COHERENT CATEGORIFICATION OF QUANTUM LOOP ALGEBRAS: 
THE $SL(2)$ CASE

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Abstract. We construct an equivalence of graded Abelian categories from a category of representations of the quiver-Hecke algebra of type $A_1^{(1)}$ to the category of equivariant perverse coherent sheaves on the nilpotent cone of type $A$. We prove that this equivalence is weakly monoidal. This gives a representation-theoretic categorification of the preprojective K-theoretic Hall algebra considered by Schiffmann-Vasserot. Using this categorification, we compare the monoidal categorification of the quantum open unipotent cells of type $A_1^{(1)}$ given by Kang-Kashiwara-Kim-Oh-Park in terms of quiver-Hecke algebras with the one given by Cautis-Williams in terms of equivariant perverse coherent sheaves on the affine Grassmannians.

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1. Introduction

1.1. Main results of the paper. Let \( Q = (I, \Omega) \) be a quiver of Kac-Moody type. For each dimension vector \( \beta \), let \( X_\beta \) be the variety of all \( \beta \)-dimensional representations of the path algebra of \( Q \) and \( X_\beta \) be the corresponding moduli stack. Lusztig introduced a graded additive subcategory \( C(X_\beta) \) of the bounded constructible derived category \( D^b(X_\beta) \) whose split Grothendieck group is isomorphic to the \( \beta \)-weight subspace of the quantum unipotent enveloping algebra \( U_q(n) \) of type \( Q \). Then, Khovanov-Lauda and Rouquier showed that \( U_q(n) \) is the Grothendieck group of the monoidal category of all projective graded modules over the quiver-Hecke algebra \( R \) of \( Q \). According to Rouquier and Varagnolo-Vasserot, this isomorphism lifts to an equivalence of additive graded monoidal categories between \( \bigoplus_\beta C(X_\beta) \), equipped with Lusztig’s geometric induction bifunctor and the grading given by the cohomological shift, and the category \( C^{proj} \) of all finitely generated graded projective \( R \)-modules, equipped with the algebraic induction bifunctor \( \circ \). Further, the quantum unipotent coordinate algebra \( A_q(n) \) of \( Q \) is isomorphic to the Grothendieck group of the monoidal category of all finite dimensional graded modules over \( R \), in such a way that the irreducible self-dual graded \( R \)-modules are identified with the dual canonical basis elements in \( A_q(n) \).

Now, let us assume that the quiver \( Q \) is of type \( A_1^{(1)} \). The graded category \( C \) of all finitely generated graded \( R \)-modules is affine properly stratified. The simple self-dual modules are labelled by Kostant partitions of a dimension vector \( \beta = n\alpha_1 + m\alpha_0 \) of \( Q \), for some non-negative integers \( n, m \). There is a monoidal structure on \( C \) given by the algebraic induction bifunctor \( \circ \). Let \( D \) be the full graded subcategory of \( C \) consisting of all objects whose composition factors are graded shifts of the simple self-dual modules labelled by the Kostant partitions supported on the set of all real positive roots of \( Q \) of the form \( \alpha_0 + n\delta \) with \( n \in \mathbb{N} \). It is a graded monoidal subcategory of \( C \) which is polynomial highest weight.

For any integer \( r \geq 0 \), the Lie algebra \( \mathfrak{gl}_r \) carries the adjoint action of \( GL_r \) and a \( \mathbb{G}_m \)-action by dilation. Write \( GL_r^\times = GL_r \times \mathbb{G}_m \) and consider the quotient stack \([\mathfrak{gl}_r/GL_r^\times]\). Let
\( \text{D}^b \text{Coh}([\mathfrak{gl}_r/GL_r^c]) \) be the bounded derived category of all \( \mathfrak{gl}_r^c \)-equivariant finitely generated graded \( \mathfrak{sl}^c(\langle -2 \rangle) \)-modules, where \( \langle \bullet \rangle \) is the grading shift functor. Let \( \text{D}^b \text{Coh}([\mathfrak{g}/G^c])_{\Lambda^+} \) be the graded triangulated subcategory of the direct sum

\[
\text{D}^b \text{Coh}([\mathfrak{g}/G^c])_{\Lambda^+} = \bigoplus_{r \in \mathbb{N}} \text{D}^b \text{Coh}([\mathfrak{gl}_r/GL_r^c])
\]
generated by all objects of the form \( (V \otimes \mathcal{O}_{gl_r}) \langle a \rangle \) where \( V \) is a polynomial representation of \( GL_r \) and \( a \) is an arbitrary integer. There is a graded triangulated monoidal structure on \( \text{D}^b \text{Coh}([\mathfrak{g}/G^c])_{\Lambda^+} \) given by the convolution bifunctor \( \circ \).

In this paper, we study relations between (variants of) the category \( \mathcal{D} \) and \( G^c \)-equivariant coherent sheaves on \( \mathfrak{g} \). In particular, we propose the following conjecture, see Conjecture 6.3.3 and Remark 6.3.4 below.

**Conjecture A.** There is an equivalence of triangulated graded monoidal categories

\[
E : (\text{D}^b(\mathcal{D}), \circ) \rightarrow (\text{D}^b \text{Coh}([\mathfrak{g}/G^c])_{\Lambda^+}, \circ^{\text{op}}).
\]

Our main result is a proof of a slightly modified version of this conjecture. To explain this, let \( \text{D}^b \text{Coh}([\mathcal{N}/G^c])_{\Lambda^+} \) denote the triangulated subcategory of \( \text{D}^b \text{Coh}([\mathfrak{g}/G^c])_{\Lambda^+} \) consisting of all complexes of coherent sheaves supported on the nilpotent cone of \( \mathfrak{gl}_r \) for some \( r \geq 0 \). This triangulated category is equipped with the perverse t-structure whose heart is denoted by \( \mathcal{P} \text{Coh}([\mathcal{N}/G^c])_{\Lambda^+} \). The convolution yields a graded Abelian monoidal structure on \( \mathcal{P} \text{Coh}([\mathcal{N}/G^c])_{\Lambda^+} \). Our main theorem, Theorem 6.3.2 is the following analogue of Conjecture A.

**Theorem B.** There is a graded Abelian and Artinian monoidal subcategory \( \mathcal{D}^* \) of \( \mathcal{D} \) containing all simple objects, with an equivalence of graded Abelian categories

\[
E^* : \mathcal{D}^* \rightarrow \mathcal{P} \text{Coh}([\mathcal{N}/G^c])_{\Lambda^+}.
\]

Moreover, both categories \( \mathcal{D}^* \) and \( \mathcal{P} \text{Coh}([\mathfrak{g}/G^c])_{\Lambda^+} \) are graded stratified, and we prove that the equivalence \( E^* \) respects these structures. In particular it takes proper standard modules to proper standard ones. Note that the proper standard modules are monomials in the simple ones, this statement can be viewed as a weak form of the monoidality of the functor \( E^* \), see Remark 6.3.4. To keep this paper in a reasonable length, we do not prove here that \( E^* \) is a monoidal equivalence

\[
(\mathcal{D}^*, \circ) \rightarrow (\mathcal{P} \text{Coh}([\mathcal{N}/G^c])_{\Lambda^+}, \circ^{\text{op}}).
\]

This is stated as Conjecture 6.3.3. We will prove it in a sequel paper [13].

The proof of Theorem B consists of constructing a chain of graded triangulated equivalences

\[
\text{D}^b \text{perf}(\mathcal{D}^*) \xrightarrow{A^*} \text{D}^b_+^{\text{op}}(\text{Gr}_{\Lambda^+,S}) \xrightarrow{B^*} \text{D}^b_+(\text{Gr}_{\Lambda^+,S}) \xrightarrow{C^*} \text{D}^b \text{perf} \text{Coh}([\mathcal{N}/G^c])_{\Lambda^+},
\]

and then checking the t-exactness. Here, the two categories in the middle are some mixed categories on think/thin affine Grassmannians, the functor \( A^* \) is given by the composition of some localization of quiver-Hecke algebras and the tilting equivalence between the module category of the Kronecker quiver and coherent sheaves on \( \mathbb{P}^1 \). The functor \( B^* \) is a Radon
transform. The functor $C^\sharp$ is the derived geometric Satake equivalence. Both $A^\sharp$ and $B^\sharp$ use mixed geometry.

One of our motivations comes from the recent work of Cautis-Williams \cite{16}. To explain the link, let $U_q(n)_{\Lambda^+}$ be the subalgebra of the quantum unipotent enveloping algebra $U_q(n)$ generated by the root vectors whose weights are of the form $\alpha_0 + n\delta$ for some integer $n \geq 0$. The theorem of Khovanov-Lauda and Rouquier implies that the split Grothendieck group $K_0(D^\text{proj})$ is isomorphic to $U_q(n)_{\Lambda^+}$. There is a perfect pairing between $K_0(D^\text{proj})$ and $G_0(D^\sharp)$. Hence, there is a ring isomorphism between $G_0(D^\sharp)$ and the quantum unipotent coordinate algebra $A_q(n)_{\Lambda^+}$ of $U_q(n)_{\Lambda^+}$. Now, given a positive integer $N$, let $A_q(n^{wN})$ be the quantum open unipotent cell of type $Q$ associated with the element $w = (s_0s_1)^N$ in the Weyl group of $Q$. This quantum open unipotent cell is a localization of the quantum unipotent coordinate subalgebra $A_q(n)_{\Lambda^+}$ of $A_q(n)_{\Lambda^+}$. It admits a quantum cluster algebra structure. Cautis-Williams proved that the category of $GL_N(O) \ltimes \mathbb{G}_m$-equivariant perverse coherent sheaves $\mathcal{PC}oh([Gr/G_N^c(O)])$ on the affine Grassmanian of $GL_N$, with the monoidal structure given by the convolution product, is a monoidal categorification of $A_q(n^{wN})$.

In \cite{22, 24} a localization $\mathcal{C}^\text{mon}_w$ of a graded monoidal Serre subcategory of $\mathcal{C}$ is introduced. It is proved there that $\mathcal{C}^\text{mon}_w$ is also a monoidal categorification $A_q(n^{wN})$. It is natural to compare it with the monoidal categorification of Cautis-Williams. To do that we introduce a localization $\mathcal{D}^\sharp_w$ of a Serre subcategory $D^\sharp_w$ of $D^\sharp$ by mimicking the construction in \cite{24}. The equivalence $E^\sharp$ yields a faithful graded exact functor

$$\Psi_w : D^\sharp_w \to \mathcal{PC}oh([Gr/G_N^c(O)])$$

which induces an isomorphism of Grothendieck groups $G_0(D^\sharp_w) = G_0(\mathcal{PC}oh([Gr/G_N^c(O)]))$. Our Conjecture 6.4.2 is the following.

**Conjecture C.** The functor $\Psi_w$ is a graded monoidal equivalence of categories.

### 1.2. Background and perspectives.

Let $\Pi_Q$ be the preprojective algebra of the quiver $Q$. A geometric construction of affine quantum groups is given by the K-theoretic Hall algebra of the category of $\Pi_Q$-modules considered by Schiffmann-Vasserot. There, the category of constructible sheaves on the moduli stack of representations of $Q$ is replaced by the category of coherent sheaves on the derived moduli stack of representations of the preprojective algebra. Our goal is to compare this category of coherent sheaves with a module category of the quiver-Hecke algebra of affine type $Q^{(1)}$ when $Q$ is of finite type.

The stack $X_\beta$ is the quotient of the variety $X_\beta$ by a linear group $G_\beta$. The group $G_\beta^c = G_\beta \ltimes \mathbb{G}_m$ acts on $T^*X_\beta$ so that $\mathbb{G}_m$ has weight 1. The $G_\beta$-action is Hamiltonian. The moment map $\mu_\beta : T^*X_\beta \to \mathfrak{g}_\beta$ is $G_\beta^c$-equivariant, with $\mathbb{G}_m$ of weight 2 on $\mathfrak{g}_\beta$. The cotangent dg-stack of $X_\beta$ is the derived fiber product

$$T^*X^c_\beta = [T^*X_\beta \times^R_{\mathfrak{g}_\beta} 0]/G^c_\beta.$$  

The truncation of this dg-stack is the moduli stack $A^c_\beta = [\mu_\beta^{-1}(0)/G^c_\beta]$ of $\beta$-dimensional representations of $\Pi_Q$. Let $\text{dgQCoh}(T^*X_\beta)$ be the category of all $G^c_\beta$-equivariant sheaves of dg-modules over $T^*X_\beta \times^R_{\mathfrak{g}_\beta} 0$ whose cohomology is a quasi-coherent sheaf over $A^c_\beta$. Let
D(dgQCoh(\(T^*\mathcal{X}_0^p\))) be its derived category. Since the dg-stack \(T^*\mathcal{X}_0^p\) is affine, this graded triangulated category is the derived category of \(G_0^p\)-equivariant modules over a \(G_0^p\)-equivariant dg-algebra which can be described as follows.

Recall that a \(G_\beta^p\)-equivariant dg-algebra is a \(G_\beta\)-equivariant \(\mathbb{Z}^2\)-graded algebra with a differential of bi-degree \((1, 0)\) satisfying the Leibniz rule. For each bi-degree \((i, j)\), we call \(i\) the cohomological degree and \(j\) the internal degree. The corresponding grading shift functors are denoted by \([\bullet]\) and \((\bullet)\) respectively. The internal degree is the weight of the \(G_m\)-action. For any \(\mathbb{Z}^2\)-graded vector space \(V\), let \(S(V)\) be the graded-symmetric algebra of \(V\), i.e., the quotient of the tensor algebra by the relations \(x \otimes y - (-1)^{|x||y|} y \otimes x\). Here \(|x|\) is the cohomological degree an homogeneous element \(x\). The graded triangulated category \(D(dgQCoh(T^*\mathcal{X}_0^p))\) is equivalent to the derived category of all \(G_\beta^p\)-equivariant graded-commutative non-positively graded dg-algebra \(\beta\) whose underlying \(G_\beta^p\)-equivariant \(\mathbb{Z}^2\)-graded algebra is \(S(T^*X_\beta(1) \oplus g_3^p[1][2])\). To describe the differential on \(C_\beta\), we consider the \(\mathbb{Z}\)-graded Lie superalgebra

\[ L_\beta = T^*X_\beta[-1][-1] \oplus g_\beta[-2][-2], \]

whose bracket is the extension by 0 of the map

\[ S^2(T^*X_\beta) \rightarrow g_\beta, \quad a \otimes b \mapsto \sum_{i \in \Omega} (a_{i}, b_{i\cdot 0}) + [b_{i}, a_{i\cdot 0}]. \]

Then \(C_\beta\) is the Chevalley-Eilenberg complex which computes the extension group \(\text{Ext}^*_\beta(k, k)\).

As a \(\mathbb{Z}^2\)-graded algebra we have

\[ C_\beta = S(L^*_\beta[-1]) = S(T^*X_\beta(1) \oplus g_3^p[1][2]). \]

The differential is the unique derivation which vanishes on the subspace \(T^*X_\beta\) and is given on \(g_\beta\) by the map \(g_\beta \rightarrow S^2(T^*X_\beta)\) dual to \([1, 1]\). Let \(D(dgcoh(T^*\mathcal{X}_0^p))\) denote the derived category of \(G_\beta^p\)-equivariant dg-modules of \(C_\beta\) whose cohomology is finitely generated over the graded algebra \(H^*(C_\beta)\). Note that \(H^*(C_\beta)\) is a finitely generated nilpotent extension of \(H^0(C_\beta)\), and that the latter is isomorphic to the function ring over the variety \(\mu_\beta^{-1}(0)\).

Now, fix a triple of dimension vectors \(\alpha, \beta, \gamma\) such that \(\beta = \alpha + \gamma\). Fix a parabolic subgroup \(P\) of \(G_\beta\) with a Levi subgroup \(L\) isomorphic to \(G_\alpha \times G_\gamma\). A \(C\)-point of \(T^*X_\beta\) is the same as a representation of the double quiver \(Q = (I, \Omega)\) such that \(\Omega = \Omega \cup \Omega^{op}\). Let \(T^*X_P\) be the subspace of all representations in \(T^*X_\beta\) which preserve a fixed flag of vector spaces of type \((\alpha, \gamma)\) whose stabilizer in \(G\) is \(P\). One defines as above a \(P^c\)-equivariant graded-commutative dg-algebra structure on the symmetric graded-commutative algebra

\[ C_P = S(T^*X_P(1) \oplus p^*[1][2]). \]

The obvious projection and inclusion

\[ T^*X_L \oplus t^* \rightarrow T^*X_P \oplus p^*, \quad T^*X_G \oplus g^* \rightarrow T^*X_P \oplus p^* \]

yield \(P^c\)-equivariant dg-algebra homomorphisms \(q : C_L \rightarrow C_P\) and \(p : C_G \rightarrow C_P\). Composing the extension of scalars relative to \(q\), the restriction of scalars relative to \(p\), the restriction from \(L^c\) to \(P^c\) and the induction from \(P^c\) to \(G^c\), we get a triangulated functor

\[ R^*_L \leq P : D(dgcoh(T^*\mathcal{X}_0^c)) \times D(dgcoh(T^*\mathcal{X}_0^c)) \rightarrow D(dgcoh(T^*\mathcal{X}_0^c)) \]
which yields a triangulated graded monoidal structure $\circ$ on the category

$$D(\text{dgCoh}(T^*\mathcal{X}^c)) = \bigoplus_{\beta} D(\text{dgCoh}(T^*\mathcal{X}^c_{\beta})).$$

Let $G_0(\text{dgCoh}(T^*\mathcal{X}^c))$ be the Grothendieck group of the graded triangulated category $D(\text{dgCoh}(T^*\mathcal{X}^c))$. It coincides with the Grothendieck group of the graded Abelian category of coherent sheaves on the stack $\bigsqcup_{\beta} \Lambda_{\beta}$. We equip it with the multiplication given by the monoidal structure $\circ$. This ring is the K-theoretic Hall algebra of the category of $\Pi_{\mathbb{Q}}$-modules. It was considered by Schiffmann-Vasserot in [15], [14] for a one vertex quiver $Q$. Then it was generalized to several other settings, see [46] and the references there. Let $U_q(Ln)$ be the quantum enveloping algebra of the loop algebra of $\mathfrak{n}$. The following theorem holds. The proof will be given elsewhere.

**Theorem D.** Assume that the quiver $Q$ is of finite type. Then, there is a $\mathbb{Z}[q, q^{-1}]$-algebra isomorphism $G_0(\text{dgCoh}(T^*\mathcal{X}^c)) = U_q(Ln)$.

The conjecture A is a lift of the isomorphism in Theorem D to a monoidal graded triangulated equivalence between $D(\text{dgCoh}(T^*\mathcal{X}^c))$ and the derived category of a module category of the quiver-Hecke algebra of affine type $Q^{(1)}$ for $Q$ of type $A_1$. Let us explain this in more details. The general case will be considered elsewhere.

From now on, let us assume that the quiver $Q = A_1$. The dimension vector $\beta$ is simply a non-negative integer $r$. The group $G_{\beta}$ is $GL_r$, the graded Lie superalgebra $\mathfrak{g}_\beta$ is $\mathfrak{gl}_r[-2] \langle -2 \rangle$ with the zero bracket, and the dg-algebra $C_{\beta}$ is the exterior algebra $S(\mathfrak{gl}_r[1] \langle 2 \rangle)$ with the zero differential. In particular $D(\text{dgCoh}(T^*\mathcal{X}^c_{\beta}))$ is the triangulated category $D(\text{mod}(S(\mathfrak{gl}_r[1] \langle 2 \rangle) \rtimes GL_r))$ of all $GL_r$-equivariant finitely generated dg-modules over $S(\mathfrak{gl}_r[1] \langle 2 \rangle)$. The derived category $D^b\text{Coh}([\mathfrak{gl}_r/GL_r])$ is $D(\text{mod}(S(\mathfrak{gl}_r[-2] \langle -2 \rangle) \rtimes GL_r))$, which is the same as the triangulated category $D(\text{mod}(S(\mathfrak{gl}_r[-2] \langle -2 \rangle) \rtimes GL_r))$. So the Koszul duality yields an equivalence

$$D(\text{dgCoh}(T^*\mathcal{X}^c_{\beta})) \rightarrow D^b\text{Coh}([\mathfrak{gl}_r/GL_r]).$$

Thus, we must compare the triangulated category $D^b\text{Coh}([\mathfrak{gl}_r/GL_r])$ with a module category of the quiver-Hecke algebra $R$ of affine type $Q^{(1)}$. Let $U_q(n^{(1)})_{\mathbb{A}^+}$ be the current subalgebra of $U_q(Ln)$. Theorem D implies that the Grothendieck group of the triangulated category $D^b\text{Coh}([\mathfrak{gl}_r/GL_r])_{\mathbb{A}^+}$ is $U_q(n^{(1)})_{\mathbb{A}^+}$. On the other hand, the split Grothendieck group $K_0(D^{\text{proj}})$ is isomorphic to $U_q(n^{(1)})_{\mathbb{A}^+}$. Thus, the Grothendieck group of the triangulated category $K_0(D^{\text{proj}})$ is also isomorphic to $U_q(n^{(1)})_{\mathbb{A}^+}$. Since the category $D$ has finite homological dimension, we deduce that the Grothendieck group of the graded triangulated $D^b(D)$ is also isomorphic to $U_q(n^{(1)})_{\mathbb{A}^+}$. Thus it is natural to compare the graded triangulated categories $D^b(D)$ and $D^b\text{Coh}([\mathfrak{g}/\mathcal{G}^c])_{\mathbb{A}^+}$. This is precisely the Conjecture A above.

**1.3. Relation to previous works.** In [12], Bezrukavnikov proved that the realization of the affine Hecke algebra of a reductive group, both as the Grothendieck group of a monoidal category of equivariant coherent sheaves on the Steinberg variety à la Ginzburg-Kazhdan-Lusztig and as the Grothendieck group of a monoidal category of equivariant perverse sheaves on the affine flag manifold of the Langlands dual group à la Kazhdan-Lusztig, lifts to a graded monoidal triangulated equivalence between corresponding derived categories. Here, we prove that the realization of a positive piece of the quantum enveloping algebra of affine type $A_1^{(1)}$,
both as the preprojective K-theoretic Hall algebra of the category of \( \Pi Q \)-modules of type \( A_1 \) à la Schiffmann-Vasserot and as the Grothendieck group of a category of semisimple complexes on a quiver moduli stack of affine type \( A_1^{(1)} \) à la Lusztig (or rather its algebraic equivalent description via quiver-Hecke algebras of type \( Q^{(1)} \)) lifts to a graded monoidal triangulated equivalence between the corresponding derived categories.

According to [33], the Langlands correspondence for \( \mathbb{P}^1 \) can be viewed as the composition of a chain of equivalences

\[
\begin{align*}
D^b(Bun) & \xrightarrow{F} D^b(Gr^+) \xrightarrow{B} D^b(Gr^-) \xrightarrow{C} D^b(\text{mod}(S(g^*[(-2)(-2)] \times G^c))) \\
& \Downarrow D \\
D^b(Coh(Loc_G)) & \xrightarrow{\text{D}} D^b(\text{mod}(S(g[1](2)) \times G^c))
\end{align*}
\]

where \( D \) is the Koszul duality mentioned above and \( F \) is the tilting equivalence in §4.2. Thus, our equivalence can be viewed as a mixed non-equivariant version of the Langlands correspondence for \( \mathbb{P}^1 \) and for the linear group.

1.4. Convention. We’ll say that a square of functors is commutative if it commutes up to a natural isomorphism. For any algebraic group \( G \) or ring \( R \), we denote by \( Z\) the centers of \( G \) and \( R \). For any commutative ring \( k \), let \( H^\ast_G \) be the cohomology of the classifying space of \( G \) with \( k \)-coefficients. Unless specified otherwise, all modules are left modules. We refer to Appendix A for all conventions related to mixed geometry.

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2. Quiver-Hecke algebras

Let \( k \) be a field which is algebraically closed of characteristic zero. Let \( F \) be any field.

2.1. Graded categories. A graded k-linear category \( C \) is a k-linear additive category with a compatible system of self-equivalences \( \langle a \rangle \), with \( a \in \mathbb{Z} \), called grading shift functors. A graded functor of graded k-linear additive categories is a k-linear functor \( E \) with a compatible system of natural equivalences \( E \langle a \rangle \rightarrow \langle a \rangle E \). All the additive categories we’ll consider here are k-linear. To simplify, we’ll omit the word \( k \)-linear everywhere. A functor of additive categories is assumed to be \( k \)-linear. If the category \( C \) is triangulated, as well as the graded shift functors, then we’ll say that \( C \) is a graded triangulated category. A monoidal structure on an additive category \( C \) is the datum of a bifunctor \( \otimes : C \times C \rightarrow C \), an associativity constraint \( a \) and an object \( 1 \), called the unit, with an isomorphism \( \varepsilon : 1 \otimes 1 \rightarrow 1 \) satisfying the pentagon axiom and such that the functors \( 1 \otimes \bullet \) and \( \bullet \otimes 1 \) are fully faithful, see, e.g., [21] App. A]. If all functors involved in this definition are graded, we’ll say that \( (C, \otimes, a, 1, \varepsilon) \) is a graded monoidal category. If the category \( C \) is Abelian and the bifunctor \( \otimes \) is exact, we says that the monoidal category is exact. If the category \( C \) is triangulated and the bifunctor \( \otimes \) is triangulated, we says that the monoidal category is triangulated. If \( \otimes, 1, a \) or \( \varepsilon \) are obvious from the context, we may omit them and we write simply \((C, \otimes) \text{ or } C \). A monoidal functor of monoidal categories...
is a functor $E$ with a natural isomorphism of bifunctors $E(\bullet \otimes \bullet) \to E(\bullet) \otimes E(\bullet)$ and an isomorphism $E(1) \to 1$ satisfying some well-known axioms.

For any objects $A, B \in \mathcal{C}$, the graded-homomorphisms space is the graded vector space

\[ \mathbb{H} \text{Hom}_\mathcal{C}(A, B) = \bigoplus_{a \in \mathbb{Z}} \text{Hom}_\mathcal{C}(A, B\langle a \rangle)\langle -a \rangle. \]

A projective object $P$ of a graded $k$-linear Abelian category $\mathcal{C}$ is called a projective graded-generator if any object of $\mathcal{C}$ is a quotient of a finite sum of $P\langle a \rangle$’s with $a \in \mathbb{Z}$. We’ll also say that $\mathcal{C}$ is graded-generated by $P$. Then, the functor $M \mapsto \mathbb{H} \text{Hom}_\mathcal{C}(P, M)$ is an equivalence from $\mathcal{C}$ to the graded category of all finitely generated graded modules over the graded $k$-algebra $R^\text{op}$ opposite to $R = \text{End}_\mathcal{C}(P)$, see, e.g., [6, prop. E.4].

Let $\mathcal{C}^\text{fl}$, $\mathcal{C}^{\text{proj}}$ be the full graded subcategories of $\mathcal{C}$ consisting of all finite length objects and all projective objects. Let $G_0(\mathcal{C}^\text{fl})$ and $K_0(\mathcal{C}^{\text{proj}})$ be the Grothendieck group and the split Grothendieck group of the Abelian category $\mathcal{C}^\text{fl}$ and the additive category $\mathcal{C}^{\text{proj}}$. For each object $M$, let $[M]$ be its class in the Grothendieck group. The ring $A = \mathbb{Z}[q, q^{-1}]$ acts on $G_0(\mathcal{C}^\text{fl})$, $K_0(\mathcal{C}^{\text{proj}})$ with $q = \langle 1 \rangle$.

**Example 2.1.1.**

(a) Given a graded Noetherian $k$-algebra $R$, let $\mathcal{C} = R\text{-mod}$ be the category of all finitely generated graded $R$-modules. Both are equipped with the grading shift functors $\langle a \rangle$ such that $(M\langle 1 \rangle)_a = M_{a+1}$ for each integer $a$. We’ll abbreviate $D^b(R) = D^b(R\text{-mod})$ and $K^b(R) = K^b(R\text{-mod})$.

(b) Let $K^b(\mathcal{C})$ be the bounded homotopy category of a graded additive category $\mathcal{C}$ with grading shift functor $\langle \bullet \rangle$. The category $K^b(\mathcal{C})$ is graded triangulated with the grading shift functor given by $\langle \bullet \rangle$. Let $[\bullet]$ be the cohomological shift functor. Recall that

\[ \mathbb{H} \text{Hom}_{K^b(\mathcal{C})}(A, B) = \bigoplus_{a \in \mathbb{Z}} \text{Hom}_{K^b(\mathcal{C})}(A, B\langle a \rangle)\langle -a \rangle. \]

We write

\[ \text{Hom}_{K^b(\mathcal{C})}^\bullet(A, B) = \bigoplus_{a \in \mathbb{Z}} \text{Hom}_{K^b(\mathcal{C})}^{a}(A, B)[-a] \quad \text{and} \quad \text{Hom}_{K^b(\mathcal{C})}^{a}(A, B) = \text{Hom}_{K^b(\mathcal{C})}(A, B[a]). \]

(c) Let $D^b(\mathcal{C})$ be the bounded derived category of a graded Abelian category $\mathcal{C}$ with grading shift functor $\langle \bullet \rangle$. The category $D^b(\mathcal{C})$ is graded triangulated with the grading shift functor given by $\langle \bullet \rangle$. We define $\text{Hom}_{D^b(\mathcal{C})}^\bullet(A, B)$ and $\text{Hom}_{D^b(\mathcal{C})}^{a}(A, B)$ as in (b). Let $\mathcal{C} = R\text{-mod}$ be as in (a). A complex of graded $R$-modules is perfect if it is quasi-isomorphic to a bounded complex of finitely generated projective graded $R$-modules. The category of perfect complexes is the full graded triangulated subcategory of $D^b(\mathcal{C})$ given by $D^\text{perf}(\mathcal{C}) = K^b(\mathcal{C}^{\text{proj}})$.

### 2.2. Polynomial highest weight and affine properly stratified categories.

Let $\mathcal{C}$ be a graded $k$-linear Abelian category with a finite set $\{L(\pi) : \pi \in \text{KP}\}$ of simple objects which is complete and irredundant up to isomorphisms and grading shifts. We’ll assume that $\mathcal{C} = R\text{-mod}$, where $R$ is a Schurian Noetherian Laurentian graded algebra as in [30, §2.1]. Hence, each simple object $L(\pi)$ admits a projective cover $P(\pi) \to L(\pi)$ with kernel $M(\pi)$,
and for any object $M \in \mathcal{C}$ the composition multiplicity of $L(\pi)$ in $M$ is the formal series in $\mathbb{Z}((q))$ given by $[M : L(\pi)] = \dim \text{Hom}_{\mathcal{C}}(P(\pi), M)$.

We assume that the set $\mathcal{KP}$ is equipped with a partial preorder $\leq$, and with a map $\rho : \mathcal{KP} \to \Pi$ to a partial ordered set $\Pi$ such that we have
\begin{equation}
\pi \leq \pi' \iff \rho(\pi) \leq \rho(\pi').
\end{equation}
For each $\pi \in \mathcal{KP}$, we define the standard and proper standard objects $\Delta(\pi)$ and $\bar{\Delta}(\pi)$ such that $\Delta(\pi)$ is the largest quotient of $P(\pi)$ such that all its composition factors $L(\sigma)$ satisfy $\sigma \leq \pi$, and $\bar{\Delta}(\pi)$ is the largest quotient of $P(\pi)$ which has $L(\pi)$ with multiplicity 1 and such that all its other composition factors $L(\sigma)$ satisfy $\sigma < \pi$. Let $K(\pi)$ be the kernel of the surjection $P(\pi) \to \Delta(\pi)$.

We say that an object $M \in \mathcal{C}$ has a $\Delta$-filtration if it has a separated filtration whose subquotients are isomorphic to $\Delta(\pi)(a)$ for some $\pi \in \mathcal{KP}$, $a \in \mathbb{Z}$. Let $\mathcal{C}^\Delta$ be the full graded additive subcategory of $\mathcal{C}$ consisting of all $\Delta$-filtered objects. For all $\xi \in \Pi$ we write $\Delta(\xi) = \bigoplus_{\rho(\pi) = \xi} \Delta(\pi)$.

