EXCITATIONS OF THE STATIC QUARK-ANTIQUARK SYSTEM IN SEVERAL GAUGE THEORIES*

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The spectrum of gluons in the presence of a static quark-antiquark pair is studied using Monte Carlo simulations on anisotropic space-time lattices. For very small quark-antiquark separations $R$, the level orderings and approximate degeneracies disagree with the expectations from an effective string theory. As the quark-antiquark separation $R$ increases, a dramatic rearrangement of the energies occurs, and above 2 fm, all of the levels studied show behavior consistent with an effective string description. The energy spacings are nearly $\pi/R$, but a tantalizing fine structure remains. In addition to 4-dimensional SU(3) gauge theory, results from 3-dimensional SU(2) and compact U(1) gauge theories are also presented.

1. Introduction

An important part of understanding confinement in quantum chromodynamics (QCD) is understanding the low-lying physics of the confining gluon field. The spectrum of gluons in the presence of a static quark-antiquark pair provides valuable clues about the nature of the low-lying stationary states of the gluon field. Innumerable lattice QCD simulations have confirmed that the energy of the ground state rises linearly with the separation $R$ between the quark and antiquark, naively suggesting that the gluon field forms a string-like confining object connecting the quark and the antiquark.

*Talk presented by C. Morningstar.
However, it should be noted that the spherical bag model also predicts a linearly rising energy for moderate $R$, and hence, the linearly rising ground-state energy is not conclusive evidence of string formation. Computations of the gluon action density surrounding a static quark-antiquark pair in $SU(2)$ gauge theory also hint at flux tube formation.

Adopting the viewpoint that the nature of the confining gluon field is best revealed in its excitation spectrum, we have embarked on a series of studies employing recent advances in lattice simulation technology, including anisotropic lattices, improved gauge actions, and large sets of creation operators, to investigate the onset of string-like behavior in the gluon field surrounding a quark-antiquark pair for a wide range of separations $R$ from 0.1 to 3 fm. Energy gaps given by multiples of $\pi/R$ and a well-defined pattern of degeneracies and level orderings among the different symmetry channels form a very distinctive and robust signature of the onset of the Goldstone modes of the effective QCD string. Non-universal details of the underlying string description, such as higher order interactions and their couplings, are encoded in the fine structure of the spectrum at large separations.

In this talk, results from this series of studies clearly demonstrating the onset of string formation for large $R$ are presented. The spectra of three-dimensional $SU(2)$ and compact $U(1)$ gauge theories are also presented to address questions about the dependence of results on the gauge group and the dimensionality of space-time. First, the classification of the states is discussed in Sec. 2. The expected level orderings at large $R$ from an effective string description are detailed in Sec. 3. The spectrum in four-dimensional $SU(3)$ is discussed in Sec. 4, and three-dimensional $SU(2)$ and compact $U(1)$ results are presented in Sec. 5. A complementary study of the spectrum and Casimir energy in three-dimensional $Z(2)$ gauge theory is presented elsewhere.

2. Classification of states

The first step in determining the energies of the stationary states of gluons in the presence of a static quark and antiquark, fixed in space some distance $R$ apart, is to classify the levels in terms of the symmetries of the problem. Such a system has cylindrical symmetry about the axis $\hat{R}$ passing through the quark and the antiquark (the molecular axis). The total angular momentum $\vec{J}_g$ of the gluons is not a conserved quantity, but its projection $\vec{J}_g \cdot \hat{R}$ onto the molecular axis is and can be used to label the
energy levels of the gluons. Here, we adopt the standard notation from the physics of diatomic molecules and denote the magnitude of the eigenvalue of $\vec{J}_g \cdot \hat{R}$ by $\Lambda$. States with $\Lambda = 0, 1, 2, 3, 4, \ldots$ are typically denoted by the capital Greek letters $\Sigma, \Pi, \Delta, \Phi, \Gamma, \ldots$, respectively. The energy of the gluons is unaffected by reflections in a plane containing the molecular axis; since such reflections interchange states of opposite handedness, given by the sign of the eigenvalue of $\vec{J}_g \cdot \hat{R}$, such states must necessarily be degenerate ($\Lambda$ doubling). However, this doubling does not apply to the $\Sigma$ states; $\Sigma$ states which are even (odd) under a reflection in a plane containing the molecular axis are denoted by a superscript $+ (-)$. Another symmetry is the combined operation of charge conjugation and spatial inversion about the midpoint between the quark and the antiquark. Here, we denote the eigenvalue of this transformation by $\eta_{CP}$ which can take values $\pm 1$. States which are even (odd) under this parity–charge-conjugation operation are indicated by subscripts $g$ ($u$). Thus, the low-lying levels in four space-time dimensions are labeled $\Sigma_g^+, \Sigma_g^-, \Sigma_u^+, \Sigma_u^-, \Pi_g, \Pi_u, \Delta_g, \Delta_u$, and so on.

