Quantum global structure of de Sitter space

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Abstract

I study the global structure of de Sitter space in the semi-classical and one-loop approximations to quantum gravity. The creation and evaporation of neutral black holes causes the fragmentation of de Sitter space into disconnected daughter universes. If the black holes are stabilized by a charge, I find that the decay leads to a necklace of de Sitter universes (‘beads’) joined by near-extremal black hole throats. For sufficient charge, more and more beads keep forming on the necklace, so that an unbounded number of universes will be produced. In any case, future infinity will not be connected. This may have implications for a holographic description of quantum gravity in de Sitter space.

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1 Introduction

1.1 Why study de Sitter space?

In an explicit realization of the holographic principle [1–3], it has recently been argued that type IIB string theory in anti-de Sitter space is dual to a conformal field theory on its boundary [4–6]. Anti-de Sitter space is the maximally symmetric solution of the vacuum Einstein equations with a negative cosmological constant $\Lambda < 0$. One would expect different implementations of the holographic principle to apply to Minkowski ($\Lambda = 0$) and de Sitter space ($\Lambda > 0$), but explicit prescriptions have yet to be found.

An important open question in the latter cases is the location of the surfaces onto which the gravitational theory should be projected. de Sitter space contains no boundaries at spatial infinity, but it has spacelike boundaries at past and future infinity (see Fig. 1). Witten has suggested [7] that these boundaries may provide the holographic surfaces for de Sitter space.

![Penrose diagram of classical de Sitter space. The vertical lines are the origins of polar coordinates on opposite poles of the spatial three-spheres. The diagonal lines are the event horizons of observers on these poles.](image)

Figure 1: Penrose diagram of classical de Sitter space. The vertical lines are the origins of polar coordinates on opposite poles of the spatial three-spheres. The diagonal lines are the event horizons of observers on these poles.

It is therefore crucial to understand whether the simple global structure of de Sitter space persists when quantum effects are included. Because de Sitter space is at a non-zero temperature, black holes will necessarily form through
non-perturbative quantum fluctuations of the metric \[8, 9\]. Therefore the future endpoints of some geodesics fail to lie on future infinity; instead, some will end on black hole singularities. Moreover, the evaporation of some of these black holes will cause the spacelike surfaces to fragment into large, disconnected pieces \[10\]. This means that future infinity will not even be connected. Since the daughter universes are locally de Sitter, one might expect the process to repeat there. This would cause the proliferation into an infinite number of separate de Sitter universes, and thus the complete fragmentation of future infinity.

The proliferation effect exploits a local instability of the maximum-size neutral black holes created in de Sitter space. The present paper will expose a similar instability in maximal black holes carrying magnetic charge. They also nucleate spontaneously in de Sitter space. Similar to the uncharged case, the instability leads to a necklace of de Sitter regions connected through black hole throats. Since the black holes will be stabilized near the extremal limit, however, they will not evaporate completely. In the charged case, therefore, the necklace will not break up into separate de Sitter spaces; instead, the daughter universes will remain connected through near-extremal black holes. If the process repeats in them, a network of such necklaces will emerge. Moreover, if the black holes are sufficiently charged, they will not evaporate at all. In this case the near-black hole regions will remain forever classically unstable. They will produce an unbounded number of black hole interiors and de Sitter ‘beads.’ The inclusion of charge thus makes the semi-classical global structure of de Sitter space even richer.

In order to avoid cluttering the equations with factors of \((D - i)\), only four-dimensional de Sitter space will be considered here. Maximal black hole solutions exist for all de Sitter spaces with spacetime dimension \(D \geq 4\), and should lead to the same instabilities and global structure.

de Sitter space is an important solution of Einstein’s equations, and the semiclassical investigation of its quantum properties is useful both in its own right and as a potential guide in the search for a complete quantum description. An additional motivation was emphasized in Refs. \[9–13\]: The universe presumably underwent a primordial inflationary era, during which it behaved much like de Sitter space. If inflation was sufficiently long (and it is, in fact, generically eternal \[14\]), such quantum effects will determine the global structure of the universe. Recent supernova measurements \[15, 16\] indicate, moreover, that the universe has once again entered a vacuum-dominated era, albeit with a much smaller (effective) cosmological constant.
If this vacuum energy is stable, the results found here will be of relevance also for the distant future of the universe.

1.2 Proliferation

The fragmentation of de Sitter space was found in Ref. [10] for four-dimensional Einstein gravity with a positive cosmological constant and no Maxwell field. Related effects were later investigated for other theories in Refs. [17–19]. It involves three steps, which can be qualitatively described as follows (see Fig. 2):

1. In de Sitter space, a non-perturbative, instanton-mediated fluctuation of the gravitational field changes the spatial topology from $S^3$ to $S^1 \times S^2$. This transition occurs at a rate of $e^{-\pi/\Lambda}$ and leads to the maximal Schwarzschild-de Sitter solution (a.k.a. Nariai), in which the $S^2$ radius is independent of the $S^1$ coordinate.

2. Quantum fluctuations will break the degeneracy of the two-sphere radius. The Nariai geometry is unstable under such perturbations. Regions along the $S^1$ in which the two-spheres are smaller than the degenerate size will collapse and form interiors of black holes. Where the two-spheres are greater, they will grow into exponentially large, asymptotically de Sitter regions. The geometry can now be thought of as a necklace, with large de Sitter beads connected through black hole interiors. The number of beads depends on the number of oscillations of the initial perturbation.

3. The inclusion of quantum radiation at the one-loop level shows that the black holes radiate and decrease in size. For neutral black holes, there is no known obstruction to their complete evaporation. This pinches the necklace, leaving only the disconnected beads, i.e., separate de Sitter universes.

The causal structure that results from this process is shown in Fig. 3 for $n = 2$.

1.3 Outline and Summary

I will assume that there are no magnetically charged particles, or at least none of smaller mass than the black holes. This means that magnetic black
Figure 2: The three-step process by which de Sitter space proliferates into disconnected daughter universes. On a de Sitter background with $S^3$ spacelike sections, a neutral Nariai solution ($S^1 \times S^2$) nucleates (the arrows indicate that opposite ends should be identified, to form the $S^1$). By a classical instability, the degenerate Nariai geometry decays into a necklace of $n$ black holes and $n$ de Sitter regions (‘beads’). Here $n = 2$, and the spatial section was chosen so that it penetrates neither the black hole interiors nor the de Sitter regions; this means that the minimal two-spheres correspond to the black hole horizons (b), and the maximal two-spheres correspond to the cosmological horizons (c). When the black holes evaporate, the beads disconnect, and $n$ separate de Sitter universes remain. The corresponding Penrose diagram is shown in Fig. 3.

holes cannot radiate away their charge. If magnetic flux was threaded around the $S^3$, the black holes could no longer evaporate completely, because they would have to retain at least the minimum mass necessary to support their charge. In other words, the necklace could not be pinched. The large de Sitter universes would have to remain connected through near-extremal black hole throats.

My aim is to show that ordinary de Sitter space decays into this type
of stable ‘necklace’ configuration. This will be an additional, ‘charged’ decay mode complementing the proliferation effect that is mediated by neutral black holes. For this purpose, I will investigate the stability of a class of charged black holes in de Sitter space. The two-parameter family of Reissner-Nordström-de Sitter solutions will be reviewed in Sec. 2.1. For a black hole to nucleate spontaneously (step 1), its geometry must contain a smooth Euclidean sector (see Ref. [20] for a discussion and possible exceptions from this rule). This restricts to a solution subspace comprising three one-parameter families, the ‘cold’, ‘lukewarm’, and ‘Charged Nariai’ black holes [21].

