(\sqrt{T \|A\| \log(1/\epsilon)})$.

4.2.3 Inhomogeneous case

Now we consider Equation (1) with a time-independent inhomogeneous term $b$. The idea is the same as Lemma 25, and to block encode $\int_0^T e^{A(T-s)} ds$ using QSVT with lower-degree polynomial.

**Lemma 26.** For any $\beta > 0$ and $0 < \epsilon < 1$, there exists and efficiently computable even polynomial $Q(x)$ such that
\[
\sup_{x \in [-1,1]} \left| \int_0^1 e^{-\beta \tau x^2} d\tau - Q(x) \right| \leq \epsilon
\]
and has degree $O(\sqrt{\max(\beta, \log(1/\epsilon)) \log(1/\epsilon)})$. 27
Proof. According to [53, Theorem 4.1], $e^{-y}$ on the interval $[0, \beta]$ can be approximated using a polynomial of degree $O(\sqrt{\max(\beta, \log(1/\epsilon)) \log(1/\epsilon)})$. Formally, there exists a polynomial

$$
\sum_{j=0}^{J} q_j(\beta, \epsilon) y^j
$$

such that

$$
\sup_{y \in [0, \beta]} \left| e^{-y} - \sum_{j=0}^{J} q_j(\beta, \epsilon) y^j \right| \leq \epsilon,
$$

and $J = O(\sqrt{\max(\beta, \log(1/\epsilon)) \log(1/\epsilon)})$. Let $y = \beta x^2$, then we have

$$
\sup_{x \in [-1,1], \tau \in [0,1]} \left| e^{-\beta x^2} - \sum_{j=0}^{J} q_j(\beta, \epsilon) \beta^j \tau^j x^{2j} \right| \leq \epsilon,
$$

which implies (using $\int_0^1 \tau^j d\tau = 1/(j+1)$)

$$
\sup_{x \in [-1,1]} \left| \int_0^1 e^{-\beta x^2} d\tau - \sum_{j=0}^{J} q_j(\beta, \epsilon) \beta^j \frac{x^{2j}}{j+1} \right| \leq \epsilon.
$$

So we may define

$$
Q(x) = \sum_{j=0}^{J} q_j(\beta, \epsilon) \beta^j \frac{x^{2j}}{j+1},
$$

which is the desired estimate and has degree $2J = O(\sqrt{\max(\beta, \log(1/\epsilon)) \log(1/\epsilon)})$. \qed

Lemma 27. Let $A = -H^2$ for a Hermitian $H$. Suppose that we are given a $(\alpha_H, n_H, 0)$-block-encoding of $H$, denoted by $U_H$. Then for any $T > 0$ and $\epsilon < 1/4$, a $(3T, n_H + 2, \epsilon)$-block-encoding of $\int_0^T e^{A(T-s)} ds$ can be constructed using $O(\sqrt{\max(T \alpha_H^2, \log(T/\epsilon)) \log(T/\epsilon)})$ queries to $U_H$, its inverse and controlled versions, and $O(n_H \sqrt{\max(T \alpha_H^2, \log(T/\epsilon)) \log(T/\epsilon)})$ additional one- or two-qubit gates.

Proof. Notice that

$$
\frac{1}{T} \int_0^T e^{A(T-s)} ds = \frac{1}{T} \int_0^T e^{-H^2(T-s)} ds = \int_0^1 e^{-T \alpha_H^2 s (H/\alpha_H)^2} ds.
$$

Let $\beta = T \alpha_H^2$, $Q(x)$ be the polynomial in Lemma 26 with error $\epsilon'$, and $d = O(\sqrt{\max(\beta, \log(1/\epsilon')) \log(1/\epsilon')})$. Then, according to Lemma 9, a $(1, n_H + 2, 0)$-block-encoding of $Q(H/\alpha_H)/3$ can be constructed using $O(d)$ applications of $U_H$, its inverse and controlled versions, and $O(n_H d)$ additional one- and two-qubit gates. Notice that this is also a $(1, n_H + 2, \epsilon'/3)$-block-encoding of $\frac{1}{3T} \int_0^T e^{A(T-s)} ds$, and equivalently a $(3T, n_H + 2, T\epsilon')$-block-encoding of $\int_0^T e^{A(T-s)} ds$. We can choose $\epsilon' = \epsilon/T$. \qed

The result for the inhomogeneous case is a direct consequence of Lemma 25, Lemma 27 and Lemma 22.

Theorem 28. Consider solving the ODE system Equation (1) with time-independent $b$ up to time $T$, where $A$ is a negative semi-definite Hermitian matrix such that $A = -H^2$ for a Hermitian $H$. Suppose the oracles described in Section 4.2.1. Then for any $T, \epsilon$ such that $0 < \epsilon < 1/4$, there exists a quantum algorithm that outputs an $\epsilon$-approximation of $[u(T)]$ with $\Omega(1)$ success probability, using

$$
Q \left( \frac{\|u(0)\| + T\|b\|}{\|u(T)\|} \right) \sqrt{\max \left( T \alpha_H^2, \log \left( \frac{\|u(0)\| + T\|b\|}{\|u(T)\|} \right) \right)} \log \left( \frac{\|u(0)\| + T\|b\|}{\|u(T)\|} \right)
$$

queries to $U_H$, its inverse and controlled version,
2. \[ \mathcal{O} \left( \frac{\|u(0)\| + T\|b\|}{\|u(T)\|} \right) \] queries to the controlled versions of \( O_u \) and \( O_b \).

3. \((n_H + 2)\) ancilla qubits,

4. \[ \mathcal{O} \left( n_H \frac{\|u(0)\| + T\|b\|}{\|u(T)\|} \right) \sqrt{\max \left( T\alpha_H^2, \log \left( \frac{\|u(0)\| + T\|b\|}{\|u(T)\|} \right) \right)} \log \left( \frac{\|u(0)\| + T\|b\|}{\|u(T)\|} \right) \] extra one or two-qubit gates.

Compared to Lemma 10, we quadratically improve the dependence on \( \|A\| \) in query complexity. In the worst case, the query complexity of our algorithm is \( \mathcal{O}(T^{3/2}) \), which is even worse than the generic algorithm. However, it is possible and typical that \( \|u(T)\| \) grows linearly in \( T \) and cancels a linear-in-\( T \) term in the numerator, yielding a quadratic improvement in both \( T \) and \( \|A\| \). To see this, let us consider the case where \( \lambda_0 = 0 \) and, for technical simplicity, assume \( \|u(0)\| = \|b\| = 1 \). Recall that this represents the inhomogeneous heat equation and the advection-diffusion equation with periodic boundary condition and a time-independent source term. Let \( |\psi_j\rangle \) be the eigenstates of \( A \). Then

\[ u(T) = e^{AT}u(0) + \int_0^T e^{A(T-s)}bds \]

\[ = \sum_{j=0}^{N-1} e^{\lambda_j T} \langle \psi_j|u(0)\rangle |\psi_j\rangle + \sum_{j=0}^{N-1} T f(\lambda_j, T) \langle \psi_j|b\rangle |\psi_j\rangle \]

\[ = (\langle \psi_0|u(0)\rangle + T \langle \psi_0|b\rangle) |\psi_0\rangle + \sum_{j=1}^{N-1} c_j |\psi_j\rangle. \]

Therefore, provided \( |\langle \psi_0|b\rangle| = \Omega(1) \), we have \( \|u(T)\| \geq |\langle \psi_0|u(0)\rangle + T \langle \psi_0|b\rangle| = \Omega(T) \) and thus the query complexity becomes \( \mathcal{O}(\sqrt{T}) \). Here the assumption \( |\langle \psi_0|b\rangle| = \Omega(1) \) is natural in the inhomogeneous heat equation, which corresponds to the trivial assumption that the heat source has a non-zero total heat.

### 4.3 Negative semi-definite coefficient matrix with known eigenvalues and eigenstates

#### 4.3.1 Oracles and notation

Let \( A \) be a negative semi-definite matrix. Specifically, \( A \) can be represented as

\[ A = U \Lambda U^\dagger. \]

Here \( U \) is a known unitary in the sense that a quantum circuit for efficiently implementing \( U \) is given. With an abuse of notation, we still use \( U \) to denote this circuit. The matrix \( \Lambda = \text{diag}(\lambda_0, \ldots, \lambda_{N-1}) \) is a diagonal matrix such that \( \lambda_j \leq 0 \) for all \( j \).

Suppose that the time \( T \) and the eigenvalues \( \lambda_j \) are binarily encoded through the oracles \( O_T : |0\rangle_v \rightarrow |T\rangle_v \) and \( O_\Lambda : |0\rangle_v \otimes |j\rangle_v \rightarrow |\lambda_j\rangle_v \otimes |j\rangle_v \). Let the function \( e^{\lambda t} \) be given through the oracle \( O_{e^{\lambda t}} : |0\rangle_f \otimes |\lambda\rangle_e |t\rangle_f \rightarrow |e^{\lambda t}\rangle_f |\lambda\rangle_e |t\rangle_f \), which can be constructed by classical arithmetic. Furthermore, let another classical-efficiently computable function \( f(\lambda, t) \) to be

\[ f(\lambda, t) = \frac{1}{t} \int_0^t e^{\lambda(t-s)}ds = \begin{cases} 1, & \text{if } \lambda = 0, \\ \frac{1}{\lambda^2} (e^\lambda - 1), & \text{else}, \end{cases} \]

and it is given by the oracle \( O_f : |0\rangle_f \otimes |\lambda\rangle_e |t\rangle_f \rightarrow |f(\lambda, t)\rangle_f |\lambda\rangle_e |t\rangle_f \). The oracle \( O_{e^{\lambda t}} \) will be used for constructing \( e^{AT} \), and the oracle \( O_f \) is associated with \( \int_0^T e^{A(T-s)}ds \) and will be discussed later. Here the meanings of the subscripts are: \( t \) for time, \( e \) for eigenvalue, \( f \) for function, and \( v \) for vector.

For the initial condition and the inhomogeneous term, we assume that \( O_u \) and \( O_b \) are the oracles such that

\[ O_u |0\rangle = \frac{1}{\|u(0)\|} \sum_{j=0}^{N-1} u_j(0) |j\rangle \] and \( O_b |0\rangle = \frac{1}{\|b\|} \sum_{j=0}^{N-1} b_j |j\rangle \), and assume that \( \|u(0)\|, \|b\| \) are known.
4.3.2 Homogeneous case

According to the equation $e^{AT} = U e^{AT} U^\dagger$, it suffices to focus on the diagonal transform $e^{AT}$, which can be implemented by controlled rotations. Notice that sequentially applying $O_T$, $O_\Lambda$, $O_{exp}$, $O_\Lambda^\dagger$ and $O_T^\dagger$ on corresponding registers can map $|j\rangle_v |0\rangle_f |0\rangle_t |0\rangle_e$ to $|j\rangle_v |e^{AT} \rangle_f |0\rangle_t |0\rangle_e$, i.e., encode the information of $e^{AT}$ into the function register. Then, starting with an quantum state $\sum_j v_j |j\rangle_v$ encoding a normalized vector $v$, we first append the information of $e^{AT}$ through the previous operator to get $\sum_{j=0}^{N-1} v_j |j\rangle_v |e^{AT} \rangle_f$, then append another control register with single qubit and apply a rotation conditioned by the function register to get $\sum_{j=0}^{N-1} v_j |j\rangle_v (e^{AT} |0\rangle_e + \sqrt{1 - e^{AT}} |1\rangle_v) |e^{AT} \rangle_f$. Notice that, after uncomputing, the part with $|0\rangle_r$ exactly encodes the vector $e^{AT} v$ in the amplitude.

The entire circuit for constructing a block-encoding of $e^{AT}$ is given in Figure 7, and we demonstrate its effectiveness and cost as follows.

**Lemma 29.** Let $A = UAU^\dagger$ where $U$ is a known unitary and $\Lambda = diag(\lambda_0, \cdots, \lambda_{N-1})$ such that $\lambda_j \leq 0$ for all $j$. Suppose we are given the oracles described in Section 4.3.1, and the time, eigenvalue and function registers consist of $n_t$, $n_e$ and $n_f$ qubits, respectively. Then a $(1, n_t + n_e + n_f + 1, 0)$-block-encoding of $e^{AT}$ can be constructed, using 6 queries to $O_T$, $O_\Lambda$, $O_{exp}$ and their inverses, 2 queries to $U$ and its inverse, and an extra controlled one-qubit rotation gate.

**Proof.** Let $|v\rangle_v = \sum_{j=0}^{N-1} v_j |j\rangle_v$ be an arbitrary state, and we start with $\sum_{j=0}^{N-1} v_j |j\rangle_v |0\rangle_r |0\rangle_f |0\rangle_e |0\rangle_t$. We first apply $U^\dagger$ on the vector register to get

$$
\sum_{j=0}^{N-1} (U^\dagger v_j) |j\rangle_v |0\rangle_r |0\rangle_f |0\rangle_e |0\rangle_t.
$$

(113)

Here for a matrix $M$ and a vector $x$, $(Mx)_j$ represents its $j$-th component.

We implement the diagonal transform $e^{AT}$. Applying $O_T$ and $O_\Lambda$ on the corresponding registers gives

$$
\sum_{j=0}^{N-1} (U^\dagger v_j) |j\rangle_v |0\rangle_r |0\rangle_f |\lambda_j \rangle_e |T\rangle_t.
$$

(114)

Apply $O_{exp}$ on the corresponding registers to get

$$
\sum_{j=0}^{N-1} (U^\dagger v_j) |j\rangle_v |0\rangle_r |e^{\lambda_j T} \rangle_f |\lambda_j \rangle_e |T\rangle_t.
$$

(115)

Notice that $e^{\lambda_j T} \leq 1$ since $\lambda_j \leq 0$, then we apply a rotation on the rotation register conditioned by the
Proof. Applying $e^{\lambda_j T}$ corresponding to $A$ is that $|\lambda_j\rangle_e |T\rangle_t$. Therefore the first part after successful measurement is $\lambda$ and it is possible that the scalings are independent of $T$ depend on $\Omega(1)$ is $T$ and we get

$$\sum_{j=0}^{N-1} (e^{\lambda_j T} |j\rangle_v |0\rangle_e |0\rangle_f |0\rangle_e |T\rangle_t + |\perp\rangle).$$

Then we uncompute the function, eigenvalue and time registers by applying $O^\dagger_{\exp}$, $O^\dagger_{\Lambda}$ and $O^\dagger_{T}$, and we get

$$\sum_{j=0}^{N-1} (\exp^{\lambda_j T} |j\rangle_v |0\rangle_r |0\rangle_f |0\rangle_e |0\rangle_t + |\perp\rangle).$$

Finally, applying $U$ gives

$$\sum_{j=0}^{N-1} (U e^{\lambda_j T} v_j |j\rangle_v |0\rangle_r |0\rangle_f |0\rangle_e |0\rangle_t + |\perp\rangle).$$

According to the definition of the block-encoding and the equation $e^{AT} = U e^{\lambda_j T} U^\dagger$, such a circuit is exactly a $(1, n_t + n_e + n_f + 1, 0)$-block-encoding of $e^{AT}$.

