RIGIDITY AND THE CHESS BOARD
THEOREM FOR CUBE PACKINGS

Andrzej P. Kisielewicz and Krzysztof Przesławski

Wydział Matematyki, Informatyki i Ekonometrii, Uniwersytet Zielonogórski
ul. Z. Szafrana 4a, 65-516 Zielona Góra, Poland
A.Kisielewicz@wmie.uz.zgora.pl
K.Przeslawski@wmie.uz.zgora.pl

Abstract

Each packing of $\mathbb{R}^d$ by translates of the unit cube $[0,1)^d$ admits a decomposition into at most two parts such that if a translate of the unit cube is covered by one of them, then it also belongs to such a part.

Key words: cube packing, tiling, rigidity.

Let $I = [0,1)^d$, and $S \subset \mathbb{R}^d$ be a non-empty set. We say that the set $I + S = \{I + s: s \in S\}$ is a packing of $\mathbb{R}^d$ by (half-open) unit cubes if the members of $I + S$ are pairwise disjoint. Clearly, $I + S$ is a packing if and only if for every two vectors $t, t' \in S$ there is $i \in [d]$ such that $|t_i - t'_i| \geq 1$. The cube-packing $I + S$ of $\mathbb{R}^d$ is a tiling of $F \subseteq \mathbb{R}^d$ if $F = \bigcup(I + S)$. The set $F$ is said to be rough if for each $u \in \mathbb{R}^d$ the inclusion $I + u \subseteq F$ implies that $u \in S$. Moreover, $F$ is rigid if $I + S$ is a unique tiling of $F$.

Lagarias and Shor conjectured in [LS] that if $I + T$ is a 2-extremal cube-tiling of $\mathbb{R}^d$, then $T$ decomposes into two explicitly defined parts $T^0$ and $T^1$ such that each of them determines $T$. Clearly, the conjecture means simply that the sets $F^0 := \bigcup(I + T^0)$ and $F^1 := \bigcup(I + T^1)$ are rigid in the sense defined previously. In [KP], we confirmed the conjecture proving a more general result on the rigidity of polyboxes. (We believe that the notion of a polybox appeared therein for the first time.) One of the referees of our work acting for the Discrete and Computational Geometry asked us whether there is a more straightforward approach to the rigidity of tilings. We hope that the present work answers this question in the positive. We offer two proofs of our main result (Theorem 1). Interestingly enough, they resemble two of the fourteen proofs (collected by S. Wagon [W]) of the result that if a rectangle can be tiled by rectangles each of which has at least one integer side, then the tiled rectangle has an integer side. Our first proof resembles that of A. Douady, while the other, that of R. Rochberg, and S. K. Stein. The conjecture of Lagarias and Shor is an immediate consequence of Theorem 3 which is formulated for packings rather than for tilings. Since their work relates to Keller’s conjecture on cube tilings, we give applications of our results to this sort of problems.
THEOREM 1 (RIGIDITY THEOREM) Let $I + S$ be a packing of $\mathbb{R}^d$. Suppose that for every two vectors $t, t' \in S$ if $t - t' \in (-1,0,1)^d$, then the number $|\{i : |t_i - t_i'| = 1\}|$ is even. Then $F = \bigcup (I + S)$ is rough. In particular, $F$ is rigid.

Clearly, the rigidity theorem can be rephrased as follows:

THEOREM 2 Let $I + S$ be a packing of $\mathbb{R}^d$ and let $u \not\in S$. If $I + u \subseteq \bigcup (I + S)$, then there are $t, t' \in S$ such that $t - t' \in (-1,0,1)^d$ and the number $|\{i : |t_i - t_i'| = 1\}|\ $ is odd.

Proof 1. We let $u = 0$ without loss of generality. In addition, we may assume that each of the cubes $I + s$, $s \in S$, intersects $I$. For every $i \in [d]$, let

$$A_i = \{a > 0 : \text{there is } s \in S \text{ such that } a = s_i\}$$

and

$$B_i = \{b < 0 : \text{there is } s \in S \text{ such that } b = s_i\}.$$ 

Let us remark that if $b \in B_i$, then $b + 1 \in A_i$. In order to show this fact, assume for simplicity that $i = d$. Fix any $v \in S$ such that $v_d = b$. Pick $y \in I \cap (I + s)$. Let $z = (y_1, \ldots, y_{d-1})$. Clearly, the line segment $J = \{z\} \times [0,1)$ intersects with at most two of the sets belonging to $I + S$. Since $I + v$ is one of them and $J \not\subseteq I + v$, there is yet another one, say $I + w$. These two cubes cover $J$. Since $b < 0$, we get $J \cap (I + v) = \{z\} \times [0,b + 1)$. Consequently, $J \cap (I + w) = \{z\} \times [b + 1,1)$, which implies $w_d = b + 1 \in A_i$.

