“Catalan Traffic” and Integrals on the Grassmannian of Lines∗

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Abstract

We prove that certain numbers occurring in a problem of paths enumeration, studied by Niederhausen in [7] (see also [10]), are top intersection numbers in the cohomology ring of the grassmannian of the lines in the complex projective \((n + 1)\)-space.

1 Introduction

1.1 The Catalan’s numbers

\[ C_n = \frac{1}{n+1} \binom{2n}{n}, \text{ for all } n \in \mathbb{N} \]

occur in several combinatorial situations (see e.g. [9]), in particular in lattice path enumeration. It is well known, for instance, that \(C_n\) is the number of lattice paths contained in \(S := \{(m, n) \in \mathbb{Z} \times \mathbb{Z} \mid 0 \leq m \leq n\}\) from \((0, 0)\) to \((n, n)\) \(\in S\), allowing unitary steps only, along the “horizontal” or “vertical” directions.

Within this context, the aim of this paper is to make some remarks on the occurrence of Catalan’s numbers in a traffic game (“Catalan traffic at the beach”) constructed by Niederhausen [7]. One is given of a city map \(C\) (a lattice in \(\mathbb{Z}^2\)) with some gates and road blocks (null traffic points). The traffic rules are as follows. First, no path can cross and go beyond the beach (the line \(m - n = 0\) in the \((m, n)\) \(\mathbb{Z}\)-plane). Furthermore:

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1. At lattice points strictly “below” the line $2m + n = 0$, only North (↑) or West (←) directions are allowed;

2. At lattice points strictly “above” the line $2m + n = 0$, only East (→) or NE (↗) directions are allowed;

3. All the points $(m, n) \in \mathbb{Z}^2$ lying on the line $2m + n = 1$ are inaccessible (road blocks ■).

4. On the line $2m + n = 0$ (gates), allow $W(←)$, $E(→)$, and $NE(↗)$ (because of the road blocks at $2m + n = 1$) (see [7], p. 2)

The diagram of the city map is depicted below: it is the same as in [7] after a harmless counterclockwise rotation of 90 degrees.

![City Map Diagram]

Fig. 1. City Map.

The problem, solved by Niederhausen, consists in finding the number of all distinct paths joining the origin to any point in the domain of $C$, compatibly with the constraints. Attaching to each point of the lattice the number of such paths one gets the following diagram:
Fig. 2. Detours preserving Catalan traffic.

The main result of [7] is that the numbers along the “beach” are Catalan’s number. This is proven in three different ways. One of them relies on the following recursion:

\[
\begin{align*}
\Upsilon(m, n) &= \Upsilon(m + 1, n) - \Upsilon(m, n - 1) \\
\Upsilon(n, n) &= C_n
\end{align*}
\]  

(1)

holding in the domain \(\{(m, n) \in \mathbb{Z}^2 | -n \leq 2m \leq 2n\}\), where \(\Upsilon(m, n)\) denotes the number of paths to get the point \((m, n)\) starting from the origin.

1.2 The Catalan number \(C_n\) has also a beautiful geometric interpretation (see e.g. [6]): in fact, it is the Plücker degree

\[\kappa_{2n,0} = \int_{G_1(\mathbb{P}^{n+1})} \sigma_1^{2n},\]

of the grassmannian of lines \(G_1(\mathbb{P}^{n+1})\). The main result of this paper is that for each \((m, n)\), such that \(n \geq m \geq 0\), \(\Upsilon(m, n) = \kappa_{2m,n-m}\) where

\[\kappa_{2m,n-m} = \int_{G_1(\mathbb{P}^{n+1})} \sigma_1^{2m} \sigma_2^{n-m},\]

is the top intersection number computed in the integral cohomology ring \(H^*(G_1(\mathbb{P}^{n+1}), \mathbb{Z})\) of \(G_1(\mathbb{P}^{n+1})\), generated (as a \(\mathbb{Z}\)-algebra) by the special Schubert cycles \(\sigma_1\) and \(\sigma_2\). The proof consists in using the formalism introduced in [2] (see also [3] and [8]) to show that the recursion (1) holds for \(K(m, n) := \kappa_{2m,n-m}\).

