SYMPLECTIC COHOMOLOGICAL RIGIDITY VIA TORIC DEGENERATIONS

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Abstract. In this paper we study whether symplectic toric manifolds are symplectically cohomologically rigid. Here we say that symplectic cohomological rigidity holds for some family of symplectic manifolds if the members of that family can be distinguished by their integral cohomology rings and the cohomology classes of their symplectic forms. We show how toric degenerations can be used to produce the symplectomorphisms necessary to answer this question. As a consequence we prove that symplectic cohomological rigidity holds for the family of symplectic Bott manifolds with rational symplectic form whose rational cohomology ring is isomorphic to $\mathrm{H}^*(\mathbb{C}P^1)^n; \mathbb{Q}$ for some $n$. In particular, we classify such manifolds up to symplectomorphism. Moreover, we prove that any symplectic toric manifold with rational symplectic form whose integral cohomology ring is isomorphic to $\mathrm{H}^*(\mathbb{C}P^1)^n; \mathbb{Z}$ is symplectomorphic to $(\mathbb{C}P^1)^n$ with a product symplectic structure.

1. Introduction

One powerful method to study (smooth) manifolds is to calculate their invariants. In particular, if two manifolds are diffeomorphic, then their integral cohomology rings are isomorphic. We say that a family of manifolds is cohomologically rigid if the converse holds, that is, if any two manifolds in the family with isomorphic integral cohomology rings are diffeomorphic.

Most families of manifolds cannot be classified by their cohomology ring in this way, but there are some important exceptions. For example, cohomological rigidity holds for 2 dimensional manifolds. More interestingly, Freedman proved that closed simply connected 4 manifolds are classified up to homeomorphism by their cohomology ring. In contrast, there are infinite families of closed simply connected 4 manifolds that are homeomorphic (and so have same cohomology ring) but not diffeomorphic.

1In the literature, the term “cohomologically rigid” is sometimes defined in terms of homeomorphism, rather than diffeomorphism.
Masuda and Suh asked if cohomological rigidity holds for another important family: smooth toric varieties. Here, by toric variety we mean a compact algebraic variety $X$ of complex dimension $n$ equipped with an action of the algebraic torus $(\mathbb{C}^\times)^n$ with a dense orbit. Smooth toric varieties are classified by, and can be studied in terms of, a purely combinatorial invariant, called a fan. As Masuda shows, this implies that smooth toric varieties are equivariantly diffeomorphic exactly if their integral equivariant cohomology rings are isomorphic as algebras over the equivariant cohomology of a point $[M]$. In contrast, the classification of smooth toric varieties up to diffeomorphism is still poorly understood; in particular, Masuda and Suh’s question is still open. Nevertheless, there have been some significant partial positive results.

To state these results, we need several definitions. First, a Bott manifold is the total space of a sequence of $\mathbb{CP}^1$ bundles starting with $\mathbb{CP}^1$, where each $\mathbb{CP}^1$ bundle is the projectivization of the Whitney sum of two holomorphic line bundles $[GK]$. Bott manifolds are smooth toric varieties: at each stage the torus action on the base lifts to the line bundles, and hence to their projectivizations; moreover, there is an action of the multiplicative group $\mathbb{C}^\times$ on the total space that restricts on each fiber to the standard action on $\mathbb{CP}^1$. Second, we say that strong cohomological rigidity holds for a family of manifolds if every isomorphism between the cohomology rings of any two manifolds in the family is induced by a diffeomorphism.

Choi and Masuda proved that strong cohomological rigidity holds for the family of Bott manifolds whose rational cohomology ring is isomorphic to that of a product of projective planes $[CM]$. This reproves an earlier result of Masuda and Panov stating that a Bott manifold $X$ is diffeomorphic to $\prod \mathbb{CP}^1$ if and only if $H^*(X;\mathbb{Z}) = H^*(\prod \mathbb{CP}^1;\mathbb{Z})$ $[MP]$ Theorem 5.1$. In $[CMS10]$, Choi, Masuda, and Suh considered a generalization of Bott manifolds where the base and fibers can be projective spaces of any dimension. They proved that such a manifold is diffeomorphic to $\prod_j \mathbb{CP}^{n_j}$ if and only if its integral cohomology ring is isomorphic to $H^*(\prod_j \mathbb{CP}^{n_j};\mathbb{Z})$. Additionally, strong cohomological rigidity holds for Bott manifolds of dimension at most $8$ $[C]$, and for the family Bott manifolds for which at most one bundle in the fibration is not trivial $[CS]$. For a more detailed survey, see $[CMS11]$.

In this paper, we study the symplectic counterpart of the question posed by Masuda and Suh. We say that a family of symplectic manifolds is symplectically cohomologically rigid if two symplectic manifolds $(M, \omega)$ and $(\tilde{M}, \tilde{\omega})$ in the family are symplectomorphic exactly if there is isomorphism $H^*(M;\mathbb{Z}) \to H^*(\tilde{M};\mathbb{Z})$ sending the class
$\omega$ to the class $[\tilde{\omega}]$. Our main goal is to determine if the family of symplectic toric manifolds is symplectically cohomologically rigid. Here, by **symplectic toric manifold** we mean a $2n$-dimensional closed connected symplectic manifold $(M, \omega)$ equipped with an effective $(S^1)^n$ action with **moment map** $\mu: M \to \mathbb{R}^n$, i.e., $\iota_{\xi_i} \omega = -d\mu_i$ for each $i \in \{1, \ldots, n\}$, where $\xi_i$ is the vector field induced on $M$ by the $i$'th copy of $(S^1)$. The moment polytope $\Delta := \mu(M)$ of $M$ is a **smooth polytope**, i.e., $\Delta$ is a rational polytope and the primitive integral vectors perpendicular to the facets meeting at any one vertex of the **moment polytope** $\Delta$ form a $\mathbb{Z}$ basis for $\mathbb{Z}^n$. As Delzant proved, these polytopes classify symplectic toric manifolds: If two symplectic toric manifolds have the same moment polytope, then there exists an equivariant symplectomorphism between them that intertwines the moment maps. Moreover, every smooth polytope arises as the moment polytope of an $(S^1)^n \subset (\mathbb{C}^\times)^n$ invariant Kähler form on a smooth toric variety. As a consequence, every symplectic toric manifold is equivariantly diffeomorphic to a smooth toric variety.

In contrast, not much is known about the classification of symplectic toric manifolds up to symplectomorphism. Karshon, Kessler and Pinsonnault proved that any 4-dimensional symplectic toric manifold admits only finitely many inequivalent toric action, that is, it is only symplectomorphic to a finite number of distinct symplectic toric manifolds [KKP]. This result was later generalized to all dimensions by McDuff [McD]. Even more relevantly for our paper, she also proved that the family of symplectic toric manifolds whose cohomology ring is isomorphic to that of a product of projective spaces is symplectically cohomologically rigid [McD, Section 2.4].

In this work, we prove that strong symplectic cohomological rigidity holds for the family of symplectic Bott manifolds whose rational cohomology ring is isomorphic to that of a product of $\mathbb{CP}^1$'s. More precisely, we prove the theorem below, which also appears in Section 5 as Theorem 5.1. As we explain in Section 3, a symplectic Bott manifold is a Bott manifold equipped with an $(S^1)^n$ invariant Kähler form. Our main tool is a new method of constructing symplectomorphism between symplectic toric manifolds using toric degeneration, which is described in more detail below.

**Theorem.** Let $(M, \omega)$ and $(\tilde{M}, \tilde{\omega})$ be symplectic Bott manifolds with rational symplectic forms\(^2\) such that $$H^*(M; \mathbb{Q}) \simeq H^*(\tilde{M}; \mathbb{Q}) \simeq H^*((\mathbb{CP}^1)^n; \mathbb{Q}).$$

\(^2\)The result still holds if some multiples of the symplectic forms are rational.
Given a ring isomorphism $F: H^*(M;\mathbb{Z}) \to H^*(\tilde{M};\mathbb{Z})$ such that $F([\omega]) = [\tilde{\omega}]$, there exists a symplectomorphism $f$ from $(M, \omega)$ to $(M, \omega)$ so that the induced map $f^*: H^*(M;\mathbb{Z}) \to H^*(\tilde{M};\mathbb{Z})$ is equal to $F$.

Given $\lambda \in \mathbb{R}^n$ such that $\lambda_i \neq 0$ for all $i$, let $\omega_{\lambda} \in \Omega^2((\mathbb{C}P^1)^n)$ be the symplectic form $\sum_i \lambda_i \pi_i^*(\omega_{FS})$, where $\pi_i: (\mathbb{C}P^1)^n \to \mathbb{C}P^1$ denotes the projection onto the $i$-th factor, and $\omega_{FS}$ is the Fubini-Study form on $\mathbb{C}P^1$ with area 1.

**Corollary 1.1.** Let $(M, \omega)$ be a symplectic toric manifold with rational symplectic form, and let $F: H^*(M;\mathbb{Z}) \to H^*((\mathbb{C}P^1)^n;\mathbb{Z})$ be an isomorphism. Then there exists $\lambda \in \mathbb{Q}^n$ with $\lambda_i \neq 0$ for all $i$ and a symplectomorphism $f$ from $((\mathbb{C}P^1)^n, \omega_{\lambda})$ to $(M, \omega)$ so that $f^* = F$.

**Proof.** As a consequence of results of Masuda and Panov in [MP], any symplectic toric manifold whose integral cohomology ring is isomorphic to $H^*((\mathbb{C}P^1)^n;\mathbb{Z})$ is symplectomorphic to a symplectic Bott manifold; see Proposition 3.7 in this paper for more details. Therefore, we may assume that $(M, \omega)$ is a symplectic Bott manifold.

Since $\omega$ is rational, $F([\omega]) = [\sum_i \lambda_i \pi_i^*(\omega_{FS})]$ for some $\lambda_i \in \mathbb{Q}$ for all $i$. Since $\omega$ is symplectic $[\omega]^n \neq 0$, and so $\lambda_i \neq 0$ for all $i$. Since complex conjugation induces a symplectomorphism from $(\mathbb{C}P^1, \omega_{FS})$ to $(\mathbb{C}P^1, -\omega_{FS})$, we may assume without loss of generality that $\lambda_i > 0$ for all $i$; hence, $(\mathbb{C}P^1)^n, \omega_{\lambda})$ is also a symplectic Bott manifold.

The claim now follows immediately from the theorem above.

Another corollary can be obtained by building on a result of Illinskii [I] (reproved in [MP]): Let $X$ be a smooth toric variety of complex dimension $n$. If a circle subgroup of $(\mathbb{C}^\times)^n$ acts semifreely with isolated fixed points, then $X$ is diffeomorphic to $(\mathbb{C}P^1)^n$.

**Corollary 1.2.** Let $(M, \omega)$ be a $2n$-dimensional symplectic toric manifold with rational symplectic form. If a circle subgroup of $(S^1)^n$ acts semifreely with isolated fixed points, then $(M, \omega)$ is symplectomorphic to $((\mathbb{C}P^1)^n, \omega_{\lambda})$ for some $\lambda \in \mathbb{Q}^n$ such that $\lambda_i > 0$ for all $i$.

Finally, as a corollary of our proof, we obtain a complete classification (up to symplectomorphism) of symplectic Bott manifolds with rational symplectic forms whose rational cohomology rings are isomorphic to $H^*((\mathbb{C}P^1)^n;\mathbb{Q})$ up to symplectomorphism; see Corollary 5.7.

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3By a slight abuse of notation, we also denote the induced map from $H^*(M;\mathbb{R})$ to $H^*(\tilde{M};\mathbb{R})$ by $F$. 
We will now briefly explain how we use toric degenerations to construct the symplectomorphisms required for the proof of the main theorem. The main tool for constructing symplectomorphisms is a combination of results of Anderson and Harada-Kaveh. Given a smooth projective variety $X$, of complex dimension $n$, equipped with a very ample line bundle, satisfying certain conditions, Anderson constructs a toric degeneration, i.e., a flat family $\pi: \mathfrak{X} \to \mathbb{C}$ with generic fiber $X$ so that the special fiber $X_0 = \pi^{-1}(0)$ is a (not necessarily normal) toric variety ([A, Theorem 2]). Harada and Kaveh observed that one can build a symplectic form $\tilde{\omega}$ on $\mathfrak{X}$ and use the flow of a certain vector field to obtain a surjective continuous map $\phi: (X, \omega) \to (X_0, \omega_0 := \tilde{\omega}|_{X_0})$, which is a symplectomorphism when restricted to a particular dense open subset $U$ of $X$ ([HK, Theorem A]). In especially nice situations this map is actually a symplectomorphism from $X$ to $X_0$.

**Organization** The paper is organized as follows. In Section 2 we explain how to obtain symplectomorphisms between symplectic toric manifolds using toric degenerations. Section 3 is devoted to symplectic Bott manifolds. We construct the symplectomorphism between specific symplectic Bott manifolds in Section 4. Finally, the main result is proved in Section 5.

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2. Symplectomorphisms coming from toric degenerations

In this section we prove that two symplectic toric manifolds are symplectomorphic if their moment polytopes are related by a “slide” (Proposition 2.7). Our main tool is a theorem of Harada and Kaveh which, in many cases, gives a map, from a given smooth projective variety $X$ with integral symplectic form to toric variety $X_0$, which is a symplectomorphism when restricted to an open dense subset.

Consider a smooth projective variety $X$ of complex dimension $n$ equipped with a very ample Hermitian line bundle $\mathcal{L}$. Let $L := H^0(X, \mathcal{L})$ denote the vector space of holomorphic sections. The Hermitian structure on $\mathcal{L}$ induces a Hermitian structure on $L$, and hence a Fubini-Study form $\omega_{FS}$ on $\mathbb{P}(L^*)$. Since the Kodaira embedding $\Phi_L: X \to \mathbb{P}(L^*)$ is
holomorphic, the pull-back $\omega := \Phi_\ast^L(\omega_{FS})$ is a Kähler form on $X$; moreover, $[\omega]$ is the first Chern class of $L$.

Fix a coordinate system $(u_1, \ldots, u_n)$ near a point $p \in X$, that is, assume that $u_i \in \mathbb{C}(X)$ satisfies $u_i(p) = 0$ for all $i$, and that $du_1, \ldots, du_n$ are linearly independent. Here, $\mathbb{C}(X)$ denotes the ring of rational functions on $X$. We define a function $\nu : \mathbb{C}(X) \setminus \{0\} \to \mathbb{Z}^n$ as follows: If a nonzero function $f \in \mathbb{C}(X)$ is regular at $p$, then nearby it can be represented as a power series $\sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} c_\alpha u_\alpha$. Define $\nu(f) := \min \{ \alpha | c_\alpha \neq 0 \}$, where the minimum is taken with respect to the lexicographical order. More generally, if $f, g \in \mathbb{C}(X)$ are regular at $p$, define $\nu(f/g) = \nu(f) - \nu(g)$. We call $\nu$ the valuation associated to the coordinate system. As noted in [HK, Example 3.2], this function satisfies the axioms for a valuation with one-dimensional leaves, as described in [HK, Definition 3.1]. In particular, $\nu(fg) = \nu(f) + \nu(g)$ for all $f, g \in \mathbb{C}(X) \setminus \{0\}$. Moreover, given any finite dimensional subspace $E \subset \mathbb{C}(X)$, the cardinality of the image $\nu(E \setminus \{0\})$ is the dimension of $E$ [HK, Proposition 3.4].

Let $L^m$ denote the image of $L^\otimes m$ in $H^0(X, \mathcal{L}^\otimes m)$, and let $R = \mathbb{C}[X] = \oplus_{m \geq 0} L^m$ be the homogeneous coordinate ring of $X$ with respect to the embedding Kodaira embedding $\Phi_L$. Choose a non-zero element $h \in L$ and identify $L^m$ with a subspace of $\mathbb{C}(X)$ by sending $f \in L^m$ to $f/h^m \in \mathbb{C}(X)$. Then we can define an additive semigroup

$$S = S(\nu, \mathcal{L}) = \bigcup_{m \geq 0} \{ (m, \nu(f/h^m) \mid f \in L^m \setminus \{0\} \} = \bigcup_{m \geq 0} \{m\} \times S_m,$$

where $S_m := \{ \nu(f/h^m) \mid f \in L^m \setminus \{0\} \}$. The dependence on $h$ is minor: replacing $h$ by $h'$ simply shifts $S_m$ by $m \nu(h/h')$. Hence, we will usually omit $h$. The Okounkov body associated to $S$ is

$$\Delta = \Delta(S) = \text{conv}(\bigcup_{m > 0} \{x/m \mid (m, x) \in S\}) \subset \mathbb{R}^n.$$

If $S$ is finitely generated, then the Okounkov body $\Delta$ is a rational polytope.

