ON A NON-LINEAR $p$-ADIC DYNAMICAL SYSTEM

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Abstract. We investigate the behavior of trajectories of a $(3, 2)$-rational $p$-adic dynamical system in the complex $p$-adic field $\mathbb{C}_p$, when there exists a unique fixed point $x_0$. We study this $p$-adic dynamical system by dynamics of real radiuses of balls (with the center at the fixed point $x_0$). We show that there exists a radius $r$ depending on parameters of the rational function such that: when $x_0$ is an attracting point then the trajectory of an inner point from the ball $U_r(x_0)$ goes to $x_0$ and each sphere with a radius $> r$ (with the center at $x_0$) is invariant. When $x_0$ is a repeller point then the trajectory of an inner point from a ball $U_r(x_0)$ goes forward to the sphere $S_r(x_0)$. Once the trajectory reaches the sphere, in the next step it either goes back to the interior of $U_r(x_0)$ or stays in $S_r(x_0)$ for some time and then goes back to the interior of the ball. As soon as the trajectory goes outside of $U_r(x_0)$ it will stay (for all the rest of time) in the sphere (outside of $U_r(x_0)$) that it reached first.

1. Introduction

What is the main difference between real and $p$-adic space-time? It is the Archimedean axiom. According to this axiom any given large segment on a straight line can be surpassed by successive addition of small segments along the same line. This axiom is valid in the set of real numbers and is not valid in the field of $p$-adic numbers $\mathbb{Q}_p$. However, it is a physical axiom which concerns the process of measurement. To exchange a number field $\mathbb{R}$ to $\mathbb{Q}_p$ is the same as to exchange axiomatics in quantum physics (12, 23).

The representation of $p$-adic numbers by sequences of digits gives a possibility to use this number system for coding of information. Therefore $p$-adic models can be used for the description of many information processes. In particular, they can be used in cognitive sciences, psychology and sociology. Such models based on $p$-adic dynamical systems 2, 3.

The study of $p$-adic dynamical systems arises in Diophantine geometry in the constructions of canonical heights, used for counting rational points on algebraic varieties over a number field, as in 10. There most recent monograph on $p$-adic dynamics is Anashin and Khrennikov, 4; nearly a half of Silvermans monograph 22 also concerns $p$-adic dynamics.

Here are areas where $p$-adic dynamics proved to be effective: computer science (straight line programs), numerical analysis and simulations (pseudorandom numbers), uniform distribution of sequences, cryptography (stream ciphers, $T$-functions), combinatorics (Latin squares), automata theory and formal languages, genetics. The monograph 4 contains the corresponding survey. For a newer results see recent papers and references therein: 1, 5, 8, 9, 11, 12, 13 - 18, 21. Moreover, there are studies in computer
science and cryptography which along with mathematical physics stimulated in 1990-th intensive research in \( p \)-adic dynamics since it was observed that major computer instructions (and therefore programs composed of these instructions) can be considered as continuous transformations with respect to the \( 2 \)-adic metric, see [6], [7].

In this paper we investigate the behavior of trajectories of an arbitrary \((3,2)\)-rational \( p \)-adic dynamical system in complex \( p \)-adic filed \( \mathbb{C}_p \). The paper is organized as follows: in Section 2 we give some preliminaries. Section 3 contains the definition of the \((3,2)\)-rational function and main results about behavior of trajectories of the \( p \)-adic dynamical system.

2. Preliminaries

2.1. \( p \)-adic numbers. Let \( \mathbb{Q} \) be the field of rational numbers. The greatest common divisor of the positive integers \( n \) and \( m \) is denoted by \( (n,m) \). Every rational number \( x \neq 0 \) can be represented in the form \( x = p^r \frac{a}{m} \), where \( r, n \in \mathbb{Z}, \) \( m \) is a positive integer, \( (p,n) = 1 \), \( (p,m) = 1 \) and \( p \) is a fixed prime number.

