Probabilistic approach to calculating the measure of uncertainty in computational problems of management, economics and finance

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Abstract. An approach to calculating the measure of uncertainty in computational problems of management, Economics and Finance is proposed. The approach consists of direct and inverse problems. In the direct problem, the values of the uncertainty measure of the desired quantities are calculated from the known values of the uncertainty measure of known quantities. In the inverse problem, the values of the vagueness measure of the source data are searched for, at which the specified values of the vagueness measure of the desired values are provided. The inverse problem is Hadamard ill-posed problem, and an additional condition is involved for its solution in the form of the principle of equal influences. A definition of "soft" computing is given within the framework of the paradigms of econometrics and the theory of errors [1]. The approach proposed in this paper to calculating the uncertainty measure in a direct problem is compared with the approach of fuzzy set theory and soft computing by L. Zadeh [2]. Two examples of calculating the measure of vagueness of the desired values are discussed: in the dealer problem (training example) and in the problem of evaluating the attractiveness of a real investment project. It is concluded that when conducting financial, economic and managerial calculations in a situation of vagueness, the discussed probabilistic approach should be used, which delivers elegant and easy-to-interpret results. The paper concludes with the application of a probabilistic approach to estimating the measure of vagueness of endogenous variables in descriptive models.

1. Problems of calculating the measure of uncertainty in the management, financial and economic spheres

The structure of any calculation task in the management, financial and economic spheres consists of three elements. 1) known data of the problem, which are known; 2) the desired values, the values of which need to be determined in the course of calculations; 3) the relationship $F$ of the initial data and the desired values. We will denote the true (usually unknown) values of the source data with symbols

$$x = (x_1, x_2, \ldots, x_n).$$

We will denote the true values of the desired values with symbols

$$y = (y_1, y_2, \ldots, y_m).$$
Approximate source data are indicated below
\[ \hat{x} = (\hat{x}_1, \hat{x}_2, ..., \hat{x}_n). \]  
(1.3)
The relationship between the true values of the source data and the desired unknowns can be represented by the equation
\[ F(x, y) = 0, \]  
(1.4)
defining the desired quantities (1.2) as implicitly defined functions of the source data (1.1), where the map \( F \) satisfies, by assumption, the implicit function theorem [3] and, consequently, each desired quantity \( y_i \) can be represented as an explicitly defined differentiable function of variables (1.1):
\[ y_i = f_i(x_1, x_2, ..., x_n), \quad i = 1, 2, ..., m. \]  
(1.5)
The system of equations (1.5) can be interpreted as a calculation scheme of the financial-economic or managerial problem being solved.

Definition 1. Calculations of the desired values (1.2) are called "soft" if the impact on the desired unknown uncertainties of the source data (1.3) is taken into account. Consequently, the uncertainty measure is evaluated in the calculated approximate values
\[ \hat{y}_i = f_i(\hat{x}_1, \hat{x}_2, ..., \hat{x}_n), \quad i = 1, 2, ..., m. \]  
(1.5)
the desired values of the problem to solve.

Let's go back to the approximate (in other words, undefined) source data (1.3). It is always possible to represent each value \( \hat{x}_j \) in the form
\[ \hat{x}_j = x_j + \Delta_j, \]  
(1.6)
where \( \Delta_j \) is an unobservable error in the value \( \hat{x}_j \), the presence of which, in fact, means uncertainty \( \hat{x}_j \). Errors \( \Delta = (\Delta_1, ..., \Delta_n)^T \) in the source data (1.3) this causes the uncertainty of the desired values \( \hat{y} = (\hat{y}_1, ..., \hat{y}_m)^T \). \( \hat{y} \) The vector \( \hat{y} \) must always be represented as
\[ \hat{y} = y + \varepsilon, \]  
(1.6)
where \( \varepsilon = (\varepsilon_1, ..., \varepsilon_n)^T \) is the error vector of values (1.5).

We will interpret the errors \( \Delta_j \) as random uncorrelated values with zero mathematical expectations,\n\[ E(\Delta_j) = 0, \]  
and with average square deviations
\[ \sigma_1, ..., \sigma_n. \]  
(1.7)
In this interpretation of errors \( \Delta_j \) values \( \hat{x}_j \), it is the set of constants (1.7) that serve as a measure of the uncertainty of the initial data (1.3). Average square deviations of the required values (1.5) serve as their measure of uncertainty:
\[ (\sigma_{y_1}, \sigma_{y_2}, ..., \sigma_{y_m}). \]  
(1.8)

Direct problem of calculating the uncertainty measure in "soft" computing the desired quantities \( \hat{y}_i \) call the definition of their measures of uncertainty (1.8) for given values of the measure of uncertainty (1.7) the initial data (1.3). Accordingly, the inverse task of calculating measures of uncertainty in soft computing values \( \hat{y}_i \) call the definition of a measure of uncertainty (1.7) of the original data \( \hat{x}_i \) for given values of the measure of uncertainty (1.8) desired quantities \( \hat{y}_i \).

