On a critical Leray–α model of turbulence

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Abstract

This paper aims to study a family of Leray-α models with periodic boundary conditions. These models are good approximations for the Navier-Stokes equations. We focus our attention on the critical value of regularization “θ” that guarantees the global well-posedness for these models. We conjecture that θ = 1/4 is the critical value to obtain such results. When alpha goes to zero, we prove that the Leray-α solution, with critical regularization, gives rise to a suitable solution to the Navier-Stokes equations. We also introduce an interpolating deconvolution operator that depends on “θ”. Then we extend our results of existence, uniqueness and convergence to a family of regularized magnetohydrodynamics equations.

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1 Introduction

The dynamics of fluids provide various highly challenging theoretical, as well as experimental and computational problems for engineers, physicists and mathematicians. It is widely believed that all the informations about turbulence are contained in the dynamics of the solutions of the Navier-Stokes equations (NSE) for viscous, incompressible, homogenous fluids.

The NSE for a homogenous incompressible fluid are usually written as

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \cdot \nabla u - \nu \Delta u + \nabla p &= f, \\
\nabla \cdot u &= 0, \\
u_t &= u_0.
\end{align*}
\]

The unknowns are the velocity vector field \( u \) and the scalar pressure \( p \). The viscosity \( \nu \), the initial velocity vector field \( u_0 \) and the external force \( f \) are given.

The dynamics of several conducting incompressible fluids in presence of a magnetic field are described by the magnetohydrodynamics equations (MHD). The MHD involve coupling Maxwell’s equations governing the magnetic field and the NSE governing the fluid motion. The system has the following form

\[
\begin{align*}
\frac{\partial u}{\partial t} - \nu_1 \Delta u + (u \cdot \nabla) u - (B \cdot \nabla) B + \nabla p + \frac{1}{2} \nabla |B|^2 &= 0, \\
\frac{\partial B}{\partial t} - \nu_2 \Delta B + (u \cdot \nabla) B - (B \cdot \nabla) u &= 0, \\
\nabla \cdot B &= \nabla \cdot u = 0, \\
u_t &= B_0, \\
u_0 &= u_0.
\end{align*}
\]

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The unknowns are the velocity vector field $u$, the magnetic vector field $B$ and the scalar pressure $p$. The kinematic viscosity $\nu_1$, the magnetic diffusivity $\nu_2$, the initial velocity vector field $u_0$ and the initial magnetic vector field $B_0$ are given.

The problem of the global existence and uniqueness of the solutions of the three-dimensional Navier-Stokes equations are among the most challenging problems of contemporary mathematics. The first attempt was done to Leray [25] who established the existence of global weak dissipative solutions to NSE. Such solutions are called “turbulent” and allow singularities in the velocity fields. The extension to no-slip boundary condition was done by Hopf [18]. Short time regularity of solutions has also been known for many years, as they have various interesting partial and conditional regularity results [20] [35] [39]. Historically, the partial regularity result of Scheffer [34] proving that the potentially singular set in time has zero half-dimensional Hausdorff measure, led to that of Caffarelli, Khon and Nirenberg [4]. In [4] they worked with a particular class of weak solutions of the Navier-Stokes equations called suitable weak solutions. By a suitable weak solution they mean a weak solution of the NSE such that for all $T \in (0, +\infty]$ and for all non negative fonction $\phi \in C^\infty$ compactly supported in space and time, the following inequality is valid:

\[ 2\nu \int_0^T \int_{T^3} |\nabla u|^2 \phi \, dx \, dt \leq \int_0^T \int_{T^3} |u|^2 (\phi_t + \nu \Delta \phi) + (|u|^2 u + 2pu) \cdot \nabla \phi \, dx \, dt + 2 \int_0^T \int_{T^3} f u \phi \, dx \, dt. \]  

(1.3)

The inequality (1.3) is known as local energy inequality. Caffarelli, Khon and Nirenberg [4] proved that for the NSE the singular set of a suitable weak solution has parabolic Hausdorff dimension at most equal to 1. This implies that if singularities do exist, they must be relatively rare.

Let us mention that it is not known whether the weak solutions satisfy the local energy inequality. Whereas it is remarkable that the weak solutions constructed by Leray [25], by regularizing the nonlinear term, or by adding hyperviscosity are actually suitable [3] [10] [16]. These problems are also open and very important for the MHD equations [11] [31] although the MHD equations are not on the Clay Institute’s list of prize problems.

In order to study various aspects of turbulence, and motivated by the difficulties related to the complexity of the 3D Navier-Stokes equations, simpler models and simplifications of hydrodynamics equations have been proposed over the years for exemple [29] [21] [6] [14] [13] [7] [19] [5] [33] [24] [22] [27].

In this paper, we study a Leray-\( \alpha \) approximation for the Navier-Stokes equation subject to space-periodic boundary conditions.

The Leray-\( \alpha \) equations we are considering are

\[
\begin{aligned}
&\frac{\partial u}{\partial t} + u \cdot \nabla u - \nu \Delta u + \nabla p = f & \text{in } \mathbb{R}^+ \times T_3, \\
&u = \alpha^{2\theta} (-\Delta)^{\theta} u + \bar{u} & \text{in } T_3, \\
&\nabla \cdot u = \nabla \cdot \bar{u} = 0 \quad \int_{T_3} u = \int_{T_3} \bar{u} = 0, \\
&u(t, x + L e_j) = u(t, x), \\
&u_{t=0} = u_0.
\end{aligned}
\]  

(1.4)

Where \(\{e_j, j=1,2 \text{ or }3\}\) is the canonical basis of \(\mathbb{R}^3\), \(L > 0\).

We consider these equations on the three dimensional torus \(T_3 = (\mathbb{R}^3/T_3)\) where \(T_3 = 2\pi \mathbb{Z}^3/L\), \(x \in T_3\), and \(t \in [0, +\infty[\). The unkowns are the velocity vector field \(u\) and the scalar pressure \(p\). The viscosity \(\nu\), the initial velocity vector field \(u_0\) and the external force.
$f$ with $\nabla \cdot f = 0$ are given.

Let $\theta$ be a non-negative parameter. The nonlocal operator $(-\Delta)^\theta$ is defined through the Fourier transform

$$(-\Delta)^\theta u(k) = |k|^{2\theta} \hat{u}(k).$$

(1.5)

Fractional order Laplace operator has been used in another $\alpha$ model of turbulence in [33]. Existence and uniqueness of solutions of other modifications of the Navier-Stokes equations have been studied by Ladyzhenskaya [21], Lions [29], Málek et al. [32].

Motivated by the above work [33], we study here the case $\theta = \frac{1}{4}$ of (1.4).

We would like to point out that after having finished writing this paper, we were informed that there exists an analysis of a general family of regularized Navier-Stokes and MHD models [17]. In our paper, we note that the Leray-$\alpha$ family considered is a particular case of the general study in [17] where the results do not recover the critical case $\theta = \frac{1}{4}$. The regularization critical values of the other $\alpha$ models studied in [17] will be reported in a forthcoming paper.

Therefore, the initial value problem we consider in particular is:

\[
\begin{cases}
\frac{\partial u}{\partial t} + \nabla \cdot u - \nu \Delta u + \nabla p = f & \text{in } \mathbb{R}^+ \times \mathbb{T}_3, \\
u = a \frac{1}{2} (-\Delta)^{\frac{1}{2}} u + \|u\|_\infty & \text{in } \mathbb{T}_3, \\
\nabla \cdot u = \nabla \cdot u = 0, \int_{\mathbb{T}_3} u = \int_{\mathbb{T}_3} u = 0, \\
u_{t=0} = u_0,
\end{cases}
\]

(1.6)

or equivalently (1.4) with $\theta = \frac{1}{4}$ and we are working with periodic boundary conditions.

One of the main results of this paper is to establish the global well-posedness of the solution to equations (1.4) in $L^2(\mathbb{T}_3)^3$ for $\theta = \frac{1}{4}$. We conjecture that $\theta = \frac{1}{4}$ is the critical value to obtain the above result to eqs. (1.4). Therefore, smooth solutions of eqs. (1.4) with $\theta \geq \frac{1}{4}$ do not develop finite-time singularities.

When $\alpha = 0$, eqs. (1.4) are reduced to the usual Navier-Stokes equations for incompressible fluids.

We recall that $H^\frac{1}{2}(\mathbb{T}_3)^3$ is a scale-invariant space for Navier-Stokes equations, i.e. if $u(x,t)$ is a solution to the Navier-Stokes equations, then so is $u_\lambda(x,t) := \lambda u(\lambda x, \lambda^2 t)$ for any $\lambda > 0$. Kato [40, 41] shows the importance for a functional space invariance by scaling. A lot of works followed which can be summed up in the following way: if the initial data are small in an invariant norm with respect to the scaling then the solution is smooth for all time.

Another result of this paper is to establish the global well-posedness of the solution to eqs. (1.4) in $H^\frac{1}{2}(\mathbb{T}_3)^3$ for $\theta = \frac{1}{4}$ and without smallness conditions on the initial data.

We also discuss the relation between the Leray-$\alpha$ equations with $\theta \geq \frac{1}{4}$ and the NSE. The first result of convergence of a $\alpha$ model to the Navier-Stokes equations is proved in [13] where the authors show the existence of a subsequence of weak solutions of the $\alpha$ model that converge to a Leray-Hopf weak solution of NSE. We improve their result by proving an $L^p$ convergence property in order to show the convergence of the weak solution to a suitable weak solution to the NSE. Moreover, we extend the theory to other models where we consider a deconvolution type regularisation to the NSE and an extension to the MHD equations is also given. The deconvolution operator was introduced by Layton and Lewandowski [23]. We will define the interpolating deconvolution operator in order to regularize the NSE in section 7.1 and the MHD equations in section 8. As the Leray-$\alpha$ regularization we will show an $L^p$ convergence property to the deconvolution operator,
in order to deduce that the deconvolution regularization will give rise to suitable weak solutions.

The critical values of regularization \( \theta = \frac{1}{4} \) holds for the MHD equations and the Navier-Stokes equations so it is natural to ask if the singularities in MHD equations are similar to the singularities in the Navier-Stokes equations. A partial answer is given in [42], where an analog of the known Caffarelli, Khon and Nirenberg result is established to the MHD equations. The extending of the partial regularity result of Scheffer [34] to the MHD equation is studied in [2].

The paper is organized as follows. In section 2 we recall some preliminary results used later in the proof. In section 3, we present the main existence results (Theorem 3.1 and Theorem 3.2 below). Sections 4 and 5 are devoted to the proof of these two results where we use the standard Galerkin approximation. In section 6 we show an \( L^p \) convergence result in order to prove that the Leray–\( \alpha \) equations give rise to a suitable solution to the Navier-Stokes equations. In section 7 we introduce the interpolating deconvolution operator and we give applications to the Navier-Stokes equations. Finally, we study a deconvolution regularization to the MHD equations.

2 Functional setting and preliminaries

In this section, we introduce some preliminary material and notations which are commonly used in the mathematical study of fluids, in particular in the study of the Navier-Stokes equations (NSE). For a more detailed discussion of these topics, we refer to [8, 9, 15, 26, 39]. We denote by \( L^p(\mathbb{T}_3)^3 \) and \( H^s(\mathbb{T}_3)^3 \) the usual Lebesgue and Sobolev spaces over \( \mathbb{T}_3 \). For the \( L^2 \) norm and the inner product, we write \( \| \cdot \|_{L^2} \) and \( (\cdot, \cdot)_{L^2} \). The norm in \( H^1(\mathbb{T}_3)^3 \) is written as \( \| \cdot \|_{H^1} \) and its scalar product as \( (\cdot, \cdot)_{H^1} \). We denote by \( (H^1(\mathbb{T}_3)^3)' \) the dual space of \( H^1(\mathbb{T}_3)^3 \). Note that we have the continuous embeddings

\[
H^1(\mathbb{T}_3)^3 \hookrightarrow L^2(\mathbb{T}_3)^3 \hookrightarrow (H^1(\mathbb{T}_3)^3)'.
\]

Moreover, by the Rellich-Kondrachov compactness theorem (see, e.g., [12]), these embeddings are compact.

Since we work with periodic boundary conditions, we can characterize the divergence-free spaces by using the Fourier series on the 3D torus \( \mathbb{T}_3 \). We expand the velocity in Fourier series as:

\[
u(t, x) = \sum_{k \in I} \hat{u}_k \exp \{ik \cdot x\},
\]

where \( I = \{k \text{ such that } k = \frac{2\pi a}{L}, a \in \mathbb{Z}^3, a \neq 0\} \),

where for the vanishing space average case, we have the condition

\[
\hat{u}_0 = \int_{\mathbb{T}_3} u = 0.
\]

Since the vector \( u(t, x) \) is real-valued, we have

\[
\hat{u}_{-k} = \hat{u}_k^* \text{ for every } k,
\]

where \( \hat{u}_k^* \) denotes the complex conjugate of \( \hat{u}_k \).

In the Fourier space, the divergence-free condition is

\[
\hat{u}_k \cdot k = 0 \text{ for all } k.
\]
For $s \in \mathbb{R}$, the usual Sobolev spaces $H^s(\mathbb{T}_3)^3$ with zero space average can be represented as

$$H^s = \{u = \sum_{k \in I} \hat{u}_k \exp \{ik \cdot x\}, \ \hat{u}_{-k} = \hat{u}_k^*, \ ||u||^2_{H^s} < \infty\},$$

where

$$||u||^2_{H^s} = \sum_{k \in I} |k|^{2s} |\hat{u}_k|^2.$$

Let

$$V^s = \{u \in H^s, \ \hat{u}_k \cdot k = 0\},$$

we identify the continuous dual space of $V^s$ as $V^{-s}$ with the pairing given by

$$(u, v)_{V^s} = \sum_{k \in I} |k|^{2s} \hat{u}_k \cdot \hat{v}_k.$$

We denote $V^0$ by $H$ and $V^1$ by $V$, where the norm $||\cdot||_{H^s}^2 \equiv ||\cdot||_{L^2}$ and the scalar product as $(\cdot, \cdot)_{H^s} \equiv (\cdot, \cdot)_{L^2}$. The norm in $V$ is written as $||\cdot||_{V^2}^2 \equiv ||\cdot||_{H^1}^2$ and its scalar product as $(\cdot, \cdot)_{H^1}$.

