Universality classes for single parameter scaling in 1D Anderson localisation

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The one-dimensional Schrödinger equation is studied for a variety of disordered potentials with finite second moment \((V^2) < \infty\) or distributed with power law tail \(P(V) \sim |V|^{-1-\alpha}\) for \(0 < \alpha < 2\). The fluctuations of the wave function \(\psi(x)\) are characterised through the generalised Lyapunov exponent (GLE), i.e. the cumulant generating function of \(\ln|\psi(x)|\). In the high energy/weak disorder limit, a universal expression of the GLE is derived. For \((V^2) < \infty\), one recovers Gaussian fluctuations with the variance equal to the mean value: \(\gamma_2 \simeq \gamma_1\) (single parameter scaling). For \((V^2) = \infty\), one finds \(\gamma_2 \simeq (2/\alpha) \gamma_1\) and non Gaussian large deviations, related to the universal limiting behaviour of the conductance distribution \(W_L(g) \sim g^{-1+\alpha/2}\) for \(g \to 0\).

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Consider the sum \(S = \sum_{n=1}^{N} X_n\) of \(N\) independent random variables, distributed according to the same distribution \(p(x)\). It is well-known that the determination of the distribution of the sum \(P_N(s)\) in the \(N \to \infty\) limit leads to consider different universality classes, depending on the second moment. When \(\langle X_n^2 \rangle < \infty\), the distribution \(P_N(s)\) is given by the universal Gaussian law (central limit theorem). When the distribution presents a power law tail, \(p(x) \sim |x|^{-1-\alpha}\) for \(x \to \pm \infty\), with \(0 < \alpha < 2\), the second moment is infinite \((\langle X_n^2 \rangle = \infty\) and the problem belongs to a different universality class. Then, \(P_N(s)\) is given by a Lévy law of index \(\alpha\), irrespectively of the details of the distribution \(p(x)\). In this letter, it is argued that similar considerations apply to the problem of wave localisation in a random medium in one dimension, which can be notoriously described in terms of random matrix products, i.e. non commuting objects. If the wave is the solution of the 1D Schrödinger equation, one can equivalently analyze the transfer matrix or the wave \(\psi(x)\) solution of the initial value (Cauchy) problem. The goal is here to study the generalised Lyapunov exponent (GLE) \(\Lambda(q) = \lim_{x \to \infty} \ln \langle |\psi(x)|^q \rangle / x = \sum_{n=1}^{\infty} \gamma_n/q! q^n\), where \(\langle \cdots \rangle\) denotes disorder averaging. The GLE is the cumulant generating function of \(\ln|\psi(x)|\). In particular, \(\gamma_1 = \lim_{x \to \infty} \ln \langle |\psi(x)| \rangle / x\) is the Lyapunov exponent (inverse localisation length). The different universality classes can be revealed by inspection of the “single parameter scaling” (SPS) property, a corner stone of localisation in a random medium in one dimension, which can arise effectively due the presence of flat bands. The popularity of the Lloyd model comes from the possibility to get several analytical results like the Lyapunov exponent [12], its variance [4] or higher cumulants [16]. In particular, the relation between the two cumulants was shown to present an additional factor of two, \(\gamma_2 \simeq 2 \gamma_1\), whose origin was not explained, to the best of my knowledge. It is the aim of this letter to provide a framework explaining in particular this factor, and showing that the SPS relation can be extended to a very broad class of 1D disordered potentials uncorrelated in space, with distribution with power law tail \(P(V) \sim |V|^{-1-\alpha}\), where \(\alpha \in [0, 2]\). The main result of the letter is the expression of the generalised Lyapunov exponent

\[
\Lambda(q) \simeq \gamma_1 \frac{2^{\alpha} \Gamma(\frac{1+\alpha}{2})}{\pi^{1/2} \Gamma(\alpha) \Gamma(1+\frac{\alpha}{2})} \times \Gamma\left(\frac{\alpha+2+q}{2}\right) \Gamma\left(\frac{\alpha-q}{2}\right) \sin\left(\frac{\pi q}{2}\right),
\]

for \(q \in [-2-\alpha, \alpha]\). This result, valid in the high energy/weak disorder limit, is universal. The case with finite moment \((V^2) < \infty\) is described by setting \(\alpha = 2\) in [12], leading to \(\Lambda(q) \simeq \gamma_1 q (1+q/2)\), characterizing Gaussian fluctuations. Expansion of [14] gives the variance

\[
\gamma_2 \simeq \frac{2}{\alpha} \gamma_1 \quad \text{for } \alpha \in [0, 2],
\]

\[\text{(3)}\]
which extends \([1]\). For the Cauchy case \((\alpha = 1)\), the ratio of \(\gamma_2/\gamma_1 \approx 2\) is thus related to the exponent controlling the disorder distribution.

