Detection of Bidirectional System-Environment Information Exchanges

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Quantum memory effects can be related to a bidirectional exchange of information between an open system and its environment, which in turn modifies the state and dynamical behavior of the last one. Nevertheless, non-Markovianity can also be induced by environments whose dynamics is not affected during the system evolution, implying the absence of any physical information exchange. An unsolved open problem in the formulation of quantum memory measures is the apparent impossibility of discerning between both paradigmatic cases. Here, we present an operational scheme that, based on the outcomes of successive measurements processes performed over the system of interest, allows to distinguishing between both kinds of memory effects. The method accurately detects bidirectional information flows in diverse dissipative and dephasing non-Markovian open system dynamics.

I. INTRODUCTION

In its modern conception, quantum non-Markovianity is related to a twofold exchange of information between an open system and its environment. Over the basis of unitary system-environment models, it is commonly assumed that this bidirectional information flow (BIF) is mediated by physical processes that modify the state and dynamical behavior of the environment. In spite of the consistence of this picture, it is well known that memory effects can also be induced by reservoirs whose state and dynamical behavior are not affected at all by the interaction with the open system. Evidently, this feature implies the absence of any physical system-environment backflow of information exchange. Stochastic Hamiltonians, incoherent bath fluctuations, collisional models, and (system) unitary dynamics characterized by random parameters are some examples of this “casual bystander” (non-Markovian) environment action. The environment affects the system dynamics but its (statistical) state is never influenced by the system.

An open problem in the formulation of quantum non-Markovianity is the lack of an underlying prescription (based only on system information) able to discriminate between the previous two complementary cases. In fact, even when a wide variety of memory witnesses (defined from the system propagator properties) has been proposed and implemented experimentally, even in absence of BIFs most of them may inaccurately detect an “environment-to-system backflow of information”. This incongruence emerges because quantum master equations with very similar structures describe the (non-Markovian) system dynamics in presence or absence of BIFs.

The previous limitation implies a severe constraint on the classification and interpretation of memory effects in quantum systems. For example, there exist non-Markovian dynamics whose underlying memory effects are classified as “extreme” ones. Nevertheless, these dynamics emerge from simple classical statistical mixtures of (memoryless) Markovian system evolutions. Added to the absence of any physical BIF, the reading of memory effects as quantum ones becomes meaningless in this situation. Remarkable cases are quantum master equations with an ever (time-dependent) negative rate as “maximally non-Markovian dynamics” where the stationary state may recover the initial condition. On the other hand, the interpretation of this kind of dynamics in terms of measurement-based stochastic wave vector evolutions may becomes ambiguous (Markovian or non-Markovian) by taking into account or not the underlying statistical mixture. In fact, for each Markovian system evolution in the statistical ensemble one can associate a Markovian stochastic wave vector evolution. Hence, there is not any memory effect at the level of single realizations. Alternatively, a non-Markovian wave vector evolution that in average recovers the system evolution may also be proposed. These examples confirm that a procedure capable to determine when memory effects rely or not on physically mediated BIFs is in general highly demanded.

The aim of this work is introduce an operational technique that accurately detects the presence of physically mediated system-environment BIFs. Consistently with the operational character, instead of a definition in the system Hilbert space, the approach relies on a probabilistic condition that indicates when an environment is unaffected by its coupling with the system. Correspondingly, memory effects emerge from a statistical average of a Markovian system dynamics that parametrically depends on the (unaffected) bath degrees of freedom. It is shown that these conditions can be checked by performing a minimal number of three system measurement processes, added to an intermediate (random) update of the system state that may depends on previous outcomes. Similarly to operational memory approaches based on causal breaks, a generalized conditional past-future (CPF) correlation defined between the first and last (past-future) measurement outcomes, conditioned to the intermediate updated
system-state, becomes an indicator of BIFs.

The three-joint outcome probabilities and its associated
generalized CPF correlation are calculated for both
quantum and classical environmental fluctuations. Con-
sistently, for classical noise fluctuations, or in general,
when memory effects can be associated to environments
with an invariant dynamics, the generalized CPF corre-
lation vanishes. This property furnishes a novel and ex-


tic experimental test for detecting BIFs. Its feasibility
is explicitly demonstrated through its characterization in
ubiquitous dissipative and dephasing non-Markovian dy-
namics that admit an exact treatment.

II. PROBABILISTIC APPROACH

Our aim is to distinguish between memory effects that
occur with and without BIFs. These opposite cases are
related to the dependence or independence of the reser-
voir dynamics on system degrees of freedom. This prop-
erty can be explicitly defined by means of the following
scheme, which is valid in both classical and quantum
realms.

We assume that both the system and the environment
are subjected to a set of (bipartite separable) measure-
ments at successive times $t_1 < t_2 \cdots < t_n$. The set of
strings $s \equiv (s_1, s_2 \cdots s_n)$ and $e \equiv (e_1, e_2, \cdots e_n)$ denote
the respective outcomes, which in turn label the cor-
responding system and environment post-measurement
states. The outcome statistics is set by a joint invari-

t probability $P(s,e)$). This object in general depends on which
measurement processes are performed.

In agreement with our definition, in absence of BIFs
the environment probability $P(e) = \sum_s P(s,e)$ must be
an invariant object that is independent of the system
initialization and dynamics. Bayes rule allows to write
$P(e) = \sum_s P(e|s)P(s)$, where $P(e|s)$ is the condi-
tional probability of $e$ given $s$, while $P(s)$ gives the probability
of $s$. Hence, the absence of BIFs can be expressed by the
condition

$$P(e|s) = P(e),$$

which guarantees that the environment statistics is in-
dependent of the system state and dynamics.

The marginal probability for the system outcomes
can always be written as $P(s) = \sum_e P(s,e) = \sum_e P(s|e)P(e)$, where $P(s|e)$ is the conditional proba-

tility of $s$ given $e$. When condition (1) is fulfilled, we
can affirm that any possible memory effect in the system
measurements follows from an (invariant) environmental
average $\langle \cdots \rangle_e \equiv \sum_e \cdots P(e)$ of a (system) joint proba-
bility $P(s|e) \equiv P(s|e)$ that parametrically depends on the
bath states,

$$P(s) = \langle P(s|e) \rangle_e.$$  

Notice that $P(s|e)$ denotes the conditional probability
$P(_e(s))$ given that condition (1) is fulfilled.

In the present approach Eqs. (1) and (2) define the
absence of any physical system-environment BIF. Sys-

tem memory effects emerge due to the conditional action
of the bath. Our problem now is to detect these proba-
bility structures by taking into account only the system
outcome statistics. Before this step, we introduce one
extra assumption.

As usual in open quantum systems, we assume that the
system-bath bipartite dynamics (without interventions)
adopts an underlying semigroup (memoryless) descrip-
tion. Hence, $P(s)$ fulfills a Markovian property with
respect to system outcomes,

$$P(s) = P(s_1) \cdots P(s_n|s_{n-1})P(s_{n-1})P(s_{n-2}) \cdots P(s_1).$$  

For notational convenience, the parametric dependence
of the conditional probabilities $P(s|s')$ on the bath
states is written through the supra index ($e$). This de-

pendence must be consistent with causality, meaning that
$P(s|s')$ cannot depend on (non-selected) future bath
outcomes.

