Antibunching of scalar particles in a 4D asymptotically flat rotating black hole solution including supertranslation

Shingo Takeuchi

Institute of Research and Development, Duy Tan University
P809, 3 Quang Trung, Hai Chau, Da Nang, Vietnam

Abstract

In this study, we obtain a four-dimensional asymptotically flat rotating black hole solution including the linear order supertranslation correction and arbitrary black hole rotation. Then, we analyze the effect of those in antibunching of scalar particles at its infinitely large $r$ region. This would be interesting as an observational measure for supertranslation corrections, aside from the memory effect.
1 Introduction

BMS symmetry \[1, 2\] is a kind of the general coordinate transformation in the vicinity of the infinite null region in the 4D asymptotically flat spacetimes, which could be said as the diffeomorphism to change the shape of its infinite null region in the range the falloff conditions allow, with the Bondi gauge is kept. In this paper, we refer to the BMS symmetry as the asymptotic symmetry.

It has been argued in \[3\] that its symmetry in the vicinity of the infinite null region is given by the direct product of the following two symmetries: 1) Virasoro symmetry (and its antiholomorphic part) on the conformal spheres and 2) supertranslations acting on the retarded/advanced time direction to which these conformal spheres attach.

After \[3\], the central charges of the asymptotic symmetry have been analyzed \[4, 5\]. On the other hand, the following 2 directions are investigated: i) the relation between the gravitational $S$-matrix and gravitational soft-theorem \[6, 7, 8, 9, 10, 11, 12, 13, 14\], and investigation for the CFT with the $S$-matrix equivalent to the gravitational $S$-matrix in the 4D asymptotically flat spacetime \[15, 16, 17, 18, 19, 20\] and ii) the black hole information paradox by the consideration that the evaporated information would be finally reflected in the spacetime configuration deformed by the asymptotic symmetry \[21, 22, 23\].

Memory effect \[24\], change of the distance between two objects by propagations of gravitational waves, are also studied, such as the spin memory effect \[25\], the color memory effect \[26\], and the electromagnetic memory effect \[27, 28, 29\] (as for the connection between the memory effect and asymptotic symmetry, see \[30, 31\] for example). Studies on the gravitational wave from the merging binary black hole based on some kinds of the memory effect are also progressing as \[32, 33, 34, 35, 36, 37, 38, 39\].

Extending the black hole spacetime to the rotating one would be one of the important issues in the current context of the asymptotic symmetry. Thus, we in this paper obtain a rotating black hole solution with the supertranslation correction. Then, we will study the antibunching in the $\phi^4$ scalar field theory in the spatially infinite region of the rotating supertranslated black hole spacetime we obtain in this paper.

One of the interesting problems in the literature of the asymptotic symmetry would be the issues concerning the soft theorem, the memory effect and the information paradox.

The region in the spacetime where the soft theorem and the gravitational memory effect will become important would be the vicinity of the infinite null region. However, the effect of the rotation of the black hole would no longer appear, as it gets weaken in that region. If one would involve it, one would have to expand the analysis to some subleading orders with regard to $r$ from the infinity. However, since it is the analysis of perturbations on the rotating black hole spacetime, its equation of motion would get very long, and the analysis would get very difficult.

Another issue would be the effect of the asymptotic symmetry in the Hawking fluxes aiming at the solution of the information paradox. However, it has been shown that the effect of the supertranslation is not involved in the Hawking fluxes in the case of the non-rotating black hole \[42, 43\]. This situation would be the same if the black hole became the rotating one.
Hence we in this paper study the antibunching in the scalar field theory with the \( \phi^4 \)-term in the spacetime we obtain in this paper.

Antibunching is a quantum phenomenology appearing in correlation function of intensities (CFI). Its analytical behaviours are remarkably different in the classical and quantum theories for the non-commutativity of the quantum field, which is a phenomenology rooted in the quantum nature, as well as T.Young’s interference experiment of light occurring for the difference of phases of the quantum fields.

More specifically, considering some experimental equipment like Fig.2. 1) first, separate a light from a light source to two lights, 2) then set these two lights to reach each detector at different time by adjusting the lengths of the pass of lights, 3) then detecting lights at each detector, calculate the value of CFI based on the data of those.

Then, while performing the experiment, if we analyzed CFI, we would find its result is remarkably different between the classical and quantum theories. As to this point, see (77) and (87) in this paper. Such a difference occurs for non-commutativity of quantum operators (see footnote of (80)). It has been confirmed that the experimental result agrees to the prediction obtained from the quantum theory [44, 45, 46]. Such a behaviour obtained in the quantum theory is called “antibunching”.

Actually, since the intensity is given by a square of a field, intensities in the quantum theory mean quantum particles. Accordingly, CFI can be considered as the correlation function of those. T.Young’s interference experiment of light could be considered for the difference of phases of quantum fields. Toward this, antibunching could be considered for the non-commutativity of quantum fields. In this sense, antibunching is a phenomenology rooted in quantum nature, as well.

Then, why we study the antibunching is that first of all we expect we could calculate the corrections of the supertranslation and rotation of our spacetime toward the field solution perturbatively, then we expect we might be able to contain some part of those into the leading free field part by a renormalization. If things went well along this expectation, we expect those spacetime effects would appear in the measurement of the antibunching at the infinite region, since it is determined by the leading free field part. Currently, memory effect is considered as the measure way for the deformation of supertranslations. Toward this, our analysis would be interesting as well, since it would indicate a future possibility that the antibunching can be another way for it.

Lastly, antibunching itself is already a well-known phenomenology in laboratories with photons. New and interesting point in our considering antibunching is the following 3 points: 1) to consider it in a curved spacetime, 2) then to show that the effect of the curvature of the spacetime is involved into it at the \( r \to \infty \) region, 3) and to point out that there is a possibility that the antibunching could be used for the measure of the deformation for supertranslations. Here, although we in this study consider a supertranslated rotating black hole spacetime as the background spacetime, there is no need at all that it is black hole spacetimes.

We mention the organization of this paper. First of all, we sketch the outline of what will be done in Sec.2 to obtain a black hole solution with the linear order supertranslation correction arbitrary black hole rotation, in Fig.1.
Then, in Sec.\ref{sec:rewrite} we begin with the rotating black hole spacetime in the Boyer-Lindquist (BL) coordinate system without supertranslation corrections. Then, we rewrite it to the Schwarzschild type coordinate system by taking its black hole rotation to the linear order, then rewrite it to the isotropic coordinate system. Then, in Sec.\ref{sec:supertranslation} we involve the supertranslation correction in that isotropic coordinate system using some coordinate transformation rule to involve the supertranslation correction. Why we once transform the coordinate system to the isotropic is that the coordinate transformation rule to involve the supertranslation correction is given in the isotropic coordinate system.

Then, we get back it to the Schwarzschild coordinate system in Sec.\ref{sec:back} and \ref{sec:back2}. Then, in Sec.\ref{sec:supertranslation2} getting back it to the rotating spacetime, we obtain a rotating supertranslated black hole spacetime with the linear order supertranslation correction and the linear order black hole rotation.

In Sec.\ref{sec:coordinate} we obtain a coordinate transformation rule which enables us to directly transform a Schwarzschild to the supertranslated Schwarzschild without going through the isotropic coordinate system unlike the one above, which is “C.T. (Eq.(19) and (24))” in Fig.\ref{fig:coordinate} from the process above,. Then, in Sec.\ref{sec:process} using that rule in the first rotating black hole spacetime in the BL coordinate system before the supertransformation is performed, we obtain a rotating supertranslated black hole spacetime with the linear order supertranslation correction and arbitrary black hole rotation, which is “C.T. (Eq.(19), (24a) and (25))” in Fig.\ref{fig:coordinate}.

Then, in Sec.\ref{sec:check} we check the relation between the metrics we have obtained as the ones and the angular momentum of the spacetime. In Sec.\ref{sec:analyze}, we analyze the antibunching in the $\phi^{4}$ scalar field theory in the spatially infinite region of the rotating supertranslated black hole spacetime we obtain in this paper. In Sec.\ref{sec:summary} we summarize this study.

\section{Rotating supertranslated black hole solution}

In this section, we first obtain a 4D black hole spacetime solution with the linear order corrections of the supertranslation and the black hole rotation in Sec.\ref{sec:rewrite}. In that process, we obtain a transformation rule which enables us to directly transform a Schwarzschild coordinate system to the supertranslated Schwarzschild, in Sec.\ref{sec:coordinate} Using that rule, we obtain a 4D black hole spacetime solution with the linear order supertranslation correction and arbitrary black hole rotation, in Sec.\ref{sec:process}.

\subsection{Rewrite BL to isotropic coordinate systems, before performing supertranslation}

We begin with the 4D rotating black hole spacetime given by the Boyer-Lindquist (BL) coordinate system,

\begin{equation}
\begin{aligned}
ds^2 &= -(1 - \frac{2mr}{r^2 + a^2 \cos^2 \theta})dt^2 + \frac{r^2 + a^2 \cos^2 \theta}{r^2 - 2mr + a^2}dr^2 + (r^2 + a^2 \cos^2 \theta) d\theta^2 \\
&+ (r^2 + a^2 + \frac{2mra^2 \sin^2 \theta}{r^2 + a^2 \cos^2 \theta}) \sin^2 \theta d\phi^2 - \frac{4mra \sin^2 \theta}{r^2 + a^2 \cos^2 \theta} dtd\phi.
\end{aligned}
\end{equation}
Figure 1: This is the outline of what we will do in Sec.2. “B.L.” and “C.T.” are the abbreviations of “Boyer-Lindquist coordinates” and “coordinate transformation”, respectively. As the point in this outline, in the process to obtain (22) by once going through the isotropic coordinate system, which is (2) → (3) → (7) → (18), we can get a coordinate transformation rule which can transform a coordinate system of a Schwarzschild to the one of the supertranslated Schwarzschild, which is “C.T. (Eq.(19) and (24)).” Then using it in the first (11), we obtain (29), which is “C.T. (Eq.(19) and (24a), (25)).”

Figure 2: Experimental equipment to finally obtain values for correlation function of intensities (CFI) from a light from a light source.

We involve the correction for the diffeomorphism of the supertranslation, to the linear order. For this purpose, let us check the following facts:

- in [41], one way to involve the supertranslation is given in the non-rotating isotropic coordinate system,
the non-rotating isotropic coordinate system can be obtained from Schwarzschild coordinate system,

• (1) can be apparently rewritten to the Schwarzschild type coordinate system by taking $a$ to the linear order as

$$ds^2 = -(1 - 2m/r)dt^2 + (1 - 2m/r)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2 + O(a^2),$$

where $a$ to the linear order is contained in $d\varphi$ as $d\varphi = d\phi - 2ma/r^3dt$.

Thus, let us apparently rewrite (1) to the isotropic coordinate system via (2) as

$$ds^2 = -\frac{(1 - m/2\rho)^2}{(1 + m/2\rho)^2}dt^2 + (1 + m/2\rho)^4(d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\varphi^2)) + O(a^2).$$

where $r = \rho(1 + m/2\rho)^2$.

### 2.2 Isotropic coordinate system (3) after the supertranslation

The coordinate transformations, which realize the deformation of the spacetime by the diffeomorphism of the supertranslation, is given in the isotropic coordinates as

$$x_s = (\rho - C)\sin\theta\cos\varphi + \sin\varphi\csc\theta\partial_\varphi C - \cos\theta\cos\varphi\partial_\theta C,$$

$$y_s = (\rho - C)\sin\theta\sin\varphi - \cos\varphi\csc\theta\partial_\varphi C - \cos\theta\sin\varphi\partial_\theta C,$$

$$z_s = (\rho - C)\cos\theta + \cos\theta\cos\varphi\partial_\theta C,$$

where $t_s = t$, and the $C(\theta, \phi)$ is some function to characterize the change of the shape of the spacetime by the supertranslation, which we can take arbitrarily. We review how (4) is obtained in Appendix A.

We take $C$ as follows:

$$C = m\varepsilon Y_2^0.$$

We list the 3 points in (1) and (5) in what follows:

• $\varepsilon$ is some infinitesimal dimensionless parameter, with regard to which we perform our analysis to the linear order.

• $m$ is that in (3), which we involve to have $C$ have the dimension same with $\rho$ (where $G/c^2$ is 1 in this paper).

• Why we consider $Y_2^0$ is that it would be the most dominant mode in the quasi-normal modes (ringdown waves) emitted from the process that a soft-hairy black hole is formed.

