THE PERIODIC TABLE OF STATIC FERMION BAGS
IN THE GROSS-NEVEU MODEL

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We study the spectrum of stable static fermion bags in the 1 + 1 dimensional Gross-Neveu model with $N$ flavors of Dirac fermions, in the large $N$ limit. In the process, we discover a new kink, heavier than the Callan-Coleman-Gross-Zee (CCGZ) kink, which is marginally stable (at least in the large $N$ limit). The connection of this new kink and the conjectured $S$ matrix of the Gross-Neveu model is obscured at this point. After identifying all stable static fermion bags, we arrange them into a periodic table, according to their $O(2N)$ and topological quantum numbers.

1. Introduction

A central concept in particle physics states that fundamental particles acquire their masses through interactions with vacuum condensates. Thus, a massive particle may carve out around itself a spherical region or a shell in which the condensate is suppressed, thus reducing the effective mass of the particle at the expense of volume and gradient energy associated with the condensate. This picture has interesting phenomenological consequences.

This phenomenon may be studied non-perturbatively in model field theories in 1 + 1 space-time dimensions such as the Gross-Neveu (GN) model and the multi-flavor Nambu-Jona-Lasinio (NJL) model, in the large $N$ limit. These models are particularly appealing, since they exhibit, among other things, asymptotic freedom and dynamical mass generation, like more realistic four dimensional models.

Many years ago, Dashen, Hasslacher and Neveu (DHN), and following them Shei, used inverse scattering analysis to find static fermion-bag soliton solutions to the large-$N$ saddle point equations of the GN and of the 1 + 1 dimensional, multi-flavor NJL models. In the GN model, with its discrete chiral symmetry, a topological soliton, the so called Callan-Coleman-Gross-Zee (CCGZ) kink, was discovered prior to the work of DHN. In this report we will concentrate exclusively on the GN model.

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One version of writing the action of the 1 + 1 dimensional GN model is
\[ S = \int d^2 x \ \left\{ \sum_{a=1}^{N} \bar{\psi}_a \left( i\frac{\partial}{\partial x} - \sigma \right) \psi_a - \frac{1}{2g} \sigma^2 \right\}, \]
where the \( \psi_a \) (\( a = 1, \ldots, N \)) are \( N \) flavors of massless Dirac fermions, with Yukawa coupling to the scalar auxiliary field \( \sigma(x) \).

This action is evidently symmetric under the simultaneous transformations \( \sigma \to -\sigma \) and \( \psi \to \gamma_5 \psi \), which generate the so-called discrete (or \( \mathbb{Z}_2 \)) chiral symmetry of the GN model. The GN action has also flavor symmetry \( O(2N) \), which can be seen by breaking the \( N \) Dirac spinors into \( 2N \) Majorana spinors. Related to this is the fact that the model is also invariant under charge-conjugation.

Performing functional integration over the Grassmannian variables in the GN action leads to the partition function
\[ Z = \int D\sigma \exp\left\{ iS_{\text{eff}}[\sigma] \right\} \]
where the bare effective action is
\[ S_{\text{eff}}[\sigma] = -\frac{1}{2g^2} \int d^2 x \sigma^2 - iN \text{Tr} \log (i\gamma^0 - \sigma) \]
and the trace is taken over both functional and Dirac indices.

The theory [1] has been studied in the limit \( N \to \infty \) with \( Ng^2 \) held fixed. In this limit the partition function \( Z \) is governed by saddle points of \( S_{\text{eff}}[\sigma] \) and the small fluctuations around them. The most general saddle point condition reads
\[ \frac{\delta S_{\text{eff}}}{\delta \sigma(x,t)} = -\sigma(x,t) + iN \text{tr} \left( (x,t) \frac{1}{i\gamma^0 - \sigma} (x,t) \right) = 0. \]

