Total Dominating Energy of Some Graphs

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Abstract: Let G be a finite, connected and not oriented graph with the vertex set V(G) and edge set E(G). We have estimated the total dominating energy of the complete, complete bipartite, doublestar, Barbell graph, and chemical structure of "acetaminophen". [10]

Keywords: total dominating set, total domination number, minimum total dominating matrix, minimum total dominating eigenvalues, total dominating energy of a graph.

I. INTRODUCTION

Gutman, I [5] presented the idea of “energy of the graph” during 1978. Let G be a graph with n vertices, m edges and that A= (a_{ij}) is the adjacency matrix of the graph. The eigen values λ_1, λ_2, λ_3,…….,λ_n of A, taken in descending order λ_1 ≥ λ_2 ≥ λ_3 ≥ .... ≥ λ_m ≥ λ_n are the eigenvalues of graph G. Since the adjacency matrix A of G is real and symmetric, its eigenvalues are real numbers. The “energy” E(G) of the graph is the sum of the absolute values of the eigenvalues of the graph G i.e., E(G)=\sum_{i=1}^{n} |\lambda_i| . [4]

II. DEFINITIONS

2.1 Total Dominating Set: A set S of vertices on a graph G(V,E) is said to be a “total dominating set” if every vertex v\in V is an adjacency component of S. The “total domination number” γ(G) is the smallest number of vertices detected on all the minimal total dominating set in a graph G.[9],[3]

2.2 Energy: “Energy of the graph” is the sum of the absolute values of the eigenvalues of the adjacency matrix A. It is represented by E(G)=\sum_{i=1}^{n} |\lambda_i| where \lambda_i is an eigenvalues of A i=1,2,…….,n. [4],[6],[7]

2.3 Total Dominating Energy: Let G be a simple graph with set of vertices V= \{v_1,v_2,…….,v_n\} and the set of edges E. Let MTDS be the minimum total dominating set of graph G. “The minimum total dominating matrix” of G is A_{MTDS}(G)=(a_{ij}) where,

A_{MTDS}(G) = \begin{cases} 1; & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 1; & \text{if } i=j, v_i \in MTDS \\ 0; & \text{otherwise} \end{cases}

The “characteristic polynomial” of A_{MTDS} (G) is indicated by Det (A_{MTDS} (G) – λI). The “minimum total dominating eigenvalues” of graph G are the eigenvalues of A_{MTDS} (G). They are λ_1, λ_2,……., λ_m in decreasing order λ_1 ≥ λ_2 ≥ .... ≥ λ_m. The ‘total dominating energy’ of G is given by E_{TD} (G)=\sum_{i=1}^{m} |\lambda_i| . [9]

Note that the trace of (a_{ij})= Total Domination Number=k

Example:

Consider a graph G with V( G) = \{v_1, v_2, v_3, v_4, v_5\}.

MTDS = \{v_2, v_3\}

γ_t (G)=2.

A_{MTDS}(G)= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}

The characteristic equation is

x^6 - 2x^5 - 6x^4 + 7x^3 + 3x^2 = 0

The eigen values are λ_1=3.5454, λ_2 =0.7341, λ_3 = -1.5224, λ_4=0, λ_5=0.

E_{TD}(G) = 6.559.

III. DEFINITIONS OF SOME GRAPHS

Definition 3.1 “Complete Graph” K_n: It is a graph in which each pair of vertices is connected by an edge.[11]

Definition 3.2 “Complete Bipartite Graph” K_{m,n}: It is a graph whose vertices are partitioned into two disjoint sets V_1 and V_2 such that every pair of vertices in the two sets are adjacent and no two vertices within the same set are adjacent.[11]

Definition 3.3 “Double Star Graph” S_{m,n}: It is obtained by joining the center of two stars K_{1,n} and K_{1,m} with an edge. [8]

Definition 3.4 “Barbell Graph” B_{m,n}: It is the graph obtained by connecting two copies of the complete graph with a cut edge. [11]

Definition 3.5 “Book Graph” B_n: It is a graph of Cartesian product of star graph and two - node path.[11]

IV. RESULTS ON TOTAL DOMINATING ENERGY OF SOME GRAPHS

Theorem 4.1:

If n ≥ 2 then E_{TD}( K_n) is equal to

\[|n \cdot 3| + \left| \frac{(-1)^{n-2} \sqrt{n^2 - 2n + 9}}{2} \right| \]

Proof: K_n is the complete graph with V= \{x_1, x_2, \ldots., x_n\}. The MTDS(K_n) = \{x_1, x_2\} and γ(K_n) = \{x_1,x_2\}

Then,
The characteristic polynomial is
\[
\det(A_{MTD}(K_n) - \lambda I) = \begin{vmatrix}
1 - \lambda & 1 & 1 & 1 & 1 & \cdots & 1 \\
1 & 1 - \lambda & 1 & 1 & 1 & \cdots & 1 \\
1 & 1 & 1 - \lambda & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & 1 & 1 - \lambda & \cdots & 1 \\
1 & 1 & 1 & 1 & 1 & \cdots & 0 \\
\end{vmatrix}
\]

