On the Existence and Uniqueness of Solutions to Stochastic Differential Equations Driven by $G$-Brownian Motion with Integral-Lipschitz Coefficients

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Abstract In this paper, we study the existence and uniqueness of solutions to stochastic differential equations driven by $G$-Brownian motion (GSDEs) with integral-Lipschitz coefficients.

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Introduction

Motivated by uncertainty problems, risk measures and the super-hedging in finance, Peng$^{[9,10]}$ introduced a framework of $G$-expectation, in which a new type of Brownian motion was constructed and the related stochastic calculus has been established. As a counterpart in the classical framework, stochastic differential equations driven by $G$-Brownian motion (GSDEs) have been studied by Gao$^{[4]}$ and Peng$^{[10]}$. In these works, the solvability of GSDEs under Lipschitz conditions has been obtained by the contraction mapping theorem.

Typically, a GSDE is of the following form:

$$X(t) = x + \int_0^t b(s, X(s))ds + \int_0^t h(s, X(s))d\langle B, B \rangle_s + \int_0^t g(s, X(s))dB_s, \quad 0 \leq t \leq T, \quad (1.1)$$

where $x \in \mathbb{R}^n$ is the initial value, $B$ is the $G$-Brownian motion and $\langle B, B \rangle$ is the quadratic variation process of $B$.

In this paper, we study the solvability of the GSDE (1.1) under a so-called integral-Lipschitz condition:

$$|b(t, x_1) - b(t, x_2)|^2 + |h(t, x_1) - h(t, x_2)|^2 + |g(t, x_1) - g(t, x_2)|^2 \leq \rho(|x_1 - x_2|^2), \quad (1.2)$$

where $\rho : (0, +\infty) \to (0, +\infty)$ is a continuous increasing and concave function that vanishes at 0+ and satisfies

$$\int_0^1 \frac{dr}{\rho(r)} = +\infty.$$
A typical example of (1.2) is
\[ |b(t, x_1) - b(t, x_2)|^2 + |h(t, x_1) - h(t, x_2)|^2 + |g(t, x_1) - g(t, x_2)|^2 \leq |x_1 - x_2|^2 \ln \frac{1}{|x_1 - x_2|}. \]
Furthermore, we consider the GSDE (1.1) under a “weaker” condition on \( b \) and \( h \):
\[ |b(t, x_1) - b(t, x_2)| + |h(t, x_1) - h(t, x_2)| \leq \rho(|x_1 - x_2|), \quad (1.3) \]
where \( \rho \) satisfies the same conditions as in (1.2). A typical example of (1.3) is
\[ |b(t, x_1) - b(t, x_2)| + |h(t, x_1) - h(t, x_2)| \leq |x_1 - x_2| \ln \frac{1}{|x_1 - x_2|}. \]

In the classical framework, Watanabe and Yamada\textsuperscript{[16,19]} and Fang and Zhang\textsuperscript{[3]} proved the pathwise uniqueness of solutions to finite-dimensional SDEs under some non-Lipschitz condition. In addition to that, Yamada\textsuperscript{[19]} found an explicit way to construct the solutions by successive approximation. On the other hand, Hu and Lerner\textsuperscript{[6]} worked on the SDEs in infinite dimension under the integral-Lipschitz conditions (1.2) and (1.3). They established both the pathwise uniqueness and successive approximations of the solutions. Corresponding to the result in Watanabe and Yamada\textsuperscript{[19]}, Lin\textsuperscript{[8]} obtained a pathwise uniqueness result for non-Lipschitz GSDEs when the coefficient \( g \) is bounded.

In this article, we present both the existence and uniqueness results for GSDE (1.1) under the integral-Lipschitz conditions (1.2) and (1.3). These results are obtained by a technique similar to that in Hu and Lerner\textsuperscript{[6]}. This paper is organized as follows: Section 2 gives the necessary preliminaries in the \( G \)-framework. Section 3 proves the existence and uniqueness theorem for GSDEs with integral-Lipschitz coefficients and Section 4 studies the case for \( G \)-backward stochastic differential equations (GBSDEs).

## 2 Preliminaries

The main purpose of this section is to recall some results in the \( G \)-framework. The reader interested in a more detailed description of these notions is referred to Denis et al.\textsuperscript{[2]}, Gao\textsuperscript{[4]} and Peng\textsuperscript{[10]}.

### 2.1 \( G \)-Brownian Motion and \( G \)-expectation

Adapting the approach in Peng\textsuperscript{[10]}, let \( \Omega \) be a given nonempty fundamental space and \( \mathcal{H} \) a linear space of real functions defined on \( \Omega \) such that (1) \( 1 \in \mathcal{H} \); (2) \( \mathcal{H} \) is stable with respect to bounded Lipschitz functions, i.e., for all \( n \geq 1 \), \( X_1, \cdots, X_n \in \mathcal{H} \) and \( \varphi \in C_{b,lip}(\mathbb{R}^n) \), it holds also \( \varphi(X_1, \cdots, X_n) \in \mathcal{H} \).

**Definition 2.1.** A sublinear expectation \( \mathbb{E}[\cdot] \) on \( \mathcal{H} \) is a functional \( \mathbb{E}[\cdot] : \mathcal{H} \to \mathbb{R} \) with the following properties: for each \( X, Y \in \mathcal{H} \), we have

1. **Monotonicity:** if \( X \geq Y \), then \( \mathbb{E}[X] \geq \mathbb{E}[Y] \);
2. **Preservation of constants:** \( \mathbb{E}[c] = c \), for all \( c \in \mathbb{R} \);
3. **Sub-additivity:** \( \mathbb{E}[X] - \mathbb{E}[Y] \leq \mathbb{E}[X - Y] \);
4. **Positive homogeneity:** \( \mathbb{E}[\lambda X] = \lambda \mathbb{E}[X] \), for all \( \lambda \in \mathbb{R}^+ \).

The triple \( (\Omega, \mathcal{H}, \mathbb{E}) \) is called a sublinear expectation space.

**Definition 2.2.** A random vector \( Y = (Y_1, \cdots, Y_n) \in \mathcal{H}^n \) is said to be independent of \( X \in \mathcal{H}^m \) under \( \mathbb{E}[\cdot] \) if for each test function \( \varphi \in C_{b,lip}(\mathbb{R}^{n+m}) \) we have
\[ \mathbb{E}[\varphi(X, Y)] = \mathbb{E}[\varphi(x, Y)]_{x=X}. \]
Definition 2.2. Let $X = (X_1, \cdots, X_n) \in \mathcal{H}^n$ be a given random vector. We define

$$F_X[\varphi] := E[\varphi(X)], \quad \varphi \in C_{b,lip}(\mathbb{R}^n).$$

Then, the functional $F_X[\cdot]$ is called the distribution of $X$ under $E[\cdot]$.

Now we begin to introduce the definition of $G$-Brownian motion and $G$-expectation.

Definition 2.3. A $d$-dimensional random vector $X$ in a sublinear expectation space $(\Omega, \mathcal{H}, E)$ is called $G$-normal distributed if for each $\varphi \in C_{b,lip}(\mathbb{R}^d)$,

$$u(t, x) := E[\varphi(x + \sqrt{t}X)], \quad t \in \mathbb{R}^+, \quad x \in \mathbb{R}^d,$

is the viscosity solution to the following PDE defined on $\mathbb{R}^+ \times \mathbb{R}^d$:

$$\begin{cases}
\frac{\partial u}{\partial t} - G(D^2 u) = 0; \\
u|_{t=0} = \varphi,
\end{cases}$$

where $G = G_X(A) : \mathbb{S}^d \rightarrow \mathbb{R}$ is defined by

$$G_X(A) := \frac{1}{2} \mathbb{E}[\langle AX, X \rangle]$$

and $D^2 u = (\partial^2_{x_i x_j} u)_{i,j=1}^d$.

In particular, $E[\varphi(X)] = u(1, 0)$ defines the distribution of $X$. By Theorem 2.1 in Chapter I of Peng [10], there exists a bounded and closed subset $\Gamma$ of $\mathbb{R}^d$, such that for each $A \in \mathbb{S}^d$, $G_X(A)$ can be represented as

$$G_X(A) = \frac{1}{2} \sup_{\gamma \in \Gamma} \text{tr}[\gamma \gamma^T A].$$

Defining a subset $\Sigma := \{\gamma \gamma^T : \gamma \in \Gamma\}$ in $\mathbb{S}^d$, the $G$-normal distribution can be denoted by $N(0, \Sigma)$.

Let $\Omega$ be the space of all $\mathbb{R}^d$-valued continuous paths $(\omega_t)_{t \geq 0}$ that start from 0 and $B$ the canonical process. We assume moreover that $\Omega$ is a metric space equipped with the following distance:

$$\rho(\omega^1, \omega^2) := \sum_{N=1}^{\infty} 2^{-N} \left( \max_{0 \leq t \leq N} (\|\omega^1_t - \omega^2_t\|_1) \wedge 1 \right).$$

For a fixed $T \geq 0$, we set

$$L^0_{ip}(\Omega_T) := \{\varphi(B_{t_1}, \cdots, B_{t_n}) : n \geq 1, 0 \leq t_1 \leq \cdots \leq t_n \leq T, \varphi \in C_{b,lip}(\mathbb{R}^{d \times n})\}.$$
where \( \psi(x_1, \ldots, x_j) := \mathbb{E}[\phi(x_1, \ldots, x_j, B_{t_{j+1}} - B_{t_j}, \ldots, B_{t_n} - B_{t_{n-1}})] \). Moreover, the mapping \( \mathbb{E}[,|\Omega_j| : L^0(\Omega_T) \to L^0(\Omega_j) \) can be continuously extended to \( \mathbb{E}[,|\Omega_j| : L^1_G(\Omega_T) \to L^1_G(\Omega_j) \).

