**Dynamics of Cohomological Expanding Mappings I:**
First and Second Main Results

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**ABSTRACT**

Let $f : V \rightarrow V$ be a Cohomological Expanding Mapping\(^1\) of a smooth complex compact homogeneous manifold with $\text{dim}_{\mathbb{C}}(V) = k \geq 1$ and Kodaira Dimension $\leq 0$. We study the dynamics of such mapping from a probabilistic point of view, that is, we describe the asymptotic behavior of the orbit $O_f(x) = \{f^n(x), n \in \mathbb{N} \text{ or } \mathbb{Z}\}$ of a generic point. Using pluripotential methods, we construct a natural invariant canonical probability measure of maximum Cohomological Entropy $\mu_f$ such that $\chi_{\mu_{f}}^m(f^m)^* \Omega \rightarrow \mu_f$ as $m \rightarrow \infty$ for each smooth probability measure $\Omega$ on $V$. Then we study the main stochastic properties of $\mu_f$ and show that $\mu_f$ is a measure of equilibrium, smooth, ergodic, mixing, K-mixing, exponential-mixing and the unique measure with maximum Cohomological Entropy. We also conjectured that $\dim_H(\mu_f) = \Psi h_\chi(f)$ and $\dim_H(\text{Supp}T_{T^+}^{l}) \geq 2(k - l) + \frac{\log \chi}{\chi}$.

**Keywords** Complex Dynamics · Cohomological Expanding Mapping · Cohomological Degree · Cohomological Entropy · Cohomological Quotient ·

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\(^1\)cf Definition 1.6.
1 Introduction

Let $\mu$ be the equilibrium measure associated with an endomorphism $f$. If $\varphi$ is an observable, $(\varphi \circ f^n)_{n \geq 0}$ can be seen as a sequence of dependent random variables. As the measure is invariant, these variables are distributed in an identical way, that is, the Borel sets $\{\varphi \circ f^n < t\}$ have the same measure $\mu$ for any fixed constant $t$. We recall some general facts of ergodic theory and probability theory. We refer to \[47, 49\] for the general theory.

Consider a dynamic system associated with a map $g : X \to X$, measurable against a $\sigma$-algebra $\mathcal{F}$ on $X$. The direct image of a probability measure $\nu$ by $g$ is the measure of probability $g_* (\nu)$ defined by

$$g_* (\nu)(A) := \nu(g^{-1}(A))$$

for each measurable set $A$. Likewise, for any positive measurable function $\varphi$, we have

$$(g_* (\nu), \varphi) := \langle \nu, \varphi \circ g \rangle.$$

The measure $\nu$ is invariant if $g_* (\nu) = \nu$. When $X$ is a compact metric space and $g$ is continuous, the set $\mathcal{M}(g)$ of invariant probability measures is convex, compact and not empty: for any sequence of probability measures $\nu_n$, the limit values of

$$\frac{1}{N} \sum_{j=0}^{N-1} (g^n)_* (\nu_N)$$

are invariant probability measures.

A measurable set $A$ is totally invariant if $\nu(A \setminus g^{-1}(A)) = \nu(g^{-1}(A) \setminus A) = 0$. An invariant probability measure $\nu$ is ergodic if any totally invariant set is of measure $\nu$ zero or complete. It is easy to show that $\nu$ is ergodic if and only if $\varphi \circ g = \varphi$, for $\varphi \in L^1(\nu)$, then $\varphi$ is constant. Here, we can replace $L^1(\nu)$ by $L^p(\nu)$ with $1 \leq p \leq +\infty$. The ergodicity of $\nu$ is also equivalent to the fact that it is extremal on $\mathcal{M}(g)$. We remember Birkhoff’s ergodic theorem, which is the analogue of the law of large numbers for independent random variables \[49\].

**Theorem 1.1** (Birkhoff). Let $g : X \to X$ be a measurable map as above. Suppose that $\nu$ is an invariant ergodic probability measure. Let $\varphi$ be a function on $L^1(\nu)$. Then

$$\frac{1}{N} \sum_{n=0}^{N-1} \varphi(g^n(x)) \to \langle \nu, \varphi \rangle$$

almost everywhere in relation to $\nu$.

When $X$ is a compact metric space, we can apply Birkhoff’s theorem to continuous functions $\varphi$ and deduce that for $\nu$ almost all $x$

$$\frac{1}{N} \sum_{n=0}^{N-1} \delta_{g^n(x)} \to \nu,$$

where $\delta_x$ indicates the mass of Dirac at $x$. The sum

$$\text{St}_N(\varphi) := \sum_{n=0}^{N-1} \varphi \circ g^n$$

is called **Birkhoff sum**. Therefore, Birkhoff’s theorem describes the behavior of $\frac{1}{N} \text{St}_N(\varphi)$ for an observable $\varphi$.

A stronger notion than ergodicity is the notion of mixing. An invariant probability measure $\nu$ is mixing if for each measurable set $A, B$

$$\lim_{n \to \infty} \nu(g^{-n}(A) \cap B) = \nu(A) \nu(B).$$

Clearly, mixing implies ergodicity. It is not difficult to see that $\nu$ is mixing if, and only if, for any test functions $\varphi, \psi$ on $L^\infty(\nu)$ or on $L^2(\nu)$, we have

$$\lim_{n \to \infty} \langle \nu, (\varphi \circ g^n) \psi \rangle = \langle \nu, \varphi \rangle \langle \nu, \psi \rangle.$$

The Quantity

$$I_n(\varphi, \psi) := |\langle \nu, (\varphi \circ g^n) \psi \rangle - \langle \nu, \varphi \rangle \langle \nu, \psi \rangle|$$
is called the correlation on time \( n \) of \( \varphi \) and \( \psi \). Thus, mixing is equivalent to the convergence of \( I_n(\varphi, \psi) \) to 0. We say that \( \nu \) is \( K \)-mixing if for each \( \psi \in L^2(\nu) \)
\[
\sup_{\|\varphi\|_{L^2(\nu)} \leq 1} I_n(\varphi, \psi) \to 0.
\]

Note that \( K \)-mixing is equivalent to the fact that the \( \sigma \)-algebra \( \mathcal{F}_\infty := \cap q^{-n}(\mathcal{F}) \) contains only sets zero and complete measures. This is the strongest form of mixing for observables on \( L^2(\nu) \). However, it is of interest to obtain quantitative information about the mixing speed for more regular observables, such as smooth functions or H"older continuous.

Now consider an endomorphism \( f \) of degree \( d \geq 2 \) of \( \mathbb{P}^k \) as above and its equilibrium measure \( \mu \). We know that \( \mu \) is totally invariant: \( f^*(\mu) = d^k\mu \). If \( \varphi \) is a continuous function, so
\[
\langle \mu, \varphi \circ f \rangle = \langle d^{-k}f^*(\mu), \varphi \circ f \rangle = \langle \mu, d^{-k}f_*\varphi \rangle = \langle \mu, \varphi \rangle.
\]

We use the obvious fact that \( f_*\varphi = d^k\varphi \). Thus, \( \mu \) is invariant.

Mixing for measure \( \mu \) was proved in [45].

**Theorem 1.2.** Let \( f \) be an endomorphism of degree \( d \geq 2 \) of \( \mathbb{P}^k \). So its measure of Green \( \nu \) is \( K \)-mixing.

The equilibrium measure \( \mu \) satisfies remarkable stochastic properties that are quite difficult to obtain in the real dynamic systems scenario. Pluripotential methods replace the delicate estimates used in some real dynamic systems.

Consider a dynamic system \( g : (X, \mathcal{F}, \nu) \to (X, \mathcal{F}, \nu) \) as above, where \( \nu \) is an invariant probability measure. Therefore, \( g^* \) defines a linear operator of norm 1 on \( L^2(\nu) \). We say that \( g \) has the Jacobian limited if there is a constant \( \kappa > 0 \) such that \( \nu(g(A)) \leq \kappa \nu(A) \) for each \( A \in \mathcal{F} \).

When \( X \) is a complex manifold, it is necessarily orientable.

Let \( \mathbb{V} \) be a smooth complex compact homogeneous manifold with \( \text{dim}_\mathbb{C}(\mathbb{V}) = k \geq 1 \) and Kodaira dimension \( \leq 0 \) and \( f : \mathbb{V} \to \mathbb{V} \) be a dominant surjective meromorphic endomorphism, that is, whose Jacobian is not identically null in any local chart. Let \( \omega \) be a \((1, 1)\)-strictly positive Hermitian form on \( \mathbb{V} \). Let \( \ell \) be a prime number.

**Definition 1.3.** The \( \ell \)-th Cohomological Degree \( \chi_\ell(f) \) of \( f \) is defined as the spectral radius of the pullback action \( f^* \) in the cohomology group \( \ell \)-adic étale \( \mathbb{H}_\ell^\dagger(\mathbb{V}, \mathbb{Q}_\ell) \) independent of \( \ell \) by: (cf [3] [6] [1] [4] [5] for more details)
\[
\chi_\ell(f) = \rho(f^*|_{\mathbb{H}_\ell^\dagger(\mathbb{V}, \mathbb{Q}_\ell)})
\]

**Definition 1.4.** We define the \((l, n)\)-th Cohomological Quotient \( \xi_l^n(f) \) of \( f \) as follows:
\[
\xi_l^n(f) = \left[ \frac{\chi_{2l-1}(f)}{\chi_{2l}(f)} \right]^n
\]

**Definition 1.5.** The Cohomological Entropy of \( f \) is defined by
\[
h_x(f) = \max_i \log \chi_{2i}(f).
\]

**Definition 1.6.** We say that \( f \) is a Cohomological Expanding Mapping when \( f \) is dynamically compatible (that is \( (f^n)^* = (f^*)^n \)) and there is \( l \in \{1, \ldots, k\} \) such that :
\[
\xi_l^{-1}(f) > 1.
\]

We will write \( \chi_l \) for \( \chi_1(f) \) and \( \xi_l^n \) for \( \xi_l^n(f) \) if there is no confusion.

Let \( (M, \mathcal{F}, m) \) be a probability space and \( g : M \to M \) be a measurable map that preserves \( m \), that is, \( m \) is \( g_* \)-invariant: \( g_*m = m \). The measure \( m \) is ergodic if for any measurable set \( A \) such that \( g^{-1}(A) = A \), we have \( m(A) = 0 \) or \( m(A) = 1 \). This is equivalent to the property that \( m \) is extremal on the convex set of invariant probability measures (if \( m \) is mixing, so it is ergodic). When \( m \) is ergodic, Birkhoff’s theorem implies that if \( \psi \) is an observable on \( L^1(m) \) then
\[
\lim_{n \to \infty} \frac{1}{n} \left[ \psi(x) + \psi(g(x)) + \cdots + \psi(g^{n-1}(x)) \right] = \langle m, \psi \rangle
\]
for \( m \) - almost all \( x \).
Suppose now that \( \langle m, \psi \rangle = 0 \). Then, the previous limit is equal to 0. The theorem of limit central (TLC), when it occurs, provides the speed of this convergence. We say that \( \psi \) satisfies the TLC if there is a constant \( \sigma > 0 \) such that
\[
\frac{1}{\sqrt{n}} \left[ \psi(x) + \psi(g(x)) + \cdots + \psi(g^{n-1}(x)) \right]
\]
converges in distribution for the Gaussian random variable \( N(0, \sigma) \) of mean 0 and variance \( \sigma \). Remember that \( \psi \) is a coboundary whether there is a function \( \psi' \) on \( L^2(\mu) \) such that \( \psi = \psi' - \psi' \circ g \). In that case, it is easy to see that
\[
\lim_{n \to \infty} \frac{1}{\sqrt{n}} \left[ \psi(x) + \psi(g(x)) + \cdots + \psi(g^{n-1}(x)) \right] = \lim_{n \to \infty} \frac{1}{\sqrt{n}} \left[ \psi'(x) - \psi'(g^n(x)) \right] = 0
\]
in distribution. Therefore, \( \psi \) does not satisfy the TLC (sometimes it is said that \( \psi \) satisfies the TLC by \( \sigma = 0 \)).

