Stopping Times and Related Itô’s Calculus with 
G-Brownian Motion

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Abstract. Under the framework of G-expectation and G-Brownian motion, We have introduced a Itô’s integral for stochastic processes without the condition of quasi-continuous. We then can obtain Itô’s integral on stopping time interval. This formulation help us to obtain Itô’s formula for a general 𝐶₁,₂-function, which generalizes the previous results of Peng [15, 16, 18] and it’s improved version of Gao [7].

1 Introduction

G-Brownian motion is a continuous process \((B_t)_{t \geq 0}\) defined on a sublinear expectation space \((Ω, ℋ, \hat{E})\) (see Definition 2.1) with stable and independent increments. It was proved that each increment \(X = B_{t+s} - B_t\) of \(B\) is G-normal distributed, namely

\[ aX + b\bar{X} \overset{d}{=} \sqrt{a^2 + b^2}X, \quad \text{for} \ a, b \geq 0, \]

where \(\bar{X}\) is an independent copy of \(X\). A new type of stochastic integral and the related Itô’s calculus has been introduced in [15, 16, 18]. For example, if \(ϕ\) is a \(C^2\)-function such that \(ϕ_{xx}(x)\) satisfies polynomial growth function, then we have

\[ ϕ(B_t) - ϕ(B_{t_0}) = \int_{t_0}^{t} ϕ_x(B_s)dB_s + \frac{1}{2} \int_{0}^{t} ϕ_{xx}(B_s)d⟨B⟩_s. \]

A interesting problem is how to extend the above formulation to the situation where \(ϕ\) is simply a \(C^2\)-function. The main obstacle to treat this situation is that the notion of stopping times and the related properties have not yet been well-understood and studied within the framework of G-expectation and

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G-Brownian motions. A difficulty hidden behind is that until now the theory is mainly based on the space of random variables \( X = X(\omega) \) which are quasi-continuous with respect to the natural Choquet capacity \( \hat{c}(A) := \hat{E}[I_A], A \in \mathcal{B}(\Omega) \). It is not yet clear that the martingale properties still hold for random variables without quasi-continuous condition. On the other hand, stopping times are closely related to random variables without quasi-continuous properties. Recently Gao [7] has improved the Itô’s formula of Peng. But the problem of (1) for \( C^2 \)-function is still open.

In this paper we will face this difficulty by introducing Itô’s stochastic integrals \( \int_0^t \eta_s dB_s \) where, for each \( t \), the integrand \( \eta_t \) needs not to be a quasi-continuous random variable. Within this framework we can treat a fundamentally important Itô’s integral \( \int_0^{\tau \wedge t} \eta_s dB_s \) for a stopping time \( \tau \) and then obtain some important properties for the related stochastic calculus. A very general form of Itô’s formula with respect to G-Brownian motion has been obtained. In particular (1) is proved to be true for \( \varphi \in C^2 \). Many important and interesting problems still open under this new framework, e.g., under what condition \( \int_0^\cdot \eta_s dB_s \) is a martingale or a local martingale?

This paper is organized as follows: In the next section we recall some basic notions an results of G-Brownian motion under a G-expectation and the related space of random variables. In Section 3 we introduce a new space \( M^2_\ast(0,T) \) of stochastic processes which are not necessarily quasi-continuous and then define the related Itô’s integral on this space. In Section 4 we discuss Itô’s integral defined on \([0,\tau]\) where \( \tau \) is a stopping time. This allows us to have a Itô’s integral for a space larger that \( M^2_\ast(0,T) \). Finally in Section 5, we prove the mentioned general form of Itô’s formula.

We believe that some notions and properties of this paper will become important and basic tools in the further development of G-Brownian motion and the corresponding nonlinear expectation analysis.

2 Basic settings

We present some preliminaries in the theory of sublinear expectations and the related G-Brownian motions. More details can be found in Peng [15], [16] and [18].

Definition 2.1 Let \( \Omega \) be a given set and let \( \mathcal{H} \) be a linear space of real valued functions defined on \( \Omega \) with \( c \in \mathcal{H} \) for all constants \( c \), and \( |X| \in \mathcal{H} \), if \( X \in \mathcal{H} \). \( \mathcal{H} \) is considered as the space of our “random variables”. A sublinear expectation \( \tilde{E} \) on \( \mathcal{H} \) is a functional \( \tilde{E} : \mathcal{H} \mapsto \mathbb{R} \) satisfying the following properties: for all \( X, Y \in \mathcal{H} \), we have

(a) Monotonicity: \( \text{If } X \geq Y \text{ then } \tilde{E}[X] \geq \tilde{E}[Y] \).

(b) Constant preserving: \( \tilde{E}[c] = c \).

(c) Sub-additivity: \( \tilde{E}[X] - \tilde{E}[Y] \leq \tilde{E}[X - Y] \).

(d) Positive homogeneity: \( \tilde{E}[\lambda X] = \lambda \tilde{E}[X], \forall \lambda \geq 0 \).
The triple \((\Omega, \mathcal{H}, \hat{E})\) is called a sublinear expectation space. \(X \in \mathcal{H}\) is called a random variable in \((\Omega, \mathcal{H})\). We often call \(Y = (Y_1, \cdots, Y_d)\), \(Y_i \in \mathcal{H}\) a \(d\)-dimensional random vector in \((\Omega, \mathcal{H})\). Let us consider a space of random variables \(\mathcal{H}\) satisfying: if \(X_i \in \mathcal{H}, i = 1, \cdots, d\), then
\[
\phi(X_1, \cdots, X_d) \in \mathcal{H}, \quad \text{for all } \phi \in C_{b, \text{Lip}}(\mathbb{R}^d),
\]
where \(C_{b, \text{Lip}}(\mathbb{R}^d)\) is the space of all bounded and Lipschitz continuous functions on \(\mathbb{R}^d\). An \(m\)-dimensional random vector \(X = (X_1, \cdots, X_m)\) is said to be independent from another \(n\)-dimensional random vector \(Y = (Y_1, \cdots, Y_n)\) if
\[
\hat{E}[\phi(X, Y)] = \hat{E}[\hat{E}[\phi(X, y)|Y = y]], \quad \text{for } \phi \in C_{b, \text{Lip}}(\mathbb{R}^m \times \mathbb{R}^n).
\]
Let \(X_1\) and \(X_2\) be two \(n\)-dimensional random vectors defined respectively in sublinear expectation spaces \((\Omega_1, \mathcal{H}_1, \hat{E}_1)\) and \((\Omega_2, \mathcal{H}_2, \hat{E}_2)\). They are called identically distributed, denoted by \(X_1 \sim X_2\), if
\[
\hat{E}_1[\phi(X_1)] = \hat{E}_2[\phi(X_2)], \quad \forall \phi \in C_{b, \text{Lip}}(\mathbb{R}^n).
\]
If \(X, \bar{X}\) are two \(m\)-dimensional random vectors in \((\Omega, \mathcal{H}, \hat{E})\) and \(\bar{X}\) is identically distributed with \(X\) and independent from \(X\), then \(\bar{X}\) is said to be an independent copy of \(X\).

**Definition 2.2 (G-normal distribution)** A \(d\)-dimensional random vector \(X = (X_1, \cdots, X_d)\) in a sublinear expectation space \((\Omega, \mathcal{H}, \hat{E})\) is called \(G\)-normal distributed if for each \(a, b \geq 0\) we have
\[
aX + b\bar{X} \sim \sqrt{a^2 + b^2}X, \quad (2)
\]
where \(\bar{X}\) is an independent copy of \(X\). Here the letter \(G\) denotes the function
\[
G(A) := \frac{1}{2}\hat{E}[(AX, X)] : \mathbb{S}_d \mapsto \mathbb{R}.
\]
It is also proved in Peng [16, 17] that, for each \(a \in \mathbb{R}^d\) and \(p \in [1, \infty)\)
\[
\hat{E}[|aX|^p] = \frac{1}{\sqrt{2\pi a^2}} \int_{-\infty}^{\infty} |x|^p \exp \left(\frac{-x^2}{2a^2}\right) dx,
\]
where \(a^2 = 2G(aa^T)\).

**Definition 2.3** A \(d\)-dimensional stochastic process \(\xi_t(\omega) = (\xi^1_t, \cdots, \xi^d_t)(\omega)\) defined in a sublinear expectation space \((\Omega, \mathcal{H}, \hat{E})\) is a family of \(d\)-dimensional random vectors \(\xi_t\) parametrized by \(t \in [0, \infty)\) such that \(\xi^i_t \in \mathcal{H}\), for each \(i = 1, \cdots, d\) and \(t \in [0, \infty)\).