### 2.2 G-capacity

Derived in Denis et al.\([2]\), G-expectation can be formulated as an upper expectation of a weakly compact family of probability measures. This family is related to the set \( \Gamma \) mentioned in the last subsection, which is a bounded and closed subset of \( \mathbb{R}^d \) that characterizes the G-function \( G(\cdot) \).

Let \( \mathbb{P}_0 \) be the Wiener measure on \( \Omega, \mathcal{F} \) the filtration generated by the canonical process \( B \) and \( \mathcal{A}_{[0, +\infty)} \) the collection of all \( \Gamma \)-valued progressively measurable processes. For each \( \theta \in \mathcal{A}_{[0, +\infty)} \), let \( \mathbb{P}_\theta \) be the probability measure introduced by the following strong formulation:

\[
\mathbb{P}_\theta := \mathbb{P}_0 \circ (X_\theta)^{-1},
\]

where \( X_\theta := (\int_0^t \theta_s dB_s)_{t \geq 0}, \mathbb{P}_0 \)-a.s. We set \( \mathcal{P} := \{ \mathbb{P}_\theta : \theta \in \mathcal{A}_{[0, +\infty)} \} \) and denote by \( \mathcal{P}_G \) the closure of \( \mathcal{P} \) under the topology of weak convergence.

Consider a capacity formulated by upper probability:

\[
\mathcal{C}(A) := \sup_{\mathbb{P} \in \mathcal{P}_G} \mathbb{P}(A), \quad A \in \mathcal{B}(\Omega).
\]

By Proposition 50 in Denis et al.\([2]\), \( \mathcal{P}_G \) is weakly compact and thus, \( \mathcal{C}(\cdot) \) is a Choquet capacity. Then, we have the following notion of “quasi-surely” (q.s.).

**Definition 2.7.** A set \( A \in \mathcal{B}(\Omega) \) is called polar if \( \mathcal{C}(A) = 0 \). A property is said to hold quasi-surely if it holds outside a polar set.

On the other hand, we set for each \( X \in L^0(\Omega_T), \)

\[
\overline{\mathbb{E}}[X] := \sup_{\mathbb{P} \in \mathcal{P}_G} \mathbb{E}_\mathbb{P}[X].
\]

In (2.1), \( \mathbb{E}_\mathbb{P}[X] \) exists under each \( \mathbb{P} \in \mathcal{P}_G \), so \( \overline{\mathbb{E}}[X] \) is well defined. By Theorem 52 in Denis et al.\([2]\), this upper expectation \( \overline{\mathbb{E}}[\cdot] \) is consistent with G-expectation \( \mathbb{E}[\cdot] \) on \( L^1_G(\Omega_T) \), i.e.,

\[
\overline{\mathbb{E}}[X] = \mathbb{E}[X], \quad \text{for all } X \in L^1_G(\Omega_T).
\]

Thus, from now on, we do not distinguish these two notations \( \mathbb{E}[\cdot] \) and \( \overline{\mathbb{E}}[\cdot] \).

By the definitions of \( \mathbb{E}[\cdot] \) and \( \mathcal{C}(\cdot) \), we can easily deduce the following Markov inequality and the upwards monotone convergence theorem in the G-framework:

**Lemma 2.8.** Let \( X \in L^0(\Omega_T) \) and for some \( p > 0 \), \( \mathbb{E}[|X|^p] < +\infty \). Then, for each \( M > 0 \),

\[
\mathcal{C}(|X| > M) \leq \frac{\mathbb{E}[|X|^p]}{M^p}.
\]

**Theorem 2.9.** Let \( \{X^n\}_{n \in \mathbb{N}} \subset L^0(\Omega_T) \) be a sequence such that \( X^n \uparrow X, \text{ q.s.} \), and there exists a \( \mathbb{P} \in \mathcal{P}_G, \mathbb{E}_\mathbb{P}[X^n] > -\infty \), then \( \overline{\mathbb{E}}[X^n] \uparrow \overline{\mathbb{E}}[X] \).

Unlike the classical downward monotone convergence theorem, the one in the G-framework only holds true for a sequence from a subset of \( L^0(\Omega_T) \) (cf. Theorem 31 in [2]).

**Theorem 2.10.** Let \( \{X^n\}_{n \in \mathbb{N}} \subset L^1_G(\Omega_T) \) be a sequence such that \( X^n \downarrow X, \text{ q.s.} \), then \( \overline{\mathbb{E}}[X^n] \downarrow \overline{\mathbb{E}}[X] \).
Moreover, by a classical argument, we have the following Fatou’s lemma and the inequality of Jensen type in the $G$-framework.

**Lemma 2.11.** Assume that $\{X^n\}_{n \in \mathbb{N}}$ is a sequence in $L^0(\Omega_T)$ and there exists a $Y \in L^0(\Omega_T)$ that satisfies $\mathbb{E}[|Y|] < +\infty$ such that for all $n \in \mathbb{N}$, $X^n \geq Y$, q.s., then

$$\mathbb{E}\left[\liminf_{n \to +\infty} X^n\right] \leq \liminf_{n \to +\infty} \mathbb{E}[X^n].$$

**Proof.** From (2.1) and by the classical Fatou-Lebesgue theorem, we have for each $P \in \mathcal{P}_G$,

$$E_P\left[\liminf_{n \to +\infty} X^n\right] \leq \liminf_{n \to +\infty} E_P[X^n] \leq \liminf_{n \to +\infty} \sup_{P \in \mathcal{P}_G} E_P[X^n] = \liminf_{n \to +\infty} \mathbb{E}[X^n].$$

Taking the supremum of the left-hand side over all $P \in \mathcal{P}_G$, we can easily obtain the desired result. $\square$

**Lemma 2.12.** Let $\rho : \mathbb{R} \to \mathbb{R}$ be an increasing and concave function, then for each $X \in L^0(\Omega_T)$, the following inequality holds:

$$\mathbb{E}[\rho(X)] \leq \rho(\mathbb{E}[X]).$$

A representation theorem for $L^p_G(\Omega_T)$ can also be found in [2]:

**Theorem 2.13.**

$$L^p_G(\Omega_T) = \{X \in L^0(\Omega_T) : X \text{ has a q.c. version, } \lim_{N \to +\infty} \mathbb{E}[|X|^p 1_{|X| > N}] = 0\}.$$

This definition of $L^p_G(\Omega_T)$ is more explicit to verify than the original one given by the completion of $L^0_{ip}(\Omega_T)$.

### 2.3 $G$-stochastic Calculus

In Peng [10], generalized Itô integrals with respect to $G$-Brownian motion and a generalized Itô formula are established.

**Definition 2.14.** A partition of $[0, T]$ is a finite ordered subset $\pi^N_{[0,T]} = \{t_0, t_1, \ldots, t_N\}$ such that $0 = t_0 < t_1 < \cdots < t_N = T$. We set

$$\mu(\pi^N_{[0,T]}) := \max_{k=0,1,\ldots,N-1} |t_{k+1} - t_k|.$$

For each $p \geq 1$, we define

$$M^p_G([0,T]):= \left\{\eta \in \left(\sum_{k=0}^{N-1} \xi_k 1_{[t_k,t_{k+1})}(t) : \xi_k \in L^0_{ip}(\Omega_{t_k})\right)\right\},$$

and we denote by $M^p_G([0,T])$ the completion of $M^p_G([0,T])$ under the norm:

$$\|\eta\|_{M^p_G([0,T])} := \left(\frac{1}{T} \int_0^T \mathbb{E}[|\eta_t|^p] \, dt\right)^{\frac{1}{p}}. \quad (2.2)$$
Remark 2.15. By Definition 2.14, if η is an element in $M^2_G([0,T])$, then there exists a sequence $\{\eta^n\}_{n \in \mathbb{N}}$ in $M^0_G([0,T])$, such that $\lim_{n \to +\infty} \int_0^T \mathbb{E}|\eta_t^n - \eta_t| dt = 0$. It is readily observed that for $t \in [0,T]$, $\lambda$-a.e., $\mathbb{E}|\eta_t^n - \eta_t| \to 0$ and thus, $\eta_t$ is an element in $L^2_B(\Omega_t)$, $\lambda$-a.e.

Let $a = (a_1, \ldots, a_d)^T$ be a given vector in $\mathbb{R}^d$ and $B^a = (a,B)$, where $(a,B)$ denotes the scalar product of $a$ and $B$.

Definition 2.16. For each $\eta \in M^2_G([0,T])$ with the form:

$$\eta_t = \sum_{k=0}^{N-1} \xi_k 1_{(t_k,t_{k+1})}(t),$$

we define

$$\mathcal{I}_{[0,T]}(\eta) = \int_0^T \eta_t dB^a_t := \sum_{k=0}^{N-1} \xi_k (B^a_{t_{k+1}} - B^a_{t_k}),$$

doing the mapping can be continuously extended to $\mathcal{I}_{[0,T]} : M^2_G([0,T]) \to L^2_B(\Omega_T)$. Then, for each $\eta \in M^2_G([0,T])$, the stochastic integral is defined by

$$\int_0^T \eta_t dB^a_t := \mathcal{I}_{[0,T]}(\eta).$$

Let $\langle B^a \rangle$ denote the quadratic variation process of $B^a$, which is formulated in $M^2_G([0,T])$ by

$$\langle B^a \rangle_t := \lim_{\mu(\pi^n_{[0,T]}) \to 0} \sum_{k=0}^{N-1} (B^a_{t_{k+1}} - B^a_{t_k})^2 = (B^a)^2 - 2 \int_0^t B^a_s dB^a_s.$$

We define

$$\sigma_{aaTv} := \sup_{\gamma \in T} (\gamma\gamma_{\mathbb{R}^dTv}).$$

By Corollary 5.7 in Chapter III of [9], we have

$$\langle B \rangle_t \in \mathcal{T}_\Sigma := \{t \times \gamma_{\mathbb{R}^dTv} : \gamma \in \Gamma\}, \quad 0 \leq t \leq T.$$

Therefore, for each $0 \leq s \leq t \leq T$,

$$\langle B^a \rangle_t - \langle B^a \rangle_s \leq \sigma_{aaTv}(t-s). \quad (2.3)$$

Definition 2.17. We define the mapping $Q_{[0,T]} : M^1_G([0,T]) \to L^1_B(\Omega_T)$ as follows:

$$Q_{[0,T]}(\eta) = \int_0^T \eta_t d(B^a)_t := \sum_{k=0}^{N-1} \xi_k ((B^a)^{t_{k+1}}_{t_{k+1}} - (B^a)^{t_{k}}_{t_{k+1}}),$$

and we extend it to $Q_{[0,T]} : M^1_G([0,T]) \to L^1_B(\Omega_T)$. This extended mapping defines $\int_0^T \eta_t d(B^a)_t$ for each $\eta \in M^1_G([0,T])$.

