Selection Rules in Minisuperspace Quantum Cosmology

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Abstract

The existence of a Noether symmetry for a given minisuperspace cosmological model is a sort of selection rule to recover classical behaviours in cosmic evolution since oscillatory regimes for the wave function of the universe come out. The so called Hartle criterion to select correlated regions in the configuration space of dynamical variables can be directly connected to the presence of a Noether symmetry and we show that such a statement works for generic extended theories of gravity in the framework of minisuperspace approximation. Examples and exact cosmological solutions are given for nonminimally coupled and higher–order theories.

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I. INTRODUCTION

Several points of view can be adopted in order to define quantum cosmology. It can be considered as the first step toward the construction of a full theory of quantum gravity. Besides, it concerns finding initial conditions from which our classical universe is started. However, with respect to other theories of physics as electromagnetism, general relativity or ordinary quantum mechanics, boundary conditions for the evolution of the system “universe” cannot be set from outside. There we need a fundamental dynamical law (e.g. Maxwell’s or Einstein’s equations or Schrödinger’s equation) and then we impose, from the outside, the initial conditions. In cosmology, by definition, there is no rest of the universe so that boundary conditions must be a fundamental law of physics. In this sense, a part the fact that quantum cosmology is a workable scheme to achieve quantum gravity, it can be considered as an autonomous branch of physics due to the issue of finding initial conditions [1].

However, not only the conceptual difficulties, but also mathematical ones make quantum cosmology hard to handle. For example, the superspace of geometrodynamics [2] has infinite degrees of freedom so that it is practically impossible to integrate the full Wheeler–DeWitt (WDW) equation. Furthermore, a Hilbert space of states describing the universe is not available [3]. Finally, it is not clear how to interpret the solutions of WDW equation in the framework of probability theory. Several interpretative schemes have been proposed but the concepts of probability and unitarity are in any case approximate. Their validity is limited by the accuracy of the semiclassical approximation and strictly depends on the suitable definition of probability current [3], [4].

Despite these still unsolved shortcomings, several positive results have been obtained and quantum cosmology has become a sort of paradigm in theoretical physics researches. For example the infinite–dimensional superspace can be restricted to opportune finite–dimensional configuration spaces called minisuperspaces. In this case, the above mathematical difficulties can be avoided and the WDW equation can be integrated. The so called no boundary condition by Harte and Hawking [3] and the tunneling from nothing by Vilenkin [4] give reasonable laws for initial conditions from which our classical universe could be started.

The Hartle criterion [3] is an interpretative scheme for the solutions of the WDW equation. Hartle proposed to look for peaks of the wave function of the universe: If it is strongly peaked, we have correlations among the geometrical and matter degrees of freedom; if it is not peaked, correlations are lost. In the first case, the emergence of classical trajectories (i.e. universes) is expected. The analogy to the non–relativistic quantum mechanics is immediate. If we have a potential barrier and a wave function, solution of the Schrödinger equation, we have an oscillatory regime on and outside the barrier; we have a decreasing exponential behaviour under the barrier. The system behaves classically in the oscillatory regime while it does not in the exponential case. The situation is analogous in quantum cosmology: Now the potential barrier has to be replaced by the superpotential $U(h^{ij}, \varphi)$, where $h^{ij}$ are the components of the three–metric of geometrodynamics and $\varphi$ is a generic scalar field describing the matter content. More precisely, the wave function of the universe can be written as

$$\Psi[h_{ij}(x), \varphi(x)] \sim e^{im_p^2S}, \quad (1.1)$$

where $m_p$ is the Planck mass and $S$ is an action. A state with classical correlations must be
a superposition of states of the form (1.1). This type of state can be expressed as a coherent superposition of eigenstates of operators that commute with the constraints and correspond to constants of the motion. A superposition of this kind can be approximated by a WKB state where

\[ S \equiv S_0 + m_P^2 S_1 + O(m_P^4), \]  

(1.2)

is the expansion of the action. We have to note that there is no normalization factor due to the lack of a probability interpretative scheme. However, in this approximation, it is possible to define a localized prefactor as shown in [7] and we can define a quasiclassical state to describe an approximate classical behaviour and a semiclassical one to describe a product of a part that is quasiclassical and a part that is not. This quasiclassical state can be a coherent one if it is a superposition of states in the sense discussed in [7]. Transition amplitudes of instantaneous eigenstates are discussed in [8].

