GENERALIZED CALABI-YAU MANIFOLDS
AND THE MIRROR OF A RIGID MANIFOLD

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ABSTRACT

The \( \mathcal{Z} \) manifold is a Calabi–Yau manifold with \( b_{21} = 0 \). At first sight it seems to provide a counter example to the mirror hypothesis since its mirror would have \( b_{11} = 0 \) and hence could not be Kähler. However by identifying the \( \mathcal{Z} \) manifold with the Gepner model \(^9\) we are able to ascribe a geometrical interpretation to the mirror, \( \mathcal{Z} \), as a certain seven-dimensional manifold. The mirror manifold \( \tilde{\mathcal{Z}} \) is a representative of a class of generalized Calabi–Yau manifolds, which we describe, that can be realized as manifolds of dimension five and seven. Despite their dimension these generalized Calabi–Yau manifolds correspond to superconformal theories with \( c = 9 \) and so are perfectly good for compactifying the heterotic string to the four dimensions of space-time. As a check of mirror symmetry we compute the structure of the space of complex structures of the mirror \( \tilde{\mathcal{Z}} \) and check that this reproduces the known results for the Yukawa couplings and metric appropriate to the Kähler class parameters on the \( \mathcal{Z} \) orbifold together with their instanton corrections. In addition to reproducing known results we can calculate the periods of the manifold to arbitrary order in the blowing up parameters. This provides a means of calculating the Yukawa couplings and metric as functions also to arbitrary order in the blowing up parameters which is difficult to do by traditional methods.

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1. Introduction

The existence of mirror symmetry among Calabi–Yau manifolds is the fact that, roughly speaking, Calabi–Yau manifolds come in so-called mirror pairs. The mirror operation can be thought of as a reflection of the Hodge diamond for a manifold $\mathcal{M}$

\[
\begin{array}{ccc}
1 & 0 & 0 \\
0 & b_{11} & 0 \\
0 & b_{21} & b_{21} & 1 \\
0 & b_{11} & 0 \\
0 & 0 & 0 \\
1
\end{array}
\]

about a diagonal axis. The effect is to exchange the values of $b_{11}$ and $b_{21}$ so that the corresponding Hodge numbers for the mirror $\mathcal{W}$ are given by

\[
b_{11}(\mathcal{W}) = b_{21}(\mathcal{M}), \quad b_{21}(\mathcal{W}) = b_{11}(\mathcal{M}).
\]  

A better statement of mirror symmetry is that Calabi–Yau manifolds are realizations of N=2 superconformal field theories and that a given SCFT can be realized as a Calabi–Yau manifold in two different ways $\mathcal{M}, \mathcal{W}$, whose Hodge numbers are related by (1.1). In the underlying SCFT there is no natural way to decide which operators correspond to (1,1)-forms and which to (2,1)-forms in an associated Calabi–Yau manifold. It was this that led to the conjecture of the existence of mirror symmetry whereby each SCFT would correspond to a pair of Calabi–Yau manifolds in which the roles of these two types of forms would be exchanged [1,2].

It is an essential fact that Calabi–Yau manifolds have parameters corresponding to the possible deformations of the complex structure and of the Kähler class. Infinitesimal deformations of the complex structure of $\mathcal{M}$ are in one-one correspondence with the elements of the cohomology group $H^{21}(\mathcal{M})$ while the infinitesimal deformations of the Kähler class are in one-one correspondence with the elements of $H^{11}(\mathcal{M})$. Under mirror symmetry, these two parameter spaces are exchanged. The realization that both types of parameter space have the same structure, both being described by special geometry [3,4], lent strong evidence to the mirror symmetry hypothesis. A construction of a large class of Calabi–Yau manifolds revealed that the great bulk of the manifolds so constructed occur in mirror pairs [5]. Contemporaneous with this was the construction of Greene and Plesser [6] who, by means of exploiting the correspondence between certain manifolds and the Gepner models, were able to provide a construction of the mirrors for these cases. A subsequent calculation by Aspinwall, Lütken and Ross [7] identified a large complex
structure limit in which, for a certain mirror pair, the Yukawa coupling appropriate to the complex structures of the mirror goes over to the topological value of the Yukawa coupling appropriate to the Kähler class of the original manifold.

The precise circumstances under which mirror symmetry is true are not known and there is currently no generally applicable procedure for constructing the mirror of a given manifold, though a number of procedures are applicable to special classes of manifolds [5,6,7,8]. There is also, at first sight, an immediate class of counter examples furnished by the rigid manifolds. These are manifolds for which $b_{21}(\mathcal{M}) = 0$. The mirror would seemingly have to have $b_{11}(\mathcal{W}) = 0$ and hence could not be Kähler. The prototypical example of a rigid Calabi–Yau manifold is the so-called $\mathcal{Z}$ manifold [9] for which $b_{21} = 0$ and $b_{11} = 36$. It turns out, however, that we can construct a mirror for the $\mathcal{Z}$ as a seven–dimensional manifold with positive first Chern class. We will see that the parameter space of complex structures of the mirror $\tilde{\mathcal{Z}}$ is described by special geometry and that we can obtain the quantum corrections to the Yukawa couplings and the kinetic terms of the low energy theory that results from the compactification of string theory on the $\mathcal{Z}$ manifold by studying the space of complex structures of $\tilde{\mathcal{Z}}$. The $\mathcal{Z}$ manifold has an orbifold limit which has been studied extensively and the instanton contributions to the Yukawa couplings have already been computed [10-12]. For the orbifold limit we derive nothing new except that we check that we recover the standard results by means of mirror symmetry. We can also go further than has hitherto proved possible with traditional methods since mirror symmetry allows us to find the Yukawa couplings, say, even away from the orbifold limit. We do not attempt a full treatment involving all 27 parameters associated with the resolution of the singularities; rather, we allow just one of these 27 parameters to vary away from the orbifold limit. It is clear that more complicated cases involving more parameters are amenable to study however our main interest here is to check that mirror symmetry is applicable and gives correct answers.

The paper is organized as follows: in Section 2 we describe the $\mathcal{Z}$ manifold, its mirror $\tilde{\mathcal{Z}}$ and the class of generalized Calabi–Yau manifolds to which it belongs. In Section 3 we calculate the Yukawa couplings of $\tilde{\mathcal{Z}}$ by means of a calculation in the ring of the defining polynomial following a method described in [13]. The issue of the normalization of the couplings is addressed via the techniques of special geometry. For this we need to find an integral basis for the periods on the manifold. In Section 5 we calculate periods both at and away from the orbifold limit. In the orbifold limit the close relation between $\tilde{\mathcal{Z}}$ and the product of tori can be used to find the integral basis. Away from this limit we must check that the modular group acts on the period vector by integral matrices. This requirement however does not fix the basis uniquely and we are obliged to describe explicitly a homology basis. Being somewhat technical this part of the analysis is relegated to an appendix. We investigate the actions of the modular group in Section 4, and we settle on a proper basis for the periods in Section 5.5. Finally in Section 6 we calculate the normalized Yukawa couplings on $\tilde{\mathcal{Z}}$ and compare to previous calculations. The reason that we make the comparison with the orbifold limit only after this lengthy discussion of the properties of
the manifold away from the orbifold limit is that the most involved part of the calculation is the computation of the normalization factors. For example, if $s$ denotes a parameter associated with blowing up the orbifold singularities then the proper normalization of the Yukawa coupling $y_{sss}$, say, involves a knowledge of the metric component $g_{ss}$ and this in turn requires the computation of the Kähler potential and hence the periods as functions of $s$.

2. The $\mathcal{Z}$ Manifold, its Mirror and a Class of Generalized Calabi-Yau Manifolds

Recall the construction of the $\mathcal{Z}$ manifold [9,14]. Let $\mathcal{T}$ be the product of 3 tori

$$\mathcal{T} = \mathcal{T}_1 \times \mathcal{T}_2 \times \mathcal{T}_3,$$

with each torus formed by making the identifications

$$z_i \simeq z_i + 1 \simeq z_i + \omega^{1/2}, \quad \omega = e^{2\pi i/3}.$$

The Euler number of $\mathcal{T}$ is zero. If we divide $\mathcal{T}$ by the $\mathbb{Z}_3$ generated by $z_i \to \omega z_i$, we obtain the $\mathcal{Z}$ orbifold. Each torus has the three fixed points $r\sqrt{3}\omega^{1/4}$, $r = 0, 1, 2$, and $\mathcal{T}$ has the 27 fixed points

$$f_{mnp} = (m, n, p)\frac{\omega^{1/4}}{\sqrt{3}}, \quad m, n, p = 0, 1, 2.$$

We delete the fixed points and glue in appropriate Eguchi-Hansen balls (which have $\chi = 3$) to obtain the $\mathcal{Z}$ manifold. The Euler number of the $\mathcal{Z}$ is therefore

$$\chi(\mathcal{Z}) = \frac{0 - 27}{3} + 27 \cdot 3 = 72.$$

It has Hodge numbers $b_{11}(\mathcal{Z}) = 36$ and $b_{21}(\mathcal{Z}) = 0$. The counting is that 9 of the (1,1)–forms can be thought of as the forms $e_i = dz^i \wedge d\bar{z}^j$ that descend from $\mathcal{T}$. The other 27 (1,1)–forms come from blowing up the fixed points $f_{mnp}$.

The $\mathcal{Z}$ manifold corresponds to the $1^9$ Gepner model [15-17] and corresponds to a Landau-Ginzburg potential

$$p = \sum_{k=1}^{9} y_k^3.$$

It is natural to think of the nine $y_k$ as the homogeneous coordinates of $\mathbb{P}_8$ and therefore to think of the mirror manifold $\tilde{\mathcal{Z}}$ as $\mathbb{P}_8[3]$, a hypersurface of degree 3 in $\mathbb{P}_8$. This is a seven–dimensional manifold. It is of interest to compute the Hodge decomposition of the middle cohomology, $H^7$. In the following we shall be concerned with the quotient of $\mathbb{P}_8[3]$ by various groups $G$. The Hodge decomposition of $H^7$ for $\mathbb{P}_8[3]/G$ is

$$H^7 : 0 \quad 0 \quad 1 \quad \beta \quad \beta \quad 1 \quad 0 \quad 0 \quad .$$
The dimension of $H^{5,2}$ is always 1 while the dimension of $H^{4,3}$ assumes various values $\beta$ depending on $G$. For the case of $\mathbb{P}_8[3]$ itself $\beta = 84$. The important point is that the boxed entries have the same form as the Hodge decomposition of $H^3$ for a Calabi–Yau manifold. In particular, $\mathbb{P}_8[3]$ has a unique $(5,2)$–form $\Omega_{5,2}$ which is the analogue of the holomorphic three–form $\Omega_{3,0}$ familiar from the theory of Calabi–Yau manifolds. In the study of the parameter space of the complex structures of a Calabi–Yau manifold interest focusses on the variation of the periods of the holomorphic three–form $\Omega_{3,0}$ [3,4]. In this way one learns that the space of complex structures is described by special geometry. The same is true for the space of complex structures of our sevenfold $\mathbb{P}_8[3]$ owing to the fact that there is a unique $(5,2)$–form and the $(4,3)$–forms correspond to the variation of $\Omega_{5,2}$ with respect to the complex structure. To understand the existence of $\Omega_{5,2}$ recall the following construction [5] of the holomorphic three–form for a Calabi–Yau manifold that is presented as $\mathbb{P}_d^{(k_1,\ldots,k_5)}[d]$. This is the vanishing locus of a polynomial $p$ in a weighted projective space $\mathbb{P}_d$ with coordinates $(y_1, \ldots, y_5)$ that have weights $(k_1, \ldots, k_5)$ i.e. the coordinates are identified

$$(y_1, y_2, \ldots, y_5) \simeq (\lambda^{k_1}y_1, \lambda^{k_2}y_2, \ldots, \lambda^{k_5}y_5) \quad (2.1)$$

for any $\lambda \neq 0$. We set

$$\mu = \frac{1}{4!}\epsilon_{A_1A_2A_3A_4A_5}y^{A_1}dy^{A_2}dy^{A_3}dy^{A_4}dy^{A_5}$$

and construct $\Omega_{3,0}$ as a residue by dividing $\mu$ by $p$ and taking an integral around a one-dimensional contour $C_p$ that winds around the hypersurface $p = 0$ in the embedding $\mathbb{P}_d$

$$\Omega_{3,0} = \frac{1}{2\pi i} \int_{C_p} \frac{\mu}{p}.$$

This construction makes sense because $\mu/p$ is invariant under the scaling (2.1). Invariance under scaling requires the degree of $p$ to be related to the weights by

$$\sum_{j=1}^{5} k_j = d$$

but this is precisely the condition of vanishing first Chern class.

