On the resolvability of the dynamics of one fluid flow via a testing fluid in a two-fluids flow model

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Abstract

In this work, by considering an isentropic fluid-fluid interaction model with a large symmetric drag force, a commonly used simplified two-fluids flow model is justified as the asymptotic limit. Equations for each fluid component with an interaction term are identified in addition to the simplified two-fluids flow model, which can be used to resolve the density of one fluid specie based on information on the density and the velocity of the other fluid specie, i.e., the testing flow.

Keywords: Asymptotic limit; Two-fluids flow model; Fluid-fluid interaction.

MSC2020: 35Q30, 76N06, 76T17.

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1 Introduction

The goal of this paper is to investigate the asymptotic limit $\sigma \to \infty$ of system

\[
\begin{aligned}
\partial_t \rho_{\sigma, \pm} + \text{div} \left( \rho_{\sigma, \pm} u_{\sigma, \pm} \right) &= 0, \\
\partial_t (\rho_{\sigma, \pm} u_{\sigma, \pm}) + \text{div} \left( \rho_{\sigma, \pm} u_{\sigma, \pm} \otimes u_{\sigma, \pm} \right) + \nabla p_{\sigma, \pm} &= \text{div} S_{\sigma, \pm} \\
&+ \sigma \rho_{\sigma, \pm} \rho_{\sigma, \pm} (u_{\sigma, \pm} - u_{\sigma, \pm}),
\end{aligned}
\]  

(1.1)

in $\Omega = T^3$ or $T^2 \times (0, 1)$ with proper boundary conditions to be specified, where

\[
\begin{aligned}
p_{\sigma, \pm} &= R_{\pm} \gamma_{\pm}, \\
S_{\sigma, \pm} &= \mu_{\pm} (\nabla u_{\sigma, \pm} + \nabla^T u_{\sigma, \pm}) + \lambda_{\pm} \text{div} u_{\sigma, \pm} I_3.
\end{aligned}
\]  

(1.2)

Here

\[
R_{\pm} > 0, \quad \gamma_{\pm} > 1, \quad \mu_{\pm} > 0, \quad \frac{2}{3} \mu_{\pm} + \lambda_{\pm} > 0, \quad \sigma > 0,
\]  

(1.3)

are constants, representing the gas constants, the adiabatic constants, the shear viscosity coefficients, the bulk viscosity coefficients, and the interaction coefficients, respectively.

We verify rigorously that, the asymptotic limit of solutions $(\rho_{\sigma, \pm}, u_{\sigma, \pm})$ to (1.1), as $\sigma \to \infty$, will converge to $(\rho_{\pm}, u)$, which is the solution to the following two-fluids flow model,

\[
\begin{aligned}
\partial_t \rho_{\pm} + \text{div} (\rho_{\pm} u) &= 0, \\
\partial_t ((\rho_{\pm} + \rho_{-}) u) + \text{div} ((\rho_{\pm} + \rho_{-}) u \otimes u) + \nabla p &= \text{div} S,
\end{aligned}
\]  

(1.4)

where, formally,

\[
\rho_{\pm} := \lim_{\sigma \to \infty} \rho_{\sigma, \pm}, \quad u_{\pm} := \lim_{\sigma \to \infty} u_{\sigma, \pm},
\]
under some appropriate assumptions, and

\[ p = \sum_{j \in \{+, -\}} R_j \rho_j^\gamma, \]
\[ \mathcal{S} = \sum_{j \in \{+, -\}} (\mu_j (\nabla u + \nabla^\top u) + \lambda_j \text{div} u I_3). \]

As a byproduct, we discover a method to resolve the density of one fluid from the density and the velocity of the other fluid, i.e., the testing fluid, in the mixture of two fluids. More precisely, the equations,

\[ \partial_t (\rho \pm u) + \text{div} (\rho \pm u \otimes u) + \nabla (R \pm \rho\gamma \pm \pm) = \text{div} (\mu \pm (\nabla u + \nabla^\top u) + \lambda \pm \text{div} u I_3) \pm \rho \pm \rho \mathcal{S}, \]

are identified, where \( \mathcal{S} \) is a quantity relating the two densities \( \rho_+, \rho_- \) and the velocity \( u \) (see (2.14), below). Thus, provided with information of the testing fluid, \( (\rho_+, u) \) for instance, one can calculate \( \rho_- \mathcal{S} \) from (1.6). From there, one can identify the density of the other flow, i.e., \( \rho_- \) in this example.

System (1.4) is a simplified two-fluids flow model. The idea of multi-components of the fluid sharing the same aligned velocity is commonly used in a lot of applications. For instance, in the study of atmosphere dynamics, dry air, water vapor, and cloud water are driven by the same velocity, which is determined by a single momentum equation similar to (1.4). See, e.g., [15, 17]. This system also serves as a plausible model for the study of particle/fluid interaction. See, e.g., [11, 26], and the references therein.

In particular, in [26], starting from the Vlasov-Fokker-Planck/Navier-Stokes system, the authors investigate the asymptotic with a strong drag force and a strong Brownian motion, where the limiting system is similar to (1.4). However, the drag force is taken to be asymmetric. In fact, as explained in Remark 1.1 in [26], the drag force was taken as \( F_d = F_0 (u - v) \), where \( F_0 \) is a constant, and the more physical relevant one should be taken as \( F_d = \rho (u - v) \), which we will refer to as the symmetric drag force.

Formally taking the hydrodynamic limit of the Vlasov-Fokker-Planck/Navier-Stokes system, with a symmetric drag force, one will end up with system (1.1), while in the case of an asymmetric drag force in the kinetic-hydrodynamic system, it leads to a two-phase fluid model similar to (1.1) but with the drag force equal to \( \sigma \rho \delta_{\pm} (u_{\sigma, \pm} - u_{\sigma, \pm}) \), i.e., an asymmetric drag force. See [10] for the formal derivation of the hydrodynamic limit. We will use the terminology of symmetric or asymmetric drag forces for hydrodynamic systems as well as kinetic-hydrodynamic systems.
In addition, from the point of view of hydrodynamics, system (1.1) can be seen as a model of the mixture of two fluids, while each one of the fluids acts as a porous media to the other, and neither of the fluids is dominating the other. Therefore, the drag force should be symmetric.

Due to the low regularity of solutions studied in [26], the lower bound of density is not \textit{a priori} known. Therefore only large asymmetric drag force was considered. In this work, we want to investigate the large drag force limit in the setting of hydrodynamic systems, with the symmetric drag force, in a more regular functional setting. We remark that it would be interesting to investigate the large, symmetric drag force limit in the setting of kinetic-hydrodynamic systems.

Among a large amount of literatures concerning two-phase fluid models, we will only mention a few in the following, and refer interested readers to the references therein. Modeled by the Vlasov-Fokker-Planck/Navier-Stokes system, Mellet and Vasseur construct the global weak solution in [24], while the asymptotic limit of a large drag force and a large Brownian motion is investigated in [26]. Without fluid viscosities, the local well-posedness of strong solutions is studied in [1]. Global classical solutions near equilibrium, as well as the decaying rates of perturbations, are studied in [6, 18] in the presence of fluid viscosities.

Concerning the limiting equations in [26], the existence of global weak solutions is established in [28]. See [30] for the recent improvement. We also refer to [11, 13], as well as the references therein, for early mathematical developments as well as the physical importance of this two-fluids flow model.

Near equilibrium but with an asymmetric drag force, global existence of classical solutions is investigated in [10]. We refer readers to [2, 4, 5, 12, 27] for a more general two-fluids flow model, where volumetric rates are taken into account, and related studies.

On the other hand, the mathematical study of mono-fluid, i.e., with one specie of fluid, has developed fruitful results. To name a few, starting with [14, 21], the authors construct the well-known Lions-Feireisl weak solutions to compressible Navier-Stokes equations. The theory of local well-posedness of strong solutions with vacuum density profiles is established in [7, 8]. Global existence in the framework of perturbation is obtained in [22, 23]. The blow-up and non-existence to full compressible Navier-Stokes system with vacuum and bounded entropy are shown in [19, 31, 32]. We also refer readers to [3, 16, 20, 25, 29] for other important developments in this direction.

The rest of this paper is organized as follows. In the next section, we
collect the notations we will be used in this paper and state the main theorems. Section 3 and section 4 are devoted to establish the uniform-in-$\sigma$ estimates, which will be the centerpiece of our analysis. In section 5 we pass the limit $\sigma \to \infty$, which yields the results in this paper.

2 Preliminaries and main theorems

We investigate our problem in domain $\Omega \subset \mathbb{R}^3$, where

$$\Omega = \{(x, y, z)^T \in \mathbb{T}^3\} \quad \text{or} \quad \{(x, y, z)^T | (x, y)^T \in \mathbb{T}^2, z \in (0, 1)\}.$$ 

$\nabla$, $\text{div}$, and $\Delta$ represent the gradient, the divergence, and the Laplace operators, respectively. Meanwhile, $\nabla_h$, $\text{div}_h$, and $\Delta_h$ are the gradient, the divergence, and the Laplace operators in the horizontal (first two) variables, respectively, i.e.,

$$\nabla_h := \left( \frac{\partial x}{\partial y} \right), \quad \text{div}_h := \nabla_h \cdot, \quad \Delta_h := \text{div}_h \nabla_h.$$ 

We use $\| \cdot \|_X$ to denote the norm of functional space $X$. We shorten the notation $\|f\|_X + \|g\|_X + \cdots$ to $\|f, g, \cdots\|_X$ for norms of multiple functions. $\int f \, dx := \int_\Omega f \, dx$ represents the integration in the spatial variables.

$(\rho_{\sigma, \pm, 0}, \mathbf{u}_{\sigma, \pm, 0})$ are used to represent the initial data of (1.1), i.e.,

$$(\rho_{\sigma, \pm}, \mathbf{u}_{\sigma, \pm}) \big|_{t=0} = (\rho_{\sigma, \pm, 0}, \mathbf{u}_{\sigma, \pm, 0}).$$

Let

$$\rho := \inf_{x \in \Omega^1} \{\rho_{\sigma, +, 0}, \rho_{\sigma, -, 0}\} > 0,$$

be the strict positive lower bound of initial densities.

In the case of $\Omega = \mathbb{T}^2 \times (0, 1)$, we impose the impermeable and complete slip boundary conditions:

$$\tau \cdot s_{\sigma, \pm} \mathbf{n} \big|_{z=0, 1} = 0, \quad \mathbf{u}_{\sigma, \pm} \cdot \mathbf{n} \big|_{z=0, 1} = 0,$$

where

$$\tau \in \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}, \quad \mathbf{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$ 

Equivalently, (2.3) can be written as

$$\partial_z \mathbf{v}_{\sigma, \pm} \big|_{z=0, 1} = 0, \quad w_{\sigma, \pm} \big|_{z=0, 1} = 0.$$
where \( v_{\sigma,+} \) and \( w_{\sigma,+} \) are the horizontal and vertical components of \( u_{\sigma,+} \), respectively, i.e., \( u_{\sigma, \pm} := (v_{\sigma, \pm}, w_{\sigma, \pm})^\top \).

