SOME NON-CONTRACTING AUTOMATA GROUPS

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ABSTRACT. We add to the classification of groups generated by 3-state automata over a 2 letter alphabet given by Bondarenko et al. [1], by showing that a number of the groups in the classification are non-contracting. We show that the criterion we use to prove a self-similar action is non-contracting also implies that the associated self-similarity graph introduced by Nekrashevych is non-hyperbolic.

1. Introduction

In [1] a list of automata groups generated by 3-state automata over a 2-letter alphabet is given and a great deal of information is listed for each. Amongst the data given for each group was whether the group was contracting or non-contracting. For ten automata the classification did not determine whether or not the group was contracting. In the numbering system of [1, page 14] the ten automata are:

749, 861, 882, 887, 920, 969, 2361, 2365, 2402, 2427.

Later Muntyan [3] showed that three of these are isomorphic to other groups in the classification, specifically 920 ∼ 2401, 2361 ∼ 939, and 2365 ∼ 939, where the groups 939 and 2401 are listed as non-contracting in [1].

The purpose of this note is to show that all of the automata groups listed above are non-contracting. We first establish a criterion for a group to be non-contracting, and then apply it in each case.

We refer to [4] for the basic definitions of self-similar actions and automata groups. Automata groups are examples of self-similar actions. The automata to be considered all have three states, labelled a b and c, and a two letter alphabet $X = \{0, 1\}$. The automata are represented by a Moore diagram, which is given below for each automaton. Each state defines an element in Aut$(X^\omega)$ and the group defined by the automaton is $G = \langle a, b, c \rangle \subset$ Aut$(X^\omega)$. For $g \in G$ and a finite word $v \in X^*$, the restriction of $g$ to $v$ is denoted $g|_v$. The basic properties of the action that we will make use of are:

\[(gh)|_v = g|_{h(v)}h|_v \quad (uv)g = g(u)g|_v \quad g|_{uv} = (g|_u)|_v\]

Recall the following definition from [4],

Definition 1. A self-similar action $G \leq$ Aut$(X^\omega)$ is called contracting if there exists a finite subset $\mathcal{N} \subseteq G$ such that for all $g \in G$ there exists $k \in \mathbb{N}$ such that...
\( g|_v \in \mathcal{N} \) for all \( v \in X^* \) with \( \ell(v) \geq k \). The minimum such \( \mathcal{N} \) is called the nucleus of the action.

We make use of the following criterion, which was used in [1]. For example, it is used to show that 744 is non-contracting, and many times after that.

**Lemma 2.** Let \( G \leq \text{Aut}(X^\omega) \) be a self-similar action. Suppose that there exist \( g \in G \) and \( v \in X^* \) such that

1. \( g|_v = g \)
2. \( g(v) = v \)
3. \( g \) has infinite order

Then \( G \) is non-contracting.

**Proof.** Assume for induction that \( g|_{v^k} = g \) and \( g(v^k) = v^k \) for \( k \geq 1 \). Then

\[
g|_{v^{k+1}} = g|_{v^k}g = g|_v = g \quad \text{and} \quad g(v^{k+1}) = g(v)g(v^k) = vg(v^k) = vv^k.
\]

Next assume for induction that \( g^n|_{v^k} = g^n \) for \( n \geq 1 \) and fixed \( k \). Then \( g^{n+1}|_{v^k} = g|_{g^{n}(v^k)}g^n|_{v^k} = g|_{v^k}g^n = gg^n \).

It follows that a nucleus must contain \( g^n \) for infinitely many \( n \) and so, since \( g \) has infinite order, the action is not contracting.

Alternatively, though less directly, the lemma follows from Proposition 8 below and Theorem 3.8.6 of [4]. □

In the next section we apply this criterion to the ten automata listed above. In Section 3 we prove that a self-similar group satisfying this criterion has a non-hyperbolic self-similarity graph.