For the case of $\mathbb{P}_8[3]$ we take

$$\mu = \frac{1}{8!}\epsilon_{A_1A_2\ldots A_9}y^{A_1}dy^{A_2}\ldots dy^{A_9}$$

so $\mu$ scales with $\lambda^9$. Since the cubic $p$ scales with $\lambda^3$ we must divide $\mu$ by $p^3$ to obtain a form invariant under the scaling. Thus we construct

$$\Omega_{5,2} = \frac{1}{2\pi i} \int_{C_p} \frac{\mu}{p^3} \quad (2.2)$$
It remains to explain why $\Omega_{5,2}$ is a (5,2)–form. This is due to the fact that we have a third order pole. A first order pole would have produced a (7,0)–form and a second order pole would have produced a (6,1)–form. A more correct statement is that the residue formula (2.2) is true in cohomology and that the (7,0) and (6,1) parts of the residue are exact.

The chiral ring can now be built up as in the previous case. If $q$, $r$, and $s$ are three cubics then $\frac{1}{2\pi i} \int \frac{\mu qr s}{p^6}$ is a (4,3)–form, $\frac{1}{2\pi i} \int \frac{\mu qr}{p^5}$ is a (3,4)–form and $\frac{1}{2\pi i} \int \frac{\mu qr s}{p^6}$ is a multiple of the unique (2,5)–form. If we write $\langle \Omega, \overline{\Omega} \rangle$ for $-i \int \Omega \wedge \overline{\Omega}$ then we have the familiar expression for the Yukawa coupling, $\kappa(q,r,s)$, of the polynomials

$$\int_{C_p} \frac{\mu qr s}{p^6} \bigg|_{2,5} = \kappa(q,r,s) \frac{\overline{\Omega}}{\langle \Omega, \overline{\Omega} \rangle}.$$ 

The point that is being made is a general one not restricted to the mirror of the $\mathcal{Z}$ manifold. Consider $\mathbb{P}(k_1, \ldots, k_{2m+3})[d]$, a hypersurface in a weighted projective space of dimension $2m+2$ defined by a polynomial $p$ of degree $d$ with the degree related to the weights by

$$\sum_{j=1}^{2m+3} k_j = md. \tag{2.3}$$

The middle dimensional cohomology of this manifold has the structure

$$H^{2m+1} : \begin{array}{ccccccc} 0 \cdots 0 & 1 & \beta & \beta & \overline{1} & 0 \cdots 0 \\ m-1 & & & & m-1 & & \\ \end{array}$$

so we again find the special geometry structure sitting within it. The reason is as before: we have a unique $(m+2, m-1)$–form given by the residue formula $\Omega = \frac{1}{2\pi i} \int \frac{\mu}{p^m}$. In the language of SCFT, Equation (2.3) is the statement that these theories have $c = 9$ and so are consistent string compactifications (even though the dimension of the manifold is $2m+1$ which is not 3 unless we are dealing with the traditional case of $m = 1$). There are many such manifolds. In [18,19] the authors compile an exhaustive list of 3,284 such spaces for the cases $m = 2, 3$ and observe that higher values of $m$ yield essentially nothing new. The reason is the fact that the extra variables enter the defining polynomial only as quadratic terms and these are trivial if the polynomial is thought of as a Landau–Ginzburg potential. A caveat to this is that the extra quadratic terms can introduce $\mathbb{Z}_2$ torsion[20]. It is an interesting and important question whether it is possible to associate a three–dimensional manifold with each member of this class. For recent results along these lines see [21,22]. We have come across this class of vacua by considering the mirror of the $\mathcal{Z}$ manifold. Only a few members of this class are the mirrors of rigid manifolds but it is not surprising to find that the mirror of a rigid manifold is one of the generalized Calabi–Yau manifolds since it could not, after all, be a traditional Calabi–Yau manifold. The notion that this class of
manifold is important to the consideration of the mirrors of rigid Calabi–Yau manifolds was known independently to Vafa [23].

In virtue of the identification of the $\mathcal{Z}$ manifold with the SCFT $1^9$, together with the work of [24], we know that the mirror of the $\mathcal{Z}$ is the quotient of $\mathbb{P}_8[3]$ by a group $G$ which is the $\mathbb{Z}_3$ generated by $\zeta = (111 222 000)$, where the notation means that

$$\zeta : y_i \rightarrow \begin{cases} \omega y_i, & 1 \leq i \leq 3 \\ \omega^2 y_i, & 4 \leq i \leq 6 \\ y_i, & 7 \leq i \leq 9 \end{cases}$$

(Quotients of $1^9$ and their relation to the $\mathcal{Z}$ were also discussed in [25]..) In this case, the seventh cohomology decomposes in the same way as for $\mathbb{P}_8[3]$, but with $\beta = 36$; 30 elements are polynomial deformations while the other 6 come from smoothing the quotient singularities. To find the 30 elements which correspond to polynomial deformations we recall that two deformations are the same if they differ by an element of the ideal generated by the partial derivatives of the defining polynomial, $\frac{\partial p}{\partial y_k}$ [13]. In our case the generators of the elements in the ideal are $y_k^2$, hence the deformations must be of the form $y_i y_j y_k$ with $i, j,$ and $k$ all different. There are $\frac{9!}{6! 3!} = 84$ such monomials. In order to discuss the action of the $\mathbb{Z}_3$ generator $\zeta$ it is convenient to think of the 9 coordinates as the elements of a 3 by 3 matrix, so we set $x_{ij} = y_{3i+j-3}$, where $i$ and $j$ now take the values 1, 2 and 3. Now we have

$$p = \sum_{i,j=1}^{3} x_{ij}^3.$$ 

The monomial deformations of $p$ are displayed in Table 2.1 which also indicates which are invariant under the action of the $\mathbb{Z}_3$ generator $\zeta : x_{ij} \rightarrow \omega^i x_{ij}$. Note that there are indeed 84 total deformations before dividing by $G$ and that 30 of them are invariant under $G$.

We can now identify these 30 monomials invariant under $G$ with $(1,1)$–forms on the $\mathcal{Z}$ manifold.

$$e_i \simeq x_{i1} x_{i2} x_{i3} , \quad f_{mnp} \simeq x_{1m} x_{2n} x_{3p} . \tag{2.4}$$

The off–diagonal $e_{ij}$’s from the $\mathcal{Z}$ manifold correspond to blow–ups on the mirror manifold, and cannot be represented by polynomial deformations. With this caveat, we write the polynomial defining the mirror, $\tilde{\mathcal{Z}}$, as

$$p = \sum_{i,j} x_{ij}^3 - 3 \sum_{k} \phi_k e_k - 3 \sum_{m,n,p} s_{mnp} f_{mnp} .$$

It would be of considerable interest to rectify this situation by finding a way to represent the ‘missing’ parameters. For our immediate purposes however this limitation is not serious since we can discuss the theory with $(b_{11}, b_{21}) = (84, 0)$ for which all the parameters may be represented by polynomial deformation or we may discuss the $\mathcal{Z}/\mathbb{Z}_3$ manifold which has $(b_{11}, b_{21}) = (12, 0)$ for which the same is true.
3. The Yukawa Couplings

We would like to calculate the Yukawa couplings as a function of the $\phi_i$’s and $s_{mnp}$’s. Initially let us simplify the problem by taking all $s_{mnp} = 0$. The calculation is more complicated when we consider $s_{mnp} \neq 0$, though there is conceptually no difference.

The classical or topological values of the couplings are given by the intersection cubic on $H^2(Z)$ \[14\]

$$y_0(a, b, c) = \int a \wedge b \wedge c .$$

$y_0$ counts the number of points of intersection of the three four-surfaces that are dual to the two-forms $a$, $b$ and $c$. In the orbifold limit the two-forms are the $e_\tau = dz^i \wedge d\bar{z}^\bar{j}$ and the 27 two-forms associated with the resolution of the fixed points $f_{mnp}$. In this limit these forms are supported on the fixed points and we will, with a slight abuse of notation, denote these forms by $f_{mnp}$ also. We denote by $e_i$ the three diagonal forms $dz^i \wedge d\bar{z}^i$ which have a direct correspondence with monomials in the mirror, and by considering the intersections of the associated hypersurfaces we immediately see that

$$e_1 e_2 e_3 \neq 0$$
$$e_i^2 = 0$$
$$e_i f_{mnp} = 0$$

\[3.1\]

$$f_{m_1 n_1 p_1} f_{m_2 n_2 p_2} f_{m_3 n_3 p_3} = 0 \text{ unless } (m_1, n_1, p_1) = (m_2, n_2, p_2) = (m_3, n_3, p_3) .$$

It is instructive to compare the topological couplings with the same couplings derived from the mirror. These couplings will contain all the sigma model corrections to (3.1). We now think of the $e_i$ and the $f_{mnp}$ as the monomials (2.4) and we are to calculate the

| monomial       | total number | invariant under $G$ |
|----------------|--------------|---------------------|
| $x_{i1} x_{i2} x_{i3}$ | 3            | 3                   |
| $x_{im} x_{in} x_{jp}$ | 54           | 0                   |
| $x_{1m} x_{2n} x_{3p}$ | 27           | 27                  |

**Table 2.1:** The enumeration of the polynomial deformations for the mirror of the $Z$ manifold.
products modulo the ideal generated by $\frac{\partial p}{\partial x_{ij}} = 0$, i.e., by the equations

$$x_{ij}^2 = \phi_i x_{i,j+1} x_{i,j+2} .$$

(3.2)

Let us first calculate $e_i^2$. Suppressing the $i$ index we have

$$e = x_1 x_2 x_3$$

and the generators of the ideal are the equations

$$x_j^2 = \phi x_{j+1} x_{j+2} .$$

(3.3)

We find

$$e^2 = x_1^2 x_2^2 x_3^2$$

$$= (\phi x_2 x_3)(\phi x_3 x_1)(\phi x_1 x_2)$$

$$= \phi^3 e^2 .$$

Thus $e^2$ vanishes since $\phi^3 \neq 1$ in general. In a similar way we see that $e_i f_{mnp}$ also vanishes. Thus we have shown that the second and third of Equations (3.1) are not corrected by instantons.

Consider now the $f^3$ couplings and label the indices on the three $f$'s such that $m_{jk}$ is the $k$'th index on the $j$'th $f$, i.e., the quantity of interest is

$$f_{m_{11}m_{12}m_{13}} f_{m_{21}m_{22}m_{23}} f_{m_{31}m_{32}m_{33}} .$$

(3.4)

To calculate this coupling we first gather together the factors $x_{1m_{11}} x_{1m_{21}} x_{1m_{31}}$ associated with the first index on each $f$. Again suppressing the first index on each $x$ we are to calculate a product of the form $x_j x_k x_l$. There are three possibilities:

$$x_1 x_2 x_3$$

$$x_j^2 x_k = 0 , \ (j \neq k)$$

$$x_3^2 = \phi x_1 x_2 x_3 ,$$

the right-hand sides of these relations following by virtue of (3.3). Proceeding in this way we find that the cubic $f$ coupling (3.4) is proportional to $e_1 e_2 e_3$ with a constant of proportionality given by the product of three factors, one corresponding to each triple of corresponding indices, that is to each column of the matrix $m_{jk}$. These are given in Table 3.1 and agree with the results in [12] and [26], which are calculated by means of an instanton sum in the SCFT of the orbifold.
| Condition                                      | Factor |
|------------------------------------------------|--------|
| If all the \( m_{jk} \), \( j = 1, 2, 3 \), are distinct | 1      |
| If precisely two of the \( m_{jk} \) are distinct         | 0      |
| If all three are equal                               | \( \phi_k \) |

**Table 3.1:** The factor for the Yukawa coupling for each value of \( k \).

As an example of the application of the rules of the Table, we see that

\[
f_{mnp}^3 = \phi_1 \phi_2 \phi_3 \ e_1 e_2 e_3 \quad \text{and} \quad f_{111} f_{122} f_{133} = \phi_1 \ e_1 e_2 e_3 .
\]

In order to examine the corrections to the large radius limit and to make contact with previous work we need to change our parametrization from the three \( \phi_i \) to three flat coordinates \( \tau_i \), whose imaginary parts may be thought of as the three “radii”. The relation between the \( \tau_i \) and the \( \phi_i \) is given by

\[
\phi_i = \phi(\tau_i)
\]

\[
\frac{\phi^3 (\phi^3 + 8)^3}{4^3 (\phi^3 - 1)^3} = J(\tau)
\]

with \( J(\tau) \) the absolute modular invariant of automorphic function theory [27]. The relation (3.5) is the same relation as that for a one-dimensional torus \( T \) presented as a cubic in \( \mathbb{P}_2 \)

\[
x_1^3 + x_2^3 + x_3^3 - 3\phi x_1 x_2 x_3 = 0
\]

and the usual \( \tau \)-parameter. This will be explained in Section 5.

