MEASURES ON PROJECTIONS IN A $W^*$-ALGEBRA OF TYPE $I_2$

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Abstract. It is shown that for every measure $m$ on projections in a $W^*$-algebra of type $I_2$, there exists a Hilbert-valued orthogonal vector measure $\mu$ such that $\|\mu(p)\|^2 = m(p)$ for every projection $p$. With regard to J. Hamhalter’s result (Proc. Amer. Math. Soc., 110 (1990), 803–806) it means that the assertion is valid for an arbitrary $W^*$-algebra.

It is well known that the problem of the extension of a measure on projections to a linear functional was positively solved for $W^*$-algebras without type $I_2$ direct summand. (A lucid exposition of Gleason-Christensen-Yeadon’s results see in [2].) In view of this, it became a good tradition to exclude the $W^*$-algebras with direct summand of type $I_2$ when measures on projections are investigated. In this respect, there is an interesting paper by J. Hamhalter [1] which describes the connection between measures on projections in conventional sense and $H$-valued ($H$ is a complex Hilbert space) orthogonal vector measures. Specifically, it has been proved in [1] (though expressed in a slightly different form) that if $m$ is a measure on projections in a $W^*$-algebra $A$ without type $I_2$ direct summand, then there exists a $H$-valued orthogonal vector measure $\mu$ on projections in $A$ such that $\|\mu(p)\|^2 = m(p)$ for every $p \in A$. The mentioned proof (in a few lines) of this assertion is based on Gleason-Christensen-Yeadon’s result.

In this paper we give a construction allowing to obtain a proof of this assertion for $W^*$-algebras of type $I_2$ and therefore for arbitrary $W^*$-algebras. The author is greatly indebted to Lugovaya G.D. for useful discussions.

Preliminaries

Let $A$ be a $W^*$-algebra, and $A^{pr}$, $A^{un}$, $A^+$ denote the sets of orthogonal projections, unitaries, positive elements in $A$, respectively. We will denote by $rp(x)$ the range projection of $x \in A^+$. It is the least projection of all projections $p \in A^{pr}$ such that $px = x$. It should be noted that $rp(x) = rp(x^{1/2})$. The basic notions those we talk about in this paper are described by the following

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definitions. (Monograph [3] gives further details of problems related to measures on projections in von Neumann algebras.)

**Definition 1.** Let $\mathcal{A}$ be a $W^*$-algebra. A mapping $m : \mathcal{A}^{pr} \to \mathbb{R}_+$ is called a *measure on projections* if the following condition is satisfied:

$$p = \sum_{i \in I} p_i \quad (p, p_i \in \mathcal{A}^{pr}, p_i p_j = 0 \ (i \neq j)) \Rightarrow m(p) = \sum_{i \in I} m(p_i).$$

Here, the series are understood as limits of the nets of finite sums (in $w^*$-topology for projections).

**Definition 2.** Let $\mathcal{A}$ be a $W^*$-algebra, $H$ be a complex Hilbert space. A mapping $\mu : \mathcal{A}^{pr} \to H$ is called an *orthogonal vector measure* if for any set $(p_j)_{j \in J} \subset \mathcal{A}^{pr}$ of mutually orthogonal projections the following two conditions are satisfied:

(i) the set $(\mu(p_j))_{j \in J}$ is orthogonal in $H$,
(ii) $\mu(\sum_{j \in J} p_j) = \sum_{j \in J} \mu(p_j),$

where the series on the right hand side are understood as the limit of the net of finite partial sums (in the norm topology on $H$).

