Counting the number of vertexes labeled connected graphs of order five with minimum five edges and maximum ten parallel edges

Amanto, Notiragayu, F C Puri, Y Antoni and Wamiliana
Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Lampung

Corresponding author: pakamanto@gmail.com

Abstract. If given a graph G(V,E) with n vertices and m edges many graphs can be constructed. The graphs constructed maybe connected graphs (there exists at least one path connecting every pair of vertices in the graph) or disconnected; either simple a (contains loop or parallel edges) or not simple. In this paper we will discuss the formula for counting the number of connected vertex labelled graph of order five (n=5) without loops, with h minimum five edges and maa y contaiof n maximum ten parallel edges.

1. Introduction
Graph theory is a branch of mathematics which discussess about discrete objects and the relationship among those objects. A point in a graph represents an object and an edge (a line) connecting a pair of vertices in graph represents the relationship between that pair of vertices. Points in a graph can represent cities, stations, houses, etc, while edges can represent roads, train track, fresh water pipeline, and so on. There is no a single correct way to draw a graph. A line can be drawn as a straight line, a curve, etc. Therefore, because of its flexibility, graph theoretical concepts are used to represent many real-life problems.

Not only can it represent the real-life problem, graph theoretical concept also can be used to solve it. A comprehensive study about the application of graph theory in networks design is given in [1], while in [2] some applications in operations research problem are given. Enumerative or counting graph is pioneered by Cayley in 1874 who interested to count the isomer of hydrocarbon CnH2n+2 and found that counting problem is related with counting rooted tree in graph. Inspired by the work of Cayley, Slomenski investigated the additive structural properties of hydrocarbon using the concept of graph [3]. However, theoretically enumerative graph is given in [4-6].

2. Literature review
If given graph G(V,E) with n number of vertices and m edges, there are a lot of graphs that can be formed, either simple graph or not simple. A simple graph is a graph which not contains loops nor parallel edges. Moreover, a graph constructed maybe connected or disconnected. To distinguished one graph with another graph with the same form or structure we can use labelling. If the label is given on every vertices, it is called vertex labelling, if the label is given on every edge than is is called as edge labelling, and if the label is given on every vertex and edge then is is called as total labelling. The formula for counting vertex labelled graphs is given in [7], but in that formula there are no distinction if the graphs
are connected or disconnected. In that formula, all graphs are included except loops which are not allowed to be in the graph. Based on [7], in 2017 the number of disconnected vertex labelled graphs of order four is investigated [8], and the number of disconnected vertex labelled graphs of order five without parallel edges is investigated in [9]. The formula for counting disconnected vertex labelled graphs of order five with maximum six 3-parallel edges and containing no loops is investigated in [10], and the formula for counting the number of connected vertex labelled graph of order five with maximum 5 parallel edges is investigated in [11].

3. Graphs construction and the patterns obtained

Given \(G(V,E)\) with the number of vertices is \(n = 5\) and the number of edges is \(m\), \(5 \leq m \leq 20\), some patterns obtained are as shown in Table 1, where \(t\) is the number of edges that connect different pairs of vertices (parallel edges are counted as one). There are many graphs can be obtained, but due to space limitation, we only provide some patterns and the number of graphs that can be constructed using that pattern.

| Graph info | Patterns | The number of graphs | Graph info | Patterns | The number of graphs |
|------------|----------|----------------------|------------|----------|----------------------|
| \(n = 5\) | \(t = 5\) | \(m = 5\) | \(n = 5\) | \(t = 5\) | \(m = 6\) |
| \(v_1\) | \(v_2\) | \(v_3\) | \(v_4\) | \(v_5\) | \(v_1\) | \(v_2\) | \(v_3\) | \(v_4\) | \(v_5\) |
| 12 | 60 | | 300 | 60 |
Notate \( t \) as the number of edges that connect different pairs of vertices (two or more edges that connect the same pair of vertices are counted as one), \( p_i \) is the number of \( i \)-parallel edges, and \( g \) is the number of non-parallel edges, then 
\[
\begin{align*}
t & = \sum p_i + g, \\
m & = \sum j p_i + g; j \in N, 
\end{align*}
\]
where \( n \) is the order of the graph and \( m \) is the number of edges in the graph. From Figure 1 we can see that \( n = 5 \), \( p_2 = 2 \), \( g = 5 \), \( t = p_2 + g = 2 + 5 = 7 \) and \( m = 2 p_2 + g = 2 (2) + 5 = 9 \).

Figure 1. An example of graph with 2-parallel edges

4. Results and discussion

From construction and observation we get Table 2 as follows:

| \( m \) | \( 5 \) | \( 6 \) | \( 7 \) | \( 8 \) | \( 9 \) | \( 10 \) |
|---|---|---|---|---|---|---|
| \( t \) | 222 | 1110 | 3330 | 1230 | 110 |
The following table is derived from Table 2 by observing the patterns obtained on the column of the table.