Considering the action (1.2) and inserting it into the WDW equation and equating similar power of \( m_p \), one obtains the Hamilton–Jacobi equation for \( S_0 \). Similarly, one gets equations for \( S_1, S_2, \ldots \), which can be solved considering results of previous orders. We need only \( S_0 \) to recover the semi–classical limit of quantum cosmology [9]. If \( S_0 \) is a real number, we get oscillating WKB modes and \( \Psi \) is peaked on a phase–space region defined by

\[ \pi_{ij} = m_P^2 \frac{\delta S_0}{\delta h_{ij}}, \quad \pi_\varphi = m_P^2 \frac{\delta S_0}{\delta \varphi}, \]  

(1.3)

where \( \pi_{ij} \) and \( \pi_\varphi \) are classical momenta conjugates to \( h_{ij} \) and \( \varphi \). The semi–classical region of superspace, where \( \Psi \) has an oscillating structure, is the Lorentz one otherwise it is Euclidean. In the latter case, we have \( S = iI \) and

\[ \Psi \sim e^{-m_P^2 I}, \]  

(1.4)

where \( I \) is the action for the Euclidean solutions of classical field equations (istantons). Given an action \( S_0 \), Eqs. (1.3) imply \( n \) free parameters (one for each dimension of the configuration space \( Q \equiv \{ h_{ij}, \varphi \} \)) and then \( n \) first integrals of motion. However the general solution of the field equations involves \( 2n - 1 \) parameters (one for each Hamilton equation of motion except the energy constraint). Consequently, the wave function oscillates on a subset of the general solution. In this sense, the boundary conditions on the wave function (e.g. Harte–Hawking, Vilenkin, others) imply initial conditions for the classical solutions.

To be more precise, if the wave function \( \Psi \) is sufficiently peaked about some region in configuration space, we predict that we will observe the correlations between observables which characterize this region. If \( \Psi \) is small in some region, we predict that observations of the correlations which characterize this region are precluded. Where \( \Psi \) is neither small nor sufficiently peaked, we do not predict anything.

For example, given the measured value of the Hubble constant and mass density, we would like a ”good” wave function for the universe to be peaked around a distribution of galaxies consistent with that which is observed. It is crucial to recognize that the wave function does not predict a specific value for \( H_0 \), or specific locations for the galaxies, but rather a ”correlation” between these observables.
Halliwell [9] has shown that an oscillatory wave function of the form (1.1) predicts a correlation between the canonical coordinate \( q \) and the momentum \( \pi_q \) of the above form \( \pi_q = m_P^2 \partial S/\partial q \). In other words, taking into account minisuperspace models, if oscillatory regimes of the WDW wave function exist, we are able to recover correlations among variables and then classical behaviours emerge. Again, in the classically allowed region, the semiclassical approximation to the WDW wave function yields just such oscillatory solutions.

A simple minisuperspace example can be constructed. The ansatz for the wave function is

\[ \Psi(a) = e^{im_2^2 S(a)}, \]  

(1.5)

where the canonical variables \( q \) coincides to the scale factor of the universe \( a \) and the phase is a slowly varying function of the scale factor. Since we wish to investigate the classical limit \( m_2^2 \to \infty \) (which corresponds to \( \hbar \to 0 \) of ordinary quantum mechanics), we use the expansion (1.2). Inserting such an ansatz into the WDW equation

\[ \left[ \frac{\partial^2}{\partial a^2} - \left( \frac{3\pi}{2G} \right)^2 a^2 \left( 1 - \frac{a^2}{a_0^2} \right) \right] \Psi(a) = 0, \]  

(1.6)

deduced by the action

\[ A = \frac{3\pi}{4G} \int dt \left[ -a^2 \dot{a} + a \left( 1 - \frac{a^2}{a_0^2} \right) \right], \]  

(1.7)

of a Friedman–Robertson–Walker (FRW) closed universe (\( a_0 \) is a constant) with the canonical momentum \( \pi_a \) given by

\[ \pi_a = -\frac{3\pi}{2G} a \dot{a}, \]  

(1.8)

we get a set of differential equations, one for any order of \( m_2^2 \), which as we said above, can be solved sequentially. The semiclassical approximation to the wave function obtains by working only to first order. We get

\[ S_0 = \int^a da' \sqrt{\left( \frac{3\pi}{2G} \right)^2 a_0^2 \left( \frac{a^4}{a_0^4} - \frac{a^2}{a_0^2} \right)}, \]  

(1.9)

and

\[ S_1 = \frac{1}{2} \ln \left( \frac{\partial S_0}{\partial a} \right). \]  

(1.10)

Thus the oscillatory semiclassical wave function \( \Psi \propto \exp(iS_0) \) is peaked about a region of minisuperspace (every point of which represents a closed FRW model) in which the correlation between the coordinate and momentum (scale factor and expansion rate), \( \pi_a = \partial S_0/\partial a \), holds good.

