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Hailiang Zhang\textsuperscript{1,2,*} and Guangting Chen\textsuperscript{1}

Abstract: In this note, we study the largest matching roots of unicyclic graphs with a given number of fixed matching number. We also characterize the extremal graph with respect to the largest matching roots. In addition, we also study this problem on the trees with a given number of fixed matching number.

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1. Introduction

All graphs considered in this paper are undirected and simple (i.e. loops and multiple edges are not allowed). Let $G = (V(G), E(G))$ be a graph with a vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and an edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$, where $|V(G)| = n$ is the order and $|E(G)| = m$ is the size of $G$. $\Gamma_G(v)$ be the neighbor set of vertex $v$ of $G$. Two edges are called adjacent if they have a common vertex. A matching $M$ is a subset of $E(G)$ where no two edges of $M$ are adjacent. Denoted by $\mathcal{M}(G, k)$, a matching with

ABOUT THE AUTHOR

Hailiang Zhang (male 1976.03–) graduated from East China Normal University with a PhD degree in applied mathematics in 2013. With Professor Guangting Chen and some other younger teachers now we have an operational research group in Taizhou University. We mainly study the graph theory and combinatorial problems and their applications. Our manuscript have been published in the Journal of Information Processing Letters, Discrete Mathematics, Ars Combinatorial, Discrete Applied Mathematics, etc. Graph polynomials are intensively studied in recent years, one reason is that it is related to the graph structure and combinational problems, but also some of the index of graphs be used in chemistry to explain the properties of material. Such as the sum of its matching polynomial are called the Hosoya index, it is widely studied in chemistry. Also the number of unsaturated number of vertices is equal the multiplicity of 0 as a root of matching polynomial. We use the information found in the polynomials of graphs to explain the structural problems. Our research is focused on the matching polynomial by far we have obtained some interesting results, such as we characterized all the graphs which only have six distinct matching roots. The largest matching root of unicyclic graphs, etc.
$k$ edges, and $m(G, k)$ the number of matchings with $k$ edges. Cvetković, Doob, Gutman, and Torgašev (1988) denote the matching polynomial as,

$$M_G(x) = \sum_{k=0}^{n/2} (-1)^k m(G, k)x^{n-2k}. \quad (1)$$

In addition, we set $m(G, 0) = 1$ by convention. Obviously, $|E(G)| = m(G, 1)$, is the numbers of edges. Clearly, $m(G, k) = 0$ if $k > n/2$. Let $M(G)$ be the largest matching root of $M_G(x)$.

For a simple connect graph. If $m = n - 1$, then we call $G$ a tree, denoted by T. Gutman (1982) shows that for trees the largest matching roots and the adjacency spectral radius are the same. Gutman and Zhang (1986) have ordered the graphs by matching numbers. If $m = r$, then we call $G$ an unicyclic graph. Zhang (2013) gives the largest matching root of unicyclic graphs and characterizes the extremal graph. Let $\mathcal{U}_n^g(r)$ be the set of unicyclic graphs on $n$ vertices, which has a cycle of length $g$ and has a matching number $r$. Let $\mathcal{U}_n^g(s_1, t_1; \ldots; s_i, t_i)$ be a graph obtained by attaching $s_i$ pendant edges and $t_i$ paths of length $2, 1 \leq i \leq g$ to the cycle of $G$ at $v_i$. In this paper, we give the largest matching roots of unicyclic graphs and trees with a fixed matching number. Furthermore, we characterize the extremal graphs with respect to it.

2. Preliminaries

Let $G - v$ be the graph obtained by deleting a vertex $v$ with its incident edges form $G$. $G - e$ be the graph obtained by deleting an edge $e$ from $G$. The following Lemmas 2.1 and 2.2 are often used to calculate the matching polynomial of a graph.

**Lemma 2.1** (Cvetković et al., 1988) Let $G$ be a graph with $u \in V(G)$, and suppose the neighborhood of $u$ is $\Gamma(u) = \{v_1, v_2, \ldots, v_r\}, uv \in E(G)$. Then

1. $M_G(x) = M_{G-v}(x) - M_{G-u-v}(x),$
2. $M_G(x) = xM_{G-u}(x) - \sum_{uv \in E(G)} M_{G-u-v}(x), v_i \in \Gamma(u)$.