Let $X \subset \mathcal{A}^{pr}$ has the property

(iii) $p, q \in X, \ pq = 0 \Rightarrow p + q \in X.$

We call $\mu : X \to H$ a *finitely additive orthogonal vector measure on* $X$ if the following condition is satisfied

$$p, q \in X, \ pq = 0 \ \Rightarrow \langle \mu(p), \mu(q) \rangle = 0, \ \mu(p + q) = \mu(p) + \mu(q).$$

We are interested here in $W^*$-algebras of type $I_2$. It is known that the every $W^*$-algebra $\mathcal{N}$ of type $I_2$ can be expressed in the form $\mathcal{N} = \mathcal{M} \otimes \mathcal{M}_2$ where $\mathcal{M}$ is a commutative $W^*$-algebra and $\mathcal{M}_2$ is the algebra of all $2 \times 2$ matrices over $\mathbb{C}$.

We turn our attention to the structure of projections in algebra $\mathcal{N}$. We will consider projections in $\mathcal{N}^{pr}$ defined as follows:

$$\pi_1 \oplus \pi_2 \equiv \begin{pmatrix} \pi_1 & 0 \\ 0 & \pi_2 \end{pmatrix}, \quad \pi_1, \pi_2 \in \mathcal{M}^{pr},$$

$$p(x, v, \pi) \equiv \begin{pmatrix} x & v(x(\pi - x))^{1/2} \\ v^*(x(\pi - x))^{1/2} & \pi - x \end{pmatrix},$$

where $\pi \in \mathcal{M}^{pr}$, $v \in \mathcal{M}^{un}$, $0 \leq x \leq \pi$, $\text{rp}(x(\pi - x)) = \pi$. In particular, $p(0, v, 0) = 0$.

The following two lemmas are fairly straightforward from equalities $p = p^2 = p^*$ for a projection $p$. 
Lemma 1. Every projection \( p \in \mathcal{N}^{pr} \) can be expressed in the form:
\[
p = \pi_1 \oplus \pi_2 + p(x, v, \pi),
\]
where \( \pi_i \leq 1 - \pi, \ i = 1, 2 \).

We will denote \( \pi \setminus \rho \equiv \pi - \pi \rho, \ \pi \Delta \rho \equiv (\pi \setminus \rho) + (\rho \setminus \pi), \ \pi, \rho \in \mathcal{M}^{pr} \).

Let us observe some useful properties of the mentioned representation for projections.

Lemma 2. \( p(x, v, \pi)p(y, w, \rho) = 0 \) if and only if
\[
y \pi \rho = (1 - x) \pi \rho, \ \ w \pi \rho = -v \pi \rho.
\]
In addition,
\[
p(x, v, \pi) + p(y, w, \rho) = \pi \rho \oplus \pi \rho + p(z, u, \pi \Delta \rho)
\]
where \( z = x(\pi \setminus \rho) + y(\rho \setminus \pi) \) and \( u \in \mathcal{M}^{un} \) satisfies equations: \( u(\pi \setminus \rho) = v(\pi \setminus \rho), \ u(\rho \setminus \pi) = w(\rho \setminus \pi) \).

Specifically, \( p(x, v, \pi)p(y, w, \pi) = 0 \) if and only if \( y \pi = (1 - x) \pi, \ \ w \pi = -v \pi. \)
In addition,
\[
p(x, v, \pi) + p(1 - x, -v, \pi) = \pi \oplus \pi.
\]

Lemma 3. Let \( \mathcal{A} \) be a \( \mathcal{W}^* \)-algebra, \( m : \mathcal{A}^{pr} \to \mathbb{R}_+ \) be a measure on projections and \( \mu : \mathcal{A}^{pr} \to H \) be a finitely additive orthogonal vector measure with
\[
\| \mu(p) \|^2 = m(p), \quad p \in \mathcal{A}^{pr}.
\]
Then \( \mu \) is the orthogonal vector measure.

