Fractional quantum Hall states of a few bosonic atoms in geometric gauge fields

B Juliá-Díaz¹,²,⁴, T Graß², N Barberán¹ and M Lewenstein²,³

¹ Departament d’Estructura i Constituents de la Matèria, Facultat de Física, Universitat de Barcelona, E-08028, Barcelona, Spain
² ICFO-Institut de Ciències Fotòniques, Parc Mediterrani de la Tecnologia, 08860 Barcelona, Spain
³ ICREA-Institució Catalana de Recerca i Estudis Avançats, 08010 Barcelona, Spain
E-mail: bruno.julia@icfo.es

New Journal of Physics 14 (2012) 055003 (27pp)
Received 11 November 2011
Published 1 May 2012
Online at http://www.njp.org/
doi:10.1088/1367-2630/14/5/055003

Abstract. We use the exact diagonalization method to analyze the possibility of generating strongly correlated states in two-dimensional clouds of ultracold bosonic atoms that are subjected to a geometric gauge field that was created by coupling two internal atomic states to a laser beam. On tuning the gauge field strength, the system undergoes stepwise transitions between different ground states (GSs), which we describe by using analytical trial wave functions, including the Pfaffian (Pf), the Laughlin and a Laughlin quasiparticle many-body state. Whereas for an infinitely strong laser field, the internal degree of freedom of the atoms can adiabatically follow their center-of-mass movement, a finite laser intensity gives rise to non-adiabatic transitions between the internal states, which are shown to break the cylindrical symmetry of the Hamiltonian. We study the influence of the asymmetry on the GS properties of the system. The main effect is to reduce the overlap of the numerical solutions with the analytical trial expressions by occupying states with higher angular momentum. Thus, we propose generalized wave functions arising from the Laughlin and Pf wave functions by including components where extra Jastrow factors appear while preserving important features of these states. We analyze quasihole excitations over the Laughlin and generalized Laughlin states and show that they possess effective fractional charge and obey anyonic statistics. Finally, we discuss the observability of the Laughlin state for increasing numbers of particles.

Author to whom any correspondence should be addressed.
1. Introduction

Spectacular progress on the manipulation and control of cold atomic clouds has been achieved since the first experimental realization of a Bose–Einstein condensed system by Anderson et al [1] and Davis et al [2] nearly simultaneously in 1995. These systems provide us with a toolbox for studying the principles of quantum mechanics, as has for instance been done in experiments involving the interference of two condensates, leading to the manifestation of coherence [3]. In recent years, cold atoms have been used to simulate interesting phenomena appearing in condensed-matter physics [4], such as the Mott-insulator-to-superfluid transition [5].

However, a severe restriction stems from the atom’s electro-neutrality, which hinders a direct implementation of phenomena involving electromagnetic forces. An important example of the latter is the physics of the integer and the fractional quantum Hall effect (FQHE), occurring in two-dimensional (2D) systems under the presence of a strong perpendicular magnetic field. The ground states (GSs) of fractional quantum Hall (FQHE) systems are highly correlated states, such as the Pfaffian (Pf) state proposed by Moore and Read [6] or the celebrated Laughlin state [7] whose bosonic analogues are found to be the exact eigenstates of a Hamiltonian with a three-body (3B) or two-body (2B) [8, 9] contact interaction, respectively. In addition, the quasihole excitations over the Laughlin state have fractional effective charge and fractional statistics [10]. Excitations of the Pf state may even obey non-Abelian braiding statistics [11–13], which makes them interesting from both fundamental and technological points of view [14].

Several proposals of experimental routes to obtain these types of states have appeared in the literature, ranging from the use of rotating traps to simulate magnetic fields acting on charges

References

1. Anderson P W, et al. ...
to the use of laser-beam configurations acting on atoms with several internal states [15–22]. In this paper, we analyze the appropriate conditions to realize some strongly correlated states within a simple configuration of a single laser beam shining on a cloud of cold atoms with two internal states. If the internal dynamics of the atoms, governed by the Rabi frequency of the atom–laser coupling, is fast enough with respect to the slow variation of center-of-mass position, then the atoms adiabatically follow the local internal eigenstate. They remain in one definite space-dependent superposition of the internal bare states, and the accumulation of Berry phase [23] during its movement mimics an effective magnetic field [24, 25]. An important goal is to go further and analyze the effect of a small amount of non-adiabaticity, which we treat in a perturbative way.

Through the controlled variation of external parameters, different strongly correlated states appear in the spectrum, i.e. a Laughlin-like state, a Pf-like state and the quasiparticle-like state obtained from the Laughlin. Our main aim is to map the regions in parameter space where the exact, numerically obtained GS has a large overlap with explicit analytical expressions provided for these relevant strongly correlated states. To remain close to possible experimental implementations, we study the effect of small perturbations that create non-adiabaticities, which are unavoidable for finite values of the laser intensity. The small perturbation produces a deformation of the atomic cloud preserving most of the notable properties of the original states, e.g. entropy, internal energy or anyonic character of excitations. These slightly asymmetric GSs are well represented by generalized analytic wave functions.

Focusing additional laser beams on the atomic cloud, it is possible to pierce holes in the system. As long as the asymmetric perturbation is small, the resulting states can be well described by an analytical quasihole wave function, which can be obtained from the Laughlin or the generalized Laughlin (GL) wave function. In both cases, the effective charge and statistical phase of the quasihole excitations are found to attain fractional values, demonstrating the possibility of observing anyons within the proposed setup.

In this work, we concentrate on the physics of few-body systems, independently of their attainability in the thermodynamic limit, which is beyond the scope of this paper. This approach is meaningful as there are nowadays a number of groups capable of manipulating small bosonic clouds using several techniques [26, 27]. In particular, Gemelke et al [27] have presented experimental evidence for the production of quantum states of FQH type for small atom systems ($N < 10$). These experimental developments have triggered a number of theoretical proposals focusing on the production of strongly correlated quantum states in small atomic clouds [17, 20, 21]. We analyze the dependence on the number of particles of some of our main results. Firstly, we show that observing the anyonic character of quasihole excitations becomes possible when $N \gtrsim 5$. Secondly, we show the behavior of the bulk gap of the Laughlin state when $N$ is increased.

The paper is organized as follows. In section 2, we present the system and derive an effective Hamiltonian to describe it. In section 3, we provide the analytical expressions for the relevant FQH states and their generalizations. Our results are presented in sections 4 and 5 in the adiabatic/symmetric and non-adiabatic/asymmetric cases, respectively. In section 6, we study the fractional charge and anyonic statistics of quasiholes, produced by means of additional lasers. In section 7, we analyze the behavior of the energy gap above the Laughlin state as $N$ increases and its evolution as a function of the system’s parameters. Finally, in section 8, we present our conclusions.
2. Description of the system

We consider a setup for producing artificial gauge fields in a small cloud of ultracold atoms closely following the configuration described in [22]. The system is confined in a harmonic trap. The confinement in the $z$-direction is assumed to be strong enough to achieve effectively a 2D system. The cloud is illuminated by a single laser beam with wave number $k$ and frequency $\omega_L$, which propagates in the $y$-direction and is close to the resonance with a transition between two internal atomic states, $|g\rangle$ and $|e\rangle$, $\omega_L = \omega_k$. The interaction between the electric field of the laser and the induced electric dipole is modeled by the atom–laser Hamiltonian, which in the rotating-wave approximation and in the rotating frame is given by [28, 29]

$$H_{AL} = E_g |g\rangle \langle g| + (E_e - \hbar \omega_L) |e\rangle \langle e| + \frac{\hbar \Omega_0}{2} e^{i ky} |e\rangle \langle g| + \frac{\hbar \Omega_0}{2} e^{-i ky} |g\rangle \langle e|,$$

where $E_g$ and $E_e$ are the energies of the bare atomic ground and excited states and $\Omega_0$ is the Rabi frequency, which is proportional to the laser intensity. No spontaneous emission of photons from the excited state is considered. This assumption is justified as long as the lifetime of the excited state is longer than the typical duration of an experiment. Lifetimes of the order of several seconds, as found for ytterbium or some alkaline earth metals, should be sufficient.

In order to obtain a nontrivial gauge potential from the coupling in equation (1), we still have to make it dependent on $x$. This can be achieved via a real magnetic field, which via the Zeeman effect makes the internal energy levels vary linearly with $x$. Introducing a parameter $w$, setting the length scale of this shift, we have

$$E_g = -\frac{\hbar \Omega_0}{2} \frac{x}{w}, \quad E_e = \hbar \omega + \frac{\hbar \Omega_0}{2} \frac{x}{w}.$$

Then, the single-particle Hamiltonian is given by

$$H_{sp} = \frac{p^2}{2M} + V(\vec{r}) + \frac{\hbar \Omega}{2} \left( \begin{array}{cc} \cos \theta & e^{i \phi} \sin \theta \\ e^{-i \phi} \sin \theta & -\cos \theta \end{array} \right),$$

where the third term is the atom–laser Hamiltonian represented in the $\{|e\rangle, |g\rangle\}$ basis. Here, $M$ is the atomic mass, $\Omega = \Omega_0 \sqrt{1 + x^2/w^2}$, $\sin \theta = w/\sqrt{w^2 + x^2}$, $\cos \theta = x/\sqrt{w^2 + x^2}$ and $\phi = ky$. Note that $V(\vec{r})$ is the trap potential that will be fixed below.