For two given vectors $a, \overline{a} \in \mathbb{R}^d$, the mutual variation process of $B^a$ and $B^{\overline{a}}$ is defined by

$$\langle B^a, B^{\overline{a}} \rangle_t := \frac{1}{4} ((B^a + B^{\overline{a}})^t - (B^a - B^{\overline{a}})^t).$$

Then, for each $\eta \in M^2_G([0,T]),$

$$\int_0^T \eta_t d(B^a, B^{\overline{a}})_t := \frac{1}{4} \left( \int_0^T \eta_t d(B^a + B^{\overline{a}})_t - \int_0^T \eta_t d(B^a - B^{\overline{a}})_t \right).$$
In view of the formulation of G-expectation (2.1) and the property of the quadratic variation process \(<B>_t\) in the G-framework (2.3), the following BDG type inequalities are obvious (cf. Theorems 2.1 and 2.2 in [3]).

**Lemma 2.18.** Let \( p \geq 1, \ a, \bar{a} \in \mathbb{R}^d, \ \eta \in M^p_G([0, T]) \) and \( 0 \leq s \leq t \leq T \). Then,

\[
\mathbb{E} \left[ \sup_{s \leq u \leq t} \left| \int_s^u \eta_r d\langle B^a, B^{\bar{a}} \rangle_r \right|^p \right] \leq \left( \frac{\sigma(a+a, \eta + \bar{a}, \eta + \bar{a})}{4} \right)^p (t-s)^{p-1} \int_s^t \mathbb{E}[|\eta_r|^p] \, dr.
\]

**Lemma 2.19.** Let \( p \geq 2, \ a \in \mathbb{R}^d, \ \eta \in M^p_G([0, T]) \) and \( 0 \leq s \leq t \leq T \). Then,

\[
\mathbb{E} \left[ \sup_{s \leq u \leq t} \left| \int_s^u \eta_r dB_r^a \right|^p \right] \leq C_p \sigma_{ab}^{p/2} (t-s)^{p-1} \left( \int_s^t \mathbb{E}[|\eta_r|^p] \, dr \right),
\]

where \( C_p > 0 \) is a constant independent of \( a, \eta \) and \( \Gamma \).

At the end of this subsection, we introduce the following G-Itô formula that can be found as Proposition 6.3 in Chapter III of [9]. For each \( 0 \leq s \leq t \leq T \), consider an \( n \)-dimensional G-Itô process:

\[
X^i_t = X^i_s + \int_s^t b^i_u \, du + \sum_{i,j=1}^d \int_s^t h^{ij}_u \, dB^i_u + \sum_{i,j=1}^d \int_s^t g^{ij}_u \, dB^j_u, \quad \nu = 1, \ldots, n.
\]

**Lemma 2.20.** Let \( \Phi \in C^2(\mathbb{R}^n) \) be a real function with bounded derivatives such that \( \{ \partial^2_{x^i x^j} \Phi \}_{i,j=1}^n \) are uniformly Lipschitz. Let \( b^i, h^{ij} \) and \( g^{ij} \in M^2_G([0, T]), \ \nu = 1, \ldots, n, \ i, j = 1, \ldots, d \) be bounded processes. Then, we have

\[
\Phi(X_t) - \Phi(X_s) = \int_s^t \partial_x \Phi(X_u) b^i_u \, du + \int_s^t \partial_x \Phi(X_u) h^{ij}_u \, dB^i_u + \int_s^t g^{ij}_u \, dB^j_u \bigg|_s^t + \frac{1}{2} \int_s^t \partial^2_{x^i x^j} \Phi(X_u) g^{ij}_u \, dB^i_u \bigg|_s^t,
\]

in which the equality holds in the sense of \( L^2_G(\Omega_t) \).

**Remark 2.21.** In (2.4), we adopt the Einstein convention, i.e., the repeated indices \( \nu, \mu, i \) and \( j \) imply summation.

### 3 Solvability of GSDEs with Integral-Lipschitz Coefficients

In this section, we give our main result of this paper, that is, the existence and uniqueness theorems for GSDEs with integral-Lipschitz coefficients. From now on, \( C \) denotes a positive constant whose value may vary from line to line.

#### 3.1 Formulation to GSDEs and Assumptions

We rewrite (1.1) into the following form:

\[
X(t) = x + \int_0^t b(s, X(s)) \, ds + \sum_{i,j=1}^d \int_0^t h_{ij}(s, X(s)) \, dB^i_s \bigg|_0^t + \sum_{j=1}^d \int_0^t g_j(s, X(s)) \, dB^j_s, \quad 0 \leq t \leq T,
\]

(3.1)
Assumption 3.2. For each \( t \in [0, T] \) and \( x, x_1, x_2 \in \mathbb{R}^n \),

\[
(H1) \left| b(t, x_1) - b(t, x_2) \right|^2 + \left| h(t, x_1) - h(t, x_2) \right|^2 + \left| g(t, x_1) - g(t, x_2) \right|^2 \leq \beta(t)^2 \rho_1(|x_1 - x_2|^2);
\]

\[
(H2) \left| b(t, x) \right|^2 + \left| h(t, x) \right|^2 + \left| g(t, x) \right|^2 \leq \beta(t)^2 + \beta_2^2 |x|^2,
\]

where \( \beta : [0, T] \to \mathbb{R}^+ \) is square integrable, \( \beta_1 \in M^2_G([0, T]) \), \( \beta_2 \in \mathbb{R}^+ \) and \( \rho : (0, +\infty) \to (0, +\infty) \) is a continuous increasing and concave function that vanishes at 0+ and satisfies

\[
\int_0^1 \frac{dr}{\rho(r)} = +\infty.
\]  

(3.2)

Assumption 3.3. For each \( t \in [0, T] \) and \( x, x_1, x_2 \in \mathbb{R}^n \),

\[
(H1') \left\{ \begin{array}{l}
\left| b(t, x_1) - b(t, x_2) \right| + \left| h(t, x_1) - h(t, x_2) \right| \leq \beta(t) \rho_1(|x_1 - x_2|); \\
\left| g(t, x_1) - g(t, x_2) \right|^2 \leq \beta(t)^2 \rho_2(|x_1 - x_2|^2);
\end{array} \right.
\]

(\(H2'\)) \( |b(t, x)|^p + |h(t, x)|^p + |g(t, x)|^p \leq \beta_1(t)^p + \beta_2^p |x|^p,
\)

where \( \beta : [0, T] \to \mathbb{R}^+ \) is square integrable, for some \( p > 2 \), \( \beta_1 \in M^p_G([0, T]) \), \( \beta_2 \in \mathbb{R}^+ \) and both \( \rho_1, \rho_2 : (0, +\infty) \to (0, +\infty) \) are continuous increasing and concave functions that vanish at 0+ and satisfy (3.2). We assume moreover that

\[
\rho_3(r) := \frac{\rho_2(r^2)}{r}, \quad r \in (0, +\infty),
\]

is also a continuous increasing and concave function that vanishes at 0+ and satisfies

\[
\int_0^1 \frac{dr}{\rho_1(r) + \rho_3(r)} = +\infty.
\]

Remark 3.3. We give an example to show that (H1') is “weaker” than (H1). If we set

\[
\left\{ \begin{array}{l}
\rho_1(r) = r \ln \frac{r}{1}; \\
\rho_2(r) = r \ln \frac{1}{r},
\end{array} \right. \quad r \in (0, +\infty),
\]

then (H1') is satisfied but (H1) is not.

To ensure that (3.1) is well defined, all the integrands in (3.1) should be in \( M^2_G([0, T]; \mathbb{R}^n) \). Thus, we need the following lemma:

Lemma 3.4. For some \( q \geq 1 \), \( \zeta \) is a function that satisfies for each \( x \in \mathbb{R}^n \), \( \zeta(\cdot, x) \in M^q_G([0, T]; \mathbb{R}^n) \). We assume moreover that, for each \( x, x_1, x_2 \in \mathbb{R}^n \):

(\(A1\)) \( \left| \zeta(t, x_1) - \zeta(t, x_2) \right| \leq \beta(t) \gamma(|x_1 - x_2|) \);

(\(A2\)) \( \left| \zeta(t, x) \right| \leq [\beta_1(t) + \beta_2|x|] \),

where \( \beta : [0, T] \to \mathbb{R}^+ \) is \( q \)-integrable, \( \beta_1 \in M^q_G([0, T]) \), \( \beta_2 \in \mathbb{R}^+ \) and \( \gamma : (0, +\infty) \to (0, +\infty) \) is an increasing function vanishes at 0+. Then, for each \( X \in M^q_G([0, T]; \mathbb{R}^n) \), \( \zeta(\cdot, X) \) is an element in \( M^q_G([0, T]; \mathbb{R}^n) \).