**Lemma 2.2** (Cvetković et al., 1988) Let $G_1, G_2, \ldots, G_k$ be $k$ components of $G$. Then

$$M_G(x) = \prod_{i=1}^{k} M_{G_i}(x).$$

**Lemma 2.3** (Gutman, 1982) Let $G$ be a connected simple graph, $v$ be a vertex of $G$ and $e$ be an edge of $G$. Then $M(G) > M(G - v)$ and $M(G) > M(G - e)$.

**Lemma 2.4** (Gutman, 1982) Let $v_1, \ldots, v_n$ be the vertices of a graph $G$, Let $G - v_i$ be the subgraphs of $G$ obtained by deleting the vertex $v_i$, then

$$\frac{d}{dx} M_G(x) = \sum_{i=1}^{n} M_{G-v_i}(x) = \sum_{k=0}^{n/2} (-1)^k (n - 2k)m(G, k)x^{n-2k-1}. \quad (2)$$

**Lemma 2.5** Let $G'$ be a spanning subgraph of $G$. $M(G')$ be the largest matching root of $G$. If $x \geq M(G')$, then $M_{G'}(x) \geq M(x)$. If $G'$ is a proper spanning subgraph of $G$ and $x > M(G)$, then $M_{G'}(x) > M(x)$.

**Proof** Let $G'$ be spanning subgraph of $G$ and $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of $G$. $G' = G - v_r$ $G'_r = G' - v_r$. For $x \in (M(G), \infty)$, we have

$$f(x) = M_{G'}(x) - M_{G}(x). \quad (3)$$
We need to prove that \( f(x) \geq 0 \), when \( x \geq M(G) \). By applying differentiation on Equation (3) and with Lemma 2.4 (2), we have

\[
f'(x) = M'_{G'}(x) - M'_G(x) = \sum_i (M_{G'}(x) - M_G(x)) = \sum_i (M_{G'}(x) - M_G(x)).
\]  

(4)

Now, we apply induction on the number of vertices.

Case 1  When \( n = 1 \), the result is trivial.

Case 2  When \( n = 2 \), \( G \cong K_2 \) and \( G' \) is empty graph of order 2. Obviously, by the Lemma 2.1 and the Lemma 2.2, we have

\[ M_G(x) = x^2 - 1, \quad M_{G'}(x) = x^2. \]

When \( x \geq 1 \), \( x^2 > x^2 - 1 \) holds, so the Lemma 2.5 holds.

Case 3  Assume that the Lemma 2.5 holds when the order of \( G \) is less than \( n \). We will show that Lemma 2.5 holds when the order of \( G \) is \( n \).

When \( n \geq 3 \), \( |V(G_i)| = n - 1 \). For every \( G_i \) there exists a spanning subgraph \( G'_i \) correspond to it. By our assumption, \( M_{G_i'}(x) \geq M_{G_i}(x) \) when \( x \geq M(G_i) \). By the Lemma 2.3, \( M(G) \geq \max \{M(G_1), \ldots, M(G_n)\} \). Then for \( x \geq M(G) \),

\[
f'(x) = M'_{G'}(x) - M'_G(x) = \sum_i (M_{G'}(x) - M_G(x)) \geq 0,
\]

holds. If \( G' \) is a proper spanning subgraph of \( G \), without loss generality, let \( G'_j \) be the proper spanning subgraph of \( G_j \) (1 \( \leq j \leq n \)) by our assumption, when \( x = M(G) > M(G_j) \),

\[ M_{G'_j}(x) > M_{G_j}(x), \]

holds. Therefore:

\[
f'(x) = M'_{G'}(x) - M'_G(x) = \sum_i M_{G'}(x) - \sum_i M_G(x) > 0.
\]

That is \( f'(M(G)) > 0 \), and

\[ f(M(G)) = M_{G'}(M(G)) - M_G(M(G)) = M_{G'}(M(G)) > 0.\]

It shows that when \( x \geq M(G) \), \( f(x) \) is a monotonic increasing function. Hence

\[ M_{G'}(x) > M_{G}(x), \quad \text{when} \quad x \geq M(G).\]

Furthermore, we have \( M(G^*) < M(G) \). \( \square \)

Since for every subgraph \( H \) of \( G \) we can add some isolated vertices to \( H \) let it be a spanning subgraph of \( G \). Hence we have for any subgraph \( H \) of \( G \), we have the following proposition.

Proposition 2.6  For any subgraph \( H \) of \( G \), if \( x \geq M(G) \) then \( M_H(x) \geq 0 \).
Definition 2.7 (Csikvári, 2011) Let \( u, v \) be two vertices of the graph \( G \), we obtain the Kelmans transformation of \( G \) as follows: we erase all edges between \( v \) and \( N(v) - (N(u) \cup \{ u \}) \) and add all edges between \( u \) and \( N(v) - (N(u) \cup \{ u \}) \) (see Figure 1).

