Relativistic second-order dissipative hydrodynamics from Zubarev’s non-equilibrium statistical operator

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Abstract
We present a new derivation of relativistic second-order dissipative hydrodynamics for quantum systems using Zubarev’s non-equilibrium statistical-operator formalism. This is achieved by a systematic expansion of the energy-momentum tensor and the charge current to second order in deviations from equilibrium. As a concrete example, we obtain the relaxation equations for the shear-stress tensor, the bulk-viscous pressure, and the charge-diffusion currents required to close the set of equations of motion for relativistic second-order dissipative hydrodynamics. We also identify new transport coefficients which describe the relaxation of dissipative processes to second-order and express them in terms of equilibrium correlation functions, thus establishing new Kubo-type formulas for second-order transport coefficients.

Keywords: Hydrodynamics, Statistical operator, Transport coefficients, Correlation functions

1. Introduction
Hydrodynamics is a powerful tool to describe low-frequency and long-wavelength phenomena in statistical systems by performing an averaging (coarse-graining) over scales that are large compared to those characteristic for kinetic phenomena, \textit{e.g.}, the mean free path of a quasiparticle \cite{1}. It finds numerous applications in astrophysics and cosmology \cite{2}, heavy-ion physics \cite{3,4}, and other areas. During the last decade, relativistic hydrodynamics has been successfully applied to describe the behavior of strongly interacting hot and dense matter created in heavy-ion collision experiments at the Relativistic Heavy-Ion Collider (RHIC) and the Large Hadron Collider (LHC). In these experiments, a new state of matter, the quark–gluon plasma (QGP), was discovered, which behaves almost like a perfect fluid (for reviews, see Refs. \cite{5–7}). Another area of applications of relativistic hydrodynamics is the physics of compact stars. In particular, binary neutron-star mergers and gravitational waves emitted in these events are modeled by the coupled dynamics of fluid and space-time in general relativity (for reviews, see Refs. \cite{8,9}).

This work aims to obtain the hydrodynamical equations of strongly correlated matter, such as the non-perturbative QGP in heavy-ion collisions or hadronic matter in compact...
stars, by adopting Zubarev’s formalism, known also as the method of the non-equilibrium statistical operator [10–12]. This method is based on a generalization of the Gibbs canonical ensemble to non-equilibrium states, \(i.e.,\) the statistical operator is promoted to a non-local functional of the thermodynamic parameters and their space-time derivatives. Assuming that the thermodynamic parameters vary sufficiently smoothly over the correlation lengths characterizing the system, the statistical operator is then expanded in a series in gradients of these parameters, commonly keeping only the first-order terms in the expansion. Then, the hydrodynamical equations for the dissipative currents emerge after statistical averaging of the relevant quantum operators. At weak coupling, this method is equivalent to hydrodynamics obtained from moments of the Boltzmann equation for the distribution function [13–15].

An advantage of Zubarev’s formalism is that the transport coefficients of the system are automatically obtained in the form of Kubo-type relations, \(i.e.,\) they are related to certain correlation functions of the underlying field theory, valid also in the strong-coupling limit.

The application of Zubarev’s formalism to quantum fields and relativistic hydrodynamics is due to the pioneering works of Zubarev et al. [12] and Hosoya et al. [16], see also Ref. [17]. In recent years, there has been a renewed interest in applications of this method to relativistic hydrodynamics. The novel developments include a formulation of anisotropic magnetohydrodynamics in strong magnetic fields [18], reformulations and reinterpretations suitable for applications to the QGP [19], second-order expansions of the statistical operator, which lead to second-order hydrodynamics [20], and hydrodynamics with anomalies [21, 22].

In this work, we provide a new, alternative derivation of relativistic second-order dissipative hydrodynamics within Zubarev’s formalism, which extends our previous work [20]. To motivate our strategy, we recall that hydrodynamics describes the state of a fluid in terms of its energy-momentum tensor and currents of conserved charges, which, in the relevant low-frequency and long-wavelength limit, can be expanded around their equilibrium values in powers of gradients of thermodynamic parameters (so-called thermodynamic forces). The zeroth-order expansion corresponds to ideal (non-dissipative) hydrodynamics. At first order, the relativistic generalization of Navier–Stokes hydrodynamics emerges from a truncation that keeps the first-order terms in gradients [1, 23]. Relativistic second-order dissipative theories, which keep the next-to-leading dissipative terms in the above expansion were first constructed by Israel and Stewart [24, 25]. Such theories were demonstrated to avoid acausalities of the first-order theory and numerical instabilities associated with it. \(^1\)

In the second-order theories, the dissipative currents satisfy relaxation equations, which include finite timescales of their relaxation towards their (first-order) Navier–Stokes values at asymptotically large times [28–40], for reviews see Refs. [41–43].

The transport coefficients entering dissipative hydrodynamics can be obtained at weak coupling either in the relaxation-time approximation or, more systematically, via the method of moments [13], or at arbitrary values of the coupling through Kubo formulas. Second-order transport coefficients were obtained via an expansion of the curved-space metric around its flat-space limit through Kubo formulas in Ref. [33]. Here we will show how to obtain these transport coefficients from the Zubarev formalism. In doing so, we apply an expansion of

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\(^1\)Recently it was shown that these acausalities and instabilities are a consequence of the matching procedure to the local-equilibrium reference state. Generalizing this matching, causal and stable first-order dissipative hydrodynamic theories have been derived [26, 27].
the non-equilibrium statistical operator up to second order in thermodynamic forces, which allows us to generate in a systematic way all second-order terms in hydrodynamics after statistical averaging the dissipative currents over the non-equilibrium statistical operator. A concise discussion of our approach and some results were reported earlier [20], see also Ref. [44].

This work is structured as follows. To set the stage, in Section 2 we briefly review Zubarev’s formalism [10, 11]. Section 3 recapitulates Navier–Stokes theory and the Kubo formulas for the first-order transport coefficients. The second-order transport equations for the dissipative currents and the relevant transport coefficients are derived in Section 4. In Section 5 we discuss the structure of the transport equations and compare them to those of earlier studies. A summary of the results is given in Section 6. Appendix A discusses the Eckart and Landau frame choices for the fluid velocity. Appendix B computes the rate of entropy generation within the Zubarev formalism. In Appendix C we derive Kubo formulas for transport coefficients. Some properties of projection operators are listed in Appendix D. We work in flat space-time described by the metric \( g_{\mu\nu} = \text{diag}(+, -, -, -) \) and use natural units throughout.

2. The non-equilibrium statistical-operator formalism

In this section, we describe Zubarev’s non-equilibrium statistical-operator formalism for a generic quantum system which is in the hydrodynamic regime [10–12]. Our starting point is the operator-valued conservation laws for energy-momentum and charges

\[
\partial_\mu \hat{T}^{\mu\nu} = 0, \quad \partial_\mu \hat{N}^\mu_a = 0, \tag{1}
\]

where \( a = 1, 2, \ldots, \ell \) labels the possible conserved charges (e.g., baryonic, electric, etc.); \( \ell \) is the total number of conserved charges. To obtain the corresponding conservation laws in hydrodynamics one needs to take the statistical averages of the operators \( \hat{T}^{\mu\nu} \) and \( \hat{N}^\mu_a \) with respect to the full non-equilibrium statistical operator. This operator should be found from the quantum Liouville equation with an infinitesimal source term [10–12] and can then be expanded in a series with respect to the thermodynamic forces up to the required order. Statistical averaging of the dissipative currents leads then to constitutive relations for the latter and provides explicit expressions for the transport coefficients via certain equilibrium correlation functions of the system.

2.1. Local-equilibrium statistical operator

As is well known, the thermodynamic state of a macroscopic quantum system is described by the statistical operator \( \hat{\rho}(t) \) (i.e., the density matrix). It obeys the Liouville–von Neumann equation, which reads in the Schrödinger representation [45]

\[
\frac{\partial \hat{\rho}_S(t)}{\partial t} + i[\hat{K}, \hat{\rho}_S(t)] = 0, \tag{2}
\]

where \( \hat{K} = \hat{H} - \sum_a \mu_a \hat{N}_a \); here \( \hat{H} \) is the Hamiltonian of the system, \( \hat{N}_a \) are the operators of the conserved charges, \( \mu_a \) are the corresponding chemical potentials, and \( [\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A} \)
denotes the commutator of the two operators \( \hat{A}, \hat{B} \). In this representation the operators acting on the quantum states of the system are independent of time, therefore the explicit time dependence in \( \hat{\rho}_S(t) \) arises from the time dependence of the thermodynamic parameters caused by a perturbation. In the Heisenberg picture, which we will adopt in this work, the operators evolve in time, whereas the statistical operator is time-independent,

\[
\frac{d\hat{\rho}_H(t)}{dt} = 0, \quad \hat{\rho}_H(t) = \hat{\rho}_H(0) = \text{Const.,} \tag{3}
\]

where \( \hat{\rho}_H(0) \) refers to the initial value of \( \hat{\rho}_H(t) \). The choice of the initial statistical operator is discussed below in Sec. 2.3. In Eq. (3), the differentiation with respect to time acts simultaneously onto the operators and the thermodynamic parameters. The two representations are related via the transformation \( \hat{\rho}_S(t) = e^{-i\hat{K}t}\hat{\rho}_He^{i\hat{K}t} \). From now on, for the sake of simplicity we will omit the index \( H \) for all quantum operators in the Heisenberg representation.

In addition to the equation of motion (3), the statistical operator should also satisfy the normalization condition \( \text{Tr} \hat{\rho} = 1 \). The knowledge of the statistical operator allows one to compute the thermal expectation value of any quantum operator \( \hat{F}(x) \) via the formula

\[
\langle \hat{F}(x) \rangle = \text{Tr}[\hat{\rho}\hat{F}(x)], \tag{4}
\]

where \( x \equiv (x, t) \) denotes a point in space-time.

In thermodynamic equilibrium, the statistical operator depends only on the integrals of motion. For a system in contact with a heat bath with temperature \( T = \beta^{-1} \) and a charge reservoir with chemical potentials \( \mu_a \), the equilibrium statistical operator is given by the grand-canonical (Gibbs) distribution

\[
\hat{\rho}_{\text{eq}} = e^{\Omega - \beta \hat{K}}, \quad e^{-\Omega} = \text{Tr} e^{-\beta \hat{K}}. \tag{5}
\]

In this case both the Heisenberg and Schrödinger representations for \( \hat{\rho}_{\text{eq}} \) coincide. Note that the distribution (5) is written in the frame where the system as a whole is at rest. We can generalize the equilibrium distribution to an arbitrary reference frame via a Lorentz transformation \( \hat{H} \rightarrow \hat{P}_\nu U^\nu \), where \( U^\nu \) is the 4-velocity of the system in the chosen frame, \( \hat{P}_\nu \) is the 4-momentum operator and \( \hat{H} \equiv \hat{P}_0 \) in the fluid rest frame. In terms of the energy-momentum tensor and the charge currents, the operators \( \hat{P}_\nu \) and \( \hat{N}_a \) are given by

\[
\hat{P}_\nu = \int d^3x \hat{T}_{0\nu}(x), \quad \hat{N}_a = \int d^3x \hat{N}_{a0}(x). \tag{6}
\]

Substituting Eq. (6) into Eq. (5) we obtain the Lorentz-covariant form of the Gibbs distribution

\[
\hat{\rho}_{\text{eq}} = \exp \left\{ \Omega - \int d^3x \beta \left[ U^\nu \hat{T}_{0\nu}(x) - \sum_a \mu_a \hat{N}_{a0}(x) \right] \right\}, \tag{7}
\]

\[
e^{-\Omega} = \text{Tr} \exp \left\{ -\int d^3x \beta \left[ U^\nu \hat{T}_{0\nu}(x) - \sum_a \mu_a \hat{N}_{a0}(x) \right] \right\}. \tag{8}
\]

\[\text{Note that Eqs. (7) and (8) can be generalized to a fully Lorentz-invariant form, where the spatial integral is replaced by an integration over an arbitrary space-time hypersurface instead of a fixed-time hypersurface [12].}\]
Now we consider a system that is out of global thermodynamic equilibrium, but each small (but still macroscopic) portion of the system has reached its local-equilibrium state, i.e., the system is in the hydrodynamic regime. Local equilibrium implies that each fluid element can be ascribed local values of the hydrodynamic parameters such as temperature $\beta^{-1}(x)$, chemical potentials $\mu_a(x)$, and a macroscopic 4-velocity $u^\nu(x)$, which vary slowly in space and time. In this case the local-equilibrium distribution given by Eqs. (7) and (8) is replaced by a local-equilibrium statistical operator via

\[ \hat{\rho}_l(t) = \exp \left\{ \Omega_l(t) - \int d^3x \left[ \beta^\nu(x) \hat{T}_{0\nu}(x) - \sum_a \alpha_a(x) \hat{N}_{a0}(x) \right] \right\}, \]

\[ e^{-\Omega_l(t)} = \text{Tr} \exp \left\{ -\int d^3x \left[ \beta^\nu(x) \hat{T}_{0\nu}(x) - \sum_a \alpha_a(x) \hat{N}_{a0}(x) \right] \right\}, \]

where

\[ \beta^\nu(x) = \beta(x) u^\nu(x), \quad \alpha_a(x) = \beta(x) \mu_a(x). \]

The operator $\hat{\rho}_l$ is determined from the principle of maximum entropy for given values of $u_\nu(\hat{T}^{\mu\nu})$ and $u_\mu \langle \hat{N}_a^{\mu} \rangle$ [10–12]. The fluid 4-velocity $u^\nu$ can be defined according to either Landau or Eckart, see Appendix A. The local-equilibrium distribution (9) is also referred to as relevant statistical operator [10, 11].

Next, we define the operators of energy and charge densities in the comoving frame via $\hat{\epsilon} = u_\mu u_\nu \hat{T}^{\mu\nu}$ and $\hat{n}_a = u_\mu \hat{N}_a^{\mu}$. The local values of the Lorentz-invariant thermodynamic parameters $\beta$ and $\alpha_a$ are then fixed by the given average values of the operators $\hat{\epsilon}$ and $\hat{n}_a$ via the following matching conditions [10–12]

\[ \langle \hat{\epsilon}(x) \rangle = \langle \hat{\epsilon}(x) \rangle_t, \quad \langle \hat{n}_a(x) \rangle = \langle \hat{n}_a(x) \rangle_t, \]

where we introduced the notation

\[ \langle \hat{F}(x) \rangle_t = \text{Tr} [\hat{\rho}_l(t) \hat{F}(x)]. \]

Note that the conditions (12) define the temperature and the chemical potentials in general [46] as non-local functionals of

\[ \langle \hat{\epsilon}(x) \rangle \equiv \epsilon(x), \quad \langle \hat{n}_a(x) \rangle \equiv n_a(x). \]

However, in the hydrodynamic description of the fluid one needs to define thermodynamic parameters as local functions of the energy and charge densities, as it is the case in global thermodynamic equilibrium. This can be done by assuming that all fluid elements where local equilibrium is already established are statistically independent of each other [47]. In other words, the local-equilibrium values $\langle \hat{\epsilon} \rangle_t$ and $\langle \hat{n}_a \rangle_t$ in Eq. (12) should be evaluated formally at constant values of $\beta$ and $\mu_a$, which are then determined by matching $\langle \hat{\epsilon} \rangle_t$ and $\langle \hat{n}_a \rangle_t$ to the real values $\langle \hat{\epsilon} \rangle$ and $\langle \hat{n}_a \rangle$ of these quantities at the given point $x$ in space-time. This assigns a fictitious local-equilibrium state to any given point, such that it reproduces the local values of the energy and charge densities. The relevant statistical operator also fixes the local values of the 3-momentum or one of the charge currents, when adopting the Landau or the Eckart...
definition of the fluid velocity, respectively, see Appendix A. Using Eq. (A.2) we can write the matching conditions in the Landau frame (hereafter L-frame) as follows

\[ u_\mu \langle \tilde{T}^{\mu\nu} \rangle = u_\mu \langle \tilde{T}^{\mu\nu} \rangle_t, \quad u_\mu \langle \tilde{N}_a^{\mu} \rangle = u_\mu \langle \tilde{N}_a^{\mu} \rangle_t. \]  

(15)

In the Eckart frame (hereafter E-frame) connected to the current \( \tilde{N}_b^{\mu} \) we have instead [see Eq. (A.16)]

\[ u_\mu u_\nu \langle \tilde{T}^{\mu\nu} \rangle = u_\mu u_\nu \langle \tilde{T}^{\mu\nu} \rangle_t, \quad u_\mu \langle \tilde{N}_b^{\mu} \rangle = u_\mu \langle \tilde{N}_b^{\mu} \rangle_t, \quad \langle \tilde{N}_a^{\mu} \rangle = \langle \tilde{N}_a^{\mu} \rangle_t, \]  

(16)

where in the second relation \( b \) runs over all values \( 1, \ldots, \ell \) except \( a \). We recall that all local-equilibrium averages in Eqs. (15) and (16) should be evaluated formally at constant values of \( \beta, \mu_a \), and \( u^\mu \), as explained above.

It is useful to also express the relevant distribution in terms of the 4-scalars \( \epsilon \) and \( \tilde{n}_a \). Going to the local rest frame for each fluid element we can write Eqs. (9) and (10) in the following form

\[ \hat{\rho}_t(t) = \exp \left\{ \Omega_t(t) - \int d^3\tilde{x} \beta(x) \left[ \hat{\epsilon}(x) - \sum_a \mu_a(x) \tilde{n}_a(x) \right] \right\}, \]  

(17)

\[ e^{-\Omega_t(t)} = \text{Tr} \exp \left\{ - \int d^3\tilde{x} \beta(x) \left[ \hat{\epsilon}(x) - \sum_a \mu_a(x) \tilde{n}_a(x) \right] \right\}, \]  

(18)

where \( d^3\tilde{x} = u^0(x)d^3x \) is the proper volume of a fluid element.

2.2. Thermodynamic relations

In this section we derive the thermodynamic relations for the local thermodynamic parameters starting from the relevant distribution (9) or (17). Following Zubarev we first define the entropy operator as [11, 12]

\[ \hat{S}(t) = -\ln \hat{\rho}_t(t) = -\Omega_t(t) + \int d^3x \left[ \beta^\nu(x) \tilde{T}_{\nu\nu}(x) - \sum_a \alpha_a(x) \tilde{N}_{ao}(x) \right] \]  

\[ = -\Omega_t(t) + \int d^3\tilde{x} \beta(x) \left[ \hat{\epsilon}(x) - \sum_a \mu_a(x) \tilde{n}_a(x) \right], \]  

(19)

which allows one to write the relevant statistical operator as

\[ \hat{\rho}_t(t) = e^{-\hat{S}(t)}. \]  

(20)

The thermodynamic entropy in local equilibrium is defined as the statistical average of the entropy operator

\[ S(t) = \langle \hat{S}(t) \rangle_t = -\Omega_t(t) + \int d^3\tilde{x} \beta(x) \left[ \langle \hat{\epsilon}(x) \rangle - \sum_a \mu_a(x) \langle \tilde{n}_a(x) \rangle \right] \equiv \langle \hat{S}(t) \rangle, \]  

(21)

where we used the matching conditions (12).

The thermodynamic relations we seek can now be derived using Eq. (18). For that purpose we consider small variations \( \delta\epsilon(x) \) and \( \delta\tilde{n}_a(x) \) in the local energy and charge densities.
These variations induce small changes in temperature, $\delta \beta(x)$, and chemical potentials, $\delta \mu_a(x)$, respectively. The corresponding change in $\Omega_l(t)$ is 

$$\delta \Omega_l(t) = \int d^3 \tilde{x} \left[ \frac{\delta \Omega_l(t)}{\delta \beta(x)} \delta \beta(x) + \sum_a \frac{\delta \Omega_l(t)}{\delta \mu_a(x)} \delta \mu_a(x) \right],$$

where the derivatives in the square brackets are Lorentz-invariant functional derivatives of $\Omega_l(t)$. From Eqs. (17) and (18) we obtain

$$\frac{\delta \Omega_l(t)}{\delta \beta(x)} = \epsilon(x) - \sum_a \mu_a(x)n_a(x),$$

$$\frac{\delta \Omega_l(t)}{\delta \mu_a(x)} = -\beta(x)n_a(x),$$

The infinitesimal change in the entropy can then be found from Eqs. (21)–(24),

$$\delta S(t) = -\delta \Omega_l(t) + \int d^3 \tilde{x} \left[ \delta \beta \left( \epsilon - \sum_a \mu_a n_a \right) - \beta \sum_a n_a \delta \mu_a + \beta \left( \delta \epsilon - \sum_a \mu_a \delta n_a \right) \right]$$

$$= \int d^3 \tilde{x} \beta(x) \left[ \delta \epsilon(x) - \sum_a \mu_a(x) \delta n_a(x) \right].$$

In the next step, we define the invariant entropy density $s(x)$ such as

$$S(t) = \int d^3 \tilde{x} \, s(x).$$

Then Eq. (25) results in

$$\int d^3 \tilde{x} \left\{ \beta(x) \left[ \delta \epsilon(x) - \sum_a \mu_a(x) \delta n_a(x) \right] - \delta s(x) \right\} = 0.$$ 

Because $\delta \epsilon(x)$ and $\delta n_a(x)$ are arbitrary variations and the entropy density $s(x)$ is assumed to be a local function of $\epsilon(x)$ and $n_a(x)$, [i.e., $s(\epsilon(x), n_a(x)) \equiv s(x)$], we derive from Eq. (27) the relation

$$T(x) \delta s(x) = \delta \epsilon(x) - \sum_a \mu_a(x) \delta n_a(x),$$

which is the first law of thermodynamics for local variables.

To obtain other thermodynamic relations we recall that the grand potential in global thermodynamic equilibrium is defined as $\Omega_l = -\beta p V$, with $p$ being the pressure and $V$ the volume of the system. In local equilibrium, $\Omega_l(t)$ given by Eq. (18) is a functional of $\epsilon(x)$ and $n_a(x)$, therefore we can define a scalar function $p(\epsilon(x), n_a(x)) \equiv p(x)$, such that

$$\Omega_l(t) = - \int d^3 \tilde{x} \beta(x)p(x).$$
The form of the function $p(\epsilon(x), n_a(x))$ should be established from Eqs. (18) and (29) and the matching conditions (12) (which determine the temperature and the chemical potentials); it is called the equation of state (EoS). Using Eq. (29) we can also write Eq. (21) as

$$S(t) = \int d^3\vec{x} \beta(x) \left[ \epsilon(x) + p(x) - \sum_a \mu_a(x) n_a(x) \right],$$

which in combination with Eq. (26) leads to the well-known thermodynamic relation

$$\epsilon(x) + p(x) = T(x)s(x) + \sum_a \mu_a(x) n_a(x) \equiv h(x),$$

where $h$ is the enthalpy density. From Eqs. (28) and (31) we obtain the Gibbs–Duhem relation

$$\delta p(x) = s(x)\delta T(x) + \sum_a n_a(x)\delta \mu_a(x).$$

Thus, using the relevant statistical operator, we constructed a full set of thermodynamic variables (i.e., a fictitious local-equilibrium state) for given energy-momentum tensor and charge current densities, as it is required for the hydrodynamic description of the system.

2.3. Deriving the non-equilibrium statistical operator

As discussed in Sec. 2.1, the relevant statistical operator $\hat{\rho}_l$ defined by Eqs. (9) and (10) reproduces the local values of the macroscopic observables $u_\nu\langle T^{\mu\nu} \rangle$ and $u_\mu \langle \dot{N}^\mu_a \rangle$. However, the operator $\hat{\rho}_l$ does not satisfy the Liouville equation (3), and, therefore, cannot describe non-equilibrium thermodynamic processes.