Multiple black hole interiors and de Sitter regions (step 2) can only form if the spatial topology contains an $S^2$ factor of constant radius. In the cold and lukewarm solutions, the two-sphere size will vary. Its single minimum will be distinctly smaller than its single maximum. When one starts with classical
solutions of this type, small quantum fluctuations can have no influence on
the number of oscillations about the degenerate two-sphere size. The necklace
will contain only one bead, whose opposite ends will be connected by a single
black hole interior. The production of such black holes is certainly interesting;
it affects the quantum structure of de Sitter space in the sense that not all
world-lines will end on future infinity. But it is not new. Here, therefore,
the focus will be on the Charged Nariai solutions, for which the two-sphere
size is classically constant, and the number of oscillations is determined only
by quantum fluctuations. Their metric and causal structure are presented in
Sec. 2.2 (In the neutral case, both step 1 and step 2 restrict the Schwarzschild-
de Sitter solutions to the one with the largest black hole size, the neutral
Nariai solution.)

It is important to distinguish between the classical and the quantum in-
stability of the Charged Nariai solutions. Classically, they are perturba-tively
unstable to the transition into a Reissner-Nordström-de Sitter solution rep-
resenting a black hole of nearly maximal mass (Sec. 3.1). This solution, in
turn, may decay into a lower mass black hole by emitting quantum radia-
tion. In Sec. 3.2 I will argue, on thermodynamic grounds only, that max-
imal black holes of sufficiently large, yet sub-extremal charge will not possess the
quantum instability.

Classically, the constraint equations allow only fluctuations that lead to
the formation of a single black hole. In a one-loop effective model that
includes quantum radiation (Sec. 4), however, this restriction is lifted. In
Sec. 5 I will show that higher-mode perturbations of a Charged Nariai solu-
tion first oscillate, then freeze out and form multiple black holes and de Sitter
regions. The model includes the back-reaction to the quantum radiation. It
predicts that the horizons do not, at first, behave according to expectations
developed in Sec. 3.2. At later times, however, they do follow the thermody-
namic behavior. In fact, the one-loop model turns out to reproduce exactly
the value of the critical charge, beyond which maximal black holes no longer
evaporate.

The different evolution of sub- and supercritically charged black holes
leads to a drastic difference in their global structure. Black holes of subcritical
charge simply radiate mass away until they are nearly extremal. The final
picture is similar to neutral proliferation, except that the large de Sitter beads
will still be connected through small black holes; see Fig. 1 (left branch). The
corresponding Penrose diagram is shown in Fig. 2. Like in the neutral case,
the number of final de Sitter universes is determined by the mode number of
Figure 4: Charged Nariai black holes nucleate semiclassically in de Sitter space. Their spacelike sections are $S^1 \times S^2$. The one-sphere expands exponentially, and the two-sphere has a constant radius. The magnetic field loops around the one-sphere and prevents it from being pinched. Quantum fluctuations lead to the formation of $n$ black holes and $n$ de Sitter regions (here $n = 2$). If the charge is small, the black holes evaporate until they are nearly extremal (see Fig. 5 for a Penrose diagram). For supercritical charge, the black holes will grow, approaching the radius of the Charged Nariai solution. The regions between a black hole and a cosmological horizon remain nearly degenerate. Small perturbations can produce more black hole interiors and de Sitter regions there. Iteratively, an infinite number of beads will form. The corresponding Penrose diagram is shown in Fig. 6.

the dominant perturbation when the Charged Nariai solution nucleates.

For Charged Nariai solutions of supercritical charge, an amusing new effect occurs. They will still form large de Sitter beads as well as black hole interiors with singularities. But the black holes do not evaporate. Instead, their horizon size asymptotically approaches the Charged Nariai value. Therefore there will always be regions where the two-sphere size is nearly constant, namely the regions between each black hole horizon and the cosmological horizon ‘surrounding’ it. Random quantum fluctuations of the two-sphere
Figure 5: *Penrose diagram for the decay of de Sitter space into n de Sitter regions (dark grey) separated by n lukewarm black holes (light gray); here n = 2. Singularities are indicated by dashed lines. The upper part of the diagram is a sequence of n Reissner-Nordström-de Sitter diagrams (Fig. 7). Future infinity fragments, but in contrast to the neutral case (Fig. 3), space remains topologically connected. This diagram corresponds to the subcritical branch of Fig. 4.*

Size in such regions will include perturbations which have a maximum on the black hole side and a minimum on the side of the cosmological horizon. If the fluctuation is strong enough, the two-spheres will be larger than the Charged Nariai radius at the maximum, and smaller at the minimum. As it crosses the apparent cosmological horizon, the minimum will freeze out and collapse, forming a new black hole interior. Similarly, the maximum will freeze out as it crosses the apparent black hole horizon; it then grows exponentially, seeding a new de Sitter region. Effectively, a new bead will have been inserted into the necklace. The process continues iteratively, so that infinitely many black holes and de Sitter regions will form around the $S^1$. The result is an unbounded number of de Sitter universes, topologically
(but not causally) connected by charged wormholes (see Fig. 4, right branch). The corresponding Penrose diagram is a fractal; it is sketched in Fig. 6.

Figure 6: Nucleation of a supercritically charged Nariai solution. Black holes form but do not evaporate. Between a black hole and a cosmological horizon, the $S^2$ size remains nearly constant, and small fluctuations can produce more black holes (light grey) and de Sitter regions (dark grey). In this way, an infinite number of beads form on the $S^1$. With better resolution, one would see more and more diamonds and triangles towards the top. This Penrose diagram corresponds to the supercritical branch of Fig. 4.

One might have expected the proliferation of de Sitter space via charged black holes to offer nothing new beyond the additional rule that the daughter universes never quite disconnect. The above argument shows, however, that in fact a genuinely new phenomenon arises. When a neutral or subcritically charged Nariai solution nucleates semiclassically, at most a finite number of de Sitter universes can develop. Infinite proliferation occurs only if the process repeats iteratively inside the daughter universes. This is plausible, but not entirely obvious (see Sec. 6). But in the supercritical case, an infinite number of de Sitter beads forms on a single $S^1$ necklace.
In Sec. 3, I will question the global viewpoint that pervades this paper. I will argue that at least some of my results are independent of its validity, and examine how its abandonment may affect the location of holographic surfaces.

2 Reissner-Nordström-de Sitter black holes

2.1 General Lorentzian solution

The four-dimensional Lorentzian Einstein-Hilbert action with a cosmological constant, Λ, and a Maxwell field, \( F_{\mu\nu} \), is given by:

\[
S = \frac{1}{16\pi} \int d^4x \left( -g^{IV} \right)^{1/2} \left[ R^{IV} - 2\Lambda - F_{\mu\nu}F^{\mu\nu} - \frac{1}{2} \sum_{i=1}^{N} (\nabla^{IV} f_i)^2 \right],
\]  

(2.1)

where \( R^{IV} \) and \( g^{IV} \) are the four-dimensional Ricci scalar and metric determinant. The scalar fields \( f_i \) will be needed later to carry the quantum radiation; classically, they can be set to zero.

