OPEN FROBENIUS CLUSTER TILTED ALGEBRAS

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Abstract. In this paper, we compute the Frobenius dimension of any cluster tilted algebra of finite type. Moreover, we give conditions on the bounded quiver of a cluster tilted algebra $\Lambda$ such that $\Lambda$ has non-trivial open Frobenius structures.

Introduction

Originally, the term Frobenius algebra referred to an algebra $A$ with the property that $A \simeq A^*$ as right $A$-modules. Later, Nakayama provided many equivalent definitions of Frobenius algebras. One equivalent definition, motivated by topological considerations, defines a Frobenius algebra as an algebra $A$ equipped with a coalgebra structure where the comultiplication is a map of $A$-modules. This topologically motivated definition arose in order to rigorously establish the theorem that a two-dimensional topological quantum field theory is essentially the same as a commutative Frobenius algebra. The Frobenius algebras without counit are called open Frobenius algebras. The open Frobenius algebras (also known as nearly Frobenius algebras [18]) give a natural generalization of Frobenius algebras. They were considered for the first time in 2004 by R. Cohen et V. Godin while they were studying the topological quantum field theory of the loop space of a closed oriented manifold [15]. Their main result states that the homology of the free loop space of a closed oriented manifold is an open Frobenius algebra. Although open Frobenius algebras generalize Frobenius algebras, they behave quite differently.

In [18] the authors proved that the direct sums, tensor and quotient of open Frobenius algebras admit natural open Frobenius structures. They showed that the family of open Frobenius structures over an algebra is a $k$-vector space and they defined the Frobenius dimension (or Frobdim for short) of an algebra as the dimension of this vector space. Later, in [2] an algorithm to compute the Frobenius dimension for gentle algebras without oriented cycles is found.

Recently, the Hochschild homology and cohomology of open Frobenius algebras, and the algebraic structures that they have, have been studied by H. Abbaspour [1]. In particular, the Hochschild homology of an open Frobenius algebra has a natural structure of open Frobenius algebra, which provides a way to produce new open Frobenius algebras from the already known ones. Another example of open Frobenius algebra is the Hochschild cohomology $HH^*(A)$ of a closed Frobenius algebra $A$ [21].

On the other hand, cluster categories were introduced in [11] as a representation theoretic framework for the cluster algebras of Fomin and Zelevinski [16]. Given a hereditary finite dimensional algebra $H$ over an algebraically closed field $k$ the cluster category is defined to be $\mathcal{C}(H) := \mathcal{D}(H)[1]$, where $[1]$ denotes the shift functor and $\tau$ is the Auslander- Reiten translation in $\mathcal{D}(H)$. By a result of Keller [19], the cluster category is triangulated. For the cluster category, cluster tilting objects have been defined also in [11], where in addition the authors showed that the clusters correspond to the tilting objects in the cluster category. The endomorphism algebras of the cluster tilting objects are called cluster-tilted algebras and were introduced by Buan, Marsh and Reiten [12] and, independently in [14] for type $A$. Since then, cluster-tilted algebras have been the subject of several investigations, see, for instance, [13] [4, 5, 6, 7, 10, 9, 22]. In [13] the authors gave an explicit description for the quivers of cluster-tilted algebras of finite representation type and as a consequence they showed that a (basic) cluster-tilted algebra of finite type is uniquely determined by its quiver. In [10] Buan and Vatne gave

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a criterion to decide whether two cluster-tilted algebras of type \(A\) are themselves derived equivalent or not. For type \(E\) the same work was done by Bastian, Holm and Ladkani in [9]. For type \(D\) Vatne gave a complete description of the cluster-tilted algebras [22].

Our purpose in this paper is to study the open (or nearly) Frobenius structures that a cluster-tilted algebra of finite type can admit. This include examples of gentle algebras having oriented cycles which generalize the main result of [2].

We now state the main results of this paper (for the definitions of the terms used, we refer the reader to sections 1 and 2 below).

**Theorem A.** If \(A\) is a cluster tilted algebra of type \(A\), then \(A\) has finite Frobenius dimension. Moreover,

\[
\text{Frobdim}(A) = 2B + \sum_{b \in \mathbb{V}_d} \ell_{\rightarrow b} \ell_{\leftarrow b}
\]

**Theorem B.** If \(A\) is a cluster tilted algebra of type \(D\), then \(A\) has finite Frobenius dimension.

**Theorem C.** If \(A\) is a non hereditary cluster tilted algebra of type \(E_6\), then \(\text{Frobdim}(A) \geq 1\).

In particular we found a large number of examples of algebras that can be endowed with open Frobenius structures.

The paper is organized as follows: In section 1 we recall facts about quivers and algebras, cluster-tilted algebras and open Frobenius algebras. We use this section to fix some notation. Also we establish the facts about cluster tilted algebras of finite type that will be used in the sequel. In section 2 we compute the Frobenius dimension for cluster tilted algebra of type \(A\). We finish section 2 with some consequences, among which, we show that the Frobenius dimension is not invariant under sink/source mutations and the fact of have non-zero Frobenius dimension neither. Section 3 is devoted to compute the Frobenius dimension for cluster tilted algebras of type \(D\) or find a lower bound. Finally, in section 4 we prove that cluster tilted algebras of type \(E_6\) admit at least one non-trivial open Frobenius structure.

## 1. Preliminaries

### 1.1. Quivers and Algebras.**

While we briefly recall some concepts concerning bound quivers and algebras, we refer the reader to [3] or [8], for instance, for unexplained notions.

Let \(k\) be an algebraically closed field. A quiver \(Q\) is the data of two sets, \(Q_0\) (the vertices) and \(Q_1\) (the arrows) and two maps \(s, t: Q_1 \to Q_0\) that assign to each arrow \(\alpha\) its source \(s(\alpha)\) and its target \(t(\alpha)\). We write \(\alpha: s(\alpha) \to t(\alpha)\). If \(\beta \in Q_1\) is such that \(t(\alpha) = s(\beta)\) then the composition of \(\alpha\) and \(\beta\) is the path \(\alpha \beta\). This extends naturally to paths of arbitrary positive length. The path algebra \(kQ\) is the \(k\)-algebra whose basis is the set of all paths in \(Q\), including one stationary path \(e_x\) at each vertex \(x \in Q_0\), endowed with the multiplication induced from the composition of paths. In case \(|Q_0|\) is finite, the sum of the stationary paths - one for each vertex - is the identity.

If the quiver \(Q\) has no oriented cycles, it is called acyclic. A relation in \(Q\) is a \(k\)-linear combination of paths of length at least 2 sharing source and target. A relation which is a path is called monomial, and the relation is quadratic if the paths appearing in it have all length 2. Let \(\mathcal{R}\) be a set of relations. Given \(\mathcal{R}\) one can consider the two-sided ideal of \(kQ\) it generates \(I = \langle \mathcal{R} \rangle \subseteq \langle Q_1 \rangle^2\). It is called admissible if there exists a natural number \(r \geq 2\) such that \(\langle Q_1 \rangle^r \subseteq I\). The pair \((Q, I)\) is a bound quiver, and associated to it is the algebra \(A = kQ/I\). It is known that any finite dimensional basic algebra over an algebraically closed field is obtained in this way, see [3], for instance.

We are interested in a particular family of path algebras, called cluster-tilted algebras.
1.2. Cluster-tilted Algebras. Let $k$ be an algebraically closed field. We consider connected hereditary finite dimensional $k$-algebras. Any such algebra $H$ is Morita equivalent to a path algebra $kQ$ for some finite quiver $Q$. Furthermore, we assume $H$ is of finite representation type, that is, there is only a finite number of indecomposable objects, up to isomorphism, in the category $\text{mod} \ H$ of finitely generated $H$-modules. It is well known that this is the case if and only if the underlying graph of $Q$ is a Dynkin graph $\mathcal{A}, \mathcal{D}$ or $\mathcal{E}$.

Let $\mathcal{D}$ be the bounded derived category $\mathcal{D}^b(H)$. It is equipped with a shift functor $[1]$ and a translation functor $\tau$ with quasi-inverse $\tau^{-1}$. The composition $F = \tau^{-1}[1]$ is an auto-equivalence in $\mathcal{D}$. Then we can define the cluster category as the orbit category $\mathcal{C} := \mathcal{D}/F$. An object $T$ of $\mathcal{C}$ is called a tilting object if $\text{Ext}^1_\mathcal{C}(T, T) = 0$ and $T$ is maximal with respect to this property. The endomorphism algebra $\text{End}_\mathcal{C}(T)$ is called a cluster-tilted algebra. See [11, 12] for more details.