The \( p \)-adic norm of \( x \) is given by

\[
|x|_p = \begin{cases} \frac{1}{p^{-r}}, & \text{for } x \neq 0, \\ 0, & \text{for } x = 0. \end{cases}
\]

It has the following properties:
1) \( |x|_p \geq 0 \) and \( |x|_p = 0 \) if and only if \( x = 0 \),
2) \( |xy|_p = |x|_p |y|_p \),
3) the strong triangle inequality

\[
|x + y|_p \leq \max\{|x|_p, |y|_p\},
\]

3.1) if \( |x|_p \neq |y|_p \) then \( |x + y|_p = \max\{|x|_p, |y|_p\} \),
3.2) if \( |x|_p = |y|_p \) then \( |x + y|_p \leq |x|_p \),

this is a non-Archimedean one.

The completion of \( \mathbb{Q} \) with respect to \( p \)-adic norm defines the \( p \)-adic field which is denoted by \( \mathbb{Q}_p \) (see [19], [20]).

The well-known Ostrovsky’s theorem asserts that norms \( |x| = |x|_\infty \) and \( |x|_p, \) \( p = 2, 3, 5... \) exhaust all nonequivalent norms on \( \mathbb{Q} \). Any \( p \)-adic number \( x \neq 0 \) can be uniquely represented in the canonical series:

\[
x = p^{\gamma(x)}(x_0 + x_1 p + x_2 p^2 + ...), \tag{2.1}
\]

where \( \gamma = \gamma(x) \in \mathbb{Z} \) and \( x_j \) are integers, \( 0 \leq x_j \leq p - 1, \) \( x_0 > 0, j = 0, 1, 2, ... \). Observe that in this case \( |x|_p = p^{-\gamma(x)} \).

The algebraic completion of \( \mathbb{Q}_p \) is denoted by \( \mathbb{C}_p \) and it is called complex \( p \)-adic numbers. For any \( a \in \mathbb{C}_p \) and \( r > 0 \) denote

\[
U_r(a) = \{ x \in \mathbb{C}_p : |x - a|_p \leq r \}, \quad V_r(a) = \{ x \in \mathbb{C}_p : |x - a|_p < r \}, \quad S_r(a) = \{ x \in \mathbb{C}_p : |x - a|_p = r \}.
\]

A function \( f : U_r(a) \to \mathbb{C}_p \) is said to be \emph{analytic} if it can be represented by

\[
f(x) = \sum_{n=0}^{\infty} f_n (x-a)^n, \quad f_n \in \mathbb{C}_p,
\]
which converges uniformly on the ball $U_r(a)$.

2.2. Dynamical systems in $\mathbb{C}_p$. In this section we recall some known facts concerning dynamical systems $(f, U)$ in $\mathbb{C}_p$, where $f : x \in U \to f(x) \in U$ is an analytic function and $U = U_r(a)$ or $\mathbb{C}_p$.

Now let $f : U \to U$ be an analytic function. Denote $x_0 = f^n(x_0)$, where $x_0 \in U$ and $f^n(x) = f \circ \cdots \circ f(x)$.

Recall some the standard terminology of the theory of dynamical systems. If $f(x_0) = x_0$ then $x_0$ is called a fixed point. The set of all fixed points of $f$ is denoted by Fix$(f)$. A fixed point $x_0$ is called an attractor if there exists a neighborhood $V(x_0)$ of $x_0$ such that for all points $y \in V(x_0)$ it holds $\lim_{n \to \infty} y_n = x_0$. If $x_0$ is an attractor then its basin of attraction is $A(x_0) = \{ y \in \mathbb{C}_p : y_n \to x_0, \ n \to \infty \}$.

A fixed point $x_0$ is called repeller if there exists a neighborhood $V(x_0)$ of $x_0$ such that $|f(x) - x_0|_p > |x - x_0|_p$ for $x \in V(x_0), x \neq x_0$. Let $x_0$ be a fixed point of a function $f(x)$. The ball $V_r(x_0)$ (contained in $U$) is said to be a Siegel disk if each sphere $S_p(x_0), \ r < r$ is an invariant sphere of $f(x)$, i.e. if $x \in S_p(x_0)$ then all iterated points $x_n \in S_p(x_0)$ for all $n = 1, 2, \ldots$. The union of all Siegel desks with the center at $x_0$ is said to a maximum Siegel disk and is denoted by $SI(x_0)$.

In complex geometry, the center of a disk is uniquely determined by the disk, and different fixed points cannot have the same Siegel disks. In non-Archimedean geometry, a center of a disk is a point which belongs to the disk. Therefore, different fixed points may have the same Siegel desk.