Finally, given a finitely generated semigroup $S \subset \mathbb{Z} \times \mathbb{Z}^n$, there is a natural $(\mathbb{C}^\times)^n$ action on the variety $X_0 := \text{Proj} \mathbb{C}[S]$, making it into a (not necessarily normal) toric variety.

We will need a corollary of the following theorem, which is a slight variant of Theorem 3.25 in [HK].

**Theorem 2.1.** Let $X$ be a smooth projective variety of complex dimension $n$, equipped with a very ample Hermitian line bundle $\mathcal{L}$, and
let \( \omega \) be the associated Kähler form. Let \( S \) be the semigroup associated to a valuation derived from a coordinate system on \( X \). Assume that \( S \) is finitely generated; let \( \Delta \) and \( X_0 := \Proj \mathbb{C}[S] \) be the associated Okounkov body and toric variety, respectively.

1. There’s a linear \((\mathbb{C}^*)^n\) action on \( \mathbb{C}^{n+1} \), an \((S^1)^n\) invariant symplectic form \( \Omega \) on \( \mathbb{C}P^N \) with moment map \( \mu_\Omega : \mathbb{C}P^N \to \mathbb{R}^n \), and an equivariant embedding \( X_0 \hookrightarrow \mathbb{C}P^N \) such that \( \mu_\Omega(X_0) = \Delta \).

2. There’s a surjective continuous map \( \phi : X \to X_0 \) that restricts to a symplectomorphism from a dense open subset \( U \subset X \) to \( (X_0)_{\text{smooth}} \), the set of smooth points in \( X_0 \), with the symplectic form induced from \( \Omega \).

The proof of Theorem 3.25 given in [HK] actually proves the above statement. Unfortunately, this fact may not be immediately obvious to the casual reader, and so we will sketch the proof.

**Proof.** Anderson constructs a toric degeneration (in the sense of [HK, Definition 2]), i.e. a flat family \( \pi : X \to \mathbb{C} \) with generic fiber \( X \) and a special fiber \( X_0 = \pi^{-1}(0) \), a (not necessarily normal) toric variety; see [A, Corollary 5.2] or [HK, Corollary 3.14].

In [HK, Section 3.3], Harada and Kaveh fix a positive multiple \( \Omega \) of the Fubini-Study Kähler form on \( \mathbb{C}P^N \), construct an embedding \( X \hookrightarrow \mathbb{C}P^N \times \mathbb{C} \) inducing a symplectic structure \( \tilde{\omega} \) on the smooth part of \( X \) as a pull back of \( \Omega \times Id \) \( d\bar{z} \wedge dz \), and show that these satisfy the assumptions (a)-(d) of [HK, Theorem 2.19]. See [HK, Corollary 3.15, Remarks 3.17 and 3.19, and Proposition 3.24] for details. In particular, assumption (b) implies that there exists a linear action of \((\mathbb{C}^*)^n\) on \( \mathbb{C}^{n+1} \) so that the induced embedding of \( X_0 \hookrightarrow \mathbb{C}P^N \times \{0\} \) is equivariant. The weights that occur with multiplicity at least one in this representation are \( S_d := \{u \in \mathbb{Z}^n \mid (d, u) \in S\} \) for some natural number \( d \); moreover, \( \text{conv}(S_d) = d\Delta \); (see [HK, Section 3.3.5]). Moreover, \( \Omega \) is chosen so that \( \mu_\Omega(\mathbb{C}P^N) = \Delta \), where \( \mu_\Omega : \mathbb{C}P^N \to \mathbb{R}^n \) is the moment map for \( \Omega \). Furthermore, [HK, Remark 3.17] implies that \( X_0 \) embeds in \( \mathbb{C}P^N \) as the closure of the \((\mathbb{C}^*)^n\) orbit of \((1, \ldots, 1)\), and so \( \mu_\Omega(X_0) = \Delta \). This proves (1).

Furthermore, by part (1) of [HK, Theorem 2.19], there exists a surjective continuous map \( \phi : X \to X_0 \) which is a symplectomorphism when restricted to a dense open subset \( U \subset X \). Reading the proof of [HK, Theorem 2.19, part (1)], we see that \( \phi(U) = U_0 \), the set defined just before [HK, Lemma 2.9]. Finally, by [HK, Corollary 2.10], \( U_0 \) is the smooth locus of \( X_0 \). This proves (2).

\( \square \)
Remark 2.2. In the setting of Theorem 2.1, the facts that $X$ is Hausdorff, $\phi$ is continuous, and $\phi$ restricts to a homeomorphism from $U$ to $(X_0)_{\text{smooth}}$, together imply that $U = \phi^{-1}((X_0)_{\text{smooth}}) \cap \overline{U}$. Since $U$ is dense in $X$, this implies that $U = \phi^{-1}((X_0)_{\text{smooth}})$.

We will only need the following corollary.

Corollary 2.3. Let $X$ be a smooth projective variety of complex dimension $n$, equipped with a very ample Hermitian line bundle $L$, and let $\omega$ be the associated Kähler form. Let $S$ be the semigroup associated to a valuation derived from a coordinate system on $X$. Finally, let $(X_\Delta, \omega_\Delta)$ be a symplectic toric manifold with integral moment polytope $\Delta$. If

$$S = \text{Cone}(\{1\} \times \Delta) \cap (\mathbb{Z} \times \mathbb{Z}^n),$$

then $(X, \omega)$ is symplectomorphic to $(X_\Delta, \omega_\Delta)$.

Proof. Because $S = \text{Cone}(\{1\} \times \Delta) \cap (\mathbb{Z} \times \mathbb{Z}^n)$ is the set of integral points in a rational polyhedral cone, $S$ is finitely generated by Gordon’s Lemma. Hence, we can apply Theorem 2.1. Moreover, since $\Delta$ is a smooth polytope, the toric variety $X_0 := \text{Proj} \mathbb{C}[S]$ is smooth. Therefore, since $\Delta$ is the Okounkov body associated to $S$, Theorem 2.1 and Remark 2.2 together imply that $(X, \omega)$ is symplectomorphic to $(X_0, \Omega|_{X_0})$, which is a symplectic toric manifold with moment polytope $\Delta$. The claim then follows from Delzant’s theorem. □

2.1. Examples of toric degenerations. In this subsection, we give a concrete description of the semigroup associated to specific coordinate system on smooth toric varieties. We then combine this description and the results above to give a criterion which guarantees that certain symplectic toric manifolds are symplectomorphic – even though, in general, there is no affine transformation between their moment polytopes. As an immediate consequence, we recover the known symplectomorphisms between different Hirzebruch surfaces.

Fix a smooth full dimensional polytope $P \subset \mathbb{R}^n$; let $X_P$ be the smooth toric variety associated to $P$, and let $\omega_P$ be an invariant symplectic form with moment map $\mu_P$ so that $\mu_P(X_P) = P$. Assume also that $P$ is integral, that is, the vertices lie in $\mathbb{Z}^n$. Then there exists a holomorphic line bundle $L_P$ over $X_P$ such that $c_1(L_P) = [\omega_P]$. Moreover, $L_P$ is very ample; see, for example, [CLS, Propositions 2.4.4 and 6.1.4]. Finally, the $(\mathbb{C}^\times)^n$ action on $X_P$ lifts to an action on $L_P$; the induced representation on $L_P := H^0(X_P, L_P)$ has one dimensional weight spaces with weights $P \cap \mathbb{Z}^n$ ([1]), see also [11].

Assume additionally that $P$ is aligned with the positive orthant $\mathbb{R}^n_{\geq 0}$, that is, $P$ is equal to $\mathbb{R}^n_{\geq 0}$ on some open neighborhood of 0. (In
particular, this implies that \( P \) is full dimensional.) Let \( h \in L_P \) be a non-zero section in the zero weight space, and let \( g_i \in L_P \) be a non-zero section in the weight space corresponding to \( e_i \) for all \( 1 \leq i \leq n \), where \( e_1, \ldots, e_n \) is the standard basis for \( \mathbb{R}^n \). We can then define \( f_i \in \mathbb{C}(X_P) \) by \( f_i = g_i/h \) for all \( i \). Using, for example, the description of \( L_P \) given in [H], it is easy to check that \( f_1, \ldots, f_n \) defines a coordinate system on an open dense subset of \( X_P \) containing the point \( \mu_P^{-1}(0) \).

Moreover, the monomials \( \{f_1^{p_1} \cdots f_n^{p_n}\}_{p \in \mathbb{P}} \) form a basis for the subspace \( \{g/h \mid g \in L_P\} \subset \mathbb{C}(X_P) \).

Finally, assume that the polytope \( P \) is normal, that is, for every positive integer \( m \) and every \( p \in mP \cap \mathbb{Z}^n \) there exist \( p_1, \ldots, p_m \in P \cap \mathbb{Z}^n \) such that \( p = p_1 + \cdots + p_m \). Given an integer \( m > 1 \), let \( L_P^m \) be the image of \( L_P^{\otimes m} \) in \( H^0(X_P, L_P^{\otimes m}) \). Then the monomial \( f_1^{p_1} \cdots f_n^{p_n} \) lies in the subspace \( \{g/h^m \mid g \in L_P^m\} \subset \mathbb{C}(X_P) \) exactly if \( p \) can be expressed as a sum of \( m \) integral points of \( P \); moreover, these monomials give a basis for that subspace. In general (i.e. if the polytope \( P \) is not necessarily normal), it is hard to describe this basis explicitly. However, since \( P \) is normal, the monomials \( \{f_1^{p_1} \cdots f_n^{p_n}\}_{p \in mP} \) form a basis for the subspace \( \{g/h^m \mid g \in L_P^m\} \subset \mathbb{C}(X_P) \).

**Example 2.4.** Let \( \nu \) be the valuation derived from the coordinate system \( f_1, \ldots, f_p \). In this case, \( \{\nu(g/h^m) \mid g \in L_P^m \setminus \{0\}\} = mP \cap \mathbb{Z}^n \), that is, the semigroup \( S \) associated to \( \nu \) is \( \text{Cone} \{1\} \times P \cap \mathbb{Z}^n \times \mathbb{Z}^n \).

As we see from the example above, if we use the valuation derived from the coordinate system \( f_1, \ldots, f_n \), then Proposition 2.3 only yields the trivial statement that \( (X_P, \omega_P) \) is symplectomorphic to itself. Therefore, we will modify this system of coordinates. Fix an non-negative integer \( c \) and integers \( 1 \leq k < \ell \leq n \). Consider the coordinate system \( u_1, \ldots, u_n \) near \( p \), where

\[
(1) \quad u_i = \begin{cases} 
    f_i & i \neq k \\
    f_k - f_{\ell} & i = k.
  \end{cases}
\]

As we show below, the semigroup associated to the valuation derived from this coordinate system can be obtained by sliding the integral points of \( \text{Cone} \{1\} \times P \) as far possible in the direction \(-e_k + ce_{\ell}\), while staying in \( \mathbb{Z}^n \times \mathbb{Z}_{\geq 0}^n \).

**Definition 2.5.** Given a subset \( Q \) of \( \mathbb{Z}_{\geq 0}^n \) and a vector \( w \in \mathbb{Z}^n \setminus \mathbb{Z}_{\geq 0}^n \), let \( \mathcal{F}_w(Q) \) be the set formed by sliding the points of \( Q \) as far as possible in the direction \( w \), while staying in \( \mathbb{Z}_{\geq 0}^n \). More formally, let \( \mathcal{F}_w(Q) \) be the unique subset of \( \mathbb{Z}_{\geq 0}^n \) such that
(1) \( Q \cap l \) and \( F_w(Q) \cap l \) have the same cardinality for every affine line \( l \) that is parallel to \( w \).

(2) If \( p \in F_wQ \) and \( p + w \in \mathbb{Z}^n_{\geq 0} \), then \( p + w \in F_w(Q) \).

**Lemma 2.6.** Consider the toric variety \( X_P \) and holomorphic line bundle \( L_P \) associated to a smooth normal integral polytope \( P \subset \mathbb{R}^n \) that’s aligned with \( \mathbb{R}^n_{\geq 0} \). Fix a non-negative integer \( c \) and integers \( 1 \leq k < \ell \leq n \); let \( \nu \) be the valuation derived from the local coordinate system defined in \( \Pi \), and let \( S \) be the associated semigroup. For each \( m > 0 \),

\[
S_m = F_{-e_k + ce_\ell}(mP \cap \mathbb{Z}^n).
\]

**Proof.** Fix an affine line \( l \) parallel to \(-e_k + ce_\ell\) such that \( P \cap \mathbb{Z}^n \cap l \neq \emptyset \). Since \( P \) is convex, \( mP \cap l \) is an interval. Hence,

\[
mP \cap \mathbb{Z}^n \cap l = \{ p + t(e_k - ce_\ell) \mid t \in [0, N] \cap \mathbb{Z} \},
\]

for some \( N \in \mathbb{Z}_{\geq 0} \) and \( p = (p_1, \ldots, p_n) \in P \cap \mathbb{Z}^n \). Fix \( t \in [0, N] \cap \mathbb{Z} \).

On the one hand, the function

\[
f := (f_k - f_\ell)^t f_\ell^{p_i - c_i} \prod_{i \neq \ell} f_i^{p_i} = \sum_{j=0}^{t} (-1)^{t-j} \binom{t}{j} f_k^{p_k + j} f_\ell^{p_\ell - c_\ell} \prod_{i \neq k, \ell} f_i^{p_i}
\]

is of the form \( g/h \) for some \( g \in L_P^m \) because \( P \) is normal and the exponent of each monomial in the final sum lies in \( mP \cap \mathbb{Z}^n \cap l \). On the other hand,

\[
f = u_k^t u_\ell^{p_\ell - c_\ell} (u_k + u_\ell)^{p_k} \prod_{i \neq k, \ell} u_i^{p_i} = \sum_{j=0}^{p_k} \binom{p_k}{j} u_k^{t+j} u_\ell^{p_\ell - c_\ell + p_k - c_\ell} \prod_{i \neq k, \ell} u_i^{p_i},
\]

and so \( \nu(f) = p + (t-p_k)(e_k - ce_\ell) \). Since this is true for all \( t \in [0, N] \cap \mathbb{Z} \),

\[
\{ p + (t-p_k)(e_k - ce_\ell) \mid t \in [0, N] \cap \mathbb{Z} \} \subseteq \{ \nu(g/h^m) \mid g \in L_P^m \setminus \{0\} \}.
\]

Finally, as explained in the beginning of this section, the cardinality of the set \( \{ \nu(g/h^m) \mid g \in L_P^m \setminus \{0\} \} \) is equal to the dimension of \( L_P^m \), which in turn is equal to the cardinality of the intersection \( mP \cap \mathbb{Z}^n \). Hence, there are no additional points in the set \( S_m = \{ \nu(g/h^m) \mid g \in L_P^m \setminus \{0\} \} \).

The claim then follows immediately from Definition 2.5. \( \square \)

**Proposition 2.7.** Let \( P \) and \( \Delta \) be integral smooth polytopes in \( \mathbb{R}^n \) that are aligned with \( \mathbb{R}^n_{\geq 0} \). Let \( (X_P, \omega_P) \) and \( (X_\Delta, \omega_\Delta) \) be the symplectic toric manifolds associated to \( P \) and \( \Delta \), respectively. Assume that there exists a non-negative integer \( c \) and integers \( 1 \leq k < \ell \leq n \) such that

\[
F_{-e_k + ce_\ell}(mP \cap \mathbb{Z}^n) = m\Delta \cap \mathbb{Z}^n
\]

for all \( m \in \mathbb{Z}_{>0} \). Then \( (X_P, \omega_P) \) is symplectomorphic to \( (X_\Delta, \omega_\Delta) \).
Lemma 2.8. The conditions of the above proposition are satisfied.

Proof. First observe that without loss of generality we can assume that $P$ is a normal polytope. To see this, recall that, since $n \geq n - 1$, the $n$-th dilate of any smooth $n$-dimensional polytope is normal; see, for example, [CLS, Theorem 2.2.12]. Obviously the assumption (2) still holds if we replace $P$ by $nP$ and $\Delta$ by $n\Delta$. Moreover there is an equivariant biholomorphism from $X_P$ to $X_{nP}$ that identifies the equivariant holomorphic line bundle $L_{nP} \to X_{nP}$ with the tensor product $L_P^\otimes n \to X_P$, and identifies the symplectic form $\omega_{nP} \in \Omega^2(X_{nP})$ with the form $n\omega_P \in \Omega^2(X_P)$. Therefore, if $(X_{nP}, \omega_{nP})$ is symplectomorphic to $(X_{n\Delta}, \omega_{n\Delta})$ then also $(X_P, \omega_P)$ is symplectomorphic to $(X_{\Delta}, \omega_{\Delta})$.