When $Q$ is a Dynkin quiver of types $\mathcal{A}$, $\mathcal{D}$ or $\mathcal{E}$, the corresponding cluster-tilted algebras are said to be of Dynkin type. These algebras have been investigated in [12], where it is shown that they are algebras and moreover they can be defined by using only zero and commutativity relations that can be extracted from their quivers in an algorithmic way. The possible quivers are precisely the quivers in the mutation class of the Dynkin quiver. By a result of Fomin and Zelevinsky [17], the mutation class of a Dynkin quiver is finite. Moreover, the quivers in the mutation classes of Dynkin quivers are explicitly known; for type $\mathcal{A}$ they can be found in [10], for type $\mathcal{D}$ in [22] and for type $\mathcal{E}$ they can be enumerated using a computer, for example by the Java applet [20].

1.3. Open Frobenius algebras. The Frobenius algebras without counit are called open Frobenius algebras. They give a natural generalization of Frobenius algebras. We briefly recall some concepts concerning open Frobenius algebras.

An algebra $A$ with multiplication $m : A \otimes A \to A$ is an open Frobenius algebra if it admits a linear map $\Delta : A \to A \otimes A$ such that $\Delta$ is an $A$-bimodule morphism, i.e.,

$$(m \otimes 1)(1 \otimes \Delta) = \Delta \circ m = (1 \otimes m)(\Delta \otimes 1)$$

Observe that any open Frobenius coproduct in $A$ is determined by its value on the unit $1$ of $A$. That is, if $\Delta : A \to A \otimes A$ is an open Frobenius coproduct, then $\Delta(x) = (x \otimes 1)\Delta(1) = \Delta(1)(1 \otimes x)$ for all $x \in A$.

If $A = kQ/I$ is a path algebra, the stationary paths $e_x$ with $x \in Q_0$ are idempotents, then $\Delta(e_x) = (e_x \otimes 1)\Delta(e_x) = \Delta(e_x)(1 \otimes e_x)$. If in addition $|Q_0|$ is finite the sum $\sum_{x \in Q_0} e_x = 1$.

Following [15] the Frobenius space associated to an algebra $A$ is the $k$-vector space $\mathcal{E}$ of all possible coproducts $\Delta$ that make $A$ into an open Frobenius algebra. Its dimension over $k$ is called the Frobenius dimension of $A$, i.e., $\text{Frobdim}(A) = \dim_k(\mathcal{E})$.

2. Open Frobenius Cluster-Tilted Algebras of type $\mathcal{A}$

Let $\mathcal{M}_A^A$ be the mutation class of $\mathcal{A}_k$. This is the set of all connected quivers with $k$ vertices that satisfy the following [10]:

- all non-trivial cycles are 3-cycles,
- every vertex has valency at most four,
- if a vertex has valency four, then two of its adjacent arrows belong to one 3-cycle, and the other two belong to another 3-cycle,
- if a vertex has valency three, then two of its adjacent arrows belong to a 3-cycle, and the third arrow does not belong to any 3-cycle.

By a 3-cycle we mean an oriented cycle of length 3. Then $A = kQ/I$ is a cluster-tilted algebra of type $\mathcal{A}_k$ if and only if $Q \in \mathcal{M}_A^A$ and every 3-cycle is saturated (i.e., the composition of two consecutive arrows belong to the ideal $I$) [12].

2.1. Definition. Given $Q \in \mathcal{M}_A^A$ a subquiver of $Q$ is said to be a tail if is a Dynkin diagram of type $\mathcal{A}_n$ with $n \geq 2$ maximal and any orientation.
Throughout this section \(Q\) is a quiver in \(M^0_{\mathbb{A}}\), every 3-cycle is saturated and \(\Delta\) is any open Frobenius structure for the cluster tilted algebra associated.

A tail with the orientation \(v_1 \xrightarrow{\alpha_1} v_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n-1} v_n\) will be called a lineal tail. According to [2] Lemma 1] if \(Q\) is a lineal tail, \(A = kQ\) admits only one open Frobenius structure (called the lineal structure) given by:

\[
\begin{align*}
\Delta(e_{v_1}) &= \alpha_1 \cdots \alpha_{n-1} \otimes e_{v_1} \\
\Delta(e_{v_i}) &= \alpha_i \cdots \alpha_{n-1} \otimes \alpha_1 \cdots \alpha_{i-1} \\
\Delta(e_{v_n}) &= e_{v_n} \otimes \alpha_1 \cdots \alpha_{n-1}
\end{align*}
\]

2.2. **Lemma.** Let \(\mathcal{T}\) be a tail of \(Q\) having a source or a sink of valency two. Then \(\Delta(e_v) = 0\) for all \(v \in \mathcal{T}_0\).

**Proof.** It is a direct consequence of [2] Lemma 2]. \(\square\)

Summarizing we have the following result for hereditary cluster tilted algebras of type \(\mathbb{A}\).

2.3. **Proposition.** An hereditary cluster tilted algebra \(A = kQ\) of type \(\mathbb{A}\) has non zero Frobenius dimension if and only if \(Q\) is a linearly oriented tail.

**Proof.** It is enough to observe that the quiver of an hereditary cluster tilted algebras of type \(\mathbb{A}\) is a tail. Then the result follows from [2] Lemma 1] and previous lemma. \(\square\)

Now, we are interested in compute the Frobenius dimension for non hereditary cluster tilted algebras of type \(\mathbb{A}\).

2.4. **Definition.** Given \(A = kQ/I\) we say that

a) a vertex \(v \in Q_0\) is a special vertex if \(v\) has valency two and there are \(\alpha, \beta \in Q_1\) with \(s(\alpha) = t(\beta) = v\) and \(\alpha \beta \in I\). Let denote by \(V_S\) the set of all special vertices.

b) a basis path \(P: v \rightsquigarrow v'\) is a path \(P: v \rightsquigarrow v'\) such that \(v, v'\) are or a special vertex or a source or a sink of valency one. We denote by \(B\) the set of all basis paths.

The following figure show the three types of basis paths \(P: 1 \rightsquigarrow n\).

![Types of basis paths](image)

**Figure 1.** Types of basis paths

Given a vertex \(b\) in a quiver \(Q\) we will use the symbol \(\ell_{\rightarrow b}\) to denote the length of the largest path with target \(b\) and \(\ell_{\leftarrow b}\) the length of the largest path with source \(b\).

We start describing a coproduct \(\Delta\) over the vertices of an arbitrary basis path of any type and computing the corresponding Frobenius dimension.

2.5. **Lemma.** Let \(A\) be the algebra given by the bound quiver

![Diagram of quiver](image)

Then \(\text{Frobdim}(A) = 2 + \ell_{\rightarrow b} \ell_{\rightarrow c} + \ell_{\leftarrow b} \ell_{\leftarrow c}\).
Proof. Every idempotent \( e_x \) satisfies \( \Delta(e_x) = (e_x \otimes 1)\Delta(e_x)(1 \otimes e_x) \). Then, the coproduct in \( e_1 \) and \( e_b \) are given by:

\[
\Delta(e_1) = B e_1 \otimes e_1 + \sum_{i=1}^{n} B e_1 \otimes \alpha_i \otimes e_1 + F \alpha_1 \cdots \alpha_n \gamma_1 \otimes e_1 \quad \text{and} \\
\Delta(e_b) = A e_b \otimes e_b + B \gamma_2 \otimes e_b + C e_b \otimes \gamma_1 + \sum_{i=1}^{n} C e_b \otimes \alpha_i \cdots \alpha_n \gamma_1 + \sum_{i=1}^{n} D \gamma_2 \otimes \alpha_i \cdots \alpha_n \gamma_1 + D \gamma_2 \otimes \gamma_1.
\]