We denote by $P_\sigma$ the Leray-Helmholtz projection operator and define the Stokes operator $A := -\sigma \Delta$ with domain $\mathcal{D}(A) := H^2(\mathbb{T}_3)^3 \cap V$. For $u \in \mathcal{D}(A)$, we have the norm equivalence $||Au||_{L^2} \equiv ||u||_{V^2}$. Furthermore, in our case it is known that $A = -\Delta$ due to the periodic boundary conditions (see, e.g., [8, 15]).

It is natural to define the powers $A^s$ of the Stokes operator in the periodic case as

$$A^s u = \sum_{k \in I} |k|^{2s} \hat{u}_k \exp \{ik \cdot x\}.$$

For $u \in \mathcal{D}(A^{s/2})$, we have the norm equivalence $||A^{s/2} u||_{L^2} \equiv ||u||_{V^s}$.

In particular for $s = 1$ we recover the space $V$, hence $\mathcal{D}(A^{1/2}) = V$, (see, e.g., [15]).

Note also that $H^{s+\epsilon}$ is compactly embedded in $H^s$ (resp. $V^{s+\epsilon}$ is compactly embedded in $V^s$) for any $\epsilon > 0$, and we have the following Sobolev embedding Theorem (see [1]).

**Theorem 2.1** The space $H^{1/2}$ is embedded in $L^3(\mathbb{T}_3)^3$ and the space $L^{3/2}(\mathbb{T}_3)^3$ is embedded in $H^{-1/2}$.

The following result deals with some interpolation inequalities.

**Lemma 2.1** Let $T > 0, 1 \leq p_1 < p < p_2 \leq \infty, s_1 < s < s_2$ and $\eta \in [0, 1]$ such that

$$\frac{1}{p} = \frac{\eta}{p_1} + \frac{1-\eta}{p_2} \quad \text{and} \quad s = \eta s_1 + (1-\eta)s_2.$$

Let $u \in \bigcap_{i=1}^{2} L^p([0, T], H^{s_i})$, then $u \in L^p([0, T], H^s)$ and

$$||u||_{L^p([0, T], H^s)} \leq C||u||_{L^p([0, T], H^{s_1})}^{\eta} ||u||_{L^p([0, T], H^{s_2})}^{1-\eta}.$$

The result holds true when we work with the spaces $L^p([0, T], V^{s_i})$. The regularization effect of the nonlocal operator involved in the relation between $\overline{u}$ and $u$ is described by the following lemma.
Lemma 2.2 Let \( \theta \in \mathbb{R}^+ \), \( s \in \mathbb{R} \) and assume that \( u \in H^{s} \). Then \( \overline{u} \in H^{s+2\theta} \)
and
\[
\| \overline{u} \|_{H^{s+2\theta}} \leq \frac{1}{\alpha^{2\theta}} \| u \|_{H^s}.
\]

Proof. When \( u = \sum_{k \in I} \hat{u}_k \exp \{ik \cdot x\} \), then
\[
\overline{u} = \sum_{k \in I} \frac{\hat{u}_k}{1 + \alpha^{2\theta} |k|^{2\theta}} \exp \{ik \cdot x\}.
\]
Formula (2.3) easily yields the estimate (2.2).

Remark 2.1 For \( \theta = \frac{1}{4} \), \( \overline{u} \in H^{s+\frac{1}{2}} \) and \( \| \overline{u} \|_{H^{s+\frac{1}{2}}} \leq \alpha^{-\frac{1}{2}} \| u \|_{H^s} \).

Following the notations of the Navier-Stokes equations, we set
\[
B(u, v) := P_\sigma(u \cdot \nabla v)
\]
for any \( u, v \) in \( V \).

We list several important properties of \( B \) which can be found for example in [15].

Lemma 2.3 The operator \( B \) defined in (2.4) is a bilinear form which can be extended as a continuous map \( B : V \times V \to V' \).

For \( u, v, w \in V \),
\[
\langle B(u, v), w \rangle_{V^{-1}} = -\langle B(u, w), v \rangle_{V^{-1}},
\]
and therefore
\[
\langle B(u, v), v \rangle_{V^{-1}} = 0.
\]
The following result holds true.

Lemma 2.4 The bilinear form \( B \) defined in (2.7) satisfies the following:
(i) Assume that \( u \in V^{\frac{1}{2}} \), \( v \) and \( w \in V \), Then the following inequality holds
\[
\| (B(u, v), w) \| \leq C \| u \|_{V^{\frac{1}{2}}} \| v \| \| w \|_V.
\]
(ii) \( B \) can be extended as a continuous map \( B : V^{\frac{1}{2}} \times V^{\frac{1}{2}} \to V^{-\frac{1}{2}} \). In particular, for every \( u \in V^{\frac{1}{2}}, v \in V^{\frac{1}{2}} \) the bilinear form \( B \) satisfies the following inequalities:
\[
\| B(u, v) \|_{V^{-\frac{1}{2}}} \leq C \| u \|_{V^{\frac{1}{2}}} \| v \|_{V^{\frac{1}{2}}}.
\]

Proof (i) We have
\[
\| (B(u, v), w) \| \leq \left| \int_{T_3} u \otimes v : \nabla w \right|.
\]
By Hölder inequality combined with Sobolev embedding theorem we get
\[
\| (B(u, v), w) \| \leq \| u \otimes v \|_{L^2} \| w \|_{H^1}
\leq \| u \|_{L^3} \| v \|_{L^6} \| w \|_{H^1}
\leq C \| u \|_{V^{\frac{1}{2}}} \| v \|_V \| w \|_V.
\]
(ii) We have
\begin{equation}
\|B(u, v)\|_{V^{-1}} \leq c \|u\|_{L^3} \|\nabla v\|_{L^3} \\
\leq c \|u\|_{V^{1/2}} \|v\|_{V^{3/2}}.
\end{equation}
Where he have used Hölder inequality and the Sobolev injection given in Theorem 2.1.

The following result is given for any \( u \in L^\infty([0, T]; H) \cap L^2([0, T]; V) \).

**Lemma 2.5** If \( u \in L^\infty([0, T]; H) \cap L^2([0, T]; V) \), then \( \overline{u} \in L^\infty([0, T]; V^{2/3}) \cap L^2([0, T]; V^{3/2}) \) and \( B(\overline{u}, u) \in L^2([0, T]; V^{-1}) \).

**Proof.** The regularity on \( \overline{u} \) is obtained from Lemma 2.2.

Then we use Lemma 2.4 to obtain
\begin{equation}
\|(B(\overline{u}, u))\|_{V^{-1}} \leq C \|\overline{u}\|_{V^{2/3}} \|u\|_{V^{3/2}}
\end{equation}
Now, \( \|\overline{u}\|_{V^{2/3}} \in L^\infty(0, T) \) and \( \|u\|_{V^{3/2}} \in L^2(0, T) \), implies that \( \|\overline{u}\|_{V^{2/3}} \|u\|_{V^{3/2}} \in L^2(0, T) \) (that is \( L^\infty([0, T]; V^{2/3}) \cap L^2([0, T]; V^{3/2}) \) \( \subseteq L^2([0, T]; H) \) ).

We remove the pressure from further consideration by projecting (1.4) by \( P_\sigma \) and searching for solutions in the space \( V^s \).

Thus, we obtain the infinite dimensional evolution equation
\begin{equation}
\begin{cases}
\frac{du}{dt} + \nu A u + B(\overline{u}, u) = f, \\
u = \alpha^{2g} A^g \overline{\nu} + \overline{u}, \\
t_{t=0} = u_0.
\end{cases}
\end{equation}

Note that the pressure may be reconstructed from \( \overline{u} \) and \( u \) by solving the elliptic equation
\( \Delta p = \nabla \cdot (\overline{u} \cdot \nabla u) \).

Later in Sect.4 and Sect. 5, we will prove that eqs. (1.4) have unique regular solutions \( (u, p) \) such that the velocity part \( u \) solve the eqs. (2.12).

We conclude this section with the interpolation lemma of Lions and Magenes [30] and a compactness Theorem see [36].

**Lemma 2.6** Let \( s > 0 \) and suppose that
\( u \in L^2([0, T]; V^s) \) and \( \frac{du}{dt} \in L^2([0, T]; V^{-s}) \).

Then
(i) \( u \in C([0, T]; H) \), with \( \sup_{t \in [0, T]} \|u(t)\|_H \leq c \left( \|u\|_{L^2([0, T]; V^s)} + \frac{\|du}{dt}\|_{L^2([0, T]; V^{-s})} \right) \)
and \( \frac{d}{dt}\|u\|_H^2 = 2 \left< \frac{du}{dt}, u \right>_{V^{-s}, V^s} \).
To set more regularity for the unique solution, we also need the following similar result.

**Corollary 2.1** Assume that for some \( k \geq 0 \) and \( s > 0 \),

\[
  u \in L^2([0,T];V^{k+s}) \quad \text{and} \quad \frac{du}{dt} \in L^2([0,T],V^{k-s}).
\]

Then
(i) \( u \in C([0,T];V^k) \), and
(ii) \( \frac{d}{dt} \|A^{k/2}u\|^2_H = 2 \langle A^{k/2} \frac{du}{dt}, A^{k/2}u \rangle_{V^{-s},V^s} \).

The result holds true in the space \( H^s \).

**Theorem 2.2** Let \( X \subset\subset H \subset Y \) be Banach spaces, where \( X \) is reflexive. Suppose that \( u_n \) is a sequence such that it is uniformly bounded in \( L^2([0,T];X) \), and \( \frac{du_n}{dt} \) is uniformly bounded in \( L^2([0,T],Y) \). Then there is a subsequence which converges strongly in \( L^2([0,T],H) \).

The above Theorem is a special case of the following more general result:

**Theorem 2.3** (Aubin-Lions) Let \( X \subset\subset H \subset Y \) be Banach spaces, where \( X \) is reflexive. Suppose that \( u_n \) is a sequence such that it is uniformly bounded in \( L^p([0,T];X) \), \( (1 < p < \infty) \), and \( \frac{du_n}{dt} \) is uniformly bounded in \( L^q([0,T],Y) \), \((1 \leq q \leq \infty)\). Then \( u_n \) is relatively compact in \( L^p([0,T],H) \).

### 3 Main existence theorems

In the following, we will assume that \( \alpha > 0, T > 0 \).

One of the aims of this paper is to establish the global well-posedness of the solution to eqs. (1.4) in \( L^2(\mathbb{T}_3)^3 \) for \( \theta = \frac{1}{4} \). It must be mentioned that the result holds true for \( \theta \geq \frac{1}{4} \).

We conjecture that \( \theta = \frac{1}{4} \) is the critical value to obtain the following result to eqs. (1.4).

**Theorem 3.1** For any \( T > 0 \), let \( f \in L^2([0,T],V^{-1}) \) and \( u_0 \in H \). Assume that \( \theta = \frac{1}{4} \).

Then there exists a unique solution \((u,p) := (u_\alpha,p_\alpha)\) to eqs. (1.4) that satisfies \( u \in C([0,T];H) \cap L^2([0,T];V) \) and \( \frac{du}{dt} \in L^2([0,T],V^{-1}) \) and \( p \in L^2([0,T],L^2(\mathbb{T}_3)) \).

Such that \( u \) verifies

\[
  \left\langle \frac{du}{dt} + \nu Au + B(\overline{u},u) - f, \phi \right\rangle_{V^{-1},V^1} = 0
\]

for every \( \phi \in V \) and almost every \( t \in (0,T) \). Moreover, this solution depends continuously on the initial data \( u_0 \) in the \( L^2 \) norm.

Therefore, smooth solutions of eqs. (1.4) with \( \theta \geq \frac{1}{4} \) do not develop finite-time singularities.

We also prove the global well-posedness of the solution to eqs. (1.4) in \( H^\frac{1}{2}(\mathbb{T}_3)^3 \) for \( \theta = \frac{1}{4} \) and without smallness conditions on the initial data.
Theorem 3.2 For any $T > 0$, let $f \in L^2([0,T], V^{-\frac{1}{2}})$ and $u_0 \in V^{\frac{1}{2}}$. Assume that $\theta = \frac{1}{4}$. Then there exists a unique solution $(u, p) := (u_\alpha, p_\alpha)$ to eqs. (1.4) that satisfies $u \in C([0,T]; V^{\frac{1}{2}}) \cap L^2([0,T]; V^{\frac{3}{2}})$, $\frac{du}{dt} \in L^2([0,T], V^{-\frac{1}{2}})$ and $p \in L^2([0,T], H^{\frac{1}{2}}(T^3))$.

Such that $u$ satisfies
\[
\left\langle \frac{du}{dt} + \nu Au + B(\overline{u}, u) - f, \phi \right\rangle_{V^{-\frac{1}{2}}, V^{\frac{1}{2}}} = 0
\]
for every $\phi \in V^{\frac{1}{2}}$ and almost every $t \in (0,T)$. Moreover, this solution depends continuously on the initial data $u_0$ in the $V^{\frac{1}{2}}$ norm.

We note that the above result holds true for any $s > 0$ and without smallness conditions on the initial data.

4 Proof of Theorem 3.1

The proof is divided into four steps.