The charged, static, spherically symmetric solutions of the vacuum Einstein equations with a cosmological constant \( \Lambda \) are given by the Reissner-Nordström-de Sitter metric

\[
ds^2 = -V(r)dt^2 + V(r)^{-1}dr^2 + r^2 d\Omega^2, 
\]

(2.2)

where

\[
V(r) = 1 - \frac{2\mu}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3} r^2; 
\]

(2.3)

d\( \Omega^2 \) is the metric on a unit two-sphere, \( Q \) is the charge, and \( \mu \) is a mass parameter. The black holes can be either magnetically charged,

\[
F = Q \sin \theta \ d\theta \wedge d\phi 
\]

(whence \( F^2 = 2Q^2/r^4 \)), or electrically charged,

\[
F = \frac{Q}{r^2} dt \wedge dr 
\]

(whence \( F^2 = -2Q^2/r^4 \)). Electrically charged black holes can lose their charge by emitting electrically charged particles. But the aim here is to
Figure 7: Penrose diagram of a generic Reissner-Nordström-de Sitter solution. Every point represents a two-sphere of radius $r$. Thus the spatial topology is $S^1 \times S^2$. The spacetime contains an asymptotically de Sitter region (dark grey, IV), bounded by the cosmological horizon at $r_3$. The black hole interior (light grey) is bounded by an ‘outer’ horizon at $r_2$. Back-reaction truncates the diagram at the ‘inner’ horizon, $r_1$. The fully extended classical diagram is indicated by thin lines.

stabilize the black holes through their charge. Therefore only magnetically charged black holes will be considered below.

Within appropriate ranges for the black hole mass and charge, $V$ has three positive roots, $r_1 < r_2 < r_3$, which may be interpreted as the inner and outer black hole horizons and the cosmological horizon. Figure 7 shows the causal structure of a generic Reissner-Nordström-de Sitter solution. In the fully extended classical solution the black hole is traversable. It contains a naked singularity at $r = 0$ in region III, beyond the inner horizon ($r_1$). Consider an observer in region I, between the black hole and the cosmological
horizon, and located where the cosmological acceleration and the black hole attraction balance exactly. This observer exists for an infinite proper time, during which he always sees radiation coming from the cosmological horizon and falling into the black hole. (At late times, he also sees an equal amount of radiation coming out of the black hole, which explains why the back-reaction remains small in region I.) An observer in region II, however, will see only the radiation emitted by the cosmological horizon, and it will take him only a finite proper time to cross this infinite amount of radiation. This means that back-reaction to the quantum radiation effectively places a null singularity on the inner horizon, and truncates the spacetime there.

Figure 8: Reissner-Nordström-de Sitter solutions exist for all points on or within the thick line. The plot is of the dimensionless quantities $\mu^2 \Lambda$ vs. $Q^2 \Lambda$. The lukewarm solutions lie on the dashed line, $\mu^2 = Q^2$. They are stable endpoints of black hole evaporation. They meet the Charged Nariai solutions at the point $C$, corresponding to the critical charge $Q_C$. Maximal black holes with higher charge are stable.

The Reissner-Nordström-de Sitter solution space is shown in Fig. 8. It is a roughly triangular region in the charge-mass plane. The square of the charge is obviously bounded from below by zero; this boundary corresponds to the neutral, Schwarzschild-de Sitter solutions. From above, it is bounded by the extremal (or “cold”) Reissner-Nordström-de Sitter solutions. These
solutions also form a lower bound on the mass at any given charge. As the mass is increased at fixed charge, the black hole radius, \( r_2 \), grows, until the black hole is as large as the cosmological horizon that “surrounds” it. This bound corresponds to the Charged Nariai solutions. The usual Reissner-Nordström-de Sitter metric, Eq. (2.2), breaks down in this limit. As this paper is concerned mainly with the peculiar stability properties of the Charged Nariai solutions, their metric will be given explicitly below.

### 2.2 Charged Nariai solution

The Charged Nariai solutions are the black holes of maximal mass for a given charge. Geometrically, they are obtained in the limit as the outer black hole horizon radius approaches the radius of the cosmological horizon. Therefore \( r \) will no longer be a suitable coordinate to describe the region between the two horizons. Instead, one may set \( r_2 = r_0 - \epsilon, r_3 = r_0 + \epsilon. \) With the transformation

\[
    r = r_0 + \epsilon \cos \chi, \quad t = \frac{1}{V(r_0)} \epsilon \psi, \quad (2.6)
\]

the Reissner-Nordström-de Sitter metric, Eq. (2.2), becomes the Charged Nariai metric,

\[
    ds^2 = \frac{1}{A} \left( -\sin^2 \chi d\psi^2 + d\chi^2 \right) + \frac{1}{B} d\Omega_2^2, \quad (2.7)
\]

in the limit \( \epsilon \to 0. \) (This limiting procedure is discussed in more detail, for the neutral case, in the appendix of Ref. [11].) Here \( A \) and \( B \) are given by

\[
    A = \lim_{\epsilon \to 0} \frac{V(r_0)}{\epsilon^2}, \quad B = \lim_{\epsilon \to 0} \frac{1}{r_0^2}. \quad (2.8)
\]

The Maxwell field is still given by Eq. (2.4), and therefore \( F^2 = 2B^2Q^2 \) in the magnetic case. The parameters \( A \) and \( B \) are given by

\[
    A = \frac{1}{2Q^2} \left( 1 - \sqrt{1 - 4\Lambda Q^2} \right), \quad B = 2\Lambda - A. \quad (2.9)
\]

Therefore \( A < B \) except in the neutral case, when \( A = B. \) The geometry can be visualized as a direct product of 1+1-dimensional de Sitter space with a two-sphere.

Note that \( 0 \leq Q^2 < (4\Lambda)^{-1} \equiv Q_{\text{max}}^2. \) At the upper bound the inner and outer black hole horizons would coincide, but the metric will be different
from Eq. (2.9) \cite{21,22}. Thus all Charged Nariai solutions are subextremal; they do, however, closely approach extremality as $Q^2 \to Q_{\text{max}}^2$. Their causal structure is shown in Fig. 9.

![Penrose diagram for a Charged Nariai solution.](image)

**Figure 9:** Penrose diagram for a Charged Nariai solution. It is identical to the diagram for 1+1-dimensional de Sitter space, except that every point represents a two-sphere of equal radius, $B^{-1/2}$. There is no black hole. Instead, any observer will see two cosmological horizons: one in each direction on the $S^1$. They are shown here for observers at $\pi/2$ and $3\pi/2$. The solution is classically unstable to perturbations of the two-sphere size. This causes two-spheres to collapse into black hole interiors or grow into de Sitter regions. If $n$ black holes are formed, the upper part of the diagram will be a sequence of $n$ Reissner-Nordström-de Sitter diagrams (Fig. 7). This is shown in Fig. 3.

For the decay of Charged Nariai black holes to have an effect on the global structure of de Sitter space, they must first be produced. This occurs through gravitational tunneling and can be described using instantons, or Euclidean solutions of the Einstein-Maxwell equations \cite{8,9,11,21,23}. Some issues specific to compact, disconnected instantons are discussed in Refs. \cite{23} and \cite{24}. For general values of $Q$ and $\mu$, the Reissner-Nordström-de Sitter metric, Eq. (2.2), has no regular Euclidean section. Unless special boundary conditions are selected \cite{20}, such black holes cannot nucleate semiclassically on a de Sitter background. A smooth instanton does exist, however, for the Charged Nariai solutions (as well as for the cold, and the ‘lukewarm’ solutions, which will be discussed later) \cite{11,21,22,23,20}.

The Euclidean Charged Nariai solution can be obtained straightforwardly
by Wick-rotation: With $\xi = i\psi$, Eq. (2.7) describes a Euclidean metric corresponding to the direct product of two round two-spheres of radii $A^{-1/2}$ and $B^{-1/2}$. This solution has a Euclidean action of $-2\pi/B$. In order to obtain the rate at which Charged Nariai black holes are produced in de Sitter space, this action must be normalized by subtracting off the de Sitter action \[8, 9, 23, 27\].

