Gaussian classical-quantum channels: gain of entanglement-assistance

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Abstract

In the present paper we introduce and study Bosonic Gaussian classical-quantum (c-q) channels; the embedding of the classical input into quantum is always possible and therefore the classical entanglement-assisted capacity $C_{ea}$ under appropriate input constraint is well defined. We prove a general property of entropy increase for weak complementary channel, that implies the equality $C = C_{ea}$ (where $C$ is the unassisted capacity) for certain class of c-q Gaussian channel under appropriate energy-type constraint. On the other hand, we show by explicit example that the inequality $C < C_{ea}$ is not unusual for constrained c-q Gaussian channel.

1 Introduction

In finite dimension a classical-quantum or quantum-classical channel can always be represented as a quantum channel, by embedding the classical input or output into quantum system. Then it makes sense to speak about entanglement-assisted capacity $C_{ea}$ [1], [2] of such a channel, in particular, to compare it with the unentangled classical capacity $C$. An interesting observation in [1] was that entanglement-assisted communication may be advantageous even for entanglement-breaking channels such as depolarizing channel with sufficiently high error probability. In the paper [6] we considered the case of quantum-classical (measurement) channels, showing that generically $C < C_{ea}$ for such channels. For infinite dimensional (in particular, continuous variable) systems an embedding of the classical output into quantum is not always possible, however entanglement-assisted transmission still makes
sense \([3]\); in particular this is the case for Bosonic Gaussian q-c channels. The measurement channels demonstrate the gain of entanglement assistance in the most spectacular way.

On the contrary, as shown in \([9]\), finite dimensional c-q channels (preparations) are essentially characterized by the property of having no gain of entanglement assistance, in this sense being “more classical” than measurements. In the present paper we study Bosonic Gaussian c-q channels; we observe that the embedding of the classical input into quantum is always possible and \(C_{ea}\) under the input constraint is thus well defined. We prove a general property of entropy increase for the weak complementary channel, that implies equality \(C = C_{ea}\) for certain class of c-q Gaussian channel under appropriate energy-type constraint. On the other hand, we show by explicit example that the inequality \(C < C_{ea}\) is not unusual for constrained c-q Gaussian channels.

### 2 Bosonic Gaussian Systems

The main applications of infinite-dimensional quantum information theory are related to Bosonic systems, for detailed description of which we refer to Ch. 12 in \([4]\). Let \(\mathcal{H}_A\) be the representation space of the Canonical Commutation Relations (CCR)

\[
W(z_A)W(z'_A) = \exp \left( -\frac{i}{2} z'_A \Delta_A z'_A \right) W(z'_A + z_A)
\]
with a coordinate symplectic space \((Z_A, \Delta_A)\) and the Weyl system \(W_A(z) = \exp(iR_A \cdot z_A); z_A \in Z_A\). Here \(R_A\) is the row-vector of the canonical variables in \(\mathcal{H}_A\), and \(\Delta_A\) is the canonical skew-symmetric commutation matrix of the components of \(R_A\),

\[
\Delta = \text{diag} \left[ \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right]_{j=1,\ldots,s}.
\]

Let \((Z_A, \Delta_A), (Z_B, \Delta_B)\) be the symplectic spaces of dimensions \(2s_A, 2s_B\), which will describe the input and the output of the channel (here \(\Delta_A, \Delta_B\) have the canonical form \([2]\)), and let \(W_A(z_A), W_B(z_B)\) be the Weyl operators in the Hilbert spaces \(\mathcal{H}_A, \mathcal{H}_B\) of the corresponding Bosonic systems. A centered Gaussian channel \(\Phi : \mathcal{F}(\mathcal{H}_A) \to \mathcal{F}(\mathcal{H}_B)\) is defined via the action of its dual
Φ* on the Weyl operators:

\[ \Phi^*[W_B(z_B)] = W(Kz_B) \exp \left[ -\frac{1}{2}z_B^t \alpha z_B \right], \quad (3) \]

where \( K \) is a matrix of a linear operator \( Z_B \to Z_A \), and \( \alpha \) is a real symmetric matrix satisfying

\[ \alpha \geq \pm \frac{i}{2} (\Delta_B - K^t \Delta_A K), \quad (4) \]

where \( \Delta_B - K^t \Delta_A K = \Delta_K \) is a real skew-symmetric matrix.

