BIGRADED DIFFERENTIAL ALGEBRA FOR VERTEX ALGEBRA COMPLEXES

A. ZUEVSKY

Abstract. For the bicomplex structure of grading-restricted vertex algebra cohomology defined in \[6\], we show that the orthogonality and double grading conditions taken into account endow it with the structure of a bigraded differential algebra with respect to a natural multiplication. The generators and commutation relations of the bigraded differential algebra form a continual Lie algebra \( G(V) \) with the root space provided by a grading-restricted vertex algebra \( V \). We prove that the differential algebra generates non-vanishing cohomological invariants associated to a vertex algebra \( V \). Finally, we provide examples associated to various choices of the vertex algebra bicomplex subspaces.

AMS Classification: 53C12, 57R20, 17B69

1. Data availability statement

The author confirms that:

1.) All data generated or analysed during this study are included in this published article.

2.) Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

2. Introduction

The cohomology theory for vertex operator algebras is an important and attractive theme for studies. In \[6\] the cohomology theory for a grading-restricted vertex algebra \[9\] was introduced. The definition of bicomplex spaces and coboundary operators uses an interpretation of vertex algebras in terms of rational functions constructed from matrix elements \[8\] for a grading-restricted vertex algebra \[4\]. The notion of composability with a number of vertex operators for bicomplex space elements is essentially involved in the formulation. The cohomology of such complexes defines a cohomology of a grading-restricted vertex algebras in the standard way. It is an important problem to study possible cohomological classes for vertex algebras.

In this paper we show that the orthogonality condition with respect to a commutator, and double grading conditions assumed for elements of the bicomplex spaces associated to a grading-restricted vertex algebra endow the bicomplex spaces with

Key words and phrases. Vertex algebras; cohomological invariants; bi-differential algebras; continual Lie algebras.
the structure of a bigraded differential algebra with respect the commutator of bicomplex mappings. The orthogonality condition for elements of bicomplex spaces is motivated by geometrical construction of cohomological invariants for foliated manifolds [5]. Originally, the orthogonality comes from consideration of differential forms. Here we implement similar idea in the case of rational functions associated to grading-restricted vertex algebras. Using the above mentioned conditions we then find further explicit examples of continual Lie algebras [12] associated to vertex algebras. We derive also cohomological classes for the bicomplex for a grading-restricted vertex algebra. Such cohomological classes are non-vanishing and independent of the choice of the bicomplex space elements. Our main motivation was to show that the bicomplex construction originating from algebraic properties of vertex algebras, and formulated in terms of rational functions with specific properties possesses deeper cohomological structure similar to that of differential forms in algebraic topology.

As for possible applications of the material presented in this paper, we would like to mention computations of higher cohomologies for grading-restricted vertex algebras [7], search for more complicated cohomological invariants, and applications to differential geometry. In particular, since vertex algebras is a useful computational tool, it would be interesting to study possible relations to cohomology of manifolds. One can show that such cohomological invariants possess analytical (with respect to the notion of composability) as well as geometrical meaning. In addition to the natural orthogonality condition, one can consider variations of multiplications defined for bicomplex spaces, and, therefore, more advanced examples of graded differential algebras. In differential geometry there exist various approaches to the construction of cohomological classes (cf., in particular, [11]). We hope to use these techniques to derive counterparts in the cohomology theory of vertex algebras.

3. The grading-restricted vertex algebra

In this Section, we recall [6] properties of grading-restricted vertex algebras and their grading-restricted generalized modules over the base field \( \mathbb{C} \) of complex numbers. A vertex algebra \((V, \mathcal{Y}_V, \mathbf{1}_V)\), cf. [9, 1], consists of a \( \mathbb{Z} \)-graded complex vector space

\[
V = \bigoplus_{n \in \mathbb{Z}} V(n),
\]

for each \( n \in \mathbb{Z} \), and linear map

\[
\mathcal{Y}_V : V \to \text{End}(V)[[z, z^{-1}]],
\]

for a formal parameter \( z \) and a distinguished vector \( \mathbf{1}_V \in V \). The evaluation of \( \mathcal{Y}_V \) on \( v \in V \) is the vertex operator

\[
\mathcal{Y}_V(v, z) = \sum_{n \in \mathbb{Z}} v(n)z^{-n-1},
\]

with components

\[
(\mathcal{Y}_V(v))_n = v(n) \in \text{End}(V),
\]
A grading-restricted vertex algebra \([6]\), defining is subject to the following conditions:

1. Grading-restriction condition: \(V(n)\) is finite dimensional for all \(n \in \mathbb{Z}\), and \(V(n) = 0\), for \(n \leq 0\).
2. Lower-truncation condition: For \(u, v \in V\), \(Y_V(u, z)_v \in V((z))\) (the space of formal Laurent series in \(z\) with coefficients in \(V\)).
3. Identity property: Let \(\text{Id}_V\) be the identity operator on \(V\). Then
   \[Y_V(1_V, z) = \text{Id}_V.\]
4. Creation property: For \(u \in V\), \(Y_V(u, z)_1 \in V[[z]]\), and
   \[
   \lim_{z \to 0} Y_V(u, z)_1 = u.
   \]
5. Duality: For \(u_1, u_2, v \in V\), \(v' \in V' = \bigoplus_{n \in \mathbb{Z}} V^*_n\) \((V^*_n)\) denotes the dual vector space to \(V_n\) and \(\langle \cdot, \cdot \rangle\) the evaluation pairing \(V' \otimes V \to \mathbb{C}\), the series \(\langle v', Y_V(u_2, z_2)Y_V(u_1, z_1)v \rangle\), and \(\langle v', Y_V(u_1, z_1 - z_2)u_2, z_2)v \rangle\), are absolutely convergent in the regions \(|z_1| > |z_2| > 0, |z_2| > |z_1| > 0, |z_2| > |z_1 - z_2| > 0\), respectively, to a common rational function in \(z_1\) and \(z_2\) with the only possible poles at \(z_1 = 0, z_2\) and \(z_1 = z_2\).

One assumes the existence of Virasoro vector \(\omega \in V\): its vertex operator
\[
Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n)z^{-n-2},
\]
is determined by Virasoro operators \(L(n) : V \to V\) fulfilling
\[
[L(m), L(n)] = (m - n)L(m + n) + \frac{c}{12}(m^3 - m) \delta_{m+n,0} \text{Id}_V,
\]
\((c\) is called the central charge of \(V\)). The grading operator is given by
\[
L(0)u = nu, \quad u \in V(n),
\]
\((n\) is called the weight of \(u\) and denoted by \(\text{wt}(u)\)).

6. \(L_V(0)\)-bracket formula: Let \(L_V(0) : V \to V\) be defined by \(L_V(0)v = nv\) for \(v \in V(n)\). Then
   \[
   [L_V(0), Y_V(v, z)] = Y_V(L_V(0)v, z) + z \frac{d}{dz} Y_V(v, z),
   \]
   for \(v \in V\).

7. \(L_V(-1)\)-derivative property: Let \(L_V(-1) : V \to V\) be the operator given by
   \[
   L_V(-1)v = \text{Res}_z z^{-2} Y_V(v, z)_1 = Y_V(-1)(v)1_V,
   \]
   for \(v \in V\). Then for \(v \in V\),
   \[
   \frac{d}{dz} Y_V(u, z) = Y_V(L_V(-1)u, z) = [L_V(-1), Y_V(u, z)].
   \]
A grading-restricted generalized $V$-module is a vector space $W$ equipped with a vertex operator map

$$Y_W : V \otimes W \rightarrow W[[z, z^{-1}]],$$

$$u \otimes w \mapsto Y_W(u, w) = \sum_{n \in \mathbb{Z}} (Y_W)_n(u, w)z^{-n-1},$$

and linear operators $L_W(0)$ and $L_W(-1)$ on $W$, satisfying conditions similar as in the definition for a grading-restricted vertex algebra. In particular,

1. Grading-restriction condition: The vector space $W$ is $\mathbb{C}$-graded, i.e., $W = \bigoplus_{\alpha \in \mathbb{C}} W(\alpha)$, such that $W(\alpha) = 0$ when the real part of $\alpha$ is sufficiently negative.
2. Lower-truncation condition: For $u \in V$ and $w \in W$, $Y_W(u, z)w$ contains only finitely many negative power terms, i.e., $Y_W(u, z)w \in W((z))$.
3. Identity property: Let $\text{Id}_W$ be the identity operator on $W$, $Y_W(1_V, z) = \text{Id}_W$.
4. Duality: For $u_1, u_2 \in V$, $w \in W$, $w' \in W' = \bigoplus_{n \in \mathbb{Z}} W^*_n$ ($W'$ is the dual $V$-module to $W$), the series $(w', Y_W(u_1, z_1)Y_W(u_2, z_2)w)$, $(w', Y_W(u_2, z_2)Y_W(u_1, z_1)w)$, and $(w', Y_W(Y_V(u_1, z_1 - z_2)u_2, z_2)w)$, are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_1| > |z_2| > 0$, $|z_1 - z_2| > 0$, respectively, to a common rational function in $z_1$ and $z_2$ with the only possible poles at $z_1 = 0 = z_2$ and $z_1 = z_2$.
5. $L_W(0)$-bracket formula: For $v \in V$, $[L_W(0), Y_W(v, z)] = Y_W(L(0)v, z) + z\frac{d}{dz}Y_W(v, z)$.
6. $L_W(0)$-grading property: For $w \in W(\alpha)$, there exists $N \in \mathbb{Z}_+$ such that $(L_W(0) - \alpha)^N w = 0$.
7. $L_W(-1)$-derivative property: For $v \in V$, $\frac{d}{dz}Y_W(v, z) = Y_W(L(-1)v, z) = [L_W(-1), Y_W(v, z)]$.

