Testing for spherical and elliptical symmetry

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Abstract

We construct new testing procedures for spherical and elliptical symmetry based on the characterization that a random vector $X$ with finite mean has a spherical distribution if and only if $\mathbb{E}[u^\top X | v^\top X] = 0$ holds for any two perpendicular vectors $u$ and $v$. Our test is based on the Kolmogorov-Smirnov statistic, and its rejection region is found via the spherically symmetric bootstrap. We show the consistency of the spherically symmetric bootstrap test using a general Donsker theorem which is of some independent interest. For the case of testing for elliptical symmetry, the Kolmogorov-Smirnov statistic has an asymptotic drift term due to the estimated location and scale parameters. Therefore, an additional standardization is required in the bootstrap procedure. In a simulation study, the size and the power properties of our tests are assessed for several distributions and the performance is compared to that of several competing procedures.

Keywords. bootstrap; elliptical symmetry; empirical process; Kolmogorov-Smirnov test; spherical symmetry

1 Introduction

The distribution of a random vector $X \in \mathbb{R}^d$ is said to be spherically symmetric if it stays invariant under orthogonal transformations, that is for

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any $d \times d$ matrix $\Gamma$ such that $\Gamma^{-1} = \Gamma^\top$ we have $\Gamma X \overset{\text{d}}{=} X$, where $\overset{\text{d}}{=}$ denotes equality in distribution. The random vector $X$ is called elliptically symmetric if it is spherically symmetric up to translation and rescaling. Spherically and elliptically symmetric distributions have drawn substantial interest as they present important and natural generalizations to the multivariate Gaussian distribution (Fang, 2018). For example, in the simple linear model $Y = \mu + \epsilon$, where the expectation $\mu \in \mathbb{R}^d$ is assumed to belong to a $k$-dimensional linear subspace for some $k \leq d$ and $\epsilon$ has a zero mean and a spherically symmetric distribution, the conclusion of the Gauss-Markov theorem on the optimality of the least squares estimator can be strengthened substantially (Berk and Hwang, 1989). The classical $t$-test also extends to spherically symmetric distributions, see e.g. Cacoullos (2014). In a single index regression model, where a response $Y \in \mathbb{R}$ is linked to a given covariate $X \in \mathbb{R}^d$ via the model $Y = \psi(\alpha^\top X) + \epsilon$ where $\mathbb{E}[\epsilon|X] = 0$, $\psi$ is some unknown ridge function and $\alpha$ is the unknown regression vector or index, Brillinger (1983) noted that if the covariate $X$ has a non-degenerate Gaussian distribution, and $\text{cov}(\psi(\alpha^\top X), \alpha^\top X) \neq 0$ then the usual least squares estimator of $\alpha$ converges to a vector that is co-linear with $\alpha$, and hence a very simple estimator of $\alpha$ can be constructed. The same facts continue to hold when Gaussianity is replaced by non-degenerate elliptical symmetry, see Olive (2014, Lemma 2.28). The main feature is that if $X \in \mathbb{R}^d$ admits an elliptically symmetric distribution with covariance matrix $\Sigma$, then for any $\beta \in \mathbb{R}^d$ the conditional expectation

$$\mathbb{E}[X|\beta^\top X] = \mu + \frac{\beta^\top (X - \mu)}{\beta^\top \Sigma \beta} \Sigma \beta$$

(1.1)

is linear in $\beta^\top X$, a well-known property for Gaussian distributions, see Cambanis et al. (1981). Duan and Li (1991) use (1.1) in their study of inverse regression. See also Baringhaus (1991, Section 5) for an account on some interesting applications of testing spherical symmetry related to animal navigation, wind speed or paleomagnetic studies. Thus, testing for spherical or elliptical symmetry is an important problem, and various methods have been proposed in the literature. Building up on the work by Smith (1977) for two-dimensional vectors, Baringhaus (1991) presents a family of tests for spherical symmetry in higher dimensions which exploit the fact that $X$ is spherically symmetric if and only if $\|X\|$ and $X/\|X\|$ are independent and that $X/\|X\|$ is uniformly distributed on $S_{d-1}$, the $d - 1$-dimensional unit sphere. Baringhaus (1991) shows that the limiting distribution of the test statistics does not involve the unknown distribution of $X$. The test suggested
by Fang et al. (1993) uses the fact that $X$ has a spherically symmetric distribution if and only if the distribution of $u^\top X$ is the same for any $u \in S_{d-1}$. Fang et al. (1993) then compares the distributions of $u_k^\top X$ for finitely many $u_k$ by using a two-sample Wilcoxon-type test. Koltchinskii and Sakhanenko (2000) suggest tests for spherical and elliptical symmetry by comparing the distribution of a random vector with its spherically symmetric projection using a Kolmogorov-Smirnov-type statistic.

Liang et al. (2008) proposed a test using the fact that if $X$ is spherically symmetric such that 0 is not an atom of its distribution, then $X/\|X\|$ is uniformly distributed on the sphere and hence a suitable transformation of $X/\|X\|$ to a $(d-1)$-dimensional vector involving the Beta distribution function has independent coordinates which are uniformly distributed on $[0,1]$. The method is easily implementable but does not yield a test which is universally consistent.

Henze et al. (2014) proposed a test based on the characterization that $X$ has a spherically symmetric distribution if and only if the characteristic function of $X$ takes the form $\phi_X(t) = E[e^{it^\top X}] = g(\|t\|^2)$ for $t \in \mathbb{R}^d$ and some function $g$. The test is based on the empirical characteristic function, does not require moment assumptions, and the resulting test statistics are of Kolmogorov-Smirnov- as well as of Cramér-von Mises-type. The critical value is determined by using the spherically symmetric bootstrap, however without theoretical justification.

Spherical and elliptical symmetry are also relevant notions in dynamic models as assumptions on the innovations. Francq et al. (2017) extend the testing methodology from Henze et al. (2014) to testing for spherical symmetry of the innovation distribution in multivariate GARCH models. To derive the asymptotic distribution they use the central limit theorem for martingale difference sequences. In combination with a parametric bootstrap for the volatility matrix, they also extend the spherically symmetric bootstrap to this setting. Extensions of the methodology proposed in this paper to such testing problems might also be of some interest.

**1.1 Our contributions and organization of the manuscript**

In this paper, we propose tests for spherical and elliptical symmetry based on the characterization that a random vector $X \in \mathbb{R}^d$, $d \geq 2$ with finite expectation has a spherically symmetric distribution if and only if

$$E[u^\top X|v^\top X] = 0 \quad \forall \ u, v \in \mathbb{R}^d \quad \text{with} \ u^\top v = 0,$$

(1.2)
see Eaton (1986, Theorem 1). In Section 2 we consider spherical symmetry. For an appropriate modification of the characterization (1.2) which does not require conditioning but only uses moment equations, we propose a consistent Kolmogorov-Smirnov-type asymptotic test in Section 2.1. Since it is not asymptotically distribution free, we develop a bootstrap version in Section 2.2. Indeed, in this section we give a consistency result for the spherically symmetric bootstrap for general VC-classes, which is of some independent interest and may be applied for other testing procedures. Our proof does not use Poissonization but is rather based on a Donsker theorem for general processes (van der Vaart and Wellner, 1996, Theorem 2.11.1). A discretized, implementable version of the Kolmogorov-Smirnov statistic is suggested in Section 2.3. In Section 3 we propose extensions of our methodology to testing for elliptical symmetry. Finally, in Section 4, we investigate the level and power of the tests in finite samples for several distributions and compare the performance of our test to that of tests constructed by Baringhaus (1991), Liang et al. (2008), Henze et al. (2014) and Koltchinskii and Sakhanenko (2000). Our simulation results show that the new procedure performs very reasonably under both the null and alternative hypotheses. Proofs are deferred to an Appendix.

2 The new test for spherical symmetry

In this section, for a random vector \( X \in \mathbb{R}^d \) we consider testing

\[
H_0 : X \text{ is spherically symmetric} \quad \text{versus} \quad H_1 : X \text{ is not.}
\]

Given \( n \in \mathbb{N} \) we let \( X_1, \ldots, X_n \) be i.i.d. random vectors with the same distribution \( P \) as \( X \). The empirical distribution of \( X_1, \ldots, X_n \) is denoted by \( P_n \). We let \( \mathbb{P} \) and \( \mathbb{E} \) be the probability measure and the expected value operator on the probability space on which the \( X_j \) are defined. We shall write \( Pf := \int f dP \) if the integral exists.

2.1 Characterizing spherical symmetry

Our test will be based on the following adaptation of Eaton (1986, Theorem 1).

**Lemma 2.1.** Let \( X \in \mathbb{R}^d \) be a random vector with finite expectation. Then \( X \) is spherically symmetric if and only if for any \( c \in \mathbb{R} \) and \( u, v \in S_{d-1} \) we have that

\[
\mathbb{E}[(v - (v^\top u)u)^\top X1_{u^\top X \geq c}] = 0.
\]
Define the family of functions on \( \mathbb{R}^d \) by
\[
\mathcal{F} = \left\{ f_{u,v,c}(x) = (v - (v^\top u)u)^\top x \mathbb{1}_{u^\top x \geq c} \mid u, v \in \mathcal{S}_{d-1}, c \in \mathbb{R} \right\}.
\] (2.3)

By Lemma 2.1, a random \( d \)-dimensional vector \( X \) is not spherically symmetric (and hence the alternative hypothesis \( H_1 \) holds true) if and only if
\[
\sup_{f \in \mathcal{F}} |Pf| = \Delta_0(X) > 0.
\] (2.4)

Consider the empirical process
\[
G_nf = \sqrt{n}(P_n - P)f
\] (2.5)
indexed by \( f \in \mathcal{F} \). By Lemma 2.1,
\[
G_nf = \sqrt{n}P_nf \quad \text{for all} \quad f \in \mathcal{F} \quad \text{if and only if} \quad H_0 \text{ holds true.} \quad (2.6)
\]

**Theorem 2.2.** Assume that \( X \) is spherically symmetric and satisfies \( \mathbb{E}[\|X\|_2^2] < \infty \). Consider the space \( \ell^\infty(\mathcal{F}) \) of all uniformly bounded functions on \( \mathcal{F} \) with respect to the supremum norm. Then there exists a tight centered Gaussian process \( G \) with covariance function
\[
\mathbb{E}[Gf_{u,v,c}Gf_{\tilde{u},\tilde{v},\tilde{c}}] = (v - (v^\top u)u)^\top \mathbb{E}[XX^\top \mathbb{1}_{u^\top X \geq c, \tilde{u}^\top X \geq \tilde{c}}] (\tilde{v} - (\tilde{v}^\top \tilde{u})\tilde{u}),
\] (2.7)
such that the weak convergence
\[
G_n \Rightarrow G, \quad n \to \infty,
\]
in \( \ell^\infty(\mathcal{F}) \) holds true.

The proof of Theorem 2.2 is based on showing that \( \mathcal{F} \) is a VC-subgraph class of functions, that is a collection of real-valued functions for which the subgraphs have a finite VC-index, see van der Vaart and Wellner (1996, Section 2.6.2).

In the sequel, we adopt the usual notation \( \|\nu_n\|_D := \sup_{f \in D} |\nu_n(f)| \) for a stochastic process \( \nu_n \) defined on a class of functions \( D \). Consider the statistic
\[
T_n := \sqrt{n}\|P_n\|_F.
\] (2.8)

Given \( \alpha \in (0, 1) \) let \( q_\alpha(G) \) denote the \( \alpha \)-quantile of \( \|G\|_F \), which will depend on the distribution of \( X \). An asymptotic test based on \( T_n \) rejects \( H_0 \) if \( T_n > q_{1-\alpha}(G) \). By (2.6) and Theorem 2.2 such a test has asymptotic level \( \alpha \). We show that the test is also consistent against fixed alternatives.

**Proposition 2.3.** Under the alternative hypothesis, that is if \( X \) does not have a spherically symmetric distribution, we have that
\[
\lim_{n \to \infty} \mathbb{P}(T_n > q_{1-\alpha}(G)) = 1.
\]
2.2 The spherically symmetric bootstrap

Asymptotic quantiles of tests for spherical symmetry can be estimated by using the spherically symmetric bootstrap. Indeed, if \( \mathbb{P}(X = 0) = 0 \), \( X \) is spherically symmetric if and only if the random variable \( X/\|X\| \) is uniformly distributed on \( S^{d-1} \), and is independent of \( \|X\| \); see for example Cambanis et al. (1981). Hence, we have the representation

\[
X \overset{d}{=} RU,
\]

where \( U \sim \mathcal{U}(S_{d-1}) \) is uniformly distributed on \( S_{d-1} \) and \( R \overset{d}{=} \|X\| \) is a positive random variable which is independent of \( U \). Thus, in order to provide new data under the null hypothesis of spherical symmetry, we shall employ the spherically symmetric bootstrap as investigated in Romano (1989, Example 3) and in Koltchinskii and Li (1998): We bootstrap the observed norms \( \|X_1\|, \ldots, \|X_n\| \) and multiply the obtained values with vectors that are independently sampled from the unit sphere.

More precisely, write \( L^*_n \) for the empirical distribution of \( R_1 = \|X_1\|, \ldots, R_n = \|X_n\| \). In the bootstrap procedure, we draw independent samples \( (R^*_1, \ldots, R^*_n) \) from \( L^*_n \) and \( (W_1, \ldots, W_n) \) from the uniform distribution \( \mathcal{U}(S_{d-1}) \) on the \( (d - 1) \)-dimensional unit sphere. For a class of functions \( f \in \mathcal{F}_s \) for which \( Pf = 0 \) vanishes if \( P \) is a spherically symmetric distribution, the spherically symmetric bootstrapped empirical process is defined as

\[
G^*_n = \sqrt{n} \hat{\mathbb{P}}^*_n, \quad \text{where} \quad \hat{\mathbb{P}}^*_n := n^{-1} \sum_{i=1}^n \delta_{R^*_i W_i}.
\]

(2.9)

The following result extends and complements previous consistency results for the spherically symmetric bootstrap by Romano (1989) and Koltchinskii and Li (1998).

**Theorem 2.4.** Suppose that \( X \) is spherically symmetric and that \( \mathbb{P}(X = 0) = 0 \). Consider a VC-subgraph class of functions \( \mathcal{F}_s \) with a square-integrable envelope function \( F \), such that \( Pf = 0 \) for all \( f \in \mathcal{F}_s \).

