Generalization of Kimberling’s Concept of Triangle Center for Other Polygons

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Abstract. C. Kimberling defined the concept of triangle center function in order to describe centers of triangles as points associated to some functions depending on the sidelengths, instead of in terms of geometrical properties. In this article we provide two definitions of \(n\)-gon center function for \(n \geq 3\): one of them in terms of the coordinates of the vertices and the other one by means of the lengths of the sides and the diagonals. Both of them are natural ways to generalize the concept of triangle center function, and we prove that they are equivalent. Moreover, we use \(n\)-gon center functions to associate to each polygon a point in the plane, that we call center. We also explore the problem of characterization of families of polygons in terms of these \(n\)-gon center functions and we study the relation between our new definitions and other approaches arising from Applied Mathematics.

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1. Introduction
Kimberling [8,9] in the second half of the twentieth century decided to give a unified and formal definition for triangle center, including the classical centers (incenter, barycenter, circumcenter and orthocenter) and many others (Steiner Point, Fermat point, \ldots). Moreover, he created an encyclopedia [10] trying to contain all known triangle centers. His new idea was to consider triangle centers as points defined in terms of coordinates which are functions of the sidelengths, instead of loci.
Following this spirit, we provide two different definitions of \( n \)-gon center function, for \( n \geq 3 \). For each polygon, we also give a geometric interpretation of this construction as a point in the plane. We call this point the \( \text{center} \) of the polygon associated to the corresponding \( \text{center function} \). The main obstacle for generalizing the definition of \( \text{center function} \) is that \( n \)-gons are not determined by their sidelegths for \( n \geq 4 \). So first, we define those \( \text{center functions} \) as functions of the coordinates of the vertices (Definition 5). Next, we provide an alternative definition of \( \text{center function} \) involving, not only sidelegths, but also the lengths of the diagonals (Definition 9), which is closer to the original definition by C. Kimberling. Thus, \( 3 \)-gon \( \text{center functions} \), in the sense of Definition 9, are \( \text{triangle center functions} \). Moreover, this second approach seems to be fruitful to encompass some of the well known examples of polygon centers. Anyway, we proof that both definitions are equivalent.

We can find some works in the literature studying “centers” of polygons (see for example [1,7,13]). In particular, it is remarkable the effort done by C. van Tienhoven in [16]. In his web page a collection of a lot of centers of polygons can be found. But as far as we know, there is no previous formal definition for the concept of “polygon center”, which is key to develop a theoretical study of the subject.

In [1] the authors already studied the problem of exploring “the degree of regularity implied by the coincidence of two or more” centers for quadrilaterals. In that article only four centers are considered, namely, the centroid of the four vertices, the centroid of the four sides, the centroid of the whole figure considered as a lamina of uniform density and the Fermat–Torricelli center. This problem is related to the fact that, for squares, those four centers coincide. In our new general setting all the \( \text{centers} \) of polygons also coincide with the centroid of the vertices for regular polygons (Proposition 8). In relation to this, we study the problem of \( \text{characterization} \) of equiangular, equilateral and regular polygons by means of one or more centers (Theorems 16, 17 and Corollary 18).

We also explore other possibilities for defining \( \text{center} \) of a polygon (Definitions 19, 20), that would include some interesting examples arising from other areas of Mathematics. We also briefly study their relation to our concept of \( \text{center of a polygon in terms of \( n \)-gon center functions} \).

This article is structured as follows. In Sect. 2 we review the main aspects about Kimberling’s definition of triangle center function. Our first definition of \( n \)-gon center function and its geometric interpretation appear in Sect. 3, together with a discussion of its suitability: we justify that it satisfies some properties that we would expect for a point to be called \( \text{center} \). In Sect. 4 we provide the second definition of \( n \)-gon center function, in tems of sidelegths and lengths of diagonals, and we prove the equivalence between both definitions. Section 5 includes some examples of classic polygon centers that are subsumed in our new definitions of \( \text{center} \). The problem of characterization of
equiangular, equilateral and regular polygons by means of one or more centers is studied in Sect. 6. The other approaches for a definition of center are discussed in Sect. 7. Finally in Sect. 8 we briefly comment on some topics related to this concept of polygon center and propose some future worklines (in relation to equivariant maps, computational geometry and others). We also include some Open Questions to close up this work.

2. Basics About Triangle Centers

As we explained in the introduction, C. Kimberling in his works [8–10] decided to define triangle centers by means of functions (the so-called triangle center functions) instead of loci in the plane.

To be coherent with the notation used later in this article, let us denote by $P_3$ the set of all triangles with the vertices labelled 1, 2, 3 in $\mathbb{R}^3$. We use the identification $P_3 \approx (\mathbb{R}^2)^3$. Unlike what happens for $n$-gons for $n \geq 4$, triangles are determined by their sidelengths:

**Remark 1.** Every triangle can be identified with the tuple of its three sidelengths $(a, b, c)$ (placed in clockwise order), up to congruences.

The set of all possible sidelengths of triangles is:

$$T_3 = \{(a, b, c) \in \mathbb{R}^3_+: a + b > c, \ a + c > b, \ b + c > a\}.$$  

Before proceeding with the definition, let us say that we denote by $[a : b : c]$ the points in the real projective plane $P_2\mathbb{R}$ with the usual convention $[a : b : c] = [a' : b' : c'] \iff \frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$. Then we have:

**Definition 2.** (Kimberling’s definition of triangle center [8–10]) A real-valued function $g$ of three real variables $a, b, c$ is a triangle center function if it satisfies the following properties:

(i) Homogeneity: there exists some constant $k \in \mathbb{N}$ such that for all $t \in \mathbb{R}_{\geq 0}$ we have $g(ta, tb, tc) = t^k g(a, b, c)$.