The characteristic equation is
\[
\lambda (\lambda + 1)^{n-3} (\lambda^2 - (n-1)\lambda - 2) = 0
\]
The eigenvalues are:
\[
\lambda = 0, \quad \lambda = -1, \quad (n-3) \text{ times}.
\]

Then, the Total Dominating Energy is
\[
E_{TD}(K_n) = |(n-3)| + \frac{|(1-n)^2 + \sqrt{n^2 - 2n + 9}}{2}.
\]

**Theorem 4.2:**
If \( n \geq 2 \) then, \( E_{TD}(K_{n,n}) \) is equal to
\[
\sqrt{n^2 + 2n - 3} + (n + 1)
\]

**Proof:** \( K_{n,n} \) is the complete bipartite graph with \( V = \{r_1, s_1, r_2, s_2, \ldots, r_n, s_n\} \). The MTDS(\( K_{n,n} \)) is \( (r_1, s_1) \) and is \( \gamma_t(K_{n,n}) = (r_1, s_1) \). Then,
\[
A_{MTD}(K_{n,n}) = \begin{vmatrix}
1 & 0 & 0 & \cdots & 1 & 1 & 1 \\
0 & 0 & 0 & \cdots & 1 & 1 & 1 \\
0 & 0 & 0 & \cdots & 1 & 1 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
1 & 1 & 1 & \cdots & -\lambda & 0 & 0 \\
1 & 1 & 1 & \cdots & 0 & -\lambda & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
1 & 1 & 1 & \cdots & 0 & -\lambda & 0 \\
\end{vmatrix}
\]

The characteristic polynomial is
\[
\det(A_{MTD}(K_{n,n}) - \lambda I) = \begin{vmatrix}
1 - \lambda & 1 & 1 & 1 & 1 & \cdots & 1 \\
0 & -\lambda & 0 & 1 & 1 & \cdots & 1 \\
0 & 0 & -\lambda & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & -\lambda & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & -\lambda & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
1 & 1 & 1 & 1 & 0 & \cdots & 0 \\
\end{vmatrix}
\]

The characteristic equation is
\[
\lambda^2 (\lambda - 2) (\lambda + 1)^{n-3} (\lambda^2 - (n-1)\lambda - 2) = 0
\]
The eigenvalues are:
\[
\lambda = 0, \lambda = -1, \lambda = 1, \lambda = -\sqrt{n-1}.
\]

Then, the Total Dominating Energy is
\[
E_{TD}(S_{r,t}) = 2 \left( \lambda + 1 \right) + \left( \frac{1}{2} \right) + \left( 1 - \frac{n}{2} \right).
\]

**Theorem 4.3:**
If \( n \geq 2 \) then, \( E_{TD}(S_{r,t}) \) is equal to
\[
\frac{3n}{2} + \frac{4}{1 + \sqrt{n^2 - 9n^4 + 48}}.
\]

**Proof:** \( S_{r,t} \) be the Double Star graph with \( V = \{u_1, w_1, u_2, w_2, \ldots, u_n, w_n\} \). The MTDS(\( S_{r,t} \)) = \( \{u_1, w_1\} \) and \( \gamma_t(S_{r,t}) = \{u_1, w_1\} \). Then,
\[
A_{MTD}(S_{r,t}) = \begin{vmatrix}
1 & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 1 & 1 & 1 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 1 & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\end{vmatrix}
\]

The characteristic polynomial is
\[
\det(A_{MTD}(S_{r,t}) - \lambda I) = \frac{3n}{2} + \frac{4}{1 + \sqrt{n^2 - 9n^4 + 48}}
\]

The characteristic equation is
\[
\lambda^2 - (\lambda - n) (\lambda + 1)^{n-3} (\lambda^2 - (n-1)\lambda - 2) = 0
\]
The eigenvalues are:
\[
\lambda = 0, \lambda = -1, \lambda = 1, \lambda = -\sqrt{n-1}.
\]

Then, the Total Dominating Energy is
\[
E_{TD}(B_{p,n}) = 2 \left( \lambda + 1 \right) + \left( \frac{1}{2} \right) + \left( 1 - \frac{n}{2} \right).
\]

**Theorem 4.4:**
If \( n \geq 2 \) then, \( E_{TD}(B_{p,n}) \) is equal to
\[
\frac{3n}{2} + \frac{4}{1 + \sqrt{n^2 - 9n^4 + 48}}.
\]

**Proof:** \( B_{p,n} \) is a Barbell graph with \( V = \{w_1, w_2, w_3, w_4, \ldots, w_{2n}\} \). The MTDS(\( B_{p,n} \)) = \( \{w_1, w_2\} \) and \( \gamma_t(B_{p,n}) = \{w_1, w_2\} \). Then,
\[
A_{MTD}(B_{p,n}) = \begin{vmatrix}
1 & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 1 & 1 & 1 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 1 & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\end{vmatrix}
\]