Thus we assume that $P$ is normal. Use $c, \ell$ and $k$ to define a coordinate system as in (1), and hence define a valuation $\nu$ and semigroup $S$. By Lemma 2.6 and (2) we have that

$$S = \text{Cone} \{ \{1\} \times \Delta \} \cap (\mathbb{Z} \times \mathbb{Z}^n).$$

The claim now follows immediately from Proposition 2.3. $\square$

Unfortunately, even just the condition that $F_{-e_k + c\ell}(P \cap \mathbb{Z}^n)$ must be the set of integral points in a smooth polytope is quite strong; it may not be the set of integral points of any convex polytope. For example, consider the polytopes

$$P = \{ p \in \mathbb{R}^2 \mid 0 \leq \langle p, e_1 \rangle \leq 2, \ 0 \leq \langle p, e_2 \rangle \leq 2 \},$$
$$P' = \{ p \in \mathbb{R}^2 \mid 0 \leq \langle p, e_1 \rangle \leq 1, \ 0 \leq \langle p, e_2 \rangle, \ \langle p, 4e_1 + e_2 \rangle \leq 6 \}.$$

In this case, $\text{conv}(F_{(-1,2)}(P \cap \mathbb{Z}^2)) = P'$; however, $F_{(-1,2)}(P \cap \mathbb{Z}^2) \neq P' \cap \mathbb{Z}^2$ because the former does not contain the point $(1,1)$. (This doesn’t violate Lemma 2.8 below because it doesn’t satisfy the correct inequalities.) Luckily, there are some interesting examples where the conditions of the above proposition are satisfied.

Lemma 2.8. Given integers $A_2^1$, $\lambda_1$, and $\lambda_2$, such that $\lambda_1 > 0$, $\lambda_2 > 0$, and $\lambda_2 > A_2^1 \lambda_1$, consider the trapezoid

$$P = \{ p \in \mathbb{R}^2 \mid 0 \leq \langle p, e_1 \rangle \leq \lambda_1, \ 0 \leq \langle p, e_2 \rangle, \ \langle p, e_2 + A_2^1 e_1 \rangle \leq \lambda_2 \}. $$

Given an integer $c$ such that $c \geq A_2^1$ and $\lambda_2 > c \lambda_1$,

$$F_{(-1,c)}(P \cap \mathbb{Z}^2) = \tilde{P} \cap \mathbb{Z}^2,$$

where $\tilde{A}_2^1 = 2c - A_2^1$, $\tilde{\lambda}_1 = \lambda_1$, and $\tilde{\lambda}_2 = \lambda_2 + (c - A_2^1) \lambda_1$, and

$$\tilde{P} = \{ p \in \mathbb{R}^2 \mid 0 \leq \langle p, e_1 \rangle \leq \tilde{\lambda}_1, \ 0 \leq \langle p, e_2 \rangle, \ \langle p, e_2 + \tilde{A}_2^1 e_1 \rangle \leq \tilde{\lambda}_2 \}.$$  

Proof. The polytope $\tilde{P} \cap \mathbb{Z}^2$ satisfies condition (2) of Definition 2.5 because $\tilde{A}_2^1 \geq c$. Moreover, if $\tilde{A}_2^1 = A_2^1$ then $P = \tilde{P}$, and so condition
(1) is obvious. Thus, we may assume that \( \tilde{A}^1_2 > c > A^1_2 \) and focus on proving condition (1).

First, consider the half-plane \( H^\leq := \{ p \in \mathbb{R}^2 \mid \langle p, e_2 + ce_1 \rangle \leq \lambda_2 \} \). Since \( c > A^1_2 \), every \( p \in H^\leq \) with \( \langle p, e_1 \rangle \geq 0 \) satisfies \( \langle p, e_2 + A^1_2 e_1 \rangle \leq \lambda_2 \); moreover, every \( p \in H^\leq \) with \( \langle p, e_1 \rangle \leq \lambda_1 \) satisfies \( \langle p, e_2 + A^1_2 e_1 \rangle \leq \lambda_2 \). Therefore, the intersections \( P \cap H^\leq \) and \( \tilde{P} \cap H^\leq \) are both equal to the trapezoid \( Q \) with vertices \((0,0), (\lambda_1,0), (\lambda_1, \lambda_2 - c\lambda_1)\), and \((0, \lambda_2)\). Hence, condition (1) holds for lines in this half plane.

Next, consider the half-plane \( H^\geq := \{ p \in \mathbb{R}^2 \mid \langle p, e_2 + ce_1 \rangle \geq \lambda_2 \} \). Since \( c > A^1_2 \), every \( p \in H^\geq \) with \( \langle p, e_1 + A^1_2 e_2 \rangle \leq \lambda_2 \) satisfies \( \langle p, e_1 \rangle \geq 0 \); moreover, every \( p \in H^\geq \) with \( \langle p, e_1 \rangle \leq \lambda_1 \) satisfies \( \langle p, e_2 \rangle \geq 0 \). Since \( c > A^1_2 \) and \( \lambda_1 > 0 \), this implies that the intersection \( P \cap H^\geq \) is the triangle \( T \) with vertices \((0, \lambda_2), (\lambda_1, \lambda_2 - c\lambda_1), \) and \((\lambda_1, \lambda_2)\). Define an affine transformation \( B : \mathbb{R}^2 \to \mathbb{R}^2 \) by

\[
B(p_1, p_2) = (\lambda_1 - p_1, 2cp_1 + p_2 - c\lambda_1).
\]

By inspection, \( B(T) = \tilde{T} \). Additionally, since \( B \) is the composition of a unimodular integral linear transformation with translation by an integral vector, \( B(\mathbb{Z}^2) = \mathbb{Z}^2 \). Finally, \( B(l) = l \) for every affine line \( l \) parallel to \(-e_1 + ce_2\). Hence, \( B \) induces a bijection from \( P \cap \mathbb{Z}^2 \cap l \) to \( \tilde{P} \cap \mathbb{Z}^2 \cap l \) for every such \( l \) in \( H^\geq \), and so condition (1) also holds for lines in this half plane. \( \square \)

**Example 2.9.** Given an integer \( m \), let \( \mathcal{O}(m) \to \mathbb{CP}^1 \) be the holomorphic line bundle \( \mathcal{O}(m) := (\mathbb{C}^2 \setminus \{0\} \times \mathbb{C})/\mathbb{C}^\times \), where \( \mathbb{C}^\times \) acts by \( \alpha \cdot (x_1, x_2, z) = (\alpha x_1, \alpha x_2, \alpha^m z) \). Given an non-negative integer \( m \), consider the Hirzebruch surface

\[
\Sigma_m := \mathbb{P}(\mathcal{O}(0) \oplus \mathcal{O}(-m)) \simeq (\mathbb{C}^2 \setminus \{0\})^2/(\mathbb{C}^\times)^2,
\]

where \( (\alpha_1, \alpha_2) \cdot (x_1, x_2, y_1, y_2) = (\alpha_1 x_1, \alpha_1 x_2, \alpha_2 y_1, \alpha_1^{-m} \alpha_2 y_2) \). Hirzebruch surfaces are the simplest non-trivial Bott manifolds; the torus \((\mathbb{C}^\times)^2 \) acts on \( \Sigma_m \) by \( (\alpha_1, \alpha_2) = [\alpha_2 x_1, x_2, \alpha_1 y_1, y_2] \). Given integers \( A^2_1 \geq 0, \tilde{A}^2_1 \geq 0, \lambda_1 > 0, \tilde{\lambda}_1, \lambda_2 > A^1_2 \lambda_1 \geq 0, \) and \( \lambda_2 > \tilde{A}^1_2 \tilde{\lambda}_1 \geq 0, \) there exist \( (S^1)^2 \subset (\mathbb{C}^\times)^2 \) invariant integral symplectic forms \( \omega \in \Omega^2(\Sigma_{A^2_1}) \) and \( \tilde{\omega} \in \Omega^1(\Sigma_{\tilde{A}^2_1}) \), and moment maps \( \mu : \Sigma_{A^2_1} \to \mathbb{R}^n \) and \( \tilde{\mu} : \Sigma_{\tilde{A}^2_1} \to \mathbb{R}^n \), so that \( \mu(\Sigma_{A^2_1}) = P \) and \( \tilde{\mu}(\Sigma_{\tilde{A}^2_1}) = \tilde{P} \), where \( P \) and \( \tilde{P} \) are the polytopes described in [3] and [4]. Then \( (\Sigma_{A^2_1}, \omega) \) and \( (\Sigma_{\tilde{A}^2_1}, \tilde{\omega}) \) are symplectomorphically exactly if \( A^2_1 \) and \( \tilde{A}^2_1 \) have the same parity, \( \tilde{\lambda}_1 = \lambda_1 \), and
When these equations are satisfied, Proposition 2.7 and Lemma 2.8 together immediately give another proof that these manifolds are symplectomorphic.

We will use the following easy consequence of the previous lemma.

**Lemma 2.10.** Given $A_1^1, \lambda_1, \lambda_2 \in \mathbb{Z}$ and $\lambda', \lambda'' \in \mathbb{Q} \cup \{\pm \infty\}$, let $Q$ be the polytope consisting of $p \in \mathbb{R}^2$ such that

\[
0 \leq \langle p, e_1 \rangle \leq \lambda_1, \quad 0 \leq \langle p, e_2 \rangle, \quad \langle p, e_2 + A_1^1 e_1 \rangle \leq \lambda_2,
\]

\[
\lambda' \leq \langle p, e_2 + ce_1 \rangle \leq \lambda''.
\]

Given a non-negative integer $c$ such that $\lambda_2 > c\lambda_1$, let $\tilde{Q}$ be the polytope consisting of $p \in \mathbb{R}^2$ such that

\[
0 \leq \langle p, e_1 \rangle \leq \tilde{\lambda}_1, \quad 0 \leq \langle p, e_2 \rangle, \quad \langle p, e_2 + \tilde{A}_1^1 e_1 \rangle \leq \tilde{\lambda}_2,
\]

\[
\lambda' \leq \langle p, e_2 + ce_1 \rangle \leq \lambda'\prime\prime,
\]

where $\tilde{A}_1^1 = 2c - A_1^1$, $\tilde{\lambda}_1 = \lambda_1$, and $\tilde{\lambda}_2 = \lambda_2 + (c - A_1^1)\lambda_1$. If $\tilde{A}_1^1 \geq A_1^1$, then

\[
\mathcal{F}_{(-1,c)}(Q \cap \mathbb{Z}^2) = \tilde{Q} \cap \mathbb{Z}^2.
\]

**Proof.** Assume that $\tilde{A}_1^1 \geq A_1^1$, and so $c \geq A_1^1$. If $\lambda_1 < 0$, the claim is obvious because $Q$ and $\tilde{Q}$ are both empty. Similarly, if $\lambda_1 = 0$, then $Q = \tilde{Q} \subset \mathbb{R} \times \{0\}$. Hence, we may assume that $\lambda_1 > 0$. Therefore, since $c \geq 0$ and $c \geq A_1^1$, the assumption that $\lambda_2 > c\lambda_1$ implies that $\lambda_2 > A_1^1\lambda_1$ and $\lambda_2 > 0$ as well. Therefore, by Lemma 2.8

\[
\mathcal{F}_{(-1,c)}(P \cap \mathbb{Z}^2) = \tilde{P} \cap \mathbb{Z}^2,
\]

where $P$ and $\tilde{P}$ are the polytopes defined in that lemma. Moreover,

\[
Q = P \cap \{ p \in \mathbb{R}^2 \mid \lambda' \leq \langle p, e_2 + ce_1 \rangle \leq \lambda'' \},
\]

\[
\tilde{Q} = P' \cap \{ p \in \mathbb{R}^2 \mid \lambda' \leq \langle p, e_2 + ce_1 \rangle \leq \lambda'' \}.
\]

Therefore, the claim follows immediately from the fact that $\langle -e_1 + ce_2, e_2 + ce_1 \rangle = 0$. \qed

### 3. Symplectic Bott manifolds

In this section, we formally define symplectic Bott manifolds and study some elementary properties of these manifolds. In particular, we describe the cohomology ring of Bott manifolds and determine which cohomology classes admit invariant Kähler forms. We also use Delzant’s theorem to construct symplectomorphisms between certain symplectic Bott manifolds.
To simplify notation, let \([n]\) denote the set \([1, \ldots, n]\) for any \(n \in \mathbb{Z}_{>0}\), and let \(M_n(\mathbb{Z})\) denote the set of \(n \times n\) integral matrices. Let \(A \in M_n(\mathbb{Z})\) be a strictly upper-triangular matrix, that is, let \(A^i_j\) be an integer for all \(i, j \in [n]\) such that \(A^i_j = 0\) if \(i \geq j\). Fix \(\lambda \in \mathbb{R}^n\), and define a polytope \(\Delta = \Delta(A, \lambda)\) as follows:

\[
\Delta = \{ p \in \mathbb{R}^n \mid \langle p, e_j \rangle \geq 0 \text{ and } \langle p, e_j + \sum_i A^i_j e_i \rangle \leq \lambda_j \ \forall \ 1 \leq j \leq n \}.
\]

Given subsets \(J, K \subseteq [n]\), define (possibly empty) faces

\[
\hat{F}_J = \bigcap_{j \in J} \{ \langle p, e_j \rangle = 0 \} \cap \Delta \quad \text{and} \quad F_K = \bigcap_{k \in K} \{ \langle p, e_k + \sum_i A^i_k e_i \rangle = \lambda_k \} \cap \Delta.
\]

If \(A\) is the zero matrix and \(\lambda = (1, \ldots, 1)\), then \(\Delta\) is the hypercube \([0, 1]^n\), and the face \(F_J \cap \hat{F}_K\) of \([0, 1]^n\) is non-empty exactly if \(J \cap K = \emptyset\). More generally, we say that \(\Delta = \Delta(A, \lambda)\) is a \textbf{Bott polytope} if

\[
F_J \cap \hat{F}_K \neq \emptyset \iff J \cap K = \emptyset \quad \forall \ J, K \subseteq [n].
\]

In this case, each vertex of \(\Delta\) lies on exactly \(n\) defining hyperplanes; moreover, the primitive integral vectors perpendicular to these hyperplanes form a \(\mathbb{Z}\) basis for \(\mathbb{Z}^n\). Therefore, \(\Delta\) is an \(n\)-dimensional smooth polytope, and \(F_i\) and \(\hat{F}_i\) are facets for each \(i\). (In particular, \(\Delta\) is combinatorially equivalent to the hypercube \([0, 1]^n\).)

Fix a strictly upper triangular matrix \(A \in M_n(\mathbb{Z})\). For each \(j \in [n]\), let \(\hat{v}_j = e_j\) and \(v_j = -e_j - \sum_i A^i_j e_i\). Define \(\pi: \mathbb{R}^{2n} \to \mathbb{R}^n\) by

\[
\pi(\hat{\alpha}_1, \alpha_1, \ldots, \hat{\alpha}_n, \alpha_n) = \sum_j \hat{\alpha}_j \hat{v}_j + \alpha_j v_j.
\]

Let \(\iota: \mathfrak{t} \to \mathbb{R}^{2n}\) be the natural inclusion the kernel of \(\pi\), let \(K \subseteq (S^1)^{2n}\) be the subtorus with Lie algebra \(\mathfrak{t}\), and let \(K_C \subseteq ((\mathbb{C}^\times)^2) \simeq (\mathbb{C}^\times)^{2n}\) be the complexification of \(K\). Then the quotient

\[
M_A := (\mathbb{C}^2 \setminus \{0\})^n/K_C
\]

is a Bott manifold; the \((\mathbb{C}^\times)^n\) action on \(M_A\) is induced by the inclusion \((\mathbb{C}^\times)^n \simeq (\mathbb{C}^\times \times \{1\})^n \hookrightarrow (\mathbb{C}^\times)^{2n}\). We call \(M_A\) the \textbf{Bott manifold} associated to \(A\). Conversely, if \(M\) is any Bott manifold then – for the appropriate identification of the torus acting on \(M\) with \((\mathbb{C}^\times)^N\) – there exists a strictly upper triangular matrix \(A \in M_n(\mathbb{Z})\) such that \(M\) is equivariantly biholomorphic to \(M_A\).