Remark 3.5. When \( q = 2 \), all the coefficients in the GSDE (3.1) satisfy both (A1) and (A2) under either Assumption 3.1 or 3.2. Therefore, the \( G \)-stochastic integrals in the GSDE (3.1)
are well defined for any solution \( X \in M^2_G([0,T];\mathbb{R}^n) \). We postpone the proof of this lemma to the appendix.

### 3.2 Main Result

As a starting point, we first refer to an inequality in Bihari\(^1\) (Bihari’s inequality). Then, we prove the existence and uniqueness theorem for the GSDE (3.1) under Assumption 3.1.

**Lemma 3.6.** Let \( \rho : (0, +\infty) \to (0, +\infty) \) be a continuous and increasing function that vanishes at 0+ and satisfies (3.2). Let \( u \) be a measurable and non-negative function defined on \((0, +\infty)\) that satisfies

\[
 u(t) \leq a + \int_0^t \kappa(s)\rho(u(s))\,ds, \quad t \in (0, +\infty),
\]

where \( a \in \mathbb{R}^+ \) and \( \kappa : [0,T] \to \mathbb{R}^+ \) is Lebesgue integrable. We have

1. If \( a = 0 \), then \( u(t) = 0, \quad t \in (0, +\infty) \), \( \lambda \)-a.e.;
2. If \( a > 0 \), we define

\[
 v(t) := \int_{t_0}^t \frac{1}{\rho(s)}\,ds, \quad t \in \mathbb{R}^+,
\]

where \( t_0 \in (0, +\infty) \), then

\[
 u(t) \leq v^{-1}(v(a) + \int_0^t \kappa(s)\,ds).
\]

**Theorem 3.7.** Under Assumption 3.1, there exists a unique process \( X \in M^2_G([0,T];\mathbb{R}^n) \) that satisfies the GSDE (3.1).

**Proof.** We begin with the proof of the uniqueness. Suppose \( X(\cdot; x_i) \in M^2_G([0,T];\mathbb{R}^n) \) is a solution to the GSDE (3.1) with initial value \( x_i, \quad i = 1, 2 \), then we calculate

\[
 |X(t; x_1) - X(t; x_2)|^2 \leq C \left( |x_1 - x_2|^2 + \left| \int_0^t (b(s, X(s; x_1)) - b(s, X(s; x_2)))\,ds \right|^2 \right.
\]
\[
 + \left. \left| \sum_{i,j=1}^d \int_0^t (h_{ij}(s, X(s; x_1)) - h_{ij}(s, X(s; x_2)))d(B^i_t, B^j_t) \right|^2 \right.
\]
\[
 + \left. \left| \sum_{i,j=1}^d \int_0^t (g_{ij}(s, X(s; x_1)) - g_{ij}(s, X(s; x_2)))dB^i_t \right|^2 \right).
\]

By the BDG type inequalities and (H1), we deduce

\[
 \mathbb{E}\left[ \sup_{0 \leq s \leq t} \left| \int_0^s (b(r, X(r; x_1)) - b(r, X(r; x_2)))\,dr \right|^2 \right] \leq C \int_0^t |\beta(s)|^2 \mathbb{E}[\rho(|X(s; x_1) - X(s; x_2)|^2)] \,ds;
\]

\[
 \mathbb{E}\left[ \sup_{0 \leq s \leq t} \left| \int_0^s (h_{ij}(r, X(r; x_1)) - h_{ij}(r, X(r; x_2)))d(B^i_t, B^j_t) \right|^2 \right] \leq C \int_0^t |\beta(s)|^2 \mathbb{E}[\rho(|X(s; x_1) - X(s; x_2)|^2)] \,ds
\]

\[
 \mathbb{E}\left[ \sup_{0 \leq s \leq t} \left| \int_0^s (g_{ij}(r, X(r; x_1)) - g_{ij}(r, X(r; x_2)))dB^i_t \right|^2 \right] \leq C \int_0^t |\beta(s)|^2 \mathbb{E}[\rho(|X(s; x_1) - X(s; x_2)|^2)] \,ds
\]
and

\[
E \left[ \sup_{0 \leq s \leq t} \left| \int_0^s (g_j(r, X(r; x_1)) - g_j(r, X(r; x_2))) dB_t^j \right|^2 \right] \leq C \int_0^t |\beta(s)|^2 E[\rho(|X(s; x_1) - X(s; x_2)|^2)]\, ds.
\]

Set

\[
u(t) := \sup_{0 \leq s \leq t} E[|X(s; x_1) - X(s; x_2)|^2],
\]

then

\[
u(t) \leq C \left( |x_1 - x_2|^2 + \int_0^t |\beta(s)|^2 E[\rho(|X(s; x_1) - X(s; x_2)|^2)]\, ds \right).
\]

As \( \rho \) is an increasing and concave function, by Lemma 2.12, we have

\[
u(t) \leq C \left( |x_1 - x_2|^2 + \int_0^t |\beta(s)|^2 E[\rho(\sup_{0 \leq r \leq s} E[|X(r; x_1) - X(r; x_2)|^2])]\, ds \right)
\]

\[
\leq C \left( |x_1 - x_2|^2 + \int_0^t |\beta(s)|^2 (\rho(u(s)))\, ds \right).
\]

By Lemma 3.6, we obtain

\[
u(t) \leq v^{-1} \left( v(C|x_1 - x_2|^2) + C \int_0^t |\beta(s)|^2\, ds \right).
\]

In particular, if \( x_1 = x_2 \), then \( \nu(t) = 0, \ 0 \leq t \leq T \), which implies the pathwise uniqueness.

Now we start to prove the existence. We define a Picard sequence \( \{X^m(\cdot)\}_{m \in \mathbb{N}} \) by the following procedure:

\[
X^0(t) = x, \quad 0 \leq t \leq T,
\]

and

\[
X^{m+1}(t) = x + \int_0^t b(s, X^m(s))\, ds + \sum_{i,j=1}^d \int_0^t h_{ij}(s, X^m(s))\, dB_t^j + \sum_{j=1}^d \int_0^t g_j(s, X^m(s))\, dB_t^j, \quad 0 \leq t \leq T. \tag{3.3}
\]

By Lemma 3.4, the sequence \( \{X^m(\cdot)\}_{m \in \mathbb{N}} \) is well defined in \( M^2_G([0, T]; \mathbb{R}^n) \).

First, we establish an a priori estimate for \( \{E[|X^m(t)|^2]\}_{m \in \mathbb{N}} \). From (3.3), by the BDG type inequalities, we may deduce

\[
E[|X^{m+1}(t)|^2] \leq C \left( |x|^2 + \int_0^t E[|\beta_1(s)|^2 + \beta_2^2 |X^m(s)|^2]\, ds \right)
\]

\[
\leq C \left( |x|^2 + \int_0^t E[|\beta_1(s)|^2]\, ds + \beta_2^2 \int_0^t E[|X^m(s)|^2]\, ds \right).
\]

Set

\[
p(t) := C e^{C \beta_2 t} \left( |x|^2 + \int_0^t E[|\beta_1(s)|^2]\, ds \right).
\]
then \( p(\cdot) \) is the solution to the following ordinary differential equation:

\[
p(t) = C \left( |x|^2 + \int_0^t \mathbb{E}[|\beta_s|^2] ds + \beta_2^2 \int_0^t p(s) ds \right).
\]

By recurrence, it is easy to verify that for each \( m \in \mathbb{N} \),

\[
\mathbb{E}[|X^m(t)|^2] \leq p(t),
\]

the right-hand side of which is continuous and therefore, bounded on \([0, T]\).

Secondly, for each \( k, m \in \mathbb{N} \), we define

\[
u_{k+1, m}(t) := \sup_{0 \leq s \leq t} \mathbb{E}[|X^{k+1+m}(s) - X^{k+1}(s)|^2].
\]

By the definition of the sequence \( \{X^m(\cdot)\}_{m \in \mathbb{N}} \), we have

\[
X^{k+1+m}(t) - X^{k+1}(t) = \int_0^t (b(s, X^{k+m}(s)) - b(s, X^{k}(s))) ds
\]

\[
+ \sum_{i,j=1}^d \int_0^t (h_{ij}(s, X^{k+m}(s)) - h_{ij}(s, X^{k}(s))) d\langle B^i, B^j \rangle_s
\]

\[
+ \sum_{j=1}^d \int_0^t (g_j(s, X^{k+m}(s)) - g_j(s, X^{k}(s))) dB^j_s.
\]

By an argument similar to the one in the proof of the uniqueness, we obtain

\[
u_{k+1, m}(t) \leq C \int_0^t |\beta(s)|^2 \rho(u_{k,m}(s)) ds.
\]

Set

\[
u_k(t) := \sup_{m \in \mathbb{N}} u_{k,m}(t), \quad 0 \leq t \leq T,
\]

then

\[
0 \leq \nu_{k+1}(t) \leq C \int_0^t |\beta(s)|^2 \rho(\nu_k(s)) ds. \tag{3.4}
\]

Finally, we define

\[
\alpha(t) := \limsup_{k \to +\infty} \nu_k(t), \quad 0 \leq t \leq T,
\]

which is uniformly bounded by \( 4p(t) \). Applying the Fatou-Lebesgue theorem to (3.4), we have

\[
0 \leq \alpha(t) \leq C \int_0^t \beta^2(s) \rho(\alpha(s)) ds.
\]