Proof. It should be enough to verify the property (ii) in Definition 2. Let \( p = \sum_{j \in J} p_j = \text{w}^*\text{-lim} \sum_{\sigma} p_j \) (the limit of the net of finite partial sums). Since (ii) is fulfilled for finite sums, we have
\[
\|\mu(p) - \sum_{j \in \sigma} \mu(p_j)\|^2 = \|\mu(p - \sum_{j \in \sigma} p_j)\|^2 = m(p - \sum_{j \in \sigma} p_j)
= m(p) - \sum_{j \in \sigma} m(p_j).
\]
As \( m \) is completely additive, it follows \( \lim_{\sigma} [m(p) - \sum_{j \in \sigma} m(p_j)] = 0 \). \( \square \)

We need also the following elementary lemma.

\(^1\)Here and subsequently 1 denote the identity element in \( \mathcal{M} \).
Lemma 4. A system of equations
\[
\begin{cases}
\lambda_1 + \mu_1 = \lambda_0, \\
\lambda_2 + \mu_2 = \mu_0, \\
\lambda_1^2 + \lambda_2^2 = \lambda^2, \\
\mu_1^2 + \mu_2^2 = \mu^2
\end{cases}
\]
with respect to unknowns \(\lambda_i, \mu_i\) \((i = 1, 2)\) where
\[
\lambda_0^2 + \mu_0^2 = \lambda^2 + \mu^2,
\]
is solvable in \(\mathbb{R}\). In this case
\[
\lambda_1 \mu_1 + \lambda_2 \mu_2 = 0.
\]

A construction of the orthogonal vector measure

Now we examine some maximal commutative \(W^*-\)subalgebras in \(\mathcal{N}\) that will useful for us. One such subalgebra is \(\mathcal{M} \oplus \mathcal{M}\), the direct sum of two copies of \(\mathcal{M}\),
\[
\mathcal{M} \oplus \mathcal{M} = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x, y \in \mathcal{M} \right\}.
\]
Next, every pair \((x, v)\) where
\[
x \in \mathcal{M}^+, \ x \leq 1, \ \text{rp}(x(1 - x)) = 1, \ v \in \mathcal{M}^{\text{un}},
\]
can be associated with a maximal commutative \(W^*-\)subalgebra \(\mathcal{N}_{x,v}\) in \(\mathcal{N}\) described by the set of its projections
\[
\mathcal{N}^{\text{pr}}_{x,v} = \{ p(x\pi_1, v, \pi_1) + p((1 - x)\pi_2, -v, \pi_2) : \pi_1 \in \mathcal{M}^{\text{pr}}, \ i = 1, 2 \}.
\]
It is easily seen that \(\mathcal{N}_{x,v}\) is maximal. Note that \(\mathcal{N}_{x,v} = \mathcal{N}_{1-x,-v}\).

Let us to index the set of all such pairs, and associate to each \(\gamma = (x, v)\) the set \(\mathcal{N}^{\text{pr}}_{\gamma} = \mathcal{N}^{\text{pr}}_{x,v}\) and associate to 0 the set \(\mathcal{N}^{\text{pr}}_0 = \{ \pi_1 \oplus \pi_2 : \pi_1 \in \mathcal{M}^{\text{pr}}, \ i = 1, 2 \}\). Then we totally order the set \(\Gamma\) of all indices \(\gamma\), taking \(0 = \min \Gamma\).

It is known (\[4\), Proposition 1.18.1]) that a commutative \(W^*-\)algebra \(\mathcal{M}\) may be realized as \(C^*-\)algebra \(L^\infty(\Omega, \nu)\) of all essentially bounded locally \(\nu\)-measurable functions on a localizable measure space \((\Omega, \nu)\) (i. e. \(\Omega\) is direct sum of finite measure spaces, see \[5\]). In this case, the Banach space \(L^1(\Omega, \nu)\) is the predual of \(L^\infty(\Omega, \nu)\): \(L^1(\Omega, \nu)^* = L^\infty(\Omega, \nu)\). Now we shall identify \(\mathcal{M}\) with \(L^\infty(\Omega, \nu)\). In this case the characteristic functions
\[
\pi(\omega) \equiv \chi_\pi(\omega) = \begin{cases} 1, & \text{if } \omega \in \pi, \\ 0, & \text{if } \omega \notin \pi, \end{cases} \pi \subset \Omega,
\]
correspond to projections \(\pi \in \mathcal{M}^{\text{pr}}\). (The reader will note to his regret that we use the same letter \(\pi\) to designate three objects: a projection in \(\mathcal{M}^{\text{pr}}\), a
\[ \sigma(\pi) = \int \pi(\omega)h(\omega)\nu(d\omega) = \int h(\omega)\nu(d\omega). \]