A. Detection scheme

The developing of BIFs, that is, departures with re-
spect to the structure defined by Eqs. (2)-(3), can be
detected with the following minimal scheme. Three mea-

surements processes performed at times $t \rightarrow t + \tau$, 

deliver the successive system outcomes $x \rightarrow (y \rightarrow \bar{y}) \rightarrow \bar{z}$. After the intermediate measurement, the system
state—labelled by $y$—is externally (and instantaneously)
updated to a renewed state—labelled by $\bar{y}$, while the
bath state is unaffected. Each $\bar{y}$-state is chosen with an
arbitrary conditional probability $\psi(\bar{y}|x, y)$. The scheme
is closed after specifying $\psi(\bar{y}|x, y)$ and calculating the
marginal probability $P(z, \bar{y}, y, x) = \sum_y P(z, \bar{y}, y, x)$. In ad-
dition, it is assumed that system and environment are
uncorrelated before the first measurement. A “deter-
mistic scheme” (d) corresponds to $\psi(\bar{y}|x, y) = \delta_{\bar{y},y}$. 

Hence, not any change is introduced after the interme-
diate measurement. A “random scheme” (r) is defined by
$\psi(\bar{y}|x, y) = \psi(\bar{y}|x)$. These two cases are motivated by the
following features.

In absence of BIFs, the joint probability for the four
events, from Eqs. (2) and (3), reads

$$P(z, \bar{y}, y, x) = \langle P(s) | z|\bar{y}\rangle \psi(\bar{y}|y, x)P(s|y|x)P(x)\rangle_e.$$  

Notice that this result also relies on Eq. (1), which guar-
antees that $\langle \cdots \rangle_e$ remains invariant even when changing
the system state at a given time, $(y \rightarrow \bar{y})$. On the other
hand, by assumption $\psi(\bar{y}|x, y)$ and $P(x)$ do not depend
on the environmental degrees of freedom. In the deter-
meministic scheme, Eq. (4) leads to

$$P(z, \bar{y}, x) = \langle P(s) | z|\bar{y}\rangle \psi(\bar{y}|x)P(x),$$  

while in the random case, using $\sum_y P(s|y|x) = 1$, 

$$P(z, \bar{y}, x) = \langle P(s) | z|\bar{y}\rangle \psi(\bar{y}|x)P(x).$$  


The deterministic scheme [Eq. (5)], given that $P(z, \hat{y}, x)$ does not fulfill a Markov property, shows that memory effects may in fact develop even in absence of BIFs. Nevertheless, due to the structure defined by Eqs. (2) and (3), they are completely “washed out” in the random scheme, which delivers a Markovian joint probability [Eq. (6)]. Taking into account the derivation of Eq. (4), this last property break down when Eq. (4) is not fulfilled. Thus, in the random scheme departure of $P(z, \hat{y}, x)$ from Markovianity witnesses BIFs, which solves our problem.

B. System and environment observables

In contrast to classical systems, in a quantum regime the previous results have an intrinsic dependence of which system and environment observables are considered.

For quantum systems, the absence of BIFs is defined by the validity of the probability structures Eqs. (5) and (6) for any kind of system measurement processes. Thus, arbitrary system observables are considered.

On the other hand, we only consider environment observables that allow to read $\langle \cdots \rangle_e$ as an unconditional average over the bath degrees of freedom. This extra assumption is completely consistent with the developed approach. Furthermore, this election (due to the conditional character) implies that $P(z, \hat{y}, x)$ can be measured without involving any explicit environment measurement process. This important feature is valid for both classical and quantum environmental fluctuations.

When the environment is defined by classical stochastic degrees of freedom with a fixed statistics [Sec. IIIA], given that classical systems are not affected by a measurement process, the previous assumption applies straightforwardly. When the reservoir must be described in a quantum regime, the previous constraint implies observables whose non-selective measurement transformations do not affect the environment state at each stage [Sec. IIIB]. Thus, independently of the environment nature, the detection of BIFs can always be performed without measuring explicitly the environment.

C. BIF witness

Independently of the nature (incoherent or quantum) of both the system and the environment, as an explicit witness of BIF we consider a generalized CPF correlation that takes into account the intermediate system state update operation (deterministic $\leftrightarrow d$ or random $\leftrightarrow r$). It measures the correlation between the initial and final (past-future) outcomes conditioned to the intermediate system state ($\hat{y}$)

$$C_{pf}^{(d/r)} | \hat{y} \rangle \equiv \sum_{z,x} O_z O_x [P(z, x|\hat{y}) - P(z|\hat{y}) P(x|\hat{y})].$$  

Here, all conditional probabilities follow from $P(z, \hat{y}, x)$ [61], while the sum indexes run over all possible outcomes at each stage. The scalar quantities $\{O_z\}$ and $\{O_x\}$ define the system observables for each outcome.

In the deterministic scheme, similarly to Ref. [56], $C_{pf}^{(d)} | \hat{y} \rangle$ detects memory effects independently of its underlying origin. In the random scheme, the condition $C_{pf}^{(r)} | \hat{y} \rangle \neq 0$ provides the desired witness of BIFs. This result follows directly from the Markovian property Eq. (6), which leads to $P(z, x|\hat{y}) = P(z|\hat{y}) P(x|\hat{y}) = C_{pf}^{(r)} | \hat{y} \rangle = 0$.

For quantum systems, the three system measurement processes are defined by a set of operators $\{\Omega_z\}$, $\{\Omega_y\}$, and $\{\Omega_x\}$, with normalization $\sum_z \Omega_z^\dagger \Omega_z = \sum_y \Omega_y^\dagger \Omega_y = \sum_x \Omega_x^\dagger \Omega_x = I$, where $I$ is the system identity operator. The intermediate $y$-measurement in taken as a projective one, $\Omega_y = |y\rangle \langle y|$. Thus, in the random scheme the system state transformation reads $\rho_y \equiv |y\rangle \langle y| \rightarrow \rho_y$, where the states $\{\rho_y\}$ (independently of outcome $y$) are randomly chosen with probability $\psi(|y\rangle x)$. This operation can be implemented, for example, as $\rho_y = U(|\hat{y}\rangle y)|\rho_y\rangle$, where the (conditional) unitary operator $U(|\hat{y}\rangle y)$ leads to the state $\rho_y$ independently of the obtained $y$-outcome [62].

III. APPLICATION TO DIFFERENT SYSTEM-ENVIRONMENT MODELS

The consistence of the developed approach is supported by studying fundamental system-reservoir models that leads to memory effects.

A. Classical noise environmental fluctuations

Here the open system is coupled to classical stochastic degrees of freedom. Its density matrix is written as $\rho_t = \mathcal{E}_{t+\tau,\tau}^{\epsilon t}[\rho_0]$, where the overbar symbol denotes an average over the environmental realizations. For each noise realization the stochastic propagator fulfills $\mathcal{E}_{t+\tau,\tau}^{\epsilon t} = \mathcal{E}_{t+\tau,\tau}^{\epsilon t,0}$, property consistent with the assumption [3].

Stochastic Hamiltonians [4, 12] as well as random unitary evolutions [20] fall in this category. As usual in these models, the statistics of the noise realizations is independent of the system dynamics. Hence, not any BIF should be detected in this case.

