EQUIVARIANT BUNDLES AND CONNECTIONS

INDRANIL BISWAS AND ARJUN PAUL

Abstract. Let $X$ be a connected complex manifold equipped with a holomorphic action of a complex Lie group $G$. We investigate conditions under which a principal bundle on $X$ admits a $G$–equivariance structure.

1. Introduction

Let $G$ a complex Lie group and $X$ a connected complex manifold equipped with a holomorphic action of $G$
$$\rho : G \times X \rightarrow X.$$ Let $E_H$ be a holomorphic principal $H$–bundle on $X$, where $H$ is a connected complex Lie group. Given these, we construct a short exact sequence of holomorphic vector bundles
$$0 \rightarrow \text{ad}(E_H) \rightarrow \text{At}_\rho(E_H) \rightarrow q : X \times g \rightarrow 0,$$
where $g$ is the Lie algebra of $G$ and $\text{ad}(E_H)$ is the adjoint vector bundle for $E_H$. The above vector bundle $\text{At}_\rho(E_H)$ is a subbundle of the vector bundle $\text{At}(E_H) \oplus (X \times g)$, where $\text{At}(E_H)$ is the Atiyah bundle for $E_H$ (see (2.9)). A holomorphic $G$–connection on $E_H$ is defined to be a holomorphic splitting of the above exact sequence. In other words, a holomorphic $G$–connection on $E_H$ is a holomorphic homomorphism of vector bundles from the trivial vector bundle with fiber $g$
$$h : X \times g \rightarrow \text{At}_\rho(E_H)$$
such that $q \circ h = \text{Id}_{X \times g}$.

Consider the pullback $\rho^*E_H$ as a holomorphic family of principal $H$–bundles on $X$ parametrized by $G$. Let
$$\mu : g = T_eG \rightarrow H^1(X, \text{ad}(E_H))$$
be the infinitesimal deformation map for this family, where $e \in G$ is the identity element. We prove that $E_H$ admits a holomorphic $G$–connection if and only if $\mu = 0$ (see Lemma 2.2).

Let $G$ be the complex Lie group consisting of all pairs of the form $(y, z)$, where

- $z \in G$ such that the holomorphic principal $H$–bundle $(\rho^*E_H)|_{\{z\} \times X} \rightarrow X$ is holomorphically isomorphic to $E_H$, and
- $y : E_H \rightarrow (\rho^*E_H)|_{\{z\} \times X}$ is a holomorphic isomorphism of principal $H$–bundles.

2010 Mathematics Subject Classification. 32M10, 32L05, 14M17.

Key words and phrases. Group action, principal bundle, Atiyah bundle, $G$–connection.
This group $G$ has natural actions on both $X$ and $E_H$. We show that $E_H$ has a tautological holomorphic $G$–connection whose curvature vanishes identically (see Proposition 3.3). From this proposition it follows that every $G$–equivariant principal $H$–bundle on $X$ has a holomorphic $G$–connection whose curvature vanishes identically (Lemma 4.1).

Finally, assume that $G$ is semisimple and simply connected. Then $E_H$ admits a $G$–equivariant structure if and only if $E_H$ admits a holomorphic $G$–connection (Theorem 4.3). Theorem 4.3 implies that $E_H$ admits a $G$–equivariant structure if and only if the holomorphic principal $H$–bundle $(\rho^*E_H)|_{\{z\}\times X}$ over $X$ is holomorphically isomorphic to $E_H$ for all $z \in G$ (Corollary 4.4).

Equivariant bundles with invariant connections are investigated in [BU] using a Jordan algebraic approach.

2. Atiyah exact sequence and group action

2.1. Atiyah exact sequence. Let $H$ be a connected complex Lie group; its Lie algebra will be denoted by $\mathfrak{h}$. Let $X$ be a connected complex manifold; its holomorphic tangent bundle will be denoted by $TX$. Take a holomorphic principal $H$–bundle on $X$

$$p : E_H \longrightarrow X.$$ (2.1)

Let

$$\psi : E_H \times H \longrightarrow E_H$$ (2.2)

be the action of $H$ on $E_H$. We note that $p \circ \psi = p \circ p_{E_H}$, where $p_{E_H} : E_H \times H \longrightarrow E_H$ is the natural projection, and the action of $H$ on each fiber of $p$ is transitive and free. Let

$$dp : TE_H \longrightarrow p^*TX$$ (2.3)

be the differential of $p$, where $TE_H$ is the holomorphic tangent bundle of $E_H$. Its kernel

$$T_{rel} := \text{kernel}(dp) \subset TE_H$$

is known as the relative tangent bundle for $p$. So we get a short exact sequence of holomorphic vector bundles on $E_H$

$$0 \longrightarrow T_{rel} \longrightarrow TE_H \xrightarrow{dp} p^*TX \longrightarrow 0.$$ (2.4)

The differential of $\psi$ in (2.2) produces a homomorphism from the trivial vector bundle on $E_H$ with fiber $\mathfrak{h}$

$$E_H \times \mathfrak{h} \longrightarrow TE_H$$

which identifies $T_{rel}$ with $E_H \times \mathfrak{h}$.