Using Eq. (1.8) for \( \pi_a \), the correlation reduces to

4
\[
\dot{a} = \sqrt{\frac{a^2}{a_0^2} - 1}, \quad (1.11)
\]

which is nothing else but the (0, 0) Einstein equation for a FRW spacetime. If

\[
a_0 = \sqrt{\frac{3}{\Lambda}}, \quad \Lambda = 8\pi G \rho_{\text{vac}}, \quad (1.12)
\]

where \(\rho_{\text{vac}}\) is a constant density, we get the solution

\[
a(t) = a_0 \cosh(a_0^{-1}t), \quad (1.13)
\]

which is an inflationary behaviour for a closed FRW model. Thus, by this simple example, in the region of minisuperspace where the wave function oscillates, a classical FRW spacetime, obeying the (classical) Einstein equation emerges.

The issue is now if there exists some method capable of selecting such constants of motion which, being first integrals of motion, allow to find correlations between classical variables and conjugate momenta in minisuperspace models. In other words, can the emergence of classical trajectories be implemented by some general approach without arbitrarily choosing regions of the phase–space where momenta \((1.3)\) are constant? Achieving this result means to obtain oscillatory subsets of WDW wave function where one gets correlations. Consequently classical regime are recovered and the Hartle criterion holds (at least in the framework of the minisuperspace approximation). For the full theory, i.e. without considering simple minisuperspace models, the issue is more delicate since we have to ask for superpositions of the form \((1.1)\) which yield peaked wave packets so that the Hartle criterion holds [7]. In this case, also the issue of universe “creation”, as a particle creation problem, has to be faced considering the way in which the quantum–classical transition is achieved [8].

In this paper, we want to restrict to a more specific (and simple) question. We want to show, for general extended gravity minisuperspace models, that the existence of a Noether symmetry implies, at least, a subset of the general solution of the WDW equation where the oscillating behaviour is recovered. Viceversa, the presence of a Noether symmetry gives rise to the emergence of classical trajectories. This analysis is performed in the context of the minisuperspace approximation and, for classical trajectories, we mean solutions of the ordinary Einstein equations. The existence of a Noether symmetry for a dynamical model is a general criterion to search for constants (first integrals) of motion so that, given a minisuperspace model exhibiting such symmetries we obtain certainly correlations and then classical behaviours. This statement, in our knowledge, has never been done also if constants of motion have been systematically used in quantum cosmology since at least fifteen years.

The layout of the paper is the following. Sect. 2 is devoted to the Noether Symmetry Approach and to its connection to quantum cosmology. In Sect. 3, we apply the method to minisuperspace models of nonminimally coupled theories of gravity, while the same is done in Sects. 4 and 5 for higher–order theories. Discussion and conclusions are drawn in Sect. 6.
II. THE NOETHER SYMMETRY APPROACH AND QUANTUM COSMOLOGY

Minisuperspaces are restrictions of the superspace of geometrodynamics. They are finite-dimensional configuration spaces on which point-like Lagrangians can be defined. Cosmological models of physical interest can be defined on such minisuperspaces (e.g. Bianchi models).

Before taking into account specific models, let us remind some properties of the Lie derivative and the derivation of the Noether theorem [10]. Let \( L_X \) be the Lie derivative

\[
(L_X \omega) \xi = \frac{d}{dt} \omega(g^t \xi),
\]

where \( \omega \) is a differential form of \( \mathcal{R}^n \) defined on the vector field \( \xi \), \( g^t \) is the differential of the phase flux \( \{g_t\} \) given by the vector field \( X \) on a differential manifold \( \mathcal{M} \). Let \( \rho_t = \rho_{g^{-t}} \) be the action of a one-parameter group able to act on functions, vectors and forms on the vector spaces \( \mathcal{C}^\infty(\mathcal{M}), \mathcal{D}(\mathcal{M}), \lambda(\mathcal{M}) \) constructed starting from \( \mathcal{M} \). If \( g_t \) takes the point \( m \in \mathcal{M} \) in \( g_t(m) \), then \( \rho_t \) takes back on \( m \) the vectors and the forms defined on \( g_t(m) \); \( \rho_t \) is a pull back [11]. Then the property

\[
\rho_{t+s} = \rho_t \rho_s
\]

holds since

\[
g_{t+s} = g_t \circ g_s.
\]