Given that each noise realization labels the environment state, we can take the equivalence $\langle \cdots \rangle_e \leftrightarrow \langle \cdots \rangle$. By using the standard formulation of quantum measurement theory, the joint probability associated to the measurement scheme can be written as (see Appendix A)

$$\frac{P(z, \hat{y}, x)}{\overline{\rho(\hat{y}|y, x)}} = \text{Tr}_x (E_z \mathcal{E}_{t+\tau,\tau}^{\epsilon t}[\rho_y]) \text{Tr}_x (E_y \mathcal{E}_{t,0}^{\epsilon t}[\hat{p}_x]),$$  

where $E_i \equiv \Omega_i^\dagger \Omega_i$ ($i = x, y, z$) and $\hat{p}_x \equiv \Omega_x \rho_0 \Omega_x^\dagger$ is the (unnormalized) system state after the first $x$-measurement. $\text{Tr}_x(\cdots)$ denotes a trace operation in the
system Hilbert space. \( \rho_y \) is the (updated) system state after the second \( y \)-measurement, while \( t \) and \( \tau \) are the elapsed times between consecutive measurements.

In the deterministic scheme \( [\phi(y|y, x) = \delta_{xy}] \) using that \( P(z, y, x) = \sum_y P(z, y, y, x) \), Eq. (5) leads to

\[
P(z, y, x) = \frac{\Tr_s(E_s E_{t+t-t}[\rho_y]) \Tr_s(E_y E_{t,0}[\rho_x])}{\Tr(E_y E_{t,0}[\rho_x])}.
\] (9)

In general, this joint probability does not fulfill a Markov condition. Thus, \( C_{pf}^{(d)}|y \neq 0 \) [Eq. (7)] detects memory effects. On the other hand, in the random scheme \( [\phi(y|y, x) = \phi(y|x)] \) from Eq. (5) it follows

\[
P(z, y, x) = \frac{\Tr_s(E_s E_{t+t-t}[\rho_y]) \phi(y|x) \Tr_s(\tilde{\rho}_y)}{\Tr(E_y E_{t,0}[\rho_x])}.
\] (10)

which recovers the Markovian result Eq. (3) with \( \langle P(e)(z|y) \rangle_e \leftrightarrow \Tr(E_s E_{t+t-t}[\rho_y]) = P(z|y) \) and \( P(x) = \Tr_s(\tilde{\rho}_y) = \Tr_s(E_x \rho_0) \). Thus, independently of the chosen system measurement observables it follows \( C_{pf}^{(r)}|y = 0 \) [Eq. (7)], indicating, as expected, the absence of any BIF.

B. Completely positive system-environment dynamics

Alternatively, system-environment (s-e) dynamics can be described in a bipartite Hilbert space. Their density matrix \( \rho^{se}_y = \mathcal{E}_{t,0}[\rho^{se}_0] \) is set by a bipartite propagator that satisfies \( \mathcal{E}_{t+t,0} = \mathcal{E}_{t+t,0} \mathcal{E}_{t,0} \). This property also supports assumption B. We consider separable initial conditions \( \rho_0^{se} = \rho_0 \otimes \sigma_0 \). Hence, \( \mathcal{E}_{t,0} \) leads to a completely positive system dynamics \( \rho_t = \Tr_s(\mathcal{E}_{t,0}[\rho_0^{se}]) \). Unitary system-environment models [1] as well as bipartite (time-irreversible) Lindblad dynamics fall in this category. As system and environment are intrinsically coupled, the developing of BIFS is expected in this model.

Here, we take the equivalence \( \{ \cdot \cdot \cdot \}_s \leftrightarrow \Tr_s(\{ \cdots \}) \). This unconditioned environment average applies when the successive (non-selective) measurements of the environment do not modify its state at each stage (the bath state remains the same after each non-selective measurement).

Due to the dynamics induced by \( \mathcal{E}_{t,0} \), in general it is not possible to know explicitly which physical reservoir observables fulfill this condition. Nevertheless, the demanded invariance straightforwardly allows to read and to obtain \( \{ \cdot \cdot \}_s \) from the bath trace operation \( \Tr_s(\{ \cdots \}) \) (see also Appendix A). Hence, similarly to the previous environment model the validity (or not) of Eqs. (5) and (6) can be checked without performing any explicit reservoir measurement process. From standard quantum measurement theory, the joint probability of system outcomes here reads (Appendix A)

\[
P(z, y, y, x) = \Tr(E_z E_{t+t-t}[\rho_y \otimes \Tr(E_y E_{t,0}[\rho^{se}_x])]),
\] (11)

where \( \rho^{se}_x \equiv \Omega_x \rho_0 \Omega^\dagger_x \otimes \sigma_0 = \tilde{\rho}_x \otimes \sigma_0 \) is the bipartite state after the first \( x \)-measurement and, as before, \( \tilde{\rho}_y \) is the updated system state.

In the deterministic scheme \( [\phi(y|y, x) = \delta_{xy}] \), the previous expression \( P(z, y, x) = \sum_y P(z, y, y, x) \) leads to

\[
P(z, y, x) = \Tr(E_z \mathcal{E}_{t+t,t}[\rho_y \otimes \Tr(E_y E_{t,0}[\rho^{se}_x])]) \] (12)

As expected, a Markovian property is not fulfilled in general implying the presence of memory effects, \( C_{pf}^{(d)}|y \neq 0 \). In the random scheme \( [\phi(y|y, x) = \phi(y|x)] \) it follows

\[
P(z, y, x) = \Tr(E_z \mathcal{E}_{t+t,t}[\rho_y \otimes \Tr(E_y E_{t,0}[\rho^{se}_x])])\phi(y|x).
\] (13)

In contrast to Eq. (10), here in general a Markov property is not fulfilled. Thus, \( C_{pf}^{(r)}|y \neq 0 \). Nevertheless, there are bipartite dynamics than in fact occur without a BIF. Below, we found the conditions that guarantee \( C_{pf}^{(r)}|y = 0 \) for arbitrary system measurement processes.

1. Invariant environment dynamics

The environment state follows by tracing out the system degrees of freedom, \( \sigma_t \equiv \Tr_y(\mathcal{E}_{t,0}[\rho^{se}_0]) \), where \( \rho^{se}_0 = \rho_0 \otimes \sigma_0 \). When this state is independent of the system initialization

\[
\sigma_t = \Tr_s(\mathcal{E}_{t,0}[\rho^{se}_0]) = \Tr_s(\mathcal{E}_{t,0}[\mathcal{M}_s[\rho^{se}_0]]),
\] (14)

where \( \mathcal{M}_s \) represents an arbitrary (trace-preserving) system transformation, a Markovian property is immediately recovered in the random scheme. In fact, introducing \( \Tr(E_y E_{t,0}[\rho^{se}_x]) = P(x)\sigma_t \), Eq. (13) becomes

\[
P(z, y, x) = \Tr(E_z \mathcal{E}_{t+t,t}[\rho_y \otimes \sigma_t])\phi(y|x)P(x),
\] (15)

which recovers the structure (5). Thus, environments with an invariant dynamics do not induce any BIF [C_{pf}^{(r)}|y = 0]. Notice that this property supports the complete consistence of the proposed approach.