The most typical stochastic process in a sublinear expectation space is the so-called \(G\)-Brownian motion.
Furthermore, it is proved in [4] that \( L \) that a weakly compact family \( B \) canonical process \( \| \) under the natural norm, sapce \((\Omega, \mathcal{H}, \mathbb{E})\) ,

\[ \rho(\omega^1, \omega^2) := \sum_{i=1}^{\infty} 2^{-i}[(\max_{t \in [0, t]} |\omega_i^1 - \omega_i^2|) \wedge 1]. \]

\( B(\Omega) \) denotes the \( \sigma \)-algebra generated by all open sets. Let \( \Omega = C_0(\mathbb{R}^+) \) be the space of all \( \mathbb{R} \)-valued continuous paths \( (\omega_t)_{t \in \mathbb{R}^+} \) with \( \omega_0 = 0 \), equipped with the distance

\[ \rho(\omega^1, \omega^2) := \sum_{i=1}^{\infty} 2^{-i}[(\max_{t \in [0, t]} |\omega_i^1 - \omega_i^2|) \wedge 1]. \]

We denote by \( B(\Omega) \) the Borel \( \sigma \)-algebra of \( \Omega \) and by \( \mathcal{M} \) the collection of all probability measure on \( (\Omega, B(\Omega)) \).

We also denote, for each \( t \in [0, \infty) \):

- \( \Omega_t := \{ \omega : \omega, \wedge_t : \omega \in \Omega \} \),
- \( \mathcal{F}_t := B(\Omega_t) \),
- \( L^0(\Omega) \) : the space of all \( B(\Omega) \)-measurable real functions,
- \( L^0(\Omega_t) \) : the space of all \( B(\Omega_t) \)-measurable real functions,
- \( B_h(\Omega) \) : all bounded elements in \( L^0(\Omega) \), \( B_b(\Omega_t) := B_b(\Omega_t) \cap L^0(\Omega_t) \),
- \( C_b(\Omega) \) : all continuous elements in \( B_b(\Omega) \); \( C_b(\Omega_t) := B_b(\Omega_t) \cap L^0(\Omega_t) \).

In [16, 18], a \( G \)-Brownian motion is constructed on a sublinear expectation space \((\Omega, \mathbb{L}_G^p(\Omega), \mathbb{E})\) for \( p \geq 1 \), with \( \mathbb{L}_G^p(\Omega) \) such that \( \mathbb{L}_G^p(\Omega) \) is a Banach space under the natural norm \( \|X\|_p := \mathbb{E}[|X|^p]^{1/p} \). In this space the corresponding canonical process \( B_t(\omega) = \omega_t, t \in [0, \infty) \), for \( \omega \in \Omega \) is a \( G \)-Brownian motion. Furthermore, it is proved in [1] that \( L^0(\Omega) \supset \mathbb{L}_G^p(\Omega) \supset C_b(\Omega) \), and there exists a weakly compact family \( \mathcal{P} \) of probability measures defined on \( (\Omega, B(\Omega)) \) such that

\[ \mathbb{E}[X] = \sup_{p \in \mathcal{P}} E_p[X], \text{ for } X \in C_b(\Omega). \]
We introduce the natural Choquet capacity

\[ \hat{c}(A) := \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega). \]

The space \( L^2_G(\Omega) \) was also introduced independently in [3] in a quite different framework.

**Definition 2.5** A set \( A \subseteq \Omega \) is polar if \( \hat{c}(A) = 0 \). A property holds “quasi-surely” (q.s.) if it holds outside a polar set.

\( L^p_G(\Omega) \) can be characterized as follows:

\[ L^p_G(\Omega) = \{ X \in L^0(\Omega) : \sup_{P \in \mathcal{P}} E_P[|X|^p] < \infty, \text{ and } X \text{ is } \hat{c}-\text{quasi surely continuous} \}. \]

We also denote, for \( p > 0 \),

- \( L^p := \{ X \in L^0(\Omega) : \hat{E}[|X|^p] = \sup_{P \in \mathcal{P}} E_P[|X|^p] < \infty \}; \)
- \( N^p := \{ X \in L^0(\Omega) : \hat{E}[|X|^p] = 0 \}; \)
- \( N := \{ X \in L^0(\Omega) : X = 0, \text{ and } \hat{c}\text{-quasi surely (q.s.)} \}. \)

It is seen that \( L^p \) and \( N^p \) are linear spaces and \( N^p = N \), for each \( p > 0 \). We denote by \( \mathbb{L}^p := L^p/N^p \). As usual, we do not make the distinction between classes and their representatives.

Now, we give the following two propositions which can be found in [4].

**Proposition 2.6** For each \( \{X_n\}_{n=1}^{\infty} \in C_b(\Omega) \) such that \( X_n \downarrow 0 \) on \( \Omega \), we have \( \hat{E}[X_n] \downarrow 0 \).

**Proposition 2.7** We have

1. For each \( p \geq 1 \), \( L^p \) is a Banach space under the norm \( \|X\|_p := \left( \hat{E}[|X|^p] \right)^{1/p} \).
2. \( L^p \) is the completion of \( B_b(\Omega) \) under the Banach norm \( \hat{E}[|X|^p]^{1/p} \).
3. \( L^p_G \) is the completion of \( C_b(\Omega) \).

The following Proposition is obvious.

**Proposition 2.8** We have

1. \( L^p \subseteq L^q \subseteq L^2 \subseteq \mathbb{L}_q \subseteq \mathbb{L}_q^{q,c} \subseteq \mathbb{L}^2, 0 < p \leq q \leq \infty; \)
2. \( \|X\|_p \uparrow \|X\|_\infty \), for each \( X \in \mathbb{L}^\infty \);
3. $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Then $X \in L^p$ and $Y \in L^q$ implies

$$XY \in L^1 \text{ and } E[|XY|] \leq (E[|X|^p])^{\frac{1}{p}} (E[|Y|^q])^{\frac{1}{q}}$$

Moreover $X \in L^p_\infty$ and $Y \in L^q_\infty$ implies $XY \in L^1_\infty$.

**Proposition 2.9** For a given $p \in (0, +\infty]$, let $\{X_n\}_{n=1}^\infty$ be a sequence in $L^p$ which converges to $X$ in $L^p$. Then there exists a subsequence $(X_{n_k})$ which converges to $X$ quasi-surely in the sense that it converges to $X$ outside a polar set.

We also have

**Proposition 2.10** For each $p > 0$,

$$L^p = \{X \in L^p : \lim_{n \to \infty} E[|X|^p 1_{\{|X| > n\}}] = 0\}.$$  

We introduce the following properties. They are important in this paper:

**Proposition 2.11** For each $0 \leq t < T$, $\xi \in L^2(\Omega_t)$, we have

$$\hat{E}[\xi(B_T - B_t)] = 0.$$  

**Proof.** Let $P \in \mathcal{P}$ be given. If $\xi \in C_b(\Omega_t)$, then we have

$$0 = -\hat{E}[-\xi(B_T - B_t)] \leq E_P[\xi(B_T - B_t)] \leq \hat{E}[\xi(B_T - B_t)] = 0.$$  

In the case when $\xi \in L^2(\Omega_t)$, we have $E_P[|\xi|^2] \leq \hat{E}[|\xi|^2] < \infty$. Since it is known that $C_b(\Omega_t)$ is dense in $L^2_P(\Omega_t)$, we then can choose a sequence $\{\xi_n\}_{n=1}^\infty$ in $C_b(\Omega_t)$ such that $E_P[|\xi - \xi_n|^2] \to 0$. Thus

$$E_P[\xi(B_T - B_t)] = \lim_{n \to \infty} E_P[\xi_n(B_T - B_t)] = 0.$$  

The proof is complete. $\blacksquare$

**Proposition 2.12** For each $0 \leq t \leq T$, $\xi \in B_b(\Omega_t)$, we have

$$\hat{E}[\xi^2(B_T - B_t)^2 - \sigma_t^2 \xi^2(T - t)] \leq 0. \quad (3)$$

**Proof.** If $\xi \in C_b(\Omega_t)$, then by [Peng], we have the following Itô’s formula:

$$\xi^2([B_T - B_t]^2 - (B_T - B_t))] = 2 \int_t^T \xi^2 B_s dB_s.$$  

It follows that $\hat{E}[\xi^2(B_T - B_t)^2 - \xi^2(B_T - B_t))] = 0$. On the other hand, we have $\langle B_T - B_t \rangle \leq \sigma^2 (T - t)$, quasi surely. Thus (3) holds for $\xi \in C_b(\Omega_t)$. It follows that, for each fixed $P \in \mathcal{P}$, we have

$$E_P[\xi^2(B_T - B_t)^2 - \xi^2(B_T - B_t))] \leq 0. \quad (4)$$  

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In the case when \( \xi \in B_b(\Omega_t) \), we can find a sequence \( \{\xi_n\}_{n=1}^\infty \) in \( C_b(\Omega_t) \), such that \( \xi_n \to \xi \) in \( L^p(\Omega, F_t, P) \), for some \( p > 2 \). Thus we have
\[
E_P[\xi_n^2(B_T - B_t)^2 - \xi_n^2(\langle B_T \rangle - \langle B_t \rangle)] \leq 0,
\]
and then, by letting \( n \to \infty \), obtain \( \text{(I)} \) for \( \xi \in B_b(\Omega_t) \). Thus \( \text{(II)} \) follows immediately for \( \xi \in B_b(\Omega_t) \).

