A Nonlinear Moment Model for Radiative Transfer Equation in Slab Geometry

Yuwei Fan,† Ruo Li,‡ and Lingchao Zheng

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Abstract

This paper is concerned with the approximation of the radiative transfer equation for a grey medium in the slab geometry by the moment method. We develop a novel moment model inspired by the classical $P_N$ model and $M_N$ model. The new model takes the ansatz of the $M_1$ model as the weight function and follows the primary idea of the $P_N$ model to approximate the specific intensity by expanding it around the weight function in terms of orthogonal polynomials. The weight function uses the information of the first two moments, which brings the new model the capability to approximate an anisotropic distribution. Mathematical properties of the moment model are investigated, and particularly the hyperbolicity and the characteristic structure of the Riemann problem of the model with three moments are studied in detail. Some numerical simulations demonstrate its numerical efficiency and show its superior in comparison to the $P_N$ model.

Keywords: Radiative transfer equation; slab geometry; grey medium; moment method; anisotropic; hyperbolicity.

1 Introduction

In kinetic theory, the radiative transfer equation (RTE), which describes the particle propagation and interaction with a background medium, has applications in a wide variety of subjects, such as neutron transport in reactor physics [34, 14], light transport in atmospheric radiative transfer [29], heat transfer [25] and optical imaging [24, 37]. The RTE is a high-dimensional integro-differential kinetic transport equation for the specific intensity of radiation, and its high-dimensionality makes it particularly challenging to solve numerically. Currently, numerical methods for solving the RTE can be categorized into two groups: the probabilistic approaches, for example, the direct simulation Monte Carlo methods [3, 22, 12] and the deterministic schemes [4, 26, 35, 23, 11, 13, 32, 1, 16]. The Monte Carlo method solves the RTE by simulating a lot of individual particles and determine the intensity by taking a statistical average over particles. This method has made remarkable successes in solving the RTE, but the statistical noise is an important issue for its accuracy.

Among the deterministic methods, two major approaches are the discrete ordinates method [4, 26, 35] and the moment methods [23, 11, 13, 1]. As the most popular deterministic method in solving the RTE, the discrete-ordinates method ($S_N$) solves the transfer equation along a discrete set of angular directions. The main flaw of this method is the so-called ray effects [26] because the number of the discrete angular directions is finite and the particles are only allowed to move along these directions.

Moment method depicts the evolution of a finite number of moments of the specific intensity. Typically, the governing equations of a lower order moments depend on higher order moments. Hence a moment closure is required to close the moment system. A common method for the moment closure is to construct an ansatz to approximate the specific intensity. Based on this idea, the two most popular moment methods are the spherical harmonics method ($P_N$) [34] and the maximum entropy method ($M_N$) [28, 13, 32]. The $P_N$ model constructs the ansatz by expanding the specific intensity around the equilibrium in terms of spherical harmonics in the velocity direction. The resulting model is a linear symmetric hyperbolic system and easy to implement, but it may lead to nonphysical oscillations, which
may lead to negative particle concentration \cite{6, 7, 31}. The \( M_N \) model constructs the ansatz is based on the principle of maximum entropy \cite{28, 13}. The resulting model retains fundamental properties of the underlying kinetic equations such as hyperbolicity, entropy dissipation, and positivity of the intensity. However, for the the case \( N \geq 2 \), there is no algebraic expression of the closed moments, and one has to solve an ill-conditioned optimization problem to obtain the closed moments in the implementation, which strongly limits the applications of the \( M_N \) model.

In this paper, we first propose two points of view — the entropy-based viewpoint and the weighted expansion viewpoint — on the relationship between the \( P_N \) model and the \( M_N \) model. Based on the entropy-based viewpoint, we show that both the models can be attributed to a minimization problem with specific object functions. The weighted expansion viewpoint reveals that the two models can also be treated as approximating the intensity by expanding it around a given weight function in terms of orthogonal polynomials. These viewpoints indicate that one can construct a new moment model by choosing a new weight function.

Although there is no algebraic expression of the closed moment for the \( M_N \) model for \( N \geq 2 \), the expression of the \( M_1 \) model is simple. We take the ansatz of the \( M_1 \) model as the weight function to develop a novel arbitrary order moment model (we call it \( MP_N \) here and put the explanation in Section 3 for the RTE. Since the weight function contains the information of the zeroth-order moment and the first-order moment, the ansatz of the new model is expected to have the capability to approximate an anisotropic distribution. The new moment model has a simple algebraic expression and easy to implement. We study the hyperbolicity and the characteristic structure of the \( MP_2 \) model in detail. This model is hyperbolic, and the judging criteria on the wave-type are investigated. Comparison with the \( P_2 \) and \( M_2 \) models, the \( MP_2 \) lies in the betweenness of these two models and can be viewed as an approximation of the \( M_2 \) model. Numerical simulations are performed to study the numerical behavior of the new moment model and show that the \( MP_N \) model has the power to simulate strong anisotropic intensity, and has superiority on the \( P_N \) model.

The rest of this paper is arranged as follows. In Section 2 we briefly introduce the radiative transfer equation and moment method, and present two viewpoints on the relationship between the \( P_N \) and \( M_N \) models. In Section 3, we derive the arbitrary order \( MP_N \) model and investigate the \( MP_2 \) model in detail. Numerical issues, including the numerical scheme, details on implementation, and numerical results, are presented in Section 4. The paper ends with a conclusion in Section 5.

\section{Radiative transfer equation and moment method}

In this paper, we study the time-dependent radiative transfer equation (RTE) for a grey medium in the slab geometry as

\begin{equation}
\frac{1}{c} \frac{\partial I}{\partial t} + \frac{\partial I}{\partial z} = S(I),
\end{equation}

where \( c \) is the speed of light, \( I = I(z,t,\mu) \) is the \textit{specific intensity} of radiation, and \( \mu \in [-1,1] \) is the velocity related variable such that \( \arccos(\mu) \) represents the angle between the photon velocity and the \( z \)-axis. The right hand side \( S(I) \) denotes the actions by the background medium on the photons. Here we adopt a common form of \( S(I) \) given in \cite{5, 30} as

\begin{equation}
S(I) = -\sigma_t I + \frac{1}{2} \sigma_a \sigma_s T^4 + \frac{1}{2} \sigma_s \int_{-1}^{1} I \, d\mu + \frac{s}{2},
\end{equation}

where \( a \) is the radiation constant, and \( s = s(z) \) is an isotropic external source of radiation. The scattering coefficient \( \sigma_s \), and the absorption coefficient \( \sigma_a \) depend on the position \( z \) and the material temperature \( T(z,t) \), and the total opacity coefficient is \( \sigma_t = \sigma_a + \sigma_s \). Denote the \( k \)-th moment of the specific intensity by

\begin{equation}
(I)_k \triangleq \int_{-1}^{1} \mu^k I(\mu) \, d\mu, \quad k \in \mathbb{N}.
\end{equation}

For simplicity of notations, in \cite{23} and the following discussion, the explicit dependence of the specific intensity and the moments on spatial coordinate and time has been suppressed (i.e., \( I_k = (I)_k(z,t) \), \( I(\mu) = I(z,t,\mu) \)). The evolution equation of the internal energy \( e \) of the background medium is

\begin{equation}
\frac{\partial e}{\partial t} = \sigma_a (I_0 - acT^4).
\end{equation}
The relationship between the temperature $T$ and the internal energy $e$ is problem dependent. We will assign it in the numerical examples when necessary.

For the whole system, in the absence of any external source of radiation i.e. $s = 0$, the total energy $e + \langle I \rangle_0/e$ is conserved:

$$\frac{\partial e}{\partial t} + \frac{1}{c} \frac{\partial (I)_0}{\partial t} + \frac{\partial (I)_1}{\partial z} = 0. \quad (2.5)$$

Moreover, for later usage, we introduce the equilibrium of the RTE as

$$I_{eq} = \frac{(I)_0}{2}. \quad (2.6)$$

### 2.1 Moment method for RTE

Multiplying (2.1) by $\mu^k$ and integrating it with respect to $\mu$ over $[-1, 1]$ yields the following equations

$$\frac{1}{c} \frac{\partial (I)_k}{\partial t} + \frac{\partial (I)_{k+1}}{\partial z} = \langle S(I) \rangle_k, \quad k \in \mathbb{N}, \quad (2.7)$$

where

$$\langle S(I) \rangle_k = -\sigma_t (I)_k + \frac{1 - (-1)^{k+1}}{2k+2} (\sigma_s acT^4 + \sigma_s (I)_0 + s).$$

In (2.7) the governing equation of $\langle I \rangle_k$ also depends on $\langle I \rangle_{k+1}$, which means that the full system contains infinite number of equations. To derive a moment model for (2.1), we first truncate the system by discarding all the governing equations of $\langle I \rangle_k, k > N$, for a given integer $N \in \mathbb{N}$. Clearly, the truncated system is not closed, due to its dependence on the $(N + 1)$-th moment $\langle I \rangle_{N+1}$, thus we have to apply a so-called moment closure to this system. Normally, the moment closure is to find an approximation for $\langle I \rangle_{N+1}$ formulated as

$$\langle I \rangle_{N+1} \approx E_{N+1} = E_{N+1}(\langle I \rangle_0, \cdots, \langle I \rangle_N). \quad (2.8)$$

To achieve this goal, a popular method is to construct an ansatz for the specific intensity. Precisely, let $E_k, k = 0, \cdots, N$, be the known moments for a certain unknown specific intensity $I$. Then one proposes an approximation $\hat{I}(\mu; E_0, \cdots, E_N)$, which is called ansatz of $I$, such that

$$\langle \hat{I}(\mu; E_0, \cdots, E_N) \rangle_k = E_k, \quad k = 0, \cdots, N, \quad (2.9)$$

and meanwhile $\hat{I}$ is uniquely determined by (2.9). For the $(N + 1)$-th moment of $I$, it is then directly approximated by the $(N + 1)$-th moment of $\hat{I}$, i.e.,

$$E_{N+1} = \langle \hat{I}(\mu; E_0, \cdots, E_N) \rangle_{N+1}. \quad (2.10)$$

The resulting moment system is

$$\frac{1}{c} \frac{\partial E_k}{\partial t} + \frac{\partial E_{k+1}}{\partial z} = \langle S(\hat{I}) \rangle_k, \quad k = 0, \cdots, N.$$  

In the following, we briefly review the two most popular moment models for the RTE: the $P_N$ model [23] and the $M_N$ model [13, 28, 92], by specifying a certain ansatz $\hat{I}$.