In this approach, the \( W^* \)-algebra \( N \) is realized as von Neumann algebra of \( 2 \times 2 \)-matrices \((x_{ij})\), \( x_{ij} \in L^\infty(\Omega, \nu) \) acting on the orthogonal sum of two copies of Hilbert space \( L^2(\Omega, \nu) \):

\[ H = L^2(\Omega, \nu) + L^2(\Omega, \nu) = \left\{ \left( \begin{array}{c} f \\ g \end{array} \right) : f, g \in L^2(\Omega, \nu) \right\}. \]

Next, let \( m : N^{pr} \to \mathbb{R}_+ \) be a given measure on projections on \( W^* \)-algebra \( L^\infty(\Omega, \nu) \otimes M_2 \). Let \( 0 \leq h_0, k_0 \in L^2(\Omega, \nu) \) such that

\[ m(\pi_1 \oplus \pi_2) = \int_{\pi_1} h_0^2(\omega)\nu(d\omega) + \int_{\pi_2} k_0^2(\omega)\nu(d\omega), \quad \pi_1 \in \mathcal{M}^{pr}. \]

Similarly, there are \( 0 \leq h_\gamma, k_\gamma \in L^2(\Omega, \nu) \) such that

\[ m(p(x, v, \pi)) = \int_{\pi} h_\gamma^2 d\nu, \quad m(p((1 - x)\pi, -v, \pi)) = \int_{\pi} k_\gamma^2 d\nu, \quad \pi \in \mathcal{M}^{pr}. \quad (2) \]

In addition, Lemma 2 and the Radon-Nykodim theorem give

\[ h_\gamma^2(\omega) + k_\gamma^2(\omega) = h_0^2(\omega) + k_0^2(\omega) \quad \text{a. e.} \]

We will now state the main result of this paper.

**Theorem 5.** Let \( m : N^{pr} \to \mathbb{R}_+ \) be a measure on projections in \( W^* \)-algebra \( N \) of type \( I_2 \). Then there exist a Hilbert space \( H \) and an orthogonal vector measure \( \mu : N^{pr} \to H \) with property

\[ \| \mu(p) \|^2 = m(p), \quad p \in N^{pr}. \]

**Proof.** Define an orthogonal vector measure \( \mu \) on the set \( [0] \equiv N_0^{pr} \) via

\[ \mu(\pi_1 \oplus \pi_2) = \left( \begin{array}{c} \pi_1 h_0 \\ \pi_2 k_0 \end{array} \right), \quad \pi_1, \pi_2 \in \mathcal{M}^{pr}. \]

We next extend \( \mu \) to an orthogonal vector measure on the set \([0, 1]\) of all projections in the form

\[ p = \pi_1 \oplus \pi_2 + p(x, v, \pi_3) + p((1 - x), -v, \pi_4), \quad \pi_i \in \mathcal{M}^{pr}, \quad (3) \]
where \((x, v)\) is a pair in (1) corresponding to index 1 \(\equiv \min(\Gamma \setminus \{0\})\). According to Lemma 2, it is possible to assume that \(\pi_3 \pi_4 = 0\). Thus,

\[
\pi_1 \pi_3 = \pi_1 \pi_4 = \pi_2 \pi_3 = \pi_2 \pi_4 = \pi_3 \pi_4 = 0.
\]