Let $x_0$ be a fixed point of an analytic function $f(x)$. Put

$$\lambda = \frac{d}{dx} f(x_0).$$

The point $x_0$ is attractive if $0 \leq |\lambda|_p < 1$, indifferent if $|\lambda|_p = 1$, and repelling if $|\lambda|_p > 1$.

3. $(3, 2)$-rational $p$-adic dynamical systems with a unique fixed point

A function is called a $(n, m)$-rational function if and only if it can be written in the form $f(x) = \frac{P_n(x)}{Q_m(x)}$, where $P_n(x)$ and $Q_m(x)$ are polynomial functions with degree $n$ and $m$ respectively, $Q_m(x)$ is not the zero polynomial.

In this paper we consider the dynamical system associated with the $(3, 2)$-rational function $f : \mathbb{C}_p \to \mathbb{C}_p$ defined by

$$f(x) = \frac{x^3 + ax^2 + bx + c}{x^2 + ax + d}, \quad a, b, c, d \in \mathbb{C}_p, b \neq d$$

where $x \neq \hat{x}_{1,2} = \frac{a \pm \sqrt{a^2 - 4d}}{2}$. Denote

$$\mathcal{P} = \{ x \in \mathbb{C}_p : \exists n \in \mathbb{N}, f^n(x) = \hat{x}_{1,2} \}.$$

The function $f$ has a unique fixed point $x_0 = \frac{c}{d-b}$. 

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For any \( x \in \mathbb{C}_p \setminus \mathcal{P} \), by simple calculations we get
\[
|f(x) - x_0|_p = |x - x_0|_p \cdot \frac{|((x - x_0) + (x_0 + \frac{a}{2}))^2 + \alpha(x_0)|_p}{|((x - x_0) + (x_0 + \frac{a}{2}))^2 + \beta(x_0)|_p},
\] (3.2)
where
\[
\alpha(x) = \frac{8x^4 + 16ax^3 + 6(a^2 + d)x^2 + (6ad + a^2 - c)x + a^2d + bd - ac}{4(x^2 + ax + d)},
\]
\[
\beta(x) = 2x^2 + 2ax + d + \frac{a^2}{4}.
\]
For
\[
\alpha = |\alpha(x_0)|_p, \quad \beta = |\beta(x_0)|_p,
\]
and
\[
\delta = |x_0 + \frac{a}{2}|_p
\]
we have that \( \alpha, \beta \) and \( \delta \) satisfy one of the following relations:
I. \( \delta \leq \min\{\sqrt{\alpha}, \sqrt{\beta}\} \).
II. \( \sqrt{\alpha} \leq \min\{\delta, \sqrt{\beta}\} \).
III. \( \sqrt{\beta} \leq \min\{\sqrt{\alpha}, \delta\} \).
Consider the following functions:
For \( \delta \leq \min\{\sqrt{\alpha}, \sqrt{\beta}\} \) define the functions \( \varphi_i : [0, +\infty) \to [0, +\infty), (i = 1, 2, 3, 4, 5, \) as the following
1. If \( \delta < \sqrt{\alpha} < \sqrt{\beta} \), then
\[
\varphi_1(r) = \begin{cases} \frac{\alpha}{\beta}r, & \text{if } r < \sqrt{\alpha}, \\ \alpha^*, & \text{if } r = \sqrt{\alpha}, \\ \beta^*, & \text{if } \sqrt{\alpha} < r < \sqrt{\beta}, \\ r, & \text{if } r > \sqrt{\beta}, \end{cases}
\]
where \( \alpha^* \) and \( \beta^* \) are some given numbers with \( \alpha^* \leq \frac{\alpha\sqrt{\alpha}}{\beta}, \beta^* \geq \sqrt{\beta} \).
2. If \( \delta < \sqrt{\alpha} = \sqrt{\beta} \), then
\[
\varphi_2(r) = \begin{cases} r, & \text{if } r \neq \sqrt{\alpha}, \\ \hat{\alpha}, & \text{if } r = \sqrt{\alpha}, \end{cases}
\]
where \( \hat{\alpha} \) is a given number.
3. If \( \delta = \sqrt{\alpha} < \sqrt{\beta} \), then