**Definition 2.2.1** ([29]). The category $\mathcal{C}$ is
(a) affine properly stratified if, for each $\pi \in \mathcal{KP}$, $\xi \in \Pi$,
1. $K(\pi)$ has a $\Delta$-filtration with subquotients of the form $\Delta(\sigma)(a)$ for $\sigma > \pi$ and $a \in \mathbb{Z}$,
2. $\Delta(\xi)$ is finitely generated and flat as a module over the algebra $\text{End}_\mathcal{C}(\Delta(\xi))^\text{op}$,
3. $\text{End}_\mathcal{C}(\Delta(\xi))^\text{op}$ is a finitely generated commutative graded $k$-algebra,
(b) polynomial highest weight if the map $\rho$ is bijective and, for each $\pi \in \mathcal{KP}$,
1. $K(\pi)$ has a $\Delta$-filtration with subquotients of the form $\Delta(\sigma)(a)$ for $\sigma > \pi$ and $a \in \mathbb{Z}$,
2. $\Delta(\pi)$ is finitely generated and free as a module over the algebra $\text{End}_\mathcal{C}(\Delta(\pi))^\text{op}$,
3. $\text{End}_\mathcal{C}(\Delta(\pi))^\text{op}$ is a graded polynomial $k$-algebra.

Since $\mathcal{C}$ is an affine properly stratified category with a finite number of isomorphism classes of simple objects, it is graded equivalent to $R$-mod for some affine properly stratified algebra $R$ by [29, cor. 6.8]. Let $I(\pi)$ denote the injective hull of $L(\pi)$ in the category of all graded $R$-modules (not necessarily finitely generated). We define $\nabla(\pi)$ to be the largest submodule of $I(\pi)$ such that all its composition factors $L(\sigma)$ satisfy $\sigma \leq \pi$, and $\nabla(\pi)$ to be the largest submodule of $I(\pi)$ which has $L(\pi)$ with multiplicity 1 and such that all its other composition factors $L(\sigma)$ satisfy $\sigma < \pi$. The objects $\nabla(\pi)$ and $\nabla(\pi)$ are called the costandard and the proper costandard objects. Note that $\nabla(\pi)$ does not necessarily belong to $\mathcal{C}$ in general.

For each subset $\Gamma \subset \mathcal{KP}$, let $\mathcal{C}_\Gamma \subset \mathcal{C}$ be the full subcategory of all objects whose composition factors are isomorphic to graded shifts of simple objects $L(\sigma)$ with $\sigma \in \Gamma$. Let
\[(f_\Gamma)_* : \mathcal{C}_\Gamma \to \mathcal{C} \quad , \quad (f_\Gamma)^* : \mathcal{C} \to \mathcal{C}_\Gamma\]
bethe obvious full embedding and its left adjoint. The functor $(f_\Gamma)_*$ sends projectives to projectives because $(f_\Gamma)_*$ is exact. For each object $M$ we have $(f_\Gamma)^*(M) = M / \sigma^*(M)$, where $\sigma^*(M)$ is the minimal subobject $N \subset M$ such that $M/N \in \mathcal{C}_\Gamma$. We have the following exact functors between derived categories
\[(f_\Gamma)_* : \mathcal{D}^b(\mathcal{C}_\Gamma) \to \mathcal{D}^b(\mathcal{C}) \quad , \quad L(f_\Gamma)^* : \mathcal{D}^-(\mathcal{C}) \to \mathcal{D}^-(\mathcal{C}_\Gamma)\]
We say that $\Gamma$ is an order ideal (resp. order coideal) if for each $\sigma \in \Gamma$, $\pi \in \mathcal{KP}$ we have $\sigma \geq \pi \implies \pi \in \Gamma$ (resp. $\sigma \leq \pi \implies \pi \in \Gamma$).
If $\Gamma$ is a finite order ideal, then the following hold [29, lem. 7.18, prop. 7.20]

(a) $(f_{r})_{*} : D^b(C_{\Gamma}) \to D^b(C)$ is fully faithful,
(b) $L(f_{r})^{*}M = (f_{r})^{*}M$ if $M \in C^{\Delta}$,
(c) $(f_{r})^{*}P(\pi)$ is a projective cover of $L(\pi)$ in $C_{\Gamma}$ if $\pi \in \Gamma$,
(d) $\Delta(\pi) = (f_{r})_{*}(f_{r})^{*}P(\pi)$ if $\Gamma = \{ \leq \pi \}$.

2.3. Kostant partitions. Let $Q$ be a finite quiver associated with a Kac-Moody algebra of affine type. Let $I$ be the set of vertices and $\Omega$ the set of arrows. Let $\Phi$ be the root system of $Q$. We identify $I$ with a fixed set of simple roots $\{\alpha_{i} : i \in I\}$ in the obvious way. Let $\{\Lambda_{i} : i \in I\}$ be the set of fundamental weights. Let $\Phi_{+} \subset \Phi$ the subset of positive roots and $Q_{+} = \bigoplus_{i \in I} \mathbb{N} \alpha_{i}$. Let $ht(\beta)$ be the height of an element $\beta$ of $Q_{+}$. Fix $0 \in I$ such that $\{\alpha_{i} : i \in I, i \neq 0\}$ is the set of simple roots of a root system $\Delta$ of finite type. Let $\delta \in \Phi_{+}$ be the null root. The set of real positive roots is $\Phi_{+}^{re} = \Phi_{+-} \sqcup \Phi_{++}$, where

$$
\Phi_{+-} = \{\beta + n\delta ; \beta \in \Delta_{+}, n \in \mathbb{N}\}, \quad \Phi_{++} = \{-\beta + n\delta ; \beta \in \Delta_{+}, n \in \mathbb{Z}_{\geq 0}\}.
$$

We fix a total convex preorder on the set $\Phi_{+}$ such that for each $i \in I \setminus \{0\}$ we have

(a) $\alpha_{i} > \alpha_{i} + \delta > \alpha_{i} + 2\delta > \cdots > \mathbb{Z}_{\geq 0}\delta > \cdots > -\alpha_{i} + 2\delta > -\alpha_{i} + \delta$,
(b) $m\delta > n\delta$ for all $m, n > 0$,
(c) each root in $\Phi_{+}^{re} \setminus \{\alpha_{i} + n\delta, -\alpha_{i} + (n + 1)\delta ; n \in \mathbb{N}\}$ is either $> \alpha_{i}$ or $< -\alpha_{i} + \delta$.

For any subset $X \subset Q_{+}$ let $X^{\beta}$ be the set of tuples of elements of $X$ with sum $\beta$. A Kostant partition of $\beta$ is a decreasing tuple $\pi = (\pi_{1} \geq \pi_{2} \geq \cdots \geq \pi_{r})$ in $\Phi_{+}^{\beta}$. We may also write $\pi = (\beta_{l})^{p_{l}}, \ldots, (\beta_{1})^{p_{1}}, (\beta_{0})^{p_{0}}$, meaning that $l$ is a positive integer and $\beta_{l} > \cdots > \beta_{1} > \beta_{0}$ are positive roots counted with multiplicities $p_{l}, \ldots, p_{1}, p_{0}$ respectively. A Kostant partition $\pi$ is a root partition if the multiplicity of each decomposable affine positive root $n\delta$, with $n > 1$, is zero. Let $\Pi_{\beta} \subset KP_{\beta}$ be the sets of all root partitions and Kostant partitions of $\beta$. Both sets are finite. There is a unique map $\rho : KP_{\beta} \to \Pi_{\beta}$ such that the real positive roots have the same multiplicities in $\pi$ and $\rho(\pi)$. We can view a Kostant partition $\pi$ as a pair $(\rho(\pi), \mu)$ where $\mu$ is a partition of the multiplicity of $\delta$ in $\rho(\pi)$, then the map $\rho$ is just the projection on the first factor. The set $\Pi_{\beta}$ of root partitions is equipped with the bileticographic partial order, such that

$$(2.2) \quad \pi \leq \pi' \iff \pi \leq_{l} \pi' \text{ and } \pi \geq_{r} \pi', \tag{2.2}$$

where $\leq_{l}$ and $\leq_{r}$ are the left and right lexicographic orders. We equip KP$_{\beta}$ with the partial preorder defined as (2.1) using the map $\rho : KP_{\beta} \to \Pi_{\beta}$. Let $S_{\mu}$ be the symmetric group. For any Kostant partition $\pi = ((\beta_{l})^{p_{l}}, \ldots, (\beta_{1})^{p_{1}}, (\beta_{0})^{p_{0}})$ we define the tuple $\bar{\pi} = (p_{l}\beta_{l}, \ldots, p_{1}\beta_{1}, p_{0}\beta_{0})$ in $Q_{+}^{\beta}$. Set

$$(2.3) \quad S_{\pi} = S_{p_{l}} \times \cdots \times S_{p_{1}} \times S_{p_{0}}, \quad n_{\pi} = \sum_{k=1}^{l} kp_{k}, \quad r_{\pi} = \sum_{k=0}^{l} p_{k}. \tag{2.3}$$
2.4. Quiver-Hecke algebras and their module categories. Let $\beta$ be any element in $Q_+$. Let $R_\beta$ be the quiver-Hecke algebra over $k$ associated with the category of $\beta$-dimensional representations of the quiver $Q$. We choose the parameters of $R_\beta$ to be given by the family of polynomials $Q_{ij}(u, v) = (-1)^{h_{ij}}(u - v)$ where $h_{ij} = z_i \rightarrow j \in \Omega$. Let $b = \text{ht}(\beta)$ be the height of $\beta$. The quiver-Hecke algebra $R_\beta$ is the associative, unital, graded $k$-algebra generated by a complete set of orthogonal idempotents $\{e_\nu; \nu \in I^\beta\}$ and some elements $x_i$, $r_k$ with $i \in [1, b]$, $k \in [1, b - 1]$, satisfying the following relations

\begin{align*}
(a) \quad x_i x_j &= x_j x_i, \quad x_i e_\nu = e_\nu x_i, \quad r_k e_\nu = e_{sk} e_\nu r_k, \quad r_k^2 e_\nu = Q_{vk, v_{k+1}}(x_k, x_{k+1}) e_\nu, \\
(b) \quad (r_k x_l - x_{sk(l)} r_k) e_\nu &= \begin{cases} 
- e_\nu & \text{if } l = k, \nu_k = \nu_{k+1}, \\
0 & \text{otherwise}, \\
\end{cases} \\
(c) \quad (r_{k+1} r_k r_{k+1} - r_k r_{k+1} r_k) e_\nu &= \begin{cases} 
\frac{Q_{vk, v_{k+1}}(x_k, x_{k+1}) - Q_{vk, v_{k+1}}(x_{k+2}, x_{k+1})}{x_k - x_{k+2}} e_\nu & \text{if } \nu_k = \nu_{k+2}, \\
0 & \text{otherwise}. 
\end{cases}
\end{align*}

There is an antiautomorphism of $R_\beta$ which fixes all the generators.

Let $C_\beta = R_\beta\text{-mod}$ be the graded Abelian category of all finitely generated graded $R_\beta$-modules. For any tuple $\pi = (\pi_1, \ldots, \pi_r)$ in $Q_+^r$ we set

$$
R_{\pi} = R_{\pi_1} \otimes \cdots \otimes R_{\pi_{r-1}} \otimes R_{\pi_r}, \quad C_{\pi} = R_{\pi}\text{-mod}.
$$

There is an obvious inclusion $R_{\pi} \subset R_{\beta}$. It yields an adjoint pair of exact induction/restriction functors

$$
\text{Ind}_{\pi} : C_{\pi} \to C_{\beta}, \quad \text{Res}_{\pi} : C_{\beta} \to C_{\pi}.
$$

If $r = 2$ we abbreviate $\circ = \text{Ind}_{\pi}$. By [30] cor. 6.24, the category $C_{\beta}$ is affine properly stratified.

The sets of proper standard objects $\{\Delta(\pi); \pi \in KP_{\beta}\}$ and standard objects $\{\Delta(\pi); \pi \in KP_{\beta}\}$ are defined in [30] (6.5), (6.6). The map $\rho : KP_{\beta} \to \Pi_{\beta}$ is as in [23].

Let $L(\pi)$ be the top of the module $\Delta(\pi)$ and $P(\pi)$ its projective cover. For a future use, we fix an idempotent $e(\pi)$ in $R_{\beta}$ such that

$$
P(\pi) = R_{\beta} e(\pi).
$$

Assume temporarily that $\beta \in \Phi^+_\text{rel}$ is real positive root. Then, by [10] [8,9], the graded module $L(\beta)$ is the unique self-dual irreducible module in $C_{\beta}$ which is cuspidal, which means that for each $\alpha, \gamma \in Q_+$ such that $\beta = \alpha + \gamma$ we have $\text{Res}_{\alpha, \gamma}(L(\beta)) = 0$ unless $\alpha < \beta < \gamma$. The partial order is such that $\alpha < \beta$ if and only if $\alpha$ is a sum of positive roots $< \beta$ and $\beta < \gamma$ if and only if $\gamma$ is a sum of positive roots $> \beta$. For each graded module $M$ we write

$$
M(\langle p \rangle) ! = \bigoplus_{s=1}^t M(\langle k_s \rangle) \text{ where } \prod_{k=0}^{p-1} (1 + q^2 + \cdots + q^{2k}) = q^{k_1} + q^{k_2} + \cdots + q^{k_t}.
$$

For any integer $p > 0$ and $M \in C_{\beta}$, we write $M^{\text{op}} = \text{Ind}_{(\beta^p)}(M^{\text{op}})$. Then, we have

$$
L(\beta^p) = L(\beta)^{\text{op}}(p(p - 1)/2).
$$

Further, there is an indecomposable summand $\Delta(\beta^{(p)})$ of $\Delta(\beta)^{\text{op}}$ such that

$$
\Delta(\beta)^{\text{op}} = \Delta(\beta^{(p)}/p) !.
$$
By [30] (6.6), [40] §10, §24, the standard and proper standard modules in $\mathcal{C}_\beta$ are
\[
\Delta(\pi) = \Delta(\beta_1)^{(p_1)} \circ \cdots \circ \Delta(\beta_0)^{(p_0)} , \quad \bar{\Delta}(\pi) = L(\beta_1^{p_1}) \circ \cdots \circ L(\beta_0^{p_0}).
\]

Now, let $\beta$ be any element in $Q_+$. Let $\Gamma_\beta \subset KP_\beta$ be the set of all Kostant partitions supported on the set $\Phi_+$ and let $\Gamma \subset KP_\beta$ be any order ideal. We abbreviate
\[
D_\Gamma = (\mathcal{C}_\beta)_\Gamma, \quad D_\beta = D_{\Gamma_\beta}.
\]

Let $(f_\Gamma)^*$ be the left adjoint functor of the obvious exact full embedding $(f_\Gamma)_*: D_\Gamma \to C_\beta$. For any nested order ideal $\Gamma_1 \subset \Gamma$ contained in $\Gamma_\beta$ we define similarly the functor $(h_{\Gamma_1, \Gamma})_*: D_{\Gamma_1} \to D_{\Gamma}$ and its left adjoint $(h_{\Gamma_1, \Gamma})^*$. We abbreviate $h_\Gamma = h_{\Gamma_\beta, \Gamma}$. Let $Q(\pi)$ be the projective cover in $D_\beta$ of the simple graded module $L(\pi)$. We have $Q(\pi) = (f_{\Gamma_\beta})^*P(\pi)$.

**Proposition 2.4.1.** Assume that $\Gamma_1 \subset \Gamma \subset \Gamma_\beta$ are order ideals.

(a) $D_\Gamma$ is polynomial highest weight with finite global dimension.

(b) $(f_\Gamma)_*: D^b(D_\Gamma) \to D^{\text{perf}}(\mathcal{C}_\beta)$ and $(h_{\Gamma_1, \Gamma})_*: D^b(D_{\Gamma_1}) \to D^b(D_{\Gamma})$ are fully faithful functors.

**Proof.** Since the category $\mathcal{C}_\beta$ is affine properly stratified relatively to the map $\rho$, by [29] prop. 5.16 the category $D_\Gamma$ is affine properly stratified relatively to restriction $\rho_\Gamma$ of $\rho$ to the subset $\Gamma$, with the sets of proper standard and standard modules given by $\{\Delta(\pi) : \pi \in \Gamma\}$ and $\{\Delta(\pi) : \pi \in \Gamma\}$. The map $\rho_\Gamma$ is the identity of $\Gamma$. By [30] thm. 6.14, to prove that the category $D_\Gamma$ is polynomial highest weight, it is enough to check that $E\text{nd}_{\mathcal{C}_\beta}(\Delta(\pi))$ is a graded polynomial ring for each $\pi \in \Gamma$. Let $\alpha \in \Phi_\beta^+$ be any positive real root. There is a central element of degree 2
\[
z_\alpha = x_1 + \cdots + x_{\text{ht}(\alpha)} \in Z(R_{\alpha})
\]
such that the standard module $\Delta(\alpha)$ is a free module over $k[z_\alpha]$ and the action yields an isomorphism of graded rings, see [30] §6, [40] §15
\[
k[z_\alpha] = E\text{nd}_{\mathcal{C}_\alpha}(\Delta(\alpha))
\]

By [40] thm. 24.1, the functoriality of induction and (2.3) yield a graded ring isomorphism
\[
E\text{nd}_{\mathcal{C}_\beta}(\Delta(\alpha)^{(p)}) = k[z_1, z_2, \ldots, z_p]^S_p.
\]

Thus, by (2.4), a short computation yields
\[
E\text{nd}_{\mathcal{C}_\beta}(\Delta(\pi)) \cong E\text{nd}_{\mathcal{C}_\beta}(\Delta(\beta_1)^{(p_1)} \otimes \cdots \otimes \Delta(\beta_0)^{(p_0)} , \text{Res}_{\mathcal{C}_\alpha}(\Delta(\pi)))
\]
\[
\cong E\text{nd}_{\mathcal{C}_\beta}(\Delta(\beta_1)^{(p_1)} \otimes \cdots \otimes \Delta(\beta_0)^{(p_0)} )
\]
\[
\cong k[z_1, z_2, \ldots, z_p]^S_p.
\]

The first isomorphism is the adjunction, the second is [40] lem. 8.6, the third is (2.10). The global dimension of $D_\Gamma$ is finite by [29] cor. 5.25, because the set $KP_\beta$ is finite. This proves part (a).

Since $D_\Gamma$ has a finite global dimension and any object of $D^\text{proj}_\Gamma$ has a finite $\Delta$-filtration, to prove that $(f_\Gamma)_*$ maps into $D^{\text{perf}}(\mathcal{C}_\beta)$ it is enough to observe that any standard module in $D_\Gamma$ has a finite projective dimension in $\mathcal{C}_\beta$. By [29] lem. 5.17, this projective dimension is less than $\sharp KP_\beta$, which is finite. Thus, part (b) follows from [29] prop. 7.20. \qed
2.5. Quiver-Hecke algebras and the moduli stack of representations of quivers. Let \( \ell \) be a prime number and \( k = \mathbb{Q}_\ell \). Let \( F \) be an algebraically closed field of characteristic prime to \( \ell \). See §2 below for a reminder on Artin stacks and constructible sheaves on stacks.

Let \( \beta \) be any element in \( Q_+ \). Let \( FQ \) be the path algebra of the quiver \( Q \) over \( F \). Let \( X_\beta \) be the variety of all \( FQ \)-modules in an \( I \)-graded \( \beta \)-dimensional \( F \)-vector space \( V \). It is isomorphic to the affine space \( A_F^{d_\beta} \) for some \( d_\beta \in \mathbb{N} \). Let \( G_\beta \) be the group of all \( I \)-graded \( F \)-linear automorphisms of \( V \). The algebraic \( F \)-group \( G_\beta \) acts on \( X_\beta \) in the obvious way. The moduli stack of \( \beta \)-dimensional representations of \( FQ \) is the quotient stack \( X_\beta = [X_\beta/G_\beta] \) over \( F \). It is a smooth stack of finite type.

For each tuple \( \nu = (\nu_1, \ldots, \nu_r) \) in \( Q_+^r \) we fix a flag of \( I \)-graded vector \( F \)-spaces \( \nu_1 \supset \cdots \supset \nu_{r-1} \supset \nu_r = 0 \) in \( V \) such that \( \dim(V_{k-1}/V_k) = \nu_k \) for each \( k = 1, 2, \ldots, r \). Let \( X_{\nu_k} \subset X_\beta \) be the set of all representations of \( FQ \) in \( V \) which preserve the flag \( V_{\nu} \). The stabilizer of the flag \( V_{\nu} \) in \( G_\beta \) is a parabolic subgroup \( P_{\nu} \) whose action preserves the subset \( X_{\nu_k} \) of \( X_\beta \). Let \( \tilde{\nu} \) be the quotient stack. Set \( \mathcal{X}_{\nu} = X_{\nu_1} \times \cdots \times X_{\nu_{r-1}} \times X_{\nu_r} \). The stacks \( \mathcal{X}_{\nu} \), \( \tilde{\mathcal{X}}_{\nu} \) fit into an induction diagram

\[
\mathcal{X}_{\nu} \xrightarrow{q_{\nu}} \tilde{\mathcal{X}}_{\nu} \xrightarrow{p_{\nu}} \mathcal{X}_{\beta}.
\]

The map \( p_{\nu} \) is representable and projective. The map \( q_{\nu} \) is a vector bundle stack whose fiber over the object \( (M_1, \ldots, M_{r-1}, M_r) \) is isomorphic to the quotient stack, relatively to the trivial action,

\[
\bigoplus_{k<h} \Ext^1_{FQ}(M_k, M_h) / \bigoplus_{k<h} \Hom_{FQ}(M_k, M_h)\big]\]

(2.11)

See, e.g., [15, prop. 6.2] for a proof of this fact. We have the adjoint pair of functors [38, §9]

\[
\text{ind}_{\nu} : D^b(\mathcal{X}_{\nu}) \rightarrow D^b(\mathcal{X}_{\beta}) \quad \text{res}_{\nu} : D^b(\mathcal{X}_{\beta}) \rightarrow D^b(\mathcal{X}_{\nu})
\]

(2.12)

which is given by

\[
\text{ind}_{\nu}(\mathcal{E}) = (p_{\nu})!(q_{\nu})^*(\mathcal{E})[\dim q_{\nu}] \quad \forall \mathcal{E} \in D^b(\mathcal{X}_{\nu}).
\]

(2.13)

Since the induction is a shift of a proper direct image and a smooth inverse image, it commutes with the Verdier duality functor \( D \).

Let \( \mathcal{L}_\beta \) be the sum of the complexes \( \text{ind}_{\nu}(k_{\mathcal{X}_{\nu}}) \) where \( \nu \) runs over the set \( I^\beta \). It is self-dual, semisimple, and it decomposes in the following way

\[
\mathcal{L}_\beta = \bigoplus_{\pi \in K_{\mathcal{P}_\beta}} \mathcal{L}(\pi) \quad \mathcal{L}(\pi) = V(\pi) \otimes IC(\pi).
\]

(2.14)

The \( IC(\pi) \)’s are self-dual irreducible perverse sheaves which are pairwise non isomorphic. The multiplicities \( V(\pi) \)’s are nonzero complexes of vector spaces with 0 differential. We define the graded additive category \( C(\mathcal{X}_{\beta}) \) as the strictly full additive subcategory of \( D^b(\mathcal{X}_{\beta}) \) generated by the direct summands of \( \mathcal{L}_\beta \) and all their cohomological shifts. The grading shift functors \( (\bullet) \) on \( C(\mathcal{X}_{\beta}) \) are the cohomological shift functors \( [\bullet] \). The Verdier duality yields an antiautoequivalence \( D \) of \( C(\mathcal{X}_{\beta}) \). Set

\[
C(\mathcal{X}_{\nu}) = C(\mathcal{X}_{\nu_1}) \times \cdots \times C(\mathcal{X}_{\nu_{r-1}}) \times C(\mathcal{X}_{\nu_r}).
\]
The functor \( \text{ind}_\nu \) maps the subcategory \( \text{C}(\chi_\nu) \) of \( \text{D}^b(\chi_\nu) \) to \( \text{C}(\chi_\beta) \). It equips the graded additive category \( \text{C}(\chi) = \bigoplus_{\beta \in \mathbb{Q}_+} \text{C}(\chi_\beta) \) with the structure of a graded monoidal category \( (\text{C}(\chi), \otimes) \).

By [49], there is a graded k-algebra isomorphism
\[
(2.15) \quad R_\beta = \text{End}_{\text{D}^b(\chi_\beta)}(L_\beta)^{\text{op}}.
\]
Under the Yoneda composition, for each complex \( \mathcal{E} \) we view \( \text{Hom}_{\text{D}^b(\chi_\beta)}(L_\beta, \mathcal{E}) \) as a graded \( R_\beta \)-module. This yields a graded functor of graded additive categories
\[
(2.16) \quad \Phi^*_{\beta} : \text{D}^b(\chi_\beta) \to \mathcal{C}_\beta, \quad \mathcal{E} \mapsto \text{Hom}_{\text{D}^b(\chi_\beta)}(L_\beta, \mathcal{E})
\]
such that for each \( \nu \in I_\beta \) we have
\[
\Phi^*_{\beta}(\text{ind}_\nu(k_{\chi_\nu})) = R_{\nu_1} \circ R_{\nu_1 - 1} \circ \cdots \circ R_{\nu_1}.
\]
The labelling of the simple summands \( IC(\pi) \) of \( L_\beta \) is chosen such that we have
\[
(2.17) \quad \Phi^*_{\beta}(IC(\pi)) = P(\pi), \quad \forall \pi \in KP_\beta.
\]
By Remark [A.3.6] the functor \( \Phi^*_{\beta} \) gives an equivalence of graded additive categories
\[
C(\chi_\beta) \to C_{\beta}^{\text{proj}}, \quad IC(\pi) \mapsto P(\pi), \quad \forall \pi \in KP_\beta.
\]
A functor of additive categories yields a triangulated functor of the corresponding homotopy categories. Thus (2.15) yields an equivalence of graded triangulated categories
\[
\Phi^* : \text{K}^b(\text{C}(\chi_\beta)) \to \text{D}^{\text{per}(\mathcal{C}_\beta)}
\]
such that \( \Phi^*_{\beta}(\text{ind}_\nu(k_{\chi_\nu})) = R_{\nu_1} \circ \Phi^*_{\nu_{1-1}} \circ \cdots \circ \Phi^*_{\nu_1} \). More generally, the following is easy to prove.

**Proposition 2.5.1.** The graded functor \( \Phi^*_{\beta} \) extends to a graded functor of graded monoidal categories \( (\text{C}(\chi_\beta), \otimes) \to (\mathcal{C}_{\beta}^{\text{proj}}, \otimes^{\text{op}}) \), i.e., there is an isomorphism of functors \( \Phi^*_{\beta} \text{ind}_\nu = \Phi^*_{\nu} \circ \Phi^*_{\nu_{1-1}} \circ \cdots \circ \Phi^*_{\nu_1} \).

### 2.6. Relation with quantum groups.

Let \( U_q(n) \) be the Lusztig \( A \)-form of the positive half of the quantized enveloping algebra of type \( Q \). Let \( U_q(\beta) \) be the \( \beta \)-weight subspace of \( U_q(n) \) and \( B_\beta \) its canonical basis. Let \( A_q(\beta) \) be the dual \( A \)-module of \( U_q(\beta) \) and \( B_\beta \) its dual canonical basis. We view \( A_q(\beta) \) as an \( A \)-submodule of \( U_q(\beta) \otimes Q(q) \) via the Lusztig pairing on \( U_q(\beta) \). Set \( A_q(\beta) = \bigoplus_{\beta \in \mathbb{Q}_+} A_q(\beta) \). By [49], for each \( \beta \in \mathbb{Q}_+ \) there is an \( A \)-linear isomorphism \( g : K_0(\mathcal{C}_\beta^{\text{proj}}) \to U_q(\beta) \) which takes \( \{P(\pi) : \pi \in KP_\beta\} \) to the canonical basis \( B_\beta \). The transpose yields an isomorphism \( \hat{g} : G_0(\mathcal{C}_\beta^{\text{proj}}) \to A_q(\beta) \) which takes \( \{L(\pi) : \pi \in KP_\beta\} \) to the dual canonical basis \( B_\beta^* \). Since any module in \( \mathcal{C}_\beta \) has a (maybe infinite) composition series, this map extends to a \( \mathbb{Z}((q^{-1})) \)-linear map
\[
\hat{g} : G_0(\mathcal{C}_\beta) \to A_q(\beta) \otimes A \mathbb{Z}((q^{-1})).
\]
If \( \beta \) is a real positive root let \( E(\beta) \) be the root vector of weight \( \beta \). The dual root vector is \( E(\beta)^* = (1 - q^{-2})E(\beta) \). By [10] thm. 18.2, we have
\[
\hat{g}(\Delta(\beta)) = E(\beta), \quad \hat{g}(L(\beta)) = E(\beta)^*.
\]
Next, there is a canonical $\mathbb{Z}[q,q^{-1}]$-algebra isomorphism $f : K_0(C(\mathcal{X})) \to U_q(n)$ such that $\{ f(I(C(\pi)) : \pi \in KP_\beta \} = B_\beta$. Consider the following diagram

$$
\begin{array}{ccc}
K_0(C(\mathcal{X}_\beta)) & \xrightarrow{\phi_\beta^*} & K_0(C^\text{proj}_\beta) \\
\downarrow{\Phi_\beta} & & \downarrow{g} \\
U_q(n)_\beta & \xrightarrow{\phi_\beta} & A_q(n)_\beta \otimes \mathbb{A} Z((q^{-1}))
\end{array}
$$

The right square commutes by definition. The left triangle commutes by Proposition 2.5.1 because the quantum divided powers $E(\alpha_i)^{(p)}$ with $i = 0,1$ and $p > 0$ generate $U_q(n)$ and we have

$$f(k_{X_{\mu_{\alpha_i}}}) = E(\alpha_i)^{(p)} , \quad g(P(\alpha_i^p)) = E(\alpha_i)^{(p)} , \quad \Phi_\beta^*(k_{X_{\mu_{\alpha_i}}}) = (P(\alpha_i^p)).$$

### 3. The Kronecker Quiver $Q$

From now on, let $Q$ be the Kronecker quiver. We have $I = \{0,1\}$ and $\Omega = \{x,y\}$ with both arrows oriented from 1 to 0. We have $\delta = \alpha_0 + \alpha_1$ and

$$\Phi_{+-} = \{ \gamma_n : n \in \mathbb{N} \} , \quad \Phi_{++} = \{ \beta_n : n \in \mathbb{N} \} , \quad \gamma_n = \alpha_1 + n\delta , \quad \beta_n = \alpha_0 + n\delta.$$

We fix the total convex preorder on the set $\Phi_+$ such that we have

$$\gamma_0 > \gamma_1 > \gamma_2 > \cdots > Z > 0 \delta > \cdots > \beta_0 \delta > \beta_1 > \beta_2 > \cdots , \quad m\delta > n\delta , \quad \forall m,n > 0.$$

Fix an element $\beta$ in the set $Q_{++} = \{ n\alpha_1 + m\alpha_0 \in Q_+ : n,m,m-n \geq 0 \}$. We write

$$\beta = n\alpha_1 + m\alpha_0 = r\alpha_0 + n\delta , \quad r = m - n.$$

We have $d_\beta = 2nm$ and $G_\beta = GL_{m,F} \times GL_{n,F}$. The set $KP_\beta$ is equipped with the partial order $\preceq$ defined in (2.1), (2.2). Let $\Gamma_\beta$ be the order ideal consisting of all Kostant partitions of $\beta$ supported on the set $\Phi_{++}$. An element of $\Gamma_\beta$ is a decreasing sequence $\pi$ in $\Phi_{++}$ of the form $\pi = ((\beta_1)^{p_1}, \ldots, (\beta_t)^{p_t}, (\beta_0)^{p_0})$ such that $r_\pi = r$ and $n_\pi = n$.

