THE BDF2-MARUYAMA SCHEME FOR STOCHASTIC EVOLUTION EQUATIONS WITH MONOTONE DRIFT

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Abstract. We study the numerical approximation of stochastic evolution equations with a monotone drift driven by an infinite-dimensional Wiener process. To discretize the equation, we combine a drift-implicit two-step BDF method for the temporal discretization with an abstract Galerkin method for the spatial discretization. After proving well-posedness of the BDF2-Maruyama scheme, we establish a convergence rate of the strong error for equations under suitable Lipschitz conditions. We illustrate our theoretical results through various numerical experiments and compare the performance of the BDF2-Maruyama scheme to the backward Euler–Maruyama scheme.

1. Introduction

In this paper, we investigate a spatio-temporal discretization of a class of nonlinear stochastic evolution equations with monotone drift. To be more precise, let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})\) be a filtered probability space satisfying the usual conditions for fixed \(T \in (0, \infty)\). By \(W\) we denote an infinite-dimensional Wiener process with covariance operator \(Q\) which is \((\mathcal{F}_t)_{t \in [0,T]}\)-adapted and takes values in a separable Hilbert space \(U\). The stochastic evolution equation under consideration then reads

\[ dX(t) + A(X(t))\, dt = B(X(t))\, dW(t) \quad \text{on } [0,T], \quad X(0) = X_0, \]

where the operators \(A: V \times \Omega \to V^*\) and \(B: V \times \Omega \to L_2(Q^{1/2}(U), H)\) are defined on a Gelfand triple \(V \hookrightarrow H \cong H^* \hookrightarrow V^*\) for a real, reflexive, separable Banach space \(V\) and a real, separable Hilbert space \(H\). The initial value satisfies \(X_0 \in L^2(\Omega, \mathcal{F}_0; \mathbb{P}; H)\), while the stochastic integral in (1) is understood in the sense of the stochastic Itô-calculus. This setting allows us to treat several linear, semi-linear, and quasi-linear stochastic partial differential equations in a unified analytical framework, see [30, Chapter 1] for more explicit examples.

Throughout this paper we employ the variational approach from [24, 26, 30, 31] to analyze the solution to the stochastic evolution equation (1) and its numerical approximation. We essentially impose the same assumptions on the operators \(A\) and \(B\) as in [24, 30], which are sufficient to ensure the existence of a unique strong solution to (1). In particular, we assume that the operators \(A\) and \(B\) satisfy a monotonicity condition and a coercivity condition (see (9) and (10)). We refer to Section 3 for a full account of all imposed conditions and the precise definition of the exact solution to (1).

The numerical approximation of (1) with time-dependent operators \(A\) and \(B\) was studied under similar assumptions in [19, 20]. It was proven in [19] that the spatio-temporal approximations arising from the forward and backward Euler–Maruyama scheme for stochastic evolution equation, BDF2-Maruyama scheme, backward differentiation formula, mean-square error, convergence rate.

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method combined with an abstract Galerkin method converge weakly to the exact solution. Furthermore, convergence rates for the strong error of these methods were derived in [20] under additional regularity assumptions on the exact solution and the operator A. Notice that the spatial semi-discretization of (1) can lead to a high-dimensional stiff system of stochastic ordinary differential equations. In practical simulations it is therefore beneficial to use an A-stable numerical method for the temporal discretization, such as the backward Euler–Maruyama method. These methods typically avoid severe step size restrictions as, for instance, [19, condition (2.16)] for the forward Euler–Maruyama method. We refer to [6, 22] for a general discussion of A-stable numerical schemes for stiff stochastic differential equations.

In this paper we focus on the BDF2 method with an equidistant step size for the temporal discretization. The BDF2 method and the backward Euler method both belong to the family of backward differentiation formulas (BDF) which have proved effective for the approximation of stiff ordinary differential equations, see, e.g., [21, 33]. In particular, if applied to ODEs the BDF2 method has the same computational cost and enjoys the same stability properties as the backward Euler method, while having the advantage of a higher order of convergence.

The discretization of stochastic ordinary differential equations (SODE) by means of the BDF2 method has already been studied in the literature. The mean-square convergence of drift-implicit linear two-step Maruyama methods on equidistant time grids was investigated in [7] under a global Lipschitz condition on the coefficients. Moreover, higher convergence rates of such methods were derived for problems driven by small noise. The mean-square stability and convergence for general drift-implicit linear multi-step methods on non-equidistant time grids were further examined in [32]. In addition, the mean-square convergence of the BDF2-Maruyama scheme was proven under a monotonicity condition on the coefficients in [1]. However, to the best of our knowledge, multi-step methods for the temporal discretization of nonlinear stochastic evolution equations have not been investigated in detail yet.

To formulate the numerical approximation of (1), we consider an equidistant temporal grid with step size $h = \frac{T}{N}$, $N_k \in \mathbb{N}$, and grid points $t_n = nk$ for $n \in \{0, \ldots, N_k\}$. In addition, let $V_h \subset V$ be a finite dimensional subspace depending on some parameter $h \in (0,1)$. For given initial values $(X^n_{0,k,h})_{n=0,1}$ the BDF2-Maruyama scheme is defined by

\begin{equation}
\left(\frac{3}{2}X^{n}_{k,h} - 2X^{n-1}_{k,h} + \frac{1}{2}X^{n-2}_{k,h}, v\right)_H + k\langle A(X^n_{k,h}), v\rangle_{V^* \times V} = \left(\frac{3}{2}B(X^{n-1}_{k,h})\Delta_k W^n - \frac{1}{2}B(X^{n-2}_{k,h})\Delta_k W^{n-1}, v\right)_H \quad \mathbb{P}\text{-a.s.}
\end{equation}

for all $v \in V_h$ and $n = 2, \ldots, N_k$, where we define the Wiener increments by $\Delta_k W^n := W(t_n) - W(t_{n-1})$ for $n \in \{1, \ldots, N_k\}$. In order to generate suitable initial values for the scheme (2), the backward Euler–Maruyama method will come in handy and is defined for given initial value $X^0_{k,h}$ by

\begin{equation}
\left(X^n_{k,h} - X^{n-1}_{k,h}, v\right)_H + k\langle A(X^n_{k,h}), v\rangle_{V^* \times V} = \left(B(X^{n-1}_{k,h})\Delta_k W^n, v\right)_H \quad \mathbb{P}\text{-a.s.}
\end{equation}

for all $v \in V_h$ and $n = 1, \ldots, N_k$.

As our first main result we show that the discrete process $(X^n_{k,h})_{n=0}^{N_k}$ is indeed well-defined by (2) under essentially the assumptions used in [24, 30]. Further, under additional conditions on $A$ and $B$ and the temporal regularity of the exact
solution to (1), cf. Assumption 4.1 to Assumption 4.3, we also show that \( (X_{t,x}^k)_{n=0}^{N_s} \) is convergent to the exact solution \( X \) in the following sense: There exist \( C \geq 0 \), \( q \in [2, \infty) \) and \( \gamma \in (0, \infty) \), where \( q \) is determined by the regularity of \( X \) and \( \gamma \) arises from an approximation error related to the initial values, such that for every sufficiently small temporal step size \( k \) and every \( h \in (0, 1) \) it holds

\[
\begin{align*}
\max_{n \in \{2, \ldots, N_s\}} \|X_{t,x}^k - X(t_n)\|_{L^2(\Omega; H)} &+ \left( k \sum_{n=2}^{N_s} \|X_{t,x}^k - X(t_n)\|_{L^2(\Omega; V)}^2 \right)^{1/2} \\
\leq C \left[ k^\gamma + h^{2\gamma} + \max_{n \in \{2, \ldots, N_s\}} \|P_n - \text{id})X(t_n)\|_{L^2(\Omega; H)} + (1 + \|P_n\|_{L(V)}) \left( k \sum_{n=2}^{N_s} \|(R_n - \text{id})X(t_n)\|_{L^2(\Omega; V)}^2 \right)^{1/2} \right].
\end{align*}
\]

Hereby, \( P_n: H \rightarrow V_h \) denotes the orthogonal projection on \( V_h \) with respect to the inner product in \( H \) and \( R_n: V \rightarrow V_h \) is mapping to the best approximation in \( V_h \) with regard to the norm in \( V \). This error estimate is precisely stated in Theorem 4.7. Notice that the order of convergence also depends on the chosen Galerkin method.

Let us emphasize some important features of our error analysis: First, we do not apply Itô’s formula since we want to avoid the difficult task to interpolate the approximation of the two-step BDF2-Maruyama scheme to continuous time. Second, in contrast to [20], we also do not require a priori knowledge of higher spatial regularity of the exact solution beyond the Gelfand triple \((V, H, V^*)\) since such regularity results are often not available in the literature and difficult to verify for nonlinear stochastic evolution equations. Finally, as already mentioned above, we cannot avoid imposing additional assumptions on the temporal regularity of the exact solution. However, we only require that the exact solution has a finite \( q \)-variation norm (see (8)) instead of the (slightly) stronger Hölder continuity condition typically used in the literature.

As it was observed in [15] for deterministic evolution equations, the following identity plays an important role in the error and stability analysis of the BDF2 scheme. For all \( x_1, x_2, x_3 \in H \) it holds true that

\[
\begin{align*}
4 \left( \frac{3}{2} x_3 - 2 x_2 + \frac{1}{2} x_1, x_3 \right)_H &= \|x_3\|_H^2 - \|x_2\|_H^2 + \|2 x_3 - x_2\|_H^2 \\\n&\quad - \|2 x_2 - x_1\|_H^2 + \|x_3 - 2 x_2 + x_1\|_H^2.
\end{align*}
\]

This identity also has been utilized in [1] to derive a strong convergence rate of the BDF2-Maruyama scheme applied to SODEs. It will also be crucial to prove (4).

The paper is structured as follows. In Section 2, we introduce some notation and recall important concepts related to the abstract analytical framework, the stochastic integration and the approximation in infinite-dimensional spaces. Section 3 is devoted to establishing sufficient conditions for the existence of a unique solution to (1) and showing the well-posedness of the BDF2-Maruyama scheme (2) under these conditions. Moreover, we present the stochastic heat equation as an applicable example. In Section 4 we prove the error estimate (4) under additional regularity assumptions. Finally, in Section 5, we provide two numerical experiments to illustrate our theoretical results and discuss aspects of their implementation. In particular, the comparison of the schemes (2) and (3) in the temporal error analysis.
indicates that the BDF2-Maruyama scheme is favourable for problems driven by noise with higher spatial regularity or noise with small intensity.

2. Preliminaries

In this section, we briefly recall some basic concepts from functional analysis, stochastic analysis, and numerical analysis which are used throughout this paper. Mostly, we employ the same notation as in [26, Chapter 2] and [30, Chapter 2].

Let \((H, \langle \cdot, \cdot \rangle_H, \| \cdot \|_H)\) be a real, separable Hilbert space and let \((V, \| \cdot \|_V)\) be a real, reflexive and separable Banach space that is continuously and densely embedded in \(H\). We denote the dual spaces of \(H\) and \(V\) by \(H^*\) and \(V^*\), respectively, and use \(\langle \cdot, \cdot \rangle_{V^* \times V}\) for the dual pairing between \(V\) and its dual \(V^*\). We consider the Gelfand triple \((V, H, V^*)\) which satisfies \(V \hookrightarrow H \cong H^* \hookrightarrow V^*\) with \(\hookrightarrow\) denoting dense and continuous embeddings and \(\cong\) the identification of \(H\) with its dual space in terms of the Riesz isomorphism. In particular, there exists \(\beta_{V \hookrightarrow H} \in (0, \infty)\) such that for every \(v \in V\) the inequality \(\|v\|_H \leq \beta_{V \hookrightarrow H} \|v\|_V\) holds. In addition, we recall that

\[
\langle u, v \rangle_{V^* \times V} = \langle u, v \rangle_H
\]

holds for all \(u \in H\) and all \(v \in V\).

For \(T \in (0, \infty)\) let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})\) be a filtered probability space satisfying the usual conditions. For \(p \in [1, \infty)\) we denote by \(L^p(\Omega; V) := L^p(\Omega, \mathcal{F}, \mathbb{P}; V)\) and \(L^p([0,T] \times \Omega; V) := L^p([0,T] \times \Omega, \mathbb{B}([0,T]) \otimes \mathcal{F}, dt \otimes \mathbb{P}; V)\) the Bochner–Lebesgue spaces which are, respectively, endowed with the norms

\[
\|X\|_{L^p(\Omega; V)} := \left(\mathbb{E}\left[\|X\|_V^p\right]\right)^{\frac{1}{p}} \quad \text{and} \quad \|X\|_{L^p([0,T] \times \Omega; V)} := \left(\mathbb{E}\left[\int_0^T \|X(t)\|_V^p \, dt\right]\right)^{\frac{1}{p}}.
\]

For an introduction to Bochner–Lebesgue spaces we refer, e.g., to [10, Appendix E] and [29, Section 4.2].

Next, let \((U, \langle \cdot, \cdot \rangle_U)\) be a further separable Hilbert space and denote by \(\mathcal{L}(U, H)\) the Banach space of all linear, bounded operators from \(U\) to \(H\). By \(\mathcal{L}_2(U, H)\) we then denote the Hilbert space of all operators \(B \in \mathcal{L}(U, H)\) with finite Hilbert– Schmidt norm \(\|B\|_{\mathcal{L}_2(U, H)}^2 := \text{Tr}(B^* B)\). Moreover, for every non-negative, symmetric operator \(Q \in \mathcal{L}(U) = \mathcal{L}(U, U)\) there exists a unique operator \(Q^{\frac{1}{2}} \in \mathcal{L}(U)\) satisfying \(Q = Q^{\frac{1}{2}} \otimes Q^{\frac{1}{2}}\). Then, \(U_0 := Q^{\frac{1}{2}}(U)\) defines a Hilbert space if endowed with the inner product

\[
\langle u, v \rangle_{U_0} := \langle Q^{-\frac{1}{2}} u, Q^{-\frac{1}{2}} v \rangle_U \quad \forall u, v \in U_0,
\]

where \(Q^{-\frac{1}{2}}\) denotes the pseudo-inverse of \(Q^{\frac{1}{2}}\). For further details, we refer to [26, Section 2.3] and [30, Section 2.3].

For a given symmetric and non-negative operator \(Q \in \mathcal{L}(U)\) we then denote by \(W\) a Hilbert space valued Wiener process with respect to the filtration \((\mathcal{F}_t)_{t \in [0,T]}\) with covariance operator \(Q\) as defined in [30, Section 2.1]. If the covariance operator \(Q \in \mathcal{L}(U)\) is, in addition, of finite trace, then we recall that the Wiener process \(W\) takes values in \(U\) almost surely and it has the representation

\[
W(t) = \sum_{j=1}^{\infty} \sqrt{\beta_j} \chi_j(t), \quad t \in [0,T], \quad \mathbb{P}\text{-a.s.}
\]

Hereby, \(\{\chi_j\}_{j \in \mathbb{N}}\) is an orthonormal basis of \(U\) which consists of eigenvectors of \(Q\) with summable eigenvalues \(q_j \geq 0\) and \(\{\beta_j\}_{j \in \mathbb{N}}\) is a family of independent scalar Brownian motions.
Further, for a given stochastically integrable process $\Phi: [0, T] \times \Omega \rightarrow L_2(U_0, H)$ we denote the stochastic Itô-integral of $\Phi$ by

$$\int_0^T \Phi(t) \, dW(t).$$

For the construction of a Hilbert space valued Wiener process, the stochastic Itô-integral and their properties we again refer to [26, Chapter 2], [30, Chapter 2], as well as [12, Chapter 4]. Moreover, we recall from [30, Section 2.5] and [12, Section 4.3] that the construction of the stochastic Itô-integral can be extended to the case of a cylindrical Wiener process whose covariance operator $Q$ is not necessarily of finite trace.

