Generalized Bochner formulas and Ricci lower bounds for sub-Riemannian manifolds of rank two

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Abstract

We study a new class of rank two sub-Riemannian manifolds encompassing Riemannian manifolds, CR manifolds with vanishing Webster-Tanaka torsion, orthonormal bundles over Riemannian manifolds, and graded nilpotent Lie groups of step two. These manifolds admit a canonical horizontal connection and a canonical sub-Laplacian. We construct on these manifolds an analogue of the Riemannian Ricci tensor and prove Bochner type formulas for the sub-Laplacian. As a consequence, it is possible to formulate on these spaces a sub-Riemannian analogue of the so-called curvature dimension inequality. Sub-Riemannian manifolds for which this inequality is satisfied are shown to share many properties in common with Riemannian manifolds whose Ricci curvature is bounded from below.

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1 Introduction

A sub-Riemannian manifold is a smooth Riemannian manifold $M$ equipped with a fiber inner product $g_R(\cdot, \cdot)$ on the tangent bundle $TM$ and a non-holonomic, or bracket generating, sub-bundle $\mathcal{H} \subset TM$. This means that if we denote by $L(\mathcal{H})$ the Lie algebra of the vector fields generated by the global $C^\infty$ sections of $\mathcal{H}$, then $\text{span}\{X(x) \mid X \in L(\mathcal{H})\} = T_x(M)$ for every $x \in M$. A piecewise smooth curve $\gamma : [a, b] \to M$ is called admissible, or horizontal, if it is tangent to $\mathcal{H}$, i.e. if $\gamma'(t) \in \mathcal{H}_{\gamma(t)}$, whenever $\gamma'(t)$ is defined. The horizontal length of $\gamma$ is defined as in Riemannian geometry

$$\ell_\mathcal{H}(\gamma) = \int_a^b g_R(\gamma'(t), \gamma'(t))dt.$$ 

Denoting by $\mathcal{H}(x, y)$ the collection of all horizontal curves joining $x, y \in M$, one defines a distance $d(x, y)$ between $x$ and $y$ by minimizing on the length of all $\gamma \in \mathcal{H}(x, y)$, i.e.

$$d(x, y) = \inf_{\gamma \in \mathcal{H}(x, y)} \ell_\mathcal{H}(\gamma).$$ 

Such distance was introduced by Carathéodory in his seminal paper [16] on formalization of the classical thermodynamics. In such framework horizontal curves correspond loosely speaking to adiabatic processes.

To be precise, in [16] the question of whether $d(x, y)$ be a true distance was left open. This question was answered by the fundamental connectivity theorem of Chow [20] and Rashevsky [54].
which states that if $\mathbb{M}$ is connected and $\mathcal{H}$ is bracket generating, then $\mathcal{H}(x, y) \neq \emptyset$ for every $x, y \in \mathbb{M}$. As a consequence, $d(x, y)$ is finite and therefore it is a true distance. Such metric is nowadays known as the control, or Carnot-Carathéodory distance on $\mathbb{M}$ (after Gromov, Lafontaine and Pansu [36], see also [34], [35]). Besides the cited references the reader should consult E. Cartan’s pioneering address [18] at the Bologna International Congress of Mathematicians in 1928, as well as the articles by Rayner [55] (where sub-Riemannian manifolds are called parabolic spaces), by Mitchell [50] and Strichartz [64], see also [65]. One should also consult the monographs [10], [51], [1] and [8].

We note that when $\mathcal{H} = T\mathbb{M}$, then the distance $d(x, y)$ is simply the Riemannian distance associated with the inner product $g_R(\cdot, \cdot)$, and thus sub-Riemannian manifolds encompass Riemannian ones. However, some aspects of the geometry of sub-Riemannian manifolds are considerably less regular than their Riemannian ancestors. Some of the major differences between the two geometries are the following:

1. The Hausdorff dimension of the metric space $(\mathbb{M}, d)$ is usually greater than the manifold dimension;
2. The exponential map defined by the geodesics of the metric space $(\mathbb{M}, d)$ is in general not a local diffeomorphism in a neighborhood of the point at which it is based (see [55]);
3. The space of horizontal paths joining two fixed points may have singularities (the so-called abnormal geodesics, see [51]).

During the last two decades there have been several advances in the study of sub-Riemannian spaces and the closely connected theory of sub-elliptic pde’s, see [62], [28], [58], [26], [52], [61], [39], [38], [27], [41], [42], [71]. However, these developments are of a local nature. As a consequence, the theory presently lacks a body of results which, similarly to the Riemannian case, connect properties of solutions of the relevant pde’s to the geometry of the ambient manifold.

The purpose of this paper is to begin a program aimed at filling this gap. Precisely, in a sub-Riemannian manifold of rank two we introduce a new notion of Ricci curvature tensor, and with such notion we obtain various results which parallel those of Riemannian manifolds with Ricci tensor bounded from below.

To put our work in the proper perspective we recall that in Riemannian geometry a first point of view on the Ricci tensor is to understand it as a measure of volume distortion in geodesic normal coordinates. This point of view has recently led several authors ( Lott-Villani [49], Sturm [66], [67], Ollivier [53]) to introduce on metric spaces more general than the Riemannian ones a suitable notion of Ricci curvature based on the theory of optimal transport (see [72]). However, as pointed out in [40], this notion of Ricci curvature bound may not be applied in a sub-Riemannian framework. In sub-Riemannian geometry, the quantification of volume distortion properties is particularly difficult to handle because of the singular nature of the exponential map. To the authors’ best knowledge, the only significant results in this direction are at the moment known only in the special case of three-dimensional contact manifolds (see [59], [40] and [2]).
There is a second point of view on the Riemannian Ricci tensor and this is the one that we have adopted in the present paper. On a Riemannian manifold, there is a canonical second order differential operator: the Laplace-Beltrami operator $\Delta$. Properties of this operator and of the associated heat flow are intimately related to the geometry and the topology of the underlying manifold. One of the cornerstones of the interplay between the analysis of the Laplace-Beltrami operator and geometry is given by the celebrated Bochner formula:

$$\Delta(|\nabla f|^2) = 2||\nabla^2 f||^2 + 2 < \nabla f, \nabla(\Delta f) > + 2 \text{Ric}(\nabla f, \nabla f),$$

This is where the Ricci curvature tensor appears in the study of the Laplace-Beltrami operator, and it is then seen that a lower bound assumption on the Ricci curvature is equivalent to a coercivity property of a canonical bilinear differential form associated to $\Delta$. More precisely, associated with $\Delta$ are the two following differential bilinear forms on smooth functions $f, g : \mathbb{M} \to \mathbb{R}$,

$$\Gamma(f, g) = \frac{1}{2}(\Delta(f g) - f \Delta g - g \Delta f) = (\nabla f, \nabla g),$$

and

$$\Gamma_2(f, g) = \frac{1}{2} \left[ \Delta \Gamma(f, g) - \Gamma(f, \Delta g) - \Gamma(g, \Delta f) \right].$$

As an application of the Bochner’s formula, which we can re-write as

$$\Delta \Gamma(f, f) = 2||\nabla^2 f||^2 + 2 \Gamma(f, \Delta f) + 2 \text{Ric}(\nabla f, \nabla f),$$

one obtains

$$\Gamma_2(f, f) = ||\nabla^2 f||^2 + \text{Ric}(\nabla f, \nabla f).$$

With the aid of Schwarz inequality, which gives $||\nabla^2 f||^2 \geq \frac{1}{d}(\Delta f)^2$, the assumption that the Riemannian Ricci tensor on $\mathbb{M}$ is bounded from below by $\rho \in \mathbb{R}$, translates then into the so-called curvature-dimension inequality:

$$\Gamma_2(f, f) \geq \frac{1}{d}(\Delta f)^2 + \rho \Gamma(f, f). \quad (1.1)$$

The inequality (1.1) perfectly captures the notion of Ricci curvature lower bound, see example 5.2 below. In the hands of D. Bakry, M. Ledoux and their co-authors it has proven to be a powerful tool in recovering most of the well-known theorems which, in Riemannian geometry, are obtained under the assumption that the Ricci curvature be bounded from below (see for instance [3], [45], [47]).

The purpose of this work is to generalize this point of view in a sub-Riemannian setting. More precisely, we assume that on $\mathbb{M}$ we are given smooth vector fields $X_1, \ldots, X_d$ which generate the horizontal subbundle $\mathcal{H}$. The commutators $[X_i, X_j]$ are supposed to satisfy some structural assumptions, see (2.3) and (2.4) below, with $Z_{mn}, m, n = 1, \ldots, h$, being the non-horizontal, or vertical directions. Setting

$$\mathcal{H}(x) = \text{span}\{X_1(x), \ldots, X_d(x)\}, \quad \mathcal{V}(x) = \text{span}\{Z_{mn}(x) \mid 1 \leq m < n \leq h\},$$
we assume that
\[ T_x M = \mathcal{H}(x) \oplus \mathcal{V}(x), \quad x \in M. \]
We also assume that \( \mathcal{H} \) is bracket-generating of rank two, i.e.
\[ T_x M = \text{span}\{X_i(x), [X_j, X_k](x)\}, \quad x \in M. \]
The first step will be to equip \( M \) with a canonical subelliptic operator
\[ L = X_0 + \sum_{i=1}^d X_i^2, \]
which shall play the role of the Laplace-Beltrami operator. We then equip \( M \) with a degenerate metric tensor \( g(\cdot, \cdot) \) for which \( \{X_1(x), ..., X_d(x)\} \) is orthonormal at every \( x \in M \), the spaces \( \mathcal{H}(x) \) and \( \mathcal{V}(x) \) are orthogonal, and such that \( g|_{\mathcal{V}} = 0 \). In such a manifold we introduce a canonical connection \( \nabla \) generalizing the Levi-Civita connection. One of the fundamental assumptions in this work is that the torsion be vertical. Besides the Riemannian case, basic examples covered by our framework are CR manifolds with vanishing Webster-Tanaka torsion (Sasakian manifolds), orthonormal bundles over Riemannian manifolds, and graded nilpotent Lie groups of step two.

An essential tool in our analysis are two sub-Riemannian Bochner formulas for \( L \), one in the horizontal direction and one in the vertical one. Such formulas are established in section \[4\]. We mention here that their technical complexity is the main \textit{raison d'être} of the above rank two assumption (in this regard, one should see the closing comments at the end of this introduction).

By means of these formulas we identify a tensor \( R \) which plays the role of the Riemannian Ricci tensor. We show in Proposition \[3.1\] that for every \( f \in C^\infty(M) \) one has
\[ R(f, f) = \sum_{\ell, k=1}^d \text{Ric}(X_\ell, X_k)X_\ell fX_k f - ((\nabla X_\ell T)(X_\ell, X_k)f)(X_k f) + \frac{1}{4}(T(X_\ell, X_k)f)^2. \]
where \( T \) is the torsion of the canonical connection \( \nabla \) and \( \text{Ric} \) its Ricci curvature. In fact, in the Riemannian case, one has
\[ R(f, f) = \text{Ric}(\nabla f, \nabla f), \quad f \in C^\infty(M). \]
A lower bound assumption on the tensor \( R \) will translate into a generalized curvature dimension inequality for \( L \) that writes in the form:
\[ \Gamma_2(f, f) + \nu \Gamma_2^Z(f, f) \geq \frac{1}{d}(Lf)^2 + \left(\rho_1 - \frac{\kappa}{\nu}\right) \Gamma(f, f) + \rho_2 \Gamma_Z^2(f, f), \quad \nu > 0, \quad (1.2) \]
where \( \Gamma^Z \) and \( \Gamma_2^Z \) are two bilinear differential forms on the vertical subbundle. The parameters of this inequality are \( \rho_1, \rho_2, \kappa \), and a recurrent theme of our work is that all estimates solely depend in an explicit quantitative way on these parameters. As shown in the following summary of the main results, the parameter \( \rho_1 \) is of particular importance:

1. \textbf{Dodziuk-Yau type theorem:} If the inequality \[(1.2)\] holds for some constants \( \rho_1 \in \mathbb{R}, \rho_2 > 0, \kappa > 0 \), then the heat semigroup \( P_t = e^{tL} \) is stochastically complete and bounded solutions of the heat equation are characterized by their initial condition;
2. **Li-Yau type inequality:** If the inequality (1.2) holds for some constants \( \rho_1 \in \mathbb{R}, \rho_2 > 0, \kappa > 0 \), then the heat semigroup \( P_t \) satisfies a Li-Yau type gradient estimate. Exploiting the latter, we prove that the heat kernel \( p(x,y,t) \) of \( P_t \) satisfies a uniform Harnack inequality and a Gaussian upper bound estimate.

3. **Yau-Liouville type theorem:** If the inequality (1.2) holds for some constants \( \rho_1 \geq 0, \rho_2 > 0, \kappa > 0 \), then there is no non constant bounded harmonic function.

4. **Volume and isoperimetry:** If the inequality (1.2) holds for some constants \( \rho_1 \geq 0, \rho_2 > 0, \kappa > 0 \), then the volume growth of the geodesic balls is at most polynomial. Moreover, under the assumption that the volume of balls is at least polynomial, we obtain a global isoperimetric inequality.

5. **Myers type theorem:** If the inequality (1.2) holds for some constants \( \rho_1 > 0, \rho_2 > 0, \kappa > 0 \), then the metric space \((M,d)\) is compact in the metric topology with a Hausdorff dimension less than \( d \left( 1 + \frac{3\kappa}{2\rho_2} \right) \) and we have

\[
diam M \leq 2\sqrt{3\pi} \sqrt{\frac{\kappa + \rho_2}{\rho_1 \rho_2} \left( 1 + \frac{3\kappa}{2\rho_2} \right)} d;
\]

6. **\( L^1 \) Poincaré inequality:** If the inequality (1.2) holds for some constants \( \rho_1 > 0, \rho_2 > 0, \kappa > 0 \), then the following inequality is satisfied

\[
\inf_{c \in \mathbb{R}} \int_M |f - c| d\mu \leq 6d \left( 1 + \frac{3\kappa}{2\rho_2} \right) \int_M \sqrt{\Gamma(f)} d\mu.
\]

7. **Lichnerowicz type estimate:** If the inequality (1.2) holds for some constants \( \rho_1 > 0, \rho_2 > 0, \kappa > 0 \), then the first non zero eigenvalue \( \lambda_1 \) of \(-L\) satisfies the estimate

\[
\lambda_1 \geq \frac{\rho_1 \rho_2}{d-1} d\rho_2 + \kappa.
\]

The basic source of the above listed results is the study of monotone entropy type functionals of the heat semigroup. More precisely, generalizing [5], [7] and [9], one of the most important observations is that in our framework, the inequality (1.2) implies that for the functionals

\[
\Phi_1(t) = P_t \left( (P_{T-t}f) \Gamma(\ln P_{T-t}f) \right),
\]

\[
\Phi_2(t) = P_t \left( (P_{T-t}f) \Gamma^Z(\ln P_{T-t}f) \right).
\]

we have the following differential inequality

\[
\left( -\frac{b'}{2\rho_2} \Phi_1 + b \Phi_2 \right)' \geq -\frac{2b' \gamma}{d\rho_2} L P_T f + \frac{b' \gamma^2}{d\rho_2} P_T f,
\]

where \( b \) is any smooth, positive and decreasing function on the time interval \([0,T]\) and

\[
\gamma = \frac{d}{4} \left( \frac{b''}{b'} + \frac{\kappa b'}{\rho_2 b} + 2\rho_1 \right).
\]
Depending on the value of \( \rho_1 \), a careful choice of the function \( b \) leads then to a generalized Li-Yau type inequality from which it is possible to deduce the above results.

Some final comments are in order. We have mentioned above that the choice of working with sub-Riemannian manifolds of rank two is closely connected with the complexity of the Bochner type formulas in section 4. On one hand, the rank two setting is rich enough to encompass at one time the case of Riemannian and CR (Sasakian) manifolds. On the other hand, following the program in section 4 for manifolds of arbitrary rank one would need to establish \( r \) Bochner type formulas (of decreasing complexity, the most difficult one being the horizontal one). Whereas it would be desirable to treat sub-Riemannian manifolds of arbitrary rank, we have felt that the increased technical difficulties connected with this endeavor would distract from the main ideas, and have consequently decided to defer the treatment of manifolds of rank \( \geq 3 \) to a future study. We should also mention that most of the constants appearing in the main results in this paper are not optimal. This can be seen by considering the special setting of graded nilpotent Lie groups where one can use the underlying non-isotropic dilations to obtain sharper constants. Finally, we mention that our methods strongly rely on the assumption that the torsion of the canonical connection is vertical.

In closing we mention that a pseudo-hermitian version of the Bonnet-Myers theorem for contact manifolds of dimension three was proved by Rumin, see Theorem 16 in [59]. We thank S. Webster for bringing this to our attention.

2 The framework and its geometrical interpretation

2.1 Preliminaries

Henceforth in this paper, \( M \) will be a smooth connected Riemannian manifold. We assume that \( X_1, \ldots, X_d \) are given smooth vector fields on \( M \) satisfying the following commutation relations:

\[
[X_i, X_j] = \sum_{\ell=1}^{d} \omega^\ell_{ij} X_\ell + \sum_{m,n=1}^{\mathfrak{h}} \gamma^m_{ij} Z_m, \quad (2.3)
\]

\[
[X_i, Z_{mn}] = \sum_{\ell=1}^{d} \delta^\ell_{imn} X_\ell, \quad (2.4)
\]

for some smooth vector fields \( \{Z_{mn}\}_{1 \leq m,n \leq \mathfrak{h}} \) and smooth functions \( \omega^\ell_{ij}, \gamma^m_{ij} \) and \( \delta^\ell_{imn} \). By convention \( Z_{mn} = -Z_{nm}, \omega^\ell_{ij} = -\omega^\ell_{ji} \) and \( \gamma^m_{ij} = -\gamma^m_{ji} \). We will assume that

\[
\delta^\ell_{imn} = -\delta^\ell_{imn}, \quad i, \ell = 1, \ldots, d, \text{ and } m,n = 1, \ldots, \mathfrak{h}. \quad (2.5)
\]

Note that (2.5) implies \( \delta^i_{imn} = 0 \) for \( i = 1, \ldots, d, \) and \( m,n = 1, \ldots, \mathfrak{h} \). The assumption (2.5) plays a pervasive role in the results of this paper.

We shall moreover assume that the vector fields \( X_i \)'s satisfy Hörmander’s finite rank condition of step two [37]: for every \( x \in M \),

\[
\text{span} \{X_i(x), [X_j, X_k](x), 1 \leq i \leq d, 1 \leq j < k \leq d \} = T_x M.
\]
We denote by
\[ H(x) = \text{span} \{X_1(x), ..., X_d(x)\}, \quad x \in \mathbb{M}, \]
the set of horizontal directions at \( x \), and by
\[ V(x) = \text{span} \{Z_{mn}(x), 1 \leq m < n \leq h\}, \]
that of vertical directions. We shall assume that
\[ \dim H(x) = d, \quad \dim V(x) = \frac{h(h-1)}{2}, \]
at each point \( x \in \mathbb{M} \), and that
\[ H(x) \oplus V(x) = T_x \mathbb{M}. \]
The dimension of \( \mathbb{M} \) is therefore \( d + \frac{h(h-1)}{2} \), but such number will never explicitly appear in the results in this paper. We indicate with \( H = \bigcup_{x \in \mathbb{M}} H(x) \), and \( V = \bigcup_{x \in \mathbb{M}} V(x) \), respectively the horizontal and vertical subbundles of \( T \mathbb{M} \).

Our goal is to study the second order subelliptic operator \( L \) given by
\[ L = \sum_{i=1}^{d} X_i^2 + X_0, \quad (2.6) \]
where
\[ X_0 = - \sum_{i,k=1}^{d} \omega_{ik} X_i. \quad (2.7) \]
We will assume that with respect to the Riemannian measure \( \mu \) of \( \mathbb{M} \),
\[ Z_{mn}^* = -Z_{mn}, \quad L^* = L, \quad (2.8) \]
where \( Z_{mn}^* \) denotes the formal adjoint of \( Z_{mn} \) and \( L^* \) the formal adjoint of \( L \). We note explicitly that (2.8) means that, if we set for every \( \phi, \psi \in C_0^\infty(\mathbb{M}) \)
\[ \langle \phi, \psi \rangle = \int_{\mathbb{M}} \phi \psi d\mu, \]
then one has
\[ \langle Z_{mn} \phi, \psi \rangle = -\langle \phi, Z_{mn} \psi \rangle, \quad \langle L \phi, \psi \rangle = \langle \phi, L \psi \rangle. \quad (2.9) \]
We recall that, thanks to Hörmander’s hypoellipticity theorem \[37\], distributional solutions to \( Lf = 0 \) in \( \mathbb{M} \) are \( C^\infty \) functions. We stress that, thanks to the second identity in the assumption (2.8), the operator \( L \) can be realized as
\[ L = - \sum_{i=1}^{d} X_i^* X_i. \quad (2.10) \]
We also assume that $\mathbb{M}$ is endowed with a Levi-Civita connection with respect to which the Laplace-Beltrami operator is given by

$$\Delta = \sum_{i=1}^{d} X_i^2 + \sum_{1 \leq m < n \leq h} Z_{mn}^2 + X_0. \quad \text{(2.11)}$$

Let us now give some examples that, at least locally, fit into the previous framework and that constitute a basic motivation for our study.

**Example 2.1** *(Laplace-Beltrami operator on a Riemannian manifold)* Let $(\mathbb{M}, g)$ be a $d$-dimensional connected Riemannian manifold with Levi-Civita connection $\nabla$. Let $X_1, \ldots, X_d$ be a local orthonormal frame around a point $x_0 \in \mathbb{M}$. In that case, we have

$$[X_i, X_j] = \nabla_{X_i} X_j - \nabla_{X_j} X_i = \sum_{k=1}^{d} \left( \Gamma^k_{ij} - \Gamma^k_{ji} \right) X_k$$

where $\Gamma^k_{ij}$ are the Christoffel symbols of the Levi-Civita connection. Thus, in this particular case we have $\gamma_{ij}^{mn} = 0$ in (2.3). The Laplace-Beltrami operator on $\mathbb{M}$ reads

$$\Delta = \sum_{i=1}^{d} X_i^2 + X_0,$$

where

$$X_0 = - \sum_{i,k=1}^{d} \left( \Gamma^k_{ik} - \Gamma^k_{ki} \right) X_i.$$

Finally, we can observe that $L$ is symmetric with respect to the Riemannian measure on $\mathbb{M}$.

**Example 2.2** *(Graded nilpotent Lie groups of step two)* Let $G$ be a connected and simply connected nilpotent Lie group of step two. This means that its Lie algebra can be written as $\mathfrak{g} = V_1 \oplus V_2$, where $[V_1, V_1] = V_2$, and $[V_1, V_2] = \{0\}$. Let $L_x(y) = xy$ be the operator of left-translation on $G$, and indicate with $dL_x$ its differential. If $e_1, \ldots, e_d$ is an orthonormal basis of $V_1$, we indicate with $X_1, \ldots, X_d$, where $X_j(x) = dL_x(e_j)$, the corresponding system of left-invariant vector fields on $G$. We assume that $G$ is endowed with a left-invariant Riemannian inner product with respect to which $\{X_1, \ldots, X_d\}$ constitutes a global orthonormal frame. In this framework, we see that (2.2) holds with

$$\omega^{\ell}_{ij} = 0,$$

$$Z_{mn} = [X_m, X_n],$$

$$\begin{cases} 
\gamma_{mn} = -\gamma_{nm} = \frac{1}{2}, & m \neq n, \\
\gamma_{ij} = 0, & \text{otherwise}, \\
\delta^l_{mn} = 0.
\end{cases}$$

In view of (2.10) we see that $L = \sum_{i=1}^{d} X_i^2$ is the sub-Laplacian associated with $X_1, \ldots, X_d$, see [28], [64]. In this case $L$ is symmetric with respect to the bi-invariant Haar measure on $\mathbb{M}$. 