#### 3.1. The Degeneration Order for $Q$.

The indecomposable $FQ$-modules are partitioned into the preprojective, preinjective and regular modules. For each $k \in \mathbb{N}$, there is a unique indecomposable preprojective $P_k$ with dimension vector $\beta_k$ and an unique indecomposable preinjective $I_k$ with dimension vector $\gamma_k$. The regular indecomposables have dimension $(k+1)\delta$, and any such representation, denoted by $R_{k,z}$, is labelled by a point $z \in \mathbb{P}^1(F)$. It is well-known that for any $k \leq h$, we have

$$\text{Hom}_{FQ}(P_k, P_h) \simeq k^{h-k+1} , \quad \text{Hom}_{FQ}(P_{h+1}, P_k) = 0 , \quad \text{Ext}_{FQ}^1(P_k, P_h) = 0.$$

Any representation $M \in X_\beta(F)$ is isomorphic to a representation of the form

$$M = \bigoplus_{k \geq 0} \bigoplus_{i=0}^M (I_k)^{\oplus i} \oplus \bigoplus_{k \geq 0, z \in \mathbb{P}^1(F)} (R_{k,z})^\oplus r_{M,k,z} \oplus \bigoplus_{k \geq 0} (P_k)^\oplus r_{M,k} , \quad p_{M,k}, r_{M,k,z}, i_{M,k} \in \mathbb{N}.$$

Set $r_{M,k} = \sum_{z \in \mathbb{P}^1(F)} r_{M,k,z}$ for each $k$. We define the partitions

$$p_M = (p_{M,\geq 0}, p_{M,\geq 1}, \ldots) , \quad i_M = (i_{M,\geq 0}, i_{M,\geq 1}, \ldots),$$

$$r_M = (r_{M,\geq 0}, r_{M,\geq 1}, \ldots) , \quad r_{M,z} = (r_{M,\geq 0,z}, r_{M,\geq 1,z}, \ldots).$$
The rank of $M$ is the integer $\rho_M = n - i_{M,\geq 0} = m - p_{M,\geq 0}$, and its type is the Kostant partition
\[
\pi = ((\gamma_0)^{i_{M,\geq 0}}, (\gamma_1)^{i_{M,\geq 1}}, \ldots, \delta_{M,\geq 0}, (2\delta)^{i_{M,\geq 1}}, \ldots, (\beta_1)^{p_{M,\geq 1}}, (\beta_0)^{p_{M,0}}).
\]
Note that $\pi \in \Gamma_\beta$ if and only if $i_M = r_M = \emptyset$. For each partitions $\lambda = (\lambda_a), \mu = (\mu_a)$ and integer $s$ put
\[
\lambda + s = (\lambda_1 + s, \lambda_2 + s, \ldots), \quad n_\lambda = \sum_{a \geq 1} \lambda_a, \quad \lambda \equiv \mu \iff \sum_{a=1}^b \lambda_a \leq \sum_{a=1}^b \mu_a, \forall b.
\]
Let $Y_M \subset X_\beta$ be the $G_\beta$-orbit of $M$. Necessary and sufficient conditions for the inclusion of orbits closures are given by Pokrzywa’s theorem, see, e.g., [17, thm. 3.1] for a more recent formulation.

**Proposition 3.1.1.** For each $M, N \in X_\beta(F)$, we have
\[
Y_N \subset Y_M \iff \begin{cases}
p_N + \rho_N \leq p_M + \rho_M, \\
r_{N,z} + i_{N,\geq 0} \geq r_{M,z} + i_{M,\geq 0}, \quad \forall z \in \mathbb{P}^1(F), \\
i_N + \rho_N \leq i_M + \rho_M.
\end{cases}
\]

To each Kostant partition $\pi = ((\beta_1)^{p_1}, \ldots, (\beta_0)^{p_0})$ in $\Gamma_\beta$ we associate the preprojective $F\mathcal{Q}$-module of type $\pi$ given by $\mathcal{P}_\pi = (\mathcal{P}_1)^{\otimes p_1} \oplus \cdots \oplus (\mathcal{P}_l)^{\otimes p_l} \oplus (\mathcal{P}_0)^{\otimes p_0}$. For any subset $\Gamma \subset \Gamma_\beta$ let
\[
Y_\Gamma = \bigcup_{\pi \in \Gamma} Y_{\pi}, \quad Y_\Gamma = [Y_\Gamma / G_\beta].
\]

We’ll need the following locally closed inclusions
\[
i_\Gamma : Y_\Gamma \rightarrow Y_\beta, \quad j_\beta : Y_\beta \rightarrow X_\beta, \quad j_\Gamma = j_\beta \circ i_\Gamma : Y_\Gamma \rightarrow X_\beta.
\]
Write $\mathcal{M}_\Gamma = (j_\Gamma)^* \mathcal{L}_\beta$ and $\mathcal{M}_\beta = \mathcal{M}_{\Gamma_\beta}$. We abbreviate $i_\pi = i_{(\pi)}, j_\pi = j_{(\pi)}$ and
\[
Y_\beta = Y_{\beta_\Gamma}, \quad Y_\beta = Y_{\beta_\Gamma}, \quad Y_\pi = Y_{\pi_\Gamma}, \quad Y_\pi = Y_{\pi_\Gamma}.
\]

**Proposition 3.1.2.**
(a) $i_\Gamma, j_\Gamma$ are open for each order ideal $\Gamma \subset \Gamma_\beta$.
(b) $Y_\sigma \subset Y_\pi \iff \sigma \geq \pi$ for each $\pi, \sigma \in \Gamma_\beta$.

**Proof.** To prove (b), let $\pi, \sigma \in \Gamma_\beta$ and $M, N \in X_\beta(F)$ be of type $\pi, \sigma$. Then, we have $\rho_N = \rho_M$. By Proposition 3.1.1 we have $Y_N \subset Y_M$ if and only if $p_N \equiv p_M$. Fix an integer $l$ such that $p_{M,>l} = p_{N,>l} = 0$ and consider the partitions
\[
\lambda_M = (p_{M,1}^{M,1}, 1^{p_{M,1}}, 0^{p_{M,0}}), \quad \lambda_N = (p_{N,1}^{N,1}, 1^{p_{N,1}}, 0^{p_{N,0}}).
\]
The partitions $1 + \lambda_M, 1 + \lambda_N$ are transpose of the partitions $p_M, p_N$. Note that we have $\sum_k p_{M,k} = \sum k p_{N,k}$. We deduce that
\[
Y_N \subset Y_M \iff \lambda_N \equiv \lambda_M \iff \lambda_N - \lambda_M \text{ is a sum of positive roots} \iff \sigma \geq \pi.
\]
By (b), to prove (a) it is enough to check that the inclusion $Y_\beta \subset X_\beta$ is open. To do so, note that a representation $M \in X_\beta(F)$ is preprojective if and only if $\text{Hom}_{F\mathcal{Q}}(R, M) = 0$ for
any regular representation $R$ of dimension $\delta$. Note also that $\text{Hom}_{\mathcal{F}}(R, M)$ is the fiber at $(R, M)$ of the projection

$$\{ (\phi, R, M) : \phi \in \text{Hom}_{\mathcal{F}}(R, M), R \in X_\delta, M \in X_\beta \} \to X_\delta \times X_\beta,$$

and the left hand side is the set of closed points of a closed $F$-subset of $\mathbb{A}_{F}^{m+n} \times X_\delta \times X_\beta$. Chevalley’s semi-continuity theorem implies that the set of all $(R, M)$ such that $\text{Hom}_{\mathcal{F}}(R, M) = 0$ is open. Since the moduli space of the regular representations of dimension $\delta$ is isomorphic to $\mathbb{P}_{F}$, hence is projective, we deduce that the set of all $M \in X_\beta(F)$ such that $\text{Hom}_{\mathcal{F}}(R, M) = 0$ for any regular representation $R$ of dimension $\delta$ is also open. \qed

**Corollary 3.1.3.** $(j_\beta)^* \mathcal{L}(\pi) = 0$ unless $\pi \in \Gamma_\delta$, hence $\mathcal{M}_\beta = \bigoplus_{\pi \in \Gamma_\delta} (j_\beta)^* \mathcal{L}(\pi)$.

**Proof.** For each root partition $\sigma \in \Pi_\beta$, let $X_\sigma = \bigcup M Y_M$ be the locally closed subset of $X_\beta$ consisting of the union of the orbits of all representations $M$ of type $\sigma$ such that the regular part of $M$ is semisimple. By [36], each perverse sheaves in the set $\{IC(\pi) : \pi \in K\Pi_\beta\}$ is the intermediate extension of a local system on a dense open subset of the stratum $X_\sigma$ for some root partition $\sigma$. Thus the corollary follows from Proposition 3.1.2(a). \qed

### 3.2. The category $\mathcal{D}_\beta$ and the moduli stack of representations of $Q$.

Let $\Gamma$ be any order ideal of $\Gamma_\beta$. The additive category $C(Y_\Gamma)$ is generated by the summands of the complex $\mathcal{M}_\Gamma$ and their cohomological shifts. Passing to the homotopy categories, we get the graded triangulated category $D^b_\mu(Y_\Gamma) = K^b(C(Y_\Gamma))$. The maps $i_\Gamma$ and $j_\Gamma$ are open inclusions. Thus, we have the restriction functor $(i_\Gamma)^* : C(Y_\beta) \to C(Y_\Gamma)$ which yields the graded triangulated functor $(i_\Gamma)^* : D^b_\beta(Y_\beta) \to D^b_\mu(Y_\Gamma)$. For each element $\pi \in \Gamma$, let $IC(\pi)_\mu$ be the complex $(j_\beta)^* IC(\pi)$ in $C(Y_\beta)$ viewed as an object of $D^b_\beta(Y_\beta)$. In [24] we have introduced graded subcategories $\mathcal{D}_\beta, \mathcal{D}_\Gamma$ of the graded module category $\mathcal{C}_\beta$ and the graded triangulated functor $L(h_\Gamma)^* : D^b(\mathcal{D}_\beta) \to D^b(\mathcal{D}_\Gamma)$.

**Proposition 3.2.1.** Assume that $\Gamma \subset \Gamma_\beta$ is an order ideal.

(a) There is an equivalence of graded triangulated categories $\Lambda_\mu^\mu : D^b(\mathcal{D}_\Gamma) \to D^b(\mathcal{Y}_\Gamma)$ such that $L(h_\Gamma)^* Q(\pi) \to (i_\Gamma)^* IC(\pi)_\mu$ for all $\pi \in \Gamma$.

(b) There is a right adjoint triangulated functor $(i_\Gamma)^\star$ to $(i_\Gamma)^*$ yielding a commutative diagram

$$\begin{array}{ccc}
D^b(\mathcal{D}_\beta) & \xrightarrow{(i_\Gamma)^*} & D^b(\mathcal{Y}_\beta) \\
\Lambda_\mu^\mu \downarrow & & \Lambda_\Gamma^\mu \\
D^b(\mathcal{D}_\Gamma) & \xrightarrow{(i_\Gamma)^\star} & D^b(\mathcal{Y}_\Gamma).
\end{array}$$

The proof of the proposition consists of checking that the graded $k$-algebra

$$\mathcal{S}_\Gamma = \text{End}^\bullet_{D^b(\mathcal{Y}_\Gamma)}(\mathcal{C}_\Gamma)$$

(3.4)
Claim 3.2.4. If \( a,\beta \in \mathcal{A}_\ast \) and it coincides with the open subset \( \Delta \) of the proof of Lemma 3.2.3, taking the homotopy categories. Applying Lemma 3.2.3 both to \( \Gamma \) and \( \Gamma' \) modules, because \( \text{Hom}_{\mathcal{A}}(\mathcal{A}_\ast, \mathcal{A}_\ast) \) is preinjective or regular, and the module \( \mathcal{M}_\ast \) for each \( \pi \in \Gamma_\beta \). We consider the composed functor

\[
\Psi^\bullet_{\Gamma} = \Phi^\bullet_{\Gamma}(\pi) : \mathcal{D}^b(\mathcal{Y}_\Gamma) \to \mathcal{C}_\beta, \quad \mathcal{E} \mapsto \text{Hom}^\bullet_{\mathcal{D}^b(\mathcal{Y}_\beta)}(\mathcal{L}_\beta, (\pi)_s\mathcal{E}) = \text{Hom}^\bullet_{\mathcal{D}^b(\mathcal{Y}_\beta)}(\mathcal{M}_\beta, \mathcal{E}).
\]

**Lemma 3.2.2.** For each \( \pi \in \Gamma_\beta \) we have the following isomorphisms of graded \( \mathcal{R}_\beta \)-modules

(a) \( \Phi^\bullet_{\Gamma}(\mathcal{C}(\pi)) \cong \mathcal{P}(\pi) \),
(b) \( \Phi^\bullet_{\Gamma}(\mathcal{E}(\pi)) \cong \Delta(\pi) \),
(c) \( \Phi^\bullet_{\Gamma}(\mathcal{E}(\pi)) \cong \mathcal{Q}(\pi) \).

**Lemma 3.2.3.** We have an equivalence of graded Abelian categories \( \mathcal{D}_\Gamma \to \mathcal{S}_\Gamma \)-mod. The functor \( \Psi^\bullet_{\Gamma} \) gives an equivalence of graded additive categories \( \mathcal{D}(\mathcal{Y}_\Gamma) \to \mathcal{D}^b(\mathcal{Y}_\beta) \) such that \( (\pi)^*\mathcal{C}(\pi) \to (\pi)^*\mathcal{P}(\pi) \) for each \( \pi \in \Gamma \).

**Proof of Proposition 3.2.1** Part (a) of the proposition follows from Proposition 2.4.1 and Lemma 3.2.3 taking the homotopy categories. Applying Lemma 3.2.3 both to \( \Gamma \) and \( \Gamma_\beta \), we get two functors \( A^\pi_{\beta} \) and \( A^\pi_{\beta} \) such that \( A^\pi_{\beta} \mathcal{L}(\pi) = (\pi)^* A^\pi_{\beta} \). We define \( (\pi)_s \) as the unique functor such that the square in part (b) of the proposition commutes. \( \square \)

**Proof of Lemma 3.2.4** By [36, thm. 6.16], the complexes \( \mathcal{C}(\pi) \) with \( \pi \in \Gamma_\beta \) belong to the canonical basis for each \( \beta \in \mathcal{Q}_+ \), i.e., there is a map \( \Gamma_\beta \to \mathcal{K}_\beta \), \( \pi \to \pi \) such that \( \mathcal{C}(\pi) = \mathcal{C}(\pi) \).

**Step 1 :** We prove (a), (b) in the case \( r = 1 \).

Let \( r = 1 \). Hence \( \beta \) is the positive root \( \beta_n \). Further, we have \( \Gamma_\beta = \{(\beta)\} \) and \( \pi \geq (\beta) \) for all \( \pi \in \mathcal{K}_\beta \). We abbreviate \( k_\beta = k_{X_\beta} \). The stratum \( Y(\beta) \) is the open dense \( G_\beta \)-orbit in \( X_\beta \), and it coincides with the open subset \( Y_\beta \). We deduce that \( k_\beta = k_{Y_\beta} \) and

\[
\Delta(k_\beta) = (j_\beta)_s k_\beta[d_\beta], \quad \nabla(k_\beta) = (j_\beta)_x k_\beta[d_\beta], \quad \mathcal{C}(k_\beta) = k_\beta[d_\beta].
\]

To prove (a) for \( r = 1 \) we must check that \( (\beta)^0 = (\beta) \).

**Claim 3.2.4.** If \( \alpha, \gamma \neq 0 \) and the condition \( \alpha < \beta < \gamma \) does not hold, then the perverse sheaf \( \mathcal{C}(\pi) \) is not a direct summand of \( \mathcal{C}(\pi) \) whenever \( \pi \in \mathcal{K}_\gamma \), \( \gamma \in \mathcal{K}_\alpha \) and \( a \in \mathbb{Z} \). \( \square \)

Indeed, for each \( FQ \)-modules \( M \in \mathcal{X}(F) \), \( N \in \mathcal{X}_\gamma(F) \) with an exact sequence

\[
0 \to M \to \mathcal{P}_n \to N \to 0,
\]

the module \( M \) is a sum of \( \mathcal{P}_m \)'s for some integers \( m < n \) because \( \text{Hom}_{\mathcal{FQ}}(Q, \mathcal{P}_n) = 0 \) whenever \( Q \) is preinjective or regular, and the module \( N \) is a sum of regular and preinjective modules, because \( \text{Hom}_{\mathcal{FQ}}(\mathcal{P}_n, \mathcal{P}_k) = 0 \) for all \( k < n \) by (3.2). Hence, we have \( \alpha < \beta < \gamma \).
By Proposition 2.5.1 for all \( \sigma, \pi \) as above we have \( \text{Hom}_{C_\beta}(P(\sigma) \circ P(\pi), L((\beta)^\delta)) = 0 \), and, by adjunction, that

\[
\text{Hom}_{C_\alpha \times C_\gamma}(P(\sigma) \otimes P(\pi), \text{Res}_{\alpha, \gamma} L((\beta)^\delta)) = 0.
\]

Thus we have \( \text{Res}_{\alpha, \gamma} L((\beta)^\delta) = 0 \), hence the module \( L((\beta)^\delta) \) is cuspidal, so it is \( L(\beta) \).

Now, we prove (b) for \( r = 1 \). The obvious morphism of complexes \( IC(k_{(\beta)}) \to \nabla(k_{(\beta)}) \) yields an \( R_\beta \)-module homomorphism

\[
(3.6) \quad \Phi_\beta(I\!C(k_{(\beta)})) \to \Phi_\beta(\nabla(k_{(\beta)})).
\]

We have

\[
\Phi_\beta^*(I\!C(k_{(\beta)})) = \Phi_\beta^*(k_{\beta}([d_\beta])),
\]

\[
= H^\bullet_{G_\beta}(X_\beta, L_\beta)[-d_\beta],
\]

\[
= (V(\beta) \otimes H^\bullet_{G_\beta}(X_\beta, k)) \oplus \bigoplus_{\pi \neq (\beta)} (V(\pi) \otimes H^\bullet_{G_\beta}(X_\beta, IC(\pi))[-d_\beta],
\]

and

\[
(3.7) \quad \Phi_\beta^*(\nabla(k_{(\beta)})) = H^\bullet_{G_\beta}(Y_\beta, \mathcal{M}_\beta)[-d_\beta] = V(\beta) \otimes H^\bullet_{G_\beta}(Y_\beta, k).
\]

Under the isomorphisms (3.7) and (3.8) the map (3.6) is identified with the restriction \( H^\bullet_{G_\beta}(X_\beta, \mathcal{L}_\beta) \to H^\bullet_{G_\beta}(Y_\beta, \mathcal{M}_\beta) \). The restriction to \( Y_\beta \) gives also a map

\[
(3.9) \quad H^\bullet_{G_\beta}(X_\beta, k) \to H^\bullet_{G_\beta}(Y_\beta, k).
\]

Let \( D_\beta \) be the diagonal copy of \( \mathbb{G}_m \) in \( G_\beta \). It is the stabilizer of any point of \( Y(\beta) \). Hence, we have \( H^\bullet_{G_\beta}(X_\beta, k) = H^\bullet_{G_\beta}(D_\beta) \) and \( H^\bullet_{G_\beta}(Y_\beta, k) = H^\bullet_{D_\beta} \). We deduce that the map (3.9) is surjective, hence (3.6) is also surjective. So, to prove that \( \Phi_\beta^*(\nabla(k_{(\beta)})) = \Delta(\beta) \) it is enough to check that

\[
\text{Ext}^1_{G_\beta}(\Phi_\beta^*(\nabla(k_{(\beta)})), L(\pi)) = 0, \quad \forall \pi \leq (\beta),
\]

or, equivalently, that \( \text{Ext}^1_{G_\beta}(\Phi_\beta^*(\nabla(k_{(\beta)})), L(\beta)) = 0 \). To do that, we apply the functor \( \text{Hom}_{C_\beta}(\bullet, L(\beta)) \) to the short exact sequence in \( C_\beta \) given by

\[
0 \to J(\beta) \to P(\beta) \to \Phi_\beta^*(\nabla(k_{(\beta)})) \to 0.
\]

We must check that

\[
(3.10) \quad \text{Hom}_{C_\beta}(J(\beta), L(\beta)) = 0.
\]

Let \( H^\bullet_+ \) be the kernel of the restriction map \( H^\bullet_{G_\beta} \to H^\bullet_{D_\beta} \). We have

\[
J(\beta) = (V(\beta) \otimes H^\bullet_+) \oplus \bigoplus_{\pi \neq (\beta)} (V(\pi) \otimes H^\bullet_{G_\beta}(X_\beta, IC(\pi))[-d_\beta].
\]

Since \( H^0_+ = 0 \), we do have (3.10) for degree reasons.

**Step 2**: We prove (b) for any \( r \).
Fix a tuple \( \pi = ((\beta_0)^{p_0}, \ldots, (\beta_{l-1})^{p_{l-1}}, (\beta_l)^{p_l}) \) in \( Q_\beta^0 \). Taking the parts of \( \pi \) in the reverse order we get a Kostant partition \( \pi^{op} \) in \( \Gamma_\beta \). Consider the open subsets \( X_\pi \subset X_{\pi} \) and \( \tilde{X}_\pi \subset \tilde{X}_\pi \) given by

\[
X_\pi = (Y_{(\beta_0)})^{p_0} \times \cdots \times (Y_{(\beta_l)})^{p_l}, \quad \tilde{X}_\pi = q_\pi^{-1}X_\pi.
\]

Comparing (2.12) with the third equality in (3.2) we deduce that the restriction of the map \( q_\pi \) to \( X_\pi \) is a gerbe with group the unipotent radical of \( P_\pi \). The second equality in (3.2) implies that the restriction of the map \( p_\pi \) to \( \tilde{X}_\pi \) is a closed embedding with image \( \gamma_\pi \). In other words, any extension of \( P_\pi, \pi \) is a surjective map that \( \pi \) is a quotient object (meaning that \( \pi \) is a subobject of \( \pi \) a quotient object) is necessarily trivial, hence isomorphic to \( \pi \). Further, the representation \( \pi \) preserves a unique flag in \( V \) which is conjugate to \( V \), under the action of \( G_\beta \). Thus, for each \( k = 0, \ldots, l \), we get

\[
\text{ind}_{(\beta_0)^{p_0}}^\pi \big( \Delta(\beta_0)_{\pi^{op}} \big) = \Delta(\beta_0)^{(\beta_0)^{p_0}}(p_0)_!,
\]

(3.11)

\[
\text{ind}_{\pi}^\pi \big( \Delta(\beta_0)^{(\beta_0)^{p_0}}(p_0) \otimes \cdots \otimes \Delta(\beta_0)^{(\beta_{l-1})^{p_{l-1}}} \otimes \Delta(\beta_0)^{(\beta_l)^{p_l}} \big) = \Delta(\pi).
\]

From (2.6), (2.7), (5.11), Proposition 2.5.1 and the isomorphism \( \Phi_{\beta}^\pi(\nabla(\beta_0)) \cong \Delta(\beta_0) \) proved in Step 1 for each \( k \in [0, l] \), we deduce that \( \Phi_{\beta}^\pi(\nabla(\pi)) = \Delta(\pi) \), proving the part (b) of the lemma for any \( r \).

**Step 3**: We prove (a) for any \( r \).

Step 2 implies that for each \( \sigma \in \Gamma_\beta \) we have

\[
\Delta(\sigma) = \Phi_{\beta}^\pi(\nabla(\pi)) = \bigoplus_{\pi \in KP_{\beta}} V(\pi) \otimes H^\pi_{G_\beta}(\sigma, (j_\sigma)^*IC(\pi)).
\]

We must check that for all \( \pi \in \Gamma_\beta \) we have \( \pi^\beta = \pi \). We have

\[
[\Delta(\sigma) : L(\pi^\beta)] = \dim e(\pi^\beta) \Delta(\sigma),
\]

\[
= \dim H^\pi_{G_\beta}(\sigma, (j_\sigma)^*IC(\pi)),
\]

\[
= \dim H^\pi_{G_\beta}(\sigma, (j_\sigma)^*IC(k_\pi)),
\]

\[
= 0 \quad \text{if } Y_\sigma \not\subseteq \bar{Y}_\pi,
\]

\[
\neq 0 \quad \text{if } \sigma = \pi.
\]

The first relation and Proposition 3.1.2(b) yield

\[
[\Delta(\pi^\beta) : L(\pi^\beta)] \neq 0 \implies Y_\pi \subseteq \bar{Y}_\pi \implies \pi^\beta \geq \pi.
\]

The second relation yields \( \pi \geq \pi^\beta \). Hence, we have \( \pi = \pi^\beta \) as wanted.

**Step 4**: We construct a surjective \( R_{\beta} \)-module homomorphism \( P(\pi) \to \Phi_{\beta}^\pi(Q(k_\pi)) \).

We’ll need mixed analogues of the complexes above. Let \( X_{\beta,0} \) be the variety of all \( \beta \)-dimensional representations of the path algebra \( F_0\mathcal{Q} \). The linear algebraic \( F_0 \)-group \( G_{\beta,0} \) acts on \( X_{\beta,0} \) and we can consider the quotient \( F_0 \)-stack \( X_{\beta,0}^\beta = [X_{\beta,0} / G_{\beta,0}] \). Since the group \( G_{\beta} \) is connected, by Lang’s theorem we have \( X_{\beta} = X_{\beta,0} \otimes_{F_0} F \). We consider the mixed complex

\[
IC(\pi)_m = IC(k_\pi)_m = (j_\pi)_*k_\pi \langle d_\pi \rangle.
\]
It is pure of weight 0. We abbreviate \( E_m = (j_{\beta})_* (j_{\beta})^* IC(\pi)_m \). The complex \( E = \omega(E_m) \) is \( Q(k_\pi) \). The adjunction yields a canonical map \( IC(\pi)_m \to E_m \), hence by functoriality a map

\[
\Phi^*_\beta(\pi) : \Phi^*_\beta(\pi) \to \Phi^*_\beta(E).
\]

**Claim 3.2.5.** There is a distinguished triangle \( \xymatrix{ IC(\pi)_m \ar[r] & E_m \ar[r] & (E_m > 0) \} \). Proof. By [5.1.14, prop. 1.4.12], the mixed complex \( E_m \) is in \( D_{\geq 0}(X_\beta, 0) \cap \mathcal{D}^b(X_\beta, 0) > 0 \).

By Proposition A.2.1, we have a distinguished triangle

\[
\xymatrix{ (\mathcal{E}m)_0 \ar[r] & (\mathcal{E}m) \ar[r] & (\mathcal{E}m) > 0. }
\]

Write \( \mathcal{E} > 0 = \omega((\mathcal{E}m)_0), \mathcal{E} = \omega((\mathcal{E}m)_0) \) and \( f = \omega(f_m) \). We have short exact sequences

\[
0 \longrightarrow p^H_I(\mathcal{E}0) \longrightarrow p^H_I(\mathcal{E}) \longrightarrow p^H_I(\mathcal{E} > 0) \longrightarrow 0, \quad \forall a \in \mathbb{Z}.
\]

By [82] ex. III.10.3 we have

\[
p^H_I(\mathcal{E}0) = IC(\pi).
\]

Since the mixed complex \( (\mathcal{E}m)_0 \) is pure, we have

\[
\mathcal{E}0 = IC(\pi) \oplus \bigoplus_{a > 0} p^H_I(\mathcal{E}0)[a].
\]

Let \( a > 0 \). If \( p^H_I(\mathcal{E}0) \neq 0 \) then the restriction of \( f \) to the summand \( p^H_I(\mathcal{E}0)[-a] \) is nonzero. Since \( \mathcal{E} = (j_{\beta})_* (j_{\beta})^* IC(\pi), \) this yields a nonzero map

\[
(j_{\beta})_* p^H_I(\mathcal{E}0)[-a] \to (j_{\beta})^* IC(\pi).
\]

This is absurd by (3.13) because, by definition of \( \mathcal{E}, \) we have

\[
\sum_{a \in \mathbb{Z}} [p^H_I(\mathcal{E}) : IC(\pi)] \leq \sum_{a \in \mathbb{Z}} [p^H_I(\mathcal{E}m) : IC(\pi)_m] = 1.
\]

We deduce that \( (\mathcal{E}m)_0 = IC(\pi)_m, \) and (3.13) yields Claim 3.2.5. \( \square \)

**Claim 3.2.6.** The map (3.12) is surjective.

Proof. By [37] §5.3, the complex \( L_\beta \) has a canonical mixed structure \( L_{\beta, m} \) which is pure of weight 0. Set \( M_{\beta, m} = (j_{\beta})_* L_{\beta, m}. \) The mixed complex \( M_{\beta, m} \) on \( Y_\beta \) is pure of weight 0. We consider the following mixed complex of vector spaces in \( D^b_m(\text{Spec} F_0) \)

\[
\Phi^*_{\beta, m}(\mathcal{F}m) = \mathcal{H}om_{D^b_m(Y_\beta, 0)}(L_{\beta, m}, \mathcal{F}m), \quad \forall \mathcal{F}m \in D^b_m(Y_\beta, 0).\]

By [47] cor. 3.10], we have

\[
\Phi^*_{\beta, m}(\mathcal{E}m) > 0 \in D^b_m(\text{Spec} F_0).
\]

Next, we consider the mixed complex \( \Phi^*_{\beta, m}(\mathcal{E}m). \) We have

\[
\Phi^*_{\beta, m}(\mathcal{E}m) = \mathcal{H}om_{D^b_m(Y_\beta, 0)}(M_{\beta, m}, (j_{\beta})_* IC(\pi)_m), \quad \omega \Phi^*_{\beta, m}(\mathcal{E}m) = \Phi^*_{\beta}(\mathcal{E}).
\]

We’ll use the notation \( H_\beta, Gr_\beta^+ \), \( Gr_\beta^- \) and \( S_\beta^+ \) introduced in 4 below, to which we refer for more details. In particular, we have \( Gr_\beta^+ = [Gr_\beta^+ / H_\beta] \) as a stack, and the stratification \( T^+_\beta \) is even affine by Proposition 4.1.1(b). The image of the mixed complexes \( M_{\beta, m} \) and
We prove Step 5: in (3.12) is surjective.

Note that (3.18)

Now, we apply the functor Φ^*_β,m to the triangle in Claim 3.2.5

We get a long exact sequence of mixed vector spaces

From (3.16), (3.17), we deduce that the map in (3.19) is surjective. Hence, taking the sum over all integers a, we get that the map

in (3.12) is surjective. □

Step 5: We prove (c) for any r.

We must prove that Φ^*_β(Q(k_π)) = Q(π). For any Kostant partition σ ∈ KPβ, we have

\[
[Φ^*_β(Q(k_π)) : L(σ)] = \text{dim} \text{Hom}_R(\sigma, Φ^*_β(Q(k_π))),
\]

\[
= \text{dim} e(σ)Φ^*_β(Q(k_π)),
\]

\[
= \text{dim} \text{Hom}^*_D(DQ(k_π), IC(σ)),
\]

\[
= \text{dim} \text{Hom}^*_D((j_β)^*IC(π), (j_β)^*IC(σ)),
\]

\[
= 0 \text{ if } σ \notin Γ_β.
\]

Note that (j_β)^* = (j_β)^! because j_β is an open immersion. We deduce that Φ^*_β(Q(k_π)) ∈ D_β.

By Step 4 the graded R_β-module P(π) maps onto Φ^*_β(Q(k_π)). Since the graded R_β-module Q(π) is the largest quotient of P(π) which lies in D_β, we get a surjective graded R_β-module homomorphism

(3.19) \[ Q(π) → Φ^*_β(Q(k_π)). \]

Since the category C_β is affine properly stratified, the module P(π) has a Δ-filtration for all π ∈ KPβ. By Propositions 4.1.1, 4.2.1 and A.3.3, the extension algebra S_β in (3.4) satisfies the conditions in 28 thm. 4.1, hence the graded S_β-module Φ^*_β(Q(k_π)) has an increasing filtration whose layers are isomorphic to the modules Φ^*_β(∇(k_μ)) with σ ∈ Γ_β. Note that Φ^*_β(∇(k_μ)) = Δ(σ) by Step 2 above. Hence, we must check that the multiplicity of Δ(σ) in P(π) and Φ^*_β(Q(k_π)) are the same for all σ ∈ Γ_β. This follows from the relations

\[
P(π) = \bigoplus_{τ ∈ KPβ} V(τ) ⊗ \text{Hom}^*(IC(π), IC(τ)),
\]

\[
Φ^*_β(Q(k_π)) = \bigoplus_{τ ∈ Γ_β} V(τ) ⊗ \text{Hom}^*((j_β)^!(j_β)^*IC(π), IC(τ)),
\]

\[
Δ(σ) = \bigoplus_{τ ∈ Γ_β} V(τ) ⊗ \text{Hom}^*((j_σ)^!(k_μ[d_σ]), IC(τ)).
\]

□
Proof of Lemma 3.2.3. First, we prove the lemma for \( \Gamma = \Gamma_\beta \). Consider the graded \( R_\beta \)-modules

\[
P_\beta = \bigoplus_{\pi \in \mathbb{K}P_\beta} V(\pi) \otimes P(\pi), \quad Q_\beta = \bigoplus_{\pi \in \mathbb{K}P_\beta} V(\pi) \otimes Q(\pi).
\]

The isomorphism \( (2.17) \) and Steps 3 and 5 above yield the following isomorphisms

\[
P_\beta = \Phi_\beta^*(\mathcal{L}_\beta) = R_\beta, \quad Q_\beta = \Psi_\beta^*(\mathcal{M}_\beta) = S_\beta.
\]

We have a surjective graded \( R_\beta \)-module homomorphism \( P_\beta \to Q_\beta \). So, the k-algebra homomorphism

\[
(\beta)^* : R_\beta \to S_\beta
\]

given by the restriction functor \((\beta)^*\) is surjective. The \( R_\beta \)-action on \( Q_\beta \) factorizes through \( (3.20) \) to an \( S_\beta \)-action. Hence, we have \( \mathbb{E}nd_{R_\beta}(Q_\beta) = S_\beta \). The graded \( R_\beta \)-module \( Q_\beta \) is a projective graded-generator of the category \( \mathcal{D}_\beta \), because \( Q(\pi) \) is the projective cover of \( L(\pi) \) in \( \mathcal{D}_\beta \) for each \( \pi \in \Gamma_\beta \). We have \( \mathbb{E}nd_{\mathcal{D}_\beta}(Q_\beta) = S_\beta \), hence the category \( \mathcal{D}_\beta \) is equivalent to \( (S_\beta)_{\text{proj}} \)-mod. Since \( S_\beta = (S_\beta)_{\text{proj}} \), we get

\[
\mathcal{D}_\beta = S_{\beta|-\text{mod}}.
\]

By Remark A.3.6(a), (3.4) and (3.21), the functor \( \Psi_\beta^* \) yields an equivalence of graded additive categories

\[
C(\mathcal{Y}_\beta) = S_{\beta|-\text{proj}} = \mathcal{D}_\beta^{\text{proj}}
\]

which maps \((\beta)^*IC(\pi)\) to \( Q(\pi) \).

