Gröbner-Shirshov Bases for Commutative Algebras with Multiple Operators and Free Commutative Rota-Baxter Algebras

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Abstract: In this paper, the Composition-Diamond lemma for commutative algebras with multiple operators is established. As applications, the Gröbner-Shirshov bases and linear bases of free commutative Rota-Baxter algebra, free commutative $\lambda$-differential algebra and free commutative $\lambda$-differential Rota-Baxter algebra are given, respectively. Consequently, these three free algebras are constructed directly by commutative $\Omega$-words.

Key words: commutative, Rota-Baxter algebras, $\lambda$-differential algebras, $\lambda$-differential Rota-Baxter algebras, commutative algebras with multiple operators, Gröbner-Shirshov bases.

AMS 2000 Subject Classification: 16S15, 13P10, 16W99, 17A50, 13XX

1 Introduction

Let $K$ be a unitary commutative ring and $\lambda \in K$. A Rota-Baxter algebra of weight $\lambda$ is a $K$-algebra $R$ with a linear operator $P : R \to R$ satisfying the Rota-Baxter relation:

$$P(u)P(v) = P(uP(v)) + P(vP(u)) + \lambda P(uv), \forall u, v \in R.$$  

The Rota-Baxter operator first occurred in the paper of G. Baxter [11] to solve an analytic problem, and the algebraic study of this operator was started by G.-C. Rota [19].

There have been some constructions of free (commutative) Rota-Baxter algebras. In this aspect, G.-C. Rota [19] and P. Cartier [9] gave the explicit constructions of free commutative Rota-Baxter algebras of weight $\lambda = 1$, which they called shuffle Baxter and standard Baxter algebras, respectively. Recently L. Guo and W. Keigher [13, 14] constructed the free commutative Rota-Baxter algebras (with unit or without unit) for any $\lambda \in K$ using the mixable shuffle product. These are called the mixable shuffle product algebras, which generalize the classical construction of shuffle product algebras.

*Supported by the Young Project on Natural Science Fund of Zhanjiang Normal University (No.QL0902).
K. Ebrahimi-Fard and L. Guo [11] further constructed the free associative Rota-Baxter algebras by using the Rota-Baxter words. In [12], K. Ebrahimi-Fard and L. Guo use rooted trees and forests to give explicit construction of free noncommutative Rota-Baxter algebras on modules and sets.

A differential algebra of weigh $\lambda$, also called $\lambda$-differential algebra is a $K$-algebra $R$ with a linear operator $D : R \rightarrow R$ such that

$$D(uv) = D(u)v + uD(v) + \lambda D(u)D(v), \forall u, v \in R.$$  

Such an operator $D$ is called a $\lambda$-differential operator. E. Kolchin [16] considered the differential algebra and constructed free differential algebra of weight $\lambda = 0$. L. Guo and W. Keigher [15] dealt with a generalization of this algebra and using the same way to construct the free differential algebra of weight $\lambda$ in both commutative and associative case.

Similar to the relation of integral and differential operators, L. Guo and W. Keigher [15] introduced the notion of $\lambda$-differential Rota-Baxter algebra which is a $K$-algebra $R$ with a $\lambda$-differential operator $D$ and a Rota-Baxter operator $P$ such that $DP = Id_R$. In the same paper [15], they construct the free $\lambda$-differential Rota-Baxter (commutative and associative).

The Gröbner-Shirshov bases theory for Lie algebras was introduced by A. I. Shirshov [4, 20]. Shirshov [20] defined the composition of two Lie polynomials, and proved the Composition lemma for the Lie algebras. L. A. Bokut [3] specialized the approach of Shirshov to associative algebras, see also Bergman [2]. For commutative polynomials, this lemma is known as the Buchberger’s Theorem in [7, 8].

The multi-operators algebras ($\Omega$-algebras) were introduced by A. G. Kurosh [17] and the Gröbner-Shirshov bases for $\Omega$-algebras were given in the paper by V. Drensky and R. Holtkamp [10]. Composition-Diamond lemma for associative algebras with multiple linear operators (associative $\Omega$-algebras) is established in a recent paper by L. A. Bokut, Y. Chen and J. Qiu [5]. Also, the Gröbner-Shirshov bases for Rota-Baxter algebras is established by L. A. Bokut, Y. Chen and X. Deng [6] and the Composition-Diamond lemma for $\lambda$-differential associative algebras with multiple operators is constructed by J. Qiu and Y. Chen [18].

In this paper, we deal with commutative algebras with multiple linear operators. We construct free commutative algebras with multiple linear operators and establish the Composition-Diamond lemma for such algebras. As applications, we obtain Gröbner-Shirshov bases of free commutative Rota-Baxter algebra, commutative $\lambda$-differential algebra and commutative $\lambda$-differential Rota-Baxter algebra, respectively. Then, by using the Composition-Diamond lemma, linear bases of these three free algebras are obtained respectively.

The author would like to express his deepest gratitude to Professors L. A. Bokut and Yuqun Chen for their kind guidance, useful discussions and enthusiastic encouragements.

2 Free commutative algebras with multiple operators

In this section, we construct free commutative algebras with multiple linear operators.
Let $K$ be a unitary commutative ring. A commutative algebra with multiple operators is a commutative $K$-algebra $R$ with a set $\Omega$ of multi-linear operators.