A direct consequence of Lemma 29 is the cost of solving homogeneous ODE. We can directly apply the block-encoding of $e^{AT}$ to the input state, and then measure all the ancilla qubits. If the outcome is all 0, then the remaining state is the desired $|u(T)\rangle$.

Theorem 30. Consider solving the ODE system Equation (17), where $b = 0$ and $A$ is a negative semi-definite Hermitian matrix with known eigenvalues and eigenstates. Suppose that we are given the oracles described in Section 4.3.1. Then for any $T > 0$, there exists a quantum algorithm that outputs $|u(T)\rangle$ with $\Omega(1)$ success probability, using

1. $O(|u(0)||///u(T)||)$ queries to $O_u$, $O_T$, $O_{\Lambda}$, $O_{\exp}$, $U$ and their inverses,
2. $(n_t + n_e + n_f + 1)$ ancilla qubits,
3. $O(|u(0)||///u(T)||)$ extra one-qubit gates.

Proof. Applying $O_u$ prepares $|u(0)\rangle_r |0\rangle_f |0\rangle_e |0\rangle_t$. Lemma 29 tells that applying the block-encoding of $e^{AT}$ gives

$$\frac{1}{\|u(0)\|} \sum_{j=0}^{N-1} (e^{\lambda_j T} u(0))_j |j\rangle_v |0\rangle_r |0\rangle_f |0\rangle_e |0\rangle_t + |\perp\rangle.$$

Therefore the first part after successful measurement is $|u(T)\rangle$, and the number of repeats to boost the success probability to $\Omega(1)$ is $O(|u(0)||///u(T)||)$ with amplitude amplification.

According to Theorem 30, the query complexity of our algorithm is $O(|u(0)||///u(T)||)$, which still might depend on $T$ and $\|A\|$. Therefore, the scalings on $T$ and $\|A\|$ heavily depends on the decay of the solution, and it is possible that the scalings are independent of $T$ and $\|A\|$. More specifically, a sufficient condition is that $A$ has 0 eigenvalue and $u(0)$ has non-trivial overlap with the 0 eigenstate. To see this, suppose that $\lambda_0 = 0$ and $|\langle u(0)|\psi_0\rangle| \geq \Omega(1)$ where $|\psi_0\rangle$ is the eigenstate corresponding to $\lambda_0$. Then, let $|\psi_j\rangle$ denote the eigenstates corresponding to $\lambda_j$, and we get

$$\frac{u(T)}{\|u(0)\|} = U e^{\lambda_j T} U^\dagger |u(0)\rangle$$

$$= U e^{\lambda_j T} U^\dagger (\sum_{j=0}^{N-1} \langle \psi_j | u(0) \rangle |\psi_j\rangle)$$

$$= \sum_{j=0}^{N-1} e^{\lambda_j T} \langle \psi_j | u(0) \rangle |\psi_j\rangle.$$
Lemma 31. Let \( A = U \Lambda U^\dagger \) where \( U \) is a known unitary and \( \Lambda = \text{diag}(\lambda_0, \ldots, \lambda_{N-1}) \) such that \( \lambda_j \leq 0 \) for all \( j \). Suppose we are given the oracles described in Section 4.3.1, and the time, eigenvalue and function registers consist of \( n_t \), \( n_e \) and \( n_f \) qubits, respectively. Then a \((T, n_t + n_e + n_f + 1, 0)\)-block-encoding of \( \int_0^T e^{A(T-s)} ds \) can be constructed, using 6 queries to \( O_T \), \( O_\Lambda \), \( O_f \) and their inverses, 2 queries to \( U \) and its inverse, and an extra controlled one-qubit rotation gate.

Combining Lemma 29, Lemma 31 and Lemma 22, we can solve the inhomogeneous ODE using the circuit in Figure 6, and the cost is summarized as follows.

Theorem 32. Consider solving the ODE system Equation (1) with time-independent \( b \) up to time \( T \), where \( A \) is a negative semi-definite Hermitian matrix with known eigenvalues and eigenstates. Suppose we are given the oracles described in Section 4.3.1, and the time, eigenvalue and function registers consist of \( n_t \), \( n_e \) and \( n_f \) qubits, respectively. Then for any \( T > 0 \), there exists a quantum algorithm that outputs the normalized solution \( |u(T)\rangle \) with \( \Omega(1) \) success probability, using

1. \( \mathcal{O}\left(\frac{\|u(0)\| + T\|b\|}{\|u(T)\|}\right) \) (121)
   queries to the given oracles, their inverses and controlled versions,

2. \((n_t + n_e + n_f + 2)\) ancilla qubits,

3. \( \mathcal{O}\left(\frac{\|u(0)\| + T\|b\|}{\|u(T)\|}\right) \) (122)
   extra one-qubit gates.

Compared to Lemma 10, we avoid the linear dependence on \( ||A|| \) in query complexity. In the worst case, the query complexity of our algorithm is still linear in \( T \), but a possible linear growth of \( ||u(T)|| \) may cancel it and leads to an \( \mathcal{O}(1) \) scalings in \( T \) as well (see the discussions after Theorem 28). We also remark that an \( \mathcal{O}(1) \) query complexity can be obtained in many more scenarios if we allow the eigenvalues to be complex and use their knowledge in constructing the quantum circuit. This will be discussed in the next subsection.

4.4 Generalization to matrices with known complex eigenvalues and eigenstates

We generalize the results in Section 4.3 to the case where the eigenvalues are not assumed to be real and non-positive. Most techniques are the same as those in Section 4.3 with two main differences: we need to perform another controlled rotation to deal with the imaginary parts of the eigenvalues, and we use the real-part difference to perform the rotation associated with the real parts to deal with possibly positive real parts of the eigenvalues.
4.4.1 Oracles and notation

Let \( A = U\Lambda U^\dagger \), \( \Lambda = \text{diag}(\lambda_0, \ldots, \lambda_{N-1}) \), and \( \lambda_j = \alpha_j + i\beta_j \) where \( \alpha_j \) and \( \beta_j \) represent the real part and the imaginary part of \( \lambda_j \), respectively. Let the time \( T \) be given by \( O_T : |0\rangle \rightarrow |T\rangle \). We assume that the eigenvalues are given by the oracles \( O_{\Lambda,r} : |0\rangle_{e} |j\rangle_{v} \rightarrow |\alpha_j\rangle_{e} |j\rangle_{v} \) and \( O_{\Lambda,i} : |0\rangle_{e} |j\rangle_{v} \rightarrow |\beta_j\rangle_{e} |j\rangle_{v} \), and the product operation between eigenvalues and time is given by \( O_{\prod} : |0\rangle_{f} |\beta_j\rangle_{e} |j\rangle_{t} \rightarrow |\beta_j\rangle_{f} |\beta_j\rangle_{e} |j\rangle_{t} \).

Assume two parameters \( \alpha \) and \( \beta \) are known, where \( \alpha \) is the largest \( \alpha_j \), and \( \beta \) is a lower bound of \( \min_{\alpha_j = 0} |\beta_j| \) (i.e., the minimum of the lengths of purely imaginary eigenvalues). Notice that \( \alpha \) is the logarithmic norm of \( A \) associated with the matrix 2-norm.

Associated with the homogeneous operator \( e^{AT} \), we assume access to the oracle \( O_{\exp,\alpha} : |0\rangle_{f} |\alpha_j\rangle_{e} |t\rangle_{t} \rightarrow |e^{(\alpha_j - \alpha)t}\rangle_{f} |\alpha_j\rangle_{e} |t\rangle_{t} \). Associated with the inhomogeneous \( \int_{0}^{T} e^{i\lambda(T-s)} ds \), we define

\[
C(\alpha, \beta, T) := \begin{cases} 
T, & \text{if } \alpha = \beta = 0, \\
\frac{2}{\beta}, & \text{if } \alpha = 0 \text{ and } \beta \neq 0, \\
\frac{1}{\alpha}(e^{\alpha T} - 1), & \text{else,}
\end{cases}
\]

and let

\[
C(\alpha, \beta, T)^{-1} \int_{0}^{T} e^{i\lambda_j(T-s)} ds = f(\alpha_j, \beta_j, T) + ig(\alpha_j, \beta_j, T),
\]

where \( f \) and \( g \) represent the real and imaginary parts, respectively. Notice that these two functions \( f \) and \( g \) are classically computable, and thus we assume access to the oracles \( O_f : |0\rangle_{f} |\alpha_j\rangle_{e} |\beta_j\rangle_{e} |t\rangle_{t} \rightarrow |f(\alpha_j, \beta_j, T)\rangle_{f} |\alpha_j\rangle_{e} |\beta_j\rangle_{e} |t\rangle_{t} \), and \( O_g : |0\rangle_{f} |\alpha_j\rangle_{e} |\beta_j\rangle_{e} |t\rangle_{t} \rightarrow |g(\alpha_j, \beta_j, T)\rangle_{f} |\alpha_j\rangle_{e} |\beta_j\rangle_{e} |t\rangle_{t} \). Here the subscript “e” refers to eigenvalue’s real part, and “e, i” refers to eigenvalue’s imaginary part.

Same as before, for the vectors \( u(0) \) and \( b \), we assume \( O_u |0\rangle = \frac{1}{||u(0)||} \sum_{j=0}^{N-1} u_j(0) |j\rangle \) and \( O_b |0\rangle = \frac{1}{||b||} \sum_{j=0}^{N-1} b_j |j\rangle \), and assume that \( ||u(0)||, ||b|| \) are known.

4.4.2 Homogeneous case

Notice that \( ||e^{AT}|| \) is now bounded by \( e^{\alpha T} \), we need to perform the normalized operator \( e^{AT}/e^{\alpha T} \). As a result, we replace the oracle \( O_{\exp} \) in Section 4.3 by the oracle \( O_{\exp,\alpha} : |0\rangle_{f} |\alpha_j\rangle_{e} |t\rangle_{t} \rightarrow |e^{(\alpha_j - \alpha)t}\rangle_{f} |\alpha_j\rangle_{e} |t\rangle_{t} \).

Then the real parts can be implemented using the same approach in Section 4.3. The imaginary parts only introduce another \( e^{i\beta T} \) factor on the basis, which can be implemented by the controlled phase shift gate. The entire circuit is summarized in Figure 8, and we claim its effectiveness and cost as follows. The proof is very similar with that of Lemma 29 and is given in Appendix D.

**Lemma 33.** Let \( A = U\Lambda U^\dagger \) where \( U \) is a known unitary and \( \Lambda = \text{diag}(\lambda_0, \ldots, \lambda_{N-1}) \) such that \( \lambda_j = \alpha_j + i\beta_j \) for all \( j \). Suppose the oracles described in Section 4.4.1, and the time, eigenvalue and function registers consist of \( n_t, n_e \) and \( n_f \) qubits, respectively. Then an \( (e^{AT}, n_t + n_e + n_f + 1, 0) \)-block-encoding of \( e^{AT} \) can be constructed, using 10 queries to aforementioned oracles and their inverses, 2 queries to \( U \) and its inverse, and 4 extra (controlled) one-qubit gates.

**Theorem 34.** Consider solving the ODE system Equation (1), where \( b = 0 \) and \( A \) is a unitarily diagonalizable matrix with known eigenvalues and eigenstates. Suppose that we are given the oracles as described.
in Section 4.4.1, and that $\alpha$ is the largest eigenvalues’ real part. Then for any $T > 0$, there exists a quantum algorithm that outputs $|u(T)\rangle$ with $\Omega(1)$ success probability, using

1. $\mathcal{O}(e^{\alpha T}||u(0)||/||u(T)||)$ queries to $O_u$, $O_T$, $O_{\lambda r}$, $O_{\lambda i}$, $O_{\exp,\alpha}$, $O_{\text{prod}}$, $U$ and their inverses,

2. $(n_t + n_c + n_f + 1)$ ancilla qubits,

3. $\mathcal{O}(e^{\alpha T}||u(0)||/||u(T)||)$ extra one-qubit gates.

There are two generalizations of Theorem 34 compared to Theorem 30. First, as defined, Theorem 34 applies to more general matrix $A$ with possibly complex eigenvalues. An important category among those is the fast-forwarding for time-independent Hamiltonian simulation, in which $||u(0)|| = ||u(T)||$ and $\alpha = 0$. Therefore, Theorem 34 implies an $\mathcal{O}(1)$ query complexity of simulating Hamiltonians with known eigenvalues and eigenstates, including diagonal Hamiltonians and more general 1-sparse Hamiltonians. Second, thanks to the a priori knowledge on $\alpha$, we could avoid possible exponential dependence caused by either positive real parts of the eigenvalues or norm decay. Specifically, let $\alpha = \alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_{N-1}$. According to Equation (120), we have $||u(T)||/||u(0)|| \geq e^{\alpha_0 T}||\langle u(0)|\psi_0 \rangle| = e^{\alpha T}||\langle u(0)|\psi_0 \rangle|$. Once $||\langle u(0)|\psi_0 \rangle| \geq \Omega(1)$, we have $e^{\alpha T}||u(0)||/||u(T)|| = \mathcal{O}(1)$ and thus the corresponding differential equation is solved with query complexity independent of $T$ and $||A||$. We remark that the underlying reason for this efficient fast-forwarding algorithm is again the shifting equivalence discussed in Section 3.3.3: instead of the original ODE, we indeed solve the ODE after shifting by $\alpha$. In the shifted ODE, all the eigenvalues of the coefficient matrix have non-positive real parts and at least one of them has zero real part, and thus the two sources of exponential computational overhead are avoided.

### 4.4.3 Inhomogeneous case

General inhomogeneous ODE with possibly complex eigenvalues can also be solved using the same approach based on Lemma 22. The key is still to construct a block-encoding of the operator $\int_0^T e^{A(T-s)}ds$, and this can be implemented by diagonal transformation as well.

