INVARIANT FUNCTIONALS ON SPEH REPRESENTATIONS

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Abstract. We study $\text{Sp}_{2n}(\mathbb{R})$-invariant functionals on the spaces of smooth vectors in Speh representations of $\text{GL}_{2n}(\mathbb{R})$.

For even $n$ we give expressions for such invariant functionals using an explicit realization of the space of smooth vectors in the Speh representations. Furthermore, we show that the functional we construct is, up to a constant, the unique functional on the Speh representation which is invariant under the Siegel parabolic subgroup of $\text{Sp}_{2n}(\mathbb{R})$. For odd $n$ we show that the Speh representations do not admit an invariant functional with respect to the subgroup $U_n$ of $\text{Sp}_{2n}(\mathbb{R})$ consisting of unitary matrices.

Our construction, combined with the argument in [GOSS12], gives a purely local and explicit construction of Klyachko models for all unitary representations of $\text{GL}_{2n}(\mathbb{R})$.

1. Introduction

In recent years, there has been considerable interest in periods of automorphic forms in relation to the Langlands program and equidistribution problems ([SV] [Ven10]). The study of periods admits a local counterpart: the study of invariant linear functionals and the concomitant notion of distinction of a representation $\pi$ of a reductive group $G$ with respect to a subgroup $H \subset G$. We recall that a representation $\pi$ is called distinguished with respect to a subgroup $H \subset G$ if the multiplicity space $\text{Hom}_H(\pi^\infty, C)$ of $H$-invariant continuous functionals on the space $\pi^\infty$ of smooth vectors of $\pi$ is non-zero. In many interesting cases the pair $(G, H)$ is a Gelfand pair, which means that the dimension of the multiplicity space is at most one for any irreducible admissible representation $\pi$ of $G$. This allows one to connect the global period integral to local linear functionals. Motivated by the work of Jacquet-Rallis [JR92] and Heumos-Rallis [HR90], the third author together with O. Offen classified in [OS07, OS08a, OS08b, OS09] those unitary representations of $\text{GL}_{2n}(F)$ that are distinguished with respect to the subgroup $\text{Sp}_{2n}(F)$, in the case that $F$ is a non-archimedean local field. The case of archimedean $F$ was treated subsequently in [GOSS12] [AOS12]. We remark that the pair $\text{Sp}_{2n}(F) \subset \text{GL}_{2n}(F)$ is a Gelfand pair (see [OS08a, AOS12, Say]).

The classification of $\text{Sp}_{2n}(\mathbb{R})$-distinguished unitary representations of $\text{GL}_{2n}(\mathbb{R})$ involves the family of unitary representations discovered by B. Speh ([Sp83]). We recall that these unitary representations and their generalizations to $\text{GL}_n(F)$, where $F$ is a local field, play a central role in the Tadic-Vogan classification of the unitary dual of $\text{GL}_n(F)$. To describe this classification we use the Bernstein-Zelevinsky notation $\pi_1 \times \pi_2$ for (normalized) parabolic induction from $\text{GL}_{n_1}(F) \times \text{GL}_{n_2}(F)$ to $\text{GL}_{n_1+n_2}(F)$. For a discrete series representation $\sigma$ of $\text{GL}_r(F)$ we denote by $U(\sigma, n)$ the corresponding Speh representation of $\text{GL}_n(F)$, and by

$$\pi(\sigma, n, \alpha) := U(\sigma, n)|^\alpha \times U(\sigma, n)|^{-\alpha}, \quad 0 < \alpha < \frac{1}{2}$$

the corresponding Speh complementary series representation.

Then any irreducible unitary representation of $\text{GL}_n(F)$ can be written in the form

$$\pi = \pi_1 \times \cdots \times \pi_k,$$

where each $\pi_i$ is either $U(\sigma_i, n_i)$ or a $\pi(\sigma_i, n_i, \alpha_i)$, and such an expression is unique up to reordering of the $\pi_i$ (see [Tad86, Vog86]). The answer to the distinction is summarized in the next theorem, which is in the archimedean case is a combination of [GOSS12, Theorem A] and [AOS12, Theorem 1.1].
Remark. If \( \pi \) is an irreducible unitary representation of \( \text{GL}_{2n}(F) \) as in \cite{J}, then \( \pi \) is \( \text{Sp}_{2n}(F) \)-distinguished if and only if all the \( n_i \) are even.

One of the key steps in the proof is to show that the generalized Speh representations \( U(\sigma, n) \) with even \( n \) are distinguished by the symplectic group. The proof of this result in \cite{OS07} and \cite{GOSS12} is based on a global argument involving periods of residues of automorphic Eisenstein series.

Recall that for archimedean \( F \) we have \( r \leq 2 \), and if \( F = \mathbb{C} \) then \( r = 1 \). If \( r = 1 \) then \( U(\sigma, n) \) is a character of \( \text{GL}_n(F) \), and \( \pi(\sigma, n, \alpha) \) is a Stein complementary series representation of \( \text{GL}_{2n}(F) \). We denote by \( D_m \) the discrete series representations of \( \text{GL}_2(\mathbb{R}) \) and by \( \delta_m \) the corresponding Speh representations of \( \text{GL}_{2n}(\mathbb{R}) \). In \cite{SaSt90} the Speh representations \( \delta_n \) of \( \text{GL}_{2n}(\mathbb{R}) \) have been constructed explicitly as natural Hilbert spaces of distributions on matrix space. The paper \cite{SaSt90} also describes and uses a construction of the Speh representations as quotients of degenerate principal series representations induced from characters of the \((n, n)\) standard parabolic subgroup (see \cite{2.2} below).

In the present paper we use the explicit constructions of \cite{SaSt90} and give a direct proof that the spaces of \( \text{Sp}_{2n}(\mathbb{R}) \)-invariant functionals on the Speh representations of \( \text{GL}_{2n}(\mathbb{R}) \) are zero if \( n \) is odd and one-dimensional if \( n \) is even. We also analyze functionals invariant with respect to subgroups of \( \text{Sp}_{2n}(\mathbb{R}) \).

To describe our result we need some further notation. Let \( G := G_{2n} \) denote the group \( \text{GL}_{2n}(\mathbb{R}) \). Let \( \omega_{2n} \) be the standard symplectic form on \( \mathbb{R}^{2n} \). More explicitly \( \omega_{2n} \) is given by \( \begin{pmatrix} 0 & \text{Id}_n \\ -\text{Id}_n & 0 \end{pmatrix} \) and let \( H := H_{2n} = \text{Sp}_{2n}(\mathbb{R}) \subset G_{2n} \) denote the stabilizer of this form. Let \( P := \left\{ \begin{pmatrix} g & X \\ 0 & (g^t)^{-1} \end{pmatrix} \mid g \in \text{GL}_n(\mathbb{R}), X \in \text{Mat}_{n \times n}(\mathbb{R}), X = X^t \right\} \subset H \)
denote the Siegel parabolic subgroup. Let \( U_n \subset H_{2n} \subset G_{2n} \) be the unitary group.

In this paper we prove the following result.

**Theorem A.**

(i) If \( n \) is even then

\[
\text{Hom}_H(\delta^\infty_m, \mathbb{C}) = \text{Hom}_P(\delta^\infty_m, \mathbb{C}) \simeq \mathbb{C}
\]

(ii) If \( n \) is odd then

\[
\text{Hom}_H(\delta^\infty_m, \mathbb{C}) = \text{Hom}_{U_n}(\delta^\infty_m, \mathbb{C}) = \{0\}.
\]

It is known that the restriction of \( \delta_n \) to \( \text{SL}_{2n}(\mathbb{R}) \) decomposes as a direct sum of two irreducible components \( \delta^\infty_n \). It follows from Theorem A that exactly one of them admits an \( H \)-invariant functional. In Lemma 1.2 we determine that \( \delta^\infty_m \) does not.