The TLC can be deduced from strong mixing, see [11, 46, 48]. In the following result, \( ET(\psi|\mathcal{F}_n) \) indicates the expectation of \( \psi \) in relation to \( \mathcal{F}_n \), that is, \( \psi \mapsto ET(\psi|\mathcal{F}_n) \) is the orthogonal projection of \( L^2(m) \) in the subspace generated by the measurable functions \( \mathcal{F}_n \).

**Theorem 1.7 (Gordin).** Consider the decreasing sequence \( \mathcal{F}_n := g^{-n}(\mathcal{F}) \), \( n \geq 0 \), of algebras. Let \( \psi \) be a function with real value on \( L^2(m) \) such that \( \langle m, \psi \rangle = 0 \). Suppose that
\[
\sum_{n \geq 0} \| ET(\psi|\mathcal{F}_n) \|_{L^2(m)} < \infty.
\]
So, the positive number \( \sigma \) defined by
\[
\sigma^2 := \langle m, \psi^2 \rangle + 2 \sum_{n \geq 1} \langle m, \psi (\psi \circ g^n) \rangle
\]
is finite. It vanishes if and only if \( \psi \) is a coboundary. Furthermore, when \( \sigma \neq 0 \), then \( \psi \) satisfies the TLC with variance \( \sigma \).

Note that \( \sigma \) is equal to the limit of \( n^{-1/2} \| \psi + \cdots + \psi \circ g^{n-1} \|_{L^2(m)} \). The last expression is equal to \( \| \psi \|_{L^2(m)} \) if the family \( \{ \psi \circ g^n \}_{n \geq 0} \) is orthogonal on \( L^2(m) \).

We refer to [47][49] for the notion of Lyapunov exponent.

**Definition 1.8.** An invariant positive measure is hyperbolic if its Lyapunov exponents are non-zero.

A function quasi-p.s.h. on \( V \) is a function of \( V \) on \( [−\infty, \infty) \), which is locally the sum of a plurisubharmonic function and a smooth function. For a given \( (1, 1) \)-continuous form \( \eta \), denote by \( \text{PSH}_0(\eta) \) the set of quasi-p.s.h. functions \( \varphi \) such that \( dd^c \varphi + \eta \geq 0 \) and \( \sup_{V} \varphi = 0 \). Equip \( \text{PSH}_0(\eta) \) with induced distance of \( L^1(V) \) using natural inclusion \( \text{PSH}_0(\eta) \subset L^1(V) \).

Remember from [22] that a complex measure \( \mu \) on \( V \) is considered PC if each quasi-p.s.h. function is \( \mu \)-integrable and for each sequence \( \{ \varphi_n \}_{n \in \mathbb{N}} \) of quasi-p.s.h. functions converging to \( \varphi \) on \( L^1 \), so that \( dd^c \varphi_n + \eta \geq 0 \) for some smooth form \( \eta \) independent of \( n \), we have \( \langle \mu, \varphi_n \rangle \to \langle \mu, \varphi \rangle \).

A pluripolar set on \( V \) is a subset of \( V \) contained on \( \{ \varphi = -\infty \} \) for some quasi-p.s.h. function \( \varphi \). By [29], any locally pluripolar set on \( V \) is pluripolar. This result implies in particular that there are abundantly quasi-p.s.h. singular functions on \( V \). Note that every PC measure has no mass on pluripolar sets.

Next, we will consider the dynamics of \( f \) with \( \xi^{-1}_f(f) > 1 \).

**Here is the first Main Result.**

**Theorem 1.9.** Let \( V \) be a smooth compact complex homogeneous manifold with \( \text{dim} \mathbb{C}(V) = k \geq 1 \) and Kodaira dimension \( \leq 0 \) and \( f : V \to V \) a Cohomological Expanding Mapping. Let \( \nu \) be a complex measure with density \( L^{k+1} \) on \( V \) such that \( \nu(V) = 1 \). Let \( \omega \) be a \( (1, 1) \)-strictly positive Hermitian form on \( V \). So the sequence \( \frac{1}{\lambda^{k+1}} (f^m)^* \nu \) converges weakly to a measure of probability \( PC \mu_f \) with Cohomological Entropy \( \geq \log \lambda_{k+1} \) independent of \( m \) as \( m \to \infty \) so that \( \lambda_{k+1}^{-1} f^* \mu_f = \mu_f \) if \( f \) is holomorphic, then for each Hermitian metric \( \omega \) on \( V \), \( \mu_f \) is Hölder continuous on \( \text{PSH}_0(\omega) \).

The Hölder continuity of \( \mu_f \) on \( \text{PSH}_0(\omega) \) for \( f \) holomorphic implies that \( \mu_f \) is moderate in the sense that there are constants \( \varepsilon, M > 0 \) such that for each \( \varphi \in \text{PSH}_0(\omega) \), we have
\[
\int_V e^{-\varepsilon M} d\mu_f \leq M.
\]
We remember a new class of functions called weakly d.s.h. that replace the role of d.s.h functions (differences of two functions quasi-psh) in case of Kähler. These functions enjoy a compactness property similar to that of the d.s.h functions and the pull-backs of d.s.h functions by meromorphic maps are weakly d.s.h. We obtain the property of exponential mixing of $\mu_f$

Here is the second Main Result.

**Theorem 1.10.** Let $V, f, \chi_{2l}, \mu_f$ be as in Theorem 1.9. So $\mu_f$ is exponential mixing in the sense that for each constant $0 < \alpha \leq 1$, there is a constant $A_\alpha$ such that

$$|\langle \mu_f, (\psi \circ f^m)\varphi \rangle - \langle \mu_f, \psi \rangle | \leq A_\alpha \xi_l^{\frac{m}{2}} \|\psi\|_{C^\alpha}$$

for each $m \geq 0$, each $\psi \in L^\infty(V)$ and every function Hölder continuous $\varphi$ of order $\alpha$. In particular, $\mu_f$ is K-mixing.

If a real function Hölder continuous $\varphi$ is not a coboundary, i.e., there is not $\psi \in L^2(V)$ with $\varphi = \psi \circ f - \psi$, and satisfies $|\langle \mu, \varphi \rangle | = 0$, then $\mu_f$ satisfies the central limit theorem, which means that there is a constant $\sigma > 0$ such that for each interval $I \subset \mathbb{R}$, we have

$$\lim_{p \to \infty} \mu_f \left\{ \frac{1}{\sqrt{p}} \sum_{j=0}^{p-1} \varphi \circ f^j \in I \right\} = \frac{1}{\sqrt{2\pi}\sigma} \int_I e^{-x^2/(2\sigma^2)} dx.$$

The expression $|\langle \mu_f, (\psi \circ f^m)\varphi \rangle - \langle \mu_f, \psi \rangle |$ is called the Correlation of order $m$ between the observables $\varphi$ and $\psi$. The measure $\mu_f$ is said mixing if this correlation converges to 0, when $m$ tends to infinity, for smooth observables (or equivalently, observables continuous, limited or $L^2(\mu_f)$).

Remember that $f_* \varphi$ is defined by

$$f_* \varphi(x) := \sum_{y \in f^{-1}(x)} \varphi(y)$$

where the points on $f^{-1}(x)$ are counted with multiplicities (there are exactly $\chi_{2l}$ points). Also define the Perron-Frobenius Operator by

$$\Lambda \varphi := \chi_{2l}^{-1} f_* \varphi.$$

As $\mu_f$ is totally invariant, this is the adjoint operator of $f^*$ on $L^2(\mu_f)$.

### 2 First Main Result

In this section, we will prove the Theorem 1.9. For a current $T$ of order 0 defined in a manifold $V$, we denote by $\|T\|_V$ the mass of $T$ on $V$. Let’s write $\lesssim$ (resp. $\gtrsim$) for $\leq$ (resp. $\geq$) modulo a multiplicative constant independent of involving terms in inequality.

**Theorem 2.1 (Theorem 1.9 “First Main Result”).** Let $V$ be a smooth compact complex homogeneous manifold with dimension $\dim_{\mathbb{C}}(V) = k \geq 1$ and Kodaira dimension $\leq 0$ and $f : V \to V$ a Cohomological Expanding Mapping. Let $\nu$ be a complex measure with density $L^{2k+1}$ on $V$ such that $\nu(V) = 1$. Let $\omega$ be a $(1, 1)$-strictly positive Hermitian form on $V$. So the sequence $\frac{1}{\chi_{2l}} (f^m)^* \nu$ converges weakly to a measure of probability PC $\mu_f$ with Cohomological Entropy $\geq \log \chi_{2l}$ independent of $\nu$ as $m \to \infty$ so that

$$\chi_{2l}^{-1} f^* \mu_f = \mu_f = f_* \mu_f$$

and if $f$ is holomorphic, then for each Hermitian metric $\omega$ on $V$, $\mu_f$ is Hölder continuous $\mu_f$ is $C^\alpha$ on $\text{PSH}_0(\omega)$.

Let $B_r$ be the ball centered on 0 of radius $r$ of $\mathbb{C}^k$, where $r \in \mathbb{R}^+$. For $r := 1$ we put $B := B_1$. The following result is very important.

**Lemma 2.2. (Classical)** Let $r \in (0, 1)$. So, for each $(1, 1)$-closed real current $R$ of order 0 defined on $B$, there is a function $U_R$ on $L^{1+1/(2k)}(B_r)$ so that the following three properties are verified:

(i) $R = dd^c U_R$

(ii) $\|U_R\|_{L^{1+1/(2k)}(B_r)} \leq c \|R\|_B$

for some constant $c_r$ independent of $R$. 

Proof. The new point is the estimate for the norm \( L^{1+1/(2k)} \) of the potential \( U_R \) and its continuity on \( R \). These properties will be obtained by carefully examining the steps in the usual construction of \( U_R \), cf \[15\] p. 135 \[5\], \[3\], \[8\], \[7\] for example.