We first study real and imaginary parts of the diagonal components. Let $\lambda_j = \alpha_j + i\beta_j$ be a complex number, and the diagonal element $\int_0^T e^{\lambda_j(T-s)}ds$ can be bounded as follows. If $\alpha_j = \beta_j = 0$, then

$$\int_0^T e^{\lambda_j(T-s)}ds = T. \quad (125)$$

If $\alpha_j = 0$ and $\beta_j \neq 0$, then

$$\left|\int_0^T e^{\lambda_j(T-s)}ds\right| = \left|\int_0^T e^{i\beta_j(T-s)}ds\right| = \left|\frac{1}{i\beta_j}(e^{i\beta_j T} - 1)\right| \leq \frac{2}{|\beta_j|}. \quad (126)$$

If $\alpha_j \neq 0$, then

$$\left|\int_0^T e^{\lambda_j(T-s)}ds\right| \leq \int_0^T |e^{\lambda_j(T-s)}|ds = \int_0^T e^{\alpha_j(T-s)}ds = \frac{1}{\alpha_j}(e^{\alpha_j T} - 1). \quad (127)$$

In summary, by the definition of the parameters $\alpha$, $\beta$ in Section 4.4.1 and the constant $C(\alpha, \beta, T)$ in Equation (123), we always have

$$\left|\int_0^T e^{\lambda_j(T-s)}ds\right| \leq C(\alpha, \beta, T). \quad (128)$$

Then the magnitude of $C(\alpha, \beta, T)^{-1} \int_0^T e^{\lambda_j(T-s)}ds$ is bounded by 1, and we can implement the corresponding diagonal transformation by controlled rotation and phase shift. More specifically, $\int_0^T e^{\lambda_j(T-s)}ds$ can be block-encoded using a circuit similar to that in Figure 8, with the following differences:

1. The single eigenvalue register is extended to two registers to encode real and imaginary parts of the eigenvalues separately and simultaneously.
2. The procedure of encoding and uncomputing the eigenvalues now becomes: (after applying $U^\dag$) we first encode both the real and the imaginary parts by applying both $O_{\Lambda,r}$ and $O_{\Lambda,i}$ on the corresponding registers, then only uncompute at the very end, right before applying $U$.

3. The oracle $O_{\exp,\alpha}$ is replaced by $O_f$.

4. The oracle $O_{\text{prod}}$ is replaced by $O_g$.

**Lemma 35.** Let $A = U\Lambda U^\dag$ where $U$ is a known unitary and $\Lambda = \text{diag}(\lambda_0, \cdots, \lambda_N)$ with $\lambda_j = \alpha_j + i\beta_j$ for all $j$. Let $C(\alpha, \beta, T)$ be defined in Equation (123). Suppose we are given the oracles described in Section 4.4.1, and the time, eigenvalue and function registers consist of $n_t, n_e$ and $n_f$ qubits, respectively. Then a $(C(\alpha, \beta, T), n_t + 2n_e + n_f + 1, 0)$-block-encoding of $\int_0^T e^{A(T-s)}ds$ can be constructed, using 10 queries to aforementioned oracles and their inverses, 2 queries to $U$ and its inverse, and 4 extra one-qubit gates.

A combination of Lemma 33, Lemma 35 and Lemma 22 gives the following result. This generalizes Theorem 32 to general complex eigenvalues with possibly positive real parts.

**Theorem 36.** Consider solving the ODE system Equation (1) with time-independent $b$ up to time $T$, where $A$ is a unitarily diagonalizable matrix with known eigenvalues and eigenstates. Let $\alpha$ be the largest $\alpha_j$, and $C(\alpha, \beta, T)$ be defined in Equation (123). Suppose we are given the oracles described in Section 4.4.1, and the time, eigenvalue and function registers consist of $n_t, n_e$ and $n_f$ qubits, respectively. Then for any $T > 0$, there exists a quantum algorithm that outputs the normalized solution $|u(T)\rangle$ with $\Omega(1)$ success probability, using

1. $\mathcal{O}\left(\frac{e^{\alpha T}||u(0)|| + C(\alpha, \beta, T)||b||}{||u(T)||}\right)$

queries to the given oracles, their inverses and controlled versions,

2. $(n_t + 2n_e + n_f + 2)$ ancilla qubits,

3. $\mathcal{O}\left(\frac{e^{\alpha T}||u(0)|| + C(\alpha, \beta, T)||b||}{||u(T)||}\right)$

extra one-qubit gates.

Similar as the discussion after Theorem 32, one can achieve $\mathcal{O}(1)$ scaling in most scenarios. Suppose that all the eigenvalues of $A$ have non-positive real parts. Then

1. If $\alpha < 0$, i.e., all the eigenvalues have negative real parts and we regard $\alpha$ as a constant, then the query complexity becomes $\mathcal{O}(||u(0)|| + ||b||/||u(T)||)$. Furthermore, in this case $A$ is invertible, and thus $||u(T)|| = \Omega(1)$ following the discussion after Theorem 23, which indicates that the query complexity is independent of both $T$ and $||A||$.

2. If $\alpha = 0$ and $\beta > 0$, i.e., all the eigenvalues of $A$ have non-positive real parts and are non-zero, then the query complexity is also independent of both $T$ and $||A||$. The reason is similar with the previous case, and notice that $||u(T)||$ is still $\Omega(1)$ for components corresponding to purely imaginary eigenvalues analog to the Hamiltonian simulation.

3. If $\alpha = 0$ and $\beta = 0$, i.e., there exists a 0 eigenvalue. Then the query complexity becomes $\mathcal{O}(||u(0)|| + T||b||/||u(T)||)$. This is the same as Theorem 32, which is always independent of $||A||$ and can be independent of $T$ if $b$ has non-trivial overlap with the eigenstate of $A$ corresponding to 0 eigenvalue.

### 4.5 Generalization to time-dependent inhomogeneous terms

Consider the most general ODE Equation (1) time-dependent inhomogeneous term $b(t)$. Here we still assume that the matrix $A$ is normal with known eigenvalues and eigenstates, following the assumptions in Section 4.4. We discuss how to solve this system with potential exponential fast-forwarding in $||A||$ and $T$. 

35
4.5.1 Oracles

We state the oracles first, and their usages will be clear in later discussions. For the matrix $A$, let $\tilde{\alpha} = \max_{0 \leq j \leq N-1} \{0, \text{Re}(\lambda_j)\}$ be a known parameter. Suppose we are given the oracles: $O_T : |0\rangle_t \rightarrow |T\rangle_t$, $O_{\lambda,t} : |0\rangle_e |j\rangle_v \rightarrow |\alpha_j\rangle_e |j\rangle_v$, $O_{\lambda,i} : |0\rangle_e |j\rangle_v \rightarrow |\beta_j\rangle_e |j\rangle_v$, $O_{\exp,\tilde{\alpha},t} : |0\rangle_f |\alpha_j\rangle_e |t\rangle_t \rightarrow |e^{(\alpha_j-\tilde{\alpha})t}|\alpha_j\rangle_e |t\rangle_t$, $O_{\exp,\tilde{\alpha},t} : |0\rangle_f |\alpha_j\rangle_e |k\rangle_t \rightarrow |e^{\alpha_j(T(1-k/M)-\tilde{\alpha}T)}|\alpha_j\rangle_e |k\rangle_t$, $O_{\text{prod},t} : |0\rangle_f |\beta_j\rangle_e |t\rangle_t \rightarrow |e^{\beta_jT(1-k/M)}|\beta_j\rangle_e |t\rangle_t$, and $O_{\text{prod},t} : |0\rangle_f |\beta_j\rangle_e |k\rangle_t \rightarrow |e^{\beta_jT(1-k/M)}|\beta_j\rangle_e |k\rangle_t$.

For the vector $u(0)$, we assume as usual the oracle $O_u : |0\rangle_v \rightarrow |u(0)\rangle_v$ and that $\|u(0)\|$ is known. The difference lies in the input models for $O_{\max}$.

4.5.2 Results

The solution of Equation (1) is given as

$$u(T) = e^{AT}u(0) + \int_0^T e^{A(T-s)}b(s)ds.$$  

(131)

Our approach is still to prepare both vectors as quantum states and then perform a linear combination of the state, similar as sketched in Lemma 22. The first part is the same as the time-independent case and can be implemented according to Lemma 33. However, the second part is no longer an operator acting on a vector and thus requires further treatment.

The idea is to first discretize the integral by numerical quadrature and then apply another linear combination of states. In particular, we use the first-order numerical quadrature (the same as the left Riemann sum) with $M$ equi-distant nodes to get

$$\int_0^T e^{A(T-s)}b(s)ds \approx \frac{T}{M} \sum_{k=0}^{M-1} e^{A(T-kT/M)}b(kT/M).$$  

(132)

Then the right hand side is again a linear combination of vectors and can be implemented following the idea of Lemma 22 and Lemma 6.

We first bound the quadrature error and estimate the scaling of $M$, which is a direct consequence of the standard quadrature error bound [60] and the chain rule.

**Lemma 37.** Suppose $b(t)$ is continuously differentiable. Then the numerical quadrature error can be bounded as

$$\left\| \int_0^T e^{A(T-s)}b(s)ds - \frac{T}{M} \sum_{k=0}^{M-1} e^{A(T-kT/M)}b(kT/M) \right\| \leq \frac{T^2e^{\tilde{\alpha}T}}{2M} \sup_{t \in [0,T]} (\|A\|\|b(t)\| + \|db(t)/dt\|),$$  

(133)

where $\tilde{\alpha} := \max_{0 \leq j \leq N-1} \{0, \text{Re}(\lambda_j)\}$. As a result, in order to bound the quadrature error by $\epsilon'$ > 0, it suffices to choose

$$M = \Theta \left( \frac{T^2e^{\tilde{\alpha}T}}{2\epsilon'} \sup_{t \in [0,T]} (\|A\|\|b(t)\| + \|db(t)/dt\|) \right).$$  

(134)

We are now ready to discuss the quantum implementation and its complexity for solving Equation (1).
Theorem 38. Consider solving the ODE system Equation (1) up to time $T$, where $A = UΛU^\dagger$ is a unitarily diagonalizable matrix with known eigenvalues and eigenstates, and $b(t)$ is a continuously differentiable vector-valued function. Let $\tilde{\alpha} = \max_{0 \leq j \leq N-1} \{0, \Re(\lambda_j)\}$ be a known parameter, and

$$M = \Theta \left( \frac{T^2 e^{\alpha T}}{\epsilon \|u(T)\|} \sup_{\|e\| \leq 0.1} \left( \|A\| \|b(t)\| + \|db(t)/dt\| \right) \right)$$

(135)

Suppose we are given the oracles described in Section 4.5.1, and the time, eigenvalue and function registers consist of $n_t$, $n_e$ and $n_f$ qubits, respectively. Then, for any $T > 0$, there exists a quantum algorithm that outputs an $\epsilon$-approximation of the normalized solution $|u(T)\rangle$ with $\Omega(1)$ success probability, using

1. $O\left(e^{\tilde{\alpha} T} \sqrt{2\|u(0)\|^2 + 2T^2 \|b\|^2_{\text{avg}}} \right) / \left(1 - \epsilon/2\|u(T)\| \right)$

(136)

for $\epsilon$ queries to the given oracles, their inverses and controlled versions, where $\|b\|^2_{\text{avg}} := M^{-1} \sum_{k=0}^{M-1} \|b(kT/M)\|^2$,

2. $(n_t + n_e + n_f + 2)$ ancilla qubits,

3. $O\left(e^{\tilde{\alpha} T} \sqrt{2\|u(0)\|^2 + 2T^2 \|b\|^2_{\text{avg}}} \right) / \left(1 - \epsilon/2\|u(T)\| \right)$

(137)

extra one-qubit gates.

Proof. We start with $|0\rangle_c |0\rangle_r |0\rangle_f |0\rangle_t |0\rangle_e |0\rangle_v$, where the subscripts represent control, rotation, function, time, eigenvalue and vector registers, respectively. For notation simplicity, we let

$$\|b\|_{\text{avg}} \equiv \frac{1}{M} \sum_{k=0}^{M-1} \|b(kT/M)\|^2 .$$

(138)

We first perform a single qubit rotation on the control register to get

$$\frac{1}{e^{\tilde{\alpha} T} \sqrt{\|u(0)\|^2 + T^2 \|b\|^2_{\text{avg}}}} \left(e^{\tilde{\alpha} T} |u(0)\rangle |0\rangle_c + e^{\tilde{\alpha} T} T \|b\|_{\text{avg}} |1\rangle_c \right) |0\rangle_r |0\rangle_f |0\rangle_t |0\rangle_e |0\rangle_v .$$

(139)

For the first part $|0\rangle_c |0\rangle_r |0\rangle_f |0\rangle_t |0\rangle_e |0\rangle_v$, we can directly apply Theorem 34 using the circuit in Figure 8 with the only difference being that all the operations are controlled by the control register and only apply when it is $|0\rangle_c$. Then we map the first part to

$$e^{-\tilde{\alpha} T} |0\rangle_c |0\rangle_r |0\rangle_f |0\rangle_t |0\rangle_e e^{\tilde{\alpha} T} u(0)\rangle_v + |0\rangle_c \langle \bot| .$$

(140)

The second part $|1\rangle_c |0\rangle_r |0\rangle_f |0\rangle_t |0\rangle_e |0\rangle_v$ will encode the approximation of the integral, and all the related operations are controlled by the control register and only apply when it is $|1\rangle_c$. We first apply $O\|b\|$ to get

$$\frac{1}{\sqrt{\sum_{k=0}^{M-1} \|b(kT/M)\|^2}} \sum_{k=0}^{M-1} \|b(kT/M)\| |1\rangle_c |0\rangle_r |0\rangle_f |k\rangle_t |0\rangle_e |0\rangle_v .$$

(141)

Notice that, with an abuse of notation, here the time register encodes the index of the time step rather than the binary encoding of the exact time. Applying $O_{b,t}$ gives

$$\frac{1}{\sqrt{\sum_{k=0}^{M-1} \|b(kT/M)\|^2}} \sum_{k=0}^{M-1} \|b(kT/M)\| |1\rangle_c |0\rangle_r |0\rangle_f |k\rangle_t |0\rangle_e |b(kT/M)\rangle_v .$$

(142)

This state gives the weighted superposition of the input states with different information on the time, and thus we can again apply Theorem 34 using a circuit similar to that in Figure 8 with the following differences:
1. All the operators are controlled by the control register and only apply when it is $|1\rangle_c$.

2. We do not perform $O_T$ and $O_1^u$ since here the time information has already been encoded.

3. $O_{\exp,a}$ and $O_{\prod}$ are replaced by $O_{\exp,a,t}$ and $O_{\prod,t}$.

