Equidistribution of periodic points for modular correspondences

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Abstract

Let \( T \) be an exterior modular correspondence on an irreducible locally symmetric space \( X \). In this note, we show that the isolated fixed points of the power \( T^n \) are equidistributed with respect to the invariant measure on \( X \) as \( n \) tends to infinity. A similar statement is given for general sequences of modular correspondences.

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1 Introduction

Let \( G \) be a connected Lie group and \( \Gamma \subset G \) be a torsion-free lattice. Let \( \hat{\lambda} \) denote the probability measure on \( \hat{X} := \Gamma \backslash G \) induced by the invariant measure on \( G \). Consider also an element \( g \in G \) such that \( g^{-1}\Gamma g \) is commensurable with \( \Gamma \), that is, \( \Gamma_g := g^{-1}\Gamma g \cap \Gamma \) has finite index in \( \Gamma \). Denote by \( d_g \) this index.

The map \( x \mapsto (x, gx) \) induces a map from \( \Gamma \backslash G \) to \( \hat{X} \times \hat{X} \). Let \( \hat{Y}_g \) be its image. The natural projections \( \hat{\pi}_1, \hat{\pi}_2 \) from \( \hat{Y}_g \) onto the factors of \( \hat{X} \times \hat{X} \) define two coverings of degree \( d_g \). Both of them are Riemannian with respect to every left-invariant Riemannian metric on \( G \). The correspondence \( \hat{T}_g \) on \( \hat{X} \) associated with \( \hat{Y}_g \) is called irreducible modular.

A general modular correspondence \( \hat{T} \) on \( \hat{X} \) is a finite sum of irreducible ones, i.e. \( \hat{T} \) is associated with a sum \( \hat{Y} = \hat{Y}_{g_1} + \cdots + \hat{Y}_{g_m} \) that we call the graph of \( \hat{T} \). The degree \( d \) of \( \hat{T} \) is the sum of the degrees of \( \hat{T}_g \). We refer the reader to ClozelOtal, ClozelUllmo, Margulis [4, 5, 10] for more details.

If \( a \) is a point in \( \hat{X} \), define \( \hat{T}(a) := \hat{\pi}_2(\hat{\pi}_1^{-1}(a)) \) and \( \hat{T}^{-1}(a) := \hat{\pi}_1(\hat{\pi}_2^{-1}(a)) \). They are sums of \( d \) points which are not necessarily distinct. If \( \hat{U} \) is a small neighbourhood of \( a \), the restriction of \( \hat{T} \) to \( \hat{U} \) can be identified to \( d \) local isometries \( \hat{\tau}_i : \hat{U} \rightarrow \hat{U}_i \) from \( \hat{U} \) to neighbourhoods \( \hat{U}_i \) of points \( a_i \) in \( \hat{T}(a) \). All these isometries
are induced by left-multiplication by elements of $G$. If $a$ is a fixed point of $\hat{\tau}$, i.e. $a = a_i$, we say that $a$ is a fixed point of $\hat{T}$. When $a$ is an isolated fixed point of $\tau$, we also say $a$ is an isolated fixed point of $T$. These points are repeated according to their multiplicities.

The composition $\hat{T} \circ \hat{S}$ of two modular correspondences $\hat{T}$ and $\hat{S}$ can be obtained by composing the above local isometries. This is also a modular correspondence. Its degree is equal to $\deg(\hat{T}) \deg(\hat{S})$. Even when $\hat{T}$ and $\hat{S}$ are irreducible, their composition is not always irreducible. Denote by $\hat{T}_n := \hat{T} \circ \cdots \circ \hat{T}$, $n$ times, the iterate of order $n$ of $\hat{T}$. Periodic points of order $n$ of $\hat{T}$ are fixed points of $\hat{T}_n$.