Let \( R \) be a \((1, 1)\)-current closed on \( B \). Let \( x \in \mathbb{C}^k \) be the canonical coordinate system. Let \( \rho \) be a smooth function supported compactly on \( B \) and \( \int_B \rho dx = 1 \). For \( y \in B \), let \( A_y : B \to B \) be the diffeomorphism defined by

\[
A_y(x) := x + \frac{1}{2}(1 - \|x\|)y
\]

for \( x \in B \). Since \( A_y \) is homotopic to \( A_0 := \text{id} \) through homotopy \( H_y : [0, 1] \times B \to B \) defined by \( H_y(t, x) := A_{ty}(x) \) for \( t \in [0, 1] \), the average

\[
R' := \int_B (A_y^*R)\rho(y)dy
\]

is a smooth closed form that is cohomologous to \( R \). Precisely, by the formula of homotopy, we have

\[
R - R' = dL_1, \quad \text{where} \quad L_1 = L_1(R) := \int_B (H_y)_*, ([0, 1] \otimes R)\rho(y)dy.
\]

Note that

\[
\|R'\|_{L^\infty(B)} \lesssim \|R\|_B, \quad \|L_1\|_B \lesssim \|R\|_B. \quad (2.1)
\]

Since \( R' \) is a smooth closed form on \( B \), we can use an explicit formula (cf \[15\] p. 13) to define a smooth form \( L_2 = L_2(R') \) on \( B \) such that

\[
R' = dL_2, \quad \|L_2\|_{L^\infty(B)} \lesssim \|R'\|_{L^\infty(B)}.
\]

This combined with (2.1) shows that for \( L_3 := L_1 + L_2 \), we have

\[
R = dL_3, \quad \|L_3\|_B \lesssim \|R\|_B \quad (2.2)
\]

and \( L_3 \) continuously depends on \( R \). So if \((R_n)_{n \in \mathbb{N}} \) is a sequence of \((1, 1)\)-currents of order \( 0 \) with uniformly limited mass, converging towards \( R \) so \( L_3(R_n) \) is also of uniformly limited mass and converges to \( L_3(R) \).

Since \( R \) is a \((1, 1)\)-real form, \( L_3 \) is a \(1\)-real form. We decompose \( L_3 \) in the sum of one \((1, 0)\)-form and a \((0, 1)\)-form as

\[
L_3 = L_3^{(1,0)} + L_3^{(0,1)} \quad (2.3)
\]

such that \( L_3^{(1,0)} = \overline{L_3^{(0,1)}} \) and \( L_3^{(0,1)} \) are currents of order \( 0 \). We deduce from (2.2) that

\[
\|L_3^{(0,1)}\|_B \lesssim \|R\|_B \quad (2.4)
\]

For a bidirectional reason and the fact that \( R = dL_3 \), we have \( \overline{\partial L_3^{(0,1)}} = 0 \). It is known that there is a distribution \( v \) defined in an open neighborhood of \( \overline{B}_r \) with \( \partial v = L_3^{(0,1)} \). We will briefly remember how to build such a \( v \) as a function of \( L_3^{(0,1)} \). The reference is \[15\] p. 28.

Let \( \rho \) be the function as above. We can assume \( \rho \equiv 1 \) on an open neighborhood of \( \overline{B}_r \). By the Koppelman formula, we have

\[
\rho L_3^{(0,1)}(x) = \partial \int_B K_1(x, y) \wedge \rho(y)L_3^{(0,1)}(y) + \int_B K_2(x, y) \wedge \partial \rho(y) \wedge L_3^{(0,1)}(y). \quad (2.5)
\]

We do not give explicit formulas here for \( K_1, K_2 \) but we emphasize only that \( K_1, K_2 \) are the products of \( \|x - y\|^{-2k+1} \) with smooth forms on \( \mathbb{C}^k \).
Denote by $I_1$, $I_2$ the first and second integrals, respectively, on the right side of (2.5). We have
\[ \partial I_1 + I_2 = \rho L_3^{(0,1)} \]
which is equal to $L_3^{(0,1)}$ on $B_r$.

By the type of singularity of $K_1$ and the fact that $L_3^{(0,1)}$ is of order 0, we see that $I_1$ is a form with coefficients in $L^{1+1/(2k)}(\mathcal{B})$ with
\[ \|I_1\|_{L^{1+1/(2k)}(\mathcal{B})} \lesssim \|L_3^{(0,1)}\|_B \lesssim \|R\|_B \tag{2.6} \]
by (2.4). On the other hand, as $\partial \rho \equiv 0$ on an open neighborhood of $\bar{\mathcal{B}}_r$, the current $I_2$ is smooth on $\bar{\mathcal{B}}_r$ for some $r' > r$. Following exactly the arguments in [15, p. 29], we get a smooth function $I_3$ on $\bar{\mathcal{B}}_{r'}$ for some $r' > r$ such that $I_2 = \partial I_3$ on $\bar{\mathcal{B}}_r$ and
\[ \|I_3\|_{L^\infty(\mathcal{B}_r)} \leq \|L_3^{(0,1)}\|_B \lesssim \|R\|_B \tag{2.7} \]
by (2.4) and $I_3 : R \mapsto I_3(R) \in L^\infty(\mathcal{B}_r)$ is continuous. So if $v := (I_1 + I_3)$ then
\[ L_3^{(0,1)} = \partial v \]
on $\mathcal{B}_r$. This together with (2.3) gives
\[ L_3 = \partial v + \partial \bar{v}. \]
We deduce from this and (2.2) that
\[ R = dL_3 = \partial \bar{\partial} (v - \bar{v}). \]
Consequently $U_R := 2 \pi \Im v$ satisfies $R = dd^c U_R$ (remember that $dd^c = (i/\pi) \partial \bar{\partial}$) and
\[ \|U_R\|_{L^{1+1/(2k)}(\mathcal{B}_r)} \lesssim \|I_1\|_{L^{1+1/(2k)}(\mathcal{B}_r)} + \|I_3\|_{L^{1+1/(2k)}(\mathcal{B}_r)} \lesssim \|R\|_B \]
by (2.6) and (2.7).

It remains to prove the property of continuity of $U_R$. We saw that $I_3, L_3$ are continuous on $R$. We just need to check this property to $I_1$. Let $(R_n)$ be the sequence as defined above. Let’s show that $I_1(R_n) \to I_1(R)$ on $L^{1+1/(2k)}(\mathcal{B})$. For the continuity property above of $L_3$, we have that $S_n := \rho L_3^{(0,1)}(R_n)$ is of uniformly limited mass and converges to $S := \rho L_3^{(0,1)}(R)$ when $n \to \infty$. Write
\[ K_1(x,y) = \|x - y\|^{-2k+1} K_1'(x,y), \]
where $K_1'(x,y)$ is a smooth form. For every small constant $\varepsilon > 0$, let
\[ K_{1,\varepsilon}(x,y) := \max\{\|x - y\|, \varepsilon\}^{-2k+1} K_1'(x,y) \]
which is a continuous form. Since $\varepsilon \to 0$, we have $K_{1,\varepsilon}(\cdot, y) \to K_1(\cdot, y)$ on $L^{1+1/(2k)}(\mathcal{B})$ uniformly on $y \in \mathcal{B}$. So when $n \to \infty$,
\[ \int_{\{y \in \mathcal{B}\}} (K_{1,\varepsilon}(x,y) - K_1(x,y)) \wedge (S_n(y) - S(y)) \to 0 \]
on $L^{1+1/(2k)}(\mathcal{B})$ because the mass of $S_n$ is uniformly limited. On the other hand,
\[ \int_{\{y \in \mathcal{B}\}} K_{1,\varepsilon}(x,y) \wedge (S_n(y) - S(y)) \]
converges uniformly to 0 as $\varepsilon$ is fixed because $K_{1,\varepsilon}$ is continuous. We deduce that $I_1(R_n) \to I_1(R)$ on $L^{1+1/(2k)}(\mathcal{B})$. This completes the proof.

**Definition 2.3.** Let $\mathcal{V}$ be a complex manifold. A function of $\mathcal{V}$ to $[-\infty, \infty)$ is said function quasi-p.s.h. if it can be written locally as the sum of a plurisubharmonic function (p.s.h.) and other smooth. For each (1,1)-continuous form $\eta$, a function quasi-p.s.h. $\varphi$ is $\eta$-p.s.h. if $dd^c \varphi + \eta \geq 0$. Through the partition of the one unit, each function quasi-p.s.h. is $\eta$-p.s.h. for some smooth form $\eta$. For a given form $\eta$, denote by $PSH(\eta)$ the set of functions quasi-p.s.h. $\varphi$ for which $dd^c \varphi + \eta \geq 0$.

**Definition 2.4.** A locally integrable function $\varphi$ on $\mathcal{V}$ is said weakly d.s.h. if $dd^c \varphi$ is a current of order 0 on $\mathcal{V}$. Let $W$ be the complex vector space of all functions weakly d.s.h. on $\mathcal{V}$.
Definition 2.5. Every function quasi-p.s.h is weakly d.s.h.. A subset of $\mathcal{V}$ is a pluripolar set if it is contained on $\{ \varphi = -\infty \}$ for some function quasi-p.s.h. $\varphi$. If $\mathcal{V}$ is compact, each locally pluripolar set is pluripolar by [29]. We use a specific case of this result: each analytic proper subset of a compact manifold $\mathcal{V}$ is pluripolar, cf Lemma 2.11 above.

Now consider that $\mathcal{V}$ is compact. Let $\mu_0$ be a smooth probability measure on $\mathcal{V}$. We use this measure to define norms $L^p$ on $\mathcal{V}$. For $\varphi \in \mathcal{W}$, put
\[
\| \varphi \|_{\mathcal{W}} := \left( \int_{\mathcal{V}} |\varphi| d\mu_0 \right) + \|dd^c \varphi\|_{\mathcal{W}},
\]
(2.8)
where $\| \cdot \|_{\mathcal{W}}$ is the mass of a current on $\mathcal{V}$. Let’s write from now $\| \cdot \|_{\mathcal{W}}$ instead of $\| \cdot \|_{\mathcal{V}}$ if there is no confusion.

The function $\| \cdot \|_{\mathcal{W}}$ is a norm on $\mathcal{W}$ because if $dd^c \varphi = 0$ then $\varphi$ must be a constant. The norm $\| \cdot \|_{\mathcal{W}}$ is similar to the norm of the space of functions d.s.h. in case of Kähler introduced by Dinh-Sibony [22]. However, we do not know whether these two norms are equivalent in this case.

We introduce the topology on $\mathcal{W}$ in the following way: we say that $\varphi_n \in \mathcal{W}$ converges to $\varphi \in \mathcal{W}$ when $n \to \infty$ if $\varphi_n \to \varphi$ as current and $\| \varphi_n \|_{\mathcal{W}}$ is uniformly limited.

We have the following compactness result.

Lemma 2.6. Let $\mathcal{V}$ be a compact complex manifold. There is a constant $c$ so that for each function weakly d.s.h. $\varphi$ on $\mathcal{V}$ with $\int_{\mathcal{V}} \varphi d\mu_0 = 0$, we have
\[
\|\varphi\|_{L^{1+1/(2k)}(\mathcal{V})} \leq c \|dd^c \varphi\|_{\mathcal{W}}.
\]
(2.9)
Furthermore, given a positive constant $A$, the set $W_0$ of functions weakly quasi-p.s.h. $\varphi$ with $\int_{\mathcal{V}} \varphi d\mu_0 = 0$ such that $\|dd^c \varphi\|_{\mathcal{W}} \leq A$ is compact on $L^{1+1/(2k)}(\mathcal{V})$.

