Integration time adjusted completeness

Dean Keithly, Dmitry Savransky, and Corey Spohn

Abstract. Future, large-scale, exoplanet direct-imaging missions will be capable of discovering and characterizing Earth-like exoplanets. These mission designs can be evaluated using completeness, the fraction of planets from some population that are detectable by a telescope at an arbitrary observation time. However, the original formulation of completeness uses instrument visibility limits and ignores additional integration time and planetary motion constraints. Some of the sampled planets used to calculate completeness may transit in and out of an instrument’s geometric and photometric visibility limits while they are being observed, thereby causing the integration time agnostic calculation to overestimate completeness. We present a method for calculating completeness that accounts for the fraction of planets that leave the visibility limits of the telescope during the integration time period. We define completeness using the aggregate fraction of an orbital period during which planets are detectable, calculated using the specific times that planets enter and leave an instrument’s visibility limits and the integration time. To perform this calculation, we derive analytical methods for finding the planet-star projected separation extrema, times past periastron that these extrema occur, and times past periastron that the planet-star projected separation intersects a specific separation circle. We also provide efficient numerical methods for calculating the planet-star difference in magnitude extrema and times past periastron corresponding to specific values Δmag. Our integration time adjusted completeness shows that, for a planned star observation at 25 pc with 1-day and 5-day integration times, integration time adjusted completeness of Earth-like planets is reduced by 1% and 5% from the integration time agnostic completeness, respectively. Integration time adjusted completeness calculated in this manner also provides a computationally inexpensive method for finding dynamic completeness—the completeness change on subsequent observations. © The Authors. Published by SPIE under a Creative Commons Attribution 4.0 Unported License. Distribution or reproduction of this work in whole or in part requires full attribution of the original publication, including its DOI. [DOI: 10.1117/1.JATIS.7.3.037002]

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1 Introduction

Direct imaging blind search mission schedules can be optimized by maximizing completeness—the fraction of exoplanets from an assumed planet population that are detectable by a particular instrument at an arbitrary observation time. Completeness is typically parameterized by a limiting planet-star brightness difference (Δmaglim), the inner working angle (IWA) of the instrument, and the outer working angle (OWA) of the instrument. The original Monte Carlo approach developed by Brown involved creating a cloud of synthetic planets by sampling the underlying Keplerian orbital elements (KOE) and physical parameters of a planet population and determining the fraction of those individually simulated planets within the visible limits of the instrument. Multiplying completeness by the exoplanet occurrence rate gives the expected exoplanet yield for observing a given star. Although completeness is a good metric for predicting instrument performance, the calculation described above only captures an instant in time and
does not include whether the time that a planet is within the visible region of the telescope is sufficient to actually make a detection.

The original method of calculating completeness was developed by Brown. Our adaptation of this method in EXOSIMS is extensively outlined in Ref. 1 and involves randomly generating planet KOE, planet geometric albedo (\( p \)), and planetary radius (\( R \)) sampled from NASA’s Exoplanet Program Analysis Group Study Analysis Group 13 (SAG13) probability density functions, generating a finely binned 2D histogram of \( \Delta \text{mag} \) versus \( s \) (planet-star separation), and fitting a 2D spline to the histogram bins. In all current methods, completeness is calculated as the double integral of \( s \) and \( \Delta \text{mag} \) over the joint probability density function as in Eq. (17) of Ref. 1 and Eq. (7) in Ref. 6. The limits of integration define a detectable planet as one in which \( \Delta \text{mag} < \Delta \text{mag}_{\text{lim}} \) and \( d_{IWA} < s < d_{OWA} \), where \( d_i \) is the distance of the host star from the spacecraft. While the \( \Delta \text{mag}_{\text{lim}} \) in this paper is used to describe a general upper limit of integration for calculating completeness, this \( \Delta \text{mag}_{\text{lim}} \) can be formulated as a function of integration time as in Eq. (12) of Ref. 1 by making assumptions about a multitude of instrument parameters and external noise sources. This approach, as conventionally implemented, is a good estimator for completeness, but it does not take planet motion in time into account.

Searches like the Gemini Planet Imager and Nancy Grace Roman Space Telescope (Roman) Coronagraphic Instrument (CGI) are sensitive to larger planets with larger planet-star separations and longer orbital periods. They make use of Brown completeness to plan blind searches. However, future telescope concepts like the Habitable Exoplanet Observatory (HabEx) and the Large UV/Optical/IR Surveyor (LUVOIR) seek to find smaller Earth-like exoplanets, with shorter orbital periods around stars farther away. Their search criteria will result in Brown completeness based yield overestimations due to planet motion out of instrument visibility limits. Figure 1 shows a schematic of a direct imaging observation; it demonstrates how a planet can be within the photometric and geometric visibility limits of the instrument and still not be detected. The red regions of the planet’s orbit indicate where the planet is within the visibility limits of the instrument but is not detectable due to integration time constraints. The dashed portions of the orbit (c) indicate times when the planet is too faint to be observed. The shaded gray regions represent the projected inner and OWAs of the instrument. The planet is unobservable on portions of the orbit intersecting these gray regions. Green dots indicate transitions in and out of instrument visibility limits.

Fig. 1 Schematic of a direct imaging observation. The line represents the projection of a planet’s orbit about its parent star (yellow) into the plane of the sky as seen by a distant observer. The arrow indicates the planet’s direction of motion. Blue portions of the orbit (a) indicate times when the planet is detectable. Red portions of the orbit (b) indicate times when the planet is within the visibility limits of the instrument, but it is not detectable due to integration time constraints. The dashed portions of the orbit (c) indicate times when the planet is too faint to be observed. The shaded gray regions represent the projected inner and OWAs of the instrument. The planet is unobservable on portions of the orbit intersecting these gray regions. Green dots indicate transitions in and out of instrument visibility limits.
instrument long enough to be directly imaged, we need a new method that only counts targets within the instrument visibility long enough to be observed or characterized.

Figure 2 demonstrates this phenomenon in the separation versus $\Delta mag$ phase space commonly used to define instrument contrast curves and completeness. Figure 2(b) shows a short-period, fast-moving, Mars-like planet that transitions into and out of the assumed instrument’s visible limits before it can be detected. Figure 2(c) shows the effect for a Uranus-like planet that crosses through and exits the visible region in less time than it takes for it to be detected. In these cases, Brown’s completeness calculation would include simulated planets in these regions, thus overestimating the overall detection yield. Additionally, Fig. 2(d) shows that the last possible moment a Neptune-like planet could be imaged before it leaves the visible limits of the telescope. This demonstrates a portion of the orbit where the planet is within the visible limits of the telescope but is not detectable.

This paper presents a method, implemented in the EXOSIMS modeling software, for calculating integration time adjusted completeness that allows us to investigate exactly how much the original completeness definition overestimates planet yields. EXOSIMS is an exoplanet direct imaging mission modeling software used to simulate the Roman CGI and future mission concepts such as HabEx and LUVOIR. EXOSIMS simulates populations of planets, observatories, instruments, and underlying dynamics to create full, end-to-end mission simulations. We derive distributions of potential mission yield from ensembles of these mission simulations. EXOSIMS utilizes completeness as a heuristic for target selection and observation scheduling.

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**Fig. 2** The $\Delta mag$ versus $s$ curve for Neptune (blue) and Mars (red). The instrument’s visible region (white) is bounded by 0.066 and 0.517 arc sec, and $\Delta mag = 31$ at $d_i = 22.87$ pc (we selected instrument parameters to make a demonstration on solar system planets about a Sun-like star, but any instrument will have planets with similar effects). Any planet in the grayed region is not visible by the instrument. Dots indicate calculated intersection points between the planet and the visible region bounds. The black dashed portions of planet orbits indicate portions of the orbit where the planet is not detectable. (b) A portion of Mars’s orbit where the planet enters and exits the visible limits of the instrument in less time than the required integration time. (c) A portion of Neptune’s orbit where the planet enters and exits the visible region of the instrument in less time than the required integration time. (d) Neptune’s detectable region adjusted for integration time. If the observation of Neptune in (d) is started when it is left of the blue $\times$ and in the visible region, then it will be detected; if it is started right of the blue $\times$, the planet will not be detected because the required integration time is larger than the time the planet will be in the visible region of the instrument.
implementing both Brown’s Monte Carlo approach to completeness\textsuperscript{2} as well as the analytical completeness formulation from Ref. 6. However, EXOSIMS determines whether actual detections occur by evaluating the effective signal-to-noise ratio (SNR) of individual simulated observations on simulated planets. Keithly et al.\textsuperscript{1} discussed the full optimization process for a single-visit blind search and showed that the ideal yield from an EXOSIMS simulation ensemble (when mission constraints are discounted) for a limited blind search mission of Roman is equivalent to the yield calculated via Brown Completeness.

Comparisons between Brown completeness-based exoplanet yield estimations and Monte Carlo of full mission simulations have been done in Ref. 10 and the Standard Definitions and Evaluation Team Final Report.\textsuperscript{11} The yields calculated by completeness are higher than the average yield from a Monte Carlo of full missions simulated in EXOSIMS as noted in Table 6 of Ref. 11. These works attribute the difference in yields to the additional mission constraints captured by the full mission simulation approach, inefficiencies in the optimization algorithms, and inefficiencies in the scheduling algorithms implemented in EXOSIMS.\textsuperscript{12} The unmentioned assumption is that Brown completeness-based yield estimates are accurate, but this paper demonstrates how the omission of planet motion and integration time reduces the resulting completeness yield estimates.

To know the amount of time that an individual planet spends within the observable limits of an instrument, we need to calculate the times in a planet’s orbit when a planet enters or exits these limits. This means that we need methods for calculating when a planet has a given planet-star separation and Δmag. Once we know all of the times when a planet enters or exits an instrument’s visibility limits, we know the fraction of time that the planet is able to be detected with that instrument. Averaging over these visibility fractions for a large number of samples evaluates completeness in a fundamentally different way from its original formulation. These same methods also allow for the evaluation of integration time adjusted completeness as well as a new method of calculating dynamic completeness.\textsuperscript{13}

Section 2 presents the derivation of this chain of calculations and discusses practical aspects of their implementation. Section 3 provides detailed validation of the methods and presents results of the various calculations enabled by them. Finally, in Section 4, we discuss various aspects of the algorithms and results, and we lay out future applications for this methodology.

2 Methods

In this section, we present our detailed process for calculating integration time adjusted completeness. We do this by finding the time windows in which a planet is within the separation and Δmag visibility limits of an instrument, discounting each time window by the integration time. The general overview of this process is as follows.

1. Calculate locations of apparent intersections between the projection of the 3D orbit into the plane of the sky and the $s_{WA}$ circle about the star.
   
   (a) Parameterize the 3D orbit as an ellipse in the plane of the sky.
   
   (b) Formulate the planet-star separation equation and its derivative.
   
   (c) Solve for the magnitude and locations of planet-star separation extrema.
   
   (d) Identify and assign the subset of algebraic solutions that are separation extrema.
   
   (e) Classify the expected number of intersections between the projected ellipse and the $s_{WA}$ circle.
   
   (f) Solve for intersections between the projection of the orbital ellipse and the $s_{WA}$ circle.
   
   (g) Identify and assign the subset of algebraic solutions that form intersections.

2. Calculate locations and magnitudes of Δmag intersections.

   (h) Formulate an expression isolating the Δmag and $\nu$ terms and its derivative.
   
   (i) Express this formulation as a polynomial and solve for the algebraic solutions.
   
   (j) Identify and assign the subset of algebraic solutions that are Δmag extrema.
(k) Classify the expected number of $\Delta \text{mag}$ and $\Delta \text{mag}_{\text{lim}}$ intersections.
(l) Solve for $\Delta \text{mag}$ and $\Delta \text{mag}_{\text{lim}}$ intersections.
(m) Identify and assign the subset of algebraic solutions that form the intersections.

3. Calculate $\nu$ from $X$ and $Y$ for each intersection and extrema.
4. Calculate $t$ from $\nu$ of each intersection and extrema.
5. Combine times of $s_{\text{WA}}$ and $\Delta \text{mag}_{\text{lim}}$ intersections to create time windows between intersections.
6. Identify time windows in which the planet is visible or not visible.
7. Calculate integration time adjusted completeness averaging the orbital fraction of time that a planet is visible discounted by the integration time.

The general equations for $\Delta \text{mag}$ and $s$ used in these derivations are

$$\Delta \text{mag} = -2.5 \log_{10}\left(p \left(\frac{R}{|r_{k/i}|}\right)^2 \Phi(\beta)\right),$$

and

$$s = \|r_{k/i} - (r_{k/i} \cdot \hat{r}_{i/\text{SC}})\hat{r}_{i/\text{SC}}\|.$$  

Here $R$ is the planet radius, $p$ is the geometric albedo of the planet, and $\Phi(\beta)$ is the planet phase function. The other variables are defined in Fig. 3, where $\beta$ is the star-planet-observer angle (also called the phase angle), $r_{k/i}$ is the vector from star $i$ to planet $k$, and $\hat{r}_{i/\text{SC}}$ is the unit vector from the spacecraft (SC) to the star ($\hat{r}$ is the unit vector of $r$). The plane of the sky for a given observation lies in the $\hat{x}$ and $\hat{y}$ plane, where $\hat{r}_{i/\text{SC}}$ defines $\hat{z}$ of the target system. For our purposes, the direction of $\hat{x}$ is arbitrary, but is typically taken to be a well-defined, inertially fixed direction, such as the ICRS mean equinox or pole direction. Here and throughout this paper, $i$ in a subscript refers to the $i$'th target star and $i$ (not subscripted) refers to the orbit inclination.

### 2.1 Projected Orbit and Separation Intersection

In this section, we derive an analytical expression for the true anomaly ($\nu$) of $s$-orbit intersection points between a circle in the plane of the sky and the projection of a 3D Keplerian orbit on the plane of the sky. A practical example of a circle in the plane of the sky is the projected inner or outer edge of a planet's light curve.

![Fig. 3](image-url)

*Fig. 3* The orbital path of planet $k$ (black) in a general XYZ Cartesian coordinate system. $r_{k/i}$ (straight blue arrow) describes the location of the planet $k$ (blue circle) relative to star $i$ (yellow circle). The plane of the sky is noted by the translucent red parallelogram entirely in the $\hat{x}$ and $\hat{y}$ plane. The dashed ellipse (red) is the projection of the planet orbit (black) onto the plane of the sky. $\hat{h}$ is the orbit angular momentum vector. $\hat{e}$ is the orbit eccentricity vector. $\hat{n}$ is the line of nodes. The star $*$ is generally referred to in subscripts as $*$ and often referred to in subscripts as the $i$'th star. This is not to be confused with the variable $i$, which is the inclination of the planet orbit. $\nu$ is the true anomaly of the planet. $\omega$ is the argument of periapsis of the planet. $\Omega$ is the longitude of the ascending node of the planet.
OWA of the instrument, equal to IWAd$_i$ (or OWAd$_i$) for target star distance $d_i$. An orbit defined by KOE takes the shape of an ellipse in the orbital plane. The perpendicular projection of a 3D elliptical orbit onto the plane of the sky is given by the $\hat{x}$ and $\hat{y}$ components of the 3D orbit. However, the resulting expression is not easily solvable for the true anomalies at intersection points. Instead, we can express the orbit projection in the plane of the sky as another ellipse. This simplifies the difficult intersection problem into analytically solvable subproblems and yields a relatively simple solution to the intersection between a circle in the plane of the sky and the projected orbit in the plane of the sky. Here we include the brief outline of this process, with the full procedure detailed in Appendix B.

1. Project the 3D orbital ellipse into a 2D projected ellipse and formulate the separation equation.
2. Find the expected number of $s$ intersection points.
3. Calculate $s$ intersection point coordinates.

### 2.1.1 Projection of the elliptical orbit

Given the KOE of a 3D orbit and a plane to project it on, we calculate the semimajor and semiminor axes of the projected orbit ($a_p$ and $b_p$, respectively) as well as the angle between the projected semimajor axis and $\hat{x}$, which we call ($\theta$). We first convert the KOE into orbital radius components in the Cartesian $XYZ$ coordinate system as in Fig. 3:

\[
egin{align*}
X &= r(\cos(\Omega) \cos(\omega + \nu) - \sin(\Omega) \sin(\omega + \nu) \cos(i)) \\
Y &= r(\sin(\Omega) \cos(\omega + \nu) + \cos(\Omega) \sin(\omega + \nu) \cos(i)) \\
Z &= r \sin(i) \sin(\omega + \nu),
\end{align*}
\]

where $r$ is the orbital radius (magnitude of $\mathbf{r}_{j|i}$) given by

\[
r = \frac{a(1 - e^2)}{1 + e \cos(\nu)}. \tag{4}
\]

$\Omega$ is the longitude of the ascending node, $\omega$ is the argument of periapsis, $\nu$ is the true anomaly, and $e$ is the eccentricity of the planet’s orbit.

Figure 4 shows a schematic view of the projection of the orbit onto the plane of the sky. There are two particularly important points associated with the projected orbit ellipse. First, $F$ is the filled focus (star location) of the orbit and retains the same coordinates in the plane of the sky. When observing a star, $F$ is the center of all working angle circles and the origin of the $XYZ$ coordinate system. Note that in Fig. 4 the projected ellipse is located well below the original 3D coordinates of the 3D orbit for clarity, but the points $F$ indicated by the orange circle and $F$ indicated by the orange $\times$ are in fact coincident.

Second, $O$ is the geometric center of the 3D orbit and is found most efficiently by averaging the $XYZ$ locations of the planet at apoaapsis and periapsis:

\[
\mathbf{FO} = \frac{1}{2}(\mathbf{r}_{j|i}(\nu = 0) + \mathbf{r}_{j|i}(\nu = \pi)), \tag{5}
\]

where $\mathbf{FO}$ is the line segment (equivalently Euclidean vector) from $F$ to $O$ and $\mathbf{r}_{j|i}(\nu)$ is the evaluation of Eq. (3) for the given value of $\nu$. As $\hat{x}$ and $\hat{y}$ define the plane of the sky, the $XYZ$ coordinates of the geometric center of the projected ellipse ($O'$) are given by

\[
O' = (\mathbf{FO} \cdot \hat{x}, \mathbf{FO} \cdot \hat{y}, 0). \tag{6}
\]

Appendix J provides a proof that any generic 3D ellipse projects to a 2D ellipse, as shown graphically in Fig. 4. The projection of the semimajor axis and semiminor axis from the 3D orbit onto the 2D plane of the sky form conjugate diameters of the projected ellipse. Any two diameters of an ellipse are conjugate diameters if and only if the tangent line to the ellipse at an
Fig. 4 The original 3D elliptical orbit of a planet (black line) containing points A, B, C, and D; the gray endpoints of the purple semimajor axis \( \overline{AB} \) and semiminor axis \( \overline{CD} \). The planet orbits about the star, which is located at focus \( F \) (orange circle). The orange dot and orange \( x \) denoted as \( F \) are the same point in space, but the red projection is shown as offset from the original orbit for clarity.

The projection of the original 3D elliptical orbit onto the \( XY \) plane is given by the red ellipse containing points \( A', B', C', \) and \( D' \) (blue diamonds; perpendicular projections of \( A, B, C, \) and \( D \)). Point \( O \) is the geometric center of the 3D elliptical orbit and projects to \( O' \) in the plane of the sky. Point \( P \) (green circle) is any arbitrary point along the original 3D ellipse and maps to the semimajor axis and semiminor axis \( H \) and \( K \), respectively (green \( x \)). \( P' \) (pink circle) is the perpendicular projection of \( P \), and \( H' \) and \( K' \) (pink \( x \)) are projections of \( H \) and \( K \), respectively. The components of these perpendicular projections preserve the ratios of their values to the semimajor and semiminor axes, which, given the equation for an ellipse, can be used to prove the projection of an ellipse is itself an ellipse. The blue lines \( \overline{AB} \) and \( \overline{CD} \) form conjugate diameters of the red ellipse and are the projection of the semimajor axis and semiminor axis of the original 3D ellipse onto the plane of the sky. These conjugate diameters can then be used to find the semimajor axis and semiminor axis of the red projected ellipse \( \overline{IR} \) and \( \overline{ST} \), respectively. \( x_{dr}, y_{dr}, z_{dr} \) are the components of the derotated reference frame \( (dr) \) as defined in the text.

The endpoint of one diameter is parallel to the other diameter. \(^{14}\) Each pair of conjugate diameters of an ellipse has a corresponding tangent parallelogram, sometimes called a bounding parallelogram. In Fig. 4, the pair of blue lines \( (\overline{A'B'} \) and \( \overline{C'D'} \) are specific examples of conjugate diameters. The pair of purple lines \( (\overline{TR} \) and \( \overline{ST} \) are also examples of conjugate diameters, and they form the semimajor and semiminor axes of the projected ellipse.