It is easy to see that if \( n \) is odd and \( m \) is even then there are no functionals on \( \delta^\infty_m \) invariant with respect to \( -\text{Id} \in P \cap U_n \), and thus neither \( P \)-invariant nor \( U_n \)-invariant functionals exist (see Remark 6.1).

**Remark.** Although the pair \((G, P)\) is not a Gelfand pair for simple geometric reasons, we show that the Speh representation \( \delta_n \) still admits at most one \( P \)-invariant functional (at least for even \( n \)). The reason we suspected this result to hold is that, as shown in \cite{SaSt90}, Speh representations stay irreducible when restricted to a standard maximal parabolic subgroup \( Q \subset G \) satisfying \( Q \cap H = P \). It is possible that \((Q, P)\) is a generalized Gelfand pair, i.e. the space of \( P \)-invariant functionals on the space of smooth vectors of any irreducible unitary representation of \( Q \) is at most one dimensional. However, this statement would still not imply our uniqueness result, since the space of \( G \)-smooth vectors of \( \delta_n \) could a priori afford more continuous functionals.

1.1. Related results. The present work was motivated by our previous results on Klyachko models for unitary representations of \( \text{GL}_n(\mathbb{R}) \). For any \( n \), any even \( k \leq n \) and any field \( F \), \cite{Kly84} defines a subgroup \( K_k \) of \( \text{GL}_n(F) \) and a generic character \( \psi_k \) of \( K_k \). In particular, \( K_0 \) is the group of upper unitriangular matrices and \( K_n = \text{Sp}_n(F) \) (if \( n \) is even). It is shown in \cite{Kly84, ES91, HZ00} for finite fields \( F \) and in \cite{HR90, OS07, OS08a, OS08b, OS09, GOSS12, AOS12} for local fields \( F \) that for any irreducible unitary representation \( \pi \) of \( \text{GL}_n(F) \), there exists a non-zero \((K_k, \psi_k)\)-equivariant functional on \( \pi^\infty \) for exactly one \( k \). The uniqueness of such functional is known only over non-archimedean fields (see \cite{OS08d}).
The proof of existence of $k$ for $F = \mathbb{R}$, given in \cite{GOSS12}, is achieved by reduction to the statement that certain representations of $G = \text{GL}_{2n}(\mathbb{R})$ are $H = \text{Sp}_{2n}(\mathbb{R})$-distinguished. This statement is further reduced, using the Vogan classification of the unitary dual, to an existence statement of $H$-invariant functionals on the Speh representations (for even $n$). Finally, the existence statement is proved using a global (adelic) argument. In the present paper we give an explicit local construction of such a functional. Together with \cite{GOSS12} this gives a proof of existence of Klyachko models which uses only the representation theory of $\text{GL}_n(\mathbb{R})$ (and the theory of distributions).

The study of invariant functionals in this paper, and more broadly the study of multiplicity spaces belongs to the long and classical tradition of branching laws (see e.g. \cite{GW09} Chapter 8). In the context of symmetric pairs and more generally in the context of spherical spaces, the basic result is that these multiplicity spaces are finite dimensional (\cite{KO13}, cf. \cite{KrSch}). Granted this qualitative result, one turns to the question of precisely determining the dimension. We note that in many interesting cases these spaces are at most one-dimensional (see e.g. \cite{vD86, AG09, AGRS10, SZ12}). This multiplicity one phenomenon has important consequences in number theory (\cite{Gross91}).

In some situations there are precise conjectures as to the dimensions of these multiplicity spaces (see e.g. \cite{CCP12, Wald12}) but in general these dimensions are hard to determine, even in the context of symmetric pairs. Another important task, motivated in part by the theory of automorphic forms, is to construct a basis for these multiplicity spaces. Recently, there has been a considerable interest in these aspects of the theory under the title of symmetry breaking. The general theory of branching laws attempts the description of symmetry breaking operators occurring in the general context of restrictions of representations as in \cite{KoSp, KoSp14}. In particular, it is interesting to compare our main result with \cite{KoSp, KoSp14}.

Another related result is the exact branching of the representations $\delta_m^\pm$ of $\text{SL}(4, \mathbb{R})$ to $\text{Sp}(4, \mathbb{R})$ as analyzed in \cite{OrSp08}. It is shown there that the decomposition of $\delta_m^+$ is discrete and multiplicity free, while the decomposition of $\delta_m^-$ is continuous.

### 1.2. Structure of the proof.

We use the realization of $\delta_m^\pm$ as the image of a certain intertwining differential operator $\Box^m : \pi_m^\mp \to \pi_m^\pm$, where $\pi_m^\mp$ and $\pi_m^\pm$ are degenerate principal series representations induced from certain characters of a fixed $(n, n)$-parabolic subgroup $\mathcal{Q} \subset G$ (see \cite{vD86}).

The study of the even case is divided into two parts. In \S 3 we first use the realization of $\delta_m^\pm$ as a quotient of the degenerate principal series $\pi_m^\pm$ to lift a linear $P$-invariant functional on $\delta_m^\pm$ to an equivariant distribution on $G$. More precisely, we study $P \times \mathcal{Q}$-equivariant distributions on $G$. The technical heart is Corollary 3.3 which states that such distributions do not vanish on the open cell $N\mathcal{Q}$. This is based on the techniques of \cite{AGS08}, classical invariant theory and a careful analysis of the double cosets $P \backslash G/\mathcal{Q}$, which is postponed to \S 4. Then we analyze the space of distributions on the open cell $N\mathcal{Q}$ by identifying it with the space of distributions on $N$ with a certain equivariance property. Identifying $N$ with its Lie algebra and using the Fourier transform we show that this space is at most one-dimensional for even $n$. This finishes the proof of Proposition 3.1 which states that there exists at most one $P$-invariant functional in the $n$ even case.

In the second part (\S 5) we construct an $H$-invariant functional as an $H \times \mathcal{Q}$-equivariant distribution on $G$. For that we fix an explicit $H \times \mathcal{Q}$-equivariant non-negative polynomial $p$ consider the meromorphic family of distributions $p^\lambda$ (cf. \cite{Ber72}) and take the principal part of this family at $\lambda = (n - m)/2$, i.e. the lowest non-zero coefficient in the Laurent expansion. This distribution defines an $H$-invariant functional on $\pi_m^\pm$. To show that the restriction of this functional to $\delta_m^\pm$ is non-zero (Lemma 5.1) we use Corollary 3.5 along with another lemma from \S 5 on non-existence of equivariant distributions with certain support. The uniqueness of $P$-invariant functionals and the existence of $H$-invariant ones imply that the two spaces are equal. Our proof shows that the spaces of such functionals are equal and one-dimensional also for the (reducible) representations $\pi_m^\pm$ and $\pi_m^{-\mp}$.

For odd $n$ we prove that already a $U_n$-invariant functional does not exist (Corollary 5.4). We do that by analyzing the $O_{2n}(\mathbb{R})$-types of $\delta_m$ described in \cite{HL99, Sah95} and showing that none of those have a $U_n$-invariant vector.
To summarize, Theorem 2.1 follows from Proposition 3.1 on uniqueness of $P$-invariant functionals for even $n$, Lemma 4.1 on existence of $H$-invariant functionals for even $n$ and Corollary 6.3 on non-existence of $U_n$-invariant functionals for odd $n$.