2. Probabilistic approach to solving the direct and inverse problem of calculating the measure of uncertainty in soft computing

Given the above assumptions about the error properties of \( \Delta_i \) quantities \( \hat{x}_i \), the solution of the direct problem of calculating the measure of uncertainty in "soft" calculations is found by the method of probability theory:
\[ \sigma_{y_i}^2 = \sum_{j=1}^{n} \left( \frac{\partial f_i}{\partial x_j} \right)^2 \cdot \sigma_j^2, \quad i = 1, 2, ..., m. \]  
(2.1)

Note 2.1. Rule (2.1) can easily be generalized to the situation where errors \( \Delta_j \) in the source data (1.3) are correlated and the correlation coefficients \( p_{jk} \) of these errors are known:
\[ \sigma_{y_i}^2 = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial f_i}{\partial x_j} \cdot \frac{\partial f_i}{\partial x_k} \cdot \sigma_j \cdot \sigma_k \cdot p_{jk}, \quad i = 1, 2, ..., m. \]  
(2.1)

For the practice of financial, economic and managerial calculations, the formula (2.1) is more useful.
In the inverse problem of calculating the measure of uncertainty, the left parts of the equations (2.1) are considered known, and the values (1.7) located in the right parts are the ones that are sought. In the situation \( m < n \), the inverse problem of calculating the uncertainty measure (1.7) has many solutions, being, as often happens, an incorrect problem [4]. Solutions to this problem will be obtained using the principle of equal influences using a two-step algorithm.

Step 1. Fix the number \( i \) of the equation from the system (2.1) and determine the required values \( \sigma_j \) according to the principle of equal influences:

\[
\left( \sigma_j \right)^2 = \frac{\sigma y_i^2}{n \left( \frac{\partial f_i}{\partial x_j} \right)}, \quad j = 1, 2, \ldots, n, \tag{2.2}
\]

assuming that \( \left( \frac{\partial f_i}{\partial x_j} \right)^2 > 0 \). It is obvious that the values \( \sigma_j \) found by rule (2.2), where \( j=1,2,\ldots,n \), depend on the number of the equation \( i \), where \( i = 1, 2, \ldots, m \), and we denote this fact \( \sigma_j(i) \).

Step 2. Define the desired values (1.7) according to the rule (the minimum is taken by \( i \)):

\[
\sigma_j = \min \{ \sigma_j(i) \} \tag{2.3}
\]

If the initial data (1.3) of the computational problem will have the levels of the uncertainty measure (1.7) found by this algorithm, then the solution (1.5) will knowingly provide the specified levels of the uncertainty measure (1.8). In the next paragraph, we compare the probabilistic approach we compare this approach with the soft computing approach of L. Zadeh [2].

3. An illustrative example of calculating the uncertainty measure using two approaches

The example of a financial calculation problem, the conditions of which are discussed below, is illustrative in nature, allowing you to clearly compare both campaigns. Here is the wording of this problem [5] (let’s call it the dealer’s problem). "The dealer bought 1,000 items at $150 apiece. The dealer plans to sell approximately 40% of the purchased product with an approximate margin of 25%. The question is, at what price should the dealer sell the product to ensure an approximate return of 40%?"

We denote: 1) \( N = 1000 \) - the number of units purchased at the price \( p_o = 150 \); 2) \( \tilde{q} \approx 40\% = 0.40 \) – the approximate share of purchased units sold at an approximate markup \( \tilde{\varepsilon} x \approx 25\% = 0.25 \); 3) \( \tilde{r} \approx 40\% = 0.40 \) – the approximate desired yield, 4) \( \tilde{p} \) - the desired sale price of the purchased product, providing the desired yield \( r \). The wave above the symbol indicates the approximate nature of the value.

Let’s write down the dealer’s problem in mathematical language. The dealer's income from the transaction is as follows:

\[
\tilde{Y} = N \cdot \tilde{q} \cdot (1 + \tilde{\varepsilon} x) \cdot p_o + (N - N \cdot \tilde{q}) \cdot \tilde{p}. \tag{3.1}
\]

The dealer's costs \( C \) are exactly known: \( C = N \cdot p_o \), so the yield \( \tilde{r} \) of the transaction is calculated as follows:

\[
\tilde{r} = \frac{\tilde{Y} - C}{\tilde{Y}} = \frac{N \cdot \tilde{q} (1 + \tilde{\varepsilon} x) \cdot p_o + (N - N \cdot \tilde{q}) \cdot \tilde{p}}{N \cdot p_o}. \tag{3.2}
\]