A graded-restricted vertex algebra is endowed with the unique symmetric invertible invariant bilinear form $\langle \cdot, \cdot \rangle$ with normalization

$$\langle 1_V, 1_V \rangle = 1,$$

where [2, 10]

$$\langle Y^\dagger(a, z) b, c \rangle = \langle b, Y(a, z)c \rangle,$$

for

$$Y^\dagger(a, z) = \sum_{n \in \mathbb{Z}} a^\dagger(n)z^{-n-1} = Y\left(e^{zL_V(1)}(-z^{-2})^{L_V(0)}a, z^{-1}\right).$$

4. **$\mathcal{W}$-valued rational functions**

In this Section we recall [6] the notion of a special rational functions which form spaces for the chain-cochain construction. Let $V$ be a grading-restricted vertex algebra, and $W$ a grading-restricted generalized $V$-module. One defines the configuration spaces [6]:

$$F_n\mathbb{C} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j, i \neq j\},$$
for \( n \in \mathbb{Z}_+ \). By \( \mathcal{W} \) we denote the algebraic completion of \( W' \),

\[
\mathcal{W} = \prod_{n \in \mathbb{C}} W_{(n)} = (W')^*.
\]

A \( \mathcal{W} \)-valued rational function in \((z_1, \ldots, z_n)\) with the only possible poles at \(z_i = z_j, \ i \neq j\), is a map

\[
f : F_n \mathbb{C} \rightarrow \mathcal{W},
\]

\[(z_1, \ldots, z_n) \rightarrow f(z_1, \ldots, z_n),\]

such that for any \( w' \in W' \),

\[
(w', f(z_1, \ldots, z_n)),
\]

is a rational function in \((z_1, \ldots, z_n)\) with the only possible poles at \(z_i = z_j, \ i \neq j\). Such map one calls \( \mathcal{W} \)-valued rational function in \((z_1, \ldots, z_n)\) with possible other poles. Denote the space of all \( \mathcal{W} \)-valued rational functions in \((z_1, \ldots, z_n)\) by \( \mathcal{W}_{z_1, \ldots, z_n} \). One defines a left action of \( S_n \) on \( \mathcal{W}_{z_1, \ldots, z_n} \) by

\[
(\sigma(f))(z_1, \ldots, z_n) = f(z_{\sigma(1)}, \ldots, z_{\sigma(n)}),
\]

for \( f \in \mathcal{W}_{z_1, \ldots, z_n} \).

### 4.1. The requirements for \( \mathcal{W} \)-valued functions.

In [6] the following properties for linear maps

\[
\Phi : V^\otimes n \rightarrow \mathcal{W}_{z_1, \ldots, z_n},
\]

are required in order to be able to construct the bicomplexes of \( \mathcal{W}_{z_1, \ldots, z_n} \)-valued functions. Such a linear map is called to have the \( L(0) \)-conjugation property if for \((v_1, \ldots, v_n) \in V, \ w' \in W', \ (z_1, \ldots, z_n) \in F_n \mathbb{C} \) and \( z \in \mathbb{C}^\times \), so that \((zz_1, \ldots, zz_n) \in F_n \mathbb{C} \),

\[
\langle w', z^{L(0)}(\Phi(v_1 \otimes \cdots \otimes v_n))(z_1, \ldots, z_n) \rangle
\]

\[
= \langle w', (\Phi(z^{L(0)}v_1 \otimes \cdots \otimes z^{L(0)}v_n))(zz_1, \ldots, zz_n) \rangle. \tag{4.2}
\]

For \( n \in \mathbb{Z}_+ \), a linear map \( \Phi \) is called to have the \( L(-1) \)-derivative property if (i)

\[
\frac{\partial}{\partial z_i} \langle w', (\Phi(v_1 \otimes \cdots \otimes v_n))(z_1, \ldots, z_n) \rangle
\]

\[
= \langle w', (\Phi(v_1 \otimes \cdots \otimes v_{i-1} \otimes L_V(-1)v_i \otimes v_{i+1} \otimes \cdots \otimes v_n))(z_1, \ldots, z_n) \rangle, \tag{4.3}
\]

for \( i = 1, \ldots, n, \ (v_1, \ldots, v_n) \in V, \) and \( w' \in W' \) and (ii)

\[
\left( \frac{\partial}{\partial z_1} + \cdots + \frac{\partial}{\partial z_n} \right) \langle w', (\Phi(v_1 \otimes \cdots \otimes v_n))(z_1, \ldots, z_n) \rangle
\]

\[
= \langle w', L_V(-1)(\Phi(v_1 \otimes \cdots \otimes v_n))(z_1, \ldots, z_n) \rangle, \tag{4.4}
\]

and \((v_1, \ldots, v_n) \in V, \ w' \in W' \).

One defines an action of \( S_n \) on the space of linear maps \( \Phi \) by

\[
(\sigma(\Phi))(v_1 \otimes \cdots \otimes v_n) = \sigma(\Phi(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)})).
\]

for \( \sigma \in S_n \) and \((v_1, \ldots, v_n) \in V \). We will use the notation \( \sigma_1, \ldots, i_n \in S_n \) to denote the the permutation given by \( \sigma_{i_1, \ldots, i_n}(j) = i_j \), for \( j = 1, \ldots, n \).
4.2. Matrix elements for coboundary operators. As we will in Section 3 the coboundary operators were introduced in [6] in terms of $E$-elements defined as follows. For $w \in W$, the $\mathcal{W}$-valued function $E^{(n)}_{W}(v_1 \otimes \cdots \otimes v_n; w)$ is defined by

$$E^{(n)}_{W}(v_1 \otimes \cdots \otimes v_n; w)(z_1, \ldots, z_n) = E(Y_W(v_1, z_1) \cdots Y_W(v_n, z_n)w),$$

where an element $E(\cdot) \in \mathcal{W}$ is given by

$$\langle w', E(\cdot) \rangle = R((w', \cdot)).$$

If a meromorphic function $f(z_1, \ldots, z_n)$ on a region in $\mathbb{C}$ which can be analytically extended to a rational function in $(z_1, \ldots, z_n)$, we denote such rational functions by $R(f(z_1, \ldots, z_n))$. We introduce

$$E^{W: (n)}_{WV}(w; v_1 \otimes \cdots \otimes v_n) = E^{(n)}_{W}(v_1 \otimes \cdots \otimes v_n; w),$$

where $E^{W: (n)}_{WV}(w; v_1 \otimes \cdots \otimes v_n)$ is an element of $\mathcal{W}_{z_1, \ldots, z_n}$. Next, we define

$$\Phi \circ \left(E^{(l_1)}_{V_1} \otimes \cdots \otimes E^{(l_n)}_{1V} \right) : V^\otimes m+n \to \mathcal{W}_{z_1, \ldots, z_{m+n}},$$

by

$$(\Phi \circ \left(E^{(l_1)}_{V_1} \otimes \cdots \otimes E^{(l_n)}_{1V} \right))(v_1 \otimes \cdots \otimes v_{m+n-1}) = E(\Phi(E^{(l_1)}_{V_1}v_1 \otimes \cdots \otimes v_{l_1}) \otimes \cdots \otimes E^{(l_n)}_{V_1}v_{l_1+\cdots+l_{n-1}+1} \otimes \cdots \otimes v_{l_1+\cdots+l_{n-1}+l_n})), $$

and

$$E^{(m)}_W \circ_{m+1} \Phi : V^\otimes m+n \to \mathcal{W}_{z_1, \ldots, z_{m+n-1}},$$

which is given by

$$(E^{(m)}_W \circ_{m+1} \Phi)(v_1 \otimes \cdots \otimes v_{m+n}) = E(E^{(m)}_W(v_1 \otimes \cdots \otimes v_{m}; \Phi(v_{m+1} \otimes \cdots \otimes v_{m+n}))).$$