3 A generalized Ito’s Integral

For notational simplification, in the rest of the paper we only discuss 1-dimensional Brownian motion, i.e., \( d = 1 \). But all the results can be generalized to multi-dimensional situation. We refer to [16, 18] for the corresponding techniques.

For \( p \geq 1 \) and \( T \in \mathbb{R}_+ \) be fixed, we first consider the following simple type of processes:
\[
M_{b,i}(0, T) = \{\eta: \eta(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega)I_{[t_j, t_{j+1})}(t), \forall N > 0, 0 = t_0 < \cdots < t_N = T, \xi_j(\omega) \in B_b(\Omega_{t_j}), j = 0, \cdots, N-1\}.
\]

**Definition 3.1** For an \( \eta \in M_{b,0}(0, T) \) with \( \eta(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega)I_{[t_j, t_{j+1})}(t), \) the related Bochner integral is
\[
\int_0^T \eta(\omega)dt = \sum_{j=0}^{N-1} \xi_j(\omega)(t_{j+1} - t_j).
\]

For each \( \eta \in M_{b,0}(0, T) \) we set
\[
\hat{E}[\eta] := \frac{1}{T} \hat{E}[\int_0^T |\eta(\omega)|^p dt] = \frac{1}{T} \hat{E}[\sum_{j=0}^{N-1} \xi_j(\omega)(t_{j+1} - t_j)].
\]

We can introduce a natural norm \( ||\eta||_{M^p(0, T)} = \{\hat{E}[\int_0^T |\eta(\omega)|^p dt]\}^{1/p} \). Under this norm, \( M_{b,0}(0, T) \) can be continuously extended to a Banach space.

**Definition 3.2** For each \( p \geq 1 \), we denote by \( M^p_{b}(0, T) \) the completion of \( M_{b,0}(0, T) \) under the norm
\[
||\eta||_{M^p(0, T)} = \{\hat{E}[\int_0^T |\eta(\omega)|^p dt]\}^{1/p}.
\]

We have \( M^p_{b}(0, T) \supset M^q_{b}(0, T) \), for \( p \leq q \). The following process
\[
\eta(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega)I_{[t_j, t_{j+1})}(t), \xi_j \in L^p(\Omega_{t_j}), j = 1, \cdots, N
\]
is also in \( M^p_{b}(0, T) \).
**Definition 3.3** For each $\eta \in M_{\text{b},0}(0,T)$ with the form

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) I_{[t_j,t_{j+1})}(t),$$

we define Itô’s integral

$$I(\eta) = \int_0^T \eta_s dB_s := \sum_{j=0}^{N-1} \xi_j(B_{t_{j+1}} - B_{t_j}).$$

**Lemma 3.4** The mapping $I : M^2_{\text{b}}(0,T) \to \mathbb{L}^2(\Omega_T)$ is a linear continuous mapping and thus can be continuously extended to $I : M^2(0,T) \to \mathbb{L}^2(\Omega_T)$. We have

$$\hat{\mathbb{E}}\left[\int_0^T \eta_s dB_s\right] = 0,$$  \hspace{1cm} (5)

$$\hat{\mathbb{E}}\left[\int_0^T \eta_s dB_s\right]^2 \leq \sigma^2 \hat{\mathbb{E}}\left[\int_0^T \eta_s^2 dt\right].$$  \hspace{1cm} (6)

**Proof.** We only need to prove (5) and (6). From Proposition 2.11, for each $j$,

$$\hat{\mathbb{E}}[\xi_j(B_{t_{j+1}} - B_{t_j})] = \hat{\mathbb{E}}[-\xi_j(B_{t_{j+1}} - B_{t_j})] = 0.$$  

Thus we have

$$\hat{\mathbb{E}}[\int_0^T \eta_s dB_s] = \hat{\mathbb{E}}[\int_0^{t_{N-1}} \eta_s dB_s + \xi_{N-1}(B_{t_N} - B_{t_{N-1}})]$$

$$= \hat{\mathbb{E}}[\int_0^{t_{N-1}} \eta_s dB_s] = \cdots = \hat{\mathbb{E}}[\xi_0(B_{t_1} - B_{t_0})] = 0.$$  

We now prove (6), we first apply Proposition 2.11 to derive

$$\hat{\mathbb{E}}[\left(\int_0^T \eta_t dB_t\right)^2] = \hat{\mathbb{E}}\left[\left(\int_0^{t_{N-1}} \eta_t dB_t + \xi_{N-1}(B_{t_N} - B_{t_{N-1}})\right)^2\right]$$

$$= \hat{\mathbb{E}}\left[\left(\int_0^{t_{N-1}} \eta_t dB_t\right)^2 + 2\left(\int_0^{t_{N-1}} \eta_t dB_t\right)\xi_{N-1}(B_{t_N} - B_{t_{N-1}})\right]$$

$$+ 2\left(\int_0^{t_{N-1}} \eta_t dB_t\right)\xi_{N-1}(B_{t_N} - B_{t_{N-1}})$$

$$= \hat{\mathbb{E}}\left[\left(\int_0^{t_{N-1}} \eta_t dB_t\right)^2 + \xi_{N-1}^2(B_{t_N} - B_{t_{N-1}})^2\right]$$

$$= \cdots = \hat{\mathbb{E}}\left[\sum_{i=0}^{N-1} \xi_i^2(B_{t_{i+1}} - B_{t_i})^2\right].$$

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Then by Proposition 2.12, we have
\[ \hat{E}[\xi^2(B^*_j - B^*_j)^2 - \sigma^2 \xi^2(t_{j+1} - t_j)] \leq 0. \]

Thus
\[ \hat{E}[\int_0^T \eta_s dB_s]^2 = \hat{E}[\sum_{i=0}^{N-1} \xi_i^2(B_{iN} - B_{iN-1})^2] \]
\[ \leq \hat{E}[\sum_{i=0}^{N-1} \xi_i^2[(B_{iN} - B_{iN-1})^2 - \sigma^2(t_{i+1} - t_i)]] + \hat{E}[\sum_{i=0}^{N-1} \sigma^2 \xi_i^2(t_{i+1} - t_i)] \]
\[ \leq \sum_{i=0}^{N-1} \hat{E}[\xi_i^2(B_{i+1} - B_i)^2 - \sigma^2 \xi_i^2(t_{i+1} - t_i)] + \sum_{i=0}^{N-1} \hat{E}[\sigma^2 \xi_i^2(t_{i+1} - t_i)] \]
\[ \leq \sigma^2 \hat{E}[\int_0^T \eta_s^2 ds]. \]

The following Proposition can be verified directly by the definition of Itô’s integral with respect to $G$-Brownian motion.

**Proposition 3.5** Let $\eta, \theta \in M_2^2(0, T)$, and let $0 \leq s \leq r \leq t \leq T$. Then we have
1. $\int_s^r \eta_u dB_u = \int_s^r \eta_u dB_u + \int_s^t \eta_u dB_u$, q.s.,
2. $\int_s^r (\alpha \eta_u + \theta_u) dB_u = \alpha \int_s^r \eta_u dB_u + \int_s^r \theta_u dB_u$, where $\alpha \in B_b(\Omega_s)$.