We will make use of the unitary dilation of the channel \( \Phi \) constructed in [3] (see also [4]). Consider the composite Bosonic system \( AD = BE \) with the Hilbert space \( \mathcal{H}_A \otimes \mathcal{H}_D \cong \mathcal{H}_B \otimes \mathcal{H}_E \) corresponding to the symplectic space \( Z = Z_A \oplus Z_D = Z_B \oplus Z_E \), where \( (Z_E, \Delta_E) \simeq (Z_A, \Delta_A) \). Thus \( [R_A R_D] = [R_B R_E] \) describe two different splits of the set of canonical observables for the composite system. Here \( A \) and \( B \) refer to input and output, while \( D \) and \( E \) to input and output environments. The channel \( \Phi \) is then described by the linear input-output relation (preserving the commutators)

\[ R'_B = R_A K + R_D K_D, \quad (5) \]

where the system \( D \) is in a centered Gaussian state \( \rho_D \) with the covariance matrix \( \alpha_D \) such that

\[ \alpha = K_D^t \alpha D K_D. \]

(for simplicity of notations we write \( R_A, \ldots \) instead of \( R_A \otimes I_D, \ldots \)). It is shown that the commutator-preserving relation \( (5) \) can be complemented to the full linear canonical transformation by putting

\[ R'_E = R_A L + R_D L_D, \quad (6) \]

where \( (2s_A) \times (2s_E) - \) matrix \( L \) and \( (2s_D) \times (2s_A) - \) matrix \( L_D \) are such that the square \( 2 (s_A + s_D) \times 2 (s_B + s_E) - \) matrix

\[ T = \begin{bmatrix} K & L \\ K_D & L_D \end{bmatrix} \quad (7) \]

is symplectic, i.e. satisfies the relation

\[ T^t \begin{bmatrix} \Delta_A & 0 \\ 0 & \Delta_D \end{bmatrix} T = \begin{bmatrix} \Delta_B & 0 \\ 0 & \Delta_E \end{bmatrix}, \]
which is equivalent to
\[ \Delta_B = K^t \Delta_A K + K_D^t \Delta_D K_D, \]
\[ 0 = K^t \Delta_A L + K_D^t \Delta_D L_D, \]
\[ \Delta_E = L^t \Delta_A L + L_D^t \Delta_D L_D. \]

Denote by the $U_T$ the unitary operator in $\mathcal{H}_A \otimes \mathcal{H}_D \simeq \mathcal{H}_B \otimes \mathcal{H}_E$ implementing the symplectic transformation $T$ so that
\[ [R_B' R_E'] = U_T^*[R_B R_E] U_T = [R_A R_D]T. \]

Then we have the unitary dilation
\[ \Phi^*[W_B(z_B)] = \text{Tr}_D (I_A \otimes \rho_D) U_T^* (W_B(z_B) \otimes I_E) U_T. \]

The weakly complementary channel is then
\[ \left( \Phi^w \right)^* [W_E(z_E)] = \text{Tr}_D (I_A \otimes \rho_D) U_T^* (I_B \otimes W_E(z_E)) U_T. \]

The equation (6) is nothing but the input-output relation for the weakly complementary channel which thus acts as
\[ \left( \Phi^w \right)^* [W_E(z_E)] = W_A(L z_E) \exp \left[ - \frac{1}{2} z_E^t L_D^t \alpha_D L_D z_E \right]. \]

In the case of pure state $\rho_D = |\psi_D\rangle\langle \psi_D|$ the relation (12) amounts to the Stinespring representation for the channel $\Phi$ with the isometry $V = U_T |\psi_D\rangle$, implying that $\Phi^w$ is the complementary channel $\tilde{\Phi}$ (see e.g. [4]).

3 A property of Gaussian classical-quantum channels

Usually classical-quantum (c-q) channel is understood as a mapping $x \to \rho_x$ of the classical alphabet $\mathcal{X} = \{ x \}$ into density operators in a Hilbert space. In the case of continuous alphabet there is no problem with embedding c-q channel into a quantum channel (as distinct from q-c channel, see [6]). Intuitively, let $\mathcal{X}$ be a continual domain with measure $dx$, then the required embedding is
\[ \Phi[\rho] = \int_{\mathcal{X}} \langle x | \rho | x \rangle \rho_x dx, \]
where \( \{|x\}; x \in \mathcal{X} \} \) is a Dirac’s system satisfying \( \langle x|x' \rangle = \delta(x - x') \). Here \( \Phi \) maps density operators into density operators. Notice that the range of the dual channel \( \Phi^* \) consists of bounded operators diagonal in the \( x \)-representation.

In general, we call a quantum channel \( \Phi \) classical-quantum \((c-q)\) if the range of \( \Phi^* \) consists of commuting operators. By using a structure theorem for Abelian algebras of operators in a Hilbert space, it is then not difficult to see that such a definition is essentially equivalent to the usual understanding. It follows from (1) that the necessary and sufficient condition for a Bosonic Gaussian channel (3) to be \( c-q \) is

\[
K^t \Delta_A K = 0. \tag{14}
\]

Thus \( \Delta_K = \Delta_B \) and therefore \( \det \Delta_K \neq 0 \). Under this condition it was shown in [7] that in the unitary dilation described above one can take \( s_E = s_A, s_D = s_B \) (and in fact \( E = A, D = B \)). We call such a dilation “minimal” as it is indeed such at least in the case of the pure state \( \rho_D \), as follows from [3]. The condition (14) then amounts to

\[
\alpha \geq \pm \frac{i}{2} \Delta_B, \tag{15}
\]

saying that \( \alpha \) is a covariance matrix of a centered Gaussian state \( \rho_D \). We say that the channel has minimal noise if \( \rho_D \) is a pure state, which is equivalent to the fact that \( \alpha \) is a minimal solution of the inequality (15). In quantum optics such channels are called quantum-limited.