**Table 3.** The patterns obtained in every column of the table.

| m   | t       |   |   |   |   |
|-----|---------|---|---|---|---|
| 5   | 1 x 222 | 6 | 1 x 205 |   |   |
| 6   | 5 x 222 | 7 | 6 x 205 | 1 x 110 |   |
| 7   | 15 x 222 | 8 | 21 x 205 | 7 x 110 | 1 x 45 |
| 8   | 35 x 222 | 9 | 56 x 205 | 28 x 110 | 8 x 45 | 1 x 10 |
| 9   | 70 x 222 | 10 | 126 x 205 | 84 x 110 | 36 x 45 | 9 x 10 | 1 x 1 |
| 10  | 126 x 222 | 11 | 210 x 205 | 210 x 110 | 120 x 45 | 45 x 10 | 10 x 1 |
| 11  | 210 x 222 | 12 | 462 x 205 | 462 x 110 | 330 x 45 | 165 x 10 | 55 x 1 |
| 12  | 462 x 222 | 13 | 924 x 205 | 792 x 110 | 792 x 45 | 495 x 10 | 220 x 1 |
| 13  | 924 x 222 | 14 | 1716 x 110 | 1716 x 45 | 1287 x 10 | 715 x 1 |
| 14  | 1716 x 222 | 15 | 3003 x 110 | 3432 x 45 | 3003 x 10 | 2002 x 1 |
| 15  | 3003 x 222 | 16 | 6435 x 45 | 6435 x 10 | 5005 x 1 |
| 16  | 6435 x 222 | 17 | 11440 x 45 | 12870 x 10 | 11440 x 1 |
| 17  | 11440 x 222 | 18 | 24310 x 10 | 24310 x 1 |
| 18  | 24310 x 222 | 19 | 437580 x 10 | 48620 x 1 |
| 19  | 437580 x 222 | 20 | 92378 x 10 | 92378 x 1 |

By grouping the graphs that can be constructed in terms of m and t, we found that the number in every column of the table formed patterns. In every column the numbers can be rewrite as a product of fixed constant with a number, and the numbers in every column makes a sequences.

For t = 5 : 1, 5, 15, 35, 70, 126, 210
The fixed difference occurs on the fourth level, therefore the polynomial that can represent that sequence is polynomial of order four:  
\[ P_4(m) = A_4 m^4 + A_3 m^3 + A_2 m^2 + A_1 m + A_0 \]

**Result 1:** Given \( G(V,E) \), \(|V| = 5\), \(|E| = m\), \(5 \leq m \leq 20\), \(t = 5\), \(t\) is the number of edges that connect different vertices (parallel edges are counted as one), then the number of connected vertex labelled graphs of order five with minimum five edges and maximum ten parallel edges is \( N(G_{5,m,t}) = 222 \binom{m-1}{4} \).

**Proof:**
Substituting the value of the number of graph obtained in the first column we get:
\[
\begin{align*}
222 & = 625 A_4 + 125 A_3 + 25 A_2 + 5 A_1 + A_0 \\
1110 & = 1296 A_4 + 216 A_3 + 36 A_2 + 6 A_1 + A_0 \\
3330 & = 2401 A_4 + 343 A_3 + 49 A_2 + 7 A_1 + A_0 \\
7770 & = 4096 A_4 + 512 A_3 + 64 A_2 + 8 A_1 + A_0 \\
15540 & = 6561 A_4 + 729 A_3 + 81 A_2 + 9 A_1 + A_0 \\
\end{align*}
\]
Solving the above system of equations we get:
\[
\begin{align*}
A_4 & = \frac{2664}{288}, \quad A_3 = -\frac{26640}{288}, \quad A_2 = \frac{93240}{288}, \quad A_1 = -\frac{133200}{288}, \\
A_0 &= \frac{63936}{288}.
\end{align*}
\]
Therefore:
\[
\begin{align*}
P_4(m) & = \frac{2664}{288} m^4 - \frac{26640}{288} m^3 + \frac{93240}{288} m^2 - \frac{133200}{288} m + \frac{63936}{288} \\
& = \frac{222}{24} (m^4 - 10m^3 + 35m^2 - 50m + 24) \\
& = \frac{222}{24} (m - 1)(m - 2)(m - 3)(m - 4) \\
& = 222 \binom{m-1}{4}.
\end{align*}
\]
Therefore \( N(G_{5,m,t}) \) for \( t = 5 \) is \( 222 \binom{m-1}{4} \).
For \( t = 6 \), the sequence of number is 1, 6, 21, 56, 126, 252, 462, 792.

\[
\begin{array}{cccccccc}
1 & 6 & 21 & 56 & 126 & 252 & 462 & 792 \\
5 & 15 & 35 & 70 & 126 & 210 & 330 \\
10 & 20 & 35 & 56 & 84 & 120 \\
10 & 15 & 21 & 7 & 8 \\
5 & 1 & 1 & 1 & 1 \\
\end{array}
\]

The fixed difference occurs on the fourth level, therefore the polynomial that can represent that sequence is polynomial of order five:  
\[ P_5(m) = A_5 m^5 + A_4 m^4 + A_3 m^3 + A_2 m^2 + A_1 m + A_0 \]

**Result 2:** Given \( G(V,E) \), \(|V| = 5\), \(|E| = m\), \(5 \leq m \leq 20\), \(t = 6\), \(t\) is the number of edges that connect different vertices (parallel edges are counted as one), then the number of connected vertex labelled graphs of order five with minimum five edges and maximum ten parallel edges is \( N(G_{5,m,t}) = 205 \binom{m-1}{5} \).