One can easily see that the set \([0, 1]\) satisfies (iii) in Definition 2. In view of Lemma 3 there are real functions \(0 \leq h_1, k_1 \in L^2(\Omega, \nu), i = 1, 2\) such that equalities

\[
\begin{align*}
(4) & \quad h_{11}(\omega) + k_{11}(\omega) = h_0(\omega), \\
(5) & \quad h_{12}(\omega) + k_{12}(\omega) = k_0(\omega), \\
(6) & \quad h_{21}^2(\omega) + h_{12}^2(\omega) = h_1^2(\omega), \\
(7) & \quad k_{11}^2(\omega) + k_{12}^2(\omega) = k_1^2(\omega), \\
(8) & \quad h_{11}(\omega) k_{11}(\omega) + h_{12}(\omega) k_{12}(\omega) = 0.
\end{align*}
\]

hold a. e. (Here the functions \(h_1, k_1\) in (2) correspond to index \(\gamma = 1\).)

We now extend the function \(\mu\) to projections in the form (3) putting

\[
\mu(p) \equiv \frac{\pi_1 h_0 + \pi_3 h_{11} +\pi_4 k_{11}}{\pi_2 k_0 + \pi_3 h_{12} + \pi_4 k_{12}},
\]

(27)

where \(h_{11}, k_{11}\) are solutions of the system (4) — (7). Direct computations with application (4) – (8) show that \(\mu\) is the finitely additive orthogonal vector measure on \([0, 1]\).

Suppose (inductive hypothesis) that \(\mu\) is extended to a finitely additive orthogonal vector measure on \([0, \gamma]\),

\[
[0, \gamma) \equiv \{p_1 + \ldots + p_s : p_j \in N_{\gamma_j}^\mathrm{pr}, p_j p_k = 0 (j \neq k), \gamma_j < \gamma\},
\]

and \(\gamma = (y, w)\). Let \(0 \leq h, k \in L^2(\Omega, \nu)\) such that

\[
m(p_\gamma) = \int_{\rho_1} h^2_\gamma d\nu + \int_{\rho_2} k^2_\gamma d\nu,
\]

(9)

where

\[
p_\gamma = p(y \rho_1, w, \rho_1) + p((1 - y) \rho_2, -w, \rho_2), \quad \rho_1 \rho_2 \in \mathcal{M}^\mathrm{pr}.
\]

(10)

Let

\[
\mathcal{P} = \{\pi \in \mathcal{M}^\mathrm{pr} : \exists (x, v) < (y, w) (x \pi = y \pi, v \pi = w \pi)\},
\]

and \((\pi_j)_{j \in J} \subset \mathcal{P}\) be a maximal set of pairwise orthogonal projections in \(\mathcal{P}\) (it exists by Zorn’s theorem). Define \(\pi_0 \equiv \sum_j \pi_j (= \sup \mathcal{P})\).
With the above notations we have
\[ p(y, w, 1) = p(y(1 - \pi_0), w, 1 - \pi_0) + \sum_j p(y\pi_j, w, \pi_j) \]
\[ = p(y(1 - \pi_0), w, 1 - \pi_0) + \sum_j p(x_j\pi_j, v_j, \pi_j), \]
\[ p(1 - y, -w, 1) = p((1 - y)(1 - \pi_0), -w, 1 - \pi_0) + \sum_j p((1 - x_j)\pi_j, -v_j, \pi_j), \]
where \((x_j, v_j) < (y, w)\).

By inductive hypothesis there defined the functions \(h_{j1}, k_{j1}, h_{j2}, k_{j2} \in L^2(\Omega, \nu)\) satisfying equalities
\[ h_{j1}(\omega) + k_{j1}(\omega) = h_0(\omega), \]
\[ h_{j2}(\omega) + k_{j2}(\omega) = k_0(\omega), \]
\[ h^2_{j1}(\omega) + h^2_{j2}(\omega) = h^2_j(\omega), \]
\[ k^2_{j1}(\omega) + k^2_{j2}(\omega) = k^2_j(\omega), \]
\[ h_{j1}(\omega)k_{j1}(\omega) + h_{j2}(\omega)k_{j2}(\omega) = 0. \]
where the density functions \(h_j, k_j\) correspond to pairs \((x_j, v_j)\) according to (2).