In particular, the non-perturbative vacuum of the GN model is governed by the simplest large \( N \) saddle point of the path integral associated with it, where the composite scalar operator \( \bar{\psi}\psi \) develops a space-time independent expectation value. Thus, the fermions acquire mass \( m = \Lambda \exp\left[ -\frac{\pi}{Ng^2(\Lambda)} \right] \) (\( \Lambda \) is an ultraviolet cut-off). This mass is associated with the dynamical breakdown of the discrete chiral symmetry by the non-perturbative vacuum. Associated with this breakdown of the discrete symmetry is a kink solution [9, 6, 10], the CCGZ kink mentioned above, \( \sigma(x) = m \tanh(mx) \). It is topology which insures the stability of these kinks. The GN model bears also non-topological solitons. These non-topological solitons are stabilized dynamically, by trapping fermions and releasing binding energy. Since the works of DHN and of Shei, these fermion bags were discussed in the literature several other times, using alternative methods [11]. For a recent review on these and related matters, see [13].

The remarkable discovery DHN made was that all the physically admissible static, space-dependent solutions of \[ S_{\text{eff}}[\sigma] \], i.e., the static bag configurations in the GN model (the CCGZ kink being a non-trivial example of which), were reflectionless. That is, the static \( \sigma(x) \)'s that solve the saddle point equations of the GN model are such that the reflection coefficient of the Dirac equation
\[ \left[ i\gamma^0 - \sigma(x) \right] \psi = 0, \]
associated with the GN action, vanishes identically.
A reflectionless configuration $\sigma(x)$ which supports $K$ bound states of the Dirac equation (3) is parametrized by a a discrete set of $2K$ real parameters $0 \leq \omega_1 < \omega_2 < \cdots < \omega_K < m$; $c_1, c_2, \cdots, c_K$, where the $\pm \omega_i$ are the bound state energies, and the $c_i$ determine the asymptotic behavior of the bound state wave functions.

(Note that the energy spectrum of our one-dimensional static Dirac equation cannot be degenerate, and thus no two $\omega_i$'s are allowed to be equal.) Then, the effective action (1), evaluated on this reflectionless $\sigma(x)$ configuration, becomes an ordinary function of these parameters. Minus that function, per unit time, is the rest mass $\omega$ be degenerate, and thus no two

In this talk we classify the entire spectrum of stable static fermion bags in the GN model, thus extending the “static part” of the work of DHN, who considered fermion bags for which the Dirac equation (3) had a single bound state. In the narrow space of these pages we can only summarize the results. The technical details of our analysis will be elaborated elsewhere.

First, let us consider the local extrema of $\mathcal{M}(\omega_i, c_i)$. One immediate result is that $\mathcal{M}$ is independent of the parameters $c_i$. The $c_i$'s are flat, collective coordinates of the soliton bag. Consider next the $K$ ordered bound state frequencies $0 \leq \omega_1 < \omega_2 < \cdots < \omega_K < m$.

Assume first that $\omega_1 > 0$. Due to invariance under charge-conjugation, the Dirac equation (3) has $2K$ bound states at $\pm \omega_i$. Imagine populating these bound states with fermions. Due to Pauli’s principle, each energy level (positive or negative) may be occupied by up to $N$ fermions, with different flavor indices. Consider a given pair of charge conjugate energy levels $\pm \omega_i$, which bind $N - h_i$ fermions in $-\omega_i$ and $p_i$ fermions in $+\omega_i$ (with $0 \leq h_i, p_i \leq N$). We will refer to such a $\pm \omega_i$ pair as containing $h_i$ holes (or anti-particles) and $p_i$ particles. Thus, it contains $n_i = p_i + h_i$ bounded fermions or antifermions in all, where, of course, $0 \leq n_i \leq 2N$. The total (valence) fermion number of such a configuration is obviously $N_{F, val} = p_i - h_i$. It turns out that the contribution of these bound states to the total mass of the soliton depends only on $n_i$, i.e., we obtain a multiplet of states degenerate in energy. By simple counting, for a fixed $n_i$, we find that the multiplet contains $(2N)!/n_i!(2N - n_i)!$ states, which we immediately identify as an $O(2N)$ anti-symmetrical tensor multiplet of rank $n_i$. This multiplet contains states with different numbers of particles and antiparticles, hence with different values of the fermion number. In fact, it is a straightforward calculation to show that in this $O(2N)$ multiplet $-n_i \leq N_{F, val} \leq n_i$ if $n_i \leq N$, and $-(2N - n_i) \leq N_{F, val} \leq 2N - n_i$ if $N \leq n_i \leq 2N$. That states with different fermion number are degenerate in energy arises here because the fermion number $N_F$ is a generator of the $O(2N)$ algebra which commutes with the $U(N)$ flavor subgroup, but has non-trivial commutation relations with the other generators.