On the functions \( f, g \in \mathcal{C}^\infty(\mathcal{M}) \) we have

\[
\rho_t(fg) = (\rho_t f)(\rho_t g);
\]

on the vectors \( X, Y \in \mathcal{D}(\mathcal{M}) \),

\[
\rho_t[X, Y] = [\rho_t X, \rho_t Y];
\]

on the forms \( \omega, \mu \in \Lambda(\mathcal{M}) \)

\[
\rho_t(\omega \wedge \mu) = (\rho_t \omega) \wedge (\rho_t \mu).
\]

\( L_X \) is the infinitesimal generator of the one-parameter group \( \rho_t \), and, being a derivative on the algebras \( \mathcal{C}^\infty(\mathcal{M}), \mathcal{D}(\mathcal{M}), \lambda(\mathcal{M}) \), the following properties have to hold

\[
L_X(fg) = (L_X f)g + f(L_X g),
\]

\[
L_X[Y, Z] = [L_X Y, Z] + [Y, L_X Z],
\]

\[
L_X(\omega \wedge \mu) = (L_X \omega) \wedge \mu + \omega \wedge (L_X \mu),
\]

which are nothing else but the Leibniz rules for functions, vectors and differential forms, respectively. Furthermore,
\[ L_X f = X f , \quad (2.10) \]
\[ L_X Y = adX(Y) = [X, Y] , \quad (2.11) \]
\[ L_X d\omega = dL_X \omega , \quad (2.12) \]

where \( ad \) is the self–adjoint operator and \( d \) is the external derivative by which a \( p \)-form becomes a \( (p+1) \)-form.

The discussion can be specified by considering a Lagrangian \( \mathcal{L} \) which is a function defined on the tangent space of configurations \( T\mathcal{Q} \equiv \{q_i, \dot{q}_i\} \). In this case, the vector field \( X \) is

\[ X = \alpha^i(q) \frac{\partial}{\partial q^i} + \dot{\alpha}^i(q) \frac{\partial}{\partial \dot{q}^i} , \quad (2.13) \]

where dot means derivative with respect to \( t \), and

\[ L_X \mathcal{L} = X\mathcal{L} = \alpha^i(q) \frac{\partial \mathcal{L}}{\partial q^i} + \dot{\alpha}^i(q) \frac{\partial \mathcal{L}}{\partial \dot{q}^i} . \quad (2.14) \]

The condition

\[ L_X \mathcal{L} = 0 \quad (2.15) \]

implies that the phase flux is conserved along \( X \): This means that a constant of motion exists for \( \mathcal{L} \) and the Noether theorem holds. In fact, taking into account the Euler–Lagrange equations

\[ \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}^i} - \frac{\partial \mathcal{L}}{\partial q^i} = 0 , \quad (2.16) \]

it is easy to show that

\[ \frac{d}{dt} \left( \alpha^i \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \right) = L_X \mathcal{L} . \quad (2.17) \]

If \( (2.13) \) holds,

\[ \Sigma_0 = \alpha^i \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \quad (2.18) \]

is a constant of motion. Alternatively, using the Cartan one–form

\[ \theta_\mathcal{L} \equiv \frac{\partial \mathcal{L}}{\partial q^i} dq^i \quad (2.19) \]

and defining the inner derivative

\[ i_X \theta_\mathcal{L} = \langle \theta_\mathcal{L}, X \rangle , \quad (2.20) \]

we get, as above,

\[ i_X \theta_\mathcal{L} = \Sigma_0 \quad (2.21) \]
if condition (2.13) holds. This representation is useful to identify cyclic variables. Using a point transformation on vector field (2.13), it is possible to get

\[ \tilde{X} = (i_X dQ^k) \frac{\partial}{\partial Q^k} + \left[ \frac{dt}{d\dot{q}^i} (i_X dQ^k) \right] \frac{\partial}{\partial \dot{Q}^k}. \]  

(2.22)

If \( X \) is a symmetry also \( \tilde{X} \) has this property, then it is always possible to choose a coordinate transformation so that

\[ i_X dQ^1 = 1, \quad i_X dQ^i = 0, \quad i \neq 1, \]  

(2.23)

and then

\[ \tilde{X} = \frac{\partial}{\partial Q^1}, \quad \frac{\partial \tilde{L}}{\partial Q^1} = 0. \]  

(2.24)

It is evident that \( Q^1 \) is the cyclic coordinate and the dynamics can be reduced [10]. However, the change of coordinates is not unique and a clever choice is always important. Furthermore, it is possible that more symmetries are found. For example, if \( X_1, X_2 \) are the Noether vector fields and they commute, \([X_1, X_2] = 0\), we obtain two cyclic coordinates by solving the system

\[ i_{X_1} dQ^1 = 1, \quad i_{X_2} dQ^2 = 1, \]  

(2.25)

\[ i_{X_1} dQ^i = 0, \quad i \neq 1; \quad i_{X_2} dQ^i = 0, \quad i \neq 2. \]

If they do not commute, this procedure does not work since commutation relations are preserved by diffeomorphisms. In this case

\[ X_3 = [X_1, X_2] \]  

(2.26)

is again a symmetry since

\[ L_{X_3} \mathcal{L} = L_{X_1} L_{X_2} \mathcal{L} - L_{X_2} L_{X_1} \mathcal{L} = 0. \]  

(2.27)

If \( X_3 \) is independent of \( X_1, X_2 \) we can go on until the vector fields close the Lie algebra [12].