**Proposition 3.6** For each $\eta \in M_2^2(0, T)$, we have
\[ \hat{E}[\sup_{0 \leq t \leq T} |\int_0^t \eta_s dB_s|^2] \leq 2\sigma^2 \hat{E}[\int_0^T \eta_s^2 ds]. \quad (7) \]

**Proof.** Since for each $\alpha \in B_b(\Omega_t)$, we have
\[ \hat{E}[\alpha \int_0^T \eta_s dB_s] = 0, \]
thus, for each fixed $P \in \mathcal{P}$, the process $\int_0^t \eta_s dB_s$ is a $P$-martingale. It follows from the classical Doob’s martingale inequality that
\[ P[\sup_{0 \leq t \leq T} |\int_0^t \eta_s dB_s|^2] \leq 2E_P[|\int_0^T \eta_s dB_s|^2] \leq 2\sigma^2 E_P[\int_0^T \eta_s^2 ds] \leq 2\sigma^2 \hat{E}[\int_0^T \eta_s^2 ds]. \]
Thus (7) holds. □

**Proposition 3.7** For any $\eta \in M_2^2(0, T)$ and $0 \leq t \leq T$, $\int_0^t \eta_s dB_s$ is continuous in $t$ quasi-surely.
Proof. The claim is true for $\eta \in M_{b,0}(0,T)$ since $(B_t)_{t \geq 0}$ is quasi-surely continuous. In the case when $\eta \in M^*_c(0,T)$, there exists $\eta^n \in M_{b,0}(0,T)$, such that $\hat{E}[\int_0^T (\eta_s - \eta^n_s)^2 ds] \to 0$. By Proposition 3.6 we have

$$\hat{E}\left[ \sup_{0 \leq t \leq T} |\int_0^t (\eta_s - \eta^n_s) dB_s|^2 \right] \leq 2\pi^2 \hat{E}\left[ \int_0^T (\eta_s - \eta^n_s)^2 ds \right] \to 0.$$ 

This implies that, quasi-surely, the sequence of processes $\int_0^t \eta^n_s dB_s$ uniformly converges to $\int_0^t \eta_s dB_s$ on $[0,T]$. Thus $\int_0^t \eta_s dB_s$ is continuous in $t$ quasi-surely.

We have also the following

**Proposition 3.8** Let $X \in M^{p+\varepsilon}_c(\Omega \times [0,T])$ with $p \geq 1$ and $\varepsilon > 0$. Then we have

$$\hat{E}\left[ \int_0^T |X_t|^p I_{\{|X_t| > n\}} dt \right] \to 0, \text{ as } n \to \infty.$$ 

Proof. It is clear that $X \in M^p_c(0,T)$. Moreover we have

$$\hat{E}\left[ \int_0^T |X_t|^p I_{\{|X_t| > n\}} dt \right] \leq \left[ \hat{E}\left[ \int_0^T |X_t|^{p+\varepsilon} dt \right] \right]^{\frac{p}{p+\varepsilon}} \cdot \left[ \hat{E}\left[ \int_0^T I_{\{|X_t| > n\}} dt \right] \right]^{\frac{\varepsilon}{p+\varepsilon}} \leq \left[ \hat{E}\left[ \int_0^T |X_t|^{p+\varepsilon} dt \right] \right]^{\frac{p}{p+\varepsilon}} \cdot \left[ \hat{E}\left[ \int_0^T n^{-p} |X_t|^p dt \right] \right]^{\frac{\varepsilon}{p+\varepsilon}} \to 0,$$

as $n \to \infty$. ■

**Proposition 3.9** For each $p \geq 1$ and $X \in M^p_c(0,T)$ we have

$$\lim_{n \to \infty} \hat{E}\left[ \int_0^T |X_t|^p I_{\{|X_t| > n\}} dt \right] = 0. \quad (8)$$

Proof. For each $X \in M^p_c(0,T)$, we can find a sequence $\{Y^{(n)}\}_{n=1}^{\infty}$ in $M_{b,0}(0,T)$ such that $\hat{E}\int_0^T [|X_t - Y^{(n)}_t|^p] dt \to 0$. Let $y_n = \sup_{\omega \in \Omega, t \in [0,T]} |Y^{(n)}_t(\omega)|$ and $X^{(n)} = (X \wedge y_n) \vee (-y_n)$. Since $|X - X^{(n)}| \leq |X - Y^{(n)}|$, we have $\hat{E}\int_0^T [|X_t - X^{(n)}_t|^p] dt \to 0$. This also implies that for any sequence $\{\alpha_n\}$ tending to $\infty$,

$$\lim_{n \to \infty} \hat{E}\left[ \int_0^T [|X_t - (X_t \wedge \alpha_n) \vee (-\alpha_n)|^p] dt \right] = 0.$$

Now we have for all $n \in \mathbb{N}$,

$$\hat{E}\left[ \int_0^T |X_t|^p I_{\{|X_t| > n\}} dt \right] = \hat{E}\left[ \int_0^T (|X_t| - n + n)^p I_{\{|X_t| > n\}} dt \right] \leq (1 + 2^{p-1}) \left( \hat{E}\left[ \int_0^T [|X_t| - n]^p I_{\{|X_t| > n\}} dt \right] + n^p \hat{E}\left[ \int_0^T I_{\{|X_t| > n\}} dt \right] \right).$$

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The first term of the right hand side tends to 0 since
\[ \hat{E} \int_0^T (|X_t| - n)^p I_{\{|X_t| > n\}} dt = \hat{E} \int_0^T |X_t - (X_t \wedge n) \vee (-n)|^p dt \to 0. \]

For the second term, since
\[ \frac{n^p}{2^p} I_{\{|X_t| > n\}} \leq (|X_t| - \frac{n}{2})^p I_{\{|X_t| > \frac{n}{2}\}} \leq \left( \frac{n}{2} \right)^p I_{\{|X_t| > \frac{n}{2}\}}, \]
thus we have
\[ \frac{n^p}{2^p} \hat{E} \int_0^T I_{\{|X_t| > n\}} dt = \hat{E} \int_0^T (|X_t| - n)^p I_{\{|X_t| > \frac{n}{2}\}} dt \to 0. \]

Consequently \( \square \) holds true for \( X \in M^p_2(0, T) \). ■

**Corollary 3.10** For each \( \eta \in M^p_2(0, T) \), let \( \eta^p = (-n) \vee (\eta \wedge n) \), then we have \( \int_0^T \eta^p dB_s \to \int_0^T \eta dB_s \) in \( M^p_2(0, T) \) for each \( t \leq T \).

**Proposition 3.11** Let \( X \in M^p_2(0, T) \). Then for each \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that for all \( \eta \in M^0_{p, 0}(0, T) \) satisfying \( \hat{E} \int_0^T |\eta_t| dt \leq \delta \) and \( |\eta_t(\omega)| \leq 1 \), we have \( \hat{E} \int_0^T |X_t|^p |\eta_t|| \leq \varepsilon. \)

**Proof.** For each \( \varepsilon > 0 \), by Proposition \( \square \) there exists \( N > 0 \) such that \( \hat{E} \int_0^T |X|^p I_{\{|X| > N\}} \leq \frac{\varepsilon}{2} \). Take \( \delta = \frac{\varepsilon}{2N^pT} \). Then we have
\[
\hat{E} \int_0^T |X_t|^p |\eta_t|| dt \leq \hat{E} \int_0^T |X_t|^p |\eta_t| I_{\{|X_t| > N\}} dt + \hat{E} \int_0^T |X_t|^p |\eta_t| I_{\{|X_t| \leq N\}} dt
\leq \hat{E} \int_0^T |X_t|^p I_{\{|X_t| > N\}} dt + N^p \hat{E} \int_0^T |\eta_t| dt \leq \varepsilon.
\]

■

**Lemma 3.12** If \( p \geq 1 \), \( X, \eta \in M^p_2(0, T) \) such that \( \eta \) is bounded, then \( X \eta \in M^p_2(0, T) \).

**Proof.** Let \( c > 0 \) be such that \( |\eta_t(\omega)| \leq c \), for \( \omega \in \Omega, t \in [0, T] \). Then we have
\[
\hat{E} \int_0^T |\eta_t| X_t I_{\{|\eta_t X_t| > N\}} dt \leq c^p \hat{E} \int_0^T |X_t| I_{\{|\eta_t X_t| > N\}} dt
\leq c^p \hat{E} \int_0^T |X_t| I_{\{|X_t| > \frac{N}{c}\}} dt \to 0, \text{ as } N \to \infty.
\]
It follows that the bounded \( \left\{ (\omega) \right\}_{n=1}^\infty := \{(-n) \vee (n \wedge (\eta X))\}_{n=1}^\infty \) is a Cauchy sequence in \( M^p_2(0, T) \). ■

**Remark 3.13** It is easy to prove that if \( \eta \in M^2_2(0, T) \), then \( \int_0^T \eta_t dB_s \in M^2_2(0, T) \).
4 Ito’s integral with stopping times

In this section we study Ito’s integral on an interval \([0, \tau]\), where \(\tau\) is a stopping time. Reader can see that, thanks to Propositions 3.9, 3.11 and Lemma 3.12, the techniques used in this section is very similar to the classical situation.

Definition 4.1 A stopping time \(\tau\) relative to the filtration \((\mathcal{F}_t)\) is a map on \(\Omega\) with values in \([0, T]\), such that for every \(t\), \(\{\tau \leq t\} \in \mathcal{F}_t\).

Lemma 4.2 For each stopping time \(\tau\), we have \(\int_0^{\tau}(\cdot) \in M^p(0, T)\), for each \(X \in M^p(0, T)\).

