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Observation of topological Uhlmann phases with superconducting qubits

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Topological insulators and superconductors at finite temperature can be characterized by the topological Uhlmann phase. However, a direct experimental measurement of this invariant has remained elusive in condensed matter systems. Here, we report a measurement of the topological Uhlmann phase for a topological insulator simulated by a system of entangled qubits in the IBM Quantum Experience platform. By making use of ancilla states, otherwise unobservable phases carrying topological information about the system become accessible, enabling the experimental determination of a complete phase diagram including environmental effects. We employ a state-independent measurement protocol which does not involve prior knowledge of the system state. The proposed measurement scheme is extensible to interacting particles and topological models with a large number of bands.

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INTRODUCTION

The search for topological phases in condensed matter1-8 has triggered an experimental race to detect and measure topological phenomena in a wide variety of quantum simulation experiments.9-15 In quantum simulators the phase of the wave function can be accessed directly, opening a whole new way to observe topological properties9,11,16 beyond the realm of traditional condensed matter scenarios. These quantum phases are very fragile, but when controlled and mastered, they can produce very powerful computational systems like a quantum computer.17,18 The Berry phase19 is a special instance of quantum phase, that is purely geometrical20 and independent of dynamical contributions during the time evolution of a quantum system. In addition, if that phase is invariant under deformations of the path traced out by the system during its evolution, it becomes topological. Topological Berry phases have also acquired a great relevance in condensed matter systems. The now very active field of topological insulators (TIs) and superconductors (TSCs)1-3 ultimately owes its topological character to Berry phases21 associated to the special band structure of these exotic materials. However, if the interaction of a TI or a TSC with its environment is not negligible, the effect of the external noise in the form of, e.g., thermal fluctuations, makes these quantum phases very fragile,22-24 and they may not even be well defined. For the Berry phase acquired by a pure state, this problem has been successfully addressed for one-dimensional systems25 and extended to two-dimensions later.26-28 The key concept behind this theoretical characterization is the notion of “Uhlmann phase”,29-42 a natural extension of the Berry phase for density matrices. In analogy to the Berry phase, when the Uhlmann phase for mixed states remains invariant under deformations, it becomes topological. Although this phase is gauge invariant and thus, in principle, observable, a fundamental question remains: how to measure a topological Uhlmann phase in a physical system? To this end, we employ an ancillary system as a part of the measurement apparatus. By encoding the temperature (or mixedness) of the system in the entanglement with the ancilla, we find that the Uhlmann phase appears as a relative phase that can be retrieved by interferometric techniques. The difficulty with this type of measurement is that it requires a high level of control over the environmental degrees of freedom, beyond the reach of condensed matter experiments. On the contrary, this situation is especially well-suited for a quantum simulation scenario. Specifically, in this work we report: (i) the measurement of the topological Uhlmann phase on a quantum simulator based on superconducting qubits,48-50 in which we have direct control over both system and ancilla, and (ii) the computation of the topological phase diagram for qubits with an arbitrary noise degree. A summary and a comparison with pure state topological measures are shown in Fig. 1. In addition, we construct a state independent protocol that detects whether a given mixed state is topological in the Uhlmann sense. Our proposal also provides a quantum simulation of the AIII class51,52 of TIs (those with chiral symmetry) in the presence of disturbing external noise. Other cases of two-dimensional TIs, TSCs and interacting systems can also be addressed by appropriate modifications as mentioned in the conclusions.

RESULTS

Topological Uhlmann phase for qubits

We briefly present the main ideas of the Uhlmann approach for a two-band model of TIs and TSCs simulated with a qubit. Let \( \theta(t) \) define a closed trajectory along a family of single qubit
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\[ \rho = \frac{1}{2} (1 + R \mathbf{n} \cdot \sigma) \]

between pure density matrices. The computation of the unitary \( U \onlinebreak \) in Eq. (2) for a transportation in time of \( \theta \) according to the Hamiltonian (3) yields

\[ U_S(t) = e^{-i \int_{t_0}^{t} h(t') dt' \mathbf{d} \theta} \mathbf{e} \]

where \( h(t) = H_S \) is the Hamiltonian parameterized by \( \theta \). This implements the eigenstate transport \( |\phi(t)\rangle = U_S(t)|\phi(0)\rangle \) and \( |\phi(t)\rangle = U_S(t)|\phi(0)\rangle \).