Now, we prove the lemma for any order ideal \( \Gamma \subset \Gamma_\beta \). By functoriality, we have the k-algebra homomorphisms

\[
(j_\Gamma)^* = (i_\Gamma)^*(\beta)^* : R_\beta \to S_\beta \to S_\Gamma.
\]

Under the equivalence of triangulated categories \( D^b(\mathcal{Y}_\beta) \to D^b(Gr_\beta^+) \) given by Proposition 1.2.1, the complex \( \mathcal{L}_\beta \) on \( \mathcal{Y}_\beta \) is identified with an \( H_\beta \)-equivariant complex on \( Gr_\beta^+ \) which is parity, because the stratification \( S_{\beta}^+ \) is even by Proposition 4.1.1(b). Hence, by [20, cor. 2.9], we have \((i_\Gamma)^*S_\beta = S_\Gamma\). Since \((\beta)^*R_\beta = S_\beta\), we deduce that \((j_\Gamma)^*\) is surjective.

For each \( \pi \in \Gamma \), the projective module \((h_\Gamma)^*Q(\pi)\) in \( \mathcal{D}_\Gamma \) is the cover of \( L(\pi) \). Since \( \mathcal{D}_\Gamma \) is a subcategory of \( \mathcal{D}_\beta \), we can view \((h_\Gamma)^*Q(\pi)\) as a graded \( S_\beta \)-module. Let \( e(\pi) \) denote the image by the algebra homomorphism \((j_\Gamma)^*\) of the idempotent \( e(\pi) \) in \( (2.5) \). We claim that the \( S_\beta \)-module \((h_\Gamma)^*Q(\pi)\) is isomorphic to the pullback by the algebra homomorphism \((i_\Gamma)^*\) of the projective \( S_\Gamma \)-module \( S_\Gamma e(\pi) \). Since \( \bigoplus_{\pi \in \Gamma}(h_\Gamma)^*Q(\pi) \) is a projective graded-generator of \( \mathcal{D}_\Gamma \), this yields an equivalence of graded Abelian categories \( \mathcal{D}_\Gamma = S_{\Gamma|-\text{mod}} \). We deduce that there is an equivalence of graded additive categories \( D_{\Gamma}^{\text{proj}} = S_{\Gamma|-\text{proj}} \) and \( S_{\Gamma|-\text{proj}} = C(\mathcal{Y}_\Gamma) \), proving the lemma.

Now, we prove the claim. As graded \( R_\beta \)-modules, we have

\[
Q(\pi) = \text{Hom}^\bullet_{D^b(\mathcal{Y}_\Gamma)}((\beta)^*IC(\pi), \mathcal{M}_\beta) = S_\beta e(\pi).
\]

Consider the \( S_\beta \)-module \( Q(\pi) \gamma = \Psi_\beta^*((j_\Gamma)^*IC(\pi)) \). We have

\[
Q(\pi) \gamma = \text{Hom}^\bullet_{D^b(\mathcal{Y}_\Gamma)}((j_\Gamma)^*IC(\pi), \mathcal{M}_\Gamma) = S_{\Gamma} e(\pi).
\]
Thus the functor $(i_\Gamma)^*\#$ yields a $S_\beta$-module homomorphism $Q(\pi) \to Q(\pi)_\Gamma$. It is surjective because the algebra homomorphism $(i_\Gamma)^*\#: S_\beta \to S_\Gamma$ is surjective. We also have a surjective $S_\beta$-module homomorphism $Q(\pi) \to (h_\Gamma)^*Q(\pi)$. We claim that $(h_\Gamma)^*Q(\pi) = Q(\pi)_\Gamma$ as graded $S_\beta$-module. This is proved as in Step 5 above. We first observe that both modules belong to the subcategory $D_\Gamma$ of $D_\beta$ and have $\Delta$-filtrations, using [28]. Further, the multiplicities of the standard modules of $D_\Gamma$ in $(h_\Gamma)^*Q(\pi)$, $Q(\pi)_\Gamma$ are the same. □

4. The affine Grassmannians

Recall that $F$ is the algebraic closure of the finite field $F_0$. Write $O = F[[t]]$, $K = F((t))$ and $O^- = F[t^{-1}]$. Let $\beta \in Q_{++}$ be as in [33.1]. We abbreviate $G_\Gamma = GL_{r,F}$. Let $\text{Alg}_F$ be the category of commutative $F$-algebras. Let $G_r(O)$, $G_r(O^-)$, $G_r(K)$ be the presheaves of groups defined, for any $R \in \text{Alg}_F$, by

$$G_r(O)(R) = G_r(R[[t]]) , \quad G_r(O^-)(R) = G_r(R[t^{-1}]) , \quad G_r(K)(R) = G_r(R((t))).$$

Let $(T_r, B_r)$ be the standard Borel pair in $G_r$. If there is no confusion, we may abbreviate $T = T_r$ and $B = B_r$. We denote the sets of characters of $T$, of dominant characters and of dominant characters with non negative entries by $\Lambda_+ \subset \Lambda_r \subset \mathbb{Z}^r$. We equip $\Lambda_r$ with the partial order such that $\lambda \geq \mu$ whenever $\lambda - \mu$ is a sum of positive roots. An interval of $\Lambda_r$ is a subset of the form

$$[\lambda, \mu] = \{ \lambda \cap \{ \leq \mu \} , \lambda, \mu \in \Lambda_r,$$

where $\{ \leq \lambda \} = \{ \nu \in \Lambda_r : \mu \leq \lambda \}$ and $\{ \geq \mu \} = \{ \nu \in \Lambda_r : \nu \geq \mu \}$. Consider the order ideal $\Lambda_\beta = \{ \leq n\omega_1 \}$ where $\omega_1 = (1, \ldots, 1, 0, \ldots, 0)$ has $i$ entries equal to 1 for each $i \in [1, r]$. We have

$$\Lambda_\beta = \{ \lambda \in \Lambda_+ ; n_\lambda = n \}$$

where $n_\lambda = \lambda_1 + \cdots + \lambda_r$ for each $\lambda = (\lambda_1, \ldots, \lambda_r)$. We’ll identify the sets $\Gamma_\beta$ and $\Lambda_\beta$ via the bijection

$$\Gamma_\beta \to \Lambda_\beta , \quad \pi = ((\beta_1)^{p_1}, \ldots, (\beta_1)^{p_1}, (\beta_0)^{p_0}) \mapsto \lambda_\pi = (p_1, \ldots, 1^{p_1}, 0^{p_0}).$$

Let $W$ be the Weyl group of $G_r$, let $\widehat{W}$ be its affine Weyl group, and $M$ the set of maximal length representatives in $\widehat{W}$ of the left cosets in $\widehat{W} / W$. For each weight $\lambda \in \Lambda_r$, let $P_\lambda \subset G_r$ be the standard parabolic subgroup whose Levi factor is the centralizer $G_\lambda$ of the character $\lambda$ in $G_r$.

4.1. The affine Grassmannians. We’ll call affine Grassmannian $\text{Gr}$ the reduced algebraic $F$-scheme underlying the $F$-scheme which is sheafification of the presheaf $\text{Alg}_F \to \text{Sets}$ taking $R$ to $G_r(K)(R) / G_r(O)(R)$. It is represented by a formally smooth reduced ind-projective $F$-scheme with a left action of the $F$-group $G_r(K)$. The Cartan decomposition

$$G_r(K) = \bigsqcup_{\lambda \in \Lambda_r} G_r(O) \cdot^\lambda G_r(O)$$

yields a partition into $G_r(O)$-orbits $\text{Gr} = \bigsqcup_{\lambda \in \Lambda_r} \text{Gr}_\lambda$. We have $\text{Gr}_\lambda \subset \text{Gr}_\mu$ if and only if $\lambda \leq \mu$. Define an irreducible projective $G_r(O)$-equivariant $F$-scheme of finite type by setting $\text{Gr}_\beta = \text{Gr}_{\beta\omega_1}$. The set of $F$-points $\text{Gr}_\beta(F)$ is the set of all $O$-sublattices of codimension $n$ in the standard lattice $L_0 = O^{gr}$.
We’ll call \( \text{thick affine Grassmannian} \) \( \text{Gr} \) the reduced algebraic \( F \)-space underlying the \( F \)-space which is the sheafification of the presheaf \( \mathcal{A}(G) \rightarrow \mathcal{S}ets \) taking \( R \) to \( \mathcal{G}_r(K)(R) / \mathcal{G}_r(O^\circ)(R) \). It is represented by a reduced separated \( F \)-scheme of infinite type with a left action of the \( F \)-group \( G_r(K) \), see \cite{25}. The decomposition

\[
G_r(K) = \bigsqcup_{\lambda \in \Lambda_r} G_r(O) \cdot t^\lambda G_r(O^-)
\]

yields a partition of \( \text{Gr} \) into \( G_r(O) \)-orbits \( \text{Gr} = \bigsqcup_{\lambda \in \Lambda_r} \text{Gr}_\lambda \) such that \( \text{Gr}_\lambda \subset \overline{\text{Gr}_\mu} \) if and only if \( \lambda \geq \mu \). The open subset \( \text{Gr}_\beta = \bigsqcup_{\lambda \in \Lambda_\beta} \text{Gr}_\lambda \) is an irreducible \( G_r(O) \)-equivariant quasi-compact \( F \)-scheme of infinite type.

Fix a principal congruence subgroup \( K_\beta \) of \( G_r(O) \) which is contained into \( \bigcap_{\lambda \in \Lambda_\beta} t^\lambda G_r(O) t^{-\lambda} \). The group \( K_\beta \) acts trivially on \( \text{Gr}_\beta \) and freely on \( \text{Gr}_\beta \). We define

\[
\text{Gr}_\beta^- = \text{Gr}_\beta \quad \text{and} \quad \text{Gr}_\beta^+ = \text{Gr}_\beta / K_\beta.
\]

The \( F \)-scheme \( \text{Gr}_\beta^+ \) is smooth of finite type, because the \( K_\beta \)-action on \( \text{Gr}_\beta \) is locally free. It is separated, see, e.g., \cite{31, A6, 25} lem. 6.3. Since \( K_\beta \) is a normal subgroup of \( G_r(O) \), we may consider the action of the affine algebraic \( F \)-group \( H_\beta = G_r(O) / K_\beta \) on \( \text{Gr}_\beta^- \). The \( H_\beta \)-orbits give a partition

\[
\text{Gr}_\beta^\pm = \bigsqcup_{\lambda \in \Lambda_\beta} \text{Gr}_\lambda^\pm.
\]

Let \( S_\beta^\pm = \{ \text{Gr}^\pm_\lambda : \lambda \in \Lambda_\beta \} \) be the corresponding stratification of \( \text{Gr}_\beta^\pm \).

Let \( I \subset G_r(O) \) be an Iwahori subgroup containing \( K_\beta \). Consider the subgroup \( I_\beta = I / K_\beta \) of \( H_\beta \). Let \( T_\beta^\pm = \{ \text{Gr}^\pm_w : w \in M_\beta \} \) be the stratification of \( \text{Gr}_\beta^\pm \) by the \( I_\beta \)-orbits. Here \( M_\beta \) is a subset of \( M \).

For each subset \( \Gamma \subset \Lambda_\beta \), we consider the quotient stack \( \mathcal{G}r_\Gamma^\pm = [\text{Gr}_\Gamma^\pm / H_\beta] \), where

\[
\text{Gr}_\Gamma^\pm = \bigsqcup_{\lambda \in \Gamma} \text{Gr}_\lambda^\pm.
\]

Let \( i_\Gamma^\pm \) be the locally closed inclusion \( \text{Gr}_\Gamma^\pm \subset \text{Gr}_\beta^\pm \). We abbreviate \( \mathcal{G}r_\Gamma^\pm = \mathcal{G}r_\Lambda^\pm \) and \( \mathcal{G}r_\lambda^\pm = \mathcal{G}r_{(\lambda)}^\pm \). The complex \( IC(\lambda)^\pm \) defined by

\[
IC(\lambda)^\pm = (i_\lambda)_*k_\lambda[d_\lambda^\pm] \quad d_\lambda^\pm = \dim \text{Gr}_\lambda^\pm
\]

can be viewed either as an object of \( D^b(\mathcal{G}r_\beta^\pm) \) or as an object of \( D^b_\mu(\mathcal{G}r_\beta^+) \). In the latter case we write \( IC(\lambda)^\pm_\mu \). Applying the forgetful functor \( \text{For} \), we may also view them as objects of the categories \( D^b(\mathcal{G}r_\beta^+, S) \) or \( D^b_\mu(\mathcal{G}r_\beta^+, S) \). Recall from Definition \cite{31} the notion of good and even stratifications.

**Proposition 4.1.1.**

(a) \( T_\beta^\pm \) is a good stratification.

(b) \( S_\beta^\pm \) is an even stratification.

(c) \( H^\bullet(\text{Gr}_\lambda^\pm, k) = H^\bullet(\text{Gr}_r/P_\lambda, k) \) for each \( \lambda \in \Lambda_r \).
Proof. The strata of $S^\pm_\beta$ are connected, because they are $H_\beta$-orbits. The simply connectedness and the statement in (c) follow from the fact that $\text{Gr}^\pm_\Lambda$ is an affine bundle over $G/P_\Lambda$, see, e.g., [35, §5]. So (a) implies (b). The strata of $T^+_\beta$ are affine. The conditions [50, §4.1(a)-(d)] are proved in [50, §5.2]. So to prove (a), it is enough to prove that the strata of $T^+_\beta$ satisfy the conditions (2), (3) in Definition A.3.1.

For $T^+_\beta$ this is well-known, the proof consists of checking the conditions in [8, lem. 4.4.1].

For $T^-_\beta$, the condition (2) follows from [27] and the odd vanishing of the Kazhdan-Lusztig polynomials. Let us concentrate on the condition (3) for $T^-_\beta$. A well-known argument of Kazhdan-Lusztig implies that $H^\ast((i_+^\ast)^\ast IC(w)^m_\ast)$ is pure of weight $a$. See [19, lem. 3.5] or [14, lem. 3.1.3] for an easier proof which generalizes to our setting. The condition (3) requires in addition that the mixed vector space $H^\ast((i_+^\ast)^\ast IC(w)^m_\ast)$ is semisimple. By [8, lem. 4.4.1], it is enough to check that the mixed vector space $\mathcal{H}^a(\text{Gr}^+_\beta,0, IC(w)^m_\ast)$ is a sum of copies of $k(-a/2)$ for each $w \in M_\beta$ and $a \in \mathbb{N}$.

To prove this, we consider Kashiwara’s thick affine flag manifold $F_l$. Let $G$, $B^+$ and $B^-$ be as in the proof of Lemma A.3.1 below. The scheme $F_l$ is separated of infinite type represented by the quotient $G/B^-$ with a decomposition into $B^+$-orbits $F_l = \bigsqcup F_{l_v}$ labelled by the affine Weyl group $\widehat{W}$ of $G$. The $B^+$-orbit $F_{l_v}$ is an affine space of codimension equal to the length of $v$. Let $W_\beta \subset \widehat{W}$ be an order ideal. Let $K_\beta$ be a congruence subgroup which acts freely on $F_{l_\beta} = \bigsqcup_{v \in W_\beta} F_{l_v}$. We have a smooth (separated) scheme $F_{l_\beta} = F_{l_\beta} / K_\beta$ with an affine stratification $F_{l_\beta} = \bigsqcup_{v \in W_\beta} F_{l_v}$ such that $F_{l_v} = F_{l_v} / K_\beta$. Here $W_\beta$ is the inverse image of $M_\beta$ by the obvious projection $\widehat{W} \to M$. The bundle $F_l \to \text{Gr}$ yields a bundle $q : F_{l_\beta,0} \to \text{Gr}^+_\beta,0$ over $F_0$ with smooth, projective fibers with affine stratifications. See Section A.2 for the convention. For each element $v \in W_\beta$ let $IC(v)_m \in D^b_m(F_{l_\beta,0})$ be the intermediate extension of the mixed complex $k_{F_{l_v}}(d_v)$, where $d_v = \text{dim } F_{l_v}$. The map $q$ restricts to an isomorphism $F_{l_v,0} \to \text{Gr}^+_v,0$. Thus, we have $q_\ast IC(v) = IC(v)_\ast \oplus F$ for some complex $F \in D^b(\text{Gr}^+_\beta)$. The following is well-known.

Claim 4.1.2. Let $E \in D^b_m(Z_0)$ be a pure complex and $F \in D^b_m(Z_0)$ be a subquotient of some perverse cohomology sheaf $\mathcal{P}H^\ast(E)$. For each $a \in \mathbb{Z}$, the mixed vector space $\mathcal{H}^a(Z_0, F)$ is a subquotient of $\mathcal{H}^a(Z_0, E)$. $\square$

Thus the mixed vector space $\mathcal{H}^a(\text{Gr}^+_\beta,0, IC(w)^m_\ast)$ is a subquotient of $\mathcal{H}^a(F_{l_\beta,0}, IC(v)_m)$ for each $a \in \mathbb{N}$. Hence, it is enough to check that $\mathcal{H}^a(F_{l_\beta,0}, IC(v)_m)$ is a sum of copies of $k(-a/2)$ for each $v \in W_\beta$. If $v$ is maximal in the poset $W_\beta$, then the stratum $F_{l_v}$ is closed in $F_{l_\beta}$, hence we have $IC(v)_m = k_{F_{l_v}}(d_v)$, thus the claim is obvious. For an arbitrary element $v \in W_\beta$, we argue by decreasing induction on the length of $v$.

For each simple reflection $s_i \in \widehat{W}$, we consider the parabolic subgroup $P^-_i = B^- s_i B^- \cup B^-$. Let $F_l^i$ be the partial thick affine flag manifold $F_l^i = G/P^-_i$. We define as above a smooth $F_0$-scheme $F_{l_\beta}^i$ with a $P^i_{F_0}$-bundle $p : F_{l_\beta,0} \to F_{l_\beta,0}$ such that

$$F_{l_v,0} \cup F_{l_{vs_i,0}} = p^{-1}p(F_{l_v,0}) \quad , \quad p(F_{l_{vs_i,0}}) = p(F_{l_v,0}) \quad , \quad \forall v \in W_\beta.$$
Proposition 4.2.1.

Now, assume that $v s_i < v$. The map $p$ restricts to an isomorphism $Fl_v \rightarrow p(Fl_v)$. Consider the endofunctor $S_i$ of $D^b_m(Fl_{\beta,0})$ given by

$$S_i(\mathcal{E}) = p^* p_* \mathcal{E} \langle 1 \rangle, \quad \forall \mathcal{E} \in D^b_m(Fl_{\beta,0}).$$

There is a complex $\mathcal{F}$ supported on the closure of the stratum $Fl_v$ in $Fl_{\beta}$ such that

$$\omega S_i(IC(v)_m) = IC(v_{s_i}) \oplus \mathcal{F}.$$ 

Hence Claim 4.1.2 implies that $H^a(Fl_{\beta,0}, IC(v_{s_i})_m)$ is a subquotient of the mixed vector space $H^a(Fl_{\beta,0}, S_i(IC(v)_m))$. Now, consider the following Cartesian diagram of $F_0$-schemes

$$\begin{array}{ccc}
Z_{\beta,0}^i & \xrightarrow{m} & Fl_{\beta,0} \\
\pi_1 \downarrow & & \downarrow p \\
Fl_{\beta,0} & \xrightarrow{p} & Fl_{\beta,0}.
\end{array}$$

All maps are $\mathbb{P}^{k}_{F_0}$-bundles. By proper base change, we have

$$S_i(IC(v)_m) = m_*(\pi_1)^*(IC(v)_m) \langle 1 \rangle.$$ 

We deduce that

$$H^a(Fl_{\beta,0}, S_i(IC(v)_m)) = H^a(Z_{\beta,0}^i, (\pi_1)^*(IC(v)_m) \langle 1 \rangle).$$

The $F_0$-scheme $Z_{\beta,0}^i$ has an affine stratification given by the product of Bruhat cells on $G/B^-$ and $\mathbb{P}^{k}_{F}/B^-$. The morphism $\pi_1$ is stratified. The mixed complex $(\pi_1)^*(IC(v)_m) \langle 1 \rangle$ in $D^b_m(Z_{\beta,0}^i)$ is the intermediate extension of the constant sheaf on the open dense stratum in $(\pi_1)^{-1}(Fl_{v,0})$. It satisfies the conditions (2) and (3) in Definition A.3.1 because the mixed complex $IC(v)_m$ in $D^b_m(Fl_{\beta,0})$ satisfy them by the induction hypothesis and both conditions are preserved by a pullback by a smooth stratified morphism.

4.2. The thick affine Grassmannian and the quiver $Q$. Let $Coh_{\beta}$ be the $F$-stack classifying coherent sheaves on $\mathbb{P}^r_F$ of rank $r$ and degree $n$. Let $Bun_{\beta}$ be the open substack parametrizing locally free coherent sheaves. We abbreviate $Coh = \bigsqcup_{\beta} Coh_{\beta}$ and $Bun = \bigsqcup_{\beta} Bun_{\beta}$. Both stacks are smooth locally quotient stacks. Consider the vector bundle $\mathcal{O}(\lambda)$ on $\mathbb{P}^r_F$ given by $\mathcal{O}(\lambda) = \mathcal{O}(\lambda_1) \oplus \cdots \oplus \mathcal{O}(\lambda_r)$ with $\lambda \in \Lambda_{\beta}$. Let $Bun^+_{\beta}$ be the full substack of $Bun_{\beta}$ classifying all vector bundles isomorphic to $\mathcal{O}(\lambda)$ for some $\lambda \in \Lambda_{\beta}$.

Proposition 4.2.1.

(a) There are $F$-stack isomorphisms $Gr^+_{\beta} \simeq Bun^+_{\beta} \simeq \mathcal{Y}_{\beta}$ taking $Gr^+_{\lambda_{\pi}}$ to the isomorphism classes of $\mathcal{O}(\lambda_{\pi})$ and $\mathcal{P}_{\pi}$ for each $\pi \in \Gamma_{\beta}$.

(b) $\lambda_\sigma \geq \lambda_\pi \iff Y_\sigma \subseteq Y_\pi \iff \sigma \geq \pi$ for each $\pi, \sigma \in \Gamma_{\beta}$.

Proof. Let $D^b(Coh)$ denote the derived category of the Abelian category $Coh$. The vector bundle $\mathcal{T} = \mathcal{O} \oplus \mathcal{O}(1)$ over $\mathbb{P}^r_F$ is a tilting generator of $D^b(Coh)$, i.e., it is a generator of $D^b(Coh)$ as a triangulated category such that $\text{End}^{>0}_{D^b(Coh)}(\mathcal{T}) = 0$. We have an $F$-algebra isomorphism $\text{End}_{D^b(Coh)}(\mathcal{T})^{op} = FQ$ such that the elements $x, y \in FQ$ span the $F$-subspace
The thick affine Grassmannian and the category $D_\beta$. Let $\Gamma \subset \Gamma_\beta$ be any order ideal. We define the triangulated functor

$$A_\Gamma = A'_\Gamma A''_\Gamma : D^b(D_\Gamma) \to D^b(Gr^+_\Gamma).$$

Conjugating the functor $(i^+_\Gamma)^* \circ$ with the Verdier duality yields the functor $(i^+_\Gamma)^*$ which is left adjoint to $(i^{-}_\Gamma)^*$. We define the following complexes in $D^b(Gr^+_\Gamma)$

$$\Delta(\lambda)_\mu = (i^{-}_\lambda)^*(i^+_\lambda)^* IC(\lambda)_\mu^+, \quad \nabla(\lambda)_\mu = (i^+_\lambda)^*(i^{-}_\lambda)^* IC(\lambda)_\mu^+. $$
(b) The following diagram of functors commutes

\[
\begin{array}{ccc}
\text{D}^b(\mathcal{D}_\beta) & \xrightarrow{L(h\Gamma)^*} & \text{D}^b(\mathcal{D}_\Gamma) \\
\Lambda_\beta & \downarrow & \downarrow \Lambda_\Gamma \\
\text{D}^b(\mathcal{G}_{\beta}^+(-)) & \xrightarrow{(i^+_\beta)^*} & \text{D}^b(\mathcal{G}_{\Gamma}^+(-)).
\end{array}
\]

**Proof.** The proposition follows from Propositions 3.2.11.4.2.1.1 except the last claim in (a).

We may assume \( \Gamma = \Gamma_\beta \). By part (b), for each \( \pi \in \Gamma_\beta \) we have

\[\nabla(\lambda_\pi)^+_\mu = (i^+_{\leq \lambda_\mu})_*(i^+_{\leq \lambda_\mu})^*IC(\lambda_\pi)^+_\mu = \Lambda_\beta(h_{\leq \pi})_*L(h_{\leq \pi})^*Q(\pi) = \Lambda_\beta(\Delta(\pi)).\]

\( \square \)

4.4. **The non equivariant case.** Fix an order ideal \( \Gamma \subset \Gamma_\beta \). For the Radon transform studied in the next section, we need to consider the mixed category \( \text{D}^b(\mathcal{G}_{\beta}^+(-)) \) rather than the equivariant mixed category \( \text{D}^b(\mathcal{G}_{\Gamma}^+(-)) \). Let us give some properties of \( \text{D}^b(\mathcal{G}_{\Gamma}^+(-)) \).

We write \( V(\lambda_\pi) = V(\pi) \) for each \( \pi \in \Gamma \). Set

\[\mathcal{M}_\Gamma^\pm = \bigoplus_{\lambda \in \Gamma} V(\lambda) \otimes IC(\lambda)^\pm.\]

We view \( \mathcal{M}_\Gamma^\pm \) either as an object of \( \text{D}^b(\mathcal{G}_{\beta}^+(-)) \), or as an object of \( \text{D}^b(\mathcal{G}_{\Gamma}^+(-)) \) via the forgetful functor \( \text{For} : \text{D}^b(\mathcal{G}_{\beta}^+(-)) \to \text{D}^b(\mathcal{G}_{\Gamma}^+(-)) \). We define

\[S_\Gamma^\pm = \text{End}^*_{\text{D}^b(\mathcal{G}_{\beta}^+(-))}(\mathcal{M}_\Gamma^\pm), \quad S_\Gamma^{\pm,\sharp} = \text{End}^*_{\text{D}^b(\mathcal{G}_{\Gamma}^+(-))}(\mathcal{M}_\Gamma^\pm).\]

**Proposition 4.4.1.** Let \( \Gamma \subset \Lambda_\beta \) be any order ideal.

(a) \( \text{C}(\mathcal{G}_{\Gamma}^+(-)) \cong S_\Gamma^{-,\sharp} \)-proj and \( \text{C}(\mathcal{G}_{\Gamma}^+(-)) \cong S_\Gamma^{\pm,\sharp} \)-proj as graded additive categories.

(b) \( \text{D}^b(\mathcal{G}_{\Gamma}^+(-)) = \text{D}^\text{perf}(S_\Gamma^\pm) \) and \( \text{D}^b(\mathcal{G}_{\Gamma}^+(-)) = \text{D}^\text{perf}(S_\Gamma^{\pm,\sharp}) \) as graded triangulated categories.

(c) \( S_\Gamma^\pm \) is free of finite rank as an \( H_{\Gamma}^\ast \)-module, and \( k \otimes_{H_{\Gamma}^\ast} S_\Gamma^\pm \cong S_\Gamma^{\pm,\sharp} \) as graded \( k \)-algebras.

**Proof.** Parts (a), (b) follow from Remark \[A.3.6\](a). To prove (c), note that \[20\] prop. 2.6 yields

\[\text{Hom}^*_{\text{D}^b(\mathcal{G}_{\Gamma}^+(-))}(\mathcal{E}, \mathcal{F}) = k \otimes_{H_{\Gamma}^\ast} \text{Hom}^*_{\text{D}^b(\mathcal{G}_{\Gamma}^+(-))}(\mathcal{E}, \mathcal{F}), \quad \forall \mathcal{E}, \mathcal{F} \in \text{C}(\mathcal{G}_{\Gamma}^+(-)),\]

because the stratification \( S \) is even. \( \square \)

By Proposition 4.4.1(c), there is an obvious surjective algebra homomorphism \( \xi^\pm : S_\Gamma^\pm \to S_\Gamma^{\pm,\sharp} \). The restriction of scalars relatively to the morphism \( \xi^\pm \) is exact and yields a functor of triangulated categories \( \xi^+_\pm : \text{D}^b(\mathcal{G}_{\Gamma}^+(-), S) \to \text{D}^b(\mathcal{G}_{\Gamma}^+(-)) \). The left adjoint functor \( L(\xi^\pm)^*: \text{D}^b(\mathcal{G}_{\Gamma}^+(-)) \to \text{D}^b(\mathcal{G}_{\Gamma}^+(-), S) \) is the functor \( \text{For} \). It is given by derived induction relatively to the morphism \( \xi^\pm \).

For any \( \alpha \in Q_+ \), an element of the representation variety \( X_\alpha \) is a pair of matrices \((x, y)\) corresponding to the actions of the generators \( x, y \) of the path algebra \( \text{FQ} \). In particular,
setting \( \alpha = \delta \) and \( x = 1, \ y = 0 \), we have the element \((1,0) \in X_\delta \). Let \( \mathcal{E}_\Gamma \) be the quotient stack over \( Y_\Gamma \) given by
\[
\mathcal{E}_\Gamma = \left\{ (x,y,\varphi) : (x,y) \in Y_\Gamma, \ \varphi \in \text{Hom}_{FQ}(y, (1,0)) \right\} / G_\beta.
\]
The morphism \( \mathcal{E}_\Gamma \to \mathcal{Y}_\Gamma \) such that \( (x,y,\varphi) \mapsto (x,y) \) is a vector bundle of rank \( r \), because
\[
\text{Hom}_{FQ}(y, (1,0)) = \text{coker}(y)^* \quad \text{and coker}(y) \cong F^r \text{ since } y \text{ is injective.}
\]
Hence \( \mathcal{E}_\Gamma \) yields a stack homomorphism \( \mathcal{Y}_\Gamma \to [\bullet/G_\Gamma] \).
The pullback by this map is a graded \( k \)-algebra homomorphism
\[
H^*_G, r \to H^*(\mathcal{Y}_\Gamma, k).
\]
By (3.3) there is a graded algebra homomorphism
\[
H^*(\mathcal{Y}_\Gamma, k) \to S_\Gamma
\]
such that the elements in the image graded-commute with the elements of \( S_\Gamma \). Composing
\[
(4.8)
\]
with (4.7) yields a graded algebra homomorphism
\[
H^*_G, r \to Z(S_\Gamma).
\]
We define \( S^+_\Gamma = k \otimes_{H^*_G, r} S_\Gamma \) and \( D^+_\Gamma = S^+_\Gamma \text{-mod} \). Let \( \xi : S_\Gamma \to S^+_\Gamma \) be the specialization homomorphism.

By Lemma 4.4.1 we have \( D_\Gamma = S_\Gamma \text{-mod} \). The restriction of scalars relatively to the map \( \xi \) yields a full embedding of graded Abelian categories
\[
(4.10) \quad \xi_* : D^+_\Gamma \to D_\Gamma.
\]
It is exact and gives a functor of triangulated categories \( \xi^* : D^{\text{perf}}(D^+_\Gamma) \to D^b(D_\Gamma) \), which is not fully faithful in general. Let \( L_\xi^*, R_\xi^* \) be the left adjoint functors, which are given by induction and derived induction relatively to \( \xi \).

**Lemma 4.4.2.** Let \( \Gamma \subset \Lambda_\beta \) be any order ideal.

(a) \( S^+_\Gamma = S^{\text{unip}}_\Gamma \) as graded \( k \)-algebras.

