Phase space structure and the path integral for gauge theories on a cylinder

Sergey V. SHABANOV

Service de Physique Theorique de Saclay
91191 Gif-sur-Yvette Cedex, France

Abstract

The physical phase space of gauge field theories on a cylindrical spacetime with an arbitrary compact simple gauge group is shown to be the quotient $\mathbb{R}^{2r}/W_A$, $r$ a rank of the gauge group, $W_A$ the affine Weyl group. The PI formula resulting from Dirac’s operator method contains a symmetrization with respect to $W_A$ rather than the integration domain reduction. It gives a natural solution to Gribov’s problem.

Some features of fermion quantum dynamics caused by the nontrivial phase space geometry are briefly discussed.

1. A main feature of gauge theories is the existence of unphysical variables whose evolution is determined by arbitrary functions of time [1], while physical quantities appear to be independent of the gauge arbitrariness. Dynamics of physical variables occurs in a configuration (or phase) space and, therefore, geometry of the physical configuration (or phase) space (below denoted as $CS_{ph}$ or $PS_{ph}$, respectively) plays an important role in the dynamical description. For example, compare particle dynamics on a circle or in a line, or on $PS_{ph}$ being a sphere. The classical as well as quantum theories are obviously different.

In the present letter, we analyze $PS_{ph}$ for 2D topological gauge theories [2] on a cylindrical spacetime, $\mathbb{R} \otimes S^1$, [3]. We include also fermion fields in the theory [4],[5] and observe that their $PS_{ph}$ is also modified, which leads to some dynamical consequences. A main purpose of our work is to construct PI over $PS_{ph}$ (being different from an Euclidean space) which results from the Dirac operator approach.

$PS_{ph}$ can be determined as the quotient of the constraint surface in the whole PS by gauge transformations $G$ generated by all first-class constraints $\sigma_a$

$$PS_{ph} = PS|_{\sigma_a=0}/G.$$ (1)

As has been pointed out in [3], $PS_{ph}$ in gauge models may differ from an ordinary Euclidean space and be, for example, a cone unfoldable into a half-plane (for a review see [4]). The path integral representation of quantum theory depends on the $PS_{ph}$ geometry [7]-[9], which leads to physical consequences for gauge field dynamics [10], [11] and a minisuperspace cosmology [12].
Though the definition (1) is independent of a coordinate (parametrization) choice and explicitly gauge-invariant, we need to introduce coordinates on $PS_{ph}$ upon the PI construction (an attempt to define a "coordinate-free" PI has been proposed in [12] for non-constrained systems). The parametrization choice is motivated by physical reasons. For instance, in gauge theories one may describe physical degrees of freedom by transverse potentials $A^\perp$ and their conjugated momenta $E^\perp$. In QED, there is a one-to-one correspondence between $PS_{ph}$ and $[A^\perp] \otimes [E^\perp] \equiv PS^\perp$ ($[A^\perp]$ implies the functional space of all configurations $A^\perp$), i.e. $PS_{ph} \sim PS^\perp$. However, for non-Abelian theories $PS_{ph}$ does not coincide with $PS^\perp$ because there are gauge equivalent configurations in $[A^\perp]$, Gribov’s copies [13]. Moreover, this parametrization (or gauge fixing) ambiguity always arises and has a geometric nature [14] related to topological properties of $PS_{ph}$. Notice that Gribov’s copies themselves do not have much physical meaning because they are strongly connected with a concrete choice of gauge fixing condition (or parametrizing $PS_{ph}$) which is rather arbitrary, while a topology of $PS_{ph}$ is gauge-independent.

2D topological gauge theories in the Hamiltonian approach (meaning a cylindrical spacetime) have a finite number of physical degrees of freedom and may serve as good toy models for verifying some ideas and methods invented for 4D gauge theories. Recently, we have proposed a PI construction method for any reasonable parametrization of $PS_{ph}$ (for any gauge fixing condition) which is based on the Dirac formalism of quantizing first-class constrained systems [10], [15]. Below we shall describe its main points for the Yang-Mills theory and then apply it to 2D gauge theories. The path integral appears to be modified as compared with the path integral formalism constructed by means of the Faddeev-Popov trick, but it recovers all results obtained by the loop (gauge-fixing free) approach [3],[16].

2. Let us turn directly to establishing $PS_{ph}$ in the Yang-Mills theory on a cylindrical spacetime. The Hamiltonian and constraint read

\[
H = \frac{1}{2} \int_0^{2\pi l} dx (E, E) , \quad (2)
\]

\[
\sigma = \nabla (A) E = \partial E + g[A, E] = 0 , \quad (3)
\]

respectively. Here $l$ is a radius of space, $E$ and $A$ are the colour electric field and potential, they are elements of a Lie algebra $X$ of a simple compact group $G$; components of $E$ and $A$ serve as canonically conjugated variables. The brackets $(,)$ and $[,]$ stand for an invariant inner product and a commutator in $X$, respectively; $\partial \equiv \partial/\partial x$, and $g$ is a coupling constant. The constraint (3) generates gauge transformations

\[
E \rightarrow \Omega E \Omega^{-1} , \quad A \rightarrow \Omega A \Omega^{-1} + g^{-1} \Omega \partial \Omega^{-1} = A^\Omega \quad (4)
\]

with $\Omega = \Omega(x)$ being an element of the gauge group $G$.