4.1 Galerkin approximation

Let us define
\[ H_m \equiv \text{span} \{ \exp \{ i k \cdot x \} : |k| \leq m \} . \]

We look at the finite-dimensional equation obtained by keeping only the first $m$ Fourier modes. In order to use classical tools for systems of ordinary differential equations we need that $f$ belongs to $C([0,T], V^{-1})$. To do so, we extend $f$ outside $[0,T]$ by zero and we set $f_\epsilon = \rho_\epsilon * f$ where $\rho_\epsilon(t) = \frac{1}{\epsilon} \rho(\frac{t}{\epsilon})$, $0 \leq \rho(s) \leq 1$, $\rho(s) = 0$ for $|s| \geq 1$, and $\int_\mathbb{R} \rho = 1$. So the approximate sequence $f_\epsilon$ is very smooth with respect to time for $\rho$ smooth and converges to $f$ in the sense that
\[
(4.1) \quad f_\epsilon \to f \text{ strongly in } L^2([0,T]; V^{-1}) \text{ when } \epsilon \to 0.
\]

The Galerkin approximation of eqs. (2.12) with $\theta = \frac{1}{4}$ is given by
\[
\begin{aligned}
\frac{du^m}{dt} + \nu P_m Au^m + P_m B(P_m \overline{u}^m, P_m u^m) = P_m f_\epsilon, \\
u \frac{d}{dt} \nu P_m u^m = \nu^2 A P_m u^m + B(P_m \overline{u}^m, P_m u^m), \\
u u^m(0) = P_m u_0. 
\end{aligned}
\]

Where for some $m \in \mathbb{N}$ and all $-1 \leq s \leq 2$, $P_m(w) \equiv \sum_{|k| \leq m} \hat{w}_k \exp \{ i k \cdot x \} : V^s \rightarrow H_m$ is the orthogonal projector onto the first $m$ Fourier modes that verifies (see in [32] for more details):
\[
(4.3) \quad \| P_m \|_{L(V^s, V^s)} \leq 1, \quad \text{and for all } v \in V^s : P_m v \to v \text{ strongly in } V^s \text{ when } m \to \infty.
\]

The classical theory of ordinary differential equations implies that eqs. (4.2) have a unique $C^1$ solution $u^m$ for a given time interval that, a priori, depends on $m$, such that $u^m = P_m u$ and $\nabla \cdot u^m = 0$. Our goal is to show that the solution remains finite for all positive times, which implies that $T_m = \infty$.

In the next subsection, we find uniform energy estimates for this solution with respect to $m$. 

9
4.2 Energy estimates in $H$

We follow here a similar method to the one used for the Navier-Stokes equations (see [39]).

**Lemma 4.1** Let $f \in L^2([0,T], V^{-1})$ and $u_0 \in H$, there exists $K_1(T)$ and $K_2(T)$ independent of $m$ such that the solution $u^m$ to the Galerkin truncation satisfies

$$\|u^m\|^2_{L^2([0,T], V)} \leq K_1(T), \text{ for all } T \geq 0,$$

where $K_1(T) = \frac{1}{\nu} \left( \|u_0\|^2_H + \frac{1}{\nu} \|f\|^2_{L^2([0,T], V^{-1})} \right)$ and

$$\|u^m\|^2_{L^\infty([0,T], H)} \leq K_2(T), \text{ for all } T \geq 0,$$

where $K_2(T) = \nu(K_1(T))$.

**Proof** Taking the $L^2$-inner product of the first equation of (4.2) with $u$ and integrating by parts, using (2.6), the incompressibility of the velocity field and the duality relation we obtain

$$\frac{1}{2} \frac{d}{dt} \|u^m\|^2_H + \nu \| \nabla u^m \|^2_H = \int_{T_3} P_m f \cdot u^m dx \leq \|f\|_{V^{-1}} \|u^m\|_V.$$

Using Young inequality we get

$$\frac{d}{dt} \|u^m\|^2_H + \nu \| \nabla u^m \|^2_H \leq \frac{1}{\nu} \|f\|^2_{V^{-1}}.$$

In particular, this leads to the estimate

$$\sup_{t \in [0,T_m]} \|u^m\|^2_H \leq \|u_0\|^2_H + \frac{1}{\nu} \|f\|^2_{L^2([0,T], V^{-1})}$$

This implies that $T_m = T$. Indeed, consider $[0, T_m)$ the maximal interval of existence. Either $T_m^\text{max} = T$ and we are done, or $T_m^\text{max} < T$ and we have $\limsup_{t \to T_m^\text{max}^-} \|u^m(t)\|^2_H = \infty$ a contradiction to (4.5). Hence we have global existence of $u^m$ for $f \in L^2([0, \infty), V^{-1})$ and $u_0 \in H$ and hereafter we take an arbitrary interval $[0, T]$ and we assume that $f \in L^2([0,T], V^{-1})$.

Integrating (4.7) with respect to time gives the desired estimate (4.4) for all $t \in [0, T]$.

Now we use the regularization effect described in Lemma 2.2 to get the following estimates on $u^m$.

**Lemma 4.2** Under the notation of Lemma 4.1, we have

$$\|\overline{u}^m\|^2_{L^2([0,T], V^\frac{1}{2})} \leq \left( \frac{1}{\alpha} \right) K_1(T), \text{ for all } T \geq 0,$$

and

$$\|\overline{u}^m\|^2_{L^\infty([0,T], V^\frac{1}{2})} \leq \left( \frac{1}{\alpha} \right) K_2(T), \text{ for all } T \geq 0.$$

By interpolation (see Lemma 2.1) between $L^2([0,T], V^\frac{1}{2})$ and $L^\infty([0,T], V^\frac{1}{2})$, we can deduce the following result
Corollary 4.1 Under the notation of Lemma 4.2, we deduce

\begin{equation}
\| \mathbf{u}^n \|_{L^2([0,T],V)}^2 \leq \left( \frac{1}{\alpha} \right) K_1(T)^{1/2}K_2(T)^{1/2}, \text{ for all } T \geq 0.
\end{equation}

Our next result provides an estimate on the time derivative of $\mathbf{u}^m$.

Lemma 4.3 Let $f \in L^2([0,T],V^{-1})$ and $\mathbf{u}_0 \in H$, there exists $K_3(T)$ independent of $m$ such that the time derivative of the solution $\mathbf{u}^m$ to the Galerkin truncation (4.2) satisfies

\begin{equation}
\| \frac{d\mathbf{u}^m}{dt} \|_{L^2([0,T],V^{-1})}^2 \leq K_3(T),
\end{equation}

where

\begin{equation}
K_3(T) = 4 (\nu^2 + C (1/\alpha) K_2(T)) K_1(T) + 2 \| f \|_{L^2([0,T],V^{-1})}^2.
\end{equation}

Proof. Taking the $H^{-1}$ norm of (4.2), one obtains:

\begin{equation}
\| \frac{d\mathbf{u}^m}{dt} \|_{V^{-1}} \leq \nu \| \mathbf{u}^m \|_{V^1} + \| B(\mathbf{u}^m, \mathbf{u}^m) \|_{V^{-1}} + \| f \|_{V^{-1}}
\end{equation}

where we note that

\begin{equation}
\| B(\mathbf{u}^m, \mathbf{u}^m) \|_{V^{-1}} = \sup \{(B(\mathbf{u}^m, \mathbf{u}^m), w); w \in V^1, \| w \|_{V^1} \leq 1 \}.
\end{equation}

It remains to show that $\| B(\mathbf{u}^m, \mathbf{u}^m) \|_{V^{-1}}$ is bounded. Indeed, we have from Lemma 2.4 that

\begin{equation}
\| B(\mathbf{u}^m, \mathbf{u}^m) \|_{V^{-1}} \leq C \| \mathbf{u}^m \|_{V^1} \| \mathbf{u}^m \|_{V}.
\end{equation}

It follows that

\begin{equation}
\| \frac{d\mathbf{u}^m}{dt} \|_{V^{-1}}^2 \leq 4 (\nu^2 + C \| \mathbf{u}^m \|_{V^1}^2) \| \mathbf{u}^m \|_{V}^2 + 2 \| f \|_{V^{-1}}^2.
\end{equation}

Integrating with respect to time, and recalling that $\mathbf{u}^m \in L^\infty([0,T],V^1)$ and $\mathbf{u}^m \in L^2([0,T],V)$, it follows from Lemma 4.1 and Lemma 4.2 that

\begin{equation}
\int_0^t \| \frac{d\mathbf{u}^m}{dt} \|_{V^{-1}}^2 \leq 4 (\nu^2 + C (1/\alpha) K_2(T)) K_1(T) + 2 \| f \|_{L^2([0,T],V^{-1})}^2.
\end{equation}

4.3 Passing to the limit $m \to +\infty$

From Lemmas 4.1 and 4.3 and from the reflexivity of the appearing Banach spaces we can extract a subsequence of $\mathbf{u}^m$ and $\frac{d\mathbf{u}^m}{dt}$ such that $\mathbf{u}^m$ converge weakly to some $\mathbf{u}$ in $L^2([0,T],V^1)$ and $\frac{d\mathbf{u}^m}{dt}$ converge weakly to some $\frac{d\mathbf{u}}{dt}$ in $L^2([0,T],V^{-1})$ respectively. Now the interpolation Lemma of Lions and Magenes (Lemma 2.6) implies that $\mathbf{u} \in C([0,T];H)$. In order to show the convergence of $\mathbf{u}^m$ to $\mathbf{u}$ in $C([0,T],H) \cap L^2([0,T],V)$, we need to show that for $\mathbf{u}_0 \in H$, the sequence $\mathbf{u}^m$ is a Cauchy sequence in $C([0,T],H) \cap L^2([0,T],V)$. We know that $\mathbf{u}^m \in C([0,T],H) \cap L^2([0,T],V)$. The difference $\mathbf{u}^{m+n} - \mathbf{u}^m$ satifies

\begin{equation}
\frac{d(\mathbf{u}^{m+n} - \mathbf{u}^m)}{dt} + P_m B((\mathbf{u}^{m+n} - \mathbf{u}^m), \mathbf{u}^m) + P_m B(\mathbf{u}^{m+n}, (\mathbf{u}^{m+n} - \mathbf{u}^m)) + \nu A(\mathbf{u}^{m+n} - \mathbf{u}^m) = 0.
\end{equation}
By taking \( u^{m+n} - u^m \) as test function in (4.17) we get
\[
\frac{1}{2} \frac{d}{dt} \| u^{m+n} - u^m \|^2_H + \nu \| u^{m+n} - u^m \|^2_V \leq \|(B((\overline{u}^{m+n} - \overline{u}^m), u^m), u^{m+n} - u^m) \| + \left< P_{m+n} f_{\frac{1}{m+n}} - P_m f_{\frac{1}{m}}, u^{m+n} - u^m \right>_V^{-1, V}
\]  
(4.18)

Lemma 2.4 combined with Young inequality give
\[
\|(B((\overline{u}^{m+n} - \overline{u}^m), u^m), u^{m+n} - u^m) \| \leq \frac{1}{\nu} \| \overline{u}^{m+n} - \overline{u}^m \|^2_V \frac{u^m}{2} \| u^m \|^2_V + \frac{\nu}{4} \| u^{m+n} - u^m \|^2_V
\]  
(4.19)

The duality norm combined with Young inequality give
\[
\left< P_{m+n} f_{\frac{1}{m+n}} - P_m f_{\frac{1}{m}}, u^{m+n} - u^m \right>_V^{-1, V} \leq \frac{1}{\nu} \| P_{m+n} f_{\frac{1}{m+n}} - P_m f_{\frac{1}{m}} \|^2_{V^{-1}} + \frac{\nu}{4} \| u^{m+n} - u^m \|^2_V
\]  
(4.20)

Thus we get
\[
\frac{d}{dt} \| u^{m+n} - u^m \|^2_H + \nu \| u^{m+n} - u^m \|^2_V \leq C \alpha^{-1} \nu^{-1} \| u^{m+n} - u^m \|^2_H \| u^m \|^2_V + \frac{1}{\nu} \| P_{m+n} f_{\frac{1}{m+n}} - P_m f_{\frac{1}{m}} \|^2_{V^{-1}}
\]  
(4.21)

By Grönwall inequality we get
\[
\| u^{m+n} - u^m \|^2_H \leq \left( \| u^{m+n}(0) - u^m(0) \|^2_H + \frac{1}{\nu} \int_0^T \| P_{m+n} f_{\frac{1}{m+n}} - P_m f_{\frac{1}{m}} \|^2_{V^{-1}} \right) \exp \int_0^T \alpha^{-1} \rho^{-1} \| u^m \|^2_V dt
\]  
(4.22)

We know that \( u^m \in L^2([0, T], V) \), thus there exists \( C(\alpha, \nu) \geq 0 \) such that
\[
\exp \int_0^T \alpha^{-1} \rho^{-1} \| u^m \|^2_V dt \leq C(\alpha, \nu).
\]

We observe that
\[
\int_0^T \left\| P_{m+n} f_{\frac{1}{m+n}} - P_m f_{\frac{1}{m}} \right\|^2_{V^{-1}} \leq \underbrace{\int_0^T \left\| P_{m+n} f_{\frac{1}{m+n}} - P_m f_{\frac{1}{m}} \right\|^2_{V^{-1}}}_{I} + \underbrace{\int_0^T \left\| P_m f_{\frac{1}{m+n}} - P_m f_{\frac{1}{m}} \right\|^2_{V^{-1}}}_{II}
\]
(4.23)

For \( I \) we have
\[
\left\| P_{m+n} f_{\frac{1}{m+n}} - P_m f_{\frac{1}{m+n}} \right\|^2_{V^{-1}} \to 0
\]
because \( f_{\frac{1}{m+n}} \in L^2([0, T], V^{-1}) \) and
\[
\left\| P_{m+n} f_{\frac{1}{m+n}} - P_m f_{\frac{1}{m+n}} \right\|^2_{V^{-1}} \leq \left\| P_{m+n} f_{\frac{1}{m+n}} \right\|^2_{V^{-1}} \leq \| f \|^2_{V^{-1}} \in L^1[0, T].
\]
Using the dominate convergence theorem, we conclude that $I$ tends to zero when $m$ tends to $\infty$.