Notice, System (1.1) with either of the above boundary conditions, admits the following conservation/balance laws:

\[
\text{Conservation of mass: } \frac{d}{dt} \int \rho_{\sigma, \pm} \, dx = 0, \tag{2.4}
\]

\[
\text{Conservation of total momentum: } \frac{d}{dt} \int \sum_{j \in \{+, -\}} \rho_{\sigma,j} u_{\sigma,j} \, dx = 0, \tag{2.5}
\]

\[
\text{Balance of total energy: } \frac{d}{dt} E_{\sigma} + D_{\sigma} + \sigma \int \rho_{\sigma,+} \rho_{\sigma,-} |u_{\sigma,+} - u_{\sigma,-}|^2 \, dx = 0, \tag{2.6}
\]

where the energy and the dissipation are defined as

\[
E_{\sigma} := \int \sum_{j \in \{+, -\}} \left( \frac{1}{2} \rho_{\sigma,j} |u_{\sigma,j}|^2 + \frac{R_j}{\gamma_j - 1} \rho_{\sigma,j}^\gamma \right) \, dx, \tag{2.7}
\]

\[
D_{\sigma} := \int \sum_{j \in \{+, -\}} \left( \frac{H_j}{2} |\nabla u_{\sigma,j} + \nabla^\top u_{\sigma,j}|^2 + \lambda_j |\text{div} u_{\sigma,j}|^2 \right) \, dx. \tag{2.8}
\]

Now we state our first main theorem in this paper:

**Theorem 2.1. Case 1.** In the case of \( \Omega = \mathbb{T}^3 \), assume that

\[
\rho > 0, \quad \rho_{\sigma, \pm, 0} \in H^2(\Omega), \quad u_{\sigma, \pm, 0} \in H^2(\Omega). \tag{2.9}
\]

There exists \( T \in (0, \infty) \), depending only on the initial data and independent of \( \sigma \), such that \( \inf_{\Omega, 0 \leq t \leq T} \rho_{\sigma, \pm} \geq \rho/2 \) and

\[
\rho_{\sigma, \pm} \in L^\infty(0, T; H^2(\Omega)), \quad \partial_t \rho_{\sigma, \pm} \in L^\infty(0, T; L^2(\Omega)), \quad u_{\sigma, \pm} \in L^\infty(0, T; H^3(\Omega)), \quad \partial_t u_{\sigma, \pm} \in L^\infty(0, T; H^1(\Omega)). \tag{2.10}
\]
Moreover, it holds
\[
\sup_{0 \leq s \leq T} \left\{ \| \rho_{\sigma, \pm}(s) \|_{H^2(\Omega)}^2 + \| \partial_t \rho_{\sigma, \pm}(s) \|_{L^2(\Omega)}^2 + \| u_{\sigma, \pm}(s) \|_{H^2(\Omega)}^2 \right. \\
\left. + \| \partial_t u_{\sigma, \pm}(s) \|_{L^2(\Omega)}^2 + \sigma^2 \| (u_{\sigma, +} - u_{\sigma, -})(s) \|_{L^2(\Omega)}^2 \right. \\
\left. + \sigma \| (u_{\sigma, +} - u_{\sigma, -})(s) \|_{H^1(\Omega)}^2 \right\} + \sigma^2 \int_0^T \| (u_{\sigma, +} - u_{\sigma, -})(s) \|_{H^1(\Omega)}^2 ds \\
+ \int_0^T \left\{ \| u_{\sigma, \pm}(s) \|_{H^3(\Omega)}^2 + \| \partial_t u_{\sigma, \pm}(s) \|_{H^1(\Omega)}^2 \right\} ds \\
+ \sigma \int_0^T \left\{ \| (u_{\sigma, +} - u_{\sigma, -})(s) \|_{H^2(\Omega)}^2 + \| \partial_t (u_{\sigma, +} - u_{\sigma, -})(s) \|_{L^2(\Omega)}^2 \right\} ds \\
\leq C_{\text{in}, T},
\]
where $C_{\text{in}, T} \in (0, \infty)$ is a constant depending only on the initial data and $T$, but independent of $\sigma$.

**Case 2.** In the case of $\Omega = \mathbb{T}^2 \times (0, 1)$, the same conclusions as in Case 1 hold, provided that:

- (2.9) holds;
- In addition to (1.3), 
  \[ \mu_\pm = \mu, \quad \lambda_\pm = \lambda; \]
- $u_{\sigma, \pm, 0}$ satisfies (2.3);
- $\sigma \geq \sigma^*$, for some $\sigma^*$ depending only on the initial data.

**Proof of Theorem 2.1.** The local well-posedness of strong solutions to (1.1), where the life span may depend on $\sigma$, follows straightforward from standard fixed point arguments. Through a continuity argument, with the uniform estimates established in sections 3 and 4 below, respectively, one can extend the life span of strong solutions to a finite time, which is strictly positive, and uniform in $\sigma$. In particular, (2.11) follows from (3.19), (3.26), (3.29), and (4.30). This finishes the proof. 

The goal of this paper is to study the asymptote of system (1.1) as
\[
\sigma \to \infty,
\]
which is stated in the following:
Theorem 2.2. Under the conditions in Theorem 2.1, in either case, there exists \((\rho_\pm, u)\), as the limit of \((\rho_{\sigma, \pm}, u_{\sigma, \pm})\) as \(\sigma \to \infty\) in the sense of (2.13), below, such that it solves the two-phase fluid model (1.4) in \(T^3\) or \(T^2 \times (0,1)\) with boundary conditions (2.3), respectively.

In addition, (1.6) holds with \(S\) given by

\[
S := \frac{1}{\rho_+ + \rho_-} \left[ \frac{R_+ \gamma_+}{\gamma_+ - 1} \nabla \rho_+^{\gamma_+ - 1} - \frac{R_- \gamma_-}{\gamma_- - 1} \nabla \rho_-^{\gamma_- - 1} \right] + \left( \frac{\mu_+}{\rho_+} - \frac{\mu_-}{\rho_-} \right) \text{div} (\nabla u + \nabla^\top u) + \left( \frac{\lambda_+}{\rho_+} - \frac{\lambda_-}{\rho_-} \right) \text{div} u. \tag{2.14}
\]

Proof of Theorem 2.2. The arguments in section 5 directly yield the theorem. \(\square\)

Remark 1. The assumption (2.9) implies that

\[
\|\nabla \log \rho_{\sigma, \pm, 0}\|_{L^3(\Omega)}, \quad \|\rho_{\sigma, \pm, 0}^{1/2}\|_{L^2(\Omega)}, \quad \|\partial_t \rho_{\sigma, \pm, 0}|_{t=0}\|_{L^2(\Omega)},
\]

\[
\|u_{\sigma, \pm, 0}\|_{H^2(\Omega)}, \quad \|\partial_t u_{\sigma, \pm, 0}|_{t=0}\|_{L^2(\Omega)},
\]

are bounded, which will be used in sections 3.3, 3.4, and 4.4.

Remark 2. The method in this paper can be generated to more general domain \(\Omega \in \mathbb{R}^3\), for instance, any bounded domain with smooth boundary, by applying standard localizing-in-space arguments. Also, with minor modifications, instead of (2.3), one can also show similar results with no-slip boundary conditions, i.e., \(u_{\sigma, \pm}|_{\partial \Omega} = 0\).

3 Uniform estimates: \(\Omega = T^3\)

In this section, we study (1.1) in periodic domain \(T^3\). The goal is to obtain uniform in \(\sigma\) estimates in order to pass the limit (2.13).

3.1 Temporal derivative estimates

After applying \(\partial_t\) to system (1.1), we end up with:

\[
\begin{aligned}
\partial_t \rho_{\sigma, \pm} + \text{div} \left( \partial_t \rho_{\sigma, \pm} u_{\sigma, \pm} \right) + \text{div} \left( \rho_{\sigma, \pm} \partial_t u_{\sigma, \pm} \right) &= 0, \\
\rho_{\sigma, \pm} \partial_t u_{\sigma, \pm} + \rho_{\sigma, \pm} u_{\sigma, \pm} \cdot \nabla \partial_t u_{\sigma, \pm} + \partial_t \rho_{\sigma, \pm} \partial_t u_{\sigma, \pm} \\
&+ \partial_t (\rho_{\sigma, \pm} u_{\sigma, \pm}) \cdot \nabla u_{\sigma, \pm} + \nabla \partial_t p_{\sigma, \pm} = \text{div} \partial_t S_{\sigma, \pm} \\
&+ \sigma \partial_t (\rho_{\sigma, \pm} p_{\sigma, \pm}) (u_{\sigma, \mp} - u_{\sigma, \pm}) + \sigma \rho_{\sigma, \pm} \rho_{\sigma, \mp} \partial_t (u_{\sigma, \mp} - u_{\sigma, \pm}) = 0.
\end{aligned}
\tag{3.1}
\]
Taking the $L^2$-inner product of (3.1) with $2\partial_t u_{\sigma,\pm}$, after applying integration by parts and summing up the $+$-estimate with the $-$-estimate, leads to
\[
\frac{d}{dt} \sum_{j \in \{+,-\}} \|\rho_{\sigma,j}^{1/2} \partial_t u_{\sigma,j}\|_{L^2(\Omega)}^2 \\
+ \sum_{j \in \{+,-\}} \left\{ \mu_j \|\nabla \partial_t u_{\sigma,j} + \nabla^\top \partial_t u_{\sigma,j}\|_{L^2(\Omega)}^2 + 2\lambda_j \|\text{div} \partial_t u_{\sigma,j}\|_{L^2(\Omega)}^2 \right\} \\
+ 2\sigma \int \rho_{\sigma,\pm} \partial_t (u_{\sigma,\pm} - u_{\sigma,-})^2 dx \\
= \sum_{j=1}^4 I_j,
\]
where
\[
I_1 := -2 \sum_{j \in \{+,-\}} \int \partial_t \rho_{\sigma,j} [\partial_t u_{\sigma,j}]^2 dx, \\
I_2 := -2 \sum_{j \in \{+,-\}} \int (\partial_t (\rho_{\sigma,j} u_{\sigma,j}) \cdot \nabla) u_{\sigma,j} \cdot \partial_t u_{\sigma,j} dx, \\
I_3 := 2 \sum_{j \in \{+,-\}} R_j \gamma_j \int \rho_{\sigma,j}^{-1} \partial_t \rho_{\sigma,j} \text{div} \partial_t u_{\sigma,j} dx, \\
I_4 := 2\sigma \int \partial_t (\rho_{\sigma,\pm} \rho_{\sigma,-}) (u_{\sigma,-} - u_{\sigma,+}) \cdot \partial_t (u_{\sigma,\pm} - u_{\sigma,-}) dx.
\]
Applying Hölder’s inequality, one can obtain that
\[
I_4 \leq \sigma \int \rho_{\sigma,\pm} \rho_{\sigma,-} |\partial_t (u_{\sigma,\pm} - u_{\sigma,-})|^2 dx \\
+ 4\sigma \|\partial_t (\rho_{\sigma,\pm} \rho_{\sigma,-})^{1/2}\|_{L^2(\Omega)}^2 \|u_{\sigma,\pm} - u_{\sigma,-}\|_{L^\infty(\Omega)}^2.
\]
The estimates of $I_1, \cdots, I_3$ are standard, which we omit the details. Thus
\[ (3.2) \] yields,

\[
\frac{d}{dt} \sum_{j\in\{+,-\}} \|\rho_{\sigma,j} \|_{L^2(\Omega)}^2 + \sum_{j\in\{+,-\}} \left\{ \frac{H_j}{2} \| \nabla \partial_t u_{\sigma,j} + \nabla^T \partial_t u_{\sigma,j} \|_{L^2(\Omega)}^2 + \lambda_j \| \text{div} \partial_t u_{\sigma,j} \|_{L^2(\Omega)}^2 \right\} \\
+ \sigma \int \rho_{\sigma,+} \rho_{\sigma,-} |\partial_t (u_{\sigma,+} - u_{\sigma,-})|^2 \, dx \\
\leq 4\sigma \| \partial_t (\rho_{\sigma,+} \rho_{\sigma,-}) \|_{L^2(\Omega)}^{1/2} \| u_{\sigma,+} - u_{\sigma,-} \|_{L^2(\Omega)}^{1/2} \\
+ \mathcal{H}(\|\rho_{\sigma,\pm}\|_{L^\infty(\Omega)}; \|\nabla \rho_{\sigma,\pm}\|_{L^p(\Omega)}; \|u_{\sigma,\pm}\|_{L^\infty(\Omega)}; \|\nabla u_{\sigma,\pm}\|_{L^p(\Omega)}), \\
\|\rho_{\sigma,\pm} \|_{L^2(\Omega)} \}
\]

where we have used \( (3.1) \) to substitute \( \partial_t \rho_{\sigma,\pm} \), and we have applied the fact that, for any vector field \( \mathbf{v} : \Omega = \mathbb{T}^3 \mapsto \mathbb{R}^3 \), one has

\[
2 \|\nabla \mathbf{v}\|_{L^2(\Omega)}^2 + 2 \|\text{div} \mathbf{v}\|_{L^2(\Omega)}^2 = \|\nabla \mathbf{v} + \nabla^T \mathbf{v}\|_{L^2(\Omega)}^2. 
\]

