Convergence of the spline method for solving the optimal dynamic measurement problem

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Abstract. The problem of optimal dynamic measurement consists in finding the dynamically distorted input signal by observing the output signal. This problem is solved as an optimal control problem to minimize the difference between the values obtained from the output data and the model values obtained in the course of a computational experiment. It was shown earlier that this problem has a unique solution. For a complex form of the input signal, various numerical methods can be used to find the optimal dynamic measurement. This article suggests to use the spline method in order to solve this problem. Namely, we present the algorithm of this method and show that the solution obtained by the spline method converges to the exact solution of this problem.

1. Introduction

The problem of optimal dynamic measurement is one of the mathematical statements of the problem on recovering a dynamically distorted input signal (measurement) from the observed data (output signal, i.e. an observation). The problem of optimal dynamic measurement is reduced to the optimal control problem, in which the measurement is understood as a control action, and the penalty functional allows finding a measurement that ensures the proximity of the observed and simulated output signal, as well as the proximity of the derivatives of these functions [1]. To date, the problem of optimal dynamic measurement [2] was studied under various conditions (see, for example, [3-5]).

In the theory of optimal dynamic measurements [6], a significant direction is the development of effective numerical methods [7,8]. Initially, the developed algorithms required large computing resources or time, therefore, as part of the development of existing algorithms, it was proposed to use the decomposition method and spline functions. This article presents a spline method for solving the problem of optimal dynamic measurement and shows its convergence.

2. Optimal dynamic measurement problem and convergence of spline method for its solution

2.1. Statement of problem on optimal dynamic measurement

Consider the space of states \( \mathbb{N} = \left\{ x \in L_2(0, \tau, \mathbb{R}^n) : \dot{x} \in L_2((0, \tau), \mathbb{R}^n) \right\} \), the space of observations \( \mathbb{Y} = C[\mathbb{N}] \) and the space of measurements \( \mathcal{A} = \left\{ u \in L_2((0, \tau), \mathbb{R}^n) : u^{(p+1)} \in L_2((0, \tau), \mathbb{R}^n) \right\} \). The measuring device is simulated by the system [9]
\[
\begin{align*}
L\dot{x} &= Ax + Bu, \\
y &= Cx,
\end{align*}
\]
(1)

and the initial Showalter – Sidorov condition [1]
\[
\left((\alpha L - A)^{-1} L\right)^{p+1} (x(0) - x_0) = 0
\]
(2)

where \( L \) and \( A \) are square matrices of the order \( n \) (note that there exist measuring systems such that \( \det L = 0 \) [10]), the matrix \( A \) is \((L; p)\)- regular [6], \( x(t) \), \( y(t) \), \( u(t) \) are vector functions of the states of the system, the observed signal and the input signal, respectively, \( x_0 \in \mathbb{R}^n, \ \alpha \in \mathbb{R}^p(M) \), the matrix \( B \) characterizes the relationship between the input of the system and its state, the matrix \( C \) characterizes the relationship between the state of the system and observations.

In \( \mathcal{A} \), consider a closed convex set of admissible measurements \( \mathcal{A}_\beta \subseteq \mathcal{A} \)
\[
\mathcal{A}_\beta = \left\{ u \in \mathcal{A} : \sum_{q=1}^{\beta} \int_0^t \left\| y^{(q)}(t) \right\|^2 dt \leq d \right\}.
\]
(3)

It is required to find the optimal dynamic measurement \( v \in \mathcal{A}_\beta \) at which the minimum value
\[
J(v) = \min_{u \in \mathcal{A}_\beta} J(u)
\]
(4)
of the functional
\[
J(u) = \sum_{q=1}^r \int_0^t \left\| y^{(q)}(u, t) - y_0^{(q)}(t) \right\|^2 dt
\]
(5)
is achieved, where \( y_0(t), \ t \in [0, T] \), is a continuously differentiable function (we consider the function to be a «real observation» constructed on the basis of either the discrete values \( Y_{0t} \) observed at the nodal points \( t \) at the output of the measuring system, or the discrete values \( y_{0t} \) observed at the nodal points \( t \) and smoothed by digital filters.

**Theorem 1** [6]. Let \( L \) and \( A \) be square matrices of the order \( n \), the matrix \( A \) be \((L; p)\)- regular, and \( \det A \neq 0 \). Then, for any \( x_0 \in \mathbb{R}^n \), there exists a unique solution \( v \in \mathcal{A}_\beta \) to problem (1) – (5), which is an optimal dynamic measurement, and \( x(v) \in \mathbb{N} \) satisfies system (1) under initial condition (2) and has the form
\[
x(t) = \lim_{k \to \infty} x_k(t) = \lim_{k \to \infty} \left[ \sum_{q=0}^{\beta} A^q \left( (kL_k^\perp(A))^{p+1} - I_n \right) \left( L - \frac{t-s}{k} A \right)^{-1} \left( L - \frac{t-s}{k} A \right)^{-1} (Bu)^{(q)} \right] + \frac{1}{k} \left( \sum_{q=0}^{\beta} A^q \left( (kL_k^\perp(A))^{p+1} - I_n \right) \left( L - \frac{t-s}{k} A \right)^{-1} \left( L - \frac{t-s}{k} A \right)^{-1} (Bu)^{(q)} \right) x_0 + \int_0^T \left( \sum_{q=0}^{\beta} A^q \left( (kL_k^\perp(A))^{p+1} - I_n \right) \left( L - \frac{t-s}{k} A \right)^{-1} \left( L - \frac{t-s}{k} A \right)^{-1} (Bu)^{(q)} \right) x_0 + \int_0^T (Bu)^{(q)}(s) ds,
\]
(6)