The desired price \( \tilde{p} \) is the root of this equation, which is rewritten as (1.4):

\[
N \cdot \tilde{q} \cdot p_o \cdot \tilde{\varepsilon} x + (N - N \cdot \tilde{q}) \cdot (\tilde{p} - p_o) = N \cdot p_o \cdot \tilde{r} = 0. \tag{3.2'}
\]

Solution of equation (3.2) by L. Zadeh's soft computing method in the form of an \( \alpha \)-slice \( \tilde{p} \alpha = [p_1(\alpha), p_2(\alpha)] \) a fuzzy number \( \tilde{p} \) has the form [5]

\[
p_1(\alpha) = 150 + 150 \cdot \frac{0.35 + 0.05 \cdot \alpha - (0.3 + 0.1 \cdot \alpha) \cdot (0.35 - 0.1 \cdot \alpha)}{0.7 - 0.1 \cdot \alpha},
\]

\[
p_2(\alpha) = 150 + 150 \cdot \frac{0.45 + 0.05 \cdot \alpha - (0.5 - 0.1 \cdot \alpha) \cdot (0.15 + 0.1 \cdot \alpha)}{0.7 - 0.1 \cdot \alpha}. \tag{3.3}
\]

We emphasize that the solution (3.3) is obtained under the assumption that the indeterminate values \( \tilde{q}, \tilde{\varepsilon} x, \tilde{r} \) are interpreted as fuzzy triangular numbers:

\[
\tilde{q} = < 0.3; 0.4; 0.5 >, \quad \tilde{\varepsilon} x = < 0.15; 0.25; 0.35 >, \quad \tilde{r} = < 0.35; 0.40; 0.45 >. \tag{3.4}
\]
Let’s interpret the solution (3.3). Let’s set any value $\alpha \in (0,1]$, say $\alpha = 0.6$. Then the truth estimate of the statement "the price $\bar{p}$ at which the dealer should sell the product belongs to the interval [214,6, 240.8]" satisfies the inequality $\mu_{\bar{p}}(x) \geq 0.6$.

Now we calculate the measure of uncertainty of the desired value $\bar{p}$ using the probabilistic approach, that is, we solve the direct problem (2.1) of calculating the measure of uncertainty. Let us turn to equation (3.1) From this equation, we find $p$ for $(q <1)$ as a function of the arguments $\bar{q}$, $\bar{e\bar{x}}$, $\bar{r}$

$$\bar{p} = \frac{p e^{(1+r-q(1+e\bar{x}))}}{1-q} = \varphi(\bar{q}, \bar{e\bar{x}}, \bar{r}).$$ (3.5)

For the values $\bar{q}$, $\bar{e\bar{x}}$, $\bar{r}$ we assume a probabilistic model similar to the model (3.4). We assume that the values $\bar{q}$, $\bar{e\bar{x}}$, $\bar{r}$ are independent random variables with a triangular distribution over the intervals of the carriers of the values $\bar{q}$, $\bar{e\bar{x}}$, $\bar{r}$ in their interpretation (3.4) as triangular fuzzy numbers. Within the framework of this probabilistic model, the desired indeterminate value calculated by rule (3.5) $\bar{p}$ is also interpreted as a random variable, and the measure of its uncertainty will now be the av.sq. deviation $\sigma$.

Recall that a continuous random variable $\Delta$ is distributed by a triangular law (Simpson’s law) on the segment $[a-l,a+l]$ if the probability density of this variable has the equation

$$f_{\Delta}(x) = \begin{cases} 0 & \text{if } x \notin [a-l,a+l] \\ \frac{x-a+l}{l^2} & \text{if } x \in [a-l,a] \\ \frac{l-x+a}{l^2} & \text{if } x \in [a,a+l]. \end{cases}$$ (3.6)

We add that the main quantitative characteristics of the random variable $\Delta$ distributed by law (3.6) are correspondingly equal:

$$E(\Delta) = \alpha, \quad Var(\Delta) = \sigma_{\Delta}^2 = l^2/6.$$ (3.7)

The probability of a random event is equal:

The fact that the random variable $\Delta$ is distributed according to (3.6), we will denote as follows: $\Delta \in \mathcal{S}_l[a-l,a+l]$.