---

†Author is writing this hearing from Vitor Cardoso in [40].
Using (4) with (5), we can obtain $\rho_s$ and the space part in (3) after the supertranslation as

$$\rho_s = (x_s^2 + y_s^2 + z_s^2)^{1/2} = \rho - \frac{\varepsilon}{8} \sqrt{\frac{5}{\pi}} m(3\cos(2\theta) + 1) + O(\varepsilon^2) \equiv \rho + \delta \rho, \quad (6a)$$

$$d\rho_s^2 + \rho_s^2 (d\theta_s^2 + \sin^2 \varphi_s d\varphi_s^2) = dx_s^2 + dy_s^2 + dz_s^2. \quad (6b)$$

With these, we write (3) to the linear order with regard to $\varepsilon$ as

$$ds^2 = -\frac{(1 - m/2\rho_s)^2}{(1 + m/2\rho_s)^2} dt^2 + (1 + m/2\rho_s)^2 (dx_s^2 + dy_s^2 + dz_s^2) + O(\varepsilon^2) + O(a^2) \equiv g_{tt} dt^2 + g_{\rho\rho} d\rho^2 + g_{\theta\theta} d\theta^2 + g_{\varphi_s\varphi_s} d\varphi_s^2, \quad (7)$$

where

- $g_{tt} = -\frac{(m - 2\rho)^2}{(m + 2\rho)^2} - \frac{5}{\pi} \varepsilon m^2 \frac{(m - 2\rho)}{(m + 2\rho)^3} (3\cos(2\theta) + 1) + O(\varepsilon^2),$
- $g_{\rho\rho} = \frac{(m + 2\rho)^4}{16\rho^4} + \frac{5}{\pi} \varepsilon m \frac{(m + 2\rho)^3}{32\rho^5} (3\cos(2\theta) + 1) + O(\varepsilon^2),$
- $g_{\theta\theta} = \frac{(m + 2\rho)^4}{16\rho^4} + \frac{5}{\pi} \varepsilon m (m + 2\rho)^3 \frac{3\cos(2\theta)(5m + 6\rho) + m - 2\rho}{64\rho^3} + O(\varepsilon^2),$
- $g_{\varphi_s\varphi_s} = \frac{(m + 2\rho)^4 \sin^2 \theta}{16\rho^2} + \frac{5}{\pi} \varepsilon m \frac{(m + 2\rho)^3}{64\rho^3} \sin^2 \theta (\cos(2\theta)(9m + 6\rho) + 7m + 10\rho).$

As for $d\varphi_s$ above, as can be seen by looking at $d\varphi$ given under (2), $d\varphi$ is also the quantity to have been displaced under the supertranslation. Once formally denoting it as $d\varphi_s$ as in (7), we proceed with the calculation for now. We make it clear explicitly in Sec.2.6.

We use the following notations in what follows:

- $g_{\mu\nu}$ for matrices of the supertranslated isotropic coordinate system,
- $j_{\mu\nu}$ for matrices of the supertranslated Schwarzschild coordinate system,
- $J_{MN}$ for matrices of the supertranslated BL coordinate system with linear order $a$,
- $K_{MN}$ for matrices of the supertranslated BL coordinate system with arbitrary $a$,

where $\mu, \nu = t, \rho, \theta, \varphi$ and $M, N = t, \rho, \theta, \varphi$. In addition, the coordinate with the lower index ‘s’ (e.g. ‘$\varphi_s$’) means the displaced coordinate by the supertranslation.

### 2.3 Reason that calculating to $\varepsilon^1$-order would be enough

The metrics we obtain in this study involve the supertranslation correction to the linear order. As for this “linear order”, it is considered it would be enough. We explain its reason in what follows.

1) Our supertranslation is the one defined with $\zeta$ in (5.2.3) in [30], which is the quantity to the linear order. (This can be seen from the fact that Lie derivatives concerning the
The diffeomorphism of the supertranslation are defined with this $\zeta$ as in (5.2.2).)

2) Hence, $f$ in (5.2.3) is the quantity to the linear order as well, and that $f$ defines $C$ as $L_fC = f$ as in (5.2.6) in [30]. Therefore $C$ is also the quantity to the linear order.

3) The $C$ in (1) in this study is the $C$ in (5.2.6) in [30], and defined as [5] with $\varepsilon$ in this study.

4) Hence, it would be enough if we performed analysis to the linear order of $\varepsilon$.

2.4 Rewrite the supertranslated isotropic coordinate system (7) to supertranslated Schwarzschild coordinate system. 1

Since we have obtained the metrices in the supertranslated isometric coordinate system as in (7), we rewrite it to the following Schwarzschild coordinate system:

$$ds^2 = -(1 - 2\mu(\rho)/r)dt^2 + (1 - 2\mu(\rho)/r)^{-1}dr^2 + j_{\theta\theta}d\theta^2 + j_{\varphi\varphi}, d\varphi_s^2.$$

(10)

In what follows, we obtain 1) a relation between $r$ and $\rho$, and 2) $\mu(\rho)$, by solving the following relations:

- $-(1 - 2\mu(\rho)/r) = g_{tt}$, (11a)
- $\frac{1}{1 - 2\mu(\rho)/r}\left(\frac{dr}{d\rho}\right)^2 = g_{\rho\rho}$, (11b)

where $g_{tt}$ and $g_{\rho\rho}$ are given in (7).

Next, let us obtain the $\mu(\rho)$. For this, let us look at (11b). Then, plugging (12) into the $r$ in (11b), solve it regarding $\mu(\rho)$ order by order to $\varepsilon^1$-order. As a result, we obtain

$$\mu(\rho) = m + \frac{c_1 \rho \varepsilon}{(m + 2\rho)^2} + O(\varepsilon^2),$$

(13)

where we took the integral constant at $\varepsilon^0$-order so that $\varepsilon^0$-order becomes $m$. $c_1$ is the integral constant at $\varepsilon^1$-order, which we put as

$$c_1 = 0,$$

(14)

otherwise (13) gets diverged at $\rho = 0$ when $m = 0$. As a result, $\mu(\rho)$ is given just $m$. Hence, we denote $\mu(\rho)$ as $\mu$ in what follows.

Now that we have obtained the relation between $\rho$ and $r$ in the form “$r = \cdots$” as in (12), using it we can rewrite the supertranslated Schwarzschild to the supertranslated isotropic coordinate system to $\varepsilon^1$-order as

$$ds^2 = -(1 - 2\mu/r)dt^2 + (1 - 2\mu/r)^{-1}dr^2 + j_{\theta\theta}d\theta^2 + j_{\varphi\varphi}, d\varphi_s^2 \rightarrow g_{tt}dt^2 + g_{\rho\rho}d\rho^2 + \left(j_{\theta\theta} + \frac{1}{1 - 2\mu/r}\left(\frac{dr}{d\theta}\right)^2\right)d\theta^2 + \frac{2}{1 - 2\mu/r}\frac{\partial r}{\partial \rho}\frac{\partial r}{\partial \theta}d\rho d\theta + g_{\varphi\varphi}, d\varphi_s^2.$$

(15)

However what we want to do is rewriting from the supertranslated isotropic to the supertranslated Schwarzschild coordinate system. Therefore, using what we have obtained in this subsection, we obtain it in the next subsection.
2.5 Rewriting the supertranslated isotropic coordinate system (7) to supertranslated Schwarzschild coordinate system.

We will obtain the relation between $\rho$ and $r$ in the form “$\rho = \cdots$” to $\varepsilon^1$-order to perform the rewriting of (15) in the opposite direction. For this, there are two ways: 1) to solve (11b) or 2) to solve (12). We can confirm that the same $\rho$ can be obtained from either of them.

Writing what we will do on 2), plugging $\mu(\rho)$ in (13) into the $\mu$ in (12), expand it to $\varepsilon^1$-order. As a result, we obtain $\rho$ as follows:

$$\rho^{(\pm)} = \frac{1}{2}(\pm \sqrt{r(r-2m)} - m + r) + \frac{1}{8}\sqrt{\frac{5}{\pi}}\varepsilon m(3\cos(2\theta) + 1) + O(\varepsilon^2).$$

(16)

Since $\rho^{(-)}$ gets diverged in the large $r$ region, we discard $\rho^{(-)}$ and consider only $\rho^{(+)}$. Therefore, we denote $\rho^{(+)}$ just as $\rho$ in what follows.

Using $\rho$ in (16) and $\mu$ in (13), we can rewrite the supertranslated isotropic coordinate system to the supertranslated Schwarzschild coordinate system to $\varepsilon^1$-order as

$$ds^2 = g_{\mu\nu}dt^2 + g_{\rho\rho}d\rho^2 + g_{\theta\theta}d\theta^2 + g_{\varphi_s\varphi_s}d\varphi_s^2$$

$$\rightarrow -(1 - \frac{2\mu}{r})dt^2 + (1 - \frac{2\mu}{r})^{-1}dr^2 + (g_{\theta\theta} + g_{\rho\rho}(\frac{\partial\rho}{\partial\theta})^2)d\theta^2 + 2g_{\rho\rho}\frac{\partial\rho}{\partial r}\frac{\partial\rho}{\partial\theta}d\rho d\theta$$

$$+ j_{\varphi_s\varphi_s}d\varphi_s^2,$$

(17)

where $g_{\mu\nu}$ are given in (7). We denote (17) as

$$\equiv j_{tt}dt^2 + j_{rr}dr^2 + j_{\theta\theta}d\theta^2 + 2j_{r\theta}dr d\theta + j_{\varphi_s\varphi_s}d\varphi_s^2,$$

(18)

where

- $j_{tt} = -(1 - \frac{2m}{r}) + O(\varepsilon^2)$,
- $j_{rr} = (1 - \frac{2m}{r})^{-1} + O(\varepsilon^2)$,
- $j_{\theta\theta} = r^2 + \frac{3\sqrt{\frac{5}{\pi}}m\cos(2\theta)(\sqrt{r(r-2m)} + r)^4}{2(\sqrt{r(r-2m)} - m + r)^3}\varepsilon + O(\varepsilon^2)$,
- $j_{r\theta} = -\frac{3\sqrt{\frac{5}{\pi}}m\sin(2\theta)(r - \sqrt{r(r-2m)})^4}{8\sqrt{r(r-2m)}(\sqrt{r(r-2m)} - m + r)^3}\varepsilon + O(\varepsilon^2)$,
- $j_{\varphi_s\varphi_s} = r^2\sin^2\theta + \frac{3\sqrt{\frac{5}{\pi}}m\sin^2(2\theta)(\sqrt{r(r-2m)} + r)^4}{8(\sqrt{r(r-2m)} - m + r)^3}\varepsilon + O(\varepsilon^2)$.

2.6 A rotating supertranslated black hole solution with linear order $\varepsilon$ and $a$

In the previous subsection, we have obtained a supertranslated Schwarzschild spacetime with the linear order of $\varepsilon$ as in (18). In this subsection, we obtain a rotating supertranslated black hole solution with the linear order $\varepsilon$ and $a$ by getting back the coordinate $\varphi_s$ in (18) to $\phi$. 

8
Let us denote the $r$ and $a$ after the supertranslation as $r_s$ and $a_s$. Then, $r_s$ has been already obtained in (6a). On the other hand, that $J$ and $ho$ respectively, and $\delta r$ is defined in (19), the top and bottom ones are written in terms of $r$ and $\rho$ as

$$r_s = \rho_s(1 + m/2\rho_s)^2|_{eq.(19)} = r + \mathcal{O}(\varepsilon^2) \equiv r + \delta r,$$

where we have written the correction part as $\delta r$ formally, but $\delta r = 0 + \mathcal{O}(\varepsilon^2)$.

Then, $d\varphi_s$ can be obtained as

$$d\varphi_s = d\phi - \frac{2ma}{(r + \delta r)^3} dt \equiv d\phi + \Theta_r dt,$$

$$= d\phi - \frac{2ma}{(\rho_s(1 + m/2\rho_s)^2)^3} dt \equiv d\phi + \Theta_\rho dt,$$

where $\delta r$ is defined in (19), and $\Theta_r$ and $\Theta_\rho$ are given using

$$\Theta_r \equiv -\frac{2a}{r^3} + \mathcal{O}(\varepsilon^2),$$

$$\Theta_\rho \equiv -\frac{128am\rho^3}{(m + 2\rho)^6} + \frac{48\sqrt{\frac{\pi}{2}}a\varepsilon m^2 \rho^2(3\cos(2\theta) + 1)(m - 2\rho)}{(m + 2\rho)^7} + \mathcal{O}(\varepsilon^2).$$

Plugging (20) into (18), we can get a black hole spacetime with the linear order $\varepsilon$ and $a$ as

$$ds^2 = J_{tt}dt^2 + J_{rr}dr^2 + J_{\theta\theta}d\theta^2 + J_{\phi\phi}d\phi^2 + 2J_{\phi\theta}d\phi d\theta + \mathcal{O}(a^2) + \mathcal{O}(\varepsilon^2),$$

where $J_{MN}$ are given using $j_{\mu\nu}$ and $\Theta_r$ as

$$J_{MN} = \begin{pmatrix}
 j_{tt} & 0 & 0 & j_{\varphi\varphi}, \Theta_r \\
 0 & j_{rr} & j_{r\theta} & 0 \\
 0 & j_{r\theta} & j_{\theta\theta} & 0 \\
 j_{\varphi\varphi}, \Theta_r & 0 & 0 & j_{\varphi\varphi}
 \end{pmatrix} + \mathcal{O}(a^2) + \mathcal{O}(\varepsilon^2).$$

- $J_{tt} = -1 + 2m/r$,
- $J_{t\varphi} = -1 + 2m/r - \frac{3\sqrt{\frac{\pi}{2}}am^2 \sin^2(2\theta)(\sqrt{r(r - 2m)} + r)^4}{4r^3(\sqrt{r(r - 2m)} - m + r)^3} \varepsilon$,
- $J_{rr} = (1 - 2m/r)^{-1}$,
- $J_{r\theta} = -\frac{3\sqrt{\frac{\pi}{2}}m \sin(2\theta)(\sqrt{r(r - 2m)} + r)^4}{8\sqrt{r(r - 2m)}(\sqrt{r(r - 2m)} - m + r)^3} \varepsilon$,
- $J_{\theta\theta} = r^2 + \frac{3\sqrt{\frac{\pi}{2}}m \cos(2\theta)(\sqrt{r(r - 2m)} + r)^4}{2(\sqrt{r(r - 2m)} - m + r)^3} \varepsilon$,
- $J_{\varphi\varphi} = r^2 \sin^2 \theta + \frac{3\sqrt{\frac{\pi}{2}}m \sin^2(2\theta)(\sqrt{r(r - 2m)} + r)^4}{8(\sqrt{r(r - 2m)} - m + r)^3} \varepsilon$. 