where \( \lim_{k \to \infty} (kL_k^\perp(A))^{p+1} \) is the projector, \( L_k^\perp(A) \) is the left resolvent of the operator \( A \) [6].

Let us find the optimal measurement in the form \( u^l = \text{col} \left( \sum_{j=1}^r a_{1j} t^j, \sum_{j=1}^r a_{2j} t^j, \ldots, \sum_{j=1}^r a_{nj} t^j \right) \). Taking into account (6), we obtain
\[
J_k(x_k^l) = \min_{u^l} J_k(u^l) = \min_{u^l} \sum_{q=1}^r \int_0^t \left\| Cx_k^{(q)}(u^l, t) - y_0^{(q)}(t) \right\|^2 dt.
\]
(7)

Therefore, a pair \( (v_k^l, x_k^l) = (v_k^l, x_k(v^l)) \) denotes an approximate solution to the problem of optimal dynamic measurement if, in addition to the optimal dynamic measurement, it is required also to find an approximate state of the system.

The following theorem is true.
Theorem 2 [11]. Let the matrix $A$ be $(L, p)$-regular, moreover $\det A \neq 0$, $p \in \{0\} \cup \mathcal{N}$, functional (5) be a strongly convex function on a compact and convex set $\mathcal{A}_v \subset \mathcal{A}$. Then

$$J_k (v'_k) \rightarrow J(v), \quad v'_k \rightarrow v \quad \text{for } k \geq K \text{ and } \ell > p,$$

and there exists $T > 0$ such that the following inequality holds:

$$T \|v'_k - v\|^2 \leq J_k (v'_k) - J(v).$$

2.2. Spline method algorithm and its convergence

Let the following be given: the matrices included in the system (1); the initial value $x_0 \in \mathbb{R}^n$; the array of the output signal values $y_{0i}$ observed at the nodal points $t_i$, $i = 0, 1, \ldots, n$, and $t_{i+1} - t_i = \delta$, $t_0 = 0$, $t_n = \tau$.

Step 1. Divide the segment $[0, \tau]$ into $M$ segments $[\tau_{m-1}, \tau_m]$, where $m = 1, 2, \ldots, M$, and $t_0 = \tau_0 = 0$, $t_n = \tau_M$.

Step 2. On each segment $[\tau_{m-1}, \tau_m]$, construct the interpolation function $v'_{0m}(t)$ in the form of a polynomial of the degree $\ell \leq (n - 1)/M$.

Step 3. Sequentially, for $m = 1, 2, \ldots, M$, on $[\tau_{m-1}, \tau_m]$, solve the problem of optimal dynamic measurement (1)-(5) for $u \in \mathcal{A}_{0m}$, where $\mathcal{A}_{0m} \subset \mathcal{A}_v$ is a closed convex subset $\mathcal{A}_v$. Find the approximate value of the optimal measurement $v'_{km}(t)$ in the form of a polynomial of the degree $\ell$ imposing the continuity condition

$$v'_{km}(\tau_m) = v'_{k,m+1}(\tau_m). \quad (8)$$

Step 4. As a result, obtain the spline function $\tilde{v}_k(t) = \bigcup_m v'_{km}(t)$ continuous on $[0, \tau]$.

Let us formulate a statement about the convergence of the obtained spline function to the exact solution of the optimal dynamic measurement problem.

Theorem 3. The spline function $\tilde{v}_k(t)$ obtained using the described algorithm converges to the exact solution $\tilde{v}(t)$ of problem (1)-(5) on $[0, \tau]$.

The validity of the statement of Theorem 3 follows from the following facts:

1) the interpolation process converges when constructing the spline function $\tilde{y}_0(t) = \bigcup_m v'_{0m}(t)$, i.e. for $M \rightarrow \infty$ $\|\tilde{y}_0(t) - y_0(t)\| \rightarrow 0$;

2) the approximate solution $v'_{km}(t)$ converges to the exact one $v_m(t)$ on the segment $[\tau_{m-1}, \tau_m]$ for $m = 1, 2, \ldots, M$ due to Theorem 2;

3) the spline function $\tilde{v}_k(t)$ obtained under condition (8) is continuous on $[0, \tau]$.

3. Conclusion

The report presents a spline method for solving the problem of optimal dynamic measurement and shows its convergence. In the future, we intend to investigate the possibility of using the spline collocation method to solve the problem of optimal dynamic measurement, and to consider other types of digital filters with a similar statement of the problem.

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