HIGH POINTS OF A RANDOM MODEL OF THE RIEMANN-ZETA FUNCTION AND GAUSSIAN MULTIPLICATIVE CHAOS
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ABSTRACT. We study the total mass of high points in a random model for the Riemann-Zeta function. We consider the same model as in [8, 2], and build on the convergence to ‘Gaussian’ multiplicative chaos proved in [14]. We show that the total mass of points which are a linear order below the maximum divided by their expectation converges almost surely to the Gaussian multiplicative chaos of the approximating Gaussian process times a random function. We use the second moment method together with a branching approximation to establish this convergence.

1. INTRODUCTION

1.1. The model. Let \( P \) denote the set of all prime numbers. Let \( (\theta_p)_{p \in P} \) be independent identically distributed random variables, being uniformly distributed on \([0, 2\pi]\). For \( N \in \mathbb{N} \), a good model for the large values of the logarithm of the Riemann-zeta function on a typical interval of length 1 of the critical line as proposed in [8] is

\[
X_N(x) = \sum_{j=1}^{N} \frac{1}{\sqrt{p_j}} \left( \cos(x \ln p_j) \cos(\theta_{p_j}) + \sin(x \ln p_j) \sin(\theta_{p_j}) \right) \quad x \in [0, 1].
\]

By Theorem 7 in [14], the process \( X_N \) can be well approximated by a log-correlated Gaussian field \( G_N(x), x \in [0, 1] \). Namely, take

\[
G_N(x) = \sum_{j=1}^{N} \frac{1}{2 \sqrt{p_j}} \left( W_j^{(1)} \cos(x \ln p_j) + W_j^{(2)} \sin(x \ln p_j) \right),
\]

where \( (W_j^{(i)})_{j \in \mathbb{N}, i \in \{1, 2\}} \) are i.i.d. standard normal distributed. It is shown in [14] that

\[
E_N(x) = G_N(x) \equiv E_N(x), \quad x \in [0, 1],
\]

where \( E_N(x) \) converges almost surely uniformly to a random function \( E(x) \). Moreover, the error \( E_N(x) \) has uniform exponential moments

\[
\mathbb{E} \left( e^{\lambda \sup_{x \in [0, 1]} E_N(x)} \right) < \infty,
\]

where \( \mathbb{E} \) denotes expectation with respect to the \( \theta_{p_j} \)'s.

Some of the behavior of the large values of the process \( X_N(x), x \in [0, 1] \) is captured by the random measure

\[
M_{\alpha,N}(dx) = \frac{e^{\alpha X_N(x)}}{\mathbb{E} e^{\alpha X_N(x)}} dx.
\]
By the independence of the $\theta_p$’s, it is not hard to see that $M_{a,N}$ converges almost surely as $N \to \infty$. By Theorem 4 in [14], the almost sure weak limit of $M_{a,N}(dx)$ is non-trivial for $0 < \alpha < 2$. We denote the limit of the total mass by $M_{a}$

\[
M_{a} = \lim_{N \to \infty} \int_{0}^{1} M_{a,N}(dx) \text{ a.s.}
\]

For log-correlated Gaussian field the analogous limiting measure is called Gaussian multiplicative chaos and $M_{a}$ corresponds to the total mass of the limiting measure. For Gaussian multiplicative chaos it was first proven by [9] that the limit is non-trivial for small $\alpha$ and was recently revisited (see [13, 12]). Note that in our case the limit of $M_{a,N}(dx)$ is almost a Gaussian multiplicative measure (see [14]). The connection between the Riemann-zeta function and Gaussian multiplicative chaos has been further analysed in [15].

The fact that the Riemann-zeta function (or a random model of it) can be well approximated by a log-correlated field have recently been used to study the extremes on a random interval \[4, 11, 2\].

1.2. Main result. Consider the Lebesgue measure of $\alpha$-high points:

\[
W_{a,N} = \text{Leb}\{X_N(x) > \frac{\alpha}{2} \ln \ln N\}
\]

The main result of this note is to relate the limit $M_{a}$ to the Lebesgue measure of high points building on the ideas of [7]:

**Theorem 1.1.** For any $0 < \alpha < 2$ and $M_{a}$ as in (1.6), we have

\[
\frac{W_{a,N}}{\mathbb{E}(W_{a,N})} \to M_{a},
\]

in probability as $N \to \infty$.

It was proved in [2] that the maximum of $X_N(x)$ on $[0, 1]$ is $\ln \ln N - (3/4 \pm \epsilon) \ln \ln \ln N$ with large probability. In view of this and of Theorem 1.1, it is not surprising to see that the $M_{a}$ is non-trivial for $\alpha < 2$. The critical case where $\alpha \to 2$ is interesting as it is related to the fluctuations of the maximum of $X_N$. It is reasonable to expect that our approach can be adapted to the method of [5] to prove the critical case. Another upshot of the proof is that it highlights the fact that $M_{a}$ depends on small primes, cf. Lemma 2.1.

The problem for the Riemann-zeta function is trickier. We expect that the equivalent of Theorem 1.1 still holds:

**Conjecture 1.2.** Let $\tau$ be a uniform random variable on $[T, 2T]$. Let $W_{a,T} = \text{Leb}\{h \in [0, 1] : \ln |\zeta(1/2 + i(\tau + h))| > \frac{\alpha}{2} \ln \ln T\}$. Then we have

\[
\lim_{T \to \infty} \frac{W_{a,T}}{\mathbb{E}[W_{a,T}]} = \lim_{T \to \infty} \frac{\int_{0}^{1} |\zeta(1/2 + i(\tau + h))|^{\alpha}}{\mathbb{E}[|\zeta(1/2 + i\tau)|^{\alpha}]} \text{ a.s.}
\]

This would be consistent with the conjecture of Fyodorov & Keating for the Lebesgue measure of high points, see Section 2.5 in [6]. One issue is that it is not obvious that a result akin to Equation (1.3) holds, mainly because of the singularities of $\ln \zeta$ at the zeros. One way around this would be to restrict to Gaussian comparison to one-point and two-point large deviation estimates. This seems doable in view of Lemmas 3.2 and 3.3 and the Gaussian comparison theorem proved for the zeta function in [1].
1.3. **Outline of the proof.** The proof of Theorem 1.1 is based on a first and a second moment estimate and follow the global strategy proposed in [7] for branching Brownian motion. First, we prove convergence of a conditional first moment to the desired limiting object in Lemma 2.1. Its proof builds on results on the Gaussian comparison and convergence to Gaussian multiplicative chaos established in [14]. Next, a localisation result is established in Lemma 2.2. Finally, we turn to the proof of Proposition 3.4 which is based on a second moment computation. We use a branching approximation similar to the one employed in [2]. Using the obtained first and second moment estimates we are finally in the position to prove Theorem 1.1.