We can check these can satisfy Einstein equation to $\varepsilon^1$- and $a^1$-orders.

### 2.7 A rotating supertranslated black hole solution with linear order $\varepsilon$ and arbitrary $a$

In this subsection, we first obtain a coordinate transformation rule which enables us to directly transform (3) to (18), where (3) and (18) are respectively

- (3) : Schwarzschild BH spacetime in the isotropic coordinate system,
- (18) : Schwarzschild BH spacetime in the isotropic coordinate system with $\varepsilon^1$.

Using that rule, we actually do it above in Sec.2.7.1.

Then, in Sec.2.7.2, using that rule in (1), the rotating black hole spacetime in the BL coordinate system before the supertranslation is performed, we obtain a rotating supertranslated black hole spacetime with the linear order supertranslation correction and arbitrary black hole rotation. We can check it can satisfy the Einstein equation to $\varepsilon^1$-order for arbitrary $a$, where we describe the reason that calculating to $\varepsilon^1$-order would be enough in Sec.2.3.

First of all, let us obtain $\theta_s$ and $\varphi_s$ in terms of $z$. For this purpose, we use the stereographic map, $z_s = e^{i\varphi_s}\cot\theta_s/2$, then $\theta_s$ and $\varphi_s$ can be written in terms of $z_s$ as $\theta_s = 2\cot^{-1}|z_s|$, $\varphi_s = \frac{1}{2}\ln z_s z_s^{-1}$. The expression of $z_s$ in terms of $z$ is given as

$$z_s = \frac{(z\bar{z} - 1)(\rho - C) + (z\bar{z} + 1)(\rho_s - \bar{z}\partial_z C - z\partial_z C)}{2\bar{z}(\rho - C) + (z\bar{z} + 1)(\bar{z}\partial_z C - \partial_z C)},$$  \hspace{1cm} (23)

where the one above is given in (101c), and $C$ and $\rho_s$ are (5) and (19).

We plug $\rho_s$ in (19) and $C$ in (5) into the $z_s$ above. Here, these $\rho_s$ and $C$ are given in terms of $z$ by rewriting $\theta$ and $\varphi$ in these in terms of $z$ using the stereographic map. Then, we can obtain $\theta_s$ and $\varphi_s$ as

$$\theta_s = \theta + \frac{3m}{2(r + \sqrt{(r - 2m)r - m})} \sqrt{\frac{5}{\pi}} \sin(2\theta) + O(\varepsilon^2) \equiv \theta + \delta\theta,$$ \hspace{1cm} (24a)

$$\varphi_s = \varphi + O(\varepsilon^3),$$ \hspace{1cm} (24b)

where no corrections with regard to $\varepsilon$ at $\varepsilon^1$- and $\varepsilon^2$-orders are involved in $\varphi_s$. From (24b), (19) and $d\varphi$ given under (2), we can see $\phi_s$ is obtained as

$$\phi_s = \phi + O(\varepsilon^3).$$  \hspace{1cm} (25)

### 2.7.1 Directly obtain $j_{MN}$ in (18) from (2)

We have obtained $j_{\mu\nu}$ in (18) from (2) once rewriting the coordinate system to the isotropic coordinate system (3) as shown in Fig.1, however now we can directly obtain $j_{MN}$ from (2) by performing the following substitution,

$$(r, \theta, \varphi) \rightarrow (r_s, \theta_s, \varphi_s),$$  \hspace{1cm} (26)

\footnote{Obtaining $J_{MN}$ to $\varepsilon^2$-order (the order of $a$ is linear) by using the results in [12], author has checked that these can satisfy the Einstein equation to $a^1$ and $\varepsilon^2$-orders.}
as
\[
\text{ds}^2 = K_{\text{MN}} \text{dx}^M \text{dx}^N + O(\varepsilon^2),
\] (29)

where \( K_{\text{MN}} \) above can satisfy the Einstein equation to \( \varepsilon^1 \)-order for arbitrary \( a \). We can also check that \( K_{\text{MN}} \) above can be identical to the slowly rotating metrics \([22]\) if expanding these to the linear order with regard to \( a \).
3 Relation with angular momentum of spacetime

In this section, we would like to give attention to the relation between the metrics we have obtained (29) and the angular momentum of the spacetime. For this purpose, we first check that (29) can be regarded as a rotating spacetime. Then, we can see that (29) are invariant under the translation of \( t \) and the rotation of \( \phi \), and the \( \varepsilon^0 \)-order of these are the same with the Kerr black hole metrics in the Boyer-Lindquist coordinates. Thus, (29) would be some kinds of the rotating black hole spacetime around \( \phi \)-direction. Actually, \( a \) appearing in (29) is closely related with the angular momentum of the rotating spacetime around the \( \phi \)-direction, which we show in what follows.

To this purpose, we first consider some small gravitational perturbations from a stationary axisymmetric rotating object, in general. As its expression at the far region, let us write as

\[
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},
\]

where \( \eta_{\mu\nu} \) are the metrics for the flat Minkowski spacetime and \( h_{\mu\nu}(t, r, \theta) \) are the perturbations from these. Note that \( h_{\mu\nu} \) are independent of \( \phi \) refactoring our assumption that the spacetime is axisymmetric toward \( \phi \).

Now, let us consider the Einstein equation,

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}.
\]

in the spherical coordinates \((t, r, \theta, \phi)\) (therefore \( \eta_{\mu\nu} \) is given as \( \text{diag}(-1, 1, r^2, r^2 \sin^2 \theta) \)). Then from its \( t\phi \)-component, we can obtain the following equation:

\[
\partial^2 r h_{t\phi} + \frac{\sin \theta}{r^2} \partial_\theta (\frac{\partial_\theta h_{t\phi}}{\sin \theta}) = -\frac{16\pi G}{c^4} T_{t\phi}.
\]

This equation can be rewritten as

\[
\Delta (\frac{\cos \phi h_{t\phi}}{\sin \theta / r}) = -\frac{16\pi G}{c^4} \frac{\cos \phi}{r \sin \theta} T_{t\phi},
\]

where \( \Delta \) means the Laplacian on the flat space: \( \Delta = \partial_x^2 + \partial_y^2 + \partial_z^2 = r^{-2} \partial_r (r^2 \partial_r) + r^{-2} \sin^{-1} \theta \partial_\theta (\sin \theta \partial_\theta) + r^{-2} \sin^{-2} \theta \partial_\phi^2 \). Therefore, using the Green function toward \( \Delta \), which is \( -\frac{1}{4\pi |x-x'|} \) (here, \( \Delta = \frac{1}{|x-x'|} = -4\pi \delta^3(\vec{x} - \vec{x}') \)), we can get

\[
\frac{\cos \phi h_{t\phi}}{\sin \theta / r} = \frac{4G}{c^4} \int_\Omega d^3\vec{x}' \frac{1}{|\vec{x} - \vec{x}'|} \frac{\cos \phi'}{r' \sin \theta'} T_{t\phi}(\vec{x}'),
\]

where \( \Omega \) means the space where the rotating object exists, and \( r = |\vec{x}|, r' = |\vec{x}'| \). Then, supposing that the stationary axisymmetric rotating object is very small compared with the \( r \), (33) can be written as (we note the calculation process from (33) to (34) in Appendix B)

\[
(33) = \frac{2G \sin \theta \cos \phi}{c^4} \int_\Omega d^3\vec{x}' (1 + \frac{3r'}{r} \cos \theta \cos \theta') T_{t\phi}(\vec{x}') + O(r^{-4}).
\]
In general, the angular momentum for an astronomical body can be written as follows:

\[ J = -c^{-1} \int_{\Omega} d^3\vec{x} T_{t\phi}. \] (35)

Using this, we can obtain \( h_{t\phi} \) as

\[ h_{t\phi} = -2G \frac{\sin^2 \theta}{c^3} J' + \mathcal{O}(r^{-3}), \quad J' = J - \frac{3}{c} \cos \theta \int_{\Omega} d^3\vec{x}' r' \cos \theta' T_{t\phi}(\vec{x}'). \] (36)

Let us stress that the \( h_{t\phi} \) above is the general gravitational field made by a stationary axisymmetric rotating object, at the far region.

Now, we compare (36) with our metrices (29). Then, since the \( t\phi \)-component of our metrices (29) can be expanded around infinite \( r \) as

\[ K_{t\phi} = -2am \sin^2 \theta \frac{J'}{r} - 6\sqrt{\frac{5}{\pi}} am \sin^2 \theta \cos^2 \theta \frac{\varepsilon}{r^2} + \mathcal{O}(r^{-3}), \] (37)

(37) can agree to (36) by the following identification:

\[ a = \frac{GJ'}{c^3 m} - \frac{3\sqrt{\frac{5}{\pi}} G J \cos^2 \theta \varepsilon}{c^3 m} r + \mathcal{O}(r^{-2}). \] (38)

where \( J' \) is given in (36). In conclusion, the \( a \) appearing in our metrices (29) is related with the angular momentum of the spacetime as above, including \( \varepsilon^1 \).

4 Antibunching of scalar particles in supertranslated rotating black hole spacetime we have obtained

In this section, we discuss the effect of the supertranslation and rotation in our spacetime (29) in the antibunching of scalar particles at the \( r \to \infty \) region. We explain what the antibunching is in Sec.\( \frac{1}{4} \\text{7} \). Currently the main measure for the supertranslation would be the memory effect. What we discuss in this section would become a new way for that.
4.1 Metrices we consider

In this study we perform our analysis in the large $r$ region of the spacetime. Therefore we consider the metrices of (2.7.2) expanded from the inifity to $r^{-1}$-order as

- $K_{\alpha\alpha} = \left( -1 + \frac{2m}{r} + O(r^{-2}) \right) + \varepsilon O(r^{-4}) + O(\varepsilon^2), \quad (39a)$
- $K_{\phi\phi} = -\frac{2am \sin^2(\theta)}{r} + O(r^{-2}) + \varepsilon O(r^{-3}) + O(\varepsilon^2), \quad (39b)$
- $K_{rr} = 1 + \frac{2m}{r} + O(r^{-2}) + \varepsilon O(r^{-3}) + O(\varepsilon^2), \quad (39c)$
- $K_{r\theta} = -3\sqrt{\frac{5\pi}{m}} \varepsilon \sin(2\theta) \frac{r}{4} + O(r^{-2}) + O(\varepsilon^2), \quad (39d)$
- $K_{\theta\theta} = \left( r^2 + a^2 \cos^2(\theta) + O(r^{-2}) \right) + \varepsilon \left( 3\sqrt{\frac{5\pi}{m}} \cos(2\theta)(r + m) \frac{2a^2 + 5m^2}{16r} \cos(2\theta) + O(r^{-2}) \right) + O(\varepsilon^2), \quad (39e)$
- $K_{\phi\phi} = r^2 \sin^2(\theta) + a^2 \sin^2(\theta) + \frac{2a^2m \sin^4(\theta)}{r} + O(r^{-2}), \quad (39f)$

Other components not written above are all zero in the analysis in this section.

The contravariant metrices of (39) to $r^{-2}$- and $\varepsilon$-orders are given as follows:

- $K^{\alpha\alpha} = diag(-1, 1, 1, 1) + O(r^{-3})$

It is held $K^{\mu\nu}K_{\mu\nu} = diag(-1, 1, 1, 1) + O(r^{-3})$.

4.2 Field theory we consider

We consider the following model,

$$L = \sqrt{-K} \left( \frac{1}{2} K^{MN} \nabla_M \phi \nabla_N \phi + \frac{\omega^2}{2} \phi^2 - \frac{\lambda}{4!} \phi^4 \right), \quad (41)$$

where $\phi$ is the scalar field and $\nabla_M$ are the covariant derivatives given with the metrices (39).
4.3 Classical solution with classical correction

We can write the equation of motion obtained from (41) as follows:

\[
(K^{MN} \nabla_M \nabla_N - \omega^2) \phi = -\frac{\lambda}{3!} \phi^3, \tag{42}
\]

where \(\Delta\) is defined under (32). Note that our metrics \(K_{MN}\) becomes to \(\text{diag}(- + + +)\) at \(r \to \infty\), so the Einstein relation for \(\phi\) is given as \(-k_0^2 + \vec{k}^2 = -m^2\) at \(r \to \infty\).

