Resonant growth of stellar oscillations by incident gravitational waves

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Abstract

Stellar oscillation under the combined influences of incident gravitational wave and radiation loss is studied in a simple toy model. The star is approximated as a uniform density ellipsoid in the Newtonian gravity including radiation damping through quadrupole formula. The time evolution of the oscillation is significantly controlled by the incident wave amplitude $h$, frequency $\nu$ and damping time $\tau$. If a combination $h\nu\tau$ exceeds a threshold value, which depends on the resonance mode, resonant growth is realized.

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1. Introduction

Stellar oscillation coupled to gravitational waves is one of the important issues for the gravitational wave observatories such as LIGO, VIRGO, GEO600 and TAMA. The current status of these detectors is given elsewhere (e.g., [1]). Detection of gravitational radiation from compact stars would provide valuable information about their interior. The frequency and damping time are uniquely affected by the stellar structure. Gravitational wave asteroseismology in the future is also discussed (e.g., [2, 3]). Various oscillation modes in neutron stars have been studied theoretically so far, but little is known of the excitation and damping processes. The most likely event that may excite the oscillations is the birth of the neutron stars after supernova. Some oscillation modes may also be induced by instabilities. An accurate description of these processes requires numerical solution of the coupled system of hydrodynamics and the Einstein equations. Recently, the numerical simulations have been developing in several approaches, e.g., 2D Newtonian simulation [4], 3D smooth particle code simulation [5], Newtonian MHD simulation [6], GR simulation [7] and the references therein.

The approximate approach is also helpful to get some insight into the physical processes. For example, excitation of neutron star oscillations in a close binary is studied by using the following approximation [8–11]. One of the two stars is an extended body, and the other is
approximated by a pointlike particle. The stellar perturbations in general relativity are solved for the extended body, while the pointlike mass moves on a geodesic of the spacetime around the star. The oscillation mode can be excited by the particle, whose orbit is close enough to satisfy the resonance condition between normal mode frequency of the star and orbital Kepler frequency. Detailed discussion concerning excitation of neutron star oscillations in this approach may be found in [11] and references therein. This is one of the examples showing that oscillations are excited by an external almost periodic disturbance. The resonant growing mode is, in general, significantly affected by viscous damping, which may come from physical origins or numerical truncation errors. It is therefore better to examine the resonant property in a simplified model. In this paper, we consider an idealized situation to study the resonant behaviour. Incident gravitational waves are regarded as an external periodic disturbance. The purpose of our study is the following. What is the resonance condition? How does the amplitude grow? The external disturbance is characterized by the frequency, duration and amplitude of the waves. These quantities are closely related to the resonance condition and the amplification of growing mode. The resonance condition is a relation between intrinsic and external frequencies. The amplification factor depends on the growth rate of the mode and the duration of the disturbances.

This paper is organized as follows. In section 2, we briefly summarize the resonance appeared in a harmonic oscillator, which is helpful to examine the stellar model. In section 3, we consider the stellar oscillation with ellipsoidal model under Newtonian gravity. The gravitational radiation loss is included by the quadrupole formula. We show the numerical results in section 4. Section 5 is devoted to discussion.

2. Resonance in a harmonic oscillator

We here briefly consider the resonance in a harmonic oscillator, which is a useful model to understand the more realistic case below. There are two kinds of resonances (e.g., [12]). One type occurs when the frequency \( \omega / (\Omega/2\pi) \) of external perturbation matches the intrinsic frequency \( \omega / (2\pi) \), i.e., \( \Omega = \omega \). The relevant equation is given by

\[
\ddot{X} + \omega^2 X = f \cos(\Omega t).
\]

(1)

In this case, the amplitude \( X \) increases linearly with time, \( X \propto (f/\Omega) t \sin(\Omega t) \).