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2 Preliminaries

2.1 The grassmannian $G_1(\mathbb{P}^{n+1})$ is a complete projective variety of complex dimension $2n$. It is known (see e.g. [1], [5]) that its integral cohomology (or Chow intersection ring) $H^*(G_1(\mathbb{P}^n), \mathbb{Z})$ is generated by the special Schubert cycles $\sigma_1$ and $\sigma_2$. The cycle $\sigma_1 \in H^*(G_1(\mathbb{P}^{n+1}), \mathbb{Z})$ is the cohomology class represented by the subvariety which is the closure of all the lines incident a codimension 2 linear subspace of $\mathbb{P}^{n+1}$, while $\sigma_2 \in H^*(G_1(\mathbb{P}^{n+1}), \mathbb{Z})$ is represented by the closure of all the lines which are incident a codimension 3 linear subspace of $\mathbb{P}^{n+1}$. A top intersection number in $G_1(\mathbb{P}^{n+1})$ is the degree of the product $\sigma_1^a \sigma_2^b$ in $H^*(G_1(\mathbb{P}^{n+1}), \mathbb{Z})$, with $a + 2b = 2n$.

2.2 The cohomology (or the intersection) theory of complex grassmannian varieties can be described by Schubert Calculus. The latter can be phrased in purely algebraic terms using the formalism introduced in [2] (see [3] for more details). For the grassmannian of lines of $\mathbb{P}^{n+1}$ this works, in short, as follows. Let $\bigwedge^2 M$ be the 2nd exterior power of a free module of rank $n+2$. If $M$ is spanned by $(\epsilon^{n+1}, \epsilon^n, \ldots, \epsilon^1, \epsilon^0)$, then $\bigwedge^2 M$ is freely generated by $\{\epsilon^i \wedge \epsilon^j \mid 0 \leq i < j \leq n+1\}$.

Let $D_1 : M \to M$ such that $D_1 \epsilon^i = \epsilon^{i-1}$ if $i > 1$ and $D_1 \epsilon^0 = 0$. Extend $D_1$ to an endomorphism of $\bigwedge^2 M$ by setting:

$$D_1(\epsilon^i \wedge \epsilon^j) = D_1 \epsilon^i \wedge \epsilon^j + \epsilon^i \wedge D_1 \epsilon^j,$$

and let $D_2 : \bigwedge^2 M \to \bigwedge^2 M$ such that:

$$D_2(\epsilon^i \wedge \epsilon^j) = D_1^2 \epsilon^i \wedge \epsilon^j + D_1 \epsilon^i \wedge D_1 \epsilon^j + \epsilon^i \wedge D_2 \epsilon^j.$$

In other words $D_1$ behaves as a “first derivative” and $D_2$ as a “second derivative”. The main result in [2] is that the endomorphism $D_1$ and $D_2$ generate a commutative subalgebra $\mathcal{A}^*$ of the $\mathbb{Z}$-algebra $\text{End}_\mathbb{Z}(\bigwedge^2 M)$ which is isomorphic to $H^*(G_1(\mathbb{P}^n), \mathbb{Z})$. The isomorphism is explicitly obtained from sending $\sigma_1 \mapsto D_1$ and $\sigma_2 \mapsto D_2$. From this point of view, it turns out that the degree of a top intersection product $\sigma_1^a \sigma_2^b$ ($a + 2b = 2n$) in $H^*(G_1(\mathbb{P}^{n+1}), \mathbb{Z})$ is nothing else than the coefficient $\kappa_{a,b}$ in the equality:

$$D_1^a D_2^b (\epsilon^{n+1} \wedge \epsilon^n) = \kappa_{a,b} \cdot \epsilon^1 \wedge \epsilon^0.$$

3 The result

In this section we will prove the main result of this paper: the connection between the numbers in the Catalan Traffic and top intersection numbers in the integral
cohomology ring of the Grassmannian $G_1(\mathbb{P}^3)$ of lines in $\mathbb{P}^3$. This connection is a consequence of the following:

### 3.1 Theorem

Let $K(m, n) := \kappa_{2m, n-m}$ be the coefficient of $\epsilon^1 \wedge \epsilon^0$ in the expansion of $D_1^{2m} D_2^{n-m} (\epsilon^{n+1} \wedge \epsilon^n)$. Thus, for all $0 \leq m \leq n$ the following recursion holds:

$$K(m, n) = K(m+1, n) - K(m, n-1).$$

**Proof.** Let $\Delta_{11}$ be the endomorphism of $M$ defined by:

$$\Delta_{11}(D)(\epsilon^i \wedge \epsilon^j) = D_1 \epsilon^i \wedge D_1 \epsilon^j = (D_1^2 - D_2)(\epsilon^i \wedge \epsilon^j).$$