A direct consequence of Lemma 2.6 is that if $\varphi_n \to \varphi$ on $\mathcal{W}$ then $\varphi_n \to \varphi$ on $L^{1+1/(2k)}$. In case of Kähler, a similar version of inequality (2.9) for functions d.s.h. with norm $L^p$ in place of norm $L^{1+1/(2k)}$ and $\| \cdot \|_{\mathcal{W}}$ in place of $\| \cdot \|_{\mathcal{V}}$ was proven on [22] using cohomological tools for functions d.s.h. His proof uses cohomological arguments that are not applicable to prove (2.9) for weakly functions quasi-p.s.h.

Proof. Consider a function weakly quasi-p.s.h. $\varphi$ with $\|dd^c \varphi\|_{\mathcal{W}} \leq A$. Let $(W_j)$ be an open (finite) cover of $\mathcal{V}$ where the $W_j$ are local charts of $\mathcal{V}$ biholomorph to the unit ball of $\mathbb{C}^k$. Since $\|dd^c \varphi\|_{\mathcal{W}} \leq A$, by Lemma 2.2 we have $\tau_j \in L^{1+1/(2k)}(W_j)$ for which $dd^c \tau_j = dd^c \varphi$ on $W_j$ and
\[
\|\tau_j\|_{L^{1+1/(2k)}(W_j)} \lesssim A.
\]
(2.10)
Therefore, $\varphi - \tau_j$ can be represented by a pluriharmonic function on $W_j$. For simplicity, we identified this function with $(\varphi - \tau_j)$. We deduce that $\varphi \in L^{1+1/(2k)}(\mathcal{V})$.

We now assume, on the contrary, that (2.9) is not valid, it means that there is a sequence of non-null functions weakly quasi-p.s.h. $\varphi_n$ with $\int_{\mathcal{V}} \varphi_n d\mu_0 = 0$ and
\[
\infty > \|\varphi_n\|_{L^{1+1/(2k)}(\mathcal{V})} \geq n\|dd^c \varphi_n\|_{\mathcal{W}}.
\]
Multiplying $\varphi_n$ by a positive constant, we can assume that
\[
\|\varphi_n\|_{L^{1+1/(2k)}(\mathcal{V})} = 1.
\]
(2.11)
So we have
\[
\|dd^c \varphi_n\|_{\mathcal{W}} \leq 1/n.
\]
(2.12)
Note that we still have $\int_{\mathcal{V}} \varphi_n d\mu_0 = 0$. Let $\tau^n_j$ be the function $\tau_j$ for $\varphi_n$ in place of $\varphi$. Put $T^n := dd^c \varphi_n$. These currents of order 0 are of uniformly limited mass and converge to 0 by (2.12). The Lemma 2.2 tells us that $\tau^n_j$ converges to 0 on $L^{1+1/(2k)}(W_j)$, for each $W_j \subset W_j$. We can also provide that $(W_j)$ continue to be a cover of $\mathcal{V}$. For simplicity, we can assume that $W_j = W_j$ for each $j$.

Now remember that $\varphi_n - \tau^n_j$ is pluriharmonic on $W_j$. The last function is of $L^{1+1/(2k)}$-norm limited on $W_j$ because of (2.10) and (2.11). The average equality for pluriharmonic functions implies that $(\varphi_n - \tau^n_j)$ is of $\mathcal{C}^0$-norm uniformly limited on compact subsets of $W_j$ on $n \in \mathbb{N}$ for each $l \in \mathbb{N}$. We deduce that, extracting a subsequence, we can assume that $\varphi_n - \tau^n_j$ converging uniformly to a pluriharmonic function $\tau^\infty_j$ on compact subsets of $W_j$ when $n \to \infty$. Since $\|\tau^n_j\|_{L^{1+1/(2k)}(W_j)} \to 0$, we get that
\[
\varphi_n \to \tau^\infty_j \text{ in } L^{1+1/(2k)}(W_j).
\]
To prove the second desired statement, we again use the function $\varphi_j$ above. We have that $\varphi - \tau_j$ is pluri-harmonic on $W_j$ and by (2.9), the $L^{1+1/(2k)}$-norm of $\varphi$ is also $\lesssim A$. Then the $L^{1+1/(2k)}$-norm of the pluri-harmonic function $(\varphi - \tau_j)$ is $\lesssim A$. It follows that its $C^0$-norm is also $\lesssim A$. Therefore, we can extract a convergent subsequence of $(\varphi - \tau_j)$ for $\varphi \in W$ on $C^0$. This combined with the $L^{1+1/(2k)}$ continuity of $\tau_j$ on $T$ implies the desired statement. This completes the proof.

We equip the vector space $B$ of Borel’s measurable functions on $V$ with the pointwise convergence topology: $h_n \to h$ if $h_n$ converges pointwise to $h$ at almost all points (with respect to the Lebesgue measure). Let $P$ be a continuous linear endomorphism of the last vector space. Define $W_P$ to be the set of $\varphi \in W$ for which $P\varphi \in W$.

**Lemma 2.7.** There is a constant $c$ such that

$$
\|P\varphi\|_{L^{1+1/(2k)}} \leq c(\|\varphi\|_W + \|dd^c(P\varphi)\|),
$$

(2.13)

for any $\varphi \in W_P$. In particular, there is a constant $c'$ such that

$$
\|P\varphi\|_{L^{1+1/(2k)}} \leq c'(\|dd^c\varphi\| + \|dd^c(P\varphi)\|)
$$

(2.14)

for each $\varphi \in W_P \cap W_0$. Furthermore, if $\varphi_n \in W_P \cap W_0 \to \varphi$ as currents when $n \to \infty$ such that

$$
(\|dd^c\varphi_n\| + \|dd^c(P\varphi_n)\|) \text{ are uniformly bounded, then } P\varphi_n \to P\varphi \text{ on } L^{1+1/(2k)}.
$$

**Proof.** The Inequality (2.14) is a direct consequence of (2.13) and of Lemma 2.6. Now suppose there is a sequence $(\varphi_n) \subset W_P$ for which

$$
\|P\varphi_n\|_{L^{1+1/(2k)}} = 1, \quad \|\varphi\|_W + \|dd^c(P\varphi)\| \leq 1/n.
$$

(2.15)

Applying compactness property in Lemma 2.6 for the sequence $(P\varphi_n)_{n\in\mathbb{N}}$, we see that by extracting a subsequence from $\varphi_n$ if necessary, the sequence $P\varphi_n$ converges on $L^{1+1/(2k)}$ for a function weakly $d.s.h \varphi'$. Consequently,

$$
\|\varphi'\|_{L^{1+1/(2k)}} = 1, \quad \|dd^c\varphi'\| = 0.
$$

(2.16)

Therefore $\varphi'_\infty$ is a constant. As the convergence on $L^1$ implies the convergence almost always of a subsequence, we can also assume that $P\varphi_n$ converges almost always to $\varphi'_\infty$.

On the other hand, the inequality of (2.15) allows us to use the compactness property in the Lemma 2.6 again for $(\varphi_n)$. Therefore, we can extract a subsequence of $(\varphi_n)$ converging to $\varphi'_{\infty} := 0$ on $L^{1+1/(2k)}$ and almost always. Thus $P\varphi_n$ converges almost always to $P\varphi_{\infty}$ because of the continuity of $P$. It follows that $\varphi'_{\infty} = P\varphi_{\infty} = 0$, note here $P(0) = 0$ by the linearity of $P$. This is a contradiction because of (2.16). Thus (2.13) follows. The last desired statement follows directly from the arguments above. This completes the proof.

Let $a \in \mathbb{C}^*$, $r$ be a constant on $(0, |a|]$ and $\delta > 0$ a constant. Assume that $P(1) = a$, where 1 is the constant function equal to 1 on $V$. Define $W_{P,r,\delta}^\infty$ to be the set of all $\varphi \in B$ such that $P^n\varphi \in W$ for each $n \geq 0$ and

$$
\|dd^c(P^n\varphi)\| \leq \delta r^n
$$

for each $n \geq 0$, here $P^n$ denotes the identity map. By the linearity of $P$, every constant function belongs to $W_{P,r,\delta}^\infty$. We equip $W_{P,r,\delta}^\infty$ with the topology induced from there on $W$. Note that $W_{P,r,\delta}^\infty$ is closed on $W$ and

$$
\left\{r^{-m}P^m(W_{P,r,\delta}^\infty) \subset W_{P,r,\delta}^\infty \right\}
$$

for every positive integer $m$. So $W_{P,r,\delta}^\infty \cap W_0$ is compact and $P^m(W_{P,r,\delta}^\infty)$ is contained in the complex vector subspace $W_{P,r,\delta}^\infty \cap W_0$ generated by $W_{P,r,\delta}^\infty$.

**Proposition 2.8.** There is a continuous linear functional function $\mu_P : \mathbb{W}_{P,r,\delta}^\infty \to \mathbb{C}$ such that for each complex measure $\nu$ with density $L^{2k+1}$ on $V$, $\nu(V) = 1$ and for each $\varphi \in \mathbb{W}_{P,r,\delta}^\infty$, we have

$$
\langle a^{-n}(P^n)_\ast \nu, \varphi \rangle \to \langle \mu_P, \varphi \rangle.
$$

(2.17)
Here for $Q : \mathcal{B} \to \mathbb{B}$, by definition, $\langle Q, \nu, \varphi \rangle := \langle \nu, Q\varphi \rangle$ for $\varphi \in \mathbb{B}$ such that $Q\varphi$ is $\nu$-integrable.

**Proof.** Remember that $\mu_0$ is a form of smooth probability volume on $\mathcal{V}$. We just need to construct $\mu_P$ on $\mathcal{W}_{\mathcal{F}, r, \delta}^\infty$ and prove (2.17) for $\varphi \in \mathcal{W}_{\mathcal{F}, r, \delta}^\infty$. The extension of $\mu_P$ to $\mathcal{W}_{\mathcal{F}, r, \delta}^\infty$ is done automatically using the linearity of $(P^n)_{n, \nu}$ and (2.17).