Proof. For the given stopping time \(\tau\), let

\[
\tau_n = \sum_{k=0}^{2^n-1} \frac{k}{2^n} I_{[\frac{2^n}{k+1}T \leq \tau < \frac{2^n}{k}T]} + T I_{[\tau \geq T]}.
\]

Then we have \(2^{-n} \geq \tau_n - \tau \geq 0\). It is clear that, for \(m \geq n\),

\[
\hat{E} \int_T |I_{[\tau_n]}(t) - I_{[\tau_m]}(t)| dt \leq \hat{E} \int_T |I_{[\tau_n]}(t) - I_{[\tau_m]}(t)| dt = \hat{E}[\tau_n - \tau] \leq 2^{-n} T.
\]

It follows from Proposition 3.11 that For each \(\tau_n\), it is easy to check that \(\{I_{[\tau_n]}X\}_{n=1}^{\infty}\) is a Cauchy sequence in \(M^p(0, T)\). Thus \(I_{[\tau]}X \in M^p(0, T)\).

Lemma 4.3 For each \(\eta \in M^p(0, T)\) and \(\tau\) be a stopping time, then

\[
\int_0^{t \wedge \tau} \eta_s dB_s = \int_0^t I_{[\tau]}(s) \eta_s dB_s, \ \text{quasi-surely.} \quad (9)
\]

Proof. For each \(n \in \mathbb{N}\), let

\[
\tau_n := \sum_{k=1}^{2^n} \frac{k}{2^n} I_{[\frac{t}{2^n} < \tau \leq \frac{t}{2^{n-1}}]} + t I_{[\tau \geq t]} = \sum_{k=1}^{2^n} I_{A^k_n t_n}.
\]

where \(t_n^k = k2^{-n} t\), \(A^k_n = [t^{k-1}_n < \tau \leq t^k_n]\), for \(k < 2^n\), and \(A^{2^n}_n = [\tau \geq t]\). \(\{\tau_n\}_{n=1}^{\infty}\) is a decreasing sequence of stopping times which converges q.s. to \(t \wedge \tau\).

We first prove that

\[
\int_{\tau_n}^t \eta_s dB_s = \int_0^t I_{[\tau_n]}(s) \eta_s dB_s. \quad (10)
\]
Remark 4.5

But by Proposition 3.5 we have

\[
\int_{\tau_n}^{t} \eta_s dB_s = \sum_{k=1}^{2^n} I_{A_{n,k}} \eta_s dB_s = \sum_{k=1}^{2^n} I_{A_{n,k}}^{\infty} \int_{\tau_n}^{t} \eta_s dB_s
\]

\[
= \sum_{k=1}^{2^n} I_{A_{n,k}} \eta_s dB_s
\]

\[
= \int_{0}^{t} \sum_{k=1}^{2^n} I_{(t_n, t]}(\omega) I_{A_{n,k}} \eta_s dB_s,
\]

from which (10) follows. We thus have

\[
\int_{0}^{t} \eta_s dB_s = \int_{0}^{\tau_n} \eta_s dB_s.
\]

Observe that \(0 \leq \tau_n - \tau_m \leq \tau_n - t \wedge \tau \leq 2^{-m} t\) for \(n \leq m\), this with Proposition 3.11 it follows that \(I_{[0, \tau_n]} \eta\) converges in \(M^p_\mathbb{L}(0, T)\) to \(I_{[0, T \wedge \tau]} \eta\) and thus \(I_{[0, T \wedge \tau]} \eta \in M^p_\mathbb{L}(0, T)\). Consequently,

\[
\lim_{n \to \infty} \int_{0}^{\tau_n} \eta_s dB_s = \int_{0}^{t \wedge \tau} \eta_s dB_s, \quad \text{ quasi-surely}
\]

and (9) holds as well. \(\blacksquare\)

Definition 4.4

Let \(p > 0\) be fixed. A stochastic process \((\eta_t)_{t \geq 0}\) with \(\eta_t \in L^p(\Omega, \mathbb{L})\) is said to be in \(M^p_\mathbb{L}(0, T)\) if there exists a sequence of increasing stopping times \(\{\sigma_m\}_{m=1}^{\infty}\), with \(\sigma_m \uparrow T\), quasi-surely, such that \(\eta I_{[0, \sigma_m]} \in M^p_\mathbb{L}(0, T)\) and

\[
\inf_{P \in \mathcal{P}} \mathbb{E} \left( \int_{0}^{T} |\eta_s|^p ds \right) < \infty = 1.
\]

Remark 4.5

In the rest of this paper the notation \(\{\sigma_m\}_{m=1}^{\infty}\) is used to denote the sequence of the corresponding process \(\eta \in M^p_\mathbb{L}(0, T)\). Let \(\{\tau_m\}_{m=1}^{\infty}\) be another sequence of increasing stopping times with \(\tau_m \uparrow T\), quasi-surely. Then it is easy to check that \(\eta I_{[0, \sigma_m \wedge \tau_m]} \in M^p_\mathbb{L}(0, T)\) and \(\sigma_m \wedge \tau_m \uparrow T\), quasi-surely. Thus we can as well use \(\sigma_m \wedge \tau_m\) in the place of \(\sigma_m\). For example, when we consider two processes \(\eta, \tilde{\eta} \in M^p_\mathbb{L}(0, T)\) with two sequences of stopping times \(\{\sigma_m\}_{m=1}^{\infty}\) and \(\{\tilde{\sigma}_m\}_{m=1}^{\infty}\), we may only use one sequence \(\{\sigma_m \wedge \tilde{\sigma}_m\}_{m=1}^{\infty}\) for both \(\eta\) and \(\tilde{\eta}\).

Lemma 4.6

Let \(\eta \in M^p_\mathbb{L}(0, T)\) be given and let

\[
\tau_n = \inf\{t \geq 0, \int_{0}^{t} |\eta_s| ds > n\} \wedge \sigma_n.
\]

Then \(\eta I_{[0, \tau_n]} \in M^p_\mathbb{L}(0, T)\) and \(\int_{0}^{t} I_{[0, \tau_n]}(\omega) \eta_s ds\) and \(\int_{0}^{t} I_{[0, \tau_n]}(\omega) \eta_s dB_s\) are well-defined processes which are continuous on \([0, T]\) quasi-surely.
The proof is similar to that of the following proposition.

**Proposition 4.7** let \( \tau_n = \inf \{ t \geq 0, \int_0^t |\eta_s|^2 ds > n \} \wedge \sigma_n \) and \( \Omega_n = \{ \tau_n = T \} \). Then \( \eta I_{[0, \tau_n]} \in M^2_\ast(0, T) \) and the stochastic process \( (\int_0^t \eta_s dB_s)_{t \in [0, T]} \) is a well-defined quasi-surely continuous process defined on \( \Omega \).

**Proof.** Since, for each \( n = 1, 2, \cdots, \eta I_{[0, \tau_n]} \in M^2_\ast(0, T) \), so the Itô’s integral \( \int_0^t I_{[0, \tau_n]}(s)\eta_s dB_s \) is well-defined. On the other hand, on the subset \( \Omega_n = \{ \tau_n = T \} \) and for each \( m > n \), we have \( \tau_m = \tau_n = T \). Thus

\[
\lim_{m \to \infty} I_{\Omega_n} \int_0^{\tau_m \wedge t} \eta_s dB_s = I_{\Omega_n} \int_0^{\tau_n \wedge t} \eta_s dB_s = I_{\Omega_n} \int_0^t I_{[0, \tau_n]}(s)\eta_s dB_s, \ t \in [0, T].
\]

Thus on \( \Omega_n \) the process \( (\int_0^t \eta_s dB_s)_{t \in [0, T]} \) is a well-defined process which is continuous in \( t \) quasi-surely. Since \( \Omega_n \uparrow \bar{\Omega} \subset \Omega \), with \( c(\bar{\Omega}^c) = 0 \). It follows \( (\int_0^t \eta_s dB_s)_{t \in [0, T]} \) can is a well-defined process which is continuous in \( t \) quasi-surely. \( \blacksquare \)

**Corollary 4.8** We assume that \( \varphi \in C^{1,2}([0, \infty) \times \mathbb{R}) \) and all first and second order derivatives of \( \varphi \) with respect to \((t, x)\) are bounded. Let \( \alpha, \eta, \beta \in M^2_\ast(0, T) \) and \( X_t = \int_0^t \alpha_s ds + \int_0^t \eta_s d\langle B \rangle_s + \int_0^t \beta_s dB_s, \ t \in [0, T] \). Then, for each \( \varphi \in C(\mathbb{R}) \) and \( \gamma \in M^2_\ast(0, T) \), \( \varphi(X)\gamma \in M^2_\ast(0, T) \).