Let us also introduce

$$E^{W: (m)}_{WV} \circ_0 \Phi : V^\otimes m+n \to \mathcal{W}_{z_1, \ldots, z_{m+n-1}},$$

which is defined by

$$(E^{W: (m)}_{WV} \circ_0 \Phi)(v_1 \otimes \cdots \otimes v_{m+n}) = E(E^{W: (m)}_{WV}(\Phi(v_1 \otimes \cdots \otimes v_{m}); v_{n+1} \otimes \cdots \otimes v_{n+m})).$$

For

$$l_1 = \cdots = l_{i-1} = l_{i+1} = 1,$$

$$l_i = m - n - 1,$$

for some $1 \leq i \leq n$, we write $\Phi \circ \left(E^{(l_1)}_{V_1} \otimes \cdots \otimes E^{(l_n)}_{1V} \right)$ for $\Phi \circ \left(E^{(l_1)}_{V_1} \otimes \cdots \otimes E^{(l_n)}_{1V} \right)$.
4.3. Maps composable with vertex operators. In order to increase or decrease the number of vertex operators included in a map \( \Phi \), one has to assure convergence of resulting matrix elements. The compositability notion introduced in [6, 4] determines the number of vertex operators that can be added to a map \( \Phi \) by means of coboundary operators. For a \( V \)-module \( W \), let

\[
P_m : W \rightarrow W(m),
\]

be the projection from \( W \) to \( W(m) \). For \( m \in \mathbb{N} \), \( \Phi \) is called to be composable with \( m \) vertex operators when it satisfies:

1. For \( l_1, \ldots, l_n \in \mathbb{Z}_+ \),

\[
l_1 + \cdots + l_n = m + n,
\]

\( (v_1, \ldots, v_{m+n}) \in V \) and \( w' \in W' \), introduce

\[
\Psi_i = E_V^{(l_i)}(v_{k_1} \otimes \cdots \otimes v_{k_i}; 1_V)(z_{k_1}, \ldots, z_{k_i}),
\]

where

\[
k_1 = l_1 + \cdots + l_{i-1} + 1, \quad \ldots \quad v_{k_i} = l_1 + \cdots + l_{i-1} + l_i,
\]

for \( i = 1, \ldots, n \). Then it is assumed that there exist positive integers \( N_m^n(v_i, v_j) \) depending only on \( v_i \) and \( v_j \) for \( i, j = 1, \ldots, k, i \neq j \) such that

\[
\sum_{r_1, \ldots, r_n \in \mathbb{Z}} \langle w', (\Phi(P_{r_1} \Psi_1 \otimes \cdots \otimes P_{r_n} \Psi_n))(\zeta_1, \ldots, \zeta_n) \rangle,
\]

(4.5)

is absolutely convergent on the domains

\[
|z_{1} + \cdots + l_i - 1 + p - \zeta_i| + |z_{i+1} + \cdots + l_{i-1} + q - \zeta_i| < |\zeta_i - \zeta_j|,
\]

for \( i, j = 1, \ldots, k, i \neq j \), and for \( p = 1, \ldots, l_i \) and \( q = 1, \ldots, l_j \). It is assumed that (4.5) is analytically extended to a rational function in \( (z_1, \ldots, z_{m+n}) \), independent of \( (\zeta_1, \ldots, \zeta_n) \), with the only possible poles at \( z_i = z_j \), of order less than or equal to \( N_m^n(v_i, v_j) \), for \( i, j = 1, \ldots, k, i \neq j \).

2. For \( (v_1, \ldots, v_{m+n}) \in V \), it is assumed that there exist positive integers \( N_m^n(v_i, v_j) \), depending only on \( v_i \) and \( v_j \), for \( i, j = 1, \ldots, k, i \neq j \), such that for \( w' \in W' \),

\[
\sum_{q \in \mathbb{C}} \langle w', (E_W^{(m)}(v_1 \otimes \cdots \otimes v_m; P_q((\Phi(v_{1+m} \otimes \cdots \otimes v_{n+m}))(z_{1+m}, \ldots, z_{n+m})))) \rangle,
\]

(4.6)

is absolutely convergent when \( z_i \neq z_j, i \neq j \) \( |z_i| > |z_k| > 0 \) for \( i = 1, \ldots, m \), and \( k = m + 1, \ldots, m + n \). It is assumed that (4.6) can be analytically extended to a rational function in \( (z_1, \ldots, z_{m+n}) \) with the only possible poles at \( z_i = z_j \), of orders less than or equal to \( N_m^n(v_i, v_j) \), for \( i, j = 1, \ldots, k, i \neq j \).
5. VERTEX ALGEBRA BICOMPLEXES

In this Section we recall, following [6], the notion of the chain-cochain bicomplex associated to a grading-restricted vertex algebra $V$. In Section 4 we determine the structure of a bigraded differential algebra for these complexes.

Let start with recall the definition of shuffles. For $l \in \mathbb{N}$ and $1 \leq s \leq l - 1$, let $J_{l,s}$ be the set of elements of $S_l$ which preserve the order of the first $s$ numbers and the order of the last $l - s$ numbers, i.e.,

$$J_{l,s} = \{ \sigma \in S_l \mid \sigma(1) < \cdots < \sigma(s), \ \sigma(s + 1) < \cdots < \sigma(l) \}.$$

The elements of $J_{l,s}$ are called shuffles. Let us denote also

$$J_{l,s}^{-1} = \{ \sigma \mid \sigma \in J_{l,s} \}.$$

The shuffles are used in the definition of $C^n_m(V,W)$ spaces in what follows, and play an essential role in the chain-cochain conditions [5,6] proof in [6].

Now we are on a position to define the space $C^n_m(V,W)$ of $\mathbb{W}_{z_1,\ldots,z_n}$-valued functions.

**Definition 1.** For $n \in \mathbb{Z}_+$, let $C^n_0(V,W)$ be the vector space of all linear maps from $V^\otimes n$ to $\mathbb{W}_{z_1,\ldots,z_n}$ satisfying the $L(-1)$-derivative property and the $L(0)$-conjugation property. For $m, n \in \mathbb{Z}_+$, let $C^n_m(V,W)$ be the vector spaces of all linear maps from $V^\otimes n$ to $\mathbb{W}_{z_1,\ldots,z_n}$ composable with $m$ vertex operators, and satisfying the $L(-1)$-derivative property, the $L(0)$-conjugation property, and such that

$$\sum_{\sigma \in J_{l,s}^{-1}} (-1)^{|\sigma|} \sigma(\Phi(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(l)})) = 0. \quad (5.1)$$

Let us set $C^n_m(V,W) = W$. Then in [6] is was proven that

$$C^n_m(V,W) \subset C^n_{m-1}(V,W),$$

for $m \in \mathbb{Z}_+$. For $\Phi \in C^n_m(V,W)$, the coboundary operator

$$\delta^n_m : C^n_m(V,W) \to C^n_{m-1}(V,W). \quad (5.2)$$

for the bicomplex $(C^n_m(V,W), \delta^n_m)$ has the form:

$$\delta^n_m(\Phi) = E^{(1)}_W \circ \Phi + \sum_{i=1}^n (-1)^i \Phi \circ_i E^{(2)}_{V(1)\cdots (i)} + (-1)^{n+1} \sigma_{n+1,1,\ldots,n} E^{(1)}_W \circ \Phi, \quad (5.3)$$

where $\circ_i$ was defined in Section 2. Explicitly, for $(v_1,\ldots,v_{n+1}) \in V$, $w' \in W'$ and $(z_1,\ldots,z_{n+1}) \in F_{n+1} \mathbb{C}$,

$$\langle w', ((\delta^n_m(\Phi))(v_1 \otimes \cdots \otimes v_{n+1}))(z_1,\ldots,z_{n+1}) \rangle$$

$$= R(\langle w', Y_W(v_1, z_1) (\Phi(v_2 \otimes \cdots \otimes v_{n+1}))(z_2,\ldots,z_{n+1}) \rangle)$$

$$+ \sum_{i=1}^n (-1)^i R(\langle w', (\Phi(v_1 \otimes \cdots \otimes v_{i-1} \otimes Y_V(v_i, z_i - z_{i+1}) v_{i+1} \otimes \cdots \otimes v_{n+1}))(z_1,\ldots,z_{i-1},z_{i+1},\ldots,z_{n+1})) \rangle)$$

$$+ (-1)^{n+1} R(\langle w', Y_W(v_{n+1}, z_{n+1}) (\Phi(v_1 \otimes \cdots \otimes v_n))(z_1,\ldots,z_n) \rangle).$$
The following particular case allows to introduce another short bicomplex. For \( n = 2 \), there exists a subspace of \( C_0^2(V, W) \) containing \( C_0^2(V, W) \) for all \( m \in \mathbb{Z}_+ \) such that \( \delta_m^2 \) remains defined on this subspace. Let \( C^2_+(V, W) \) be the subspace of \( C_0^2(V, W) \) consisting of elements \( \Phi \) such that for \( (v_1, v_2, v_3) \in V, \ w' \in W' \),