Let $X$ be a set, $CS(X)$ the free commutative semigroup on $X$ and

$$\Omega = \bigcup_{t=1}^{\infty} \Omega_t$$

where $\Omega_t$ is the set of $t$-ary operators.

Define

$$\Gamma_0 = X, \ Q_0 = CS(\Gamma_0)$$

and

$$\Gamma_1 = X \cup \Omega(Q_0), \ Q_1 = CS(\Gamma_1)$$

where

$$\Omega(Q_0) = \bigcup_{t=1}^{\infty} \{\theta(u_1, u_2, \ldots, u_t) | \theta \in \Omega_t, u_i \in Q_0, i = 1, 2, \ldots, t\}.$$ 

For $n > 1$, define

$$\Gamma_n = X \cup \Omega(Q_{n-1}), \ Q_n = CS(\Gamma_n)$$

where

$$\Omega(Q_{n-1}) = \bigcup_{t=1}^{\infty} \{\theta(u_1, u_2, \ldots, u_t) | \theta \in \Omega_t, u_i \in Q_{n-1}, i = 1, 2, \ldots, t\}.$$ 

Let

$$Q(X) = \bigcup_{n \geq 0} Q_n.$$ 

Then it is easy to see that $Q(X)$ is a commutative semigroup such that $\Omega(Q(X)) \subseteq Q(X)$.

Let $K[X; \Omega]$ be the commutative $K$-algebra spanned by $Q(X)$. Extend linearly each $\sigma \in \Omega_t, t \geq 1$,

$$\sigma : Q(X)^t \rightarrow Q(X), \ (x_1, x_2, \ldots, x_n) \mapsto \sigma(x_1, x_2, \ldots, x_n)$$

to

$$K[X; \Omega]^t \rightarrow K[X; \Omega].$$ 

Then, it is easy to see that $K[X; \Omega]$ is a free commutative algebra with multiple operators $\Omega$ on set $X$.

### 3 Composition-Diamond lemma for commutative algebras with multiple operators

In this section, we introduce the notions of Gröbner-Shirshov bases for the commutative algebras with multiple operators and establish the Composition-Diamond lemma for such algebras.
The element in $Q(X)$ and $K[X; \Omega]$ are called commutative $\Omega$-word and commutative $\Omega$-polynomial, respectively. For any $u \in Q(X)$, $u$ has a unique expression

$$u = u_1 u_2 \cdots u_n$$

where each $u_i \in X \cup \Omega(Q(X))$. If this is the case, we define $bre(u) = n$. Let $u \in Q(X)$. Then

$$dep(u) = \min\{n|u \in Q_n\}$$

is called the depth of $u$.

Let $\star \notin X$. By a commutative $\star$-$\Omega$-word we mean any expression in $Q(X \cup \{\star\})$ with only one occurrence of $\star$. We define the set of all commutative $\star$-$\Omega$-words on $X$ by $Q^\star(X)$. Let $u \in Q^\star(X)$.

Then $dep(u) = \min\{n|u \in Q_n\}$ is called the depth of $u$.

Let $\star_1, \star_2 \notin X$. By a commutative $(\star_1, \star_2)$-$\Omega$-word we mean any expression in $Q(X \cup \{\star_1, \star_2\})$ with only one occurrence of $\star_1$ and only one occurrence of $\star_2$. Let us denote by $Q^{\star_1,\star_2}(X)$ the set of all commutative $(\star_1, \star_2)$-$\Omega$-words. Let $u \in Q^{\star_1,\star_2}(X)$, $s_1, s_2 \in K[X; \Omega]$. Then we call

$$u|_{s_1, s_2} = u|_{\star_1 \rightarrow s_1, \star_2 \rightarrow s_2}$$

a commutative $s_1$-$s_2$-$\Omega$-word.

Now, we assume that $Q(X)$ is equipped with a monomial order $>$. This means that $>$ is a well order on $Q(X)$ such that for any $w, v \in Q(X), u \in Q^\star(X)$,

$$w > v \Rightarrow u|_w > u|_v.$$  

Note that such an order on $Q(X)$ exists, for example, the order (1) in the next section.

For every commutative $\Omega$-polynomial $f \in K[X; \Omega]$, let $\bar{f}$ be the leading term of $f$. If the coefficient of $\bar{f}$ is 1, then we call $f$ monic.

Let $f, g$ be two monic $\Omega$-polynomials. Then there are two kinds of compositions.

(I) If there exists a commutative $\Omega$-word $w = a\bar{f} = b\bar{g}$ for some $a, b \in Q(X)$ such that $bre(w) < bre(\bar{f}) + bre(\bar{g})$, then we call $(f, g)_w = af - bg$ the intersection composition of $f$ and $g$ with respect to $w$.

(II) If there exists a commutative $\Omega$-word $w = \bar{f} = u|_\bar{g}$ for some $u \in Q^\star(X)$, then we call $(f, g)_w = f - u|_\bar{g}$ the including composition of $f$ and $g$ with respect to $w$.  

4
In the above definition, \( w \) is called the ambiguity of the composition. Clearly,

\[(f, g)_w \in Id(f, g) \quad \text{and} \quad (f, g)_w < w\]

where \( Id(f, g) \) is the ideal of \( K[X; \Omega] \) generated by \( f \) and \( g \).