Up to this point, the Hamiltonian is given by physically measurable parameters. In the next step, we choose particular expressions for the single-particle states that diagonalize the atom–laser Hamiltonian. This corresponds in fact to the selection of a specific gauge for the vector and scalar potentials that will drive the center-of-mass dynamics, as will be shown below. We consider the eigenfunctions of $H_{AL}$ given by

$$|\Psi_1\rangle = e^{-i G} \left( \begin{array}{c} C \\ S \end{array} \right), \quad |\Psi_2\rangle = e^{i G} \left( \begin{array}{c} -S \\ C \end{array} \right),$$

in the $\{|e\rangle, |g\rangle\}$ basis, where $C = \cos \theta/2$, $S = \sin \theta/2$ and $G = \frac{kvy}{4w}$. The atomic state can then be expressed as

$$\chi(\vec{r}) = a_1(\vec{r}) \otimes |\Psi_1\rangle + a_2(\vec{r}) \otimes |\Psi_2\rangle,$$

where $a_i$ captures the external dynamics and $|\Psi_1\rangle$ provides internal degree of freedom. The single-particle Hamiltonian is then expressed in the $\{|\Psi_1\rangle, |\Psi_2\rangle\}$ basis as

$$H_{sp} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix},$$

New Journal of Physics 14 (2012) 055003 (http://www.njp.org/)
acting on the spinor \([a_1(\vec{r}), a_2(\vec{r})]\). Defining a vector potential \(\vec{A}\),

\[
\vec{A}(\vec{r}) = \hbar k \left[ \frac{y}{4w} - \frac{x}{4w} + \frac{x}{2\sqrt{x^2 + w^2}} \right],
\]

and a scalar potential \(U\),

\[
U(\vec{r}) = \frac{\hbar^2 w^2}{8M (x^2 + w^2)} \left( k^2 + \frac{1}{x^2 + w^2} \right),
\]

we obtain

\[
H_{11} = \frac{[\vec{p} - \vec{A}(\vec{r})]^2}{2M} + U(\vec{r}) + V(\vec{r}) + \frac{\hbar \Omega(\vec{r})}{2}
\]

and

\[
H_{22} = \frac{[\vec{p} + \vec{A}(\vec{r})]^2}{2M} + U(\vec{r}) + V(\vec{r}) - \frac{\hbar \Omega(\vec{r})}{2},
\]

which are the Hamiltonians driving the external dynamics of atoms being in the internal state \(|\Psi_1\rangle\) and \(|\Psi_2\rangle\), respectively. By expanding the \(H_{ij}\) terms up to second order in \(x\) and \(y\), which is justified by choosing \(w\) to be larger than the extension of the cloud, we find that

the energy distance between these two manifolds is given by \(\hbar \Omega_0\). For convenience, we make the Hamiltonian for the low-energy manifold, \(H_{22}\), independent of \(\Omega_0\) by adding the constant term \(\frac{\hbar \Omega_0}{2}\) to the diagonal of \(\hat{H}_{sp}\). Further, we note that with the explicit selection of equation (4) and for \(x, y \ll w\), \(A\) is in the symmetric gauge: \(\vec{A} \approx \frac{\hbar k}{4w} (y, -x)\). This allows us to make \(H_{22}\) fully symmetric by a proper choice of the trapping frequency. Equation (10) then takes the form

\[
H_{22} = \frac{p^2}{2M} + \frac{\vec{p} \cdot \vec{A}}{M} + \frac{M}{2} \omega_\perp (x^2 + y^2),
\]

and we can take advantage of knowledge of its single-particle eigenfunctions, namely the Fock–Darwin states \([30]\). This final expression is formally equal to the Hamiltonian driving a system of charges trapped by a harmonic potential of frequency \(\omega_\perp\), under a constant magnetic field along the \(z\)-direction, or equivalently, a system of neutral atoms trapped by a rotating harmonic potential of frequency \(\omega_\perp\), expressed in the rotating frame of reference \([31, 32]\).

We stress that this equivalence to the rotating case holds only for \(H_{22}\). For the upper manifold, described by \(H_{11}\), it does not. Due to the off-diagonal terms in \(\hat{H}_{sp}\), the two manifolds are coupled. Typical expected values of \(H_{12}\) and \(H_{21}\) are of the order of the recoil energy \(E_R = \frac{\hbar k^2}{2M}\), which gives the scale for the kinetic energy of the atomic center-of-mass motion when it absorbs or emits a single photon. If we consider \(\hbar \Omega_0 \gg E_R\), this coupling is small, and we can restrict ourselves to the low-energy manifold. Namely, we are in the situation where the internal dynamics is much faster than the center-of-mass motion and can follow the external variations in a quasadiabatic way \([33]\).

To go beyond the adiabatic approximation, we consider the influence of the high-energy manifold as a small perturbation. Using the procedure appropriate to systems that show two significantly different energy scales as explained in \([28]\) and detailed in appendix, we calculate an effective Hamiltonian up to second order in the perturbation, which reads

\[
H_{22}^{\text{eff}} = H_{22} - \frac{H_{21}H_{12}}{\hbar \Omega_0},
\]

\(\text{New Journal of Physics 14} (2012) 055003 \) (http://www.njp.org/)
where the explicit expression for the perturbative term $H_{21}H_{12}/\Omega_0$ is given in appendix. Although mathematically more involved and physically richer, this term is reminiscent of the anisotropic potential that is applied to set an atomic cloud in rotation. Usually, the expression used to model the stirring laser is given by $\alpha M \omega_\perp (x^2 - y^2)$ [34, 35], where $\alpha$ measures the strength of the deformation. Similarly, the term $H_{21}H_{12}/\hbar\Omega_0$ can only produce changes in $L$ of $\Delta L = 0, \pm 2, \pm 4$ (see appendix). In what follows we will identify ‘deformation’ with ‘coupling’, indicating that a large connection with $H_{11}$ implies deformation since $H_{11}$ is not cylindrically symmetric.

The many-body Hamiltonian which finally models our system is obtained by adding a contact interaction term to the effective Hamiltonian from equation (12):

$$H = \sum_{i=1}^{N} H_{22}^{\text{eff}}(i) + \frac{\hbar^2 g}{M} \sum_{i<j} \delta(\vec{r}_i - \vec{r}_j),$$

where $g$ is a dimensionless parameter fixing the contact interaction strength. From now on, we will consider $gN = 6$ in the numerical calculations. The Hamiltonian $H$ acts only on the low-energy manifold, which effectively is perturbed by the second manifold. The two important parameters in $H$ are the dimensionless ratio $\eta \equiv \frac{\hbar k}{M \omega_\perp}$ and the degree of the perturbation given by $\Omega_0$. The expression $\eta \omega_\perp$ plays the role of the rotating frequency. The effective magnetic field strength $B_0$ is given by

$$B_0 \equiv \frac{\hbar k}{2\omega} = 2M \omega_\perp \eta.$$  

The largest possible value of $\eta$ is given by 1 in order to keep the system confined. We consider that the artificial magnetic field is strong enough to work exclusively in the lowest Landau level (LLL) regime. To this end, our parameters must fulfill the following condition: the energy difference between Landau levels is larger than both the kinetic energy of a single particle within a Landau level and the interaction energy per particle. Note that the strength of the atom–laser coupling, characterized by $\Omega_0$, is different from the strength of the magnetic field, characterized by $\eta$. Within the adiabatic approximation $\eta$ is independent of the atom–laser coupling strength. The latter is a consequence of the geometric origin of the effective magnetic field.

3. Analytical many-body states

In this section, we give an overview of the analytical wave functions discussed in the literature which will turn out to be relevant for describing our system.