Next, we consider the Hamiltonian of a two-band TI in the AII chiral-unitary class,

\[ H_k = \frac{\Delta}{2} n_k \cdot \sigma \]

where \( \Delta \) is the detuning \( \Delta = 2(\cos \theta + M) \) between qubit and drive is parameterized in terms of \( \theta \) and a hopping amplitude \( M \), whereas the coupling strength between the qubit and the incident microwave field is given by \( \Omega = 2 \sin \theta \).

The non-trivial topology of pure quantum states \( \langle \psi | \psi \rangle \in (0, 1) \) of this class of topological materials can be witnessed by the winding number. This is defined as the angle swept out by \( n_\theta \) as \( \theta \) varies from \( 0 \) to \( 2\pi \), namely,

\[ \omega_1 = \frac{1}{2\pi} \int \frac{\mathbf{d} \theta}{n_\theta} \int \mathbf{d} \theta \]

Then, using Eqs. (3) and (4), the system is topological \( (\omega_1 = 1) \) when the hopping amplitude is less than unity \( (M < 1) \) and trivial \( (\omega_1 = 0) \) if \( M > 1 \). In fact, the topological phase diagram coincides with the one given by the Berry phase acquired by the “ground” state \( |\psi\rangle \) or the “excited” state \( |\phi\rangle \) of Hamiltonian (3) when \( \theta \) varies from \( 0 \) to \( 2\pi \), (see Supplementary Note 2).

The computation of the unitary \( U \) in Eq. (2) for a transportation in time of \( \theta \) according to the Hamiltonian (3) yields

Along a trajectory \( \theta(t) \), the induced purity evolution \( \rho(t) \) is given by

\[ \rho_\theta = \frac{1}{2} (1 + R \mathbf{n} \cdot \sigma) \]

where \( \mathbf{n} \) is the actual gap between the valence and conduction bands in the TI, and \( n_\theta \) is a unit vector called winding vector. We now map the crystalline momentum \( k \) of the TI to a tunable time-dependent parameter \( \theta \) of the quantum simulator. When invoking the rotating wave approximation this model also describes, e.g., the dynamics of a driven transmon qubit.

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In addition, we can consider a similar form for the unitary \( U \) in Eq. (2),

\[ U(t) = |U(t)| e^{-i \int_{t_0}^{t} h(t') dt' \mathbf{d} \theta} \mathbf{e} \]

The detailed technical derivation is provided in Supplementary Notes 1 and 2. Now, from Eq. (2) it is possible to define the relative phase \( \Phi_M \) between the initial \( |\psi(0)\rangle \) and the final state, i.e., \( |\psi(\theta)\rangle \). For Hamiltonian (3), density matrix (1) and purification (2), we find

\[ \Phi_M := |\langle \psi(0) | \psi(\theta) \rangle | = \rho_{\theta} = \rho_{\theta} + \rho_{\theta}^* \]

where \( \rho_{\theta} = \int_{t_0}^{t} h(t') dt' \mathbf{d} \theta \). As commented before, by assuming \( \rho_{\theta} = \rho_{\theta} + \rho_{\theta}^* \), the purification precisely follows Uhlmann parallel transport and the relative phase \( \Phi_M \) becomes the Uhlmann phase \( \Phi_M \) associated to the trajectory. For a closed path \( t_1, t_1 \), the integral \( \int_{t_0}^{t_1} h(t') dt' \mathbf{d} \theta \) becomes the topological Berry phase. In that case, the Uhlmann phase simplifies to

\[ \Phi_M = \arg \{ \rho_{\theta} \} = \arg \{ \rho_{\theta} \} = \rho_{\theta}^* \]

