A Leading–Order, But More Than One–Loop,
Calculation of Structure Functions

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Abstract. I present a full leading–order calculation of $F_2(x, Q^2)$ and $F_L(x, Q^2)$, including contributions not only from leading order in $\alpha_s$, but also from the leading power of $\alpha_s$ for each order in $\ln(1/x)$. The calculation is ordered according to the inputs and evolution of the structure functions, and the perturbative form of the inputs is determined. I compare the results of fits to data to those using conventional LO and NLO order calculations, and the correct inclusion of leading $\ln(1/x)$ terms is clearly preferred. A prediction for $F_L(x, Q^2)$ is produced which is smaller at small $x$ than that obtained from the conventional approach.

INTRODUCTION

There has recently been a great deal of new data obtained at HERA for the structure function $F_2(x, Q^2)$ at small $x$ [1,2], and consequently a great deal of theoretical activity. The main theoretical question is whether one should include leading $\ln(1/x)$ terms which cause small $x$ enhancement of terms at high orders in $\alpha_s$, and if so, then in precisely what manner. Common wisdom was that such terms should not be included because standard ways of doing so showed that the terms were indeed important for the ranges of $x$ and $Q^2$ probed at HERA, but that they worsened global fits to data obtained from conventional LO and NLO approaches, rather than improved them. Indeed, the best global fits to data seemed to come from NLO evolution starting at $Q_0^2 \sim 2\text{GeV}^2$ where the input for the singlet quark distribution (weighted by $x$), and hence the structure function, behaved like $x^{-0.25}$ at small $x$ [3]. Thus, despite the good quality of the global fits, the situation was unsatisfactory since terms were ignored because they were inconvenient, rather than for any sound theoretical reason, and the required input for the quark distribution was of a rather steep, unjustified form.

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Recently I proposed that a correct theoretical treatment of the calculation of structure functions demands that the leading powers of $\alpha_s$ for given powers of $\ln(1/x)$ must be included, but in expressions for physical quantities [4]. This latter point was inspired by, and is similar to the idea of using physical anomalous dimensions proposed by Catani [5]. However, the structure imposed by an ordered expansion within a given renormalization scheme has a number of other consequences for the form of the expressions for structure functions. I will discuss the leading–order–renormalization–scheme–consistent (LORSC) calculation of structure functions briefly before looking at comparison with experiment.

THE LORSC CALCULATION.

The principles underlying the calculation and the consequences are:

1. The quantities one calculates to a given order are directly observable. Hence rather than coefficient functions and parton distributions the structure function is factorized into inputs at some scale $Q^2_I$ and evolution away from this scale. One obvious consequence of this is factorization scheme independence.

2. For a physical quantity the leading–order expression for each independent component begins at its lowest power of $\alpha_s$, whatever power that happens to be, e.g. if a term of $\ln^m(1/x)$ appears for the first time at order $\alpha^m_s$ this is the leading–order term for $x$-dependence of this type. In practice this is implemented in moment space where the leading power in $\alpha_s$ for each inverse power of the moment variable $N$ is part of the full leading–order expression. As it must, this results in leading–order expressions which are renormalization scheme independent and which are therefore compatible with the use of the one–loop running coupling.

3. The inputs for the structure functions are two fundamental nonperturbative functions, one each for $F_2$ and $F_L$, convoluted with perturbative contributions. The nonperturbative functions are taken to be flat at small $x$. The perturbative parts are determined so that the total expression for the structure function is independent of the choice of starting scale, i.e. the LORSC expression is invariant under changes of $Q^2_I$ up to changes beyond leading order. This determines the inputs uniquely up to a scale, $Q^2_{NP}$, which should be typical of the onset of nonperturbative physics. For both $F_2$ and $F_L$ the inputs are required to behave roughly like $x^{-0.28}$ for $Q^2_I \sim 20 - 100$GeV$^2$ for $10^{-2} \leq x \leq 10^{-5}$. Moreover, the forms of $F_L(Q^2_I)$, $F_2(Q^2_I)$ and $(d F_2(Q^2)/d \ln Q^2)_{Q^2_I}$ are all related at small $x$, and once one of them is set (in practice $F_2(Q^2_I)$) by fitting to data there is very little freedom in the others, a constraint largely absent in conventional approaches.