(b) \( C(\text{Gr}_{\Gamma, r}^+, S) \cong S^+_\Gamma \text{-proj} \) as graded additive categories.

**Proof.** The module \( Q_\Gamma = (h_\Gamma)^* Q_\beta \), is a projective graded-generator of \( D_\Gamma \). By Proposition 4.3.1 there is an equivalence of graded triangulated categories \( A_\beta : D^b(D_\beta) \to D^b(\text{Gr}_{\beta}^+) \) such that \( A_\beta Q_\Gamma = M^+_\Gamma \). Hence, we have
\[
(4.11) \quad S_\Gamma = \text{End}_{D_\Gamma}(Q_\Gamma) = \text{End}_{D^b(D_\Gamma)}(Q_\Gamma) = \text{End}_{D^b(\text{Gr}_{\Gamma}^+)}(M^+_\Gamma) = \text{End}^*_D(\text{Gr}_{\Gamma}^+)(M^+_\Gamma) = S^+_\Gamma.
\]
The vector bundle \( E_\Gamma \) over \( Y_\Gamma \) is isomorphic to the pull-back of the universal bundle over \( \text{Bun}_{\beta}^+ \times \mathbb{P}_F \) by the embedding
\[
Y_\Gamma \subset Y_\beta = \text{Bun}_{\beta}^+ \times \{0\} \subset \text{Bun}_{\beta}^+ \times \mathbb{P}_F.
\]
Therefore, under the stack isomorphism \( \mathcal{Y}_\Gamma \cong \mathcal{G}_{\Gamma}^+ \) in Proposition 4.2.1 the \( G_r \)-torsor associated with \( E_\Gamma \) is identified with the \( G_r \)-torsor \( \{\text{Gr}_{\Gamma}^+/U_\beta\} \) over \( \text{Gr}_{\Gamma}^+ \), where \( U_\beta \) is the unipotent
radical of $H_{\beta}$ with Levi complement $G_{r}$. We deduce that the obvious graded $k$-algebra homomorphism
\begin{equation}
H_{G_{r}}^{\ast} \to H^{\ast}(G_{r}^{+}, k) \to Z(S_{\Gamma}^{+}),
\end{equation}
is identified with the central homomorphism \((4.10)\) under the algebra isomorphism $S_{\Gamma} = S_{\Gamma}^{+}$. This implies that $S_{\Gamma}^{+} = S_{\Gamma}^{+}$. Part (b) follows from (a) and Proposition 4.4.1(a). \hfill \Box

For each Kostant partition $\pi \in \Lambda_{\beta}$ we define the module $\Delta(\pi)^{\sharp} \in D_{\beta}^{\sharp}$ by setting
\begin{equation}
\Delta(\pi)^{\sharp} = \xi^{\ast} \Delta(\pi).
\end{equation}

**Proposition 4.4.3.** Let $\Gamma \subset \Lambda_{\beta}$ be any order ideal.

(a) We have an equivalence of graded triangulated categories $A_{\Gamma}^{\sharp} : D^{\text{perf}}(D_{\Gamma}^{\sharp}) \to D^{b}_{\mu}(\text{Gr}_{\Gamma}^{+}, S)$ such that the following diagram of functors commutes
\[
\begin{array}{ccc}
D^{\text{perf}}(D_{\Gamma}^{\sharp}) & \xrightarrow{\xi_{\ast}} & D^{b}(D_{\Gamma}) \\
\downarrow{L_{\ast}} & & \downarrow{L_{\ast}} \\
A_{\Gamma}^{\sharp} & \xrightarrow{\xi_{\ast}} & D^{b}_{\mu}(\text{Gr}_{\Gamma}^{+}).
\end{array}
\]

(b) For each $\pi \in \Gamma$, we have $A_{\Gamma}^{\sharp}L(h_{\Gamma})^{\ast}Q(\pi)^{\sharp} = (i_{\Gamma}^{\ast})^{\ast}IC(\lambda_{\mu})^{+}_{\mu}$ and $A_{\Gamma}^{\sharp}\Delta(\pi)^{\sharp} = \nabla(\lambda_{\mu})^{+}_{\mu}$.

**Proof.** By Lemma 4.4.2, we have an equivalence of graded triangulated categories
\[
A_{\Gamma}^{\sharp} : D^{\text{perf}}(D_{\Gamma}^{\sharp}) = K^{b}(S_{\Gamma}^{\sharp} \text{-proj}) \to K^{b}(C(\text{Gr}_{\Gamma}^{+}, S)) = D^{b}_{\mu}(\text{Gr}_{\Gamma}^{+}, S)
\]
such that $A_{\Gamma}^{\sharp}L(h_{\Gamma})^{\ast}Q(\pi)^{\sharp} = (i_{\Gamma}^{\ast})^{\ast}IC(\lambda_{\mu})^{+}_{\mu}$ for all $\pi \in \Gamma$. The isomorphism $A_{\Gamma}^{\sharp}\Delta(\pi)^{\sharp} = \nabla(\lambda_{\mu})^{+}_{\mu}$ is proved as in Proposition 4.3.1. \hfill \Box

For a future use, we now describe explicitly the map $\xi : S_{\Gamma} \to S_{\Gamma}^{\sharp}$. To simplify we set $\Gamma = \Gamma_{\beta}$. Let $c_{\ell}(E_{\beta})$ be the Chern polynomial of the rank $r$ vector bundle $E_{\beta} \to Y_{\beta}$. Let $J_{\beta} \subset Z(S_{\beta})$ be the ideal generated by the image by \((4.8)\) of the non-constant coefficients of $c_{\ell}(E_{\beta})$.

Let $\Lambda$ be the ring of symmetric functions in two sets of variables $x_{1}, x_{2}, \ldots$ and $y_{1}, y_{2}, \ldots$ and coefficients in $k$. Let $e_{i}(x), e_{j}(y)$ be the $i$th elementary symmetric functions and $E_{i}(x), E_{j}(y)$ be the corresponding generating series. Since $X_{\beta}$ is an affine space, we have
\begin{equation}
H^{\ast}(X_{\beta}, k) = H_{G_{\beta}}^{\ast} = \Lambda / (e_{i}(x), e_{j}(y) ; i > m, j > n).
\end{equation}

Let $I_{\beta} \subset H_{G_{\beta}}^{\sharp}$ be the ideal generated by the non-constant coefficients of the formal series
\begin{equation}
E_{\beta}(t) = E_{-t}(x) / E_{-t}(y).
\end{equation}
The restriction $(j_{\beta})^{\ast} : H^{\ast}(X_{\beta}, k) \to H^{\ast}(Y_{\beta}, k)$ maps the coefficient of $t^{i}$ in $E_{\beta}(t)$ to 0 for each $i > r$, due to the exact sequence \((4.10)\). Let $I_{\beta} \subset Z(S_{\beta})$ be the ideal generated by the image by \((4.8)\) of $(j_{\beta})^{\ast}I_{\beta}$.

**Proposition 4.4.4.** We have $S_{\beta}^{\sharp} = S_{\beta}/J_{\beta}S_{\beta}$ and $J_{\beta} = I_{\beta}$. 


Proof. The first claim is the definition of $S^\pm_\Gamma$. We now concentrate on the second one. The tautological representations of the group $G_\beta$ in $F^m$ and $F^n$ yield two vector bundles $\mathcal{U}_0$, $\mathcal{U}_1$ over $X_\beta$ of rank $m$, $n$ respectively, with an exact sequence of vector bundles over $Y_\beta$

$$0 \longrightarrow (j_\beta)^*\mathcal{U}_1 \longrightarrow (j_\beta)^*\mathcal{U}_0 \longrightarrow (\mathcal{E}_\beta)^* \longrightarrow 0. \tag{4.16}$$

Since the k-algebra homomorphism $(j_\beta)^*: R_\beta \rightarrow S_\beta$ in (4.19) is onto, it gives an algebra homomorphism $Z(R_\beta) \rightarrow Z(S_\beta)$, which fits in the following commutative diagram

$$\begin{array}{c}
H^*_\mathcal{G}_\beta \ar@{->}[r] \ar@{->}[d]_{(j_\beta)^*} & H^*(Y_\beta, k) \ar@{->}[d]_{(j_\beta)^*} \ar@{->}[r] & Z(S_\beta) \\
H^*_\mathcal{G}_\beta \ar@{->}[r] & H^*(X_\beta, k) \ar@{->}[r]_{(j_\beta)^*} & Z(R_\beta)
\end{array} \tag{4.17}$$

Let $c_i(\mathcal{U}_0)$ and $c_i(\mathcal{U}_1)$ be the Chern polynomials. By (4.16) we have

$$c_i(\mathcal{E}_\beta) = (j_\beta)^*c_{-i}(\mathcal{U}_0)/(j_\beta)^*c_{-i}(\mathcal{U}_1). \tag{4.18}$$

The identification (4.14) takes the formal series $c_{-i}(\mathcal{U}_0)/c_{-i}(\mathcal{U}_1)$ to $E_\beta(t)$, hence $J_\beta = I_\beta^\prime$. □

Remark 4.4.5. The k-algebra $S^\pm_\Gamma$ is finite dimensional by Lemma 4.4.2 because $S_\Gamma^{+,\pm}$ is finite dimensional by (4.3). Hence the category $D^b_\Gamma$ is Artinian.

Example 4.4.6. Assume that $\beta = \beta_n$. Hence $G_\beta = G_n \times G_{n+1}$ and the $G_\beta$-variety $Y_\beta$ contains a single orbit with stabilizer $D_\beta = G_{m,F}$. The constant sheaf is a graded-generator of $C(Y_\beta)$, hence $C(\mathcal{G}r^+_\beta) = C(Y_\beta) = S_\beta$-proj with $S_\beta = H^*(Y_\beta, k) = H^*_{D_\beta} = k[z]$. Further, we have $C(\mathcal{G}r^+_\beta, S) = S^\pm_\beta$-proj with $S^\pm_\beta = k$.

4.5. The Radon transform. For each subset $\Gamma$ of $\Lambda_\beta$ or $M_\beta$ we have

$$\text{Gr}_\Gamma^\pm = \coprod_{\omega \in \Lambda} \text{Gr}_\lambda^\pm, \quad \text{Gr}_\Gamma^\pm = \coprod_{\omega \in \Gamma} \text{Gr}_w^\pm, \quad d^\pm_\lambda = \dim \text{Gr}_\lambda^\pm, \quad d^\pm_w = \dim \text{Gr}_w^\pm.$$

We equip the F-scheme $\text{Gr}_\Gamma^\pm$ with the stratifications given by $S^\pm_\Gamma = \text{Gr}_\Gamma^\pm \cap S_\beta^\pm$ and $T^\pm_\Gamma = \text{Gr}_\Gamma^\pm \cap T_\beta^\pm$. We’ll abbreviate $S = S^\pm_\Gamma$ and $T = T^\pm_\Gamma$. The F-scheme $\text{Gr}_\Gamma^\pm$ has an obvious $F_0$-structure $\text{Gr}_{\Gamma,0}^\pm$. The stratifications $S^\pm$ and $T^\pm$ are defined over $F_0$.

First, let us recall the Radon transform, following [50]. Let $U_0$ be the diagonal orbit of $G_r(F_0((t)))$ acting on the origin of $\text{Gr}_0 \times \text{Gr}_0$. The diagonal action of the $F_0$-group $K_{\beta,0}$ on $U_0 \cap (\text{Gr}_{\Gamma,0} \times \text{Gr}_{\Gamma,0})$ is free. Let $U_{\Gamma,0}$ be the quotient by $K_{\beta,0}$. It is open and dense in $\text{Gr}_{\Gamma,0} \times \text{Gr}_{\Gamma,0}$ and may view as a correspondence of $F_0$-schemes

$$\begin{array}{c}
\text{Gr}_{\Gamma,0} \ar@{->}[r]^{j^-} \ar@{->}[r]^{j^+} & U_{\Gamma,0} \ar@{->}[r]^{j^+} & \text{Gr}_{\Gamma,0}
\end{array} \tag{4.19}$$

We write $U_{\Gamma} = U_{\Gamma,0} \otimes_{F_0} F$. From now on we’ll always assume that $\Gamma$ is an interval in $\Lambda_\beta$ or in $M_\beta$. Then, the scheme $\text{Gr}_{\Gamma,0}^\pm$ is irreducible, and we may set $d^\pm_\Gamma = \dim \text{Gr}_{\Gamma,0}^\pm$. 

The Radon transform denotes both a pair of adjoint functors \((R^-_\Gamma, R^+_\Gamma)\) between the categories \(\text{D}^b(\text{Gr}_\Gamma, T)\) and a pair of adjoint functors \((R^-_{\Gamma,m}, R^+_{\Gamma,m})\) between the categories of mixed complexes \(\text{D}^b_m(\text{Gr}_{\Gamma,0}, T)\). Both pairs of functors are given by

\[
\begin{pmatrix}
(f^+_\Gamma)\circ (f^-_\Gamma)^* & (f^+_\Gamma)^* \\
(f^-_\Gamma)\circ (f^+_\Gamma) & (f^-_\Gamma)^*
\end{pmatrix}.
\]

Since the correspondence \(U_\Gamma\) is \(I_\beta\)-equivariant, the functors \(R^\pm_{\Gamma,m}\) preserve the full subcategories \(\text{D}^b_m(\text{Gr}_{\Gamma,0}, T)\) of \(\text{D}^b_m(\text{Gr}_{\Gamma,0}, T)\). By [50, cor. 4.1.5, §5.4], they yield a commutative square of functors

\[
\begin{array}{ccc}
\text{D}^b_m(\text{Gr}_{\Gamma,0}, T) & \rightarrow & \text{D}^b(\text{Gr}_{\Gamma}, T) \\
\downarrow_{\omega} & & \downarrow_{\omega} \\
\text{D}^b(\text{Gr}_{\Gamma}, T) & \rightarrow & \text{D}^b(\text{Gr}_{\Gamma}, T).
\end{array}
\]

(4.18)

In this diagram the two pairs of horizontal arrows are adjoint equivalences. Since the correspondence \(U_\Gamma\) is \(H_\beta\)-equivariant, the Radon transform preserves the \(S\)-constructible complexes. Hence, it yields a commutative square of equivalences

\[
\begin{array}{ccc}
\text{D}^b_m(\text{Gr}_{\Gamma,0}, S) & \rightarrow & \text{D}^b(\text{Gr}_{\Gamma}, S) \\
\downarrow_{\omega} & & \downarrow_{\omega} \\
\text{D}^b(\text{Gr}_{\Gamma}, S) & \rightarrow & \text{D}^b(\text{Gr}_{\Gamma}, S).
\end{array}
\]

(4.19)

Given a nested pair of subsets \(\Gamma_1 \subset \Gamma\) in \(\Lambda_\beta\), we consider the following diagram

\[
\begin{array}{ccc}
\text{Gr}_{\Gamma_1} & \xrightarrow{f^-_{\Gamma_1}} & \text{Gr}_{\Gamma} \\
\downarrow_{i^-_{\Gamma_1,r}} & & \downarrow_{i^-_{\Gamma,r}} \\
\text{Gr}_{\Gamma_1} & \xrightarrow{f^+_{\Gamma_1}} & \text{Gr}_{\Gamma}.
\end{array}
\]

(4.20)

Lemma 4.5.1.

(a) If \(\Gamma_1 = \{\geq \lambda\} \cap \Gamma\), then the right square in (4.20) is Cartesian.

(b) If \(\Gamma_1 = \{\leq \mu\} \cap \Gamma\), then the left square in (4.20) is Cartesian.

Proof. Let \(x^\pm\) be the origin in \(\text{Gr}_\beta^\pm\). We abbreviate

\[G = G_r(K), \quad P^+ = G_r(O), \quad P^- = G_r(O^\circ), \quad B^+ = I_\beta.\]

Let \(B^- \subset P^-\) be the co-Iwahori subgroup opposite to \(B^+\). It is the preimage under the projection \(P^- \rightarrow G_r(F)\) of the Borel subgroup opposite to the image of \(B^+\) by the projection
The original diagram (4.20) is a particular case of this new diagram. Note that we have

(a) the right square is Cartesian \( \iff \) \( B^+ \Gamma_1 P^- x^- \cap B^+ \Gamma x^- \subset B^+ \Gamma_1 x^- \),
(b) the left square is Cartesian \( \iff \) \( B^+ \Gamma_1 P^+ x^+ \cap B^+ \Gamma x^+ \subset B^+ \Gamma_1 x^+ \).

Therefore, to prove the lemma it is enough to check that

(c) \( B^+ \{ \geq w \} P^- x^- \subset B^+ \{ \geq w \} x^- \),
(d) \( B^+ \{ \leq w \} B^+ x^+ \subset B^+ \{ \leq w \} x^+ \).

To prove (d) it is enough to prove that \( B^+ w B^+ x^+ \subset B^+ \{ \leq w \} x^+ \). We’ll argue by induction on the length of \( w \). If \( w = 1 \) this is obvious. Let \( U_i \subset B^+ \) be the root subspace associated with the simple root \( \alpha_i \). If \( w = s_i v > v \), then \( U_i w \subset w P^- \), hence

\[
(4.21) \quad B^+ s_i B^+ w x^+ = B^+ s_i U_i w x^+ = B^+ v x^+.
\]

Using (4.21) and the equality \( B^+ s_i B^+ B^+ = B^+ s_i B^+ B^+ \), we deduce that

\[
(4.22) \quad B^+ s_i B^+ v x^+ = B^+ s_i B^+ s_i B^+ w x^+ = B^+ w x^+ \cup B^+ s_i B^+ w x^+ = B^+ w x^+ \cup B^+ v x^+.
\]

Using (4.21), (4.22), we deduce that

\[
B^+ w B^+ x^+ = B^+ s_i v B^+ x^+,
\]

\[
\subset \bigcup_{u \leq v} B^+ s_i B^+ u x^+,
\]

\[
\subset \bigcup_{u \leq v} (B^+ u x^+ \cup B^+ s_i u x^+),
\]

\[
\subset B^+ \{ \leq w \} x^+.
\]

To prove (c), note that

(c) \( \iff \) \( B^+ w B^- x^- \cap B^+ v x^- = \emptyset, \forall v < w, \)
(d) \( \iff \) \( B^+ w B^+ x^+ \cap B^+ v x^+ = \emptyset, \forall v > w, \)

Thus, the proof of (d) above implies that \( B^+ w B^- \cap B^+ v B^- = \emptyset \) in \( G \) for all \( v > w \), from which we deduce that \( B^+ w B^- \cap B^+ v B^- = \emptyset \) in \( G \) for all \( v < w \), and this implies (c). \( \Box \)

Let \( \Gamma \) be an interval in \( \Lambda_\beta \) or in \( M_\beta \). For each \( \lambda, \mu \) we abbreviate \( i_{\leq \mu}^\pm = i_{\leq \mu}^\pm |_{\{ \leq \mu \} \cap \Gamma} \) and \( i_{\geq \lambda}^\pm = i_{\geq \lambda}^\pm |_{\{ \geq \lambda \} \cap \Gamma} \).

**Lemma 4.5.2.** In \( \text{D}^b_{\cap} (\text{Gr}_{G,0}^\pm, S) \) or \( \text{D}^b_{\cap} (\text{Gr}_{\Gamma,0}^\pm, T) \) the following hold

(a) \( (i_{\leq \mu}^\pm)^! \circ R_{\Gamma, m}^{\pm} = R_{\{ \leq \mu \} \cap \Gamma, m}^{\pm} \circ (i_{\leq \mu}^\pm)^! \) and \( (i_{\geq \lambda}^\pm)^* \circ R_{\Gamma, m}^{-\pm} = R_{\{ \geq \lambda \} \cap \Gamma, m}^{-\pm} \circ (i_{\geq \lambda}^\pm)^* \),
(b) \( (i_{\geq \lambda}^\pm)^* \circ R_{\Gamma, m}^{\pm} = R_{\{ \geq \lambda \} \cap \Gamma, m}^{\pm} \circ (i_{\geq \lambda}^\pm)^* \) and \( (i_{\leq \mu}^\pm)^! \circ R_{\Gamma, m}^{-\pm} = R_{\{ \leq \mu \} \cap \Gamma, m}^{-\pm} \circ (i_{\leq \mu}^\pm)^! \),
(c) \( R_{\Gamma, m}^{\pm} \circ (i_{\geq \lambda}^\pm)^* = (i_{\geq \lambda}^\pm)^* \circ R_{\{ \geq \lambda \} \cap \Gamma, m}^{\pm} \) and \( R_{\Gamma, m}^{-\pm} \circ (i_{\geq \lambda}^\pm)^* = (i_{\geq \lambda}^\pm)^* \circ R_{\{ \geq \lambda \} \cap \Gamma, m}^{-\pm} \),
(d) \( R_{\Gamma, m}^{\pm} \circ (i_{\leq \mu}^\pm)^! = (i_{\leq \mu}^\pm)^! \circ R_{\{ \leq \mu \} \cap \Gamma, m}^{\pm} \) and \( R_{\Gamma, m}^{-\pm} \circ (i_{\leq \mu}^\pm)^! = (i_{\leq \mu}^\pm)^! \circ R_{\{ \leq \mu \} \cap \Gamma, m}^{-\pm} \).
Proof. Part (a) follows from Lemma 4.5.1 and proper base change. Part (c) follows from (a) by adjunction. Since \( R_{\Gamma,m}^\alpha \) and \( R_{\Gamma,m}^\beta \) are quasi-inverse, part (d) follows from (c), and (b) from (a).

**Lemma 4.5.3.**

(a) If \( \Gamma = \{ \lambda \} \) in \( \Lambda_\beta \), then we have \( R_{\lambda,m}^\alpha(\kappa_\lambda(d_\lambda^\alpha)) = \kappa_\lambda(-d_\lambda^-) \) in \( D_{\nu,m}^\beta(\Gr_{\lambda,0}^\alpha, S) \).

(b) If \( \Gamma = \{ w \} \) in \( \mathcal{M}_\beta \), then we have \( R_{w,m}^\alpha(\kappa_w(d_w^\alpha)) = \kappa_w(-d_w^-) \) in \( D_{\nu,m}^\beta(\Gr_{w,0}^\beta, S) \).

Proof. Consider the diagram

\[
\begin{array}{ccc}
\Gr_{\lambda}^\alpha & \xrightarrow{f_\lambda^\alpha} & U_\lambda \\
\downarrow & & \downarrow \\
\Gr_{\lambda}^\beta & \xrightarrow{f_\lambda^\beta} & \Gr_{\lambda}^\alpha
\end{array}
\]

Since \( U_\lambda \) is open dense in \( \Gr_{\lambda}^\alpha \times \Gr_{\lambda}^\beta \), it has dimension \( d_\lambda^- + d_\lambda^\beta \). Since the map \( f_\lambda^\beta \) is \( H_\beta \)-equivariant over an \( H_\beta \)-orbit, it is a fibration whose fiber at \( t^\lambda x^\beta \) is isomorphic to \( G_r(O)t^\lambda x^\beta \cap t^\lambda G_r(O^\beta)x^\beta \), which is an affine space (here we write \( O^\beta = O \)). Thus, the map \( f_\lambda^\beta \) is a (étale locally trivial) fibration whose fibers are affine spaces of dimension \( d_\lambda^- \). Part (a) follows, because \( \Gr_{\lambda}^\beta \) is simply connected. Part (b) is proved in a similar way.

Now, we consider the mixed category \( D_{\mu}^b(\Gr_{\beta}^\lambda, S) \). The functors \( (\delta_\beta^\alpha)^* \), \( (\delta_\beta^\alpha)^! \), \( (\delta_\beta^\alpha)^* \) and \( (\delta_\beta^\alpha)^! \) between \( D_{\mu}^b(\Gr_{\beta,0}^\lambda, S), D_{\mu}^b(\Gr_{\beta,0}^\beta, S) \) lift to the mixed categories \( D_{\mu}^b(\Gr_{\beta}^\lambda, S), D_{\mu}^b(\Gr_{\beta}^\beta, S) \) under the functor \( \iota \), and they enjoy all the usual adjunction properties, see Proposition A.3.3

We consider the objects in \( D_{\mu}^b(\Gr_{\beta}^\lambda, S) \) given by

\[
\Delta(\lambda)^\alpha_\mu = (i_{\leq 0}^\mu)^!(i_{\leq 0}^\mu)^*IC(\lambda)_\mu^\alpha , \quad \nabla(\lambda)_\mu^\alpha = (i_{\geq 0}^\mu)^!(i_{\geq 0}^\mu)^*IC(\lambda)_\mu^\alpha ,
\]

\[
\Delta(\lambda)^\beta_\mu = (i_{\geq 0}^\mu)^!(i_{\geq 0}^\mu)^*IC(\lambda)_\mu^\beta , \quad \nabla(\lambda)_\mu^\beta = (i_{\geq 0}^\mu)^!(i_{\leq 0}^\mu)^*IC(\lambda)_\mu^\beta .
\]

The Verdier duality yields an involution \( D \) of the category \( D_{\mu}^b(\Gr_{\beta}^\lambda, S) \) such that \( D(\Delta(\lambda)_\mu^\beta) = \nabla(\lambda)_\mu^\beta \) and \( D(\nabla(\lambda)_\mu^\beta) = IC(\lambda)_\mu^\beta \). The complexes \( \Delta(\lambda)_\mu^\beta, \nabla(\lambda)_\mu^\beta \) are non-equivariant analogues of the objects of \( D_{\mu}^b(\Gr_{\beta}^\lambda, S) \) in (4.4).

**Proposition 4.5.4.**

(a) We have an equivalence of graded triangulated categories \( B_\beta^\alpha : D_{\mu}^b(\Gr_{\beta}^\lambda, S) \to D_{\mu}^b(\Gr_{\beta}^\beta, S) \)

(b) \( B_\beta^\alpha \nabla(\lambda)_\mu^\beta = \Delta(\lambda)_\mu^\beta \) for each \( \lambda \in \Lambda_\beta \).

Proof. Step 1 : We construct an equivalence \( S_{\Gamma,\mu} : D_{\mu}^b(\Gr_{\beta}^\lambda, S) \to D_{\mu}^b(\Gr_{\beta}^\beta, S) \)

Let \( P_{\omega,m}(\Gr_{\Gamma,0}^\beta, T)^{\text{proj}} \) be the full subcategory of \( P_{\omega,m}(\Gr_{\Gamma,0}^\beta, T) \) consisting of the objects whose image by \( \omega \) belongs to \( P(\Gr_{\Gamma}^\beta, T)^{\text{proj}} \). Let \( P_{\omega,m}(\Gr_{\Gamma,0}^\beta, T)^{\text{tilt}} \) be the full subcategory of \( P_{\omega,m}(\Gr_{\Gamma,0}^\beta, T) \) consisting of the tilting objects. By Proposition 4.1.1, the stratification \( T \) on \( \Gr_{\Gamma}^\beta \) is good. Hence, by Propositions A.3.3(a) and A.4.3(b), the functor \( \iota \) restricts to a fully faithful functor \( P_{\mu}(\Gr_{\Gamma}^\beta, T) \to P_{\omega,m}(\Gr_{\Gamma,0}^\beta, T) \) and to an equivalence

\[
P_{\mu}(\Gr_{\Gamma}^\beta, T)^{\text{tilt}} \to P_{\omega,m}(\Gr_{\Gamma,0}^\beta, T)^{\text{tilt}}.
\]
In particular, the category $P_{\tilde{0},m}(\text{Gr}_{\Gamma,0}^{-}, T)^{\text{tilt}}$ is the full subcategory of $P_{\tilde{0},m}(\text{Gr}_{\Gamma,0}^{-}, T)$ consisting of the objects whose image by $\omega$ belongs to $P(\text{Gr}_{\Gamma}^{-}, T)^{\text{tilt}}$. We define the functors $S_{\Gamma}^{-} = R_{\Gamma}^{-}[\pm 1_{\tilde{0}}]$ and $S_{\Gamma,m}^{-} = R_{\Gamma,m}^{-}[\pm 1_{\tilde{0}}]$. By [9], [50], the functors $S_{\Gamma}^{-}$ and $S_{\Gamma}^{+}$ yield inverse equivalences of additive categories

\[ P(\text{Gr}_{\Gamma}^{-}, T)^{\text{tilt}} \cong P(\text{Gr}_{\Gamma}^{+}, T)^{\proj}. \]

Hence, by (4.30), we have the diagram of equivalences of additive categories

\[ P_{\tilde{0},m}(\text{Gr}_{\Gamma,0}^{-}, T)^{\text{tilt}} \cong P_{\tilde{0},m}(\text{Gr}_{\Gamma,0}^{+}, T)^{\proj}. \]

(4.26)

Thus, the set of indecomposable objects in $P_{\tilde{0},m}(\text{Gr}_{\Gamma,0}^{+}, T)^{\proj}$ is

\[ \{ S_{\Gamma,m}^{-} T(w)_\mu(a/2) ; w \in W_{\Gamma}, a \in \mathbb{Z} \}. \]

On the other hand, the set of indecomposable objects in $P_{\mu}(\text{Gr}_{\Gamma}^{+}, T)^{\proj}$ is

\[ \{ P(w)_\mu(a/2) ; w \in W_{\Gamma}, a \in \mathbb{Z} \}. \]

Since the objects $\iota P(w)_\mu(a/2)$ and $S_{\Gamma,m}^{-} T(w)_\mu(a/2)$ have isomorphic images by $\omega$, we deduce that the functor $\iota$ restricts to an equivalence

\[ P_{\mu}(\text{Gr}_{\Gamma}^{+}, T)^{\proj} \cong P_{\tilde{0},m}(\text{Gr}_{\Gamma,0}^{+}, T)^{\proj}. \]

(4.27)

From (4.26), (4.25), (4.27), we deduce that there are equivalences of graded additive categories

\[ P_{\mu}(\text{Gr}_{\Gamma}^{+}, T)^{\text{tilt}} \cong P_{\mu}(\text{Gr}_{\Gamma}^{+}, T)^{\proj}. \]

(4.28)

Taking the homotopy categories, by Propositions A.4.1, A.4.3 we get quasi-inverse equivalences of graded triangulated categories $S_{\Gamma,\mu}$ yielding the following commutative diagram

\[ \begin{diagram}
\text{D}^{b}_{\mu}(\text{Gr}_{\Gamma}^{-}, T) & \cong & \text{D}^{b}_{\mu}(\text{Gr}_{\Gamma}^{+}, T), \\
\text{D}^{b}_{\tilde{0},m}(\text{Gr}_{\Gamma,0}^{-}, T) & \cong & \text{D}^{b}_{\tilde{0},m}(\text{Gr}_{\Gamma,0}^{+}, T).
\end{diagram} \]

(4.29)

In other words, the functors $S_{\Gamma,m}^{-}$ between the categories $\text{D}^{b}_{\tilde{0},m}(\text{Gr}_{\Gamma,0}^{-}, T)$ are genuine. Thus, the functors $S_{\Gamma,m}^{+}$ between the categories $\text{D}^{b}_{\tilde{0},m}(\text{Gr}_{\Gamma,0}^{+}, S)$ are also genuine by [2] lem. 7.21,
i.e., we have quasi-inverse equivalences of graded triangulated categories $S^\pm_{\Gamma,\mu}$ yielding the following commutative diagram

\[
\begin{array}{cccc}
D^b_{\mu}(\text{Gr}^-_{\Gamma}, S) & \xrightarrow{S^+_{\Gamma,\mu}} & D^b_{\mu}(\text{Gr}^+_{\Gamma}, S) \\
\uparrow & & \uparrow \\
D^b_{\mu,m}(\text{Gr}^-_{\Gamma,0}, S) & \xrightarrow{S^+_{\Gamma,m}} & D^b_{\mu,m}(\text{Gr}^+_{\Gamma,0}, S).
\end{array}
\]

\((4.30)\)

**Step 2:** We prove that $S_{\beta,\mu}^-(\nabla(\lambda)^-) = \Delta(\lambda)^+_{\mu}.$

Since the map $\tilde{\iota}_{\leq \mu}$ is a closed embedding, the functor $(\tilde{\iota}_{\leq \mu})!$ preserves tilting mixed perverse sheaves. Hence, by Lemma 4.5.2 and (4.26), the inclusion $\{ \leq \mu \} \cap \Gamma \subset \Gamma$ yields the commutative square of functors

\[
\begin{array}{cccc}
P_{\mu,m}(\text{Gr}^-_{\{ \leq \mu \} \cap \Gamma,0}, T)^\text{tilt} & \xrightarrow{S^-_{\{ \leq \mu \} \cap \Gamma,m}} & P_{\mu,m}(\text{Gr}^+_{\{ \leq \mu \} \cap \Gamma,0}, T)^\text{proj} \\
(\tilde{\iota}_{\mu})! & & (\mu^+_{\{ \leq \mu \}})! \\
P_{\mu,m}(\text{Gr}^-_{\mu}, T)^\text{tilt} & \xrightarrow{S^-_{\mu,m}} & P_{\mu,m}(\text{Gr}^+_{\mu}, T)^\text{proj}.
\end{array}
\]

