Numerical computation of one-photon mazer resonances for arbitrary field modes

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We present a novel approach for solving numerically one-dimensional scattering problems and apply it for computing the emission probability of an ultracold atom interacting with an arbitrary field mode of a high-$Q$ cavity. Our method is efficient, stable and succeeds when other numerical integration methods fail. It also replaces and improves advantageously the WKB approximation. The cases of sinusoidal, sech$^2$ and Gaussian field modes are studied and compared. Divergences with previous works, where WKB was used, are found.

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\section{I. INTRODUCTION}

Some recent works have been devoted to the interaction of ultracold atoms with microwave cavities \cite{1,2,3}. These studies treated the interaction between an incident atom in an excited state and a cavity field containing $n$ photons, taking the quantum mechanical CM motion of the atom into account. This interaction leads to a new kind of induced emission intimately associated with the quantization of the CM motion, named the mazer action \cite{4}.

After interaction with the cavity, the atom can be found transmitted in the excited state or in the lower state, or reflected as well. It has been shown \cite{4} that these event probability amplitudes (denoted respectively $T_{a,n}$, $T_{b,n+1}$, $R_{a,n}$ and $R_{b,n+1}$) are given by

\begin{equation}
T_{a,n} = \frac{1}{2}(t_{+n} + t_{-n}) \quad T_{b,n+1} = \frac{1}{2}(t_{+n} - t_{-n}) \quad R_{a,n} = \frac{1}{2}(r_{+n} + r_{-n}) \quad R_{b,n+1} = \frac{1}{2}(r_{+n} - r_{-n})
\end{equation}

where $t_{\pm n}$ and $r_{\pm n}$ are, respectively, the transmission and reflection amplitudes of the elementary scattering process of the atom incident upon the potential

\begin{equation}
V_n^\pm(x) = \pm\hbar g\sqrt{n+1} u(x)
\end{equation}

where $x$ is the atom traveling direction, $g$ is the atom – field coupling strength and $u(x)$ is the cavity field mode function.

The induced emission probability of the atom interacting with the cavity field is then given by

\begin{equation}
P_{em} = |T_{b,n+1}|^2 + |R_{b,n+1}|^2
\end{equation}

All these probabilities are strongly dependent on the cavity mode profile. Their calculation needs to solve the one-dimensional time-independent Schrödinger equation. Löffler et al. \cite{5} have calculated them as a function of the interaction length $\kappa_n L$ ($L$ is the cavity length and $\kappa_n = \kappa\sqrt{n+1}$ with $\kappa = \sqrt{2mg/\hbar}$, and $m$ the atomic mass) for various cavity mode functions: mesa, sech$^2$ and sinusoidal modes. For the two first modes, analytical results of the probability $P_{em}(\kappa_n L)$ have been given. For the sinusoidal modes, detailed Wentzel-Kramers-Brillouin solutions have been presented. Nevertheless, Retamal et al. \cite{6} have shown that the WKB approximation may lead to inaccurate predictions for actual interaction and cavity parameters.

The consideration of actual interaction and cavity parameters is a difficult numerical task as they correspond to very large values of $\kappa_n L$. When no analytical solutions of the Schrödinger equation are available, usual numerical integration methods do not converge rapidly or do not converge at all.

In this paper, we present a novel approach for solving the one-dimensional time-independent Schrödinger equation. Our method is described in Sec. II. It may be applied for any potential function. It is efficient, stable and succeeds when other numerical integration methods fail. It may replace and improve advantageously the WKB approach. Details of the implementation of our method and validity tests are presented in Sec. IV.

We apply our method for calculating some new $P_{em}(\kappa_n L)$ curves. In particular, the Gaussian potential has been considered, thinking in open cavities in the microwave or optical field regime. These results are presented in Sec. V.

