Categorification of some level two representations of $\mathfrak{sl}_n$

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1 Introduction

Let $W$ be the fundamental $n$-dimensional representation of the Lie algebra $\mathfrak{sl}_n$. Let $\omega_1, \ldots, \omega_{n-1}$ be the fundamental dominant weights of $\mathfrak{sl}_n$, the highest weights of the exterior powers $\Lambda^k W$ of $W$. For the rest of this paper fix $k$ between 1 and $n-1$ and denote by $V$ the irreducible representation with the highest weight $2\omega_k$. This representation is a direct summand of $S^2(\Lambda^k W)$.

Decompose $V$ into weight spaces, $V = \oplus \lambda V_\lambda$. We call $\lambda$ admissible if $V_\lambda \neq 0$. Admissible weights are enumerated by sequences

$$\lambda = (\lambda_1, \ldots, \lambda_n), 0 \leq \lambda_i \leq 2, \sum_{i=1}^n \lambda_i = 2k.$$

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For an admissible \( \lambda \) let \( m = m(\lambda) \) be one-half of the number of 1s in the sequence \((\lambda_1, \ldots, \lambda_{2n})\). \( m(\lambda) \) is an integer between 0 and \( \min(k, n - k) \). The dimension of \( V_\lambda \) depends only on \( m \) and is the \( m \)-th Catalan number

\[
c_m = \frac{1}{m+1} \binom{2m}{m}.
\]

\( E_i \in \mathfrak{sl}_n \) maps the weight space \( V_\lambda \) to \( V_{\lambda + \epsilon_i} \) where \( \epsilon_i = (0, \ldots, 0, 1, -1, 0, \ldots, 0) \), and 1, -1 are the \( i \)-th and \((i + 1)\)-th entries. \( E_i V_\lambda = 0 \) if \( \lambda + \epsilon_i \) is not admissible. \( F_i \in \mathfrak{sl}_n \) maps \( V_\lambda \) to \( V_{\lambda - \epsilon_i} \).

A level two representation is an irreducible representation with the highest weight \( \omega_i + \omega_j \). In particular, \( V \) is a level two representation. Green [G] found a graphical interpretation of Lusztig-Kashiwara canonical basis [L1], [Ka], [L2] in level two representations of \( U_q(\mathfrak{sl}_n) \) via a calculus of planar diagrams similar to the one of Temperley-Lieb.

In this paper we categorify \( V \) and Green’s construction of the canonical basis in \( V \) with the help of rings \( H^m \) introduced in [K]. Let \( \mathcal{C} \) be the direct sum of categories of \( H^m(\lambda) \)-modules, over admissible \( \lambda \). The Grothendieck group of \( H^m(\lambda) \)-mod is naturally isomorphic to the weight space \( V_\lambda \) (more precisely, to a \( \mathbb{Z} \)-lattice in the latter), and the rank of the Grothendieck group is the \( m \)-th Catalan number (the dimension of \( V_\lambda \)). We construct exact functors \( \mathcal{E}_i \) and \( \mathcal{F}_i \) in the category \( \mathcal{C} \) that in the Grothendieck group descend to \( E_i \) and \( F_i \) acting on \( V \). Various structures in \( V \) lift to fancier structures in \( \mathcal{C} \).

The symmetric group action on \( V \) lifts to a braid group action in the derived category of \( \mathcal{C} \). The contravariant symmetric bilinear form on \( V \) is given by dimensions of Hom spaces between projective modules in \( \mathcal{C} \). Contravariance, meaning \((E_i v, w) = (v, F_i w) \) for \( v, w \in V \), turns into the property that the functor \( \mathcal{E}_i \) is both left and right adjoint to \( \mathcal{F}_i \).

Rings \( H^m \) are naturally graded and throughout the paper we work with the categories of graded \( H^m \)-modules. The Grothendieck groups are then \( \mathbb{Z}[q, q^{-1}] \)-modules (the grading shift functor descends to the multiplication by a formal variable \( q \) in the Grothendieck group), and assemble into a representation (also denoted \( V \)) of the quantum enveloping algebra \( U = U_q(\mathfrak{sl}_n) \). Indecomposable projective modules in \( \mathcal{C} \) descend to the canonical basis in \( V \).

All of our results specialize easily to the \( q = 1 \) case, by working with the category of ungraded modules. This specialization was sketched two paragraphs above, the details are left out. Some results become simpler when the grading is ignored. The Hom form on the Grothendieck group is semilinear with the grading, and bilinear without it. Functors \( \mathcal{E}_i \) and \( \mathcal{F}_i \) are left and right adjoint in the category of ungraded modules, while in the category of graded modules \( \mathcal{E}_i \) and \( \mathcal{F}_i \) are left and right adjoint only up to
shifts in the grading that depend on \( \lambda \) (see proposition 3).

The category \( H^m\)-mod is a bicategorification of the \( m \)-th Catalan number
\[
c_m = \frac{1}{m+1} \binom{2m}{m}.
\]
Informally, to categorify is to upgrade a number to a vector space, or to upgrade a vector space to a category. The number becomes the dimension of the vector space; the vector space becomes the Grothendieck group of the category (we should tensor the Grothendieck group with a field to get a vector space). To bicategorify is to upgrade a number to a category, so that the number becomes the rank of the Grothendieck group of the category.

\[
\begin{array}{ccc}
\text{Number} & \xleftrightarrow{\text{Categorification}} & \text{Vector space} \\
& \xrightarrow{\text{dimension}} & \\
& \xleftrightarrow{\text{Categorification}} & \text{Category} \\
& \xrightarrow{\text{Grothendieck group}} & \\
\end{array}
\]

Any nonnegative integer is, of course, a dimension of some vector space, but just picking a vector space is not a categorification. What we want is a vector space that appears naturally and comes with a bonus: an algebra structure, a group action, etc. Some examples:

I) Categorifications of \( n! \)

- Cohomology ring of the flag variety of \( \mathbb{C}^n \). Benefits include grading, commutative multiplication, the basis of Schubert cells, action of the symmetric group.

- Group algebra of the symmetric group \( S_n \).

- Other categorifications: the Hecke algebra, nilCoxeter and nilHecke algebras, quantum cohomology ring of the flag variety.

An example of a bicategorification of \( n! \) is a regular block \( \mathcal{O}_{\text{reg}} \) of the highest weight category of \( \mathfrak{sl}_n \)-modules:

\[
\begin{array}{ccc}
n! & \xleftrightarrow{\text{Categorification}} & \text{Regular representation} \\
& \xrightarrow{\text{dimension}} & \text{of the symmetric group} \\
& \xleftrightarrow{\text{Categorification}} & \mathcal{O}_{\text{reg}} \\
& \xrightarrow{\text{Grothendieck group}} & \\
\end{array}
\]

The action of the symmetric group on the regular representation lifts to a braid group action in the derived category of \( \mathcal{O}_{\text{reg}} \). The braid group action extends to a representation of the braid cobordism category (objects are braids with \( n \) strands and morphisms are cobordisms in \( \mathbb{R}^4 \) between braids) in \( D^b(\mathcal{O}_{\text{reg}}) \), by assigning certain natural transformations to braid cobordisms (see Rouquier [R]).