If we denote by \( \hat{e}_i \) the monomials corresponding to the \( \tau_i \),

\[
\hat{e}_i \overset{\text{def}}{=} -\frac{\partial p}{\partial \tau_i} = \phi_i' e_i ,
\]

then in the orbifold limit

\[
f_{mnr}^3 = \frac{\phi_1 \phi_2 \phi_3}{\phi_1' \phi_2' \phi_3'} \hat{e}_1 \hat{e}_2 \hat{e}_3 .
\]

In the limit that \( \tau \to i \infty \) the asymptotic form of \( \phi \) is \( \phi \sim \frac{e^{-2\pi i \tau/3}}{3} \) so we find that this coupling has a finite nonzero limit as all \( \tau_j \to i \infty \). All the other couplings tend to zero in this limit, and hence we recover the topological couplings (3.1). The new couplings, however, also contain the instanton corrections to the topological couplings. In order to make a detailed comparison with known results we must first settle some issues involving the choice of basis and the normalization of the couplings. We do this in Section 5, and we return to the couplings in Section 6. The final results are given in Table 6.1 and do indeed agree with the results of Hamidi and Vafa [10].
3. 1. The Yukawa Couplings for One $s$ Nonzero

To calculate the couplings with one $s$ nonzero, we use the polynomial

$$p = \sum x^3_{ij} - 3 \phi_k e_k - 3 s f,$$

with only one $f$ (so we may take, for example, $f = x_{11}x_{21}x_{31}$). We find that $e_i^2$ no longer vanishes; rather we find the relation

$$e_i^2 = \frac{s \phi_i^2}{1 - \phi_i^3} e_i f.$$  \hfill (3.6)

Note that this relation is exact, i.e., valid to all orders in $s$.

The essential results for the couplings are:

$$e_i^3 = \frac{s^3 \phi_i^3 \phi_1 \phi_2 \phi_3 \left((1 - \phi_1^3)^2(1 - \phi_2^3)(1 - \phi_k^3) - s^6\right)}{\left(1 - \phi_i^3\right)^2 \left(1 - \phi_i^3 - s^3\right) \left((1 - \phi_i^3)(1 - \phi_j^3) - s^3\right) \left((1 - \phi_i^3)(1 - \phi_k^3) - s^3\right)} e_1 e_2 e_3$$

$$e_i^2 e_j = \frac{s^3 \phi_i^3 \phi_k}{\left(1 - \phi_i^3\right) \left((1 - \phi_i^3)(1 - \phi_j^3) - s^3\right)} e_1 e_2 e_3$$

$$f^3 = \frac{\phi_1 \phi_2 \phi_3}{1 - s^3} \left\{1 + \frac{s^3}{1 - \phi_1^3 - s^3} + \frac{s^3}{1 - \phi_2^3 - s^3} + \frac{s^3}{1 - \phi_3^3 - s^3}ight\} + \frac{s^3}{(1 - \phi_1^3)(1 - \phi_2^3) - s^3} \left\{1 + \frac{s^3}{1 - \phi_1^3 - s^3} + \frac{s^3}{1 - \phi_2^3 - s^3}\right\} + \frac{s^3}{(1 - \phi_1^3)(1 - \phi_3^3) - s^3} \left\{1 + \frac{s^3}{1 - \phi_1^3 - s^3} + \frac{s^3}{1 - \phi_3^3 - s^3}\right\} + \frac{s^3}{(1 - \phi_2^3)(1 - \phi_3^3) - s^3} \left\{1 + \frac{s^3}{1 - \phi_2^3 - s^3} + \frac{s^3}{1 - \phi_3^3 - s^3}\right\} e_1 e_2 e_3,$$

from which the remaining couplings, $e_i e_j f$, $e_i^2 f$, and $e_i f^2$ may be obtained via (3.6).

The procedure for obtaining these results is to use the ideal generated by $\frac{\partial p}{\partial x_{ij}} = 0$ to find relations among the products of monomials. Since these equations are

$$x^2_{ij} = \phi_i x_{i,j+1} x_{i,j+2} + \delta_{ij} s x_{i+1,j} x_{i+2,j},$$

we see that one cannot simplify the calculation by considering the two indices separately as was the case in the orbifold limit. Regardless, a dogged application of these relations
leads first to the discovery of relationships among the Yukawa couplings such as:

\[
(1 - s^3) f^3 = \phi_1 \phi_2 \phi_3 e_1 e_2 e_3 + s \phi_1 \phi_2 e_1 e_2 f + s \phi_1 \phi_3 e_1 e_3 f + s \phi_2 \phi_3 e_2 e_3 f \\
+ s^2 \phi_1 e_1 f^2 + s^2 \phi_2 e_2 f^2 + s^2 \phi_3 e_3 f^2 \\
(1 - \phi_i^3 - s^3) e_i f^2 = s^2 \phi_j e_i e_j f + s^2 \phi_i e_i f + s \phi_j \phi_k e_1 e_2 e_3
\]

where \( i, j, \) and \( k \) are distinct and finally to the ability to express all couplings in terms of a single one, which we have taken to be \( e_1 e_2 e_3 \).

4. The Modular Group

The modular group consists of those transformations of the periods under which the theory is invariant. The modular transformations for orbifolds are a well-studied subject [12,28]. For the \( \mathbb{Z} \) orbifold the modular group is known to be \( \frac{SU(3,3)}{SU(3,\mathbb{Z})} \) [29,30]. For the case of the \( \mathbb{Z}/\mathbb{Z}_3 \) orbifold the modular group has been studied by Shevitz [31] who has shown the the modular group is \( SL(2,\mathbb{Z}) \otimes SL(2,\mathbb{Z}) \otimes SL(2,\mathbb{Z}) \) each \( SL(2,\mathbb{Z}) \) acting in a familiar way on parameters \( \tau_i, i = 1, 2, 3 \).

It is an interesting question whether this group is preserved when the singularities of the orbifold are resolved, \( i.e., \) the \( s_{mnp} \) become nonzero. We will not settle this issue here. On first undertaking this investigation our expectation was that this would not be the case; however, it appears that the parameters \( s_{mnp} \) are able to accomodate the effect of a modular transformation on the \( \tau \)'s by becoming automorphic functions. Our primary concern here is to use the modular transformations to select a basis for the periods. When \( s \) becomes nonzero the number of periods increases from eight to ten and we need to find a new period \( z \) and its dual \( \partial G / \partial z \) in order to accomodate the two new periods. This we do by demanding that the modular transformations act on the period vector via integral matrices. We again begin our discussion by setting \( s_{mnp} = 0 \).

When the \( s_{mnp} \) vanish the polynomial \( p \) separates into the sum \( p_1 + p_2 + p_3 \) with each \( p_i \) corresponding to one torus. The analysis is therefore largely similar to that for each torus separately. There are however some subtleties that prevent the analogy from being complete.

It is clear that the replacement

\[ A_i : \phi_i \rightarrow \omega \phi_i \]

is a modular transformation since it can be undone by a coordinate transformation that multiplies one of the coordinates by \( \omega^{-1} \). Another that is less obvious but is nevertheless well-known is

\[ B_i : \phi_i \rightarrow \frac{\phi_i + 2}{\phi_i - 1}. \]
This also can be undone by a coordinate transformation

\[ x_{im} = \lambda_i \sum_n \omega^{mn} \hat{x}_{in} , \]

with

\[ \lambda_i \overset{\text{def}}{=} -3^{-\frac{1}{3}} (\phi_i - 1)^{-\frac{1}{3}} . \] (4.1)

If we were dealing with just one torus we would not need to correct an overall scale by means of the factor \( \lambda_i \) but since our polynomial is in fact \( p_1 + p_2 + p_3 \) we do have to allow for this factor. The operations \( A_i \) and \( B_i \) have the properties that \( A_i^3 = 1, B_i^2 = 1 \), and between them they generate the tetrahedral group.

It is instructive to introduce a variable \( \gamma \) defined by

\[
\gamma(\phi) = i \left( \frac{Z_1(\phi) - \omega^2 Z_2(\phi)}{Z_1(\phi) + \omega^2 Z_2(\phi)} \right)
= 2\sqrt{3} \left( \tau(\phi) - \frac{1}{2} \right) .
\]

For the moment we simplify the notation by dropping the subscript which tells us which torus we are working on. Let \( C \) be the operation that transports \( \phi \) around the branch point at \( \phi = 1 \). Then \( A \) and \( C \) together act on the upper half \( \gamma \)-plane and generate the triangle group corresponding to the angles \((0, 0, \pi/3)\). In fact we have a representation by matrices given by

\[
A = \begin{pmatrix}
\cos \frac{\pi}{3} & \sin \frac{\pi}{3} \\
-\sin \frac{\pi}{3} & \cos \frac{\pi}{3}
\end{pmatrix}, \quad CA = \begin{pmatrix}
1 & 2 \tan \frac{2\pi}{3} \\
0 & 1
\end{pmatrix} .
\]

Adding the operation \( B \) to \( A \) and \( C \) gives us the group \( \text{PSL}(2, \mathbb{Z}) \) with the operations having the actions

\[ A\tau = \frac{\tau - 1}{3\tau - 2} , \quad B\tau = \frac{3\tau - 2}{5\tau - 3} , \quad C\tau = -\frac{\tau}{3\tau - 1} . \]

Verifying the action of \( B \) requires use of the remarkable identities

\[
Z_1 \left( \frac{\phi + 2}{\phi - 1} \right) = -\frac{1}{3} (\phi - 1)(2Z_1(\phi) + Z_2(\phi))
\]
\[
Z_2 \left( \frac{\phi + 2}{\phi - 1} \right) = \frac{1}{3} (\phi - 1)( Z_1(\phi) + 2Z_2(\phi)) .
\]

The standard \( \text{SL}(2, \mathbb{Z}) \) generators

\[
S : \tau \rightarrow -\frac{1}{\tau} , \quad T : \tau \rightarrow \tau + 1
\]
are given in terms of \(A, B\) and \(C\) by the relations
\[
S = BA^2CA\ , \quad T = C^{-1}A^{-1}.
\]

It may be shown that \(\phi\) is related to \(\tau\) by
\[
J(\tau) = \frac{\phi^3 (\phi^3 + 8)^3}{4^3 (\phi^3 - 1)^3}.
\] (4.2)

This relation has 12 branches since if \(\phi\) is one branch then
\[
A\phi = \omega \phi
\]
and
\[
B\phi = \frac{\phi + 2}{\phi - 1}
\]
are others and successive applications of \(A\) and \(B\) give the other branches. For definiteness we choose a particular branch; we require \(\phi \to \infty\) as \(\tau \to i\infty\). This still leaves us with a phase ambiguity \(\phi \to \omega \phi\) but we know that
\[
J \sim \frac{1}{12^3q} \ ; \quad q = e^{2\pi i\tau}
\]
as \(\tau \to \infty\). So we fix the phase by requiring that
\[
\phi \sim \frac{1}{3q^{1/3}} \quad \text{as} \quad \tau \to i\infty.
\]

Following [12] and [26] we introduce the characters of the level 1 \(SU(3)\) Kac-Moody algebra \(\chi_i(\tau)\)
\[
\chi_i(\tau) = \frac{1}{\eta^2(\tau)} \sum_{\nu \in \Gamma_i} q^{\frac{1}{2}|\nu|^2}
\]
where \(\eta\) is the Dedekind function and the \(\Gamma_i\) denote the conjugacy classes of the \(SU(3)\) weight lattice (\(\Gamma_0\) is the root lattice and \(\Gamma_{1,2}\) are \(\Gamma_0\) shifted by the fundamental dominant weights). Note that in fact \(\chi_1 = \chi_2\). On writing out the sums we find
\[
\eta^2 \chi_0 = \sum_{m,n} q^{(m^2+mn+n^2)}
\]
\[
\eta^2 \chi_1 = q^{1/3} \sum_{m,n} q^{(m^2+mn+n^2+m+n)}.
\] (4.3)

In [12] it was shown that under the modular transformation \(\tau \to -1/\tau\) the \(\chi_i\) transform according to the rule
\[
\begin{pmatrix}
\chi_0 \\
\chi_1 \\
\chi_2
\end{pmatrix}
(-1/\tau) = \frac{1}{\sqrt{3}}
\begin{pmatrix}
1 & 1 & 1 \\
1 & \omega & \omega^2 \\
1 & \omega^2 & \omega
\end{pmatrix}
\begin{pmatrix}
\chi_0 \\
\chi_1 \\
\chi_2
\end{pmatrix}(\tau).
\] (4.4)
It can be shown that
\[ \phi(\tau) = \frac{\chi_0(\tau)}{\chi_1(\tau)} \] (4.5)
in virtue of (4.2). We observe that we are on the correct branch since \( \eta^2 \chi_0 \to 1 \) as \( \tau \to i\infty \) while \( \eta^2 \chi_1 \sim 3q^{1/3} \), the factor of 3 coming from the three terms for which \( (m,n) = (0,0), (-1,0), \) and \( (0,-1) \).