Similarly by using the dominate convergence theorem combined with the fact that
\[
\|P_m f \frac{1}{m} - P_m f \frac{1}{m} \|^2_{V^{-1}} \to 0
\]
and
\[
\|P_m f \frac{1}{m+n} - P_m f \frac{1}{m} \|^2_{V^{-1}} \leq \|f \frac{1}{m} - f \frac{1}{m} \|^2_{V^{-1}} \leq \|f\|^2_{V^{-1}} \in L^1[0, T],
\]
we obtain that $II$ tends to zero when $m$ tends to $\infty$.

Since $u_0 \in H$ then $\|u^{m+n}(0) - u^m(0)\|^2_H$ converge to zero when $m$ goes to $\infty$. By integrating (4.21), we deduce that $u^{m+n} - u^m$ tend to zero in $C([0, T], H) \cap L^2([0, T], V)$. This implies that $u^m$ is a Cauchy sequence in $C([0, T], H) \cap L^2([0, T], V)$.

Concerning the initial data, we can check that $u(0) = u_0$. Thanks to the result above we have $u^m$ converge to $u$ in $C([0, T]; H)$, in particular $u^m(0)$ converge to $u(0)$ in $H$. In the other hand, we have that $u^m(0) = P_n u_0$ and $u^m(0)$ converge to $u_0$ in $H$. The unicity of the limit in $H$ allows us to deduce the result.

It is obvious from Lemma (2.2) that $\overline{u}^m$ converges to $\overline{u}$ in $C([0, T]; V^{1/2})$.

Moreover,
\[
\int_0^T (A u^m, \phi) dt = - \int_0^T (A^{1/2} u^m, A^{1/2} \phi) dt \to - \int_0^T (A^{1/2} u, A^{1/2} \phi) dt = \int_0^T (A u, \phi) dt,
\]
for all $\phi \in L^2([0, T], V^1)$. Thus, $A u^m$ converges weakly to $A u$ in $L^2([0, T], V^{-1})$ as $m \to \infty$.

We finish by showing that the non linear term $B(\overline{u}^m, u^m)$ converges weakly to $B(\overline{u}, u)$ in $L^2([0, T], V^{-1})$ as $m \to \infty$. From the properties of the trilinear form we have
\[
\left| \int_0^T (B(\overline{u}^m, u^m), \phi) dt - \int_0^T (B(\overline{u}, u), \phi) dt \right| \leq \int_0^T \|B(\overline{u}^m - \overline{u}, u^m), \phi) \| dt + \int_0^T \|B(\overline{u}, u^m - u), \phi) \| dt,
\]
and by using Lemma (2.4) combined with Hölder inequality we get
\[
\int_0^T \|B(\overline{u}^m - \overline{u}, u^m), \phi) \| dt \leq \|\overline{u}^m - \overline{u}\|_{L^\infty([0,T],V^{1/2})}\|u^m\|_{L^2([0,T],V^1)}\|\phi\|_{L^2([0,T],V^1)},
\]
and
\[
\int_0^T \|B(\overline{u}, u^m - u), \phi) \| dt \leq \|\overline{u}\|_{L^\infty([0,T],V^{1/2})}\|u^m - u\|_{L^2([0,T],V^1)}\|\phi\|_{L^2([0,T],V^1)}.
\]
Thus $B(\overline{u}^m, u^m)$ converges weakly to $B(\overline{u}, u)$ in $L^2([0, T], V^{-1})$ as $m \to \infty$. This implies that $P_m B(\overline{u}^m, u^m)$ converges weakly to $B(\overline{u}, u)$ in $L^2([0, T], V^{-1})$ as $m \to \infty$.

The convergence of $P_m f \underline{v}$ to $f$ in $L^2([0, T], V^{-1})$ is obvious because $f \in L^2([0, T], V^{-1})$.

We have shown that $u$ satisfies (2.12) viewed as a functional equality in $V^{-1}$.
4.4 Uniqueness

The solution constructed above is unique. Next, we will show the continuous dependence of the solutions on the initial data and in particular the uniqueness.

Let \( u \) and \( v \) be any two solutions of (2.12) on the interval \([0, T]\), with initial values \( u_0 \in H \) and \( v_0 \in H \), respectively. Let us denote by \( w = u - v \) and \( \overline{w} = \overline{u} - \overline{v} \). Then, we can write the evolution equation for \( w \) as an equality in \( V^{-1} \) given by

\[
\frac{d}{dt} w + \nu A w + B(\overline{w}, u) + B(\overline{w}, w) = 0
\]

We take the inner product of (4.24) with \( w \). Applying the Lemma 2.6 of Lions-Magenes and by the properties of the trilinear form we get

\[
\frac{1}{2} \frac{d}{dt} \|w\|^2_H + \nu \|\nabla w\|^2_H + (B(\overline{w}, u), w) = 0
\]

Using Lemma 2.4 combined with Young’s inequality and Lemma 2.2, we obtain

\[
\frac{d}{dt} \|w\|^2_H + \nu \|\nabla w\|^2_H \leq \frac{1}{4 \alpha} \|w\|^2_H \|u\|^2_V
\]

Using Grönwall’s inequality we conclude that

\[
\|w\|^2_{L^\infty([0, T], H)} \leq \|w_0\|^2_H \exp \frac{1}{\alpha \nu} K_1.
\]

We have shown the continuous dependence of the solutions on the initial data in the \( L^\infty([0, T], H) \) norm. In particular, if \( w_0 = 0 \) then \( w = 0 \) and the solutions are unique for all \( t \in [0, T] \).

**Remark 4.1** Since \( T > 0 \) is arbitrary, the solution above may be uniquely extended for all time.

**Remark 4.2** The pressure is absent in equations (2.12) once we can recover it by using the de Rham Theorem [38]. Generally, the existence of the pressure is not obvious and the pressure may not exist [37].

In our situation, as we work with periodic boundary conditions, we can proceed differently. We take the divergence of (1.4), this yields the following equation for the pressure

\[
\Delta p = \nabla \cdot (\nabla \cdot \overline{u} u).
\]

where \( \overline{u} u \) is the tensor \((\overline{u} u^j)_{1 \leq i, j \leq 3}\).

We recall that for any divergence-free field \( u \), we have

\[
(\overline{u} \cdot \nabla) u = \nabla \cdot (\overline{u} u).
\]

Since \((\overline{u} \cdot \nabla) u \in L^2([0, T], H^{-1}) \). We conclude from the classical elliptic theory that \( p \) is bounded in \( L^2([0, T], L^2(T_3)) \).

**Lemma 4.4** (Uniqueness of the pressure) The pressure is uniquely determined by the velocity.
Proof. Let \((u, p)\) and \((u, q)\) be two weak solutions of (1.1) (with \(\theta \geq \frac{1}{2}\)) such that 
\[ \int_{T_3} p = \int_{T_3} q = 0. \]
Then for every \(\phi \in H^1\) and for almost every \(t \in \mathbb{R}^+\) we have
\[ (\nabla (p - q), \phi) = 0, \]
combined with the condition \(\int_{T_3} p = \int_{T_3} q = 0\) gives \(p = q\).
This concludes the proof of Theorem 3.1.

5 Proof of Theorem 3.2

5.1 Energy estimates in \(V_T^{1/2}\)

Let us write the energy inequality in the space \(V_T^{1/2}\). Taking the \(V_T^{1/2}\) scalar product of eqs. (4.2) with \(u_m\) it turns out, due to the divergence-free condition, that
\[ \frac{1}{2} \frac{d}{dt} \|u_m\|_{V_T^{1/2}}^2 + \nu \|\nabla u_m\|_{V_T^{1/2}}^2 = (P_m B(\overline{u}_m, u_m), u_m)_{V_T^{1/2}} + (P_m f_{\frac{1}{m}}, u_m)_{V_T^{1/2}}. \]

By definition of the scalar product on \(V_T^{1/2}\), we get
\[ \left| (P_m B(\overline{u}_m, u_m), u_m)_{V_T^{1/2}} \right| \leq \|B(\overline{u}_m, u_m)\|_{V_T^{-1/2}} \|\nabla u_m\|_{V_T^{1/2}}. \]

The Sobolev embedding in Theorem 2.1 together with a Hölder estimate give
\[ |(B(\overline{u}_m, u_m), |\nabla u_m|)_{L^2}| \leq c \|B(\overline{u}_m, u_m)\|_{V_T^{-1/2}} \|\nabla u_m\|_{V_T^{1/2}} \leq c \|\overline{u}_m\|_{L^6} \|\nabla u_m\|_{L^3} \|\nabla u_m\|_{V_T^{1/2}} \leq c \|\overline{u}_m\|_{V_T^{1/2}} \|\nabla u_m\|_{V_T^{1/2}}. \]

By the interpolation inequality between \(V_T^{1/2}\) and \(V_T^{3/2}\) we get
\[ \|B(\overline{u}_m, u_m), |\nabla u_m|\|_{L^2} \leq \|\overline{u}_m\|_{V_T^{1/2}} \|\nabla u_m\|_{V_T^{3/2}}. \]

Using the fact that
\[ (P_m f_{\frac{1}{m}}, u_m)_{V_T^{1/2}} \leq \|f\|_{V_T^{-1/2}} \|\nabla u_m\|_{V_T^{1/2}} \]
combined with the Young inequality, we infer
\[ \frac{d}{dt} \|u_m\|_{V_T^{1/2}}^2 + \nu \|\nabla u_m\|_{V_T^{1/2}}^2 \leq \frac{9}{2\nu} \|\overline{u}_m\|_{V_T^{1/2}}^4 \|u_m\|_{V_T^{1/2}}^2 + \frac{2}{\nu} \|f\|_{V_T^{-1/2}}^2. \]

The following Lemmas are crucial to the proof of Theorem 1.1.

Lemma 5.1 Let \(f \in L^2([0, T], V_T^{-1/2})\) and \(u_0 \in V_T^{1/2}\), there exists \(M_1(T)\) and \(M_2(T)\) independent of \(m\) such that the solution \(u_m\) to the Galerkin truncation (4.3) satisfies
\[ \|u_m\|_{L^2([0, T], V_T^{3/2})}^2 \leq M_1(T), \text{ for all } T \geq 0, \]
and
\[ \|u_m\|_{L^\infty([0, T], V_T^{1/2})}^2 \leq M_2(T), \text{ for all } t \geq 0. \]
\textbf{Proof.} We proceed by two steps: 

First step: From (5.6) and by using Grönwall’s inequality we conclude that

\begin{equation}
\|u^m\|^2_{L^\infty([0,T], V^{1/2})} \leq M_2(T),
\end{equation}

where $M_2(T)$ is given by

\[ M_2(T) = \left(\|u_0\|^2_{V^{1/2}} + \frac{2}{\nu} \|f\|^2_{L^2([0,T], V^{-1/2})}\right) \exp \frac{9}{2
\nu^3 \alpha} K_1(T)^{1/2} K_2(T)^{1/2}. \]

Second step: Integrate the original inequality (5.6) on $[0, T]$. We get

\begin{equation}
\|u^m(T, x)\|^2_{V^{1/2}} + \nu \int_0^T \|\nabla u^m(t, x)\|^2_{V^{1/2}} dt \leq \|u_0\|^2_{V^{1/2}} + \frac{2}{\nu} \|f\|^2_{L^2([0,T], V^{-1/2})} + \frac{9}{2 \nu^3 \alpha} M_2(T) K_1(T)^{1/2} K_2(T)^{1/2}. \]

We set

\[ M_1(T) = \|u_0\|^2_{V^{1/2}} + \frac{2}{\nu} \|f\|^2_{L^2([0,T], V^{-1/2})} + \frac{9}{2 \nu^3 \alpha} M_2(T) K_1(T)^{1/2} K_2(T)^{1/2}. \]

Thus $u^m \in L^\infty([0,T], V^{1/2}) \cap L^2([0,T], V^{3/2})$ for all $T > 0$.

We deduce from Lemma 2.2 that

\begin{equation}
\|u^m\|^2_{L^2([0,T], V^2)} \leq (1/\alpha) M_1(T), \text{ for all } T \geq 0
\end{equation}

and

\begin{equation}
\|u^m\|^2_{L^\infty([0,T], V^1)} \leq (1/\alpha) M_2(T), \text{ for all } T \geq 0
\end{equation}

Our next result provides an estimate on the time derivative of $u^m$.

\textbf{Lemma 5.2} Let $f \in L^2([0,T], V^{-1/2})$ and $u_0 \in V^{1/2}$, there exists $M_3(T)$ independent of $m$ such that the time derivative of the solution $u^m$ to the Galerkin truncation (4.2) satisfies

\begin{equation}
\left\| \frac{du^m}{dt} \right\|^2_{L^2([0,T], V^{-1/2})} \leq M_3(T), \text{ for all } T \geq 0,
\end{equation}

where

\begin{equation}
M_3(T) = \left(4(\nu^2 + C M_1(T)) M_2(T) + 2 \|f\|^2_{L^2([0,T], V^{-1/2})}\right).
\end{equation}

\textbf{Proof.} Taking the $V^{-1/2}$ norm of (4.2), one obtains:

\begin{equation}
\left\| \frac{du^m}{dt} \right\|_{V^{-1/2}} \leq \nu \|u^m\|_{V^{1/2}} + \|B(\bar{u}^m, u^m)\|_{V^{-1/2}} + \|f\|_{V^{-1/2}}
\end{equation}

where we note that

\[ \|B(\bar{u}^m, u^m)\|_{V^{-1/2}} = \sup \left\{ (B(\bar{u}^m, u^m), w); w \in V^{1/2}, \|w\|_{1/2} \leq 1 \right\}. \]

It remain to show that $\|B(\bar{u}^m, u^m)\|_{V^{-1/2}}$ is bounded.