\section{II. DESCRIPTION OF THE METHOD}

The one-dimensional time-independent Schrödinger equation can be written in atomic units in the form

\begin{equation}
\left( \frac{1}{2} \frac{d^2}{dx^2} + (E - V(x)) \right) \phi(x) = 0
\end{equation}

We assume here that the potential $V(x)$ has non zero values only in a given region of the $x$ axis, say
\[ [x_a, x_b]. \] This region is divided into \( J \) grid points, denoted \( x_1, \ldots, x_J \) such that \( x_a = x_1 < x_2 < \cdots < x_J < x_b = x_J \). Let us note \( I_j (0 < j < J) \) the region \([x_j, x_{j+1}], \)
\( I_0 = [−\infty, x_1], \) and \( I_J = [x_J, +\infty]. \)

The potential \( V(x) \) is approximated in each region \( I_j \) by a straight line connecting \( V(x_j) \) and \( V(x_{j+1}) \). The approximated potential is noted \( V_{\text{approx}}(x) \). Then Schrödinger equation takes the simple form in \( I_j (0 \leq j \leq J): \)

\[
\left( \frac{d^2}{dx^2} + (a_j + b_j x) \right) \phi_j(x) = 0 \quad (5)
\]
with \( a_0 = a_J = 2E, b_0 = b_J = 0, \) and for \( 0 < j < J \)
\[
\begin{align*}
a_j &= 2 \left( E - \frac{x_{j+1} - x_j}{x_{j+1} - x_j} V_j \right) \\
b_j &= -2 \frac{V_{j+1} - V_j}{x_{j+1} - x_j} \quad \text{with} \quad V_i = V(x_i), i = j, j + 1 \quad (6)
\end{align*}
\]

The most general solutions of Eq. (3) are given by

\[
\phi_j(x) = C_j f_j^+(x) + D_j f_j^-(x) \quad (7)
\]
where \( C_j \) and \( D_j \) are two complex constants and \( f_j^+(x) \) and \( f_j^-(x) \) 2 functions depending on the \( a_j \) and \( b_j \) values. These functions are given in Table II.

As shown in Table II, the functions \( f_j^+(x) \) and \( f_j^-(x) \) depend on the sign of \( a_j + b_j x \). To avoid a change of this sign in a given region \( I_j \), the set \( x_1, \ldots, x_J \) must contain the roots of the equation \( V(x) = E \). Thus, the condition \( a_j + b_j x > 0 \) (resp. \( a_j + b_j x < 0 \)) is equivalent to that \( E > V(x) \) (resp. \( E < V(x) \)) for \( x \in I_j \).

The complex constants \( C_j \) and \( D_j \) in Eq. (5) are determined by use of the wavefunction asymptotic behaviour knowledge \( (C_0, D_0) \) or \( (C_j, D_j) \) (supposed to be known) and of the set of conditions imposing the continuity of the wavefunction and its derivative along the x axis:

\[
\begin{align*}
\phi_j(x_J) &= \phi_{j-1}(x_j) \\
\phi_j'(x_J) &= \phi_{j-1}'(x_j) \quad (0 < j \leq J). \quad (8)
\end{align*}
\]

If \( (C_0, D_0) \) are given, Eq. (8) may be written as

\[
\begin{pmatrix} C_j \\ D_j \end{pmatrix} = A_j(x_J) \begin{pmatrix} C_{j-1} \\ D_{j-1} \end{pmatrix} \quad (for \quad 0 < j \leq J), \quad (9)
\]
with

\[
A_j = \frac{1}{f_j g_{j-1} - f_{j-1} g_j} \times \begin{pmatrix} f_j g_{j-1} - f_{j-1} g_j & f_{j-1} g_{j-1} - f_j g_j \\ f_j g_{j-1} - f_{j-1} g_j & f_{j-1} g_{j-1} - f_j g_j \end{pmatrix} \quad (10)
\]

and

\[
g_j^\pm(x) = \frac{df_j^\pm}{dx}, i = j, j - 1. \quad (11)
\]

If \( (C_j, D_j) \) are known, Eq. (8) should be written

\[
\begin{pmatrix} C_{j-1} \\ D_{j-1} \end{pmatrix} = B_j(x_J) \begin{pmatrix} C_j \\ D_j \end{pmatrix} \quad (for \quad J \geq j > 0), \quad (12)
\]
with

\[
B_j = \frac{1}{f_j g_{j-1} - f_{j-1} g_j} \times \begin{pmatrix} f_j g_{j-1} - f_{j-1} g_j & f_{j-1} g_{j-1} - f_j g_j \\ f_j g_{j-1} - f_{j-1} g_j & f_{j-1} g_{j-1} - f_j g_j \end{pmatrix} \quad (13)
\]

The knowledge of \( \phi_j(x) \) in each region \( I_j \) defines an approximated wavefunction of the Schrödinger equation (\( I \)). The grid point number \( J \) fixes the accuracy of the method. The higher \( J \) is, the more accurate is the approximated potential \( V_{\text{approx}}(x) \) and the better is the approximated wavefunction. The actual number \( J \) to be considered for a given accuracy depends on various parameters (potential form, energy \( E \), size of the region \([x_a, x_b]) \). We show in Sec. III and IV that a few hundreds are typical numbers.