Let \( f, g \) be commutative \( \Omega \)-polynomials and \( g \) monic with \( \tilde{f} = u|_{\bar{g}} \) for some \( u \in Q^*(X) \). Then the transformation

\[ f \rightarrow f - \alpha u|_{\bar{g}} \]

is called elimination of the leading commutative \( \Omega \)-word (ELW) of \( f \) by \( g \), where \( \alpha \) is the coefficient of the leading commutative \( \Omega \)-word of \( f \).

Let \( S \) be a set of monic commutative \( \Omega \)-polynomials. Then the composition \((f, g)_w\) is called trivial modulo \((S, w)\), if

\[(f, g)_w = \sum \alpha_i u_i|_{s_i}\]

where each \( \alpha_i \in K, u_i \in Q^*(X), s_i \in S \) and \( u_i|_{\bar{s}_i} < w \). If this is the case, we write

\[(f, g)_w \equiv 0 \mod (S, w).\]

In general, for any two commutative \( \Omega \)-polynomials \( p \) and \( q \), \( p \equiv q \mod (S, w) \) means

\[ p - q = \sum \alpha_i u_i|_{s_i} \]

where each \( \alpha_i \in K, u_i \in Q^*(X), s_i \in S \) and \( u_i|_{\bar{s}_i} < w \).

Then \( S \) is called a Gröbner-Shirshov basis in \( K[X; \Omega] \) if any composition \((f, g)_w\) of \( f, g \in S \) is trivial modulo \((S, w)\).

**Lemma 3.1** Let \( S \) be a Gröbner-Shirshov basis in \( K[X; \Omega] \) and \( u_1, u_2 \in Q^*(X), s_1, s_2 \in S \). If \( w = u_1|_{\bar{s}_1} = u_2|_{\bar{s}_2} \), then

\[ u_1|_{s_1} \equiv u_2|_{s_2} \mod (S, w). \]

**Proof:** There are three cases to consider.

(I) The commutative \( \Omega \)-words \( \bar{s}_1 \) and \( \bar{s}_2 \) are disjoint. Then there exits a commutative \((\ast_1, \ast_2)\)-\( \Omega \)-words \( \Pi \) such that

\[ \Pi|_{\bar{s}_1, \bar{s}_2} = u_1|_{\bar{s}_1} = u_2|_{\bar{s}_2}. \]

Then

\[ u_2|_{s_2} - u_1|_{s_1} = \Pi|_{\bar{s}_1, s_2} - \Pi|_{s_1, \bar{s}_2} = (\Pi|_{s_1, s_2} - \Pi|_{s_1, \bar{s}_2}) = -\Pi|_{s_1, s_2} + \Pi|_{s_1, \bar{s}_2} \]

Let

\[ -\Pi|_{s_1, \bar{s}_2} = \sum \alpha_{s_2} u_2|_{s_2} \quad \text{and} \quad \Pi|_{s_1, s_2 \bar{s}_2} = \sum \alpha_{s_1} u_1|_{s_1}. \]

Since \( s_1 - \bar{s}_1 < \bar{s}_1 \) and \( s_2 - \bar{s}_2 < \bar{s}_2 \), we have

\[ u_2|_{\bar{s}_2}, u_1|_{\bar{s}_1} < w. \]
Therefore
\[ u_2|s_2 - u_1|s_1 = \sum \alpha_2, u_2|s_2 + \sum \alpha_1, u_1|s_1 \]
with each \( u_2|s_2, u_1|s_1 < w \). It follows that
\[ u_1|s_1 \equiv u_2|s_2 \ mod(S, w). \]

(II) The commutative \( \Omega \)-words \( \overline{s_1} \) and \( \overline{s_2} \) have nonempty intersection but do not include each other. For example,
\[ as_1 = b\overline{s_2} \]
for some commutative \( \Omega \)-words \( a, b \). Then there exists a commutative \( *-\Omega \)-word \( \Pi \) such that
\[ \Pi|a|s_1 = u_1|\overline{s_1} = u_2|\overline{s_2} = \Pi|b|\overline{s_2}. \]
Then we have
\[ u_2|s_2 - u_1|s_1 = \Pi|b|s_2 - \Pi|a|s_1 = -\Pi|a|s_1 - b|s_2. \]
Since \( S \) is a Gröbner-Shirshov basis in \( K[X; \Omega] \), we have
\[ as_1 - bs_2 = \sum \alpha_j v_j|s_j \]
where each \( \alpha_j \in K, v_j \in Q^*(X) \), \( s_j \in S \) and \( v_j|\overline{s_1} < a|\overline{s_1} \). Let
\[ \Pi|v_j|s_j = \Pi_j|s_j, \]
Then
\[ u_2|s_2 - u_1|s_1 = \sum \alpha_j \Pi_j|s_j \]
with
\[ \Pi_j|\overline{s_j} < w. \]
It follows that
\[ u_1|s_1 \equiv u_2|s_2 \ mod(S, w). \]

(III) One of commutative \( \Omega \)-words \( \overline{s_1}, \overline{s_2} \) is contained in the other. For example, let
\[ \overline{s_1} = u|\overline{s} \]
for some commutative \( *-\Omega \)-word \( u \). Then
\[ w = u_2|\overline{s} = u_1|u|\overline{s}, \]
and
\[ u_2|s_2 - u_1|s_1 = u_1|u|s_2 - u_1|s_1 = -u_1|s_1 - u|s_2. \]
Similarly to (II), we can obtain the result. ■

The following theorem is an analogy of Shirshov’s composition lemma for Lie algebras [20], which was specialized to associative algebras by Bokut [3]. For commutative algebras, this lemma is known as the Buchberger’s Theorem in [7, 8].
**Theorem 3.2** (Composition-Diamond lemma)  Let $S$ be a set of monic commutative $\Omega$-polynomials in $K[X;\Omega]$ and $>\,$ a monomial order on $Q(X)$. Then the following statement are equivalent:

(I) $S$ is a Gröbner-Shirshov basis in $K[X;\Omega]$.