As all field variables are functions on $S^1$, they should be periodic with a period $2\pi l$. Also, $\Omega(x + 2\pi l) = \Omega(x)$ (modulo the group center). We denote the space of functions on $S^1$ as $\mathcal{F}[S^1]$; any element $f \in \mathcal{F}[S^1]$ can be decomposed into the Fourier series

\[
f(x) = f_0 + \sum_{n=1}^{\infty} \left( f_{s,n} \sin \frac{n x}{l} + f_{c,n} \cos \frac{n x}{l} \right) . \quad (5)
\]
Formally, any configuration $A$ might be reduced to zero by the transformation (4) with
\[ \Omega^{-1} = \Omega_W(x) = P \exp(-g \int_0^x dy A(y)) \] since $(\partial + gA)\Omega_W = 0$. But the group element $\Omega_W$ does not belong to the gauge group because
\[ \Omega_W(x + 2\pi l) = W[A]\Omega_W(x), \] (6)
where $W[A] = \Omega_W(2\pi l)$ is the Wilson loop. Put, for example, $A = A_0 = const$ then,
\[ W[A_0] = \exp(-2\pi glA_0) \neq 1. \] Therefore, the system possesses physical degrees of freedom.
To separate them, we first reduce fields $E(x)$ and $A(x)$ on the constraint surface (3) to constant configurations $E_0$ and $A_0$ by means of a gauge transformation [3]. The residual continuous gauge arbitrariness consists of constant gauge transformations of $E_0$, $A_0 \in X$.

The surface (3) is reduced to $[E_0, A_0] = 0$.

Any element of $X$ can be represented in the form [17] $A_0 = \Omega_A h \Omega_A^{-1}$, $h$ an element of the Cartan subalgebra $H$ in $X$, $\Omega_A \in G$. Therefore, any phase-space point $E_0, A_0$ on the surface $[E_0, A_0] = 0$ can be obtained from the pair $p_h, h \in H$ by a gauge transformation $\Omega = \Omega_A$. Indeed, $[\Omega_A^{-1}E_0\Omega_A, h] \equiv [p_h, h] = 0$, i.e., $p_h \in H$. The element $h$ has a stationary group being the Cartan subgroup $G_H$ in $G$. This means that not all of the constraints (3) are independent (Eq.(3) implies an infinite number of constraints since $\sigma \in F[\mathbb{S}^1]$).
Namely, there are just $N - r$ independent components amongst $\sigma_0 = \int_0^{2\pi l} \sigma dx$ where $N = \dim G$, $r = rank G = \dim H$.

Thus, the system has $r$ physical degrees of freedom. However, $PS_{ph}$ does not coincide with $\mathbb{R}^{2r}$ because there remain discrete gauge transformations which cannot decrease a number of physical degrees of freedom, but they do reduce their phase space.

Consider a root system in $H$. Let $P$ be a subset of simple roots [14]. A number of simple roots is equal to $r$. $P$ forms a non-orthogonal basis in $H$ [17]. All transformations of $h$ being compositions of reflections $s_\omega$ in hyperplanes orthogonal to simple roots, $(h, \omega) = 0$ $\in P$, $\omega \in P$, form a subgroup of $G$ called the Weyl group $W$ [17], [3],
\[ s_\omega h = \Omega_\omega h \Omega_\omega^{-1} = h - \frac{2(h, \omega)}{(\omega, \omega)}\omega, \quad \Omega_\omega \in G. \] (7)
Therefore, the points $s_\omega p_h$, $s_\omega h$ in $\mathbb{R}^{2r}$ (meaning $H \sim \mathbb{R}^r$) should be identified in accordance with (1) [9]. The Weyl group simply transitively acts on the set of Weyl chambers [17], p.458. Any element of $H$ can be obtained from an element of the Weyl chamber $K^+$ ($h \in K^+$ if $(h, \omega) > 0$ for all $\omega \in P$) by a certain transformation from $W$. The group $W$ does not cover the whole admissible discrete gauge arbitrariness.

Set $E = p_h$ and $A = h$ in (4) and consider gauge transformations with $\Omega = \Omega_\eta = \exp x\eta/l$, $\eta \in H$. The element $\eta$ cannot be arbitrary since the periodicity requires $\exp(2\pi \eta) = e$, $e$ the group unit. This yields
\[ \eta = \sum_{\omega \in P} \frac{2n_\omega}{(\omega, \omega)}\omega, \quad n_\omega \in \mathbb{Z} \] (8)
(a consequence of Lemma 7.6 in [17], p.317). Gauge transformations with $\Omega = \Omega_\eta$ transfer $h$ to $h - a_0 \eta$, $a_0 = (gl)^{-1}$, and leave $p_h$ untouched. Semidirect product of a group of these
\[ ^1 \text{As } H \sim \mathbb{R}^r, \text{ one can assume } (\omega, h) = \omega_i h_i, \text{ } i \text{ labels Cartesian coordinates of the vectors } \omega \text{ and } h. \]
translations and the Weyl group is called the affine Weyl group \( W_A \). The discrete gauge arbitrariness is exhausted by \( W_A \). Any element of \( W_A \) is a composition of reflections \( \hat{s}_{\omega,n} \) in hyperplanes \( (h,\omega) = a_0 n_\omega \). Thus, the physical phase space is the quotient