By Lemma 3.6, we deduce

\[
\alpha(t) = 0, \quad 0 \leq t \leq T,
\]

which implies that \( \{X^m(\cdot)\}_{m \in \mathbb{N}} \) is a Cauchy sequence under the norm \( \sup_{0 \leq t \leq T} (\mathbb{E}[|\cdot|^2])^{\frac{1}{2}} \), which is stronger than the \( M_2^d([0, T]; \mathbb{R}^n) \) norm (2.2). Therefore, one can find a process \( X \in M_2^d([0, T]; \mathbb{R}^n) \) that satisfies

\[
\sup_{0 \leq t \leq T} \mathbb{E}[|X^m(t) - X(t)|^2] \to 0, \quad \text{as } m \to +\infty.
\]
Moreover, it is readily observed that
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t (b(s, X^m(s)) - b(s, X(s))) ds \right|^2 \right]
\]
\[+ \sum_{i,j=1}^d \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t (h_{ij}(s, X^m(s)) - h_{ij}(s, X(s))) d(B^i_s, B^j_s) \right|^2 \right]
\]
\[+ \sum_{i=1}^d \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t (g_j(s, X^m(s)) - g_j(s, X(s))) dB^j_s \right|^2 \right]
\]
\[\leq C \int_0^T |\beta(t)|^2 \rho(\mathbb{E}[|X^m(t) - X(t)|^2]) \, dt
\]
\[\leq C \rho \left( \sup_{0 \leq t \leq T} \mathbb{E}[|X^m(t) - X(t)|^2] \right). \tag{3.5}
\]

By the continuity of \(\rho\) and \(\rho(0+) = 0\), we know that \(\rho(\sup_{0 \leq t \leq T} \mathbb{E}[|X^m(t) - X(t)|^2]) \to 0\) and the left-hand side of (3.5) converges to 0. Thus, \(\{X^m(\cdot)\}_{m \in \mathbb{N}}\) is a successive approximation to \(X\), which is a solution to the GSDE (3.1) in \(M^2_G([0, T]; \mathbb{R}^n)\).

In what follows, we establish the existence and uniqueness theorem to GSDE (3.1) under Assumption 3.2 instead of Assumption 3.1.

**Theorem 3.8.** Under Assumption 3.2 there exists a unique process \(X \in M^2_G([0, T]; \mathbb{R}^n)\) that satisfies GSDE (3.1).

**Proof.** We start with the proof of existence. Similar to (3.3), we define a sequence of processes \(\{X^m\}_{m \in \mathbb{N}}\) as follows:
\[X^0(t) = x, \quad 0 \leq t \leq T;\]
and
\[X^{m+1}(t) = x + \int_0^t b(s, X^m(s)) ds + \sum_{i,j=1}^d \int_0^t h_{ij}(s, X^m(s)) d(B^i_s, B^j_s)
\] \[+ \sum_{j=1}^d \int_0^t g_j(s, X^{m+1}(s)) dB^j_s, \quad 0 \leq t \leq T. \tag{3.6}\]

Owing to Theorem 3.7, the sequence \(\{X^m\}_{m \in \mathbb{N}}\) is well defined in \(M^2_G([0, T]; \mathbb{R}^n)\).

We notice that the coefficients in (3.6) cannot not be bounded. In order to apply the G-Itô formula, we shall firstly construct, for each \(m \in \mathbb{N}\), a sequence of G-Itô processes that approximates \(X^m\), and whose coefficients are all truncated. These sequences are given by the following steps:

Step 1: For each \(N \in \mathbb{N}\), we set
\[\zeta^N(t, x) = \begin{cases} \zeta(t, x), & \text{if } |\zeta(t, x)| \leq N; \\ N\zeta(t, x)/|\zeta(t, x)|, & \text{if } |\zeta(t, x)| > N, \end{cases} \tag{3.7}\]
where \(\zeta = b, \ h_{ij}\) or \(g_j\), \(i, j = 1, \cdots, d\), respectively. It is easy to verify that \(b^N, h^N_{ij}\) and \(g^N_j\) still satisfy (H1’) and (H2’).
Step 2: For each $m \in \mathbb{N}$, we define

$$X^{m+1,N}(t) = x + \int_0^t b^N(s, X^m(s)) \, ds + \sum_{i,j=1}^d \int_0^t h^N_{ij}(s, X^m(s)) \, d(B^i, B^j)_s$$

$$+ \sum_{j=1}^d \int_0^t g^N_j(s, X^{m+1}(s)) \, dB^j_s, \quad 0 \leq t \leq T.$$  

By Lemma 3.4, the sequence $\{X^{m,N}(\cdot)\}_{N \in \mathbb{N}}$ is also well defined in $M^2_{\mathcal{F}}([0, T]; \mathbb{R}^n)$.

Let us now establish an a priori estimate for $\{\mathbb{E}[|X^m(t)|^p]\}_{m \in \mathbb{N}}$. By (H2) and the BDG type inequalities,

$$\mathbb{E}[|X^{m+1}(t)|^p] \leq C \left( |x|^p + \int_0^t \mathbb{E}[|\beta_1(s)|^p] \, ds + \beta_2^p \int_0^t \mathbb{E}[|X^m(s)|^p] \, ds \right).$$

By induction, we obtain that $\mathbb{E}[|X^m(t)|^p] \leq p'(t)$, where $p'(\cdot)$ is the solution to the following ordinary differential equation with coefficients in (3.8):

$$p'(t) = C \left( |x|^p + \int_0^t \mathbb{E}[|\beta_1(s)|^p] \, ds + \beta_2^p \int_0^t p'(s) \, ds \right).$$

Since $p'(\cdot)$ is continuous and bounded on $[0, T]$, we have

$$\sup_{m \in \mathbb{N}} \sup_{0 \leq t \leq T} \mathbb{E}[|X^m(t)|^p] \leq M < +\infty. \quad (3.9)$$

Fixing $m > 0$, we calculate

$$\sup_{0 \leq t \leq T} \mathbb{E}[|X^{m,N}(t) - X^m(t)|]$$

$$\leq \mathbb{E} \left[ \int_0^T |b^N(t, X^m(t)) - b(t, X^m(t))| \, dt \right]$$

$$+ \sum_{i,j=1}^d \mathbb{E} \left[ \int_0^T |h^N_{ij}(t, X^m(t)) - h_{ij}(t, X^m(t))| \, d(B^i, B^j)_s \right]$$

$$+ \sup_{0 \leq t \leq T} \sum_{j=1}^d \mathbb{E} \left[ \int_0^t (g^N_j(s, X^{m+1}(s)) - g_j(s, X^{m+1}(s))) \, dB^j_s \right].$$

By the definition of the truncated coefficients and the BDG type inequalities, we deduce that

$$\sup_{0 \leq t \leq T} \mathbb{E}[|X^{m,N}(t) - X^m(t)|]$$

$$\leq \int_0^T \mathbb{E}[|b(t, X^m(t))| \mathbf{1}_{|b(t, X^m(t))| > N}] \, dt$$

$$+ C \left( \sum_{i,j=1}^d \int_0^T \mathbb{E}[|h_{ij}(t, X^m(t))| \mathbf{1}_{|h_{ij}(t, X^m(t))| > N}] \, dt \right)$$

$$+ \sum_{j=1}^d \left( \int_0^T \mathbb{E}[|g_j(t, X^{m+1}(t))|^{2 \mathbf{1}_{|g_j(t, X^{m+1}(t))| > N}}] \, dt \right)^{\frac{1}{2}}. \quad (3.10)$$
By Lemma 3.4, for each \( m \in \mathbb{N} \), \( h_i(\cdot, X^m) \), \( h_{ij}(\cdot, X^m) \), \( g_j(\cdot, X^m) \in M^2_G([0, T]; \mathbb{R}^n) \), \( i = 1, \ldots, d \). Then, by Remark 2.15 and Theorem 2.13 along with Lebesgue’s dominated convergence theorem, the right-hand side of (3.10) converges to 0. Therefore,

\[
\sup_{0 \leq t \leq T} \mathbb{E}[|X^{m,N}(t) - X^m(t)|] \longrightarrow 0, \quad \text{as } N \to +\infty. \tag{3.11}
\]

Since \(|x|\) is not a \( C^2(\mathbb{R}^n) \) function, we have to approximate \(|x|\) by a sequence of \( C^2(\mathbb{R}^n) \) functions, i.e., \( \{F_\varepsilon(x)\}_{\varepsilon>0} \), where

\[
F_\varepsilon(x) := (|x|^2 + \varepsilon)^{\frac{1}{2}}, \quad x \in \mathbb{R}^n.
\]

We notice that

\[
F_\varepsilon(x) \geq \varepsilon^{\frac{1}{2}}, \quad \left| \frac{\partial F_\varepsilon(x)}{\partial x_i} \right| \leq 1; \quad \left| \frac{\partial^2 F_\varepsilon(x)}{\partial x_i \partial x_j} \right| \leq \frac{2}{F_\varepsilon(x)},
\]

and thus, \( \frac{\partial F_\varepsilon(x)}{\partial x_i}, \frac{\partial^2 F_\varepsilon(x)}{\partial x_i \partial x_j}, \ i, j = 1, \ldots, n \), are uniformly Lipschitz.