Now fix \(\lambda \in \mathbb{R}^n\) such that \(\Delta = \Delta(A, \lambda)\) is a Bott polytope. In this case, \(\hat{v}_j\) and \(v_j\) are the inward primitive integral vectors perpendicular to the facets \(\hat{F}_j\) and \(F_j\), respectively. The standard \((S^1)^{2n}\) action on \(\mathbb{C}^{2n}\) is Hamiltonian with moment map

\[
(\hat{z}_1, z_1, \ldots, \hat{z}_n, z_n) \mapsto \left(\frac{1}{2} |\hat{z}_1|^2, \frac{1}{2} |z_1|^2, \ldots, \frac{1}{2} |\hat{z}_n|^2, \frac{1}{2} |z_n|^2\right).
\]
Let \((M_{\Delta}, \omega_{\Delta})\) be the symplectic reduction of \(\mathbb{C}^{2n}\) by \(K \subset (S^1)^{2n}\) at \(\iota^*(0, \lambda_1, \ldots, 0, \lambda_n) \in \mathfrak{t}^*\), where \(\iota^*: \mathbb{R}^{2n} \to \mathfrak{t}^*\) is the dual map. Then the inclusion \((S^1)^n \simeq (S^1 \times \{1\}) \hookrightarrow (S^1)^{2n}\) induces a Hamiltonian \((S^1)^n\) action on \((M_{\Delta}, \omega_{\Delta})\) with moment polytope \(\Delta\). Moreover, since \(\Delta\) is a Bott polytope, the manifold \(M_{\Delta}\) is equivariantly diffeomorphic to the Bott manifold \(M_A\) defined above, Moreover, \(\omega_{\Delta}\) is a Kähler form with respect to this complex structure. (See, e.g., [H].) Conversely, if \(\omega\) is any \((S^1)^n\) invariant Kähler form on \(M_A\), then the \((S^1)^n\) action on \(M_A\) is Hamiltonian with moment image \(\Delta(A, \lambda)\) for some \(\lambda \in \mathbb{R}^n\). This justifies the following definition.

**Definition 3.1.** A symplectic Bott manifold is the symplectic toric manifold \((M, \omega, \mu)\) associated to a Bott polytope \(\Delta(A, \lambda)\) via the above construction. In this case we say that \((A, \lambda)\) defines \((M, \omega, \mu)\).

We need the following special case of a classical result of Danilov [D].

**Lemma 3.2.** Let \((M, \omega, \mu)\) be the symplectic Bott manifold associated to a strictly upper triangular matrix \(A \in M_n(\mathbb{Z})\) and \(\lambda \in \mathbb{R}^n\). Then

\[
H^*(M; \mathbb{Z}) \simeq \mathbb{Z}[x_1, \ldots, x_n]/(x_i^2 + \sum_j A_{ij}x_jx_i).
\]

Moreover, \(x_i \in H^2(M; \mathbb{Z})\) is Poincaré dual to the moment preimage \(\mu^{-1}(F_i)\), \(x_k + \sum_i A_{ik}x_i\) is Poincaré dual to \(\mu^{-1}(\hat{F})\), and \([\omega] = \sum_i \lambda_i x_i\).

Let \(M_A\) be the Bott manifold associated to a strictly upper triangular matrix \(A \in M_n(\mathbb{Z})\), and fix \(\lambda \in \mathbb{R}^n\). Our next proposition give a criteria for whether \((A, \lambda)\) defined a symplectic Bott manifold, or equivalently, whether there is an \((S^1)^n\) invariant Kähler form on \(M_A\) in the cohomology class \(\operatorname{\sum}_i \lambda_i x_i\). Note, in particular, that we can always find some \(\lambda \in \mathbb{R}^n\) that satisfies these conditions: Choose \(\lambda_{i+1} >> \lambda_i\) for all \(i\).

Given a non-empty subset \(I \subseteq [n]\), let \(\max(I)\) denote the maximal element of \(I\).

**Proposition 3.3.** Let \(A \in M_n(\mathbb{Z})\) be strictly upper-triangular and fix \(\lambda \in \mathbb{R}^n\). Then \((A, \lambda)\) defines a symplectic Bott manifold exactly if \(\Xi(I) > 0\) for every non-empty subset \(I \subseteq [n]\), where

\[
\Xi(I) := \sum_{i_0 < \cdots < i_m} (-1)^m \lambda_{i_0} A_{i_0}^{i_0} \cdots A_{i_m}^{i_m - 1},
\]

where the sum is taken over subsets \(\{i_0, \ldots, i_m\} \subseteq I\) such that \(i_0 < \cdots < i_m = \max(I)\).

\(^4\) In particular, the summand associated to \(\{\max(I)\}\) is \(\lambda_{\max(I)}\).
Proof. Let \( \Delta \) be the polytope associated to \((A, \lambda)\). Given a (possibly empty) subset \( I \subseteq [n] \), let \( p_I \) be the unique point in the intersection of hyperplanes
\[
\{ \eta \in \mathbb{R}^n \mid \langle \eta, e_j \rangle = 0 \ \forall j \in I^c \text{ and } \langle \eta, e_j + \sum_i A_{ij} e_i \rangle = \lambda_j \ \forall j \in I \}.
\]
By induction on \( j \),
\[
(5) \quad \langle p_I, e_j \rangle = \Xi([j] \cap I) \quad \forall j \in I.
\]
Thus,
\[
(6) \quad \langle p_I, e_j + \sum_i A_{ij} e_i \rangle = \lambda_j - \Xi([j] \cap I \cap \{j\}) \quad \forall j \in I^c.
\]

Assume first that \( \Delta \) is a Bott polytope. Then \( F_I \cap \tilde{F}_I \neq \emptyset \) for every non-empty subset \( I \subseteq [n] \), and so \( p_I \in \Delta \). By (5), this implies that \( \langle p_I, e_{\max(I)} \rangle = \Xi(I) \geq 0 \). In contrast \( F_I \cap \tilde{F}_I \cap \tilde{F}_{\max(I)} = \emptyset \), and so \( \langle p_I, e_{\max(I)} \rangle = \Xi(I) \neq 0 \). Therefore, \( \Xi(I) > 0 \).

So assume instead that \( \Xi(I) > 0 \) for every non-empty subset \( I \subseteq [n] \). By (5) and (6), this implies that \( \{p_I\} = F_I \cap \tilde{F}_I \) is a vertex of \( \Delta \) for every \( I \subseteq [n] \); moreover, \( p_I \) doesn’t lie in \( \tilde{F}_j \) for any \( j \in I \) or in \( F_j \) for any \( j \in I^c \). Hence, \( F_j \cap \tilde{F}_I \neq \emptyset \) for all \( J, K \subseteq [n] \) such that \( J \cap K = \emptyset \). Moreover, since the vertex \( p_I \) lies on exactly \( n \) defining hyperplanes, there are exactly \( n \) vertices that are joined to \( p_I \) by an edge; namely, \( p_{I \setminus \{i\}} \) for \( i \in I \), and \( p_{I \cup \{i\}} \) for \( i \in I^c \). Therefore, every vertex of \( \Delta \) is equal to \( p_I \) for some \( I \). (By Balinski’s Theorem, the graph of vertices and edges of any polytope is connected.) Since every face contains a vertex, this implies that \( F_J \cap \tilde{F}_K = \emptyset \) for all \( J, K \subseteq [n] \) such that \( J \cap K \neq \emptyset \). Hence, \( \Delta \) is a Bott polytope. \( \square \)

Example 3.4. A strictly upper triangular matrix \( A \in M_3(\mathbb{Z}) \) and \( \lambda \in \mathbb{R}^3 \) define a symplectic Bott manifold exactly if the following quantities are positive: \( \lambda_1, \lambda_2, \lambda_3 - \lambda_1 A_1^1, \lambda_3 - \lambda_1 A_1^2, \lambda_3 - \lambda_2 A_2^3, \lambda_3 - \lambda_2 A_2^1 - \lambda_1 A_1^2 - \lambda_1 A_1^2 A_2^2 \).

One straightforward way to construct a symplectomorphism between symplectic toric manifolds is to find an affine transformation between their moment polytopes. More specifically, let \((M, \omega, \mu)\) and \((\tilde{M}, \tilde{\omega}, \tilde{\mu})\) be \(2n \)-dimensional symplectic toric manifolds with moment polytope \( \Delta \) and \( \tilde{\Delta} \), respectively. Assume that there exists a unimodular matrix \( \Lambda \in M_n(\mathbb{Z}) \) and \( v \in \mathbb{R}^n \) such that \( \Lambda(\tilde{\Delta}) + v = \Delta \). Then Delzant’s theorem implies that there exists a symplectomorphism \( f \) from \((\tilde{M}, \tilde{\omega})\) to \((M, \omega)\) that covers the affine transformation \( \Lambda + v \), i.e., the following
diagram commutes:

\[ \begin{array}{ccc}
\tilde{M} & \xrightarrow{f} & M \\
\tilde{\mu} & \downarrow & \mu \\
\tilde{\Delta} & \xrightarrow{\Lambda + v} & \Delta 
\end{array} \]

In particular, if \( \Lambda + v \) takes a facet \( \tilde{G} \) of \( \tilde{\Delta} \) to a facet \( G \) of \( \Delta \), then \( f \) takes the moment preimage \( \tilde{\mu}^{-1}(\tilde{G}) \) to the moment preimage \( \mu^{-1}(G) \). Therefore, \( f^*: H^*(M; \mathbb{Z}) \to H^*(\tilde{M}; \mathbb{Z}) \) takes the Poincaré dual of \( \mu^{-1}(G) \) to the Poincaré dual of \( \tilde{\mu}^{-1}(\tilde{G}) \).

**Lemma 3.5.** Let \((M, \omega)\) and \((\tilde{M}, \tilde{\omega})\) be the symplectic Bott manifolds associated to strictly upper triangular matrices \( A \) and \( \tilde{A} \) in \( M_n(\mathbb{Z}) \) and \( \lambda \) and \( \tilde{\lambda} \) in \( \mathbb{R}^n \). Let \( F: H^*(M; \mathbb{Z}) \to H^*(\tilde{M}; \mathbb{Z}) \) be a ring isomorphism such that \( F([\omega]) = [\tilde{\omega}] \) and \( F(x_i) = \tilde{x}_{\sigma(i)} \) for all \( i \), where \( \sigma \) is a permutation of \([n]\). Then there exists a symplectomorphism \( f \) from \((\tilde{M}, \tilde{\omega})\) to \((M, \omega)\) so that \( f^* = F \).

**Proof.** Let \( \Delta \) and \( \tilde{\Delta} \) be the moment polytopes of \((M, \omega)\) and \((\tilde{M}, \tilde{\omega})\), respectively. Let \( \Lambda \in M_n(\mathbb{Z}) \) be the unimodular matrix taking \( e_i \) to \( e_{\sigma(i)} \). Since \( F(x_i) = \tilde{x}_{\sigma(i)} \) for all \( i \) and \( F(\sum \lambda_i x_i) = \sum \tilde{\lambda}_i \tilde{x}_i \),

\[ \tilde{A}_{\sigma(i)}^i = A_j^i \quad \text{and} \quad \tilde{\lambda}_{\sigma(i)} = \lambda_i \quad \forall \; i, j. \]

Therefore, \( \Lambda^T \tilde{\Delta} = \Delta \); moreover, \( \Lambda^T \) takes the facet \( \{ (p, e_{\sigma(j)} + \sum_i A_j^i e_i) = \lambda_{\sigma(j)} \} \cap \tilde{\Lambda} \) to the facet \( \{ (p, e_j + \sum_i A_j^i e_i) = \lambda_j \} \cap \Delta \). The moment preimages of these facets are Poincaré dual to \( \tilde{x}_{\sigma(i)} \) and \( x_i \), respectively. Therefore, as we saw above, the claim follows from Delzant’s theorem.

**Lemma 3.6.** Let \( A, \tilde{A} \in M_n(\mathbb{Z}) \) be strictly upper-triangular, and fix \( \lambda, \tilde{\lambda} \) in \( \mathbb{R}^n \). Assume that there exists \( k \in [n] \) so that

\[ \tilde{A}_j^k = -A_j^k \quad \forall \; j, \]

\[ \tilde{A}_j^j = A_j^j - A_k^j A_j^k \quad \text{and} \quad \tilde{\lambda}_j = \lambda_j - \lambda_k A_j^k \quad \forall \; j, \quad \forall \; i \neq k. \]

If \((A, \lambda)\) defines a symplectic Bott manifold \((M, \omega)\) then \((\tilde{A}, \tilde{\lambda})\) defines a symplectic Bott manifold \((\tilde{M}, \tilde{\omega})\); moreover, there exists a symplectomorphism \( f \) from \((M, \omega)\) to \((\tilde{M}, \tilde{\omega})\) such that \( f^* \) takes \( \tilde{x}_k \) to \( x_k + \sum_j A_j^k x_j \) and \( \tilde{x}_j \) to \( x_j \) for all \( j \neq k \).
Proof. Let $\Delta$ be the polytope associated to $(A, \lambda)$ and $\tilde{\Delta}$ be the polytope associated to $(\tilde{A}, \tilde{\lambda})$. Let $\Lambda \in M_n(\mathbb{Z})$ be the unimodular matrix that takes $e_k$ to $-e_k - \sum_i A_i^k e_i$ and takes $e_j$ to itself for all $j \neq k$. Then $\Lambda^T(\Delta) + \lambda_k e_k = \tilde{\Delta}$. Therefore, if $(A, \lambda)$ defines a symplectic Bott manifold $(\tilde{M}, \tilde{\omega})$, then $(\tilde{A}, \tilde{\lambda})$ defines a symplectic Bott manifold $(\tilde{M}, \tilde{\omega})$. Moreover, in this case the affine transformation takes the facet $\{ \langle p, e_k \rangle = 0 \} \cap \Delta$, whose moment preimage is Poincaré dual to $x_k + \sum_i A_i^k x_i$, to the facet $\{ \langle p, e_k + \sum_i \tilde{A}_i^k e_i \rangle = \tilde{\lambda}_k \} \cap \tilde{\Delta}$, whose moment preimage is Poincaré dual to $\tilde{x}_k$. Similarly, for all $j \neq k$ the transformation takes $\{ \langle p, e_j + \sum_i A_i^j e_i \rangle = \lambda_j \} \cap \Delta$, whose moment preimage is Poincaré dual to $x_j$, to the facet $\{ \langle p, e_j + \sum_i \tilde{A}_i^j e_i \rangle = \tilde{\lambda}_j \} \cap \tilde{\Delta}$, whose moment preimage is Poincaré dual to $\tilde{x}_j$. Therefore, the claim follows from Delzant’s theorem. □

Our last application was used to deduce Corollary 1.1.

**Proposition 3.7.** Let $\tilde{M}$ be a Bott manifold, and let $(M, \omega)$ be a symplectic toric manifold with $H^\ast(M; \mathbb{Z}) \cong H^\ast(\tilde{M}; \mathbb{Z})$. Then $(M, \omega)$ is symplectomorphic to a symplectic Bott manifold.

**Proof.** Clearly $M$ and $\tilde{M}$ must have the same dimension. Call it $2n$. Since $\tilde{M}$ is a Bott manifold it’s quotient polytope is an $n$-cube. Hence, by Theorem 5.5 in [MP], the fact that $H^\ast(M; \mathbb{Z}_2) = H^\ast(\tilde{M}; \mathbb{Z}_2)$ implies that that quotient polytope of $M$ is also an $n$-cube (because the cohomology rings with $\mathbb{Z}_2$ coefficients are BQ-algebras in the sense of [MP]). By Corollary 3.5 in [MP], this implies that there exists a unimodular matrix in $M_n(\mathbb{Z})$ taking the moment polytope of $\Delta$ to a Bott polytope. Therefore, the claim follows from Delzant’s theorem. □

**4. Constructing symplectomorphisms between Bott manifolds**

The main goal of this section is to use the toric degenerations discussed in Section 2 to construct symplectomorphisms between certain symplectic Bott manifolds. More concretely, symplectic Bott manifolds $(M, \omega)$ and $(\tilde{M}, \tilde{\omega})$ are symplectomorphic whenever there exists integers $k, \ell$, and $\gamma$ with $1 \leq k < \ell \leq n$, and an isomorphism from $H^\ast(M)$ to $H^\ast(\tilde{M})$ that takes $x_k$ to $\tilde{x}_k - \gamma \tilde{x}_\ell$, $x_i$ to $\tilde{x}_i$ for all $i \neq k$, and $[\omega]$ to $[\tilde{\omega}]$. This result, described in Proposition 4.8, directly generalizes the symplectomorphisms between Hirzebruch surfaces. In particular, in general there is no unimodular transformation between their moment polytopes, so these Bott manifolds are not equivariantly symplectomorphic. In the next section, we will use this result to prove that, up
to symplectomorphism, every symplectic Bott manifold \((M, \omega)\) has a simplified form. To prepare for Proposition 4.8, we give a criteria for such isomorphisms between cohomology rings.