In three space-time dimensions, there is no longer a rotational symmetry about the molecular axis since there are only two spatial dimensions. Instead, the analogous symmetry is a reflection in the molecular axis, and states are either symmetric $S$ or antisymmetric $A$ under this transformation. The combined operation of charge conjugation and spatial inversion about the midpoint between the quark and the antiquark is still a symmetry in three space-time dimensions. Once again, states which are even (odd) under this parity–charge-conjugation operation are indicated by the subscripts $g$ ($u$). To summarize, the low-lying states in three space-time dimensions are labeled by $S_g, A_g, S_u, A_u$.

One last note concerning the classification of states should be made. In a gauge theory based on the group $SU(2)$, the subscript $g$ and $u$ refers only to spatial inversion about the midpoint between the static sources, without charge conjugation. This is due to the fact that in $SU(2)$, the complex conjugate representation $\Gamma$ is equivalent to the $\Gamma$ representation.

3. String modes

The ground-state energy of gluons in the presence of a static quark-antiquark pair rises linearly with the quark-antiquark separation $R$. This fact has led to the general belief that at sufficiently large $R$, the chromoelectric and chromomagnetic fields become confined to a long tube-like region of space connecting the quark and the antiquark. A treatment of
the gluon field in terms of the collective degrees of freedom associated with
the position of the long flux might then be sufficient for reproducing the
long-wavelength physics. If true, one then hopes that the oscillating flux
can be well described in terms of an effective string theory. In such a case,
the lowest-lying excitations are expected to be the Goldstone modes asso-
ciated with the spontaneously broken transverse translational symmetry.
These modes are a universal feature of any low-energy description of the
effective QCD string and have energy separations above the ground state
given by multiples of $\frac{\pi}{R}$. For the gluonic excitations at small $R$, no robust
expectations from string theory presently exist. In this section, the pattern
of degeneracies and level orderings of the expected string modes for large
$R$ in both three and four space-time dimensions are deduced.

The excitations of long flux lines are expected to be described by a
local derivative expansion of a massless vector field $\mathbf{\xi}$ with two transverse
components in four-dimensional space-time. Assume that the quark is
located at $(0, 0, 0)$ and the antiquark is at $(0, 0, R)$, then $\mathbf{\xi}(x_3, x_4)$ represents
the transverse displacement in the $x_1$ and $x_2$ directions of the thin flux line
from its equilibrium position. We further assume that this displacement
field is continuous and single-valued, so that string configurations which
do not double-back on themselves or overhang the ends are not allowed. Symmetry
arguments then require that the effective QCD string action in Euclidean
space-time should have a leading term given by

$$S_{\text{eff}}^{(0)} = \frac{1}{2} c_0 \int dx_4 \int_0^R dx_3 \partial_\mu \mathbf{\xi} \cdot \partial_\mu \mathbf{\xi}, \quad \mathbf{\xi}(0, x_4) = \mathbf{\xi}(R, x_4) = 0,$$  \hspace{1cm} (1)

where the derivatives are taken with respect to the two worldsheet coor-
dinates $x_3$ and $x_4$, and $c_0$ has the dimension of a mass squared and is
proportional to the string tension. The stationary states are found by
expressing the displacement field $\mathbf{\xi}$ in terms of normal modes. For fixed
ends, the normal modes are standing waves $\sin(m\pi x_3/R)$. These modes
have energies $m\omega$ for positive integer $m$ and $\omega = \pi/R$. For two transverse
dimensions, one defines right ($+$) and left ($-$) circularly polarized ladder
operators $a_{m \pm}^\dagger$, then the string eigenmodes are

$$\prod_{m=1}^{\infty} \left( (a_{m+}^\dagger)^{n_{m+}} (a_{m-}^\dagger)^{n_{m-}} \right) |0\rangle,$$  \hspace{1cm} (2)