(ii) Bisymmetry in the second and third variables: for all $(a, b, c) \in T_3$, we have $g(a, b, c) = g(a, c, b)$.

Define also the coordinate map $\varphi_g : \mathcal{D}_3 \subset T_3 \to P_2\mathbb{R}$ given by:

$$\varphi_g(a, b, c) = [g(a, b, c) : g(b, c, a) : g(c, a, b)].$$

The set $\mathcal{D}_3$ is the subset of elements in $T_3$ such that $g(a, b, c) + g(b, c, a) + g(c, a, b) \neq 0$.

The motivation of this abstract definition is the geometric interpretation of the center function:
Remark 3. In the original works concerning triangle centers [8–10], \( \varphi_g(a, b, c) \) is geometrically interpreted as the set of trilinear coordinates of a point in the plane called center. In our generalization, this geometric interpretation is different, since we consider the coordinate map as a set of barycentric coordinates with respect to the vertices. The reason is explained at the beginning of Sect. 3. We then have a map:

\[
\Phi_g : D_3 \subset P_3 \rightarrow \mathbb{R}^2
\]

that associates to each triangle its center.

There is a correspondence between trilinear coordinates and barycentric coordinates in a triangle given by the following relation: if \([t_1 : t_2 : t_3]\) are the trilinear coordinates of a point \(P\) (with respect to the sides of a triangle), then \([a t_1 : b t_2 : c t_3]\) are the barycentric coordinates of this point (with respect to the vertices of this triangle). So, for any coordinate map \(\varphi_g(a, b, c) = [t_1 : t_2 : t_3]\) there exists another triangle center function \(g_2(a, b, c) = a g(a, b, c)\) whose coordinate map is \(\varphi_{g_2}(a, b, c) = [a t_1 : b t_2 : c t_3]\). In this sense, both geometric interpretations are equivalent.

Example 4. (circumcenter, see [10]) Consider the triangle center functions \(g_1(a, b, c) = a(b^2 + c^2 - a^2)\) and \(g_2(a, b, c) = a^2(b^2 + c^2 - a^2)\). The circumcenter is the point whose trilinear coordinates are:

\[
\varphi_{g_1} = [a(b^2 + c^2 - a^2) : b(c^2 + a^2 - b^2) : c(a^2 + b^2 - c^2)],
\]

and whose barycentric coordinates are:

\[
\varphi_{g_2} = [a^2(b^2 + c^2 - a^2) : b^2(c^2 + a^2 - b^2) : c^2(a^2 + b^2 - c^2)].
\]

We see that if we have an element \((a_1, a_2, a_3) \in T_3\) and a permutation \(\sigma\) of the set \(\{1, 2, 3\}\), then if:

\[
\varphi_g(a_1, a_2, a_3) = [t_1 : t_2 : t_3] = [g(a_1, a_2, a_3) : g(a_2, a_3, a_1) : g(a_3, a_1, a_2)],
\]

we have that \(\varphi_g(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}) = [t_{\sigma(1)} : t_{\sigma(2)} : t_{\sigma(3)}]\).

In this setting, Kimberling’s definition of triangle center function ensures that:

Property 1. (the coordinate map is well defined) The coordinate map (if defined) of a triangle center function is associated to the triangle independently of the labelling of the sides (but obviously re-ordered).

Property 2. (the definition is coherent with respect to similarities) Let \(g\) be a triangle center function. Let \(T_1, T_2\) be two similar triangles in \(P_3\), with side-lengths \((a, b, c)\) and \((a', b', c')\) and such that \(S\) is the similarity \(T_1 = S(T_2)\). Then \(\Phi_g(a, b, c) = S(\Phi_g(a', b', c'))\).
3. First Definition of Polygon Center Functions

First we will fix the notation. For us, a $n$-gon consists of a set of points $V_1, V_2, \ldots, V_n$ in the plane called vertices and a set of $n$ segments called edges or sides, each of them connecting two adjacent vertices $V_i, V_{i+1 \mod n}$, for $i = 1, \ldots, n$. We also call triangles, quadrilaterals, pentagons, … to the corresponding 3-gons, 4-gons, 5-gons, … A segment joining two vertices that are not adjacent (with respect to the labelling) is called diagonal.

For a fixed $n$, we denote the set of all $n$-gons with vertices labelled $\{1, \ldots, n\}$ by $P_n \approx (\mathbb{R}^2)^n$. Sometimes we will restrict ourselves to subsets of $P_n$. We say that the polygon is non-degenerate if none of the vertices coincide and that it is simple if two sides only intersect when they are adjacent and they do in a vertex.

Consider the dihedral group $D_n = \langle \rho, \sigma : \rho^n = id, \sigma^2 = id, \sigma \rho \sigma = \rho^{-1} \rangle$ (it has $2n$ elements). It can be viewed as a subset of the permutation group of the set $\{1, \ldots, n\}$, determined by

$$\rho(i) = i + 1 \mod n, \quad \sigma(i) = 2 + n - i \mod n. \quad (1)$$

But it can also be viewed as an action in $P_n$, interpreted as a vertex relabelling in the set of all $n$-gons:

$$\forall \alpha \in D_n, \quad \alpha(V_1, \ldots, V_n) = (V_{\alpha(1)}, \ldots, V_{\alpha(n)}).$$

We want to define first the $n$-gon center function as a real function defined on the vertices $f(V_1, \ldots, V_n)$ (it may not be defined in all $P_n$) and then the coordinate map $\varphi_f(V_1, \ldots, V_n)$. Trilinear coordinates are not a good option to provide a geometric interpretation of $\varphi_f(V_1, \ldots, V_n)$ since they do not extend in a natural way from triangles to $n$-gons for $n \geq 4$. So, we will use “barycentric coordinates” instead (as announced in Remark 3 and in the sense specified in Remark 6).