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2. **First moment estimates**

For $R \leq N$, we define $\mathcal{F}_R$ to be the $\sigma$-algebra generated by $(\theta_p)_{p \leq R}$. We will often condition on $\mathcal{F}_R$ to fix the dependence on the small primes. The variance of $G_N(x) - G_R(x)$, $x \in [0, 1]$ is by definition

\[ \sigma^2_R(N) \equiv \text{Var}(G_N(x) - G_R(x)) = \frac{1}{2} \sum_{R < p \leq N} p^{-1} \]

The prime number theorem, see e.g. [10], implies that the density of the primes goes like $(\ln p)^{-1}$. More precisely, we have

\[ \sigma^2_R(N) = \left| \sum_{R < p \leq N} p^{-1} - \frac{1}{2} (\ln \ln N - \ln \ln R) \right| = o(1) \text{ as } N \to \infty \text{ and } R \to \infty. \]

It turns out that the non-trivial contribution to Theorem 1.1 comes from the small primes.

**Lemma 2.1.** For $W_{\alpha,N}$ as in (1.7), we have for $0 < \alpha < 2$

\[ \lim_{R \to \infty} \lim_{N \to \infty} \frac{\mathbb{E}(W_{\alpha,N}|\mathcal{F}_R)}{\mathbb{E}(W_{\alpha,N})} = M_{\alpha} \text{ a.s.} \]

**Proof.** We start by computing $\mathbb{E}(W_{\alpha,N}|\mathcal{F}_R)$. Using Fubini’s Theorem we can write the left-hand side of (2.3) as

\[ \int_0^1 \mathbb{P}\left(X_N(x) > \frac{\alpha}{2} \ln \ln N \mid \mathcal{F}_R\right) dx \]

\[ = \int_0^1 \mathbb{P}\left(G_N(x) - G_R(x) + (E_N(x) - E_R(x)) > \frac{\alpha}{2} \ln \ln N - G_R(x) - E_R(x) \mid \mathcal{F}_R\right) dx, \]

where we used (1.3). Moreover, again for each $\epsilon > 0$ there is $R_0$ such that for all $R \geq R_0$ $|E_R(x) - E_N(x)| < \epsilon$ almost surely and uniformly in $x$. Hence, we can again upper bound (2.4) by

\[ \int_0^1 \mathbb{P}\left(G_N(x) - G_R(x) > \frac{\alpha}{2} \ln \ln N - G_R(x) - E_R(x) - \epsilon \mid \mathcal{F}_R\right) dx, \]

and a corresponding lower by replacing $\epsilon$ by $-\epsilon$. Next, observe that by definition of $X_N(x)$ and $E_N(x)$, $G_N(x) - G_R(x)$ are independent of $\mathcal{F}_R$. We have that the probability in (2.5) is
bounded from above by
\[
\frac{\sigma_r(N)}{\sqrt{2\pi(\alpha \ln N - G_R(x) - E_R(x) - \epsilon)}} \exp \left( - \frac{(\frac{1}{2} \ln \ln N - G_R(x) - E_R(x) - \epsilon)^2}{2\sigma^2_r(N)} \right)
\]
(2.6)  
\[= \frac{\sigma_r(N)}{\sqrt{2\pi(\alpha \ln N)}} \exp \left( - \frac{\alpha^2(\ln \ln N)^2}{8\sigma^2_r(N)} + \alpha(E_R(x) + G_R(x) + \epsilon) \right) (1 + o(1)), \]

Next, we turn to \( \mathbb{E}(W_{a,N}) \). We have that
\[
\mathbb{E}(W_{a,N}) = \mathbb{E}(\mathbb{E}(W_{a,N}|\mathcal{F}_R)) \leq \frac{\sigma_r(N)}{\sqrt{2\pi(\alpha \ln N)}} \exp \left( - \frac{\alpha^2(\ln \ln N)^2}{8\sigma^2_r(N)} \right)
\]
(2.7) 
\[
\times \int_0^1 \mathbb{E}(\exp(\alpha(E_R(x) + G_R(x) - \epsilon))) \, dx (1 + o(1))
\]
A corresponding lower bound we obtain by replacing \( \epsilon \) by \(-\epsilon\). Taking the quotient of (2.6) and (2.7) and integrating with respect to \( x \) we get
\[
\frac{\int_0^1 \exp(\alpha(E_R(x) + G_R(x) + \epsilon)) \, dx}{\mathbb{E}(\mathbb{E}(W_{a,N}|\mathcal{F}_R))} \leq \frac{\int_0^1 \exp(\alpha(E_R(x) + G_R(x) - \epsilon)) \, dx}{\int_0^1 \mathbb{E}(\exp(\alpha(E_R(x) + G_R(x) + \epsilon))) \, dx} (1 + o(1)),
\]
(2.8)
Pulling the terms involving \( \epsilon \) out of the integral and noting the normalization of \( M_{a,R} \) is chosen such that \( \mathbb{E}M_{a,R} = 1 \) and noting that
\[
\mathbb{E} \left( \frac{\int_0^1 \exp(\alpha(E_R(x) + G_R(x))) \, dx}{\int_0^1 \mathbb{E}(\exp(\alpha(E_R(x) + G_R(x)))) \, dx} \right) = 1,
\]
we can rewrite (2.7) as
\[
M_{a,R} e^{2\alpha \epsilon} (1 + o(1)) \leq \frac{\mathbb{E}(W_{a,N}|\mathcal{F}_R)}{\mathbb{E}(W_{a,N})} \leq M_{a,R} e^{-2\alpha \epsilon} (1 + o(1)).
\]
(2.10)
Note that (2.10) holds for all \( \epsilon > 0 \). When taking \( N, R \uparrow \infty \) \( M_{a,R} \) converges a.s. to \( M_a \) hence we have a.s.
\[
M_{a} e^{2\alpha \epsilon} (1 + o(1)) \leq \liminf_{N,R \to \infty} \frac{\mathbb{E}(W_{a,N}|\mathcal{F}_R)}{\mathbb{E}(W_{a,N})} \leq \limsup_{N,R \to \infty} \frac{\mathbb{E}(W_{a,N}|\mathcal{F}_R)}{\mathbb{E}(W_{a,N})} \leq M_{a} e^{-2\alpha \epsilon} (1 + o(1)).
\]
As (2.11) does not depend on \( r \) and \( N \) anymore, we can take the limit as \( \epsilon \to 0 \) and obtain
\[
\lim_{R \to \infty} \lim_{N \to \infty} \frac{\mathbb{E}(W_{a,N}|\mathcal{F}_R)}{\mathbb{E}(W_{a,N})} = M_a.
\]
(2.12)