Let $U = \{1\} \times A_1 \cup \cdots \cup \{d\} \times A_d$. Then, by the above remark, for every $s \in S$, we can define the mapping $f_s : \mathbb{R}^U \to \mathbb{R}^d$ by the formula

$$(f_s(x))_i = \begin{cases} x(i,s_i + 1) - 1 & \text{if } s_i < 0, \\ 0 & \text{if } s_i = 0, \\ x(i,s_i) & \text{if } s_i > 0. \end{cases}$$

Let us define $p \in \mathbb{R}^U$ so that $p(i,a) = a$, $(i,a) \in U$. Observe that $f_s(p) = s$ for all $s \in S$. Let us define $\varepsilon > 0$ so that for each $i$ and each $a \in A_i$, the interval $(a - \varepsilon, a + \varepsilon)$ does not contain any element of the set $\{0,1\} \cup (A_i \setminus \{a\})$. Let

$$V = \{q \in \mathbb{R}^U : |q(i,a) - p(i,a)| < \varepsilon, \text{ for every } (i,a) \in U\}.$$ 

Let $W$ be the set of all these $q \in V$ for which $I + f_s(q)$, $s \in S$, are disjoint cubes intersecting $I$, whose union contains $I$. As $p \in W$, it follows that $W$ is non-empty. We prove now that $V = W$.

Let $q^0 \in W$. Fix $(i,a) \in U$ and pick any element $q^1 \in V$ such that $q^0(j,c) = q^1(j,c)$, whenever $(j,c) \neq (i,a)$. Let $R^\tau = \{f_s(q^\tau) : s \in S\}$, where $\tau \in \{0,1\}$. Define

$$R^\tau_a = \{f_s(q^\tau) : (f_s(q^\tau))_i = q^\tau(i,a)\}.$$ 

By the definition of $\varepsilon$, $q^\tau(i,a) \in (0,1)$. For the sake of brevity, assume that $i = d$. Therefore, since $I + R^0$ is a packing of $\mathbb{R}^d$ and at the same time a covering of $I$, it is easily observed that there is a set $X \subseteq [0,1)^{d-1}$ such that

$$I \cap \bigcup (I + R^0_d) = X \times [0,1) \text{ and } I \cap \bigcup (I + (R^0_d \setminus R^0_0)) = ([0,1)^{d-1} \setminus X) \times [0,1).$$

Again, by the definition of $\varepsilon$,

$$R^0_a = R^0_0 + (q^1(i,a) - q^0(i,a))e_i,$$
where \( e_i \) is the \( i \)-th vector of the standard basis of \( \mathbb{R}^d \). This equality together with the already mentioned relation \( q^* (i, a) \in (0, 1) \) imply that \( I \cap \bigcup \{ I + R_i^\varepsilon \} = X \times [0, 1) \). Moreover, by the definition of \( q^* \), the set \( R \setminus R_i^\varepsilon \) is independent of \( \tau \). Consequently, \( I + R_i^\varepsilon \) is a covering of \( I \), and a packing of \( \mathbb{R}^d \). Equivalently, \( q^1 \in W \). Since \( V \) is a cube in \( \mathbb{R}^d \), what we have shown implies \( V = W \).