We are now ready for generalizing the definition of triangle center function by C. Kimberling for polygons:

**Definition 5.** ($n$-gon center function in terms of the vertices) For $n \geq 3$, we say that a real-valued function $f(V_1, \ldots, V_n)$ is a $n$-gon center function if it satisfies the following properties:

1. Preservation with respect to relabellings: for the symmetry $\sigma \in D_n$ (defined in (1)),

$$f(V_1, V_2, \ldots, V_n) = f(V_1, V_{\sigma(2)}, \ldots, V_{\sigma(n)}).$$

2. Homogeneity: there exists some $k \in \mathbb{N}$ such that, for all $t \in \mathbb{R}_{\geq 0}$ we have that

$$f(tV_1, \ldots, tV_n) = t^k f(V_1, \ldots, V_n).$$

3. Preservation with respect to motions: for every rigid motion $T$ in the plane, $f(T(V_1), \ldots, T(V_n)) = f(V_1, \ldots, V_n)$.

Define also the coordinate map $\varphi_f : C_n \subset P_n \rightarrow P_{n-1}\mathbb{R}$ given by:

$$\varphi_f(V_1, \ldots, V_n) = [f(V_1, \ldots, V_n) : f(V_2, \ldots, V_n, V_1) : \ldots : f(V_n, V_1, \ldots, V_{n-1})].$$
The set $C_n$ is the subset of elements in $P_n$ such that:

$$f(V_1, \ldots, V_n) + f(V_2, \ldots, V_n, V_1) + \ldots + f(V_n, V_1, \ldots, V_{n-1}) \neq 0.$$ 

Concerning the geometric interpretation:

**Remark 6.** For a given $n$-gon center function $f$ and for $(V_1, \ldots, V_n) \in C_n$, we can define the normalized coordinates $\tilde{f}(V_1, \ldots, V_n), \ldots, \tilde{f}(V_n, \ldots, V_1)$ as the ones satisfying:

$$\begin{cases}
\varphi_f(V_1, \ldots, V_n) = [\tilde{f}(V_1, \ldots, V_n) : \ldots : \tilde{f}(V_n, V_1, \ldots, V_{n-1})], \\
\tilde{f}(V_1, \ldots, V_n) + \ldots + \tilde{f}(V_n, V_1, \ldots, V_{n-1}) = 1.
\end{cases} \quad (2)$$

So we can give a geometric interpretation of the center of $(V_1, \ldots, V_n)$ associated to $f$ via the map $\Phi_f : C_n \subset P_n \rightarrow \mathbb{R}^2$ given by

$$P = \tilde{f}(V_1, \ldots, V_n)V_1 + \ldots + \tilde{f}(V_n, \ldots, V_{n-1})V_n. \quad (3)$$

We refer to those normalized coordinates as “barycentric coordinates” although they are only the coefficients of an affine combination. In fact, when there are more than 3 vertices, they are not even “coordinates” (the same point can be obtained as an affine combination of the vertices in several ways).

The approach followed in [16] is more careful with the use of (proper) coordinates but this would lead to a lack of symmetry in the definition of the coordinate map. This is one of the reasons why he does not provide a general definition of center of a polygon.

For any center function $f$, we want the map $\Phi_f$ to satisfy an analogue of Properties 1 and 2. The next result ensures that and may help us to clarify the notation and ideas behind this definition.

**Theorem 7.** Definition 5 and the geometric interpretation described in (3) provide an analogue to Properties 1 and 2 for $n$-gon center functions, i.e.,

- **P1** (the coordinate map is well defined) The coordinates given by the coordinate map $\varphi_f$ are associated to each polygon independently of the labelling of the vertices (but obviously re-ordered).
- **P2** (the definition is coherent with respect to similarities) Let $f$ be a $n$-gon center function. Let $N_1 = (V_1, \ldots, V_n)$ and $N_2 = (V'_1, \ldots, V'_n)$ be two similar $n$-gons such that $S$ is the similarity $N_1 = S(N_2)$. Then $\Phi_f(N_1) = S(\Phi_f(N_2))$.

**Proof.** To prove P1, see that for every $\alpha \in D_n$

$$\varphi_f(\alpha(V_1, \ldots, V_n)) = \alpha(\varphi_f(V_1, \ldots, V_n)),$$

where $\alpha([t_1 : \ldots : t_n]) = [t_{\alpha(1)} : \ldots : t_{\alpha(n)}]$. To prove P2 see that any similarity $S$ can be obtained as a composition $T \circ H$ of a rigid motion $T$ and an homothethy $H = \lambda \cdot id$ fixing the origin. Hence,

$$\varphi_f(S(V_1), \ldots, S(V_n)) = \lambda^k \varphi_f(T(V_1), \ldots, T(V_n)) = \varphi_f(T(V_1), \ldots, T(V_n))$$
\[ = \varphi_f(V_1, \ldots, V_n). \]

We conclude this section with the following result, which states an important property of regular polygons. Recall that we say that an \(n\)-gon is \textit{regular} if it is equiangular and equilateral. Regular \(n\)-gons can be either convex or star. □

**Proposition 8.** For any center function \(f\), a regular \(n\)-gon (convex or star) \((V_1, \ldots, V_n)\) satisfies

\[ f(V_1, \ldots, V_n) = f(V_2, \ldots, V_n, V_1) = \ldots = f(V_n, V_1, \ldots, V_{n-1}), \]

so, if defined, \(\varphi_f(V_1, \ldots, V_n) = [1 : \ldots : 1]\) and \(\Phi_f(V_1, \ldots, V_n)\) coincides with the centroid of the vertices (see Example 11 below).