Using that \( \Delta(\alpha_1 \cdots \alpha_n \gamma_1) = \Delta(e_1 \alpha_1 \cdots \alpha_n \gamma_1) = \Delta(\alpha_1 \cdots \alpha_n \gamma_1 e_b) \) and \( \gamma_1 \gamma_2 = 0 \) we deduce that \( F = C_1 \) and \( A = A' = C = B_i = C_j = 0 \) for \( i = 1, \ldots, n \) and \( j = 2, \ldots, n \). Therefore

\[
\Delta(e_1) = Fa_1 \cdots \alpha_n \gamma_1 \otimes e_1 \\
\Delta(e_b) = B \gamma_2 \otimes e_b + F e_b \otimes \alpha_1 \cdots \alpha_n \gamma_1 + \sum_{i=1}^{n} D \gamma_2 \otimes \alpha_i \cdots \alpha_n \gamma_1 + D \gamma_2 \otimes \gamma_1
\]

A general expression for \( \Delta(e_i) \) is \( \Delta(e_i) = \sum_{\rho, \eta} G_{\rho, \eta} \rho \otimes \eta \) with \( s(\rho) = t(\eta) = i \). Using that \( \Delta(\alpha_1 \cdots \alpha_i) = \Delta(e_1 \alpha_1 \cdots \alpha_i) = \Delta(\alpha_1 \cdots \alpha_i e_i) \) we deduce that \( \Delta(e_i) = Fa_i \cdots \alpha_n \gamma_1 \otimes \alpha_1 \cdots \alpha_{i-1} \) for \( i = 2, \ldots, n \). Similarly, \( \Delta(e_a) = F \gamma_1 \otimes \alpha_1 \cdots \alpha_n \).

Finally \( \Delta(e_c) = \sum_{\rho, \eta} G_{\rho, \eta} \rho \otimes \eta \) with \( s(\rho) = t(\eta) = c \). Comparing the expressions \( \Delta(\gamma_2 e_c) \) and \( \Delta(e_c \gamma_2) \) we obtain that \( \Delta(e_c) = Be_c \otimes \gamma_2 + E \gamma_3 \otimes \gamma_2 \).

Observe that \( \ell_{+} \otimes \ell_{-} = 1 = \sharp\{E\}, \ell_{\rightarrow} \otimes \ell_{\leftarrow} = \sharp\{D, D\} \) and we have two independent linear structures associated to the scalars \( B \) and \( F \) which completes the proof. \( \square \)

2.6. Corollary. Let \( A \) be an algebra as above such that in the tail \( T: 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\cdots} n \xrightarrow{\alpha_n} a \) there is a vertex \( 2 \leq i \leq n \) which is a source or a sink. Then \( \Delta_{ij} \equiv 0 \) and Frobdim(\( A \)) = \( 1 + \ell_{+} + \ell_{-} \).

Proof. From lemma \ref{2.10} it is clear that \( \Delta(e_i) = 0 \) for \( i = 1, \ldots, n \). The coproduct \( \Delta \) could change for the vertices \( a, b \) and \( c \). There is no loss of generality in assuming \( \alpha_n : n \rightarrow a \), then \( 0 = \Delta(e_n a_n) = \Delta(\alpha_n) = \Delta(\alpha_n e_a) \) implies \( \Delta(e_a) = 0 \) because \( \alpha_n \mu \neq 0 \) for all \( \mu \) starting at \( a \). Similarly we can see that \( \Delta(e_c) \) does not change and \( \Delta(e_b) = B \gamma_2 \otimes e_b + \sum_{\mu, \eta} D_{\mu, \eta} \gamma_2 \otimes \mu \gamma_1 \); with \( \mu \neq \gamma_3 \). \( \square \)

2.7. Corollary. Let \( A \) be the algebra given by the bound quiver

\[
\begin{array}{ccccccccccc}
1 & \xrightarrow{\alpha_1} & 2 & \xrightarrow{\alpha_2} & 3 & \cdots & n & \xrightarrow{\alpha_n} & a & \xrightarrow{\gamma_1} & b & \xrightarrow{\gamma_2} & \gamma_3 & \xrightarrow{\delta_1} & n+1 & \cdots & n+m-1 & \xrightarrow{\delta_m} & n+m
\end{array}
\]

Then Frobdim(\( A \)) = \( 2 + \ell_{\rightarrow} + \ell_{\leftarrow} \).

Proof. Analysis similar to that in the proof of lemma \ref{2.10} shows that

\[
\Delta(e_1) = Fa_1 \cdots \alpha_n \gamma_1 \otimes e_1 \\
\Delta(e_i) = Fa_i \cdots \alpha_n \gamma_1 \otimes \alpha_1 \cdots \alpha_{i-1}; \quad \text{for} \quad i = 2, \ldots, n. \\
\Delta(e_a) = F \gamma_1 \otimes \alpha_1 \cdots \alpha_n \\
\Delta(e_b) = B \gamma_2 \delta_1 \cdots \delta_m \otimes e_b + F e_b \otimes \alpha_1 \cdots \alpha_n \gamma_1 + \sum_{\mu, \eta} D_{\mu, \eta} \gamma_2 \otimes \mu \gamma_1; \quad \text{with} \quad \mu, \eta \neq \gamma_3 \\
\Delta(e_c) = B \delta_1 \cdots \delta_m \otimes \gamma_2 \\
\Delta(e_{n+j}) = B \delta_{j+1} \cdots \delta_m \otimes c_2 \delta_1 \cdots \delta_j; \quad \text{for} \quad j = 1, \ldots, m-1. \\
\Delta(e_{n+m}) = B \gamma_2 \delta_1 \cdots \delta_m \otimes e_{n+m}
\]

Then we have a linear structure for both basis path \( \alpha_1 \cdots \alpha_n \gamma_1 \) and \( \gamma_2 \delta_1 \cdots \delta_m \) (associated to the scalars \( F \) and \( B \) respectively). The special vertex \( b \) add \( \sharp\{D_{\mu, \eta}\} = \ell_{\rightarrow} + \ell_{\leftarrow} \). \( \square \)

2.8. Lemma. Let \( A \) be the algebra given by the bound quiver
Then $\text{Frobdim}(\mathcal{A}) = 3 + \ell \rightarrow \ell \rightarrow b + \ell \rightarrow_1 \ell \rightarrow_1 + \ell \rightarrow_c \ell \rightarrow c + \ell \rightarrow_d \ell \rightarrow d$.

**Proof.** First observe that if we delete the arrows $\beta$ and $\gamma$ we get an algebra like the lemma above. Clearly $\Delta$ only can change for the vertices $e_1$ and $e_2$. Then
\[
\Delta(e_i) = F\alpha_i \cdots \alpha_n \gamma_1 \otimes \alpha_1 \cdots \alpha_{i-1}, \text{ for } i = 3, \ldots, n.
\]
\[
\Delta(e_1) = F\gamma_1 \otimes \alpha_1 \cdots \alpha_n.
\]
\[
\Delta(e_b) = B\gamma_2 \otimes e_b + Fe_b \otimes \alpha_1 \cdots \alpha_n \gamma_1 + \sum_{i=1}^n D_i \gamma_2 \otimes \alpha_i \cdots \alpha_n \gamma_1 + E\gamma_2 \otimes \gamma_1.
\]
\[
\Delta(e_c) = Be_c \otimes \gamma_2 + D\gamma_3 \otimes \gamma_2.
\]

The equality $\Delta(e_2 \alpha_2) = \Delta(\alpha_2 e_3) = F\alpha_2 \cdots \alpha_n \gamma_1 \otimes \alpha_1 \alpha_2$ implies $\Delta(e_2) = F\alpha_2 \cdots \alpha_n \gamma_1 \otimes \alpha_1$.

Similarly, comparing $\Delta(e_1)$ and $\Delta(e_b)$ using the path $\alpha_1 \cdots \alpha_n \gamma_1$ we obtain that
\[
\Delta(e_1) = F\alpha_1 \cdots \alpha_n \gamma_1 \otimes e_1 + \sum_{i=1}^n D'_\alpha \alpha_1 \cdots \alpha_i \otimes \beta + I\alpha_1 \cdots \alpha_n \gamma_1 \otimes \beta + H e_1 \otimes \beta.
\]

A general expression for $\Delta(e_d) = J\beta \otimes \gamma + K\beta \otimes e_d + L e_d \otimes \gamma$. Comparing with $\Delta(e_2)$ using the arrow $\gamma$ we obtain $L = 0$ and comparing with $\Delta(e_1)$ using the arrow $\beta$ we get $H = K$. Finally $\Delta(e_d) = J\beta \otimes \gamma + H\beta \otimes e_d$. We conclude that $\text{Frobdim}(\mathcal{A}) = 5 + 2(n + 1) = 3 + \ell \rightarrow_1 \ell \rightarrow b + \ell \rightarrow_1 \ell \rightarrow_1 + \ell \rightarrow_c \ell \rightarrow c + \ell \rightarrow_d \ell \rightarrow d$.