A reduction procedure by cyclic coordinates can be implemented in three steps: i) we choose a symmetry and obtain new coordinates as above. After this first reduction, we get a new Lagrangian \( \tilde{\mathcal{L}} \) with a cyclic coordinate; ii) we search for new symmetries in this new space and apply the reduction technique until it is possible; iii) the process stops if we select a pure kinetic Lagrangian where all coordinates are cyclic. This case is not very

\[ 1 \]We shall indicate the quantities as Lagrangians and vector fields with a tilde if the non–degenerate transformation

\[ Q^i = Q^i(q), \quad \dot{Q}^i(q) = \frac{\partial Q^i}{\partial q^j} \dot{q}^j \]  

is performed. However the Jacobian determinant \( \mathcal{J} = \| \partial Q^i / \partial q^j \| \) has to be non–zero.
common and often it is not physically relevant. Going back to the point of view interesting in quantum cosmology, any symmetry selects a constant conjugate momentum since, by the Euler–Lagrange equations

$$\frac{\partial \tilde{L}}{\partial Q_i} = 0 \iff \frac{\partial \tilde{L}}{\partial \dot{Q}_i} = \Sigma_i.$$  \hspace{1cm} (2.28)

Vice versa, the existence of a constant conjugate momentum means that a cyclic variable has to exist. In other words, a Noether symmetry exists.

Further remarks on the form of the Lagrangian $L$ are necessary at this point. We shall take into account time–independent, non–degenerate Lagrangians $L = L(q^i, \dot{q}^i)$, i.e.

$$\frac{\partial L}{\partial t} = 0, \quad \det H_{ij} \equiv \det \left| \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \right| \neq 0,$$  \hspace{1cm} (2.29)

where $H_{ij}$ is the Hessian. As in usual analytic mechanics, $L$ can be set in the form

$$L = T(q^i, \dot{q}^i) - V(q^i),$$  \hspace{1cm} (2.30)

where $T$ is a positive–defined quadratic form in the $\dot{q}^i$ and $V(q^i)$ is a potential term. The energy function associated with $L$ is

$$E_L \equiv \frac{\partial L}{\partial \dot{q}^i} \dot{q}^i - L(q^i, \dot{q}^i)$$  \hspace{1cm} (2.31)

and by the Legendre transformations

$$\mathcal{H} = \pi_j \dot{q}^j - L(q^j, \dot{q}^j), \quad \pi_j = \frac{\partial L}{\partial \dot{q}^j},$$  \hspace{1cm} (2.32)

we get the Hamiltonian function and the conjugate momenta.

Considering again the symmetry, the condition (2.15) and the vector field $X$ in Eq.(2.13) give a homogeneous polynomial of second degree in the velocities plus an inhomogeneous term in the $q^i$. Due to (2.15), such a polynomial has to be identically zero and then each coefficient must be independently zero. If $n$ is the dimension of the configuration space (i.e. the dimension of the minisuperspace), we get $\{1 + n(n + 1)/2\}$ partial differential equations whose solutions assign the symmetry, as we shall see below. Such a symmetry is over–determined and, if a solution exists, it is expressed in terms of integration constants instead of boundary conditions.

In the Hamiltonian formalism, we have

$$[\Sigma_j, \mathcal{H}] = 0, \quad 1 \leq j \leq m,$$  \hspace{1cm} (2.33)

as it must be for conserved momenta in quantum mechanics and the Hamiltonian has to satisfy the relations

$$L_t \mathcal{H} = 0,$$  \hspace{1cm} (2.34)

in order to obtain a Noether symmetry. The vector $\Gamma$ is defined by \textbf{[11]}. \[9\]
\[ \Gamma = \dot{q}^{i} \frac{\partial}{\partial q^{i}} + \ddot{q}^{i} \frac{\partial}{\partial \dot{q}^{i}}. \]  

(2.35)

Let us now go to the minisuperspace quantum cosmology and to the semi-classical interpretation of the wave function of the universe.