The wavefunctions yielded by our method are not divergent at the classical turning points (where \( E = V(x) \)), on the contrary of the WKB method. Although some Bessel functions of Table II are divergent at these points (as \( z_j (x) \equiv a_j + b_j x = 0 \)), all the functions \( f_j^\pm(x) \) and their derivative \( g_j^\pm(x) \) admit finite limits there (see Table II).

The transmission and reflection complex amplitudes (denoted respectively \( t \) and \( r \)) of a particle incident upon a potential \( V(x) \) may be calculated as well (and consequently the induced emission probability (3)). If we consider the outgoing wavefunction \( \phi_J(x) = e^{ikx} \) with \( k = \sqrt{2E} \), we can calculate the corresponding incoming wavefunction \( \phi_0(x) = C_0 \cos(kx) + D_0 \sin(kx) \) by use of relations (12). We then have

\[
t = \frac{2}{C_0 - iD_0}, \quad r = \frac{C_0 + iD_0}{C_0 - iD_0} \quad (14)
\]

**III. METHOD IMPLEMENTATION**

We have implemented the method described above on a PC Pentium based computer. Bessel functions of Table II have been coded using algorithms given by Zhang and Jin [3]. The evaluation of the Mathematica® Bessel functions has been discarded due to poor performances (thousand times slower than direct computation). For usual cavity parameters, \( C_0 \) and \( D_0 \) coefficients of Eq. (14) can rapidly become very large. To avoid computation overflow errors, we have adopted a custom number representation. Our internal range of values was \( 9.9 \times 10^{-9} \text{E} \pm (\sim 9 \times 10^8) \) with 15 significant digits.

As a first test of our method, we checked its ability to converge when the grid point number \( J \) increases. We have calculated the induced emission probability \( P_{\text{em}} \) of an atom interacting with a sech\(^2 \) mode profile cavity
potential so that the area under it remained constant
with the number
FWHM of the potential function).

In order to improve the convergence of our method with the number J, we renormalized the approximated potential so that the area under it remained constant whatever value J had. Instead of using \( V_i = V(x_i) \) in Eq. (11), we have used \( V_i = a V(x_i) \) with \( a \) equal to the ratio of the area under \( V(x) \) over the area under \( V_{\text{approx}}(x) \).

The study of the sech\(^2\) mode profile is a good test for a numerical method as there exists an analytical expression for \( P_{em}(\kappa_n L) \). In this case we have (see L"offler et al. [3])

\[
\begin{align*}
\tau_{+}^{n} &= \frac{\Gamma \left[ 1/2 - i(kL + \xi_{+}^{2}) \right]}{\Gamma \left[ -i kL \right] \cdot \Gamma \left[ 1 - i kL \right]} \cdot \frac{\Gamma \left[ 1/2 - i(kL - \xi_{+}^{2}) \right]}{\Gamma \left[ -i kL \right] \cdot \Gamma \left[ 1 - i kL \right]} \\
\tau_{-}^{n} &= \frac{\Gamma \left[ i kL \right]}{\Gamma \left[ 1/2 + i \xi_{-}^{2} \right]} \cdot \frac{\Gamma \left[ 1 - i kL \right]}{\Gamma \left[ 1/2 - i \xi_{-}^{2} \right]} \cdot \tau_{+}^{n}
\end{align*}
\]

with \( \xi_{\pm}^{2} = \pm(\kappa_n L)^2 - 1/4 \).

Fig. 2 shows two \( P_{em}(\kappa_n L) \) curves calculated for \( k/\kappa_n = 0.01 \) and \( k/\kappa_n = 0.1 \) in the case of the sech\(^2\) mode profile. On this figure, the solid lines represent the analytical results deduced from formulas (11), (12) and (13) while the dotted ones represent those obtained using our method (the grid point number \( J \) was fixed to 200). The agreement between these results is very good.