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2. **Preliminaries**

2.1. **Notation.** Recall the notation $G = G_{2n} = GL_{2n}(\mathbb{R})$, and $H = H_{2n} = Sp_{2n}(\mathbb{R}) \subset G$. Let

$$Q := \left\{ \begin{pmatrix} a & c \\ 0 & d \end{pmatrix} \in G \right\}, \quad \overline{Q} := \left\{ \begin{pmatrix} a & 0 \\ b & d \end{pmatrix} \in G \right\}, \quad N := \left\{ \begin{pmatrix} \text{Id}_n & c \\ 0 & \text{Id}_n \end{pmatrix} \in G \right\}.$$  

Recall that $P$ denotes $Q \cap H$ and let

$$M := \left\{ \begin{pmatrix} g & 0 \\ 0 & (g^t)^{-1} \end{pmatrix} \right\} \quad \text{and} \quad U := \left\{ \begin{pmatrix} \text{Id}_n & B \\ 0 & \text{Id}_n \end{pmatrix} \mid B = B^t \right\},$$

denote the Levi subgroup and the unipotent radical of $P$.

For $g \in \text{Mat}_{n \times 1}(\mathbb{R})$ we denote $|g| := |\det(g)|$ and $\text{sgn}(g) := \text{sgn}(\det(g))$.

For $q = \begin{pmatrix} A & 0 \\ B & D \end{pmatrix} \in \overline{Q}$ we denote $\gamma(q) := |A||D|^{-1}$ and $\epsilon(q) := \text{sgn}(D)$.

For any integer $m$ let $L_m$ denote the character of $\overline{Q}$ given by $L_m := \epsilon^{m+1} \gamma^{-(n+m)/2}$. Let $\pi^\infty_m$ denote the (unnormalized) induced representation $\text{Ind}_{\overline{Q}}^G(L_m)$, with the topology of uniform convergence on $G/\overline{Q}$ together with all the derivatives. Considering $N$ as an open subset of $G/\overline{Q}$, one can restrict smooth vectors of $\pi^\infty_m$ to $N$. This restriction is an embedding since $N$ is an open subset of $G/\overline{Q}$. We sometimes identify $N$ and its Lie algebra $\mathfrak{n}$ with $\text{Mat}_{n \times n}(\mathbb{R})$ by

$$\begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \mapsto X \text{ and } \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} \mapsto X.$$ 

This enables us to define the Fourier transform on $\mathfrak{n}$. Denote by $M^+_n$ (respectively $M^-_n$) the subset of $\text{Mat}_{n \times n}(\mathbb{R})$ consisting of matrices with nonnegative (resp. nonpositive) determinant. For $f \in \pi^\infty_m$ we denote its restriction to $\mathfrak{n}$ by $f|_{\mathfrak{n}}$. We denote the space of all smooth functions obtained in this way by $\pi^\infty_m|_{\mathfrak{n}}$.

2.2. **Sahi-Stein realization of the Speh representations.** For any $m \in \mathbb{Z}_{\geq 0}$ define

$$\hat{H}_m := \{ f \in S^*(\mathfrak{n}) \mid \hat{f} \in L^2(\mathfrak{n}, |x|^{-m}dx) \} \quad \text{and} \quad \hat{H}^\pm_m := \{ f \in \hat{H}_m \mid \text{Supp} \hat{f} \subset M^\pm_n \},$$

where $S^*(\mathfrak{n})$ denotes the space of tempered distributions $\mathfrak{n}$. The $\hat{H}_m$ and $\hat{H}^\pm_m$ are Hilbert spaces with the scalar product

$$\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle_{L^2(\mathfrak{n}, |x|^{-m}dx)}.$$ 

Define an action of $Q$ on $\hat{H}_m$ by

$$\delta_m(q)f(x) := L_m(q)f(a^{-1}(c + xd)),$$

for $q = \begin{pmatrix} a & c \\ 0 & d \end{pmatrix}$, or equivalently on the Fourier transform side by

$$\delta_m(q)f(\xi) = \exp(2\pi i \text{Tr}(cd^{-1}\xi))L^{-1}_m(q)\hat{f}(d^{-1}\xi a).$$

Summarizing the main results of [SaSt90] we obtain

**Theorem 2.1** (**SaSt90**). Let $m \in \mathbb{Z}_{\geq 0}$. Then
(i) The action of $Q$ extends to a unitary representation $\delta_m$ of $G$ on $\hat{H}_m$.
(ii) $(G, \delta_m, \hat{H}_m)$ is isomorphic to the Speh representation of $G$.
(iii) There exists an epimorphism $\pi_m \to \delta_m$ and an embedding $\delta_m \subset \pi_m^\infty$. The latter is defined by the inclusion $\delta_m^\infty \subset \pi_m^\infty$.
(iv) The restriction of $\delta_m$ to $\text{SL}(2n,\mathbb{R})$ is a direct sum of two irreducible representations $\delta_m^\pm$, realized on the subspaces $\hat{H}_m^\pm$.

Consider the determinant as a polynomial on $\mathfrak{n}$ and let $\Box$ denote the corresponding differential operator.

**Theorem 2.2.** The operator $\Box^m$ defines a continuous $G$-equivariant map $\pi_m^\infty \to \pi_m^\infty$ with image $\delta_m^\infty$.

**Proof.** By [KV77, Proposition 2.3] (see also [Bec85]), the operator $\Box^m$ defines a continuous $G$-equivariant map $\pi_m^\infty \to \pi_m^\infty$, which is non-zero by [SaS99]. By [HL99] Theorems 3.4.2-3.4.4, $\pi_m^\infty$ has unique composition series in the strong sense, meaning that any quotient of $\pi_m^\infty$ has a unique irreducible subrepresentation, and all these irreducible subquotients are pairwise non-isomorphic. It is easy to see that $\pi_m^\infty$ is dual to $\pi_m^\infty$ and thus their composition series are opposite. Hence, the image of any nonzero intertwining operator from $\pi_m^\infty$ to $\pi_m^\infty$ is the unique irreducible subrepresentation of $\pi_m^\infty$. Since $\delta_m^\infty$ is an irreducible subrepresentation of $\pi_m^\infty$, the image of $\Box^m$ is a dense subspace of $\delta_m^\infty$. By the result of Casselman and Wallach (see [CasS9] or [Wal92, Chapter 11]), the category of smooth admissible Fréchet representations of moderate growth is abelian and any morphism in it has closed image. Thus the image of $\Box^m$ is $\delta_m^\infty$. \qed

**Remark 2.3.** One can deduce Theorem 2.2 also from [KS93], which computes the action of $\Box^m$ on every $K$-type, where $K = \text{O}(2n,\mathbb{R})$. From the formula in [KS93] and the description of the $K$-types of the composition series of $\pi_m^\infty$ in [HL99, Sah95] one can see that $\Box^m$ does not vanish precisely on the $K$-types of $\delta_m^\infty$.

### 2.3. Invariant distributions.

We will now recall some generalities on Schwartz functions and tempered distributions.

**Definition 2.4.** For an affine algebraic manifold $M$ we denote by $\mathcal{S}(M)$ the space of Schwartz functions on $M$, that is smooth functions $f$ such that $df$ is bounded for any differential operator $d$ on $M$ with algebraic coefficients. We endow this space with a Fréchet topology using the sequence of seminorms $N_d(f) := \sup_{x \in M} |df(x)|$, where $d$ is a differential operator on $M$ with algebraic coefficients. Also, for an algebraic vector bundle $E$ over $M$ we denote by $\mathcal{S}(M, E)$ the space of Schwartz sections of $E$. We denote by $\mathcal{S}^*(M, E)$ the space of continuous linear functionals on $\mathcal{S}(M, E)$ and call its elements tempered distributional sections. For a closed subvariety $Z \subset M$ we denote by $\mathcal{S}_M(Z, E) \subset \mathcal{S}^*(M, E)$ the subspace of tempered distributional sections supported in $Z$. For the theory of Schwartz functions and distributions on general semi-algebraic manifolds we refer the reader to [AG08].