**Proof.** Let \((V_1, \ldots, V_n)\) be a regular \(n\)-gon and \(f\) a center function for this \(n\)-gon. Then, the \(n\)-gon \((V_i, V_{i+1}, \ldots, V_n, V_1, \ldots, V_{i-1})\) is also regular and corresponds to a rotation \(T_i\) of \(\frac{2\pi(i-1)}{n}\) rad of \((V_1, \ldots, V_n)\), i.e.,

\[ T_i(V_1, \ldots, V_n) = (T_i(V_1), \ldots, T_i(V_n)) = (V_i, V_{i+1}, \ldots, V_n, V_1, \ldots, V_{i-1}). \]

Since a rotation is a rigid motion in the plane, by property (3) of Definition 5 we have that \(f(T_i(V_1), \ldots, T_i(V_n)) = f(V_1, \ldots, V_n)\). Thus, if defined,

\[ \varphi_f(V_1, \ldots, V_n) = [f(V_1, \ldots, V_n) : f(V_2, \ldots, V_n, V_1) : \ldots : f(V_n, V_1, \ldots, V_{n-1})] \]

\[ = [f(V_1, \ldots, V_n) : f(T_2(V_1), \ldots, T_2(V_n)) : \ldots : f(T_n(V_1), \ldots, T_n(V_n))] \]

\[ = [f(V_1, \ldots, V_n) : f(V_1, \ldots, V_n) : \ldots : f(V_1, \ldots, V_n)] = [1 : 1 : \ldots : 1]. \]

\[ \square \]

### 4. Second Definition of Polygon Center Function

Definition 5 is, in some sense, not comfortable to use. The first reason is that it may not be immediate to verify condition (3) to decide if a given \(f\) is a center function or not. And the second one is that, traditionally, some of the most useful triangle centers are described in terms of the sidelengths, not in terms of the coordinates of the vertices. Also some of the already known centers of polygons are described in terms of the sidelengths and the length of the diagonals. So in this section we provide another definition of center function by means of these lengths, equivalent, in some sense, to Definition 5 (Theorem 10).

We need again to establish some conventions. Let \((V_1, \ldots, V_n)\) be an \(n\)-gon. We will denote by \(d_{ij}\) the length of the segment with endpoints \(V_i, V_j\). It is obvious that \(d_{ij} = d_{ji}\). If \(j = i + 1 \mod n\), then \(d_{ij}\) is a sidelength. We will write \(e_{ij}\) instead of \(d_{ij}\) when we want to emphasize that we are referring to sidelengths.

An \(n\)-gon is not completely determined by its sidelengths up to congruence: the lengths of the diagonals are also required to determine it.
The set of all the sidelengths and of the lengths of the diagonals of an \( n \)-gon must satisfy some compatibility conditions. For example, consider a quadrilateral with sidelengths \( e_{12}, e_{23}, e_{34}, e_{41} \) and diagonals \( d_{13}, d_{24} \). According to the Cayley-Menger determinant formula for the volume of a 3-dimensional tetrahedron (see [15]) we have that:

\[
\begin{vmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 0 & e_{12}^2 & d_{13}^2 & e_{14}^2 \\
1 & e_{21}^2 & 0 & e_{23}^2 & d_{24}^2 \\
1 & d_{31}^2 & e_{32}^2 & 0 & e_{34}^2 \\
1 & e_{41}^2 & d_{42}^2 & e_{43}^2 & 0 \\
\end{vmatrix} = 0.
\]

Anyway we prefer to consider \( n \cdot (n - 1) \)-tuples \( (d_{ij}) \) with \( i \neq j \), taking in mind that some of the entries \( d_{ij} \) are redundant. For a given \( n \) let \( T_n \subset \mathbb{R}^{n(n-1)} \) be the set of all possible \( n \cdot (n - 1) \)-tuples corresponding to some \( n \)-gon. \( T_n \) can be viewed as a quotient of \( \mathcal{P}_n \). Let denote the projection map \( \pi_n : \mathcal{P}_n \to T_n \).

In this setting we can define the \( n \)-gon center function as a real-valued function \( g \) depending on the sidelengths and the lengths of the diagonals as follows:

**Definition 9.** (\( n \)-gon center function in terms of lengths) For \( n \geq 3 \), we say that a real-valued function \( g(d_{ij}), i, j = 1, \ldots, n \), is a \( n \)-gon center function if it satisfies the following properties:

1. Preservation with respect to relabellings: for the symmetry \( \sigma \in D_n \) (defined in (1)):
   \[
g(d_{ij}) = g(d_{\sigma(i), \sigma(j)}).
   \]

2. Homogeneity: there exists some \( k \in \mathbb{N} \) such that, for all \( t \in \mathbb{R}_{\geq 0} \), we have that \( g(t \cdot d_{ij}) = t^k \cdot g(d_{ij}) \).

Define also the coordinate map \( \varphi_g : D_n \subset T_n \to \mathbb{R}^{n-1} \) given by:

\[
\varphi_g(d_{ij}) = [g(d_{ij}) : g(d_{\rho(i), \rho(j)}) : \ldots : g(d_{\rho^{n-1}(i), \rho^{n-1}(j)})],
\]

for \( \rho \in D_n \) (defined also in (1)). The set \( D_n \) is the set of elements in \( T_n \) such that

\[
g(d_{ij}) + g(d_{\rho(i), \rho(j)}) + \ldots + g(d_{\rho^{n-1}(i), \rho^{n-1}(j)}) \neq 0.
\]

See that triangle center functions (in the sense of Definition 2) are 3-gon center functions in the sense of Definition 9. The bridge between both notations is \( (a, b, c) = (d_{23}, d_{13}, d_{12}) \).