#### 2.1.1 $P_N$ model

The $P_N$ model is a counterpart of the Grad’s moment method [19] in the RTE. It expands the specific intensity around the equilibrium in terms of orthogonal polynomials with respect to $\mu$. For the RTE, the normalized equilibrium is a constant, hence the corresponding orthogonal polynomials are the Legendre polynomials. Denote the monic Legendre polynomial of degree $m$ by $P^{(m)}(\mu)$, then the ansatz for the $P_N$ model, denoted by $\hat{I}_P$, is

$$\hat{I}_P(\mu; E_0, \cdots, E_N) = \sum_{m=0}^{N} f_m(E_0, \cdots, E_N) P^{(m)}(\mu). \quad (2.11)$$

Due to the moment constraints (2.9), we have

$$\sum_{m=0}^{N} (P^{(m)})_k f_m = E_k, \quad k = 0, \cdots, N. \quad (2.12)$$
Hence, the expansion coefficients $f_m, m = 0, \ldots, N$ are uniquely determined by the moments $E_k, k = 0, \ldots, N$ through the linear system (2.12). The moment closure is then given as

$$E_{N+1} = \sum_{m=0}^{N} J_m \langle P^{(m)} \rangle_{N+1},$$

which is a linear function of $E_k, k = 0, \ldots, N$.

The $P_N$ model is widely used in the numerical simulations of RTE due to its good mathematical properties. For example, it has a simple analytical form such that the evaluation of the flux is fast. Its system can be transformed into a symmetric hyperbolic system [18]. Moreover, the $P_N$ model formally converges in an $L^2$ setting to the solution of the transport equation as $N \to \infty$ [11].

Meanwhile, the $P_N$ model also suffers some drawbacks. Its solution may have undesirable oscillations, which may lead to negative particle concentration [7, 31]. Its approximation rate to the RTE is low, such that a lot of moments are required in numerical simulations [2, 15].

### 2.1.2 $M_N$ model

The $M_N$ model takes the solution of the entropy minimization problem, denoted by $\hat{I}_M$, to close the system. Precisely, we have

$$\hat{I}_M = \arg\min \int_{-1}^{1} \eta(\hat{I}) \, d\mu,$$

s.t. $\langle \hat{I} \rangle_k = E_k, \quad k = 0, 1, \ldots, N$. (2.13)

Here $\eta : \mathbb{R} \to \mathbb{R}$ is the Bose-Einstein entropy

$$\eta(f) = f \log f - (1 + f) \log(1 + f),$$

for photon. The intensity takes the form [28, 13]

$$\frac{2h\nu^3}{c^2} \left( \exp \left( \frac{h\nu}{k_B \sum_{i=0}^{N} \alpha_i \mu^i} \right) - 1 \right)^{-1},$$

where $\alpha_i, i = 0, \ldots, N$ are the Lagrange multipliers to be determined, $\hbar$ is the Planck constant, and $k_B$ is the Boltzmann constant. Integrating (2.15) with respect to the frequency $\nu$ on $[0, \infty]$, we obtain the intensity for the grey medium case

$$\hat{I}_M = \frac{\sigma}{\left( \sum_{i=0}^{N} \alpha_i \mu^i \right)^{\frac{3}{4}}},$$

where $\sigma$ is the Stefan-Boltzmann constant. Clearly, the Lagrange multipliers $\alpha_i, i = 0, \ldots, N$ are uniquely determined by the moments $E_k, k = 0, \ldots, N$, and the moment closure $E_{N+1} = \langle \hat{I}_M \rangle_{N+1}$ follows.

Particularly, if $N = 0$, the ansatz is the equilibrium, i.e.,

$$\hat{I}_M = I_{eq} = \frac{E_0}{2}.$$ (2.17)

If $N = 1$, the Lagrange multipliers $\alpha_i, i = 0, 1$ can be directly solved, and the solution of the minimizing entropy problem is given as

$$\hat{I}_M(\mu) = E_0 \frac{\varepsilon}{(1 + \alpha \mu)^{\frac{3}{2}}},$$

where

$$\alpha = -\frac{3E_1/E_0}{2 + \sqrt{4 - 3(E_1/E_0)^2}}.$$ (2.18)

The corresponding moment closure is

$$E_2 = E_0 \frac{3 + 4(E_1/E_0)^2}{5 + 2\sqrt{4 - 3(E_1/E_0)^2}}.$$ (2.19)
However, for $N \geq 2$, there is no algebraic expression of the Lagrange multipliers $\alpha_i$ with respect to the moments $E_k$. Thus, an expensive iterative procedure is required to solve the Lagrange multipliers. This drawback of the $M_N$ model strongly limited its applications with $N \geq 2$, though it has been demonstrated that the $M_N$ models yield promising results [20].

As for the properties of the system, the $M_N$ model retains many fundamental properties from the kinetic formalism. The characteristic speed of this model is no larger than the speed of light, which agrees with the fact that information cannot travel faster than the speed of light. The ansatz $I_M$ is always positive, the $M_N$ model is equipped with entropy, and the resulting system of equations can be transformed into a symmetric hyperbolic system.

For the $M_N$ model, though the drawback in the numerical simulations hinders its application, its praised properties motivate researchers to construct new moment models. The moment model developed in the next section is partially inspired by the $M_N$ model.

2.1.3 Relationship between $P_N$ and $M_N$ model

In this subsection, we propose two points of view — the entropy-based viewpoint and the weighted expansion viewpoint — on the relationship between the $P_N$ and $M_N$ model, and show our motivation on the novel moment model developed in this paper. All the discussion in this subsection is formal, not rigorous.

Entropy-based viewpoint If the specific intensity $I$ is close to its equilibrium, i.e., $I \sim I_{eq}$, using the Taylor expansion on $\ln(1 + x)$, we can approximate the Bose-Entropy entropy (2.14) as

$$\eta(I) \approx \eta_{app}(I) \overset{\Delta}{=} I_{eq} \ln(I_{eq}) + (1 + \ln(I_{eq}))(I - I_{eq}) - (1 + I_{eq})\ln(1 + I_{eq})$$

$$- (1 + \ln(1 + I_{eq}))(I - I_{eq}) + \frac{(I - I_{eq})^2}{2I_{eq}(1 + I_{eq})},$$

by discarding high order term with respect to $O((I - I_{eq})^3)$. Let $\eta^P(I) = \eta_{app}(I)$ or equivalently $I_{eq}^2$, then one can easily check that the ansatz of the $P_N$ model can be obtained by minimizing $\eta^P$ as

$$\hat{I}_P = \arg\min \int_{-1}^{1} \eta^P(\hat{I}) d\mu,$$

$$\text{s.t. } \langle \hat{I} \rangle_k = E_k, \quad k = 0, 1, \ldots, N.$$  \hspace{1cm} (2.21)

This indicates that the $P_N$ model can also be brought into the framework of the “entropy” minimization problem by choosing a proper object function $\eta$. It motivates us to construct new moment models by selecting a different object function. On the other hand, the start point of the $P_N$ model is expanding the velocity distribution around the equilibrium, where the distribution is implicitly assumed to be close to the equilibrium. It is consistent with the assumption in the approximation of $\eta_{app}$.

Weighted expansion viewpoint For the $P_N$ model, the approach to construct its ansatz can be generalized as an expansion of the velocity distribution function around a weight function $\omega(\mu)$ in terms of orthogonal polynomials. Precisely, given a weight function $\omega(\mu)$ satisfying

$$\omega(\mu) \geq 0, \quad \int_{-1}^{1} \omega(\mu) d\mu = 1,$$

we denote the monic orthogonal polynomial of degree $m$ by $\phi_m(\mu)$ such that

$$\int_{-1}^{1} \omega(\mu) \phi_m(\mu) \phi_n(\mu) d\mu = \delta_{mn}c_m,$$  \hspace{1cm} (2.23)

where $\delta$ is the Kronecker delta and $c_m$ are non-zero constants. It is worth to point out that the weight function $\omega$ is allowed to depend on the moments $E_k$. Then we construct the ansatz as

$$\hat{I} = \omega(\mu) \sum_{m=0}^{N} f_m(E_0, \ldots, E_N) \phi_m(\mu).$$  \hspace{1cm} (2.24)
Due to the moment constraints (2.19), we obtain the system
\[ \sum_{m=0}^{N} \int_{-1}^{1} \omega(\mu)\phi_m(\mu)\mu^k \, d\mu = E_k, \quad k = 0, \ldots, N, \] (2.25)
which uniquely determines the ansatz (2.24).