The semimajor and semiminor axes of the projected ellipse may be found from any conjugate diameters. We take the conjugate diameters \( \overline{A'B'} \) and \( \overline{C'D'} \) and draw line \( \overline{QQ'} \) through \( B' \), perpendicular to \( \overline{D'C'} \) as shown by the gray line in Fig. 5. Points \( Q \) and \( Q' \) are chosen such that \( \overline{B'Q} = \overline{B'Q'} = \overline{O'D'} \). The principal axes \( \overline{TR} \) and \( \overline{ST} \) lie on the bisectors of the angles formed by lines \( \overline{O'Q} \) and \( \overline{O'Q'} \). From this construction, we calculate the projected ellipse semimajor axis, semiminor axis, and angular offset of the semimajor axis from \( \dot{x} \). \(^{16}\) \( \overline{TR} \) and \( \overline{ST} \) are given by

\[
\overline{TR} = \overline{O'Q'} + \overline{O'Q}, \tag{7}
\]

\[
\overline{TS} = \overline{O'Q'} - \overline{O'Q}. \tag{8}
\]

Defining the angle between \( \overline{O'B'} \) and \( \overline{O'D'} \) as \( \phi \), the cosine rule applied to triangles \( \overline{O'B'Q} \) and \( \overline{O'B'Q'} \) yielding

\[
|\overline{O'Q'}|^2 = |\overline{O'B'}|^2 + |\overline{O'D'}|^2 - 2|\overline{O'B'}||\overline{O'D'}| \sin \phi, \tag{9}
\]

\[
|\overline{O'Q'}|^2 = |\overline{O'B'}|^2 + |\overline{O'D'}|^2 + 2|\overline{O'B'}||\overline{O'D'}| \sin \phi, \tag{10}
\]

where \( |\overline{O'B'}| \) denotes the length of \( \overline{O'B'} \). Inserting these into Eq. (7), we obtain
\[ O'R \cdot O'S = |O'B'||O'D'| \sin \phi, \]  
\[ |O'R|^2 + |O'S|^2 = |O'B'|^2 + |O'D'|^2. \]

\[ |O'R| \] must be twice the semimajor axis of the projected ellipse \((a_p)\), so

\[ a_p = \frac{|O'R|}{2} = \frac{|O'R|}{2}, \]  
and \(|O'S|\) must be twice the semiminor axis of the projected ellipse, so

\[ b_p = \frac{|O'S|}{2} = \frac{|O'S|}{2}. \]

The angle of the semimajor axis of the projected ellipse from \(\hat{x}\) is calculated using the average of the angles between \(O'Q\) and \(\hat{x}\) and \(O'Q'\) and \(\hat{x}\):

\[ \theta = \frac{1}{2} \left( \tan^{-1} \left( \frac{O'Q \cdot \hat{x}}{|O'Q|} \right) + \tan^{-1} \left( \frac{O'Q' \cdot \hat{x}}{|O'Q'|} \right) \right). \]

Appendix E provides the full expressions for \(a_p, b_p,\) and \(\theta\) via full expansions of Eqs. (13)–(15), respectively.

Now that we know all of the parameters necessary to describe the projected ellipse, we can standardize this ellipse into a simpler form to simplify subsequent calculations. We define a new frame (\(dr\) as in Fig. 4) as the derotation and geometric centering of the projected orbit such that the semimajor axis of the projected ellipse \((O'T)\) is aligned with \(\hat{x}_{dr}\), the semimajor axis of the projected ellipse \((O'S)\) is aligned with \(\hat{y}_{dr}\), and \(O'\) is the origin of the \(dr\) coordinate system. The \(dr\) coordinates of the star location \((x_s, y_s)\) are given by a simple rotation of the projection of \(O'F\) onto the \(\hat{x}, \hat{y}\) plane by angle \(\theta\):

\[
\begin{bmatrix}
  x_s \\
  y_s
\end{bmatrix}
= - \begin{bmatrix}
  \cos(\theta) & \sin(\theta) \\
  -\sin(\theta) & \cos(\theta)
\end{bmatrix}
\begin{bmatrix}
  O'F \cdot \hat{x} \\
  O'F \cdot \hat{y}
\end{bmatrix}_{XYZ}. \]

2.1.2 Global and local extrema of planet-star separation

Before solving for the true anomalies where the orbit’s projected separation \(s\) is equal to \(s_{WA}\) (general working angle separation \(WA_{Adl}\)), we first need to know how many of these \(s_{WA}\)-orbit
intersections we are looking for. We find the expected number of solutions by finding the $s$

extrema throughout the orbit. We do so by solving for the roots of the derivative of the projected

planet-star separation.

We start with the general equation for an ellipse:

$$
\left(\frac{x_e}{a_p}\right)^2 + \left(\frac{y_e}{b_p}\right)^2 = 1, \tag{17}
$$

where $x_e$ and $y_e$ are the coordinates of any point on the ellipse and $a_p$, $b_p$ are the semimajor and

semiminor axes, respectively. We rewrite this in terms of $x_e$, giving

$$
y_e = b_p \sqrt{1 - \frac{x_e^2}{a_p^2}}. \tag{18}
$$

The projected separation is given by

$$
s^2 = (-x_e + x_e)^2 + (-y_e + y_e)^2. \tag{19}
$$

Taking the derivative of Eq. (19) with respect to $x_e$, substituting Eq. (18) (and its derivative), and

setting it equal to zero, we have

$$
0 = \frac{\partial s^2}{\partial x_e} = -2x_e + 2x_e + \frac{2b_p x_e y_e}{a_p \sqrt{a_p^2 - x_e^2}} - \frac{2b_p^2 x_e}{a_p^2}, \tag{20}
$$

Isolating the square root term to one side and squaring both sides of the equation gives

$$
\left(-2x_e + 2x_e - \frac{2b_p^2 x_e}{a_p^2}\right)^2 = \left(\frac{-2b_p x_e y_e}{a_p \sqrt{a_p^2 - x_e^2}}\right)^2, \tag{21}
$$

which is expanded with coefficients of $x_e$, to get the polynomial expression

$$
0 = x_e^4 + \frac{-6a_p^2 x_e + 8b_p^2 x_e}{(4a_p^4 - 8a_p^2 b_p^2 + 4b_p^4)/a_p^2} x_e^3 + \frac{-4a_p^4 + 8a_p^2 b_p^2 - 4b_p^4 + 4b_p^2 y_e^2}{(4a_p^4 - 8a_p^2 b_p^2 + 4b_p^4)/a_p^2} x_e^2
$$

$$
+ \frac{8a_p^2 x_e - 8a_p^2 b_p^2 x_e}{(4a_p^4 - 8a_p^2 b_p^2 + 4b_p^4)/a_p^2} x_e + \frac{-4a_p^4 x_e^2}{(4a_p^4 - 8a_p^2 b_p^2 + 4b_p^4)/a_p^2} \tag{22}
$$

This expression is a fourth-order polynomial, which we write in standard quartic form as

$$
0 = x_e^4 + A_0 x_e^3 + B_0 x_e^2 + C_0 x_e + D_0. \tag{23}
$$

where $A_0$, $B_0$, $C_0$, and $D_0$ are the constants and functions of $a_p$, $b_p$, $x_e$, and $y_e$, defined in

Appendix F. We now apply the analytical solutions of the general quartic expression in standard

form given in Appendix K.

Solving this quartic gives us a set of four $x_e$ solutions ($x_e$) corresponding to two global

extrema and two local extrema (if the latter exist for a particular orbit). Figure 6 shows a schematic

representation of an orbit with four extrema. We use the imaginary components of the

solutions, geometry of the ellipse, and magnitude of the higher order terms in the quartic solutions

to identify which solutions belong to which extrema. We first use the magnitude of the

imaginary components of solutions to determine how many extrema there are and filter out sol-

utions that are not extrema. The algebraic solutions of the quartic polynomial all give values in

the first quadrant (quadrants 1 to 4 are numbered counter-clockwise such that quadrant 1 has

strictly positive coordinates, see the four quadrants of Fig. 6), so we must use geometry of the

problem to determine the proper sign of each extremum’s true coordinates $x_{e,g}$ and $y_{e,g} [g refer-

ences solutions 0 to 3 Eq. (91)–Eq. (94)].
All numerical solutions to the quartic have some degree of imaginary component due to accumulation of numerical errors. Quartic solution sets with only two extrema will have two real solutions (solutions with small imaginary components only due to numerical error) and two solutions with large imaginary components (algebraic solutions that are artifacts to be thrown away). The majority of KOE have only two \( s \) extrema. The only case in which no extrema exist is in a circular face-on orbit. In this specific case, all solutions to the quartic will be nearly identical and have large imaginary components, and the resulting \( s \) extrema will be identical \((s_{\text{min}} = s_{\text{max}})\). We assume that all solutions are real if \(|I(x)_{e,g}| < 10^{-5}\) for only two solutions. We define a new ordered set containing either two or four elements depending on magnitude of the imaginary components as

\[
x_R = \{|\Re(x_{e,g})|: |\Im(x_{e,g})| < 10^{-5} \ \forall \ g \in \{0,1,2,3\}\},
\]

where the absolute value is due to the algebraic solutions of the quartic being only defined in the first quadrant.

We now know the number of expected solutions from the dimension of \( x_R \) but need to leverage the quartic algebraic solution and geometry to determine which components belong to which quadrant. For orbits with \( x_R \) containing only two solutions, the first two solutions to the quartic \((x_{R,0} \text{ and } x_{R,1})\) produce the largest and smallest \(x_{R,g} = x_s\), respectively. Due to the shape of an ellipse, these must necessarily produce \( s_{\text{min}} \) and \( s_{\text{max}} \). We define the set \( y_R \) as the application of Eq. (18) to each element of \( x_R \). We define four separation quantities from the possible sign combinations of the coordinate magnitudes in \( x_R \) and \( y_R \) as

\[
s_{-\pm 0} = \sqrt{(x_{R,0} - x_s)^2 + (y_{R,0} \pm y_s)^2},
\]

\[
s_{+\pm 0} = \sqrt{(x_{R,0} + x_s)^2 + (y_{R,0} \pm y_s)^2},
\]

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We now know the number of expected solutions from the dimension of \( x_R \) but need to leverage the quartic algebraic solution and geometry to determine which components belong to which quadrant. For orbits with \( x_R \) containing only two solutions, the first two solutions to the quartic \((x_{R,0} \text{ and } x_{R,1})\) produce the largest and smallest \(x_{R,g} = x_s\), respectively. Due to the shape of an ellipse, these must necessarily produce \( s_{\text{min}} \) and \( s_{\text{max}} \). We define the set \( y_R \) as the application of Eq. (18) to each element of \( x_R \). We define four separation quantities from the possible sign combinations of the coordinate magnitudes in \( x_R \) and \( y_R \) as

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s_{-\pm 0} = \sqrt{(x_{R,0} - x_s)^2 + (y_{R,0} \pm y_s)^2},
\]

\[
s_{+\pm 0} = \sqrt{(x_{R,0} + x_s)^2 + (y_{R,0} \pm y_s)^2},
\]

Fig. 6 Planet orbit (black) in the \( dr \) frame and planet-star separation extrema. The minimum separation (cyan) always occurs in the same quadrant as the star in the \( dr \) frame (orange \( x \)). The maximum separation (red) always occurs in the quadrants opposite the star. The local minimum separation (magenta) and local maximum separation (gold) occur in the same half-plane about the \( y \) axis as the star, but in opposite half-planes about the \( x \) axis as the star (fourth quadrant). This configuration applies to the vast majority of orbits. A small number of edge cases exist, including circular orbits and some edge on orbits. The blue dots represent the foci of the projected ellipse in the \( dr \) frame, the orange dot is the ellipse center, and the purple dashed lines are the semimajor and semiminor axes.


\[ s_{\pm 1} = \sqrt{(x_{\text{rel},1} - x_c)^2 + (y_{\text{rel},1} \pm y_*)^2}, \]

\[ s_{\pm l} = \sqrt{(x_{\text{rel},1} + x_c)^2 + (y_{\text{rel},1} \pm y_*)^2}. \]

Typically, \( s_{-1} \) is smaller than \( s_{-0} \), but in \( \approx 0.01 \% \) of cases, \( s_{-0} \) can be the smallest separation due to numerical error. (In this case, we ensure \( |s_{-1} - s_{-0}| < 10^{-8} \) and swap values.) In addition to \( s_{-1} < s_{-0} \), we also know that \( s_{+0} < s_{+1}, s_{+l} < s_{+0}, \) and \( s_{+l} < s_{+0} \) for all cases. Using this knowledge, we reduce the number of comparisons that we need to find the minimum planet-star separation to

\[ s_{\text{min}} = \begin{cases} 
  s_{-1} & \text{where } (s_{-1} < s_{-0}) \text{ and } (s_{-1} < s_{+1}) \text{ and } (s_{-1} < s_{+1}), \\
  s_{+0} & \text{where } (s_{+0} < s_{-1}) \text{ and } (s_{+0} < s_{+1}) \text{ and } (s_{+0} < s_{+l}), \\
  s_{+l} & \text{where } (s_{+1} < s_{+0}) \text{ and } (s_{+1} < s_{-1}) \text{ and } (s_{+1} < s_{+1}), \\
  s_{+l} & \text{where } (s_{+l} < s_{+0}) \text{ and } (s_{+l} < s_{-1}) \text{ and } (s_{+l} < s_{-1}). 
\end{cases} \]

The maximum separation similarly is found as

\[ s_{\text{max}} = \begin{cases} 
  s_{-0}, & \text{where } (s_{-0} > s_{+0}) \text{ and } (s_{-0} > s_{+1}) \text{ and } (s_{-0} > s_{+0}), \\
  s_{+0}, & \text{where } (s_{+0} > s_{-1}) \text{ and } (s_{+0} > s_{+1}) \text{ and } (s_{+0} > s_{+0}), \\
  s_{+l}, & \text{where } (s_{+1} > s_{+0}) \text{ and } (s_{+1} > s_{-0}) \text{ and } (s_{+1} > s_{+0}), \\
  s_{+0}, & \text{where } (s_{+0} > s_{+0}) \text{ and } (s_{+0} > s_{+1}) \text{ and } (s_{+0} > s_{-0}). 
\end{cases} \]

We are able to find \( x_c \) and \( y_* \) of the minimum and maximum separation diamonds drawn in Fig. 6 using the same logic as for finding \( s_{\text{min}} \) and \( s_{\text{max}} \).

For the orbits with four solutions in \( x_{\text{rel}} \), the first two elements will always be the global extrema, and the last two elements will be the local extrema, which are given by

\[ s_{\text{min}} = s_{-1}, \]

\[ s_{\text{max}} = s_{+0}, \]

\[ s_{\text{min}} = \begin{cases} 
  s_{-3}, & \text{if } s_{-2} > s_{-3}, \\
  s_{-2}, & \text{else} 
\end{cases} \]

\[ s_{\text{max}} = \begin{cases} 
  s_{-2}, & \text{if } s_{-2} > s_{-3}, \\
  s_{-3}, & \text{else} 
\end{cases} \]

where \( s_{-2} \) and \( s_{-3} \) are calculated in the same manner as \( s_{+0} \) and \( s_{-0} \). We are able to find the coordinates of all four using the same logic as for finding \( s_{\text{min}}, s_{\text{max}}, s_{\text{min}}, \) and \( s_{\text{max}} \). This procedure yields the coordinates of all existing extrema in the \( dr \) frame. To find their locations on the projection of orbit in the plane of the sky, we apply the inverse of Eq. (16).

### 2.1.3 Intersections between a circle and an ellipse

We find \( s_{\text{WA}} \)-orbit intersections by formulating the circle-ellipse intersections as another quartic, solving this, and assigning the algebraic solutions to intersections. We assign solutions using the number of \( s \) extrema, the size of the \( s_{\text{WA}} \) intersecting circle relative to these \( s \) extrema, and the ellipse geometry. To formulate the \( s_{\text{WA}} \)-orbit intersections as a quartic, we start with Eq. (19) and substitute in Eq. (18). This gives us a separation equation solely as a function of \( x_c \) and star location. We expand and transform this into a general polynomial of \( x_c \) with a general \( s \).
We always have four algebraic solutions to \( x \) solutions that may or may not correspond to actual
solutions to Eq. (36) that we have to choose from, we know that there will be two intersections, then two of the four algebraic solutions must be real solutions, and the other two are some combination of repeated roots or non-physical imaginary solutions.) In each of these cases, we must handle the assignment of solutions to the correct quadrants. Figure 7 shows a schematic representation of two cases corresponding to four total intersections.

For KOE with four extrema and \( s_{\min} < s_{WA} < s_{\max} \), we know that there will be two intersections on the same \( y \) side of the ellipse as the star (quadrants 1 or 2 in the \( dr \) frame). Of the four quartic solutions to Eq. (36) that we have to choose from, we know that \( x_0 \) is one of them. The other solution could either be \( x_1 \) if \( 3(x_1) < 10^{-9} \) or \( x_3 \) if \( 3(x_1) > 10^{-9} \).

For KOE with four extrema and \( s_{\max} < s_{WA} < s_{\max} \), we know that there will be two intersections on the opposite \( x \) side of the ellipse as the star (quadrants 2 and 3 in the \( dr \) frame). In all cases, \( x_0 \) and \( x_1 \) are the intersection solutions. This is because the first two solutions have the largest term in the quartic solution [Eqs. (91) and (92)]. \( x_1 \) is slightly smaller than \( x_0 \) because it subtracts the second largest term. The relative magnitudes of \( x_0 \) and \( x_1 \) determine that \( y_0 \) occurs in the same side of the ellipse as the star and \( y_1 \) must occur on the opposite side of the ellipse as the star. Therefore, \( (x_0, y_0) \) occurs in quadrant 2, and \( (x_1, y_1) \) occurs in quadrant 3.

For KOE with four extrema and \( s_{\min} < s_{WA} < s_{\max} \), we know that there will be four intersections. Unlike in Sec. 2.1.2 and the rest of this paper where \( x_0 \) through \( x_3 \) are ordered as in Appendix K, we order \( x_0 \) based on \( \Delta x_h = |x_h - x_0| \). We order the quartic solutions from \( x_0 \) to \( x_3 \) such that \( x_0 \) is where \( \min(\{\Delta x_h \forall h \in \{0,1,2,3\}\}) \) and \( x_3 \) is where \( \max(\{\Delta x_h \forall h \in \{0,1,2,3\}\}) \). These newly ordered \( \Delta x_h \) correspond to those shown in Fig. 7. The \((x_2, y_2)\) intersection occurs in either quadrant 1 or 2, but it is always above and to the left of the star in the \( dr \) frame and has the second largest \( \Delta x_h \) component. The \((x_3, y_3)\) intersection occurs in either quadrant 1 or 4, but it always has the largest \( \Delta x_h \) component. We resolve the sign of \( y_3 \) by testing it in both quadrants. In >99.99% of cases, we assign \( y_3 \) to quadrant 1 as in Fig. 7(a), but for a minority of cases, its correct assignment is quadrant 4 as in Fig. 7(b). The \((x_1, y_1)\) intersection always occurs in quadrant 4. The \((x_0, y_0)\) intersection occurs in either quadrant 3 or quadrant 4, but it is always below and to the left of the star in the \( dr \) frame and has the smallest \( \Delta x_h \) component.

For KOE with two extrema and \( s_{\min} < s_{WA} < s_{\max} \), we know that there will be two circle-ellipse intersections. The KOE determine where the star is located in the \( dr \) frame. The location of the star in the \( dr \) frame relative to the vertices of the projected ellipse \( \{(0, b_p), (0, -b_p), (-a_p, 0), \text{and} (a_p, 0)\} \) determines the star-vertex separation ordering and subsequently to which quadrants the two intersection solutions belong. Instead of calculating the star-vertex separation for each orbit, we divide the first quadrant into four regions that specify the star location type [types 0 to 3 as indicated in Fig. 8(a)]. This means that any KOE with the star in location type 2 has the associated star-vertex separation ordering.

Using the equidistant lines between ellipse vertices, we divide the first quadrant into four regions (types 0 to 3) as shown in Fig. 8(a). Regions 0 and 2 are divided by the line defined by
where a projected ellipse and star is of type 0 if $y > y_{d, p, p, y} (x_0)$ and one of types 1, 2, or 3 otherwise depending on the other equidistant lines. Similarly, regions 1 and 2 are divided by the line defined by

$$y_{a, p, p, y} = \frac{a_p}{b_p} x + \frac{a_p^2}{2b_p} - \frac{b_p}{2},$$

(38)

regions 2 and 3 are divided by the line defined by

$$y_{a, p, p, y} = \frac{a_p}{b_p} x - \frac{a_p^2}{2b_p} + \frac{b_p}{2},$$

(39)

and the star-ellipse classification type is calculated similar to type 0.
An example star is shown in Fig. 8(a). This star location is type 2 because it is in the region bounded by the three dashed lines (gray, pink, and teal). This type 2 star has the separation ordering indicated in Fig. 8(a) and Table 1. Let us consider a separation circle such that:

\[ s_{WA} - x/C_3; y/C_3 < s < s_{WA} + x/C_3; y/C_3. \]

We, therefore, know that the two intersections must occur in quadrant 2 and quadrant 4 of the \( dr \) frame.

The distances \( s_{x_p}; y; b_p; \pm y/C_3 \), \( s_{a_p}; x_p; y; \pm x/C_3 \), and \( s_{a_p}; x; y; \pm x/C_3 \) in Fig. 8(a) are the distances of the star to each of the ellipse vertices and are calculated by:

\[ s_{x_p}; y; b_p; \pm y/C_3 = \sqrt{x_p^2 + (b_p \pm y/C_3)^2}, \]  
\[ s_{a_p}; x_p; y; \pm x/C_3 = \sqrt{(a_p \pm x_p)^2 + y^2/C_3}. \]  

Figure 8(a) and Table 1 define orbit types sorted by vertex distances from smallest to largest based on star location type in the first quadrant of the \( dr \) frame.
Finally, after determining the correct number of intersection solutions and the proper quadrants to which these solutions belong, we rerotate and translate the intersection solution locations back into the 2D projection of the 3D orbit.