**Notation 2.5.**
\begin{itemize}
  \item For a manifold $M$ and closed submanifold $Z \subset M$ we denote by $N^M_Z := T^M|_Z/T^Z$ the normal bundle to $Z$ in $M$ and by $CN^{MN}_Z \subset T^*M$ its dual bundle, i.e. the conormal bundle to $Z$ in $M$.
  \item For a point $z \in Z$ we denote by $N^M_{Z,z}$ the normal space at $z$ to $Z$ in $M$ and by $CN^{MN}_{Z,z}$ the conormal space at $z$ to $Z$ in $M$.
  \item For a group $K$ acting on a vector space $V$ we denote by $V^K$ the subspace of $K$-invariant vectors and by $V^{K,\chi}$ the subspace of vectors that change by the character $\chi$.
  \item If $K$ acts on a manifold $M$ we denote by $\mathcal{S}^*(M)^{K,\chi}$ the space of distributions on $M$ that change by the character $\chi$ under the action of $K$.
  \item For a real algebraic group $K$ we denote by $\Delta_K$ its modular character.
\end{itemize}

**Theorem 2.6** ([AG08, B.2]). Let a real algebraic group $K$ act on a real algebraic manifold $M$. Let $Z \subset M$ be a Zariski closed subset. Let $Z = \bigcup_{i=1}^l Z_i$ be a $K$-invariant stratification of $Z$. Let $\chi$ be a character of $K$. Suppose that for any $k \in \mathbb{Z}_{\geq 0}$ and $1 \leq i \leq l$,
\[ \mathcal{S}^*(Z_i, \text{Sym}^k(CN^M_{Z_i}))^{K,\chi} = \{0\} \]
then $\mathcal{S}^*_M(Z)^{K,\chi} = \{0\}$.
Theorem 2.7 (Frobenius descent, see [AGM] Appendix B). Let a real algebraic group $K$ act on a real algebraic manifold $M$. Let $Z$ be a real algebraic manifold with a transitive action of $K$. Let $\phi : M \to Z$ be a $K$-equivariant map. Let $z \in Z$ be a point and $M_z := \phi^{-1}(z)$ be its fiber. Let $K_z$ be the stabilizer of $z$ in $K$. Let $E$ be a $K$-equivariant algebraic vector bundle over $M$.

Then there exists a canonical isomorphism

$$\text{Fr} : (S^*(M_z, E)|_{M_z}) \otimes \Delta_K|_{K_z} : \Delta_{K_z}^{-1}|_{K_z} \cong S^*(M, E)^K.$$  

From those two theorems we obtain the following corollary.

Corollary 2.8. Let a real algebraic group $K$ act on a real algebraic manifold $M$. Let $Z \subset M$ be a Zariski closed subset. Suppose that $Z$ has a finite number of orbits: $Z = \bigcup_{i=1}^l K z_i$. Let $\chi$ be a character of $K$.

Suppose that for any $1 \leq i \leq l$ we have

$$\text{Sym}^\ast(N_{K z_i}^M)^{K_{z_i}} \chi_{\Delta K|_{K_{z_i}}} \Delta_{K_{z_i}}^{-1} = \{0\},$$

where $\text{Sym}^\ast$ denotes the symmetric algebra. Then $\mathcal{S}^\ast_{M}(Z)^{K \chi} = \{0\}$.

Lemma 2.9. Let $K$ be a real algebraic group, and $R$ be a (closed) algebraic subgroup. Consider the right action of $R$ on $K$ and suppose that $K/R$ is compact. Let $\xi$ be a character of $R$. Then we have a natural isomorphism of left $K$ - representations

$$(C^\infty(K, \xi)^R)^\ast \cong S^\ast(K, \xi \Delta_R^{-1})^R \cong S^\ast(K)^{(R, \xi^{-1}(\Delta) R)}.$$ 

Proof. Let $\mathfrak{n} \mathfrak{d}(\xi)$ be the bundle on $K/R$ corresponding to $\xi$. Consider the surjective submersion $\pi : K \to K/R$. It defines an isomorphism $C^\infty(K, \xi)^R \cong C^\infty(K/R, \mathfrak{n} \mathfrak{d}(\xi))$.

Since $K/R$ is compact, we have $C^\infty(K/R, \mathfrak{n} \mathfrak{d}(\xi))^\ast \cong S^\ast(K/R, \mathfrak{n} \mathfrak{d}(\xi))$. Consider the diagonal action of $K$ on $K \times K/R$ and the projections $p_1, p_2$ of $K \times K/R$ on both coordinates. From Theorem 2.7 we obtain

$$S^\ast(K/R, \mathfrak{n} \mathfrak{d}(\xi)) \cong S^\ast(K \times K/R, p_1(\xi)) \cong S^\ast(K, \xi \Delta_R^{-1})^R.$$ 

The isomorphism $S^\ast(K, \xi \Delta_R^{-1})^R \cong S^\ast(K)^{(R, \xi^{-1}(\Delta) R)}$ is straightforward. $\square$

3. Uniqueness of $P$ - invariant functionals

In this section we assume that $n$ is even. The goal of this section is to prove the following proposition.

Proposition 3.1. For any integer $m$ we have

$$\dim((\pi^\infty_m)^\ast)^P \leq 1.$$ 

Recall the character $L_m$ of $\overline{Q}$ from [2.1] and note that $L_m^{-1} = \gamma^{m+1}(n-m)/2$. Since $\Delta_{\overline{Q}} = \gamma^{-n}$, we obtain from the definition of $\pi^\infty_m$ and Lemma 2.9

$$\left(\pi^\infty_m\right)^\ast \cong S^\ast(G)\overline{Q} L_m^{-1},$$

and thus in order to prove Proposition 3.1 we have to show that for even $n$

$$\dim S^\ast(G)^{P \times \overline{Q}, 1 \times L_m^{-1}} \leq 1.$$ 

We will need the following proposition, which we will prove in section 5.

Proposition 3.2. Denote $K := P \times \overline{Q}$, and let $x \notin N \overline{Q}$. Then

$$\text{Sym}^\ast(N_{\overline{Q}, x}^G)^{K_{x}, L_m^{-1}} \Delta_K|_{K_{x}} \Delta_{L_m}^{-1} = \{0\}.$$ 

From this proposition and Corollary 2.8 we obtain

Corollary 3.3.

$$\mathcal{S}^\ast_{\overline{Q}}(G - N \overline{Q})^{P \times \overline{Q}, 1 \times L_m^{-1}} = \{0\}.$$

By this corollary it is enough to analyze $S^\ast(N \overline{Q})^{P \times \overline{Q}, 1 \times L_m^{-1}}$. Let $S$ denote the space of symmetric $n \times n$ matrices, and $A$ denote the space of anti-symmetric $n \times n$ matrices. Identify $M \cong \text{GL}_n(\mathbb{R})$ and let it act on $S$ and on $A$ by $x \mapsto gxg^t$. 


Lemma 3.4. We have
\[ S^* (Q)^* \simeq S^* (A)^{GL_n (R), \text{det}^m} \approx S^* (A)^{GL_n (R), \text{sgn}^m} \mid |m| - n \]

Proof. Identify \( U \cong S \) and let it act on itself by translations. Then \( N \mathcal{Q} \) is isomorphic to \( A \times S \times \mathcal{Q} \), where \( \mathcal{Q} \) acts on the third coordinate (by right translations), \( U \) acts on the second coordinate and \( M \) acts on the first and the second coordinates. Note that the action of \( P \times \mathcal{Q} \) on \( S \times \mathcal{Q} \) is transitive and that \( \Delta \mathcal{Q} = \gamma^n \) and \( \Delta (g (g^{-1} t)) = |g|^n + 1 \). The first isomorphism follows now from Frobenius descent.