II) Categorifications of the \( m \)-th Catalan number
• $\text{Inv}_{\mathfrak{sl}_2}(L^\otimes 2m)$, the space of $\mathfrak{sl}_2$-invariants in the $2m$-th tensor power of the two-dimensional "defining" representation $L$ of $\mathfrak{sl}_2$.

• Irreducible representation of $S_{2m}$ associated to the partition $(m, m)$. This categorification is equivalent to the previous one.

• $\text{Inv}_{U_q(\mathfrak{sl}_2)}(L^\otimes 2m)$, the space of invariants in the $2m$-th tensor power of the two-dimensional "defining" representation of the quantum group $U_q(\mathfrak{sl}_2)$, for generic $q$.

• The Temperley-Lieb algebra (this categorification is equivalent to the previous one).

• The weight space $V_\lambda$ with $m = m(\lambda)$.

• The quotient of $\mathbb{C}[x_1, \ldots, x_m]$ by the ideal generated by all quasisymmetric functions in the variables $x_1, \ldots, x_n$ with 0 constant term [ABB].

• the subspace of $S_n$-alternating elements in the space of diagonal harmonics [H].

The Grothendieck group of the category of $H^m$-modules (without grading) is naturally isomorphic (after tensoring with $\mathbb{C}$) to the space of $\mathfrak{sl}_2$-invariants in the $2m$-th tensor power of the fundamental representation $L$ of $\mathfrak{sl}_2$:

$$K(H^m-\text{mod}) \otimes_\mathbb{Z} \mathbb{C} \cong \text{Inv}_{\mathfrak{sl}_2}(L^\otimes 2m)$$

The category of $H^m$-modules can be viewed as a bicategorification of the $m$-th Catalan number:

$$\begin{pmatrix} 2m \\ m \end{pmatrix} \overset{\text{Categorification}}{\xrightarrow{\text{dimension}}} \overset{\text{Categorification}}{\xRightarrow{\text{Grothendieck group}}} \text{Inv}_{\mathfrak{sl}_2}(L^\otimes 2m) \overset{\text{Categorification}}{\xleftarrow{\text{dimension}}} H^m-\text{mod}$$

## 2 Flat tangles, rings $H^m$, and bimodules

We recall some definitions from [K]. Denote by $\mathcal{A}$ the cohomology ring $H^*(S^2, \mathbb{Z})$ of the 2-sphere. $\mathcal{A} \cong \mathbb{Z}[X]/(X^2)$, where $X$ is a generator of $H^2(S^2, \mathbb{Z})$.

$\mathcal{A}$ is a commutative Frobenius ring, with the nondegenerate trace form

$$\text{tr} : \mathcal{A} \to \mathbb{Z}, \quad \text{tr}(1) = 0, \quad \text{tr}(X) = 1,$$
We make $\mathcal{A}$ into a graded ring, by placing $1 \in \mathcal{A}$ in degree $-1$ and $X$ in degree $1$. The multiplication map $\mathcal{A}^\otimes 2 \to \mathcal{A}$ has degree one.

We assign to $\mathcal{A}$ a 2-dimensional topological quantum field theory $\mathcal{F}$, a functor from the category of oriented $(1+1)$-cobordisms to the category of abelian groups. $\mathcal{F}$ associates

- $\mathcal{A}^\otimes i$ to a disjoint union of $i$ circles,
- the multiplication map $\mathcal{A}^\otimes 2 \to \mathcal{A}$ to the three-holed sphere viewed as a cobordism from two circles to one circle.
- the comultiplication
  \[
  \Delta : \mathcal{A} \to \mathcal{A}^\otimes 2, \quad \Delta(1) = 1 \otimes X + X \otimes 1, \quad \Delta(X) = X \otimes X
  \]
to the three-holed sphere viewed as a cobordism from one circle to two circles.
- either trace or the unit map to the disk (depending on whether we consider the disk as a cobordism from one circle to the empty manifold or vice versa).

Let $B^m$ be the set of matchings of integers from 1 to $2m$ without any quadruple $i < j < l < p$ such that $i$ is matched with $l$ and $j$ with $p$. $B^m$ has a geometric interpretation as the set of crossingless matchings of $2m$ points. Figure 1 shows elements of $B^2$.

![Crossingless Matchings](image)

Figure 1: crossingless matchings $\{(12), (34)\}$ and $\{(14), (23)\}$

For $a, b \in B^m$ denote by $W(b)$ the reflection of $b$ about the horizontal axis and by $W(b)a$ the closed 1-manifold obtained by gluing $W(b)$ and $a$ along their boundaries, see figure 2.

$\mathcal{F}(W(b)a)$ is a graded abelian group, isomorphic to $\mathcal{A}^\otimes I$, where $I$ is the set of connected components (circles) of $W(b)a$. For $a, b, c \in B^m$ there is a canonical cobordism from $W(c)bW(b)a$ to $W(c)a$ given by ”contracting” $b$ with $W(b)$, see figure 3 for an example.

This cobordism induces a homomorphism of abelian groups

\[
\mathcal{F}(W(c)b) \otimes \mathcal{F}(W(b)a) \to \mathcal{F}(W(c)a) \quad (1)
\]
If $M$ is a graded abelian group, denote by $M\{i\}$ the graded abelian group obtained by shifting the grading of $M$ up by $i$. Let

$$H^m \overset{\text{def}}{=} \bigoplus_{a,b \in B^m} b(H^m)_a, \quad b(H^m)_a \overset{\text{def}}{=} \mathcal{F}(W(b)a\{m\}).$$

Homomorphisms (II), over all $a,b,c$, define an associative multiplication in $H^m$ (the product $d(H^m)_c \otimes b(H^m)_a \rightarrow d(H^m)_a$ is set to zero if $b \neq c$). The grading shift $\{m\}$ the multiplication grading-preserving.

$a(H^m)_a$ is a subring of $H^m$, isomorphic to $A^\otimes m$. Its element $1_a \overset{\text{def}}{=} 1^\otimes n \in A^\otimes m$ is an idempotent in $H^m$. The sum $\sum_a 1_a$ is the unit element of $H^m$. Notice that $b(H^m)_a = 1_b H^m 1_a$.

Suppose we are given a diagram of a system of disjoint arcs and circles in a horizontal plane strip, see figure 4.