To proceed further we write the Liouville equation (3) in the form

$$\frac{d\hat{\sigma}(t)}{dt} = 0,$$

where $\hat{\sigma}(t) = -\ln \hat{\rho}(t)$. Equation (33) simply implies that $\hat{\sigma}(t)$ is the statistical operator at some initial moment in time $t'$. Following Refs. [10, 11, 48–51], we take the initial condition for Eq. (33) in the form $\hat{\sigma}(t') = \hat{S}(t')$, i.e., the statistical operator coincides with its local-equilibrium counterpart at the moment of time $t'$ (note that the matching conditions (12) are automatically satisfied by this choice of the initial condition). According to Eq. (33), $\hat{\sigma}$ remains unchanged for all subsequent times, i.e.,

$$\hat{\sigma}(t) = \hat{S}(t'), \quad t \geq t'.$$

Our next step is to modify the expression (34) in such a way that it can incorporate the irreversibility of the thermodynamic processes. This can be achieved by averaging the right-hand side of Eq. (34) over the initial states in some time interval $t_0 \leq t' \leq t$ around $t'$ [10, 11], i.e.,

$$\hat{\sigma}^*(t) \equiv \frac{1}{t - t_0} \int_{t_0}^{t} dt' \hat{S}(t') = \frac{1}{t - t_0} \int_{-(t - t_0)}^{0} dt' \hat{S}(t' + t).$$
Note that information is lost in the averaging process, therefore this procedure is irreversible. The time interval \( t - t_0 \) should be sufficiently large for the details of the initial state (correlations) to become inessential. Therefore, it is natural to take the limit \( t_0 \to -\infty \). Using Abel’s theorem [see Ref. [11], Eq. (2.3.9)]

\[
\lim_{T \to \infty} \frac{1}{T} \int_{-T}^{0} dt' f(t') = \lim_{\varepsilon \to 0^+} \varepsilon \int_{-\infty}^{0} dt' e^{\varepsilon t'} f(t') ,
\]

which is valid if the function \( f(t) \) is sufficiently smooth and at least one of the above limits exists. We obtain for Eq. (35)

\[
\hat{\sigma}^*(t) = \lim_{\varepsilon \to 0^+} \varepsilon \int_{-\infty}^{0} dt' e^{\varepsilon t'} \hat{S}(t' + t) = \lim_{\varepsilon \to 0^+} \varepsilon \int_{-\infty}^{t} dt' e^{\varepsilon(t' - t)} \hat{S}(t') .
\]

It is easy to show that this operator satisfies the Liouville equation (33). Let

\[
\hat{\sigma}_\varepsilon(t) \equiv \varepsilon \int_{-\infty}^{t} dt' e^{\varepsilon(t' - t)} \hat{S}(t') ,
\]

such that, according to Eq. (37), \( \lim_{\varepsilon \to 0^+} \hat{\sigma}_\varepsilon(t) = \hat{\sigma}^*(t) \). Then

\[
\frac{d\hat{\sigma}_\varepsilon(t)}{dt} = \varepsilon \hat{S}(t) - \varepsilon^2 \int_{-\infty}^{t} dt' e^{\varepsilon(t' - t)} \hat{S}(t') = -\varepsilon \left[ \hat{\sigma}_\varepsilon(t) - \hat{S}(t) \right] ,
\]

with the right-hand side tending to zero as \( \varepsilon \to 0 \). According to Eq. (39) the statistical operator

\[
\hat{\rho}_\varepsilon(t) \equiv \exp \left[ -\hat{\sigma}_\varepsilon(t) \right]
\]

satisfies the Liouville equation up to a source term \( \sim \varepsilon \). The latter breaks the reversibility of the Liouville equation by choosing its retarded solution. Hence we draw the important conclusion that the limit \( \varepsilon \to 0^+ \) should be performed only after the thermodynamic limit is taken. Due to this procedure, one maintains irreversibility in the evolution of the system until the end of the calculations. Thus, the statistical average of any operator \( \hat{F}(x) \) should be computed according to the following rule [11]

\[
\langle \hat{F}(x) \rangle = \lim_{\varepsilon \to 0^+} \lim_{V \to \infty} \text{Tr} \left[ \hat{\rho}_\varepsilon(t) \hat{F}(x) \right] ,
\]

where \( V \) is the volume of the system. Furthermore, because the statistical operator (40) incorporates memory effects, the equations of motion obtained from Eq. (40) are expected

\[\text{Eq. (36) is a property of the Laplace transform which is frequently written as}
\]

\[
\lim_{\tau \to \infty} \frac{1}{\tau} \int_{0}^{\tau} f(t) dt = \lim_{\varepsilon \to 0^+} \varepsilon \int_{0}^{\infty} f(t) e^{-\varepsilon t} dt
\]

and follows from the general theory of Tauber’s theorems [52–54].
to be causal \cite{46, 56-58}. Thus, we constructed a causal non-equilibrium statistical operator starting from the relevant statistical operator.

Now we substitute the explicit expression for \( \dot{S}(t) \) given by Eq. (19) into Eq. (40) to obtain (omitting the index \( \varepsilon \) for the sake of brevity)

\[
\dot{\hat{S}}(t) = Q^{-1}(t) \exp \left\{ -\int d^3x \, \hat{Z}(x, t) \right\}, \quad Q(t) = \text{Tr} \exp \left\{ -\int d^3x \, \hat{Z}(x, t) \right\}, \tag{42}
\]

where

\[
\hat{Z}(x, t) = \varepsilon \int_{-\infty}^{t} dt_1 e^{\varepsilon(t_1-t)} \left[ \beta^\nu(x, t_1) \hat{T}_{0\nu}(x, t_1) - \sum_a \alpha_a(x, t_1) \hat{N}_{a0}(x, t_1) \right]. \tag{43}
\]

In the case of one type of conserved charge (\( \ell = 1 \)) Eqs. (42) and (43) coincide with Eqs. (42) – (44) of Ref. \cite{18}, and in the case of \( \ell = 0 \) we recover Eqs. (2.4) and (2.8) of Ref. \cite{16}.

We can separate the local-equilibrium contribution from Eq. (43) by an integration by parts:

\[
\hat{Z}(x, t) = \beta^\nu(x, t) \hat{T}_{0\nu}(x, t) - \sum_a \alpha_a(x, t) \hat{N}_{a0}(x, t) \\
- \int_{-\infty}^{t} dt_1 e^{\varepsilon(t_1-t)} \frac{d}{dt_1} \left[ \beta^\nu(x, t_1) \hat{T}_{0\nu}(x, t_1) - \sum_a \alpha_a(x, t_1) \hat{N}_{a0}(x, t_1) \right],
\]

where we assumed that the exponential factor \( e^{\varepsilon(t_1-t)} \) guarantees the limiting behavior

\[
\lim_{t_1 \to -\infty} e^{\varepsilon(t_1-t)} \hat{F}(t_1) = 0,
\]

with the operator \( \hat{F}(t_1) \) defined as the term in square brackets in Eq. (43). The conservation laws (1) imply the relations \( \partial^\mu \hat{T}_{\mu\nu} = \partial_0 \hat{T}_{0\nu} + \partial^\nu \hat{T}_{0\nu} = 0 \) and \( \partial^\mu \hat{N}_{a\mu} = \partial^0 \hat{N}_{a0} + \partial^a \hat{N}_{ai} = 0 \), which give

\[
\partial^0 \left( \beta^\nu \hat{T}_{0\nu} - \sum_a \alpha_a \hat{N}_{a0} \right) = \hat{T}_{\mu\nu} \partial^\mu \beta^\nu - \sum_a \hat{N}^\mu \partial_\mu \alpha_a - \partial^a \left( \beta^\nu \hat{T}_{0\nu} - \sum_a \alpha_a \hat{N}_{ai} \right). \tag{44}
\]

The last term in Eq. (44) gives a surface term after integration over the spatial volume, which vanishes if the surface is taken at infinity, and we are left with

\[
\int d^3x \, \hat{Z}(x, t) = \int d^3x \left[ \beta^\nu(x, t) \hat{T}_{0\nu}(x, t) - \sum_a \alpha_a(x, t) \hat{N}_{a0}(x, t) \right] \\
- \int d^3x \int_{-\infty}^{t} dt_1 e^{\varepsilon(t_1-t)} \left[ \hat{T}_{0\nu}(x, t_1) \partial^\mu \beta^\nu(x, t_1) - \sum_a \hat{N}^\mu \partial_\mu \alpha_a(x, t_1) \right]. \tag{45}
\]

where the 4-gradients are implied to act on \((x, t_1)\). The first term in this expression corresponds to the local-equilibrium part of the statistical operator. The integrand of the second term is a thermodynamic “force” as it involves gradients of temperature, chemical potentials,
and the velocity field. Naturally, the second term in Eq. (45) is then identified with the non-equilibrium part of the statistical operator. Using Eqs. (42) and (45) we can write the full statistical operator as

$$\hat{\rho}(t) = Q^{-1}e^{-\hat{A} + \hat{B}}, \quad Q = \text{Tr}e^{-\hat{A} + \hat{B}},$$

where

$$\hat{A}(t) = \int d^3x \left[ \beta^\nu(x, t)\hat{T}_{\mu\nu}(x, t) - \sum_a \alpha_a(x, t)\hat{N}_{a0}(x, t) \right],$$

$$\hat{B}(t) = \int d^3x \int_{-\infty}^t dt_1e^{\epsilon(t_1-t)}\hat{C}(x, t_1),$$

$$\hat{C}(x, t) = \hat{T}_{\mu\nu}(x, t)\partial^\mu\beta^\nu(x, t) - \sum_a \hat{N}_a^\mu(x, t)\partial_\mu\alpha_a(x, t).$$

The statistical operator given by Eq. (46) can now be used to derive the equations of motion for the dissipative currents. For this purpose, one treats the non-equilibrium part (48) as a perturbation. Keeping only the first-order terms in the Taylor expansion of $\hat{\rho}(t)$ with respect to the operator $\hat{B}(t)$ yields the usual first-order dissipative hydrodynamic theory [16, 18]. In this work, we will include all second-order terms in the Taylor expansion to obtain second-order dissipative hydrodynamics within the Zubarev formalism.

2.4. Second-order expansion of the statistical operator

We start from the following formula for two arbitrary operators $\hat{A}$ and $\hat{B}$ [55],

$$e^{-\hat{A} + \hat{B}}e^{\hat{A}} = 1 + \int_0^1 d\tau e^{\tau(-\hat{A} + \hat{B})}\hat{B}e^{\tau\hat{A}},$$

which can be obtained by integrating the identity

$$\frac{d}{d\tau}e^{\tau(-\hat{A} + \hat{B})}e^{\tau\hat{A}} = e^{\tau(-\hat{A} + \hat{B})}\hat{B}e^{\tau\hat{A}}$$

over the variable $\tau$ from 0 to 1. Keeping only the linear term in $\hat{B}$ under the integral we obtain from Eq. (50)

$$e^{-\hat{A} + \hat{B}} = e^{-\hat{A}} + \int_0^1 d\lambda e^{-\lambda\hat{A}}\hat{B}e^{\lambda\hat{A}}e^{-\hat{A}} + \mathcal{O}(\hat{B}^2).$$

Our next task is to derive the second-order term in $\hat{B}$ in the expansion of the non-equilibrium statistical operator given by Eq. (46). We start with the numerator and expand the identity (50) up to the second order. The integrand in Eq. (50) is already of order $\mathcal{O}(\hat{B})$, therefore it is sufficient to evaluate the operator $e^{\tau(-\hat{A} + \hat{B})}$ using the first-order expansion (51). Replacing $\hat{A} \rightarrow \tau\hat{A}$, $\hat{B} \rightarrow \tau\hat{B}$ in Eq. (51) and inserting the result into Eq. (50) we find

$$e^{-\hat{A} + \hat{B}} = (1 + \hat{\alpha}_1 + \hat{\alpha}_2)e^{-\hat{A}},$$
with

\[ \hat{\alpha}_1 = \int_0^1 d\tau \hat{B}_\tau, \quad \hat{\alpha}_2 = \frac{1}{2} \int_0^1 d\tau \int_0^1 d\lambda \hat{T} \{ \hat{B}_\lambda \hat{B}_\tau \}, \]  

(53)

where we introduced the short-hand notation

\[ \hat{X}_\tau = e^{-\tau A} \hat{X} e^{\tau A} \]

(54)

for any operator \( \hat{X} \) and \( \hat{T} \) is the anti-chronological time-ordering operator with respect to the variables \( \tau \) and \( \lambda \).

In the next step, the trace \( Q \) in Eq. (46) is expanded up to second order in a Taylor series, giving

\[ Q^{-1} = \frac{1}{\text{Tr} e^{-A+B}} = \frac{1}{\text{Tr} e^{-A}} \left[ 1 - \langle \hat{\alpha}_1 \rangle_l - \langle \hat{\alpha}_2 \rangle_l + \langle \hat{\alpha}_1 \rangle_l^2 \right], \]

(55)

where we used the notation (13). Substituting Eqs. (52), (53), and (55) into Eq. (46) and dropping higher-order terms we obtain

\[ \hat{\rho} = \hat{\rho}_l + \hat{\rho}_1 + \hat{\rho}_2, \]

(56)

with

\[ \hat{\rho}_1 = \int_0^1 d\tau \left( \hat{B}_\tau - \langle \hat{B}_\tau \rangle_l \right) \hat{\rho}_l, \]

(57)

\[ \hat{\rho}_2 = \frac{1}{2} \int_0^1 d\tau \int_0^1 d\lambda \left[ \hat{T} \{ \hat{B}_\lambda \hat{B}_\tau \} - \langle \hat{T} \{ \hat{B}_\lambda \hat{B}_\tau \} \rangle_l \right. \\
\left. \quad - \hat{B}_\tau \langle \hat{B}_\lambda \rangle_l + \hat{B}_\lambda \langle \hat{B}_\tau \rangle_l + 2 \langle \hat{B}_\tau \rangle_l \langle \hat{B}_\lambda \rangle_l \right] \hat{\rho}_l. \]

(58)

Equation (57) coincides with the result given in Ref. [18].

In the last step, we substitute the explicit expression for the operator \( \hat{B} \) from Eq. (48), which gives for the first-order correction

\[ \hat{\rho}_1(t) = \int d^4x_1 \int_0^1 d\tau \left[ \hat{C}_\tau(x_1) - \langle \hat{C}_\tau(x_1) \rangle_l \right] \hat{\rho}_l, \]

(59)

and for the second-order correction

\[ \hat{\rho}_2(t) = \frac{1}{2} \int d^4x_1 d^4x_2 \int_0^1 d\tau \int_0^1 d\lambda \left[ \hat{T} \{ \hat{C}_\lambda(x_1) \hat{C}_\tau(x_2) \} - \langle \hat{T} \{ \hat{C}_\lambda(x_1) \hat{C}_\tau(x_2) \} \rangle_l \right. \\
\left. \quad - \langle \hat{C}_\lambda(x_1) \rangle_l \hat{C}_\tau(x_2) - \hat{C}_\lambda(x_1) \langle \hat{C}_\tau(x_2) \rangle_l + 2 \langle \hat{C}_\lambda(x_1) \rangle_l \langle \hat{C}_\tau(x_2) \rangle_l \right] \hat{\rho}_l, \]

(60)

where we introduced the abbreviation

\[ \int d^4x_1 \equiv \int d^3x_1 \int_{-\infty}^t dt_1 e^{(t_1-t)}. \]

(61)
Given the generic expansions above, we can now write down the statistical average of an arbitrary operator $\hat{X}(x)$ with the help of Eqs. (4), (56), (59), and (60) as

$$\langle \hat{X}(x) \rangle = \langle \hat{X}(x) \rangle_l + \int d^4 x_1 \left( \hat{X}(x), \hat{C}(x_1) \right) + \int d^4 x_1 d^4 x_2 \left( \hat{X}(x), \hat{C}(x_1), \hat{C}(x_2) \right), \quad (62)$$

where we defined the two-point correlation function

$$\left( \hat{X}(x), \hat{Y}(x) \right) \equiv \int_0^1 d\tau \left( \hat{X}(x) \left[ \hat{Y}_\tau(x) - \langle \hat{Y}_\tau(x) \rangle_l \right] \right)_l, \quad (63)$$

and the three-point correlation function

$$\left( \hat{X}(x), \hat{Y}(x), \hat{Z}(x) \right) \equiv \frac{1}{2} \int_0^1 d\tau \int_0^1 d\lambda \left\{ \hat{T}(x) \left[ \hat{Y}_\lambda(x) \hat{Z}_\tau(x) \right. \right.$$  

$$\left. - \langle \hat{Y}_\lambda(x) \rangle_l \hat{Z}_\tau(x) - \hat{Y}_\lambda(x) \langle \hat{Z}_\tau(x) \rangle_l \right.$$  

$$\left. - 2 \langle \hat{Y}_\lambda(x) \rangle_l \langle \hat{Z}_\tau(x) \rangle_l \right] \right\}_l. \quad (64)$$

From Eq. (64) it is straightforward to find the symmetry relation

$$\int d^4 x_1 d^4 x_2 \left( \hat{X}(x), \hat{Y}(x), \hat{Z}(x) \right) = \int d^4 x_1 d^4 x_2 \left( \hat{X}(x), \hat{Z}(x), \hat{Y}(x) \right), \quad (65)$$

which we will utilize below.

2.5. Hydrodynamic equations

To separate the dissipative processes related to the viscous and diffusion currents we need to decompose the energy-momentum tensor and the charge currents in terms of their equilibrium and dissipative parts. The most general decompositions are

$$\hat{T}^{\mu\nu} = \hat{\varphi}_{\nu} u^\mu - \hat{\varrho}_{\nu} \Delta^{\mu\nu} + \hat{q}_{\nu} u^\mu + \hat{\tau}_{\nu} u^\mu + \hat{\pi}^{\mu\nu}, \quad (66)$$

$$\hat{N}^a = \hat{n}_a u^\mu + \hat{j}_a^\mu, \quad (67)$$

where $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ is the projection operator onto the 3-space orthogonal to $u^\mu$. The shear-stress tensor $\hat{\pi}^{\mu\nu}$, the heat flux $\hat{\varphi}_{\nu}$, and the diffusion currents $\hat{j}_a^\mu$ are orthogonal to $u_\mu$, and $\hat{\tau}^{\mu\nu}$ is traceless:

$$u_{\nu} \hat{q}_{\nu} = 0, \quad u_{\nu} \hat{j}_a^\nu = 0, \quad u_{\nu} \hat{\tau}^{\mu\nu} = 0, \quad \hat{\pi}_\mu^\mu = 0. \quad (68)$$

Note that here we did not separate the equilibrium part of the pressure from the bulk-viscous pressure. The statistical average of the operator $\hat{\varphi}$ gives the actual isotropic (non-equilibrium) pressure, which in general differs from the equilibrium pressure $p(\langle \hat{\varphi} \rangle, \langle \hat{n}_a \rangle)$ introduced in Sec. 2.2. The latter is obtained by averaging the operator $\hat{\varphi}$ over the local-equilibrium distribution (evaluated formally at constant values of the thermodynamic parameters). The
bulk-viscous pressure is defined as the difference between these two averages, see Sec. 3.2 for details.

The operators on the right-hand sides of Eqs. (66) and (67) are given by projections of $\hat{T}^{\mu\nu}$ and $N^\mu_a$,

\begin{align}
\dot{\epsilon} &= u_\mu u_\nu \hat{T}^{\mu\nu}, \quad \hat{n}_a = u_\mu \hat{N}^\mu_a, \quad \hat{p} = -\frac{1}{3} \Delta_{\mu\nu} \hat{T}^{\mu\nu}, \\
\hat{\pi}^{\mu\nu} &= \Delta_{\alpha\beta}^{\mu\nu} \hat{T}^{\alpha\beta}, \quad \hat{\theta}^\mu = u_\alpha \Delta_\mu^\alpha \hat{T}^{\alpha\beta}, \quad \hat{j}_a^\nu = \Delta_{\mu\nu} \hat{N}^\mu_a,
\end{align}

where the following relations have been used

\[ u_\mu \Delta^{\mu\nu} = \Delta^{\mu\nu} u_\nu = 0, \quad \Delta^{\mu\nu} \Delta^{\nu\lambda} = \Delta^{\mu\lambda}, \quad \Delta_\mu^\nu = 3. \]

In Eq. (70) we also introduced the rank-4 traceless projector orthogonal to $u^\mu$ via

\[ \Delta_{\mu\nu\rho\sigma} = \frac{1}{2} (\Delta_{\mu\rho} \Delta_{\nu\sigma} + \Delta_{\mu\sigma} \Delta_{\nu\rho}) - \frac{1}{3} \Delta_{\mu\nu} \Delta_{\rho\sigma}, \]

which has the properties

\[ \Delta_{\mu\nu\rho\sigma} = \Delta_{\nu\mu\rho\sigma} = \Delta_{\rho\mu\nu\sigma}, \quad u_\mu \Delta_{\mu\nu\rho\sigma} = 0, \quad \Delta_{\alpha\beta}^\mu \Delta_{\mu\nu\rho\sigma} = \Delta_{\alpha\nu\rho\sigma}, \]

\[ \Delta_\mu^\nu = 0, \quad \Delta_{\nu\mu}^\rho = \frac{5}{3} \Delta_{\nu\sigma}, \quad \Delta_{\mu\nu}^\rho = 5, \quad \Delta_{\mu\nu\rho\sigma} \Delta_{\alpha\beta}^\rho = \Delta_{\mu\nu\alpha\beta}. \]

In the local rest frame, one finds from Eqs. (69) and (70)

\begin{align}
\dot{\epsilon} &= \hat{T}^{00}, \quad \hat{n}_a = \hat{N}^0_a, \quad \hat{p} = -\frac{1}{3} \hat{T}^k_k, \\
\hat{\pi}_{kl} &= \left( \delta_{ki} \delta_{lj} - \frac{1}{3} \delta_{kl} \delta_{ij} \right) \hat{T}^{ij}, \quad \hat{\theta}^i = \hat{T}^{0i}, \quad \hat{j}_a^i = \hat{N}^i_a.
\end{align}

Averaging Eqs. (66) and (67) over the non-equilibrium statistical operator and substituting them into Eq. (1) leads to the equations of dissipative hydrodynamics,

\begin{align}
D n_\alpha + n_\alpha \theta + \partial_\mu j^\mu_a &= 0, \quad (77) \\
D \epsilon + (h + \Pi) \theta + \partial_\mu q^\mu - q^\mu D u_\mu + \pi^{\mu\nu} \sigma_{\mu\nu} &= 0, \quad (78) \\
(h + \Pi) D u_\alpha - \nabla_\alpha (p + \Pi) + \Delta_{\alpha\mu} D q^\mu + q^\mu \partial_\mu u_\alpha + q_\alpha \theta + \Delta_{\alpha\nu} \partial_\nu \pi^{\mu\nu} &= 0, \quad (79)
\end{align}

where $\epsilon \equiv \langle \dot{\epsilon} \rangle$, $n_\alpha \equiv \langle \hat{n}_a \rangle$, $q^\mu \equiv \langle \hat{\theta}^\mu \rangle$, $\pi^{\mu\nu} \equiv \langle \hat{\pi}^{\mu\nu} \rangle$, and $j^\mu_a \equiv \langle \hat{j}_a^\mu \rangle$ are the statistical averages of the corresponding operators; $p \equiv p(\epsilon, n_\alpha)$ is the pressure in local equilibrium, i.e., the pressure given by the EoS, whereas $\Pi$ is the non-equilibrium part of the pressure (see Sec. 3.2 for details); $D \equiv u^\mu \partial_\mu$ is the comoving derivative (equal to the time derivative in the local rest frame), $\nabla_\alpha \equiv \Delta_{\alpha\beta} \partial^\beta$ is the covariant spatial derivative, $\sigma_{\mu\nu} \equiv \Delta_{\mu\nu} \partial_\alpha u_\beta$ is the shear tensor, and $\theta \equiv \partial_\mu u^\mu$ is the expansion scalar. The latter quantity is a measure for the rate of the fluid expansion (for $\theta > 0$) or contraction (for $\theta < 0$) and is therefore also called the fluid expansion rate. Equations (78) and (79) are obtained by contracting the first equation (1) by $u_\nu$ and $\Delta_{\nu\alpha}$, respectively. The system of Eqs. (77) – (79) contains $\ell + 4$ equations, whereas the number of independent variables is $4\ell + 10$. In order to close the system we
need additional equations of motion. In the L-frame, where \( q^\mu = 0 \), these are 3\( \ell \) equations for the independent components of the diffusion currents, 5 equations for the independent components of the shear-stress tensor, and one equation for the bulk-viscous pressure (recall that the equilibrium pressure is fixed by the EoS). In the E-frame, one of the diffusion currents can be eliminated, but then the determination of the independent components of the heat flux requires 3 additional equations of motion.