\[
\sum_{r \in C} (\langle w', E^{(1)}_W (v_1; P_r((\Phi(v_2 \otimes v_3))(z_2 - \zeta, z_3 - \zeta))(z_1, \zeta) \rangle \\
+ \langle w', (\Phi(v_1 \otimes P_r((E^{(2)}_V(v_2 \otimes v_3; 1_V)))(z_2 - \zeta, z_3 - \zeta))(z_1, \zeta) \rangle),
\]

\[
\sum_{r \in C} (\langle w', (\Phi(P_r((E^{(2)}_V(v_1 \otimes v_2; 1_V)))(z_1 - \zeta, z_2 - \zeta))(z_3) \rangle \\
+ \langle w', E^{W:(1)}_W (P_r((\Phi(v_1 \otimes v_2))(z_1 - \zeta, z_2 - \zeta); v_3))(z_3) \rangle),
\]

are absolutely convergent in the regions

\[
|z_1 - \zeta| > |z_2 - \zeta|, \\
|z_2 - \zeta| > 0, \\
|\zeta - z_3| > |z_1 - \zeta|, \\
|z_2 - \zeta| > 0,
\]

respectively, and can be analytically extended to rational functions in \( z_1 \) and \( z_2 \) with the only possible poles at \( z_1, z_2 = 0 \) and \( z_1 = z_2 \). It is clear that

\[
C_0^2(V, W) \subset C^2_+(V, W),
\]

for \( m \in \mathbb{Z}_+ \). The coboundary operator

\[
\delta_+^2 : C^2_+(V, W) \rightarrow C_0^2(V, W),
\]

is defined in \( \Box \) by

\[
\delta_+^2(\Phi) = E^{(1)}_W \circ_2 \Phi + \sum_{i=1}^2 (-1)^i E^{(2)}_{V, 1_V} \circ_1 \Phi + E^{W:(1)}_{W, V} \circ_2 \Phi,
\]

\[
\langle w', ((\delta_+^2(\Phi))(v_1 \otimes v_2 \otimes v_3))(z_1, z_2, z_3) \rangle \\
= R(\langle w', (E^{(1)}_W (v_1; \Phi(v_2 \otimes v_3))(z_1, z_2, z_3) \rangle \\
+ \langle w', (\Phi(v_1 \otimes E^{(2)}_V(v_2 \otimes v_3; 1_V)))(z_1, z_2, z_3) \rangle) \\
- R(\langle w', (\Phi(E^{(2)}_V(v_1 \otimes v_2; 1_V))(z_1, z_2, z_3) \rangle \\
+ \langle w', (E^{W:(1)}_{W, V}(\Phi(v_1 \otimes v_2; v_3))(z_1, z_2, z_3) \rangle)
\]

for \( w' \in W' \), \( \Phi \in C^2_+(V, W) \), \( (v_1, v_2, v_3) \in V \) and \( (z_1, z_2, z_3) \in F_3 \mathbb{C} \).

In \( \Box \) one has
Proposition 1. For $n \in \mathbb{N}$ and $m \in \{Z_+ + 1\}$,

$$0 \rightarrow C^0_m(V, W) \overset{\delta_0}{\rightarrow} C^1_m(V, W) \overset{\delta_1}{\rightarrow} \cdots \overset{\delta_{m-1}}{\rightarrow} C^m_0(V, W) \rightarrow 0,$$  

(5.6)

$$0 \rightarrow C^0_3(V, W) \overset{\delta_0}{\rightarrow} C^1_3(V, W) \overset{\delta_1}{\rightarrow} C^2_3(V, W) \overset{\delta_2}{\rightarrow} C^3_0(V, W) \rightarrow 0.$$  

(5.7)

\[ \delta^m_{n+1} \circ \delta^m_n = 0, \]

(5.8)

\[ \delta^2_1 \circ \delta^2_2 = 0. \]  

(5.9)

are chain-cochain bicomplexes with respect to the coboundary operators (5.3) and (5.4).

6. The multiplication of elements of $C^m_m(V, W)$-spaces

In this Section we introduce the definition of the simplest variant of multiplication * of elements of two bicomplex spaces with the image in another bicomplex space coherent with respect to the original differential (5.2), and satisfying the symmetry (5.1), $LV(0)$-conjugation (4.2), and $LV(-1)$-derivative (4.3)–(4.4) properties described in Section [2]. It is assumed that $(x_1, \ldots, x_k) \in F_k \mathbb{C}$ and $(y_1, \ldots, y_n) \in F_n \mathbb{C}$. Therefore, according to the definition of the configuration space $F_n \mathbb{C}$, coinciding formal parameters are dropped from $F_{k+n} \mathbb{C}$. A number $r$ of common formal parameters $(x_1, \ldots, x_k)$ of $C^k_m(V, W)$ with $r$ formal parameters in $(y_1, \ldots, y_n)$ of $C^m_n(V, W)$ appear in $F_{k+n} \mathbb{C}$ only once. Thus we obtain

$$(z_1, \ldots, z_{k+n-r}) = (x_1, \ldots, x_{i_l}, \ldots, x_k; y_1, \ldots, y_{i_r}), \quad (6.1)$$

and the exclusion of corresponding formal parameter for $x_{i_l} = y_{i_r}, 1 \leq l \leq r$ in $F_{k+n-r} \mathbb{C}$. The operation of exclusion will be denotes by

\[ \check{R} \Phi(v_1 \otimes \cdots \otimes v_k; v'_1 \otimes \cdots \otimes v'_n)(x_1, \ldots, x_k; y_1, \ldots, y_n). \]  

(6.2)

6.1. The multiplication of matrix elements. The simplest possible multiplication of elements of two $C^m_m(V, W)$-spaces is defined by multiplications of matrix elements of the form (4.1) summed over a $V_{(i)}$-basis for $l \in \mathbb{Z}$. For $\Phi(v_1 \otimes \cdots \otimes v_k)(x_1, \ldots, x_k) \in C^k_m(V, W)$, and $\Psi(v'_1 \otimes \cdots \otimes v'_n)(y_1, \ldots, y_n) \in C^m_n(V, W)$, the multiplication

\[ \Phi(v_1 \otimes \cdots \otimes v_k)(x_1, \ldots, x_k) \ast \Psi(v'_1 \otimes \cdots \otimes v'_n)(y_1, \ldots, y_n) \]

\[ \mapsto \check{R} \Theta(v_1 \otimes \cdots \otimes v_k \otimes v'_1 \otimes \cdots \otimes v'_n) \quad (x_1, \ldots, x_k; y_1, \ldots, y_n). \]  

(6.2)

is a $\mathbb{W}_{z_1, \ldots, z_{k+n-r}}$-valued rational form

\[ \langle u', \check{R} \Theta (v_1 \otimes \cdots \otimes v_k \otimes v'_1 \otimes \cdots \otimes v'_n)(x_1, \ldots, x_k; y_1, \ldots, y_n) \rangle \]

\[ = \sum_{l \in \mathbb{Z}} c_l \sum_{u_l \in V_{(i)}} \langle u', Y^W_{W'V} (\Phi(v_1 \otimes \cdots \otimes v_k)(x_1, \ldots, x_k), \zeta_1) \quad u_l \rangle \]

\[ \langle u', Y^W_{W'V} (\Psi(v'_1 \otimes \cdots \otimes v'_1 \otimes \cdots \otimes v'_n)(y_1, \ldots, y_n), \zeta_2) \quad \pi_l \rangle, \]  

(6.3)

parametrized by $\zeta_1, \zeta_2 \in \mathbb{C}$. The sum is taken over any $V_l$-basis $\{u_l\}$, where $\pi_l$ is the dual of $u_l$ with respect to a non-degenerate bilinear form $\langle \cdot, \cdot \rangle$, (3.1) over $V$. The
operation \( \widehat{R} \) eliminates \( r \) formal parameters from \( \Theta \) from the set \( (y_1, \ldots, y_n) \) coinciding with \( r \) formal parameter of the set \( (x_1, \ldots, x_k) \), and excludes all monomials \( (x_{i_l} - y_{j_l}), 1 \leq l \leq r \), from (6.3). By the standard reasoning [2, 18], (6.3) does not depend on the choice of a basis of \( u \in V_l \), \( l \in \mathbb{Z} \). The form of the multiplication defined above is natural in terms of the theory of characters for vertex operator algebras [15] [18] [18].