\((4.31)\)

From (4.25), (4.27), we deduce that there is a commutative square of functors

\[
\begin{array}{cccc}
P_{\mu}(\text{Gr}^-_{\{ \leq \mu \} \cap \Gamma}, T)^\text{tilt} & \xrightarrow{S^-_{\{ \leq \mu \} \cap \Gamma,m}} & P_{\mu}(\text{Gr}^+_{\{ \leq \mu \} \cap \Gamma}, T)^\text{proj} \\
(\tilde{\iota}_{\mu})! & & (\mu^+_{\{ \leq \mu \}})! \\
P_{\mu}(\text{Gr}^-_{\mu}, T)^\text{tilt} & \xrightarrow{S^-_{\mu,m}} & P_{\mu}(\text{Gr}^+_{\mu}, T)^\text{proj}.
\end{array}
\]

\((4.32)\)

Taking the homotopy categories, by Propositions A.4.1, A.4.3 we get the following commutative square of functors

\[
\begin{array}{cccc}
D^b_{\mu}(\text{Gr}^-_{\{ \leq \mu \} \cap \Gamma}, S) & \xrightarrow{S^-_{\{ \leq \mu \} \cap \Gamma,m}} & D^b_{\mu}(\text{Gr}^+_{\{ \leq \mu \} \cap \Gamma}, S) \\
(\tilde{\iota}_{\mu})! & & (\mu^+_{\{ \leq \mu \}})! \\
D^b_{\mu}(\text{Gr}^-_{\mu}, S) & \xrightarrow{S^-_{\mu,m}} & D^b_{\mu}(\text{Gr}^+_{\mu}, S).
\end{array}
\]

\((4.33)\)

Similarly, since the map $\tilde{\iota}_{\geq \lambda}$ is an open embedding, the functor $(\tilde{\iota}_{\geq \lambda})^*$ preserves tilting mixed perverse sheaves. Hence, the inclusion $\{ \geq \lambda \} \cap \Gamma \subset \Gamma$ yields the commutative square
of functors

\[
P_{\mu,m}(\text{Gr}^{-}_{[\geq \lambda] \cap \Gamma,0}, T)^{\text{tilt}} \xrightarrow{S_{\Gamma,m}^{-}} P_{\mu,m}(\text{Gr}^{+}_{[\geq \lambda] \cap \Gamma,0}, T)^{\text{proj}}
\]

(4.34)

\[
(i_{\geq \lambda})^* \begin{array}{c} \downarrow \pi_\mu \end{array}
\]

\[
D_{\mu}^{b}(\text{Gr}^{-}_{[\geq \lambda] \cap \Gamma, S}) \xrightarrow{S_{\mu,\lambda}} D_{\mu}^{b}(\text{Gr}^{+}_{[\geq \lambda] \cap \Gamma, S})
\]

(4.35)

\[
(i_{\leq \lambda})^* \begin{array}{c} \downarrow \pi_\mu \end{array}
\]

\[
D_{\mu}^{b}(\text{Gr}^{-}_{\leq \lambda}, S) \xrightarrow{S_{\mu,\lambda}} D_{\mu}^{b}(\text{Gr}^{+}_{\leq \lambda}, S)
\]

(4.36)

where \( d = d_{[\geq \lambda] \cap \Gamma}^{+} - d_{[\leq \lambda]}^{+} \). Taking the adjoint functors and using a similar argument as above, we obtain the commutative square of functors

\[
D_{\mu}^{b}(\text{Gr}^{-}_{[\geq \lambda] \cap \Gamma}, S) \xrightarrow{S_{\mu,\lambda}} D_{\mu}^{b}(\text{Gr}^{+}_{[\geq \lambda] \cap \Gamma}, S)
\]

In particular, applying (4.33) and (4.35) to the chain of inclusions \( \{ \lambda \} \subset \{ \leq \lambda \} \subset \Lambda_\beta \) and using the identity \( d_{\lambda}^{-} + d_{\lambda}^{+} = d_{\beta}^{+} \), we get the commutative diagram of functors

\[
D_{\mu}^{b}(\text{Gr}^{-}_{\leq \lambda}, S) \xrightarrow{S_{\mu,\lambda}} D_{\mu}^{b}(\text{Gr}^{+}_{\leq \lambda}, S)
\]

(4.36)

Lemma (4.5.3, 4.24) and (4.36) yield an isomorphism

\[
S_{\beta,\mu}^{-}(\nabla(\lambda)_\mu^{\lambda}) = S_{\beta,\mu}^{-}(i_{[\geq \lambda]}^\lambda)^*(IC(\lambda)_\mu^{\lambda}),
\]

\[
= S_{\beta,\mu}^{-}(i_{[\geq \lambda]}^\lambda)^*(k(d_{\lambda}^{-})),
\]

\[
= S_{\beta,\mu}^{-}(i_{[\leq \lambda]}^\lambda)^*(k(d_{\lambda}^{+})),
\]

\[
= (i_{[\leq \lambda]}_{\beta})^!(S_{\lambda,\mu}^{\beta,\mu}(k(2d_{\lambda}^{+}))),
\]

\[
= (i_{[\leq \lambda]}_{\beta})^!(k(d_{\lambda}^{+})),
\]

\[
= \Delta(\lambda)^{\mu}_{\mu}.
\]

Thus, the functor \( B_{\beta}^\lambda = D \circ S_{\beta,\mu}^{+} \circ D \) satisfies the conditions in the proposition. \( \square \)
5. Coherent sheaves on the nilpotent cone

5.1. Perverse coherent sheaves on the nilpotent cone. From now on, we write \( G_r = GL_{r,k} \). This notation differs from the notation used in the beginning of Section 4.1. Let \( (T_r, B_r, N_r) \) be the standard Borel triple. We may abbreviate \( T = T_r, B = B_r \) or \( N = N_r \). The group of characters of \( T \) is identified with \( \mathbb{Z}^r \). Let \( g_r, b_r, n_r \) be the Lie algebra of \( G_r, B_r, N_r \) and \( i : N_r \to g_r \) the embedding of the nilpotent cone. We’ll abbreviate \( G^c_r = G_r \times \mathbb{G}_{m,k} \). The group \( G^c_r \) acts on \( g_r, N_r \) via

\[
(g, z) \cdot \xi = z^{-2} \text{Ad}(g)(\xi) \quad , \quad \forall \xi \in g_r \quad , \quad (g, z) \in G^c_r.
\]

For \( X = g_r \) or \( N_r \), let \( \mathcal{C}oh^{G^c_r}(X) \) be the category of \( G^c_r \)-coherent sheaves on \( X \). We equip this Abelian category with the grading shift functors \( T \) given by \( A \times y \)-Abelian category with the grading shift functors \( T \) be the standard Borel triple. We may abbreviate \( \text{tă} \) the constituents of the objects in \( \mathcal{T} \) be the category of \( \mathcal{A} \) graded Abelian subcategory of perverse coherent sheaves \( \mathcal{P} \).

(5.1) The proper standard and costandard objects are given by \( \Delta(\lambda)^\sharp, \nabla(\lambda)^\sharp, \Delta(\lambda)^\flat, \nabla(\lambda)^\flat, \mathcal{L}(\lambda)^\sharp \) with \( \lambda \in \Lambda_r \). Let us recall briefly their definitions. Consider the diagram of \( G^c_r \)-equivariant k-schemes

\[
\begin{array}{ccc}
N_r & \xrightarrow{q} & G_r \times_{B_r} n_r \xrightarrow{p} & G_r/B_r.
\end{array}
\]

For each weight \( \lambda \in \mathbb{Z}^r \), the Andersen-Jantzen sheaf \( A(\lambda) \) is the complex in \( \mathbb{D}^b \text{Coh}(\mathcal{N}_r/G^c_r) \) given by \( A(\lambda) = Rq_*p^*\mathcal{O}_{G_r/B}(\lambda) \). It is perverse. Set \( \delta_\lambda = \min\{\ell(w) : w w_0 \lambda \in \Lambda_r\} \). If \( \lambda = \lambda_x \) as in (4.1), then we have

\[
\delta_{\lambda_x} = r(r-1)/2 - \sum_{k=0}^{i} p_k(p_k-1)/2.
\]

The proper standard and costandard objects are given by

\[
\Delta(\lambda)^\sharp = A(u_0 \lambda)\langle \delta_\lambda \rangle \quad , \quad \nabla(\lambda)^\sharp = A(\lambda)\langle -\delta_\lambda \rangle \quad , \quad \lambda \in \Lambda_r.
\]

Let \( V(\lambda) \) be the simple rational \( G_r \)-module with highest weight \( \lambda \). We define

\[
\mathcal{T}(\lambda) = V(\lambda) \otimes \mathcal{O}_{g_r} \quad , \quad \mathcal{T}(\lambda)^\sharp = \mathcal{L}^*\mathcal{T}(\lambda) = V(\lambda) \otimes \mathcal{O}_{N_r}.
\]

The set of indecomposable tilting objects is \( \{\mathcal{T}(\lambda)^\sharp(a) : \lambda \in \Lambda_r, a \in \mathbb{Z}\} \). For any order ideal \( \Gamma \subset \Lambda_r \) we consider the Serre subcategory \( \mathcal{P} \text{Coh}(\mathcal{N}_r/G^c_r)_{\Gamma} \) of \( \mathcal{P} \text{Coh}(\mathcal{N}_r/G^c_r) \) generated by the constituents of the elements in \( \{\mathcal{T}(\lambda)^\sharp(a) : \lambda \in \Gamma, a \in \mathbb{Z}\} \). We have

\[
\mathbb{D}^b \text{Coh}(\mathcal{N}_r/G^c_r) = \mathbb{D}^b \mathcal{P} \text{Coh}(\mathcal{N}_r/G^c_r).
\]

Setting \( \Gamma = \{< \lambda\} \) for some weight \( \lambda \in \Lambda_r \), we may consider the quotient functor

\[
(\pi_{>\lambda})^*: \mathbb{D}^b \text{Coh}(\mathcal{N}_r/G^c_r) \to \mathbb{D}^b(\mathcal{P} \text{Coh}(\mathcal{N}_r/G^c_r))/\mathcal{P} \text{Coh}(\mathcal{N}_r/G^c_r)_{<\lambda}).
\]

Let \( (\pi_{>\lambda})_! \) and \( (\pi_{>\lambda})^* \) be the left and right adjoints, see [4] for details. We define

\[
\Delta(\lambda)^\sharp = (\pi_{>\lambda})_!(\pi_{>\lambda})^*(\mathcal{T}(\lambda)^\sharp)\langle -\delta_\lambda \rangle \quad , \quad \nabla(\lambda)^\sharp = (\pi_{>\lambda})_!(\pi_{>\lambda})^*(\mathcal{T}(\lambda)^\sharp)\langle \delta_\lambda \rangle.
\]
Both objects belong to the category $\mathcal{P}Coh([N_r/G^c_r])_{\leq \lambda}$. They are called the standard and costandard objects respectively.

Being the Serre subcategory associated with the order ideal $\Gamma$ of $\Lambda_r$, the Abelian category $\mathcal{P}Coh([N_r/G^c_r])_\Gamma$ is graded properly stratified, with the standard, costandard, proper standard and proper costandard objects given by $\Delta(\lambda)^\pm, \nabla(\lambda)^\pm, \Delta(\lambda)^\pm, \nabla(\lambda)^\pm$ with $\lambda \in \Gamma$.

Consider the graded triangulated subcategories
\[
\text{D}^b\text{Coh}([g_r/G^c_r])_\Gamma \subset \text{D}^b\text{Coh}([g_r/G^c_r]), \quad \text{D}^{\text{perf}}\text{Coh}([N_r/G^c_r])_\Gamma \subset \text{D}^{\text{perf}}\text{Coh}([N_r/G^c_r])
\]
generated by the sets $T_{\Gamma} = \{T(\lambda)\langle a \rangle; \lambda \in \Gamma, a \in \mathbb{Z}\}$ and $T^\Gamma_{\Gamma} = \{T(\lambda)^\pm\langle a \rangle; \lambda \in \Gamma, a \in \mathbb{Z}\}$. We have
\[
\text{D}^{\text{perf}}\text{Coh}([N_r/G^c_r])_\Gamma = \text{D}^{\text{perf}}(\mathcal{P}Coh([N_r/G^c_r])_\Gamma).
\]

### 5.2. The derived Satake equivalence
Recall the graded triangulated categories $\text{D}^b(\text{Gr}_{\beta}^-) = K^b(C(\text{Gr}_{\beta}^-))$ and $\text{D}^b(\text{Gr}_{\beta}^+, S) = K^b(C(\text{Gr}_{\beta}^+, S))$. The following is an instance of the derived Satake equivalence of Bezrukavnikov-Finkelberg.

**Proposition 5.2.1.**

(a) There is an equivalence of graded triangulated categories
\[
\text{C}_\beta : \text{D}^b(\text{Gr}_{\beta}^-) \rightarrow \text{D}^b\text{Coh}([g_r/G^c_r])_{\lambda, \beta}
\]
such that $\text{C}_\beta IC(\lambda)_{\mu} = T(\lambda)$ for all $\lambda \in \Lambda_{\beta}$.

(b) There is an equivalence of graded triangulated categories
\[
\text{C}^\beta : \text{D}^b(\text{Gr}_{\beta}^+, S) \rightarrow \text{D}^{\text{perf}}\text{Coh}([N_r/G^c_r])_{\lambda, \beta}
\]
such that the following diagram commutes
\[
\begin{array}{ccc}
\text{D}^b(\text{Gr}_{\beta}^-) & \xrightarrow{\xi_\beta} & \text{D}^b(\text{Gr}_{\beta}^+) \\
\text{C}_\beta & \downarrow{\xi_\beta^*} & \text{C}^\beta \\
\text{D}^{\text{perf}}\text{Coh}([N_r/G^c_r])_{\lambda, \beta} & \xrightarrow{\lambda_\beta^*} & \text{D}^b\text{Coh}([g_r/G^c_r])_{\lambda, \beta}
\end{array}
\]

(c) For each $\lambda \in \Lambda_{\beta}$ we have $\text{C}^\beta_\beta \Delta(\lambda)_{\mu} = \Delta(\lambda)^\pm, \text{C}^\beta_\beta \nabla(\lambda)_{\mu} = \nabla(\lambda)^\pm$ and $\text{C}^\beta_\beta IC(\lambda)_{\mu} = T(\lambda)^\pm$.

**Proof.** Since the group $K_{\beta}$ is unipotent, the obvious functor
\[
\text{D}^b(\text{Gr}_{\beta}^-) = \text{D}^b([\text{Gr}_{\beta}/H_{\beta}]) \rightarrow \text{D}^b([\text{Gr}_{\beta}/G_r(O)])
\]
is a triangulated equivalence, compare [19] thm. 3.7.3. Hence, part (a) follows from [42] thm. 5.5.1. Part (b) is [1] prop. 5.7. More precisely, let $\text{Add}(T_\beta)$ be the additive full subcategory of $\text{Coh}([g_r/G^c_r])$ generated by the set of objects $T_\beta$. By [11] cor. 5.5.4, we have an equivalence of triangulated categories $\text{D}^b\text{Coh}([g_r/G^c_r])_{\lambda, \beta} = K^b(\text{Add}(T_\beta))$. The equivalence $C_{\beta}$ follows from the isomorphism [42] (5.6.2)
\[
\text{Hom}_{\text{D}^b([g_r/G^c_r])}(IC(\lambda)^-, IC(\mu)^-\langle a \rangle) = \text{Hom}_{\text{D}^b\text{Coh}([g_r/G^c_r])}(T(\lambda), T(\mu)^{\langle a \rangle}), \quad \forall \lambda, \mu \in \Lambda_{\beta}
\]
which yields an equivalence of graded additive categories $C(\text{Gr}_{\beta}^-) = \text{Add}(T_{\beta})$ which commutes with the grading shift functors $\langle \cdot \rangle$. The equivalence $C^\beta_\beta$ is proved in a similar way in [1] prop. 5.5, thm. 2.16.
The commutativity of the equivalences $C_{β}, C^\sharp_{β}$ with the functors $Lξ^*, Lι^*$ is a consequence of the isomorphisms (4.6) and the isomorphism

$$(5.3) \quad k \otimes H^*_G \text{Hom}_{D^b\text{Coh}([g_r//G_r])}(T(λ), T(μ)(a)) = \text{Hom}_{D^\text{perf}\text{Coh}([N_r/Γ_r])}(T(λ), T(μ)^{\sharp}(a)),$$

where the $H^*_G$-module structure comes from the identification of graded rings $H^*_G = k[g_r//G_r]$. The commutativity with $ξ_*, ι_*$ follows by adjunction.

More precisely, in the derived category of $O_\beta//G_r$-modules, we have

$$(5.3) \quad \text{RHom}_{D^b\text{Coh}([g_r//G_r])}(T(λ), T(μ)(a)) = \text{Inv}^{G_r}(g_r//G_r) \text{RHom}_{D^\text{perf}\text{Coh}(G_r)}(T(λ), T(μ)(a)),$$

where $\text{Inv}^{G_r}(g_r//G_r)$ is the derived functor of invariants relatively to the flat affine group scheme $G_r \times (g_r//G_r)$ over $g_r//G_r$. Derived invariants commute with the derived base change functor $M \to k \otimes H^*_G M$, see, e.g., [39, App. A]. We deduce that

$$\text{RHom}_{D^b\text{Coh}([g_r//G_r])}(T(λ), T(μ)(a)) = \text{Inv}^{G_r}(k \otimes H^*_G \text{RHom}_{D^\text{perf}\text{Coh}(G_r)}(T(λ), T(μ)(a))) = \text{RHom}_{D^\text{perf}\text{Coh}([N_r/Γ_r])}(T(λ)^\sharp, T(μ)^{\sharp}(a)).$$

Now, since $T(λ), T(λ)^\sharp$ are locally free and $G_r$ is reductive, we have

$$\text{RHom}^{>0}_{D^b\text{Coh}([g_r//G_r])}(T(λ), T(μ)(a)) = \text{RHom}^{>0}_{D^\text{perf}\text{Coh}([N_r/Γ_r])}(T(λ)^\sharp, T(μ)^{\sharp}(a)) = 0.$$

The isomorphism (5.3) follows. Part (c) of the proposition is [5, cor. 3.4].

We can now prove the following theorem.

**Theorem 5.2.2.** There is an equivalence of graded triangulated categories

$$E^\sharp_β : D^\text{perf}(D^\dag_β) \to D^\text{perf}\text{Coh}([N_r/Γ_r])_{λ_β}$$

such that $E^\sharp_β(Δ(π)^\sharp) = Δ(λ_π)^\sharp$ for all $π ∈ Γ_β$.

**Proof.** Let first observe that

$$(5.4) \quad Δ(π)^\sharp ∈ D^\text{perf}(D^\dag_β), \quad Δ(λ_π)^\sharp ∈ D^\text{perf}\text{Coh}([N_r/Γ_r])_{λ_β}, \quad ∀ π ∈ Γ_β.$$

The second identity follows from [11, prop. 5.4], the first one from Lemma 6.3.1(b) below, and the fact that the category $D_β$ has finite global dimension. Now, by Propositions 4.5.4, 4.4.3, 5.2.1 there is a chain of equivalences of graded triangulated categories

$$E^\sharp_β : D^\text{perf}(D^\dag_β) → A^\dag_β D^b(Gr^+_β, S) → B^\dag_β D^b(Gr^-_β, S) → C^\dag_β D^\text{perf}\text{Coh}([N_r/Γ_r])_{λ_β}$$

such that

$$Δ(π)^\sharp(α) → A^\sharp_β(π)^\sharp(α) → B^\sharp_β(π)^\sharp(α) → C^\sharp_β(π)^\sharp(α).$$

□
Conjecture 5.2.3. There is an equivalence of graded triangulated categories

$$E_\beta : D^b(D_\beta) \to D^bCoh([g_r/G_r^\ast])_{\Lambda_\beta}$$

such that the following square of functors commutes

$$
\begin{array}{ccc}
D^\text{perf}(D^\perp_\beta) & \xrightarrow{\xi_*} & D^b(D_\beta) \\
E^\perp_\beta & \downarrow{L\xi^*} & \downarrow{E_\beta} \\
D^\text{perf}Coh([N_r/G_r^\ast])_{\Lambda_\beta} & \xrightarrow{i_*} & D^bCoh([g_r/G_r^\ast])_{\Lambda_\beta} \\
\end{array}
$$

Further, we have $E_\beta \Delta(\pi) = \Delta(\lambda_\pi)$ for each $\pi \in \Gamma_\beta$.

Remark 5.2.4. Given any weight dominant $\lambda$, let $P_\lambda$ be the largest parabolic subgroup such that the line bundle $O_{G_r/B}(\lambda)$ on $G_r/B$ is the pull-back of a line bundle on $G_r/P_\lambda$. Let $O_{G_r/P_\lambda}(\lambda)$ denote this line bundle. Let $p_\lambda$ be the Lie algebra of $P_\lambda$. The diagram of $G_r$-equivariant $k$-schemes

$$
g_r \xrightarrow{q_\lambda} G_r \times P_\lambda \xrightarrow{p_\lambda} G_r/P_\lambda$$

gives rise to the following complexes

$$\Delta(\lambda) = (Rq_{w_0 \lambda})_* (p_{w_0 \lambda})^* O_{G_r/P_{w_0 \lambda}}(w_0 \lambda), \quad \nabla(\lambda) = (Rq_\lambda)_* (p_\lambda)^* O_{G_r/P_\lambda}(\lambda).$$

The standard and costandard objects should be made more explicit in the following way:

$$\Delta(\lambda)^\ast = Li^* \Delta(\lambda) \quad \text{and} \quad \nabla(\lambda)^\ast = Li^* \nabla(\lambda).$$

This should follows from the computation in [11 (29)-(30)].

6. The equivalence of monoidal categories

In Theorem 5.2.2 we have constructed an equivalence of graded triangulated categories

$$E^\perp_\beta : D^\text{perf}(D^\perp_\beta) \to D^\text{perf}Coh([N_r/G_r^\ast])_{\Lambda_\beta}.$$ 

Our next goal is to prove that it yields indeed an equivalence of graded Abelian categories

$$\bigoplus_{\beta \in Q^+} D^\perp_\beta \to \bigoplus_{\beta \in Q^+} PCoh([N_r/G_r^\ast])_{\Lambda_\beta}.$$ 

6.1. The monoidal structure on $D^\perp$. Consider the graded Abelian categories

$$D^\perp = \bigoplus_{\beta \in Q^+} D^\perp_\beta, \quad D = \bigoplus_{\beta \in Q^+} D_\beta, \quad C = \bigoplus_{\beta \in Q^+} C_\beta.$$ 

We have obvious fully faithful functors $D^\perp \subset D \subset C$. The category $D^\perp_\beta$ is Artinian by Remark 4.4.5 for each $\beta \in Q^+$. Hence we have $D^\perp \subset D^\text{fl}$. For each $\alpha, \gamma \in Q^+$ with $\beta = \alpha + \gamma$, there is an obvious inclusion $R_\alpha \otimes R_\gamma \subset R_\beta$. The induction functors relative to those embeddings equip the categories $C$ and $C^\text{fl}$ with an exact monoidal structure $\circ$.

Lemma 6.1.1. The bifunctor $\circ$ preserves the subcategory $D^\text{fl}$ of $C^\text{fl}$. 

Proof. For each Kostant partitions $\pi$, $\sigma$ supported on $\Phi_{++}$, let us check that the induced module $L(\pi) \circ L(\sigma)$ belongs to $\mathcal{D}$. By the associativity of the induction, we may assume that $\pi = (\beta_n)$ and $\sigma = (\beta_m)$ with $n, m \in \mathbb{N}$. If $n \geq m$ then $L(\beta_n) \circ L(\beta_m)$ is the proper standard module $\Delta_1(\beta_n, \beta_m)$ by (2.7). If $n < m$ then $L(\beta_n) \circ L(\beta_m)$ is the proper costandard module $\nabla(\beta_n, \beta_m)$, up to some grading shift, by [10] (5.2), thm. 10.1(2)] and the definition of the proper costandard modules in [10] §24. Hence it lies again in $\mathcal{D}$. □

Proposition 6.1.2. The bifunctor $\circ$ preserves the subcategory $\mathcal{D}^\sharp$ of $\mathcal{C}^\mathbb{A}$, yielding an exact graded monoidal full subcategory $(\mathcal{D}^\sharp, \circ)$ of $(\mathcal{C}^\mathbb{A}, \circ)$.

Proof. The $Z(R_\beta)$-action on $R_\beta$ by multiplication preserves the subalgebra $R_\alpha \otimes R_\gamma$. The action on the unit of $R_\alpha \otimes R_\gamma$ gives an inclusion

\[(6.1) \quad Z(R_\beta) \subset Z(R_\alpha) \otimes Z(R_\gamma)\]

which fits into the following commutative diagram

\[
\begin{array}{ccc}
Z(R_\beta) & \xrightarrow{(6.1)} & Z(R_\alpha) \otimes Z(R_\gamma) \\
H^*_\alpha & \xrightarrow{\Delta_{\alpha,\gamma}} & H^*_\alpha \otimes H^*_\gamma
\end{array}
\]

where $\Delta_{\alpha,\gamma}$ is the diagonal map. The formal series $E_\beta(t)$ in (4.15) is such that

\[(6.3) \quad \Delta_{\alpha,\gamma} E_\beta(t) = E_\alpha(t) \otimes E_\gamma(t).\]

Thus, under the inclusion (6.1), we have

\[(6.4) \quad J_\beta \subset (J_\alpha \otimes 1) + (1 \otimes J_\gamma).\]

We deduce that $J_\beta$ annihilates the $R_\beta$-module

\[M \circ N = R_\beta \otimes_{R_\alpha \otimes R_\gamma} (M \otimes N)\]

for any modules $M \in \mathcal{D}_\alpha$, $N \in \mathcal{D}_\gamma$ killed by $J_\alpha$, $J_\gamma$ respectively. The category $\mathcal{D}_\beta^\sharp$ consists of the modules in $\mathcal{D}_\beta$ which are killed by $J_\beta$. From the discussion above, we deduce that the monoidal structure $\circ$ preserves the subcategory $\mathcal{D}^\sharp$ of $\mathcal{D}$, i.e., the functor $\xi_\circ$ in (4.10) extends to a monoidal functor $(\mathcal{D}^\sharp, \circ) \rightarrow (\mathcal{D}, \circ)$. □

6.2. The monoidal structure on perverse coherent sheaves on the nilpotent cone.

Fix $\alpha, \beta, \gamma \in Q_{++}$ with $\beta = \alpha + \gamma$. Write $\alpha = u\alpha_0 + k\delta$ and $\gamma = v\alpha_0 + l\delta$. Let $P_{u,v}^c$ be the standard parabolic subgroup of $G_v^c$ with Levi subgroup $G_u \times G_v \times G_m$. Let $\mathcal{N}_{u,v}$ be the nilpotent cone of $P_{u,v}^c$. We consider the following diagram of Artin stacks

\[(6.5) \quad [\mathcal{N}_u/G_u^c] \times [\mathcal{N}_v/G_v^c] \xrightarrow{q} [\mathcal{N}_{u,v}/P_{u,v}^c] \xrightarrow{p} [\mathcal{N}_v/G_v^c].\]

Since the morphism $q$ is smooth, we have a triangulated bifunctor

\[\circ = Rp_\circ q^* : \text{D}^b\text{Coh}([\mathcal{N}_u/G_u^c]) \times \text{D}^b\text{Coh}([\mathcal{N}_v/G_v^c]) \rightarrow \text{D}^b\text{Coh}([\mathcal{N}_v/G_v^c]).\]
We consider the graded Abelian category given by
\[ \mathcal{P}\text{Coh}([\mathcal{N}/G^c])_{\Lambda^+} = \bigoplus_{r > 0} \mathcal{P}\text{Coh}([\mathcal{N}_r/G^c_r])_{\Lambda^+} = \bigoplus_{\beta \in Q_{++}} \mathcal{P}\text{Coh}([\mathcal{N}_r/G^c_r])_{\Lambda_\beta}. \]

**Proposition 6.2.1.** The functor \( \circ \) restricts to an exact bifunctor of graded Abelian categories
\[ \circ : \mathcal{P}\text{Coh}([\mathcal{N}_u/G^c_u]) \times \mathcal{P}\text{Coh}([\mathcal{N}_v/G^c_v]) \to \mathcal{P}\text{Coh}([\mathcal{N}_r/G^c_r]), \]
giving rise to an exact graded monoidal category \( \mathcal{P}\text{Coh}([\mathcal{N}/G^c])_{\Lambda^+}, \circ \).

**Proof.** For any characters \( \mu, \nu \) of \( T_u, T_v \) respectively, let \( (\mu, \nu) \) be the character of the torus \( T_r = T_u \times T_v \) obtained by glueing \( \mu \) with \( \nu \). The base change theorem yields the following isomorphism of Andersen-Jantzen sheaves \( A(\mu) \circ A(\nu) = A(\mu, \nu) \). Hence, for all dominant \( \mu, \nu \) the complex \( \overline{\Delta}(\mu)^\dagger \circ \overline{\Delta}(\nu)^\ddagger \) is perverse. We deduce that the complex \( L(\mu)^\dagger \circ L(\nu)^\ddagger \) is perverse as well, and the proposition follows. \( \square \)

**Remark 6.2.2.** By Proposition 6.2.1 the triangulated category
\[ \text{D}^{\text{perf}}\text{Coh}([\mathcal{N}/G^c])_{\Lambda^+} = \bigoplus_{\beta \in Q_{++}} \text{D}^{\text{perf}}\text{Coh}([\mathcal{N}_r/G^c_r])_{\Lambda_\beta} \]
is monoidal. Considering instead of (6.5) the following diagram of Artin stacks
\[ [\mathfrak{g}_u/G^c_u] \times [\mathfrak{g}_v/G^c_v] \leftarrow [\mathfrak{p}_{u,v}/\mathcal{P}_{\mathfrak{g},\mathfrak{p}_{u,v}}] \rightarrow [\mathfrak{g}_r/G^c_r], \]
we get a triangulated monoidal structure on the triangulated category defined by
\[ \text{D}^b\text{Coh}([\mathfrak{g}/G^c])_{\Lambda^+} = \bigoplus_{\beta \in Q_{++}} \text{D}^b\text{Coh}([\mathfrak{g}_r/G^c_r])_{\Lambda_\beta}. \]
The functor \( i_u \) in (6.2) yields a monoidal functor \( \text{D}^{\text{perf}}\text{Coh}([\mathcal{N}/G^c])_{\Lambda^+} \to \text{D}^b\text{Coh}([\mathfrak{g}/G^c])_{\Lambda^+} \). The left adjoint functor \( L i^* \) is not monoidal.

### 6.3. The monoidal equivalence
The category \( \mathcal{P}\text{Coh}([\mathcal{N}_r/G^c_r])_{\Lambda_\beta} \) is graded properly stratified. The graded Abelian categories \( \text{D}^\dagger_{\beta}, \text{D}_\beta \) are equipped with the pair of adjoint functors \( (\xi^*, \xi_*) \). By Definition 6.1.2 and the definition of the proper standard and costandard modules \( \overline{\Delta} (\pi), \overline{\nabla} (\pi) \) in \( \text{D}_\beta \) in [30] (6.5), (6.7), we deduce that \( \overline{\Delta} (\pi), \overline{\nabla} (\pi) \in \text{D}^\dagger_{\beta} \) for all \( \pi \in \Gamma_\beta \). We may write \( \overline{\Delta} (\pi)^\dagger = \overline{\Delta} (\pi) \) and \( \overline{\nabla} (\pi)^\dagger = \overline{\nabla} (\pi) \) to indicate that we view them as objects in \( \text{D}^\dagger_{\beta} \). In [14.13] we have defined \( \overline{\Delta} (\pi)^\ddagger = \xi^* \overline{\Delta} (\pi) \) for all \( \pi \in \Gamma_\beta \). Hence, we can consider the subcategory \( \text{D}^\dagger_{\beta} \Delta \subset \text{D}^\dagger_{\beta} \) consisting of the objects with a finite filtration whose quotients are isomorphic to some \( \overline{\Delta} (\pi)^\ddagger \) with \( \pi \in \Gamma_\beta \).