Let us explain how this notion of \( c-q \) channel agrees with the usual one in the case of Bosonic Gaussian channels. The condition (14) means that the components of the operator \( R_A K \) all commute, hence their joint spectral measure is a sharp observable, and their probability distribution \( \mu_\rho(d^2n_z) \) can be arbitrarily sharply peaked around any point \( z = E_\rho(R_A K)^t = K^t m \) in the support \( \mathcal{X} \) of this measure by appropriate choice of the state \( \rho \). Here \( E_\rho \) denotes expectation with respect to \( \rho \) and \( m = E_\rho(R_A)^t \), hence \( \mathcal{X} = \text{Ran}K^t \subseteq Z_B \). Thus in this case it is natural to identify \( \Phi \) as \( c-q \) channel determined by the family of states \( z \mapsto W(z)\rho_B W(z)^*; z \in \mathcal{X} \).

**Proposition 1.** Let \( \Phi \) be a Gaussian \( c-q \) channel, then the weak complementary \( \Phi^w \) in the minimal unitary dilation has nonnegative entropy gain:

\[
S(\Phi^w[\rho]) - S(\rho) \geq 0 \quad \text{for all} \quad \rho.
\]
In particular if $\Phi$ has minimal noise, then this holds for the complementary channel $\tilde{\Phi}$, implying
\[
I(\rho, \Phi) \leq S(\Phi[\rho]),
\]
where
\[
I(\rho, \Phi) = S(\rho) + S(\Phi[\rho]) - S(\tilde{\Phi}[\rho])
\]
is the quantum mutual information.

Proof. Taking into account (14), the relation (8) becomes
\[
\Delta_B = K_D^t \Delta_D K_D.
\]
We consider the minimal dilation for which $\Delta_D = \Delta_B$, $\Delta_E = \Delta_A$, hence $K_D$ is a symplectic $2s_B \times 2s_B$ matrix. Then (9) implies
\[
L_D = - (K_D^t \Delta_D)^{-1} K^t \Delta_A L.
\]
Substituting (10) gives $\Delta_E = L^t M L$, where
\[
M = \Delta_A + \Delta_A K (\Delta_D K_D)^{-1} \Delta_D (K_D^t \Delta_D)^{-1} K^t \Delta_A
\]
\[
= \Delta_A + \Delta_A K \Delta_B^{-1} \Delta_D^{-1} (K_D^t)^{-1} K^t \Delta_A
\]
\[
= \Delta_A + \Delta_A K \Delta_B^{-1} K^t \Delta_A.
\]
Therefore $1 = (\det L)^2 \det M$, where
\[
\det M = \det (\Delta_A + \Delta_A K \Delta_B^{-1} K^t \Delta_A)
\]
\[
= \det \left( I_{2s_A \times 2s_A} + K \Delta_B^{-1} K^t \Delta_A \right).
\]
Due to (14) the matrix $N = K \Delta_B^{-1} K^t \Delta_A$ satisfies $N^2 = 0$, hence it has only zero eigenvalues. Therefore $I_{2s_A \times 2s_A} + N$ has only unit eigenvalues, implying $\det M = 1$ and hence $|\det L| = 1$.

By relation (13), the channel $\tilde{\Phi}^w$ is the Gaussian channel with the operator $L$ playing the role of $K$. By using a result of [5], we have
\[
S(\tilde{\Phi}^w[\rho]) - S(\rho) \geq \log |\det L| = 0.
\]

Proposition 2. Let $\Phi$ be a Gaussian c-q channel with minimal noise $\alpha$, such that $\text{Ran} K^t = Z_B$, satisfying the input constraint\footnote{The trace here is understood in the sense of extended expectation, as in [5].}
\[
\text{Tr} \rho H \leq E,
\]
where \( H = RK\epsilon K^t \) and \( \epsilon \) is real symmetric strictly positive definite matrix.

Then denoting \( C(E) \) (resp. \( C_{ea}(E) \)) the classical (resp. entanglement-assisted) capacity of the channel under the constraint (18),

\[
C(E) = C_{ea}(E) = \sup_{\rho: \text{Tr} \rho H \leq E} S(\Phi[\rho]).
\] (19)

An important condition here is \( \text{Ran} K^t = Z_B \), as we shall see in the next Section. The form of the operator \( H = RK\epsilon K^t \) is such that the constraint is expressed only in terms of the input observables of the c-q channel. Without it one could hardly expect the equality (19), although this requires further investigation. On the other hand, assumption of minimality of the noise seems to be related to the method of the proof and probably could be relaxed, with the last expression in (19) replaced by the supremum of \( \chi \)-function.