The Euclidean de Sitter solution is given by Eq. (2.2), with $\mu = Q = 0$ and $\tau = it$. This yields a Euclidean four-sphere of radius $(\Lambda/3)^{-1/2}$. Its action is $-3\pi/\Lambda$; therefore the Charged Nariai nucleation rate, neglecting a prefactor, is given by

$$
\exp\left(-\frac{\pi}{B} \frac{1 + 2Q^2B}{1 - Q^2B}\right).
$$

This is less than one, and in fact ranges from $\exp(-\pi/\Lambda)$ to $\exp(-2\pi/\Lambda)$ as the charge increases from 0 to $Q^2_{\text{max}}$. Therefore, black hole production is suppressed, and more suppressed the higher the charge. The suppression is weak if the cosmological constant is close to the Planck value. If it is smaller, the suppression becomes huge. In an eternal de Sitter space, however, the process is bound to occur nevertheless.

### 3 Preliminary stability analysis

The aim of this section is to gain an understanding of the type of instabilities that may be expected in a Charged Nariai geometry. Because the spacelike sections are arbitrarily similar to the submaximal Reissner-Nordström-de Sitter solutions, the metric is classically unstable to the formation of black hole interiors and large de Sitter regions. The resulting nearly maximal Reissner-Nordström-de Sitter black holes are classically stable. However, thermodynamic considerations suggest that some (but not all) of them will decay by emitting Hawking radiation.

In Sec. 4 these expectations will be verified, and additional instabilities found, by considering a variety of metric perturbations in a model that explicitly includes quantum radiation and back-reaction.

#### 3.1 Classical instability

The limit leading to Eq. (2.7) has made the $S^1 \times S^2$ topology of the spacelike sections of Reissner-Nordström-de Sitter spaces manifest; $\psi$ is the coordinate
along the one-sphere. The Charged Nariai solutions are precisely those for which a slicing can be found in which the two-sphere radius, $r$, is independent of the $S^1$-coordinate, $x$. For other Reissner-Nordström-de Sitter black holes, the $S^2$ size varies around the $S^1$; one can find a slicing in which the maximal two-sphere corresponds to the cosmological horizon, and the minimal two-sphere to the outer black hole horizon. In this sense, it will be helpful to think of the spacelike section of a Charged Nariai solution as a ‘perfect doughnut,’ while it would be a ‘wobbly doughnut’ for a generic Reissner-Nordström-de Sitter solution.

In terms of its space-time geometry, the Charged Nariai solution is the direct product of 1+1-dimensional de Sitter space with a round two-sphere. There is no black hole interior; instead the black hole horizon has become a second cosmological horizon. Looking in either direction of the $S^1$, an observer will see a cosmological horizon. This is shown in the Penrose diagram, Fig. 9. However, the Charged Nariai solutions are classically unstable. Any region in which the two-spheres are slightly smaller will collapse to form the interior of a black hole. Larger two-spheres will grow exponentially to form asymptotically de Sitter regions.

A simple way to understand this instability is to remember that the Charged Nariai solutions form a set of measure zero in the Reissner-Nordström-de Sitter solution space; yet they have a completely different causal structure. For all $\epsilon$, except $\epsilon = 0$, the coordinates given in Eq. (2.6) describe a generic Reissner-Nordström-de Sitter solution with the causal structure of Fig. 7. Even if at a given moment a spacelike section has $\epsilon = 0$ exactly, a small $\epsilon$-perturbation would destroy the degeneracy and lead to Reissner-Nordström-de Sitter space.

This perturbation is really the only one allowed by the classical constraint equations. When quantum matter will be included in Sec. 4, however, more general perturbations with multiple minima and maxima will become possible (a ‘doughnut with many wobbles’). In Sec. 5 I will show that this leads to necklace configurations: Along the $S^1$, the Charged Nariai spacelike section will develop a number of asymptotically de Sitter regions separated by black hole interiors.

### 3.2 Thermodynamics

Gibbons and Hawking have shown that the cosmological horizon emits thermal radiation in de Sitter space \[28\]. Its temperature is given by the horizon’s
surface gravity, over $2\pi$, as it would be for black holes. Indeed, if a black hole is present, it will exchange radiation with the cosmological horizon. As a result, it will grow or shrink, depending on the net influx.

In the particular case of Charged Nariai solutions, the two horizons are of equal temperature, and will be in thermodynamic equilibrium. One would not expect this equilibrium to be stable. As discussed above, a small perturbation will turn the Charged Nariai solution into a submaximal Reissner-Nordström-de Sitter geometry. For the uncharged case, the following argument was given in Ref. [29]: In the perturbed geometry, the black hole will be slightly smaller than the cosmological horizon. Correspondingly, it will be hotter, and it will lose more radiation than it absorbs. Thus it will continue to shrink at an ever-increasing rate.

One would expect the Charged Nariai solutions to contain a similar quantum instability. There are some important differences, however. For example, the black hole mass cannot become smaller than the extremal value. Below, the thermodynamic argument is re-examined for the charged case. It will turn out that for sufficient charge, black holes actually anti-evaporate, asymptotically approaching the Charged Nariai size.

In flat space, a charged black hole does indeed evaporate until it becomes extremal. In the early stages of evaporation, the charge will be negligible, and the temperature will increase as the black hole loses mass. In the extremal limit, however, the temperature approaches zero and evaporation ceases. Therefore there is a turnaround point: As the black hole evaporates, the temperature first increases, then decreases to zero.

The difference in de Sitter space is that the black hole cannot become arbitrarily cold. As it approaches the extremal limit, its temperature eventually drops to that of the cosmological horizon. At this level the black hole will be stabilized by the Gibbons-Hawking radiation bath it is immersed in. Indeed there are two families within the Reissner-Nordström-de Sitter solution space for which the black hole is in thermodynamic equilibrium with the cosmological horizon. The first is the unstable Charged Nariai family, for which the black holes are large. The second is the family of ‘lukewarm’ solutions [25], which are characterized by the condition $Q^2 = \mu^2$ in the metric, Eq. (2.2). As shown in Fig. 8, the lukewarm solutions meet the Charged Nariai solutions at the point $C$, corresponding to a critical charge $Q^2_C = 3/(16\Lambda) = 3Q^2_{\text{max}}/4$.

First consider black holes with subcritical charge ($Q^2 < Q^2_C$). If $Q^2 < \mu^2$, the black hole is hotter than the cosmological horizon, and will evaporate until it becomes a lukewarm solution. This includes the case of a perturbed
Charged Nariai solution. Subcritical black holes with $Q^2 > \mu^2$ are colder than the lukewarm solution and will absorb radiation from the cosmological horizon until they reach the lukewarm level. Thus the lukewarm black holes form the stable endpoints of the evolution of subcritically charged black holes.

Now consider the black holes of supercritical charge ($Q^2 > Q^2_C$). There are no equilibrium solutions between the Charged Nariai solutions, in which the black hole is as hot as the cosmological horizon, and the cold solutions, in which the black hole temperature (but not the cosmological temperature) vanishes. Therefore the black hole is colder than the cosmological horizon for all but the Charged Nariai solutions. It will absorb cosmological radiation and grow, asymptotically approaching the Charged Nariai limit.

In summary, all Charged Nariai solutions are expected to be classically unstable to the formation of a black hole interior. If $Q^2 < Q^2_C$, the black hole is expected to be thermodynamically unstable; for larger charges, it should be stable.