\[
\varphi_3(r) = \begin{cases} 
\lambda r, & \text{if } r < \delta, \\
\delta^*, & \text{if } r = \delta, \\
\frac{\alpha^3}{\beta^3}, & \text{if } \delta < r < \sqrt{\beta}, \\
\beta^*, & \text{if } r = \sqrt{\beta}, \\
r, & \text{if } r > \sqrt{\beta},
\end{cases}
\]

where \( \lambda \leq \frac{\delta^2}{\beta} < 1 \), \( \delta^* \leq \frac{\delta^3}{\beta} \) and \( \beta^* \geq \sqrt{\beta} \).

4. If \( \delta < \sqrt{\beta} < \sqrt{\alpha} \), then

\[
\varphi_4(r) = \begin{cases} 
\frac{\alpha^3}{\beta^2} r, & \text{if } r < \sqrt{\beta}, \\
\beta^*, & \text{if } r = \sqrt{\beta}, \\
\frac{\alpha}{\beta}, & \text{if } \sqrt{\beta} < r < \sqrt{\alpha}, \\
\alpha^*, & \text{if } r = \sqrt{\alpha}, \\
r, & \text{if } r > \sqrt{\alpha},
\end{cases}
\]

where \( \alpha^* \leq \sqrt{\alpha} \), \( \beta^* \geq \frac{\alpha}{\sqrt{\beta}} \).

5. If \( \delta = \sqrt{\beta} < \sqrt{\alpha} \), then

\[
\varphi_5(r) = \begin{cases} 
\lambda r, & \text{if } r < \delta, \\
\delta^*, & \text{if } r = \delta, \\
\frac{\alpha^3}{\beta^2}, & \text{if } \delta < r < \sqrt{\alpha}, \\
\alpha^*, & \text{if } r = \sqrt{\alpha}, \\
r, & \text{if } r > \sqrt{\alpha},
\end{cases}
\]

where \( \lambda \geq \frac{\delta^2}{\beta} > 1 \), \( \delta^* \geq \frac{\alpha}{\delta} \) and \( \alpha^* \leq \sqrt{\alpha} \).

For \( \sqrt{\alpha} \leq \min\{\delta, \sqrt{\beta}\} \) define the function \( \phi_j : [0, +\infty) \rightarrow [0, +\infty) \) \( (j = 1, 2, 3) \) as the following

1. If \( \sqrt{\alpha} < \delta < \sqrt{\beta} \), then

\[
\phi_1(r) = \begin{cases} 
\frac{\alpha^3}{\beta^3} r, & \text{if } r < \delta, \\
\delta', & \text{if } r = \delta, \\
\frac{\alpha^3}{\beta^3}, & \text{if } \delta < r < \sqrt{\beta}, \\
\beta', & \text{if } r = \sqrt{\beta}, \\
r, & \text{if } r > \sqrt{\beta},
\end{cases}
\]

where \( \delta' \) and \( \beta' \) some positive numbers with \( \delta' \leq \frac{\delta^3}{\beta}, \beta' \geq \sqrt{\beta} \).
2. If $\sqrt{\alpha} < \delta = \sqrt{\beta}$, then

$$
\phi_2(r) = \begin{cases} 
\lambda r, & \text{if } r < \delta, \\
\delta', & \text{if } r = \delta, \\
r, & \text{if } r > \delta,
\end{cases}
$$

where $\lambda \geq 1$, $\delta' \leq \delta$.

3. If $\sqrt{\alpha} \leq \delta < \sqrt{\beta}$, then

$$
\phi_3(r) = \begin{cases} 
r, & \text{if } r \neq \delta, \\
\hat{\delta}, & \text{if } r = \delta,
\end{cases}
$$

where $\hat{\delta}$ is a given number.

For $\sqrt{\beta} \leq \min\{\sqrt{\alpha}, \delta\}$ we define the function $\psi_k : [0, +\infty) \rightarrow [0, +\infty)$ ($k = 1, 2, 3$) as the following

1. If $\sqrt{\beta} < \delta < \sqrt{\alpha}$, then

$$
\psi_1(r) = \begin{cases} 
\frac{\alpha}{2^2} r, & \text{if } r < \delta, \\
\delta^*, & \text{if } r = \delta, \\
\alpha^*, & \text{if } \delta < r < \sqrt{\alpha}, \\
r, & \text{if } r > \sqrt{\alpha},
\end{cases}
$$

where $\delta^* \geq \frac{\alpha}{2}$, $\alpha^* \leq \sqrt{\alpha}$.