**2.9. Corollary.** Let $\mathcal{A}$ be an algebra as above such that the tail $\mathcal{T}$: $2 \overset{\alpha_1 \cdots \alpha_n}{\longrightarrow} a$ has a vertex $3 \leq i \leq n$ which is a source or a sink. Then $\Delta|_\gamma \equiv 0$ and $\text{Frobdim}(\mathcal{A}) = \ell \rightarrow b + \ell \rightarrow_1 + 4$.

**Proof.** Analysis similar to that in the proof of corollary 2.6 shows that $\Delta|_\beta \equiv 0$, $\Delta(e_c)$ and $\Delta(e_d)$ do not change and
\[
\Delta(e_b) = B \gamma_2 \otimes e_b + \sum_{\mu / (\mu) = a} D_{\mu} \gamma_2 \otimes \mu \gamma_1 ; \ \mu \neq \gamma_3
\]
\[
\Delta(e_1) = H e_1 \otimes \beta + \sum_{\eta / (\eta) = 2} D_{\eta} \alpha_1 \eta \otimes \beta ; \ \eta \neq \gamma_3
\]

which completes the proof.

In some cases we do not have basis paths and we can find the following situation.

**2.10. Lemma.** Let $\mathcal{A}$ be the algebra given by the bound quiver:

![Diagram](attachment:image.png)

Then $\Delta \equiv 0$ and $\text{Frobdim}(\mathcal{A}) = 0$.

**Proof.** It follows by comparing $\Delta(e_1)$ with $\Delta(e_{n+1})$ using the path $\alpha_n \cdots \alpha_1$. By symmetry we obtain $\Delta(e_{n+2}), \Delta(e_{n+m+1}), \Delta(e_c)$ and $\Delta(e_{c_{i+1}})$. Then, compare $\Delta(e_{n+2})$ with $\Delta(e_{n+1})$ using the arrow $\alpha_{n+1}$ to find that $\Delta(e_i) = 0$ for all $1 \leq i \leq n + m + 1$. Finally, conclude that $\Delta(e_c) = \Delta(e_{c_j}) = 0$ for all $3 \leq j \leq t + 1$. □
Observe that if \((Q, I)\) is the bound quiver of a cluster tilted algebra of type \(A\) without basis paths and without sources or sinks of valency two, then \((Q, I)\) has a subquiver as above (or its dual).

Now we want to see what happens if we add arrows to a basis path.

2.11. Definition. Let \(\alpha_i : i \rightarrow i+1\) be an arrow of a quiver \(Q\). We say that \(Q'\) is obtained by adding to \(Q\) a tail through the arrow \(\alpha_i\) if \(Q'\) has the form

\[
\begin{array}{c}
\bullet \bullet \bullet i-1 \\
\alpha_i \\
\bullet \bullet \bullet i+1 \\
\beta \\
\bullet \bullet \bullet i \rightarrow i+1 \\
\gamma \\
\bullet \bullet \bullet i+2 \\
\beta_2 \\
\bullet \bullet \bullet i+1 \\
\beta_{t-1} \\
\bullet \bullet \bullet i \\
\beta_t \\
\end{array}
\]

with the cycle \(\alpha_i \gamma \beta\) saturated.

2.12. Lemma. Let \(A\) be the algebra given by the quiver \(Q\):

\[
\begin{array}{c}
\bullet \bullet \bullet i-1 \\
\alpha_i \\
\bullet \bullet \bullet i+1 \\
\beta \\
\bullet \bullet \bullet i \rightarrow i+1 \\
\gamma \\
\bullet \bullet \bullet i+2 \\
\beta_2 \\
\bullet \bullet \bullet i+1 \\
\beta_{t-1} \\
\bullet \bullet \bullet i \\
\beta_t \\
\end{array}
\]

with \(t \geq 2\). If \(\Delta(\beta) = \Delta(\gamma) = 0\), then \(\Delta(e_{d_i}) = 0\) for all \(i = 1, \ldots, t\).

Proof. By 2.2 it suffices to consider the case where the tail \(d_1 \beta_2 \cdots d_{t-1} \beta_t \rightarrow i \rightarrow i+1 \rightarrow i+2\) is linearly oriented. We can assume that there is a path \(\beta_2 \cdots \beta_t\) from \(d_1\) to \(d_t\). Since \(\Delta(\gamma) = 0\) we have \(0 = \Delta(\gamma \beta_2 \cdots \beta_t) = \Delta(\gamma \beta_2 \cdots \beta_t e_{d_i})\) which implies \(\Delta(e_{d_i}) = 0\) for all \(i \neq 1\). For \(d_1\) compare \(\Delta(e_{d_i})\) with \(\Delta(e_{d_i}) = 0\) using the arrow \(\beta_2\).

If an algebra as above has \(t = 1\) we have the following particular case:

\[
\begin{array}{c}
\bullet \bullet \bullet i-1 \\
\alpha_i \\
\bullet \bullet \bullet i+1 \\
\beta \\
\bullet \bullet \bullet i \rightarrow i+1 \\
\gamma \\
\bullet \bullet \bullet i+2 \\
\end{array}
\]

2.13. Lemma. Let \(A\) be the algebra as above with \(\Delta(\beta) = \Delta(\gamma) = 0\), then

\[
\Delta(e_{d_i}) = \sum_{s(\rho) = i} \sum_{t(\mu) = i+1} A_{\rho \mu} \beta \rho \otimes \mu \gamma.
\]

In particular, if \(i \rightarrow i \rightarrow i \rightarrow i+2\) is linearly oriented \(\Delta(e_{d_i}) = A \beta \otimes \gamma\).

Proof. The coproduct \(\Delta\) at the vertex \(e_d\) has the general expression \(\Delta(e_{d_i}) = \sum_{\sigma, \eta} G_{\sigma, \eta} \sigma \otimes \eta\) with \(s(\sigma) = t(\eta) = d_1\). As \(0 = \Delta(\gamma) = \Delta(\gamma e_{d_i})\) and \(\gamma \beta = 0\) only the terms with \(\sigma = \beta \rho\) where \(s(\rho) = i\) have a non zero \(G_{\sigma, \eta}\). Analogously \(0 = \Delta(\beta)\) implies that only the terms with \(\eta = \mu \gamma\) where \(t(\mu) = i + 1\) have a non zero \(G_{\sigma, \eta} = A_{\rho \mu}\).
2.14. Remark. Since $\mathcal{A}_{pq} : s(p) = i, t(\mu) = i + 1 = \ell \rightarrow d_1 \ell + \ell d_i$, every special vertex $d_1$ add $\ell \rightarrow d_1 \ell + \ell d_i$ to the Frobenius dimension.

2.15. Proposition. Assume that we know $\Delta$ for the algebra $A$ given by the quiver $Q$:

\[ \begin{array}{c}
\alpha_i \rightarrow \alpha_{i+1} \\
\beta_i \rightarrow \beta_{i+1}
\end{array} \]

with $i, i + 1$ vertices of valency 2. Let $Q'$ be the quiver obtained by adding to $Q$ a tail through the arrow $\alpha_i$. Let $\Delta'$ be a coproduct for the algebra $A'$ associated to $Q'$. Then $\Delta'(e_v) = 0$ for all $v \in Q'_0 \setminus Q_0$.

Moreover, Frobdim($A$) = Frobdim($A'$).

Proof. If we prove that $\Delta'(\gamma) = 0$, the first assertion follows from lemma 2.12. We have $\Delta'(e_{i+1}) = \sum G_{\sigma,\eta} \sigma \otimes \eta$ with $s(\sigma) = t(\eta) = i + 1$. Since the vertex $i + 1$ has valency two in $Q$, we have $\eta = \eta' \alpha_i$ with $t(\eta') = i$ or $\eta = e_{i+1}$. We claim that every $G_{\sigma,\eta}$ has to be zero. Indeed, $\Delta'(\alpha_i) = \sum G_{\sigma,\eta} \alpha_i \otimes \eta$. Since $t(\eta') = i$, which do not have any term $\alpha_i \otimes \eta' \alpha_i$ and, in consequence, $\Delta'(\gamma) = 0$. Similarly we prove that $\Delta'(\beta) = 0$.