3.1. The Laughlin state

The well-known Laughlin state has the analytical form [7, 36, 37]

$$\Psi_L(z_1, \ldots, z_N) = \mathcal{N}_L \prod_{i<j} (z_i - z_j)^{1/\nu} e^{-\sum z_i^2 / 2\lambda_\perp^2},$$

where $\mathcal{N}_L$ is a normalization constant, $z = x + iy$ and $\lambda_\perp = \sqrt{\hbar/M \omega_\perp}$. The inverse of the exponent, $\nu$, fixes both the density of the system and the symmetry of the wave function. For bosons, $1/\nu$ must be even. The Laughlin regime discussed here will be at half-filling, so for the rest of this paper we set $\nu = \frac{1}{2}$.
The analysis of the squared overlap $|⟨Ψ_L|ψ_{GS}⟩|^2$ of the Laughlin state with the exact GS as a function of the artificial magnetic field strength $η$ and the atom–laser coupling $Ω_0$ shows that the overlap is reduced as $hΩ_0/E_R$ decreases, even for large values of $η$. Larger overlap with the GS can be obtained by adding an admixture of the Laughlin state with additional Jastrow factors that allow for an increase of total angular momentum, which in the Laughlin state is given by

$$L = \frac{N(N-1)}{2η}.$$ 

Based on these observations, in [22] an analytical ansatz for the GS in the Laughlin-like region was proposed:

$$Ψ_L = αΨ_L + βΨ_{L1} + γΨ_{L2},$$

(16)

hereafter referred to as the GL state, with $Ψ_{L1} = N_LΨ_L · ∑_{i=1}^N z_i^2$, $Ψ_{L2} = N_L(Ψ_{L2} − ⟨Ψ_{L1}|Ψ_{L2}⟩Ψ_{L1})$ and $Ψ_{L2} = N_LΨ_L · ∑_{i,j} z_i z_j$, such that we ensure $⟨Ψ_L|Ψ_{Lj}⟩ = 0$ and $⟨Ψ_L|Ψ_{Lj}⟩ = δ_{ij}$. This ansatz involves components of angular momentum $L = N(N-1)$ and $L = N(N-1) + 2$, and yields zero contact interaction energy. The values of $α$, $β$ and $γ$ are computed as $α = ⟨Ψ_L|GS⟩/√N$, $β = ⟨Ψ_{L1}|GS⟩/√N$ and $γ = ⟨Ψ_{L2}|GS⟩/√N$, with $N = |⟨Ψ_L|GS⟩|^2 + |⟨Ψ_{L1}|GS⟩|^2 + |⟨Ψ_{L2}|GS⟩|^2$.

3.2. Pfaffian (Moore–Read) state

While the Laughlin and the GL state turn out to be good trial states for strong magnetic fields $η \lesssim 1$, for smaller field strengths the Laughlin quasiparticle (LQP) state and the Pf state become relevant. The Pf state has $L = N(N-2)/2$ for even $N$ and $L = (N-1)^2/2$ for odd $N$ and its analytical expression reads

$$Ψ_P = N_pΨ_P(z) \prod_{i<j} (z_i - z_j)$$

(17)

with $N_p$ being a normalization constant, and

$$Ψ_P(z) = Α \left[ \frac{1}{(z_1 - z_2)} \frac{1}{(z_3 - z_4)} \cdots \frac{1}{(z_{N-1} - z_N)} \right],$$

(18)

where $Α$ is an antisymmetrizer of the product. As explained in [8, 9], the Pf state can also be computed as

$$Ψ_P = S \prod_{i<j \in \sigma_1} (z_i - z_j)^2 \prod_{k<l \in \sigma_2} (z_k - z_l)^2,$$

(19)

where $σ_1$ and $σ_2$ are two subsets containing $N/2$ particles each if $N$ is even and $(N+1)/2$ and $(N-1)/2$ if $N$ is odd. $S$ symmetrizes the expression. The Pf state has been shown to be the lowest-energy eigenstate of a Hamiltonian which contains only 3B contact interactions [6, 21],

$$H_{int} = ∑_{i<j<k} δ(z_i - z_j)δ(z_j - z_k),$$

(20)

and, similarly to the Laughlin with the 2B interaction, it has zero 3B interaction energy. Remarkably, however, as shown in [8, 17] for a 2B interaction Hamiltonian and for some values of $η$, there is a sizeable overlap between the GS of the system and this state.

5 Defined as the parameter domain where $⟨L⟩ \geq N(N-1)$. 

New Journal of Physics 14 (2012) 055003 (http://www.njp.org/)
As we did before for the Laughlin state in equation (16), we can define a generalized Pfaffian (GPf) state as
\[ \Psi_{GP} = \alpha \Psi_p + \beta \Psi_{p1} + \gamma \Psi_{p2}, \] (21)
with \( \Psi_{p1} = N_{p1} \Psi_p \cdot \sum_{i=1}^{N} z_i^2, \) \( \Psi_{p2} = N_{p2} (\tilde{\Psi}_{p2} - \langle \Psi_{p1} | \tilde{\Psi}_{p2} \rangle \Psi_{p1}), \) and \( \tilde{\Psi}_{p2} = \tilde{N}_{p2} \Psi_p \cdot \sum_{i<j} z_i z_j. \) Again, the parameters \( \alpha, \beta \) and \( \gamma \) are fixed to maximize the overlap of the numerical GS with \( \Psi_{GP}. \)

### 3.3. Laughlin-quasiparticle state

The LQP state arises from the Laughlin state by increasing the density at the origin and thereby decreasing its angular momentum, \( L_{qp} = (N(N-1) - N). \) The latter formula holds if the quasiparticle is at the origin. Otherwise, it also carries angular momentum and the total expected value of the angular momentum of the system is no longer an integer. The wave function is written as
\[ \Psi_{L_{qp}} = N_{L_{qp}}(\xi, \xi^*) (\partial_{\xi_1} - \xi) \cdots (\partial_{\xi_N} - \xi) \Psi_L, \] (22)
with \( N_{L_{qp}}(\xi, \xi^*) \) being a normalization constant that depends on the position \( \xi \) and \( \xi^* \) of the excitation. Also for the LQP state we define a generalized version (GLQP), built up from the same Jastrow factors used in equation (16), i.e.
\[ \Psi_{GL_{qp}} = \alpha \Psi_{L_{qp}} + \beta \Psi_{L_{qp1}} + \gamma \Psi_{L_{qp2}}, \] (23)
with \( \Psi_{L_{qp1}} = N_{L_{qp1}} \Psi_{L_{qp}} \cdot \sum_{i=1}^{N} z_i^2, \) \( \Psi_{L_{qp2}} = N_{L_{qp2}} (\tilde{\Psi}_{L_{qp2}} - \langle \Psi_{L_{qp1}} | \tilde{\Psi}_{L_{qp2}} \rangle \Psi_{L_{qp1}}) \) and \( \tilde{\Psi}_{qp2} = \tilde{N}_{L_{qp2}} \Psi_{L_{qp}} \cdot \sum_{i<j} z_i z_j. \)

### 3.4. Laughlin-quasihole state

As an alternative to increasing the homogeneous density of the Laughlin state locally, one might also decrease it by piercing a hole in the atomic cloud. Formally, this is achieved by introducing an additional zero into the wave function, multiplying it with \( \prod_{i}(\xi - z_i). \) The resulting quasihole state pierced at \( \xi \) reads
\[ \Psi_{L_{qh}} = N_{L_{qh}}(\xi, \xi^*) \prod_{i=1}^{N} (\xi_i - z_i) \Psi_L, \] (24)
where \( N_{L_{qh}}(\xi, \xi^*) \) is a normalization constant that explicitly depends on the position of the quasihole. To test the anyonic nature of the quasiholes, we need to do the same operation twice, such that the presence of two quasiholes is described by
\[ \Psi_{L_{2qh}} = N_{L_{2qh}}(\xi_1, \xi_1^*, \xi_2, \xi_2^*) \prod_{i=1}^{N} (\xi_1 - z_i)(\xi_2 - z_i) \Psi_L. \] (25)

The state with one quasihole has a fixed total angular momentum, which is \( N \) quanta above the GS if the quasihole is at the origin, \( \xi = 0. \) For off-centered quasihole positions, the average angular momentum is slightly reduced and non-integer. The state with two quasiholes has an angular momentum close to \( 2N \) quanta above the GS.
Figure 1. Upper panel: interaction energy in units of $\hat{h}\omega_{\perp}$ (black circles) and one-body entanglement entropy (red squares) of the GS as a function of $\eta$. Middle panel: angular momentum in units of $\hat{h}$ of the GS and of the first excited state as a function of $\eta$. Lower panel: squared overlap between the GS of the system and the exact Laughlin, Pf and LQP states. The plots correspond to the case $H_{22}^{\text{eff}} = H_{22}$.

We may also apply the same operation to the GL state and define

$$\Psi_{GL_{\text{qh}}} = N_{GL_{\text{qh}}}(\xi, \xi^*) \prod_{i=1}^{N} (\xi - z_i) \left[ \alpha \Psi_{\mathcal{E}} + \beta \Psi_{L_1} + \gamma \Psi_{L_2} \right]$$ (26)

for the state with one quasihole and

$$\Psi_{GL_{2\text{qh}}} = N_{GL_{2\text{qh}}}(\xi_1, \xi_1^*, \xi_2, \xi_2^*) \prod_{i=1}^{N} (\xi_1 - z_i)(\xi_2 - z_i) \left[ \alpha \Psi_{\mathcal{E}} + \beta \Psi_{L_1} + \gamma \Psi_{L_2} \right]$$ (27)

for the state with two quasiholes. As explained in section 5, we always find $\alpha, \beta \gg \gamma$; thus in practice we will consider always $\gamma \equiv 0$.