The other type of resonance is parametric resonance, which occurs for time-dependent frequency. One example known as Mathieu’s equation is

\[
\ddot{X} + \omega^2 (1 + b \cos(\Omega t)) X = 0.
\]

(2)

The exponential growth for instability appears when \( \omega / \Omega = n/2 \) (n = 1, 2, 3, ...). There is an allowed range of frequency for each resonance condition, but the width of resonance range decreases rapidly with increasing n. The matching for large n is hardly satisfied. The strongest instability occurs for the fundamental one \( n = 1 \). Therefore the parametric resonance with \( n = 1 \) is very important. The growth rate s in \( X \propto \exp(st) \) for the fundamental mode can be calculated for small b as

\[
s = -\frac{1}{4} \left[ -\left(\Omega - 2\omega\right)^2 + \frac{1}{4} b^2 \omega^2 \right]^{1/2} \sim \frac{1}{4} b \omega
\]

(3)

for the unstable region

\[-\frac{1}{4} b \omega < \Omega - 2\omega < \frac{1}{2} b \omega.\]

(4)

In the presence of frictional damping, the amplitude decreases with time as \( \exp(-\gamma t) \). The damping counteracts the parametric resonance, so that the condition is more constrained. The growing solution of the resonance is possible only when b exceeds a threshold \( 4\gamma / \omega \).
3. Stellar pulsation

Now we consider the pulsation of the star based on Newtonian gravity. The equation of motion for the pulsation driven by external acceleration $\vec{g}$ is

$$\frac{d\vec{v}}{dt} = -\frac{1}{\rho} \vec{v} p - \vec{\nabla} \phi + \vec{g}. \quad (5)$$

As an external perturbation, we consider a plane parallel wave with amplitude $h$ and angular frequency $\Omega$, propagating along the $z$-axis. The tidal acceleration by the incident gravitational wave is expressed as [13]

$$\vec{g} = \left[ -\frac{1}{2} \frac{h}{\Omega^2} x \cos \Omega (t - z), \; \frac{1}{2} \frac{h}{\Omega^2} y \cos \Omega (t - z), \; 0 \right]. \quad (6)$$

We here consider + mode only. Another polarization mode $\times$ is given simply by $\pi/4$ rotation about the $z$-axis, and may be ignored without loss of generality. Deceleration due to quadrupole radiation loss is given by [13]

$$\vec{g} = -\vec{\nabla} G \frac{5}{5c^5} I_{ij} x^i x^j, \quad (7)$$

where $I_{ij}$ is the five time derivative of reduced quadrupole moment. We incorporate both effects by equations (6) and (7).

We are interested in the global motion of a star, and therefore approximate it by the ellipsoidal model. In the approximation, the dynamical motion is limited to uniform expansion and compression along three axes. Thus, the fluid motion is described by three functions of time $a_i(t)$ ($i = 1, 2, 3$). The ellipsoidal approximation is often used as a simplified model to extract dynamical features in more realistic cases (see, e.g., [14–18]). Assuming incompressible fluid with uniform density $\rho$, the dynamics can be determined by

$$\ddot{a}_1 = -2\pi G \rho a_1 A_1 + \frac{K}{a_1} - \frac{1}{2} \frac{h}{\Omega^2} a_1 \cos \Omega t = \frac{2G}{5c^5} I_{ij} a_1, \quad (8)$$

$$\ddot{a}_2 = -2\pi G \rho a_2 A_2 + \frac{K}{a_2} + \frac{1}{2} \frac{h}{\Omega^2} a_2 \cos \Omega t = -\frac{2G}{5c^5} I_{ij} a_2, \quad (9)$$

$$\ddot{a}_3 = -2\pi G \rho a_3 A_3 + \frac{K}{a_3} - \frac{2G}{5c^5} I_{ij} a_3, \quad (10)$$

where the function $A_i$ depends on the shape of the ellipsoid, and is described in [19]. The term $K$ associated with pressure is algebraically determined from the condition $d(\ln(a_1 a_2 a_3))/dt = 0$.

The system of equations (8)–(10) depends on three dimensionless parameters, $Q = \omega \tau / 2\pi = \nu \tau, \omega / \Omega$ and $h$, where $\nu (= \omega / 2\pi)$ is the intrinsic frequency of the stellar oscillation, and $\tau$ is the damping time due to gravitational radiation. The amplitude becomes $e^{-1}$ after $Q$ time oscillations. These values are physically related to the mass $M$ and radius $R$ of the star as

$$\omega = \left( \frac{4GM}{5R^3} \right)^{1/2}, \quad \tau^{-1} = \frac{2GMR^2 \omega^4}{25c^5}. \quad (11)$$