where $|0\rangle$ denotes the ground state of the string, and $n_{m+}$ and $n_{m-}$ are the
occupation numbers which take values 0, 1, 2, . . . . If $E_0$ denotes the energy
of the ground state, then the eigenvalues $E$ (energy), $\Lambda$, and $\eta_{CP}$ associated
The circular polarizations yields a superscript + or −. For the Σ states, the evenness or oddness under exchange (with the string eigenstates are given by

| N = 0: | Σ_0^+ | [0] |
| N = 1: | Π_u | a_{1,1+}(0) | a_{1,1-}(0) |
| N = 2: | Σ_0^+ | a_{1,1+}a_{1,1-}(0) | a_{1,1-}(0) |
| | Π_g | a_{2,1+}(0) | a_{1,1-}(0) |
| | \Delta_\eta | (a_{1,1+})^2(0) | (a_{1,1-})^2(0) |
| N = 3: | Σ_0^+ | (a_{1,1+}a_{1,1-} + a_{1,1-}a_{1,1+})(0) | a_{1,1-}(0) |
| | Π_u | a_{3,1+}(0) | a_{1,1-}(0) |
| | Π''_u | (a_{1,1+})^2a_{1,1-}(0) | a_{1,1-}^2(0) |
| | \Delta_u | a_{1,1+}a_{1,1-}(0) | a_{1,1-}a_{1,1-}(0) |
| | \Phi_u | (a_{1,1+})^3(0) | (a_{1,1-})^3(0) |
| N = 4: | Σ_0^+ | a_{2,1+}a_{1,1-}(0) |
| | Σ_0^{(iv)} | (a_{1,1+})^2(a_{1,1-})^2(0) |
| | \Sigma_0^{(iv)} | (a_{1,1+})^4 + (a_{1,1-})^4(0) |
| | \Sigma''_g | (a_{1,1+})^4a_{1,1-} + (a_{1,1-})^4a_{1,1+}(0) |
| | \Sigma''_g | (a_{1,1+})^4a_{1,1-} - (a_{1,1-})^4a_{1,1+}(0) |
| | \Pi_g | a_{4,1+}(0) | a_{1,1-}(0) |
| | \Pi''_g | (a_{1,1+})^2a_{1,1-}(0) | (a_{1,1-})^2a_{1,1-}(0) |
| | \Pi''''_g | a_{1,1+}a_{1,1-}a_{1,1-}(0) | a_{1,1-}a_{1,1-}a_{1,1-}(0) |
| | \Delta_\eta | a_{1,1+}a_{1,1-}(0) | a_{1,1-}a_{1,1-}(0) |
| | \Delta_\eta | (a_{1,1+})^2(0) | (a_{1,1-})^2(0) |
| | \Delta_\eta | (a_{1,1+})^3a_{1,1-}(0) | a_{1,1-}^3(0) |
| | \Phi_\eta | (a_{1,1+})^4a_{1,1-}(0) | (a_{1,1-})^4a_{1,1-}(0) |
| | \Gamma_\eta | (a_{1,1+})^4(0) | (a_{1,1-})^4(0) |

with the string eigenstates are given by

\[ E = E_0 + \frac{N \pi}{R}, \quad N = \sum_{m=1}^{\infty} m (n_{m+} + n_{m-}), \]

\[ \eta_{CP} = (-1)^N, \quad \Lambda = \sum_{m=1}^{\infty} \left| n_{m+} - n_{m-} \right|. \]  

For the Σ states, the evenness or oddness under exchange (−) ↔ (+) of the circular polarizations yields a superscript + or −, respectively. Using
Table 2. Low-lying string levels for fixed ends in three space-time dimensions. The \( N = 1 \) level is nondegenerate, and the \( N = 2, 3, 4 \) levels are 2, 3, 5-fold degenerate, respectively. The positive integers indicate the standing wave normal modes.

\[
\begin{array}{c|cc}
N = 0: & S_g & |0\rangle \\
N = 1: & A_u & a_1^0|0\rangle \\
N = 2: & S'_g & (a_1^0)^2|0\rangle \\
 & A_g & a_1^1|0\rangle \\
N = 3: & A'_u & (a_1^1)^3|0\rangle \\
 & A''_u & a_3^0|0\rangle \\
 & S_u & a_1^1 a_2^0|0\rangle \\
N = 4: & S''_g & (a_1^2)^2|0\rangle \\
 & S''_g & a_1^1 a_4^0|0\rangle \\
 & S''_g & (a_1^2)^4|0\rangle \\
 & A'_g & a_4^0|0\rangle \\
 & A''_g & (a_1^2)^2 a_2^0|0\rangle \\
\end{array}
\]

these properties, the orderings and degeneracies of the Goldstone string energy levels and their symmetries are as shown in Table 1. One sees that the \( N\pi/R \) behavior and a well-defined pattern of degeneracies and level orderings among the different channels form a very distinctive signature of the onset of the Goldstone modes for the effective QCD string.