### 2.2 Δmag Intersections

In this section, we present our method for calculating the values of $\nu$ on a planet’s orbit where the planet has a specific value of $\Delta mag_{lim}$, called Δmag intersections. As in Sec. 2.1, to compute these solutions, we first need to calculate all Δmag extrema ($\Delta mag_{min}$, $\Delta mag_{max}$, $\Delta mag_{lim}$, and $\Delta mag_{max}$). The process for calculating Δmag extrema is detailed in Appendix C, but briefly included as follows:

1. express $\Delta mag$ as a polynomial in $\cos(\nu)$;
2. find the values of $\nu$ and $\Delta mag$ for all polynomial roots; and
3. remove invalid and duplicate solutions.

The general process for calculating the true anomalies where a planet has $\Delta mag_{lim}$ is similar to the extrema-finding process but with some minor modifications. After calculating the Δmag extrema, we determine how many Δmag intersections a given orbit should have with a particular $\Delta mag_{lim}$ value. If $\Delta mag_{lim} < \Delta mag_{min}$ or $\Delta mag_{lim} > \Delta mag_{max}$, there are no intersections. If $\Delta mag_{min} < \Delta mag_{lim} < \Delta mag_{lim}$ or $\Delta mag_{lim} < \Delta mag_{lim} < \Delta mag_{max}$, then there are exactly two intersections. If $\Delta mag_{lim} < \Delta mag_{lim} < \Delta mag_{lim} < \Delta mag_{lim}$, then there are exactly four intersections.

When the orbit does not contain local Δmag extrema, there are just two Δmag intersections if $\Delta mag_{lim} < \Delta mag_{lim} < \Delta mag_{max}$. Knowing the number of solutions to expect, we follow the same steps as above: finding a governing polynomial equation, solving for its roots (represented by the ordered set $x$), and filtering out the relevant solutions. We additionally throw out any solutions with large errors from the input $\Delta mag_{lim}$. The full process outline is included in Appendix D and discussed in depth below.

We start with the definition of Δmag given in Eq. (1). Although this expression contains multiple terms that are fully defined by an orbit’s KOE, it is also a function of other planet properties, including the planet’s radius, geometric albedo, and phase function. To make the mathematical development presented below tractable, we assume that the quasi-Lambert phase function\(^{17}\) is a sufficient approximation of any planet’s phase function. In general, the quasi-Lambert phase function is a better representation of the Earth’s phase function than the Lambert phase function.

This quasi-Lambert phase function is given by

$$\Phi_L(\beta) = \cos^4\left(\frac{\beta}{2}\right).$$  \hspace{1cm} (42)

We take advantage of this function’s form by substituting the half-angle formula:

---

**Table 1** Separation order from smallest to largest by star location type.

| Type | Condition | Separation order from smallest to largest |
|------|-----------|------------------------------------------|
| 0    | $s_{x_0-x_1-y} < s_{x_0-x_2-y}, \ s_{x_1-x_0-y}, \ s_{x_2-x_0-y}$ | $s_{x_0-x_1-y}, \ s_{x_0-x_2-y}, \ s_{x_1-x_0-y}, \ s_{x_2-x_0-y}$ |
| 1    | $s_{x_1-x_0-y} < s_{x_0-x_2-y}, \ s_{x_2-x_0-y}, \ s_{x_0-x_1-y}$ | $s_{x_1-x_0-y}, \ s_{x_0-x_2-y}, \ s_{x_2-x_0-y}, \ s_{x_0-x_1-y}$ |
| 2    | $s_{x_1-x_0-y} < s_{x_0-x_2-y}, \ s_{x_0-x_1-y}, \ s_{x_2-x_0-y}, \ s_{x_0-x_1-y}$ | $s_{x_1-x_0-y}, \ s_{x_0-x_2-y}, \ s_{x_2-x_0-y}, \ s_{x_0-x_1-y}$ |
| 3    | $s_{x_1-x_0-y} < s_{x_0-x_2-y}, \ s_{x_0-x_1-y}, \ s_{x_2-x_0-y}, \ s_{x_0-x_1-y}$ | $s_{x_1-x_0-y}, \ s_{x_0-x_2-y}, \ s_{x_2-x_0-y}, \ s_{x_0-x_1-y}$ |
At the same time, from the orbital geometry defined in Fig. 3 and Eq. (3), we write
\[ \beta = \cos^{-1}(\sin(i) \sin(\nu + \omega)), \]  
where the final term is expanded by the angle addition formula as
\[ \sin(\nu + \omega) = \sin(\nu) \cos(\omega) + \cos(\nu) \sin(\omega). \]  
Substituting in \( \beta \) from Eq. (44) expanded with Eq. (45) into Eq. (43) allows us to reduce the order of the fully substituted Eq. (42). Note that this expression for \( \beta \) depends on making the approximation that the observer-star and observer-planet vectors are parallel, which introduces minor error given the large distances to even the nearest stars.\(^{18}\)

After making these substitutions, as well as taking Eq. (4) for the planet-star distance term \( |l_{x,i}| \), simplifying, and collecting all of the non-orbital planet parameters on the left side, we find
\[ \frac{a^2(1 - e^2)^2}{p R^2} 10^{-2.5 \Delta \text{mag}} = \frac{1}{4} (e \cos(\nu) + 1)^2 (1 + \sin(i) \cos(\omega) \sin(\nu) + \sin(i) \sin(\omega) \cos(\nu))^2. \]  
We now have an expression that isolates terms containing \( \nu \) and can be decomposed into a numerically solvable polynomial. To improve solving efficiency and determine which subset of planets should have \( \nu \) solutions for a given \( \Delta \text{mag} \), we need to first calculate the \( \Delta \text{mag} \) extrema over the full orbit. The outline of the process for calculating these \( \Delta \text{mag} \) extrema is included in Appendix C.

To calculate these \( \Delta \text{mag} \) extrema, we first find the derivative of Eq. (46) and multiply by two (for simplification purposes) to get
\[ 0 = -e^2 \sin^2(i) \sin^3(\nu) \cos(\nu) \cos^2(\omega) - 3e^2 \sin^2(i) \sin^2(\nu) \sin(\omega) \cos^2(\nu) \cos(\omega) \\
- 2e^2 \sin^2(i) \sin(\omega) \sin^3(\nu) \cos^2(\nu) + e^2 \sin^2(i) \sin(\nu) \cos^2(\nu) \cos^2(\omega) \\
+ e^2 \sin^2(i) \sin(\omega) \cos^4(\nu) \cos(\omega) - 2e^2 \sin(i) \sin^2(\nu) \cos(\nu) \cos(\omega) \\
- 3e^2 \sin(i) \sin(\omega) \cos^2(\nu) \cos^2(\nu) + 2e^2 \sin(i) \sin(\nu) \cos^2(\nu) \cos^2(\omega) \\
+ 2e^2 \sin^2(i) \sin(\omega) \cos^2(\nu) \cos(\omega) - 2e \sin(i) \sin^2(\nu) \cos(\omega) \\
- 4e \sin(i) \sin(\omega) \cos(\nu) + 2e \sin(i) \cos^3(\nu) \cos(\nu) - e \sin(\nu) \\
- \sin^2(i) \sin^3(\nu) \sin(\omega) \cos(\omega) - \sin^2(i) \sin(\nu) \sin^2(\omega) \cos(\nu) \\
+ \sin^2(i) \sin(\nu) \cos(\nu) \cos^2(\omega) + \sin^2(i) \sin(\omega) \cos^2(\nu) \cos(\omega) \\
- \sin(i) \sin(\nu) \sin(\omega) + \sin(i) \cos(\nu) \cos(\omega), \]  
which is a function of the \( \sin(\nu) \) and \( \cos(\nu) \) terms. We now fully expand this expression and substitute in
\[ \sin(\nu) = \sqrt{1 - \cos^2(\nu)} \]  

to get an expression in \( \cos(\nu) \) only. We define \( x = \cos(\nu) \), isolate the \( \sqrt{1 - x^2} \) term, square both sides, and expand to get an eighth degree polynomial in \( x \) of the form
\[ 0 = A_2 x^8 + B_2 x^7 + C_2 x^6 + D_2 x^5 + E_2 x^4 + F_2 x^3 + G_2 x^2 + H_2 x + I_2. \]  
The coefficients of this expression are included in Eq. (75) in Appendix H.
Although eighth degree polynomials do not have analytical solutions, a numerical root solver can determine the eight roots of this function denoted as the ordered set $x$. We discard any $x_p > 1 \forall p \in \{0 \ldots 8\}$, $x_p < -1 \forall p \in \{0 \ldots 8\}$, or solutions with large imaginary components. We then calculate the remaining true anomaly solutions by $\nu_0 = \cos^{-1}(x)$ and $\nu_1 = 2\pi - \nu_0$. Of the remaining valid solutions, we identify whether each solution is an extremum by evaluating whether $\Delta\text{mag}(\nu_0 \pm \delta \nu)$ are both larger or both smaller than $\Delta\text{mag}(\nu_0)$. If identified as a potential extremum, the smallest and largest extrema are assigned to $\Delta\text{mag}_{\text{min}}$ and $\Delta\text{mag}_{\text{max}}$. The remaining extrema are checked for duplicates, which are identified by solutions with $(\nu, \Delta\text{mag})$ values close to existing extrema. If any solutions are remaining, an additional check and assignment is then made for the local extrema. Through this process, a solution identified in $\nu_0$ has the associated solution in $\nu_1$ removed.

To calculate the $\Delta\text{mag}$ intersections, we apply the same process for turning Eq. (46) into a polynomial as in the $\Delta\text{mag}$ extremum calculation. The full outline for calculating the $\Delta\text{mag}$ intersections is given in Appendix D. By following this process, we arrive at

$$\xi = A_3 x^8 + B_3 x^7 + C_3 x^6 + D_3 x^5 + E_3 x^4 + F_3 x^3 + G_3 x^2 + H_3 x + I_3,$$

(50)

where $\xi$ is the collection of constants on the left side of Eq. (46). The coefficients of this polynomial are given in Eq. (76) of Appendix I.

We again use a numerical root solver to find the roots of this function $x$ and again discard all $x_p > 1 \forall p \in \{0 \ldots 8\}$, $x_p < -1 \forall p \in \{0 \ldots 8\}$, and solutions with large imaginary components. We then calculate the true anomalies of all remaining solutions by $\nu_0 = \cos^{-1}(x)$ and $\nu_1 = 2\pi - \nu_0$. We then evaluate $\Delta\text{mag}_{\text{lim}} = \Delta\text{mag}(\nu_0)$ and $\Delta\text{mag}_{\text{lim}} = \Delta\text{mag}(\nu_1)$ and remove solutions where $|\Delta\text{mag}_{\text{lim}}|^2 > 0.01$ and $|\Delta\text{mag}_{\text{lim}}|^2 > 0.01$. (Note that 0.01 is $< \pm 0.08\%$ error on $\Delta\text{mag}$.) We do an iterative process of selecting $(\nu, \Delta\text{mag})$, which are unique (not duplicate solutions) and are closest to the expected $\Delta\text{mag}_{\text{lim}}$, until we have the expected number of solutions. In the majority of cases in which solutions exist, there are generally only two viable solutions, the assignment of which is simple. In some cases, the solution selection is more ambiguous as double roots are possible.

### 2.3 $\nu$ from $X$ and $Y$

Sections 2.1.2, 2.1.3, and 2.2 give locations of extrema at intersections in the plane of the sky of the form $(x, y)$, but we need to know the true anomalies of the orbit where these intersections occur.

We start with $X$ and $Y$ in Eq. (3) and solve for $(1 + e \cos(\nu))/(a(1 - e^2))$, resulting in

$$\frac{1 + e \cos(\nu)}{a(1 - e^2)} = \frac{1}{Y} [\sin \Omega \cos(\omega + \nu) + \cos \Omega \sin(\omega + \nu) \cos i],$$

(51)

and

$$\frac{1 + e \cos(\nu)}{a(1 - e^2)} = \frac{1}{X} [\cos(\Omega) \cos(\omega + \nu) - \sin(\Omega) \sin(\omega + \nu) \cos i].$$

(52)

In both of these expressions, we substitute angle addition formulas of $\sin(\omega + \nu)$ and $\cos(\omega + \nu)$ and subsequently set the two equations equal to each other.

We set Eqs. (51) and (52) equal to each other, expand, isolate a $\cos(\nu)$ term and $\sin(\nu)$ term, solve for $\nu$, and rearrange to get

$$\nu = \tan^{-1} \left[ -\frac{Y}{X} \sin \Omega \cos \omega - \frac{X}{Y} \cos \Omega \cos i \sin \omega + \cos \Omega \cos \omega - \sin \Omega \sin i \cos \omega \right].$$

(53)

We now have an analytical expression for $\nu$ solely as a function of the KOE, $X$, and $Y$ of a particular point on the orbital ellipse.
By combining the original X and Y equations to solve for \( \nu \) analytically, we have created two potential solutions at \( \nu \) and \( \nu / C \pi \). One of these is the correct \( \nu \) value and the other is not. To calculate the correct intersection point, we calculate the separations at both \( \nu \) and \( \nu / C \pi \) to find absolute error in \( s \) and use the smaller error of the two.

\[ E = \tan^{-1}\left(\frac{\sqrt{1 - e^2} \sin \nu}{e + \cos \nu}\right), \]  

(54)

which gives the corresponding time

\[ t = \frac{E - e \sin(E)T}{2\pi}, \]  

(55)

where \( T \) is the orbital period of a general planet.

### 2.4 Calculate \( t \) from \( \nu \)

We have \( \nu \) for the locations where \( s \) extrema, \( \Delta \text{mag} \) extrema, \( s_{\text{WA}} \)-orbit intersections, and \( \Delta \text{mag} \) intersections occur, but we need them in terms of time. We calculate eccentric anomaly \( E \) of these events directly from \( \nu \) as

\[ E = \tan^{-1}\left(\frac{\sqrt{1 - e^2} \sin \nu}{e + \cos \nu}\right), \]  

(54)

which gives the corresponding time

\[ t = \frac{E - e \sin(E)T}{2\pi}, \]  

(55)

where \( T \) is the orbital period of a general planet.

### 2.5 Converting from Star to Star

The planet-star intersection points and visibility ranges can be saved as either true anomalies or as specific times for a reference star. It makes the most sense to store the values as times and use \( 1 M_\odot \) as the reference time. The times are scaled to any star mass \( M \) by

\[ T = T_\odot \sqrt{\frac{M_\odot}{M}}, \]  

(56)

### 2.6 Calculating Completeness

We calculate integration time adjusted completeness using the aggregated fraction of time that planets are detectable. Using the methods described in Sec. 2 on the instrument’s photometric visibility limit (\( \Delta \text{mag}_{\text{lim}} \)) and astrometric visibility limits (IWA and OWA), we calculate the specific times that the planet enters or exists the instruments visibility limits. By collecting these times and inspecting intermediate test points, we identify the time windows in which any planet is detectable by the instrument. Given an integration time (\( t_{\text{max}} \)) required to reach some SNR consistent with a clear detection (typically a value \( > 5 \)), we throw out all visibility time windows less than \( t_{\text{max}} \) and discount all other visibility windows by \( t_{\text{max}} \). Dividing the sum of all integration time discounted visibility windows for a planet by its orbital period gives us the fraction of time that a planet is detectable by the instrument. Aggregating the fractions of time that planets are detectable by the instrument gives us integration time adjusted completeness. In mathematical form, this is given as

\[ C_{\text{max}} = \frac{1}{N} \left( \sum_{j,k} \sum_{j} (\delta t_{j,k} - t_{\text{max}} U_{k}) \right), \]  

(57)

Here \( \delta t_{j,k} \) is the \( j \)'th time window larger than \( t_{\text{max}} \) in which the \( k \)'th planet is visible, \( U_{k} \) is a boolean that indicates that the planet is always visible \( 0 \) or at least sometimes visible \( 1 \), \( N \) is the total number of planets, and \( T_k \) is the orbital period for the \( k \)'th planet.

We also calculate the completeness for an Earth-like exoplanet of the Earth-like exoplanets subpopulation:

\[ C_{\oplus, t_{\text{max}}} = \frac{1}{N_{\oplus}} \left( \sum_{j,k} \sum_{j} (\delta t_{j,k} - t_{\text{max}} U_{k}) \right), \]  

(58)

where \( N_{\oplus} \) is the total number of Earth-like planets simulated.
Similarly, this capability can be extended to the entirety of the Kopparapu classification scheme. Our method uses orders of magnitude fewer exoplanets and an equivalent memory but fills in the \( \Delta \text{mag} \) space using an order of magnitude better accounting with strategically calculated true anomalies.

### 2.7 Calculating Dynamic Completeness

The methods presented in this paper can also be used to calculate dynamic completeness. Dynamic completeness extends the original concept of completeness by considering the fraction of planets observable on subsequent observations of the same target. For the original formulation of completeness, which relies on simulating a cloud of planets, this requires propagating every simulated planet along its orbits. Dynamic completeness is frequently used to compute the fraction of planets in a population that are initially undetectable but become detectable upon a second observation some time later.

We define \( P \) as the unordered set of all planets, \( P_{\text{detected}1} \) as the set of planets detected at observation time one, and \( P_{\text{undetected}1} \) as the set of planets not detected at observation time one. To find the sets of detected planets at the first observation, we start by generating a random observation time for the \( k \)th planet \((t_{\text{start};k})\) between 0 and \( T_k \). We then determine which planets are within the instrument’s visibility limits at \( t_{\text{start};k} \) to create \( P_{\text{detected}1} \) and its complement \( P_{\text{undetected}1} \).

For a subsequent observation some time later \((t_{\text{wait}})\), we find the time past periastron at which the observation occurs by \([\text{mod}(t_{\text{start}} + t_{\text{wait}}, T_k)]\) and determine which planets are within the instrument’s visibility limits at that time. This defines the unordered set of planets detected at observation time two as \( P_{\text{detected}2} \). Its complement is \( P_{\text{undetected}2} \), the set of planets not detected at observation time two.

Dynamic completeness for the second observation of a target star is given by the fraction of planets undetected in the initial observation but detected on the second one:

\[
C_2 = \frac{1}{N} n(P_{\text{undetected}1} \cap P_{\text{detected}2}),
\]

where \( n(P) \) is a function giving the number of elements in set \( P \) and \( N \) is the total number of planets.

Dynamic completeness of the \( m \)th visit is similarly given by the fraction of previously undetected planets that are detected on that visit:

\[
C_m = \frac{1}{N} n \left( \left( \bigcap_{j=0}^{m-1} P_{\text{undetected}j} \right) \cap P_{\text{detected}m} \right).
\]

### 3 Results

#### 3.1 \( \nu \) from \( s \)

We apply the methods developed Sec. 2 to calculate the planet-star separation extrema, the true anomalies at which they occur, the times past periastron at which the extrema occur, the true anomalies where the separation circle intersects with the projected orbital ellipse, and the times at which these intersections occur for a planet orbit. Figure 9 shows the projected separation of a sample orbit with global and local extrema calculated via Sec. 2 methods. This figure also demonstrates the ability of our methods to find all true anomalies (and times) when the projected separation takes a specific value (in this case \( s_{\text{WA}} = 1 \text{ AU} \)).

We now determine the accuracy of the methods implemented. To do this, we calculate the true anomalies of \( s_{\text{WA}} \)-orbit intersections for \( 10^5 \) orbits using the method in Sec. 2.1.3 and \( s_{\text{WA}} = 1 \text{ AU} \). We then calculate the planet-star separation by substituting these true anomalies back into Eq. (2) and plot the error between these separations and the input \( s_{\text{WA}} \) in Fig. 10. Of the \( 10^6 \) planets orbits simulated, \( \sim 25,000 \) orbits produced two or more intersections. The largest
intersection error observed is $<10^{-5}$. Machine precision error is $\sim 10^{-16}$, and the square root of this is $10^{-8}$, which indicates that we have approached machine precision in our method.

### 3.2 $\nu$ from $\Delta \text{mag}$

We apply the methods developed Sec. 2 to calculate the $\Delta \text{mag}$ extrema, the true anomalies at which they occur, the times past periastron at which the extrema occur, the true anomalies where the $\Delta \text{mag}$ intersections occur, and the times at which these $\Delta \text{mag}$ intersections occur for a planet orbit. Figure 11 shows the $\Delta \text{mag}$ of a sample orbit with global and local extrema calculated via Sec. 2 methods. This figure also demonstrates the ability to find all true anomalies (and times) when the $\Delta \text{mag}$ takes a specific value (in this case $\Delta \text{mag}_{\text{lim}} = 25.0$).

To check the error in true anomaly produced by the $\Delta \text{mag}$ intersection method in this paper, we compare the $\nu$ of $\Delta \text{mag}$ intersections calculated with alternative numerical solving methods. The method presented in this paper finds $\Delta \text{mag}$ intersections of $10^5$ planets within $\sim 6.3$ s. The first error checking method uses a cubic spline fit to 300 ($\nu$, $\Delta \text{mag}$) points along each planet’s orbit. We then subtract $\Delta \text{mag}_{\text{lim}} = 29$ (our test point) from the spline and find the roots. This univariate spline root solving method is capable of executing in 419 s on the 510,120 planets in the population that produced two intersections, a rate of $8.1 \times 10^{-4}$ s per planet. The univariate spline root solving method is $100\times$ slower than the $\Delta \text{mag}$ intersection method in Sec. 2.2. The separation error between the univariate spline method and the $\Delta \text{mag}$ intersection method is $<6 \times 10^{-5}$ for $>99.98\%$ of planets and $<10^{-4}$ for the other $0.02\%$. These high-error targets are planets with low variation in $\Delta \text{mag}(\nu)$ orbits. The second-error checking method solves for the ($\nu$, $\Delta \text{mag}$) intersections using a numerical minimization method on a random subset of $10^5$ planets. This method is far more inefficient, taking 1646 s on $10^4$ planets with two intersections at a rate of 0.1646 s per planet. Using a minimization function allows us to determine the error in $\Delta \text{mag}$ of an intersection point to within $10^{-8}$. This numerical minimization method independently confirms the accuracy of solutions to the univariate spline roots method on the limited number of targets tested. We plotted the normalized frequency of true anomaly error of both the numerical minimization method and univariate spline root solving method compared against the $\Delta \text{mag}$ intersection method in Fig. 12. Note that the total number of incidences of a given error is normalized by the bin width and total number of targets, so the non-normalized frequency of the $3 \times 10^{-11}$ bin is incredibly large compared with the $5 \times 10^{-4}$ bin. Both methods indicate that the resulting true anomalies of the $\Delta \text{mag}$ intersections are within $10^{-4}$ rad of each other.