We take \(\phi^\text{as}\) as follows:

\[
\phi^\text{as} = \cos \omega t. \tag{43}
\]

\(\phi^\text{as}\) is the solution for (42) at \(r \to \infty\) and \(\lambda = 0\), as follows:

\[
(-\partial_t^2 + \Delta - \omega^2) \phi^\text{as} = 0, \tag{44}
\]

With this, let us formally write the solution of \(\phi\) perturbatively with regard to \(\lambda\) as

\[
\phi = \phi_0 + \phi_1 + \cdots, \tag{45}
\]

where \(\phi_0 = \phi^\text{as}\). With the \(\phi\) above, we write the equation of motion (42) as follows:

\[
(-\partial_t^2 + \Delta - \omega^2) \phi = -\frac{\lambda}{3!} \phi^3 - \frac{2m}{r^2} \partial_r \phi + \left(\frac{2m}{r} - \frac{4m^2}{r^2}\right) \partial^2_r \phi + \left(\frac{2m}{r} + \frac{4m^2}{r^2}\right) \partial^2_t \phi
- \frac{3}{4r^2} \sqrt{\frac{5}{\pi}} \varepsilon \sin(2\theta) + O(r^{-3}) + O(\varepsilon^2) \equiv J(\phi). \tag{46}
\]

Then, we can write \(\phi_1\) as

\[
\phi_1(x) = -\int d^4 y \sqrt{-g(y)} G_R(x - y) J(\phi_0(y)). \tag{47}
\]

where \(G_R(x - y)\) is defined at the \(r \to \infty\) region as

\[
(-\partial_t^2 + \Delta - \omega^2) G_R(x - y) = -\delta^4(x - y), \quad \lim_{x^0 \to -\infty} G_R(x - y) = 0, \tag{48}
\]

and is given generally as

\[
G_R(x - y) = \frac{1}{4\pi} \left( \left(\frac{1}{r_{xy}} - \frac{1}{r'_{xy}}\right) - \frac{mJ_1(m \sqrt{t^2 - r_{xy}^2})}{\sqrt{t^2 - r_{xy}^2}} \Theta(t_{xy} - r_{xy}) \right). \tag{49}
\]

\(\Theta\) and \(J_1\) are the step and Bessel function of the first kind, and \(x\) and \(y\) are the coordinates of the positions of the field we want to get and fields to affect it. \(t_{xy}\) and \(r_{xy}\) mean \(x^0 - y^0\) and \(\sqrt{(x^1 - y^1)^2 + (x^2 - y^2)^2 + (x^3 - y^3)^2}\), respectively.
### 4.4 Evaluation of classical correction

Among all the terms in $\phi_1$ in (47), we will evaluate the contribution to be finally contained into $\phi_0$ and deform the original $\phi_0$. This can be considered a classical renormalization.

We write (45) to $\phi_1$ explicitly in what follows:

$$
\phi(x) = \phi^0 - \int d^4y \sqrt{-K(y)} G_F(x - y) \left( \frac{\lambda}{3!} \cos^3(\omega y^0) + \frac{2m}{r_{xy}} + \frac{4m^2}{r_{xy}^2} \omega^2 \cos^2(\omega y^0) \right)
$$

$$
= \phi^0 + \frac{\lambda}{3!} \int d^4y \sqrt{-K(y)} \frac{\delta(t_{xy} - r_{xy})}{4\pi r_{xy}} \cos^3(\omega y^0) + \frac{2m}{r_{xy}} + \frac{4m^2}{r_{xy}^2} \omega^2 \cos^2(\omega y^0) + \cdots,
$$

where

- $t_{xy}$ and $r_{xy}$ are given under (43),
- “...” in the second line means terms coming from the second term in the r.h.s. in (49), which would not be contained into $\phi_0$ in (58). Therefore these are finally irrelevant in the analysis in this section, and we disregard in what follows,
- “...” in the third line means terms at $r^{-2}$ and $r^{-3}$-orders, which would also not be contained into $\phi_0$ in (58), as these would be corrections in (55) for negative powers of $r$ of those. Thus we disregard these in what follows as well.

Taking the spherical polar coordinates, we will write (50) again but explicitly as

$$
\phi(t, r, \theta, \phi) = \phi^0(t, r, \theta, \phi) + \frac{\lambda}{3!} \int_{-\infty}^{\infty} dt' \int_{r}^{r+\Delta r} dr' \int_{0}^{\pi} d\theta' \int_{0}^{2\pi} d\phi' \sqrt{-K_{\mu\nu}(t', r', \theta', \phi')} \frac{\delta(t - t' - |r - r'|) \cos^3(\omega t')}{4\pi |r - r'|} \cos^3(\omega^0) \delta(t - t' - |r - r'|) \cos^3(\omega t') + \cdots,
$$

where we suppose that interactions of fields separated by the radius more than $\Delta r$ toward $\phi$ at $r$ are ignorable.

We evaluate the part $\sqrt{-K}$ in (51) by the following steps; 1) first, expand it to $a^2$- and $\varepsilon^2$-orders, 2) then expand it with regard to $r$ from the infinity to $r^{-1}$-order. As a result, we get

$$
(51) = \phi^0 + \frac{\lambda}{3!} \int dt' dr' d\theta' d\phi' (\Sigma_2 t' + \Sigma_1 r' + \Sigma_0 + \frac{\Sigma_{-1}}{r'} + \frac{\Sigma_{-2}}{r'^2}) \cos^3(\omega t') \delta(t - t' - |r - r'|) + \cdots,
$$

where “...” are subleading terms in the expansions mentioned above, which are irrelevant.
in the analysis in this section and we ignore in what follows. \( \Sigma_i (i = \pm 2, \pm 1, 0) \) mean

\[
\Sigma_2 = 3 \sqrt{\frac{5}{\pi}} \varepsilon \sin(\theta')(3 \cos(2\theta') + 1)(4a^2 \cos^2(\theta') + 69m^2) + 16a^2 m \sin^3(\theta'), \tag{53a}
\]

\[
\Sigma_1 = \sin(\theta')(m(32m - 3 \sqrt{\frac{5}{\pi}} \varepsilon (3 \cos(2\theta') + 1)) - a^2 (\cos(2\theta') + 3)), \tag{53b}
\]

\[
\Sigma_0 = 3 \sqrt{\frac{5}{\pi}} \varepsilon \sin(\theta')(3 \cos(2\theta') + 1)(27m^2 - 4a^2 \cos^2(\theta')) - 16a^2 m \sin^3(\theta'), \tag{53c}
\]

\[
\Sigma_{-1} = m \sin(\theta')(a^2 (\cos(2\theta') + 3) + 3 \sqrt{\frac{5}{\pi}} m (3 \cos(2\theta') + 1)), \tag{53d}
\]

\[
\Sigma_{-2} = m \sin(\theta')(3 \sqrt{\frac{5}{\pi}} \varepsilon (3 \cos(2\theta') + 1) (4a^2 \cos^2(\theta') + 5m^2) + 16a^2 m \sin^2(\theta')). \tag{53e}
\]

The important term in (52) is \( \Sigma_2 r'^2 \) in the sense its contribution is finally contained into \( \phi_0 \) and deform the original \( \phi_0 \) (it turns out in the actual calculation that other terms are not affect \( \phi_0 \) for the insufficiency of the power of \( r' \)). Therefore focusing \( \Sigma_2 r'^2 \), we perform integrals of \( \theta' \) and \( \phi' \). As a result we get as

\[
\int 54 = \phi^0 + \frac{a^2}{32m^3} \left( \frac{\varepsilon}{\sqrt{5\pi}} + m \right) \lambda \frac{1}{3!} r \int dt' dr' r'^2 \cos^3(\omega t') \delta(t - t' - |r - r'|) + \cdots, \tag{54}
\]

where “\( \cdots \)” are terms other than \( \Sigma_2 r'^2 \), which are ignorable in the sense mentioned above.

In (54), it should be held, \( t - t' - (r' - r) = 0 \). Therefore, it is calculated

\[
\int_{r - \Delta r}^{r + \Delta r} dt' dr' r'^2 \left( -3r \omega \cos(3\omega t) - 81r \omega \cos(\omega t) + \omega \{81 \cos(\omega (\Delta r + t)) + 3 \cos(3\omega (\Delta r + t))\} + \{82 - 45 \omega^2 r^2 + (2 - 9 \omega^2 r^2) \cos(2\omega t)\} \sin(\omega t) + \{82 + 45 (\Delta r)^2 \omega^2 + (-2 + 9 \omega^2 (r + \Delta r)^2) \cos(2\omega (\Delta r + t))\} \sin(\omega (\Delta r + t))\}
\]

\[
= -45r \frac{\sin(\omega t)}{54r} + \cdots, \tag{55}
\]

where “\( \cdots \)” are terms not to be contained into \( \phi_0 \) in (58) and irrelevant in the analysis in this section. With (55), it is led

\[
\int 54 = \phi^0 - \frac{5a^2 \lambda}{1152m^3} \left( \frac{\varepsilon}{\sqrt{5\pi}} + \frac{m}{3} r \right) \sin(\omega t) + \cdots
\]

\[
= \cos(\omega t) - \Sigma(a, \varepsilon) \lambda r \sin(\omega t) + \cdots, \tag{56}
\]

\[
\Sigma(a, \varepsilon) \equiv \frac{5a^2}{1152m^3} \left( \frac{\varepsilon}{\sqrt{5\pi}} + \frac{m}{3} \right), \quad \text{and } \phi^0 \text{ is given in (43).}
\]
We suppose
\[ r = \gamma t. \] (57)
This means that the field we observe moves at a constant rate with pass of time in the radial direction. Then we can finally write (56) as
\[
\phi = \sqrt{1 + (-\Sigma(a, \varepsilon) \lambda r)^2} \cos(\omega t - \tan^{-1}(-\Sigma(a, \varepsilon) \lambda \gamma t)) + \cdots
\]
\[ = \cos(\Omega t) + \cdots, \quad \Omega \equiv \omega + \delta \omega, \quad \delta \omega \equiv \Sigma(a, \varepsilon) \lambda \gamma, \] (58)
where
\[ A \cos \theta + B \sin \theta = \sqrt{A^2 + B^2} \cos(\theta - \delta), \quad \delta = \tan^{-1} B/A \approx B/A \quad \text{for} \quad B/A \ll 1, \]
and “\cdots” are corrections toward the leading free field, \( \cos(\Omega t) \), which will vanish at \( r \to \infty \) region. In terms of observational practicality, it is supposed \( \gamma \ll c \). The point in the one above is that the term \(-\Sigma(a, \varepsilon) \lambda r \sin(\omega t)\) in (56), a part of the correction term \( \phi_1 \) in (47), is contained into the free leading field \( \phi_0 \) and deform it.

Let us comment about what we have done in this subsection. We have shown that some of the interaction effects deform the leading free field as it is involved into that, at the classical level, with suppositions (43) and (57). This is a classical renormalization, and the realizing classical behaviour of the leading free field would be such a classically renormalized one given with the mass \( \Omega \). We will study the quantum theory of such a deformed leading free field in the next subsection.

### 4.5 Quantum solution with classical correction in the \( r \to \infty \) region

In this subsection, we will discuss the quantum theory of the leading free field modified for the classical corrections. For this purpose, we have to obtain the Hamiltonian corresponding to our Lagrangian (41), which we can obtain as
\[
\hat{\mathcal{H}} = -\frac{1}{2} \hat{\pi}^2 + \mathcal{K}_{ii} \hat{\pi} \partial_i \hat{\phi} + \frac{1}{2} \mathcal{K}_{ij} \partial_i \hat{\phi} \partial_j \hat{\phi} + \frac{m^2}{2} \hat{\phi}^2 + \frac{\lambda}{4!} \hat{\phi}^4, \] (59)
where \( \hat{\pi} \) is the quantum theory version of the canonical momentum obtained as
\[
\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \mathcal{K}^{tt} \partial_0 \phi \] (60)
and
\[
[\hat{\phi}(x), \hat{\pi}(y)]|_{x^0 = y^0} = i \frac{\delta^3(x-y)}{\sqrt{-g}}, \quad [\hat{\phi}(x), \hat{\phi}(y)]|_{x^0 = y^0} = [\hat{\pi}(x), \hat{\pi}(y)]|_{x^0 = y^0} = 0. \] (61)
From (59), we can obtain the equation of motion as
\[
(-\partial_t^2 + g^{ij} \partial_i \partial_j - \omega^2)\hat{\phi}(x) = \frac{\lambda}{3!} \delta^3(x) \equiv \hat{J}(\hat{\phi}(x)). \] (62)
This equation of motion is the same one obtained from our Lagrangian (11), and the equation of motion \( \hat{\phi}^{as} \) should satisfy is given as
\[
(-\partial_t^2 + g^{ij} \partial_i \partial_j - \omega^2) \hat{\phi}^{as} = 0. \tag{63}
\]

Note that (62) agrees to the one obtained our Lagrangian (41), however since our metrices (39) is not the so-to called 1+3 decomposition form for \( K_t \phi \) not ignorable, there is a concern that the equations of motion obtained from Lagrangian and Hamiltonian get different each other, generally speaking. Therefore, we show the derivation process of (62) in Appendix C.