Now, we want to show that $\Delta(e_v) = \Delta'(e_v)$ for all $v \in Q_0$. Clearly, since $\beta \alpha_i = \alpha_i \gamma = 0$, $\Delta(e_v) = \Delta'(e_v)$ for all $v \in Q_0 \setminus \{i, i + 1\}$. For $v = i$ observe that $\Delta'(e_i) = \Delta(e_i) + \sum G_{\sigma,\eta} \sigma \otimes \eta \beta$ with $s(\sigma) = i$ and $t(\eta) = s(\beta)$. Then,

\[
\Delta(\alpha_{i-1}) = \Delta(e_{i-1})(1 \otimes \alpha_{i-1}) = \Delta'(e_{i-1})(1 \otimes \alpha_{i-1})
\]

\[
= \Delta'(\alpha_{i-1}) = (\alpha_{i-1} \otimes 1) \Delta'(e_i)
\]

\[
= \Delta(\alpha_{i-1}) + \sum_{\sigma,\eta} G_{\sigma,\eta} \alpha_{i-1} \sigma \otimes \eta \beta
\]

which implies that every scalar $G_{\sigma,\eta} = 0$ and consequently $\Delta'(e_i) = \Delta(e_i)$. Similar arguments applies for $v = i + 1$. Finally, Frobdim($A$) = Frobdim($A'$).

Recall that a gluing of two quivers with relations $Q$ and $Q'$ with respect to a vertex $v$ in $Q$ and $v'$ in $Q'$ is obtained by taking the disjoint union $Q \sqcup Q'$ and identifying the vertices $v$ and $v'$. The relations are induced from those of $Q$ and $Q'$ (i.e. there are no additional relations).

The following lemma show what happens if we glue a saturated cycle at the 'final' vertex of a tail.

2.16. Lemma. Let $A$ be the algebra given by the bound quiver

\[ \begin{array}{c}
\alpha \rightarrow \beta \\
\gamma \rightarrow \delta
\end{array} \]

Assume that $\Delta(\beta) = \Delta(\gamma) = 0$, then $\Delta(e_i) = 0$ for all $i = 1, \cdots, m$; $\Delta(e_{m+1}) = \sum G_{\eta} \varepsilon \otimes \eta$ and $\Delta(e_d) = J \mu \otimes \varepsilon + G_{\varepsilon m+1} e_d \otimes \varepsilon$.

Proof. As in lemma 2.12 $\Delta(e_i) = 0$ for all $i = 1, \cdots, m$. For $e_{m+1}$,

\[
\Delta(e_{m+1}) = \sum_{t(\eta)=m+1} G_{\eta} \varepsilon \otimes \eta + \sum_{t(\eta)=m+1} G'_{\eta} \varepsilon_{m+1} \otimes \eta.
\]
Comparing with $\Delta(e_m) = 0$ using the arrow $\beta_{m+1}$ and the fact that $\beta_{t+1} = 0$, we deduce that every scalar $G_{\eta}$ has to be zero. Then $\Delta(e_m) = \sum_{t(\eta) = m+1} G_{\eta} \varepsilon \otimes \eta$.

A general expression for $\Delta(e_d) = J\mu \otimes \varepsilon + K\mu \otimes e_d + Le_d \otimes \varepsilon$. Comparing with $\Delta(e_{m+1})$ using the arrow $\varepsilon$ we obtain $L = G_{m+1}$ and comparing with $\Delta(e_{m+1})$ using the arrow $\mu$ we get $K = 0$. Finally $\Delta(e_d) = J\mu \otimes \varepsilon + G_{m+1} e_d \otimes \varepsilon$.

\[ \square \]

2.17. Corollary. Let $Q$ be the following quiver with $a, a + 1$ vertices of valency 2.

\[
\begin{array}{cccc}
& a & \alpha_a & a + 1 & \ldots \\
\end{array}
\]

Let $Q'$ be the quiver obtained by adding to $Q$ a tail 1 \ldots \ m through the arrow $\alpha_a$ and gluing a saturated cycle $m$ \rightarrow through the arrow of the tail.

Name $A$ the algebra given by the quiver $Q$ and $A'$ the algebra associated to $Q'$. Then,

\[ \text{Frobdim}(A') = \text{Frobdim}(A) + \ell_{\rightarrow m+1} + 2. \]

Proof. According to proposition 2.15 and lemma 2.16 (or its dual) it is enough to observe that the number of different $G_{\eta}$ at lemma 2.16 is $|kQ' e_{m+1}| = \ell_{\rightarrow m+1} + 1$ and $\Delta(e_d)$ add one more for the scalar $J$.

Observe that $\ell_{\rightarrow m+1} + 2 = \ell_{\rightarrow m+1} \ell_{\rightarrow m+1} + \ell_{\rightarrow d} \ell_{\rightarrow d} + 1$.

Now we are ready to state and prove the main result of this section.

2.18. Theorem. If $A$ is a cluster tilted algebra of type $\Delta$, then $A$ has finite Frobenius dimension. Moreover,

\[ \text{Frobdim}(A) = 2B + \sum_{b \in V_\Delta} \ell_{\rightarrow b, \ell_{\rightarrow b}}. \]

Proof. We first claim that for every basis path we have a linear structure. Let $P : 1 \rightsquigarrow n$ be a basis path. If $P$ is of the first type 2.15 Lemma 1 implies that $\Delta|_P$ is the linear structure. If $P$ is of the second type (or third type) lemma 2.16 (or lemma 2.18 respectively) shows that the coproduct $\Delta$ restricted to $P$ is the linear structure. Observe that if $A$ does not have basis paths, according to lemmas 2.2 and 2.10 $\Delta$ restricted to $Q_0 \setminus V_\Delta$ is zero.

Now assume that we have two basis paths $P$ and $P'$ that share a vertex $b$, then we have two cases:

1) $b = t(P) = s(P')$ is an special vertex: using corollary 2.7 we conclude that we have two independent linear structures, one for each basis path. Moreover, from the same result we obtain that the special vertex $b$ add $\ell_{\rightarrow b, \ell_{\rightarrow b}}$ to the Frobdim($A$).

2) $b$ is a vertex of valency four: we can assume that $P : \cdots \rightarrow c_1 \rightarrow b \rightarrow c_3 \cdots$ is of any type and let $\Delta_{P}$ be the coproduct restricted to the algebra $A'$ associated to the quiver of $P$ (i.e. a linear structure). Then we have the following three different situations depending on the type of $P'$.

a) $P' : f \rightsquigarrow d_2 \rightarrow b \rightarrow d_1 \rightsquigarrow g$ of the first type:

Then, a simple computation gives
\[ \Delta(e_f) = A\alpha\delta_3\beta_2\alpha' \otimes e_f \]
\[ \vdots \]
\[ \Delta(e_{d_2}) = A\delta_3\beta_2\alpha' \otimes \alpha \]
\[ \Delta(e_b) = A\beta_2\alpha' \otimes \alpha\delta_3 + \Delta_P(e_b) \]
\[ \Delta(e_{d_1}) = A\alpha' \otimes \alpha\delta_3\beta_2 \]
\[ \vdots \]
\[ \Delta(e_g) = A\varepsilon \otimes \alpha\delta_3\beta_2\alpha' \]

b) \( P' : f \sim d_2 \rightarrow b \rightarrow d_1 \) of the second type: \[
\begin{array}{ccccccc}
& & c_1 & & b & & c_3 & & \\
& \delta_2 & & \beta_2 & & \delta_3 & & \beta_3 \\
& & & d_1 & & & & \\
& & & d_2 & & & & \\
\end{array}
\]

Assume that \( \alpha : f \sim d_2 = \alpha_1 \cdots \alpha_n \). Then,
\[ \Delta(e_f) = A\alpha\delta_3\beta_2 \otimes e_f \]
\[ \vdots \]
\[ \Delta(e_{d_2}) = A\delta_3\beta_2 \otimes \alpha \]
\[ \Delta(e_b) = A\beta_2 \otimes \alpha\delta_3 + \Delta_P(e_b) \]
\[ \Delta(e_{d_1}) = A\varepsilon \otimes \alpha\delta_3\beta_2 + B_1\delta_2 \otimes \beta_2 + B_2\delta_2 \otimes \delta_3\beta_2 + \sum_{i=1}^{n} B_{3+n-i}\delta_2 \otimes \alpha_i \cdots \alpha_n\delta_3\beta_2 \]

A simple computation gives
\[ \Delta(e_b) = A\beta_2 \otimes \delta_3 + \Delta_P(e_b) \]
\[ \Delta(e_{d_1}) = A\varepsilon \otimes \delta_3\beta_2 + B\delta_2 \otimes \beta_2 + C\delta_2 \otimes \delta_3\beta_2 \]
\[ \Delta(e_{d_2}) = A\delta_3\beta_2 \otimes \varepsilon + D\delta_3 \otimes \beta_2 + E\delta_3\beta_2 \otimes \beta_3 \]

Then we conclude that \( P' \) add an independent linear structure. Also, we obtain that in the cases where \( d_i \) is an special vertex it add \( \ell_{-d_i}, \ell_{-d_i} \) to the Frobdim(\( A \)). In general, according to corollary \ref{cor:2.17} every special vertex \( d \) add \( \ell_{-d}, \ell_{-d} \) to the Frobdim(\( A \)).

