External field dependence of the correlation lengths in the three-dimensional \( O(4) \) model *

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We investigate numerically the transverse and longitudinal correlation lengths of the three-dimensional \( O(4) \) model as a function of the external field \( H \). In the low-temperature phase we verify explicitly the \( H^{-1/2} \)-dependence of the transverse correlation length, which is expected due to the Goldstone modes of the model. On the critical line we find the universal amplitude ratio \( \xi_T^*/\xi_L^* = 1.99(1) \). From our data we derive the universal scaling function for the transverse correlation length. The \( H \)-dependencies of the correlation lengths in the high temperature phase are discussed and shown to be in accord with the scaling functions.

1. INTRODUCTION

In \( O(N) \) spin models with \( N > 1 \) two types of correlation lengths appear, corresponding to the transverse and longitudinal spin components. Like for the magnetization the behaviour of these correlation lengths in the critical region is described by asymptotic scaling functions and critical exponents, which characterise the underlying universality class. In addition, there are predictions \(^{12}\) for the correlation lengths, which are related to the presence of massless Goldstone modes. The measurement of the correlation lengths as functions of the external field \( H \) enables us to verify these predictions and to determine the critical parameters and scaling functions. We consider the three-dimensional \( O(4) \) model because it is believed to belong to the same universality class as QCD with two degenerate light-quark flavours at its chiral transition in the continuum limit. The variant of the model, which we study here, is defined by

\[
\beta \mathcal{H} = -J \sum_{\langle x,y \rangle} \mathbf{S}_x \cdot \mathbf{S}_y - H \cdot \mathbf{S}_x ,
\]

where \( x \) and \( y \) are nearest neighbours on a hypercubic lattice with \( L \) points per direction, and \( \mathbf{S}_x \) is a four-component unit vector at site \( x \). From \(^*\)We thank for support by DFG Grant No. FOR 339/2-1. the lattice averages \( S^\parallel \) and \( S^\perp \) of the components parallel (longitudinal) and perpendicular (transverse) to \( \mathbf{H} \) we find

\[
M = \langle S^\parallel \rangle , \quad \langle S^\perp \rangle = 0 ,
\]

where \( M \) is the magnetization. Correspondingly, there are two susceptibilities

\[
\chi_L = \frac{\partial M}{\partial H} = V \langle (S^\parallel)^2 \rangle - M^2 ,
\]

\[
\chi_T = \frac{V}{3} \langle S^\perp^2 \rangle = \frac{M}{H} , \quad \text{with} \quad V = L^3 .
\]

The connected correlation functions of the longitudinal and transverse spins are defined by

\[
G_L(x) = \langle S^\parallel_x S^\parallel_0 \rangle - M^2 , \quad G_T(x) = \frac{1}{3} \langle S^\perp_x S^\perp_0 \rangle .
\]

The large distance behaviour of \( G_{L,T} \) is governed by the exponential correlation lengths \( \xi_{L,T} \), except for \( H = 0, T \leq T_c \).

The spontaneous breaking of the rotational symmetry for \( T < T_c \) gives rise to spin waves or massless Goldstone modes and leads to the following divergencies for \( H \to 0 \) at all fixed \( T < T_c \)

\[
\chi_L \sim H^{-1/2} , \quad \chi_T \sim H^{-1} , \quad \xi_T \sim H^{-1/2} .
\]

The prediction for \( \xi_T \) comes from the relation \( \xi_T^2 \sim \chi_T \), in accord with the mass interpretation \( m_T^2 = \chi_T^{-1} \) and \( m_x \sim \xi_T^{-1} \); the relation \( \xi^2 L \sim \chi_L \) (here, \( m = m_x \)) does not hold for \( T < T_c \).\(^\dagger\)
2. NUMERICAL RESULTS

Our simulations were done on lattices with periodic boundary conditions and linear extensions \( L = 48, 72, 96 \) and 120. We used the Wolff single cluster algorithm and made 20000 measurements for each fixed \((H, J)\)-pair. Between two measurements 100-3000 cluster updates were performed. In order to determine the correlation lengths we first calculated spin averages over planes and their respective correlation functions \( \bar{G} (\tau) \). From the correlators at \( \tau \) and \( \tau + 1 \) we then derived an effective correlation length \( \xi_{\text{eff}} (\tau) \). When \( \xi_{\text{eff}} (\tau) \) reached a plateau inside its error bars we used the corresponding \( \tau \)-range to find \( \xi \) from a fit to

\[
\bar{G} (\tau) = A[\exp(-\tau/\xi) + \exp(-(L - \tau)/\xi)] .
\]

2.1. Critical behaviour

In the thermodynamic limit \((V \to \infty)\) and close to \( T_c \) critical observables show power law behaviour in the reduced temperature \( t = (T - T_c)/T_c \): For \( H = 0 \)

\[
M = B (-t)^{\beta} \quad t < 0 \quad (8)
\]

\[
\chi_L = C^+ t^{-\gamma} \quad t > 0 \quad (9)
\]

\[
\xi = \xi^+ t^{-\nu} \quad t > 0 \quad (10)
\]

and at \( T = T_c \) for \( H > 0 \)

\[
M = B^c H^{1/\delta} \quad (11)
\]

\[
\xi_{L,T} = \xi_{L,T}^c H^{-\nu_c} \quad \nu_c = \nu/\beta \delta . \quad (12)
\]