Concerning Definition 9, the geometric interpretation of the center is done by means of another map \( \Phi_g : \pi_n^{-1}(D_n) \subset \mathcal{P}_n \to \mathbb{R}^2 \) given by

\[
(V_1, \ldots, V_n) \mapsto \tilde{g}(d_{ij})V_1 + \tilde{g}(d_{\rho(i), \rho(j)})V_2 + \ldots + \tilde{g}(d_{\rho^{n-1}(i), \rho^{n-1}(j)})V_n,
\]

where \( \tilde{g}(d_{ij}), \ldots, \tilde{g}(d_{\rho^{n-1}(i), \rho^{n-1}(j)}) \) are the normalized coordinates as done in (2).
The next result ensures that both definitions of center function are equivalent.

**Theorem 10.** (equivalence between Definitions 5 and 9) Given a $n$-gon center function $f(V_1, \ldots, V_n)$ (in the sense of Definition 5), it is possible to find an $n$-gon center function $g(d_{ij})$ (in the sense of Definition 9) such that the following diagram commutes

\[
P_n \xrightarrow{f} \mathbb{R}
\]

\[
P_n \xrightarrow{g} \mathbb{R}
\]

\[
\pi_n \downarrow
\]

\[
T_n
\]

and so $\Phi_f(V_1, \ldots, V_n) = \Phi_g(V_1, \ldots, V_n)$, and vice versa.

**Proof.** First, see that if $f(V_1, \ldots, V_n)$ is a center function in the sense of Definition 5, then we can define the center function $g(d_{ij})$ in the sense of Definition 9

\[
g(d_{ij}) = f(V_1(d_{ij}), \ldots, V_n(d_{ij}))
\]

for some element $(V_1(d_{ij}), \ldots, V_n(d_{ij})) \in \pi^{-1}(d_{ij})$. This map is well defined, since $f$ assigns the same value to all of the elements in the same class of equivalence $\pi^{-1}(d_{ij})$. We just have to prove that if $f$ satisfies properties (1), (2) and (3), then $g$ satisfies properties (1') and (2'). For $\sigma \in D_n$ as defined in (1) we have:

\[
g(d_{\sigma(i),\sigma(j)}) = f(V_1, V_{\sigma(2)}(d_{\sigma(i),\sigma(j)}), \ldots, V_{\sigma(n)}(d_{\sigma(i),\sigma(j)}))
\]

\[
= f(V_1(d_{ij}), \ldots, V_n(d_{ij})) = g(d_{ij}),
\]

by properties (1) and (3). So, property (1') holds. On the other hand:

\[
g(t \cdot d_{ij}) = f(V_1(t \cdot d_{ij}), \ldots, V_n(t \cdot d_{ij})) = f(t \cdot V_1(d_{ij}), \ldots, t \cdot V_n(d_{ij}))
\]

\[
= t^k \cdot f(V_1(d_{ij}), \ldots, t_n(V_n(d_{ij}))) = t^k \cdot g(d_{ij}),
\]

and so property (2') also holds.

Next, see that if $g(d_{ij})$ is a center function in the sense of Definition 9, then it is easy to find an associated center function $f(V_1, \ldots, V_n)$ in the sense of Definition 5 just by $f = g \circ \pi_n$. We just have to prove that if $g$ satisfies properties (1') and (2'), then this $f$ satisfies properties (1), (2), and (3). First see that, for $\sigma \in D_n$ as defined in (1) we have that:

\[
f(V_1, V_{\sigma(2)}, \ldots, V_{\sigma(n)}) = g(\|V_{\sigma(i)} - V_{\sigma(j)}\|) = g(\|V_i - V_j\|) = f(V_1, \ldots, V_n),
\]

by property (1'). Hence, property (1) holds. Now, see that:

\[
f(t \cdot V_1, \ldots, t \cdot V_n) = g(\|t \cdot V_i - t \cdot V_j\|) = g(t \cdot \|V_i - V_j\|) = t^k \cdot g(\|V_i - V_j\|)
\]

\[
= t^k \cdot f(V_1, \ldots, V_n),
\]

by property (2'). So, property (2) also holds. Finally, the proof of property (3) is immediate: all congruent $n$-gons are mapped to the same class by $\pi_n$. \qed
See that, if $f = g \circ \pi_n$, then $\Phi_f = \Phi_g \circ \pi_n$.

5. Some Examples of Polygon Centers

In this section we present some of the more relevant centers for polygons, already appearing in the literature. Most of them arise from important problems in Applied Mathematics, and can be naturally defined as an affine combination of the vertices. The coefficients of this combination are functions of either the vertices or the sidelengths and the lengths of the diagonals and so can be considered to be center functions in the sense of Definitions 5 and 9, respectively. Some of those examples can be found in [1] in the particular case $n = 4$ (quadrilaterals).

Example 11. (centroid, barycenter or center of mass of the vertices) The barycenter of a polygon with vertices $V_1, \ldots, V_n$, or the center of mass of the vertices (provided that all the vertices have the same weight) is the point:

$$\Phi_{f_0}(V_1, \ldots, V_n) = \frac{1}{n} V_1 + \ldots + \frac{1}{n} V_n.$$  

(4)

So, the associated center function can be chosen to be $f_0(V_1, \ldots, V_n) = 1$ and the coordinate map is $\varphi_{f_0}(V_1, \ldots, V_n) = [1 : \ldots : 1]$ (recall that the coefficients for the affine combination are the normalized ones).