A relevant situation where Eq. (14) applies is the case of systems coupled to incoherent degrees of freedom governed by a (invariant) classical master equation [13]. While these dynamics lead to memory effects [14], our approach correctly identify the absence of any BIF. Random unitary evolutions [20], as well as quantum Markov chains [49, 50] fall in this case.

It is important to remark that environments developing quantum features (coherences) may also fulfill condition (11). This is the case, for example, of some collisional models [18] whose underlying description can be formulated with bipartite Lindblad equations [19].

2. Unitary system-environment dynamics

When modeling open quantum dynamics from an underlying bipartite Hamiltonian dynamics, the unitary propagator reads \( \mathcal{E}_{t,0}[\cdot] = \exp(-it \hat{H}_T) \cdot \exp(+it \hat{H}_T) \), where \( \hat{H}_T \) is

\[
\hat{H}_T = \hat{H}_s + \hat{H}_e + \hat{H}_f.
\] (15)
The first two terms define respectively the system and bath Hamiltonians, while the last one introduces their interaction. Given the system-environment mutual interaction, for nearly all Hamiltonians $H_T$ it is expected that the developing of memory effects [Eq. (12)] rely on BIFs [Eq. (13)].

One exception to the previous rule arises when the bath and interaction Hamiltonians commute,

$$[H_e, H_I] = 0.$$  \hspace{1cm} (16)

Under this condition, denoting the bath eigenvectors as $H_e |e\rangle = e |e\rangle$, the system density matrix reads $\rho_t = \text{Tr}_e (\rho_e^t) = \sum_e w_e \exp(-iH_s^t) \rho_0 \exp(iH_s^t)$, where the weights are $w_e = \langle e | \sigma_0 | e\rangle$ and $H_s^t = H_s + (e | H_I | e\rangle$. Thus, the system dynamics can be represented by a random unitary map $\hat{U}$. For arbitrary dynamics, this property does not guarantee the absence of BIFs. In fact, here the environment invariance property (14) is not fulfilled in general. Nevertheless, after a straightforward calculation, the probabilities of the deterministic and random schemes, Eqs. (14) and (15), can be written as in Eqs. (19) and (20) (valid in absence of BIFs) respectively. In fact, under the replacement $(\cdots)_e \rightarrow \sum_e w_e (\cdots)$, the conditional probabilities are $P^\sigma(e) (\cdots | y) \rightarrow \text{Tr}_e (E_y G^\sigma_\tau [\rho_0])$ and $P^\sigma_y (\cdots | x) \rightarrow \text{Tr}_x (E_y G^\sigma_\tau [\rho_x])$, where $G_\tau^\sigma[\cdot] = \exp(-iH_s^\tau) \cdot \exp(iH_s^\tau)$ and $\rho_x \equiv \rho_0 / \text{Tr}_x (\rho_x)$. Thus, from these expressions we conclude that the condition guarantees that the joint probabilities, for arbitrary system measurement processes, can also be obtained from a statistical mixture (with invariant weights $\{w_e\}$) of unitary system evolutions (with propagators $\{G_\tau^\sigma[\cdot]\}$), which consistently implies $C^{(r)}_{pf} | \tilde{y} = 0$.

\textbf{IV. EXAMPLES}

Here, different explicit examples that admit an exact treatment are studied.

\textbf{A. Eternal non-Markovianity}

As a first explicit example we consider the non-Markovian system evolution

$$\frac{d\rho_t}{dt} = \frac{1}{2} \sum_{\alpha = \hat{x}, \hat{y}, \hat{z}} \gamma_\alpha(t) (\sigma_\alpha \rho_t \sigma_\alpha - \rho_t),$$ \hspace{1cm} (17)

where $\{\sigma_\alpha\}$ are the $\alpha$-Pauli matrices (directions in Bloch sphere are denoted with a hat symbol). The time-dependent rates are $\gamma_\alpha(t) = \gamma_r(t) = \gamma_r$ and $\gamma_\tau(t) = -\gamma \tanh[\gamma t]$. As demonstrated in Ref. [49] this kind of \textit{eternal} non-Markovian evolution $[\gamma_\alpha(t) < 0 \forall t]$ is induced by the coupling of the system with a statistical mixture of classical random fields. In fact, the system state can be written as $\rho_t = \sum_{\alpha = \hat{x}, \hat{y}, \hat{z}} \rho_0 \exp[\gamma t L_{\alpha}] [\rho_0]$, where $L_\alpha[\cdot] \equiv (\sigma_\alpha \cdot \sigma_\alpha - \cdot)$ is induced by each random field, whose (mixture) weights are $q_\alpha = q_{\tilde{g}} = 1/2$, and $q_\alpha = 0$. This underlying “microscopic” description allows to calculating multi-time statistics in an exact way. In particular, the CPF correlations follow straightforwardly from Eqs. (9) and (10), \textit{\textbf{(10)} $\to \sum_{\alpha = \hat{x}, \hat{y}, \hat{z}} q_\alpha (\cdots)$}, where the (time-independent) “noise environmental realizations” only assumes the values $\alpha = \hat{x}, \hat{y}, \hat{z}$, each with probability $q_\alpha$.

Assuming that the three measurements processes are performed in the Bloch directions $\hat{x}$-$\hat{y}$-$\hat{z}$, where $\hat{b}$ is an arbitrary direction in the $\hat{z}$-$\hat{x}$ plane (with azimutal angle $\theta$), for the deterministic scheme it follows (see Appendix B)

$$C^{(d)}_{pf} | \tilde{y} = \pm 1 = \sin^2(\theta) [c(t + \tau) - c(t) c(\tau)],$$ \hspace{1cm} (18)

where $c(t) \equiv q_\alpha + (q_{\tilde{g}} + q_\alpha) \exp[-2\gamma t]$. The initial system state was taken as $\rho_0 = |\pm\rangle \langle |\pm\rangle$, where $|\pm\rangle$ denotes the eigenvectors of $\sigma_z$. In Fig. 1(a) we plot $C^{(d)}_{pf} | \tilde{y} = \mp 0$ [Eq. (15)] and $C^{(r)}_{pf} | \tilde{y} = 0$ for equal measurement time intervals, $t = \tau$. The property $\lim_{\tau \to \infty} C^{(d)}_{pf} | \tilde{y} \neq 0$ indicates that the environment correlation do not decay in time [50]. On the other hand, independently of the election of the renewed (pure) states $\rho_0 = |\pm\rangle$ and $\Psi(y|x)$, we get $C^{(r)}_{pf} | \tilde{y} = 0$ (see Appendix B). As expected from Eq. (10), this result indicates the absence of any BIF.

\textbf{B. Interaction with a bosonic bath}

As a second example, we consider a two-level system coupled to a bosonic bath,

$$H_T = \frac{\omega_0}{2} \sigma_z + \sum_k \omega_k b_k^\dagger b_k + \sum_k g_k S_l b_k^\dagger + g_k^* S_l^\dagger b_k.$$ \hspace{1cm} (19)

Each contribution defines the system, bath, and interaction Hamiltonians respectively [Eq. (15)]. The bosonic operators satisfy $[b_k, b_k^\dagger] = \delta_{k,k'}$. Taking the system operators $S_l = |+\rangle \langle -|$ and $S = |-\rangle \langle +|$ as the raising and lowering operators in the natural basis $|\pm\rangle$, the system dynamics is dissipative [1], while in the case $S = S_l^\dagger = \sigma_z$ a dephasing dynamics is recovered. We assume the bipartite initial state $|\Psi_0^{lr}\rangle = |\psi_0\rangle \otimes \bigotimes_k |0\rangle_k$, where $\{|0\rangle_k\}$ are the ground states of each bosonic mode. In this case, by working the observables in their interaction representation, similarly to Refs. [51, 52], the joint probabilities (12) and (13) can be calculated in an exact way [64].