Next, we want to control
\[
W_{a,N} = \text{Leb}\{x \in [0,1] : X_N(x) \geq \alpha \ln \ln N; \exists k \in [R,N] : X_k(x) > (\alpha + \epsilon) \ln \ln k\}.
\]
(2.13)
The idea is that, at high points, the value \( X_N(x) \) is most likely shared equally by the increments as defined in (3.1) below.

\( \square \)
Lemma 2.2. For all \( \epsilon > 0 \) there exists \( R_0 \) such that for all \( R = o(N) \) and \( R, N > R_0 \) such that for all \( c > 0 \)

\[
\mathbb{P} \left( W_{a, N}^c > c\mathbb{E}W_{a, N} \right) \leq e^{-er},
\]

where \( r = \ln \ln R \).

Proof. We want to use Markov’s inequality to bound the probability on (2.14). Hence, we need to bound \( \mathbb{E}W_{a, N}^c \) from above. First, we bound \( \mathbb{E} \left( W_{a, N}^c \big| \mathcal{F}_R \right) \) from above by

\[
\int_0^1 \mathbb{P} \left( \{ G_N(x) - G_R(x) \geq \alpha \ln \ln N - X_R(x) - \epsilon \} \right.
\]

\[
\times \mathbb{P} \left( \exists K \in [R, N] : G_K(x) - G_R(x) > (\alpha + \epsilon) \ln \ln K - X_R(x) - \epsilon' \right) \bigg| \mathcal{F}_R \bigg),
\]

where we used (1.3) and the fact that \( E_R(x) \) converges a.s. uniformly to a continuous function \( E(x) \). Hence, for all \( \epsilon' > 0 \) there is \( R_0 \) such that for all \( K \geq R_0 \) and all \( x \) we have \( |E_K(x) - E_R(x)| < \epsilon' \).

Similarly as in (2.1), the variable \( G_K(x) - G_R(x) \) is Gaussian with mean 0 and variance

\[
\sigma_K^2 = \frac{1}{2} \sum_{k<p\leq K} p^{-1}.
\]

Let

\[
B_K(x) = G_K(x) - G_R(x) - \frac{\sigma_K^2(K)}{\sigma_K^2}(G_N(x) - G_R(x)),
\]

then \( (B_K(x))_{k=1}^n \) are points on a time-changed brownian bridge from zero to zero in time \( \sigma_K^2(N)^2 \). As a Brownian bridge is independent from its endpoint, Equation (2.15) is equal to

\[
\int_0^1 \int_{\frac{2}{\ln \ln N - X_R(x) - \epsilon'}}^\infty \mathbb{P} \left( G_N(x) - G_R(x) \in \mathcal{F}(dy) \right)
\]

\[
\times \mathbb{P} \left( \exists K \in [R, N] : B_K(x) > \frac{\alpha + \epsilon}{2} \ln \ln K - X_R(x) - \epsilon' - \frac{\sigma_K^2(K)}{\sigma_K^2(N)^2} y \bigg| \mathcal{F}_R \right) dx
\]

as \( |\sum_{p<k} p^{-1} - \ln \ln K| < C \). Let \( r = \ln \ln R \) and \( n = \ln \ln N \). Next, let us control the probability that \( X_R(x) \) is too large.

\[
\mathbb{P} \left( X_R(x) \geq \frac{er}{3} \right) \leq \mathbb{P} \left( G_R(x) \geq \frac{er}{4} \right) + \mathbb{P} \left( E_R(x) \geq \frac{er}{12} \right).
\]

The second probability in (2.18) is bounded by \( Ce^{-\frac{er}{2}} \) by (1.4). For the first probability in (2.18) is bounded by \( Ce^{-\frac{er}{2}} \) by Gaussian tail asymptotics and the variance estimate for \( G_R(x) \) for \( r \) large enough and uniformly in \( x \). On the event that \( \{X_R(x) \leq \frac{er}{8} \} \) we can bound the second probability in (2.17) from above by

\[
\mathbb{P} \left( \exists s \in [0, \sigma^2(N)] : b(s) > (\alpha + \epsilon) \left( (s + \sigma_0(R^2) - C) - \frac{er}{3} - \epsilon' - \frac{s}{\sigma^2(N)} y \right) \right),
\]

where \( b(s) \) is a Brownian bridge from zero to zero in time \( \sigma^2(N) \). Consider the line \( l \) from