Let $\varphi \in \mathcal{W}_{\mathcal{F}, r, \delta}^\infty$. Put $b_0 := \int_X \varphi d\mu_0$ and $\varphi_0 := \varphi - b_0$. We define two sequences $\varphi_n, b_n$ as follows. Put

\[ b_n = b_n(\varphi) := \int_X (P\varphi_{n-1}) d\mu_0, \quad \varphi_n := P\varphi_{n-1} - b_n \]

for $n \geq 1$. We have $\epsilon^n\varphi_n \in \mathcal{W}_0 \cap \mathcal{W}_{\mathcal{F}, r, \delta}^{\mathcal{F}}$ and $d\varphi(P^m\varphi_n) = d\varphi(P^m\varphi_{n-1})$ for each $n, m$. By Lemma 2.7, we have

\[ ||\varphi_n||_{L^{1+1}/(2k)} \leq c(||d\varphi(P\varphi_{n-1})|| + ||d\varphi(P\varphi_{n-1})||), \quad |b_n| \leq c(||d\varphi(P\varphi_{n-1})|| + ||d\varphi(P\varphi_{n-1})||) \]

(2.18)

for some constant $c$ independent of $n, \varphi$. It follows that

\[ ||\varphi_n||_{L^{1+1}/(2k)} \leq c(||d\varphi(P\varphi)|| + ||d\varphi(P^{m-1}\varphi)||) \]

(2.19)

for $n \geq 1$. Since $P(1) = a$ we have $P(b_n) = ab_n$ for each $n$. Using this, it gives

\[ a^{-n}P^n\varphi = b_0 + a^{-n}P^n\varphi_0 = b_0 + a^{-n}P^{-1}(P\varphi_0) = b_0 + a^{-1}b_1 + a^{-n}P^{-1}\varphi_1 \]

(2.20)

\[ = \cdots = b_0 + a^{-1}b_1 + \cdots a^{-n}b_n + a^{-n}\varphi_n. \]

(2.21)

Put $b'_n = b'_n(\varphi) := b_0 + a^{-1}b_1 + \cdots a^{-n}b_n$ that converges to a number $b'_\infty$ (depending on $\varphi$) by (2.19) and the fact that $|a| > r$. We deduce from (2.20) that

\[ |a^{-n}P^n\varphi - b'_n| \leq |a|^{-n}||\varphi_n||. \]

(2.22)

This combined with the first inequality of (2.19) implies that $a^{-n}P\varphi$ converges to $b'_\infty$ on $L^{1+1}/(2k)$. Precisely, we have

\[ ||a^{-n}P^n\varphi - b'_n||_{L^{1+1}/(2k)} \lesssim \delta|a|^{-n}. \]

Since $\nu(X) = 1$, we have

\[ \langle a^{-n}(P^n)_{n, \nu}, \varphi \rangle = b'_n = \langle \nu, a^{-n}P^n\varphi - b'_n \rangle. \]

Using this, (2.22) and Hölder’s inequality imply that $\langle a^{-n}(P^n)_{n, \nu}, \varphi \rangle$ converges to $b'_\infty = b'_\infty(\varphi)$ because $\nu$ has $L^{2k+1}/(2k)$ density. Define $\langle \mu_P, \varphi \rangle := b'_\infty(\varphi)$ that is independent of $\nu$. Then, we obtain the desired convergence for $\mu_P$.

Consider a sequence $\tilde{\varphi}_m \to \varphi$ on $\mathcal{W}_{\mathcal{F}, r, \delta}^\infty$. Let $b_{nm}, \tilde{\varphi}_{nm}$ respectively the $b_n$ and $\varphi_n$ for $\tilde{\varphi}_m$ in place of $\varphi$. By the last statement of the Lemma 2.7, $b_{nm} \to b_n$ when $m \to \infty$ for each $n$ and (2.19) still applies to $b_{nm}, \tilde{\varphi}_{nm}$ in place of $b_n, \varphi_n$. We infer that $b'_{nm} \to b'_n$ and $a^{-n}\tilde{\varphi}_{nm} \to 0$ on $L^{1+1}/(2k)$ when $m \to \infty$. Thus, $\langle \mu_P, \tilde{\varphi}_m \rangle \to \langle \mu_P, \varphi \rangle$ when $m \to \infty$. In other words, $\mu_P$ is continuous. This completes the proof.

Let $\mathcal{V}$ be a complex compact manifold and $f$ be a meromorphic self-map on $\mathcal{V}$. Denote by $\Gamma$ the graph of $f$ on $\mathcal{V} \times \mathcal{V}$ and $\pi_1, \pi_2$ the restrictions to $\Gamma$ of natural projections of $\mathcal{V} \times \mathcal{V}$ for the first and second components respectively.

Let $\Phi$ be a form with measurable coefficients on $\mathcal{V}$. We say that $\Phi \in L^1$ if its coefficients are $L^1$ functions (in relation to the Lebesgue measure on $\mathcal{V}$). If $\Omega$ is a dense open subset of Zariski of $\mathcal{V}$ such that $\pi_2$ is a unrestricted cover on $\Omega$, the form $f_*\Phi := (\pi_2|_{\pi_2^{-1}(\Omega)})^*\pi_1^*\Phi$ is a measurable form on $\Omega$. Consequently $f_*\Phi$ is a measurable form on $\mathcal{V}$ independent of $\Omega$. We can verify that $f_* : \mathcal{B} \to \mathcal{B}$ is continuous. Consequently, $f_*$ is an example of the map $P$ considered above.

If $f_*\Phi \in L^1$, then we can define $f_*\Phi$ to be a current of order 0 induced by $f_*\Phi$ on $\mathcal{V}$. This definition is independent of the choice of $\Omega$. Note that the pull-back by $f$ of smooth functions or smooth forms is always on $L^1$. The following is similar to the results on $\mathcal{V}$.
Lemma 2.9. For each quasi-p.s.h. function \( \varphi \) on \( \mathbb{V} \), we have \( f_* \varphi \in L^1 \) and if \( dd^c \varphi + \eta \geq 0 \) for some \((1,1)\)-continuous form \( \eta > 0 \), then \( dd^c (f_* \varphi) + f_* \eta \geq 0 \). In particular:

\[
(f^n)_* \varphi \in \mathcal{W}_{f_*} \cap \mathcal{W}.
\] (2.23)

The inclusion (2.23) explains the crucial roles of \( \mathcal{W}_{f_*} \), \( \mathcal{W} \) in this study.

Proof. Let \( \sigma : \Gamma' \rightarrow \Gamma \) be a desingularization of \( \Gamma \). Let \( \Omega \) be as above. Put \( \pi'_j := \pi_j \circ \sigma \) for \( j = 1, 2 \). Since \( \varphi \) is quasi-p.s.h., \( \varphi \circ \pi'_1 \) is also. Thus, \( \varphi \circ \pi_1 = \sigma_*(\varphi \circ \pi'_1) \) is on \( L^1(\Gamma_f) \). Since

\[
\| f_* \varphi \|_{L^1(\Omega)} = \| (\pi_2)_*(\varphi \circ \pi_1) \|_{L^1(\Omega)} \lesssim \| \varphi \circ \pi_1 \|_{L^1(\Gamma)},
\]

we get the first desired statement.

By [2], [4] and the fact that \( \eta > 0 \), there is a decreasing sequence of smooth functions quasi-p.s.h \( \varphi_n \) converging pointwise to \( \varphi \) such that \( dd^c \varphi_n + \eta \geq 0 \) for each \( n \). By Lebesgue’s dominated convergence theorem, the sequence \( \varphi_n \circ \pi'_1 \) converges on \( L^1 \) to \( \varphi \circ \pi'_1 \). It follows that the sequence of positive smooth forms \( dd^c (\varphi_n \circ \pi'_1) + \pi'_1 \eta \) converges weakly to \( dd^c (\varphi \circ \pi'_1) + \pi'_1 \eta \). Thus, the last current is also positive. Now note that

\[
(\pi'_2)_*(dd^c (\varphi \circ \pi'_1) + \pi'_1 \eta) = dd^c (\pi'_2)_*(\varphi \circ \pi_1) + (\pi'_2)_* \pi'_1 \eta = dd^c ((\pi_2)_* \pi'_1 \varphi) + (\pi_2)_* \pi'_1 \eta
\]

because \( \pi'_1 \varphi \) and \( \pi'_1 \eta \) have no mass in zero Lebesgue measure sets. Therefore \( dd^c (f_* \varphi) + f_* \eta \geq 0 \).

Note that \( f_* \eta \) has finite mass on \( \mathbb{V} \). We infer that \( f_* \varphi \in \mathcal{W} \). In other words, \( \varphi \in \mathcal{W}_{f_*} \cap \mathcal{W} \). Applying this to \( f^n \) instead of \( f \) and using the formula that \( (f^n)_* \varphi = f_*(f^{n-1})_* \varphi \) as functions in some suitable open dense subset of \( \mathbb{V} \), we get (2.23). This completes the proof.

Lemma 2.10. Let \( \mathbb{V} \) be a compact complex manifold of dimension \( k \) and \( f : \mathbb{V} \rightarrow \mathbb{V} \) be a Cohomological Expanding Mapping. Let \( \varphi \) be a function quasi-p.s.h. on \( \mathbb{V} \) with \( dd^c \varphi + \eta \geq 0 \) for some \((1,1)\)-continuous form \( \eta \). So there is a constant \( A \) independent of \( \varphi, \eta \) for which

\[
\| dd^c (f^n)_* \varphi \| \leq A \chi_{2l-1}(\| \eta \|_{L^\infty})
\] (2.24)

for each \( n \geq 1 \).

Proof. Replacing \( \eta \) by a strictly positive smooth form that dominates it, we can assume that \( \eta > 0 \). Let \( \omega \) be a metric of Gauduchon on \( \mathbb{V} \), this means that \( \omega \) is a Hermitian metric and \( dd^c \omega^{k-1} = 0 \), cf [?]. Let \( \Gamma_n' \) be the graph of \( f^n \) and \( \pi_{1,n}, \pi_{2,n} \) the natural maps of \( \Gamma_n' \) for the first and second components of \( \mathbb{V} \times \mathbb{V} \). By Lemma 2.9, the current \( dd^c (f^n)_* \varphi + (f^n)_* \eta \) is positive. So, using \( dd^c \omega^{k-1} = 0 \) gives

\[
\| dd^c (f^n)_* \varphi + (f^n)_* \eta \| \lesssim \langle dd^c (f^n)_* \varphi + (f^n)_* \eta, \omega, \omega^{k-1} \rangle = \langle (f^n)_* \eta, \omega, \omega^{k-1} \rangle \lesssim \langle f^n \varphi, \omega, \omega^{k-1} \rangle
\]

This combined with the definition of \( \chi_{2l-1} \) gives

\[
\| dd^c (f^n)_* \varphi + (f^n)_* \eta \| \leq A (\chi_{2l-1})^n \| \eta \|_{L^\infty}.
\]

The desired inequality follows immediately. This completes the proof.

We come now to the end of the proof of the first main result.