Comparing the graph of the function (3.6) with the graph of the membership function of fuzzy triangular numbers (3.4), we state the identity of these graphs. So, for example, in the interpretation that $\bar{q} =< 0.3; 4.0; 0.5 > 0.3; 4.0; 0.5 >$ — this is a fuzzy triangular number with a carrier $[0.3, 0.5]$, the constant $a = 0.4$ is the mode of this fuzzy number. In the interpretation of the same, that $\bar{q} \in \mathcal{S}_l[0.3, 0.5]$, constant $a = 0.4$ it makes sense of the expected value of the random variable $\bar{q}$. In practice, the difference between these concepts is hardly noticeable. In the situation of the model $\bar{q} \in \mathcal{S}_l[0.3, 0.5]$, the measure of uncertainty of the value $\bar{q}$ is $\sigma_{\bar{q}} = \frac{l}{\sqrt{6}} = \frac{0.1}{\sqrt{6}} = 0.041$.

So, within the framework of the probabilistic model accepted for the arguments $\bar{q}$, $\bar{e\bar{x}}$, $\bar{r}$ of the function (3.5)

$$\bar{q} \in \mathcal{S}_l[0.3, 0.5], \quad \bar{e\bar{x}} \in \mathcal{S}_l[0.15, 0.35], \quad \bar{r} \in \mathcal{S}_l[0.35, 0.45],$$ (3.9)

the main quantitative characteristics of $\bar{q}$, $\bar{e\bar{x}}, \bar{r}$:

$$E(\bar{q}) = q = 0.4, \quad \sigma_{\bar{q}} = 0.041; E(\bar{e\bar{x}}) = ex = 0.25, \quad \sigma_{\bar{e\bar{x}}} = 0.041; E(\bar{r}) = r = 0.4, \quad \sigma_{\bar{r}} = 0.02.$$ (3.10)

These characteristics are sufficient to determine the main quantitative characteristics of the desired value $\bar{p}$. Indeed, in the neighborhood of the point $(q, ex, r) = (0.4, 0.25, 0.4)$, we decompose the function (3.5) into a Taylor series, bounded by linear terms:

$$\bar{p} \approx \varphi(q, ex, r) + \frac{\partial \varphi}{\partial q} \cdot (\bar{q} - q) + \frac{\partial \varphi}{\partial ex} \cdot (\bar{e\bar{x}} - ex) + \frac{\partial \varphi}{\partial r} \cdot (\bar{r} - r).$$ (3.11)

We compute the constants $\varphi(q, ex, r)$, $\frac{\partial \varphi}{\partial q}$, $\frac{\partial \varphi}{\partial ex}$, $\frac{\partial \varphi}{\partial r}$: $\varphi(q, ex, r) = 225; \quad \frac{\partial \varphi}{\partial q} = 62.5; \quad \frac{\partial \varphi}{\partial ex} = -100; \quad \frac{\partial \varphi}{\partial r} = 250.$ (3.12)

Taking into account (3.10), (3.11) and (2.1) find the main quantitative characteristics of the value $\bar{p}$:

$$E(\bar{p}) \approx \varphi(q, ex, r) = 225; \quad \sigma_{\bar{p}}^2 = (\frac{\partial \varphi}{\partial q})^2 \cdot \sigma_{\bar{q}}^2 + (\frac{\partial \varphi}{\partial ex})^2 \cdot \sigma_{\bar{e\bar{x}}}^2 + (\frac{\partial \varphi}{\partial r})^2 \cdot \sigma_{\bar{r}}^2 = 48.4.$$ (3.13)
Hence, the expected value $E(\widehat{p})$ and the uncertainty measure (mean square deviation) of the random variable $\widehat{p}$ are $225$ and $\sigma_{\widehat{p}}=7$, respectively. According to (3.11), the law of distribution of the random variable $p$ is close to Gauss’ law and the probability of a random event
\[
\{\widehat{p} \in [225 - \sigma_{\widehat{p}}, 225 + \sigma_{\widehat{p}}]\} = \{218 \leq \widehat{p} \leq 232\} \quad (3.14)
\]
approximately equal to 0.68. Taking into account this fact we will write down the solution of the dealer’s problem using the probabilistic approach concisely
\[
\widehat{p} = 225 \pm 7 \quad (3.15)
\]
and compare with the solution (3.3) by L. Zadeh’s soft computing method. It is clear that these decisions do not contradict each other, although they are not identical. It can also be seen that the solution (3.15) more elegant and better interpreted. This conclusion is supported by the assumption of the probability approach to the inverse problem of calculating the measure of uncertainty in the dealer’s problem.