By a straightforward canonical quantization procedure, we have

\[ \pi_{j} \rightarrow \hat{\pi}_{j} = -i \partial_{j}, \]  

(2.36)

\[ \mathcal{H} \rightarrow \hat{\mathcal{H}}(q^{j}, -i \partial_{q}^{i}). \]  

(2.37)

It is well known that the Hamiltonian constraint gives the WDW equation, so that if \( |\Psi > \) is a state of the system (i.e. the wave function of the universe), dynamics is given by

\[ \mathcal{H}|\Psi > = 0. \]  

(2.38)

If a Noether symmetry exists, the reduction procedure outlined above can be applied and then, from (2.28) and (2.32), we get

\[ \pi_{1} \equiv \frac{\partial L}{\partial \dot{Q}_{1}} = i x_{1} \theta_{L} = \Sigma_{1}, \]  

\[ \pi_{2} \equiv \frac{\partial L}{\partial \dot{Q}_{2}} = i x_{2} \theta_{L} = \Sigma_{2}, \]  

\[ \ldots \ldots \ldots, \]  

depending on the number of Noether symmetries. After quantization, we get

\[ -i \partial_{1}|\Psi > = \Sigma_{1}|\Psi >, \]  

\[ -i \partial_{2}|\Psi > = \Sigma_{2}|\Psi >, \]  

\[ \ldots \ldots \ldots, \]  

(2.40)

which are nothing else but translations along the \( Q^{j} \) axis singled out by corresponding symmetry. Eqs. (2.40) can be immediately integrated and, being \( \Sigma_{j} \) real constants, we obtain oscillatory behaviours for \( |\Psi > \) in the directions of symmetries, i.e.

\[ |\Psi > = \sum_{j=1}^{m} e^{i \Sigma_{j} Q^{j}} |\chi(Q^{j}) >, \quad m < l \leq n, \]  

(2.41)

where \( m \) is the number of symmetries, \( l \) are the directions where symmetries do not exist, \( n \) is the total dimension of minisuperspace. It is worthwhile to note that the component \( |\chi > \) of the wave function could also depend on \( \Sigma_{j} \) but it is not possible to state “in general” if it is oscillating.

Viceversa, dynamics given by (2.38) can be reduced by (2.40) if and only if it is possible to define constant conjugate momenta as in (2.39), that is oscillatory behaviours of a subset of solutions \( |\Psi > \) exist only if Noether symmetry exists for dynamics.

The \( m \) symmetries give first integrals of motion and then the possibility to select classical trajectories. In one and two–dimensional minisuperspaces, the existence of a Noether symmetry allows the complete solution of the problem and to get the full semi–classical limit.
of minisuperspace quantum cosmology. By these arguments, the Halliwell request that an oscillatory wave function predict correlations between coordinates and canonical conjugate momenta [9] is fully recovered.

In conclusion, we can set out the following

**Theorem**: In the semi–classical limit of quantum cosmology and in the framework of minisuperspace approximation, the reduction procedure of dynamics, due to the existence of Noether symmetries, allows to select a subset of the solution of WDW equation where oscillatory behaviours are found. As consequence, correlations between coordinates and canonical conjugate momenta emerge so that classical cosmological solutions can be recovered. Viceversa, if a subset of the solution of WDW equation has an oscillatory behaviour, due to Eq.(2.40), conserved momenta have to exist and Noether symmetries are present. In other words, Noether symmetries select classical universes.

In what follows, we shall give realizations of such a statement for minisuperspace cosmological models derived from extended gravity theories.

### III. SCALAR–TENSOR GRAVITY COSMOLOGIES

Let us take into account a nonminimally coupled theory of gravity of the form

\[
A = \int d^4x \sqrt{-g} \left[ F(\varphi)R + \frac{1}{2}g^{\mu\nu}\varphi_\mu\varphi_\nu - V(\varphi) \right],
\]

(3.1)

where \( F(\varphi) \) and \( V(\varphi) \) are respectively the coupling and the potential of a scalar field [13].

We are using, from now on, physical units \( 8\pi G = c = \hbar = 1 \), so that the standard Einstein coupling is recovered for \( F(\varphi) = -1/2 \).

Let us restrict, for the sake of simplicity, to a FRW cosmology. The Lagrangian in (3.1) becomes

\[
\mathcal{L} = 6a\dot{a}^2 F + 6a^2 \dot{\varphi} F - 6k a F + a^3 \left[ \frac{\dot{\varphi}^2}{2} - V \right],
\]

(3.2)

in terms of the scale factor \( a \).