4 Including radiation

4.1 Model construction

The production of a Charged Nariai geometry, described in the previous section, requires only semiclassical techniques. The process can be thought of as a non-perturbative fluctuation of the gravitational field, mediated by instantons. To continue with the description of the fragmentation process, however, it is necessary to work in a model that includes the quantum radiation exchanged between the black hole and cosmological horizons.

Black holes in asymptotically flat space have been shown to radiate \cite{30}; they will lose mass, shrink, and unless they are stabilized by charge, they will eventually disappear. For black holes in de Sitter space, the consequences of this radiation are less obvious, since the cosmological horizon also radiates \cite{28}. For the Nariai solution the two horizons are initially of equal size and temperature, so the black hole suffers no net loss of energy. In Ref. \cite{29} we investigated the stability of this equilibrium, considering only perturbations corresponding to the formation of a single black hole. For this purpose, a spherically symmetric model was introduced that included the one-loop effective action of $N$ scalar fields. The same model was employed in Ref. \cite{10} to demonstrate the proliferation effect arising from higher mode perturba-
tions of the Nariai solution. Here it will be used to investigate whether the Charged Nariai solutions contain a similar instability. (The stability of the neutral Nariai solution was investigated using more elaborate models in Refs. [31, 32].)

Restricting to spherically symmetric fields and quantum fluctuations, the metric may be written as

\[ ds^2 = e^{2\phi} \left( -dt^2 + dx^2 \right) + e^{-2\phi} d\Omega^2, \]  

where \( x \) is the coordinate on the \( S^1 \), with period \( 2\pi \). Using this ansatz, and the on-shell condition for magnetic fields,

\[ F_{\mu\nu} F^{\mu\nu} = 2Q^2 e^{4\phi}, \]  

the angular coordinates and the Maxwell field can be integrated out in Eq. (2.1), which reduces the action to

\[ S = \frac{1}{16\pi} \int d^2x (-g)^{1/2} e^{-2\phi} \left[ R + 2(\nabla\phi)^2 + 2e^{2\phi} - 2\Lambda - 2Q^2 e^{4\phi} - \frac{1}{2} \sum_{i=1}^{N} (\nabla f_i)^2 \right], \]  

The scalars couple to the dilaton in the two-dimensional action. Thus, to include the back-reaction from quantum radiation, one should find the classical solutions to the action \( S + W^* \), where \( W^* \) is the scale-dependent part of the one-loop effective action for dilaton coupled scalars [33–41]:

\[ W^* = -\frac{1}{48\pi} \int d^2x (-g)^{1/2} \left[ \frac{1}{2} R R - 6(\nabla\phi)^2 - w\phi R \right]. \]  

In the large \( N \) limit, the contribution from the quantum fluctuations of the scalars dominates over that from the metric fluctuations. In order for quantum corrections to be small, one should take \( N\Lambda \ll 1 \). For small perturbations of the charged Nariai solutions, the \((\nabla\phi)^2\) term may be neglected [29]. The coefficient \( w \) will not be specified here. It will later drop out of the calculation; thus, the \( \phi R \) term will not affect any results at the present level of approximation.

One can obtain a local form of this action by introducing an independent scalar field \( Z \) which mimics the trace anomaly. With the classical solution \( f_i = 0 \) the \( N \) scalars can be integrated out. Thus the action of the one-loop
model will be given by:

\[ S = \frac{1}{16\pi} \int d^3x (-g)^{1/2} \left[ \left( e^{-2\phi} + \frac{N}{3}(Z + w\phi) \right) R - \frac{N}{6}(\nabla Z)^2 + 2 + 2e^{-2\phi} (\nabla \phi)^2 - 2e^{-2\phi} \Lambda - 2Q^2 e^{2\phi} \right]. \] (4.5)

4.2 Equations of motion

Differentiation with respect to \( t \) (\( x \)) will be denoted by an overdot (a prime). For any functions \( f \) and \( g \), define:

\[ \partial f / \partial g \equiv -\dot{f} \dot{g} + f' g', \quad \partial^2 g \equiv -\ddot{g} + g''. \] (4.6)

\[ \delta f / \delta g \equiv \dot{f} \dot{g} + f' g', \quad \delta^2 g \equiv \ddot{g} + g''. \] (4.7)

Variation with respect to \( \rho \), \( \phi \) and \( Z \) yields the following equations of motion:

\[- \left( 1 - wN \frac{6}{e^{2\phi}} \right) \partial^2 \phi + 2(\partial \phi)^2 + \frac{N}{6}e^{2\phi} \partial^2 Z + e^{2\phi} (\Lambda e^{-2\phi} + Q^2 e^{2\phi} - 1) = 0; \] (4.8)

\[- \left( 1 - wN \frac{6}{e^{2\phi}} \right) \partial^2 \phi^2 + (\partial \phi)^2 + \Lambda e^{2\phi} - Q^2 e^{2\phi} + 4\phi = 0; \] (4.9)

\[ \partial^2 Z - 2\partial^2 \rho = 0. \] (4.10)

The constraint equations are:

\[- \left( 1 - wN \frac{6}{e^{2\phi}} \right) \left( \delta^2 \phi - 2\delta \phi \delta \rho \right) - (\delta \phi)^2 = \frac{N}{12}e^{2\phi} \left[ (\delta Z)^2 + 2\delta^2 Z - 4\delta Z \delta \rho \right]; \] (4.11)

\[- \left( 1 - wN \frac{6}{e^{2\phi}} \right) \left( \dot{\phi}' - \dot{\rho} \phi' - \rho' \phi \right) - \dot{\phi} \phi' = \frac{N}{12}e^{2\phi} \left[ \dot{Z} Z' + 2\dot{Z}' - 2(\dot{\rho} Z' + \rho' \dot{Z}) \right]. \] (4.12)

From Eq. (4.10), it follows that \( Z = 2\rho + \eta \), where \( \eta \) satisfies \( \partial^2 \eta = 0 \). The remaining freedom in \( \eta \) can be used to satisfy the constraint equations for any choice of \( \rho, \dot{\rho}, \phi \) and \( \dot{\phi} \) on an initial spacelike section \[29\].

5 Linear perturbations and back-reaction
5.1 Quantum Charged Nariai solution

Using the model established in the previous section, the stability of the Charged Nariai solutions can now be investigated. The analysis will follow the procedure set forth (for neutral black holes) in Refs. [10, 29], where more details can be found. Here the emphasis will be on the effects of charge.

The Charged Nariai solution, Eq. (2.7), may be rewritten in the form of Eq. (4.1). This yields

\[ e^{2\rho} = \frac{1}{A \cos^2 \phi}, \quad e^{2\phi} = B, \quad (5.1) \]

with \( A \) and \( B \) given by Eq. (2.9).

The next step is to find the quantum corrections to the unperturbed solution. They take the form of small corrections to \( A \) and \( B \), and can be obtained by substituting Eq. (5.1) into the one-loop equations of motion, Eqs. (4.8) and (4.9). This yields the following cubic equation for \( B \):

\[ B \left[ \left( 1 - \frac{wN}{6} B \right) - Q^2 B \left( 1 - \frac{(w - 2)N}{6} B \right) \right] = \left[ 1 - \frac{(w + 2)N}{6} B \right] \Lambda, \quad (5.2) \]

while \( A \) is given by

\[ A = \frac{\Lambda - Q^2 B^2}{1 - wNB/6}. \quad (5.3) \]

Expanding in the small parameter \( N\Lambda \), one obtains to first order:

\[ A = A_0 \left[ 1 + \frac{NB_0}{3} \left( \frac{2Q^2 B_0}{1 - 2Q^2 B_0} + \frac{w}{2} \right) \right], \quad (5.4) \]

\[ B = B_0 \left( 1 - \frac{NB_0}{3} \right), \quad (5.5) \]

where \( A_0 \) and \( B_0 \) refer to the classical solution, Eq. (2.9). Note that in the near-extremal limit, one has \( 2Q^2 B_0 \to 1 \), and the expansion in \( N\Lambda \) breaks down.