\( (0, \frac{er}{3}) \) to \( (\sigma^2(N), (\alpha + \epsilon)(n/2 - C) - \epsilon' - y) \). One checks that \( l(s) \leq (\alpha+\epsilon) \left((s + \sigma_0(R^2) - C) - \frac{er}{3} - \epsilon' - \frac{s}{\sigma^2(N)} y \right) \).
By definition, we have
\[ \mathbb{P}(\exists s \in [0, \sigma^2_t(N)] : b(s) > k(s)) = \exp\left(-2k(0)k\left(\frac{\sigma^2_t(N)}{\sigma^2_t(N)}\right)\right) \]

Hence, on the event we can bound the expectation of (2.17) by
\[ \int_0^1 \mathbb{E}\left(\int_{\frac{1}{2} \ln \ln N}^{\infty} \mathbb{P}(G_N(x) - G_R(x) < dy)e^{-\frac{\|r(2\ln(\frac{1}{2}) + \sigma^2_t(N))\|}{2(\ln - x)}} \right) \theta + o(1))dx + Ce^{-\frac{1}{2}O(W_{\sigma,N})} \]

Using the Gaussian tail asymptotics for \(G_N(x) - G_R(x)\) together with (2.7), Equation (2.21) is bounded above by
\[ \mathbb{E}(W_{\sigma,N})e^{-\frac{1}{2}O(W + o(1))} \]
This implies the claim of Lemma 2.2. \(\square\)

3. Branching approximation and second moment estimates

3.1. Definition of the increments. The goal is to use a branching approximation similar to [2] to compute the necessary second moments. To this end, we define for \(k \in \mathbb{N}\) and \(x \in (0, 1)\)
\[ Y_k(x) = \sum_{\epsilon_1 < \epsilon_2 \leq \epsilon_3} \frac{1}{2\sqrt{p_j}} \left( W_j^{(1)} \cos(x \ln p_j) + W_j^{(2)} \sin(x \ln p_j) \right) \]
By definition, we have
\[ G_N(x) = \sum_{k=1}^{n} Y_k(x) \]
where for the rest of the section we set \(n = \ln \ln N\). The increments \(Y_k\) are such that
\[ \rho_k(x, x') = \mathbb{E}(Y_k(x)Y_k(x')) = \sum_{\epsilon_1 < \epsilon_2 \leq \epsilon_3} \frac{1}{2p} \cos(|x - x'| \ln p_j). \]
The covariances can be computed again by the prime number theorem. This is done in Lemma 2.1 in [2]. It is convenient to state the result to introduce branching point of \(x, x' \in (0, 1)\) by
\[ x \wedge x' \equiv \lfloor \ln |x - x'|^{-1} \rfloor. \]

**Lemma 3.1** (Lemma 2.1 in [2]). For \(k \geq 1\) and \(x, x' \in (0, 1)\) we have
\[ \mathbb{E}(Y_k^2(x)) = \frac{1}{2} + O(e^{-c \sqrt{x}}). \]
and
\[ \rho_k(x, x') = \begin{cases} \frac{1}{2} + O\left(e^{-2(k \wedge x' - k)}\right) + O\left(e^{-c \sqrt{x}}\right) & \text{if } k \leq x \wedge x', \\ O\left(e^{-2(k \wedge x' - k)}\right) + O\left(e^{-c \sqrt{x}}\right) & \text{if } k > x \wedge x' \end{cases} \]

There is a fast decoupling between the increments after the branching point where the distribution of \(Y_k(x)\) and \(Y_k(x')\) is very close to independent Gaussians with mean zero and variance 1/2. We introduce a parameter \(\Lambda\) that gives some room before and after the branching point to ensure uniform estimates.
Lemma 3.2. Let $\Delta > 0$. Let $x, x' \in (0, 1)$ and $m > x \wedge x' + \Delta$. Then we have

$$
\mathbb{P}
\left(
\bigg| \sum_{k=m+1}^{n} Y_k(x) - \sum_{k=m+1}^{n} Y_{k}(x') \bigg|
\right)

\leq
\mathbb{P}
\left(
\bigg| \sum_{k=m+1}^{n} Y_k \bigg|
\right)

\leq
\mathbb{P}
\left(
\bigg| \sum_{k=m+1}^{n} Y_{k} \bigg|
\right)

\left(1 + O(e^{-\epsilon})\right),

$$

where $(Y_i)_{i \in \mathbb{N}}$ are iid Gaussians with mean zero and variance $\sigma^2$.

Proof. As $(\sum_{k=m+1}^{n} Y_k(x), \sum_{k=m+1}^{n} Y_{k}(x'))$ is a Gaussian process and its covariance is controlled in Lemma 3.1 it suffices to compare densities. This follows the same lines starting from Eq. (61) in [2] only that in our setting $\mu = 0$.

Before the branching point we want to show that $Y_k(x)$ and $Y_k(x')$ are almost fully correlated. This is specified in the lemma below.

Lemma 3.3. Let $\Delta > 0$. Let $x, x' \in (0, 1)$ and $r < m < x \wedge x' - \Delta$. Then we have

$$
\mathbb{P}
\left(
\bigg| \sum_{k=r}^{m} Y_k(x) - \sum_{k=r}^{m} Y_{k}(x') \bigg|
\right)

\leq
\mathbb{P}
\left(
\bigg| \sum_{k=r}^{m} Y_k \bigg|
\right)

\leq
\mathbb{P}
\left(
\bigg| \sum_{k=r}^{m} Y_{k} \bigg|
\right)

\left(1 + O(e^{-\epsilon})\right),

$$

Proof. As $G$ is a Gaussian process this follows from the density estimates in Lemma 3.1.

3.2. Second moment computation. The main result of this section is:

Proposition 3.4. There exists $\kappa_\alpha > 0$ such that for $R = o(\ln \ln N)$ as $N \to \infty$ we have

$$
\mathbb{P}
\left(
\frac{W_{\alpha,N} - \mathbb{E}(W_{\alpha,N} | F_R)}{\mathbb{E}(W_{\alpha,N})}
> c
\right)

\leq
\left(1 + o(1)\right)e^{-\kappa_\alpha r},

$$

where $r \equiv \ln \ln R$ and $C > 0$ a constant depending on $c$.