The corresponding circuit computes $e^{A(T-kT/M)}b(kT/M)$ in parallel in time and gives the state

$$e^{-\tilde{\alpha}T} \sqrt{\sum_{k=0}^{M-1} ||b(kT/M)||^2} \sum_{k=0}^{M-1} ||b(kT/M)|| |1\rangle_c |0\rangle_r |0\rangle_f |k\rangle_t |0\rangle_e e^{A(T-kT/M)} |b(kT/M)\rangle_v + |1\rangle_e |\perp\rangle.$$  

(143)

Applying $\otimes^n H$ on the time register gives

$$e^{-\tilde{\alpha}T} \sqrt{\sum_{k=0}^{M-1} ||b(kT/M)||^2} |1\rangle_c |0\rangle_r |0\rangle_f |0\rangle_t |0\rangle_e \left( \sum_{k=0}^{M-1} ||b(kT/M)|| e^{A(T-kT/M)} |b(kT/M)\rangle_v \right) + |1\rangle_e |\perp\rangle.$$  

(144)

Now we combine Equation (140) and Equation (144) together, and the state in Equation (139) is transformed to

$$\frac{1}{e^{\tilde{\alpha}T} \sqrt{||u(0)||^2 + T^2 ||b||^2_{\text{avg}}}} (|0\rangle_r |0\rangle_f |0\rangle_t |0\rangle_e \otimes \left| \begin{array}{c} \langle u(0)\| e^{AT} |u(0)\rangle_v \end{array} \right| + |1\rangle_e \left( \frac{T}{M} \sum_{k=0}^{M-1} ||b(kT/M)|| e^{A(T-kT/M)} |b(kT/M)\rangle_v \right) + |\perp\rangle).$$  

(145)

The final step is to apply a Hadamard gate on the control register to add up the two states, which yields

$$\frac{||\tilde{u}(T)||}{e^{\tilde{\alpha}T} \sqrt{2(||u(0)||^2 + T^2 ||b||^2_{\text{avg}})}} (|0\rangle_c |0\rangle_r |0\rangle_f |0\rangle_t |0\rangle_e |\tilde{u}(T)\rangle_v + |\perp\rangle),$$  

(146)

where

$$\tilde{u}(T) = e^{AT} u(0) + \frac{T}{M} \sum_{k=0}^{M-1} e^{A(T-kT/M)} b(kT/M).$$  

(147)

According to Lemma 37 and Equation (135), we have $||\tilde{u}(T) - u(T)|| \leq \frac{1}{2} \epsilon ||u(T)||$. Therefore, according to Lemma 1, $|\tilde{u}(T)\rangle$ is an $\epsilon$-approximation of $|u(T)\rangle$. $|u(T)\rangle$ can be obtained by measuring the state in Equation (146) and obtaining all 0 in the ancilla registers which indicates success. Since the state in Equation (146) takes $O(1)$ queries and extra one-qubit gates to prepare, to prepare $|u(T)\rangle$ with $\Omega(1)$ success probability, we can use a number of queries and extra one-qubit gates of order

$$\frac{e^{\tilde{\alpha}T} \sqrt{2(||u(0)||^2 + T^2 ||b||^2_{\text{avg}})}}{||\tilde{u}(T)||} \leq \frac{e^{\tilde{\alpha}T} \sqrt{2(||u(0)||^2 + T^2 ||b||^2_{\text{avg}})}}{||u(T)||} \leq \frac{e^{\tilde{\alpha}T} \sqrt{2(||u(0)||^2 + T^2 ||b||^2_{\text{avg}})}}{1 - \epsilon/2}.$$  

(148)

As remarked after Theorem 36, our generalized Theorem 38 can yield an $O(1)$ query complexity under typical scenarios when most $||b(t)||$’s are $\Omega(1)$ and the overlaps between $b(t)$’s and the eigenstates of $A$ corresponding to 0 eigenvalue are non-degenerate. The key feature of our algorithm to achieve fast-forwarding is the efficient implementation of the numerical quadrature, i.e., linear combination of quantum states. We note that, although the number of the quadrature nodes $M$ scales badly in $T$ and $\epsilon$, it does not contribute to the overall query complexity, and only introduces poly log($M$) overhead through the number of qubits needed in the time register and the constructions of the oracles $O_{\delta,t}$ and $O_{\gamma(b(t))}$. Similar idea of fast quantum implementation of numerical integration has been used and is also the key reason for speedups in the interaction picture Hamiltonian simulation [18] and the qHOP method for time-dependent Hamiltonian simulation [56].
4.6 Application: parabolic PDEs

We apply our fast-forwarded algorithms for coefficient matrices with known eigenvalues and eigenstates to solving various high-dimensional evolutionary PDEs of parabolic type. A standard approach for solving evolutionary PDEs is the method of lines: we first discretize all the spatial variables and transform the PDE to an ODE, then apply ODE solvers to obtain numerical solutions. The key observation is that many classes of PDEs involve the spatial divergence and Laplacian operators, which, after suitable spatial discretization, can be simultaneously diagonalized under the Fourier basis and the eigenvalues have closed-form expressions. This enables fast-forwarding.

We consider PDEs with time variable \( t \in [0, T] \) and spatial variable \( x = (x_0, \cdots, x_{d-1}) \in [0, 1]^d \). We impose periodic boundary conditions on each spatial dimension. The gradient operator is defined as
\[
\nabla = \left( \frac{\partial}{\partial x_0}, \cdots, \frac{\partial}{\partial x_{d-1}} \right)^T,
\]
and the spatial Laplacian operator is defined as
\[
\Delta = \sum_{j=0}^{d-1} \frac{\partial^2}{\partial x_j^2}.
\]

More generally, we also consider generalized gradient and Laplacian operators defined, for a \( d \)-dimensional real vector \( a \), as
\[
\nabla^a = \left( a_0 \frac{\partial}{\partial x_0}, \cdots, a_{d-1} \frac{\partial}{\partial x_{d-1}} \right)^T,
\]
and
\[
\Delta^a = \sum_{j=0}^{d-1} a_j \frac{\partial^2}{\partial x_j^2}.
\]

The general class of PDEs we consider is of the form
\[
\frac{\partial}{\partial t} u(x, t) = \Delta^a u(x, t) + \nabla^{a'} \cdot u(x, t) + cu(x, t) + b(x, t), \quad t \in [0, T], x \in [0, 1]^d,
\]
\[
u(x, 0) = u_0(x).
\]

Here \( a, a' \) are two bounded real vectors such that all the elements of \( a \) are non-negative, \( c \) is a non-positive real number and \( b(x, t) \) is a smooth scalar function. We first show how to spatially discretize the spatial divergence and Laplacian operators, and then discuss various classes of PDEs where our fast-forwarded algorithms can apply.

4.6.1 Spatial discretization

We discretize space by the central difference method, with \( n \) grid points in each coordinate. For the Laplacian, it can be approximated by the \( d \)-dimensional discrete Laplacian \( A_L^a \), defined as
\[
A_L^a = a_0 D_h \otimes I \otimes \cdots \otimes I + a_1 I \otimes D_h \otimes I \otimes \cdots \otimes I + \cdots + a_{d-1} I \otimes \cdots \otimes I \otimes D_h.
\]

Here \( D_h \) is the one-dimensional discrete Laplacian operator defined as
\[
D_h := \frac{1}{h^2} \begin{pmatrix}
-2 & 1 & 1 \\
1 & -2 & 1 \\
\ddots & \ddots & \ddots \\
1 & 1 & -2
\end{pmatrix},
\]
\[
(155)
\]
with $h = 1/n$. The eigenvalues of $D_h$ are $\mu_k = -4n^2 \sin^2(k\pi/n)$, and the corresponding eigenstate is $\frac{1}{\sqrt{n}}(1, \omega_n^k, \omega_n^{2k}, \ldots, \omega_n^{(n-1)k})^T$ where $\omega_n = e^{2\pi i/n}$. Similarly, for the divergence operator, it can be approximated by the $d$-dimensional discrete divergence $A_G^d$, defined as

$$A_G^d := a_0 V_h \otimes I \otimes \cdots \otimes I + a_1 I \otimes V_h \otimes I \otimes \cdots \otimes I + \cdots + a_{d-1} I \otimes \cdots \otimes I \otimes V_h,$$  

(156)

where

$$V_h := \frac{1}{2h} \begin{pmatrix} 0 & 1 & \cdots & -1 \\ -1 & 0 & 1 & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ 1 & \cdots & -1 & 0 \end{pmatrix},$$  

(157)

with $h = 1/n$. The eigenvalues of $V_h$ are $\mu_k = in \sin(2k\pi/n)$, and the corresponding eigenstate is the same as that of $D_h$, i.e., $\frac{1}{\sqrt{n}}(1, \omega_n^k, \omega_n^{2k}, \ldots, \omega_n^{(n-1)k})^T$. Therefore, for any two vectors $a$ and $a'$, the linear combination $A_L^a + A_G^a + cI$ has the eigenbasis $F_h^{\otimes d}$, where $F_h$ denotes the discrete Fourier transform matrix and can be efficiently implemented by the (inverse) quantum Fourier transform. The eigenvalues of $A_L^a + A_G^a + cI$ are given as

$$\mu = c - 4n^2 \sum_{j=0}^{d-1} a_j \sin^2(k_j \pi/n) + in \sum_{j=0}^{d-1} a'_j \sin(2k_j \pi/n), \quad k_j \in [n].$$  

(158)

Given Equation (158), we can use classical arithmetic to construct an oracle encoding the eigenvalues of $A_L^a + A_G^a + cI$. However, we remark that the gate complexity of constructing such an oracle typically scales as $\mathcal{O}(d)$, since computing the eigenvalues of $A_L^a + A_G^a + cI$ requires computing the sum of $2d$ terms, which generally cannot be expressed in a simple closed form.

### 4.6.2 Fast-forwarding

After spatial discretization, the PDE Equation (153) can be transferred to the ODE system

$$\frac{d}{dt} \vec{u}(t) = (A_L^a + A_G^a + cI) \vec{u}(t) + \vec{b}(t), \quad t \in [0, T],$$  

$$\vec{u}(0) = \vec{u}_0.$$  

(159)

Here, $\vec{u}(t)$ and $\vec{b}(t)$ are $n^d$-dimensional vectors containing the function values of $u(\cdot, t)$ and $b(\cdot, t)$ evaluated at all the grid points $(k_1h, k_2h, \ldots, k Dh)$ for $k_j \in [n]$, respectively.

Notice that generic quantum ODE solvers are not efficient for solving Equation (159) due to the large spectral norm of the coefficient matrix. Specifically, $\|D_h\| \sim 1/h^2$ and $\|V_h\| \sim 1/h$, and thus $\|A_L^a\| = \Theta(dn^2)$ and $\|A_G^a\| = \Theta(dn^2)$. Therefore the coefficient matrix $A_L^a + A_G^a + cI$ has a spectral norm $\Theta(dn^2)$, which can be very large in high-dimensional equations or when the number of the grid points for each coordinate is large for accurate spatial discretization. If we apply best existing generic ODE solver as in Lemma 10 and Lemma 11, the overall query complexity scales

$$\mathcal{O} \left( \frac{\sup_{t \in [0, T]} \|u(t)\|}{\|u(T)\|} Tdn^2 \text{ poly log}(T/\epsilon) \right).$$  

(160)

We can apply our fast-forwarding result, namely Theorem 30 for homogeneous case and Theorem 38 for general inhomogeneous case, because Equation (153) satisfies the assumptions in Theorem 30 and Theorem 38, of which the most important one is that the coefficient matrix has known eigenvalues and eigenstates as discussed before.

**Corollary 39.** Consider solving the semi-discretized PDE Equation (153) up to time $T$. Then there exists a quantum algorithm that outputs the normalized solution $|\vec{u}(T)\rangle$ with $\Omega(1)$ success probability, using

$$\mathcal{O} \left( \frac{\sqrt{2\|\vec{u}(0)\|^2 + 2T^2\|\vec{b}\|^2_{\text{avg}}}}{\|\vec{u}(T)\|} \right).$$  

(161)
queries to the prepare oracles of $\vec{u}(0)$ and $\vec{b}(t)$, and

$$\tilde{O} \left( \frac{\sqrt{2\|\vec{u}(0)\|^2 + 2T^2\|\vec{b}\|_{\text{avg}}^2}}{\|\vec{u}(T)\|} d \log^2(n) \text{ poly log}(T/\epsilon) \right)$$

(162)

additional gates. Here $\|\vec{b}\|_{\text{avg}}^2 := M^{-1} \sum_{k=0}^{M-1} \|\vec{b}(kT/M)\|^2$.

Proof. The number of queries to the state preparation oracles is a direct consequence of Theorem 38. The additional gates are mainly due to the implementation of the oracles mentioned in Section 4.5.1 and the $d$-dimensional QFT operator $F_{\mathbb{R}^d}$. Since the eigenvalues of the matrix $A_{L}^T + A_{G}^T + cI$ have closed-form expression Equation (158) containing summation of $(2d + 1)$ terms, constructing oracles $O_{\Lambda,r}$, $O_{\Lambda,i}$ would require $\tilde{O}(d \text{ poly log}(1/\epsilon))$ elementary gates by performing classical arithmetic operations with reversible computational model [61, Chapter 6]. Similarly, constructing all other oracles requires $\tilde{O}(\text{poly log}(T/\epsilon))$ gates, and implementing QFT operator $F_{\mathbb{R}^d}$ requires $\mathcal{O}(d \log^2(n))$ gates. These together contribute to an $\tilde{O} \left( d \log^2(n) \text{ poly log}(T/\epsilon) \right)$ multiplicative factor.

Compared to generic ODE solvers, Corollary 39 avoids the dependence on the spectral norm of the coefficient matrix, and thus can be more efficient for solving high-dimensional PDEs with a large degree of freedom in spatial discretization. Specifically, our result exponentially improves the dependence on $n$. Our result also exponentially improves the number of state preparation oracles in $d$, though the number of additional elementary gates required by other oracles and Fourier transform still scales linearly in $d$. Following the discussions after Theorem 36, the linear scaling in $T$ can also be cancelled if $c = 0$ and the inhomogeneous term $\vec{b}(t)$ has non-trivial overlap with the eigenstate of $A_L^T + A_G^T$, corresponding to the 0 eigenvalue, since in this scenario $\|\vec{u}(T)\|$ grows linearly in $T$ and cancels the $T$ factor in the numerator. In the continuous analog, such a condition means that $b(x,t)$ has a non-trivial overlap with constant functions, since the eigenfunction of the Laplacian and divergence operators corresponding to their 0 eigenvalue is the constant function. We remark that, in the heat equation, this condition is natural and corresponds to non-zero total heat of the heat source.