The configuration space of such a Lagrangian is \( Q \equiv \{a, \varphi\} \), i.e. a two–dimensional minisuperspace. A Noether symmetry exists if (2.15) holds. In this case, it has to be

\[
X = \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial \varphi} + \dot{\alpha} \frac{\partial}{\partial \dot{a}} + \dot{\beta} \frac{\partial}{\partial \dot{\varphi}},
\]

(3.3)

where \( \alpha, \beta \) depend on \( a, \varphi \). The system of partial differential equation given by (2.15) is
\begin{align}
F(\varphi) \left[ \alpha + 2a \frac{\partial \alpha}{\partial a} \right] + a F'(\varphi) \left[ \beta + a \frac{\partial \beta}{\partial a} \right] &= 0, \\
3\alpha + 12 F'(\varphi) \frac{\partial \alpha}{\partial \varphi} + 2a \frac{\partial \beta}{\partial \varphi} &= 0, \\
a \beta F''(\varphi) + \left[ 2\alpha + a \frac{\partial \alpha}{\partial a} + \frac{\partial \beta}{\partial \varphi} \right] F'(\varphi) + \frac{2}{6} a^2 \frac{\partial \beta}{\partial a} &= 0, \\
[3aV(\varphi) + a \beta V'(\varphi)]a^2 + 6k[\alpha F(\varphi) + a \beta F'(\varphi)] &= 0.
\end{align}

(3.4) 

(3.5) 

(3.6) 

(3.7)

The prime indicates the derivative with respect to \( \varphi \). The number of equations is 4 as it has to be, being \( n = 2 \). Several solutions exist for this system \([13–15]\). They determine also the form of the model since the system (3.4)–(3.7) gives \( \alpha, \beta, F(\varphi), \) and \( V(\varphi) \). For example, if the spatial curvature is \( k = 0 \), a solution is

\[ \alpha = -\frac{2}{3} p(s) \beta_0 a^{s^1} \varphi^{m(s)-1}, \quad \beta = \beta_0 a^{s} \varphi^{m(s)}, \]

(3.8)

\[ F(\varphi) = D(s) \varphi^2, \quad V(\varphi) = \lambda \varphi^{2p(s)}, \]

(3.9)

where

\[ D(s) = \frac{(2s + 3)^2}{48(s + 1)(s + 2)}, \quad p(s) = \frac{3(s + 1)}{2s + 3}, \quad m(s) = \frac{2s^2 + 6s + 3}{2s + 3}, \]

(3.10)

and \( s, \lambda \) are free parameters. The change of variables \((2.23)\) gives

\[ w = \sigma_0 a^3 \varphi^{2p(s)}, \quad z = \frac{3}{\beta_0 \lambda(s)} a^{-s} \varphi^{1-m(s)}, \]

(3.11)

where \( \sigma_0 \) is an integration constant and

\[ \chi(s) = -\frac{6s}{2s + 3}. \]

(3.12)

Lagrangian (3.2) becomes, for \( k = 0 \),

\[ \mathcal{L} = \gamma(s) w^{s/3} \dot{z} \dot{w} - \lambda w, \]

(3.13)

where \( z \) is cyclic and

\[ \gamma(s) = \frac{2s + 3}{12 \sigma_0^2 (s + 2)(s + 1)}. \]

(3.14)

The conjugate momenta are

\[ \pi_z = \frac{\partial \mathcal{L}}{\partial \dot{z}} = \gamma(s) w^{s/3} \dot{w}, \quad \pi_w = \frac{\partial \mathcal{L}}{\partial \dot{w}} = \gamma(s) w^{s/3} \dot{z}, \]

(3.15)

and the Hamiltonian is
The Noether symmetry is given by
\[ \pi_z = \Sigma_0. \] (3.17)
Quantizing Eqs. (3.15), we have
\[ \pi \rightarrow -i\partial_z, \quad \pi_w \rightarrow -i\partial_w, \] (3.18)
and then the WDW equation
\[ [(i\partial_z)(i\partial_w) + \tilde{\lambda}w^{1+s/3}]|\Psi\rangle = 0, \] (3.19)
where \( \tilde{\lambda} = \gamma(s)\lambda. \)