End of Proof of Theorem 7.9. Let \( \xi_{-1} = 1. \) Put \( P := f_* \), \( a := \chi_{2l} \), \( r := \chi_{2l-1} \), \( \delta := A \), \( \mathbb{P} := \chi_{2l} \), \( \mathbb{P} := \chi_{2l-1} \), \( \delta := A \),

where \( A \) is the constant on Lemma 2.10. Let \( \varphi \) be a function quasi-p.s.h. with \( dd^c \varphi + \eta \geq 0 \) for some \((1,1)\)-continuous form \( \eta > 0 \) such that \( \| \eta \|_{L^\infty} \leq 1 \). We have \( \mathbb{P}(1) = a \) and \( \varphi \in \mathcal{W}_{f_r, \delta} \) by Lemma 2.10. Every function quasi-p.s.h. is on \( \mathcal{W}_{f_r, \delta} \). Since \( \nu \) does not have mass in proper analytical subsets of \( \mathbb{V} \), Note that

\[
\langle (f^n)_* \nu, \varphi \rangle = \langle \nu, (f^n)_* \varphi \rangle = \langle \nu, P^m \varphi \rangle
\] (2.25)
because we only need to consider integrals on a dense open subset of Zariski of \( \mathcal{V} \). Applying Proposition \ref{prop:2.8} for \( P \), we get a continuous functional \( \mu_P \) on \( \mathbb{W}_{P,r,r}^\infty \) such that

\[
\langle \chi_{2l}^{-1}(f^n)\nu, \varphi \rangle \to \langle \mu_P, \varphi \rangle,
\]

for each \( \varphi \in \mathbb{W}_{P,r,r}^\infty \). Choosing \( \nu \geq 0 \), we see that \( \langle \mu_P, \varphi \rangle \geq 0 \) if \( \varphi \geq 0 \). Let \( \mu_f \) be the probability measure on \( \mathcal{V} \) defined by \( \langle \mu_f, \varphi \rangle := \langle \mu_P, \varphi \rangle \) for each smooth function \( \varphi \). Remember here that smooth functions are quasi-p.s.h. on \( \mathcal{V} \). Let’s prove that \( \mu_f = \mu_P \) for each function quasi-p.s.h. \( \varphi \).

Consider a sequence of smooth functions quasi-p.s.h. \( \varphi_m' \) with \( \text{dd}^c \varphi_m' + \eta \geq 0 \) decreasing to \( \varphi \), we have

\[
\langle \mu_f, \varphi_m' \rangle = \langle \mu_P, \varphi_m' \rangle \text{ and } \langle \mu_f, \varphi_m' \rangle \to \langle \mu_f, \varphi \rangle \text{ by Lebesgue’s monotonous convergence theorem. This} \]

combined with the continuity of \( \mu_P \) gives (2.26) \( \langle \mu_f, \varphi \rangle = \langle \mu_P, \varphi \rangle \). So we have

\[
\lim_{m \to \infty} \langle \chi_{2l}^{-1}(f^n)^*\nu - \mu_f, \varphi \rangle = 0
\]

for each function quasi-p.s.h. \( \varphi \) on \( \mathcal{V} \).

As the functions quasi-p.s.h. are \( f^* \)-integrable, \( \mu_f \) has no mass on pluripolar sets. By Lemma \ref{lem:2.11} below, proper analytic subsets of \( \mathcal{V} \) are pluripolar. This implies that \( \mu_f \) has no mass on proper analytic subsets of \( \mathcal{V} \). We deduce that the pull-back \( f^*\mu_f \) is well defined. Here we just take the pull-back of \( \mu_f \) on an open subset of Zariski \( \Omega \) of \( \mathcal{V} \) where \( \pi_\Omega \) is a non-branched cover. It can be seen that this definition is independent of the choice of \( \Omega \) and if \((\Phi_m)_{m \in \mathbb{N}}\) is a sequence of positive measures without mass on the proper analytical subsets of \( \mathcal{V} \) and converging to \( \mu_f \), then \( f^*\Phi_m \) converges to \( f^*\mu_f \) because the mass of \( f^*\Phi_m \) converges to that of \( f^*\mu_f \), cf. example \ref{ex:2.8} Lema 3.6]. The Equality

\[
\chi_{2l}^{-1} f^* \mu_f = \mu_f
\]

is obtained by applying the pull-back \( f^* \) for convergence \( \chi_{2l}^{-1}(f^n)^*\nu \to \mu_f \), where \( \nu \) is a smooth measure of probability. Once we have \( f_*f^* = \chi_{2l} \) on Borel’s measurable functions, we get \( f_*\mu_f = \mu_f \), in other words, \( \mu_f \) is invariant by \( f \).

Let \( I_f \) be the indeterminacy set of \( f \). Put \( Z := \bigcup_{n \in \mathbb{N}} f^n(I_f) \). The measure \( \mu_f \) has no mass on \( Z \). The cohomological entropy of \( \mu_f \) is by definition \( 1_{\mathcal{V}} \vee 2f^*H_f \) in relation to \( f|_{\mathcal{V}\setminus Z} \). For Parry’s inequality \ref{parry} \ref{parry1}, using \( f^*\mu_f = \chi_{2l}\mu_f \), we deduce that the cohomological entropy of \( \mu_f \) is at least \( \log \chi_{2l} \).

Suppose now that \( f \) is holomorphic. To prove that \( \mu_f \) is Hölder continuous on \( \text{PSH}(\omega) \), we use a known idea of \ref{parry}. Without loss of generality, we can assume that \( ||\omega||_{L^1} \leq 1 \). Let \( \varphi, \psi \) be two functions quasi-p.s.h. on \( \text{PSH}(\omega) \). Remember that they are on \( \mathbb{W}_{P,r,r}^\infty \).

Let \( b_n(\varphi), b_n(\psi) \) be as in the proof of the proposition \ref{prop:2.8}. Let \( J_f \) be the Jacobian of \( f \). We have

\[
|f_*\varphi - f_*\psi|_{L^1} = \sup_{||h||_{L^1} \leq 1} \langle |(f_*\varphi - f_*\psi, h|\mu_0)\rangle = \sup_{||h||_{L^1} \leq 1} \langle |\varphi - \psi, (h \circ f)|\mu_0)\rangle
\]

what is

\[
\leq ||J_f||_{L^\infty} ||\varphi - \psi||_{L^1}.
\]

Applying the latest inequality to \( f^n \) in place of \( f \) gives

\[
|b_n(\varphi) - b_n(\psi)| \leq 2^n ||J_f||_{L^\infty} ||\varphi - \psi||_{L^1}.
\]

Put

\[
A_1 := \sum_{n=0}^{M+1} \chi_{2l}^{-n}[b_n(\varphi) - b_n(\psi)], \quad A_2 := \sum_{n=M+1}^{\infty} \chi_{2l}^{-n}[b_n(\varphi) - b_n(\psi)].
\]

Using \ref{prop:2.20} gives

\[
\langle \mu_f, \varphi - \psi \rangle = A_1 + A_2, \quad |A_1| \leq \sum_{n=0}^{M} \chi_{2l}^{-n}2^n ||J_f||_{L^\infty} ||\varphi - \psi||_{L^1}, \quad |A_2| \leq (\chi_{2l}^{-M})^M \chi_{2l}^{-M}.
\]

Consider the case where \( 2||J_f||_{L^\infty} \leq \chi_{2l} \). We have \( |A_1| \leq M||\varphi - \psi||_{L^1} \). Choosing \( M \) to be the smallest integer for which \( M \geq - \log ||\varphi - \psi||_{L^1}/\log \tau \), where \( \tau := \chi_{2l}/(\chi_{2l}^{-1}) \), we get that

\[
|\langle \mu_f, \varphi - \psi \rangle| \leq |A_1| + |A_2| \leq ||\varphi - \psi||_{L^1}^{-\varepsilon}
\]

which implies that \( \mu_f \) is Hölder continuous in that case. It remains to treat the case \( 2||J_f||_{L^\infty} \geq \chi_{2l} \). We have

\[
|A_1| \leq 2^M \chi_{2l}^{-M} ||J_f||_{L^\infty} ||\varphi - \psi||_{L^1} + \tau^{-M}.
\]

Choose \( M := - \log ||\varphi - \psi||_{L^1}/\log(2\chi_{2l}^{-1} ||J_f||_{L^\infty}) \). We see that

\[
|A_1| + |A_2| \leq - \log ||\varphi - \psi||_{L^1} \log ||\varphi - \psi||_{L^1} / \log(2\chi_{2l}^{-1} ||J_f||_{L^\infty}).
\]

Consequently, \( \mu_f \) is also Hölder continuous in this case. This completes the proof. □
Now we would like to say something about Theorem 1.10. If we try to imitate the arguments in the proof of [22] Theorem 1.3 to prove Theorem 1.10, we are led to estimate \(|\mu_f, | \phi_n|\). The measure \(\mu_f\) still satisfies the property that for each \(\omega\)-p.s.h. function \(\varphi\) with \(supp\, \varphi = 0\) is of \(L^1(\mu_f)\)-norm uniformly limited, cf. [22] Proposition 2.3. But unlike the case of Kähler, we don’t know if \(\varphi_p\) is the difference of two \(\omega\)-p.s.h. functions. So this explains why we cannot directly apply the approach in [22] to obtain a correlation decay for \(\mu_f\).

**Lemma 2.11.** Any proper analytic subset \(V\) of a complex compact manifold \(\mathcal{V}\) is a pluripolar set on \(\mathcal{V}\).

**Proof.** We use here the idea in [22] where the authors prove the same result when \(\mathcal{V}\) is Kähler. Suppose now that \(V\) is smooth and \(\text{codim} V \geq 2\) (otherwise the problem is trivial). Let \(\sigma : \hat{V} \to V\) be the explosion of \(V\) along \(V\). Denote by \(\hat{V}\) the exceptional hypersurface.

Let \(\omega\) be a positive-defined Hermitian form on \(\mathcal{V}\). Let \(\tilde{\omega}_h\) be a form of Chern of \(\mathcal{O}(−\hat{V})\) whose restriction to each fiber of \(\tilde{\omega} \approx \mathcal{P}(E)\) is strictly positive. Choosing \(\omega\) if necessary, we can assume that \(\tilde{\omega} := \sigma^* \omega + \tilde{\omega}_h > 0\). Since \(\sigma_\ast \tilde{\omega}_h = \sigma_\ast \omega - \omega\), the closed current \(\sigma_\ast \tilde{\omega}_h\) is quasi positive. Thus, there is a function quasi-p.s.h. \(\varphi\) on \(\mathcal{V}\) such that

\[
\sigma_\ast \tilde{\omega}_h = dd^c \varphi + \eta
\]

for some smooth closed form \(\eta\). Multiplying \(\tilde{\omega}_h\) by a strictly positive constant, we have \(\sigma^* \sigma_\ast \tilde{\omega}_h = \tilde{\omega}_h + [\hat{V}]\).

Thus \(|\varphi \circ \sigma(x) − \log \text{dist}(x, V)|\) is a limited function on \(\hat{V}\). As a consequence,

\[
|\varphi(x) − \log \text{dist}(x, V)| \leq 1
\]

on compact subsets of \(\mathcal{V}\). Consequently, \(V\) is contained in \(\{\varphi = −\infty\}\). Thus \(V\) is pluripolar in this case.

By the construction above, we can build a Hermitian metric in the explosion \(\hat{V}\) of \(\mathcal{V}\) along \(V\) as the sum of a pull-back of a Hermitian on \(\mathcal{V}\) and an appropriate form of Chern of \(\mathcal{O}(−\hat{V})\). Thus, if \(\sigma' : \hat{V}' \to V\) is a composition of explosions along smooth submanifolds, so there’s a form (1, 1) closed and smooth \(\eta'\) on \(\hat{V}'\) and a Hermitian metric \(\omega\) on \(\mathcal{V}\) such that \(\sigma' = \sigma' \omega + \eta'\) is a Hermitian metric on \(\hat{V}'\).