Let \( x \in V \), and \( s \in S \). It follows from the definition of \( \varepsilon \) and \( f_s \) that \( s_i \) and \( (f_s (x))_i \) have the same sign. Let \( \text{vol} \) denote the standard volume measure in \( \mathbb{R}^d \). Therefore, we have

\[
\text{vol}(I \cap (I + f_s (x))) = \prod_{i: \ s_i > 0} (1 - x(i, s_i)) \cdot \prod_{i: \ s_i < 0} x(i, s_i + 1) =: P_s (x).
\]

Summing up with respect to \( s \) gives us

\[
1 = \sum_s P_s (x) =: P (x).
\]

Thus, \( P \) is a polynomial on \( \mathbb{R}^d \) which is constant on \( V \). Since the latter set is open, \( P \) is constant. We have assumed that \( 0 \notin S \). This implies that each of the polynomials \( P_s \), \( s \in S \), is of a positive degree. It is clear that in the expansion of \( P_s \) into monomials there is only one leading term, that is the term of the greatest degree,

\[
Q_s (x) = (-1)^{|\{ i : s_i > 0 \}|} \prod_{i : s_i > 0} x(i, s_i) \cdot \prod_{i : s_i < 0} x(i, s_i + 1).
\]

Let us pick \( t \in S \) so that \( P_t \) is of maximal degree. Since \( P \) is constant, we deduce that there is \( t' \in S \setminus \{ t \} \) such that \( Q_t + Q_{t'} = 0 \). It is easily seen that this equation is equivalent to saying that \( t - t' \in \{ -1, 0, 1 \}^d \) and the number \(|\{ i : |t_i - t'_i| = 1 \}| \) is odd.

**Proof 2.** As before, we assume that \( u = 0 \) and \( I + s \) intersects \( I \), whenever \( s \in S \). Choose \( t \in S \) so that it has the maximum number of non-zero coordinates. As we can change the order of coordinates if necessary, we may assume that there is \( k \in [d] \) such that \( t_i \neq 0 \), if \( i \leq k \), otherwise \( t_i = 0 \). Let us identify \( \mathbb{R}^k \) with the subspace \( \mathbb{R}^k \times \{ 0 \}^{d-k} \). For each \( x \in \mathbb{R}^d \), let \( x^t = (x_1, \ldots, x_k) \) be the projection of \( x \) onto \( \mathbb{R}^k \). Let us define \( v \in \mathbb{R}^k \) as follows

\[
v_i = \begin{cases} t_i + 1, & \text{if } t_i < 0; \\ t_i, & \text{otherwise.} \end{cases}
\]

Clearly, \( v \) is an interior point of the \( k \)-dimensional unit cube \( I^t = [0, 1)^k \). For sufficiently small \( \varepsilon > 0 \), the cube \( \varepsilon I^t + v \) is contained in \( I^t \). In particular, it is covered by the cubes \( I^t + s^t \), for \( s \in S \). Let us define

\[
T = \{ s^t : s \in S, \ I^t + (\varepsilon I^t + v) \neq \emptyset, \text{ for every } \varepsilon > 0 \}.
\]

(We identify here \( I^t \) with \( I^t \times \{ 0 \}^{d-k} \) and we interpret \( v \) as an element of \( \mathbb{R}^d \) according to the convention we have made.) It is clear that \( I^t + T \) is a packing of \( \mathbb{R}^k \) by unit cubes. Moreover, there is \( \gamma > 0 \) such that \( B = \gamma I^t + v \) is covered by \( I^t + T \), and, at the same time, included in \( I^t \). Let us split each factor \( B_t \) of the cube \( B \) into segments \( B_t^0 = [-\gamma, 0) + v_t \) and \( B_t^1 = [0, \gamma) + v_t \). Then \( B \) decomposes into \( 2^n \) cubes \( B^\sigma = B_t^{\sigma t_1} \times \cdots \times B_t^{\sigma t_d} \), where \( \sigma \in \{ 0, 1 \}^d \). Let us
subordinate to each $B^*$ its sign $\text{sgn}(B^*) = (-1)^\sum_i \sigma_i$. For every set $C \subseteq B$ that can be represented as a union of a non-empty family $\mathcal{F} \subseteq \mathcal{B} := \{B^*: \sigma \in \{0,1\}^d\}$, we define the index of $C$ $\text{ind}(C) = \sum_{B^* \subseteq C} \text{sgn}(B^*)$. Observe that if $C = (I^t + z) \cap B$, where $z \in T$, then $C$ is a box and for each $i \leq k$, the factor $C_i$ is equal to one of the three sets $B_i, B_i^0, B_i^1$. In particular, the index of $C$ is well-defined, and is equal to zero if and only if there is $i$ such that $C_i = B_i$, which in turn is equivalent to saying that $z_i = 0$. Furthermore, if $C_i \neq B_i$ for each $i$, then there is $\kappa \in \{0,1\}^k$ such that $C = B\kappa$ and