The second isomorphism is given by Fourier transform on \( A \) defined using the trace form. \( \square \)

Let \( O \subset Z \) denote the open dense subset of non-degenerate matrices and \( Z \) denote its complement. The following lemma is a straightforward computation.

Lemma 3.5.

(i) Every orbit of \( GL_n (R) \) in \( Z \) includes an element of the form \( x := \begin{pmatrix} 0_k \times k & 0 \\ 0 & \omega_{n-k} \end{pmatrix} \), for some even \( k \).

(ii) \( N^A_{GL_n (R), x} \cong \left\{ \begin{pmatrix} 0_k \times k & b \\ 0 & 0 \end{pmatrix} \right\} \) and \( GL_n (R)_x = \left\{ \begin{pmatrix} a_k \times k & 0 \\ c & d \end{pmatrix} \text{ such that } d \in Sp(n-k) \right\} \).

(iii) \( \Delta_{GL_n (R), x} = | \cdot |^{- (n-k)} \).

Corollary 3.6. For any \( x \in Z \) we have
\[ \text{Sym}^\times (N^A_{GL_n (R), x})^{GL_n (R), \text{sgn}^m} \mid |m| - n \Delta^{-1}_{GL_n (R), x} = \{ 0 \} \]

Proof. From the previous lemma \( \text{sgn}^m \cdot \Delta^{-1}_{GL_n (R), x} = \text{sgn}^{k+1} \text{det}^{-k} = \text{sgn} \text{det}^{-k} \). This is not an algebraic character of \( GL_n (R)_x \) and thus there are no tensors that change under this character. \( \square \)

Corollary 3.7.
\[ \dim S^* (A)^{GL_n (R), \text{sgn}^m} \mid |m| - n \leq 1 \]

Proof. By Corollary 3.6 and Corollary 2.8
\[ \text{Sym}^\times (N^A_{GL_n (R), x})^{GL_n (R), \text{sgn}^m} \mid |m| - n \Delta^{-1}_{GL_n (R), x} = \{ 0 \} \]

Therefore, the restriction of equivariant distributions to \( O \) is an embedding. Now,
\[ \dim S^* (O)^{GL_n (R), \text{sgn}^m} \mid |m| - n \leq 1 \]

since \( O \) is a single orbit.

\( \square \)

Proposition 3.1 follows now from Corollary 3.4, Lemma 3.3, Corollary 3.6 and 2.

Remark 3.8. Corollary 3.3 does not extend to the case of odd \( n \). For example, in this case the closed \( P \times \mathcal{Q} \)-orbit \( \mathcal{Q} \) does support an equivariant distribution.

4. Construction of the \( H \)-invariant functional

Let \( n \) be even. In this section we construct an \( H \)-invariant functional \( \phi \) on \( \pi^\infty_m \) for any \( m \in \mathbb{Z} \geq 0 \) and show that its restriction to \( \delta_m \) is non-zero. Define a polynomial \( p \) on \( Mat_{2n \times 2n} (\mathbb{R}) \) by
\[ p \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) := \text{det}(D^t B - B^t D) = \text{Pfaffian}^2 (D^t B - B^t D) \]

Note that \( p \) is non-negative, \( H \)-invariant on the left and changes under the right multiplication by \( \mathcal{Q} \) by the character \( | \cdot |^{-1} \). Consider the meromorphic family of distributions on \( Mat_{2n \times 2n} (\mathbb{R}) \) given by \( p^\lambda \). This family is defined for \( \text{Re} \lambda > 0 \) and by \( \text{Res} [2] \) has a meromorphic continuation (as a family of distributions) to the entire complex plane. For \( \text{Re} \lambda > 0 \), the restriction of this distribution to \( G = GL_{2n} (\mathbb{R}) \) is a non-zero smooth function, and thus the restriction of the family to \( G \) is not identically zero. Define
\[ \eta^m_\lambda : = (p^\lambda |G|) \mid |^{-1} \lambda \text{sgn}^m \]
This is a tempered distribution, since $|\cdot|^\lambda$ is a smooth function on $G$ of moderate growth. Note that

$$n^{m}_\lambda \in S^*(G)^{(H \times \mathbb{Q},1 \times e^{n+m+1} \gamma^\lambda)}.$$ 

Let $\alpha \in S^*(G)$ be the principal part of this family at $\lambda = \frac{n-m}{2}$, i.e. the lowest non-zero coefficient in the Laurent expansion. By (2) $\alpha$ defines a non-zero $H$-invariant functional $\phi$ on $\pi^n_m$.

**Lemma 4.1.** $|\phi|_m^{\infty} \neq 0$.

**Proof.** By Theorem 2.2 it is enough to show that $\Box^m \phi \neq 0$. By Corollary 3.3 $\alpha|_{N_\mathbb{Q}} \neq 0$. It is enough to show that $|\Box^m \alpha|_{N_\mathbb{Q}} \neq 0$. As in 3.4 let $A \subset N$ denote the subspace of anti-symmetric matrices and $O \subset A$ the open subset of non-degenerate matrices. Note that $\alpha|_{N_\mathbb{Q}} \neq 0$ is $P \times \mathbb{Q}$-equivariant and let $\beta \in S^*(A)^{\text{GL}_n(\mathbb{R}),\text{det}^{1-m}}$ be the distribution on $A$ corresponding to $\alpha$ by the Frobenius descent (see Lemma 3.4). Note that $\mathcal{F}(\Box^m \beta)$ is $\mathcal{F}(\beta)$ multiplied by a polynomial. Thus it is enough to show that $\mathcal{F}(\beta)$ has full support, i.e. $\mathcal{F}(\beta)|_O \neq 0$. This follows from the equivariance properties of $\mathcal{F}(\beta)$ by (3). \hfill \Box

This argument in fact proves slightly more.

**Lemma 4.2.** $|\phi|_{(s_m^n)^\infty} \neq 0$.

**Proof.** If $g$ is a Schwartz function on $M^+ \subset N$ then its Fourier transform $\widehat{g}$ defines a vector in $(\delta_m^n)^\infty$ by Theorem 2.4. Thus it is enough to find such a $g$ for which $\zeta(\widehat{g}) \neq 0$, where $\zeta$ denotes the $P$-invariant distribution on $N$ corresponding to $\alpha$.

Let $f$ be a compactly supported smooth function on $O$ such that $\beta(\mathcal{F}(f)) \neq 0$. Since the determinant is positive on $O$, there exists a compact neighborhood $Z$ of zero in the space $S$ of symmetric $n$ by $n$ matrices such that $\text{Supp}(f) + Z \subset M^+$. Let $h$ be a smooth function on $S$ which is supported on $Z$ and s.t. $h(0) = 1$. Let $g := f \mathcal{F} h$ be the function on $N$ defined by $g(X + Y) := f(X)h(Y)$ where $X \in A$ and $Y \in S$. Let $\mathcal{F}_S$ denote the Fourier transform on $S$. Then we have

$$\zeta(\widehat{g}) = \zeta(\mathcal{F}(f) \mathcal{F}_S(h)) = \beta(\mathcal{F}(f)) \neq 0.$$ \hfill \Box

**Remark 4.3.**

(i) For odd $n$, the polynomial $p$ is identically zero, since the matrix $D'B - B'D$ is an anti-symmetric matrix of size $n$.