To prove Proposition 3.4 we essentially need to control the second moment of

$$
W_{\alpha,N}^2 = \text{Leb}\{x \in [0, 1] : \sum_{j \leq n} Y_j(x) \geq \alpha n/2; \forall k \in [2r,n] : \sum_{j \geq k} Y_j(x) \leq (\alpha + \epsilon)k/2\}.
$$

Remark. Throughout the proof we restrict our computations to $R$ and $N$ such that $r = \ln \ln R$ and $n = \ln \ln N$ are natural numbers. The general case follows in the same way by considering the last resp. first summands in the representation in (3.2) of $G_N$ separately. The desired estimates carry over by minor adjustments but would require a more involved notation. To keep the computations that follow as clear as possible and not to burden the reader with heavier notations we restrict ourselves to the case where $r, n \in \mathbb{N}$.

Indeed, Markov’s inequality and Lemma 2.2 imply

$$
\mathbb{P}
\left(
\left|\frac{W_{\alpha,N} - \mathbb{E}(W_{\alpha,N} | F_R)}{\mathbb{E}(W_{\alpha,N})}\right|
> c
\right)

\leq
\left(\mathbb{E}(W_{\alpha,N}^2 - \mathbb{E}(W_{\alpha,N}^2 | F_R))^{2} \mathbb{E}(W_{\alpha,N})^2 \right)

\leq
\frac{c^2}{4} + Ce^{-R(c)}.
$$

Clearly, we have

$$
(W_{\alpha,N}^2)^2 = \text{Leb}^2\{x, x' \in [0, 1] : \forall y \in [x, x'] \sum_{k \leq n} Y_k(y) \geq \alpha n/2; \forall k \in [r,n] \sum_{j \leq k} Y_j(y) \leq (\alpha + \epsilon)k/2\}
$$

Let $0 < \Delta < r$. We divide the right side into four terms depending on the branching point:

(I) : $x \wedge x' > n - \Delta$ \hspace{1cm} (II) : $r + \Delta < x \wedge x' \leq n - \Delta$ \hspace{1cm} (III) : $r - \Delta < x \wedge x' \leq r + \Delta$ \hspace{1cm} (IV) : $x \wedge x' \leq r - \Delta$.

The term (IV) is controlled in the following Lemma.
Lemma 3.5. For $R = o(\ln \ln N)$ we have

\begin{equation}
\lim_{\Delta \to \infty} \lim_{N \to \infty} \frac{\mathbb{E}((IV)\mathcal{F}_R) - \left( \mathbb{E} \left( W_{a,N}^\sigma \mathcal{F}_R \right) \right)^2}{\mathbb{E} \left( W_{a,N}^\sigma \right)^2} = 0 \quad \text{a.s.}
\end{equation}

Proof of Lemma 3.5 As $x \land x' < r - \Delta$ and by a similar rewriting in (2.5) we have by Lemma 3.2 that it is bounded from above by

\begin{equation}
\int_{[x \land x' < r - \Delta]} \prod_{y \in [x,x']} \mathbb{P} \left( \sum_{k=1}^{n} Y_k(\mathcal{F}_R) > \frac{\alpha}{2} n - G_R(y) - E_R(y) - \varepsilon \right) dx' dx \left( 1 + O \left( e^{-c_\Delta} \right) \right)
\end{equation}

We now compare (3.13) with $\mathbb{E} \left( W_{a,N}^\sigma \mathcal{F}_R \right)^2$ which is bounded below by

\begin{equation}
\int_{[0,1]^2} \prod_{y \in [x,x']} \mathbb{P} \left( G_N(y) - G_R(y) > \frac{\alpha}{2} n - G_R(y) - E_R(y) + \varepsilon \right) dx' dx
\end{equation}

for any $\varepsilon > 0$. By (2.1) and the Gaussian approximation given in (3.1) the absolute value of the difference of (3.13) and (3.14) is bounded by

\begin{equation}
M^2 e^{-2\varepsilon} \mathbb{E}(W_{a,N})^2 e^{-2\varepsilon} \left( 1 + O \left( e^{-c_\Delta} \right) \right)
\end{equation}

Hence (3.15) divided by $\mathbb{E}(W_{a,N})^2$ converges almost surely to

\begin{equation}
M^2 e^{-2\varepsilon} \left( e^{-2\varepsilon} - e^{2\varepsilon} + e^{-2\varepsilon} O \left( e^{-c_\Delta} \right) \right)
\end{equation}

for all $\varepsilon, \Delta > 0$. Note that (3.15) converges to zero as $\varepsilon \to 0$ and $\Delta \to \infty$. \hfill \tiny \Box

Lemma 3.6. Let $0 < \alpha < 2$. There exists $\kappa_\alpha > 0$ such that for $R = o(\ln \ln N)$ as $N \to \infty$ we have

\begin{equation}
\mathbb{E} \left( (I) + (II) + (III) \right) \leq \mathbb{E} \left( W_{a,N} \right)^2 e^{-\kappa_\alpha r}.
\end{equation}

Proof of Lemma 3.6 We bound $\mathbb{E}(I)\mathcal{F}_R$ from above by

\begin{equation}
e^{-n + \Delta} \int_0^1 \mathbb{P} \left( \sum_{k \leq n} Y_k(x) > \frac{\alpha}{2} n \mathcal{F}_R \right) dx = e^{-n + \Delta} \mathbb{E}(W_{a,N}^\sigma \mathcal{F}_R),
\end{equation}

by (2.4). Hence,

\begin{equation}
\mathbb{E}(I) \leq \mathbb{E}(W_{a,N})^2 \frac{e^{-n + \Delta}}{\mathbb{E}(W_{a,N})^2} = \mathbb{E}(W_{a,N})^2 o(1),
\end{equation}

as $\mathbb{E}(W_{a,N}) = cn^{-1/2} e^{-2n^2}$ and $0 < \alpha < 2$.