Let $\mu$ be a probability measure on $\hat{X}$. Define a positive measure $\hat{T}_*(\mu)$ of mass $d$ on $\hat{X}$ by

$$\hat{T}_*(\mu) := (\hat{\pi}_2)_*(\hat{\pi}_1)^*(\mu).$$

A sequence of correspondences $\hat{T}_n$ of degree $d_n$ is said to be equidistributed if for any $a \in \hat{X}$ the sequence of probability measures $d_n^{-1}(\hat{T}_n)_*(\delta_a)$ converges weakly to $\hat{\lambda}$ as $n$ tends to infinity. Here, $\delta_a$ denotes the Dirac mass at $a$.

Let $K$ be a compact Lie subgroup of $G$. Since the left-multiplication on $G$ commutes with the right-multiplication, a modular correspondence $\hat{T}$ as above, induces a modular correspondence $T$ on $X := \hat{X}/K$ with the same degree. Its graph is the projection $Y$ of $\hat{Y}$ on $X \times X$. The above notion and description of $\hat{T}$ can be extended to $T$ without difficulty. We call $T$ the lift of $T$ to $\hat{X}$. Consider on $X$ the probability measure $\lambda$ induced by the invariant measure on $G$, i.e. the direct image of $\hat{\lambda}$ in $X$. Here is our main result.

**Theorem 1.1.** Let $T_n$ be a sequence of modular correspondences on $X$ and let $\hat{T}_n$ be the lifts of $T_n$ to $\hat{X}$. Assume that the sequence $\hat{T}_n$ is equidistributed. Then the isolated fixed points of $T_n$ are equidistributed. More precisely, there is a constant $s \geq 0$, depending only on $G$ and $K$, such that if $d_n$ is the degree of $T_n$ and $P_n$ is the set of isolated fixed points of $T_n$ counted with multiplicity, we have

$$\lim_{n \to \infty} \frac{1}{d_n} \sum_{a \in P_n} \delta_a = s\lambda.$$

The last convergence is equivalent to the following property. If $W$ is an open subset of $X$ such that its boundary has zero $\lambda$ measure, then

$$\lim_{n \to \infty} \frac{|P_n \cap W|}{d_n} = s\lambda(W).$$

We can of course replace $W$ with $\overline{W}$.

Now, assume moreover that $G$ is semi-simple, $K$ is a maximal compact Lie subgroup of $G$ and $\Gamma$ is an irreducible lattice. An irreducible correspondence $T$ associated with an element $g \in G$ as above is exterior if the group generated by $g$ and $\Gamma$ is dense in $G$. For such a correspondence, Clozel-Otal proved in [4] that the iterate sequence $\hat{T}_n$ is equidistributed (their proof given for $T$ is also valid for $\hat{T}$), see also Clozel-Ullmo [5]. We deduce from Theorem 1.1 the following result.
Corollary 1.2. Let $T$ be an exterior correspondence on an irreducible locally symmetric space $X$ as above. Then the isolated periodic points of order $n$ of $T$ are equidistributed with respect to $\lambda$ as $n$ tends to infinity.

The proof of our main result will be given in Section 2. In Section 3, we will give similar results related to the Arnold-Krylov-Guivarch theorem [1, 8]. We refer to Benoist-Oh [2] and Clozel-Oh-Ullmo [3] for other sequences of modular correspondences for which our main result can be applied. The reader will also find in Clozel-Ullmo [5], Dinh-Sibony [6, 7] and Mok-Ng [11, 12] some related topics.

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2 Proof of the main result

Fix a Riemannian metric on $G$ which is invariant under the left-action of $G$ and the right-action of $K$. It induces Riemannian metrics on $\hat{X}$ and $X$. We normalize the metric so that the associated volume form on $X$ is a probability measure. So, it is equal to $\lambda$. If $\Pi : \hat{X} \to X$ is the canonical projection, we have $\Pi_*(\lambda) = \lambda$.

Let $l$ and $m$ denote the dimension of $G$ and $X$ respectively.