**Lemma 4.1.** Let \(M\) and \(\tilde{M}\) be the Bott manifolds associated to distinct strictly upper-triangular matrices \(A\) and \(\tilde{A}\) in \(M_n(\mathbb{Z})\), respectively. Fix integers \(1 \leq k < \ell \leq n\). The following are equivalent:

1. There exists \(\gamma \in \mathbb{Z}\) and an isomorphism from \(H^*(M)\) to \(H^*(\tilde{M})\) that takes \(x_k\) to \(\tilde{x}_k - \gamma \tilde{x}_\ell\) and \(x_i\) to \(\tilde{x}_i\) for all \(i \neq k\).
2. The following equations hold:

\[
\tilde{A}^k_\ell = A^k_\ell \pmod{2}, \\
\tilde{A}^i_\ell = A^i_\ell - \frac{1}{2} A^i_k (A^k_\ell - \tilde{A}^k_\ell) \quad \forall i \neq k, \\
\tilde{A}^j_j = A^j_j \quad \forall i \neq j, \ell, \\
A^j_j = \frac{1}{2} (A^j_j + \tilde{A}^j_j) A^j_j \quad \forall j \neq \ell.
\]

Moreover, when (1) is satisfied, \(\gamma = \frac{1}{2} (A^k_\ell - \tilde{A}^k_\ell)\). Hence, given \(\lambda, \tilde{\lambda} \in \mathbb{R}^n\), the isomorphism takes \(\sum_i \lambda_i x_i\) to \(\sum_i \tilde{\lambda}_i \tilde{x}_i\) exactly if

\[
\tilde{\lambda}_\ell = \lambda_\ell - \frac{1}{2} \lambda_k (A^k_\ell - \tilde{A}^k_\ell) \quad \text{and} \quad \tilde{\lambda}_j = \lambda_j \quad \forall j \neq \ell.
\]

**Proof.** Fix \(\gamma \in \mathbb{Z}\), and define a ring homomorphism

\[
\Psi: \mathbb{Z}[x_1, \ldots, x_n] \to H^*(\tilde{M}; \mathbb{Z}) = \mathbb{Z}[\tilde{x}_1, \ldots, \tilde{x}_n]/(\tilde{x}_\ell^2 + \sum_j \tilde{A}^j_j \tilde{x}_j \tilde{x}_i)
\]

by \(\Psi(x_k) = \tilde{x}_k - \gamma \tilde{x}_\ell\) and \(\Psi(x_i) = \tilde{x}_i\) for all \(i \neq k\). Then

\[
\Psi \left( x_k^2 + \sum_j A^k_j x_k x_j \right) = \Psi(x_k)^2 + \sum_j A^k_j \Psi(x_k) \Psi(x_j) \\
= (\tilde{x}_k - \gamma \tilde{x}_\ell)^2 + \sum_j A^k_j (\tilde{x}_k - \gamma \tilde{x}_\ell) \tilde{x}_j \\
= \tilde{x}_k^2 - 2 \gamma \tilde{x}_k \tilde{x}_\ell + \sum_j A^k_j \tilde{x}_k \tilde{x}_j + (\gamma^2 - \gamma A^k_\ell) \tilde{x}_\ell^2 - \sum_{j \neq \ell} \gamma A^k_j \tilde{x}_j \tilde{x}_\ell \\
= - \sum_j A^k_j \tilde{x}_k \tilde{x}_j - 2 \gamma \tilde{x}_k \tilde{x}_\ell + \sum_j A^k_j \tilde{x}_k \tilde{x}_j - \sum_j (\gamma^2 - \gamma A^k_\ell) \tilde{A}^j_\ell \tilde{x}_j \tilde{x}_\ell - \sum_{j \neq \ell} \gamma A^k_j \tilde{x}_j \tilde{x}_\ell \\
= (A^k_\ell - \tilde{A}^k_\ell - 2 \gamma) \tilde{x}_k \tilde{x}_\ell + \sum_{j \neq \ell} (A^k_j - \tilde{A}^k_j) \tilde{x}_k \tilde{x}_j - \sum_{j \neq \ell} \gamma \left( A^k_j - A^k_\ell \tilde{A}^j_\ell + \gamma \tilde{A}^j_\ell \right) \tilde{x}_j \tilde{x}_\ell.
\]
Moreover, for all $i \neq k$, we have
\[
\Psi(x_i^2 + \sum_j A_j^i x_i x_j) = \Psi(x_i)^2 + \sum_j A_j^i \Psi(x_i) \Psi(x_j)
\]
\[
= -\sum_j \tilde{A}_j^i \tilde{x}_i \tilde{x}_j + \sum_{j \neq k} A_j^i \tilde{x}_i \tilde{x}_j + A_k^i \tilde{x}_i (\tilde{x}_k - \gamma \tilde{x}_\ell) \]
\[
= \sum_{j \neq \ell} (A_j^i - \tilde{A}_j^i) \tilde{x}_i \tilde{x}_j + (A_k^i - \tilde{A}_k^i - \gamma A_k^i) \tilde{x}_i \tilde{x}_\ell.
\]
This homomorphism $\Psi$ induces a homomorphism
\[
H^*(M) = \mathbb{Z}[x_1, \ldots, x_n]/(x_i^2 + \sum_j A_j^i x_i x_j) \to H^*(\tilde{M})
\]
exactly if the right hand sides of these equations vanish, or equivalently, if each term of these expressions vanish. If $\gamma = 0$, this is equivalent to $A = \tilde{A}$. Otherwise, it is straightforward to check that these terms vanish exactly if $\gamma = \frac{1}{2} (A_k^i - \tilde{A}_k^i)$ and the equations in (2) are satisfied. Finally, it’s clear that in this case there’s also a ring homomorphism from $H^*(\tilde{M})$ to $H^*(M)$ that takes $\tilde{x}_k$ to $x_k + \gamma x_\ell$ and $\tilde{x}_i$ to $x_i$ for all $i \neq k$; since this map is the inverse the homomorphism from $H^*(M)$ to $H^*(\tilde{M})$ constructed above, both maps are isomorphisms. (Alternatively, the homomorphism $H^*(M) \to H^*(\tilde{M})$ is a surjective map of free modules of the same dimension.)

The following observations will be useful later.

**Corollary 4.2.** Let $M$ and $\tilde{M}$ be Bott manifolds satisfying the equivalent statements of Lemma 4.1. Then
\[
\tilde{A}_i^i = A_i^i \quad \forall \ i > k,
\]
\[
\tilde{A}_j^k = A_j^k = 0 \quad \forall \ j < \ell.
\]

**Proof.** The matrix $A$ is strictly upper triangular. Hence, the first claim follows from the facts that $\tilde{A}_i^i = A_i^i - \frac{1}{2} A_k^i (A_k^i - \tilde{A}_k^i)$ for all $i \neq k$, and $A_k^i = 0$ for all $i > k$. Similarly, the next claim follows from the facts $\tilde{A}_j^k = A_j^k = \frac{1}{2} (A_k^k + \tilde{A}_k^k) A_k^i$ for all $j \neq \ell$, and $A_j^\ell = 0$ for all $j < \ell$.

**Remark 4.3.** Let $A \in M_n(\mathbb{Z})$ be strictly upper-triangular, and fix integers $1 \leq k < \ell \leq n$. Given an integer $c$, there exists a distinct strictly upper-triangular matrix $\tilde{A} \in M_n(\mathbb{Z})$ with $c = \frac{1}{2} (A_k^k + \tilde{A}_k^k)$ satisfying the statements in Lemma 4.1 exactly if $A_j^\ell = c A_j^\ell$ for all $j \neq \ell$. In particular, there exist infinitely many such matrices exactly if $A_j^\ell = A_j^\ell = 0$ for all $j \neq \ell$; c.f. Definition 5.3.
Our next result shows that if there is an isomorphism of the type considered in Lemma 4.1 between the cohomology ring of two Bott manifolds, and the “more skew” polytope is a Bott polytope, then the “less skew” polytope is as well. The converse if not true, as can be seen by considering trapezoids.

**Proposition 4.4.** Let $M$ and $\widetilde{M}$ be the Bott manifolds associated to strictly upper triangular matrices $A$ and $\widetilde{A}$ in $M_n(\mathbb{Z})$, respectively. Fix $\lambda, \widetilde{\lambda} \in \mathbb{R}^n$ and integers $1 \leq k < \ell \leq n$. Assume that there exists an isomorphism from $H^*(M)$ to $H^*(\widetilde{M})$ that takes $x_k$ to $\widetilde{x}_k - \frac{1}{2}(A^k_\ell - \widetilde{A}^k_\ell)\widetilde{x}_\ell$, $x_i$ to $\widetilde{x}_i$ for all $i \neq k$, and $\sum \lambda_ix_i$ to $\sum \widetilde{\lambda}_i\widetilde{x}_i$. If $|A^k_\ell| \geq |\widetilde{A}^k_\ell|$ and $(A, \lambda)$ defines a symplectic Bott manifold, then $(\widetilde{A}, \widetilde{\lambda})$ also defines a symplectic Bott manifold.

**Proof.** By Proposition 3.3 it suffices to show that $\Xi(I) > 0$ for every non-empty subset $I \subset \{n\}$, assuming that such inequalities hold for $\widetilde{\Xi}$. This follows straightforwardly from Lemma 4.5 below and the following observation: if $|A^k_\ell| \geq |\widetilde{A}^k_\ell|$, then $\widetilde{A}^k_\ell + A^k_\ell \geq 0$ or $\widetilde{A}^k_\ell - A^k_\ell \geq 0$; similarly, either $-\widetilde{A}^k_\ell + A^k_\ell \geq 0$ or $-\widetilde{A}^k_\ell - A^k_\ell \geq 0$.

**Lemma 4.5.** Let $M$ and $\widetilde{M}$ be the Bott manifolds associated to strictly upper triangular matrices $A$ and $\widetilde{A}$ in $M_n(\mathbb{Z})$, respectively. Fix $\lambda, \widetilde{\lambda} \in \mathbb{R}^n$ and integers $1 \leq k < \ell \leq n$. Assume that there exists an isomorphism from $H^*(M)$ to $H^*(\widetilde{M})$ that takes $x_k$ to $\widetilde{x}_k - \frac{1}{2}(A^k_\ell - \widetilde{A}^k_\ell)\widetilde{x}_\ell$, $x_i$ to $\widetilde{x}_i$ for all $i \neq k$, and $\sum \lambda_ix_i$ to $\sum \widetilde{\lambda}_i\widetilde{x}_i$. Let $\Xi$ and $\widetilde{\Xi}$ be associated to $(A, \lambda)$ and $(\widetilde{A}, \widetilde{\lambda})$, respectively. Then for any subset $I \subset \{n\} \setminus \{k, \ell\}$:

(a) $\Xi(I) = \widetilde{\Xi}(I)$ and $\Xi(I \cup \{k\}) = \widetilde{\Xi}(I \cup \{k\})$.

(b) If $\max(I) < \ell$, let $\Theta := \Xi([k] \cap I \cup \{k\})$. Then

$$
\Xi(I \cup \{\ell\}) - \frac{A^k_\ell \Theta}{2} = \Xi(I \cup \{k, \ell\}) + \frac{A^k_\ell \Theta}{2}
$$

$$
= \Xi(I \cup \{\ell\}) - \frac{\widetilde{A}^k_\ell \Theta}{2} = \widetilde{\Xi}(I \cup \{k, \ell\}) + \frac{\widetilde{A}^k_\ell \Theta}{2}.
$$

(c) If $\max(I) > \ell$, then $\Xi(I \cup \{\ell\}) = \widetilde{\Xi}(I \cup \{k, \ell\})$ and $\Xi(I \cup \{k, \ell\}) = \widetilde{\Xi}(I \cup \{\ell\})$.

---

5. To simplify exposition, we adopt the conventions that $\Xi(\emptyset) = \Xi(\emptyset)$ and $\max(\emptyset) = -\infty$.

6. By Claim (a), $\Xi([k] \cap I \cup \{k\}) = \widetilde{\Xi}([k] \cap I \cup \{k\})$. 
Proof. By Lemma 4.1 and Corollary 4.2, all the displayed equations in both results hold. Before proceeding further, define
\[ \Lambda(J) := (-1)^m \lambda_{i_0} A_{i_1}^{i_0} \cdots A_{i_m}^{i_{m-1}} \]
for any non-empty subset \( J = \{i_0, \ldots, i_m\} \subseteq [n] \) with \( i_0 < \cdots < i_m \).
By definition, for any non-empty subset \( I \subseteq [n] \),
\[ \Xi(I) = \sum_{\max(J) \in J \subseteq I} \Lambda(J). \]
Define \( \tilde{\Lambda} \) analogously. Now fix \( J \subseteq [n] \setminus \{k, \ell\} \). Then
\[ \Lambda(J) = \tilde{\Lambda}(J) \text{ and } \Lambda(J \cup \{k\}) = \tilde{\Lambda}(J \cup \{k\}) \]
because \( \tilde{A}_j^i = A_j^i \) and \( \tilde{\lambda}_j = \lambda_j \) for all \( i \) and \( j \neq \ell \). Moreover, because also \( \tilde{A}_k^i = A_k^i \) for all \( i > k \),
\[ \Lambda(J \cup \{\ell\}) = \tilde{\Lambda}(J \cup \{\ell\}) \text{ and } \Lambda(J \cup \{k, \ell\}) = \tilde{\Lambda}(J \cup \{k, \ell\}) \]
if \( J \cap \{k+1, \ldots, \ell-1\} \neq \emptyset \).
Similarly, the fact that \( \tilde{A}_j^k = A_j^k = 0 \) for all \( j < \ell \) implies that
\[ \Lambda(J \cup \{k\}) = \tilde{\Lambda}(J \cup \{k\}) = \Lambda(J \cup \{k, \ell\}) = \tilde{\Lambda}(J \cup \{k, \ell\}) = 0 \]
if \( J \cap \{k+1, \ldots, \ell-1\} \neq \emptyset \).
Moreover, by definition,
\[ \Lambda(J \cup \{k, \ell\}) = -A_k^\ell \Lambda(J \cup \{k\}) \text{ and } \tilde{\Lambda}(J \cup \{k, \ell\}) = -\tilde{A}_k^\ell \tilde{\Lambda}(J \cup \{k\}) \]
if \( \max(J) < k \).
Next, we claim that
\[ \Lambda(J \cup \{\ell\}) + \frac{1}{2} \Lambda(J \cup \{k, \ell\}) = \tilde{\Lambda}(J \cup \{\ell\}) + \frac{1}{2} \tilde{\Lambda}(J \cup \{k, \ell\}). \]
If \( J \cap \{k+1, \ldots, \ell-1\} \neq \emptyset \), this follows from (8). Otherwise, since \( \tilde{A}_j^i = A_j^i \) and \( \tilde{\lambda}_j = \lambda_j \) for all \( i \) and \( j \neq \ell \), depending on whether or not \( \ell \) is smaller than every element of \( J \), it either follows from the fact that \( \tilde{\lambda}_\ell = \lambda_\ell - \frac{1}{2} \tilde{\lambda}_k (A_k^\ell - \tilde{A}_k^\ell) \), or from the fact that \( \tilde{A}_k^\ell = A_k^\ell - \frac{1}{2} A_k^1 (A_k^k - \tilde{A}_k^k) \) for all \( i \neq k \). Finally, we claim that
\[ \Lambda(J \cup \{k\}) = \tilde{\Lambda}(J \cup \{k\}) = -\frac{1}{2} \Lambda(J \cup \{k, \ell\}) - \frac{1}{2} \tilde{\Lambda}(J \cup \{k, \ell\}) \]
if \( \max(J) > \ell \).

\(^7\)To simplify exposition, we adopt the that convention \( \Lambda(\emptyset) = \tilde{\Lambda}(\emptyset) \).
Indeed, if \( J \cap \{ k+1, \ldots, \ell - 1 \} \neq \emptyset \), every term of this equation vanishes by (20). Otherwise, it follows from the facts that \( \tilde{\lambda}_k = \lambda_k \), and \( \tilde{A}_j^k = A_j^k \) and \( \tilde{A}_j^\ell = \frac{1}{2} (A_j^\ell + \tilde{A}_j^\ell) A_j^\ell \) for all \( i \) and \( j \neq \ell \), and so \( A_j^k = \tilde{A}_j^k = \frac{1}{2} A_j^\ell A_j^\ell = \frac{1}{2} \tilde{A}_j^\ell \tilde{A}_j^\ell \).

We are now ready to prove the main claims. So fix a subset \( I \subseteq [n] \setminus \{ k, \ell \} \).

First, claim (a) follows immediately from (7).

Next, assume that \( \max(I) < \ell \). By (9) and (10),

\[
\Xi(I \cup \{ k, \ell \}) = \sum_{J \subseteq I} \left( \Lambda(J \cup \{ \ell \}) + \Lambda(J \cup \{ k \}) \right) - \sum_{J \subseteq [k] \cap I} A_k^\ell \Lambda(J \cup \{ k \}) = \Xi(I \cup \{ \ell \}) - A_k^\ell \Xi([k] \cap I \cup \{ k \}).
\]

Similarly,

\[
\tilde{\Xi}(I \cup \{ k, \ell \}) = \tilde{\Xi}(I \cup \{ \ell \}) - \tilde{A}_k^\ell \tilde{\Xi}([k] \cap I \cup \{ k \}).
\]

Moreover, by (11),

\[
\Xi(I \cup \{ \ell \}) \Xi(I \cup \{ k, \ell \}) = \sum_{J \subseteq I} \left( 2\Lambda(J \cup \{ \ell \}) + \Lambda(J \cup \{ k \}) \right) - \sum_{J \subseteq [k] \cap I} (2\tilde{\Lambda}(J \cup \{ \ell \}) + \tilde{\Lambda}(J \cup \{ k \})) = \Xi(I \cup \{ \ell \}) + \tilde{\Xi}(I \cup \{ k, \ell \}).
\]

Claim (b) follows immediately from Claim (a) and equations (13), (14), and (15).