We can now deduce the topological properties of these phases in the presence of external noise, as measured by the parameter \( \rho \).
(Eq. (1)). This is depicted in Fig. 1. Namely, if $M > 1$ then $\omega_i = 0$, and $\Phi_U = 0$ (trivial phase) for every mixedness parameter $r$. If $M < 1$ then $\omega_i = 1$ and one obtains $\Phi_U = arg\{-cos(2n\sqrt{r(1-r)})\}$. If the state is pure ($r = 0$), then $\Phi_U = \pi$, recovering the same topological phase given by the winding number and the Berry phase. However, for $r \neq 0$ there are critical values of the mixedness $r_c$ at which the Uhlmann phase, according to Eq. (8), jumps from $\pi$ to zero (see Fig. 1). The first $r_c = \frac{1}{3}(2 - V3) \approx 0.067$ signals the mixedness at which the system loses the topological character of the ground state. Moreover, there exists another $r_c = 1 - r_c$, at which the system becomes topological again due to the topological character of the excited state ($r = 1$). Notice that at $r = 1$ the system becomes a pure state again (the excited state), which is also topologically non-trivial according to the Berry phase. Actually, provided that the weight $p_i c_p = 0.5$, the system is topological in the Uhlmann sense as long as $M < 1$. This reentrance in the topological phase at $r_c$ was absent in previous works.35–37

Experimental realization

Measuring the topological Uhlmann phase is a very challenging task since its definition in terms of purifications implies precise control over auxiliary/environmental degrees of freedom (the ancilla). In an experiment, we therefore include an extra ancilla qubit representing the environment. We also include a third qubit acting as a probe system $P$, such that by measuring qubit $P$ we retrieve the accumulated phase by means of interferometric techniques. The measurement protocol is described in Fig. 2:

**Step 1.** Following Eq. (2), we prepare the initial state $|\Psi_{\theta(0)}\rangle \otimes |0\rangle_P$ (red block of Fig. 2) using single qubit rotations $R_{y}$ about the y-axis for an angle $\gamma = 2arccos\sqrt{1 - r}$ and a two-qubit controlled not gate. For superconducting qubits, the latter can be performed, e.g., by implementing a controlled phase gate for frequency-tunable transmons54 or by a cross-resonance gate.55

**Step 2.** We apply the bi-local unitary $U_S(t) \otimes U_A(t)$ on $S \otimes A$ conditional to the state of the probe $P$. This is accomplished by single qubit rotations about an angle $\beta_1$ or $\beta_2$ determined by $h(t)$ and $\rho_{\sigma}$ (blue block of Fig. 2), and two-qubit gates. This decomposition is based on the fact that any controlled unitary gate can be always decomposed as a product of unitary single-qubit gates and two-qubit CNOT gates.18 Figure 2 shows the final result after the decomposition of the Uhlmann transport, conditional to the probe qubit $P$, is performed. As a result, the three qubits ($S$, $A$, $P$) are in the superposition

$$|\Phi_t\rangle_{SA} = \frac{1}{\sqrt{2}}(|\Psi_{\theta(0)}\rangle \otimes |0\rangle_P + |\Psi_{\theta(0)}\rangle \otimes |1\rangle_P).$$

(9)

**Step 3.** After the holonomic evolution has been completed, we read out $\Phi_M$ from the state of the probe qubit. Tracking out the system and ancilla in Eq. (9), the reduced state for the probe qubit is

$$\rho_P = \frac{1}{2}(\cdot + \text{Re}(\langle\Psi_{\theta(0)}|\langle\Psi_{\theta(0)}\rangle)\sigma_x + i \text{Im}(\langle\Psi_{\theta(0)}|\langle\Psi_{\theta(0)}\rangle)\sigma_y).$$

(10)

Thus, by measuring the expectation values $\langle \sigma_x \rangle$ and $\langle \sigma_y \rangle$ (green block of Fig. 2), we can retrieve $\Phi_M$ in the form

$$\Phi_M = \arg\{\langle \sigma_x \rangle + i \langle \sigma_y \rangle\} = \arg\{\langle\Psi_{\theta(0)}|U_S(t) \otimes U_A(t)|\Psi_{\theta(0)}\rangle\}.\$$