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Example 2.3 [Horizontal Bochner Laplace operator] Let \((\mathbb{M}, g)\) be a \(d\)-dimensional connected smooth Riemannian manifold endowed with the Levi-Civita connection. Let us consider the orthonormal frame bundle \(\mathcal{O}(\mathbb{M})\) over \(\mathbb{M}\). For each \(x \in \mathbb{R}^d\) we can define a horizontal vector field \(H_x\) on \(\mathcal{O}(\mathbb{M})\) by the property that at each point \(u \in \mathcal{O}(\mathbb{M})\), \(H_x(u)\) is the horizontal lift of \(u(x)\) from \(u\). If \((e_1, \ldots, e_d)\) is the canonical basis of \(\mathbb{R}^d\), the fundamental horizontal vector fields are then defined by
\[
H_i = H_{e_i}.
\]
Now, for every \(M \in \mathfrak{o}_d(\mathbb{R})\) (space of \(d \times d\) skew symmetric matrices), we can define a vertical vector field \(V_M\) on \(\mathcal{O}(\mathbb{M})\) by
\[
(V_M F)(u) = \lim_{t \to 0} \frac{F(ue^{tM}) - F(u)}{t},
\]
where \(u \in \mathcal{O}(\mathbb{M})\) and \(F : \mathcal{O}(\mathbb{M}) \to \mathbb{R}\). If \(E_{ij}\), \(1 \leq i < j \leq d\) denote the canonical basis of \(\mathfrak{o}_d(\mathbb{R})\) (\(E_{ij}\) is the matrix whose \((i, j)\)-th entry is \(1/2\), \((j, i)\)-th entry is \(-1/2\) and all other entries are zero), then the fundamental vertical vector fields are given by
\[
V_{ij} = V_{E_{ij}}.
\]
It can be shown that we have the following Lie bracket relations:
\[
[H_i, H_j] = -2 \sum_{k<l} \Omega_{kl}^{ij} V_{kl},
\]
\[
[H_i, V_{jk}] = -\delta_{ij} \frac{1}{2} H_k + \delta_{ik} \frac{1}{2} H_j,
\]
where \(\delta_{ij} = 1\) if \(i = j\) and \(0\) otherwise, and where \(\Omega\) is the Riemannian curvature form:
\[
\Omega(X, Y)(u) = u^{-1} R(\pi_* X, \pi_* Y) u, \quad X, Y \in T_u \mathcal{O}(\mathbb{M}),
\]
\(R\) denoting the Riemannian curvature tensor on \(\mathbb{M}\) and \(\pi\) the canonical projection \(\mathcal{O}(\mathbb{M}) \to \mathbb{M}\).
In this setting, the Bochner’s horizontal Laplace operator is by definition the operator on \(\mathcal{O}(\mathbb{M})\) given by
\[
\Delta_{\mathcal{O}(\mathbb{M})} = \sum_{i=1}^d H_i^2.
\]
Its fundamental property is that it is the lift of the Laplace-Beltrami operator \(\Delta_\mathbb{M}\) of \(\mathbb{M}\). That is, for every smooth \(f : \mathbb{M} \to \mathbb{R}\),
\[
\Delta_{\mathcal{O}(\mathbb{M})}(f \circ \pi) = (\Delta_\mathbb{M} f) \circ \pi.
\]
For the reader unfamiliar with the above construction, we refer for instance to Chapter 3 in [8] for further details. It is also proved in the last reference that if the curvature form \(\Omega\) is everywhere non degenerate, then \(\Delta_{\mathcal{O}(\mathbb{M})}\) is subelliptic. Under this last assumption, it is then readily checked that the study of \(\Delta_{\mathcal{O}(\mathbb{M})}\) falls into our framework.
Example 2.4 [The subelliptic Laplace operator on CR manifolds] Let $M$ be a non degenerate CR manifold of real hypersurface type and dimension $d + 1$, where $d = 2n$. Let $\theta$ be a contact form on $M$ with respect to which the Levi form $L_\theta$ is positive definite. Let us assume that the pseudo-Hermitian torsion of the Tanaka-Webster connection of $(M, \theta)$ is zero. We denote by $T$ the characteristic direction of $\theta$ and consider a local orthonormal frame $T_1, \ldots, T_n$, that is $L_\theta(T_i, \overline{T_j}) = \epsilon_{ij}$ where $\epsilon_{ij}$ is the Kronecker symbol. The following commutations properties hold:

$$
[T_i, T_j] = \sum_{k=1}^{n} \left( \Gamma^k_{ij} - \Gamma^k_{ji} \right) T_k,
$$

$$
[T_i, \overline{T}_j] = -2\sqrt{-1}\epsilon_{ij} T + \sum_{k=1}^{n} \Gamma^k_{ij} T_k - \Gamma^k_{ji} \overline{T}_k,
$$

$$
[T, T_i] = \sum_{k=1}^{n} \Gamma^k_{0i} T_k,
$$

where the $\Gamma^k_{ij}$ are the Christoffel symbols of the Tanaka-Webster connection (see [25] pp. 32). The sub-Laplacian (locally) reads:

$$
\Delta = \sum_{k=1}^{n} T_i^* T_i + \overline{T}_i^* \overline{T}_i,
$$

where $T_i^*$ is the adjoint of $T_i$ with respect to the volume form $\theta \wedge (d\theta)^n$. If we denote

$$
X_i = \frac{1}{\sqrt{2}}(T_i + \overline{T}_i), \quad X_{i+n} = \frac{\sqrt{-1}}{\sqrt{2}}(T_i - \overline{T}_i), \quad 1 \leq i \leq n,
$$

then we fall into the previous framework.

2.2 Canonical connection

On $M$ there is a canonical connection associated with the geometry of the differential system generated by the vector fields $X_1, \ldots, X_d$. We consider the degenerate metric tensor $g$ on $M$, such that $\{X_1(x), \ldots, X_d(x)\}$ is orthonormal at each point $x \in M$, and for which the spaces $\mathcal{H}(x)$ and $\mathcal{V}(x)$ are orthogonal, and $g_{/\mathcal{V}(x)} = 0$.

Proposition 2.5 On $M$, there is a unique affine connection $\nabla$ that satisfies the following properties:

- $\nabla g = 0$;
- $\nabla X_i X_j$ is horizontal for $1 \leq i, j \leq d$;
- $\nabla Z_{mn} = 0$, $1 \leq m, n \leq d$;
- If $X, Y$ are horizontal vector fields, the torsion field $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ is vertical and $T(X_i, Z_{mn}) = 0$, $1 \leq i \leq d, 1 \leq m, n \leq h$. 

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This connection is characterized by the formulas:

\[ \nabla_{X_i} X_j = \sum_{k=1}^{d} \Gamma^k_{ij} X_k = \sum_{k=1}^{d} \frac{1}{2} \left( \omega^k_{ij} + \omega^j_{ki} - \omega^k_{jk} \right) X_k, \quad (2.12) \]

\[ \nabla_{Z_{mn}} X_i = -\sum_{\ell=1}^{d} \delta^\ell_{imn} X_\ell, \quad (2.13) \]

\[ \nabla Z_{mn} = 0, \quad (2.14) \]

where we have denoted by \( \Gamma^k_{ij} \) the Christoffel symbols of the connection.

**Proof.** First, it is easy to check that the connection given by the formulas (2.12), (2.13), (2.14) satisfies the above properties.

Let now \( \nabla \) be an affine connection that satisfies these properties. One first has

\[ \nabla_{X_i} X_j = \sum_{k=1}^{d} g(\nabla_{X_i} X_j, X_k) X_k \]

Next, by using the fact that \( \nabla g = 0 \) and that the torsion tensor has to be vertical, we easily obtain that the Koszul identity holds for \( \nabla \), that is

\[ g(\nabla_X Y, Z) = \frac{1}{2} \left\{ X g(Y, Z) + Y g(Z, X) - Z g(X, Y) \right\} \]

\[ + g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y) \}. \quad (2.15) \]

If we use the orthonormality assumptions on the \( X_i \)'s and the \( Z_{mn} \)'s, and (2.3), we obtain

\[ \Gamma^k_{ij} = g(\nabla_{X_i} X_j, X_k) = \frac{1}{2} \left\{ g([X_i, X_j], X_k) - g([X_j, X_k], X_i) + g([X_k, X_i], X_j) \right\} \]

\[ = \frac{1}{2} \left\{ \omega^k_{ij} + \omega^j_{ki} - \omega^k_{jk} \right\}. \]

Similarly, since \( T(X, Z_{mn}) = 0 \) and \( \nabla X_i Z_{mn} = 0 \) one has

\[ \nabla_{Z_{mn}} X_i = \sum_{\ell=1}^{d} g(\nabla_{Z_{mn}} X_i, X_\ell) X_\ell. \]

Using (2.15) again, along with (2.4) and (2.5), we find

\[ g(\nabla_{Z_{mn}} X_i, X_\ell) = \frac{1}{2} \left\{ g([Z_{mn}, X_i], X_\ell) - g([X_i, X_\ell], Z_{mn}) + g([X_\ell, Z_{mn}], X_i) \right\} \]

\[ = \frac{1}{2} \left\{ -\delta^\ell_{imn} + \delta^i_{\ell mn} \right\} = -\delta^\ell_{imn}. \]

In the sequel it will be useful to have the expression of the torsion tensor on the vector fields \( X_1, \ldots, X_d \).
Proposition 2.6  For every $k, \ell = 1, \ldots, d$ one has

$$T(X_\ell, X_k) = -\sum_{m,n=1}^{h} \gamma_{\ell k}^{mn} Z_{mn}.$$  

Proof. One has

$$T(X_\ell, X_k) = \nabla_{X_\ell} X_k - \nabla_{X_k} X_\ell - [X_\ell, X_k]$$

$$= \sum_{s=1}^{d} (\Gamma_{\ell k}^{s} - \Gamma_{k \ell}^{s} - \omega_{\ell k}^{s}) X_s - \sum_{m,n=1}^{h} \gamma_{\ell k}^{mn} Z_{mn}.$$  

Using (2.12) and the skew-symmetry of the matrix $[\omega_{ij}^{s}]_{i,j=1,\ldots,d}$ we now easily conclude that

$$\Gamma_{\ell k}^{s} - \Gamma_{k \ell}^{s} - \omega_{\ell k}^{s} = 0.$$  

We also record the following consequence of (2.12)

$$\nabla_{X_i} X_i = -\sum_{j=1}^{d} \omega_{ij} X_j.$$  

(2.16)

Remark 2.7  We can observe that

$$L = \sum_{i=1}^{d} X_i^2 - \nabla_{X_i} X_i$$

Example 2.8  For the Example 2.1, it is readily checked that $\nabla$ is the Levi-Civita connection.

Example 2.9  Let us now identify $\nabla$ for the Example 2.3, whose notations are in force in what follows. Let us recall (see for instance Chapter 3 in [8]) that the Ehresmann connection form $\alpha$ on $O(M)$ is the unique skew-symmetric matrix $\alpha$ of one forms on $O(M)$ such that:

1. $\alpha(X) = 0$ if and only if $X \in HO(M)$;
2. $V_{\alpha(X)} = X$ if and only if $X \in VO(M)$,

where $HO(M)$ denotes the horizontal bundle and $VO(M)$ the vertical bundle. It is then easily checked that for a vector field $Y$ on $O(M)$,

$$\nabla_Y H_i = \sum_{k=1}^{d} \alpha_{ij}^{k}(Y) H_k.$$  

Let us observe for later use that if $X, Y$ are smooth horizontal vector fields then we have for the torsion:

$$T(X, Y) = -V_{\Omega(X,Y)}.$$  

Example 2.10  For the Example 2.4, it is an immediate consequence of Theorem 1.3 in [25] that $\nabla$ is the Webster-Tanaka connection.
2.3 Sub-Riemannian distance and local volume growth

In sub-Riemannian geometry the Riemannian distance $d_R$ of $\mathbb{M}$ is most of the times confined to
the background (see in this regard the discussion in section 0.1 of Gromov’s Carnot-Carathéodory
spaces seen from within in [10]). There is another distance on $\mathbb{M}$, that was introduced by
Carathéodory in his seminal paper [16], which plays a central role. A piecewise $C^1$ curve
$\gamma: [0, T] \to \mathbb{M}$ is called subunitary at $x$ if for every $\xi \in T_x\mathbb{M}$ one has

$$g_R(\gamma'(t), \xi)^2 \leq \sum_{i=1}^{d} g_R(X_i(\gamma(t)), \xi)^2.$$  

We define the subunit length of $\gamma$ as $\ell_s(\gamma) = T$. If we indicate with $S(x, y)$ the family of subunit
curves such that $\gamma(0) = x$ and $\gamma(T) = y$, then thanks to the fundamental accessibility theorem
of Chow-Rashevsky the connectedness of $\mathbb{M}$ implies that $S(x, y) \neq \emptyset$ for every $x, y \in \mathbb{M}$, see [20], [54]. This allows to define the sub-Riemannian distance on $\mathbb{M}$ as follows

$$d(x, y) = \inf \{\ell_s(\gamma) \mid \gamma \in S(x, y)\}.$$  

We refer the reader to the cited contribution of Gromov to [10], and to the opening article by Bellaïche in the same volume.

We next recall that a metric space $(S, \rho)$ is called a length-space, or intrinsic, if for any $x, y \in S$

$$\rho(x, y) = \inf \ell(\gamma_{xy}),$$

where the infimum is taken over all continuous, rectifiable curves $\gamma_{xy}$, joining $x$ to $y$. For a
continuous curve $\gamma: [a, b] \to S$, one defines $\ell(\gamma) = \sup \sum_{i=1}^{p} \rho(\gamma(t_i), \gamma(t_{i+1}))$, the supremum
being taken on all finite partitions $a = t_1 < t_2 < \cdots < t_p < t_{p+1} = b$ of the interval $[a, b]$. Since
by the triangle inequality we trivially have $\rho(x, y) \leq \ell(\gamma_{xy})$ for any continuous curve joining $x$
to $y$, it follows that in any metric space $(S, \rho)$,

$$\rho(x, y) \leq \inf \ell(\gamma_{xy}), \quad x, y \in S.$$  

In particular, such inequality is therefore valid in the space $(\mathbb{M}, d)$. Now it is proved in Proposition 2.2 in [22] that in any sub-Riemannian space the opposite inequality is also valid. Therefore,
every sub-Riemannian space $(\mathbb{M}, d)$ is a length-space.

Another elementary consequence of the Chow-Rashevsky theorem is that $i: (\mathbb{M}, d) \hookrightarrow (\mathbb{M}, d_R)$
is continuous. On the other hand, it was proved in [52] that for any connected set $\Omega \subset \mathbb{M}$ which
is bounded in the distance $d_R$ there exist $C = C(\Omega) > 0$, and $\epsilon = \epsilon(\Omega) > 0$, such that

$$d(x, y) \leq Cd_R(x, y)^{\epsilon}, \quad x, y \in \Omega.$$  

This implies that also the inclusion $i: (\mathbb{M}, d_R) \hookrightarrow (\mathbb{M}, d)$ is continuous, and thus, the topologies
of $d_R$ and $d$ coincide. In particular, compact sets are the same in either topology. The metric
space $(\mathbb{M}, d)$ is locally compact and, furthermore, for every compact set $K \subset \mathbb{M}$ there exists
$r_0(K) > 0$ such that for $x \in K$ and $0 < r < r_0$, the closed balls $B(x, r)$ in the metric $d$ are
compact, see Proposition 1.1 in [31]. If $\mathbb{M}$ is unbounded and the vector fields grow too fast at
infinity, then balls of large radii may not be bounded in $d_R$ in general, and the space $(M, d)$ may fail to be complete. The lack of completeness and the fact that not all metric balls are bounded in the metric $d_R$ are two equivalent properties in view of the following powerful generalization of the classical Theorem of Hopf-Rinow due to Cohn-Vossen, see [12].

**Theorem 2.11 (of Hopf-Rinow type)** In any locally compact length-space $(S, \rho)$, the completeness of the metric space is equivalent to the compactness of the closed balls.

Since as we have mentioned $(M, d)$ is a locally compact length-space, if we want to guarantee the compactness of closed balls (in the $d$ metric) of arbitrary radii, we need to assume that $(M, d)$ is a complete metric space. By what has been said, this is equivalent to requiring that the space $(M, d_R)$ is complete.

In section 12 it will be expedient to work with yet another distance on $M$. For every $x, y \in M$, we set

$$
\rho(x, y) = \sup_{f \in C^\infty(M), \Gamma(f) \leq 1} |f(x) - f(y)|,
$$

(2.17)

where $\Gamma(f) = \sup_M \Gamma(f, f)$, and the quantity $\Gamma(f, f)$ is defined in (4.26) below. Thanks to Lemma 5.43 in [17] we know that

$$
d(x, y) = \rho(x, y), \quad x, y \in M,
$$

hence we can work indifferently with either one of the distances $d$ or $\rho$.

The following fundamental result proved by Nagel, Stein and Wainger in [52] provides a uniform local control of the growth of the metric balls in $(M, d)$.

**Theorem 2.12** For any $x \in M$ there exist constants $C(x), R(x) > 0$ such that with $Q(x) = \log_2 C(x)$ one has

$$
\mu(B(x, tr)) \geq C(x)^{-1} Q(x) \mu(B(x, r)), \quad 0 \leq t \leq 1, \quad 0 < r \leq R(x).
$$

Given any compact set $K \subset \mathbb{M}$ one has

$$
\inf_{x \in K} C(x) > 0, \quad \inf_{x \in K} R(x) > 0.
$$

### 2.4 Geodesics

In this section we prove a first variation formula for the geodesics of the metric space $(M, d)$. More precisely, we will describe with our connection $\nabla$ the local minima of the energy functional. To this end, let us first introduce a family of natural skew-symmetric linear operators of the horizontal bundle. For $V \in \mathcal{H}$, we define:

$$
J_{mn}(V) = \sum_{i, j=1}^{d} g(X_j, V) \gamma_{ij}^{mn} X_i, \quad 1 \leq m, n \leq h.
$$

With this in hands, we can now extend the method of Rumin [59] to provide the equation of the geodesics. We first have the following result.
Lemma 2.13 For $1 \leq m, n \leq h$, let us denote by $\theta_{mn}$ the one-form on $\mathbb{M}$ such that $\theta_{mn}(Z_{mn}) = 1$, $\theta_{mn}(Z_{pq}) = 0$ if $\{m, n\} \neq \{p, q\}$ and $\theta(X_i) = 0$. If $V_1$ is a smooth vector field on $\mathbb{M}$, and $V_2$ is a smooth horizontal vector field, we have

$$d\theta_{mn}(V_1, V_2) = -g(V_1, J_{mn}V_2).$$

Proof. Let us first assume that $V_1$ and $V_2$ are both horizontal. From Cartan’s formula

$$d\theta_{mn}(V_1, V_2) = V_1\theta_{mn}(V_2) - V_2\theta_{mn}(V_1) - \theta_{mn}([V_1, V_2]).$$

Therefore

$$d\theta_{mn}(V_1, V_2) = -\theta_{mn}([V_1, V_2]).$$

¿From the definition of $\theta_{mn}$, we easily check that for $1 \leq i, j \leq d$,

$$\theta_{mn}([X_i, X_j]) = \gamma_{ij}^{mn},$$

so that

$$d\theta_{mn}(V_1, V_2) = -g(V_1, J_{mn}V_2).$$

Now, if $V_1 = Z_{pq}$, since from our assumption (2.4) the bracket $[Z_{pq}, X_i]$ is always a horizontal vector field, again from Cartan’s formula we obtain

$$d\theta_{mn}(V_1, V_2) = 0.$$

□

Proposition 2.14 Let $x, y \in \mathbb{M}$ and denote by $\mathcal{H}(x, y)$ the set of horizontal curves $\gamma(t)$, $0 \leq t \leq 1$, going from $x$ to $y$. Then $\gamma \in \mathcal{H}(x, y)$ is a critical point of the energy functional

$$E(\gamma) = \int_0^1 g(\gamma'(t), \gamma'(t))dt$$

if and only if there exist constants $a_{mn}$, $1 \leq m, n \leq h$ such that

$$\nabla_{\gamma'(t)}\gamma'(t) = \sum_{m,n=1}^h a_{mn}J_{mn}\gamma'(t).$$

Proof. Let us consider a family of curve $\gamma_u(t)$, $-\epsilon \leq u \leq \epsilon$, $0 \leq t \leq 1$, such that $\gamma_0(t) \in \mathcal{H}(x, y)$ and $V = \frac{\partial \gamma_u}{\partial u}|_{u=0}$ is a Legendre vector field of the constrained problem corresponding to the variational problem we are looking at. We denote $X = \gamma'(t)$. We have

$$V(x) = V(y) = 0,$$

and

$$[V, X] \in \mathcal{H}.$$
Since the torsion of $\nabla$ is vertical, we now compute

$$\frac{\partial E(\gamma_u)}{\partial u} \bigg|_{u=0} = \int_0^1 V g(X, X) dt$$

$$= 2 \int_0^1 g(\nabla_V X, X) dt$$

$$= 2 \int_0^1 g(\nabla_X V, X) dt$$

$$= 2 \int_0^1 X g(V, X) - g(V, \nabla_X X) dt$$

$$= -2 \int_0^1 g(V, \nabla_X X) dt$$

Therefore, if $\gamma$ is a critical point, we must have $g(X, \nabla_X X) = 0$. We now write

$$\nabla_X X = \sum_{m,n=1}^{h} a_{mn}(t) J_{mn} X + Y,$$

where $Y$ is orthogonal to $X$ and to $\sum_{m,n=1}^{h} a_{mn}(t) J_{mn} X$. We have then

$$g(V, \sum_{m,n=1}^{h} a_{mn}(t) J_{mn} X) = \sum_{m,n=1}^{h} a_{mn}(t) g(V, J_{mn} X).$$

But, from the previous lemma and Cartan’s formula

$$g(V, J_{mn} X) = -d\theta_{mn}(V, X) = X\theta(V).$$

Therefore,

$$\frac{\partial E(\gamma_u)}{\partial u} \bigg|_{u=0} = -2 \int_0^1 \sum_{m,n=1}^{h} a_{mn}(t) X \theta(V) + g(V, Y) dt$$

Integrating by parts, we obtain

$$\frac{\partial E(\gamma_u)}{\partial u} \bigg|_{u=0} = -2 \int_0^1 \sum_{m,n=1}^{h} -a'_{mn}(t) \theta(V) + g(V, Y) dt,$$

so that $\gamma_0$ is a critical point if and only if $a_{mn}$ is constant and $Y = 0$. \qed
3 The curvature tensor

In this section we introduce a first-order differential quadratic form which plays a pervasive role in the results of this paper. If \( f \in C^\infty(M) \) we define

\[
R(f,f) = \sum_{k,\ell=1}^{d} \left\{ \left( \sum_{j=1}^{d} \sum_{m,n=1}^{h} \gamma_{k_{j}}^{m_{n}} \delta_{j_{m_{n}}}^{k_{\ell}} \right) + \sum_{j=1}^{d} (X_{\ell} \omega_{f_{k_{j}}}^{j} - X_{j} \omega_{f_{\ell_{j}}}^{k}) \right\} X_{f_{k}} f_{X_{\ell}} f_{X_{k}} f_{X_{f}} + \sum_{i,j=1}^{d} \left( \sum_{\ell,j=1}^{d} \omega_{f_{\ell_{j}}}^{k_{j}} + \sum_{1 \leq \ell < j \leq d} \left( \omega_{f_{\ell_{j}}}^{k_{j}} - (\omega_{f_{j_{\ell}}}^{i_{j}} + \omega_{f_{i_{\ell}}}^{j_{i}}) (\omega_{f_{j_{\ell}}}^{i_{j}} + \omega_{f_{i_{\ell}}}^{j_{i}}) \right) \right) Z_{f_{m_{n}}} f_{X_{f_{k}} f_{X_{f_{k}}}} f^{2}.
\]

The geometric meaning of the differential quadratic form \( R(f,f) \) will become fully clear in section 4. The main purpose of the present section is to prove its tensorial nature. In fact, in Proposition 3.1 below we show that \( R(f,f) \) can be expressed solely in terms of the curvature and torsion tensors with respect to the canonical connection \( \nabla \) introduced in section 2.2. At the end of this section we work out the formulas for \( R(f,f) \) in some special examples. In example 3.2 we show that in the Riemannian case this form coincides with the Riemannian Ricci tensor.

In what follows \( \nabla \) indicates the canonical connection on \( M \) introduced in section 2.2. Let us recall that the torsion tensor is given by

\[
T(X,Y) = \nabla_{X} Y - \nabla_{Y} X - [X,Y],
\]

the curvature tensor is given by

\[
R(X,Y)Z = \nabla_{X} \nabla_{Y} Z - \nabla_{Y} \nabla_{X} Z - \nabla_{[X,Y]} Z,
\]

and the Ricci tensor by

\[
\text{Ric}(X,Y) = \sum_{i=1}^{d} g(R(X_{i},X)Y,X_{i}),
\]

where \( X,Y,Z \) are smooth vector fields.

**Proposition 3.1** If \( f : M \to \mathbb{R} \) is a smooth function, then

\[
R(f,f) = \sum_{\ell,k=1}^{d} \text{Ric}(X_{\ell},X_{k}) X_{\ell} f X_{k} f - ((\nabla_{X_{\ell}} T)(X_{k})f)(X_{k} f) + \frac{1}{4} (T(X_{\ell},X_{k}) f)^{2}.
\]

As a consequence, \( R \) is a \((0,2)\) tensor on \( M \) and therefore it is coordinate free.
Proof. We begin by writing the quantity \( \mathcal{R}(f, f) \) in (3.18) as follows
\[
\mathcal{R}(f, f) = \mathcal{R}_I(f, f) + \mathcal{R}_{II}(f, f) + \mathcal{R}_{III}(f, f),
\]
where
\[
\mathcal{R}_I(f, f) = \sum_{k, \ell=1}^{d} \left\{ \left( \sum_{j=1}^{d} \sum_{m,n=1}^{b} \gamma_{kj}^{mn} \delta_{jmn}^\ell \right) + \sum_{j=1}^{d} (X_{\ell} \omega_{kj}^j - X_j \omega_{\ell j}^k) \right\} X_k f X_{\ell} f,
\]
\[
\mathcal{R}_{II}(f, f) = \sum_{k=1}^{d} \sum_{m,n=1}^{b} \left( \sum_{\ell,j=1}^{d} \omega_{\ell j}^k \gamma_{kj}^{mn} + \sum_{1 \leq \ell < j \leq d} (\omega_{\ell j}^k \gamma_{kj}^{mn} - \omega_{\ell j}^{k} \gamma_{kj}^{mn}) \right) Z_{mn} f X_k f,
\]
\[
\mathcal{R}_{III}(f, f) = \frac{1}{2} \sum_{1 \leq \ell < j \leq d} \left( \sum_{m,n=1}^{b} \gamma_{\ell j}^{mn} Z_{mn} f \right)^2.
\]
The proof will be completed if we show that
\[
\mathcal{R}_I(f, f) = \sum_{k, \ell=1}^{d} \text{Ric}(X_k, X_\ell) X_k f X_{\ell} f, \tag{3.19}
\]
\[
\mathcal{R}_{II}(f, f) = - \sum_{\ell,k=1}^{d} ((\nabla_{X_{\ell}} T)(X_\ell, X_k) f)(X_k f), \tag{3.20}
\]
\[
\mathcal{R}_{III}(f, f) = \frac{1}{4} \sum_{\ell,k=1}^{d} (T(X_\ell, X_k) f)^2. \tag{3.21}
\]
To establish (3.19) we observe that
\[
\text{Ric}(X_k, X_\ell) = \sum_{i=1}^{d} g(R(X_i, X_k) X_{\ell}, X_i) = \sum_{j=1}^{d} \sum_{m,n=1}^{b} \gamma_{kj}^{mn} \delta_{jmn}^\ell + \sum_{j=1}^{d} \left( X_j \Gamma_j^{k \ell} - X_k \Gamma_j^{j \ell} \right) \tag{3.22}
\]
\[
+ \sum_{i,j=1}^{d} \left( \Gamma_{k \ell}^i \Gamma_{ij}^i - \Gamma_{i \ell}^i \Gamma_{kj}^i - \omega_{ik}^j \Gamma_{j \ell}^i \right).
\]
The identity (3.22) can be verified in a standard fashion. Our goal is thus proving that
\[
\sum_{k, \ell=1}^{d} \left( \sum_{j=1}^{d} \left( X_j \Gamma_j^{k \ell} - X_k \Gamma_j^{j \ell} \right) + \sum_{i,j=1}^{d} \left( \Gamma_{k \ell}^i \Gamma_{ij}^i - \Gamma_{i \ell}^i \Gamma_{kj}^i - \omega_{ik}^j \Gamma_{j \ell}^i \right) \right) (X_k f)(X_\ell f)
\]
\[
= \sum_{k, \ell=1}^{d} \left( \sum_{j=1}^{d} (X_{\ell} \omega_{kj}^j - X_j \omega_{\ell j}^k) + \sum_{i,j=1}^{d} \omega_{ij}^j \omega_{kj}^j - \sum_{i=1}^{d} \omega_{\ell i}^j \omega_{ki}^j \right.
\]
\[
+ \frac{1}{2} \sum_{1 \leq i < j \leq d} \left( \omega_{ij}^j \omega_{\ell j}^i - (\omega_{ij}^j + \omega_{\ell j}^i)(\omega_{ij}^j + \omega_{\ell i}^j) \right) \right) (X_k f)(X_\ell f).
\]
To show this we use the formula (2.12)

$$\Gamma^k_{ij} = \frac{1}{2} \left( \omega^k_{ij} + \omega^j_{ki} - \omega^i_{jk} \right),$$

which allows to express the Christoffel symbols \( \Gamma^k_{ij} \) in terms of the structural constants \( \omega^k_{ij} \).