Although the $\Delta \text{mag}$ intersection method featured in this paper is orders of magnitude faster than the univariate-spline-roots method and just as accurate, the quasi-Lambert phase function required for its use is not the best phase function for all planets; however, it does fit Earth-like planet phase function quite well. At a substantial time cost, the univariate-spline-roots method can be used for planets with any phase function.

### 3.3 Convergence and Validation

We test convergence of our method for calculating completeness by repeatedly calculating completeness on $10^5$ planets. We test convergence of Brown’s method by repeatedly calculating completeness over a set of a logarithmically increasing number of planets. We use the same SAG13 planet population parameters from Ref. 1 but make the substitution of a quasi-Lambert phase function to define the underlying planet population. Since this technique will be most relevant to future highly capable telescopes, we are using the HabEx IWA of 0.045 arc sec, OWA of 6 arc sec, and upper $\Delta \text{mag}$ limit of 25 on a star that is 10 pc away. This resulting instrument visibility limits and planet population cover a well populated region of the $\Delta \text{mag}$ versus $s$ joint probability density function [see Fig. 1 in Ref. 1 for a comparable example of the joint probability density function]. Convergence of the two completeness methods is demonstrated in Fig. 13.

Both methods presented in Fig. 13 demonstrate similar percent convergence to their own converged means. Completeness repeatedly calculated using the method in Sec. 2.6 with $10^5$ planets has a mean of 0.25785 with a standard deviation of 0.0010 for $4.28 \times 10^8$ planets. In comparison, the Brown completeness method converges to 0.25783. This error in the converged
mean of the two methods is consistent with the completeness error identified in Fig. 14. We expect that completeness calculated using the method in this paper will have comparatively better convergence in sparsely populated regions of the $\Delta \text{mag}$ versus $s$ joint probability density function (see Fig. 1 in Ref. 1).

Since completeness for an individual planet is the fraction of time that a planet is visible to the instrument, we validate individual planet visibility windows using test points. To calculate a ground truth fraction of time that a planet is visible, we create $10^5$ test points, evenly spaced in time, and calculate both the planet-star separation and planet-star $\Delta \text{mag}$ at each of these test points. We then determine whether each point is within the visibility limits of the instrument. The fraction of visible points is the fraction of time that a planet is visible. We repeated this for 25,000 planets due to computational cost of the test point method. The histogram of error in completeness for individual planets calculated using the test point method and method in this paper is shown in Fig. 14. The maximum error in completeness is observed to be near $5 \times 10^{-5}$, which is consistent with the converged error between Brown completeness and integration time adjusted completeness.

The test point method that we used to validate integration time adjusted completeness calculations took over 52 h on only 25,000 planets (less than a quarter of the planets needed to compute completeness using integration time adjusted completeness). The computation time for the integration time adjusted completeness calculation on $10^5$ planets is 21.34 s with a standard deviation of 0.4 s over 1000 calculations. Brown completeness, including the joint probability density function generation, has a mean execution time of 3.19 s and standard deviation of 0.13 s over 1000 calculations. This execution time of all of these methods scales linearly with the number of planets. Some room for optimization exists in integration time adjusted completeness.

### 3.4 Completeness Versus Integration Time

To evaluate the effects of integration time on completeness, we calculate integration time adjusted completeness for various integration times, star distances, and planet populations (SAG13 and Earth-like planet population). Figure 15 shows the integration time adjusted completeness for the SAG13 planet population\(^1\) and the Earth-like planet subpopulation defined in Appendix L. The decrease in completeness for longer integration times is most prominent for nearby stars for both populations. As expected, longer integration times decrease completeness.

For the assumed observatory parameters, the Brown completeness of an Earth-like population for a target 5 pc away is 0.583. As we increase integration time to 1, 2, and 5 days, the associated integration time adjusted completeness decreases by 0.63%, 1.27%, and 3.15%, respectively. Recalculating Brown completeness for a target at 25 pc in Fig. 15(b) gives 0.583. As we increase integration time to 1, 2, and 5 days on this 25 pc target, the associated integration time adjusted completeness decreases by 0.989%, 1.97%, and 4.92%, respectively. Our integration time adjustment of completeness shows Brown completeness overestimates exoplanet yield for any observation. Brown completeness applied to stars farther away results in a substantial overestimation of exoplanet yield.

Technically, the $\Delta \text{mag}_{\text{lim}}$ used as the upper limit for calculating completeness is a function of integration time and will approach a theoretical upper limit as the noise floor is reached. By calculating $\Delta \text{mag}_{\text{lim}}$ using Eq. (12) of Ref. 1, we can calculate completeness at each integration time to get the completeness versus integration time curve. We created a 4-m telescope as in Appendix M and evaluated completeness at each integration time to create an example demonstrating how Brown completeness and integration time adjusted completeness vary with integration time in Fig. 16. We can see that the completeness of both methods tracks very closely until they diverge. Brown completeness approaches its asymptotic limit as expected, whereas integration time adjusted completeness approaches a maximum and begins decreasing past $\sim$0.2 day.

### 3.5 Dynamic Completeness

We compute dynamic completeness for the example test case in Fig. 1 of Ref. 13 using the method described in this paper and replicating the approach of the original work. This computation is only for a second epoch. We replicated the original work by sampling a large number of planets, finding the planets initially visible, propagating these planets to some time past initial
observation, and determining what fraction of these planets were not detected the first time but were detected the second time. We replicated Brown’s work using both the Lambert phase function (orange line in Fig. 17) to match the original work as well as the quasi-Lambert phase function (red line in Fig. 17).

We compared the dynamic completeness computation time for 1000 dynamic completeness calculations testing 1000 individual points in time past the initial observation. Brown’s dynamic completeness had an average execution time of 34.62 s with a standard deviation of 0.99 s. Calculating dynamic completeness using the method presented in this work resulted in an average execution time of 4.757 s with a standard deviation of 0.036 s. The dynamic completeness calculation presented in this paper is ≈7x faster than the traditional Brown completeness method, enabling its use in optimization.

4 Discussion

4.1 Convergence

Both methods of calculating completeness suffer from sparse sampling of the exoplanet population parameter space. Brown completeness is most affected as it needs planets sampled over \( \Omega, \omega, a, e, i, \nu, p, \) and \( R \). The current implementation of our method needs the same parameters except for \( \nu \). We are therefore surprised that Brown’s method and our method share such similar completeness convergence. We hypothesize that the similar convergence results are because the integration bounds used for calculating completeness are over a dense region of the joint probability density function. If we examined a more sparse region of the joint probability density function such as high \( \Delta \text{mag} \) or larger separation (where the larger, lower occurrence rate planets occur), we expect our completeness methods to have marginally better convergence for low numbers of planets. Regardless, integration time adjusted completeness easily reaches completeness errors below \( 10^{-3} \) (translates to ≈0.8% where \( C = 0.12 \), the largest completeness of an observation in Ref. 1), which is a sufficient error for use in optimization.

4.2 Reducing Parameter Spaces

The integration time adjusted completeness method implemented in EXOSIMS is not cached for computational efficiency like the Brown completeness is. Although we are able to store the joint probability density function of \( \Delta \text{mag} \) versus \( s \) with Brown’s method, integration time completeness requires the KOE of each simulated planet to be stored. The curse of dimensionality prohibits us from finely sampling the entire subspace of exoplanets, making the caching of integration time adjusted completeness prohibitive, but not all our methods require all 6 KOE, \( p \), and \( R \). So, in the future, it may be possible to create a representative subsampling of planets weighted by occurrence rate based on the parameters needed for different calculations. For example, \( s \) extrema, \( s_{WA} \)-orbit intersections, and \( \Delta \text{mag} \) extrema calculations only require \( \omega, a, e, \) and \( i \) (Note that the \( \nu \) locations of the \( \Delta \text{mag} \) extrema are independent of \( p \) and \( R \), but the magnitude of \( \Delta \text{mag} \) extrema are dependent upon them). The photometric property \( p \times R \) could then be sampled and independently combined with individual \( (\omega, a, e, i) \) combinations. The primary benefit is that, given some instrument parameters and star properties, we could determine the subset of parameters that can physically be observed prior to calculating completeness, thus making the per-star completeness calculations more efficient.

4.3 Limitations

The approach we implemented in this paper did not vary the \( \Delta \text{mag}_{\text{lim}} \) as we varied the integration time. If properly executed, \( \Delta \text{mag}_{\text{lim}} \) should increase with increased integration time. However, calculating the planet true anomaly intersections with a given \( \Delta \text{mag}_{\text{lim}} \) is the most expensive calculation, so repeatedly calculating this to convergence is not desirable.

The \( \Delta \text{mag}_{\text{lim}} \) of coronagraph designs are working angle dependent and therefore separation dependent, but we assume that this limit is a fixed quantity. If there is substantial variation in the
limiting $\Delta \text{mag}$, it is better to use the conservative value with this method. If a higher level of predictive accuracy is desired, these methods could be used to ascertain if any intersection with the $\Delta \text{mag}_{\text{lim}}$ exists and subsequently another method could be used on the subset of planets with known intersections.

### 4.4 Impact of the Integration Time Adjusted Completeness

As we showed in this paper, integration time adjusted completeness and Brown completeness converge to the same value to within 0.00002 when evaluated at $t_{\text{max}} = 0$ day. The integration time adjusted completeness for an observation of an Earth-like planet population on a 1-$M_\odot$ star 5 pc away and integration time of $\approx 1$ day is 0.64\% lower than the comparable Brown completeness. For reference, the Roman target list in Table 9 of Ref. 1 has a maximal integration time of 1.71 days but most are $< 0.6$ day. This means that we could expect an average reduction in yield below 0.64\% when observing a population of Earth-like planets around Sun-like stars. This integration time completeness adjustment is within the 3.19\%, 3$\sigma$, margin of error from 1000 Monte Carlo simulations of the cycle 6 Roman in Ref. 1 and cannot be considered statistically significant. However, a $\approx 4$-day integration time on a Sun-like star 25 pc away observing an Earth-like population has a decrease in completeness above the Ref. 1 3$\sigma$ margin of error. However, the threshold of statistical significance should not deter the widespread use of integration time adjusted completeness as the adoption of this method can shore up the differences between completeness-based yields and simulation-based yields. We can also say that integration time has a muted effect when observing stars that are farther away. This can most likely be attributed to the increase in $s_{\text{min}}$ and resulting decrease in the total time-fraction that smaller period planets spend within the visible limits of the telescope. Big planets with the orbital radius of Jupiter will move more slowly and be less affected by integration times.

In Appendix M, we optimized an exoplanet direct imaging mission maximizing single-visit Brown completeness yield and included the resulting Design Reference Mission (DRM) in Table 2. The resulting DRM has a Brown completeness yield of 387.16 exoplanet detections in a single-visit detection survey on average, but the integration time adjusted completeness yield expects only 354.67 exoplanets on average. This means that Brown completeness for the particular instrument parameters optimized over the SAG13 planet population with the particular mission parameters overpredicts the actual exoplanet yield by 32.49 exoplanets on average, 9.16\% more than the integration time adjusted completeness yield. When calculating completeness, we are careful to scale the completeness limits of integration by the star’s luminosity. For the integration time adjusted completeness method, we additionally scale the planet’s orbital periods based off the mass of the host star. The percentage difference above and beyond that expected from Fig. 15 can be found by looking at these two adjustments applied to each star and the target list in Table 2. The average star in the target list has a larger mass and a brighter luminosity than that of the Sun. The brighter luminosity results in a smaller $s_{\text{min}}$ and $s_{\text{max}}$, which serves to incorporate more smaller semimajor axis planets into the completeness calculations, and the larger star masses result in shorter periods, meaning that the planet visibility windows all decrease in duration.

Integration time adjusted completeness is crucial for determining the ability of an Earth-like planet to be spectrally characterized. A spectral characterization with a coronagraph could take between a few days and 60 days. Because of the long integration time, the planners of a HabEx use the “characterization completeness” of 10\% and maximum integration time of 60 days as a filter on stars to consider observing. The calculation of this critical filter could be substantially improved by considering integration time adjusted completeness if the spectrum of the planet must be taken all at once (i.e., observation spanning multiple weeks) and not spread across multiple epochs. Just because a planet can be detected does not mean that it can be spectrally characterized.

### 4.5 Dynamic Completeness and Computation Cost

The greatest benefit of using the methods in this paper to calculate completeness is the marginal additional cost of calculating dynamic completeness. Generally, dynamic completeness requires
the computation of true anomalies from time, which is prohibitive for >10^5 orbits at >100 different times in the future. With our methods, calculating visibility windows allows us to use boolean operations to compute dynamic completeness about 7x faster than Brown’s dynamic completeness method. The computational cost of calculating completeness using the method described in this paper is an order of magnitude larger than Ref. 2, but it can use orders of magnitude fewer planets to do so. Unlike Brown’s method, within the computation time of our method, we also get additional desirable information about the detectable planets such as their \( s \) extrema and \( \Delta \text{mag} \) extrema.

4.6 Revisiting the Same Exoplanet

Another limitation present in the planning of exoplanet direct-imaging missions is the telescope keep-out angles. Figure 3 of Ref. 1 contains a keep-out map for a subset of the DRM created in that paper. The smallest percent of time that a target star is visible for the Roman is nominally 28%. Due to symmetry of the keep-out region, this translates into two separate time windows of visibility of \( \sim 51 \) days. Instead of considering the integration time an integration time input, it can also be considered a revisit time input for determining the probability of being able to observe a planet twice in the same target-star visibility window.

4.7 Exoplanet Classification

The underlying methods in this paper are used to find locations along a planet’s orbit where \( s \) and \( \Delta \text{mag} \) intersections occur. The methods in this paper, with some modification, can also be used to probabilistically classify an exoplanet subtype. If an exoplanet is detected with a particular \( (s, \Delta \text{mag}) \) and an uncertainty region of \( s \pm \sigma_s \) and \( \Delta \text{mag} \pm \sigma_{\Delta \text{mag}} \), then the methods in this paper can be applied to each of these four bounding lines. By finding the average orbital time-fraction that exoplanets of each type spend in the bounding uncertainty box, we can find the probability that the exoplanet belongs to a specific exoplanet subtype. This requires additional work beyond the scope of this paper.

5 Conclusion

We have demonstrated an accurate method for calculating integration time adjusted completeness and its adaptation to calculating dynamic completeness. In the process, we also created fast and accurate methods for calculating the true anomalies where a planet’s orbit has specific values of projected separation \( \Delta \text{mag} \) and their extrema. We demonstrated how to use these methods to calculate a more accurate integration time adjusted completeness using the fractions of time that a planet is detectable by an instrument. We demonstrate that traditional methods of calculating completeness overestimate the number of observable planets because they do not subtract the integration time used in observing the target. For a Sun-like star at 25 pc with 1-day and 5-day integration times, integration time adjusted completeness of Earth-like planets is reduced by 1% and 5%, respectively. We applied integration time adjusted completeness to a target list optimized using the Brown completeness method and found that Brown completeness overestimated yields by 9.61%. We also demonstrated that our methods can be used to quickly calculate dynamic completeness for determining the optimal time to revisit a target star.

6 Appendix A: Common Notation

There are many common variable forms used in this paper that are simply summarizable. Any variable \( x \) refers to a 3D vector in \( X, Y, \) and \( Z \) coordinates of something. We use \( x \) to indicate an array of variables, specifically used when referring to multiple roots of a polynomial. \( |\overrightarrow{AB}| \) refers to the length of line segment \( \overrightarrow{AB} \). Line segment \( \overrightarrow{AB} \) is treated as a vector. \( C' \) is a projection of point \( C \).

There are multiple different subscripts with different meanings used in this paper. A subscript with \( x_i \) refers to the index of a host star (out of some whole, non-descript, star catalog).
A subscript $x_k$ refers to an individual planet out of the whole large set of planets. Subscripts of $x_{\text{min}}$, $x_{\text{max}}$, $x_{\text{lmin}}$, and $x_{\text{lmax}}$ are descriptors on the individual variable $x$ that indicate that the variable is associated with the minimum, maximum, local minimum, or local maximum, respectively. When solving for the distance between points inside an ellipse and the vertices or co-vertices of that ellipse with semimajor axis $a$, we use the shorthand notation of the form $s_{a+x,y}$. [This is specifically the distance between point $(x, y)$ and $(-a, 0)$].

We denote coefficients of the polynomials of the four methods that solve the quartic in this paper as $A_i$.

In Fig. 4, we reference many points on the 3D elliptical orbit and 2D projection of this orbit into the plane of the sky. Points on the original 3D ellipse are labeled $B, H, P, C, K, O, D, A, B', H', P', C', K', O', D'$, and $A'$. In Appendix K, we use $p_0$ to $p_{11}$ as intermediate constants for simplifying the full quartic expression. $P, D$, and $\Delta$ (by itself and only in this section) are intermediate constants derived from quartic coefficients for determining the sign of the quartic solutions.

$K_0$ to $K_9$ are intermediate simplifying coefficients used in representing Eqs. (13)–(15) in Sec. 2.1.1. The full expansion of these equations is included in Appendix E.

![Fig. 9](image-url) Planet-star separation of the planet is plotted in black. The separation extrema are indicated by diamonds, where the maximum is red, local maximum is yellow, local minimum is magenta, and minimum is teal. Lines are drawn at the minimum and maximum separations. For the input separation of $s = 1$ AU, the green dots are the analytically calculated orbit intersections. (a) The separation versus true anomaly and (b) the separation versus time past periastron for a planet orbit with $a = 1.82$ AU, $e = 0.09$, $\Omega = 3.37$ rad, $\omega = 4.86$ rad, and $i = 1.25$ rad.
Fig. 10 $s_{WA}$-orbit intersection error histogram calculated for a separation of $s_{WA} = 1$ AU for $10^5$ planet orbits generated from the SAG13 planet population. Of these $10^5$ planet orbits, 6201 orbits have two intersections with the $s_{WA}$ circle, and 17,952 have four intersections with the $s_{WA}$ circle. This results in a total of 84,210 planet-star intersections. After calculating and identifying true anomalies of intersections using the methods described in this paper, we evaluated the planet-star separation of each orbit at the true anomalies and calculated error from the input $s_{WA}$.

Fig. 11 (a) The $\Delta$mag of a planet (black line) plotted against $\nu$ and (b) time past periastron. $\Delta$mag extrema are indicated by diamonds, where the maximum is red, local maximum is yellow, local minimum is magenta, and minimum is teal. Separation minimum and maximum are indicated by horizontal lines. For an input $\Delta$mag$_{\text{lim}} = 25$ (green line), we calculated the specific true anomalies (green dots) where the planet’s $\Delta$mag intersects this line. The specific planet KOE are $a = 5.36$ AU, $e = 0.56$, $\omega = 5.06$ rad, $\Omega = 0.69$ rad, $i = 0.81$ rad, $p = 0.3$, and $R = 4R_\odot$. 

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Fig. 12 A histogram of $\nu(\Delta \text{mag})$ error between the root solving method described in this paper and a root solver of a cubic spline fit method (purple) for 510, 120 planet orbits that produce two intersections (totaling 1,020,240 datapoints). We also compare the error between the root solving method described in this paper and an optimization method (blue) for $10^4$ planet subset of the planet orbits that produce intersections. The optimization method investigates fewer planets because it is orders of magnitude more computationally expensive, but it is a fundamentally different approach to validating our method than a root solver.

Fig. 13 Convergence of % error in completeness for varying numbers of planets used in its computation for Brown’s method (green) and the method presented in this work with $t_{\text{max}} = 0$ day (purple). Completeness at the maximum number of planets is assumed to be the converged value of completeness of the respective methods. The converged mean of Brown’s method is 0.25783 compared with the converged mean of the method in this paper of 0.25785. The standard deviation of the method in this paper is 0.0010.
Fig. 14  Per planet completeness error histogram between the integration time adjusted completeness method with $t_{\text{max}} = 0$ day and the test point method for 25,000 planets.

Fig. 15  Integration time adjusted completeness values of (a) SAG13 planet population and (b) Earth-like planet population with $\text{IWA} = 0.045$ arc sec, $\text{OWA} = 6$ arc sec, and $\Delta \text{mag} = 25$ for varying integration times and star distances.

Fig. 16  Brown completeness and integration time adjusted completeness versus integration time for HIP 32,279 assuming a mass of $1.564 \ M_\odot$, luminosity of $7.12 \ L_\odot$, and distance of $d_f = 3.51$ pc. The assumed telescope is a 4-m monolith with $\text{IWA} = 0.045$ arc sec and $\text{OWA} = 6$ arc sec and $\Delta \text{mag}_{\text{lim}}$ computed using the instrument noise model as in Ref. 1.
Appendix B: True Anomalies of s Intersection Process

The process outlined here is used in Sec. 2.1.2.