The averages of the dissipative operators over the local-equilibrium distribution vanish \([12]\):

\[
\langle \hat{q}^\mu \rangle_t = 0, \quad \langle \hat{j}_a^\mu \rangle_t = 0, \quad \langle \hat{\pi}^{\mu\nu} \rangle_t = 0. \tag{80}
\]

Indeed, the relevant distribution given by Eqs. (17) and (18) depends only on the scalar operators \( \hat{\epsilon} \) and \( \hat{n}_a \), which are not correlated with vector and tensor quantities due to Curie’s theorem \([12, 16, 59, 60]\). As a result, averaging Eqs. (66) and (67) over the relevant distribution and substitution into Eqs. (1) leads to the equations of ideal hydrodynamics, namely

\[
Dn_a + n_a \theta = 0, \quad D\epsilon + h \theta = 0, \quad hDu_\alpha = \nabla_\alpha p. \tag{81}
\]

The first two equation in Eq. (81) are the covariant forms of the charge- and the energy-conservation laws, respectively. The third equation is the relativistic Euler equation. We see that the rest-mass density is replaced here by the enthalpy density \( h \), which therefore is the appropriate measure of inertia for relativistic fluids. To include dissipative phenomena, one needs to take into account the deviation of the statistical operator from its local-equilibrium form.

3. First-order dissipative hydrodynamics

The expansion of the non-equilibrium statistical operator obtained in the previous section allows us to derive hydrodynamics of a dissipative fluid order by order. We have seen that at zeroth order this expansion leads to ideal hydrodynamics given by Eq. (81). In this section, we review the derivation of the relativistic Navier–Stokes equations by exploiting the first-order expansion of the statistical operator in the thermodynamic forces \([12, 16, 18, 22]\). This procedure will allow us to obtain Kubo formulas for the transport coefficients entering first-order dissipative hydrodynamics.

3.1. Decomposition into different dissipative processes

For our further computations it is convenient to decompose the operator \( \hat{C} \) given by Eq. (49) into the different dissipative quantities entering Eqs. (66) and (67). Similar decompositions were performed in Refs. \([12, 16, 18]\). Using Eqs. (68) and (71), as well as \( \beta^\nu \equiv \beta u^\nu \), we obtain

\[
\hat{C} = \hat{\epsilon} D \beta - \hat{\rho} \beta \theta - \sum_a \hat{n}_a D \alpha_a + \hat{q}^\mu (\beta Du_\alpha + \partial_\sigma \beta) - \sum_a \hat{j}_a^\sigma \partial_\sigma \alpha_a + \beta \hat{\pi}_{\rho\sigma} \partial^\rho u^\sigma. \tag{82}
\]

The first three terms correspond to the scalar, the next two terms to the vector, and the last term to the tensor dissipative processes. To lowest order in dissipative currents and gradients,
the terms \( D\beta, D\alpha_a, \) and \( Du^\sigma \) can be eliminated using the equations of ideal hydrodynamics (81). (Higher-order corrections from the fully dissipative hydrodynamic equations (77) – (79) are necessary, and will be taken into account, in the derivation of second-order dissipative hydrodynamics, see Sec. 4.1.) Choosing \( \epsilon \) and \( n_a \) as independent thermodynamic variables and using the first two equations (81) we find

\[
D\beta = \frac{\partial \beta}{\partial \epsilon}\Bigg|_{n_a} D\epsilon + \sum_a \frac{\partial \beta}{\partial n_a}\Bigg|_{\epsilon,n_b\neq n_a} Dn_a = -\theta \left( h \frac{\partial \beta}{\partial \epsilon}\Bigg|_{n_a} + \sum_a n_a \frac{\partial \beta}{\partial n_a}\Bigg|_{\epsilon,n_b\neq n_a} \right),
\]

\[
D\alpha_c = \frac{\partial \alpha_c}{\partial \epsilon}\Bigg|_{n_a} D\epsilon + \sum_a \frac{\partial \alpha_c}{\partial n_a}\Bigg|_{\epsilon,n_b\neq n_a} Dn_a = -\theta \left( h \frac{\partial \alpha_c}{\partial \epsilon}\Bigg|_{n_a} + \sum_a n_a \frac{\partial \alpha_c}{\partial n_a}\Bigg|_{\epsilon,n_b\neq n_a} \right).
\]

For the sake of convenience we write the thermodynamic relations (28), (31), and (32) in the following form

\[
ds = \beta d\epsilon - \sum_a \alpha_a d\epsilon, \quad \beta dp = -hd\beta + \sum_a n_a d\alpha_a.
\]

We obtain from the first equation the set of Maxwell relations

\[
\frac{\partial \beta}{\partial n_a}\Bigg|_{\epsilon,n_b\neq n_a} = -\frac{\partial \alpha_a}{\partial \epsilon}\Bigg|_{n_b}, \quad \frac{\partial \alpha_c}{\partial n_a}\Bigg|_{\epsilon,n_b\neq n_a} = \frac{\partial \alpha_a}{\partial \epsilon}\Bigg|_{n_b},
\]

and from the second equation we immediately read off

\[
h = -\beta \frac{\partial p}{\partial \beta}\Bigg|_{\alpha_a}, \quad n_a = \beta \frac{\partial p}{\partial \alpha_a}\Bigg|_{\beta,\alpha_b\neq \alpha_a}.
\]

Substituting Eqs. (86) and (87) into Eqs. (83) and (84) we obtain

\[
D\beta = \beta \theta \left( \frac{\partial p}{\partial \beta}\Bigg|_{\alpha_a} \frac{\partial \beta}{\partial \epsilon}\Bigg|_{n_a} + \sum_a \frac{\partial p}{\partial \alpha_a}\Bigg|_{\beta,\alpha_b\neq \alpha_a} \frac{\partial \alpha_a}{\partial \epsilon}\Bigg|_{n_b} \right) \equiv \beta \theta \gamma,
\]

\[
D\alpha_c = -\beta \theta \left( \frac{\partial p}{\partial \beta}\Bigg|_{\alpha_a} \frac{\partial \beta}{\partial n_c}\Bigg|_{\epsilon,n_b\neq n_c} + \sum_a \frac{\partial p}{\partial \alpha_a}\Bigg|_{\beta,\alpha_b\neq \alpha_a} \frac{\partial \alpha_a}{\partial n_c}\Bigg|_{\epsilon,n_b\neq n_c} \right) \equiv -\beta \theta \delta_c,
\]

where

\[
\gamma \equiv \frac{\partial p}{\partial \epsilon}\Bigg|_{n_a}, \quad \delta_a \equiv \frac{\partial p}{\partial n_a}\Bigg|_{\epsilon,n_b\neq n_a}.
\]

The first three terms in Eq. (82) can then be combined as follows,

\[
\hat{\epsilon} D\beta - \hat{p} \beta \theta - \sum_a \hat{n}_a D\alpha_a = -\beta \theta \hat{p}^*,
\]

where

\[
\hat{p}^* = \hat{p} - \gamma \hat{\epsilon} - \sum_a \delta_a \hat{n}_a.
\]
Note that the coefficient $\gamma$ coincides with the square of the speed of sound $c_s$ in the case where conserved charges are absent. However, in general $\gamma \neq c_s^2$. (For a single conserved charge $n$, we have $c_s^2 = \partial p/\partial \epsilon|_{s/n}$.)

Using the second relation (85) and the third equation (81) we obtain

$$T \nabla_\sigma \beta + Du_\sigma = T \sum_a \frac{n_a}{h} \nabla_\sigma \alpha_a. \quad (93)$$

Using Eq. (93), as well as $\hat{q}^\sigma \nabla_\sigma = \hat{q}^\sigma \partial_\sigma$ and $\hat{j}_a^\sigma \nabla_\sigma = \hat{j}_a^\sigma \partial_\sigma$, which is a consequence of Eq. (68), we modify the vector terms in Eq. (82) as follows,

$$\hat{q}^\sigma (\beta Du_\sigma + \partial_\sigma \beta) = \sum_a \frac{n_a}{h} \hat{q}^\sigma \nabla_\sigma \alpha_a, \quad \hat{j}_a^\sigma \partial_\sigma \alpha_a = \hat{j}_a^\sigma \nabla_\sigma \alpha_a. \quad (94)$$

Finally, using the properties of the shear-stress tensor $\hat{\pi}_{\rho\sigma}$ we can replace $\partial_\rho u^\sigma \rightarrow \sigma^{\rho\sigma} = \Delta^{\rho\sigma} \partial^\mu u_\mu$ in the last term in Eq. (82). Now combining Eqs. (82), (91), and (94) we obtain the final form of the operator $\hat{C}$ to first order in gradients

$$\hat{C} = -\beta \theta \hat{p}^* + \beta \hat{\pi}_{\rho\sigma} \sigma^{\rho\sigma} - \sum_a \hat{J}_a^\sigma \nabla_\sigma \alpha_a, \quad (95)$$

where

$$\hat{J}_a^\sigma = \hat{j}_a^\sigma - \frac{n_a}{h} \hat{q}^\sigma \quad (96)$$

are the charge-diffusion currents with respect to the energy current, as shown in Appendix A.

As seen from Eq. (95), the operator $\hat{C}$ depends linearly on the thermodynamic forces $\theta$, $\sigma^{\rho\sigma}$, and $\nabla_\sigma \alpha_a$ which correspond to the bulk-viscous, the shear-viscous, and the flavor-diffusion effects, respectively. We will proceed below to obtain the linear Navier–Stokes relations between these thermodynamic forces and the dissipative currents.

3.2. Computing the dissipative quantities

According to Curie’s theorem, in an isotropic medium the correlations between operators of different rank vanish [59, 60]. Using this fact, we obtain from Eqs. (62) and (95) for the shear-stress tensor to leading order

$$\langle \hat{\pi}_{\mu\nu}(x) \rangle_1 = \int d^4 x_1 \left( \hat{\pi}_{\mu\nu}(x), \hat{\pi}_{\rho\sigma}(x_1) \right) \beta(x_1) \sigma^{\rho\sigma}(x_1). \quad (97)$$

The main contribution to the integrand in Eq. (97) comes from the range $|x_1 - x| \lesssim \lambda$, where $\lambda$ is a typical microscopic length scale over which the shear-stress correlation function decays. This correlation length characterizes the range of the interaction and is thus proportional to the mean free path between particle scatterings. On the other hand, in the hydrodynamic regime, the thermodynamic parameters and the fluid velocity vary over a macroscopic length scale $L \gg \lambda$. The ratio $Kn = \lambda/L \ll 1$ is commonly called the Knudsen number. Corrections to ideal hydrodynamics can be formally arranged in a power series in $Kn$. First-order dissipative hydrodynamics takes into account terms of linear order in $Kn$.  

17
Since the thermodynamic force $\beta \sigma^{\rho\sigma}$ varies over the scale $L$, while the integrand in Eq. (97) contributes only over a range $\sim \lambda$, we may take $\beta \sigma^{\rho\sigma}$ as a constant and factor it out from the integral. In doing so, we take $\beta \sigma^{\rho\sigma}$ at the space-time point $x$ [12, 16, 18], in accordance with the mean-value theorem of integral calculus. This leads to a local, linear relation between the shear-stress tensor and the shear tensor

$$\pi_{\mu\nu}(x) \equiv \left\langle \pi_{\mu\nu}(x) \right\rangle_1 = \beta(x) \sigma^{\rho\sigma}(x) \int d^4 x_1 \left( \tilde{\pi}_{\mu\nu}(x), \tilde{\pi}_{\rho\sigma}(x_1) \right).$$  \hspace{1cm} (98)

Since the effective range of integration in Eq. (97) is of order $\lambda$ and since $|\sigma^{\rho\sigma}| \simeq |u^\rho|/L$, we observe that the right-hand side of Eq. (98) is of order $\lambda/L = Kn$, i.e., of order one in the Knudsen number. However, as we will show in the next section, the non-locality of the thermodynamic forces is crucial for the derivation of causal equations of motion for the dissipative currents [46, 56–58]. Therefore, we need to go beyond this approximation to maintain the causality of the theory.

Next we turn to the bulk-viscous pressure $\Pi$, which is defined as the deviation of the actual isotropic pressure $\langle \hat{p} \rangle = \langle \hat{p} \rangle_1 + \langle \hat{p} \rangle_1$ from its equilibrium value $p(\epsilon, n_a)$ given by the EoS as a result of fluid expansion or compression

$$\Pi = \langle \hat{p} \rangle - p(\epsilon, n_a) = \langle \hat{p} \rangle_1 + \langle \hat{p} \rangle_1 - p(\epsilon, n_a).$$  \hspace{1cm} (99)

We have also $\epsilon = \langle \hat{\epsilon} \rangle_1 + \langle \hat{\epsilon} \rangle_1$, $n_a = \langle \hat{n}_a \rangle_1 + \langle \hat{n}_a \rangle_1$, therefore at first order in gradients we have

$$\langle \hat{p} \rangle_1 \equiv p(\langle \hat{\epsilon} \rangle_1, \langle \hat{n}_a \rangle_1) = p(\epsilon - \langle \hat{\epsilon} \rangle_1, n_a - \langle \hat{n}_a \rangle_1) = p(\epsilon, n_a) - \gamma \langle \hat{\epsilon} \rangle_1 - \sum_a \delta_a \langle \hat{n}_a \rangle_1,$$  \hspace{1cm} (100)

where the coefficients $\gamma, \delta_a$ are defined in Eq. (90). Note that the corrections $\langle \hat{\epsilon} \rangle_1$ and $\langle \hat{n}_a \rangle_1$ vanish if the matching conditions (12) are imposed. We prefer to keep them for the sake of generality, so that the final expressions will be independent of the choice of the matching conditions. Substituting Eq. (100) in Eq. (99) for the bulk-viscous pressure we obtain

$$\Pi = \langle \hat{p} - \gamma \hat{\epsilon} - \sum_a \delta_a \hat{n}_a \rangle_1 = \langle \hat{p}^* \rangle_1,$$  \hspace{1cm} (101)

where we used the definition (92) of $\hat{p}^*$. From Eqs. (62) and (95) we now obtain the first-order correction to the bulk viscous pressure (applying similar arguments as in the derivation of Eq. (98))

$$\Pi(x) = \left\langle \hat{p}^*(x) \right\rangle_1 = -\beta(x) \theta(x) \int d^4 x_1 \left( \hat{p}^*(x), \hat{p}^*(x_1) \right).$$  \hspace{1cm} (102)

Thus, the bulk-viscous pressure is expressed in terms of a symmetric correlator between two $\hat{p}^*$ operators.

Finally, again using Curie’s theorem, we find for the charge-diffusion currents

$$\mathcal{J}_a^\mu(x) \equiv \left\langle \hat{\mathcal{J}}_a^\mu(x) \right\rangle_1 = -\sum_b [\nabla_\sigma \alpha_b(x)] \int d^4 x_1 \left( \hat{\mathcal{J}}_a^\mu(x), \hat{\mathcal{J}}_b^\sigma(x_1) \right),$$  \hspace{1cm} (103)

which are expressed in terms of symmetric correlators as well.
3.3. Transport coefficients for an isotropic medium

The isotropy of the medium together with the conditions (68) further implies [16]

\[
\left( \mathcal{J}^{\mu}_{a}(x), \mathcal{J}^{\nu}_{b}(x_1) \right) = \frac{1}{3} \Delta^{\mu\nu}(x) \left( \mathcal{J}^{\lambda}_{a}(x), \mathcal{J}^{b\lambda}(x_1) \right),
\]

\[
\left( \tilde{\pi}_{\mu\nu}(x), \tilde{\pi}_{\rho\sigma}(x_1) \right) = \frac{1}{5} \Delta_{\mu\nu\rho\sigma}(x) \left( \tilde{\pi}^{\lambda\eta}(x), \tilde{\pi}^{\lambda\eta}(x_1) \right).
\]

(104)

(105)

Defining the shear and the bulk viscosities as

\[
\eta(x) \equiv \frac{\beta(x)}{10} \int d^{4}x_{1} \left( \tilde{\pi}_{\mu\nu}(x), \tilde{\pi}^{\mu\nu}(x_1) \right),
\]

\[
\zeta(x) \equiv \beta(x) \int d^{4}x_{1} \left( \hat{p}^{*}(x), \hat{p}^{*}(x_1) \right).
\]

(106)

(107)

we obtain from Eqs. (98), (102), and (105)

\[
\pi_{\mu\nu} = 2 \eta \sigma_{\mu\nu}, \quad \Pi = -\zeta \theta.
\]

(108)

To write down \(\eta(x)\) in the local rest frame, we use the first relation in Eq. (70) in Eq. (106) and make the transition (indicated by the arrow)

\[
\eta(x) = \frac{\beta(x)}{10} \Delta_{\alpha\beta\gamma\delta}(x) \int d^{4}x_{1} \left( \hat{T}^{\alpha\beta}(x), \hat{T}^{\gamma\delta}(x_1) \right) \rightarrow \frac{\beta(x)}{10} \int d^{4}x_{1} \left( \hat{\pi}_{ij}(x), \hat{\pi}^{ij}(x_1) \right),
\]

(109)

where the \(\Delta(x_1)\) projector is taken out of the integral at the value \(x\) (which is correct up to the required accuracy) and

\[
\Delta_{\alpha\beta\gamma\delta}(x) \rightarrow \Delta_{ijkl} = \frac{\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}}{2} - \frac{1}{3} \delta_{ij}\delta_{kl}
\]

in the local rest frame.

The two-point correlators in Eqs. (106) and (107) should be evaluated at constant values of the thermodynamic parameters, which, nevertheless, can change from space-time point to space-time point.

Using Eqs. (103) and (104) we obtain for the diffusion currents

\[
\mathcal{J}^{\mu}_{a} = \sum_{b} \chi_{ab} \nabla^{\mu}\alpha_{b},
\]

(110)

where we defined the matrix of diffusion coefficients

\[
\chi_{ab}(x) = -\frac{1}{3} \int d^{4}x_{1} \left( \mathcal{J}^{\lambda}_{a}(x), \mathcal{J}^{b\lambda}(x_1) \right).
\]

(111)

In the case of a single conserved charge species we define the heat-flux operator as (see Appendix A for details)

\[
\hat{h}^{\mu} = \hat{q}^{\mu} - \frac{\hat{h}}{n} \hat{\mathcal{J}}^{\mu} = -\frac{\hat{h}}{n} \mathcal{J}^{\mu},
\]

(112)
the averaging of which gives

$$h^\mu = -\kappa \frac{\eta T^2}{\hbar} \nabla^\mu \alpha, \quad (113)$$

where the thermal conductivity is defined as

$$\kappa(x) = -\frac{\beta^2(x)}{3} \int d^4x_1 \left\langle \hat{h}^\lambda(x), \hat{h}_\lambda(x_1) \right\rangle. \quad (114)$$

Equations (108), (110), and (113) establish the required linear relations between the dissipative currents and the thermodynamic forces. We thus conclude that the non-equilibrium statistical operator correctly reproduces the Navier–Stokes limit of relativistic dissipative hydrodynamics.

The shear viscosity, the bulk viscosity, and the thermal conductivity given by Eqs. (106), (107), and (114), respectively, are positive quantities, and the matrix $\chi_{ab}$ is positive semidefinite in flavor space due to the fact that all diffusion currents are spatial. These properties guarantee the entropy increase in dissipative fluids, see Appendix B.

As shown in Appendix C, the transport coefficients defined in Eqs. (106), (107), (111), and (114) can be expressed via two-point retarded Green’s functions as follows

$$\eta = -\frac{1}{10} \frac{d}{d\omega} \text{Im} G_{\hat{h}_\mu, \hat{h}_\mu}^R (\omega) \bigg|_{\omega=0}, \quad \zeta = -\frac{d}{d\omega} \text{Im} G_{\hat{p}^\rho, \hat{p}^\rho}^R (\omega) \bigg|_{\omega=0}, \quad (115)$$

$$\chi_{ab} = \frac{T}{3} \frac{d}{d\omega} \text{Im} G_{\hat{J}^\lambda a, \hat{J}^\lambda b}^R (\omega) \bigg|_{\omega=0}, \quad \kappa = \frac{1}{3T} \frac{d}{d\omega} \text{Im} G_{\hat{h}^\lambda, \hat{h}^\lambda}^R (\omega) \bigg|_{\omega=0}, \quad (116)$$

where

$$G_{\hat{X}\hat{Y}}^R (\omega) = -i \int_0^\infty dt e^{i\omega t} \int d^3x \left\langle [\hat{X}(x,t), \hat{Y}(0,0)] \right\rangle, \quad (117)$$

is the Fourier transform of the retarded two-point correlator taken in the zero-wavenumber limit and the square brackets denote the commutator. Some of these relations were obtained within the Zubarev formalism in Refs. [16–18]. The relations (115) and (116) are known as the Kubo formulas for the transport coefficients [61, 62].