**Models.**— I am interested here in universal properties of Anderson localisation, hence there is some freedom on the choice of the model. I consider a continuous model: the Schrödinger equation with a random potential \(V(x)\),

\[
- \psi''(x) + V(x) \psi(x) = E \psi(x).
\]

The only crucial assumption here is the absence of spatial correlations in the disorder. As a consequence, the disorder can be fully characterised by the Lévy exponent \(\mathcal{L}(s)\) of the stochastic process obtained by integration of the random potential \([17]\)

\[
\left\{ e^{-is \int_0^x dt V(t)} \right\} = e^{-x \mathcal{L}(s)}.
\]

It is common to introduce the generating functional, related to the Lévy exponent by \(\langle \exp \left\{ - i \int dx h(x) V(x) \right\} \rangle = \exp \left\{ - i \int dx \mathcal{L}(h(x)) \right\} \) \([18]\). In particular, one can deduce the generating function \(\langle V(x)V(x') \rangle - \langle V(x) \rangle^2 = \mathcal{L}''(0) \delta(x-x')\). For example, the model with Gaussian white noise potential \([19]\), \(\rho \delta(x-x_n)\), for \(\delta\)-impurities at random uncorrelated positions with mean density \(\rho\) corresponds to \(\mathcal{L}(s) = (\sigma/2) s^2\). The Frisch-Lloyd model \([20]\), \(V(x) = \sum_n v_n \delta(x-x_n)\), for \(\delta\)-impurities at random uncorrelated positions with mean density \(\rho\) corresponds to \(\mathcal{L}(s) = \rho \left[ 1 - \tilde{p}(s) \right]\), where \(\tilde{p}(s)\) is the Fourier transform of the weight distribution \(p(v)\).

From now on, I assume that \(V(x)\) has a symmetric distribution around zero, leading to a real symmetric Lévy exponent, \(\mathcal{L}(-s) = \mathcal{L}(s)\). In the universal regime, only the \(s \to 0\) behaviour of the Lévy exponent is important as I will show. One has to distinguish two cases

— The Lévy exponent has an analytic behaviour for \(s \to 0\), precisely \(\mathcal{L}(s) \simeq c s^2\), where \(c\) is some nonuniversal constant (disorder strength).

— The Lévy exponent has a non-analytic behaviour at the origin, \(\mathcal{L}(s) \simeq c |s|^\alpha\) for \(s \to 0\), with \(\alpha \in ]0,2[\).

Both cases can be treated on the same footing by considering \(\alpha \in ]0,2[\). These two situations have a clear interpretation within the Frisch-Lloyd model. In the first case \(c = \rho \langle v_n^2 \rangle / 2 < \infty\), while in the second, the weight distribution presents a power law tail \(p(v) \sim |v|^{-1-\alpha}\) leading to \(\langle v_n^2 \rangle = \infty\) and \(\tilde{p}(s) \simeq 1 - b |s|^\alpha\) for \(s \to 0\) \([21]\). The strict equality \(\mathcal{L}(s) = c |s|^\alpha\) describes the case where \(\int_0^\infty dt V(t)\) is a \(\alpha\)-stable symmetric Lévy process \([17]\).

**Formalism.**— The analysis is based on the formalism of Ref. \([22]\), where the question of fluctuations of random matrix products was addressed. The GLE is the largest eigenvalue of a certain linear operator \([22]\). The spectral problem (Eq. 7.17 of Ref. \([22]\)) can be formulated as follows: denote by \(\phi(s;\Lambda)\) the solution of

\[
\left[ -\frac{d^2}{ds^2} + \frac{q}{s} \frac{d}{ds} + E - \frac{\mathcal{L}(s) + \Lambda}{is} \right] \phi(s;\Lambda) = 0
\]

vanishing for \(s \to +\infty\). One can argue that it behaves as

\[
\phi(s;\Lambda) \simeq 1 - \frac{i\Lambda}{q} s + \omega(\Lambda) s^{q+1} + \cdots \text{ for } s \to 0^+.
\]

The GLE is the solution \(\Lambda = \Lambda(q)\) of the secular equation \(\text{Im}[\omega(\Lambda)] = 0\).