![Figure 4: A flat (2,1)-tangle](image_url)

We only consider diagrams with even number of bottom endpoints, and will refer to such a digram with $2m$ bottom and $2l$ top endpoints as a flat
(l, m)-tangle. To a flat (l, m)-tangle $T$ we associate an $(H^l, H^m)$-bimodule $\mathcal{F}(T)$:

$$\mathcal{F}(T) \overset{\text{def}}{=} \bigoplus_{b \in B^l, a \in B^m} \mathcal{F}(W(b)Ta)\{m\}.$$ 

Since the diagram $W(b)Ta$ is a closed 1-manifold, we can apply $\mathcal{F}$ to it. $\mathcal{F}(W(b)Ta) \cong A^\otimes r$, where $r$ is the number of circles in $W(b)Ta$. Notice that $r$ depends on the choice of $a$ and $b$. The ring $H^l$ acts on $T$ on the left via maps

$$\mathcal{F}(W(c)b) \otimes \mathcal{F}(W(b)Ta) \longrightarrow \mathcal{F}(W(c)Ta)$$

where $c, b \in B^l$ and $a \in B^m$, and the map is induced by the cobordism from $W(c)bW(b)Ta$ to $W(c)Ta$ which contracts $b$ with $W(b)$ (see [K, Section 2.7] for more details).

A flat $(r, l)$-tangle $T_1$ can be composed with a flat $(l, m)$-tangle $T_2$ by identifying the bottom endpoints of $T_1$ with the top endpoints of $T_2$ to produce a flat $(r, m)$-tangle $T_1 T_2$. We recall the following result [K, Theorem 1].

**Proposition 1** There is a canonical isomorphism of $(H^r, H^m)$-bimodules

$$\mathcal{F}(T_1 T_2) \cong \mathcal{F}(T_1) \otimes_{H^l} \mathcal{F}(T_2).$$

When defining the set $B^m$ we did not specify the positions of the arc’s endpoints on the horizontal line. We do so now. For each sequence $s = (s_1, \ldots, s_{2m}), s_1 < s_2 < \cdots < s_{2m}$ of $2m$ points on the real line we can consider crossingless matchings of $s_1, \ldots, s_{2m}$. The set of such matchings is canonically isomorphic to $B^m$, and we can repeat our definition of $H^m$ and get a ring, $H(s)$, canonically isomorphic to $H^m$.

$\mathcal{F}(T)$, associated to a flat tangle $T$ with a bottom endpoints sequence $s = (s_1, \ldots, s_{2m})$ and a top endpoints sequence $t = (t_1, \ldots, t_{2l})$, is naturally an $(H(t), H(s))$-bimodule. $\mathcal{F}(T)$ is also, of course, an $(H^l, H^m)$-bimodule. Working with sequences $s$ and $t$ is simply a way to keep track of the real coordinates of the endpoints.

Consider an admissible weight $\lambda$. Let $\lambda_{s_1} = \lambda_{s_2} = \cdots = \lambda_{s_{2m}} = 1$, that is, $s_1, \ldots, s_{2m}$ are the indices of those coefficients of $\lambda$ that are equal to 1. Let $s(\lambda) = (s_1, \ldots, s_{2m})$. We denote the ring $H(s(\lambda))$ simply by $H_\lambda$. Notice that $H_\lambda \cong H^{m(\lambda)}$.

**Example:** If $\lambda = (0, 2, 1, 1, 0, 1)$ then $s(\lambda) = (3, 4, 5, 7)$ and $H_\lambda \cong H^2$. 

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Suppose that \( \lambda_i = 1 \) and \( \lambda_{i+1} \neq 1 \) (so that \( \lambda_{i+1} \) is either 0 or 2). Let \( \mu = (\mu_1, \ldots, \mu_{2m}) \) be the transposition of the \( i \)-th and \( i+1 \)-th coefficients of \( \lambda \) (so that \( \mu_j = \lambda_j \) if \( j \neq i, i+1, \mu_{i+1} = \lambda_i \), and \( \mu_i = \lambda_{i+1} \)). Note that \( s(\mu) \) is obtained from \( s(\lambda) \) by changing \( i \in s(\lambda), i = s_j \) for some \( j \), to \( i+1 \).

To such \( \lambda \) and \( i \) we assign an \((H_\mu, H_\lambda)\) bimodule \( F(\text{Id}_{i+1}^i) \) where \( \text{Id}_{i+1}^i \) is the flat tangle depicted on the left of figure 5. The bimodule \( F(\text{Id}_{i+1}^i) \) defines the obvious isomorphism of rings \( H_\lambda \) and \( H_\mu \).

![Figure 5: Flat tangles Id\_i\^i+1 and Id\_i+1\^i](image)

For \( \lambda \) and \( i \) such that \( \lambda_{i+1} = 1 \) and \( \lambda_i \neq 1 \) we similarly construct an \((H_\mu, H_\lambda)\)-bimodule \( F(\text{Id}_{i+1}^i) \), where \( \mu = (\lambda_1, \ldots, \lambda_{i+1}, \lambda_i, \ldots, \lambda_n) \).

For \( \lambda \) and \( i \) such that \( \lambda_i = \lambda_{i+1} = 1 \) we have an \((H_\mu, H_\lambda)\)-bimodule \( F(\cap_{i,i+1}) \) where \( \cap_{i,i+1} \) is the diagram on the right of figure 6 and \( \mu = (\lambda_1, \ldots, \lambda_{i-1}, 0, 2, \lambda_{i+2}, \ldots, \lambda_n) \) or \( \mu = (\lambda_1, \ldots, \lambda_{i-1}, 2, 0, \lambda_{i+2}, \ldots, \lambda_n) \). Bimodules \( \cup_{i,i+1} \) are defined likewise.

![Figure 6: Flat tangles \cup\_i\^i+1 and \cap\_i\^i+1](image)

3 Category \( \mathcal{C} \) and functors \( \mathcal{E}_i, \mathcal{F}_i \)

For an admissible weight \( \lambda \) let \( \mathcal{C}(\lambda) \) be the category \( H_\lambda\)-mod of graded finitely-generated \( H_\lambda \)-modules. \( \mathcal{C}(\lambda) \) is equivalent to the category of (graded finitely-generated) \( H^{m(\lambda)} \)-modules where, recall, \( 2m(\lambda) \) is the number of 1’s in \( \lambda \). For instance, if \( \lambda \) consists entirely of 0’s and 2’s, then \( H_\lambda \cong \mathbb{Z} \) and \( \mathcal{C}(\lambda) \) is equivalent to the category of finitely-generated graded abelian groups.
Let the category $\mathcal{C}$ be the direct sum of $\mathcal{C}(\lambda)$, over all admissible $\lambda$:

$$\mathcal{C} \equiv \bigoplus_{\lambda} \mathcal{C}(\lambda).$$

Define the functor $E_i : \mathcal{C} \rightarrow \mathcal{C}$ as the sum, over all admissible $\lambda$, of the following functors $\mathcal{C}(\lambda) \rightarrow \mathcal{C}(\lambda + \epsilon_i)$:

- the zero functor if $\lambda + \epsilon_i$ is not admissible,
- tensoring with the bimodule $\mathcal{F}(\text{Id}^{i+1}_i)$ if $(\lambda_i, \lambda_{i+1}) = (1, 2),$
- tensoring with the bimodule $\mathcal{F}(\text{Id}^{i+1}_i)$ if $(\lambda_i, \lambda_{i+1}) = (0, 1),$
- tensoring with the bimodule $\mathcal{F}(\cup^{i+1}_i)$ if $(\lambda_i, \lambda_{i+1}) = (0, 2),$
- tensoring with the bimodule $\mathcal{F}(\cap^{i+1}_i)$ if $(\lambda_i, \lambda_{i+1}) = (1, 1).$

For instance, the flat tangle $\text{Id}^{i+1}_i$ shifts a bottom endpoint with coordinate $i$ to the top endpoint with coordinate $i + 1$, so that $\mathcal{F}(\text{Id}^{i+1}_i)$ is an $(H_{\lambda + \epsilon_i}, H_\lambda)$-bimodule for any admissible $\lambda$ with $(\lambda_i, \lambda_{i+1}) = (1, 2)$.

Define the functor $F_i : \mathcal{C} \rightarrow \mathcal{C}$ as the sum, over all admissible $\lambda$, of the following functors $\mathcal{C}(\lambda) \rightarrow \mathcal{C}(\lambda - \epsilon_i)$:

- the zero functor if $\lambda - \epsilon_i$ is not admissible,
- tensoring with the bimodule $\mathcal{F}(\text{Id}^{i+1}_i)$ if $(\lambda_i, \lambda_{i+1}) = (1, 0),$
- tensoring with the bimodule $\mathcal{F}(\text{Id}^{i+1}_i)$ if $(\lambda_i, \lambda_{i+1}) = (2, 1),$
- tensoring with the bimodule $\mathcal{F}(\cup^{i+1}_i)$ if $(\lambda_i, \lambda_{i+1}) = (2, 0),$
- tensoring with the bimodule $\mathcal{F}(\cap^{i+1}_i)$ if $(\lambda_i, \lambda_{i+1}) = (1, 1).$

**Warning:** Although the notations $\mathcal{F}$ and $\mathcal{F}_i$ are similar, the two are no more related than $\mathcal{F}$ and $E_i$. The similarity is the negative side effect of making our notations compatible with those of both [K] and [KH].

Let $K_i : \mathcal{C} \rightarrow \mathcal{C}$ be the functor that shifts the grading of $M \in \mathcal{C}(\lambda)$ up by $\lambda_i - \lambda_{i+1}$:

$$K_i(M) \equiv M\{\lambda_i - \lambda_{i+1}\}.$$
Proposition 2  There are functor isomorphisms
\[ K_i K_i^{-1} \cong \text{Id} \cong K_i^{-1} K_i, \]
\[ K_i E_j \cong K_j K_i, \]
\[ K_i E_j \cong E_i K_i \{c_{i,j}\}, \]
\[ K_i F_j \cong F_j K_i \{c_{i,j}\}, \]
\[ E_i F_j \cong F_j E_i \text{ if } i \neq j, \]
\[ E_i E_j \cong E_j E_i \text{ if } |i - j| > 1, \]
\[ F_i F_j \cong F_j F_i \text{ if } |i - j| > 1, \]
\[ E_i^2 E_j \oplus E_i E_j^2 \cong E_i E_j E_i \{1\} \oplus E_i E_j E_i \{-1\} \text{ if } j = i \pm 1 \]
\[ F_i^2 F_j \oplus F_j F_i^2 \cong F_i F_j F_i \{1\} \oplus F_i F_j F_i \{-1\} \text{ if } j = i \pm 1 \]

where
\[ c_{i,j} = \begin{cases} 
2 & \text{if } j = i, \\
-1 & \text{if } j = i \pm 1, \\
0 & \text{if } |j - i| > 1.
\end{cases} \]

Proposition 3  For any admissible \( \lambda \) there is an isomorphism of functors in the category \( C(\lambda) \)
\[ E_i F_i \cong F_i E_i \oplus \text{Id}\{1\} \oplus \text{Id}\{-1\} \text{ if } (\lambda_i, \lambda_{i+1}) = (2, 0), \]
\[ E_i F_i \cong F_i E_i \oplus \text{Id} \text{ if } \lambda_i - \lambda_{i+1} = 1, \]
\[ E_i F_i \cong F_i E_i \text{ if } \lambda_i = \lambda_{i+1}, \]
\[ E_i F_i \oplus \text{Id} \cong F_i E_i \text{ if } \lambda_i - \lambda_{i+1} = -1, \]
\[ E_i F_i \oplus \text{Id}\{1\} \oplus \text{Id}\{-1\} \cong F_i E_i \text{ if } (\lambda_i, \lambda_{i+1}) = (0, 2). \]

Proof of Proposition 2: The top four isomorphisms in (2) are obvious. The next three isomorphisms are clear if \(|i - j| > 1\), since functors \( E_i \) and \( F_i \) (respectively \( E_j \) and \( F_j \)) come from bimodules assigned to flat tangles that are nontrivial only in the area with the \( x \)-coordinate between \( i \) and \( i + 1 \) (respectively \( j \) and \( j + 1 \)). Composition of such flat tangles is commutative (see example in figures 7 and 8).

To check commutativity \( E_i F_{i+1} \cong F_{i+1} E_i \) (and its variations) one considers all possible triples \((\lambda_i, \lambda_{i+1}, \lambda_{i+2})\) and draws flat tangles that define functors \( E_i F_{i+1} \) and \( F_{i+1} E_i \) in each case. Unless \( \lambda_i < 2, \lambda_{i+1} = 2, \) and \( \lambda_{i+2} > 0 \) the sequence \( \lambda + \epsilon_i - \epsilon_{i+1} \) is not admissible, so that there are only four nontrivial cases. The case \((0, 2, 1)\) is depicted in figure 9, other cases are similar and left to the reader.
The last two isomorphisms in (2) are also checked case by case. To illustrate, we verify the isomorphism
\[
\mathcal{E}_i^2 \mathcal{E}_{i+1} \oplus \mathcal{E}_{i+1} \mathcal{E}_i^2 \cong \mathcal{E}_i \mathcal{E}_{i+1} \mathcal{E}_i \{1\} \oplus \mathcal{E}_i \mathcal{E}_{i+1} \mathcal{E}_i \{-1\}
\]
when \((\lambda_i, \lambda_{i+1}, \lambda_{i+2}) = (0,1,2)\). The functor \(\mathcal{E}_{i+1} \mathcal{E}_i^2\) is zero since \(\lambda + 2\epsilon_i\) is not admissible. Functors \(\mathcal{E}_i^2 \mathcal{E}_{i+1}\) and \(\mathcal{E}_i \mathcal{E}_{i+1} \mathcal{E}_i\) are assigned to diagrams in figure 10.