For the \textit{dissipative} dynamics $S = |-\rangle \langle +|$ in Eq. (19) the CPF correlation in the random scheme reads [64]

$$C^{(r)}_{pf} | \tilde{y} = -1 = \frac{1}{\pm 2} |G(t, \tau)|^2, \hspace{1cm} C^{(r)}_{pf} | \tilde{y} = -1 = -\text{Re}[G(t, \tau)].$$ \hspace{1cm} (20)

Here, we consider two different measurement possibilities, $\hat{z}$-$\hat{z}$-$\hat{z}$ and $\hat{x}$-$\hat{z}$-$\hat{x}$ directions, both with conditional
FIG. 1: CPF correlation [Eq. (2)] for the deterministic and random schemes, left and right columns respectively, for equal measurement time intervals \( t = \tau \). (a) Eternal non-Markovianity, measurements \( \hat{z} - \hat{n} - \hat{x} \). (b) Decay in a bosonic bath, measurements \( \hat{z} - \hat{z} - \hat{x} \) and \( \hat{x} - \hat{x} - \hat{x} \). (c) Dephasing in a bosonic bath, measurements \( \hat{n} - \hat{y} - \hat{x} \). In all cases, the \( \hat{n} - \) direction is defined by the angle \( \theta \). The renewed states \( \rho_{\hat{y}=\pm 1} \) are described in the main text.

\( \hat{y} = -1 \). The renewed states are \( \rho_{\hat{y} = \pm} = |\pm\rangle \langle \pm | \) and we take \( \psi(\hat{y}|x) = 1/2 \). The initial system state \( |\psi_0\rangle \) is chosen such that \( P(x) = 1/2 \). Under this condition, for both measurement directions, in the deterministic scheme we get \( C_{pf}^{(d)}|\hat{y} = -1 \rangle = |1 - |G(t)|^2/2|^{-2}C_{pf}^{(r)}|\hat{y} = -1 \rangle \). In these expressions, \( G(t, \tau) \equiv \int_0^t dt\int_0^\tau dt' f(t')G(t - t')G(\tau - t') \), where \( G(t) \) is defined by the evolution \( \langle d\psi/dt\rangle G(t) = -\int_0^1 f(t - t')G(t')dt' \), \( G(0) = 1 \). The memory kernel is the bath correlation \( f(t) = \sum_k |g_k|^2 \exp[i(\omega_0 - \omega_k)t] \).

In Fig. 1(b), for a Lorentzian spectral density \([58]\), \( f(t) = (\gamma/2\tau_c)\exp(-|t|/\tau_c) \), with \( \gamma \tau_c = 5 \), we plot the CPF correlations. In contrast to the previous case, here for both the deterministic and random schemes, the CPF correlations do not vanish. Thus, memory effects rely on BIFs, which are present independently of the bath correlation time \( \tau_c \).

In the dephasing case \( S = \sigma_z \) in Eq. (19), the CPF correlation in the random schemes is \([64]\), the CPF correlation in the random schemes is

\[ C_{pf}^{(r)}|\hat{y} \rangle = \tilde{y} \cos(\theta) \exp(-\gamma r) \sin(\Phi_{t,\tau}). \]

Here, we consider the successive measurements in Bloch directions \( \hat{n} - \hat{y} - \hat{x} \). Furthermore, we take \( \psi(\hat{y}|x) = 1/2 \), and pure states \( \rho_{\hat{y}} \) corresponding to the eigenvectors of \( \sigma_\hat{y} \). The initial condition \( |\psi_0\rangle \) is such that independently of \( \hat{n}, P(x) = 1/2 \). Under this condition the CPF correlation of the deterministic scheme can be written as

\[ C_{pf}^{(d)}|\hat{y} \rangle = \tilde{y} \cos(\theta) \exp(-\gamma r) \sin(\Phi_{t,\tau}). \]

In these expressions, \( \gamma_{t,\tau} = \gamma + \gamma_{t,\tau} - \gamma_{t+\tau} \) and \( \Phi_{t,\tau} \) is defined by the evolution \( \langle d\psi/dt\rangle G(t) = -\int_0^1 f(t - t')G(t')dt' \), \( G(0) = 1 \).

V. CONCLUSIONS

Memory effects in open quantum systems may underlay or not on a bidirectional system-environment physical exchange of information. We introduced an operational scheme that allow to distinguishing between both situations, solving a long standing problem in the theory of non-Markovian open quantum systems. The method is based on a probabilistic relation that relates the developing of BIFs with the modification of the environmental dynamical behavior. We showed that BIFs can be detected with a minimal number of three system measurement processes added to an intermediate system update operation.

A generalized CPF correlation, defined between the first and last measurement outcomes, witnesses memory effects. Depending on the system state update scheme, deterministic vs. random, it witnesses memory effects independently of its underlying origin or restricted to the presence of BIFs respectively. Consistently, for environments modeled by classical noise fluctuations or when the environment dynamics (incoherent or quantum) is not affected during the system evolution, not any BIFs is detected. The presence of BIFs for decay and dephasing dynamics modeled through unitary system-environment interactions also support the consistency of the developed approach.

Given the operational character of the proposed scheme, it can be implemented, for example, in quantum optical arrangements \([57, 58]\), providing in general a valuable experimental tool for studying the underlying origin of quantum memory effects. Generalizations for an arbitrary number of measurement processes can also be worked out in a similar way. The proposed theoretical ground may also shed light on the possibility of classifying memory effects in classical and quantum ones \([65]\), and may also provide an explicit test for different (causal) structures arising in quantum causal modelling \([66]\).
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Appendix A: Joint probabilities

The system is subjected to three measurement processes performed at times $0 \rightarrow t \rightarrow t + \tau$. The corresponding measurement operators are denoted as $\{\Omega_x\}$, $\{\Omega_y\}$, and $\{\Omega_z\}$. The intermediate $y$-measurement is taken as a projective one, $\Omega_y = |y\rangle\langle y|$. The corresponding post-measurement system state is $\rho_y = |y\rangle\langle y|$. After this step, the state transformation $\rho_y \rightarrow \rho_{\bar{y}}$ is externally applied. Each of the possible states $\{\rho_y\}$ is chosen with conditional probability $\psi(\bar{y}|y,x)$, which only depends on the previous particular measurement outcomes $x$ and $y$.

The relevant joint probability $P(z, \bar{y}, x)$ for the present proposal can be obtained as

$$P(z, \bar{y}, x) = \sum_y P(z, \bar{y}, y, x). \quad (A1)$$

The joint probability for the four events $P(z, \bar{y}, y, x)$ follows from standard quantum measurement theory after knowing the open system dynamics. The CPF probability $P(z,x|\bar{y})$, which determine the CPF correlation [Eq. (7) 56], can straightforwardly be obtained as

$$P(z, x|\bar{y}) = P(z, \bar{y}, x)/P(\bar{y}), \quad (A2)$$

where $P(\bar{y}) = \sum_z P(z, \bar{y}, x) = \sum_z P(z, \bar{y}, y, x)$. In addition, $P(z|\bar{y}) = \sum_x P(z, x|\bar{y})$ and $P(x|\bar{y}) = \sum_x P(z, x|\bar{y})$.