Finally, assume that \( i = \max(I) > \ell \). By (7), (11), and (12),

\[
\Xi(I) = \sum_{i \in J \subseteq I} \left( \Lambda(J) + \Lambda(J \cup \{ \ell \}) \right) - \sum_{i \in J \subseteq I} \left( \tilde{\Lambda}(J) + \tilde{\Lambda}(J \cup \{ \ell \}) + \frac{1}{2} \tilde{\Lambda}(J \cup \{ k \}) \right) = \tilde{\Xi}(I \cup \{ k \}).
\]

By a similar argument \( \Xi(I \cup \{ k \}) = \tilde{\Xi}(I) \); this proves claim (c). \( \square \)

We are now ready to give a sequence of results culminating in the main result of this section, Proposition 4.8. We begin with the following technical consequence of the above lemma.
Lemma 4.6. Let $M$ and $\tilde{M}$ be the Bott manifolds associated to distinct strictly upper triangular matrices $A$ and $\tilde{A}$ in $M_n(\mathbb{Z})$, respectively. Fix $\lambda, \tilde{\lambda} \in \mathbb{R}^n$ and integers $1 \leq k < \ell \leq n$. Assume that there exists an isomorphism from $H^*(M)$ to $H^*(\tilde{M})$ that sends $x_k$ to $\tilde{x}_k - \frac{1}{2}(A^k_\ell - \tilde{A}^k_\ell)\tilde{x}_\ell$, $x_i$ to $\tilde{x}_i$ for all $i \neq k$, and $\sum \lambda_ix_i$ to $\sum \tilde{\lambda}_i\tilde{x}_i$. If $(\tilde{A}, \tilde{\lambda})$ defines a symplectic Bott manifold, then

\begin{equation}
\lambda_\ell - \langle p, \sum_{i \neq k} A^i_\ell e_i \rangle > \frac{1}{2} (A^k_\ell + \tilde{A}^k_\ell) (\lambda_k - \langle p, \sum_i A^i_k e_i \rangle)
\end{equation}

for each $p \in \mathbb{R}^n$ that satisfies the inequalities

\begin{equation}
0 \leq \langle p, e_j \rangle \quad \text{and} \quad \langle p, e_j + \sum_i A^i_\ell e_i \rangle \leq \lambda_j \text{ for all } j \in [\ell] \setminus \{k, \ell\}.
\end{equation}

Proof. Both sides of the inequality in (16) are linear functions on $\mathbb{R}^n$. Hence, to show that (16) holds for all points in the polyhedron defined by the inequalities in (17), it is enough to prove it holds on the minimal faces of that polyhedron. So assume that $p$ lies on such a minimal face. Then there exists $I \subseteq [\ell] \setminus \{k, \ell\}$ such that $p \in \mathbb{R}^n$ satisfies

\[ \langle p, e_j \rangle = \begin{cases} 
\lambda_j - \langle p, \sum_i A^i_\ell e_i \rangle & \text{for all } j \in I \\
0 & \text{for all other } j \in [\ell] \setminus \{k, \ell\}.
\end{cases} \]

Since $A$ is strictly upper triangular, $A^i_j = 0$ unless $i < j$. Moreover, Corollary 4.2 implies that $A^j_j = 0$ for all $j < \ell$. Hence, by induction on $j$, for all $j \in I$

\[ \langle p, e_j \rangle = \sum_{\{i_0, \ldots, i_m\} \subseteq I} (-1)^m \lambda_{i_0} A^{i_0}_{i_1} \ldots A^{i_{m-1}}_{i_m} = \Xi([j] \cap I), \]

with the sum taken over subsets $\{i_0, \ldots, i_m\} \subseteq I$ with $i_0 < \cdots < i_m = j$. Hence,

\[ \lambda_\ell - \langle p, \sum_{i \neq k} A^i_\ell e_i \rangle = \sum_{\{i_0, \ldots, i_m\} \subseteq I \cup \{\ell\}} (-1)^m \lambda_{i_0} A^{i_0}_{i_1} \ldots A^{i_{m-1}}_{i_m} = \Xi(I \cup \{\ell\}); \]

\[ \lambda_k - \langle p, \sum_i A^i_k e_i \rangle = \sum_{\{i_0, \ldots, i_m\} \subseteq I \cup \{k\}} (-1)^m \lambda_{i_0} A^{i_0}_{i_1} \ldots A^{i_{m-1}}_{i_m} = \Xi([k] \cap I \cup \{k\}). \]

By part (b) of Lemma 4.5,

\[ \Xi(I \cup \{\ell\}) = \frac{1}{2} (A^k_\ell + \tilde{A}^k_\ell) \Xi([k] \cap I \cup \{k\}) + \tilde{\Xi}(I \cup \{k, \ell\}). \]

Finally, by Proposition 3.3, if $(\tilde{A}, \tilde{\lambda})$ defines a symplectic Bott manifold then $\tilde{\Xi}(I \cup \{k, \ell\})$ is positive. The claim follows immediately. \qed
We can now show how the cohomology ring isomorphisms we are considering here relate to the “slide” operator defined in Definition 2.5.

Lemma 4.7. Let \((M, \omega)\) and \((\tilde{M}, \tilde{\omega})\) be the symplectic Bott manifolds associated to strictly upper triangular matrices \(A\) and \(\tilde{A}\) in \(M_n(\mathbb{Z})\) and \(\lambda\) and \(\tilde{\lambda}\) in \(\mathbb{Z}^n\), respectively. Fix integers \(1 \leq k < \ell \leq n\). Assume that there exists an isomorphism from \(H^*(\tilde{M})\) to \(H^*(\tilde{M})\) that takes \(x_k\) to \(\tilde{x}_k - \frac{1}{2}(A_k^k - \tilde{A}_k^k)\tilde{x}_t\), \(x_i\) to \(\tilde{x}_i\) for all \(i \neq k\), and \(\sum \lambda_ix_i\) to \(\sum \tilde{\lambda}_i\tilde{x}_i\). If \(\tilde{A}_k^k \geq |A_k^k|\) and \(c := \frac{1}{2}(A_k^k + \tilde{A}_k^k)\), then

\[
\mathcal{F}_{-e_k+ce\ell}(\Delta(A, \lambda) \cap \mathbb{Z}^n) = \Delta(\tilde{A}, \tilde{\lambda}) \cap \mathbb{Z}^n.
\]

Proof. By assumption \(c \geq 0\). By Lemma 4.1, \(c\) is an integer; moreover,

\[
\tilde{A}_k^i = A_k^i + A_k^j(c - A_k^j) \quad \forall \; i \neq k
\]

\[
\tilde{A}_j^i = A_j^i \quad \forall \; i \neq j, \; \forall \; j \neq \ell
\]

\[
\tilde{A}_j^k = cA_j^k \quad \forall \; j \neq \ell.
\]

Given \(p' \in \mathbb{Z}^n\), consider the affine two-plane

\[
V(p') := \{p \in \mathbb{R}^n \mid \langle p, e_j \rangle = \langle p', e_j \rangle \text{ for all } j \notin \{k, \ell\}\}.
\]

Below, we will fix an arbitrary \(p' \in \mathbb{Z}^n\), and prove that

\[
\mathcal{F}_{-e_k+ce\ell}(\Delta(A, \lambda) \cap \mathbb{Z}^n \cap V(p')) = \Delta(\tilde{A}, \tilde{\lambda}) \cap \mathbb{Z}^n \cap V(p').
\]

Since the vector \(-e_k + ce\ell\) is parallel to the plane \(V\),

\[
\mathcal{F}_{-e_k+ce\ell}(\Delta(A, \lambda) \cap \mathbb{Z}^n) \cap V(p') = \mathcal{F}_{-e_k+ce\ell}(\Delta(A, \lambda) \cap \mathbb{Z}^n \cap V(p')).
\]

Hence, since both equations hold for all \(p' \in \mathbb{Z}^n\), we obtain

\[
\mathcal{F}_{-e_k+ce\ell}(\Delta(A, \lambda) \cap \mathbb{Z}^n) = \Delta(\tilde{A}, \tilde{\lambda}) \cap \mathbb{Z}^n,
\]

which is exactly the required equation (18).

We now fix \(p' \in \mathbb{Z}^n\) and turn to proving (19).

Assume first that \(0 > \langle p', e_j \rangle\) for some \(j \notin \{k, \ell\}\). In this case, \(\Delta(A, \lambda) \cap V(p')\) and \(\Delta(\tilde{A}, \tilde{\lambda}) \cap V(p')\) are both empty; hence, (19) is satisfied. Similarly, assume that \(\langle p', e_j + \sum_i A_j^i e_i \rangle > \lambda_j\) for some \(j \notin \{k, \ell\}\) such that \(A_j^j = 0\). Then, \(A_j^j = cA_j^j = 0\) as well. Therefore, \(\langle p, e_j + \sum_i A_j^i e_i \rangle = \langle p', e_j + \sum_i A_j^i e_i \rangle > \lambda_j\) for all \(p \in V(p')\), and so \(\Delta(A, \lambda) \cap V(p')\) is empty. Since \(\tilde{\lambda}_j = \lambda_j\) and \(\tilde{A}_j^j = A_j^j\) for all \(i\), \(\Delta(\tilde{A}, \tilde{\lambda}) \cap V(p')\) is also empty. Once again, (19) is satisfied.

Therefore, we may assume that \(0 \leq \langle p', e_j \rangle\) for all \(j \notin \{k, \ell\}\), and that \(\langle p', e_j + \sum_i A_j^i e_i \rangle \leq \lambda_j\) for all \(j \notin \{k, \ell\}\) such that \(A_j^j = 0\). Since
these inequalities are also satisfied if we replace $p'$ by any $p \in V(p')$, a point $p \in V(p')$ lies in $\Delta(A, \lambda)$ exactly if it satisfies the inequalities

$$0 \leq \langle p, e_k \rangle \leq \lambda_k - \langle p', \sum_{i} A^k_i e_i \rangle, \quad 0 \leq \langle p, e_\ell \rangle,$$

$$\langle p, A^k_\ell e_k + e_\ell \rangle \leq \lambda_\ell - \langle p', \sum_{i \neq k} A^k_i e_i \rangle, \quad \lambda' \leq \langle p, c e_k + e_\ell \rangle \leq \lambda'',$$

where

$$\lambda' = \max_{\{ j : A^j_\ell < 0 \}} \frac{\lambda_j - \langle p', e_j + \sum_{i \in \{ k, \ell \}} A^j_i e_i \rangle}{A^j_\ell},$$

$$\lambda'' = \min_{\{ j : A^j_\ell > 0 \}} \frac{\lambda_j - \langle p', e_j + \sum_{i \in \{ k, \ell \}} A^j_i e_i \rangle}{A^j_\ell}.$$

Here, we have used the fact that $A^k_j = c A^j_\ell$ for any $j \neq \ell$, and adopted the convention that $\min(\emptyset) = \infty$ and $\max(\emptyset) = -\infty$. Moreover, since $A^j_\ell = 0$ for all $j < \ell$, $0 \leq \langle p', e_j \rangle$ and $\langle p', e_j + \sum_i A^j_i e_i \rangle \leq \lambda_j$ for all $j \in [\ell] \setminus \{ k, \ell \}$. Hence, Lemma 4.6 implies that

$$\lambda_\ell - \langle p', \sum_{i \neq k} A^k_\ell e_i \rangle > c (\lambda_k - \langle p', \sum_i A^k_i e_i \rangle).$$

Therefore, by Lemma 2.10 (with $(\lambda_k - \langle p', \sum_i A^k_i e_i \rangle)$ and $(\lambda_\ell - \langle p', \sum_{i \neq k} A^k_i e_i \rangle)$ playing the roles of $\lambda_1$ and $\lambda_2$, respectively) an integral point $p \in \mathbb{Z}^n \cap V(p')$ lies in $\mathcal{F}_{-e_k + ce_\ell} (\Delta(A, \lambda) \cap \mathbb{Z}^n)$ exactly if it satisfies the inequalities

$$0 \leq \langle p, e_k \rangle \leq \lambda_k - \langle p', \sum_i A^k_i e_i \rangle, \quad 0 \leq \langle p, e_\ell \rangle,$$

$$\langle p, (2c - A^k_\ell) e_k + e_\ell \rangle \leq (c - A^k_\ell) (\lambda_k - \langle p', \sum_{i \neq k} A^k_i e_i \rangle) + \lambda_\ell - \langle p', \sum_{i \neq k} A^k_i e_i \rangle,$$

$$\lambda' \leq \langle p, c e_k + e_\ell \rangle \leq \lambda''.$$

By the equations displayed at the beginning of this proof, we can rewrite these inequalities as

$$0 \leq \langle p, e_k \rangle \leq \lambda_k - \langle p', \sum_i A^k_i e_i \rangle, \quad 0 \leq \langle p, e_\ell \rangle,$$

$$\langle p, A^k_\ell e_k + e_\ell \rangle \leq \bar{\lambda}_\ell - \langle p', \sum_{i \neq k} A^k_i e_i \rangle, \quad \lambda' \leq \langle p, c e_k + e_\ell \rangle \leq \lambda'',$$

Finally, a point $p \in \mathbb{Z}^n \cap V(p')$ lies in $\Delta(\tilde{A}, \tilde{\lambda})$ exactly if it satisfies the above inequalities. Hence, as required, (19) is satisfied. \qed
We now have all the tools we need to prove our main proposition.

**Proposition 4.8.** Let \((M, \omega)\) and \((\tilde{M}, \tilde{\omega})\) be the symplectic Bott manifolds associated to strictly upper triangular matrices \(A\) and \(\tilde{A}\) in \(M_n(\mathbb{Z})\) and \(\lambda\) and \(\tilde{\lambda}\) in \(\mathbb{Z}^n\), respectively. Fix integers \(1 \leq k < \ell \leq n\). Assume that there exists an isomorphism from \(H^*(M)\) to \(H^*(\tilde{M})\) that takes \(x_k\) to \(\tilde{x}_k - \frac{1}{2}(A^k_k - \tilde{A}^k_k)\tilde{x}_\ell\), \(x_i\) to \(\tilde{x}_i\) for all \(i \neq k\), and \(\sum \lambda_ix_i\) to \(\sum \tilde{\lambda}_i\tilde{x}_i\). Then \((M, \omega)\) and \((\tilde{M}, \tilde{\omega})\) are symplectomorphic.

**Proof.** Suppose first that \(c := \frac{1}{2}(A^k_k + \tilde{A}^k_k) \geq 0\). By reversing the isomorphism if necessary, we may assume that \(\tilde{A}^k_k \geq A^k_k\), and so \(\tilde{A}^k_k \geq |A^k_k|\). By Lemma 4.7,

\[
\mathcal{F}_{-k+c\ell}(\Delta(A, \lambda) \cap \mathbb{Z}^n) = \Delta(\tilde{A}, \tilde{\lambda}) \cap \mathbb{Z}^n.
\]

Moreover, the only properties of \(\lambda\) and \(\tilde{\lambda}\) we needed are the fact that this isomorphism takes \(\sum \lambda_ix_i\) to \(\sum \tilde{\lambda}_i\tilde{x}_i\), and the fact that \((A, \lambda)\) and \((\tilde{A}, \tilde{\lambda})\) define symplectic Bott manifolds. Therefore, (18) still holds if we replace \(\lambda\) by \(m\lambda\) and \(\tilde{\lambda}\) by \(m\tilde{\lambda}\) for some positive integer \(m\). Since \(m\Delta(A, \lambda) = \Delta(A, m\lambda)\) and \(m\Delta(A, \tilde{\lambda}) = \Delta(A, m\tilde{\lambda})\), this implies that

\[
\mathcal{F}_{-k+c\ell}(m\Delta(A, \lambda) \cap \mathbb{Z}^n) = m\Delta(\tilde{A}, \tilde{\lambda}) \cap \mathbb{Z}^n
\]

for all \(m \in \mathbb{Z}_{>0}\). Hence, since \(\Delta(A, \lambda)\) and \(\Delta(\tilde{A}, \tilde{\lambda})\) are smooth polytopes, Proposition 2.7 implies that \((M, \omega)\) and \((\tilde{M}, \tilde{\omega})\) are symplectomorphic.