5.2 Metric perturbation

Quantum fluctuations will destroy the degeneracy of the Charged Nariai solution. This means that the two-sphere radius, \( e^{-\phi} \), will no longer be constant. Instead, it will be a function of the one-sphere coordinate, \( x \),
and time, $t$. It can be decomposed into Fourier modes with time-dependent coefficients. It simplifies the analysis considerably to assume that one mode dominates over all others; in this case the black holes will be distributed evenly around the one-sphere. The probability for the $n$-th mode to dominate will be of the order of $\exp(-n^2)$. The corresponding perturbation may be written as

$$e^{2\phi} = B \left[ 1 + 2\epsilon \sigma_n(t) \cos nx \right], \quad (5.6)$$

This simple ansatz is sufficient to illustrate various evolutionary possibilities for black hole-de Sitter solutions which ultimately affect the global structure of de Sitter space, including the formation of one or several charged or uncharged black holes on a necklace, their evaporation or anti-evaporation, and their fragmentation. Nevertheless, it has limitations. Most importantly, the perturbation can be considered linear only as long as the two-sphere size does not differ significantly from the Charged Nariai value anywhere on the $S^1$. Yet the perturbations soon become non-linear in the black hole interiors, where the two-spheres collapse to zero size, and in the de Sitter regions, where they expand indefinitely. Even considering only the regions between the black hole and cosmological horizon, the linear approximation eventually breaks down, if the black hole evaporates. (This is only a problem if its charge is small; otherwise a large lukewarm endpoint will preclude further evaporation, or, for supercritical charge, evaporation never begins.) It is fair to assume, of course, that a very small black hole will hardly notice the cosmological horizon and can be treated like a Schwarzschild black hole. Numerical work is currently underway to verify the smooth transition between the perturbative and non-perturbative regimes [42]. We will also investigate more general perturbations and examine the conditions for domination of certain modes.

I will not consider the constant mode perturbation ($n = 0$) in detail, but most of the analysis for general $n$ will apply to it. If the two-spheres are all larger than the Charged Nariai value, they will expand exponentially everywhere. The resulting spacetime will be locally de Sitter, with no black holes and a connected future infinity. Globally its topology will be $S^1 \times S^2$. If the two-spheres are smaller, they will collapse to a singularity everywhere, unless they are stabilized by quantum radiation at a small radius. It is not clear whether this should be interpreted as the collapse of the entire de Sitter space; related questions are discussed in Sec. 6.

In order to obtain an equation of motion for the metric perturbation, $\sigma_n$,
one may eliminate $\partial^2 \rho$ and $\partial^2 Z$ from Eq. (4.8). This yields

$$
- \left[ 1 - (w + 1) \frac{N}{3} e^{2\phi} + \left( \frac{wN}{6} e^{2\phi} \right)^2 \right] \partial^2 \phi + \left[ 2 - (w + 1) \frac{N}{3} e^{2\phi} \right] (\partial \phi)^2 =
$$

$$
= e^{2\rho} \left\{ \left( 1 - \frac{wN}{6} e^{2\phi} \right) e^{2\phi} - \left[ 1 - (w - 2) \frac{N}{6} e^{2\phi} \right] Q^2 e^{4\phi} - \left[ 1 - (w + 2) \frac{N}{6} e^{2\phi} \right] \Lambda \right\}. 
$$

(5.7)

Note that the left- and right-hand sides of this equation vanish separately in the unperturbed case ($\epsilon = 0$). Therefore, perturbations of $\rho$ will drop out at first order in $\epsilon$ and will not be considered here. Insertion of the perturbation ansatz, Eq. (5.6), yields

$$
\ddot{\sigma}_n = \frac{a}{\cos^2 t} - n^2,
$$

(5.8)

where

$$
a = \frac{2 \left[ 6 - 2wNB - 12Q^2B + 3(w - 2)NQ^2B^2 + (w + 2)NA \right] (6 - wNB)B}{[36 - 12(w + 1)NB + w^2 B^2](\Lambda - Q^2B^2)}. 
$$

(5.9)

The value of $a$ is crucial in determining whether a black hole grows or shrinks; in the neutral case it can be expressed in terms of $\Lambda$, $N$, and $w$ only [29]. In order to evaluate $a$ for $Q^2 > 0$, however, one would need to solve the cubic equation (5.2) for $B$. It seems preferable to evaluate $a$ only to first order in $NA$. This gives a surprisingly simple result:

$$
a = 2 + \left( \frac{2NB_0}{3} \right) \left( \frac{1 - 4Q^2B_0}{1 - 2Q^2B_0} \right). 
$$

(5.10)

As advertised, this expression is independent of $w$. The first factor will be small as long as $NA \ll 1$, which must be assumed anyway for the one-loop effective action to make sense. The second factor will be of the order of 1, except in the near-extremal limit, when $2Q^2B_0 \approx 1$. There the expansion in $NA$ cannot be trusted, and should be supplemented by a numerical calculation of $a$. Fig. [11] shows a plot of $a$ as a function of the black hole charge. Note that $4Q^2B_0 < 1$ for $Q < Q_c^2 = 3/(16\Lambda)$, and $4Q^2B_0 > 1$ for $Q > Q_c^2$. Therefore, $a$ will be greater (less) than 2 for subcritical (supercritical) black hole charge.
Figure 10: A typical plot for the parameter $a$ as a function of $Q^2\Lambda$. $a > 2$ for small charges. $a$ passes through 2 at the critical charge, near $Q^2\Lambda = 3/16$. Black holes with larger charge will anti-evaporate. $a$ becomes zero exactly in the limit where the Charged Nariai solutions become extremal, near $Q^2\Lambda = 1/4$.

5.3 Horizon perturbation

The metric perturbations considered above lead to the formation of black hole and cosmological horizons. The condition for an apparent horizon is $(\nabla \phi)^2 = 0$. Eq. (5.6) yields

$$\frac{\partial \phi}{\partial t} = \epsilon \dot{\sigma}_n \cos nx, \quad \frac{\partial \phi}{\partial x} = -\epsilon \sigma_n n \sin nx.$$  (5.11)

Therefore, there will be $2n$ black hole horizons, and $2n$ cosmological horizons, located at

$$x^{(k)}_b(t) = \frac{1}{n} \left( 2\pi k + \arctan \left| \frac{\dot{\sigma}_n}{n\sigma_n} \right| \right),$$  (5.12)
\[ x^{(n+k)}_b(t) = \frac{1}{n} \left( -2\pi k + \arctan \frac{\dot{\sigma}_n}{n\sigma_n} \right), \quad (5.13) \]
\[ x^{(l)}_c(t) = x^{(l)}_b(t) + \frac{\pi}{n}, \quad (5.14) \]

where \( k = 0 \ldots n - 1 \) and \( l = 0 \ldots 2n - 1 \).

By inserting these values of \( x \) back into the metric perturbation, Eq. (5.6), one finds the size of the black hole horizons:

\[ r_b(t)^{-2} = e^{2\phi(t,x^{(l)}_b(t))} = B [1 + 2\epsilon \delta(t)], \quad (5.15) \]

where

\[ \delta \equiv \sigma_n \cos nx^{(l)}_b = \sigma_n \left( 1 + \frac{\dot{\sigma}_n^2}{n^2\sigma_n^2} \right)^{-1/2}. \quad (5.16) \]

will be called the horizon perturbation.