(2)

Recalling that

$$D_1^{2m} D_2^{n-m}(\epsilon^{n+1} \wedge \epsilon^n) = K(m, n) \cdot \epsilon^1 \wedge \epsilon^0,$$

by definition of $K(m, n)$, one has

$$K(m, n) \cdot \epsilon^1 \wedge \epsilon^0 = D_1^{2m} D_2^{n-m}(\epsilon^{n+1} \wedge \epsilon^n) =$$

$$= D_1^{2m} D_2^{n-m-1}(D_1^2 - \Delta_{11})(\epsilon^{n+1} \wedge \epsilon^n) =$$

$$= D_1^{2m+2} D_2^{n-m-1}(\epsilon^{n+1} \wedge \epsilon^n) - D_1^{2m} D_2^{n-m-1}(\epsilon^n \wedge \epsilon^{n-1}).$$

(3)

Now, on the r.h.s of formula (3), the former summand is precisely $K(m+1, n) \epsilon^1 \wedge \epsilon^0$ while the latter, using (2), is equal to $D_1^{2m} D_2^{n-m-1}(\epsilon^n \wedge \epsilon^{n-1})$ which in turn equals $\kappa_{2m, n-m-1} \epsilon^1 \wedge \epsilon^0$. Hence, keeping in mind that $\kappa_{2m, n-m-1} = K(m, n-1)$, one has, using (3):

$$K(m, n) \cdot \epsilon^1 \wedge \epsilon^0 = (K(m+1, n) - K(m, n-1)) \cdot \epsilon^1 \wedge \epsilon^0.$$

As a conclusion $K(m+1, n) = K(m, n) + K(m, n-1).$  

### 3.2 Proposition

For all $n \geq 0$, $K(n, n) = C_n$.  

**Proof.** In fact one has:

$$D_1^{2n}(\epsilon^{n+1} \wedge \epsilon^n) = \sum_{i=0}^{2n} \binom{2n}{i} D_1^i \epsilon^{n+1-i} \wedge D_1^{2n-i} \epsilon^n$$

$$= \sum_{i=0}^{2n} \binom{2n}{i} \epsilon^{n+1-i} \wedge \epsilon^{n-i}.$$  

(4)
Since $n \leq i \leq n+1$, $i = n$ or $i = n+1$. Hence only the sum
\[
\binom{2n}{n} e^1 \wedge e^0 + \binom{2n}{n+1} e^0 \wedge e^1
\]
can survive in expression (4). Therefore:
\[
D^{2n} e^{n+1} \wedge e^n = \left[\binom{2n}{n} - \binom{2n}{n+1}\right] e^1 \wedge e^0
\]  
(5)

so that
\[
K(n,n) = \binom{2n}{n} - \binom{2n}{n+1} = \frac{(2n)!}{(n+1)!n!} = C_n.
\]

3.3 Corollary. For all $0 \leq m \leq n$, the number $\Upsilon(m,n)$ (see equation (1)) coincides with the number $\kappa_{2m,n-m}$ of lines in $\mathbb{P}^{n+1}$ incident $2m$ linear subspaces of codimension 2 and $n-m$ subspaces of codimension 3 in general position in $\mathbb{P}^{n+1}$.

Proof. This follows from Theorem 3.1, Proposition 3.2 and the remarks in Section 2.

3.4 Remark. A formula for the number $\kappa_{a,b}$, with $a + 2b = 2n$ is computed in [8]. For $a = 2m$ and $b = n - m$, it coincides with the following combinatorial expression:
\[
\kappa_{2m,n-m} = \sum_{i=0}^{n-m} \sum_{j=0}^{2m} \binom{2m}{j} \frac{(n-m)!(m+n-2j-3i+1)}{i!(n-j-2i+1)!(i+j-m)!},
\]
(6)

holding for each $n \geq m \geq 0$.

Further manipulations involving other properties of the operators $D_i$, studied in [8] and not mentioned here, lead to the following simplified form:
\[
\kappa_{2m,n-m} = \sum_{i=0}^{n-m} (-1)^i \binom{n-m}{i} C_{n-i},
\]
holding for each $n \geq m \geq 0$. 

6
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