For the \( P_N \) model, the weight function is \( \omega_p = 1/2 \), and the orthogonal polynomials are the Legendre polynomials. For the \( M_N \) model, the weight function is the normalization of the ansatz \( I_M \) itself, and the coefficients satisfy \( f_m = 0, m = 1, \ldots, N \). To check it is not difficult but rather complex. We refer readers to [17, Section 5.4] and [9, Section 4.2.2] for details.

In the above, we also bring the \( M_N \) model in the framework of the \( P_N \) model. The key point of this framework is the weight function. Once the weight function is given, one can directly obtain the corresponding moment model following the above routines.

These viewpoints build a bridge between the \( P_N \) and \( M_N \) models and also present two methods to construct new models. The following part of this paper will focus on the new model and its analysis.

### 3 M₁-based moment model

The viewpoints in Subsection 2.1.3 provide methods to construct new moment models. The remaining issue is how to choose the weight function. A weight function, which contains much information of the moments, usually resulting in a strong nonlinear system, for instance, the \( M_N \) model. Such a system is expected to have a good approximation to the RTE, but the evaluation of its flux is, in general, expensive. Hence, one has to make a trade-off between the numerical efficiency and the approximation rate.

#### 3.1 M₁-based moment system

The ansatz of the \( M_N \) model with \( N = 1 \) (we call it \( M_1 \) hereafter) contains the zeroth-order moment \( E_0 \) and the first order moment \( E_1 \). This brings it the capability to approximate an anisotropic distribution. If we take the ansatz of the \( M_1 \) model as the weight function to construct a new model, the corresponding ansatz is expected to have a better approximation on the anisotropic distributions than that of the \( P_N \) model. In this section, based on this idea, we develop a novel moment model and study its properties in detail.

Let the weight function be
\[ \omega^{[\alpha]}(\mu) = \frac{\varepsilon}{(1 + \alpha \mu)^\alpha}, \quad \alpha \in (-1, 1), \quad \text{such that} \quad \int_{-1}^{1} \omega^{[\alpha]}(\mu) \, d\mu = 1, \] (3.1)
so \( \varepsilon = \frac{3(1 - \alpha^2)^3}{2(3 + \alpha^2)} > 0 \). Here \( \alpha \) is a parameter to be determined. Denote by \( \phi_j^{[\alpha]}(\mu) \) the monic orthogonal polynomial of \( \mu \) with degree \( j \) respect to the weight function \( \omega^{[\alpha]}(\mu) \). We introduce the moments of the weight function and the inner product as
\[ \mathcal{E}_k \equiv \langle \omega^{[\alpha]}(\mu) \rangle_k, \quad \langle f, g \rangle^{[\alpha]} \equiv \int_{-1}^{1} f(\mu)g(\mu)\omega^{[\alpha]}(\mu) \, d\mu. \] (3.2)

Then by Gram-Schmidt orthogonalization, the polynomials \( \phi_j^{[\alpha]}(\mu) \) are obtained recursively as
\[ \phi_0^{[\alpha]}(\mu) = 1, \quad \phi_j^{[\alpha]}(\mu) = \mu^j - \sum_{k=0}^{j-1} \frac{A_{j,k}}{A_{k,k}} \phi_k^{[\alpha]}(\mu), \quad j \geq 1, \] (3.3)
where \( A_{j,k} = \langle \mu^j, \phi_k^{[\alpha]}(\mu) \rangle^{[\alpha]} \). The orthogonality of \( \phi_j^{[\alpha]} \) indicates that
\[ A_{k,k} = \langle \phi_k^{[\alpha]}, \phi_k^{[\alpha]} \rangle^{[\alpha]} = \delta_{k,j}, \quad A_{k,j} = 0, \quad k < j. \]

Applying the inner product on (3.3) and \( \mu^i \) yields the relationship between \( A_{i,j} \) and \( \mathcal{E}_k \) as
\[ A_{0,0} = 1, \quad A_{i,j} = \mathcal{E}_{i+j} - \sum_{k=0}^{j-1} \frac{A_{j,k}A_{i,k}}{A_{k,k}}, \quad 1 \leq j \leq i. \]
The corresponding ansatz \( \hat{I} \) is defined as

\[
\hat{I}(\mu; E_0, \cdots, E_N) \equiv \sum_{i=0}^{N} f_i \Phi_i^{[\alpha]}(\mu),
\]

where \( \Phi_i^{[\alpha]}(\mu) = \phi_i^{[\alpha]}(\mu)\omega^{[\alpha]}(\mu), i = 0, 1, \cdots, N \) are the basis functions, and \( f_i \) are the expansion coefficients to be determined by the moment constraints (2.25). Thanks to the orthogonality of \( \phi_i^{[\alpha]} \), we have

\[
f_i = \frac{1}{A_{i,i}} \int_{-1}^{1} \phi_i^{[\alpha]}(\mu) \hat{I}(\mu) \, d\mu.
\]

Substituting the recursive relationship (3.3) into the upper equation yields the following recursive formulation for \( f_i \), which are functions dependent on \( E_i \).

\[
f_0 = E_0, \quad f_i = \frac{1}{A_{i,0}} \left( E_i - \sum_{j=0}^{i-1} A_{i,j} f_j \right), \quad 0 \leq i \leq N.
\]

Thus the explicit formation of the ansatz (3.4) is obtained. The moment closure is then given as

\[
E_{N+1} = \sum_{k=0}^{N} f_k A_{N+1,k}.
\]

If \( \alpha \equiv 0 \), then \( \omega^{[\alpha]} = 1/2 \), the orthogonal polynomials are the Legendre polynomials, and the resulting system is the \( P_N \) model. If we set

\[
\alpha = -\frac{3E_1/E_0}{2 + \sqrt{4 - 3(E_1/E_0)^2}},
\]

the weight function is the normalization of the ansatz of the \( M_1 \) model. Some calculations yield

\[
f_0 = E_0, \quad f_1 = 0, \quad E_0 = 1, \quad E_1 = \frac{E_1}{E_0}.
\]

In the following part of the paper, we always use this setup.

Notice that the moment model uses the ansatz of the \( M_1 \) model as the weight function, and generates the arbitrary order models following the idea of the \( P_N \) model. We call the moment model as the \( MP_N \) model in the following.

In the Subsection 2.1.3, we have put the \( P_N \) model into the framework of the entropy minimization form with the corresponding object function \( \eta_P(\hat{I}) = \frac{\hat{I}^2}{I_{eq}} \). For the \( MP_N \) model, let \( \eta_\omega = \frac{I^2}{\omega_{\alpha}} \), then one can easily check that the ansatz (3.4) is also the minimizer of the following problem

\[
\text{argmin} \int_{-1}^{1} \eta_\omega(\hat{I}) \, d\mu, \quad \text{s.t.} \quad \langle \hat{I} \rangle_k = E_k, \quad k = 0, 1, \cdots, N.
\]

We compare the approximation efficiency of the \( MP_N \) model and the \( P_N \) model, and select the following two intensity as examples:

\[
I^{(1)} = \frac{6}{5\pi} \frac{1}{1 - \frac{8}{3} \mu + \mu^2}, \quad I^{(2)} = \frac{3}{2 \sinh(3)} \exp(3\mu).
\]

The corresponding profiles of the intensity are presented in Figure 1. Clearly, the ansatz of the \( MP_N \) model shows advantage in the approximating such anisotropic distributions, because of its specific weight function.

### 3.2 \( MP_2 \) model

The complex form of \( MP_N \) (3.5) makes it not easy to investigate the \( MP_N \) model with arbitrary order. Here we provide a perspective on the \( MP_N \) model by studying the simplest non-trivial case, i.e., \( N = 2 \) case in detail.
3.2.1 $MP_2$ moment system

Direct calculation on $\ref{3.3}$ yields the orthogonal polynomials

$$
\phi_0^{[\alpha]}(\mu) = 1, \\
\phi_1^{[\alpha]}(\mu) = \mu - E_1, \\
\phi_2^{[\alpha]}(\mu) = \mu^2 - E_2 - \beta(\mu - E_1), \quad \beta = \frac{E_3}{E_2 - E_1^2}.
$$

The expansion coefficients are given as

$$
f_0 = E_0, \quad f_1 = 0, \quad f_2 = \frac{E_2 - E_0E_2}{E_4 - E_2^2 - \beta(E_4 - E_1E_2)},
$$

and the ansatz $\hat{I}(\mu)$ is

$$
\hat{I}(\mu) = \omega^{[\alpha]}(\mu)(f_0 + f_2\phi_2^{[\alpha]}(\mu)).
$$

Then, the moment closure is directly obtained by the third moment of the intensity

$$
E_3 = \langle \hat{I}(\mu) \rangle_3 = E_0E_3 + f_2E_5 - E_2E_3 - \beta(E_4 - E_1E_3).
$$

For the RTE, the positivity of the specific intensity $I$ provides constraints on the moments. Based on this, we introduce realizable domain for the RTE.

**Definition 1.** The realizable domain is the set of moments where each point corresponds to a positive intensity, i.e.,

$$
\Omega_R \triangleq \{(E_0, E_1, E_2)^T : \exists I(\mu) > 0, \langle I \rangle_k = E_k, k = 0, 1, 2\}.
$$

By the Cauchy-Schwarz inequality, $|E_1| < E_0$, and $E_1^2 < E_0E_2$ have to be fulfilled for a positive intensity $I$. Direct calculations indicate that the realizable region is

$$
\Omega_R = \{(E_0, E_1, E_2)^T : E_0 > 0, E_2 < E_0, E_1^2 < E_0E_2\}.
$$

Hereafter we focus on the $MP_2$ model in the realizable domain $\Omega_R$. Let $sgn(x)$ be the sign function

$$
sgn(x) = \begin{cases} 
-1, & x < 0, \\
0, & x = 0, \\
1, & x > 0.
\end{cases}
$$

Then we have the following properties on the closure of $MP_2$.