Let \( R^*_1, \ldots, R^*_n \) be an i.i.d. sample from the empirical distribution of \( R_1 = \|X_1\|, \ldots, R_n = \|X_n\| \). Also, let \( (W_1, \ldots, W_n) \) be a random sample from \( \mathcal{U}(S_{d-1}) \) taken to be independent of \( (R^*_1, \ldots, R^*_n) \). Then, for almost all \( X_1, X_2, \ldots \) it holds that

\[
G^*_n \Rightarrow G
\]

in \( \ell^\infty(\mathcal{F}_s) \), where \( G \) is a tight Gaussian process on \( \ell^\infty(\mathcal{F}_s) \) with covariance function \( \mathbb{P}(f_1 f_2), f_i \in \mathcal{F}_s \).
2.3 Discretization

For an actual implementation, the test based on the Kolmogorov-Smirnov statistic in (2.8) needs to be discretized, that is, the supremum has to be computed over a finite number of elements from $F$ in (2.3). We shall choose the points on the sphere uniformly at random, while we use a discretization of a sufficiently large interval as values for the parameter $c$. More precisely, for an integer $N_u \geq 1$ sample $U_1, \ldots, U_{N_u}, V_1, \ldots, V_{N_u}$ independently from the uniform distribution on $S_{d-1}$ such that they are also independent of $X_1, \ldots, X_n$. Write

$$U_{N_u} = \{U_1, \ldots, U_{N_u}\}$$

and similarly for $V_{N_u}$. Let also $c_0 \geq 2$ and $N_c \in \mathbb{N}$, and consider the (random) family of functions

$$F_{N_u,N_c,c_0} := \{ f_{u,v,c} \mid u \in U_{N_u}, v \in V_{N_u}, c \in \{-c_0 + 2c_0(j/N_c) \mid j = 0, \ldots, N_c\} \}. \tag{2.10}$$

The discretized version of our test statistic $T_n$ in (2.8) will be

$$\tilde{T}_{n,N_u,N_c,c_0} := \sqrt{n}\|P_n\|_{F_{N_u,N_c,c_0}}.$$

We shall derive the asymptotic distribution and consistency when $N_u, N_c, c_0 \to \infty$ as $n \to \infty$. To this end, consider the full index set $\Theta = S_{d-1} \times S_{d-1} \times \mathbb{R}$ for the function class $\mathcal{F} = \mathcal{F}_\Theta$ in (2.3). Given $\theta \in \Theta$ we let $n(\theta)$ denote the element $(u,v,c) \in U_{N_u} \times V_{N_u} \times \{-c_0 + 2c_0(j/N_c) \mid j = 0, \ldots, N_c\}$ of minimal distance to $\theta$.

**Theorem 2.5.** Assume that $\mathbb{E}[\|X\|_2^2] < \infty$, that $X$ has a continuous distribution, and that $N_u, N_c, c_0 \to \infty$ such that $c_0/N_c \to 0$ as $n \to \infty$. Then for the empirical process $G_n$ in (2.5) we have that

$$\{G_n(f_{n(\theta)}) \mid \theta \in \Theta\} \Rightarrow \{G(f_\theta) \mid \theta \in \Theta\}, \quad n \to \infty$$

in $\ell^\infty(\Theta)$, where $G$ is the Gaussian process from (2.7).

From this result we deduce consistency of the test based on the discretized test statistic.

**Corollary 2.6.** Under the assumptions of Theorem 2.5, if $X$ is spherically symmetric then for $\alpha > 0$ we have that

$$\lim_{n \to \infty} \mathbb{P}\left(\tilde{T}_{n,N_u,N_c,c_0} > Q_{1-\alpha}(\mathbb{G})\right) = \alpha,$$

as $n \to \infty$.\[7]
where as above \( q_{1-\alpha}(G) \) is the \( 1-\alpha \)-Quantile of \( \|G\|_F \). In contrast, if \( X \) is not spherically symmetric then
\[
\lim_{n \to \infty} \mathbb{P}(\tilde{T}_{n,N_u,N_c,c_0} > q_{1-\alpha}(G)) = 1.
\]
The bootstrap version of the discretized test statistic is given by
\[
\tilde{T}_{n,N_u,N_c,c_0}^* = \|G_n^*\|_{F_{N_u,N_c,c_0}},
\]
where \( G_n^* \) is defined in (2.9) and \( F_{N_u,N_c,c_0} \) in (2.10). The results in Corollary 2.6 extend to \( \tilde{T}_{n,N_u,N_c,c_0}^* \). We refrain from providing the formal details, which would require an extension of Theorem 2.4 to the situation of changing functions classes as in Theorem 2.5.

Algorithm 1 describes how to simulate from the distribution of \( \tilde{T}_{n,N_u,N_c,c_0}^* \), conditionally on \( X_1,\ldots,X_n \), to obtain a critical value.

## 3 Extensions to testing elliptical symmetry

Elliptical symmetry is tightly connected to spherical symmetry. If the random vector \( X \in \mathbb{R}^d \) has a non-singular covariance matrix \( \Sigma_0 \) and mean \( \mu_0 \), then it is elliptically symmetric if and only if the representation
\[
X = \mu_0 + \Sigma_0^{1/2} Y
\]
holds true for a spherically symmetric \( Y \). In the following, we consider testing the hypothesis
\[
H_0' : X \text{ is elliptically symmetric} \quad \text{versus} \quad H_1' : X \text{ is not}.
\]
We shall restrict ourselves to investigating the asymptotic distribution of the proposed test statistics under the null hypothesis, and hence will always assume that \( X \) is elliptically symmetrically distributed, and \( Y \) will correspond to the representation in (3.11). We let \( X_1,\ldots,X_n \) be i.i.d. with the distribution of \( X \) and denote the associated empirical probability measure by \( \mathbb{P}_n \). We shall write \( P \) for the distribution of \( X \). If \( (f_a) \) is a family of \( P \)-integrable functions, and \( \hat{a}_n \) depends on \( X_1,\ldots,X_n \), we write
\[
P f_{\hat{a}_n} := E[f_{\hat{a}_n}(X)] := \int_{\mathbb{R}^d} f_{\hat{a}_n}(x) \, dP(x),
\]
a random variable depending on \( X_1,\ldots,X_n \). As above, \( \mathbb{G}_n f = \sqrt{n}(\mathbb{P}_n - P)f \) denotes the empirical process. Finally, \( \mathbb{P} \) and \( \mathbb{E} \) are probability and expected value operator on the space where \( X, X_1, X_2, \ldots \) are defined.
Data: $X_1, \ldots, X_n, N_u, N_c, c_0$ and $B$

Result: $\tilde{T}_{n,N_u,N_c,c_0}^{1*}, \ldots, \tilde{T}_{n,N_u,N_c,c_0}^{B*}$

$b = 0$;

while $b < B$ do

$\begin{align*}
&b \leftarrow b + 1; \\
&\text{draw } W_1^b, \ldots, W_n^b \text{ i.i.d. } \sim \mathcal{U}(S_{d-1}); \\
&\text{sample with replacement } R_{1b}^{bs}, \ldots, R_{nb}^{bs} \text{ from } \{\|X_1\|, \ldots, \|X_n\|\} \\
&\text{independently of } (W_1^b, \ldots, W_n^b); \\
&\text{define } X_{1b}^{bs} := R_{1b}^{bs} W_1^b, \ldots, X_{nb}^{bs} := R_{nb}^{bs} W_n^b; \\
&\text{draw } U_1^b, \ldots, U_{N_u}^b, V_1^b, \ldots, V_{N_u}^b \text{ i.i.d. } \sim \mathcal{U}(S_{d-1}); \\
&\text{compute } \tilde{T}_{n,N_u,N_c,c_0}^{bs} \text{ in the same way as } \tilde{T}_{n,N_u,N_c,c_0}^{n*} \text{ replacing } \\
&(X_1, \ldots, X_n) \text{ with } (X_{1b}^{bs}, \ldots, X_{nb}^{bs}) \text{ and } (U_1, \ldots, U_{N_u}, V_1, \ldots, U_{N_u}) \text{ with } (U_1^b, \ldots, U_{N_u}^b, V_1^b, \ldots, V_{N_u}^b); \\
&\text{store } \tilde{T}_{n,N_u,N_c,c_0}^{bs}; \\
\end{align*}$

end

compute the empirical $(1 - \alpha)$-quantile $q_{1-\alpha,n,N_u,N_c,c_0}^B$ based on the sample $(\tilde{T}_{n,N_u,N_c,c_0}^{1*}, \ldots, \tilde{T}_{n,N_u,N_c,c_0}^{B*})$.

**Algorithm 1:** Bootstrap procedure for producing $\tilde{T}_{n,N_u,N_c,c_0}^{bs}$, $b = 1, \ldots, B$ and the empirical quantile $q_{1-\alpha,n,N_u,N_c,c_0}^B$.

### 3.1 Convergence of the empirical process

Consider the empirical mean and covariance matrix

$$
\bar{X}_n = n^{-1} \sum_{i=1}^{n} X_i \quad \text{and} \quad \hat{\Sigma}_n = n^{-1} \sum_{i=1}^{n} (X_i - \bar{X}_n) (X_i - \bar{X}_n)^\top.
$$

In the following we shall assume that $X$ is absolutely continuously distributed, in which case $\hat{\Sigma}_n$ is non-singular for $n \geq d + 1$ with probability 1, see Gupta (1971). Therefore, roughly speaking we can reduce tests for elliptical symmetry to testing for spherical symmetry of the standardized random variables

$$
\tilde{X}_i = \hat{\Sigma}_n^{-1/2} (X_i - \bar{X}_n), \quad i = 1, \ldots, n.
$$

(3.12)
We start with the following theorem, which is based on results from van der Vaart and Wellner (2007) for dealing with empirical processes involving estimated functions. We let \( \eta = (A, \mu) \in \mathcal{E} \), where \( A \) ranges through positive-definite \( d \times d \)-matrices, \( \mu \in \mathbb{R}^d \), and as before we let \( \theta = (u, v, c) \in \Theta = \mathcal{S}_{d-1} \times \mathcal{S}_{d-1} \times \mathbb{R} \).

Consider functions on \( \mathbb{R}^d \) defined by
\[
f_{\theta, \eta}(x) = (v - (v^\top u) u)^\top A (x - \mu) 1_{u^\top A (x - \mu) \geq c},
\]
where \( \theta = (u, v, c) \in \Theta \) and \( \eta = (A, \mu) \in \mathcal{E} \). Recall the empirical process \( G_n \) from (2.6).

**Theorem 3.1.** If \( X \) is absolutely continuously distributed with \( \mathbb{E}[\|X\|^2] < \infty \), setting \( \eta_0 = (\Sigma_0^{-1/2}, \mu_0) \) and \( \tilde{\eta}_n = (\tilde{\Sigma}_n^{-1/2}, \tilde{X}_n) \), we have as \( n \to \infty \) that
\[
\sup_{\theta \in \Theta} |G_n(f_{\theta, \tilde{\eta}_n} - f_{\theta, \eta_0})| \to 0 \quad \text{in probability.}
\]

If \( X \) is elliptically symmetric, then \( Pf_{\theta, \eta_0} = 0 \) for all \( \theta \in \Theta \), and hence an asymptotic test for elliptical symmetry can again be based on a Kolmogorov-Smirnov type statistic
\[
T_{n,e} = \sup_{\theta \in \Theta} |\sqrt{n} \mathbb{P}_n f_{\theta, \tilde{\eta}_n}|.
\]
However, estimation of the parameters in \( \eta_0 \) induces an additional drift term in the limit process, as described in the following result.

**Corollary 3.2.** Suppose that \( X \) is elliptically symmetric such that \( \mathbb{E}[\|X\|^4] < \infty \) and that \( X \) admits a continuous Lebesgue density. Furthermore, assume that the density \( g \) of any coordinate of \( Y = \Sigma_0^{-1/2}(X - \mu_0) \) satisfies \( \sup_{v \in \mathbb{R}} |v| g(v) < \infty \). Then, as \( n \to \infty \)
\[
\{\sqrt{n} \mathbb{P}_n f_{\theta, \tilde{\eta}_n} \mid \theta \in \Theta\} \Rightarrow \{G f_{\theta, \eta_0} + \mathbb{L}(\theta) \mid \theta \in \Theta\},
\]
in the space \( \ell^\infty(\Theta) \), a Gaussian process in which \( G \) is defined in Theorem 2.2, and
\[
\mathbb{L}(\theta) = -(w^\top \Sigma_0^{-1/2} \mathbb{D}) \mathbb{P}(V > c) + w^\top (\Sigma \Sigma_0^{1/2} + \Sigma_0^{1/2} \mathbb{S}) u \mathbb{E}[V 1_{V > c}],
\]
where \( w = v - (v^\top u) u \), \( V \) is distributed as any coordinate of \( Y \) in (3.11), and \( \mathbb{D} \) and \( \mathbb{S} \) are as in Lemma 5.2.
3.2 Modifying the spherically symmetric bootstrap

Corollary 3.2 shows that we cannot simply apply the spherically symmetric bootstrap from Section 2.2 to the standardized variables \( \hat{X}_i \). Rather, as in Koltchinskii and Sakhanenko (2000) the bootstrapped sample needs to be standardized again before computing the Kolmogorov-Smirnov statistic. More precisely, sample with replacement from the empirical distribution of the norms of \( \tilde{R}_i = \| \tilde{X}_i \|, i = 1, \ldots, n \) to obtain \( \tilde{R}_1^*, \ldots, \tilde{R}_n^* \). Then, draw a random sample \((W_1, \ldots, W_n)\) from the unit sphere \( S_{d-1} \), independently of \((\tilde{R}_1^*, \ldots, \tilde{R}_n^*)\) and set

\[
\tilde{X}_i^* = \tilde{R}_i^* W_i \quad \text{and} \quad \hat{X}_i^* = \hat{\Sigma}_n^{-1/2} (\tilde{X}_i^* - \bar{X}_n^*),
\]

where \( \bar{X}_n^* \) and \( \hat{\Sigma}_n^* \) are empirical mean and covariance matrix of \( \tilde{X}_1^*, \ldots, \tilde{X}_n^* \). The following theorem implies consistency of this modified procedure we have just described.