If flat tangle \(T_2\) is obtained from a flat tangle \(T_1\) by adding an extra circle, then there is an isomorphism of bimodules
\[
\mathcal{F}(T_2) \cong \mathcal{F}(T_1) \otimes A \cong \mathcal{F}(T_1) \{1\} \oplus \mathcal{F}(T_1) \{-1\}.
\]

The right diagram in figure 10 after the circle is removed is isotopic to the left diagram. Hence,
\[
\mathcal{E}_i^2 \mathcal{E}_{i+1} \cong \mathcal{E}_i \mathcal{E}_{i+1} \mathcal{E}_i \{1\} \oplus \mathcal{E}_i \mathcal{E}_{i+1} \mathcal{E}_i \{-1\}
\]
Figure 9: Flat tangles for functors $F_{i+1}E_i$ and $E_iF_{i+1}$ and the triple $(0, 2, 1)$

Figure 10: Flat tangles for functors $E_i^2E_{i+1}$ and $E_iE_{i+1}E_i$ and the triple $(0, 1, 2)$

when $(\lambda_i, \lambda_{i+1}, \lambda_{i+2}) = (0, 1, 2)$. Other cases, as well as proof of proposition 3, are left to the reader. □

Functor isomorphisms of proposition 3 categorify the quantum group relation $E_iF_i - F_iE_i = K_i^{-1} - K^{-1}_i$, while those of proposition 2 category all other defining relations in $U_q(\mathfrak{sl}_2)$. We recall the defining relations of $U = U_q(\mathfrak{sl}_2)$:

\[
\begin{align*}
K_iK_i^{-1} &= 1 = K_i^{-1}K_i, \\
K_iK_j &= K_jK_i, \\
K_iE_j &= q^{\epsilon_{i,j}}E_jK_i, \\
K_iF_j &= q^{-\epsilon_{i,j}}F_jK_i, \\
E_iF_j - F_jE_i &= \delta_{i,j}\frac{K_i - K_i^{-1}}{q - q^{-1}}, \\
E_iE_j &= E_jE_i \text{ if } |i - j| > 1, \\
F_iF_j &= F_jF_i \text{ if } |i - j| > 1, \\
E_i^2E_{i\pm1} - (q + q^{-1})E_iE_{i\pm1}E_i + E_{i\pm1}E_i^2 &= 0, \\
F_i^2F_{i\pm1} - (q + q^{-1})F_iF_{i\pm1}F_i + F_{i\pm1}F_i^2 &= 0.
\end{align*}
\]
The quantum divided powers are defined by

\[ E_i^{(j)} = \frac{E_i^j}{[j]!} \quad \text{and} \quad F_i^{(j)} = \frac{F_i^j}{[j]!}, \]

where \([j]! = [1][2] \ldots [j]\) and \([j] = \frac{q^j - q^{-j}}{q - q^{-1}}\). In the representation \(V\) operators \(E_i^j, F_i^j\) are zero for \(j > 2\).

Quantum divided powers \(E_i^{(2)}\) and \(F_i^{(2)}\) admit the following categorification. Note that \(E_i^{(2)} : V_\lambda \to V_{\lambda+2\epsilon_i}\) is nonzero only if \((\lambda_i, \lambda_{i+1}) = (0, 2)\). In the latter case rings \(H_\lambda\) and \(H_{\lambda+2\epsilon_i}\) are canonically isomorphic and we define

\[ E_i^{(2)} : C(\lambda) \to C(\lambda + 2\epsilon_i), \quad F_i^{(2)} : C(\lambda + 2\epsilon_i) \to C(\lambda) \]

as the mutually inverse equivalences of categories induced by this isomorphism. For other \(\lambda\)'s we set the functors to zero.

**Proposition 4** There are functor isomorphisms

\[ E_i^2 \cong E_i^{(2)}\{1\} \oplus E_i^{(2)}\{-1\}, \quad (5) \]
\[ F_i^2 \cong F_i^{(2)}\{1\} \oplus F_i^{(2)}\{-1\}, \quad (6) \]
\[ E_iE_jE_i \cong E_i^{(2)}E_j \oplus E_jE_i^{(2)} \quad \text{if} \quad j = i \pm 1, \quad (7) \]
\[ F_iF_jF_i \cong F_i^{(2)}F_j \oplus F_jF_i^{(2)} \quad \text{if} \quad j = i \pm 1. \quad (8) \]

Proof is straightforward. Isomorphisms (5) and (6) simplify the last two isomorphisms in (4).

**4 The structure of \(C\)**

**Grothendieck group**

The Grothendieck group of \(C\) is \(\mathbb{Z}[q, q^{-1}]\)-module, with multiplication by \(q\) corresponding to the grading shift, \([M\{1\}] = q[M]\), where we denote by \([M]\) the image of the module \(M\) in the Grothendieck group \(K(C)\).

Functors \(E_i, F_i\) and \(K_i\) are exact and commute with \(\{1\}\). Therefore, they descend to \(\mathbb{Z}[q, q^{-1}]\)-linear endomorphisms \([E_i], [F_i]\) and \([K_i]\) of \(K(C)\). Functor isomorphisms (5) and (6) descend to quantum group relations (4) between \([E_i], [F_i]\) and \([K_i]\) in the Grothendieck group \(K(C)\). Therefore, the Grothendieck group is naturally a \(U\)-module. For accuracy, let’s view \(U\) as an algebra over \(\mathbb{Q}(q)\), the field of rational functions in an indeterminate \(q\) with rational
coefficients. To make $K(C)$ into a $U$-module we tensor it with $Q(q)$ over $\mathbb{Z}[q, q^{-1}]$. Recall that $V$ denotes the irreducible representation of $U$ with the highest weight $2\omega_k$. Choose a highest weight vector $\eta \in V$ (its weight is $2\omega_k = (2^k 0^{n-k})$.)

**Proposition 5** The Grothendieck group of $C$ is isomorphic to the irreducible representation $V$ of $U$ with highest weight $2\omega_k$:

$$K(C) \otimes_{\mathbb{Z}[q, q^{-1}]} Q(q) \cong V.$$  \hfill (9)

Clearly, $K(C) \otimes_{\mathbb{Z}[q, q^{-1}]} Q(q)$ is a representation of $U$. Why is it irreducible and isomorphic to $V$? For instance, because dimensions of its weight spaces equal dimensions of weight spaces of $V$ (equal Catalan numbers). Dimensions of weight spaces of $V$ can be computed via the Weyl character formula, or extracted from [G] which explicitly describes all level 2 representations of $\mathfrak{sl}_n$, including $V$.