### 3.2 Spatial derivative estimates

After applying \( \partial^2 = \partial \partial \) to system \( (3.1) \), with \( \partial \in \{\partial_x, \partial_y, \partial_z\} \), one can write down:

\[
\begin{cases}
\partial_t \partial^2 \rho_{\sigma,\pm} + \text{div} (\partial^2 \rho_{\sigma,\pm} u_{\sigma,\pm}) + 2\text{div} (\partial \rho_{\sigma,\pm} \partial u_{\sigma,\pm}) \\
+ \text{div} (\rho_{\sigma,\pm} \partial^2 u_{\sigma,\pm}) = 0, \\
\rho_{\sigma,\pm} \partial_t \partial u_{\sigma,\pm} + \rho_{\sigma,\pm} u_{\sigma,\pm} \cdot \nabla \partial^2 u_{\sigma,\pm} + 2\partial \rho_{\sigma,\pm} \partial_t u_{\sigma,\pm} \\
+ 2\partial (\rho_{\sigma,\pm} u_{\sigma,\pm}) \cdot \nabla \partial u_{\sigma,\pm} + \partial^2 \rho_{\sigma,\pm} \partial_t u_{\sigma,\pm} + \partial^2 (\rho_{\sigma,\pm} u_{\sigma,\pm}) \cdot \nabla u_{\sigma,\pm} \\
+ \nabla \partial^2 p_{\sigma,\pm} = \text{div} \partial^2 \mathbf{u}_{\sigma,\pm} + \sigma \rho_{\sigma,\pm} \rho_{\sigma,\pm} \partial^2 (u_{\sigma,\pm} - u_{\sigma,\pm}) \\
+ 2\sigma \partial (\rho_{\sigma,\pm} u_{\sigma,\pm}) \partial (u_{\sigma,\pm} - u_{\sigma,\pm}) + \sigma \partial^2 (\rho_{\sigma,\pm} u_{\sigma,\pm}) (u_{\sigma,\pm} - u_{\sigma,\pm}).
\end{cases}
\]

\[ (3.5) \]

**Remark 3.** For any two functions \( f \) and \( g \), we have used

\[
\partial^2 (fg) = \partial^2 f \times g + 2\partial f \partial g + f \partial^2 g
\]

to represent \( \partial_t \partial_j (fg) = \partial_t \partial_j f \times g + \partial_t f \partial_j g + \partial_j f \partial g + f \times \partial \partial_j g, i, j \in \{x, y, z\} \)

to the sake of simplifying the notations.

Taking the \( L^2 \)-inner product of \( (3.5) \) with \( 2\partial^2 u_{\sigma,\pm} \), after applying integration by parts and summing up the \(+\) estimate and the \(-\) estimate, leads
to
\[
\frac{d}{dt} \sum_{j \in \{+, -\}} \|\rho_{\sigma,j}^{1/2} \partial^2 u_{\sigma,j}\|^2_{L^2(\Omega)}
\]
\[
+ \sum_{j \in \{+, -\}} \left\{ \mu_j \|\nabla \partial^2 u_{\sigma,j} + \nabla \cdot \partial^2 u_{\sigma,j}\|^2_{L^2(\Omega)} + 2 \lambda_j \|\text{div} \partial^2 u_{\sigma,j}\|^2_{L^2(\Omega)} \right\}
\]
\[
+ 2\sigma \int_{\Omega} \rho_{\sigma,+} \rho_{\sigma,-} |\partial^2(u_{\sigma,+} - u_{\sigma,-})|^2 \ dx
\]
\[
= \sum_{j=5}^{11} I_j,
\]
(3.6)

where

\[
I_5 := -4 \sum_{j \in \{+, -\}} \int \partial \rho_{\sigma,j} \partial_t \partial u_{\sigma,j} \cdot \partial^2 u_{\sigma,j} \ dx,
\]

\[
I_6 := -4 \sum_{j \in \{+, -\}} \int \partial \rho_{\sigma,j} \partial u_{\sigma,j} \cdot \nabla \partial^2 u_{\sigma,j} \ dx,
\]

\[
I_7 := -2 \sum_{j \in \{+, -\}} \int \partial^2 \rho_{\sigma,j} \partial_t \partial u_{\sigma,j} \cdot \partial^2 u_{\sigma,j} \ dx,
\]

\[
I_8 := -2 \sum_{j \in \{+, -\}} \int \partial^2 \rho_{\sigma,j} \partial u_{\sigma,j} \cdot \nabla \partial^2 u_{\sigma,j} \ dx,
\]

\[
I_9 := 2 \sum_{j \in \{+, -\}} R_j \int \partial^2 \rho_{\sigma,j} \partial^2 u_{\sigma,j} \ dx,
\]

\[
I_{10} := 4\sigma \int \partial \rho_{\sigma,+} \partial \rho_{\sigma,-} \partial (u_{\sigma,-} - u_{\sigma,+}) \cdot \partial^2(u_{\sigma,+} - u_{\sigma,-}) \ dx,
\]

\[
I_{11} := 2\sigma \int \partial^2 \rho_{\sigma,+} \partial \rho_{\sigma,-} (u_{\sigma,-} - u_{\sigma,+}) \cdot \partial^2(u_{\sigma,+} - u_{\sigma,-}) \ dx.
\]

Applying Hölder’s inequality leads to

\[
I_{10} + I_{11} \leq \sigma \int \rho_{\sigma,+} \rho_{\sigma,-} |\partial^2(u_{\sigma,+} - u_{\sigma,-})|^2 \ dx
\]
\[
+ \sigma (2||\partial \log(\rho_{\sigma,+} \rho_{\sigma,-})||_{L^3(\Omega)}^2 ||\partial(\rho_{\sigma,+} \rho_{\sigma,-})^{1/2}||_{L^{6}(\Omega)}^2
\]
\[
+ 8||\partial^2(\rho_{\sigma,+} \rho_{\sigma,-})^{1/2}||_{L^2(\Omega)}^2 \times ||u_{\sigma,+} - u_{\sigma,-}||_{L^\infty(\Omega)}^2
\]
\[
+ 32\sigma ||\partial(\rho_{\sigma,+} \rho_{\sigma,-})^{1/2}||_{L^3(\Omega)}^2 ||\partial(u_{\sigma,+} - u_{\sigma,-})||_{L^3(\Omega)}^2.
\]
Thus (3.6) yields, for any $\delta \in (0, 1)$,
\[
\frac{d}{dt} \sum_{j \in \{+,-\}} \|\rho_{\sigma,j}^{1/2} \partial^2 u_{\sigma,j}\|^2_{L^2(\Omega)} \\
+ \sum_{j \in \{+,-\}} \left\{ \frac{\mu_j}{2} \|\nabla \partial^2 u_{\sigma,j} + \nabla^T \partial^2 u_{\sigma,j}\|^2_{L^2(\Omega)} + \lambda_j \|\text{div} \partial^2 u_{\sigma,j}\|^2_{L^2(\Omega)} \right\} \\
+ \sigma \int \rho_{\sigma,+} \rho_{\sigma,-} |\partial^2 (u_{\sigma,+} - u_{\sigma,-})|^2 \, dx \\
\leq \sigma (2\|\partial \log (\rho_{\sigma,+} \rho_{\sigma,-})\|^2_{L^2(\Omega)} \|\partial (\rho_{\sigma,+} \rho_{\sigma,-})^{1/2}\|^2_{L^2(\Omega)} \\
+ 8\|\partial^2 (\rho_{\sigma,+} \rho_{\sigma,-})^{1/2}\|^2_{L^2(\Omega)} \times \|u_{\sigma,+} - u_{\sigma,-}\|^2_{L^2(\Omega)} \\
+ 32\sigma \|\partial (\rho_{\sigma,+} \rho_{\sigma,-})^{1/2}\|^2_{L^2(\Omega)} \|\partial (u_{\sigma,+} - u_{\sigma,-})\|^2_{L^2(\Omega)} \\
+ \delta \sum_{j \in \{+,-\}} \|\nabla \partial_t u_{\sigma,j}\|^2_{L^2(\Omega)} + \mathcal{H}(C_\delta, \|\rho_{\sigma,\pm}\|_{L^\infty(\Omega)}, \|\nabla \rho_{\sigma,\pm}\|_{L^2(\Omega)}), \\
\|\nabla \log \rho_{\sigma,\pm}\|_{L^1(\Omega)}, \|\nabla^2 \rho_{\sigma,\pm}\|_{L^2(\Omega)}; \|u_{\sigma,\pm}\|_{L^\infty(\Omega)}; \\
\|\nabla u_{\sigma,\pm}\|_{L^6(\Omega)}, \|\rho_{\sigma,\pm}\|_{L^6(\Omega)}; \|\nabla \rho_{\sigma,\pm}\|_{L^6(\Omega)}; \|\partial_t u_{\sigma,\pm}\|_{L^6(\Omega)}). \\
(3.7)
\]

Similar estimates also hold for $\|\rho_{\sigma,\pm}^{1/2} \partial u_{\sigma,\pm}\|_{L^2(\Omega)}$. We omit the detail and only record the result here:
\[
\frac{d}{dt} \sum_{j \in \{+,-\}} \|\rho_{\sigma,j}^{1/2} \partial u_{\sigma,j}\|^2_{L^2(\Omega)} \\
+ \sum_{j \in \{+,-\}} \left\{ \frac{\mu_j}{2} \|\nabla \partial u_{\sigma,j} + \nabla^T \partial u_{\sigma,j}\|^2_{L^2(\Omega)} + \lambda_j \|\text{div} \partial u_{\sigma,j}\|^2_{L^2(\Omega)} \right\} \\
+ \sigma \int \rho_{\sigma,+} \rho_{\sigma,-} |\partial (u_{\sigma,+} - u_{\sigma,-})|^2 \, dx \\
\leq 4\sigma \|\partial (\rho_{\sigma,+} \rho_{\sigma,-})^{1/2}\|^2_{L^2(\Omega)} \|u_{\sigma,+} - u_{\sigma,-}\|^2_{L^2(\Omega)} \\
+ \mathcal{H}(\|\rho_{\sigma,\pm}\|_{L^\infty(\Omega)}, \|\nabla \rho_{\sigma,\pm}\|_{L^2(\Omega)}; \|u_{\sigma,\pm}\|_{L^\infty(\Omega)}, \|\nabla u_{\sigma,\pm}\|_{L^6(\Omega)}; \\
\|\rho_{\sigma,\pm}\|_{L^6(\Omega)}; \|\partial_t u_{\sigma,\pm}\|_{L^6(\Omega)}). \\
(3.8)
\]