As such, \( \hat{\phi}^{as} \) would be considered to be given with \( \omega \) according to (63), however \( \hat{\phi}^{as} \) finally realized would be, denoting it as \( \hat{\phi}^{def} \), given as follows:
\[
\hat{\phi}^{def}(x) = \int \frac{d^3 \vec{p}}{(2\pi)^3 E^{def}(\vec{p})} (a^{def}(\vec{p}) e^{-i p^{def} x} + a^{def\dagger}(\vec{p}) e^{i p^{def} x}), \tag{64}
\]
\[
p^{def} x = -E^{def}(\vec{p}) x^0 + \vec{p} \cdot \vec{x}, \quad E^{def}(\vec{p}) = \sqrt{p^2 + \Omega^2} = E^{as}(\vec{p}) + \frac{\delta \omega}{\sqrt{p^2 + \omega^2}},
\]
where
- \( \Omega \) is defined in (58), and \( E^{as}(\vec{p}) \equiv \sqrt{p^2 + \omega^2} \),
- “def” in \( E^{def} \) means “deformed”,
- \( a^{def}(\vec{p}) \) and \( a^{def\dagger}(\vec{p}) \) are the annihilation and creation operators for the excited state of \( \hat{\phi}^{def}(x) \) satisfying the following commutation relations:
\[
[a^{def}(\vec{p}), a^{def\dagger}(\vec{q})] = \delta^3(\vec{p} - \vec{q}), \quad [a^{def}(\vec{p}), a^{def}(\vec{q})] = [a^{def\dagger}(\vec{p}), a^{def\dagger}(\vec{q})] = 0. \tag{65}
\]

4.6 Number operator of deformed leading free field in terms of leading free field

In this subsection, we calculate the number operator of deformed leading free field \( \hat{\phi}^{def} \) in terms of leading free field \( \hat{\phi}^{as} \) with the classical correction \( \delta \omega \).

We write \( \hat{\phi}^{def} \) and \( \hat{\phi}^{as} \) with \( f^{def}_p \) and \( f^{as}_p \) in the following:
\[
\hat{\phi}^{def}(x) = \int d^3 \vec{p} (a^{def}(\vec{p}) f^{def}_p(x) + a^{def\dagger}(\vec{p}) f^{def\dagger}_p(x)), \tag{66a}
\]
\[
\hat{\phi}^{as}(x) = \int d^3 \vec{p} (a^{as}(\vec{p}) f^{as}_p(x) + a^{as\dagger}(\vec{p}) f^{as\dagger}_p(x)), \tag{66b}
\]
where
\[
f^{def}_p(x) \equiv \frac{e^{-ip^{def} x}}{\sqrt{(2\pi)^3 E^{def}(\vec{p})}}, \quad f^{as}_p(x) \equiv \frac{e^{-ip^{as} x}}{\sqrt{(2\pi)^3 E^{as}(\vec{p})}}. \tag{67}
\]
We can obtain the following one:\footnote{It can be written: }$i\hat{a}_{\text{def}}(\hat{p}) = \int d^3x \hat{\phi}_{\text{def}}(x) \hat{\partial}_0 f_{\text{def}}^+(x)$ and $-i\hat{a}_{\text{def}}(\hat{p}) = \int d^3x \hat{\phi}_{\text{def}}(x) \hat{\partial}_0 f_{\text{def}}^-(x)$. Then if we calculated the r.h.s. of these expanding these to the linear order of $\delta \omega$, the results of these should be identical to $i\hat{a}_{\text{def}}(\hat{p})$ and $-i\hat{a}_{\text{def}}(\hat{p})$, respectively. We show this in Appendix.  

\begin{align}
E_{\text{def}}(\hat{p}) & = \sqrt{\hat{p}^2 + (\omega + \delta \omega)^2} = E_{\text{as}}(\hat{p}) + \delta \omega, \quad (68a) \\
\frac{1}{\sqrt{E_{\text{def}}(\hat{p})}} & = \frac{1}{\sqrt{E_{\text{as}}(\hat{p})}} (1 - \frac{\delta \omega}{2E_{\text{as}}(\hat{p})}), \quad (68b) \\
p^\text{def} x = -E_{\text{def}}(\hat{p}) x^0 + \hat{p} \cdot \vec{x} = p_{\text{as}} x - x^0 \delta \omega \quad (p_{\text{as}} x = -E_{\text{as}}(\hat{p}) x^0 + \hat{p} \vec{x}), \quad (68c) \\
e^{-ip_{\text{def}} x} & = e^{-ip_{\text{as}} x} (1 + i\delta x^0 \omega), \quad (68d) \\
f_{\hat{p}}^\text{def}(x) & = \frac{e^{-ip_{\text{def}} x}}{\sqrt{(2\pi)^3 E_{\text{def}}(\hat{p})}} f_{\hat{p}}^{\text{as}}(x) (1 + i\beta_+ \delta \omega), \quad (\beta_+ \equiv x^0 \pm \frac{i}{2 E_{\text{as}}(\hat{p})}). \quad (68e)
\end{align}

With these, $\hat{\phi}_{\text{def}}(x)$ to the linear order of $\delta \omega$ can be given as follows:

\begin{align}
\hat{a}_{\text{as}}(\hat{p}) & = \int d^3\vec{p} (a_{\text{def}}(\hat{p})(1 + i\beta_+ \delta \omega) f_{\hat{p}}^{\text{as}}(x) + a_{\text{def}}^\dagger(\hat{p})(1 - i\beta_+ \delta \omega) f_{\hat{p}}^{\text{as}}(x)), \quad (69)
\end{align}

Comparing this with (66b), we can obtain

\begin{align}
a_{\text{def}}^\dagger(\hat{p}) & = a_{\text{def}}(\hat{p})(1 + i\beta_+ \delta \omega), \quad a_{\text{as}}(\hat{p}) = a_{\text{def}}^\dagger(\hat{p})(1 - i\beta_+ \delta \omega). \quad (70)
\end{align}

From this, we can obtain the number operator mentioned in the beginning of this subsection as

\begin{align}
\hat{n}_{\text{def}}(\hat{p}) & = \frac{a_{\text{as}}(\hat{p})}{(1 + i\beta_+ \delta \omega)(1 - i\beta_+ \delta \omega)} = (1 + \frac{\delta \omega}{E_{\text{as}}(\hat{p})}) \hat{n}_{\text{as}}(\hat{p}) + \mathcal{O}(\delta \omega^2), \quad (71)
\end{align}

where $\hat{n}_{\text{def}}(\hat{p}) = a_{\text{def}}(\hat{p}) a_{\text{def}}^\dagger(\hat{p})$ and $\hat{n}_{\text{as}}(\hat{p}) = a_{\text{as}}(\hat{p}) a_{\text{as}}^\dagger(\hat{p})$, and $\delta \omega$ and $E_{\text{as}}(\hat{p})$ are defined in (58) and (64).

### 4.7 General theory of bunching and antibunching

Considering an experimental equipment as in Fig. 2, we try to detect a particle in some interval $0 \leq t \leq t_0 + dt_1$ and in some interval $t_0 + \tau \leq t \leq t_0 + \tau + dt_1$. The probability we can get it is proportional to

\begin{align}
I(t)I(t + \tau)dt_1dt_2, \quad (72)
\end{align}

where $I(t)$ means the intensity of the beam at $t$. We denote its average for $t$ as

\begin{align}
\langle I(t)I(t + \tau) \rangle. \quad (73)
\end{align}

This is the correlator of the intensities, and we consider that normalized as

\begin{align}
g^{(2)}(\tau) = \frac{\langle I(t)I(t + \tau) \rangle}{\langle I(t) \rangle \langle I(t + \tau) \rangle}. \quad (74)
\end{align}

(This \footnote{} is called “second-order normalized correlation function” or “correlation function of intensities” in the quantum optics.)
4.7.1 Bunching (classical theory)

Let us first discuss $g^{(2)}(\tau)$ in (74) in the classical theory. In this case, the intensity $I(t)$ is c-number. Then, since it is supposed, $\langle I(t)I(t + \tau)\rangle = \langle I(t)\rangle\langle I(t + \tau)\rangle$ at $\tau \to \infty$, $g^{(2)}(\tau)$ at $\tau \to \infty$ would be given as

$$\lim_{\tau \to \infty} g^{(2)}(\tau) = 1.$$ (75)

On the other hand, at $\tau \to 0$, $g^{(2)}(\tau)$ can be written as

$$g^{(2)}(0) = \frac{\langle I(t)^2 \rangle}{\langle I(t) \rangle^2} = 1 + \frac{\langle (I(t) - \langle I(t) \rangle)^2 \rangle}{\langle I(t) \rangle^2} \geq 1.$$ (76)

where it is always held that $\langle (I(t) - \langle I(t) \rangle)^2 \rangle, \langle I(t) \rangle^2 \geq 0$. With these, we can get

$$g^{(2)}(\tau) \geq 1 \text{ for any } \tau, \text{ and } \lim_{\tau \to \infty} g^{(2)}(\tau) = 1.$$ (77)

The result (77) means that $g^{(2)}(\tau)$ gets larger than or equal to 1 for two lights we observe putting a time interval $\tau$. This phenomenology, is called “bunching” (which is a key phenomenology in the quantum optics).

However, this bunching is the one in the classical theory, and gets changed remarkably when we consider it in the quantum theory, which we show in the following.

4.7.2 Antibunching (quantum theory)

In general (if we considered the following discussion in our case, consider replacing as $\hat{a}$ with $\hat{a}^{\text{def}}$), the intensity $I(t)$ can be given as $a(t)a(t)$ at the quantum theory. Therefore, $g^{(2)}(\tau)$ can be given as

$$g^{(2)}(\vec{k}, \tau) = \frac{\langle \hat{a}^{\dagger}(\vec{k}, t)\hat{a}^{\dagger}(\vec{k}, t + \tau)\hat{a}(\vec{k}, t + \tau)\hat{a}(\vec{k}, t) \rangle}{\langle \hat{I}(\vec{k}, t) \rangle\langle \hat{I}(\vec{k}, t + \tau) \rangle},$$ (78)

where $\langle \cdots \rangle$ in the one above means the inner product of the quantum theory.

At $\tau \to \infty$, since $\hat{a}(\vec{k}, t + \tau)$ and $\hat{a}(\vec{k}, t)$ are supposed to be possible to commute each other, it is led,

$$\lim_{\tau \to \infty} g^{(2)}(\vec{k}, \tau) = 1.$$ (79)

On the other hand, $g^{(2)}(\tau)$ at $\tau \to 0$ at the quantum theory can be written as

$$\lim_{\tau \to 0} g^{(2)}(\vec{k}, \tau) = \lim_{\tau \to 0} \frac{\langle \hat{a}^{\dagger}(\vec{k}, t)\hat{a}^{\dagger}(\vec{k}, t + \tau)\hat{a}(\vec{k}, t + \tau)\hat{a}(\vec{k}, t) \rangle}{\langle \hat{I}(\vec{k}, t) \rangle\langle \hat{I}(\vec{k}, t) \rangle} = 1 + \frac{F - 1}{\langle \hat{n}(\vec{k}) \rangle}, \quad F \equiv \frac{\langle \hat{n}^2(\vec{k}) \rangle - \langle \hat{n}(\vec{k}) \rangle^2}{\langle \hat{n}(\vec{k}) \rangle^2},$$ (80)

We show the calculation to (80), to show that the expression is independent of states, bra and ket.

$$\frac{\langle \hat{a}^{\dagger}(\vec{k}, t)\hat{a}^{\dagger}(\vec{k}, t)\hat{a}(\vec{k}, t)\hat{a}(\vec{k}, t) \rangle}{\langle \hat{a}^{\dagger}(\vec{k}, t)\hat{a}(\vec{k}, t) \rangle^2} = \frac{\langle \hat{a}^{\dagger}(\vec{k})\hat{a}(\vec{k})\hat{a}(\vec{k})\hat{a}(\vec{k}) \rangle}{\langle \hat{a}^{\dagger}(\vec{k})\hat{a}(\vec{k}) \rangle^2} = \frac{\langle \hat{n}^2(\vec{k}) \rangle - \langle \hat{n}(\vec{k}) \rangle^2}{\langle \hat{n}(\vec{k}) \rangle^2} = \langle \hat{n}(\vec{k}) \rangle^2 = 0.$$ (81)
where \( \hat{n}(\vec{k}) = \hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) \) and \( \hat{a}(\vec{k}, t) = \hat{a}(\vec{k})e^{-imt} \) (\( m \) means the mass of the field we are considering here generally. If it came to our case, it would be considered exchanging \( m \) with \( \Omega \). See Appendix E for derivation of this). Note that the order of creation and annihilation operators are given by the normal ordered product.