The characteristic polynomial is
\[
\det(A_{MTD}(B_{p,n}) - \lambda I) = \frac{3n}{2} + \frac{4}{1 + \sqrt{n^2 - 9n^4 + 48}}
\]

The characteristic equation is
\[
\lambda^2 - (\lambda - n) (\lambda + 1)^{n-3} (\lambda^2 - (n-1)\lambda - 2) = 0
\]
The eigenvalues are:
\[
\lambda = 0, \lambda = -1, \lambda = 1, \lambda = -\sqrt{n-1}.
\]

Then, the Total Dominating Energy is
\[
E_{TD}(\gamma(B_{p,n})) = 2 \left( \lambda + 1 \right) + \left( \frac{1}{2} \right) + \left( 1 - \frac{n}{2} \right).
\]
Theorem 4.5:
The $E_{TD}(B_m)$ is equal to
\[ (n - 4) + \frac{3\sqrt{n^2 - 8n + 48}}{2} + \left| -1 + \sqrt{2n - 3} \right| \]

Proof: $B_m$ is a Book graph with $V = \{ m_1, m_2, \ldots, m_n \}$. The MTDS $B_m$ is equal to $\gamma_t = (m_1, m_2)$. Then,
\[
A_{MTDS}(B_m) =
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
1 & 1 - \lambda & 0 & 1 & 0 & \cdots & 0 \\
0 & 1 & 1 - \lambda & 1 & 1 & \cdots & 0 \\
0 & 0 & 1 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & 0 & 0 & \cdots & 1 - \lambda \\
0 & 1 & 0 & 0 & 0 & \cdots & 1 - \lambda \\
\end{pmatrix}
\]
The Characteristic polynomial is
\[
\det(A_{MTDS}(B_m) - \lambda I) = (\lambda + 1)^2 (\lambda - 1)^2 ([\lambda^2 + \lambda - (n/2 - 3)]) = 0
\]
The eigenvalues are
\[
\lambda = -1 \left( \frac{n}{2} - 2 \right) \text{ times}, \quad \lambda = 1 \left( \frac{n}{2} - 2 \right) \text{ times}.
\]
Then,
\[
E_{TD}(B_m) = (n - 4) + \frac{3\sqrt{n^2 - 8n + 48}}{2} + \left| -1 + \sqrt{2n - 3} \right|
\]

Theorem 6.1:
Let us take the graph with $p$ vertices, and $q$ edges. Then, $\lambda_t(G)$ are the eigen values of minimum total dominating matrix $A_{TD}(G)$ and $E_{TD}(G) \geq \sqrt{p(2q + k)}$.

Proof: Cauchy Schwarz inequality is
\[
(S_{i=1}^p a_i b_i)^2 \leq (S_{i=1}^p a_i^2) (S_{i=1}^p b_i^2)
\]
If $a_i = 1, b_i = |\lambda_t|$, then,
\[
(S_{i=1}^p |\lambda_t|)^2 \leq (S_{i=1}^p a_i^2) (S_{i=1}^p b_i^2)
\]
If $E_{TD}(G) \leq p$ (2q+k) [Theorem 5.1] then,
\[
E_{TD}(G) \geq \sqrt{p(2q + k)}.
\]
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Where \( k \) is the total domination number.

**Proof:** Cauchy Schwarz inequality is

\[
\sum_{i=1}^{p} (a_i b_i) \leq \left( \sum_{i=1}^{p} a_i^2 \right)^{1/2} \left( \sum_{i=1}^{p} b_i^2 \right)^{1/2},
\]

Put \( a_i = 1, b_i = |\lambda_i| \) then

\[
\left( \sum_{i=1}^{p} |\lambda_i| \right)^{1/2} \leq \left( \sum_{i=1}^{p} \lambda_i^2 \right)^{1/2}.
\]

\[
\implies \left[ E_{TD}(G) - \lambda_1 \right] \leq (p-1) \left( 2q + k - \lambda_1^2 \right)
\]

Let \( f(x) = x + \sqrt{(p-1)(2q + k - x^2)} \)

For decreasing function,

\[
f'(x) \leq 0 \implies 1 - \frac{(p-1)(2q + k - x^2)}{\sqrt{(p-1)(2q + k - x^2)}} \leq 0
\]

\[
\implies x \geq \sqrt{\frac{2q+k}{p}}
\]

Since \( 2q+k \geq p \), we have

\[
\frac{2q+k}{p} \leq \lambda_1
\]

\[
f(-\lambda_2) \leq f \left( \frac{2q+k}{p} \right)
\]

i.e., \( E_{TD} \leq f(\lambda_2) \leq \left( \frac{2q+k}{p} \right) \)

i.e., \( E_{TD} \leq f \left( \frac{2q+k}{p} \right) \)

i.e., \( E_{TD} \leq \frac{(2q+k)}{p} + \sqrt{(p-1) \left( (2q+k) - \frac{(2q+k)^2}{p} \right)} \).

**VII. CONCLUSION**

We evaluated \( E_{TD} \) of complete, complete bipartite, double star, Barbell, Book graph and chemical structure of acetaminophen. In future, we will work on double domination.

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