**Property 2.** For the $MP_2$ moment system, the closed moment $E_3 = E_3(E_0, E_1, E_2), (E_0, E_1, E_2) \in \Omega_R$, satisfies the following properties:
a). \( \text{sgn} \left( \frac{\partial E_3}{\partial E_0} \right) = -\text{sgn}(E_1); \)

b). \( \frac{\partial E_3}{\partial E_1} \bigg|_{E_1=0} > 0; \)

c). \( \text{sgn} \left( \frac{\partial E_3}{\partial E_2} \right) = \text{sgn}(E_1); \)

d). \( E_3 \) is linear dependent on \( E_2 \), i.e. \( \frac{\partial^2 E_3}{\partial E_2^2} = 0; \)

e). \( \text{sgn} \left( E_3 - E_3^1 - (E_2 - E_2^1) \frac{\partial E_3}{\partial E_2} \right) = \text{sgn}(E_1). \)

These properties depict the behavior of the closed moment \( E_3 \). One can directly check these properties with the help of (3.12) and Figure 2. But the rigorous proof is tedious, and we put it in the Appendix A.

Figure 2: Contour of \( \frac{E_3}{E_0} \) of the \( P_2 \) model, \( MP_2 \) model, and \( M_2 \) model. The range of \( \frac{E_3}{E_0} \) for the \( MP_2 \) and \( M_2 \) model is \((-1, 1)\) in the realizable domain, while that for the \( P_2 \) model is \((-3/5, 3/5)\).

Figure 3: Profile of \( E_3/E_0 \) with respect to \( E_1/E_0 \) for given \( E_2/E_0 \) for the \( P_2 \) model, the \( MP_2 \) model, and the \( M_2 \) model.

Now we compare the \( MP_2 \) model with the \( P_2 \) model and the \( M_2 \) model. Figure 2 presents the contour of \( \frac{E_3}{E_0} \) in \( \Omega_R \) for the three models, and Figure 3 presents some cross sections with respect to \( \frac{E_3}{E_0} = 1/5, 1/3, 81/100 \). The \( P_2 \) model is a linear system, hence its closed moment \( E_3 \) is linearly dependent on the moments \( E_0, E_1, \) and \( E_2 \). The \( M_2 \) model is a nonlinear system due to its complex ansatz. The \( MP_2 \) model is also a nonlinear system because of its weight function, which contains the information of \( E_0 \) and \( E_1 \). Compared with the \( P_2 \) and \( M_2 \) model, the \( MP_2 \) model falls in the middle. In the sense of the closed moment, the \( MP_2 \) can be treated as an approximation of the \( M_2 \) model.
3.2.2 Hyperbolicity

Denote the relevant moments and the flux by
\[ U = (E_0, E_1, E_2)^T, \quad F(U) = (E_1, E_2, E_3)^T, \]
then the moment equation is given by
\[ \frac{1}{c} \frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial z} = S, \]
with \( S = (S_0, S_1, S_2)^T \), and \( S_k = \langle S(I) \rangle_k \). We declare the main conclusion of this subsection in the following theorem.

**Theorem 3.** The \( MP_2 \) system (3.16) is hyperbolic for any \( U \in \Omega_R \).

Before the proof of the theorem, we introduce the following notations and lemma. The Jacobian of the \( M_1-P_2 \) model is
\[ J = \frac{\partial F}{\partial U} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{\partial E_3}{\partial E_0} & \frac{\partial E_3}{\partial E_1} & \frac{\partial E_3}{\partial E_2} \end{pmatrix}. \]

Its characteristic polynomial is
\[ p(\lambda) = \lambda^3 - \frac{\partial E_3}{\partial E_2} \lambda^2 - \frac{\partial E_3}{\partial E_1} \lambda - \frac{\partial E_3}{\partial E_0}, \]
which satisfies the following property:

**Lemma 4.** For any \( U \in \Omega \), we have
\[ \text{sgn} \left( p \left( \frac{E_1}{E_0} \right) \right) = -\text{sgn}(E_1). \]

**Proof.** If \( E_1 = 0 \), then \( p(0) = -\frac{\partial E_3}{\partial E_0} = 0 \) thanks to Property 2a). So (3.19) holds.

Next we assume \( E_1 \neq 0 \). Noticing that \( \frac{E_3}{E_0} \) is a function of \( \frac{E_1}{E_0} \) and \( \frac{E_2}{E_0} \), we have
\[ \frac{\partial E_3}{\partial E_0} = \frac{E_3}{E_0} - \frac{\partial E_3}{\partial E_2} \frac{E_2}{E_0} - \frac{\partial E_3}{\partial E_1} \frac{E_1}{E_0}. \]

Therefore, \( p \left( \frac{E_1}{E_0} \right) \) can be simplified as
\[ p \left( \frac{E_1}{E_0} \right) = \left( \frac{E_1}{E_0} \right)^3 - \frac{\partial E_3}{\partial E_2} \left( \frac{E_1}{E_0} \right)^2 - \frac{\partial E_3}{\partial E_1} \frac{E_1}{E_0} - \frac{\partial E_3}{\partial E_0} = \left( \frac{E_1}{E_0} \right)^3 - \frac{E_3}{E_0} - \left( \frac{E_1}{E_0} \right)^2 \frac{E_2}{E_0} \frac{\partial E_3}{\partial E_2}. \]

According to Property 2b), i.e., \( \frac{\partial^2 E_3}{\partial E_2^2} = 0 \), we obtain
\[ \frac{\partial p \left( E_1/E_0 \right)}{\partial E_2} = -\frac{1}{E_0} \frac{\partial E_3}{\partial E_2} + \frac{1}{E_0} \frac{\partial E_3}{\partial E_2} = 0, \]
which indicates that \( p \left( \frac{E_1}{E_0} \right) \) is independent of \( E_2 \). Hence, we can set \( E_2 \) as any available value, for instance \( E_2 = E_0 \), where \( f_2 = 0 \), and then \( E_3 = E_0 E_3 \). In this case, we have
\[ p \left( \frac{E_1}{E_0} \right) = E_3^3 - E_3 - (E_1^2 - E_2) \frac{\partial E_3}{\partial E_2}, \]
whose sign is different from that of \( E_1 \), due to Property 2b). This completes the proof. \( \square \)
Proof of Theorem \[3.16\]. To prove the system \(3.16\) is hyperbolic, that is, the matrix \(J\) is real diagonalizable, we need only to show that the characteristic polynomial \(p(\lambda)\) has three distinct real zeros.

If \(E_1 = 0\), then \(p(\lambda) = \lambda^3 - \frac{\partial E_3}{\partial E_1} \lambda\). Property \((4.9)\) shows \(\frac{\partial E_3}{\partial E_1} |_{E_1=0} > 0\), so \(p(\lambda)\) has three distinct zeros.

If \(E_1 \neq 0\), without loss of generality, we can assume that \(E_1 < 0\). According to Lemma \[4], \(p\left(\frac{E_1}{E_0}\right) > 0\). Property \((4.10)\) shows that \(p(0) = -\frac{\partial E_3}{\partial E_0} < 0\). Hence \(p(\lambda)\) has three distinct zeros, which satisfy \(\lambda_1 < \frac{E_1}{E_0} < \lambda_2 < 0 < \lambda_3\). This completes the proof. \(\blacksquare\)

Denote the three distinct zeros of \(p(x)\) by \(\lambda_i(U), i = 1, 2, 3\) satisfying \(\lambda_1(U) < \lambda_2(U) < \lambda_3(U)\). One can directly deduce the following conclusion from Theorem \[3\] and its proof.

Deduction 5. For any \(U \in \Omega_R\), we have

\[
\lambda_1(U) < \frac{E_1}{E_0} < \lambda_3(U).
\]

(3.21)

3.2.3 Riemann problem

The characteristic structure of the moment model is fundamental for further investigations into the behavior of the solution of the system. Meanwhile, the solution structure of the Riemann problem is instructional for studying the approximate Riemann solver, which is the basis of the numerical methods using Godunov type schemes. Here we investigate the characteristic structure of the MP\(_2\) model by the following Riemann problem:

\[\begin{aligned}
&1 \frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial z} = 0, \\
&U(z, t = 0) = \begin{cases} U^L, & \text{if } z < 0, \\
U^R, & \text{if } z > 0.
\end{cases}
\end{aligned}\]

(3.22)

Recall that the Jacobian \(J\) has three distinct eigenvalues. For the eigenvalue \(\lambda_k, k = 1, 2, 3\), the corresponding eigenvector is

\[r_k = (1, \lambda_k, \lambda_k^2)^T.\]

(3.23)

We have the following conclusion on the wave type of each characteristic field.

Theorem 6. The 1- and 3-characteristic field are genuinely nonlinear, and the 2-characteristic field is neither genuinely nonlinear nor linearly degenerate.