\[
PS_{ph} = \mathbb{R}^{2r}/W_A ,
\]

where the action of \( W_A \) on \( \mathbb{R}^{2r} \) is determined by all possible compositions of

\[
\hat{s}_{\omega,n} p_h = \hat{s}_{\omega} p_h , \quad \hat{s}_{\omega,n} h = \hat{s}_{\omega} h + \frac{2n_\omega a_0}{(\omega,\omega)} h ,
\]

where \( \hat{s}_\omega \in W \) (cf. (7)), \( \omega \) ranges over \( P, n_\omega \in \mathbb{Z} \).

The fundamental modular domain \( K_A^+ = H/W_A \) being \( CS_{ph} \) in the model is compact and coincides with the Weyl cell \([17]\), a cell of a lattice vertices of which are intersection points of hyperplanes \( (h,\alpha) = a_0 n, \) \( n \) an integer, \( \alpha \) runs over positive roots. This lattice is dual to the root lattice with vertices \( a_0 \sum_{P} n_\omega \omega \). For example, for \( SU(3) \) \( K_A^+ \) is an equal-side triangle with side length \( a_0 \). Notice that the size of the fundamental modular domain non-perturbatively depends on the coupling constants \( a_0 \sim 1/g \), i.e., the geometry of the gauge orbit space \( CS_{ph} \) becomes important in a non-perturbative region \([13],[19]\).

Let \( G = SU(2) \), then \( r = 1 \), \( W = \mathbb{Z}_2 \), \( H \sim \mathbb{R}, (\omega,\omega) = 1 \). Let us construct \( PS_{ph} \). After identification points \( p_h, h + 2a_0 n, \) \( n \in \mathbb{Z} \), on the phase plane, we get a strip \( p_h \in \mathbb{R}, h \in [-a_0, a_0] \) with identified lines \( h = \pm a_0 \) (a cylinder). On this strip, one should stick together points \( p_h, h \) and \( -p_h, -h \). This turns the strip into a half-cylinder ended by two conic horns (the points \( p_h = 0, h = 0, a_0 \)).

So, \( PS_{ph} \) in 2D Yang-Mills theories is a non-homogeneous (hypercylinder-like) symplectic space with (hyper)conic singular points ("hyperhorns"). Notice that in the Abelian case \( PS_{ph} \) is a cylinder with no singular points at all. For groups of rank 2, all singular points of \( PS_{ph} \) lie on a triangle being the boundary \( \partial K_A^+ \) of the Weyl cell. In a neighbourhood of each point \( h_0, p_h = 0 \in \partial K_A^+ \) except the triangle vertices, \( PS_{ph} \) locally coincides with \( \mathbb{R}^2 \otimes cone \), where \( \mathbb{R}^2 \) is spanned by coordinates varying along lines \( (\alpha, h - h_0) = a_0 \delta_\alpha, (\alpha, p_h) = 0, \alpha \) a root orthogonal to \( \partial K_A^+ \), \( \delta_\alpha = 0 \) if \( \alpha \in P \) and \( \delta_\alpha = 1 \) otherwise, while the cone is spanned by coordinates ranging over lines perpendicular to the above ones. It can be easily seen from (10) that the second pair of the local canonical coordinates changes its sign under the reflection \( \hat{s}_{\alpha,\delta_\alpha} \) in the straight line containing a part of \( \partial K_A^+ \) at \( h = h_0 \), whereas \( \hat{s}_{\alpha,\delta_\alpha} \) leaves the first canonical pair untouched. At the triangle vertices, two conic singularities going along two edges stick together. If those edges are orthogonal, \( PS_{ph} \) looks locally like \( cone \otimes cone \), if not, we get a 4D-hypercone as \( PS_{ph} \) in a neighbourhood of the triangle vertices.

A generalization of this singular point pattern in \( PS_{ph} \) to groups of an arbitrary rank is trivial. The Weyl cell is an \( rD \)-polyhedron. \( PS_{ph} \) at the polyhedron vertices has the most singular \( 2rD-hypercone \) structure. On the polyhedron edges, it is locally viewed as \( \mathbb{R}^2 \otimes 2(r-1)D-hypercone \). Further, on the polyhedron faces, being polygons, the local \( PS_{ph} \) structure is \( \mathbb{R}^4 \otimes 2(r-2)D-hypercone \), etc.

A quantization of such symplectic spaces with singular points might give rise to difficulties \([24]\). Fortunately, we know the origin of the singularities — constraints and gauge
symmetry. Therefore, quantization before reducing phase space (Dirac’s operator method) looks preferable.

3. This point is to describe briefly the PI formula for 4D Yang-Mills theory which takes into account a true geometry of $PS_{ph}$. Then we shall apply it to the 2D case to verify our general recipe.

