COMPLETENESS OF RANK ONE PERTURBATIONS OF NORMAL
OPERATORS WITH LACUNARY SPECTRUM

ANTON D. BARANOV AND DMITRY V. YAKUBOVICH

Abstract. Suppose $A$ is a compact normal operator on a Hilbert space $H$ with certain
lacunarity condition on the spectrum (which means, in particular, that its eigenvalues
go to zero exponentially fast), and let $L$ be its rank one perturbation. We show that
either infinitely many moment equalities hold or the linear span of root vectors of $L$,
corresponding to non-zero eigenvalues, is of finite codimension in $H$. In contrast to
classical results, we do not assume that the perturbation is weak.

M.S.C.(2000): Primary: 42A65; Secondary: 42C30
Keywords: selfadjoint operator, rank one perturbation, completeness of eigenvectors,
Pólya peaks

1. Introduction and main results

1.1. The main result. Let $L$ be a compact operator on a separable Hilbert space $H$.
We will say that $L$ is complete if its root vectors, corresponding to non-zero eigenvalues,
are complete in $H$. (Notice that the point 0 of the spectrum of a compact operator
plays a special role.) We will say that $L$ is nearly complete if the root vectors of $L$,
corresponding to non-zero eigenvalues, span a subspace of $H$ of finite codimension.
One can observe that for any positive integer $N$, the closed linear span $M(L)$ of root vectors
of $L$, corresponding to non-zero eigenvalues, is contained in $(\ker L^* N)^\perp$. So, whenever
$\ker L^* \neq 0$, $L$ cannot be complete and we only can expect the near completeness.

Let $\{\lambda_n\}$ be a sequence of complex numbers. We will say that this sequence is
lacunary if there is a positive constant $\varepsilon$ such that $|\lambda_n - \lambda_m| \geq \varepsilon \max(|\lambda_m|, |\lambda_n|)$ for
all indices $n \neq m$. This is equivalent to the condition that for some $\delta > 0$, the discs
$B(\lambda_n, \delta |\lambda_n|)$ are disjoint. Such sequence can accumulate only to 0 and to $\infty$. If $\lambda_n$ tend
to zero as $n \to +\infty$ and are numbered so that the moduli $|\lambda_n|$ decrease, then they decay
exponentially fast, moreover, there exist some $\sigma \in (0, 1)$ and some $B \geq 1$ and such that
$|\lambda_m/\lambda_n| \leq B \sigma^{m-n}$ whenever $m \geq n$.

Suppose that $A$ is a compact normal operator with trivial kernel. By the spectral
theorem,

$$A = \sum_{n \in \mathbb{N}} s_n P_n,$$

where $\mathbb{N} = \{1, 2, \ldots\}$, $s_n \neq 0$, $s_n \to 0$, and $P_n$ are finite dimensional orthogonal
projections in $H$ such that $P_n P_m = 0$ for $m \neq n$ and $\sum_n P_n = I$.

Our main object of study will be a one-dimensional perturbation of $A$ of the following
form:

$$L x = Ax + \langle x, b \rangle a, \quad a, b \in H.$$
To formulate our results, we need to introduce the following sequence of “moment”
equations
\[ \sum_n s_n^{-1} \langle P_n a, b \rangle = -1, \quad (M_1) \]
\[ \sum_n s_n^{-k} \langle P_n a, b \rangle = 0, \quad (M_k) \]
k = 2, 3, \ldots. Note that, for a general one-dimensional perturbation, the above series
need not converge.

Our first main result is Theorem 1.3 below. It might be instructive to precede it with
two simpler statements.

**Theorem 1.1.** Let \( A \) be a normal operator given by (1.1) which belongs to a Schatten
ideal \( S_p, 0 < p < \infty \), and whose spectrum is contained in a finite union of rays \( \arg z = \alpha_k, 1 \leq k \leq n \). Let \( L \) be a one-dimensional perturbation of \( A \), given by (1.2). Assume
that, for some \( k \in \mathbb{N} \), we have
\[ \sum_n |s_n|^{-k} |\langle P_n a, b \rangle| < \infty, \]
but the equality \((M_k)\) does not hold. Then \( L \) and \( L^* \) are nearly complete.

Moreover, for any \( \varepsilon > 0 \) there is a radius \( r > 0 \) such that the intersection of non-zero
spectrum \( \sigma(L) \setminus \{0\} \) with the disc \( B(0, r) \) is contained in the union of angles \( \alpha_k - \varepsilon < \arg z < \alpha_k + \varepsilon, 1 \leq k \leq n \).

This assertion can be obtained by an application of standard methods based on re-
solvent estimates, see Section 2.

Another simple observation is that \( \cup_n \ker L^* \) is orthogonal to \( M(L) \). So, if the linear
manifold \( \cup_n \ker L^* \) is infinite dimensional, then, obviously, \( L \) is not nearly complete.
It is easy to see that the following fact holds.

**Proposition 1.2.** \( \cup_n \ker L^* \) is infinite dimensional if and only if \( b \in \text{Ran} A^n \) for any
integer \( n > 0 \) and the equalities \((M_k)\) hold for all \( k \geq 1 \).

Here is our first main result.

**Theorem 1.3.** Let \( L \) given by (1.2) be a one-dimensional perturbation of a compact
normal operator \( A \), given by (1.1), whose spectrum is lacunary. If \( L \) is not nearly
complete, then the equalities \((M_k)\) are valid for all \( k \geq 1 \).

This theorem shows that a stronger statement than Theorem 1.1 holds for the case
of a lacunary spectrum (with arbitrary geometry). At the same time, it can be seen as
a partial converse of Proposition 1.2.

**Remarks.** 1. Since \( L^* x = A^* x + \langle x, a \rangle b \), it follows that the same criterion holds for
nearly completeness of \( L^* \), where equalities \((M_k)\) are literally the same.

2. In the case when all moment equalities \((M_k), k \geq 1,\) are fulfilled, the operator \( L \)
may be complete or incomplete. In [3, Theorem 1.3], for any compact selfadjoint operator
\( A \) with simple point spectrum and trivial kernel a bounded rank one perturbation \( L \) of
\( A \) with real spectrum was constructed such that \( L \) is complete and \( \ker L = 0 \), while \( L^* \)
is even not nearly complete. Therefore the near completeness of \( L \) is not equivalent to
the near completeness of \( L^* \), even for rank one perturbations of normal operators with
lacunary spectrum we are considering here.

It is essential for the construction in [3] that all moment equalities hold. For the
lacunary spectra this follows also from our Theorem 1.3.
1.2. Sharpness of Theorem 1.3 In this section we will see that the lacunarity condition in Theorem 1.3 cannot be weakened much. Namely, if the spectrum of $\mathcal{A}$ is more dense than a lacunary one (but still much sparser than any power spectrum), then there exists a rank one perturbation, which is not nearly complete, but already the first moment does not exist, moreover, 

\begin{equation}
\sum_n |s_n|^{-1}|\langle P_n a, b \rangle| = \infty.
\end{equation}

To make the conditions on the spectrum clearer it is better to pass to the inverses $t_n = s_n^{-1}$. Note that the lacunarity implies that $n_T(r) = O(\log r)$, $r \to \infty$. Here $n_T$ is the counting function of the sequence $\{t_n\}$: $n_T = \#\{n : |t_n| < r\}$. We show that rank one perturbations satisfying (1.3), which fail to be nearly complete, always exist unless $n_T(r) = O(\log^2 r)$, $r \to \infty$.

The precise formulation of our second main result is as follows:

**Theorem 1.4.** Let $\mathcal{A}$ be a compact selfadjoint operator given by (1.1), where $s_n \in \mathbb{R}$, $s_n \neq 0$, and $t_n = s_n^{-1}$. Suppose that rank $P_n = 1$ for all $n$. Assume also that $\inf_{n \neq k} |t_n - t_k| > 0$ and that, for any $N > 0$, we have

\begin{equation}
\liminf_{|n| \to \infty} |t_n|^N \prod_{k: \frac{1}{2} \leq \frac{|t_k|}{t_n} \leq 2, k \neq n} \frac{|t_k - t_n|}{t_k} = 0.
\end{equation}

Then there exists a rank one perturbation $\mathcal{L}$ of $\mathcal{A}$ such that (1.3) holds, but the eigenvectors of $\mathcal{L}$ are not complete with an infinite defect. It is true, in particular, if

\begin{equation}
\limsup_{r \to \infty} \frac{n_T(r)}{\log^2 r} = +\infty.
\end{equation}

We will show, in fact, that (1.3) implies (1.4). One can express the condition (1.4) in equivalent ways, see the remark in Subsection 6.1

1.3. Relations to known results. Most general abstract sufficient conditions for completeness are due to Keldyš [18, 19] and Macaev [21, 22]. A good exposition of these results by Macaev and their generalizations to operator pencils can be found in [23].

**Theorem A** (Keldyš, 1951). Let $A, S$ be compact Hilbert space operators. Suppose $A$ is normal, belongs to a Schatten ideal $\mathfrak{S}_p$, $0 < p < \infty$, and its spectrum is contained in a finite union of rays $\arg z = \alpha_k$, $1 \leq k \leq n$. Suppose $\ker A = \ker(I + S) = 0$. Put $L = A(I + S)$. Then the operators $L$ and $L^*$ are complete.

The original statement by Keldyš referred only to the case of a selfadjoint operator $A$; the above formulation appears, for instance, in [20]. A perturbation of a compact operator $A$ of the form $A(I + S)$ or $(I + S)A$, with $S$ compact, is called a weak perturbation. Macaev’s theorems also concern weak perturbations. In [24], Macaev and Mogul’skii give an explicit condition on the spectrum of $A$, equivalent to the property that all weak perturbations of $A$ with $\ker(I + S) = 0$ are complete.