4. Second-order dissipative hydrodynamics

In this section, we systematically compute all second-order corrections to the dissipative currents. Along with the thermodynamic forces these currents are regarded as first-order quantities in deviations from equilibrium. The second-order terms are those which involve either space-time derivatives of the dissipative currents or products of the dissipative currents with the thermodynamic forces. It is easy to see that such second-order contributions arise not only from the three-point correlators in Eq. (62), which are quadratic in the operator $\hat{C}$, but also from the two-point correlators, where one should take into account the second-order corrections to the operator $\hat{C}$, which were neglected in the derivation of Eq. (95). Apart from that, additional second-order gradient terms arise from the non-locality of the thermodynamic forces in the two-point correlators, i.e., when the difference between the space-time arguments $x$ and $x_1$ in the dissipative currents and the thermodynamic forces are taken into account. As we will show below in Secs. 4.3 – 4.5, these non-local effects generate relaxation terms in the transport equations which are required to maintain causality.
4.1. Decomposing the thermodynamic forces up to second order

In this subsection, we repeat the decomposition of the operator \( \hat{C} \), keeping all second-order corrections that were neglected in Sec. 3.1. Our starting point is Eq. (82), which on account of Eq. (68) we can write in the form

\[
\hat{C} = \hat{\epsilon} D\beta - \hat{p}\beta\theta - \sum_a \hat{n}_a D\alpha_a + \hat{q}^\sigma (\beta Du_\sigma + \nabla_\sigma \beta) - \sum_a \hat{j}_a^\sigma \nabla_\sigma \alpha_a + \beta \pi_{\rho\sigma} \sigma_{\rho\sigma}. \tag{118}
\]

We now use the dissipative hydrodynamical equations of motion (77), (78) to eliminate the terms \( D\beta, D\alpha_a \) in Eq. (118). Instead of Eqs. (83), (84), (88), and (89) we now have

\[
D\beta = \beta \theta \gamma - (\Pi \theta + \partial_\mu q^\mu - q^\mu Du_\mu - \pi^{\mu\nu} \sigma_{\mu\nu}) \frac{\partial \beta}{\partial \epsilon} \bigg|_{n, n_b \neq n_a}, \tag{119}
\]

\[
D\alpha_c = -\beta \theta \delta_c - (\Pi \theta + \partial_\mu q^\mu - q^\mu Du_\mu - \pi^{\mu\nu} \sigma_{\mu\nu}) \frac{\partial \alpha_c}{\partial \epsilon} \bigg|_{n_a} - \sum_a \hat{j}_a^\mu \frac{\partial \alpha_c}{\partial \epsilon} \bigg|_{n_b \neq n_a}. \tag{120}
\]

The first three terms in Eq. (118) can be combined as follows

\[
\hat{\epsilon} D\beta - \hat{p}\beta\theta - \sum_a \hat{n}_a D\alpha_a = -\beta \theta \hat{\beta}^* - \hat{\beta}^* (\Pi \theta + \partial_\mu q^\mu - q^\mu Du_\mu - \pi^{\mu\nu} \sigma_{\mu\nu}) + \sum_a \hat{\alpha}_a^* \partial_\mu j_a^\mu, \tag{121}
\]

where we used Eqs. (86) and (92) and defined new operators

\[
\hat{\beta}^* = \hat{\epsilon} \frac{\partial \beta}{\partial \epsilon} \bigg|_{n_a} + \sum_a \hat{n}_a \frac{\partial \beta}{\partial \epsilon} \bigg|_{n_b \neq n_a}, \tag{122}
\]

\[
\hat{\alpha}_a^* = \hat{\epsilon} \frac{\partial \alpha_a}{\partial \epsilon} \bigg|_{n_b} + \sum_c \hat{n}_c \frac{\partial \alpha_a}{\partial \epsilon} \bigg|_{n_b \neq n_c}. \tag{123}
\]

Next we use Eq. (79) in the form [the gradient of pressure is modified according to the second relation in Eq. (85)]

\[
hDu_\sigma = -hT \nabla_\sigma \beta + T \sum_a n_a \nabla_\sigma \alpha_a + \nabla_\sigma \Pi
\]

\[
- \Pi Du_\sigma - \Delta_\sigma \mu Dq^\mu - q^\mu \partial_\mu u_\sigma - q_\sigma \theta - \Delta_\sigma \nu \partial_\nu \pi^{\mu\nu}, \tag{124}
\]

to modify the vector term involving \( \hat{q}^\sigma \) in Eq. (118),

\[
\hat{q}^\sigma (\beta Du_\sigma + \nabla_\sigma \beta) = \sum_a \frac{n_a}{h} \hat{q}^\sigma \nabla_\sigma \alpha_a
\]

\[
- \hat{q}^\sigma \beta h^{-1} (-\nabla_\sigma \Pi + \Pi Du_\sigma + Dq_\sigma + q^\mu \partial_\mu u_\sigma + q_\sigma \theta + \partial_\mu \pi^{\mu\nu}). \tag{125}
\]

Combining Eqs. (118), (121), and (125) we obtain

\[
\hat{C}(x) = \hat{C}_1(x) + \hat{C}_2(x), \tag{126}
\]
where \( \hat{C}_1 \) and \( \hat{C}_2 \) are the first- and the second-order contributions, respectively:

\[
\begin{align*}
\hat{C}_1(x) &= -\beta \theta \hat{p}^* + \beta \hat{\pi} \rho \sigma \sigma^\rho \sigma - \sum_a \hat{\mathcal{J}}_a^\sigma \nabla_\sigma \alpha_a, \\
\hat{C}_2(x) &= -\hat{\beta}^* (\Pi \theta + \partial_\mu q^\mu - q^\mu D u_\mu - \pi^\mu \sigma_{\mu \nu}) + \sum_a \hat{\alpha}_a^\sigma \partial_\mu j_a^\mu \\
&\quad - \hat{q}^\sigma \beta h^{-1} \left(- \nabla_\sigma \Pi + \Pi D u_\sigma + D q_\sigma + q^\mu \partial_\mu u_\sigma + q_\sigma \theta + \partial_\mu \pi_\sigma^\mu \right).
\end{align*}
\] (127)

We observe that the operator \( \hat{C}_2 \) contains only scalar and vector terms, and, therefore, contributes only to the bulk-viscous pressure and the diffusion currents. The reason for this is that \( \hat{C}_2 \) originates from the dissipative terms of the hydrodynamic equations (77) – (79), which have either scalar or vector structure. It is natural to denote the expressions contained in parentheses in Eq. (128) as generalized or extended thermodynamic forces. They involve either space-time derivatives of the dissipative currents or their products with the “ordinary” thermodynamic forces. However, in these generalized thermodynamic forces the comoving derivatives (i.e., time derivatives in the local rest frame)

\[
\dot{\Pi} \equiv D \Pi, \quad \dot{\pi}_{\mu \nu} \equiv \Delta_{\mu \nu \rho \sigma} D \pi_{\rho \sigma}, \quad \dot{\mathcal{J}}_{a \mu} \equiv \Delta_{\mu \nu} D \mathcal{J}^\nu_{a},
\] (129)
do not appear, but should be present in a causal theory. As we will show below, these terms arise from the non-locality of the thermodynamic forces involved in Eq. (127). [Note that in Eq. (128) we have a comoving derivative for the energy flow, i.e., \( D q_\sigma \). This term, however, is absent in the L-frame. Therefore, we actually have only space-like gradients of the dissipative currents in Eq. (128).]

Now using Eq. (62) for the statistical average of an arbitrary operator \( \hat{X}(x) \) we can write up to second order

\[
\langle \hat{X}(x) \rangle = \langle \hat{X}(x) \rangle_1 + \langle \hat{X}(x) \rangle_2 + \langle \hat{X}(x) \rangle_3.
\] (130)

The first-order correction is given by

\[
\langle \hat{X}(x) \rangle_1 = \int d^4x_1 \left( \hat{X}(x), \hat{C}_1(x_1) \right) \bigg|_{\text{loc}},
\] (131)

where the index “loc” indicates that the thermodynamic forces in the integrand are approximated by their local values at the point \( x \), i.e., the non-local effects are neglected, as explained in Sec. 3.2.

The second-order correction \( \langle \hat{X}(x) \rangle_2 \) can be decomposed into three terms,

\[
\langle \hat{X}(x) \rangle_2 = \langle \hat{X}(x) \rangle_1^2 + \langle \hat{X}(x) \rangle_2^2 + \langle \hat{X}(x) \rangle_3^3,
\] (132)

with

\[
\begin{align*}
\langle \hat{X}(x) \rangle_1^2 &= \int d^4x_1 \left( \hat{X}(x), \hat{C}_1(x_1) \right) - \langle \hat{X}(x) \rangle_1, \\
\langle \hat{X}(x) \rangle_2^2 &= \int d^4x_1 \left( \hat{X}(x), \hat{C}_2(x_1) \right), \\
\langle \hat{X}(x) \rangle_3^3 &= \int d^4x_1 d^4x_2 \left( \hat{X}(x), \hat{C}_1(x_1), \hat{C}_1(x_2) \right).
\end{align*}
\] (133)
The first term in Eq. (132) collects those corrections which arise from the non-locality of the thermodynamic forces involved in the operator $\hat{C}(x_1)$. These corrections are of second order, because they involve the differences of a thermodynamic force, e.g., $\sigma^{\mu\nu}$, at points $x_1$ and $x$, as seen from Eqs. (127), (131), and (133). Therefore, we can approximate $\sigma^{\mu\nu}(x_1) - \sigma^{\mu\nu}(x) \approx (x_1 - x)^{\alpha} \partial_\alpha \sigma^{\mu\nu}(x) \sim \text{Kn} \, \sigma^{\mu\nu}(x)$, because $x_1 - x \sim \lambda$ and $\partial \sim L^{-1}$, as mentioned in Sec. 3.2. Thus, the corrections of the type (133) contain an additional power of the Knudsen number as compared to the first-order expression (131), and, therefore, are at least of second order in the hydrodynamic expansion.

The second term in Eq. (132) includes the corrections from the generalized thermodynamic forces. Finally, the third term stands for the corrections which are nonlinear (quadratic) in the three thermodynamic forces $\theta$, $\sigma_{\rho\sigma}$, and $\nabla_\sigma \alpha_a$, which appear in the Navier–Stokes limit.

In order to properly derive the non-local corrections (133) to the dissipative currents, we should first generalize the expressions for the two-point correlators given by Eqs. (104) and (105), which were initially written to provide first-order accuracy only.

4.2. Non-local generalization of two-point correlators

The two-point correlators (104) and (105) can be generalized in a straightforward manner to incorporate the non-locality of the spatial projectors

\[
\left( \hat{\mathcal{J}}_a^{\mu}(x), \hat{\mathcal{J}}_b^{\nu}(x_1) \right) = \frac{1}{3} \Delta^{\mu\nu}(x, x_1) \left( \hat{\mathcal{J}}_a^{\lambda}(x), \hat{\mathcal{J}}_b^{\lambda}(x_1) \right),
\]

\[
\left( \hat{\pi}_{\mu\nu}(x), \hat{\pi}_{\rho\sigma}(x_1) \right) = \frac{1}{5} \Delta_{\mu\nu\rho\sigma}(x, x_1) \left( \hat{\pi}_{\lambda\lambda}(x), \hat{\pi}_{\lambda\lambda}(x_1) \right),
\]

Here the new projectors

\[
\Delta_{\mu\nu}(x, x_1) = \Delta_{\mu\lambda}(x)\Delta^{\lambda\nu}(x_1),
\]

\[
\Delta_{\mu\nu\rho\sigma}(x, x_1) = \Delta_{\mu\rho\sigma\beta}(x)\Delta^{\alpha\beta}(x_1)
\]

are the natural non-local generalizations of the second-rank $\Delta_{\mu\nu}$ and fourth-rank $\Delta_{\mu\nu\rho\sigma}$ projectors, respectively. The normalization of the right-hand sides of Eqs. (136) and (137) is performed at the leading order in velocity gradients, see Appendix D for details. The non-local form of the projectors (138) and (139) guarantees that the orthogonality conditions (68) are satisfied for the correlation functions given by Eqs. (136) and (137) at both points $x$ and $x_1$.

For our calculations, it is sufficient to keep only the linear terms in the difference $x_1 - x$ of the expansion of the non-local projectors around $x_1 = x$. As shown in Appendix D, the first derivative of the fourth-rank projector is

\[
\frac{\partial}{\partial x_1^\alpha} \Delta_{\mu\nu\rho\sigma}(x, x_1) \bigg|_{x_1 = x} = -\left( \Delta_{\mu\rho\sigma\beta} u_\sigma + \Delta_{\mu\nu\sigma\beta} u_\nu \right) \partial_\alpha u^\beta,
\]

which we will utilize below. In addition, we will assume that Curie’s theorem holds also in this approximation, i.e., the two-point correlations between tensors of different rank vanish.

4.3. Second-order corrections to the shear-stress tensor

According to Eqs. (132) – (135), the second-order corrections to the dissipative currents arise from three different sources. In this subsection we compute all these corrections separately for the shear-stress tensor. In the following two subsections we do this for the bulk-viscous pressure and the diffusion currents.
4.3.1. Non-local corrections from the two-point correlation function

Substituting the decomposition (127) into Eq. (133) and recalling Curie’s theorem we obtain

$$\langle \hat{\pi}_{\mu\nu}(x) \rangle_{\frac{1}{2}} = \int d^{4}x_{1} \left( \hat{\pi}_{\mu\nu}(x), \hat{\pi}_{\rho\sigma}(x_{1}) \right) \beta(x_{1}) \sigma^{\rho\sigma}(x_{1}) - 2\eta(x)\sigma_{\mu\nu}(x), \tag{141}$$

where we used the first-order relation for $\langle \hat{\pi}_{\mu\nu}(x) \rangle_{1}$ given by Eq. (108). The thermodynamic force $\beta(x_{1})\sigma^{\rho\sigma}(x_{1})$ cannot be factored out from the integral with its value at $x_{1}$, but should be expanded around that value to first order in the difference $x_{1} - x$. Here we note that, in order to obtain all second-order corrections which have non-local origin, we should take into account also the non-locality of the fluid velocity $u^{\lambda}$ in the expression $\hat{\pi}^{\lambda\eta}(x_{1}) = \Delta_{\gamma\delta}^{\lambda\eta}(x_{1})\hat{T}^{\gamma\delta}(x_{1})$ [see Eq. (70)]. In contrast to $\Delta_{\gamma\delta}^{\lambda\eta}(x_{1})$, which is a hydrodynamic quantity, the energy-momentum tensor $\hat{T}^{\gamma\delta}(x_{1})$ is a microscopic quantity, and, therefore, does not require an expansion.

Now we substitute the two-point correlation function given by Eqs. (137) and (139) into Eq. (141) and use the definition of the shear-stress tensor given in Eq. (70) as well as $\Delta_{\mu\nu}^{\rho\sigma}\sigma^{\rho\sigma} \equiv \sigma^{\alpha\beta}$ to obtain,

$$\langle \hat{\pi}_{\mu\nu}(x) \rangle_{\frac{1}{2}} = \frac{1}{5} \Delta_{\mu\nu;\rho\sigma}(x) \int d^{4}x_{1} \left( \hat{\pi}_{\lambda\eta}(x), \hat{\pi}_{\lambda\eta}(x_{1}) \right) \beta(x_{1}) \sigma^{\rho\sigma}(x_{1}) - 2\eta(x)\sigma_{\mu\nu}(x)
= \frac{1}{5} \Delta_{\mu\nu;\rho\sigma}(x) \int d^{4}x_{1} \left( \hat{T}^{\alpha\beta}(x), \hat{T}^{\gamma\delta}(x_{1}) \right) \Delta_{\alpha\beta\gamma\delta}(x, x_{1}) \beta(x_{1}) \sigma^{\rho\sigma}(x_{1}) - 2\eta(x)\sigma_{\mu\nu}(x).$$

In the following step we expand $\Delta_{\alpha\beta\gamma\delta}(x, x_{1}) \beta(x_{1}) \sigma^{\rho\sigma}(x_{1})$ around $x$ up to first order in $x_{1} - x$. Using Eq. (140), as well as $\Delta_{\alpha\beta\gamma\delta}(x, x) = \Delta_{\alpha\beta\rho\sigma}(x) \Delta_{\gamma\delta}^{\rho\sigma}(x) = \Delta_{\alpha\beta\gamma\delta}(x)$, we obtain

$$\langle \hat{\pi}_{\mu\nu}(x) \rangle_{\frac{1}{2}} = \frac{1}{5} \Delta_{\mu\nu;\rho\sigma}(x) \int d^{4}x_{1} \left( \hat{T}^{\alpha\beta}(x), \hat{T}^{\gamma\delta}(x_{1}) \right) \left\{ \Delta_{\alpha\beta\gamma\delta}(x) \beta(x) \sigma^{\rho\sigma}(x) \right\}
+ (x_{1} - x)\tau \left[ \Delta_{\alpha\beta\gamma\delta} \partial_{\tau} (\beta \sigma^{\rho\sigma}) - \beta \sigma^{\rho\sigma}(\Delta_{\alpha\beta\gamma\delta} u_{\delta} + \Delta_{\alpha\beta\gamma\delta} u_{\gamma}) \partial_{\tau} u^{\lambda} \right] - 2\eta(x)\sigma_{\mu\nu}(x)
= 2a^{\tau}(x)\beta^{\alpha\beta}(x) \Delta_{\mu\nu;\rho\sigma}(x) \partial_{\tau} [\beta(x) \sigma^{\rho\sigma}(x)] - 4b^{\tau}_{\lambda}(x)\sigma_{\mu\nu}(x) \partial_{\tau} u^{\lambda}(x). \tag{142}$$

Here we used the fact that the first term in curly brackets cancels with the last term $-2\eta\sigma_{\mu\nu}$, on account of Eqs. (70) and (106). We also defined

$$a^{\tau}(x) \equiv \Delta_{\alpha\beta\gamma\delta}(x) I^{\alpha\beta\gamma\delta,\tau}(x), \quad b^{\tau}_{\lambda}(x) \equiv \Delta_{\alpha\beta\gamma\delta}(x) u_{\delta}(x) I^{\alpha\beta\gamma\delta,\tau}(x), \tag{143}$$

with

$$I^{\alpha\beta\gamma\delta,\tau}(x) = \frac{\beta(x)}{10} \int d^{4}x_{1} \left( \hat{T}^{\alpha\beta}(x), \hat{T}^{\gamma\delta}(x_{1}) \right) (x_{1} - x)^{\tau}. \tag{144}$$

Recalling the relations (69) and (70) we can write the expressions (143) and (144) in the following form

$$a^{\tau}(x) = \frac{\beta(x)}{10} \int d^{4}x_{1} \left( \hat{\pi}_{\alpha\beta}(x), \hat{\pi}_{\alpha\beta}(x_{1}) \right) (x_{1} - x)^{\tau}, \tag{145}$$
$$b^{\tau}_{\lambda}(x) = \frac{\beta(x)}{10} \int d^{4}x_{1} \left( \hat{\pi}_{\alpha\lambda}(x), \hat{q}^{\alpha}(x_{1}) \right) (x_{1} - x)^{\tau} = 0, \tag{146}$$
where we have approximated \( u^\mu(x) \simeq u^\mu(x_1) \), because the quantities (145) and (146) are already multiplied with second-order terms in Eq. (142). The tensor \( \dot{b}_\lambda \) vanishes on account of Curie’s theorem, and the vector \( a^\tau \) can be written in the following form [see Appendix C, Eqs. (C.24) – (C.27)]

\[
a^\tau = -\eta \tau_\pi u^\tau,
\]

where we defined

\[
\eta \tau_\pi = -i \frac{d}{d\omega}\eta_\pi(\omega) \bigg|_{\omega=0} = \frac{1}{20} \frac{d^2}{d\omega^2} \text{Re} G^R_{\pi, \pi}(\omega) \bigg|_{\omega=0}.
\]

Here the retarded Green’s function is given by Eq. (117), and the frequency-dependent shear viscosity \( \eta(\omega) \) is defined by analogy with Eq. (C.11). As seen from Eq. (148), the new coefficient \( \tau_\pi \) has the dimension of time and can be regarded as a relaxation time for the shear-stress tensor.

Combining Eqs. (142), (146), and (147) we obtain

\[
\langle \hat{\pi}_{\mu\nu} \rangle_2^1 = -2\eta \tau_\pi \beta^{-1} \Delta_{\mu\nu\rho\sigma} D(\beta \sigma^{\rho\sigma}) = -2\eta \tau_\pi (\Delta_{\mu\nu\rho\sigma} D\sigma^{\rho\sigma} + \gamma \theta \sigma_{\mu\nu}),
\]

where we used Eq. (88) in the second step and for the sake of brevity omitted the argument \( x \) at all quantities.

### 4.3.2. Corrections from extended thermodynamic forces

Since the operator \( \hat{C}_2 \) given by Eq. (128) does not have a tensor part, the correction of \( \hat{\pi}_{\mu\nu} \) from this term vanishes due to Curie’s theorem

\[
\langle \hat{\pi}_{\mu\nu}(x) \rangle_2^2 = \int d^4x_1 \langle \hat{\pi}_{\mu\nu}(x), \hat{C}_2(x_1) \rangle = 0.
\]

### 4.3.3. Corrections from the three-point correlation function

From Eqs. (127) and (135) we have

\[
\langle \hat{\pi}_{\mu\nu}(x) \rangle_3^2 = \int d^4x_1 d^4x_2 \langle \hat{\pi}_{\mu\nu}(x), \left[ -\beta \theta \hat{p}^* + \beta \hat{\pi}_{\rho\sigma} \sigma^{\rho\sigma} - \sum_a \hat{\mathcal{J}}_a^\sigma \nabla_\sigma \alpha_a \right]_{x_1}, \left[ -\beta \theta \hat{p}^* + \beta \hat{\pi}_{\alpha\beta} \sigma^{\alpha\beta} - \sum_b \hat{\mathcal{J}}_b^\alpha \nabla_\alpha \alpha_b \right]_{x_2} \rangle.
\]

In contrast to the two-point correlators, where only operators of the same rank are coupled, in the three-point correlators, one may have a mixing between operators of different ranks. The mixed three-point correlators in Eq. (151) which in general do not vanish are

\[
\left( \hat{\pi}_{\mu\nu}(x), \hat{\pi}_{\alpha\beta}(x_1), \hat{\pi}_{\gamma\delta}(x_2) \right) = \frac{1}{5} \Delta_{\mu\nu\alpha\beta}(x) \left( \hat{\pi}_{\gamma\delta}(x), \hat{\pi}_{\alpha\beta}(x_1), \hat{\pi}_{\gamma\delta}(x_2) \right),
\]

\[
\left( \hat{\pi}_{\mu\nu}(x), \hat{\mathcal{J}}_{\alpha\gamma}(x_1), \hat{\mathcal{J}}_{\beta\delta}(x_2) \right) = \frac{1}{5} \Delta_{\mu\nu\sigma\alpha}(x) \left( \hat{\pi}_{\gamma\delta}(x), \hat{\mathcal{J}}_{\alpha\gamma}(x_1), \hat{\mathcal{J}}_{\beta\delta}(x_2) \right),
\]

where we exploited the symmetries of the relevant operators and of the three-point correlator, cf. Eq. (65). The odd-rank mixed three-point correlator \( \left( \hat{\pi}_{\mu\nu}(x), \hat{p}^*(x_1), \hat{\mathcal{J}}_{\alpha\gamma}(x_2) \right) \) vanishes.
on account of the space-time symmetry of the operators involved. For similar reasons, the

\[ (\hat{\pi}_{\mu\nu}(x), \hat{p}^*(x_1), \hat{p}^*(x_2)) \]

can only be proportional to \( \Delta_{\mu\nu}(x) \), but this is not trace-
less, so cannot contribute to the constitutive relation for \( \langle \hat{\pi}_{\mu\nu}(x) \rangle \). Because the correlators

(152) and (153) are accompanied by second-order terms in the thermodynamic forces, it is

sufficient to evaluate all \( \Delta \) projectors on the right-hand sides of these relations at the point

\( x \).