The proof is easy since \( (X_t)_{t \geq 0} \) is a quasi-surely continuous process.

## 5 Itô’s Formula

**Lemma 5.1** We assume that \( \varphi \in C^2(\mathbb{R}^n) \) and all first and second order derivatives of \( \varphi \) with respect to \( x \) are bounded. Let \( X = (X^1, \cdots, X^n) \) and

\[
X_t^i = X_0 + \int_0^t \alpha_s^i ds + \int_0^t \eta_s^i d\langle B \rangle_s + \int_0^t \beta_s^i dB_s, \ i = 1, \cdots, n.
\]

where \( \alpha, \beta, \eta \) are bounded elements in \( M^2_\ast(0, T) \). Then for each \( t \geq 0 \), we have

\[
\varphi(X_t) - \varphi(X_0) = \int_0^t \partial_{x_i} \varphi(X_u) \beta_u^i dB_u + \int_0^t \partial_{x_i} \varphi(X_u) \alpha_u^i du
\]

\[
+ \int_0^t \partial_{x_i} \varphi(X_u) \eta_u^i + \frac{1}{2} \partial_{x_i x_j} \varphi(X_u) \beta_u^i \beta_u^j dB_u.
\]

Here and in the rest of this paper we use the Einstein convention, i.e., the above repeated indices of \( i \) and \( j \) within one term imply the summation from 1 to \( n \).

The proof will be given in the appendix.
Lemma 5.2 Let $\varphi \in C^2(\mathbb{R}^n)$ and its first and second derivatives are in $C_{b,Lip}(\mathbb{R}^n)$. Let $X_t^1 = X_0^1 + \int_0^t \alpha_s^1 ds + \int_0^t \eta_s^1 dB_s + \int_0^t \beta_s^1 d\beta_s$, where $\alpha, \eta$ in $M_1^1(0, T)$, $\beta$ in $M_2^2(0, T)$. Then for each $t \geq 0$, we have

$$\varphi(X_t) - \varphi(X_0) = \int_0^t \partial_x \varphi(X_u) \beta_u^1 dB_u + \int_0^t \partial_x \varphi(X_u) \alpha_u^1 du + \int_0^t \left[ \partial_x \varphi(X_u) \eta_u^1 + \frac{1}{2} \partial_{xx} \varphi(X_u) \beta_u^1 \beta_u^1 \right] dB_u.$$  

(11)

Proof. For simplicity, we only state for the case where $n = 1$. Let $\alpha^{(k)}$, $\beta^{(k)}$, and $\eta^{(k)}$ bounded processes such that, as $k \to \infty$,

$$\alpha^{(k)} \to \alpha, \eta^{(k)} \to \eta \text{ in } M_1^1(0, T) \text{ and } \beta^{(k)} \to \beta, \text{ in } M_2^2(0, T)$$

and let

$$X_t^{(k)} = X_0 + \int_0^t \alpha_u^{(k)} ds + \int_0^t \eta_u^{(k)} dB_s + \int_0^t \beta_u^{(k)} d\beta_s.$$

Then we have

$$\lim_{k \to \infty} \hat{E}[ \sup_{0 \leq t \leq T} |X_t - X_t^{(k)}|^2 ] = 0.$$

We see that

$$\hat{E} \int_0^T |\partial_x \varphi(X_t^{(k)}) \beta_t^{(k)} - \partial_x \varphi(X_t) \beta_t|^2 dt \leq \hat{E} \int_0^T |\partial_x \varphi(X_t^{(k)}) \beta_t^{(k)} - \partial_x \varphi(X_t^{(k)}) \beta_t|^2 dt$$

$$+ \hat{E} \int_0^T |\partial_x \varphi(X_t^{(k)}) \beta_t - \partial_x \varphi(X_t) \beta_t|^2 dt$$

$$\leq C \hat{E} \int_0^T |\beta_t^{(k)} - \beta_t|^2 dt$$

$$+ \hat{E} \int_0^T |\beta_t^2 |\partial_x \varphi(X_t^{(k)}) - \partial_x \varphi(X_t)|^2 dt].$$

But we have $\sup_{0 \leq t \leq T} |\partial_x \varphi(X_t^{(k)}) - \partial_x \varphi(X_t)|^2 \leq c$ and

$$\hat{E} \int_0^T |\partial_x \varphi(X_t^{(k)}) - \partial_x \varphi(X_t)|^2 dt \to 0, \text{ as } k \to \infty.$$

Thus we can apply Proposition 3.11 to prove that $\partial_x \varphi(X^{(k)}) \beta^{(k)} \to \partial_x \varphi(X) \beta$ in $M_2^2(0, T)$. Similarly, $\partial_x \varphi(X^{(k)}) \alpha^{(k)} \to \partial_x \varphi(X) \alpha$, $\partial_x \varphi(X^{(k)}) \eta^{(k)} \to \partial_x \varphi(X) \eta$ and $\partial_x \varphi(X^{(k)}) \beta^{(k)} \to \partial_x \varphi(X) \beta$ in $M_1^1(0, T)$. But from the above lemma we have

$$\varphi(X_t^{(k)}) - \varphi(X_0^{(k)}) = \int_0^t \partial_x \varphi(X_u^{(k)}) \beta_u^{(k)} dB_u + \int_0^t \partial_x \varphi(X_u^{(k)}) \alpha_u^{(k)} du$$

$$+ \int_0^t \left[ \partial_x \varphi(X_u^{(k)}) \eta_u^{(k)} + \frac{1}{2} \partial_{xx} \varphi(X_u^{(k)}) \beta_u^{(k)} \beta_u^{(k)} \right] dB_u.$$
Lemma 5.3 Let $X$ be given as the above lemma and let $\varphi \in C^{1,2}([0, \infty) \times \mathbb{R}^n)$ such that $\varphi$, $\partial_t \varphi$, $\partial_x \varphi$ and $\partial^2_{xx} \varphi$ are bounded and uniformly continuous on $[0, \infty) \times \mathbb{R}^n$. Then we have

$$
\varphi(t, X_t) - \varphi(0, X_0) = \int_0^t \partial_x \varphi(u, X_u) \beta_u^1 dB_u + \int_0^t [\partial_t \varphi(u, X_u) + (\partial_x \varphi(X_u) \alpha_u^i)] du
$$

$$
+ \int_0^t [\partial_x \varphi(X_u) \eta_u^i + \frac{1}{2} \partial^2_{x,x} \varphi(X_u) \beta_u^1 \beta_u^1] d(B)_u.
$$

Proof. We can take $\{\varphi_k\}_{k=1}^\infty$ such that, for each $k$, $\varphi_k$ and all its first order and second order derivatives are in $C^{2,2}_{b, Lip}((-\infty, \infty) \times \mathbb{R}^n)$ and such that, as $n \to \infty$, $\varphi_k$, $\partial_t \varphi_k$, $\partial_x \varphi_k$ and $\partial^2_{xx} \varphi_k$ converge respectively to $\varphi$, $\partial_t \varphi$, $\partial_x \varphi$ and $\partial^2_{xx} \varphi$ uniformly on $[0, \infty) \times \mathbb{R}$. We then use the above Itô’s formula to $\varphi_k(X_t^0, X_t)$, with $Y_t = (X_t^0, X_t)$, with $X_t^0 \equiv t$:

$$
\varphi_k(t, X_t) - \varphi_k(0, X_0) = \int_0^t \partial_x \varphi_k(u, X_u) \beta_u^1 dB_u + \int_0^t [\partial_t \varphi_k(u, X_u) + (\partial_x \varphi_k(u, X_u) \alpha_u^i)] du
$$

$$
+ \int_0^t [\partial_x \varphi_k(u, X_u) \eta_u^i + \frac{1}{2} \partial^2_{x,x} \varphi_k(u, X_u) \beta_u^1 \beta_u^1] d(B)_u.
$$

It follows that, as $k \to \infty$, we have uniformly

$$
|\partial_x \varphi_k(u, X_u) - \partial_x \varphi(u, X_u)| \to 0, \quad |\partial^2_{x,x} \varphi_k(u, X_u) - \partial^2_{x,x} \varphi(u, X_u)| \to 0,
$$

$$
|\partial_t \varphi_k(u, X_u) - \partial_t \varphi(u, X_u)| \to 0.
$$

We then can apply the above Lemma to $\varphi_k(t, X_t) - \varphi_k(0, X_0)$ and pass to the limit as $k \to \infty$ to obtain the desired result. $
$