Our results deal with the situation which is much more special than in the Keldyš and Macaev theorems, but they concern perturbations that are not necessarily weak and also treat the case of nontrivial kernels. It would be interesting to know whether an analogue of Theorem 1.3 holds true for finite rank perturbations. A result on completeness (up to a finite defect) for an abstract compact operator in Schatten class $\mathfrak{S}_p$ is contained in [12, Theorem XI.9.29], where power estimates of the resolvent in certain angles are assumed. We remark that, although Keldyš and Macaev’s theorems can be deduced from this theorem, the condition $\ker(I + S) = 0$ is crucial for getting these power estimates (see the Example 4.3 below).
We refer to [1], [28], [29] for some recent abstract results about completeness and bases of eigenvectors and to the books [13], [15], [12], [20] and the review [26] for an extensive exposition. The papers [17], [14], [11] contain some general results on spectral properties of finite dimensional perturbations of diagonal operators (in particular, on the existence of invariant subspaces). It seems that not much is known in general for weak perturbations with $\ker(I + S) \neq 0$ and for non-weak perturbations.

This work can be considered as a continuation of our papers [3], [4], where the completeness and related properties (e.g., the spectral synthesis) were studied for similar class of operators by using a kind of functional model involving de Branges spaces. In these works, we considered perturbations of selfadjoint operators without a condition on lacunarity of the spectrum. Here we consider perturbations of general compact normal operators with lacunary spectrum. In contrast to the proof of Theorem 1.4 in the proof of Theorem 1.3 we do not use the techniques of the functional model and deal directly with the resolvent estimates.

As in [3], [4], here we also will consider (singular) rank one perturbations of unbounded normal operators with discrete spectrum and obtain parallel completeness results for them.

The functional model (see Subsection 6.2 below) translates the completeness or spectral synthesis problems for one-dimensional perturbations to the completeness of systems which are biorthogonal to the systems of reproducing kernels in some spaces of entire (or meromorphic) functions as well as to completeness problems of ”mixed” systems (for the corresponding results see the recent papers [8, 5, 6, 7]). In particular, in [8], Riesz bases of reproducing kernels in spaces of Cauchy transforms of discrete lacunary measures in $\mathbb{C}$ has been described. It seems that there are some relations between this work and the questions we study here.

1.4. Method of the proof. The spectral analysis of the perturbation $\mathcal{L}$ leads to a consideration of the function

$$\beta(z) = 1 + \langle (A - z^{-1})^{-1}a, b \rangle = 1 - z\langle (I - zA)^{-1}a, b \rangle = 1 + \sum_n c_n \left( \frac{1}{t_n - z} - \frac{1}{t_n} \right),$$

where $t_n = 1/s_n$ and $c_n = -s_n^{-2} \langle P_n a, b \rangle$. This function is meromorphic in $\mathbb{C}$. It is easy to see that the zero set of $\beta$ coincides with the set $\{\lambda^{-1} : \lambda \in \sigma(\mathcal{L}), \lambda \neq 0 \}$.

We will adopt the following notation. If $A$ is a measurable subset of $[0, +\infty)$, its linear density is defined as $\lim_{R \to +\infty} R^{-1} m([0, R] \cap A)$. Given a function $f$ on $\mathbb{C}$, we will write $\lim_{z \to \infty}^* f(z) = w$ if there exists a closed set $A \subset [0, +\infty)$ of linear density one such that

$$\lim_{z \to \infty, |z| \in A} f(z) = w.$$

Our main complex variable tool will be the following theorem.

**Theorem 1.5.** Suppose that a complex sequence $\{t_n\}$ goes to infinity, is lacunary and $t_n \neq 0$ for all $n$. Let $\alpha \in \mathbb{C}$ and let $c_n$ be any complex coefficients, not all equal to zero, such that $\sum_n |c_n/t_n^2| < +\infty$. Put

$$\beta(z) = \alpha + \sum_n c_n \left( \frac{1}{t_n - z} - \frac{1}{t_n} \right).$$

If for any $s \in \mathbb{N}$,

$$\lim_{z \to \infty}^* z^s \beta(z) = 0,$$

then $\sum_n t_n^{-1} c_n = \alpha$ and $\sum_n t_k^k c_n = 0$ for $k \in \mathbb{Z}$, $k \geq 0$.

It is easy to see that the conditions $\sum_n t_n^{-1} c_n = \alpha$ and $\sum_n t_k^k c_n = 0$, $k \geq 0$, imply that $\lim z^s \beta(z) = 0$ for any $s$ as $|z| \to \infty$ and $\dist (z, \{t_n\}) \geq 1$. Theorem 1.5 shows that in the lacunary case the converse is true. The proof of this theorem uses a lemma on
“peaks” of numerical sequences by Pólya [25] which enables us to obtain some estimates of the function \( \beta \) from below.

Let us mention that in [27], Shkalikov obtained lower estimates for meromorphic functions outside small “exceptional” sets, which he applied in [28] to get a new criterion for eigenvectors of a perturbed selfadjoint operator to form a basis with parenthesis.

The paper is organized as follows. In Section 2 we collect some preliminary results relating the near completeness to the properties of some entire (meromorphic) functions defined in terms of resolvents. Section 3 contains the proof of our main result, Theorem 1.3. Some additional remarks on the general properties of nearly complete operators are given in Section 4. In Section 5 we introduce (unbounded) singular rank one perturbations and state the counterpart of the main result for this case. Theorem 1.4 about completeness of perturbations of normal operators 5

Proposition 2.1. Let \( \mathcal{L} \) be a general compact operator on a Hilbert space \( H \). If \( \mathcal{L} \) is nearly complete, then the linear set \( \cup_{n} \ker \mathcal{L}^{n} \) is finite dimensional, there is some \( k \geq 0 \) such that \( \ker \mathcal{L}^{n} = \ker \mathcal{L}^{k} \) for all \( n > k \) and \( M(\mathcal{L})^{\perp} = \cup_{n} \ker \mathcal{L}^{n} = \ker \mathcal{L}^{k} \).

The proof is immediate from the fact that \( \mathcal{L}^{\ast} | M(\mathcal{L})^{\perp} \) is always quasinilpotent.

Next, let us make the following observation. Let \( H^{1} \) be the smallest reducing subspace of \( \mathcal{A} \) containing the vectors \( a, b \), and put \( H^{2} = H \ominus H^{1} \). Then \( \mathcal{L} \) decomposes as \( \mathcal{L} = \mathcal{L}^{1} \ominus \mathcal{A}^{2} \), where \( \mathcal{L}^{1}x = \mathcal{A}x + \langle x, b \rangle a, x \in H^{1} \) and \( \mathcal{A}^{2} = \mathcal{A}|H^{2} \). This implies that it suffices to prove Theorem 1.3 for the case when the smallest reducing subspace of \( \mathcal{A} \) containing vectors \( a, b \) coincides with \( H \). From now on, we will assume that this property is fulfilled. It follows that all spectral projections \( P_{n} \) in (1.1) have at most rank two.

Put \( M = M(\mathcal{L}) \). Given any \( x \in H \) and any \( y \in M^{\perp} \), the function

\[
fx,y(\lambda) = \langle (I - \lambda \mathcal{L})^{-1}x, y \rangle
\]

is entire.

We will use an easy formula

\[
(\mathcal{L} - z)^{-1} = (\mathcal{A} - z)^{-1} - (\mathcal{A} - z)^{-1}a \beta^{-1}(z^{-1}) b^{\ast}(\mathcal{A} - z)^{-1},
\]

where \( \beta(z) \) is defined in (1.6). To check it, one can write down the second resolvent identity

\[
(\mathcal{L} - z)^{-1} - (\mathcal{A} - z)^{-1} = -(\mathcal{A} - z)^{-1}ab^{\ast}(\mathcal{L} - z)^{-1}.
\]

By multiplying it by \( b^{\ast} \) on the left, one gets \( b^{\ast}(\mathcal{L} - z)^{-1} = \beta^{-1}(z^{-1}) b^{\ast}(\mathcal{A} - z)^{-1} \), which gives (2.1). Then, for any \( s \in \mathbb{N} \), \( x \in H \) and \( y \in M^{\perp} \) we have

\[
z^{-s-3}fx,y(z) = z^{-s-4}y^{\ast}(z^{-1} - \mathcal{L})^{-1}x
\]

\[
= z^{-s-4}y^{\ast}(z^{-1} - \mathcal{A})^{-1}x
\]

\[
+ [z^{-2}y^{\ast}(\mathcal{A} - z^{-1})^{-1}a] \cdot z^{-s} \beta^{-1}(z) \cdot [z^{-2}b^{\ast}(\mathcal{A} - z^{-1})^{-1}x].
\]

Proposition 2.2. Suppose \( N \in \mathbb{Z} \), \( N \geq 0 \) and \( \mathcal{L} \) is a compact operator. Then the following statements are equivalent:

(i) \( M^{\perp} \subset \ker \mathcal{L}^{N} \);

(ii) For any \( x \in H \) and any \( y \in M^{\perp} \), \( fx,y(\lambda) \) is a polynomial in \( \lambda \) of degree less than \( N \).
Proof. Since $f_{x,L^*y}(\lambda) = \lambda^{-1}[f_{x,y}(\lambda) - f_{x,y}(0)]$, it follows that (ii) is equivalent to the condition $f_{x,L^*y}=0$ for all $x \in H$, $y \in M^\perp$, that is, to the condition $M^\perp \subset \ker L^*N$. This gives this statement.