2. If $\sqrt{\beta} < \delta = \sqrt{\alpha}$ and $|\delta^2 + \alpha|_p < \delta^2$, then

$$
\psi_2(r) = \begin{cases} 
\lambda r, & \text{if } r < \delta, \\
\hat{\delta}, & \text{if } r = \delta, \\
r, & \text{if } r > \delta,
\end{cases}
$$

where $\hat{\delta} \geq \delta$, $\lambda = \frac{|\delta^2 + \alpha|_p}{\delta^2}$.

3. If $\sqrt{\beta} < \sqrt{\alpha} \leq \delta$ and $|\delta^2 + \alpha|_p = \delta^2$, then

$$
\psi_3(r) = \begin{cases} 
r, & \text{if } r \neq \delta, \\
\hat{\delta}, & \text{if } r = \delta,
\end{cases}
$$

where $\hat{\delta}$ is a given number.

Using the formula (3.2) we easily get the following:
Lemma 3.1. If \( x \in S_r(x_0) \), then the following formula holds

\[
|f^n(x) - x_0|_p = \begin{cases} 
\varphi^1_n(r), & \text{if } \delta < \sqrt{\alpha} < \sqrt{\beta} \\
\varphi^2_n(r), & \text{if } \delta < \sqrt{\alpha} = \sqrt{\beta} \\
\varphi^3_n(r), & \text{if } \delta = \sqrt{\alpha} < \sqrt{\beta} \\
\varphi^4_n(r), & \text{if } \delta < \sqrt{\beta} < \sqrt{\alpha} \\
\varphi^5_n(r), & \text{if } \delta = \sqrt{\beta} < \sqrt{\alpha} \\
\end{cases}
\]

Thus the \( p \)-adic dynamical system \( f^n(x), n \geq 1, x \in \mathbb{C}_p \setminus \mathbb{P} \) is related to the real dynamical systems generated by \( \varphi_i, \phi_j \) and \( \psi_k \). Now we are going to study these (real) dynamical systems.

Lemma 3.2. The dynamical system generated by \( \varphi_i(r), (i = 1, 2, 3, 4, 5) \) has the following properties:

1. \( \text{Fix}(\varphi_i) = \{0\} \cup \begin{cases} 
\{r : r > \sqrt{\beta}\} \cup \left\{ \beta^* : \text{if} \ \sqrt{\beta} = \beta^* \right\}, & \text{for } \alpha < \beta, \ i = 1, 3, \\
\{r : r \neq \sqrt{\alpha}\} \cup \left\{ \hat{\alpha} : \text{if} \ \hat{\alpha} = \sqrt{\alpha} \right\}, & \text{for } \alpha = \beta, \ i = 2, \end{cases} \)

2. If \( \alpha < \beta \), then for functions \( \varphi_i(r), i = 1, 3 \) we have

\[
\lim_{n \to \infty} \varphi^n_i(r) = \begin{cases} 
0, & \text{for all } r < \sqrt{\beta}, \\
r, & \text{for all } r > \sqrt{\beta}, \\
\varphi_i(\beta^*), & \text{if } r = \sqrt{\beta} \\
\end{cases}
\]

3. If \( \alpha = \beta \), then for function \( \varphi_2(r) \)

\[
\lim_{n \to \infty} \varphi^n_2(r) = \begin{cases} 
r, & \text{for all } r \neq \sqrt{\alpha}, \\
\varphi_2(\hat{\alpha}), & \text{if } r = \sqrt{\alpha}, \\
\end{cases}
\]

4. If \( \alpha > \beta \), then for functions \( \varphi_i(r), i = 4, 5 \) we have

\[
\lim_{n \to \infty} \varphi^n_i(r) = \begin{cases} 
0, & \text{if } r = 0, \\
\in C_i, & \text{for all } 0 < r \leq \sqrt{\alpha}, \\
r, & \text{for all } r > \sqrt{\alpha}, \\
\end{cases}
\]
where \( C_4 = (\sqrt{\alpha}, \alpha/\sqrt{\beta}) \cup \{\beta^*\} \), \( C_5 = (\sqrt{\alpha}, \alpha/\delta) \cup \{\delta^*\} \).