4. Results for the adiabatic/symmetric case, $H_{22}^{\text{eff}} = H_{22}$

In the symmetric case in which the perturbation $H_{21}H_{12}/(\hat{h}\Omega_0)$ is not included, and for $N = 4$, four distinct regions are detected depending on the value of $\eta$ as obtained previously in [8, 17]: condensed, Pf-like state, LQP and Laughlin region. We analyze them in figures 1 and 2. We must remark that we use the term ‘Pf’ in the second region as usually referred to in the literature.
show that the first excitation is of quadrupolar character up
(1) The first excitation corresponds to a fully condensed system with zero angular momentum and vanishing one-body entanglement entropy\(^6\), see figure 1. The GS can be well described by a wave function given by \(\Psi_C = \mathcal{N}_C e^{-\sum_i z_i^2/2 \lambda} \), with \(\mathcal{N}_C\) being a normalization constant.

Figures 1 (middle panel) and 2 show that the first excitation is of quadrupolar character up to \(\eta = 0.76\). Lower values of \(\eta\) lie beyond our LLL approximation, where a larger Hilbert space including more Landau levels has to be considered.

At the critical value \(\eta_1 = 1 - g N / (8 \pi) \approx 0.76\) \((g N = 6)\), degeneracy between states with \(L = 0, 2, 3\) and 4 occurs, see figure 2. At this \(\eta_1\) a state with broken symmetry, a combination of several \(L\)-eigenstates, is a precursor of the nucleation of the first vortex state [35]. For increasing \(\eta\) the GS recovers the cylindrical symmetry and the angular momentum is \(L = 4\). All this phenomenology can be inferred from the Yrast line displayed in figure 3. The Yrast line is constructed by plotting the interaction energy contribution of the lowest-energy state for each \(L\). From this line, the addition of the kinetic energy, which reads (up to a term independent of \(L\) and \(\eta\)) \(E_{\text{kin}} = (1 - \eta) L h \omega_\perp\), produces the total energy with its minimum at the angular momentum of the GS, \(L_{\text{GS}}\), as exemplified for \(\eta = 0.85\) and 0.94 in figure 3.

The GS with \(L = N (= 4)\) is known to be exactly \(\Psi_{L=N}\). As the sizeable entanglement entropy indicates (see figure 1), this state is not fully condensed. In the \(N \to \infty\) limit, the center of mass is pinned to the origin [38], and the state becomes fully condensed, i.e. see figure 4 of [40]. In our finite system, the single vortex state given by \(\Psi_{1vx} = \mathcal{N}_{1vx} \prod_{i=1}^N z_i e^{-\sum_i z_i^2/2 \lambda} \) (\(\mathcal{N}_{1vx}\) being a normalization constant) is found only as the most occupied eigenstate of the one-body density matrix, which plays the role of the order parameter in a mean-field approach. However, due to the significant non-condensed fraction for this low value of \(N\), this function turns out to be a poor representation of the GS. Analytically [41], we find the squared overlap with the single-vortex state \(\Psi_{1vx}\) to be \(15/32 \approx 0.47\) for \(N = 4\). A much higher squared overlap of \(32/35 \approx 0.91\) is found between the exact GS and the Pf state, which has \(L = N\) in the case of \(N = 4\). We note that the overlap between the \(\Psi_{L=N}\) state and the Pf is non-zero only for

---

\(^6\) The entropy is defined here from the one-body density matrix, as \(S = - \sum n_i \ln(n_i)\), where \(n_i\) are the eigenvalues of the one-body density matrix. For a more detailed discussion of the entropy, see, for example, [22].
Figure 3. The Yrast line for \( N = 4 \), red solid circles, which corresponds to the interaction energy contribution of the lowest eigenstates for each value of \( L \). The triangles and diamonds depict the sum of the interaction energy and the kinetic contribution for \( \eta = 0.85 \) and \( \eta = 0.94 \), respectively. The arrows mark the value of \( L \) that corresponds to the GS in each case. The energies are given in units of \( \hbar \omega_\perp \).

\( N = 4 \), while for larger particle numbers the Pf has always \( L > N \) and the single-vortex region, \( L = N \), is thus distinct from the Pf region \([8, 17]\), which is where the GS has a sizeable overlap with the Pf state. In the Pf region we find three different kinds of excitations, \( L = 3, 6 \) and 8, as can be seen in figure 2. The latter has a large overlap with the LQP state, as can be seen in figure 4.

For \( 0.92 \leq \eta \leq 0.96 \), the GS has \( L = 8 \), a higher entanglement entropy and a smaller interaction energy. The GS has a large overlap with the LQP state, as shown in figures 1 and 4. This quasiparticle region has four different excitations, \( L = 4, 6, 9 \) and 12. The \( L = 9 \) corresponds to a center-of-mass excitation of the system as dictated by Kohn’s theorem \([30, 42]\).

Finally, for \( \eta \geq \eta_2 \simeq 0.96 \) (\( \eta_2 \) will be defined below), the GS wave function is the Laughlin wave function, with zero interaction energy. Its excitations are the LQP \( L = 8 \) and a center-of-mass excitation, the Kohn mode, with \( L = 13 \), whose analytical form is \( \Psi = N(z_1 + z_2 + z_3 + z_4)\Psi_L \).

5. Effects of the non-adiabaticity/asymmetry, \( H_{22}^{\text{eff}} = H_{22} - H_{21} H_{12} / (\hbar \Omega_0) \)

As discussed in section 2 and appendix, the considered setup can be mapped onto a symmetric Hamiltonian, \( H_{22} \), equivalent to the one of rotating atomic clouds in symmetric traps, plus a term, \( H_{21} H_{12} / (\hbar \Omega_0) \), whose importance can be controlled by tuning the laser coupling, \( \Omega_0 \). As discussed in \([22]\), the first effect of the perturbation in the Laughlin-like region is to increase the angular momentum of the GS by populating the states \( \Psi_{L1} \), defined in equation (16). One can consider a fairly small coupling \( \hbar \Omega_0 / E_R \sim 40 \) and still get Laughlin-like GSs of the form of equation (16), which retain most of the known properties of Laughlin states, namely a large entanglement entropy and vanishing interaction energy \([22]\). Now we extend the previous study to the effect of the perturbation on the Pf and LQP regions.
In figure 5 we show the squared overlap between the GS and the three original correlated states (left panel) and their generalized versions (right panel) identified as the generalized Pf, generalized Laughlin-quasiparticle (GLQP) and generalized Laughlin (GL), see equations (16), (21) and (23). It turns out that in all three cases the state that is proportional to $\gamma$ is much less populated than the states proportional to $\alpha$ and $\beta$. We thus neglect the contribution of this term, for simplicity. As shown in figure 5, overviewing all three regimes, the largest improvement by using the generalized versions occurs in the Laughlin region. Here, the total angular momentum increases continuously with $\eta$ \cite{22}, leading to substantial occupation of the state proportional to $\beta$.

Detailed information about the effect of the perturbation is given in figures 6–8 for each of the three regions separately. First, in figure 6 we consider the overlap with the Pf and GPf, exploring fairly low values of $\hbar \Omega_0/E_R$. Lower values of $\hbar \Omega_0/E_R$ require the consideration of higher-order terms in the expansion of $H_{22}^{\text{eff}}$ not included in our calculations. There one can see how by decreasing the value of $\hbar \Omega_0$, the $\eta$ that provides maximum overlap becomes smaller. Thus, while in the symmetric case, the only region with non-negligible squared overlap with the Pf was $0.75 \lesssim \eta \lesssim 0.92$, with, e.g., $\hbar \Omega_0/E_R = 20$, the region is roughly displaced to $0.73 \lesssim \eta \lesssim 0.89$.

Also, the squared overlap with the Pf is reduced, going from about 0.9 in the symmetric case, to 0.7 for $\hbar \Omega_0/E_R = 10$. As occurred with the Laughlin, the main effect of the perturbation is to populate states that are of the GPf type, i.e. a Pf core with appropriate Jastrow factors. The GPf state has a large overlap with the GS, of the same order as the Pf itself with the symmetric
Figure 5. Left panel: squared overlap between the GS and the original strongly correlated states considered, namely the Pf, Laughlin and LQP states, as a function of $\eta$ for $\hbar \Omega_0/E_R = 40$ and 100. Right panel: squared overlap between the GS and the generalized correlated states considered, namely GPf, GL and GLQP, as a function of $\eta$ for $\hbar \Omega_0/E_R = 40$ and 100.

Figure 6. Squared overlap between the GS and the Pf and GPf states defined in the text, upper and lower panels, respectively, as a function of $\eta$. The different lines correspond to different values of $\hbar \Omega_0/E_R$. The solid line is obtained with $H_{22}^{\text{eff}} = H_{22}$. 
Figure 7. Squared overlap between the exact GS and the LQP and GLQP states defined in the text, upper and lower panels, respectively, as a function of $\eta$. The different lines correspond to different values of $\hbar \Omega_0 / E_R$. The solid line is obtained with $H_{22}^{\text{eff}} = H_{22}$.

GS. Interestingly, large values, $> 0.8$, of the squared overlap with the GPf state can be found already for $\hbar \Omega_0 / E_R > 20$, which is relevant from the experimental point of view as it increases the window of observability.