Proof. Denote \(\Delta_k := \nabla U \cdot r_k, k = 1, 2, 3\). Noticing that \(\lambda_k\) is a zero of \(p(\lambda) = 0\), we can obtain the following formula by the implicit differentiation

\[\Delta_k = \frac{1}{p'(\lambda_k)} \sum_{i,j=0}^2 \frac{\partial^2 E_3}{\partial E_i \partial E_j} \lambda_k^{i+j}, \quad k = 1, 2, 3.\]

The fact that \(p(\lambda)\) has three distinct zeros indicates \(p'(\lambda_1) > 0, p'(\lambda_2) < 0\) and \(p'(\lambda_3) > 0\). Hence, we need only to study the sign of

\[q(\lambda) = \sum_{i,j=0}^2 \frac{\partial^2 E_3}{\partial E_i \partial E_j} \lambda^{i+j},\]

(3.24)

on \(\lambda = \lambda_k, k = 1, 2, 3\). Property \((4.9)\) shows \(\frac{\partial^2 E_3}{\partial E_2^2} < 0\), so \(q(\lambda)\) is a polynomial of degree 3.

For the 1- and 3-characteristic fields, we can directly solve the values of \(\lambda_1\) and \(\lambda_3\) because the degree of \(p(\lambda)\) is 3, and then substitute them into \(q(\lambda)\) to check their sign for \(U \in \Omega_R\). However, the calculation is too complex. With the help of the cylindrical algebraic decomposition in computer algebraic [10] and its implementation in Maple, we validate that \(q(\lambda_1) < 0\) and \(q(\lambda_3) > 0\) for all \(U \in \Omega_R\).

For the 2-characteristic field, if we set \(E_1 = 0\), then \(q(\lambda_2) = \frac{\partial^2 E_3}{\partial E_0^2} = 0\). If we set \(E_1/E_0 = \pm 1/10\) and \(E_2/E_0 = 1/3\), then \(q(\lambda_2) \approx \pm 0.0293 > 0\). Thus the sign of \(q(\lambda_2)\) varies over \(\Omega_R\), which indicates that the 2-characteristic field is neither genuinely nonlinear nor linearly degenerate. \(\blacksquare\)
For the system (3.22), if two states \( U^L \) and \( U^R \) are connected by a rarefaction wave in a genuinely nonlinear field, then the following two conditions must be satisfied:

1. Constancy of the generalised Riemann invariants across the wave. That is to say, the integral curve \( \tilde{U}(\zeta) = (\tilde{E}_0(\zeta), \tilde{E}_1(\zeta), \tilde{E}_2(\zeta))^T \) satisfies
   \[
   \tilde{U}''(\zeta) = C(\zeta) r_k(\tilde{U}(\zeta)), \quad k = 1, 3, \tag{3.25}
   \]
   where \( C(\zeta) \) is a nonzero scalar factor with a fixed sign for any \( \zeta \).

2. Consistent condition:
   \[ \lambda_k(U^L) < \lambda_k(U^R). \tag{3.26} \]

Since the characteristic speed is \( \lambda_k = \lambda_k(\tilde{U}) \), along the integral curve, we have
\[
\frac{1}{C(\zeta)} \frac{d\lambda_k}{d\zeta} = \nabla \lambda_k \cdot r_k \left\{ \begin{array}{ll}
< 0, & k = 1, \\
> 0, & k = 3,
\end{array} \right.
\]
according to the proof of the Theorem 6. Noticing \( U^L \) and \( U^R \) lie in an integral curve, we let \( U^L = \tilde{U}(0) \), then there exists a \( \zeta_* \) such that \( U^R = \tilde{U}(\zeta_*) \). According to (3.26), we have
\[
C(\zeta_*) \zeta_* \left\{ \begin{array}{ll}
< 0, & k = 1, \\
> 0, & k = 3.
\end{array} \right. \tag{3.27}
\]

For the 3-rd characteristic field, (3.25) and (3.27) indicates \( \frac{d\tilde{E}_0(\zeta)}{d\zeta} = C(\zeta) \), and thus
\[
E_0^L < E_0^R.
\]

Let \( \tilde{u}(\zeta) = \frac{\tilde{E}_1(\zeta)}{\tilde{E}_0(\zeta)} \) and \( \tilde{p}(\zeta) = \frac{\tilde{E}_2(\zeta)}{\tilde{E}_0(\zeta)} \). By (3.25), we obtain
\[
\frac{d\tilde{u}(\zeta)}{d\zeta} = \frac{\tilde{E}_0(\zeta) \frac{d\tilde{E}_1(\zeta)}{d\zeta} - \tilde{E}_1(\zeta) \frac{d\tilde{E}_0(\zeta)}{d\zeta}}{\tilde{E}_0^2(\zeta)} = C(\zeta) (\lambda_3 - \tilde{u}),
\]
\[
\frac{d\tilde{p}(\zeta)}{d\zeta} = C(\zeta)(\lambda_3 - \tilde{u})^2.
\]

Figure 4: Contour of \( q(\lambda) \) with \( \lambda = \lambda_k, k = 1, 2, 3.. \)

To present a visualization on the sign of \( q(\lambda) \) in (3.24), we plot the contour of \( q(\lambda_k), k = 1, 2, 3 \) with \( E_0 = 1 \) as functions of \( E_1 \) and \( E_2 \) in Figure 4. The contour lines in the figures agree with the conclusion in the Theorem 6.

The 1- and 3-characteristic fields are genuinely nonlinear, thus each field associates with one wave, whose type is either rarefaction wave or shock. However, for the 2-characteristic fields, it corresponds to the nonconvex flux [27, Chapter 16.1], and one field may associate with more than one wave. Investigation on nonconvex wave requires too many tools, so we will not discuss the 2-field too much. Below, we study the rarefaction waves and shocks for the 1- and 3-fields.

**Rarefaction waves** For the system (3.22), if two states \( U^L \) and \( U^R \) are connected by a rarefaction wave in a genuinely nonlinear field, then the following two conditions must be satisfied:

1. Constancy of the generalised Riemann invariants across the wave. That is to say, the integral curve \( \tilde{U}(\zeta) = (\tilde{E}_0(\zeta), \tilde{E}_1(\zeta), \tilde{E}_2(\zeta))^T \) satisfies
   \[
   \tilde{U}''(\zeta) = C(\zeta) r_k(\tilde{U}(\zeta)), \quad k = 1, 3, \tag{3.25}
   \]
   where \( C(\zeta) \) is a nonzero scalar factor with a fixed sign for any \( \zeta \);

2. Consistent condition:
   \[ \lambda_k(U^L) < \lambda_k(U^R). \tag{3.26} \]

Since the characteristic speed is \( \lambda_k = \lambda_k(\tilde{U}) \), along the integral curve, we have
\[
\frac{1}{C(\zeta)} \frac{d\lambda_k}{d\zeta} = \nabla \lambda_k \cdot r_k \left\{ \begin{array}{ll}
< 0, & k = 1, \\
> 0, & k = 3,
\end{array} \right.
\]
according to the proof of the Theorem 6. Noticing \( U^L \) and \( U^R \) lie in an integral curve, we let \( U^L = \tilde{U}(0) \), then there exists a \( \zeta_* \) such that \( U^R = \tilde{U}(\zeta_*) \). According to (3.26), we have
\[
C(\zeta_*) \zeta_* \left\{ \begin{array}{ll}
< 0, & k = 1, \\
> 0, & k = 3.
\end{array} \right. \tag{3.27}
\]

For the 3-rd characteristic field, (3.25) and (3.27) indicates \( \frac{d\tilde{E}_0(\zeta)}{d\zeta} = C(\zeta) \), and thus
\[
E_0^L < E_0^R.
\]

Let \( \tilde{u}(\zeta) = \frac{\tilde{E}_1(\zeta)}{\tilde{E}_0(\zeta)} \) and \( \tilde{p}(\zeta) = \frac{\tilde{E}_2(\zeta)}{\tilde{E}_0(\zeta)} \). By (3.25), we obtain
\[
\frac{d\tilde{u}(\zeta)}{d\zeta} = \frac{\tilde{E}_0(\zeta) \frac{d\tilde{E}_1(\zeta)}{d\zeta} - \tilde{E}_1(\zeta) \frac{d\tilde{E}_0(\zeta)}{d\zeta}}{\tilde{E}_0^2(\zeta)} = C(\zeta) (\lambda_3 - \tilde{u}),
\]
\[
\frac{d\tilde{p}(\zeta)}{d\zeta} = C(\zeta)(\lambda_3 - \tilde{u})^2.
\]
According to (3.21), \( \lambda_3 - \bar{u} > 0 \). Thus we can obtain

\[ u^L < u^R, \quad p^L < p^R. \]

Analogously, for the 1-st characteristic wave, we have

\[ E_0^L > E_0^R, \quad u^L < u^R, \quad p^L > p^R. \]

**Shock waves** If two states \( U^L \) and \( U^R \) are connected by a shock in a genuinely nonlinear field, then we have the following two relationships:

1. Rankine-Hugoniot condition:

\[
\begin{align*}
E_1^R - E_1^L &= s_k (E_0^R - E_0^L), \\
E_2^R - E_2^L &= s_k (E_1^R - E_1^L), \\
E_3^R - E_3^L &= s_k (E_2^R - E_2^L), \quad k = 1, 3,
\end{align*}
\]  

(3.28)

where \( s_k \) is the speed of the shock;

2. Entropy condition:

\[ \lambda_k^L > s_k > \lambda_k^R, \quad k = 1, 3. \]  

(3.29)

The first two equations in (3.28) indicate

\[
s_k = \frac{E_1^R - E_1^L}{E_0^R - E_0^L} = \frac{E_2^R - E_2^L}{E_1^R - E_1^L}. 
\]  

(3.30)

Let \( u = \frac{E_1}{E_0} \) and \( p = \frac{E_2}{E_0} \). For the 3-rd characteristic field, noticing \( s_3 > \lambda_3^R > u^R \), we can obtain

\[
(E_0^R - E_0^L)(u^R - u^L) = \frac{(E_0^R - E_0^L)^2}{E_0^L}(s_3 - u^R) > 0, 
\]  

(3.31)

and

\[
(E_3^R - E_3^L)(p^R - p^L) = (E_1^R - E_1^L)^2 - (E_0^R - E_0^L)^2 \left( \frac{(E_1^R)^2}{E_1^R} - \frac{(E_1^L)^2}{E_1^L} \right) 
\]

\[
= \frac{1}{E_0^L E_0^R} (E_0^R E_1^L - E_0^L E_1^R)^2 > 0. 
\]  

(3.32)

The remaining work is to study the sign of \( E_0^R - E_0^L \) for the 3-characteristic field.