Consider quantum Yang-Mills theory in the Hamiltonian functional representation, i.e., eigenstates $\Phi_n$ of the Hamiltonian $H = \langle E, E \rangle/2 + \langle B, B \rangle/2$, $E = -i\delta/\delta A$, $B$ the colour magnetic field, $\langle \cdot, \cdot \rangle = \int d^3x \{\cdot, \cdot\}$, are functionals of the vector potential. The constraint operator (3) (where $A, E \rightarrow A, E$) must annihilate physical states $\sigma \Phi_n = 0$ [1], which means that $\Phi_n[A^{\Omega}] = \Phi_n[A]$ (see (4)).

Let $[A]$ be a functional space where $\Phi_n$ are defined. Then $CS_{ph} = [A]/G = K$, $G$ acts in $[A]$ as (4). Suppose we wish to parametrize $K$ by fields satisfying a gauge condition $F[A] = 0$, $A \in [A]_F \subset [A]$. The gauge condition is assumed to be complete, it fixes all continuous gauge arbitrariness. Due to the Gribov ambiguity, there is no one-to-one correspondence between $K$ and $[A]_F$. The space $[A]_F$ contains gauge-equivalent configurations $A^s = \Omega A \Omega^{-1} + g^{-1} \Omega \partial \Omega^{-1}$ with $A, A^s \in [A]_F$. The set of residual gauge transformation $S_F = \{\Omega_s\}$ is analogous to $W_A$ in p.2, but $S_F$ is not always a group [15] since a composition of two $\Omega_s$’s does not always give a new copy of $A$. Obviously, $K \sim [A]_F / S_F$.

The condition $\sigma \Phi_n = 0$ guarantees that $\Phi_n$ are functionals on $K$. Therefore, one should incorporate somehow the gauge condition $F = 0$ in the Dirac operator method to make sure that a projection of $\Phi_n$ on the space $[A]_F \neq K$ does not break down the gauge invariance. To reduce the Schrödinger equation $H \Phi_n = E_n \Phi_n$ on $[A]_F$ in a gauge-invariant way, we propose to introduce curvilinear coordinates [10]

$$A_j = A_j [a, w] = \Omega \tilde{A}_j \Omega^{-1} + g^{-1} \Omega \partial \Omega^{-1},$$

(11)

where $\Omega = \Omega[w]$ and $\tilde{A}_j = \tilde{A}_j [a]$ such that $F[A] \equiv 0$ for all $a \in [a]$, i.e., variables $a$ parametrize the space $[A]_F$; by definition $\delta w = \Omega^{-1} \delta \Omega$, $\delta$ stands for a functional variation. One should emphasize that gauge transformations leave $a$ invariant, while $w$ is transferred by them. Thus, we do not fix a gauge at all, but we do choose a parametrization of the orbit space by gauge-invariant variables $a$.

The metric tensor in new variables reads [10], [15]

$$\langle \delta A, \delta A \rangle = \langle \delta q^c, \dot{g}_{cb} [a] \delta q^b \rangle, \quad c, b = 1, 2,$$

(12)

where $\dot{g}_{cb}$ is a linear operator depending on $F$, $\delta q^1 = \delta a$, $\delta q^2 = \delta w$. Rewriting $\delta/\delta A$ via $\delta/\delta q^b$ one can find that $\sigma \sim \delta/\delta w$ and, therefore, $\Phi_n[A] = \Phi_n[\tilde{A}] = \Phi_n^F [a]$, $\Phi_n^F$ are regular solutions to the Schrödinger equation

$$\hat{H}_{ph} \Phi_n^F = \left( \frac{1}{2} \langle p_a, \dot{g}^{11} p_a \rangle + V_q [a] + \frac{1}{2} \langle B, B \rangle \right) \Phi_n^F = E_n \Phi_n^F.$$

(13)

Here $p_a = -i \mu^{-1/2} \delta/\delta a \circ \mu^{1/2}$, $\dot{g}_{cb} \dot{g}^{bd} = \delta^c_\mu$ and $\mu = (\det \dot{g}_{cb})^{1/2}$; an operator ordering correction to the potential is $V_q = 1/2 \mu^{-1/2} \langle \delta/\delta a, \dot{g}^{11} \delta/\delta a \mu^{1/2} \rangle$. 

5
The scalar product is defined in a standard way (the Jacobian $\mu[a]$ has to be taken into account in the scalar product measure $\mu[a]$)
\[
\int_{[A]} DA |\Phi_n|^2 = \int_G Dw \int_K Da \mu |\Phi_n^F|^2 \to \int_K Da \mu |\Phi_n^F|^2 ,
\]
where an infinite constant $\int_G Dw$ can be removed by renormalizing physical states, which is symbolized by the arrow in (14). The integration domain for $a$ has to coincide with $K$.