Fix a point $c \in X$. Denote by $B(c, r)$ the ball of center $c$ and of radius $r$ in $X$. In order to prove the main result, we will consider the following quantity

$$\frac{|P_n \cap B(c, r)|}{d_n}$$

Let $\Phi$ denote the natural projection from $G$ to $\hat{X} := G/K$. The image of $K$ by $\Phi$ is a point that we denote by $0$. Denote by $B(0, r)$ the ball of center $0$ and of radius $r$ in $\hat{X}$. Define $K_r := \Phi^{-1}(B(0, r))$. So, $K_r$ is a union of classes $xK$ with $x \in G$. Fix also a constant $r_0 > 0$ small enough so that $B(0, r')$ is convex for every $r' \leq 3r_0$. Here, the convexity is with respect to the Riemannian metric induced by the one on $G$. From now on, assume that $r < r_0$.

Lemma 2.1. Let $g$ be an element of $G$. If $g$ admits a fixed point in $B(0, r)$ then $g$ belongs to $K_{2r}$. The set of fixed points of $g$ in $B(0, r_0)$ is a convex submanifold of $B(0, r_0)$. Moreover, a fixed point $e \in B(0, r_0)$ of $g$ is isolated if and only if $1$ is not an eigenvalue of the differential of $g$ at $e$.

Proof. Assume that $g$ admits a fixed point $e$ in $B(0, r)$. Since $g$ is locally isometric, $g(0)$ belongs to $B(0, 2r)$. It follows that $g$ belongs to $K_{2r}$. If $e, e'$ are two different fixed points in $B(0, r_0)$ then every point of the geodesic in $B(0, r_0)$ containing $e, e'$ is fixed. We deduce that the set of fixed points in $B(0, r_0)$ is a
convex submanifold. If 1 is an eigenvalue of the differential of \( g \) at \( e \), the associated tangent vector at \( e \) defines a geodesic of fixed points. This implies the last assertion in the lemma.

Recall that a semi-analytic set in a real analytic manifold \( W \) is locally defined by a finite family of inequalities \( f > 0 \) or \( f \geq 0 \) with \( f \) real analytic. A set in \( W \) is subanalytic if locally it is the projection on \( W \) of a bounded semi-analytic set in \( W \times \mathbb{R}^n \). The boundary of a subanalytic open set is also subanalytic with smaller dimension. We refer the reader to [9] for further details. We will need the following lemma.

**Lemma 2.2.** Let \( M_r \) denote the set of all \( g \in G \) which admit exactly one fixed point in \( B(0, r) \). Then \( M_r \) is a subanalytic open set contained in \( K_{2r} \).

**Proof.** The last assertion in Lemma 2.1 implies that \( M_r \) is open. The first assertion of this lemma implies that \( M_r \) is contained in \( K_{2r} \).

Denote by \( M' \) the set of points \((g, x)\) in \( K_{2r_0} \times B(0, r_0) \) such that \( g(x) = x \). This is an analytic subset of \( K_{2r_0} \times B(0, r_0) \). So, it is a semi-analytic set in \( G \times \tilde{X} \). Let \( M \) be the set of points \((g, x)\) in \( M' \) such that the differential of \( g \) at \( x \) does not have 1 as eigenvalue. So, \( M \) is also a semi-analytic set.

If \( \sigma_1, \sigma_2 \) are the natural projections from \( M' \) to \( G \) and to \( \tilde{X} \) respectively, we deduce from Lemma 2.1 that \( M_r \) is equal to \( \sigma_1(M \cap \sigma_2^{-1}(B(0, r))) \). Moreover, \( \sigma_1 \) defines a bijection from \( M \cap \sigma_2^{-1}(B(0, r)) \) to \( M_r \). It is now clear that \( M_r \) is a subanalytic set.