### 3.3 Estimates on the densities

First, we derive the estimate of $\|\rho_{\sigma,\pm}^{1/2}\|_{H^2(\Omega)}$. Recall from (1.1), one has
\[
2\partial_t \rho_{\sigma,\pm}^{1/2} + 2u_{\sigma,\pm} \cdot \nabla \rho_{\sigma,\pm}^{1/2} + \rho_{\sigma,\pm}^{1/2} \text{div} u_{\sigma,\pm} = 0. \\
(3.9)
\]
Then performing standard $H^s$-estimates yields
\[
\|\rho_{\sigma, \pm}(t)\|_{H^2(\Omega)} \leq \|\rho_{\sigma, \pm, 0}\|_{H^2(\Omega)} e^{C \int_0^t \|\nabla u_{\sigma, \pm}(s)\|_{H^2(\Omega)} \, ds},
\]  
(3.10)
and therefore
\[
\|\rho_{\sigma, \pm}\|_{L^\infty(\Omega)}, \|\nabla \rho_{\sigma, \pm}\|_{L^p(\Omega)} \leq C\|\rho_{\sigma, \pm, 0}\|_{H^2(\Omega)} e^{C \int_0^t \|\nabla u_{\sigma, \pm}(s)\|_{H^2(\Omega)} \, ds}.
\]  
(3.11)

Next, we shall derive the estimate of $\|\nabla \log \rho_{\sigma, \pm}\|_{L^3(\Omega)}$. $\log \rho_{\sigma, \pm}$ satisfies
\[
\partial_t \log \rho_{\sigma, \pm} + u_{\sigma, \pm} \cdot \nabla \log \rho_{\sigma, \pm} + \text{div} u_{\sigma, \pm} = 0.
\]  
(3.12)
Therefore, one can derive
\[
\frac{d}{dt} \|\nabla \log \rho_{\sigma, \pm}\|_{L^3(\Omega)} \leq 2 \|\nabla u_{\sigma, \pm}\|_{L^\infty(\Omega)} \|\nabla \log \rho_{\sigma, \pm}\|_{L^3(\Omega)} + \|\nabla^2 u_{\sigma, \pm}\|_{L^3(\Omega)}
\]
\[
\leq C\|\nabla u_{\sigma, \pm}\|_{H^2(\Omega)} \|\nabla \log \rho_{\sigma, \pm}\|_{L^3(\Omega)} + C\|\nabla u_{\sigma, \pm}\|_{H^2(\Omega)},
\]
which yields
\[
\|\nabla \log \rho_{\sigma, \pm}(t)\|_{L^3(\Omega)} \leq (C + \|\nabla \log \rho_{\sigma, \pm, 0}\|_{L^3(\Omega)}) e^{C \int_0^t \|\nabla u_{\sigma, \pm}(s)\|_{H^2(\Omega)} \, ds} - C.
\]  
(3.13)
In addition, applying $\partial_t$ to (3.9) leads to
\[
2\partial_t^{1/2} \rho_{\sigma, \pm} + 2u_{\sigma, \pm} \cdot \nabla \partial_t \rho_{\sigma, \pm} + \partial_t \rho_{\sigma, \pm} \text{div} u_{\sigma, \pm}
\]
\[
+ 2\partial_t u_{\sigma, \pm} \cdot \nabla \rho_{\sigma, \pm} + \rho_{\sigma, \pm} \text{div} \partial_t u_{\sigma, \pm} = 0.
\]  
(3.14)
Then after taking the $L^2$-inner product of (3.14) with $\partial_t \rho_{\sigma, \pm}$ and applying integration by parts in the resultant, one has
\[
\frac{d}{dt} \|\partial_t \rho_{\sigma, \pm}\|_{L^2(\Omega)}^2 = -2 \int (\partial_t u_{\sigma, \pm} \cdot \nabla) \rho_{\sigma, \pm} \cdot \partial_t \rho_{\sigma, \pm} \, dx
\]
\[
- \int \rho_{\sigma, \pm} \text{div} \partial_t u_{\sigma, \pm} \cdot \partial_t \rho_{\sigma, \pm} \, dx
\]
\[
\leq 2 \|\partial_t u_{\sigma, \pm}\|_{L^3(\Omega)} \|\nabla \rho_{\sigma, \pm}\|_{L^6(\Omega)} \|\partial_t \rho_{\sigma, \pm}\|_{L^2(\Omega)}
\]
\[
+ \|\rho_{\sigma, \pm}\|_{L^\infty(\Omega)} \|\text{div} \partial_t u_{\sigma, \pm}\|_{L^2(\Omega)} \|\partial_t \rho_{\sigma, \pm}\|_{L^2(\Omega)},
\]
which implies
\[
\|\partial_t \rho_{\sigma, \pm}(t)\|_{L^2(\Omega)} \leq \|\partial_t \rho_{\sigma, \pm, 0}\|_{L^2(\Omega)}
\]
\[
+ \sup_{0 \leq s \leq t} \|\nabla \rho_{\sigma, \pm}(s)\|_{L^6(\Omega)} \times \int_0^t \|\partial_t u_{\sigma, \pm}(s)\|_{L^3(\Omega)} \, ds
\]
\[
+ \sup_{0 \leq s \leq t} \|\rho_{\sigma, \pm}(s)\|_{L^\infty(\Omega)}^{1/2} \times \int_0^t \|\text{div} \partial_t u_{\sigma, \pm}(s)\|_{L^2(\Omega)} \, ds,
\]  
(3.15)
where \( \partial_t \rho_{\sigma, \pm}^{1/2} := \partial_t \rho_{\sigma, \pm}^{1/2} \big|_{t=0} \) is the initial data of \( \partial_t \rho_{\sigma, \pm}^{1/2} \) defined by (3.19).

Last but not least, we will need to derive the lower bounds of \( \rho_{\sigma, \pm} \). From (1.1), one can write down

\[
\partial_t (\rho - M(t) - \rho_{\sigma, \pm}) + \mathbf{u}_{\sigma, \pm} \cdot \nabla (\rho - M(t) - \rho_{\sigma, \pm}) + (\rho - M(t) - \rho_{\sigma, \pm}) \mathbf{u}_{\sigma, \pm} = -\partial_t M(t) + (\rho - M(t)) \text{div} \mathbf{u}_{\sigma, \pm} \leq 0,
\]

where

\[
M(t) := \rho \int_0^t \| \text{div} \mathbf{u}_{\sigma, \pm}(s) \|_{L^\infty(\Omega)} \, ds \times e^{\int_0^t \| \text{div} \mathbf{u}_{\sigma, \pm}(s) \|_{L^\infty(\Omega)} \, ds}.
\]

Therefore, testing the above equation with \((\rho - M(t) - \rho_{\sigma, \pm})^+\) leads to

\[
\| (\rho - M(t) - \rho_{\sigma, \pm})^+ \|^2_{L^2(\Omega)} \leq 0,
\]

and hence

\[
\inf_{\Omega} \rho_{\sigma, \pm}(t) \geq \rho (1 - \int_0^t \| \text{div} \mathbf{u}_{\sigma, \pm}(s) \|_{L^\infty(\Omega)} \, ds \times e^{\int_0^t \| \text{div} \mathbf{u}_{\sigma, \pm}(s) \|_{L^\infty(\Omega)} \, ds}),
\]

(3.16)

### 3.4 Uniform-in-\( \sigma \) estimates

In this subsection, we summarize the estimates above and derive the uniform-in-\( \sigma \) estimates. Let \( T > 0 \) be the uniform-in-\( \sigma \) life span of solutions to (1.1), and denote by

\[
\sup_{0 \leq s \leq T} \| \partial_t \mathbf{u}_{\sigma, \pm}(s) \|^2_{L^2(\Omega)} + \int_0^T \left\{ \| \nabla \mathbf{u}_{\sigma, \pm}(s) \|^2_{H^2(\Omega)} + \| \nabla \partial_t \mathbf{u}_{\sigma, \pm}(s) \|^2_{L^2(\Omega)} \right\} \, ds \leq \mathcal{M},
\]

(3.17)

which will be shown to be independent of \( \sigma \).

To shorten the notations, we will use \( C_{in} \in (0, \infty) \) through out this section to represent a generic constant depending only on

\[
\rho > 0, \| \nabla \log \rho_{\sigma, \pm, 0} \|_{L^3(\Omega)}, \| \rho_{\sigma, \pm, 0}^{1/2} \|_{H^2(\Omega)}, \| \mathbf{u}_{\sigma, \pm, 0} \|_{H^2(\Omega)}, \| \mathbf{u}_{\sigma, \pm, 1} \|_{L^2(\Omega)},
\]

but independent of \( \sigma \), which might be different from line to line, where \( \mathbf{u}_{\sigma, \pm, 1} = \partial_t \mathbf{u}_{\sigma, \pm} \big|_{t=0} \) are the initial data of \( \partial_t \mathbf{u}_{\sigma, \pm} \) defined by equation (1.1)2.
Applying Hölder’s inequality, one can obtain that, for \( T \) small enough and any \( t \in (0, T) \),

\[
\int_0^t \| \partial_t u_{\sigma, \pm}(s) \|_{L^3(\Omega)} \, ds \leq CT \sup_{0 \leq s \leq T} \| \partial_t u_{\sigma, \pm}(s) \|_{L^2(\Omega)}
\]

\[
+ CT^{1/2} \left( \int_0^T \| \nabla \partial_t u_{\sigma, \pm} \|_{L^2(\Omega)}^2 \right)^{1/2} \leq C(T + T^{1/2}) M^{1/2} \leq 1,
\]

\[
\int_0^t \| \text{div} \, \partial_t u_{\sigma, \pm}(s) \|_{L^2(\Omega)} \, ds \leq CT^{1/2} \left( \int_0^T \| \nabla \partial_t u_{\sigma, \pm} \|_{L^2(\Omega)}^2 \right)^{1/2}
\]

\[
\leq CT^{1/2} M^{1/2} \leq 1,
\]

\[
\int_0^t \| \nabla u_{\sigma, \pm}(s) \|_{H^2(\Omega)} \, ds \leq T^{1/2} \left( \int_0^T \| \nabla u_{\sigma, \pm}(s) \|_{H^2(\Omega)}^2 \, ds \right)^{1/2}
\]

\[
\leq T^{1/2} M^{1/2} \leq c_1,
\]

with some uniform constant \( c_1 \), independent of \( \sigma \) and small enough such that

\[
\int_0^t \| \text{div} \, u_{\sigma, \pm}(s) \|_{L^\infty(\Omega)} \, ds \times e^\int_0^t \| \text{div} \, u_{\sigma, \pm}(s) \|_{L^\infty(\Omega)} \, ds
\]

\[
\leq C \int_0^t \| \nabla u_{\sigma, \pm}(s) \|_{H^2(\Omega)} \, ds \times e^{C \int_0^t \| \nabla u_{\sigma, \pm}(s) \|_{H^2(\Omega)} \, ds}
\]

\[
\leq Cc_1 \times e^{Cc_1} \leq 1/2.
\]