**Theorem 3.3.** Let \( \hat{P}_n^* \) be the empirical process based on \( \tilde{X}_1^*, \ldots, \tilde{X}_n^* \). Then, under the same conditions of Corollary 3.2, we have conditionally of \( X_1, X_2, \ldots \)

\[
\{ \sqrt{n} \hat{P}_n^* f_\theta \mid \theta \in \Theta \} \Rightarrow \{ G f_\theta, \eta_0 + L(\theta) \mid \theta \in \Theta \}
\]

in the space \( \ell^\infty(\Theta) \). Here, \( G \) and the drift \( L \) are the same Gaussian process and drift as in Corollary 3.2, \( f_\theta = f_{u,v,c} \) is defined in (2.3) and \( f_{\theta,\eta_0} \) in (3.13).

We present the main steps of the proof in the Appendix. Together with a discretization as in Section 2.3 this results in Algorithm 2.

4 Simulations

In this section we present the results of an extensive simulation study. We focus on testing for spherical symmetry, and in Section 4.1 investigate the type I error for the bootstrap test proposed in Section 2.3, while in Section 4.2 we investigate its power under various alternatives, and compare it with the procedures of Baringhaus (1991) and Liang et al. (2008). In Section 4.3, we consider the alternatives presented in Henze et al. (2014) for a power comparison of their procedure with our test. Finally, in Section 4.4 we briefly consider the finite sample performance of our test for elliptical symmetry, and compare it with six methods which were investigated in Sakhanenko (2008).
Data: \(X_1, \ldots, X_n, N_u, N_c, c_0\) and \(B\)

Result: \(\tilde{T}_{n,N_u,N_c,c_0}^{1*}, \ldots, \tilde{T}_{n,N_u,N_c,c_0}^{B*}\)

compute the empirical mean \(\bar{X}_n\) and empirical covariance matrix \(\hat{\Sigma}_n\);
compute the standardized variables \(\hat{X}_1, \ldots, \hat{X}_n\):
\[
\hat{X}_i = \hat{\Sigma}_n^{-1/2} (X_i - \bar{X}_i); 
\]
\(b = 0\);
while \(b < B\) do
\[
\text{draw } W_1^b, \ldots, W_n^b \text{ i.i.d. } \sim \mathcal{U}(S_{d-1}); 
\]
sample with replacement \(\tilde{R}_1^{bs}, \ldots, \tilde{R}_n^{bs}\) from \(\{\|\hat{X}_1\|, \ldots, \|\hat{X}_n\|\}\)
individually of \((W_1^b, \ldots, W_n^b)\);
\[
\tilde{X}_1^{bs} := \tilde{R}_1^{bs} W_1, \ldots, \tilde{X}_n^{bs} := \tilde{R}_n^{bs} W_n; 
\]
standardize \(\tilde{X}_1^{bs}, \ldots, \tilde{X}_n^{bs}\) resulting in \(\hat{X}_1^{bs}, \ldots, \hat{X}_n^{bs}\);
\[
\text{draw } U_1^b, \ldots, U_{N_u}^b, V_1^b, \ldots, V_{N_c}^b \text{ i.i.d. } \sim \mathcal{U}(S_{d-1}); 
\]
compute \(\tilde{T}_{n,N_u,N_c,c_0}^{bs}\) in the same way as described in Algorithm 1;
store \(\tilde{T}_{n,N_u,N_c,c_0}^{bs}\)
end
compute the empirical \((1 - \alpha)\)-quantile \(q_{B,\alpha,n,N_u,N_c,c_0}^B\) based on the
sample \(\tilde{T}_{n,N_u,N_c,c_0}^{1*}, \ldots, \tilde{T}_{n,N_u,N_c,c_0}^{B*}\).

**Algorithm 2**: Bootstrap procedure for producing \(\tilde{T}_{n,N_u,N_c,c_0}^{bs}, b = 1, \ldots, B\) and the empirical quantile \(q_{B,\alpha,n,N_u,N_c,c_0}^B\) for the extended test to elliptical symmetry.

All simulations were performed for dimensions \(d \in \{3, 6, 10\}\) and with the choice of \(N_u = 1000, N_c = 500\) and \(c_0 = 10\). For estimating the unknown critical point of our statistic, \(B = 100\) bootstrap replications were used. For the nominal level we always chose \(\alpha = 0.05\).

### 4.1 Type I Error

Table 1 displays the empirical levels of the test for spherical symmetry in Section 2.3 obtained with five spherically symmetric distributions and with
1000 replications. These null distributions are as follows:

- “G”: Multivariate Gaussian with mean 0 and correlation matrix equal to the identity $I_d$.
- “Cauchy ”: Cauchy distribution with location parameter 0 and scale parameter 1,
- “MVt, df ”: Multivariate $t$-distribution with $df$ degrees of freedom,
- “Kotz rq”: Kotz type distribution with $N = 2$, $r = 1$ and $s = 0.5$,
- “PVII ”: Pearson type VII distribution with $N = 10$, $m = 2$.

For the definitions of the multivariate $t$-distribution, the Kotz and Pearson type VII distributions we refer to Fang et al. (1990). The results show that for all distributions and dimensions we are near the theoretical level of 0.05, except for the Cauchy distribution. The latter does not have a finite expectation, and hence our method is not applicable in this case. Note also that neither the dimension nor the sample size seem to strongly influence the type I error.

Table 1: Finite-sample level of the proposed bootstrap-test for spherical symmetry for nominal level $\alpha = 0.05$. See the text for description of the parameters of the test, and for details on the distributions under which data are generated.

| Distr.  | n     | Dim. 3 | Dim. 6 | Dim. 10 |
|---------|-------|--------|--------|---------|
| G       | 100   | 0.053  | 0.057  | 0.05    |
|         | 200   | 0.063  | 0.056  | 0.067   |
| Cauchy  | 100   | 0.003  | 0.008  | 0.009   |
|         | 200   | 0.015  | 0.008  | 0.005   |
| MVt, df = 5 | 100  | 0.048  | 0.047  | 0.052   |
|         | 200   | 0.055  | 0.045  | 0.047   |
| Kotz    | 100   | 0.065  | 0.062  | 0.051   |
|         | 200   | 0.063  | 0.064  | 0.063   |
| PVII    | 100   | 0.061  | 0.054  | 0.059   |
|         | 200   | 0.058  | 0.062  | 0.062   |
4.2 Type II Error

Next, to study the power properties of our test for spherical symmetry in Section 2.3, we consider the following distributions:

- “G, ρ”: Multivariate Gaussian with mean μ = 0 and correlation matrix Σ, where Σ_{ij} = ρ if i ≠ j and Σ_{ii} = 1, with ρ ∈ {0.4, 0.6},
- “MG, μ”: Mixture of Gaussian distributions with mean μ and −μ and correlation matrix I_d. An observation from this distribution has probability 0.5 to be sampled from N(μ, I_d), otherwise, it is sampled from N(−μ, I_d). The μ chosen are (μ_1, 0, ..., 0), with μ_1 ∈ {1, 1.5, 2},
- “NCG, μ”: Multivariate Gaussian N(μ, I_d) with mean μ = (μ_1, 0, ..., 0), μ_1 ∈ {1, 2}. NC in the abbreviation stands for “not centered”,
- “MTt”: Meta-Type normal distribution obtained from a multivariate t-distribution with 5 degrees of freedom. Definitions and theory on Meta-Type distributions can be found in Fang et al. (2002) or Liang et al. (2008),
- “Cube”: Uniform distribution on the Hypercube [−1, 1]^d.

The results can be found in Table 2 under the columns “T_n”. The outcomes from two other spherical symmetry tests, labeled with A_n and B_n, are also integrated into the table. The first one, A_n, denotes the test introduced in Liang et al. (2008), while the second test, B_n, represents the test proposed by Baringhaus (1991). Our test has the best performance in dimension d = 3 for the first seven distributions. Under the mixture of Gaussian distribution, MG, the test looses power as the dimension increases. This is mainly caused by the decreasing importance of the deviation. For the Meta-type normal and cubic distributions neither our test nor the test by Baringhaus (1991) exhibit substantial power above the level. However, additional simulations (not displayed) show that the power tends to 1 with increasing sample size.

4.3 Some comparison based on Henze et al. (2014)

Next, we investigate power properties of the tests for spherical symmetry under the following alternatives from the simulation study in Henze et al. (2014),

- “H^{(2)}_1”: the distribution of the random vector X = (X_1, X_2, X_3) such that X_1, X_2 and X_3 are independent, X_1 and X_2 are standard Gaussian, while X_3 has an Exponential distribution with rate 1.
Table 2: Finite-sample power of the proposed bootstrap-test $T_n$ for spherical symmetry, together with results for two competing procedures $A_n$ and $B_n$, see text for details.

| Distr. | n  | $T_n$ | $A_n$ | $B_n$ | $T_n$ | $A_n$ | $B_n$ | $T_n$ | $A_n$ | $B_n$ |
|--------|----|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| G, $\rho = 0.4$ | 100 | 0.772 | 0.142 | 0.504 | 0.986 | 0.328 | 0.941 | 0.998 | 0.484 | 0.998 |
|                    | 200 | 0.991 | 0.235 | 0.928 | 1     | 0.541 | 1     | 1     | 0.732 | 1     |
| G, $\rho = 0.6$   | 100 | 1     | 0.615 | 0.969 | 1     | 0.928 | 1     | 1     | 0.995 | 1     |
|                    | 200 | 1     | 0.854 | 1     | 1     | 0.996 | 1     | 1     | 1     | 1     |
| MG, $\mu_1 = 1$   | 100 | 0.34  | 0.063 | 0.264 | 0.191 | 0.063 | 0.188 | 0.096 | 0.063 | 0.142 |
|                    | 200 | 0.761 | 0.071 | 0.596 | 0.481 | 0.06  | 0.414 | 0.191 | 0.058 | 0.234 |
| MG, $\mu_1 = 1.5$| 100 | 0.976 | 0.315 | 0.95  | 0.883 | 0.137 | 0.847 | 0.574 | 0.102 | 0.644 |
|                    | 200 | 1     | 0.528 | 1     | 1     | 0.165 | 1     | 0.975 | 0.124 | 0.981 |
| NCG, $\mu_1 = 1$  | 100 | 1     | 0.067 | 1     | 1     | 0.056 | 1     | 1     | 0.063 | 1     |
|                    | 200 | 1     | 0.06  | 1     | 1     | 0.053 | 1     | 1     | 0.047 | 1     |
| NCG, $\mu_1 = 2$  | 100 | 1     | 0.913 | 1     | 1     | 0.566 | 1     | 1     | 0.369 | 1     |
|                    | 200 | 1     | 0.998 | 1     | 1     | 0.833 | 1     | 1     | 0.594 | 1     |
| MTt $df = 5$       | 100 | 0.053 | 0.21  | 0.056 | 0.07  | 0.413 | 0.071 | 0.051 | 0.651 | 0.05  |
|                    | 200 | 0.068 | 0.349 | 0.062 | 0.042 | 0.727 | 0.062 | 0.053 | 0.901 | 0.062 |
| Cube              | 100 | 0.086 | 0.818 | 0.07  | 0.067 | 0.998 | 0.062 | 0.057 | 1     | 0.07  |
|                    | 200 | 0.102 | 0.985 | 0.074 | 0.063 | 1     | 0.066 | 0.05  | 1     | 0.072 |

- $H_1^{(4)}$: the distribution of the random vector $X = (X_1, X_2, X_3)$ such that $X_1$ and $X_2$ are generated from a uniform distribution on an equilateral triangle centered at the origin, where the length of each side is set to $\sqrt{12}$, while $X_3$ is independent of $X_1$ and $X_2$ and has a uniform distribution on the interval $(-\sqrt{12}, \sqrt{12})$.

Table 3 gives the power properties of the tests for these alternatives for sample sizes $n \in \{100, 200\}$. In the column “Henze et al ”, we display the largest power from the family of tests (various tuning parameters, Kolmogorov-Smirnov and Cramer-von-Mises statistic) as obtained by Henze et al. (2014) in their simulation study.

For $n = 100$, the necessary test of Liang et al. (2008) based on the statistic $A_n$ has the lowest power. For $n = 200$, the four tests have comparable
performances with slight advantage for the test based on $T_n$ in the case of $H_1^{(4)}$.

Table 3: Power of the tests based on $T_n$, $A_n$, $B_n$ and the (best) test of Henze et al. (2014) for additional alternative hypotheses.

| Distr. | n   | $T_n$ | $A_n$ | $B_n$ | Henze et al. |
|--------|-----|-------|-------|-------|--------------|
| $H_1^{(2)}$ | 100 | 1 0.177 | 1     | 0.73  |              |
|         | 200 | 1 0.304 | 1     | 1     |              |
| $H_1^{(4)}$ | 100 | 1 0.993 | 1     | 0.72  |              |
|         | 200 | 1 1     | 1     | 0.99  |              |

4.4 Simulations for testing elliptical symmetry

In this section we conduct a small simulation study for the bootstrap test for elliptical symmetry as suggested in Algorithm 2 in Section 3.2. To investigate the level, we simulated from the distributions “G, $\rho$” with $\rho \in \{0, 0.4, 0.6\}$ and “Kotz” under the null hypothesis. The results for sample sizes $n \in \{100, 200\}$ under the null hypothesis are shown in Table 4.