1. Classical noise environmental fluctuations

For classical noisy environments the outcomes probabilities are obtained for each realization, while an ensemble average is performed at the end of the calculation.

Let $\rho_0$ denotes the initial system state. After performing the first system measurement, with operators $\{\Omega_x\}$, it occurs the transformation $\rho_0 \rightarrow \rho_x$, where

$$\rho_x = \frac{\Omega_x \rho_0 \Omega_x^\dagger}{\text{Tr}_s(E_x \rho_0)} \quad (A3)$$

Here, $E_x = \Omega_x^\dagger \Omega_x$. The probability of each outcome is

$$P(x) = \text{Tr}_s(E_x \rho_0) \quad (A4)$$

During the time interval $0 \rightarrow t$, the system evolves with a (completely positive) dynamics defined by the stochastic propagator $\mathcal{E}_{t,0}^\pi$. After the second $y$-measurement, with operators $\{\Omega_y\}$, it follows the transformation $\mathcal{E}_{t,0}^\pi|\rho_x| \rightarrow \rho_y$, where

$$\rho_y = \frac{\Omega_y \mathcal{E}_{t,0}^\pi|\rho_x| \Omega_y^\dagger}{\text{Tr}_s(E_y \mathcal{E}_{t,0}^\pi|\rho_x|)} = |y\rangle\langle y|, \quad (A5)$$

and $E_y = \Omega_y^\dagger \Omega_y$. Here, we used that the $y$-measurement is a projective one, $\Omega_y = |y\rangle\langle y|$. The conditional probability $P^{st}(y|x)$ of outcome $y$ given that the previous one was $x$ is

$$P^{st}(y|x) = \text{Tr}_s(E_y \mathcal{E}_{t,0}^\pi|\rho_x|). \quad (A6)$$

At this stage, independently of the outcome $y$, the system state is updated as $\rho_y \rightarrow \rho_{\bar{y}}$. The states $\{\rho_y\}$ are chosen with conditional probability $\psi(\bar{y}|y,x)$, which does not depend on the particular noise realization.

In the final steps ($t \rightarrow t + \tau$), the system evolves with the propagator $\mathcal{E}_{t+\tau,t}^\pi$ and the last $z$-measurement, with operators $\{\Omega_z\}$, is performed ($\tau$ is the time interval between the measurements). Thus, $\mathcal{E}_{t+\tau,t}^\pi \mathcal{E}_{t,0}^\pi|\rho_{\bar{y}}| \rightarrow \rho_{\bar{y}}^z$, where

$$\rho_{\bar{y}}^z = \frac{\Omega_z \mathcal{E}_{t+\tau,t}^\pi|\rho_{\bar{y}}^z| \Omega_z^\dagger}{\text{Tr}_s(E_z \mathcal{E}_{t+\tau,t}^\pi|\rho_{\bar{y}}^z|)} \quad (A7)$$

with $E_z = \Omega_z^\dagger \Omega_z$. The conditional probability of outcome $z$ given that the previous ones were $x$ and $y$, and given that the state $\rho_0$ was imposed, is

$$P^{st}(z|\bar{y}, y, x) = \text{Tr}_s(E_z \mathcal{E}_{t+\tau,t}^\pi|\rho_{\bar{y}}^z|). \quad (A8)$$

For each noise realization, this object does not depend on outcomes $y$ and $x$.

The joint probability of the four events $P(z, \bar{y}, y, x)$ can be obtained as an average over an ensemble of realizations. Denoting the average operation with the overbar symbol, Bayes rule leads to

$$P(z, \bar{y}, y, x) = P^{st}(z|\bar{y}, y, x) \psi(\bar{y}|y,x) P^{st}(y|x) P(x). \quad (A9)$$

From Eqs. (A1), (A8), and (A9), we get

$$P(z, \bar{y}, y, x) = \frac{\text{Tr}(E_z \mathcal{E}_{t+\tau,t}^\pi|\rho_{\bar{y}}^z|) \text{Tr}(E_y \mathcal{E}_{t,0}^\pi|\rho_x|)}{\psi(\bar{y}|y,x)}, \quad (A10)$$

where $\tilde{\rho}_x \equiv \Omega_x \rho_0 \Omega_x^\dagger$, which recovers Eq. (8).

2. Completely positive system-environment dynamics

Let $\rho_{0e}^x = \rho_0 \otimes \sigma_0$ denotes the bipartite state at the initial time. After performing the first system measurement, with operators $\{\Omega_x\}$, it occurs the transformation $\rho_{0e}^x \rightarrow \rho_{xe}^x$, where the post-measurement state is

$$\rho_{xe}^x = \frac{\Omega_x \rho_{0e}^x \Omega_x^\dagger}{\text{Tr}_{se}(E_x \rho_{0e}^x)} \quad (A11)$$
with $E_x = \Omega_x\Omega_x$. The probability of each outcome is

$$P(x) = Tr_s(E_x\rho_0).$$  \hspace{1cm} (A12)

During the time interval $0 \to t$, the bipartite arrangement evolves with a completely positive dynamics defined by the propagator $\mathcal{E}_{t,0}$. After the second $y$-measurement, it follows the transformation $\mathcal{E}_{t,0}\rho_{y\omega} \to \rho_{y\omega}^{se}$, where

$$\rho_{y\omega}^{se} = \frac{\Omega_y\mathcal{E}_{t,0}[\rho_{y\omega}^{se}]\Omega_y^\dagger}{Tr_{se}(E_y\mathcal{E}_{t,1}[\rho_{y\omega}^{se}])} = \rho_y \otimes \sigma_y^{se}.$$  \hspace{1cm} (A13)

Here, $E_y = \Omega_y\Omega_y$. In the last equality we used that the second measurement is a projective one, $\Omega_y = |y\rangle\langle y|$ and $\rho_y = |y\rangle\langle y|$. The environment state is

$$\sigma_y^{se} = \frac{Tr_s(E_y\mathcal{E}_{t,0}[\rho_{y\omega}^{se}])}{Tr_{se}(E_y\mathcal{E}_{t,1}[\rho_{y\omega}^{se}])}.$$  \hspace{1cm} (A14)

The conditional probability $P(y|x)$ of outcome $y$ given that the previous one was $x$ is

$$P(y|x) = Tr_{se}(E_y\mathcal{E}_{t,1}[\rho_{y\omega}^{se}]).$$  \hspace{1cm} (A15)

At this stage, independently of the outcome $y$, the system is initialized in an independently chosen state $\rho_y$, with conditional probability $\psi(y|x)$. Thus, the bipartite state $[\text{Eq. (A13)}]$ becomes

$$\rho_y^{se} \to \rho_y^{se} = \rho_y \otimes \sigma_y^{se}. $$  \hspace{1cm} (A16)