A similar behavior was found when studying the squared overlap of the LQP state with the exact GS of the system. As shown in figure 7 the region with sizeable overlap with the LQP state gets shifted toward lower values of $\eta$ when we decrease $\Omega_0$, peaking at $\eta = 0.85$ for $\hbar \Omega_0 / E_R = 10$. Also, a sizeable overlap is found with the GLQP state. It is, however, clear that large values of the squared overlap, $>0.8$, can only be found for values of $\hbar \Omega_0 / E_R > 30$.

In figure 8 we present the corresponding figure for the case of the Laughlin and generalized Laughlin states. Firstly, we note that again the region where the L and GL are most populated gets shifted toward lower values of $\eta$ when we decrease the value of $\hbar \Omega_0 / E_R$. For instance, the maximum value obtained is around $\eta = 0.91$ for $\hbar \Omega_0 / E_R = 15$, and this maximum is of 0.4 in the case of GL. Squared overlaps larger than 0.8 are obtained only for $\hbar \Omega_0 / E_R > 40$. Squared overlaps larger than 0.5 can however be obtained with $\hbar \Omega_0 / E_R$ as low as 20 for $\eta \sim 0.92$.

6. Fractional charge and anyonic statistics of quasihole excitations in the Laughlin regime

An important property of the strongly correlated $N$-body states discussed in the previous sections is their excitations that might behave as particles with fractional charge and obey anyonic statistics, as is the case for the quasihole excitations over the Laughlin GS [10, 16].
Figure 8. Squared overlap between the exact GS and the Laughlin (L) and GL states defined in the text (upper and lower panels, respectively) as a function of $\eta$. The different lines correspond to different values of $\hbar\Omega_0/E_R$. The solid line is obtained with $H_{22}^{\text{eff}} = H_{22}$.

An experimentally feasible way of creating quasihole excitations in our system is by focusing a laser beam onto the atomic cloud. This can be described by adding the following potential to the single-particle Hamiltonian of equation (13) [16]:

$$\hat{V}(\xi) \propto I \sum_i \delta(\xi - z_i), \quad (28)$$

where $I$ is the laser intensity and $\xi$ is the position onto which the beam is focused. With two such potentials we should be able to create states with two quasiholes, according to equation (25). In the following, we will first study quasiholes in the Laughlin state, which can be created when the system is in the adiabatic regime. Then we will also consider a slightly non-adiabatic situation, where we find quasiholes in the GL state, as defined by equation (26). We analyze the quasiholes in both the Laughlin and the GL state with respect to their fractional character.

6.1. Quasiholes in the adiabatic case

As already discussed in section 4, for $\hbar\Omega_0 \gg E_R$, the system’s GS squared overlap with the Laughlin state is effectively 1, above the critical field strength $\eta_2$. Now we consider the system with the additional term (28) and find that there is also a region of $\eta$ where the overlap of the GS of the system with the analytical quasihole state is effectively 1; see figure 9. This shows that the potential of equation (28) is able to produce quasiholes described by (24). Similarly, adding two such lasers we also find a region of $\eta$ where the overlap between the exact GS and the analytical state with two holes, equation (25), is very close to 1. However, we note that the
values of \( \eta \) at which the overlap for one or two quasiholes reaches 1 differ from each other, and both are found for values larger than \( \eta^2 \); see figure 9. These features are essentially independent of the laser strength \( I \), for sufficiently large \( I \).

6.1.1. Fractional charge. The most interesting property of these excitations is their fractionality, i.e. fractional charge and statistics. To study the fractional charge of the quasiholes, we first note that in our electro-neutral system subjected to an artificial magnetic field, there exists the analogue of an electric charge which can be defined via the behavior of a particle or quasiparticle evolving within the gauge field. To this end, we consider the phase a quasihole picks up while being adiabatically displaced following a closed trajectory. The general expression for the Berry phase on a closed loop \( C \) is given by \([43]\)

\[
\gamma_C = i \oint_C \left( \Psi(\vec{R}) \nabla_{\vec{R}} \bar{|\Psi(\vec{R})|} \right) \cdot d\vec{R},
\]

with \( |\Psi(\vec{R})| \) being the state of the system, characterized by a parameter \( \vec{R} \), which in our case is the position of the quasihole. For simplicity, we now assume that the quasihole is fixed at a radial position \( |\xi| = R \lambda_L \), but is moved along a circle centered at the origin; see figure 10, parameterized by the angle \( \phi \). This is sufficient to test the fractional behavior.

For general contours, one can extract the acquired phases from the normalization factor of the quasihole state, as described in \([44]\). For our circularly symmetric contour, however, the situation is simpler, as we can re-write equation (29) as

\[
\gamma_C = i \int_0^{2\pi} \langle \Psi_{\text{qh}}(\phi) | \partial_\phi | \Psi_{\text{qh}}(\phi) \rangle \, d\phi \equiv \int_0^{2\pi} f(R) \, d\phi.
\]

Here we note that, due to the circular symmetry of the Laughlin state, the integrand does not depend on the angular position of the quasihole. The function \( f(R) \) can be calculated by decomposing the Laughlin quasihole state into the Fock–Darwin basis \([41]\), which we have
Figure 10. Moving one or two quasiholes at fixed radial positions on circles around the origin allows us to determine the effective charge and the statistical phase angle. The quasiholes are always fixed at a radial position $R = 1$ (in units of $\lambda_{\perp}$).

done analytically for particle numbers up to 6. For compactness, we explicitly give here only the result for $N = 4$:

$$\gamma_C = 2\pi f(R) = 2\pi \frac{4\left(10128R^2 + 5313R^4 + 1659R^6 + 553R^8\right)}{85572 + 40512R^2 + 10626R^4 + 2212R^6 + 553R^8}.$$  \hspace{1cm} (31)

If we assume that the quasihole is moved sufficiently close to the center, i.e. $R \lesssim 1$, we can expand this expression in $R$ and find $\gamma_C = 2\pi (0.473426R^2 + 0.0242202R^4 + O(R^6)) \approx \pi R^2$. Thus, the acquired phase is approximately given by the enclosed area in units of $\lambda_{\perp}^2$.

To obtain the effective charge of the quasihole, we must compare this result with the geometric phase acquired by a particle moved along the same closed contour due to the gauge field. In the Laughlin regime, where $\eta \approx 1$, we find with equation (14) $B_0 \approx 2\hbar/\lambda_{\perp}$; thus the acquired phase $\phi$ is two times the enclosed area in units of $\lambda_{\perp}^2$, i.e. $\phi = \frac{1}{2}B_0(R\lambda_{\perp})^2\pi \approx 2R^2\pi$.

From this it follows that the effective charge of the quasihole is $q_{\text{eff}} = \frac{\hbar}{\phi} \approx 0.47$, close to the expected value for the Laughlin state at half-filling in the thermodynamic limit, $1/2$ \cite{7, 10}. We have conducted a similar study for $N = 5$ and $N = 6$, finding that for $N = 5$, the effective charge is about 0.48, and for $N = 6$, it is found to be 0.49, i.e. by increasing the particle number the value $1/2$ is approached.

6.1.2. Fractional statistics. To prove the fractional statistics of the quasihole excitations we now consider the system with two quasiholes at $\xi_1 = |\xi_1|e^{i\phi_1}$ and $\xi_2 = |\xi_2|e^{i\phi_2}$, which we assume to sit on opposite radial positions, i.e. $|\xi_1| = |\xi_2| = R\lambda_{\perp}$ and $\phi_2 - \phi_1 = \pi$. We now consider the simultaneous adiabatic movement of the two quasiholes on two half-circles, in such a way that, at the end, the quasiholes interchange position (see figure 10). This differs from a more common setup to test the statistical angle, where one quasihole is fixed in the center, while the other is encircling it, but it has the advantage that it maximizes the distance between the two quasiholes.
Figure 11. (a) Squared overlap between the quasihole wave functions, $|\langle \Psi_{Lqh}(\xi_1) | \Psi_{Lqh}(\xi_2) \rangle|^2$, at opposite angular positions, $\phi_1 - \phi_2 = \pi$, as a function of their distance to the center, $|\xi_1| = |\xi_2| = R\lambda_\perp$. (b) Statistical angle of two quasiholes at opposite angular positions and radial position $R\lambda_\perp$. In both panels we present the results for $N = 4$ (black solid), $N = 5$ (red dotted) and $N = 6$ (green dashed).

Note that in figure 10, the radial position is chosen at $R = 1$, i.e. the distance between the center of the quasiholes is $2\lambda_\perp$, which seems to be the minimum distance needed for not having a significant overlap ($\ll 10\%$) between the two quasiholes, see figure 11(a). On the other hand, in this small system of just four particles, larger radial positions lead to quasiholes overlapping with the systems edge.