Fixing an \( \varepsilon \in (0, +\infty) \), we define

\[
\Delta F^{k,m,N}_\varepsilon(t) := F_\varepsilon(\Delta X^{k,m,N}(t)) - F_\varepsilon(\Delta X^{k,m}(t)),
\]

where

\[
\Delta X^{k,m,N}(t) = X^{k+m,N}(t) - X^{k,N}(t)
\]

and

\[
\Delta X^{k,m}(t) = X^{k+m}(t) - X^{k}(t).
\]

We apply the \( G \)-Itô formula to \( F_\varepsilon(\Delta X^{k+1,m,N}(t)) \) and take \( G \)-expectation on both sides. Then, from (3.12) and by the BDG type inequalities, it is easy to show that

\[
\mathbb{E}[F_\varepsilon(\Delta X^{k+1,m,N}(t))]
\]

\[
\leq \int_0^t \mathbb{E}[|b^N(s, X^{k+m,N}(s)) - b^N(s, X^{k,N}(s))|] \, ds
\]

\[
+ C \left( \sum_{i,j=1}^d \int_0^t \mathbb{E}[|h^N_{ij}(s, X^{k+m,N}(s)) - h^N_{ij}(s, X^{k,N}(s))|] \, ds \right.
\]

\[
+ \left. \sum_{j=1}^d \int_0^t \mathbb{E} \left[ \frac{|g_j^N(s, X^{k+m+1,N}(s)) - g_j^N(s, X^{k+1,N}(s))|^2}{F_\varepsilon(\Delta X^{k+1,m,N}(s))} \right] \, ds \right), \quad 0 \leq t \leq T. \tag{3.13}
\]

By (H1’) and Lemma 2.12, we deduce from (3.13) that

\[
\mathbb{E}[F_\varepsilon(\Delta X^{k+1,m,N}(t))] \leq C \int_0^t \beta(s) \left( \rho_1(\mathbb{E}[|\Delta X^{k,m,N}(s)|]) + \mathbb{E} \left[ \rho_2(\frac{|\Delta X^{k+1,m,N}(s)|^2}{F_\varepsilon(\Delta X^{k+1,m,N}(s))}) \right] \right) \, ds. \tag{3.14}
\]

From (3.12), we know that \( F_\varepsilon(x) \) is uniformly Lipschitz. Based on this fact and (3.11), we obtain

\[
\sup_{0 \leq t \leq T} \mathbb{E}[|\Delta F^{k+1,m,N}_\varepsilon(t)|]
\]

\[
\leq \sup_{0 \leq t \leq T} \mathbb{E}[|\Delta X^{k+1,m,N}(t) - \Delta X^{k+1,m}(t)|]
\]

\[
\leq \sup_{0 \leq t \leq T} \mathbb{E}[|X^{k+1,m,N}(t) - X^{k+1,m,N}(t)|]
\]

\[
\quad + \sup_{0 \leq t \leq T} \mathbb{E}[|X^{k+1,m,N}(t) - X^{k+1,m+N}(t)|] \longrightarrow 0, \quad \text{as } N \to +\infty. \tag{3.15}
\]
Since \( \rho_1, \rho_2 : (0, +\infty) \to (0, +\infty) \) are concave and vanish at 0+, for each \( \delta \in (0, +\infty) \), we can find a positive constant \( K_\delta \) such that for each \( x \in [\delta, +\infty) \), \( \rho_1(x), \rho_2(x) \leq K_\delta x \). Fixing a \( \delta > 0 \) and \( M \in (\delta, +\infty) \), we calculate

\[
\sup_{0 \leq t \leq T} \mathbb{E} \left[ \frac{\rho_2(\Delta X^{k+1,m}(t))^2}{F_\varepsilon(\Delta X^{k+1,m,N}(t))} \right] \leq 2^{t} K_\delta \sup_{0 \leq t \leq T} \mathbb{E}[|\Delta X^{k+1,m}(t)|^21_{|\Delta X^{k+1,m}(t)|^2 > M}] + \varepsilon^{-1} \rho_2(M) \sup_{0 \leq t \leq T} \mathbb{E}[|\Delta F_\varepsilon^{k+1,m,N}(t)|].
\]

On account of (3.15) and by Hölder’s inequality and Lemma 2.8,

\[
\limsup_{N \to +\infty} \sup_{0 \leq t \leq T} \mathbb{E} \left[ \frac{\rho_2(\Delta X^{k+1,m}(t))^2}{F_\varepsilon(\Delta X^{k+1,m,N}(t))} - \frac{\rho_2(\Delta X^{k+1,m}(t))^2}{F_\varepsilon(\Delta X^{k+1,m}(t))} \right] \leq \frac{2K_\delta}{\varepsilon^{2} M^{p-2}} \sup_{0 \leq t \leq T} \mathbb{E}[|\Delta X^{k+1,m}(t)|^p].
\]

As \( M \) can be arbitrary large and \( \mathbb{E}[|\Delta X^{k+1,m}(t)|^p] \) is finite from (3.9), we deduce

\[
\lim_{N \to +\infty} \sup_{0 \leq t \leq T} \mathbb{E} \left[ \frac{\rho_2(\Delta X^{k+1,m}(t))^2}{F_\varepsilon(\Delta X^{k+1,m,N}(t))} \right] = 0.
\]

Due to (3.15) again, the left-hand side of (3.14) converges to \( \mathbb{E}[F_\varepsilon(\Delta X^{k+1,m}(t))], \) as \( N \to +\infty \). Then, by the Fatou-Lebesgue theorem, we have

\[
\mathbb{E}[F_\varepsilon(\Delta X^{k+1,m}(t))] \leq C \limsup_{N \to +\infty} \int_0^t \beta(s) \left( \rho_1(\mathbb{E}[\Delta X^{k,m}(s)]) + \mathbb{E} \left[ \frac{\rho_2(\Delta X^{k+1,m}(s))^2}{F_\varepsilon(\Delta X^{k+1,m,N}(s))} \right] \right) ds \\
\leq C \int_0^t \beta(s) \left( \rho_1(\mathbb{E}[\Delta X^{k,m}(s)]) + \limsup_{N \to +\infty} \mathbb{E} \left[ \frac{\rho_2(\Delta X^{k+1,m}(s))^2}{F_\varepsilon(\Delta X^{k+1,m,N}(s))} \right] \right) ds \\
= C \int_0^t \beta(s) \left( \rho_1(\mathbb{E}[\Delta X^{k,m}(s)]) + \mathbb{E} \left[ \frac{\rho_2(\Delta X^{k+1,m}(s))^2}{F_\varepsilon(\Delta X^{k+1,m}(s))} \right] \right) ds.
\]

Letting \( \varepsilon \to 0 \), \( F_\varepsilon(\Delta X^{k+1,m}(t)) \downarrow |\Delta X^{k+1,m}(t)| \). By Remark 2.15, for \( t \in [0, T] \), \( \lambda \)-a.e., \( \Delta X^{k+1,m}(t) \) belongs to \( L_p^0(\Omega) \). One the other hand, for each \( \varepsilon > 0 \), \( F_\varepsilon(x) \) is Lipschitz in \( x \), then \( F_\varepsilon(\Delta X^{k+1,m}(t)) \) is also an element in \( L_p^0(\Omega) \). By Theorem 2.10, (H1’) and Lemma 2.12, we obtain \( \mathbb{E}[F_\varepsilon(\Delta X^{k+1,m}(t))] \right) \leq \mathbb{E}[\Delta X^{k+1,m}(t)] \) and the following inequality:

\[
\mathbb{E}[|\Delta X^{k+1,m}(t)|] \leq C \int_0^t \beta(s) \left( \rho_1(\mathbb{E}[\Delta X^{k,m}(s)]) + \mathbb{E} \left[ \frac{\rho_2(\Delta X^{k+1,m}(s))^2}{|\Delta X^{k+1,m}(s)|} \right] \right) ds \\
\leq C \int_0^t \beta(s) \left( \rho_1(\mathbb{E}[\Delta X^{k,m}(s)]) + \mathbb{E} \left[ \frac{\rho_2(\Delta X^{k+1,m}(s))^2}{|\Delta X^{k+1,m}(s)|} \right] \right) ds \\
= C \int_0^t \beta(s) \left( \rho_1(\mathbb{E}[\Delta X^{k,m}(s)]) + \rho_3(\mathbb{E}[\Delta X^{k+1,m}(s)]) \right) ds.
\]

Borrowing the notations in the proof of Theorem 3.7, we rewrite (3.17) into a simpler form:

\[
u_{k+1,m}(t) \leq C \int_0^t \beta(s)(\rho_1(u_{k,m}(s)) + \rho_3(u_{k+1,m}(s))) ds.
\]
Taking the supremum of the left-hand side over all \( m \in \mathbb{N} \), we have
\[
0 \leq v_{k+1}(t) \leq C \int_0^t \beta(s)(\rho_1(v_k(s)) + \rho_3(v_{k+1}(s)))ds,
\]
and it follows that
\[
0 \leq \alpha(t) \leq C \int_0^t \beta(s)(\rho_1(\alpha(s)) + \rho_3(\alpha(s)))ds.
\]
By (H1') and Lemma 3.6, we deduce that the last equality in (3.18) may be easily deduced by Lemma 2.8. Letting
\[
\lim_{T \to +\infty} \sup_{0 \leq t \leq T} \mathbb{E}|X^m(t) - X(t)|^p = 0,
\]
as \( N \to +\infty \).