(II) $f \in Id(S) \Rightarrow \bar{f} = u|_{\pi}$ for some $u \in Q^*(X)$ and $s \in S$.

(III) $\operatorname{Irr}(S) = \{ w \in Q(X) | w \neq u|_{\pi} \text{ for any } u \in Q^*(X) \text{ and } s \in S \}$ is a $K$-basis of $K[X;\Omega]/Id(S) = K[X;\Omega][S]$.

**Proof:** (I) $\implies$ (II) Let $0 \neq f \in Id(S)$. Then

\[ f = \sum_{i=1}^{n} \alpha_i u_i|_{s_i} \]

where each $\alpha_i \in K$, $u_i \in Q^*(X)$ and $s_i \in S$. Let $w_i = u_i|_{\pi}$ and we arrange this leading commutative $\Omega$-words in non-increasing order by

\[ w_1 = w_2 = \cdots = w_m > w_{m+1} \geq \cdots \geq w_n. \]

We prove the result by induction on $m$.

If $m = 1$, then $\bar{f} = u_1|_{\pi}$.

Now we assume that $m \geq 2$. Then

\[ u_1|_{\pi} = w_1 = w_2 = u_2|_{\pi}. \]

We prove the result by induction on $w_1$. If $\bar{f} = w_1$, there is nothing to prove. Clearly, $w_1 > \bar{f}$. Since $S$ is a Gröbner-Shirshov basis in $K[X;\Omega]$, by Lemma 3.1 we have

\[ u_2|_{s_2} - u_1|_{s_1} = \sum \beta_j v_j|_{s_j} \]

where $\beta_j \in K$, $s_j \in S$, $v_j \in Q^*(X)$ and $v_j|_{\pi} < w_1$. Therefore, since

\[ \alpha_1 u_1|_{s_1} + \alpha_2 u_2|_{s_2} = (\alpha_1 + \alpha_2) u_1|_{s_1} + \alpha_2 (u_2|_{s_2} - u_1|_{s_1}), \]

we have

\[ f = (\alpha_1 + \alpha_2) u_1|_{s_1} + \sum \alpha_2 \beta_j v_j|_{s_j} + \sum_{i=3}^{n} \alpha_i u_i|_{s_i}. \]

If either $m > 2$ or $\alpha_1 + \alpha_2 \neq 0$, then the result follows from induction on $m$. If $m = 2$ and $\alpha_1 + \alpha_2 = 0$, then the result follows from induction on $w_1$.

(II) $\implies$ (III) For any $f \in K[X;\Omega]$, by ELWs, we can obtain that $f + Id(S)$ can be expressed by the elements of $\operatorname{Irr}(S)$. Now suppose $\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n = 0$ in $K[X;\Omega][S]$ with $u_i \in \operatorname{Irr}(S)$, $u_1 > u_2 > \cdots > u_n$ and $\alpha_i \neq 0$. Then, in $K[X;\Omega]$,

\[ g = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n \in Id(S). \]

By (II), we have $u_1 = \bar{g} \notin \operatorname{Irr}(S)$, a contradiction. So $\operatorname{Irr}(S)$ is $K$-linearly independent. This shows that $\operatorname{Irr}(S)$ is a $K$-basis of $K[X;\Omega][S]$. 

7
(III)$\Rightarrow$(II) Let $0 \neq f \in \text{Id}(S)$. Suppose that $\bar{f} \in \text{Irr}(S)$. Then
\[
f + \text{Id}(S) = \alpha(\bar{f} + \text{Id}(S)) + \sum \alpha_i(u_i + \text{Id}(S))
\]
where $u_i \in \text{Irr}(S)$ and $\bar{f} > u_i$. Therefore, $f + \text{Id}(S) \neq 0$, a contradiction. So $\bar{f} = u|s_i$ for some $s \in S$ and $u \in \mathcal{Q}^*(X)$.

(II)$\Rightarrow$(I) By the definition of the composition, we have $(f, g)_w \in \text{Id}(S)$. If $(f, g)_w \neq 0$, then by (II), $(f, g)_w = u_1|s_1 < w$ for some $s_1 \in S$ and $u_1 \in \mathcal{Q}^*(X)$. Let
\[
h = (f, g)_w - \alpha_1 u_1|s_1
\]
where $\alpha_1$ is the coefficient of $(f, g)_w$. Then $h < w$ and $h \in \text{Id}(S)$. By induction on $w$, we can get the result. $lacksquare$

4 Gröbner-Shirshov bases for free commutative Rota-Baxter algebras

In this section, a Gröbner-Shirshov basis for free commutative Rota-Baxter algebra was obtained. By using the Composition-Diamond lemma (Theorem 3.2), a linear basis of such algebra was given and the free commutative Rota-Baxter algebra was directly constructed by commutative $\Omega$-words.