From Equation (4.5) we have
\[ \phi(\tau + 1) = \omega^2 \phi(\tau) \]
and
\[ \phi(-1/\tau) = \frac{\phi(\tau) + 2}{\phi(\tau) - 1}. \]

The inverse relation to Equation (4.5) is
\[ \tau(\phi) = \frac{i}{\sqrt{3}} \frac{Z_1(\phi) - Z_2(\phi)}{Z_1(\phi) + \omega^2 Z_2(\phi)} \]
\[ = \frac{i}{2\pi} \left\{ \log(\phi^3) + \sum_{n=0}^{\infty} \frac{\Gamma(\frac{1}{3}+n) \Gamma(\frac{2}{3}+n)}{(n!)^2 \phi^{3n}} \left(2\Psi(1+n) - \Psi(\frac{2}{3}+n) - \Psi(\frac{1}{3}+n)\right) \right\}, \] (4.6)
which holds for \( 0 < \Re\tau < 1 \) and the second equality being true for sufficiently large \( \phi \).

4.1. \( s \neq 0 \) and the Action of the Modular Group on \( \mu \) and \( p \)

We proceed to study the action of the modular transformations on the differential form \( \mu \) that was used to construct the holomorphic \((5,2)\)-form and the polynomial \( p \), and we now allow \( s_{mnp} \) to be nonzero. Consider first the coordinate transformation
\[ x_{11} = \omega^2 \tilde{x}_{11}, \quad x_{im} = \tilde{x}_{im}, \] for \( (i,m) \neq (1,1). \) (4.7)

If we redefine the parameters appropriately
\[ \tilde{\phi}_1 = \omega^2 \phi_1, \quad \tilde{s}_{1nr} = \omega^2 s_{1nr} \]
with the other parameters unchanged then the polynomial is invariant in the sense that \( p(\tilde{x}|\tilde{\phi}, \tilde{s}) = p(x|\phi, s) \). This is just a restatement of the fact that \( (\tilde{\phi}, \tilde{s}) \) and \( (\phi, s) \) define the same manifold and we know from our previous discussion that this change corresponds to the modular transformation \( \tau_1 \to \tau_1 + 1 \). Under this transformation \( \mu = \)]
$\frac{1}{3!} \epsilon_{A_1 A_2 \ldots A_9} x^{A_1} dx^{A_2} \ldots dx^{A_9}$ is not invariant. In fact $\tilde{\mu} = \omega \mu$. The modular transformation $\tau_1 \to -1/\tau_1$ corresponds to the more complicated coordinate transformation

$$x_{1m} = \lambda_1 \sum_n \omega^{mn} \hat{x}_{1n}, \quad (4.8)$$

with $\lambda_1$ as in (4.1). These transformations of the parameters are summarized in Table 4.1.

| Action | $\tau_1$ | $\phi_1$ | $s_{1np}$ | $\mu$ |
|--------|----------|----------|-----------|-------|
| $T_1$  | $\tau_1 + 1$ | $\omega^2 \phi_1$ | $\omega^2 s_{1np}$ | $\omega \mu$ |
| $S_1$  | $-\frac{1}{\tau_1}$ | $\frac{\phi_1 + 2}{\phi_1 - 1}$ | $\lambda_1 \sum_k \omega^{mk} s_{knp}$ | $-3^{-\frac{1}{3}} i(\phi_1 - 1) \mu$ |

**Table 4.1:** The transformation of the parameters under $T_1$ and $S_1$.

We can simplify these transformation rules by a redefinition of variables. Consider first the transformation rule for $s_{mnp}$ under $\tau_1 \to -1/\tau_1$. This involves the awkward factor of $\lambda_1 = -3^{-\frac{1}{3}} (\phi_1 - 1)^{-\frac{1}{3}}$. Notice that under $B_1$

$$\left( \eta^2 \chi_1 \frac{d\phi_1}{dJ} \right)^{\frac{1}{3}} \to -\frac{3^{\frac{1}{3}}}{(\phi_1 - 1)^{\frac{1}{3}}} \left( \eta^2 \chi_1 \frac{d\phi_1}{dJ} \right)^{\frac{1}{3}}. \quad (4.9)$$

So if we set

$$s_{mnp} = \left( \eta^2 \chi_1 \frac{d\phi_1}{dJ} \right)^{\frac{1}{3}} v_{mnp}$$

we find that $v$ transforms according to the much simpler rule

$$v_{mnp} \to \frac{1}{\sqrt{3}} \sum_k \omega^{mk} v_{knp}.$$

The question arises as to what is meant by the cube roots that appear in (4.9) and whether they can be consistently defined. The answer is that the right hand side of (4.9) is defined only up to multiplication by cube roots of unity. However this is of no consequence since all the $v_{mnp}$ are multiplied by a common phase which can be absorbed by a simultaneous shift $\tau_i \to \tau_i + k$. This leaves $\mu$ invariant. Note that $v_{mnp}$ does not return to itself under $B_1^2$ but rather to $v_{-m,n,p}$. Thus iterating the operation $\tau_1 \to -1/\tau_1$ induces $x_{1m} \to x_{1,-m}$ which changes the sign of $\mu$.
We can also absorb the $\phi_1$ dependent factor that appears in the transformation rule for $\mu$ by noting that under $B_1$

$$\left(\frac{d\phi_1}{dJ}\right)^\frac{1}{2} \to \frac{3^\frac{1}{2} i}{(\phi_1 - 1)} \left(\frac{d\phi_1}{dJ}\right)^\frac{1}{2}.$$  

Thus

$$m \overset{\text{def}}{=} J^{-\frac{1}{3}} \left(\frac{d\phi_1}{dJ}\right)^\frac{1}{2} \mu$$

is invariant under $\tau_1 \to -1/\tau_1$ up to a sign ambiguity introduced by the square roots in (4.10) ($J^{-\frac{1}{3}}$, on the other hand, can be unambiguously defined in virtue of (4.2)). It is also invariant under $\tau_1 \to \tau_1 + 1$ since the factor $J^{-\frac{1}{3}}$ acquires a factor $\omega$. Note also that under $\tau_1 \to \tau_1 + 1$ the factor $\left(\chi_1 \frac{d\phi_1}{dJ}\right)^\frac{1}{2}$ associated with $v$ is invariant.

The burden of these observations is that $\frac{m_1}{v}$ can be rendered invariant under modular transformations up to a sign ambiguity. However the sign ambiguity is of no consequence for the prepotential or the Yukawa couplings since these have charge two.

Note that $B_i$ is a symmetry only in the limit where all $s_{mnp} = 0$, or if we take a certain subset of the $s_{mnp}$’s. For example, if $s_{mnp}$ is nonzero then $s_{m+1np}$ and $s_{m+2np}$ must also be nonzero in order for the defining polynomial to be invariant under $B_1$ since $B_1(s_{mnp}) = \lambda \sum_k \omega^{mk} s_{knp}$. The action of the $B_i$ on the $s_{mnp}$ is to transform an $s$ into a linear combination of $s$’s so this action will not in general preserve a hypersurface of the parameter space corresponding to setting some but not all of the $s$’s to zero.

A complete discussion of the modular group would entail letting all $s_{mnp}$ be nonzero, and investigating the transformation properties of the flat coordinate $\rho$ associated with each $s$. The interested reader should consult Reference [29].

5. The Periods

The complex structure of a generalized Calabi–Yau manifold $\mathcal{M}$ of, say, seven dimensions can be described by giving the periods of the holomorphic form, $\Omega_{5,2}$, over a canonical homology basis. By virtue of the Hodge decomposition displayed in Section 2, we see that the dimension of $H_7(\mathcal{M}, \mathbb{Z})$ is $2(b_{4,3} + 1)$. We proceed in a familiar way by choosing a symplectic basis $(A^a, B_b)$, $a, b = 1, \ldots, b_{4,3} + 1$ for $H_7(\mathcal{M}, \mathbb{Z})$ such that

$$A^a \cap B_b = \delta^a_b \ , \ A^a \cap A^b = 0 \ , \ B_a \cap B_b = 0 ,$$  

and we denote by $(\alpha_a, \beta^b)$ the cohomology basis dual to this so that

$$\int_{A^a} \alpha_b = \delta^a_b , \ \int_{B_a} \beta^b = \delta^b_a ,$$

16
with other integrals vanishing. Being a seven–form, \( \Omega_{5,2} \) may be expanded in terms of the basis

\[
\Omega_{5,2} = z^a \alpha_a - \mathcal{G}_a \beta^a .
\]

The coefficients \((z^a, \mathcal{G}_b)\) are the periods of \( \Omega_{5,2} \), and are given by the integrals of \( \Omega_{5,2} \) over the homology basis

\[
z^a = \int_{A^a} \Omega_{5,2} \quad , \quad \mathcal{G}_b = \int_{B_b} \Omega_{5,2} .
\] (5.2)

A comment is in order about the holomorphicity of \( \Omega_{5,2} \). Let \( t \) be one of the \( b_{4,3} \) parameters on which the complex structure depends, and denote by \( x^\mu \) the coordinates of \( \mathcal{M} \), then it is a fundamental observation in the theory of variation of complex structure \([32]\) that

\[
\frac{\partial (dx^\mu)}{\partial t}
\]

is a linear combination of one–forms of type (1,0) and (0,1) and that

\[
\frac{\partial (dx^\mu)}{\partial t} = 0 .
\] (5.3)

It follows from (5.3) that

\[
\frac{\partial \Omega}{\partial t} \in H^{5,2} .
\]

There could, in principle, be a part in \( H^{6,1} \) but \( H^{6,1} \) is trivial. In other words,

\[
\frac{\partial \Omega}{\partial t} = h(t, \overline{t}) \Omega
\]

for some function of the parameters \( h \). Thus by redefining \( \Omega \) by multiplication by a function of the parameters

\[
\Omega \rightarrow \Omega \exp \left( -\int \tau h(t, \overline{t}) d\overline{t} \right)
\]

we can ensure that

\[
\frac{\partial \Omega}{\partial t} = 0 .
\]

Thus we can assume that \( \Omega \) varies holomorphically with the parameters. It is now apparent that the formalism of special geometry can be applied. We know that the complex structure depends on \( b_{4,3} \) parameters so we may take the \( \mathcal{G}_a \) to be functions of the \( z^a \). The \( z^a \) are homogeneous coordinates on the space of complex structures and we have that

\[
\Omega(\lambda z) = \lambda \Omega(z) \quad \text{and} \quad \mathcal{G}_b(\lambda z) = \lambda \mathcal{G}_b(z)
\]

in the familiar way. We also observe that

\[
\frac{\partial \Omega}{\partial z^a} \in H^{5,2} \oplus H^{4,3}
\]
so we have the relation

\[ \int \Omega \wedge \frac{\partial \Omega}{\partial z^a} = 0 \]

which has the consequence that the \( G_a \) are the derivatives of a prepotential \( G \) of homogeneity degree two:

\[ G_a = \frac{\partial G}{\partial z^a}. \]

The multiplication of \( \Omega \) by a holomorphic function of the moduli

\[ \Omega \rightarrow f \Omega \] (5.4)

has no effect on the metric on moduli space or on the invariant Yukawa couplings. However, it has a direct effect on the periods and their modular transformations. We shall choose a particular gauge in Section 5.5 in order to find an integral basis for the periods.