IV. NEW RESULTS

A. Fundamental sinusoidal mode

The fundamental sinusoidal mode profile

\[
u(x) = \begin{cases} 
\sin(\pi x/L) & \text{for } 0 < x < L \\
0 & \text{elsewhere}
\end{cases}
\]

has been studied by L"offler et al. [3] and Retamal et al. [3]. Their conclusions are not in agreement with regard to the behaviour of the induced emission probability for ultracold atoms (\( k/\kappa_n = 0.01 \)) interacting with large cavities (\( \kappa_n L \) of the order of \( 10^5 \)). Such cavity parameters are of a great interest as they correspond to realistic values for Rydberg atoms interacting with microwave cavities in recent experiments performed by the Ecole Normale Supérieure Group [3] (see discussion about the orders of magnitude in [3]). Retamal et al. [3] have predicted well resolved resonances in the curve \( P_{em}(\kappa_n L) \) for the parameters cited above, whereas the curve predicted on the basis of the results of L"offler et al. [3] looks very different.

We have calculated the \( P_{em}(\kappa_n L) \) curve for an atom interacting with this mode profile by use of our method. Fig. 3 presents this curve for \( \kappa_n L \) comprised between 100000 and 100010 (\( k/\kappa_n = 0.01 \) and \( J = 100 \)). Our curve shows the same well resolved resonances as those predicted by Retamal et al. [3]. So we confirm their predictions that the WKB approximation cannot be used when considering ultracold atoms with \( k/\kappa_n = 0.01 \) and realistic interaction lengths of the order of \( 10^5 \). Our calculations confirm also that these resonances have got smeared at \( \kappa_n L = 10^5 \) if we consider warmer atoms characterized by \( k/\kappa_n = 0.1 \). For this last value of \( k/\kappa_n \) the WKB approximation still holds and \( P_{em} \) is then invariably equal to 1/2.

B. First excited sinusoidal mode

The first excited sinusoidal mode profile

\[
u(x) = \begin{cases} 
\sin(2\pi x/L) & \text{for } 0 < x < L \\
0 & \text{elsewhere}
\end{cases}
\]

has been studied by L"offler et al. [3]. These authors have derived an expression for \( P_{em} \) on the basis of a detailed WKB calculation. They have shown that, for large interaction lengths (without being very explicit about the word “large”), the behaviour of the induced emission probability is described by \( P_{em}(\kappa_n L) = \sin^2(\Delta_n) \) with

\[
\Delta_n = \kappa_n L \cdot \frac{1}{\pi} \int_0^{\pi/2} \sqrt{(k/\kappa_n)^2 + \cos(x)} dx
\]

which can be written \( \Delta_n = \kappa_n L \cdot \text{Const.} \) if \( k/\kappa_n \) is a fixed ratio.

In Fig. 2, we present two curves of \( P_{em}(\kappa_n L) \) calculated for \( \kappa_n L \) comprised between 100000 and 100020 (for \( k/\kappa_n = 0.01 \) and \( k/\kappa_n = 0.1 \)). The first curve does not exhibit a square sine dependence over the interaction length, whereas the second one does. This indicates that \( \kappa_n L = 10^5 \) cannot be considered as “large” in the sense of L"offler et al. [3] for \( k/\kappa_n = 0.01 \). For \( k/\kappa_n = 0.1 \), \( \Delta_n = 2\pi \kappa_n L / T \), with the period \( T \) equals to \( \sim 16.3 \) (according to Eq. (18)). This period is well reproduced by our calculations (see curve (b) on Fig. 4).

It is a remarkable result of L"offler et al. [3] the possibility of building state-changing and state-preserving mirrors for atoms by modifying the length of the cavity using
this first excited sinusoidal mode. From Fig. 4 we conclude that resonances in $P_{em}$ for the first excited mode are even narrower than those predicted by a square sine function $\sin^2(\Delta_n)$ . This is a very convenient result as we are considering actual interaction and cavity parameters.