Consider equation (3.13). Now we will treat its right part as a known quantity. Required are now considered to be the levels of a measure of uncertainty
\[
(s_{\widehat{q}}, s_{\widehat{e}}, s_{\widehat{r}}) \quad (3.16)
\]
source data for the dealer’s task $\widehat{q}, \widehat{e}, \widehat{r}$. Since the dealer’s problem has only one desired value $\widehat{p}$ (i.e. $m = 1$), then the algorithm for determining the values discussed above (3.13) is implemented in one step. According to (2.2) taking into account (3.12) and $n = 3$, we find
\[
\sigma_{\widehat{q}}^2 = \frac{\sigma_{\widehat{p}}^2}{\frac{100}{3(\sigma_{\widehat{p}}^2)}^2} = \frac{484}{3(62,5)^2} = 0,00413; \quad \sigma_{\widehat{e}}^2 = \frac{\sigma_{\widehat{p}}^2}{\frac{100}{3(\sigma_{\widehat{p}}^2)}^2} = \frac{-484}{3(-100)^2} = 0,00166;
\]
\[
\sigma_{\widehat{r}}^2 = \frac{\sigma_{\widehat{p}}^2}{\frac{100}{3(\sigma_{\widehat{p}}^2)}^2} = \frac{484}{3(250)^2} = 0,000258. \quad (3.17)
\]
From here we get the desired levels (3.16) of the measure of uncertainty of the source data in the dealer’s problem:
\[
s_{\widehat{q}} = 0,064, \quad s_{\widehat{e}} = 0,040; \quad s_{\widehat{r}} = 0,016. \quad (3.19)
\]
Comment. Calculated according to the principle of equal influences (2.2) in the process of solving ill-defined inverse problem (3.13) levels a measure of uncertainty (3.19) initial data $\widehat{q}, \widehat{e}, \widehat{r}$ in the problem to the dealer, of course, do not coincide with standard deviations (3.10) of the initial data $\widehat{q}, \widehat{e}, \widehat{r}$ values $\widehat{q}, \widehat{e}, \widehat{r}$. The reason is not the uniqueness of the solution of equation (3.13). We emphasize that the solution (3.19) is consistent with the quantities (3.10), which indicates the acceptability of the principle of equal influences in the process of solving the inverse problem by the proposed approach calculating the measure of uncertainty by a probabilistic approach in computational problems in management, Economics and Finance. The following is an example of practical importance, reinforces this conclusion.

4. Calculation of the uncertainty measure in the problem of evaluating the attractiveness of an investment project using a probabilistic approach

Below is an example of calculating the uncertainty measure of the modified rate of return of an investment project. The modified internal rate of return (MIRR) [6, p. 92] is defined by the equation
\[
\sum_{t=0}^{T} COF_t \frac{COF_t}{(1+r)^t} = \frac{\sum_{t=0}^{T} CIF_t(1+r)^{T-t}}{(1+MIRR)^T}, \quad (4.1)
\]
The symbol $COF_t$ indicates investment levels, the symbol $CIF_t$ is the return on the asset for the period $t$. Further, the left side of equation (4.1) is the discounted PVO value of all investments in the asset. On the right side of equation (4.1) is the terminal cost of TVI, discounted at the MIRR rate. For TVI>PVO, the modified rate of return $MIRR > 0$ has a single value. Let us add that equation (4.1) is a concrete example of equation (1.4).

So that the investment project turns out to be economically feasible if the corresponding discount factor $r$ satisfies the inequality
\[
r \leq MIRR \quad (4.2)
\]
Below we will construct \( MIRR \) an uncertainty interval for the MIRR value

\[
MIRR^- \leq MIRR \leq MIRR^+
\]  
(4.3)

and, in accordance with this interval, our principle of selecting investment projects in a set of acceptable projects will be based on inequality

\[
r \leq MIRR^-.
\]  
(4.4)

Note that equation (4.1) you can rewrite it as follows

\[
\ln(1 + MIRR) = \frac{1}{r} \cdot (\ln(TVI) - \ln(PVO)).
\]  
(4.5)

In turn, the terminal cost of the TVI project is more convenient to present as follows:

\[
TVI = (1 + r)^T \cdot \sum_{t=0}^{T} \frac{CIF_t}{(1+r)^t}.
\]  
(4.6)

It is known [7, p. 208] what are the project-related components \((COF_t, CIF_t)\) cash flow is estimated in budgeting. The future is uncertain, so the components of the cash flow \((COF_t, CIF_t)\) are always approximate. Below under symbols \((COF_t, CIF_t)\) are kept for the exact values of the cash flow element for the exact values of the cash flow element. In turn, the symbols \((\overline{COF_t}, \overline{CIF_t})\) denote a known approximate value of the values \((COF_t, CIF_t)\). Consequently, when analyzing an investment project, the approximate values \(\overline{NPV}(r) = MIRR\). The direct task of calculating the uncertainty measure is to evaluate the accuracy of the \(MIRR\) value, i.e., to calculate its av. sq. deviation \(\sigma_{MIRR}\). The following statement is true.