We define the action of

\[
\partial_p = \partial_{z_p} = \partial/\partial_{z_p},
\]

for \( 1 \leq p \leq k + n - r \), the differentiation of

\[
\Theta(v_1 \otimes \ldots \otimes v_k \otimes v'_1 \otimes \ldots \otimes \widehat{v}'_l \otimes \ldots \otimes v'_n)(x_1, \ldots, x_k; y_1, \ldots; \widehat{y}_i, \ldots, y_n),
\]

for \( 1 \leq l \leq r \), with respect to the \( p \)-th entry of \( (x_1, \ldots, x_k; y_1, \ldots; \widehat{y}_i, \ldots, y_n) \) as follows

\[
\begin{align*}
\langle w', \partial_p \Theta(v_1 \otimes \ldots \otimes v_k \otimes v'_1 \otimes \ldots \otimes \widehat{v}'_l \otimes \ldots \otimes v'_n) & = \sum_{q \in \mathbb{Z}} \sum_{u_q \in V_q} \langle w', \partial^{p,1}_u Y^W_{1 \overline{W}} (\Phi(v_1 \otimes \ldots \otimes v_k)(x_1, \ldots, x_k), \zeta_l) \ u_q \\
& = \langle w', \partial^{p,1}_u Y^W_{1 \overline{W}} (\Phi(v_1 \otimes \ldots \otimes v_k)(x_1, \ldots, x_k), \zeta_l) \ u_q \\
& \quad \langle \Phi(v_1 \otimes \ldots \otimes v_k)(x_1, \ldots, x_k), \zeta_l) \ u_q \rangle \prod_{q_2} u_q \rangle, \\
\end{align*}
\]

(6.4)

We define the action of an element \( \sigma \in S_{k+n-r} \) on the multiplication of \( \Phi(v_1 \otimes \ldots \otimes v_k)(x_1, \ldots, x_k) \in C^m_m(V, W) \) and \( \Psi(v'_1 \otimes \ldots \otimes v'_n)(y_1, \ldots, y_n) \in C^{n'}_m(V, W) \), as

\[
\begin{align*}
\langle w', \sigma(\widehat{R} \mathcal{F})(v_1 \otimes \ldots \otimes v_k \otimes v'_1 \otimes \ldots \otimes v'_n)(x_1, \ldots, x_k; y_1, \ldots, y_n) \rangle & = \langle w', \Theta(\widehat{v}_{\sigma(1)} \otimes \ldots \otimes \widehat{v}_{\sigma(k+n-r)})(z_{\sigma(1)}, \ldots, z_{\sigma(k+n-r)}) \rangle \\
& = \sum_{l \in \mathbb{Z}} \sum_{u_l \in V_l} \langle w', Y^W_{1 \overline{W}} (\Phi(\widehat{v}_{\sigma(1)} \otimes \ldots \otimes \widehat{v}_{\sigma(k+n-r)})(z_{\sigma(1)}, \ldots, z_{\sigma(k+n-r)}), \zeta_l) \ u_l \\
& \quad \langle \Phi(\widehat{v}_{\sigma(1)} \otimes \ldots \otimes \widehat{v}_{\sigma(k+n-r)})(z_{\sigma(1)}, \ldots, z_{\sigma(k+n-r)}), \zeta_l) \ u_l \rangle \prod_{q_2} u_l \rangle, \\
\end{align*}
\]

(6.5)

where by \( (\widehat{v}_{\sigma(1)}, \ldots, \widehat{v}_{\sigma(k+n-r)}) \) we denote a permutation of

\[
(\widehat{v}_1, \ldots, \widehat{v}_{k+n-r}) = (v_1, \ldots, v_k; v'_1, \ldots, \widehat{v}'_l, \ldots, v'_n). 
\]

(6.6)

Let \( t \) be the number of common vertex operators the mappings \( \Phi(v_1 \otimes \ldots, v_k)(x_1, \ldots, x_k) \in C^k_m(V, W) \) and \( \Psi(v'_1 \otimes \ldots \otimes v'_n)(y_1, \ldots, y_n) \in C^{n'}_m(V, W) \), are composable with. Using the definition of \( C^m_m(V, W) \)-space and the definition of mappings composable with vertex operators, we then have

**Proposition 2.** For \( \Phi(v_1 \otimes \ldots \otimes v_k)(x_1, \ldots, x_k) \in C^m_m(V, W) \) and \( \Psi(v'_1 \otimes \ldots \otimes v'_n)(y_1, \ldots, y_n) \in C^{n'}_m(V, W) \), the multiplication

\[
\widehat{R} \Theta (v_1 \otimes \ldots \otimes v_k \otimes v'_1 \otimes \ldots \otimes v'_n)(x_1, \ldots, x_k; y_1, \ldots, y_n),
\]

(6.3) belongs to the space \( C^{k+n-r}_{m+m'-l}(V, W) \), i.e.,

\[
* : C^k_m(V, W) \times C^{n'}_m(V, W) \to C^{k+n-r}_{m+m'-l}(V, W).
\]

(6.7)
Proof. We show that (6.3) converges to a $W$-valued rational function defined on the configuration space $FC_{k+n-r}$, for formal variables with only possible poles at

$$(z_1, \ldots, z_{k+n-r}) = (x_1, \ldots, x_k; y_1, \ldots, \hat{y}_i, \ldots, y_n),$$

satisfies (5.1), $L_V(0)$-symmetry (1.2), and $L_V(-1)$-derivative (1.3) conditions, and composable with $m + m' - t$ vertex operators. In order to prove convergence of a multiplication of elements of two spaces $C^k_m(V, W)$ and $C^m_{m'}(V, W)$ we use a geometrical interpretation [8, 17]. Recall that a $C^k_m(V, W)$-space is defined by means of matrix elements of the form (1.1), and satisfying $L(0)$-conjugation, $L(-1)$-derivative conditions, (5.1), and composable with $m$ vertex operators. For a vertex algebra $V$, and it module $W$, satisfying certain extra conditions [16], one associate elements of a space $C^k_m(V, W)$ with the data on the Riemann sphere. In particular, formal parameters of $C^k_m(V, W)$-elements and vertex operators they are composable to, are identified with local coordinates of marked points on a sphere. For a pair of spaces $C^k_m(V, W)$ and $C^m_{m'}(V, W)$, we consider data on two Riemann spheres. Two extra points are chosen for centers of annuli used in order to sew spheres [17] [8] to obtain another sphere. The resulting multiplication (6.3) represents a sum of multiplications of matrix elements originated from two original Riemann spheres.

Two complex parameters $\zeta_1, \zeta_2$ of (6.3) are identified with coordinates on annuli. After identification of annuli $r$ coinciding coordinates may occur. This takes into account case of coinciding formal parameters.

The sewing parameter condition is $17$, $\zeta_1 \zeta_2 = \epsilon$. In two sphere $\epsilon$-sewing formulation, the complex parameters $\zeta_a$, $a = 1, 2$ are coordinates inside identified annuluses, and $|\zeta_a| \leq r_a$. The multiplication (6.3) converges for various cases of $V$ and $W$. In particular, it is converges in case of a vertex algebra decomposable into Heisenberg vertex operator algebras [16]. Some further converging examples of $V$ and it modules $W$ will be considered elsewhere. The matrix elements in (6.3) are absolutely convergent in powers of $\epsilon$ with some radii of convergence $R_a \leq r_a$, with $|\zeta_a| \leq R_a$. By expanding the multiplication (6.3) as power series in $\epsilon$ for $|\zeta_a| \leq R_a$, where $|\epsilon| \leq r$ for $r < r_1 r_2$. By using Cauchy's inequality to coefficients for $x$- and $y$-depending parts of the multiplication we find that (6.3) is absolute convergent as a formal series in $\epsilon$ is defined for $|\zeta_a| \leq r_a$, and $|\epsilon| \leq r$ for $r < r_1 r_2$, with extra poles only at $z_i$, $1 \leq i \leq k + n - r$.

When (6.3) is convergent, using geometrical procedure [8, 17] of sewing of two Riemann spheres, we prove that the limiting function is analytically extendable to a $W$-valued function defined on the configuration space

$$FC_{k+n-r} = \{(z_1, \ldots, z_{k+n-r}) : z_i \neq z_j, i \neq j\}.$$
two original Riemann spheres to the sphere formed by sewing. The construction of (6.3) provides that it gives the \( W \)-valued rational function on the configuration space \( F \mathbb{C}_{k+n-r} \). Similar, the construction of the multiplication (6.3) provides that the limiting function is a \( \mathbb{W} \)-valued rational function with the only possible poles at \( z_i = z_j, 1 \leq i < j \leq k + n - r \). By direct substitution we prove that the multiplication (6.3) satisfies the \( L_V(-1) \)-derivative (1.20) and \( L_V(0) \)-conjugation (1.22) properties. Using the definition of the action of an element \( \sigma \in S_{k+n-r} \) on the multiplication (6.3), we prove (6.4) for (6.3).