The action $\psi$ in (2.2) produces an action of $H$ on the total space of $TE_H$. The quotient

$$\text{At}(E_H) := (TE_H)/H$$

is a holomorphic vector bundle on $E_H/H = X$, which is known as the Atiyah bundle $\text{At}$. Let

$$\text{ad}(E_H) := E_H \times^H \mathfrak{h} \longrightarrow X$$ (2.5)
be the adjoint vector bundle for $E_H$ which is associated to it for the adjoint action of $H$ on $\mathfrak{h}$. The action of $H$ on $TE_H$ preserves the subbundle $T_{rel}$. Using the above identification of $T_{rel}$ with $E_H \times \mathfrak{h}$, we have

$$T_{rel}/H = \text{ad}(E_H).$$

So after taking quotient by $H$, the exact sequence in (2.4) produces a short exact sequence of holomorphic vector bundles on $X$

$$0 \longrightarrow \text{ad}(E_H) \xrightarrow{\iota} \text{At}(E_H) \xrightarrow{d'p} TX \longrightarrow 0,$$

which is known as the Atiyah exact sequence $\text{At}$; the above homomorphism $d'p$ is given by $dp$ in (2.3).

A $C^\infty$ connection on $E_H$ compatible with its holomorphic structure is a $C^\infty$ splitting of the exact sequence in (2.6). A holomorphic connection on $E_H$ is a holomorphic splitting of this exact sequence $\text{At}$.

2.2. Atiyah bundle for group action. Let $G$ be a complex Lie group acting holomorphically on the left of $X$ with

$$\rho : G \times X \longrightarrow X$$

being the map giving the action. The Lie algebra of $G$ will be denoted by $\mathfrak{g}$. The differential of $\rho$ in (2.7) produces a $\mathcal{O}_X$–linear homomorphism from the trivial vector bundle on $X$ with fiber $\mathfrak{g}$

$$d'\rho : X \times \mathfrak{g} \longrightarrow TX,$$

where $\mathcal{O}_X$ is the sheaf of holomorphic functions on $X$. We note that the image of $d'\rho$ need not be a subbundle of $TX$.

Consider the holomorphic homomorphism of vector bundles

$$\rho' : \text{At}(E_H) \oplus (X \times \mathfrak{g}) \longrightarrow TX,$$

where $d'p$ and $d'\rho$ are constructed in (2.6) and (2.8) respectively. We note that $\rho'$ is surjective because $d'p$ is surjective. Define the subsheaf

$$\text{At}_\rho(E_H) := (\rho')^{-1}(0) \subset \text{At}(E_H) \oplus (X \times \mathfrak{g})$$

which is in fact a subbundle because $\rho'$ is surjective.

We have two homomorphisms

$$\iota_0 : \text{ad}(E_H) \longrightarrow \text{At}_\rho(E_H), \quad v \longmapsto (\iota(v), 0),$$

where $\iota$ is constructed in (2.6), and

$$q : \text{At}_\rho(E_H) \longrightarrow X \times \mathfrak{g}, \quad (v, w) \longmapsto w,$$

where $v \in \text{At}(E_H)$ and $w \in X \times \mathfrak{g}$. Consequently, there is a short exact sequence of holomorphic vector bundles on $X$

$$0 \longrightarrow \text{ad}(E_H) \xrightarrow{\iota_0} \text{At}_\rho(E_H) \xrightarrow{q} X \times \mathfrak{g} \longrightarrow 0.$$

A holomorphic splitting of (2.10) is a holomorphic homomorphism of vector bundles

$$h : X \times \mathfrak{g} \longrightarrow \text{At}_\rho(E_H)$$
such that $q \circ h = \text{Id}_{X \times \mathfrak{g}}$.

**Definition 2.1.** A holomorphic $G$–connection on $E_H$ is a holomorphic splitting of (2.10).

A holomorphic section of $\text{At}(E_H)$ defined over an open subset $U \subset X$ is a $H$–invariant holomorphic vector field on $p^{-1}(U)$, where $p$ is the projection in (2.11). The Lie bracket of two $H$–invariant holomorphic vector fields on $p^{-1}(U)$ is again a $H$–invariant holomorphic vector field on $p^{-1}(U)$. Therefore, the sheaf of holomorphic sections of $\text{At}(E_H)$ has the structure of a Lie algebra. This and the complex Lie algebra structure of $\mathfrak{g}$ together produce a complex Lie algebra structure on the sheaf of holomorphic sections of $\text{At}_\rho(E_H)$.