In order to clarify the dynamics of this system, we assume small deviation from spherically symmetric equilibrium ($a_i = 1$), and linearize equations (8)–(10). We neglect the radiation loss for simplicity and limit the oscillation to the toroidal motion in the $x$–$y$ plane. The dynamics can be expressed by a single function $X$ as

$$\ddot{X} + \omega^2 \left( 1 + \frac{h}{2} \left( \frac{\Omega}{\omega} \right)^2 \cos \Omega t \right) X = -\frac{1}{2} \frac{h}{\Omega^2} \cos \Omega t, \quad (12)$$
where $X$ represents the deviations $\delta a_i (= a_i - 1)$ from the equilibrium state:

$$\delta a_1 = -\delta a_2 = X, \quad \delta a_3 = 0.$$  \hfill (13)

Equation (12) is the Mathieu’s one with periodic source term. As considered in section 2, it is clear that two kinds of resonances are possible in equation (12), and hence in the system of equations (8)–(10). In the astrophysical situation, periodic external perturbation may be expected, e.g., in a close binary system, where the standard resonance due to the periodic driving force is relevant. On the other hand, we do not definitely know the situation in which parametric resonance occurs, but the relevant elements may be involved in a complicated system like the hydrodynamical core collapse.

4. Numerical calculation

4.1. Off-resonant case

Before examining the resonant properties of the system (8)–(10), we first of all show the behaviour of the non-resonant case. This is necessary for checking and comparing models. As for initial conditions of numerical calculations, we adopt a static state, i.e., $(a_1 = a_2 = a_3 = 1, \dot{a}_1 = \dot{a}_2 = \dot{a}_3 = 0)$. The oscillation amplitudes $\delta a_1$, $\delta a_3$ as a function of time are shown in figure 1. The amplitude $\delta a_2$ is always well approximated as $\delta a_2 \approx -\delta a_1$, and is omitted in the figure. The parameters adopted for the calculation are $Q = 90$, $\omega/\Omega = 0.6$ and $h = 0.1$.

The damping time $\tau$ of the model corresponds to $\tau = 2\pi Q/\omega = 150 \times (2\pi/\Omega)$. In the early stage of the oscillation, the temporal profiles are a mixture of two modes with frequencies $\omega (= 0.6\Omega)$ and $\Omega$. There is a modulation of two modes, which causes larger amplitudes of the oscillations due to interference of two waves. As time goes on, the amplitudes of both modes are damped by the radiation loss. Since the external perturbation with $\Omega$ is always supplied, the mode becomes dominant around $\tau (= 150 \times (2\pi/\Omega))$, and the oscillation is eventually enforced to match it at the late phase, typically at several times $\tau$, as shown in the right panel of figure 1. From the numerical results, we found that the motion along $a_3$ is also induced by the incompressible condition $a_3 = 1/(a_1 a_2)$. This differs from linear analysis, in which $a_3$ is always constant. The oscillation of $\delta a_3$ is of order $h^2$ in the amplitude, and is induced by a mixture of stellar intrinsic oscillation, $\omega$ and external perturbation, $\Omega$. The frequencies of $\delta a_3$ are therefore described by $2\Omega$, $2\omega$ and $\Omega \pm \omega$.

We have also calculated the oscillations for different parameters, and found that the general behaviour is almost the same unless the resonance condition is satisfied. The temporal behaviour may be easily understood by analysing the numerical data with Fourier
transformation. It is found that the oscillation is approximated by the following curve consisting of two frequencies
\[
\delta a_1 \approx -\delta a_2 \approx \frac{\Omega^2 h}{2(\Omega^2 - \omega^2)} (\cos \Omega t - \exp(-t/\tau) \cos \omega t), \quad \delta a_3 \propto h^2.
\]  
This formula can be derived from the linearized system equation (12) and the accuracy is checked by fitting the numerical results for a wide range of parameters.

In the limit of \( t \to \infty \), the oscillation with \( \omega \) in \( a_1 \) or \( a_2 \) is completely damped, and the mode matched with the incident wave survives with the amplitude \( \Omega^2 h/(2|\Omega^2 - \omega^2|) \). The amplitude of the steady state realized at \( t \to \infty \) becomes larger when \( \Omega \to \omega \). The amplitude apparently diverges at the resonance, but does not actually, because even small dissipation is important and suppresses the divergence in that case. See section 4.3.