First of all, we define an order on $\mathcal{Q}(X)$, which will be used in this section. Let $X$ and $\Omega$ be well ordered. We define an order on $\mathcal{Q}(X) = \bigcup_{n \geq 0} \mathcal{Q}_n$ by induction on $n$. For any $u, v \in \mathcal{Q}_0 = \text{CS}(X)$, we have
\[
u = x_{i_1}^{j_1}x_{i_2}^{j_2} \cdots x_{i_t}^{j_t} \quad \text{and} \quad v = x_{i_1}^{j_1}x_{i_2}^{j_2} \cdots x_{i_t}^{j_t}
\]
where each $x_i \in X$, $i_k, j_t \geq 0$ and $x_i > x_{i+1}$. Then we define
\[
u > v \iff (\text{bre}(u), i_1, i_2, \ldots, i_t) > (\text{bre}(v), j_1, j_2, \ldots, j_t) \quad \text{lexicographically}.
\]
Assume the order on $\mathcal{Q}_{n-1}$ has been defined. Now, we define an order on $\Gamma_n = X \cup \Omega(\mathcal{Q}_{n-1})$. Let $v_1, v_2 \in \Gamma_n$. Then $v_1 > v_2$ means one of the following holds:

- (a) $v_1, v_2 \in X$ and $v_1 > v_2$;
- (b) $v_1 \in \Omega(\mathcal{Q}_{n-1})$ and $v_2 \in X$;
- (c) $v_1 = \theta_k(v_1'), v_2 = \theta_l(v_2'), v_1' \in \mathcal{Q}_k, v_2' \in \mathcal{Q}_l$ with $\theta_k > \theta_l$ or $\theta_k = \theta_l$, $v_1' > v_2'$ lexicographically.

For any $u, v \in \mathcal{Q}_n$, we have
\[
u = w_{i_1}^{j_1}w_{i_2}^{j_2} \cdots w_{i_t}^{j_t} \quad \text{and} \quad v = w_{i_1}^{j_1}w_{i_2}^{j_2} \cdots w_{i_t}^{j_t}
\]
where each $w_i \in \Gamma_n$, $i_k, j_t \geq 0$ and $w_i > w_{i+1}$. Here, $\text{bre}(u) = i_1 + i_2 + \cdots + i_t$. Define
\[
u > v \iff (\text{bre}(u), i_1, i_2, \ldots, i_t) > (\text{bre}(v), j_1, j_2, \ldots, j_t) \quad \text{lexicographically} \quad (1)
\]
Then the Order (II) is a monomial order on $\mathcal{Q}(X)$. 


Let $K$ be a commutative ring with unit and $\lambda \in K$. A commutative Rota-Baxter algebra of weight $\lambda$ (see [1,13,19]) is a commutative $K$-algebra $R$ with a linear operator $P : R \to R$ satisfying the Rota-Baxter relation:

$$P(u)P(v) = P(P(u)v + P(uP(v)) + \lambda P(uv), \forall u,v \in R.$$  

It is obvious that any commutative Rota-Baxter algebra is a commutative algebra with multiple operators $\Omega$, where $\Omega = \{K\}$ and $S$ is a $K$-Gröbner-Shirshov basis in $R$.

In this section, we assume that $\Omega = \{1\}$. Let $Q(X)$ be defined as before with $\Omega = \{1\}$ and $K[X;P]$ be the free commutative algebra with operator $\Omega = \{P\}$ on set $X$.

**Theorem 4.1** With the order $[1]$ on $Q(X)$,

$$S = \{P(u)P(v) - P(P(u)v + P(uP(v)) - \lambda P(uv)| u, v \in Q(X)\}$$

is a Gröbner-Shirshov basis in $K[X;P]$.

**Proof:** The ambiguities of all possible compositions of the commutative $\Omega$-polynomials in $S$ are:

(i) $P(u)P(v)P(w) \quad$ (ii) $P(z|P(v)P(w))P(u)$

where $u,v,w \in Q(X), z \in Q^*(X)$. It is easy to check that all these compositions are trivial. Here, for example, we just check (i). For any $u,v \in Q(X)$, let

$$f(u,v) = P(u)P(v) - P(P(u)v + P(uP(v)) - \lambda P(uv).$$

Then

$$(f(u,v), f(v,w))_{P(u)P(v)P(w)}$$

$$= -P(P(u)v)P(w) - P(uP(v)P(w) - \lambda P(uv)P(w)$$

$$+P(u)P(v)P(w) + P(uP(vP(w))) + \lambda P(uP(vw))$$

$$\equiv -P(u)P(vP(w)) - P(uP(vP(w))) - \lambda P(uvP(w))$$

$$-P(P(u)P(w)) + P(vP(w)) + \lambda P(uP(vP(w)))$$

$$+\lambda P(P(vP(w)) + \lambda P(uP(vP(w))$$

$$\equiv 0 \mod(S, P(uP(vP(w)))$$

Define

$$\Phi_0 = CS(X),$$

$$\Phi_1 = \Phi_0 \cup P(\Phi_0) \cup \Phi_0 P(\Phi_0),$$

$$\vdots$$

$$\Phi_n = \Phi_0 \cup P(\Phi_{n-1}) \cup \Phi_0 P(\Phi_{n-1}),$$

$$\vdots$$
where 
\[ \Phi_0 P(\Phi_{n-1}) = \{ uP(v) | u \in \Phi_0, v \in \Phi_{n-1} \} \].

Let 
\[ \Phi(X) = \bigcup_{n \geq 0} \Phi_n \].