To measure the regularity of the trajectories of a continuous $V$-valued stochastic process, we use the concept of finite $q$-variation for a given $q \in [1, \infty)$. A continuous function $f: [0, T] \rightarrow V$ is of finite $q$-variation with respect to the norm $\| \cdot \|_V$ if

$$\| f \|_{q\text{-var}, V} := \left( \sup_{P} \sum_{n=1}^{N_P} \| f(t_n) - f(t_{n-1}) \|_V^{q} \right)^{\frac{1}{q}} \leq \infty,$$

where the supremum is taken over the set of all finite partitions $P = \{t_0, \ldots, t_{N_P}\}$ of the interval $[0, T]$. For further details on the concept of the $q$-variation we refer to [18, Section 5] and [28, Section 1].

Finally, we recall some properties of abstract Galerkin methods. Let $(V_h)_{h \in (0, 1)}$ be a family of finite dimensional subspaces of the Banach space $V \subset H$ such that for every $v \in H$ it holds $\inf_{v_h \in V_h} \| v_h - v \|_H \rightarrow 0$ as $h \rightarrow 0$. Such a family of subspaces is called an (abstract) Galerkin scheme. By $N_h \in \mathbb{N}$ we denote the dimension of the subspace $V_h$. In addition, the parameter $h \in (0, 1)$ governs the granularity of the Galerkin scheme. In particular, if $H$ is of infinite dimensions then $N_h = \dim(V_h) \rightarrow \infty$ as $h \rightarrow 0$.

Further, we define $P_h: H \rightarrow V_h$ as the orthogonal projection map onto $V_h$ with respect to the inner product $(\cdot, \cdot)_H$. Therefore, for each $v \in H$ the element $P_h v \in V_h$ is the best approximation of $v$ in $V_h$ with respect to the norm in $H$, see, e.g., [5, Theorem 5.2]. Hence, it holds

$$\| P_h v - v \|_H = \text{dist}_H(v, V_h) := \inf_{v_h \in V_h} \| v_h - v \|_H \quad \forall v \in H.$$

If the Banach space $V$ is uniformly convex then for each $v \in V$ there also exists a unique element $R_h v \in V_h$ that is the best approximation of $v$ in $V_h$ with respect to the norm in $V$, see [5, Exercise 3.32]. This defines a (possibly nonlinear) map $R_h: V \rightarrow V_h, v \mapsto R_h v$ satisfying

$$\| R_h v - v \|_V = \text{dist}_V(v, V_h) := \inf_{v_h \in V_h} \| v_h - v \|_V \quad \forall v \in V.$$

Recall that if $V$ is itself a Hilbert space, then it is also uniformly convex, see [5, Section 3.7]. In this case, the mapping $R_h$ coincides with the orthogonal projector of $V$ onto $V_h$ with respect to the inner product of $V$.

3. Discretization: a priori estimates and well-posedness

The goal of this section is to establish sufficient conditions on the operators $A$ and $B$ and the initial conditions to ensure the well-posedness of the numerical scheme (2). For this, we first establish an a priori estimate for solutions to the numerical scheme for any value of the spatial refinement parameter $h \in (0, 1)$ and
every sufficiently small temporal step size \( k = \frac{T}{N_k} \), \( N_k \in \mathbb{N} \). Afterwards we also discuss existence and uniqueness of a solution to this scheme.

Throughout this section, we fix \( p \in (1, \infty) \) and a Gelfand triple \((V, H, V^*)\) as in Section 2.

**Assumption 3.1.** The operators \( A: V \times \Omega \to V^* \) and \( B: V \times \Omega \to L_2(U_0, H) \) are measurable with respect to \( \mathcal{B}(V) \otimes \mathcal{F}_0/\mathcal{B}(V^*) \) and \( \mathcal{B}(V) \otimes \mathcal{F}_0/B(L_2(U_0, H)) \), respectively, where \( \mathcal{B}(V) \) denotes the Borel \( \sigma \)-algebra on \( V \). In addition, the operator \( A \) is hemicontinuous, i.e. the mapping \( z: [0, 1] \to \mathbb{R}, \lambda \mapsto (A(u + \lambda v, \omega), w)^{V^*} \) is continuous for all \( u, v, w \in V \) and \( \omega \in \Omega \). Moreover, there are \( \kappa, c \in [0, \infty) \), \( \mu \in (0, \infty) \) and \( \nu \in [1, \infty) \) such that the operators \( A \) and \( B \) satisfy the monotonicity and coercivity condition
\[
2 \langle A(u) - A(v), u - v \rangle_{V^*} + \kappa \|u - v\|^2_H \geq \|B(u) - B(v)\|^2_{L_2(U_0, H)} \quad \text{on } \Omega
\]
for all \( u, v \in V \). Furthermore, the growth condition
\[
\|A(v)\|_{V^*} \leq c(1 + \|v\|_{V^*})^{p-1} \quad \text{on } \Omega
\]
is satisfied for all \( v \in V \).

Before we turn to the numerical scheme (2), we mention that Assumption 3.1 is sufficient to ensure the existence of a uniquely determined exact solution to (1), which we define in the same way as in [30, Definition 4.2.1]. More precisely, let \( X_0 \in L^2(\Omega, \mathcal{F}_0; H) \) be the initial value. Then, we call a continuous, \( H \)-valued and \((\mathcal{F}_t)_{t \in [0, T]}\)-adapted process \( X \in L^p([0, T] \times \Omega; V) \cap L^2([0, T] \times \Omega; H) \) a solution of (1) if
\[
X(t) = X_0 + \int_0^t A(X(s)) \, ds + \int_0^t B(X(s)) \, dW(s)
\]
holds in \( V^* \) for all \( t \in [0, T] \) almost surely, where \( X \) is a \( V \)-valued, progressively measurable modification of \( X \). Such a solution is said to be unique if any two solutions \( X \) and \( Y \) to (1) are indistinguishable, i.e.,
\[
\mathbb{P}\left( \sup_{t \in [0, T]} \|X(t) - Y(t)\|_H = 0 \right) = 1.
\]
For a proof of the following result, we refer to [24, Section 3] and [30, Chapter 4].

**Proposition 3.2.** Let Assumption 3.1 be satisfied for some \( p \in (1, \infty) \) and let \( X_0 \in L^2(\Omega, \mathcal{F}_0; H) \). Then the stochastic evolution equation (1) admits a unique solution.

We now turn to the question of well-posedness of the numerical scheme (2). As for every two-step scheme it is first necessary to find two suitable initial values. The following assumption is required to ensure adaptedness and square-integrability of the numerical solution.

**Assumption 3.3.** The initial values \((X_{k,h}^n)_{n=0,1}\) satisfy
\[
X_{k,h}^n \in L^2(\Omega, \mathcal{F}_{k,n}; H) \quad \text{and} \quad B(X_{k,h}^n) \in L^2(\Omega, \mathcal{F}_{k,n}; L_2(U_0, H)), \quad n \in \{0, 1\},
\]
and \( \mathbb{P}(\{\omega \in \Omega : X_{k,h}^n(\omega) \in V_h\}) = 1 \) for each \( n \in \{0, 1\} \).
The BDF2-Maruyama scheme is well-defined if there exists a unique discrete stochastic process \((X^n_{k,h})_{n=0}^{N_k}\), which is \((\mathcal{F}_k)_{n=0}^{N_k}\)-adapted, \(\mathbb{P}\)-almost surely \(V_h\)-valued and solves the recursion (2). We call such a solution unique if any two solutions \((X^n_{k,h})_{n=0}^{N_k}\) and \((Y^n_{k,h})_{n=0}^{N_k}\) to (2) are indistinguishable, which is understood in the same way as in (13). For the purpose of readability, we omit the dependence of the discrete solution on the parameters \(k\) and \(h\) by writing \(X^n := X^n_{k,h}\) throughout the proofs presented in Section 3 and Section 4.

Before we prove the existence of a unique solution to (2), we first derive the following useful \emph{a priori estimate}.

**Theorem 3.4.** Let Assumption 3.1 be satisfied for some \(p \in (1, \infty)\). Let \(h \in (0, 1)\) and \(k = \frac{r}{N_k}\), \(N_k \in \mathbb{N}\), be fixed with \(2k \mu < 1\). Let \((X^n_{k,h})_{n=0}^{N_k}\) be an arbitrary \((\mathcal{F}_k)_{n=0}^{N_k}\)-adapted and \(\mathbb{P}\)-almost surely \(V_h\)-valued process satisfying Assumption 3.3 and (2). Then, it holds

\[
\max_{n \in \{1, \ldots, N_k\}} \left( \mathbb{E}[\|X^n_{k,h}\|^2_{\mathbb{H}}] + 2k \mu \mathbb{E}[\|B(X^n_{k,h})\|^2_{L_2(U_0, \mathbb{H})}] \right) + 2k \mu \sum_{n=2}^{N_k} \mathbb{E}[\|X^n_{k,h}\|^2_{\mathbb{H}}] + \frac{\nu}{\nu - 1} \sum_{n=2}^{N_k} \mathbb{E}[\|X^n_{k,h} - 2X^n_{k-1} + X^n_{k-2}\|^2_{\mathbb{H}}] \leq C_k \left( T + \sum_{n=0}^{1} \mathbb{E}[\|X^n_{k,h}\|^2_{\mathbb{H}}] + k \sum_{n=0}^{1} \mathbb{E}[\|B(X^n_{k,h})\|^2_{L_2(U_0, \mathbb{H})}] \right)
\]

where \(C_k = 2\max\{2c, 11, 2\nu\}(1 - 2k \mu)^{-1}e^{2\nu T(1 - 2k \mu)^{-1}}\).

**Proof.** By an inductive argument we will show that for every \(j \in \{1, \ldots, N_k\}\) it holds

\[
\mathbb{E}[\|X^j\|^2_{\mathbb{H}}] + 2k \mu \mathbb{E}[\|B(X^j)\|^2_{L_2(U_0, \mathbb{H})}] + 2k \mu \sum_{n=2}^{j} \mathbb{E}[\|X^n\|^2_{\mathbb{H}}] + \frac{\nu}{\nu - 1} \left( \mathbb{E}[\|2X^j - X^{j-1}\|^2_{\mathbb{H}}] + \sum_{n=2}^{j} \mathbb{E}[\|X^n - 2X^{n-1} + X^{n-2}\|^2_{\mathbb{H}}] \right) \leq C_k \left( T + \sum_{n=0}^{1} \mathbb{E}[\|X^n\|^2_{\mathbb{H}}] + k \sum_{n=0}^{1} \mathbb{E}[\|B(X^n)\|^2_{L_2(U_0, \mathbb{H})}] \right),
\]

where we set the two sums on the left-hand side equal to zero in the case \(j = 1\). Observe that (15) directly implies the estimate (14). Moreover, it immediately follows from Assumption 3.3 and the choice of \(C_k\) that (15) holds true for \(j = 1\).

Next, let us assume that the estimate (15) holds true for some fixed \(j - 1 \in \{1, \ldots, N_k - 1\}\). In addition, since \((X^n)_{n=0}^{N_k}\) satisfies (2) for all \(v \in \mathbb{V}\), we obtain \(\mathbb{P}\)-almost surely with \(v = 4X^n\) that

\[
4 \left( \frac{3}{2} X^n - 2X^{n-1} + \frac{1}{2} X^{n-2}, X^n \right)_{\mathbb{H}} + 4k \langle A(X^n), X^n \rangle_{\mathbb{V} \times \mathbb{V}} = 4 \left( \frac{3}{2} B(X^{n-1}) \Delta_k W^n - \frac{1}{2} B(X^{n-2}) \Delta_k W^{n-1}, X^n \right)_{\mathbb{H}}
\]
for each $n = 2, \ldots, N_k$. By applying the identity (5), summing over $n$ from 2 to $j$ and taking expectation, we see that

\begin{equation}
\mathbb{E}[\|X^j\|_H^2] + \mathbb{E}[\|2X^j - X^{j-1}\|_H^2] + \sum_{n=2}^{j} \mathbb{E}[\|X^n - 2X^{n-1} + X^{n-2}\|_H^2] \\
= \mathbb{E}[\|X^1\|_H^2] + \mathbb{E}[\|2X^1 - X^0\|_H^2] + 2 \sum_{n=2}^{j} (-2k) \mathbb{E}[\langle A(X^n), X^n \rangle_{V' \times V}] \\
+ 2 \sum_{n=2}^{j} \mathbb{E}\left[(3B(X^{n-1}) \Delta_k W^n - B(X^{n-2}) \Delta_k W^{n-1}, X^n)_{H}\right].
\end{equation}

(16)

An application of the coercivity condition (10) shows that

\begin{equation}
-2k \mathbb{E}[\langle A(X^n), X^n \rangle_{V' \times V}] \leq k \kappa \mathbb{E}[\|X^n\|_H^2] - k \nu \mathbb{E}[\|B(X^n)\|_{L_2(U_n, H)}^2] \\
- k \mu \mathbb{E}[\|X^n\|_{V'}^2] + kc.
\end{equation}

(17)

After some elementary calculations, we obtain the following decomposition

\begin{equation}
(3B(X^{n-1}) \Delta_k W^n - B(X^{n-2}) \Delta_k W^{n-1}, X^n)_{H} \\
= (B(X^{n-1}) \Delta_k W^n - B(X^{n-2}) \Delta_k W^{n-1}, X^n - 2X^{n-1} + X^{n-2})_{H} \\
+ (B(X^{n-1}) \Delta_k W^n, 2X^n - X^{n-1})_{H} \\
- (B(X^{n-2}) \Delta_k W^{n-1}, 2X^{n-1} - X^{n-2})_{H} \\
+ (B(X^{n-1}) \Delta_k W^n, 3X^{n-1} - X^{n-2})_{H}.
\end{equation}

(18)

Since the random variables $(3X^{n-1} - X^{n-2})$ and $B(X^{n-2}) \Delta_k W^{n-1}$ are $\mathcal{F}_{t_{n-1}}$ measurable and integrable for every $n \leq j$, we use the martingale property of the stochastic integral to deduce

\[ \mathbb{E}\left[(B(X^{n-1}) \Delta_k W^n, 3X^{n-1} - X^{n-2})_{H}\right] = 0 \]

as well as

\[ \mathbb{E}\left[(B(X^{n-1}) \Delta_k W^n, B(X^{n-2}) \Delta_k W^{n-1})_{H}\right] = 0. \]