By a classical argument, there exists a process \( X(\cdot) \in M_G^1([0, T]; \mathbb{R}^n) \) and a subsequence \( \{X^{m_k}(\cdot)\}_{k \in \mathbb{N}} \subset \{X^m(\cdot)\}_{m \in \mathbb{N}} \) such that for each \( t \in [0, T] \),
\[
X^{m_k}(t) \to X(t), \text{ q.s., as } l \to +\infty.
\]
By the a priori estimate (3.9) and Lemma 2.11, we know
\[
\lim_{l \to +\infty} \sup_{0 \leq t \leq T} \mathbb{E}|X^{m_l}(t) - X(t)|^p = \sup_{l \to +\infty} \mathbb{E}\liminf_{l \to +\infty} |X^{m_l}(t) - X^{m_l}(t)|^p.
\]
Fixing a \( \delta \in (0, +\infty) \), we calculate
\[
\lim_{m \to +\infty} \sup_{0 \leq t \leq T} \mathbb{E}|X^m(t) - X(t)|^2
\]
\[
\leq \delta^2 + \limsup_{m \to +\infty} \sup_{0 \leq t \leq T} \mathbb{E}|X^m(t) - X(t)|^2 1_{\{|X^m(t) - X(t)| > \delta\}}
\]
\[
\leq \delta^2 + \limsup_{m \to +\infty} \sup_{0 \leq t \leq T} \left( \frac{\mathbb{E}|X^m(t) - X(t)|^p}{M^p} \right) \delta \left( \mathbb{E}1_{\{|X^m(t) - X(t)| > \delta\}} \right)^{\frac{p-2}{p}}
\]
\[
\leq \delta^2 + M\sup_{0 \leq t \leq T} \left( \mathbb{E}1_{\{|X^m(t) - X(t)| > \delta\}} \right)^{\frac{p-2}{p}}.
\]
Because
\[
\lim_{m \to +\infty} \sup_{0 \leq t \leq T} \mathbb{E}|X^m(t) - X(t)| = 0,
\]
the last equality in (3.18) may be easily deduced by Lemma 2.8. Letting \( \delta \to 0 \), we obtain
\[
\lim_{m \to +\infty} \sup_{0 \leq t \leq T} \mathbb{E}|X^m(t) - X(t)|^2 = 0.
\]
On the other hand, fixing a \( \delta \in (0, +\infty) \), we have the following inequality in a way similar to (3.16):
\[
\limsup_{m \to +\infty} \int_0^t \beta^2(s) \mathbb{E}(|\rho_1(|X^m(t) - X(t)|)|^2) \, dt
\]
\[
\leq C(\rho_1(\delta^2))^2 + K\sup_{0 \leq t \leq T} \mathbb{E}|X^m(t) - X(t)|^2
\]
\[
= C(\rho_1(\delta^2))^2.
\]
As $\delta$ may be chosen to be arbitrarily small, by (H1'), we deduce
\[
\lim_{m \to \infty} E \left[ \sup_{0 \leq r \leq T} \left| \int_0^t (b(s, X^m(s)) - b(s, X(s))) ds \right|^2 \right] = 0
\]
and
\[
\lim_{m \to \infty} E \left[ \sup_{0 \leq r \leq T} \left| \int_0^t (h_{ij}(s, X^m(s)) - h_{ij}(s, X(s))) d(B^i, B^j)_s \right|^2 \right] = 0, \quad i, j = 1, \ldots, d.
\]
Moreover, by the BDG type inequalities and Lemma 2.12, we may also deduce
\[
\limsup_{m \to +\infty} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t (g_j(s, X^m(s)) - g_j(s, X(s))) dB^j_s \right|^2 \right] = 0, \quad j = 1, \ldots, d.
\]
In view of the above arguments, we conclude that $X \in M^2_{\mathbb{L}}([0, T]; \mathbb{R}^n)$ is a solution to the GSDE (3.1).

Now we turn to the proof of uniqueness. Suppose $X_1, X_2 \in M^2_{\mathbb{L}}([0, T]; \mathbb{R}^n)$ are two solutions that satisfy the GSDE (3.1), borrowing the notations in the proof of existence, we define for each $N \in \mathbb{N}$,
\[
(X^1)^N(t) = x + \int_0^t b^N(s, X^1(s)) ds + \sum_{i,j=1}^d \int_0^t h_{ij}^N(s, X^1(s)) dB^i_s \quad 0 \leq t \leq T;
\]
\[
(X^2)^N(t) = x + \int_0^t b^N(s, X^2(s)) ds + \sum_{i,j=1}^d \int_0^t h_{ij}^N(s, X^2(s)) dB^i_s \quad 0 \leq t \leq T.
\]
Following a similar procedure in the proof of the existence, we know that $\{(X^1)^N\}_{N \in \mathbb{N}}$ and $\{(X^2)^N\}_{N \in \mathbb{N}}$ converge to $X^1$ and $X^2$, respectively in $M^1_{\mathbb{L}}([0, T]; \mathbb{R}^n)$ and we have
\[
E[F_{\varepsilon}((X^1)^N(t) - (X^2)^N(t))] \leq C \int_0^t \beta(s) \left( \rho_1(E[|X^1(s) - X^2(s)|]) \right) + \rho_2(E[|X^1(s) - X^2(s)|]) ds.
\]
Letting $N \to +\infty$ and $\varepsilon \to 0$, we deduce
\[
E[|X^1(t) - X^2(t)|] \leq C \int_0^t \beta(s) \left( \rho_1(\mathbb{E}[|X^1(s) - X^2(s)|]) \right) + \rho_3(\mathbb{E}[|X^1(s) - X^2(s)|]) ds.
\]
Thus,
\[
\sup_{0 \leq s \leq t} E[|X^1(s) - X^2(s)|] \leq C \int_0^t \beta(s) \left( \rho_1 + \rho_3 \left( \sup_{0 \leq u \leq s} \mathbb{E}[|X^1(u) - X^2(u)|] \right) \right) ds.
\]
Finally, Lemma 3.6 gives the uniqueness result.

\begin{remark}
Fang and Zhang\cite{3} proved a pathwise uniqueness result for the classical SDEs by a stopping time technique, where $\rho$ is not necessary to be concave. Although we do have a
\end{remark}
similar stopping time technique, Lemma 3.3 in [16] is not true in the \( G \)-framework, because for an \( M^1_G([0,T];\mathbb{R}^n) \) process \( \xi \), it is difficult to verify whether \( \Phi(\xi) \) (using the notations in that paper) satisfies Definition 4.4 in [6] or not. This means that the \( G \)-stochastic integrals in the proof of that lemma, whose upper limit involves a stopping time, cannot be well defined. Fang and Zhang\(^{[7]}\) also derived an existence result by the well-known Yamada-Watanabe theorem, which says that the existence of weak solution and pathwise uniqueness imply the existence of strong solution. In the \( G \)-framework, the corresponding Yamada-Watanabe theorem are unfortunately not available.

4 Solvability of \( G \)-backward Stochastic Differential Equations

In this section, we prove the existence and uniqueness theorem for the following GBSDE:

\[
Y_t = \mathbb{E} \left[ \xi + \int_t^T f(s, Y_s)ds + \sum_{i,j=1}^d \int_t^T h_{ij}(s, Y_s)dB^i(s) \bigg| \Omega_t \right], \quad 0 \leq t \leq T, \tag{4.1}
\]

where \( \xi \in L^1_\mathbb{F}(\Omega_T;\mathbb{R}^n) \) and \( f, g_{ij} \) are given functions that satisfy for each \( x \in \mathbb{R}^n \), \( f(\cdot, x) \), \( h_{ij}(\cdot, x) \in M^1_G([0,T];\mathbb{R}^n) \), \( i, j = 1, \ldots, d \).

We assume moreover that, for each \( t \in [0,T] \) and \( y, y_1, y_2 \in \mathbb{R}^n \):

\begin{enumerate}
  \item[(H1\(^*\))] \( |f(t, y_1) - f(t, y_2)| + |h(t, y_1) - h(t, y_2)| \leq |\beta(t)| \rho(|y_1 - y_2|) \);
  \item[(H2\(^*\))] \( |f(t, y)| + |h(t, y)| \leq \beta_1(t) + \beta_2|y| \),
\end{enumerate}

where \( \beta : [0,T] \rightarrow \mathbb{R}^+ \) is Lebesgue integrable, \( \beta_1 \in M^1_G([0,T]) \), \( \beta_2 \in \mathbb{R}^+ \) and \( \rho : (0, +\infty) \rightarrow (0, +\infty) \) is a continuous increasing and concave function that vanishes at 0+ and satisfies (3.2).

**Theorem 4.1.** Under the assumptions above, (4.1) admits a unique solution \( Y \in M^1_G([0,T];\mathbb{R}^n) \).

**Proof.** Let \( Y_1, Y_2 \in M^1_G([0,T];\mathbb{R}^n) \) be two solutions of (4.1), then

\[
Y_1^t - Y_2^t = \mathbb{E} \left[ \xi + \int_t^T f(s, Y_1^s)ds + \sum_{i,j=1}^d \int_t^T h_{ij}(s, Y_1^s)dB^i(s) \bigg| \Omega_t \right] - \mathbb{E} \left[ \xi + \int_t^T f(s, Y_2^s)ds + \sum_{i,j=1}^d \int_t^T h_{ij}(s, Y_2^s)dB^i(s) \bigg| \Omega_t \right].
\]

Due to the sub-additivity of \( \mathbb{E}|.| \Omega_t \), we obtain

\[
|Y_1^t - Y_2^t| \leq \mathbb{E} \left[ \left| \int_t^T (f(s, Y_1^s) - f(s, Y_2^s))ds \right| \bigg| \Omega_t \right] + \sum_{i,j=1}^d \mathbb{E} \left[ \left| \int_t^T (h_{ij}(s, Y_1^s) - h_{ij}(s, Y_2^s))dB^i(s) \bigg| \Omega_t \right].
\]

Taking \( G \)-expectation on both sides and using the BDG type inequalities and Lemma 2.12, we have

\[
\mathbb{E}|Y_1^t - Y_2^t| \leq \mathbb{E} \left[ \int_t^T (f(s, Y_1^s) - f(s, Y_2^s))ds \right] + \sum_{i,j=1}^d \mathbb{E} \left[ \int_t^T (h_{ij}(s, Y_1^s) - h_{ij}(s, Y_2^s))dB^i(s) \bigg| ds \right] \leq C \int_t^T \rho(\mathbb{E}|Y_1^s - Y_2^s|)ds.
\]
Set
\[ u(t) = \mathbb{E}[|Y_t^1 - Y_t^2|], \]
then
\[ u(t) \leq K \int_t^T \rho(u(s))ds. \]
By Lemma 3.6, we deduce
\[ u(t) = 0, \quad 0 \leq t \leq T, \]
which yields the pathwise uniqueness.