2. The automata

For each of the ten automata not listed as contracting or non-contracting in [1] we give an element \( g \in G \) and a word \( v \in \{0,1\}^* \) with the (easily verifiable) property that \( g(v) = v \) and \( g|_v = g \). The Moore diagram of the automaton is given for reference. Active states are shaded in the diagram. Then an argument is given to prove that \( g \) has infinite order. The criterion of the lemma above then applies. The approach to showing that \( g \) has infinite order is to find another string \( v' \) that is not fixed by any power of \( g \). We found the candidates for suitable elements and strings using some simple computer code and observing various patterns.

It is convenient to introduce the equivalence relation on \( \{0,1\}^\omega \) given by left shift equivalence, that is, \( u \sim v \) if there are finite prefixes \( u' \) and \( v' \) of \( u \) and \( v \) respectively, and \( w \in \{0,1\}^\omega \) such that \( u = u'w \) and \( v = v'w \).

For a finite word \( u \in \{0,1\}^* \), we denote by \( u^\omega \) the element of \( \{0,1\}^\omega \) formed by repeating \( u \) infinitely many times.

2.1. Automaton 749.

\[
\text{a}^2bc(0100) = 0100
\]

\[
(a^2bc)|_{0100} = a^2bc
\]
To see that $g = a^2bc$ has infinite order we consider the string $0^\infty$. Observe that since $g|_{000} = babc$, $babc(000) = 101$, and $(babc)|_{000} = babc$, we have $g(0^\infty) = 011(101)^\infty$. Then note that $a|_{101} = b|_{101} = c|_{101} = a$ and $a^4(101) = 101$. It follows that for any $n \geq 1$, $g^n(0^\infty) = u_n(101)^\infty$ where $u_1 = 001$ and $u_n = g(u_{n-1}101)$. In other words $g^n(0^\infty)$ is left-shift equivalent to $(101)^\infty$. We now note that $g^{-1}(0^\infty)$ is not of this form, which establishes that $g$ has infinite order. Observe that

$$g^{-1}|_{0000} = a^{-1}b^{-1}a^{-2} \quad g^{-1}(0000) = 0011$$

Therefore $g^{-1}(0^\infty) = 0011(1011)^\infty$.

### 2.2. Automaton 861.

$$c(010) = 010$$
$$c|_{010} = c$$

Since $x|_{11} = b$ for any $x \in \{a, b, c\}$ and $b(1^\infty) = 1^\infty$, it follows that $c^n(1^\infty) \sim 1^\infty$ for any $n \geq 0$. But $c^{-1}(1^\infty) = (10)^\infty$, so $c$ has infinite order.

### 2.3. Automaton 882.

$$acacbc(11) = 11$$
$$(acacbc)|_{11} = acacbc$$

To show that $g = acacbc$ has infinite order we use the following lemma to conclude that $g^{2n}(0^\infty) = 0^{2n+1}110^\infty$ for all $n \leq 1$.

**Lemma 3.**

1. $g^{2n}(0^{2n+1}) = 0^{2n+1}$
2. $g^{2n}|_{a^{2n+1}} = cacb$
3. $cacb(0^\infty) = 110^\infty$

**Proof.** For the third part

$$cacb(0^\infty) = cacb(00)(cacb)|_{00} (0^\infty) = 11cbbh(0^\infty)$$
$$= 11cb(0^\infty) = 11cb(0)(cb)|_{0} (0^\infty) = 110b^2(0^\infty) = 110^\infty$$

We prove the first and second by induction on $n$. Note first that $b^2$ is the identity in the group, as can be seen from the automaton for $b^2$. 