1. Define KOE.
2. \( \frac{r_k}{i} \) from KOE.
3. Orbiting foci \( F \) from KOE.
4. Calculate 3D orbit ellipse center.
5. Project 3D ellipse to 2D ellipse.
   (a) Prove that 3D ellipses project to 2D ellipses.
   (b) Project 3D ellipse center to 2D ellipse center.
   (c) Project ellipse orbiting foci into plane of the sky.
   (d) Project semiminor axis line of 3D ellipse to 2D ellipse.
   (e) Project semimajor axis line of 3D ellipse to 2D ellipse.
   (f) Calculate \( OQ \) construction line from semiminor axis and semiminor axis.
   (g) Calculate projected ellipse semimajor axis and semiminor axis from \( O'Q' \) and \( O'Q \).
   (h) Calculate projected semimajor axis angular offset from \( X \) axis of plane of the sky.
6. Derotate the projected ellipse.
7. Center the projected ellipse.
8. Find minimum, maximum, local minimum, and local maximum of projected planet-star separation.
   (i) Formulate projected planet-star separation equation.
   (j) Reformat and set \( \delta \dot{x}^2 / \delta x_e \) equal to 0.
   (k) Combine coefficients into standard quartic form.
   (l) Use standard general quartic solutions.
   (m) Take the absolute value and only real component of \( x_e \) solutions.
   (n) Calculate \( y_e \) associated with each solution.
   (o) Assign solutions to minimum, maximum, local minimum, and local maximum planet-star separations.

Fig. 17 Dynamic completeness assuming the same parameters as in Ref. 13 (\( \Delta \text{mag} = 26, M_{HIP29271} = 1.103 M_\odot, d_i = 10.215 \text{pc}, \text{OWA} = 600 \text{arc sec}, \text{IWA} = 0.075 \text{arc sec}, 0.7 \sqrt{L} \leq a \leq 1.5 \sqrt{L}, 0 \leq e \leq 0.35, p = 0.26, R = 1R_\odot, \text{and } L = 0.83L_\odot). In this work (blue), we assume a quasi-Lambert phase function, but in Ref. 13, a Lambert phase function is assumed. Work was done to replicate Brown’s original work (Brown Lambert, orange line) and replicate Brown’s work using the quasi-Lambert phase function (Brown quasi-Lambert, red line).

7 Appendix B: True Anomalies of \( s \) Intersection Process

The process outlined here is used in Sec. 2.1.2.
i. For all solutions (all real and only 2 real):
   A. Calculate $s_{\min}$ from $x_1$ and assign $(x_{\min}, y_{\min})$ of quadrant 1.
   B. Calculate $s_{\max}$ from $x_0$ and assign $(x_{\max}, y_{\max})$ of quadrant 1.

ii. All real solutions (where all solutions have $\Im(y) < 10^{-5}$, additionally assign).
   C. The larger of $s(x_2)$ and $s(y_1)$ becomes $s_{\text{max}}$ and assign $(x_{\max}, y_{\max})$ of quadrant 1.
   D. The smaller of $s(x_1)$ and $s(y_1)$ becomes $s_{\text{min}}$ and assign $(x_{\min}, y_{\min})$ of quadrant 1.

iii. Assign solution signs to proper quadrants.

9. Find circle and projected derotated centered ellipse intersection points.

(p) Formulate circle-ellipse intersection equation.
(q) Reformat into standard quartic form.
(r) Use standard general quartic solutions.
(s) Classify intersection solutions.
   iv. Inside outer separation bounds, $s_{\min} < s < s_{\max}$.
   v. Inside local min/max separation bounds, $s_{\text{min}} < s < s_{\text{max}}$.
   vi. Outside outer separation bounds (no intersections).

(t) Calculate intersection solutions for planets with $s_{\min}$, $s_{\max}$, $s_{\text{min}}$, and $s_{\text{max}}$.

vii. Two intersections on same $y$ side of ellipse as the star, $s_{\min} < s < s_{\text{min}}$.

\[ \begin{cases} x_3, \Re(x_1) > 10^{-10} \\ x_1, \Re(x_1) < 10^{-10} \end{cases} \]

\[ x_0 \]

viii. Four intersections where, $s_{\text{min}} < s < s_{\text{max}}$

- Calculate $\Delta x = x_k - |x|$, ($k \in \{0, 1.2.3\}$).
- Sort $\Delta x_k$ from min to max and rearrange the associated $x_k$ to match that order.
- Same XY points is $x_3$.
- Same X opposite Y is $x_2$.
- Opposite X Same Y is $x_0$.
- Opposite X opposite Y is $x_1$.

ix. Two intersections on opposite $x$ side of ellipse as the star, $s_{\text{max}} < s < s_{\max}$.

- $x_0$ and $x_1$.
- $y_1$ is opposite sign of $|y|$.

(u) Calculate intersection solutions for planets with only $s_{\min}$ and $s_{\max}$.

x. Calculate projected ellipse quadrant separation bounds $s_{x,b+y}$, $s_{x,b-y}$, $s_{a+x,y}$, and $s_{a-x,y}$.

xi. Identify star location type and quadrant order.

- Type 0 occurs where $s_{x+d,y} < s_{x,y+b}$. Smallest to largest order: $s_{x,b-y}$, $s_{a-x,y}$, $s_{x+y,b}$, $s_{x+a,y}$, $s_{x,y+b}$.
- Type 1 occurs where $s_{x,y+b} < s_{a-x,y}$. Smallest to largest order: $s_{x,b-y}$, $s_{x+y,b}$, $s_{a-x,y}$, $s_{x+a,y}$.
- Type 2 occurs where $s_{a-x,y} < s_{x,y+b}$. Smallest to largest order: $s_{a-y,b}$, $s_{a-x,y}$, $s_{x,y+b}$, $s_{x+y,b}$.
- Type 3 occurs where $s_{a-y,b} < s_{x,y+b}$ Smallest to largest order: $s_{a-x,y}$, $s_{x,b-y}$, $s_{x,y+b}$, $s_{x+a,y}$.
8 Appendix C: $\nu$ from $\Delta \text{mag}$ Extrema process

Here we present the full outline of our process for calculating true anomalies where the planet’s orbit has a $\Delta \text{mag}$ extrema. These steps are discussed in Sec. 2.2.

1. Substitute into Eq. (1) components and expand until all terms are functions of KOE.
2. Manipulate the $\Delta \text{mag}$ equation until all $\nu$ terms are isolated on the right side.
3. Take the derivative.
4. Replace $\sin(\nu)$ with $\sqrt{1 - \cos(\nu)^2}$.
5. Replace all $\cos(\nu)$ with $x$.
6. Isolate all square root terms to general form $A^2\sqrt{1 - x^2} = B^2$.
7. Square both sides.
8. Find polynomial coefficients of $x$.
9. Solve with root solver.
10. Filter out $x$ solutions with imaginary components ($>10^{-7}$) ($10^{-8}$ is approximately $\sqrt{10^{-16}}$, machine precision).
11. Filter out invalid solutions, $1 < x$ or $-1 > x$.
12. Compute $\nu_0 = \cos^{-1}(x)$ and $\nu_1 = 2\pi - \nu_0$.
13. Compute associated $\Delta \text{mag}_0$ and $\Delta \text{mag}_1$.
14. Remove solutions that are not extrema.
15. Assign solutions $\Delta \text{mag}_{\text{max}} = \min(\Delta \text{mag}_0, \Delta \text{mag}_1)$.
16. Assign solutions $\Delta \text{mag}_{\text{min}} = \max(\Delta \text{mag}_0, \Delta \text{mag}_1)$
17. Remove duplicate solutions.
18. Remove assigned solutions.
19. Check if solution is extrema.
20. Assign $\Delta \text{mag}_{\text{lim}}$ and $\Delta \text{mag}_{\text{max}}$.

9 Appendix D: $\nu$ from $\Delta \text{mag}_{\text{lim}}$ Intersection Process

Here we present the full outline of our process for calculating true anomalies where the planet’s orbit has a specified $\Delta \text{mag}$, which we call an intersection. These steps are discussed in Sec. 2.2.

1. Find $\Delta \text{mag}$ extrema.
2. Find planets where $\Delta \text{mag} < \Delta \text{mag}_{\text{max}}$, $\Delta \text{mag} > \Delta \text{mag}_{\text{min}}$, and $\Delta \text{mag} > \Delta \text{mag}_{\text{lim}}$ or $\Delta \text{mag} < \Delta \text{mag}_{\text{lim}}$ (these planets have only two intersections).
3. Substitute into $\Delta \text{mag}$ equation components and expand until all terms are functions of KOE.
4. Manipulate the equation until all $\nu$ terms are isolated on the right side.
5. Expand.
6. Replace $\sin(\nu)$ with $\sqrt{1 - \cos(\nu)^2}$.
7. Replace all $\cos(\nu)$ with $x$.
8. Isolate all square root terms to general form $A^2\sqrt{1 - x^2} = B^2$.
9. Square both sides.
10. Expand.
11. Find polynomial coefficients of $x$.
12. Solve with root solver.
13. Filter out solutions with imaginary components ($>10^{-7}$).
14. Filter out solutions $1 < x_k$ or $-1 > x_k$.
15. Take arrays $\nu_0 = \cos^{-1}(x)$ and $\nu_1 = 2\pi - \nu_0$.
16. Solve for associated arrays $\Delta \text{mag}_0$ and $\Delta \text{mag}_1$. 
17. Remove solutions where $|\Delta \text{mag}_0 - \Delta \text{mag}| > 0.01$ and $|\Delta \text{mag}_1 - \Delta \text{mag}| > 0.01$ (Note that 0.01 is $\pm 0.08\%$ error on $\text{dmag}$).
18. Remove duplicate solutions.
19. Verify that there are only two viable solutions.
20. Assign solutions to $\nu$.

10 Appendix E: Projected Ellipse Semimajor Axis and Semiminor Axis, and $\theta$

We write the analytical expressions for $a_p$, $b_p$, and $\theta$ using full expansions of Eqs. (13)–(15). These expressions are far too long to be practically conveyed, so we use the following intermediary parameters here and only here to simplify these expressions:

$$K_0 = e(1 - e^2)$$
$$K_1 = \sin(\Omega) \cos(\omega) + \sin(\omega) \cos(\Omega) \cos(i)$$
$$K_2 = \sqrt{\frac{e + 1}{1 - e}}$$
$$K_3 = a(1 - e^2)$$
$$K_4 = -\sin(\Omega) \sin(\omega) \cos(i) + \cos(\Omega) \cos(\omega)$$
$$K_5 = \sqrt{K_2^2 K_3^2 + K_2^2 K_4^2 + K_3^2 \sin^2(i) \sin^2(\omega)}$$
$$K_6 = K_3(\sin(\Omega) \cos(\omega) - \sin(\omega) \cos(\Omega) \cos(i))/(1 - e)$$
$$K_7 = K_3 \sin(\Omega) \sin(\omega) \cos(i) - \cos(\Omega) \cos(\omega)/(1 - e)$$
$$K_8 = K_3(\sin(\Omega) \cos(\omega) + 2 \tan^{-1}(K_2)) + \sin(\omega) + 2 \tan^{-1}(K_2) \cos(\Omega) \cos(i))/(\sin(2 \tan^{-1}(K_2)) + 1)$$
$$K_9 = K_3(-\sin(\Omega) \sin(\omega) + 2 \tan^{-1}(K_2)) \cos(i) + \cos(\Omega) \cos(\omega) + 2 \tan^{-1}(K_2))/(\sin(2 \tan^{-1}(K_2)) + 1).$$

(61)

The simplified expression for the semimajor axis of the projected ellipse is

$$a_p = \frac{1}{2} \sqrt{\frac{K_0 \rho^2 K_1}{K_5} - \frac{K_0 \rho^2 K_4}{K_5} + K_6 - K_9} + \frac{K_0 \rho^2 K_1}{K_5} + \frac{K_0 \rho^2 K_4}{K_5} + K_7 + K_8 \right)^2 + \frac{K_0 \rho^2 K_1}{K_5} + \frac{K_0 \rho^2 K_4}{K_5} + K_6 + K_9 \right)^2.\right)

(62)

The simplified expression for the projected ellipse semiminor axis is

$$b_p = \frac{1}{2} \sqrt{\frac{K_0 \rho^2 K_1}{K_5} - \frac{K_0 \rho^2 K_4}{K_5} + K_6 - K_9} + \frac{K_0 \rho^2 K_1}{K_5} + \frac{K_0 \rho^2 K_4}{K_5} + K_7 + K_8 \right)^2 + \frac{K_0 \rho^2 K_1}{K_5} + \frac{K_0 \rho^2 K_4}{K_5} + K_6 + K_9 \right)^2.\right)

(63)

The simplified expression for the angle between the projected ellipse semimajor axis and $X$ axis is

$$\theta = \frac{1}{2} \left[ \tan^{-1} \left( \frac{K_0 \rho^2 K_1}{K_5} - \frac{K_0 \rho^2 K_4}{K_5} + K_6 - K_9 \right) + \tan^{-1} \left( \frac{K_0 \rho^2 K_1}{K_5} + \frac{K_0 \rho^2 K_4}{K_5} + K_6 + K_9 \right) \right]. \right)

(64)
11 Appendix F: s Extrema Polynomial

We reduced Eq. (22) from Sec. 2.1.2 into the standard reduced form of a quartic expression:

\[ 0 = x^4 + \frac{-2a_p^2 x_s + 2b_p^2 x_e}{(a_p^4 - 2a_p^2 b_p^2 + b_p^4)/a_p^2} x^3 + \frac{-a_p^4 + 2a_p^2 b_p^2 + a_p^2 x_s^2 - b_p^4 + b_p^2 x_e^2}{(a_p^5 - 2a_p^3 b_p^2 + b_p^4)/a_p^2} x^2 + \frac{2a_p^2 x_s - 2a_p^2 b_p^2 x_e}{(a_p^5 - 2a_p^3 b_p^2 + b_p^4)/a_p^2} x + \frac{-a_p^4 x_e^2}{(a_p^5 - 2a_p^3 b_p^2 + b_p^4)/a_p^2}. \] (65)

The coefficients of the quartic expression are

\[ A_0 = \frac{-2a_p^2 x_s + 2b_p^2 x_e}{(a_p^4 - 2a_p^2 b_p^2 + b_p^4)/a_p^2}, \] (66)

\[ B_0 = \frac{-a_p^4 + 2a_p^2 b_p^2 + a_p^2 x_s^2 - b_p^4 + b_p^2 x_e^2}{(a_p^5 - 2a_p^3 b_p^2 + b_p^4)/a_p^2}, \] (67)

\[ C_0 = \frac{2a_p^4 x_s - 2a_p^2 b_p^2 x_e}{(a_p^5 - 2a_p^3 b_p^2 + b_p^4)/a_p^2}, \] (68)

\[ D_0 = \frac{-a_p^4 x_e^2}{(a_p^5 - 2a_p^3 b_p^2 + b_p^4)/a_p^2}. \] (69)

12 Appendix G: s Intersection Polynomial

After dividing by the coefficient of \( x^4 \) in Eq. (35) from Sec. 2.1.3, we get

\[ 0 = x^4 + \frac{-4a_p^2 x_s}{a_p^2 - b_p^2} x^3 + \frac{2a_p^2 (a_p b_p - a_p^2 s^2 + 3a_p^2 x_s^2 + a_p^2 y_s^2 - b_p^4 + b_p^2 s^2 - b_p^2 x_s^2 + b_p^2 y_s^2)}{a_p^4 - 2a_p^2 b_p^2 + b_p^4} x^2 + \frac{4a_p^2 x_s (-b_p^2 + s^2 - x^2 - y^2)}{a_p^4 - 2a_p^2 b_p^2 + b_p^4} x + \frac{a_p^4 (b_p^4 - 2b_p^2 s^2 + 2b_p^2 x^2 - 2b_p^2 y^2 + s^4 - 2s^2 x^2 - 2s^2 y^2 + 2x^2 y^2 + x^4 + 2x^2 y^2 + y^4)}{a_p^4 - 2a_p^2 b_p^2 + b_p^4}. \] (70)

The coefficients of the general quartic expression in Eq. (36) are

\[ A_1 = -\frac{4a_p^2 x_s}{a_p^2 - b_p^2}, \] (71)

\[ B_1 = \frac{2a_p^2 (a_p b_p - a_p^2 s^2 + 3a_p^2 x_s^2 + a_p^2 y_s^2 - b_p^4 + b_p^2 s^2 - b_p^2 x_s^2 + b_p^2 y_s^2)}{a_p^4 - 2a_p^2 b_p^2 + b_p^4}, \] (72)

\[ C_1 = \frac{4a_p^2 x_s (-b_p^2 + s^2 - x^2 - y^2)}{a_p^4 - 2a_p^2 b_p^2 + b_p^4}, \] (73)

and

\[ D_1 = \frac{a_p^4 (b_p^4 - 2b_p^2 s^2 + 2b_p^2 x^2 - 2b_p^2 y^2 + s^4 - 2s^2 x^2 - 2s^2 y^2 + 2x^2 y^2 + x^4 + 2x^2 y^2 + y^4)}{a_p^4 - 2a_p^2 b_p^2 + b_p^4}. \] (74)
13 Appendix H: Δmag Extrema Polynomial

Here we include the coefficients of Eq. (49) from Sec. 2.2 as

\[ A_2 = e^4 \sin^4(i), \]
\[ B_2 = 3e^3(e \sin(\omega) + \sin(i))\sin^3(i), \]
\[ C_2 = \frac{1}{4} e^2(4e^2 \sin^2(i) \sin^2(\omega) - 8e^2 \sin^2(i) + 8e^2 \sin^2(\omega) + 5e^2 + 34e \sin(i) \sin(\omega) + 13\sin^3(i))\sin^2(i), \]
\[ D_2 = \frac{1}{2} e^2(2e^3 \sin^2(i) \sin^2(\omega) - 8e^2 \sin^2(i) \sin(\omega) + 3e^3 \sin(\omega) + 5e^2 \sin^3(i) \sin^2(\omega) - 11e^2 \sin^3(i) + 10e^2 \sin(i) \sin^2(\omega) + 7e^2 \sin(i) + 17e^2 \sin(i) \sin(\omega) + 3\sin^3(i)) \sin(i), \]
\[ E_2 = -\frac{5}{4} e^4 \sin^4(i) \sin^2(\omega) + \frac{5}{4} e^4 \sin^4(i) - \frac{7}{4} e^4 \sin^2(i) \sin^2(\omega) - \frac{3}{2} e^4 \sin^2(i) \]
\[ + \frac{e^4}{4} + \frac{3}{2} e^3 \sin^3(i) \sin^3(\omega) - 10e^3 \sin^3(i) \sin(\omega) + \frac{7}{2} e^3 \sin(i) \sin(\omega) \]
\[ + 2e^2 \sin^4(i) \sin^2(\omega) - \frac{21}{4} e^2 \sin^4(i) + 4e^2 \sin^2(i) \sin^2(\omega) + \frac{7}{2} e^2 \sin^2(i) \]
\[ + \frac{7}{2} e^2 \sin^3(i) \sin(\omega) + \frac{\sin^2(i)}{4}, \]
\[ F_2 = -\frac{3}{2} e^4 \sin^4(i) \sin^2(\omega) + \frac{3}{2} e^4 \sin^4(i) \sin^2(\omega) - \frac{3}{2} e^4 \sin(i) \sin(\omega) - \frac{e^4}{2} \sin^4(i) \sin^4(\omega) \]
\[ - \frac{5}{2} e^4 \sin^2(i) \sin^2(\omega) + \frac{3}{2} e^4 \sin^2(i) - \frac{5}{2} e^3 \sin^2(i) \sin^2(\omega) - \frac{7}{2} e^3 \sin^2(i) + \frac{e^3}{2} \]
\[ - \frac{e^2 \sin^3(i) \sin^3(\omega)}{2} - 8e^2 \sin^3(i) \sin(\omega) + \frac{5}{2} e^2 \sin(i) \sin(\omega) + \frac{e^2 \sin^4(i) \sin^2(\omega)}{2}, \]
\[ G_2 = -\frac{e^4}{2} \sin^4(i) \sin^4(\omega) + \frac{e^4}{2} \sin^4(i) \sin^2(\omega) - \frac{e^4}{2} \sin^2(i) \sin^2(\omega) - \frac{e^4}{2} \sin^2(i) \sin^2(\omega) \]
\[ + \frac{e^4 \sin^2(i)}{2} - \frac{e^4}{4} - \frac{7}{2} e^3 \sin^3(i) \sin^3(\omega) + \frac{7}{2} e^3 \sin(i) \sin(\omega) - \frac{7}{2} e^3 \sin(i) \sin(\omega) \]
\[ - \frac{5}{4} e^2 \sin^4(i) \sin^4(\omega) - \frac{e^2 \sin^4(i) \sin^2(\omega)}{4} + \frac{9}{4} e^2 \sin^4(i) \sin^2(\omega) - \frac{5}{2} e^2 \sin^2(i) \sin^2(\omega) \]
\[ - \frac{5}{2} e^2 \sin^2(i) + \frac{3}{2} e^2 \sin^3(i) \sin^3(\omega) - 2e \sin^3(i) \sin(\omega) + \frac{e^2 \sin(i) \sin(\omega)}{2} \]
\[ - \frac{\sin^4(i)}{4} + \frac{\sin^2(i)}{4}, \]
\[ H_2 = -\frac{3}{2} e^2 \sin(i) \cos^2(\omega) + \frac{e^3}{2} \sin^2(i) \cos^2(\omega) - \frac{e^3}{2} + \frac{5}{2} e^2 \sin^3(i) \sin(\omega) \cos^2(\omega) \]
\[ - \frac{5}{2} e^2 \sin(i) \sin(\omega) - 3e(\cos(2i) - 1)^2(\cos(4w) - 1) + \frac{e \sin^4(i) \cos^2(\omega)}{2} \]
\[ - \frac{5}{2} e^2 \sin^2(i) \sin^2(\omega) - e \sin^2(i) \cos^2(\omega) - \frac{\sin^3(i) \sin(i)}{4}, \text{ and} \]
\[ I_2 = -\frac{e^2}{2} \sin^2(i) \cos^2(\omega) + \frac{e^2}{2} \sin^2(i) \cos^2(\omega) - \frac{e^2}{4} + \frac{e \sin^3(i) \sin(\omega) \cos^2(\omega)}{2} \]
\[ - \frac{e \sin(i) \sin(\omega)}{2} - \frac{(\cos(2i) - 1)^2(\cos(4w) - 1)}{128} - \frac{\sin^2(i) \sin^2(\omega)}{4}. \quad (75) \]
14 Appendix I: \( \Delta \text{mag} \) Intersection Polynomial