Indeed, consider transformations of $w$ and $a$ induced by $S_F$, $\Omega[w] \to \Omega \Omega^{-1}[w]_s$ and $\hat{A}[a] \to \hat{A}^*[a] = \hat{A}[a_s]$ (as $\hat{A}^* \in [A]_F$, there exists $a_s = a_s[a] \equiv \hat{s}a$, $\hat{s} \in S_F$, such that $\hat{A}^*[a] = \hat{A}[a_s]$). Obviously, $A_j[a, w] = A_j[a_s, w_s]$. Hence, the mapping (11) is one-to-one (i.e., it defines a change of variables) if $a \in [a]/S_F \sim K$.

Any regular solution to (13) must be automatically $S_F$-invariant because $\Phi_n^F[a_s] = \Phi_n[A^*[a]] = \Phi_n[\hat{A}[a]] = \Phi_n[a]$ where $\Phi_n$ are gauge-invariant regular functionals in $[A]$. So, we do not need to require additionally the $S_F$-invariance of physical states.

For reasonable gauges $F$ the operator $\hat{g}_{cb}$ is invertible [19]. The main goal of our construction is that the scalar product (14) (amplitudes) and the spectrum $E_n$ do not depend on the choice of $F$, a change of $F$ corresponds to a change of variables $a \to \hat{a}[a]$. Quantum theories with different $F$’s are unitary equivalent [13].

To obtain the PI representation for the evolution operator kernel
\[
U_{t_{\text{ph}}}^p[a, a'] = \langle a|e^{-itH_{\text{ph}}^F}|a'\rangle = \sum_n \Phi_n^F[a] \Phi_n^F[a'] e^{-itE_n} ,
\]
one can use the standard slice approximation procedure with the scalar product (14). A naive implement of this scheme leads to PI with the integration domain reduced to $K \subset [a]$. After removing the slice regularization, we get the problem of treating (or calculating) PI over $K$. For example, in the model considered in p.2 $K = K_\Lambda$ is compact. Needless to say, even a finite dimensional Gaussian integral cannot be explicitly done over a compact part of an Euclidean space.

The idea to reduce the integration domain in PI for Yang-Mills theory to the fundamental modular domain has been proposed in [13] and developed in [19]. One can consider such a recipe as a quantization postulate. However, the analysis of exact solvable gauge models [7]-[9] has shown that the Dirac operator formalism leads to another PI representation. Based on this, we propose the following PI formula [10]
\[
U_{t_{\text{ph}}}^p[a, a'] = \int_{[a]} \frac{Da''}{(\mu''\mu)^{1/2}} U_{t_{\text{eff}}}^p[a, a'', Q[a'', a']] ,
\]
\[
Q[a'', a'] = \sum_{S_F} \delta[a'' - \hat{s}a'] , \quad a'' \in [a] , \quad a' \in K ,
\]
\[
U_{t_{\text{eff}}}^p[a, a''] = \int_{[a]} \prod_{\tau=0}^t (Da(\tau)Dp_a(\tau)) \exp i \int_0^t d\tau (\langle p_a, \hat{a} \rangle - H^{\text{eff}}) ,
\]
where $\mu'' = \mu[a'']$, $\int_{[a]} Da'' \delta[a - \hat{s}a'] \Phi[a'] = \Phi[a]$, and $H^{\text{eff}}$ is obtained from the operator $H^F_{ph}$ by replacing the operator $p_a$ by a c-number and by adding an operator ordering term $-i(\delta/\delta a) \hat{g}_{11}(p_a)/2$. Some details of deriving (16)-(18) may be found in [13].
As follows from (16)-(18), instead of solving the problem of definition of PI over \( K \), one has to calculate PI (18) with the ordinary measure and then to symmetrize the result with respect to \( S_F \)-transformations. The bellow application of (16)-(18) to 2D Yang-Mills theory allows us to verify precisely our proposal. All functional integrals entering into (16)-(18) can be explicitly done. The result therefore can be compared with the known (gauge-invariant) operator solution of the problem [3]. We demonstrate in p.4 that they coincide.

4. For sequent calculations we shall use the Cartan-Weyl basis in \( X \) [17] \( [e_\alpha, e_{-\alpha}] = \alpha \), \( \alpha \) a positive root, \( [e_\alpha, e_\beta] = N_{\alpha\beta} e_{\alpha+\beta} \) and

\[
[h, e_{\pm\alpha}] = \pm(h, \alpha)e_{\pm\alpha},
\]

where \( h \in H \) and \( N_{\alpha\beta} \) are non-zero numbers if \( \alpha + \beta \) is a root. As follows from the analysis of p.2, the conditions \( \partial A = 0 \) (the Coulomb gauge) and \( (e_{\pm\alpha}, A) = 0 \) fix all continuous gauge arbitrariness. Therefore one can set \( A = h \in H \) in (11), i.e. \( h \) plays the role of gauge-invariant variables spanning \( CS_p \). The mapping (11) determines a change of variables if \( h \in K_A^+ \) and the Cartan subalgebra components of the homogeneous part of \( \delta w \) (or \( w \)) identically vanish, \( (h, \delta w_0) \equiv 0 \) (we use the notation introduced in (5)). Thus, \( S_F = W_A \) and \( K = K_A^+ \) in our system.