Table 4: Rejection rates for testing elliptical symmetry under the null hypothesis for nominal level of $\alpha = 0.05$.

| Distr. | n   | Dim. 3 | Dim. 6 | Dim.10 |
|--------|-----|--------|--------|--------|
| G, $\rho = 0$ | 100 | 0.061  | 0.058  | 0.068  |
|         | 200 | 0.066  | 0.069  | 0.063  |
| G, $\rho = 0.4$ | 100 | 0.065  | 0.057  | 0.069  |
|         | 200 | 0.051  | 0.054  | 0.064  |
| G, $\rho = 0.6$ | 100 | 0.065  | 0.052  | 0.071  |
|         | 200 | 0.067  | 0.053  | 0.082  |
| Kotz    | 100 | 0.068  | 0.069  | 0.056  |
|         | 200 | 0.068  | 0.07   | 0.063  |

Given the moderate sample sizes, the actual levels are reasonably close to the nominal one of $\alpha = 0.05$. To investigate the power, we shall compare our test to those studied in Sakhanenko (2008). There, three tests from the class of tests of Koltchinskii and Sakhanenko (2000), which are denoted
by $S_n$, $C_n$ and $H_n$, are compared with the tests proposed by Beran (1979); Huffer and Park (2007) and Manzotti et al. (2002), denoted $B^0_n$, $Q_n$ and $HP_n$. As in Sakhanenko (2008) we consider the following alternative distributions in 2 and 3 dimensions:

- $M_2$: a mixture of two bivariate Gaussian distributions: the first one a standard normal and the second with mean $\mu_2 = (1, 2)$ and covariance matrix $\Sigma_2 = (5, -4; -4, 5)$, with mixing probabilities equal to $1/2$,

- $M_3$: a mixture of two three-dimensional Gaussian distributions: the first again a standard normal, the other with mean $\mu = (1, 2, 3)$ and covariance $\Sigma = (5, -4, 1; -4, 6, -4; 1, -4, 5)$, with mixing probabilities equal to $1/2$,

- $\Gamma + N_{d-1}$: a $d$-dimensional random vector, with the first component a $\Gamma(2, 3)$ distribution and the other $d-1$ coordinates independent of the first one and distributed according to a $(d-1)$-dimensional standard normal distribution,

- $B(\beta)$: a multivariate Burr distribution, which has the distribution of $(1 + X/Y)^{-\beta}$, where $Y$ is a univariate gamma distribution with shape parameter $1$ and scale parameter $\beta$, that we’ll set equal to $0.5$ in both dimension 2 and 3, while the coordinates of $X$ are all mutually independent and identically gamma-distributed with shape parameter 2 and scaling parameter 3,

- $U(C)$: the uniform distribution on the unit cube in dimensions 2 and 3 (this corresponds to “Cube” used in Section 4.2),

- $U(A)$: the uniform distribution on the set $A = [0, 1] \times [0, \pi/2] \cup [0, 1] \times [\pi, 3\pi/2] \cup [1, \sqrt{2}] \times [\pi/2, \pi] \cup [1, \sqrt{2}] \times [3\pi/2, 2\pi]$ in dimension 2.

We run our test with the parameters $N_u = 1000$, $N_c = 500$ and $c_0 = 10$ as above, but the number of replications is set to be 5000 to match the setting in Sakhanenko (2008).

The results obtained for our test $T_n$, together with those from the paper Sakhanenko (2008) for the tests mentioned above, are given in Table 5 for dimension 2 and in Table 6 for dimension 3. The test $T_n$ compares very favorably to the competing procedures in terms of power.
Table 5: Rejection rates for testing elliptical symmetry under the selected two-dimensional alternatives for nominal level of $\alpha = 0.05$. The table contains the results from (Sakhanenko, 2008, Table 2).

| Distr. | $n$ | $T_n$ | $S_n$ | $C_n$ | $H_n$ | $B^0_n$ | $Q_n$ | $HP_n$ |
|--------|-----|-------|-------|-------|-------|---------|-------|--------|
| $M_2$  | 50  | 0.575 | 0.333 | 0.241 | 0.258 | 0.060   | 0.295 | 0.103  |
|        | 100 | 0.922 | 0.728 | 0.490 | 0.510 | 0.101   | 0.609 | 0.261  |
|        | 200 | 0.999 | 0.984 | 0.880 | 0.869 | 0.478   | 0.936 | 0.623  |
| $\Gamma + N_1$ | 50  | 0.316 | 0.095 | 0.447 | 0.494 | 0.061   | 0.077 | 0.111  |
|        | 100 | 0.591 | 0.172 | 0.809 | 0.860 | 0.115   | 0.117 | 0.285  |
|        | 200 | 0.928 | 0.375 | 0.991 | 0.996 | 0.350   | 0.203 | 0.692  |
| $U(C)$ | 50  | 0.204 | 0.091 | 0.058 | 0.068 | 0.051   | 0.116 | 0.029  |
|        | 100 | 0.410 | 0.159 | 0.058 | 0.086 | 0.058   | 0.221 | 0.039  |
|        | 200 | 0.756 | 0.343 | 0.059 | 0.166 | 0.098   | 0.454 | 0.042  |
| $B(0.5)$ | 50  | 0.521 | 0.487 | 0.250 | 0.338 | 0.081   | 0.419 | 0.152  |
|        | 100 | 0.910 | 0.839 | 0.508 | 0.699 | 0.358   | 0.746 | 0.363  |
|        | 200 | 1     | 0.991 | 0.906 | 0.979 | 0.930   | 0.975 | 0.737  |
| $U(A)$ | 50  | 0.977 | 0.199 | 0.222 | 0.194 | 0.423   | 0.074 | 0.536  |
|        | 100 | 1     | 0.547 | 0.399 | 0.413 | 0.950   | 0.069 | 0.824  |
|        | 200 | 1     | 0.961 | 0.726 | 0.843 | 1       | 0.078 | 0.998  |

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5 Appendix: Proofs

5.1 Proofs for Section 2

Proof of Lemma 2.1. First we show that a random vector $X$ in $\mathbb{R}^d$ with zero expectation is spherically symmetric if and only if we have that

$$\mathbb{E}[(v - (v^\top u)u)^\top X | u^\top X] = 0 \quad \forall \ u, v \in S_{d-1}. \quad (5.15)$$
Table 6: Rejection rates for testing elliptical symmetry under the selected three-dimensional alternatives for nominal level of $\alpha = 0.05$. The table contains the results from (Sakhanenko, 2008, Table 3).

| Dim. 3 |  |  |  |  |  |  |
|--------|---|---|---|---|---|---|
| Distr. | $n$ | $T_n$ | $S_n$ | $B_n^1$ | $B_n^2$ | $Q_n$ | $HP_n$ |
| $M_3$  | 100| 1  | 0.997 | 0.058 | 0.076 | 0.954 | 0.995 |
|        | 200| 1  | 1     | 0.052 | 0.079 | 1     | 1     |
| $\Gamma + N_1$ | 100| 0.551 | 0.198 | 0.056 | 0.064 | 0.135 | 0.188 |
|        | 200| 0.932 | 0.414 | 0.054 | 0.078 | 0.260 | 0.444 |
| U(C)   | 100| 0.399 | 0.317 | 0.052 | 0.059 | 0.179 | 0.033 |
|        | 200| 0.797 | 0.711 | 0.058 | 0.068 | 0.375 | 0.036 |
| B(0.5) | 100| 0.938 | 0.972 | 0.096 | 0.145 | 0.855 | 0.686 |
|        | 200| 1  | 1     | 0.116 | 0.228 | 0.994 | 0.969 |

Indeed, (5.15) immediately implies (1.2), and the converse is equally clear since $v - (v^T u) u$ and $u$ are perpendicular. Now Lemma 2.1 follows from the characterization (5.15) and the following lemma.

**Lemma 5.1.** Let $Y, Z$ be random variables with $E|Y| < \infty$. Then $E[Y 1_{Z \geq c}] = 0$ for all $c \in \mathbb{R}$ if and only if $E[Y | Z] = 0$.

**Proof of Lemma 5.1.** If $E[Y | Z] = 0$ then $E[Y 1_{Z \geq c}] = E[1_{Z \geq c} E[Y | Z]] = 0$ for every $c$. Conversely, $E[Y | Z] = 0$ if $E[Y 1_{Z \in B}] = 0$ for all Borel-subsets $B$ of $\mathbb{R}$. But here it suffices to consider sets $B$ in an intersection-stable generator (which contains $\Omega$ resp. $\mathbb{R}$). Now, the collection of sets $\{ (c, \infty), c \in \mathbb{R} \}$ is an intersection-stable generator of the Borel-$\sigma$-field of $\mathbb{R}$, and hence the events $\{ Z \in [c, \infty), c \in \mathbb{R} \} = \{ \{ Z \geq c \}, c \in \mathbb{R} \}$ are an intersection-stable generator of the $\sigma$-field generated by $Z$. By letting $c \to -\infty$ and using dominated convergence, from the assumption we obtain that $E[Y] = E[Y 1_{\Omega}] = 0$, so we can add the full space $\Omega$ to this system of sets.

**Proof of Theorem 2.2.** We shall show that $\mathcal{F}$ in (2.3) is a VC-class of functions with an integrable envelope, see (van der Vaart and Wellner, 1996, Theorem 2.5.2 and Section 2.6). We have that

\[
\mathcal{F} \subset \mathcal{G} := \left\{ g(x) = w^T x 1_{u^T x \geq c} \mid (u, w) \in \mathbb{R}^d \times \mathbb{R}^d, c \in \mathbb{R} \right\},
\]

(5.16)
and it suffices to show the assertions for $G$. First, the function $F(x) = 2\|x\|$ is an 
envelope for $G$ which is square-integrable by assumption. We show now that $G$ is a VC-class; i.e., we need to show that the collection of all subgraphs 
$$
C = \left\{ \{(x, t) \in \mathbb{R}^{d+1} \mid g(x) > t\} \mid g \in G \right\}
$$
forms a VC-class in $\mathbb{R}^{d+1}$; see Section 2.6 in van der Vaart and Wellner (1996). We 
have that 
$$
C = \left\{ \{(x, t) \in \mathbb{R}^{d+1} \mid w^\top x > t, u^\top x \geq c\} \mid c \in \mathbb{R}, u, w \in \mathbb{R}^d \right\} \cup \left\{ \{(x, t) \in \mathbb{R}^{d+1} \mid t < 0, u^\top x < c\} \mid c \in \mathbb{R}, u \in \mathbb{R}^d \right\} = C_1 \cup C_2.
$$
From (van der Vaart and Wellner, 1996, Lemma 2.6.27) it suffices to show that $C_i$ are VC-classes of sets, which follows from (Pollard, 2012, Section II.4, Lemma 18).

Proof of Proposition 2.3. From Theorem 2.2 it follows that as $n \to \infty$, 
$$
\|G_n\|_F = \|\sqrt{n}(P_n - P)f\|_F \to_d \|G\|_F.
$$
On the other hand, we have that 
$$
T_n = \sqrt{n}\|P_n f\|_F = \sqrt{n}\|P f\|_F \geq \sqrt{n}\|P f\|_F - \sqrt{n}\|P_n f\|_F.
$$
Thus, 
$$
\Pr(T_n > q_{1-\alpha}(G)) \geq \Pr\left(\sqrt{n}\|P f\|_F - \sqrt{n}\|P_n f\|_F > q_{1-\alpha}(G)\right) \geq \Pr\left(\sqrt{n}\|P_n f\|_F < -q_{1-\alpha}(G) + \sqrt{n}\Delta_0(X)\right) \to 1
$$
as $n \to \infty$, where $\Delta_0(X) > 0$ is as in (2.4).

Proof of Theorem 2.4. We shall use (van der Vaart and Wellner, 1996, Theorem 2.11.1). To this end, we shall check that the assumptions of this theorem hold conditionally on the sample $X_1, X_2, \ldots$ almost surely, and that the limit process always is $G$. We write $L_n$ for the distribution of $R$, $L_n^*$ for the empirical distribution $n^{-1} \sum_{j=1}^n \delta_{R_j}$, and $\mathcal{U}$ for the uniform distribution $\mathcal{U}(S_{d-1})$. When taking the expected value conditionally on $X_1, X_2, \ldots$, we write $\mathbb{E}_{L_n^* \otimes \mathcal{U}}$ and $\mathbb{E}_{\mathcal{U}}$ if the expression only involves the $W_j$ and not the $R_j^*$. We write 
$$
Z_{ni} = n^{-1/2}\delta_{W_i R_i^*}, \quad i = 1, \ldots, n.
$$
Since $W_i R_i^*$ is spherically symmetric under $L_n^* \otimes \mathcal{U}$, we have that $\mathbb{E}_{L_n^* \otimes \mathcal{U}}[f(W_i R_i^*)] = 0$ for $f \in \mathcal{F}_s$ by assumption, so that 
$$
G_n^*(f) = \sum_{i=1}^n (Z_{ni}(f) - \mathbb{E}_{L_n^* \otimes \mathcal{U}}Z_{ni}(f)) = \sum_{i=1}^n Z_{ni}(f), \quad f \in \mathcal{F}_s,
$$
Consider the variance semimetric
\[ \rho^2(f, g) = E_{L^0 \otimes U}[(f(RW) - g(RW))^2] = E_P[(f(X) - g(X))^2] \]
for \( f, g \in \mathcal{F}_s \). By the assumptions on the class of functions \( \mathcal{F}_s \), it is \( P \)-Donsker (for any \( P \) for which the majorant is square-integrable), and hence by (van der Vaart and Wellner, 1996, Corollary 2.3.12 ) it follows that the space \((\mathcal{F}_s, \rho)\) is totally bounded. Given \( \eta > 0 \) we have that
\[
\sum_{i=1}^{n} E_{L_n^* \otimes U} \left[ \|Z_{ni}\|^2 1_{\|Z_{ni}\| \geq \eta} \right] \leq \sum_{j=1}^{n} E_U \left[ F^2(R_jW) 1_{\{|F(R_jW)| > n^{1/2}\eta\}} \right],
\]
where \( F \) is the envelope of \( \mathcal{F}_s \) and \((R^*, W)\) is distributed as \( L_n^* \otimes U \). Expanding the expected value w.r.t. \( L_n^* \) yields
\[
E_{L_n^* \otimes U} \left[ F^2(R^*W) 1_{\{|F(R^*W)| > n^{1/2}\eta\}} \right] = \frac{1}{n} \sum_{j=1}^{n} E_U \left[ F^2(R_jW) 1_{\{|F(R_jW)| > n^{1/2}\eta\}} \right].
\]
Since \( E[F^2(RW)] < \infty \) by assumption, from the dominated convergence theorem it follows that
\[
E\left[ E_U[ F^2(RW) 1_{\{|F(RW)| > n^{1/2}\eta\}}] \right] \to 0, \quad n \to \infty,
\]
and the strong law implies that as \( n \to \infty \),
\[
\sum_{i=1}^{n} E_{L_n^* \otimes U} \left[ \|Z_{ni}\|^2 1_{\|Z_{ni}\| \geq \eta} \right] \leq \frac{1}{n} \sum_{j=1}^{n} E_U \left[ F^2(R_jW) 1_{\{|F(R_jW)| > n^{1/2}\eta\}} \right] \to 0.
\]
This shows the first assumption in (van der Vaart and Wellner, 1996, Theorem 2.11.1).