$$a^2$$
$$|_{[00]}$$
$$b^2$$
$$|_{[00]}$$
$$c^2$$
$$|_{[00]}$$
$$a^2$$

$$0|1$$
$$1|0$$
$$0|0$$

$$0|1$$
$$1|0$$
$$0|0$$

$$1|1$$
$$0|0$$

$$0|1$$

$$1|1$$

$$1|1$$

$$1|1$$

$$0|0$$

$$0|0$$

$$0|0$$

$$0|0$$

$$0|0$$
We have \( g(0) = 0 \) and \( g(0) = cacbb = cacb \). Then inductively,
\[
g^{2n+1}(0^{2n+3}) = g^n g^{2n}(0^{2n+1}00) \\
= g^n(0^{2n+1}cacb(00)) = g^n (0^{2n+1}11) \\
= 0^{2n+1}cacb(11) = 0^{2n+1}00
\]
\[
g^{2n+1} \mid_{0^{2n+3}} = (g^{2n} g^{2n}) \mid_{0^{2n+3}} = g^{2n} \mid_{0^{2n+3}} g^{2n} \mid_{0^{2n+3}} \\
= g^{2n} \mid_{0^{2n+1}11} g^{2n} \mid_{0^{2n+3}} = (g^{2n} \mid_{0^{2n+1}}) \mid_{11} (g^{2n} \mid_{0^{2n+1}}) \mid_{00} \\
= (cacb) \mid_{11} (cacb) \mid_{00} = bb cacbb = cacb
\]

\( \square \)

2.4. Automaton 887.

\[
bc(00) = 00 \\
(bc) \mid_{00} = bc
\]

To establish that \( bc \) has infinite order we prove the following.

**Lemma 4.** For all \( n \geq 1 \), \( (bc)^n(1^\infty) \neq 1^\infty \).

**Proof.** Since \( bc(1) = 1 \) and \( (bc) \mid_1 = ca \), we have \( (bc)^n(1^\infty) = 1(ca)^n(1^\infty) \) and it suffices to show that \( (ca)^n(1^\infty) \neq 1^\infty \). We show that for \( n \geq 2 \)

1. \( (ca)^4(111) = 111 \) and \( (ca)^4(110) = 110 \)
2. \( (ca)^{2n} \mid_{111} = (ca)^{2n-1} \)
3. \( (ca)^{2n} (1^\infty) = (111)^{n-1}(1010)1^\infty \)

It’s clear that the third claim implies that \( (ca)^n(1^\infty) \neq 1^\infty \) for all \( n \geq 1 \).

The first claim follows from \( (ca)^4(111) = 111 \) and \( (ca)^4(110) = 110 \).

For the second claim, note first that \( a, b \) and \( c \) all have order 2, as can be seen from the automaton for \( \langle a^2, b^2, c^2 \rangle \).

\[
\begin{align*}
\text{Then } (ca) \mid_{111} &= aa = 1 \text{ and } (ca) \mid_{110} = bb = 1. \\
&\text{Also } \\
(ca)^2 \mid_{111} &= (ca) \mid_{ca(111)} (ca) \mid_{111} = (ca) \mid_{011} (ca) \mid_{111} = ca \\
(ca)^4 \mid_{111} &= (ca)^2 \mid_{(ca)^2(111)} (ca)^2 \mid_{111} = (ca)^2 \mid_{101} ca = caaca = caca
\end{align*}
\]
Inductively, for $n \geq 3$,
\[
(ca)^{2n} |_{111} = (ca)^{2n-1} (ca) |_{111} = (ca)^{2n-2} (ca)^{2n-2} = (ca)^{2n-1}
\]
For the third claim, note that $ca(1^{\infty}) = 0(bb)(1^{\infty}) = 01^{\infty}$ and
\[
(ca)^4(1^{\infty}) = (ca)^4(111)(ca)^4 |_{111} (1^{\infty}) = 111(ca)^2(1^{\infty})
\]
\[
= 111101(ca)^2 |_{111} (1^{\infty}) = 111101(ca)(1^{\infty}) = 1111011^{\infty}
\]
Then for $n \geq 3$
\[
(ca)^{2n} (1^{\infty}) = (ca)^{2n} (111)(ca)^{2n} |_{111} (1^{\infty}) = 111(ca)^2 |_{111} (1^{\infty})
\]
\[
= 111(111)^{n-1}(1010)^{1^{\infty}} = (111)^{n}(1010)^{1^{\infty}}
\]

2.5. Automaton 920.

\[
b(1) = 1
\]
\[
b |_{1} = b
\]

To show that $b$ has infinite order, consider the inverse automaton:

Since $a^{-1} |_{1} = b^{-1} |_{1} = b^{-1}$ and $b^{-1}(1) = 1$, it follows that $b^{-n}(01^{\infty}) \sim 1^{\infty}$. But $b(01^{\infty}) = 0^{\infty}$, so $b$ has infinite order.