Keeping in mind that \( \omega^k_{\ell ij} = -\omega^k_{ij \ell} \), one easily recognizes that

$$\sum_{k,\ell=1}^d \left( \sum_{j=1}^d \left( X_j \Gamma^j_{k\ell} - X_k \Gamma^j_{j\ell} \right) \right) X_k f X_\ell f = \sum_{k,\ell=1}^d \left( \sum_{j=1}^d \left( X_j \omega^j_{k\ell} - X_k \omega^j_{\ell j} \right) \right) X_k f X_\ell f.$$  

To complete the proof of (5.19) we are thus left with proving that

$$\sum_{k,\ell=1}^d \sum_{i,j=1}^d \left( \Gamma^j_{k\ell} \Gamma^i_{ij} - \Gamma^j_{i\ell} \Gamma^i_{kj} - \omega^j_{ik} \Gamma^i_{j\ell} \right) X_k f X_\ell f = \sum_{k,\ell=1}^d \left\{ \sum_{i,j=1}^d \omega^j_{ij} \omega^i_{kj} - \sum_{i=1}^d \omega^i_{k\ell} \omega^i_{\ell i} + \frac{1}{2} \sum_{1 \leq i < j \leq d} \left( \omega^i_{ij} \omega^j_{kj} - (\omega^i_{kj} + \omega^j_{ik})(\omega^i_{ij} + \omega^j_{ij}) \right) \right\} X_k f X_\ell f.$$  

We now notice that again from (2.12) and the skew-symmetry of \( \omega^f_{jk} \) in the lower indices we obtain

$$\sum_{k,\ell=1}^d \sum_{i,j=1}^d \Gamma^j_{k\ell} \Gamma^i_{ij} X_k f X_\ell f = \sum_{k,\ell=1}^d \sum_{i,j=1}^d \frac{\Gamma^j_{k\ell} + \Gamma^j_{i\ell}}{2} \Gamma^i_{ij} X_k f X_\ell f = \sum_{k,\ell=1}^d \sum_{i,j=1}^d \frac{\omega^j_{k\ell} + \omega^j_{i\ell}}{2} \omega^i_{ij} X_k f X_\ell f = \sum_{k,\ell=1}^d \sum_{i,j=1}^d \omega^j_{ij} \omega^i_{kj} X_k f X_\ell f.$$  

Next, we have

$$- \sum_{k,\ell=1}^d \sum_{i,j=1}^d \left( \Gamma^j_{i\ell} \Gamma^i_{kj} + \omega^j_{ik} \Gamma^i_{j\ell} \right) X_k f X_\ell f = - \sum_{k,\ell=1}^d \sum_{i,j=1}^d \Gamma^j_{i\ell} \left( \Gamma^i_{kj} + \omega^i_{jk} \right) X_k f X_\ell f.$$  

By (2.12) one has

$$\Gamma^i_{kj} + \omega^i_{jk} = \frac{1}{2} \left( \omega^i_{ik} + \omega^i_{ij} + \omega^i_{jk} \right),$$
and thus
\[ - \sum_{k,\ell=1}^{d} \sum_{i,j=1}^{d} \left( \Gamma_{k\ell}^{ij} \Gamma_{kj}^{\ell i} + \omega_{ik}^{j} \Gamma_{kj}^{\ell i} \right) X_{k} f X_{\ell} f \]
\[ = \frac{1}{4} \sum_{k,\ell=1}^{d} \sum_{i,j=1}^{d} \left( \omega_{ij}^{i} - \omega_{i\ell}^{j} \right) \left( \omega_{ij}^{k} + \omega_{ik}^{j} + \omega_{i\ell}^{\ell} \right) X_{k} f X_{\ell} f \]
\[ = \sum_{k,\ell=1}^{d} \sum_{i=1}^{d} \omega_{ij}^{i} \omega_{i\ell}^{j} X_{k} f X_{\ell} f + \frac{1}{2} \sum_{k,\ell=1}^{d} \sum_{1 \leq i < j \leq d} \omega_{ij}^{\ell} \omega_{ij}^{k} X_{k} f X_{\ell} f \]
\[ - \frac{1}{2} \sum_{k,\ell=1}^{d} \sum_{1 \leq i < j \leq d} \left( \omega_{ij}^{i} + \omega_{i\ell}^{j} \right) \left( \omega_{ij}^{k} + \omega_{i\ell}^{\ell} \right) X_{k} f X_{\ell} f, \]

where the last equality is obtained by expanding the product and canceling equal terms. This proves (3.23), thus completing the proof of (3.19).

Next, we turn to the proof of (3.20). One has
\[ \sum_{\ell,k=1}^{d} (\nabla_{X_{\ell}} T(X_{\ell}, X_{k})) f X_{k} f = \sum_{\ell,k=1}^{d} \nabla_{X_{\ell}} (T(X_{\ell}, X_{k})) f X_{k} f \] (3.24)
\[ - \sum_{\ell,k=1}^{d} T(X_{\ell}, X_{k}) f X_{k} f - \sum_{\ell,k=1}^{d} T(X_{\ell}, \nabla_{X_{\ell}} X_{k}) f X_{k} f. \]

Proposition 2.6 and the fact that \( \nabla Z = 0 \) now give
\[ \sum_{\ell,k=1}^{d} \nabla_{X_{\ell}} (T(X_{\ell}, X_{k})) f X_{k} f = - \sum_{\ell,k=1}^{d} \sum_{m,n=1}^{b} (X_{\ell} \gamma_{mn}^{k}) Z_{mn} X_{k} f = \sum_{k=1}^{d} \sum_{m,n=1}^{b} \sum_{j=1}^{d} (X_{j} \gamma_{mn}^{k}) Z_{mn} X_{k} f. \]

Next, using (2.16) and Proposition 2.6 again one obtains
\[ - \sum_{\ell,k=1}^{d} T(X_{\ell}, X_{k}) f X_{k} f = \sum_{\ell,k=1}^{d} \sum_{j=1}^{d} \omega_{ij}^{\ell} T(X_{j}, X_{k}) f X_{k} f \]
\[ = - \sum_{k=1}^{d} \sum_{m,n=1}^{b} \sum_{j=1}^{d} \omega_{ij}^{\ell} \gamma_{mn}^{k} Z_{mn} f X_{k} f. \]
Finally,
\[- \sum_{\ell,k=1}^{d} T(X_{\ell}, \nabla X_{\ell} X_{k}) f X_{k} f = - \sum_{j,\ell,k=1}^{d} \Gamma_{\ell k}^{j} T(X_{\ell}, X_{j}) f X_{k} f \]
\[= \sum_{k=1}^{d} \sum_{m,n=1}^{b} \sum_{j,\ell,k=1}^{d} \gamma_{\ell j}^{mn} \Gamma_{\ell k}^{j} Z_{mn} f X_{k} f = \frac{1}{2} \sum_{k=1}^{d} \sum_{m,n=1}^{b} \sum_{j,\ell,k=1}^{d} \gamma_{\ell j}^{mn} (\omega_{\ell k}^{j} + \omega_{j k}^{\ell} + \omega_{j k}^{\ell}) Z_{mn} f X_{k} f \]
\[= \frac{1}{2} \sum_{k=1}^{d} \sum_{m,n=1}^{b} \sum_{1 \leq \ell < j \leq d} \gamma_{\ell j}^{mn} \omega_{\ell j}^{k} Z_{mn} f X_{k} f.\]

Substitution in (3.24) gives
\[- \sum_{\ell,k=1}^{d} \left( (\nabla X_{\ell} T)(X_{\ell}, X_{k}) f \right) (X_{k} f) = - R_{I I I}(f, f),\]
which proves (3.20). In order to complete the proof we are left with establishing (3.21). From Proposition 2.6 and the skew-symmetry of $\gamma_{\ell k}^{mn}$ in the lower indices, we easily find
\[\frac{1}{4} \sum_{\ell,k=1}^{d} \left( (\nabla X_{\ell} T)(X_{\ell}, X_{k}) f \right)^{2} = \frac{1}{2} \sum_{\ell,k=1}^{d} \left( \sum_{m,n=1}^{b} \gamma_{\ell k}^{mn} Z_{mn} f \right)^{2} = R_{I I I}(f, f).\]
This finishes the proof.

We can now compute $\mathcal{R}$ in the geometric examples introduced in section 2.1.

Example 3.2 For a Riemannian manifold $\mathcal{M}$, see Example 2.1, we have
\[\mathcal{R}(f, f) = \text{Ric}(\nabla f, \nabla f).\]
This follows immediately from Proposition 3.1.

Example 3.3 When $\mathcal{M}$ is a graded nilpotent Lie group of step two, see Example 2.2, we immediately obtain from (3.18)
\[\mathcal{R}(f, f) = \frac{1}{4} \sum_{i,j=1}^{b} (Z_{ij} f)^{2} = \frac{1}{4} \sum_{i,j=1}^{b} ([X_{i}, X_{j}] f)^{2}.\]
We note here that, in terms of the quadratic form introduced in (4.29) below, one has
\[\mathcal{R}(f, f) = \frac{1}{4} \mathcal{P}(f, f).\]
Example 3.4 In the Example 2.3 we have

\[ \mathcal{R}(f,f) = \sum_{j,k=1}^{d} \text{Ric}^{*}(H_j,H_j)(H_j f)(H_k f) + V_{\nabla_H \Omega(H_j,H_k)} f H_k f + \frac{1}{4} \left( V_{\Omega(H_j,H_k)} f \right)^2, \]

where for horizontal vector fields \( X \) and \( Y \),

\[ \text{Ric}^{*}(X,Y) = \text{Ric}(\pi_* X, \pi_* Y), \]

with \( \text{Ric} \) denoting the Ricci tensor of \( M \).

4 Bochner type formulas

Our goal in this section is to prove sub-Riemannian Bochner type formulas which play a central role in this paper. Since the operator \( L \) is two-step generating, we prove two types of such formulas, one involving the commutation between the horizontal gradient and \( L \), the other one involving the commutation between a vertical gradient and \( L \). As it is to be surmised, the horizontal one will prove quite involved technically.

In the sequel, we will use the following differential bilinear forms:

\[ \Gamma(f,g) = \frac{1}{2}(L(fg) - fLg -gLf) = \sum_{i=1}^{d} X_i f X_i g, \tag{4.26} \]

\[ \Gamma_2(f,g) = \frac{1}{2} \left[ L\Gamma(f,g) - \Gamma(f,Lg) - \Gamma(g,Lf) \right], \tag{4.27} \]

\[ \Gamma^Z(f,g) = \sum_{i,j=1}^{b} (Z_{ij} f)(Z_{ij} g), \tag{4.28} \]

\[ \Gamma^Z_2(f,g) = \frac{1}{2} \left[ L\Gamma^Z(f,g) - \Gamma^Z(f,Lg) - \Gamma^Z(g,Lf) \right]. \tag{4.29} \]

4.1 The first Bochner formula

Henceforth, we adopt the notation

\[ f_{,ij} = \frac{X_i X_j f + X_j X_i f}{2} \]

for the entries of the symmetrized Hessian of \( f \) with respect to the vector fields \( X_1, \ldots, X_d \).

Noting that

\[ X_i X_j f = f_{,ij} + \frac{1}{2} [X_i, X_j] f, \]

using (2.3) we obtain the useful formula

\[ X_i X_j f = f_{,ij} + \frac{1}{2} \sum_{t=1}^{d} \omega_{ij}^t X_t f + \frac{1}{2} \sum_{m,n=1}^{b} \gamma_{ij}^{mn} Z_{mn} f. \tag{4.30} \]

Our principal result of this section is the following:
Theorem 4.1 (Horizontal Bochner formula) For every smooth function \( f : \mathbb{M} \rightarrow \mathbb{R} \),
\[
\Gamma_2(f, f) = \sum_{\ell=1}^{d} \left( f, \ell \ell - \sum_{i=1}^{d} \omega_{i\ell} X_i f \right)^2 + 2 \sum_{1 \leq \ell < j \leq d} \left( f, \ell j - \frac{\omega_{ij}}{2} X_i f \right)^2 \tag{4.31}
\]

\[
- 2 \sum_{i,j=1}^{d} \sum_{m,n=1}^{d} \gamma_i^{mn} (X_j Z_{mn} f)(X_i f) + \mathcal{R}(f, f),
\]

where \( \mathcal{R}(f, f) \) is the quadratic form defined in (3.18).

Proof. We begin by observing that for any smooth function \( F \) on \( \mathbb{M} \)
\[
L(F^2) = 2FLF + 2\Gamma(F, F),
\]
This and (4.26) give
\[
L\Gamma(f, f) = \sum_{i=1}^{d} L((X_i f)^2) = 2 \sum_{i=1}^{d} X_i f L(X_i f) + 2 \sum_{j=1}^{d} \Gamma(X_i f, X_i f).
\]

We now have
\[
L(X_i f) = X_0 X_i f + \sum_{j=1}^{d} X_j^2 X_i f = X_i X_0 f + [X_0, X_i] f + \sum_{j=1}^{d} X_j (X_i X_j f) + X_j [X_j, X_i] f
\]
\[
= X_i X_0 f + [X_0, X_i] f + \sum_{j=1}^{d} \left\{ X_i (X_j X_j f) + [X_j, X_i] X_j f + X_j [X_j, X_i] f \right\}
\]
\[
= X_i (L f) + [X_0, X_i] f + \sum_{j=1}^{d} \left\{ [X_j, X_i] X_j f + X_j [X_j, X_i] f \right\}
\]
\[
= X_i (L f) + [X_0, X_i] f + 2 \sum_{j=1}^{d} [X_j, X_i] X_j f + \sum_{j=1}^{d} [X_j, [X_j, X_i]] f.
\]
Using this identity we find
\[
L\Gamma(f, f) = 2 \sum_{i=1}^{d} X_i f \left\{ X_i (L f) + [X_0, X_i] f + 2 \sum_{j=1}^{d} [X_j, X_i] X_j f + \sum_{j=1}^{d} [X_j, [X_j, X_i]] f \right\}
\]
\[
+ 2 \sum_{i,j=1}^{d} (X_j X_i f)^2
\]
\[
= 2\Gamma(f, L f) + 2 \sum_{i=1}^{d} X_i f [X_0, X_i] f + 4 \sum_{i,j=1}^{d} X_i f [X_j, X_i] X_j f + 2 \sum_{i,j=1}^{d} X_i f [X_j, [X_j, X_i]] f
\]
\[
+ 2 \sum_{i,j=1}^{d} (X_j X_i f)^2.
\]
Since, thanks to the skew-symmetry of the matrix \(\{[X_i, X_j] f\}_{i,j=1,...,d}\), we have

\[
\sum_{i,j=1}^{d} f_{ij} [X_i, X_j] f = 0,
\]

we find

\[
\sum_{i,j=1}^{d} (X_j X_i f)^2 = \sum_{i,j=1}^{d} f_{ij}^2 + \frac{1}{d} \sum_{i,j=1}^{d} ([X_i, X_j] f)^2 + \sum_{i,j=1}^{d} f_{ij} [X_i, X_j] f
\]

\[
= \sum_{i,j=1}^{d} f_{ij}^2 + \frac{1}{d} \sum_{i,j=1}^{d} ([X_i, X_j] f)^2.
\]

We thus obtain

\[
\frac{1}{2} [L \Gamma(f, f) - 2 \Gamma(f, L f)] = \sum_{i=1}^{d} X_i f [X_0, X_i] f
\]

\[
+ 2 \sum_{i,j=1}^{d} X_i f [X_j, X_i] X_j f + \sum_{i,j=1}^{d} X_i f [X_j, [X_j, X_i]] f
\]

\[
+ \sum_{i,j=1}^{d} f_{ij}^2 + \frac{1}{d} \sum_{i,j=1}^{d} ([X_i, X_j] f)^2.
\]

Since (4.27) gives \(\Gamma_2(f, f) = \frac{1}{2} [L \Gamma(f, f) - 2 \Gamma(f, L f)]\), we conclude

\[
\Gamma_2(f, f) = \sum_{i,j=1}^{d} f_{ij}^2 - 2 \sum_{i,j=1}^{d} X_i f [X_i, X_j] X_j f
\]

\[
+ \frac{1}{4} \sum_{i,j=1}^{d} ([X_i, X_j] f)^2 + \sum_{i=1}^{d} X_i f [X_0, X_i] f + \sum_{i,j=1}^{d} X_i f [[X_i, X_j], X_j] f.
\]

To complete the proof we need to recognize that the right-hand side in (4.32) coincides with
that in (4.31). With this objective in mind, using (2.3) and (4.30) we obtain

\[
\sum_{i,j=1}^{d} f_{ij}^2 - 2 \sum_{i,j=1}^{d} X_i f[X_i, X_j] X_j f
\]

\[
= \sum_{\ell=1}^{d} f_{\ell\ell}^2 + 2 \sum_{1 \leq \ell < j \leq d} f_{j\ell}^2 - 2 \sum_{i,j=1}^{d} X_i f \left( \sum_{\ell=1}^{d} \omega_{ij}^\ell X_\ell + \sum_{m,n=1}^{h} \gamma_{ij}^{mn} Z_{mn} \right) X_j f
\]

\[
= \sum_{\ell=1}^{d} f_{\ell\ell}^2 + 2 \sum_{1 \leq \ell < j \leq d} f_{j\ell}^2
\]

\[
- 2 \sum_{i,j=1}^{d} \sum_{\ell=1}^{d} \omega_{ij}^\ell X_\ell X_j f X_i f - 2 \sum_{i,j=1}^{d} \sum_{m,n=1}^{h} \gamma_{ij}^{mn} Z_{mn} X_j f X_i f
\]

\[
= \sum_{\ell=1}^{d} f_{\ell\ell}^2 + 2 \sum_{1 \leq \ell < j \leq d} f_{j\ell}^2 - 2 \sum_{i,j=1}^{d} \sum_{\ell=1}^{d} \omega_{ij}^\ell f_{\ell j} X_i f
\]

\[
- \sum_{i,j=1}^{d} \sum_{\ell=1}^{d} \omega_{ij}^\ell \omega_{kj}^\ell X_k f X_i f - \sum_{i,j=1}^{d} \sum_{\ell=1}^{d} \sum_{m,n=1}^{h} \omega_{ij}^\ell \gamma_{ij}^{mn} Z_{mn} f X_i f
\]

\[
- 2 \sum_{i,j=1}^{d} \sum_{m,n=1}^{h} \gamma_{ij}^{mn} Z_{mn} X_j f X_i f
\]

\[
= \sum_{\ell=1}^{d} \left( f_{\ell\ell}^2 - 2 \sum_{i=1}^{d} \omega_{ij}^\ell X_i f \right) f_{\ell\ell}
\]

\[
+ 2 \sum_{1 \leq \ell < j \leq d} \left( f_{j\ell}^2 - 2 \sum_{1 \leq \ell < j \leq d} \sum_{i=1}^{d} \frac{\omega_{ij}^\ell + \omega_{ij}^\ell}{2} X_i f \right) f_{j\ell}
\]

\[
- \sum_{i,j=1}^{d} \sum_{\ell=1}^{d} \omega_{ij}^\ell \omega_{kj}^\ell X_k f X_i f - \sum_{i,j=1}^{d} \sum_{\ell=1}^{d} \sum_{m,n=1}^{h} \omega_{ij}^\ell \gamma_{ij}^{mn} Z_{mn} f X_i f
\]

\[
- 2 \sum_{i,j=1}^{d} \sum_{m,n=1}^{h} \gamma_{ij}^{mn} Z_{mn} X_j f X_i f.
\]
If we now complete the squares we obtain

\[
\sum_{i,j=1}^{d} f_{ij}^2 - 2 \sum_{i,j=1}^{d} X_i f[X_i, X_j] X_j f
\]

(4.33)

\[
= \sum_{\ell=1}^{d} \left( f_{\ell \ell} - \sum_{i=1}^{d} \omega_{i\ell} X_i f \right)^2 + 2 \sum_{1 \leq \ell < j \leq d} \left( f_{\ell j} - \sum_{i=1}^{d} \frac{\omega_{ij} + \omega_{i\ell}}{2} X_i f \right)^2
\]

\[
- \sum_{\ell=1}^{d} \left( \sum_{i=1}^{d} \omega_{i\ell} X_i f \right)^2 - 2 \sum_{1 \leq \ell < j \leq d} \left( \sum_{i=1}^{d} \omega_{ij} + \omega_{i\ell} \right)^2 X_i f
\]

\[
- \sum_{i,j,k,\ell=1}^{d} \omega_{ij} \omega_{k\ell} X_k f X_i f - \sum_{i,j=1}^{d} \sum_{\ell=1}^{d} \sum_{m,n=1}^{b} \omega_{ij} \gamma_{mn} Z_{mn} f X_i f
\]

\[
- 2 \sum_{i,j=1}^{d} \sum_{m,n=1}^{b} \gamma_{ij} Z_{mn} f X_i f - 2 \sum_{i,j=1}^{d} \sum_{m,n=1}^{b} \gamma_{ij} [Z_{mn}, X_j] f X_i f.
\]

Next, we have from (2.7)

\[
\sum_{i=1}^{d} X_i f[X_0, X_i] f = \sum_{i,j,k,\ell=1}^{d} \omega_{jk} \omega_{ik} X_{ik} f X_i f
\]

(4.34)

\[
+ \sum_{i=1}^{d} \sum_{j,k=1}^{d} \sum_{m,n=1}^{b} \omega_{jk} \gamma_{ij} Z_{mn} f X_i f + \sum_{i=1}^{d} \sum_{j,k=1}^{d} (X_i \omega_{jk}) X_i f X_j f,
\]

and also

\[
\sum_{i,j=1}^{d} X_i f[[X_i, X_j], X_j] f = \sum_{i,j=1}^{d} \sum_{i,j=1}^{d} \omega_{ij} X_i f[X_i, X_j] f X_i f + \sum_{i,j=1}^{d} \sum_{m,n=1}^{b} \gamma_{ij} Z_{mn}, X_j] f X_i f
\]

(4.35)

\[
= \sum_{i,j=1}^{d} \sum_{i,j=1}^{d} \omega_{ij} X_i f[X_i, X_j] f X_i f - \sum_{i,j=1}^{d} \sum_{i,j=1}^{d} (X_i \omega_{ij}) X_i f X_j f
\]

\[
+ \sum_{i,j=1}^{d} \sum_{m,n=1}^{b} \gamma_{ij} Z_{mn}, X_j] f X_i f - \sum_{i,j=1}^{d} \sum_{i,j=1}^{d} (X_j \gamma_{ij}) Z_{mn} f X_i f.
\]

Using (2.33) we find

\[
\sum_{i,j=1}^{d} X_i f[[X_i, X_j], X_j] f = \sum_{i,j=1}^{d} \sum_{i,j=1}^{d} \omega_{ij} X_i f[X_i, X_j] f X_k f + \sum_{i,j=1}^{d} \sum_{i,j=1}^{d} \omega_{ij} \gamma_{mn} Z_{mn} f X_i f
\]

(4.35)

\[
+ \sum_{i,j=1}^{d} \sum_{i,j=1}^{d} \gamma_{ij} Z_{mn}, X_j] f X_i f - \sum_{i,j=1}^{d} \sum_{i,j=1}^{d} (X_j \gamma_{ij}) Z_{mn} f X_i f
\]

\[
- \sum_{i,j=1}^{d} \sum_{i,j=1}^{d} (X_j \omega_{ij}) X_i f X_j f.
\]
Again by (2.3) we have

\[
\frac{1}{4} \sum_{i,j=1}^{d} ([X_i, X_j]_f)^2 = \frac{1}{2} \sum_{1 \leq i < j \leq d} \left( \sum_{\ell=1}^{d} \omega_{ij}^{\ell} X_{\ell f} \right)^2 + \frac{1}{2} \sum_{1 \leq i < j \leq d} \left( \sum_{m,n=1}^{h} \gamma_{ij}^{mn} Z_{mn} f \right)^2 
\] \tag{4.36}

+ \sum_{1 \leq i < j \leq d} \sum_{\ell=1}^{d} \omega_{ij}^{\ell} \gamma_{ij}^{mn} Z_{mn} f X_{\ell f}.