**Lemma 6.3.1.**
(a) \( \xi^* \) restricts to an exact functor \( \text{D}^\dagger_{\beta} \Delta \to \text{D}^\dagger_{\beta} \).
(b) \( \text{D}^\dagger_{\beta} \text{proj} = \{ P \in \text{D}^\dagger_{\beta} \Delta : \text{Ext}^1_{\text{D}^\dagger_{\beta}} (P, \overline{\Delta} (\pi)^\dagger) = 0, \forall \pi \in \Gamma_\beta \} \).
(c) \( \mathcal{P}\text{Coh}([\mathcal{N}_r/G^c_r])_{\Lambda_\beta}^{\text{proj}} = \{ P \in \mathcal{P}\text{Coh}([\mathcal{N}_r/G^c_r])_{\Lambda_\beta} : \text{Ext}^1_{\mathcal{P}\text{Coh}([\mathcal{N}_r/G^c_r])} (P, \overline{\Delta} (\lambda)^\ddagger) = 0, \forall \lambda \in \Lambda_\beta \} \).
Proof. Fix \( \pi = ((\beta_l)^{pr}, \ldots, (\beta_0)^{pr}) \) in \( \Gamma_\beta \). The map (6.1) yields an inclusion
\[
Z(R_\beta) \subseteq Z(R_{\beta_l})^{\otimes pr} \otimes \cdots \otimes Z(R_{\beta_0})^{\otimes pr}.
\]
Since \( \pi \) is a Kostant partition of \( \beta \), we have \( r_\pi = r \), hence the central elements \( z_{\beta_1}, \ldots, z_{\beta_l} \) in (2.8) yield a k-algebra embedding
\[
(6.6) \quad k[z_1, z_2, \ldots, z_r] \subseteq Z(R_{\beta_l})^{\otimes pr} \otimes \cdots \otimes Z(R_{\beta_0})^{\otimes pr}
\]
which fits into the following commutative diagram of inclusions
\[
\begin{array}{ccc}
Z(R_\beta) & \xrightarrow{(6.1)} & Z(R_{\beta_l})^{\otimes pr} \otimes \cdots \otimes (R_{\beta_0})^{\otimes pr} \\
\downarrow & & \downarrow (6.6) \\
k[z_1, z_2, \ldots, z_r]^S & \rightarrow & k[z_1, z_2, \ldots, z_r]
\end{array}
\]
Now, for each \( k = 1, \ldots, r \), the map \( H_{G,\beta_k}^* \rightarrow Z(S_{\beta_k}) \) in (4.9) is the composition of the map \( H_{G,\beta_k}^* = k[z_{\beta_k}] \rightarrow Z(R_{\beta_k}) \in (2.8) \) with the restriction \( (j_{\beta_k})^*: Z(R_{\beta_k}) \rightarrow Z(S_{\beta_k}) \). Hence, the proof of (6.1) implies that the map \( H_{G,\beta}^* \rightarrow Z(S_\beta) \) in (4.9) is the composition of the chain of maps
\[
H_{G,\beta}^* \xrightarrow{k[z_1, z_2, \ldots, z_r]^S} Z(R_\beta) \xrightarrow{(j_\beta)^*} Z(S_\beta).
\]
Since \( \Delta(\beta_k) \) is a free module over \( k[z_{\beta_k}] \) for each \( k \), the induced module
\[
(6.7) \quad \Delta(\beta_l)^{\otimes pr} \circ \cdots \circ \Delta(\beta_1)^{\otimes pr} \circ \Delta(\beta_0)^{\otimes pr}
\]
is free over \( k[z_1, z_2, \ldots, z_r] \), hence over \( H_{G,\beta}^* \). By (2.3), (2.7), the standard module \( \Delta(\pi) \) is a direct summand of the induced module (6.7), hence it is also free over \( H_{G,\beta}^* \). Part (a) follows, because \( \xi^*(M) = k \otimes_{H_{G,\beta}^*} M \) for each module \( M \in D_\beta \).

Now, let us concentrate on part (b). Since \( \xi^* \) is left adjoint to \( \xi_* \), which is exact, we deduce that \( \xi^*(D_{\beta}^{\text{proj}}) \subseteq D_{\beta}^{\sharp \text{proj}} \). Using the fact that the categories \( D_\beta, D_{\beta}^{\sharp} \) are both Krull-Schmidt, we get indeed an equality \( \xi^*(D_{\beta}^{\text{proj}}) = D_{\beta}^{\sharp \text{proj}} \). Since the category \( D_\beta \) is affine highest weight, we have
\[
D_{\beta}^{\text{proj}} = \{ P \in D_\beta^A; Ext^1_{D_\beta}(P, \Delta(\pi)) = 0, \forall \pi \in \Gamma_\beta \},
\]
hence part (a) implies that
\[
D_{\beta}^{\sharp \text{proj}} \subseteq \{ P \in D_{\beta}^A; Ext^1_{D_{\beta}^A}(P, \Delta(\pi)^\sharp) = 0, \forall \pi \in \Gamma_\beta \}.
\]
To prove the reverse inclusion, by part (a) and adjunction we have, for all \( i > 0 \),
\[
Ext^i_{D_{\beta}^A}(\Delta(\sigma)^\sharp, \nabla(\pi)^\sharp) = Ext^i_{D_{\beta}^A}(L\xi^*\Delta(\sigma), \nabla(\pi)^\sharp)
\]
\[
= Ext^i_{D_{\beta}^A}(\Delta(\sigma), \nabla(\pi)) = 0,
\]
from which we deduce that
\[
D_{\beta}^{A, \Delta} \subseteq \{ P \in D_{\beta}^A; Ext^{i > 0}_{D_{\beta}^A}(P, \nabla(\pi)^\sharp) = 0, \forall \pi \in \Gamma_\beta \}.
\]
For each \( \beta \in \mathbb{Q}^+ \) there is an equivalence of graded Abelian categories

\[
E^\delta_\beta : D^\Delta_\beta \rightarrow \mathcal{P}\mathcal{Coh}(\mathcal{N}_r/G^r_c)_{\Lambda_\beta}
\]

which takes the graded \( S^\delta_\beta \)-modules \( \Delta(\pi)^\sharp, \bar{\Delta}(\pi)^\sharp, \bar{\nabla}(\pi)^\sharp, L(\pi)^\sharp \) to the perverse coherent sheaves \( \Delta(\lambda)^\sharp, \bar{\Delta}(\lambda)^\sharp, \bar{\nabla}(\lambda)^\sharp, L(\lambda)^\sharp \) respectively, for all \( \pi \in \Gamma_\beta \).

**Proof.** Given subsets \( A, B \subset \text{Isom}(\mathcal{D}) \) of the set of isomorphism classes of objects of a triangulated category \( \mathcal{D} \), let \( A \ast B \subset \text{Isom}(\mathcal{D}) \) be the set of classes of all objects \( Z \) for which there exists an exact triangle \( X \rightarrow Z \rightarrow Y \rightarrow X[1] \) with \( X \in A \) and \( Y \in B \). The octahedron axiom implies that the operation \( \ast \) is associative, hence for each \( A \) as above we may consider the strictly full subcategory \( \langle A \rangle \subset \mathcal{D} \) such that \( \text{Isom}(\langle A \rangle) = \bigcup_{n>0} A \ast A \ast \cdots \ast A \), where \( A \) appears \( n \) times. By (5.3), we may define the additive subcategories

\[
D^\text{perf}(D^\Delta_\beta)^\Delta = \langle \{\Delta(\pi)^\sharp \langle a \rangle : \pi \in \Gamma_\beta, a \in \mathbb{Z}\}\rangle,
\]

\[
D^\text{perf}(\mathcal{Coh}(\mathcal{N}_r/G^r_c))_{\Lambda_\beta}^\Delta = \langle \{\Delta(\lambda)^\sharp \langle a \rangle : \lambda \in \Lambda_\beta, a \in \mathbb{Z}\}\rangle
\]

of \( D^\text{perf}(D^\Delta_\beta) \) and \( D^\text{perf}(\mathcal{Coh}(\mathcal{N}_r/G^r_c))_{\Lambda_\beta} \) respectively. The equivalence of triangulated categories \( E^\delta_\beta \) in Theorem 5.2.2 restricts to an equivalence of graded additive categories

\[
(D^\text{perf}(D^\Delta_\beta)^\Delta, \langle \bullet \rangle) \rightarrow (D^\text{perf}(\mathcal{Coh}(\mathcal{N}_r/G^r_c))_{\Lambda_\beta}^\Delta, \langle \bullet \rangle)
\]

Since we have \( D^\text{perf}(D^\Delta_\beta)^\Delta = D^\Delta_\beta \) and \( D^\text{perf}(\mathcal{Coh}(\mathcal{N}_r/G^r_c))_{\Lambda_\beta}^\Delta = \mathcal{P}\mathcal{Coh}(\mathcal{N}_r/G^r_c)_{\Lambda_\beta}^\Delta \), it is indeed an equivalence of graded additive categories

\[
E^\delta_\beta : (D^\Delta_\beta, \langle \bullet \rangle) \rightarrow (\mathcal{P}\mathcal{Coh}(\mathcal{N}_r/G^r_c)_{\Lambda_\beta}^\Delta, \langle \bullet \rangle).
\]
By Lemma 6.3.1 it restricts further to an equivalence of graded additive categories
\[(D^\mathbf{c}_\beta \mathcal{P}_{\text{proj}}, \{\circ\}) \to (\mathcal{P}Coh([\mathcal{N}_r/\mathcal{G}^c_r])_{\mathcal{A}_\beta}, \{\circ\})\]
which takes the projective cover of \(L(\pi)^c\) to the projective cover of \(\mathcal{L}(\lambda_\pi)^c\). Therefore, it yields an equivalence of graded Abelian categories \(D^\mathbf{c}_\beta \to \mathcal{P}Coh([\mathcal{N}_r/\mathcal{G}^c_r])_{\mathcal{A}_\beta}\) as in the theorem. \(\square\)

Let \(\circ^\text{op}\) denote monoidal structure opposite to \(\circ\). Set \(E^c_\beta = \bigoplus_{\beta \in \mathbb{Q}^+} \mathcal{E}_\beta^c\).

**Conjecture 6.3.3.** The equivalence of graded Abelian categories \(E^c : D^c \to \mathcal{P}Coh([\mathcal{N}/\mathcal{G}^c])_{\mathcal{A}^+}\) extends to an equivalence of exact graded monoidal categories \((D^c, \circ) \to (\mathcal{P}Coh([\mathcal{N}/\mathcal{G}^c])_{\mathcal{A}^+}, \circ^\text{op})\).

**Remark 6.3.4.**
(a) From [27], Theorem 6.3.2 and the conjecture we deduce that
\[
\mathbf{\Delta}(\pi)^c = E^c(\mathbf{\Delta}(\pi)^c) = (\mathbf{L}(\beta_0)^c)^{\circ^\text{op}} \circ \cdots \circ (\mathbf{L}(\beta_1)^c)^{\circ^\text{op}} (\sum_{k=0}^{l} p_k(p_k - 1)/2)
\]
for any Kostant partition \(\pi = ((\beta_1)^{p_1}, \ldots, (\beta_1)^{p_1}, (\beta_0)^{p_0})\) in \(\Gamma_\beta\). This is precisely the formula [5.1]. So, the isomorphism \(\mathbf{\Delta}(\pi)^c = E^c(\mathbf{\Delta}(\pi)^c)\) is a weak form of monoidality of the functor \(E^c\).

(b) By [40, §24], [49] the \(A\)-module isomorphism \(G_0(\mathcal{C}^\beta) = A_{\mathbb{Q}}(\mathfrak{n})\) maps the proper standard modules and the simple ones to the elements of the dual PBW basis and the dual canonical basis respectively. More precisely, if \(\pi = ((\beta_1)^{p_1}, \ldots, (\beta_1)^{p_1}, (\beta_0)^{p_0})\) it maps the graded module \(\mathbf{\Delta}(\pi)\) to the element
\[
E^*(\pi) = q^{\sum_{k=0}^{l} p_k(p_k - 1)/2} E^*(\beta_1)^{p_1} \cdots E^*(\beta_2)^{p_2} E^*(\beta_0)^{p_0}.
\]

(c) We expect that the functor \(E = \bigoplus_{\beta \in \mathbb{Q}^+} E_\beta\) in Conjecture 5.2.3 extends also to an equivalence of exact graded monoidal categories \((D, \circ) \to (\mathcal{P}Coh([\mathfrak{g}/\mathcal{G}^c])_{\mathcal{A}^+}, \circ^\text{op})\).

6.4. Perverse coherent sheaves on the affine Grassmannian. Fix a positive integer \(N\). Consider the element \(w = (s_0s_1)^N\) in the Weyl group of the affine root system \(\Phi\), hence
\[\Phi_+ \cap w(-\Phi_+) = \{\beta_0, \beta_1, \ldots, \beta_{2N-1}\}.
\]
Let \(\mathcal{C}_w^\mathbf{c}\) be the graded monoidal full subcategory of \(\mathcal{C}^\mathbf{c}\) generated by \(L(\beta_0), L(\beta_1), \ldots, L(\beta_{2N-1})\). The quantum unipotent coordinate algebra \(A_{\mathbb{Q}}(\mathfrak{n}(w))\) is the \(A\)-subalgebra of \(A_{\mathbb{Q}}(\mathfrak{n})\) generated by the dual root vectors \(E^*(\beta_0), E^*(\beta_1), \ldots, E^*(\beta_{2N-1})\). The graded monoidal category \(\mathcal{C}_w^\mathbf{c}\) is a categorification of \(A_{\mathbb{Q}}(\mathfrak{n}(w))\). The quantum open unipotent cell \(A_{\mathbb{Q}}(\mathfrak{n}(w))\) is a localization of \(A_{\mathbb{Q}}(\mathfrak{n}(w))\). In [21] a localization \(\mathcal{C}_w^\mathbf{c}\) of the category \(\mathcal{C}_w^\mathbf{c}\) is introduced. It is a graded monoidal category. It is proved there that \(\mathcal{C}_w^\mathbf{c}\) categorifies \(A_{\mathbb{Q}}(\mathfrak{n}(w))\).

From now on, let \(Gr\) denote the affine Grassmannian of \(G_N\) over \(k\). We define \(\mathcal{O} = k[[t]]\) and \(G^c_N(\mathcal{O}) = GL_N(\mathcal{O}) \rtimes \mathbb{G}_{m,k}\). The multiplication in \(G^c_N(\mathcal{O})\) is given by
\[(1, z) \cdot (g(t), 1) = (g(z^4t), 1).
\]

The group \(G^c_N(\mathcal{O})\) acts on \(Gr\) so that the subgroup \(\mathbb{G}_{m,k}\) is the 4-fold cover of the loop rotation. Let \(D^\mathbf{c}Coh([Gr/G^c_N(\mathcal{O})])\) be the derived category of \(G^c_N(\mathcal{O})\)-equivariant coherent
sheaves on Gr. It is a $\mathbb{L}_{\mathbb{Z}}$-graded triangulated category with the grading shift functors such that $\langle -1/2 \rangle$ is the tensor product with the tautological 1-dimensional $\mathbb{G}_m$-module (of weight 1). In other words $q = \langle 1 \rangle$ corresponds to shifting the weight by -1 with respect to the double cover of loop rotation. Let $\mathcal{P}Coh([Gr/G^c_N(O)])$ be the heart of the perverse $t$-structure in $\mathbb{D}^bCoh([Gr/G^c_N(O)])$. We equip both categories with the convolution product $\odot$ denoted by the symbol $\ast$. In \cite{13} or \cite{16}. This yields an exact graded monoidal category

$$ (\mathcal{P}Coh([Gr/G^c_N(O)]), \odot). $$

For each integer $r \geq 0$, we set $\alpha_r = N\alpha_0 + r\delta$ and $Gr_{\alpha_r} = \overline{Gr_{\omega_1}}$. Consider the $G^c_r$-invariant open subset $N_{\alpha_r} \subset N_r$ consisting of all nilpotent $r$ by $r$ matrices with at most $N$ Jordan blocks. We define the graded categories

$$ \mathcal{P}Coh([Gr/Gr_{\alpha_r}^c]) = \bigoplus_{r \geq 0} \mathcal{P}Coh([Gr_{\alpha_r}/Gr_{\alpha_r}^c]); $$

$$ \mathcal{P}Coh([N/G^c_r]) = \bigoplus_{r \geq 0} \mathcal{P}Coh([N_r/G^c_r]). $$

The graded shift functors are defined above and in \cite{5}. The $G_r^N(O)$-orbit $Gr_{\omega_r}$ in Gr is

$$ \mathcal{G}_{\omega_r} = \{ L \subset L_0 : tL_0 \subset L, \dim(L_0/L) = r \}. $$

Its dimension is $d_{\omega_r} = r(N - r)$. Let $\det_{\omega_r} = \bigwedge^r(L_0/L)$ be the determinant bundle over $Gr_{\omega_r}$. Let $\hat{\omega}_r$ be the embedding of $Gr_{\omega_r}$ in $Gr$. Consider the object of $\mathcal{P}Coh([Gr/Gr_{\hat{\alpha}_r}^c])$ given by

$$ \mathcal{P}_{r, \ell} = \langle \hat{\omega}_r \rangle \odot \langle \det_{\omega_r} \rangle \odot \langle -d_{\omega_r}/2 - r\ell \rangle, \quad \forall r \in [1, N], \quad \forall \ell \in \mathbb{Z}. $$

**Proposition 6.4.1** (\cite{18}).

(a) There is a flat stack homomorphism $\psi_r : [Gr_{\alpha_r}/Gr_{\omega_r}^c] \to [N_{\alpha_r}/G_r^c]$.  

(b) Set $d_r = r(N - 1)/2$. The triangulated functor

$$ \Psi_r = \psi_r^* : [d_r] \langle d_r \rangle : \mathbb{D}^bCoh([N_{\alpha_r}/G_r^c]) \to \mathbb{D}^bCoh([Gr_{\alpha_r}/Gr_{\omega_r}^c]) $$

is graded and $t$-exact with respect to the perverse $t$-structures of both sides.

(c) Composing $\Psi_r$ with the restriction to the open subset $N_{\alpha_r} \subset N_r$ and taking the sum over all $r > 0$, we get a graded monoidal functor $\Psi_+$ yielding the following commutative diagram of Abelian graded monoidal categories

$$ \mathcal{P}Coh([N/G^c_r]) \xrightarrow{i} \mathcal{P}Coh([N/G^c_r]) $$

$$ \mathcal{P}Coh([Gr/Gr_{\omega_r}^c]) \xrightarrow{j} \mathcal{P}Coh([Gr_{\alpha_r}/Gr_{\omega_r}^c]) $$

such that, for each $k$ in $[0, 2N - 1] \subset \mathbb{N} = \Lambda_1^+$, we have $\Psi \mathbb{E}^k(L(\beta_k)^{\ell}) = \Psi(L(k)^{\ell}) = \mathcal{P}_{1,N-k}^{t-1/2}$.  

**Proof.** Part (a) is \cite{18, lem. 4.9}. Part (b) is \cite{18, prop. 4.12}. Part (c) is proved as \cite{18, lem. 4.15}. The functors $i, j$ are the obvious inclusions. The functor $i$ is a monoidal functor, relatively to the monoidal structure $\odot$. The functor $j$ is a monoidal functor, relatively to the monoidal structure $\circ$. \hfill $\square$
We equip the \(\mathcal{A}\)-modules \(G_0(D^\mathbb{A})\) and \(G_0(C^\mathbb{A})\) with the multiplications given by the exact bifunctor \(\circ\). Let \(\mathcal{A}_q(n)_{\mathbb{A}+}\) be the \(\mathcal{A}\)-subalgebra of \(\mathcal{A}_q(n)\) generated by the dual root vectors \(E(\beta)^*\) with \(\beta \in \Phi_+\). We have \(G_0(C^\mathbb{A}) = \mathcal{A}_q(n)\) and \(G_0(D^\mathbb{A}) = \mathcal{A}_q(n)_{\mathbb{A}+}\). Consider the graded monoidal Serre subcategory of \(D^\mathbb{A}\) given by \(D^\mathbb{A} = C^\mathbb{A}_{w} \cap D^\mathbb{A}\). We have \(G_0(D^\mathbb{A})_{w} = \mathcal{A}_q(n(w))\). The equivalence \(E^\mathbb{A} : D^\mathbb{A} \rightarrow \mathcal{P} \text{Coh}([N/G^\mathbb{A}]_{\mathbb{A}+})\) gives an \(\mathcal{A}\)-module isomorphism
\[
\mathcal{A}_q(n)_{\mathbb{A}+} \rightarrow G_0(\mathcal{P} \text{Coh}([N/G^\mathbb{A}]_{\mathbb{A}+})).
\]
This isomorphism identifies the elements of the dual PBW basis and the dual canonical basis of \(\mathcal{A}_q(n)_{\mathbb{A}+}\) with the classes of the proper standard modules and the simple ones in \(\mathcal{P} \text{Coh}([N/G^\mathbb{A}]_{\mathbb{A}+})\) respectively.

Let \(\mathcal{D}(w\Lambda_i, \Lambda_i), i = 0, 1\), be the unipotent quantum minors in \(\mathcal{A}_q(n)\). See, e.g., \([23]\) def. 1.5) for the notation. They belong to \(\mathcal{A}_q(n(w))\). By \([23]\) §4, there are unique simple self-dual objects \(M(w\Lambda_i, \Lambda_i), i = 0, 1\) in \(C^\mathbb{A}_{w}\) such that the isomorphism \(G_0(C^\mathbb{A}_{w}) = \mathcal{A}_q(n(w))\) identifies the class of \(M(w\Lambda_i, \Lambda_i)\) with \(D_i\). We abbreviate \(C_i = M(w\Lambda_i, \Lambda_i)\). With the terminology in \([24]\), the objects \(C_0, C_1\) are non-degenerate graded braiders. Let \(C^\mathbb{A}_{w}\) be the localization of the graded monoidal category \(C^\mathbb{A}_{w}\) by \(C_0, C_1\). It is an exact graded monoidal category. We abbreviate \(C^\mathbb{A}_{w} = C^\mathbb{A}_{w}(C_0^{\mathbb{A}^{-1}}, C_1^{\mathbb{A}^{-1}})\). Since the modules \(C_0, C_1\) are simple and \(G_0(D^\mathbb{A}_{w}) = G_0(C^\mathbb{A}_{w})\), they belong to the subcategory \(D^\mathbb{A}_{w}\). Let \(\tilde{D}^\mathbb{A}_{w}\) be the localization of the graded monoidal category \(D^\mathbb{A}_{w}\) by \(C_0, C_1\), i.e., we set
\[
\tilde{D}^\mathbb{A}_{w} = D_{w}^{\mathbb{A}^{-1}}(C_0^{\mathbb{A}^{-1}}, C_1^{\mathbb{A}^{-1}}).
\]
It is an exact graded monoidal category.

Composing the functors \(E^\mathbb{A}\), \(\Psi\) and the inclusion \(D^\mathbb{A}_{w} \subset D^\mathbb{A}\), we get a graded functor
\[
\tilde{\Psi}_w : \tilde{D}^\mathbb{A}_{w} \rightarrow \mathcal{P} \text{Coh}([\text{Gr}/G^\mathbb{A}_N(O)]).
\]
By \([14]\), we have \(\Psi_w(C_i) = P_{N, i-1}\) for \(i = 0, 1\). In particular, both objects \(\Psi_w(C_0)\) and \(\Psi_w(C_1)\) are invertible in the monoidal category \([6, 14]\). Now, assume that Conjecture 6.3.3 holds. Then, the functor \(\Psi_w\) is graded monoidal. Thus, the universal property in \([24]\) thm. 2.7] yields a commutative triangle of exact monoidal functors
\[
\begin{array}{c}
\tilde{D}^\mathbb{A}_{w} \\
\Psi_w \\
\mathcal{P} \text{Coh}([\text{Gr}/G^\mathbb{A}_N(O)])
\end{array}
\xymatrix{
\tilde{D}^\mathbb{A}_{w} \ar[r]^{\Psi_w} & \mathcal{P} \text{Coh}([\text{Gr}/G^\mathbb{A}_N(O)]) \\
\Psi_w \\
\mathcal{P} \text{Coh}([\text{Gr}/G^\mathbb{A}_N(O)])
}

**Conjecture 6.4.2.** The functor \(\tilde{\Psi}_w\) is a graded monoidal equivalence
\[
(\tilde{D}^\mathbb{A}_{w}, \circ) \rightarrow (\mathcal{P} \text{Coh}([\text{Gr}/G^\mathbb{A}_N(O)]), \circ).
\]

The functor \(\tilde{\Psi}_w\) is exact by definition. Since \(G_0(D^\mathbb{A}_{w}) = G_0(C^\mathbb{A}_{w})\), we have \(G_0(\tilde{D}^\mathbb{A}_{w}) = G_0(C^\mathbb{A}_{w}) = \mathcal{A}_q(n^w)\). By \([16]\) we have \(\mathcal{A}_q(n^w) = G_0(\mathcal{P} \text{Coh}([\text{Gr}/G^\mathbb{A}_N(O)]))\). Hence, taking the Grothendieck groups, the functor \(\tilde{\Psi}_w\) yields a group homomorphism
\[
G_0(\tilde{D}^\mathbb{A}_{w}) \rightarrow G_0(\mathcal{P} \text{Coh}([\text{Gr}/G^\mathbb{A}_N(O)]))
\]
which is invertible. Now, recall that if \(F : \mathcal{A} \rightarrow \mathcal{B}\) is an exact functor of Abelian Artinian categories such that the induced map \(G_0(\mathcal{A}) \rightarrow G_0(\mathcal{B})\) is injective, then \(F\) is faithful. We
deduce that $\hat{\Psi}_w$ is a faithful functor. To prove the conjecture we must prove that the functor $\hat{\Psi}_w$ is full.

Appendix A. Reminders on Artin stacks and mixed geometry

A.1. Schemes and stacks. Let $k = \mathbb{Q}_ℓ$ and $F$ be any field of characteristic prime to $ℓ$. All $F$-schemes are assumed to be separated. An Artin $F$-stack is a stack over the category of affine $F$-schemes with a smooth atlas and a representable, quasi-compact, quasi-separated diagonal. Given any algebraic $F$-space $Z$ with an action of an affine algebraic $F$-group $G$, the quotient $F$-stack $[Z/G]$ is the Artin stack whose set of $F$-points consists of $G$-torsors on $\text{Spec} R$ with $G$-equivariant map to $Z$. A locally quotient stack is an Artin stack which is locally equivalent to a quotient stack.

Let $Z$ be an Artin $F$-stack of finite type. Let $\text{D}^b(Z)$ be the bounded constructible derived category of étale sheaves of $k$-modules on $Z$, in the sense of [34]. Let $\text{P}(Z)$ be the subcategory of perverse sheaves. For each complex $\mathcal{E} \in \text{D}^b(Z)$ and each integer $a$, let $\mathcal{H}^a \mathcal{E} \in \text{P}(Z)$ be the $a$-th perverse cohomology complex. Let $\text{C}(Z) \subset \text{D}^b(Z)$ be the additive full subcategory of semisimple complexes.

Let $Z$ be an $F$-scheme of finite type with an action of an affine algebraic $F$-group $G$. Let $\text{D}^b_G(Z)$ denote the $G$-equivariant derived category of complexes of étale sheaves of $k$-vector spaces on $Z$ with bounded constructible cohomology, in the sense of Bernstein-Lunts [10]. The triangulated category $\text{D}^b_G(Z)$ depends only on the quotient stack $[Z/G]$, and not on the action groupoid $\{G \times Z \rightrightarrows Z\}$ representing it. It is equivalent to the triangulated category $\text{D}^b([Z/G])$.

By a stratification $S = \{Z_w : w \in W\}$ of an $F$-scheme of finite type $Z$ we’ll mean a finite algebraic Whitney stratification. Let $\text{D}^b(Z, S) \subset \text{D}^b(Z)$ be the full subcategory whose objects have constructible cohomology with respect to $S$. We write $\text{P}(Z, S) = \text{P}(Z) \cap \text{D}^b(Z, S)$ and $\text{C}(Z, S) = \text{C}(Z) \cap \text{D}^b(Z, S)$.

Unless specified otherwise, we’ll assume that $Z$ is the quotient stack $[Z/G]$ of an $F$-scheme $Z$ of finite type by the action of an affine algebraic $F$-group $G$ with a finite number of orbits. Then, the $G$-orbits define a stratification $S$ of $Z$ and the categories $\text{P}(Z)$, $\text{C}(Z)$ are $G$-equivariant analogues of the categories $\text{P}(Z, S)$, $\text{C}(Z, S)$. Let For $: \text{D}^b(Z) \to \text{D}^b(Z, S)$ be the forgetful functor. The categories $\text{C}(Z)$ and $\text{C}(Z, S)$ are graded. The category grading is given by the cohomological shift functor, i.e., we set

$$\langle \bullet \rangle = [\bullet].$$

A.2. Mixed complexes. From now on we assume that $F$ is the algebraic closure of a finite field $F_0$ of characteristic prime to $ℓ$. We’ll use the following convention : objects over $F_0$ are denoted with a subscript 0, and suppression of the subscript means passing to $F$ by extension of scalars. For instance, we may write $Z = F \otimes_{F_0} Z_0$ for some $F_0$-scheme $Z_0$. Then, we’ll assume that the stratification $S$ of $Z$ is the extension of scalars of a stratification $\{Z_{w,0} : w \in W\}$ of $Z_0$. Let $\text{D}^b_m(Z_0)$ be the full triangulated subcategory of mixed complexes in $\text{D}^b(Z_0)$. The extension of scalars yields a triangulated $t$-exact functor

$$\omega : \text{D}^b_m(Z_0) \to \text{D}^b(Z).$$
We may write a mixed complex with a subscript $\bullet_m$, and abbreviate $E = \omega(E_m)$. Let $D^b_m(Z_0, S)$ be the full triangulated subcategory of $D^b_m(Z_0)$ such that $D^b_m(Z_0, S) = \omega^{-1}D^b(Z_0, S)$. Let $P_m(Z_0, S)$ be the category of mixed perverse sheaves in $D^b_m(Z_0)$. Recall that $Z$ is an $F$-stack of the form $Z = [Z / G]$. Assume further that $Z$ is isomorphic to $F \otimes F_0 Z_0$, with $Z_0 = [Z_0 / G_0]$ and some affine algebraic $F_0$-group $G_0$ such that $G = F \otimes F_0 G_0$. Let $D^b_m(Z_0)$ be the full triangulated subcategory of mixed complexes in $D^b(Z_0)$. See [34] and [17] for the definition and the basic properties of the category $D^b_m(Z_0)$. Let For : $D^b_m(Z_0) \rightarrow D^b_m(Z_0, S)$ be the forgetful functor. For each weight $w$, we consider the full subcategories $D^b_{\leq w}(Z_0)$, $D^b_{\geq w}(Z_0)$ of $D^b_m(Z_0)$ consisting of the mixed complexes of weight $w$ or $w \geq w$. The category of pure complexes of weight $w$ is

$$D^b_w(Z_0) = D^b_{\leq w}(Z_0) \cap D^b_{\geq w}(Z_0).$$

By [17] prop. 5.1.15, we have

\[(A.2) \quad \text{Hom}_{D^b(Z_0)}(E, F) = 0, \quad \forall E \in D^b_{\leq w}(Z_0), \quad \forall F \in D^b_{\geq w}(Z_0).\]

We also have the following refinement of (A.2)

\[(A.3) \quad \omega(f) = 0, \quad \forall f \in \text{Hom}_{D^b(Z_0)}(E, F), \quad \forall E \in D^b_{\leq w}(Z_0), \quad \forall F \in D^b_{\geq w}(Z_0).\]

Let $P_m(Z_0)$ be the category of mixed perverse sheaves in $D^b_m(Z_0)$. A mixed complex $E$ is pure of weight $w$, $w \leq w$ or $w \geq w$ if and only if the perverse sheaf $p^H a E$ is pure of weight $w + a$, $w \leq w + a$ or $w \geq w + a$ for each $a \in \mathbb{Z}$, by [2] thm. 5.4.1. A mixed perverse sheaf $E$ has a unique finite increasing weight filtration $W w E$, $a \in \mathbb{Z}$, such that the subquotient $Gr^w E = W w E / W w - 1 E$ is a pure mixed perverse sheaf of weight $a$, which may not be semisimple, see [4] thm. 5.3.5.