Approval 4.1. Let’s assume that estimates are defined in the process of budgeting investments elements \((\overline{COF_t}, \overline{CIF_t})\) the cash flow be determined with relative accuracy \(\delta\) (for example, \(\delta = 0.05\) or \(0.1\) ) and the true errors \(\Delta t = (\Delta COF_t, \Delta CIF_t)\) of these elements are uncorrelated in pairs. Then av. square deviation \(\sigma_{MIRR}\) of the value \(\overline{MIRR}\), calculated according to (2.1) has the form

\[
\sigma_{MIRR} = \delta \cdot R,
\]  
(4.7)

where

\[
R = \left(\frac{1 + MIRR}{T}\right) \cdot \sqrt{\left(\frac{(1+r)^T}{TVI}\right)^2 \cdot \sum_{t=0}^{T} \left(\frac{\overline{CIF_t}}{(1+r)^t}\right)^2 + \frac{1}{PV(O)} \cdot \sum_{t=0}^{T} \left(\frac{\overline{COF_t}}{(1+r)^t}\right)^2}.
\]  
(4.8)

The investigation. The boundaries of the simplest uncertainty interval of the modified internal rate of return can be defined by the rule

\[
MIRR^- = \overline{MIRR} - \delta \cdot R \quad \text{and} \quad MIRR^+ = \overline{MIRR} + \delta \cdot R.
\]  
(4.9)

Here is a proof of equality (4.7), which is based on rule (2.1) for solving the direct problem. Let’s return to equation (4.5) and, reasoning in differentials, we obtain on the basis of this equation the relationship of true errors of the quantities included in this equation:

\[
\frac{\Delta MIRR}{(1+MIRR)} = \frac{1}{T} \cdot \left(\frac{\Delta TVI}{TVI} - \frac{\Delta PV(O)}{PV(O)}\right).
\]  
(4.10)

Here symbols \(\Delta MIRR, \Delta TVI, \Delta PV(O)\) marked the true errors in the corresponding values generated by the true error \(\Delta t = (\Delta COF_t, \Delta CIF_t)\) elements of the cash flow. Now we define it based on the equation

\[
PV(O) = \sum_{t=0}^{T} \frac{\overline{COF_t}}{(1+r)^t}
\]  
(4.11)

rule for calculating the value \(\Delta PV(O)\):

\[
\Delta PV(O) = \sum_{t=0}^{T} \frac{\Delta COF_t}{(1+r)^t}.
\]  
(4.12)

Reasoning similarly with respect to equation (4.6), we find the formula for calculating the value \(\Delta TVI\):

\[
\Delta TVI = (1 + r)^T \cdot \sum_{t=0}^{T} \frac{\Delta CIF_t}{(1+r)^t}.
\]  
(4.13)

We assume that the true errors \(\Delta t\) of elements \((\overline{COF_t}, \overline{CIF_t})\) of financial flows for the project are centered and pairwise uncorrelated random variables, i.e.
\[ E(\Delta_t) = 0, \quad \rho(\Delta_t, \Delta_t) = 0 \]  
for \( t \neq \tau \), and, in addition, all elements \((\text{COF}_t, \text{CIF}_t)\) have the same relative accuracy \(\delta\), i.e.  
\[ \Delta_c = \delta \cdot (\text{COF}_t, \text{CIF}_t). \]  
(4.15)

Note 4.1. For example, if \(\delta=0.1\), it means that the accuracy of budgeting is 10%.

Taking into account (2.1) and equations (4.10), (4.12), and (4.13) we get the rule (4.7) – (4.8) calculation of the average square deviation of the \(\text{MI}RR\) value. Below, this rule applies to a specific investment project.

Calculation of the uncertainty measure of the \(\text{MI}RR\) value according to rule (4.7) – (4.8) for the regional investment project "Development of the residential area "Chistye Prudy" in Kirov" [8], see table 1 [9]. The project is characterized by the following indicators: \(NPV = 25.14\) million rubles, \(\text{IRR} = 0.1929\), \(\text{WACC} = 0.1298\).

Calculated from the values \(\{\text{COF}_t, \text{CIF}_t\}\) from table 1, the values \(\text{MI}RR\), \(R\) and \(\sigma_{\text{MI}RR}\) were found to be at \(r = \text{WACC} = 0.1298\):

\[ \text{MI}RR = 0.1307, \quad R = 0.1588, \quad \sigma_{\text{MI}RR} = 0.016. \]  
(4.16)

| \(t\) | 2009   | 2010   | 2011   | 2012   | 2013   |
|------|--------|--------|--------|--------|--------|
| \(\text{COF}_t\) | 1812,29 | 2047,52 | 2313,29 | 2613,55 | —      |
| \(\text{CIF}_t\) | 1677,71 | 2071,63 | 2283,16 | 2696,41 | 175,62 |
| \(\text{CF}_t\)  | -134,58 | 24,11  | -30,13  | 82,86  | 175,62 |

\[ \text{MI}RR = 0.1307 \]

\[ \text{MI}RR^- = \text{MI}RR - 0.1 \cdot R = 0.1307 - 0.1 \cdot 0.1588 = 0.1148 \]

\[ \text{MI}RR^+ = \text{MI}RR + 0.1 \cdot R = 0.1307 + 0.1 \cdot 0.1588 = 0.1466 \]

Table 1. Financial flows for an investment project and calculation of the uncertainty measure of the \(\text{MI}RR\) value at \(\delta=0.1\).

