INJECTIVITY AND SURJECTIVITY OF THE DRESS MAP

RICARDO G ROJAS-ECHENIQUE

Abstract. For a nontrivial finite Galois extension $L/k$ (where the characteristic of $k$ is different from 2) with Galois group $G$, we prove that the Dress map $h_{L/k} : A(G) \to GW(k)$ is injective if and only if $L = k(\sqrt{\alpha})$ where $\alpha$ is not a sum of squares in $k^\times$. Furthermore, we prove that $h_{L/k}$ is surjective if and only if $k$ is quadratically closed in $L$. As a consequence, we give strong necessary conditions for faithfulness of the Heller-Ormsby functor $c^*_{L/k} : SH_G \to SH_k$, as well as strong necessary conditions for fullness of $c^*_{L/k}$.

1. Introduction

Let $k$ be a field of characteristic different from 2 and let $L/k$ be a finite Galois extension with Galois group $G$. Let $A(G)$ denote the Burnside ring of $G$ and let $GW(k)$ denote the Grothendieck-Witt ring of $k$. Recall that, as abelian groups, $A(G)$ is freely generated under disjoint union by cosets $G/H$ where $H$ runs through a set of representatives for conjugacy classes of subgroups, and $GW(k)$ is generated by 1-dimensional quadratic forms $\langle a \rangle$, where $a$ runs through the group of square classes $k^\times/(k^\times)^2$, under orthogonal sum $\langle a \rangle + \langle b \rangle = \langle a, b \rangle$. Multiplication in $A(G)$ is given by cartesian product with identity $G/G$ and multiplication in $GW(k)$ is given by the Kronecker product $\langle a \rangle \langle b \rangle = \langle ab \rangle$ with identity $\langle 1 \rangle$. Following the construction in [1, Appendix B], the Dress map $h_{L/k} : A(G) \to GW(k)$ is a ring homomorphism that takes the coset $G/H$ to the trace form $\text{tr}_{L/H/k} \langle 1 \rangle_{L/H}$, the quadratic form $x \mapsto \text{tr}_{L/H/k} x^2$. (Our restriction on the characteristic of $k$ is necessary for $h_{L/k}$ to be well defined.)

A particular point of interest is that the Dress map appears naturally in the study of equivariant and motivic stable homotopy theory. Heller and Ormsby [2, §4] construct a strong symmetric monoidal triangulated functor $c^*_{L/k} : SH_G \to SH_k$ from the stable $G$-equivariant homotopy category to the stable motivic homotopy category over $k$. This functor induces a homomorphism between the endomorphism rings of the unit objects in each category, which are in fact $A(G)$ and $GW(k)$, respectively. In [2, Proposition 3.1], Heller and Ormsby show that this homomorphism agrees with $h_{L/k}$. In particular, fullness and faithfulness of $c^*_{L/k}$ are obstructed by surjectivity and injectivity of $h_{L/k}$ respectively.

The main goal of this note is to investigate when the Dress map, and thereby $c^*_{L/k}$, is injective or surjective. While Heller and Ormsby have resolved the investigation when $h_{L/k}$ is an isomorphism [2, Theorem 3.4], we proceed by examining injectivity and surjectivity separately.

When $L = k$ it is obvious that $h_{L/k}$ is injective. The following theorem gives a complete account of when $h_{L/k}$ is injective in the remaining cases.
Theorem 1. For a finite nontrivial Galois extension \(L/k\), \(h_{L/k}\) is injective if and only if \(L = k(\sqrt{\alpha})\) where \(\alpha \in k^\times\) is not a sum of squares in \(k^\times\).

The proof of Theorem 1 is given in \(\S2\). Note that Theorem 1, taken with \(\ref{prop:trivial-extension}\), immediately gives the following corollary.

Corollary 2. If \(c^*_{L/k}\) is faithful, then either \(L/k\) is the trivial extension or of the form described in Theorem 1.