Consider a general modular correspondence \( T \) as above. Let \( \pi_1, \pi_2 \) denote the natural projections from \( Y \) to \( X \). If \( r \) is small enough, the ball \( B(c, r) \) is simply connected and \( \pi_1^{-1}(B(c, r)) \) is the union of \( d \) balls \( B(c_i', r) \) of center \( c_i' \) in \( Y \). The restriction of \( \pi_1 \) to \( B(c_i', r) \) is injective. The projection \( \pi_2 \) sends \( B(c_i', r) \) to the ball \( B(c_i, r) \) of center \( c_i := \pi_2(c_i') \) in \( X \). So, the restriction of \( T \) to \( B(c, r) \) is identified with the family of \( d \) maps \( \tau_i : B(c, r) \to B(c_i, r) \).

Fix a point \( b \in \tilde{X} \) such that \( \Pi(b) = c \). Let \( \tilde{T} \) denote the lift of \( T \) to \( \tilde{X} \) as above. The restriction of \( \tilde{T} \) to \( B(b, r) \) can be identified with a family of \( d \) maps \( \tilde{\tau}_i : B(b, r) \to B(b_i, r) \) which are the lifts of \( \tau_i \) to \( \tilde{X} \), i.e. we have \( \Pi \circ \tilde{\tau}_i = \tau_i \circ \Pi \).

Fix also a point \( a \in G \) such that \( \Psi(a) = b \) where \( \Psi : G \to \tilde{X} \) is the natural projection. The left-multiplication by \( a \) induces the map \( x \mapsto \Psi(ax) \) from \( M_r \) to \( \tilde{X} \). Its image is independent of the choice of \( a \) and is denoted by \( M_{b,r} \). Since \( \Gamma \) is torsion-free, its intersection with \( K \) is trivial. Therefore, when \( r \) is small enough, the above map is injective on \( K_{2r} \). So, it defines a bijection from \( M_r \) to \( M_{b,r} \). This is an isometry since the metric on \( G \) is invariant.

**Lemma 2.3.** The map \( \tau_i \) admits exactly one fixed point in \( B(c, r) \) if and only if \( b_i \) belongs to \( M_{b,r} \).
Proof. Without loss of generality, we can assume that $T$ and $\hat{T}$ are irreducible and given by an element $g \in G$ such that $g^{-1}\Gamma g$ is commensurable with $\Gamma$. Choose $d$ elements $\delta_1, \ldots, \delta_d$ of $\Gamma$ which represent the classes of $\Gamma g \backslash \Gamma$. Then, up to a permutation, $\hat{\tau}_i$ and $\tau_i$ are induced by the maps $x \mapsto g_i x$ where $g_i := g \delta_i$.

Assume that $\tau_i$ has a unique fixed point in $B(c, r)$. This point can be written as $\Theta(ae)$ for some point $e \in B(0, r)$, where $\Theta$ is the canonical projection from $\tilde{X}$ to $X$. So, we have $g_i ae = \gamma ae$ for some $\gamma \in \Gamma$. The maps $\hat{\tau}_i$ and $\hat{\tau}$ are also induced by $x \mapsto g'_i x$ where $g'_i := \gamma^{-1} g \gamma$ since $\gamma^{-1} \in \Gamma$. We have $g'_i ae = ae$ and $(a^{-1} g'_i a) e = e$. By Lemma 2.1, $a^{-1} g'_i a$ belongs to $M_r$. Since $b_i = \Psi(g'_i a)$, we deduce that $b_i \in \Psi(aM_r) = M_{br}$. We see in the above arguments that the converse is also true.