(11)

In Fig. 3 we present the results of phase measurements performed on the IBM Quantum Experience platform,36 using three transmon qubits coupled through co-planar waveguide resonators (see Methods). In Fig. 3a, we show the measurement of the Uhlmann phase $\Phi_U$ for different values of the mixedness parameter $r$, where we set $M = 0.2$ and $\rho_{\sigma} = \rho_n$, i.e., fulfilling the parallel transport condition. The critical jump from $\Phi_U = \pi$ (topological) to $\Phi_U = 0$ (trivial) is clearly observed following the previous protocol. Additionally, we can check whether the Uhlmann parallel transport condition is satisfied at every time interval during the experiment. By partitioning the closed trajectory in small time steps $\delta t$, the relative phase between the state at time $n\delta t$ and at $(n + 1)\delta t$ must be close to zero if the condition is fulfilled. This is the case in the experiment as shown in Fig. 3b. During the state preparation (Step 1), we need to include two additional single qubit rotations $R_{y}^{\beta_1}$ and $R_{y}^{\beta_2}$ acting on the system and ancilla qubits respectively, where $\alpha_1^{\delta t} = 2\pi^{\delta t}$ and $\alpha_2^{\delta t} = \rho_{\sigma}\alpha_1^{\delta t}$. These two unitaries make the entangled state between system and ancilla evolve until the state $|\Psi_{\delta t}\rangle$ is reached. In Step 2, the state evolves to $|\Psi_{\delta t}\rangle$ conditional to the state of the probe $P$. The
is symmetric around the \( \Phi \leq \) function of the applied phase \( \phi \).

Hence, we present a modified state of the Uhlmann parallel condition, which implies the ancillary weight \( \rho_\text{a} \) in Eq. (2), or the time step in the simulation model given in Methods.

State-independent protocol

The application of \( U_\text{a}(t) \) and \( U_\text{t}(t) \) with \( \rho_\text{a} = \rho_\text{T} \) to the purification \( |\Psi_\text{f}(t)\rangle \) implements the Uhlmann parallel transport and hence \( \Phi_\text{M} = \Phi_\text{T} \). However, this would imply some knowledge about the mixedness parameter \( r \) beforehand, which is not always possible. Hence, we present a modification of the previous protocol to measure the topological Uhlmann phase without prior knowledge of the state \( \rho \) and its mixedness parameter \( r \).

Firstly, we fix \( \theta = 2\pi t \) and consider open holonomies \( \frac{1}{2} < t < 1 \) covering more than one half of the complete path. No previous knowledge of the state is assumed to perform the evolution. Hence, the ancillary weight \( \rho_\text{a} \) can be different than \( \rho_\text{T} \) in Eq. (2), but still satisfying \( 0 \leq \rho_\text{a} \leq 1 \). From Eq. (7), the overlap \( \langle \Psi_\text{b}|\Psi_\text{a}\rangle = \text{real} \) is always real and thus the phase \( \Phi_\text{M} \) is either 0 or \( \pi \), depending on both the weight \( \rho_\text{a} \) associated to the state \( \rho_\text{b} \) [Eq. (1)] and the ancillary weight \( \rho_\text{a} \).

We aim to find an \( r \)-independent value for \( \rho_\text{a} \) such that the observed phase \( \Phi_\text{M} \) takes on the same value as the Uhlmann phase for a Hamiltonian with the form of (3). By studying \( \rho_\text{a} \) as a function of the applied \( \rho_\text{a} \), we conclude that if we tune the ancillary weight

\[
\rho_\text{a} = \rho_\text{T} := \frac{1}{\ln 2} \arctan \left( \frac{2}{\tan \left( \frac{\ln 2}{2} \right)} \right) \tag{12}
\]

the value of the observed phase \( \Phi_\text{M}(\rho_\text{a} = \rho_\text{T}) \) coincides with the topological Uhlmann phase \( \Phi_\text{T} \). Algebraic details are provided in Methods.