4.2. Parametric resonance of fundamental mode

In this section, we consider the parametric resonance of the fundamental mode by setting \( \omega/\Omega = 1/2 \). In figure 2, we show time evolution of the model with \( h = 0.1, Q = 45 \). The damping time \( \tau \) corresponds to \( \tau = 90 \times (2\pi/\Omega) \). In the early phase of the time evolution, the amplitude of intrinsic stellar oscillation mode \( \omega (= \Omega/2) \) grows exponentially. After hundreds of oscillations, the damping effect becomes important around \( t \sim \tau \). Eventually the mode \( \omega \) is damped, and oscillation is enforced to external mode with \( \Omega \) at the late phase \( t \gg \tau \), in which the amplitude of \( \delta a_1 \) becomes \( \delta a_1 \approx 2h/3 \). This behaviour in the final steady state is the same as that in the off-resonant case. Compare the right panel in figure 1 \( (t \approx 1000 \times (2\pi/\Omega)) \) with that in figure 2 \( (t \approx 700 \times (2\pi/\Omega)) \). Both figures show behaviour corresponding to several times \( \tau \). Thus, the parametric excitation decays away in this case.

We consider a case in which the radiation loss is less effective. The results for the parameters \( h = 0.1, Q = 75 \) are shown in figure 3. The damping time \( \tau \) corresponds to \( \tau = 150 \times (2\pi/\Omega) \). In the early stage, a mixture of two frequencies \( \omega (= \Omega/2) \) and \( \Omega \) can be seen. There are no drastic differences between the left panels in figures 2 and 3 as for the early stages \( t < \tau \). By comparing the right panels in figures 2 and 3, it is found that parametric excitation occurs in figure 3 since the oscillation amplitudes of both \( \delta a_1 \) and \( \delta a_3 \) are indeed enhanced. The dominant oscillation frequency at the later stage is determined not by the external one \( \Omega \), but by the intrinsic one \( \omega \). The frequency of \( \delta a_3 \) is also \( \omega \), which comes from quadratic coupling between \( \Omega \) and \( \omega \), as \( \Omega - \omega (= \omega) \). The other higher frequencies such as \( 2\omega, \Omega + \omega (= 3\omega) \) and \( 2\Omega (= 4\omega) \) are less excited. The amplitude at the later stage is saturated around the value \( \delta a_1 \approx \delta a_3 \approx 4h \), which is several times larger than that of off-resonant oscillation. This saturation property is quite different from the linear theory, in which the amplitude of the unstable mode exponentially increases so far as the resonant condition is
satisfied. In the nonlinear ellipsoidal model, the motion along the longitudinal direction $a_3$ plays a key role in the saturation as explained below. The amplitudes of $\delta a_1$ and $\delta a_2$ grow in proportion to $h$, while that of $\delta a_3$, which is formally $\sim h^2$, also grows due to nonlinear coupling. The amplitude of $\delta a_3$ is given by $\delta a_3 \sim (\delta a_1)^2$. When the oscillation amplitude becomes large enough, say $\delta a_1 \sim 0.5$, $\delta a_3$ is also large enough. The motion of $\delta a_3$ is no longer neglected and significantly affects the dynamics of $\delta a_1$ and $\delta a_2$. In the presence of the dynamics of $\delta a_3$, the growing amplitudes are saturated around the same order $\delta a_1 \sim \delta a_2 \sim \delta a_3 \sim 0.5$.

So far we have shown two different results of the resonance in figures 2 and 3. Parametric oscillation is excited for the weak damping case, i.e., large Q, while it decays away for the strong damping case, i.e., small Q. The excitation condition should also depend on the amplitude $h$ of the incident gravitational wave. In order to study the resonance condition for $\omega/\Omega = 1/2$, we have performed numerical calculations of equations (8)–(10) for various parameters $(h, Q)$, typically in the range $20 < Q < 200$. The results are summarized in figure 4.

We found that a combination of the parameters, $hQ$, is a very important factor to determine the evolution. As shown in figure 4, there is a critical curve discriminating between growth and damped cases. The curve is empirically given by $hQ \approx 7$. Below the critical curve, the damping effect is so strong that the parametric excitation is suppressed as in figure 2. On the other hand, above the critical curve, the damping effect is less effective. In this case, parametric excitation of the intrinsic oscillation with $\omega$ occurs and the amplitudes can grow. Eventually,  

\footnote{Note that $hQ$ corresponds to $b\omega/\gamma$ in the harmonics oscillator shown in section 2. The importance of this factor is reasonable.}
the nonlinear effect becomes important and the amplitudes of the unstable oscillations are saturated around finite values.