The most general expression for the correlator of three shear-stress tensors which satisfies

the orthogonality condition \( u^\mu \hat{\pi}_{\mu\nu} = 0 \) is

\[
\left( \hat{\pi}_{\mu\nu}(x), \hat{\pi}_{\rho\sigma}(x_1), \hat{\pi}_{\alpha\beta}(x_2) \right) = a(\Delta_{\mu\nu}, \Delta_{\rho\sigma}, \Delta_{\mu\nu} \Delta_{\alpha\beta}) + b\Delta_{\mu\nu} \Delta_{\rho\sigma} \Delta_{\alpha\beta} + c(\Delta_{\mu\nu}, \Delta_{\rho\sigma}, \Delta_{\mu\nu} \Delta_{\alpha\beta}) + d(\Delta_{\mu\nu}, \Delta_{\rho\sigma}, \Delta_{\mu\nu} \Delta_{\alpha\beta})
\]

where the coefficients \( a, b, c \) are functions of \( x, x_1, \) and \( x_2 \), while the \( \Delta \) projectors can all be
taken at point \( x \), since the three-point correlator is accompanied by second-order terms. We

now determine the coefficients \( a, b, c \). Since \( \hat{\pi}_{\alpha} = 0 \), we obtain by contracting the indices \( \alpha \) and \( \beta \) in Eq. (154) and using the properties (71), (73), and (74),

\[
\left( \hat{\pi}_{\mu\nu}(x), \hat{\pi}_{\rho\sigma}(x_1), \hat{\pi}_{\alpha\beta}(x_2) \right) = 3a\Delta_{\mu\nu}, + b\Delta_{\rho\sigma} + c(\Delta_{\mu\nu} \Delta_{\rho\sigma})
\]

\[
= (3a + 8c)\Delta_{\mu\nu} \Delta_{\rho\sigma} + \left( 3b + \frac{8c}{3} \right) \Delta_{\mu\nu} = 0,
\]

which implies that \( b = a/3, c = -3a/8 \). In order to determine \( a \) we compute the mixed

contraction

\[
\left( \hat{\pi}_{\mu\nu}(x), \hat{\pi}_{\rho\sigma}(x), \hat{\pi}_{\alpha\beta}(x_2) \right) = 5a + b + 22c \Delta_{\mu\beta} = -\frac{35}{12} a \Delta_{\mu\beta}
\]

\[
\implies a = -\frac{4}{35} \left( \hat{\pi}_{\mu\nu}(x), \hat{\pi}_{\rho\sigma}(x_1), \hat{\pi}_{\alpha\beta}(x_2) \right).
\]

Then Eq. (154) can be cast into the compact form

\[
\left( \hat{\pi}_{\mu\nu}(x), \hat{\pi}_{\rho\sigma}(x_1), \hat{\pi}_{\alpha\beta}(x_2) \right) = \frac{1}{35} \left[ 3(\Delta_{\rho\sigma} \Delta_{\mu\nu}, + \Delta_{\rho\beta} \Delta_{\mu\nu}, \Delta_{\sigma\alpha} + \Delta_{\rho\beta} \Delta_{\mu\nu}, \Delta_{\sigma\beta} \Delta_{\mu\nu}) \right]
\]

\[
-4(\Delta_{\rho\sigma} \Delta_{\mu\nu}, + \Delta_{\sigma\beta} \Delta_{\rho\nu}) \left( \hat{\pi}_{\mu\nu}(x), \hat{\pi}_{\rho\sigma}(x_1), \hat{\pi}_{\alpha\beta}(x_2) \right).
\]

Inserting the correlation functions given by Eqs. (152), (153), and (155) into Eq. (151),

factoring out the thermodynamic forces from the integral with their values at \( x \), taking into
account the symmetry property (65), and defining the set of transport coefficients

\[
\lambda_{\gamma} = \frac{12}{35} \beta^2 \int d^4x_1d^4x_2 \left( \hat{\pi}_{\gamma}(x), \hat{\pi}_{\delta}(x_1), \hat{\pi}_{\delta}(x_2) \right),
\]

\[
\lambda_{\gamma\delta} = -\frac{\beta^2}{5} \int d^4x_1d^4x_2 \left( \hat{\pi}_{\gamma}(x), \hat{\pi}_{\delta}(x_1), \hat{p}^*(x_2) \right),
\]

\[
\lambda_{\gamma\delta}^ab = \frac{1}{5} \int d^4x_1d^4x_2 \left( \hat{\pi}_{\gamma}(x), \hat{\pi}_{\delta}(x_1), \hat{\pi}_{\delta}(x_2) \right),
\]

26
we finally obtain
\[ \langle \pi_{\mu\nu} \rangle^3 = 2 \lambda_{\pi} \theta_{\sigma_{\mu\nu}} + \lambda_{\pi} \sigma_{\alpha<\mu} \sigma_{\nu>}^\alpha + \sum_{ab} \lambda_{\pi}^{ab} \nabla_{\mu} \alpha_a \nabla_{\nu} \alpha_b, \] (159)
where we again suppressed the \( x \)-dependence for the sake of brevity, employed the notation
\[ A_{\mu\nu} \equiv \Delta_{\alpha\beta} A_{\alpha\beta}, \] (160)
and applied the identities \( \sigma_{<\alpha\beta>} = \sigma_{\alpha\beta}, \sigma_{\alpha\beta} \Delta_{\beta\lambda} = \sigma_{\alpha\lambda}, \) and \( \sigma_{\alpha\alpha} = 0. \)

### 4.3.4. Final result for the shear-stress tensor

Combining all contributions from Eqs. (108), (149), (150), and (159) and using Eqs. (80), (130), and (132) we obtain the complete second-order expression for the shear-stress tensor
\[ \pi_{\mu\nu} = 2 \eta_{\sigma_{\mu\nu}} + \bar{\lambda}_{\pi} \sigma_{\alpha<\mu} \sigma_{\nu>}^\alpha + \sum_{ab} \lambda_{\pi}^{ab} \nabla_{\mu} \alpha_a \nabla_{\nu} \alpha_b. \] (161)

Here, the second-order terms in the first line represent the non-local corrections, whereas the second line collects the nonlinear corrections from the three-point correlations. The physical interpretation of the various terms in Eq. (161) is discussed in Sec. 5.2.

In order to derive a relaxation-type equation for \( \pi_{\mu\nu} \) from expression (161), we follow Refs. [28, 35, 38] and use the first-order Navier–Stokes relation (108) to replace \( 2 \sigma_{\rho\sigma} \rightarrow \eta^{-1} \pi_{\rho\sigma} \) in the second term of the right-hand-side of Eq. (161). This substitution is justified because that term is already of second order in space-time gradients, so any correction to the Navier–Stokes result would be at least of third order. We then have
\[ -2 \eta \tau_{\pi} \Delta_{\mu\rho\sigma} D_{\sigma^\rho} \simeq -\tau_{\pi} \pi_{\mu\nu} + \tau_{\pi} \beta \eta^{-1} \left( \frac{\partial \eta}{\partial \beta} - \sum_a \delta_a \frac{\partial \eta}{\partial \alpha_a} \right) \theta_{\pi_{\mu\nu}}, \] (162)
where we used Eqs. (88), (89), and (129). Combining Eqs. (161) and (162) and introducing the coefficients
\[ \lambda = 2(\lambda_{\pi} - \gamma \eta \tau_{\pi}), \] (163)
\[ \bar{\lambda}_{\pi} = \tau_{\pi} \beta \eta^{-1} \left( \frac{\partial \eta}{\partial \beta} - \sum_a \delta_a \frac{\partial \eta}{\partial \alpha_a} \right), \] (164)
we finally obtain the following relaxation equation for the shear-stress tensor,
\[ \tau_{\pi} \pi_{\mu\nu} + \pi_{\mu\nu} = 2 \eta \sigma_{\mu\nu} + \bar{\lambda}_{\pi} \theta_{\pi} \sigma_{\mu\nu} + \lambda_{\pi} \sigma_{\alpha<\mu} \sigma_{\nu>}^\alpha + \sum_{ab} \lambda_{\pi}^{ab} \nabla_{\mu} \alpha_a \nabla_{\nu} \alpha_b. \] (165)

### 4.4. Second-order corrections to the bulk-viscous pressure

In order to evaluate the bulk-viscous pressure to second order, we should include also the second-order corrections to Eqs. (100) and (101). Denoting \( \Delta \epsilon = \langle \dot{\epsilon} \rangle_1 + \langle \dot{\epsilon} \rangle_2 \) and \( \Delta \eta_a = \langle \dot{n}_a \rangle_1 + \langle \dot{n}_a \rangle_2 \) we obtain
\[ \langle \dot{\epsilon} \rangle_1 = p(\epsilon - \Delta \epsilon, n_a - \Delta n_a) = p(\epsilon, n_a) - \gamma \Delta \epsilon - \sum_a \delta_a \Delta n_a \]
\[ + \psi_{\epsilon a} \Delta \epsilon^2 + 2 \sum_a \psi_{\epsilon a} \Delta \epsilon \Delta n_a + \sum_{ab} \psi_{ab} \Delta n_a \Delta n_b, \] (166)
Then we obtain for the bulk-viscous pressure

\[ \Pi = \langle \hat{p} \rangle_1 + \langle \hat{p} \rangle_2 - p(\epsilon, n_a) = \langle \hat{p} \rangle_1 + \langle \hat{p} \rangle_2 - \gamma \Delta \epsilon - \sum_a \delta_a \Delta n_a + \psi_{\epsilon \epsilon} \Delta \epsilon^2 + 2 \sum_a \psi_{\epsilon a} \Delta \epsilon \Delta n_a + \sum_{ab} \psi_{ab} \Delta n_a \Delta n_b. \]  

(168)

Substituting \( \Delta \epsilon \) and \( \Delta n_a \) and dropping higher-order terms we obtain

\[ \Pi = \langle \hat{p}^* \rangle_1 + \langle \hat{p}^* \rangle_2 + \psi_{\epsilon \epsilon} \langle \hat{\epsilon} \rangle_1^2 + 2 \sum_a \psi_{\epsilon a} \langle \hat{\epsilon} \rangle_1 \langle \hat{n}_a \rangle_1 + \sum_{ab} \psi_{ab} \langle \hat{n}_a \rangle_1 \langle \hat{n}_b \rangle_1, \]  

(169)

where we used the definition (92) of \( \hat{p}^* \).

Upon introducing the coefficients [see Eq. (C.20) of Appendix C]

\[ \zeta_\epsilon = \beta \int d^4 x_1 \left( \hat{\epsilon}(x), \hat{p}^*(x_1) \right) = -\frac{d}{d\omega} \text{Im} G_{\hat{p}^*}^R (\omega) \bigg|_{\omega=0}, \]  

(170)

\[ \zeta_a = \beta \int d^4 x_1 \left( \hat{n}_a(x), \hat{p}^*(x_1) \right) = -\frac{d}{d\omega} \text{Im} G_{\hat{n}_a \hat{p}^*}^R (\omega) \bigg|_{\omega=0}, \]  

(171)

according to Eqs. (127) and (131) the averages \( \langle \hat{\epsilon} \rangle_1 \) and \( \langle \hat{n}_a \rangle_1 \) can be written as

\[ \langle \hat{\epsilon} \rangle_1 = -\zeta_\epsilon \theta, \quad \langle \hat{n}_a \rangle_1 = -\zeta_a \theta, \]  

(172)

where we have used Curie’s theorem, i.e., \( \left( \hat{\epsilon}(x), \hat{\pi}_{\rho\sigma}(x_1) \right) = \left( \hat{\epsilon}(x), \hat{J}_\alpha^\sigma(x_1) \right) = 0 \). Then we have from Eqs. (108), (169), and (172)

\[ \Pi = -\zeta \theta + \left( \psi_{\epsilon \epsilon} \zeta_\epsilon^2 + 2 \zeta_\epsilon \sum_a \psi_{\epsilon a} \zeta_a + \sum_{ab} \psi_{ab} \zeta_a \zeta_b \right) \theta^2 + \langle \hat{p}^* \rangle_2. \]  

(173)

4.4.1. Non-local corrections from the two-point correlation function

From Eqs. (108), (127), and (133) we have

\[ \langle \hat{p}^*(x) \rangle_2^1 = -\int d^4 x_1 \left( \hat{p}^*(x), \hat{p}^*(x_1) \right) \beta(x_1) \theta(x_1) + \zeta(x) \theta(x). \]  

(174)

Now we need to expand all hydrodynamic quantities which are evaluated at the point \( x_1 \) around the point \( x \), as explained in Sec. 4.3. For this purpose we use Eqs. (69) and (92) to express the operator \( \hat{p}^* \) in terms of the operators \( \hat{T}^{\mu\nu} \) and \( \hat{N}_a^\mu \)

\[ \hat{p}^*(x_1) = -\frac{1}{3} \Delta_{\mu\nu}(x_1) \hat{T}^{\mu\nu}(x_1) - \gamma(x_1) u_\mu(x_1) u_\nu(x_1) \hat{T}^{\mu\nu}(x_1) - \sum_a \delta_a(x_1) u_\mu(x_1) \hat{N}_a^\mu(x_1). \]  

(175)

Expanding the hydrodynamic quantities, i.e., \( u_\mu(x_1) \), \( \gamma(x_1) \), and \( \delta_a(x_1) \) in Eq. (175) in terms of Taylor series around \( x_1 = x \) and keeping only the linear terms we obtain

\[ \hat{p}^*(x_1) = \hat{p}^*(x_1)_x + (x_1 - x)^T \partial_r \hat{p}^*(x), \]  

(176)
where \( \dot{p}^*(x_1) \) is obtained from \( \dot{p}^*(x) \), Eq. (175), by replacing the arguments \( x_1 \) of all hydrodynamic quantities (but not of the microscopic operators \( T^{\mu \nu} \) and \( \tilde{N}_a^\mu \)) with \( x \), and

\[
\partial_\tau \dot{p}^* = 2 \left( \frac{1}{3} - \gamma \right) \dot{q}^\mu \partial_\tau u_\mu - 2 \dot{\epsilon} \left( \psi_{\epsilon a} \partial_\tau n_\alpha + \sum_a \psi_{\epsilon a} \partial_\tau n_\alpha \right)
- 2 \sum_a \hat{n}_a \left( \psi_{\epsilon a} \partial_\tau \epsilon + \sum_b \psi_{ab} \partial_\tau n_\beta \right) - \sum_a \hat{j}_{a}^{\mu} \delta_{a} \partial_\tau u_\mu, \quad (177)
\]

where we have used Eqs. (90) and (167), as well as Eqs. (69) and (70), at the same time approximating \( u_\mu(x) \approx u_\mu(x_1) \), since the difference is of higher order. All operators in Eq. (177) are evaluated at \( x_1 \), while all hydrodynamic quantities are evaluated at \( x \).

Substituting Eq. (176) into Eq. (174) and expanding also the thermodynamic force \( \beta \theta \) around \( x_1 = x \) we obtain up to the second order in gradients

\[
\langle \dot{p}^*(x) \rangle^1_2 = - \int d^4x_1 \left( \dot{p}^*(x), \dot{p}^*(x_1) \right) \left[ \beta \theta + \partial_\tau (\beta \theta) \right]_x + \zeta(x) \theta(x)
\]

\[
= - \partial_\tau (\beta \theta) \int d^4x_1 \left( \dot{p}^*(x), \dot{p}^*(x_1) \right) (x_1 - x)^\tau
- \beta(x) \theta(x) \int d^4x_1 \left( \dot{p}^*(x), \partial_\tau \dot{p}^*(x) \right) (x_1 - x)^\tau\quad (178)
\]

where we canceled the zeroth-order term of the expansion with \( \zeta \theta \) noting that the definition (107) in the present context is rewritten as

\[
\zeta = \beta \int d^4x_1 \left( \dot{p}^*(x), \dot{p}^*(x_1) \right), \quad (179)
\]

where the index \( x \) means that the slowly varying thermodynamic quantities (but not the operators) are taken at the position \( x \).

Now inserting Eq. (177) into Eq. (178), taking into account Curie’s theorem, and using Eqs. (C.24) – (C.27) in Appendix C we obtain

\[
\langle \dot{p}^*(x) \rangle^1_2 = - \partial_\tau (\beta \theta) \int d^4x_1 \left( \dot{p}^*(x), \dot{p}^*(x_1) \right) (x_1 - x)^\tau
+ 2 \theta \left( \psi_{\epsilon a} \partial_\tau \epsilon + \sum_a \psi_{\epsilon a} \partial_\tau n_\alpha \right) \beta \int d^4x_1 \left( \dot{p}^* (x), \dot{\epsilon} (x_1) \right) (x_1 - x)^\tau
+ 2 \theta \sum_a \left( \psi_{\epsilon a} \partial_\tau \epsilon + \sum_b \psi_{ab} \partial_\tau n_\beta \right) \beta \int d^4x_1 \left( \dot{p}^* (x), \dot{n}_a (x_1) \right) (x_1 - x)^\tau
= D(\beta \theta) \beta^{-1} \zeta \tau_\Pi - 2 \theta \left( \psi_{\epsilon a} D \epsilon + \sum_a \psi_{\epsilon a} D n_\alpha \right) \zeta \tau_\epsilon
- 2 \theta \sum_a \left( \psi_{\epsilon a} D \epsilon + \sum_b \psi_{ab} D n_\beta \right) \zeta a \tau_a, \quad (180)
\]
introduced in Eq. (180) are given by

\[
\begin{align*}
\zeta_{\tau_\Pi} &= -i \frac{d}{d\omega} \zeta(\omega) \bigg|_{\omega=0} = \frac{1}{2} \frac{d^2}{d\omega^2} \text{Re} G^R_{\bar{\rho} \rho^*}(\omega) \bigg|_{\omega=0}, \\
\zeta_\epsilon \tau_\epsilon &= -i \frac{d}{d\omega} \zeta_\epsilon(\omega) \bigg|_{\omega=0} = \frac{1}{2} \frac{d^2}{d\omega^2} \text{Re} G^R_{\bar{\rho} \rho^*}(\omega) \bigg|_{\omega=0}, \\
\zeta_a \tau_a &= -i \frac{d}{d\omega} \zeta_a(\omega) \bigg|_{\omega=0} = \frac{1}{2} \frac{d^2}{d\omega^2} \text{Re} G^R_{\bar{\rho} \rho^*}(\omega) \bigg|_{\omega=0},
\end{align*}
\]

(181, 182, 183)

where \( \zeta_\epsilon \) and \( \zeta_a \) in the limit \( \omega \to 0 \) are defined in Eqs. (170) and (171). In the case of \( \omega \neq 0 \) the formula (C.11) should be used with the relevant choices of the operators \( \hat{X} \) and \( \hat{Y} \). [Note that there is no summation on the left-hand side of Eq. (183).]

The derivatives \( D\beta, D\epsilon, \) and \( Dn_a \) can be eliminated from Eq. (180) by employing Eqs. (81), (88), and (89). Denoting

\[
\zeta^* = \gamma \zeta_{\tau_\Pi} + 2 \zeta_\epsilon \tau_\epsilon \left( \psi_\epsilon h + \sum_a n_a \psi_\epsilon a \right) + 2 \sum_a \zeta_a \tau_a \left( \psi_\epsilon h + \sum_b \psi_\epsilon b n_b \right),
\]

(184)

we obtain from Eqs. (180) – (184)

\[
\langle \hat{p}^* \rangle^1_2 = \zeta_{\tau_\Pi} D\theta + \zeta^* \theta^2.
\]

(185)

4.4.2. Corrections from extended thermodynamic forces

Inserting Eq. (128) into Eq. (134) and taking into account Curie’s theorem we obtain

\[
\langle \hat{p}^* (x) \rangle^2_2 = -\zeta_\beta \left( \Pi \theta + \partial_\mu q^\mu - q^\mu D\mu - \pi^{\mu\nu} \sigma_{\mu\nu} \right) + \sum_a \zeta_a \left( \partial_\beta j_a^\mu \right),
\]

(186)

where we factored out all thermodynamic forces from the integral with their values at \( x \) and introduced short-hand notations

\[
\begin{align*}
\zeta_\beta &= \int d^4x_1 \left( \hat{p}^* (x), \hat{\beta}^* (x_1) \right) = T \frac{\partial}{\partial \epsilon} \zeta_\epsilon + \sum_c T \frac{\partial}{\partial n_c} \zeta_c, \\
\zeta_\alpha_a &= \int d^4x_1 \left( \hat{p}^* (x), \hat{\alpha}^*_a (x_1) \right) = T \frac{\partial}{\partial \epsilon} \zeta_\epsilon + \sum_c T \frac{\partial}{\partial n_c} \zeta_c.
\end{align*}
\]

(187, 188)

In the second step we used the definitions (122), (123), (170) and (171).

It is convenient to modify the second term in Eq. (186) using Eq. (A.7)

\[
\partial_\mu j_a^\mu = \partial_\mu \mathcal{J}_a^\mu + n_a h^{-1} \partial_\mu q^\mu + q^\mu \partial_\mu (n_a h^{-1}).
\]

Using this and defining

\[
\tilde{\zeta}_\beta = \zeta_\beta - h^{-1} \sum_a n_a \zeta_\alpha_a,
\]

(189)

we obtain for Eq. (186)

\[
\langle \hat{p}^* \rangle^2_2 = \sum_a \zeta_\alpha_a \partial_\mu \mathcal{J}_a^\mu - \zeta_\beta \left( \Pi \theta - \pi^{\mu\nu} \sigma_{\mu\nu} \right) - \tilde{\zeta}_\beta \partial_\mu q^\mu + q^\mu \left( \zeta_\beta D\mu + \sum_a \zeta_\alpha_a \nabla_\mu (n_a h^{-1}) \right).
\]

(190)
4.4.3. Corrections from the three-point correlation function

From Eqs. (127) and (135) we have

\[
\langle \hat{p}^* (x) \rangle^3_2 = \int d^4 x_1 d^4 x_2 \left( \hat{p}^* (x), \left[ -\beta \theta \hat{p}^* + \beta \hat{\pi}_\rho \sigma^\rho - \sum_a \partial_a \nabla_\sigma \alpha_a \right]_{x_1}, \right. \\
\left. \left[ -\beta \theta \hat{p}^* + \beta \hat{\pi}_\alpha \sigma^\alpha \rho - \sum_a \partial_a \nabla_\alpha \beta_2 \right]_{x_2} \right). \tag{191}
\]

The non-vanishing correlators in Eq. (191) are \( \left( \hat{p}^* (x), \hat{p}^* (x_1), \hat{p}^* (x_2) \right) \) and

\[
\left( \hat{p}^* (x), \hat{\pi}_{\rho \sigma} (x_1), \hat{\pi}_{\alpha \beta} (x_2) \right) = \frac{1}{3} \Delta_{3 \sigma} (x) \left( \hat{p}^* (x), \hat{\pi}_{\alpha \gamma} (x_1), \hat{\pi}_{\beta \gamma} (x_2) \right), \tag{192}
\]

\[
\left( \hat{p}^* (x), \hat{\pi}_\rho (x_1), \hat{\pi}_{\alpha \beta} (x_2) \right) = \frac{1}{5} \Delta_{5 \rho \alpha \beta} (x) \left( \hat{p}^* (x), \hat{\pi}_{\gamma \delta} (x_1), \hat{\pi}_{\gamma \delta} (x_2) \right). \tag{193}
\]

Inserting these expressions into Eq. (191) we obtain

\[
\langle \hat{p}^* \rangle^3 = \lambda_{\Pi} \theta^2 - \lambda_{\Pi \pi} \sigma^\alpha \sigma^\beta + T \sum_{\alpha \beta} \zeta_{\Pi}^{ab} \nabla_\sigma \alpha_a \nabla_\beta \alpha_b, \tag{194}
\]

where we defined the coefficients

\[
\lambda_{\Pi} = \beta^2 \int d^4 x_1 d^4 x_2 \left( \hat{p}^* (x), \hat{p}^* (x_1), \hat{p}^* (x_2) \right), \tag{195}
\]

\[
\lambda_{\Pi \pi} = - \frac{\beta^2}{5} \int d^4 x_1 d^4 x_2 \left( \hat{p}^* (x), \hat{\pi}_{\gamma \delta} (x_1), \hat{\pi}_{\gamma \delta} (x_2) \right), \tag{196}
\]

\[
\zeta_{\Pi}^{ab} = \frac{\beta}{3} \int d^4 x_1 d^4 x_2 \left( \hat{p}^* (x), \hat{\pi}_{\alpha \gamma} (x_1), \hat{\pi}_{\beta \gamma} (x_2) \right). \tag{197}
\]

4.4.4. Final result for the bulk-viscous pressure

Combining all contributions from Eqs. (185), (190), and (194) we obtain according to Eq. (132)

\[
\langle \hat{p}^* \rangle_2 = \zeta_{\Pi} \theta + \zeta_{\beta} (\Pi \theta - \pi^\mu \sigma_{\mu \nu}) - \zeta_{\beta} \theta q^\mu + (\lambda_{\Pi} + \zeta^+) \theta^2 \\
- \lambda_{\Pi \pi} \sigma^\alpha \sigma^\beta + \sum_a \zeta_{\alpha a} \partial_\mu \pi_\mu + T \sum_{ab} \zeta_{\Pi}^{ab} \nabla_\sigma \alpha_a \nabla_\beta \alpha_b \\
+ q^\mu \left[ \zeta_{\beta} Du_\mu + \sum_a \zeta_{\alpha a} \nabla_\mu (n_a h^{-1}) \right]. \tag{198}
\]

A relaxation equation for the bulk-viscous pressure can be obtained by approximating in the first term \( \theta \simeq -\zeta^{-1} \Pi \), as we did in the case of the shear-stress tensor. We thus obtain

\[
\zeta_{\Pi} \theta \Pi = -\tau_{\Pi} \Pi + \tau_{\Pi} \Pi \zeta^{-1} D \zeta \\
- \tau_{\Pi} \Pi + \tau_{\Pi} \beta \zeta^{-1} \left( \frac{\partial \zeta}{\partial \beta} - \sum_a \partial_a \frac{\partial \zeta}{\partial \alpha_a} \right) \theta \Pi, \tag{199}
\]

31
where we used Eqs. (88) and (89).