The form \( \Omega \) and its derivatives with respect to the complex structure parameters are all seven-forms. There are at most \( 2(b_{4,3} + 1) \) linearly independent such quantities. It follows that \( \Omega \) satisfies linear differential equations: the Picard-Fuchs equations. For the mirror of the \( Z \) manifold these provide a straightforward way of computing the periods. Derivations of the Picard-Fuchs equations intended for physicists are given in [33-35], the method having derived its origin in the work of Griffiths [36].

We shall employ this method to find the periods in the orbifold limit when all \( s_{mnp} = 0 \). We relate this to a set of periods calculated by choosing a homology basis and directly evaluating the periods via (5.2). We also exploit the relationship of \( \tilde{Z} \) in the orbifold limit to a product of tori: the homology of \( \tilde{Z} \) factorizes, and so we use our understanding of the homology of the torus to define a basis satisfying (5.1). Thus we find an integral basis for the periods when all \( s_{mnp} = 0 \).

When we allow one \( s \) to be nonzero the homology no longer factorizes. To determine a suitable basis for the periods we insist that the modular group act on the period vector by integral matrices. We find first a basis of periods that satisfy the Picard-Fuchs equations and which reduce to the periods of the integral basis in the orbifold limit. The next step is to investigate the effect of modular transformations on the periods. We do this in Section 5.5.

5.1. The Periods on the Mirror when All \( s_{mnp} = 0 \)

When all \( s_{mnp} \) vanish we expect 8 independent periods. In this limit the defining polynomial is

\[ p = \sum_{i=1}^{9} y_i^3 - 3 \sum_{k=1}^{3} \phi_k e_k, \] (5.5)

which is suggestively like three copies of the equation used to define a torus as a \( \mathbb{P}_2[3] \) with a polynomial

\[ p_1 = x^3 + y^3 + z^3 - 3 \phi xyz. \] (5.6)
Both [33] and [35] derive the differential equation satisfied by the periods for a torus; they find that each period $q_i$ is a solution of the equation

$$\left[ \phi^3 (1 - \phi^3) \frac{d^2}{d(\phi^3)^2} + \left( \frac{2}{3} - \frac{5}{3} \phi^3 \right) \frac{d}{d\phi^3} - \frac{1}{9} \right] q_i = 0 . \quad (5.7)$$

The Picard-Fuchs equations for $\tilde{Z}$ turns out to be just three copies of this, with $\phi$ replaced by $\phi_i$. We can show directly that the periods on the manifold must solve Equation (5.7) for each $\phi_i$ by following the procedure outlined in [35], using $\phi_1$ as an example. A period $q_i$ is an integral of $\Omega$ (2.2) over a seven-cycle, so we write

$$q_i = \int \frac{\mu}{p^3} .$$

Then two derivatives with respect to $\phi_1$ yields

$$\frac{\partial^2 q_i}{\partial \phi_1^2} = 4 \cdot 3^3 \int \frac{\mu (y_1 y_2 y_3)^2}{p^5}$$

$$= 4 \cdot 3^3 \int \frac{\mu (y_1 y_2)^2}{p^5} \left( \frac{1}{3} \frac{\partial p}{\partial y_3} + \phi_1 y_1 y_2 \right)$$

$$= 4 \cdot 3^3 \int \mu \left( \frac{\phi_1 (y_1 y_2)^3}{p^5} - \frac{1}{12} \frac{\partial}{\partial y_3} \left( \frac{y_1^2 y_2^2}{p^4} \right) \right) .$$

In the second line we have used $y_3 = \frac{1}{3} \frac{\partial p}{\partial y_3} + \phi_1 y_1 y_2$. The second term of the third line is zero. We then make similar substitutions in the first term, first with $\frac{\partial p}{\partial y_2}$, then with $\frac{\partial p}{\partial y_1}$, and set to zero all integrals of total derivatives. The final result is that

$$q'' = \phi_1 q + 3 \phi_1^2 q' + \phi_1^3 q'' ,$$

where $q' = \frac{\partial q_i}{\partial \phi_1}$. If we change variables to $\phi_1^3$ we find Equation (5.7). Thus the periods on $\tilde{Z}$ must solve this equation for each $i$. This is a surprise on first acquaintance since our period $q$ contains an integral over $\frac{1}{p^3}$ rather than the integrals over $\frac{1}{p^1}$ that arise for the torus. The reason that the periods nevertheless satisfy the same equations is that the homology is carried by the three tori $T_i$. We shall explain this presently.

The Equation (5.7) is of course a hypergeometric equation for which the solutions can be represented by a Riemann P-symbol

$$P\left\{ \begin{array}{ccc} 0 & \infty & 1 \\ 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 \end{array} \right\} .$$
There are two independent solutions; we define

\[
Z_1(\phi) = \frac{\Gamma^2(\frac{3}{4})}{\Gamma(\frac{5}{4})} \text{$_2$F$_1$}(\frac{3}{4}; \frac{3}{4}; \frac{\phi}{3})
\]

\[
Z_2(\phi) = \phi \frac{\Gamma^2(\frac{3}{4})}{\Gamma(\frac{5}{4})} \text{$_2$F$_1$}(\frac{3}{4}; \frac{3}{4}; \frac{\phi}{3})
\]

and conclude that there is a basis of 8 independent periods given by the product functions \(Z_i(\phi_1)Z_j(\phi_2)Z_k(\phi_3)\), where \(i, j,\) and \(k\) are independently chosen to be 1 or 2.

Another method of calculating the periods is to proceed by choosing a basis for \(H_7\) and directly integrating \(\Omega\) over these seven-cycles (5.2). We leave the details for the Appendix; the resulting periods factorize, as they must, into products of solutions of (5.7).

Our solutions are products of solutions of the Picard-Fuchs equation for a torus. Hence it is intuitive that we may calculate a basis by using the homology cycles on the torus. In this basis we can easily find the intersection of the cycles, and so it is this basis which we will use to define the integral periods. To explain this we turn to a discussion of the periods on the torus and their relevance before calculating the periods on \(\tilde{Z}\) away from the orbifold point.

5.2. Three Tori from One Seven–fold

Let us examine why the periods on the mirror of the \(Z\) break into products of three periods on tori in the orbifold limit. We note that from Equation (5.5), we can break \(p\) into a sum of three polynomials \(p = p_1 + p_2 + p_3\), where

\[
p_i = \sum_{k=3i-2}^{3i} y_k^3 - 3\phi_i e_i .
\]

For a seven-cycle \(H\),

\[
\int_H \Omega_{5,2} = \frac{1}{2\pi i} \int_H \int_{C_p} \frac{y_9 dy_1 \ldots dy_8}{p^3}
\]

\[
= \frac{1}{(2\pi i)^2} \int_H \int_{C_p} \int_{C_{y_9}} \frac{dy_1 \ldots dy_9}{p^3}
\]

\[
= \frac{1}{(2\pi i)^2} \int_{H \times C_p \times C_{y_9}} \frac{dy_1 \ldots dy_9}{6p_1p_2p_3} \sum_{n=0}^{\infty} \left(\frac{\Delta}{6p_1p_2p_3}\right)^n ,
\]

where \(\Delta = (p_1 + p_2 + p_3)^3 - 6p_1p_2p_3\) and the second equality follows due to the fact that the integral is actually independent of the value of \(y_9\) in virtue of the homogeneity of the integrand. We may therefore introduce a factor of \(\frac{1}{2\pi iy_9}\) and integrate around a loop \(C_{y_9}\).
without changing the value of the integral. Suppose our cycle \( \tilde{H} = H \times C_p \times C_y \) contains the circles \( C_{p_1} \times C_{p_2} \times C_{p_3} \); then it is easy to see that we get a contribution from only the zero’th order term in the sum.

The period is proportional to

\[
\int \frac{dy_1 \ldots dy_9}{\tilde{H} p_1 p_2 p_3}
\]

and we see that the integrand obviously breaks into the product of three integrands of the form in Equations (5.8) and (5.9) above. The integral becomes separable when \( \tilde{H} \) is separable. For example, suppose we take a seven-cycle \( H = \{ C_{y_1} \times \ldots \times C_{y_7} \} \). Then if we solve \( p = 0 \) for \( y_8 \), choosing the branch \( y_8 \to 0 \) as \( \phi_3 \to \infty \), we see that we have three copies of the integral in Equation (5.8) (provided we are in the regime where all \( \phi_i \) are large).

5.3. The Periods on a Torus

Consider the torus as a cubic in \( \mathbb{P}_3 \) given by (5.6) and define the contour \( A_3 = C_{p_1} \times C_x \times C_y \), where \( C_f \) is a contour that winds around the hypersurface \( f = 0 \). In the limit \( \phi \to \infty \) and on the branch of \( p_1 = 0 \) for which \( z \to 0 \) as \( \phi \to \infty \), we calculate the period \( R \) to be

\[
R = \int_{A_3} \frac{dx dy dz}{p_1} = -\frac{1}{3\phi} (2\pi i)^3 \frac{\Gamma(\frac{1}{2}, \frac{x}{2}; \phi^{-3})}{\Gamma(\frac{3}{2})} \]

\[
= \frac{(2\pi)^2}{3} (\omega - 1)(Z_1(\phi) + \omega^2 Z_2(\phi)) .
\]

(5.8)

Alternatively, we could choose to integrate \( \frac{dy}{dp_1/dz} \) over the one–cycle

\[
A = \{ x = 1, C_y, \}
\]

\[
z \text{ given by the branch of } p_1 = 0 \text{ such that } z \to 0 \text{ as } \phi \to \infty \}.
\]

We then find that

\[
R = (2\pi i)^2 \int_A \frac{dy}{\partial p_1/\partial z} .
\]

In order to find a second solution, we should examine \( p_1 \) as \( \phi \to 1 \). There is a singularity near \( (x, y, z) = (1, 1, 1) \). In the neighborhood of this singularity we let

\[
x = 1 , \quad z = 1 + u_3 ,
\]

\[
y = 1 + u_2 , \quad \phi = 1 + e^2 .
\]
Substituting this into \( p_1 \) we arrive at
\[
p_1 = u_2^2 + u_3^2 - u_2 u_3 - \epsilon^2 .
\]
Thus for \( u_2 \) and \( u_3 \) real, there is a circle that shrinks to zero as \( \epsilon \to 0 \). Since this is a one-cycle, we can integrate a one–form over it. If we define
\[
B = \{ x = 1, \ y = 1 + u_2, \ z \text{ follows the } S^1 \text{ that shrinks to zero as } \phi \to 1 \} ,
\]
then the period \( Q \) is
\[
Q = (2\pi i)^2 \int_B \frac{dz}{\partial p_1/\partial y} = -\frac{(2\pi)^3}{3\sqrt{3}} 2F1(\frac{1}{3}, \frac{1}{3}, 1; 1 - \phi^3) \]
\[
= \frac{(2\pi)^2}{3} (-Z_1(\phi) + Z_2(\phi)) .
\]
(5.9)

We remark that we are integrating the same one form as before, because \( \frac{dz}{\partial p_1/\partial y} = -\frac{dy}{\partial p_1/\partial z} \).

The importance of the periods \( Q \) and \( R \) lies in their relation to the \( \tau \)-parameter. The relation is
\[
\tau = \frac{Q}{R} .
\]

Of course, we could have defined a three cycle to be integrated over a three form as well:
\[
B_3 = \{ C_x \times C_{p_1} \times the \ S^1 \text{ near } (1,1,1) \text{ that shrinks to zero as } \phi \to 1 \} ,
\]
so
\[
Q = \int_{B_3} \frac{dx dy dz}{p_1} .
\]

We know that we can define a period on the 7–dimensional manifold by a product of these periods. We arrange the 8 independent periods into a column vector, let \( \varpi_{QQQ} \) stand for \( Q(\phi_1)Q(\phi_2)Q(\phi_3) \), etc., and define
\[
\Pi_8 = \begin{pmatrix}
-\varpi_{QQQ} \\
\varpi_{RQQ} \\
\varpi_{QQR} \\
\varpi_{QRQ} \\
\varpi_{QRR} \\
\varpi_{RQR} \\
\varpi_{RRQ}
\end{pmatrix}
= \frac{1}{R(\phi_1)R(\phi_2)R(\phi_3)}
\begin{pmatrix}
-\tau_1 \tau_2 \tau_3 \\
\tau_2 \tau_3 \\
\tau_1 \tau_3 \\
\tau_1 \tau_2 \\
\tau_1 \\
\tau_2 \\
\tau_3
\end{pmatrix} .
\]
(5.10)

It remains to discuss the intersection of the cycles. Consider again the neighborhood of the singularity. Locally the singularity approximates a \( S^0 = \mathbb{Z}_2 \) bundle over an \( S^1 \).
consists of two points so we have a double cone. What is happening is that a short cycle
on the torus is being shrunk to a point, and so the torus becomes a sphere with two points
identified. The basis we have chosen is the short cycle that shrinks to zero \( B \) and the
long cycle that passes through the node \( A \). It is clear that the cycles \( B \) and \( A \) intersect
in a point. It is this fact which allows us to define elements of the integral homology in the
seven–fold. The cycles on the 7–manifold that produce the periods that are products of the
periods \( Q \) and \( R \) on the tori have intersection numbers that are just given by the product
of the intersections of the cycles \( B \) and \( A \) on the tori. Thus the cycles that produce the
periods in (5.10) have intersection numbers (5.1), and we recover in this way the basis of
Shevitz \[31\].