The elements in \( \Phi(X) \) are called commutative Rota-Baxter words.

By Theorem 4.1 and Theorem 5.2, we have the following theorems.

**Theorem 4.2** \((\text{[6]})\) \( \text{Irr}(S) = \Phi(X) \) is a basis of \( K[X; \Omega|S] \).

**Theorem 4.3** \( K[X; \Omega|S] \) is a free commutative Rota-Baxter algebra of weight \( \lambda \) on set \( X \) with a basis \( \Phi(X) \).

By using ELWs, we have the following algorithm. In fact, it is an algorithm to compute the product of two commutative Rota-Baxter words in the free commutative Rota-Baxter algebra \( K[X; \Omega|S] \).

**Algorithm** Let \( u, v \in \Phi(X) \). We define \( u \circ v \) by induction on \( n = \text{dep}(u) + \text{dep}(v) \).

1. If \( n = 0 \), then \( u, v \in CS(X) \) and \( u \circ v = uv \).
2. If \( n \geq 1 \), then \( u \circ v = \begin{cases} uv & \text{if } u \in CS(X) \text{ or } v \in CS(X) \\ u_1v_1(P(u' \circ v') + u' \circ P(v')) + \lambda P(u' \circ v') & \text{if } u = u_1P(u'), v = v_1P(v') \end{cases} \)

5 **Gröbner-Shirshov bases for free commutative \( \lambda \)-differential algebras**

In this section, we give a Gröbner-Shirshov basis for a free commutative \( \lambda \)-differential algebra and then by using the Composition-Diamond lemma (Theorem 5.2), we obtain a linear basis of such an algebra, which is the same as the one in \([15]\). Consequently, we construct the free \( \lambda \)-differential algebra on set \( X \) directly by commutative \( \Omega \)-words.

Let \( K \) be a commutative unitary ring and \( \lambda \in K \). A commutative \( \lambda \)-differential algebra \((\text{[15, 16]}\) over \( K \) is a commutative \( K \)-algebra \( R \) together with a linear operator \( D : R \to R \) such that
\[ D(uv) = D(u)v + uD(v) + \lambda D(u)D(v), \; \forall u, v \in R. \]

It is obvious that any commutative \( \lambda \)-differential algebra is a commutative algebra with multiple operators \( \Omega \) on \( X \), where \( \Omega = \{ D \} \).

In this section, we assume that \( \Omega = \{ D \} \). Let \( K[X; D] \) be the free commutative algebra with operator \( \Omega = \{ D \} \). Let \( X \) be well ordered. For any \( u \in Q(X) \), \( u \) has a unique expression
\[ u = u_1u_2 \cdots u_n \]
where \( n \geq 1 \) and each \( u_i \in X \cup D(Q(X)) \). Define \( \text{deg}(u) \) the number of \( x \in X \) in \( u \). For example, if \( u = x_1D(D(x_2))D(x_3) \), then \( \text{deg}(u) = 3 \).
We define an order on $\mathcal{Q}(X) = \bigcup_{n \geq 0} \mathcal{Q}_n$ by induction on $n$. For any $u, v \in \mathcal{Q}_0 = CS(X)$, we have
\[ u = x_1^{i_1} x_2^{i_2} \cdots x_t^{i_t} \quad \text{and} \quad v = x_1^{j_1} x_2^{j_2} \cdots x_t^{j_t} \]
where each $x_i \in X$, $i_k, j_l \geq 0$ and $x_i > x_{i+1}$. Then we define
\[ u > v \iff (\deg(u), i_1, i_2, \ldots, i_t) > (\deg(v), j_1, j_2, \ldots, j_t) \quad \text{lexicographically}. \]

Assume the order on $\mathcal{Q}_{n-1}$ has been defined. Now, we define an order on $\Gamma_n = X \cup \Omega(\mathcal{Q}_{n-1})$ firstly. Let $v_1, v_2 \in \Gamma_n$. Then $v_1 > v_2$ means one of the following holds:
(a) $v_1, v_2 \in X$ and $v_1 > v_2$;
(b) $v_1 \in D(\mathcal{Q}_{n-1})$ and $v_2 \in X$;
(c) $v_1 = D(v_1')$, $v_2 = D(v_2')$ and $v_1' > v_2'$.

For any $u, v \in \mathcal{Q}_n$, we have
\[ u = w_1^{i_1} w_2^{i_2} \cdots w_t^{i_t} \quad \text{and} \quad v = w_1^{j_1} w_2^{j_2} \cdots w_t^{j_t} \]
where each $w_i \in \Gamma_n$, $i_k, j_l \geq 0$ and $w_i > w_{i+1}$. Define
\[ u > v \iff (\deg(u), i_1, i_2, \ldots, i_t) > (\deg(v), j_1, j_2, \ldots, j_t) \quad \text{lexicographically} \quad (2) \]

Then the Order (2) is a monomial order on $\mathcal{Q}(X)$.

**Theorem 5.1** With the Order (2) on $\mathcal{Q}(X)$,
\[ S = \{ D(uv) - D(u)v - uD(v) - \lambda D(u)D(v) \mid u, v \in \mathcal{Q}(X) \} \]
is a Gröbner-Shirshov basis in $K[X; D]$.