**Lemma.** Under the assumption (14) there exists a sequence of real symmetric \( (2s_A) \times (2s_A) \) -matrices \( \gamma_n \) satisfying the conditions:

1. \( \gamma_n \geq \pm \frac{i}{2} \Delta_A \);
2. \( K^t \gamma_n K \to 0 \).

**Proof.** The assumption (14) means that the subspace \( \mathcal{N} = \text{Ran} K \subset Z_A \) is isotropic, i.e. such that \( \Delta_A \) is degenerate on it. From the linear algebra it is known that there is a symplectic basis in \( Z_A \) of the form \( \{ e_1, \ldots, e_k, h_1, \ldots, h_k, g_1, \ldots \} \), where \( \{ e_1, \ldots, e_k \} \) is a basis in \( \mathcal{N} \), \( \{ h_1, \ldots, h_k \} \) span the isotropic subspace \( \mathcal{N}' \) and are such that \( e^t_i \Delta_A h_j = \delta_{ij} \), and \( \{ g_1, \ldots \} \) span the symplectic orthogonal complement of \( \mathcal{N} + \mathcal{N}' \). Then \( \Delta_A \) has the block matrix form in this basis

\[
\Delta_A = \begin{bmatrix}
0 & I_k & 0 \\
-I_k & 0 & 0 \\
0 & 0 & \Delta_g
\end{bmatrix}.
\]

Let \( \varepsilon_n \) be a sequence of positive numbers converging to zero, then

\[
\gamma_n = \begin{bmatrix}
\varepsilon_n I_k & 0 & 0 \\
0 & \frac{1}{4\varepsilon_n} I_k & 0 \\
0 & 0 & \gamma_g
\end{bmatrix},
\]
where $\gamma_0 \geq \pm \frac{1}{2}\Delta g$, satisfies the condition 1, and $K^t\gamma_n K = \varepsilon_n K^t K \to 0$. □

Proof of Proposition 2. According to the general version of the finite-dimensional result of [2] proven in [8],

$$C_{ea}(E) = \sup_{\rho : \text{Tr}\rho H \leq E} I(\rho, \Phi). \quad (20)$$

This version makes the only assumption that $H$ is positive self-adjoint operator, allowing the constraint set to be non-compact, which is important for our considerations in Sec. 4. Due to (16), it is then sufficient to show that

$$C(E) \geq \sup_{\rho : \text{Tr}\rho H \leq E} S(\Phi[\rho]).$$

We first consider the supremum in the right-hand side. Since the constraint operator $H = RK\epsilon K^t K^t$ is quadratic in the canonical variables $R$, the supremum can be taken over (centered) Gaussian states. Since the entropy of Gaussian state with covariance matrix $\alpha$ is equal to

$$\frac{1}{2}\text{Sp} g(\text{abs}(\Delta^{-1}\alpha) - I/2) = \frac{1}{2} \sum_{j=1}^{2s} g(|\lambda_j| - \frac{1}{2}), \quad (21)$$

where $g(x) = (x + 1) \log(x + 1) - x \log x$, Sp denotes trace of the matrices as distinct from that of operators in $\mathcal{H}$, and $\lambda_j$ are the eigenvalues of $\Delta^{-1}\alpha$ (see e.g. [4], Sec. 12.3.4), we have

$$\sup_{\rho : \text{Tr}\rho H \leq E} S(\Phi[\rho]) = \frac{1}{2} \sup_{\beta : \text{Sp} K\epsilon K^t K^t \beta \leq E} \text{Sp} g(\text{abs}(\Delta^{-1}_{\beta}(K^t\beta K + \alpha)) - I/2)$$

$$= \frac{1}{2} \max_{\mu : \text{Sp} \mu \leq E} \text{Sp} g(\text{abs}(\Delta^{-1}_{\mu}(\mu + \alpha)) - I/2). \quad (22)$$

Here in the first equality we used the formula (21) for the output state with the covariance matrix $K^t\beta K + \alpha$, and in the second we denoted $\mu = K^t\beta K$ and used the fact that for every $\mu$ such a $\beta$ exists due to the condition $\text{Ran} K^t = Z_B$. In the second expression the supremum is attained on some $\mu_0$ due to nondegeneracy of $\epsilon$ (see [4], Sec. 12.5). Denote by $\beta_0$ a solution of the equation $\mu_0 = K^t\beta_0 K$.