For the proof of existence, we define a sequence of processes \( \{Y^m\}_{m \in \mathbb{N}} \) as follows:
\[ Y^0(t) = 0, \quad 0 \leq t \leq T; \]
and
\[ Y^{m+1}(t) = \mathbb{E}\left[ \zeta + \int_t^T f(s, Y^m_s)ds + \sum_{i,j=1}^d \int_t^T h_{ij}(s, Y^m_s)d\langle B^i, B^j \rangle_s \bigg| \Omega_t \right], \quad 0 \leq t \leq T. \]
The rest of the proof goes in a similar way to the proof of Theorem 3.7, so we omit it. \( \Box \)

**Remark 4.2.** We notice that the definition of the GBSDE above is not the typical one (cf. (3.1) in [4]), in which the generator \( f \) involves \( Z \), i.e., the integrand of the Itô type G-stochastic integral with respect to \( G \)-Brownian motion. Based on the great efforts of many authors, such as Xu and Zhang\(^{[17]} \), Soner et al.\(^{[13]} \) and Song\(^{[14,15]} \), Peng et al.\(^{[11]} \) have given a complete theory for \( G \)-martingale representation. Subsequently, Hu et al.\(^{[5]} \) have derived a complete existence and uniqueness theorem for nonlinear GBSDEs with a generator \( f \) that is uniformly Lipschitz in both \( y \) and \( z \).

An extensive study to GBSDEs is meaningful because there will be numerous possible applications of GBSDEs in finance, for example, pricing and robust utility maximization in a model with a non-dominated class of probability measures.

**Appendix**

In the appendix, we give the proof of Lemma 3.4 in three steps. First of all, we consider the simplest case when \( \zeta \) is uniformly Lipschitz in \( x \). Then, we prove this lemma for all \( \zeta \) that is uniformly bounded. To generalize the result to the case that \( \zeta \) is unbounded, we need to define a sequence of truncated functions \( \{\zeta^N\}_{N \in \mathbb{N}} \) as (3.7) and complete the proof with the help of Theorem 2.13. Now, we begin with the following lemmas.

**Lemma 5.1.** For some \( p \geq 1 \), \( \zeta \) is a function that satisfies \( \zeta(\cdot, x) \in \mathcal{M}_G^p([0, T]; \mathbb{R}^n) \) for each \( x \in \mathbb{R}^n \). We assume moreover that \( \zeta(\cdot, x) \) satisfies the Lipschitz condition, i.e., for each \( t \in [0, T] \) and each \( x_1, x_2 \in \mathbb{R}^n \), \( |\zeta(t, x_1) - \zeta(t, x_2)| \leq C_L|x_1 - x_2| \). Then, for each \( X \in \mathcal{M}_G^p([0, T]; \mathbb{R}^n) \), \( \zeta(\cdot, X) \) is an element in \( \mathcal{M}_G^p([0, T]; \mathbb{R}^n) \).

**Proof.** Without loss of the generality, we only give the proof of the one dimensional case. Suppose that \( X \) can be approximated by a sequence \( \{X^N\}_{N \in \mathbb{N}} \subset \mathcal{M}_G^p([0, T]) \) of the form below:
\[ X^N_t := \sum_{k=0}^{N-1} \xi_k 1_{(t_k, t_{k+1})}(t), \]
where \( \xi_k \in L^0_{ip}(\Omega_{t_k}) \), then
\[ \int_0^T \mathbb{E}[|\zeta(t, X^N_t) - \zeta(t, X_t)|^p]dt \leq C_L \int_0^T \mathbb{E}[|X^N_t - X_t|^p]dt \to 0, \quad \text{as} \ N \to +\infty. \]
Lemma 5.2. Fixing a $T \geq 1$, $\eta$ is an element in $L^p_{ip}(\Omega_1)$, then $\zeta(\cdot, \eta)1_{[1,T]}(\cdot) \in M^p_G([0,T])$. In what follows, we prove this assertion.

Proof. Suppose $\phi \in \mathcal{C}_0^\infty$ such that for all $i \in I$, $\supp(\phi_i) \subset N_\eta([0,T])$. It suffices to prove that $\zeta(\cdot, \eta)\phi_i(\eta)1_{[1,T]}(\cdot) \in M^p_G([0,T])$, $i = 1, \ldots, N(n)$, which is given by the following lemma.

**Lemma 5.2.** Fixing a $T \geq 1$, let $X$ be an element in $M^p_G([0,T])$ and $\eta$ is an element in $L^p_{ip}(\Omega_1)$, then $\eta X 1_{[1,T]}(\cdot) \in M^p_G([0,T])$.

**Proof.** Suppose $X$ can be approximated by a sequence $\{X^N\}_{N \in \mathbb{N}} \subset M^p_G([0,T])$ of the form below:

$$X^N_t := \sum_{k=0}^{N-1} \xi_k 1_{[t_k, t_{k+1})}(t),$$

then $X 1_{[1,T]}(\cdot)$ can be approximated by a sequence $\{\overline{X}^N\}_{N \in \mathbb{N}} \subset M^p_G([0,T])$:

$$\overline{X}^N_t := \sum_{k=0}^{N-1} \xi_k 1_{[t_k \vee 1, t_{k+1} \vee 1]}(t),$$

where $\xi_k \in L^p_{ip}(\Omega_{t_k})$. We define a sequence $\{\tilde{X}^N\}_{n \in \mathbb{N}}$ by

$$\tilde{X}^N_t := \sum_{k=0}^{N-1} \alpha_k 1_{[t_k, t_{k+1})}(t),$$

where

$$\alpha_k := \begin{cases} 0, & \text{if } t_{k+1} < 1; \\ \eta \xi_k, & \text{if } t_{k+1} \geq 1. \end{cases}$$

Since $L^p_{ip}(\Omega_1) \subset L^0_{ip}(\Omega_{t \vee 1})$ and $L^0_{ip}(\Omega_{t \vee 1})$ is closed under multiplication, we deduce that $\{\tilde{X}^N\}_{N \in \mathbb{N}} \subset M^p_G([0,T])$. Moreover,

$$|\tilde{X}^N_t - \eta X_t 1_{[1,T]}(t)| \leq |\eta|X^N_t - \eta X_t 1_{[1,T]}(t) \leq M|X^N_t - X_t|, \quad 0 \leq t < T,$$
where $M$ is the bound of $\eta$. This implies that $\eta X \mathbf{1}_{[1,T]}(\cdot)$ is the limit of $\bar{X}^N$ under the $M^p_G([0,T])$ norm (2.2).

**Proof of Lemma 3.4.** Let $J \in C^\infty(\mathbb{R}^n)$ be a non-negative function satisfies $\text{supp}(J) \subset B(0,1)$ and

$$\int_{\mathbb{R}^n} J(x) dx = 1.$$ 

For each $\lambda > 0$, we set

$$J_\lambda(x) = \frac{1}{\lambda^n} J(\frac{x}{\lambda})$$

and

$$\zeta_\lambda(t, x) = \int_{\mathbb{R}^n} J_\lambda(x-y) \zeta(t, y) dy.$$

We assume that $\zeta$ is uniformly bounded, then $\zeta_\lambda$ is uniformly Lipschitz in $x$. By Lemma 5.1, we have $\zeta_\lambda(\cdot, X) \in M^p_G([0,T]; \mathbb{R}^n)$. To deduce the desired result, we only need to show that $\zeta(\cdot, X)$ is the limit of $\zeta_\lambda(\cdot, X)$ under the $M^p_G([0,T]; \mathbb{R}^n)$ norm (2.2).

Fixing a $\lambda > 0$, we calculate

$$|\zeta_\lambda(t, x) - \zeta(t, x)| \leq \int_{\mathbb{R}^n} J_\lambda(y) |\zeta(t, x-y) - \zeta(t, x)| dy.$$

Therefore,

$$\int_0^T \mathbb{E}[|\zeta_\lambda(t, X(t)) - \zeta(t, X(t))|^q] dt \leq \int_0^T \mathbb{E} \left[ \left| \int_{\mathbb{R}^n} J_\lambda(y) |\zeta(t, X(t)-y) - \zeta(t, X(t))| dy \right|^q \right] dt \leq \int_0^T \|\beta(t)\|^q \left( \int_{\mathbb{R}^n} J_\lambda(y) \gamma(|y|) dy \right)^q dt \leq \|\gamma(\lambda)\|^q \int_0^T \|\beta(t)\|^q dt \leq C \|\gamma(\lambda)\|^q \rightarrow 0, \quad \text{as } \lambda \rightarrow 0.$$

For an unbounded function $\zeta$, we construct a sequence of processes $(\zeta^N)_{n \in \mathbb{N}}$ as (3.7). Fixing an $N \in \mathbb{N}$, we have

$$\int_0^T \mathbb{E}[|\zeta^N(t, X(t)) - \zeta(t, X(t))|^q] dt \leq \int_0^T \mathbb{E}[|\zeta(t, X(t))|^q \mathbf{1}_{\{|\zeta(t, X(t))| > N\}}] dt \leq \int_0^T \mathbb{E}[|\beta_1(t) + \beta_2[X(t)]|^{q} \mathbf{1}_{\{|\beta_1(t) + \beta_2[X(t)]| > N\}}] dt \leq C \left( \int_0^T \mathbb{E}[|\beta_1(t)|^q] dt + \int_0^T \mathbb{E}[|\beta_2[X(t)]|^{q} \mathbf{1}_{\{|\beta_2[X(t)]| > N\}}] dt \right) \leq C \left( \int_0^T \mathbb{E}[|\beta_1(t)|^q] dt + \int_0^T \mathbb{E}[|X(t)|^q] dt \right). \quad (5.1)$$

Since $\beta_1$ and $X$ are $M^p_G([0,T])$ processes, by Remark 2.15 and Theorem 2.13 along with Lebesgue’s dominated convergence theorem, the right-hand side of (5.1) converges to 0. This yields the desired result.

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