Theorem 5.4 Let $\varphi \in C^{1,2}([0, \infty) \times \mathbb{R})$ and $X_t = X_0 + \int_0^t \alpha_s ds + \int_0^t \eta_s dB_s$, where $\alpha, \eta \in M^2(0, T)$ and $\beta \in M^2(0, T)$. Then for each $t \geq 0$, we have

$$
\varphi(t, X_t) - \varphi(0, X_0) = \int_0^t \partial_x \varphi(u, X_u) \beta_u^1 dB_u + \int_0^t [\partial_t \varphi(u, X_u) + \partial_x \varphi(u, X_u) \alpha_u^i] du
$$

$$
+ \int_0^t [\partial_x \varphi(u, X_u) \eta_u^i + \frac{1}{2} \partial^2_{x,x} \varphi(u, X_u) \beta_u^1 \beta_u^1] d(B)_u.
$$

Proof. We set, for $k = 1, 2, \ldots$,

$$
\gamma_t = |X_t - X_0| + \int_0^t (|\beta_u|^2 + |\alpha_u| + |\eta_u|) du
$$

and $\tau_k := \inf\{t \geq 0|\gamma_t > k\} \wedge \sigma_k$. Let $\varphi_k$ be a $C^{1,2}$-function on $[0, \infty) \times \mathbb{R}^n$ such that $\varphi$, $\partial_t \varphi$, $\partial_x \varphi$ and $\partial^2_{xx} \varphi$ are uniformly bounded and such that $\varphi_k = \varphi$, for $|x| \leq 2k, t \in [0, T]$. It is clear that

$$
\mathbf{I}_{[0, \tau_k]} \beta \in M^2(0, T), \quad \mathbf{I}_{[0, \tau_k]} \alpha, \quad \mathbf{I}_{[0, \tau_k]} \eta \in M^1(0, T)
$$
and we have

\[ X^{t}_{t \wedge \tau_{k}} = X^{t}_{0} + \int_{0}^{t} \alpha_{s}^{t}_{[0, \tau_{k}]} ds + \int_{0}^{t} \eta_{s}^{t}_{[0, \tau_{k}]} dB_{s} + \int_{0}^{t} \beta_{s}^{t}_{[0, \tau_{k}]} dB_{s} \]

We then can apply the above lemma to \( \varphi_{k}(s, X_{s \wedge \tau_{k}}) \), \( s \in [0, t] \) to obtain

\[ \varphi(t, X_{t \wedge \tau_{k}}) - \varphi(0, X_{0}) = \int_{0}^{t} \partial_{s} \varphi(u, X_{u}) \beta_{s}^{t}_{[0, \tau_{k}]} dB_{s} + \int_{0}^{t} \partial_{s} \varphi(u, X_{u}) \alpha_{s}^{t}_{[0, \tau_{k}]} dB_{s} + \int_{0}^{t} \partial_{s} \varphi(u, X_{u}) \eta_{s}^{t}_{[0, \tau_{k}]} dB_{s} \]

Passing to the limit as \( k \to \infty \) and applying Corollary 4.8 we then obtain the desired result.

**Example 5.5** For a given \( \varphi \in C^{2}(\mathbb{R}) \) we have

\[ \varphi(B_{t}) - \varphi(B_{t_{0}}) = \int_{t_{0}}^{t} \varphi_{x}(B_{s}) dB_{s} + \frac{1}{2} \int_{0}^{t} \varphi_{xx}(B_{s}) d\langle B \rangle_{s} . \]

6  **Appendix: Proof of Lemma 5.1**

The proof is of Lemma 5.1 is very similar to those of Lemma 4.6 and proposition 48 in Peng [16] (see also [18]). We first consider the following simple case.

**Lemma 6.1** Let \( \Phi \in C^{2}(\mathbb{R}^{n}) \) with \( \partial_{x_{\nu}^{1}} \Phi, \partial_{x_{\nu}^{2}}^{2} \Phi \in C_{b, Lip}(\mathbb{R}^{n}) \) for \( \mu, \nu = 1, \ldots, n \). Let \( s \in [0, T] \) be fixed and let \( X = (X^{1}, \cdots, X^{n})^{T} \) be an \( n \)-dimensional process on \([s, T]\) of the form

\[ X^{j}_{s} = X^{j}_{s_{0}} + \alpha^{j}(t - s) + \eta^{j}(\langle B \rangle_{t} - \langle B \rangle_{s}) + \beta^{j}(B_{t} - B_{s}) , \]

where, for \( j = 1, \ldots, n \), \( \alpha^{j} \), \( \eta^{j} \) and \( \beta^{j} \) are bounded elements in \( L_{s}^{2}(\Omega_{s}) \) and \( X_{s} = (X^{1}_{s}, \cdots, X^{n}_{s})^{T} \) is a given random vector in \( L_{s}^{2}(\Omega_{s}) \). Then we have, in \( L_{s}^{2}(\Omega_{s}) \)

\[ \Phi(X_{t}) - \Phi(X_{s}) = \int_{s}^{t} \partial_{x^{j}} \Phi(X_{u}) \beta^{j} dB_{u} + \int_{s}^{t} \partial_{x^{j}} \Phi(X_{u}) \alpha^{j} du + \int_{s}^{t} \partial_{x^{j}} \Phi(X_{u}) \eta^{j} + \frac{1}{2} \partial_{x_{\nu}^{2}}^{2} \Phi(X_{u}) \beta^{j} \beta^{j} d \langle B \rangle_{u} . \]  

**Proof.** For each positive integer \( N \) we set \( \delta = (t - s)/N \) and take the partition

\[ \pi_{s, t}^{N} = \{t_{0}^{N}, t_{1}^{N}, \cdots, t_{N}^{N}\} = \{s, s + \delta, \cdots, s + N \delta = t\} . \]
We have
\[
\Phi(X_t) - \Phi(X_s) = \sum_{k=0}^{N-1} [\Phi(X_{i_k}^N) - \Phi(X_{i_k}^N)]
\]
\[
= \sum_{k=0}^{N-1} \{ \partial_{x^k} \Phi(X_{i_k}^N)(X_{i_k}^N - X_{i_k}^N) + \frac{1}{2} \partial_{x^k,x^j} \Phi(X_{i_k}^N)(X_{i_k}^N - X_{i_k}^N)(X_{i_k}^N - X_{i_k}^N) + \eta_{k}^N \},
\]
where
\[
\eta_{k}^N = [\partial_{x^k,x^l} \Phi(X_{i_k}^N + \theta_k(X_{i_k}^N - X_{i_k}^N)) - \partial_{x^k,x^l} \Phi(X_{i_k}^N)](X_{i_k}^N - X_{i_k}^N)(X_{i_k}^N - X_{i_k}^N)
\]
with \( \theta_k \in [0, 1] \). We have
\[
\hat{E}[|\eta_{k}^N|^2] = \hat{E}[|\partial_{x^k,x^l} \Phi(X_{i_k}^N + \theta_k(X_{i_k}^N - X_{i_k}^N)) - \partial_{x^k,x^l} \Phi(X_{i_k}^N)|^2]
\]
\[
\leq c\hat{E}[|X_{i_k}^N - X_{i_k}^N|^2] \leq C[\delta^6 + \delta^3],
\]
where \( c \) is the Lipschitz constant of \( \{ \partial_{x^k,x^l} \Phi \}_{i,j=1}^N \) and \( C \) is a constant independent of \( k \). Thus
\[
\hat{E}[\sum_{k=0}^{N-1} \eta_{k}^N] \leq N \sum_{k=0}^{N-1} \hat{E}[|\eta_{k}^N|^2] \to 0.
\]

The rest terms in the summation of the right side of (13) are \( \xi_{t_k}^N + \zeta_{t_k}^N \) with
\[
\xi_{t_k}^N = \sum_{k=0}^{N-1} \{ \partial_{x^k} \Phi(X_{i_k}^N)[\alpha^i(t_{k+1}^N - t_k^N) + \eta_i((B)_{t_{k+1}^N} - (B)_{t_k^N})]
\]
\[
+ \beta^j(B_{t_{k+1}^N}^N - B_{t_k^N}^N) + \frac{1}{2} \partial_{x^k,x^j} \Phi(X_{i_k}^N)\beta^j(B_{t_{k+1}^N}^N - B_{t_k^N}^N)^2 \}
\]
and
\[
\zeta_{t_k}^N = \sum_{k=0}^{N-1} \partial_{x^k,x^l} \Phi(X_{i_k}^N)[\{ \alpha^l(t_{k+1}^N - t_k^N) + \eta_i((B)_{t_{k+1}^N} - (B)_{t_k^N})]
\]
\[
\times [\alpha^l(t_{k+1}^N - t_k^N) + \eta_i((B)_{t_{k+1}^N} - (B)_{t_k^N})]
\]
\[
+ 2[\alpha^l(t_{k+1}^N - t_k^N) + \eta_i((B)_{t_{k+1}^N} - (B)_{t_k^N})] \beta^l(B_{t_{k+1}^N}^N - B_{t_k^N}^N) \}.
\]

We observe that, for each \( u \in [t_k^N, t_{k+1}^N) \)
\[
\hat{E}[|\partial_{x^k} \Phi(X_u) - \partial_{x^k} \Phi(X_{i_k}^N)|^2] \leq c^2 \hat{E}[|X_u - X_{i_k}^N|^2] \leq C[\delta^6 + \delta^3],
\]

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where \(c\) is the Lipschitz constant of \(\{\partial_x \Phi\}_{j=1}^n\) and \(C\) is a constant independent of \(k\). Thus \(\sum_{k=0}^{N-1} \partial_{x^j} \Phi(X_{\hat{k}}^N) 1_{[t_{\hat{k}}^N, t_{\hat{k}+1}^N)}(\cdot)\) tends to \(\partial_{x^j} \Phi(X)\) in \(M^2_* (0,T, \Omega)\).