By applying Young’s inequality with weight $\nu$ to the decomposition (18) and taking expectation, we conclude that

\begin{align*}
\mathbb{E}\left[(3B(X^{n-1}) \Delta_k W^n - B(X^{n-2}) \Delta_k W^{n-1}, X^n)_{H}\right] \\
\leq \frac{\nu}{2} \left( \mathbb{E}[\|B(X^{n-1}) \Delta_k W^n\|_H^2] + \mathbb{E}[\|B(X^{n-2}) \Delta_k W^{n-1}\|_H^2] \right) \\
+ \frac{1}{2\nu} \mathbb{E}[\|X^n - 2X^{n-1} + X^{n-2}\|_H^2] + \mathbb{E}[\langle B(X^{n-1}) \Delta_k W^n, 2X^n - X^{n-1} \rangle_{H}] \\
- \mathbb{E}[\langle B(X^{n-2}) \Delta_k W^{n-1}, 2X^{n-1} - X^{n-2} \rangle_{H}].
\end{align*}
Inserting this and (17) into equation (16) then gives
\[
\begin{align*}
\mathbb{E}[\|X^j\|_{H}^2] + \mathbb{E}[\|2X^j - X^{j-1}\|_{H}^2] + \frac{\nu - 1}{\nu} \sum_{n=2}^{j} \mathbb{E}[\|X^n - 2X^{n-1} + X^{n-2}\|_{H}^2] \\
= \mathbb{E}[\|X^1\|_{H}^2] + \mathbb{E}[\|2X^1 - X^0\|_{H}^2] + 2k\nu \sum_{n=2}^{j} \mathbb{E}[\|X^n\|_{V}^2] \\
- 2k\nu \sum_{n=2}^{j} \mathbb{E}[\|B(X^n)\|_{L^2(U_0, H)}^2] - 2k\mu \sum_{n=2}^{j} \mathbb{E}[\|X^n\|_{V}^2] + 2k(c-j-1) \\
+ \nu \sum_{n=2}^{j} \left( \mathbb{E}[\|B(X^{n-1})\|_{D} W^n\|_{H}^2] + \mathbb{E}[\|B(X^{n-2})\|_{D} W^n\|_{H}^2] \right) \\
+ 2\mathbb{E}[\|B(X^{j-1})\|_{D} W^j, 2X^j - X^{j-1}|_H] - 2\mathbb{E}[\|B(X^{0})\|_{D} W^1, 2X^1 - X^{0}|_H].
\end{align*}
\]
Applying a discrete version of Gronwall’s inequality, see, e.g., \[E_{15}\], and hence the result.

Next, due to the Itô isometry the last two sums on the right-hand side almost cancel each other up to two summands. Moreover, a further application of Young’s inequality yields
\[
\|2X^1 - X^0\|_{H}^2 = 4\|X^1\|_{H}^2 - 4(X_1, X_0)_H + \|X^0\|_{H}^2 \leq 5\|X^1\|_{H}^2 + \|X^0\|_{H}^2.
\]
Since \(1 > 1 - 2k\nu > 0\) and \(\frac{\nu - 1}{\nu} \leq 2\) by assumption, we obtain
\[
\begin{align*}
\mathbb{E}[\|X^j\|_{H}^2] + 2k\nu \mathbb{E}[\|B(X^j)\|_{L^2(U_0, H)}^2] + 2k\mu \sum_{n=2}^{j} \mathbb{E}[\|X^n\|_{V}^2] \\
+ \frac{\nu - 1}{\nu} \left( \mathbb{E}[\|2X^j - X^{j-1}\|_{H}^2] + \sum_{n=2}^{j} \mathbb{E}[\|X^n - 2X^{n-1} + X^{n-2}\|_{H}^2] \right) \\
\leq \frac{2k\nu}{1 - 2k\nu} \sum_{n=2}^{j} \mathbb{E}[\|X^n\|_{H}^2] + \frac{2Tc}{1 - 2k\nu} \\
+ \frac{1}{1 - 2k\nu} \left( \sum_{n=0}^{j} \mathbb{E}[\|X^n\|_{H}^2] + 2k\nu \sum_{n=0}^{j} \mathbb{E}[\|B(X^n)\|_{L^2(U_0, H)}^2] \right).
\end{align*}
\]
Applying a discrete version of Gronwall’s inequality, see, e.g., \[E_{15}\], yields the estimate (15) and hence the result. \(\square\)

Under the assumptions stated in this section, the existence and uniqueness of a solution to implicit methods such as the BDF2-Mayurama scheme (2) and the BEM scheme (3) can be proven through techniques from nonlinear PDE theory.
These techniques rely on the monotonicity condition (9) and have been used to show well-posedness of the one-step BEM scheme applied to nonlinear stochastic evolution equations, see, e.g., [16, Theorem 3.3] or [19, Theorem 2.9]. Here, we adapt this approach to the multi-step BDF2-Mayurama scheme in order to prove well-posedness.

Theorem 3.5. Let Assumption 3.1 be satisfied for some \( p \in (1, \infty) \). Let \( h \in (0,1) \) and \( k = \frac{2}{\tau} \), \( N_h \in \mathbb{N} \), be fixed with \( k \kappa \leq 3 \) and let some initial values \( (X_{k,h}^n)_{n=0,1} \) satisfy Assumption 3.3. Then the numerical scheme (2) has a unique \( (F_{i,h})_{n=0}^\tau \)-adapted and \( \mathbb{P} \)-almost surely \( V_h \)-valued solution \( (X_{k,h}^n)_{n=0}^\tau \). In addition, if \( 2k\kappa < 1 \) holds, then the random variables \( X_{k,h}^n \) are \( L^p(\Omega; V) \cap L^2(\Omega; H) \)-integrable for every \( n \in \{ 2, \ldots, N_h \} \).

The following lemmas are needed to show existence as well as adaptedness of a discrete solution to the numerical scheme (2). A proof of each result can be found, respectively, in [17, Section 9.1] and [13, Lemma 4.3].

Lemma 3.6. Let \( R \in (0, \infty) \) and let \( f : \mathbb{R}^N \to \mathbb{R}^N, N \in \mathbb{N} \), be a continuous function. If \( f(x) \cdot x \geq 0 \) holds for every \( x \in \mathbb{R}^N \) with \( \| x \|_2 = R \), where \( \| \cdot \|_2 \) denotes the Euclidean norm, then there exists \( x_0 \in \mathbb{R}^N \) with \( \| x_0 \|_2 \leq R \) satisfying \( f(x_0) = 0 \).

Lemma 3.7. Let \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P}) \) be a filtered probability space. Let \( \mathcal{F}_t \) be a complete sub-\( \sigma \)-algebra of \( \mathcal{F} \) for fixed \( t \in [0,T] \) and let \( \Omega' \in \mathcal{F}_t \) with \( \mathbb{P}(\Omega') = 1 \). Further, let the function \( f : \Omega \times \mathbb{R}^N \to \mathbb{R}^N, N \in \mathbb{N} \), be \( \mathcal{F}_t \)-measurable in the first argument for every \( x \in \mathbb{R}^N \) and continuous in the second argument for every \( \omega \in \Omega' \). Moreover, assume for each \( \omega \in \Omega' \) that the equation \( f(\omega, x) = 0 \) has a unique solution \( x(\omega) \in \mathbb{R}^N \). Then the mapping

\[
x : \Omega \to \mathbb{R}^N, \quad \omega \mapsto \begin{cases} x(\omega) & \text{for } \omega \in \Omega', \\ 0 & \text{otherwise,} \end{cases}
\]

is \( \mathcal{F}_t \)-measurable.

Proof of Theorem 3.5. First, we will show by an inductive argument over \( n \in \{ 1, \ldots, N_h \} \) the existence and the \( \mathbb{P} \)-almost sure uniqueness of random variables \( X^n \) which solve (2) and are \( \mathcal{F}_{t_n} \)-measurable as well as almost surely \( V_h \)-valued. Notice that this assertion follows for \( n = 1 \) from Assumption 3.3. Hence we assume that \( \mathcal{F}_{t_n} \)-measurable and \( \mathbb{P} \)-almost surely \( V_h \)-valued random variables \( X^n \) satisfying (2) exist for \( j = 0, \ldots, n - 1, \ n \geq 2 \).

Let \( (\phi_i)_{i=1}^{N_h} \) be a basis of the \( N_h \)-dimensional subspace \( V_h \subset V \). We will identify uniquely every \( X \in V_h \) with a vector \( X = (X_1, \ldots, X_{N_h})^T \in \mathbb{R}^{N_h} \) by the relation \( X = \sum_{i=1}^{N_h} X_i \phi_i \) and define a norm on \( \mathbb{R}^{N_h} \) by \( \| X \|_{\mathbb{R}^{N_h}} := \| X \|_H \). Since the filtered probability space satisfies the usual conditions, \( \mathcal{F}_0 \) contains all \( \mathbb{P} \)-null sets. Hence, we can choose \( \Omega' \in \mathcal{F}_{t_n} \) with \( \mathbb{P}(\Omega') = 1 \) such that the Wiener process \( W \) is \( U \)-valued and the random variables \( X^0, \ldots, X^{n-1} \) are \( V_h \)-valued on \( \Omega' \).

For any \( \omega \in \Omega' \) and \( X \in \mathbb{R}^{N_h} \) with associated \( X \in V_h \) we define the function \( f : \Omega \times \mathbb{R}^{N_h} \to \mathbb{R}^{N_h} \) componentwise for \( i = 1, \ldots, N_h \) by

\[
f(\omega, X)_i := (3X - 4X^{n-1}(\omega) + X^{n-2}(\omega), \phi_i)_H + 2k(A(X, \omega), \phi_i)_V \times V
\]

\[- (3B(X^{n-1}(\omega), \omega) \Delta_h W^n(\omega) - B(X^{n-2}(\omega), \omega) \Delta_h W^{n-1}(\omega), \phi_i)_H.\]
Notice that $X$ fulfills the equation $f(\omega, X) = 0$ if and only if $X^n(\omega) := X = \sum_{i=1}^{N_n} X_i \phi_i$ solves the equation (2) for given $\omega \in \Omega'$.

In the following, let $\omega \in \Omega'$ be arbitrary, but fixed. To prove the existence of a zero of the function $f(\omega, \cdot)$, we will show that the mapping $f(\omega, \cdot) : \mathbb{R}^{N_n} \to \mathbb{R}^{N_n}$ is continuous and satisfies $f(\omega, X) \cdot X \geq 0$ for some $R \in (0, \infty)$ and all $X \in \mathbb{R}^{N_n}$ with $\|X\|_2 = R$. The hemicontinuity of $A$ and the monotonicity condition (9) imply the demicontinuity of the operator $A(\cdot, \omega)$. Since weak and strong convergence are equivalent in finite-dimensional spaces, the function $f(\omega, \cdot)$ is continuous. Moreover, we observe that

$$
f(\omega, X) \cdot X = (3X - 4X^{n-1}(\omega) + X^{n-2}(\omega), X)_H + 2k(A(X, \omega), X)_{V', V}$$

$$- (3B(X^{n-1}(\omega), \omega)\Delta_k W^n(\omega) - B(X^{n-2}(\omega), \omega)\Delta_k W^{n-1}(\omega), X)_H$$

holds for every $X \in \mathbb{R}^{N_n}$. Applying the Cauchy–Schwarz inequality and the coercivity condition (10) leads to

$$f(\omega, X) \cdot X \geq \|X\|_H \left(2\|\cdot\|_{V', H}^0 k \|X\| - \|4X^{n-1}(\omega) - X^{n-2}(\omega)\|_H \|X\|_Hight)^{p-1}$$

$$- \|3B(X^{n-1}(\omega), \omega)\Delta_k W^n(\omega) - B(X^{n-2}(\omega), \omega)\Delta_k W^{n-1}(\omega)\|_H \|X\|_H.$$  

Using the assumption $3 - k\kappa \geq 0$ and the continuity of the embedding $V \hookrightarrow H$, we derive the estimate

$$f(\omega, X) \cdot X \geq C \|X\|_2 \left(2\|\cdot\|_{V', H}^0 k \|X\| - \|4X^{n-1}(\omega) - X^{n-2}(\omega)\|_H \|X\|_Hight)^{p-1}$$

$$- \|3B(X^{n-1}(\omega), \omega)\Delta_k W^n(\omega) - B(X^{n-2}(\omega), \omega)\Delta_k W^{n-1}(\omega)\|_H \|X\|_H - c.$$  

Since $\|X\|_H = \|X\|_{\mathbb{R}^{N_n}}$ and norms on the finite-dimensional space $\mathbb{R}^{N_n}$ are equivalent, there is some constant $C \in (0, \infty)$ such that

$$f(\omega, X) \cdot X \geq C \|X\|_2 \left(2\|\cdot\|_{V', H}^0 k \|X\| - \|4X^{n-1}(\omega) - X^{n-2}(\omega)\|_H \|X\|_Hight)^{p-1}$$

$$- \|3B(X^{n-1}(\omega), \omega)\Delta_k W^n(\omega) - B(X^{n-2}(\omega), \omega)\Delta_k W^{n-1}(\omega)\|_H \|X\|_H - c.$$  

Now we choose $R(\omega) \in (0, \infty)$ sufficiently large such that $f(\omega, X) \cdot X \geq 0$ holds for all $X \in \mathbb{R}^{N_n}$ with $\|X\|_2 = R(\omega)$. From Lemma 3.6 it then follows that a zero of the function $f(\omega, \cdot)$ exists.

To prove the uniqueness of a zero of the function $f(\omega, \cdot)$ for fixed $\omega \in \Omega'$, assume that two distinct solutions $X, Y \in \mathbb{R}^{N_n}$ with associated $X, Y \in V_h$, respectively, exist such that $f(\omega, X) = f(\omega, Y) = 0$. The monotonicity condition (9) and the condition $k\kappa \leq 3$ imply that

$$0 = (f(\omega, X) - f(\omega, Y), X - Y)_2$$

$$= 3\|X - Y\|_H^2 + k(A(X, \omega) - A(Y, \omega), X - Y)_{V', V} \geq (3 - k\kappa)\|X - Y\|_H^2 \geq 0.$$  

This shows that $X$ and $Y$ coincide in $V_h$ and hence $X$ and $Y$ coincide in $\mathbb{R}^{N_n}$. Therefore, $f(\omega, \cdot)$ has a unique zero for every $\omega \in \Omega'$.

Now, we set $X^n(\omega) := X \in V_h$ for every $\omega \in \Omega'$ and $X^n(\omega) := 0$ for each $\omega \in \Omega \setminus \Omega'$. To prove the $F_t\omega$-measurability of $X^n$, recall that $X^j(\cdot)$ is assumed to be measurable with respect to $F_t^j \subset F_{t_n}$ for each $j = 0, \ldots, n - 1$. Moreover, Assumption 3.1 and the measurability properties of the Wiener process $W$ imply the $F_t\omega$-measurability of $A(v, \cdot)$ and $B(X^j(\cdot), \cdot)\Delta_k W^n(\cdot)$ for every $v \in V_h$ and $j = 0, \ldots, n - 1$. Therefore, the function $f(\cdot, X)$ is $F_t\omega$-measurable for every fixed $X \in$
\[ \mathbb{R}^{N_q}. \] Since the \( \sigma \)-algebra \( \mathcal{F}_{t_n} \) contains all \( \mathbb{P} \)-null sets, we deduce from Lemma 3.7 the measurability of the mapping \( \omega \mapsto X^n(\omega) \) with respect to \( \mathcal{F}_{t_n} \).