Now combining Eqs. (173), (198), and (199), denoting \( \dot{u}_\mu = Du_\mu \), and defining

\[
\zeta = \lambda_\Pi + \zeta^* + \psi_\kappa \zeta^2 + 2 \zeta \sum_a \psi_{\kappa a} \zeta_a + \sum_{ab} \psi_{ab} \zeta_a \zeta_b, \tag{200}
\]

\[
\tilde{\lambda}_\Pi = \tau_\Pi \beta \zeta^{-1} \left( \gamma \frac{\partial \zeta}{\partial \beta} - \sum_a \delta_a \frac{\partial \zeta}{\partial \alpha a} \right), \tag{201}
\]

we obtain the final evolution equation for the bulk-viscous pressure,

\[
\tau_\Pi \dot{\Pi} + \Pi = -\zeta \theta + \tilde{\lambda}_\Pi \theta \Pi + \zeta_\beta (\sigma_{\mu \nu} \Pi^{\mu \nu} - \theta \Pi) + \zeta \theta^2 - \lambda_\Pi \sigma_{\mu \nu} \sigma^{\mu \nu} - \tilde{\zeta}_\beta \partial_\mu q^\mu + \sum_a \zeta_{\alpha a} \partial_\mu J^\mu_a + T \sum_{ab} \zeta^{ab} \nabla^\mu \alpha_a \nabla_\mu \alpha_b + q^\mu \left[ \zeta_\beta \dot{u}_\mu + \sum_a \zeta_{\alpha a} \nabla_\mu (n_a h^{-1}) \right]. \tag{202}
\]

### 4.5. Second-order corrections to the diffusion currents

#### 4.5.1. Non-local corrections from the two-point correlation function

Using Eqs. (110), (127), and (133) and again applying Curie’s theorem we obtain

\[
\langle \hat{J}_{c\mu}(x) \rangle_2^1 = - \sum_a \int d^4x_1 \left( \hat{J}_{c\mu}(x), \hat{J}_{a\sigma}(x_1) \right) \nabla^\sigma_{x_1} \alpha_\sigma(x_1) - \sum_a \chi_{ca}(x) \nabla_\mu \alpha_a(x). \tag{203}
\]

Now we substitute the correlation function given by Eqs. (136) and (138) into Eq. (203)

\[
\langle \hat{J}_{c\mu}(x) \rangle_2^1 = - \frac{1}{3} \sum_a \Delta_{\alpha a}(x) \int d^4x_1 \left( \hat{J}_{c\mu}(x), \hat{J}_{a\lambda}(x_1) \right) \nabla^\lambda_{x_1} \alpha_\lambda(x_1) - \sum_a \chi_{ca}(x) \nabla_\mu \alpha_a(x). \tag{204}
\]

Next we use Eqs. (70) and (96) to express the operators \( \hat{J}_{a\lambda}^{\lambda}(x) \) in terms of the energy-momentum tensor and the charge currents, as we did in the case of the bulk-viscous pressure,

\[
\hat{J}_{a\lambda}(x_1) = \Delta_{\alpha \mu}(x_1) \hat{N}_{\alpha}^\mu(x_1) - \frac{n_a(x_1)}{h(x_1)} \Delta_{\alpha \mu}(x_1) \hat{u}_\nu(x_1) \hat{T}^\mu_\nu(x_1). \tag{205}
\]

Expanding all hydrodynamic quantities in Eq. (205) around their values at \( x_1 = x \), keeping only the first-order terms in gradients, and using the decompositions (66) and (67) we obtain

\[
\hat{J}_{a\lambda}(x_1) = \hat{J}_{a\lambda}(x_1) + (x_1 - x)^\tau \partial_\tau \hat{J}_{a\lambda}(x), \tag{206}
\]

where \( \hat{J}_{a\lambda}(x_1) \) is obtained from \( \hat{J}_{a\lambda}(x_1) \) via replacing the arguments \( x_1 \) of all hydrodynamic quantities with \( x \), while the operators are taken at \( x_1 \), and where

\[
\partial_\tau \hat{J}_{a\lambda} = -u_\lambda (\partial_\tau u_\mu) \hat{J}_{a}^\mu - 2 \hat{p}_\lambda - \hat{\pi}_{a \lambda} = \frac{n_a h^{-1}}{h^2} \hat{u}_\nu - \hat{\pi}^{(a)}_{\lambda} \partial_\tau u_\nu. \tag{207}
\]

Here, we have made use of Eq. (70) and the fact that to this order we are allowed to approximate \( u_\mu(x) \approx u_\mu(x_1) \). As in Eq. (177), all hydrodynamic quantities are evaluated at \( x \), while
all operators are evaluated at $x_1$, the difference being of higher order. Substituting Eq. (206)
into Eq. (204) and expanding also the thermodynamic force $\nabla^\beta x_1 \alpha_a (x_1)$ around $x$ we obtain

$$\langle \mathcal{J}^{\mu}_{\alpha} (x) \rangle_{\frac{1}{2}} = - \frac{1}{3} \sum_a \Delta_{\mu\beta} (x) \int d^4 x_1 \left( \mathcal{J}^{\lambda}_{\beta} (x), \mathcal{J}_{\alpha\lambda} (x_1) + \partial_\tau \mathcal{J}_{\alpha\lambda} (x_1) (x_1 - x)^\tau \right)$$

$$\times \left[ \nabla^\beta \alpha_a (x) + \partial_\tau (\nabla^\beta \alpha_a) (x_1 - x)^\tau \right] - \sum_a \chi_{ca} (x) \nabla_\mu \alpha_a (x)$$

$$= - \frac{1}{3} \sum_a \Delta_{\mu\beta} (x) \partial_\tau (\nabla^\beta \alpha_a) \int d^4 x_1 \left( \mathcal{J}^{\lambda}_{\beta} (x), \mathcal{J}_{\alpha\lambda} (x_1) (x_1 - x)^\tau \right)$$

$$- \frac{1}{3} \sum_a \nabla_\mu \alpha_a (x) \int d^4 x_1 \left( \mathcal{J}^{\lambda}_{\beta} (x), \partial_\tau \mathcal{J}_{\alpha\lambda} (x_1) (x_1 - x)^\tau \right).$$  \hspace{1cm} (208)

Here the first-order terms cancel each other in the same way as in the case of bulk viscosity, see Eqs. (178) and (179).

Now substituting Eq. (207) into Eq. (208), taking into account Curie’s theorem, employing
the orthogonality condition $u^\lambda \mathcal{J}_{c\lambda} = 0$, and using Eqs. (C.24) – (C.27) in Appendix C we obtain

$$\langle \mathcal{J}^{\mu}_{\alpha} (x) \rangle_{\frac{1}{2}} = - \frac{1}{3} \sum_a \Delta_{\mu\beta} (x) \partial_\tau (\nabla^\beta \alpha_a) \int d^4 x_1 \left( \mathcal{J}^{\lambda}_{\beta} (x), \mathcal{J}_{\alpha\lambda} (x_1) \right) (x_1 - x)^\tau$$

$$+ \frac{1}{3} \sum_a \partial_\tau (n_a h^{-1}) \nabla_\mu \alpha_a (x) \int d^4 x_1 \left( \mathcal{J}^{\lambda}_{\beta} (x), \hat{q}_{\alpha \lambda} (x_1) \right) (x_1 - x)^\tau$$

$$= \sum_a \tilde{\chi}_{ca} \Delta_{\mu\beta} (x) D (\nabla^\beta \alpha_a) - \tilde{\chi}_{cq} \sum_a D (n_a h^{-1}) \nabla_\mu \alpha_a (x),$$ \hspace{1cm} (209)

where

$$\tilde{\chi}_{ac} = \left. \frac{d}{d\omega} \chi_{ac} (\omega) \right|_{\omega = 0} = \frac{T}{6} \frac{d^2}{d\omega^2} \text{Re} G^R_{\alpha \beta} \mathcal{J}^{\alpha \beta} (\omega) \right|_{\omega = 0},$$  \hspace{1cm} (210)

$$\tilde{\chi}_{cq} = \left. \frac{d}{d\omega} \chi_{cq} (\omega) \right|_{\omega = 0} = \frac{T}{6} \frac{d^2}{d\omega^2} \text{Re} G^R_{\alpha \beta} \mathcal{J}^{\alpha \beta} (\omega) \right|_{\omega = 0},$$  \hspace{1cm} (211)

and $\chi_{cq}$ is defined in the next subsection. Because the expression (209) is already of second
order, we can use Eq. (81) to replace $D (n_a h^{-1}) = - n_a h^{-2} Dp$. From Eqs. (81) and (90) we find

$$Dp = \gamma D \epsilon + \sum_d \delta_d Dn_d = - \left( \gamma h + \sum_d \delta_d n_d \right) \theta.$$

Substituting these results into Eqs. (209) we obtain

$$\langle \mathcal{J}^{\mu}_{\alpha} \rangle_{\frac{1}{2}} = \sum_a \tilde{\chi}_{ca} \Delta_{\mu\beta} D (\nabla^\beta \alpha_a) - \tilde{\chi}_{cq} h^{-2} \left( \gamma h + \sum_d \delta_d n_d \right) \theta \sum_a n_a \nabla_\mu \alpha_a,$$ \hspace{1cm} (212)

where we suppressed the $x$-dependence for the sake of brevity.
4.5.2. Corrections from extended thermodynamic forces

Using Eqs. (128) and (134) and again taking into account Curie’s theorem we obtain

\[ \langle \hat{J}_{c\mu} \rangle^2 = \chi_{eq} \beta h^{-1} \left( -\nabla_\mu \Pi + \Pi D u_\mu + \Delta_{\mu\nu} D q^\nu + q^\nu \partial_\nu u_\mu + q_\mu \theta + \Delta_{\mu\sigma} \partial_\nu \pi^{\nu\sigma} \right), \]  

(213)

where we used a relation analogous to the one given by Eq. (104)

\[ \left( \hat{J}_{c\mu}(x), \hat{q}^\sigma(x_1) \right) = \frac{1}{3} \Delta^{\sigma}_{\mu}(x) \left( \hat{J}_{c\alpha}(x), \hat{q}^\alpha(x_1) \right), \]

and defined new coefficients via

\[ \chi_{eq} = -\frac{1}{3} \int d^4x_1 \left( \hat{J}_{c\alpha}(x), \hat{q}^\alpha(x_1) \right) = \frac{T}{3} \frac{d}{d\omega} \text{Im} G^R_{\hat{J}_{c\alpha} \hat{q}^\alpha}(\omega) \bigg|_{\omega=0}. \]

(214)

Denoting

\[ \hat{u}_\mu = Du_\mu, \quad \hat{q}_\mu = \Delta_{\mu\nu} D q^\nu, \]

Eq. (213) is written as

\[ \langle \hat{J}_{c\mu} \rangle^2 = \chi_{eq} \beta h^{-1} \left( -\nabla_\mu \Pi + \Pi \hat{u}_\mu + \hat{q}_\mu + q^\nu \partial_\nu u_\mu + q_\mu \theta + \Delta_{\mu\sigma} \partial_\nu \pi^{\nu\sigma} \right). \]

(215)

4.5.3. Corrections from the three-point correlation function

Substituting Eq. (127) into Eq. (135) we obtain

\[ \langle \hat{J}_{c\mu}(x) \rangle^3 = \int d^4x_1 d^4x_2 \left( \hat{J}_{c\mu}(x), \left[ -\beta \theta \hat{p}^\ast + \beta \hat{\pi}_{\rho\sigma} \sigma^{\rho\sigma} - \sum_a \hat{J}_{a}^{\sigma} \nabla_\sigma \alpha_a \right]_{x_1}, \left[ -\beta \theta \hat{p}^\ast + \beta \hat{\pi}_{\alpha\beta} \sigma^{\alpha\beta} - \sum_b \hat{J}_{b}^{\alpha} \nabla_\alpha \alpha_b \right]_{x_2} \right). \]

(216)

The nonvanishing correlators in this case are

\[ \left( \hat{J}_{c\mu}(x), \hat{J}_{a\sigma}(x_1), \hat{p}^\ast(x_2) \right) = \frac{1}{3} \Delta_{\mu\sigma}(x) \left( \hat{J}_{c\beta}(x), \hat{J}_{a}^{\beta}(x_1), \hat{p}^\ast(x_2) \right), \]

(217)

\[ \left( \hat{J}_{c\mu}(x), \hat{J}_{a\sigma}(x_1), \hat{\pi}_{\alpha\beta}(x_2) \right) = \frac{1}{5} \Delta_{\mu\sigma\alpha\beta}(x) \left( \hat{J}_{c}^{\gamma}(x), \hat{J}_{a}^{\delta}(x_1), \hat{\pi}_{\gamma\delta}(x_2) \right). \]

(218)

We now define the following coefficients

\[ \zeta_{J}^{ca} = \frac{2\beta}{3} \int d^4x_1 d^4x_2 \left( \hat{J}_{c\gamma}(x), \hat{J}_{a}^{\gamma}(x_1), \hat{p}^\ast(x_2) \right), \]

(219)

\[ \lambda_{J}^{ca} = \frac{2\beta}{5} \int d^4x_1 d^4x_2 \left( \hat{J}_{c}^{\gamma}(x), \hat{J}_{a}^{\delta}(x_1), \hat{\pi}_{\gamma\delta}(x_2) \right). \]

(220)

Then, from Eqs. (216) – (220) and from the symmetry property (65) we obtain

\[ \langle \hat{J}_{c\mu} \rangle^3 = \sum_a \left( \zeta_{J}^{ca} \theta \nabla_\mu \alpha_a - \lambda_{J}^{ca} \sigma_{\mu\nu} \nabla_\nu \alpha_a \right). \]

(221)
4.5.4. Final result for the diffusion currents

Combining Eqs. (110), (132), (212), (215), and (221) we obtain the diffusion currents up to second order,

\[
\mathcal{J}_{\mu} = \sum_b \chi_{cb} \nabla_{\mu} \alpha_b + \sum_a \tilde{\chi}_{ca} \Delta_{\mu\beta} D(\nabla^\beta \alpha_a) - \tilde{\chi}_{cq} h^{-2} \left( \gamma h + \sum_d \delta_d n_d \right) \theta \sum_b n_b \nabla_{\mu} \alpha_b \\
+ \chi_{cq} \beta h^{-1} \left( - \nabla_{\mu} \Pi + \Pi \dot{u}_{\mu} + \dot{q}_\mu + q^\nu \partial_{\nu} u_{\mu} + q_{\mu} \theta + \Delta_{\mu\sigma} \partial_{\nu} \pi^{\nu\sigma} \right) \\
+ \sum_b \left( \zeta_{cb} \theta \nabla_{\mu} \alpha_b - \lambda_{cb} \sigma_{\mu\nu} \nabla^{\nu} \alpha_b \right). 
\]

(222)

In order to obtain relaxation equations for the diffusion currents we invert Eq. (110) as

\[
\nabla^{\beta} \alpha_a = \sum_b (\chi^{-1})_{ab} \mathcal{J}_b, 
\]

and employ it to modify the second term in Eq. (222). Using Eqs. (88), (89), and (129) we obtain

\[
\sum_a \tilde{\chi}_{ca} \Delta_{\mu\beta} D(\nabla^\beta \alpha_a) = - \sum_b \tau^{cb} \mathcal{J}_{b\mu} \\
+ \beta \theta \sum_{ab} \tilde{\chi}_{ca} \left[ \gamma \frac{\partial (\chi^{-1})_{ab}}{\partial \beta} - \sum_d \delta_d \frac{\partial (\chi^{-1})_{ab}}{\partial \alpha_d} \right] \mathcal{J}_{b\mu}, 
\]

(223)

where we defined the relaxation-time matrix

\[
\tau^{cb} = - (\tilde{\chi} \chi^{-1})_{cb} = - \sum_a \tilde{\chi}_{ca} (\chi^{-1})_{ab}. 
\]

(224)

Introducing the coefficients

\[
\tilde{\chi}^{cb}_{\mathcal{J}} = \beta \sum_a \tilde{\chi}_{ca} \left[ \gamma \frac{\partial (\chi^{-1})_{ab}}{\partial \beta} - \sum_d \delta_d \frac{\partial (\chi^{-1})_{ab}}{\partial \alpha_d} \right], 
\]

(225)

\[
\chi^{cb}_{\mathcal{J}} = \zeta_{cb} \theta - \tilde{\chi}_{cq} n_b h^{-2} \left( \gamma h + \sum_d \delta_d n_d \right), 
\]

(226)

we obtain

\[
\sum_b \tau^{ab} \mathcal{J}_{b\mu} + \mathcal{J}_{a\mu} = \sum_b \left[ \chi_{ab} \nabla_{\mu} \alpha_b + \tilde{\lambda}^{ab}_{\mathcal{J}} \theta \mathcal{J}_{b\mu} + \chi^{ab}_{\mathcal{J}} \theta \nabla_{\mu} \alpha_b - \lambda^{ab}_{\mathcal{J}} \sigma_{\mu\nu} \nabla^{\nu} \alpha_b \right] \\
+ \chi_{aq} \beta h^{-1} \left( - \nabla_{\mu} \Pi + \Pi \dot{u}_{\mu} + \dot{q}_\mu + q^\nu \partial_{\nu} u_{\mu} + q_{\mu} \theta + \Delta_{\mu\sigma} \partial_{\nu} \pi^{\nu\sigma} \right). 
\]

(227)

If we have only one conserved charge species, then Eq. (227) simplifies to

\[
\tau_{\mathcal{J}} \dot{\mathcal{J}}_{\mu} + \mathcal{J}_{\mu} = \chi \nabla_{\mu} \alpha + \tilde{\lambda}_{\mathcal{J}} \theta \mathcal{J}_{\mu} + \chi^{*} \theta \nabla_{\mu} \alpha - \lambda^{*}_{\mathcal{J}} \sigma_{\mu\nu} \nabla^{\nu} \alpha \\
+ \chi_{q} \beta h^{-1} \left( - \nabla_{\mu} \Pi + \Pi \dot{u}_{\mu} + \dot{q}_\mu + q^\nu \partial_{\nu} u_{\mu} + q_{\mu} \theta + \Delta_{\mu\sigma} \partial_{\nu} \pi^{\nu\sigma} \right), 
\]

(228)
where the current relaxation time is given by [see Eqs. \((210)\) and \((224)\)]

\[
\chi \tau_J = -i \frac{d}{d\omega} \chi(\omega) \bigg|_{\omega=0} = -\frac{T}{6} \frac{d^2}{d\omega^2} \text{Re} G^R_{\mu\nu}(\omega) \bigg|_{\omega=0},
\]

and

\[
\tilde{\lambda}_J = \tau_J \beta \chi^{-1} \left( \frac{\partial \chi}{\partial \beta} - \delta \frac{\partial \chi}{\partial \alpha} \right),
\]

\[
\chi^* = \zeta_J - \bar{\chi}_q n h^{-2} (\gamma h + \delta n).
\]

Recalling the relation

\[
\kappa = \left( \frac{h}{nT} \right)^2 \chi
\]

between the diffusion coefficient and the thermal conductivity and Eq. \((112)\), we can write the relaxation time \(\tau_J \equiv \tau_h\) also in the following form

\[
\kappa \tau_h = -i \frac{d}{d\omega} \kappa(\omega) \bigg|_{\omega=0} = -\frac{\beta}{6} \frac{d^2}{d\omega^2} \text{Re} G^R_{h\mu h\nu}(\omega) \bigg|_{\omega=0}.
\]

The frequency-dependent coefficients \(\chi\) and \(\kappa\) in Eqs. \((229)\) and \((233)\) are defined according to the formula \((C.11)\) in Appendix C with the pertaining choice of operators.

5. Discussion

5.1. General structure of the second-order dissipative hydrodynamic equations

The complete set of evolution equations for the dissipative currents obtained in the previous section reads [see Eqs. \((165)\), \((202)\), and \((227)\)]

\[
\tau_\pi \dot{\pi}_{\mu\nu} + \pi_{\mu\nu} = 2\eta \sigma_{\mu\nu} + \tilde{\lambda}_\pi \theta \pi_{\mu\nu} + \lambda \theta \sigma_{\mu\nu}
+ \lambda \pi \rho_{\mu\nu} + \sum_{ab} \lambda_{ab} \sigma_{\rho\nu} \sigma_{\mu\nu} \nabla_{\nu} \alpha_a \nabla_{\rho} \alpha_b,
\]

\[
\tau_\Pi \dot{\Pi} + \Pi = -\zeta_\theta + \lambda_{\Pi} \theta \Pi + \zeta_\theta^2 + \zeta_\beta (\sigma_{\mu\nu} \pi_{\mu\nu} - \theta \Pi) - \lambda_{\Pi} \sigma_{\mu\nu} \sigma_{\mu\nu}
+ \sum_{a} \zeta_{\alpha a} \partial_{\mu} \mathcal{J}_a - \bar{\zeta}_\beta \partial_{\mu} \xi^a + \xi^a \left[ \bar{\zeta}_\beta \partial_{\mu} \xi + \sum_{a} \zeta_{\alpha a} \nabla_{\mu} (n_a h^{-1}) \right]
+ T \sum_{ab} \zeta_{ab} \nabla_{\mu} \alpha_a \nabla_{\mu} \alpha_b,
\]

\[
\sum_b \tau^b \dot{\mathcal{J}}_{b\mu} + \mathcal{J}_{a\mu} = \sum_b \left[ \chi_{ab} \nabla_{\mu} \alpha_{b\mu} + \tilde{\lambda}^b_{ab} \theta \mathcal{J}_{b\mu} + \chi^b_{ab} \theta \nabla_{\mu} \alpha_{b\mu} - \lambda^b_{ab} \sigma_{\mu\nu} \nabla_{\nu} \alpha_0 \right]
+ \chi_{ab} \beta h^{-1} (- \nabla_{\mu} \Pi + \Pi \dot{u}_{\mu} + \dot{q}_{\mu} + q_{\nu} \partial_{\nu} \Pi + q_{\mu} \theta + \Delta_{\mu\sigma} \partial_{\nu} \pi^{\nu\sigma}),
\]

where the dot denotes the comoving derivative

\[
\dot{\Pi} = D_\Pi, \quad \dot{\pi}_{\mu\nu} = \Delta_{\mu\nu\rho\sigma} D_{\pi^{\rho\sigma}}, \quad \dot{u}_{\mu} = D u_{\mu},
\]

\[
\dot{q}_{\mu} = \Delta_{\mu\nu} D q^{\nu}, \quad \dot{\mathcal{J}}_{a\mu} = \Delta_{\mu\nu} D \mathcal{J}_{a\nu}.
\]
The first terms on the right-hand sides of Eqs. (234) – (236) represent the corresponding Navier–Stokes contributions. The first-order coefficients \( \eta, \zeta, \) and \( \chi_{ab} \) are the shear viscosity, the bulk viscosity, and the matrix of diffusion coefficients, respectively; these coefficients are expressed in terms of retarded two-point correlation functions via the Kubo formulas (115) – (117).