(ii) The polynomial $p$ defines the open orbit of $H$ on $G/\overline{\mathbb{Q}}$. In general, one can show that if a linear complex algebraic group $K$ acts with finitely many orbits on a complex affine algebraic manifold $M$, both defined over $\mathbb{R}$, $W$ is a basic open subset of $M$ defined by a $K$-equivariant polynomial $p$ with real coefficients, $\chi$ is a character of the group of real points $K$ of $K$ and there exists a non-zero $(K,\chi)$-equivariant tempered distribution $\xi$ on $W$ then there exists a non-zero $(K,\chi)$-equivariant tempered distribution on $M$. Here, $W$ and $M$ denote the real points of $W$ and $M$.

To prove that consider the analytic family of distributions $|p|^\lambda \xi$ on $W$. For $\Re \lambda$ big enough, it can be extended to a family $\eta_\lambda$ on $M$. By [Her72] the family $\eta_\lambda$ has a meromorphic continuation to the entire complex plane. Note that the distributions in this family are equivariant with a character that depends analytically on $\lambda$. Thus taking the principal part at $\lambda = 0$ we obtain a non-zero $(K,\chi)$-equivariant tempered distribution on $M$.

Note that since this construction involves taking principal part, the obtained distribution is not necessary an extension of the original $\xi$. This can already be seen in the case when $M = \mathbb{C}$ is the affine line, $W$ is the complement to 0 and $K$ is the multiplicative group $\mathbb{C}^\times$.

5. **Proof of Proposition 3.2**

We start from the description of the double cosets $P \backslash G/\overline{\mathbb{Q}}$. Let $r_1, r_2, s, t$ be non-negative integers such that $r_1 + r_2 + 2s + 2t = n$. We will view $2n \times 2n$ matrices as $10 \times 10$ block matrices in the following way. First of all, we view them as $2 \times 2$ block matrices with each block of size $n \times n$. Now, we divide each block to $5 \times 5$ blocks of sizes $r_1, r_2, s, s, 2t$ in correspondence. Denote by $\sigma_{16}$ the permutation matrix that permutes blocks 1 and 6, by $\sigma_{39}$ the permutation matrix that permutes blocks 3 and 9, and by $\tau_{5,10}$ the...
matrix which has \( \begin{pmatrix} \text{Id}_{2t} & \omega_{2t} \\ 0 & \text{Id}_{2t} \end{pmatrix} \) in blocks 5 and 10 and is equal to the identity matrix in other blocks.

Recall the notation \( \omega_{2t} := \begin{pmatrix} 0 & \text{Id}_t \\ -\text{Id}_t & 0 \end{pmatrix} \). Denote

\[
x_{r_1,r_2,s,t} := \sigma_{16,39,75,10}.
\]

**Lemma 5.1.** Each double coset in \( P \setminus \text{GL}_{2n}(\mathbb{R})/Q \) includes a unique element of the form \( x_{r_1,r_2,s,t} \). The orbits in \( N_Q \) correspond to \( r_1 = s = 0 \).

**Proof.** Consider the Lagrangian subspaces \( L := \text{Span}\{e_1, \ldots, e_n\} \subset \mathbb{R}^{2n} \) and \( L' := \text{Span}\{e_{n+1}, \ldots, e_{2n}\} \subset \mathbb{R}^{2n} \). Note that \( Q \) preserves \( L \) and \( Q \) preserves \( L' \). Identify \( G/\overline{Q} \) with the Grassmannian of \( n \)-dimensional subspaces of \( \mathbb{R}^{2n} \) by \( g \mapsto gL' \). To an \( n \)-dimensional subspace \( W \subset \mathbb{R}^{2n} \) we associate the following invariants:

\[
r_1 := \dim L \cap W \cap W^\perp, \quad r_2 := \dim W^\perp \cap W - r_1, \quad s := \dim L \cap W - r_1, \quad t := (n - r_1 - r_2)/2 - s.
\]

Note that \( n - r_1 - r_2 \) is even since it is the rank of \( \omega|_W \). Clearly, \( W \in NL' \) if and only if \( r_1 = s = 0 \).

Note the equality of vectors

\[
(v_1,0,v_2,0,\omega_{2t}u|0,w_2,w_1,0,0) = x_{r_1,r_2,s,t}(0,0,0,0|v_1,w_2,w_1,v_2,u)t.
\]

It is enough to show that \( W \) can be transformed, using the action of \( P \), to a space of vectors of the form \( \ref{eq:ourform} \).

Let us first show that \( W \) can be transformed to a space of vectors of the form

\[
(v,Aw+Bv|Cw,w,Dw)t, \quad \text{where size}(v) + \text{size}(w) = n \text{ and } A \text{ is a square matrix}.
\]

There exists a set \( S \) of \( n \) coordinates such that the projection of \( W \) on the space of vectors that have zero coordinates from \( S \) is an isomorphism. Suppose that \( S \) has \( k \) of the coordinates \( 1 \ldots n \), and thus \( n - k \) of the coordinates \( n + 1 \ldots 2n \). Note that acting by \( M \) we can perform any permutation of the first \( n \) coordinates followed the same permutation on the last \( n \) coordinates. Using such permutations we can transform \( S \) to the set \( \{n - k + 1, \ldots, n, n + 1 \ldots n + l, n + k + l + 1, \ldots, 2n\} \) for some \( l \leq n - k \). Then \( W \) will have the form \( \ref{eq:ourform} \).

Let us now rewrite \( \ref{eq:ourform} \) in more detailed form, using four blocks of the same sizes \( y_i \) in the first \( n \) coordinates and last coordinates:

\[
(v_1, v_2, A_{11}w_1 + A_{12}w_2 + B_{11}v_1 + B_{12}v_2, A_{21}w_1 + A_{22}w_2 + B_{21}v_1 + B_{22}v_2 | C_1w_1+C_2w_2, w_1, w_2, D_1w_1+D_2w_2)t
\]

Denote the first four blocks by \( e_i \) and the last by \( f_i \). For any \( i,j \in \{1,2,3,4\} \) with \( i \neq j \), \( M = \text{GL}_n(\mathbb{R}) \) allows us to do the following operations:

\[
(1)_i \quad e_i \mapsto ge_i, \quad f_i \mapsto (g')^{-1}f_i, \quad \text{where } g \in \text{GL}_{y_i}(\mathbb{R})
\]

\[
(2)_{ij} \quad e_i \mapsto e_i + ae_j, \quad f_j \mapsto f_j - a^t f_i, \quad \text{where } a \in \text{Mat}_{y_i \times y_j}(\mathbb{R})
\]

Similarly, \( U \) allows us to do two more operations:

\[
(3)_{ij} \quad e_i \mapsto e_i + b f_j, \quad e_j \mapsto e_j + b^t f_i, \quad \text{where } b \in \text{Mat}_{y_i \times y_j}(\mathbb{R})
\]

\[
(4)_i \quad e_i \mapsto e_i + (c + c^t) f_i, \quad \text{where } c \in \text{Mat}_{y_i \times y_i}(\mathbb{R})
\]

Using \( (2)_{31} \) and \( (2)_{41} \), and redefining \( C \) and \( D \) we get \( B = 0 \). Using \( (2)_{21} \) and \( (2)_{34} \), and redefining \( A \) we get \( C = 0 \) and \( D = 0 \).