Next, we show that (6.3) is composable with \( m + m' - t \) vertex operators. Recall that \( \Phi(v_1 \otimes \ldots \otimes v_k)(x_1, \ldots, x_k) \) is composable with \( m \) vertex operators, and \( \Psi(v_i' \otimes \ldots \otimes v_{i'}')(y_1, \ldots, y_{i'}) \) is composable with \( m' \) vertex operators. We consider the first condition for compositability of (6.3) with a number of vertex operators. We redefine the notations for the set

\[
(v_1''', \ldots, v_i''', v_{i+1}''', \ldots, v_k''', v_{k+1}''', \ldots, v_{k+n-1}''', v_{k+n-1}'', \ldots, v_{k+n-r-1}''', v_{k+n-r-1}'', \ldots, v_{k+n-r}''', \ldots, v_{k+n-r+m'}''', \ldots, v_{k+n-r+m'-t}''')
\]

(6.9)

for \( i'' = 1, \ldots, k + n - r, \) and we take

\[
(\zeta_1''', \ldots, \zeta_{k+n-r}'') = (\zeta_1, \ldots, \zeta_k; \zeta_1', \ldots, \zeta_{i'}').
\]

Then, we consider

\[
\mathcal{I}^{k+n-r}_{m+m'-t}(\hat{R} \Theta) = \sum_{r_1''', \ldots, r_{k+n-r}'''} \langle w', \hat{R} \Theta(P_{v_1'''} \Psi_1'''' \ldots \otimes P_{v_{k+n-r}'''} \Psi''_{k+n-r}) (\zeta_1'', \ldots, \zeta_{k+n-r}'') \rangle,
\]

(6.10)

and prove it is absolutely convergent with some conditions. The condition

\[
|s_{i''}''' + t_{i''}' + u_{i''}''' - \zeta_i'''| + |s_{j''}''' + t_{j''}' + u_{j''}''' - \zeta_j'''| < |\zeta_i''' - \zeta_j''|
\]

(6.11)

of absolute convergence for (6.10) for \( i'' \neq j, i = 1, \ldots, k + n - r, \) and for \( p'' = 1, \ldots, l_{i''}''', q'' = 1, \ldots, l_{j''}''', \) follows from corresponding conditions for \( \Phi \) and \( \Psi \). We obtain

\[
\left| \mathcal{I}^{k+n-r}_{m+m'-t}(\hat{R} \Theta) \right| \leq \left| \mathcal{I}^{k}_{m}(\Phi) \right| \left| \mathcal{I}^{n}_{m'}(\Psi) \right|.
\]

Thus, we infer that (6.10) is absolutely convergent. Recall that the maximal orders of possible poles of (6.10) are \( N_m^k(v_i, v_j), N_m^{n'}(v_i', v_j') \) at \( x_i = x_j, y_i' = y_j' \). From the last expression we deduce that there exist positive integers \( N^{k+n-r}_{m+m'-t}(v_i'', v_j'') \) for \( i, j = 1, \ldots, k, i \neq j, i', j' = 1, \ldots, n, i' \neq j' \), depending only on \( v_i'' \) and \( v_j'' \) for \( i'' \), \( j'' = 1, \ldots, k + n, i'' \neq j'' \) such that the series (6.10) can be analytically extended to
a rational function in \((x_1, \ldots, x_k; y_1, \ldots, y_n)\), independent of \((\zeta''_i, \ldots, \zeta''_{n+r})\), with extra possible poles at and \(x_i = y_j\), of order less than or equal to \(N_{m+m'-t}^{k+n-r}(v''_i, v''_j)\), for \(i'', j'' = 1, \ldots, n, i'' \neq j''\).

Let us proceed with the second condition of composability. For the multiplication \((6.3)\) we obtain \((v''_1, \ldots, v''_{k+n-r+m+m'-t}) \in V\), and \((z_1, \ldots, z_{k+n-r+m+m'-t}) \in \mathbb{C}\), we find positive integers \(N_{m+m'-t}^{k+n-r}(v''_i, v''_j)\), depending only on \(v''_i\) and \(v''_j\), for \(i'', j'' = 1, \ldots, k+n-r, i'' \neq j''\), such that for arbitrary \(w' \in W\). Under conditions

\[
z_{i''} \neq z_{j''}, \quad i'' \neq j'', \quad |z_{i''}| > |z_{k''}| > 0,
\]

for \(i'' = 1, \ldots, m + m' - t\), and \(k'' = m + m' - t + 1, \ldots, m + m' - t + k + n - r\), let us introduce

\[
J_{m+m'-t}^{k+n-r}(\hat{R} \Theta) = \sum_{q \in \mathcal{C}} (v''_i, v''_j, \ldots, v''_{m+m'-t}) z_{i''} \cdots \rho_{m+m'-t+1}^{m+m'-t+k+n-r} (z_{i''}, \ldots, z_{m+m'-t+k+n-r}).
\]

We then obtain

\[
|J_{m+m'-t}^{k+n-r}(\hat{R} \Theta)| \leq |J_m^k(\Phi)| |J_{m'}^{n'}(\Psi)|,
\]

where we have used the invariance of \((6.3)\) with respect to \(\sigma \in \mathbb{S}_{m+m'-t+k+n-r}. J_m^k(\Phi)\) and \(J_{m'}^{n'}(\Psi)\) in the last expression are absolute convergent. Thus, we infer that \(J_{m+m'-t}^{k+n-r}(\hat{R} \Theta)\) is absolutely convergent, and the sum \((6.10)\) is analytically extendable to a rational function in \((z_1, \ldots, z_{k+n-r+m+m'-t})\), with the only possible poles at \(x_i = x_j, y_i = y_j\), and \(x_i = y_j\), i.e., the only possible poles at \(z_{i''} = z_{j''}\), of orders less than or equal to \(N_{m+m'-t}^{k+n-r}(v''_i, v''_j)\), for \(i'', j'' = 1, \ldots, k'', i'' \neq j''\). This finishes the proof of the proposition.

The multiplication admits the action of the differential operator \(\delta_{m+m'-t}^{k+n-r}\) defined in \((5.3)\) and \((5.5)\), where \(r\) is the number of common formal parameters, and \(t\) the number of common composite vertex operators for \(\Phi \in C_m^{k}(V, W)\) and \(\Psi \in C_m^{n'}(V, W)\). The co-boundary operators \((5.3)\) and \((5.5)\) possess a variation of Leibniz law with respect to the \(*\)-multiplication.

**Proposition 3.** For arbitrary \(w' \in W\), and for \(\Phi(v_1 \otimes \ldots \otimes v_k)(x_1, \ldots, x_k) \in C_m^{k}(V, W)\) and \(\Psi(v'_1 \otimes \ldots \otimes v'_n)(y_1, \ldots, y_n) \in C_m^{n'}(V, W)\), the action of \(\delta_{m+m'-t}^{k+n-r}\) on their multiplication \((6.3)\) is given by

\[
\langle w', \delta_{m+m'-t}^{k+n-r}(\Phi(v_1 \otimes \ldots \otimes v_k)(x_1, \ldots, x_k) \ast \Psi(v'_1 \otimes \ldots \otimes v'_n)(y_1, \ldots, y_n) \rangle
\]

\[
= \langle w', \delta_{m}^{k}(\Phi(v_1 \otimes \ldots \otimes v_k)(z_1, \ldots, z_k) \ast \Psi(v'_1 \otimes \ldots \otimes v'_n)(z_{k+1}, \ldots, z_{k+n-r}) \rangle
\]

\[
+ (-1)^{k} \langle w', \Phi(v_1 \otimes \ldots \otimes v_k)(z_1, \ldots, z_k) \ast \delta_{m'-t}^{n-r}(\Psi(v'_1 \otimes \ldots \otimes v'_{k+n-r})(z_{k+1}, \ldots, z_{k+n-r}) \rangle.
\]

\[
(6.14)
\]

□
The multiplication (6.3) extends the chain-cochain complex structure of Proposition 11 to all multiplications $C^k_m(V,W) \times C^n_{m'}(V,W)$, $k, n \geq 0$, $m, m' \geq 0$.

7. **Bigraded differential algebras associated to a vertex algebra**

For purposes of introduction of a bigraded differential structure associated to chain-cochain bicomplexes (5.3) and (5.5), we introduce the commutator $\Phi \cdot \Psi$ with respect to $*$-multiplication of elements $\Phi$ and $\Psi$ as

$$\Phi \cdot \Psi = \Phi * \Psi - \Psi * \Phi,$$

(7.1)

with respect to the multiplication $*$ introduced in Section 7.1 for elements of the spaces $C^m_m(V,W)$ and $C^m_{m'}(V,W)$.

7.1. **The structure of bigraded algebras associated to vertex algebras.** In this Section we provide the main results of the paper by deriving relations for the double graded differential algebras associated to bicomplexes (5.6) and (5.7) for a grading-restricted vertex algebra. By analogy with the notion of integrability for differential forms on foliated manifolds [5], we introduce here the notion of orthogonality for elements of bicomplex spaces.