End of the proof of Theorem 1.1. Denote by $\lambda'$ the volume form on $G$ which induces on $\hat{X}$ the form $\hat{\lambda}$. By Lemma 2.2, $M_r$ and $M_{br}$ are subanalytic sets. So, their boundaries are of dimension $\leq l - 1$. Since the sequence $\hat{T}_n$ is equidistributed, using Lemma 2.3, we obtain

$$\lim_{n \to \infty} \frac{|P_n \cap B(c, r)|}{d_n} = \lim_{n \to \infty} \frac{\hat{T}_n(b) \cap M_{br}}{d_n} = \hat{\lambda}(M_{br}) = \lambda'(M_r).$$

It follows that the sequence of positive measures

$$\frac{1}{d_n} \sum_{x \in P_n} \delta_x$$

converges to a measure $\mu$ which satisfies $\mu(B(c, r)) = \lambda'(M_r)$ for $r$ small enough. Since $M_r$ is contained in $K_{2r}$, the last quantity is of order $O(r^m)$. Hence, $\mu = s \lambda$ where $s \geq 0$ is a function. Finally, the fact that $\lambda'(M_r)$ is independent of $c$ implies that $s$ is constant. It depends only on $G$ and $K$. 

Remark 2.4. The constant $s$ is an invariant depending only on $G$ and $K$. So, it can be computed using a particular case, e.g. when $\Gamma$ is co-compact and $T_n$ have only isolated fixed points. So, Lefschetz’s fixed points formula may be used here. We have for example $s = 2$ when $G = \text{PSL}(2, \mathbb{R})$ and $K = \text{SO}(2)$. We can also obtain a speed of convergence in our main theorem in term of the speed of convergence in the equidistribution property of $\hat{T}_n$.

3 On the Arnold-Krylov-Guivarc’h theorem

Consider now the case where $G$ is a compact connected semi-simple Lie group, $\Gamma$ is trivial and $K$ a connected compact subgroup of $G$. Define $X := G/K$. Let $\hat{\lambda}$ be the invariant probability measure of $G$ and $\lambda$ its direct image in $X$. 

Let $H \subset G$ be a semi-group generated by a finite family of elements $g_1, \ldots, g_d$ of $G$. Denote by $H_n$ the set of words of length $n$ in $H$. We say that $H$ is equidistributed on $G$ if for every point $a \in G$, the sequence of probability measures

$$d^{-n} \sum_{g \in H_n} \delta_{ga}$$

converges to $\hat{\lambda}$ as $n$ tends to infinity.

The left-multiplication by $g_i$ defines a self-map $\hat{T}_{g_i}$ on $G$. Their sum $\hat{T}$ can be seen as a correspondence of degree $d$ on $G$. It induces a correspondence $T$ on $X$ of the same degree. So, $H$ is equidistributed if and only if the sequence $\hat{T}^n$ is equidistributed. We deduce from our main result the following theorem.

**Theorem 3.1.** Let $G, K, X, \lambda, H$ and $H_n$ be as above. Assume that $H$ is equidistributed on $G$. Then the isolated fixed points in $X$ of the elements of $H_n$ are equidistributed with respect to $\lambda$ when $n$ tends to infinity.

Assume that $d = 2$ and that the first Betti number of $X$ vanishes. A result by Guivarc’h [8] says that if the group generated by $H$ is dense in $G$ then $H$ is equidistributed, see also Arnold-Krylov [1]. So, Theorem 3.1 can be applied in this case.

A similar result holds for groups. Let $H \subset G$ be a group generated by a finite family $\{g_1, \ldots, g_{2d}\}$ where $g_i = g_{2d-i}^{-1}$. Let $H_n$ denote the family of reduced words of length $n$ in $H$. We say that $H$ is equidistributed if the sequence of probability measures

$$\mu_n := \frac{1}{|H_n|} \sum_{g \in H_n} \delta_{ga}$$

converges to $\hat{\lambda}$ for every $a \in G$. There are also correspondences $\hat{T}_n$ and $T_n$ of degree $|H_n|$ such that $(\hat{T}_n)_* (\delta_a) = |H_n| \mu_n$. So, Theorem 3.1 holds for equidistributed groups $H$.

Another result by Guivarc’h [8] says that if $d = 2$ and if $H$ is dense in $G$ then it is equidistributed. Therefore, our result can be applied under these conditions.

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