Next, we turn to $\mathbb{E}(II)\mathcal{F}_R$. Using that uniformly in $y$ for all $R, N$ large enough $|E_N(y) - E_R(y)| \leq \varepsilon$, we can bound $\mathbb{E}(II)\mathcal{F}_R$ from above by

\begin{equation}
\int_{[r + \Delta \leq x \land x' \leq n - \Delta]} \mathbb{P} \left( \forall_{y \in [x,x']} \sum_{j=r+1}^{n} Y_j(y) > \frac{\alpha}{2} n - X_R(y) - \varepsilon, \forall_{k \leq n} \sum_{j=r}^{k} Y_j(y) \leq \frac{\alpha + \varepsilon}{2} k \mathcal{F}_R \right) dx' dx
\end{equation}
Dropping the barrier constraint except at \( x \land x' - \Delta \) and \( x \land x' + \Delta \) we can bound the probability in (3.20) from above by

\[
(3.21) \quad P \left( \bigvee_{y \in \{x, x'\}} \sum_{j=r+1}^{n} Y_j(y) > \frac{\alpha}{2} n - X_R(y) - \epsilon, \ \forall_{k \in \{x, x' - \Delta, x, x' + \Delta\}} \sum_{j=1}^{k} Y_j(y) \leq \frac{\alpha + \epsilon}{2} k \right). 
\]

We evaluate the probability in the integral at a fixed \( x \land x' = m \), and sum the contributions over \( m \) afterwards. We introduce an extra conditioning. Let \( F^{-\gamma}_k = \sigma(Y', j \leq k) \). We condition on \( F^{-\gamma}_{m+\Delta} \), slightly after the branching point. Lemma 3.2 applied to (3.21) then yields

\[
(3.22) \quad \left(1 + e^{-c\Delta}\right) \mathbb{P} \left( \bigvee_{y \in \{x, x'\}} \sum_{k=m+\Delta+1}^{n} Y_k(y) > \frac{\alpha}{2} n - X_R(y) - \epsilon - \sum_{r < j < m+\Delta} Y_j(y) \right) 
\]

We distinguish two cases. First, consider the case when for \( y = x \) or \( y = x' \),

\[
(3.23) \quad \frac{\alpha}{2} n - X_R(y) - \epsilon - \sum_{r < j < m+\Delta} Y_j(y) \leq 0.
\]

Note that due to the barrier in (3.22) this can only happen jointly with the barrier event if \( m \geq \frac{\alpha}{a+\epsilon} n - C'\epsilon \) for some constant \( C' > 0 \) independent of \( \epsilon \). In this case we bound the probabilities above by one and bound (3.22) from above by

\[
(3.24) \quad \left(1 + e^{-c\Delta}\right) \mathbb{P} \left( \frac{\alpha}{2} n - X_R(y) - \epsilon - \sum_{r < j < m+\Delta} Y_j(y) \leq 0 : \bigvee_{y \in \{x, x'\}} \sum_{j=m+\Delta} Y_j(y) \leq \frac{\alpha + \epsilon}{2} (m + \Delta) \right). 
\]

As for an upper bound we can drop all constraints in the expectation with respect \( x' \) (if \( y = x \)) and \( x \) otherwise, let us assume without loss of generality that \( y = x \). We need to distinguish whether \( \frac{\alpha}{2} n - X_R(x) - \epsilon > 0 \) or not. On the event \( \frac{\alpha}{2} n - X_R(x) - \epsilon \leq 0 \) we bound the expectation in (3.24) by one and obtain that the expectation of (3.24) from above by

\[
(3.25) \quad \mathbb{P} \left( X_R(x) \geq \frac{\alpha}{2} n - \epsilon \right) \leq \mathbb{E} \left( e^{\alpha X_R(x) - \alpha \left( \frac{\alpha}{2} n - \epsilon \right)} \right)
\]

by the exponential Chebyshev inequality. Hence, integrating over \( x, x' \) in (3.28) we get

\[
(3.26) \quad e^{-\frac{\alpha}{a+\epsilon} n - C'\epsilon} \int_{0}^{\frac{\alpha}{2} n - \epsilon} \mathbb{E} \left( e^{\alpha X_R(x) - \alpha \left( \frac{\alpha}{2} n - \epsilon \right)} \right) dx 
\]

by (2.7). When \( \frac{\alpha}{2} n - X_R(x) - \epsilon > 0 \), we bound (3.24) from above using Gaussian tail asymptotics by

\[
(3.27) \quad \left(1 + e^{-c\Delta}\right) \mathbb{P} \left( \sum_{r < j < m+\Delta} Y_j(x) \geq \frac{\alpha}{2} n - X_R(y) - \epsilon \right) \leq \left(1 + e^{-c\Delta}\right) e^{-\frac{(x - X_R(y) - \epsilon)^2}{4\sigma^2(x,m+\Delta)}}.
\]
The integral of (3.27) with respect to $x$ and $x'$ can be bounded from above by

\[
(1 + e^{-c\Delta}) \sum_{\frac{n-m-\Delta}{n} - C \leq m \leq n} e^{-m} \int_0^1 e^{-\left(\frac{\alpha X_r(x) - \alpha}{2y(m+\Delta)}\right)} dx 
\]

\[
\leq (1 + e^{-c\Delta}) \sum_{\frac{n-m-\Delta}{n} - C \leq m \leq n} e^{-m} \int_0^1 e^{-\left(\alpha \sigma_r(m+\Delta)\right)} e^{\left(2\alpha \sigma_r(m+\Delta)\right)} dx 
\]

\[
\leq (1 + e^{-c\Delta}) \sum_{\frac{n-m-\Delta}{n} - C \leq m \leq n} e^{-m} \int_0^1 e^{-\left(\alpha \sigma_r(m+\Delta)\right)} e^{\left(2\alpha \sigma_r(m+\Delta)\right)} dx
\]