Indeed, we have from Lemma 2.3 that
We are now ready to take limits along subsequences of \( u^m \) to show the existence of solutions that satisfy (2.12) viewed as a functional equality in \( V^{-rac{1}{2}} \).

From Lemmas 5.1 and 5.2 combined with Alaoglu compactness theorem we can extract a subsequence of \( u^m \) and \( \frac{du^m}{dt} \) that converge weakly to some \( u \) in \( L^2([0, T], V^\frac{3}{2}) \) and \( \frac{du}{dt} \) in \( L^2([0, T], V^{-\frac{1}{2}}) \) respectively.

Now the Lions-Magenes Lemma 2.6 implies that \( u \in C([0, T]; V^\frac{1}{2}) \) and this subsequence \( u^m \) converge to \( u \) in \( C([0, T]; H) \) and in particular we have \( u(0) = u_0 \). Thus it is obvious that \( \overline{u}^m \) converge to \( \overline{u} \) in \( C([0, T]; V^\frac{1}{2}) \).

Fix \( \epsilon \) such that \( 0 < \epsilon < 1/2 \). Since \( V^\frac{1}{2} \) is compactly embedded in \( V^{\frac{3}{2} - \epsilon} \) then Theorem 2.2 implies that there exists a further subsequence of \( u^m \), denoted by \( u^{m_i} \), that converges strongly in \( L^2([0, T]; V^{\frac{3}{2} - \epsilon}) \). Moreover,

\[
\int_0^T (Au^{m_i}, \phi) dt = \int_0^T (u^{m_i}, A\phi) dt \to \int_0^T (u, A\phi) dt = \int_0^T (Au, \phi) dt,
\]

for all \( \phi \in L^2([0, T], V^\frac{1}{2}) \). Thus, \( Au^{m_i} \) converge weakly to \( Au \) in \( L^2([0, T], V^{-\frac{1}{2}}) \) as \( i \to \infty \).

We finish by showing that the non linear term \( B(\overline{u}^m, u^{m_i}) \) converge weakly to \( B(\overline{u}, u) \) in \( L^2([0, T], V^{\frac{3}{2} - \epsilon}) \) as \( i \to \infty \). From the properties of the trilinear form we have

\[
\left| \int_0^T (B(\overline{u}^m, u^{m_i}), \phi) dt - \int_0^T (B(\overline{u}, u), \phi) dt \right| \leq \int_0^T \left| \left( B(\overline{u}^m - \overline{u}, u^{m_i}), \phi \right) \right| dt + \int_0^T \left| \left( B(\overline{u}, u^{m_i} - u), \phi \right) \right| dt,
\]

by using Hölder inequality combined with Sobolev injection Theorem and Poincare inequality we get

\[
\int_0^T \left| \left( B(\overline{u}^m - \overline{u}, u^{m_i}), \phi \right) \right| dt \leq \int_0^T \left\| \phi \right\|_{L^2([0, T], V^{\frac{3}{2}})} \left\| \overline{u}^m - \overline{u} \right\|_{L^\infty([0, T], V^\frac{3}{2})} \left\| u^{m_i} \right\|_{L^2([0, T], V^{\frac{3}{2}})} dt \leq \left\| \phi \right\|_{L^2([0, T], V^{\frac{3}{2}})} C_p \left\| \overline{u}^m - \overline{u} \right\|_{L^\infty([0, T], V^\frac{3}{2})} \left\| u^{m_i} \right\|_{L^2([0, T], V^{\frac{3}{2}})} dt.
\]
Thus $B(\overline{w}^m, w^m_i)$ converge weakly to $B(\overline{w}, u)$ in $L^2([0, T], V^{-1/2})$ as $i \to \infty$.

We have shown that $u$ satisfies (2.12) viewed as a functional equality in $V^{-1/2}$.

5.3 Uniqueness

The solution constructed above is unique. Next, we will show the continuous dependence of the solutions on the initial data and in particular the uniqueness.

Let $u$ and $v$ any two solutions of (1.4) on the interval $[0, T]$, with initial values $u_0 \in V^{1/2}$ and $v_0 \in V^{1/2}$, respectively. Let us denote by $w = u - v$ and $\overline{w} = \overline{u} - \overline{v}$. Then from (2.12) we can write the evolution equation for $w$ as an equality in $V^{-1/2}$ given by

\[ \frac{d}{dt} w + \nu A w + B(\overline{w}, u) + B(\overline{v}, w) = 0 \]

We take the inner product of (5.19) with $w$, applying the Lemma of Lions-Magenes 2.6 and by the bilinearity of $B$ we have

\[ \frac{1}{2} \frac{d}{dt} \|w\|_{V^{1/2}}^2 + \nu \|\nabla w\|_{V^{1/2}}^2 + (B(\overline{w}, u), |\nabla| w)_{L^2} + (B(\overline{v}, w), |\nabla| w)_{L^2} = 0 \]

The first non linear term is estimated by

\[ |(B(\overline{w}, u), |\nabla| w)_{L^2}| \leq C \|\overline{w} \nabla u\|_{V^{-1/2}} \|\nabla w\|_{V^{1/2}} \]

and by Young’s inequality, we obtain

\[ |(B(\overline{w}, u), |\nabla| w)_{L^2}| \leq C \|\overline{w}\|_{V^{1/2}}^2 \|u\|_{V^{3/2}}^2 + \frac{\nu}{4} \|\nabla w\|_{V^{1/2}}^2. \]

The second non linear term is estimated by

\[ |(B(\overline{v}, w), |\nabla| w)_{L^2}| \leq C \|\overline{v} \nabla w\|_{V^{-1/2}} \|\nabla w\|_{H} \]

and by Young’s inequality, we obtain

\[ |(B(\overline{v}, w), |\nabla| w)_{L^2}| \leq C \|v\|_{V^{3/2}}^{3/2} \|w\|_{V^{1/2}}^2 \]

From the above inequalities we get

\[ \frac{d}{dt} \|w\|_{V^{1/2}}^2 + \nu \|\nabla w\|_{V^{1/2}}^2 \leq \frac{C}{\nu} \|w\|_{V^{1/2}}^2 \left( \|v\|_{V^3}^1 + \|u\|_{V^{3/2}}^2 \right) \]

Using Grönwall’s inequality we conclude that

\[ \|w\|_{L^2([0, T], V^{1/2})}^2 \leq \|w_0\|_{V^{1/2}}^2 \exp \left( C \nu \left( \int_0^T \|v\|_{V^3}^1 dt + \int_0^T \|u\|_{V^{3/2}}^2 dt \right) \right). \]

Note that the above estimate does not depend on $\alpha$, thus we have not used the regularization effect to get the uniqueness in $V^{3/2}$ and this is not surprising because the solutions of the Navier-Stokes equations with initial data in $V^{3/2}$ are unique.

We have shown the continuous dependence of the solutions on the initial data in the $L^2([0, T], V^{1/2})$ norm. In particular, if $w_0 = 0$ then $w = 0$ and solutions are unique for all $t \in [0, T]$. Since $T > 0$ is arbitrary this solution may be uniquely extended for all time.
Remark 5.1 The pressure is absent in eqs. (2.12) once we can reconstruct it from $\bar{\pi}$ and $u$ (up to a constant), if necessary, see Lemma 4.4 and Remark 4.2. We take the divergence of (1.4). This yields the following equation for the pressure

$$(5.26) \quad -\Delta p = \nabla \cdot (\nabla \cdot \bar{u}u).$$

Where $\bar{u}u$ is the tensor $(\bar{u}^i u^j)_{1 \leq i, j \leq 3}$.

We recall that for any field $u$ with divergence equal to zero, we have

$$(\bar{u} \cdot \nabla)u = \nabla \cdot (\bar{u}u).$$

Since $(\bar{u} \cdot \nabla)u \in L^2([0, T], H^{-1/2})$. One concludes from the classical elliptic theory that $p$ is bounded in $L^2([0, T], H^{1/2}(T_3))$.

This finish the proof of Theorem 3.2.

6 Relation between NSE and Leray-$\alpha$: construction of suitable solutions

The regularized solution constructed above with $\theta \geq \frac{1}{4}$ is unique. So the solution is suitable. In this section we will construct a suitable weak solution to the Navier-Stokes equations by taking the limit when $\alpha$ tends to zero to the regularized solution.

We start by the following Lemma:

Lemma 6.1 Let $(u_\alpha, p_\alpha)$ be the unique solution of (1.4) with $\theta \geq \frac{1}{4}$ then $(u_\alpha, p_\alpha)$ verifies the following local inequality

$$(6.1) \quad 2\nu \int_0^T \int_{T_3} |\nabla u_\alpha|^2 \phi \ dx dt = \int_0^T \int_{T_3} |u_\alpha|^2 (\phi_t + \nu \Delta \phi) + (|u_\alpha|^2 \bar{u}_\alpha + 2p u_\alpha) \cdot \nabla \phi \ dx dt + 2 \int_0^T \int_{T_3} fu_\phi \ dx dt,$$

for all $T \in (0, +\infty]$ and for all non negative function $\phi \in C^\infty$ and supp $\phi \subset \subset T_3 \times (0, T)$.

Proof. We take $2\phi u_\alpha$ as test function in (1.4). We note that the condition $\theta \geq 1/4$ ensure that all the tems are well defined. In particular the integral

$$2 \int_0^T \int_{T_3} \bar{u}_\alpha \nabla u_\alpha \cdot u_\alpha \phi \ dx dt$$

is finite by using the fact that $\bar{u}_\alpha \nabla u_\alpha \in L^2([0, T]; H^{-1})$ and $2\phi u_\alpha \in L^2([0, T]; H^1)$.

An integration by part combined with $\phi(T, \cdot) = \phi(0, \cdot) = 0$ and the following identity

$$(6.2) \quad 2 \int_{T_3} \bar{u}_\alpha \nabla u_\alpha \cdot u_\alpha \phi \ dx = \int_{T_3} \bar{u}_\alpha |u_\alpha|^2 \cdot \nabla \phi \ dx$$

yield that $(u_\alpha, p_\alpha)$ verifies (6.1).

In order to take the limits $\alpha \to 0$ over (6.1), we need first to show that for all $u_\alpha \in L^p(0, T; L^p(T_3)^3)$, $2 \leq p < 10/3$, we have:

$$(6.3) \quad \bar{u}_\alpha \to u \quad \text{strongly in } L^p(0, T; L^p(T_3)^3) \text{ for all } 2 \leq p < 10/3.$$

This is the aim of the two following Lemmas.
Lemma 6.2 Let $\theta \in \mathbb{R}^+, 0 \leq \beta \leq 2\theta, s \in \mathbb{R}$ and assume that $u \in H^s$. Then $\mathbf{\mathcal{F}} \in H^{s+\beta}$ and

$$\| \mathbf{\mathcal{F}} \|^r_{H^{s+\beta}} \leq \frac{1}{\alpha r^2} \| u \|^r_H.$$  

**Proof.** When $u = \sum_{k \in I} \hat{u}_k \exp \{ ik \cdot x \}$, then

$$\mathbf{\mathcal{F}} = \sum_{k \in I} \frac{\hat{u}_k}{1 + \alpha^2 \beta^2 |k|^{2\theta}} \exp \{ ik \cdot x \}.$$  

Formula (6.5) easily yields the estimate

$$\| \mathbf{\mathcal{F}} \|^r_{H^{s+\beta}} \leq \left( \sup_{k \in I} \frac{|k|^{2\beta}}{(1 + \alpha^2 \beta^2 |k|^{2\theta})^2} \right)^{\frac{r}{2}} \| u \|^r_H,$$

where we have used the fact that

$$\sup_{k \in I} \left( \frac{\alpha^\beta |k|^{\beta}}{1 + \alpha^2 \beta^2 |k|^{2\theta}} \right)^2 \leq 1, \text{ for all } 0 \leq \beta \leq 2\theta.$$  

Lemma 6.3 Assume $\mathbf{\mathcal{F}}_\alpha$ belongs to the energy space of solutions of the Navier-Stokes equations, then

$$\mathbf{\mathcal{F}}_\alpha \rightarrow u \quad \text{strongly in } L^p(0, T; L^p(\mathbb{T}_3)^3 \text{ for all } 2 \leq p < \frac{10}{3}).$$

**Proof.** From the Sobolev injection $H^{\frac{3p}{2p-6}} \hookrightarrow L^p(\mathbb{T}_3)^3$, it is sufficient to show that

$$\int_0^T \| \mathbf{\mathcal{F}}_\alpha - u_\alpha \|_{H^{\frac{3p-6}{2p}}}^p \, dt \rightarrow 0, \text{ when } \alpha \rightarrow 0.$$  

From the relation between $\mathbf{\mathcal{F}}_\alpha$ and $u_\alpha$ we have

$$\| \mathbf{\mathcal{F}}_\alpha - u_\alpha \|_{H^{\frac{3p-6}{2p}}}^p \leq \alpha^{2p} \| u_\alpha \|_{H^{\frac{3p-6+2\theta}{2p}}}^p.$$  

Lemma 6.2 implies that

$$\int_0^T \| \mathbf{\mathcal{F}}_\alpha - u_\alpha \|_{H^{\frac{3p-6}{2p}}}^p \, dt \leq \alpha^{\frac{10-3p-6p}{2p}} \int_0^T \| u_\alpha \|_{H^{\frac{3p}{2p}}}^p \, dt.$$  

Recall that

$$\int_0^T \| u_\alpha \|_{H^{\frac{3p}{2p}}}^p \, dt < \infty \text{ for any } 2 \leq p \leq \infty.$$  

This yields the desired result for any $2 \leq p < 10/3$. In order to show that $(u_\alpha, p_\alpha)$ gives rise to a suitable solution of the Navier-Stokes equations, it is necessary to take the limit $\alpha \rightarrow 0$. We have the following theorem:
**Theorem 6.1** Let \((u_\alpha, p_\alpha)\) be the solution of (1.4), in the \(\alpha \to 0\) limit, \((u_\alpha, p_\alpha)\) tends to a suitable solution to the Navier-Stokes equations.