5. 4. The Periods for One \( s_{mnp} \) Nonzero

It is of interest to examine the periods on the manifold as a function of the \( s_{mnp} \). Suppose
we let only one \( s_{mnp} \equiv s \) be non-zero, and so we take

\[
p = \sum_{i=1}^{9} y_i^3 - 3 \sum_{k=1}^{3} \phi_k e_k - 3sf_{mnp} .
\]

The homology no longer factors, but our method of defining a set of cycles and integrating
to find the periods is still available, and yields the periods as power series in \( s \). We
compute the periods in this way in the Appendix and find, as we must, that there are now
10 independent periods. These periods are:

- A set of eight periods of the form
  \[
  \varpi_{ijk} = \sum_{r=0}^{\infty} \frac{(3s)^{3r}}{(3r)!} Z_i(\phi_1, r)Z_j(\phi_2, r)Z_k(\phi_3, r)
  \]
  where the functions
  \[
  Z_1(\phi, r) = \frac{\Gamma\left(\frac{1}{3}\right) \Gamma\left(r + \frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} \, {}_{2}F_{1}\left(\frac{1}{3}, r + \frac{4}{3}; \frac{2}{3}; \phi^3\right) \\
  Z_2(\phi, r) = \frac{\Gamma\left(\frac{1}{3}\right) \Gamma\left(r + \frac{2}{3}\right)}{\Gamma\left(\frac{4}{3}\right)} \, \phi \, {}_{2}F_{1}\left(\frac{2}{3}, r + \frac{4}{3}; \frac{4}{3}; \phi^3\right)
  \]
  generalise the functions \( Z_1(\phi) \) and \( Z_2(\phi) \) defined previously which correspond to
  \( Z_1(\phi, 0) \) and \( Z_2(\phi, 0) \) respectively. These periods reduce to the eight periods of the
  orbifold limit as \( s \to 0 \).
- A period that is \( O(s) \) as \( s \to 0 \)
  \[
  \varpi = \sum_{r=0}^{\infty} \frac{(3s)^{3r+1}}{(3r + 1)!} Z_3(\phi_1, r)Z_3(\phi_2, r)Z_3(\phi_3, r) , \quad Z_3(\phi, r) \overset{\text{def}}{=} Z_1(\phi, r + \frac{1}{3}) .
  \]
• A period that is $O(s^2)$ as $s \to 0$

$$\widehat{\omega} = \sum_{r=0}^{\infty} \frac{(3s)^{3r+2}}{(3r+2)!} Z_4(\phi_1, r) Z_4(\phi_2, r) Z_4(\phi_3, r) \quad , \quad Z_4(\phi, r) \overset{\text{def}}{=} Z_2(\phi, r + \frac{2}{3}) .$$

The Picard-Fuchs equations are still relatively simple. One may use the same method as in the orbifold case to derive the differential equations satisfied by the holomorphic form $\Omega$. As a practical matter, though, it seems simpler to find the periods by direct integration as in Appendix A and then find the differential equations that they satisfy.

It is easy to see that each of the ten periods above satisfies the equation

$$\left\{ \phi_a^3 (1 - \phi_a^3) \frac{\partial^2}{\partial (\phi_a^3)^2} + \left( \frac{2}{3} - \frac{5}{3} \phi_a^3 \right) \frac{\partial}{\partial \phi_a^3} - \left( \frac{\phi_a^3}{3} \frac{\partial}{\partial \phi_a^3} + \frac{1}{9} \right) s \frac{\partial}{\partial s} - \frac{1}{9} \right\} \Omega = 0 \quad (5.11)$$

for each $a = 1, 2, 3$. A further equation follows from the observation that

$$Z_i(\phi, r + 1) = \left[ \phi_a^3 \frac{d}{d \phi^3} + \left( r + \frac{1}{3} \right) \right] Z_i(\phi, r) \quad , \quad i = 1, 2,$$

from which it follows that each of the periods satisfies the relation

$$\left\{ L_1 L_2 L_3 - \left( \frac{1}{3} \frac{\partial}{\partial s} \right)^3 \right\} \Omega = 0 \quad (5.12)$$

with $L_a$ denoting the operators

$$L_a = \phi_a^3 \frac{\partial}{\partial \phi_a^3} + \frac{s}{3} \frac{\partial}{\partial s} + \frac{1}{3} .$$

It remains to extend the integer basis that we had for $s = 0$ to include the two new elements. Our criterion will be to demand that the modular transformations act on the basis by means of integral matrices. We may choose eight of the ten basis elements to be the linear combinations of the $\varpi_{ijk}$ that reduce to the periods (5.10) in the limit $s \to 0$. these are periods $\varpi_{QQQ}$, $\varpi_{QQR}$ etc. where now $\varpi_{QQQ}$, for example, denotes $\sum_{r=0}^{\infty} \frac{(3s)^{3r}}{(3r)!} Q(\phi_1, r) Q(\phi_2, r) Q(\phi_3, r)$ and $Q(\phi, r)$ and $R(\phi, r)$ are defined as the obvious generalizations of the corresponding quantities with $r = 0$

$$\begin{pmatrix} Q(\phi, r) \\ R(\phi, r) \end{pmatrix} = \frac{(2\pi)^2}{3} \begin{pmatrix} \frac{1}{i \sqrt{3} \omega^2} & 1 \\ -i \sqrt{3} \omega & -i \sqrt{3} \omega \end{pmatrix} \begin{pmatrix} Z_1(\phi, r) \\ Z_2(\phi, r) \end{pmatrix} . \quad (5.13)$$
5.5. An Integral Basis for the Periods

We are finally ready to find an integral, symplectic basis for the periods. First recall that, by construction, the period vector $\Pi_8$ of (5.10) corresponds to a symplectic basis for the corresponding cycles and it possesses the right intersection numbers (5.1). Thus we may use it to calculate the invariant couplings as described in [37] and Section 6. We would like to show explicitly that there is a gauge in which the periods transform by integers under the modular group.

The modular group is in this context the group generated by the $A_i$ and the $C_i$ since the transformations generated by the $B_i$ do not respect the condition that only one $s$ is nonzero. The generators $A$ and $C$ do not change powers of $s$ so it is clear that we may take eight of the ten basis elements to correspond to the eight elements of $\Pi_8$. The issue reduces to how to choose linear combinations of the new periods $\psi$ and $\hat{\psi}$ in order to obtain the remaining periods. The action of $A$ and $C$ on the functions $Z_1(\phi, r)$ and $Z_2(\phi, r)$ is independent of $r$ if $r$ is integral:

$$A: \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}, \quad C: \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \rightarrow \begin{pmatrix} -2\omega & i\sqrt{3} \\ -i\sqrt{3} & -2\omega^2 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}. \quad (5.14)$$

We see, in virtue of (5.13) that

$$A: \begin{pmatrix} Q \\ R \end{pmatrix} \rightarrow \omega^2 \begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} Q \\ R \end{pmatrix}, \quad C: \begin{pmatrix} Q \\ R \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} Q \\ R \end{pmatrix}. \quad (5.15)$$

The matrices that appear in this basis are the matrices $A$ and $C$ of Section 4, apart from a factor of automorphy of $\omega^2$ that appears in relation to $A$. It is clear from these matrices that, apart from this factor of automorphy, the period vector $\Pi_8$ transforms integrally under $A$ and $C$. Turning now to the new periods we find that

$$A: \begin{pmatrix} \psi \\ \hat{\psi} \end{pmatrix} \rightarrow \omega^2 \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix} \begin{pmatrix} \psi \\ \hat{\psi} \end{pmatrix}, \quad C: \begin{pmatrix} \psi \\ \hat{\psi} \end{pmatrix} \rightarrow \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix} \begin{pmatrix} \psi \\ \hat{\psi} \end{pmatrix}. \quad (5.16)$$

Note that the same SL(2) matrix appears in each case and that $A$ is accompanied by the same factor of automorphy as in (5.14).

We are to find a change of basis

$$\begin{pmatrix} G_4 \\ z^4 \end{pmatrix} = M \begin{pmatrix} \psi \\ \hat{\psi} \end{pmatrix}, \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

such that the matrix that represents $C$ becomes integral. In (5.14) $C$ is represented by a matrix whose cube is the identity. It is well known that the the only matrices in $\text{Sp}(2, \mathbb{Z}) = \text{Sp}(2, \mathbb{Z})$.
SL(2, \mathbb{Z}) with this property are, up to conjugation, the matrix \(\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\) and its inverse \(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\). Thus we require

\[
M \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix} M^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \quad \text{or} \quad M \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix} M^{-1} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.
\]

A consideration of the the periods calculated in the Appendix by integrating over combinations of integral cycles leads us to

\[
M = c \begin{pmatrix} \omega^2 & \omega \\ 1 & 1 \end{pmatrix}, \quad \text{where} \quad c = \frac{(2\pi)^6}{3^{5/2}},
\]

which satisfies the second equality. In this way we obtain the ten–component period vector

\[
\Pi \overset{\text{def}}{=} \begin{pmatrix} \mathcal{G}_0 \\ \mathcal{G}_1 \\ \mathcal{G}_2 \\ \mathcal{G}_3 \\ \mathcal{G}_4 \\ z^0 \\ z^1 \\ z^2 \\ z^3 \\ z^4 \end{pmatrix} = f \begin{pmatrix} -\omega_{QQQ} \\ \omega_{RQQ} \\ \omega_{QRR} \\ \omega_{QQR} \\ c(\omega^2 \omega + \omega \hat{\omega}) \\ \omega_{RRR} \\ \omega_{QRR} \\ \omega_{RRQ} \\ \omega_{RRQ} \\ c(\omega + \hat{\omega}) \end{pmatrix},
\]

Modular transformations induce factors of automorphy in the periods as has been noted above. These factors are the same as those that we encountered in the transformation of \(\mu\) in Section 4. We have therefore introduced a gauge factor

\[
f = \prod_{i=1}^{3} J^{-\frac{i}{2}}(\phi_i) \left( \frac{d\phi_i}{dJ} \right)^{1/2}.
\]

Thus defined \(\Pi\) transforms with integral symplectic transformations and without any factors of automorphy. The reader interested in a further discussion of the modular properties of the periods is referred to the works of \[38,39\].

### 6. The Normalized Yukawa Couplings

We are now able to return to the calculation of the Yukawa couplings and the metric on the moduli space with a view to comparing with known results. In virtue of the relations given by special geometry, the Yukawa couplings are given by

\[
y_{ABC} = \int \Omega \wedge \partial_{ABC} \Omega, \\
y_{ABC} = \Pi^T \Sigma \partial_{ABC} \Pi,
\]

where \(\Sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\).
The \( y_{ABC} \) are dependent upon the gauge choice (5.4). To make the comparison with previous work we consider the invariant Yukawa coupling (see [37])

\[
\kappa_{ABC} = \sqrt{g_{A,\bar{A}}} \sqrt{g_{B,\bar{B}}} \sqrt{g_{C,\bar{C}}} e^K |y_{ABC}|.
\]

Here \( K \) is the Kähler potential and \( g_{A,\bar{B}} \) the metric on the parameter space which are given in terms of the period vector by

\[
e^{-K} = -i \Pi^\dagger \Sigma \Pi \quad g_{A,\bar{B}} = \partial_A \partial_{\bar{B}} K.
\]

The moduli space is \( b_{4,3} \)-dimensional and the \( \phi_i \)’s together with the \( s_{mnp} \)’s form a set of coordinates on the moduli space. We can also use ratios of the integral periods as coordinates. These correspond to the ‘flat coordinates’[40]. In these coordinates many expressions take a simple form, as we will see. Some of the quantities of interest may be obtained by setting \( s = 0 \) \textit{ab initio} and working with the periods (5.10). This will not yield all the Yukawa couplings in the orbifold limit; the coupling \( \kappa_{sss} \) for example receives contributions from the order \( s^3 \) terms in the periods that survive the limit \( s \to 0 \). In order to calculate the metric components \( g_{s,\bar{s}} \) we must also differentiate the Kähler potential before setting \( s = 0 \).