So assume instead that \(A^k_k + \tilde{A}^k_k < 0\). Let \(F\) be the isomorphism from \(H^*(M)\) to \(H^*(\tilde{M})\) that takes \(x_k\) to \(\tilde{x}_k - \frac{1}{2}(A^k_k - \tilde{A}^k_k)\tilde{x}_\ell\), \(x_i\) to \(\tilde{x}_i\) for all \(i \neq k\), and \(\sum \lambda_ix_i\) to \(\sum \tilde{\lambda}_i\tilde{x}_i\). Define \(\tilde{\lambda}, \tilde{A} \in M_n(\mathbb{Z})\) and \(\tilde{\lambda}, \tilde{\lambda} \in \mathbb{R}^n\) by

\[
\tilde{A}^j_k = -A^j_k \quad \text{and} \quad \tilde{A}^k_k = -\tilde{A}^k_k \quad \forall \ j,
\]

\[
\tilde{A}^j_k = A^j_k - A^i_k A^j_k, \quad \tilde{\lambda}^j_k = \tilde{\lambda}^j_k - \tilde{\lambda}^k_k \tilde{A}^j_k \quad \forall \ j \quad \text{and} \quad \forall \ i \neq k.
\]

By Lemma 3.6, \((\tilde{A}, \tilde{\lambda})\) and \((\tilde{\lambda}, \tilde{\lambda})\) define symplectic Bott manifolds \((\tilde{M}, \tilde{\omega})\) and \((\tilde{M}, \tilde{\lambda})\); moreover, there exist symplectomorphisms \(f : M \to \tilde{M}\) and \(\tilde{f} : \tilde{M} \to \tilde{M}\) such that the induced maps in cohomology take \(\tilde{x}_k\) to \(x_k + \sum_j A^j_k x_j\) and \(\tilde{x}_k\) to \(\tilde{x}_k + \sum_j \tilde{A}^j_k \tilde{x}_j\), and take \(\tilde{x}_i\) to \(x_i\) and \(\tilde{\tilde{x}}_i\) to \(\tilde{x}_i\) for all \(i \neq k\). Moreover, by Lemma 4.10, \(\tilde{A}^k_k = A^k_k\) for all \(j \neq \ell\). Therefore, a simple computation shows that the composition
Remark 4.9. We believe that the symplectomorphism from \((\tilde{M}, \omega)\) to \((M, \omega)\) constructed in Proposition 4.8 induces the isomorphism in cohomology described in that proposition. However, we have not proved this claim here because we didn't need it to prove our main results.

One consequence of this proposition is that, given a symplectic Bott manifold, we can often find a different symplectic Bott manifold that is symplectomorphic.

Corollary 4.10. Let \((M, \omega)\) be the symplectic Bott manifold associated to a strictly upper triangular matrix \(A \in \mathbb{M}_n(\mathbb{Z})\) and \(\lambda \in \mathbb{Z}^n\). Assume that there exists \(c \in \mathbb{Z}\) and integers \(1 \leq k < \ell \leq n\) such that \(|2c - A^k_{\ell}| \leq |A^k_{\ell}|\) and \(A^i_{j} = cA^i_{j} \) for all \(j \neq \ell\). Then \((M, \omega)\) is symplectomorphic to the symplectic Bott manifold associated to a strictly upper triangular matrix \(\tilde{A} \in \mathbb{M}_n(\mathbb{Z})\) and \(\tilde{\lambda} \in \mathbb{Z}^n\) with

- \(\tilde{A}^k_{\ell} = 2c - A^k_{\ell}\), and otherwise
- \(\tilde{A}^i_{j} = A^i_{j}\) for all \(j \) and all \(i \geq k\).

Proof. Define a strictly upper triangular matrix \(\hat{A} \in \mathbb{M}_n(\mathbb{Z})\) and \(\tilde{\lambda} \in \mathbb{Z}^n\) by \(\hat{A}^k_{\ell} = 2c - A^k_{\ell}\), \(\hat{\lambda}^k_{\ell} = \lambda^k_{\ell} - \lambda_k (A^k_{\ell} - c)\), \(\hat{A}^i_{k} = A^i_{k} - A^k_{k} (A^k_{\ell} - c)\) for all \(i \neq k\), and \(\hat{A}^i_{j} = A^i_{j}\) and \(\hat{\lambda}^i_{j} = \lambda^i_{j}\) for all \(i\) and \(j \neq \ell\). Let \(\tilde{M}\) be the Bott manifold associated to \(\hat{A}\). By Corollary 4.2, \(\hat{A}\) satisfies the equations above; moreover, by Lemma 4.1, there’s an isomorphism from \(H^*(M)\) to \(H^*(\tilde{M})\) that takes \(x_k\) to \(x_k - (A^k_{\ell} - c)x_{\ell}\), \(x_i\) to \(\tilde{x}_i\) for all \(i \neq k\), and \(B^k_{\ell}\) to \(\tilde{B}^k_{\ell}\) and \(\tilde{\lambda}_i\) to \(\tilde{\lambda}_i\). Hence, since \(|A^k_{\ell}| \geq |\hat{A}^k_{\ell}|\) and \((A, \lambda)\) defines a symplectic Bott manifold, Propositions 4.4 and 4.8 together imply that \((\hat{A}, \tilde{\lambda})\) defines a symplectic Bott manifold that is symplectomorphic to \((\tilde{M}, \omega)\).

5. Proof of main theorem

The main goal of this section is to prove that strong symplectic cohomological rigidity holds for the family of \(\mathbb{Q}\)-trivial symplectic Bott manifolds with rational symplectic forms. Here, a \(2n\)-dimensional Bott manifold \(M\) is \(\mathbb{Q}\)-trivial if its rational cohomology ring \(H^*(M; \mathbb{Q})\) is isomorphic to \(H^*((\mathbb{C}P^1)^n; \mathbb{Q}) \simeq \mathbb{Q}[x_1, \ldots, x_n]/(x_1^2, \ldots, x_n^2)\). In the process, we obtain a complete classification of \(\mathbb{Q}\)-trivial symplectic Bott
manifolds with rational symplectic forms up to symplectomorphism; see Corollary 5.7.

**Theorem 5.1.** Let \((M, \omega)\) and \((\tilde{M}, \tilde{\omega})\) be \(\mathbb{Q}\)-trivial symplectic Bott manifolds with rational symplectic forms. Given a ring isomorphism \(F^*: H^*(M; \mathbb{Z}) \to H^*(\tilde{M}; \mathbb{Z})\) such that \(F(\omega) = \tilde{\omega}\), there exists a symplectomorphism \(f\) from \((\tilde{M}, \tilde{\omega})\) to \((M, \omega)\) so that \(f^* = F\).

We begin by constructing some examples of \(\mathbb{Q}\)-trivial Bott manifolds. Given a positive integer \(n\), define a strictly upper triangular matrix \(A \in M_n(\mathbb{Z})\) by \(A_{i,j} = -1\) for all \(i < n\), and \(A_{j,i} = 0\) for all other \(i, j \in [n]\). Let \(\mathcal{H}_n\) be the associated Bott manifold. By the discussion in the beginning of Section 3

\[
\mathcal{H}_n = \left( (\mathbb{C}P^1)^{n-1} \times (\mathbb{C}^2 \setminus \{(0,0)\}) \right) / \mathbb{C}^\times,
\]

where \(\mathbb{C}^\times\) acts diagonally on \(\mathbb{C}^2\), and acts on each \(\mathbb{C}P^1\) by multiplication on the second coordinate. Hence, \(\mathcal{H}_n\) is a \((\mathbb{C}P^1)^{n-1}\) bundle over \(\mathbb{C}P^1\). More specifically, \(\mathcal{H}_2\) is the Hirzebruch surface \(\mathbb{P}(\mathcal{O}(0) \oplus \mathcal{O}(1)) \cong \mathbb{P}(\mathcal{O}(-1) \oplus \mathcal{O}(0))\), and \(\mathcal{H}_n\) is the fiber product of \(\mathcal{H}_{n-1}\) with \(\mathcal{H}_1\) for all \(n \geq 2\). By Lemma 3.2 the cohomology of \(\mathcal{H}_n\) is given by

\[
H^*(\mathcal{H}_n; \mathbb{Z}) = \mathbb{Z}[x_1, \ldots, x_n]/(x_1^2 - x_1x_n, \ldots, x_{n-1}^2 - x_{n-1}x_n, x_n^2).
\]

Define \(y_i \in H^*(\mathcal{H}_n; \mathbb{Q})\) by \(y_i = x_i - \frac{1}{2}x_n\) for all \(i < n\), and \(y_n = x_n\). The rational cohomology ring \(H^*(\mathcal{H}_n; \mathbb{Q})\) is isomorphic to \(\mathbb{Q}[y_1, \ldots, y_n]/(y_1^2, \ldots, y_n^2)\), as required.

Given \(\lambda \in \mathbb{R}^n\), the associated polytope \(\Delta(A, \lambda)\) is the set of \(p \in \mathbb{R}^n\) such that

\[
\langle p, e_j \rangle \geq 0 \forall j, \quad \langle p, e_j \rangle \leq \lambda_j \forall j < n, \quad \text{and} \quad \langle p, e_n - \sum_{i<n} e_i \rangle \leq \lambda_n.
\]

By Proposition 3.3, \(\Delta(A, \lambda)\) is a Bott polytope exactly if \(\lambda \in (\mathbb{R}_{>0})^n\). Assume this holds, and let \(\mathcal{H}_n = \mathcal{H}(\lambda_1, \ldots, \lambda_n)\) be the associated symplectic Bott manifold. More generally, given a partition \(\sum_{s=1}^m l_s = n\) of \(n\), and \(\lambda \in (\mathbb{R}_{>0})^n\), define a \(\mathbb{Q}\)-trivial symplectic Bott manifold

\[
\mathcal{H}(\lambda_1, \ldots, \lambda_l) \times \cdots \times \mathcal{H}(\lambda_{n-l+1}, \ldots, \lambda_n).
\]

Our first lemma, when combined with Lemma 3.5, shows that strong symplectic cohomological rigidity holds for the family of symplectic Bott manifolds that are products of the \(\mathcal{H}_i\).

**Lemma 5.2.** Fix a positive integer \(n\). Let \(\sum_{s=1}^m l_s = \sum_{s=1}^{\tilde{m}} \tilde{l}_s = n\) be partitions of \(n\), and fix \(\lambda, \tilde{\lambda} \in (\mathbb{R}_{>0})^n\). Consider the symplectic Bott
Thus, let positive numbers \( \lambda_1, \ldots, \lambda_l \) this implies that every zero element of \( H \), where

\[
(M, \omega) := H(\lambda_1, \ldots, \lambda_l) \times \cdots \times H(\lambda_{n-l+1}, \ldots, \lambda_n),
\]

\[
(\tilde{M}, \tilde{\omega}) := H(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_l) \times \cdots \times H(\tilde{\lambda}_{n-\tilde{l}+1}, \ldots, \tilde{\lambda}_n).
\]

Given a ring isomorphism \( F: H^*(M; \mathbb{Z}) \to H^*(\tilde{M}; \mathbb{Z}) \) such that \([\omega] = [\tilde{\omega}]\), there exists a permutation \( \sigma \) of \( [n] \) such that \( F(x_i) = \tilde{x}_{\sigma(i)} \) for all \( i \in [n] \).

**Proof.** We start with the observation that \( F \) must map \( H^2(M; \mathbb{Z}) \) to \( H^2(\tilde{M}; \mathbb{Z}) \) because the cohomology of a symplectic Bott manifold is generated by elements of degree 2. (In fact, the same is true for any symplectic toric manifold.) Assume first that both partitions of \( n \) are trivial, that is,

\[
(M, \omega) = H(\lambda_1, \ldots, \lambda_n) \quad \text{and} \quad (\tilde{M}, \tilde{\omega}) = H(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n).
\]

By inspection, there are exactly \( 2n \) primitive classes in \( H^2(M; \mathbb{Z}) \) with trivial square: \( \pm z_1, \ldots, \pm z_n \), where \( z_i = x_i - x_{n-i+1} \) for all \( i < n \). Similarly, \( \pm \tilde{z}_1, \ldots, \pm \tilde{z}_n \) are the primitive classes in \( H^2(\tilde{M}; \mathbb{Z}) \) with trivial square, where \( \tilde{z}_i = \tilde{x}_i - x_{n-i+1} \) for all \( i < n \). Therefore, since \( F \) is a ring isomorphism, there exists \( \epsilon \in \{-1, 1\}^i \) and a permutation \( \sigma \) of \( [n] \) such that \( F(z_i) = \epsilon_i \tilde{x}_{\sigma(i)} \) for all \( i \). (Conversely, this defines an isomorphism for any such \( \epsilon \) and \( \sigma \).) Clearly \( z_1, \ldots, z_n \) is a basis for \( H^2(M; \mathbb{R}) \). In this basis, we can write \([\omega] = \sum_{i=1}^n \eta_i z_i\), where \( \eta_i = \frac{\lambda_i}{2} \) for all \( i < n \) and \( \eta_n = \lambda_n + \sum_{i<n} \frac{\lambda_i}{2} \). Similarly, \([\tilde{\omega}] = \sum_{i=1}^n \tilde{\eta}_i \tilde{z}_i\), where \( \tilde{\eta}_i = \frac{\tilde{\lambda}_i}{2} \) for all \( i < n \) and \( \tilde{\eta}_n = \tilde{\lambda}_n + \sum_{i<n} \frac{\tilde{\lambda}_i}{2} \). Since \( F([\omega]) = [\tilde{\omega}] \), this implies that \( \epsilon_i \eta_i = \tilde{\eta}_{\sigma(i)} \) for all \( i \). Since \( \lambda_i \) and \( \tilde{\lambda}_i \) are positive for all \( i \), \( \eta_i \) and \( \tilde{\eta}_i \) are positive for all \( i \), and so this implies that \( \epsilon_i = 1 \) for all \( i \). Moreover, since \( \eta_n > \eta_i \) and \( \tilde{\eta}_n > \tilde{\eta}_i \) for all \( i < n \), we have \( \sigma(n) = n \). Thus, \( F(x_i) = \tilde{x}_{\sigma(i)} \) for all \( i \).

We now turn to the general case. To simplify notation, let \( i_s = l_1 + \cdots + l_s \) for each \( s \in [m] \), let \( i_0 = 0 \), and let \( \lambda^{i_s} \) denote the \( l_s \)-tuple of positive numbers \( (\lambda_{i_{s-1}+1}, \ldots, \lambda_{i_s}) \). By inspection, the primitive square zero elements of \( H^*(M; \mathbb{Z}) \) are precisely

\[
\pm x_{i_s} \quad \text{and} \quad \pm (2x_i - x_{i_s}) \quad \text{for} \quad s \in [m] \quad \text{and} \quad i_{s-1} < i < i_s.
\]

In particular, each primitive square zero element sits in some subring \( H^*(H(\lambda^{i_s}); \mathbb{Z}) \subseteq H^*(M; \mathbb{Z}) \) and every primitive square zero element in \( H^*(H(\lambda^{i_s}); \mathbb{Z}) \) are equal to \( x_{i_s} \) modulo 2. Therefore, for each \( s \in [m] \) there exists \( \tilde{s} \) with \( \tilde{l}_s = l_s \) such that \( F \) restricts to an isomorphism from \( H^*(H(\lambda^{i_s}); \mathbb{Z}) \) to \( H^*(H(\tilde{\lambda}^{i_{\tilde{s}}}); \mathbb{Z}) \) and takes \( \sum_{i=i_{s-1}+1}^{i_s} \lambda_i x_i \) to
hence, by the previous paragraph, there is a bijection 
\[ \sigma_i : \{ i_{s-1} + 1, \ldots, i_s \} \rightarrow \{ \tilde{i}_{s-1} + 1, \ldots, \tilde{i}_s \} \]
with \( \sigma(i_s) = \tilde{i}_s \) so that 
\[ F(x_i) = \tilde{x}_{\sigma_i(i)} \]
for all \( i \in \{ i_{s-1} + 1, \ldots, i_s \} \). Combining \( \sigma_1, \ldots, \sigma_m \), we
construct the required permutation \( \sigma \) of \( [n] \).