**Proof:**
Substituting the value of the number of graph obtained in the first column we get:
\[
205 = 7776 A_5 + 1296 A_4 + 216 A_3 + 36 A_2 + 6 A_1 + A_0
\]
1230 = 16807A5 + 4012 A4 + 343 A3 + 49A2 + 7A1 + A0
4305 = 32768A5 + 4096A4 + 512A3 + 64A2 + 8A1 + A0
11480 = 59049A5 + 6561A4 + 729 A3 + 81A2 + 9A1 + A0
25850 = 100000A5 + 10000A4 + 10000A3 + 1000A2 + 10A1 + A0
51660 = 161051A5 + 114641A4 + 1331A3 + 121A2 + 11A1 + A0

Solving the above system of equations we get

\[A1 = \frac{16176960}{34560}, \quad A0 = \frac{-7084800}{34560}.\]

Therefore:

\[P_5(m) = \frac{59040}{34560}m^5 - \frac{885600}{34560}m^4 + \frac{5016400}{34560}m^3 - \frac{13284000}{34560}m^2 + \frac{16176960}{34560}m - \frac{7084800}{34560} \cdot \]

\[= \frac{205}{120} (m^5 - 15m^4 + 85m^3 - 225m^2 + 274m - 120) \]

\[= \frac{205}{120} (m - 1)(m - 2)(m - 3)(m - 4)(m - 5) \]

\[= 205 \left(\frac{m - 1}{5}\right).\]

Therefore \(N(G_{5,m,t})\) for \(t = 6\) is 205 \(\left(\frac{m - 1}{5}\right)\).

Based on Results 1 and 2, we get the following results:

**Result 3**: Given \(G(V,E), |V| = 5, |E| = m, 5 \leq m \leq 20\), \(t = 7\), \(t\) is the number of edges that connect different vertices (parallel edges are counted as one), then the number of connected vertex labelled graphs of order five with minimum five edges and maximum ten parallel edges is \(N(G_{5,m,t}) = 110 \left(\frac{m - 1}{6}\right)\).

**Result 4**: Given \(G(V,E), |V| = 5, |E| = m, 5 \leq m \leq 20\), \(t = 8\), \(t\) is the number of edges that connect different vertices (parallel edges are counted as one), then the number of connected vertex labelled graphs of order five with minimum five edges and maximum ten parallel edges is \(N(G_{5,m,t}) = 45 \left(\frac{m - 1}{7}\right)\).

**Result 5**: Given \(G(V,E), |V| = 5, |E| = m, 5 \leq m \leq 20\), \(t = 9\), \(t\) is the number of edges that connect different vertices (parallel edges are counted as one), then the number of connected vertex labelled graphs of order five with minimum five edges and maximum ten parallel edges is \(N(G_{5,m,t}) = 10 \left(\frac{m - 1}{8}\right)\).

**Result 6**: Given \(G(V,E), |V| = 5, |E| = m, 5 \leq m \leq 20\), \(t = 10\), \(t\) is the number of edges that connect different vertices (parallel edges are counted as one), then the number of connected vertex labelled graphs of order five with minimum five edges and maximum ten parallel edges is \(N(G_{5,m,t}) = \left(\frac{m - 1}{9}\right)\).

**5. Conclusion**

Based on the above discussion we can conclude that given \(G(V,E), |V| = 5, |E| = m, 5 \leq m \leq 20\), and \(t, t\) is the number of edges that connect different vertices (parallel edges are counted as one), then the number of connected vertex labelled graphs of order five with minimum five edges and maximum ten parallel edges \(N(G_{5,m,t})\) is:

- For \(t = 5\), \(N(G_{5,m,t}) = 222 \left(\frac{m - 1}{4}\right)\).
- For \(t = 6\), \(N(G_{5,m,t}) = 205 \left(\frac{m - 1}{5}\right)\).
- For \(t = 7\), \(N(G_{5,m,t}) = 110 \left(\frac{m - 1}{6}\right)\).
- For \(t = 8\), \(N(G_{5,m,t}) = 45 \left(\frac{m - 1}{7}\right)\).
- For \(t = 9\), \(N(G_{5,m,t}) = 10 \left(\frac{m - 1}{8}\right)\).
- For \(t = 10\), \(N(G_{5,m,t}) = \left(\frac{m - 1}{9}\right)\).
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