**GLE in the universal regime.**— The solution of the spectral problem is now found in the high energy/weak disorder limit. For \(E = k^2 \to +\infty\), one expects the GLE to be of the order of the disorder strength, \(\Lambda(q) = \mathcal{O}(c)\). The idea is to solve the differential equation \([3]\) by a perturbation method by considering \(\mathcal{L}(s) + \Lambda / (is)\) as the perturbation. One writes \(\phi(s;\Lambda) = \phi_0(ks;\Lambda) + \phi_2(ks;\Lambda) + \cdots\) where \(\phi_0 = \mathcal{O}(c^\alpha)\). Correspondingly, the coefficient \(\omega(\Lambda)\) can be expanded in powers of \(c\) as well \(\omega(\Lambda) = \omega_0 + \omega_1(\Lambda) + \omega_2(\Lambda) + \cdots\).

At order \(c^0\) the differential equation \(-\phi_0''(y) + (q/y)\phi_0(y) + \phi_0(y) = 0\) has solution \(\phi_0(y) = y^\nu K_{\nu}(y)\) with \(\nu = (q+1)/2\), where \(K_{\nu}(z)\) is the MacDonald function \([24]\). The order \(c^0\) contribution solves

\[
\left[ -\frac{d^2}{dy^2} + \frac{q}{y} \frac{d}{dy} + 1 \right] \phi_n(y;\Lambda) = \frac{\mathcal{L}(y/k) + \Lambda}{iky} \phi_{n-1}(y;\Lambda)
\]

Since \(\phi_0\) is real, one deduces that \(\phi_1 \in \mathbb{R}, \phi_2 \in \mathbb{R}\), etc. Hence \(\omega_0 \in \mathbb{R}, \omega_1 \in \mathbb{R}, \omega_2 \in \mathbb{R}\), etc., and the secular equation takes the form \(\omega_1(\Lambda) + \omega_2(\Lambda) + \cdots = 0\). In the \(E \to +\infty\) limit, one can simply truncate the equation as \(\omega_1(\Lambda) \simeq 0\).

The solution at order \(c^1\) is

\[
\phi_1(y;\Lambda) = \frac{i}{k} y^{\nu} \left\{ K_{\nu}(y) \int_0^{\nu} du \left( \mathcal{L}(u/k) + \Lambda \right) I_0(u) K_{\nu}(u) + I_{\nu}(y) \int_0^{\infty} du \left( \mathcal{L}(u/k) + \Lambda \right) K_{\nu}(u)^2 \right\}
\]

which vanishes exponentially at infinity, as \(\sim e^{-y}\). The problem is now to identify the term \(\omega_1(\Lambda) y^{2\nu+1}\) for \(y \to 0\).

Eq. \([3]\) makes clear that, in the limit \(k \to \infty\), the solution is fully controlled by the \(s \to 0\) behaviour of the Lévy exponent \(\mathcal{L}(s)\). Thus, all the results derived below are completely universal, controlled only by the exponent \(\alpha \in ]0,2[\) of the Lévy exponent.

It is easy to see that the first term of \([4]\) only provides contributions \(\mathcal{O}(y)\) and \(\mathcal{O}(y^{2\nu+2})\), to lowest order in \(y\), hence do not contribute to \(\omega_1(\Lambda)\). The leading order of the second term of \([4]\) is easily obtained

\[
y^{\nu} I_{\nu}(y) \int_y^{\infty} du \left( \frac{\mathcal{L}(u/k) + \Lambda}{\Lambda} + 1 \right) K_{\nu}(u)^2
\]

\[
= \frac{2^{\nu} T(\nu)}{4\nu(2\nu-1)} y + \Omega(\Lambda) y^{2\nu} + \cdots
\]