$C(2\omega_k)$ is isomorphic to the category of graded finitely-generated abelian groups. Let $Q_{2\omega_k}$ be the object of $C(2\omega_k)$ which is $\mathbb{Z}$ is degree 0.

We fix isomorphism (9) such that $[Q_{2\omega_k}]$ is taken to $\eta \in V$.

For $a \in B^m$ denote by $\mathbb{Z}(a)$ the graded $H^m$-module isomorphic as a graded abelian group to $\mathbb{Z}$ (placed in degree 0), with the idempotent $1_a \in H^m$ acting as identity and $1_b \mathbb{Z}(a) = 0$ for $b \neq a$. These modules are analogous to simple modules for finite-dimensional algebras over a field, in the sense that after tensoring $H^m$ and $\mathbb{Z}(a)$ with a field these modules become simple. Images of $\mathbb{Z}(a)$’s make a basis in $K(H^m$-mod), see [K, Proposition 20].

For an admissible $\lambda$ and $a \in B^m(\lambda)$ denote by $\mathbb{Z}(\lambda, a)$ the $H_\lambda$ module isomorphic to $\mathbb{Z}(a)$ under the canonical isomorphism $H_\lambda \cong H^m(\lambda)$. Proposition 20 in [K] implies

**Proposition 6** The Grothendieck group of $C$ is a free $\mathbb{Z}[q, q^{-1}]$-module with a basis $\{[\mathbb{Z}(\lambda, a)]\}_{\lambda, a}$ over all admissible $\lambda$ and $a \in B^m(\lambda)$.

**Projective Grothendieck group**

For $a \in B^m$ we denote by $P_a$ the left $H^m$-module $H^m1_a$, see [K, Section 2.5]. Let $Q_a = P_a\{-m\}$ be the indecomposable projective graded $H^m$-module given by shifting the grading of $P_a$ down by $m$,

$$Q_a = \bigoplus_{b \in H^m} F(W(b)a).$$
The grading of $Q_a$ is balanced, in the sense that its nontrivial graded components are in degrees $-m, -m+2, \ldots, m$. We will refer to $Q_a$’s as balanced indecomposable projectives.

Similarly, for any admissible $\lambda$ and $a \in B^{m(\lambda)}$ we define projective $H_\lambda$-module $Q_{\lambda,a}$ as the image of $Q_a$ under the canonical ring isomorphism $H_\lambda \cong H^{m(\lambda)}$.

**Proposition 7** Any indecomposable projective in $\mathcal{C}$ is isomorphic to $Q_{\lambda,a}\{i\}$ for some (and unique) admissible $\lambda, a \in B^{m(\lambda)}$ and $i \in \mathbb{Z}$.

Let $K_P(\mathcal{C})$ be the projective Grothendieck group of $\mathcal{C}$, the subgroup of $K(\mathcal{C})$ generated by images of projective modules. $K_P(\mathcal{C})$ is a free $\mathbb{Z}[q, q^{-1}]$-module with the basis $[Q_{\lambda,a}]$ over all $\lambda, a$ as above. The inclusion $K_P(\mathcal{C}) \subset K(\mathcal{C})$ is proper but turns into an isomorphism when tensored with the field $\mathbb{Q}(q)$:

$$K(\mathcal{C}) \otimes_{\mathbb{Z}[q, q^{-1}]} \mathbb{Q}(q) \cong K_P(\mathcal{C}) \otimes_{\mathbb{Z}[q, q^{-1}]} \mathbb{Q}(q).$$ (10)

$K_P(\mathcal{C})$ is stable under the action of $[\mathcal{E}_i], [\mathcal{F}_i]$, and $[\mathcal{K}_i]$, since functors $\mathcal{E}_i$, $\mathcal{F}_i$, and $\mathcal{K}_i$ take projectives to projectives.

$Q_{2\omega_k}$ is the unique (up to isomorphism) balanced indecomposable projective in $\mathcal{C}(2\omega_k)$.

**Proposition 8** Any balanced indecomposable projective in $\mathcal{C}$ is isomorphic to $\mathcal{F}_{i_1}^{(j_1)} \cdots \mathcal{F}_{i_r}^{(j_r)} Q_{2\omega_k}$ for some sequences $(i_1, \ldots, i_r)$ and $(j_1, \ldots, j_r)$ where $j_1, \ldots, j_r \in \{1, 2\}$.

The following example makes it clear. Let $n = 6$, $k = 3$, $\lambda = (1, 1, 1, 0, 2, 1)$ and $Q$ be the balanced indecomposable projective given by the flat tangle in figure 11. You can check using figure 12 that

$$Q \cong \mathcal{F}_2 \mathcal{F}_4^{(2)} \mathcal{F}_3^{(2)} \mathcal{F}_5 \mathcal{F}_1 \mathcal{F}_4 \mathcal{F}_2 \mathcal{F}_3 Q_{2\omega_3}$$

![Figure 11: Flat tangle for one of the two balanced indecomposable projectives in $\mathcal{C}(1, 1, 1, 0, 2, 1)$](image)
Figure 12: A presentation of $Q$

Proposition 8 implies that $K_P(C) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q)$ is a cyclic $U_-$-module generated by $[Q_{2\omega_k}]$. Therefore, $K_P(C) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q)$ is an irreducible $U$-module with highest weight $2\omega_k$. This and (10) gives another proof of Proposition 5.

Biadjoint functors

Let $\tau$ be the $\mathbb{Q}$-linear involution of $\mathbb{Q}(q)$ which changes $q$ into $q^{-1}$. $U$ has an antiautomorphism $\tau : U \to U^{\text{op}}$ described by

$$
\tau(E_\alpha) = q F_\alpha K_\alpha^{-1}, \quad \tau(F_\alpha) = q E_\alpha K_\alpha, \quad \tau(K_\alpha) = K_\alpha^{-1},
\tau(fx) = f\tau(x), \quad \text{for } f \in \mathbb{Q}(q) \text{ and } x \in U,
\tau(xy) = \tau(y)\tau(x), \quad \text{for } x, y \in U.
$$

(11)

**Proposition 9** The functor $E_i$ is left adjoint to $F_i K_i^{-1}\{1\}$, the functor $F_i$ is left adjoint to $E_i K_i\{1\}$ and $K_i$ is left adjoint to $K_i^{-1}$. 