Lemma 2.8 (Csikvári, 2011) Assume that \( G^* \) is a graph obtained from \( G \) by some Kelmans transformation, then \( M(G^*) \geq M(G) \).

In the following proof of our main results, we need to move pendant edges and pendant paths of length two in \( G \) to construct another graph \( G^* \). This is a special case of the Kelmans transformation. We give as a Corollary 2.9.

Corollary 2.9 Let \( G \) be a simple graph, \( u, v \in V(G) \) with \( s \) and \( t \) pendant paths of length 1, or pendant paths of length 2 on each. \( G^* \) denote the graph obtained by moving all pendant paths to one vertex (see Figure 2). Then

\[ M(G^*) > M(G). \]

Let \( G \) be a simple graph. If \( e = (v_1, v_2) \) is not an edge of \( C_3 \) of \( G \), then \( G \diamond e \) denotes the graph obtained by contracting edge \( e \). \( G^* \) denotes the graph obtained by adding a pendant edge to the contracted vertex \( u \) of \( G \diamond e \) (see Figure 3). For the largest matching root of \( G \) and \( G^* \), we have the following Lemma 2.10.

Lemma 2.10 Let \( H_1 \) and \( H_2 \) be two graphs with distinguished vertices \( u_1, u_2 \) of \( H_1 \) and \( H_2 \) respectively. Let \( G \) be the graph connecting \( u_1 \) and \( u_2 \) by an edge \( e \). Let \( H_1 \cup H_2 \) be the graph obtained from \( H_1 \) and \( H_2 \) by identifying the vertices of \( u_1 \) and \( u_2 \) to a new vertex \( u \) (see Figure 3). Let \( G^* \) be the graph obtained by attaching a pendant to \( u \) of \( G_1 \) (see \( G^* \) in Figure 3). Then

\[ M(G) < M(G^*). \]

Proof By Lemma 2.1 and Lemma 2.2, we calculate the matching polynomial of \( G \) and \( G^* \) as:

![Figure 1. Kelmans transformation.](image1)

![Figure 2. G and G*.](image2)
Theorem 3.1 We also give the extremal graphs for the unicyclic graphs and trees with a fixed matching number.

In right-hand side of Equation (7) all graph \( H \) are subgraphs of \( G \). By Lemma 2.3 and Proposition 2.6, for all \( x \geq M(G) \), \( M_{H \cup u}(x) > 0 \). Then \( M_G(x) - M_{G'}(x) > 0 \). Hence

\[ M(G) < M(G'). \]

In particular, if \( e \) is an pendant edge of \( G \) then \( G' \cong G \), and

\[ M(G') = M(G). \]

3. Main results of this paper

In this section, we present the extremal graphs in \( \mathcal{H}_n^3(r) \), by using the Lemmas obtained in Section 2. We also give the extremal graphs for the unicyclic graphs and trees with a fixed matching number.

**Theorem 3.1.** Let \( \gamma^3_1(s_0,t_1,s_2,s_3) \) be the unicyclic graph attaching \( s_1, s_2, s_3 \) pendants edges on the vertex \( v_1, v_2, v_3 \) and \( t_1, t_2, t_3 \) paths of length 2 to \( v_1, v_2, v_3 \), respectively. Then

\[
M_G(x) = x^3 + x^2 + x - 3 \left( x^2 - 1 \right) \sum_{i=1}^{3} \left( x^2(x^2 - 1) - s_i(x^2 - 1) - t_i x^2 \right)
\]

**Corollary 3.2.** Let \( U_g^3(s_0,t_1,\ldots,s_n,t_n) \) be the unicycle graphs (on \( n \) vertices and length of cycle \( g \) with matching number \( r \) defined in Section 1). Then
\[ M_{\text{tr}}(n - 2r + 1, r - 2; 0; 0, 0, 0)(x) = x^{n-2r}(x^2 - 1)^{r-2} \]
\[ [x^4 - (n - r + 2)x^2 + (n - 2r + 1)]. \]

Corollary 2.9 and Lemma 2.10 guarantee the largest matching root will not decrease under those graph transformations. To finish our proof, we also need to prove that the matching number also does not change under those transformations. Let us first deal with the trees.