Denote the Hugoniot curves by \( \bar{U}(\tau) \), with \( \bar{U}(0) = U^L \) and \( \bar{U}(1) = U^R \). For a given \( \tau_0 \in [0, 1] \) and a sufficiently small \( \epsilon \), let \( \bar{U}^L = \bar{U}(\tau_0) \) and \( \bar{U}^r = \bar{U}(\tau_0 + \epsilon) \), then \( \bar{U}^L \) and \( \bar{U}^r \) also satisfy the Rankine-Hugoniot condition (3.28) and the entropy condition (3.29).

Let \( d = E_0^R - E_0^L \), then \( d \ll 1 \) and

\[
E_1^r = E_1^L + s_3 d, \quad E_2^r = E_2^L + s_3^2 d. 
\]

According to the Taylor expansion, we have

\[
\lambda_3^r - \lambda_3^L = d \left( \frac{\partial \lambda_3}{\partial E_0} \right)^1 s_3 + \left( \frac{\partial \lambda_3}{\partial E_1} \right)^1 s_3^2 + O(d^2). 
\]  

(3.33)

Thus, \( |\lambda_3^r - \lambda_3^L| \ll 1 \), and \( |s_3 - \lambda_3^L| \ll 1 \). With

\[
\left( \frac{\partial \lambda_3}{\partial E_0} \right)^1 + \left( \frac{\partial \lambda_3}{\partial E_1} \right)^1 \lambda_3^L + \left( \frac{\partial \lambda_3}{\partial E_2} \right)^1 (\lambda_3^L)^2 = (\nabla U \lambda_3 \cdot r_3)|_{U = U^L} = \Delta_3|_{U = U^L} > 0, 
\]  

(3.34)

and the entropy condition, we obtain that

\[ d < 0, \quad E_0^L < E_0^r. \]
Because the upper relation holds for any $\tau \in [0, 1)$, we have that for the 3-characteristic field
\[ E_0^L > E_0^R, \quad u^L > u^R, \quad p^L > p^R. \]  
(3.35)

Analogously, for the 1-characteristic field, we have
\[ E_0^L < E_0^R, \quad u^L > u^R, \quad p^L < p^R. \]  
(3.36)

We summarize all the conclusions on genuinely nonlinear waves in the following theorem to close this section.

**Theorem 7.** For the 1- and 3-characteristic fields, the variables $E_0, u = \frac{E_0}{c^2}$ and $p = E_2 - \frac{E_0^2}{c^2}$ on both sides of the wave have the relationship with the wave type as in the Table 2.

| Wave type     | $E_0$          | $u$          | $p$          |
|---------------|---------------|--------------|--------------|
| Rarefaction   | 1-wave $E_0^L > E_0^R$ | $u^L < u^R$ | 1-wave $p^L > p^R$ |
|               | 3-wave $E_0^L < E_0^R$ | $u^L < u^R$ | 3-wave $p^L < p^R$ |
| Shock wave    | 1-wave $E_0^L < E_0^R$ | $u^L > u^R$ | 1-wave $p^L < p^R$ |
|               | 3-wave $E_0^L > E_0^R$ | $u^L > u^R$ | 3-wave $p^L > p^R$ |

Table 1: The relationship between the wave type and the variables $E_0$, $u$, and $p$.

## 4 Numerical simulation

In this section, we discuss the numerical scheme for the MP$_N$ model, and perform numerical simulations on some typical examples to demonstrate its numerical efficiency.

### 4.1 Numerical scheme

Because the convection part of the MP$_N$ model is hyperbolic conservation laws in the sense of balance laws, we discretize it by the finite volume method. The source term and the governing equation of internal energy (3.36) both contain the term $T^s$, which is usually a stiff term. Hence, an implicit scheme is adopted to deal with the stiff terms.

Precisely, we assume the spatial domain is $[z_i, z_{i+1}]$, and the number of discretization cell is $N_{cell}$. A uniform discretization yields the spatial step $\Delta z = \frac{z_{i+1} - z_i}{N_{cell}}, i = 1, \cdots, N$, and mesh cells $[z_{i-1/2}, z_{i+1/2}], i = 1, \cdots, N$ with $z_{i+1/2} = z_i - \Delta z/2$.

Denote the approximation of the solution in $i$-cell at time step $t_n$ by $U^n_i$, and analogous for the source $S$ and the internal energy $c$. The numerical scheme for the MP$_N$ system is
\[
\frac{U^{n+1}_i - U^n_i}{c \Delta t} + \frac{F_{i+1/2} - F_{i-1/2}}{\Delta z} = \frac{S^{n+1}_i}{\Delta t},
\]
(4.1)
where the $k$-th element of the source term $S^{n+1}_i$ has the form
\[
\frac{E^{n+1}_i}{\Delta t} = \frac{e^{n+1}_i}{\Delta t} - c^{n+1}_i = \sigma_{a,i} \left( E^{n+1}_{0,i} - ac(T^{n+1}_{i})^4 \right),
\]
(4.2)

Here we adopt the Lax-Friedrich scheme in the numerical flux $F_{i+1/2}$. Due to the fact that the source term and the governing equation of the internal energy is implicitly discretized, the constraints of the time step is the CFL condition
\[
\Delta t = \text{CFL} \cdot \min_i \frac{\Delta z}{\max \left| \lambda_{k}(U^n_i) \right|}.
\]
(4.3)

We set the CFL number to be 0.3 in all the numerical simulations. It is worth to point out that in the absence of any external source of radiation, i.e. $s = 0$, adding the first equation of the discretization of moments in (4.1) and the discretization of the internal energy in (4.1) yields
\[
\frac{e^{n+1}_i}{\Delta t} - \frac{e^n_i}{\Delta t} + \frac{E^{n+1}_{0,i} - E^n_{0,i}}{c \Delta t} + \frac{F_{0,i+1/2} - F_{0,i-1/2}}{\Delta z} = 0,
\]
(4.4)

which is the discretization version of the conservation of the total energy (2.5).
4.1.1 Boundary condition

The ansatz of the \( MP_N \) provides an injective function between the concerned moments in the \( MP_N \) model and the intensity in the form (3.4). This allows us to construct the boundary condition of the \( MP_N \) model based on the boundary condition of the RTE. Without loss of generality, we take the left boundary as an example. For the RTE, the intensity on the boundary is

\[
I^B(t,\mu) = \begin{cases} 
I(z = z_l, t, \mu), & \mu < 0, \\
I_{\text{out}}(t, \mu), & \mu > 0, 
\end{cases} 
\]

(4.5)

where \( I_{\text{out}} \), which is the intensity outside of the domain, is problem dependent. For instance, the intensity outside the domain for the common used reflective boundary condition is

\[
I_{\text{out}}(t, \mu) = I(z = z_l, t, -\mu), \quad \mu > 0, 
\]

(4.6)

and for the vacuum boundary condition, we have

\[
I_{\text{out}}(t, \mu) = 0, \quad \mu > 0. 
\]

(4.7)

Due to the ansatz (3.4), for the boundary condition of the \( MP_N \) model, the intensity close to the domain \( I(z = z_l, t, \mu) \) is replaced by the ansatz constructed by the moments close to the domain \( \hat{I}(\mu; U(z = z_l, t)) \). Then, one can directly evaluate the moments on the boundary. Particularly, the flux across the boundary for the \( k \)-th moment is

\[
F^B_k = \int_0^1 \mu^{k+1} \hat{I}(\mu; U(z = z_l, t)) \, d\mu + \int_0^1 \mu^{k+1} I_{\text{out}}(t, \mu) \, d\mu. 
\]

(4.8)

4.1.2 Implementation

The implementation of the numerical scheme (4.1) is straightforward except the moment closure, where one has to evaluate \( E_i, i = 0, \ldots, 2N + 1 \) fast and accurately. However, a naive implementation on \( E_i \) will lose the accuracy when \( \alpha \) varies in \((-1, 1)\). In the following, we discuss the details of the implementation.

Note that \( E_k = \langle \omega^{(\alpha)}(\mu) \rangle_k \). If \( |\alpha| \) is close to 0, using Taylor expansion on the weight function \( \omega^{(\alpha)}(\mu) \), one can obtain

\[
E_k = \int_{-1}^1 \varepsilon \sum_{m=0}^{\infty} \frac{(m + 1)(m + 2)(m + 3)}{6} (-\alpha)^m \mu^{m+k} \, d\mu 
= \sum_{m+k \text{ is even}} \varepsilon \frac{(m + 1)(m + 2)(m + 3)}{3} (-\alpha)^m \frac{1}{m + k + 1}.
\]

Hence, we have

\[
E_k = \begin{cases} 
\sum_{j=0}^{\infty} \frac{2(2j+1)(j+1)(2j+3)\alpha^{2j}}{3} \frac{\varepsilon}{2^{2j+2}}, & k \text{ is even,} \\
-\varepsilon \sum_{j=0}^{\infty} \frac{4(j+1)(2j+3)(j+2)\alpha^{2j}}{3} \frac{\varepsilon}{2^{2j+2}}, & k \text{ is odd.} 
\end{cases} 
\]

(4.9)

We truncate the summation of series as \( j = 0, \ldots, n_s \), then its error is \( O((n_s + 3)^3 \alpha^{2n_s+2}) \). Clearly, the formula (4.9) is efficient for the evaluation of \( E_k \) if \( |\alpha| \) is small.