We construct a sequence of suboptimal ensembles as follows. Using the condition 1 of the Lemma, we let $\rho_n$ be a centered Gaussian state in $\mathcal{H}_A$ with the covariance matrices $\gamma_n$ and $\rho_n(z) = D(z)\rho_n D(z)^*, z \in Z_A$, be the family
of the displaced states, where $D(z)$ are the displacement operators obtained
by re-parametrization of the Weyl operators $W(z)$. Define the Gaussian
probability density $p_n(z)$ with zero mean and the covariance matrix $k_n\beta_0$,
where $k_n = 1 - \text{Sp}\gamma_n K\epsilon K^t/E > 0$ for large enough $n$ by the condition 2.
The average state of this ensemble is centered Gaussian with the co-
variance matrix $\gamma_n + k_n\beta_0$. Taking into account that $S(\rho_n(z)) = S(\rho_n)$, the $\chi$–quantity
of this ensemble is equal to

$$\chi_n = \frac{1}{2}\text{Sp} g \left( \text{abs} \left( \Delta^{-1}_B (K^t \gamma_n K + k_n K^t \beta_0 K + \alpha) \right) \right) - I/2)$$

$$-\frac{1}{2}\text{Sp} g \left( \text{abs} \left( \Delta^{-1}_B (K^t \gamma_n K + \alpha) \right) \right) - I/2).$$

By the condition 2 this converges to

$$\frac{1}{2}\text{Sp} g \left( \text{abs} \left( \Delta^{-1}_B (K^t \beta_0 K + \alpha) \right) \right) - I/2) - \frac{1}{2}\text{Sp} g \left( \text{abs} \left( \Delta^{-1}_B \alpha \right) \right) - I/2).$$

By minimality of the noise the second term is entropy of a pure state, equal
to zero, and the first term is just the maximum in (22). Thus

$$C(E) \geq \limsup_{n \to \infty} \chi_n = \sup_{\rho : \text{Tr}\rho H \leq E} S(\Phi[\rho]). \quad \square$$

4 One mode

Let $q, p$ be a Bosonic mode, $W(z) = \exp i(xq + yp)$ the corresponding Weyl
operator and $D(z) = \exp i(yq - xp)$ the displacement operator. We give two
examples where the channel describes classical signal with additive Gaussian
(minimal) quantum noise, in the first case the signal being two-dimensional
while in the second – one-dimensional. As we have seen, a c-q channel can
be described in two equivalent ways: as a mapping $m \to \rho_m$, where $m$ is the
classical signal, and as an extended quantum channel satisfying (14).

1. We first consider the minimal noise c-q channel with two-dimensional
real signal and show the coincidence of the classical entanglement-assisted
and unassisted capacities of this channel under appropriate input constraint,
by using result of Sec. 3. Such a coincidence is generic for unconstrained
finite-dimensional channels [2], but in infinite dimensions, as we will see in
the second example, situation is different. Some sufficient conditions for the
equality $C = C_{ea}$ were given in [9], however they do not apply here.
Let \( m = (m_q, m_p) \in \mathbb{R}^2 \) and consider the mapping \( m \to \rho_m \), where \( \rho_m \) is the state with the characteristic function

\[
\text{Tr} \rho_m W(z) = \exp \left[ i(m_q x + m_p y) - \frac{(N + \frac{1}{2})}{2} (x^2 + y^2) \right],
\]

so that

\[ \rho_m = D(m) \rho_0 D(m)^*. \]

The mapping \( m \to \rho_m \) can be considered as transmission of the two-dimensional classical signal \( m = (m_q, m_p) \) with the additive quantum Gaussian noise \( q, p \) with the average number of quanta \( N \). The minimal noise corresponds to \( N = 0 \).

The classical capacity of this channel with the input constraint

\[
\frac{1}{2} \int \|m\|^2 \ p(m) \ d^2m \leq E
\]

is given by the expression (see e.g. [4], Sec. 12.1.4)

\[
C(E) = g(N + E) - g(N),
\]

with the optimal distribution

\[
p(m) = \frac{1}{2\pi E} \exp \left( -\frac{\|m\|^2}{2E} \right)
\]

in the ensemble of coherent states \( |m\rangle\langle m| \). In particular, for the minimal noise channel \( (N = 0) \),

\[
C(E) = g(E) = S(\bar{\rho}),
\]

where \( \bar{\rho} \) is the Gaussian state with

\[
\text{Tr}\bar{\rho} W(z) = \exp \left[ -\frac{(E + \frac{1}{2})}{2} (x^2 + y^2) \right].
\]

Let us now embed this channel into quantum Gaussian channel \( \Phi \) in the spirit of previous Section. Since the input \( m = (m_q, m_p) \) is two-dimensional classical, one has to use two Bosonic input modes \( q_1, p_1, q_2, p_2 \) to describe it.
quantum-mechanically, so that e.g. \( m_q = q_1, m_p = q_2 \). The environment is one mode \( q, p \) in the Gaussian state \( \rho_0 \) so the output is given by the equations

\[
\begin{align*}
q' &= q + q_1 = q + m_q; \\
p' &= p + q_2 = p + m_p,
\end{align*}
\]

and the channel \( \Phi \) parameters are

\[
K = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \alpha = \left( N + \frac{1}{2} \right) I_2.
\]