Proof. 1. This is the result of a simple analysis of the equation \( \varphi_i(r) = r \).

Proofs of parts 2, 3 follow from the property that \( \varphi_i(r), i = 1, 2, 3 \) is an increasing function (except at points of discontinuity). The part 4 easily can be proved using graphs of the corresponding functions \( \varphi_i, i = 4, 5 \).

We note that for any \( a \in C_i \) there exists \( x \in (0, \sqrt{\alpha}) \) such that \( \varphi_i^m(x) = a, i = 4, 5 \).

Lemma 3.3. The dynamical system generated by \( \phi_j(r), (j = 1, 2, 3) \) has the following properties:

A. Fix(\( \phi_j \)) = \( \{0\} \cup \{r : r > \sqrt{\beta} \cup \{\beta^* : if \ \beta^* = \beta\}, for \ \sqrt{\alpha} < \delta < \sqrt{\beta}; \}
\{r : r \neq \delta \cup \{\delta^*: if \ \delta^* = \delta\}, for \ \sqrt{\alpha} \leq \sqrt{\beta} < \delta; \}
\{r : r \neq \delta \cup \{\delta^* : if \ \delta^* = \delta\}, for \ \sqrt{\alpha} < \sqrt{\beta} = \delta and \ \delta^2 = |\delta^2 + \beta|_p; \}
\{r : r > \delta \cup \{\delta^* : if \ \delta^* = \delta\}, for \ \sqrt{\alpha} < \sqrt{\beta} = \delta and \ \delta^2 > |\delta^2 + \beta|_p. \}

B. For function \( \phi_1(r) \), we have
\[ \lim_{n \to \infty} \phi_1^n(r) = \begin{cases} 0, & for \ all \ r < \sqrt{\beta}, \\ r, & for \ all \ r > \sqrt{\beta}; \end{cases} \]
\[ \phi_1(\beta^*), \ if \ r = \sqrt{\beta}. \]

C. For function \( \phi_2(r) \):
C.a If \( \delta^2 = |\delta^2 + \beta|_p \), then
\[ \lim_{n \to \infty} \phi_2^n(r) = \begin{cases} r, & for \ all \ r \neq \delta, \\ \phi_2(\delta^*), & if \ r = \delta; \end{cases} \]
C.b If \( \delta^2 > |\delta^2 + \beta|_p \), then
\[ \lim_{n \to \infty} \phi_2^n(r) = \begin{cases} 0, & if \ r = 0, \\ \in B, & for \ all \ 0 < r \leq \delta; \\ r, & for \ all \ r > \delta, \end{cases} \]
where \( B = (\delta, \delta^3/|\delta^2 + \beta|_p) \).

D. For function \( \phi_3(r) \), we have
\[ \lim_{n \to \infty} \phi_3^n(r) = \begin{cases} r, & for \ all \ r \neq \delta, \\ \phi_3(\delta), & if \ r = \delta. \end{cases} \]

Proof. The proof consists simple analysis of the functions \( \phi_j(r) \) using their graphs.

The following lemma is obvious:

Lemma 3.4. The dynamical system generated by \( \psi_k(r), (k = 1, 2, 3) \) has the following properties:
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(i) Fix($\psi_k$) = \{0\} $\cup$

\[
\begin{cases}
{\{r : r > \sqrt{\alpha}\} \cup \{\alpha^* : \text{if } \alpha^* = \alpha\}, \text{ for } \sqrt{\beta} < \delta < \sqrt{\alpha},} \\
{\{r : r > \delta\} \cup \{\hat{\delta} : \text{if } \hat{\delta} = \delta\}, \text{ for } \sqrt{\beta} < \delta = \sqrt{\alpha} \text{ and } \delta^2 > |\delta^2 + \alpha |_p,} \\
{\{r : r \neq \delta\} \cup \{\hat{\delta} : \text{if } \hat{\delta} = \delta\}, \text{ for } \sqrt{\beta} < \sqrt{\alpha} \leq \delta \text{ and } \delta^2 = |\delta^2 + \alpha |_p,}
\end{cases}
\]

(ii) For function $\psi_1(r)$, we have

\[
\lim_{n \to \infty} \psi_1^n(r) = \begin{cases}
0, & \text{if } r = 0, \\
E, & \text{for all } 0 < r \leq \sqrt{\alpha}, \\
r, & \text{for all } r > \sqrt{\alpha},
\end{cases}
\]

where $E = (\sqrt{\alpha}, \alpha/\delta) \cup \{\delta^*\}$.