It remaind to proof that \( \Delta(e_v) = 0 \) for all non special vertex \( v \) not belonging to any basis path. It follows from lemma \ref{lem:2.10} proposition \ref{prop:2.15} and corollary \ref{cor:2.17}.

\[ \square \]

2.19. Corollary. Let \( A \) be a cluster tilted algebra of type \( \mathbb{A} \). Then Frobdim(\( A \)) \( \geq 1 \) if and only if \( \not\exists \mathcal{V}_0 \neq \emptyset \).

2.20. Example. The algebra \( A \) given by the following bound quiver has Frobdim(\( A \)) = 0.
2.1. **Further consequences.** Given a quiver $Q$ without loops and 2-cycles and a vertex $k$, we denote by $\mu_k(Q)$ the Fomin-Zelevinsky quiver mutation [16] of $Q$ at $k$. Two quivers are called *mutation equivalent* if one can be reached from the other by a finite sequence of quiver mutations. In particular, two quivers are called *sink/source equivalent* if one can be obtained from the other by performing mutations only at vertices which are sinks or sources.

2.21. **Example.** This example shows that the Frobenius dimension is not invariant under sink/source mutations.

\[
\begin{array}{c}
1 \rightarrow 2 \rightarrow 3 \leftarrow 4 \\
\mu_4 \rightarrow
\end{array}
\]

\[
\text{Frobdim}(A) = 4 \quad \text{Frobdim}(\mu_4(A)) = 6
\]

2.22. **Example.** Moreover, the fact of having non-zero Frobenius dimension neither is an invariant under mutation as we can see at this example where the algebras $A$ and $A'$ are mutation equivalent.

\[
\begin{array}{c}
1 \leftarrow 2 \rightarrow 3 \rightarrow 4 \\
\mu_4 \rightarrow
\end{array}
\]

\[
\text{Frobdim}(A) = 0 \quad \text{Frobdim}(A') = 7
\]

In particular, we conclude that the Frobenius dimension is not a derived invariant.

From the examples we can observe that the exact number of the Frobenius dimension is not really interesting but we are still interested in knowing whatever or not a cluster tilted algebra has non-zero Frobenius dimension and equivalently admit at least one open Frobenius structure.

3. **Open Frobenius Cluster-Tilted Algebras of type $D$**

Following [22], we will describe the quivers of cluster-tilted algebras of type $D$. Let $\mathcal{M}_k^D$ be the mutation class of $A_k$. The union of all $\mathcal{M}_k^D$ for all $k$ will be denoted by $\mathcal{M}^D$. For a quiver $\Gamma$ in $\mathcal{M}^A$, we will say that a vertex $v$ is a *connecting vertex* if $v$ has valency 1 or if $v$ has valency 2 and $v$ belongs to a 3-cycle in $\Gamma$. Let $Q$ be a quiver with $n$ vertices having a full subquiver $\hat{Q}$ of one of the following four types:

| Type I | Type II | Type III | Type IV |
|--------|---------|----------|--------|
| ![Type I Diagram] | ![Type II Diagram] | ![Type III Diagram] | ![Type IV Diagram] |

For $\hat{Q}$ of type I, II or III, let $Q' = (Q \setminus \hat{Q}) \cup \{c\}$ and $Q'' = (Q \setminus \hat{Q}) \cup \{d\}$. For $\hat{Q}$ of type IV the cycle $\alpha_1 \alpha_2 \cdots \alpha_k$ is a directed $k$-cycle (called the *central cycle*), with $k \geq 3$. For each arrow $\alpha : a \rightarrow b$ in the central cycle, there may (and may not) be a vertex $c_{\alpha}$ which is not on the central cycle, such
that there is an oriented 3-cycle \( a \rightarrow b \rightarrow c_a \rightarrow a \). Such a 3-cycle will be called a spike. For each spike let \( Q^* \) be the quiver \((\hat{Q} \setminus \hat{Q}) \cup \{c_a\}\).

We say that \( Q \) is of:

- **Type I** if \( Q \) has a full subquiver \( \hat{Q} \) of type I, the vertices \( a \) and \( b \) have valency one and both are a sink (or a source), \( Q' \) is in \( \mathcal{M}_{t-2}^A \) and \( c \) is a connecting vertex for \( Q' \).
- **Type II** (or **Type III**) if \( Q \) has a full subquiver \( \hat{Q} \) of type II (or type III, respectively), the vertices \( a \) and \( b \) have valency 2, \( Q' \) and \( Q'' \) are both in \( \mathcal{M}^A \) and have \( c \) and \( d \) as connecting vertices respectively.
- **Type IV** if \( Q \) has a full subquiver \( \hat{Q} \) of type IV, \( Q^* \) is in \( \mathcal{M}^A \) and has the vertex \( c_a \) as a connecting vertex.

Observe that in types II, III and IV, the subquivers \( Q', Q'', Q^* \) can be in the set \( \mathcal{M}^A \), i.e. they can have only one vertex. We define \( \mathcal{M}_{t}^D \) (\( i \in \{ I, II, III, IV \} \)) to be the set of quivers \( Q \) belonging to the type \( i \) described above.

3.1. **Cluster-tilted algebras of type \( D \) and sub-type I.** According to [12], \( A = kQ/I \) is a cluster-tilted algebra of type \( D \) and sub-type I if and only if \( Q \in \mathcal{M}_{t}^D \) and every 3-cycle is saturated.

Given a vertex \( c \in Q_0 \) let \( \mathcal{B}_c \) be the set of all basis paths \( \mathcal{P} : v_1 \rightsquigarrow v_t \) such that \( v_i = c \) for some \( 1 \leq i \leq t \).

3.1. **Proposition.** Let \( A = kQ/I \) be a cluster-tilted algebra of type \( D \) and sub-type I, then

\[
\text{Frobenius dim}(A) = \sharp(\mathcal{B} \setminus \mathcal{B}_c) + \sum_{b \in \mathcal{V}_S} \ell_{\rightarrow b} \ell_{\leftarrow b}
\]

**Proof.** It is enough to observe that according to [2] Lemma 2 \( \Delta(e_c) = 0. \) Then \( \Delta |_{\mathcal{P}} = 0, \) for every basis path \( \mathcal{P} \in \mathcal{B}_c. \) Moreover, \( \Delta(e_a) = \Delta(e_b) = 0. \)

3.2. **Cluster-tilted algebras of type \( D \) and sub-type II.** \( A = kQ/I \) is a cluster-tilted algebra of type \( D \) and sub-type II if and only if \( Q \in \mathcal{M}_{t}^D \), \( \alpha \delta = \beta \gamma, \mu \alpha = \mu \beta = \delta \mu = \gamma \mu = 0 \) and every 3-cycle in \( Q' \cup Q'' \) is saturated [12].

We adapt the definition 2.4 to this case. We say that

- a vertex \( v \in Q_0 \) is an **special vertex** if \( v \) has valency two and there are \( \alpha, \beta \in Q_1 \) with \( s(\beta) = t(\alpha) = v \) and \( \alpha \beta \in I \) or \( v \in \{c, d\} \) and has valency three. Let denote by \( \mathcal{V}_S^* \) the new set of special vertices.
- an **extended basis path** is a path \( \mathcal{P} : v \rightsquigarrow v' \) such that \( v, v' \) are or an special vertex or a source or a sink of valency one. We denote by \( \mathcal{B}^* \) the set of all extended basis paths.