C. Gaussian mode

Up to-date, the Gaussian cavity mode profile has not been studied exactly in the quantum theory of the mazer. Löffler et al. [3] have argued that the sech$^2$ mode profile could be used as a good approximation of the Gaussian one. To verify this assumption, we have calculated various $P_{em}(\kappa_n L)$ curves for the profile

$$u(x) = e^{-\frac{x^2}{\sigma^2}}$$

The parameter $\sigma$ was fixed to $\sqrt{2/\pi L}$ in order to adopt the same normalization factor for the two profiles (identical area under the modes).

Fig. 5 shows our results for $k/\kappa_n = 0.1$ in the range $\kappa_n L = 0$ to $\kappa_n L = 20$. Qualitatively both profiles exhibit the same behaviour: the resonances in the curves get smeared with increasing values of $\kappa_n L$. But this phenomenon is not so marked in the case of the Gaussian profile. Resonances still exist for longer interaction lengths. This is not a surprising result as the Gaussian profile is growing more abruptly than the sech$^2$ one. Thus it is in some sense “closer” to the mesa mode, which exhibits resonances at infinity.

We have also considered the case $k/\kappa_n = 0.01$. Our calculations have shown that 90% damped oscillations are still present in the $P_{em}(\kappa_n L)$ curve for interaction lengths approximately 3 times larger in comparison with the sech$^2$ mode case. For these calculations, the region $[x_n, x_b]$ was limited to 16 times the cavity length $L$ and the grid point number $J$ was fixed to 300.

As it was pointed out in Ref. [3] the mazer effect is not restricted to the microwave domain and it might be tested more efficiently in the optical domain. Note that typical optical cavities have a Gaussian mode profile and it is possible to consider large coupling constants, fact that will help for testing the mazer effect. We want to call attention to the fact that Hood et al., in Ref. [3], presented the first experimental result for which the interaction energy $\hbar g$ is greater than the atomic kinetic energy ($k/\kappa_n \ll 1$).

V. SUMMARY

We have developed a new method for solving one-dimensional scattering problems that may be applied advantageously instead of the WKB approach. This has enabled us to calculate efficiently the induced emission probability $P_{em}$ of an atom interacting with a high-$Q$ cavity for various mode profiles. Two sinusoidal modes have been considered. For these cases, we have been able to assert that the WKB approximation cannot be used in the computation of $P_{em}$ for ultracold atoms ($k/\kappa_n = 0.01$) interacting with actual cavities characterized by $\kappa_n L$ of the order of $10^5$. Significant and convenient different physical predictions are found, if we compare our results with previous works [3].

The Gaussian mode profile has also been considered and we have shown that, although it exhibits a similar behaviour in comparison with the sech$^2$ mode profile, the resonances in the $P_{em}(\kappa_n L)$ curves exist for significantly larger values of $\kappa_n L$. The Gaussian mode is relevant when considering open cavities in the microwave or optical domains.

The presented numerical method will be helpful for computing the induced emission probability in the case of the recently studied two-photon mazer [3] when considering field mode profiles different from the mesa function. This and other applications of this numerical method will be presented elsewhere.

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[1] M. O. Scully, G. M. Meyer, and H. Walther, Phys. Rev. Lett. 76, 4144 (1996).
[2] G. M. Meyer, M. O. Scully, and H. Walther, Phys. Rev. A 56, 4142 (1997).
[3] M. Löeffler, G. M. Meyer, M. Schröder, M. O. Scully, and H. Walther, Phys. Rev. A 56, 4153 (1997).
[4] M. Schröder, K. Vogel, W. P. Schleich, M. O. Scully, and H. Walther, Phys. Rev. A 56, 4164 (1997).
[5] J. C. Retamal, E. Solano, and N. Zagury, Optics Comm. 154, 28 (1998).
[6] Z.-M. Zhang, Z.-Y. Lu, and L.-S. He, Phys. Rev. A 59, 808 (1999).
[7] S. Zhang and J. Jin, Computation of Special Functions (Wiley-InterScience, 605 Third Avenue, New York, NY 10158-0012, 1996).
[8] M. Brune, E. Hagley, J. Dreyer, X. Maître, A. Maali, C. Wunderlich, J. Raimond, and S. Haroche, Phys. Rev. Lett. 77, 4887 (1996).
[9] C. J. Hood, M. S. Chapman, T. W. Lynn, and H. J. Kimber, Phys. Rev. Lett. 80, 4157 (1998).
TABLE I. $f_+^j(x)$ and $f_-^j(x)$ functions defining the general solutions [2]. In this table, $J_{1/3}(x)$ and $Y_{1/3}(x)$ denote respectively the first and second kind Bessel functions of order 1/3, $I_{1/3}(x)$ and $K_{1/3}(x)$ the first and second kind modified Bessel functions of order 1/3, $k_j = \sqrt{2(E - V_j)}$, $\rho_j = \sqrt{2(V_j - E)}$, and $z_j(x) = a_j + b_j x$.