Note that there is an intuitive reason why we can get topological information out of a phase associated to an open path longer than one half of a non-trivial topological loop. Indeed, \( \theta_\text{r}(t) \) is symmetric around \( t = \frac{1}{2} \). Then, once we have covered one half of the path, we know about the topology of the whole system thanks to this symmetry. Therefore, even an open path for \( \frac{1}{2} < t < 1 \) can be considered as global.

In terms of the experimental protocol, we only need to modify Step 2 by fixing \( \rho_\text{a} = \rho_\text{T} \) for the unitary \( U_\text{A}(t) \). In Fig. 3c, we present the results for the state-independent protocol recovering the topological Uhlmann phase without prior knowledge of the state, for \( M = 0.6 \) and \( t_f = 0.6 \). These are qualitatively the same as in Fig. 3a, but the state-independent protocol is more sensitive to errors mainly around the transition point. The mismatch between experiment and simulations is most likely caused by small calibration-dependent systematic errors in the cross-resonance gates.

**DISCUSSION**

We have successfully measured the topological Uhlmann phase, originally proposed in the context of TIs and superconductors, making use of ancilla-based protocols. The experiment is realized within a minimal quantum simulator consisting of three superconducting qubits. We have exploited the quantum simulator to realize a controlled coupling of the system to an environment represented by the ancilla degrees of freedom. Moreover, we have proposed and tested a state-independent protocol that allows us to classify states of topological systems according to the Uhlmann measure. To our knowledge, this is the first time that a noise/temperature-induced topological transition in a quantum phase has been observed. Recently, these transitions have been addressed in connection to new thermodynamical properties of these systems. The fact that these effects can be experimentally observed opens the possibility for the search of warm topological matter in the lab. Due to the intrinsic geometric character of the Uhlmann phase, our results may find application in generalizations of holonomic quantum protocols for general, possibly mixed, states.

In addition, an increase of experimental resources such as the number of qubits, the speed and fidelity of the quantum gates, etc. will allow us to study additional topological phenomena with superconducting qubits. In particular, by including interactions in the model Hamiltonian we can test different features: quantum simulations of thermal topological transitions in 2D TIs and TSCs, the interplay between noise and interactions within a topological phase, etc. These effects can be achieved since a system with more interacting qubits can be mapped onto models for interacting fermions with spin. Further details can be found in the Supplementary Note 5. Although such a proposal would be experimentally more demanding, it represents a clear outlook that would need precise controllability of more qubits and the ability to perform more gates with high fidelity.
METHODS
Superconducting qubit realization of a controllable Uhlmann phase
The experiments on the topological Uhlmann phase have been realized on the IBM Quantum Experience (ibmqx2), a quantum computing platform with online user-access based on five fixed-frequency transmon-type qubits coupled via on-chip waveguide resonators. We have used three qubits, qubit Q0 as the probe qubit, Q1 as the system qubit and Q2 as the ancilla qubit. This choice is motivated by the connectivity required for the measurement protocol and the superior T1 and T2 times of this set of qubits when compared to the set (Q2, Q3, Q4) at the time of the experiment. We have used the open-source python SDK Qiskit (https://www.qiskit.org) to program the quantum computer and retrieve the data. The explicit quantum algorithm to measure the expectation values of $\sigma_{x}$ and $\sigma_{z}$ is provided in Supplementary Note 4 using the OPENQASM intermediate representation (https://github.com/QISKit/openqasm). The phase is then extracted from the measured data by evaluating $\Phi_{g} = \arg \left( \sigma_{x} - i \left( \sigma_{z} \right) \right)$.

For all experiments we have measured 8192 repetitions providing a single value for the phase. For the measurement of the topological Uhlmann phase (Fig. 3a) we vary the initial mixedness of the system state $\rho = |0\rangle\langle 0 |$ by setting the rotation angle $\gamma = 2 \arccos \sqrt{1 - r}$. The transport of the state according to Uhlmann's parallel transport condition is set by the value $\beta_{1,2} = (0, 1) = \pi$ for $M = 1$ and $\beta_{1,2} = (0, 1) = 2 \sqrt{r} (\gamma - 1)$, as defined in Eq. (7). The energy relaxation times of the qubits are $T_{1}^{(i)} = (45 \mu s, 31 \mu s, 46 \mu s)$ and the decoherence times $T_{2}^{(i)} = (40 \mu s, 27 \mu s, 80 \mu s)$ as stated in the calibrated data.