- **Case for the \( n \)-particle states:**

We denote \( n \)-particle states of the leading free field as \( |n(\vec{k})\rangle \), where those are defined as

\[
|n(\vec{k})\rangle \equiv \frac{1}{\sqrt{n!}}(\hat{a}^\dagger(\vec{k}))^n|0\rangle.
\] (81)

Then it is held, \( \hat{n}(\vec{k})|n(\vec{k})\rangle = n(\vec{k})|n(\vec{k})\rangle \), therefore \( \langle \hat{n}^2(\vec{k}) \rangle = n^2(\vec{k}) \) and \( F = 0 \). Hence,

\[
g^{(2)}(\vec{k}, \tau) = 1 - 1/n(\vec{k}) \leq 1 \tag{82}
\]

in the \( n \)-particle states, where \( \tau \) is some finite values.

- **Case for the coherent states:**

We denote coherent states as \( |\alpha\rangle \), where those are defined as

\[
|n(\vec{k})\rangle \equiv e^{-|\alpha|^2/2}\sum_{n=1}^{\infty} \frac{\alpha^n(\vec{k})}{\sqrt{n!}}|n(\vec{k})\rangle,
\] (83)

and \( \hat{a}(\vec{k}, t)|\alpha\rangle = \alpha(\vec{k}, t)|\alpha\rangle \). Then, it is led that \( \langle \hat{n}(\vec{k}) \rangle = |\alpha(\vec{k}, 0)|^2 \) and \( \langle \hat{n}^2(\vec{k}) \rangle = |\alpha(\vec{k}, 0)|^4 + |\alpha(\vec{k}, 0)|^2 \). Therefore, \( F = 1 \) and

\[
g^{(2)}(\vec{k}, \tau) = 1. \tag{84}
\]

Hence, \( g^{(2)}(\vec{k}, \tau) \) gets smaller than or equal to 1 at the quantum theory in the \( n \)-particle state, contrary to the classical theory given in (77). This is called “antibunching”, which has been confirmed in experiments [44, 45, 46].

### 4.8 Antibunching in our case at \( r \to \infty \) region

The \( r \to \infty \) region of our supertranslated rotating black hole spacetime is flat, where it can be supposed that there is no field interactions. Therefore it can be written as

\[
\lim_{r \to \infty} \phi = \lim_{r \to \infty} (\phi_0 + \phi_1 + \cdots) \simeq \phi_0. \tag{85}
\]

However, the number operator is deformed as in (71) in that region.
Therefore, if we measure $g^{(2)}(\vec{k}, \tau)$ for the scalar particles at the $r \to \infty$ region taking a $n$-particle state composed with not-deformed creation operators,

$$|n(\vec{k})\text{:as} \rangle = \frac{1}{\sqrt{n!}} (\hat{a}^{\text{as}}(\vec{k}))^{n} |0\rangle,$$  

(86)

it would be predicted that some antibunching, which is deformed for the effect of the spacetime compared with that in the flat spacetime for a field theory, would appear in the measurement result:

$$g^{(2)}(\vec{k}, \tau) = 1 - \frac{1}{n} (1 - \frac{\delta \omega}{E^{\text{as}}(\vec{k})}) \leq 1,$$  

(87)

where $\delta \omega$ and $E^{\text{as}}(\vec{p})$ are given in (58) and (64).

5 Summary and Comment

In this study, we have obtained the metrics of a rotating supertranslated black hole spacetime containing the linear order supertranslation corrections and arbitrary black hole rotation as in (29). Since (29) is not obtained by solving Einstein equation, these would not be the general solution, however since these can satisfy the Einstein equation in the range of the linear order concerning the supertranslation corrections for arbitrary black hole rotation as mentioned under (29), these would be a solution of the rotating supertranslated black hole spacetime. After that, we have analyze the effect of those in antibunching in Sec.4, which would be interesting as an observational measure for supertranslation corrections, aside from the memory effect.

We have shown the outline of our analysis to obtain (29) in Fig.1. The point in our obtaining the metrics (29) would be to have used the supertranslation correction given as the coordinate transformations such as (4). In addition, the point in our finding the solution containing the arbitrary black hole rotation would be that we have found the transformation rules including the linear order supertranslation correction applicable to the coordinates in the Boyer-Lindquist coordinate system as in (28).

In the studies until now, rotating black hole spacetimes deformed by the supertranslations would have not been obtained basically. Its reason would be that studies until now would have stuck to the supertranslations given by the vector fields in the neighborhood of the null infinity, and not considered to involve these by the coordinate transformation such as (4) which enables us to consider in the inside region from the null infinity.

In the actual analysis to obtain the metrics (29), one of ideas author has tried to involve the rotating solution is to use Newman-Janis (NJ) algorithm [47, 48]. Concretely, author has followed the way given in (6) in [49]. However, it could not work in our metrics. Its reason is that there are $r$ in the metrics, and the powers of some of them are odd numbers. As a result, the imaginary numbers appear in the metrics.

Author has also tried the idea to obtain the rotating solution by using (4) in the Kerr-Schild (KS) form. However it has also not worked for the situation that when
author transforms the metrics obtained from the KS form into the ones in the BL coordinate system, these cannot agree to the ones originally obtained in the BL coordinate system, namely Essential point in this problem would be that \( \text{(4)} \) could be available in the isotropic coordinate system and could not be used in the KS form\(^1\).

Let us turn to our analysis of the antibunching in the correlation function of intensities (CFI) in our spacetime in Sec\(\text{[4]}\). In there, we have shown that classical corrections of the supertranslation, black hole rotation and \( \phi^4 \) term get contained into the free leading field considering the fundamental \( \phi^4 \) theory and suppositions (43), (57) and (86), which is a classical renormalization. Then, we have shown that those will appear in the antibunching of scalar particles at the \( r \to \infty \) region where the effect of the interaction is disappeared and the spacetime is flat, which leads to a possibility that the antibunching might be able to be a measure for the deformation of supertranslations, up to the future development.

We will mention the future development of the analysis.

- The first is to consider more general and realistic model by adding fermions, gauge fields and some interaction terms constructed phenomenologically.

- Secondly, although we have analyzed the classical corrections in this study, originally we should have analyzed the quantum effect as well. If we do it, we should proceed with the analysis along the renormalization theory, and as a result, the amplitude, mass and coupling constant of the field would be changed. We want to analyze the antibunching with those. However, if we treated some curved spacetime, it would be unclear whether or not we can get the results we want to get.

- There are also practical problems. The antibunching is basically the artificial phenomenology which we make occur with photons in laboratories. Therefore, we should consider the corrections with photons, in future.

- Next, if we address the antibunching in the spacetime, we would come to regard a celestial body as a light source, however which is possible? How should we consider passes of lights (see Fig\(\text{[2]}\) ? Further, although we have considered \( n \)-particle state as the quantum state to observe CFI, we would need to consider some other states as well from the practical point of view. We will address these practical problems by discussing with experimentalists and phenomenologists.

- Also, we should calculate the numerical values of the corrections (probably we would find it is so tiny and very hard for us to detect those).

\(^1\)The papers author has mainly refereed in the analysis by the KS form are \([50, 51, 52, 53]\). As a result, author has found that the coordinate transformation from the KS form to the BL form is given in the following 3: (ia) (7.3) in \([50]\), (ib) (6) in \([51]\), and (ic) (34)-(36) in \([52]\). However, each and every is slightly different each other.

Next, the KS form is given in the following 2: (iia) (1.2) in \([53]\), and (iib) (32) in \([52]\). However, these are also slightly different each other.

Finally, author has confirmed that the combination of (ic) and (iia) can agree to the BL form (e.g. (3) in \([52]\)).
Finally, we just point out that we have treated two kinds of corrections in this section: 1) the curved spacetime (rotation and supertranslation) and 2) the field interaction ($\phi^4$ term). The correction with regard to the latter would be appearing in current experiments already.

Here, if we look at the correction (58), we can see it vanishes for $a = 0$. Therefore, we might consider corrections would not appear if we considered non-rotating spacetimes. However, it is the result in the setting in this study, and such a situation might be changed up to the future setting and model we consider.

A Review for how to obtain (4)

In this appendix, we note how to obtain (4). This Appendix is the review of [41]. In order to make clear the origin of (4) and (23) in the body text and to make this paper self-consistent, we describe this Appendix. Of course, all the equations in this Appendix are written after confirmed.

The squared line element before supertranslations in the flat Minkowski spacetime in the retarded coordinates $(r_s, u_s, z_s, \bar{z}_s)$ can be written as follows:

$$ds^2 = -du^2 - 2du dr + \frac{4r^2}{(1 + z \bar{z})^2} dz d\bar{z},$$

where this can reduce to the Cartesian flat Minkowski coordinate $(t, r, \theta, \phi)$ by $u = t - r$ and $z = e^{i\phi} \cot \theta/2$ (these $(t, r, \theta, \phi)$ are the coordinates in [41]). (88) can be obtained from

$$ds^2 = -2du_c dr_c + 2r_c^2 dz_c d\bar{z}_c$$

by the coordinate transformations:

$$r_c = \sqrt{2r} \frac{1 + z \bar{z}}{1 + \sqrt{2} u}, \quad u_c = \frac{1 + z \bar{z}}{\sqrt{2}} u - \frac{z \bar{z}}{2r_c} u^2, \quad z_c = z - \frac{z}{\sqrt{2} r_c} u.$$  

Now, let us suppose the supertranslation is performed toward the spacetime of (89). Then, the $ds^2$ of the spacetime obtained as a result of that is considered to be obtained by the following replacement toward (90):

$$(r, u, z) \rightarrow (r_c, u_c, z_c)$$

where

$$r_c = \frac{\partial_r \partial_z W}{\partial_z G \partial_z G} + \sqrt{\frac{r^2}{(\partial_u W)^2} + \frac{(\partial_r^2 G \partial_z W - \partial_z G \partial_r^2 W)(\partial_r^2 \bar{G} \partial_z W - \partial_z \bar{G} \partial_r^2 W)}{(\partial_r G)^2 (\partial_z G)^2}},$$  

$$u_c = W - \frac{1}{r_c} \frac{\partial_r W \partial_z W}{\partial_z G \partial_z G},$$  

$$z_c = G - \frac{1}{r_c} \frac{\partial_r W}{\partial_z G}, \quad \bar{z}_c = \bar{G} - \frac{1}{r_c} \frac{\partial_r W}{\partial_z G}.$$  

25
Here, \( W(u, z, \bar{z}) \) characterizes the Weyl rescalings and supertranslations, and \( G(z) \) characterizes the Lorentz transformations.

Let us take the \( G \) and \( W \) as follows:

\[
G = z, \quad W = \sqrt{\gamma_{zz}}(u + C),
\]

(93)

where \( C(z, \bar{z}) \) means the effect of the supertranslation, which we can take arbitrarily. Here, if we take \( C \) as 0, (93) can reduce to (90).

Performing the replacement of (91) with \( G \) and \( W \) in (93), we can obtain the following one:

\[
ds^2 = -du^2 - 2dudF + Hz^Adz^B, \quad F = \sqrt{r^2 + U} + \frac{1}{2}(D^2 + 2)C, \quad H = (r^2 + 2U)\gamma_{AB} + \sqrt{r^2 + UC_{AB}},
\]

(94)

where

\[
z^A = (z, z^*), \quad C_{AB} = -(2D_AD_B - \gamma_{AB}D^2)C, \quad U = \frac{1}{8}C_{AB}C^{AB}, \quad \gamma_{AB} = \gamma_{zz}^{-2}
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

and \( D_A \) is the covariant derivative for the space of the coordinates \( z^A \) given by the metric \( \gamma_{AB} \). Note \( D^2C = \gamma^{AB}D_AD_BC = \gamma^{AB}D_A\partial_BC \). With the following coordinate transformations:

\[
t = u + \rho, \quad \rho = \sqrt{r^2 + U} + E, \quad E = \frac{1}{2}D^2C + C.
\]

(95)

(94) can be further rewritten into the following one:

\[
ds^2 = -dt^2 + dr^2 + (((\rho - E)^2 + U)\gamma_{AB} + (\rho - E)C_{AB})dz^Adz^B.
\]

(96)

Here, we can check (96) can reduce to the Bondi gauge:

\[
ds^2 = e^{2\beta}V_r du^2 - 2e^{2\beta}dudr + g_{AB}(dz^A - U^Adu)(dz^B - U^Bdu)
\]

(97)

with \( z^A = (z, \bar{z}) \), where \( \partial_r(r^{-2}\det(g_{AB})) = 0, g_{rr} = g_{rz} = g_{z\bar{z}} = 0 \) and

\[
g_{AB} = r^2\gamma_{AB}\gamma_{zz}^{-2}
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} + \mathcal{O}(r).
\]

(98)