Substituting (4.33)-(4.36) in (4.32) we obtain

\[
\Gamma_2(f, f) = \sum_{\ell=1}^{d} \left( f_{,\ell} - \sum_{i=1}^{d} \omega_{i\ell}^{f} X_{i f} \right)^2 + 2 \sum_{1 \leq \ell < j \leq d} \left( f_{,\ell} - \sum_{i=1}^{d} \frac{\omega_{i\ell}^{f} + \omega_{j\ell}^{f}}{2} X_{i f} \right)^2
\]

\[ - 2 \sum_{i,j=1}^{d} \sum_{m,n=1}^{h} \gamma_{ij}^{mn} X_j Z_{mn} f X_i f + \text{Monster} \]

where we have let

\[
\text{Monster} = - \sum_{\ell=1}^{d} \left( \sum_{i=1}^{d} \omega_{i\ell}^{f} X_{i f} \right)^2 - 2 \sum_{1 \leq \ell < j \leq d} \left( \sum_{i=1}^{d} \frac{\omega_{i\ell}^{f} + \omega_{j\ell}^{f}}{2} X_{i f} \right)^2 \tag{4.37}
\]

+ \sum_{i,j,k,\ell=1}^{d} \omega_{ij}^{k} \omega_{i\ell}^{f} X_{k f} X_{i f} - \sum_{i,j,k,\ell=1}^{d} \omega_{ij}^{k} \omega_{i\ell}^{f} X_{k f} X_{i f} - \sum_{i,j=1}^{d} \sum_{f=1}^{d} \sum_{m,n=1}^{h} \omega_{ij}^{f} \gamma_{ij}^{mn} Z_{mn} f X_i f

- \sum_{i,j=1}^{d} \sum_{m,n=1}^{h} \gamma_{ij}^{mn} [Z_{mn}, X_j]_f X_i f + \sum_{i=1}^{d} \sum_{j,k=1}^{d} \sum_{m,n=1}^{h} \omega_{jk}^{f} \gamma_{ij}^{mn} Z_{mn} f X_i f

+ \sum_{i=1}^{d} \sum_{j,k=1}^{d} (X_i \omega_{jk}^{f}) X_{j f} X_{j f} + \sum_{i,j=1}^{d} \sum_{\ell,\ell=1}^{d} \omega_{ij}^{f} \omega_{i\ell}^{k} X_{i f} X_{k f} + \sum_{i,j,\ell=1}^{d} \sum_{m,n=1}^{h} \omega_{ij}^{f} \gamma_{ij}^{mn} Z_{mn} f X_i f

- \sum_{i,j=1}^{d} \sum_{m,n=1}^{h} (X_j \gamma_{ij}^{mn}) Z_{mn} f X_i f - \sum_{i,j=1}^{d} \sum_{\ell=1}^{d} (X_j \omega_{ij}^{f}) X_{i f} X_{\ell f}

+ \frac{1}{2} \sum_{1 \leq i < j \leq d} \left( \sum_{\ell=1}^{d} \omega_{ij}^{f} X_{\ell f} \right)^2 + \frac{1}{2} \sum_{1 \leq i < j \leq d} \left( \sum_{m,n=1}^{h} \gamma_{ij}^{mn} Z_{mn} f \right)^2

+ \sum_{1 \leq i < j \leq d} \sum_{\ell=1}^{d} \sum_{m,n=1}^{h} \omega_{ij}^{f} \gamma_{ij}^{mn} Z_{mn} f X_{\ell f}.
Simplifying the expression we obtain

\[ \text{Monster} = - \sum_{k,\ell=1}^{d} \sum_{i=1}^{d} \omega_i^k \omega_i^\ell X_k f X_\ell f \]

\[ - \frac{1}{2} \sum_{k,\ell=1}^{d} \sum_{1 \leq i < j \leq d} (\omega_i^k \omega_i^\ell + \omega_j^k \omega_j^\ell) X_k f X_\ell f \] (4.38)

\[ + \sum_{k,\ell=1}^{d} \sum_{j=1}^{d} (X_k \omega_j^k \omega_j^\ell - X_j \omega_j^k \omega_j^\ell) X_k f X_\ell f + \sum_{i,j,k,\ell=1}^{d} \omega_i^j \omega_i^\ell X_k f X_\ell f \]

\[ + \frac{1}{2} \sum_{k,\ell=1}^{d} \sum_{1 \leq i < j \leq d} \omega_i^\ell \omega_i^k X_k f X_\ell f + \sum_{k,j=1}^{d} \sum_{m,n=1}^{b} \gamma_{kj}^{mn} [X_j, Z_{mn}] f X_k f \]

\[ + \sum_{i=1}^{d} \sum_{j=1}^{d} \sum_{k,j=1}^{d} \sum_{m,n=1}^{b} \omega_i^k \gamma_{ij}^{mn} Z_{mn} f X_i f + \sum_{1 \leq i < j \leq d} \left( \sum_{m,n=1}^{b} \gamma_{ij}^{mn} Z_{mn} f \right)^2. \]

To complete the proof we need to recognize that the right-hand side of (4.38) coincides with \( R(f, f) \) defined by (3.18). If we now use (2.4) we obtain

\[ \sum_{k,j=1}^{d} \sum_{m,n=1}^{b} \gamma_{kj}^{mn} [X_j, Z_{mn}] f X_k f = \sum_{k,j=1}^{d} \sum_{m,n=1}^{b} \sum_{\ell=1}^{d} \gamma_{kj}^{mn} \delta_{m,n} \delta_{j,mn} X_k f X_\ell f \]

\[ = \sum_{k,\ell=1}^{d} \left( \sum_{m,n=1}^{b} \gamma_{kj}^{mn} \delta_{m,n} \right) X_k f X_\ell f, \]

Substituting in (4.38) we find

\[ \text{Monster} = \sum_{k,\ell=1}^{d} \left\{ \left( \sum_{j=1}^{d} \sum_{m,n=1}^{b} \gamma_{kj}^{mn} \delta_{j,mn} \right) + \sum_{j=1}^{d} (X_k \omega_j^k - X_j \omega_j^k) \right\} X_k f X_\ell f \]

\[ + \sum_{i,j=1}^{d} \omega_i^j \omega_i^\ell - \sum_{i=1}^{d} \omega_i^k \omega_i^\ell + \frac{1}{2} \sum_{1 \leq i < j \leq d} \left( \omega_i^j \omega_i^\ell - (\omega_i^j + \omega_i^\ell) (\omega_i^k + \omega_i^\ell) \right) \}

\[ + \sum_{i=1}^{d} \sum_{j,k=1}^{d} \sum_{m,n=1}^{b} \omega_j^k \gamma_{ij}^{mn} Z_{mn} f X_i f + \sum_{1 \leq i < j \leq d} \sum_{\ell=1}^{d} \sum_{m,n=1}^{b} \omega_i^\ell \gamma_{ij}^{mn} Z_{mn} f X_\ell f \]

\[ - \sum_{i,j=1}^{d} \sum_{m,n=1}^{b} (X_j \gamma_{ij}^{mn}) Z_{mn} f X_i f + \frac{1}{2} \sum_{1 \leq i < j \leq d} \left( \sum_{m,n=1}^{b} \gamma_{ij}^{mn} Z_{mn} f \right)^2. \]
Rearranging the indices we can rewrite (4.39) as follows

\[
\mathcal{M} = \sum_{k,\ell=1}^{d} \left\{ \left( \sum_{j=1}^{d} \sum_{m,n=1}^{b} \gamma^{mn}_{kj} \delta_{jmn} \right) + \sum_{j=1}^{d} \left( X_{\ell} \omega^{k}_{kj} - X_{j} \omega^{k}_{kj} \right) \right\} + \sum_{i,j=1}^{d} \left( X_{i} \omega^{k}_{ij} - X_{j} \omega^{k}_{ij} \right) + \sum_{1 \leq \ell < j \leq d}^{d} \gamma^{mn}_{ij} Z_{mn} f X_{\ell} f \]

This completes the proof of Theorem 4.1. \(\square\)

**Remark 4.2** When \(M\) is a graded nilpotent Lie group of step two, then using the structural constants in Example 2.2 we obtain from formula (4.31)

\[
\Gamma_{2}(f, f) = \sum_{\ell=1}^{d} f_{\ell}^{2} + 2 \sum_{1 \leq \ell < j \leq d} f_{\ell} f_{j} - 2 \sum_{i,j=1}^{d} \sum_{m,n=1}^{b} \gamma^{mn}_{ij} (X_{j} Z_{mn} f)(X_{i} f) + \mathcal{R}(f, f).
\]

Keeping (3.25) in mind, we thus obtain from the latter equation

\[
\Gamma_{2}(f, f) = ||\nabla_{H} f||^{2} + \frac{1}{4} \sum_{i,j=1}^{d} (X_{i} X_{j} f)^{2} + 2 \sum_{i,j=1}^{d} |X_{i} X_{j} f| X_{i} f X_{j} f,
\]

which is Proposition 3.3 in [30] (the latter, however, is relative to groups of arbitrary step). Here, \(\nabla_{H} f = [f, ij]\) represents the symmetrized horizontal Hessian of \(f\).

### 4.2 The second Bochner formula

To prove our next result we will need the following lemma.

**Lemma 4.3** For every \(m, n = 1, \ldots, \mathfrak{h}\) one has

\[
[L, Z_{mn}] = 0.
\]

**Proof.** We begin by observing that, as consequence of the assumption (2.5), for every \(m, n = 1, \ldots, \mathfrak{h}\), the commutator \([L, Z_{mn}]\) is a vector field. Indeed, given a smooth function \(f\), using the
formulas (2.4), (2.7) and (4.30), one obtains by elementary computations

\[
[L, Z_{mn}]f = [X_0, Z_{mn}]f + \sum_{i=1}^{d} X_i[X_i, Z_{mn}]f + [X_i, Z_{mn}]X_i f
\]

\[
= - \sum_{i,k,\ell=1}^{d} \omega_{ik}^d \delta_{imn}^{\ell} X_{\ell} f + \sum_{i,k=1}^{d} Z_{mn}(\omega_{ik}^d) X_i f + \sum_{i,\ell=1}^{d} (X_i \delta_{imn}^{\ell}) X_{\ell} f + \delta_{imn}^{\ell} (X_{\ell} X_i f + X_i X_{\ell} f)
\]

\[
= - \sum_{i,k,\ell=1}^{d} \omega_{ik}^d \delta_{imn}^{\ell} X_{\ell} f + \sum_{i,k=1}^{d} Z_{mn}(\omega_{ik}^d) X_i f + \sum_{i,\ell=1}^{d} (X_i \delta_{imn}^{\ell}) X_{\ell} f
\]

where we used the crucial fact that

\[
\delta_{imn}^{\ell} = - \delta_{i\ell mn}.
\]

To complete the proof let \(\phi, \psi \in C^\infty_0(M)\), then (2.9) gives

\[
< [L, Z_{mn}]^* \phi, \psi > = < \phi, [L, Z_{mn}] \psi > = < \phi, L(Z_{mn} \psi) > = < \phi, Z_{mn} (L \psi) >
\]

\[
= < L \phi, Z_{mn} \psi > + < Z_{mn} \phi, L \psi > = < L(Z_{mn} \phi, \psi) > = < Z_{mn} (L \phi), \psi >
\]

and thus \([L, Z_{mn}]^* = [L, Z_{mn}]\), i.e. \([L, Z_{mn}]\) is a symmetric vector field. Such vector fields must vanish. Indeed, if \(V\) is symmetric vector field, one has for any \(\phi, \psi \in C^\infty_0(M)\)

\[
0 = < V^1, \phi \psi > = < V^* 1, \phi \psi > = < 1, V(\phi \psi) > = < 1, \phi \psi > + < 1, \psi \phi >
\]

\[
= < \phi, V \psi > + < \psi, V \phi > = 2 < V \phi, \psi >.
\]

Taking \(\psi = V \phi\) we conclude that it must be \(V \phi = 0\). By the arbitrariness of \(\phi \in C^\infty_0(M)\) we conclude \(V = 0\).

Our second Bochner type formula is expressed by the following proposition.

**Proposition 4.4 (Vertical Bochner formula)**  For every smooth function \(f : M \to \mathbb{R}\),

\[
\Gamma_Z^2(f, f) = \sum_{m,n=1}^{h} \Gamma(Z_{mn} f, Z_{mn} f)
\]

**Proof.** From (4.29) we obtain

\[
\Gamma_Z^2(f, f) = \frac{1}{2} \left[ L \Gamma_Z^2(f, f) - 2 \Gamma_Z^2(f, Lf) \right].
\]
We now have from (4.28) and from Lemma 4.3
\[
L \Gamma^Z(f, f) = \sum_{m,n=1}^{b} L((Z_{mn} f)^2) = 2 \sum_{m,n=1}^{b} Z_{mn} f L(Z_{mn} f) + 2 \sum_{m,n=1}^{b} \sum_{k=1}^{d} (X_k(Z_{mn} f))^2
\]
\[
= 2 \sum_{m,n=1}^{b} Z_{mn} f Z_{mn}(Lf) + 2 \sum_{m,n=1}^{b} \sum_{k=1}^{d} (X_k(Z_{mn} f))^2
\]
\[
= 2 \Gamma^Z(f, Lf) + 2 \sum_{m,n=1}^{b} \Gamma(Z_{mn} f, Z_{mn} f),
\]
where in the last term we have used (4.26). □

5 A sub-Riemannian curvature dimension inequality

As a consequence of our two Bochner’s formulas we obtain a generalization of the curvature dimension inequality of Bakry. A new feature of such inequality is the presence in the left hand-side of the vertical quadratic form \(\Gamma^Z(f, f)\). For a smooth function \(f\), we define the quantity

\[
T(f, f) = \sum_{j=1}^{d} \sum_{m,n=1}^{b} \left( \sum_{i=1}^{d} \gamma_{ij}^{mn} X_i f \right)^2.
\]

(5.40)

We observe that with the notations of Section 2.4

\[
T(f, f) = \sum_{m,n=1}^{b} g(J_{mn}(\nabla f), J_{mn}(\nabla f)).
\]

We notice explicitly that, since in the Riemannian case \(\gamma_{ij}^{mn} = 0\), see Example 2.1 in that case we have

\[
T(f, f) = 0.
\]

(5.41)

When instead \(M\) is a graded nilpotent Lie group of step two, then from the expressions of the \(\gamma_{ij}^{mn} = 0\) in example 2.2 we find

\[
T(f, f) = \frac{d-1}{2} \Gamma(f, f).
\]

(5.42)

**Proposition 5.1 (Sub-Riemannian curvature-dimension inequality)** For every smooth function \(f : M \to \mathbb{R}\) and every \(\nu > 0\),

\[
\Gamma_2(f, f) + \nu \Gamma^Z(f, f) \geq \frac{1}{d} (Lf)^2 + \mathcal{R}(f, f) - \frac{1}{\nu} T(f, f).
\]

Proof. From (2.6) and (2.7) and Schwarz inequality we find

\[
Lf = \sum_{\ell=1}^{d} \left( f, \ell - \sum_{i=1}^{d} \omega_{\ell i} X_i f \right) \leq \sqrt{d} \left( \sum_{\ell=1}^{d} \left( f, \ell - \sum_{i=1}^{d} \omega_{\ell i} X_i f \right)^2 \right)^{1/2}
\]
From this inequality and from Theorem 4.1, using Schwarz inequality again we obtain for every $\nu > 0$

$$\frac{1}{d}(Lf)^2 \leq \Gamma_2(f, f) + 2 \sum_{j=1}^{d} \sum_{m,n=1}^{b} \left( \sum_{i=1}^{d} \gamma_{ij}^{mn} X_i f \right) X_j(Z_{mn} f) - \mathcal{R}(f, f)$$

$$- 2 \sum_{1 \leq \ell < j \leq d} \left( f_{\ell j} - \sum_{i=1}^{d} \omega_{ij}^{\ell} + \omega_{ij} f_X \right)^2$$

$$\leq \Gamma_2(f, f) + \nu \sum_{j=1}^{d} \sum_{m,n=1}^{b} (X_j(Z_{mn} f))^2 + \frac{1}{\nu} \sum_{j=1}^{d} \sum_{m,n=1}^{b} \left( \sum_{i=1}^{d} \gamma_{ij}^{mn} X_i f \right)^2 - \mathcal{R}(f, f).$$

Using Proposition 4.4 and the definition (5.40) we find

$$\frac{1}{d}(Lf)^2 \leq \Gamma_2(f, f) + \nu \Gamma_2^Z(f, f) + \frac{1}{\nu} \mathcal{T}(f, f) - \mathcal{R}(f, f),$$

which gives the desired conclusion. □

**Example 5.2** On a Riemannian manifold $\mathbb{M}$ with Laplace-Beltrami operator $L$ we have $\Gamma_2^Z(f, f, \mathcal{T}(f, f) = 0$ (see (5.11)), and $\mathcal{R}(f, f) = \text{Ric}(\nabla f, \nabla f)$ (see Example 3.2). Proposition 5.1 thus gives

$$\Gamma_2(f, f) \geq \frac{1}{d}(Lf)^2 + \text{Ric}(\nabla f, \nabla f).$$

When

$$\text{Ric}(V, V) \geq \rho_1 |V|^2, \quad \rho_1 \in \mathbb{R},$$

we thus recover the Riemannian curvature-dimension inequality

$$\Gamma_2(f, f) \geq \frac{1}{d}(Lf)^2 + \rho_1 \Gamma(f, f).$$

(5.44)

**Remark 5.3** It is important to keep in mind that, in view of Theorem 1.3 in [57] and Proposition 3.3 in [4], the Riemannian curvature dimension inequality (5.44) is, in fact, equivalent to (5.43).

**Example 5.4** Let us assume that $L$ is the sub-Laplacian on a two-step graded nilpotent Lie group, then for every smooth function $f$ and every $\nu > 0$,

$$\Gamma_2(f, f) + \nu \Gamma_2^Z(f, f) \geq \frac{1}{d}(Lf)^2 - \frac{d-1}{2\nu} \Gamma(f, f) + \frac{1}{4} \Gamma^Z(f, f).$$

6 Gradient estimates for the heat semigroup

In this whole section we assume that the metric space $(\mathbb{M}, d)$ is complete, where $d$ denotes the Carathéodory distance associated to $L$ (see section 2.2). Recall that this is equivalent to
assuming that $\mathbb{M}$ is complete with respect to its Riemannian distance $d_R$. We also suppose that there exist constants $\rho_1 \in \mathbb{R}$, $\rho_2 > 0$ and $\kappa > 0$ such that for every smooth function $f : \mathbb{M} \to \mathbb{R}$:

$$
\begin{cases}
\mathcal{R}(f,f) \geq \rho_1 \Gamma(f,f) + \rho_2 \Gamma^Z(f,f), \\
\mathcal{T}(f,f) \leq \kappa \Gamma(f,f).
\end{cases}
$$

(6.45)

We emphasize that, as we have seen in example 5.2 and remark 5.3, the assumptions (6.45) should be considered as the sub-Riemannian analogue of (5.43). In particular, the requirement $\rho_1 \geq 0$ in (6.45) corresponds to the Riemannian $\text{Ric} \geq 0$.

According to Proposition 5.1 the assumptions (6.45) imply, for every smooth function $f : \mathbb{M} \to \mathbb{R}$ and every $\nu > 0$,

$$
\Gamma_2(f,f) + \nu \Gamma^Z_2(f,f) \geq \frac{1}{d}(Lf)^2 + \left(\rho_1 - \frac{\kappa}{\nu}\right) \Gamma(f,f) + \rho_2 \Gamma^Z(f,f).
$$

(6.46)

This is a sub-Riemannian version of the curvature-dimension inequality (1.1).

6.1 The heat semigroup

As an operator defined on $C_0^\infty(\mathbb{M})$ the operator $L$ is symmetric with respect to the measure $\mu$ and non positive: For $f \in C_0^\infty(\mathbb{M})$, $<Lf,f> \leq 0$. Therefore, it admits a self-adjoint extension to $L^2(\mathbb{M},\mu)$, the Friedrichs extension. Following an argument of Strichartz [64], Theorem 7.3 p. 246 and p. 261, we now prove that $L$ is essentially self-adjoint on $C_0^\infty(\mathbb{M})$ (the completeness of the metric space $(\mathbb{M}, d)$ is crucial here).

In what follows, to distinguish it from the canonical connection $\nabla$ introduced in section 2.2, we use the notation $\nabla^R$ for the Riemannian connection on $\mathbb{M}$. We recall that, thanks to the assumption (2.11), $\|\nabla^R f\|^2 = \Gamma(f) + \frac{1}{2} \Gamma^Z(f)$, $f \in C^\infty(\mathbb{M})$.

Lemma 6.1 There exists an increasing sequence $h_n \in C_0^\infty(\mathbb{M})$ such that $h_n \not\rightarrow 1$ on $\mathbb{M}$, and $\|\nabla^R h_n\|_\infty \to 0$, as $n \to \infty$.

Proof. As in [32], if we fix a base point $x_0 \in \mathbb{M}$, we can find an exhaustion function $\rho \in C^\infty(\mathbb{M})$ such that $|\rho - d_R(x_0, \cdot)| \leq L$, $|\nabla^R \rho| \leq L$ on $\mathbb{M}$.

By the completeness of $(\mathbb{M}, d_R)$ and the Hopf-Rinow theorem, the level sets $\Omega_s = \{x \in \mathbb{M} | \rho(x) < s\}$ are relatively compact and, furthermore, $\Omega_s \not\rightarrow \mathbb{M}$ as $s \to \infty$. We now pick an increasing sequence of functions $\phi_n \in C^\infty([0, \infty))$ such that $\phi_n \equiv 1$ on $[0, n]$, $\phi_n \equiv 0$ outside $[0, 2n]$, and $|\phi_n'| \leq \frac{2}{n}$. If we set $h_n(x) = \phi_n(\rho(x))$, then we have $h_n \in C^\infty(\mathbb{M})$, $h_n \not\rightarrow 1$ on $\mathbb{M}$ as $n \to \infty$, and

$$
\|\nabla^R h_n\|_\infty \leq \frac{2L}{n}.
$$

In what follows we define the action of the operator $-L$ on $C_0^\infty(\mathbb{M})$ by the equation

$$
< -L \phi, \psi > = \int_{\mathbb{M}} \Gamma(\phi, \psi) d\mu, \quad \psi \in C_0^\infty(\mathbb{M}).
$$

We recall our assumption (2.9). As a corollary of the previous lemma, we obtain:
Proposition 6.2 The operator $L$ is essentially self-adjoint on $C_0^\infty(M)$.

Proof. Denote by $\mathcal{T}$ the Friedrichs extension of $L$ initially defined on $C_0^\infty(M)$. If $L^*$ is the adjoint of $\mathcal{T}$, then according to Reed-Simon [56], p. 137, it is enough to prove that the eigenvalues of the adjoint $L^*$ are negative. We thus have to show that if $L^*f = \lambda f$ with $\lambda > 0$, then $f = 0$.

From the hypoellipticity of $L$, we first deduce that $f$ has to be a smooth function. Now, for $h \in C_0^\infty(M)$,

$$< \Gamma(f, h^2 f) > = -< f, L(h^2 f) > = -< L^* f, h^2 f > = -\lambda < f, h^2 f > = -\lambda < f^2, h^2 > \leq 0.$$  

Since

$$\Gamma(f, h^2 f) = h^2 \Gamma(f, f) + 2 fh \Gamma(f, h),$$

we deduce that

$$< h^2, \Gamma(f, f) > + 2 < fh, \Gamma(f, h) > \leq 0.$$  

Therefore, by Schwarz inequality

$$< h^2, \Gamma(f, f) > \leq 4\|f\|^2_2 \|\Gamma(h, h)\|_\infty.$$  

If we now use the sequence $h_n$ constructed in Lemma 6.1 and let $n \to \infty$, we obtain $\Gamma(f, f) = 0$ and therefore $f = 0$, as desired. \hfill \Box

If $L = -\int_0^{+\infty} \lambda dE_\lambda$ denotes the spectral decomposition of $L$ in $L^2(M, \mu)$, then by definition, the heat semigroup $(P_t)_{t \geq 0}$ is given by $P_t = \int_0^{+\infty} e^{-\lambda t} dE_\lambda$. It is a family of bounded operators on $L^2(M, \mu)$. Since the quadratic form $-< f, Lf >$ is a Dirichlet form in the sense of Fukushima [29], we deduce from the above result that $C_0^\infty(M)$ is dense in the domain of $P_t$ and that $(P_t)_{t \geq 0}$ is a sub-Markov semigroup: it transforms positive functions into positive functions and satisfies

$$P_t 1 \leq 1. \quad (6.47)$$

This property implies in particular

$$\|P_t f\|_{L^1(M)} \leq \|f\|_{L^1(M)}, \quad \|P_t f\|_{L^\infty(M)} \leq \|f\|_{L^\infty(M)}, \quad (6.48)$$

and therefore by the Theorem of Riesz-Thorin

$$\|P_t f\|_{L^p(M)} \leq \|f\|_{L^p(M)}, \quad 1 \leq p \leq \infty. \quad (6.49)$$

Moreover, it can be shown as in [63], Theorem 3.9:

Proposition 6.3 The unique solution of the Cauchy problem

$$\left\{ \begin{array}{l}
\frac{\partial u}{\partial t} - Lu = 0, \\
u(x, 0) = f(x), \quad f \in L^p(M), 1 < p < +\infty,
\end{array} \right.$$

that satisfies $\|u(\cdot, t)\|_p \leq Ce^{Mt}$, for some constants $C$ and $M$, is given by $u(x, t) = P_t f(x)$.  