Given a vertex \( z \) we will denote by \( \sharp_{\rightarrow z} = \dim_k(e_z(kQ/I)) - 1 \) (or \( \sharp_{\leftarrow z} = \dim_k((kQ/I)e_z) - 1 \)) the number of non trivial paths with source (or target, respectively) \( z \).

3.2. **Proposition.** Let \( A = kQ/I \) be a cluster-tilted algebra of type \( D \) and sub-type II, then

\[
\text{Frobenius dim}(A) = \sharp(\mathcal{B}^* + \sum_{b \in \mathcal{V}_S^*} (\sharp_{\rightarrow b} \sharp_{\leftarrow b})
\]

**Proof.** First observe that in this case we possible add the vertices \( c \) and \( d \) to the set of special vertices that we already had for types \( A \) and \( D \) sub-type I. In this way we possible have new basis paths (called extended basis paths) that contribute also with linear structures. Then, as for the \( A \)-case, for every extended basis path we have a linear structure.

When we compute the Frobenius dimension and we look at an special vertex \( z \), we were interested in the number of non trivial paths with source (or target) \( z \). In other words, the numbers \( \dim_k(e_z(kQ/I)) - 1 \) and \( \dim_k((kQ/I)e_z) - 1 \). Cluster tilted algebras of types \( A \) are gentle which implies that those numbers
coincide with the length of the largest path with source (or target, respectively) \( z \). Finally, an adaptation of the formula for the \( A \)-case to this one gives our claim.

\[ \square \]

We illustrate the case with the following example.

3.3. Example. Let \( A \) be the algebra given by the following quiver

![Quiver Diagram]

bounded by the relations \( \alpha \delta = \beta \gamma, \mu \alpha = \mu \beta = \delta \mu = \gamma \mu = \phi_1 \alpha_1 = \alpha_1 \beta_1 = \beta_1 \phi_1 = 0 \).

A simple computation gives

\[
\begin{align*}
\Delta(e_b) &= 0 \\
\Delta(e_a) &= 0 \\
\Delta(e_{y_i}) &= 0; \quad 1 \leq i \leq n \\
\Delta(e_{x_j}) &= A \beta_{j+1} \cdots \beta_m \otimes \mu \beta_1 \cdots \beta_j \\
\Delta(e_c) &= A \beta_1 \cdots \beta_m \otimes \mu \\
\Delta(e_d) &= A \mu \beta_1 \cdots \beta_m \otimes e_d + B \mu \otimes \delta + C \mu \otimes \gamma + D \mu \otimes \alpha \delta + E \mu \otimes \alpha_1 \alpha \delta + \sum_{i=1}^{m} B_i \mu \beta_1 \cdots \beta_i \otimes \delta \\
&\quad + \sum_{i=1}^{m} C_i \mu \beta_1 \cdots \beta_i \otimes \gamma + \sum_{i=1}^{m} D_i \mu \beta_1 \cdots \beta_i \otimes \alpha \delta + \sum_{i=1}^{m} E_i \mu \beta_1 \cdots \beta_i \otimes \alpha_1 \alpha \delta
\end{align*}
\]

Then \( \text{Frobdim}(A) = 1 + 4(m + 1) \).

For the last two sub-types we are going to find a lower bound for the Frobenius dimension.

3.3. Cluster-tilted algebras of type \( D \) and sub-type III. \( A = kQ/I \) is a cluster-tilted algebra of type \( D \) and sub-type III if and only if \( Q \in \mathcal{M}_{III}^{D} \), \( \alpha \beta \gamma = \beta \gamma \delta = \gamma \delta \alpha = \delta \alpha \beta = 0 \) and every 3-cycle in \( Q' \cup Q'' \) is saturated [12].

3.4. Proposition. Let \( A = kQ/I \) be a cluster-tilted algebra of type \( D \) and sub-type III, then

\[ \text{Frobdim}(A) \geq 2 \]

Proof. It is enough to see that \( \Delta(e_a) = A \delta \alpha \otimes \beta \gamma + \sum_{\rho, \eta} G_{\rho, \eta} \rho \otimes \eta \) with \( s(\rho) = t(\eta) = a \) and \( \rho \neq \delta \alpha, \eta \neq \beta \gamma \). Since \( \beta \gamma \delta = \gamma \delta \alpha = 0 \) we are sure that \( A \neq 0 \).

In the same way, we can affirm that \( \Delta(e_b) = B \beta \gamma \otimes \delta \alpha + \sum_{\eta, \rho} G_{\eta, \rho} \eta \otimes \rho \) with \( t(\rho) = s(\eta) = b \) and \( \rho \neq \delta \alpha, \eta \neq \beta \gamma \). Since \( \alpha \beta \gamma = \delta \alpha \beta = 0 \) we are sure that \( B \neq 0 \). \( \square \)

The following example shows an algebra with Frobenius dimension exactly two.

3.5. Example. Let \( A \) be the algebra given by the following quiver
bounded by the relations $\alpha\beta\gamma = \beta\gamma\delta = \gamma\delta\alpha = \delta\alpha\beta = 0$, $\delta_1\gamma_1 = \gamma_1\phi_2 = \phi_2\delta_1 = 0$, $\phi_1\alpha_1 = \alpha_1\beta_1 = \beta_1\phi_1 = 0$.

A simple computation gives

$$
\begin{align*}
\Delta(e_d) &= \Delta(e_c) = 0 \\
\Delta(e_{y_i}) &= 0 ; 1 \leq i \leq n \\
\Delta(e_{x_i}) &= 0 ; 1 \leq i \leq m \\
\Delta(e_i) &= 0 ; 1 \leq i \leq n' \\
\Delta(e_z) &= 0 ; 1 \leq i \leq m' \\
\Delta(e_b) &= B\beta\gamma \otimes \delta\alpha \\
\Delta(e_a) &= A\delta\alpha \otimes \beta\gamma
\end{align*}
$$

Then $\text{Frobdim}(A) = 2$.

3.4. Cluster-tilted algebras of type $\mathbb{D}$ and sub-type IV. $A = kQ/I$ is a cluster-tilted algebra of type $\mathbb{D}$ and sub-type IV if and only if $Q \in M^D_{IV}$, every 3-cycle in $\cup_i Q^i$ is saturated and for each arrow $\alpha_i$ in the central cycle $\alpha_i+1 \cdots \alpha_k\alpha_1 \cdots \alpha_{i-1} = \beta_i\gamma_i$ if there exits $c_{\alpha_i}$ or $\alpha_i+1 \cdots \alpha_k\alpha_1 \cdots \alpha_{i-1} = 0$ otherwise. Indices are read modulo $k$. [12].

3.6. Proposition. Let $A = kQ/I$ be a cluster-tilted algebra of type $\mathbb{D}$ and sub-type IV with a central cycle of length $k$, then

$$
\text{Frobdim}(A) \geq k
$$

Proof. Let us start with the case where there are not spikes. Then, for each arrow $\alpha_i$ in the central cycle we have $\alpha_i+1 \cdots \alpha_k\alpha_1 \cdots \alpha_{i-1} = 0$. We affirm that $\Delta(e_2)$ has the term $A\alpha_2 \cdots \alpha_k \otimes \alpha_1$ with $A \neq 0$. Since $\alpha_1 \cdots \alpha_{k-1} = 0$ and $\alpha_4 \cdots \alpha_6\alpha_1\alpha_2 = 0$, the scalar $A$ does not appear in the expression of $\Delta(\alpha_1)$ and $\Delta(\alpha_2)$ and, in consequence, does not appear in $\Delta(e_i)$ with $i \neq 1$. The same reasoning applied to any vertex $i$ gives $\text{Frobdim}(A) \geq k$.

We now turn to the case where there are spikes. We can assume that there exists an spike $c_{\alpha_i}$. Then, $\alpha_1 \cdots \alpha_k = \beta_1\gamma_1$ and $\alpha_1\beta_1 = \gamma_1\alpha_1 = 0$. We affirm that $\Delta(e_2)$ has the term $A\alpha_2 \cdots \alpha_k \otimes \alpha_1$ with $A \neq 0$. Since $\alpha_1\beta_1 = 0$ and $\alpha_2 \cdots \alpha_k = \beta_1\gamma_1$, the scalar $A$ does not appear in the expression of $\Delta(\beta_1)$ and $\Delta(\alpha_1)$. On the other hand the term $A\alpha_2 \cdots \alpha_k \otimes \alpha_1\alpha_2$ has to appear in the expression of $\Delta(\alpha_2)$ and consequently $A\alpha_3 \cdots \alpha_k \otimes \alpha_1\alpha_2\alpha_3$ appears in the expression of $\Delta(e_3)$.