This concludes the proof for the existence and uniqueness of a solution \((X^n)_{n=0}^{N_q}\) to the numerical scheme (2). The \( L^p(\Omega; V) \cap L^2(\Omega; H) \)-integrability of the discrete solution follows for sufficiently small temporal step size \( k \) from Theorem 3.4. \( \square \)

**Remark 3.8.** Consider the BEM scheme (3) with an initial value \( X_{k,h}^0 \) satisfying

\[
X_{k,h}^0 \in L^2(\Omega, \mathcal{F}_{t_0}, \mathbb{P}; H) \quad \text{and} \quad B(X_{k,h}^0) \in L^2(\Omega, \mathcal{F}_{t_0}, \mathbb{P}; L^2(U_0, H))
\]

such that \( X_{k,h}^0 \) is \( \mathbb{P} \)-almost surely \( V_h \)-valued. Under Assumption 3.1, the BEM scheme admits for every temporal step size \( k = \frac{T}{N_q} \) with \( k \kappa < 1 \) a unique solution \((X^n_{k,h})_{n=0}^{N_q}\), which is \((\mathcal{F}_{t_n})_{n=0}^{N_q}\)-adapted, \( \mathbb{P} \)-almost surely \( V_h \)-valued and \( L^p(\Omega; V) \cap L^2(\Omega; H) \)-integrable. This result can be proven for \( p \in (1, \infty) \) with similar techniques as used in the proof of Theorem 3.5. In the case of \( p \in [2, \infty) \), an alternative proof can be found in [19].

**Remark 3.9.** Let \( h \in (0, 1) \) and \( V_h \neq \{0\} \) be fixed. The initialization of the BDF2-Maruyma scheme (2) requires two initial values \((X^n_{k,h})_{n=0,1}\), which are \((\mathcal{F}_{t_n})_{n=0,1}\)-adapted and \( \mathbb{P} \)-almost surely \( V_h \)-valued. A typical choice for the first initial value is \( X_{k,h}^0 = P_h(X_0) \), where \( P_h : H \rightarrow V_h \) denotes the orthogonal projector onto \( V_h \) with respect to the inner product in \( H \). In Section 5, we also consider an interpolation operator as an alternative to \( P_h \). Further, one iteration of the BEM scheme (3) with \( X_{k,h}^0 \) as the initial value yields an \( \mathcal{F}_{t_1} \)-measurable and \( \mathbb{P} \)-almost surely \( V_h \)-valued random variable \( X_{k,h}^1 \) which is an admissible choice for the second initial value. Compare further with Remark 3.8.

We close this section with a simple example of a stochastic partial differential equation, which fits into the framework of Assumption 3.1. For further examples of stochastic evolution equations we refer to [30, Section 4.1] and Section 5 below.

**Example 3.10.** We consider the stochastic heat equation

\[
\begin{align*}
du(t, x) - u_{xx}(t, x) \, dt &= \sigma \, dW(t, x), \quad (t, x) \in (0, T) \times (0, 1), \\
u(t, 0) &= u(t, 1) = 0, \quad t \in (0, T), \\
u(0, x) &= \sin(\pi x), \quad x \in (0, 1),
\end{align*}
\]

with additive noise determined by the scalar \( \sigma \in \mathbb{R} \), Dirichlet boundary conditions and a smooth deterministic initial value.

In the context of our abstract setting, we make use of the Gelfand triple induced by the spaces \( V = H^1_0(0, 1) \) and \( H = L^2(0, 1) \) and identify \( X \) as the abstract function of \( u \) such that

\[ X: [0, T] \times \Omega \rightarrow V, \quad (t, \omega) \mapsto u(t, \cdot, \omega). \]

The deterministic initial value \( X_0 = \sin(\pi \cdot) \) is smooth and equal to zero on the boundary. Hence, we have \( X_0 \in V \). The Wiener process \( W \) is assumed to take values in \( U = L^2(0, 1) \) and its covariance operator \( Q \) to have finite trace. If \( \{\chi_j\}_{j \in \mathbb{N}} \) is an orthonormal basis of \( U \) consisting of eigenfunctions of \( Q \) with eigenvalues \( q_j \geq 0 \), then \( \{Q^{1/2} \chi_j\}_{j \in \mathbb{N}} \) is an orthonormal basis of \( U_0 = Q^{1/2}(U) \subset H \) and it follows

\[
\|\sigma \mathbf{id}_H\|_{L^2(U_0, H)}^2 = \sigma^2 \sum_{j \in \mathbb{N}} \|Q^{1/2} \chi_j\|_H^2 = \sigma^2 \sum_{j \in \mathbb{N}} q_j = \sigma^2 \text{Tr}(Q) < \infty.
\]
In addition, the operators
\[ A: V \to V^*, \; v \mapsto A(v) \]
\[ B: V \to \mathcal{L}_2(U_0, H), \; v \mapsto \sigma \text{id}_H \]
are well-defined, where \( A(v) \in V^* = H^{-1}(0, 1) \) is the linear functional given by
\[ \langle A(v), w \rangle_{V^* \times V} = \int_0^1 v'(x)w'(x) \, dx = (v, w)_V \]
for all \( v, w \in V = H^1_0(\Omega) \). Altogether, this allows us to reformulate problem (19) as a stochastic evolution equation of the form (1).

The operators \( A \) and \( B \) are both deterministic and \( \mathcal{B}(V) \)-measurable. In particular, the linear operator \( A: V \to V^* \) is semi-continuous, bounded and, hence, of linear growth. The monotonicity condition (9) and the coercivity condition (10) are satisfied with \( \kappa = 0, \mu = 1, p = 2, \) and \( c = \nu \sigma^2 \text{Tr}(Q) \) for any \( \nu \in [1, \infty) \).

For fixed \( h > 0 \) and some finite-dimensional subspace \( \{0\} \neq V_h \subset V \), we generate the initial values for the BDF2-Maruyama scheme as discussed in Remark 3.9. Further, the initial values \( (X_{n,k,h})_{n=0}^{N_k} \) are \( L^2(\Omega; H) \)-integrable by construction. Since the operator \( B \) is constant, the terms \( B(X_{n,k,h}), n \in \{0,1\} \), also fulfill the integrability condition in Assumption 3.3.

Consequently, Assumption 3.1 and Assumption 3.3 are satisfied and Theorem 3.5 guarantees for every sufficiently small step size \( k \) that the BDF2-Maruyama scheme is well-defined for the given problem (19).

4. Convergence of the BDF2-Maruyama method

In this section, we derive an estimate for the strong error between the approximate solution \( (X_{n,k,h})_{n=0}^{N_k} \) of (2) and the exact solution \( X \) of (1). In order to determine a lower bound for the order of convergence, we have to impose additional conditions on the operators \( A \) and \( B \) for the error analysis.

Throughout this section, we fix \( p = 2 \) and a Gelfand triple \((V, H, V^*)\) with \( V \) being uniformly convex as discussed in Section 2.

**Assumption 4.1.** Let the operators \( A \) and \( B \) satisfy Assumption 3.1 for \( p = 2 \). Moreover, there are \( \kappa \in (0, \infty), \nu \in (1, \infty) \) and \( L, K \in (0, \infty) \) such that the operators \( A \) and \( B \) satisfy \( \mathbb{P} \)-almost surely on \( \Omega \) for all \( v, u \in V \) the monotonicity condition
\[ 2\langle A(u) - A(v), u - v \rangle_{V^* \times V} + \kappa \|u - v\|_H^2 \geq \nu \|B(u) - B(v)\|_{L_2(U_0, H)}^2 + K\|u - v\|_V^2 \]
and the Lipschitz condition
\[ \|A(v) - A(u)\|_{V^*} \leq L\|v - u\|_V. \]

In order to determine an order of convergence, we require the consistency of the initial values for the numerical method (2).

**Assumption 4.2.** Let the initial values \( (X_{n,k,h})_{n=0}^{N_k} \) satisfy Assumption 3.3. In addition, there exist \( C_1 \in (0, \infty) \) and \( \gamma \in (0, \infty) \) such that
\[ \sum_{n=0}^{1} \mathbb{E}[\|X_{n,k,h} - X(t_n)\|_H^2 + k\|B(X_{n,k,h}) - B(X(t_n))\|_{L_2(U_0, H)}^2] \leq C_1(k + h^\gamma) \]
holds for all \( k = \frac{T}{N_k}, N_k \in \mathbb{N}, \) and \( h \in (0, 1) \).
We also need to impose the following additional temporal regularity condition on the exact solution. To this end, we recall the definition of the $q$-variation norm from (8).

**Assumption 4.3.** Let the initial value $X_0$ be $V$-valued $\mathbb{P}$-almost surely and let the solution to (1) satisfy $X \in C([0,T]; L^2(\Omega; V))$. In addition, there exists $q \in [2,\infty)$ with $\|X\|_{q-\text{var}, L^2(\Omega, V)} < \infty$, i.e. $X$ is of finite $q$-variation with respect to the $L^2(\Omega; V)$-norm.

Evidently, if the exact solution $X \in C([0,T]; L^2(\Omega; V))$ is Hölder continuous with exponent $\gamma = \frac{\rho}{\rho'} \in (0, \frac{2}{q})$ then Assumption 4.3 is satisfied. The following two lemmas show how the $q$-variation norm is applied in the error analysis.

**Lemma 4.4.** Let $X \in C([0,T]; L^2(\Omega; V))$ be a stochastic process of finite $q$-variation with respect to the $L^2(\Omega; V)$-norm for some $q \in [2,\infty)$. Then it holds for every finite partition $P = \{t_0 = 0, \ldots, t_N = T\}$, $N \in \mathbb{N}$, of the interval $[0,T]$ with maximal step size $k := \max_{n=1,\ldots,N}(t_n - t_{n-1})$ that

$$\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \mathbb{E}[\|X(s) - X(t_n)\|_{L^2}^2] \, ds \leq (1 + T) k^\frac{2}{\gamma} \|X\|_{q-\text{var}, L^2(\Omega, V)}^2.$$}

**Proof.** The assumption $X \in C([0,T]; L^2(\Omega; V))$ implies that the real-valued function $s \mapsto \mathbb{E}[\|X(s) - X(t_n)\|_{L^2}^2]$ is continuous on the interval $[0,T]$. By the intermediate value theorem, there exist for each $n \in \{1, \ldots, N\}$ a point $\xi_n \in [t_{n-1}, t_n]$ independent of $\Omega$ such that

$$\int_{t_{n-1}}^{t_n} \mathbb{E}[\|X(s) - X(t_n)\|_{L^2}^2] \, ds = (t_n - t_{n-1}) \mathbb{E}[\|X(\xi_n) - X(t_n)\|_{L^2(\Omega, V)}^2].$$

A summation over $n$ from 1 to $N$ shows that

$$\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \mathbb{E}[\|X(s) - X(t_n)\|_{L^2}^2] \, ds = \sum_{n=1}^{N} (t_n - t_{n-1}) \frac{2}{\gamma} \mathbb{E}[\|X(\xi_n) - X(t_n)\|_{L^2(\Omega, V)}^2].$$

Finally, applying Hölder’s inequality with exponents $\rho = \frac{2}{\gamma}$ and $\rho' = \frac{2}{2-\gamma}$ yields

$$\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \mathbb{E}[\|X(s) - X(t_n)\|_{L^2}^2] \, ds \leq \left( \sum_{n=1}^{N} (t_n - t_{n-1}) \right)^{\frac{2}{\gamma}} \left( \sum_{n=1}^{N} (t_n - t_{n-1}) \mathbb{E}[\|X(\xi_n) - X(t_n)\|_{L^2(\Omega, V)}^2] \right)^{\frac{\gamma}{2}} \leq (1 + T) k^\frac{2}{\gamma} \|X\|_{q-\text{var}, L^2(\Omega, V)}^2,$$

where we use $T^\frac{2-\gamma}{\gamma} \leq (1 + T)$ for $T \in (0,\infty)$ and recall the definition of the $q$-variational norm from (8) in the last step. \hfill \Box

**Lemma 4.5.** Let Assumption 4.1 and Assumption 4.3 be satisfied with $q \in [2,\infty)$. Then, it holds for every finite partition $P = \{t_0 = 0, \ldots, t_N = T\}$, $N \in \mathbb{N}$, of the interval $[0,T]$ with maximal step size $k := \max_{n=1,\ldots,N}(t_n - t_{n-1})$ that

$$\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \mathbb{E}[\|A(X(s)) - A(X(t_n))\|_{L^2}^2] \, ds \leq L_A k^\frac{2}{\gamma} \|X\|_{q-\text{var}, L^2(\Omega, V)}^2$$

with constant $L_A = (1 + T) L^2$. \hfill \Box
Proof. The Lipschitz condition (21) yields for every \( n \in \{1, \ldots, N\} \) and \( s \in [t_{n-1}, t_n] \) that
\[
\mathbb{E}\left[ \|A(X(s)) - A(X(t_n))\|^2_{L^2} \right] \leq L^2 \mathbb{E}\left[ \|X(s) - X(t_n)\|^2_{L^2} \right].
\]
The assertion follows now from an application of Lemma 4.4. \( \square \)

**Lemma 4.6.** Let Assumption 4.1 and Assumption 4.3 with \( q \in [2, \infty) \) be satisfied. Then, it holds for every finite partition \( P = \{ t_0 = 0, \ldots, t_N = T \} \), \( N \in \mathbb{N} \), of the interval \([0, T]\) with maximal step size \( k := \max_{n=1, \ldots, N} (t_n - t_{n-1}) \) that
\[
\sum_{n=1}^{N} \mathbb{E}\left[ \left\| \int_{t_{n-1}}^{t_n} B(X(s)) - B(X(t_{n-1})) \, dW(s) \right\|^2_{L^2} \right] \leq L_B k^\frac{q}{2} \|X\|_{q-\text{var}, L^2(\Omega; V)}^2
\]
with constant \( L_B = (1 + T)(\kappa \beta_{V,H}^2 + 2L - K) \geq 0 \).

**Proof.** First, observe that the monotonicity-like condition (20) with \( \nu \in (1, \infty) \) and the Lipschitz continuity of \( A \) imply for every \( u, v \in V \) that
\[
\|B(u) - B(v)\|_{L^2(\mathbb{R}_0; H)}^2 \leq 2\|A(u) - A(v)\|_{L^2(\mathbb{R}_0; H)} \|u - v\|_V + \kappa \|u - v\|_H^2 - K \|u - v\|_V^2
\]
\[
\leq \kappa \|u - v\|_H^2 + (2L - K) \|u - v\|_V^2
\]
\[
\leq (\kappa \beta_{V,H}^2 + 2L - K) \|u - v\|_V^2
\]
\[
= \frac{L_B}{1 + T} \|u - v\|_V^2 \quad \mathbb{P}\text{-a.s.},
\]
where we also used that \( V \) is continuously embedded into \( H \), i.e. for all \( v \in V \) it holds \( \|v\|_H \leq \beta_{V,H} \|v\|_V \). In particular, it also follows from the above estimate that \( L_B \geq 0 \).