The equations for the environment modes describing the weakly complementary channel \( \tilde{\Phi}^w \) are

\[
\begin{align*}
q'_1 &= q_1, \\
p'_1 &= p_1 - p - q_2/2, \\
q'_2 &= q_2, \\
p'_2 &= p_2 + q + q_1/2.
\end{align*}
\]

In fact, the set of equations (27), (28) is the same as for the quantum channel with additive classical Gaussian noise (see [4], Ex. 12.42), but in the latter case the input variables are \( q, p \) while in the former – \( q_1, p_1, q_2, p_2 \) (in both cases the output is \( q', p' \)). If \( N = 0 \) so that \( \rho_0 \) is pure, these equations describe the complementary channel \( \tilde{\Phi} \).

Having realized the c-q channel as a quantum one (i.e. a channel with quantum input and output), it makes sense to speak of its entanglement-assisted capacity. Under the same constraint it is given by the expression

\[
C_{ea}(E) = \sup_{\rho_{12} \in \mathcal{S}_E} I(\rho_{12}, \Phi),
\]

where

\[
\mathcal{S}_E = \left\{ \rho_{12} : \Tr\rho_{12} \left( \frac{q_1^2 + q_2^2}{2} \right) \leq E \right\}
\]

corresponds to the constraint (24). Notice that the constraint operator \( H = \frac{q_1^2 + q_2^2}{2} \) is unusual in that it is given by degenerate quadratic form in the input variables \( q_1, p_1, q_2, p_2 \). In this case the set \( \mathcal{S}_E \) is not compact, the supremum
in (29) is not attained and to obtain this formula we need to use a result from [8].

Now assume the minimal noise $N = 0$ and let us show that

$$C_{ea}(E) = C(E) = g(E). \tag{30}$$

Proposition 1 of Sec. 3 implies

$$C_{ea}(E) \leq \sup_{\rho_{12} \in \mathcal{E}_E} S(\Phi[\rho_{12}]).$$

But

$$\Phi[\mathcal{E}_E] = \{ \bar{\rho}_p : p \in \mathcal{P}_E \},$$

where $\mathcal{P}_E$ is defined by (25), as can be seen from the equations of the channel (27) and the identification of the probability density $p(m_q, m_p)$ with that of observables $q_1, q_2$ in the state $\rho_{12}$. Invoking (26) gives $\sup_{\rho_{12} \in \mathcal{E}_E} H(\Phi[\rho_{12}]) = g(E)$ and hence the equality (30). This example is a special case of Proposition 2 in Sec. 3, all the conditions of which are fulfilled with $\text{Ran} K^t = Z_B = \mathbb{R}^2$ and

$$\gamma_n = \begin{bmatrix} \varepsilon_n & 0 & 0 & 0 \\ 0 & 4\varepsilon_n & 0 & 0 \\ 0 & 0 & \varepsilon_n & 0 \\ 0 & 0 & 0 & \frac{4\varepsilon_n}{4} \end{bmatrix}.$$ 

2. Now we give an example with $C(E) < C_{ea}(E)$. Let $m \in \mathbb{R}$ be a real one-dimensional signal and the channel is $m \rightarrow \rho_m$, where $\rho_m$ is the state with the characteristic function

$$\text{Tr} \rho_m W(z) = \exp \left[ imx - \frac{1}{2}(\sigma^2 x^2 + \frac{1}{4\sigma^2} y^2) \right], \tag{31}$$

so that

$$\rho_m = D(x, 0)\rho_0 D(x, 0)^*.$$ 

The mapping $m \rightarrow \rho_m$ can be considered as transmission of the classical signal $m$ with the additive noise arising from the $q$-component of quantum Gaussian mode $q, p$ with the variances $D_q = \sigma^2, D_p = \frac{1}{16\sigma^2}$ and zero covariance between $q$ and $p$. The state $\rho_0$ is pure (squeezed vacuum) corresponding to a minimal noise.

The constraint on the input probability distribution $p(m)$ is defined as

$$\int m^2 p(m) \, dm \leq E, \tag{32}$$
where $E$ is a positive constant. As the component $p$ is not affected by the signal, from information-theoretic point of view this channel is equivalent to the classical additive Gaussian noise channel $m \to m + q$, and its capacity under the constraint (32) is given by the Shannon formula

$$C(E) = \frac{1}{2} \log (1 + r),$$  \hspace{1cm} (33)

where $r = E/\sigma^2$ is the signal-to-noise ratio.