Suppose, in particular, that $\ker L^*N$ is finite dimensional for any $N \in \mathbb{N}$ (it is true for the operator $L$, given by (1.2)). Then, by the Propositions 2.2 and 2.1 $L$ is nearly complete if and only if there is an integer $N > 0$ such that for any $x \in H$ and any $y \in M^\perp$, $f_{x,y}(\lambda)$ is a polynomial in $\lambda$ of degree less than $N$.

We precede the proof of Theorem 1.1 with the following observation.

Remark. If the series in $(M_1)$ converges absolutely, then $L$ is what we called in [3] a generalized weak perturbation of $\mathcal{A}$. So, the case when the hypotheses of Theorem 1.1 hold for $k=1$ is very close to the sufficient condition for completeness, which was given in [3]. Theorem 3.3. This case can be derived from the results of Macaev in the same way as was made in [3], Proposition 1.1 (where rank $n$ perturbations of compact selfadjoint operators were considered).

The statement about the geometry of the spectrum of a weakly perturbed operator in the case when $(M_1)$ is not satisfied is standard in the context of Keldysh’s theorem. As we will see now, Theorem 1.1 and its proof are variations of these ideas.

Proof of Theorem 1.1. Assume that $k \in \mathbb{N}$ is the smallest positive integer such that $\sum_n |s_n|^{−k}|⟨P_n a, b⟩| < \infty$ but the equality $(M_k)$ does not hold. By the above remark, assume that $k \geq 2$. Put $t_n = s_n^{−1}$ and define $\beta(z)$ by (1.6), where $c_n = t_n^2 |⟨P_n a, b⟩|$. Then we have $\sum_n |t_n|^k |c_n| < \infty$ and $\sum_n t_n |c_n| = \gamma \neq 0$. By the obvious formula

$$
\frac{1}{z-t_n} = \sum_{j=0}^{k} \frac{t_n^j}{z^{j+1}} + \frac{t_n^{k+1}}{z^{k+1}(z-t_n)}
$$

we have

$$
\beta(z) = -\frac{1}{z^{k+1}} \sum_n t_n^k c_n + \frac{1}{z^{k+1}} \sum_n \frac{t_n^{k+1} c_n}{t_n - z} = \frac{\gamma}{z^{k+1}} + \frac{1}{z^{k+1}} \sum_n \frac{t_n^{k+1} c_n}{t_n - z}.
$$

Recall that $\{t_n\}$ is contained in a finite union of rays $\arg z = \alpha_k$, $1 \leq k \leq n$. Hence, for any $\varepsilon > 0$, we have

$$
|\beta(z)| > \frac{c_\varepsilon}{|z|^{k+1}}
$$

when $|\arg z - \alpha_k| \geq \varepsilon$, $1 \leq k \leq n$ and $|z|$ is sufficiently large. Clearly, $\|(|A - z|^{−1})\|$ also admits a power estimate for such values of $z$ and we conclude, by (2.2), that $|f_{x,y}(z)|$ admits a power estimate for $|\arg z - \alpha_k| \geq \varepsilon$. Since $A \in \mathcal{G}_0$, we get that $f_{x,y}$ is a function of finite order and, by the Phragmén–Lindelöf principle, we conclude that $f_{x,y}$ is a polynomial of degree less than some fixed $N$ for any $x \in H$, $y \in M^\perp$. □

Now we pass to the analysis of the case when the spectrum is lacunary.

Given an entire function $F$, we use notations $M_F(r) = \max_{|z|=r} |F(z)|$, $m_F(r) = \min_{|z|=r} |F(z)|$, and put $n_F(r)$ to be the number of zeros of $F$ in the disc $|z| < r$, counted with multiplicities. In what follows, we will say that an entire function $F$ is of class Slow if it is of zero order and $\log M_F(r) = \mathcal{O}\left((\log r)^2\right)$ as $r \to \infty$; the last condition can be replaced by the condition $n_F(r) = \mathcal{O}(\log r)$. We will use the following version of [10] Theorem 3.6.1.

**Theorem B.** For any entire function $F$ of the class Slow, which is not a polynomial, and any $N \in \mathbb{N}$, one has \( \lim_{r \to \infty} |z|^{-N} |F(z)| = +\infty \).

**Lemma 2.3.** Let $x \in H$ and $y \in M^\perp$. Then the entire function $f_{x,y}$ belongs to the class Slow.
Proof. It is well-known that $\mathcal{L}^*|M^\perp$ is quasinilpotent (see, for instance, [20]). Given a linear operator $B$ on a Hilbert space, we denote by $\{\mu_j(B)\}_{j \geq 1}$ the sequence of its singular numbers. The lacunarity of the spectrum of $\mathcal{A}$, the property that rank $P_n \leq 2$ for all $n$ and the well-known estimates for singular numbers of the sum of two operators [12 Corollary XI.9.3] imply that
\[
\mu_j(\mathcal{L}^*|M^\perp) \leq \mu_j(\mathcal{L}^*) \leq C\sigma^j,
\]
where $\sigma < 1$ is a constant. In particular, $\mathcal{L}^*|M^\perp$ is a trace class operator. By applying the arguments employed in the proof of [12 Theorem XI.9.26], we get that
\[
\|(I - \tilde{\lambda}\mathcal{L}^*)^{-1}|M^\perp\| = \|\det((I - \tilde{\lambda}\mathcal{L}^*)|M^\perp)\cdot (I - \tilde{\lambda}\mathcal{L}^*)^{-1}|M^\perp\|
\leq \prod_{j=1}^{\infty} \mu_j((I - \tilde{\lambda}\mathcal{L}^*)|M^\perp) \leq \prod_{j=1}^{\infty} (1 + |\lambda|\mu_j(\mathcal{L}^*|M^\perp)) \leq \exp(C(\log|\lambda|)^2)
\]
(notice that $\det((I - \tilde{\lambda}\mathcal{L}^*)|M^\perp) \equiv 1$ for all $\lambda$). Since
\[
|f_{x,y}(\lambda)| = \|x, (I - \tilde{\lambda}\mathcal{L}^*)^{-1}y\| \leq \|x\| \cdot \|y\| \cdot \|(I - \tilde{\lambda}\mathcal{L}^*)^{-1}|M^\perp\|
\]
the assertion of Lemma follows. \hfill\Box

**Lemma 2.4.** For any normal compact operator $\mathcal{A}$ with lacunary spectrum and any $\delta > 1$, one has $\lim_{z \to \infty} z^{-\delta} \|(\mathcal{A} - z^{-1})^{-1}\| = 0$.

**Proof.** Let $1 < \delta_1 < \delta$, and consider the discs $B_n := B(s_n, |s_n|^\delta_1)$. Assume that $|z| \geq 1$. If $z^{-1} \notin \bigcup_n B_n$, then $|z^{-1} - s_n| \geq |z|^{-\delta_1}$ for all $n$, where $\varepsilon > 0$ is some constant (consider the cases $|z^{-1}| \geq 2|s_n|$ and $|z^{-1}| < 2|s_n|$). Therefore $|z|^{-\delta_1}||z^{-1} - \mathcal{A}|| \leq \varepsilon^{-1}$ for all $z$ such that $z^{-1} \in \mathbb{C} \setminus \bigcup_n B_n$. One gets from the lacunarity of the spectrum that the set $\{|z|^{-1} : z \in \bigcup_n B_n\}$ has linear density zero, which implies the statement. \hfill\Box

**Lemma 2.5.** If $\mathcal{L}$ is not nearly complete, then for any $s \in \mathbb{N}$,
\[
\lim_{z \to \infty} z^s \beta(z) = 0,
\]
where $\beta(z)$ is defined in (1.6).

**Proof.** Suppose $\mathcal{L}$ is not nearly complete, so that $M^\perp$ is infinite dimensional. Fix some $s \in \mathbb{N}$. By Proposition 2.2, Lemma 2.3 and Theorem A, there exist $x \in H$ and $y \in M^\perp$ such that $\lim_{z \to \infty} z^{-s-3}|f_{x,y}(z)| = +\infty$. By Lemma 2.4
\[
\lim_{z \to \infty} z^{-2}u^*(\mathcal{A} - z^{-1})^{-1}v = 0
\]
for any pair of vectors $u, v \in H$. Since the limit $\lim^*$ of the modulus of the left hand part in (2.2) equals +$\infty$ and the finite intersection of any subsets of $[0, +\infty)$ of linear density one has linear density one, the assertion of the lemma follows. \hfill\Box

### 3. Proof of Theorems 1.5 and 1.3

First we show how to deduce Theorem 1.3 from Theorem 1.5

**Proof of Theorem 1.5 assuming Theorem 1.2.** Let $\mathcal{L}$ have the form given in the Theorem. Put $t_n = s_n^{-1}$ and define $\beta(z)$ by (1.6), where $c_n = -t_n^2\langle P_n a, b \rangle$. Assume $\mathcal{L}$ is not nearly complete; then by Lemma 2.5 equality (1.8) holds for any positive integer $s$. Therefore Theorem 1.5 gives us the conclusions of Theorem 1.3. \hfill\Box

The rest of this section is devoted to the proof of Theorem 1.5

**Lemma 3.1.** Let $r$ be a positive integer and let $f \in C^r[a,b]$ be a real function. If $|f^{(r)}| > \varepsilon > 0$ on $[a,b]$, then there exists a subinterval $[c,d]$ of $[a,b]$ of length $\frac{b-a}{3}$ such that $|f(x)| \geq \left(\frac{x-a}{b-x}\right)^r \varepsilon$ for all $x \in [c,d]$. 

Proof. Consider first the case when \( r = 1 \). Then we can assume without loss of generality that \( f' > \varepsilon \) on \([a, b]\). Let \([a, b] = I_1 \cup I_2 \cup I_3\) be the subdivision of \([a, b]\) into three subsequent equal intervals. Then one can take \([c, d] = I_1\) if \( f\left(\frac{x_k}{2}\right) < 0\) and \([c, d] = I_3\) in the opposite case.