**Proof:** case by case verification for each pair $(\lambda_i, \lambda_{i+1})$. Suppose $(\lambda_i, \lambda_{i+1}) = (1,1)$. Then $F_i E_i$ is given by the left diagram in figure 13, while the identity functor in $C(\lambda)$ is given by the right diagram. There are standard cobordisms (embedded surfaces in $\mathbb{R}^3$) between these two flat tangles that give rise to natural transformations of functors $F_i E_i \Rightarrow \text{id}$ and $\text{id} \Rightarrow F_i E_i$. Similar cobordisms provide natural transformations $E_i F_i \Rightarrow \text{id}$ and $\text{id} \Rightarrow E_i F_i$ for functors in $C(\lambda + \epsilon_i)$. Isotopies of surfaces translate into relations between these four natural transformations which imply that, up to grading shifts, $F_i$
Figure 13: Flat tangles for functors $F_i E_i$ and id when $(\lambda_i, \lambda_{i+1}) = (1, 1)$

is left and right adjoint to $E_i$ (the natural transformations come from bimodule maps that change grading, thus grading shifts appear). Shifts are taking care of by composing with $K_{i \pm 1}{1}$. Details and other cases are left to the reader. □

The moral: our categorification lifts the automorphism $\tau$ of $U$ to the operation of taking the right adjoint functor.

**Semilinear form**

We say that a form $V \times V \to \mathbb{Q}(q)$ is semilinear if it is $q$-antilinear in the first variable and $q$-linear in the second:

$$\langle fv, w \rangle = f \langle v, w \rangle, \quad \langle v, fw \rangle = f \langle v, w \rangle.$$ 

$V$ has a unique semilinear form subject to conditions

$$\langle \eta, \eta \rangle = 1, \quad \langle xv, w \rangle = \langle v, \tau(x)w \rangle \quad x \in U, v, w \in V. \quad (12)$$

On the other hand we have a semilinear form $\langle , \rangle$

$$K_P(C) \times K(C) \longrightarrow \mathbb{Z}[q, q^{-1}]$$

which measures the graded rank of Hom:

$$\langle [P], [M] \rangle \overset{\text{def}}{=} \text{rkHom}_C(P, M),$$

where $P$ is a projective module in $C$, while $M$ is any module, and the rank of a finitely-generated graded abelian group is a Laurent polynomial in $q$ with coefficients given by ranks of graded components of the group. After tensoring with $\mathbb{Q}(q)$ we obtain a semilinear form $\langle , \rangle$ on $K(C) \otimes_{\mathbb{Z}[q, q^{-1}]} \mathbb{Q}(q)$ with values in $\mathbb{Q}(q)$. Under the isomorphism \(^9\) this form coincides with the form $\langle , \rangle$ on $V$. 

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Relation (13) comes from interpreting $\tau$ via the right adjoint functor, for instance
\[ \text{Hom}_C(E_i P, M) \cong \text{Hom}_C(P, \mathcal{F}_i \mathcal{K}_i^{-1} M \{1\}) \]
descends to $\langle E_i v, w \rangle = \langle v, \tau(E_i) w \rangle$.

**Symmetric bilinear form**

A left $H^m$-module $M$ can be made into a right $H^m$-module $\psi M$ by twisting the action of $H^m$ by the antiinvolution $\chi$. For $H^m$-modules $M, N$ we can form $\psi M \otimes_{H^m} N$, which is a graded abelian group. We similarly define $\psi M$ for $M \in \text{Ob}(C)$ and graded abelian group $\psi M \otimes_H N$ (or simply $\psi M \otimes N$) for $M, N \in \text{Ob}(C)$. Notice that $\psi M \otimes N = 0$ if $M \in \text{Ob}(C(\lambda)), N \in \text{Ob}(C(\mu))$ and $\lambda \neq \mu$.

**Proposition 10** There are natural isomorphisms
\[
\begin{align*}
\psi(E_i M) \otimes N & \cong \psi M \otimes (\mathcal{K}_i \mathcal{F}_i N) \{1\}, \\
\psi(\mathcal{F}_i M) \otimes N & \cong \psi M \otimes (\mathcal{K}_i^{-1} E_i N) \{1\}, \\
\psi(K_i M) \otimes N & \cong \psi M \otimes (K_i N).
\end{align*}
\]

Introduce a bilinear form $(\cdot, \cdot)$ on $K_P(C)$ with values in $\mathbb{Z}[q, q^{-1}]$ by
\[
([P], [Q]) \overset{\text{def}}{=} \text{rk}(\psi P \otimes Q) \in \mathbb{Z}[q, q^{-1}]
\]
where $P$ and $Q$ are projectives in $C$ and $\text{rk}$ is the graded rank. Form $(\cdot, \cdot)$ is $\mathbb{Z}[q, q^{-1}]$-linear in each variable, since
\[
\psi P \{1\} \otimes Q \cong (\psi P \otimes Q) \{1\} \cong \psi P \otimes Q \{1\}.
\]

Since $\psi^2 = 1$, graded abelian groups $\psi P \otimes Q$ and $\psi Q \otimes P$ are isomorphic and the form is symmetric.

We have
\[
\psi Q_{\lambda, a} \otimes Q_{\mu, b} = 0 \quad \text{if} \quad \lambda \neq \mu;
\]
\[
\psi Q_{\lambda, a} \otimes Q_{\lambda, b} \cong \mathcal{F}(W(a)b) \{-m\}.
\]

Therefore,
\[
([Q_{\lambda, a}], [Q_{\lambda, b}]) = (q + q^{-1})^r q^{-m} = (1 + q^{-2})^r q^{-m},
\]
where $r$ is the number of connected components of $W(a)b$. Notice that $r < m$ unless $a = b$. 18
Corollary 1

\([Q_{\lambda,a}, [Q_{\mu,b}]] = 0 \text{ if } \lambda \neq \mu,\)

\([Q_{\lambda,a}, [Q_{\lambda,b}]] \in q^{-1}Z[q^{-1}] \text{ if } a \neq b,\)

\([Q_{\lambda,a}, [Q_{\lambda,a}]] \in 1 + q^{-1}Z[q^{-1}] \text{ for all } a \in B^m.\)

Bilinear form \((,\) extends to \(\mathbb{Q}(q)\)-bilinear form on \(K_P(C) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q).\) We turn it into a bilinear form on \(V\) via isomorphisms (9) and (10). This form is the unique bilinear form on \(V\) such that

\((\eta, \eta) = 1,\)

\((xv, w) = (v, \rho(x)w) \text{ for all } v, w \in V \text{ and } x \in U,\)

where \(\rho\) is a \(\mathbb{Q}(q)\)-linear antiinvolution of \(U\) defined on generators by

\(\rho(E_i) = qK_iF_i, \quad \rho(F_i) = qK_i^{-1}E_i, \quad \rho(K_i) = K_i.\)

Canonical basis

Let \(\psi\) be the following \(\mathbb{Q}\)-algebra involution of \(U:\)

\(\psi(E_a) = E_a, \quad \psi(F_a) = F_a, \quad \psi(K_a) = K_a^{-1},\)

\(\psi(fx) = \mathcal{f}x \text{ for } f \in \mathbb{Q}(q) \text{ and } x \in U.\) \hspace{1cm} (14)

There is a unique \(\mathbb{Q}\)-linear involution \(\psi_V\) of \(V\) such that

\(\psi_V(\eta) = \eta, \quad \psi_V(xv) = \psi(x)\psi_V(v) \text{ for } x \in U, v \in V.\) \hspace{1cm} (15)

Involutions \(\psi\) and \(\psi_V\) are denoted by \(-\) in [L2].