The following theorem gives a complete account of when the Dress map is surjective.

Theorem 3. For a finite Galois extension \(L/k\), \(h_{L/k}\) is surjective if and only if \(k\) is quadratically closed in \(L\).

The proof of Theorem 3 is given in \(\S3\). The following corollary is immediate.

Corollary 4. If \(c^*_{L/k}\) is full, then \(L/k\) is of the form described in Theorem 3.

Theorems 1 and 3 combine to replicate Heller and Ormsby’s result that for a finite Galois extension \(L/k\), \(h_{L/k}\) is an isomorphism if and only if either \(k\) is quadratically closed and \(L = k\), or \(k\) is euclidean and \(L = k(i)\). If \(L/k\) is the trivial extension then Theorem 3 requires that \(k\) be quadratically closed, otherwise Theorem 1 requires that \(L = k(\sqrt{\alpha})\) and \(k^\times/(k^\times)^2\) contains an element that is not a sum of squares. In the latter case, \(k\) must be formally real and then Theorem 3 requires that \(k^\times/(k^\times)^2\) contains \(\alpha\), i.e. \(|k^\times/(k^\times)^2| = 2\), so \(k\) is euclidean and \(\alpha = -1\).

Acknowledgements. I thank Kyle Ormsby for advising and editing this writeup and Irena Swanson for reviewing an earlier draft. Additionally, I thank the referee for suggesting several helpful improvements to the exposition. I gratefully acknowledge that this research was conducted with support under NSF grant DMS-1406327.

2. Proof of Theorem 1

We begin by stating a number of results that are necessary in the proof of Theorem 1. Many of these results are standard and are stated without proof.

Proposition 5. Let \(L/k\) be a finite Galois extension.

1. If \(L = k\), then \(\text{tr}_{L/k}(1)_L = (1)\).
2. If \(L = k(\sqrt{\alpha})\), then \(\text{tr}_{L/k}(1)_L = (2, 2\alpha)\).
3. If \(L = k(\sqrt{\alpha_1}, \sqrt{\alpha_2})\), then \(\text{tr}_{L/k}(1)_L = (1, \alpha_1, \alpha_2, \alpha_1\alpha_2)\).

The following is a standard result from Galois theory.

Proposition 6. Let \(L/k\) be a finite Galois extension with Galois group \(G\). If \(G \cong \mathbb{Z}/4\mathbb{Z}\), then there is a field \(E\) between \(L\) and \(k\) such that \(E = k(\sqrt{\alpha})\) where \(\alpha = a^2 + b^2\) for some \(a, b \in k^\times\).

The following theorem is taken directly from Lam \[3, Proposition 6.14\].

Proposition 7. Let \(L/k\) be a finite Galois extension, and let \(E\) be any field between \(k\) and \(L\) with \([L : E] = 2r + 1\). Then

\[
\text{tr}_{L/k}(1)_L = (2r + 1)\text{tr}_{E/k}(1)_E.
\]

The following lemma is stated in different terms elsewhere. For a proof of this version see \[3\].
Lemma 8. For \( \alpha \in k^\times \), there are positive integers \( a, b \) such that \( a(1) = b(2, 2\alpha) \) if and only if \( \alpha \) is a sum of squares in \( k^\times \).

We are now ready to prove Theorem 1. Suppose that \( L = k(\sqrt{\alpha}) \) where \( \alpha \in k^\times \) is not a sum of squares in \( k^\times \). Then the only subgroups of \( \text{Gal}(L/k) \) are the trivial subgroup and the entire group. Thus, by Proposition 5, the image of \( h_{L/k} \) consists of elements of the form

\[ a(1) + b(2, 2\alpha) \quad \text{where } a, b \in \mathbb{Z}. \]

Now suppose for contradiction that \( h_{L/k} \) is not injective. Then \( \ker(h_{L/k}) \) is nontrivial. That is,

\[ a'(1) - b(2, 2\alpha) = 0 \quad \text{for some } a, b \in \mathbb{Z}^+. \]

It follows from Lemma 8 that \( \alpha \) is a sum of squares in \( k^\times \). This contradicts the hypothesis so we are done with one direction.