The case of general \( r \) now follows by an obvious induction argument. \( \square \)

We will use the following result by G. Pólya (1923).

**Lemma B** (see Pólya [25], p. 170). Let \( \{p(n)\}, \{\alpha(n)\}\) and \( \{q(n)\}\) \((n \in \mathbb{N})\) be sequences such that \( p(n) \geq 0, q(n) \geq 0, \alpha(n) > 0, \) and \( q(n) = \alpha(n)p(n) \) for all \( n \). Suppose that \( \limsup p(n) = +\infty, \lim q(n) = 0, \) and that the sequence \( \{\alpha(n)\} \) decreases and tends to 0. Then there exists an increasing index sequence \( \{m_k\} \) such that

1. \( p(m_k) = \max \{p(s) : 1 \leq s \leq m_k\} \) for all \( k \);
2. \( q(m_k) = \max \{q(s) : s \geq m_k\} \) for all \( k \);
3. \( \lim_k p(m_k) = +\infty. \)

The main step in the proof of Theorem 1.5 will be the following statement.

**Lemma 3.2.** Suppose that the sequences \( \{t_n\}, \{c_n\}\) and a complex number \( \varkappa \) meet all the conditions of the above Theorem 1.5 but instead of (1.8), we only require that

\[
(3.1) \quad \lim_{|z| \to +\infty} z\beta(z) = 0.
\]

Then \( \sum_n |t_n^{-1} c_n| < +\infty \) and \( \sum_n t_n^{-1} c_n = \varkappa. \)

**Proof.** Since \( \{t_n\}\) is lacunary, it follows that for some constant \( \alpha > 0, |t_m - t_n| \geq \alpha \max(|t_m|, |t_n|) \) for all \( m \neq n \). Also, there are some constants \( g, B > 1 \) such that \( |t_n/t_m| \leq Bg^{n-m} \) for all \( n < m \). We assume that the sequence \( \{|t_n|\}\) increases.

First let us prove that \( \sum_n |t_n^{-1} c_n| < \infty \). Assume it is not so. Choose some \( u \in (1, g) \) close to 1. Put \( p(n) = u^n |c_n||t_n|^{-1}, \alpha(n) = u^{-n}|t_n|^{-1}, q(n) = |t_n|^{-2}|c_n| \). Since the series \( \sum_n |t_n^{-1} c_n| \) diverges and the series \( \sum_n |t_n^{-2} c_n| \) converges, it follows that all the hypotheses of Pólya’s lemma are satisfied. Let \( \{m_k\}\) be an index sequence that has properties (1)–(3).

Properties (1) and (2) imply that

\[
|c_n| \leq u^{m_k-n} \frac{|t_n|}{|t_m|} |c_{m_k}| \leq B(g^{-1}u)^{m_k-n} |c_{m_k}| \quad \text{for } n < m_k;
\]

\[
\frac{|c_n|}{|t_n|^2} \leq \frac{|c_{m_k}|}{|t_{m_k}|^2} \quad \text{for } n > m_k.
\]

We have

\[
\frac{\beta''(z)}{2} = \sum_{n=1}^\infty \frac{c_n}{(z-t_n)^3}.
\]

Let \( |z-t_{m_k}| \leq \varepsilon |t_{m_k}|, \) where \( \varepsilon \) is a small positive constant, which will be chosen later. If \( n < m_k \), then

\[
|z-t_n| \geq |t_n-t_{m_k}| - |z-t_{m_k}| \geq (\alpha - \varepsilon) |t_{m_k}|.
\]
Similarly, if \( n > m_k \), then \( |z - t_n| \geq |t_n - t_{m_k}| - |z - t_{m_k}| \geq (\alpha - \varepsilon)|t_n| \). Hence inequalities (3.2) imply the estimates

\[
\left| \frac{\beta''(z)}{2} - \frac{c_{m_k}}{(z - t_{m_k})^3} \right| \\
\leq \sum_{n=1}^{m_k-1} \frac{|c_n|}{(\alpha - \varepsilon)^3 |t_{m_k}|^3} + \sum_{n=m_k+1}^{\infty} \frac{|c_n|}{(\alpha - \varepsilon)^3 |t_n|^3} \\
\leq \frac{B|c_{m_k}|}{|t_{m_k}|^3 (\alpha - \varepsilon)^3} \sum_{n=1}^{m_k-1} \left( \frac{u}{g} \right)^{m_k-n} + \frac{B}{(\alpha - \varepsilon)^3} \sum_{n=m_k+1}^{\infty} \frac{|c_{m_k}|}{|t_{m_k}|^3} g^{-n+m_k} \\
\leq K(\varepsilon) \left| \frac{c_{m_k}}{(z - t_{m_k})^3} \right| \quad \text{for } 0 < |z - t_{m_k}| < \varepsilon |t_{m_k}|,
\]

where

\[
K(\varepsilon) \overset{\text{def}}{=} B \left( \frac{\varepsilon}{\alpha - \varepsilon} \right)^3 \left[ \frac{g}{g - u} + \frac{1}{g - 1} \right].
\]

Choose a small \( \varepsilon \in (0, \alpha) \) such that \( K(\varepsilon) < \frac{1}{2} \). Assume now that \( z \in [(1 + \varepsilon_1)|t_{m_k}|, (1 + \varepsilon)|t_{m_k}|) \), where \( \varepsilon_1 \in (0, \varepsilon) \) is a constant. Then the inequality \( |a^3 - b^3| \leq 3|a - b| (\min(|a|, |b|))^{-1} \) and (3.3), together with the triangle inequality, give

\[
\left| \frac{\beta''(z)}{2} - \frac{c_{m_k}}{(1 + \varepsilon_1)|t_{m_k} - t_{m_k}|} \right| \leq \frac{2}{3} \left| \frac{c_{m_k}}{(1 + \varepsilon_1)|t_{m_k} - t_{m_k}|} \right|
\]

if \( \varepsilon_1 \) is sufficiently close to \( \varepsilon \). By property (3) from Lemma B, \( |c_{m_k}| = u^{-m_k} t_{m_k} p(m_k) \geq \varepsilon_2 > 0 \) for all \( k \). Now it follows from (3.1) that there are constants \( \zeta \in \mathbb{C}, |\zeta| = 1 \) and \( \rho > 0 \) such that

\[
|\beta''(t)| \geq \rho |t_{m_k}|^{-3} |c_{m_k}| \geq \rho \varepsilon_2 |t_{m_k}|^{-3}
\]

for all \( k \) and all \( t \in [(1 + \varepsilon_1)|t_{m_k}|, (1 + \varepsilon)|t_{m_k}|] \), where \( f(t) = \text{Re} (\zeta \beta(t \cdot t_{m_k} / |t_{m_k}|)) \). So Lemma 3.1 yields that there is a subinterval of \([(1 + \varepsilon_1)|t_{m_k}|, (1 + \varepsilon)|t_{m_k}|] \) of length \( (\varepsilon - \varepsilon_1)|t_{m_k}|/9 \) on which \( |f(t)| \geq \varepsilon_3 t^{-1} \), where \( \varepsilon, \varepsilon_1, \varepsilon_3 > 0 \) do not depend on \( k \). This contradicts the assumption (3.1).

We conclude that the sum \( \sum_n |t_n^{-1} c_n| \) converges. Take some \( \tau \in (0, \frac{\alpha}{2}) \). Then the discs \( B(t_n, \tau |t_n|) \) are pairwise disjoint. Let \( U = U(\tau) \) be their union. Choose \( \tau \) so small that the set

\[
A = \{ r > 0 : \partial B(0, r) \subset \mathbb{C} \setminus U(\tau) \}
\]

satisfies \( \limsup_{R \to \infty} R^{-1} |A \cap [0, R]| > 0 \). Now one can apply the Lebesgue dominated convergence theorem to the sum \( \beta(z) = \zeta + \sum_n \frac{c_n z}{t_n(t_n - z)} \). Since \( |z/(t_n - z)| \leq C(\tau) < \infty \) for all \( z \notin U \) and all \( n \), one gets that the limit of \( \beta(z) \) as \( |z| \to \infty, |z| \in A \) exists and equals \( \zeta - \sum_n t_n^{-1} c_n \). Hence \( \sum_n t_n^{-1} c_n = \zeta \).

\( \square \)

**Proof of Theorem 3.5** Given an integer \( \ell \geq 1 \), consider the statements:

1. \( \sum_n |c_n t_n^{-\ell-2}| < \infty; \)
2. \( \sum_n c_n t_n^{-\ell-2} = 0 \) if \( \ell \geq 2 \) and \( \sum_n c_n t_n^{-\ell-2} = \zeta \) if \( \ell = 1; \)
3. \( \beta(z) = z^{-\ell} \beta_\ell(z) \), where

\[
\beta_\ell(z) = \sum_n c_n t_n^\ell \left( \frac{1}{t_n - z} - \frac{1}{t_n} \right) = z \sum_n c_n t_n^{\ell-1} \frac{1}{t_n - z}.
\]

Lemma 3.2 gives (1)_\ell, (2)_1 and (3)_1. If for some \( \ell \geq 1, (1)_\ell, (2)_\ell \) and (3)_\ell have been obtained, one applies Lemma 3.2 to \( \tilde{c}_n = c_n t_n^{\ell} \) and to \( \beta_\ell \) and gets properties (1)_{\ell+1} and (2)_{\ell+1}, which imply that \( \beta_{\ell+1}(z) = z \beta_\ell(z) \). This gives (3)_{\ell+1}.