We also find the functions \(\tilde{h}_{\gamma1}, \tilde{k}_{\gamma1}, \tilde{h}_{\gamma2}, \tilde{k}_{\gamma2} \in L^2(\Omega, \nu)\) that are solutions of equations
\[ \tilde{h}_{\gamma1}(\omega) + \tilde{k}_{\gamma1}(\omega) = h_0(\omega), \]
\[ \tilde{h}_{\gamma2}(\omega) + \tilde{k}_{\gamma2}(\omega) = k_0(\omega), \]
\[ \tilde{h}^2_{\gamma1}(\omega) + \tilde{h}^2_{\gamma2}(\omega) = h^2_\gamma(\omega), \]
\[ \tilde{k}^2_{\gamma1}(\omega) + \tilde{k}^2_{\gamma2}(\omega) = k^2_\gamma(\omega), \]
where \(h_\gamma, k_\gamma\) are defined by (9). Therefore, there are defined the functions
\[ h_{\gamma1}(\omega) \equiv (1 - \pi_0)\tilde{h}_{\gamma1}(\omega) + \sum_j \pi_j(\omega)h_{j1}(\omega), \]
\[ h_{\gamma2}(\omega) \equiv (1 - \pi_0)\tilde{h}_{\gamma2}(\omega) + \sum_j \pi_j(\omega)h_{j2}(\omega), \]
\[ k_{\gamma1}(\omega) \equiv (1 - \pi_0)^2(\omega)\tilde{k}_{\gamma1}(\omega) + \sum_j \pi_j(\omega)k_{j1}(\omega), \]
\[ k_{\gamma2}(\omega) \equiv (1 - \pi_0)\tilde{k}_{\gamma2}(\omega) + \sum_j \pi_j(\omega)k_{j2}(\omega). \]
In this case
\[ h_{\gamma_1}^2(\omega) + h_{\gamma_2}^2(\omega) = h_{\gamma_1}^2(\omega), \quad k_{\gamma_1}^2(\omega) + k_{\gamma_2}^2(\omega) = k_{\gamma_1}^2(\omega) \quad \text{a.e.}, \]
\[ h_{\gamma_1}(\omega)k_{\gamma_1}(\omega) + h_{\gamma_2}(\omega)k_{\gamma_2}(\omega) = 0 \quad \text{a.e.} \]

Now we put
\[ \mu(p + p_\gamma) \equiv \mu(p) + \left( \rho_1 h_{\gamma_1} + \rho_2 k_{\gamma_1} \right), \]
where \( p \in [0, \gamma) \) and \( p_\gamma \) is defined by (10). Again, direct computations show that \( \mu \) is the finitely additive orthogonal vector measure on \([0, \gamma]\).

In view of Lemma 1 it follows that \( \mu \) turned out extended to \( \mathcal{N}^{pr} \). Applying Lemma 3 we complete the proof.

**Corollary 6.** Let \( m : \mathcal{A}^{pr} \to \mathbb{R}_+ \) be a measure on projections in an arbitrary \( W^* \)-algebra \( \mathcal{A} \). Then there exist complex Hilbert space \( H \) and an orthogonal vector measure \( \mu : \mathcal{A}^{pr} \to H \) such that
\[ \| \mu(p) \|^2 = m(p), \quad p \in \mathcal{A}^{pr}. \]

**Proof.** Because an orthogonal vector measure is uniquely defined by its restrictions to direct summands of a \( W^* \)-algebra, the statement follows by Theorem 5 and the proof of Theorem in [1].

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