**Proof:** By a classical compactness argument \[13\], we deduce that \(u_\alpha\) approaches \(u\) strongly in \(L^2([0, T], H)\) for all \(T > 0\) when \(\alpha\) tends to zero, where \(u\) is a weak solution to the Navier-Stokes equations. It remains to show that \((u, p)\) verifies the following local energy inequality

\[
2\nu \int_0^T \int_{\mathbb{T}_3} |\nabla u|^2 \phi \, dx \, dt \leq \int_0^T \int_{\mathbb{T}_3} |u|^2 (\phi_t + \nu \Delta \phi) + (|u|^2 u + 2pu) \cdot \nabla \phi \, dx \, dt + 2 \int_0^T \int_{\mathbb{T}_3} f u \phi \, dx \, dt.
\]

To do so, we need to find estimates that are independent from \(\alpha\). Using the fact that \((u_\alpha, p_\alpha)\) belong to the energy space: \(L^\infty([0, T]; H) \cap L^2([0, T]; V) \cap L^\frac{10}{3}([0, T]; L^\frac{10}{3}(\mathbb{T}_3))^3\) and from the Aubin-Lions compactness Lemma (the same arguments as in section 4) we can find a not relabeled subsequence \((u_\alpha, p_\alpha)\) and \((u, p)\) such that when \(\alpha\) tends to zero we have:

\[
\begin{align*}
(6.14) & \quad u_\alpha \rightharpoonup^* u \quad \text{weakly}^* \text{ in } L^\infty([0, T]; H), \\
(6.15) & \quad u_\alpha \rightharpoonup u \quad \text{weakly in } L^2([0, T]; V) \cap L^\frac{10}{3}([0, T]; L^\frac{10}{3}(\mathbb{T}_3))^3, \\
(6.16) & \quad p_\alpha \rightharpoonup p \quad \text{weakly in } L^\frac{10}{3}([0, T]; L^\frac{10}{3}(\mathbb{T}_3)), \\
(6.17) & \quad u_\alpha \to u \quad \text{strongly in } L^p([0, T]; L^p(\mathbb{T}_3)^3) \text{ for all } 2 \leq p < \frac{10}{3}, \\
(6.18) & \quad \overline{u}_\alpha \to u \quad \text{strongly in } L^p([0, T]; L^p(\mathbb{T}_3)^3) \text{ for all } 2 \leq p < \frac{10}{3}, \\
(6.19) & \quad p_\alpha \rightharpoonup p \quad \text{strongly in } L^p([0, T]; L^p(\mathbb{T}_3)) \text{ for all } 1 < p < \frac{5}{3}.
\end{align*}
\]

These convergence results allow us to take the limit in (6.1) in order to obtain, by using the weak lower semicontinuity of the norm in \(L^2([0, T]; V)\)

\[
\liminf_{\alpha \to 0} \int_0^T \|\nabla u_\alpha\|_H^2 \, dt \geq \int_0^T \|\nabla u\|_H^2 \, dt,
\]

that \((u, p)\) verifies the local energy inequality (6.13).

**7 The deconvolution case**

**7.1 The modified deconvolution operator**

In this section, we introduce the modified deconvolution operator which interpolate the usual deconvolution operator introduced in \[23\].

Let \(\alpha \geq 0\), \(s \geq -1\), \(0 \leq \theta \leq 1\), \(u \in H^s\) and let \(\overline{u} \in H^{s+2\theta}\) be the unique solution to the equations

\[
\begin{align*}
(7.1) \quad & \quad \alpha^2 (-\Delta) \theta \overline{u} + \overline{u} = u, \\
(7.2) \quad & \quad \nabla \cdot u = \nabla \cdot \overline{u} = 0
\end{align*}
\]
We also shall denote by $\mathcal{G}$ the operator

$$\mathcal{G} : H^{s+2\theta} \to H^s, \quad w \mapsto \alpha^{2\theta} (-\Delta)^{\theta} w + w.$$  

(7.3)

Therefore, one has

$$\mathcal{P} = \mathcal{G}^{-1} u.$$  

(7.4)

Let us consider the operators

$$D_N = \sum_{n=0}^{N} (I - \mathcal{G}^{-1})^n.$$  

and

$$H_N(u) = D_N(\mathcal{P}).$$  

(7.5)

A straightforward calculation yields

$$H_N \left( \sum_{k \in \mathbb{Z}} u_k e^{ikx} \right) = \sum_{k \in \mathbb{Z}} \left( 1 - \left( \frac{\alpha^{2\theta} |k|^{2\theta}}{1 + \alpha^{2\theta} |k|^{2\theta}} \right)^{N+1} \right) u_k e^{ikx}. \quad \text{(7.6)}$$

One can prove the following (see in [23] [27] with $\theta = 1$):

- Assume $u \in H^s$. Then $H_N(u) \in H^{s+2\theta}$ and $\|H_N(u)\|_{H^{s+2\theta}} \leq C\langle N, \alpha \rangle\|u\|_{H^s}$, where $\langle N, \alpha \rangle$ blows up when $\alpha$ goes to zero and/or $N$ goes to infinity. This is due to the fact

$$\left( 1 - \left( \frac{\alpha^{2\theta} |k|^{2\theta}}{1 + \alpha^{2\theta} |k|^{2\theta}} \right)^{N+1} \right) \approx \frac{N + 1}{\alpha^{2\theta} |k|^{2\theta}} \quad \text{as} \quad |k|_{\infty} \to \infty.$$

- The operator $H_N$ maps continuously $H^s$ into $H^s$ and $\|H_N\|_{L(H^s, H^s)} = 1$.

- Assume $u \in H^s$. Then the sequence $(\mathcal{P}_n)_{n>0}$ converges strongly to $u$ in the space $H^s$.

- Let $u \in H^s$. Then the sequence $(H_N(u))_{N \in \mathbb{N}}$ converges strongly to $u$ in $H^s$ when $N$ goes to infinity.

### 7.2 The deconvolution model: well-posedness results

We consider here a family of equations interpolating between the Navier-Stokes equations [25] and the Leray-deconvolution model [23] with periodic boundary conditions.

$$\begin{align*}
\frac{\partial u}{\partial t} + H_N(u) \cdot \nabla u - \nu \Delta u + \nabla p &= f \quad \text{in} \ \mathbb{R}^+ \times \mathbb{T}_3, \\
\nabla \cdot u &= \nabla \cdot H_N(u) = 0, \quad \int_{\mathbb{T}_3} u = \int_{\mathbb{T}_3} H_N(u) = 0, \\
u(t,x + Le_j) &= u(t,x), \\
u_{t=0} &= u_0.
\end{align*}$$

(7.7)

Where $H_N(u)$ is an interpolating deconvolution operator introduced above.

When $N = 0$, we obtain the equations (1.4). Similarly, we obtain the same results of well-posedness for the interpolating model (7.7).
Theorem 7.1 For any $T > 0$, let $f \in L^2([0, T], V^{-1})$ and $u_0 \in H$. Assume that $\theta = 1/4$. Then there exists a unique solution $(u, p) := (u_{\alpha,N}, p_{\alpha,N})$ to eqs. (7.7) that satisfies $u \in C([0, T]; H) \cap L^2([0, T]; V)$ and $\frac{du}{dt} \in L^2([0, T]; V^{-1})$ and $p \in L^2([0, T], L^2(\mathbb{T}_3))$. Such that $u$ verifies

$$\left\langle \frac{du}{dt} + \nu Au + B(H_N(u), u) - f, \phi \right\rangle_{V^{-1}, V^1} = 0$$

for every $\phi \in V$ and almost every $t \in (0, T)$. Moreover, this solution depends continuously on the initial data $u_0$ in the $L^2$ norm.

And

Theorem 7.2 For any $T > 0$, let $f \in L^2([0, T], V^{-\frac{1}{2}})$ and $u_0 \in V^{\frac{1}{2}}$. Assume that $\theta = \frac{1}{4}$. Then there exists a unique solution $(u, p) := (u_{\alpha,N}, p_{\alpha,N})$ to eqs. (7.7) that satisfies $u \in C([0, T]; V^{\frac{1}{2}}) \cap L^2([0, T]; V^\frac{3}{2})$, $\frac{du}{dt} \in L^2([0, T], V^{-\frac{1}{2}})$ and $p \in L^2([0, T], H^{\frac{1}{2}}(\mathbb{T}_3))$. Such that $u$ satisfies

$$\left\langle \frac{du}{dt} + \nu Au + B(H_N(u), u) - f, \phi \right\rangle_{V^{-\frac{1}{2}}, V^{\frac{1}{2}}} = 0$$

for every $\phi \in V^{\frac{1}{2}}$ and almost every $t \in (0, T)$. Moreover, this solution depends continuously on the initial data $u_0$ in the $H^{\frac{1}{2}}$ norm.

Since the proof of Theorem 7.1 and 7.2 are similar to that of Theorem 3.1 and 3.2 we omit their proof here.

7.3 Convergence to a suitable weak solution to the NSE

By a similar argument to that in [23], it is possible to prove that up to a subsequence the solution for the deconvolution model (with $\theta = \frac{1}{4}$) converges when $\alpha \to 0$ or/and $N \to +\infty$ to a weak Leray solution to the Navier-Stokes equations. Next, we will show that the deconvolution regularization gives rise to a suitable weak solution to the Navier-Stokes equations. We need first

Lemma 7.1 Let $(u_{\alpha,N}, p_{\alpha,N})$ be the unique solution of (7.7) with $\theta \geq \frac{1}{4}$ then $(u_{\alpha,N}, p_{\alpha,N})$ verifies the following local inequality

$$2\nu \int_0^T \int_{\mathbb{T}_3} |\nabla u_{\alpha,N}|^2 \phi \, dx \, dt = \int_0^T \int_{\mathbb{T}_3} |u_{\alpha,N}|^2 (\phi_t + \nu \Delta \phi) \, dx \, dt$$

$$+ \int_0^T \int_{\mathbb{T}_3} (u_{\alpha,N})^2 H_N(u_{\alpha,N}) + 2p_{\alpha,N} u_{\alpha,N}) \cdot \nabla \phi \, dx \, dt + 2 \int_0^T \int_{\mathbb{T}_3} f u_{\alpha,N} \phi \, dx \, dt,$$

for all $T \in (0, +\infty]$ and for all non negative function $\phi \in C^\infty$ and supp $\phi \subset \subset \mathbb{T}_3 \times (0, T)$.

Proof. See Lemma 6.1

Then we have
Lemma 7.2 Assume $u_{\alpha,N}$ belong to the energy space of the solutions of the Navier-Stokes equations, then

\begin{equation}
H_N(u_{\alpha,N}) \to u \quad \text{strongly in } L^p(0,T;L^p(T_3)^3) \quad \text{for all } 2 \leq p < \frac{10}{3}.
\end{equation}

Proof. From the sobolev injection $H^{3p-6}_{\alpha,N} \hookrightarrow L^p(T_3)^3$, it is sufficient to show that

\begin{equation}
\int_0^T \|H_N(u_{\alpha,N}) - u_{\alpha,N}\|^p_{H^{3p-6}_{\alpha,N}} \, dt \to 0, \quad \text{when } \alpha \to 0 \text{ or/and } N \to \infty.
\end{equation}

From the relation between $H_N(u_{\alpha,N})$ and $u_{\alpha,N}$ we have

\begin{equation}
\|H_N(u_{\alpha,N}) - u_{\alpha,N}\|^p_{H^{3p-6}_{\alpha,N}} = \sum_{k \in I} \left( |k|^{\frac{3p-6}{p}} \left( \frac{\alpha^{2\theta}|k|^{2\theta}}{1 + \alpha^{2\theta}|k|^{2\theta}} \right)^{2(N+1)} \right)^{\frac{p}{2}}
\end{equation}

\begin{equation}
\leq \left( \sup_{k \in I} |k|^{\frac{3p-10}{p}} \left( \frac{\alpha^{2\theta}|k|^{2\theta}}{1 + \alpha^{2\theta}|k|^{2\theta}} \right)^{2(N+1)} \right)^{\frac{p}{2}} \|u\|^p_{H^2^p}.
\end{equation}

Using the fact that $I^m \hookrightarrow I^\infty$ for any $m \geq 1$ we get for $p < \frac{10}{3}$ that

\begin{equation}
\|H_N(u_{\alpha,N}) - u_{\alpha,N}\|^p_{H^{3p-6}_{\alpha,N}} \leq \left( \sum_{k \in I} \frac{1}{|k|^{\frac{10-3p}{p}} \left( \frac{\alpha^{2\theta}|k|^{2\theta}}{1 + \alpha^{2\theta}|k|^{2\theta}} \right)^{2m(N+1)}} \right)^{\frac{p}{2}} \|u\|^p_{H^2^p}.
\end{equation}

Now, we choose $m$ such that $m^{10-3p}/p > 3$, and recall that

\begin{equation}
\int_0^T \|u_{\alpha,N}\|^p_{H^2^p} \, dt < \infty \quad \text{for any } 2 \leq p \leq \infty,
\end{equation}

and

\begin{equation}
\frac{\alpha^{2\theta}|k|^{2\theta}}{1 + \alpha^{2\theta}|k|^{2\theta}} \to 0 \quad \text{or/and} \quad \sup_{k \in I} \left( \frac{\alpha^{2\theta}|k|^{2\theta}}{1 + \alpha^{2\theta}|k|^{2\theta}} \right)^{2m} \leq 1.
\end{equation}

Using the dominate convergence theorem, we conclude the desired result for any $2 \leq p < 10/3$.