So, by induction, the equalities (1)_\ell, (2)_\ell and (3)_\ell hold for any \( \ell \geq 1. \) \( \square \)
4. Some additional remarks

First, we give an explicit form of the space \( \bigcup_n \ker L^{sn} \) whenever this space is finite dimensional.

**Proposition 4.1.** Let \( L \) be given by (1.2) and not all conditions \((M_n)\) are fulfilled. Choose the integer \( k \geq 0 \) so that \((M_1), \ldots, (M_k)\) are fulfilled and \((M_{k+1})\) either is not fulfilled or has no sense (that is, the sum diverges). Let \( \ell \geq 0 \) be the largest integer such that \( b \in \text{Ran} \mathcal{A}^{s\ell} \), and put \( b = \mathcal{A}^{s\ell}b_s \), where \( b_s \in H \) \((s = 1, \ldots, \ell)\). Then

\[
\bigcup_n \ker L^{sn} = \ker L^{sm} = \text{span}\{b_1, \ldots, b_m\},
\]

where \( m = \min(k, \ell) \).

We omit the proof, which is completely straightforward. The same calculations imply Proposition 1.2.

Let \( L_1, L_2 \) be two bounded operators on Hilbert spaces \( H_1, H_2 \), respectively. In [3], we used the following definition: \( L_2 \) is said to be \( d\)-subordinate to \( L_1 \) \((L_1 \preceq L_2)\) if there exists a bounded linear operator \( Y : H_1 \to H_2 \), which intertwines \( L_1 \) with \( L_2 \) and has a dense range:

\[
Y L_1 = L_2 Y; \quad \text{clos} \text{Ran} Y = H_2.
\]

As it was mentioned there, if \( L_1 \preceq L_2 \) and \( L_1 \) is complete then \( L_2 \) is complete. In connection with the present article, we can also mention the following fact.

**Proposition 4.2.** If \( L_1 \) and \( L_2 \) are compact, \( M(L_1) = \text{clos} L_1^d H \) and \( L_2 \) is \( d\)-subordinate to \( L_1 \) then \( M(L_2) = \text{clos} L_2^d H \).

The proof is straightforward, and we leave it to an interested reader.

As it follows from (2.1) and Theorem 1.5, if some of the infinite sequence of moment equalities (1.1) fail, then there is an estimate

\[
\| (L - z)^{-1} \| \leq |z|^{-s}
\]

for the resolvent of \( L \) for a set of the form \( \{ z : |z|^{-1} \in A \} \), where \( A \subset [0, +\infty) \) is a closed subset of linear density 1 at infinity. This can be compared with the estimates, which are used in the proof of the Keldyš theorem: in the conditions of this theorem, for any \( \varepsilon > 0 \), an estimate \( \| (L - z)^{-1} \| \leq C_\varepsilon |z|^{-1} \) holds for sufficiently small \( |z| \) in the complement of the union of the angles \( \alpha_k - \varepsilon \leq \arg z \leq \alpha_k + \varepsilon \) (see [20], Lemma 3.2). In particular, for each \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that all non-zero spectrum of \( L \), which is contained in the disc \( |z| < \delta \), lies in the union of the above angles.

The next example shows that even for weak perturbations \( L = \mathcal{A}(I + S) \) that do not satisfy the requirement \( \ker(I + S) = 0 \), this geometric property of the spectrum does not hold in general.

**Example 4.3.** There is an operator \( L = \mathcal{A}(I + S) \), which has the form (1.2), where \( \mathcal{A} \) is a cyclic selfadjoint operator with lacunary spectrum \( \{ 2^{-n} : n \in \mathbb{N} \} \) such that \( \sigma(L) = \{ 0 \} \cup \{ i2^{-n} : n \in \mathbb{N}, n \geq 2 \} \).

To construct this operator, consider the function

\[
\psi(z) = \frac{2}{2 - z} \prod_{n=2}^{\infty} \varphi_n(z), \quad \text{where} \quad \varphi_n(z) = \frac{2^n + iz}{2^n - z}.
\]

Notice that, given any constant \( K \in (0, 1) \), there exists \( C_K > 0 \) such that \( |\varphi_n(z)| \leq C_K 2^{-n}|z| \) if \( 2^{-n}|z| < K \) and \( |\varphi_n(z) + i| \leq C_K 2^n|z|^{-1} \) if \( 2^n|z|^{-1} < K \). It follows that the above product converges for any \( z \neq 2^n, n \in \mathbb{N} \) and defines a meromorphic function on \( \mathbb{C} \). The residues \( c_n = -\text{Res}_{2^n} \psi \) satisfy \( |c_n| \asymp 1 \). The above estimates for \( \varphi_n(z) \) imply that \( \max_{|z|=3.2^n} |\psi(z)| \leq C 2^{-k} \). Put \( t_n = 2^n \). By writing down the residue theorem for
the function \( \psi(\cdot) \) on the contours \(|z| = 3 \cdot 2^k\), where \( \zeta \) is fixed and letting \( k \to \infty \), one gets
\[
\psi(\zeta) = \sum_{n\in\mathbb{N}} \frac{c_n}{t_n - \zeta} = 1 + \sum_{n\in\mathbb{N}} c_n \left( \frac{1}{t_n - \zeta} - \frac{1}{t_n} \right)
\]
(here we use that \( \psi(0) = 1 \)). Take any sequences \( a = \{a_n\} \) and \( b = \{b_n\} \) such that \( \{2^n a_n\} \) and \( \{b_n\} \) are in \( \ell^2 \) and \( c_n = -t_n^2 a_n b_n \). (For instance, one can put \( a_n = t_n^{-3/2} \) and \( b_n = -c_n t_n^{-1/2} \).) The operator \( A \) on \( H = \ell^2 \), defined by \( A \{x_n\} = \{2^{-n} x_n\} \), is cyclic, compact, selfadjoint and has trivial kernel. Since \( \{2^n a_n\} \) is in \( \ell^2 \), the operator \( \mathcal{L} = A + ab^* \) on \( \ell^2 \) has the form \( \mathcal{L} = \mathcal{A}(I + S) \), where \( S \) is a rank one operator, so \( \mathcal{L} \) is a weak perturbation of \( \mathcal{A} \). Since for this perturbation, \( \beta(z^{-1}) = \psi(z) \), we get that the spectrum of \( \mathcal{L} \) is \( \{0\} \cup \{2^{-n} : n \in \mathbb{N}, n \geq 2\} \).

Notice that in this example, we get that \( \sum a_n b_n s_n^{-1} = - \sum c_n t_n^{-1} = -1 \), so that the first moment equation \((M_1)\) holds and one has absolute convergence in its left hand part. The general term of the sum \( \sum a_n b_n s_n^{-2} \) in \((M_2)\) does not tend to zero. So the hypotheses or Theorem \(1.1\) do not hold. This example shows that in this case, although the perturbation is weak, it spectrum is not contained in the union of angles, given by Theorem \(1.1\). By our Theorem \(1.3\) \( \mathcal{L} \) and \( \mathcal{L}^* \) are nearly complete.

In fact, it is easy to get that \( |\beta(z)| \geq C|z|^{-1} \) for \( |z| \geq 1 \), \( z \) outside arbitrarily small angles around the two coordinate axes. So in this particular case, it is easy to get that \( \mathcal{L} \) and \( \mathcal{L}^* \) are nearly complete either by using the argument of the proof of Theorem \(1.1\) or the criterion of completeness, given in \(3\), Proposition 3.1.

5. The case of singular perturbations of unbounded normal operators

Let \( \mathcal{A} \) be an unbounded normal operator on a Hilbert space \( H \) such that \( \mathcal{A}^{-1} \) exists and is bounded. Let us explain first, what we mean by its rank one singular perturbations (here we follow \(3\)).

We define the extrapolation Hilbert space \( \mathcal{A}H \) as the set of formal expressions \( Ax \), where \( x \) ranges over the whole space \( H \). Put \( ||Ax||_{\mathcal{A}H} = ||x||_H \) for all \( x \in H \). The formula \( x = \mathcal{A}(A^{-1}x) \) allows one to interpret \( H \) as a linear submanifold of \( \mathcal{A}H \). We consider the scale of spaces
\[
\mathcal{D}(\mathcal{A}) \subseteq H \subseteq \mathcal{A}H.
\]
Notice that \( \mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A}^*) \). The pairing \( \langle x, y \rangle \overset{\text{def}}{=} \langle Ax, A^*y \rangle \), \( x \in \mathcal{D}(\mathcal{A}), y \in \mathcal{A}H \), gives rise to a natural identification \( \mathcal{D}(\mathcal{A}) = (\mathcal{A}H)^* \).

A rank 1 singular balanced perturbation of \( \mathcal{A} \) can be described in terms of what we will call 1-data for \( \mathcal{A} \). By 1-data we mean a triple \( (a, b, \varkappa) \), where \( a, b \in \mathcal{A}H \) are non-zero, \( \varkappa \in \mathbb{C} \) and the following condition is fulfilled:

(A) If \( \varkappa \in H \), then \( \varkappa \neq \langle A^{-1}a, b \rangle \).

Given 1-data \( (a, b, \varkappa) \), the corresponding rank 1 singular balanced perturbation \( \mathcal{L} = \mathcal{L}(a, b, \varkappa) \) of \( \mathcal{A} \) is defined as follows:
\[
\mathcal{D}(\mathcal{L}) = \{y = y_0 + cA^{-1}a : c \in \mathbb{C}, y_0 \in \mathcal{D}(\mathcal{A}), \varkappa c + b^* y_0 = 0\};
\]
\[
\mathcal{L} y \overset{\text{def}}{=} Ay_0, \quad y \in \mathcal{D}(\mathcal{L}).
\]
Condition (A) is equivalent to the uniqueness of the decomposition \( y = y_0 + cA^{-1}a \) for \( y \in \mathcal{D}(\mathcal{L}) \) and hence to the correctness of the definition of \( \mathcal{L} \).