Since the adjoint action of $H$ on $\mathfrak{h}$ preserves its Lie algebra structure, every fiber of the vector bundle $\text{ad}(E_H)$ in (2.5) is a Lie algebra isomorphic to $\mathfrak{h}$. We note that the above Lie algebra structure on the fibers of $\text{ad}(E_H)$ coincides with the one given by the Lie bracket of sections of $T\text{rel}$ (see (2.4) for $T\text{rel}$).

The homomorphism $\iota_0$ in (2.10) is compatible with the Lie bracket operations on the sections of $\text{ad}(E_H)$ and $\text{At}_\rho(E_H)$. Similarly, the homomorphism $q$ is also compatible with the Lie bracket operations on the sections of $\text{At}_\rho(E_H)$ and $X \times \mathfrak{g}$.

Let $h : X \times \mathfrak{g} \longrightarrow \text{At}_\rho(E_H)$ be a holomorphic $G$–connection on $E_H$. For any two holomorphic sections $s$ and $t$ of the trivial holomorphic vector bundle $X \times \mathfrak{g}$ defined over an open subset $U \subset X$, consider

$$K(h)(s, t) := [h(s), h(t)] - h([s, t]) \in \Gamma(U, \text{At}_\rho(E_H)) .$$

Since the homomorphism $q$ in (2.10) is compatible with the Lie algebra structures, it follows that $q(K(h)(s, t)) = 0$. Hence $K(h)(s, t)$ lies in the image of $\text{ad}(E_H)$. We have

$$K(h)(f s, t) = f K(h)(s, t)$$

for any holomorphic function $f$ defined on $U$. Also, clearly we have

$$K(h)(s, t) = -K(h)(t, s) .$$

Combining all these it follows that

$$K(h) \in H^0(X, \text{ad}(E_H) \otimes \bigwedge^2 (X \times \mathfrak{g})^*) = H^0(X, \text{ad}(E_H)) \otimes \bigwedge^2 \mathfrak{g}^* . \quad (2.11)$$

This section $K(h)$ will be called the curvature of the holomorphic $G$–connection $h$ on $E_H$.

### 2.3. Criterion for connection

Henceforth, we will always assume that the complex manifold $X$ is compact.

The space of all infinitesimal deformations of the holomorphic principal $H$–bundle $E_H$ are parametrized by $H^1(X, \text{ad}(E_H))$. Therefore, given any holomorphic principal $H$–bundle $\widetilde{E}_H$ on $T \times X$ with $T$ being a complex manifold, and a holomorphic isomorphism of $E_H$ with $\widetilde{E}_H|_{\{t\} \times X}$, where $t \in T$ is a fixed point, we have the infinitesimal deformation homomorphism

$$T\text{rel} \longrightarrow H^1(X, \text{ad}(E_H)) ,$$

where $T\text{rel}$ is the fiber at $t$ of the holomorphic tangent bundle $TT$ of $T$. 
Let $\rho^*E_H \longrightarrow G \times X$ be the pulled back holomorphic principal $H$–bundle, where $\rho$ is the map in (2.7). Considering it as a holomorphic family of principal $H$–bundles on $X$ parametrized by $G$, we have the infinitesimal deformation homomorphism

$$\mu : \mathfrak{g} = T_eG \longrightarrow H^1(X, \text{ad}(E_H)).$$

(2.12)

Lemma 2.2. The principal $H$–bundle $E_H$ admits a holomorphic $G$–connection if and only if $\mu = 0$.

Proof. Let

$$H^0(X, \text{At}_\rho(E_H)) \xrightarrow{\mu_1} H^0(X, X \times \mathfrak{g}) = \mathfrak{g} \xrightarrow{\mu_2} H^1(X, \text{ad}(E_H))$$

(2.13)

be the long exact sequence of cohomologies associated to the short exact sequence in (2.10).

We will show that the above homomorphism $\mu_2$ coincides with $\mu$ in (2.12). This requires recalling the construction of $\mu$. Take any $v \in \mathfrak{g}$. Let $\tilde{v}$ be a holomorphic vector field defined around the identity element $e \in G$ such that $\tilde{v}(e) = v$. Take open subsets $\{U_i\}_{i \in I}$ of $G \times X$ such that

1. $\{e\} \times X \subset \bigcup_{i \in I} U_i$, and
2. for each $i \in I$, the vector field $(\tilde{v}, 0)$ on $U_i$ lifts to a $H$–invariant vector field $\tilde{v}_i$ on $(\rho^*E_H)|_{U_i}$; we choose such a vector field for each $i \in I$. (Here 0 denotes the zero vector field on $X$.)

Now for each ordered pair $(i, j) \in I \times I$, consider the vector field

$$\tilde{v}_i - \tilde{v}_j$$

on $p^{-1}(U_i \cap U_j \cap \{e\} \times X) \subset E_H$, where $p$ is the projection in (2.11). They form a 1–cocycle with values in $\text{ad}(E_H)$. The corresponding cohomology class in $H^1(X, \text{ad}(E_H))$ is $\mu(v)$. From this it is straight–forward to check that $\mu$ coincides with $\mu_2$ in (2.13).