Then from (3.10), (3.11), (3.13), (3.15), and (3.16), it follows that, for \( t \in (0, T) \),

\[
\inf_{\Omega} \rho_{\sigma, \pm}(t) \geq \rho/2,
\]

\[
\| \rho_{\sigma, \pm}^{1/2}(t) \|_{H^2(\Omega)}, \| \rho_{\sigma, \pm}(t) \|_{L^\infty(\Omega)}, \| \nabla \rho_{\sigma, \pm}^{1/2}(t) \|_{L^6(\Omega)},
\]

\[
\| \nabla \log \rho_{\sigma, \pm}(t) \|_{L^3(\Omega)}, \| \partial_t \rho_{\sigma, \pm}^{1/2}(t) \|_{L^2(\Omega)} \leq C_{in}.
\]

Meanwhile, applying the Gagliardo–Nirenberg inequality, one has

\[
\| u_{\sigma, +} - u_{\sigma, -} \|_{L^\infty(\Omega)} \leq C \| u_{\sigma, +} - u_{\sigma, -} \|_{H^2(\Omega)}^{3/4} \| u_{\sigma, +} - u_{\sigma, -} \|_{L^2(\Omega)}^{1/4},
\]

\[
\| \nabla (u_{\sigma, +} - u_{\sigma, -}) \|_{L^2(\Omega)} \leq C \| u_{\sigma, +} - u_{\sigma, -} \|_{H^2(\Omega)}^{3/4} \| u_{\sigma, +} - u_{\sigma, -} \|_{L^2(\Omega)}^{1/4}.
\]

Therefore, after collecting (3.3), (3.7), and (3.8), with suitably small \( \delta \in (0, 1) \), substituting inequalities (3.19) and (3.20), and applying (3.4),
Sobolev embedding inequalities, and Young’s inequality, one can obtain

\[
\frac{d}{dt} E_{\sigma,1} + D_{\sigma,1} + \sigma \int \rho_{\sigma,+} \rho_{\sigma,-} \left( |\partial_t(u_{\sigma,+} - u_{\sigma,-})|^2 + |\nabla(u_{\sigma,+} - u_{\sigma,-})|^2 \right) dx \\
\quad \leq c_2 \sigma \int \rho_{\sigma,+} \rho_{\sigma,-} |u_{\sigma,+} - u_{\sigma,-}|^2 dx + \mathcal{H}(E_{\sigma}, E_{\sigma,1}, C_{in}),
\]

(3.21)

where \(c_2 \in (0, \infty)\) is a constant depending on \(C_{in}\), but independent of \(\sigma\), and

\[
E_{\sigma,1} := \sum_{j \in \{+,-\}} \left\{ \|\rho_{\sigma,j}^{1/2} \partial_t u_{\sigma,j}\|_{L^2(\Omega)}^2 + \|\rho_{\sigma,j}^{1/2} \nabla u_{\sigma,j}\|_{L^2(\Omega)}^2 \right\},
\]

(3.22)

\[
D_{\sigma,1} := \sum_{j \in \{+,-\}} \left\{ \mu_j \|\nabla \partial_t u_{\sigma,j}\|_{L^2(\Omega)}^2 + (\lambda_j + \mu_j) \|\partial_t u_{\sigma,j}\|_{L^2(\Omega)}^2 \right\}.
\]

(3.23)

Consequently, (2.6) and (3.21) yield that, there is some \(T^* \in (0, T]\), independent of \(\sigma\), such that for any \(t \in (0, T^*]\),

\[
\begin{align*}
\sup_{0 \leq s \leq t} \{E_{\sigma}(s) + E_{\sigma,1}(s)\} &+ \int_0^t \{D_{\sigma}(s) + D_{\sigma,1}(s)\} ds \\
&\quad + \sigma \int_0^t \left\{ \|\nabla u_{\sigma,+} - u_{\sigma,-}\|^2_{H^2(\Omega)} + \|\partial_t(u_{\sigma,+} - u_{\sigma,-})\|^2_{L^2(\Omega)} \right\} ds \\
&\leq C_{in},
\end{align*}
\]

(3.24)

which, in particular, implies

\[
\begin{align*}
\sup_{0 \leq s \leq T^*} \|\partial_t u_{\sigma,\pm}(s)\|^2_{L^2(\Omega)} &+ \int_0^{T^*} \left\{ \|\nabla u_{\sigma,\pm}(s)\|^2_{H^2(\Omega)} + \|\nabla \partial_t u_{\sigma,\pm}(s)\|^2_{L^2(\Omega)} \right\} ds \\
&\leq \mathfrak{M}^* (C_{in}).
\end{align*}
\]

(3.25)

Now, we update \(\mathfrak{M}\) to \(\mathfrak{M}^* (C_{in})\), and \(T\) to \(T^*\) accordingly. To this end,
we conclude this section with (3.19) and the following uniform estimates:

\[
\sup_{0 \leq s \leq T} \left\{ \left\| u_{\sigma, \pm}(s) \right\|_{H^2(\Omega)}^2 + \left\| \partial_t u_{\sigma, \pm}(s) \right\|_{L^2(\Omega)}^2 \right\} \\
+ \int_0^T \left\{ \left\| u_{\sigma, \pm}(s) \right\|_{H^2(\Omega)}^2 + \left\| \partial_t u_{\sigma, \pm}(s) \right\|_{H^1(\Omega)}^2 \right\} ds \\
+ \sigma \int_0^T \left\{ \left\| (u_{\sigma, +} - u_{\sigma, -})(s) \right\|_{H^2(\Omega)}^2 + \left\| \partial_t (u_{\sigma, +} - u_{\sigma, -})(s) \right\|_{L^2(\Omega)}^2 \right\} ds \\
\leq C_{\text{in}, T}.
\]

(3.26)

On the other hand, we can rewrite (1.1) as

\[
\partial_t u_{\sigma, \pm} + u_{\sigma, \pm} \cdot \nabla u_{\sigma, \pm} + \frac{\nabla p_{\sigma, \pm} - \text{div} S_{\sigma, \pm}}{\rho_{\sigma, \pm}} = \sigma \rho_{\sigma, \pm} (u_{\sigma, +} - u_{\sigma, -}).
\]  

(3.27)

Then after subtracting the + equation by the − equation of (5.3), one obtains

\[
\partial_t (u_{\sigma, +} - u_{\sigma, -}) + u_{\sigma, +} \cdot \nabla u_{\sigma, +} - u_{\sigma, -} \cdot \nabla u_{\sigma, -} \\
+ \frac{\nabla p_{\sigma, +} - \text{div} S_{\sigma, +}}{\rho_{\sigma, +}} - \frac{\nabla p_{\sigma, -} - \text{div} S_{\sigma, -}}{\rho_{\sigma, -}} = \sigma (\rho_{\sigma, +} + \rho_{\sigma, -}) (u_{\sigma, +} - u_{\sigma, -}).
\]

(3.28)

Therefore, utilizing (3.19) and (3.26), one can derive from (3.28) that

\[
\sup_{0 \leq s \leq T} \left\{ \sigma^2 \left\| (u_{\sigma, +} - u_{\sigma, -})(s) \right\|_{L^2(\Omega)}^2 + \sigma \left\| (u_{\sigma, +} - u_{\sigma, -})(s) \right\|_{H^1(\Omega)}^2 \right\} \\
+ \sigma^2 \int_0^T \left\| (u_{\sigma, +} - u_{\sigma, -})(s) \right\|_{H^1(\Omega)}^2 ds \leq C_{\text{in}, T},
\]

(3.29)

where we have applied Cauchy’s inequality and the fact that

\[
\left\| u_{\sigma, +} - u_{\sigma, -} \right\|_{H^1(\Omega)}^2 \leq \left\| u_{\sigma, +} - u_{\sigma, -} \right\|_{L^2(\Omega)} \left\| u_{\sigma, +} - u_{\sigma, -} \right\|_{H^2(\Omega)}.
\]

4 Uniform estimates: \( \Omega = \mathbb{T}^2 \times (0, 1) \)

In this section, we study (1.1) in domain \( \mathbb{T}^2 \times (0, 1) \). The goal is to obtain uniform estimates in order to pass the limit (2.13).
4.1 Known estimates

First, we shall record the estimates from section 3 that still work, with minor modifications. In order to do so, we remind readers that, instead of using (3.4), we will replace it by Korn’s inequality in this section.

Then estimate (3.3) holds, and all estimates in section 3.3 hold.

Next, directly, one can check

\[
\frac{d}{dt}\|\nabla u_{\sigma,\pm}\|_{L^2(\Omega)}^2 \leq 2\|\nabla u_{\sigma,\pm}\|_{L^2(\Omega)}\|\nabla \partial_t u_{\sigma,\pm}\|_{L^2(\Omega)}.
\]

(4.1)

Thus, we only need to obtain the uniform \(H^2\)– and \(H^3\)–estimates of \(u_{\sigma,\pm}\) to close the uniform estimates. This will be done in the next two subsections.

4.2 Normal derivative estimates: \(H^2\)

We start by rewriting

\[
\text{div } S_{\sigma,\pm} = \mu \Delta u_{\sigma,\pm} + (\mu + \lambda) \text{div } u_{\sigma,\pm}
\]

\[
= \mu \Delta_h u_{\sigma,\pm} + (\mu + \lambda) \left( \nabla_h \text{div } u_{\sigma,\pm} \right) + \left( \mu \partial_{zz} v_{\sigma,\pm} \right) + (2\mu + \lambda) \partial_{zz} w_{\sigma,\pm},
\]

(4.2)

and denote by

\[
A_{\sigma,\pm} := \rho_{\sigma,\pm} \partial_t u_{\sigma,\pm} + \rho_{\sigma,\pm} u_{\sigma,\pm} \cdot \nabla u_{\sigma,\pm} + \nabla p_{\sigma,\pm}.
\]

(4.3)

Thus (1.1) can be written as

\[
\mu \Delta u_{\sigma,\pm} + (\mu + \lambda) \text{div } u_{\sigma,\pm} + \sigma \rho_{\sigma,\pm} \rho_{\sigma,\mp} (u_{\sigma,\mp} - u_{\sigma,\pm}) = A_{\sigma,\pm}.
\]

Therefore, after subtracting and adding the \(+\)–equation and \(\mp\)–equation, respectively, one has

\[
\mu \Delta u_{\sigma,++} + (\mu + \lambda) \text{div } u_{\sigma,++} = A_{\sigma,+} + A_{\sigma,-},
\]

(4.4)

\[
\mu \Delta u_{\sigma,+-} + (\mu + \lambda) \text{div } u_{\sigma,--} - 2\sigma \rho_{\sigma,\pm} \rho_{\sigma,\mp} u_{\sigma,\mp} = A_{\sigma,+} - A_{\sigma,-},
\]

(4.5)

where

\[
u_{\sigma,++} := u_{\sigma,\mp} + u_{\sigma,\pm}, \quad u_{\sigma,+-} := u_{\sigma,\mp} - u_{\sigma,\pm}.
\]