Next, for every \( n \in \{1, \ldots, N\} \) an application of the Itô isometry yields
\[
\mathbb{E}\left[ \left\| \int_{t_{n-1}}^{t_n} B(X(s)) - B(X(t_{n-1})) \, dW(s) \right\|^2_{L^2} \right]
\]
\[
= \mathbb{E}\left[ \int_{t_{n-1}}^{t_n} \|B(X(s)) - B(X(t_{n-1}))\|^2_{L^2(\mathbb{R}_0; H)} \, ds \right]
\]
\[
\leq \frac{L_B}{1 + T} \int_{t_{n-1}}^{t_n} \mathbb{E}\left[ \|X(s) - X(t_{n-1})\|^2_{L^2} \right] \, ds.
\]
Then, the assertion follows from an application of a slightly modified version of Lemma 4.4. \( \square \)

We are now prepared to state the main result of this section.

**Theorem 4.7.** Let Assumption 4.1 and Assumption 4.3 be satisfied with \( q \in [2, \infty) \). Let \( h \in (0, 1) \) and \( k = \frac{\nu}{\gamma} \), \( N_k \in \mathbb{N} \), be fixed with \( 2\kappa k < 1 \). Further, let some initial values \( (X_{n, h})_{n=1, \ldots, N_k} \) satisfy Assumption 4.2 for \( \gamma \in (0, \infty) \). Then, it holds the error estimate
\[
\max_{n \in \{2, \ldots, N_k\}} \|X_{n, h} - X(t_n)\|_{L^2(\Omega; H)}^2 + k \sum_{n=2}^{N_k} \|X_{n, h} - X(t_n)\|_{L^2(\Omega; V)}^2
\]
\[
\leq C_k \left( k^{\frac{q}{2}} \|X\|_{q-\text{var}, L^2(\Omega; V)}^2 + k + h^\gamma + \max_{n \in \{2, \ldots, N_k\}} \mathbb{E}\left[ \text{dist}_H(X(t_n), V_h)^2 \right] \right)
\]
\[
+ k \left( 1 + \|P_h\|_{L(V)} \right)^2 \sum_{n=2}^{N_k} \mathbb{E}\left[ \text{dist}_V(X(t_n), V_h)^2 \right].
\]
where the constant $C_k$ is defined for $\tilde{C}_k = \min\{1-2k\kappa, K\}$ by

$$ C_k = 2e^{2T\kappa\tilde{C}_k^{-1}}\max\left\{1, \frac{L^2}{K} + K, 16C I, 2\kappa C I, 32\frac{L A}{K} + \frac{2\nu L B}{\nu - 1}\right\}. $$

Proof. In the following, all equalities and inequalities involving random variables are assumed to hold $\mathbb{P}$-almost surely, unless stated otherwise. For $n = 0, \ldots, N_k$, we denote the error of the discretization scheme (2) at time $t_n$ by $E^n = X^n - X(t_n)$. Using the orthogonal projection $P_h : H \to V_h$, we split the error into two parts by writing

$$ E^n = P_h E^n + (\text{id} - P_h) E^n $$

$$ = (X^n - P_h X(t_n)) + (P_h - \text{id})X(t_n) =: \Theta^n + \Xi^n, \quad n = 0, \ldots, N_k. $$

By definition, $\Theta^n$ and $\Xi^n$ are orthogonal with respect to the inner product $(\cdot, \cdot)_H$ and, hence,

$$ \|E^n\|^2_H = \|\Theta^n\|^2_H + \|\Xi^n\|^2_H. $$

Let us fix $n \in \{2, \ldots, N_k\}$ for now. Recalling the identity (5), it holds that

$$ \|E^n\|^2_H - \|E^{n-1}\|^2_H + \|2E^n - E^{n-1}\|^2_H - \|2E^{n-1} - E^{n-2}\|^2_H $$

$$ + \|E^n - 2E^{n-1} + E^{n-2}\|^2_H = 4\left(\frac{3}{2}E^n - 2E^{n-1} + \frac{1}{2}E^{n-2}, E^n\right)_H =: \Gamma^n. $$

We insert the $H$-orthogonal decomposition $E^n = \Theta^n + \Xi^n$ to obtain

$$ \Gamma^n = 4\left(\frac{3}{2}E^n - 2E^{n-1} + \frac{1}{2}E^{n-2}, \Theta^n + \Xi^n\right)_H $$

$$ = 4\left(\frac{3}{2}X^n - 2X^{n-1} + \frac{1}{2}X^{n-2}, \Theta^n\right)_H $$

$$ - 4\left(\frac{3}{2}X(t_n) - 2X(t_{n-1}) + \frac{1}{2}X(t_{n-2}), \Theta^n\right)_H $$

$$ + 2(3\Xi^n - 4\Xi^{n-1} + \Xi^{n-2}, \Xi^n)_H. $$

Using the definitions of the numerical scheme (2) and of the exact solution (12) to equation (1), we deduce further

$$ \Gamma^n = -4k\left\langle A(X^n), \Theta^n\right\rangle_{V \times V} $$

$$ + 2\left\langle 3B(X^{n-1})\Delta_k W^n - B(X^{n-2})\Delta_k W^{n-1}, \Theta^n\right\rangle_H $$

$$ + 2\left\langle 3\int_{t_{n-1}}^{t_n} A(X(s)) ds - \int_{t_{n-2}}^{t_{n-1}} A(X(s)) ds, \Theta^n\right\rangle_{V \times V} $$

$$ - 2\left\langle 3\int_{t_{n-1}}^{t_n} B(X(s)) dW(s) - \int_{t_{n-2}}^{t_{n-1}} B(X(s)) dW(s), \Theta^n\right\rangle_H $$

$$ + 2(3\Xi^n - 4\Xi^{n-1} + \Xi^{n-2}, \Xi^n)_H, $$

where we do not distinguish notationally between the solution $X$ and its modification appearing in (12), since we eventually take expectations of these terms.
After rearranging the terms, we arrive at $\Gamma_n = \Gamma_1^n + \Gamma_2^n + \Gamma_3^n + \Gamma_4^n + \Gamma_5^n$ with

\[
\Gamma_1^n := -4k(A(X^n) - A(X(t_n)), \Theta^n)_{V \times V}, \\
\Gamma_2^n := 2\left[ \left( B(X^{n-1}) - B(X(t_{n-1})) \right) \Delta_k W^n, \Theta^n \right]_H \\
= -2\left( \left( B(X^{n-2}) - B(X(t_{n-2})) \right) \Delta_k W^{n-1}, \Theta^n \right)_H, \\
\Gamma_3^n := 2\left[ 3 \int_{t_{n-1}}^{t_n} A(X(s)) \, ds - \int_{t_{n-2}}^{t_{n-1}} A(X(s)) \, ds - 2kA(X(t_n)), \Theta^n \right]_{V \times V}, \\
\Gamma_4^n := 2\left[ 3 \int_{t_{n-1}}^{t_n} B(X(t_{n-1})) - B(X(s)) \, dW(s), \Theta^n \right]_H \\
= -2\left( \int_{t_{n-2}}^{t_{n-1}} B(X(t_{n-2})) - B(X(s)) \, dW(s), \Theta^n \right)_H, \\
\Gamma_5^n := 2\left[ 3\Xi^n - 4\Xi^{n-1} - \Xi^{n-2}, \Xi^n \right]_H.
\]

We will further estimate each $\Gamma_i^n$ for $i \in \{1, \ldots, 5\}$ separately.

The assumption (20) is essential to estimate $\Gamma_1^n$ appropriately. Together with the Lipschitz continuity of the operator $A$ and an application of Young’s inequality, we conclude that

\[
\Gamma_1^n = -4k(A(X^n) - A(X(t_n)), E^n)_{V \times V} + 4k(A(X^n) - A(X(t_n)), \Xi^n)_{V \times V} \\
\leq -2kK||E^n||^2_V - 2k\nu\|B(X^n) - B(X(t_n))\|_{L_2(U_0,H)} + 2k\kappa||E^n||^2_H \\
+ 4kL||E^n||_V||\Xi^n||_V \\
\leq \frac{3}{2}kK||E^n||^2_V - 2k\nu\|\Delta B^n\|^2_{L_2(U_0,H)} + 2k\kappa||E^n||^2_H + 8k\frac{L^2}{K}||\Xi^n||^2_V,
\]

where we also made use of the notation $\Delta B^n := B(X^n) - B(X(t_n))$ for $n \in \{0, \ldots, N_k\}$. Regarding $\Gamma_2^n$, some elementary calculations yield that

\[
\Gamma_2^n = 2(3\Delta B^{n-1} - \Delta B^{n-2} \Delta_k W^{n-1}, \Theta^n)_{H} \\
= 2(\Delta B^{n-1} - \Delta B^{n-2} \Delta_k W^{n-1}, \Theta^n - 2\Theta^{n-1} + \Theta^{n-2})_{H} \\
+ 2(\Delta B^{n-1} - \Delta B^{n-2} \Delta_k W^{n-1}, 2\Theta^n - \Theta^{n-1})_{H} - 2(\Delta B^{n-2} \Delta_k W^{n-1}, 2\Theta^{n-1} - \Theta^{n-2})_{H} \\
+ 2(\Delta B^{n-2} \Delta_k W^{n-1}, 3\Theta^{n-1} - \Theta^{n-2})_{H}.
\]

By using the martingale property of the stochastic integral, it follows that

\[
\mathbb{E}[\Delta B^{n-1} \Delta_k W^n \mid F_{t_{n-1}}] = \mathbb{E}\left[ \int_{t_{n-1}}^{t_n} B(X^{n-1}) - B(X(t_{n-1})) \, dW(s) \mid F_{t_{n-1}} \right] = 0.
\]

This fact and the $F_{t_{n-1}}$-measurability of the random variable $(3\Theta^{n-1} - \Theta^{n-2})$ imply

\[
\mathbb{E}[\left( (\Delta B^{n-1} \Delta_k W^n, 3\Theta^{n-1} - \Theta^{n-2})_{H} \right)] = 0.
\]

By applying Young’s inequality to the decomposition of $\Gamma_2$ and taking expectation, we conclude that

\[
\mathbb{E}[\Gamma_2^n] \leq \nu\mathbb{E}[\|\Delta B^{n-1} \Delta_k W^n - \Delta B^{n-2} \Delta_k W^{n-1}\|_H^2] \\
+ \frac{1}{\nu}\mathbb{E}[\|\Theta^n - 2\Theta^{n-1} + \Theta^{n-2}\|_H^2] + 2\mathbb{E}[\left( (\Delta B^{n-1} \Delta_k W^n, 2\Theta^n - \Theta^{n-1})_{H} \right)] \\
- 2\mathbb{E}[\left( (\Delta B^{n-2} \Delta_k W^{n-1}, 2\Theta^{n-1} - \Theta^{n-2})_{H} \right)].
\]
Next, observe that

\[
E[\|\Delta B^{n-1} \Delta_k W^n - \Delta B^{n-2} \Delta_k W^{n-1}\|^2_H]
= E[\|\Delta B^{n-1} \Delta_k W^n\|^2_H] + E[\|\Delta B^{n-2} \Delta_k W^{n-1}\|^2_H]
\]

which follows again from the martingale property of the stochastic integral. Hence, after summing over \(n\) from 2 to \(j \in \{2, \ldots, N_k\}\) we arrive at

\[
\sum_{n=2}^{j} E[\Gamma_n^2] \leq \nu \sum_{n=2}^{j} \left( E[\|\Delta B^{n-1} \Delta_k W^n\|^2_H] + E[\|\Delta B^{n-2} \Delta_k W^{n-1}\|^2_H] \right)
+ 2E[\langle \Delta B^{j-1} \Delta_k W^j, 2\Theta^j - \Theta^{j-1} \rangle_H] - \langle \Delta B^0 \Delta_k W^1, 2\Theta^1 - \Theta^0 \rangle_H
+ \frac{1}{\nu} \sum_{n=2}^{j} E[\|\Theta^n - 2\Theta^{n-1} + \Theta^{n-2}\|^2_H].
\]

From applications of the Cauchy–Schwarz inequality, Young’s inequality and the Itô isometry we obtain

\[
2E[\langle \Delta B^{j-1} \Delta_k W^j, 2\Theta^j - \Theta^{j-1} \rangle_H]
\leq \frac{1}{\nu} E[\|2\Theta^j - \Theta^{j-1}\|^2_H] + \nu E[\|\Delta B^{j-1} \Delta_k W^j-1\|^2_H]
= \frac{1}{\nu} E[\|2\Theta^j - \Theta^{j-1}\|^2_H] + k\nu E[\|\Delta B^{j-1}\|^2_{\mathcal{Z}_2(U_n, H)}].
\]

The term \(E[\langle \Delta B^0 \Delta_k W^1, 2\Theta^1 - \Theta^0 \rangle_H]\) can be estimated in the same way. Inserting this into the estimate of \(\Gamma_n^2\), applying again the Itô isometry and recalling that \(\nu \in (1, \infty)\) then finally yields the estimate

\[
\sum_{n=2}^{j} E[\Gamma_n^2] \leq \frac{1}{\nu} \left( E[\|2\Theta^j - \Theta^{j-1}\|^2_H] + \sum_{n=2}^{j} E[\|\Theta^n - 2\Theta^{n-1} + \Theta^{n-2}\|^2_H] \right)
+ 2k\nu \sum_{n=1}^{j} E[\|\Delta B^{n-1}\|^2_{\mathcal{Z}_2(U_n, H)}] + E[\|2\Theta^1 - \Theta^0\|^2_H].
\]