Example 12. (center of mass of the perimeter of a convex polygon) The center of mass of the perimeter of a convex polygon (provided that all the points in the perimeter have the same weight) with vertices $V_1, \ldots, V_n$ is the point (see [7]):

$$\Phi_{g_1}(V_1, \ldots, V_n) = \sum_{i=1}^{n} \left( \frac{e_{i-1(\text{mod } n),i} + e_{i,i+1(\text{mod } n)}}{2 \sum_{j=1}^{n} e_{j,j+1(\text{mod } n)}} \right) \cdot V_i.$$  

(5)

Example 13. (centroid of the polygonal lamina) The centroid of a polygonal lamina with vertices $V_1, \ldots, V_n$ is the point (see [4]):

$$\Phi_{f_2}(V_1, \ldots, V_n) = \sum_{i=1}^{n} \left( \frac{V_{i-1(\text{mod } n)} \land V_i \lor V_{i+1(\text{mod } n)}}{3 \sum_{j=1}^{n} V_j \land V_{j+1(\text{mod } n)}} \right) \cdot V_i,$$  

(5)

where $V_m \land V_k = x_m y_k - x_k y_m$, for $V_m = (x_m, y_m)$ and $V_k = (x_k, y_k)$. This is not an affine combination since

$$\sum_{i=1}^{n} \left( \frac{V_{i-1(\text{mod } n)} \land V_i \lor V_{i+1(\text{mod } n)}}{3 \sum_{j=1}^{n} V_j \land V_{j+1(\text{mod } n)}} \right) = \frac{2}{3} \neq 1,$$
so, this is not the geometric interpretation of a center in our sense.

To include this important center in the setting of center functions we are going to modify (5). If the $n$-gon $(V_1, \ldots, V_n)$ is convex, $\Phi_{f_2}(V_1, \ldots, V_n)$ can be computed via “geometric decomposition” as

$$
\sum_{i=1}^{n} \left( \Phi_{f_2}(V_i, V_{i+1(\text{mod } n)}, B) \cdot \text{AREA}(V_i, V_{i+1(\text{mod } n)}, B) \right) \cdot \text{AREA}(V_1, \ldots, V_n),
$$

for $B = \Phi_{f_0}(V_1, \ldots, V_n)$ (see (4)), which, according to the Shoelace Formula (see [14]) and to the formula of the centroid of a triangular lamina (the classical centroid of the triangle), equals to

$$
\sum_{i=1}^{n} \left( \frac{1}{3} V_i + \frac{1}{3} V_{i+1(\text{mod } n)} + \frac{1}{3} B \right) \cdot \frac{1}{2} \| (B - V_i) \wedge (B - V_{i+1(\text{mod } n)}) \| \cdot \frac{1}{2} \sum_{i=1}^{n} \| (B - V_i) \wedge (B - V_{i+1(\text{mod } n)}) \|
$$

$$
= \sum_{i=1}^{n} \left( C_1(i) + C_2(i) + C_3(i) \right) \cdot \frac{3}{n} \sum_{i=1}^{n} \| (B - V_i) \wedge (B - V_{i+1(\text{mod } n)}) \|.
$$

Expression (6) does correspond to the geometric interpretation of the center function

$$
f_2(V_1, \ldots, V_n) = \| (B - V_1) \wedge (B - V_2) \| + \sum_{j=1}^{n} \| (B - V_j) \wedge (B - V_{j+1(\text{mod } n)}) \|. \quad (6)
$$

where:

$$
C_1(i) = \| (B - V_i) \wedge (B - V_{i+1(\text{mod } n)}) \|,
$$

$$
C_2(i) = \| (B - V_{i-1(\text{mod } n)}) \wedge (B - V_i) \|,
$$

$$
C_3(i) = \frac{1}{n} \sum_{j=1}^{n} \| (B - V_j) \wedge (B - V_{j+1(\text{mod } n)}) \|.
$$

Example 14. (medoid) The medoid of the set of vertices $V_1, \ldots, V_n$ is the point $G_3$ such that (see, for example, the recent work [2]):

$$
\Phi_{f_3} = \arg \min_{V \in \{V_1, \ldots, V_n\}} \sum_{i=1}^{n} \| V - V_i \|.
$$

The medoid is not well defined for every $n$-gon (this minimum may be reached by two or more of the vertices). When defined, it can also be considered as a center in our sense. In this case the center function is:

$$
f_3(V_1, \ldots, V_n) = \begin{cases} 
1 & \text{if } V_1 = \min_{V \in \{V_1, \ldots, V_n\}} \sum_{i=1}^{n} \| V - V_i \|, \\
0 & \text{in other case.}
\end{cases}
$$
6. Characterization of $n$-Gons Using Centers

The idea of characterizing regular polygons using $n$-gon center functions was one of the main reasons of our interest in this topic, in connection to other geometric problems. This study was already started in [1] for quadrilaterals. We say that:

**Definition 15.** Let $\mathcal{A} \subset \mathcal{P}_n$ be a family of polygons (for example convex polygons). A set of center functions $f_1, \ldots, f_k$ (resp. $g_1, \ldots, g_k$) with associated coordinate maps $\varphi_{f_1}, \ldots, \varphi_{f_k}$ (resp. $\varphi_{g_1}, \ldots, \varphi_{g_k}$) characterizes a family $\mathcal{F} \subset \mathcal{A}$ of $n$-gons if, for $(V_1, \ldots, V_n) \in \mathcal{A}$:

$$
(V_1, \ldots, V_n) \in \mathcal{F} \iff \left\{ \begin{array}{l}
\varphi_{f_1}(V_1, \ldots, V_n) = [1 : \ldots : 1], \\
\vdots \\
\varphi_{f_k}(V_1, \ldots, V_n) = [1 : \ldots : 1].
\end{array} \right.
$$

See that, in this case, $\Phi_{f_1}(V_1, \ldots, V_n) = \ldots = \Phi_{f_k}(V_1, \ldots, V_n) = P$, for $P$ being the barycenter of $(V_1, \ldots, V_n)$.