For the state-independent protocol (Fig. 3c, main text) we set $M = 0.6$ and the final time $t_{f} = 0.6$. The system is rotated about $\beta_{1} = \beta_{2} = \beta_{3} = 0.954407$. In this measurement energy relaxation and decoherence times are $T_{1}^{(i)} = (41 \mu s, 52 \mu s, 62 \mu s)$ and $T_{2}^{(i)} = (31 \mu s, 37 \mu s, 87 \mu s)$. Note, that here the error bars are larger as compared to the state-dependent measurement described above, because the expectation values $\langle \sigma_{x} \rangle$ and $\langle \sigma_{z} \rangle$ are closer to zero leading to larger statistical errors in the phase. Also, we notice a systematic offset of $\delta_{\Phi} = 0.098 \pm 0.014$ from the expected value $\langle \sigma_{y} \rangle_{m} = 0$. Here, $\delta_{\Phi}$ is the average over all r values and repetitions. This offset is subtracted from the phase data $\Phi_{g} = \arg \left( \left( \sigma_{x} - i \left( \sigma_{z} \right) \right) \right)$ and the result is plotted in Fig. 3c.

We consider accumulated phase shifts during two-qubit operations as the main reason for this mismatch. We have also noticed that this value changes for different calibrations of the IBM Quantum Experiment and when taking different sets of qubits.

Finally, for the measurement of the parallel transport condition we modify the algorithm to prepare the intermediate state $|\Psi_{0}(n_{0},t_{0})\rangle$ by applying $U_{0}(n_{0})$ to system and ancilla qubit. For the measurement of the Uhlmann phase, the same circuit as above is used to obtain a state evolution $|\Psi_{0}(n_{0},t_{0})\rangle$. The complete protocol to measure the parallel transport condition is shown in the Supplementary Fig. 1. In the experiment, we choose $M = 0.2$ and $r = 0.02$ to stay within the topological sector. The mixedness angle evolves to $\gamma = 2 \arccos \sqrt{0.95} = 0.2838$. The angles for the intermediate state preparation are determined by $\alpha_{1}(n) = n \alpha_{0}$ and $\alpha_{2}(n) = n \alpha_{0} = 2 \sqrt{r} (\gamma - 1) = 0.2868 \pi$. The evolution from $n_{0}$ to $n + 1 \Delta t$ is determined by the angles $\beta_{1}(n) = \beta_{1}^{(n_{0}=1,0)}$ and $\beta_{2}(n) = \beta_{2}^{(n_{1}=1,0)} = 0.2868 \pi$. The recorded data shown in Fig. 3b, main text, shows that the measured phase difference $|\Phi_{g}(n_{0})\rangle - |\Phi_{g}(n_{0})\rangle_{m} = 0.07 \pm 0.2$ is zero within the statistics. However, the residuals do not follow a normal distribution which hints at systematic gate errors instead of stochastic errors.

State-independent derivation
The derivation of the value for $p_{c}$ [Eq. (12)] is as follows. From Eq. (7) we find the value $p_{c} = p_{c}^{0}$ (where the superindex c stands for critical) at which $\Phi_{g}$ goes abruptly from $\pi$ to 0 as a function of $p_{c}$ and $\rho_{0}$.

$$p_{c}^{0} = \frac{1}{\sqrt{2}} \arctan \left( \frac{1}{\rho_{1} \tan \left( \frac{\gamma}{2} \right)} \right).$$

(13)

If we set $\frac{1}{2} c t < 1$, then $p_{c}$ is a monotonically decreasing function of $\rho_{0}$.

$$\frac{dp_{c}}{d\rho_{0}} = \frac{\tan \left( \frac{\gamma}{2} \right)}{\sqrt{1 + p_{c}^{2} \tan^{2} \left( \frac{\gamma}{2} \right)}} < 0.$$  