Remark 7.1 Note that the above result holds true for $p = 10/3$.

The above $L^p$ convergence combined with the fact that $u_{\alpha,N}$ belong to the energy space of solutions of the Navier-Stokes equations and the Aubin-Lions compactness Lemma allow us to take the limit $\alpha \to 0$ or/and $N \to \infty$ in (1.3) and to deduce the following Theorem:

Theorem 7.3 Let $(u_{\alpha,N},p_{\alpha,N})$ be the solution of (1.7), Then when $\alpha \to 0$ and/or $N \to \infty$, $(u_{\alpha,N},p_{\alpha,N})$ tends to a suitable weak solution $(u,p)$ to the Navier-Stokes equations. The convergence to $u$ is weak in $L^2([0,T];V) \cap L^{10}([0,T];L^{10}(T_3)^3)$ and strong in $L^q([0,T]; L^q(T_3)^3)$ for all $2 \leq q < \frac{10}{3}$ and the convergence to $p$ is strong in $L^q([0,T]; L^q(T_3))$ for all $1 < q < \frac{5}{3}$ and weak in $L^{\frac{5}{3}}([0,T]; L^{\frac{5}{3}}(T_3))$. 

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8 The MHD Case

In this section, we consider a deconvolution-type regularization of the magneto-hydrodynamic (MHD) equations, given by

\begin{align}
(8.1a) & \quad \partial_t \mathbf{u} - \nu_1 \Delta \mathbf{u} + (H_N(\mathbf{u}) \cdot \nabla) \mathbf{u} - (H_N(\mathbf{B}) \cdot \nabla) \mathbf{B} + \nabla p + \frac{1}{2} \nabla |\mathbf{B}|^2 = 0, \\
(8.1b) & \quad \partial_t \mathbf{B} - \nu_2 \Delta \mathbf{B} + (H_N(\mathbf{u}) \cdot \nabla) \mathbf{B} - (H_N(\mathbf{B}) \cdot \nabla) \mathbf{u} = 0, \\
(8.1c) & \quad \nabla \cdot \mathbf{B} = \nabla \cdot H_N(\mathbf{B}) = \nabla \cdot \mathbf{u} = \nabla \cdot H_N(\mathbf{u}) = 0, \\
(8.1d) & \quad B(0) = B_0, \quad u(0) = u_0,
\end{align}

where the boundary conditions are taken to be periodic, and we also assume as before that \( \int_{T_3} \mathbf{u} \, dx = \int_{T_3} \mathbf{B} \, dx = 0. \)

Here, the unknowns are the fluid velocity field \( \mathbf{u}(t, \mathbf{x}) \), the fluid pressure \( p(t, \mathbf{x}) \), and the magnetic field \( \mathbf{B}(t, \mathbf{x}) \). Note that when \( \alpha = 0 \), we formally retrieve the MHD equations and the MHD-Leray-\( \alpha \) equations are obtained when \( N = 0 \) and \( \theta = 1 \).

Existence and uniqueness results for MHD equations are established by G. Duvaut and J.L. Lions in [11]. These results are completed by M. Sermange and R. Temam in [31]. They showed that the classical properties of the Navier-Stokes equations can be extended to the MHD system. The use of Leray-\( \alpha \) regularization to the MHD equations has received many studies see [28]. The idea to use the deconvolution operator from [23] to MHD equations is a new feature for the present work.

8.1 Existence, unicity and convergence results

First, we establish the global existence and uniqueness of solutions for the MHD-Deconvolution equations [51] for \( \theta = 1/4 \).

We have the following theorem:

**Theorem 8.1** For \( \theta = 1/4 \). Assume \( \mathbf{u}_0 \in H \) and \( \mathbf{B}_0 \in H \). Then for any \( T > 0 \), (8.1) has a unique regular solution \( (\mathbf{u}, \mathbf{B}, p) := (\mathbf{u}_{\alpha,N}, \mathbf{B}_{\alpha,N}, p_{\alpha,N}) \) such that, \( \mathbf{u}, \mathbf{B} \in C((0,T], H) \cap L^2([0,T], V^1) \), and \( p \in L^2([0,T], L^2(T_3)) \).

Furthermore the solution verifies

\begin{equation}
2\nu \int_0^T \int_{T_3} (|\nabla \mathbf{u}|^2 + |\nabla \mathbf{B}|^2) \phi \, dx \, dt \\
= \int_0^T \int_{T_3} |\mathbf{u}|^2 (\phi_t + \nu_1 \Delta \phi) + |\mathbf{B}|^2 (\phi_t + \nu_2 \Delta \phi) \, dx \, dt \\
+ \int_0^T \int_{T_3} \left( (|\mathbf{u}|^2 + |\mathbf{B}|^2) H_N(\mathbf{u}) + 2p \mathbf{u} \cdot \nabla \phi \right) \, dx \, dt \\
- 4 \int_0^T \int_{T_3} (\mathbf{u} \nabla \phi) H_N(\mathbf{B}) \cdot \nabla \phi \, dx \, dt
\end{equation}

for all \( T \in (0, +\infty) \) and for all non negative fonction \( \phi \in C^\infty \) and \( \text{supp} \ \phi \subset \subset T_3 \times (0,T) \).

**Proof.** We only sketch the proof since it is similar to the Navier-Stokes equations case. The proof is obtained by taking the inner product of (8.1a) with \( \mathbf{u} \), (8.1b) with \( \mathbf{B} \) and then adding them, the existence of a solution to Problem (8.1) can be derived thanks to the Galerkin method. Notice that \( \mathbf{u}, \mathbf{B} \) satisfy the following estimates

\begin{equation}
\frac{1}{2} \frac{d}{dt} \left( \|\mathbf{u}(t,x)\|_H^2 + \|\mathbf{B}(t,x)\|_H^2 \right) + \min(\nu_1, \nu_2) \left( \|\nabla \mathbf{u}(t,x)\|_H^2 + \|\nabla \mathbf{B}(t,x)\|_H^2 \right) \leq 0
\end{equation}

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The pressure $p$ is reconstructed from $u, H_N(u), B, H_N(B)$ (as we work with periodic boundary conditions) and its regularity results from the fact that $(H_N(u) \cdot \nabla) u, (H_N(B) \cdot \nabla) B \in L^2([0,T], H^{-1})$.

It remains to prove the uniqueness. Let $(u_1, B_1, p_1)$ and $(u_2, B_2, p_2)$, be two solutions, $\delta u = u_2 - u_1, \delta B = B_2 - B_1, \delta p = p_2 - p_1$. Then one has

(8.4a)  
$$\partial_t \delta u + (H_N(u_1) \nabla) \delta u - (H_N(B_1) \nabla) \delta B - \nu_1 \Delta \delta u + \nabla \delta p = -(H_N(\delta u) \nabla) u_2 + (H_N(\delta B) \nabla) B_2,$$

(8.4b)  
$$\partial_t \delta B + (H_N(u_1) \nabla) \delta B - (H_N(B_1) \nabla) \delta u - \nu_2 \Delta \delta B = (H_N(\delta B) \nabla) u_2 - (H_N(\delta u) \nabla) B_2,$$

and $\delta u = 0, \delta B = 0$ at initial time. One can take $\delta u \in L^\infty((0,T], H) \cap L^2([0,T], V)$ as test in (8.4a) and $\delta B \in L^\infty((0,T], H) \cap L^2([0,T], V)$ as test in (8.4b). Since $H_N(u_1)$ is divergence-free, one has

(8.5a)  
$$\int_0^T \int_{\mathbb{T}_3} (H_N(u_1) \nabla) \delta u. \delta u = 0,$$

(8.5b)  
$$\int_0^T \int_{\mathbb{T}_3} (H_N(u_1) \nabla) \delta B. \delta B = 0.$$

Therefore,

(8.6)  
$$\frac{d}{2dt} \int_{\mathbb{T}_3} |\delta u|^2 + \nu_1 \int_{\mathbb{T}_3} |\nabla \delta u|^2 - \int_{\mathbb{T}_3} (H_N(B_1) \nabla) \delta B. \delta u = - \int_{\mathbb{T}_3} (H_N(\delta u) \nabla) u_2. \delta u + \int_{\mathbb{T}_3} (H_N(\delta B) \nabla) B_2. \delta u,$$

and

(8.7)  
$$\frac{d}{2dt} \int_{\mathbb{T}_3} |\delta B|^2 + \nu_2 \int_{\mathbb{T}_3} |\nabla \delta B|^2 - \int_{\mathbb{T}_3} (H_N(B_1) \nabla) \delta u. \delta B = \int_{\mathbb{T}_3} (H_N(\delta B) \nabla) u_2. \delta B - \int_{\mathbb{T}_3} (H_N(\delta u) \nabla) B_2. \delta B.$$

One has by a integration by parts,

(8.8)  
$$- \int_{\mathbb{T}_3} (H_N(B_1) \nabla) \delta B. \delta u = \int_{\mathbb{T}_3} (H_N(B_1) \nabla) \delta u. \delta B.$$

One has by adding (8.6), (8.7) and using (8.8)

(8.9)  
$$\frac{d}{2dt} \int_{\mathbb{T}_3} |\delta u|^2 + \frac{d}{2dt} \int_{\mathbb{T}_3} |\delta B|^2 + \nu_1 \int_{\mathbb{T}_3} |\nabla \delta u|^2 + \nu_2 \int_{\mathbb{T}_3} |\nabla \delta B|^2
$$

$$= - \int_{\mathbb{T}_3} (H_N(\delta u) \nabla) u_2. \delta u + \int_{\mathbb{T}_3} (H_N(\delta B) \nabla) B_2. \delta u + \int_{\mathbb{T}_3} (H_N(\delta B) \nabla) u_2. \delta B - \int_{\mathbb{T}_3} (H_N(\delta u) \nabla) B_2. \delta B.$$
One has by integration by parts,

\[ (8.10a) \quad - \int_{T_3} (H_N(\delta u) \nabla) u_2, \delta u = \int_{T_3} H_N(\delta u) \otimes u_2 : \nabla \delta u, \]

\[ (8.10b) \quad \int_{T_4} (H_N(\delta B) \nabla) B_2, \delta u = - \int_{T_3} H_N(\delta B) \otimes B_2 : \nabla \delta u, \]

\[ (8.10c) \quad \int_{T_3} (H_N(\delta B) \nabla) u_2, \delta B = - \int_{T_3} H_N(\delta B) \otimes u_2 : \nabla \delta B, \]

\[ (8.10d) \quad - \int_{T_3} (H_N(\delta u) \nabla) B_2, \delta B = \int_{T_3} H_N(\delta u) \otimes B_2 : \nabla \delta B. \]

By Young's inequality,

\[ (8.11a) \quad | \int_{T_3} (H_N(\delta u) \nabla) u_2, \delta u | \leq \frac{\nu_1}{4} \int_{T_3} |\nabla \delta u|^2 + \frac{1}{\nu_1} \int_{T_3} |H_N(\delta u)|^2 |u_2|^2, \]

\[ (8.11b) \quad | \int_{T_3} (H_N(\delta B) \nabla) B_2, \delta u | \leq \frac{\nu_1}{4} \int_{T_3} |\nabla \delta u|^2 + \frac{1}{\nu_1} \int_{T_3} |H_N(\delta B)|^2 |B_2|^2, \]

\[ (8.11c) \quad | \int_{T_3} (H_N(\delta B) \nabla) u_2, \delta B | \leq \frac{\nu_2}{4} \int_{T_3} |\nabla \delta B|^2 + \frac{1}{\nu_2} \int_{T_3} |H_N(\delta B)|^2 |u_2|^2, \]

\[ (8.11d) \quad | \int_{T_3} (H_N(\delta u) \nabla) B_2, \delta B | \leq \frac{\nu_2}{4} \int_{T_3} |\nabla \delta B|^2 + \frac{1}{\nu_2} \int_{T_3} |H_N(\delta u)|^2 |B_2|^2. \]

By Hölder inequality combined with Sobolev injection

\[ (8.12a) \quad \frac{1}{\nu_1} \int_{T_3} |H_N(\delta u)|^2 |u_2|^2 \leq \frac{1}{\nu_1} \|H_N(\delta u)\|^2_{L^2} \|u_2\|^2_{L^6}, \]

\[ \leq \frac{1}{\nu_1} \|H_N(\delta u)\|^2_{V^\frac{1}{2}} \|u_2\|^2_{V}, \]

\[ (8.12b) \quad \frac{1}{\nu_1} \int_{T_4} |H_N(\delta B)|^2 |B_2|^2 \leq \frac{1}{\nu_1} \|H_N(\delta B)\|^2_{L^2} \|B_2\|^2_{L^6}, \]

\[ \leq \frac{1}{\nu_1} \|H_N(\delta B)\|^2_{V^\frac{1}{2}} \|B_2\|^2_{V}, \]

\[ (8.12c) \quad \frac{1}{\nu_2} \int_{T_3} |H_N(\delta B)|^2 |u_2|^2 \leq \frac{1}{\nu_2} \|H_N(\delta B)\|^2_{L^2} \|u_2\|^2_{L^6}, \]

\[ \leq \frac{1}{\nu_2} \|H_N(\delta B)\|^2_{V^\frac{1}{2}} \|u_2\|^2_{V}, \]

\[ (8.12d) \quad \frac{1}{\nu_2} \int_{T_3} |H_N(\delta u)|^2 |B_2|^2 \leq \frac{1}{\nu_2} \|H_N(\delta u)\|^2_{L^2} \|B_2\|^2_{L^6}, \]

\[ \leq \frac{1}{\nu_2} \|H_N(\delta u)\|^2_{V^\frac{1}{2}} \|B_2\|^2_{V}. \]

Hence,

\[ (8.13) \quad \frac{d}{2 dt} \int_{T_3} |\delta u|^2 + \frac{d}{2 dt} \int_{T_3} |\delta B|^2 + \frac{\nu_1}{2} \int_{T_3} |\nabla \delta u|^2 + \frac{\nu_2}{2} \int_{T_3} |\nabla \delta B|^2 \]

\[ \leq \frac{1}{\min (\nu_1, \nu_2) C(\alpha, N)} \left( \|H_N(\delta u)\|^2_{V^\frac{1}{2}} + \|H_N(\delta B)\|^2_{V^\frac{1}{2}} \right) \left( \|u_2\|^2_{V} + \|B_2\|^2_{V} \right) \]

\[ \leq \frac{1}{\min (\nu_1, \nu_2)} \left( \|u_2\|^2_{H} + \|B_2\|^2_{H} \right) \left( \|\delta u\|^2_{H} + \|\delta B\|^2_{H} \right). \]

Therefore,

\[ \frac{d}{2 dt} \left( \|\delta u\|^2_{H} + \|\delta B\|^2_{H} \right) \leq C(t) \left( \|\delta u\|^2_{H} + \|\delta B\|^2_{H} \right), \]
where \( C(t) = \frac{C(\alpha, N)}{\min (\nu_1, \nu_2)} \left( \|u_2\|_{V}^2 + \|B_2\|_{V}^2 \right) \in L^1([0, T]). \) We conclude that \( \delta u = \delta B = 0 \) thanks to Grönwall’s Lemma.