Considering the results obtained, we can conclude that with the average weighted cost of capital \(r=\text{WACC} = 0.1298\) and budgeting accuracy \(\delta = 0.1\), the project may not be economically feasible. Indeed, we will solve the inverse problem and determine the accuracy of the budgeting system, which ensures the economic attractiveness of the project.

Here the inverse problem has a unique solution. Indeed, in order for the project to be commercially attractive, it is sufficient that the equity of the inequality (4.4). Taking into account (4.9) this inequality takes the form

\[ r \leq \frac{\text{MI}RR}{\Delta} \cdot R. \]  
(4.17)

This results in an inequality for the desired budgeting accuracy \(\delta\) of a given project:

\[ \delta \leq \frac{\text{MI}RR - r}{R} = \frac{0.1307 - 0.1298}{0.1588} = 0.006 = 0.6\%. \]  
(4.18)

Consequently, the investment project "development of the residential area" Chistye Prudy "in Kirov" will be commercially attractive, if the accuracy of budgeting cash flows for this project is not worse than 0.6%.

The discussed examples illustrate, in our opinion, the flexibility and good interpretability of the results of "soft" calculations using a probabilistic approach in problems of management, Economics and Finance. Moreover, the model on which the proposed approach is based is more natural on which L. Zadeh's "soft" calculations are based [2]. In the next paragraph, we will discuss the natural application of the probabilistic approach to estimating the uncertainty measure of endogenous variables calculated using descriptive economic and mathematical models.
5. Probabilistic approach to the estimation of uncertainty measures of the values of endogenous variables in descriptive models

Let us return to equation (1.4). We can look at it as a structural form of a descriptive model with a priori known parameters (coefficients) \( p = (p_1, \ldots, p_n)^T \). We will include these parameters in the equation entry (this will be required later):

\[
F(x, y; p) = 0. \tag{5.1}
\]

Here \( x = (x_1, \ldots, x_n)^T \) - exogenous variables (explaining the characteristics of the object), \( y = (y_1, \ldots, y_m)^T \) - endogenous variables (explaining the characteristics of the object). The two examples of computational problems discussed above fit into this scheme. Thus, in the problem from point 4, the discount factor \( r = \text{WACC} = 0.1298 \) is a priori a known parameter.

In this (basic version) model, it is assumed that the model (5.1) is absolutely adequate to reality, its parameters \( p \) are infallibly defined, and the only source of uncertainty in the calculated values \( \tilde{y} = y + \varepsilon \) of endogenous variables is the uncertainty in the known values \( \tilde{x} = x + \Delta \) of exogenous variables. We will expand this scheme later.

According to the probabilistic approach, we can interpret the error vector \( \Delta \) of known values of exogenous variables as a centered random vector with a known covariance matrix \( \text{Cov}(\Delta) \). This matrix also serves as a measure of the uncertainty of the values \( \tilde{x} \). In turn, the covariance matrix \( \text{Cov}(\varepsilon) \) of the error vector \( \varepsilon \) of the values of endogenous variables \( \tilde{y} \) will serve as a measure of their uncertainty.

We find the relationship of the matrices \( \text{Cov}(\Delta) \) and \( \text{Cov}(\varepsilon) \), assuming that the mapping \( F \) in the model (5.1) satisfies the implicit function theorem [3]. Based on this theorem, we linearize the mapping \( F \) in the vicinity of the point \( (x, y) \in \mathbb{R}^{n+m} \):

\[
F(\tilde{x}, \tilde{y}; p) \approx F(x, y; p) + F'_x \cdot \Delta + F'_y \cdot \varepsilon. \tag{5.2}
\]

From here we find the vector of random errors \( \varepsilon \) calculated in the given form (1.5) models (5.1) values \( \tilde{y} \) endogenous variables

\[
\varepsilon = -(F'_y)^{-1} \cdot F'_x \cdot \Delta = A \cdot \Delta \tag{5.3}
\]

and measures at the uncertainty of the vector \( \tilde{y} \):

\[
\text{Cov}(\varepsilon) = A \cdot \text{Cov}(\Delta) \cdot A^T. \tag{5.4}
\]

Limitations of the basic version (5.1) the descriptive model can be easily removed, and then in the most General case, the specification of the descriptive model looks like this:

\[
F(x, y, p) = u. \tag{5.1}
\]

Here \( p \) – parameters of the model, whose values are estimated by econometrics methods based on statistical information [10], so that their estimation is available instead of exact parameters

\[
\tilde{p} = p + \xi. \tag{5.5}
\]

Further, \( u \) is an unobservable centered vector of random perturbations that reflects the influence of unrecorded factors on endogenous variables of the model. We assume that the covariance matrix \( \text{Cov}(u) \) of this vector is known.