In the final steps ($t \to t+\tau$), the bipartite system arrangement evolves with the propagator $\mathcal{E}_{t+\tau,t}$, and the last $z$-measurement is performed. Hence, $\mathcal{E}_{t+\tau,t}[\rho_y \otimes \sigma_y^{se}] \to \rho_z^{se}$, where

$$\rho_z^{se} = \frac{\Omega_z\mathcal{E}_{t+\tau,t}[\rho_y \otimes \sigma_y^{se}]\Omega_z^\dagger}{Tr_{se}(E_z\mathcal{E}_{t+1,t}[\rho_y \otimes \sigma_y^{se}])},$$  \hspace{1cm} (A17)

with $E_z = \Omega_z\Omega_z$. The conditional probability of outcome $z$ given that the previous ones were $x$ and $y$, and given that the state $\rho_y$ was imposed, is

$$P(z|\tilde{y},y,x) = Tr_{se}(E_z\mathcal{E}_{t+\tau,t}[\rho_y \otimes \sigma_y^{se}]).$$  \hspace{1cm} (A18)

From Bayes rule, the joint probability $P(z, \tilde{y}, y, x)$ of the four events can be written as

$$P(z, \tilde{y}, y, x) = P(z|\tilde{y}, y, x)\psi(\tilde{y}|y, x)P(y|x)P(x).$$ \hspace{1cm} (A19)

From Eqs. (A12), (A15), and (A18), it follows

$$P(z, \tilde{y}, y, x) = Tr_{se}(E_z\mathcal{E}_{t+\tau,t}[\rho_y \otimes \sigma_y^{se}])\psi(\tilde{y}|y, x)Tr_s(E_y\mathcal{E}_{t,0}[\rho_{y\omega}^{se}]),$$  \hspace{1cm} (A20)

where $\tilde{\rho}_x^{se} \equiv \Omega_x\rho_0^{se}\Omega_x^\dagger$. Using Eq. (A14) for $\sigma_y^{se}$, finally we get

$$P(z, \tilde{y}, y, x) = \frac{Tr_{se}(E_z\mathcal{E}_{t+\tau,t}[\rho_y \otimes \rho_{y\omega}^{se}])\psi(\tilde{y}|y, x)}{\psi(\tilde{y}|y, x)} = Tr_{se}(E_z\mathcal{E}_{t+\tau,t}[\rho_y \otimes \rho_{y\omega}^{se}])$$  \hspace{1cm} (A21)

which recovers Eq. (A11).

3. Unconditional environment average

The calculus of $P(z, \tilde{y}, y, x)$ in the previous section relies on the association $(\cdots)_e \leftrightarrow Tr_e(\cdots)$. This unconditional environment average emerges when the successive (non-selective) measurement of the environment do not modify its state at each stage. While this result follows straightforwardly from quantum measurement theory, here it is explicitly confirmed.

We consider three measurement processes but now they provide information of both the system and the environment. The successive outcomes are denoted as $x \to (y \to \tilde{y}) \to z$ and $X \to \mathcal{Y} \to \mathcal{Z}$ (Latin and Fraktur letters respectively). Introducing the notation $X = (x, \mathcal{X})$, $Y = (y, \mathcal{Y})$, and $Z = (z, \mathcal{Z})$, the measurement operators are denoted as $\{\Omega_X\}$, $\{\Omega_Y\}$, and $\{\Omega_Z\}$, where $\Omega_X = \Omega_x \otimes \Omega_X$, $\Omega_Y = \Omega_y \otimes \Omega_Z$, and $\Omega_Z = \Omega_z \otimes \Omega_Z$. As before, the intermediate system measurement is taken as a projective one, $\Omega_x = |y\rangle\langle y|$.

From Bayes rule, the probability of all measurements and preparation events can be written as

$$P(Z, \tilde{y}, y, X) = P(Z|\tilde{y}, y, X)\psi(\tilde{y}|y, x)P(Y|X)P(X).$$  \hspace{1cm} (A22)

By performing the same calculus steps as in the previous section, from Eqs. (A20) straightforwardly we obtain

$$P(Z, \tilde{y}, y, X) = Tr_{se}(E_Z\mathcal{E}_{t+\tau,t}[\rho_y \otimes \sigma_y^{se}])\psi(\tilde{y}|y, x)Tr_s(E_Y\mathcal{E}_{t,0}[\rho_{y\omega}^{se}]).$$  \hspace{1cm} (A23)

where $E_J = \Omega_J\Omega_J$ ($J = X, Y, Z$), and $\tilde{\rho}_X^{se} = \Omega_X\rho_0^{se}\Omega_X^\dagger$. Furthermore,

$$\sigma_y^{se} = \frac{\Omega_Y Tr_s(E_Y\mathcal{E}_{t,0}[\rho_{y\omega}^{se}])\Omega_Y^\dagger}{Tr_{se}(E_Y\mathcal{E}_{t,1}[\rho_{y\omega}^{se}])},$$  \hspace{1cm} (A24)

where $\rho_X^{se} = \tilde{\rho}_X^{se}/Tr_s(E_X\rho_0^{se})$. Similarly, Eq. (A23) can be rewritten as

$$P(Z, \tilde{y}, y, X) = \frac{Tr_{se}(E_Z\mathcal{E}_{t+\tau,t}[\rho_y \otimes \sigma_y^{se}])Tr_s(E_Y\mathcal{E}_{t,0}[\rho_{y\omega}^{se}])\Omega_Y^\dagger}{\psi(\tilde{y}|y, x)}.$$  \hspace{1cm} (A25)

The probability for the environment outcomes follows by marginating the system outcomes,

$$P(\mathcal{Z}, \mathcal{Y}, X) = \sum_{\tilde{y}, y, x} P(\mathcal{Z}, \tilde{y}, y, X).$$  \hspace{1cm} (A26)

Similarly, the probability for the system outcomes follows by marginating the outcomes corresponding to the reservoir measurements,

$$P(z, \tilde{y}, y, x) = \sum_{\mathcal{Z}, \mathcal{Y}} P(z, \tilde{y}, y, X).$$  \hspace{1cm} (A27)

This result for $P(z, \tilde{y}, y, x)$ relies on explicit environment measurements. In contrast, the results of the previous section were derived assuming that the environment is not observed at all. Nevertheless, both kind of results can
be put in one-to-one correspondence. In fact, Eqs. (A20) and (A21) can be recovered from Eqs. (A23) and (A25), via the margination (A27), under the conditions
\[ \sigma_0 = \sum_{\chi} \Omega_{\chi} \sigma_0 \Omega_{\chi}^\dagger, \quad \sigma_{e z}^{\mu z} = \sum_{\gamma} \Omega_{\gamma} \sigma_{e z}^{\mu z} \Omega_{\gamma}^\dagger, \]  
(A28)
where \( \sigma_0 \) is the initial bath state and \( \sigma_{e z}^{\mu z} \) is defined by Eq. (A14). As expected, these equalities imply that the bath states at each stage are not modified by the corresponding reservoir (non-selective) measurement processes. Thus, the unconditional environment average of the previous section [Eq. (A21)] relies on this kind of observables, which allow us to formulate the full approach without performing any explicit reservoir measurement.

For projective environment measurements, the relations (A28) implies the commutation relations \([\sigma_0, \Omega_{\chi}] = 0, \quad [\sigma_{e z}^{\mu z}, \Omega_{\gamma}] = 0\). In classical (incoherent) reservoirs, where the bath state is diagonal in (a unique) privileged basis, these conditions define the corresponding “classical environment observables.”