As for the second, we need to show that for any sequence \( \delta_n > 0 \) with \( \delta_n \to 0 \) we have that
\[
\sup_{\rho(f, g) < \delta_n} \sum_{i=1}^{n} E_{L_n^* \otimes U} \left[ (Z_{ni}(f) - Z_{ni}(g))^2 \right] \leq \sup_{\rho(f, g) < \delta_n} \frac{1}{n} \sum_{i=1}^{n} E_U \left[ (f(R_iW) - g(R_iW))^2 \right] \to 0 \quad (5.17)
\]
for almost all \( X_1, X_2, \ldots \), where \( f, g \in \mathcal{F}_s \) in the supremum. To this end, we show below that the class of functions defined as
\[
\mathcal{C} := \left\{ r \mapsto E_U[(f - g)^2(Wr)] \mid r \in (0, \infty), \text{ and } f, g \in \mathcal{F}_s \right\}
\]
is Glivenko-Cantelli for the law $L_0$ of $R$. Then we have in particular that

$$
\sup_{\rho(f,g) < \delta_n} \left[ \sum_{i=1}^{n} E_L \left[ (f(R_iW) - g(R_iW))^2 \right] - E_{L_0 \otimes U} \left[ (f - g)^2(WR) \right] \right] \to 0 \quad (5.19)
$$

almost surely as $n \to \infty$. Since

$$
E_{L_0 \otimes U} \left[ (f - g)^2(WR) \right] = \rho^2(f,g)
$$

we evidently have

$$
\sup_{\rho(f,g) < \delta_n} \left| E_{L_0 \otimes U} \left[ (f - g)^2(WR) \right] \right| \to 0,
$$

which together with (5.19) implies (5.17). To show that (5.19) is a Glivenko-Cantelli class, we shall apply (van der Vaart and Wellner, 1996, Theorem 2.4.3). We first note that $\tilde{\varepsilon}$ is Glivenko-Cantelli for the law $L$ and the law of large numbers, this implies that $\log N(\varepsilon, \mathcal{C}_M, L_2(L_0^* \otimes U)) = o_{L_0}(n)$, as required in (van der Vaart and Wellner, 1996, Theorem 2.4.3).

To check the third condition of (van der Vaart and Wellner, 1996, Theorem 2.11.1), since $\mathcal{F}_s$ is a VC-class of functions, we obtain the estimate

$$
N(\varepsilon, \mathcal{C}_M, L_2(L_0^*)) \leq K \left( \frac{4 \sqrt{M} \left\| \tilde{F} \right\|_{L_0^*}^2}{\varepsilon} \right) V
$$

for positive constants $K, V > 0$. By integrability of $\tilde{F}$ under $L_0$ and the law of large numbers, this implies that $\log N(\varepsilon, \mathcal{C}_M, L_2(L_0^*)) = o_{L_0}(n)$, as required in (van der Vaart and Wellner, 1996, Theorem 2.4.3).

Since $\mathcal{F}_s$ is a VC-class, there are constants $K', V' > 0$ for which

$$
N(\varepsilon, \mathcal{F}_s, L_2(Q_n)) \leq K' \left( \frac{\left\| F \right\|_{Q_2}}{\varepsilon} \right)^{V'},
$$

where $Q_n$ is the empirical distribution of $(W_1, R_1), \ldots, (W_n, R_n)$. The distance in $L_2(Q_n)$ is

$$
d_n^2(f,g) = \sum_{i=1}^{n} (Z_{ni}(f) - Z_{ni}(g))^2.
$$
as required in (van der Vaart and Wellner, 1996, p. 206). Hence we estimate the entropy integral as

\[
\int_0^{\delta_n} \sqrt{\log N(\epsilon, F, d_{\text{a}})} d\epsilon \leq \int_0^{\delta_n} \sqrt{\log(K') + V'(\|F\|_{Q_n,2}) + V' \log(1/\epsilon)} d\epsilon
\]

\[
= \delta_n \sqrt{\log(K') + V'(\|F\|_{Q_n,2}) + 2\gamma \int_{1/\delta_n}^{\infty} \frac{\sqrt{\log t}}{t^2} dt}
\]

\[\to 0, \quad \text{as} \ \delta_n \to 0.\]

Finally, for the limiting covariance we have that

\[
\text{cov}_{L^*_n \otimes U} \left( \sum_{i=1}^{n} Z_{ni}(f), \sum_{i=1}^{n} Z_{ni}(g) \right) = n \text{cov}_{L^*_n \otimes U}(Z_{n1}(f), Z_{n1}(g))
\]

\[
= E_{L^*_n \otimes U}[f(WR^*)g(WR^*)]
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} E_U[f(WR_i)g(WR_i)]
\]

\[\to E_{L \otimes U}[f(WR)g(WR)]\]

almost surely by the strong law of large numbers. The latter equals

\[E[f(X)g(X)] = \text{cov}(f(X), g(X)),\]

the covariance of the limiting process \(G\). We conclude that \(G^*_n\) also converges weakly to this process, given almost all sequences \(X_1, X_2, \ldots\).

**Proof of Theorem 2.5.** We shall apply the changing classes central limit theorem, Theorem 19.28 in van der Vaart (1998), conditionally on almost all sequences \(U_1, U_2, \ldots\) and \(V_1, V_2, \ldots\). As the limiting process will be the same almost surely, the convergence then is also unconditional.

In the proof of Theorem 2.2 we showed that \(\mathcal{F}_\Theta\) is a VC-class of functions, and hence the same is true for each \(\mathcal{F}^{(n)}_{\Theta} = \{f_{n(\theta)} \mid \theta \in \Theta\}\) with the same bound on the entropy numbers. Therefore, the condition on the entropy integral in van der Vaart (1998, Theorem 19.28) is satisfied. Further, \(\|X\|_2\) is a majorant for \(\mathcal{F}_\Theta\) and hence also for each function class \(\mathcal{F}^{(n)}_{\Theta}\), which satisfies the Lindeberg-condition by the assumed integrability of \(\|X\|_2^2\). Finally, for each \(\theta \in \Theta\), by continuity of the distribution of \(X\) we have that \(f_{n(\theta)}(x) \to f_\theta(x)\) for almost all \(x \in \mathbb{R}^d\), conditionally on almost all sequences \(U_1, U_2, \ldots\) and \(V_1, V_2, \ldots\). By dominated convergence we obtain convergence of the covariances \(P(f_{n(\theta)} f_{n(\theta)}) - P(f_{n(\theta)}) P(f_{n(\theta)}) \to P(f_\theta f_\theta) - P(f_\theta) P(f_\theta)\), conditionally on almost all sequences \(U_1, U_2, \ldots\) and \(V_1, V_2, \ldots\). \(\square\)
Proof of Corollary 2.6. Again we set \( F^{(n)}_\Theta = \{ f_{n(\theta)} \mid \theta \in \Theta \} \). First suppose that \( X \) is spherically symmetric. Then by Theorem 2.5, since \( Pf = 0, f \in F_\Theta \),
\[
\tilde{T}_{n,N,u,c_0} = \| G_n \|_{F^{(n)}_\Theta} \Rightarrow \| G \|_{F_{\Theta}}
\]
which implies the first statement by continuity of the distribution of \( \| G \|_{F_{\Theta}} \).

Now suppose that \( X \) is not spherically symmetric, so that in (2.4), \( \Delta_0(X) > 0 \). By continuity of the distribution of \( X \) we have that
\[
\| Pf \|_{F^{(n)}_\Theta} \uparrow \Delta_0(X)
\]
along almost all sequences \( U_1, U_2, \ldots \) and \( V_1, V_2, \ldots \). Let \( A_{U,V} \) denote the \( \sigma \)-algebra generated by \( U_1, U_2, \ldots \) and \( V_1, V_2, \ldots \). Then arguing as in the proof of Proposition 2.3,
\[
P \left( \tilde{T}_{n,N,u,c_0} > q_{1-\alpha}(G) \mid A_{U,V} \right)
\]
\[
\geq P \left( \sqrt{n} \| Pf \|_{F^{(n)}_\Theta} - \sqrt{n} \| (P_n - P) f \|_{F^{(n)}_\Theta} > q_{1-\alpha}(G) \mid A_{U,V} \right) \to 1
\]
almost surely, since from the proof of Theorem 2.5 we also have convergence of \( \sqrt{n} \| (P_n - P) f \|_{F^{(n)}_\Theta} \) to \( \| G \|_{F_{\Theta}} \) conditionally on \( A_{U,V} \). Taking the expected value of the conditional probability yields the statement of the corollary.

5.2 Proofs for Section 3

Proof of Theorem 3.1. As discussed in the beginning of Section 3.1, we have that \( P(\hat{\eta}_n \in E) = 1, n \geq d + 1 \). We shall use Theorem 2.1 in van der Vaart and Wellner (2007), and for that we need to show that
\[
\sup_{\theta \in \Theta} P (f_{\theta,\hat{\eta}_n} - f_{\theta,\eta_0})^2 \to 0 \tag{5.20}
\]
in probability as \( n \to \infty \) and that the class
\[
\left\{ f_{\theta,\eta} \mid (\theta, \eta) \in \Theta \times E \right\} \quad \text{is P-Donsker.} \tag{5.21}
\]
To show (5.21), we write
\[
f_{\theta,\eta}(x) = w^\top A(x - \mu) 1_{u^\top A(x - \mu) \geq c}
\]
\[
= (A^\top w)^\top x 1_{(A^\top u)^\top x \geq c + u^\top A \mu} + (A^\top w)^\top \mu 1_{(A^\top u)^\top x \geq c + u^\top A \mu}
\]
\[
= k^\top x 1_{s^\top x \geq c'} + c'' 1_{s^\top x \geq c'}
\]
with \( k = A w, s = A^\top u, c' = c + u^\top A \mu \), and \( c'' = (A^\top w)^\top \mu \), which implies that
\[
\left\{ f_{\theta,\eta} \mid (\theta, \eta) \in \Theta \times H \right\} \subseteq \mathcal{G} + \mathcal{G}
\]
with \( \mathcal{G} \) the class defined in (5.16). As shown in the proof of Theorem 2.2, the class \( \mathcal{G} \) is a VC-class and hence in particular Euclidean, meaning that the covering
numbers grow polynomially as the radius decreases, see Wellner (2005, p. 64). It follows from the preservation of the Euclidean property under sums (Wellner, 2005, Proposition 8.5) that the class in (5.21) is also $P$-Donsker.

To show (5.20), it follows from law of large numbers and the continuous mapping theorem that

$$(\hat{\Sigma}_n^{-1/2}, \bar{X}_n) \rightarrow_P (\Sigma_0^{-1/2}, \mu_0).$$

Let $w = v - (v^T u)u$. We write

$$f_{\theta, \eta_n}(x) - f_{\theta, \eta_0}(x)$$

$$= w^T \left( \hat{\Sigma}_n^{-1/2}(x - \bar{X}_n) - \Sigma_0^{-1/2}(x - \bar{X}_n) \right) + w^T \Sigma_0^{-1/2}(x - \mu_0) \left( I_{u^T \Sigma_0^{-1/2}(x - \mu_0) \geq c} - I_{u^T \Sigma_0^{-1/2}(x - \mu_0) \geq c} \right)$$

$$= w^T \left( \hat{\Sigma}_n^{-1/2}(x - \bar{X}_n) - \Sigma_0^{-1/2}(x - \mu_0) \right) I_{u^T \Sigma_0^{-1/2}(x - \mu_0) \geq c}$$

$$+ w^T \Sigma_0^{-1/2}(x - \mu_0) \left( I_{u^T \Sigma_0^{-1/2}(x - \mu_0) \geq c} - I_{u^T \Sigma_0^{-1/2}(x - \mu_0) \geq c} \right)$$

$$= A_n(x) + B_n(x).$$

Since $||w|| \leq 2$, it follows that

$$P(||A_n||^2) \leq 8 |||\hat{\Sigma}_n^{-1/2} - \Sigma_0^{-1/2}|||^2 E(||X||^2) + 8 |||\hat{\Sigma}_n^{-1/2} - \Sigma_0^{-1/2}|||^2 ||\bar{X}_n||^2$$

$$+ 8 |||\Sigma_0^{-1/2}||^2 ||\bar{X}_n - \mu_0||^2 \rightarrow 0,$$

where $||| \cdot |||$ is the spectral norm of a symmetric $d \times d$ matrix. To handle the term $B_n(x)$, note that

$$\left| I_{u^T \Sigma_0^{-1/2}(x - \bar{X}_n) \geq c} - I_{u^T \Sigma_0^{-1/2}(x - \mu_0) \geq c} \right| \leq I_{u^T \Sigma_0^{-1/2}(x - \bar{X}_n) \geq c, u^T \Sigma_0^{-1/2}(x - \mu_0) < c} + I_{u^T \Sigma_0^{-1/2}(x - \mu_0) \geq c} u^T \Sigma_0^{-1/2}(x - \mu_0) \geq c.$$

Now, for any $M > 0$ we have that

$$\left\{ u^T \Sigma_0^{-1/2}(x - \mu_0) \geq c, u^T \Sigma_0^{-1/2}(x - \mu_0) < c \right\}$$

$$\subset \left\{ u^T \Sigma_0^{-1/2}(x - \mu_0) \geq c - \delta_n, u^T \Sigma_0^{-1/2}(x - \mu_0) < c, \|x\| \leq M \right\}$$

$$\cup \{\|x\| > M\}$$

with

$$|\delta_n| \leq |||\hat{\Sigma}_n^{-1/2} - \Sigma_0^{-1/2}|| M + |||\hat{\Sigma}_n^{-1/2} - \Sigma_0^{-1/2}|| ||\bar{X}_n||$$

$$+ |||\Sigma_0^{-1/2}|| \|\bar{X}_n - \mu_0\| = o_P(1),$$

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implying that for any $\nu > 0$ and $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

\[
\mathbb{P}\left( u^\top \hat{\Sigma}_n^{-1/2}(X - \bar{X}_n) \geq c, \ u^\top \Sigma_0^{-1/2}(X - \mu_0) < c \right) \\
\leq \mathbb{P}\left( u^\top \Sigma_0^{-1/2}(X - \mu_0) \geq c - \nu, \ u^\top \Sigma_0^{-1/2}(X - \mu_0) < c \right) + 2\epsilon/3
\]

by choosing $M$ such that $\mathbb{P}(\|X\| > M) \leq \epsilon/3$ and $n_0$ large enough such that $\mathbb{P}(\delta_n > \nu) \leq \epsilon/3$. Since $u \in \mathcal{S}_{d-1}$ and $Y = \Sigma_0^{-1/2}(X - \mu_0)$ is spherically symmetric, we have that $u^\top \Sigma_0^{-1/2}(X - \mu_0) \overset{d}{=} V$ with $V$ distributed as the first coordinate of $Y$. Thus, it follows that

\[
P\left( u^\top \hat{\Sigma}_n^{-1/2}(X - \bar{X}_n) \geq c, \ u^\top \Sigma_0^{-1/2}(X - \mu_0) < c \right) \\
\leq P\left( V \in [c - \nu, c) \right) + 2\epsilon/3 \\
= F_V(c) - F_V(c - \nu) + 2\epsilon/3 \\
\leq \epsilon,
\]

for all $c \in \mathbb{R}$, using (uniform) continuity of the distribution of $V$ and taking $\nu$ to be small enough. By the Cauchy-Schwarz inequality, it follows that

\[
P(\|B_n\|^2) \leq 4 \epsilon E[\|Y\|^2]
\]

for sufficiently large $n$. Since the second indicator in $B_n(x)$ can be handled similarly, (5.20) follows.