The metric tensor can be obtained from the equality

\[
\delta A = \Omega(dh - g^{-1}\nabla(h)\delta w)\Omega^{-1}.
\]

Substituting it into (12) we find \( \hat{g}_{11} = 1, \hat{g}_{12} = \hat{g}_{21} = 0 \) (since \( \nabla(h)dh \equiv 0 \)) and \( \hat{g}_{22} = -g^{-2}\nabla^2(h) \). The measure \( \mu \) in the scalar product (14) is proportional to det \( \nabla^{ab}(h) \), where \( \nabla^{ab}(h) = \delta^{ab}\partial + h^{ab}, h^{ab} = -gh_i f^{iab}, f^{abc} \) are the structure constants of \( X \), indices \( i, j \) stand for the Cartan subalgebra components. Obviously, \( h^0 = 0 \). The determinant has to be calculated on a subspace of \( F[S^1] \) defined by \( f_0^i \equiv 0 \) since \( \delta w_0^H \equiv 0 \). The operator has no zero modes on this subspace if \( h \in K_A^+ \), \((\partial K_A^+) \) is not included into \( K_A^+ \).

The operator \( \nabla(h) \) acts as an infinite dimensional matrix in the space of Fourier coefficients \( f_0^\pm, f_c^{i,sn}, f_s^{\pm,sn} \) where the upper index \( \pm \alpha \) symbolizes components corresponding to the basis elements \( e_{\pm\alpha} \). This infinite matrix has the block-diagonal form, each block is finite dimensional. We denote these blocks \( \nabla_0, \nabla_h^0, \nabla_h_{\pm}(n = 1, 2, ...) \). They act in invariant subspaces of \( \nabla(h) \) composed of coefficients \( \{f_0^{\pm}\}, \{f_c^{i,sn}\}, n \) fixed, \( \{f_s^{\pm,sn}\}, n \) fixed, respectively. Hence, \( \det \nabla = \det \nabla_0 \prod_n \det \nabla_h^0 \). Straightforward calculations in the Cartan-Weyl basis (actually, only (19) is sufficient to use) lead to the following result [1], \( \det \nabla = C(l)\mu(h), \mu = \kappa^2(h) \) and [23], p.37,

\[
\kappa(h) = \prod_{\alpha > 0} \left[ \frac{\pi(h, \alpha)}{a_0} \prod_{n=1}^\infty \left( 1 - \frac{(h, \alpha)^2}{a_0^2 n^2} \right) \right] = \prod_{\alpha > 0} \sin \frac{\pi(h, \alpha)}{a_0}.
\]

The infinite constant \( C(l) \) appears to be included into a definition of the symbol \( Dw \), i.e., into the norm of physical states.

---

[2] The Cartan-Weyl basis is not orthogonal. Its connection with a real orthogonal basis in \( X \) is given in [23], p.149. The latter is important for a correct calculation of \( h^{ab} \).
Notice also that \( \det \nabla(h) \) coincides with the Faddeev-Popov determinant in the above (Coulomb) gauge condition (cf. the \( SU(N) \) case considered in [24]). It is positive on the fundamental modular domain \( K^+_A \) and vanishes at its boundary in accordance with a general analysis [13]. Moreover, the vacuum configuration \( A = h = 0 \) coincides with the most singular points of \( \det \nabla \), it vanishes as \( |h|^{2N_+}, N_+ = (N - r)/2 \) a number of positive roots.

The physical Hamiltonian reads
\[
H_{ph} = -\frac{\hbar^2}{4\pi l} \frac{1}{\kappa(h)} (\partial_h, \partial_h) \circ \kappa(h) - E_0 ,
\]
where we restore the Planck constant and take into account \( \hat{g}^{11} = 1 \) in (13); the factor \( (2\pi l)^{-1} \) results from \( \delta/\delta A = (2\pi l)^{-1} \Omega \partial_h \Omega^{-1} + ... \) and integration over \( x \);
\[
V_q = \frac{\hbar^2}{4\pi l} \kappa^{-1} \partial^2_h \kappa = -\frac{\pi \hbar^2}{4d_0^2} \left( \sum_{\alpha > 0} \alpha \right)^2 \equiv -E_0 .
\]
The proof of (23) is analogous to that given in [9] to show vanishing the operator ordering corrections in the model considered there.

The effective Hamiltonian in (18) for the operator (22) coincides with a Hamiltonian of an \( r \)-dimensional free particle with mass \( 2\pi l \). Doing the PI (18) we derive from (16) our final result
\[
U_{t+\tau}^{ph}(h, h') = \left( \frac{l}{iht} \right)^{r/2} \sum_{\hat{s} \in W_A} (\kappa(h)\kappa(\hat{s}h'))^{-1} \exp \left( \frac{i\pi l(h - \hat{s}h')^2}{ht} + itE_0 \right) ,
\]
where we have put in (16) \([a] = R^r, \mu^{1/2} = \kappa, S_F = W_A\) and done the integral over \( a'' \equiv h'' \).