Here, the temperature \( T \) is the inverse of the coupling \( J = 1/T \) and \( J_c = 0.93590 \). From \( [3] \) we use the values \( B = 0.9916(5) \) and \( \beta = 0.380 \). We have fitted the data for \( M \) to Eq. (11) and find \( B^c = 0.721(2) \) and \( \delta = 4.824(9) \). From \( \beta, \delta \) and the hyperscaling relations all other exponents are fixed. In Fig. 1 we show the \( H \)-dependence of \( \xi_L \) and \( \xi_T \) at \( T_c \) and compare it to a fit to Eq. (12), with the critical amplitudes

\[
\xi_T^c = 0.838(1) , \quad \xi_L^c = 0.421(2) , \quad (13)
\]

which leads to the universal ratio \( U_\xi = \xi_T^c/\xi_L^c = 1.99(1) \), a result expected actually below \( T_c \) from a relation between \( \bar{G}_L \) and \( G_T \) \([12]\). At \( H = 0 \), \( T > T_c \), where \( \xi_T = \xi_L = \xi \) and \( \chi_T = \chi_L \) we have made similar fits to our data and find the amplitudes \( \xi^+ = 0.466(2) , \quad C^+ = 0.231(2) \).

![Figure 1. The correlation lengths \( \xi_T \) and \( \xi_L \) at \( T_c \) versus \( H \) from \( L = 48, 72, 96 \)-lattices. The lines are fits to Eq. (12) with the result (13).](image)

From the measured amplitudes we obtain more universal amplitude ratios \([1]\)

\[
R_\chi = C^+ (B^c)^{-\delta} B^{d-1} = 1.084(18) , \quad (14)
\]

\[
Q_\zeta = B^2 (\xi^+ d/C^+) = 0.431(9) . \quad (15)
\]

![Figure 2. The normalized transverse scaling function \( \tilde{g}_\xi^T(z) \) as \( \xi_T h^{\nu_c}/g^T_\xi(0) \) for various \( J \)-values. The line is the asymptotic form, Eq. (15).](image)

2.2. The scaling functions

In the critical region the dependence of the observables on the temperature and the external field is described by scaling functions

\[
M = h^{1/\delta} f_G(z) , \quad \chi_L = h^{1/\delta - 1} f_\chi(z)/H_o , \quad (16)
\]

\[
\xi_T = h^{-\nu_c} g^T_\xi(z) , \quad \xi_L = h^{-\nu_c} g^L_\xi(z) , \quad (17)
\]

\[
\chi_T = h^{-\nu_c} f_\chi(z) / H_o , \quad (18)
\]

\[
\xi_{L,T} = \xi_{L,T}^c H^{-\nu_c} / H_o , \quad (19)
\]
where \( z = \tilde{t} h^{-1/\delta} \), and \( \tilde{t} = t B^{1/\beta} \), \( h = H/H_0 \) with \( H_0 = (T^2c)^{-1/\delta} \), are the normalized reduced temperature and field. The asymptotic behaviour of \( g^{L,T}_\xi(z) \) for \( z \to \infty \) \((H \to 0, t > 0)\), is

\[
g^{L,T}_\xi(z) \xrightarrow{z \to \infty} \xi^+ B^{\nu/\beta} z^{-\nu}. \tag{18}
\]

In Fig. 2 we show the normalized (and universal) scaling function of the transverse correlation length. Whereas it was always possible to determine \( \xi_T \), we could only determine \( \xi_L \) above \( T_c \), because below \( T_c \) a whole spectrum of states contributes to the longitudinal correlation functions. As it turned out, the shape of \( \hat{g}^T_\xi(z) \) is similar to \( f_G(z) \) and - at least for \( z > 0 \) \((T > T_c)\) - the shape of \( \hat{g}^L_\xi(z) \) resembles that of \( f_\chi(z) \) (for example, the peak positions are the same). This similarity comes from the relations \( \xi^2_T \sim \chi_T \) and \( \xi^2_L \sim \chi_L \). For more details see Ref. 6.

2.3. The \( H \)-dependence at fixed \( T < T_c \)

In Fig. 3 we show our results for \( \xi_T \) at fixed \( T = 1/J \) below \( T_c \). The lines are fits to the form expected due to the presence of Goldstone modes

\[
\xi_T = x_0 H^{-1/2} + x_1.
\tag{19}
\]

We see a clear verification of this prediction. The increase of the slope \( x_0 = 0.481(2), 0.577(1) \) and \( 0.637(2) \) for the three \( J \)-values with decreasing temperature is in accord with the scaling function. By comparing the asymptotic behaviour of \( g^T_\xi \) to the \( H \)-dependence of \( \xi_T \) we find

\[
x_0 \sim (-\tilde{t})^q, \text{ with } q = -\nu + \beta \delta/2 = 0.1789. \tag{20}
\]

Figure 3. \( \xi_T \) in the low temperature phase at \( J = 1.0, 0.975 \) and 0.95 vs. \( H^{-1/2} \). The results for \( J = 1.0(0.975) \) are shifted upwards by 10(5).

2.4. The \( H \)-dependence at fixed \( T > T_c \)

Above \( T_c \), we have for all \( H > 0 \) the inequality \( \xi_T > \xi_L \), whereas at \( H = 0 \) the correlation lengths coincide. In Fig. 4 we show \( \xi_T \) and \( \xi_L \) at \( T > T_c \) versus \( H \). With increasing temperature the curves become flatter. The behaviour can again be explained from the asymptotic scaling function. Since \( \xi \) is an even function of \( h \) we must have

\[
\xi = \tilde{t}^{-\nu}[g_0 + g_1 \tilde{t}^{-2\beta} h^2 + \ldots], \tag{21}
\]

with constant \( g_i \). Yet, close to \( T_c \), where \(|z|\) is small, but not \( h \), the expansion of \( g_\xi \) in powers of \( z \) leads to a different \( h \)-dependence

\[
\xi = h^{-\nu}[g_0^T + g_1^T \tilde{t} h^{-1/\delta} + \ldots]. \tag{22}
\]

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