The first terms on the left-hand sides of Eqs. (234) – (236) signify the relaxation of the dissipative currents towards their leading-order (Navier–Stokes) values, if they differ from those at the initial moment. The coefficients \( \tau_\pi, \tau_\Pi, \) and \( \tau_{ab}^{\mu\nu} \) represent the characteristic time scales on which this relaxation occurs. For example, in the case of \( \sigma_{\mu\nu} = 0, \) Eq. (234) implies an exponential decay of the shear-stress tensor on the time scale given by \( \tau_\pi, \) i.e., \( \pi_{\mu\nu} \propto \exp(-t/\tau_\pi) \) (if we ignore the other second-order terms).

The relaxation times \( \tau_\pi, \tau_\Pi, \) and \( \tau_{ab}^{\mu\nu} \) are related to the corresponding first-order transport coefficients. For example, the shear relaxation time is given by a Kubo-type formula

\[
\eta \tau_\pi = -i \frac{d}{d\omega} \eta(\omega) \bigg|_{\omega=0} = \frac{1}{20} \frac{d^2}{d\omega^2} \text{Re} G_r^{\pi,\pi}(\omega) \bigg|_{\omega=0}, \tag{239}
\]

where the retarded Green’s function is defined in Eq. (117). Here \( \eta(\omega) \) is the generalization of the shear viscosity to nonvanishing frequencies and is defined via the generalization of Eq. (106) according to Eq. (C.11). The positivity of \( \tau_\pi \) can be anticipated from Eqs. (239) and (C.11), upon taking into account that the \( (\pi_{\mu\nu}, \pi^{\mu\nu}) \) correlator, which determines the shear viscosity \( \eta \), should be positive. Similar formulas as Eq. (239) hold also for the bulk and the diffusion relaxation times [see Eqs. (181), (210), (224), and (229)]. Our formulas for the shear and the bulk relaxation times are in general consistent with those obtained in Refs. [28, 30, 33], but may differ in particular details. A direct comparison is not straightforward because of the difference in methods and approaches applied in those works.

The physical meaning of the formula (239) for \( \tau_\pi \) is easy to understand. As we showed in the previous section, the relaxation terms originate from the non-local (memory) effects encoded in the non-equilibrium statistical operator. In the case where these memory effects are neglected, i.e., in the first-order theory, the proportionality between \( \pi_{\mu\nu} \) and \( \sigma_{\mu\nu} \) is given by the zero-frequency (static) limit of the shear viscosity, as seen from Eqs. (108) and (115). The effects of finite memory in the dissipation of shear stresses imply actually a dispersion (i.e., frequency dependence) in the shear viscosity, which at the leading order should be accounted for by the first frequency derivative of \( \eta(\omega) \), as seen from Eq. (239). Therefore, we conclude that the memory effects naturally generate finite relaxation time scales in the transport equations, as stressed earlier in Refs. [46, 57, 58].

The second terms on the right-hand sides of Eqs. (234) – (236) arise as a consequence of inhomogeneities in the first-order coefficients, because these are functions of the temperature and the chemical potentials, which vary in space and time. The coefficients \( \lambda_\pi, \lambda_\Pi, \) and \( \lambda_{ab}^{\mu\nu} \) involve derivatives of the corresponding first-order transport coefficients with respect to temperature and chemical potentials, see Eqs. (164), (201), and (225).

In Eqs. (234) – (236) we identify three classes of second-order terms which are not of the relaxation-type: (i) terms which contain products of the thermodynamic forces with dissipative currents [e.g., the term \( \propto \theta \Pi \) in Eq. (235)]; (ii) terms which contain space-like derivatives of the dissipative currents (e.g., \( \partial_\mu \mathcal{J}_a^{\mu} \)); and (iii) terms which include a product of two thermodynamic forces (e.g., \( \sigma_{\mu\nu} \sigma^{\mu\nu} \)). The terms of the type (i) originate either from
the non-local corrections [second terms on the right-hand sides of Eqs. (234) – (236)], or from the second-order corrections to the operator \( \hat{C} \) [see Eq. (128)]. The corrections of the type (ii) arise purely from the operator \( \hat{C}_2 \). As we discussed in Sec. 4.1, the operator \( \hat{C}_2 \) originates from the dissipative terms in the hydrodynamic equations (77) – (79) and can be viewed as an extended thermodynamic force. Thus, the coefficients \( \zeta_\beta \), \( \tilde{\zeta}_\beta \), and \( \zeta_\alpha \) in Eq. (235) and \( \chi_{aq} \) in Eq. (236) represent the mixing of the transport equations with the conservation laws. We also note that such mixed terms are absent in the relaxation equation (234) for the shear-stress tensor. The transport coefficients in the terms of type (i) and (ii) are related to two-point correlation functions [see Eqs. (187), (188), (189), and (214)].

The corrections of the type (iii) contain all possible combinations which are quadratic in the thermodynamic forces \( \sigma_{\mu\nu} \), \( \theta \), and \( \nabla_\mu \alpha_\alpha \). For example, the relevant corrections for the shear-stress tensor are given by three terms which are allowed by the symmetries: \( \theta \sigma_{\mu\nu} \), \( \sigma_{\rho<\mu} \sigma_{\rho>\nu} \), and \( \nabla_{<\mu} \alpha_\alpha \nabla_{>\nu} \alpha \). The transport coefficients coupled to these terms involve three-point correlation functions, which account for nonlinear couplings between different dissipative processes.

Note that because of the terms of the type (iii) the transport equations become parabolic, and, therefore, acausal and unstable [13, 38]. This problem can be circumvented by modifying some of the nonlinear terms using the Navier–Stokes equations, e.g., \( \lambda_\pi \sigma_{\rho<\mu} \sigma_{\rho>\nu} \) can be replaced by \( (\lambda_\pi/2\eta)\sigma_{\rho<\mu} \sigma_{\rho>\nu} \) [13, 38]. In this case, we recover most of the second-order terms derived in Ref. [32].

5.2. Comparison with other studies

In this subsection, we discuss in more detail the second-order expression for the shear-stress tensor and compare it with the results of other studies.

For the sake of simplicity, we will consider a fluid without conserved charges. Equation (90) then implies \( \gamma \equiv c_s^2 \), with \( c_s \) being the speed of sound. It is more suitable to use here the expression for the shear-stress tensor given by Eq. (161)

\[
\pi_{\mu\nu} = 2\eta\sigma_{\mu\nu} - 2\eta\tau_\pi(\tilde{\sigma}_{\mu\nu} + c_s^2 \theta \sigma_{\mu\nu}) + 2\lambda_\pi \Pi \theta \sigma_{\mu\nu} + \lambda_\pi \sigma_{\alpha<\mu} \sigma_{\nu>}^\alpha, \tag{240}
\]

where \( \tilde{\sigma}_{\mu\nu} \equiv \Delta_{\mu\nu\rho\sigma} \sigma^{\rho\sigma} \). We recall that the terms in the parentheses are those which arise from the non-locality of the thermodynamic forces in the statistical operator (see Sec. 4.3). The next two terms in Eq. (240) arise from the quadratic terms of the expansion of the statistical operator in thermodynamic forces. Thus, the second-order corrections to the shear-stress tensor in the absence of conserved charges contain three new coefficients: \( \tau_\pi \), \( \lambda_\pi \), and \( \lambda_\pi \Pi \).

The second-order terms in the parentheses in Eq. (240) have a simple physical interpretation. The effects of non-locality generate two distinct terms in the shear-stress tensor. The first term in the parentheses involves the comoving derivative of the thermodynamic force \( \sigma_{\mu\nu} \) and incorporates the effect of the acceleration of the fluid on account of the shear stresses. In other words, this term contains information about the velocity stresses \( \sigma_{\mu\nu} \) from the previous moments in time. The relaxation time \( \tau_\pi \) measures how long this information remains in the “memory” of the shear-stress tensor \( \pi_{\mu\nu} \). Thus, the first term in parentheses in Eq. (240) can be associated with the non-locality of the statistical operator in time, i.e., it is related to memory effects.
We argue that the second term in parentheses in Eq. (240) accounts for spatially non-local effects. Indeed, this term involves the product of the thermodynamic force $\sigma_{\mu\nu}$ with the fluid expansion rate $\theta = \partial_\mu u^\mu$, which can be regarded as a relevant (scalar) measure of how strong the spatial “non-locality” in the fluid velocity field is. This term describes how the shear-stress tensor is distorted by uniform expansion or contraction of the fluid.

Next, we discuss the last two terms in Eq. (240). These terms are quadratic in the thermodynamic forces $\sigma_{\mu\nu}$ and $\theta$. The relevant second-order transport coefficients are expressed via three-point correlation functions by the formulas (156) and (157). The coefficient $\lambda_\pi$ describes the nonlinear effects of two velocity stresses on $\pi_{\mu\nu}$. By analogy with the relevant linear transport coefficient $\eta$, which measures the correlation between two shear stresses, the second-order coefficient $\lambda_\pi$ measures the correlation between three shear stresses. The coefficient $\lambda_{\pi\Pi}$ describes the nonlinear coupling between the shear- and the bulk-viscous processes. Similarly, this coefficient is given by a three-point correlation function between two shear stresses and the bulk-viscous pressure.

We remark that $\lambda_{\pi\Pi}$ term in Eq. (240) has the same gradient structure $\theta \sigma_{\mu\nu}$ as the second term in the parentheses. However, despite this formal similarity, these two terms have different origins and, therefore, different physical interpretations. As explained above, the relevant term $\propto \pi_\pi$ originates from non-local effects in the statistical distribution, whereas the term $\propto \lambda_{\pi\Pi}$ stands purely for nonlinear coupling between the bulk- and the shear-viscous effects. In this sense, it is natural to regard as nonlinear only the term $\propto \lambda_{\pi\Pi}$, but not the term $\propto \pi_\pi$. A similar classification of the second-order terms was suggested also in Ref. [34].

It is instructive to compare our expression for the shear-stress tensor (240) to the one in Ref. [28] for conformal fluids. The most general second-order expression for the shear-stress tensor of a conformal fluid in flat space-time reads

$$\pi^c_{\mu\nu} = 2\eta\sigma_{\mu\nu} - 2\eta\tau_\pi \left( \sigma_{\mu\nu} + \frac{1}{3}\theta\sigma_{\mu\nu} \right) + \lambda_1 \sigma_{\alpha<\mu}\sigma^{\alpha}_{\nu>} + \lambda_2 \sigma_{\alpha<\mu}\omega^{\alpha}_{\nu>} + \lambda_3 \omega_{\alpha<\mu}\omega^{\alpha}_{\nu>}. \tag{241}$$

where $\omega_{\alpha\beta} = (\nabla_\alpha u_\beta - \nabla_\beta u_\alpha)/2$ is the vorticity tensor. Note that we neglected the vorticity tensor from the outset assuming that the fluid is irrotational. As a consequence, our transport equations (234) – (236) do not contain terms involving vorticity. These, however, can be restored if the energy-momentum tensor in the local-equilibrium distribution is extended to include non-zero vorticity [63, 64].

In the case of a conformal fluid we have $c_s^2 = 1/3$. Furthermore, because the conformal invariance implies vanishing bulk-viscous pressure, it is natural to expect that the correlations involving the relevant operator $\hat{p}^*$ [see Eq. (92)] vanish as well, i.e., $\lambda_{\pi\Pi} = 0$ in this case. We then recover from Eq. (240) the term involving $\tau_\pi$ in Eq. (241). Hence we conclude that the terms $\propto \tau_\pi$ in Eq. (241) given in Ref. [28] have a non-local origin. For the rest of the terms we identify $\lambda_1 = \lambda_\pi$, $\lambda_2 = \lambda_3 = 0$.

In the case of non-conformal fluids Eq. (241) has two additional terms (in flat space-time) [30]. One of these terms shares the same structure with the term $-2\eta\tau_\pi\theta\sigma_{\mu\nu}/3$ and can be written, after converting to our notations, as $-2\eta\tau_\pi^c\theta\sigma_{\mu\nu}/3$. Comparing with our

\footnote{Note that Ref. [28] uses a metric convention which differs by an overall sign from ours, and their definition of the shear viscosity differs from ours by a factor of 2.}
expression (240), we identify $\tau^*_\pi = \tau_\pi (3c_s^2 - 1) - 3\lambda_{\pi\Pi}/\eta$. Our formula for $\tau^*_\pi$ contains an additional term $\propto \lambda_{\pi\Pi}$ compared to the formula given in Ref. [30] for a special class of strongly coupled fluids.

6. Concluding remarks

In this work, we provided a novel derivation of relativistic second-order dissipative hydrodynamics for strongly correlated systems. We adopted Zubarev’s non-equilibrium statistical-operator formalism and extended the existing studies of relativistic dissipative hydrodynamics within this formalism up to second order in thermodynamic gradients. We considered a multicomponent quantum system in the hydrodynamic regime, where it is described utilizing the energy-momentum tensor and the currents of conserved charges.

Our starting point is Zubarev’s method of deriving the full non-equilibrium statistical operator from the quantum Liouville equation. Starting from the exact solution of the Liouville equation in the form of a non-local functional of thermodynamic parameters and their space-time derivatives we performed an expansion of the statistical operator up to second order with respect to the thermodynamic gradients to obtain approximate solutions which capture the low-frequency and the long-wavelength dynamics of the system. In this manner we derived second-order evolution equations for the shear-stress tensor, the bulk-viscous pressure, and the flavor-diffusion currents under the assumption that the fluid is irrotational. In particular, we obtained new non-local terms which do not appear in the first-order treatments of Refs. [12, 16, 18, 22]. We classified the second-order terms by observing that they arise from two different sources: (i) the quadratic terms in the Taylor expansion of the statistical operator; (ii) the linear terms of this expansion with proper inclusion of effects of memory and non-locality. The terms of the type (i) generate corrections which are quadratic (nonlinear) in thermodynamic forces. The terms of the type (ii) generate relaxation terms for the dissipative currents, which are required for reasons of causality. Apart from these non-local terms, the corrections from class (ii) include also additional second-order terms which account for the mixing of the transport equations with the conservation laws. These last types of corrections contribute only to the bulk-viscous pressure and to the diffusion currents.

We obtained also formal expressions for all second-order transport coefficients in terms of certain two- and three-point equilibrium correlation functions, the computation of which can be performed by applying standard thermal field-theory methods [65]. In particular, we derived Kubo-type formulas for the relaxation times, which are given via the frequency derivatives of the relevant first-order transport coefficients, taken in the relevant zero-frequency limit.

It was demonstrated that in the absence of diffusion currents the second-order expression for the shear-stress tensor contains in general three second-order transport coefficients: the shear relaxation time $\tau_\pi$ and two other coefficients, which are responsible for nonlinear couplings between shear and bulk viscous effects.

It would be interesting to compute the second-order transport coefficients derived in this work for strongly interacting matter, for example, in the framework of the Nambu–Jona-Lasinio model of QCD [66–68]. Of special importance are the shear and the bulk relaxation
times, which are necessary for hydrodynamic simulations of heavy-ion collisions as well as binary neutron-star mergers.

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Appendix A. Frames

In ideal hydrodynamics, the energy-momentum and the charge currents are always parallel, and the fluid rest frame is defined as the frame where all these currents vanish. The simultaneous presence of energy- and charge-diffusion currents in the case of dissipative fluids makes the definition of the fluid velocity and the fluid rest frame ambiguous. There are two natural ways to define the fluid rest frame, which we will discuss in this appendix.

**Landau frame.** One of the natural choices of the fluid rest frame is the frame where the 3-momentum current is zero (Landau frame or L-frame) [1]. In this case, \( u^\mu \) is chosen to be the time-like eigenvector of \( T^{\mu\nu} \),

\[
    u_L^\mu = \frac{u_{L\nu} T^{\mu\nu}}{\sqrt{u_{L\nu} T^{\mu\nu} u_{L\lambda} T_{\mu\lambda}}} ,
\]

which together with Eqs. (66), (68), and (69) implies

\[
    \epsilon_L = \sqrt{u_{L\nu} T^{\mu\nu} u_{L\lambda} T_{\mu\lambda}} , \quad u_{L\nu} T^{\mu\nu} = \epsilon_L u_{L}^\mu , \quad q_L^\mu = 0 ,
\]

where the index \( L \) labels the quantities evaluated according to the Landau definition of \( u^\mu \). Thus, with this choice of the velocity field the energy-diffusion current vanishes, and heat-transport phenomena are expressed via the charge-diffusion currents \( j_{L\alpha}^\mu \). The 4-momentum current

\[
    P^\mu \equiv u_\nu T^{\mu\nu} + pu^\mu = hu^\mu + q^\mu
\]

in the L-frame is parallel to the fluid velocity (because \( q_L^\mu = 0 \)), which allows one to write Eq. (A.1) in an alternative way

\[
    u_L^\mu = \frac{P_L^\mu}{\sqrt{P_L^\mu P_{L\mu}}} .
\]

To find the relation between \( u_L^\mu \) and a generic velocity \( u^\mu \) we note that in a generic fluid rest frame the current (A.3) reads \( P^\mu = (h, q^\prime) \), therefore the boost velocity from an arbitrarily defined rest frame \((u^i = 0)\) to the Landau rest frame \((u_L^i = 0)\) is \( v_L^i = q^i/h = O_1 \)
(following Refs. [24, 25], here we introduced the symbol $O_n$ to denote the quantities of $n$th order in deviations from equilibrium). The transformation of the charge currents into the L-frame then reads

$$N_{La}^i = N_a^i - v_L^i N_a^0 + O_2.$$ 

Substituting here $N_a^i = j_a^i$, $N_{La}^i = j_{La}^i$, and $N_a^0 = n_a$, we obtain the charge-diffusion currents measured in the Landau rest frame,

$$j_{La}^i = j_a^i - n_a \hbar q^i + O_2.$$ 

(A.5)

Note that the transformed current $j_{La}^i$ is evaluated at the transformed coordinate $x'$, but the difference

$$j_{La}^i(x') - j_{La}^i(x) \simeq (x' - x)^T \partial x j_{La}^i \propto v_L |\nabla j_{La}^i|$$

is already of order $O_3$ and can be ignored. Taking into account that in the fluid rest frame $j_{La}^0 = j_a^0 = q^0 = 0$, we can cast Eq. (A.5) into a covariant form

$$j_{La}^\mu = j_a^\mu - \frac{n_a}{\hbar} q^\mu + O_2,$$ 

(A.6)

which is valid in an arbitrary frame, i.e., not only in the fluid rest frame. The 4-currents

$$J_a^\mu = j_a^\mu - \frac{n_a}{\hbar} q^\mu = N_a^\mu - \frac{n_a}{\hbar} P^\mu$$ 

(A.7)

are the charge-diffusion currents with respect to the 4-momentum flow, i.e., the charge currents in the absence of energy-diffusion currents. Although the energy-diffusion current $q^\mu$ and the charge-diffusion currents $j_a^\mu$ depend on the choice of the velocity field, the combination (A.7) remains invariant under first-order changes in $u^\mu$.

The remaining thermodynamic variables entering Eqs. (66) and (67) change only in second order in thermodynamic gradients under the change of $u^\mu$ [24, 25]. We then obtain the following relation between $u$ and $u_L$

$$u_L^\mu = u^\mu + \frac{q^\mu}{\hbar} + O_2,$$ 

(A.8)

which follows immediately from a comparison of Eqs. (A.3) and (A.4).

**Eckart frame.** According to the Eckart definition the velocity field is chosen to be parallel to one of the conserved currents $N_a^\mu$ (E-frame). In the case where there is only one species of conserved charge (for example, the net particle number), we have $N^\mu = nu^\mu + j^\mu$, and the 4-velocity is defined as [23]

$$u_E^\mu = \frac{N^\mu}{\sqrt{N^\mu N_\mu}},$$ 

(A.9)

which together with Eqs. (66), (68), and (69) implies

$$n_E = \sqrt{N^\mu N_\mu}, \quad N^\mu = n_E u_E^\mu, \quad j_E^\mu = 0,$$ 

(A.10)

i.e., the particle-diffusion flux is absent in this case. The index $E$ in Eqs. (A.9) and (A.10) labels the E-frame.
The boost velocity from a generic rest frame to the Eckart rest frame is $v^i_E = j^i/n$, and the velocities $u^\mu$ and $u^\mu_E$ are related via

$$u^\mu_E = u^\mu + \frac{j^\mu}{n} + \mathcal{O}_2. \quad (A.11)$$

Transforming the 4-momentum current into the Eckart rest frame we obtain

$$P^i_E = P^i - \frac{j^i}{n} P^0 + \mathcal{O}_2, \quad (A.12)$$

therefore the energy-diffusion current in the E-frame reads

$$q^\mu_E = q^\mu - \frac{\hbar}{n} j^\mu + \mathcal{O}_2. \quad (A.13)$$

The quantity

$$h^\mu = q^\mu - \frac{\hbar}{n} j^\mu = -\frac{\hbar}{n} J^\mu \quad (A.14)$$

is the energy flow with respect to the particle flow, and, therefore, it is natural to call it heat flux. The relation (A.14) shows that heat conduction and particle diffusion are the same phenomena observed from different reference frames, in the case where only first-order deviations from equilibrium are taken into account. From Eqs. (A.8) and (A.11) we find the relation between the L-frame and the E-frame to order $\mathcal{O}_1$

$$u^\mu_L - u^\mu_E = \frac{h^\mu}{h} = -\frac{\hbar}{n} J^\mu. \quad (A.15)$$

The generalization to the case of multiple conserved charges is straightforward. We can connect a reference frame to each of these species via the definition

$$u^\mu_a = \frac{N^\mu_a}{\sqrt{N^\mu_a N_{a\mu}}}, \quad (A.16)$$

which implies that the corresponding diffusion current vanishes, i.e., $j^\mu_a = 0$.