$$\text{ind}(C) = \text{sgn}(B\kappa) = (-1)^{\sum \{i: z_i = \kappa_i\}} = (-1)^{\sum \{i: z_i > 0\}},$$

as by the definition of $T$, the sets $\{i: z_i = t_i\}$ and $\{i: z_i > 0\}$ have to be equal. Let $T'$ be the subset of $T$ consisting of all $z \in T$ such that $z_i \neq 0$, whenever $i \in [k]$. We have

$$0 = \text{ind}(B) = \sum_{z \in T} \text{ind}((I^t + z) \cap B) = \sum_{z \in T'} \text{ind}((I^t + z) \cap B) = \sum_{z \in T'} (-1)^{\sum \{i: z_i > 0\}}.$$

Since $t = t^t$ belongs to $T'$, there has to exists an additional element $w$ belonging to this set such that the sets $\{i: t_i > 0\}$ and $\{i: w_i > 0\}$ are of different parity. Let $t'$ be the element of $S$ which projects on $w$. Since $w_i \neq 0$ for each $i \leq k$, then by the definition of $k$, we deduce that $t' = w$. It is easily seen that just constructed $t$ and $t'$ satisfy our conclusion.

The main idea of the proof can be easily grasped by analyzing the picture below.

![Fig. 1](image_url)

Fig. 1. This picture corresponds to the case $d = k = 2$. The shaded areas indicate all these cubes $B^*$ in the decomposition of $B$ for which $\text{sgn}(B^*) = -1$. The set $T$ is equal here to $\{t, u, v\}$. Clearly, $\text{ind}((I + t) \cap B) = -1$, $\text{ind}((I + u) \cap B) = 0$. The element $t'$, which existence is guaranteed by our reasoning, coincides with $v$.

**Remark 1** An inspection of any of the two proofs reveals that the conclusion of Theorem 2 can be strengthened: Let $S'$ be the set of all these $s \in S$ for which the intersection $(I + s) \cap (I + u)$ is non-empty, and $\mathcal{I}$ be the family of all sets $\{s\} = \{i \in [d]: 0 < |s_i - u_i|\}$, $s \in S'$. Then for each $t \in S'$ such that $\langle t \rangle$ is a maximal element of $\mathcal{I}$ with respect to the partial order defined by the inclusion, there is $t' \in S'$ such that $\langle t' \rangle = \langle t \rangle$, $t - t' \in \{-1,0,1\}^d$ and $\{i: |t_i - t'_i| = 1\}$ is of odd cardinality.
Theorem 3 (chess board theorem) If $I + S$ is a cube packing of $\mathbb{R}^d$, then there is a decomposition $S^0$, $S^1$ of $S$ such that the sets $F^i = \bigcup(I + S^i)$, are rough. The sets $S^i$ can be defined explicitly.

Proof. Let us define two relations $\sim$ and $\approx$ in $S$ as follows:

$s \sim t$ if and only if $s - t \in \mathbb{Z}^d$,

$s \approx t$ if and only if $s \sim t$ and the number $|\{i: t_i - s_i \equiv 1 \mod 2\}|$ is even.

Both of these relations are equivalences. Either each equivalence class $A$ of the relation $\sim$ is an equivalence class of $\approx$ or it splits into exactly two such classes $A'$ and $A''$. Let us pick $S^0$ so that if $A$ does not split, then $A$ is contained in $S^0$ or is disjoint with this set; otherwise, $S^0$ includes exactly one of the classes $A'$, $A''$.

It is easily observed that the sets $S^0$ and $S^1 = S \setminus S^0$ satisfy the assumptions of the rigidity theorem, therefore they define the desired decomposition. $\square$

Fig. 2. A cube tiling and a related decomposition of $\mathbb{R}^2$ into two rough parts: $F^0$ (white) and $F^1$ (black).

Let us remark that one of the sets $S^i$ is allowed to be empty.

The stronger version of Theorem 3 described in Remark 1 can be applied to cube tilings of $\mathbb{R}^d$.

Theorem 4 Suppose $I + S$ is a cube tiling of $\mathbb{R}^d$. For every $t \in S$ and every $\varepsilon \in \{-1, 1\}^d$ there is a set $J \subseteq [d]$ of odd cardinality such that the vector $t' = t + \sum_{i \in J} \varepsilon_i e_i$, where $e_i$ are elements of the standard basis $e_1 = (1, 0, \ldots, 0), \ldots, e_d = (0, \ldots, 1)$, belongs to $S$.