where \(\Omega(\Lambda) = -(ik/\Lambda)\omega_1(\Lambda)\). It is however much more tricky to get the next leading order term \(\mathcal{O}(y^{2\nu})\), i.e. \(\mathcal{O}(y^{2\nu+1})\), and derive the coefficient \(\Omega(\Lambda)\).
For $q < \alpha$, the integral $\int_y^\infty u\,d\mathcal{L}(u/k)\,K_\nu(u)^2$ has a limit for $y \to 0$. Thus
\[ \Omega(\Lambda) = \frac{1}{2\nu\Gamma(\nu + 1)} \left[ \frac{c}{\Lambda^{\alpha}} \int_0^\infty u^\alpha\,K_\nu(u)^2 \right. \\
\left. + \lim_{y \to 0} \left\{ \int_y^\infty du\,K_\nu(u)^2 - 2^{2\nu - 2}\Gamma(\nu)^2 \frac{y^{-2\nu + 1}}{2\nu - 1} \right\} \right] \]
One uses (formula 6.576 of [24])
\[ \int_0^\infty du\,u^\alpha\,K_\nu(u)^2 = \frac{2^{\alpha - 2}}{\Gamma(1+\alpha)} \Gamma \left( \frac{1+\alpha}{2} \right) \Gamma \left( \frac{1+\alpha}{2} + \nu \right) \Gamma \left( \frac{1+\alpha}{2} - \nu \right) \]
for $\alpha > 2|\nu| - 1$, i.e. $-2 - \alpha < q < \alpha$, and
\[ \lim_{y \to 0} \left\{ \int_y^\infty du\,K_\nu(u)^2 - 2^{2\nu - 2}\Gamma(\nu)^2 \frac{y^{-2\nu + 1}}{2\nu - 1} \right\} = \frac{\pi^2}{4\cos\pi\nu} \]
for $0 < \nu < 3/2$, i.e. $-1 < q < 2$ for $0 < \nu < 1/2$, the integral converges for $y = 0$ and the equation is simply given by setting $\alpha = 0$ in [13]. From (11), one sees that the secular equation $\Omega(\Lambda) \simeq 0$ gives
\[ \Lambda(q) \simeq \frac{c\,\kappa^{-\nu}\Gamma(\frac{1+\alpha}{2})}{\pi^{3/2}\Gamma(1+\frac{\alpha}{2})} \left[ \frac{\alpha + 2 + q}{2} \right] \Gamma \left( \frac{\alpha - q}{2} \right) \sin \left( \frac{\pi q}{2} \right) \]
(14)
for $q \in [-2 - \alpha, \alpha]$. One checks the symmetry relation $\Lambda(q) = \Lambda(-2 - q)$ [22, 25].

The normal case for disorder with finite second moment corresponds to $\alpha = 2$: from (14), one gets $\Lambda(q) \simeq \left[ c/(4k^2) \right] q \left( 1 + q/2 \right)$. Thus the cumulants $\gamma_n$ with $n > 2$ are subleading in the disorder, as shown in Refs. [16, 20] by studying the first cumulants for specific models (corresponding to the Gaussian distribution). CUMULANTS.— Note that the form $\Lambda(q) \simeq (\alpha + q)f_{\text{odd}}(q)$, with $f_{\text{odd}}(-q) = -f_{\text{odd}}(q)$, implies
\[ \gamma_n \simeq \frac{n}{\alpha} \gamma_{n-1} \text{ for } n \text{ even,} \]
generalizing [3]. Expansion of $\Lambda(q)$ in powers of $q$ gives
\[ \gamma_1 \simeq \frac{c\Gamma(\alpha)}{(2k)^\alpha}, \quad \gamma_3 \simeq \left[ \frac{3}{2} \psi'(\frac{\alpha}{2} - \frac{\pi^2}{4}) \right] \gamma_1, \cdots \]
where $\psi(z)$ is the digamma function. For $\alpha = 2$, one recovers the well-known perturbative result $\gamma_1 \simeq c/(4E)$ (see [1, 27]). For $\alpha < 2$, the anomalous energy dependence $\gamma_1 \simeq k^{-\alpha}$ was obtained earlier in Ref. [28]. Rescaling the GLE [14] by $\gamma_1$ leads to the universal form [2].