**Appendix B. The entropy-production rate (H-theorem)**

It is interesting to compute also the entropy-production rate using the formalism of the non-equilibrium statistical operator. Differentiating Eq. (19) with respect to time, we obtain

$$\frac{d}{dt} \hat{S}(t) = -\frac{d}{dt} \Omega(t) + \int d^3 x \frac{d}{dt} \left[ \beta^\nu(x) \hat{T}_{0\nu}(x) - \sum_a \alpha_a(x) \hat{N}_a^0(x) \right]. \quad (B.1)$$

The first term can be computed from Eq. (10),

$$\frac{d}{dt} \Omega(t) = \int d^3 x \left\{ \frac{d}{dt} \left[ \beta^\nu(x) \hat{T}_{0\nu}(x) - \sum_a \alpha_a(x) \hat{N}_a^0(x) \right] \right\}_t. \quad (B.2)$$
where we used Eqs. (9) and (13). The time derivative of the integrand in Eqs. (B.1) and (B.2) was already computed in Eq. (44), where the surface term can be dropped. Recalling the definition of the thermodynamic force given by Eq. (49) we obtain for Eq. (B.1)

$$\frac{d}{dt} \hat{S}(t) = \int d^3x \left[ \hat{C}(x) - \langle \hat{C}(x) \rangle_l \right].$$

(A.3)

Averaging Eq. (B.3) over the full non-equilibrium statistical operator (56) and using Eq. (62) we obtain the entropy-production rate

$$\frac{d}{dt} S(t) = \int d^3x d^4x_1 \left( \hat{C}(x), \hat{C}(x_1) \right) + \int d^3x d^4x_1 d^4x_2 \left( \hat{C}(x), \hat{C}(x_1), \hat{C}(x_2) \right) + \ldots,$$

(B.4)

where the ellipsis stands for higher-order terms.

Thus, the entropy-production rate can be computed in principle at any order using equilibrium correlators between several operators $\hat{C}(x)$. We see from Eq. (B.3) that the source of the irreversible entropy production is given by the deviation of the operator $\hat{C}$ from its local-equilibrium value. The physical meaning of this result is intuitively clear: entropy can be produced only if the system deviates from local thermodynamic equilibrium.

In first-order approximation we can substitute the expression (95) into Eq. (B.3),

$$\frac{d}{dt} S(t) = \int d^3x \left( -\beta \theta \Pi + \beta \sigma_{\mu \sigma} \sigma^\mu - \sum_a J_a \nabla^\mu \alpha_a \right).$$

(B.5)

Substituting Eqs. (108) and (110) into Eq. (B.5) we obtain

$$\frac{d}{dt} S(t) = \int d^3x \left( \zeta \beta \theta^2 + 2\eta \beta \sigma_{\mu \nu} \sigma^{\mu \nu} - \sum_{ab} \chi_{ab} \nabla_{\mu} \alpha_a \nabla_{\mu} \alpha_b \right).$$

(B.6)

The integrand of Eq. (B.6) is always positive if $\eta, \zeta$ are positive and $\chi_{ab}$ is positive semidefinite, which guarantees an increase in entropy due to non-equilibrium processes.

Appendix C. Derivation of the Kubo formulas

In this Appendix, we derive the relations between the first-order transport coefficients defined in Sec. 3.3 and retarded Green’s functions, closely following similar derivations in Refs. [16, 18]. We recall that in the evaluation of the transport coefficients any non-uniformities in the thermodynamic parameters can be neglected, i.e., the local-equilibrium distribution can be replaced by a global-equilibrium distribution with some average temperature $T = \beta^{-1}$ and chemical potentials $\mu_a$.

Consider a generic two-point correlator given by Eq. (63). In equilibrium and in the fluid rest frame we have $\hat{A} = \beta \hat{K}$, $\hat{K} = \hat{H} - \sum_a \mu_a \hat{N}_a$ [see Eq. (5)], therefore from Eqs. (54) and (63) we obtain

$$(\hat{X}(x, t), \hat{Y}(x_1, t_1)) = \int_0^1 d\tau \left. \hat{X}(x, t) e^{-\beta \tau K} \hat{Y}(x_1, t_1) e^{\beta \tau K} - \langle \hat{Y}(x_1, t_1) \rangle \right|_t.$$  

(C.1)
The time evolution of any operator in the Heisenberg picture is governed by the equation

$$\hat{Y}(x, t) = e^{iKt} \hat{Y}(x, 0)e^{-iKt},$$ (C.2)

therefore we have $$\hat{Y}(x, t + \delta t) = e^{iK(t + \delta t)} \hat{Y}(x, 0)e^{-iK(t + \delta t)} = e^{iK\delta t} \hat{Y}(x, t)e^{-iK\delta t}.$$ Performing an analytic continuation $$\delta t \to i\tau'$$ we obtain

$$\hat{Y}(x, t + i\tau') = e^{-K\tau'} \hat{Y}(x, t)e^{K\tau'},$$ (C.3)

from which we obtain the relations

$$\langle \hat{Y}(x, t + i\tau') \rangle_t = \langle \hat{Y}(x, t) \rangle_t, \quad \text{(C.4)}$$

$$\langle \hat{X}(x, t)\hat{Y}(x_1, t') + i\beta \rangle_t = \langle \hat{Y}(x, t_1)\hat{X}(x, t) \rangle_t, \quad \text{(C.5)}$$

The relation (C.5) is known as Kubo–Martin–Schwinger relation.

Performing a variable change $$\beta\tau = \tau'$$ in Eq. (C.1) and employing Eqs. (C.3) and (C.4) we obtain

$$\left(\hat{X}(x, t), \hat{Y}(x_1, t_1)\right) = \frac{1}{\beta} \int^\beta_0 d\tau' \langle \hat{X}(x, t) \left[\hat{Y}(x_1, t_1 + i\tau') - \langle \hat{Y}(x_1, t_1 + i\tau') \rangle_t \right] \rangle_t. \quad \text{(C.6)}$$

Assuming that the correlations vanish in the limit $$t_1 \to -\infty$$ [16, 18], i.e.,

$$\lim_{t_1 \to -\infty} \left(\langle \hat{X}(x, t)\hat{Y}(x_1, t_1 + i\tau') \rangle_t - \langle \hat{X}(x, t) \rangle_t\langle \hat{Y}(x_1, t_1 + i\tau') \rangle_t) = 0, \quad \text{(C.7)}$$

we can modify the integrand in Eq. (C.6) as follows

$$\langle \hat{X}(x, t)\hat{Y}(x_1, t_1 + i\tau') \rangle_t - \langle \hat{X}(x, t) \rangle_t\langle \hat{Y}(x_1, t_1 + i\tau') \rangle_t$$

$$= \langle \hat{X}(x, t) \int^{-t_1}_{-\infty} dt' \frac{d}{dt'} \hat{Y}(x_1, t' + i\tau') \rangle_t - \langle \hat{X}(x, t) \rangle_t \int^{t_1}_{-\infty} dt' \frac{d}{dt'} \langle \hat{Y}(x_1, t' + i\tau') \rangle_t$$

$$= -i \int^{t_1}_{-\infty} dt' \langle \hat{X}(x, t) \frac{d}{dt'} \hat{Y}(x_1, t' + i\tau') \rangle_t + i \int^{t_1}_{-\infty} dt' \langle \hat{X}(x, t) \rangle_t \frac{d}{dt'} \langle \hat{Y}(x_1, t' + i\tau') \rangle_t.$$

Substituting this back into Eq. (C.6) and using the relations (C.4) and (C.5) we obtain

$$\left(\hat{X}(x, t), \hat{Y}(x_1, t_1)\right) = \frac{i}{\beta} \int^{t_1}_{-\infty} dt' \langle [\hat{X}(x, t), \hat{Y}(x_1, t')] \rangle_t, \quad \text{(C.8)}$$

where the square brackets denote the commutator. Taking into account that $$t' \leq t_1 \leq t$$, we can write for Eq. (C.8)

$$\left(\hat{X}(x, t), \hat{Y}(x_1, t_1)\right) = -\frac{1}{\beta} \int^{t_1}_{-\infty} dt' G_{XY}^R(x - x_1, t - t'), \quad \text{(C.9)}$$

where

$$G_{XY}^R(x - x', t - t') = -i\theta(t - t') \langle [\hat{X}(x, t), \hat{Y}(x', t')] \rangle_t \quad \text{(C.10)}$$
is the retarded two-point Green’s function for a uniform medium.

Now consider a generic transport coefficient given by the integral

\[ I[\hat{X}, \hat{Y}](\omega') = \beta \int d^3x_1 \int_{-\infty}^{t} dt_1 e^{i\omega'(t-t_1)} e^{\varepsilon(t_1-t)} \left( \hat{X}(x, t), \hat{Y}(x_1, t_1) \right), \]  

(C.11)

where we also introduced a nonzero frequency \( \omega' > 0 \) for the sake of convenience; we will take the limit \( \omega' \to 0 \) at the end of the calculations. According to Eq. (C.9) we can write Eq. (C.11) as

\[ I[\hat{X}, \hat{Y}](\omega') = -\int_{-\infty}^{0} dt'e^{(\varepsilon-\omega')t'} \int_{-\infty}^{t'} dt \int d^3x G^R_{XY}(-x, -t). \]  

(C.12)

Considering the Fourier transformation

\[ G^R_{XY}(x, t) = \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i(\omega t - k x)} G^R_{XY}(k, \omega), \]

we obtain

\[ \int d^3x G^R_{XY}(-x, -t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} G^R_{XY}(\omega), \]

where \( G^R_{XY}(\omega) \equiv \lim_{k \to 0} G^R_{XY}(k, \omega). \) In Eq. (C.12) we now encounter the integral \( \int_{-\infty}^{t'} dt e^{i\omega t} \), which we compute by a shift \( \omega \to \omega - i\delta, \delta > 0 \), taking the limit \( \delta \to 0^+ \) at the end,

\[ \int_{-\infty}^{t'} dt e^{i\omega t} = \lim_{\delta \to 0^+} \int_{-\infty}^{t'} dt e^{(\omega + \delta)t} = \lim_{\delta \to 0^+} \frac{e^{(i\omega + \delta)t}}{i\omega + \delta}. \]  

(C.13)

Then we have from Eq. (C.12)

\[ I[\hat{X}, \hat{Y}](\omega') = -\lim_{\delta \to 0^+} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} G^R_{XY}(\omega) \int_{-\infty}^{0} dt' e^{(\varepsilon-\omega')t'} \frac{e^{(i\omega + \delta)t'}}{i\omega + \delta} \]

\[ = \lim_{\delta \to 0^+} \frac{i}{\omega'} \int \frac{d\omega}{2\pi i} \left( \frac{1}{\omega - \omega' - i\varepsilon} - \frac{1}{\omega - i\delta} \right) G^R_{XY}(\omega), \]

where the integral is closed in the upper half-plane, where the retarded Green’s function is analytic. Note that the contribution from the upper half-circle at infinity vanishes if the retarded Green’s function goes to zero sufficiently rapidly, namely, not slower than \( \omega^{-1} \), which we assume to be the case here. Applying Cauchy’s integral formula and performing the limits \( \delta \to 0^+, \varepsilon \to 0^+ \) we obtain

\[ I[\hat{X}, \hat{Y}](\omega') = \frac{i}{\omega'} \left[ G^R_{XY}(\omega') - G^R_{XY}(0) \right]. \]  

(C.14)

Going to the zero-frequency limit \( \omega' \to 0 \) we obtain the final formula

\[ I[\hat{X}, \hat{Y}](0) = i \frac{d}{d\omega} G^R_{XY}(\omega) \bigg|_{\omega=0}, \]  

(C.15)
with

\[ G_{X,Y}^{R}(\omega) = -i \int_{0}^{\infty} dt e^{i\omega t} \int d^{3}x \left\langle [\hat{X}(x,t), \hat{Y}(0,0)] \right\rangle_{t}. \] (C.16)

From Eqs. (C.14) and (C.16) we find that

\[ \{ G_{X,Y}^{R}(\omega) \}^{*} = G_{X,Y}^{R}(-\omega), \quad \{ I[\hat{X}, \hat{Y}](\omega) \}^{*} = I[\hat{X}, \hat{Y}](\omega). \] (C.17)

Indeed, since \( \hat{X}(x,t) \) and \( \hat{Y}(x,t) \) are hermitian operators, we have the property

\[ \left\langle \left[ \hat{X}(x,t), \hat{Y}(x', t') \right] \right\rangle = -\left\langle \left[ \hat{X}(x,t), \hat{Y}(x', t') \right] \right\rangle, \] (C.18)

therefore the retarded Green’s function given by Eq. (C.10) is real, which is used to obtain the first relation in Eq. (C.17). From Eq. (C.17) we have also

\[ \text{Re} G_{X,Y}^{R}(-\omega) = \text{Re} G_{X,Y}^{R}(\omega), \quad \text{Im} G_{X,Y}^{R}(-\omega) = -\text{Im} G_{X,Y}^{R}(\omega), \] (C.19)

therefore from Eqs. (C.11) and (C.15) we obtain in the zero-frequency limit

\[ I[\hat{X}, \hat{Y}](0) = \beta \int d^{3}x \left( \hat{X}(x), \hat{Y}(x) \right) = -\frac{d}{d\omega} \text{Im} G_{X,Y}^{R}(\omega) \bigg|_{\omega=0}, \] (C.20)

where we used the short-hand notation defined in Eq. (61).

Now let us show that the Green’s function (C.16) is symmetric in its arguments if the operators \( \hat{X} \) and \( \hat{Y} \) have the same parity under time reversal. We have

\[ G_{Y,X}^{R}(\omega) = i \int_{0}^{\infty} dt e^{i\omega t} \int d^{3}x \left\langle \left[ \hat{X}(0,0), \hat{Y}(x,t) \right] \right\rangle_{t} = i \int_{0}^{\infty} dt e^{i\omega t} \int d^{3}x \left\langle \left[ \hat{X}(-x,-t), \hat{Y}(0,0) \right] \right\rangle_{t} = i \int_{0}^{\infty} dt e^{i\omega t} \int d^{3}x \left\langle \left[ \hat{X}(x,-t), \hat{Y}(0,0) \right] \right\rangle_{t}, \] (C.21)

where we used the uniformity of the medium. For hermitian operators, we have the following transformation rule under time reversal

\[ \hat{X}_{T}(x,t) = \eta_{X} \hat{X}(x,-t), \quad \hat{Y}_{T}(x,t) = \eta_{Y} \hat{Y}(x,-t), \]

with \( \eta_{X,Y} = \pm 1 \) for even/odd parity under time reversal. For Eq. (C.21) we then have

\[ G_{Y,X}^{R}(\omega) = i \eta_{X} \eta_{Y} \int_{0}^{\infty} dt e^{i\omega t} \int d^{3}x \left\langle \left[ \hat{X}_{T}(x,t), \hat{Y}_{T}(0,0) \right] \right\rangle_{t} = i \eta_{X} \eta_{Y} \int_{0}^{\infty} dt e^{i\omega t} \int d^{3}x \left\langle \left[ \hat{X}(x,t), \hat{Y}(0,0) \right] \right\rangle_{t}, \]

Finally, taking into account that the statistical average of a commutator of hermitian operators is purely imaginary and the operator of time reversal is antiunitary (i.e., transforms a number to its complex conjugate), we obtain

\[ G_{Y,X}^{R}(\omega) = -i \eta_{X} \eta_{Y} \int_{0}^{\infty} dt e^{i\omega t} \int d^{3}x \left\langle \left[ \hat{X}(x,t), \hat{Y}(0,0) \right] \right\rangle_{t} = \eta_{X} \eta_{Y} G_{X,Y}^{R}(\omega). \] (C.22)
Thus, if \( \eta_X = \eta_Y \), we obtain \( G^R_{YX}(\omega) = G^R_{XY}(\omega) \), and, therefore, \( I[\hat{Y}, \hat{X}] = I[\hat{X}, \hat{Y}] \), which is Onsager’s symmetry principle for transport coefficients. Using now Eq. (C.20) and the definitions of the transport coefficients given by Eqs. (106), (107), (111) and (114), we obtain the formulas (115) and (116) of the main text.

In the derivation of the second-order equations of motion for the dissipative currents we encounter integrals of the type

\[
I^r[\hat{X}, \hat{Y}](\omega) = \beta \int d^4x_1 e^{i\omega(t_1-t)} (\hat{X}(x), \hat{Y}(x_1))(x_1 - x)^r,
\]

where we used again the short-hand notation (61). The correlator \( (\hat{X}(x), \hat{Y}(x_1)) \) evaluated in the local rest frame depends on the spatial coordinates only via the difference \( |x - x_1| \), i.e., it is an even function of \( x - x_1 \). Then Eq. (C.23) implies that the spatial components of the vector \( I^r \) vanish in that frame, and for the temporal component we have

\[
I^0[\hat{X}, \hat{Y}](\omega) = \beta \int d^4x_1 e^{i\omega(t_1-t)} (\hat{X}(x_1), \hat{Y}(x_1))(t_1 - t) = i\beta \frac{d}{d\omega} \int d^4x_1 e^{i\omega(t_1-t)} (\hat{X}(x_1), \hat{Y}(x_1)) = i\frac{d}{d\omega} I[\hat{X}, \hat{Y}](\omega),
\]

where we used Eq. (C.11). From Eqs. (C.14) and (C.24) we obtain in the limit \( \omega \to 0 \)

\[
I^0[\hat{X}, \hat{Y}](0) = K[\hat{X}, \hat{Y}],
\]

where we defined

\[
K[\hat{X}, \hat{Y}] \equiv -\frac{1}{2} \frac{d^2}{d\omega^2} G^R_{X\hat{Y}}(\omega) \bigg|_{\omega=0} = -\frac{1}{2} \frac{d^2}{d\omega^2} \text{Re} G^R_{X\hat{Y}}(\omega) \bigg|_{\omega=0}. \tag{C.26}
\]

Note that in Eqs. (C.15) and (C.26) the Green’s function should be evaluated in the fluid rest frame. The relation (C.25) can also be cast into a covariant form

\[
\beta \int d^4x_1 (\hat{X}(x_1) \hat{Y}(x_1))(x_1 - x)^r = K[\hat{X}, \hat{Y}] u^r. \tag{C.27}
\]

**Appendix D. Properties of the non-local projectors**

Using Eqs. (71) and (138) we find the following properties for the second-rank projector \( \Delta_{\mu\nu}(x, x_1) \)

\[
\Delta_{\mu\nu}(x, x_1) = \Delta_{\nu\mu}(x_1, x), \quad u^\mu(x) \Delta_{\mu\nu}(x, x_1) = \Delta_{\mu\nu}(x, x_1) u^\nu(x_1) = 0, \quad \Delta_{\mu\nu}(x, x) = \Delta_{\nu\mu}(x), \quad \Delta_{\alpha\nu}(x, x_1) = \Delta_{\alpha\nu}(x_1, x).
\]

\[
\Delta_{\mu\nu\rho\sigma}(x, x_1) = \Delta_{\nu\mu\rho\sigma}(x_1, x) = \Delta_{\nu\mu\rho\sigma}(x, x_1) = \Delta_{\rho\sigma\mu\nu}(x_1, x), \quad u^\mu(x) \Delta_{\mu\nu\rho\sigma}(x, x_1) = 0, \quad \Delta_{\mu\nu\rho\sigma}(x, x_1) u^\rho(x_1) = 0, \quad \Delta_{\mu\nu\rho\sigma}(x, x_1) \Delta_{\alpha\beta}(x_1) = \Delta_{\mu\nu\alpha\beta}(x, x_1) \tag{D.2}
\]

\[
\Delta_{\mu\nu\rho\sigma}(x, x) = \Delta_{\mu\nu\rho\sigma}(x), \quad \Delta_{\mu\nu\rho\sigma}(x, x_1) = 0, \quad \Delta_{\mu\nu\rho\sigma}(x, x_1) \Delta_{\alpha\beta}(x_1) = \Delta_{\mu\nu\alpha\beta}(x, x_1).
\]

48
For our calculations, it is sufficient to expand the non-local projectors around the point $x_1 = x$ keeping only the linear terms in the difference $x_1 - x$. We thus approximate

$$u_\mu(x_1) \simeq u_\mu(x) + (x_1 - x)^\alpha \partial_\alpha u_\mu(x),$$

which due to the identity $u^\nu \partial_\alpha u_\mu = 0$ gives $u^\nu(x)u_\mu(x_1) \simeq 1$. In this approximation, we find for the projector (138) up to terms of second order in $x_1 - x$

$$\Delta_{\mu\nu}(x, x_1) = \Delta_{\mu\nu}(x) - [u_\mu(x_1) - u_\mu(x)] u_\nu(x_1) = \Delta_{\mu\nu}(x) - u_\nu(x)(x_1 - x)^\alpha \partial_\alpha u_\mu(x), \quad (D.3)$$

and from Eqs. (D.1) and (D.3) we obtain

$$\Delta_{\mu\nu}(x, x_1) = 3, \quad \Delta^\alpha_{\mu}(x) \Delta^\alpha_{\nu}(x, x_1) = \Delta_{\mu\nu}(x),$$

$$\Delta_{\mu\nu}(x, x_1) \Delta^{\mu\nu}(x, x_1) = \Delta_{\mu\lambda}(x) \Delta^{\mu\nu}(x, x_1) \Delta^\lambda_{\nu}(x_1) = 3, \quad \Delta_{\mu\nu}(x, x_1) \Delta^{\nu\mu}(x, x_1) \Delta^\lambda_{\nu}(x_1) = \Delta^{\nu\lambda}(x, x_1) \Delta^\nu_{\lambda}(x_1) = 3. \quad (D.4)$$

In the same approximation the fourth-rank projector $\Delta_{\mu\nu\rho\sigma}(x, x_1)$ can be written as

$$\Delta_{\mu\nu\rho\sigma}(x, x_1) = \frac{1}{2} \left[ \Delta_{\mu\rho}(x, x_1) \Delta_{\nu\sigma}(x, x_1) + \Delta_{\mu\sigma}(x, x_1) \Delta_{\nu\rho}(x, x_1) \right] - \frac{1}{3} \Delta_{\nu\nu}(x) \Delta_{\rho\sigma}(x_1), \quad (D.5)$$

which together with the properties (D.4) gives

$$\Delta_{\mu\nu}^{\mu\nu}(x, x_1) = 5. \quad (D.6)$$

Equation (D.6) together with the first relation in Eq. (D.4) was used in Eqs. (136) and (137) for the normalization of the corresponding correlation functions.

Using the relation $\partial_\alpha \Delta_{\gamma\delta} = -u_\gamma \partial_\alpha u_\delta - u_\delta \partial_\alpha u_\gamma$, from Eq. (72) we obtain

$$\partial_\alpha \Delta_{\gamma\delta\rho\sigma} = -\frac{1}{2} \left[ \Delta_{\gamma\rho}(u_\sigma \partial_\alpha u_\delta + u_\delta \partial_\alpha u_\sigma) + \Delta_{\delta\sigma}(u_\gamma \partial_\alpha u_\rho + u_\rho \partial_\alpha u_\gamma) + (\rho \leftrightarrow \sigma) \right]$$

$$+ \frac{1}{3} \left[ \Delta_{\gamma\delta}(u_\rho \partial_\alpha u_\sigma + u_\sigma \partial_\alpha u_\rho) + \Delta_{\rho\sigma}(u_\gamma \partial_\alpha u_\delta + u_\delta \partial_\alpha u_\gamma) \right].$$

Multiplying this by $\Delta_{\mu\nu}^{\gamma\delta}(x)$ and using the properties (73) and (74) we obtain

$$\frac{\partial}{\partial x_1^\rho} \Delta_{\mu\nu\rho\sigma}(x, x_1) \bigg|_{x_1 = x} = \Delta_{\mu\nu}^{\gamma\delta} \partial_\alpha \Delta_{\gamma\delta\rho\sigma} = - (\Delta_{\mu\nu\rho\beta} u_\sigma + \Delta_{\mu\nu\sigma\beta} u_\rho) \partial_\alpha u_\beta, \quad (D.7)$$

where we recalled Eq. (139).
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