It can further be shown that the extremal values of the frequencies are $\omega_i = m \cos \left( \frac{\pi n_i}{2N} \right)$. Thus, each energy level is determined by the number of fermions
trapped in it. In fact, such a level cannot exist unless it binds fermions. A notable exception is a bound state at \( \omega = 0 \), if it exists.

A bound state at \( \omega = 0 \) is self-charge-conjugate. It arises if and only if the extremal \( \sigma(x) \) is topologically non-trivial (i.e., a kink or an anti-kink). It arises due to topology, and is thus stable, regardless of the number \( 0 \leq n_0 \leq N \) of valence fermions it traps. All kink states fall into the (reducible) \( 2^N \) dimensional spinor representation of \( O(2N) \). The fermion number eigenvalues of states of this multiplet are \( -N/2, -N/2 + 1, \ldots, N/2 \), which demonstrate the phenomenon of fermion number fractionalization by the topological background.

Consider an extremal \( \sigma(x) \) configuration, with \( n_i \) trapped fermions occupying the pair \( \pm \omega_i \) \( (i = 1, \ldots, K) \), as described above. It corresponds to a reducible \( O(2N) \) representation made of a tensor product of the corresponding antisymmetric \( O(2N) \) tensors. If this configuration is a kink or an anti-kink, it will have in addition a representation made of a tensor product of the corresponding antisymmetric tensors (and also the spinorial representation, if \( q = 0 \) in the topologically non-trivial sector). The sum runs only over the non-zero energy levels. Note (for \( q = \pm 1 \)) that the kink mass is independent of \( n_0 \). This extremal configuration corresponds to a tensor product of the appropriate antisymmetrical tensors (and also the spinorial representation, if \( |q| = 1 \)). We verified in \[13\] (using resolvent methods) that the expectation value \( N_F \) of the fermion number operator, in the background of this bag, is simply \( N_F = n_0 + \sum_{i=1}^{K} (p_i - h_i) - \frac{|q|N}{2} \). The sum part of this expression for \( N_F \) is the contribution of the valence fermions trapped in the soliton, and the second part, which arises only in the topologically non-trivial sector, is the fractional part. Note that the binding energy \( B = m \sum_i n_i - M_{\text{soliton}} \) increases as each of the occupation numbers \( n_i \) tends to its maximal possible value \( 2N \). The more fermions the bag traps, the more stable it becomes. This is known in the physics of fermion bags as the “mattress effect”.

The most important question is which of the extremal fermion bags we have just described are stable, i.e., true local minima of \[18\], which correspond to eigenstates of the hamiltonian of the GN model (at least in the large \( N \) limit)?

Each decay process must conserve, of course, energy and momentum, the \( O(2N) \) charges (and in particular, the fermion number \( N_F \)), and the topological charge. The list of stable bags, arranged according to their \( O(2N) \) quantum numbers (the “periodic table”), is as follows [13]:

The topologically trivial sector, \( q = 0 \):