First assume that $\mu_2 = 0$. So $\mu_1$ in (2.13) is surjective. Fix a complex linear subspace $S \subset H^0(X, \text{At}_\rho(E_H))$ such that the restriction

$$\mu_0 := \mu_1|_S : S \longrightarrow \mathfrak{g}$$

is an isomorphism. Now define

$$h : X \times \mathfrak{g} \longrightarrow \text{At}_\rho(E_H), \quad (x, v) \longmapsto (\mu_0)^{-1}(v)(x).$$

Clearly, we have $q \circ h = \text{Id}_{X \times \mathfrak{g}}$, where $q$ is the homomorphism in (2.10). Hence $h$ defines a holomorphic $G$–connection on $E_H$.

Conversely, let $h : X \times \mathfrak{g} \longrightarrow \text{At}_\rho(E_H)$ be a holomorphic $G$–connection on $E_H$. Let

$$h_* : H^0(X, X \times \mathfrak{g}) \longrightarrow H^0(X, \text{At}_\rho(E_H))$$

(2.14)

be the homomorphism induced by $h$. Since $\mu_1 \circ h_* = \text{Id}_{H^0(X \times \mathfrak{g})}$, where $\mu_1$ is the homomorphism in (2.13), it follows that $\mu_1$ is surjective. Hence from the exactness of (2.13) we conclude that $\mu_2 = 0$. □
2.4. Homomorphisms and induced connection. Let 
\[ f : G_1 \longrightarrow G \]
be a holomorphic homomorphism of complex Lie groups. Using \( f \), the action of \( G \) on \( X \) produces an action of \( G_1 \) on \( X \). More precisely,
\[ \rho_1 : G_1 \times X \longrightarrow X, \quad (g, x) \mapsto \rho(f(g), x), \]
where \( \rho \) is the map in (2.7), is a holomorphic action of \( G_1 \) on \( X \). The Lie algebra of \( G_1 \) will be denoted by \( g_1 \). Let 
\[ df : g_1 \longrightarrow g \]
be the homomorphism of Lie algebras associated to \( f \). From the construction of At \( \rho_1(E_H) \) in (2.9) it follows that
\[ \text{At}_{\rho_1}(E_H) = \{(y, z) \in \text{At}_{\rho}(E_H) \oplus (X \times g_1) \mid q(y) = (\text{Id}_X \times df)(z)\}, \quad (2.15) \]
where \( q \) is the homomorphism in (2.10).

Lemma 2.3. A holomorphic \( G \)-connection \( h \) on \( E_H \) induces a holomorphic \( G_1 \)-connection \( h_1 \) on \( E_H \). The curvature \( K(h_1) \) coincides with the image of \( K(h) \) under the homomorphism
\[ H^0(X, \text{ad}(E_H)) \otimes \bigwedge^2 g^* \longrightarrow H^0(X, \text{ad}(E_H)) \otimes \bigwedge^2 g_1^* \]
given by the identity map of \( H^0(X, \text{ad}(E_H)) \) and the homomorphism
\[ \bigwedge^2 g^* \longrightarrow \bigwedge^2 g_1^* \]
induced by the dual homomorphism \((df)^* : g^* \longrightarrow g_1^*\).

Proof. Consider the description of \( \text{At}_{\rho_1}(E_H) \) in (2.15). Let 
\[ h_1 : X \times g_1 \longrightarrow \text{At}_{\rho_1}(E_H) \]
be the homomorphism defined by
\[ z \mapsto (h((\text{Id}_X \times df)(z)), z) \in \text{At}_{\rho}(E_H) \oplus (X \times g_1). \]
Then \( h_1 \) is a holomorphic \( G_1 \)-connection on \( E_H \). Its curvature \( K(h_1) \) is as described in the lemma. \( \square \)

3. Connection and lifting an action

As before, \( X \) is equipped with a holomorphic action of \( G \), and \( E_H \) is a holomorphic principal \( H \)-bundle on \( X \).

Let \( \text{Aut}(E_H) \) denote the group of all holomorphic automorphisms of the principal \( H \)-bundle \( E_H \) over the identity map of \( X \). In other words, an element \( g \in \text{Aut}(E_H) \) is a biholomorphism \( E_H \xrightarrow{g} E_H \) such that

1. \( p \circ g = p \), where \( p \) is the projection in (2.7), and
2. \( \psi(g(z), y) = g(\psi(z, y)) \) for all \((z, y) \in E_H \times H\), where \( \psi \) is the action in (2.2).
This Aut($E_H$) is a complex Lie group. Its Lie algebra is $H^0(X, \text{ad}(E_H))$; the Lie algebra structure on the fibers of ad($E_H$) produces a complex Lie algebra structure on $H^0(X, \text{ad}(E_H))$.