**Proof:** The ambiguities of all possible compositions of the commutative $\Omega$-polynomials in $S$ are
\[ D(u|_{D(xy)} v) \]
where $x, y, v \in \mathcal{Q}(X), u \in \mathcal{Q}^*(X)$. Let
\[ g(u, v) = D(uv) - D(u)v - uD(v) - \lambda D(u)D(v). \]

Then
\[
\begin{align*}
(g(u|_{D(xy)} v), g(x, y)) & D(u|_{D(xy)} v) \\
= -D(u|_{D(xy)} v) - u|_{D(xy)} D(v) + \lambda D(u|_{D(xy)} v) D(v) \\
& + D(u|_{D(xy)} v) + D(u|_{x D(y)} v) + \lambda D(u|_{x D(y)} v) D(v) \\
& \equiv -D(u|_{D(xy)} v) - D(u|_{x D(y)} v) - \lambda D(u|_{D(xy)} D(v)) v \\
& - u|_{D(xy)} D(v) - u|_{x D(y)} D(v) - \lambda u|_{D(xy)} D(v) \\
& - \lambda D(u|_{x D(y)} D(v)) - \lambda D(u|_{x D(y)} D(v)) - \lambda^2 D(u|_{D(xy)} D(v)) D(v) \\
& + D(u|_{D(xy)} v) + u|_{D(xy)} D(v) + \lambda D(u|_{D(xy)} D(v)) D(v) \\
& + D(u|_{D(xy)} v) + u|_{x D(y)} D(v) + \lambda D(u|_{x D(y)} D(v)) D(v) \\
& + \lambda D(u|_{D(xy)} D(v)) + \lambda u|_{D(xy)} D(v) + \lambda^2 D(u|_{D(xy)} D(v)) D(v) \\
& \equiv 0 \quad \text{mod}(S, D(u|_{D(xy)} v)).
\end{align*}
\]

Let $D^\omega(X) = \{ D^i(x) \mid i \geq 0, x \in X \}$, where $D^0(x) = x$ and $CS(D^\omega(X))$ the free commutative semigroup generated by $D^\omega(X)$.

11
Theorem 5.2 \((15)\) \(Irr(S) = CS(D^\omega(X))\) is a \(K\)-basis of \(K[X; D|S]\).

**Proof:** By Theorem 3.2 and Theorem 5.1.

Theorem 5.3 \((15)\) \(K[X; D|S]\) is a free commutative \(\lambda\)-differential algebra on set \(X\) with a basis \(CS(D^\omega(X))\).

**Proof:** By Theorem 5.2.

By using ELWs, we have the following algorithm.

**Algorithm** \((15)\) Let \(u = u_1u_2\cdots u_n\), where each \(u_k \in D^\omega(X), n > 0\). Define \(D(u)\) by induction on \(n\).

1. If \(n = 1\), i.e., \(u = D^i(x)\) for some \(i \geq 0, x \in X\), then \(D(u) = D^{(i+1)}(x)\).
2. If \(n \geq 1\), then

\[
D(u) = D(u_1u_2\cdots u_n) = D(u_1)(u_2\cdots u_n) + u_1D(u_2\cdots u_n) + \lambda D(u_1)D(u_2\cdots u_n).
\]

6 Gröbner-Shirshov bases for free commutative \(\lambda\)-differential Rota-Baxter algebras

In this section, we give a Gröbner-Shirshov basis for a free commutative \(\lambda\)-differential Rota-Baxter algebra on a set. By using the Composition-Diamond lemma for commutative algebras with multiple operators (Theorem 3.2), we obtain a linear basis of a free commutative \(\lambda\)-differential Rota-Baxter algebra on a set. Also, we construct the free commutative \(\lambda\)-differential Rota-Baxter algebra on a set directly by commutative \(\Omega\)-words.

Let \(K\) be a unitary commutative ring and \(\lambda \in K\). A commutative \(\lambda\)-differential Rota-Baxter algebra \((15)\) is a commutative \(K\)-algebra \(R\) with two linear operators \(P, D : R \to R\) such that, for any \(u, v \in R\),

1. \((\text{Rota-Baxter relation})\) \(P(u)P(v) = P(uP(v)) + P(P(u)v) + \lambda P(uv),\)
2. \((\lambda\text{-differential relation})\) \(D(uv) = D(u)v + uD(v) + \lambda D(u)D(v),\)
3. \((\lambda\text{-differential relation})\) \(D(P(u)) = u.\)

It is obvious that any commutative \(\lambda\)-differential Rota-Baxter algebra is a commutative algebra with multiple operators \(\Omega\) where \(\Omega = \{P, D\}\).

In this section, we assume that \(\Omega = \{P, D\}\). Let \(K[X; \Omega]\) be the free commutative algebra with multiple operators \(\Omega\) on \(X\), where \(\Omega = \{P, D\}\).

Let \(X\) be well ordered and \(D > P\). For any \(u \in Q(X)\), define \(\text{deg}_\mu(u)\) the number of \(P\) in \(u\).