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Due to the hypoellipticity of $L$, $(t, x) \to P_t f(x)$ is smooth on $\mathbb{M} \times (0, \infty)$ and

$$P_t f(x) = \int_M p(x, y, t) f(y) d\mu(y), \quad f \in C_0^\infty(\mathbb{M}),$$

where $p(x, y, t) > 0$ is the so-called heat kernel associated to $P_t$. Such function is smooth outside the diagonal of $\mathbb{M} \times \mathbb{M}$, and it is symmetric, i.e.,

$$p(x, y, t) = p(y, x, t).$$

By the semi-group property for every $x, y \in \mathbb{M}$ and $0 < s, t$ we have

$$p(x, y, t+s) = \int_M p(x, z, t)p(z, y, s)d\mu(z) = \int_M p(x, z, t)p(y, z, s)d\mu(z) = P_s(p(x, \cdot, t))(y). \quad (6.50)$$

### 6.2 A variational inequality

The goal of this section is to establish a variational inequality which is the cornerstone of all the gradient estimates we shall obtain in the sequel. Henceforth, to simplify the notation we let

$$\Gamma(f) = \Gamma(f, f), \quad f \in C^1(\mathbb{M}).$$

For the sequel it will be convenient to observe that if $\phi \in C^2(\mathbb{R})$, $f \in C^2(\mathbb{M})$, then

$$L(\phi \circ f) = \phi''(f)\Gamma(f) + \phi'(f)Lf. \quad (6.51)$$

In particular, if $f \in C_0^\infty(\mathbb{M})$ and $f \geq 0$, then for any fixed $T > 0$ and $t < T$, we obtain from (6.51)

$$\frac{LP_{T-t}f}{P_{T-t}f} = L(\log P_{T-t}f) + \Gamma(\log P_{T-t}f). \quad (6.52)$$

For such $f$'s we now introduce the two functionals

$$\Phi_1(t) = P_t ((P_{T-t}f)\Gamma(\ln P_{T-t}f)),$$

$$\Phi_2(t) = P_t ((P_{T-t}f)\Gamma^Z(\ln P_{T-t}f)).$$

Notice that for every $T > 0$ the function $u(x, t) = P_{T-t}f(x)$ satisfies the backward Cauchy problem

$$\begin{cases} Lu + \frac{\partial u}{\partial t} = 0 \quad u \geq 0 \text{ in } M \times (-\infty, T), \\
u(x, T) = f(x). \end{cases}$$

It follows that

$$\begin{cases} \Phi_1(0) = P_T(f)\Gamma(\ln P_T f), \quad \Phi_2(0) = P_T(f)\Gamma^Z(\ln P_T f), \\
\Phi_1(T) = P_T(f\Gamma(\ln f)), \quad \Phi_2(T) = P_T(f\Gamma^Z(\ln f)). \end{cases} \quad (6.53)$$

**Lemma 6.4** We have

$$\Phi_1'(t) = 2P_t ((P_{T-t}f)\Gamma_2(\ln P_{T-t}f)),$$

and

$$\Phi_2'(t) = 2P_t ((P_{T-t}f)\Gamma_2^Z(\ln P_{T-t}f)).$$
Proof. Let $V$ be a smooth vector field on $\mathbb{M}$ and let $\Phi_V$ be the functional

$$\Phi_V(t) = P_t \left( (P_{T-t}f)(V \ln P_{T-t}f)^2 \right).$$

We have

$$\Phi_V(t) = P_t \left( L((P_{T-t}f)(V \ln P_{T-t}f)^2) \right) - P_t \left( (LP_{T-t}f)(V \ln P_{T-t}f)^2 \right)$$

$$- 2P_t \left( (VP_{T-t}f)V \left( \frac{LP_{T-t}f}{P_{T-t}f} \right) \right).$$

We now compute

$$P_t \left( L((P_{T-t}f)(V \ln P_{T-t}f)^2) \right) = P_t \left( L(P_{T-t}f)(V \ln P_{T-t}f)^2 \right) + P_t \left( (P_{T-t}f)L(V \ln P_{T-t}f)^2 \right)$$

$$+ 2P_t \left( \Gamma(P_{T-t}f, (V \ln P_{T-t}f)^2) \right)$$

$$= P_t \left( L(P_{T-t}f)(V \ln P_{T-t}f)^2 \right) + 2P_t \left( (P_{T-t}f)(V \ln P_{T-t}f)(LV \ln P_{T-t}f) \right)$$

$$+ 2P_t \left( (P_{T-t}f)\Gamma(V \ln P_{T-t}f, V \ln P_{T-t}f) \right) + 4P_t \left( V \ln P_{T-t}f \Gamma(P_{T-t}f, V \ln P_{T-t}f) \right).$$

By taking into account (6.52) we obtain

$$\Phi_V(t) = 2P_t \left( (V P_{T-t}f) [L, V] \ln P_{T-t}f \right) + 2P_t \left( (P_{T-t}f)\Gamma(V \ln P_{T-t}f, V \ln P_{T-t}f) \right)$$

$$+ 4P_t \left( V \ln P_{T-t}f \Gamma(P_{T-t}f, V \ln P_{T-t}f) \right) - 2P_t \left( (V P_{T-t}f)VT \ln P_{T-t}f \ln P_{T-t}f \right).$$

We now observe that

$$V \Gamma(\ln P_{T-t}f, \ln P_{T-t}f) = 2\Gamma(\ln P_{T-t}f, Z \ln P_{T-t}f) + 2 \sum_{i=1}^{d} (X_i \ln P_{T-t}f)([Z, X_i] \ln P_{T-t}f).$$

Thus,

$$\Phi_V'(t) = 2P_t \left( (P_{T-t}f)\Gamma^V_2(\ln P_{T-t}f, \ln P_{T-t}f) \right)$$

$$- 4P_t \left( (VP_{T-t}f) \sum_{i=1}^{d} (X_i \ln P_{T-t}f)([V, X_i] \ln P_{T-t}f) \right),$$

where we have defined

$$\Gamma^V_Y(f, g) = \frac{1}{2} (L((V f)(V g)) - V f V L g - V g V L f).$$

We first apply (6.54) with $V = X_j$, and sum in $j = 1, ..., d$ to obtain

$$\Phi_V'(t) = 2P_t \left( (P_{T-t}f)\Gamma_2(\ln P_{T-t}f) \right).$$

Here, we have used to skew-symmetry of $[X_i, X_j]$, which gives

$$\sum_{j=1}^{d} (X_j P_{T-t}f) \sum_{i=1}^{d} (X_i \ln P_{T-t}f)([X_j, X_i] \ln P_{T-t}f) = 0.$$
We next apply (6.54) with $V = Z_{mn}$ and sum in $m, n = 1, ..., h$, obtaining
\[
\Phi'_2(t) = 2P_t \left( (P_{t-f}) \Gamma_2 (\ln P_{t-f}) \right) - 4 \sum_{m,n=1}^{h} P_t \left( (Z_{mn} P_{t-f}) \sum_{i=1}^{d} (X_i \ln P_{t-f}) ([Z_{mn}, X_i] \ln P_{t-f}) \right).
\]

We now observe that, thanks to the crucial assumption (2.5), one has
\[
\sum_{m,n=1}^{h} (Z_{mn} P_{t-f}) \sum_{i=1}^{d} (X_i \ln P_{t-f}) ([Z_{mn}, X_i] \ln P_{t-f}) = 0.
\]

Therefore,
\[
\Phi'_2(t) = 2P_t \left( (P_{t-f}) \Gamma_2 (\ln P_{t-f}) \right).
\]

\section{Proposition 6.5} Let $b$ be a smooth, positive and decreasing function on the time interval $[0, T]$. On $\mathbb{M} \times [0, T]$, we have
\[
\left( -\frac{b'}{2\rho_2} \Phi_1 + b\Phi_2 \right)' \geq -\frac{2b'\gamma}{d\rho_2} L P_T f + \frac{b'\gamma^2}{d\rho_2} P_T f,
\]
where
\[
\gamma = \frac{d}{4} \left( \frac{b''}{b'} + \frac{\kappa b'}{\rho_2 b} + 2\rho_1 \right).
\]

\textbf{Proof.} To prove this result we apply the sub-Riemannian curvature-dimension inequality (6.46), in combination with Lemma 6.4. If $a$ and $b$ are positive functions we thus obtain
\[
(a\Phi_1 + b\Phi_2)' \geq \left( a' + 2\rho_1 a - 2\kappa \frac{a^2}{b} \right) \Phi_1 + (b' + 2\rho_2 a)\Phi_2 + \frac{2a}{d} \left( P_t ((P_{t-f})(L \ln P_{t-f})^2) \right)
\]
But, for every $\gamma \in \mathbb{R}$,
\[
(L \ln P_{t-f})^2 \geq 2\gamma L \ln P_{t-f} - \gamma^2,
\]
and
\[
L \ln P_{t-f} = \frac{LP_{t-f}}{L_{t-f}} - \Gamma(\ln P_{t-f}).
\]
Therefore,
\[
(a\Phi_1 + b\Phi_2)' \geq \left( a' + 2\rho_1 a - 2\kappa \frac{a^2}{b} - 2\frac{a\gamma}{d} \right) \Phi_1 + (b' + 2\rho_2 a)\Phi_2 + \frac{2a\gamma}{d} \left( LP_{t-f} - 2\frac{a\gamma^2}{d} P_T f \right).
\]
By taking $a, b, \gamma$ such that
\[
a' + 2\rho_1 a - 2\kappa \frac{a^2}{b} - \frac{4}{d} a = 0,
\]
\[
b' + 2\rho_2 a = 0,
\]
we obtain the desired conclusion. \hfill \square
6.3 Li-Yau type estimates

In this section, we extend the celebrated Li-Yau inequality in [46] to the heat semigroup associated with the subelliptic operator $L$. Let us mention that, in this setting, related inequalities were obtained by Cao-Yau [14]. However, these authors work only in the case of a compact manifold and do not base their study on the analysis of a tensor like $\mathcal{R}$. As a consequence they do not obtain a control of the constants in terms of the geometry of the manifold.

**Proposition 6.6 (Gradient estimate)** Assume that (6.45) hold. Let $f \in C^\infty_0(M)$, with $f \geq 0$, then the following inequality holds for $t > 0$:

$$
\Gamma(\ln P_t f) + \frac{2\rho_2}{3} \Gamma^Z(\ln P_t f) \leq \left(1 + \frac{3\kappa}{2\rho_2} - \frac{2\rho_1}{3} \right) \frac{L P_t f}{P_t f} + \frac{2\rho_1}{3} t - \frac{\rho_1}{2} \left(1 + \frac{3\kappa}{2\rho_2}\right) + \frac{d}{2t} \left(1 + \frac{3\kappa}{2\rho_2}\right)^2.
$$

**Proof.** If we apply Proposition 6.5 with $b(t) = (T - t)^3$, we obtain:

$$
\gamma(t) = \frac{d}{4} \left(\frac{2\rho_1(T - t) - \frac{3\kappa}{\rho_2} t}{T - t} - 2\right) = \frac{d}{2} \left(\rho_1 - \frac{1}{T - t} \left(1 + \frac{3\kappa}{2\rho_2}\right)\right),
$$

$$
\int_0^T b'(t) \gamma(t) dt = -\frac{\rho_1}{2} T^3 + \frac{3d}{4} \left(1 + \frac{3\kappa}{2\rho_2}\right) T^2,
$$

and

$$
\int_0^T b'(t) \gamma(t)^2 dt = -\frac{3d^2}{16} \left(\frac{4\rho_1^2}{3} T^3 + 4 \left(1 + \frac{3\kappa}{2\rho_2}\right)^2 T - 4\rho_1 \left(1 + \frac{3\kappa}{2\rho_2}\right) T^2\right).
$$

The result easily follows. □

**Remark 6.7** When $\rho_1 > 0$, then (6.45) and (6.46) also hold in particular with $\rho_1 = 0$. As a consequence of Proposition 6.6

$$
\Gamma(\ln P_t f) + \frac{2\rho_2}{3} \Gamma^Z(\ln P_t f) \leq \left(1 + \frac{3\kappa}{2\rho_2}\right) \frac{L P_t f}{P_t f} + \frac{d}{2t} \left(1 + \frac{3\kappa}{2\rho_2}\right)^2.
$$

(6.55)

This inequality leads to a optimal Harnack inequality only when $\rho_1 = 0$ (which corresponds to $\text{Ric} \geq 0$). Sharper bounds in the strictly positive curvature case will be obtained in (12.89) by a different choice of the function $b(t)$.

**Remark 6.8** If we set

$$
D = d \left(1 + \frac{3\kappa}{2\rho_2}\right),
$$

(6.56)

then, as a consequence of (6.55), we obtain that in the case $\rho_1 \geq 0$,

$$
\frac{L P_t f}{P_t f} \geq -\frac{D}{2t}.
$$

(6.57)

The constant $-\frac{D}{2t}$ in (6.57) is, in general, not sharp, as the example of the heat semigroup on a graded nilpotent Lie group shows. In such case, in fact, one can argue as in [28] to...
show that the heat kernel \( p(x,y,t) \) is homogeneous of degree \(-\frac{Q}{2}\) with respect to the parabolic dilations \((x,t) \rightarrow (\delta(\lambda)(x), \lambda^2 t)\), where \(\delta(\lambda)\) represent the non-isotropic dilations associated with the grading of the Lie algebra of \(\mathbb{M}\), and \(Q = d + 2\eta\) indicates the corresponding homogeneous dimension of \(\mathbb{M}\). From such homogeneity of \(p(x,y,t)\), a scaling argument produces the estimate

\[
\frac{L_P f}{P_t f} \geq -\frac{Q}{2t},
\]

which, unlike (6.57), is best possible. However, the estimate (6.57) is sharp in the case of the Laplace operator in \(\mathbb{R}^d\) for which we have \(\kappa = 0\). It seems difficult to obtain sharp geometric constants by using only the inequality (6.46). In part this is due to the fact that when we apply Cauchy-Schwarz inequality in the Bochner formulas, we loose a piece of information. Another difference is in the presence of the commutator terms in (6.46). These aspects are quite different from the Riemannian case, for which the CD\((d, R)\) inequality (1.1) provides sharp geometric constants (see [4], [45]).

7 Stochastic completeness of the heat semigroup

In [75] Yau proved that if \(\mathbb{M}\) is a complete Riemannian manifold with a Ric \(\geq \rho\), with \(\rho \in \mathbb{R}\), then one has the stochastic completeness of the heat semigroup, i.e. \(P_t 1 = 1\). Under the same assumptions, Dodziuk [24] proved that bounded solutions of the heat equation are characterized by their initial condition.

In this section, we extend these theorems of Yau and Dodziuk to our setting. Throughout this section, we assume that \((\mathbb{M}, d)\) is complete and that for every smooth function \(f : \mathbb{M} \rightarrow \mathbb{R}\) and every \(\nu > 0\),

\[
\Gamma_2(f,f) + \nu \Gamma^Z_2(f,f) \geq \frac{1}{d}(Lf)^2 + \left(\rho_1 - \frac{\kappa}{\nu}\right) \Gamma(f,f) + \rho_2 \Gamma^Z(f,f),
\]

(7.58)

where \(\rho_1 \in \mathbb{R}\), whereas \(\rho_2, \kappa > 0\). As in the previous section, the idea is to study monotone increasing functionals of the heat semigroup.

**Proposition 7.1** Let \(f \in C_0^\infty(\mathbb{M})\) and \(T > 0\). If \(a, b : [0, T] \rightarrow \mathbb{R}\) are two positive functions such that

\[
a'(t) + 2\rho_1 a(t) - \frac{2\kappa a(t)^2}{b(t)} \geq 0
\]

\[
b'(t) + 2\rho_2 a(t) \geq 0,
\]

then the functional

\[
E(t) = a(t) P_t(\Gamma(P_{T-t} f)) + b(t) P_t(\Gamma^Z(P_{T-t} f))
\]

is monotone increasing.

**Proof.** We begin by noting that, with \(\Phi_1(t), \Phi_2(t)\) as in section 6.2 we have

\[
E(t) = a(t)\Phi_1(t) + b(t)\Phi_2(t).
\]
Lemma [5.4] now gives

$$E'(t) = a'(t)P_t(\Gamma(P_{T-t}f)) + b'(t)P_t(\Gamma^Z(P_{T-t}f)) + 2a(t)P_t(\Gamma_2(P_{T-t}f)) + 2b(t)P_t(\Gamma^Z_2(P_{T-t}f)).$$

Using the curvature-dimension inequality (6.46) we now find

$$a(t)P_t(\Gamma_2(P_{T-t}f)) + b(t)P_t(\Gamma^Z_2(P_{T-t}f)) \geq a(t)\left(\rho_1 - \frac{ka(t)}{b(t)}\right)P_t(\Gamma(P_{T-t}f)) + \rho_2P_t(\Gamma^Z(P_{T-t}f)).$$

Therefore,

$$E'(t) \geq \left(a'(t) + 2\rho_1a(t) - \frac{2ka(t)^2}{b(t)}\right)P_t(\Gamma(P_{T-t}f)) + (b'(t) + 2\rho_2a(t))P_t(\Gamma^Z(P_{T-t}f)) \geq 0.$$

The desired conclusion immediately follows from this inequality.

The following energy estimate represents a basic consequence of Proposition 7.1.

**Corollary 7.2** There exists \( \nu \in \mathbb{R} \) \((\nu \leq 2\min\{\rho_2, \rho_1 - \kappa\} \text{ will do})\), such that for every \( f \in C_0^\infty(M) \), one has

$$\Gamma(P_tf) + \Gamma^Z(P_tf) \leq e^{-\nu t} \left(P_t\Gamma(f) + P_t\Gamma^Z(f)\right).$$

As a consequence, for every \( 1 \leq p \leq \infty \) one obtains

$$\|\Gamma(P_tf)\|_{L^p(M)} \leq e^{-\nu t} \left(\|\Gamma(f)\|_{L^p(M)} + \|\Gamma^Z(f)\|_{L^p(M)}\right), \quad t \geq 0.$$

**Proof.** Let \( t > 0 \) and consider the interval \( 0 \leq s \leq t \). With the choice

$$a(s) = b(s) = e^{-\nu s}, \quad 0 \leq s \leq t,$$

where \( \nu \in \mathbb{R} \) satisfies \( \nu \leq 2\min\{\rho_2, \rho_1 - \kappa\} \), Proposition 7.1 gives \( E(0) \leq E(s) \). This inequality reads

$$\Gamma(P_tf) + \Gamma^Z(P_tf) \leq e^{-\nu s} \left(P_s(\Gamma(P_{t-s}f)) + P_s(\Gamma^Z(P_{t-s}f))\right).$$

Letting \( s \to t^- \) we obtain (7.59). Combining (7.59) with (6.49) we deduce (7.60).

**Theorem 7.3** For \( t \geq 0 \), one has \( P_t1 = 1 \).

**Proof.** Let \( f, g \in C_0^\infty(M) \), we have

$$\int_M (P_tf - f)g d\mu = \int_0^t \int_M \left(\frac{\partial}{\partial s} P_sf\right) g d\mu ds = \int_0^t \int_M (LP_sf) g d\mu ds = -\int_0^t \int_M \Gamma(P_sf) gd\mu ds.$$

By means of (7.60), and Cauchy-Schwarz inequality, we find

$$\left|\int_M (P_tf - f)g d\mu\right| \leq \left(\int_0^t e^{-\nu s} ds\right) \sqrt{\|\Gamma(f)\|_\infty + \|\Gamma^Z(f)\|_\infty} \int_M \Gamma(g)^{\frac{1}{2}} d\mu. \quad (7.61)$$
We now apply (7.61) with \( f = h_n \), where \( h_n \) are the functions in Lemma 6.1, and then let \( n \to \infty \). Since by Beppo Levi’s monotone convergence theorem we have \( P_t h_n(x) \to P_t 1(x) \) for every \( x \in \mathbb{M} \), we see that the left-hand side converges to \( \int_{\mathbb{M}} (P_t 1 - 1) g d\mu \). We claim that the right-hand side converges to zero. To see this observe that, thanks to the assumption (2.11), we have for any \( \phi \in C^\infty(\mathbb{M}) \)

\[
|\nabla R \phi|^2 = \Gamma(\phi) + \frac{1}{2} \Gamma^Z(\phi).
\]

Therefore,

\[
\sqrt{\|\Gamma(h_n)\|_\infty + \|\Gamma^Z(h_n)\|_\infty} \leq C ||\nabla R h_n||_\infty \to 0,
\]

as \( n \to \infty \).

We thus reach the conclusion

\[
\int_{\mathbb{M}} (P_t 1 - 1) g d\mu = 0, \quad g \in C_0^\infty(\mathbb{M}).
\]

It follows that \( P_t 1 = 1 \).

\[ \square \]

Theorem 7.3 is equivalent to the uniqueness in the Cauchy problem for initial data in \( L^\infty(\mathbb{M}) \).

**Proposition 7.4** The unique bounded solution of the Cauchy problem

\[
\begin{cases}
\frac{\partial u}{\partial t} - Lu = 0, \\
u(x, 0) = f(x), & f \in L^\infty(\mathbb{M}),
\end{cases}
\]

is given by \( u(x, t) = P_t f(x) \).

**Proof.** Since the vector fields \( X_i \)'s are locally Lipschitz (because smooth), for every \( x \in \mathbb{M} \), the stochastic differential equation

\[
Y_t^x = x + \int_0^t X_0(Y_s^x) ds + \sqrt{2} \sum_{i=1}^d \int_0^t X_i(Y_s^x) \circ dB_s^i,
\]

has a solution up to a stopping time \( e(x) \), where \( (B_t)_{t \geq 0} \) is a \( d \)-dimensional Brownian motion. It is seen that

\[
(Q_t f)(x) = \mathbb{E}(1_{t < e(x)} f(Y_t^x)), \quad f \in L^2(\mathbb{M}),
\]

defines a contraction semigroup on \( L^2(\mathbb{M}) \). By Itô’s formula its generator on \( C_0(\mathbb{M}) \) is given by \( L \). By uniqueness of the heat semigroup, we actually have \( Q_t f(x) = P_t f(x) \), for \( f \in C_0(\mathbb{M}) \). By using the definition of \( Q_t \), we deduce that for \( f \in C_0(\mathbb{M}) \),

\[
P_t f(x) = \mathbb{E}(1_{t < e(x)} f(Y_t^x))
\]

By applying the previous equality with \( f h_n \), and letting \( n \to +\infty \), we deduce from Beppo Levi’s monotone convergence theorem that

\[
P_t f(x) = \mathbb{E}(1_{t < e(x)} f(Y_t^x))
\]

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when \( f \) is a bounded and smooth function. Since \( P_t1 = 1 \), almost surely \( e(x) = +\infty \), and
\[
P_tf(x) = \mathbb{E}(f(Y^x_t)).
\]
Let now \( u(t,x) \) be a bounded solution of the Cauchy problem:
\[
\left\{
\begin{array}{l}
\frac{\partial u}{\partial t} - Lu = 0, \\
u(x,0) = f(x),
\end{array}
\right.
\]
where \( f \) is a bounded and smooth function. From Itô’s formula, for \( t > 0 \) the process \((u(t - s, Y^x_s))_{0 \leq s \leq t}\) is a local martingale and since \( u \) is bounded, it is a martingale. The expectation of \((u(t - s, Y^x_s))_{0 \leq s \leq t}\) is therefore constant. From this we deduce
\[
u(t,x) = \mathbb{E}(f(Y^x_t)) = P_tf(x).
\]
Finally, when \( f \in L^\infty(M) \), for the Cauchy problem
\[
\left\{
\begin{array}{l}
\frac{\partial u}{\partial t} - Lu = 0, \\
u(x,0) = f(x),
\end{array}
\right.
\]
we have from the regularization property of the heat semigroup and the above uniqueness result: For every \( \tau > 0 \),
\[
u(x,t + \tau) = \int_M p(x,y,t)u(y,\tau)\mu(dy).
\]
By letting \( \tau \to 0 \), we obtain \( u(x,t) = P_tf(x) \) as desired.

\section{A parabolic Harnack inequality}

In this section we generalize the celebrated Harnack inequality in \[46\] to nonnegative solutions of the heat equation \( H = L - \frac{\partial}{\partial t} \) on \( \mathbb{M} \) which are in the form \( u(x,t) = P_tf(x) \), for some \( f \in C^\infty(M) \cap L^\infty(M) \). Theorem \[8.1\] below should be seen as a generalization of (i) of Theorem 2.2 in \[46\], in the case of a zero potential \( q \). One should also see the paper \[14\], where the authors deal with subelliptic operators on a compact manifold. As we have mentioned, they do not obtain bounds which explicitly depend solely on the geometry of the underlying manifold.