4.3. Resonance at $\omega = \Omega$

In this section, we consider the resonant behaviour at $\omega = \Omega$. Two typical cases are shown in figures 5 and 6. These figures show their long-term evolution up to hundreds of cycles of the
oscillations, so that each sinusoidal curve is not clear, but the gradual change of the envelope is seen. The model parameters of figure 5 are \( Q = 100, \ h = 5 \times 10^{-2} \), while those of figure 6 are \( Q = 100, \ h = 5 \times 10^{-3} \). The behaviour is the same only for the initial stages, in which the amplitude of \( \delta a_1 \) (or \( \delta a_2 \)) increases linearly with time: \( \delta a_1 \approx h \Omega_1 \sin(\Omega_1 t)/4 \), so far as \( \delta a_1 \ll 1 \). The later behaviour significantly depends on the adopted parameters, and is exhibited in quite different ways. We have also calculated in a wide range of parameter space \((Q, h)\) and found that the most important factor is \( hQ \), as in the previous section. The reason is explained below.

The oscillation of \( \delta a_3 \) is inevitably induced by the nonlinear coupling: \( \delta a_3 \propto \delta a_1 \delta a_2 \sim (h\Omega_1)^2 \sin^2(\Omega_1 t)/16 \). The amplitude of \( \delta a_3 \) grows quadratically with time, and is therefore small only in the initial evolution. There is an epoch in which both amplitudes of \( \delta a_1 \) and \( \delta a_3 \) become the same order. This nonlinear timescale \( t_n \) is estimated as \( t_n = 4/(h\Omega_1) \) by setting \( \delta a_1 = \delta a_3 \), if the damping may be neglected. It is important to compare this timescale with that of the radiation loss \( \tau = Q \times 2\pi/\Omega_1 \). If \( \tau < t_n \), which corresponds to \( hQ < 2/\pi \approx 0.6 \), the radiation loss controls the evolution. As shown in figure 5 \((hQ = 0.5)\), the grown amplitude of \( \delta a_1 \) is saturated around the value \( \delta a_1 \sim h\Omega_1 \tau \sim hQ \). The amplitude of \( \delta a_3 \) grows over a much longer nonlinear timescale, which is not exactly but roughly given by \( t_n \). We have shown in figure 5 the time evolution for \( t < 250 \times (2\pi/\Omega_1) \), but confirmed that the subsequent evolution for \( t > 250 \times (2\pi/\Omega_1) \) becomes almost steady state with some small modulation in the amplitudes.

On the other hand, if \( \tau > t_n \), i.e., \( hQ > 2/\pi \), the nonlinear coupling is important. The behaviour in this case is highly unstable as shown in figure 6 \((hQ = 5)\). The amplitudes \( \delta a_i \) \((i = 1, 2, 3)\) become \( \mathcal{O}(1) \). The nonlinear timescale in this model corresponds to \( t_n \sim 13 \times (2\pi/\Omega_1) \), where the amplitude \( \delta a_3 \) becomes large. The time evolution exhibits chaotic behaviour\(^2\), and therefore the actual numerical integration is rather difficult for much longer time, say, up to \( 10^3 \times (2\pi/\Omega_1) \) in figure 6.

5. Summary and discussion

In this paper, we have examined resonant oscillations in an ellipsoidal stellar model with incoming gravitational wave. Resonance occurs when a certain matching condition holds between the stellar oscillation and the external periodic disturbance. In the presence of radiation loss, the evolution significantly depends on a combination \( hQ = h
\nu\tau \) consisting of the wave amplitude \( h \), frequency \( \nu \) and the damping time \( \tau \). The parametric resonance of the fundamental mode can grow only if the quantity \( hQ \) is larger than a critical value \((\approx 7)\). The amplitude of stellar oscillation increases, but is saturated due to nonlinear coupling, by which energy is transferred to the additional oscillatory motion. This point differs from a linearized system, in which the amplitude grows exponentially. If the quantity \( hQ \) is smaller than the critical value, the resonant oscillation is damped. The resonant behaviour at \( \omega = \Omega_1 \) is also determined by the parameter \( hQ \). If \( hQ \) is larger than a critical value \((\approx 1)\), then the system exhibits chaotic behaviour. We have also performed numerical calculations to explore the possibility of parametric resonance in higher modes \( \omega/\Omega_1 = n/2 \) \((n = 3, 4, \ldots)\), and found that the higher modes are much more difficult to excite.