In three space-time dimensions, there is only one transverse direction for the string, so the ladder operators are written \( a_m^\dagger \) since there are no right and left circular polarizations. The orderings and degeneracies of the Goldstone modes are given in Table 2.

4. \( SU(3) \) gauge theory in 4 dimensions

The spectrum shown in Fig. 1 provides clear evidence that the gluon field can be well approximated by an effective string theory for large separations \( R \). Energy gaps \( \Delta E \) above the ground state are compared to asymptotic string gaps for 15 excited states in Fig. 2. The quantity \( \Delta E/(N\pi/R) - 1 \) is plotted to show percentage deviations from the asymptotic string levels for string quantum number \( N = 1, 2, 3, 4 \). For small \( R < 2 \) fm, the energy gaps lie far below the null lines and are strongly split for fixed \( N \). In other words, string formation does not appear to set in until the quark and the antiquark are separated by about 2 fm. For small separations, the level
Figure 1. The spectrum of gluonic excitations in the presence of a static quark-antiquark pair separated by a distance $R$ in 4-dimensional $SU(3)$ gauge theory (from Ref. 2). Results are from one simulation for lattice spacing $a_s \sim 0.2$ fm using an improved action on a $(10^2 \times 30) \times 60$ anisotropic lattice with coupling $\beta = 2.5$ and bare aspect ratio $\xi = 5$. At large distances, all levels without exception are consistent with the expectations from an effective string theory description. A dramatic level rearrangement is observed in the crossover region between 0.5 – 2.0 fm. The dashed line marks a lower bound for the onset of mixing effects with glueball states.
Figure 2. The energy gaps $\Delta E$ above the ground state $\Sigma^+_u$ of the stationary states of gluons in the presence of a static quark-antiquark pair in 4-dimensional $SU(3)$ gauge theory. The results at lattice spacing $a_s \sim 0.2$ fm are shown against the quark-antiquark separation $R$ and are compared with the $N\pi/R$ splittings expected in an effective string theory at large $R$. The large-$R$ results for a free Nambu-Goto (NG) string are also shown.
Figure 3. One possible interpretation of the spectrum in Fig. 1. (a) For small quark-antiquark separations, the strong chromoelectric field of the quark-antiquark pair repels the physical vacuum (dual Meissner effect) creating a bubble. Explanations of the low-lying stationary states must take into account both the gluonic modes inside the bubble and oscillations of the collective coordinates describing the bubble. (b) For large quark-antiquark separations, the bubble stretches into a thin tube of flux, and the low-lying states are explained by the collective motion of the tube since the internal gluonic excitations are much higher lying.

orderings and degeneracies are not consistent with the expectations from an effective string description. More importantly, the gaps differ appreciably from $N\pi/R$ with $N = 1, 2, 3, \ldots$, as clearly shown in Fig. 2. Such deviations, as large as 50% or more, cannot be considered mere corrections, making the applicability of an effective string description problematical. Between 0.5 to 2 fm, a dramatic level rearrangement occurs. For separations above 2 fm, the levels agree without exception with the ordering and degeneracies expected from an effective string theory. The gaps agree well with $N\pi/R$, but a fine structure remains. This first glimpse of such a fine structure offers the exciting possibility of deducing details of the effective QCD string action in future higher precision simulations.

It is reasonable to expect that the first few terms in the effective string action might predominantly arise from the geometric properties of the flux tube. The Nambu-Goto (NG) action is one of the simplest geometrical string models. The spectrum of the Nambu-Goto string with fixed ends in $d$ dimensions has been calculated\(^6\), with the result

$$E_N = \sigma R \left(1 - \frac{(d-2)\pi}{12\sigma R^2} + \frac{2\pi N}{\sigma R^2}\right)^{1/2}. \quad (4)$$

For small $R$, this model has a quantization problem\(^6\) unless $d = 26$, but the problem disappears as $R$ becomes large. The energy gaps expected for a Nambu-Goto string at large $R$ are shown in Fig. 2. Deviations of
the simulation results from the Nambu-Goto gaps suggests that physical properties, such as rigidity, may be relevant for the effective string action.