We choose flat coordinates:

\[
\zeta^a = \frac{z^a}{z^0}, \quad a = 1, 2, 3, \quad \rho = \frac{z^4}{z^0}.
\]

When \( s = 0 \) the \( \zeta^a \) coincide with the \( \tau_a \) discussed previously and in this limit we shall use the \( \zeta \)'s and the \( \tau \)'s interchangeably. When \( s \neq 0 \) we shall continue to think of the \( \tau \)'s as related to the \( \phi \)'s by (4.5).

The exponential of the Kähler potential is

\[
e^{-K} = -i |f|^2 \prod_{i=1}^{3} \left( Q(\phi_i) R(\bar{\phi}_i) - R(\phi_i) Q(\bar{\phi}_i) \right)
\]

\[
= -2^3 |z^0|^2 \prod_{i=1}^{3} \Im m \tau_i,
\]

with \( f \) chosen as in (5.18). We see from (5.10) that the prepotential is given by the expected quantity [41,31]

\[
G = \frac{z^1 z^2 z^3}{z^0} = (z^0)^2 \tau_1 \tau_2 \tau_3.
\]

This leads to the metric components

\[
g_{\tau_i,\tau_j} = \frac{\delta_{ij}}{4(\Im m \tau_i)^2},
\]

27
which is standard metric on the moduli–space of the torus. The only nonzero Yukawa couplings are \( y_{\tau_1,\tau_2,\tau_3} \) and \( y_{sss} \). We see that

\[
y_{\tau_1,\tau_2,\tau_3} = (z^0)^2 ,
\]

for which the invariant coupling is seen to be

\[
k_{\tau_1,\tau_2,\tau_3} = \sqrt{g_{\tau_1,\tau_1}} \sqrt{g_{\tau_2,\tau_2}} \sqrt{g_{\tau_3,\tau_3}} e^K |y_{\tau_1,\tau_2,\tau_3}| = 1 .
\]

In the limit that \( s \to 0 \) the mixed terms of the metric \( g_{s,\bar{s}} \) or \( g_{i,\bar{s}} \) vanish, while we find that

\[
g_{s,\bar{s}} = (2\pi)^{12} 3^{-\frac{2}{3}} i \prod_{i=1}^{3} \frac{|Z_3(\phi_i)|^2}{R(\phi_j)Q(\phi_i) - Q(\phi_i)R(\phi_j)} .
\]

Apart from the coupling \( k_{\tau_1,\tau_2,\tau_3} \) the only other coupling that survives the limit \( s \to 0 \) is

\[
y_{sss} = 27 f^2 \prod_{j=1}^{3} \left( R(\phi_j, 0)Q(\phi_j, 1) - Q(\phi_j, 0)R(\phi_j, 1) \right) = \phi_1\phi_2\phi_3 y_{\phi_1,\phi_2,\phi_3} ,
\]

which is in agreement with the calculations in Section 3 using the cohomology ring, cf. Table 3.1. Changing variables, we find that to lowest order in \( s \),

\[
y_{ppp} = (z^0)^2 \prod_{k=1}^{3} 3\sqrt{2} (2\pi)^{3} \nu_3 R(\phi_k) = (z^0)^2 \prod_{k=1}^{3} \sqrt{2i} \nu_3^2 (\tau_k) \chi_0(\tau_k) ,
\]

where \( \nu = \sqrt{\frac{3}{2} \frac{\Gamma^2(\frac{3}{4})}{\Gamma(\frac{3}{4})}} \) and we have used the relation\(^1\)

\[
R(\phi) = \frac{8\pi^3 i}{3} \nu_3^2 \chi_1 ,
\]

with \( \phi(\tau) \) is given by Equation (4.5).

The normalized coupling is

\[
k_{sss} = (g^s)^{\frac{3}{2}} e^K |y_{sss}|
\]

\[
= \prod_{j=1}^{3} \left( \frac{3^{1/4}}{2\pi^3 \Gamma^3(\frac{1}{3})} |\phi_j| \left| R(\phi_j)Q(\phi_j) - Q(\phi_j)R(\phi_j) \right|^{\frac{3}{2}} \right)
\]

\[
= \frac{3^3}{(2\pi)^3} \prod_{j=1}^{3} \nu_r_j |\phi_j R(\phi_j)|
\]

\[
= \prod_{j=1}^{3} \nu_r_j |\eta^2(\tau_j)\chi_0(\tau_j)| ,
\]

\(^1\) This relation was suggested by the analysis of this section. We have checked it numerically but we do not have an analytic proof.
where $r_j = \frac{2}{\sqrt{3}} \sqrt{3m \tau_j}$. Now we can write the coupling $f_{m_1m_2m_3}f_{m_2m_3m_2}f_{m_3m_2m_3}$ as a product of three factors (one for each column of the matrix $m_{jk}$) in terms of $\chi_0$ and $\chi_1$. The results are tabulated in Table 6.1.

| Condition                        | Factor                                      |
|----------------------------------|---------------------------------------------|
| If all the $m_{jk}$, $j = 1, 2, 3$, are distinct | $r_k \nu |\eta^2(\tau_k)\chi_1(\tau_k)|$ |
| If precisely two of the $m_{jk}$ are distinct     | 0                                           |
| If all three are equal            | $r_k \nu |\eta^2(\tau_k)\chi_0(\tau_k)|$ |

**Table 6.1**: The factor for the Yukawa coupling in terms of $\chi_0$ and $\chi_1$.

Evaluating the couplings in the limit $s \to 0$ corresponds to calculating the fully corrected couplings, including instanton effects, in the orbifold limit of the $\mathcal{Z}$-manifold. This was first done in [10]. Later, in [12] and [26], the normalized rules were written for a particular basis. The Yukawa couplings we find are the absolute values of those in [12]²; we may conclude that our results agree with previous calculations in the orbifold limit. Note that the ratio of the third row to the first row is

$$\frac{\chi_0(\tau_k)}{\chi_1(\tau_k)} = \phi_k,$$

which agrees with our calculations using the cohomology ring, cf. Table 3.1. Also we see that this is indeed a sum over instantons and that without these corrections the coupling vanishes. As an example, we find that

$$f_{111}f_{122}f_{133} = \phi_1 e_1 e_2 e_3$$

$$= r_1 \nu \eta^2(\tau_1) \chi_0(\tau_1)$$

$$= r_1 \nu \sum_{m,n} e^{2\pi i(m^2+mn+n^2)\tau_1}.$$

Our purpose has been to rediscover the instanton corrections to the Yukawa couplings on the $\mathcal{Z}$ orbifold. However, our method may be used to do more. Since the periods have been found as an expansion in $s$, it is possible to examine the expansion of the Yukawa couplings in the flat coordinate $\rho$. This should follow the predictions of [38] and [39] which were based on the behaviour of the Yukawa couplings under modular transformations. One may also consider the more general case of allowing all $s_{mnp}$ to be nonzero. This would allow a more complete treatment of the modular group.

² [26] miss a factor of $\sqrt{2}$ in $\Im m \tau$ when they quote the results of [12].
A. The Periods Calculated from a Homology Basis

We wish to show in this Appendix a method of calculating the periods by directly evaluating the integral of $\Omega$ over a homology basis. We do this firstly in the orbifold limit, and then in the case where one $s_{mnp}$ is nonzero.

A. 1. The Periods on the Mirror when All $s_{mnp} = 0$

Recall that

$$\Omega_{5,2} = \frac{1}{2\pi i} \int_{C_p} \frac{\mu}{\beta^3},$$

where $C_p$ is a circle around $p = 0$, and that in the limit where all $s_{mnp} \to 0$, we have

$$p = \sum_{i=1}^{9} y_i^3 - 3 \sum_{k=1}^{3} \phi_k e_k.$$

We can create a 7–cycle

$$\Gamma = \{ y_9 = 1; \ y_8 \text{ given by the branch of } p = 0 \}

\text{for which } \arg(y_8) \to \frac{\pi}{3} \text{ as } \phi_1, \phi_2, \phi_3 \to 0;

\gamma_1 \times \ldots \times \gamma_7 \}$$

using one-cycles $\gamma_j$ that are defined in the $y_j$ plane: the cycle comes in from infinity along the line $y_j = t_j e^{2\pi i/3}$, where $t_j$ is a real number, and goes out along the real axis. Along this cycle the branch choice is unique. $C_p$ is a circle around $p = 0$ in the $y_8$ plane; this can be deformed to $\gamma_8$. Now we can compute the period in the approximation that the $\phi_i$’s are all small. In that case we can write

$$q = \int_{\Gamma} \Omega_{5,2}

= \frac{1}{2\pi i} \int_{\gamma_1 \times \ldots \times \gamma_8} \frac{dy_1 \ldots dy_8}{(\sum_{i=1}^{9} y_i^3)^3} \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} \left( \frac{3\phi_k e_k}{\sum_{i=1}^{9} y_i^3} \right)^n.$$

The integrals are of the form

$$I(k_1, \ldots, k_8) = \int_{\gamma_1 \times \ldots \times \gamma_8} \frac{\prod_{i=1}^{8} y_i^{k_i} dy_i}{(1 + \sum_{i=1}^{8} y_i^3)^{3+n}}

= \frac{1}{3^8} (1 - \omega_{r_1+1}) (1 - \omega_{r_2+1}) \prod_{i=1}^{3} \frac{(1 - \omega^{r_i+1})^2 \Gamma^3 \frac{r_{i+1}}{3}}{\Gamma(n+3)},$$

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where $r_1 = k_1 = k_2 = k_3, r_2 = k_4 = k_5 = k_6, r_3 = k_7 = k_8 = k_9$. The last equality follows by noting that $\sum_{i=1}^{9} k_i = 3n$ or $\sum_{i=1}^{3} r_i = n$. Using this we can write

$$q = \frac{1}{2 \cdot 3^8 \cdot 2\pi i} \sum_{r_1, r_2, r_3=0}^{\infty} (1 - \omega^{r_1+1}) (1 - \omega^{r_2+1}) \prod_{j=1}^{3} \left\{ (1 - \omega^{r_j+1})^2 \Gamma^3 \left( \frac{r_j + 1}{3} \right) \frac{(3\phi_j)^{r_j}}{r_j!} \right\}.$$ 

This factors into a product of three sums, one for each $\phi_i$. The sums for $\phi_1$ and $\phi_2$ are identical and will be called $S_1$; the other we will call $S_2$. So we have

$$q = \frac{1}{2 \cdot 3^8 \cdot 2\pi i} S_1(\phi_1) S_1(\phi_2) S_2(\phi_3).$$

These factors can be written as linear combinations of hypergeometric functions, as we will now show. Consider

$$S_1(\phi) = \sum_{r=0}^{\infty} (3\phi)^r (1 - \omega^{r+1})^3 \frac{\Gamma^3 (r + 1)}{r!}$$

$$= \sum_{k=0}^{\infty} (3\phi)^{3k} (1 - \omega)^3 \frac{\Gamma^3 (k + 1/3)}{\Gamma (3k + 1)} + \sum_{k=0}^{\infty} (3\phi)^{3k+1} (1 - \omega^2)^3 \frac{\Gamma^3 (k + 2/3)}{\Gamma (3k + 2)}.$$

The last line is obtained by splitting the original sum into three sums depending on the value of $r \mod 3$. Only two terms survive since $(1 - \omega^{r+1})^3 = 0$ if $r = 2 \mod 3$. Using the multiplication formula for the gamma function

$$\Gamma(z)\Gamma(z + 1/3)\Gamma(z + 2/3) = 2\pi \ 3^{1/2 - 3z} \Gamma(3z)$$

we arrive at

$$S_1(\phi) = 3 \cdot 2\pi i \left( -\sum_{k=0}^{\infty} \phi^{3k} \frac{\Gamma^2 (k + 1/3)}{\Gamma (k + 2/3) k!} + \phi \sum_{k=0}^{\infty} \phi^{3k} \frac{\Gamma^2 (k + 2/3)}{\Gamma (k + 4/3) k!} \right).$$