Using \( (3)_{32} \) and \( (3)_{42} \) and \( (3)_{43} \) we can arrange \( A_{11} = A_{21} = A_{22} = 0 \). Using \( (3)_{33} \) we make \( A_{12} \) anti-symmetric. Now, using \( (1)_3 \) we can replace \( A_{12} \) by \( gA_{12}g^t \) and thus we can bring it to the form

\[
A_{12} = \begin{pmatrix} 0 & 0 \\ 0 & \omega_{2t} \end{pmatrix}.
\]

\( \square \)
Proof of Proposition 3.2. Let \( K := P \times \mathbb{Q} \) and \( x := x_{r_1, r_2, s, t} \). Then

(i) If \( s > 0 \) then

\[
\text{Sym}^r(N_{x,P/Q}^G)^{K_x, \mathbb{Q}, \Delta K_x} = \{0\}.
\]

(ii) If \( s = 0 \) then

\[
\text{Sym}^r(N_{x,P/Q}^G)^{K_x, \mathbb{Q}, \Delta K_x} \cong \text{Sym}^r(\mathfrak{gl}_{r_1})_{GL_{r_1}, det^{t-m+1}} \otimes \text{Sym}^r(\mathfrak{o}_{r_2})_{GL_{r_2}, det^{t-m+1}}
\]

where \( \mathfrak{o}_{r_2} \) denotes the space of antisymmetric matrices and \( GL_{r_1} \) and \( GL_{r_2} \) act by \( r \mapsto gag^t \).

Lemma 5.3. Let \( k, l \in \mathbb{Z}_{>0}, r \in \mathbb{Z}_{>0} \).

(i) If \( k \neq l \pmod{2} \) then

\[
\text{Sym}^r(\mathfrak{gl}_r)_{GL_r, det^k} = \{0\}.
\]

(ii) If \( k \neq 0 \) and \( r \) is odd then

\[
\text{Sym}^r(\mathfrak{o}_r)_{GL_r, det^k} = \{0\}.
\]

Proof.

(i) The only algebraic characters of \( GL_r \) are powers of the determinant.

(ii) The stabilizer in \( GL_r \) of every matrix in \( \mathfrak{o}_r \) has an element with determinant different from 1. \( \square \)

Proof of Proposition 5.2. By Lemma 5.1, it is enough to show that for \( x = x_{r_1, r_2, s, t} \) with \( r_1 + s > 0 \) we have

\[
\text{Sym}^r(N_{x,P/Q}^G)^{K_x, \mathbb{Q}, \Delta K_x} = \{0\}.
\]

If \( s > 0 \) this follows from Lemma 5.2(i). Otherwise \( r_1 > 0 \) and, by Lemma 5.2(i), we have to show that

\[
\text{Sym}^r(\mathfrak{gl}_{r_1})_{GL_{r_1}, det^{t-m+1}} \otimes \text{Sym}^r(\mathfrak{o}_{r_2})_{GL_{r_2}, det^{t-m+1}} = \{0\}
\]

Note that since \( n \) is even, \( r_1 \) and \( r_2 \) are of the same parity. If they are even then (9) follows from Lemma 5.3(i), and otherwise from Lemma 5.3(ii). \( \square \)

5.1. Proof of Lemma 5.2. Let \( x = x_{r_1, r_2, s, t} \) be as in the lemma. We need to compute the space \( N_{x,P/Q}^G \) and its modular function. In order to do that we compute the conjugates of \( P \) and its Lie algebra \( \mathfrak{p} \) under \( x \).

Lemma 5.4. Let \( q := \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathfrak{q} \). Then \( x^{-1} qx = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \), where

\[
A = \begin{pmatrix}
    d_{11} & 0 & d_{14} & 0 & 0 \\
    b_{21} & a_{22} & b_{24} & a_{24} & a_{25} \\
    d_{41} & 0 & d_{44} & 0 & 0 \\
    b_{41} & a_{42} & b_{44} & a_{44} & a_{45} \\
    b_{51} - \omega_2 d_{51} & a_{52} & b_{54} - \omega_2 d_{54} & a_{54} & a_{55}
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
    0 & d_{12} & d_{13} & 0 & d_{15} \\
    a_{21} & b_{22} & b_{23} & a_{23} & b_{25} + a_{25} \omega_2 t \\
    0 & d_{42} & d_{43} & 0 & d_{45} \\
    a_{41} & b_{42} & b_{43} & a_{43} & b_{45} + a_{45} \omega_2 t \\
    a_{51} & b_{52} - \omega_2 d_{52} & b_{53} - \omega_2 d_{53} & a_{53} & b_{55} + a_{55} \omega_2 t - \omega_2 d_{55}
\end{pmatrix}
\]

\[
C = \begin{pmatrix}
    b_{11} & a_{12} & b_{14} & a_{14} & a_{15} \\
    d_{21} & 0 & d_{24} & 0 & 0 \\
    d_{31} & 0 & d_{34} & 0 & 0 \\
    b_{31} & a_{32} & b_{34} & a_{34} & a_{35} \\
    d_{51} & 0 & d_{54} & 0 & 0
\end{pmatrix}
\]

\[
D = \begin{pmatrix}
    a_{11} & b_{12} & a_{13} & b_{15} + a_{15} \omega_2 t \\
    0 & d_{22} & a_{23} & b_{25} + a_{25} \omega_2 t \\
    0 & d_{32} & a_{33} & b_{35} + a_{35} \omega_2 t \\
    a_{31} & b_{32} & a_{33} & b_{35} + a_{35} \omega_2 t \\
    0 & d_{52} & d_{53} & 0 & d_{55}
\end{pmatrix}
\]
Let Corollary 5.7. Denote
\[ \Delta_K(k) = \text{sgn}(a) \text{sgn}(b) \text{sgn}(c) \text{sgn}(d)^{m+1} |a|^{-m-r_1} |b|^{2s+2t+2-2m+1} |c|^{-r_1-s} |d|^{-r_1-s}. \]

From the previous lemma we obtain

**Corollary 5.5.** Recall the identification \( n \cong \text{Mat}_{n \times n}(\mathbb{R}) \) and let \( V \subset n \) denote the subspace consisting of matrices of the form

\[
\begin{pmatrix}
  n_{11} & n_{12} & 0 & n_{14} & n_{15} \\
  n_{12} & n_{22} & 0 & 0 & 0 \\
  n_{31} & 0 & 0 & n_{34} & 0 \\
  0 & 0 & 0 & 0 & 0 \\
  n_{41} & 0 & 0 & 0 & 0
\end{pmatrix},
\]

such that \( n_{22} = -n_{22} \).

Then \( V \) projects isomorphically onto \( n/(n \cap (x^{-1}px + \mathcal{Q})) \).

Now let us analyze the stabilizer \( K_x \). From Lemma 5.4 we obtain

**Corollary 5.6.**

(i) The Lie algebra \( p \cap x\mathcal{Q}x^{-1} \) consists of matrices \( \begin{pmatrix} A & B \\ 0 & -A^t \end{pmatrix} \) such that

\[
A = \begin{pmatrix}
  A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\
  0 & A_{22} & 0 & 0 & 0 \\
  0 & A_{32} & A_{33} & 0 & -\omega_2 B_{35} \\
  0 & A_{42} & 0 & A_{44} & \omega_2 B_{45} \\
  0 & A_{52} & 0 & 0 & A_{55}
\end{pmatrix},
\]

\[
B = \begin{pmatrix}
  B_{11} & B_{12} & B_{13} & B_{14} & B_{15} \\
  B_{12} & 0 & 0 & 0 & 0 \\
  B_{13} & 0 & B_{33} & 0 & B_{35} \\
  B_{14} & 0 & 0 & B_{44} & B_{45} \\
  B_{15} & B_{35} & B_{45} & 0 & 0
\end{pmatrix},
\]

\[ A_{55} \in \mathfrak{sp}(2t), B_{11} = B_{11}'^t, B_{33} = B_{33}'^t, B_{44} = B_{44}'^t. \]

(ii) Let \( p = \begin{pmatrix} A & B \\ 0 & (A^t)^{-1} \end{pmatrix} \in P \). Let \( k = (p, x^{-1}px) \in K_x \). The modular function of \( K_x \) is given by

\[ \Delta_{K_x}(k) = |A_{11}|^{2n-r_1+1} |A_{22}|^{-n+r_1+r_2} |A_{33}|^{n-r_1-s} |A_{44}|^{n-r_1-s+1}. \]

(iii) Let \( q = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \in \mathcal{Q} \cap x^{-1}Px \). Let \( k = (qx^{-1}, q) \in K_x \). Then \( k \) acts on \( V \) by

\[ k \cdot n = \text{pr}_V(AnD^{-1}), \]

where \( \text{pr}_V : n \rightarrow V \) denotes the projection.