**Definition 2.** For a complex given by

$$0 \rightarrow C_0 \xrightarrow{\delta_0} C_1 \xrightarrow{\delta_1} \ldots \xrightarrow{\delta_n} C_n \xrightarrow{\delta_{n+1}} \ldots$$

(7.2)

let us require that for a pair of a bicomplex spaces $C_i$ and $C_j$, $i, j \geq 0$. there exist subspaces $C_i' \subset C_i$ and such that for $\Phi_i \in C_i'$ and $\Phi_j \in C_j'$,

$$\Phi_i \cdot \delta_j \Phi_j = 0,$$

(7.3)

i.e., $\Phi_i$ commutes to $\delta_j \Phi_j$ with respect to the commutation (7.1). We call (7.3) the orthogonality condition for mappings of the complex (7.2).

The double grading condition on elements of bicomplex spaces occurs from the assumption that elements of both sides of equations following from orthogonality condition belong to the same bicomplex space. For differential forms considered on smooth manifolds, the Frobenius theorem for a distribution leads to the orthogonality condition. We formulate the first main result of this paper:

**Theorem 1.** The orthogonality condition (7.3) endows elements of bicomplex spaces of (5.6) and (5.7) with the structure of a double graded differential algebra with respect to the multiplication (6.3).

**Proof.** Let us consider the most general case. For non-negative $n_0$, $n$, $n_1$, $m_0$, $m$, $m_1$, let $\chi \in C^{m_0}_{m_0}(V,W)$, $\Phi \in C^n_m(V,W)$, and $\alpha \in C^{m_1}_{m_1}(V,W)$. For $\Phi$ and $\alpha$, let $r_0$ be the number of common vertex algebra elements (and formal parameters), and $t_0$ be the number of common vertex operators $\Phi$ and $\alpha$ are composable to. Note that we assume $n, n_1 \geq r_0$, $m, m_1 \geq t_0$. Taking into account the orthogonality condition

$$\Phi \cdot \delta_{m_0}^{m_0} \chi = 0,$$

implies that there exist $\alpha_1 \in C^{m_1}_{m_1}(V,W)$, such that

$$\delta_{m_0}^{m_0} \chi = \Phi \cdot \alpha_1.$$
From the last equations we obtain
\[ n_0 + 1 = n + n_1 - r_0, \]
\[ m_0 - 1 = m + m_1 - t_0. \]
Note that we have extra conditions following from the last identities: \( n_0 + 1 \geq 0, \)
\( m_0 - 1 \geq 0. \) The conditions above for indexes express the double grading condition
for the bicomplexes (5.6) and (5.7). As a result, we have a system in integer variables satisfying the grading conditions above. Consequently acting by corresponding coboundary operators we obtain the full structure of differential relations
\[
\Phi \cdot \delta^m_{m_0} \chi = 0,
\delta^m_{m_0} \chi = \Phi \cdot \alpha_1,
0 = \delta^m_{m_0} \Phi \cdot \alpha_1 + (-1)^n \Phi \cdot \delta^m_{m_1} \alpha_1,
0 = \delta^m_{m_0} \Phi \cdot \delta^m_{m_0} \chi,
\delta^m_{m_0} \chi = \delta^m_{m_0} \Phi \cdot \alpha_2,
0 = \delta^m_{m_0} \Phi \cdot \delta^m_{m_0} \alpha_i,
\delta^m_{m_1} \alpha_i = \delta^m_{m_1} \Phi \cdot \alpha_{i+1}, \quad i \geq 2,
\]
where \( \alpha_i \in C^m_{m_i}(V,W), \) and \( n_i, m_i, i \geq 2 \) satisfy relations
\[ n_i = n + n_{i+1} - r_{i+1}, \]
\[ m_i = m + m_{i+1} - t_{i+1}. \]
The sequence of relations (7.4) does not cancel until the conditions on indexes given above fulfill. □

Thus, we see that the orthogonality condition for the bicomplexes (5.6) and (5.7) together with the action of coboundary operators \( \delta^m_m \) and \( \delta^2_2 \), and the multiplication (6.3), define a differential algebra depending on vertex algebra elements and formal parameters. Next, we have the second main result of this paper

**Proposition 4.** The set of commutation relations generates a sequence of non-vanishing cohomological classes:
\[
[\delta^m_{m_0} \chi] \cdot \chi, \quad [\delta^m_{m_0} \Phi] \cdot \Phi, \quad [\delta^m_{m_i} \alpha_i] \cdot \alpha_i,
\]
for \( i = 1, \ldots, L \), for some \( L \in \mathbb{N} \), with non-vanishing \( \delta^m_{m_0} \chi \cdot \chi, \delta^m_{m_0} \Phi \cdot \Phi, \) and \( \delta^m_{m_i} \alpha_i \cdot \alpha_i \). These classes are independent on the choices of \( \chi \in C^m_{m_0}(V,W), \) \( \Phi \in C^m_{m}(V,W), \) and \( \alpha_i \in C^m_{m_i}(V,W). \)

**Proof.** Let \( \phi \) be one of generators \( \chi, \Phi, \alpha_i, \beta, 1 \leq i \leq L \). Let us show now the non-vanishing property of \( (\delta_{m_i}^n \phi) \cdot \phi \). Indeed, suppose
\[ (\delta_{m_i}^n \phi) \cdot \phi = 0. \]
Then there exists \( \gamma \in C^{m'}_{m_i}(V,W), \) such that
\[ \delta_{m_i}^n \phi = \gamma \cdot \phi. \]
Both sides of the last equality should belong to the same bicomplex space but one can see that it is not possible since we obtain \( m' = t - 1 \), i.e., the number of common
vertex operators for the last equation is greater than for one of multipliers. Thus, \((\delta^m_n \phi) \cdot \phi\) is non-vanishing.

Now let us show that \([\delta^m_n \phi \cdot \phi]\) is invariant, i.e., it does not depend on the choice of \(\Phi \in C^m_n(V,W)\). Substitute \(\phi\) by \((\phi + \eta) \in C^m_n(V,W)\). We have

\[
(\delta^m_n (\phi + \eta)) \cdot (\phi + \eta) = (\delta^m_n \phi) \cdot \phi + ((\delta^m_n \phi) \cdot \eta - \phi \cdot \delta^m_n \eta) + (\phi \cdot \delta^m_n \eta + \delta^m_n \eta \cdot \phi) + (\delta^m_n \eta) \cdot \eta.
\]

(7.5)

Since

\((\phi \cdot \delta^m_n \eta + (\delta^m_n \eta) \cdot \phi) = \phi \delta^m_n \eta - (\delta^m_n \eta) \phi + (\delta^m_n \eta) \phi - \phi \delta^m_n \eta = 0\),

then \((7.5)\) represents the same cohomology class \([\delta^m_n \phi \cdot \phi]\]. □

8. Examples

In this Section we consider particularly interesting examples of algebras and their invariants described in Proposition 5. The orthogonality condition for a bicomplex sequence (5.7), together with the action of coboundary operators \(\delta^m_n\) and \(\delta^2_1\), and the multiplication (6.3), define a differential bigraded algebra depending on vertex algebra elements and formal parameters. In particular, for the bicomplex (5.6), we obtain in this way the generators and commutation relations for a continual Lie algebra \(G(V)\) (a generalization of ordinary Lie algebras with continual space of roots, c.f. [12] described in Appendix (9)) with the continual root space represented by a grading-restricted vertex algebra \(V\).

8.1. Invariants associated with \(C^2_\frac{1}{2}(V,W)\). Due to non-trivial action of the coboundary operators

\[
\delta^2_\frac{1}{2} : C^2_\frac{1}{2}(V,W) \to C^0_0(V,W),
\]

the case \(\Phi \in C^2_\frac{1}{2}\) is exceptional among the relations coming from the double grading condition for corresponding vertex algebra bicomplex, and allows to reconstruct the classical invariants. Let \(\Phi \in C^2_\frac{1}{2}(V,W)\) and \(\chi \in C^m_n(V,W)\). Then we require the orthogonality:

\[
\chi \cdot \delta^2_\frac{1}{2} \Phi = 0.
\]

Thus, there exist \(\beta \in C^m_n(V,W)\) such that

\[
\delta^2_\frac{1}{2} \Phi = \chi \cdot \beta.
\]

We then get

\[
3 = n + n' - r,
0 = m + m' - t.
\]

Let \(n = r + \alpha, 0 \leq \alpha \leq 3, n' = 3 - \alpha; m' = t - m \geq 0\), i.e., \(t = m\), thus \(m = t = m' = 0\).