The total phase picked up during the described movement should be the sum of the phase picked up by one quasihole moved along a circle plus a phase factor due to the interchange of the two quasiholes. Again the phase gradient turns out to be independent of the angular position, but is described by a different function $\tilde{f}$ of the radial coordinate:

$$\tilde{f}(R) = \frac{8 \left( 2868120 R^4 + 461616 R^8 + 25242 R^{12} + 553 R^{16} \right)}{41660640 + 11472480 R^4 + 923232 R^8 + 33656 R^{12} + 553 R^{16}}. \quad (32)$$

The statistical phase angle is thus

$$\psi(R)_{\text{stat}} = \int_0^{2\pi} f(R) \, d\phi - \int_0^{\pi} \tilde{f}(R) \, d\phi = 2\pi f(R) - \pi \tilde{f}(R). \quad (33)$$

It is shown, as a function of $R$, in figure 11(b). First, as expected, the statistical phase is zero if both quasiholes are in the same position, and it increases linearly when the distance between the quasiholes is increased. This linear behavior then saturates once the overlap between the two quasiholes, $|\langle \Psi_{Lqh}(\xi_1) | \Psi_{Lqh}(\xi_2) \rangle|^2$, drops below 0.1 and remains mostly constant around $\pi/2$. By increasing the number of particles $N$, the phase angle become less dependent on $R$ once the two quasiholes do not overlap, meaning that it becomes a robust property of the quasiholes. It stabilizes around the expected value of $\pi/2$. At larger distances $R$, the system’s edge starts to play a role.
6.2. Non-adiabatic effects on the properties of quasiholes

To study the fractionality of quasihole excitations in the non-adiabatic case, we will again profit from the generalized analytical representations used to describe the GS of the system. Following the discussion in the previous section, we compute the squared overlap of the GS obtained with no, one and two extra lasers piercing holes into the system. First, we find a significant squared overlap for the slightly perturbed case at $\hbar \Omega_0 = 100E_R$, see figure 12. As occurred in the adiabatic case, a large overlap with the analytical one- and two-quasihole states appears only at higher field strengths than the one at which the generalized Laughlin state is reached. Our study of the properties of quasiholes in the non-adiabatic case will be restricted to the parameter domain where a fair description of the states is provided by the generalized state given in section 3.

We now test the behavior of the quasiholes in a generalized Laughlin state. This can be done as before, but now we must note that the gradient of the state will not only depend on the radial, but also on the angular position of the quasiholes. Furthermore, it will depend on the parameter $\beta$ as defined in equations (26) and (27), which is used to improve the overlap. As shown in figure 12, for given parameters $\eta$ and $\Omega_0$ the same value of $\beta$ optimizes simultaneously the GS, the quasihole state and the state with two quasiholes. We define

$$ f_\beta(R, \phi) \equiv \langle \Psi_{Lqh}(\phi) | \partial_\phi | \Psi_{Lqh}(\phi) \rangle. $$

(34)

This function $f_\beta$ is quite lengthy, so we expand it in $R$ and give only the lowest term ($\mathcal{O}(R^2)$):

$$ f_\beta(R, \phi) \simeq \frac{8115904 + 2799526 \beta^2 - 7102 \sqrt{94958 (1 - \beta^2) \beta \cos(2\phi)}}{17142924 + 4477401 \beta^2} R^2. $$

(35)

From the expression we see that for a fixed and small value of $R$, $f_\beta$ oscillates around $R^2/2$, such that the angular integration $\int_0^{2\pi} f_\beta(R, \phi) d\phi$ again will yield a Berry phase close to the encircled area, thus half of the Berry phase accumulated by a normal particle.
Figure 13. The effective charges $q_y$ (blue dotted) and $q_x$ (red dashed) and $q_{\text{eff}} = (q_x + q_y)/2$ (green solid) of quasiholes in the GL state as a function of the admixture $\beta$ of higher angular momentum to the Laughlin state, for $N = 4$ (left) and $N = 6$ (right).

Formally, we can capture this oscillating behavior by defining two effective charges $q_x$ and $q_y$, depending on the direction in which the quasihole moves. Up to linear order in the quasihole coordinates $X$ and $Y$, which is valid for small radial positions $R \lesssim 1$, the Berry connection is given by $(q_x Y - q_y X)/\lambda_\perp^2$. The Berry phase defined in equation (29) reads

$$\gamma_C = \oint \frac{1}{\lambda_\perp^2} (q_x Y - q_y X) \cdot d\vec{R} = (q_x + q_y) \frac{A}{\lambda_\perp^2},$$

where $A$ is the encircled area and Stokes’ theorem has been applied. The effective charge, defined as $q_{\text{eff}} = (q_x + q_y)/2$, and the gauge-dependent $q_x$ and $q_y$ are plotted as a function of $\beta$ in figure 13 for different $N$. In all cases the effective charges are close to $1/2$. For small values of $\beta$, which represent realistic states of the system, the value of the charges approaches $1/2$ when the number of atoms in the system is increased. In summary, the average charge $q_{\text{eff}}$ has only a minor dependence on $\beta$, which decreases as $N$ increases. Although not realized in our system, we note that in the limit $\beta \to 1$, both charges $q_x$ and $q_y$ again coincide due to the recovered cylindrical symmetry of the state $\Psi_{L1}$.

Finally, we introduce two quasiholes into the generalized Laughlin state. Following a procedure similar to the one for the adiabatic case presented in the previous section, we extract the statistical phase angle of the quasiholes. The result as a function of $\beta$ is shown in figure 14 for $N = 4$ and closed paths of different radii. While the quasiholes in the bulk, $R = 1$, remain with almost constant phase angles $\varphi_{\text{stat}} \approx 0.51$, the phase angles of quasiholes closer to the edge of the system have a stronger dependence on $\beta$. Thus, we find that the presence of a certain degree of non-adiabaticity, $\beta \leq 0.7$, does not spoil the presence of anyonic quasihole excitations of the Laughlin state.

7. Evolution with $N$ of the energy gap

The typical scenario of an experiment that has as a goal the realization of a specific strongly correlated state is to first prepare an easily attainable initial state. Then, one may follow adiabatically a route in parameter space which ends in the final desired state [17, 21]. This final
state is expected to be robust, with a mean lifetime larger than the time necessary to perform the measurements. A crucial ingredient necessary for the success of such an approach is to have energy gaps as large as possible over all the GSs involved along the route.

In this section we concentrate on the study of the energy gap over the Laughlin state. We consider first the adiabatic case and then study the effect of the perturbation on the energy gap. The gap is the energy difference between the GS and the lowest excitation in the thermodynamic limit. We will analyze the behavior of the gap for increasing \( N \) by means of our exact diagonalization calculations up to \( N = 7 \).

For the Laughlin state in the adiabatic/symmetric case, as is shown in figure 2 for \( N = 4 \) and \( L = 12 \) there are two excitations with the lowest energy with \( L = 12 - 4 = 8 \) and \( L = 13 \) depending on the value of \( \eta \). This is a general result for any \( N \), the excitations have \( L = N(N - 1) - N \) and \( L = N(N - 1) + 1 \). The excitation with \( L = 13 \) is an excitation of the center of mass of the system. This is due to the incompressibility of the Laughlin state. Namely, as shown in figure 3, the state can increase its angular momentum without changing its interaction energy. We ignore this excitation, since we are interested in bulk excitations. This means that the linear left branch in the Laughlin region in figure 2 must be extended up to \( \eta = 1 \) (where one has the largest gap). All the branches on the right that would lie below this line are edge excitations of different polarity. In addition, this largest gap at \( \eta = 1 \) where effectively there is no trap coincides with the energy difference (denoted as \( \Delta \)) between the GS and the first excitation in the subspace \( L = N(N - 1) \).

As a consequence, to see the evolution of the gap over the Laughlin state, it is sufficient to calculate the first two eigenvalues of the energy spectrum in the \( L = N(N - 1) \) subspace for each \( N \). In figure 15(a), we show the evolution of \( \Delta / g \) with \( N \) for a fixed value of \( g \) (independent of \( N \)). Had we used the assumption \( gN = \text{constant} \) we would obtain the same figure. This is a consequence of the fact that \( \Delta \) is proportional to \( g \) and independent of \( N \) for large values of \( N \). This result is in agreement with the calculations reported in [45]. The tendency up to \( N = 7 \) is to asymptotically recover the value of 0.1 that was previously obtained by Regnault and Jolicoeur [37] assuming a spherical geometry and later reproduced by Roncaglia et al [20].
Figure 15. (a) $\Delta / g$, in units of $\hbar \omega_\perp$, of the Laughlin state as a function of $N$ in the adiabatic/symmetric case. (b) Value of $\eta_2$ computed for $N = 3, 4, 5, 6$ and $7$ in the adiabatic/symmetric case keeping $gN = 6$, compared to the prediction explained in the text, $\eta_2 = 1 - 0.1(gN/N^2)$.