If \( \delta \) grows, the black hole is shrinking; this corresponds to evaporation. To see whether this happens, one needs to find solutions for \( \sigma_n \) for given initial conditions, and plug them into Eq. (5.16).

### 5.4 Early time evolution

Exact solutions to Eq. (5.8) have not been found for general values of \( a \). Various approximations can be used, however, to investigate the initial behavior of the black holes, and to determine whether they ultimately evaporate, or grow.

The possible initial conditions for \( \sigma_n \) can be parametrized by writing:

\[ \sigma_n(0) = \alpha \sin \vartheta, \quad \dot{\sigma}_n(0) = \alpha \cos \vartheta. \quad (5.17) \]

To obtain the early-time behavior of the horizons, one may solve the equation of motion, Eq. (5.8), for a power series in \( t \), and use the result in the equation for the horizon perturbation, Eq. (5.16). Writing down the general expressions for its coefficients would not be very illuminating. It suffices to say that the coefficients \( \delta_i \) can be positive or negative, depending on \( a \), \( \vartheta \), and \( n \), and not all of them will have the same sign generically.

This makes the early-time behavior quite complicated. The main point, however, is that black holes with \( Q < Q_C \), which are thermodynamically unstable, will not necessarily evaporate initially. And supercritically charged black holes, which ought to be stable according to Sec. 3.2, may initially
become smaller. This can be demonstrated by choosing $\theta = \pi/2$ and $n = 1$. (Note that this choice corresponds to vanishing time-dependence. It thus gives the initial conditions most similar to a slightly submaximal Reissner-Nordström-de Sitter black hole.)

For $\theta = \pi/2$, the linear coefficient, $\delta_1$, vanishes. The early time evolution is determined by the quadratic coefficient, and is given by

$$\delta(t) = \alpha \left[1 - \frac{1}{2}(a - 1)(a - 2)t^2\right].$$

(5.18)

By Eq. (5.10), $a > 2$ for black holes of subcritical charge. Thus the horizon perturbation begins to decrease. Therefore, contrary to thermodynamic expectations, such black holes will initially grow, or 'anti-evaporate'. On the other hand, if the charge is supercritical (but sufficiently non-extremal), $a$ will lie between 1 and 2, and the horizon perturbations begin to increase. These black holes evaporate initially.

This behavior may be counterintuitive, but it is not obviously absurd. In de Sitter space, the quantum radiation is distributed on a compact spatial manifold. Through the constraint equations, (4.11) and (4.12), the initial metric perturbation completely determines the distribution of the quantum radiation around the $S^1$. This distribution may correspond to energy heading towards the black hole even when it is thermodynamically hotter, and vice versa. This does not violate the Second Law. It means only that the radiation field is initially out of equilibrium with the horizons.

But what happens at later times? Does the radiation field relax and the thermodynamic evolution prevail? To answer this question fully a numerical study is called for, work on which is underway [42]. Analytically, two things can be done. First, I will give asymptotic solutions to the linear perturbation equation at late times. Second, I will investigate the particular initial conditions selected whenever the Nariai geometry emerges from a semiclassical transition.

5.5 Late time evolution

It will be useful to rescale the time variable in Eq. (2.7) as $\cosh v = 1/\cos t$. Then the Nariai metric takes the form

$$ds^2 = \frac{1}{A} \left(-dv^2 + \cosh^2 v \, dx^2\right) + \frac{1}{B} d\Omega^2.$$  

(5.19)
In the new time coordinate, the metric perturbation equation (5.8) becomes
\[ \frac{d^2 \sigma_n}{dv^2} + \tanh v \frac{d \sigma_n}{dv} + \left( \frac{n^2}{\cosh^2 v} - a \right) \sigma_n = 0. \quad (5.20) \]

At late times \((v \gg 1)\), one can take \(\tanh v \approx 1\) and neglect the \(n^2\) term. Asymptotic solutions are therefore given by
\[ \sigma_n(v) = \alpha_+ \exp(c_+ v) + \alpha_- \exp(c_- v), \quad (5.21) \]
where \(\alpha_{\pm}\) are arbitrary constants and \(c_{\pm}\) are the two solutions to \(c(c+1) = a\). Unless \(\alpha_+\) is exactly 0, which would be unphysical fine-tuning, the mode with the larger exponent, \(c_+\), will eventually dominate no matter how small its initial excitation.

Neglecting the weaker mode, the horizon perturbation can be found from Eq. (5.16):
\[ \delta = 2\alpha_n c_n^{-1} \exp [(c_+ - 1)v]. \quad (5.22) \]
The exponent is positive (negative) for \(c_+ > 1\) (\(c_+ < 1\)). Thus black holes will evaporate if \(a > 2\), and they will anti-evaporate if \(a < 2\). But by Eq. (5.10), \(a\) goes through 2 precisely when the charge becomes critical. This means that Charged Nariai black holes will evolve according to thermodynamic expectations at late times, even if they initially fail to do so. The conclusion holds for generic \(\vartheta\) and all \(n\). Therefore, the anti-evaporation effect found for neutral black holes in Ref. [29] is transitory, and eventually gives way to evaporation. Similarly, the evaporation found for supercritically charged black holes will give way to growth.

Equation (5.10) and the full expression for \(a\), Eq. (5.9), arise from complicated combinations of one-loop terms in the equations of motion. Yet they give perfect agreement with the thermodynamic analysis, and even predict the same critical charge beyond which evaporation is replaced by black hole growth. This is a non-trivial check. It shows that this two-dimensional effective model is capable of accurately reproducing the thermodynamic properties of the four-dimensional solutions. In addition, of course, it also incorporates back-reaction effects.

\(^1\)Before this argument was found, the black hole ‘turnaround’ was first demonstrated numerically by Jens Niemeyer [42].
5.6 Euclidean boundary conditions

So far, I have considered general linear perturbations of the Charged Nariai geometry. When this solution nucleates by semiclassical tunneling, however, it will emerge from a compact Euclidean solution. Any perturbation has to be regular in the entire Euclidean region. Otherwise the geometry would not be a solution. It would fail to dominate the path integral, and would be semiclassically forbidden.

The regularity condition effectively selects a particular value of the phase, \( \vartheta \), on the \( t = 0 \) surface. The perturbation amplitude can still be chosen freely, but the relative strength of \( \sigma_n \) and \( \dot{\sigma}_n \) is fixed. In Ref. [10], an approximate solution to Eq. (5.8) was found which is everywhere regular and extends far beyond the nucleation surface, so that the late-time evolution of the black hole can be seen. The solution depends only on \( a \) and \( n \), and was found for general \( a \), so that only a brief review will be given here.

Wick-rotation, \( u = \frac{\pi}{2} + iv \), yields the Euclidean version of the Charged Nariai metric, Eq. (5.19):

\[
ds^2 = \frac{1}{A} \left( du^2 + \sin^2 u \, dx^2 \right) + \frac{1}{B} d\Omega^2.
\]

This \( S^2 \times S^2 \) instanton describes the spontaneous nucleation of a degenerate handle in de Sitter space. A suitable nucleation path runs from the South pole of the first two-sphere, at \( u = 0 \), to \( u = \pi/2 \), and then parallel to the imaginary time axis \( (u = \pi/2 + iv) \) from \( v = 0 \) to \( v = \infty \). Geometrically, this corresponds to cutting the first two-sphere in half, and joining to it a Lorentzian 1+1-dimensional de Sitter hyperboloid.