4.6.3 Examples

Transport equation The transport equation describes phenomena where a conserved quantity, such as mass, heat, energy, momentum, or electric charge is transported in a space. Transport equations play a fundamental role in many applied areas, from wave propagation, electromagnetism, particle physics, nuclear reaction, to computer vision, medical imaging, and semiconductor conducting [62–65]. The equation has the form

$$\frac{\partial}{\partial t} u(x,t) = \nabla^a \cdot u(x,t) + b(x,t).$$

(163)

Equation (163) is a special form of Equation (153) with $a = 0$ and $c = 0$, so can be fast-forwarded using Corollary 39. We remark that the coefficient matrix of Equation (163) after spatial discretization has purely imaginary eigenvalues, and thus the homogeneous version (i.e., $b \equiv 0$) is a Hamiltonian simulation problem and can be fast-forwarded without any exponential dependence in time related to the norm decay.

A generalized version of the transport equation is the Liouville’s equation, which is used to model the evolution of the distribution of particles in a phase space. Josiah Willard Gibbs was the first to recognize the importance of this equation as the fundamental equation of statistical mechanics, with connections to Hamiltonian dynamical systems, symplectic geometry, and ergodic theory, as well as various applications to kinetic theory of gases, atmospheric dynamics, astronomy, and thermodynamics [66–68]. The equation has the form

$$\frac{\partial}{\partial t} u(x,t) = \sum_{j=0}^{d-1} a_j(t) u(x,t) + b(x,t).$$

(164)

However, our fast-forwarded algorithm requires the coefficient matrix to be constant, so we can only deal with special cases of Equation (164) with constant functions $a_j$’s. In these cases, the Liouville’s equation degenerates to the transport equation.
**Heat equation**  The heat equation, which is used to describe the conduction of heat energy. Here we consider the inhomogeneity as an external source of heat. As the prototypical parabolic PDE, the heat equation is as fundamental to the broader field of science and engineering, such as the connection with random walk and Brownian motion in probability theory, the Black-Scholes option pricing model in finance, the formulation of hydrodynamical shocks in fluid dynamics, and the edge detection model in image analysis [69–73]. The Schrödinger equation of quantum mechanics can also be regarded as a heat equation in imaginary time. The equation has the form
\[
\frac{\partial}{\partial t} u(x, t) = \Delta u(x, t) + b(x, t). \tag{165}
\]
This can be fast-forwarded using Corollary 39 since it is a special case of Equation (153) where all the entries of \(a\) are equal, \(a' = 0\) and \(c = 0\).

**Advection-diffusion equation**  The advection-diffusion equation is a combination of the heat and transport equations, and describes physical phenomena where physical quantities are transferred inside a system due to two processes: diffusion and advection. The model deals with the time evolution of chemical or biological species in water or air, exhibiting rich phenomena in chemical processes such as combustion, biological and ecological networks, and population growth [69, 74, 75]. The equation has the form
\[
\frac{\partial}{\partial t} u(x, t) = a \Delta u(x, t) + \nabla a' \cdot u(x, t) + b(x, t). \tag{166}
\]
This can be fast-forwarded using Corollary 39 since it is a special case of Equation (153) where (with an abuse of notations) all the entries of \(a\) are equal and \(c = 0\).

### 4.7 Application: hyperbolic PDEs

We now consider high-dimensional evolutionary PDEs of hyperbolic type. The main difference between hyperbolic PDEs and parabolic PDEs is that hyperbolic PDEs typically involve a second-order time derivative. In order to apply our fast-forwarded algorithm, which only applies to first-order equations, we will first extend the original single PDE to an equivalent system of two PDEs of both the solution and its time derivative. The resulting system of PDEs become first-order in time and can be fast-forwarded as in Section 4.6.

We continue using the notation introduced in Section 4.6. The hyperbolic PDEs we consider have the form
\[
\begin{align*}
\frac{\partial^2}{\partial t^2} u(x, t) &= \Delta^a u(x, t) + cu(x, t) + b(x, t), \quad t \in [0, T], x \in [0, 1]^d, \\
\frac{\partial}{\partial t} u(x, 0) &= u_0(x), \\
\frac{\partial}{\partial t} u(x, 0) &= w_0(x). \tag{167}
\end{align*}
\]
There are two differences compared to Equation (153). First, Equation (167) involves a second-order time derivative but Equation (153) only has first-order time derivatives. Second, there is no spatial divergence term in Equation (167). The reason is that the eigenvalues of the coefficient matrix must be real numbers in order to transform the second-order equation into a first-order system with a normal coefficient matrix.

#### 4.7.1 Spatial discretization and lifting

We first spatially discretize Equation (167) using the same approach as in Section 4.6. The resulting system of ODEs is
\[
\begin{align*}
\frac{d^2}{dt^2} \bar{u}(t) &= (A^a_L + cI)\bar{u}(t) + \bar{b}(t), \quad t \in [0, T], \\
\bar{u}(0) &= \bar{u}_0, \\
\frac{d}{dt} \bar{u}(0) &= \bar{w}_0. \tag{168}
\end{align*}
\]
To apply Theorem 38, we need to transform this equation to a first-order ODE system. To this end, we follow the idea in [76]. Write \( A_L^a + cI = (F_h^{\otimes d})^{-1}DF_h^{\otimes d} \) where \( D \) is a diagonal matrix with diagonal entries

\[
\mu = c - 4n^2 \sum_{j=0}^{d-1} a_j \sin^2(k_j \pi/n), \quad k_j \in [n].
\]  

(169)

Define \( B = (F_h^{\otimes d})^{-1} \sqrt{-D}F_h^{\otimes d} \) where \( \sqrt{-D} \) denotes another diagonal matrix with diagonal entries to be the positive square root of \(-D\)'s, then \( B \) is a Hermitian matrix such that \( B^2 = -(A_L^a + cI) \). Let \( \vec{v}(t) \) be another \( n^d \)-dimensional vector and consider the ODE for \((\vec{u}, \vec{v})\) in the form

\[
\frac{d}{dt} \begin{pmatrix} \vec{u}(t) \\ \vec{v}(t) \end{pmatrix} = \begin{pmatrix} 0 & iB \\ iB & 0 \end{pmatrix} \begin{pmatrix} \vec{u}(t) \\ \vec{v}(t) \end{pmatrix} + \begin{pmatrix} 0 \\ \vec{b}(t) \end{pmatrix},
\]

(170)

which becomes a first-order ODE system. It is straightforward to check that \( \vec{u}(t) \) satisfies Equation (168), and by definition, \( \vec{v}(t) \) solves the linear system \( iB\vec{v}(t) = \frac{d}{dt} \vec{u}(t) \).

4.7.2 Fast-forwarding

We discuss quantum algorithms for Equation (168), i.e., preparing a quantum state approximate to \( \vec{u}(T) \). Our approach is to first consider Equation (170) and prepare a quantum state encoding \((\vec{u}, \vec{v})\), then post-select \( \vec{u} \). Our fast-forwarding result can be summarized as follows.

**Corollary 40.** Consider the semi-discretized PDE Equation (168) up to time \( T \). Suppose that \( c \) is always non-positive and, when \( c = 0 \), an extra condition \( \sum_{j \in [n]^d} w_0(j/n) = 0 \) holds. Then there exists a quantum algorithm that outputs the normalized solution \( |\vec{u}(T)\rangle \) with \( \Omega(1) \) success probability, using

\[
\tilde{O} \left( \frac{\sqrt{2}||\vec{u}(0)||^2 + 2||\vec{v}(0)||^2 + 2T^2||\vec{b}\|^2_{\text{avg}}}{||\vec{u}(T)||} \right)
\]

queries to the prepare oracles of \( \vec{u}_0 \) and \( \vec{b}(t) \),

\[
\tilde{O} \left( \frac{||\vec{u}_0||\sqrt{2}||\vec{u}(0)||^2 + 2||\vec{v}(0)||^2 + 2T^2||\vec{b}\|^2_{\text{avg}}}{||\vec{v}(0)||||\vec{u}(T)||} \right)
\]

queries to the prepare oracle of \( \vec{u}_0 \), and

\[
\tilde{O} \left( \frac{||\vec{u}_0||\sqrt{2}||\vec{u}(0)||^2 + 2||\vec{v}(0)||^2 + 2T^2||\vec{b}\|^2_{\text{avg}}}{||\vec{v}(0)||||\vec{u}(T)||}d \log^2(n) \polylog(T/\epsilon) \right)
\]

(173)

additional gates.

**Proof.** We first analyze the eigenvalue and eigenstate of the coefficient matrix. By discrete Fourier transform and switching the blocks,

\[
\begin{pmatrix}
\frac{1}{\sqrt{2}}I & \frac{1}{\sqrt{2}}I \\
\frac{1}{\sqrt{2}}I & -\frac{1}{\sqrt{2}}I
\end{pmatrix}
\begin{pmatrix}
F_h^{\otimes d} & 0 \\
0 & F_h^{\otimes d}
\end{pmatrix}
\begin{pmatrix}
0 & iB \\
iB & 0
\end{pmatrix}
\begin{pmatrix}
(F_h^{\otimes d})^{-1} & 0 \\
0 & (F_h^{\otimes d})^{-1}
\end{pmatrix}
\begin{pmatrix}
\frac{1}{\sqrt{2}}I & \frac{1}{\sqrt{2}}I \\
\frac{1}{\sqrt{2}}I & -\frac{1}{\sqrt{2}}I
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\frac{1}{\sqrt{2}}I & \frac{1}{\sqrt{2}}I \\
\frac{1}{\sqrt{2}}I & -\frac{1}{\sqrt{2}}I
\end{pmatrix}
\begin{pmatrix}
0 & i\sqrt{-D} \\
i\sqrt{-D} & 0
\end{pmatrix}
\begin{pmatrix}
\frac{1}{\sqrt{2}}I & \frac{1}{\sqrt{2}}I \\
\frac{1}{\sqrt{2}}I & -\frac{1}{\sqrt{2}}I
\end{pmatrix}
\]

\[
= \begin{pmatrix}
i\sqrt{-D} & 0 \\
0 & -i\sqrt{-D}
\end{pmatrix}
\]

(174)
Therefore, the eigenvalues of the coefficient matrix are
\[ \pm i \sqrt{-c + 4n^2 \sum_{j=0}^{d-1} a_j \sin^2(k_j \pi/n)}, \quad k_j \in [n], \] (175)
and the corresponding eigenstates form the unitary matrix
\[
\begin{pmatrix}
(F_{\tilde{h}}^{\otimes d})^{-1} & 0 \\
0 & (F_{\tilde{h}}^{\otimes d})^{-1}
\end{pmatrix}
\begin{pmatrix}
\frac{1}{\sqrt{2}} I & \frac{1}{\sqrt{2}} I \\
\frac{1}{\sqrt{2}} I & -\frac{1}{\sqrt{2}} I
\end{pmatrix}.
\] (176)

Therefore, the coefficient matrix of Equation (170) has closed-form eigenvalues (though they require summation of \(O(d)\) terms), and the eigenstate transformation matrix can be quantum implemented by \(2d\) (inverse) quantum Fourier transform and a Hadamard gate. According to the same reasoning of Corollary 39, we may use Theorem 38 to prepare a quantum state proportional to \((\tilde{u}(T), \tilde{v}(T))\), with
\[
\mathcal{O} \left( \frac{\sqrt{2||\tilde{u}(0)||^2 + 2||\tilde{v}(0)||^2 + 2T^2 ||\tilde{b}||^2_{\text{avg}}}}{\sqrt{||\tilde{u}(T)||^2 + ||\tilde{v}(T)||^2}} \right)
\] (177)
queries to the prepare oracles of \((\tilde{u}(0), \tilde{v}(0))\) and \(\tilde{b}(t)\), and
\[
\tilde{O} \left( \frac{\sqrt{2||\tilde{u}(0)||^2 + 2||\tilde{v}(0)||^2 + 2T^2 ||\tilde{b}||^2_{\text{avg}}}}{\sqrt{||\tilde{u}(T)||^2 + ||\tilde{v}(T)||^2}} d \log^2(n) \text{ poly log}(T/\epsilon) \right)
\] (178)
additional gates.

The state preparation oracle of \((\tilde{u}(0), \tilde{v}(0))\) requires the controlled version of the prepare oracles of \(|\tilde{u}(0)\rangle\) and \(|\tilde{v}(0)\rangle\). The prepare oracle of \(|\tilde{u}(0)\rangle\) is given as assumed. For the prepare oracle of \(|\tilde{u}(0)\rangle\), using the relation \(iB\tilde{v}(t) = \frac{d}{dt} \tilde{u}(t)\), we need to solve the linear system
\[ iB\tilde{v}(0) = \tilde{w}_0. \] (179)

Equation (179) has two features. First, Equation (179) is not always solvable. Notice that the matrix \(B\) is invertible when \(c \neq 0\), and in this case \(\tilde{v}(0)\) uniquely exists for any \(\tilde{w}_0\). However, \(B\) becomes singular when \(c = 0\), and in this case we need to require \(\tilde{w}_0\) does not overlap with the eigenspace of \(B\) corresponding to 0 eigenvalue. Since the eigenstates of \(B\) form Fourier basis with periodic boundary condition, this eigenspace is span \{\(1, 1, \cdots, 1\)\}. Therefore, under the assumption \(\sum_{j \in [n]} u_0(j/n) = 0\), Equation (179) has the solution. Second, though solving general linear system of equations requires complexity linear in the condition number, Equation (179) can be solved using fast inversion proposed in [55] since \(B\) is a normal matrix with known eigenvalues and eigenstates. According to [55, Proposition 7], we need \(\mathcal{O}(||\tilde{u}_0||/||\tilde{v}(0)||)\) queries to preparing \(|\tilde{w}_0\rangle\) and \((F_{\tilde{h}}^{\otimes d})\) as such prepare an extra multiplicative factor \(||\tilde{u}(0)||/||\tilde{v}(0)||\) as such prepare an extra multiplicative factor in all the complexities.

Finally, to obtain \(|\tilde{u}(T)\rangle\) from a state proportional to \((\tilde{u}(T), \tilde{v}(T))\), we need post-selection over the correct subspace, and the average number of repeats required after amplitude amplification is of order \(\sqrt{||\tilde{u}(T)||^2 + ||\tilde{v}(T)||^2/||\tilde{u}(T)||}\). This contributes another multiplicative factor in all the complexities.