\textbf{Theorem 8.1} Let \( \mathbb{M} \) be a complete sub-Riemannian manifold such that \[6.45\] holds with \( \rho_1 \geq 0 \). Let \( f \in C^\infty(M) \) be such that \( 0 \leq f \leq M \), and consider \( u(x,t) = P_tf(x) \). For every \((x,s),(y,t) \in \mathbb{M} \times (0,\infty) \) with \( s < t \) one has with \( D \) as in \[6.50\]
\[
u(x,s) \leq u(y,t) \left( \frac{t}{s} \right) \frac{D}{d} \exp \left( \frac{D d(x,y)^2}{4(t-s)} \right).
\]
\( (8.62) \)

\textbf{Proof.} Let \( f \) be as in the statement of the theorem, and consider \( f_n = h_n f \), where \( h_n \) is the sequence in Lemma \[6.1\]. Then, \( f_n \in C^\infty_0(\mathbb{M}) \), and \( 0 \leq f_n \not\sim f \). By Beppo Levi’s monotone
convergence theorem we have \( u_n(x, t) = P_t f_n(x) \) and \( u(x, t) = P_t f(x) \) for every \((x, t) \in M \times (0, \infty)\). Since \( Lu_n = \frac{\partial u_n}{\partial t} \), in terms of \( u_n \) inequality (6.55) can be reformulated as

\[
\Gamma(\ln u_n) + \frac{2\rho_2}{3} \Gamma Z (\ln u_n) \leq (1 + \frac{3\kappa}{2\rho_2}) \frac{\partial \log u_n}{\partial t} + \frac{d \left( \frac{1 + 3\kappa}{2\rho_2} \right)^2}{2t}.
\]

In particular, this implies

\[- (1 + \frac{3\kappa}{2\rho_2}) \frac{\partial \ln u_n}{\partial t} \leq -\Gamma(\ln u_n) + \frac{d \left( \frac{1 + 3\kappa}{2\rho_2} \right)^2}{2t}.
\] (8.63)

We now fix two points \((x, s), (y, t) \in M \times (0, \infty)\), with \( s < t \). Let \( \gamma(\tau), 0 \leq \tau \leq T \) be a subunit path such that \( \gamma(0) = y, \gamma(T) = x \). Consider the path in \( M \times (0, \infty) \) defined by

\[
\alpha(\tau) = \left( \gamma(\tau), t + \frac{s - t}{T} \right), \quad 0 \leq \tau \leq T,
\]

so that \( \alpha(0) = (y, t), \alpha(T) = (x, s) \). We have

\[
\ln \frac{u_n(x, s)}{u_n(y, t)} = \int_0^T d \frac{\ln u_n(\alpha(\tau))}{d \tau} = \int_0^T \left[ < \gamma'(\tau), \nabla_R (\ln u_n)(\alpha(\tau)) > - \frac{t - s}{T} \frac{\partial \ln u_n}{\partial t}(\alpha(\tau)) \right] d \tau.
\]

Now since \( \gamma(\tau) \) is subunitary we have

\[
|< \gamma'(\tau), \nabla_R (\ln u_n)(\alpha(\tau)) >| \leq \Gamma(\ln u_n(\alpha(\tau)))^{\frac{1}{2}},
\]

and applying (8.63) for any \( \epsilon > 0 \) we find

\[
\log \frac{u_n(x, s)}{u_n(y, t)} \leq T^{\frac{1}{2}} \left( \int_0^T \Gamma(\ln u_n)(\alpha(\tau)) d \tau \right)^{\frac{1}{2}} - \frac{t - s}{T} \int_0^T \frac{\partial \ln u_n}{\partial t}(\alpha(\tau)) d \tau
\]

\[
\leq \frac{1}{2\epsilon} T + \frac{\epsilon}{2} \int_0^T \Gamma(\ln u_n)(\alpha(\tau)) d \tau - \frac{t - s}{T(1 + \frac{3\kappa}{2\rho_2})} \int_0^T \Gamma(\ln u_n)(\alpha(\tau)) d \tau
\]

\[
- \frac{d \left( \frac{1 + 3\kappa}{2\rho_2} \right)}{2T} \int_0^T \frac{d \tau}{t + \frac{2s - t}{T}}.
\]

If we now choose \( \epsilon > 0 \) such that

\[
\frac{\epsilon}{2} = \frac{t - s}{T(1 + \frac{3\kappa}{2\rho_2})},
\]

we obtain from the latter inequality

\[
\log \frac{u_n(x, s)}{u_n(y, t)} \leq \frac{\ell_s(\gamma)^2}{4(t - s)} + \frac{d \left( \frac{1 + 3\kappa}{2\rho_2} \right)}{2} \ln \left( \frac{t}{s} \right),
\]
where we have denoted by $\ell_s(\gamma)$ the subunitary length of $\gamma$. If we now minimize over all subunitary paths joining $y$ to $x$, and we exponentiate, we obtain

$$u_n(x, s) \leq u_n(y, t) \left( \frac{t}{s} \right)^{\frac{d}{2} \left( 1 + \frac{3\kappa}{2\rho_2} \right)} \exp \left( \frac{d(x, y)^2 \left( 1 + \frac{3\kappa}{2\rho_2} \right)}{4(t - s)} \right).$$

Letting $n \to \infty$ in this inequality we finally obtain \eqref{8.62}.

The following result represents an important consequence of Theorem 8.1.

**Corollary 8.2** Let $p(x, y, t)$ be the heat kernel on $\mathbb{M}$. For every $x, y, z \in \mathbb{M}$ and every $0 < s < t < \infty$ one has

$$p(x, y, s) \leq p(x, z, t) \left( \frac{t}{s} \right)^{\frac{d}{2}} \exp \left( \frac{D d(y, z)^2}{d 4(t - s)} \right).$$

*Proof.* Let $\tau > 0$ and $x \in \mathbb{M}$ be fixed. By Hörmander’s hypoellipticity theorem [37] we know that $p(x, \cdot, \cdot + \tau) \in C^\infty(\mathbb{M} \times (-\tau, \infty))$. From \eqref{6.50} we have

$$p(x, y, s + \tau) = P_s(p(x, \cdot, \tau))(y)$$

and

$$p(x, z, t + \tau) = P_t(p(x, \cdot, \tau))(z).$$

Since we cannot apply Theorem 8.1 directly to $u_n(y, t) = P_t(p(x, \cdot, \tau))(y)$, we consider as in the proof of the latter $u_n(y, t) = P_t(h_n p(x, \cdot, \tau))(y)$, where $h_n \in C^\infty_0(\mathbb{M})$, $0 \leq h_n \leq 1$, and $h_n \not\rightarrow 1$. From \eqref{8.62} we find

$$P_s(h_n p(x, \cdot, \tau))(y) \leq P_t(h_n p(x, \cdot, \tau))(z) \left( \frac{t}{s} \right)^{\frac{d}{2}} \exp \left( \frac{D d(y, z)^2}{d 4(t - s)} \right).$$

Letting $n \to \infty$, by Beppo Levi’s monotone convergence theorem we obtain

$$p(x, y, s + \tau) \leq p(x, z, t + \tau) \left( \frac{t}{s} \right)^{\frac{d}{2}} \exp \left( \frac{D d(y, z)^2}{d 4(t - s)} \right).$$

The desired conclusion follows by letting $\tau \to 0$.

\[\square\]

**9 Off-diagonal Gaussian upper bounds for $p(x, y, t)$**

Let $\mathbb{M}$ be a complete sub-Riemannian manifold such that \eqref{6.35} holds with $\rho_1 \geq 0$. Fix $x \in \mathbb{M}$ and $t > 0$. Applying Corollary 8.2 to $(y, t) \to p(x, y, t)$ for every $y \in B(x, \sqrt{t})$ we find

$$p(x, x, t) \leq 2^d e^{\frac{D}{2\rho_1}} p(x, y, 2t) = C(d, \kappa, \rho_2) p(x, y, 2t).$$

Integration over $B(x, \sqrt{t})$ gives

$$p(x, x, t) \mu(B(x, \sqrt{t})) \leq C(d, \kappa, \rho_2) \int_{B(x, \sqrt{t})} p(x, y, 2t) d\mu(y) \leq C(d, \kappa, \rho_2),$$

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where we have used $P_t 1 \leq 1$. This gives the on-diagonal upper bound
\[
p(x, x, t) \leq \frac{C(d, \kappa, \rho_2)}{\mu(B(x, \sqrt{t}))}.
\] (9.64)

The aim of this section is to establish the following off-diagonal upper bound for the heat kernel.

**Theorem 9.1** Let $\mathbb{M}$ be a complete sub-Riemannian manifold such that (6.45) holds with $\rho_1 \geq 0$. For any $0 < \epsilon < 1$ there exists a constant $C(d, \kappa, \rho_2, \epsilon) > 0$, which tends to $\infty$ as $\epsilon \to 0^+$, such that for every $x, y \in \mathbb{M}$ and $t > 0$ one has
\[
p(x, y, t) \leq \frac{C(d, \kappa, \rho_2, \epsilon)}{\mu(B(x, \sqrt{t})))}\exp \left(-\frac{d(x, y)^2}{(4 + \epsilon)t}\right).
\]

**Proof.** We suitably adapt here an idea in [14] for the case of a compact manifold without boundary. Since, however, we allow the manifold $\mathbb{M}$ to be non-compact, we need to take care of this aspect. Corollary 7.2 will prove crucial in this connection. Given $T > 0$, and $\alpha > 0$ we fix $0 < \tau \leq (1 + \alpha)T$. For a function $\psi \in C^\infty_c(\mathbb{M})$, with $\psi \geq 0$, in $\mathbb{M} \times (0, \tau)$ we consider the function
\[
f(y, t) = \int_{\mathbb{M}} p(y, z, t)p(x, z, T)\psi(z)d\mu(z), \quad x \in \mathbb{M}.
\]
Since $f = P_t(p(x, \cdot, T)\psi)$, it satisfies the Cauchy problem
\[
\begin{cases}
L f - f_t = 0 & \text{in } \mathbb{M} \times (0, \tau),
\end{cases}
\]
\[
f(z, 0) = p(x, z, T)\psi(z), \quad z \in \mathbb{M}.
\]

Notice that thanks to Hörmander’s theorem [37] we know $y \to p(x, y, T)$ is in $C^\infty(\mathbb{M})$, and therefore $p(x, \cdot, T)\psi \in L^\infty(\mathbb{M})$. Moreover, (6.49) gives
\[
||P_t(p(x, \cdot, T)\psi)||^2_{L^2(\mathbb{M})} \leq ||p(x, \cdot, T)\psi||^2_{L^2(\mathbb{M})} = \int_{\mathbb{M}} p(x, z, T)^2\psi(z)d\mu(z) < \infty,
\]
and therefore
\[
\int_0^\tau \int_{\mathbb{M}} f(y, t)^2d\mu(z)dt \leq \tau \int_{\mathbb{M}} p(x, z, T)^2\psi(z)d\mu(z)dt < \infty. \quad (9.65)
\]
Invoking (7.59) in Corollary 7.2 we have
\[
\Gamma(f)(z, t) \leq e^{-\nu t}(P_t\Gamma(p(x, \cdot, T)\psi)(z) + P_t\Gamma^Z(p(x, \cdot, T)\psi)(z)).
\]

This allows to conclude
\[
\int_0^\tau \int_{\mathbb{M}} \Gamma(f)(z, t)^2d\mu(z)dt \leq e^{\nu|\tau|} \int_{\mathbb{M}} \left\{\Gamma(p(x, \cdot, T)\psi)(z)^2 + \Gamma^Z(p(x, \cdot, T)\psi)(z)^2\right\}d\mu(z) < \infty. \quad (9.66)
\]
We now consider a function $g \in C^1([0, (1 + \alpha)T], \text{Lip}_d(\mathbb{M})) \cap L^\infty(\mathbb{M} \times (0, (1 + \alpha)T))$ such that
\[
- \frac{\partial g}{\partial t} \geq \frac{1}{2}\Gamma(g), \quad \text{on } \mathbb{M} \times (0, (1 + \alpha)T). \quad (9.67)
\]
Since
\[(L - \frac{\partial}{\partial t})f^2 = 2f(L - \frac{\partial}{\partial t})f + 2\Gamma(f) = 2\Gamma(f),\]
multiplying this identity by \(h_n^2(y)e^g(y,t)\), where \(h_n\) is the sequence in Lemma\,[6.1] and integrating by parts, we obtain
\[
0 = 2 \int_0^T \int_S h_n^2 e^g \Gamma(f) d\mu(y) dt - \int_0^T \int_S h_n^2 e^g (L - \frac{\partial}{\partial t}) f^2 d\mu(y) dt
\]
\[
= 2 \int_0^T \int_S h_n^2 e^g \Gamma(f) d\mu(y) dt + 4 \int_0^T \int_S h_n e^g \Gamma(h_n,f) d\mu(y) dt
\]
\[
+ 2 \int_0^T \int_S h_n^2 e^g f \Gamma(f,g) d\mu(y) dt - \int_0^T \int_S h_n e^g f^2 \frac{\partial g}{\partial t} d\mu(y) dt
\]
\[
- \int_S h_n e^g f^2 d\mu(y) \bigg|_{t=0}^{t=\tau} + \int_S h_n e^g f^2 d\mu(y) \bigg|_{t=\tau}^{t=\tau}
\]
\[
\geq 2 \int_0^T \int_S h_n^2 e^g \sum_{i=1}^d \left( X_i f + \frac{f}{2} X_i g \right)^2 d\mu(y) dt + 4 \int_0^T \int_S h_n e^g \Gamma(h_n,f) d\mu(y) dt
\]
\[
+ \int_S h_n e^g f^2 d\mu(y) \bigg|_{t=\tau}^{t=\tau} - \int_S h_n e^g f^2 d\mu(y) \bigg|_{t=0}^{t=0},
\]
where in the last inequality we have made use of the assumption (9.67) on \(g\). From this we conclude
\[
\int_S h_n e^g f^2 d\mu(y) \bigg|_{t=\tau}^{t=\tau} \leq \int_S h_n e^g f^2 d\mu(y) \bigg|_{t=0}^{t=0} - 4 \int_0^T \int_S h_n e^g \Gamma(h_n,f) d\mu(y) dt.
\]
We now claim that
\[
\lim_{n \to \infty} \int_0^T \int_S h_n e^g \Gamma(h_n,f) d\mu(y) dt = 0.
\]
To see this we apply Cauchy-Schwarz inequality which gives
\[
\left| \int_0^T \int_S h_n e^g \Gamma(h_n,f) d\mu(y) dt \right| \leq \left( \int_0^T \int_S h_n^2 e^g f^2 \Gamma(h_n) d\mu(y) dt \right)^{\frac{1}{2}} \left( \int_0^T \int_S e^g \Gamma(f) d\mu(y) dt \right)^{\frac{1}{2}}
\]
\[
\leq \left( \int_0^T \int_S e^g f^2 \Gamma(h_n) d\mu(y) dt \right)^{\frac{1}{2}} \left( \int_0^T \int_S e^g \Gamma(f) d\mu(y) dt \right)^{\frac{1}{2}} \to 0,
\]
as \(n \to \infty\), thanks to (9.65), (9.66). With the claim in hands we now let \(n \to \infty\) in the above inequality obtaining
\[
\int_S e^{g(y,\tau)} f^2(y,\tau) d\mu(y) \leq \int_S e^{g(y,0)} f^2(y,0) d\mu(y).
\]
(9.68)
At this point we fix \(x \in \mathbb{M}\) and for \(0 < t \leq \tau\) consider the indicator function \(1_{B(x,\sqrt{t})}\) of the ball \(B(x,\sqrt{t})\). Let \(\psi_k \in C_0^\infty(\mathbb{M})\), \(\psi_k \geq 0\), be a sequence such that \(\psi_k \to 1_{B(x,\sqrt{t})}\) in \(L^2(\mathbb{M})\), with \(\text{supp } \psi_k \subset B(x,100\sqrt{t})\). Slightly abusing the notation we now set
\[
f(y,s) = P_s(p(x,\cdot,T)1_{B(x,\sqrt{t})})(y) = \int_{B(x,\sqrt{t})} p(y,z,s)p(x,z,T) d\mu(z).
\]
Thanks to the symmetry of \( p(x,y,s) = p(y,x,s) \), we have

\[
f(x,T) = \int_{B(x,\sqrt{T})} p(x,z,T)^2 \, d\mu(z). \tag{9.69}
\]

Applying (9.68) to \( f_k(y,s) = P_s(p(x,\cdot, T) \psi_k(y), \) we find

\[
\int_{\mathbb{M}} e^{g(y,\tau)} f_k^2(y,\tau) \, d\mu(y) \leq \int_{\mathbb{M}} e^{g(y,0)} f_k^2(y,0) \, d\mu(y). \tag{9.70}
\]

At this point we observe that as \( k \to \infty \)

\[
\left| \int_{\mathbb{M}} e^{g(y,\tau)} f_k^2(y,\tau) \, d\mu(y) - \int_{\mathbb{M}} e^{g(y,\tau)} f^2(y,\tau) \, d\mu(y) \right| \\
\leq 2 ||e^{g(\cdot,\tau)}||_{L^\infty(\mathbb{M})} ||p(x,\cdot, T)||_{L^2(\mathbb{M})} ||p(x,\cdot, \tau)||_{L^\infty(B(x,110\sqrt{T}))} ||\psi_k - 1_{B(x,\sqrt{T})}||_{L^2(\mathbb{M})} \to 0.
\]

By similar considerations we find

\[
\left| \int_{\mathbb{M}} e^{g(y,0)} f_k^2(y,0) \, d\mu(y) - \int_{\mathbb{M}} e^{g(y,0)} f^2(y,0) \, d\mu(y) \right| \\
\leq 2 ||e^{g(\cdot,0)}||_{L^\infty(\mathbb{M})} ||p(x,\cdot, T)||_{L^\infty(B(x,110\sqrt{T}))} ||\psi_k - 1_{B(x,\sqrt{T})}||_{L^2(\mathbb{M})} \to 0.
\]

Letting \( k \to \infty \) in (9.71) we thus conclude that the same inequality holds with \( f_k \) replaced by \( f(y,s) = P_s(p(x,\cdot, T)1_{B(x,\sqrt{T})})(y) \). This implies in particular the basic estimate

\[
\inf_{z \in B(x,\sqrt{T})} e^{g(z,\tau)} \int_{B(x,\sqrt{T})} f^2(z,\tau) \, d\mu(z) \leq \int_{B(x,\sqrt{T})} e^{g(z,\tau)} f^2(z,\tau) \, d\mu(z) \\
\leq \int_{\mathbb{M}} e^{g(z,0)} f^2(z,0) \, d\mu(z) = \int_{B(y,\sqrt{T})} e^{g(z,0)} p(x,z,T)^2 \, d\mu(z) \\
\leq \sup_{z \in B(y,\sqrt{T})} e^{g(z,0)} \int_{B(y,\sqrt{T})} p(x,z,T)^2 \, d\mu(z).
\]

At this point we choose in (9.71)

\[
g(y,t) = g_x(y,t) = -\frac{d(x,y)^2}{2((1 + 2\alpha)T - t)}.
\]

Using the fact that \( \Gamma(d) \leq 1 \), one can easily check that (9.67) is satisfied for this \( g \). Taking into account that

\[
\inf_{z \in B(x,\sqrt{T})} e^{g_x(z,\tau)} = \inf_{z \in B(x,\sqrt{T})} e^{-\frac{d(x,z)^2}{2((1 + 2\alpha)T - \tau)}} \geq e^{\frac{1}{2(1 + 2\alpha)T}}
\]

if we now choose \( \tau = (1 + \alpha)T \), then from the previous inequality and from (9.69) we conclude that

\[
\int_{B(x,\sqrt{T})} f^2(z,(1 + \alpha)T) \, d\mu(z) \leq \left( \sup_{z \in B(y,\sqrt{T})} e^{-\frac{d(x,z)^2}{2(1 + 2\alpha)T}} + \frac{e}{\sqrt{T}} \right) \int_{B(y,\sqrt{T})} p(x,z,T)^2 \, d\mu(z). \tag{9.72}
\]
We now apply Theorem 8.1 which gives for every $z \in B(x, \sqrt{t})$

$$f(x, T)^2 \leq f(z, (1 + \alpha)T)^2(1 + \alpha)^{1 + \frac{3\alpha}{2\alpha^2}} e^{-rac{t(1 + \frac{3\alpha}{2\alpha^2})}{2\alpha^2}}.$$ 

Integrating this inequality on $B(x, \sqrt{t})$ we find

$$\left(\int_{B(y, \sqrt{t})} p(x, z, T)^2 d\mu(z)\right)^2 = f(x, T)^2 \leq \frac{(1 + \alpha)^{d(1 + \frac{3\alpha}{2\alpha^2})} e^{-rac{t(1 + \frac{3\alpha}{2\alpha^2})}{2\alpha^2}}}{\mu(B(x, \sqrt{t}))} \int_{B(x, \sqrt{t})} f^2(z, (1 + \alpha)T)d\mu(z).$$

If we now use (9.72) in the last inequality we obtain

$$\int_{B(y, \sqrt{t})} p(x, z, T)^2 d\mu(z) \leq \frac{(1 + \alpha)^{d(1 + \frac{3\alpha}{2\alpha^2})} e^{-rac{t(1 + \frac{3\alpha}{2\alpha^2})}{2\alpha^2}}}{\mu(B(x, \sqrt{t}))} \left(\sup_{z \in B(y, \sqrt{t})} e^{-\frac{d(x, z)^2}{2(1 + 2\alpha)(1 + \alpha)}}\right).$$

Choosing $T = (1 + \alpha)t$ in this inequality we find

$$\int_{B(y, \sqrt{t})} p(x, z, (1 + \alpha)t)^2 d\mu(z) \leq \frac{(1 + \alpha)^{d(1 + \frac{3\alpha}{2\alpha^2})} e^{-rac{t(1 + \frac{3\alpha}{2\alpha^2})}{2\alpha^2}}}{\mu(B(x, \sqrt{t}))} \left(\sup_{z \in B(y, \sqrt{t})} e^{-\frac{d(x, z)^2}{2(1 + 2\alpha)(1 + \alpha)}}\right).$$

We now apply Corollary 8.2 obtaining for every $z \in B(y, \sqrt{t})$

$$p(x, y, t)^2 \leq p(x, z, (1 + \alpha)t)^2(1 + \alpha)^{d(1 + \frac{3\alpha}{2\alpha^2})} \exp\left(\frac{1 + \frac{3\alpha}{2\alpha}}{2\alpha}\right).$$

Integrating this inequality in $z \in B(y, \sqrt{t})$, we have

$$\mu(B(y, \sqrt{t})) p(x, y, t)^2 \leq (1 + \alpha)^{d(1 + \frac{3\alpha}{2\alpha^2})} \exp\left(\frac{1 + \frac{3\alpha}{2\alpha}}{2\alpha}\right) \int_{B(y, \sqrt{t})} p(x, z, (1 + \alpha)t)^2 d\mu(z).$$

Combining this inequality with (9.73) we conclude

$$p(x, y, t) \leq \frac{(1 + \alpha)^{d(1 + \frac{3\alpha}{2\alpha^2})} e^{-rac{t(1 + \frac{3\alpha}{2\alpha^2})}{4\alpha(1 + \alpha)}}}{\mu(B(x, \sqrt{t}))^{\frac{1}{2}} \mu(B(y, \sqrt{t}))^{\frac{1}{2}}} \left(\sup_{z \in B(y, \sqrt{t})} e^{-\frac{d(x, z)^2}{2(1 + 2\alpha)(1 + \alpha)}}\right).$$

If now $x \in B(y, \sqrt{t})$, then

$$d(x, z)^2 \geq (d(x, y) - \sqrt{t})^2 > d(x, y)^2 - t,$$

and therefore

$$\sup_{z \in B(y, \sqrt{t})} e^{-\frac{d(x, z)^2}{2(1 + 2\alpha)(1 + \alpha)}} \leq e^{-\frac{d(x, y)^2}{2(1 + 2\alpha)(1 + \alpha)}},$$

If instead $x \notin B(y, \sqrt{t})$, then for every $\delta > 0$ we have

$$d(x, z)^2 \geq (1 - \delta)d(x, y)^2 - (1 + \delta^{-1})t.$$
Choosing $\delta = \alpha / (\alpha + 1)$ we find
\[ d(x, z)^2 \geq \frac{d(x, y)^2}{1 + \alpha} - (2 + \alpha^{-1})t, \]
and therefore
\[ \sup_{z \in B(y, \sqrt{t})} e^{-\frac{d(x, z)^2}{2(1 + 2\alpha)(1 + \alpha)^2t}} \leq e^{-\frac{d(x, y)^2}{2(1 + 2\alpha)(1 + \alpha)^2t} + \frac{2 + \alpha^{-1}}{2(1 + 2\alpha)(1 + \alpha)}} \]
For any $\epsilon > 0$ we now choose $\alpha > 0$ such that $2(1 + 2\alpha)(1 + \alpha)^2 = 4 + \epsilon$ to reach the desired conclusion.

\[ \square \]

10 A generalization of Yau’s Liouville theorem

In his seminal 1975 paper [73], by using gradient estimates, Yau proved his celebrated Liouville theorem that there exists no non-constant positive harmonic function on a complete Riemannian manifold with non-negative Ricci curvature. The aim of this section is to extend Yau’s theorem to the sub-Riemannian setting of this paper. An interesting point to keep in mind here is that, even in the Riemannian setting, our approach gives a new proof of Yau’s theorem which is not based on delicate tools from Riemann geometry such as the Hessian and the Laplacian comparison theorems for the geodesic distance. However, due to the nature of our proof at the moment we are only able to deal with harmonic functions bounded from two sides, whereas in [73] the author is able to treat functions satisfying a one-side bound.

In what follows we assume that the curvature-dimension inequality (6.46) hold with $\rho_1 = 0$, $\rho_2 > 0$ and $\kappa > 0$. We also assume that the metric space $(\mathbb{M}, d)$, or equivalently $(\mathbb{M}, d_R)$, is complete. As we mentioned it before, see Remark 5.3 in the Riemannian case of Example 2.1 our assumptions are in fact equivalent to assuming that $\mathbb{M}$ is a complete Riemannian manifold with $\text{Ric} \geq 0$. We begin with a Harnack type inequality for the operator $L$.

**Theorem 10.1** Let $\mathbb{M}$ be a complete sub-Riemannian manifold and assume (6.46) with $\rho_1 = 0$. Let $0 \leq f \leq M$ be a harmonic function on $\mathbb{M}$, then there exists a constant $C = C(d, \rho_2, \kappa) > 0$ such that for any $x_0 \in \mathbb{M}$ and any $r > 0$ one has
\[ \sup_{B(x_0, r)} f \leq C \inf_{B(x_0, r)} f. \]

**Proof.** Let $f$ be as in the statement of the theorem. By Hörmander’s theorem [37] we know that $f \in C^\infty(\mathbb{M})$. Applying Theorem 8.1 to the function $u(x, t) = P_t f(x)$, we obtain for $x, y \in B(x_0, r)$
\[ P_s f(x) \leq P_t f(y) \left( \frac{t}{s} \right)^{D_2} \exp \left( \frac{D r^2}{d(t - s)} \right), \quad 0 < s < t < \infty. \]
At this point we observe that, thanks to the assumption $Lf = 0$, the functions $u(x, t) = P_t f(x)$ and $v(x, t) = f(x)$ solve the same Cauchy problem on $\mathbb{M}$. By Proposition 7.3 we must have

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\( P_t f(x) = f(x) \) for every \( x \in \mathbb{M} \) and every \( t > 0 \). Therefore, taking \( s = r^2, t = 2r^2 \), the latter inequality gives

\[
f(x) \leq \left( \sqrt{2} e^{\frac{1}{2}} \right)^D f(y), \quad x, y \in B(x_0, r).
\]

\[\square\]

**Corollary 10.2 (of Cauchy-Liouville type)** Assume (6.46) with \( \rho_1 = 0 \), then there exist no bounded solutions to \( Lf = 0 \) on \( \mathbb{M} \), other than the constants.