(4.6)

After testing (4.4) with \(-u_{\sigma,++}, \Delta_h u_{\sigma,++},\) respectively, and applying integration by parts and Hölder’s inequality in the resultant, one can derive

\[
\mu \|\nabla u_{\sigma,++}, \nabla \nabla_h u_{\sigma,++}\|_{L^2(\Omega)}^2 + (\mu + \lambda) \|\text{div } u_{\sigma,++}, \nabla_h \text{div } u_{\sigma,++}\|_{L^2(\Omega)}^2
\]

\[
\leq \sum_{j\in\{+,\mp\}} \|A_{\sigma,j}\|_{L^2(\Omega)} \|u_{\sigma,j}, \nabla_h^2 u_{\sigma,j}\|_{L^2(\Omega)}.
\]

(4.7)
Notice, the identity (4.2) holds for $u_{\sigma,\pm}$ replaced by $u_{\sigma,++}$. Thus, one can derive, straightforwardly, that
\[
\|\nabla^2 u_{\sigma,++}\|_{L^2(\Omega)} \leq C \|\nabla \nabla h u_{\sigma,++}\|_{L^2(\Omega)} + \|\mu \Delta u_{\sigma,++} + (\mu + \lambda)\div u_{\sigma,++}\|_{L^2(\Omega)}.
\]
Hence (4.7) implies
\[
\|u_{\sigma,++}\|_{H^2(\Omega)} \leq C \sum_{j \in \{+,-\}} (\|A_{\sigma,j}\|_{L^2(\Omega)} + \|u_{\sigma,j}\|_{L^2(\Omega)}).
\]
Similarly, after testing (4.5) with $-u_{\sigma,--}$, $\Delta_h u_{\sigma,--}$, respectively, and applying integration by parts, Hölder’s inequality, and Cauchy’s inequality in the resultant, one can derive
\[
\mu \|\nabla u_{\sigma,--} \cdot \nabla \nabla h u_{\sigma,--}\|_{L^2(\Omega)} + (\mu + \lambda)\|\div u_{\sigma,--} + \nabla \div u_{\sigma,--}\|_{L^2(\Omega)}^2
\]
\[
+ \sigma \int \rho_{\sigma,+} \rho_{\sigma,-} (2 |u_{\sigma,+-}|^2 + |\nabla_h u_{\sigma,+-}|^2) \, dx
\]
\[
\leq \sum_{j \in \{+,-\}} (\|A_{\sigma,j}\|_{L^2(\Omega)} \|u_{\sigma,j}\|_{L^2(\Omega)} + \|\nabla^2 u_{\sigma,j}\|_{L^2(\Omega)} + 4\|\nabla (\rho_{\sigma,+} \rho_{\sigma,-})^{1/2} \|_{L^2(\Omega)} \|u_{\sigma,++}\|_{L^2(\Omega)}),
\]
where we have applying the following estimates,
\[
-2\sigma \int \rho_{\sigma,+} \rho_{\sigma,-} u_{\sigma,+-} \cdot \Delta_h u_{\sigma,+-} \, dx
\]
\[
= 2\sigma \int \rho_{\sigma,+} \rho_{\sigma,-} |\nabla_h u_{\sigma,+-}|^2 \, dx
\]
\[
+ 2\sigma \sum_{\partial_h \in \{\partial_x, \partial_y\}} \int \rho_{\sigma,+} \rho_{\sigma,-} u_{\sigma,+-} \cdot \partial_h u_{\sigma,+-} \, dx
\]
\[
\geq \sigma \int \rho_{\sigma,+} \rho_{\sigma,-} |\nabla_h u_{\sigma,+-}|^2 \, dx
\]
\[
- 4\sigma \|\nabla (\rho_{\sigma,+} \rho_{\sigma,-})^{1/2} \|_{L^2(\Omega)} \|u_{\sigma,++}\|_{L^2(\Omega)}.
\]
In addition, due to (4.2) and (4.5), one has
\[
\|\mu \partial_{zz} v_{\sigma,+-} - 2\rho_{\sigma,+} \rho_{\sigma,-} v_{\sigma,+-}\|_{L^2(\Omega)} + \|2(\mu + \lambda) \partial_{zz} w_{\sigma,+-} - 2\rho_{\sigma,+} \rho_{\sigma,-} w_{\sigma,+-}\|_{L^2(\Omega)}
\]
\[
\leq C \|\nabla \nabla h u_{\sigma,+-}\|_{L^2(\Omega)} + \sum_{j \in \{+,-\}} \|A_{\sigma,j}\|_{L^2(\Omega)}.
\]
In fact, \(\text{inequality and Cauchy's inequality,}
\)

Therefore, (4.9), (4.10), (4.11), and (4.12) yield, after applying Hölder’s inequality and Cauchy’s inequality,

\[
\begin{align*}
\| \mu \partial_{zz} \mathbf{v}_{\sigma,+-} - 2\sigma \rho_{\sigma,+} \rho_{\sigma,-} \mathbf{v}_{\sigma,+-} \|_{L^2(\Omega)}^2 &= \mu^2 \| \partial_{zz} \mathbf{v}_{\sigma,+-} \|_{L^2(\Omega)}^2 \\
+ 4\sigma^2 \| \rho_{\sigma,+} \rho_{\sigma,-} \mathbf{v}_{\sigma,+-} \|_{L^2(\Omega)}^2 + 4\mu \sigma \int \rho_{\sigma,+} \rho_{\sigma,-} |\partial_z \mathbf{v}_{\sigma,+-}|^2 \, dx \\
+ 4\mu \sigma \int \partial_z (\rho_{\sigma,+} \rho_{\sigma,-}) \mathbf{v}_{\sigma,+-} \cdot \partial_z \mathbf{v}_{\sigma,+-} \, dx,
\end{align*}
\]

\[
\| (2\mu + \lambda) \partial_{zz} w_{\sigma,+-} - 2\sigma \rho_{\sigma,+} \rho_{\sigma,-} w_{\sigma,+-} \|_{L^2(\Omega)}^2
= (2\mu + \lambda)^2 \| \partial_{zz} w_{\sigma,+-} \|_{L^2(\Omega)}^2 + 4\sigma^2 \| \rho_{\sigma,+} \rho_{\sigma,-} w_{\sigma,+-} \|_{L^2(\Omega)}^2
+ 4(2\mu + \lambda) \sigma \int \rho_{\sigma,+} \rho_{\sigma,-} |\partial_z w_{\sigma,+-}|^2 \, dx \\
+ 4(2\mu + \lambda) \sigma \int \partial_z (\rho_{\sigma,+} \rho_{\sigma,-}) w_{\sigma,+-} \cdot \partial_z w_{\sigma,+-} \, dx.
\]

To finish this subsection, we write down the estimates of \(\| A_{\sigma,\pm} \|_{L^2(\Omega)}\).

In fact,

\[
\begin{align*}
\| A_{\sigma,\pm} \|_{L^2(\Omega)} &\leq \| \rho_{\sigma,\pm} \|_{L^\infty(\Omega)}^{1/2} \| \rho_{\sigma,\pm} \|_{L^6(\Omega)}^{1/2} \| \partial_t u_{\sigma,\pm} \|_{L^2(\Omega)} \\
+ \| \rho_{\sigma,\pm} \|_{L^\infty(\Omega)} \| u_{\sigma,\pm} \|_{L^6(\Omega)} \| \nabla u_{\sigma,\pm} \|_{L^2(\Omega)} + \| \nabla p_{\sigma,\pm} \|_{L^2(\Omega)} \\
&\leq \| \rho_{\sigma,\pm} \|_{L^\infty(\Omega)}^{1/2} \| \rho_{\sigma,\pm} \|_{L^6(\Omega)}^{1/2} \| \partial_t u_{\sigma,\pm} \|_{L^2(\Omega)} + \| \nabla p_{\sigma,\pm} \|_{L^2(\Omega)} \\
+ \| \rho_{\sigma,\pm} \|_{L^\infty(\Omega)} \| u_{\sigma,\pm} \|_{L^6(\Omega)} \| \nabla u_{\sigma,\pm} \|_{L^2(\Omega)} + \| \nabla u_{\sigma,\pm} \|_{L^2(\Omega)} \\
\times \| \nabla u_{\sigma,\pm} \|_{L^2(\Omega)} \times \| u_{\sigma,\pm} \|_{H^2(\Omega)}.
\end{align*}
\]
Consequently, combining (4.8) and (4.13) yields

\[
\| u_{\sigma,+} \|^2_{H^2(\Omega)} + \sigma^2 \| \rho_{\sigma,+} + \rho_{\sigma,-} (u_{\sigma,+} - u_{\sigma,-}) \|^2_{L^2(\Omega)} \\
+ \sigma \int \rho_{\sigma,+} + \rho_{\sigma,-} (\| u_{\sigma,+} - u_{\sigma,-} \|^2 + | \nabla (u_{\sigma,+} - u_{\sigma,-}) |^2) \, dx \\
\leq H(\| \rho_{\sigma,\pm} \|_{L^\infty(\Omega)}, \| \nabla \rho_{\sigma,\pm} \|_{L^2(\Omega)}, \| \rho_{\sigma,\pm}^{1/2} u_{\sigma,\pm} \|_{L^2(\Omega)}, \| \nabla u_{\sigma,\pm} \|_{L^2(\Omega)}, \\
\| \rho_{\sigma,\pm} \partial_i u_{\sigma,\pm} \|_{L^2(\Omega)} \\
+ C\sigma \left( 1 + \sup_{x \in \Omega} \frac{1}{\rho_{\sigma,+} + \rho_{\sigma,-}} \right) \| \nabla_h (\rho_{\sigma,+} + \rho_{\sigma,-})^{1/2} \|_{L^6(\Omega)} \| u_{\sigma,+-} \|_{L^2(\Omega)}^2, \\
\]  

4.3 Normal derivative estimates: \( H^3 \)

Recall that \( \partial_h \in \{ \partial_x, \partial_y \} \). Applying \( \partial_h \) to (4.4) and (4.5) leads to

\[
\mu \Delta \partial_h u_{\sigma,++} + (\mu + \lambda) \nabla \text{div} \partial_h u_{\sigma,++} = \partial_h A_{\sigma,+} + \partial_h A_{\sigma,-}, \\
\mu \Delta \partial_h u_{\sigma,--} + (\mu + \lambda) \nabla \text{div} \partial_h u_{\sigma,--} = -2 \sigma \rho_{\sigma,+} + \rho_{\sigma,-} \partial_h u_{\sigma,---} \\
+ \partial_h A_{\sigma,+} - \partial_h A_{\sigma,-} + 2 \sigma \partial_h (\rho_{\sigma,+} + \rho_{\sigma,-}) u_{\sigma,---}. 
\]