One should stress that, first, the PI thus constructed obeys the convolution rule (see (14))
\[
U_{t+t'}^{ph}(h, h') = \int_{K_A} dh'' \kappa^2(h'')U_t^{ph}(h, h'')U_t^{ph}(h'', h')
\]
and, second, it gives a regular solution to Schrödinger equation \( (i\hbar \partial_t - H_{ph})U_t^{ph} = 0 \). The latter means, as has been argued in p.3, that the amplitude (24) must be an analytical gauge-invariant functional in the whole configuration space \([A]\). We shall see bellow that this is so.

As an independent test, one can also recover (24) by summing the spectral series (15). The \textit{regular} eigenfunctions of (22) are
\[
\Phi_{(n)}(h) = \frac{\text{const}}{\kappa(h)} \sum_{\hat{s} \in W} \det \hat{s} \cdot \exp \left( \frac{2\pi i}{a_0} (\gamma_{(n)}, \hat{s}h) \right) ,
\]
where the vector \( \gamma_{(n)} = \sum_P n_\omega \omega \), belongs to the root lattice \( (n_\omega \text{ integers}) \), by definition \( \det \hat{s}_\omega = -1 \) and \( \det \hat{s}' \hat{s} = \det \hat{s} \det \hat{s}' \). The sum of exponents in (26) has zeros (if it
\[3\text{For } G = SU(2) \text{ } 2\pi \text{ in the exponential has to be replaced by } \pi \text{ because the root matrix is a number, } (\omega, \omega) = 1.\]
ψ is assumed to be used \[26\]. Their bosonic part coincides with (11), while the fermionic one

\[A, \psi\] superspace \[\sigma\] should add the standard gauged fermion Hamiltonian to (2) and modify the Gauss law

spectrum would be continuous, which is wrong.

\[\sigma\] condition must be imposed on physical states (compare with \[4\], \[24\]). Thirdly, the

be meaningless \[25\]. Secondly, in Dirac’s operator approach, no addit ional

trajectory reflected from the boundary \(\partial K\) \(\in\) the model). The correct PI is obtained by summing over

that, first, one should not reduce the integration region in PI to the fundamental mod-

\(\hat{s}\) can be expanded into the spectral series (15).

The latter results in the explicit gauge invariance of the transition amplitude (24) because

it can be expanded into the spectral series (15).

Thus, the PI constructed above recover all results of the loop (gauge-fixing free) approach for 2D Yang-Mills theory \[16\], \[3\]. Lessons following from our consideration are that, first, one should not reduce the integration region in PI to the fundamental modular domain (the Weyl cell in the model). The correct PI is obtained by summing over all trajectories reflected from the boundary \(\partial K\) \(\in\) (Gribov’s horizon) and connecting the initial and final configurations. It resembles the PI quantization of a particle on a circle (or in a box). The reduction of the integration domain in PI to an interval is known to be meaningless \[23\]. Secondly, in Dirac’s operator approach, no additional \(W\)-invariance condition must be imposed on physical states (compare with \[4\], \[24\]). Thirdly, the spectrum strongly depends on the \(PS_{ph}\) geometry. If one assumes \(PS_{ph}\) to be \(\mathbb{R}^{2r}\), the spectrum would be continuous, which is wrong.

5. Including fermions in our PI approach does not meet serious difficulties. One should add the standard gauged fermion Hamiltonian to (2) and modify the Gauss law (3) \(\sigma \to \sigma + \rho(\psi, \psi^+)\) with \(\rho\) being the colour charge density of the fermion field \(\psi\) \[4\], \[3\]. To solve the constraint \(\sigma \Phi_{ph} = 0\), one has to introduce curvilinear coordinates on the superspace \([A, \psi]\) (the Grassmann holomorphic representation for fermion operators is assumed to be used) \[26\]. Their bosonic part coincides with (11), while the fermionic one is \(\psi = \Omega \xi\) where the fermion field \(\xi\) plays the role of physical fermion variables. Then,

\[\text{if for instance } G = SU(2), \text{one can always set } (\omega, \omega') = 1 \text{ and } \gamma_n = \omega n, \text{ a positive integer (}K^+\text{ is a positive semiaxis), then the (Casimir) spectrum (and the irreducible representations) is labelled by the spin } j = 0, 1/2, 1, ..., E_n = E_0(n^2 - 1) = 4E_0j(j + 1), \ E_0 = \pi h^2/(4a_0^2).\]
σΦ\text{ph} = 0 decouples into two independent parts \(\delta/\delta w\Phi\text{ph} = 0\) and \(\rho_0^H(\xi, \xi^+)\Phi\text{ph} = 0\) where \(\rho_0^H\) is a homogeneous component of \(\rho\) (cf. (5)) belonging to the Cartan subalgebra. We remind that gauge transformations from the Cartan subgroup \(G_H\) of \(G\) leave \(h\) untouched, but they do transform the fermion field \(\xi\). The \(G_H\)-invariance of physical states yields the constraint \(\rho_0^H\Phi\text{ph} = 0\).