Consider the action $\rho$ in (2.7). For any $z \in G$, let $$\rho_z : X \to X$$ (3.1) be the holomorphic automorphism defined by $x \mapsto \rho(z,x)$.

Let $G_1 \subset G$ (3.2) be the subset consisting all $z \in G$ such that the pulled back principal $H$–bundle $\rho^*_z E_H$ is holomorphically isomorphic to $E_H$ over the identity map of $X$. So $z \in G_1$ if and only if there is a holomorphic automorphism of the principal $H$–bundle $E_H$ over the automorphism $\rho_z$ of $X$. Let $\mathcal{G}$ denote the space of all pairs of the form $(y,z)$, where $z \in G_1$, and $$y : E_H \to E_H$$ is a holomorphic automorphism of the principal $H$–bundle over the automorphism $\rho_z$ of $X$. We observe that $\mathcal{G}$ is equipped with the group operation defined by $$(y', z') \cdot (y, z) = (y' \circ y, z'z),$$ while the inverse is the map $(y, z) \mapsto (y^{-1}, z^{-1})$. Therefore, $\mathcal{G}$ fits in the following short exact sequence of groups

$$0 \to \text{Aut}(E_H) \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} G_1 \to 0,$$

(3.3)

where $\beta(y, z) = z$ and $\alpha(t) = (t, e)$ with $e$ being the identity element of $G_1$. There is a complex Lie group structure on $\mathcal{G}$ which is uniquely determined by the condition that (3.3) is a sequence of complex Lie groups.

The Lie algebra structure on the sheaf of sections of $\text{At}_\rho(E_H)$ produces the structure of a complex Lie algebra on $H^0(X, \text{At}_\rho(E_H))$.

**Proposition 3.1.** The Lie algebra of $\mathcal{G}$ is canonically identified with the above Lie algebra $H^0(X, \text{At}_\rho(E_H))$.

**Proof.** The Lie algebra of $\mathcal{G}$ will be denoted by $\tilde{\mathfrak{g}}$. We will show that there is a natural homomorphism from $\tilde{\mathfrak{g}}$ to $H^0(X, \text{At}_\rho(E_H))$.

First observe that the group $\mathcal{G}$ has a tautological action on the total space $E_H$. Indeed, the action of $(y, z) \in \mathcal{G}$ sends any $x \in E_H$ to $y(x) \in E_H$. It is straIGHT-forward to check that this defines a holomorphic action of $\mathcal{G}$ on $E_H$. From the definition of $\mathcal{G}$ it follows immediately that this action of $\mathcal{G}$ commutes with the action of $H$ on $E_H$. Consequently, we get a homomorphism of complex Lie algebras

$$h' : \tilde{\mathfrak{g}} \to H^0(X, \text{At}(E_H)).$$

Now define

$$h_1 : \tilde{\mathfrak{g}} \to H^0(X, \text{At}_\rho(E_H)), \quad v \mapsto (h'(v), d\beta(v)) \in H^0(X, \text{At}(E_H)) \oplus \mathfrak{g},$$

(3.4)
where \( d\beta : \tilde{\mathfrak{g}} \rightarrow \text{Lie}(G_1) \hookrightarrow \mathfrak{g} \) is the homomorphism of Lie algebras associated to \( \beta \) in (3.3); it is straight-forward to check that
\[
(h'(v), d\beta(v)) \in H^0(X, \text{At}_\rho(E_H)) \subset H^0(X, \text{At}(E_H)) \oplus \mathfrak{g};
\]
see (2.9). Clearly, \( h_1 \) is an injective homomorphism of complex Lie algebras.

To prove that \( h_1 \) is surjective, take any \( w \in H^0(X, \text{At}(E_H)) \). Let \( t \mapsto \varphi^t_w, t \in \mathbb{C}, \) be the 1–parameter family of biholomorphisms of \( E_H \) associated to \( w \). We note that \( \varphi^t_w \) exists because \( X \) is compact and \( w \) is \( H \)–invariant. Since \( w \) is fixed by the action of \( H \) on \( E_H \), it follows immediately that the biholomorphism \( \varphi^t_w \) commutes with the action of \( H \) on \( E_H \) for every \( t \).

Now assume that there is an element \( v \in \mathfrak{g} \) such that \( (w, v) \in H^0(X, \text{At}_\rho(E_H)) \subset H^0(X, \text{At}(E_H)) \oplus \mathfrak{g} \) (see (2.9)). Let \( t \mapsto \exp(tv), t \in \mathbb{C}, \) be the 1–parameter subgroup of \( G \) associated to \( v \). Now we observe that \( (\varphi^t_w, \exp(tv)) \in \mathcal{G} \). Indeed, this follows from the fact that the vector field on \( E_H \) given by \( w \) projects to the vector field on \( X \) given by \( v \). Consequently, the above element \( (w, v) \in H^0(X, \text{At}_\rho(E_H)) \) lies in the image of the homomorphism \( h_1 \) in (3.4). Hence \( h_1 \) is surjective. \( \square \)

In the proof of Proposition 3.1 we saw that \( \mathcal{G} \) has a tautological action on \( E_H \). Let \( \eta : \mathcal{G} \times E_H \rightarrow E_H \) be this action.