If there are not others spikes, we continue with this reasoning showing that $A\alpha_1 \cdots \alpha_k \otimes \alpha_1 \cdots \alpha_{i-1}$ appears in the expression of $\Delta(e_i)$ for $i < k$ and does not appear for $i = k$ because $\alpha_1 \cdots \alpha_{k-1} = 0$. This establishes $A \neq 0$. We now suppose that there exists an spike $c_{\alpha_i}$. It follows, by the same method as before, that $A\alpha_1 \cdots \alpha_k \otimes \alpha_1 \cdots \alpha_{i-1}$ appears in the expression of $\Delta(c_{\alpha_i})$. Since $\gamma_i\alpha_i = 0$ the scalar $A$ does not appear in the expression of $\Delta(\gamma_i)$. The rest of the proof runs as before. Observe that as before we can apply this argument to any idempotent $e_j$ and then $\text{Frobdim}(A) \geq k$. $\square$
4. **Open Frobenius Cluster-Tilted Algebras of type $E$**

We start by computing the mutation class of $E_6$. This can be done, for example, by using the Java applet of Keller [20]. The mutation class of $E_6$ consists of 67 quivers. In the table below we list all the quivers in the mutation class of type $E_6$. In some quivers certain arrows are replaced by undirected lines; this has to be read that these lines can take any orientation. For each quiver in the table we compute the relations of the corresponding cluster-tilted algebra according to [12]. Since there is at most one arrow between any two vertices, we indicate a path by the sequence of vertices it traverses. A zero-relation is then indicated by a sequence of the form $(a, b, c, \ldots)$ and a commutativity relation has the form $(a, b, c, \ldots) - (a', b', c', \ldots)$.

| No. | Quiver | Relations |
|-----|--------|-----------|
| 1   | ![Quiver 1](image1) | None |
| 2   | ![Quiver 2](image2) | $(2, 3, 4), (3, 4, 2), (5, 3, 4), (3, 4, 5), (4, 2, 3) - (4, 5, 3)$ |
| 3   | ![Quiver 3](image3) | $(3, 5, 2), (5, 2, 3), (6, 5, 2), (5, 2, 6), (2, 6, 5) - (2, 3, 5)$ |
| 4   | ![Quiver 4](image4) | $(2, 3, 1), (3, 1, 2), (4, 3, 1), (3, 1, 4), (1, 2, 3) - (1, 4, 3)$ |
| 5   | ![Quiver 5](image5) | $(3, 4, 2), (4, 2, 3), (4, 2, 6), (2, 6, 5), (5, 2, 6), (2, 3, 4) - (2, 6, 4), (6, 4, 2) - (6, 5, 2)$ |
| 6   | ![Quiver 6](image6) | $(3, 4, 2), (4, 2, 3), (6, 4, 2), (5, 6, 4), (6, 4, 5), (2, 3, 4) - (2, 6, 4), (4, 2, 6) - (4, 5, 6)$ |
| No. | Quiver | Relations |
|-----|--------|-----------|
| 7   | ![Quiver 7](image) | (2, 5, 6), (6, 2, 5), (2, 5, 3), (4, 5, 3), (5, 3, 4), (5, 6, 2) − (5, 3, 2), (3, 2, 5) − (3, 4, 5) |
| 8   | ![Quiver 8](image) | (1, 2, 4), (2, 4, 1), (5, 2, 4), (5, 2, 6), (2, 6, 3), (3, 2, 6), (4, 1, 2) − (4, 5, 2), (2, 4, 5) − (2, 6, 5), (6, 3, 2) − (6, 5, 2) |
| 9   | ![Quiver 9](image) | (1, 2, 4), (2, 4, 1), (5, 2, 4), (3, 6, 5, 2), (6, 5, 2, 3), (5, 2, 3, 6), (4, 1, 2) − (4, 5, 2), (2, 4, 5) − (2, 3, 6, 5) |
| 10  | ![Quiver 10](image) | (1, 3, 2), (2, 1, 3), (1, 3, 4), (6, 3, 4), (6, 3, 5), (5, 6, 3), (3, 2, 1) − (3, 4, 1), (3, 4, 6) − (3, 5, 6), (4, 1, 3) − (4, 6, 3) |
| 11  | ![Quiver 11](image) | (3, 6, 2), (6, 2, 3), (6, 2, 5), (1, 2, 5, 4), (2, 5, 4, 1), (4, 1, 2, 5), (2, 3, 6) − (2, 5, 6), (5, 6, 2) − (5, 4, 1, 2) |
| 12  | ![Quiver 12](image) | (1, 2, 4), (2, 4, 1), (5, 2, 4), (3, 5, 2), (3, 5, 6), (6, 3, 5), (2, 4, 5) − (2, 3, 5), (4, 1, 2) − (4, 5, 2), (5, 2, 3) − (5, 6, 3) |
| 13  | ![Quiver 13](image) | (3, 4, 2), (4, 2, 3), (5, 6, 4, 2), (6, 4, 2, 5), (4, 2, 5, 6), (2, 3, 4) − (2, 5, 6, 4) |
| 14  | ![Quiver 14](image) | (2, 3, 6), (6, 2, 3), (4, 5, 2, 3), (5, 2, 3, 4), (2, 3, 4, 5), (3, 6, 2) − (3, 4, 5, 2) |
4.1. **Theorem.** Let $A$ be a non hereditary cluster tilted algebra of type $E_6$. Then $\text{Frobdim}(A) \geq 1$.

*Proof.* Observe that we are reduced to prove that every quiver on the table above, except the first one, has a vertex $a$ where any coproduct $\Delta$ satisfies $\Delta(e_a) \neq 0$. For quivers 2, · · · , 12 there is a vertex $a$ of valency 3 of the following type (or its dual)
Then, $\Delta(e_a) = A\mu \otimes \alpha + B\mu \otimes \beta + \cdots$ where the relations $\alpha\mu = 0$ and $\beta\mu = 0$ implies $A, B \neq 0$.

Quivers $13, \cdots, 16$ have the following subquiver (or its dual)

\[
\begin{array}{ccc}
\alpha_1 & \alpha_2 & \beta_1 \\
\alpha_3 & \beta_2 & \alpha_4
\end{array}
\]

with $a$ a vertex of valency 3 and $b$ a vertex of valency 2. Then $\Delta(e_a) = A\alpha_1 \otimes \alpha_2 \alpha_3 \alpha_4 + \cdots$ and $\Delta(e_b) = A\alpha_4 \alpha_1 \otimes \alpha_2 \alpha_3 + \cdots$. The relations on the quiver implies $A \neq 0$.

Quiver 17 is the following

\[
\begin{array}{ccc}
\alpha_1 & \alpha_2 & \alpha_3 \\
\alpha_4 & \alpha_5 & \alpha_6 \\
\alpha_7 & \alpha_5 & \alpha_2 \alpha_4
\end{array}
\]

Then, $\Delta(e_5) = A\alpha_7 \alpha_6 \alpha_5 \otimes \alpha_5 \alpha_2 + \cdots$ and $\Delta(e_4) = A\alpha_4 \alpha_1 \otimes \alpha_5 \alpha_2 \alpha_3 + \cdots$ with $A \neq 0$.

For quivers 18 and 19 we have the following subquiver

\[
\begin{array}{ccc}
\beta_1 & \beta_2 & \beta_1 \beta_2 = 0 \\
\alpha_1 & \alpha_2 & \alpha_3
\end{array}
\]

with $a$ a vertex of valency 2. Then, the relation $\beta_1 \beta_2 = 0$ implies that the term $A\beta_2 \otimes \beta_1$ (with $A \neq 0$) appears in the expression of $\Delta(e_a)$.

Finally, for quivers 20 and 21 we have a generalization of the case above where there is a vertex $a$ of valency 2 as above but $\beta_1 \beta_2 \neq 0$ and there are two non-zero paths $\mu$ and $\rho$ such that $s(\mu) = t(\rho) = a$ and $\rho \beta_2 = 0 = \beta_1 \mu$. Then we can guarantee that the term $A\mu \otimes \rho$ appears in the expression of $\Delta(e_a)$.

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