(14)

If $M > 1$, then $-\pi/2 < \xi < \pi/2$, which from Eq. (7) implies $\Phi_{g} = 0$ for any value of $p_{c}$ and $\rho_{0}$. Hence, for the trivial case $M = 1$, there is no critical value $p_{c}^{0}$ and $\Phi_{g} = 0$ always. This maps $\Phi_{g}$ to the Uhlmann phase $\Phi_{g}$ at least for this case. On the contrary, if $M < 1$, then $-\pi/2 < \xi < \pi/2$ which implies $\tan \left( \frac{\gamma}{2} \right) < 0$. Since $0 < p_{c} < 1$, then $-\arctan \left( \frac{1}{\rho_{1} \tan \left( \frac{\gamma}{2} \right)} \right) < \pi/2$. Thus, there is always a solution of Eq. (13) with $0 < p_{c} < 1$ for any $p_{c}$. As discussed in the main text, the state $\rho_{0}$ in Eq. (1) is topological in the Uhlmann sense $\Phi_{g} = \pi$, only if $M < 1$ and $p_{c} < 0.5$.

Now, we define $p_{c} := p_{c}^{0}(p_{c} = 0.5)$ using Eq. (13). Note that the true $p_{c}$ of the system is unknown as we have assumed no knowledge of the state. Nevertheless, if $p_{c} > 0.5$, then its associated critical value [from Eq. (13)] is $p_{c}^{0} > p_{c}$. This means that by applying $U_{0}$ with $p_{c} = p_{c}^{0}$ and measuring the associated phase $\Phi_{g}$ we can extract the following conclusions:

- If we measure $\Phi_{g}(p_{c}) = 0$, the system is within a trivial phase ($\Phi_{g} = 0$). Because this implies $p_{c}^{0} < p_{c}$ and hence $p_{c} < 0.5$ ($\Phi_{g} = 0$), as we have proven that $p_{c}^{0}$ always decreases with $p_{c}$.

- If we measure $\Phi_{g}(p_{c}) = \pi$, the system is in a topological phase ($\Phi_{g} = \pi$). Because in that case $p_{c}^{0} > p_{c}$ and then $p_{c} < 0.5$ ($\Phi_{g} = 0$).

Hence, we have just proven that $\Phi_{g}(p_{c}) = \Phi_{0}$. 

Error simulation
The detrimental effect of experimental errors is modeled by means of a Liouvillian term $-\Gamma_{err}$, so that the Liouvillian $-\Gamma$ accounting for the idealized dynamics, is in fact substituted by $-\Gamma_{err} + \Gamma$. Specifically, if a gate is performed during a time $\tau$ via a Hamiltonian $H_{0}$, i.e., $U_{\tau} = e^{-iH_{0}\tau}$, we substitute

$$e^{-iH_{0}\tau} \rightarrow e^{-iH_{0}\tau} \exp \left( -\frac{\Gamma_{err}}{2} \tau \right).$$

Note, that here the error bars are $\Gamma_{err} = \frac{\Gamma}{2}$, which from Eq. (7) implies that $0 < \xi < \frac{\pi}{2}$. The transport of the state is set by the $\Phi_{0}$, the same circuit as above is used to obtain a state $|\Psi_{f}(\tau)\rangle$. Finally, for the measurement of the parallel transport condition we use the experimental measurements of the topological Uhlmann phase $\Phi_{g}$. Despite the errors, the topological transition is clearly noticed.

Data availability
All relevant data are available from the authors on reasonable request.

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AUTHOR CONTRIBUTIONS
O.V. and M.A.M.D. conceived and developed the theoretical project. S.G., A.W. and S.F. devised the experiment. S.F. carried out the experimental measurements and analyzed the data with input from the theoretical teams. All authors wrote the manuscript.

ADDITIONAL INFORMATION
Supplementary information accompanies the paper on the npj Quantum Information website (https://doi.org/10.1038/s41534-017-0056-9).

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