In order to deduce the local energy equality (8.2), we multiply the equation (8.1a) with \( 2u\phi \), the equation (8.1b) with \( 2B\phi \), for all \( T \in (0, +\infty) \) and for all non-negative function \( \phi \in C^\infty \) with \( \text{supp} \phi \subset \subset T_3 \times (0, T) \), and then adding them. The rest can be done in exactly the way as in Lemma 6.1 so we omit the details.

With smooth initial data we may also prove the following theorem

**Theorem 8.2** For \( \theta = 1/4 \). Assume \( u_0 \in V^{\frac{1}{2}} \) and \( B_0 \in V^{\frac{1}{2}} \). Then for any \( T > 0 \), (8.1) has a unique regular solution \((u, B, p) := (u_{\alpha, N}, B_{\alpha, N}, p_{\alpha, N}) \), such that \( u, B \in C([0, T], V^{\frac{1}{2}}) \cap L^2([0, T], V^{\frac{3}{2}}) \), and \( p \in L^2([0, T], H^{\frac{2}{3}}(T_3)). \)

**Proof.** We only sketch the proof since is similar to the one for Navier-Stokes equations. The proof is obtained by taking the inner product of (8.1a) with \( u \) and then adding them. The existence of a solution to Problem (8.1) can be derived thanks to the Galerkin method. Notice that \( u, B \) satisfy the following estimates

\[
\frac{1}{2} \frac{d}{dt} \left( \|u\|_{V^{\frac{1}{2}}}^2 + \|B\|_{V^{\frac{1}{2}}}^2 \right) + \min (\nu_1, \nu_2) \left( \|\nabla u\|_{V^{\frac{1}{2}}}^2 + \|\nabla B\|_{V^{\frac{1}{2}}}^2 \right) \\
\leq \int_{T_3} H_N(u)\nabla u|\nabla|u| \, dx + \int_{T_3} H_N(B)\nabla B|\nabla|u| \, dx \\
+ \int_{T_3} H_N(u)\nabla B^{\alpha, N}|\nabla|B^{\alpha, N}| \, dx + \int_{T_3} H_N(B)\nabla u|\nabla|B| \, dx.
\] (8.14)

The first non linear term is estimated by

\[
\left| \int_{T_3} H_N(u)\nabla u|\nabla|u| \, dx \right| \leq \|H_N(u)\|_{V^{\frac{1}{2}}} \|\nabla u\|_{V^{\frac{12}}}
\leq \|H_N(u)\|_{L^2} \|\nabla u\|_{V^{\frac{5}{2}}}
\leq \|H_N(u)\|_{L^{\infty}} \|\nabla u\|_{L^{2}} \|\nabla u\|_{V^{\frac{5}{2}}}
\leq \|H_N(u)\|_{V} \|\nabla u\|_{V^{\frac{3}{2}}} \|\nabla u\|_{V^{\frac{5}{2}}}
\leq \|H_N(u)\|_{V} \|\nabla u\|_{V^{\frac{3}{2}}} \|\nabla u\|_{V^{\frac{5}{2}}}.
\] (8.15)

Where we have used the Sobolev embedding in Theorem 2.1 together with Hölder estimate and the interpolation inequality between \( V^{\frac{1}{2}} \) and \( V^{\frac{3}{2}} \).

Similarly, we can estimate the third term by

\[
\left| \int_{T_3} H_N(u)\nabla B|\nabla|B| \, dx \right| \leq \|H_N(u)\|_{V^{\frac{1}{2}}} \|B\|_{V^{\frac{12}}} \|\nabla B\|_{V^{\frac{5}{2}}}
\] (8.16)

The second non-linear term is estimated by

\[
\left| \int_{T_3} H_N(B)\nabla B|\nabla|u| \, dx \right| \leq \|H_N(B)\|_{V^{\frac{1}{2}}} \|\nabla u\|_{V^{\frac{5}{2}}}
\leq \|H_N(B)\|_{L^2} \|\nabla u\|_{V^{\frac{3}{2}}}
\leq \|H_N(B)\|_{L^{\infty}} \|\nabla u\|_{L^{2}} \|\nabla u\|_{V^{\frac{3}{2}}}
\leq \|H_N(B)\|_{V} \|\nabla u\|_{V^{\frac{1}{2}}} \|\nabla u\|_{V^{\frac{3}{2}}}
\leq C(\alpha, N) \|B\|_{V^{\frac{1}{2}}} \|B\|_{V} \|\nabla u\|_{V^{\frac{1}{2}}}.
\] (8.17)
By the same way we obtain

\begin{equation}
\left| \int_{T_3} H_N(B) \nabla u \nabla |B| \, dx \right| \leq C(\alpha, N) \|B\|_{V^\frac{1}{2}} \|u\|_{V^\frac{1}{2}} \nabla u \|_{V^\frac{1}{2}}.
\end{equation}

Therefore,

\begin{equation}
\frac{1}{2} \frac{d}{dt} \left( \|u\|^2_{V^\frac{1}{2}} + \|B\|^2_{V^\frac{1}{2}} \right) + \min(\nu_1, \nu_2) \left( \|\nabla u\|^2_{V^\frac{1}{2}} + \|\nabla B\|^2_{V^\frac{1}{2}} \right)
\end{equation}

\begin{equation}
\leq \|H_N(u)\|_{V^\frac{1}{2}} \|u\|^2_{V^\frac{1}{2}} \nabla u \|_{V^\frac{1}{2}} + \|H_N(u)\|_{V^\frac{1}{2}} \|B\|^2_{V^\frac{1}{2}} + C(\alpha, N) \|B\|_{V^\frac{1}{2}} \|u\|_{V^\frac{1}{2}} \nabla u \|_{V^\frac{1}{2}}.
\end{equation}

By Young inequality,

\begin{equation}
\frac{d}{dt} \left( \|u\|^2_{V^\frac{1}{2}} + \|B\|^2_{V^\frac{1}{2}} \right) + \min(\nu_1, \nu_2) \left( \|\nabla u\|^2_{V^\frac{1}{2}} + \|\nabla B\|^2_{V^\frac{1}{2}} \right)
\end{equation}

\begin{equation}
\leq \frac{c}{\min(\nu_1, \nu_2)} \|H_N(u)\|^4_{V^\frac{1}{2}} + \frac{c}{\min(\nu_1, \nu_2)} \|H_N(u)\|^4_{V^\frac{1}{2}} \|B\|^2_{V^\frac{1}{2}} + C(\alpha, N) \|B\|^2_{V^\frac{1}{2}} \|u\|^2_{V^\frac{1}{2}}
\end{equation}

\begin{equation}
\leq C(t) \left( \|u\|^2_{V^\frac{1}{2}} + \|B\|^2_{V^\frac{1}{2}} \right)
\end{equation}

where \( C(t) = \left( \frac{c}{\min(\nu_1, \nu_2)} \|H_N(u)\|^4_{V^\frac{1}{2}} + \frac{C(\alpha, N)}{\min(\nu_1, \nu_2)} \|B\|^2_{V^\frac{1}{2}} + \frac{C(\alpha, N)}{\min(\nu_1, \nu_2)} \|u\|^2_{V^\frac{1}{2}} \right) \in L^1([0, T]).

We conclude that \( u, B \in L^\infty((0, T], V^\frac{1}{2}) \cap L^2([0, T], V^\frac{3}{2}) \) thanks to Grönwall’s Lemma.

The pressure \( p \) is reconstructed from \( u, H_N(u), B, H_N(B) \) (as we are working with periodic boundary conditions) and its regularity results from the fact that \((H_N(u) \cdot \nabla)u, (H_N(B) \cdot \nabla)B \in L^2([0, T], H^{-\frac{1}{2}})\).

It remains to prove the uniqueness. Let \((u_1, B_1, p_1)\) and \((u_2, B_2, p_2)\) be two solutions, \( \delta u = u_2 - u_1, \delta B = B_2 - B_1, \delta p = p_2 - p_1 \). Then one has

\begin{equation}
\partial_t \delta u + (H_N(u_1) \nabla) \delta u - (H_N(B_1) \nabla) \delta B - \nu \Delta \delta u + \nabla \delta p = -(H_N(\delta u) \nabla)u_2 + (H_N(\delta B) \nabla)B_2,
\end{equation}

\begin{equation}
\partial_t \delta B + (H_N(u_1) \nabla) \delta B - (H_N(B_1) \nabla) \delta u - \nu \Delta \delta B
= (H_N(\delta B) \nabla)u_2 - (H_N(\delta u) \nabla)B_2,
\end{equation}

and \( \delta u = 0, \delta B = 0 \) at initial time. One can take \( |\nabla| \delta u \in L^\infty((0, T], V^{-\frac{1}{2}}) \cap L^2([0, T], V^\frac{1}{2}) \) as test in [8.22] and \( |\nabla| \delta B \in L^\infty((0, T], V^{-\frac{1}{2}}) \cap L^2([0, T], V^\frac{1}{2}) \) as test in [8.23].
Once we obtain by a similar way as in Theorem 8.1 and Theorem 3.2 that
\[
\frac{d}{2dt} \left( \|\delta u\|^2_{V^s} + \|\delta B\|^2_{V^s} \right) \leq C(t) \left( \|\delta u\|^2_{V^s} + \|\delta B\|^2_{V^s} \right),
\]
where \( C(t) \in L^1([0, T]) \). We conclude that \( \delta u = \delta B = 0 \) thanks to Grönwall’s Lemma.

Next, we will deduce that the deconvolution regularization give rise to a suitable weak solution to the MHD equations.

**Theorem 8.3** Let \((u_{\alpha,N}, B_{\alpha,N}, p_{\alpha,N})\) be the solution of (7.7). Then when \( \alpha \to 0 \) and/or \( N \to \infty \), \((u_{\alpha,N}, B_{\alpha,N}, p_{\alpha,N})\) tends to a weak solution \((u, B, p)\) to the MHD equations. The convergence of \( u \) and the convergence to \( B \) are weak in \( L^2([0, T]; V) \cap L_{\text{loc}}^2([0, T]; L^3(T_3)^3) \) and strong in \( L^q([0, T]; L^3(T_3)^3) \) for all \( 2 \leq q < \frac{10}{3} \) and the convergence to \( p \) is strong in \( L^q([0, T]; L^3(T_3)^3) \) for all \( 1 < q < \frac{5}{3} \) and weak in \( L^2([0, T]; L^3(T_3)^3) \). Furthermore the solution verifies in addition

\[
2\nu \int_0^T \int_{T_3} \left( |\nabla u|^2 + |\nabla B|^2 \right) \phi \, dx \, dt \leq \int_0^T \int_{T_3} g(u) (\phi_t + \nu_1 \Delta \phi) + |B|^2 (\phi_t + \nu_2 \Delta \phi) \, dx \, dt + \int_0^T \left( (|u|^2 + |B|^2) u + 2pu \right) \cdot \nabla \phi \, dx \, dt - 4 \int_0^T \int_{T_3} (uB) \cdot \nabla \phi \, dx \, dt
\]

for all \( T \in (0, +\infty) \) and for all non negative function \( \phi \in C^\infty \) and \( \text{supp} \phi \subset T_3 \times (0, T) \).

**Proof.** As in Lemma 3.2 we can show that

\[
H_N(u_{\alpha,N}) \to u \quad \text{strongly in } L^p(0, T; L^3(T_3)^3) \quad \text{for all } 2 \leq p < \frac{10}{3},
\]

\[
H_N(B_{\alpha,N}) \to B \quad \text{strongly in } L^p(0, T; L^3(T_3)^3) \quad \text{for all } 2 \leq p < \frac{10}{3}.
\]

The above \( L^p \) convergence combined with the fact that \( u_{\alpha,N} \) and \( B_{\alpha,N} \) belong to the energy space of the solutions of the Navier-Stokes equations and the Aubin-Lions compactness Lemma allow us to take the limit \( \alpha \to 0 \) or/and \( N \to \infty \) in (8.2). The rest can be done in exactly way as in [23], so we omit the details.

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