In practice there are several ways in which the thermodynamic limit can be defined; see, e.g., section 2.4 of [31]. The most constraining restriction in our case is that of remaining within the LLL approximation. As studied in [45], this imposes a constraint on the maximum value of $g$ that can be considered for a given $N$. One way to study the thermodynamic limit ensuring that the LLL assumption is always valid is to keep the chemical potential of the system constant, $gN = \text{constant} < (gN)_{\text{crit}} < 8\pi$ (with $(gN)_{\text{crit}}$ dependent on $N$ [45]). This assumption gives a well-defined positive value of $\eta_1 = 1 - gN/8\pi$ for arbitrary $N$. In this case, the bulk gap would get reduced when the atom number is increased, since $\Delta$ must compensate for the imposed $N$ dependence of $g$, and consequently approach zero as $\Delta \sim 1/N$. In an experiment, a way to approach the thermodynamic limit would be to consider a constant value of $g$, determined from the s-wave scattering length and the confinement in the $z$-direction, and to increase the number of atoms in the system. In this case, provided the value of $g$ is small enough to prevent the excitation of the next Landau level [45], the bulk gap over the Laughlin state remains constant, thus permitting the formation of moderately large Laughlin states.

Our discussion is based on the bulk gap presented above; in a real experimental situation the most likely scenario is that of a Laughlin-like state forming in the middle region of the system with the parts of the system close to the edges developing a gapless superfluid state.

For possible practical implementations, it is also important to quantify the size of the parameter region where the Laughlin can be produced. Therefore, a good estimate of $\eta_2$ can be obtained taking advantage of the above-mentioned coincidence. $\eta_2$ is the critical value where the energies of the Laughlin and $L = N(N - 1) - N$ state cross, or

$$(1 - \eta_2)(L_0 - N)\hbar \omega_\perp + V_1 = (1 - \eta_2)L_0\hbar \omega_\perp + V_0,$$

where $L_0 = N(N - 1)$ and $V_0$ and $V_1$ are $E_{\text{int}}(L_0)$ and $E_{\text{int}}(L_0 - N)$ (see figure 3), respectively. Or

$$\eta_2 = 1 - \frac{V_1}{N\hbar \omega_\perp}(V_0 = 0).$$

Or

$$\eta_2 = 1 - \frac{V_1}{N\hbar \omega_\perp}(V_0 = 0).$$

In addition, $V_1$ coincides with the energy difference $\Delta$ between the GS and the first excitation in the $L_0$ subspace which tends, as $N$ increases (see figure 15)(a), to

$$\Delta \sim 0.1 \frac{g\hbar \omega_\perp}{N}.$$
and then

$$\eta_2 = 1 - \Delta/(N\hbar\omega_\perp) \sim 1 - 0.1 \frac{gN}{N^2}. \quad (40)$$

In figure 15(b), we compare the prediction of this formula and the computed $\eta_2$ for different $N$, keeping $gN = 6$. As can be seen, the formula agrees very well with the numerically obtained values. We have also checked that the formula reproduces the values of $\eta_2$ reported in [45].

Finally, figure 16 shows the change of the gap with decreasing $\Omega_0$. There are some important differences between the symmetric and the perturbed cases. The perturbation mixes a large number of subspaces and now it is not possible to ignore the right branch. The initial ($\eta_2$) and final frequencies at the boundaries of the Laughlin region are shifted to smaller values. The largest gap, the one at the upper vertex of the triangle, is nearly constant. At $\eta_2$ the perturbation opens a gap where degeneracy occurs in the symmetric case. As a consequence, we conclude that, on the one hand, the perturbation favors the observability, and on the other hand, the detection has to be restricted to a small number of particles.

8. Summary and conclusions

We have studied the possibility of producing relevant strongly correlated quantum states as GSs of a system of ultracold two-level atoms subjected to an artificial gauge field. The focus is on the formation of the Moore–Read (Pf), Laughlin and LQP states. Considering a small number of atoms, we have shown by exact diagonalization methods that large squared overlaps between Pf-like, LQP-like and Laughlin-like states and the GS of the system are found even for fairly small values of the external laser intensity $\hbar\Omega_0/E_R$. Reducing the laser intensity, that is, reducing the Rabi frequency, deforms the system and increases the angular momentum of the states. Nevertheless, the main properties of the original underformed states remain unchanged within a broad region of parameter space. Analytical representations of the GSs can be obtained from the original states by considering an admixture containing additional Jastrow factors and thereby having higher angular momentum. Variationally we can choose the weight of this admixture such that a large overlap with the numerical results is obtained. This allowed us to get analytical insights into the fractional behavior of the quasihole excitations.

New Journal of Physics 14 (2012) 055003 (http://www.njp.org/)
We have checked that quasihole states on the Laughlin and generalized Laughlin states can be produced by means of additional laser beams. We have studied the fractional charge and anyonic statistics of such quasiholes making use of the analytical representation of the states. Both the effective charge and the statistical phase angle are close to the expected value of \(1/2\), even for the small number of particles considered here. Such a fractional behavior is found whenever quasiholes are able to evolve in the bulk of the system. Thus, the agreement with the expected behavior improves as the size of the system is increased. The admixture of higher angular momentum in the generalized Laughlin state does not modify the fractional behavior of its quasihole excitation. However, as the creation of two quasiholes requires a higher artificial field strength \(\eta\) than the one necessary to get into the (generalized) Laughlin regime, our analysis is relevant mostly in the weakly perturbed regime, whereas in the highly perturbed regime an exact numerical evolution of the adiabatic movement would be needed, falling beyond the scope of the present work.

Concerning the observability of Laughlin-like states, we find that decreasing \(\hbar \Omega_0/E_R\) shifts the spectrum to smaller values of \(\eta\), which are thus further away from the instability region \(\eta = 1\), favoring the experimental conditions.

An interesting topic for future investigations is excitations in the Pf-like regime, which might obey non-Abelian statistics. A signature of this behavior would be a degeneracy in the state with four quasiholes \([46]\). However, our present system is too small to test this physics.

Acknowledgments

The authors thank N Cooper, J Dalibard and K J Günter for useful comments and discussions and J Lapeyre for a careful reading of the manuscript. This work was supported by the EU (NAMEQUAM, AQUTE, MIDAS), ERC (QUAGATUA), Spanish MINECO (FIS2008-00784, FIS2010-16185, FIS2008-01661 and QOIT Consolider-Ingenio 2010), Alexander von Humboldt Stiftung, IFRAF, ANR (the BOFL project) and AII-Hubbard. BJ-D is partially supported by Grup Consolidat SGR 21-2009-2013 and by the Ramón y Cajal program. ML acknowledges support from a Hamburg Theory Award.

Appendix. Effective Hamiltonian

Let us consider a system described by a single-particle Hamiltonian given by \(H = H_0 + V\), where \(H_0\) is solvable and \(V\) can be treated as a perturbation. In addition, let us assume that the eigenvalues of \(H_0\) are grouped into manifolds well separated in energy, i.e. the eigenenergies \(E_{i\alpha}\) have the following property,

\[
|E_{i\alpha} - E_{j\alpha}| \ll |E_{i\alpha} - E_{j\beta}|
\]

where \(\alpha\) and \(\beta\) denote different manifolds and the index \(i\) denotes different states inside a manifold. In our case we have two manifolds, the lowest energy one \((\alpha = 2)\) described by \(H_{22}\) and the most energetic one described by \(H_{11}\) \((\beta = 1)\); both together play the role of \(H_0\),

\[
H_0 = \begin{pmatrix} H_{11} & 0 \\ 0 & H_{22} \end{pmatrix}
\]

acting on the spinor shown below equation (6).
The difference in energy $\hbar \Omega_0$ is much larger than the typical expected values of $H_{22}$, which are of the order of the recoil energy $E_R = \hbar^2 k^2 / 2M$. Within the lowest manifold we make the LLL assumption. This scenario is valid in the case when the two internal states are uncoupled and the system evolves always in $|\psi_1\rangle$ or $|\psi_2\rangle$.

Instead, if there is coupling between the manifolds, the Hamiltonian is represented by $H_0 + \lambda V$ where $\lambda$ is a dimensionless parameter and $V$, in general, has nonzero matrix elements inside each manifold as well as between them. As long as $\lambda$ is small, the structure of the well-separated manifolds and their degeneracy is preserved with slight modifications. Physically, the coupling between the two manifolds means that the motion of an atom in $|\psi_2\rangle$ is slightly modified by a sudden and short time period in the other manifold. The high-frequency dynamics driven by $(E_{i\alpha} - E_{j\beta})/\hbar$ ($\alpha \neq \beta$) is averaged by the slow dynamics driven by $(E_{i\alpha} - E_{j\alpha})/\hbar$. In a way, the wave functions of the manifold-2 are ‘influenced’ (or dressed) by the wave functions of manifold-1. In our case, the structure of $V$ is

$$V = \begin{pmatrix} 0 & H_{12} \\ H_{21} & 0 \end{pmatrix},$$

(A.3)

with no diagonal elements.

The main goal will be to obtain an effective Hermitian Hamiltonian $H'$ that acts only on the unperturbed manifold-2, although having the same eigenvalues as those of $H$, and with zero matrix elements between the two unperturbed manifolds. In this way, we will be able to consider only the FD functions and ignore manifold-1.