Note that the predicted values \( \tilde{y} \) endogenous variables are always calculated under the assumption that \( u = 0 \), since the vector \( u \) is unobservable. For this reason, expression (5.2) will now look like this:

\[
0 = F(\tilde{x}, \tilde{y}, \tilde{p}) \approx F(x, y, p) + F'_x \cdot \Delta + F'_y \cdot \varepsilon + F'_p \cdot \xi, \tag{5.6}
\]

a formula (5.3) taking into account (5.3)’ takes the form of

\[
\varepsilon = -(F'_y)^{-1} \cdot (u + F'_x \cdot \Delta + F'_p \cdot \xi). \tag{5.7}
\]

In turn, the expression (5.4) is transformed into the following rule for calculating the full measure of uncertainty calculated by the model

\[
F(\tilde{x}, \tilde{y}, \tilde{p}) = 0 \tag{5.8}
\]

forecast values of \( \tilde{y} \) endogenous variables:

\[
\text{Cov}(\varepsilon) = F'_y^{-1} \cdot (\text{Cov}(u) + F'_x \cdot \text{Cov}(\Delta) \cdot F'_x^T + F'_p \cdot \text{Cov}(\xi) \cdot F'_p^T) \cdot F'_y^{-1}. \tag{5.9}
\]
Diagonal elements $\sigma_{i}^{2}$ of the matrix $\text{Cov}(\varepsilon)$ allow constructing confidence intervals for predicted values of $y_i$ endogenous variables:

$$\left(\hat{y}_i - z_{1-\frac{\alpha}{2}} \cdot \sigma_{\varepsilon_i}, \hat{y}_i + z_{1-\frac{\alpha}{2}} \cdot \sigma_{\varepsilon_i}\right).$$

(5.10)

Here $z_{1-\frac{\alpha}{2}}$ is the quantile of level $1 - \frac{\alpha}{2}$ of the standard normal distribution.

Comment. In modern statistical applications (for example, in a statistical application R), in the process of constructing confidence intervals (5.10) in the values $\sigma_{\varepsilon_i}^{2}$ the uncertainty measures $\text{Cov}(u)$ and $\text{Cov}(\xi)$ are taken into account, but $\text{Cov}(\Delta)$ is not taken into account. This circumstance leads to the fact that the real confidence probability is lower than the declared confidence probability of $1-\alpha$.

6. Conclusion

1. A probabilistic approach to estimating the measure of uncertainty in management, Economics, and Finance tasks provides convenient and easily interpreted results.
2. The solution of the inverse problem provides a simple rule for calculating the uncertainty measure initial data, at which a given level of uncertainty of the desired values is provided.
3. The model on which the proposed approach is based is more natural on which L. Zadeh's "soft" calculations are based.

References

[1] Taylor G 1985 Introduction to the theory of errors (Moscow: Mir) 272 (in Russian)
[2] Zadeh L A 1994 Fuzzy logic, neural networks, and soft computing Communications of the ASM, 37(3) 77 - 84
[3] Kolmogorov A N and Fomin S V 1976 Elements of the theory of functions and functional analysis (Moscow: "Nauka") 492 (in Russian)
[4] Tikhonov A N and Arsenin V Ya 1979 Methods of solving ill-posed problems (Moscow: "Nauka") 284 (in Russian)
[5] Volkova E S and Gisin V B 2014 Fuzzy sets and soft computing in economics and finance (Moscow: Finuniversitet publ.) 59 (in Russian)
[6] Kovalev V V 2003 Methods of evaluating investment projects (Moscow: Finance and statistics) 144
[7] Brigham Yu and Gapenski L (ed.) 1997 Financial management vol. 1 (St. Petersburg: Ekonomicheskaya SHKOLA) 669 (in Russian)
[8] Presentations of regional investment projects: http://www.minregion.ru/invest_phound/presents_reg/ (in Russian)
[9] Byvshev V A and Mikhailova M Yu 2015 Materials of the XVIII International conference on soft computing and measurement, Moscow (in Russian)
[10] Byvshev V A 2008 Econometrica (Moscow: Finance and statistics) 480 (in Russian)