The Laplace-Beltrami operator \(\langle \delta/\delta A, \delta/\delta A \rangle\) in curvilinear supercoordinates contains the term \(V_f = \langle \tilde{\rho}, \nabla^{-2}(h)\tilde{\rho} \rangle/2\) in addition to (22), \(\tilde{\rho} = \rho - \rho_0^H\) (the terms with \(\delta/\delta w\) and \(\rho_0^H\) vanish on physical states). The operator \(\nabla^{-2}(h)\) can be obtained in the same fashion as we have calculated \(\det\nabla(h)\). Regular solutions to the functional Schroedinger equation on the physical subspace will automatically be \(W_A\)-invariant \cite{26}. Notice that the group \(W_A\) non-trivially acts on fermion fields and identifies some points on the Grassmann hyperplane \(\xi, \xi^*\) (\(PS_{ph}\) of physical fermions degrees of freedom is modified \cite{7}, \cite{26}). The PI formula has the same form \(\tilde{U}_t^{ph} = \tilde{U}_t^{eff}\tilde{Q}\) \cite{10}, \cite{15}, i.e., it contains the \(W_A\)-symmetrization provided by \(\tilde{Q}\) rather than the integration domain reduction. The PI for \(\tilde{U}_t^{eff}\) cannot be explicitly done because of the presence of the non-Gaussian term \(V_f\) in the effective action.

An important feature appeared in the mixed model is that the operator \(\tilde{Q}\) simultaneously symmetrize both gauge and fermion fields, \(h\) and \(\xi\), with respect to \(W_A\). Therefore \(\tilde{Q}\) does not commute with fermion creation and destruction operators, which might result in a modification of the fermion Green functions in a non-perturbative region \cite{10}, \cite{15}. Also, the residual transformations responsible for translations \(h \rightarrow h + a_0\eta\) mix fermion creation and destruction operators. It leads to the anomalies in the model \cite{4}.

A detail derivation of the modified PI representation for the 2D Dirac-Yang-Mills theory will be given elsewhere together with an investigation of dynamical consequences emerging due to the non-trivial geometric structure of \(PS_{ph}\).

Acknowledgments

The author is kindly grateful to Prof. J.Klauder (Florida University) and Prof. A.Di Giacomo (Pisa University) for the warm hospitality extended to him during his stay in Gainesville and Pisa where this work has been done.

References

[1] P.A.M.Dirac, Lectures on Quantum Mechanics (Academic Press, N.Y., 1965).

[2] E.Witten, Commun.Math.Phys. 117 (1988) 353.

[3] G.Rajeev, Phys.Lett.B 212 (1988) 203.

[4] E.Langmann and G.W.Semenoff, Phys.Lett.B 303 (1993) 303.

[5] J.Mickelson, Phys.Lett.B 242 (1990) 217.

[6] L.V.Prokhorov, Sov.J.Nucl.Phys. 35 (1982) 192.
[7] L.V.Prokhorov and S.V.Shabanov, Sov.Phys.Uspekhi 34(2) (1991) 108.
[8] S.V.Shabanov, Theor.Math.Phys.(USSR) 78 (1989) 341.
[9] L.V.Prokhorov and S.V.Shabanov, Phys.Lett.B 216 (1989) 341.
[10] S.V.Shabanov, Phys.Lett.B 255 (1991) 398; Mod.Phys.Lett.A 6 (1991) 909.
[11] S.V.Shabanov, Phys.Lett.B 272 (1991) 11.
[12] J.Klauder, Ann.Phys. 188 (1988) 120.
[13] V.N.Gribov, Nucl.Phys.B 139 (1978) 1.
[14] I.M.Singer, Commun.Math.Phys. 60 (1978) 7; M.A.Soloviev, Theor.Math.Phys.(USSR) 78 (1989) 117.
[15] S.V.Shabanov, Lectures on Quantization of Gauge Theories by the Path Integral Method, IFM preprint 16/92, IFM, Lisbon, 1992 (to appear in The Proceedings of the IV Hellenic School on High Energy Physics, Corfu, Greece, 1992).
[16] A.A.Migdal, Sov.Phys.-JETP 42 (1976) 413.
[17] S.Helgason, Differential Geometry, Lie Groups, and Symmetric Spaces (Academic Press, NY,1978).
[18] P.van Baal, Nucl.Phys.B 369 (1992) 259.
[19] D.Zwanziger, Nucl.Phys.B 209 (1982) 336; Nucl.Phys.B 345 (1990) 461.
[20] V.P.Maslov and M.V.Karasiov, Non-linear Poisson Brackets, Quantization, and Geometry (Nauka, Moscow, 1991) (in Russian).
[21] M.Luscher, Nucl.Phys.B 219 (1983) 233.
[22] N.Jacobson, Lie Algebras (Interscience publishers, J.Wiley and Sons, NY, 1965).
[23] I.S.Gradshteyn and I.M.Ryzhyk, Table of Integrals, Series, and Products (Academic Press, NY,1965).
[24] J.E.Hetrick and Y.Hosotani, Phys.Lett.B 230 (1989) 88.
[25] W.Pauli, Pauli Lectures on Physics (Massachusetts, NY, 1973); H.Kleinert and W.Janke, Lett.Nuovo Cim. 25 (1979) 297; L.V.Prokhorov, Sov.J.Part.Nucl. 13(5) (1982) 456.
[26] S.V.Shabanov, J.Phys.A:Math.Gen. 24 (1991) 1199.