Let \( G(\mathcal{A}) \) stand for the graph of \( \mathcal{A} \); it is a subspace of \( H \oplus H \). One can give an abstract description: \( \mathcal{L} \) is a singular balanced rank one perturbation of \( \mathcal{A} \) if \( G(\mathcal{A}) \cap G(\mathcal{L}) \) has codimension in both spaces \( G(\mathcal{A}) \) and \( G(\mathcal{L}) \). We refer to \(3\) for the definition of singular balanced rank \( n \) perturbations of a not necessarily normal operator \( \mathcal{A} \) and for
more discussion. If one takes for $A$ an ordinary differential operator on an interval and changes its defining boundary conditions without changing the formal differential expression, then one obtains this kind of perturbation of $A$.

Let us assume that $A$ has a compact resolvent. Then one can write

$$(5.2) \quad Ax = \sum_{n \in \mathbb{N}} t_n P_n x,$$

where the finite dimensional orthogonal projections $P_n$ are as above: $P_n P_m = 0$ for $m \neq n$, $\sum_n P_n = I$, but now $|t_n| \to \infty$. The domain of $A$ is the set of vectors $x \in H$, for which the above sum converges.

We will say that the singular perturbation $L(a, b, x)$ is degenerate if $\langle P_n a, b \rangle = 0$ for all $n$ and at the same time $x = 0$ (it is consistent with the condition (A) if $a \notin H$). We will say that $L(a, b, x)$ is non-degenerate in all other cases.

It is easy to check that the spectrum of $L$ coincides with its point spectrum and equals to the zero set of the meromorphic function

$$\beta_L(\lambda) = x + \lambda b^* (A - \lambda)^{-1} A^{-1} a = x + \lambda \sum_k \frac{\langle P_n a, b \rangle}{t_n (t_n - \lambda)},$$

(see [3]). So, if the operator $L(a, b, x)$ is degenerate, then each point $\lambda \in \mathbb{C}$ is its eigenvalue. If $L(a, b, x)$ is non-degenerate, then its spectrum is discrete.

Whereas the point 0 was a special point of the spectrum for compact operators, in the present context of unbounded operators, this rôle passes to the point $\infty$. So, given an operator $L$ on $H$ with compact resolvent, we adopt the following definitions. We say that $L$ is complete if its root vectors span $H$. We say that $L$ is nearly complete if its root vectors span a subspace of $H$ of finite codimension.

**Theorem 5.1.** Let $A$ be a normal operator with compact resolvent, given by (5.2), which has lacunary spectrum $\{t_n\}_{n \in \mathbb{N}}$. Suppose that $0 \notin \sigma(A)$. Let $(a, b, x)$ be 1-data for $A$, and let $L = L(a, b, x)$ be the corresponding singular perturbation of $A$, which is non-degenerate. If $L$ is not nearly complete, then the following infinite sequence of “moment” equations holds for all $k \in \mathbb{Z}$, $k \geq -1$:

$$(S_k) \quad \sum_n t_n^k \langle P_n a, b \rangle = \begin{cases} x, & k = -1, \\ 0, & k \geq 0. \end{cases}$$

**Proof.** We reduce this assertion to the case of a compact operator.

First we observe that it is suffices to consider the case when $x \neq 0$. Indeed, by [3] Proposition 1.6], for any $\lambda \in \rho(A)$, one has

$$(5.3) \quad L(A, a, b, x) - \lambda I = L(A - \lambda I, a, b, \beta_L(\lambda)).$$

Notice that the function $\beta_L(\lambda)$ has poles exactly at the points $\{t_n\}$ and its residue at $t_n$ equals $\langle P_n a, b \rangle$. If all these residues are zero for all $n$, then $x \neq 0$, by the non-degenerateness assumption.

If not all numbers $\langle P_n a, b \rangle$ are zero, then $\beta_L$ is non-constant, and therefore $\beta_L(\lambda) \neq 0$ for some $\lambda$. Operator $L - \lambda I$ is nearly complete if and only if $L$ is. A direct calculation shows that moment equations $(S_k)$ hold for $L$ if and only if they hold for $L - \lambda I$. So we may assume that $x \neq 0$, just by replacing $L$ with $L - \lambda I$, where $\lambda \in \rho(A)$ is any number such that $x(\lambda) \neq 0$.

If $x \neq 0$, we can apply [3] Proposition 1.6], which gives that $L = L_0^{-1}$, where $L_0$ is a rank one perturbation of $A^{-1}$:

$$L_0 = A^{-1} - (A^{-1} a x^{-1} (b^* A^{-1}))^{-1}.$$
Notice that $A^{-1}$ is a compact normal operator with lacunary spectrum and that $L$ is nearly complete if and only if $L_0$ is. Now the statement follows by applying Theorem 1.3 to $L_0$. \hfill \Box

6. Proof of Theorem 1.4

6.1. Preliminaries. We begin with the proof of the fact that any separated sequence $T = \{t_n\}$, for which (1.4) does not hold, satisfies $n_T(r) = O(\log^2 r)$ (and thus is sufficiently sparse).

**Lemma 6.1.** Let $R > 0$ and let the interval $[R, 2R]$ contain at least $2M$ points $t_k$. Then there exists $t_n \in [R, 2R]$ such that

$$\prod_{k: R \leq t_k \leq 2R, k \neq n} \left| \frac{t_k - t_n}{t_k} \right| \leq 2^{-M+1}.$$  

**Proof.** Clearly, we can choose $M$ points $t_{n_1}, \ldots, t_{n_M} \in \{t_k\}$ such that $|t_{n_j} - t_{n_i}| \leq R/2$, $j = 2, \ldots, M$. Hence, we have

$$\prod_{k: R \leq t_k \leq 2R, k \neq n} \left| \frac{t_k - t_{n_1}}{t_k} \right| \leq \left( \frac{R}{2} \right)^{M-1} \left( \frac{1}{R} \right)^{M-1} = 2^{-M+1}$$

(we dropped the factors for which $|t_{n_1} - t_k| > R/2$ since they are anyway smaller than 1). \hfill \Box

**Corollary 6.2.** Condition (1.5) implies (1.4).

**Proof.** Assume that (1.4) is not satisfied. Then $|t_n|^N 2^{-M} \gtrsim 1$ for some $N$ which is independent on $t_n$, and so, by Lemma 6.1, $M = O(\log |t_n|)$. Thus, we conclude that, for any $m \in \mathbb{N}$, there is always $O(m)$ points $t_n$ between $2^m$ and $2^{m+1}$, and so $n_T(r) = O(\log^2 r)$. \hfill \Box

6.2. Functional model for rank one perturbations. The proof of the main part of Theorem 1.4 uses a functional model for singular rank one perturbations of unbounded selfadjoint operators with discrete spectrum, which are essentially the algebraic inverses to rank one perturbations of compact selfadjoint operators. This model was introduced in [3]. Let us briefly recall its statement (in the generality we need here). For the details see [3] or [2, Section 4].

We consider the following objects:

- $\{t_n\}$ is, as above, a sequence of real points such that $|t_n| \to \infty$ as $|n| \to \infty$, and $t_n \neq 0$. We can assume without loss of generality that $\{t_n\}$ is an increasing sequence enumerated by $\mathbb{Z}$, $\mathbb{N}$ or $-\mathbb{N}$.
- $A$ is an entire function which is real on $\mathbb{R}$ and has simple zeros exactly at the points $t_n$.
- Two sequences $\{a_n\}$ and $\{b_n\}$, $b_n \neq 0$ for any $n$, and a complex number $\kappa \neq 0$ satisfy

$$\sum_n |a_n|^2 + |b_n|^2 \leq \infty \quad \text{and} \quad \sum_n \frac{a_n \overline{b_n}}{t_n^2} \neq \kappa$$

in the case when $\sum_n |b_n|^2 < \infty$.
- Entire function $G$ is given by

\begin{equation}
G(z) = \kappa + \sum_n \left( \frac{1}{t_n - z} - \frac{1}{t_n} \right) a_n \overline{b_n}.
\end{equation} (6.1)
- $\mathcal{H}(E)$ is a de Branges space (for the definition see [9] or [3] [2]) associated with the spectral measure $\sum_n |b_n|^2 \delta_{t_n}$. The Hermite–Biehler function $E$ is given by $E = A - iB$, where

$$
\frac{B(z)}{A(z)} = \delta + \sum_n \left( \frac{1}{t_n - z} - \frac{1}{t_n} \right) |b_n|^2,
$$

and $\delta$ is an arbitrary real constant.

- The model operator and its domain are defined by the formulas

$$
\mathcal{D}(\mathcal{T}) := \{ F \in \mathcal{H}(E) : \text{there exists } c = c(F) \in \mathbb{C} : zF - cG \in \mathcal{H}(E) \},
$$

$$
\mathcal{T} F := zF - cG, \quad F \in \mathcal{D}(\mathcal{T}).
$$

Now the functional model from [3] Theorem 4.4 combined with [3] Proposition 2.4 (see also [2] Section 4) can be stated as follows.

**Theorem C.** Any singular rank one perturbation $\mathcal{L}$ of the selfadjoint operator $A$ (with simple spectrum and trivial kernel) is unitary equivalent to some model operator $\mathcal{T}$ defined by the above parameters $\{ t_n \} = \{ s_n^{-1} \}$, $A$ and $G$. Conversely, any function $G$ as above appears in the model of some rank one perturbation of $A$.