Now consider the general situation where \(\mathcal{V}\) is an analytical subset of \(\mathcal{V}\). As a finite union of pluripolar sets is again pluripolar, it is enough to prove that the regular part \(\text{Reg} V\) of \(\mathcal{V}\) is a pluripolar set because we can write \(\mathcal{V}\) as a finite union of the regular parts of suitable analytical subsets of \(\mathcal{V}\). By Hironaka’s desingularization, there is a composition \(\sigma' : \hat{V} \to \mathcal{V}\) of explosions along smooth submanifolds that do not cross \(\text{Reg} V\) (or their inverse images) so that the strict transformation \(\mathcal{V}\) of \(V\) is smooth.

Let \(\varphi', \omega, \eta\) be as above. For the above arguments, \(\hat{V}' \subset \{\varphi' = −\infty\}\) for some function quasi-p.s.h. \(\varphi'\) on \(\hat{V}'\) and \(dd^c \varphi' + \omega > 0\). Put \(S := \sigma'_\ast (dd^c \varphi' + \eta)\) which is a (1, 1) - current closed on \(\mathcal{V}\) and \(S + \omega \geq 0\). We can write

\[
S = dd^c \varphi_S + \eta_S, \quad \sigma_\ast \eta' = dd^c \psi + \eta
\]

for some smooth closed forms \(\eta_S, \eta\). We have

\[
dd^c \varphi_S + \eta_S + \omega \geq 0, \quad dd^c \psi + \eta + \omega \geq 0.
\]

Thus \(\varphi_S, \psi\) are quasi-p.s.h. functions on \(\mathcal{V}\). In addition, we also have \(\varphi_S = \sigma'_\ast (\varphi') + \psi\) with a smooth function on an open neighborhood of \(\text{Reg} V\) in which \(\sigma'\) is biholomorph. Consequently, \(\text{Reg} V \subset \{\varphi' = −\infty\}\).

This completes the proof.

\[\square\]

### 3 Second Main Result

In this section, we prove the Theorem 1.10. Our idea is to consider suitable test functions in the Sobolev space \(W^{1,2}\). This approach is inspired by [21].

Fix a smooth volume form \(\mu_0\) on \(\mathcal{V}\) and we use this form to define the norm in space \(L^2(\mathcal{V})\). Let \(W^{1,2}\) be the function space with real value \(\varphi \in L^2(\mathcal{V})\) such that \(d\varphi\) has \(L^2\) coefficients. Remember the following inequality of Poincaré-Sobolev: for \(\varphi \in W^{1,2}\) with \(\int_{\mathcal{V}} \varphi d\mu_0 = 0\), we have

\[
\|\varphi\|_{L^2} \leq c \|d\varphi\|_{L^2}, \quad (3.1)
\]

for some constant \(c\) independent of \(\varphi\), cf. for example [26] or [25]. Note that the term \(\|d\varphi\|_{L^2}^2\) is comparable to the mass of the positive current \(i\partial \varphi \wedge \partial \varphi\). We have the following lemma.
Lemma 3.1. (27 Pro. 3.1) Let $I$ be a compact subset of $\mathcal{V}$ $(2k-1)$-Hausdorff’s zero dimensional measure. Let $\varphi$ be a function with real value $L_{1,\infty}(\mathcal{V}\\setminus I)$. Suppose that the coefficients of $d\varphi$ are in $L^2(\mathcal{V}\setminus I)$. Then $\varphi \in W^{1,2}$ and there is a compact subset $M$ of $\mathcal{V}\setminus I$ and a constant $c > 0$ both independent of $\varphi$ such that
\[
\|\varphi\|_{L^2(M)} \leq c(\|\varphi\|_{L^2(M)} + \|d\varphi\|_{L^2(\mathcal{V})}).
\]

Definition 3.2. Let $W_{*,f}^{1,2}$ be the subset of $W^{1,2}$ consisting of $\varphi$ such that there are $m_1 \in \mathbb{N}$, a $(1,1)$-continuous form $\eta$ and a function $\eta$-p.s.h. $\psi$ satisfying
\[
i\partial \varphi \wedge \bar{\partial} \varphi \leq d\varphi ((f)^m_1, \psi) + (f)^m_1, \eta \]
(3.2)
as currents. A size representative of $\varphi$ is $m := (m_0, m_1)$, where $m_0$ is an upper limit of $\|\eta\|_{L^\infty}$.

If $\mathcal{V}$ is Kähler, $W_{*,f}^{1,2}$ coincides with the space $W_{*,f}^{1,2}$ considered in [21] that is independent of $f$. In this context, the space $W_{*,f}^{1,2}$ is studied in detail in [10] and used in [16] for the study of correspondences on Riemann surfaces with two equal dynamic degrees. Let $\xi_0^{-1}(f) > 1$. We have the following observation.

Lemma 3.3. Let $\varphi \in W_{*,f}^{1,2}$ and $m = (m_0, m_1)$ be a size representative of $\varphi$. So we have
\[
|d\varphi|_{L^2} \leq A m_0^{1/2} (\chi^2_{2l-1})^{m_1/2}
\]
for some constant $A$ independent of $\varphi$.

Proof. Let $\eta$ be as on (3.2). Let $\omega$ be a Hermitian metric on $\mathcal{V}$ with $d\omega^k = 0$. Testing $d\omega^k ((f)^m_1, \psi) + (f)^m_1, \eta$ with this form, we see that the norm of $d\omega^k ((f)^m_1, \psi) + (f)^m_1, \eta$ is equal to $\int_{\mathcal{V}} (f)^m_1, \eta \wedge \omega^k$ which is limited by $A m_0 (\chi^2_{2l-1})$ for some constant $A$ independent of $\eta, m_0, m_1$. The desired inequality follows. This completes the proof.

Lemma 3.4. For each $\varphi \in W_{*,f}^{1,2}$, if $m = (m_0, m_1)$ is a size representative of $\varphi$, then $m$ is also a size representative of $|\varphi|, \varphi^+ \text{ and } \varphi^-.$

We already know that the pushforward of a function quasi-p.s.h. by $f$ is a function weakly d.s.h. The following result, which explains the role of $W_{*,f}^{1,2}$ in this study, provides a more accurate description in the case of functions quasi-p.s.h. limited.

Lemma 3.5. Each function quasi-p.s.h limited is on $W_{*,f}^{1,2}$ and $f_*$ preserves $W_{*,f}^{1,2}$. In addition, for each $\varphi \in W_{*,f}^{1,2}$, if $m = (m_0, m_1)$ is a size representative of $\varphi$, then $m' := (d_k m_0, m_1 + 1)$ is a size representative of $f_* \varphi$ and
\[
|f_* \varphi|_{L^2} \leq c(\|\varphi\|_{L^2} + \|d(f_* \varphi)\|_{L^2})
\]
(3.3)
for some constant $c$ independent of $\varphi$.

Proof. Let $\varphi$ be a function quasi-p.s.h limited and $f : \mathcal{V} \to \mathcal{V}$ a dominant meromorphic map. Using the identity
\[
2i\partial \varphi \wedge \bar{\partial} \varphi = i\partial \bar{\partial} \varphi^2 - 2\varphi i\partial \bar{\partial} \varphi
\]
we see that there is a $(1,1)$-continuous form $\eta$ and a function $\eta$-p.s.h. $\psi$ for which $i\partial \varphi \wedge \bar{\partial} \varphi \leq d\varphi + \eta$.

Consequently $\varphi \in W_{*,f}^{1,2}$.

Now let $\varphi$ be an arbitrary element of $W_{*,f}^{1,2}$. Let $\eta$ and $\psi$ be such that (3.2) holds. Fix a dense open subset of Zariski $\Omega$ of $\mathcal{V}$ in which $f_* \varphi, (f)^m_1, \psi, (f)^m_1, \eta$ are well-defined functions or forms and $\pi_1$ is an unbranched cover on $f^{-1}(\Omega)$. We have $f_* \varphi \in L_{loc}^1(\Omega)$ and
\[
|f_* \varphi|_{L^1(K)} \leq c\|\varphi\|_{L^1},
\]
(3.4)
for any compact $K$ on $\Omega$ and some constant $c$ independent of $\varphi$. Note that $\mathcal{V}\setminus\Omega$ is a proper analytical subset of $\mathcal{V}$, thus is of Hausdorff $(2k-1)$-dimensional and zero measure. On $\Omega$, by Cauchy-Schwarz inequality, we have

$$i\partial(f_*\varphi)\wedge\bar{i}\partial(f_*\varphi)\leq \chi_2 f_* (i\partial\varphi\wedge\bar{i}\partial\varphi) \leq \chi_2 f_* [dd^c ((f^{m_1})\varphi) + (f^{m_1})\eta]$$

$$= \chi_2 [dd^c ((f^{m_1})\varphi) + (f^{m_1})\eta].$$

It follows that $d(f_*\varphi) \in L^2(\Omega)$. For this and by Lemma 3.3, we get $f_*\varphi \in W^{1,2}$. Thus, $i\partial(f_*\varphi)\wedge\bar{i}\partial(f_*\varphi)$ has no mass on $\mathcal{V}\setminus\Omega$. It follows that

$$i\partial(f_*\varphi)\wedge\bar{i}\partial(f_*\varphi) \leq \chi_2 1_{\Omega}[dd^c ((f^{m_1})\varphi) + (f^{m_1})\eta] \leq \chi_2 [dd^c ((f^{m_1})\varphi) + (f^{m_1})\eta]$$

because the last current is positive by Lemma 2.9. Combining this with (3.1) and (3.4) gives (3.3). The desired statement then follows. This completes the proof.