These points are discussed in far greater detail in [4], and the way in which they are put into practice is presented in the latter of these papers.
FITS TO DATA AND PREDICTIONS

The fit using the LORSC expressions is performed with the parameters specifying the form of the nonperturbative inputs and the nonperturbative scale left free, and is repeated for a wide variety of input scales $Q_{IP}^2$. The best fit comes from $Q_{IP}^2 = 40\text{GeV}^2$, but is very insensitive to $Q_{IP}^2$ for values from $20\text{GeV}^2 \to 80\text{GeV}^2$. We find that $Q_{NP}^2 = 0.55\text{GeV}^2$, precisely the sort of value expected. The charm and bottom quarks are dealt with rather naively, treated as massless above threshold ($4\text{GeV}^2$ and $20\text{GeV}^2$ respectively) and playing no role below this. The charm threshold is chosen to give a reasonable description of data on the charm structure function. $Q^2$ is chosen as the (squared) renormalization scale, and the value of $\Lambda_{QCD}$ is held fixed at $100\text{MeV}$ (giving $\alpha_s(M_Z^2) = 0.115$ at one loop), though this is certainly close to the best fit value. The results of the fit are shown in table 1.

The conventional NLO fit is performed in the usual manner and the input scale for the parton distributions $Q_0^2$ is chosen to be the same as that for the charm threshold ($2.75\text{GeV}^2$). For the best fit we find that $\Lambda_{\overline{MS}} = 300\text{MeV}$, i.e. $\alpha_s(M_Z^2) = 0.118$. The input quark distribution $\sim x^{-1.22}$. The conventional LO fit is performed in the same way with charm threshold of $3\text{GeV}^2$, $\Lambda_{QCD} = 160\text{MeV}$, (one–loop $\alpha_s(M_Z^2) = 0.124$), and small $x$ quark $\sim x^{-1.23}$. The results of these more conventional fits are also shown in table 1.\(^1\)

In each case the values of $F_2(x, Q^2)$ used for the HERA data are not simply those published but are corrected for the values of $F_L(x, Q^2)$ predicted in each of the fits. For the LORSC fit this causes a slight lowering of $F_2(x, Q^2)$ for some points due to smaller predicted $F_L(x, Q^2)$ which in fact improves the fit to the small $x$ data by a $\chi^2$ of about 5. For the NLO fit there is essentially no correction, but for the LO fit the predicted $F_L(x, Q^2)$ is extremely large since the fit requires both a large coupling and large gluon at small $x$. Hence, the values of $F_2(x, Q^2)$ are corrected upwards, sometimes by 4%, leading to the correct LO fit being $\sim 25$ worse than a fit to uncorrected data.\(^2\)

As one can see the LORSC fit is clearly superior: although the difference

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\(^1\) All fits were constrained not only by consistency with charm structure function data, but also by consistency with high–$x$ prompt photon data.

\(^2\) For the LO and NLO fits the normalization of the H1 data is as low as I allow, i.e. 98.5%, while for the LORSC fit it is 100%. In each case the normalization of the ZEUS data is $\sim 1\%$ higher.