**Proof.** Suppose \( a \leq f \leq b \) on \( \mathbb{M} \). Consider the function \( g = f - \inf_{\mathbb{M}} f \). Clearly, \( 0 \leq g \leq M = b - a \).

If we apply Theorem 10.1 to \( g \) we find for any \( x_0 \in \mathbb{M} \) and \( r > 0 \)

\[
\sup_{B(x_0, r)} g \leq C \inf_{B(x_0, r)} g.
\]

Letting \( r \to \infty \) we reach the conclusion \( \sup_{\mathbb{M}} f = \inf_{\mathbb{M}} f \), hence \( f \equiv \text{const.} \)

\[\square\]

### 11 Volume growth and Isoperimetric inequality when \( \rho_1 = 0 \)

Throughout this section, we shall assume that \( (\mathbb{M}, d) \) is complete non compact and that (6.45) holds with \( \rho_1 \geq 0 \).

#### 11.1 Volume growth

We first derive a basic and straightforward consequence of the parabolic Harnack inequality on the volume growth of metric balls.

**Proposition 11.1** For every \( x \in \mathbb{M} \) and every \( R_0 > 0 \) there is a constant \( C(d, \kappa, \rho_2) > 0 \) such that, with \( D \) as in (6.56),

\[
\mu (B(x, R)) \leq \frac{C(d, \kappa, \rho_2)}{R_0^D p(x, x, R_0^2)} R^D, \quad R \geq R_0.
\]

**Proof.** Let \( t > \tau > 0 \). From the Harnack inequality of Corollary 8.2 we have

\[
p(x, x, t) \geq p(x, x, \tau) \left( \frac{\tau}{t} \right)^{\frac{D}{2}}
\]

On the other hand, the inequality (9.64) gives

\[
p(x, x, t) \leq \frac{C(d, \kappa, \rho_2)}{\mu (B(x, \sqrt{t}))}.
\]

This implies the desired conclusion.

\[\square\]
11.2 Isoperimetric inequality

In [31] it was proved that in a Carnot-Carathéodory space \((X, \mu, d)\) the doubling condition

\[
\mu(B(x, 2r)) \leq C_1 \mu(B(x, r)), \quad x \in X, r > 0,
\]

for the volume of the metric balls combined with a weak Poincaré inequality suffice to establish the following basic relative isoperimetric inequality

\[
\min \{\mu(E \cap B(x, r)), \mu((X \setminus E) \cap B(x, r))\} \frac{D-1}{D} \leq C_{\text{iso}} \left( \frac{\mu(B(x, r))}{\mu(B(x, r))} \right)^{1/D} P(E, B(x, r)),
\]

(11.74)

where \(P(E, B(x, r))\) represents a generalization of De Giorgi’s variational notion of perimeter, and \(E \subset X\) is any set of locally finite perimeter. In this inequality the number \(D = \log_2 C_1\), where \(C_1\) is the doubling constant, and \(C_{\text{iso}}\) is a constant which depends only on \(C_1\) and on the constant in the Poincaré inequality. If in addition the space \(X\) satisfies the maximum volume growth

\[
\mu(B(x, r)) \geq C_2 r^D, \quad x \in \mathbb{M}, r > 0,
\]

(11.75)

then (11.74) gives

\[
\min \{\mu(E \cap B(x, r)), \mu((X \setminus E) \cap B(x, r))\} \frac{D-1}{D} \leq C_{\text{iso}}^* P(E, B(x, r)),
\]

(11.76)

where \(C_{\text{iso}}^* = C_2^{-1/D} C_{\text{iso}}\).

When \(X = \mathbb{M}\) is a Riemannian manifold with \(\text{Ric} \geq 0\), then the doubling condition with \(C_1 = 2^D\), \(D = \dim \mathbb{M}\), follows from Bishop-Gromov comparison theorem (see [19], Proposition 3.3 and Theorem 3.10), whereas the Poincaré inequality was proved by Buser, see [13]. As a consequence, one obtains the relative isoperimetric inequality (11.74) in this setting. When \(\mathbb{M}\) satisfies the maximum volume growth (11.75), one also obtains from (11.74) the global isoperimetric inequality

\[
\mu(E) \frac{D-1}{D} \leq C_{\text{iso}} P(E, \mathbb{M}),
\]

(11.77)

for any measurable set of locally finite perimeter \(E \subset \mathbb{M}\).

In this subsection we investigate the sub-Riemannian counterpart of the isoperimetric estimate (11.77) under the assumption that \(\rho_1 = 0\). Here, the main obstacle is precisely the a priori lack of a global doubling condition and of a Poincaré inequality, and therefore we cannot rely on the above cited results from [31]. Instead, using our Li-Yau type estimate (11.79) we adapt some beautiful ideas of Varopoulos and Ledoux to provide a characterization of those sub-Riemannian manifolds which support an inequality such as (11.77). We stress that, due to the nature of our approach, we obtain a lower bound on the dimension \(D\) in (11.77), but in the case of a graded nilpotent Lie group our \(D\) is not optimal, see Remark 6.8.

In what follows, given an open set \(\Omega \subset \mathbb{M}\) we will indicate with

\[
\mathcal{F}(\Omega) = \{ \phi \in C^1_0(\Omega, \mathcal{H}) \mid \|\phi\|_\infty \leq 1 \}.
\]
Here, for $\phi = \sum_{i=1}^{d} \phi_i X_i$, we have $\|\phi\|_{\infty} = \sup_{\Omega} \sqrt[2]{\sum_{i=1}^{d} \phi_i^2}$. Following [15], given a function $f \in L_{\text{loc}}^{1}(\Omega)$ we define the horizontal total variation of $f$ in $\Omega$ as

$$\text{Var}_{\mathcal{H}}(f;\Omega) = \sup_{\phi \in \mathcal{F}(\Omega)} \int_{\Omega} f \left( \sum_{i=1}^{d} X_i \phi_i \right) d\mu.$$ 

The space $\mathcal{B}V_{\mathcal{H}}(\Omega) = \{f \in L^{1}(\Omega) \mid \text{Var}_{\mathcal{H}}(f;\Omega) < \infty\}$, endowed with the norm

$$\|f\|_{\mathcal{B}V_{\mathcal{H}}(\Omega)} = \|f\|_{L^{1}(\Omega)} + \text{Var}_{\mathcal{H}}(f;\Omega),$$

is a Banach space. It is well-known that $W_{\mathcal{H}}^{1,1}(\Omega) = \{f \in L^{1}(\Omega) \mid X_i f \in L^{1}(\Omega), i = 1, \ldots, d\}$ is a strict subspace of $\mathcal{B}V_{\mathcal{H}}(\Omega)$. It is important to note that when $f \in W_{\mathcal{H}}^{1,1}(\Omega)$, then $f \in \mathcal{B}V_{\mathcal{H}}(\Omega)$, and one has in fact

$$\text{Var}_{\mathcal{H}}(f;\Omega) = \|\sqrt{\Gamma(f)}\|_{L^{1}(\Omega)}.$$ 

Given a measurable set $E \subset \mathbb{M}$ we say that it has finite horizontal perimeter in $\Omega$ if $1_E \in \mathcal{B}V_{\mathcal{H}}(\Omega)$. In such case the horizontal perimeter of $E$ relative to $\Omega$ is by definition

$$P_{\mathcal{H}}(E;\Omega) = \text{Var}_{\mathcal{H}}(1_E;\Omega).$$

We say that a measurable set $E \subset \mathbb{M}$ is a Caccioppoli set if $P_{\mathcal{H}}(E;\Omega) < \infty$ for any $\Omega \subset \mathbb{M}$. We will need the following approximation result, see Theorem 1.14 in [31].

**Proposition 11.2** Let $f \in \mathcal{B}V_{\mathcal{H}}(\Omega)$, then there exists a sequence $\{f_n\}_{n \in \mathbb{N}}$ of functions in $C_{\infty}(\Omega)$ such that:

(i) $\|f_n - f\|_{L^{1}(\Omega)} \to 0$;

(ii) $\int_{\Omega} \sqrt{\Gamma(f_n)} d\mu \to \text{Var}_{\mathcal{H}}(f;\Omega)$.

If $\Omega = \mathbb{M}$, then the sequence $\{f_n\}_{n \in \mathbb{N}}$ can be taken in $C_{0}^{\infty}(\mathbb{M})$.

Our intent is to establish the following result.

**Theorem 11.3 (Isoperimetric inequality)** Suppose that $\mathbb{M}$ is not compact in the metric topology and that there exists $D > 1$ such that

$$\mu(B(x, r)) \geq C_1 r^{D}. \quad (11.78)$$

Then, there is a constant $C_{\text{iso}} = C_{\text{iso}}(d, \rho_2, \kappa, C_1, D) > 0$, such that for every Caccioppoli set $E \subset \mathbb{M}$

$$\mu(E)^{\frac{d-1}{d}} \leq C_{\text{iso}} P_{\mathcal{H}}(E, \mathbb{M}).$$

The essence of the proof of Theorem 11.3 is contained in the following result.

**Proposition 11.4** Let $D > 1$. Let us assume that $\mathbb{M}$ is not compact in the metric topology, then the following assertions are equivalent:
(1) There exists a constant $C_1 > 0$ such that for every $x \in \mathbb{M}$, $r \geq 0$,
$$\mu(B(x,r)) \geq C_1 r^D.$$ 

(2) There exists a constant $C_2 > 0$ such that for $x \in \mathbb{M}$, $t > 0$,
$$p(x,x,t) \leq \frac{C_2}{t^\frac{D}{2}}.$$ 

(3) There exists a constant $C_3 > 0$ such that for every Caccioppoli set $E \subset \mathbb{M}$ one has
$$\mu(E) \frac{D}{D-1} \leq C_3 P_H(E;\mathbb{M}).$$ 

(4) With the same constant $C_3 > 0$ as in (3), for every $f \in BV_H(\mathbb{M})$ one has
$$\left(\int_{\mathbb{M}} |f| \frac{D}{D-1} d\mu\right) \frac{D}{D-1} \leq C_3 \text{Var}_H(f;\mathbb{M}).$$ 

Proof. That (1) $\rightarrow$ (2) follows immediately from the Gaussian bound of Theorem 9.1 (observe that we may take $C_2 = \frac{1}{C_1}$).
The proof that (2) $\rightarrow$ (3) is not straightforward. First we note that (6.45) implies the Li-Yau type estimate (6.55). This enables us to adapt some beautiful ideas of Varopoulos (see [70], pp.256-58) and Ledoux (see pp. 22 in [43], see also Theorem 8.4 in [44]). Let $f \in C_0(\mathbb{M})$ with $f \geq 0$. By (6.55) we obtain
$$\Gamma(P_t f) - (1 + \frac{3\kappa}{2\rho_2}) P_t f \frac{\partial P_t f}{\partial t} \leq \frac{d}{2} \left(1 + \frac{3\kappa}{2\rho_2}\right)^2 (P_t f)^2. \quad (11.79)$$

This gives in particular, with $\nu = \frac{d}{2} \left(1 + \frac{3\kappa}{2\rho_2}\right)$,
$$\left(\frac{\partial P_t f}{\partial t}\right)^{-} \leq \frac{\nu}{2t} P_t f, \quad (11.80)$$

where we have denoted $a^+ = \sup\{a,0\}$, $a^- = \sup\{-a,0\}$. It follows that for every $0 < T_1 < T_2 < \infty$
$$\int_{T_1}^{T_2} \int_{\mathbb{M}} \left(\frac{\partial P_t f}{\partial t}\right)^{-} d\mu dt \leq \frac{\nu}{2} \ln(T_2/T_1) \|f\|_{L^1(\mathbb{M})} < \infty. \quad (11.81)$$

Using Tonelli’s theorem we now have for any $t > 0$
$$\int_{\mathbb{M}} P_t f(x) d\mu(x) = \int_{\mathbb{M}} f(y) \int_{\mathbb{M}} p(x,y,t)d\mu(x)d\mu(y) = \int_{\mathbb{M}} f(y)d\mu(y),$$
where we have used Theorem 7.33. This gives for any $t, h > 0$
$$0 = \int_{\mathbb{M}} P_{t+h} f(x) d\mu(x) - \int_{\mathbb{M}} P_{t} f(x) d\mu(x) = \int_{\mathbb{M}} \int_{t}^{t+h} \frac{\partial P_s f(x)}{\partial s} ds d\mu(x)$$
$$= \int_{t}^{t+h} \int_{\mathbb{M}} \frac{\partial P_s f(x)}{\partial s} d\mu(x) ds,$$
where the exchange of order of integration is justified by Fubini's theorem, which we can apply in view of (11.81). The latter equation implies
\[ \int_t^{t+h} \int_M \left( \frac{\partial P_s f}{\partial s}(x) \right)^+ \, d\mu(x) \, ds = \int_t^{t+h} \int_M \left( \frac{\partial P_s f}{\partial s}(x) \right)^- \, d\mu(x) \, ds, \quad t, h > 0, \]
and therefore, we have
\[ \frac{1}{h} \int_t^{t+h} \int_M \left| \frac{\partial P_s f}{\partial s}(x) \right| \, d\mu(x) \, ds = \frac{2}{h} \int_t^{t+h} \int_M \left( \frac{\partial P_s f}{\partial s}(x) \right)^- \, d\mu(x) \, ds \leq \frac{\nu}{h} \left( \int_t^{t+h} \frac{ds}{s} \right) \|f\|_{L^1(M)}. \]
Letting \( h \to 0^+ \) we finally conclude
\[ \| \frac{\partial P_t f}{\partial t} \|_{L^1(M)} \leq \frac{\nu}{t} \|f\|_{L^1(M)}, \quad t > 0. \]
By duality, we deduce that for every \( f \in C_0(M) \), \( f \geq 0 \),
\[ \| \frac{\partial P_t f}{\partial t} \|_{L^\infty(M)} \leq \frac{\nu}{t} \|f\|_{L^\infty(M)}. \]
Once we have this crucial information we can return to (11.79) and infer
\[ \Gamma(P_t f) \leq \frac{1}{t} \frac{3 \nu^2}{2d} \|f\|_{L^\infty(M)}^2, \quad t > 0. \]
Thus,
\[ \| \sqrt{\Gamma}(P_t f) \|_{L^\infty(M)} \leq \nu \sqrt{\frac{3d}{2t}} \|f\|_{L^\infty(M)}. \]
Applying this inequality to \( g \in C_0^\infty(M) \), with \( g \geq 0 \) and \( \|g\|_{L^\infty(M)} \leq 1 \), if \( f \in C_0^1(M) \) we have
\[ \int_M g(f - P_t f) \, d\mu = \int_0^t \int_M g \frac{\partial P_s f}{\partial s} \, d\mu ds = \int_0^t \int_M gLP_s f \, d\mu ds = \int_0^t \int_M LgP_s f \, d\mu ds \]
\[ = \int_0^t \int_M P_s Lg f \, d\mu ds = \int_0^t \int_M LP_s g f \, d\mu ds = -\int_0^t \int_M \Gamma(P_s g, f) \, d\mu ds \]
\[ \leq \int_0^t \| \sqrt{\Gamma}(P_s g) \|_{L^\infty(M)} \int_M \sqrt{\Gamma}(f) \, d\mu ds \leq \sqrt{6d} \nu \sqrt{t} \int_M \sqrt{\Gamma}(f) \, d\mu. \]
We thus obtain the following basic inequality: for \( f \in C_0^1(M) \),
\[ \| P_t f - f \|_{L^1(M)} \leq \sqrt{6d} \nu \sqrt{t} \| \sqrt{\Gamma}(f) \|_{L^1(M)}, \quad t > 0. \] (11.82)
Suppose now that \( E \subset M \) is a bounded Caccioppoli set. But then, \( 1_E \in BV_M(\Omega) \), for any bounded open set \( \Omega \supset E \). It is easy to see (see e.g. the proof of Lemma 2.5 in [21]) that \( \text{Var}_M(1_E; \Omega) = \text{Var}_M(1_E; M) \), and therefore \( 1_E \in BV_M(M) \). By Proposition 11.2 there exists a sequence \( \{f_n\}_{n \in \mathbb{N}} \) in \( C_0^\infty(M) \) satisfying (i) and (ii). Applying (11.82) to \( f_n \) we obtain
\[ \| P_t f_n - f_n \|_{L^1(M)} \leq \sqrt{6d} \nu \sqrt{t} \| \sqrt{\Gamma}(f_n) \|_{L^1(M)} = \sqrt{6d} \nu \sqrt{t} \text{Var}_M(f_n, M), \quad n \in \mathbb{N}. \]
Letting $n \to \infty$ in this inequality, we conclude
\[
\|P_t 1_E - 1_E\|_{L^1(M)} \leq \sqrt{6d \nu} \sqrt{t} \text{Var}_H(1_E, M) = \sqrt{6d \nu} \sqrt{t} P_H(E; M), \quad t > 0.
\]
Observe now that, using $P_t 1 = 1$, we have
\[
\|P_t 1_E - 1_E\|_{L^1(M)} = 2 \left( \mu(E) - \int_E P_t 1_E d\mu \right).
\]
On the other hand,
\[
\int_E P_t 1_E d\mu = \int_M (P_{t/2} 1_E)^2 d\mu.
\]
We thus obtain
\[
\|P_t 1_E - 1_E\|_{L^1(M)} = 2 \left( \mu(E) - \int_M (P_{t/2} 1_E)^2 d\mu \right).
\]
We now observe that (9.64) and the assumption (1) imply
\[
p(x, x, t) \leq \frac{C(d, \kappa, \rho_2)}{\mu(B(x, \sqrt{t}))} \leq \frac{C_4}{t^{D/2}}, \quad x \in M, t > 0,
\]
where $C_4 = C_1^{-1} C(d, \kappa, \rho_2)$. This gives
\[
\int_M (P_{t/2} 1_E)^2 d\mu \leq \left( \int_E \left( \int_M p(x, y, t/2)^2 d\mu(y) \right)^{\beta} d\mu(x) \right)^2
\]
\[
= \left( \int_E p(x, x, t)^{\beta} d\mu(x) \right)^2 \leq \frac{C_4}{t^{D/2}} \mu(E)^2.
\]
Combining these equations we reach the conclusion
\[
\mu(E) \leq \frac{\sqrt{6d \nu}}{2} \sqrt{t} P_H(E; M) + \frac{C_4}{t^{D/2}} \mu(E)^2, \quad t > 0.
\]
Now the absolute minimum of the function $g(t) = At^\alpha + Bt^{-\beta}$, $t > 0$, where $A, B, \alpha, \beta > 0$, is given by
\[
g_{\text{min}} = \left[ \left( \frac{\alpha}{\beta} \right)^{\frac{\beta}{\alpha + \beta}} + \left( \frac{\beta}{\alpha} \right)^{\frac{\alpha}{\alpha + \beta}} \right] A^{\frac{\beta}{\alpha + \beta}} B^{\frac{\alpha}{\alpha + \beta}}.
\]
Applying this observation with $\alpha = \frac{1}{2}, \beta = \frac{D}{2}$, we conclude
\[
\mu(E)^{\frac{D - 1}{D}} \leq C_3 P_H(E, M),
\]
with
\[
C_3 = (1 + D)^{\frac{D + 1}{D}} \left( \frac{\sqrt{6d \nu}}{2D} \right)^{\frac{1}{D}} C_4^{\frac{1}{D}}.
\]
The latter inequality proves (3).
The proof that (3) is equivalent to (4) follows the classical ideas of Fleming-Rishel and Maz’ya, and it is based on a generalization of Federer’s co-area formula for the space $BV_H$, see for instance [31].

Finally, we show that (4) $\rightarrow$ (1). We adapt an idea in [60] (see Theorem 3.1.5 on p. 58). In what follows we let $\nu = D/(D - 1)$. Let $p, q \in (0, \infty)$ and $0 < \theta \leq 1$ be such that

$$\frac{1}{p} = \frac{\theta}{\nu} + \frac{1 - \theta}{q}.$$

Hölder inequality, combined with assumption (4), gives for any $f \in Lip_d(M)$ with compact support

$$||f||_{L^p(M)} \leq ||f||_{L^q(M)}^{\theta} ||f||_{L^\nu(M)}^{1-\theta} \leq \left(C_3 ||\sqrt{f}||^\theta_{L^1(M)}\right)^\theta ||f||_{L^\nu(M)}^{1-\theta},$$

For any $x \in M$ and $r > 0$ we now let $f(y) = (r - d(y, x))^+$. Clearly such $f \in Lip_d(M)$ and supp $f = B(x, r)$. Since with this choice $||\sqrt{f}||_{L^1(M)}^\theta \leq \mu(B(x, r))^\theta$, the above inequality implies

$$\frac{r}{2} \mu(B(x, \frac{r}{2})^{\frac{1}{p}} \leq r^{1-\theta} (C_3 \mu(B(x, r))^\theta \mu(B(x, r))^{\frac{1-\theta}{q}},$$

which, noting that $\frac{1-\theta}{q} + \theta = \frac{D+\theta p}{\theta p}$, we can rewrite as follows

$$\mu(B(x, r)) \geq \left(\frac{1}{2C_3^\theta}\right)^{\frac{p}{\theta}} \mu(B(x, \frac{r}{2})) a r^{\theta p a},$$

where we have let $a = \frac{D}{D+\theta p}$. Notice that $0 < a < 1$. Iterating the latter inequality we find

$$\mu(B(x, r)) \geq \left(\frac{1}{2C_3^\theta}\right)^{\frac{p}{\theta}} r^{\theta p \sum_{j=1}^{k} a^j} 2^{-\theta p \sum_{j=1}^{k} (j-1) a^j} \mu(B(x, \frac{r}{2^k})) a^k, \quad k \in \mathbb{N}.$$

From Theorem 2.12 for any $x \in M$ there exist constants $C(x), R(x) > 0$ such that with $Q(x) = \log_2 C(x)$ one has

$$\mu(B(x, tr)) \geq C(x)^{-1} t^{Q(x)} \mu(B(x, r)), \quad 0 \leq t \leq 1, 0 < r \leq R(x).$$

This estimate implies that

$$\liminf_{k \to \infty} \mu(B(x, \frac{r}{2^k})) a^k \geq 1, \quad x \in \mathbb{M}, r > 0.$$

Since on the other hand $\sum_{j=1}^\infty a^j = \frac{D}{\theta p}$, and $\sum_{j=1}^\infty (j-1) a^j = \frac{D^2}{\theta p^2}$, we conclude that

$$\mu(B(x, r)) \geq \left(2^{-\frac{1}{\theta^2 (1+\frac{D}{p})}}C_3^{-1}\right)^{D} r^D, \quad x \in \mathbb{M}, r > 0.$$

This establishes (1), thus completing the proof.
12 A sub-Riemannian Bonnet-Myers theorem

Let \((\mathbb{M}, g)\) be a complete, connected Riemannian manifold of dimension \(d \geq 2\). It is well-known that if the Ricci tensor of \(\mathbb{M}\) satisfies the following bound for all \(V \in T\mathbb{M}\)

\[
\text{Ric}(V, V) \geq (d - 1) \rho_1 |V|^2, \quad \rho_1 > 0,
\]

(12.83)

then \(\mathbb{M}\) is compact, with a finite fundamental group, and \(\text{diam}(\mathbb{M}) \leq \frac{\pi}{\sqrt{\rho_1}}\). This is the celebrated Myer’s theorem, which strengthens Bonnet’s theorem. Like the latter, Myer’s theorem is usually proved by using Jacobi vector fields (see e.g. Theorem 2.12 in [19]).

A different approach is based on the curvature-dimension inequality

\[
\Gamma_2(f, f) \geq \frac{1}{n} (Lf)^2 + (n - 1) \rho_1 \Gamma(f, f),
\]

(12.84)

which one obtains from (5.44) assuming (12.83). By using ingenious non linear methods based on the study of the partial differential equation

\[
c(f^{p-1} - f) = -Lf, 1 \leq p \leq \frac{2d}{d-2},
\]

the inequality (12.84) implies (for \(d > 2\)) the following Sobolev inequality

\[
\frac{d}{(d-2)R^2} \left[ \left( \int_{\mathbb{M}} |f|^p \, d\mu \right)^{2/p} - \int_{\mathbb{M}} f^2 \, d\mu \right] \leq \int_{\mathbb{M}} \Gamma(f, f) \, d\mu, \quad f \in C_0^\infty(\mathbb{M}).
\]

(12.85)

where \(\mu\) is the Riemannian measure and. Using a simple iteration procedure, it is deduced from (12.85) that the diameter of \(\mathbb{M}\) is finite and bounded by \(\frac{\pi}{\sqrt{\rho_1}}\), see [15]. This non linear method seems difficult to extend in our framework. However, a weak version of the Myers theorem could be proven by Bakry in [3] by using linear methods only. This method based on entropy-energy inequalities (a strong form of log-sobolev inequalities) can be extended to our framework.

In this section we establish the following sub-Riemannian Bonnet-Myer’s compactness theorem. Here again, we assume that \((\mathbb{M}, d)\) is complete.

**Theorem 12.1** If there exist constants \(\rho_1, \rho_2 > 0\) and a constant \(\kappa > 0\) such that for every smooth function \(f : \mathbb{M} \rightarrow \mathbb{R}\):

\[
\mathcal{R}(f, f) \geq \rho_1 \Gamma(f, f) + \rho_2 \Gamma^Z(f, f)
\]

(12.86)

\[
T(f, f) \leq \kappa \Gamma(f, f),
\]

(12.87)

then the metric space \((\mathbb{M}, d)\) is compact in the metric topology with a Hausdorff dimension less than \(d \left( 1 + \frac{3\kappa}{2\rho_2} \right)\) and we have

\[
\text{diam} \mathbb{M} \leq 2\sqrt{3\pi} \frac{\kappa + \rho_2}{\rho_1 \rho_2} \left( 1 + \frac{3\kappa}{2\rho_2} \right) d.
\]

We shall proceed in several steps. Throughout this section, we assume that (12.86) and (12.87) are satisfied.
12.1 Global heat kernel bounds

Our first result is the following large-time exponential decay for the heat kernel.