| $b_j = 0$ | $a_j = 0$ | $f_+^j(x) = 1$ | $f_-^j(x) = x$ |
|-----------|-----------|----------------|----------------|
| $a_j > 0$ | $f_+^j(x) = \cos(k_j x)$ | $f_-^j(x) = \sin(k_j x)$ |
| $a_j < 0$ | $f_+^j(x) = e^{-\rho_j x}$ | $f_-^j(x) = e^{\rho_j x}$ |

| $b_j \neq 0$ | $z_j(x) > 0$ | $f_+^j(x) = \sqrt{|z_j(x)|} J_{\frac{1}{3}} \left( \frac{2}{3|b_j|} |z_j(x)|^{\frac{1}{2}} \right)$ | $f_-^j(x) = \sqrt{|z_j(x)|} Y_{\frac{1}{3}} \left( \frac{2}{3|b_j|} |z_j(x)|^{\frac{1}{2}} \right)$ |
|-----------|-----------|----------------|----------------|
| $z_j(x) < 0$ | $f_+^j(x) = \sqrt{|z_j(x)|} K_{\frac{1}{3}} \left( \frac{2}{3|b_j|} |z_j(x)|^{\frac{1}{2}} \right)$ | $f_-^j(x) = \sqrt{|z_j(x)|} I_{\frac{1}{3}} \left( \frac{2}{3|b_j|} |z_j(x)|^{\frac{1}{2}} \right)$ |

TABLE II. Behaviour of the $f_\pm^j(x)$ functions and their derivative $g_\pm^j(x)$ at the classical turning points when $b_j \neq 0$.

| $E > V(x)$ region | $E < V(x)$ region |
|-------------------|-------------------|
| $f_+^j(x) \to 0$ | $f_+^j(x) \to 0$ |
| $f_-^j(x) \to -\frac{1}{\pi} |3b_j|^{1/3} \Gamma(1/3)$ | $f_-^j(x) \to \frac{1}{2} |3b_j|^{1/3} \Gamma(1/3)$ |
| $g_+^j(x) \to \frac{b_j}{|3b_j|^{1/3} \Gamma(4/3)}$ | $g_+^j(x) \to -\frac{b_j}{|3b_j|^{1/3} \Gamma(4/3)}$ |
| $g_-^j(x) \to \frac{b_j}{\sqrt{3} |3b_j|^{1/3} \Gamma(4/3)}$ | $g_-^j(x) \to \frac{\pi b_j}{\sqrt{3} |3b_j|^{1/3} \Gamma(4/3)}$ |
FIG. 1. The induced emission probability $P_{em}$ as a function of the grid point number $J$ for $\kappa_n L = 10$ ($k/\kappa_n = 0.01$, sech² cavity mode profile).

FIG. 2. The induced emission probability $P_{em}$ as a function of the interaction length $\kappa_n L$ for $k/\kappa_n = 0.01$ (a) and $k/\kappa_n = 0.1$ (b). The cavity mode is the sech² profile. Solid lines were obtained by analytical calculations, dot ones by our method ($J = 200$).

FIG. 3. The induced emission probability $P_{em}$ as a function of the interaction length $\kappa_n L$ for $k/\kappa_n = 0.01$ ($J = 100$, fundamental sinusoidal cavity mode profile).

FIG. 4. The induced emission probability $P_{em}$ as a function of the interaction length $\kappa_n L$ for $k/\kappa_n = 0.01$ (a) and $k/\kappa_n = 0.1$ (b) ($J = 200$, first excited sinusoidal cavity mode profile).

FIG. 5. The induced emission probability $P_{em}$ as a function of the interaction length $\kappa_n L$ for the sech² mode profile (a) and the Gaussian profile (b) ($k/\kappa_n = 0.1$, $J = 300$).