Cauchy disorder.— The case of Cauchy disorder ($\alpha = 1$) has been much studied and deserves a special discussion. From (14), one gets the rescaled GLE
\[ L(q) = \lim_{E \to \infty} \frac{\Lambda(q)}{\gamma_1} = \frac{2}{\pi} (q + 1) \tan \left( \frac{\pi q}{2} \right) \text{ for } q \in [-3, 1]. \]

One can write $L(q) = \sum_{n=1}^\infty (\kappa_n/n! \alpha^n$, where $\kappa_n = \lim_{E \to \infty} \gamma_n/\gamma_1$ are the rescaled cumulants, given by
\[ \kappa_n = 4\pi^{n-2} (2^n - 1) |B_n| \text{ and } \kappa_{n-1} = \frac{\kappa_n}{n} \]
for $n$ even, where $B_n$‘s are the Bernoulli numbers [24]. In particular $\kappa_1 = 1$, $\kappa_2 = 2$, $\kappa_3 = \pi^2/2 \simeq 4.93$, $\kappa_4 = 2\pi^2 \simeq 19.7$, $\kappa_5 = \pi^4 \simeq 79.4$, $\kappa_6 = 6\pi^2$, etc. This perfectly agrees with the estimation given in [10]: $\kappa_3 \simeq 5$ [29], $\kappa_4 \simeq 20$ and $\kappa_5 \simeq 100$. In Ref. [10], the calculation was performed within a lattice model, rather different from the continuous model studied here, which emphasizes once more the universal character of the results presented in this letter.

Large deviations.— The GLE (14) diverges as $\Lambda(q) \sim 1/(\alpha - q)$ for $q \to \alpha^-$ (and diverges symmetrically for $q \to (-\alpha - 2)^+$). Correspondingly, the distribution of the modulus of the wave function $|\psi(x)|$ presents power law large deviation tails
\[ \mathcal{P}_x(\psi) \sim \left\{ \begin{array}{ll} \psi^{1+\alpha} & \text{for } \psi \to 0 \\
\psi^{-1-\alpha} & \text{for } \psi \to \infty \end{array} \right. \]
while typical fluctuations are described by the log-normal behaviour $\mathcal{P}_x(\psi) \sim \psi^{-1} \exp \left\{ -[\alpha/(4\gamma_1)](\ln \psi - \gamma_1 x)^2 \right\}$. The tail for $\psi \to \infty$ corresponds to weakly conducting samples, for which one can write $\psi \sim |\mathcal{L}|^{-2}$, where $L$ is the sample length. Hence, the distribution of the conductance presents the universal large deviation tail
\[ W_L(g) \sim g^{-1+\alpha/2} \text{ for } g \to 0. \]
In [12], a power law behaviour was identified, although the exponent was not obtained. The result (20) is in agreement with the numerics of Ref. [31].

Conclusion.— In this letter, I have derived the generalized Lyapunov exponent characterizing wave function statistics in the 1D Schrödinger equation. A broad variety of disordered models was considered, which has allowed to derive a universal form for the GLE in the high energy/weak disorder limit, Eq. (2). For disorder with finite second moment ($\alpha = 2$), one has $\Lambda(q) \simeq \gamma_1(q)(1 + q/2)$ for $q \in [-4, +2]$, which characterizes Gaussian fluctuations. I have however not proven that the GLE is still quadratic out of the interval. For the model with Gaussian white noise potential, the non universal behaviour $\Lambda(q) \sim c^{1/3} |q|^{4/5}$ for $q \to \pm \infty$ was derived in [31, 32] (see also [22]). For disorder with power law distribution (exponent $\alpha \in [0, 2]$), the universal expression (20) was derived for $q \in [-2 - \alpha, \alpha]$, with the GLE diverging at the boundaries of the interval. Interestingly, this shows that, for small but finite disorder, universality is stronger for power law disorder with $\langle V^2 \rangle = \infty$ than in the normal case with $\langle V^2 \rangle < \infty$. In this latter case, the large deviations are non universal (dominated by higher cumulants $\gamma_n$ with $n > 2$ subleading in the disorder strength), while
in the former all cumulants scale the same way and the GLE is universal over its whole interval of definition.

An interesting question could be to investigate the universality of the GLE for models within other symmetry class: in the presence of a chiral symmetry, disordered models are known to present “anomalies” which have been widely studied. The first cumulants have been determined for a lattice model in \([34]\) (see also \([35]\) for a study of \(\gamma_2\) within a continuous model), but the GLE is still unknown in this case. Finally, a more challenging issue would be extend the results to the multichannel case or higher dimensions, relevant for the problem of Ref. \([32]\).

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