**Corollary 5.7.** Denote

\[ \chi := L_{-m}^{-1} \cdot \Delta_K|_{K_x} \cdot \Delta_{K_x}^{-1} = \gamma^{m+1} \gamma^{(n-m)/2} \cdot \Delta_K|_{K_x} \Delta_{K_x}^{-1}. \]

Let

\[ q = \text{diag}(a, b, c, (c')^{-1}, \text{Id}, (a')^{-1}, (b')^{-1}, d, (d')^{-1}, \text{Id}). \]

Let \( k = (qx^{-1}, q) \in K_x \). Then

\[ \chi(k) = (\text{sgn}(a) \text{sgn}(b) \text{sgn}(c) \text{sgn}(d))^{m+1} |a|^{-m-r_1} |b|^{2s+2t-m+1} |c|^{-r_1-s} |d|^{-r_1-s}. \]

**Proof.**

\[ \gamma(q) = |a|^2 |b|^2 \quad \text{and} \quad \Delta_{\mathcal{Q}}(q) = |a|^{-2n} |b|^{-2n} \]

\[ qx^{-1} = \text{diag}((a')^{-1}, b, (d')^{-1}, (c')^{-1}, \text{Id}, a, (b')^{-1}, d, c, \text{Id}) \]

\[ \Delta_K(k) = |a|^{-3n-1} |b|^{-n+1} |c|^{-n-1} |d|^{-n-1}. \]
\[ \Delta_{K_x}(k) = |a|^{-2n+r_1-1} |b|^{-n+r_1+r_2} |c|^{-n+r_1+s-1} |d|^{-n+r_1+s-1} \]

Now we are ready to prove Lemma 6.2.

**Proof of Lemma 6.2.** If \( s > 0 \) then Sym\(^*(V)^{K\cdot X} = 0 \), since tensors cannot have negative homogeneity degrees. Otherwise, V involves only 3 blocks - the ones numbered 1, 2 and 5.

Let \( p \in \text{Sym}^{*}(V)^{K\cdot X} \). Identify \( K_x \) with \( x^{-1} P x \cap Q \) using the projection on the second coordinate.

Consider the action of the block \( A_{21} \). It can map any non-zero vector in the block \( n_{11} \) to any vector in the block \( n_{12} \). This action does not change any element in any other block of \( V \) (it does effect \( n_{22} \), but not its anti-symmetric part). Also, the character \( \chi \) does not depend on \( A_{21} \). Therefore \( p \) does not depend on the variables in the block \( n_{12} \).

In the same way, using the action of \( A_{32} \), we can show that \( p \) does not depend on the variables in the block \( n_{15} \). Therefore, \( p \) depends only on \( n_{11} \) and \( n_{12} \). Hence

\[ \text{Sym}^{*}(V)^{K\cdot X} \cong \text{Sym}^{*}(\mathfrak{gl}_{11})^{\text{GL}_{r_1}} |^{−m−r_1 \sgn m+1} \otimes \text{Sym}^{*}(\mathfrak{gl}_{22})^{\text{GL}_{r_2}} |^{21−m−1 \sgn m+1}. \]

\[ □ \]

6. Non-existence of an \( H \)-invariant functional for odd \( n \)

In this section we prove that if \( n \) is odd then there are no \( U_n \)-invariant functionals on the Speh representations and therefore there are no \( H \)-invariant functionals. We do that using \( K \)-type analysis. The maximal compact subgroup of \( G \) is \( K := O_{2n}(\mathbb{R}) \), and \( U_n = K \cap H \) is a symmetric subgroup of \( K \). We show that no \( K \)-type of \( \delta_m \) has a \( U_n \)-invariant vector.

The root system of \( K \) is of type \( D_n \), and we make the usual choice of positive roots

\[ \{ \varepsilon_i \pm \varepsilon_j : i < j \} \]

where \( \varepsilon_i \) is the \( i \)-th unit vector in \( \mathbb{R}^n \). With this choice, the highest weights of \( K \)-modules are given by integer sequences \( \mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n \) such that

\[ \mu_1 \geq \cdots \geq \mu_{n−1} \geq \mu_n \geq 0. \] (10)

**Remark 6.1.** From the definition of \( \pi^\infty_m \) we see that if \( n \) is odd and \( m \) is even then the central element \( −\text{Id} \in G \) acts by scalar \(-1\), and there are neither \( P \)-invariant nor \( U_n \)-invariant functionals on \( \delta^\infty_m \).

Since \( \delta^\infty_m \) is the irreducible quotient of \( \pi^\infty_m \), the following theorem follows from [HL99, Theorems 3.4.2 - 3.4.4] (see also [Sa09]).

**Theorem 6.2.** The \( K \)-types of \( \pi^\infty_m \) are given by sequences as in (10) with \( \mu_i \equiv m+1 \) (mod 2), while the \( K \)-types of the Speh representation \( \delta_m \) satisfy the additional condition \( \mu_n \geq m+1 \).

**Lemma 6.3.** If \( n \) is odd then no \( K \)-type \( (\mu_1, \ldots, \mu_n) \) with \( \mu_n \neq 0 \) has \( U_n \)-invariant vectors.

**Proof.** Let \( \rho \) be an irreducible representation of \( K \) with \( \mu_n \neq 0 \). Suppose that \( \rho \) has a non-zero \( U_n \)-invariant vector. Then \( \rho = \rho_1 \oplus \rho_2 \), where \( \rho_i \) are irreducible non-zero representations of \( K^0 = SO_{2n}(\mathbb{R}) \). The pair \( (K, U_n) \) is a symmetric pair of compact groups and therefore a Gelfand pair. Hence the \( U_n \)-invariant vector is unique up to a scalar and belongs to one of the \( \rho_i \). Denote it by \( v \) and say \( v \in \rho_1 \).

Consider \( g := \begin{pmatrix} \text{Id} & 0 \\ 0 & -\text{Id} \end{pmatrix} \in K \). Since \( n \) is odd, \( g \notin K^0 \). Hence \( \rho(g)v \notin \rho_1 \), since otherwise \( \rho \) would be reducible. However, \( g \) normalizes \( U_n \) and hence \( \rho(g)v \) is \( U_n \)-invariant and therefore proportional to \( v \). Contradiction. \[ □ \]

**Corollary 6.4.** If \( n \) is odd then there are no \( U_n \)-invariant functionals on \( \delta^\infty_m \).

**Proof.** By Remark 6.1 we can assume that \( m \) is odd. Then by Lemma 6.3 and Theorem 6.2 no \( K \)-type of \( \delta_m \) has a \( U_n \)-invariant vector. Therefore, the space of \( K \)-finite vectors, which decomposes to a direct sum of \( K \)-types, does not have a \( U_n \)-invariant functional. This space is dense in \( \delta^\infty_m \), hence there are no \( U_n \)-invariant functionals on \( \delta^\infty_m \) either. \[ □ \]
Remark 6.5. Using the Cartan-Helgason theorem and the table in [Kna85] Appendix C, [2] it can be shown that the $K$-types that have $U_{n^-}$-invariant vectors are of the form $\mu_{2i-1} = \mu_i$, for $1 \leq i \leq n/2$ and if $n$ is odd then $\mu_n=0$, which gives an alternative proof of Lemma 6.3.

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