**Lemma 3.2.** There is a natural holomorphic isomorphism of vector bundles
\[
\text{At}_\eta(E_H) \rightarrow \text{ad}(E_H) \oplus (X \times H^0(X, \text{At}_\rho(E_H))),
\]
where \( X \times H^0(X, \text{At}_\rho(E_H)) \) is the trivial vector bundle on \( X \) with fiber \( H^0(X, \text{At}_\rho(E_H)) \), and \( \text{At}_\eta(E_H) \) is constructed as in (2.9).

**Proof.** Since \( \text{At}_\eta(E_H) \) has a natural projection to \( X \times \tilde{\mathfrak{g}} \) (see (2.10)), and Proposition 3.1 identifies \( \tilde{\mathfrak{g}} \) with \( H^0(X, \text{At}_\rho(E_H)) \), we obtain a projection
\[
q_1 : \text{At}_\eta(E_H) \rightarrow X \times H^0(X, \text{At}_\rho(E_H)).
\]

The action of \( \mathcal{G} \) on \( X \) factors through the action of \( G \) on \( X \). Recall the description of \( \text{At}_\eta(E_H) \) given in (2.15). Consider the projection
\[
q' : \text{At}_\rho(E_H) \oplus (X \times \tilde{\mathfrak{g}}) = \text{At}_\rho(E_H) \oplus (X \times H^0(X, \text{At}_\rho(E_H))) \rightarrow \text{At}_\rho(E_H)
\]
that sends any \((z, (x, s))\), where \( x \in X, z \in \text{At}_\rho(E_H)_x \) and \( s \in H^0(X, \text{At}_\rho(E_H)) \), to \( z - s(x) \in \text{At}_\rho(E_H)_x \). From (2.15) it follows immediately, that
\[
q \circ (q'|_{\text{At}_\eta(E_H)}) = 0,
\]
where \( q \) is the projection in (2.10). Therefore, the restriction \( q'|_{\text{At}_\eta(E_H)} \) produces a homomorphism
\[
q_2 : \text{At}_\eta(E_H) \rightarrow \text{kernel}(q) = \text{ad}(E_H). \tag{3.5}
\]
Now it is straight-forward to check that the composition
\[ q_2 \oplus q_1 : \text{At}_\eta(E_H) \to \text{ad}(E_H) \oplus (X \times H^0(X, \text{At}_\rho(E_H))) , \]
is an isomorphism. \hfill \Box

**Proposition 3.3.** The principal $H$–bundle $E_H$ has a tautological holomorphic $G$–connection. The curvature of this holomorphic $G$–connection on $E_H$ vanishes identically.

**Proof.** Let $\iota' : \text{ad}(E_H) \to \text{At}_\eta(E_H)$ be the natural inclusion (see (2.10)). For the homomorphism $q_2$ in (3.5), we have
\[ q_2 \circ \iota' = \text{Id}_{\text{ad}(E_H)} . \]
Therefore, kernel($q_2$) provides a holomorphic splitting of the analog of the short exact sequence (2.10) for $\text{At}_\eta(E_H)$. In other words, kernel($q_2$) defines a holomorphic $G$–connection on $E_H$.

Since the homomorphism $q_2$ preserves the Lie algebra structure on the sheaf of sections of $\text{At}_\eta(E_H)$ and $\text{ad}(E_H)$, it follows that the sheaf of sections of kernel($q_2$) is closed under the Lie algebra structure on the sheaf of sections of $\text{At}_\eta(E_H)$. Therefore, the curvature of the holomorphic $G$–connection on $E_H$ defined by kernel($q_2$) vanishes identically. \hfill \Box

4. **Equivariant bundles and connection**

Now-onwards, we assume that the Lie group $G$ is connected.

An equivariance structure on the principal $H$–bundle $E_H$ is a holomorphic action of $G$ on the total space of $E_H$
\[ \rho_E : G \times E_H \to E_H \]
such that

1. $p \circ \rho_E = \rho \circ (\text{Id}_G \times p)$, where $p$ and $\rho$ are the maps in (2.1) and (2.7) respectively, and
2. $\rho_E \circ (\text{Id}_G \times \psi) = \psi \circ (\rho_E \times \text{Id}_H)$ as maps from $G \times E_H \times H$ to $E_H$, where $\psi$ is the action in (2.2).

An equivariant principal $H$–bundle is a principal $H$–bundle with an equivariance structure.

**Lemma 4.1.** Let $(E_H, \rho_E)$ be an equivariant principal $H$–bundle. Then $E_H$ has a tautological holomorphic $G$–connection. The curvature of this holomorphic $G$–connection vanishes identically.