\[\square\]

**Lemma 5.2.** Suppose that $X_1, \ldots, X_n$ are i.i.d. $\in \mathbb{R}^d$ such that $\mathbb{E}[\|X_1\|^4] < \infty$, with common mean $\mu_0$ and covariance matrix $\Sigma_0$ assumed to be positive definite. If $f : \mathbb{R}^d \to \mathbb{R}$ is such that $Pf^2 < \infty$ and $E[X^2 f^2(X)] < \infty$, then as $n \to \infty$

\[
\sqrt{n} \left( \begin{array}{c}
\hat{\Sigma}_n^{-1} - \Sigma_0^{-1} \\
\hat{\Sigma}_n^{-1/2} - \Sigma_0^{-1/2} \\
\bar{X}_n - \mu_0 \\
\mathbb{P}_n(f) - Pf
\end{array} \right) \Rightarrow \left( \begin{array}{c}
-\Sigma_0^{-1}K\Sigma_0^{-1} \\
-\Sigma_0^{-1}\Sigma_0^{-1}K \Sigma_0^{-1} - \Sigma_0^{-1/2}e^{-t\Sigma_0^{-1/2}} dt \overset{d}{=} \mathcal{D} \\
\mathcal{D} \\
\mathcal{G}(f)
\end{array} \right)
\]

where $K$ and $\mathcal{D}$ are a centered Gaussian $d \times d$ matrix and $d$-dimensional vector respectively such that for any vector $a \in \mathbb{R}^d$ the covariance matrix of the $(2d + 1)$-dimensional centered Gaussian vector $(Ka, \mathcal{D}, \mathcal{G}(f))^\top$ is the $(2d + 1) \times (2d + 1)$ matrix $\Gamma(a)$ given by

\[
\Gamma(a) = \begin{pmatrix}
\Gamma_{11}(a) & \Gamma_{12}(a) & \Gamma_{13}(a, f) \\
\Gamma_{12}(a)^\top & \Sigma_0 & \Gamma_{23}(f) \\
\Gamma_{13}(a, f)^\top & \Gamma_{23}(f)^\top & Pf^2 - (Pf)^2
\end{pmatrix}
\]
with
\[
\begin{align*}
\Gamma_{11}(a) &= E \left[ ((X - \mu_0)(X - \mu_0)^\top - \Sigma_0) a a^\top ((X - \mu_0)(X - \mu_0)^\top - \Sigma_0) \right], \\
\Gamma_{12}(a) &= E \left[ ((X - \mu_0)(X - \mu_0)^\top - \Sigma_0) a (X - \mu_0)^\top \right], \\
\Gamma_{13}(a, f) &= E \left[ ((X - \mu_0)(X - \mu_0)^\top - \Sigma_0) a (f(X) - P f) \right], \\
\Gamma_{23}(f) &= E \left[ (X - \mu_0) (f(X) - P f) \right].
\end{align*}
\]

Proof of Lemma 5.2. It is well-known and easy to see that \( \hat{\Sigma}_n \) is asymptotically equivalent to \( n^{-1} \sum_{i=1}^n (X_i - \mu_0)(X_i - \mu_0)^\top \). Thus, from the central limit theorem, we have
\[
\sqrt{n} \left( \frac{1}{n} \sum_{i=1}^n (X_i - \mu_0)(X_i - \mu_0)^\top - \Sigma_0 \right) \Rightarrow \left( \begin{array}{c} K \\
\mathbb{D} \\
\mathcal{G}(f) \end{array} \right)
\]
where the covariance matrix of the centered Gaussian vector \((\mathbb{K} a, \mathbb{D})^\top \in \mathbb{R}^d\) is \( \Gamma \) given above.

Now, the operator \( \Sigma \mapsto \Sigma^{-1} \) defined on the sub-space of invertible matrices in \( \mathbb{R}^{d \times d} \) is differentiable at any invertible matrix \( A \) with gradient \( H \mapsto -A^{-1} HA^{-1} \), where \( H \in \mathbb{R}^{d \times d} \). Also, it is known that the operator \( \Sigma \mapsto \Sigma^{1/2} \), when defined on the space of positive definite matrices, is differentiable with gradient at \( A \) given by
\[
H \mapsto \int_0^\infty e^{-tA^{1/2}} H e^{-tA^{1/2}} dt
\]
which can be shown to be the solution, \( S \), of the Sylvester equation \( A^{1/2} S + S A^{1/2} = H \).

Using the delta-method and the asymptotic equivalence mentioned above, it follows that
\[
\sqrt{n} \left( \begin{array}{c} \hat{\Sigma}_n^{-1} - \Sigma_0^{-1} \\
\hat{\Sigma}_n^{-1/2} - \Sigma_0^{-1/2} \\
\hat{\Sigma}_n - \mu_0 \\
\mathbb{P}_n(f) - P f \end{array} \right) \Rightarrow \left( \begin{array}{c} -\Sigma_0^{-1} \mathbb{K} \Sigma_0^{-1} \\
-\Sigma_0^{-1/2} \mathbb{K} \Sigma_0^{-1/2} \\
-\Sigma_0^{-1} \mathbb{K} \Sigma_0^{-1} e^{-t\Sigma_0^{-1/2}} dt \\
\mathbb{D} \\
\mathcal{G}(f) \end{array} \right)
\]
which completes the proof.

Proof of Corollary 3.2. Since \( Pf_{\theta, \eta_0} = 0 \) for any \( \theta \in \Theta \),
\[
\sqrt{n} P_{n, f_{\theta, \eta_0}} = \sqrt{n} (\mathbb{P}_n f_{\theta, \eta_0} - Pf_{\theta, \eta_0}) = G_n (f_{\theta, \eta_0} - f_{\theta, \eta_0}) + G_n f_{\theta, \eta_0} + \sqrt{n} Pf_{\theta, \eta_0}.
\]
In view of Theorem 3.1, the first term is a process which converges weakly to 0. By Theorem 2.2, we know that \( G_n f_{\theta, \eta_0} \) converges weakly to \( G \). Hence, it remains
to find the weak limit of the third term (jointly with $G$). To this end, first fix
$	heta = (u, v, c) \in \Theta$. As shown in the proof of Theorem 3.1, setting $w = v - (u^\top u)u$ we have that

$$\sqrt{n}P_{f_{\theta, \hat{\theta}_n}} = \sqrt{n}P(f_{\theta, \hat{\theta}_n} - f_{\theta, \theta_0}) = I_n + II_n$$

where

$$I_n = w^\top \hat{S}_n^{-1/2} \sqrt{n}(\mu_0 - \bar{X}_n)P(u^\top Y > c) + w^\top \sqrt{n}(\hat{\Sigma}_n^{-1/2} - \Sigma_0^{-1/2})\Sigma_0^{1/2}E[Y \mathbf{1}_{u^\top Y > c}]$$

and

$$II_n = w^\top \sqrt{n}(\hat{\Sigma}_n^{-1/2} - \Sigma_0^{-1/2})$$

$$\times E \left[ (X - \mu_0)(\mathbf{1}_{u^\top \hat{S}_n^{-1/2}(X - \bar{X}_n) > c} - \mathbf{1}_{u^\top \Sigma_0^{-1/2}(X - \mu_0) > c}) \right] + w^\top \sqrt{n} \Sigma_0^{-1/2} E \left[ (X - \mu_0)(\mathbf{1}_{u^\top \hat{S}_n^{-1/2}(X - \bar{X}_n) > c} - \mathbf{1}_{u^\top \Sigma_0^{-1/2}(X - \mu_0) > c}) \right]$$

$$+ w^\top \hat{\Sigma}_n^{-1/2} \sqrt{n}(\mu_0 - \bar{X}_n)$$

$$\times \left( P(u^\top \hat{\Sigma}_n^{-1/2}(X - \bar{X}_n) > c) - P(u^\top \Sigma_0^{-1/2}(X - \mu_0) > c) \right).$$

It follows from Lemma 5.2 and the continuous mapping theorem that

$$I_n \Rightarrow -w^T \mathbb{E} P(u^\top Y > c) + w^T \Sigma_0^{-1/2} \mathbb{E} [Y \mathbf{1}_{u^\top Y > c}]$$

as $n \to \infty$, where $Z = \Sigma_0^{-1/2} D$, and $D$ and $S$ are as in Lemma 5.2. Using spherical symmetry of $Y$, the weak convergence above can be given by the equivalent form

$$I_n \to_d -w^T \mathbb{E} P(V > c) + w^T \Sigma_0^{-1/2} \mathbb{E} [V \mathbf{1}_{V > c}] \quad (5.22)$$

where $V$ is distributed for example as the first coordinate of $Y$.

Note first that both the first and last term in $II_n$ converge to 0 by Lemma 5.2, the Central Limit Theorem and the arguments used below showing that $P(u^\top \hat{\Sigma}_n^{-1/2}(X - \bar{X}_n) > c) - P(u^\top \Sigma_0^{-1/2}(X - \mu_0) > c) = O_p(n^{-1/2})$ (we can also use for this part arguments similar to those used in the proof of Theorem 3.1). For the middle term, $II_{n,2}$ say, we can write

$$E \left[ (X - \mu_0)(\mathbf{1}_{u^\top \hat{S}_n^{-1/2}(X - \bar{X}_n) > c} - \mathbf{1}_{u^\top \Sigma_0^{-1/2}(X - \mu_0) > c}) \right]$$

$$= \Sigma_0^{1/2} E \left[ Y \mathbf{1}_{u^\top \hat{S}_n^{-1/2}(\Sigma_0^{1/2} Y + \mu_0 - \bar{X}_n) > c} - \Sigma_0^{1/2} E (Y \mathbf{1}_{u^\top Y > c}) \right]$$

$$= \Sigma_0^{1/2} E \left[ Y (\mathbf{1}_{u^\top Y > c} - \mathbf{1}_{u^\top Y > c}) \right]$$

$$= \Sigma_0^{1/2} \left\{ E \left[ Y \mathbf{1}_{u^\top Y > c} \right] - E \left[ Y \mathbf{1}_{u^\top Y > c} \right] \right\}$$

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with $u_n = \Sigma_0^{1/2} \Sigma_n^{-1/2} u$, $c_n = c + u^T \hat{\Sigma}_n^{-1/2}(\hat{X}_n - \mu_0)$ and $Y = \Sigma_0^{-1/2}(X - \mu_0)$. It follows that

$$ II_{n,2} = u^T \sqrt{n} \left\{ E \left[ Y I_{u_n^T Y > c_n} \right] - E \left[ Y I_{u_n^T Y > c} \right] \right\}. \quad (5.23) $$

Since $Y$ is spherically symmetric, we have that

$$ E \left[ Y I_{u_n^T Y > c_n} \right] = u_n \left[ u_n^T Y I_{u_n^T Y > c_n} \right] = u_n \left[ u_n \left[ E \left[ \tilde{u}_n^T Y I_{\tilde{u}_n^T Y > c_n} \right] \right] \right], \quad \text{with } \tilde{u}_n = u_n / \| u_n \| $$

where $V$ denotes again the first component of the vector spherically symmetric vector $Y$. Similar arguments and the fact that $\| u \| = 1$ imply that

$$ E \left[ Y I_{u_n^T Y > c_n} \right] - E \left[ Y I_{u_n^T Y > c} \right] = u_n \| u_n \left[ E \left[ V I_{V > c_n} \right] \right] - u E \left[ V I_{V > c} \right] \right\]$$

where $O(1)$ does not depend on $c$ since $v \mapsto v g(v)$ is assumed to be uniformly bounded. Now we compute

$$ \sqrt{n} h_n = \sqrt{n} \left( \frac{c + u^T \hat{\Sigma}_n^{-1/2}(\hat{X}_n - \mu_0)}{\| u_n \|} - c \right) $$

$$ = \frac{1}{\| u_n \|} \sqrt{n} \left( c + u^T \hat{\Sigma}_n^{-1/2}(\hat{X}_n - \mu_0) - c \| u_n \| \right) $$

$$ = - \frac{c}{\| u_n \|} \sqrt{n}(\| u_n \| - 1) + \frac{u^T}{\| u_n \|} \sqrt{n} \hat{\Sigma}_n^{-1/2}(\hat{X}_n - \mu_0) $$

$$ = - \frac{c}{\| u_n \|} \sqrt{n}(\| u_n \| - 1) + \frac{u^T}{\| u_n \|} \sqrt{n} \hat{\Sigma}_n^{-1/2}(\hat{X}_n - \mu_0). $$

Furthermore,

$$ \sqrt{n}(\| u_n \|^2 - 1) = - u^T \hat{\Sigma}_n^{-1/2} \sqrt{n}(\hat{\Sigma}_n - \Sigma_0) \hat{\Sigma}_n^{-1/2} u $$

so that

$$ \sqrt{n} h_n = \frac{c}{\| u_n \|(1 + \| u_n \|)} u^T \hat{\Sigma}_n^{-1/2} \sqrt{n}(\hat{\Sigma}_n - \Sigma_0) \hat{\Sigma}_n^{-1/2} u $$

$$ + \frac{u^T}{\| u_n \|} \sqrt{n} \hat{\Sigma}_n^{-1/2}(\hat{X}_n - \mu_0). $$
By Lemma 5.2, $\sqrt{n}u_n$ admits a weak limit as $n \to \infty$. Also, it follows from the same lemma that $u_n \parallel u_n \parallel$ converges to $u$ in probability as $n \to \infty$. Since $w^T u = 0$, this implies that

$$w^T u_n \parallel u_n \parallel \sqrt{n}E[V(1_{V > c_n} \parallel u_n \parallel^{-1} - 1_{V > c})] \to 0$$

in probability as $n \to \infty$. Now, we get to the second term in (5.24). We have that

$$\sqrt{n}(u_n \parallel u_n \parallel - u_n) = \sqrt{n}(\Sigma_0^{1/2} \Sigma_n^{-1/2} u_n - u_n) + \frac{u_n}{\parallel u_n \parallel + 1} \sqrt{n}(\parallel u_n \parallel^2 - 1)$$

Thus, only the first term in the preceding display will contribute using again that $w^T u_n/\parallel u_n \parallel + 1 \to_p 0$ as $n \to \infty$. Using the expression of $\Pi_{n,2}$ in (5.23) it follows from our calculations above that

$$\Pi_{n,2} \Rightarrow w^T \Sigma_0^{1/2} \Sigma u E[V 1_{V > c}],$$

where we recall that $V$ distributed as the first coordinate of $Y$. Putting this together with the weak convergence in (5.22) it follows that

$$\sqrt{n}P_{\theta, \hat{\eta}_n} \Rightarrow -w^T \Sigma u E[V 1_{V > c}] + 2w^T \Sigma_0^{1/2} u E[V 1_{V > c}].$$

Putting all pieces together, using the fact that $w^T u = 0$ and the joint weak convergence established in Lemma 5.2, the claimed weak convergence

$$\sqrt{n}P_{\theta, \hat{\eta}_n} \Rightarrow \mathbb{L}(\theta)$$

follows for that chosen $\theta$. To show that this weak convergence holds for the whole process converges in $\ell^{\infty}(\Theta)$, it is enough to show that this process is tight and apply the Prohorov’s Theorem, see e.g. Theorem 1.3.9 in van der Vaart and Wellner (1996). Indeed, uniqueness of the weak limit for a fixed $\theta$ will imply that the process converges weakly to $\mathbb{L}$. Following the calculations detailed above, it is easy to see that for any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that

$$P(\parallel \sqrt{n}P_{\theta, \hat{\eta}_n} \parallel_{\Theta} \leq C_\epsilon) \geq 1 - \epsilon.$$ 

By considering the compact set $K_\epsilon = \{ \psi \in \ell^{\infty}(\Theta) : \parallel \psi \parallel_{\infty} \leq C_\epsilon \}$, we see that the preceding inequality gives tightness. The joint Gaussianity of $\mathbb{L}(\theta)$ and $\mathbb{G}$ follows from of Lemma 5.2. This completes the proof. □
Proof of Theorem 3.3. The proof proceeds in the following steps.