Proposition 12.2 Let $0 < \nu < \frac{\rho_1 \rho_2}{\rho_2 + \kappa}$. There exist $t_0 > 0$ and $C_1 > 0$ such that for every $f \in C_0^\infty(M)$, $f \geq 0$:

$$\left| \frac{\partial}{\partial t} \ln P_t f(x) \right| \leq C_1 e^{-\nu t}, \quad x \in M, t \geq t_0.$$  

Proof. In Proposition 6.5, we choose

$$b(t) = (e^{-\alpha t} - e^{-\alpha T})^\beta, \quad 0 \leq t \leq T,$$

with $\beta > 2$ and $\alpha > 0$. With such choice a simple computation gives,

$$\gamma(t) = \frac{d}{4} \left( 2\rho_1 - \alpha \beta - \alpha \beta \frac{\kappa}{\rho_2} - e^{-\alpha T} \left( \alpha (\beta - 1) + \frac{\alpha \beta \kappa}{\rho_2} \right) b(t)^{-\frac{1}{\beta}} \right).$$

Keeping in mind that $b(T) = b'(T) = 0$, and that $b(0) = (1 - e^{-\alpha T})^\beta$, $b'(0) = -\alpha \beta (1 - e^{-\alpha T})^{\beta-1}$, we obtain from (6.53)

$$- \frac{\alpha \beta (1 - e^{-\alpha T})^{\beta-1}}{2\rho_2} \Gamma(\ln P_T f) - (1 - e^{-\alpha T})^{\beta} \Gamma^Z(\ln P_T f) \geq - \frac{2}{\rho_2} \left( \int_0^T b'(t) \gamma(t) dt \right) \left( \frac{L P_T f}{P_T f} + \frac{1}{d \rho_2} \left( \int_0^T b'(t) \gamma(t)^2 dt \right) \right). \quad (12.88)$$

Now,

$$\int_0^T b'(t) \gamma(t) dt = - \frac{d}{4} \left( 2\rho_1 - \alpha \beta - \alpha \beta \frac{\kappa}{\rho_2} \right) (1 - e^{-\alpha T})^\beta$$

$$+ \frac{d}{4} \left( \alpha \beta - \alpha + \alpha \beta \frac{\kappa}{\rho_2} \right) e^{-\alpha T} (1 - e^{-\alpha T})^{\beta-1},$$

$$\int_0^T b'(t) \gamma(t)^2 dt = - \frac{d^2}{16} \left( 2\rho_1 - \alpha \beta - \alpha \beta \frac{\kappa}{\rho_2} \right)^2 (1 - e^{-\alpha T})^\beta$$

$$+ \frac{d^2}{8} \left( 2\rho_1 - \alpha \beta - \alpha \beta \frac{\kappa}{\rho_2} \right) \left( \alpha \beta - \alpha + \alpha \beta \frac{\kappa}{\rho_2} \right) e^{-\alpha T} (1 - e^{-\alpha T})^{\beta-1}$$

$$- \frac{d^2}{16} \left( \alpha \beta - \alpha + \alpha \beta \frac{\kappa}{\rho_2} \right)^2 e^{-2\alpha T} (1 - e^{-\alpha T})^{\beta-2}.$$  

If we choose

$$\alpha = \frac{2\rho_1 \rho_2}{\beta (\rho_2 + \kappa)},$$

then

$$2\rho_1 - \alpha \beta - \alpha \beta \frac{\kappa}{\rho_2} = 0, \quad \alpha \beta - \alpha + \alpha \beta \frac{\kappa}{\rho_2} = 2\rho_1 - \alpha,$$

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and we obtain from (12.88):

\[
0 \leq \frac{\rho_1}{\rho_2 + \kappa} \Gamma(\ln P_T f) + (1 - e^{-\alpha T})\Gamma(Z(\ln P_T f)) \leq \frac{d(2\rho_1 - \alpha)}{2\rho_2 \left(1 - \frac{1}{\beta}\right)} e^{-\alpha T} L_P f + \frac{d(2\rho_1 - \alpha)^2}{16\rho_2 \left(1 - \frac{2}{\beta}\right)} e^{-2\alpha T}. \tag{12.89}
\]

Noting that \(2\rho_1 - \alpha = \frac{2\rho_1}{\beta(\rho_2 + \kappa)}((\beta - 1)\rho_2 + \beta \kappa) > 0\), and that \(\beta > 2\) implies \(\alpha < \frac{2\rho_1 \rho_2}{\beta(\rho_2 + \kappa)}\), (12.89) gives in particular the desired lower bound for \(\frac{d}{dt} \ln P_T f(x)\) with \(\nu = \alpha\).

The upper bound is more delicate. We fix \(0 < \eta = \frac{2\rho_1 \rho_2}{\beta(\rho_2 + \kappa)}\), and with \(\gamma = 2\beta \rho_1 \rho_2\) we now choose \(\alpha = \frac{2\rho_1 \rho_2 - \gamma e^{-\eta T}}{\beta(\rho_2 + \kappa)} = \eta - \frac{\gamma e^{-\eta T}}{\beta(\rho_2 + \kappa)}\).

Clearly, \(\alpha > 0\) provided that \(T\) be sufficiently large. This choice gives

\[
2\rho_1 - \alpha \beta - \alpha \beta \frac{\kappa}{\rho_2} = \frac{\gamma e^{-\eta T}}{\rho_2}, \quad \alpha \beta - \alpha + \alpha \beta \frac{\kappa}{\rho_2} = 2\rho_1 - \alpha - \frac{\gamma e^{-\eta T}}{\rho_2}.
\]

We thus have

\[
\int_0^T b'(t) \gamma(t) dt = -\frac{d}{4} e^{-\alpha T} (1 - e^{-\alpha T}) \beta^{-1} \left\{ \frac{\gamma(1 - e^{-\alpha T}) e^{-\eta T}}{\rho_2} - \frac{\beta}{\beta - 1} (2\rho_1 - \alpha - \frac{\gamma e^{-\eta T}}{\rho_2}) \right\}.
\]

Noting that \(e^{-\eta T} = e^{-\frac{\eta T}{\beta(\rho_2 + \kappa)}} \to 1\), and \(\alpha \to \frac{2\rho_1 \rho_2}{\beta(\rho_2 + \kappa)}\) as \(T \to \infty\), we obtain

\[
\frac{\gamma(1 - e^{-\alpha T}) e^{-\eta T}}{\rho_2} - \frac{\beta}{\beta - 1} (2\rho_1 - \alpha - \frac{\gamma e^{-\eta T}}{\rho_2}) \to \frac{\gamma}{\rho_2} - \frac{\beta}{\beta - 1} \left(2\rho_1 - \frac{2\rho_1 \rho_2}{\beta(\rho_2 + \kappa)}\right).
\]

Since by our choice of \(\gamma\) we have \(\frac{\gamma}{\rho_2} - \frac{\beta}{\beta - 1} \left(2\rho_1 - \frac{2\rho_1 \rho_2}{\beta(\rho_2 + \kappa)}\right) > 0\), it is clear that we have

\[
\int_0^T b'(t) \gamma(t) dt \leq -\frac{d}{8} \left( \frac{\gamma}{\rho_2} - \frac{\beta}{\beta - 1} \left(2\rho_1 - \frac{2\rho_1 \rho_2}{\beta(\rho_2 + \kappa)}\right) \right) e^{-\alpha T} (1 - e^{-\alpha T}) \beta^{-1},
\]

provided that \(T\) be large enough. We also have

\[
\int_0^T b'(t) \gamma(t)^2 dt = -\frac{d^2}{16} e^{-2\alpha T} (1 - e^{-\alpha T}) \beta^{-2} \left\{ \frac{\beta}{\beta - 2} (2\rho_1 - \alpha - \frac{\gamma e^{-\eta T}}{\rho_2})^2 
\right. 
\]
\[
+ \frac{\gamma^2}{\rho_2} (1 - e^{-\alpha T})^2 e^{-2(\eta - T)} - \frac{2}{\rho_2} \frac{\beta}{\beta - 1} (1 - e^{-\alpha T}) (2\rho_1 - \alpha - \frac{\gamma e^{-\eta T}}{\rho_2}) e^{-\eta T} \right\}.
\]

Using our choice of \(\gamma\) we see that, if we let \(T \to \infty\), the quantity between curly bracket in the right-hand side converges to

\[
\frac{\beta}{\beta - 2} 4\rho_1^2 \left( \frac{(\beta - 1)\rho_2 + \beta \kappa}{\beta(\rho_2 + \kappa)} \right)^2 + 4\beta^2 \rho_1^2 - \frac{8\beta^2 \rho_1^2 (\beta - 1)\rho_2 + \beta \kappa}{\beta - 1} \frac{\beta(\rho_2 + \kappa)}{\beta(\rho_2 + \kappa)}.
\]
This quantity is strictly positive provided that
\[
\frac{2\beta (\beta - 1) \rho_2 + \beta \kappa}{\beta - 1} \frac{\beta (\rho_2 + \kappa)}{(\beta (\rho_2 + \kappa))} < \frac{1}{\beta - 2} \left( \frac{(\beta - 1) \rho_2 + \beta \kappa}{\beta (\rho_2 + \kappa)} \right)^2 + \beta,
\]
and this latter inequality is true, as one recognizes by applying the inequality \(2xy \leq x^2 + y^2\).

From these considerations and from (12.88) we conclude the desired upper bound for \(\frac{\partial}{\partial t} \ln P_t f(x)\).

**Proposition 12.3** Let \(0 < \nu < \frac{\rho_1 \rho_2}{\kappa + \rho_2}\). There exist \(t_0 > 0\) and \(C_2 > 0\) such that for every \(f \in C_0^\infty(M)\), with \(f \geq 0\),

\[
e^{-C_2 \nu t} d(x, y) \leq \frac{P_t f(x)}{P_t f(y)} \leq e^{C_2 \nu t} d(x, y), \quad x, y \in M, \ t \geq t_0.
\]

**Proof.** If we combine (12.89) with the upper bound of Proposition 12.2, we obtain that for \(x \in M\) and \(t \geq t_0\),

\[
\Gamma(\ln P_t f)(x) \leq C_2 e^{-\nu t}.
\]

We infer that the function \(u(x) = C_2^{-1} e^{\nu t} \ln P_t f(x)\), which belongs to \(C^\infty(M)\), is such that \(\Gamma(u)(x) \leq 1, \ x \in M\). From (12.17) we obtain that

\[
|u(x) - u(y)| \leq d(x, y), \quad x, y \in M.
\]

This implies the sought for conclusion.

If we now fix \(x \in M\), and denote by \(p(x, \cdot, t)\) the heat kernel with singularity at \((x, 0)\), then according to Proposition 12.2 we obtain for \(t \geq t_0\),

\[
\left| \frac{\partial \ln p(x, y, t)}{\partial t} \right| \leq C_1 \exp(\nu t), \quad (12.90)
\]

with \(0 < \nu < \frac{\rho_1 \rho_2}{\kappa + \rho_2}\). This shows that \(\ln p(\cdot, \cdot, t)\) converges when \(t \to \infty\). Let us call \(\ln p_\infty\) this limit. Moreover, from Proposition 12.3 the limit, \(\ln p_\infty(x, \cdot)\) is a constant \(C(x)\). By the symmetry property \(p(x, y, t) = p(y, x, t)\), so that \(C(x)\) actually does not depend on \(x\). We deduce from this that the invariant measure \(\mu\) is finite. We may then as well suppose that \(\mu\) is a probability measure, in which case \(p_\infty = 1\). We assume this from now on.

We now can prove a global and explicit upper bound for the heat kernel \(p(x, y, t)\).

**Proposition 12.4** For \(x, y \in M\) and \(t > 0\),

\[
p(x, y, t) \leq \frac{1}{\left( 1 - e^{-\frac{2\beta \rho_1 \rho_2 \kappa}{\beta (\rho_2 + \kappa)} t} \right)^{\frac{1}{2}}} \left( 1 + \frac{3\kappa}{\rho_2} \right).
\]
Proof. We apply (12.89) with $\beta = 3$ and obtain

$$\frac{\rho_1}{\rho_2 + \kappa} \Gamma(\ln P_t f) + (1 - e^{-\alpha t}) \Gamma^2(\ln P_t f) \leq \frac{\rho_1}{2\rho_2} \frac{2\rho_2 + 3\kappa}{\rho_2 + \kappa} e^{-\alpha t} \frac{L P_t f}{P_t f} + \frac{d\rho_1^2}{12\rho_2} \left( \frac{2\rho_2 + 3\kappa}{\rho_2 + \kappa} \right)^2 \frac{e^{-2\alpha t}}{1 - e^{-\alpha t}},$$

(12.91)

where $\alpha = \frac{2\rho_1\rho_2}{d(\rho_2 + \kappa)}$. We deduce

$$\frac{\partial \ln P_t f}{\partial t} \geq - \frac{d\rho_1}{6} \frac{2\rho_2 + 3\kappa}{\rho_2 + \kappa} \frac{e^{-\alpha t}}{1 - e^{-\alpha t}}.$$

By integrating from $t$ to $\infty$, we obtain

$$-\ln p(x, y, t) \geq - \frac{d}{2} \left( 1 + \frac{3\kappa}{2\rho_2} \right) \ln (1 - e^{-\alpha t}).$$

This gives the desired conclusion. \qed

12.2 Diameter bound

In this subsection we conclude the proof of Theorem 12.1 by showing that the diam $\mathbb{M}$ is bounded. Since we have assumed that $(\mathbb{M}, d)$ be complete, this implies that such metric space is compact. The idea is to show that the operator $L$ satisfies an entropy-energy inequality. Such inequalities have been extensively studied by Bakry in [3] (see chapters 4 and 5).

To simplify the computations, in what follows we denote by $D$ the number defined in (6.56), and we set

$$\alpha = \frac{2\rho_1\rho_2}{3(\rho_2 + \kappa)}.$$ 

Proposition 12.5 For $f \in L^2(\mathbb{M})$ such that $\int_{\mathbb{M}} f^2 d\mu = 1$, we have

$$\int_{\mathbb{M}} f^2 \ln f^2 d\mu \leq \Phi \left( \int_{\mathbb{M}} \Gamma(f) d\mu \right),$$

where

$$\Phi(x) = D \left[ \left( 1 + \frac{2}{\alpha D} x \right) \ln \left( 1 + \frac{2}{\alpha D} x \right) - \frac{2}{\alpha D} x \ln \left( \frac{2}{\alpha D} x \right) \right].$$

Proof. From Proposition 12.4 for every $f \in L^2(\mathbb{M})$ we have

$$\|P_t f\|_{\infty} \leq \frac{1}{(1 - e^{-\alpha t})^2} \|f\|_2.$$

Therefore, from Davies theorem (Theorem 2.2.3 in [23]), for $f \in L^2(\mathbb{M})$ such that $\int_{\mathbb{M}} f^2 d\mu = 1$, we obtain

$$\int_{\mathbb{M}} f^2 \ln f^2 d\mu \leq 2t \int_{\mathbb{M}} \Gamma(f) d\mu - D \ln (1 - e^{-\alpha t}), \quad t > 0.$$
By minimizing over $t$ the right-hand side of the above inequality, we obtain
\[
\int_{\mathcal{M}} f^2 \ln f^2 d\mu \leq -\frac{2}{\alpha} x \ln \left(\frac{2x}{2x + \alpha D}\right) + D \ln \left(\frac{2x + \alpha D}{\alpha D}\right).
\]
where $x = \int_{\mathcal{M}} \Gamma(f) d\mu$. It is now an easy exercise to recognize that the right-hand side of the latter inequality is the same as $\Phi(x)$.

\[
\square
\]

With Proposition 12.5 in hands, we can finally complete the proof of Theorem 12.1.

**Proposition 12.6** One has

\[
\text{diam} \mathcal{M} \leq 2\sqrt{\frac{D}{\alpha}} = 2\sqrt{3\pi} \sqrt{\frac{\rho_2 + \kappa}{\rho_1 \rho_2}} \left(1 + \frac{3\kappa}{2\rho_2}\right) d.
\]

**Proof.** The function $\Phi$ that appears in the Proposition 12.5 enjoys the following properties:

- $\Phi'(x)/x^{1/2}$ and $\Phi(x)/x^{3/2}$ are integrable on $(0, \infty)$;
- $\Phi$ is concave;
- $\frac{1}{2} \int_0^{+\infty} \Phi'(x) x^{-3/2} dx = \int_0^{+\infty} \Phi(x) \sqrt{x} dx = -2 \int_0^{+\infty} \sqrt{x} \Phi''(x) dx < +\infty$.

We can therefore apply the beautiful Theorem 5.4 in [3] to deduce that the diameter of $\mathcal{M}$ is finite and

\[
\text{diam} \mathcal{M} \leq -2 \int_0^{+\infty} \sqrt{x} \Phi''(x) dx.
\]

Since $\Phi''(x) = -\frac{2D}{x(2x + \alpha D)}$, a routine calculation shows

\[
-2 \int_0^{+\infty} \sqrt{x} \Phi''(x) dx = 2\sqrt{3\pi} \sqrt{\frac{\rho_2 + \kappa}{\rho_1 \rho_2}} \left(1 + \frac{3\kappa}{2\rho_2}\right) d.
\]

\[
\square
\]

**Remark 12.7** The constant $2\sqrt{3\pi} \sqrt{\frac{\rho_2 + \kappa}{\rho_1 \rho_2}} \left(1 + \frac{3\kappa}{2\rho_2}\right) d$ is not sharp. For instance, in the Riemannian case $\kappa = \rho_2 = 0$, we obtain

\[
\text{diam} \mathcal{M} \leq 2\sqrt{3\pi} \sqrt{\frac{d}{\rho_1}},
\]

whereas it is known from the Bonnet-Myer’s theorem that

\[
\text{diam} \mathcal{M} \leq \pi \sqrt{\frac{d-1}{\rho_1}}.
\]
12.3 Dimension bound

We now turn to an upper bound for the Hausdorff dimension of the compact metric space \((M, d)\).

**Proposition 12.8** The Hausdorff dimension of the metric space \((M, d)\) is less than \(D\) given by (6.56).

**Proof.** Let us recall that, from our assumptions,

\[
T_x M = \mathcal{H}(x) \oplus \mathcal{V}(x), \quad x \in M,
\]

where

\[
\mathcal{H}(x) = \text{span} \{X_1(x), \ldots, X_d(x)\}, \quad x \in M,
\]

and

\[
\mathcal{V}(x) = \text{span} \{Z_{mn}(x), 1 \leq m, n \leq h\},
\]

We moreover assumed that

\[
\dim \mathcal{H}(x) = d, \quad x \in M,
\]

and this implies that also \(\dim \mathcal{V}(x) = \dim M - d\), is independent of \(x \in M\). From Theorem 2 in [50] we deduce that the Hausdorff dimension of the compact metric space \((M, d)\) is equal to \(\dim_{\text{Haus}}(M) = d + 2 \dim \mathcal{V}(x)\). Moreover, from [11] and [68] (see also Chapter 3 in [8]), there exists a smooth and positive function \(m\) on \(M\) such that

\[
\lim_{t \to 0} t^D p(x, x, t) = m(x).
\]

From the bound (6.57) we conclude that

\[
\dim_{\text{Haus}}(M) \leq D.
\]

\(\square\)

12.4 Isoperimetric bounds and \(L^1\) Poincaré inequality

We recall our assumption, following Proposition [12.3] that \(\mu(M) = 1\). Also, let \(D\) be defined by (6.56). With this in hands, we can now proceed as in section [11]

**Proposition 12.9** Let \(E \subset M\) be a Caccioppoli set. We have

\[
\mu(E)(1 - \mu(E)) \leq \frac{3}{2} D \sqrt{\frac{\kappa + \rho_2}{d \rho_1 \rho_2}} P_H(E, M).
\]

**Proof.** We proceed exactly as in the proof of Proposition [11.4] to obtain from (12.91) the inequalities

\[
\|\sqrt{\Gamma(P_t f)}\|_\infty \leq D \sqrt{\frac{\rho_1 \rho_2}{d (\kappa + \rho_2)}} \frac{e^{-at}}{\sqrt{1 - e^{-at}}} \|f\|_\infty
\]

and

\[
\|f - P_t f\|_1 \leq 3D \sqrt{\frac{\kappa + \rho_2}{d \rho_1 \rho_2}} \sqrt{1 - e^{-at}} \|f\|_1.
\]
Combining this with Proposition 12.4 gives
\[ 3D \sqrt{\frac{\kappa + \rho_2}{d\rho_1 \rho_2} \sqrt{1 - e^{-\alpha t}}} \mathcal{P}_H(E, \mathcal{M}) \geq 2 \left( \mu(E) - \frac{1}{(1 - e^{-\alpha})D/2} \mu(E)^2 \right). \]

We conclude by letting \( t \to +\infty \).

\[ \square \]

The previous isoperimetric inequality leads to the following \( L^1 \) Poincaré inequality.

**Proposition 12.10** Let \( f \in C^\infty(\mathcal{M}) \), then
\[ \inf_{c \in \mathbb{R}} \int_{\mathcal{M}} |f - c| d\mu \leq 6D \sqrt{\frac{\kappa + \rho_2}{d\rho_1 \rho_2}} \int_{\mathcal{M}} \sqrt{\Gamma(f)} d\mu. \]

**Proof.** Let \( m \) be a median for \( f \), that is
\[ \mu(f \geq m) \geq \frac{1}{2}, \quad \mu(f \leq m) \geq \frac{1}{2}. \]

Set
\[ f^+ = \max(f - m, 0), \quad f^- = -\min(f - m, 0) \]
so that \( f - m = f^+ - f^- \). We have
\[ \int_{\mathcal{M}} |f - m| d\mu = \int_{\mathcal{M}} f^+ d\mu + \int_{\mathcal{M}} f^- d\mu, \]
and thus
\[ \int_{\mathcal{M}} |f - m| d\mu = \int_0^{+\infty} \mu(f^+ > t) dt + \int_0^{+\infty} \mu(f^- > t) dt. \]

Observe that for every \( t > 0 \),
\[ \mu(f^+ > t) \leq \frac{1}{2}, \quad \mu(f^- > t) \leq \frac{1}{2}, \]
from Proposition 12.9 we obtain
\[ \mu(f^+ > t) \leq 3D \sqrt{\frac{\kappa + \rho_2}{d\rho_1 \rho_2}} \mathcal{P}_H(\{f^+ > t\}, \mathcal{M}), \]
and
\[ \mu(f^- > t) \leq 3D \sqrt{\frac{\kappa + \rho_2}{d\rho_1 \rho_2}} \mathcal{P}_H(\{f^- > t\}, \mathcal{M}). \]

This gives
\[ \int_{\mathcal{M}} |f - m| d\mu \leq 6D \sqrt{\frac{\kappa + \rho_2}{d\rho_1 \rho_2}} \left( \int_{\mathcal{M}} \sqrt{\Gamma(f^+)} d\mu + \int_{\mathcal{M}} \sqrt{\Gamma(f^-)} d\mu \right). \]

Observing that \( \sqrt{\Gamma(f^+)} + \sqrt{\Gamma(f^-)} = \sqrt{\Gamma(f^+ + f^-)} \), completes the proof.

\[ \square \]
12.5 A Lichnerowicz type theorem

A well-known theorem of Lichnerowicz asserts that on a $d$-dimensional complete Riemannian manifold whose Ricci curvature is bounded below by a non negative constant $\rho$, then the first eigenvalue of the Laplace-Beltrami operator is bounded below by $\frac{\rho}{d-1}$. In this section, we provide a similar theorem for our operator $L$. Let us observe that in [33], Greenleaf obtained a similar result for the sub-Laplacian on a CR manifold.

**Proposition 12.11** The first non zero eigenvalue $\lambda_1$ of $-L$ satisfies the estimate

$$\lambda_1 \geq \frac{\rho_1\rho_2}{d-1}\rho_2 + \kappa.$$

**Proof.** Let $f : M \to \mathbb{R}$ be an eigenfunction corresponding to the eigenvalue $-\lambda_1$. From our assumptions,

$$\Gamma_2(f, f) + \nu \Gamma_Z^2(f, f) \geq \frac{1}{d} (Lf)^2 + \left(\rho_1 - \frac{\kappa}{\nu}\right) \Gamma(f, f) + \rho_2 \Gamma_Z(f, f).$$

By integrating this inequality on the manifold $M$, we obtain

$$\int_M \Gamma_2(f, f) d\mu + \nu \int_M \Gamma_Z^2(f, f) d\mu \geq \frac{1}{d} \int_M (Lf)^2 d\mu + \left(\rho_1 - \frac{\kappa}{\nu}\right) \int_M \Gamma(f, f) d\mu + \rho_2 \int_M \Gamma_Z(f, f) d\mu.$$

Let us now recall that

$$\Gamma(f, f) = \frac{1}{2}(L(f^2) - 2Lf),$$

$$\Gamma_2(f, f) = \frac{1}{2}[L\Gamma(f, f) - 2\Gamma(f, Lf)],$$

and

$$\Gamma_Z^2(f, f) = \frac{1}{2}[L\Gamma_Z(f, f) - 2\Gamma_Z(f, Lf)].$$

Therefore, by using $Lf = -\lambda_1 f$ and integrating by parts in the above inequality, we find

$$\left(\lambda_1^2 - \frac{\lambda_1^2}{d} + \frac{\kappa \lambda_1}{\nu} - \rho_1 \lambda_1\right) \int_M f^2 d\mu \geq (\rho_2 - \nu \lambda_1) \int_M \Gamma_Z(f, f) d\mu.$$

By choosing $\nu = \frac{\rho_2}{\lambda_1}$, we obtain the desired inequality

$$\lambda_1 \geq \frac{\rho_1\rho_2}{d-1}\rho_2 + \kappa.$$

□

**Remark 12.12** We note that when $\kappa = 0$, we recover the classical theorem of Lichnerowicz.
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