**Proof.** Note that for any $g \in G$, the map
\[ \rho_E^g : E_H \to E_H , \enspace z \mapsto \rho_E(g, z) , \]
is an automorphism of the principal $H$–bundle $E_H$ over the automorphism $\rho_g$ of $X$ in (3.1). Therefore, the group $G_1$ in (3.2) coincides with $G$. In fact, the above map
\[ g \mapsto \rho_E^g \]
is a homomorphism
\[ \beta_E : G \rightarrow \mathcal{G} \]
such that \( \beta \circ \beta_E = \text{Id}_G \), where \( \beta \) is the homomorphism in (3.3).

Consider the tautological holomorphic \( \mathcal{G} \)-connection in Proposition 3.3. In view of the above homomorphism \( \beta_E \), using Lemma 2.3 it produces a holomorphic \( G \)-connection. From Lemma 2.3 it also follows that the curvature of this holomorphic \( G \)-connection vanishes identically. \( \square \)

The following is a converse of Lemma 4.1.

**Lemma 4.2.** Let \( h : X \times \mathfrak{g} \rightarrow \text{At}_\rho(E_H) \) be a holomorphic \( G \)-connection on \( E_H \) such that the curvature vanishes identically. Assume that \( G \) is simply connected. Then there is an equivariance structure
\[ \rho_E : G \times E_H \rightarrow E_H \]
such that the holomorphic \( G \)-connection associated to it by Lemma 4.1 coincides with \( h \).

**Proof.** Recall that \( H^0(X, \text{At}_\rho(E_H)) = \tilde{\mathfrak{g}} = \text{Lie}(\mathcal{G}) \) by Proposition 3.1. Let
\[ h_* : \mathfrak{g} = H^0(X, X \times \mathfrak{g}) \rightarrow H^0(X, \text{At}_\rho(E_H)) = \tilde{\mathfrak{g}} \]
be the \( \mathbb{C} \)-linear map induced by \( h \). Since the curvature of the holomorphic \( G \)-connection \( h \) vanishes identically, it follows that \( h_* \) is a homomorphism of Lie algebras. As \( G \) is simply connected, there is a unique holomorphic homomorphism of complex Lie groups
\[ \gamma : G \rightarrow \mathcal{G} \]
such that the differential \( d\gamma(e) : \mathfrak{g} \rightarrow \tilde{\mathfrak{g}} \) coincides with \( h_* \). Now \( \gamma \) produces an equivariance structure on \( E_H \); recall that \( \mathcal{G} \) acts on \( E_H \). The corresponding holomorphic \( G \)-connection given by Lemma 4.1 clearly coincides with \( h \). \( \square \)

**4.1. The group \( G \) is semisimple.** We now assume that \( G \) is a semisimple and simply connected affine algebraic group defined over \( \mathbb{C} \).

The following theorem show that holomorphic \( G \)-connections produce \( G \)-equivariance structures.

**Theorem 4.3.** Let \( E_H \) be a holomorphic principal \( H \)-bundle on \( X \) admitting a holomorphic \( G \)-connection \( h \). Then \( E_H \) admits an equivariance structure
\[ \rho_E : G \times E_H \rightarrow E_H . \]

**Proof.** Consider the homomorphisms \( \mu_1 \) and \( h_* \) constructed in (2.13) and (2.14) respectively. Since
\[ \mu_1 \circ h_* = \text{Id}_{H^0(X, X \times \mathfrak{g})} = \text{Id}_{\mathfrak{g}} , \]
we know that the Lie algebra homomorphism \( \mu_1 \) is surjective. As \( \mathfrak{g} \) is semisimple, there is a Lie subalgebra
\[ \mathcal{S} \subset H^0(X, \text{At}_\rho(E_H)) \]
such that the restriction
\[ \hat{\mu} := \mu_1|_S : S \rightarrow H^0(X, X \times g) = g \]
is an isomorphism \[\text{[Bo, p. 91, Corollaire 3]}\]. Fix a subspace \( S \) as above. Define \( \hat{h} \) to be the composition
\[ H^0(X, X \times g) = g \xrightarrow{\hat{\mu}^{-1}} S \hookrightarrow H^0(X, \text{At}_\rho(E_H)) \].

Since \( \hat{\mu} \) is a homomorphism of Lie algebras, it follows that \( \hat{h} \) is a holomorphic \( G \)–connection on \( E_H \) such that the curvature vanishes identically. Now the theorem follows from Lemma \[\text{4.2}\].

**Corollary 4.4.** Let \( E_H \) be a holomorphic principal \( H \)–bundle on \( X \) such that the pulled back principal \( H \)–bundle \( \rho^*_zE_H \) is holomorphically isomorphic to \( E_H \) for every \( z \in G \), where \( \rho_z \) is defined in \[\text{(3.1)}\]. Then \( E_H \) admits an equivariance structure.