Fig. 3 illustrates one possible interpretation of the results shown in Fig. 1. At small quark-antiquark separations, the strong chromoelectric field of the quark-antiquark pair repels the physical vacuum in a dual Meissner effect, creating a bubble surrounding the pair. Descriptions of the low-lying stationary states must take into account both the gluonic modes inside the bubble and the motion of the collective coordinates describing the bubble. For large quark-antiquark separations, the bubble stretches into a thin tube of flux, and the low-lying states could then be explained by the collective motion of the tube since the internal gluonic excitations, being typically of order 1 GeV, are now much higher lying. We caution the reader that the above interpretation is simply speculation based on observations to date. Although the simulation results rule out the usefulness at small $R$ of an effective string action constructed as a $1/R$ expansion, they do not actually rule out the unlikely possibility of a string description based on some other expansion parameter.

5. *SU*(2) and compact *U*(1) gauge theories in 3 dimensions

The spectra of three-dimensional *SU*(2) and compact *U*(1) gauge theories were also studied to address questions about the dependence of the results on the gauge group and the dimensionality of space-time. Due to the reduced dimensionality, higher statistical precision was possible in these calculations. These simulations also served to check various systematic errors.

The excitation gaps $\Delta E$ above the $S_g$ ground state of ten levels were computed and are compared with $N \pi / R$ in Fig. 4. Again, the large-$R$ results are consistent with the expectations from an effective string description without exception. A fine structure is also observed, but it is less pronounced than that in four-dimensional *SU*(3). Unlike in four-dimensional *SU*(3), no dramatic level rearrangements occur between small and large separation, but deviations from $N \pi / R$ are significant for small $R$. There is remarkable agreement between the *SU*(2) and compact *U*(1) results. A detailed examination of these results is still work in progress.

We have also pursued the spectrum in three-dimensional *Z*(2) lattice gauge theory. These results are reported elsewhere. Extremely precise determinations are possible in *Z*(2) by exploiting a duality transformation into an Ising model. In the critical region, the resulting Ising model admits
Figure 4. The energy gaps $\Delta E$ above the ground state $S_g$ of the stationary states of the gauge field in the presence of a static source pair in 3-dimensional $SU(2)$ and compact $U(1)$ gauge theories. These gaps are shown against the separation $R$ of the static sources and are compared with the $N\pi/R$ splittings expected in an effective string theory at large $R$. The results in $SU(2)$ were obtained on an $84^3$ lattice using an anistropic improved lattice action with coupling $\beta = 5.6$ and bare aspect ratio $\xi = 2$, so that $a_s \sim 0.1$ fm. The compact $U(1)$ results were obtained on a $28^2 \times 224$ lattice using an anisotropic improved lattice action for $\beta = 0.5$ and $\xi = 8$. 
a description in terms of a $\phi^4$ real scalar field theory, allowing the possibility of understanding the underlying microscopic origins of confinement in a rigorous field-theoretical setting. The details of this work in progress are presented elsewhere\(^3\).

6. Conclusion

In this talk, Monte Carlo computations of the energies of sixteen stationary states of the gluon field in the presence of a static quark-antiquark pair separated by a distance $R$ were presented for a wide range of $R$ from 0.1 to 3 fm. Striking confirmation of string-like flux formation of the gluon field surrounding a quark-antiquark pair separated by distances larger than 2 fm was presented. A tantalizing fine structure was revealed, suggesting the possibility of identifying the effective QCD string action in future higher precision simulations. A dramatic level rearrangement between small and large quark-antiquark separations was observed in a crossover region around 2 fm. The observed pattern of energy levels at small $R$ strongly challenges an effective string description.

Eleven levels in three-dimensional $SU(2)$ and compact $U(1)$ lattice gauge theory were also studied. String formation was once again confirmed at large separations, with a fine structure less pronounced than in four-dimensional $SU(3)$. No dramatic level rearrangement was found between large and small separations. These studies are ongoing, and we are also vigorously pursuing the spectrum and other observables in three-dimensional $Z(2)$ gauge theory with the goal of determining the effective string action. Future work also includes calculating the three-dimensional $SU(3)$ spectrum, torelon (flux loops winding around the lattice) spectra, and studying the spatial structures of these gluonic excitations. This work was supported by the U.S. National Science Foundation under award PHY-0099450, the U.S. DOE, Grant No. DE-FG03-97ER40546, and the European Community’s Human Potential Programme under contract HPRN-CT-2000-00145, Hadrons/Lattice QCD.

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