1) Bags with a single bound state, \( K = 1 \), with \( \omega_1 = m \cos \left( \frac{\pi n}{2N} \right) \), \( 0 \leq n_1 \leq 2N \). These are the bags discussed in [1] (and we shall refer to them as “DHN bags”). The mass of such a soliton is \( M_0 = \frac{2N \kappa}{\pi} \sin \left( \frac{\pi n_1}{2N} \right) \) and its profile is \( \sigma(x) = \sigma(<\infty) + \kappa \tanh \left[ \kappa(x - x_0) \right] - \kappa \tanh \left[ \kappa(x - x_0) + \frac{1}{2} \log \left( \frac{m + \kappa}{m - \kappa} \right) \right] \), where \( \sigma(\infty) = \sigma(-\infty) = \pm m, \, \kappa = m \sin \left( \frac{\pi n_1}{2N} \right) = \sqrt{m^2 - \omega_1^2} \) and \( x_0 \) is a translational collective mode. It
corresponds to an antisymmetric tensor representation of $O(2N)$ of rank $n$. Such a bag cannot decay into lighter DHN bags, because any such presumed process can be shown to violate $N_F$ conservation. Thus, they are stable. Note that $M_{n=N} = \frac{2Nm}{\pi} = 2M_{CCGZ}$, and more over, that as $n \to N$ (i.e., $\kappa \to m$), the profile $\sigma(x)$ tends to $\sigma(\infty) + m \tanh(mx) - m \tanh[m(x-R)]$, $R \to \infty$. Thus, the configuration at $n = N$ is that of infinitely separated CCGZ kink and anti-kink bound at threshold.

2) There are no stable, static, topologically trivial bags with $K \geq 2$ bound states. (In this case there are always decay channels which respect energy and fermion number conservation.)

The topologically non-trivial sector, $q = \pm 1$:

3) Topological bags with a single bound state, $K = 1$. These are the CCGZ kinks (or anti-kinks), $\sigma(x) = \pm m \tanh(mx-x_0)$. They are the lightest topologically non-trivial states. Thus, they are stable. The single bound state is at $\omega_0 = 0$, by topology. These kinks form a $2^N$ dimensional (reducible) spinorial representation of $O(2N)$ as was described above. They trap any number $0 \leq n_0 \leq N$ of valence fermions. All kink states are degenerate in mass at $M_{CCGZ} = \frac{Nm}{\pi}$.

4) Topological bags with $K = 2$ bound states. There is such a heavier, marginally stable multiplet of kink (or anti-kink) states. By topology, it has a zero mode $\omega_0 = 0$, and in addition, a pair of bound states at $\pm \omega_1$, with $\omega_1 = m \cos\left(\frac{\pi n_1}{2N}\right)$. This multiplet is reducible: It is the tensor product of the $2^N$ spinorial representation and an antisymmetric tensor of rank $n_1$. The mass of this kink is $M = \frac{2Nm}{\pi} \sin\left(\frac{\pi n_1}{2N}\right) + M_{CCGZ}$, and is thus degenerate in mass (in the large $N$ limit) with the sum of masses of a CCGZ kink and one of the DHN bags mentioned above. It can be shown that all presumed decay channels of this heavier kink, but two channels, which are allowed by energy conservation and topology, violate $N_F$ conservation. The two decay channels which do not violate energy and $N_F$ conservation are decays into a CCGZ kink plus a DHN bag with either $n = n_1$ or $n = 2N - n_1$, which are degenerate in mass with the decaying heavier kink. Thus, these decay processes, if not excluded by other reasons, are at threshold. Thus, the heavier kink is marginally stable. However, the profile of this bag is tightly packed: This kink has two collective coordinates, and in general, its profile is a rational function of hyperbolic functions. However, for a specific choice of the collective coordinates, and for the particular filling $n_1 = N/3$, the expression for the profile simplifies into $\sigma(x) = \pm m \tanh(mx/2)$, which is a kink as twice as extended in space as the CCGZ kink. Thus, unlike the DHN bag at $n = N$ which we mentioned above, it does not have the shape of a configuration of infinitely separated CCGZ kink and a DHN bag (whose mass is just the sums of masses of the individual lumps). Thus, we conjecture that this heavier kink is a genuine marginally stable state in the spectrum. Its connection to the conjectured $S$ matrix of the Gross-Neveu model is obscured at this point.

5) There are no stable, static, topologically non-trivial bags with $K \geq 3$ bound states.
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