Here we include the coefficients of Eq. (50) from Sec. 2.2 as

\[
A_3 = \frac{e^4 \sin^4(i)}{16},
\]
\[
B_3 = \frac{1}{4} e^4 (e \sin(\omega) + \sin(i)) \sin^3(i),
\]
\[
C_3 = \frac{1}{8} e^2 (e^2 \cos^2(i) \cos^2(\omega) - 3e^2 \cos^2(\omega) + 3e^2 + 8e \sin(i) \sin(\omega) - 3 \cos^2(i) + 3) \sin^2(i),
\]
\[
D_3 = \frac{1}{4} e^4 [-e^3 \sin^3(\omega) \cos^2(i) + e^3 \sin^2(\omega) + e^3 \sin(\omega) \cos^2(i) - 2e^3 \sin^3(i) \cos^2(\omega) - 4e^2 \sin(i) \cos^2(\omega) + 6e^2 \sin(i) - 6e \sin(\omega) \cos^2(i) + 6e \sin(\omega) + \sin^3(i) \sin(i),
\]
\[
E_3 = \frac{1}{16} e^4 \sin^4(i) \sin^4(\omega) - \frac{1}{8} e^4 \sin^4(i) \sin^2(\omega) + \frac{1}{16} e^4 \sin^4(i) + \frac{1}{8} e^4 \sin^2(i) \sin^2(\omega)
\]
\[
- \frac{1}{8} e^4 \sin^2(i) + \frac{1}{16} e^4 \sin^3(i) + e^3 \sin(i) \sin(i) \sin(\omega) + e^3 \sin(i) \sin(\omega)
\]
\[
- \frac{1}{8} e^2 \sin^2(i) \sin^2(\omega) + \frac{1}{2} e^2 \sin^2(i) \sin^2(\omega) - \frac{3}{4} e^2 \sin^4(i) - \frac{3}{4} e^2 \sin^4(i)
\]
\[
+ \frac{3}{2} e^2 \sin^2(i) \sin^2(\omega) + \frac{3}{4} e^2 \sin^2(i) \sin(\omega) + \frac{e \sin^2(\omega) + e \sin^2(i) \sin(\omega) + \frac{e \sin^2(i)}{2}
\]
\[
F_3 = \frac{1}{4} e^3 \sin^4(i) \sin^4(\omega) - \frac{1}{4} e^3 \sin^4(i) \sin^2(\omega) + \frac{1}{4} e^3 \sin^4(i) + \frac{1}{4} e^3 \sin^2(i) \sin^2(\omega)
\]
\[
- \frac{1}{4} e^3 \sin^2(i) + \frac{1}{4} e^3 \sin(i) \sin(\omega) + \frac{3}{2} e^3 \sin^3(i) \sin^3(\omega)
\]
\[
- \frac{1}{8} e^2 \sin^2(i) \sin(\omega) + \frac{1}{2} e^2 \sin^2(i) \sin(\omega) - 2e \sin^2(i) \sin^2(\omega) + e \sin^2(i)
\]
\[
+ \frac{1}{2} e \sin^2(i) \sin^2(\omega) + e \sin^2(i) \sin^2(\omega) + \frac{e \sin^2(i)}{2}
\]
\[
G_3 = \frac{1}{2} e^2 \sin^2(i) \cos^2(\omega) - \frac{1}{2} e^2 \sin^2(i) \cos^2(\omega) - \frac{3}{4} e^2 \sin^2(i) \cos^2(\omega)
\]
\[
+ \frac{3}{2} e^2 \sin^2(i) \cos^2(\omega) - 2e \sin(i) \sin(\omega) - e \sin(i) \sin(\omega) \cos^2(\omega) + e \sin(i) \sin(\omega)
\]
\[
- \frac{1}{2} e \sin^2(i) \sin^2(\omega) + \frac{1}{8} \sin^2(i) \sin^2(\omega) + \sin^2(i) \sin^2(\omega)
\]
\[
- \frac{2}{256} (\cos(2i) - 1)^2 \cos(4\omega) - 1)
\]
\[
H_3 = \frac{1}{2} e \sin(i) \sin(i) \sin(\omega) - \frac{1}{4} e \sin^2(i) \cos^2(\omega) - \frac{1}{2} e \sin^2(i) \cos^2(\omega) + \frac{e}{4}
\]
\[
- \frac{1}{4} \sin(i) \sin(i) \sin(\omega) - \frac{1}{4} \sin(i) \sin(i) \sin(i) \sin(i) \sin(\omega) + \sin(i) \sin(\omega), \quad \text{and}
\]
\[
I_3 = \frac{1}{2} - \frac{1}{2} e^2 \sin^2(i) \cos^2(\omega) + \frac{1}{8} e^2 \sin^2(i) \cos^2(\omega) - \frac{1}{16} \sin^2(i) \cos^2(\omega) + \frac{1}{16}.
\]

(76)
15 Appendix J: Proof that Ellipses Project to Ellipses

Let us suppose that we have some ellipse on an arbitrary plane indicated by the black ellipse in Fig. 4. We will say $AB$ and $CD$ are the principal axes of some this 3D ellipse. $AB$ and $CD$ intersect at point $O$, the geometric center of this 3D ellipse. We will say point $P$ is any point on the ellipse with $H$ being the projection of point $P$ onto axis $AB$ and $K$ being the projection of point $P$ onto axis $CD$. By the definition of an ellipse, we have

$$\frac{OH^2}{OB^2} + \frac{OK^2}{OD^2} = 1. \quad (77)$$

Now we let $A', B', C', D', O', K', H'$, and $P'$ be the perpendicular projections of points $A$, $B$, $C$, $D$, $O$, $K$, $H$, and $P$ onto any given plane. Since perpendicular projections preserve the ratios of segments on a line, we can now say

$$\frac{O'H'^2}{O'B'^2} + \frac{O'K'^2}{O'D'^2} = 1. \quad (78)$$

This equation means that point $P'$ belongs to the ellipse having $A'B'$ and $C'D'$ as conjugate diameters. For an ellipse, two diameters are conjugate if and only if the tangent line to the ellipse at an endpoint of one diameter is parallel to the other diameter.

16 Appendix K: Quartic Solution

The quartic function has a known analytical solution. We start with a set of useful simplifying terms

$$p_0 = \left(-\frac{3A^2}{8} + B\right)^3, \quad (79)$$
$$p_1 = \left(A\left(\frac{A^2}{8} - \frac{B}{2}\right) + C\right)^2, \quad (80)$$
$$p_2 = -A\left(A\left(\frac{3A^2}{256} - \frac{B}{16}\right) + \frac{C}{4}\right) + D, \quad (81)$$
$$p_3 = -\frac{3A^2}{8} + B, \quad (82)$$
$$p_4 = 2A\left(\frac{A^2}{8} - \frac{B}{2}\right), \quad (83)$$
$$p_5 = -\frac{p_0}{108} - \frac{p_1}{8} + \frac{p_2p_3}{3}, \quad (84)$$
$$p_6 = \sqrt[3]{\frac{p_0}{216} + \frac{p_1}{16} - \frac{p_2p_3}{6} + \frac{p_2^2}{4} + \frac{\left(-p_2 - \frac{p_3}{2}\right)^3}{27}}, \quad (85)$$
$$p_7 = \frac{A^2}{4} - \frac{2B}{3}. \quad (86)$$
\[ p_8 = \frac{2p_2 + p_6^2}{3p_6}, \quad (87) \]

\[ p_9 = \sqrt{-2\sqrt{p_5} + p_7}, \quad (88) \]

\[ p_{10} = \sqrt{2p_6 + p_7 + p_8}, \quad (89) \]

and

\[ p_{11} = \frac{A^2}{2} - \frac{4B}{3}. \quad (90) \]

The solutions are four piecewise functions:

\[
x_0 = \begin{cases} 
-\frac{A}{4} + \frac{p_6}{2} - \sqrt{\frac{p_{11} + 2\sqrt{p_5 - \frac{2C + p_4}{p_9}}}{2}} & \text{for } p_2 + \frac{p_4}{12} = 0, \\
-\frac{A}{4} - \frac{p_6}{2} - \sqrt{\frac{p_{11} - 2p_5 + p_3}{p_9}} & \text{otherwise}
\end{cases}
\]

\[
x_1 = \begin{cases} 
-\frac{A}{4} + \frac{p_6}{2} + \sqrt{\frac{p_{11} + 2\sqrt{p_5 - \frac{2C + p_4}{p_9}}}{2}} & \text{for } p_2 + \frac{p_4}{12} = 0, \\
-\frac{A}{4} - \frac{p_6}{2} + \sqrt{\frac{p_{11} - 2p_5 + p_3}{p_9}} & \text{otherwise}
\end{cases}
\]

\[
x_2 = \begin{cases} 
-\frac{A}{4} - \frac{p_6}{2} - \sqrt{\frac{p_{11} + 2\sqrt{p_5 - \frac{2C + p_4}{p_9}}}{2}} & \text{for } p_2 + \frac{p_4}{12} = 0, \\
-\frac{A}{4} + \frac{p_6}{2} - \sqrt{\frac{p_{11} - 2p_5 + p_3}{p_9}} & \text{otherwise}
\end{cases}
\]

and

\[
x_3 = \begin{cases} 
-\frac{A}{4} - \frac{p_6}{2} + \sqrt{\frac{p_{11} + 2\sqrt{p_5 - \frac{2C + p_4}{p_9}}}{2}} & \text{for } p_2 + \frac{p_4}{12} = 0, \\
-\frac{A}{4} + \frac{p_6}{2} + \sqrt{\frac{p_{11} - 2p_5 + p_3}{p_9}} & \text{otherwise}
\end{cases}
\]

Theoretically, we always have four solutions. Theoretically, we can use equations of coefficients to determine how many roots are real, complex, and double roots. The general form of the quartic defines the expressions \( \Delta, P, D_2, R \), and \( \Delta_0 \), which are

\[
\Delta = 256D^3 - 192ACD^2 - 128B^2D^2 + 144BC^2D - 27C^4 + 144A^2BD^2 - 6A^2C^2D \\
- 80AB^2CD + 18ABC^3 + 16B^3D - 4B^2C^2 - 27A^4D^2 + 18A^3BCD - 4A^3C^3 \\
- 4A^2B^3D + A^2B^2C^2, \quad (95)
\]

\[ P = 8B - 3A^2, \quad (96) \]

\[ D_2 = 64D - 16B^2 + 16A^2B - 16AC - 3A^4. \quad (97) \]
\[ R = A^3 + 8C - 4AB, \]  
\[ \Delta_0 = B^2 - 3AC + 12D. \]

After evaluation of these constants, we can determine

\[
\text{solution types} = \begin{cases} 
2 \text{ distinct } \Re \text{ roots } & \Delta < 0 \\
4 \Re \text{ distinct roots} & \Delta > 0 \land P < 0 \land D < 0 \\
2 \text{ pair complex conjugate roots} & \Delta > 0 \land (P > 0 \lor D > 0)
\end{cases}
\]

(100)

However, the use of this theoretical classification does not work when numerical errors are introduced. The accumulation of numerical errors causes the solutions to be improperly classified for a sufficiently large number of cases to prohibit the use of these classifications. The \( \Delta \) expression rarely evaluates to \( \Delta = 0 \) when it should do so quite frequently. Numerical rounding errors can also frequently result in the evaluation \( \Delta < 0 \) when in reality \( \Delta > 0 \) and vice versa.

17 Appendix L: Earth-Like Subpopulation

We use a definition of Earth-like exoplanet similar to that used in Ref. 8. We use a more conservative planetary radius range of \( 0.9R_\oplus \leq R \leq 1.4R_\oplus \). We use the flux-at-planet range of \( 0.3586 \leq L_{\text{plan}} \leq 1.1080 \). To classify a planet, we use the time-averaged incident flux on the planet calculated by

\[ L_{\text{plan}} = \frac{L_s}{(a + \frac{ac}{2})^2}. \]

(101)

18 Appendix M: 4-m Monolith Design Reference Mission

We calculated the Brown completeness and integration time adjusted completeness for a 4-m monolith telescope with a contrast of \( 10^{-10} \) on the SAG13 planet population discussed in Ref. 1. The \( \Delta_{\text{mag}} \) and separations of integration are scaled by the luminosity of each candidate target star. The orbital periods are similarly scaled by the mass of each target star in the case of integration time adjusted completeness. We optimized the mission schedule using the sequential least squares quadratic programming method discussed in Ref. 1 with the Brown completeness method. Our approach included the adoption of the use of the local zodiacal light minimum and filtering of target stars.

We assume a total telescope time of 213.67 days (~7 months) and that each observation requires 0.1 day of overhead time and 0.042 day settling time. The coronagraph parameters used are from a vortex charge six coronagraph at a wavelength of 500 nm with an IWA of 0.045 arc sec and OWA of 2.127 arc sec and a working angle of integration consistent with that used in Ref. 1.

The resulting Brown completeness optimized integration times for a DRM are included in Table 2. The summed Brown completeness for this DRM is 68.89, whereas the same summed integration time adjusted completeness is 63.11, a difference of 5.78. We can multiply either of these numbers by the SAG13 exoplanet occurrence rate of 5.62 to get the expected number of exoplanets detected in a single-visit blind search. These are 387.16 and 354.67 on average, respectively. This means that 32.49 planets are lost in a mission by not taking into account integration times when optimizing the mission schedule.
Table 2  4-m monolith DRM.

| Name | Dist (pc) | Int. time (day) | Δmag | $C_{{}_{brown}}$ | $C_{{}_{IAC}}$ |
|------|-----------|-----------------|------|------------------|----------------|
| HIP 439 | 4.34 | 0.399 | 19.557 | 0.049 | 0.041 |
| HIP 522 | 25.71 | 0.181 | 23.476 | 0.038 | 0.033 |
| HIP 544 | 13.67 | 0.6 | 23.403 | 0.16 | 0.141 |
| HIP 746 | 16.78 | 0.066 | 27.142 | 0.247 | 0.24 |
| HIP 910 | 18.75 | 0.383 | 24.916 | 0.16 | 0.146 |
| HIP 950 | 21.28 | 0.341 | 24.504 | 0.106 | 0.095 |
| HIP 1292 | 17.5 | 0.338 | 22.181 | 0.054 | 0.047 |
| HIP 1392 B | 15.19 | 0.293 | 21.362 | 0.039 | 0.033 |
| HIP 1475 | 3.57 | 0.625 | 20.237 | 0.105 | 0.089 |
| HIP 1499 | 23.25 | 0.139 | 21.736 | 0.018 | 0.016 |
| HIP 1599 | 8.59 | 0.374 | 25.793 | 0.47 | 0.438 |
| HIP 1803 | 20.86 | 0.241 | 22.209 | 0.036 | 0.031 |
| HIP 2021 | 7.46 | 0.179 | 26.942 | 0.55 | 0.525 |
| HIP 2072 | 23.81 | 0.129 | 25.773 | 0.103 | 0.097 |
| HIP 2081 | 25.97 | 0.053 | 26.432 | 0.082 | 0.078 |
| HIP 2711 | 25.48 | 0.193 | 23.752 | 0.045 | 0.039 |
| HIP 3093 | 11.06 | 0.745 | 23.731 | 0.242 | 0.213 |
| HIP 3170 | 24.96 | 0.192 | 23.257 | 0.038 | 0.033 |
| HIP 3419 | 29.53 | 0.035 | 26.545 | 0.048 | 0.046 |
| HIP 3497 | 22.06 | 0.184 | 21.928 | 0.025 | 0.022 |
| HIP 3505 | 26.75 | 0.169 | 24.171 | 0.044 | 0.039 |
| HIP 3583 | 15.16 | 0.567 | 23.919 | 0.17 | 0.151 |
| HIP 3765 | 7.45 | 0.861 | 23.974 | 0.38 | 0.339 |
| HIP 3810 | 23.45 | 0.27 | 24.446 | 0.078 | 0.07 |
| HIP 3909 | 15.75 | 0.515 | 24.685 | 0.207 | 0.187 |
| HIP 4148 | 14.17 | 0.293 | 21.192 | 0.039 | 0.033 |
| HIP 4151 | 18.74 | 0.353 | 25.07 | 0.166 | 0.152 |
| HIP 5336 | 7.55 | 0.66 | 24.859 | 0.459 | 0.418 |
| HIP 5862 | 15.11 | 0.46 | 25.001 | 0.238 | 0.217 |
| HIP 6379 | 16.81 | 0.217 | 21.188 | 0.028 | 0.024 |
| HIP 6706 | 25.21 | 0.207 | 23.99 | 0.051 | 0.046 |
| HIP 6813 | 28.62 | 0.13 | 24.535 | 0.037 | 0.033 |
| HIP 7235 | 19.05 | 0.164 | 21.17 | 0.021 | 0.018 |
| HIP 7339 | 20.66 | 0.236 | 22.106 | 0.035 | 0.03 |
Table 2 (Continued).