Define an order on \(Q(X) = \bigcup_{n\geq0} Q_n\) by induction on \(n\). For any \(u, v \in Q_0 = CS(X)\), we have

\[
u = x_1^{i_1}x_2^{i_2}\cdots x_t^{i_t} \quad \text{and} \quad v = x_1^{j_1}x_2^{j_2}\cdots x_t^{j_t}
\]

where each \(x_i \in X, i_k, j_l \geq 0\) and \(x_i > x_{i+1}\). Then we define

\[
u > v \iff (\text{deg}(u), i_1, i_2, \cdots, i_t) > (\text{deg}(v), j_1, j_2, \cdots, j_t) \quad \text{lexicographically}.
\]

12
Assume the order on \( \mathbb{Q}_{n-1} \) has been defined. For any \( u, v \in \mathbb{Q}_n \), we have

\[
u = u_1^{k_1} \cdots u_s^{k_s} \text{ and } v = u_1^{l_1} \cdots u_s^{l_s}\]

where each \( u_i \in X \cup \Omega(\mathbb{Q}_{n-1}) \), \( k_i, l_i \geq 0 \) and \( u_i > u_{i+1} \). Here, \( u_i > u_{i+1} \) means one of the following holds:

(a) \( u_i, u_{i+1} \in X \) and \( u_i > u_{i+1} \),
(b) \( u_i \in D(\mathbb{Q}_{n-1}) \) or \( u_i \in P(\mathbb{Q}_{n-1}) \) and \( u_{i+1} \in X \),
(c) \( u_i = \theta_1(u'_i), u_{i+1} = \theta_2(u'_{i+1}) \) where \( \theta_1, \theta_2 \in \{D, P\} \) and

\[
(deg(u_i), deg_P(u_i), \theta_1, u'_i) > (deg(u_i), deg_P(u_i), \theta_2, u'_{i+1}) \text{ lexicographically}
\]

Then we define \( u > v \) if and only if

\[
(deg(u), deg_P(u), bre(u), k_1, \ldots, k_s) > (deg(v), deg_P(v), bre(v), l_1, \ldots, l_s) \text{ lexicographically}
\]

Then the Order \([3]\) is a monomial order on \( \mathbb{Q}(X) \).

Let \( S \) be a set consisting of the following commutative \( \Omega \)-polynomials:

1. \( P(u)P(v) - P(uP(v)) - P(P(u)v) - \lambda P(uv), \quad u, v \in \mathbb{Q}(X); \)
2. \( D(uv) - D(u)v - uD(v) - \lambda D(u)D(v), \quad u, v \in \mathbb{Q}(X); \)
3. \( D(P(u)) - u, \quad u \in \mathbb{Q}(X). \)

**Theorem 6.1** With the Order \([3]\) on \( \mathbb{Q}(X) \), \( S \) is a Gröbner-Shirshov basis in \( K[X; \Omega] \).

**Proof.** Denote by \( i \wedge j \) the composition of \( \Omega \)-polynomials of type \( i \) and type \( j \). The ambiguities of all possible compositions of commutative \( \Omega \)-polynomials in \( S \) are only as below. In the following list, \( i \wedge j \) \( w \) means \( w \) is the ambiguity of the composition \( i \wedge j \).

\[
\begin{align*}
3 \wedge 3 & \quad D(P(u|_{D(P(u))}),) \quad 3 \wedge 2 & \quad D(P(u|_{D(uv)})) \\
3 \wedge 1 & \quad D(P(u|_{P(u)v})) & 2 \wedge 3 & \quad D(u|_{D(P(u))}w) \\
2 \wedge 2 & \quad D(u|_{D(uv)z}) & 2 \wedge 1 & \quad D(u|_{P(u)v}z) \\
1 \wedge 3 & \quad P(u|_{D(P(u))})P(w) & 1 \wedge 2 & \quad P(u|_{D(uv)})P(z) \\
1 \wedge 1 & \quad P(z)P(v)P(w) & 1 \wedge 1 & \quad P(v)P(u|_{P(v)P(u)})
\end{align*}
\]

where \( u \in \mathbb{Q}'(X), v, w, z \in \mathbb{Q}(X) \). It is easy to check that all these compositions are trivial. Similar to the proofs in Theorem \([4]\) and Theorem \([5]\), \( 1 \wedge 1 \) and \( 2 \wedge 2 \) are trivial. Others are also easily checked. Here we just check one for examples.

\[
2 \wedge 3 = -D(u|_{D(P(u))})w - u|_{D(P(u))}D(w) - \lambda D(u|_{D(P(u))})D(w) + D(u|_w)w = -D(u|_w)w - u|_wD(w) - \lambda D(u|_w)D(w) + D(u|_w)w + D(u|_w)D(w) + D(u|_w)D(w) = 0 \mod(S, D(u|_{D(P(u))}w)).
\]

\[\blacksquare\]
Let $D^ω(X) = \{D^i(x) | i \geq 0, x \in X\}$, where $D^0(x) = x$. Define

$$
\Upsilon_0 = CS(D^ω(X)),
\Upsilon_1 = \Upsilon_0 \cup P(\Upsilon_0) \cup \Upsilon_0 P(\Upsilon_0),
\vdots
\Upsilon_n = \Upsilon_0 \cup P(\Upsilon_{n-1}) \cup \Upsilon_0 P(\Upsilon_{n-1}),
\vdots
$$

and

$$
\Upsilon(D^ω(X)) = \bigcup_{n \geq 0} \Upsilon_n.
$$

**Theorem 6.2** $\text{Irr}(S) = \Upsilon(D^ω(X))$ is a basis of $K[X; \Omega|S]$.

**Proof:** By Theorem 3.2 and Theorem 6.1, we can obtain the result easily. ■

**Theorem 6.3** $K[X; \Omega|S]$ is a free commutative $\lambda$-differential Rota-Baxter algebra on set $X$ with a basis $\Upsilon(D^ω(X))$.

**Proof:** By Theorem 6.2

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