Then, by performing similar estimates from (4.7) to (4.13), one can obtain the following estimates:

\[
\| \nabla_h u_{\sigma,++} \|_{H^2(\Omega)} \leq C \sum_{j \in \{+,-\}} \left( \| \nabla_h A_{\sigma,j} \|_{L^2(\Omega)} + \| \nabla_h u_{\sigma,j} \|_{L^2(\Omega)} \right), \\
\]  

\[
\| \nabla_h u_{\sigma,--} \|_{H^2(\Omega)} + \sigma \int \rho_{\sigma,+} + \rho_{\sigma,-} (| \nabla_h u_{\sigma,--} |^2 + | \nabla_h \nabla u_{\sigma,--} |^2) \, dx \\
+ \sigma^2 \| \rho_{\sigma,+} + \rho_{\sigma,-} \nabla_h u_{\sigma,--} \|_{L^2(\Omega)}^2 \leq C \sum_{j \in \{+,-\}} \left( \| \nabla_h A_{\sigma,j} \|_{L^2(\Omega)}^2 \\
+ \| \nabla_h u_{\sigma,j} \|_{L^2(\Omega)}^2 \\
+ C\sigma \| \nabla_h (\rho_{\sigma,+} + \rho_{\sigma,-}) \|_{L^6(\Omega)} \| \nabla_h u_{\sigma,--} \|_{L^3(\Omega)}^2 \\
+ C\sigma^2 \| \nabla_h (\rho_{\sigma,+} + \rho_{\sigma,-}) \|_{L^6(\Omega)} \| u_{\sigma,--+} \|_{L^2(\Omega)}^2. 
\]  

On the other hand, after applying \( \partial_z \) to (4.31), together with identity (4.2), it follows:

\[
\begin{pmatrix}
\mu \partial_{zz \sigma} v_{\sigma,++} \\
(2\mu + \lambda) \partial_{zz \sigma} w_{\sigma,++}
\end{pmatrix} = \partial_z A_{\sigma,+} + \partial_z A_{\sigma,-}. 
\]
\[- \mu \Delta_h \partial_z u_\sigma,++ - (\mu + \lambda) \left( \nabla_h \text{div} \partial_z u_\sigma,++ - \partial_z \text{div}_h \partial_z v_\sigma,++ \right) \cdot \]

Therefore, it holds:

\[
\| \partial_z z z u_\sigma,++ - \|_{L^2(\Omega)} \leq C \sum_{j \in \{+, -\}} \| \partial_z A_{\sigma,j} \|_{L^2(\Omega)} + C \| \nabla_h u_\sigma,++ - \|_{H^2(\Omega)} \cdot (4.17)
\]

Similarly, after applying \( \partial_z \) to (4.5), together with identity (4.2), it follows:

\[
\left( \mu \partial_z z z v_\sigma,++ - 2 \sigma \rho_\sigma,+ \rho_\sigma,- \partial_z v_\sigma,++ \right) = \partial_z A_{\sigma,} - \partial_z A_{\sigma,-} + 2 \sigma \partial_z (\rho_\sigma,+) u_\sigma,++
\]

\[
- \mu \Delta_h \partial_z u_\sigma,++ - (\mu + \lambda) \left( \nabla_h \text{div} \partial_z u_\sigma,++ - \partial_z \text{div}_h \partial_z v_\sigma,++ \right) \cdot (4.18)
\]

Again, directly applying integration by parts yields

\[
\| \mu \partial_z z z v_\sigma,++ - 2 \sigma \rho_\sigma,+ \rho_\sigma,- \partial_z v_\sigma,++ - \|^2_{L^2(\Omega)} = \mu^2 \| \partial_z z z v_\sigma,++ - \|^2_{L^2(\Omega)} + 4 \sigma^2 \| \rho_\sigma,+ \rho_\sigma,- \partial_z v_\sigma,++ - \|^2_{L^2(\Omega)} + 4 \mu \sigma \int \rho_\sigma,+ \rho_\sigma,- |\partial_z z z v_\sigma,++ - \|^2 \ dx
\]

\[
+ 4 \mu \sigma \int \partial_z (\rho_\sigma,+) \partial_z v_\sigma,++ - \partial_z z z v_\sigma,++ - \ dx, \quad (4.19)
\]

\[
\| (2 \mu + \lambda) \partial_z z z w_\sigma,++ - 2 \sigma \rho_\sigma,+ \rho_\sigma,- \partial_z w_\sigma,++ - \|^2_{L^2(\Omega)} = (2 \mu + \lambda)^2 \| \partial_z z z w_\sigma,++ - \|^2_{L^2(\Omega)} + 4 \sigma^2 \| \rho_\sigma,+ \rho_\sigma,- \partial_z w_\sigma,++ - \|^2_{L^2(\Omega)}
\]

\[
+ 4(2 \mu + \lambda) \sigma \int \rho_\sigma,+ \rho_\sigma,- |\partial_z z z w_\sigma,++ - \|^2 \ dx
\]

\[
+ 4(2 \mu + \lambda) \sigma \int \partial_z (\rho_\sigma,+) \partial_z w_\sigma,++ - \partial_z z z w_\sigma,++ - \ dx
\]

\[
- 4(2 \mu + \lambda) \sigma \left( \int_{T^2} \rho_\sigma,+ \rho_\sigma,- \partial_z w_\sigma,++ - \partial_z z z w_\sigma,++ - dS \right) \bigg|_{z=0}. \quad (4.20)
\]

Moreover, applying the trace embedding inequality and the Gagliardo-Nirenberg
inequality implies

\[
\left( \int_{\mathbb{T}^2} \rho_{\sigma,+}\rho_{\sigma,-} \partial_z w_{\sigma,+} \cdots \partial_z^2 w_{\sigma,+} \cdot dS \right)_{z=0}^{1} \leq \| \rho_{\sigma,+}\rho_{\sigma,-} \|_{L^{\infty}(\Gamma)}
\]

\[
\times \| \partial_z w_{\sigma,+} \|_{L^2(\Gamma)} \| \partial_z^2 w_{\sigma,+} \|_{L^2(\Gamma)} \leq \| \rho_{\sigma,+}\rho_{\sigma,-} \|_{H^{3/2}(\Gamma)}
\]

\[
\times \| \partial_z w_{\sigma,+} \|_{H^{1/2}(\Omega)} \| \partial_z^2 w_{\sigma,+} \|_{H^{1/2}(\Omega)} \leq C \| \rho_{\sigma,+}\rho_{\sigma,-} \|_{H^2(\Omega)} \quad (4.21)
\]

\[
\times \| \partial_z w_{\sigma,+} \|_{L^2(\Omega)} \| \partial_z^2 w_{\sigma,+} \|_{L^2(\Omega)} \leq \| \rho_{\sigma,+}\rho_{\sigma,-} \|_{L^2(\Omega)}
\]

Therefore, combining (4.18)–(4.21) leads to

\[
\| \partial_z z u_{\sigma,+} \|_{L^2(\Omega)}^2 + \sigma^2 \| \rho_{\sigma,+}\rho_{\sigma,-} \partial_z u_{\sigma,+} \|_{L^2(\Omega)}^2
\]

\[
+ \sigma \int_{\Omega} \rho_{\sigma,+}\rho_{\sigma,-} | \partial_z^2 u_{\sigma,+} |^2 \, dx \leq C \sum_{j \in \{+,\cdots\} } \| \partial_z A_{\sigma j} \|_{L^2(\Omega)}^2
\]

\[
+ C \| \nabla h u_{\sigma,+} \|_{H^2(\Omega)}^2 + C \| \partial_z (\rho_{\sigma,+}\rho_{\sigma,-})^{1/2} \|_{L^6(\Omega)}^2 \| \partial_z u_{\sigma,+} \|_{L^3(\Omega)}^2
\]

\[
+ C \sigma^2 \| \partial_z (\rho_{\sigma,+}\rho_{\sigma,-})^{1/2} \|_{L^6(\Omega)}^2 \| u_{\sigma,+} \|_{L^3(\Omega)}^2
\]

\[
+ \mathcal{H}(\| \rho_{\sigma,+} \|_{H^2(\Omega)}, \| u_{\sigma,+} \|_{H^2(\Omega)}).
\]

(4.22)

To finish this subsection, we write down the estimates of \( \| \nabla A_{\sigma,\pm} \|_{L^2(\Omega)} \).

Direct calculation yields,

\[
\| \nabla A_{\sigma,\pm} \|_{L^2(\Omega)} \leq C \| \nabla \rho_{\sigma,\pm} \|_{L^6(\Omega)} \| \rho_{\sigma,\pm} \|_{L^6(\Omega)} \| \partial_z u_{\sigma,\pm} \|_{L^3(\Omega)}
\]

\[
+ C \| \rho_{\sigma,\pm} \|_{L^{\infty}(\Omega)} \| \nabla \partial_t u_{\sigma,\pm} \|_{L^2(\Omega)}
\]

\[
+ C \| \nabla \rho_{\sigma,\pm} \|_{L^6(\Omega)} \| u_{\sigma,\pm} \|_{L^3(\Omega)} \| \nabla u_{\sigma,\pm} \|_{L^3(\Omega)}
\]

\[
+ C \| \rho_{\sigma,\pm} \|_{L^{\infty}(\Omega)} \| u_{\sigma,\pm} \|_{L^3(\Omega)} \| \nabla^2 u_{\sigma,\pm} \|_{L^2(\Omega)} + C \| \nabla^2 \rho_{\sigma,\pm} \|_{L^2(\Omega)}.
\]
Consequently, combining (4.15), (4.16), (4.17), and (4.22) yields
\[\|u_{\sigma,\pm}\|_{H^3(\Omega)} + \sigma^2\|\rho_{\sigma,+,\rho_{\sigma,-}\nabla(u_{\sigma,+,u_{\sigma,-}})}\|^2_{L^2(\Omega)}\]
\[+ \sigma \int \rho_{\sigma,+,\rho_{\sigma,-}\nabla^2(u_{\sigma,+,u_{\sigma,-}})} dx\]
\[\leq C\sigma \int \rho_{\sigma,+,\rho_{\sigma,-}\nabla(u_{\sigma,+,u_{\sigma,-}})} dx\]
\[+ C\sigma \left(1 + \sup_{x\in\Omega} \frac{1}{\rho_{\sigma,+,\rho_{\sigma,-}}} \right) \|\nabla(u_{\sigma,+,u_{\sigma,-}})\|_{L^2(\Omega)} \|u_{\sigma,+,u_{\sigma,-}}\|_{L^2(\Omega)}\]
\[+ C\sigma^2 \left(1 + \sup_{x\in\Omega} \frac{1}{\rho_{\sigma,+,\rho_{\sigma,-}}} \right) \|\nabla(u_{\sigma,+,u_{\sigma,-}})\|_{L^2(\Omega)} \|u_{\sigma,+,u_{\sigma,-}}\|_{L^2(\Omega)}\]
\[+ \sum_{j\in\{+,\} -} (\|\rho_{\sigma,j}\|_{L^\infty(\Omega)} + 1) \|\nabla u_{\sigma,j}\|_{L^2(\Omega)}\]
\[+ \mathcal{H}(\|\nabla \rho_{\sigma,\pm}^1\|_{L^6(\Omega)}, \|\rho_{\sigma,\pm}\|_{H^2(\Omega)}, \|u_{\sigma,\pm}\|_{H^2(\Omega)}, \|\rho_{\sigma,\pm}^1\|_{L^2(\Omega)} + \|\nabla u_{\sigma,\pm}\|_{L^2(\Omega)}),\]
\[(4.23)\]
where we have applied interpolation inequality (4.2) and
\[\|\nabla u_{\sigma,+,\pm}\|_{L^3(\Omega)} \leq C\|\nabla u_{\sigma,+,\pm}\|_{L^2(\Omega)} (\|\nabla u_{\sigma,+,\pm}\|_{L^2(\Omega)} + \|\nabla^2 u_{\sigma,+,\pm}\|_{L^2(\Omega)}).\]

### 4.4 Uniform-in-\(\sigma\) estimates

In this subsection, we summarize the estimates above and derive the uniform-in-\(\sigma\) estimates. Again, let \(T > 0\) be the uniform-in-\(\sigma\) life span of solutions to (1.1), and denote \(M\) as in (3.17), which will be shown to be independent of \(\sigma\). Also, \(C_{in} \in (0, \infty)\) represents a generic constant depending only on
\[\rho > 0, \|\nabla \log \rho_{\sigma,\pm,0}\|_{L^3(\Omega)}, \|\rho_{\sigma,\pm,0}\|_{H^2(\Omega)}, \|u_{\sigma,\pm,0}\|_{H^1(\Omega)}, \|u_{\sigma,\pm,1}\|_{L^2(\Omega)},\]
but independent of \(\sigma\), which might be different from line to line, where \(u_{\sigma,+,1} = \partial_t u_{\sigma,\pm}\big|_{t=0}\) are the initial data of \(\partial_t u_{\sigma,\pm}\) defined by equation (1.1).