| Name  | Dist (pc) | Int. time (day) | Δmag  | \( C_{\text{brown}} \) | \( C_{\text{IAC}} \) |
|-------|-----------|-----------------|-------|-------------------------|-------------------------|
| HIP 7513 | 13.49     | 0.296           | 25.765| 0.309                   | 0.288                   |
| HIP 7601 | 27.38     | 0.129           | 23.04 | 0.023                   | 0.02                    |
| HIP 7734 | 21.37     | 0.17            | 21.643| 0.022                   | 0.019                   |
| HIP 7918 | 12.74     | 0.521           | 24.978| 0.295                   | 0.269                   |
| HIP 7978 | 17.43     | 0.483           | 24.302| 0.153                   | 0.137                   |
| HIP 7981 | 7.53      | 0.733           | 24.611| 0.44                    | 0.398                   |
| HIP 8102 | 3.65      | 0.323           | 26.364| 0.745                   | 0.704                   |
| HIP 8362 | 10.07     | 0.753           | 24.254| 0.318                   | 0.285                   |
| HIP 8433 | 27.86     | 0.119           | 23.024| 0.021                   | 0.019                   |
| HIP 8497 | 23.19     | 0.241           | 25.078| 0.098                   | 0.09                    |
| HIP 8796 | 19.42     | 0.141           | 26.116| 0.18                    | 0.17                    |
| HIP 8903 | 17.99     | 0.08            | 26.814| 0.217                   | 0.209                   |
| HIP 9007 | 17.85     | 0.199           | 25.89 | 0.207                   | 0.194                   |
| HIP 9236 | 22.01     | 0.069           | 26.565| 0.138                   | 0.132                   |
| HIP 9884 | 20.18     | 0.069           | 26.753| 0.172                   | 0.165                   |
| HIP 10138| 10.78     | 0.746           | 23.544| 0.231                   | 0.204                   |
| HIP 10306| 28.87     | 0.115           | 23.779| 0.026                   | 0.023                   |
| HIP 10644| 10.78     | 0.528           | 25.092| 0.359                   | 0.329                   |
| HIP 10723| 24.35     | 0.218           | 23.513| 0.047                   | 0.041                   |
| HIP 10798| 12.67     | 0.605           | 22.975| 0.146                   | 0.128                   |
| HIP 11029| 29.66     | 0.097           | 23.551| 0.02                    | 0.018                   |
| HIP 11783| 26.7      | 0.179           | 24.768| 0.055                   | 0.049                   |
| HIP 12114| 7.18      | 0.88            | 23.866| 0.379                   | 0.337                   |
| HIP 12186| 25.78     | 0.173           | 23.307| 0.034                   | 0.03                    |
| HIP 12444| 21.76     | 0.299           | 23.526| 0.067                   | 0.059                   |
| HIP 12623| 24.19     | 0.249           | 24.661| 0.076                   | 0.068                   |
| HIP 12653| 17.17     | 0.488           | 24.473| 0.167                   | 0.15                    |
| HIP 12777| 11.13     | 0.324           | 25.829| 0.381                   | 0.356                   |
| HIP 12843| 14.22     | 0.352           | 25.471| 0.28                    | 0.259                   |
| HIP 13402| 10.35     | 0.771           | 23.572| 0.245                   | 0.216                   |
| HIP 13665| 26.71     | 0.159           | 23.616| 0.035                   | 0.031                   |
| HIP 14146| 27.17     | 0.112           | 25.539| 0.062                   | 0.058                   |
| HIP 14150| 20.64     | 0.162           | 21.435| 0.021                   | 0.018                   |
| HIP 14632| 10.54     | 0.328           | 25.855| 0.402                   | 0.375                   |
| Name     | Dist (pc) | Int. time (day) | Δmag     | \(C_{\text{brown}}\) | \(C_{\text{IAC}}\) |
|----------|------------|-----------------|----------|----------------------|----------------------|
| HIP 14954 | 22.58      | 0.299           | 24.511   | 0.09                 | 0.08                 |
| HIP 15330 | 12.01      | 0.669           | 24.437   | 0.278                | 0.25                 |
| HIP 15371 | 12.03      | 0.592           | 24.779   | 0.303                | 0.275                |
| HIP 15457 | 9.14       | 0.567           | 25.073   | 0.413                | 0.378                |
| HIP 15510 | 6.04       | 0.426           | 25.751   | 0.581                | 0.541                |
| HIP 16245 | 21.68      | 0.264           | 25.16    | 0.121                | 0.111                |
| HIP 16537 | 3.21       | 0.377           | 26.121   | 0.768                | 0.722                |
| HIP 16852 | 13.96      | 0.339           | 25.542   | 0.289                | 0.268                |
| HIP 17378 | 9.04       | 0.286           | 26.162   | 0.466                | 0.438                |
| HIP 17420 | 13.95      | 0.326           | 21.324   | 0.044                | 0.038                |
| HIP 17439 | 16.03      | 0.286           | 21.475   | 0.039                | 0.033                |
| HIP 17651 | 17.63      | 0.25            | 25.64    | 0.206                | 0.192                |
| HIP 18859 | 18.83      | 0.429           | 24.311   | 0.132                | 0.118                |
| HIP 19076 | 16.94      | 0.466           | 23.433   | 0.114                | 0.101                |
| HIP 19205 | 27.6       | 0.147           | 24.021   | 0.036                | 0.032                |
| HIP 19335 | 21.0       | 0.341           | 23.973   | 0.09                 | 0.08                 |
| HIP 19849 | 4.98       | 0.504           | 25.516   | 0.625                | 0.579                |
| HIP 19859 | 21.33      | 0.244           | 22.41    | 0.039                | 0.034                |
| HIP 19884 | 13.04      | 0.19            | 20.397   | 0.024                | 0.02                 |
| HIP 19893 | 20.46      | 0.206           | 25.608   | 0.152                | 0.141                |
| HIP 19921 | 18.24      | 0.322           | 25.261   | 0.183                | 0.168                |
| HIP 19990 | 28.94      | 0.122           | 24.383   | 0.033                | 0.03                 |
| HIP 21421 | 20.43      | 0.039           | 27.351   | 0.17                 | 0.166                |
| HIP 21547 | 29.43      | 0.107           | 23.929   | 0.025                | 0.022                |
| HIP 21770 | 20.17      | 0.24            | 25.449   | 0.153                | 0.141                |
| HIP 21861 | 28.67      | 0.127           | 24.318   | 0.034                | 0.03                 |
| HIP 22263 | 13.28      | 0.635           | 24.402   | 0.243                | 0.218                |
| HIP 22336 | 26.42      | 0.154           | 23.183   | 0.029                | 0.026                |
| HIP 22449 | 8.07       | 0.218           | 26.617   | 0.516                | 0.491                |
| HIP 23311 | 8.71       | 0.832           | 23.26    | 0.258                | 0.226                |
| HIP 23482 | 26.08      | 0.185           | 24.074   | 0.047                | 0.042                |
| HIP 23693 | 11.65      | 0.434           | 25.351   | 0.346                | 0.319                |
| HIP 23783 | 26.29      | 0.189           | 24.587   | 0.055                | 0.049                |
| HIP 23875 | 27.4       | 0.044           | 26.48    | 0.067                | 0.064                |
| Name    | Dist (pc) | Int. time (day) | Δ mag  | $C_{\text{Brown}}$ | $C_{\text{IAC}}$ |
|---------|------------|-----------------|--------|---------------------|------------------|
| HIP 23941 | 25.46      | 0.208           | 24.364 | 0.057               | 0.051            |
| HIP 24186 | 3.91       | 0.377           | 19.298 | 0.044               | 0.037            |
| HIP 24786 | 25.03      | 0.183           | 23.045 | 0.034               | 0.029            |
| HIP 24813 | 12.63      | 0.458           | 25.207 | 0.311               | 0.285            |
| HIP 25110 | 20.89      | 0.343           | 24.749 | 0.12                | 0.108            |
| HIP 25278 | 14.39      | 0.517           | 24.847 | 0.246               | 0.224            |
| HIP 25544 | 19.2       | 0.183           | 21.347 | 0.024               | 0.02             |
| HIP 25623 | 13.02      | 0.163           | 20.148 | 0.02                | 0.017            |
| HIP 25878 | 5.68       | 0.483           | 20.34  | 0.068               | 0.058            |
| HIP 26394 | 18.32      | 0.446           | 24.074 | 0.128               | 0.114            |
| HIP 26779 | 12.28      | 0.645           | 23.215 | 0.171               | 0.15             |
| HIP 27072 | 8.93       | 0.256           | 26.326 | 0.474               | 0.448            |
| HIP 27288 | 21.61      | 0.108           | 26.165 | 0.141               | 0.134            |
| HIP 27321 | 19.44      | 0.155           | 26.012 | 0.178               | 0.167            |
| HIP 27435 | 15.18      | 0.544           | 23.539 | 0.145               | 0.128            |
| HIP 27628 | 26.73      | 0.081           | 25.936 | 0.071               | 0.066            |
| HIP 27887 | 13.0       | 0.376           | 21.413 | 0.053               | 0.045            |
| HIP 27890 | 26.25      | 0.19            | 24.754 | 0.058               | 0.053            |
| HIP 28103 | 14.88      | 0.2             | 26.143 | 0.282               | 0.266            |
| HIP 28360 | 24.87      | 0.031           | 27.217 | 0.098               | 0.096            |
| HIP 28908 | 25.62      | 0.143           | 22.489 | 0.022               | 0.019            |
| HIP 28954 | 15.27      | 0.327           | 21.575 | 0.046               | 0.039            |
| HIP 29271 | 10.2       | 0.582           | 24.95  | 0.369               | 0.337            |
| HIP 29295 | 5.75       | 0.401           | 20.008 | 0.051               | 0.043            |
| HIP 29525 | 17.95      | 0.337           | 22.3   | 0.055               | 0.048            |
| HIP 29568 | 16.72      | 0.423           | 22.752 | 0.083               | 0.073            |
| HIP 29650 | 20.82      | 0.358           | 24.425 | 0.109               | 0.097            |
| HIP 29800 | 19.25      | 0.394           | 24.751 | 0.145               | 0.131            |
| HIP 29860 | 19.25      | 0.396           | 23.768 | 0.102               | 0.09             |
| HIP 30503 | 21.88      | 0.228           | 22.401 | 0.036               | 0.031            |
| HIP 31592 | 19.75      | 0.225           | 25.568 | 0.163               | 0.151            |
| HIP 32349 | 2.63       | 0.025           | 30.177 | 0.872               | 0.861            |
| HIP 32362 | 18.0       | 0.145           | 26.221 | 0.21                | 0.199            |
| HIP 32366 | 24.38      | 0.207           | 23.238 | 0.041               | 0.036            |
Table 2 (Continued).

| Name     | Dist (pc) | Int. time (day) | Δmag  | $C_{Brown}$ | $C_{IAC}$ |
|----------|-----------|-----------------|-------|-------------|-----------|
| HIP 32439 | 17.87     | 0.465           | 24.414| 0.152       | 0.136     |
| HIP 32480 | 16.72     | 0.498           | 24.565| 0.18        | 0.163     |
| HIP 32607 | 29.6      | 0.051           | 26.098| 0.046       | 0.043     |
| HIP 32765 | 25.26     | 0.215           | 24.454| 0.061       | 0.055     |
| HIP 32984 | 8.71      | 0.777           | 22.753| 0.205       | 0.179     |
| HIP 33094 | 25.89     | 0.154           | 22.824| 0.026       | 0.023     |
| HIP 33277 | 17.24     | 0.471           | 23.721| 0.125       | 0.111     |
| HIP 33690 | 18.33     | 0.245           | 21.638| 0.033       | 0.028     |
| HIP 33817 | 14.65     | 0.425           | 22.086| 0.071       | 0.061     |
| HIP 34017 | 19.13     | 0.372           | 23.259| 0.082       | 0.072     |
| HIP 34834 | 21.43     | 0.231           | 25.368| 0.13        | 0.12      |
| HIP 35136 | 16.89     | 0.503           | 24.178| 0.156       | 0.139     |
| HIP 35296 A | 14.59   | 0.444           | 22.214| 0.077       | 0.067     |
| HIP 36366 | 18.05     | 0.245           | 25.626| 0.197       | 0.183     |
| HIP 36399 | 27.24     | 0.132           | 23.032| 0.024       | 0.021     |
| HIP 36439 | 20.24     | 0.38            | 24.354| 0.114       | 0.102     |
| HIP 36795 | 25.3      | 0.184           | 25.172| 0.076       | 0.069     |
| HIP 37279 | 3.51      | 0.063           | 28.816| 0.796       | 0.778     |
| HIP 37349 | 14.21     | 0.281           | 21.13 | 0.037       | 0.031     |
| HIP 37606 | 24.67     | 0.233           | 24.476| 0.067       | 0.06      |
| HIP 37826 | 10.36     | 0.073           | 27.756| 0.442       | 0.43      |
| HIP 38784 | 17.19     | 0.358           | 22.262| 0.059       | 0.051     |
| HIP 38908 | 16.2      | 0.532           | 24.279| 0.174       | 0.155     |
| HIP 39157 | 16.77     | 0.216           | 21.173| 0.028       | 0.024     |
| HIP 39342 | 17.31     | 0.195           | 21.121| 0.025       | 0.022     |
| HIP 39757 | 19.48     | 0.083           | 26.621| 0.184       | 0.177     |
| HIP 39780 | 23.29     | 0.266           | 24.017| 0.068       | 0.06      |
| HIP 39903 | 19.98     | 0.305           | 25.155| 0.148       | 0.135     |
| HIP 40035 | 22.38     | 0.292           | 23.955| 0.075       | 0.066     |
| HIP 40693 | 12.49     | 0.674           | 23.689| 0.205       | 0.181     |
| HIP 40702 | 19.56     | 0.196           | 25.75 | 0.171       | 0.159     |
| HIP 40706 | 28.63     | 0.124           | 25.177| 0.045       | 0.041     |
| HIP 40843 | 18.27     | 0.438           | 24.634| 0.156       | 0.141     |
| HIP 41211 | 26.62     | 0.159           | 23.493| 0.033       | 0.029     |
| Name      | Dist (pc) | Int. time (day) | Δmag   | C\textsubscript{brown} | C\textsubscript{IAC} |
|-----------|-----------|-----------------|--------|-------------------------|----------------------|
| HIP 41484 | 22.25     | 0.216           | 22.368 | 0.033                   | 0.029                |
| HIP 41926 | 12.21     | 0.621           | 22.933 | 0.151                   | 0.132                |
| HIP 42438 | 14.36     | 0.607           | 24.195 | 0.204                   | 0.182                |
| HIP 42808 | 11.14     | 0.631           | 22.574 | 0.141                   | 0.123                |
| HIP 43587 | 12.34     | 0.673           | 23.552 | 0.197                   | 0.173                |
| HIP 43726 | 17.39     | 0.442           | 23.35  | 0.105                   | 0.092                |
| HIP 43797 | 24.15     | 0.229           | 23.68  | 0.052                   | 0.046                |
| HIP 44075 | 21.03     | 0.323           | 23.609 | 0.077                   | 0.068                |
| HIP 44127 | 14.51     | 0.129           | 26.677 | 0.3                     | 0.288                |
| HIP 44143 | 26.42     | 0.182           | 24.346 | 0.049                   | 0.044                |
| HIP 44897 | 19.19     | 0.377           | 23.372 | 0.086                   | 0.076                |
| HIP 45170 | 20.36     | 0.232           | 21.979 | 0.033                   | 0.028                |
| HIP 45333 | 19.57     | 0.397           | 24.602 | 0.134                   | 0.12                 |
| HIP 45343 | 6.11      | 0.566           | 20.808 | 0.09                    | 0.076                |
| HIP 45617 | 17.27     | 0.131           | 20.537 | 0.016                   | 0.014                |
| HIP 46509 | 17.33     | 0.342           | 25.261 | 0.201                   | 0.184                |
| HIP 46580 | 12.91     | 0.281           | 20.919 | 0.036                   | 0.031                |
| HIP 46733 | 23.82     | 0.112           | 25.913 | 0.105                   | 0.099                |
| HIP 46843 | 17.79     | 0.161           | 20.93  | 0.021                   | 0.018                |
| HIP 46853 | 13.48     | 0.158           | 26.543 | 0.327                   | 0.312                |
| HIP 47080 | 11.37     | 0.689           | 24.424 | 0.295                   | 0.265                |
| HIP 47592 | 15.01     | 0.457           | 25.025 | 0.241                   | 0.22                 |
| HIP 48113 | 18.37     | 0.425           | 24.727 | 0.159                   | 0.143                |
| HIP 48331 | 11.16     | 0.237           | 20.399 | 0.03                    | 0.025                |
| HIP 48833 | 28.15     | 0.136           | 24.138 | 0.034                   | 0.031                |
| HIP 49081 | 15.05     | 0.571           | 24.431 | 0.206                   | 0.185                |
| HIP 49593 | 28.24     | 0.137           | 25.074 | 0.047                   | 0.043                |
| HIP 49669 | 24.31     | 0.029           | 27.69  | 0.106                   | 0.105                |
| HIP 49809 | 27.74     | 0.141           | 23.924 | 0.034                   | 0.03                 |
| HIP 49908 | 4.87      | 0.989           | 22.892 | 0.356                   | 0.312                |
| HIP 50075 | 22.85     | 0.198           | 22.331 | 0.03                    | 0.026                |
| HIP 50384 | 22.81     | 0.257           | 23.348 | 0.054                   | 0.047                |
| HIP 50505 | 20.24     | 0.198           | 21.651 | 0.026                   | 0.022                |
| HIP 50564 | 21.37     | 0.309           | 24.921 | 0.119                   | 0.108                |
| Name      | Dist (pc) | Int. time (day) | Δmag | $C_{\text{brown}}$ | $C_{\text{IAC}}$ |
|-----------|-----------|-----------------|------|---------------------|------------------|
| HIP 50954 | 16.22     | 0.219           | 25.922 | 0.244             | 0.229            |
| HIP 51459 | 12.78     | 0.464           | 25.177 | 0.305             | 0.28             |
| HIP 51502 | 21.5      | 0.333           | 24.57  | 0.105             | 0.094            |
| HIP 51523 | 21.81     | 0.298           | 24.914 | 0.112             | 0.102            |
| HIP 51814 | 26.53     | 0.179           | 24.362 | 0.049             | 0.044            |
| HIP 51933 | 25.08     | 0.195           | 23.413 | 0.04              | 0.035            |
| HIP 53229 | 29.09     | 0.108           | 25.297 | 0.043             | 0.04             |
| HIP 53253 | 29.13     | 0.101           | 25.391 | 0.044             | 0.041            |
| HIP 53721 | 14.06     | 0.509           | 24.905 | 0.257             | 0.234            |
| HIP 53910 | 24.45     | 0.039           | 26.951 | 0.103             | 0.1              |
| HIP 54035 | 2.54      | 1.0             | 21.604 | 0.308             | 0.263            |
| HIP 54182 | 28.99     | 0.125           | 24.696 | 0.037             | 0.033            |
| HIP 54211 | 4.86      | 0.263           | 19.093 | 0.027             | 0.023            |
| HIP 54745 | 21.93     | 0.213           | 22.22  | 0.032             | 0.027            |
| HIP 54872 | 17.91     | 0.074           | 26.903 | 0.219             | 0.212            |
| HIP 55691 A | 13.71   | 0.174           | 20.377 | 0.022             | 0.018            |
| HIP 55779 | 27.22     | 0.158           | 24.189 | 0.041             | 0.037            |
| HIP 56445 | 27.23     | 0.133           | 23.075 | 0.024             | 0.021            |
| HIP 56452 | 9.56      | 0.799           | 24.001 | 0.31              | 0.276            |
| HIP 56802 | 26.73     | 0.158           | 23.596 | 0.034             | 0.03             |
| HIP 56997 | 9.61      | 0.688           | 24.618 | 0.365             | 0.329            |
| HIP 56998 | 12.4      | 0.178           | 20.179 | 0.022             | 0.018            |
| HIP 57443 | 9.22      | 0.55            | 25.115 | 0.413             | 0.378            |
| HIP 57507 | 17.47     | 0.358           | 22.357 | 0.061             | 0.053            |
| HIP 57632 | 11.0      | 0.088           | 27.443 | 0.417             | 0.404            |
| HIP 57757 | 10.93     | 0.259           | 26.148 | 0.398             | 0.374            |
| HIP 57939 | 9.09      | 0.782           | 22.992 | 0.22              | 0.192            |
| HIP 58001 | 25.5      | 0.037           | 26.884 | 0.089             | 0.087            |
| HIP 58345 | 10.16     | 0.549           | 21.707 | 0.093             | 0.08             |
| HIP 58576 | 12.76     | 0.676           | 24.144 | 0.236             | 0.21             |
| HIP 58803 | 25.32     | 0.212           | 24.371 | 0.058             | 0.052            |
| HIP 58910 | 5.52      | 0.78            | 21.563 | 0.167             | 0.143            |
| HIP 59072 | 19.76     | 0.204           | 25.689 | 0.166             | 0.154            |
| HIP 59199 | 14.94     | 0.253           | 25.855 | 0.273             | 0.256            |
### Table 2 (Continued)