(iii) For function $\psi_2(r)$

\[
\lim_{n \to \infty} \psi_2^n(r) = \begin{cases}
0, & \text{for all } r < \delta, \\
\psi_2(\delta), & \text{if } r = \delta, \\
r, & \text{for all } r > \delta,
\end{cases}
\]

(iv) For function $\psi_3(r)$

\[
\lim_{n \to \infty} \psi_3^n(r) = \begin{cases}
r, & \text{for all } r \neq \delta, \\
\psi_3(\hat{\delta}), & \text{if } r = \delta.
\end{cases}
\]

Now we shall apply these lemmas to the study of the $p$-adic dynamical system generated by $f$.

Using Lemma 3.1 and Lemma 3.2 we obtain the following

**Theorem 3.5.** If $\delta \leq \min\{\sqrt{\alpha}, \sqrt{\beta}\}$ and $x \in S_r(x_0)$, then the $p$-adic dynamical system generated by $f$ has the following properties:

1. The following spheres are invariant with respect to $f$:

   \[
   S_r(x_0), \quad \text{if } r > \sqrt{\max\{\alpha, \beta\}} \text{ and } \alpha \neq \beta;
   \]

   \[
   S_r(x_0), \quad \text{if } r \neq \sqrt{\alpha}, \text{ and } \alpha = \beta;
   \]

2. For $\alpha < \beta$, we have

   \[
   \lim_{n \to \infty} f^n(x) = x_0, \quad \text{for all } r < \sqrt{\beta},
   \]

   \[
   f(S_r(x_0) \setminus P) \subset S_r(x_0), \quad \text{for all } r > \sqrt{\beta},
   \]

   \[
   \lim_{n \to \infty} f^n(x) \in \begin{cases}
   S_{\varphi_1(\delta^*)}(x_0), & \text{if } \delta < \sqrt{\alpha}, \ r = \sqrt{\beta};
   \\
   S_{\varphi_3(\delta^*)}(x_0), & \text{if } \delta = \sqrt{\alpha}, \ r = \sqrt{\beta};
   \end{cases}
   \]

3. If $\alpha = \beta$, then

   \[
   f(S_r(x_0) \setminus P) \subset S_r(x_0), \quad \text{for all } r \neq \sqrt{\alpha},
   \]

   \[
   \lim_{n \to \infty} f^n(x) \in S_{\varphi_2(\delta)}(x_0), \quad \text{if } r = \sqrt{\alpha};
   \]
4. If $\alpha > \beta$, then
\[
\lim_{n \to \infty} f^n(x) \in S_\rho(x_0), \quad \rho \in \begin{cases} 
C_4, & \text{if } \delta < \sqrt{\beta}, \\
C_5, & \text{if } \delta = \sqrt{\beta},
\end{cases}, \quad 0 < r \leq \sqrt{\alpha},
\]
\[
f(S_r(x_0) \setminus P) \subset S_r(x_0), \quad \text{for all } r \neq \sqrt{\alpha}.
\]

By Lemma 3.1 and Lemma 3.3 we obtain the following

**Theorem 3.6.** If $\sqrt{\alpha} \leq \min\{\delta, \sqrt{\beta}\}$ and $x \in S_r(x_0)$, then the $p$-adic dynamical system generated by $f$ has the following properties:

A. The following spheres are invariant:
\[
S_r(x_0), \quad \text{if } \sqrt{\alpha} < \delta < \sqrt{\beta}, \quad r > \sqrt{\beta},
\]
\[
S_r(x_0), \quad \text{if } \sqrt{\alpha} \leq \sqrt{\beta} < \delta, \quad r \neq \delta,
\]
\[
S_r(x_0), \quad \text{if } \sqrt{\alpha} \leq \sqrt{\beta} = \delta, \quad \delta^2 = |\delta^2 + \beta_p|, \quad \text{and } r \neq \delta,
\]
\[
S_r(x_0), \quad \text{if } \sqrt{\alpha} < \sqrt{\beta} = \delta, \quad \delta^2 > |\delta^2 + \beta_p|, \quad \text{and } r > \delta.
\]