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| $x$-range | data points | LORSC $\chi^2$ | NLO $\chi^2$ | LO $\chi^2$ |
|-----------|-------------|----------------|---------------|-------------|
| $x \geq 0.1$ | 551         | 622            | 615           | 605         |
| $x < 0.1$   | 548         | 483            | 554           | 598         |
| total      | 1099        | 1105           | 1169          | 1203        |
in quality is $\sim 0.06$ per point this is significant when considering 1099 data
points. Indeed, the difference between the quality of the LORSC fit and the
NLO fit is larger than that between the NLO and LO fits. The quality in
terms of different experiments may be found in the second of [4] (as may the
references for the experiments), but it is illustrative to present it in terms of
two bins in $x$. There is little difference in the quality of the high–$x$ fits, as we
would expect since at high $x$ the missing $\ln(1/x)$ terms in the LO and NLO
fits cause no problems (and an appropriate choice of coupling makes the LO
fit as good as the NLO fit). In fact, the LORSC calculation is missing NLO
$\alpha_s$ terms, and should perhaps be a little worse than the NLO fit. However,
at small $x$ there is a very clear deterioration in the quality of fit going from
LORSC to NLO to LO (which would be even greater if values of $\Lambda$ consistent
with the best high–$x$ fits were used). This is due to the loss of vital $\ln(1/x)$
terms as ones goes from LORSC to NLO, and also, to a lesser extent, when
going from NLO to LO. Hence, the results of the fits are precisely as we would
expect from theoretical arguments. They clearly imply that in determining
values of $\alpha_s(M_Z^2)$ and the gluon distribution in particular schemes the NLO
fit should only be trusted at relatively high $x$.

Shown in fig. 1 is the comparison of the predictions for $F_L(x, Q^2)$ made
using the best fit from the LORSC approach and the NLO approach. As one can
see there is significant difference between them, and any relatively accurate
measurement of $F_L(x, Q^2)$ should see some sign of which approach is preferred.
The “determination” of $F_L(x, Q^2)$ by H1 is no use in differentiating between
the two since it is really just a consistency check. If one assumes that the H1
NLO fit is correct and extrapolates from low $y = Q^2/xs$, where the determina-
tion of $F_2(x, Q^2)$ from the cross–section is insensitive to $F_L(x, Q^2)$ then the
difference between the measured cross–section and the extrapolation provides
a value of $F_L(x, Q^2)$ [6]. This must be consistent with the NLO prediction for
$F_L(x, Q^2)$, which it is. However, assuming that the LORSC calculation is cor-
correct the extrapolation is different, and hence so is the determined $F_L(x, Q^2)$.
Consistency between the determined $F_L(x, Q^2)$ and the prediction is also good
in this case. 4 Hence, a true measurement of $F_L(x, Q^2)$ would be an invaluable
aid to the determination of the real physics describing hadron interactions at
small $x$. An important role may also be played by improved data on the charm
structure function, and incorporating heavy quarks correctly into the LORSC
approach is a project well underway. Less inclusive quantities may well also
play an important part, but calculations remain to be done.

3) This is clear if we perform fits to high–$x$ data only: the LORSC fit only improves by a
few points, but the NLO fit improves to 578, with $\Lambda_{\overline{MS}} = 215$MeV. Such a low value of
$\Lambda_{\overline{MS}}$ is not allowed in the full NLO fit since it leads to a very poor fit at small $x$.
4) The H1 fit results in a gluon which is inconsistent with both high–$x$ prompt photon data
and moderate–$x$ charm data, and it would be illustrative to repeat the process with a more
constrained NLO fit. It would also be interesting to use all available high $y$ data, i.e. also
include ZEUS data. Such a study is being performed.
FIGURE 1. Comparison of predictions of $F_L(x, Q^2)$ using the LORSC fit and the NLO fit. In both cases $F_L(x, Q^2)$ increases with $Q^2$ at small $x$.

CONCLUSION

A comparison to all significant data gives clear evidence that the LORSC calculation of structure functions is preferred to the usual NLO–in–$\alpha_s$ approach. Thus, although the usual expansion technique seems to be acceptable at small $x$ at present, and will probably be used for most QCD calculations in the near future, when attempting to describe any data at small $x$ it should be borne in mind that this approach may well be untrustworthy: anomalies may occur, and more careful calculations may be needed. The situation will be clarified by a variety of different small $x$ measurements.

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