Corollary 40 suggests that the overall complexity in \(d, n\) and \(T\) depends on the scalings of \(||\tilde{v}(0)||\) and \(||\tilde{u}(T)||\). Here we focus on a more explicit scaling of \(||\tilde{v}(0)||\), while we postpone the discussions on \(||\tilde{u}(T)||\) to specific examples. According to Equation (170), the vector \(\tilde{v}(0)\) is determined by solving the possibly generalized linear system \(iB\tilde{v}(0) = \tilde{w}_0\). As discussed in the proof, the extra condition \(\sum_{j \in [n]} u_0(j/n) = 0\) when \(c = 0\) is to ensure that such linear system is always solvable. We remark that this condition is a discrete analog of the continuous counterpart \(\int_{[0,1]} w_0(x)dx = 0\) and is also required by some existing algorithm for wave equation [76]. The solution \(\tilde{v}(0)\), however, may be as small as \(\Theta(d^{-1/2} n^{-1} ||\tilde{w}_0||)\), leading to an extra multiplicative factor \(\mathcal{O}(\sqrt{dn})\) in the worst case. This is because the effective condition number of \(B\) is on...
the scale of \( \sqrt{dn} \). However, \( \|\vec{v}(0)\| \) can also be asymptotically comparable to \( \|\vec{w}_0\| \) in the best situations. As discussed in [55, Proposition 8], this happens when \( \vec{w}_0 \) has \( \Theta(1) \) overlap with the eigenstates corresponding to \( \Theta(1) \) eigenvalues of \( B \), which can be guaranteed if \( w_0(x) \) is a smooth function. We summarize the above discussions with explicit best- and worst-case scalings of \( \|\vec{v}(0)\| \) as follows.

**Corollary 41.** Under the same conditions as in Corollary 40,

1. the worst-case complexity can be estimated as:

\[
O \left( \sqrt{2 \|\vec{u}(0)\|^2 + 2 \|\vec{v}_0\|^2 + 2T^2 \|\vec{b}\|^2_{\text{avg}}} \right) \tag{180}
\]

queries to the prepare oracles of \( \vec{u}_0 \) and \( \vec{b}(t) \),

\[
\tilde{O} \left( \sqrt{2 \|\vec{u}(0)\|^2 + 2 \|\vec{v}_0\|^2 + 2T^2 \|\vec{b}\|^2_{\text{avg}}} \right) \sum_{t=0}^{T} \sqrt{dn} \log^2(n) \text{polylog}(T/\epsilon) \tag{182}
\]

additional gates,

2. if we further assume that \( w_0(x) \) is smooth with uniformly bounded partial derivatives, then the overall complexity can be estimated as:

\[
O \left( \sqrt{2 \|\vec{u}(0)\|^2 + 2 \|\vec{v}_0\|^2 + 2T^2 \|\vec{b}\|^2_{\text{avg}}} \right) \tag{183}
\]

queries to the prepare oracles of \( \vec{u}_0 \) and \( \vec{b}(t) \),

\[
\tilde{O} \left( \sqrt{2 \|\vec{u}(0)\|^2 + 2 \|\vec{v}_0\|^2 + 2T^2 \|\vec{b}\|^2_{\text{avg}}} \right) d^{1/2}n \log^2(n) \text{polylog}(T/\epsilon) \tag{185}
\]

additional gates.

Now we compare our algorithm with existing ones. For technical simplicity we only compare the numbers of queries to the state preparation oracles, and we focus on the scaling in \( d \) and \( n \) and postpone the discussions on potential speedup in terms of \( T \) to specific examples. Due to the second-order nature of Equation (167), there are two ways of applying existing generic algorithms. The first approach is to solve Equation (170) using Lemma 11. As shown in the proof of Corollary 40, the spectral norm of the coefficient matrix \( [0, iB; iB, 0] \) is \( O(\sqrt{dn}) \), so the query complexity of generic algorithms scales

\[
\tilde{O} \left( \frac{\sup_t \sqrt{\|\vec{u}(t)\|^2 + \|\vec{v}(t)\|^2}}{\|\vec{u}(T)\|} T \sqrt{dn} \text{polylog}(1/\epsilon) \right) \tag{186}
\]
which is always $\tilde{O}(\sqrt{dn})$. However, notice that here the query model to the initial condition is that to $(\vec{u}(0), \vec{v}(0))$, so the oracle of preparing $\vec{v}(0)$ still requires extra steps, i.e., solving the linear system $iB\vec{v}(0) = \vec{w}_0$, which is our fast-forwarded algorithm. Taken this into consideration, generic algorithm for Equation (170) uses $\tilde{O}(\sqrt{dn})$ queries to $\vec{u}_0$ and $\vec{b}(t)$, and $\tilde{O}(\sqrt{dn})$ to $\tilde{O}(dn^2)$ queries to $\vec{w}_0$ depending on the regularity of $w_0(x)$ in the same fashion of Corollary 41. Therefore our algorithm is always better than the existing one by a multiplicative factor $\sqrt{dn}$.

The second approach is to use a more natural way to lift Equation (168) to first-order equations by introducing $\vec{v}(t) = \frac{d}{dt} \vec{u}(t)$. The corresponding first-order ODE system becomes

$$
\frac{d}{dt} \begin{pmatrix} \vec{u}(t) \\ \vec{v}(t) \end{pmatrix} = \begin{pmatrix} 0 & I \\ A_L^* + cI & 0 \end{pmatrix} \begin{pmatrix} \vec{u}(t) \\ \vec{v}(t) \end{pmatrix} + \begin{pmatrix} 0 \\ \vec{b}(t) \end{pmatrix}.
$$

This is more natural because the input oracle of $\vec{u}(0)$ is exactly the prepare oracle of $w_0$, so there is no the extra linear system issue any more. Notice that our fast-forwarded algorithm cannot be applied to Equation (187) because the coefficient matrix here is no longer normal. However, one may apply the generic algorithm in Lemma 11. Notice that the coefficient matrix has spectral norm $O(\sqrt{dn})$ and condition number $O(n)$ [76], so the query complexity of generic algorithms scales

$$
\tilde{O}\left(\frac{\sup_t \sqrt{||\vec{u}(t)||^2 + ||\vec{v}(t)||^2}}{\|\vec{u}(T)\|} T\sqrt{dn^2} \text{ poly log}(1/\epsilon)\right),
$$

which is always $\tilde{O}(\sqrt{dn^2})$. Therefore our fast-forwarded algorithm always outperforms existing one in $n$, and can be better in $d$ as well if $w_0(x)$ is smooth. Nevertheless, we remark that our fast-forwarded algorithm can only solve a subclass of Equation (168) as specified in Corollary 40, while generic algorithm applied to Equation (187) works for all possible initial conditions.

Finally, we remark that one may also obtain a quantum state encoding $\frac{d}{dt} \vec{u}(T)$ from our algorithm. This can be done by first solving Equation (170) and post-select $\vec{v}(T)$ rather than $\vec{u}(T)$, then compute $\frac{d}{dt} \vec{u}(T) = iB\vec{v}(T)$. To apply the operator $iB$, one may still use the QFT and controlled rotations similar as the quantum fast inversion in [55], since the eigenvalues and eigenstates of $iB$ are known as well.

### 4.7.3 Examples

**Wave equation** The wave equation, as the prototypical hyperbolic PDE, is for the description of travelling and standing waves, such as mechanical waves (e.g., water waves, sound waves, and seismic waves) or electromagnetic waves (including light waves). It arises in fields like acoustics, electromagnetism, fluid mechanics, continuum mechanics, quantum mechanics, plasma physics, general relativity, geophysics, and many other disciplines [69, 77]. Here we consider the inhomogeneity as an external source, such as the electromagnetic wave equation. The equation has the form

$$
\frac{\partial^2}{\partial t^2} u(x, t) = \Delta u + b(x, t).
$$

This is a special case of Equation (167) where all the entries of $a$ are equal and $c = 0$, so can be fast-forwarded using Corollary 40. Since in the wave equation $c = 0$, all the eigenvalues of the corresponding coefficient matrix after spatial discretization and lifting are purely imaginary. Therefore Equation (170) corresponding to the homogeneous wave equation is just a Hamiltonian simulation problem, and the exponential decay related to the norm decay is avoided. Furthermore, the coefficient matrix has zero eigenvalues, which suggests, in the inhomogeneous case, the possibility that $\|\vec{u}(T)\|$ might be $\Theta(T)$ and thus the overall query complexity in Corollary 40 can be $O(1)$. However, noticing that the condition that $(0, \vec{b}(t))$ overlaps with eigenstates corresponding to the zero eigenvalues only implies the linear growth in time of the extended solution $(\vec{u}(t), \vec{v}(t))$, it is not clear yet whether the linear growth is contributed by $\vec{u}(t)$, so the overall query complexity to approximate $|\vec{u}(T)|$ might still be linear in $T$.  

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**Klein-Gordon equation**  The Klein-Gordon equation is a relativistic wave equation that characterizes a field whose quanta are spinless particles. It is a quantized version of the relativistic energy–momentum relation. The Klein-Gordon equation plays the role of one of the fundamental equations of quantum field theory [78–80]. The equation goes as

\[
\frac{\partial^2}{\partial t^2} u(x, t) = \Delta u - m^2 u + b(x, t).
\]

This is a special case of Equation (170) with all equal entries in \(a\). Notice that, since \(m > 0\), the eigenvalues of the coefficient matrix have strictly negative real part, so the solution \(\tilde{u}(t)\) cannot grow linearly. In this case, according to Corollary 41, provided that the solution does not decay fast, the overall query complexity is always linear in time, and our fast-forwarded algorithm achieves speedup compared to existing generic ones only in \(d\) and \(n\).

### 4.8 Application: higher-order PDEs

We have discussed how our algorithm can be applied to several parabolic and hyperbolic PDEs and achieve speedups over existing quantum algorithms. These two types of PDEs are second-order PDEs and only involve spatial derivatives up to second order. However, our algorithm can also be applied to PDEs with higher-order spatial derivatives. This is because the spatial derivative of any order (under periodic boundary conditions) has the Fourier basis functions as its eigenfunctions, and so its discretized version can be diagonalized by the QFT. In this section, we discuss two higher-order examples of PDEs that cannot be characterized as either parabolic or hyperbolic: the Airy equation and the beam equation. For technical simplicity, we only consider the one-dimensional case.

**Airy equation**  The Airy equation is the linearized version of the Korteweg–De Vries (KdV) equation, which was initially introduced to model shallow water waves and found to have various applications in modeling long internal waves in oceans [81], ion acoustic waves in a plasma [82], and the Fermi–Pasta–Ulam–Tsingou problem in chaos theory [83]. When the group velocity of the wave is at an extremum, the wave envelope is described by the Airy equation, which was initially introduced to model shallow water waves and found to have various applications in modeling long internal waves in oceans [81], ion acoustic waves in a plasma [82], and the Fermi–Pasta–Ulam–Tsingou problem in chaos theory [83]. The equation goes as

\[
\frac{\partial}{\partial t} u(x, t) + \frac{\partial^3}{\partial x^3} u(x, t) = b(x, t).
\]

We first discretize \(\frac{\partial^3}{\partial x^3}\) by the central difference method using \(n\) grid points \(x_j = j/n, j \in [n]\) and spatial step size \(h = 1/n\). According to the central difference formula \(f^{(3)}(x) \approx \frac{1}{h^3} (-f(x-2h)/2 + f(x-h) - f(x) + f(x+h) + f(x+2h)/2)\), we may define the discretized operator as \(D_{h,3}\) of which all the non-zero entries are given by

\[
D_{h,3}(j, j-2) = -\frac{1}{2h^3}, \quad D_{h,3}(j, j-1) = \frac{1}{h^3}, \quad D_{h,3}(j, j+1) = -\frac{1}{h^3}, \quad D_{h,3}(j, j+2) = \frac{1}{2h^3}.
\]

Here, under periodic boundary condition, we interpret \(D_{h,3}(j, k) = D_{h,3}(j, k + n)\) to let the index fall into \([n]\). The corresponding semi-discretized equation is given as

\[
\frac{d}{dt} \tilde{u}(t) = -D_{h,3} \tilde{u}(t) + b(t).
\]

The eigenvalues of \(D_{h,3}\) are \(\mu_k = -4i n^3 \sin(2k\pi/n) \sin^2(k\pi/n)\), and the corresponding eigenstate is \(\frac{1}{n\sqrt{n}} (1, \omega_n^k, \omega_n^{2k}, \ldots, \omega_n^{(n-1)k})^T\) where \(\omega_n = e^{2\pi i / n}\). Therefore, the coefficient matrix \(-D_{h,3}\) can be diagonalized using QFT, and its eigenvalues have closed-form expression. Theorem 38 directly implies that the overall query complexity of our fast-forwarded algorithm applied to Equation (193) is

\[
O \left( \frac{\sqrt{2\| \tilde{u}(0) \|^2 + 2T^2 \| b \|_{2\text{avg}}^2}}{\| \tilde{u}(T) \|} \right).
\]

---

6The Airy equation also commonly refers to \(\frac{d^2 y}{dx^2} - xy = 0\). The connection between this equation and Equation (191) is that if \(y(x)\) solves the Airy ODE then \(u(x, t) = t^{-1/3} y(x/(3t)^{1/3})\) solves Equation (191) with \(b \equiv 0\).
Again, this is always independent of \( n \), at most linear in \( T \), and in the best case can be independent of \( T \) as well. As a comparison, since \( \|D_{h,3}\| = \Theta(n^3) \), query complexities of best existing algorithms (Lemma 10 and Lemma 11) have qubit scaling in \( n \) and linear scaling in \( T \). So our algorithm always achieves a speedup in \( n \) and can potentially achieve a speedup in \( T \) with further assumption on \( \bar{b}(t) \).