As was expected, these are linear combinations of the solutions of Equation (5.7)

$$S_1(\phi) = 3 \cdot 2\pi i (-Z_1(\phi) + Z_2(\phi))$$

$$S_2(\phi) = -2\pi \sqrt{3} \omega (Z_1(\phi) + \omega Z_2(\phi)).$$

Other periods can be found by integrating over different 7–cycles. We can alter $\Gamma$ by changing any of the one–cycles. If we choose a different one–cycle in the $x_{ij}$’th coordinate
plane, only the function in $\phi_i$ in $q$ changes. For example, if we substitute $\hat{\gamma}_1$ for $\gamma_1$, where $\hat{\gamma}_1$ is $\gamma_1$ rotated by $e^{2\pi i/3}$, then we get

$$\frac{1}{2\pi i} \int_{\gamma_1 \times \gamma_2 \times \ldots \times \gamma_7 \times C_p} \frac{\mu}{p^3} = \frac{1}{2 \cdot 3^8 \cdot 2\pi i} \hat{S}_1(\phi_1)S_1(\phi_2)S_2(\phi_3),$$

where

$$\hat{S}_1(\phi_1) = \omega S_1(\omega \phi_1) = 3 \cdot 2\pi i\omega (\omega S_1(\phi_1) + \omega Z_2(\phi_1)).$$

Similarly if instead of $\gamma_1$, we take $\tilde{\gamma}_1$ defined by a rotation of $e^{4\pi i/3}$, the factor in $q$ involving $\phi_1$ becomes

$$\tilde{S}_1(\phi_1) = \omega^2 S_1(\omega^2 \phi_1) = 3 \cdot 2\pi i\omega^2 (\omega^2 S_1(\phi_1) + \omega^2 Z_2(\phi_1)).$$

Were we to have changed the $\gamma_7$ to $\hat{\gamma}_7$ or $\tilde{\gamma}_7$, we would change the $\phi_3$ dependence to

$$\hat{S}_2(\phi_3) = \omega S_2(\omega \phi_3) \text{ or } \tilde{S}_2(\phi_3) = \omega^2 S_2(\omega^2 \phi_3),$$

respectively. Choosing a different branch for the solution of $p = 0$ in the $y_8$ plane also changes the period; $C_p$ can be deformed to $\hat{\gamma}_8$ if we take $\text{arg}(y_8) \to \pi$, and to $\tilde{\gamma}_8$ if $\text{arg}(y_8) \to \frac{5\pi}{3}$.

We have a wealth of possible seven-cycles using the three one-cycles $\gamma$, $\hat{\gamma}$, and $\tilde{\gamma}$. However, this does not lead to an overabundance of independent periods. First of all, a simple permutation of two coordinates within the same $e_i$ has no effect: the period derived from choosing $\hat{\gamma}_1 \times \gamma_2$ is the same as that from choosing $\gamma_1 \times \hat{\gamma}_2$, for example. Furthermore, since $\gamma + \hat{\gamma} + \tilde{\gamma} = 0$, there are many other simplifying relations that are easy to find. As one more example, choosing $\hat{\gamma}_1 \times \hat{\gamma}_2$ gives the same period as choosing $\tilde{\gamma}_1 \times \gamma_2$. Of course, since all these hypergeometric functions are solutions of the same second order differential equation, each can be written as a linear combination of only 2 independent functions. We note that

$$\begin{align*}
\hat{S}_1(\phi) &= -2S_1(\phi) + 3S_2(\phi) \\
\hat{S}_1(\phi) &= S_1(\phi) - 3S_2(\phi) \\
\hat{S}_2(\phi) &= -S_1(\phi) + S_2(\phi) \\
\hat{S}_2(\phi) &= S_1(\phi) - 2S_2(\phi) 
\end{align*}$$

(A.1)

Since there are two linearly independent solutions of Equation (5.7) for each $\phi_i$, and since $2^3 = 8$, we conclude that we can choose to write the eight independent periods on the manifold as products of these solutions.
A.2. The Expansion for One \( s_{mnp} \) Nonzero

We wish to examine the periods on the manifold to some order in the \( s_{mnp} \). Suppose we let only one \( s_{mnp} \equiv s \) be non-zero. We take

\[
p = \sum_{i=1}^{9} y_i^3 - 3 \sum_{k=1}^{3} \phi_k e_k - 3sf_{mnp},
\]

and a variation of the seven–cycle

\[
\Gamma = \{ y_9 = 1; \ y_8 \text{ is a solution of } p = 0, \text{ branch chosen by } \arg(y_8) \to \frac{\pi}{3} \text{ as } \phi_1, \phi_2, \phi_3, \text{ and } s \to 0; \gamma_1 \times \cdots \times \gamma_7 \}.
\]

The 7–cycle is built out of one–cycles \( \gamma_i \) and a branch choice. As explained previously, we could rotate some of the one–cycles by \( e^{2\pi i/3} \) to \( \hat{\gamma}_i \), or choose a different branch. We use the notation \( (\delta_1, \delta_2, \ldots, \delta_8) \) to stand for the choices made: \( \delta_i = 0 \) means that we integrate \( y_i \) over \( \gamma_i \); \( \delta_i = 1 \) means that we integrate \( y_i \) over \( \hat{\gamma}_i \). We calculate the period via

\[
q = \int (\delta_1, \delta_2, \ldots, \delta_8) \Omega_{5,2}, \text{ and expand in powers of } s, \text{ using the notation that } q_i \text{ is of order } i \text{ in } s: q = q_0 + q_1 + q_2 + \ldots. \text{ We further split the period up according to the value of } r = k \mod 3, \text{ by letting } k = 3d + r.
\]

It turns out that \( q \) depends only on four phases

\[
\begin{align*}
\Delta_1 &= \delta_1 + \delta_2 + \delta_3 \\
\Delta_2 &= \delta_4 + \delta_5 + \delta_6 \\
\Delta_3 &= \delta_7 + \delta_8 \\
\Delta_s &= \delta_m + \delta_{n+3} + \delta_{p+6}.
\end{align*}
\]

Since we have set \( y_9 = 1 \), we take the nonzero \( s_{mnp} \) to be \( s_{m1} \) or \( s_{m2} \); then we can write the period in the following way:

\[
\begin{align*}
\frac{2 \cdot 2\pi i \cdot 3^8}{(2\pi \sqrt{3})^3} q &= \omega^{\Delta_1 + \Delta_2 + \Delta_3} \sum_{d=0}^{\infty} \frac{(3s)^{3d}}{(3d)!} F_1 \\
&\quad + \omega^{\Delta_1 + \Delta_2 + \Delta_3 + \Delta_s} \sum_{d=0}^{\infty} \frac{(3s)^{3d+1}}{(3d+1)!} F_2 \\
&\quad + \omega^{2(\Delta_1 + \Delta_2 + \Delta_3 + \Delta_s)} \sum_{d=0}^{\infty} \frac{(3s)^{3d+2}}{(3d+2)!} F_3 \quad (A.2)
\end{align*}
\]

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where
\[ F_1 = 3\omega [-Z_1(\phi, d) + \omega^\Delta Z_2(\phi, d)] [-Z_1(\phi, d) + \omega^\Delta Z_2(\phi, d)] \times 
\begin{align*}
&[Z_1(\phi, d) + \omega^{1+\Delta} Z_2(\phi, d)] \\
&F_2 = -3\omega[Z_3(\phi, d)] [Z_3(\phi, d)] [Z_3(\phi, d)] \\
&F_3 = -3\omega^2[Z_4(\phi, d)] [Z_4(\phi, d)] [Z_4(\phi, d)] 
\end{align*} 
\]
and
\[ Z_1(\phi, d) = \frac{\Gamma(\frac{1}{2}) \Gamma(d + \frac{1}{2})}{\Gamma(\frac{1}{2})} F_1(\frac{1}{2}, d + \frac{1}{2}; \phi^3) \]
\[ Z_2(\phi, d) = \frac{\Gamma(\frac{1}{2}) \Gamma(d + \frac{1}{2})}{\Gamma(\frac{1}{2})} \phi_2 F_1(\frac{1}{2}, d + \frac{1}{2}; \phi^3) \]
\[ Z_3(\phi, d) = Z_1(\phi, d + \frac{1}{2}), \quad Z_4(\phi, d) = Z_2(\phi, d + \frac{3}{2}). \]

Now if we take the combination
\[ q(\Delta_1, \Delta_2, \Delta_3, \Delta_s) + q(\Delta_1, \Delta_2, \Delta_3, \Delta_s + 1) + q(\Delta_1, \Delta_2, \Delta_3, \Delta_s + 2) \]
\[ = \frac{(2\pi \sqrt{3})^3}{2 \cdot 2\pi i \cdot 3^7} \omega^{\Delta_1+\Delta_2+\Delta_3} \sum_{d=0}^{\infty} \frac{(3s)^{3d}}{(3d)!} F_1 \]
we find periods that are independent of \( \Delta_s \), but they still depend on \( \Delta_1, \Delta_2 \) and \( \Delta_3 \) independently. Thus there are eight of these. They reduce to our familiar eight in the limit \( s \to 0 \). The first correction to these periods is of order \( s^3 \). The integral periods \( z^0, \ldots, z^3, G_0, \ldots, G_3 \) of Equation (5.17) are linear combinations of these. The change of basis involves a symplectic transformation times a scale factor, as can be inferred from (5.13).

If we form the combination
\[ q(\Delta_1, \Delta_2, \Delta_3, \Delta_s) + q(\Delta_1 + 1, \Delta_2, \Delta_3, \Delta_s + 2) + q(\Delta_1 + 2, \Delta_2, \Delta_3, \Delta_s + 1) \]
\[ = \frac{(2\pi \sqrt{3})^3}{2 \cdot 2\pi i \cdot 3^7} \left\{ \omega^{\Delta_1+\Delta_2+\Delta_3+\Delta_s} \sum_{d=0}^{\infty} \frac{(3s)^{3d+1}}{(3d+1)!} F_2 \right. \]
\[ + \omega^{2(\Delta_1+\Delta_2+\Delta_3+\Delta_s)} \sum_{d=0}^{\infty} \frac{(3s)^{3d+2}}{(3d+2)!} F_3 \right\} \]
we find that there are only two new independent periods, as there should be. (Recall that the number of periods is \( 2(b_{4,3} + 1) \); so for each nonzero \((4,3)\)-form there are two periods.) Choosing the periods formed when \( \Delta_1 + \Delta_2 + \Delta_3 + \Delta_s = 1 \) and \( 2 \) and applying the same change of scale as for the other periods leads us to the periods \( z^4 \) and \( G_4 \) of (5.15) and (5.16).

Altogether we now have 10 independent periods; the eight we found when all \( s_{mnp} = 0 \) get corrected by terms of order \( s^{3d} \) and the two new ones have terms of order \( s^{3d+1} \) and
\( s^{3d+2} \). With foresight we could have chosen the gauge

\[
\Omega_{5,2} = 6(2\pi i)^3 f \int \frac{\mu}{p^3}
\]

using the factor \( f \) as given in (5.18) and incorporating the change of scale mentioned above. The integral periods of (5.17) can be calculated directly by integrating this over a homology basis derived from the cycles defined above.

As a check on the relative normalization of the integral periods, we compute the value of

\[
W_s = \int \Omega \wedge \partial_s \Omega = \mathcal{G}_a \partial_s z^a - z^a \partial_s \mathcal{G}_a.
\]

This should vanish, since \( \Omega \in H^{5,2} \) and \( \partial_s \Omega \in H^{5,2} \oplus H^{4,3} \). Looking at the expansion to \( \mathcal{O}(s^2) \),

\[
W_s = \frac{3^3 s^2}{2} f^2 \left\{ \prod_j (R(\phi_j,0)Q(\phi_j,1) - Q(\phi_j,0)R(\phi_j,1)) + (\omega^2 - \omega)c^2 \prod_j Z_3(\phi_j)Z_4(\phi_j) \right\},
\]

so using

\[
R(\phi,0)Q(\phi,1) - Q(\phi,0)R(\phi,1) = \frac{(2\pi)^5 i}{3^2} \phi(1 - \phi^3)^{-1}
\]

\[
Z_3(\phi)Z_4(\phi) = \frac{2\pi}{\sqrt{3}} \phi(1 - \phi^3)^{-1}
\]

we find that \( W_s = 0 \) only when \( c = \frac{(2\pi)^6 i}{3^{5/2}} \).
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