| Name      | Dist (pc) | Int. time (day) | Δmag | $C_{\text{brown}}$ | $C_{\text{IAC}}$ |
|-----------|-----------|-----------------|------|---------------------|------------------|
| HIP 59750 | 22.35     | 0.244           | 22.816 | 0.043               | 0.038            |
| HIP 59774 | 24.69     | 0.072           | 26.26  | 0.097               | 0.092            |
| HIP 60965 | 26.63     | 0.051           | 26.407 | 0.075               | 0.071            |
| HIP 61053 | 21.78     | 0.261           | 22.834 | 0.047               | 0.041            |
| HIP 61084 | 27.15     | 0.037           | 26.765 | 0.07                | 0.068            |
| HIP 61174 | 18.28     | 0.271           | 25.475 | 0.188               | 0.174            |
| HIP 61291 | 16.18     | 0.248           | 21.274 | 0.033               | 0.028            |
| HIP 61317 | 8.44      | 0.386           | 25.755 | 0.475               | 0.442            |
| HIP 62145 | 14.88     | 0.318           | 21.442 | 0.043               | 0.037            |
| HIP 62207 | 17.38     | 0.456           | 23.567 | 0.116               | 0.102            |
| HIP 62523 | 16.93     | 0.418           | 22.789 | 0.083               | 0.073            |
| HIP 62951 | 7.53      | 0.378           | 20.357 | 0.049               | 0.041            |
| HIP 62956 | 25.31     | 0.03            | 27.46  | 0.093               | 0.091            |
| HIP 63076 | 29.29     | 0.112           | 24.085 | 0.028               | 0.025            |
| HIP 63613 | 27.87     | 0.107           | 25.502 | 0.055               | 0.051            |
| HIP 63721 | 4.62      | 0.538           | 20.255 | 0.081               | 0.068            |
| HIP 64394 | 9.13      | 0.381           | 25.732 | 0.447               | 0.416            |
| HIP 64408 | 20.67     | 0.337           | 24.857 | 0.127               | 0.115            |
| HIP 64583 | 18.2      | 0.38            | 25.015 | 0.174               | 0.159            |
| HIP 64792 | 17.56     | 0.467           | 24.586 | 0.166               | 0.15             |
| HIP 64924 | 8.56      | 0.553           | 25.147 | 0.439               | 0.403            |
| HIP 65109 | 18.02     | 0.076           | 26.859 | 0.216               | 0.209            |
| HIP 65352 | 15.45     | 0.25            | 21.164 | 0.033               | 0.028            |
| HIP 65355 | 16.14     | 0.162           | 20.661 | 0.021               | 0.017            |
| HIP 65530 | 21.17     | 0.228           | 22.163 | 0.034               | 0.029            |
| HIP 65721 | 17.99     | 0.426           | 24.802 | 0.169               | 0.153            |
| HIP 66249 | 22.71     | 0.095           | 26.18  | 0.124               | 0.117            |
| HIP 66765 | 15.65     | 0.293           | 21.439 | 0.039               | 0.034            |
| HIP 67153 | 19.4      | 0.236           | 25.54  | 0.169               | 0.157            |
| HIP 67155 | 5.41      | 0.319           | 19.535 | 0.036               | 0.03             |
| HIP 67275 | 15.62     | 0.345           | 25.386 | 0.242               | 0.224            |
| HIP 67422 A | 13.41 | 0.362          | 21.413 | 0.05               | 0.043            |
| HIP 67927 | 11.4      | 0.139           | 26.884 | 0.397               | 0.38             |
| HIP 68030 | 24.86     | 0.165           | 22.587 | 0.026               | 0.023            |
| Name    | Dist (pc) | Int. time (day) | Δmag | $C_{brown}$ | $C_{IAC}$ |
|---------|-----------|-----------------|------|-------------|-----------|
| HIP 68184 | 10.06     | 0.729           | 22.959 | 0.194       | 0.169     |
| HIP 68682 | 16.98     | 0.42            | 22.833 | 0.085       | 0.074     |
| HIP 68933 | 18.03     | 0.072           | 26.912 | 0.216       | 0.209     |
| HIP 69671 | 21.22     | 0.258           | 22.541 | 0.043       | 0.037     |
| HIP 69701 | 22.24     | 0.194           | 25.491 | 0.121       | 0.112     |
| HIP 69713 | 29.07     | 0.124           | 24.773 | 0.037       | 0.034     |
| HIP 69965 | 18.03     | 0.43            | 23.529 | 0.106       | 0.093     |
| HIP 69972 | 11.8      | 0.592           | 22.482 | 0.124       | 0.108     |
| HIP 69989 | 26.1      | 0.181           | 23.903 | 0.043       | 0.038     |
| HIP 70319 | 17.19     | 0.411           | 22.815 | 0.082       | 0.072     |
| HIP 70497 | 14.53     | 0.261           | 25.85  | 0.284       | 0.266     |
| HIP 70857 | 19.89     | 0.178           | 21.426 | 0.023       | 0.02      |
| HIP 70873 | 23.74     | 0.172           | 22.267 | 0.025       | 0.022     |
| HIP 71075 | 26.61     | 0.056           | 26.311 | 0.074       | 0.071     |
| HIP 71181 | 13.22     | 0.337           | 21.254 | 0.046       | 0.039     |
| HIP 71284 | 15.83     | 0.314           | 25.495 | 0.241       | 0.224     |
| HIP 71855 | 20.0      | 0.203           | 21.649 | 0.027       | 0.023     |
| HIP 71957 | 18.27     | 0.204           | 25.823 | 0.197       | 0.185     |
| HIP 72567 | 18.17     | 0.429           | 23.644 | 0.109       | 0.097     |
| HIP 72603 | 22.98     | 0.285           | 24.398 | 0.082       | 0.073     |
| HIP 72622 | 23.24     | 0.062           | 26.537 | 0.119       | 0.114     |
| HIP 72848 | 11.51     | 0.715           | 23.602 | 0.219       | 0.193     |
| HIP 73100 | 25.11     | 0.202           | 23.679 | 0.046       | 0.04      |
| HIP 73165 | 26.9      | 0.165           | 25.043 | 0.058       | 0.052     |
| HIP 73996 | 19.55     | 0.347           | 25.001 | 0.15        | 0.136     |
| HIP 74273 | 24.2      | 0.167           | 22.366 | 0.025       | 0.022     |
| HIP 74537 | 17.67     | 0.311           | 21.982 | 0.046       | 0.04      |
| HIP 74605 | 25.34     | 0.211           | 24.386 | 0.059       | 0.053     |
| HIP 74702 | 15.85     | 0.274           | 21.367 | 0.036       | 0.031     |
| HIP 74975 | 25.38     | 0.211           | 24.352 | 0.058       | 0.052     |
| HIP 75181 | 14.81     | 0.589           | 24.092 | 0.188       | 0.167     |
| HIP 75718 | 20.58     | 0.139           | 21.214 | 0.018       | 0.015     |
| HIP 75809 | 21.85     | 0.176           | 21.803 | 0.024       | 0.02      |
| HIP 76219 | 28.93     | 0.123           | 24.381 | 0.033       | 0.03      |
| Name    | Dist (pc) | Int. time (day) | Δmag  | $C_{\text{Brown}}$ | $C_{\text{IAC}}$ |
|---------|-----------|----------------|-------|---------------------|------------------|
| HIP 76267 | 23.01     | 0.038          | 27.087| 0.125               | 0.121            |
| HIP 76829 | 17.44     | 0.34           | 25.26 | 0.198               | 0.182            |
| HIP 77052 | 14.66     | 0.578          | 23.722| 0.165               | 0.146            |
| HIP 77070 | 22.68     | 0.084          | 26.306| 0.125               | 0.119            |
| HIP 77257 | 12.12     | 0.385          | 25.5  | 0.338               | 0.313            |
| HIP 77358 | 15.35     | 0.525          | 23.372| 0.132               | 0.117            |
| HIP 77760 | 15.89     | 0.352          | 25.336| 0.234               | 0.216            |
| HIP 77801 | 17.35     | 0.437          | 23.278| 0.102               | 0.089            |
| HIP 77952 | 12.38     | 0.126          | 26.908| 0.365               | 0.351            |
| HIP 78072 | 11.25     | 0.272          | 26.058| 0.385               | 0.361            |
| HIP 78459 | 17.24     | 0.488          | 24.429| 0.164               | 0.147            |
| HIP 78527 | 21.03     | 0.173          | 25.748| 0.145               | 0.135            |
| HIP 78775 | 14.52     | 0.451          | 22.244| 0.079               | 0.069            |
| HIP 79190 | 14.67     | 0.304          | 21.331| 0.041               | 0.035            |
| HIP 79248 | 17.57     | 0.328          | 22.113| 0.051               | 0.044            |
| HIP 79537 | 13.89     | 0.226          | 20.787| 0.029               | 0.025            |
| HIP 79578 | 21.55     | 0.168          | 21.662| 0.022               | 0.019            |
| HIP 79672 | 13.9      | 0.625          | 24.242| 0.217               | 0.194            |
| HIP 79822 | 29.74     | 0.107          | 24.346| 0.028               | 0.026            |
| HIP 80179 | 27.27     | 0.165          | 24.698| 0.049               | 0.044            |
| HIP 80331 | 28.23     | 0.053          | 26.202| 0.058               | 0.055            |
| HIP 80337 | 12.78     | 0.641          | 24.472| 0.261               | 0.235            |
| HIP 80686 | 12.12     | 0.492          | 25.116| 0.32                | 0.293            |
| HIP 81300 | 9.75      | 0.794          | 23.977| 0.302               | 0.268            |
| HIP 81935 | 14.26     | 0.166          | 20.386| 0.021               | 0.017            |
| HIP 82003 | 9.81      | 0.179          | 19.732| 0.021               | 0.017            |
| HIP 82020 | 26.11     | 0.192          | 24.816| 0.061               | 0.055            |
| HIP 82396 | 19.54     | 0.085          | 26.601| 0.183               | 0.175            |
| HIP 82587 | 29.19     | 0.111          | 23.885| 0.026               | 0.023            |
| HIP 82588 | 17.25     | 0.303          | 21.827| 0.043               | 0.037            |
| HIP 82621 | 26.94     | 0.13           | 22.785| 0.021               | 0.019            |
| HIP 82860 | 15.26     | 0.419          | 25.138| 0.241               | 0.22             |
| HIP 83000 | 28.04     | 0.084          | 25.743| 0.056               | 0.052            |
| HIP 83389 | 18.59     | 0.25           | 21.73 | 0.034               | 0.029            |
Table 2 (Continued).

| Name    | Dist (pc) | Int. time (day) | Δmag  | C\text{Brown} | C\text{IAC} |
|---------|-----------|-----------------|-------|---------------|-------------|
| HIP 83541 | 17.84     | 0.284           | 21.815| 0.04          | 0.034       |
| HIP 83591 | 10.71     | 0.218           | 20.197| 0.027         | 0.023       |
| HIP 83601 | 20.67     | 0.317           | 23.214| 0.067         | 0.058       |
| HIP 83609 | 5.62      | 0.311           | 19.561| 0.035         | 0.029       |
| HIP 83990 | 13.62     | 0.26            | 20.929| 0.034         | 0.029       |
| HIP 84143 | 22.53     | 0.104           | 26.117| 0.126         | 0.119       |
| HIP 84478 | 5.95      | 0.95            | 23.071| 0.332         | 0.29        |
| HIP 84862 | 14.33     | 0.578           | 24.57 | 0.231         | 0.208       |
| HIP 85042 | 19.52     | 0.316           | 22.628| 0.056         | 0.049       |
| HIP 85235 | 12.8      | 0.588           | 22.853| 0.136         | 0.119       |
| HIP 85295 | 7.7       | 0.45            | 20.704| 0.063         | 0.053       |
| HIP 86201 | 23.16     | 0.251           | 25.007| 0.097         | 0.088       |
| HIP 86400 | 11.0      | 0.644           | 22.624| 0.147         | 0.128       |
| HIP 86486 | 21.45     | 0.29            | 25.037| 0.121         | 0.11        |
| HIP 86614 | 22.84     | 0.226           | 25.235| 0.107         | 0.098       |
| HIP 86620 | 23.06     | 0.257           | 23.538| 0.057         | 0.05        |
| HIP 86742 | 25.09     | 0.079           | 26.132| 0.091         | 0.086       |
| HIP 86796 | 15.51     | 0.512           | 24.744| 0.215         | 0.195       |
| HIP 86974 | 8.31      | 0.253           | 26.393| 0.501         | 0.474       |
| HIP 88175 | 23.55     | 0.233           | 25.073| 0.094         | 0.085       |
| HIP 88635 | 29.7      | 0.065           | 25.833| 0.044         | 0.041       |
| HIP 88694 | 17.55     | 0.439           | 23.382| 0.104         | 0.092       |
| HIP 88771 | 26.63     | 0.088           | 25.85 | 0.071         | 0.066       |
| HIP 88972 | 11.02     | 0.686           | 23.01 | 0.178         | 0.156       |
| HIP 89042 | 17.61     | 0.476           | 24.3  | 0.15          | 0.135       |
| HIP 89348 | 22.92     | 0.28            | 24.79 | 0.094         | 0.085       |
| HIP 89474 | 22.82     | 0.213           | 22.54 | 0.034         | 0.03        |
| HIP 89962 | 18.54     | 0.154           | 26.107| 0.197         | 0.186       |
| HIP 90496 | 23.97     | 0.088           | 26.129| 0.105         | 0.099       |
| HIP 90790 | 13.25     | 0.399           | 21.587| 0.059         | 0.05        |
| HIP 91262 | 7.68      | 0.03            | 29.135| 0.559         | 0.549       |
| HIP 91768 | 3.52      | 0.41            | 19.289| 0.05          | 0.041       |
| HIP 92024 | 28.55     | 0.136           | 24.804| 0.041         | 0.037       |
| HIP 92043 | 19.21     | 0.229           | 25.603| 0.174         | 0.162       |
| Name    | Dist (pc) | Int. time (day) | Δmag | C\textsubscript{brown} | C\textsubscript{IAC} |
|---------|-----------|----------------|------|------------------------|----------------------|
| HIP 92161 | 28.89     | 0.115          | 25.259 | 0.044                  | 0.041                |
| HIP 92549 | 25.67     | 0.187          | 23.697 | 0.042                  | 0.037                |
| HIP 93747 | 25.46     | 0.055          | 26.499 | 0.088                  | 0.084                |
| HIP 93858 | 16.95     | 0.437          | 23.018 | 0.094                  | 0.082                |
| HIP 93966 | 21.43     | 0.285          | 23.049 | 0.056                  | 0.049                |
| HIP 94083 | 27.3      | 0.159          | 24.403 | 0.044                  | 0.039                |
| HIP 94376 | 29.87     | 0.06           | 25.889 | 0.043                  | 0.04                |
| HIP 95149 | 18.83     | 0.293          | 22.147 | 0.045                  | 0.039                |
| HIP 95319 | 15.76     | 0.465          | 22.822 | 0.097                  | 0.085                |
| HIP 95447 | 15.18     | 0.537          | 24.662 | 0.217                  | 0.196                |
| HIP 95501 | 15.53     | 0.158          | 26.357 | 0.269                  | 0.256                |
| HIP 95995 A | 16.96   | 0.367          | 22.278 | 0.061                  | 0.053                |
| HIP 96100 | 5.75      | 0.532          | 25.374 | 0.576                  | 0.531                |
| HIP 96258 | 25.11     | 0.198          | 23.568 | 0.043                  | 0.038                |
| HIP 96441 | 18.34     | 0.281          | 25.424 | 0.186                  | 0.172                |
| HIP 96895 | 21.08     | 0.315          | 23.431 | 0.07                   | 0.062                |
| HIP 96901 | 21.21     | 0.284          | 22.947 | 0.054                  | 0.047                |
| HIP 97649 | 5.12      | 0.061          | 28.669 | 0.689                  | 0.673                |
| HIP 97650 | 27.87     | 0.135          | 23.712 | 0.03                   | 0.027                |
| HIP 97675 | 19.19     | 0.403          | 24.679 | 0.143                  | 0.129                |
| HIP 97944 | 14.05     | 0.552          | 23.011 | 0.128                  | 0.113                |
| HIP 98036 | 13.7      | 0.257          | 25.942 | 0.308                  | 0.289                |
| HIP 98066 | 25.99     | 0.196          | 24.561 | 0.057                  | 0.051                |
| HIP 98470 | 21.25     | 0.322          | 23.721 | 0.078                  | 0.069                |
| HIP 98698 | 12.86     | 0.225          | 20.606 | 0.029                  | 0.024                |
| HIP 98767 | 15.86     | 0.539          | 23.947 | 0.159                  | 0.142                |
| HIP 98792 | 15.77     | 0.222          | 21.051 | 0.029                  | 0.025                |
| HIP 98819 | 17.77     | 0.453          | 23.791 | 0.122                  | 0.108                |
| HIP 98921 | 18.79     | 0.371          | 23.036 | 0.076                  | 0.067                |
| HIP 98959 | 17.73     | 0.425          | 23.298 | 0.098                  | 0.086                |
| HIP 99031 | 23.95     | 0.242          | 23.966 | 0.061                  | 0.054                |
| HIP 99137 | 23.43     | 0.197          | 22.533 | 0.031                  | 0.027                |
| HIP 99240 | 6.11      | 0.299          | 26.312 | 0.598                  | 0.564                |
| HIP 99461 | 6.01      | 0.782          | 24.537 | 0.499                  | 0.452                |
| Name        | Dist (pc) | Int. time (day) | Δmag | C_{brown} | C_{IAC} |
|-------------|-----------|-----------------|------|-----------|---------|
| HIP 99572   | 27.7      | 0.118           | 22.892 | 0.02      | 0.018   |
| HIP 99701   | 6.2       | 0.402           | 20.137 | 0.052     | 0.044   |
| HIP 99825   | 8.91      | 0.825           | 23.923 | 0.323     | 0.287   |
| HIP 100017  | 17.57     | 0.453           | 23.653 | 0.117     | 0.104   |
| HIP 100511  | 26.25     | 0.131           | 22.509 | 0.02      | 0.018   |
| HIP 100925  | 19.52     | 0.216           | 21.657 | 0.029     | 0.025   |
| HIP 101345  | 24.4      | 0.212           | 23.393 | 0.044     | 0.039   |
| HIP 101612  | 27.79     | 0.153           | 24.797 | 0.046     | 0.042   |
| HIP 101983  | 24.66     | 0.232           | 24.425 | 0.066     | 0.059   |
| HIP 101997  | 14.38     | 0.499           | 22.622 | 0.101     | 0.088   |
| HIP 102040  | 20.95     | 0.242           | 22.254 | 0.037     | 0.032   |
| HIP 102333  | 24.17     | 0.195           | 25.258 | 0.09      | 0.083   |
| HIP 102422  | 14.27     | 0.201           | 26.191 | 0.299     | 0.282   |
| HIP 102455  | 14.68     | 0.297           | 25.672 | 0.275     | 0.256   |
| HIP 102488  | 22.29     | 0.068           | 26.555 | 0.133     | 0.128   |
| HIP 103389  | 21.97     | 0.291           | 23.579 | 0.067     | 0.059   |
| HIP 103458  | 22.14     | 0.176           | 21.863 | 0.024     | 0.02    |
| HIP 103673  | 28.01     | 0.12            | 23.163 | 0.022     | 0.02    |
| HIP 104214  | 3.5       | 0.778           | 24.704 | 0.657     | 0.599   |
| HIP 104217  | 3.5       | 0.966           | 23.852 | 0.562     | 0.503   |
| HIP 104239 A| 17.57     | 0.13            | 20.584 | 0.016     | 0.014   |
| HIP 105090  | 3.95      | 1.019           | 22.792 | 0.39      | 0.341   |
| HIP 105199  | 15.04     | 0.077           | 27.164 | 0.291     | 0.282   |
| HIP 105858  | 9.26      | 0.361           | 25.804 | 0.445     | 0.415   |
| HIP 106440  | 4.95      | 0.288           | 19.245 | 0.031     | 0.026   |
| HIP 106559  | 27.1      | 0.14            | 23.208 | 0.027     | 0.023   |
| HIP 106696  | 14.62     | 0.283           | 21.212 | 0.037     | 0.032   |
| HIP 107089  | 21.2      | 0.174           | 25.728 | 0.142     | 0.132   |
| HIP 107350  | 17.89     | 0.43            | 23.466 | 0.104     | 0.092   |
| HIP 107556  | 11.87     | 0.142           | 26.818 | 0.38      | 0.364   |
| HIP 107649  | 15.99     | 0.539           | 24.202 | 0.173     | 0.154   |
| HIP 107975  | 27.45     | 0.143           | 23.663 | 0.031     | 0.028   |
| HIP 108036  | 26.62     | 0.173           | 24.239 | 0.046     | 0.041   |
| HIP 108870  | 3.62      | 0.625           | 25.183 | 0.688     | 0.633   |
| Name       | Dist (pc) | Int. time (day) | Δmag | Cbrown | C_IAC |
|------------|-----------|-----------------|------|--------|-------|
| HIP 109176 | 11.73     | 0.252           | 26.12| 0.371  | 0.349 |
| HIP 109378 | 21.56     | 0.164           | 21.628| 0.022  | 0.018 |
| HIP 109422 | 18.28     | 0.399           | 24.905| 0.168  | 0.153 |
| HIP 109427 | 28.3      | 0.071           | 25.894| 0.055  | 0.052 |
| HIP 109821 | 22.05     | 0.24            | 22.624| 0.04   | 0.035 |
| HIP 110649 | 20.56     | 0.369           | 24.363| 0.11   | 0.098 |
| HIP 111449 | 22.68     | 0.294           | 24.314| 0.083  | 0.074 |
| HIP 112117 | 23.64     | 0.22            | 23.07 | 0.042  | 0.037 |
| HIP 112447 | 16.3      | 0.286           | 25.586| 0.233  | 0.217 |
| HIP 112935 | 27.28     | 0.155           | 24.059| 0.038  | 0.034 |
| HIP 113137 | 26.75     | 0.131           | 22.721| 0.021  | 0.019 |
| HIP 113283 | 7.61      | 0.849           | 22.821| 0.242  | 0.211 |
| HIP 113357 | 15.61     | 0.554           | 24.306| 0.186  | 0.167 |
| HIP 113368 | 7.7       | 0.065           | 28.225| 0.554  | 0.54  |
| HIP 113421 | 19.86     | 0.31            | 22.688| 0.055  | 0.048 |
| HIP 113576 | 8.22      | 0.273           | 20.018| 0.033  | 0.028 |
| HIP 113860 | 29.42     | 0.109           | 24.09 | 0.027  | 0.024 |
| HIP 114046 | 3.28      | 1.0             | 21.937| 0.308  | 0.265 |
| HIP 114430 | 27.65     | 0.123           | 23.011| 0.022  | 0.019 |
| HIP 114570 | 24.59     | 0.192           | 25.221| 0.085  | 0.077 |
| HIP 114622 | 6.54      | 0.811           | 24.377| 0.46   | 0.415 |
| HIP 114924 | 20.5      | 0.362           | 24.042| 0.098  | 0.087 |
| HIP 114948 | 20.54     | 0.356           | 23.94 | 0.094  | 0.083 |
| HIP 114996 | 23.06     | 0.154           | 25.676| 0.112  | 0.104 |
| HIP 116085 | 16.93     | 0.296           | 21.711| 0.041  | 0.036 |
| HIP 116584 | 26.41     | 0.133           | 25.433| 0.068  | 0.063 |
| HIP 116727 | 14.1      | 0.186           | 26.3  | 0.305  | 0.29  |
| HIP 116745 | 11.42     | 0.44            | 21.427| 0.065  | 0.056 |
| HIP 116771 | 13.71     | 0.317           | 25.66 | 0.3    | 0.279 |
| HIP 120005 | 6.11      | 0.543           | 20.704| 0.083  | 0.07  |
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Dean Keithly is a PhD candidate in the Sibley School of Mechanical and Aerospace Engineering at Cornell University. He received his bachelor’s degree in mechanical engineering from Michigan Technological University, where he built the Oculus-ASR satellite thermal control subsystem. He received his master’s degree in systems engineering from Cornell University, working on projects funded by the NASA Institute of Advanced Concepts. His primary research focus is on spacecraft modeling, mission modeling, optimization, and scheduling.

Dmitry Savransky is a professor in the Sibley School of Mechanical and Aerospace Engineering at Cornell University and PI of the Space Imaging and Optical Systems...
Laboratory. He received his PhD from Princeton University in 2011 and was a postdoctoral researcher at Lawrence Livermore National Laboratory, where he worked on integration, testing and commissioning of the Gemini Planet Imager. His research focuses on the optimization of ground and space-based exoplanet imaging surveys, control of autonomous optical systems, and advanced image processing.

Corey Spohn received a Bachelor of Science in both physics and engineering science and mechanics from Virginia Tech in May 2018. During his time at Virginia Tech he focused heavily on astrophysics while conducting research on anti-frosting surfaces, inter-droplet ice bridging, and flying snakes. For his senior design project, he investigated human-robot dynamics in a crowded panic situation by having humans play the children’s game sharks and minnows with robots in a motion capture studio. He began working towards a PhD in the fall of 2018 and joined the Space Imaging and Optical Systems Lab in December 2018 where he is currently researching exoplanet mission scheduling.