Step 1.: For \( i = 1, \ldots, n \), let us define

\[
Z_{ni} = n^{-1/2} \hat{\delta}_{\hat{X}_i} = n^{-1/2} \hat{R}_i W_i,
\]

where we recall that \( \hat{R}_1^*, \ldots, \hat{R}_n^* \) are i.i.d. random variables from the empirical distribution of the norms of the standardized observed vectors, that is,

\[
\hat{R}_1 = \| \hat{\Sigma}_n^{-1/2}(X_1 - \bar{X}_n) \|, \ldots, \hat{R}_n = \| \hat{\Sigma}_n^{-1/2}(X_n - \bar{X}_n) \|.
\]

(5.25)

and \( W_1, \ldots, W_n \) are independently and uniformly sampled from the unit sphere \( S_{d-1} \), and such that they are also independent of \( \hat{R}_1, \ldots, \hat{R}_n \). Then the process \( \{ \sum_{i=1}^n Z_{ni}(f), f \in F_s \} \) converges weakly to \( G f, f \in F_s \), where \( F_s \) is as in Theorem 2.4, and the process \( G \) is defined in Theorem 2.4. We apply this with \( F_s = \mathcal{F} \), where \( \mathcal{F} \) is as in (2.3).

Step 2.: In this step we derive the limit distribution of the bootstrapped matrix \( \hat{\Sigma}_n^* \) given \( X_1, X_2, \ldots, \), and show that conditional on \( X_1, X_2, \ldots, \), almost surely,

\[
\sqrt{n}(\hat{\Sigma}_n^* - I_d) \Rightarrow \mathbb{K}_0,
\]

(5.26)

with \( I_d \) the \( d \times d \) identity matrix and \( \mathbb{K}_0 \) a \( d \times d \) matrix with centered Gaussian components such that for any vector \( a \in \mathbb{R}^d \), the covariance matrix of \( \mathbb{K}_0 a a^\top \) is given by

\[
\Gamma_0(a) = \mathbb{E} [(YY^\top - I_d)aa^\top (YY^\top - I_d)].
\]

Step 3.: To conclude, recall that the drift, \( \mathbb{L}(\theta) \), in the weak limit of Corollary 3.2 is given by

\[
\mathbb{L}(\theta) = -(w^\top \Sigma_0^{-1/2} D_0) \mathbb{P}(V > c) + w^\top (\Sigma_0^{-1/2} + \Sigma_0^{-1/2} \mathbb{S}) u \mathbb{E}[V(1_{V > c})],
\]

(5.27)

where \( \theta = (u, v, c) \in S_{d-1} \times S_{d-1} \times \mathbb{R} \), \( w = v - (v^\top u) u \), \( V \) is any component of \( Y = \Sigma_0^{-1/2}(X - \mu_0) \), and \( D_0 \) and \( \mathbb{S} \) are centered Gaussian vector in \( \mathbb{R}^d \) and matrix in \( \mathbb{R}^d \times \mathbb{R}^d \) such that

\[
\sqrt{n} \left( \begin{array}{c} X_n - \mu_0 \\ \Sigma_0^{-1/2} - \Sigma_0^{-1/2} \end{array} \right) \Rightarrow \left( \begin{array}{c} D_0 \\ \mathbb{S} \end{array} \right).
\]

Following the same steps of the proof of Corollary 3.2 while replacing \( \hat{\eta}_n = (\hat{\Sigma}_n^{-1/2}, \hat{X}_n) \) with \( \hat{\eta}_n^* = ((\hat{\Sigma}_n^*)^{-1/2}, \hat{X}_n^*) \), the associated drift has the asymptotic distribution

\[
-(w^\top D_0) \mathbb{P}(V > c) + 2w^\top S_0 u \mathbb{E}[V(1_{V > c})],
\]

(5.28)

where \( \mathcal{D}_0 \sim \mathcal{N}(0, I_d) \) and \( S_0 \) is the centered Gaussian \( d \times d \) matrix such that

\[
\sqrt{n}((\hat{\Sigma}_n^*)^{-1/2} - I_d) \Rightarrow S_0.
\]
the existence of which follows from Step 2. Hence, in order to show that the modified bootstrap is consistent, we need to show that the drift \( \mathbb{L}(\theta) \) in (5.27) is also given by the expression in (5.28), for which we provide the details below. This finishes the proof of the theorem. Details for Step 1. Since the proof of this weak convergence should go along the same lines of the proof of Theorem 2.4, one needs to check first that the three conditions of Theorem 2.11.1 in van der Vaart and Wellner (1996). Here, we show that the third condition is indeed satisfied as the first two ones involve very similar arguments as in the proof of Theorem 2.4. For this third condition, we need to show that

\[
\lim_{n \to \infty} \text{cov}_{\hat{L}_n^* \otimes \mathcal{U}} \left( \sum_{i=1}^{n} Z_{ni}(f), \sum_{i=1}^{n} Z_{ni}(g) \right) = \mathbb{E}_{\hat{L}_n^* \otimes \mathcal{U}} \left[ f(WR) g(WR) \right]
\]

where \( \hat{L}_n^* \) is the empirical distribution \( 1/n \sum_{i=1}^{n} \delta_{\hat{R}_i} \), and \( \mathcal{U} \) denotes again the uniform distribution on the sphere \( S_d \). By definition of \( Z_{ni} \) we have that

\[
\text{cov}_{\hat{L}_n^* \otimes \mathcal{U}} \left( \sum_{i=1}^{n} Z_{ni}(f), \sum_{i=1}^{n} Z_{ni}(g) \right) = n \text{cov}_{\hat{L}_n^* \otimes \mathcal{U}} (Z_{n1}(f), Z_{n1}(g))
\]

where for \( f(x) = f_{u,v,c}(x) = (v - (v^\top u)u)^\top x \mathbf{1}_{u^\top x \geq c} \), and \( g(x) = g_{u',v',c'}(x) = (v' - (v'^\top u')u')^\top x \mathbf{1}_{u'^\top x \geq c'} \) for \( (u,v),(u',v') \in S^2_d \) and \( c,c' \in \mathbb{R} \) we have that

\[
f(\hat{R}_i W) = ||\hat{\Sigma}_n^{-1/2}(X_i - \bar{X}_n)|| (v - (v^\top u)u)^\top \mathcal{W} \mathcal{I}_{||\hat{\Sigma}_n^{-1/2}(X_i - \bar{X}_n)|| u^\top \mathcal{W} \geq c}
\]

and

\[
g(\hat{R}_i W) = ||\hat{\Sigma}_n^{-1/2}(X_i - \bar{X}_n)|| (v' - (v'^\top u')u')^\top \mathcal{W} \mathcal{I}_{||\hat{\Sigma}_n^{-1/2}(X_i - \bar{X}_n)|| u'^\top \mathcal{W} \geq c'}
\]

Since we wish to replace \( \hat{R}_i \) by \( R_i = ||\Sigma_0^{-1/2}(X_i - \mu_0)|| \) for \( i = 1, \ldots, n \), we can write that

\[
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_{\mathcal{U}} \left[ f(\hat{R}_i W) g(\hat{R}_i W) \right] = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_{\mathcal{U}} \left[ f(R_i W) g(R_i W) \right] \\
+ \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_{\mathcal{U}} \left[ (f(\hat{R}_i W) - f(R_i W)) g(\hat{R}_i W) \right] \\
+ \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_{\mathcal{U}} \left[ (f(R_i W)) (g(\hat{R}_i W) - g(R_i W)) \right]
\]

\[= \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_{\mathcal{U}} \left[ f(R_i W) g(R_i W) \right] + A_n + B_n.
\]

(5.29)
Now for \( i = 1, \ldots, n \) we have that

\[
|f(\tilde{R}_i W) - f(R_i W)|
\]

\[
= \left| \|\hat{\Sigma}_n^{-1/2}(X_i - \bar{X}_n)\| (v - (v^\top u)u)^\top W \|\hat{\Sigma}_n^{-1/2}(X_i - \bar{X}_n)\| u^\top W \geq c \right|
\]

\[
- \|\Sigma_0^{-1/2}(X_i - \mu_0)\| (v - (v^\top u)u)^\top W \|\Sigma_0^{-1/2}(X_i - \mu_0)\| u^\top W \geq c \right|
\]

\[
\leq \left( \|\hat{\Sigma}_n^{-1/2}(X_i - \bar{X}_n)\| - \|\Sigma_0^{-1/2}(X_i - \mu_0)\| \right)
\cdot (v - (v^\top u)u)^\top W \|\hat{\Sigma}_n^{-1/2}(X_i - \bar{X}_n)\| u^\top W \geq c
\]

\[
+ \|\Sigma_0^{-1/2}(X_i - \mu_0)\| \times |(v - (v^\top u)u)^\top W|
\]

\[
\cdot \left| I \|\hat{\Sigma}_n^{-1/2}(X_i - \bar{X}_n)\| u^\top W \geq c - I \|\Sigma_0^{-1/2}(X_i - \mu_0)\| u^\top W \geq c \right|
\]

\[
= I_{n,i} + II_{n,i}
\]

where \( I_{n,i} \) and \( II_{n,i} \) are functions of the observed data as well as \( W \sim U \). The goal now is to find an upper bound of the expectation of these terms with respect to the distribution \( U \). We have that

\[
\|\hat{\Sigma}_n^{-1/2}(X_i - \bar{X}_n)\|
\]

\[
= \|\Sigma_0^{-1/2}(X_i - \mu_0)\| + (\hat{\Sigma}_n^{-1/2} - \Sigma_0^{-1/2})(X_i - \mu_0) + \hat{\Sigma}_n^{-1/2}(\bar{X}_n - \mu_0)
\]

and hence

\[
\left| \|\hat{\Sigma}_n^{-1/2}(X_i - \bar{X}_n)\| - \|\Sigma_0^{-1/2}(X_i - \mu_0)\| \right|
\]

\[
\leq \| (\hat{\Sigma}_n^{-1/2} - \Sigma_0^{-1/2})(X_i - \mu_0) \| + \| \hat{\Sigma}_n^{-1/2}(\bar{X}_n + \mu_0) \|
\]

\[
\leq \| \hat{\Sigma}_n^{-1/2} - \Sigma_0^{-1/2} \| \cdot \| X_i - \mu_0 \| + \| \hat{\Sigma}_n^{-1/2} \| \cdot \| \bar{X}_n - \mu_0 \|
\]

where \( \|A\| \) of a matrix \( A \) denotes the usual operator norm defined here with respect to the Euclidean norm. Using the Cauchy-Schwarz inequality and the fact that \( \|v - (v^\top u)u\| \leq 2 \) and \( \|W\| = 1 \), the latter implies that

\[
I_{n,i} \leq 2 \left( \| \hat{\Sigma}_n^{-1/2} - \Sigma_0^{-1/2} \| \cdot \| X_i - \mu_0 \| + \| \hat{\Sigma}_n^{-1/2} \| \cdot \| \bar{X}_n - \mu_0 \| \right)
\]