Using that in the range of summation in (3.28) $\sigma_r(m+\Delta)$ is bounded from above and below by $\frac{1}{2}(m - r) + C$ resp. $\frac{1}{2}(m - r) - C$, for some constant large enough, we can bound (3.28) from above by

\[
(1 + e^{-c\Delta}) \sum_{\frac{n-m-\Delta}{n} - C \leq m \leq n} \int_0^1 e^{-\left(\alpha \sigma_r(m+\Delta)\right)} e^{\left(2\alpha \sigma_r(m+\Delta)\right)} dx.
\]

As $m \geq \frac{\sigma_r}{\alpha} + C'\epsilon$, exponential term in $e$ bounded by $e^{C\epsilon}$ and as $0 < \alpha < 2$ we have that on the one hand $\frac{\sigma_r}{\alpha} - 1 < 0$ and on the other hand we can choose together with (2.7) we can bound the corresponding expectation in (3.29) from above by

\[
(1 + e^{-c\Delta}) E(W_{n,N})^2 e^{-cn} e^{cr},
\]

for some $c > 0$.

Finally, we turn to bound (3.22) for $\frac{\sigma_r}{\alpha} n - X_r(y) - \epsilon - \sum_{r < j \leq m+\Delta} Y_j(y) \geq 0$ we can bound (3.22) from above by a Gaussian tail bound and obtain

\[
\mathbb{E}\left(\frac{(n-m-\Delta)/2}{2\pi} \prod_{x \in [x',x')} \left(\frac{2}{(n-m-\Delta - \epsilon) - X_R(y) - \epsilon}\right) \mathbb{I}_{Y_j(x) \leq \sum_{j \leq m+\Delta} Y_j(y)} \right. 
\]

\[
\times \exp\left(-\sum_{y \in [x',x)} \left(\frac{\frac{\sigma_r}{\alpha} n - X_R(y) - \epsilon - \sum_{r \leq j \leq m+\Delta} Y_j(y)}{n-m-\Delta}\right)^2\right) \left| F_R \right| \right)
\]

Next, we condition on $F_{m-\Delta}^Y$. The terms depending on $\sum_{m-\Delta < j \leq m+\Delta} Y_j$ can be bounded by the moment generating function:

\[
\mathbb{E}\left(e^{C\Delta \sum_{m-\Delta < j \leq m+\Delta} Y_j(x) + Y_j(x')}\right) \leq e^{C\Delta^2}.
\]

Hence, Equation (3.31) is bounded above by

\[
e e^{C\Delta^2} \mathbb{E}\left(e^{C\Delta^2} \sum_{y \in [x',x)} \left(\frac{\frac{\sigma_r}{\alpha} n - X_R(y) - \epsilon - \sum_{r \leq j \leq m+\Delta} Y_j(y)}{n-m-\Delta}\right)^2\right) \left| F_R \right| \right)
\]

Using the fact that the variables $Y_j(x)$ and $Y_j(x')$ almost coincide for $j \leq m - \Delta$ by Lemma 3.3 we have that (3.33) is bounded above by

\[
e e^{C\Delta^2} \mathbb{E}\left(e^{C\Delta^2} \sum_{y \in [x',x)} \left(\frac{\frac{\sigma_r}{\alpha} n - X_R(y) - \epsilon - \sum_{r \leq j \leq m+\Delta} Y_j(y)}{n-m-\Delta}\right)^2\right) \left| F_R \right| \right)
\]

(3.34)
The expectation in (3.34) is equal to
\[ \int_{-\infty}^{\infty} e^{-2\left(\frac{2n}{m} - X_k(x) - \epsilon\right)^2 (m - \Delta - r)} \frac{d\epsilon}{\sqrt{\pi (m - \Delta - r)}}. \]

The integrand with respect to $\epsilon$ is minimal for
\[ z^* = 2 \left( \frac{\alpha}{n} - X_k(x) - \epsilon \right) \frac{(m - \Delta - r)}{n + m - 3\Delta - 2r}. \]

When $\frac{2n}{m} - z^* \ll 0$ which is the case when $m < (1 - \delta)n$ for some $\delta > 0$, we can use Gaussian tail asymptotics to bound (3.35) from above by
\[ \exp \left( -\frac{2\left( \frac{\alpha}{n} - X_k(x) - \epsilon - \frac{2n}{m} \right)^2}{(m - \Delta - r)} - \frac{\left( \frac{2n}{m} \right)^2}{(m - \Delta - r)} \right). \]

Plugging this bound into (3.34), summing over $m < (1 - \delta)n$, and computing the squares in the exponential, we obtain that (3.34) is bounded from above by
\[ \sum_{i = r + \Delta}^{(1 - \delta)n} e^{\frac{\alpha^2}{2} - \alpha^2 \epsilon^2} e^{C\Delta + C\epsilon} (W_{\alpha,N})^2 (1 + o(1)), \]

If $x \wedge x' < (1 - \delta)n$ we can bound the Gaussian integral by one and get that (3.38) is bounded from above by
\[ e^{C(\Delta + \epsilon)} (W_{\alpha,N})^2 \left( e^{\frac{2(\frac{\alpha}{n} - X_k(x) - \epsilon)^2}{m - \Delta + \delta}} \right). \]

Using (3.36) we can bound the expectation of (3.39) for $m > (1 - \delta)n$ by
\[ e^{C(\Delta + \epsilon)} (W_{\alpha,N})^2 \left( e^{\frac{2(\frac{\alpha}{n} - X_k(x) - \epsilon)^2}{m - \Delta + \delta}} \right). \]

Plugging this into (3.34) we can bound the contribution from above
\[ \sum_{m > (1 - \delta)n} 2^{-m} e^{C(\Delta^2 + \epsilon)} (W_{\alpha,N})^2 \left( e^{\frac{2(\frac{\alpha}{n} - X_k(x) - \epsilon)^2}{m - \Delta + \delta}} \right). \]

Noting that $2n - n\delta \leq n + m \leq 2n$ the above term can be bounded from above by
\[ \sum_{m > (1 - \delta)n} 2^{-m} e^{C(\Delta^2 + \epsilon)}. \]

Note that the exponent in (3.42) is negative for $\delta$ sufficiently small.