The Hermiticity and the coincidence on the eigenvalues with $H$ are achieved if we consider a unitary transformation from $H$ to $H'$ as

$$H' = THT^\dagger,$$

(A.4)

where $T = e^{iS}$ and $S = S^\dagger$. Or

$$H' = e^{iS}He^{-iS} = H + [iS, H] + \frac{1}{2!} [iS, [iS, H]] + \cdots = H_0 + \lambda H'_1 + \lambda^2 H'_2 + \cdots,$$

(A.5)

where we have considered the expansion $S = \lambda S_1 + \lambda^2 S_2 + \cdots$ and the condition that $S$ has zero matrix elements inside each manifold. Grouping the terms in orders of $\lambda$, it is possible to solve $S_i$ step by step as functions of known quantities. One arrives at the expression

$$\langle i\alpha | H' | j\alpha \rangle = E_{i\alpha} \delta_{ij} + \langle i\alpha | \lambda V | j\alpha \rangle + \frac{1}{2} \sum_{k,\gamma \neq \alpha} \langle i\alpha | \lambda V | k\gamma \rangle \langle k\gamma | \lambda V | j\alpha \rangle \frac{1}{E_{i\alpha} - E_{k\gamma}} + \frac{1}{E_{j\alpha} - E_{k\gamma}} + \cdots.$$

(A.6)

The first term represents the unperturbed levels in the manifold-2, the second one is the direct coupling between unperturbed levels in the manifold-2 and the third term is the contribution of the indirect coupling through the manifold-1. In our case, the last equation reduces to

$$\langle i\alpha | H' | j\alpha \rangle \simeq E_{i2} \delta_{ij} + \frac{\lambda^2}{2} \sum_k \langle i2 | V | k1 \rangle \langle k1 | V | j2 \rangle \left( -\frac{1}{\Omega_0} - \frac{1}{\Omega_0} \right),$$

(A.7)
where we have approximated $E_{k1}$ by $\hbar\Omega_0$ and considered $E_{i2} \ll \hbar\Omega_0$ or

$$
\langle i\alpha \mid H' \mid j\alpha \rangle = E_{i2}\delta_{ij} - \frac{\lambda^2}{\Omega_0} \langle i2 \mid V^2 \mid j2 \rangle ;
$$

(A.8)

thus,

$$
H' = H_{22} - \frac{H_{21}H_{12}}{\Omega_0},
$$

(A.9)

which is the result used in equation (12) of section 2. The interaction term is considered part of the non-perturbed term together with the kinetic contribution, see equation (13).

The explicit form of the perturbation term $H_{21}H_{12}$ used is

$$
H_{21}H_{12} = \left( \frac{\hbar^4}{4M^2w^4} - \frac{2x^2\hbar^4}{M^2w^6} + \frac{k^2x^2\hbar^4}{16M^2w^4} + \frac{k^4x^2\hbar^4}{64M^2w^2} + \frac{ikxy\hbar^4}{4M^2w^5} + \frac{k^2y^2\hbar^4}{64M^2w^4} \right)
$$

$$
+ \left( -\frac{ik\hbar^4}{4M^2w^3} - \frac{ik^3\hbar^4}{8M^2w} \right) \partial_y + \left( \frac{x\hbar^4}{M^2w^4} - \frac{iky\hbar^4}{8M^2w^3} \right) \partial_x
$$

$$
+ \left( -\frac{k^2\hbar^4}{4M^2} + \frac{k^2x^2\hbar^4}{4M^2w^2} \right) \partial_y^2 + \left( -\frac{\hbar^4}{4M^2w^2} + \frac{x^2\hbar^4}{2M^2w^4} \right) \partial_x^2.
$$

(A.10)

One can show that this operator does not conserve $L$, as it connects $L'$-subspaces with $L = L + \Delta$ where $\Delta = 0, \pm 2, \pm 4$.

References

[1] Anderson M H, Ensher J R, Matthews M R, Wieman C E and Cornell E A 1995 *Science* **269** 198
[2] Davis K B, Mewes M-O, van Druten N J, Durfee D S, Kurn D M and Ketterle W 1995 *Phys. Rev. Lett.* **75** 3969
[3] Andrews M R, Townsend C G, Meisner H-J, Durfee D S, Kurn D M and Ketterle W 1997 *Science* **275** 637
[4] Lewenstein M, Sanpera A, Ahufinger V, Damski B, Sen A (De) and Sen U 2007 *Adv. Phys.* **56** 243
[5] Greiner M, Mandel O, Esslinger T, H"ansch T W and Bloch I 2002 *Nature* **415** 39
[6] Moore G and Read N 1991 *Nucl. Phys. B* **360** 362
[7] Laughlin R B 1983 *Phys. Rev. Lett.* **50** 1395
[8] Wilkin N K and Gunn J M F 2000 *Phys. Rev. Lett.* **84** 6
[9] Cooper N R and Wilkin N K 1999 *Phys. Rev. B* **60** R16279
[10] Arovas D J, Schrieffer J R and Wilczek F 1984 *Phys. Rev. Lett.* **53** 722
[11] Nayak C and Wilczek F 1996 *Nucl. Phys. B* **479** 529
[12] Read N and Rezayi E H 1996 *Phys. Rev. B* **54** 16864
[13] Kapit E, Ginsparg P and Mueller E 2012 *Phys. Rev. Lett.* **108** 066802
[14] Nayak C, Simon S H, Stern A, Freedman M and Das Sarma S 2008 *Rev. Mod. Phys.* **80** 1083
[15] Madison K W, Chevy F, Bretin V and Dalibard J 2001 *Phys. Rev. Lett.* **86** 4443
[16] Paredes B, Fedichev P, Cirac J I and Zoller P 2001 *Phys. Rev. Lett.* **87** 010402
[17] Popp M, Paredes B and Cirac J I 2004 *Phys. Rev. A* **70** 053612
[18] Bloch I, Dalibard J and Zwerger W 2008 *Rev. Mod. Phys.* **80** 885
[19] Günter K J, Cheneau M, Yefsah T, Rath S P and Dalibard J 2009 *Phys. Rev. A* **79** 011604
[20] Roncaglia M, Rizzi M and Dalibard J 2011 *Sci. Rep.* **1** 43
[21] Roncaglia M, Rizzi M and Cirac J I 2010 *Phys. Rev. Lett.* **104** 096803
[22] Julià-Díaz B, Dagnino D, Günter K J, Graß T, Barberán N, Lewenstein M and Dalibard J 2011 *Phys. Rev. A* **84** 053605

*New Journal of Physics* **14** (2012) 055003 (http://www.njp.org/)
[23] Wilczek F and Zee A 1984 Phys. Rev. Lett. 52 2111
[24] Juzeliūnas G and Öhberg P 2004 Phys. Rev. Lett. 93 033602
[25] Ruseckas J, Juzeliūnas G, Öhberg P and Fleischhauer M 2005 Phys. Rev. Lett. 95 010404
[26] Kinoshita T, Wenger T and Weiss D S 2006 Nature 440 900
[27] Gemelke N, Sarajlic E and Chu S 2010 arXiv:1007.2677
[28] Cohen-Tannoudji C, Dupont-Roc J and Grynberg G 1992 Atom–Photon Interactions (New York: Wiley)
[29] Basdevant J L and Dalibard J 2006 The Quantum Mechanics Solver (The Netherlands: Springer)
[30] Jacak L, Hawrylak P and Wojs A 1998 Quantum Dots (Berlin: Springer)
[31] Cooper N R 2008 Adv. Phys. 57 539
[32] Fetter A L 2009 Rev. Mod. Phys. 81 647
[33] Dalibard J, Gerbier F, Juzeliūnas G and Öhberg P 2011 Rev. Mod. Phys. 83 1523
[34] Parke M I, Wilkin N K, Gunn J M F and Bourne A 2008 Phys. Rev. Lett. 101 110401
[35] Dagnino D, Barberán N, Lewenstein M and Dalibard J 2009 Nature Phys. 5 431
[36] Cooper N R, Wilkin N K and Gunn J M F 2001 Phys. Rev. Lett. 87 120405
[37] Regnault N and Jolicoeur T 2003 Phys. Rev. Lett. 91 030402
[38] Wilkin N K, Gunn J M F and Smith R A 1998 Phys. Rev. Lett. 80 2265
[39] Smith R A and Wilkin N K 2000 Phys. Rev. A 62 061602
[40] Dagnino D, Barberan N and Lewenstein M 2009 Phys. Rev. A 80 053011
[41] Juliá-Díaz B and Graß T 2012 Comput. Phys. Commun. 183 737
[42] Cazalilla M A 2003 Phys. Rev. A 67 063613
[43] Berry M V 1984 Proc. R. Soc. A 392 45
[44] Wen X-G 2004 Quantum Field Theory of Many-Body Systems (Oxford: Oxford University Press)
[45] Morris A G and Feder D L 2006 Phys. Rev. A 74 033605
[46] Fradkin E, Nayak C, Tsvelik A and Wilczek F 1998 Nucl. Phys. B 516 704