A different way to describe this channel is to represent it as a quantum Gaussian channel $\Phi$. Introducing the input mode $q_1, p_1$, so that $m = q_1$, with the environment mode $q, p$ in the state $\rho_0$, the output is given by the equations

$$q'_1 = q_1 + q;$$
$$p'_1 = p,$$

and the channel $\Phi$ parameters are

$$K = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \alpha = \begin{bmatrix} \sigma^2 & 0 \\ 0 & \frac{1}{4\sigma^2} \end{bmatrix}.$$  

The equations for the environment mode describing the complementary channel $\tilde{\Phi}$ are (see [4])

$$q' = q_1,$$
$$p' = p_1 - p,$$

and the set of equations (34), (35) describes the canonical transformation of the composite system = system + environment.

The classical entanglement-assisted capacity of this channel under the same constraint is given by the expression

$$C_{ea}(E) = \sup_{\rho_1 \in \mathcal{E}_E^{(1)}} I(\rho_1, \Phi),$$  \hspace{1cm} (36)

where $\mathcal{E}_E^{(1)} = \{\rho_1 : \text{Tr} \rho_1 q_1^2 \leq E\}$. As in the first example, the constraint operator $q_1^2$ is given by degenerate quadratic form in the input variables $q_1, p_1$, the set $\mathcal{E}_E^{(1)}$ is not compact and the supremum in (29) is not attained.
Let us compute the entanglement-assisted capacity. For this consider the values of \( I(\rho_A, \Phi) \) for centered Gaussian states \( \rho_A = \rho_1 \) with covariance matrices
\[
\alpha_1 = \begin{bmatrix} E & 0 \\ 0 & E_1 \end{bmatrix},
\]
satisfying the uncertainty relation \( EE_1 \geq \frac{1}{4} \) and belonging to the set \( \mathcal{G}_E^{(1)} \) with the equality.

We use the formula (21) implying
\[
S(\rho_A) = g \left( \sqrt{EE_1} - \frac{1}{2} \right),
\]
According to (34), the output state \( \rho_B = \Phi[\rho_A] \) has the covariance matrix
\[
\alpha_B = \begin{bmatrix} E + \sigma^2 & 0 \\ 0 & \frac{1}{4\sigma^2} \end{bmatrix},
\]
with the entropy
\[
S(\rho_B) = g \left( \sqrt{E + \frac{1}{4\sigma^2}} - \frac{1}{4} \right).
\]
Similarly, according to (35) the state \( \rho_E = \tilde{\Phi}[\rho_A] \) of the environment has the covariance matrix
\[
\alpha_E = \begin{bmatrix} E & 0 \\ 0 & E_1 + \frac{1}{4\sigma^2} \end{bmatrix},
\]
with the entropy
\[
S(\rho_E) = g \left( \sqrt{EE_1 + \frac{E}{4\sigma^2}} - \frac{1}{2} \right).
\]
Summing up,
\[
I(\rho_A, \Phi) = S(\rho_A) + S(\rho_B) - S(\rho_E)
= g \left( \sqrt{E + \frac{1}{4\sigma^2}} + \frac{1}{4} - \frac{1}{2} \right) - \delta_1(E_1),
\]
where
\[
\delta_1(E_1) = g \left( \sqrt{EE_1 + \frac{E}{4\sigma^2}} - \frac{1}{2} \right) - g \left( \sqrt{EE_1 - \frac{1}{2}} \right).
\]
is a positive function in the range \([\frac{1}{4E}, \infty)\), decreasing from \(g\left(\sqrt{\frac{E}{4\sigma^2} + \frac{1}{4} - \frac{1}{2}}\right)\) to 0 for \(E \to \infty\) (this follows from the asymptotic \(g(x) = \log(x/e) + o(1)\)). Thus

\[ C_{ea}(E) \geq g\left(\sqrt{\frac{E}{4\sigma^2} + \frac{1}{4} - \frac{1}{2}}\right). \]

Let us show that in fact there is equality here, by using the concavity of the quantum mutual information (see [4], Sec. 12.5). For a given input state \(\rho\) with finite second moments consider the state

\[ \tilde{\rho} = \frac{1}{2}(\rho + \rho^\top), \]

where the transposition \(^\top\) corresponds to the antiunitary conjugation \(q, p \to q, -p\). The state \(\tilde{\rho}\) has the same variances \(D_q, D_p\) as \(\rho\), and zero covariance between \(q\) and \(p\). The channel (34) is covariant with respect to the transposition; by the aforementioned concavity, \(I(\tilde{\rho}, \Phi) \geq I(\rho, \Phi)\), moreover, \(I(\tilde{\rho}_G, \Phi) \geq I(\tilde{\rho}, \Phi)\), where \(\tilde{\rho}_G\) is the Gaussian state with the same first and second moments as \(\tilde{\rho}\). Thus

\[ C_{ea}(E) = g\left(\sqrt{\frac{E}{4\sigma^2} + \frac{1}{4} - \frac{1}{2}}\right) = g\left(\frac{\sqrt{1 + r} - 1}{2}\right) \]

\[ = \frac{\sqrt{1 + r} + 1}{2} \log \frac{\sqrt{1 + r} + 1}{2} - \frac{\sqrt{1 + r} - 1}{2} \log \frac{\sqrt{1 + r} - 1}{2}, \]

where \(r = E/\sigma^2\) is signal-to-noise ratio. Comparing this with (33), one has \(C_{ea}(E) > C(E)\) for \(E > 0\) (see Appendix), with the entanglement-assistance gain \(C_{ea}(E)/C(E) \sim -\frac{1}{2} \log r\), as \(r \to 0\) and \(C_{ea}(E)/C(E) \to 1\), as \(r \to \infty\) (see Figures).