**Proof.** Since \( \rho^*_zE_H \) is holomorphically isomorphic to \( E_H \) for all \( z \in G \), it follows that the infinitesimal deformation map \( \mu \) in \[\text{(2.12)}\] is the zero homomorphism. Therefore, \( E_H \) admits a holomorphic \( G \)–connection by Lemma \[\text{2.2}\]. Now Theorem \[\text{4.3}\] completes the proof. \( \square \)

## 5. Some examples

### 5.1. Trivial action.
Consider the trivial action of \( G \) on \( X \). So \( \rho(g, x) = x \) for all \( g \in G \) and \( x \in X \). In this case
\[ \text{At}_\rho(E_H) = \text{ad}(E_H) \oplus (X \times g) \].

Therefore, a holomorphic \( G \)–connection on \( E_H \) is a holomorphic homomorphism \( X \times g \rightarrow \text{ad}(E_H) \). Note that there is a tautological holomorphic \( G \)–connection on \( E_H \) given by the zero homomorphism from \( X \times g \) to \( \text{ad}(E_H) \).

### 5.2. Trivial tangent bundle.
Let \( X \) be a compact complex manifold such that the holomorphic tangent bundle \( TX \) is holomorphically trivial. Then a theorem of Wang says that there is a complex connected Lie group \( G \) and a discrete subgroup \( \Gamma \subset G \), such that \( X \) is biholomorphic to \( G/\Gamma \) \[\text{[Wa, p. 774, Theorem 1]}\]. Fix an isomorphism of \( X \) with \( G/\Gamma \). Let \( \rho \) be the left translation action of \( G \) on \( G/\Gamma = X \). In this case, the homomorphism \( d^\rho \) in \[\text{(2.8)}\] is an isomorphism. Therefore, the Atiyah exact sequence in \[\text{(2.6)}\] coincides with the exact sequence in \[\text{(2.10)}\].

Consequently, holomorphic \( G \)–connections on \( E_H \) are same as holomorphic connections on \( E_H \).
5.3. **Smooth toric variety.** Let $X$ be a smooth complex projective toric manifold such that the subvariety $D \subset X$ where the torus action is not free is actually a simple normal crossing divisor. Then the logarithmic tangent bundle $TX(-\log D)$ is holomorphically trivial [BDP, p. 317, Lemma 3.1(2)]. Therefore, the holomorphic $G$–connections on $E_H$ are the logarithmic connections on $E_H$ singular over $D$. Hence [BDP, p. 319, Proposition 3.2] follows from Lemma 4.1. We note that [BDP, p. 324, Theorem 4.2] can also be proved modifying the proof of Theorem 4.3.

5.4. **Homogeneous manifolds.** Let $M$ be a closed subgroup of a complex connected Lie group $G$. The group $G$ acts on $X := G/M$ as left–translations. The left–translation action of $G$ on itself will be denoted by $\theta$. Consider the short exact sequence in (2.10) over $X$. Pull this exact sequence back to $G$ by the quotient map $q : G \rightarrow G/M$. Note that this pullback is of the form

$$0 \rightarrow \text{ad}(q^*E_H) \rightarrow \text{At}_\theta(q^*E_H) \rightarrow G \times \mathfrak{g} \rightarrow 0,$$

where $\mathfrak{g}$ is the Lie algebra of $G$. Observe that (5.1) coincides with the Atiyah exact sequence for $q^*E_H$.

Now consider the right–translation action of $M$ on $G$. Since $q^*E_H$ is pulled back from $G/M$, it follows that $q^*E_H$ is a $M$–equivariant principal $H$–bundle on $G$.

From the above observation that (5.1) is the Atiyah exact sequence for $q^*E_H$ it follows that the holomorphic $G$–connections on $E_H$ are precisely the $M$–equivariant holomorphic connections on the $M$–equivariant principal $H$–bundle on $q^*E_H$.

**REFERENCES**

[At] M. F. Atiyah, Complex analytic connections in fibre bundles, *Trans. Amer. Math. Soc.* **85** (1957), 181–207.

[BDP] I. Biswas, A. Dey and M. Poddar, Equivariant principal bundles and logarithmic connections on toric varieties, *Pacific Jour. Math.* **280** (2016), 315–325.

[BU] I. Biswas and H. Upmeier, Homogeneous holomorphic hermitian principal bundles over hermitian symmetric spaces, *New York Jour. Math.* **22** (2016), 21–47.

[Bo] N. Bourbaki, Éléments de mathématique. XXVI. Groupes et algèbres de Lie. Chapitre 1: Algèbres de Lie, Actualités Sci. Ind. No. 1285, Hermann, Paris 1960.

[Wa] H.-C. Wang, Complex parallisable manifolds, *Proc. Amer. Math. Soc.* **5** (1954), 771–776.

**School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India**

*E-mail address: indranil@math.tifr.res.in*

**School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India**

*E-mail address: apmath90@math.tifr.res.in*