For \(a, b \in B^m\) the diagram \(W(b)a\) is the mirror image of \(W(a)b.\) Consequently, there is a natural isomorphism of graded abelian groups \(\mathcal{F}(W(b)a) \cong \mathcal{F}(W(a)b).\) Summing over all \(a, b \in B^m\) we obtain an antiinvolution \(\chi\) of the ring \(H^m.\) Notice that \(\chi\) preserves all minimal idempotents of \(H^m, \chi(1_a) = 1_a.\) For each admissible \(\lambda\) we similarly have an antiinvolution of \(H_\lambda,\) also denoted \(\chi.\)

For what follows in this subsection we need to switch either from the base ring \(\mathbb{Z}\) to a field, or from \(\mathcal{C}\) to its full subcategory which consists of modules that are free as abelian groups. Denote the latter category by \(\mathcal{C}_f.\)

To \(M \in \text{Ob}(\mathcal{C}_f)\) we assign \(M^* = \text{Hom}_\mathbb{Z}(M, \mathbb{Z}),\) which is a right graded module over \(\bigoplus H^\lambda.\) Using antiinvolution \(\chi\) we turn \(M^*\) into a left graded \(\bigoplus H^\lambda-\) module, denoted \(\Psi M.\) Note that \(\Psi M \in \text{Ob}(\mathcal{C}_f)\) and that \(\Psi\) is a contravariant duality functor in \(\mathcal{C}_f.\)
Proposition 11 \( \Psi \) preserves indecomposable balanced projectives:

\[
\Psi Q_{\lambda,a} \cong Q_{\lambda,a}.
\]

There are equivalences of functors

\[
\Psi E_i \cong E_i \Psi, \quad \Psi F_i \cong F_i \Psi, \quad \Psi K_i \cong K_i^{-1} \Psi, \quad \Psi \{1\} \cong \{-1\} \Psi.
\]

\( \Psi \) is exact and descends to a \( q \)-antilinear automorphism of the Grothendieck group \( K(C) \) and projective Grothendieck group \( K_P(C) \). Under the isomorphism of proposition 5 involution \( [\Psi] \) corresponds to the involution \( \psi_V \) of \( V \), that is, the diagram below commutes (horizontal arrows are inclusions)

\[
\begin{array}{ccc}
K(C) & \longrightarrow & V \\
[\Psi] & \downarrow \psi_V & \\
K(C) & \longrightarrow & V
\end{array}
\]

We omit the proof. \( \square \)

Proposition 12 The basis \( \{[Q_{\lambda,a}]\}_{\lambda,a} \) of balanced indecomposable projective modules in \( K_P(C) \) is the canonical basis in \( V \).

Proof: From the previous proposition we see that \( [Q_{\lambda,a}] \) is invariant under \( \psi_V = [\Psi] \). This and proposition 3 imply that \( [Q_{\lambda,a}] \) is a canonical basis vector, using \([L2, \text{Theorem 19.3.5}]\). \( \square \)

In particular, any canonical basis vector in \( V \) can be presented as a monomial in divided powers of \( F_i \)'s applied to the highest weight vector. This is a rather special property, characteristic of representations with small or degenerate highest weight.

**Braid group action**

The \( n \)-stranded braid group \( Br_n \) acts in any finite-dimensional representation of \( U \) via

\[
\sigma_i(v) = \sum_{a, b, c \geq 0, -a + b - c = r} (-1)^b q^{b-ac} E^{(a)} F^{(b)} E^{(c)} v
\]  

(16)

where \( v \) has weight \( \lambda \) and \( r = \lambda_i - \lambda_{i+1} \). To categorify this action we look for a way to change the sum into a complex of functors \( E^{(a)} F^{(b)} E^{(c)} \{b - ac\} \). In
representation \( V \) the sums simplify and we expect similar simplifications in the categorification.

Let \( \mathcal{D} \) be either the bounded derived category of \( \mathcal{C} \) or the category of bounded complexes of objects of \( \mathcal{C} \) up to chain homotopies. Categories \( \mathcal{D}(\lambda) \) are defined similarly.

We define functors \( \Sigma_i : \mathcal{D} \to \mathcal{D} \) that take \( \mathcal{D}(\lambda) \) to \( \mathcal{D}(\pi_i \lambda) \) (where \( \pi_i \) transposes \( \lambda_i \) and \( \lambda_{i+1} \)) for all admissible \( \lambda \). Rings \( H_\lambda \) and \( H_{\pi_i \lambda} \) are naturally isomorphic, and we denote by \( \mathcal{Y}_\lambda \) the equivalence \( \mathcal{D}(\lambda) \cong \mathcal{D}(\pi_i \lambda) \) induced by this isomorphism.

The restriction of \( \Sigma_i \) to \( \mathcal{D}(\lambda) \) is the following functor:

- If \( (\lambda_i, \lambda_{i+1}) \neq (1, 1) \) then \( \Sigma_i = \mathcal{Y}_\lambda[x]\{x\} \) where \( x = \max(0, \lambda_i - \lambda_{i+1}) \).
- If \( (\lambda_i, \lambda_{i+1}) = (1, 1) \) then \( \Sigma_i \) is the complex of functors

\[
\to 0 \to \mathcal{F}_i \mathcal{E}_i \{1\} \to \text{id} \to 0 \to
\]

where \( \text{id} \) is in cohomological degree 0 and the natural transformation comes from the simplest cobordism between flat tangles that describe \( \mathcal{F}_i \mathcal{E}_i \) and \( \text{id} \), see figure 13.

The Grothendieck groups of \( \mathcal{D} \) and \( \mathcal{C} \) are isomorphic, and the functor \( \Sigma_i \) descends to the operator \( \sigma_i \) in the Grothendieck group \( K(\mathcal{C}) \).

**Proposition 13** The functors \( \Sigma_i \) are invertible and satisfy functor isomorphisms

\[
\Sigma_i \Sigma_{i+1} \Sigma_i \cong \Sigma_{i+1} \Sigma_i \Sigma_{i+1}, \\
\Sigma_i \Sigma_j \cong \Sigma_j \Sigma_i, \quad |i - j| > 1.
\]

Follows from results of [K]. □

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