As it is to be expected, Proposition 2 is not applicable, as \(\text{rank} K^t = 1 < \text{dim} Z_B\) here, while

\[ \gamma_n = \begin{bmatrix} \varepsilon_n & 0 \\ 0 & \frac{1}{4\varepsilon_n} \end{bmatrix} \]

still satisfies the conditions 1, 2 of the Lemma.

5 Appendix

1. Consider the channel (27). It is instructive to compare its unassisted classical capacity \(C(E)\) given by (30) with the values of \(I(\rho_{12}, \Phi)\) for centered
Gaussian states $\rho_{12} = \rho_A$ with the covariance matrices

$$\alpha_{12} = \begin{bmatrix} E & 0 & 0 & 0 \\ 0 & E_1 & 0 & 0 \\ 0 & 0 & E & 0 \\ 0 & 0 & 0 & E_1 \end{bmatrix},$$

satisfying the uncertainty relation $EE_1 \geq \frac{1}{4}$ and belonging to the set $\mathcal{S}_E$ with the equality.

We then find

$$S(\rho_{12}) = 2g \left( \sqrt{EE_1} - \frac{1}{2} \right).$$

According to (27), $\rho_B = \Phi[\rho_A]$ has the covariance matrix

$$\alpha_B = \begin{bmatrix} E + \frac{1}{2} & 0 \\ 0 & E + \frac{1}{2} \end{bmatrix},$$

with the entropy $g(E)$, and according to (28) the state $\rho_E$ of the environment has the covariance matrix

$$\alpha_E = \begin{bmatrix} E & 0 & 0 & E/2 \\ 0 & \tilde{E}_1 & -E/2 & 0 \\ 0 & -E/2 & E & 0 \\ E/2 & 0 & 0 & \tilde{E}_1 \end{bmatrix},$$

where $\tilde{E}_1 = E_1 + \frac{1}{2} + \frac{E}{4}$. The eigenvalues of $\Delta^{-1}_E \alpha_E$ are $\sqrt{E} \left( \sqrt{\tilde{E}_1} \pm \frac{1}{2} \sqrt{E} \right)$ and have multiplicity 2. Thus

$$S(\rho_E) = S(\Phi[\rho_{12}]) = g \left( \sqrt{E} \left( \sqrt{\tilde{E}_1} + \frac{1}{2} \sqrt{E} \right) - \frac{1}{2} \right)$$

$$+ g \left( \sqrt{E} \left( \sqrt{\tilde{E}_1} - \frac{1}{2} \sqrt{E} \right) - \frac{1}{2} \right).$$

Summing up,

$$I(\rho_{12}, \Phi) = g(E) - \delta(E_1),$$

where

$$\delta(E_1) = g \left( \sqrt{E} \left( \sqrt{\tilde{E}_1} + \frac{1}{2} \sqrt{E} \right) - \frac{1}{2} \right) + g \left( \sqrt{E} \left( \sqrt{\tilde{E}_1} - \frac{1}{2} \sqrt{E} \right) - \frac{1}{2} \right)$$

$$- 2g \left( \sqrt{EE_1} - \frac{1}{2} \right)$$

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is a positive function in the range $[\frac{1}{4E}, \infty)$, varying from $g(E)$ to 0. Hence the value (30) is attained only asymptotically for the input states $\rho_{12}$ with momentum variance $E_1 \to \infty$.

2. Introducing the new variable $x = \sqrt{1 + r} \geq 1$, we have

$$C(E) = \log x \equiv f_1(x), \quad C_{ea}(E) = \frac{x + 1}{2} \log \frac{x + 1}{2} - \frac{x - 1}{2} \log \frac{x - 1}{2} \equiv f_2(x).$$

Then $f_1(1) = f_2(1), f'_1(\infty) = f'_2(\infty)$ and $f''_1(x) > f''_2(x)$. It follows $f_1(x) < f_2(x), x > 1$.

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Figure 1: Ex.2: The classical capacities (nats) as functions of signal-to-noise ratio $r$: $C_{ea}(E)$ – solid line, $C(E)$ – dashed line.

Figure 2: Ex.2: The gain of entanglement assistance $C_{ea}(E)/C(E)$ as function of signal-to-noise ratio.