Moreover, if $\sum_n |a_n|^2 = \infty$ or $\sum_n |a_n|^2 < \infty$ and $\sum_n t_n^{-1}a_n = 0$, then the above function $G$ corresponds to the perturbation $\mathcal{L} = A - x^{-1}(\cdot, b)a$. Thus, condition (6.3) is equivalent to the condition $\sum_n |t_n^{-1}a_nb_n| = \infty$.

### 6.3. A key lemma.

Now our study reduces to a completeness problem for a system of reproducing kernels in de Branges spaces. Namely, in view of the above model, to prove Theorem 1.4 we need to construct an entire function $G$ such that

$$
\frac{G(z)}{A(z)} = x + \sum_n c_n \left( \frac{1}{t_n - z} - \frac{1}{t_n} \right), \quad \sum_n \frac{|c_n|}{t_n} < \infty, \quad \sum_n \frac{|c_n|}{t_n} = \infty,
$$

and $x \neq 0$, but the system $\{ K_\lambda \}_\lambda \in Z_2$ has an infinite dimensional defect in $\mathcal{H}(E)$.

We will construct $G$ of the form $A/S$ where $S$ is a canonical product of order less than one, whose zeros are form a subset of the set $\{ t_n \}$. Then (6.3) will take the form

$$
\frac{G(z)}{A(z)} = \frac{1}{S(z)} = \frac{1}{S(0)} - \sum_{t_n \in Z_2} \frac{1}{S'(t_n)} \left( \frac{1}{t_n - z} - \frac{1}{t_n} \right).
$$

The following lemma plays the key role.

**Lemma 6.3.** In the hypothesis of Theorem 1.4 there exists a canonical product $S$ of order less than one with zeros in the set $\{ t_n \}$ such that

$$
\sum_{t_n \in Z_2} \frac{|S'(t_n)|}{t_n^2} < \infty, \quad \sum_{t_n \in Z_2} \frac{1}{|t_n^2S'(t_n)|} = \infty.
$$

**Proof.** Without loss of generality we assume that $t_n > 0$ and that (1.4) holds. Put $S = \prod_k S_{T_k}$, where

$$
S_{T_k}(z) = \prod_{t_n \in T_k} \left( 1 - \frac{z}{t_n} \right),
$$

and $T_k$ is a set of $t_n$ such that $1/t_n < k$.
and $T_k \subset \{ t_n : n \neq 1, t_n/2 \leq t_n \leq 2t_n \}$, where $n_k$ go rapidly to infinity so that, for any $N > 0$,

$$t_{n_k}^N \prod_{l \neq n_k: t_{n_k}/2 \leq t_l \leq 2t_{n_k}} \frac{|t_{n_k} - t_l|}{t_l} \to 0, \quad k \to \infty. \tag{6.6}$$

We will show that for an appropriate choice of $\{T_k\}$ either $S(z)$ or $(z - t_1)S(z)$ is the desired function.

The sets $T_k$ will be chosen inductively. Suppose that the sets $T_1, \ldots, T_{k-1}$ have been already chosen and put $U_{k-1} = \prod_{j=1}^{k-1} S_{T_j}$. Then, clearly, $|U_{k-1}(z)| \sim q_k |z|^N, \; |\xi| \to \infty$, for some constants $q_k > 0$ and $N_k \in \mathbb{N}$.

Let us first consider the case when $T_k = \{ t_n : t_n/2 \leq t_n \leq 2t_n \}$, assuming that $n_k$ is sufficiently large. Then, by (6.6), the corresponding function $S_{T_k}$ satisfies

$$t_{n_k}^2 |U_{k-1}(t_n)S'_{T_k}(t_n)| = t_{n_k} |U_{k-1}(t_n)| \prod_{l \neq n_k: t_{n_k}/2 \leq t_l \leq 2t_{n_k}} \frac{|t_{n_k} - t_l|}{t_l} < 1.$$ 

In particular,

$$\sum_{t_n \in T_k} \frac{1}{|t_n U_{k-1}(t_n)S'_{T_k}(t_n)|} \geq t_{n_k} \gg 1.$$ 

Now consider another extreme case where $T_k$ consists only of the point $t_{n_k}$, that is, $T_k = \{ t_{n_k} \}$. Then $S_{T_k}(z) = 1 - z/t_{n_k}$, and we have

$$\sum_{t_n \in T_k} \frac{1}{|t_n U_{k-1}(t_n)S'_{T_k}(t_n)|} = \frac{1}{|U_{k-1}(t_{n_k})|} \ll 1.$$ 

Hence, there exists a (not necessarily unique) set $T_k \subset \{ t_n : t_{n_k}/2 \leq t_n \leq 2t_{n_k} \}$ such that $S_{T_k}$ satisfies

$$\sum_{t_n \in T_k} \frac{1}{|t_n U_{k-1}(t_n)S'_{T_k}(t_n)|} > 1 \tag{6.7}$$

and $T_k$ is minimal in the sense that the estimate (6.7) no longer holds if one removes any point from $T_k$.

Now let $t_j$ be any point in $T_k$ and let $\tilde{S}_j = S_{T_k \setminus \{t_j\}}$. Then, by the choice of $T_k$,

$$1 \geq \sum_{t_n \in T_k, t_n \neq t_j} \frac{1}{|t_n U_{k-1}(t_n)S'_{T_k}(t_n)|} = \sum_{t_n \in T_k, t_n \neq t_j} \frac{1}{|t_n U_{k-1}(t_n)S'_{T_k}(t_n)|} \frac{|t_j - t_n|}{|t_j|} \geq \sum_{t_n \in T_k, t_n \neq t_j} \frac{1}{t_{n_k}^2 |U_{k-1}(t_n)S'_{T_k}(t_n)|},$$

where the last inequality follows from the hypothesis $\inf_{n \neq j} |t_n - t_j| > 0$. Since $t_j \in T_k$ was arbitrary, we conclude that, uniformly with respect to $k$,

$$\sum_{t_n \in T_k} \frac{1}{t_{n_k}^2 |U_{k-1}(t_n)S'_{T_k}(t_n)|} \lesssim 1. \tag{6.8}$$

Obviously, by choosing $t_{n_k}$ to grow sufficiently fast, we may achieve that, for the function $S = \prod_k S_{T_k}$ the factors $S_{T_j}$ with $j > k$ almost do not influence the product at the points $t_n \in T_k$ so that $\frac{1}{2} \leq \prod_{j=k+1}^{\infty} |S_{T_j}(t_n)| \leq 2$ for $t_n \in [t_{n_k}/2, 2t_{n_k}]$. Then

$$\frac{1}{2} \sum_{t_n \in T_k} \frac{1}{|t_n S'(t_n)|} \leq \sum_{t_n \in T_k} \frac{1}{|t_n U_{k-1}(t_n)S'_{T_k}(t_n)|} \leq 2 \sum_{t_n \in T_k} \frac{1}{|t_n S'(t_n)|}, \tag{6.9}$$

$$\frac{1}{2} \sum_{t_n \in T_k} \frac{1}{t_{n_k}^2 |S'(t_n)|} \leq \sum_{t_n \in T_k} \frac{1}{t_{n_k}^2 |U_{k-1}(t_n)S'_{T_k}(t_n)|} \leq 2 \sum_{t_n \in T_k} \frac{1}{t_{n_k}^2 |S'(t_n)|}. \tag{6.10}$$
Also, it follows from Lemma \((6.1)\) and from \((6.8)\) that \(\#T_k\), the number of elements in \(T_k\), satisfies \(\#T_k \lesssim N_k \ln t_{n_k} + \ln q_k\). Since \(N_k\) and \(q_k\) do not depend on the choice of \(t_{n_k}\), the function \(S\) will be of zero order if \(t_{n_k}\) grow sufficiently fast.

By the construction of \(S^\prime T_k\) (namely, from \((6.7)\) and \((6.9)\)) we clearly have

\[\sum_{t_n \in Z_S} \frac{1}{|t_n S'(t_n)|} = \infty.\]

If, at the same time,

\[\sum_{t_n \in Z_S} \frac{1}{t_n^2 |S'(t_n)|} < \infty,\]

then our construction is completed. If the latter sum is also infinite, then put \(\tilde{S} = (z - t_1)S = (z - t_1)\prod_{k} S_T_k\). Then, clearly, \(|\tilde{S}'(t_n)| \asymp |t_n S'(t_n)|, t_n \in Z_S\), and so, by \((6.10)\) and \((6.8)\), we have

\[\sum_{t_n \in Z_S} \frac{1}{t_n^2 |S'(t_n)|} \lesssim \sum_{k} \frac{1}{t_{n_k}} \sum_{t_n \in T_k} \frac{1}{t_n^2 |U_{k-1}(t_n)|S_{T_k}(t_n)} \lesssim \sum_{k} \frac{1}{t_{n_k}} < \infty.\]

Thus, \(\tilde{S}\) has the required properties. \(\square\)

6.4. End of the proof of Theorem \((1.4)\). Let \(S\) be the entire function constructed in Lemma \((6.3)\) It remains to prove \((6.4)\). The series in the right-hand side of \((6.4)\) converges absolutely by the conditions on \(S\). The proof of \((6.4)\) follows by the standard interpolation series argument. Note that

\[H(z) = \frac{1}{S(z)} - \frac{1}{S(0)} - \frac{1}{S'(0)} \sum_{t_n \in Z_S} \left( \frac{1}{z - t_n} + \frac{1}{t_n} \right)\]

is an entire function (the poles disappear). Since \(S\) is of order less than one with real zeros, we conclude that \(1/S\) is of Smirnov class in the upper and in the lower half-planes, as well as the regularized Cauchy transform in the right-hand side of \((6.4)\). Hence, by the classical theorem of M.G. Krein (see, e.g., \([16\), Part II, Chapter 1\]), \(H\) is an entire function of zero exponential type. Note also that \(|H(iy)| = o(|y|), |y| \to \infty\), whence \(H\) is a constant. Since \(H(0) = 0\), we finally get that \(H \equiv 0\).