Thus, \(\chi \in C^r_0(V,W), \beta \in C^{3-\alpha}_0(V,W)\). For \(r + \alpha = 3 - \alpha = 2\) we obtain \(\alpha = 1, r = 1\). If we require \(\chi = \Phi \in C^2_\frac{1}{2}(V,W) \subset C^2_\frac{1}{2}(V,W), k > 0\), then the equation

\[
\delta^2_\frac{1}{2} \Phi = \Phi \cdot \beta,
\]
corresponds to a generalization of Gadbillon-Vey invariant [5] for differential forms. We obtain also the commutation relations:

\[
\begin{align*}
[H, X_{+2}] &= 0, \\
[H, Y_{-}] &= X_{+2}, \\
[X_{-2}, X_{+2}] &= H, \\
[Y_{+}, H_{0}] &= X_{+2},
\end{align*}
\]

for generators

\[
\begin{align*}
H &= \chi, \\
X_{+1} &= \Phi, \\
X_{+2} &= \delta_{2}^{2} \Phi, \\
Y_{-} &= \beta.
\end{align*}
\]

It is easy to see that since all mappings have zero operators composable with, then all further actions of the coboundary operators vanish. Nevertheless, recall that

\[
C_{k}^{2}(V, W) \subset C_{2}^{2}(V, W),
\]

for \( k > 0 \), thus we can consider the most general case when \( \chi \in C_{m_{0}+\alpha}^{r}(V, W) \), \( \Phi \in C_{m_{1}}^{2}(V, W) \), \( \beta \in C_{m_{2}-\alpha}^{3}(V, W) \). Then the grading condition requires \( m_{1} - 1 = m_{0} + m_{2} - t’ \), where \( t’ \) is the number of common vertex operators for \( \chi \in C_{m_{0}+\alpha}^{r}(V, W) \) and \( \beta \in C_{m_{2}-\alpha}^{3}(V, W) \). Thus on acting by coboundary operators we obtain further commutation relations of the form (7.4).

8.2. A continual Lie algebra associated to the bicomplex (5.6). Using the orthogonality condition, way we obtain the generators and commutation relations for a continual Lie algebra. We have

**Proposition 5.** For the bicomplex (5.6), the generators

\[
\{ \chi, \Phi, \alpha_{i}, \delta_{m_{0}}^{0} \chi, \delta_{m_{1}}^{1} \Phi, \delta_{m_{2}}^{2} \alpha_{i}, \}
\]

with \( i \geq 0 \), and commutation relations (7.4) form a continual Lie algebra \( \mathcal{G}(V) \) with a root space provided by the grading-restricted vertex algebra \( V \).

**Proof.** With the redefinition (we suppress here the dependence on vertex algebra elements and formal parameters),

\[
\begin{align*}
H_{0} &= \delta_{0}^{0} \chi, \\
H_{0}^{*} &= \chi, \\
X_{+1} &= \Phi, \\
X_{-i} &= \alpha_{i}, \\
Y_{+1} &= \delta_{1}^{1} \Phi, \\
Y_{-i} &= \delta_{1}^{i} \alpha_{i},
\end{align*}
\]
we arrive at the commutation relations:
\[
[H_0, X_{+1}] = 0,
\]
\[
[X_{+1}, X_{-1}] = H_0,
\]
\[
[Y_{+1}, X_{-1}] = (-1)^n [Y_{-1}, X_{+1}],
\]
\[
[Y_{+1}, X_{-2}] = 0,
\]
\[
[Y_{+1}, Y_{-1}] = 0,
\]
\[
[Y_{+1}, X_{-(i+1)}] = Y_{-i}.
\]
One easily checks Jacobi identities for generators. □

9. Appendix: Continual Lie algebras

Continual Lie algebras were introduced in [12] and then studied in [13, 14]. Suppose \( E \) is an associative algebra (which we call the base algebra) over \( \mathbb{R} \) or \( \mathbb{C} \), and \( K_0, K_\pm, K_{0,0} : E \times E \to E \), are bilinear mappings. The local Lie part of a continual Lie algebra is defined as
\[
\hat{G} = G_{-1} \oplus G_0 \oplus G_{+1},
\]
where \( G_i, i = 0, \pm1 \), are isomorphic to \( E \) and parametrized by its elements. The subspaces \( G_i \) consist of the elements
\[
\{ X_i(\phi), \phi \in E \}, i = 0, \pm1.
\]
The generators \( X_i(\phi) \) are subject to the commutation relations
\[
[X_0(\phi), X_0(\psi)] = X_0(K_{0,0}(\phi, \psi)),
\]
\[
[X_0(\phi), X_{\pm1}(\psi)] = X_{\pm1}(K_{\pm}(\phi, \psi)),
\]
\[
[X_{+1}(\phi), X_{-1}(\psi)] = X_0(K_0(\phi, \psi)),
\]
for all \( \phi, \psi \in E \). It is also assumed that Jacobi identities are satisfied. Then the conditions on mappings \( K_{0,0}, K_0, \pm \) follow:
\[
K_\pm(K_{0,0}(\phi, \psi), \chi) = K_\pm(\phi, K_\pm(\psi, \chi)) - K_{\pm}(\psi, K_{\pm}(\phi, \chi)),
\]
\[
K_{0,0}(\psi, K_0(\phi, \chi)) = K_0(K_+(\psi, \phi), \chi) + K_0(\phi, K_-(\psi, \chi)),
\]
for all \( \phi, \psi, \chi \in E \). An infinite dimensional algebra
\[
\mathcal{G}(E; K) = \mathcal{G}'(E; K)/J,
\]
is called a continual contragredient Lie algebra, where \( \mathcal{G}'(E; K) \) is a Lie algebra freely generated by \( \hat{G} \), and \( J \) is the largest homogeneous ideal with trivial intersection with \( \hat{G}_0 \) (consideration of the quotient is equivalent to imposing the Serre relations in an ordinary Lie algebra case) [13, 14].
REFERENCES

[1] Frenkel, E.; Ben-Zvi, D. Vertex algebras and algebraic curves. Mathematical Surveys and Monographs, 88. American Mathematical Society, Providence, RI, 2001. xii+348 pp.
[2] I. B. Frenkel, Y.-Z. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, preprint, 1989; Memoirs Amer. Math. Soc. 104, 1993.
[3] Ph. Francesco, P. Mathieu, and D. Senechal. Conformal Field Theory. Graduate Texts in Contemporary Physics. 1997.
[4] F. Qi. Representation theory and cohomology theory of meromorphic open string vertex algebras, Ph.D. dissertation, (2018).
[5] E. Ghys L’invariant de Godbillon-Vey. Seminaire Bourbaki, 41–eme annee, n 706, S. M. F. Asterisque 177–178 (1989)
[6] Y.-Zh. Huang A cohomology theory of grading-restricted vertex algebras. Comm. Math. Phys. 327 (2014), no. 1, 279–307.
[7] Y.-Z. Huang, The first and second cohomologies of grading-restricted vertex algebras, Comm. Math. Phys. 327 (2014), no. 1, 261–278.
[8] Y.-Z. Huang, Two-Dimensional Conformal Geometry and Vertex Operator Algebras, Progress in Mathematics, Vol. 148, Birkhäuser, Boston, 1997.
[9] V. Kac: Vertex Operator Algebras for Beginners, University Lecture Series 10, AMS, Providence 1998.
[10] Li, H.: Symmetric invariant bilinear forms on vertex operator algebras, J. Pure. Appl. Alg. 96 (1994) 279–297.
[11] M.V.Losik, On some generalization of a manifold and its characteristic classes (Russian), Functional. Anal. i Prilozhen. 24(1990), no 1, 29-37 ; English translation in Functional Anal. Appl. 24 (1990), 26–32.
[12] M. V. Saveliev, Integro-differential nonlinear equations and continual Lie algebras. Comm. Math. Phys. 121 (1989), no. 2, 283–290.
[13] M. V. Saveliev, A. M. Vershik. Continuum analogues of contragredient Lie algebras. Commun. Math. Phys. 126, 367, 1989;
[14] M. V. Saveliev, A. M. Vershik. New examples of continuum graded Lie algebras. Phys. Lett. A, 143, 121, 1990.
[15] A. Tsuchiya, K. Ueno, and Y. Yamada, Y.: Conformal field theory on universal family of stable curves with gauge symmetries, Adv. Stud. Pure. Math. 19 (1989), 459–566.
[16] M. P. Tuite, A. Zuevsky, A generalized vertex operator algebra for Heisenberg intertwiners. J. Pure Appl. Algebra 216 (2012), no. 6, 1442–1453.
[17] A. Yamada, Precise variational formulas for abelian differentials. Kodai Math.J. 3 (1980), 114–143.
[18] Y. Zhu, Modular invariance of characters of vertex operator algebras, J. Amer. Math. Soc. 9 (1996), 237–307.

INSTITUTE OF MATHEMATICS, CZECH ACADEMY OF SCIENCES, PRAHA

Email address: zuevsky@yahoo.com