**Theorem 3.3** Let \( S(s, t) \) be a graph obtained by subdivision \( t \) edges of a star \( K_{1,3+t} \). Then for every matching \( M(T, r) \) of \( T \) of order \( 2t + s + 1 \), there is a corresponding map from matching \( M(T, r) \) of \( T \) to a matching of \( M(S(s, t), r) \).

**Proof** For a tree \( T \) on \( n \) vertices, the path \( P_n \) has the maximal matching of order \( n / 2 \) when \( n \) is even, or \( (n - 1) / 2 \) when \( n \) is odd; hence we only need to construct an corresponding map from maximal matching \( M \) of \( P_n \) to maximal match \( M^* \) of \( S(s, t) \), for the small number of \( n \) we can easily construct.

Assume that \( n \) is an even number and \( n \geq 4 \). Let \( M = \{e_1, e_2, \ldots, e_{n/2}\} \) be a maximal match of \( P_n \). We can correspond \( e_1, e_2, \ldots, e_{n/2-1} \) to the pendant \( n/2 - 1 \) edges of \( P_n \) of \( S(1, n/2 - 1) \), and \( e_{n/2} \) to the one pendant edge of \( S(1, n/2 - 2) \).

Assume that \( n \) is an odd number and \( n \geq 4 \). Let \( M = \{e_1, e_2, \ldots, e_{(n-1)/2}\} \) be a maximal match of \( P_n \). We can correspond \( e_1, e_2, \ldots, e_{(n-3)/2} \) to the pendant \( (n - 1)/2 \) edges of \( P_n \) of \( S(0, (n - 1)/2) \).

By Lemma 2.10 and Theorem 3.3, for trees with matching number \( r \), we have Theorem 3.4:

**Theorem 3.4** Let \( T \) be a tree on \( n \) vertices and with a matching number \( r \), then \( S(n - 2r + 1, r - 1) \) has the largest matching root, which is

\[ M(T) = \sqrt{n - r + 1 + \sqrt{(n - r)^2 - 2(n - 3r) - 3}} \]

Now, we study the largest matching root of unicyclic graphs.

**Theorem 3.5** For any \( G \in \mathcal{W}_n^2(r) \), we have

\[ M(G) \leq \sqrt{n - r + 2 + \sqrt{(n - r)^2 + 4r}} \]

with the equality if and only if \( G \cong U_{n}(n - 2r + 1, r - 2; 0, 0, 0) \).

**Proof** Suppose that \( G \in \mathcal{W}_n^2(r) \) and \( T_1, T_2, \ldots, T_i \) be the pendant trees attaching to vertices \( v_1, \ldots, v_i \) of \( C_g \) of \( G \), we use Lemma 2.10 on the \( T_i \) and move all the pendant paths to one vertex \( u \) of \( C_g \). After this processing, the vertices of \( C_g \) are saturated by a pendant edge or an edge of \( C_g \) or both a pendant edge and an edge of \( C_g \). This situation will happen, if the \( T_i \) is a path on even number of vertices and \( g \) is on an even cycle with perfect matching.

Assume that the vertex \( v_i \) is saturated by a pendant edge, then the neighbor of \( v_i \) on the cycle \( v_{i-1} \) and \( v_{i+1} \) are not saturated. We delete \( v_i v_{i-1} \) and connect \( v_{i-1} \) and \( v_{i+1} \). This processing does not change the matching number and creates a pendant path of length 2 on \( v_{i-1} \). Then move this path to \( u \) (actually this processing is a Kelmans transformation).

If the vertex \( v_i \) is saturated by edges on cycle \( C_g \) Let this edge be \( v_i v_{i+1} \). We contract this edge and add a pendant to the path of length 1, this processing shortens the girth of \( C_g \) by one, and does not change the matching number.
Now, we consider the case that a vertex $v_i$ is saturated by an edge on cycle. Let us say $e = v_i v_{i-1}$ and another pendant edge $v_i u$ of attached tree $T$, then we contract edge $e$ and add a new edge $uw$ to the pendant edge of the tree. This processing shrinks $C_g$ by one, and have one more pendant path of length 2, which decrease the number of pendant edges by one, but without changing the matching number.

Finally, by the graph transformations shown in the Corollary 2.9, Lemma 2.10 the resulting graph should be

$$C_g^3(n - 2r + 1, n - 2; 0; 0, 0).$$

By Corollary 3.2 and solving that polynomial, we have

$$M(G) = \sqrt{\frac{n - r + 2 + \sqrt{(n-r)^2 + 4r}}{2}}.$$

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