Let $\varphi \in W^{1,2}_f$ and $m_1 = (m_0, m_1)$ be a size representative of $\varphi$. Consider $f_*$ acting on Borel’s measurable functions. Remember that $f_*$ preserves the set of constant functions. As in the last section, let $b_0 := \int_{\mathcal{V}} \varphi d\mu_0$, and $\varphi_0 := \varphi - b_0$. We define two sequences $\varphi_n, b_n$ as follows. Put

$$b_n = b_n(\varphi) := \int_{\mathcal{V}} (f_*\varphi_{n-1}) d\mu_0, \quad \varphi_n := f_*\varphi_{n-1} - b_n$$

for $n \geq 1$. Note that $\varphi_n$ differs from $(f^n)\varphi$ by a constant. Lemma 3.5 implies that $m_n := (\chi_{2k} f_0, m_1 + n)$ is a size representative of $\varphi_n$. This together with Lemma 3.4 imply that

**Lemma 3.6.** $m_n := (\chi_{2k} f_0, m_1 + n)$ is also a size representative of $\varphi_n$.

By Lemma 3.3 we get

$$\|d\varphi_n\|_{L^2} \leq Am_0^{1/2} \chi_{2k}^{-n/2} (\chi_{2k} - 1)^{(n+m_1)/2}$$

Using (3.5), (3.1) and (3.3) give

$$\|\varphi_n\|_{L^2} \leq Am_0^{1/2} \chi_{2k}^{-n/2} (\chi_{2k} - 1)^{(n+m_1)/2}, \quad |b_n| \leq Am_0^{1/2} \chi_{2k}^{-n/2} (\chi_{2k} - 1)^{(n+m_1)/2}$$

for $n \geq 1$ and some possible different constant $A$. Now we are in a situation very similar to the one in the last section. Using arguments similar to those in the last section, we can show that $\lim_{n \to \infty} \langle \chi_{2k}^{-n}(f^n)^*\omega^k, \varphi \rangle$ exists and denote by $b_\infty(\varphi)$ its limit. In fact, we have

$$b_\infty = \sum_{j=0}^{\infty} \chi_{2k}^{-j} b_j.$$

It follows that

$$|b_\infty(\varphi)| \leq \|\varphi\|_{L^2} + Am_0^{1/2} (\chi_{2k}^{-1})^{m_1/2}$$

for some constant $A$ independent of $\varphi$. Clearly, if $\varphi$ is a function quasi-p.s.h limited, $b_\infty$ is equal to the same number defined in the last section. So we have

$$(\mu_f, \varphi) = b_\infty(\varphi)$$

for function quasi-p.s.h limited $\varphi$. Let $W^{1,2}_{f,*}$ the subset of $W^{1,2}_f$ consisting of functions that are continuous outside a closed pluripolar set. Note that $f_*$ preserves $W^{1,2}_{*,f}$ because $f$ is a covering outside an analytical subset of $\mathcal{V}$. We now affirm that

**Lemma 3.7.** For $\varphi \in W^{1,2}_{*,f}$, we have $\langle \mu_f, \varphi \rangle = b_\infty(\varphi)$.

**Proof.** The proof is similar to that on [21] Lemma 5.5. We proved first that $\varphi$ is $\mu_f$-integrable. We assume for a moment that $\varphi \geq 0$. Let $V$ be a closed pluripolar set such that $\varphi$ is continuous outside of $V$. Remember that $\mu_f$ has no mass on pluripolar sets, therefore, on $V$. Since $\chi_{2k}^{-n}(f^n)^*\omega^k$ converges to $\mu_f$ as positive measures and $\mathcal{V}\setminus\Omega$ is open, we have

$$\langle \mu_f, \varphi \rangle \leq \lim_{n \to \infty} \inf \chi_{2k}^{-n}(f^n)^*\omega^k, \varphi = \lim_{n \to \infty} \sum_{j=0}^{n} \chi_{2k}^{-j} b_j + \lim_{n \to \infty} \inf \omega^k, \chi_{2k}^{-n} \varphi_n$$
which is equal to $b'_\infty(\varphi)$. Thus $\varphi$ is $\mu_f$-integrable if $\varphi \geq 0$. In general, write $\varphi = \varphi^+ - \varphi^-$ and applying the last property, show that $\varphi$ is $\mu_f$-integrable. If $m = (m_0, m_1)$ is a size representative of $\varphi$, then we also get that

$$\langle |\mu_f, \varphi| \rangle \leq |b'_\infty(\varphi^+)| + |b'_\infty(\varphi^-)| \leq A(\|\varphi\|_1 + m_0^{1/2}(\chi_2^-1)^{m_1/2}), \quad (3.8)$$

for some constant $c$ independent of $\varphi$. Now using $f^*\mu_f = \chi_{2\mu_f}$ gives

$$\langle |\mu_f, \varphi| \rangle = \langle f^{-1}(\varphi)\rangle \leq |c_n| + \langle |\mu_f, \chi_{2\mu_f}^\varphi n| \rangle,$$

where $c_n := -\sum_{j \geq n+1} \chi_{2\mu_f}^j|b_j|$. Note that the first term on the right side of the last inequality tends to 0 because of (3.6). On the other hand, by (3.8) and Lemma 3.6, the second term is limited by

$$\chi_{2\mu_f}^{-n}(\|\varphi_n\|_1 + m_0^{1/2} (\chi_2^-1)^{(m_1+n)/2})$$

which tends to 0 when $n \to \infty$. This produces the desired equality. This completes the proof.

\[\rule{1cm}{1cm}\]

**Theorem 3.8.** Let $\mathcal{V}, f, \chi_{2\mu_f} \in L^1$ be as above with $\xi_{\mu_f}^{-1}(f) > 1$. So there is a constant $A > 0$ such that

$$I_n(\psi, \varphi) := \langle |\mu_f, (\psi \circ f^n)\varphi| - \langle |\mu_f, \psi| \rangle \rangle \leq A\|\psi\|_1 A_n(\varphi),$$

where

$$A_n(\varphi) := [\|\varphi\|_1 + m_0^{1/2} (\chi_2^-1)^{m_1/2}] \chi_{2\mu_f}^{n/2}(\chi_2^-1)^{n/2},$$

for each $\psi \in L^\infty(\mu_f)$, $\varphi \in W^{1,2}$ and $(m_0, m_1)$ a size representative of $\varphi$.

Note that if $\varphi$ is a function $\eta$-p.s.h. limited for some $(1, 1)$-continuous form $\eta$ of $L^\infty$-norm $\leq 1$, then there is a constant $\tilde{m}_0$ independent of $\varphi$ such that $(\tilde{m}_0, 1)$ is a size representative of $\varphi$. Therefore, the above theorem gives a uniform correlation decay for each $\varphi$.

**Proof.** Let the annotations be as above, $I_n(\psi, \varphi + c) = I_n(\psi, \varphi)$ for each constant $c$ because of the invariance of $\mu_f$. We can assume that $\langle |\mu_f, \varphi| \rangle = 0$. By Lemma 3.7, we get $b'_\infty(\varphi) = 0$. Consequently, $\chi_{2\mu_f}^{-n}(f^n)\varphi = c_n + \chi_{2\mu_f}^{-n}\varphi_n$. Using $f^*\mu_f = \chi_{2\mu_f}$ gives

$$I_n(\psi, \varphi) = \chi_{2\mu_f}^{n}(|\mu_f, \psi(f^n)\varphi|) = \langle |\mu_f, \psi(c_n + \chi_{2\mu_f}^{-n}\varphi_n)| \rangle \leq |c_n| + \chi_{2\mu_f}^{n}(|\mu_f, |\varphi_n||) \rangle.$$  

Note that, as before, we have

$$|c_n| \leq A A_n(\varphi)$$

for some constant $A$ independent of $\varphi$. On the other hand, $f^*$ preserves $W^{1,2}$, thus $\varphi_n \in W^{1,2}$ and so is $|\varphi_n|$. By Lemma 3.6, $(\chi_{2\mu_f}^{m_0}, m_1 + n)$ is a size representative of $|\varphi_n|$ if $(m_0, m_1)$ is a size representative of $\varphi$. Arguing as in the proof of Lemma 3.7 gives that

$$\chi_{2\mu_f}^{-n}(|\mu_f, |\varphi_n|)| \leq A A_n(\varphi)$$

for some constant $A$ independent of $\varphi$. Hence the desired inequality follows. This completes the proof.

\[\rule{1cm}{1cm}\]

**End of Proof of Theorem 3.8.** The central limit theorem for $\mu_f$ is a direct consequence of its correlation decay as shown in (2.1). Therefore, it remains to prove the property of the correlation decay. By Theorem 3.8 for each $C^1$ function $\varphi$ on $\mathcal{V}$, we have

$$I(\psi, \varphi) \leq A\|\psi\|_1 \|\varphi\|_1 \chi_{2\mu_f}^{-n/2}(\chi_2^-1)^{n/2}.$$  

This combined with the interpolation inequality for functional in Banach spaces $C^1, C^0$ provides the desired correlation decay for $\mu_f$, cf (2.1).

Remember that $\mu_f$ is $K$-mixing if for each $\varphi \in L^2(\mu_f)$, we have

$$\sup_{\psi \in L^2(\mu_f)} I_n(\psi, \varphi) \to 0.$$  

Note that the operator $\chi_{2\mu_f}^{-1}f^*$ can be extended to be a continuous linear operator on $L^2(\mu_f)$ because $|f^*\psi|^2 \leq \chi_{2\mu_f}f^*(|\varphi|^2)$. As above, to prove (3.10), we can assume that $\langle |\mu_f, \varphi| \rangle = 0$. Using (3.9), we have

$$I(\psi, \varphi) \leq \|\chi_{2\mu_f}^{-n}f^*\varphi\|_{L^2(\mu_f)}.$$  

Consider now $\varphi$ to be a limited function on $W^{1,2}$. The set of these functions is dense on $L^2(\mu_f)$. We have

$$\|\chi_{2\mu_f}^{-n}f^*\varphi\|_{L^2(\mu_f)} \leq \|\varphi\|_1 \|\chi_{2\mu_f}^{-n}f^*\varphi\|_{L^1(\mu_f)}$$

that tends to 0 by proof of theorem 3.8. This combined with (3.11) gives (3.10). The proof is completed.

\[\rule{1cm}{1cm}\]
Remark 1. By inequality (3.6), we see that for each complex measure $\nu$ with density $L^2$ and $\nu(X) = 1$, $\chi_{\nu}^{2l}(f^n)^*\nu$ converges weakly to $\mu_f$.

4 Conjectures

Here is the First Conjecture.

Conjecture 4.1. Let $\mathcal{V}, f, \mu_f$ as in Theorem 1.9. Let $\psi_1, \ldots, \psi_k$ be the Lyapunov exponents of $\mu_f$ and $\Psi = \sum \frac{1}{\psi_i}$ its inverse sum. So the Hausdorff dimension of $\mu_f$ satisfies

$$\dim_H(\mu_f) = \Psi \chi_f.$$ 

Here is the Second Conjecture.

Conjecture 4.2. Let $\mathcal{V}, f, \mu_f$ be as in Theorem 1.9. So there are $T^+_l$ and $T^-_{k-l}$ such that $\mu_f$ is defined by:

$$\mu_f := T^+_l \wedge T^-_{k-l},$$

where $T^+_l$ is a positive invariant closed current of bidegree $(l, l)$, i.e.

$$\frac{1}{\chi_{2l}} (f^m)^*\omega_l \longrightarrow T^+_l$$

and $T^-_{k-l}$ designates a positive invariant closed current of $(k - l, k - l)$, i.e.

$$\frac{1}{\chi_{2(k-l)}} (f^m)^*\omega_{k-l} \longrightarrow T^-_{k-l}.$$

Here is the Third Conjecture.

Conjecture 4.3. Let $\mathcal{V}, f, \mu_f$ be as in Theorem 1.9 and $T^+_l$ as in Conjecture 4.2. Let $\psi_1, \ldots, \psi_k$ be the Lyapunov exponents of $\mu_f$ with $\psi_l = \max_{1 \leq i \leq k} \psi_i$. So the Hausdorff dimension of the Support of $T^+_l$ satisfies

$$\dim_H(\text{Supp}T^+_l) \geq 2(k - l) + \log \frac{\chi_{2l}}{\psi_l}.$$

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