and hence, using again \( \|v - (v^\top u)u\| \leq 2 \) and \( \|W\| = 1 \) in addition to the
inequality obtained in (5.30), it follows that
\[
I_{n,i} \cdot |g(\tilde{R}_i, W)|
\leq 4 \left( \|\Sigma_n^{-1/2} - \Sigma_0^{-1/2}\| \cdot \|X_i - \mu_0\| + \|\Sigma_n^{-1/2}\| \cdot \|\tilde{X}_n - \mu_0\| \right) \|\Sigma_n^{-1/2}(X_i - \tilde{X}_n)\|
\leq 4 \left( \|\Sigma_n^{-1/2} - \Sigma_0^{-1/2}\| \cdot \|X_i - \mu_0\| + \|\Sigma_n^{-1/2}\| \cdot \|\tilde{X}_n - \mu_0\| \right)
\times \left( \|\Sigma_0^{-1/2}(X_i - \mu_0)\| + \|\Sigma_n^{-1/2} - \Sigma_0^{-1/2}\| \cdot \|X_i - \mu_0\| + \|\Sigma_n^{-1/2}\| \cdot \|\tilde{X}_n - \mu_0\| \right)
\leq a_n \|X_i - \mu_0\|^2 + \beta_n \|X_i - \mu_0\| + \gamma_n
\]
where \(a_n, \beta_n\) and \(\gamma_n\) are \(O_p(n^{-1/2})\) as a consequence of Lemma 5.2 and the weak law of large numbers. Since \(\mathbb{E}[\|X - \mu_0\|^2] < \infty\) is satisfied under the assumption that \(\mathbb{E}[\|X\|^4] < \infty\) we can conclude that
\[
\frac{1}{n} \mathbb{E}_d[I_{n,i} \cdot |g(\tilde{R}_i, W)|] = O_p(n^{-1/2}).
\]
Now, we turn to the terms II_{n,i}, \(i = 1, \ldots, n\). We have that
\[
\mathbb{E}_d[II_{n,i} \cdot |g(\tilde{R}_i, W)|] \leq 4 \|\Sigma_0^{-1/2}(X_i - \mu_0)\|
\times \left( \mathbb{P}_d(\|\Sigma_n^{-1/2}(X_i - \tilde{X}_n)\|u^\top W \leq c, \|\Sigma_0^{-1/2}(X_i - \mu_0)\|u^\top W > c)
+ \mathbb{P}_d(\|\Sigma_n^{-1/2}(X_i - \tilde{X}_n)\|u^\top W > c, \|\Sigma_0^{-1/2}(X_i - \mu_0)\|u^\top W \leq c) \right).
\]
Using again the inequality obtained in (5.30), we can find \(a_n = O_p(n^{-1/2})\) and \(b_n = O_p(n^{-1/2})\) both independent of \(i\) such that \(\|\Sigma_n^{-1/2}(X_i - \tilde{X}_n)\| \leq \|\Sigma_0^{-1/2}(X_i - \mu_0)\|(1 + a_n) - b_n\). Hence,
\[
\mathbb{P}_d \left( \|\Sigma_n^{-1/2}(X_i - \tilde{X}_n)\|u^\top W \leq c, \|\Sigma_0^{-1/2}(X_i - \mu_0)\|u^\top W > c \right)
= \mathbb{P}_d \left( \|\Sigma_0^{-1/2}(X_i - \mu_0)(1 + a_n)\|u^\top W \in (c, c + b_n) \right)
\leq \mathbb{P}_d \left( \|\Sigma_0^{-1/2}(X_i - \mu_0)\|u^\top W \in (2c, 2(c + b_n)) \right)
= \frac{b_n}{\|\Sigma_0^{-1/2}(X_i - \mu_0)\|}
\]
using that \(u^\top W\) is uniformly distributed on \([-1, 1]\). Handling the second probability in (5.31) in a similar fashion gives
\[
\frac{1}{n} \sum_{i=1}^n \mathbb{E}_d[II_{n,i} \cdot |g(\tilde{R}_i, W)|] = O_p(n^{-1/2})
\]
and we conclude that
\[ A_n = O_p(n^{-1/2}). \]
Similarly, we can show that \( B_n = O_p(n^{-1/2}) \). Now, by the strong law of large numbers,
\[
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[f(R_i W)g(R_i W)] \to_{\text{a.s.}} \mathbb{E}[f(WR)g(WR)],
\]
\[
= \mathbb{E}[f(Y)g(Y)] = \text{cov}(f(Y), g(Y)),
\]
where \( L_0 \) is the true distribution of \( R \). This in turn implies that
\[
\lim_{n \to \infty} \text{cov}_{\hat{L}_n}(\sum_{i=1}^{n} Z_{ni}(f), \sum_{i=1}^{n} Z_{ni}(g)) = \text{cov}(f(Y), g(Y))
\]
which is exactly the covariance of the Gaussian process \( G \).

**Details for Step 2:** Recall that by definition
\[
\hat{\Sigma}_n = \frac{1}{n} \sum_{i=1}^{n} (\hat{X}_i^* - \hat{X}_n^*)(\hat{X}_i^* - \hat{X}_n^*)^\top
\]
\[
= \frac{1}{n} \sum_{i=1}^{n} \hat{X}_i^*(\hat{X}_i^*)^\top - \hat{X}_n^*(\hat{X}_n^*)^\top
\]
\[
= I_n + II_n
\]
where \( \hat{X}_i^* = \hat{R}_i^* W_i \), with \( \hat{R}_i^* \) is randomly drawn from \( \hat{R}_1, \ldots, \hat{R}_n \) given in (5.25) and \( W_1, \ldots, W_n \) are i.i.d. \( \sim \mathcal{U} \) and independent of \( \hat{R}_1, \ldots, \hat{R}_n \). We can write
\[
I_n = \frac{1}{n} \sum_{i=1}^{n} (\hat{R}_i^*)^2 W_i W_i^\top.
\]
Note that selecting \( \hat{R}_i^* \) from the sample \( (\hat{R}_1, \ldots, \hat{R}_n) \) is equivalent to selecting \( X_i^* \) from \( (X_1, \ldots, X_n) \) and putting \( \hat{R}_i^* = [(X_i^* - \bar{X}_n)^\top \hat{\Sigma}_n^{-1}(X_i^* - \bar{X}_n)]^{1/2} \). Thus,
\[
I_n = \frac{1}{n} \sum_{i=1}^{n} (X_i^* - \bar{X}_n)^\top \hat{\Sigma}_n^{-1}(X_i^* - \bar{X}_n) W_i W_i^\top
\]
\[
= \frac{1}{n} \sum_{i=1}^{n} (X_i^* - \bar{X}_n)^\top \Sigma_0^{-1}(X_i^* - \bar{X}_n) W_i W_i^\top
\]
\[
+ \frac{1}{n} \sum_{i=1}^{n} (X_i^* - \bar{X}_n)^\top (\hat{\Sigma}_n^{-1} - \Sigma_0^{-1})(X_i^* - \bar{X}_n) W_i W_i^\top.
\]
Since \((X_1^*, \ldots, X^*_n)\) is independent of \((W_1, \ldots, W_n)\), it can be shown using the same techniques based on Poissonization described in the proof of van de Vaart and Wellner (1996, Theorem 3.6.3) that

\[
\frac{1}{n} \sum_{i=1}^{n} (X_i^* - \bar{X}_n)^\top \Sigma_0^{-1} (X_i^* - \bar{X}_n) W_i W_i^\top
\]

has the same limit distribution as

\[
\frac{1}{n} \sum_{i=1}^{n} (X_i - \mu_0)^\top \Sigma_0^{-1} (X_i - \mu_0) W_i W_i^\top = \frac{1}{n} \sum_{i=1}^{n} R_i^2 W_i W_i^\top = \frac{1}{n} \sum_{i=1}^{n} Y_i Y_i^\top,
\]

which is

\[
\sqrt{n} \left( \frac{1}{n} \sum_{i=1}^{n} Y_i Y_i^\top - I_d \right) \Rightarrow \mathbb{K}_0.
\]

If \(P_n^*\) denotes the empirical process based on \(X_1^*, \ldots, X^*_n\), then it is known conditionally on \(X_1, X_2, \ldots\), the process \(\sqrt{n}(P_n^* - P_n)\) has the same limit distribution as \(\sqrt{n}(P_n^* - P)\); see van der Vaart and Wellner (1996, Chapter 3.6). This in turn implies that \(\frac{1}{n} \sum_{i=1}^{n} \|X_i^* - \bar{X}_n\|^2 = \mathcal{O}_p(1)\). Also,

\[
\left\| \frac{1}{n} \sum_{i=1}^{n} (X_i^* - \bar{X}_n)^\top (\hat{\Sigma}_n^{-1} - \Sigma_0^{-1})(X_i^* - \bar{X}_n) W_i W_i^\top \right\|
\]

\[
\leq \|\hat{\Sigma}_n^{-1} - \Sigma_0^{-1}\| \left\| \frac{1}{n} \sum_{i=1}^{n} X_i^* - \bar{X}_n \right\|^2 = \mathcal{O}_p(n^{-1/2}),
\]

where we have used the fact that \(\|vv^\top\| = \|v\|^2\) and \(\|W_i\| = 1\) since \(W_i\) belongs to the unit sphere. This allows us to conclude that

\[
\sqrt{n}(I_n - I_d) \Rightarrow \mathbb{K}_0
\]

as \(n \to \infty\). Finally, we have that

\[
\bar{X}_n^* = \frac{1}{n} \sum_{i=1}^{n} \hat{X}_i^*
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \left[ (X_i^* - \bar{X}_n)^\top (\hat{\Sigma}_n^{-1} - \Sigma_0^{-1})(X_i^* - \bar{X}_n) \right]^{1/2} W_i
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \left[ (X_i^* - \bar{X}_n)^\top \left( \hat{\Sigma}_n^{-1} - \Sigma_0^{-1} \right)(X_i^* - \bar{X}_n) 
+ (X_i^* - \bar{X}_n)^\top \Sigma_0^{-1} (X_i^* - \bar{X}_n) \right]^{1/2} W_i.
\]
Using the inequality \(|a + h|^{1/2} - a^{1/2}| \leq \sqrt{2}a^{-1/2}|h|\) for \(a > 0\) and \(h\) small enough so that \(a + h \geq 0\) together with the triangle inequality we have that

\[
\left\| \hat{X}_n^* - \frac{1}{n} \sum_{i=1}^{n} [(X_i^* - \bar{X}_n)\Sigma_0^{-1}(X_i^* - \bar{X}_n)]^{1/2} W_i \right\| \\
\leq \frac{\sqrt{2}}{n} \sum_{i=1}^{n} \left| (X_i^* - \bar{X}_n) \left( \hat{\Sigma}_n^{-1} - \Sigma_0^{-1} \right) (X_i^* - \bar{X}_n) \right|^{1/2} (\text{since } \|W_i\| = 1) \\
\leq \frac{\sqrt{2}}{\lambda_m} \| \hat{\Sigma}_n^{-1} - \Sigma_0^{-1} \|
\]

where \(\lambda_m > 0\) is the largest eigenvalue of \(\Sigma_0\). Using again the fact that \(\hat{\Sigma}_n^{-1} - \Sigma_0^{-1} = O_p(n^{-1/2})\) and that

\[
\frac{1}{n} \sum_{i=1}^{n} [(X_i^* - \bar{X}_n)\Sigma_0^{-1}(X_i^* - \bar{X}_n)]^{1/2} W_i
\]

has the same limit distribution as

\[
\frac{1}{n} \sum_{i=1}^{n} [(X_i - \mu_0)\Sigma_0^{-1}(X_i - \mu_0)]^{1/2} W_i = \frac{1}{n} \sum_{i=1}^{n} R_i W_i = \frac{1}{n} \sum_{i=1}^{n} Y_i
\]

conditionally on \(X_1, X_2, \ldots\) it follows that

\[
\sqrt{n}\hat{X}_n^* \Rightarrow \mathcal{N}(0, \mathcal{I}_d)
\]

conditionally on \(X_1, X_2, \ldots\). Thus, \(\sqrt{n}I_n = o_p(n^{-1/2})\) and we get finally (5.26).

**Details for Step 3.** First, note that

\[
\Sigma_0^{-1/2} D_0 \overset{d}{=} D_0 \sim \mathcal{N}(0, \mathcal{I}_d).
\]

Now, we will show that

\[
2S_0 \overset{d}{=} \hat{\Sigma}^{1/2} \Sigma_0^{1/2} + \Sigma_0^{1/2} S.
\]

To this aim, recall that conditionally on \(X_1, X_2, \ldots\) the covariance matrix \(\hat{\Sigma}_n^*\) is asymptotically equivalent to

\[
\frac{1}{n} \sum_{i=1}^{n} Y_i Y_i^\top = \Sigma_0^{-1/2} \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu_0)^\top (X_i - \mu_0)^\top \Sigma_0^{-1/2}
\]

and the right side is itself equivalent to \(\Sigma_0^{-1/2} \Sigma_n \Sigma_0^{-1/2}\). Now, note that

\[
\sqrt{n}((\hat{\Sigma}_n^*)^{-1/2} - \mathcal{I}_d)
\]

\[
= \sqrt{n}((\hat{\Sigma}_n^*)^{-1}(\hat{\Sigma}_n^*)^{1/2} - \mathcal{I}_d)
\]

\[
= \sqrt{n} \left( (\hat{\Sigma}_n^*)^{-1} - \mathcal{I}_d \right) (\hat{\Sigma}_n^*)^{1/2} + \sqrt{n} \left( (\hat{\Sigma}_n^*)^{1/2} - \mathcal{I}_d \right)
\]

\[
= \sqrt{n} \left( (\hat{\Sigma}_n^*)^{-1} - \mathcal{I}_d \right) (\hat{\Sigma}_n^*)^{1/2} - (\hat{\Sigma}_n^*)^{1/2} \sqrt{n} \left( (\hat{\Sigma}_n^*)^{-1} - \mathcal{I}_d \right)
\]

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and therefore,

\[(I_d + (\hat{\Sigma}_n^s)^{1/2}) \sqrt{n}((\hat{\Sigma}_n^s)^{-1/2} - I_d) = \sqrt{n} \left((\hat{\Sigma}_n^s)^{-1} - I_d\right)(\hat{\Sigma}_n^s)^{1/2}.\]

Now, the left side weakly converges to $2\mathbb{S}_0$. As for the right side, we have $(\hat{\Sigma}_n^s)^{1/2} \to I_d$ in probability. Hence, $\sqrt{n} \left((\hat{\Sigma}_n^s)^{-1} - I_d\right)(\hat{\Sigma}_n^s)^{1/2}$ has the same weak limit as $\sqrt{n} \left((\hat{\Sigma}_n^s)^{-1} - I_d\right)$ which in turn has the same weak limit as

\[\sqrt{n} \Sigma_0^{1/2} \left(\hat{\Sigma}_n^{-1} - \Sigma_0^{-1}\right) \Sigma_0^{1/2}\]

\[= \sqrt{n} \Sigma_0^{1/2} \left(\hat{\Sigma}_n^{-1/2}\hat{\Sigma}_n^{-1/2} - \Sigma_0^{-1/2}\Sigma_0^{-1/2}\right) \Sigma_0^{1/2}\]

\[= \Sigma_0^{-1/2} \left(\sqrt{n} \left(\hat{\Sigma}_n^{-1/2} - \Sigma_0^{-1/2}\right) \hat{\Sigma}_n^{-1/2} + \Sigma_0^{-1/2} \sqrt{n} \left(\hat{\Sigma}_n^{-1/2} - \Sigma_0^{-1/2}\right)\right) \Sigma_0^{1/2}\]

\[\Rightarrow \Sigma_0^{1/2} \mathbb{S} + \mathbb{S} \Sigma_0^{1/2},\]

from which we conclude the claimed result in (5.32).

\[\square\]

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