Next, we turn to the estimation of \(\Gamma_0^3\). We apply Young’s inequality and the Lipschitz continuity of \(A\) to deduce that

\[
\Gamma_0^3 = 4 \int_{t_{n-1}}^{t_n} \langle A(X(s)) - A(X(t_n)), \Theta^n \rangle_{V_x \times V} \, ds
+ 2 \int_{t_{n-1}}^{t_n} \langle A(X(s)) - A(X(t_{n-1})), \Theta^n \rangle_{V_x \times V} \, ds
- 2 \int_{t_{n-2}}^{t_{n-1}} \langle A(X(s)) - A(X(t_{n-1})), \Theta^n \rangle_{V_x \times V} \, ds
\leq \frac{16}{K} \int_{t_{n-1}}^{t_n} \|A(X(s)) - A(X(t_n))\|^2_{V_x} \, ds
+ \frac{8}{K} \int_{t_{n-2}}^{t_{n-1}} \|A(X(s)) - A(X(t_{n-1}))\|^2_{V_x} \, ds + \frac{1}{2} kK\|\Theta^n\|^2_{V_x}.
\]
After taking expectations and applying Fubini’s theorem, we arrive at
\[
E[\Gamma_n] \leq \frac{16}{K} \int_{t_{n-1}}^{t_n} E \left[ \|A(X(s)) - A(X(t_n))\|_2^2 \right] ds \\
+ \frac{8}{K} \int_{t_{n-2}}^{t_{n-1}} E \left[ \|A(X(s)) - A(X(t_{n-1}))\|_2^2 \right] ds \\
+ kK \left( E[\|E^n\|_2^2] + E[\|\Xi^n\|_2^2] \right).
\]
Therefore, the summation over \( n \) from 2 to \( j \in \{2, \ldots, N_k \} \) together with an application of Lemma 4.5 (and an obvious modification thereof) shows that
\[
\sum_{n=2}^{j} E[\Gamma_n] \leq kK \sum_{n=2}^{j} \left( E[\|E^n\|_2^2] + E[\|\Xi^n\|_2^2] \right) + 32\frac{L_A k^2}{K} \|X\|_{\text{var}, L^2(\Omega; V)}.
\]
To decompose the term \( \Gamma_n \), we define \( I^n := \int_{t_{n-1}}^{t_n} B(X(t_{n-1})) - B(X(s)) dW(s) \) for \( n \in \{1, \ldots, N_k \} \). An elementary calculation shows that
\[
\Gamma_n = 2(3I^n - I^{n-1}, \Theta^n)_H \\
= 2(I^n - I^{n-1}, \Theta^n - 2\Theta^{n-1} + \Theta^{n-2})_H \\
+ 2(I^{n-1}, 2\Theta^n - \Theta^{n-1})_H - 2(I^{n-1}, 2\Theta^{n-1} - \Theta^{n-2})_H \\
+ 2(I^n, 3\Theta^{n-1} - \Theta^{n-2})_H.
\]
In the same way as in the estimation of \( \Gamma_2 \), we get \( E[\langle I^n, 3\Theta^{n-1} - \Theta^{n-2} \rangle_H] = 0 \).
After applying Young’s inequality and taking expectation, we arrive at
\[
E[\Gamma_n] \leq \frac{\nu}{\nu - 1} E[\|I^n - I^{n-1}\|_H^2] + \frac{\nu - 1}{\nu} \sum_{n=2}^{j} E[\|\Theta^n - 2\Theta^{n-1} + \Theta^{n-2}\|_H^2] \\
+ 2E[\langle I^1, 2\Theta^1 - \Theta^0 \rangle_H] - 2E[\langle I^{n-1}, 2\Theta^{n-1} - \Theta^{n-2} \rangle_H].
\]
Hence, after summing over \( n \) from 2 to \( j \in \{2, \ldots, N_k \} \) we obtain
\[
\sum_{n=2}^{j} E[\Gamma_n] \leq \frac{\nu}{\nu - 1} \sum_{n=2}^{j} E[\|I^n - I^{n-1}\|_H^2] + \frac{\nu - 1}{\nu} \sum_{n=2}^{j} E[\|\Theta^n - 2\Theta^{n-1} + \Theta^{n-2}\|_H^2] \\
+ 2E[\langle I^1, 2\Theta^1 - \Theta^0 \rangle_H] - 2E[\langle I^{n-1}, 2\Theta^{n-1} - \Theta^{n-2} \rangle_H] \\
\leq \frac{\nu}{\nu - 1} \sum_{n=2}^{j} E[\|I^n - I^{n-1}\|_H^2] + \frac{\nu - 1}{\nu} \sum_{n=2}^{j} E[\|\Theta^n - 2\Theta^{n-1} + \Theta^{n-2}\|_H^2] \\
+ \frac{\nu}{\nu - 1} E[\|I^1\|_H^2] + \frac{\nu - 1}{\nu} E[\|2\Theta^1 - \Theta^0\|_H^2] \\
+ E[\|I^{n-1}\|_H^2] + E[\|2\Theta^{n-1} - \Theta^{n-2}\|_H^2]
\]
by a further application of Young’s inequality. Since \( I^n \) and \( I^{n-1} \) are uncorrelated and, hence, orthogonal with respect to the inner product in \( L^2(\Omega; H) \), it follows
\[
E[\|I^n - I^{n-1}\|_H^2] = E[\|I^n\|_H^2] + E[\|I^{n-1}\|_H^2]
\]
for all \( n \in \{2, \ldots, N_k \} \). Together with Lemma 4.6 we therefore get
\[
\frac{\nu}{\nu - 1} E[\|I^1\|_H^2] + E[\|I^1\|_H^2] + \frac{\nu}{\nu - 1} \sum_{n=2}^{j} E[\|I^n - I^{n-1}\|_H^2] \\
\leq \frac{2\nu}{\nu - 1} \sum_{n=1}^{j} E[\|I^n\|_H^2] \leq \frac{2\nu}{\nu - 1} L_B k^2 \|X\|_{\text{var}, L^2(\Omega; V)}^2.
\]
Altogether, this gives the estimate
\[
\sum_{n=2}^{j} E[\Gamma_n^j] \leq \frac{\nu - 1}{\nu} \left( E[\|2\Theta^n - \Theta^{n-1}\|_H^2] + \sum_{n=2}^{j} E[\|\Theta^n - 2\Theta^{n-1} + \Theta^{n-2}\|_H^2] \right)
\]
\[
+ \frac{2\nu - 1}{\nu} L_B k^2 \|X\|_{q-var,L^2(\Omega;V)}^2 \right) + E[\|2\Theta^j - \Theta^0\|_H^2]
\]
for every \( j \in \{2, \ldots, N_k\} \).

Finally, the term \( \Gamma_0^n \) is rewritten in terms of the identity (5) by
\[
\Gamma_0^n = \|\Xi^n\|_H^2 - \|\Xi^{n-1}\|_H^2 + \|2\Xi^n - \Xi^{n-1}\|_H^2 - \|2\Xi^{n-1} - \Xi^{n-2}\|_H^2
\]
\[+ \|\Xi^n - 2\Xi^{n-1} + \Xi^{n-2}\|_H^2.
\]
After taking expectation and summing over \( n \) from 2 to \( j \in \{2, \ldots, N_k\} \) in equation (23), we also see that
\[
E[\|E^j\|_H^2] - E[\|E^1\|_H^2] + E[\|2E^j - E^{j-1}\|_H^2] - E[\|2E^1 - E^0\|_H^2]
\]
\[\leq \sum_{n=2}^{j} \left( E[\Gamma_n^j + \Gamma_0^n + \Gamma_2^n + \Gamma_3^n + \Gamma_4^n] - E[\|E^n - 2E^{n-1} + E^{n-2}\|_H^2] \right).
\]
Now, we insert the estimates for \( \Gamma_i^n \), \( i = 1, \ldots, 5 \), and we use that
\[
E[\|E^n - 2E^{n-1} + E^{n-2}\|_H^2]
\]
\[= E[\|\Theta^n - 2\Theta^{n-1} + \Theta^{n-2}\|_H^2] + E[\|\Xi^n - 2\Xi^{n-1} + \Xi^{n-2}\|_H^2],
\]
which follows from (22). This gives for every \( j \in \{2, \ldots, N_k\} \) that
\[
E[\|E^j\|_H^2] - E[\|E^1\|_H^2] + E[\|2E^j - E^{j-1}\|_H^2] - E[\|2E^1 - E^0\|_H^2]
\]
\[\leq -\frac{1}{2} kK \sum_{n=2}^{j} E[\|E^n\|^2_H] + 2kK \sum_{n=2}^{j} E[\|E^n\|^2_H] + k \left( \frac{L^2}{K} + K \right) \sum_{n=2}^{j} E[\|\Xi^n\|^2_V] + E[\|2\Theta^j - \Theta^{j-1}\|_H^2] + 2E[\|2\Theta^1 - \Theta^0\|_H^2] + 2k\nu \sum_{n=0}^{j} E[\|\Delta B^n\|^2_{L_2(U_n,H)}] + E[\|\Xi^j\|_H^2] - E[\|\Xi^0\|_H^2] + E[\|2\Xi^j - \Xi^{j-1}\|_H^2] - E[\|2\Xi^1 - \Xi^0\|_H^2] + \left( \frac{32L^2}{K} + \frac{2\nu L_B}{\nu - 1} \right) k^2 \|X\|_{q-var,L^2(\Omega;V)}^2.
\]
Thus, after recalling (22) and some rearranging we arrive at
\[
E[\|E^j\|_H^2] + \frac{1}{2} kK \sum_{n=2}^{j} E[\|E^n\|^2_H] + E[\|E^j\|_H^2] + k \left( \frac{L^2}{K} + K \right) \sum_{n=2}^{j} E[\|\Xi^n\|^2_V] + 3E[\|2\Theta^j - \Theta^{j-1}\|_H^2] + E[\|\Theta^1\|_H^2] + 2k\nu \sum_{n=0}^{j} E[\|\Delta B^n\|^2_{L_2(U_n,H)}] + \left( \frac{32L^2}{K} + \frac{2\nu L_B}{\nu - 1} \right) k^2 \|X\|_{q-var,L^2(\Omega;V)}^2.
\]
Finally, it is necessary to prove the uniform bound on \( \|\Xi^n\|_H \) as follows:
\[
\|\Xi^n\|_H = \text{dist}_H(X(t_n), V_h) \quad \text{for each} \ n \in \{0, \ldots, N_k\}.
\]
Further, the properties of the projections $P_h$ and $R_h$ yield the estimate
\[
\|\Xi^n\|_V = \|(P_h - \text{id})X(t_n)\|_V
\leq \|(P_h(\text{id} - R_h))X(t_n)\|_V + \|(P_hR_h - \text{id})X(t_n)\|_V
\leq (\|P_h\|_{\mathcal{L}(V)} + 1)\|(R_h - \text{id})X(t_n)\|_V = (1 + \|P_h\|_{\mathcal{L}(V)})\text{dist}_V(X(t_n), V_h)
\]
for each $n \in \{0, \ldots, N_h\}$. Altogether, we conclude for $j \in \{2, \ldots, N_h\}$ that
\[
\mathbb{E}[\|\Xi^j\|_{\hat{H}}^2] + k\left(\frac{L^2}{K} + K\right)\sum_{n=2}^{j} \mathbb{E}[\|\Xi^n\|_{\hat{H}}^2]
\leq \mathbb{E}[\text{dist}_H(X(t_j), V_h)^2]
\]
\[
+ k\left(\frac{L^2}{K} + K\right)(1 + \|P_h\|_{\mathcal{L}(V)})^2 \sum_{n=2}^{j} \mathbb{E}[\text{dist}_V(X(t_n), V_h)^2].
\]
(26)

Moreover, we want to estimate further the remaining terms in (25) that depend on the time steps $t_n$ with $n \in \{0, 1\}$. It holds that $\|2\Theta^1 - \Theta^0\|_{\hat{H}}^2 \leq 5(\|\Theta^1\|_{\hat{H}}^2 + \|\Theta^0\|_{\hat{H}}^2)$. Then, it follows from the $H$-orthogonality of the decomposition $E^n = \Theta^n + \Xi^n$ that
\[
3\mathbb{E}[\|2\Theta^1 - \Theta^0\|_{\hat{H}}^2] + \mathbb{E}[\|\Theta^1\|_{\hat{H}}^2] \leq 16 \sum_{n=0}^{1} \mathbb{E}[\|E^n\|_{\hat{H}}^2].
\]

This estimate together with Assumption 4.2 yields
\[
3\mathbb{E}[\|2\Theta^1 - \Theta^0\|_{\hat{H}}^2] + \mathbb{E}[\|\Theta^1\|_{\hat{H}}^2] + 2k\nu \sum_{n=0}^{1} \mathbb{E}[\|\Delta B^n\|_{\mathcal{L}^2(U_0, H)}^2]
\leq C_t \max \{16, 2\nu\}(k + h^\gamma).
\]

By inserting the last inequality and (26) into estimate (25), we arrive at
\[
(1 - 2k\kappa)\mathbb{E}[\|E^j\|_{\hat{H}}^2] + kK \sum_{n=2}^{j} \mathbb{E}[\|E^n\|_{\hat{H}}^2]
\leq 2k\kappa \sum_{n=2}^{j-1} \mathbb{E}[\|E^n\|_{\hat{H}}^2] + \tilde{C} \mathbb{E}[\text{dist}_H(X(t_j), V_h)^2]
\]
\[
+ \tilde{C}k(1 + \|P_h\|_{\mathcal{L}(V)})^2 \sum_{n=2}^{j} \mathbb{E}[\text{dist}_V(X(t_n), V_h)^2]
\]
\[
+ \tilde{C}(k^{\frac{2}{3}}\|X\|_{L^2_{u,u}^{\nu}} + k + h^\gamma),
\]
where the constant $\tilde{C} > 0$ is defined by
\[
\tilde{C} = \max \left\{1, 8\frac{L^2}{K} + K, 16C_t, 2\nu C_t, 32\frac{L^A}{K} + 2\nu \frac{L_B}{\nu - 1}\right\}.
\]

Finally, applying a discrete version of Gronwall’s inequality, see, e.g., [9], shows
\[
\mathbb{E}[\|E^j\|_{\hat{H}}^2] + k \sum_{n=2}^{j} \mathbb{E}[\|E^n\|_{\hat{H}}^2]
\leq e^{2k\nu \tilde{C}^{-1} C_k}(k + h^\gamma + k^{\frac{2}{3}}\|X\|_{L^2_{u,u}^{\nu}} + \mathbb{E}[\text{dist}_H(X(t_j), V_h)^2]
\]
\[
+ k(1 + \|P_h\|_{\mathcal{L}(V)})^2 \sum_{n=2}^{j} \mathbb{E}[\text{dist}_V(X(t_n), V_h)^2],
\]
where $\tilde{C}_k := \min\{1 - 2k\kappa, K\}$. Taking the maximum with respect to $j \in \{2, \ldots, N_k\}$ on the right-hand side of this inequality yields an estimate for each summand on the left-hand side and completes the proof. □

**Remark 4.8.** The error estimate in Theorem 4.7 depends on the term $\|P_h\|_{L(V)}$ which is, in general, not uniformly bounded for arbitrarily small $h \in (0, 1)$. However, for many important examples of evolution equations and Galerkin schemes a uniform bound can indeed be given. For instance, for the finite element method and the typical choice of the Gelfand triple with $V = H^1_0(\mathcal{D})$ and $H = L^2(\mathcal{D})$ on a bounded domain $\mathcal{D} \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, the $H^1$-stability of the orthogonal $L^2$-projection $P_h$ has been investigated in [2, 3, 8, 11].