On the other hand, if \( |\alpha| \) is not small, we let \( J_k^{(n)} = \int_{-1}^1 \varepsilon (1 + \alpha \mu)^n \mu^k \, d\mu \), then \( E_k = J_k^{(4)} \). Direct calculation using the integral by part yields

\[
J_k^{(n)} = \frac{k}{(n-1)\alpha} J_k^{(n-1)} + \frac{\varepsilon}{(n-1)\alpha} \left( \frac{(-1)^k}{(1+\alpha)^n-1} - \frac{1}{(1+\alpha)^{n-1}} \right). 
\]

(4.10)

For the case \( n = 1 \), using the following recursive relationship

\[
J_k^{(1)} = \begin{cases} 
-\frac{1}{\alpha} J_{k-1}^{(1)}, & k \text{ is even,} \\
-\frac{1}{\alpha} J_{k-1}^{(1)} + \frac{\varepsilon}{\alpha k}, & k \text{ is odd,} 
\end{cases} 
\]
we obtain
\[ J_k^{(1)} = \varepsilon \left( \frac{\ln(1+\alpha)-\ln(1-\alpha)}{\alpha^{k+1}} + \sum_{j=1}^{\lfloor k/2 \rfloor} \frac{2}{(k+1-2j)\alpha^{j+1}} \right), \quad k \text{ is even}, \]
\[ \frac{\ln(1-\alpha)}{\alpha^{k+1}} + \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{2}{(k-2j)\alpha^{k-j+1}}, \quad k \text{ is odd}. \]  

The equations (4.10) and (4.11) provide a formula to evaluate \( E_k \) for the case \( |\alpha| \) not small. In our implementation, the algorithm for \( E_k \) is a combination of (4.9) and (4.10).

4.2 Bilateral inflow

This example is used to study the behavior of the solution of the \( MP_N \) model, hence the right hand side of (2.1) vanishes, i.e., the RTE degenerates into
\[ \frac{1}{c} \frac{\partial I}{\partial t} + \mu \frac{\partial I}{\partial z} = 0. \]  

Using the method of characteristics, we obtain the analytical solution
\[ I(z,\mu,t) = I_0(z - c\mu t,\mu), \]  
where \( I_0 \) is the initial value at \( t = 0 \). Here we choose the initial value as
\[ I_0(z,\mu) = \begin{cases} 
ac\delta(\mu - 1), & z \leq 0.2, \\
0, & 0.2 < z \leq 0.8, \\
\frac{1}{2}ac, & z > 0.8,
\end{cases} \]
which is consist of two Riemann problems. The initial intensity on the left is a Dirac delta function, which is an extremely anisotropic distribution. Generally, it is challenging to approximate such a function for the method based on the polynomial expansion, including the \( PN \) and \( MPN \) models. The initial intensity on the right is an equilibrium, however, because the initial intensity in the middle is zero, the intensity for the right Riemann problem is not continuous. In the following, we perform simulations to study the efficiency of the \( MP_N \) model on this bilateral flow.

![Graphs](image)

Figure 5: Profile of \( E_0 \) and \( E_1 \) for the \( MP_N \) model and the \( PN \) model for the bilateral inflow.

We simulate the problem by the \( MP_N \) model and the \( PN \) model till \( ct_{\text{end}} = 0.1 \). Because the speed of light is finite, we can limit the computational domain in \([0, 1]\), which is uniformly discretized with the
number of cells to be $N_{\text{cell}} = 100000$. Figure 4 presents the profile of $E_0$ and $E_1$ for the $\MP_N$ and $P_N$ models with $N = 2, 6, 10$ and the reference solution.

Clearly, from the left part ($z < 0.5$) of each figure, there is an oscillation in the results of both $E_0$ and $E_1$ of the $P_N$ model, and as $N$ increases, the oscillation frequency increases. This Gibbs phenomenon is caused by the failure of the approximation to the Dirac delta function by Legendre series. On the other hand, the results of the $\MP_N$ model agree with the reference solution well, even when $N = 2$, which indicates the $\MP_N$ has the ability to simulate the strongly anisotropic problem. This supports the argument at the end of Subsection 3.1 that the ansatz of the $\MP_N$ model has the ability to approximate anisotropic distributions because of its specific weight function. Moreover, according to Figure 4d and Figure 5a, one can observe that the fastest wave of the $P_3$ model spread much slower than the reference solution, because the characteristic speed of the $P_3$ model is less than 1. The $\MP_2$ does not have this issue due to its specific weight function, which allows the characteristic speed reaches 1.

From the right part ($z > 0.5$) of each figure, both the $\MP_N$ and $P_N$ models have a good agreement with the reference solution, and it turns better as $N$ increases. This indicates the $\MP_N$ model also has the ability to simulate the problem with the discontinuous intensity. Compared with the $P_N$ model, one can observe that the results of the $\MP_N$ model are a bit closer to the reference solution.

4.3 Gaussian source problem

This example simulates particles with an initial specific intensity that is a Gaussian distribution in space $\mathbb{R}^2$:

$$I_0(z, \mu) = \frac{ac}{\sqrt{2\pi} \theta} e^{-\frac{z^2}{2\theta^2}}, \quad \theta = \frac{1}{100}, \quad z \in (-L, L). \quad (4.15)$$

Here we set a large enough $L = c t_{\text{end}} + 1$ to ensure that the energy reaching the boundaries is negligible, and vacuum boundary conditions are prescribed at both boundaries. The medium is purely scattering with $\sigma_s = \sigma_t = 1$, thus the material coupling term vanishes. We also set the external source to be zero.

![Figure 6: Numerical results of the Gaussian source problem.](image)

In this problem, we set $c t_{\text{end}} = 1$, thus the problem domain is $[-2, 2]$, and the number of cells is $N_{\text{cell}} = 8000$, with $\Delta z = \frac{1}{2000}$. Figure 6 presents the numerical results, including the profiles of $E_0$, $E_1$, and $\frac{E_1}{E_0}$ of the $\MP_N$ model, the $P_N$ model and the reference solution, which is the solution of the $P_{31}$ model. The relative $\ell_2$ errors of $E_0$ of the $\MP_N$ and $P_N$ models are shown in Figure 7. Spurious oscillations occur in the numerical solutions of $E_0$ of both the $\MP_N$ model and the $P_N$ model, and the
oscillation amplitude decreases as the number of moments increases. In the comparison of the \( P_N \) model, the \( MP_N \) model is more effective in reducing the oscillations.

Moreover, the \( E_0 \) in this problem can be sufficiently large (close to 1) to make the distribution function anisotropic. Clearly, both the numerical results of the \( MP_N \) model and the \( P_N \) model can approximate the reference solution well. In the comparison of the \( P_N \) model, the \( MP_N \) model is more effective in reducing the oscillations. The \( MP_N \) model approximates better than the \( P_N \) model for the anisotropic distribution function.

### 4.4 Pure absorbing problem

This example is adopt to compare the approximation efficiency of \( MP_N \) model and \( P_N \) model. We use the setup in [8] as the computational domain \([-5, 5]\), the absorption coefficient \( \sigma_a \equiv 0.5 \), the scattering coefficient \( \sigma_s \equiv 0 \), and the external source term

\[
s(z) = \begin{cases} 
ac, & -1 \leq z \leq 1, \\
0, & \text{otherwise}.
\end{cases}
\]

All conclusions in Subsection 4.3 are also valid for this example. What we want to point out is that the results of the \( MP_2 \) model is good enough for this example.

### 4.5 Su-Olson problem

The Su-Olson problem [33] is a non-equilibrium radiative transfer problem with a material coupling term. The computation domain is \([0, 30]\) and the absorption and scattering opacity are \( \sigma_a = 1 \) and \( \sigma_s = 0 \), respectively. The external source terms \( s \) is given by

\[
s(z, t) = \begin{cases} 
ac, & \text{if } 0 \leq z \leq \frac{1}{2}, \text{ and } 0 \leq ct \leq 10, \\
0, & \text{otherwise}.
\end{cases}
\]

The relationship between the temperature and the internal energy is given by \( e(T) = aT^4 \). The left boundary condition is the reflective boundary condition, while the right boundary condition is the vacuum boundary condition.

The setup of the simulation is \( N_{\text{cell}} = 60000 \) with \( \Delta z = \frac{1}{2000} \). Figure [10] presents the profile of \( E_0 \) of the \( MP_N \) model and the \( P_N \) model with \( N = 2, 3, 4 \) at different end time \( ct_{\text{end}} = 1, 3.16, \text{ and } 10 \). The reference solution is the semi-analytic solution in [36]. For both the \( P_N \) and \( MP_N \) models, the results agree with the reference well as \( N \) increases. The \( MP_N \) model has the capability to simulate such benchmark a few moments, for instance, \( N = 2 \). Moreover,
Figure 8: Numerical results of the pure absorbing problem.