Finally, we want to bound $\mathbb{E}((III))$. By Lemma 3.2 we have similar to (3.22) that $\mathbb{E}((III))$ is bounded from above by $(1 + e^{-C\Delta})$ times
\[ \sum_{m = r - \Delta + 1}^{r + \Delta} \int_{x \wedge x' = m} \exp \left( \prod_{y \in [x, x']} \mathbb{P} \left( \sum_{k = m + \Delta + 1}^{n} Y_k(y) > \frac{\alpha}{2} n - X_k(y) - \epsilon - \sum_{j = r + 1}^{m + \Delta} Y_j(y) | f_{m + \Delta}^y \right) \right) dxdx'. \]

(3.43)
If $\frac{s}{2}n - X_R(y) - \epsilon - \sum_{j=r+1}^{m+\Delta} Y_j(y) > 0$ for $y \in \{x, x'\}$ we can use Gaussian tail asymptotics for the probabilities in (3.43) to bound the expectation in (3.43) from above by (3.44)

$$
\mathbb{E} \left( \frac{(n - r)/2}{2\pi \prod_{i \in \{x, x'\}} \left( \int_{y-x}^{y+x} \left( \frac{z_{x+y} - X_R(y) - \epsilon - \sum_{j=r+1}^{m+\Delta} Y_j(y)}{\sqrt{n}} \right)^2 \right)} \right) e^{-\left( \frac{z_{x+y} - X_R(y) - \epsilon - \sum_{j=r+1}^{m+\Delta} Y_j(y)}{\sqrt{n}} \right)^2} e^{-\frac{1}{4} \left( \frac{z_{x+y} - X_R(y) - \epsilon - \sum_{j=r+1}^{m+\Delta} Y_j(y)}{\sqrt{n}} \right)^2}.
$$

Noticing that the polynomial prefactor is bounded by $C/n$ and otherwise proceeding as in (3.32) we can bound (3.43) from above by

$$
ee^{C \Delta^2} \sum_{m=r-\Delta+1}^{r+\Delta} \int_{\{x \land x' = m\}} C \mathbb{E} \left( \frac{n \mathbb{E} \left( \frac{z_{x+y} - X_R(y) - \epsilon - \sum_{j=r+1}^{m+\Delta} Y_j(y)}{\sqrt{n}} \right)^2 \mathbb{E} \left( \frac{z_{x+y} - X_R(y) - \epsilon - \sum_{j=r+1}^{m+\Delta} Y_j(y)}{\sqrt{n}} \right)^2 \right) e^{\left( \frac{z_{x+y} - X_R(y) - \epsilon - \sum_{j=r+1}^{m+\Delta} Y_j(y)}{\sqrt{n}} \right)^2} (1 + \mathbb{E} \left( \frac{z_{x+y} - X_R(y) - \epsilon - \sum_{j=r+1}^{m+\Delta} Y_j(y)}{\sqrt{n}} \right)^2) \right) dx \, dx \left( 1 + O \left( e^{-\Delta} \right) \right)
$$

by (2.6) for any $\epsilon > 0$ and $\Delta > 0$. Note first that due to the barrier event in (3.43) the case $\frac{s}{2}n - X_R(y) - \epsilon - \sum_{j=r+1}^{m+\Delta} Y_j(y) \leq 0$ for at least one $y \in \{x, x'\}$ can be excluded for $m \in \{r - \Delta, r + \Delta\}$.

This completes the control of (III) and hence also the proof of Theorem 3.6. □

Proof of Proposition 3.4 We bound (3.10) from above by

$$
\mathbb{P} \left( (I) + (II) + (III) \mathbb{E} (W_{a,n})^{-2} > c^2/8 \right) + \mathbb{P} \left( \mathbb{E} ((IV) | F_R) - \mathbb{E} (W_{a,n} | F_R) > c^2/8 \right) + Ce^{-Rc(\epsilon)}
$$

$$
\leq \frac{8}{c^2} \mathbb{E} (I) + (II) + (III) \mathbb{E} (W_{a,n})^{-2} + \mathbb{P} \left( \mathbb{E} ((IV) | F_R) - \mathbb{E} (W_{a,n} | F_R) > c^2/8 \right) + Ce^{-Rc(\epsilon)}
$$

where we used Chebyshev’s inequality. By Lemma 3.6 we can bound (3.10) from above by

$$
\frac{8}{c^2} \mathbb{E} (W_{a,n})^{-2} e^{-x_{\nu^+}} + \mathbb{P} \left( \mathbb{E} ((IV) | F_R) - \mathbb{E} (W_{a,n} | F_R) > c^2/8 \right) \leq \frac{4}{c^2} e^{-x_{\nu^+}} + Ce^{-Rc(\epsilon)},
$$

which yields Proposition 3.4 by possibly modifying the constants and noting that $\epsilon$ in (3.47) is arbitrary (but fixed) as the claim of Lemma 3.5 holds almost surely in the $N \to \infty$ limit.

□

4. PROOF OF THEOREM 1.1

Finally, we are in the position to prove Theorem 1.1 using Lemma 2.1 and Proposition 3.4.

Proof of Theorem 1.1 First, we rewrite

$$
\frac{W_{a,N}}{\mathbb{E} (W_{a,n})} = \frac{\mathbb{E} (W_{a,n} | F_R)}{\mathbb{E} (W_{a,n})} + \frac{W_{a,n} - \mathbb{E} (W_{a,n} | F_R)}{\mathbb{E} (W_{a,n})}.
$$

By Proposition 3.4 the second summand on the right hand side of (4.1) converges to zero in probability when first $N \to \infty$ and then $R \to \infty$. By Lemma 2.1 the term the first summand on the right hand side of (4.1) converges almost surely to $M_\alpha$ defined in (1.6). This completes the proof of Theorem 1.1. □
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