B. For $\sqrt{\alpha} < \delta < \sqrt{\beta}$, we have
\[
\lim_{n \to \infty} f^n(x) = x_0, \quad \text{for all } r < \sqrt{\beta},
\]
\[
f(S_r(x_0) \setminus P) \subset S_r(x_0), \quad \text{for all } r > \sqrt{\beta},
\]
\[
\lim_{n \to \infty} f^n(x) \in S_{\phi_1(\beta^*)}(x_0), \quad \text{if } r = \sqrt{\beta}.
\]

C. Let $\sqrt{\alpha} < \sqrt{\beta} = \delta$.

C.a) If $\delta^2 = |\delta^2 + \beta|_p$, then
\[
f(S_r(x_0) \setminus P) \subset S_r(x_0), \quad \text{for any } r \neq \delta,
\]
\[
\lim_{n \to \infty} f^n(x) \in S_{\phi_2(\beta^*)}(x_0), \quad \text{if } r = \delta.
\]

C.b) If $\delta^2 > |\delta^2 + \beta|_p$, then
\[
\lim_{n \to \infty} f^n(x) \in S_\mu(x_0), \quad \mu \in B, \quad \text{for any } 0 < r \leq \delta,
\]
\[
f(S_r(x_0) \setminus P) \subset S_r(x_0), \quad \text{for any } r > \delta;
\]

D. If $\sqrt{\alpha} \leq \sqrt{\beta} < \delta$, then
\[
f(S_r(x_0) \setminus P) \subset S_r(x_0), \quad \text{for any } r \neq \delta,
\]
\[
\lim_{n \to \infty} f^n(x) \in S_{\phi_3(\delta)}(x_0), \quad \text{if } r = \delta.
\]

By Lemma 3.1 and Lemma 3.4 we get

**Theorem 3.7.** If $\sqrt{\beta} \leq \min\{\delta, \sqrt{\alpha}\}$, and $x \in S_r(x_0)$, then the dynamical system generated by $f$ has the following properties:
(i) The following spheres are invariant:

\[ S_r(x_0), \text{ if } \delta < \sqrt{\alpha}, \quad r > \sqrt{\alpha}, \]
\[ S_r(x_0), \text{ if } \delta = \sqrt{\alpha}, \quad \delta^2 > |\delta^2 + \alpha|_p, \quad r > \delta, \]
\[ S_r(x_0), \text{ if } \delta \geq \sqrt{\alpha}, \quad \delta^2 = |\delta^2 + \beta|_p, \quad r \neq \delta; \]

(ii) Let \( \delta < \sqrt{\alpha} \). Then

\[ \lim_{n \to \infty} f^n(x) \in S_\nu(x_0), \quad \nu \in E, \text{ for any } 0 < r \leq \sqrt{\alpha}, \]

\[ f(S_r(x_0) \setminus \mathcal{P}) \subset S_r(x_0), \text{ for any } r > \sqrt{\alpha}; \]

(iii) If \( \delta = \sqrt{\alpha} \) and \( \delta^2 > |\delta^2 + \alpha|_p \), then

\[ \lim_{n \to \infty} f^n(x) = x_0, \text{ for all } r < \delta, \]

\[ \lim_{n \to \infty} f^n(x) \in S_{\psi_2(\delta)}(x_0), \text{ if } r = \delta; \]

\[ f(S_r(x_0) \setminus \mathcal{P}) \subset S_r(x_0), \text{ for any } r > \delta; \]

(iv) If \( \delta \geq \sqrt{\alpha} \) and \( \delta^2 = |\delta^2 + \alpha|_p \), then

\[ \lim_{n \to \infty} f^n(x) \in S_{\psi_3(\delta)}(x_0), \text{ if } r = \delta; \]

\[ f(S_r(x_0) \setminus \mathcal{P}) \subset S_r(x_0), \text{ for any } r \neq \delta. \]

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