The other direction is more difficult so we separate the proof into lemmas.

Lemma 9. Let \( L/k \) be a finite Galois extension with Galois group \( G \). If there is an odd prime \( p \) such that \( p \) divides \( |G| \), then \( h_{L/k} \) is not injective.

Proof. Suppose there is an odd prime \( p \) such that \( p \) divides \( |G| \). Then by Cauchy’s theorem there is a subgroup \( H \leq G \) of order \( p \), so \( [L : L^H] = p \). It follows from Proposition 4 that

\[ h_{L/k}(G/e - pG/H) = \text{tr}_{L/k}(1)_L - p\text{tr}_{L^H/k}(1)_{L^H} = p\text{tr}_{L^H/k}(1)_{L^H} - p\text{tr}_{L^H/k}(1)_{L^H} = 0 \]

where \( e \) is the trivial subgroup. Clearly, \( e \) and \( H \) are in distinct conjugacy classes thus no linear combination of \( G/e \) and \( G/H \) is 0 in \( A(G) \). Hence, \( \ker(h_{L/k}) \) is nontrivial so \( h_{L/k} \) is not injective. \( \square \)

Lemma 10. Let \( L/k \) be a finite Galois extension with Galois group \( G \). If \( |G| = 2^n \) for \( n > 1 \) then \( h_{L/k} \) is not injective.

Proof. Suppose \( |G| = 2^n \) for \( n > 1 \). Since a group of order \( p^k \) has a normal subgroup of order \( p^k \) for each \( 0 \leq k \leq n \), \( G \) has a normal subgroup \( H \) of order \( 2^n - 2 \). We have \( |\text{Gal}(L^H/k)| = 4 \), so \( \text{Gal}(L^H/k) \cong \mathbb{Z}/4\mathbb{Z} \) or \( \text{Gal}(L^H/k) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \). We now analyze each case separately.

Suppose \( \text{Gal}(L^H/k) \cong \mathbb{Z}/4\mathbb{Z} \). Then, by Proposition 4 there is a subextension \( E = k(\sqrt{\alpha}) \) of \( L \) where \( \alpha \) is a sum of two squares. We will use the fact that if \( \alpha \) is a sum of two squares, then \( 2(\alpha) = 2(1) \) (a more general version of this fact is proved in 8). Now by Proposition 4 and since \( \alpha \) is a sum of 2 squares,

\[ h_{L/k}(4G/G - 2G/\text{Gal}(L/E)) = 4(1) - 2\text{tr}_{E/k}(1)_E \]

\[ = 4(1) - 2(2, 2\alpha) \]

\[ = 4(1) - 4(1) = 0. \]

Clearly \( G \) and \( \text{Gal}(L/E) \) are in distinct conjugacy classes. Thus \( \ker(h_{L/k}) \) is nontrivial so \( h_{L/k} \) is not injective.

Now suppose \( \text{Gal}(L^H/k) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \). Then \( L^H = k(\sqrt{\alpha_1}, \sqrt{\alpha_2}) \) and \( L^H \) has distinct subextensions \( E_1 = k(\sqrt{\alpha_1}) \), \( E_2 = k(\sqrt{\alpha_2}) \), and \( E_3 = k(\sqrt{\alpha_1\alpha_2}) \). Let
Consider $h_{L/k}(4G/G + 2G/H - 2G/H_1 - 2G/H_2 - 2G/H_3)$.