Similarly,

\[
\sum_{k=0}^{N-1} \partial_{x^j} \Phi(X_{\hat{k}}^N) 1_{[t_{\hat{k}}^N, t_{\hat{k}+1}^N)}(\cdot) \to \partial_{x^j} \Phi(X) \quad \text{in} \quad M^2_* (0,T, \Omega).
\]

Let \(N \to \infty\), from Lemma 4.6, Proposition 4.7 and Corollary 4.8 as well as the definitions of the integrations of \(dt\), \(dB\) and \(d(B)\) the limit of \(\xi^N\) in \(L^2_* (\Omega_t)\) is just the right hand side of (12). By the next Remark we also have \(\xi^N \to 0\) in \(L^2_* (\Omega_t)\). We then have proved (12).

Remark 6.2: In the proof of \(\xi^N \to 0\) in \(L^2_* (\Omega_t)\), we use the following estimates: for \(\psi^N \in M_{b,0}(0,T)\) and \(\varphi^N = \sum_{k=0}^{N-1} \xi^N \Phi_{t_{\hat{k}}^N, t_{\hat{k}+1}^N}(t)\), and \(\pi^N = \{t_{\hat{k}}^N, \cdots, t_{\hat{k}+1}^N\}\) such that \(\lim_{N \to \infty} \mu(\pi^N) = 0\) and \(\mathbb{E}[\sum_{k=0}^{N-1} |\xi^{N}_k|^2 (t_{\hat{k}+1}^N - t_{\hat{k}}^N)] \leq C\), for all \(N = 1, 2, \ldots\), we have \(\mathbb{E}[\sum_{k=0}^{N-1} \xi^N_k (t_{\hat{k}+1}^N - t_{\hat{k}}^N)^2] \to 0\), and

\[
\mathbb{E}[\sum_{k=0}^{N-1} \xi^N_k ((B)_t^N - (B)_t^N)(t_{\hat{k}+1}^N - t_{\hat{k}}^N)] \leq C \mathbb{E}[\sum_{k=0}^{N-1} |\xi^N_k|^2 (t_{\hat{k}+1}^N - t_{\hat{k}}^N)^3]
\]

\[
\mathbb{E}(N-1) \xi^N_k((B)_t^N - (B)_t^N)(t_{\hat{k}+1}^N - t_{\hat{k}}^N)^2)
\]

\[
\mathbb{E}(N-1) \xi^N_k((B)_t^N - (B)_t^N)(t_{\hat{k}+1}^N - t_{\hat{k}}^N)^3)
\]

as well as

\[
\mathbb{E}[\sum_{k=0}^{N-1} \xi^N_k (t_{\hat{k}+1}^N - t_{\hat{k}}^N)(B_{t_{\hat{k}+1}^N} - B_{t_{\hat{k}}^N})^2] \leq C \mathbb{E}[\sum_{k=0}^{N-1} |\xi^N_k|^2 (t_{\hat{k}+1}^N - t_{\hat{k}}^N)^2 |B_{t_{\hat{k}+1}^N} - B_{t_{\hat{k}}^N}|^2]
\]

\[
\mathbb{E}[\sum_{k=0}^{N-1} \xi^N_k (t_{\hat{k}+1}^N - t_{\hat{k}}^N)^2 \sigma^2(t_{\hat{k}+1}^N - t_{\hat{k}}^N)^2]
\]

\[
\mathbb{E}[\sum_{k=0}^{N-1} \xi^N_k (t_{\hat{k}+1}^N - t_{\hat{k}}^N)^2 \sigma^2(t_{\hat{k}+1}^N - t_{\hat{k}}^N)^2]
\]

\[
\mathbb{E}[\sum_{k=0}^{N-1} \xi^N_k (t_{\hat{k}+1}^N - t_{\hat{k}}^N)^2 \sigma^2(t_{\hat{k}+1}^N - t_{\hat{k}}^N)^2]
\]

\[
19
\]
and

\[
\hat{E}\left[\sum_{k=0}^{N-1} \xi_k^N (\langle B \rangle_{t_{k+1}^N} - \langle B \rangle_{t_k^N})(B_{t_{k+1}^N} - B_{t_k^N})^2\right] \\
\leq C \hat{E}\left[\sum_{k=0}^{N-1} |\xi_k^N|^2 (\langle B \rangle_{t_{k+1}^N} - \langle B \rangle_{t_k^N})^2 |B_{t_{k+1}^N} - B_{t_k^N}|^2\right] \\
\leq C \hat{E}\left[\sum_{k=0}^{N-1} |\xi_k^N|^2 \sigma^4 (t_{k+1}^N - t_k^N)^2\right] \to 0.
\]

We now give the proof of Lemma 5.1

Proof of Lemma 5.1. Let \( \Phi \in C^2(\mathbb{R}^n) \) with \( \partial_x \Phi, \partial^2_{x_i x_j} \Phi \in C_b.Lip(\mathbb{R}^n) \) for \( i, j = 1, \cdots, n \). Let \( \alpha^j, \beta^j \) and \( \eta^j \), \( j = 1, \cdots, n \), be bounded processes in \( M^2_2(0,T) \). We need to prove that

\[
\Phi(X_t^N) - \Phi(X_s^N) = \int_s^t \partial_x \Phi(X_u^N) \beta_u^i d\langle B \rangle_u + \int_s^t \partial_x \Phi(X_u^N) \alpha_u^i du \tag{14}
\]

We first consider the case where \( \alpha, \eta \) and \( \beta \) are step processes of the form

\[
\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega) I_{(t_k, t_{k+1})}(t).
\]

From the above Lemma, it is clear that (14) holds true. Now let

\[
X_{t,N}^i = X_t^i + \int_0^t \alpha_u^{i,N} ds + \int_0^t \eta_u^{i,N} d\langle B \rangle_s + \int_0^t \beta_u^{i,N} dB_s,
\]

where \( \alpha^N, \eta^N \) and \( \beta^N \) are uniformly bounded step processes that converge to \( \alpha, \eta \) and \( \beta \) in \( M^2_2(0,T) \) as \( N \to \infty \). From Lemma 6.1

\[
\Phi(X_{t,N}^i) - \Phi(X_s^N) = \int_s^t \partial_x \Phi(X_u^N) \beta_u^{i,N} d\langle B \rangle_u + \int_s^t \partial_x \Phi(X_u^N) \alpha_u^{i,N} du \tag{15}
\]

Since

\[
\hat{E}[|X_{t,N}^i - X_t^i|^2] \\
\leq C \hat{E}\left[\int_0^T [(\alpha_s^{i,N} - \alpha_s^j)^2 + |\eta_s^{i,N} - \eta_s^j|^2 + |\beta_s^{i,N} - \beta_s^j|^2] ds\right],
\]

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where \( C \) is a constant independent of \( N \). We then can prove that, in \( M^2_\mathbb{P}(0,T) \),

\[
\frac{\partial}{\partial x_j} \Phi(X^N) \eta_j^N \to \frac{\partial}{\partial x_j} \Phi(X_j) \eta_j^j,
\]

\[
\frac{\partial^2}{\partial x_i \partial x_j} \Phi(X^N) \beta^j_i^N \beta^i_j^N \to \frac{\partial^2}{\partial x_i \partial x_j} \Phi(X_j) \beta^j_i \beta^i_j,
\]

\[
\frac{\partial}{\partial x_j} \Phi(X^N) \alpha^j_j^N \to \frac{\partial}{\partial x_j} \Phi(X_j) \alpha^j_j,
\]

\[
\frac{\partial}{\partial x_j} \Phi(X^N) \beta^j_j^N \to \frac{\partial}{\partial x_j} \Phi(X_j) \beta^j_j.
\]

We then can pass limit in both sides of (15) to get (14). ■

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