**Proposition A.2.1.** For each $E \in D^b_m(Z_0)$ and $w \in \mathbb{Z}$, there is a distinguished triangle

\[(A.4) \quad E_{\leq w} \rightarrow E \rightarrow E_{> w}, \quad E_{\leq w} \in D^b_{\leq w}(Z_0), \quad E_{> w} \in D^b_{> w}(Z_0)\]

such that $E_{> w} = 0$, $E_{> - w} = E$ if $w \gg 0$, and

- (a) there is a distinguished triangle
  \[
  E_w \rightarrow E_{> w} \rightarrow E_{> w}, \quad E_w = (E_{> w})_{\leq w} \in D^b_m(Z_0),
  \]

- (b) the long exact sequence of perverse cohomologies splits into short exact sequences
  \[
  0 \rightarrow p^H a E_{\leq w} \rightarrow p^H a E \rightarrow p^H a E_{> w} \rightarrow 0, \quad \forall a \in \mathbb{Z}.
  \]

**Proof.** Any perverse sheaf in $P_m(Z_0)$ has a finite length. The construction of $E_{\leq w}$, $E_{> w}$ is by induction on the total length of $E$, i.e., on the sum of the lengths of the perverse sheaves $p^H a E$, following the lines of [2] lem. 6.7. Given $w$, let $a$ be the smallest integer such that the subobject $W w + a(p^H a E)$ of $p^H a E$ is $\neq 0$. Set $G = W w + a(p^H a E)[-a]$. The inclusion $G \subset p^H a E[-a]$ factors to a distinguished triangle

\[(A.5) \quad G \rightarrow E \rightarrow F,\]

see [2] (6.8) for details, such that $p^H b G = 0$ for all $b \neq a$ and

\[(A.6) \quad 0 \rightarrow p^H b G \rightarrow p^H b E \rightarrow p^H b F \rightarrow 0, \quad \forall b \in \mathbb{Z}.
  \]
Hence $\mathcal{F}$ has a lower total length than $\mathcal{E}$, and induction yields a distinguished triangle (A.7) \[
\mathcal{F}_{\leq w} \to \mathcal{F} \to \mathcal{F}_{> w} \to \mathcal{F}_{\leq w} \oplus \mathcal{F}_{> w} \in D^b_{\leq w}(\mathcal{Z}_0), \quad \mathcal{F}_{> w} \in D^b_{> w}(\mathcal{Z}_0).
\]
From (A.5), (A.7) and [1, lem. 1.3.10], we get distinguished triangles (A.8) \[
\mathcal{H} \to \mathcal{E} \to \mathcal{F}_{> w} \to \mathcal{H} \to \mathcal{F}_{\leq w} \to \mathcal{H} \in D^b_{\leq w}(\mathcal{Z}_0).
\]
Set $\mathcal{E}_{\leq w} = \mathcal{H}$ and $\mathcal{E}_{> w} = \mathcal{F}_{> w}$. The induction hypothesis yields short exact sequences (A.9) \[
0 \to \mathcal{H} \to \mathcal{E} \to \mathcal{F}_{> w} \to \mathcal{H} \to \mathcal{F}_{\leq w} \to \mathcal{H} \to 0, \quad \forall b \in \mathbb{Z}.
\]
From (A.6) and (A.9), we deduce that the long exact sequence (A.10) \[
H^b \mathcal{E} \to H^b \mathcal{F} \to H^b \mathcal{F}_{> w} \to H^b(\mathcal{H}) \to H^b \mathcal{F}_{\leq w} \to H^b(\mathcal{H}) \to 0
\]
 splits into short exact sequences, yielding the condition (b). Since $\text{Hom}_{D^b(\mathcal{Z}_0)}(\mathcal{E}_{\leq w}, \mathcal{E}_{> w}) = 0$ by (A.2), the map $\mathcal{E} \to \mathcal{E}_{> w}$ factors to a morphism $\mathcal{E}_{\leq w} \to \mathcal{E}_{> w}$. Completing this morphism to a distinguished triangle yields the claim (a). □

For any mixed complexes $\mathcal{E}$, $\mathcal{F}$ on $\mathcal{Z}_0$, we write \[
\text{Hom}_{D^b(\mathcal{Z}_0)}^\bullet(\mathcal{E}, \mathcal{F}) = \bigoplus_{a \in \mathbb{Z}} \text{Hom}_{D^b(\mathcal{Z}_0)}(\mathcal{E}, \mathcal{F})[-a] \quad \text{Hom}_{D^b(\mathcal{Z}_0)}^a(\mathcal{E}, \mathcal{F}) = \text{Hom}_{D^b(\mathcal{Z}_0)}(\mathcal{E}, \mathcal{F}[a]).
\]
Let $a : \mathcal{Z}_0 \to \text{Spec} F_0$ be the structure map. We define the geometric $\text{Hom}$ functor by \[
\text{Hom}_{D^b(\mathcal{Z}_0)}^\bullet(\mathcal{E}, \mathcal{F}) = \text{a}_\ast \text{RHom}_{D^b(\mathcal{Z}_0)}(\mathcal{E}, \mathcal{F}).
\]
It is a mixed complex on $\text{Spec} F_0$. We define \[
\text{Hom}_{D^b(\mathcal{Z}_0)}^a(\mathcal{E}, \mathcal{F}) = H^a(\text{Hom}_{D^b(\mathcal{Z}_0)}^\bullet(\mathcal{E}, \mathcal{F})).
\]
It is a mixed vector space consisting of a graded $k$-vector space \[
\omega \text{Hom}_{D^b(\mathcal{Z}_0)}^a(\mathcal{E}, \mathcal{F}) = \text{Hom}_{D^b(\mathcal{Z})}^a(\omega \mathcal{E}, \omega \mathcal{F})
\]
and a Frobenius operator $\text{Fr}$. We abbreviate \[
H^\bullet(\mathcal{Z}_0, \mathcal{E}) = \text{Hom}_{D^b(\mathcal{Z}_0)}^\bullet(k_{\mathcal{Z}_0}, \mathcal{E}).
\]

A.3. Even stratifications, mixed categories and parity sheaves. Let $k_w$ denote the mixed constant sheaf in $D^b(\mathcal{Z}_w, 0)$ which is pure of weight 0. Let $i_w$ be the locally closed embedding of the stratum $\mathcal{Z}_w \subset \mathcal{Z}$. Set $\dim \mathcal{Z}_w = d_w$. Let $k_w = k_{\mathcal{Z}_w}$ be the constant sheaf on $\mathcal{Z}_w$. We define the following objects in $D^b(\mathcal{Z}, S)$ (A.10) \[
\Delta(w) = (i_w)_!k_w[d_w] \quad \nabla(w) = (i_w)_!k_w[d_w] \quad IC(w) = (i_w)_!k_w[d_w].
\]
Fix a square root of the Tate sheaf. For each $a \in \mathbb{Z}$ let $(a/2)!$ be the twist by the $a$th power of this square root. We abbreviate (A.11) \[
\ast(a/2)! = (\ast/2)!\ast.
\]
We define the following mixed complexes in $D^b_{\text{in}}(\mathcal{Z}_0, S)$ (A.12) \[
\Delta(w)_m = (i_w)_!k_w<d_w> \quad \nabla(w)_m = (i_w)_!k_w<d_w> \quad IC(w)_m = (i_w)_!k_w<d_w>.
\]
Let $D^b_{\mu,m}(Z_0,S) \subset D^b_m(Z_0,S)$ and $P_{\mu,m}(Z_0,S) \subset P_m(Z_0,S)$ be the full triangulated subcategory and the Serre subcategory generated by the set of objects \( \{ IC(w)_m(a/2); w \in W, a \in \mathbb{Z} \} \). The triangulated category $D^b_{\mu,m}(Z_0,S)$ has a t-structure whose heart is $P_{\mu,m}(Z_0,S)$.

**Definition A.3.1.** The stratification $S$ is

(a) **affine** if each stratum is isomorphic to an affine space,

(b) even affine if

1. $S$ is affine,
2. $H^a((i_u)^*IC(v)_m) = 0$ for all $u,v \in W$, $a \in \mathbb{Z}$ with $a + d_u$ odd,
3. $H^a((i_u)^*IC(v)_m)$ is a sum of copies of $k_u(-a/2)$ if $a + d_u$ is even,

(c) even if

4. there is an even affine stratification $T$ of $Z$ which refines $S$,
5. the strata of $S$ are connected and simply connected.

(d) good if $S$ is even affine and satisfies the conditions [50 §4.1(a)-(d)].

Since $IC(v)_m$ is Verdier self-dual and $Z_u$ is smooth by (1), the conditions (2) and (3) are equivalent to the following conditions

6. $H^a((i_u)^*IC(v)_m) = 0$ for all $u,v \in W$, $a \in \mathbb{Z}$ with $a + d_u$ odd,
7. $H^a((i_u)^*IC(v)_m)$ is a sum of copies of $k_u(a/2)$ if $a + d_u$ is even.

The conditions (2), (6) imply that the complex $IC(v)$ is a parity sheaf of $D^b(Z,S)$ in the sense of [29]. The conditions (3), (7) imply that the complex $IC(v)$ is very pure in the sense of [14] def. 3.1.2. They tell us in addition that the mixed vector spaces $H^a((i_u)^*IC(v)_m)$ and $H^a((i_u)^*IC(v)_m)$ are semisimple.

If the stratification $S$ is even, and $T$ is an even affine stratification which refines $S$, then there is a full embedding of triangulated categories

$$D^b(Z,S) \subset D^b(Z,T), \quad D^b_{\mu,m}(Z_0,S) \subset D^b_{\mu,m}(Z_0,T).$$

Since each stratum of $S$ contains a unique dense stratum of $T$, there is a full embedding of additive categories

$$P(Z,S) \subset P(Z,T), \quad P_{\mu,m}(Z_0,S) \subset P_{\mu,m}(Z_0,T).$$

**Definition A.3.2.**

(a) $C_m(Z_0,S) \subset D^b_m(Z_0,S)$ is the full subcategory of all mixed complexes which are isomorphic to finite direct sums of objects in $\{ IC(w)_m(a); w \in W, a \in \mathbb{Z} \}$. It is a graded additive category with the graded shift functor $\langle \bullet \rangle$ in (A.1).

(b) $D^b_\mu(Z,S) = K^b(C_m(Z_0,S))$ as a graded triangulated category with the graded shift functor $\langle \bullet \rangle$ in (A.1).

(c) $P_\mu(Z,S) \subset P_{\mu,m}(Z_0,S)$ is the full subcategory of all mixed perverse sheaves $E$ such that $Gr^W E$ is semisimple. It is a graded Abelian category for the Tate shift functor $\langle \bullet \rangle/2$.

**Proposition A.3.3.** Assume that the stratification $S$ is even.

(a) $D^b_\mu(Z,S)$ has a t-structure and a triangulated $t$-exact faithful functor

$$\iota : D^b_\mu(Z,S) \rightarrow D^b_{\mu,m}(Z_0,S).$$
The heart of $D_\mu^b(Z,S)$ is equivalent to $P_\mu(Z,S)$ and the restriction of $\iota$ to this heart is the full embedding $P_\mu(Z,S) \subset D_\mu^b(Z_0,S)$.

(b) $\zeta = \omega \circ \iota$ is a $t$-exact functor $D_\mu^b(Z,S) \to D_\mu^b(Z,S)$ such that

$$\bigoplus_{a \in \mathbb{Z}} \text{Hom}_{D_\mu^b(Z)}(\mathcal{E}, F(a/2)) = \text{Hom}_{D_\mu^b(Z)}(\zeta \mathcal{E}, \zeta F), \quad \forall \mathcal{E}, F \in D_\mu^b(Z,S).$$

(c) For any inclusion $Y \to Z$ of a union of strata of $S$, the functors $h_*, h_!, h^*$, $h^!$ between the categories $D_\mu^b(Y_0, S)$ and $D_\mu^b(Z_0, S)$ lift to triangulated functors between the categories $D_\mu^b(Y, S)$ and $D_\mu^b(Z, S)$ which satisfy the usual adjointness properties.

Proof. Part (a) is proved in [2, §7.2, (2)] and Proposition A.3.3, since for any $\mathcal{E}, F \in D_\mu^b(Z, S)$, we have $D_\mu^b(Z_0,S) \subset D_\mu^b(Z,T)$ for each affine refinement $T$ of $S$, we can assume that $S$ is even affine. Part (a) follows from [2, lemma 7.8, (2)], which also implies that the mixed complex $\text{Hom}_{D_\mu^b(Z_0)}(\iota \mathcal{E}, \iota F)$ is semisimple for each objects $\mathcal{E}, F \in D_\mu^b(Z, S)$. Hence, for each $a, b \in \mathbb{Z}$, we have

\begin{equation}
\text{Hom}_{D_\mu^b(Z)}^a(\mathcal{E}, F(b/2)) = (\text{Hom}_{D_\mu^b(Z_0)}^a(\iota \mathcal{E}, \iota F(b/2)))^{Fr},
\end{equation}

and to prove (b) it is enough to check that $\text{Hom}_{D_\mu^b(Z_0)}^a(\iota \mathcal{E}, \iota F)$ is pure of weight 0 whenever $\mathcal{E}, F \in C_\mu(Z_0, S)$. The mixed Abelian category $P_\mu(Z,S)$ is Koszul by [8, theorem 4.4.4]. Hence, if $\mathcal{E}, F \in P_\mu(Z, S)$ are pure of weight zero, we have

\begin{equation}
b \neq a \implies \text{Hom}_{D_\mu^b(Z)}^a(\mathcal{E}, F(b/2)) = 0.
\end{equation}

So, the mixed vector space $\text{Hom}_{D_\mu^b(Z_0)}^a(\iota \mathcal{E}, \iota F)$ is pure of weight $a$, so the mixed complex $\text{Hom}_{D_\mu^b(Z)}^a(\iota \mathcal{E}, \iota F)$ is pure of weight 0, proving (b) because any object of $C_\mu(Z_0, S)$ is a sum of $IC(w)_m(a)$’s. Part (c) follows from (A.13) and Proposition A.3.3, since for any $\mathcal{E}, F \in C_\mu(Z_0, S)$ we have

$$\text{Hom}_{D_\mu^b(Z)}^a(\mathcal{E}, F) = \bigoplus_{a \in \mathbb{Z}} \text{Hom}_{D_\mu^b(Z)}(\mathcal{E}, F(a/2)) = \text{Hom}_{D_\mu^b(Z)}(\zeta \mathcal{E}, \zeta F).$$

For each $w \in W$, let $IC(w)_\mu$ be $IC(w)_m$ viewed as an object of $D_\mu^b(Z, S)$. Assume that the stratification $S$ is even. By Proposition A.3.3, we have $D_\mu^b(Z, S) = K^b(C(Z, S))$. This identification takes $IC(w)_\mu$ to $IC(w)$. The grading $K^b(C(Z, S))$ is given by the shift functor $\langle \bullet \rangle$ on $C(Z, S)$ in (A.1).
We define the equivariant mixed category of the stack $\mathcal{Z} = [Z / G]$ by $D^b_\mu(Z) = K^b(C(Z))$. Let $S$ be the stratification by the $G$-orbits. We have the forgetful functor $F : D^b_\mu(Z) \to D^b(Z, S)$. We do not know any equivariant analogue of Proposition A.3.4. However, the following holds, see e.g. [14, lem. 3.1.5].

Proposition A.3.5. Assume that the $G_0$-orbits in $Z_0$ are affine. If $\mathcal{E}, \mathcal{F} \in D^b_{\text{perf}}(Z_0)$ are very pure of weight 0, then the mixed complex $\text{Hom}_{D^b_{\text{perf}}(Z_0)}(\mathcal{E}, \mathcal{F})$ in $D^+(\text{Spec } F_0)$ is pure of weight 0 and it is free of finite rank as an $H^*_G$-module. \qed

Remark A.3.6.

(a) Let $S$ be any stratification of $Z$. The category $C(Z, S)$ has split idempotents, and the Verdier duality $D$ yields an equivalence $C(Z, S) \cong C(Z, S)_{\text{op}}$. Let $\mathcal{L} \in C(Z, S)$ be a graded-generator. Set $R = \text{End}_{D^b(Z, S)}(\mathcal{L})_{\text{op}}$. The functor $\mathcal{E} \mapsto \text{Hom}_{D^b(Z, S)}(\mathcal{L}, \mathcal{E})$ gives an equivalence of graded additive categories $C(Z, S) \to R_{\text{proj}}$. Taking the homotopy categories, we get a graded triangulated equivalence $K^b(C(Z, S)) \to D^b_{\text{perf}}(R)$.

(b) If $h : Y \to Z$ is a closed embedding then the functor $h_1 = h_\ast : D^b_\mu(Y) \to D^b_\mu(Z)$ in Proposition A.3.3 is given by restricting the functor $h_1 = h_\ast : D^b(Y) \to D^b(Z)$ to $C(Y) \subset C(Z)$ and taking the homotopy categories. If $h : Y \to Z$ is an open embedding then the functor $h^! = h^\ast$ is defined in a similar way. We do not know any equivariant analogue of Proposition A.3.3 which would yield functors $h_\ast, h_1, h^!, h^\ast$ between the categories $D^b_\mu(Y)$ and $D^b_\mu(Z)$ for any inclusion $h : Y \to Z$ of a union of strata. However the functors $h_\ast = h_1$ are well defined in the equivariant case if $h$ is a closed embedding, so are $h^! = h^\ast$ if $h$ is an open embedding.

(c) If the stratification $S$ is even then the set $\{IC(w)[a] : w \in W, a \in Z\}$ is a complete and irredundant set of indecomposable objects of $C(Z, S)$. It is also a complete and irredundant set of parity sheaves of $D^b(Z, S)$ in the sense of [20].

(d) A triangulated functor $\phi_m : D^b_{\text{perf}}(Y_0) \to D^b_{\text{perf}}(Z_0)$ is geometric if there is a triangulated functor $\phi : D^b(Y) \to D^b(Z)$ with a natural isomorphism $\phi_\omega = \omega \phi_m$. It is genuine if it is geometric and there is a triangulated functor $\phi_\mu : D^b_{\mu}(Y) \to D^b_{\mu}(Z)$ with a natural isomorphism $\phi_\mu = \mu \phi_m$.

(e) Let $T, V$ be even affine refinements of even stratifications $S$, $U$ of $F_0$-schemes $Y_0$, $Z_0$. By [21, lem. 7.12], there are full embedding of triangulated categories $D^b_{\text{perf}}(Y_0, S) \subset D^b_{\text{perf}}(Y_0, T)$ and $D^b_{\text{perf}}(Z_0, U) \subset D^b_{\text{perf}}(Z_0, V)$. By [21, lem. 7.21], the restriction of a genuine functor $D^b_{\text{perf}}(Y_0, T) \to D^b_{\text{perf}}(Z_0, V)$ that takes $D^b_{\text{perf}}(Y_0, S)$ into $D^b_{\text{perf}}(Z_0, U)$, is a genuine functor $D^b_{\text{perf}}(Y_0, S) \to D^b_{\text{perf}}(Z_0, U)$.

(f) By Proposition A.3.3 if the stratification $S$ is even we may view $D^b_\mu(Z, S)$ as a (non full) subcategory of $D^b_{\text{perf}}(Z_0, S)$ consisting of objects whose stalks carry a semisimple action of the Frobenius.

(g) Any object $E$ of $C(Z_0, S)$ is semisimple and pure of weight 0, that is $E \simeq \bigoplus_a p^H\mu(a)(E)[-a]$ where each mixed pervers sheaf $p^H\mu(a)(E)$ is pure of weight $a$.

(h) An even stratification $S$ is called affable in [21, def. 7.2]. Then, the category $D^b_{\text{Weil}}(Z_0, S)$ is the same as $D^b_{\text{Weil}}(Z_0)$ in [21, §6.1]. If $S$ is even affine, then $D^b_{\text{Weil}}(Z_0, S)$ is the category $D^b_{\text{Weil}}(Z_0)$ in [50, §2.1].
A.4. Even affine stratifications, projective and tilting objects. Assume that the stratification $S$ is even affine. The objects $\Delta(w)_m$ and $\nabla(w)_m$ have canonical lifts $\Delta(w)_\mu$ and $\nabla(w)_\mu$ in $P_\mu(Z, S)$ by Proposition A.3.3. We have
\[
\iota IC(w)_\mu = IC(w)_m, \quad \iota \Delta(w)_\mu = \Delta(w)_m, \quad \iota \nabla(w)_\mu = \nabla(w)_m, \quad \forall w \in W.
\]
We equip the triangulated category $D^b(P_\mu(Z, S))$ with the grading shift functors
\[
(A.15) \quad \langle \bullet \rangle = (\langle \bullet \rangle/2)\langle \bullet \rangle,
\]
where $(\langle \bullet \rangle/2)$ is the Tate shift functor on $P_\mu(Z, S)$ and $\langle \bullet \rangle$ is the cohomological shift. By [2] cor. 7.10, there is an equivalence of graded triangulated categories $D^b(P_\mu(Z, S)) \to D^b(Z, S)$ which identifies the grading shift functors (A.11) and (A.15). We’ll use two refinements of this equivalence which involve projective and tilting objects of $P_\mu(Z, S)$.

**Proposition A.4.1.** Assume that the stratification $S$ is even affine.
(a) $P_\mu(Z, S)$, $P(Z, S)$ have enough projectives and finite cohomological dimension. The sets of indecomposable objects in $P(Z, S)^{proj}$ and $P_\mu(Z, S)^{proj}$ are $\{P(w); w \in W\}$ and $\{P(w)_\mu(a/2); w \in W, a \in \mathbb{Z}\}$, where $P(w), P(w)_\mu$ are the projective covers of $\Delta(w), \Delta(w)_\mu$ in $P(Z, S)$, $P_\mu(Z, S)$.
(b) $P_\mu(Z, S)^{proj} = \zeta^{-1}(P(Z, S)^{proj})$.
(c) $P_\mu(Z, S)^{proj} \subset D^b(Z, S)$ extends to a graded triangulated equivalence $K^b(P_\mu(Z, S)^{proj}) \to D^b(Z, S)$.

**Proof.** Part (a) is [2] prop. 7.7(1),(2), part (b) is [2] prop. 7.7(2), and part (c) is proved [2] cor. 7.10, prop. 7.11.

**Definition A.4.2 (9, 50).** Assume that the stratification $S$ is even affine. A mixed perverse sheaf $\mathcal{E} \in P_{\mu, m}(Z_0, S)$ is tilting if either of the following equivalent conditions hold
(a) $(i_w)^*\mathcal{E}$ and $(i_w)_!\mathcal{E}$ are perverse for each $w \in W$.
(b) $\mathcal{E}$ has both a filtration by $\Delta(w)_m(a/2)$’s and by $\nabla(w)_m(a/2)$’s, with $w \in W$ and $a \in \mathbb{Z}$.

We define a tilting object in $P(Z, S)$ in a similar way.

Let $P(Z, S)^{tilt} \subset P(Z, S)$ and $P_{\mu, m}(Z_0, S)^{tilt} \subset P_{\mu, m}(Z_0, S)$ be the full additive subcategories of tilting objects. Let $P_\mu(Z, S)^{tilt} \subset P_\mu(Z, S)$ be the full subcategory whose objects are the complexes which map to $P_{\mu, m}(Z_0, S)^{tilt}$ by the functor $\iota$.

**Proposition A.4.3.** Assume that the stratification $S$ is even affine.
(a) For each $w \in W$, there are unique indecomposable objects $T(w), T(w)_\mu$ in $P(Z, S)^{tilt}$, $P_\mu(Z, S)^{tilt}$ supported on $X_w$ whose restriction to $X_w$ are $k_w[d_w], k_w(d_w)$ respectively. The sets of indecomposable objects in $P(Z, S)^{tilt}$, $P_\mu(Z, S)^{tilt}$ are $\{T(w); w \in W\}$ and $\{T(w)_\mu(a/2); w \in W, a \in \mathbb{Z}\}$.
(b) Assume that the stratification $S$ is good. Then, we have $P_\mu(Z, S)^{tilt} = P_{\mu, m}(Z_0, S)^{tilt}$.
(c) $P_\mu(Z, S)^{tilt} = \zeta^{-1}(P(Z, S)^{tilt})$.
(d) $P_\mu(Z, S)^{tilt} \subset D^b(Z, S)$ extends to a graded triangulated equivalence $K^b(P_\mu(Z, S)^{tilt}) \to D^b(Z, S)$. 

Proof. Part (a) is [2, prop. 10.3], [9]. Part (b) is [50, § 4.2]. Part (c) is obvious. Part (d) is [2, prop. 10.5]. □

References

[1] Achar, P. N., Perverse coherent sheaves on the nilpotent cone in good characteristic, in Recent Developments in Lie Algebras, Groups and Representation Theory, Proc. Sympos. Pure Math., 86, 1-23. Amer. Math. Soc., Providence, RI, 2012.

[2] Achar, P. N., Riche, S., Koszul duality and semisimplicity of Frobenius, Ann. Inst. Fourié, Grenoble 63 (2013), 1511-1612.

[3] Achar, P. N., Riche, S., Modular perverse sheaves on flag varieties II: Koszul duality and formality, Duke Math. J. 165 (2016), 161-215.

[4] Achar, P. N., Rider, L., Parity sheaves on the affine Grassmannian and the Mirković-Vilonen conjecture, Acta Math. 215 (2015), 183-216.

[5] Achar, P. N., Rider, L., The affine Grassmannian and the Springer resolution in positive characteristic, Compositio Math. 152 (2016), 2627-2677.

[6] Andersen, H. H., Jantzen, J. C., Soergel, W., Representations of Quantum Groups at a p-th Root of Unity and of Semisimple Groups in Characteristic p: Independence of p. Astérisque, 220 (1994), pp. 321.

[7] Beilinson, A. A., Bernstein, J., Deligne, P., Faisceaux pervers, Analysis and topology on singular spaces, I (Luminy, 1981), 5-171, Astérisque, 100, Soc. Math. France, Paris, 1982.

[8] Beilinson, A. A., Ginzburg, V., Soergel, W., Koszul duality patterns in representation theory, J. Amer. Math. Soc. 9 (1996), 473-527.

[9] Beilinson, A., Bezrukavnikov, R., Mirkovic, I., Tilting exercises, Mosc. Math. J. 4 (2004), 547-557.

[10] Bernstein, J., Lunts, V., Equivariant sheaves and functors, Lecture Notes in Mathematics 1578, Springer-Verlag, 1994.

[11] Bezrukavnikov, R., Quasi-exceptional sets and equivariant coherent sheaves on the nilpotent cone, Represent. Theory, 7 (2003), 1-18.

[12] Bezrukavnikov, R., On two geometric realizations of an affine Hecke algebra, Publ. Math. Inst. Hautes Études Sci. 123 (2016), 1-67

[13] Bezrukavnikov, R., Finkelberg, M., Mirkovic, I., Equivariant homology and K-theory of affine Grassmannians and Toda lattices, Compos. Math.141 (2005), 746-768.

[14] Bezrukavnikov, R., Yun, Z., On Koszul duality for Kac-Moody groups, Represent. Theory 17 (2013), 1-98.

[15] Bridgeland, T., An introduction to motivic Hall algebras, Adv. Math. 229 (2012), 102-138.

[16] Cautis, S., Williams, H., Cluster theory of the coherent Satake category. [arXiv:1801.08111]

[17] Edelman, A., Elmroth, E., Kagstrom, B., A geometric approach to perturbation theory of matrices and matrix pencils, part II: a stratification-enhanced staiarce algorithme, SIAM J. Matrix Anal. Appl. 20 (1999), 667-699.

[18] Finkelberg, M., Fujita, R, Coherent IC-sheaves on type An affine Grassmannians and dual canonical basis of affine type A1, [arXiv:1901.05994]

[19] Ginzburg, V., Perverse sheaves and C*-actions, J. Amer. Math. Soc. 4 (1991), 483-490.

[20] Juteau, D., Mautner, C., Williamson, G., Parity sheaves, J. Amer. Math. Soc. 27, 1169-1212 (2014).

[21] Kang, S.-J., Kashiwara, M., Kim, M., Symmetric quiver Hecke algebras and R-matrices of quantum affine algebras, Invent. Math. 211 (2018), 591-685.

[22] Kang, S.-J., Kashiwara, M., Kim, M., Oh, S.-J., Monoidal categorification of cluster algebras, J. Amer. Math. Soc. 31 (2018), 349-426.

[23] Kashiwara, M., Kim, M., Oh, S.-J., Park, E., Monoidal categories associated with strata of flag manifolds, Adv. Math. 328 (2018), 959-1009.

[24] Kashiwara, M., Kim, M., Oh, S.-J., Park, E., Localizations for quiver Hecke algebras, [arXiv:1901.09319]

[25] Kashiwara, M., The flag manifold of Kac-Moody Lie algebra, in: Algebraic Analysis, Geometry, and Number Theory (Baltimore, MD, 1988), Johns Hopkins Univ. Press, Baltimore, MD, 1989, 161-190.
[26] Kashiwara, M., Shimozono, M., Equivariant K-theory of affine flag manifolds and affine Grothendieck polynomials. Duke Math. J. 148, 501-538 (2009)

[27] Kashiwara, M., Tanisaki, T., Parabolic Kazhdan-Lusztig polynomials and Schubert varieties, J. Algebra 249 (2002), 306-325.

[28] Kato, S., An algebraic study of extension algebras, Amer. J. Math. 139 (2017), 567-615.

[29] Kleshchev, A., Affine highest weight categories and affine quasihereditary algebras, Proc. Lond. Math. Soc. 110 (2015), 841-882.

[30] Kleshchev, A., Muth, R., Stratifying KLR algebras of affine ADE types, J. Algebra 475 (2017), 133-170.

[31] Kumar, S., Positivity in T-equivariant K-theory of flag varieties associated to Kac-Moody groups, J. Eur. Math. Soc. 19 (2017), 2469-2519.

[32] Kiehl, R., Weissauer, R., Weil conjectures, perverse sheaves and l’adic Fourier transform. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 42. Springer-Verlag, Berlin, 2001.

[33] Lafforgue, V., Quelques calculs reliés à la correspondance de Langlands géométrique pour $\mathbb{P}^1$, unpublished.

[34] Laszlo, Y., Olsson, M., Perverse t-structure on Artin stacks, Math. Z. 261 (2009), 737-748.

[35] Laumon, G., Faisceaux automorphes liés aux séries d’Eisenstein. (French) [Automorphic sheaves associated with Eisenstein series] Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988), 227-281, Perspect. Math., 10, Academic Press, Boston, MA, 1990.

[36] Lusztig, G., Affine quivers and canonical bases, Publ. Math. Inst. Hautes Études Sci. 76 (1992), 111-162.

[37] Lusztig, G., Canonical bases and Hall algebras, A. Broer and A. Daigneault(eds.), Representation Theories and Algebraic Geometry, 365-399, 1998 Kluwer Academic Publishers.

[38] Lusztig, G., Introduction to quantum groups. Reprint of the 1994 edition. Modern Birkhäuser Classics. Birkhäuser/Springer, New York, 2010.

[39] Mautner, C., Riche, S., On the exotic t-structure in positive characteristic, Int. Math. Res. Notices 18 (2016), 5727-5774.

[40] McNamara, P., Representations of Khovanov-Lauda-Rouquier algebras III: symmetric affine type, Math. Z. (2017), 243-286.

[41] Minn-Thu-Aye, M., Multiplicity formulas for Perverse Coherent Sheaves on the Nilpotent Cone, Ph. D. Thesis, Louisiana State University, Baton Rouge, LA, 2013.

[42] Riche, S., Kostant section, universal centralizer, and a modular derived Satake equivalence, Math. Z. 286 (2017), 223-261.

[43] Rouquier, R., Shan, P., Varagnolo, M., Vasserot, E., Coherent categorification of quantum loop algebras: the monoidality (in preparation).

[44] Schiffmann, O., Vasserot, E., Hall algebras of curves, commuting varieties and Langlands duality, Math. Ann. 353 (2012), 1399-1451.

[45] Schiffmann, O., Vasserot, E., The elliptic Hall algebra and the K-theory of the Hilbert scheme of $A^2$, Duke Math. J. 162 (2013), 279-366.

[46] Schiffmann, O., Vasserot, E., On cohomological Hall algebras of quivers: Generators, J. reine angew. Math. (to appear), arXiv:1207.0574.

[47] Sun, S., Decomposition theorem for perverse sheaves on Artin stacks over finite fields, Duke Math. J. 161 (2012), 2297-2310.

[48] Varagnolo, V., Vasserot, E., Finite-dimensional representations of DAHA and affine Springer fibers: the spherical case, Duke Math. J. 147 (2009), 439-540.

[49] Varagnolo, V., Vasserot, E., Canonical bases and KLR-algebras. J. Reine Angew. Math. 659 (2011), 67-100.

[50] Yun, Z., Weights of mixed tilting sheaves and geometric Ringel duality. Selecta Math. (N.S.) 14 (2009), 299-320.

[51] Zhu, X., An introduction to affine Grassmannians and the geometric Satake equivalence. Geometry of moduli spaces and representation theory, 59-154, IAS/Park City Math. Ser., 24, Amer. Math. Soc., Providence, RI, 2017.