Now, let us take \( C \), the effect of the supertranslation to zero in (96), then we can obtain the (96) before the supertranslation as

\[
ds^2 = -dt^2 + dr^2 + r^2\gamma_{AB}dz^Adz^B,
\]

(99a)

\[
= -dt^2 + dx^2 + dy^2 + dz^2,
\]

(99b)
where \( z = e^{i\phi} \cot \theta/2 \). We can check that (99a) can be mapped to (96) by the following replacement:

\[
(r, \rho, z) \rightarrow (t_s, \rho_s, z_s)
\]

where

\[
t_s = t,
\]

\[
\rho_s = \sqrt{(\rho - C)^2 + D_A C D^A C},
\]

\[
z_s = \frac{(z - \bar{z}^{-1})(\rho - C) + (z + \bar{z}^{-1})(\rho_s - z \partial_z C - \bar{z} \partial_{\bar{z}} C)}{2(\rho - C) + (1 + z \bar{z})(\bar{z} \partial_z C - \bar{z}^{-1} \partial_{\bar{z}} C)}.
\]

We can check the following equalities:

\[
\rho_s^2 = x_s^2 + y_s^2 + z_s^2,
\]

\[
d\rho_s^2 + \rho_s^2 \gamma_{AB} dz_s^A dz_s^B = dx_s^2 + dy_s^2 + dz_s^2,
\]

where

\[
x_s = (\rho - C) \sin \theta \cos \varphi + \sin \varphi \csc \theta \partial_{\varphi} C - \cos \theta \cos \varphi \partial_{\theta} C,
\]

\[
y_s = (\rho - C) \sin \theta \sin \varphi - \cos \varphi \csc \theta \partial_{\varphi} C - \cos \theta \sin \varphi \partial_{\theta} C,
\]

\[
z_s = (\rho - C) \cos \theta + \cos \theta \cos \varphi \partial_{\theta} C,
\]

In conclusion, it can be considered that, if there is a three-dimensional flat space written by the Cartesian coordinate system, we can get it after the supertranslation by the following replacement:

\[
(x, y, z) \rightarrow (x_s, y_s, z_s).
\]
B Calculation process from (33) to (34)

\[
\int_{\Omega} d^3x' \frac{1}{|\vec{x} - \vec{x}'|} \cos \phi' T_{t\phi}(t', \theta')
= \int_{\Omega} d^3x' \left( \frac{1}{r} \left( 1 + \frac{\vec{x} \cdot \vec{x}'}{r' r''} + \frac{1}{r^2} \left( 3 \frac{\left( \vec{x} \cdot \vec{x}' \right)^2}{r'^2} - \frac{r'^2}{2} \right) + \cdots \right) \cos \phi' \right) \sin \theta' T_{t\phi}(\vec{x}') + O(r^{-4})
\]

\[
= \frac{1}{r^2} \int_{\Omega} d^3x' \left( \frac{\vec{x} \cdot \vec{x}'}{r' r''} + \frac{3 \left( \vec{x} \cdot \vec{x}' \right)^2}{2 r'^3 r''} \right) \cos \phi' \sin \theta' T_{t\phi}(\vec{x}') + O(r^{-4})
\]

\[
= \frac{1}{r^2} \int_{\Omega} d^3x' \left\{ \sin \theta \left( \sin \theta' \cos \phi \cos \phi' + \sin \phi \sin \phi' \right) + \cos \theta \cos \theta' \right.
+ \frac{3 r'}{2 r} \left( \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi') \right)^2 \cos \frac{\phi'}{\sin \theta'} T_{t\phi}(\vec{x}') + O(r^{-4})
\]

\[
= \frac{1}{r^2} \int_{\Omega} d^3x' \left( \sin \theta \cos \phi \cdot \cos^2 \phi' + \frac{3 r'}{r} \cos \theta \cos \theta' \sin \theta \cos \phi' \sin (\phi - \phi') \right) T_{t\phi}(\vec{x}') + O(r^{-4})
\]

\[
= \frac{\sin \theta \cos \phi}{r^2} \int dr' d\theta' r'^2 \sin \theta' T_{t\phi}(\vec{x}') \cdot \int_{0}^{2\pi} d\phi' \cdot \int_{0}^{2\pi} d\phi'' \cdot \int_{0}^{2\pi} d\phi' \cos^2 \phi' \sin \theta' \cos \theta' \cos \phi' T_{t\phi}(\vec{x}') + O(r^{-4})
\]

\[
+ \frac{3 \pi \cos \theta \sin \theta \cos \phi}{r^3} \int dr' d\theta' r'^3 \sin \theta' \cos \theta' \cos \phi' T_{t\phi}(\vec{x}') + O(r^{-4})
\]

\[
= \frac{\sin \theta \cos \phi}{2 r^2} \int_{\Omega} d^3x' T_{t\phi}(\vec{x}') + \frac{3 \cos \theta \sin \theta \cos \phi}{2 r^3} \int_{\Omega} d^3x' \frac{r'}{r^3} \cos \theta' T_{t\phi}(\vec{x}') + O(r^{-4}),
\]

where we have exploited the assumption that $T_{t\phi}$ is independent of $\phi$.

C Equation of motion obtained from Hamiltonian

We show that the equation of motion same with the one obtained from the Lagrangian (41) can be obtained from the Hamiltonian (59). Why we show this is that our metrics (39) are not in the 3+1 decomposition form, therefore it looks that the right Hamiltonian could not be obtained in the sense that the equation of motion obtained from that gets different from the one obtained from the Lagrangian.

There is a freedom in a Lagrangian to transform it leaving the equation of motion invariant. Corresponding to this, there is also a freedom in a Hamiltonian. Therefore, we may consider these are corresponding each other, if only the equations of motion obtained from the Lagrangian and the Hamiltonian can agree each other.

Therefore, we show that the equation of motion same with the one obtained from the Lagrangian (41) can be obtained from the Hamiltonian (59).
With the Hamiltonian (59), we can obtain the following time developments:

\[
-i\dot{\phi}(x) = [\dot{\phi}(x), \mathcal{H}]
\]

\[
= \int d^3y[\dot{\phi}(x), -\frac{\pi^2(y)}{2\mathcal{K}u(y)} + \mathcal{K}^{ti}(y)\pi(y)\partial_i\dot{\phi}(y)]|_{x^0=y^0}
\]

\[
= \int d^3y(-\frac{\pi(y)}{\mathcal{K}u(y)} + \mathcal{K}^{ti}(y)\partial_i\dot{\phi}(y))i\delta^3(x-y)|_{x^0=y^0}
\]

\[
= i(-\frac{\pi(x)}{\mathcal{K}u(y)} + \mathcal{K}^{ti}(y)\partial_i\dot{\phi}(x)), \quad (106a)
\]

\[-i\dot{\pi}(x) = [\dot{\pi}(x), \mathcal{H}]
\]

\[
= \int d^3y[\dot{\pi}(x), \mathcal{K}^{ti}(y)\pi(y)\partial_i\dot{\phi}(y) + \frac{1}{2}\mathcal{K}^{ij}(y)\partial_i\dot{\phi}(y)\partial_j\dot{\phi}(y) + \frac{m^2}{2}\dot{\phi}^2(y) + \frac{\lambda}{4!}\dot{\phi}^4(y)]|_{x^0=y^0}
\]

\[
= \int d^3y(-\mathcal{K}^{ti}(y)\dot{\pi}(y)\partial_i - \mathcal{K}^{ij}(y)\partial_i\dot{\phi}(y)\partial_j - (m^2\dot{\phi}(y) + \frac{\lambda}{3!}\dot{\phi}^3(y)))i\delta^3(x-y)|_{x^0=y^0}
\]

\[
= i(\mathcal{K}^{ti}\partial_i\dot{\pi}(x) + \mathcal{K}^{ij}\partial_i\dot{\phi}(x) - (m^2\dot{\phi}(x) + \frac{\lambda}{3!}\dot{\phi}^3(x))), \quad (106b)
\]

where we have used (61) and a formula: \([A, BC] = B[A, C] + [A, B]C\). \(\mathcal{K}^{MN}\) are given in (53). From these, we can obtain the equation of motion as follows:

\[
(K^{ti}\partial_t^2 + 2K^{ti}\mathcal{K}^{ti}\partial_i + \mathcal{K}^{ij}\partial_i\partial_j - m^2)\dot{\phi}(x) = \lambda \frac{3!}{3!}\dot{\phi}^3(x), \quad (107)
\]

\(\mathcal{K}^{ti}\mathcal{K}^{ti}\) in the one above is ignorable in the order we treat in our analysis.

We can rewrite (107) using the covariant derivatives as

\[
(K^{MN}\nabla_M\nabla_N - m^2)\dot{\phi}(x) = \lambda \frac{3!}{3!}\dot{\phi}^3(x). \quad (108)
\]

where the points in the rewriting from (107) to (108) are 1) \(\mathcal{K}^{ti} = 0\) and 2) \(\mathcal{K}^{MN}\) are all independent of \(t\), therefore Christoffel symbols in the one above are all vanishing. This is the quantum theory version of (12).

### D Calculation in footnote at (68)

The purpose of this Appendix is to check that, when we expand the r.h.s. of (110) with regard to \(\delta\omega\) to linear order, the corrections of \(\delta\omega\) can be exactly canceled out and these can be identical to the l.h.s. of (110).

Following relations are held:

\[
\int d^3\vec{x}f^{\text{def}}_{\vec{p}}(x)f^{\text{def}}_{\vec{q}}(x) = \frac{\delta^3(\vec{p} - \vec{q})}{2E^{\text{def}}(\vec{p})}, \quad (109a)
\]

\[
\int d^3\vec{x}f^{\text{def}}_{\vec{p}}(x)f^{\text{def}}_{\vec{q}}(x) = \frac{\delta^3(\vec{p} + \vec{q})}{2E^{\text{def}}(\vec{p})}e^{-2iE^{\text{def}}(\vec{p})x^0}, \quad (109b)
\]

\[
\partial_0f^{\text{def}}_{\vec{p}}(x) = iE^{\text{def}}(\vec{p})f^{\text{def}}_{\vec{p}}(x). \quad (109c)
\]
With these, we can write the creation and annihilation operators as

\begin{align}
  ia^{\text{def} \dagger}(\vec{p}) &= \int d^3x \left( \partial_0 f^{\text{def}}(x) \cdot \hat{\phi}^{\text{def}}(x) - \vec{f}_p^{\text{def}}(x) \cdot \partial_0 \hat{\phi}^{\text{def}}(x) \right), \\
  -ia^{\text{def}}(\vec{p}) &= \int d^3x \left( \partial_0 f^{\ast \text{def}}(x) \cdot \hat{\phi}^{\text{def}}(x) - \vec{f}_p^{\ast \text{def}}(x) \cdot \partial_0 \hat{\phi}^{\text{def}}(x) \right).
\end{align}

\hspace{1cm} (110a) \hspace{1cm} (110b)

Note that \( p^{\text{def}} \cdot x \) in \( f^{\text{def}}(x) = \frac{e^{-ip^{\text{def}} \cdot x}}{(2\pi)^{3/2}E^{\text{def}}(\vec{p})} \) is \(-p^{\text{def}}_0 x^0 + \vec{p} \cdot \vec{x}\). In what follows, we calculate the r.h.s of (1110) expanding these to the linear order of \( \delta \omega \).