Appendix B: Eternal non-Markovianity

The non-Markovian system density matrix evolution is given by Eq. (17). There exist different underlying dynamic that lead to this dynamics. The solution map \( \rho_0 \to \rho_t \) can be written as a mixture of three Markovian maps [49]
\[ \rho_t = \sum_{\alpha=x,\hat{y},\hat{z}} q_\alpha \mathcal{E}_t^{(\alpha)} [\rho_0], \]  
(B1)
with positive and normalized statistical weights \( \{q_\alpha\} \), \( \sum_{\alpha=x,\hat{y},\hat{z}} q_\alpha = 1 \). The Markovian propogators are
\[ \mathcal{E}_{t,0}^{(\alpha)} [\rho_0] = h_t^{(+)} t_0 \rho_0 + h_t^{(-)} t_0 \sigma_0 \rho_0 \sigma_0, \]  
(B2)
with scalar functions \( h_t^{(\pm)} (\equiv (1 \pm e^{-2\gamma t})/2 \). Each propagator indicates what is obtained in the \( \hat{y} \)-direction of the Bloch sphere. The intermediate one is performed in a direction \( \hat{n} = \{\sin(\theta), 0, \cos(\theta)\} \), which lies in the \( \hat{x}\hat{z} \) plane of the Bloch sphere. Thus, the measurement operators are \( \Omega_{x=\pm} = |\hat{x}_\pm \rangle \langle \hat{x}_\pm |, \quad \Omega_{y=\pm} = |\hat{y}_\pm \rangle \langle \hat{y}_\pm |, \quad \Omega_{z=\pm} = |\hat{z}_\pm \rangle \langle \hat{z}_\pm | \). Consistent with the chosen directions, we have \( |\hat{x}_\pm \rangle = (|+\rangle \pm |-\rangle) / \sqrt{2} \). jointly with \( |\hat{n}_+ \rangle = \cos(\theta / 2)|+\rangle + \sin(\theta / 2)|-\rangle \), and \( |\hat{n}_- \rangle = \sin(\theta / 2)|+\rangle - \cos(\theta / 2)|-\rangle \).

For an explicit calculation of the previous probabilities we need to calculate \( \rho_0 (\hat{n} |\hat{x} \rangle \langle \hat{x}|) = \mathcal{E}_{t,0}^{(\alpha)} [\rho_0] \) and \( \rho_0 (\hat{n} |\hat{n} \rangle \langle \hat{n}|) = \mathcal{E}_{t,0}^{(\alpha)} [\rho_0] \), where \( \mathcal{E}_{t,0}^{(\alpha)} [\rho_0] \). Thus, independently of the chosen system measurement observables it follows \( \mathcal{C}^{(\alpha)} (\hat{n} |\hat{y} \rangle \langle \hat{y}|) = 0 \), indicating consistently the absence of any BIF.

\( \hat{x}\hat{z} \)- measurements

We consider the case in which the three measurements are projective ones. The first and third ones are performed in \( \hat{x}\hat{z} \)-direction of the Bloch sphere. The intermediate one is performed in a direction \( \hat{n} = \{\sin(\theta), 0, \cos(\theta)\} \), which lies in the \( \hat{x}\hat{z} \) plane of the Bloch sphere. Thus, the measurement operators are \( \Omega_{x=\pm} = |\hat{x}_\pm \rangle \langle \hat{x}_\pm |, \quad \Omega_{y=\pm} = |\hat{y}_\pm \rangle \langle \hat{y}_\pm |, \quad \Omega_{z=\pm} = |\hat{z}_\pm \rangle \langle \hat{z}_\pm | \). Consistent with the chosen directions, we have \( |\hat{x}_\pm \rangle = (|+\rangle \pm |-\rangle) / \sqrt{2} \). jointly with \( |\hat{n}_+ \rangle = \cos(\theta / 2)|+\rangle + \sin(\theta / 2)|-\rangle \), and \( |\hat{n}_- \rangle = \sin(\theta / 2)|+\rangle - \cos(\theta / 2)|-\rangle \).

For an explicit calculation of the previous probabilities we need to calculate \( \rho_0 (\hat{n} |\hat{x} \rangle \langle \hat{x}|) = \mathcal{E}_{t,0}^{(\alpha)} [\rho_0] \) and \( \rho_0 (\hat{n} |\hat{n} \rangle \langle \hat{n}|) = \mathcal{E}_{t,0}^{(\alpha)} [\rho_0] \), where \( \mathcal{E}_{t,0}^{(\alpha)} [\rho_0] \). Thus, independently of the chosen system measurement observables it follows \( \mathcal{C}^{(\alpha)} (\hat{n} |\hat{y} \rangle \langle \hat{y}|) = 0 \), indicating consistently the absence of any BIF.
where we considered the updated states \( \rho_{\tilde{y}} = \pm = |\tilde{n}_x\rangle\langle\tilde{n}_x| \).

The generalized CPF correlation is given by Eq. \[17\],
\[
C_{pf}^{(d/r)}|_{\tilde{y}} = \sum_{zz} O_z O_x [P(z, x|y) - P(z|y)P(x|y)],
\]
where \( P(z, x|y) \) follows from Eq. \[A2\]. Furthermore, \( O_z = z = \pm 1 \) and \( O_x = x = \pm 1 \). From Eq. \[B5\], the CPF correlation in the deterministic scheme reads
\[
C_{pf}^{(d)}|_{\tilde{y}} = \sin^2(\theta) \frac{1 - \langle x \rangle^2}{4|P(y)|^2} \langle c(t + \tau) - c(t)c(\tau) \rangle,
\]
where \( P(y) = (1/2)[1 + \tilde{y}\langle x \rangle \sin(\theta)c(t)] \) and \( \langle x \rangle\equiv \sum_{x=\pm 1} xP(x) \). When \( \rho_0 = |\pm\rangle\langle\pm| \) it follows \( P(x) = 1/2 \) and consequently \( \langle x \rangle = 0 \). This case recovers Eq. \[18\].

In the random scheme, from Eq. \[B10\] consistently it follows
\[
C_{pf}^{(r)}|_{\tilde{y}} = 0.
\]

This equality is valid independently of the chosen measurement processes and updated system states [see Eq. \[B6\]].

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[61] In fact, from Bayes rule \( P(z, x|\tilde{y}) = P(z, \tilde{y}, x)/P(\tilde{y}) \) where \( P(\tilde{y}) = \sum_{x,z} P(z, \tilde{y}, x) \), jointly with \( P(z|\tilde{y}) = \sum_{x} P(z, x|\tilde{y}) \) and \( P(x|\tilde{y}) = \sum_{z} P(z, x|\tilde{y}) \).

[62] Formally, the update \( \rho_y \rightarrow \rho_{\tilde{y}} \) is equivalent to discard the system state and feeds forward an independent system state. Named as “repreparation,” this operation was introduced in Ref. \[53\] for studying “quantum Markov order.” The formulation with unitary transformations \{\( U(\tilde{y}|y) \)\} defines a feasible experimental implementation.

[63] Under the condition \([H_e, H_I] = 0\), the invariance property \[13\] is fulfilled only when the initial bath state satisfies \([\sigma_0, H_e] = 0\).

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