Note that according to [4] this group is isomorphic to that of automaton 2401 which is non-contracting.

2.6. Automaton 969.
That $c$ has infinite order follows from the next lemma.

**Lemma 5.** For $n \geq 1$, $c^n((101)\infty) \sim \begin{cases} (100)\infty & \text{n even} \\ (011)\infty & \text{n odd} \end{cases}$

**Proof.** Note that $c((101)\infty) = 11c((110)\infty) = 11(100)\infty$. If $u \sim (100)\infty$, then $c(u) \sim (011)\infty$. If $u \sim (011)\infty$, then $c(u) \sim (100)\infty$. Both statements follow from the observation that for any generator $x \in \{a, b, c\}$, $x|_0 = c$. □

Finally, observe that $c^{-1}((101)\infty) = 1\infty$. This together with the lemma imply that $c$ has infinite order.

**2.7. Automaton 2361.**

Observe that $a(0\infty) = 10\infty$ and $c(0\infty) = 0\infty$. Therefore, for all $n \geq 0$, $c^n(10\infty) \sim 0\infty$. Also, $c^{-1}(10\infty) = 1\infty$. It follows that $c$ has infinite order.

Note that according to [3] this group is isomorphic to that of automaton 939 which is non-contracting.

**2.8. Automaton 2365.**

To see that $c$ has infinite order, observe that $a^{-1}(0\infty) = 10\infty$ and $c^{-1}(0\infty) = 0\infty$. Therefore, for all $n \geq 0$, $c^{-n}(10\infty) \sim 0\infty$. As $c(10\infty) = 1\infty$, it follows that $c$ has infinite order.

Note that according to [3] this group is isomorphic to that of automaton 939 which is non-contracting.
2.9. Automaton 2402.

\[ c(0) = 0 \]
\[ c = 0 \]

Note that \( c^n(10^\infty) \approx 0^\infty \) since \( x|_{00} = c \) for any \( x \in \{a, b, c\} \). However, \( c^{-2}(10^\infty) = 101^\infty \). Therefore, \( c \) has infinite order.

2.10. Automaton 2427.

\[ c(0) = 0 \]
\[ c = 0 \]

To see that \( c \) has infinite order note that
\( a((101)^\infty) = 01(101)^\infty \)
\( b((101)^\infty) = 00(101)^\infty \)
\( c((101)^\infty) = 11(101)^\infty \).

Therefore, for all \( n \geq 1 \), \( c^n((101)^\infty) \approx (101)^\infty \). However, \( c^{-2}((101)^\infty) = (100)^\infty \).

3. Non-hyperbolic self-similarity graphs

Nekrashevych introduced the notion of a self-similarity graph of a self-similar action. He proved that if a self-similar group is contracting, the corresponding self-similarity graph (endowed with the natural metric) is hyperbolic. The converse to this result is open.

Here we provide a partial converse to this fact, which applies to self-similar actions that satisfy the criterion of Lemma. We do not know of a non-contracting self-similar group that does not satisfy the criterion.

**Definition 6** ([4] Defn. 3.7.1). The *self-similarity graph* \( \Sigma(G) \) of a self-similar group \( G \) with generating set \( S \) acting on \( X^* \) is the graph with vertex set \( X^* \) and an edge \( \{u, v\} \) whenever:
- \( u = s(v) \) for some \( s \in S \); these are the *horizontal edges*;
- \( u = xv \) for some \( x \in X \); these are the *vertical edges*.

Observe that horizontal edges connect strings in \( X^* \) of the same length, vertical edges connect strings that differ in length by 1.