**Beam equation**  The beam equation (also known as the Euler–Bernoulli equation) describes the relationship between the beam’s deflection and the applied load. It is widely used in engineering practice, especially civil, mechanical, and aeronautical engineering. Civil engineering structures often consist of an assembly or grid of beams with cross-sections having shapes such as T’s and I’s. A large number of machine parts also are beam-like structures: lever arms, shafts, etc. Several aeronautical structures such as wings and fuselages can also be treated as thin-walled beams. The beam models provide the designer with a simple tool to analyze the structures, and make it successfully to solve a variety of engineering problems [85–87]. The equation has the form

\[
\frac{\partial^2}{\partial t^2} u(x,t) + \frac{\partial^4}{\partial x^4} u(x,t) = b(x,t). \tag{195}
\]

We follow the approach for hyperbolic PDEs that we first spatially discretize the PDE to obtain a system of second-order ODEs, then transfer it to a system of first-order ones in order to apply our fast-forwarded algorithm. According to the central difference formula \( f^{(4)}(x) \approx \frac{1}{h^4}(f(x - 2h) - 4f(x - h) + 6f(x) - 4f(x + h) + f(x + 2h)) \), we may discretize the differential operator \( \frac{\partial^4}{\partial x^4} \) by \( D_{h,4} \) of which all the non-zero entries are

\[
D_{h,3}(j, j \pm 2) = \frac{1}{h^2}, \quad D_{h,3}(j, j \pm 1) = -\frac{4}{h^2}, \quad D_{h,3}(j, j) = \frac{6}{h^2}. \tag{196}
\]

The corresponding semi-discretized equation is given as

\[
\frac{d^2}{dt^2} \vec{u}(t) = -D_{h,4} \vec{u}(t) + \vec{b}(t). \tag{197}
\]

One may compute that the eigenvalues of \( D_{h,4} \) are \( \mu_k = 16n^4 \sin^4(k\pi/n) \), and the corresponding eigenstate is \( \frac{1}{\sqrt{n}}(1, \omega_n^k, \omega_n^{2k}, \ldots, \omega_n^{(n-1)k})^T \) where \( \omega_n = e^{2\pi i/n} \). So we may write \( D_{h,4} = F_h^{-1} \Lambda_4 F_h \) where \( F_h \) represents the inverse QFT and \( \Lambda_4 = \text{diag}(16n^4 \sin^4(k\pi/n)) \). Define \( B_4 = F_h^{-1}\sqrt{\Lambda_4} F_h \) where \( \sqrt{\Lambda_4} = \text{diag}(4n^2 \sin^2(k\pi/n)) \). Then \( B_4^2 = D_{h,4} \). Consider the ODE system

\[
\frac{d}{dt} \begin{pmatrix} \vec{u}(t) \\ \vec{v}(t) \end{pmatrix} = \begin{pmatrix} 0 & iB_4 \\ iB_4 & 0 \end{pmatrix} \begin{pmatrix} \vec{u}(t) \\ \vec{v}(t) \end{pmatrix} + \begin{pmatrix} 0 \\ \vec{b}(t) \end{pmatrix}, \tag{198}
\]

Then \( \vec{u}(t) \) is also the solution of Equation (197), and by definition \( iB_4 \vec{v}(t) = \frac{d}{dt} \vec{u}(t) \). Notice that Equation (198) has the same form of Equation (170), so all the discussions in Section 4.7 still hold for beam equation.

### 5 Conclusion and outlook

This work develops a theory of quantum algorithms for solving non-quantum dynamics from two opposite directions: their limitations when applied to general non-quantum dynamics and their fast-forwarding when applied to specific non-quantum dynamics. On the one hand, by proving worst-case lower bounds, we show that generic quantum algorithms for solving homogeneous ODEs suffer from computational overheads due to their “non-quantumness”, and conclude that quantum algorithms for quantum dynamics work best. On the other hand, for several specific but practically useful ODE systems, we design new “one-shot” algorithms to obtain quadratic to exponential improvements compared to the best existing generic algorithms. Remarkably, our fast-forwarded algorithm can be applied to various high-dimensional linear evolutionary PDEs.

Our lower bounds on generic quantum ODE solvers are in terms of the number of queries to the prepare oracle of the initial state. For inhomogeneous ODEs, it is not hard to prove the same lower bounds in terms of the number of queries to the prepare oracle of \( \hbar \), since the operator \( \int_0^T e^{A(T-s)} ds \) is similar to \( e^{AT} \) and can be regarded as an amplifier in the same subspace. It remains an interesting open question to lower
bound the number of queries to the coefficient matrix $A$. We also note that our lower bounds are worst-case, meaning that we proved them by allowing the initial state $u(0)$ and/or the coefficient matrix $A$ to take their worst possible values. It would be interesting to prove lower bounds in the average case as they may be more practically relevant and a better bound may be attained. For example, one concrete direction is to generalize Proposition 15 and Proposition 18 to the case when the coefficient matrix $A$ is average-case or even arbitrary.

To prove our lower bounds, we propose a general framework that establishes lower bounds on any quantum algorithm that can be regarded as an amplifier. We believe that this framework can serve as a general tool for investigating the power of quantum algorithms for solving non-quantum problems, since they may embed non-unitary transformations and thus can be regarded as amplifiers. Our framework is closely related to that of quantum state discrimination but differs from it in two ways: 1. we make the stronger assumptions that we can access the prepare oracle and its inverse whereas only access to copies of the unknown state is allowed in the standard setup, and 2. we obtain a different lower bound of $\Omega(1/\sqrt{\epsilon})$ queries compared to the standard lower bound of $\Omega(1/\epsilon)$ copies. Our result motivates the renewed study of the foundational quantum state discrimination problem but under different input access assumptions, and the investigation into why different assumptions lead to different complexities.

For several specific ODE systems, we quadratically or exponentially improve over the efficiencies of the best existing generic quantum algorithms. A natural future direction is to identify more ODE systems that can be fast-forwarded. For example, it would be interesting to generalize Theorem 23 and Theorem 28 to the case when $b = b(t)$ is time-dependent. It might be that our algorithm in Theorem 38 based on integral quadrature and LCU also works in this case, but the cost of LCU with more complicated unitaries (specifically, block-encodings of $e^{A(T-s)}$) requires careful analysis especially since the number of quadrature nodes must be large to control the error and previous work suggests that the cost of LCU may depend linearly on the number of unitaries combined in some scenarios (see, e.g., [27]). More generally, it is of independent interest to determine how the cost of LCU depends on the number and type of unitaries combined—we have seen linear dependence in [27] and logarithmic dependence in this work as well as in [18, 56].

Our algorithms for solving ODEs with a time-independent inhomogeneous term that achieve fast-forwarding are one-shot in the sense that they neither require time discretization nor solving high-dimensional linear systems. Instead, they directly map the input state to the final solution, which we believe could be a more straightforward and intuitive way of designing generic quantum ODE solvers, irrespective of whether fast-forwarding is possible. For example, for ODEs with a time-independent inhomogeneous term $b$, we can separately implement block-encodings of $e^{AT}$ and $\int_0^T e^{A(T-s)} ds$ using the contour integral representation [6, 54, 55] $f(A) = \int_{\Gamma} f(z)(z - A)^{-1} dz$ for suitable choices of $f$. The contour integral can then be discretized using sufficiently many nodes and be implemented using LCU. We leave the rigorous analysis of such an algorithm to future work.

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A Proofs of linear algebra results

Proof of Lemma 1. The first claim can be proved by the triangle inequality that
\[ \| \tilde{a} \| \geq \| a \| - \| \tilde{a} - a \| \geq \| a \| - \epsilon. \] (199)
The second claim can be proved as
\[ \left\| \frac{a}{\|a\|} - \frac{\tilde{a}}{\| \tilde{a} \|} \right\| \leq \left| \frac{a}{\|a\|} - \frac{\tilde{a}}{\| \tilde{a} \|} \right| + \left| \frac{1}{\| a \|} - \frac{1}{\| \tilde{a} \|} \right| = \left| \frac{a - \tilde{a}}{\| a \|} \right| + \left| \frac{1}{\| a \|} - \frac{1}{\| \tilde{a} \|} \right| \leq \frac{2\epsilon}{\| a \|}. \] (200)

Proof of Lemma 2. Using triangle inequality and Cauchy-Schwarz inequality, we have
\begin{align*}
| \langle \tilde{\psi} | \tilde{\phi} \rangle | &= | \langle \psi | \phi \rangle + \langle \psi | (| \tilde{\phi} \rangle - | \phi \rangle) + (| \tilde{\phi} \rangle - \langle \psi | \rangle | \tilde{\phi} \rangle | \\
&\leq | \langle \psi | \phi \rangle | + | \langle \psi | (| \tilde{\phi} \rangle - | \phi \rangle) | + | (| \tilde{\phi} \rangle - \langle \psi | \rangle | \tilde{\phi} \rangle | \\
&\leq | \langle \psi | \phi \rangle | + \| | \tilde{\phi} \rangle - | \phi \rangle \| + \| | \tilde{\psi} \rangle - | \psi \rangle \|. \tag{201}
\end{align*}

Proof of Lemma 3. The first equality is the standard relation between the trace distance and the fidelity for pure states, and the proof can be found in [88]. For the second, it can be proved as
\begin{align*}
\| | \psi \rangle - | \phi \rangle \|^2 &= (\langle \psi | - \langle \phi | \rangle) (| \psi \rangle - | \phi \rangle) \\
&= 2 - 2 \Re(\langle \psi | \phi \rangle) \\
&= | \langle \psi | \phi \rangle |^2 + 1 - | \langle \psi | \phi \rangle |^2 \geq 1 - | \langle \psi | \phi \rangle |^2. \tag{202}
\end{align*}

B Lower bound for solving linear systems of equations

We show how to use Theorem 13 to study lower bounds for quantum linear system problem. The goal of the quantum linear system problem is to prepare an approximation of the normalized solution |x⟩ = A⁻¹|b⟩ / ∥A⁻¹|b⟩∥ for an invertible matrix A with ∥A∥ = 1 and a state |b⟩. We will show that, for any invertible matrix A, generic quantum linear system solves with bounded error must take \( \Omega(\kappa) \) queries to the prepare oracle of |b⟩ or its inverse in the worst case, where \( \kappa = \| A \| \| A^{-1} \| \) is the condition number of A.

Let A = UDV† be the singular value decomposition. Here U = (|u_1⟩, ⋯, |u_N⟩) and V = (|v_1⟩, ⋯, |v_N⟩) are two unitary matrices with column vectors |uj⟩ and |vj⟩, respectively. D = diag(\( d_1, ⋯, d_N \)) is the diagonal matrix containing singular values, and without loss of generality we assume 1 = d₁ ≥ d₂ ≥ ⋯ ≥ d_N = 1/\( \kappa \). Note that the smallest singular value must be the inverse of the condition number due to the definition of the condition number.

Consider two quantum states
\[ |b_1⟩ = |u_1⟩, \quad |b_2⟩ = \sqrt{1 - 1/\kappa^2} |u_1⟩ + (1/\kappa) |u_N⟩, \tag{203} \]
and let
\[
|x_1\rangle = \frac{A^{-1}|b_1\rangle}{\|A^{-1}|b_1\|}, \quad |x_2\rangle = \frac{A^{-1}|b_2\rangle}{\|A^{-1}|b_2\|}.
\]  

(204)

On the one hand,
\[
\langle b_1|b_2\rangle = \sqrt{1 - 1/\kappa^2} \geq 1 - 1/\kappa^2.
\]

(205)

On the other hand, using \(A^{-1} = V D^{-1} U^\dagger\), for any \(1 \leq j \leq N\), we can compute
\[
A^{-1}|u_j\rangle = V D^{-1} U^\dagger |u_j\rangle = V D^{-1} |e_j\rangle = V (d_j^{-1} |e_j\rangle) = d_j^{-1} |v_j\rangle,
\]

where \(|e_j\rangle\) is an \(N\)-dimensional vector with 1 on its \(j\)-th entry and 0 on all other entries. Then
\[
A^{-1}|b_1\rangle = |v_1\rangle, \quad A^{-1}|b_2\rangle = \sqrt{1 - 1/\kappa^2} |v_1\rangle + |v_N\rangle,
\]

and thus
\[
\langle x_1|x_2\rangle = \frac{(A^{-1}|b_1\rangle)^* (A^{-1}|b_2\rangle)}{\|A^{-1}|b_1\| \|A^{-1}|b_2\|} = \frac{\sqrt{1 - 1/\kappa^2}}{\sqrt{2 - 1/\kappa^2}} \leq \frac{1}{\sqrt{2}}.
\]

(208)

Let \(|\bar{x}_1\rangle\) and \(|\bar{x}_2\rangle\) be the outputs of a generic quantum linear system solver that approximate \(|x_1\rangle\) and \(|x_2\rangle\) with 2-norm error at most 1/10. Then, according to Lemma 2 and Lemma 3, the fidelity between \(|\bar{x}_1\rangle\) and \(|\bar{x}_2\rangle\) is at most \(1/\sqrt{2} + 1/10 + 1/10 < 0.91\), and the trace distance between \(|\bar{x}_1\rangle\) and \(|\bar{x}_2\rangle\) is at least 0.41. Therefore, a generic quantum linear system solver can be viewed as the amplifier for the state pair \(|b_1\rangle\) and \(|b_2\rangle\) with overlap \(\geq 1 - 1/\kappa^2\), and Theorem 13 directly implies an \(\Omega(\kappa)\) lower bound on the number of querying \(O_b\), its inverse and controlled version.

Our lower bound recovers the well-known \(\Omega(\kappa)\) scaling for quantum linear system solvers. Notice that existing ones in e.g., [26, 39] require both the matrix \(A\) and the vector \(b\) to be the worst case, and obtain lower bounds on queries to \(A\) and \(b\). As a comparison, our result holds for arbitrary matrix \(A\) and only requires worst-case vector \(b\), but our lower bound is in terms of the prepare oracle of \(b\) and no lower bound is obtained on the input model of \(A\). Our result also suggests \(\Omega(\kappa)\) lower bound for solving linear system of equations with positive-definite matrix \(A\), since no condition on \(A\) is assumed in our result. There are two slight differences compared to [36]: we consider the algorithm for solving the linear system of equations deterministically while [36] considers the state verification problem with possible randomness, and our result applies to general matrices \(A\) while the state pair constructed in [36, Appendix A] applies to normal matrices.

**C Proofs of Proposition 17 and Proposition 18**

*Proof of Proposition 17.* The proof is similar to that of Proposition 14. Let \(\lambda_j = \alpha_j + i \beta_j\) where \(\alpha_j\) and \(\beta_j\) are the real part and the imaginary part of \(\lambda_j\), respectively. Without loss of generality, assume that
\[
\max_j \text{Re}(\lambda_j) = \text{Re}(\lambda_1) = \alpha_1
\]

and
\[
\min_j \text{Re}(\lambda_j) = \text{Re}(\lambda_2) = \alpha_2.
\]

(209)

(210)

Then
\[
\max_{i,j} \text{Re}(\lambda_i - \lambda_j) = \alpha_1 - \alpha_2.
\]

(211)

Our claim is trivial if \(\alpha_1 \leq 0\) or \(\alpha_1 - \alpha_2 = 0\), so we assume that \(\alpha_1 > 0\) and \(\alpha_1 - \alpha_2 > 0\). Let \(V = (v_1, \cdots, v_N)\) where each \(v_i\) is the eigenvector of \(A\) corresponding to the eigenvalue \(\lambda_i\) and are normalized such that \(\|v_j\| = 1\). Without loss of generality, we assume that \(\langle v_1|v_2\rangle\) is real, otherwise we may rotate and redefine \(v_2\) without changing \(A\).