Using (68) and (109), it can be calculated,

\begin{align}
  \hat{\phi}^{\text{def}}(x) &= \hat{\varphi}(x) + \delta \omega \int d^3 \vec{p} \left( i \beta_+ a^{\text{def}}(\vec{p}) f^{\text{as}}(x) - i \beta_- (a^{\text{def}}(\vec{p}) \dagger f^{\ast \text{as}}(x) \right), \\
  \hat{\varphi}(x) &\equiv \int d^3 \vec{p} \left( a^{\text{def}}(\vec{p}) f^{\text{as}}(x) + a^{\text{def} \dagger}(\vec{p}) f^{\ast \text{as}}(x) \right).
\end{align}

\hspace{1cm} (111)

With this, we can calculate as follows:

\begin{align}
  \int d^3 \vec{x} \partial_0 f^{\text{def}}(x) \cdot \hat{\phi}^{\text{def}}(x) &= \int d^3 \vec{x} \left( 1 + i \beta_+ \delta \omega \right) \partial_0 f^{\text{as}}(x) \cdot \hat{\varphi}(x) \\
  &\quad + \frac{\delta \omega}{2} \left( -\beta_- a^{\text{def}}(\vec{p}) e^{2iE^{\text{as}}(\vec{p})x^0} + \beta_+ a^{\text{def} \dagger}(\vec{p}) \right), \\
  \int d^3 \vec{x} \vec{f}^{\text{def}}(x) \cdot \partial_0 \hat{\phi}^{\text{def}}(x) &= \int d^3 \vec{x} \left( 1 + i \beta_+ \delta \omega \right) f^{\ast \text{as}}(x) \cdot \partial_0 \hat{\varphi}(x) \\
  &\quad + \frac{\delta \omega}{2} \left( -\beta_- a^{\text{def}}(\vec{p}) e^{2iE^{\text{as}}(\vec{p})x^0} - \beta_+ a^{\text{def} \dagger}(\vec{p}) \right).
\end{align}

\hspace{1cm} (112a) \hspace{1cm} (112b)

Using these, we can obtain the result for r.h.s. of (110a) which is identical to its l.h.s., the correction of \( \delta \omega \) is canceled out, as follows:

\begin{align}
  \text{r.h.s. of (110a)} &= \text{r.h.s. of (112a)} - \text{r.h.s. of (112b)} = ia^{\text{def} \dagger}(\vec{p}).
\end{align}

\hspace{1cm} (113)

Since (110b) is the complex conjugate of (110a), it can be known readily that r.h.s. of (110b) becomes \(-ia^{\text{def}}(\vec{p})\) can be known readily, if (113) is held.

\begin{align}
  E \hat{a}^{\text{as}}(\vec{k}, t) = \hat{a}^{\text{as}}(\vec{k}, 0) e^{-i\omega t} \text{ in (80)}
\end{align}

Since the equation of motion of \( \hat{\phi}^{\text{as}}(x) \) is given by the wave equation in the flat spacetime such as (63), \( \hat{\phi}^{\text{as}}(x) \) follows the formalism in the flat spacetime. Therefore, the energy and momentum of \( \hat{\phi}^{\text{as}}(x) \) can be given as \( P^\mu = \int d^3k k^\mu \hat{n}(\vec{k}) \). With this, we can obtain \([P_\mu, \hat{a}^{\text{as}}(\vec{k}, t)] = -i\partial_\mu \hat{a}^{\text{as}}(\vec{k}, t)\), from which we can obtain the equation in the subject of this Appendix. Time development equation of \( a^{\text{def}}(\vec{k}, t) \) is likewise with \( \Omega \) instead of \( \omega \).
References

[1] H. Bondi, M. G. J. van der Burg and A. W. K. Metzner, “Gravitational waves in general relativity. 7. Waves from axisymmetric isolated systems,” Proc. Roy. Soc. Lond. A A269, 21-52 (1962)

[2] R. K. Sachs, “Gravitational waves in general relativity. 8. Waves in asymptotically flat space-times,” Proc. Roy. Soc. Lond. A A270, 103-126 (1962)

[3] G. Barnich and C. Troessaert, “Symmetries of asymptotically flat 4 dimensional space times at null infinity revisited,” Phys. Rev. Lett. 105, 111103 (2010) [arXiv:0909.2617 [gr-qc]].

[4] G. Barnich and C. Troessaert, “Aspects of the BMS/CFT correspondence,” JHEP 05, 062 (2010) [arXiv:1001.1541 [hep-th]].

[5] G. Barnich and C. Troessaert, “BMS charge algebra,” JHEP 12, 105 (2011) [arXiv:1106.0213 [hep-th]].

[6] A. Strominger, “On BMS Invariance of Gravitational Scattering,” JHEP 07, 152 (2014) [arXiv:1312.2229 [hep-th]].

[7] T. He, V. Lysov, P. Mitra and A. Strominger, “BMS supertranslations and Weinberg’s soft graviton theorem,” JHEP 05, 151 (2015) [arXiv:1401.7026 [hep-th]].

[8] F. Cachazo and A. Strominger, “Evidence for a New Soft Graviton Theorem,” [arXiv:1404.4091 [hep-th]].

[9] D. Kapec, V. Lysov, S. Pasterski and A. Strominger, “Semiclassical Virasoro symmetry of the quantum gravity S-matrix,” JHEP 08, 058 (2014) [arXiv:1406.3312 [hep-th]].

[10] D. Kapec, P. Mitra, A. M. Raclariu and A. Strominger, “2D Stress Tensor for 4D Gravity,” Phys. Rev. Lett. 119, no.12, 121601 (2017) [arXiv:1609.00282 [hep-th]].

[11] T. He, D. Kapec, A. M. Raclariu and A. Strominger, “Loop-Corrected Virasoro Symmetry of 4D Quantum Gravity,” JHEP 08, 050 (2017) [arXiv:1701.00496 [hep-th]].

[12] A. Strominger, “Asymptotic Symmetries of Yang-Mills Theory,” JHEP 07, 151 (2014) [arXiv:1308.0589 [hep-th]].

[13] T. He, P. Mitra, A. P. Poryfyriadis and A. Strominger, “New Symmetries of Massless QED,” JHEP 10, 112 (2014) [arXiv:1407.3789 [hep-th]].

[14] T. He and P. Mitra, “Asymptotic symmetries and Weinberg’s soft photon theorem in Minkowski space-time,” JHEP 10, 213 (2019) [arXiv:1903.02608 [hep-th]].

[15] T. He, P. Mitra and A. Strominger, “2D Kac-Moody Symmetry of 4D Yang-Mills Theory,” JHEP 10, 137 (2016) [arXiv:1503.02663 [hep-th]].
[16] A. Bagchi, R. Basu, A. Kakkar and A. Mehra, “Flat Holography: Aspects of the dual field theory,” JHEP 12, 147 (2016) [arXiv:1609.06203 [hep-th]].

[17] S. Pasterski, S. H. Shao and A. Strominger, “Flat Space Amplitudes and Conformal Symmetry of the Celestial Sphere,” Phys. Rev. D 96, no.6, 065026 (2017) [arXiv:1701.00049 [hep-th]].

[18] C. Cardona and Y. t. Huang, “S-matrix singularities and CFT correlation functions,” JHEP 08, 133 (2017) [arXiv:1702.03283 [hep-th]].

[19] S. Pasterski and S. H. Shao, “Conformal basis for flat space amplitudes,” Phys. Rev. D 96, no.6, 065022 (2017) [arXiv:1705.01027 [hep-th]].

[20] S. Pasterski, S. H. Shao and A. Strominger, “Gluon Amplitudes as 2d Conformal Correlators,” Phys. Rev. D 96, no.8, 085006 (2017) [arXiv:1706.03917 [hep-th]].

[21] S. W. Hawking, M. J. Perry and A. Strominger, “Soft Hair on Black Holes,” Phys. Rev. Lett. 116, no.23, 231301 (2016) [arXiv:1601.00921 [hep-th]].

[22] S. W. Hawking, M. J. Perry and A. Strominger, “Superrotation Charge and Supertranslation Hair on Black Holes,” JHEP 05, 161 (2017) [arXiv:1611.09175 [hep-th]].

[23] A. Strominger, “Black Hole Information Revisited,” [arXiv:1706.07143 [hep-th]].

[24] A. Strominger and A. Zhiboedov, “Gravitational Memory, BMS Supertranslations and Soft Theorems,” JHEP 01, 086 (2016) [arXiv:1411.5745 [hep-th]].

[25] S. Pasterski, A. Strominger and A. Zhiboedov, “New Gravitational Memories,” JHEP 12, 053 (2016) [arXiv:1502.06120 [hep-th]].

[26] M. Pate, A. M. Radciariu and A. Strominger, “Color Memory: A Yang-Mills Analog of Gravitational Wave Memory,” Phys. Rev. Lett. 119, no.26, 261602 (2017) [arXiv:1707.08016 [hep-th]].

[27] L. Bieri and D. Garfinkle, “An electromagnetic analogue of gravitational wave memory,” Class. Quant. Grav. 30, 195009 (2013) [arXiv:1307.5098 [gr-qc]].

[28] L. Susskind, “Electromagnetic Memory,” [arXiv:1507.02584 [hep-th]].

[29] S. Pasterski, “Asymptotic Symmetries and Electromagnetic Memory,” JHEP 09, 154 (2017) [arXiv:1505.00716 [hep-th]].

[30] A. Strominger, “Lectures on the Infrared Structure of Gravity and Gauge Theory,” [arXiv:1703.05448 [hep-th]].

[31] G. Comp`ere and A. Fiorucci, “Advanced Lectures on General Relativity,” [arXiv:1801.07064 [hep-th]].

[32] M. S. Pshirkov, D. Baskaran and K. A. Postnov, “Observing gravitational wave bursts in pulsar timing measurements,” Mon. Not. Roy. Astron. Soc. 402, 417 (2010) [arXiv:0909.0742 [astro-ph.CO]].
[33] R. van Haasteren and Y. Levin, “Gravitational-wave memory and pulsar timing arrays,” Mon. Not. Roy. Astron. Soc. 401, 2372 (2010) [arXiv:0909.0954 [astro-ph.IM]].

[34] N. Seto, “Search for Memory and Inspiral Gravitational Waves from Super-Massive Binary Black Holes with Pulsar Timing Arrays,” Mon. Not. Roy. Astron. Soc. 400 (2009) L38 [arXiv:0909.1379 [astro-ph.GA]].

[35] J. B. Wang, G. Hobbs, W. Coles, R. M. Shannon, X. J. Zhu, D. R. Madison, M. Kerr, V. Ravi, M. J. Keith, R. N. Manchester, Y. Levin, M. Bailes, N. D. R. Bhat, S. Burke-Spolaor, S. Dai, S. Osłowski, W. van Straten, L. Toomey, N. Wang and L. Wen, “Searching for gravitational wave bursts with the Parkes Pulsar Timing Array,” Mon. Not. Roy. Astron. Soc. 446, 1657-1671 (2015) [arXiv:1410.3323 [astro-ph.GA]].

[36] Z. Arzoumanian et al. [NANOGrav], “NANOGrav Constraints on Gravitational Wave Bursts with Memory,” Astrophys. J. 810, no.2, 150 (2015) [arXiv:1501.05343 [astro-ph.GA]].

[37] P. D. Lasky, E. Thrane, Y. Levin, J. Blackman and Y. Chen, “Detecting gravitational-wave memory with LIGO: implications of GW150914,” Phys. Rev. Lett. 117, no.6, 061102 (2016) [arXiv:1605.01415 [astro-ph.HE]].

[38] M. Boyle, “Transformations of asymptotic gravitational-wave data,” Phys. Rev. D 93, no.8, 084031 (2016) doi:10.1103/PhysRevD.93.084031 [arXiv:1509.00862 [gr-qc]].

[39] K. Mitman, D. A. B. Iozzo, N. Khera, M. Boyle, T. De Lorenzo, N. Deppe, L. E. Kidder, J. Moxon, H. P. Pfeiffer and M. A. Scheel, et al. “Adding gravitational memory to waveform catalogs using BMS balance laws,” Phys. Rev. D 103, no.2, 024031 (2021) [arXiv:2011.01309 [gr-qc]].

[40] E. Berti, V. Cardoso and C. M. Will, “On gravitational-wave spectroscopy of massive black holes with the space interferometer LISA,” Phys. Rev. D 73, 064030 (2006) [gr-qc/0512160].

[41] G. Compere and J. Long, “Classical static final state of collapse with supertranslation memory,” Class. Quant. Grav. 33, no.19, 195001 (2016) [arXiv:1602.05197 [gr-qc]].

[42] F. L. Lin and S. Takeuchi, “Hawking flux from a black hole with nonlinear supertranslation hair,” Phys. Rev. D 102, no.4, 044004 (2020) [arXiv:2004.07474 [hep-th]].

[43] S. Takeuchi, “Hawking flux of 4D Schwarzschild blackhole with supertransition correction to second-order,” SciPost Phys. Proc. 4, 010 (2021) [arXiv:2104.05483 [hep-th]].
[44] R. Ghosh and L. Mandel “Observation of nonclassical effects in the interference of two photons,” Phys. Rev. Lett. 59 (1987) 1903,

[45] F. Diedrich and H. Walther “Nonclassical radiation of a single stored ion,” Phys. Rev. Lett. 58 (1987) 203,

[46] M. Koashi, K Kono, T Hirano, and M. Matsuoka “Photon antibunching in pulsed squeezed light generated via parametric amplification,” Phys. Rev. Lett. 71 (1993) 1164,

[47] E. T. Newman and A. I. Janis, “Note on the Kerr spinning particle metric,” J. Math. Phys. 6, 915-917 (1965)

[48] E. T. Newman, R. Couch, K. Chinnapared, A. Exton, A. Prakash and R. Torrence, “Metric of a Rotating, Charged Mass,” J. Math. Phys. 6, 918-919 (1965)

[49] C. Bambi and L. Modesto, “Rotating regular black holes,” Phys. Lett. B 721, 329-334 (2013) [arXiv:1302.0075 [gr-qc]].

[50] S. A. Teukolsky, “The Kerr Metric,” Class. Quant. Grav. 32, no.12, 124006 (2015) [arXiv:1410.2130 [gr-qc]].

[51] N. Arkani-Hamed, Y. t. Huang and D. O’Connell, “Kerr black holes as elementary particles,” JHEP 01, 046 (2020) [arXiv:1906.10100 [hep-th]].

[52] M. Visser, “The Kerr spacetime: A Brief introduction,” [arXiv:0706.0622 [gr-qc]].

[53] G. W. Gibbons, H. Lu, D. N. Page and C. N. Pope, “The General Kerr-de Sitter metrics in all dimensions,” J. Geom. Phys. 53, 49-73 (2005) [arXiv:hep-th/0404008 [hep-th]].