Thus, \(G\) satisfies \((6.3)\) and so it corresponds to a certain rank one perturbation \(L\) of \(A\), generated by some \(\{a_n\}\) and \(\{b_n\}\). Moreover, since \(\sum_n |a_n b_n t_n^{-1}| = \infty\), we conclude that \(L^*\) also is a well-defined singular rank one perturbation of \(A\) and the system of its eigenvectors is unitary equivalent to the system of reproducing kernels \(\{K_\lambda\}_{\lambda \in Z_G}\) in \(H(E)\). It remains to see that this system is not complete in \(H(E)\). However, it is a basic fact of de Branges theory that \(\{K_\lambda\}_{\lambda \in Z_A}\) is an orthogonal basis \(H(E)\) (see \([9\), Theorem 22\]). Hence, \(\{K_\lambda\}_{\lambda \in Z_A \setminus Z_S}\) is incomplete with infinite defect. Theorem \((1.4)\) is proved. \(\square\)

Remark. One can rewrite \((1.4)\) in equivalent ways. Recall that the Krein class consists of entire functions \(F\) which are real on \(\mathbb{R}\), have only simple and real zeros and for some positive integer \(k\) and some polynomial \(P\) of degree at most \(k\) satisfy the following absolutely convergent expansion:

\[\frac{1}{F(z)} = P(z) + \sum_n \frac{1}{F'(t_n)} \left( \frac{1}{z - t_n} + \frac{1}{t_n} + \cdots + \frac{z^{k-1}}{t_n^k} \right),\]

where \(s_n\) are zeros of \(F\).

It is not difficult to see that condition \((1.4)\) is equivalent to any of the following properties \((i)\) and \((ii)\):

\[\sum_{n} \frac{1}{s_n} = \infty,\]

\[\sum_{n} \frac{1}{s_n^2} < \infty,\]

\[\sum_{n} \frac{1}{s_n^3} < \infty,\]

\[\sum_{n} \frac{1}{s_n^{k+1}} < \infty,\]

\[\sum_{n} \frac{1}{s_n^k} < \infty,\]

\[\sum_{n} \frac{1}{s_n^{k+2}} < \infty,\]
(i) There exists an entire function $S$ of order less than one, whose zeros are simple and lie in the set $\{t_n\}$, such that for any $N$ we have $\liminf_{t_n \to \infty, t_n \in \mathbb{Z}} |t_n^N S(t_n)| = 0$ (the lower limit is taken here over those $t_n$ which are zeros of $S$).

(ii) There exists a subsequence of $\{t_n\}$, which is not the zero set of a function in the Krein class.

For instance, to prove that (1.1) implies (i), it suffices to put

$$S(z) = \prod_k \prod_{m: t_{nk}/2 \leq m \leq t_{nk}} \left(1 - \frac{z}{t_m}\right),$$

where the sequence $\{n_k\}$ grows fast enough (by the above proof of Lemma 6.3, this product defines a zero order entire function). We leave the details to the reader.

References

[1] J. Adduci, B. Mityagin, Root system of a perturbation of a selfadjoint operator with discrete spectrum. *Integral Equations Operator Theory* 73 (2012), no. 2, 153–175.

[2] A. Baranov, Yu. Belov, A. Borichev, D. Yakubovich, Recent developments in spectral synthesis for exponential systems and for non-self-adjoint operators, *Recent Trends in Analysis* (Proceedings of the Conference in Honor of Nikolai Nikolski, Bordeaux, 2011), Theta Foundation, Bucharest, 2013, pp. 17–34.

[3] A. Baranov, D. Yakubovich, Completeness and spectral synthesis of nonselfadjoint one-dimensional perturbations of selfadjoint operators. [arXiv:1212.5965](http://arxiv.org/abs/1212.5965)

[4] A. Baranov, D. Yakubovich, One-dimensional perturbations of unbounded selfadjoint operators with empty spectrum, *J. Math. Anal. Appl.* 424 (2015), no. 2, 1404–1424.

[5] A. Baranov, Yu. Belov, Systems of reproducing kernels and their biorthogonal: completeness or incompleteness? *Int. Math. Res. Notices* 2011, 22 (2011), 5076–5108.

[6] A. Baranov, Yu. Belov, A. Borichev, Hereditary completeness for systems of exponentials and reproducing kernels, *Adv. Mathematics* 235, no. 1 (2013), 525–554.

[7] A. Baranov, Yu. Belov, A. Borichev, Spectral synthesis in de Branges spaces, *Geom. Funct. Anal.* (GAFA) 25 (2015), 2, 417–452.

[8] Yu. Belov, T.Y. Mengestie, K. Seip, Discrete Hilbert transforms on sparse sequences, *Proc. London Math. Soc.* 103, no. 1 (2011), 73–105.

[9] L. de Branges, *Hilbert Spaces of Entire Functions*, Prentice–Hall, Englewood Cliffs, 1968.

[10] Boas, *Entire functions*. Academic Press Inc., New York, 1954.

[11] Q. Fang, J. Xia, Invariant subspaces for certain finite-rank perturbations of diagonal operators, *J. Funct. Anal.* 263 (2012), no. 5, 1356–1377.

[12] N. Dunford, J.T. Schwartz, *Linear Operators, Vol. 2: Spectral Theory*, Interscience, New York, 1963.

[13] I. Gohberg, M. Krein, *Introduction to the Theory of Linear Nonsselfadjoint Operators*, Amer. Math. Soc., Providence, R.I., 1969.

[14] C. Foias, I.B. Jung, E. Ko, C. Pearcy, Spectral decomposability of rank-one perturbations of normal operators, *J. Math. Anal. Appl.* 375 (2011), 602–609.

[15] I. Gohberg, M. Krein, *Theory and Applications of Volterra Operators in Hilbert Space*, Amer. Math. Soc., Providence, R.I., 1970.

[16] V. Havin, B. Jöricke, *The Uncertainty Principle in Harmonic Analysis*, Springer-Verlag, Berlin, 1994.

[17] E. Ionascu, Rank-one perturbations of diagonal operators, *Integral Equations Operator Theory* 39 (2001), no. 4, 421–440.

[18] M. V. Keldyš, On the characteristic values and characteristic functions of certain classes of non-self-adjoint equations. (Russian) *Doklady Akad. Nauk SSSR (N.S.)* 77, (1951), 11–14, Engl. transl. in [20].

[19] M. V. Keldyš, On the completeness of the eigenfunctions of some classes of non-selfadjoint linear operators, *Russ. Math. Surv.* 26 (1971), 4, 15–41.

[20] A. S. Markus, *Introduction to the spectral theory of polynomial operator pencils*. AMS Transl. Math. Monographs, 71 (1988).

[21] V.I. Macaev, On a class of completely continuous operators, *Dokl. Akad. Nauk SSSR* 139 (1961), 3, 548–551; English transl. in *Soviet Math. Dokl.* 2 (1961), 972–975.
[22] V.I. Macaev, Some theorems on completeness of root subspaces of completely continuous operators, *Dokl. Akad. Nauk SSSR* **155** (1964), 273–276. English transl. in *Soviet Math. Dokl.* **5** (1964), 396–399.

[23] V.I. Macaev, E.Z. Mogul’ski, Certain criteria for the multiple completeness of the system of eigen- and associated vectors of polynomial operator pencils, *Teor. Funkcii, Funkcional. Anal. i Prilozhen.* 13 (1971), 3–45.

[24] V.I. Macaev, E.Z. Mogul’ski, The completeness of weak perturbation of the selfadjoint operators, *Zap. Nauchn. Sem. LOMI* **56** (1976), 90–103; English transl. in *J. Sov. Math.* **14** (1980), 2, 1091–1103.

[25] G. Pólya, Bemerkungen über unendliche Folgen und ganze Funktionen. (German) *Math. Ann.* **88** (1923), no. 3–4, 169–183.

[26] G.V. Radzievskii, The problem of the completeness of root vectors in the spectral theory of operator-valued functions, *Russian Math. Surveys* **37** (1982), 2, 91–164.

[27] A. A. Shkalikov, Theorems of Tauberian type on the distribution of zeros of holomorphic functions. *Mat. Sb. (N.S.)* 123(165) (1984), no. 3, 317–347; English transl. in *Math. USSR Sb.* 51, no. 2 (1985), no. 3, 315–344.

[28] A.A. Shkalikov, On the basis property of root vectors of a perturbed self-adjoint operator, *Proc. Steklov Inst. Math.* 269 (2010), 284–298.

[29] C. Wyss, Riesz bases for p-subordinate perturbations of normal operators, *J. Funct. Anal.* **258** (2010), 1, 208–240.

DEPARTMENT OF MATHEMATICS AND MECHANICS, SAINT PETERSBURG STATE UNIVERSITY, 28, UNIVERSITETSKI PR., ST. PETERSBURG, 198504, RUSSIA

AND

NATIONAL RESEARCH UNIVERSITY HIGHER SCHOOL OF ECONOMICS, ST. PETERSBURG, RUSSIA

E-mail address: anton.d.baranov@gmail.com

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD AUTÓNOMA DE MADRID, CANTOBLANCO 28049 (MADRID) SPAIN

AND

INSTITUTO DE CIENCIAS MATEMÁTICAS (CSIC - UAM - UC3M - UCM)

E-mail address: dmitry.yakubovich@uam.es