For any \(0 < \epsilon < 1\), we consider solving Equation (17) with
\[
b = v_2,
\]

(212)
and two possible initial conditions
\begin{align*}
u(0) &= v_2, \\
w(0) &= \sqrt{v_1} + \xi v_2.
\end{align*}

Here \( \xi \) is chosen to be a real number such that \( \|w(0)\| = 1 \), i.e.,
\[
1 = |\langle w(0) | w(0) \rangle | = \epsilon + |\xi|^2 + 2\sqrt{\epsilon} \langle v_1 | v_2 \rangle.
\]

The solutions \( u(T) \) and \( w(T) \) can be solved as follows. As proved in Proposition 14, for any \( t \) and \( j \) we have
\[
e^{At} v_j = e^{\lambda_j t} v_j.
\]

Therefore, according to Equation (27)
\[
u(T) = e^{\lambda_2 T} v_2 + \int_0^T e^{\lambda_2 (T-s)} v_2 ds \sim v_2,
\]
and
\[
w(T) = \sqrt{e^{\lambda_1 T}} v_1 + \xi e^{\lambda_2 T} v_2 + \int_0^T e^{\lambda_2 (T-s)} v_2 ds := c_1 v_1 + c_2 v_2,
\]
where
\[
c_1 = \sqrt{e^{\lambda_1 T}}, \quad c_2 = \xi e^{\lambda_2 T} + \int_0^T e^{\lambda_2 (T-s)} ds.
\]

We now compute the fidelity of the input states and the output states. For the input states, the same as Equation (39), we have
\[
\langle u(0) | w(0) \rangle > 1 - \epsilon.
\]

For the output states, we have
\[
|\langle u(T) | w(T) \rangle|^2 = \frac{w(T)^\dagger u(T) u(T)^\dagger w(T)}{\|u(T)\|^2 \|w(T)\|^2}
\]
\[
= \frac{(c_1 v_1 + c_2 v_2)^\dagger v_2 \bar{c}_2 (c_1 v_1 + c_2 v_2)}{|c_1 v_1 + c_2 v_2|^2}
\]
\[
= \frac{|c_1|^2 \langle v_1 | v_2 \rangle^2 + |c_2|^2 + 2 \text{Re}(\bar{c}_1 c_2 \langle v_1 | v_2 \rangle)}{|c_1|^2 + |c_2|^2 + 2 \text{Re}(\bar{c}_1 c_2 \langle v_1 | v_2 \rangle)}.
\]

Notice that
\[
|2 \text{Re}(\bar{c}_1 c_2 \langle v_1 | v_2 \rangle)| \leq 2|c_1 c_2 \langle v_1 | v_2 \rangle| \leq |c_1|^2 |\langle v_1 | v_2 \rangle|^2 + |c_2|^2,
\]
and that function \( f(x) = \frac{ax + b}{x^2 + b} \) with \( b > a > 0 \) is monotonically increasing for \( x \geq -a \), we can further bound the final fidelity as
\[
|\langle u(T) | w(T) \rangle|^2 \leq \frac{2|c_1|^2 |\langle v_1 | v_2 \rangle|^2 + 2|c_2|^2}{|c_1|^2 (1 + |\langle v_1 | v_2 \rangle|^2)^2 + 2|c_2|^2}.
\]

According to Equation (218) and Equation (45), we can bound \( |c_2| \) as
\[
|c_2| \leq |\xi| e^{\alpha_2 T} + \int_0^T e^{\alpha_2 (T-s)} ds
\]
\[
\leq |\xi| e^{\max\{0, \alpha_2\} T} + \int_0^T e^{\max\{0, \alpha_2\} (T-s)} ds
\]
\[
\leq |\xi| e^{\max\{0, \alpha_2\} T} + \int_0^T e^{\max\{0, \alpha_2\} T} ds
\]
\[
\leq (1 + \sqrt{2} + T) e^{\max\{0, \alpha_2\} T},
\]

57
Proof of Proposition 18. Consider the example with $T$ to the choice of $\Omega$. Notice that such $T$ uniquely exists, because the function $f(T) = \sqrt{\epsilon e^{T \min(\alpha_1, \alpha_1 - \alpha_2)}} - (1 + \sqrt{2} + T)$ defined on $[0, +\infty)$ first monotonically decreases and then monotonically increases with boundary values $f(0) < 0$ and $f(+\infty) = +\infty$. Therefore,

$$|\langle u(T) \mid w(T) \rangle|^2 \leq \frac{2|c_1|^2| \langle v_1 \mid v_2 \rangle|^2 + 2(1 + \sqrt{2} + T)^2 e^{2T \max(0, \alpha_2)}}{|c_1|^2(1 + |\langle v_1 \mid v_2 \rangle|^2) + 2(1 + \sqrt{2} + T)^2 e^{2T \max(0, \alpha_2)}} = \frac{2\epsilon e^{2T \alpha_1} |\langle v_1 \mid v_2 \rangle|^2 + 2(1 + \sqrt{2} + T)^2 e^{2T \max(0, \alpha_2)}}{\epsilon e^{2T \min(\alpha_1, \alpha_1 - \alpha_2)} (1 + |\langle v_1 \mid v_2 \rangle|^2) + 2(1 + \sqrt{2} + T)^2}.$$  \hfill (224)

We choose $T$ such that

$$\epsilon e^{2T \min(\alpha_1, \alpha_1 - \alpha_2)} = (1 + \sqrt{2} + T)^2.$$  \hfill (225)

Notice that such $T$ uniquely exists, because the function $f(T) = \sqrt{\epsilon e^{T \min(\alpha_1, \alpha_1 - \alpha_2)}} - (1 + \sqrt{2} + T)$ defined on $[0, +\infty)$ first monotonically decreases and then monotonically increases with boundary values $f(0) < 0$ and $f(+\infty) = +\infty$. Therefore,

$$|\langle u(T) \mid w(T) \rangle| \leq \sqrt{\frac{2|\langle v_1 \mid v_2 \rangle|^2 + 2}{1 + |\langle v_1 \mid v_2 \rangle|^2} = C.}$$  \hfill (226)

Here $C < 1$ and only depends on $V$, and thus we regard $C$ as a constant and will absorb it into the notation $O$ and $\Omega$.

Given a black box to prepare either $|u(0)\rangle$ or $|w(0)\rangle$, we denote $|\tilde{u}(T)\rangle$ and $|\tilde{w}(T)\rangle$ as the corresponding outputs of a quantum differential equation solver with 2-norm distance at most $(1-C)/4$ of $|u(T)\rangle$ and $|w(T)\rangle$, respectively. Following the same argument as in Equation (47), we have $||\tilde{u}(T)\rangle \langle u(T)\rangle - |\tilde{w}(T)\rangle \langle w(T)\rangle||_1 = \Omega(1)$. According to Theorem 13, preparing $|\tilde{u}(T)\rangle$ and $|\tilde{w}(T)\rangle$ requires query complexity $\Omega(1/\sqrt{\epsilon})$. According to the choice of $T$ in Equation (225), the complexity with the worst-case input state is bounded from below by $\Omega(1/\sqrt{\epsilon}) = \Omega(e^{T \min(\alpha_1, \alpha_1 - \alpha_2)}/(1 + \sqrt{2} + T))$. \hfill \square

Proof of Proposition 18. Consider the example with $N = 3$, $u = (u_1, u_2, u_3)^T$,

$$A = \begin{pmatrix} -1 & -1/\delta & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1/2 \end{pmatrix},$$  \hfill (227)

and

$$b = (0, 0, 1)^T.$$  \hfill (228)

Here $\delta$ is a real positive parameter in $(0, 1)$. Notice that the matrix $A$ can be diagonalized such that $A = VDV^{-1}$ where

$$V = \begin{pmatrix} 1 & 1 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1/2 \end{pmatrix},$$  \hfill (229)

and that

$$A^\dagger A - AA^\dagger = \begin{pmatrix} -1/\delta^2 & -1/\delta & 0 \\ -1/\delta & 1/\delta^2 & 0 \\ 0 & 0 & 1/4 \end{pmatrix},$$  \hfill (230)

and

$$\mu(A) = \frac{\sqrt{1 + \delta^2}}{\delta} = \Theta\left(\frac{1}{\delta}\right).$$  \hfill (231)

We choose two initial conditions

$$u(0) = (0, 0, 1)^T,$$

$$v(0) = (0, \delta, \sqrt{1 - \delta^2})^T.$$  \hfill (232)
According to Equation (27) and noting that $e^A$ can be computed as $Ve^DV^{-1}$, we obtain
\[
\begin{align*}
  u(1) &= (0, 0, 2 - e^{1/2})^T, \\
  v(1) &= (-e^{-1} + e^{-2}, e^{-2} \delta, 2 - (2 - \sqrt{1 - \delta^2})e^{-1/2})^T := (v_1(1), v_2(1), v_3(1))^T.
\end{align*}
\] (233)

Now, suppose that we are given a black box that prepares either $|u(0)\rangle$ or $|v(0)\rangle$. Let $|\tilde{u}(1)\rangle$ and $|\tilde{v}(1)\rangle$ be the corresponding outputs of a quantum ODE solver with 2-norm distance at most $1/1000$ of $|u(1)\rangle$ and $|v(1)\rangle$, respectively. On the one hand,
\[
\langle u(0)|v(0)\rangle = \sqrt{1 - \delta^2} \geq 1 - \delta^2.
\] (234)

On the other hand,
\[
\begin{align*}
  |\langle u(1)|v(1)\rangle| &= \frac{|v_3(1)|}{\sqrt{v_1(1)^2 + v_2(1)^2 + v_3(1)^2}} \\
  &= \frac{1}{\sqrt{1 + v_1(1)^2/v_3(1)^2 + v_2(1)^2/v_3(1)^2}} \\
  &\leq \frac{1}{\sqrt{1 + (e - 1)^2/(4e^4)}},
\end{align*}
\] (235)

where in the last inequality we use
\[
\frac{|v_1(1)|}{|v_3(1)|} = \frac{1/e - 1/e^2}{2 - (2 - \sqrt{1 - \delta^2})e^{-1/2}} \geq \frac{1/e - 1/e^2}{2}. \tag{236}
\]

According to Lemma 3, we have
\[
\begin{align*}
  \| |\tilde{u}(1)\rangle \langle u(1)| - |\tilde{v}(1)\rangle \langle v(1)| \|_1 &= 2 \sqrt{1 - |\langle \tilde{u}(1)|\tilde{v}(1)\rangle|^2} \\
  &\geq 2\sqrt{1 - (1/\sqrt{1 + (e - 1)^2/(4e^4)} + 1/1000 + 1/1000)^2} \\
  &\geq 0.19 = \Omega(1).
\end{align*}
\] (237)

Therefore, Theorem 13, Equation (234) and Equation (231) imply that preparing $|\tilde{u}(1)\rangle$ or $|\tilde{v}(1)\rangle$ requires query complexity $\Omega(1/\delta) = \Omega(\mu(A))$. \hfill \Box

D \hspace{1em} \textbf{Proof of Lemma 33}

Proof. Let $|v\rangle_v = \sum_{j=0}^{N-1} v_j |j\rangle_v$ be an arbitrary state, and we start with $\sum_{j=0}^{N-1} v_j |j\rangle_v |0\rangle_r |0\rangle_f |0\rangle_e |0\rangle_t$. We first apply $U^\dagger$ on the vector register to get
\[
\sum_{j=0}^{N-1} (U^\dagger v_j |j\rangle_v |0\rangle_r |0\rangle_f |0\rangle_e |0\rangle_t).
\] (238)

Here for a matrix $M$ and a vector $x$, $(Mx)_j$ represents its $j$-th component.

Now we implement the diagonal transform $e^{AT}$. Let $\Lambda = A + iB$ where $A = \text{diag}(\alpha_0, \ldots, \alpha_{N-1})$ and $B = \text{diag}(\beta_0, \ldots, \beta_{N-1})$. Since $A$ and $B$ commute, we have $e^{AT} = e^{iBT}e^{AT}$. We can then implement $e^{AT}$ and $e^{iBT}$ sequentially. The validation of the parts regarding $e^{AT}$ is the same as the proof of Lemma 29. As a consequence, after applying $O_T$ and $O_{A,r}$, applying $O_{\exp, \alpha}$, performing the controlled rotation and uncomputing (except the time register), we arrive at
\[
e^{-\alpha T} \sum_{j=0}^{N-1} (e^{AT}U^\dagger v_j |j\rangle_v |0\rangle_r |0\rangle_f |0\rangle_e |T\rangle_t + |\perp\rangle).
\] (239)
For $e^{iT}$, we first apply $O_{\Lambda,i}$ to encode the imaginary parts and get

$$e^{-\alpha T} \sum_{j=0}^{N-1} (e^{AT} U^\dagger v)_j |j\rangle_v |0\rangle_r |0\rangle_f |\beta_j\rangle_e |T\rangle_t + |\perp\rangle.$$  \hfill (240)

Applying $O_{\text{prod}}$ to get

$$e^{-\alpha T} \sum_{j=0}^{N-1} (e^{AT} U^\dagger v)_j |j\rangle_v |0\rangle_r |\beta_j T\rangle_f |\beta_j\rangle_e |T\rangle_t + |\perp\rangle.$$  \hfill (241)

Now we append the phase factor $e^{i\beta T}$. This can be done by first flipping the rotation register, applying the phase shift gate conditioned by the function register and flipping the rotation register back. We obtain

$$e^{-\alpha T} \sum_{j=0}^{N-1} e^{i\beta T} (e^{AT} U^\dagger v)_j |j\rangle_v |0\rangle_r |\beta_j T\rangle_f |\beta_j\rangle_e |T\rangle_t + |\perp\rangle.$$  \hfill (242)

Uncomputing the last three registers gives

$$e^{-\alpha T} \sum_{j=0}^{N-1} (e^{AT} U^\dagger v)_j |j\rangle_v |0\rangle_r |0\rangle_f |0\rangle_e |T\rangle_t + |\perp\rangle.$$  \hfill (243)

Finally, applying $U$ gives

$$e^{-\alpha T} \sum_{j=0}^{N-1} (U e^{AT} U^\dagger v)_j |j\rangle_v |0\rangle_r |0\rangle_f |0\rangle_e |T\rangle_t + |\perp\rangle.$$  \hfill (244)

According to the definition of the block-encoding and the equation $e^{AT} = U e^{AT} U^\dagger$, such a circuit is exactly a $(e^{AT}, n_t + n_e + n_f + 1, 0)$-block-encoding of $e^{AT}$. \hfill \Box