Proof. Clearly, we may assume $t = 0$. Let $u = (1/2)\varepsilon$. Since $I + S$ is a tiling, $I + u$ is contained in $I + S$. Define $S'$ and $\mathcal{J}$ as in Remark 1. It is easily seen that $0 \in S'$ and $\{0\} = [d]$. Consequently, the latter set is maximal in $\mathcal{J}$. Thus, by Remark 1, there is $t' \in S'$ such that $\langle t' \rangle = [d]$, and the set $J = \{i \in [d]: |t'_i - t_i| = |t'_i| = 1\}$ is odd. Observe that $\varepsilon_i = -1$, for $i \in J$ if and only if $t'_i = -1$, as in other case $I + t'$ and $I + u$ would be disjoint. Thus, $t' = \sum_{i \in J} \varepsilon_i e_i$. $\square$

Corollary 5 If $I + S$ is a cube tiling of $\mathbb{R}^d$, then for every $t \in S$ the set $(t + \mathbb{Z}^d) \cap S$ is infinite.
PROPOSITION 6 Suppose \( I + S \) is a cube tiling of \( \mathbb{R}^d \). If \( G = S \cap \mathbb{Z}^d \) is a subgroup of \( \mathbb{Z}^d \) and there are \( k = (k_1, \ldots, k_d) \in \mathbb{Z}^d \) and a set \( L \subseteq [d] \) containing at least \( d - 2 \) elements such that

1. for every \( i \in [d] \), the multiple \( k_i e_i \) of \( e_i \) belongs to \( G \),
2. for every \( i \in [d] \) and \( l \in L \), the coordinates \( k_i \) and \( k_l \) are relatively prime, whenever \( i \neq l \),

then there is \( m \in [d] \) such that \( e_m \in G \).

Proof. By the preceding theorem and the fact that \( 0 \in S \), there is a set \( J \subseteq [d] \) of odd cardinality such that \( s = \sum_{i \in J} e_i \) belongs to \( S \). Obviously, \( s \) is also an element of \( G \). If \( J \) is a singleton, then \( s \) is a vector of the standard basis; therefore, it remains to consider the case \( |J| \geq 3 \). Then there is \( m \in J \cap L \). Let \( n = \prod_{i \in J \setminus \{m\}} k_i \). It follows from assumption (2) that \( k_m \) and \( n \) are relatively prime. Thus, there exist nonzero integers \( x \) and \( y \) such that \( xn + yk_m = 1 \).

We have \( ns = ne_m + \sum_{i \in J \setminus \{m\}} ne_i \). Since the elements \( ne_i \), \( i \in J \setminus \{m\} \), are multiples of \( k_i e_i \), they belong to \( G \). Consequently, \( ne_m \) belongs to \( G \). Now, we have \( e_m = x(ne_m) + y(k_m e_m) \) belongs to \( G \). \( \square \)

Theorem 5 is a generalization of Theorem 50 in [KP]. It should be mentioned however that it can be proved within the framework of a theory developed there. Theorem 2 relates to Lemma 31 in [KP]. These results rest upon an idea which has been already exploited in [BFF] (see also [Z]).

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References

[BFF] M. A. Berger, A. Felzenbaum and A. Fraenkel, A non-analytic proof of the Newmann-Znám result for disjoint covering systems, Combinatorica 6 (1986), 235–243.

[KP] A. P. Kisielewicz, K. Przeslawski, Polyboxes, cube tilings and rigidity, Discrete Comput. Geom. 40 (2008), 1-30. (arxiv:math.CO/0609132)
[LS] J. C. Lagarias, P. W. Shor, Cube tilings and nonlinear codes, *Discrete Comput. Geom.* 11 (1994), 359–391.

[W] S. Wagon, Fourteen Proofs of a Result About Tiling a Rectangle, *Amer. Math. Monthly* 94 (1987), 601–617.

[Z] D. Zeilberger, How Berger, Felzenbaum and Fraenkel revolutionized COV-ERING SYSTEMS the same way that George Boole revolutionized LOGIC, *Electr. J. Combin.* 8 (2001), #A1.