We use the characterization of hyperbolic geodesic metric spaces involving the divergence of geodesics, see [2, p.412].

**Proposition 7.** Let \( Y \) be a geodesic metric space. A function \( e : \mathbb{N} \rightarrow \mathbb{R} \) is called a divergence function if for all \( y \in Y \), for all \( R, r \in \mathbb{N} \) and for all geodesics \( \alpha : [0, a] \rightarrow Y \) and \( \beta : [0, b] \rightarrow Y \) with \( \alpha(0) = \beta(0) = y \), \( a > R + r \) and \( b > R + r \) the
following holds: if \(dy(\alpha(R), \beta(R)) > e(0)\) then any path from \(\alpha(R + r)\) to \(\beta(R + r)\) that stays outside the open ball of radius \(R + r\) about \(y\) has length at least \(e(r)\).

Then \(Y\) is hyperbolic if it admits an exponential divergence function.

**Proposition 8.** Let \(G\) be a self-similar group with finite generating set \(S\) acting on \(X^\ast\), and suppose that there exist \(g \in G\) and \(v \in X^\ast\) such that

1. \(g|v = g\)
2. \(g(v)v = v\)
3. \(g\) has infinite order

Then the self-similarity graph \(\Sigma(G)\) is non-hyperbolic.

**Proof.** The vertex in \(\Sigma(G)\) corresponding to the empty string is labelled \(\emptyset\). A vertex in the open ball based at \(\emptyset\) of radius \(N\) corresponds to a string in \(X^\ast\) of length less than \(N\). Note that an element of \(X^\ast\) uniquely defines a vertical geodesic emanating from \(\emptyset\) whose length is equal to that of the word. Considering such geodesics, we show that \(\Sigma(G)\) does not admit an exponential divergence function, and is therefore not hyperbolic.

Suppose for a contradiction that \(e : \mathbb{N} \rightarrow \mathbb{R}\) is a divergence function for \(\Sigma(G)\) and that it is increasing and unbounded. If the maximum size of an orbit of any \(w \in X^\ast\) under \(g\) was \(N\), then \(g^N(w) = w\) for all \(w \in X^\ast\). Since \(g\) has infinite order, it follows that there are arbitrarily large orbits under its action on \(X^\ast\). Vertices in \(\Sigma(G)\) have uniformly bounded degree. It follows that there is a bound on the number of vertices in any metric ball of fixed radius, so we can choose \(n \in \mathbb{N}\) and \(w \in X^\ast\) such that \(d_{\Sigma}(w, g^n(w)) > e(0)\).

For all \(k \in \mathbb{N}\) the vertices \(v^kw\) and \(g^n(v^kw) = v^kg^n(w)\) are connected by a horizontal path of length exactly \(n||g||_S\), and this path lies outside the open ball of radius \(|v^kw|\) centered at \(\emptyset\). Choose \(k \in \mathbb{N}\) such that \(e(|v^k|) > n||g||_S\). Since \(e\) is a divergence function, any horizontal path connecting \(v^kw\) and \(v^kg^n(w)\) must have length at least \(e(|v^k|)\). This contradiction establishes the result. \(\square\)

**References**

[1] Ievgen Bondarenko, Rostislav Grigorchuk, Rostyslav Kravchenko, Yevgen Muntyan, Volodymyr Nekrashevych, Dmytro Savchuk, and Zoran Šunić. “On classification of groups generated by 3-state automata over a 2-letter alphabet”. In: *Algebra Discrete Math.* 1 (2008), pp. 1–163. issn: 1726-3255.

[2] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*. Vol. 319. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Berlin: Springer-Verlag, 1999, pp. xxii+643. isbn: 3-540-64324-9.

[3] Yevgen Muntyan. “Automata Groups”. PhD thesis. Texas A&M University, 2009.

[4] Volodymyr Nekrashevych. *Self-similar groups*. Vol. 117. Mathematical Surveys and Monographs. Providence, RI: American Mathematical Society, 2005, pp. xii+231. isbn: 0-8218-3831-8.
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