If a family $\mathcal{F}$ is characterized by a set of center functions, then it must be closed under congruences and it must contain regular $n$-gons (convex and star, see Proposition 8).

For $n = 3$, regular triangles (recall that, for triangles, equilaterality and equiangularity are equivalent properties) are characterized by just one center function. Take for example:

$$
f(V_1, V_2, V_3) = 1 + \frac{\langle V_1 V_2, V_1 V_3 \rangle}{\|V_1 V_2\| \cdot \|V_1 V_3\|}.
$$

This is not so trivial: the cosine of two angles being equal does not imply the angles are equal but complementary. But in this case this is not a problem since the sum of the angles of a non-degenerated quadrilateral must be less or equal to $2\pi$ rad (exactly $2\pi$ rad if the quadrilateral is simple).

However, with the same choice of $\mathcal{A}$, there is no center or set of centers characterizing either equilateral quadrilaterals (rhombi) or regular quadrilaterals (squares). The following results formalize this idea:

**Theorem 16.** For $n \geq 3$, for $\mathcal{A} \subset \mathcal{P}_n$ being the family of non-degenerated and convex $n$-gons, equiangular $n$-gons can be characterized by one center function.

**Proof.** The $n$-gon center function that characterizes equiangular $n$-gons (provided that they are convex and so the angle between two adjacent sides is less than $\pi$ rad) is again

$$
f_1(V_1, \ldots, V_n) = 1 + \frac{\langle V_1 V_2, V_1 V_n \rangle}{\|V_1 V_2\| \cdot \|V_1 V_n\|}. \quad (7)
$$
In Fig. 1 we show a non-convex pentagon which, despite not being equiangular, is also included in a family of polygons characterized by the center (7).

Next, we define, for $n \geq 3$, the family $S_n$ of non-degenerated $n$-gons $(V_1, \ldots, V_n)$ such that $\forall i, j \in \{1, \ldots, n\}$, for all $k \in \mathbb{N}$,

$$\|V_{\rho^k(i)} - V_{\rho^k(j)}\| = \|V_{\sigma^k(i)} - V_{\sigma^k(j)}\|.$$  \hspace{1cm} (8) 

For any $n$-gon center function $g$, any element $(V_1, \ldots, V_n) \in S_n$ satisfies:

$$g(d_{\rho^k(i), \rho^k(j)}) = \begin{cases} g(d_{ij}) & \text{for } k \text{ even} \\ g(d_{\sigma(i), \sigma(j)}) = g(d_{ij}) & \text{for } k \text{ odd} \end{cases}$$

for $(d_{ij}) = \pi_n(V_1, \ldots, V_n)$. The equality in the $k$ odd case is a consequence of the “preservation with respect to relabellings property” ((1’) in Definition 9).

For $n$ even, this family can be considered, in some sense, a generalization of rectangles for $n \geq 6$. Elements in $S_n$ are symmetric with respect to their medians and are not necessarily equilateral (see Fig. 2). For $n$ odd, this family is just the set of regular $n$-gons.

**Theorem 17.** Let $A$ be the set of non-degenerated polygons in $P_n$.

For $n \geq 3$ being an odd number, equilateral $n$-gons can be characterized by one center function.

For $n \geq 4$ being an even number, $S_n$ can be characterized by one $n$-gon center function. But equilateral $n$-gons cannot be characterized by any number of center functions (any family $F$ characterized by $n$-gon center functions always contains $S_n$).
Figure 2. This hexagon is an element of $S_6$

Proof. The $n$-gon center function that characterizes equilateral $n$-gons for $n$ odd is

$$f_2(V_1, \ldots, V_n) = \|V_{(n+1)/2} - V_{(n+3)/2}\|.$$

The center function that characterizes $S_n$ is:

$$f_3(V_1, \ldots, V_n) = \|V_{n/2} - V_{(n/2)+2}\|.$$

□

Theorems 16 and 17 together imply:

Corollary 18. Let $\mathcal{A}$ be the set of non-degenerated and convex polygons in $\mathcal{P}_n$. For $n \geq 3$ odd, regular $n$-gons can be characterized by two center functions, provided that they are convex. For $n \geq 4$ even, regular $n$-gons cannot be characterized by any number of center functions.

See that Theorems 3.1, 3.2, 3.3, 3.4, 3.5 in [1] are compatible with the results proved here, although the authors are only interested in some particular quadrilateral centers.

7. Other Possible Definitions of Center

Some of the centers arising from Applied Mathematics are defined by an implicit equation involving the vertices, or as solution of a problem of optimization (see Example 14). Besides, the term “center” appears in a different setting for compact length spaces in [13], from a totally different approach. This leads to the two following alternative definitions of center:

Definition 19. Let $F(P, V_1, \ldots, V_n)$ be a map satisfying the following properties:

(a) For every $(V_1, \ldots, V_n)$ in $\mathcal{P}_n$ (additionally non-degeneration property can be required) $F(P, V_1, \ldots, V_n) = 0$ defines univocally $P$. 