In our idealized model, the external perturbation is assumed to be constant. This corresponds to the situation that duration \( t_e \) of the driving source is much larger than radiation damping time \( \tau \), or nonlinear timescale \( t_n \). If this condition does not hold, then

\(^2\) In general, it may be necessary to measure, e.g., the Lyapunov exponent in order to judge whether the nonlinear behaviour is chaotic or not. We did not calculate the exponent, but found that the numerical behaviour is very sensitive like a chaotic system.
the environmental effect would appear first in the evolution. For example, when the external perturbation stops at $t_c$, the enhancement by the resonance also stops there, and the amplitude is subsequently damped around $\tau (\geq t_c)$.

We now consider the astrophysical relevance of the resonant oscillations. The $f$-mode oscillation of a neutron star is estimated as the frequency $\nu = \text{a few kHz}$, the damping time $\tau = \text{a few } 10^{-1} \text{ s}$, and therefore $Q \approx 10^2$. In this case, the exotic case needs large amplitude of the gravitational wave $h > Q^{-1} \approx 10^{-2}$. The condition is realized only in the hydrodynamical regime, e.g., in core collapse of a supernova and in the final phase of binary coalescence. The duration of periodic disturbance is also an important factor. Typically, the timescale is of order $10^{-3} \text{ s}$, and may be too short for the mode to be excited. Anyway, more realistic simulation is needed.

Our result may be scaled to the excitation of the $f$-mode in white dwarfs. In this case, the frequency and damping time are given by $\nu \sim 0.1 \text{ Hz}$, $\tau \sim 10^{11} \text{ s}$, and therefore $Q \sim 10^{10}$. In this case, the condition for the exotic case becomes less severe, $h > 10^{-10}$. The region available for this large amplitude may be estimated by the total amount of radiated energy $\Delta M c^2$. The distance from the periodic gravitational wave source is still limited to $r \sim G\Delta M/(hc^2) \ll 1 \text{ pc}$.

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References

[1] Allen B and Wiseman B (ed) 2004 Proc. 8th Gravitational Wave Data Analysis Workshop (Milwaukee, WI, USA, 17–20 December 2003) Class. Quantum Grav. 21 S1575–854
[2] Andersson N and Kokkotas K D 1998 Mon. Not. R. Astron. Soc. 299 1059
[3] Benhar O, Ferrari V and Gualtieri L 2004 Phys. Rev. D 70 124015
[4] Ott C D, Burrows A, Livne E and Walder R 2004 Astrophys. J. 600 834
[5] Fryer C L, Holz D E and Hughes S A 2004 Astrophys. J. 609 288
[6] Kotake K, Yamada S, Sato K, Sumiyoshi K, Ono H and Suzuki H 2004 Phys. Rev. D 69 124004
[7] Shibata M and Sekiguchi Y 2004 Phys. Rev. D 69 084024
[8] Kojima Y 1987 Prog. Theor. Phys. 77 297
[9] Ruoff J, Laguna P and Pullin J 2001 Phys. Rev. D 63 064019
[10] Tominaga K, Saio M and Maeda K 2001 Phys. Rev. D 63 124012
[11] Pons J A, Berti E, Gualtieri L, Miniutti G and Ferrari V 2002 Phys. Rev. D 65 104021
[12] Landau L D and Lifshitz E M 1976 Mechanics (Oxford: Pergamon)
[13] Misner C M, Thorne K S and Wheeler J A 1973 Gravitation (San Francisco, CA: Freeman)
[14] Press W H and Teukolsky S A 1972 Astrophys. J. 181 513
[15] Miller B D 1974 Astrophys. J. 187 609
[16] Carter B and Luminet J P 1985 Mon. Not. R. Astron. Soc. 212 23
[17] Chandrasekhar S 1969 Ellipsoidal Figures of Equilibrium (New Haven, CT: Yale University Press)