$$h_{L/k}(4G/G + 2G/H - 2G/H_1 - 2G/H_2 - 2G/H_3)$$

$$= 4\text{tr}_{L/k}(1)_{L/k} + 2\text{tr}_{L/k}(1)_{L/u} - 2\text{tr}_{E_1/k}(1)_{E_1} - 2\text{tr}_{E_2/k}(1)_{E_2} - 2\text{tr}_{E_3/k}(1)_{E_3}$$

$$= 4(1) + 2(1)_{\alpha_1, \alpha_2, \alpha_1, \alpha_2} - 2(2, 2\alpha_1) - 2(2, 2\alpha_2) - 2(2, 2\alpha_1\alpha_2)$$

$$= 4(1) + 2(1)_{\alpha_1, \alpha_2, \alpha_1, \alpha_2} - 2(1)_{\alpha_1, \alpha_2} - 2(1, \alpha_1\alpha_2)$$

$$= 6(1) + 2(\alpha_1, \alpha_2, \alpha_1\alpha_2) - 6(1) - 2(\alpha_1, \alpha_2, \alpha_1\alpha_2) = 0.$$
Proof of Lemma Suppose that $a\langle 1 \rangle = b\langle 2, 2\alpha \rangle$ where $a, b \in \mathbb{Z}^+$. Then the set of elements $D(b\langle 2, 2\alpha \rangle)$ of $k^\times$ represented by $b\langle 2, 2\alpha \rangle$ and the set of elements $D(a\langle 1 \rangle)$ of $k^\times$ represented by $a\langle 1 \rangle$ are equal. Clearly $2\alpha \in D(b\langle 2, 2\alpha \rangle)$, so $2\alpha \in D(a\langle 1 \rangle)$. It follows that, $2\alpha$ is a sum of squares in $k^\times$. Note that the set of sums of squares in $k^\times$ is a group under multiplication so $\alpha$ is a sum of squares.

For the other direction we proceed by induction on the stronger claim:

$$\alpha \text{ is a sum of } n + 1 \text{ squares } \implies 2^n\langle 1 \rangle = 2^n(\alpha).$$

The base case $n = 0$ is clearly true. Now assume the claim holds for any sum of $n$ squares. Then, for any sum of $n + 1$ squares $\alpha = x_1^2 + \cdots + x_n^2 + x_{n+1}^2$, we have, by a standard fact about quadratic forms (see §II 4),

$$\langle x_1^2 + \cdots + x_n^2 \rangle \in D\langle 1 \rangle, \langle x_{n+1}^2 \rangle \in D\langle x_1^2 + \cdots + x_n^2 \rangle.$$

If we multiply both sides of the above equality by $2^{n-1}$, it follows from the induction hypothesis that

$$2^{n-1}\langle 1 \rangle + 2^{n-1}\langle 1 \rangle = 2^{n-1}(\alpha)(\langle 1 \rangle + \langle x_{n+1}^2 \rangle \langle x_1^2 + \cdots + x_n^2 \rangle)$$

which in turn implies

$$2^n\langle 1 \rangle = 2^n(\alpha)$$

so we have proved the claim.

Now note that for any even positive integer $a$, since $2$ is a sum of $2$ squares,

$$a\langle 1 \rangle = a\langle 2 \rangle.$$

Putting this together we see,

$$\alpha \text{ is a sum of } n \text{ squares in } k^\times \implies 2^n\langle 1 \rangle = 2^n(\langle 1 \rangle + \langle \alpha \rangle) = 2^{n-1}\langle 2, 2\alpha \rangle.$$

This proves the other direction. \qed

References

[1] Andreas W. M. Dress. *Notes on the theory of representations of finite groups*. Universität Bielefeld, Fakultät für Mathematik, Bielefeld, 1971.

[2] J. Heller and K. Ormsby. “Galois equivariance and stable motivic homotopy theory”. In: *ArXiv e-prints* (Jan. 2014). Accepted for publication in Transactions of the AMS. arXiv:1401.4728 [math.AT]

[3] T. Y. Lam. *Introduction to quadratic forms over fields*. Vol. 67. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2005.