(b) Preservation with respect to relabellings: if $F(P, V_1, \ldots, V_n) = 0$, then for any element $\alpha \in D_n$ we have

$$F(P, V_{\alpha(1)}, \ldots, V_{\alpha(n)}) = 0.$$ 

(c) Homogeneity: for every $\lambda \in \mathbb{R}_{\geq 0}$,

$$F(P, V_1, \ldots, V_n) = 0 \Rightarrow F(\lambda P, \lambda V_1, \ldots, \lambda V_n) = 0.$$ 

(d) Preservation by motions: for every rigid motion $T$ in the plane,

$$F(P, V_1, \ldots, V_n) = 0 \Rightarrow F(T(P), T(V_1), \ldots, T(V_n)) = 0.$$ 

We say that the point $P$ ensured by (a) is an implicit center of $(V_1, \ldots, V_n)$.

**Definition 20.** Let $G(P, V_1, \ldots, V_n)$ be a real function defined in $P_n$ (additionally non-degeneration property can be required for the domain) such that:

(a') For every $(V_1, \ldots, V_n)$ in $P_n$ there exists a unique

$$X = \arg \min_{P \in \mathbb{R}^2} (G(P, V_1, \ldots, V_n)).$$ 

(b') Preservation with respect to relabellings: for any element $\alpha \in D_n$ we have

$$\arg \min_{P \in \mathbb{R}^2} (G(P, V_1, \ldots, V_n)) = \arg \min_{P \in \mathbb{R}^2} (G(P, V_{\alpha(1)}, \ldots, V_{\alpha(n)})).$$ 

(c') Homogeneity: for every $\lambda \in \mathbb{R}_{\geq 0}$,

$$\lambda \cdot \arg \min_{P \in \mathbb{R}^2} (G(P, V_1, \ldots, V_n)) = \arg \min_{P \in \mathbb{R}^2} (G(P, \lambda V_1, \ldots, \lambda V_n)).$$ 

(d') Preservation by motions: for every rigid motion $T$ in the plane,

$$T (\arg \min_{P \in \mathbb{R}^2} (G(P, V_1, \ldots, V_n))) = \arg \min_{P \in \mathbb{R}^2} (G(P, T(V_1), \ldots, T(V_n))).$$ 

We say that the point $X$ ensured by (a') is a minimal center of $(V_1, \ldots, V_n)$.

We have that:

**Theorem 21.** Any implicit center in the sense of Definition 19 is a minimal center in the sense of Definition 20 and vice versa. Moreover, any center in the sense of Definition 5 is an implicit center in the sense of Definition 19 (and therefore, a minimal center in the sense of Definition 20).

**Proof.** To show that Definitions 19 and 20 are equivalent take

$$F(P, V_1, \ldots, V_n) = G(P, V_1, \ldots, V_n) - X(V_1, \ldots, V_n)$$ 

and, for $X$ the unique point satisfying $F(X, V_1, \ldots, V_n) = 0$,

$$G(P, V_1, \ldots, V_n) = \min_{P \in \mathbb{R}^2} (\text{dist}(P, X)).$$ 

The proof of the second statement is immediate taking

$$F(P, V_1, \ldots, V_n) = (\tilde{f}(V_1, \ldots, V_n)V_1 + \ldots + \tilde{f}(V_n, V_1, \ldots, V_{n-1})V_n) - P.$$ 

An important problem is: are implicit and minimal centers, centers in our sense? This has been included as Open Question 26. Some examples of well-known points usually called “centers” that could be naturally included in these different definitions could be:
Example 22. (geometric median of the vertices) The geometric median of the set of vertices $V_1, \ldots, V_n$ of an $n$-gon is the point $X$ minimizing the sum of distances to the vertices. Thus, it could be naturally considered as a minimal center defined by (see [6]):

$$X = \arg \min_{P \in \mathbb{R}^2} \sum_{j=1}^{n} \|V_j - P\|.$$ 

Provided that $X$ is distinct from any vertex, it can be also described as an implicit center by the formula:

$$\sum_{j=1}^{n} \frac{V_j - X}{\|V_j - X\|} = 0.$$ 

It is known that there is no explicit “simple” formula for $G$ or its coordinates (see [3]). So, we cannot expect to find a “simple” center function for this center.

Example 23. (Chebyshev center) The Chebyshev center of a bounded set $Q$ is the center $X$ of the minimal-radius ball enclosing the entire set $Q$ (see [5]). It is described as a minimal center by the formula:

$$X = \arg \min_{P \in \mathbb{R}^2} \left( \max_{V \in Q} \|V - P\|_2 \right).$$

8. Final Comments

During the development of this article, some questions have arisen:

Open Question 24. Can regular $n$-gons, for $n$-odd, be characterized by only one center function?

Open Question 25. What do we know about the Characterization Problem (analogue statements for Theorems 16, 17) when $A = \mathcal{P}_n$?

Open Question 26. Is any implicit center in the sense of Definition 19 a center in the sense of Definition 5? See that Example 22 shows that the corresponding center function may not be trivial at all to find.

Concerning future work, it would be very interesting to study more general concepts like central lines and in general central curves for $n$-gons with $n \geq 3$. In triangle geometry, central lines are certain special straight lines associated to a triangle, such as the Euler line. Whereas the circumcircle is an example of what we could call central curve. In [16] a lot of such constructions appear. An algebraic definition similar to the one proposed in this article for polygon centers would be desirable. On the other hand, $\mathcal{P}_n$ is naturally a $D_n$-space (a topological space endowed with a group of symmetries, see [11]). In this context, coordinate maps can be understood as $G$-maps, also called equivariant maps (see [11]). It could be interesting to explore this point of
view. In particular, this may connect $n$-gon centers with interesting problems in Plane Geometry such as the Square Peg Problem and its variants (see [12]). Finally, we would like to remark that the study of centers for $k$-dimensional polyhedra ($k \geq 3$, but specially $k = 3$) would be of great interest in different areas (computational geometry and computer vision, for instance), and is a problem still to be explored.

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