**Remark 4.9.** Under Assumption 4.1 and Assumption 4.3, the BEM scheme (3) with initial value $X_{k,h}^0 \in L^2(\Omega; V)$ admits for sufficiently small temporal step size with $k\kappa < 1$ an approximate solution $(X_{k,h}^n)_{n \geq 0}$ such that at time $t_1 = k$ the error estimate

$$
\|X_{1,h}^k - X(t_1)\|_{L^2(\Omega; H)}^2 + k\|X_{1,h}^k - X(t_1)\|_{L^2(\Omega; V)}^2 \\
\leq \frac{C}{h^k} \left( k + \|(P_h - \text{id})X(t_1)\|_{L^2(\Omega; H)}^2 + k\|(P_h - \text{id})X(t_1)\|_{L^2(\Omega; V)}^2 \\
+ \|X_{k,h}^0 - X(t_0)\|_{L^2(\Omega; H)}^2 + k\|B(X_{k,h}) - B(X(t_0))\|_{L^2(\Omega; \mathcal{L}(V_n, H))}^2 \right)
$$

holds, where $C_k := \min\{1 - k\kappa, K\}$ and $C > 0$ is a constant only depending on $\kappa, \nu, L, K, T$ and $\beta \nu \to H$. This estimate can be proven with similar techniques as used in the proof of Theorem 4.7.

5. Numerical experiments

In this section, we perform two numerical experiments to give a more practical assessment of the BDF2-Maruyama scheme (2). In Subsection 5.1 we use the scheme to simulate the stochastic heat equation with additive noise and in Subsection 5.2 we consider a stochastic partial differential equation with a quasilinear drift as well as nonlinear multiplicative noise. To better illustrate its performance, we compare the BDF2-Maruyama scheme to the BEM scheme (3).

In all numerical experiments, we use equidistant grids to discretize the time-space domain $[0, T] \times [0, 1]$. Regarding the temporal discretization, the BEM scheme (3) as well as the BDF2-Maruyama scheme (2) are applied with the equidistant temporal step size $k = \frac{T}{N_k}$, where $N_k = 2^l$ for $l = 5, \ldots, 10$. The spatial discretization is realized by using the standard finite element method. To be more precise, we consider the equidistant partition $\{x_i = ih \mid i = 0, \ldots, N_h + 1\}$ with $N_h = 2^{12}$ interior nodes and spatial step size $h = \frac{1}{N_h + 1}$. We define the space $V_h$ consisting of piecewise linear finite elements by

$$V_h = \{ v \in C([0, 1]): v|_{[x_{j-1}, x_j]} \in \mathcal{P}_1 \forall i = 1, \ldots, N_h + 1, v(0) = v(1) = 0 \},$$

where $\mathcal{P}_1$ denotes the set of all polynomials up to degree 1. By $\{\phi_i\}_{i=1}^{N_h} \subset V_h$ we denote the Lagrange basis functions of $V_h$ which are uniquely determined by $\phi_i(x_j) = \delta_{ij}$ for all $i, j = 1, \ldots, N_h$. Further, we recall from [4, Section 4.4] or [25, Section 5.1] that the family of spaces $\{V_h\}_{h \in (0, 1)}$ defines a Galerkin scheme for the Sobolev space $V = H^1_0(0, 1)$. From [11, Theorem 2] it follows that $\|P_h\|_{L(V)} < \infty$ holds uniformly in $h \in (0, 1)$. 

Moreover, we denote by
\[
I_h: C([0,1]) \rightarrow V_h, \quad v \mapsto I_h(v) = \sum_{i=1}^{N_h} v(x_i) \phi_i,
\]
the interpolation operator. An explicit calculation verifies the interpolation error estimate
\[
\|I_h(v) - v\|_{L^2(0,1)} \leq C h \|v\|_{H^2_0(0,1)}
\]
for every function \(v \in H^1_0(0,1) \hookrightarrow C([0,1])\) and some constant \(C > 0\), see, e.g., [4, Theorem 4.4.20].

Regarding the simulation of the \(U\)-valued \(Q\)-Wiener process \(W\), we follow [27, Section 10.2] and consider the Karhunen–Loève expansion (7). Notice that the decay of the eigenvalues \((q_j)_{j \in \mathbb{N}}\) determines the smoothness of the Wiener process regarding the spatial variable. In the case of \(U = L^2(0,1)\), the choice of the sine basis \(\chi_j(x) = \sqrt{2} \sin(j \pi x)\) and the eigenvalues \(q_j = j^{-2} j^{2r+1+\varepsilon}\) with \(\varepsilon > 0\) and \(r \in \mathbb{R}_+\) leads to an almost surely \(H^r_0(0,1)\)-valued Wiener process, see [27, Example 10.9]. This enables us to sample efficiently the Wiener process \(W\), since the corresponding truncated representation
\[
W^J(t,x) = \sum_{j=1}^{J} \sqrt{2} j^{-\frac{1}{2}(2r+1+\varepsilon)} \beta_j(t) \sin(j \pi x), \quad J \in \mathbb{N},
\]
can be implemented by using a discrete sine transform. The truncation parameter is chosen to be \(J = 2^{12}\) in all simulations.

In our numerical experiments, we compute the strong error between the approximate solution of the respective scheme and the exact solution of the stochastic evolution equation (1) with respect to the \(L^\infty([0,T]; L^2(\Omega; H))\)-norm. Hereby, we only take the maximum over the points of the temporal grid of the considered numerical approximation. Using also a Monte Carlo simulation with \(M = 10^4\) independent samples, we approximate the strong error by
\[
\text{error}_{k,h} = \max_{n \in \{2,\ldots,N_n\}} \left( \frac{1}{M} \sum_{m=1}^{M} \|X_{k,h}^{n,(m)} - X^{(m)}(nk)\|_H^2 \right)^{\frac{1}{2}} \approx \max_{n \in \{2,\ldots,N_n\}} \|X_{k,h}^n - X(t_n)\|_{L^2(\Omega,H)},
\]
where \(\{X_{k,h}^{n,(m)} - X^{(m)}(nk)\}_{m=1,\ldots,M}\) are independently generated samples of the error \(X_{k,h}^n - X(t_n)\). Notice that the computation of the strong error in (30) is not explicitly depending on the initial values at the two grid points \(\{t_0, t_1\}\). The reason for this is that the same initial values are used for both considered schemes, cf. Remark 3.9. Therefore, in (30) we only measure the error for all temporal grid points where the two schemes differ.

As a substitute for the exact solution in (30), we use a numerical reference solution which is computed by using the BDF2-Maruyama scheme with \(N_k = 2^{15}\) steps and the same number \(N_h = 2^{12}\) of degrees of freedom in all simulations. We mention that the numerical results reported further below are not qualitatively impacted if the BEM scheme is used for the computation of the reference solution. Moreover, to validate the statistical significance of our numerical results, we determine the asymptotically valid \((1 - \alpha)\)-confidence interval for \(\|X_{k,h}^n - X(t_n)\|_{L^2(\Omega,H)}\)
with \( \alpha = 0.05 \) for some value of the index \( n \in \{2, \ldots, N_k\} \) at which the error estimator \( \text{error}_{k,h} \) in (30) attains its maximum. In detail, we compute the confidence interval (CI) using the formula

\[
\left[ \left( \overline{Y}_M - z_{(1-\epsilon)} \frac{S_M}{\sqrt{M}} \right), \left( \overline{Y}_M + z_{(1-\epsilon)} \frac{S_M}{\sqrt{M}} \right) \right],
\]

where \( \overline{Y}_M \) as well as \( S_M \) denote the sample mean and the unbiased sample standard deviation with respect to \( M \) realizations of independent and identically distributed copies of the random variable \( Y = \|X^n_{k,h} - X(t_n)\|_H^2 \) and \( z_{(1-\epsilon)} \) denotes the \((1-\epsilon)\)-quantile of the standard normal distribution.

We also compute the experimental order of convergence (EOC) as an estimator of the temporal convergence rates. We define the EOC for successive temporal step sizes \( k_{i-1}, k_i \) and fixed spatial step size \( h \) by

\[
\text{EOC} = \frac{\log(\text{error}_{k_i,h}) - \log(\text{error}_{k_{i-1},h})}{\log(k_i) - \log(k_{i-1})}.
\]

5.1. The Stochastic Heat Equation with additive noise. We examine again the stochastic heat equation introduced in Example 3.10. As above, we consider \( U = L^2(0, 1) \) and a \( U \)-valued Wiener process which takes almost surely values in \( H^1_0(0, 1) \subset U \) for some \( r \in \mathbb{R}_+ \).

Recall from Example 3.10, that the operators \( A \) and \( B \) satisfy Assumption 3.1. Since the operator \( A : V \to V^* \) is linear and bounded, it is also Lipschitz continuous with \( \|A\| = 1 \) and the stronger monotonicity condition (20) holds with \( \kappa = 0, \nu \in (1, \infty) \) and \( K \in (0, 2] \). Therefore, Assumption 4.1 is satisfied.

Moreover, it was shown in [23, Theorem 2.31] that the linear problem (19) admits for every \( r \in (0, \infty) \) a unique mild solution \( X \) which is H"older continuous with exponent \( \gamma = \min\{2, \frac{1}{\nu}\} \) with respect to the \( L^2(\Omega; H^1_0(0, 1)) \)-norm. Since the unique solution to (19) as defined in Section 3 coincides with the mild solution, cf. [12, Chapter 6], Assumption 4.3 is satisfied.

In contrast to Example 3.10, we do not project the initial value \( X_0 := \sin(\pi \cdot) \in V = H^1_0(0, 1) \) onto the subspace \( V_h \) by applying \( P_h \). Instead we make use of the interpolation operator (27), which is easier to implement. Hence, we set \( X^0_{k,h} := I_h(X_0) \in L^2(\Omega; V_h) \) as the first initial value for both schemes. From (28) we obtain the estimate

\[
\|X^0_{k,h} - X(t_0)\|^2_{L^2(\Omega; H)} \leq Ch^2\|X_0\|^2_{L^2(\Omega; V)}.
\]

As discussed in Remark 3.9, the second initial value \( X^1_{k,h} \) required for the BDF2-Maruyama scheme is computed by performing one step with the BEM scheme. Due to Remark 4.9 and the previous estimate, it holds

\[
\|X^1_{k,h} - X(t_1)\|^2_{L^2(\Omega; H)} \leq C(k + h^2 + \|P_h - \text{id}\|X(t_1)\|_{L^2(\Omega; H)}) \leq Ck + k\|\|P_h - \text{id}\|X(t_1)\|_{L^2(\Omega; V)}\|
\]

The best approximation property of \( P_h \) in \( V_h \) with respect to the norm in \( H \) and a further application of (28) yield

\[
\|\|P_h - \text{id}\|X(t_1)\|_{L^2(\Omega; H)} \leq \|\|I_h - \text{id}\|X(t_1)\|_{L^2(\Omega; V)}\| \leq Ch^2\|X(t_1)\|_{L^2(\Omega; H)} \leq Ch^2\|X(t_1)\|_{L^2(\Omega; V)}.
\]

Moreover, we deduce

\[
k\|\|P_h - \text{id}\|X(t_1)\|_{L^2(\Omega; V)} \leq k\|\|P_h\|\|V) + 1\|X(t_1)\|_{L^2(\Omega; V)}^2.
\]
Since \( X(t_1) \in L^2(\Omega; V) \) and the operator norm \( \|P_h\|_{L(V)} \) is uniformly bounded for \( h \in (0, 1) \), Assumption 4.2 is fulfilled with \( \gamma = 2 \). In particular, both considered numerical schemes are well-defined.

Altogether, this shows that Theorem 4.7 is applicable to problem (19). Hence, assuming that the spatial step size \( h \) is chosen sufficiently small, we expect in our temporal error analysis that the strong error of the BDF2-Maruyama scheme converges at least with rate \( \frac{1}{2} \) as \( k \to 0 \). Furthermore, the BEM scheme is also expected to converge at least with the same rate of \( \frac{1}{2} \), see [20, Theorem 6.1].

Let the temporal step size \( k \) and the spatial step size \( h \) be given and fixed. For our numerical experiments we have to project the Wiener process onto the Galerkin space \( V_h \), which requires the evaluation of the terms \( \langle \sigma \Delta_k W^{n,J}, \phi_i \rangle_H \) for each \( i = 1, \ldots, N_h \). We approximate these terms by a further application of the interpolation operator \( I_h \)

\[
(\sigma \Delta_k W^{n,J}, \phi_i)_H \approx (I_h(\sigma \Delta_k W^{n,J}), \phi_i)_H = \sigma \sum_{j=1}^{N_h} (\phi_i, \phi_j)_H \cdot \Delta_k W^{n,J}(x_j).
\]

Notice that \( \Delta_k W^{n,J} \) takes values in \( C^\infty([0, 1]) \) almost surely and the application of \( I_h \) is well-defined. In particular, its approximation error is sufficiently small with respect to the \( H \)-norm.

Our goal is then to determine a discrete process \( (X^n)_{n=1}^{N_h} \) consisting of random variables \( X^n: \Omega \to \mathbb{R}^{N_h} \) such that

\[
(31) \quad X_{k,h}^n = \sum_{i=1}^{N_h} X_i^n \phi_i
\]

holds in \( V_h \) for each \( n = 0, \ldots, N_h \). For this, let \( M_h = [(\phi_i, \phi_j)_H]_{i,j=1}^{N_h} \) and \( A_h = [\langle \phi_i, \phi_j \rangle_V]_{i,j=1}^{N_h} \) denote the mass matrix and the stiffness matrix arising from the finite element method. Then the reduced discrete systems of (19) for the BEM scheme and the BDF2-Maruyama scheme are given, respectively, by

\[
M_h(X^n - X^{n-1}) + kA_hX^n = \sigma M_h \Delta_k W^{n,J}
\]

and

\[
M_h(3X^n - 4X^{n-1} + X^{n-2}) + 2kA_hX^n = \sigma M_h(3\Delta_k W^{n,J} - \Delta_k W^{n-1,J}),
\]

where \( \Delta_k W^{n,J} = [W^J(t_n, x_i) - W^J(t_{n-1}, x_i)]_{i=1}^{N_h} \). The discrete systems of both schemes are linear in \( X^n \) and can be solved efficiently by using sparse matrix solvers. Due to the representation formula (31), the \( H \)-norm of the approximation \( X_{k,h}^n \) can be computed by

\[
\|X_{k,h}^n\|_H = \sqrt{(X^n)^T M_h X^n}.
\]

We consider \( T = 1 \) in all numerical experiments. In the first experiment, we simulate the deterministic heat equation (19) with \( \sigma = 0 \). The corresponding results in Table 1 show that the numerical error of the BDF2-Maruyama scheme is significantly smaller compared to the error of the BEM scheme for each level of the temporal discretization. Further, the margin between these errors increases for larger temporal step sizes \( k \) and the BDF2-Maruyama scheme converges twice as fast as indicated by the experimental order of convergence. Theses observations are in line with the well studied deterministic case, see, e.g., [34, Theorem 10.2].
In the following, we compare the two schemes for fixed noise intensity $\sigma = 1$ and varying spatial regularity of the Wiener process $W$ determined by the parameter $r \in \{0.1, 1.5\}$. The numerical results of the BEM scheme and the BDF2-Maruyama scheme are presented for each parameter value $r$ in Table 2 to Table 4, respectively.

In Table 2 we see that the errors of the BDF2-Maruyama scheme are only slightly smaller compared to those of the BEM scheme in the case of the least regular noise with $r = 0.1$. The values for the experimental order of convergence essentially agree for both schemes. This is in line with the expectation that a higher order temporal scheme does not provide an advantage if the exact solution is not sufficiently regular.