Then following the same lines of proof, one can establish estimate (3.19), after choosing \(T\) small enough.

On the other hand, for \(\sigma \in (\sigma^*, \infty)\), with some \(\sigma^*\) large enough, depending on \(C_{in}\), such that in (4.14),
\[C\sigma \left(1 + \sup_{x\in\Omega} \frac{1}{\rho_{\sigma,+,\rho_{\sigma,-}}} \right) \|\nabla \rho_{\sigma,+,\rho_{\sigma,-}}\|_{L^6(\Omega)} \|u_{\sigma,+,\pm}\|_{L^2(\Omega)}\]
\[\leq \frac{\sigma^2}{2} \|\rho_{\sigma,+,\rho_{\sigma,-}}(u_{\sigma,+,u_{\sigma,-}})\|_{L^2(\Omega)}^2.\]
That is, for $\sigma \geq \sigma^*$, (4.14) implies
\[
\|u_{\sigma,\pm}\|_{H^2(\Omega)}^2 + \sigma^2\|\rho_{\sigma,\pm}(u_{\sigma,\pm} - u_{\sigma,-})\|_{L^2(\Omega)}^2 + \sigma \int \rho_{\sigma,\pm}(u_{\sigma,\pm} - u_{\sigma,-})^2 + |\nabla(u_{\sigma,\pm} - u_{\sigma,-})|^2 \, dx 
\leq H(\|\rho_{\sigma,\pm}\|_{L^\infty(\Omega)}, \|\nabla\rho_{\sigma,\pm}\|_{L^2(\Omega)}, \|\rho_{\sigma,\pm}^{1/2} u_{\sigma,\pm}\|_{L^2(\Omega)}, \|\nabla u_{\sigma,\pm}\|_{L^2(\Omega)}, \\
\|\rho_{\sigma,\pm}^{1/2} \partial_t u_{\sigma,\pm}\|_{L^2(\Omega)}).
\] (4.24)

Then after collecting (3.3) and (4.1), substituting (3.20) and (4.24), and applying Young’s inequality, one has
\[
\frac{d}{dt}E_{\sigma,2} + D_{\sigma,2} + \sigma \int \rho_{\sigma,\pm}(u_{\sigma,\pm} - u_{\sigma,-})^2 \, dx 
\leq c_3 \sigma \int \rho_{\sigma,\pm}(u_{\sigma,\pm} - u_{\sigma,-})^2 \, dx + H(E_{\sigma}, E_{\sigma,2}, C_{in}),
\] (4.25)

where $c_3 \in (0, \infty)$ is a constant depending on $C_{in}$, but independent of $\sigma$, and
\[
E_{\sigma,2} := \sum_{j \in \{+, -\}} \left\{ \|\rho_{\sigma,j}^{1/2} \partial_t u_{\sigma,j}\|_{L^2(\Omega)}^2 + \|\nabla u_{\sigma,j}\|_{L^2(\Omega)}^2 \right\},
\] (4.26)
\[
D_{\sigma,2} := \sum_{j \in \{+, -\}} \left\{ \mu_j \|\nabla \partial_t u_{\sigma,j}\|_{L^2(\Omega)}^2 + (\lambda_j + \mu_{j+})\|\text{div} \partial_t u_{\sigma,j}\|_{L^2(\Omega)}^2 \right\}.
\] (4.27)

Consequently, (2.6) and (4.25) yield that, there is some $T^{**} \in (0, T]$, independent of $\sigma$, such that for any $t \in (0, T^{**})$,
\[
\sup_{0 \leq s \leq t} \left\{ E_{\sigma}(s) + E_{\sigma,2}(s) \right\} + \int_0^t \left\{ D_{\sigma}(s) + D_{\sigma,2}(s) \right\} ds
+ \sigma \int_0^t \left\{ \|u_{\sigma,\pm}(s)\|_{L^2(\Omega)}^2 + \|\partial_t(u_{\sigma,\pm} - u_{\sigma,-})(s)\|_{L^2(\Omega)}^2 \right\} ds 
\leq C_{in},
\] (4.28)

which, in particular, implies, together with (4.24) and (4.23)
\[
\sup_{0 \leq s \leq T^{**}} \|\partial_t u_{\sigma,\pm}(s)\|_{L^2(\Omega)}^2 + \int_0^{T^{**}} \left\{ \|\nabla u_{\sigma,\pm}(s)\|_{H^2(\Omega)}^2 \right\} ds 
\leq M^{**}(C_{in}).
\] (4.29)
Now, while keeping $\sigma \geq \sigma^*$, we update $\mathfrak{M}$ to $\mathfrak{M}^{**}(C_{in})$, and $T$ to $T^{**}$ accordingly. To this end, we conclude this section with (3.19) and the following uniform estimates:

$$\sup_{0 \leq s \leq T} \left\{ \| u_{\sigma,\pm}(s) \|_{H^2(\Omega)}^2 + \| \partial_t u_{\sigma,\pm}(s) \|_{L^2(\Omega)}^2 + \sigma^2 \| (u_{\sigma,+} - u_{\sigma,-})(s) \|_{L^2(\Omega)}^2 \right\}$$
$$+ \int_0^T \left\{ \| u_{\sigma,\pm}(s) \|_{H^3(\Omega)}^2 + \| \partial_t u_{\sigma,\pm}(s) \|_{H^1(\Omega)}^2 \right\} \, ds$$
$$+ \sigma^2 \int_0^T \| (u_{\sigma,+} - u_{\sigma,-})(s) \|_{H^1(\Omega)}^2 \, ds$$
$$+ \sigma \int_0^T \left\{ \| (u_{\sigma,+} - u_{\sigma,-})(s) \|_{H^2(\Omega)}^2 + \| \partial_t (u_{\sigma,+} - u_{\sigma,-})(s) \|_{L^2(\Omega)}^2 \right\} \, ds$$
$$\leq C_{in,T}. \quad (4.30)$$

5 Passing the limit $\sigma \to \infty$

We are ready to pass the limit $\sigma \to \infty$ in system (1.1). From the uniform estimates in (3.19), (3.26), and (4.30) for $\Omega = \mathbb{T}^3$ and $\Omega = \mathbb{T}^2 \times (0,1)$, respectively, one can conclude that, there exist $\rho_{\pm}$ and $u$ in the corresponding spaces, such that

$$u_{\sigma,+} - u_{\sigma,-} \to 0 \quad \text{in} \quad L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)),$$
$$\partial_t (u_{\sigma,+} - u_{\sigma,-}) \to 0 \quad \text{in} \quad L^2(0,T;L^2(\Omega)),$$
$$\rho_{\sigma,\pm} \to \rho_{\pm} \quad \text{in} \quad C(0,T;H^1(\Omega)),$$
$$\partial_t \rho_{\sigma,\pm} \to \partial_t \rho_{\pm} \quad \text{weak-* in} \quad L^\infty(0,T;H^2(\Omega)),$$
$$u_{\sigma,\pm} \to u \quad \text{in} \quad C(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)),$$
$$u_{\sigma,\pm} \to u \quad \text{weakly in} \quad L^2(0,T;H^3(\Omega)),$$
$$u_{\sigma,\pm} \to u \quad \text{weak-* in} \quad L^\infty(0,T;H^2(\Omega)),$$
$$\partial_t u_{\sigma,\pm} \to \partial_t u \quad \text{weakly in} \quad L^2(0,T;H^1(\Omega)),$$
$$\partial_t u_{\sigma,\pm} \to \partial_t u \quad \text{weak-* in} \quad L^\infty(0,T;L^2(\Omega)). \quad (5.1)$$
as \( \sigma \to \infty \). Meanwhile, after adding the +\(-\)equation and the \(-\)\(+\)equation of (1.1) together, one obtains
\[
\partial_t (\rho_{\sigma,+} u_{\sigma,+} + \rho_{\sigma,-} u_{\sigma,-}) + \text{div} \left( \rho_{\sigma,+} u_{\sigma,+} \otimes u_{\sigma,+} + \rho_{\sigma,-} u_{\sigma,-} \otimes u_{\sigma,-} \right) \\
+ \nabla (p_{\sigma,+} + p_{\sigma,-}) = \text{div} \left( S_{\sigma,+} + S_{\sigma,-} \right).
\] (5.2)

Passing \( \sigma \to \infty \) and utilizing (5.1) in (1.1) and (5.2) lead to the convergences of them to (1.4) in the sense of distribution. Thus we have verified the asymptotic limit (1.4) of system (1.1).

Our remaining goal is to investigate the asymptote of
\[
\sigma \rho_{\sigma,+} \rho_{\sigma,-} (u_{\sigma,+} - u_{\sigma,-})
\] in (1.1)\( \, \)2, which, in general, does not vanish as \( \sigma \to \infty \). Notice that, from (3.28), we have
\[
\sigma (u_{\sigma,-} - u_{\sigma,+}) = \frac{1}{\rho_{\sigma,+} + \rho_{\sigma,-}} \left( \partial_t (u_{\sigma,+} - u_{\sigma,-}) + u_{\sigma,+} \cdot \nabla u_{\sigma,+} \\
- u_{\sigma,-} \cdot \nabla u_{\sigma,-} + \frac{\nabla p_{\sigma,+} - \text{div} S_{\sigma,+}}{\rho_{\sigma,+}} - \frac{\nabla p_{\sigma,-} - \text{div} S_{\sigma,-}}{\rho_{\sigma,-}} \right). 
\] (5.3)

Therefore, passing \( \sigma \to \infty \) in (5.3) implies that, in the sense of distribution,
\[
\sigma (u_{\sigma,-} - u_{\sigma,+}) \sigma \to \infty \frac{1}{\rho_+ + \rho_-} \left[ \frac{R_+}{\gamma_+ - 1} \nabla \rho_+^{\gamma_+ - 1} - \frac{R_-}{\gamma_- - 1} \nabla \rho_-^{\gamma_- - 1} \\
+ \left( \frac{\mu_+}{\rho_+} - \frac{\mu_-}{\rho_-} \right) \text{div} (\nabla u + \nabla^T u) + \left( \frac{\lambda_+}{\rho_+} - \frac{\lambda_-}{\rho_-} \right) \nabla \text{div} u \right] = \mathcal{G}.
\] (5.4)

Consequently, after passing \( \sigma \to \infty \) in (1.1)\( \, \)2 in the sense of distribution, one verifies (1.6).

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