Our next goal is to prove that every \( \mathbb{Q} \)-trivial symplectic Bott 
manifold is symplectomorphic to a product of the \( \mathcal{H}_i \); see Proposition 5.6.
Let \( M \) be the Bott manifold associated to a strictly upper triangular 
matrix \( A \in M_n(\mathbb{Z}) \). Recall that

\[ H^*(M; \mathbb{Z}) = \mathbb{Z}[x_1, \ldots, x_n]/(x_i^2 + \sum_j A^i_j x_j x_j) \]

For each \( i \in [n] \), consider the special cohomology classes

\[ \alpha_i = - \sum_j A^i_j x_j \in H^*(M; \mathbb{Z}) \quad \text{and} \quad y_i = x_i - \frac{1}{2} \alpha_i \in H^*(M; \mathbb{Q}). \]

**Definition 5.3.** We say that \( x_k \) has **even (odd) exceptional type** 
if \( \alpha_k = my_k \) for some \( \ell \in [n] \) and even (respectively, odd) integer \( m \).
In “coordinates”, this means that \( A^k_k \) is even (respectively, odd) and 
that \( A^i_j = \frac{1}{2} A^k_k A^j_i \) for \( j \neq \ell \). Additionally, we say that \( x_k \) has **special** 
exceptional type if \( \alpha_k = 0 \), that is, \( A^k_k = A^j_j = 0 \) for all \( j \neq \ell \).

The following lemma, due to Choi and Masuda [CM Proposition 3.1], shows that \( \mathbb{Q} \)-trivial Bott manifolds have many cohomology classes 
with exceptional type. We reprove it here for the reader’s convenience.

**Lemma 5.4** (Choi-Masuda). If \( M \) is an \( 2n \)-dimensional \( \mathbb{Q} \)-trivial Bott 
manifold then \( x_i \in H^*(M; \mathbb{Z}) \) has exceptional type for all \( i \in [n] \).

**Proof.** To begin, let \( u \in H^2(M; \mathbb{Z}) \) be any nonzero class with square 
zero. Then \( u = \sum_j u_j x_j \) for some nonzero \((u_1, \ldots, u_n) \in \mathbb{Z}^n; \) let \( i \in [n] \)
be the smallest number such that that \( u_i \neq 0 \). Then

\[ 0 = u^2 = u_i^2 x_i^2 + 2 \sum_{j > i} u_i u_j x_j x_j + (\text{terms involving } x_j x_k \text{ for } j, k > i) \]

\[ = - u_i^2 \sum_j A^i_j x_j x_j + 2 \sum_{j \neq i} u_i u_j x_j x_j + (\text{terms involving } x_j x_k \text{ for } j, k > i); \]

hence, \( u_i^2 A^i_j = 2u_i u_j \), and so \( u_j = \frac{1}{2} u_i A^i_j \), for all \( j \neq i \). Therefore,
\( u = u_i (x_i - \frac{1}{2} \alpha_i) = u_i y_i \) for some \( u_i \in \mathbb{Z} \) and \( i \in [n] \).

Since \( y_i \in H^*(M; \mathbb{Q}) \), there is a unique positive integer \( u_i \) such that 
\( u_i y_i \) is a primitive class in \( H^*(M; \mathbb{Z}) \) for each \( i \). Therefore, by 
the previous paragraph, there are twice as many nonzero primitive classes 
in \( H^*(M; \mathbb{Z}) \) with square zero as there are \( i \in [n] \) such that \( y_i^2 = 0 \). On
the other hand, since \( H^*(M; \mathbb{Q}) \simeq H((\mathbb{C}P^1)^n; \mathbb{Q}) \), there exist exactly \( 2n \) nonzero primitive elements in \( H^*(M; \mathbb{Z}) \) with square zero. Hence, given any \( i \in [n] \), \( 0 = y_i^2 = (x_i - \frac{1}{2}\alpha_i)^2 = \frac{1}{4}\alpha_i^2 \). By the previous paragraph, this implies that \( \alpha_i = my_\ell \) for some \( m \in \mathbb{Z} \) and \( \ell \in [n] \). Hence, \( x_i \) has exceptional type.

Moreover, we can use the symplectomorphisms constructed in Section 4 to show that every symplectic Bott manifold is symplectomorphic to a Bott manifolds with a simpler cohomology ring, in the sense that most cohomology classes with exceptional type have a standard form.

**Lemma 5.5.** Let \((M, \omega)\) be a \(2n\)-dimensional symplectic Bott manifold with integral symplectic form. There exist a strictly upper-triangular matrix \( \tilde{A} \in M_n(\mathbb{Z}) \) and \( \tilde{\lambda} \in \mathbb{Z}^n \) so that the associated symplectic Bott manifold \((\tilde{M}, \tilde{\omega})\) is symplectomorphic to \((M, \omega)\) and has the following properties for all \( i \in [n] \):

1. If \( \tilde{x}_i \) has even exceptional type, then \( \tilde{\alpha}_i = 0 \).
2. If \( \tilde{x}_i \) has odd special exceptional type, then \( \tilde{\alpha}_i = \tilde{y}_j \) for some \( j > i \).

Here \( \tilde{\alpha}_i, \tilde{y}_i \in H(\tilde{M}; \mathbb{Q}) = \mathbb{Q}[\tilde{x}_1, \ldots, \tilde{x}_n]/(\tilde{x}_i^2 + \sum_j \tilde{A}_{ij}\tilde{x}_i\tilde{x}_j) \) are defined as in (20).

**Proof.** By definition, \((M, \omega)\) is the symplectic Bott manifold associated to some strictly upper-triangular matrix \( A \in M_n(\mathbb{Z}) \) and \( \lambda \in \mathbb{Z}^n \).

Since \( \alpha_n = 0 \), the manifold \( M \) has properties (1) and (2) for \( i = n \). Assume that manifold has properties (1) and (2) for all \( i > k \). We will prove that there exits a strictly upper-triangular matrix \( \tilde{A} \in M_n(\mathbb{Z}) \) and \( \tilde{\lambda} \in \mathbb{Z}^n \) so that the associated symplectic Bott manifold \((\tilde{M}, \tilde{\omega})\) is symplectomorphic to \((M, \omega)\) and has properties (1) and (2) for all \( i \geq k \). The claim then follows by induction.

Suppose first that \( x_k \) has even exceptional type. If \( \alpha_k \neq 0 \), then \( \alpha_k = -2c\gamma_\ell \) for some nonzero \( c \in \mathbb{Z} \) and \( \ell > k \), that is, \( A_k^j = 2c \) and \( \tilde{A}_j^k = cA_j^\ell \) for all \( j \neq \ell \). Since \( A \) is strictly upper triangular, this implies that \( \tilde{A}_j^k = 0 \) for all \( j < \ell \); c.f. Corollary 4.12. Hence, by Corollary 4.10, \((M, \omega)\) is symplectomorphic to the symplectic Bott manifold \((\tilde{M}, \tilde{\omega})\) associated to a strictly upper triangular matrix \( \tilde{A} \in M_n(\mathbb{Z}) \) and \( \tilde{\lambda} \in \mathbb{Z}^n \) with

- \( \tilde{A}_j^k = 0 \) for all \( j \leq \ell \), and
- \( \tilde{A}_j^i = A_j^i \) for all \( i > k \).

The manifold \( \tilde{M} \) has properties (1) and (2) for all \( i > k \). Moreover, if \( \tilde{x}_k \) has even exceptional type and \( \tilde{\alpha}_k \neq 0 \), then \( \tilde{\alpha}_k = -2c'\tilde{y}_\ell \) for some
nonzero $c' \in \mathbb{Z}$ and $\ell' > \ell$. Hence, by repeating this process as many times as necessary, we can ensure that either $\bar{x}_k$ does not have even exceptional type or $\tilde{\alpha}_k = 0$, that is, $\tilde{M}$ has property (1) for $i = k$.

So suppose instead that $x_k$ has odd special exceptional type. Then there exists $c \in \mathbb{Z}$ and $\ell > k$ such that $A^k_\ell = 2c + 1$ and $A^k_j = A^k_j = 0$ for all $j \neq \ell$. In this case, Corollary 4.10 implies that $(M, \omega)$ is symplectomorphic to the symplectic Bott manifold associated to a strictly upper-triangular matrix $\tilde{A} \in M_n(\mathbb{Z})$ and $\tilde{\lambda} \in \mathbb{Z}^n$ with

$$\begin{itemize}
  \item $A^k_\ell = -1$ and otherwise $A^k_j = 0$ for all $j$, i.e., $\tilde{\alpha}_k = \tilde{y}_\ell$, and
  \item $A^j_j = A^k_j$ for all $i > k$.
\end{itemize}$$

Hence, $\tilde{M}$ has properties (1) and (2) for all $i \geq k$, as required. □

**Proposition 5.6.** Let $(M, \omega)$ be a $2n$-dimensional $\mathbb{Q}$-trivial symplectic Bott manifold with rational symplectic form. There exists a partition $\sum_{s=1}^m l_s = n$ of $n$ and $\lambda \in (\mathbb{Q}_{>0})^n$ so that $(M, \omega)$ is symplectomorphic to the product

$$\mathcal{H}(\lambda_1, \ldots, \lambda_{l_1}) \times \cdots \times \mathcal{H}(\lambda_{n-l_m+1}, \ldots, \lambda_n).$$

**Proof.** By rescaling if necessary, we may assume that $\omega$ is integral.

On the one hand, by Proposition 5.3 we may assume that $(M, \omega)$ is the symplectic Bott manifold associated to a strictly upper-triangular matrix $A \in M_n(\mathbb{Z})$ and $\lambda \in (\mathbb{Z}_{>0})^n$ with the following properties for all $i \in [n]$:

1. If $x_i$ has even exceptional type, then $\alpha_i = 0$.
2. If $x_i$ has odd special exceptional type, then $\alpha_i = y_j$ for some $j > i$.

On the other hand, since $M$ is $\mathbb{Q}$-trivial, Lemma 5.4 below implies that $x_i$ has exceptional type for all $i$, that is, $\alpha_i = my_j$ for some $m \in \mathbb{Z}$ and $j > i$.

So fix any $i \in [n]$. If $x_i$ has even exceptional type, then (1) implies that $\alpha_i = 0$. If $x_i$ has odd exceptional type, then $\alpha_i = (2m + 1)y_j = (2m + 1)(x_j - \frac{1}{2} \alpha_j)$ for some $m \in \mathbb{Z}$ and $j > i$. If $x_j$ also has odd exceptional type, then $\alpha_j = (2m' + 1)(x_j' - \frac{1}{2} \alpha_j')$ for some $m' \in \mathbb{Z}$ and $j' > j$, and so $A^j_j = \frac{1}{2}(2m + 1)(2m' + 1) \notin \mathbb{Z}$. Since this is impossible, $x_j$ has even exceptional type. As we’ve seen, this implies that $\alpha_j = 0$, that is, $x_i$ has special exceptional type. By (2) this implies that $\alpha_i = y_j$. In conclusion, either $\alpha_i = 0$, or there exists $j > i$ such that $\alpha_i = y_j$ and $\alpha_j = 0$.

Let $\mathcal{E} = \{i_1, \ldots, i_m\}$ be the set of $i \in [n]$ such that $\alpha_i = 0$. For each $i \in \mathcal{E}$ let

$$B_i := \{i\} \cup \{j \mid \alpha_j = y_i\}$$
and let \( l_i \) be the number of elements in \( B_i \). By the previous paragraph, \( B_1, \ldots, B_m \) is a partition of \([n]\), that is \( l_1 + \cdots + l_m = n \). Hence, there exists a permutation \( \sigma \) of \([n]\) such that, for all \( j \in [n] \) and \( s \in [m] \), we have \( \sigma(i_s-1) < \sigma(j) \leq \sigma(i_s) \) exactly if \( j \in B_i_s \). (Here, we let \( \sigma(i_0) = 0 \).) Choose \( \tilde{\lambda} \in (\mathbb{Q}_{>0})^n \) so that \( \lambda_i = \tilde{\lambda}_{\sigma(i)} \) for all \( i \), and consider the symplectic Bott manifold

\[
\tilde{H}(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{l_1}) \times \cdots \times \tilde{H}(\tilde{\lambda}_{n-l_m+1}, \ldots, \tilde{\lambda}_n).
\]

There is a ring isomorphism \( F: H^*(M) \to H^*(\tilde{M}) \) such that \( F([\omega]) = [\tilde{\omega}] \) and \( F(x_i) = \tilde{x}_{\sigma(i)} \) for all \( i \). Therefore, the claim follows from Lemma 3.5.

The proof of the main theorem is now immediate.

**Proof of Theorem 5.7** Let \((M, \omega)\) and \((\tilde{M}, \tilde{\omega})\) be \(\mathbb{Q}\)-trivial symplectic Bott manifolds with rational symplectic forms, and let \( F: H^*(M; \mathbb{Z}) \to H^*(\tilde{M}; \mathbb{Z}) \) be a ring isomorphism such that \( F([\omega]) = [\tilde{\omega}] \). Let \( n = \dim H^2(M; \mathbb{Q}) \). By assumption,

\[
\dim M = 2 \dim H^2(M; \mathbb{Q}) = 2 \dim H^2(M; \mathbb{Q}) = \dim \tilde{M}.
\]

Hence, by Proposition 5.6 there exist partitions \( \sum_{s=1}^m l_s = \sum_{s=1}^{\tilde{m}} \tilde{l}_s = n \) of \( n \) and \( \lambda, \tilde{\lambda} \in (\mathbb{Q}_{>0})^n \) such that

\[
(M, \omega) := \tilde{H}(\lambda_1, \ldots, \lambda_{l_1}) \times \cdots \times \tilde{H}(\lambda_{n-l_m+1}, \ldots, \lambda_n),
\]

\[
(\tilde{M}, \tilde{\omega}) := \tilde{H}(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{\tilde{l}_1}) \times \cdots \times \tilde{H}(\tilde{\lambda}_{n-\tilde{l}_m+1}, \ldots, \tilde{\lambda}_n).
\]

Hence, combining Lemmas 5.2 and 3.5 there exists a symplectomorphism \( f \) from \((\tilde{M}, \tilde{\omega})\) to \((M, \omega)\) so that that \( f^* = F \).

As a corollary, we obtain a complete classification up to symplectomorphism of \(\mathbb{Q}\)-trivial symplectic Bott manifolds with rational symplectic forms. Recall that a **pointed set** is a pair \((A, a)\), where \(A\) is a set and \(a \in A\).

**Corollary 5.7.** Fix a positive integer \( n \). There is a one-to-one correspondence between symplectomorphism classes of \(2n\)-dimensional symplectic Bott manifolds with rational symplectic forms and multisets of \( n \) positive rational numbers equipped with a partition into pointed multisets.

**Proof.** As we showed in the beginning of this section, there is a symplectic Bott manifold with rational symplectic form

\[
(M, \omega) = \tilde{H}(\lambda_1, \ldots, \lambda_{l_1}) \times \cdots \times \tilde{H}(\lambda_{n-l_m+1}, \ldots, \lambda_n)
\]

associated to any partition \( \sum_{i=1}^m l_i = n \) of \( n \) and \( \lambda \in (\mathbb{Q}_{>0})^n \).
If \((\tilde{M}, \tilde{\omega})\) is the symplectic Bott manifold associated to another partition \(\sum_{i=1}^{m} \tilde{l}_i = n\) of \(n\) and \(\tilde{\lambda} \in (\mathbb{Q}_{>0})^n\), then \(M\) and \(\tilde{M}\) are symplectomorphic exactly if \(\{\lambda_i\}\) and \(\{\tilde{\lambda}_i\}\) agree as multisets with partitions into pointed multisets. To see this, assume first that \(M\) and \(\tilde{M}\) are symplectomorphic. Then there exists a ring isomorphism \(F: H^*(M; \mathbb{Z}) \to H^*(\tilde{M}; \mathbb{Z})\) such that \(F([\omega]) = [\tilde{\omega}]\). Hence, by Lemma 5.2 there exists a permutation \(\sigma\) of \([n]\) such that \(F(x_i) = \tilde{x}_{\sigma(i)}\) for all \(i \in [n]\). This implies that \(\lambda_i = \tilde{\lambda}_{\sigma(i)}\) for all \(i\), and that \(A_{ij} = \tilde{A}_{\sigma(i)\sigma(j)}\) for all \(i, j \in [n]\), or equivalently, that \(\sigma\) induces an isomorphism of multisets with partitions into pointed multisets. Conversely, if a permutation \(\sigma\) of \([n]\) induces an isomorphism of multisets with partitions into pointed multisets, then it also induces a ring isomorphism \(F: H^*(M; \mathbb{Z}) \to H^*(\tilde{M}; \mathbb{Z})\) such that \(F([\omega]) = [\tilde{\omega}]\) and \(F(x_i) = F(\tilde{x}_{\sigma(i)})\) for all \(i\). Therefore, by Lemma 3.5 \(M\) and \(\tilde{M}\) are symplectomorphic.

Finally, by Proposition 5.6 every symplectic Bott manifold with rational symplectic form is symplectomorphic to the symplectic Bott manifold associated to some partition \(\sum_{i=1}^{m} l_i = n\) and some \(\lambda \in (\mathbb{Q}_{>0})^n\).

In comparison, by a theorem of Choi and Masuda, the diffeomorphism type of \(M\) is determined by the partition of \(n\) [CM, Theorem 4.1].

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