In Table 3 and Table 4, we notice that, in the case of more regular noise, the BDF2-Maruyama scheme yields significantly more accurate approximations in comparison to the BEM scheme. The observed EOC values of both schemes exceed the expected rate of $\frac{1}{2}$. However, this does not come as a surprise since we discretize an evolution equation with additive noise and both schemes coincide with their respective Milstein variants. In addition, observe that the BDF2-Maruyama scheme converges with a slightly higher rate when using coarse temporal grids with $N_k \in \{32, 64, 128\}$. Further, the accuracy of the BDF2-Maruyama scheme increases more clearly if the noise is more regular.

In conclusion, our numerical experiments indicate that the BDF2 scheme is superior to the BEM scheme, in particular, if the noise and, hence, the exact solution admit a certain regularity. Only in the case of less regular noise, both schemes perform equally well.

Table 2. Stochastic heat equation with $\sigma = 1$ and $r = 0.1$.

| $N_k$ | error  | CI ±  | EOC  | error  | CI ±  | EOC  |
|------|--------|-------|------|--------|-------|------|
| 32   | 0.067292 | 0.000396 | 0.055539 | 0.000299 |
| 64   | 0.043789 | 0.000209 | 0.036166 | 0.000163 | 0.62  |
| 128  | 0.029026 | 0.000114 | 0.024732 | 0.000094 | 0.55  |
| 256  | 0.019404 | 0.000064 | 0.016847 | 0.000055 | 0.55  |
| 512  | 0.013059 | 0.000037 | 0.011466 | 0.000031 | 0.56  |
| 1024 | 0.008803 | 0.000021 | 0.007773 | 0.000018 | 0.56  |
Table 3. Stochastic heat equation with $\sigma = 1$ and $r = 1.0$.

| $N_k$ | error   | CI ±   | EOC error | CI ±   | EOC ±   |
|-------|---------|--------|-----------|--------|---------|
| 32    | 0.048895| 0.000448| 0.034177  | 0.000305|         |
| 64    | 0.026680| 0.000226| 0.016160  | 0.000128| 1.08    |
| 128   | 0.014333| 0.000111| 0.008146  | 0.000055| 0.99    |
| 256   | 0.007569| 0.000055| 0.004345  | 0.000026| 0.91    |
| 512   | 0.003984| 0.000027| 0.002293  | 0.000012| 0.92    |
| 1024  | 0.002077| 0.000013| 0.001203  | 0.000006| 0.93    |

Table 4. Stochastic heat equation with $\sigma = 1$ and $r = 5.0$.

| $N_k$ | error   | CI ±   | EOC error | CI ±   | EOC ±   |
|-------|---------|--------|-----------|--------|---------|
| 32    | 0.044139| 0.000471| 0.029223  | 0.000356|         |
| 64    | 0.023424| 0.000249| 0.012110  | 0.000152| 1.27    |
| 128   | 0.012039| 0.000125| 0.005206  | 0.000072| 1.22    |
| 256   | 0.006154| 0.000064| 0.002579  | 0.000036| 1.01    |
| 512   | 0.003093| 0.000032| 0.001282  | 0.000017| 1.01    |
| 1024  | 0.001563| 0.000016| 0.000640  | 0.000009| 1.00    |

5.2. A Nonlinear SPDE with multiplicative noise. In this subsection, we consider the quasilinear stochastic partial differential equation

$$ \frac{du(t,x) - \left( \psi(|u_x(t,x)|) \cdot u_x(t,x) \right)_x}{2} dt = \sigma \sqrt{8u(t,x)}^2 + 1 dW(t,x), \quad (t,x) \in (0, T] \times (0,1), $$

where we again consider the Gelfand triple with $V = H^1_0(0,1)$ and $H = L^2(0,1)$. Moreover, the operator $A$ is globally Lipschitz continuous and strongly monotone.
In this numerical experiment, we choose the function \( \psi: \mathbb{R}^+_0 \to \mathbb{R} \) to be \( \psi(t) = \text{erf}(t - 2) + 2 \), where \( \text{erf} \) denotes the error function
\[
\text{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-s^2} \, ds, \quad t \in \mathbb{R}.
\]

As before, the parameter \( \sigma \in \mathbb{R} \) determines the intensity of the nonlinear multiplicative noise in (32). The Wiener process \( W \) is assumed to take values in \( H^1_0(0, 1) \) almost surely and its approximation \( W^{(J)} \) is defined in the same way as in (29). The corresponding abstract operator for the multiplicative noise is given by
\[
B: V \to L_2(U_0, H), \quad v \mapsto \sigma \sqrt{8|v|^2 + 1} \cdot \text{id}_H.
\]
It holds
\[
\|B(v)\|_{L_2(U_0, H)}^2 = \sigma^2 \cdot \sum_{j \in \mathbb{N}} \|\sqrt{8|v|^2 + 1}Q_j^2 H_j\|^2_H
\leq \sigma^2 \left( \sum_{j \in \mathbb{N}} q_j \|H_j\|_{C([0,1])}^2 \|\sqrt{8|v|^2 + 1}\|^2_{L^2(0,1)} \right)
= 2\sigma^2 \text{Tr}(Q)(8|v|^2_H + 1)
\]
for every \( v \in V \). This ensures that the operator \( B \) is well-defined and bounded. In addition, notice that the mapping \( g: \mathbb{R} \to \mathbb{R}, x \mapsto \sqrt{8x^2 + 1} \) is Lipschitz continuous with Lipschitz constant \( L = \sqrt{8} \). Therefore, we obtain
\[
\|B(v) - B(u)\|_{L_2(U_0, H)}^2 = \sigma^2 \cdot \sum_{j \in \mathbb{N}} \|\sqrt{8|v|^2 + 1} - \sqrt{8|u|^2 + 1}Q_j^2 H_j\|^2_H
\leq \sigma^2 \left( \sum_{j \in \mathbb{N}} q_j \|H_j\|_{C([0,1])}^2 \|g(v) - g(u)\|^2_{L^2(0,1)} \right)
\leq 16\sigma^2 \text{Tr}(Q)\|v - u\|_{H}^2
\]
for all \( v, u \in V \).

Since the operators \( A: V \to V^* \) and \( B: V \to L_2(U_0, H) \) are Lipschitz continuous, both are \( \mathcal{B}(V) \)-measurable. The Lipschitz continuity of the operator \( A \) also implies that \( A \) is hemi-continuous and grows linearly with \( p = 2 \). Further, it holds \( \langle A(v), v \rangle_{V^* \times V} \geq m_1\|v\|_V^2 \) for all \( v \in V \). Together with the estimate (33) this yields that the coercivity condition (10) holds with \( \kappa = 16\nu\sigma^2 \text{Tr}(Q), c = 2\nu\sigma^2 \text{Tr}(Q) \) and \( \mu = m_1 \) for any \( \nu \in (1, \infty) \). The strong monotonicity of \( A \) and the estimate (34) imply that the stronger monotonicity condition (20) is satisfied with \( K = 2m_1 > 0 \). Hence Assumption 4.1 is fulfilled.

The smooth initial value \( X_0 = \sin(\pi \cdot) \) satisfies \( X_0 \in V \). However, in case of this nonlinear problem, sufficient regularity properties of the exact solution could neither be proven nor found in the literature.

The initial values for the schemes (3) and (2) are computed in the same way as in Subsection 5.1. Notice that the operator \( B: H \to L_2(U_0, H) \) is Lipschitz continuous due to estimate (34) and hence the consistency of the initial values with respect to the \( H \)-norm is sufficient for Assumption 4.2 to be satisfied.
As before, the projection of the noise term on the Galerkin space $V_h$ is realized by applying the interpolation operator $I_h$ such that for each $i = 1, \ldots, N_h$

$$(B(X_{k,h}^{n-1}) \Delta_k W^{n,J}, \phi_i)_H \approx (I_h(B(X_{k,h}^{n-1}) \Delta_k W^{n,J}), \phi_i)_H$$

$$= \sigma \sum_{j=1}^{N_h} \langle \phi_i, \phi_j \rangle_H \cdot [B(X_{k,h}^{n-1}(x_j)) \Delta_k W^{n,J}(x_j)].$$

Since the mapping $g$ is continuously differentiable and $X_{k,h}^{n-1}$ is $V$-valued, the composition $g \circ X_{k,h}^{n-1}$ is also $V$-valued, compare, e.g., with [5, Corollary 8.11]. Moreover, the discrete Wiener increment $\Delta_k W^{n,J}$ is smooth and hence the term $B(X_{k,h}^{n-1}) \Delta_k W^{n,J}$ is $V$-valued as well as continuously embedded into $C([0,1])$. This ensures that this approximation is well-defined and, by estimate (28), the corresponding interpolation error is of order $O(h)$.

By identifying $X_{k,h}^n$ with the random $\mathbb{R}^{N_h}$-valued vector $X^n$ through the formula (31) for each $n = 0, \ldots, N_k$, we define the stiffness matrix by

$$A_h(X^n) = [(\psi((X_{k,h}^n)'_i, \phi'_j)_{L^2(0,1)})_{i,j=1}^{N_h} \text{ with } (X_{k,h}^n)' = \sum_{i=1}^{N_h} X^n_i \phi'_i,$$

and introduce the notation

$$B(X^{n-1}) \Delta_k W^{n,J} = \left[ (8|X^{n-1}|^2 + 1)^{\frac{1}{2}} (W^J(t_n, x_i) - W^J(t_{n-1}, x_i)) \right]_{i=1}^{N_h}.$$

Since $X_{k,h}^n \in V_h$ is piecewise linear, the corresponding derivative $(X_{k,h}^n)'$ is piecewise constant and can be directly implemented without applying any quadrature. The reduced discrete systems of (32) for the BEM scheme and the BDF2-Maruyama scheme are given, respectively, by

$$M_h(X^n - X^{n-1}) + k A_h(X^n) X^n = \sigma M_h(B(X^{n-1}) \Delta_k W^{n,J})$$

and

$$M_h(3X^n - 4X^{n-1} + X^{n-2}) + 2k A_h(X^n) X^n$$

$$= \sigma M_h(3B(X^{n-1}) \Delta_k W^{n,J} - B(X^{n-2}) \Delta_k W^{n-1,J}).$$

Since the discrete systems of both schemes are nonlinear, we solve for $X^n$ by applying Newton’s method with $N$ iterations in each temporal step. In more detail, we set $\tilde{X}^0 = X^{n-1}$ and compute iteratively $\tilde{X}^l$ for each $l \in \{1, \ldots, N\}$ by

$$J_{k,h}(\tilde{X}^{l-1})(\tilde{X}^l - \tilde{X}^{l-1}) = -F^n_{k,h}(\tilde{X}^{l-1}),$$

where $F^n_{k,h}$ and its Jacobian $J_{k,h}$ are given for the BEM scheme and the BDF2-Maruyama scheme, respectively, by

$$F^n_{k,h}(X) = (M_h + k A_h(X)) X - M_h(X^{n-1} + \sigma B(X^{n-1}) \Delta_k W^{n,J})$$

and

$$J_{k,h}(X) = M_h + k A_h^+(X)$$

and

$$F^n_{k,h}(X) = (3M_h + 2k A_h(X)) X - M_h(4X^{n-1} - X^{n-2})$$

$$- \sigma M_h(3B(X^{n-1}) \Delta_k W^n - B(X^{n-2}) \Delta_k W^{n-1,J})$$

$$J_{k,h}(X) = 3M_h + 2k A_h^+(X).$$
Hereby, $A^*_h$ denotes the Jacobian of the mapping $X \mapsto A_h(X)X$. A short computation yields

$$A^*_h(X) = \left[ \left( [\psi(|X'|) + \psi'(|X'|)|X'|] \phi'_i, \phi'_j \right)_{L^2(0,1)} \right]_{i,j=1}^{N_h} \quad \text{with} \quad X' = \sum_{l=1}^{N_h} X_l \phi'_l.$$

The number of iterations $N$ is at least $N_{min} = 3$ and is increased up to $N_{max} = 10$ as long as the current residual exceeds the tolerance limit $tol = 10^{-12}$.

As before, we consider $T = 1$ throughout all numerical experiments. First, we simulate (32) without noise by setting $\sigma = 0$. The corresponding results in Table 5 show that the BDF2-Maruyama scheme yields a more accurate approximation and converges faster for smaller temporal step sizes $k$ in comparison to the BEM scheme.

| BEM | BDF2 |
|-----|------|
| $N_k$ | error  | EOC | error  | EOC |
| 32  | 0.066045 | 0.040309 |
| 64  | 0.040482 | 0.71 | 0.025797 | 0.64 |
| 128 | 0.022121 | 0.87 | 0.012795 | 1.01 |
| 256 | 0.011636 | 0.93 | 0.005395 | 1.25 |
| 512 | 0.005996 | 0.96 | 0.002478 | 1.12 |
| 1024| 0.003038 | 0.98 | 0.000994 | 1.32 |

In addition, we simulate (32) with different noise intensities by choosing $\sigma \in \{0.25, 0.75\}$. In Table 6 we observe that for smaller noise intensity with $\sigma = 0.25$ the approximation results behave very similar to the deterministic case. In particular, the BDF2-Maruyama scheme provides more favourable results. On the contrary, we notice in Table 7 that the advantage of the BDF2-Maruyama scheme over the BEM scheme for larger noise intensity with $\sigma = 0.75$ is barely noticeable and diminishes as the temporal step size decreases.

In conclusion, the numerical experiments indicate that our theoretical results are indeed applicable to this nonlinear stochastic partial differential equation. In case of small noise intensity, the BDF2-Maruyama scheme performs significantly better than the BEM scheme for similar temporal refinement levels. The margin is less significant for large noise intensity though.

| BEM | BDF2 |
|-----|------|
| $N_k$ | error  | CI $\pm$ EOC | error  | CI $\pm$ EOC |
| 32  | 0.067400 | 0.000226 | 0.043252 | 0.000250 |
| 64  | 0.041681 | 0.000138 | 0.69 | 0.027827 | 0.000150 | 0.64 |
| 128 | 0.022931 | 0.000078 | 0.86 | 0.014031 | 0.000073 | 0.99 |
| 256 | 0.011636 | 0.91 | 0.005395 | 0.000027 | 1.25 |
| 512 | 0.005996 | 0.96 | 0.002478 | 0.000013 | 1.12 |
| 1024| 0.003038 | 0.98 | 0.000994 | 0.13 |
Table 7. Nonlinear stochastic PDE (32) with σ = 0.75.

| N_k | BEM error | CI ± | EOC error | CI ± | EOC |
|-----|-----------|------|-----------|------|-----|
| 32  | 0.092583  | 0.000849 | 0.081076  | 0.001099 |
| 64  | 0.061146  | 0.000686 | 0.055266  | 0.000790 | 0.55 |
| 128 | 0.038248  | 0.000505 | 0.035124  | 0.000648 | 0.65 |
| 256 | 0.024065  | 0.000338 | 0.022407  | 0.000389 | 0.65 |
| 512 | 0.015567  | 0.000239 | 0.014762  | 0.000259 | 0.60 |
| 1024| 0.010278  | 0.000159 | 0.009875  | 0.000163 | 0.58 |

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