Figure 9: Relative $\ell_2$ error (err) of $E_0$ of the pure absorbing problem in the linear scale (left figure) and the semi-logarithm scale (right figure).

in the comparison of the $P_N$ model, the $MP_N$ model shows superiority to handle such material coupling problem.

5 Conclusion

According to the two viewpoints on the $P_N$ and $M_N$ models, we proposed the $MP_N$ model by expanding the specific intensity around the ansatz of the $M_1$ model in terms of orthogonal polynomials. The certain selection of the weight function permitted the $MP_N$ model to simulate the problems with strong anisotropic distribution. The $MP_N$ model had an explicit expression of the closure for arbitrary $N$, which allowed us to directly solve it cheaply in the comparison of the $M_N$ model. In all the numerical tests, only a few moments (for instance, $N = 2$ is good enough for many tests) were required to give good numerical results. Hence, it was believed that the $MP_N$ model could be used to solve the RTE fast and accurately. The current work focused on the novel idea on the construction of the $MP_N$ model to approximate the RTE for a grey medium in the slab geometry. The extension to the general medium and 3D case was in process.
Figure 10: Profile of $E_0$ of the $MP_N$ and $P_N$ models with $N = 2, 3, 4$ (from left to right) for the Su-Olson problem at different time $c t_{end} = 1, 3.16$ and $10$ (from down to up in each figure) with the linear scale (upper figures) and logarithm scale (lower figures).

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A Proof of the Property 2

In this appendix, we prove the Property 2. Except Property 2 b), the case $E_1 = 0$ is trivial. Without loss of generality, we need only to check the case $\alpha \in (0, 1)$ for all the properties except Property 2 b). Noticing (3.12), we can obtain the following relationships with direct calculation:

\[
\frac{\partial E_3}{\partial E_2} = -\frac{C_1}{C_2}, \tag{A.1}
\]

\[
\frac{\partial \alpha}{\partial E_0} = \frac{4 - 2\sqrt{4 - 3(E_1/E_0)^2}}{E_1\sqrt{4 - 3(E_1/E_0)^2}}, \tag{A.2}
\]

\[
\frac{\partial \alpha}{\partial E_1} = \frac{E_0 - 2\sqrt{4 - 3(E_1/E_0)^2}}{E_1^2\sqrt{4 - 3(E_1/E_0)^2}}, \tag{A.3}
\]

where

\[C_1 = 2(\alpha^2(-6 + 7\alpha^2) + \alpha (12 - 13\alpha^2 + \alpha^4)\text{atanh}(\alpha) + 6(-1 + \alpha^2)\text{atanh}^2(\alpha)),\]

\[C_2 = \alpha(\alpha^2(-3 + 4\alpha^2) - 6\alpha(-1 + \alpha^2)\text{atanh}(\alpha) + (-3 + 2\alpha^2 + \alpha^4)\text{atanh}^2(\alpha)).\]

Since (A.1) is a function of $\alpha$, we let $\kappa(\alpha) := \frac{\partial E_3}{\partial E_2}$. Direct calculations yield

\[
\frac{\partial \kappa(\alpha)}{\partial \alpha} = \frac{4C_3C_4}{C_2^2}, \tag{A.4}
\]

where

\[C_3 = (\text{atanh}(\alpha) - \alpha)(\alpha(3 - 2\alpha^2) - 3(1 - \alpha^2)\text{atanh}(\alpha)),\]

\[C_4 = \alpha^2(-3 + 8\alpha^2) + (6\alpha - 8\alpha^3)\text{atanh}(\alpha) + 3(-1 + \alpha^2)\text{atanh}^2(\alpha).\]
a) To prove \( \frac{\partial E_3}{\partial E_0} > 0 \) when \( \alpha \in (0,1) \), we need to verify that \( \frac{\partial^2 E_3}{\partial E_2 \partial E_0} > 0 \) and \( \frac{\partial E_3}{\partial E_0} \big|_{E_2 = \frac{\epsilon_1^2}{\pi^2}} > 0 \).

Since \( \frac{\partial E_3}{\partial E_2} \) in (A.1) is a function of \( \alpha \), using the chain rule gives

\[
\frac{\partial^2 E_3}{\partial E_0 \partial E_2} = \frac{\partial \kappa(\alpha)}{\partial \alpha} \frac{\partial \alpha}{\partial E_0} = \frac{4C_3 C_4}{C_2} \frac{\partial \alpha}{\partial E_0}.
\]

When \( \alpha \in (0,1) \), we have \( E_1 \in (-1,0) \), so one can check that \( C_3 > 0, C_4 < 0 \) and \( \frac{\partial \alpha}{\partial E_0} < 0 \). Thus we obtain \( \frac{\partial^2 E_3}{\partial E_2 \partial E_0} > 0 \).

Then we consider the situation \( E_2 = E_0^2/E_0 \) to make \( \frac{\partial E_3}{\partial E_0} \) a function of \( \alpha > 0 \), which can be written as

\[
\frac{\partial E_3}{\partial E_0} \bigg|_{E_2 = \frac{\epsilon_1^2}{\pi^2}} = \frac{(1 - \alpha^2)C_5}{\alpha^2(-9 + \alpha^4)C_2^2},
\]

where

\[
C_5 = -3\alpha^5(-135 + 450 \alpha^2 - 537 \alpha^4 + 222 \alpha^6 + 8 \alpha^8) + \alpha^4(-2025 + 6129 \alpha^2 - 6345 \alpha^4 + 2067 \alpha^6 + 230 \alpha^8 + 24 \alpha^{10}) \tanh(\alpha)
- 2\alpha^3(-2025 + 5508 \alpha^2 - 4671 \alpha^4 + 681 \alpha^6 + 472 \alpha^8 + 51 \alpha^{10}) \tanh(\alpha)
+ 2\alpha^2(-2025 + 4887 \alpha^2 - 3033 \alpha^4 - 630 \alpha^6 + 689 \alpha^8 + 103 \alpha^{10} + 9 \alpha^{12}) \tanh^3(\alpha)
+ \alpha(2025 - 4266 \alpha^2 + 1431 \alpha^4 + 1668 \alpha^6 - 601 \alpha^8 - 218 \alpha^{10} - 39 \alpha^{12}) \tanh^4(\alpha)
+ 3(5 + \alpha^2)(-3 + 2\alpha^2 + \alpha^4)^3 \tanh^4(\alpha),
\]

is an elementary function of \( \alpha \). One can check that \( C_5 < 0 \) when \( \alpha \in (0,1) \). Therefore, \( \frac{\partial E_3}{\partial E_0} > 0 \) when \( E_2 > \frac{E_0^2}{E_0} \).

b) Let \( G(\alpha) \triangleq \frac{\partial E_3}{\partial E_1} \bigg|_{E_2 = \frac{\epsilon_1^2}{\pi^2}} \). To prove \( \frac{\partial E_3}{\partial E_1} \bigg|_{E_2 = \frac{\epsilon_1^2}{\pi^2}} > 0 \), it is sufficient to show \( \frac{\partial^2 E_3}{\partial E_1 \partial E_2} > 0 \), and \( G(0) > 0 \). Using the chain rule gives

\[
\frac{\partial^2 E_3}{\partial E_1 \partial E_2} = \frac{\partial \kappa(\alpha)}{\partial \alpha} \frac{\partial \alpha}{\partial E_2}.
\]

Since \( \frac{\partial ^2}{\partial \alpha E_2} < 0 \) when \( \alpha \in (0,1) \), we obtain \( \frac{\partial^2 E_3}{\partial E_1 \partial E_2} > 0 \), according to the proof of Property 2a).

Direct calculations yield

\[
G(\alpha) = \frac{64}{57} \alpha^{21} + O(\alpha^{23}),
\]

thus we have \( G(0) = \frac{12}{37} > 0 \).

c) Noticing (A.1), to prove \( \frac{\partial E_3}{\partial E_2} < 0 \) when \( \alpha \in (0,1) \), we need only to check that the signs of \( C_1 \) and \( C_2 \) are same. Because \( C_1 \) and \( C_2 \) are elementary function of \( \alpha \), direct calculation yields \( C_1, C_2 > 0 \) when \( \alpha \in (0,1) \).

d) Noticing that (A.1) is a function of \( \alpha \), independent on \( E_2 \), and \( \alpha \) is independent on \( E_2 \), we have that \( E_3 \) is linear dependent on \( E_2 \), i.e., \( \frac{\partial^2 E_3}{\partial E_2^2} = 0 \).

e) With the help of (A.1), we can obtain

\[
E_3 - E_1^3 - (E_2 - E_1^2) \frac{\partial E_3}{\partial E_2} = \frac{3(1 - \alpha^2)^2 C_6}{\alpha^3(3 + \alpha^2)^3 C_2},
\]

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where
\[
C_\alpha = -\alpha^3(27 - 72\alpha^2 + 39\alpha^4 + 2\alpha^6) + \alpha^2(81 - 171\alpha^2 + 69\alpha^4 + 19\alpha^6 + 2\alpha^8)\tanh(\alpha) \\
- 3\alpha(1 - \alpha^2)(27 + 12\alpha^2 + \alpha^4)\tanh(\alpha)^2 + (1 - \alpha^2)^2(3 + \alpha^2)^3\tanh^3(\alpha).
\]

One can show that \(C_\alpha > 0\) when \(\alpha \in (0, 1)\) because \(C_\alpha\) is an elementary function. This results

\[
\mathcal{E}_3 - \mathcal{E}_1^3 - (\mathcal{E}_2 - \mathcal{E}_1^2) \frac{\partial \mathcal{E}_3}{\partial \mathcal{E}_2} < 0.
\]

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