On the South pole, the \( S^1 \) factor of the spacelike sections degenerates into a point. Therefore, the perturbation may not depend on the angular variable \( x \) there. This means that the condition \( \sigma_n(u = 0) = 0 \) must be imposed when Eq. (5.8) is solved. For any \( n \geq 2 \), the family of solutions, parametrized by a real prefactor \( \alpha \), is given by [10]

\[
\sigma_n(u) = 2\alpha e^{i(c^+ - n)\pi/2} \left( \tan \frac{u}{2} \right)^n (n + \cos c u);
\]

for \( n = 1 \), it is [29]

\[
\sigma_1(u) = 2\alpha e^{i(c^+ - 1)\pi/2} \sin c u.
\]

The choice of phase ensures that \( \sigma_n \) will be real at late Lorentzian times, when measurements are made. As before, \( c^+ \) is the larger root of \( c(c + 1) = a \).
The solutions are exact only for $c_+ = 1$ (which corresponds to no quantum matter) but are a good approximation for the whole range of $c_+$ attained by charged black holes.

This solution is particularly useful because it describes the entire evolution, from the creation of the Charged Nariai geometry to the evaporation of the black holes, in a single expression. For small Lorentzian times, $0 \leq v \leq \text{arsinh } n$, the metric perturbation is contained within a single Hubble volume, and oscillates. When the $S^1$ size has expanded by a factor of $n$, at $v \approx \text{arsinh } n$, the perturbations leave the horizon and freeze out. The $n$ maxima run away to form asymptotically de Sitter regions, while the $n$ minima collapse to form black hole interiors. Thus a necklace configuration develops. For larger $v$, the solution asymptotes to

$$\sigma_n(v) = \alpha \exp(c_+ v).$$

By Eq. (5.21), this agrees with the late time attractor found above, as it should.

A discussion of the corresponding evolution of the horizon perturbation has already been given after Eq. (5.22). Subcritically charged Nariai solutions become a necklace with $n$ beads, as expected. A finite number of de Sitter universes results (see Figs. 4 and 5). Supercritically charged Nariai solutions also decay into a necklace, but the black holes do not become small. Instead they anti-evaporate and approach the Charged Nariai limit. There will be $2n$ regions in which the two-sphere size is nearly constant across a Hubble length on the $S^1$, namely each region between a black hole horizon and its neighboring cosmological horizon. Classically, the geometry is locally Nariai in such regions. Higher-mode fluctuations continue to push the two-sphere size over and under the degenerate value locally. As these perturbations freeze out, more black hole and cosmological regions will be seeded. Ultimately, this effect leads to the formation of an infinite number of de Sitter beads along the same $S^1$ (see Figs. 4 and 5).

6 Discussion: Global vs. local perspective

In Ref. [10] I showed that de Sitter space is unstable to the proliferation into disconnected daughter universes. This fragmentation is mediated by spontaneously nucleated neutral black holes. In the present paper, I have investigated the consequences of charged black hole creation in de Sitter
space, and found that it leads to the formation of a ‘necklace’ of de Sitter universes. Unlike the neutral case, the necklace does not fragment. Instead, the de Sitter universes remain connected by charged black hole throats. The most significant difference to the neutral case is an iterative effect occurring if the black holes contain more than a certain critical charge. In this case, a single nucleation event results in the formation of an unbounded number of daughter universes.

Sec. 1.3 contains a more detailed summary of the processes found in this paper. Here I will discuss some open questions.

During inflation, the universe was effectively in a de Sitter state. In most models, the proliferation effect occurs; this means that we live in one of a large or infinite number of universes that originated in the same inflationary region \[10\]. But unless the other universes lead to subtle, non-local quantum effects, we cannot verify their existence even in principle, because they will never intersect our causal past. One may ask, therefore, whether Occam’s razor should not be applied to the whole scenario. I believe this would be a mistake. Occam’s razor applies to theories, not to their solutions. A theory can be simple and successful in describing every single experiment we can perform, and yet it may predict many phenomena outside our range of observation. For quantum gravity, this is actually to be expected. One cannot consider a theory complete, and yet reject the picture it gives us for the global structure of spacetime. (Euclidean quantum gravity, of course, is neither complete nor consistent, but it may give a good semi-classical approximation to the full theory.) Otherwise it would also be wrong, for example, to speak of the ‘present universe,’ or think of it as a homogeneous constant curvature space; all we can observe, after all, is our past lightcone.

But what if genuine reasons are found for abandoning the global point of view in studying de Sitter space? Would this render the effects of fragmentation and proliferation irrelevant? A local observer cannot see all of the spacetime; half of it will be hidden behind an event horizon. If quantum effects are included and de Sitter space fragments, the daughter universes will be causally disconnected; a local observer will never know about more than one of them. She may notice an endpoint of black hole evaporation; but she has no way of telling whether the black hole throat really was a connection to a different de Sitter region, or whether it merely wrapped around to the other end of her own, as in the classical Reissner-Nordström-de Sitter solution. Is there any way, then, that a local observer can find out about the global structure? This question is one reason why the new process of infinite
bead production in the supercritically charged Nariai geometry is important. An observer in a region between a black hole and cosmological horizon will actually notice the constant production of new daughter universes. From her perspective, this corresponds to flipping the direction in which the black hole lies, i.e., in which the two-sphere radius decreases. So at least in the supercritical case, the fragmentation of future infinity can be inferred even by a local observer.

The transition from de Sitter to Charged Nariai is a topology change, $S^3 \rightarrow S^1 \times S^2$; it is like punching a hole through the three-sphere. Asymptotic de Sitter regions develop in the resulting spacetime. From a local point of view, they are indistinguishable from the de Sitter background on which the transition first happened, so one would expect more topological transitions to occur in the daughter universes. This argument is crucial for the production of an infinite number of disconnected universes in the neutral case (proliferation). But if one tries explicitly to specify surfaces onto which the de Sitter instanton can match, one finds that they necessarily cross the original nucleation surface, and are therefore globally distinct from a de Sitter background. The supercritical case of infinite bead production on the same $S^1$ circumvents this problem, which is another reason why I consider it significant.

Related questions arise for other instanton-mediated processes in de Sitter space, such as the nucleation of cosmic strings or the Hawking-Moss transition. From a global point of view, there should only ever be one such transition, occurring at a minimal three-sphere (the ‘waist’) of de Sitter space. This is because no minimal three-sphere background for additional transitions is ever available thereafter if the global picture is taken at face value. But from a local point of view there will be many Hubble volumes in which the space is asymptotically indistinguishable from the background. It would seem absurd that the process should not be allowed there. Similarly, in the Hawking-Moss case, it has been debated whether the whole space, or just one Hubble volume, tunnels into the new state. In pure de Sitter space such distinctions may just be meaningless. But for tunneling events in an inflationary universe the difference is significant. A post-inflationary observer will be able to see widely separated regions of the effective de Sitter space, and will be able to determine whether tunneling occured more than once.

As mentioned in the introduction, it has been suggested that past and

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2I am indebted to Ted Jacobson for discussions of this question.
future infinity form the holographic surfaces of de Sitter space [7]. This conjecture takes a global point of view. In Ref. [11] and the present paper I have shown that it would be inconsistent with quantum theory to assume future infinity to be connected. If future infinity is indeed a holographic surface, this result will have consequences for the implementation of the holographic principle in de Sitter space.

The conjecture was motivated by the compactness of space, and the lack of a null infinity, in fully extended de Sitter space. From a local point of view, however, one should not consider the whole of de Sitter space. The geometry is effectively bounded by a null surface, the observer’s event horizon (see Fig. 1). Could this horizon be a holographic surface? Then the fragmentation of future infinity would not cause any complications; a local observer can only reach a single point of future infinity anyway.

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