The quantum version of constraint (3.17) is
\[ -i\partial_z|\Psi\rangle = \Sigma_0|\Psi\rangle, \] (3.20)
so that dynamics results reduced. A straightforward integration of Eqs. (3.19) and (3.20) gives
\[ |\Psi\rangle = |\Omega(w)\rangle |\chi(z)\rangle \propto e^{i\Sigma_0 z} e^{-i\tilde{\lambda}w^{2+s/3}}, \] (3.21)
which is an oscillating wave function. In the semi–classical limit, we have two first integrals of motion: \( \Sigma_0 \) (i.e. the equation for \( \pi_z \)) and \( E_L = 0 \), i.e. the Hamiltonian (3.16) which becomes the equation for \( \pi_w \). Classical trajectories in the configuration space \( Q \equiv \{w, z\} \) are immediately recovered
\[ w(t) = [k_1 t + k_2]^{3/(s+3)}, \] (3.22)
\[ z(t) = [k_1 t + k_2]^{(s+6)/(s+3)} + z_0, \] (3.23)
then, going back to \( Q \equiv \{a, \varphi\} \), we get the classical cosmological behaviour
\[ a(t) = a_0(t - t_0)^l(s), \] (3.24)
\[ \varphi(t) = \varphi_0(t - t_0)^q(s), \] (3.25)
where
\[ l(s) = \frac{2s^2 + 9s + 6}{s(s + 3)}, \quad q(s) = -\frac{2s + 3}{s}. \] (3.26)

Depending on the value of \( s \), we get Friedman, power–law, or pole–like behaviours.

If we take into account generic Bianchi models, the configuration space is \( Q \equiv \{a_1, a_2, a_3, \varphi\} \) and more than one symmetry can exist as it is shown in [12]. The considerations on the oscillatory regime of the wave function of the universe and the recovering of classical behaviours are exactly the same.
IV. FOURTH-ORDER GRAVITY COSMOLOGIES

Similar arguments work for higher–order gravity cosmology. In particular, let us consider fourth–order gravity given by the action

\[ A = \int d^4x \sqrt{-g} f(R), \] (4.1)

where \( f(R) \) is a generic function of scalar curvature. If \( f(R) = R + 2\Lambda \), the standard second–order gravity is recovered. We are discarding matter contributions. Reducing the action to a point-like, FRW one, we have to write

\[ A = \int dt L(a, \dot{a}; R, \dot{R}), \] (4.2)

where dot means derivative with respect to the cosmic time. The scale factor \( a \) and the Ricci scalar \( R \) are the canonical variables. This position could seem arbitrary since \( R \) depends on \( a, \dot{a}, \ddot{a} \), but it is generally used in canonical quantization \[16–18\]. The definition of \( R \) in terms of \( a, \dot{a}, \ddot{a} \) introduces a constraint which eliminates second and higher order derivatives in action (4.2), and yields to a system of second order differential equations in \( \{a, R\} \). Action (4.2) can be written as

\[ A = 2\pi^2 \int dt \left\{ a^3 f(R) - \lambda \left[ R + 6 \left( \frac{\dddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) \right] \right\}, \] (4.3)

where the Lagrange multiplier \( \lambda \) is derived by varying with respect to \( R \). It is

\[ \lambda = a^3 f'(R). \] (4.4)

Here prime means derivative with respect to \( R \). To recover a more strict analogy with previous scalar–tensor models, let us introduce the auxiliary field

\[ p \equiv f'(R), \] (4.5)

so that the Lagrangian in (4.3) becomes

\[ \mathcal{L} = 6a\dddot{a} p + 6a^2 \dot{a} \dot{p} - 6kap - a^3 W(p), \] (4.6)

which is of the same form of (3.2) a part the kinetic term. This is an Helmholtz–like Lagrangian \[19\] and \( a, p \) are independent fields. The potential \( W(p) \) is defined as

\[ W(p) = h(p)p - r(p), \] (4.7)

where

\[ r(p) = \int f'(R)dR = \int pdR = f(R), \quad h(p) = R, \] (4.8)

such that \( h = (f')^{-1} \) is the inverse function of \( f' \). The configuration space is now \( Q \equiv \{a, p\} \) and \( p \) has the same role of the above \( \varphi \). Condition (2.15) is now realized by the vector field
\[ X = \alpha(a, p) \frac{\partial}{\partial a} + \beta(a, p) \frac{\partial}{\partial p} + \dot{\alpha} \frac{\partial}{\partial \dot{a}} + \dot{\beta} \frac{\partial}{\partial \dot{p}} \]  

(4.9)

and explicitly it gives the system

\[ p \left[ \alpha + 2a \frac{\partial \alpha}{\partial a} \right] p + a \left[ \beta + a \frac{\partial \beta}{\partial a} \right] = 0, \]  

(4.10)

\[ a^2 \frac{\partial \alpha}{\partial p} = 0, \]  

(4.11)

\[ 2\alpha + a \frac{\partial \alpha}{\partial a} + 2p \frac{\partial \alpha}{\partial p} + a \frac{\partial \beta}{\partial p} = 0, \]  

(4.12)

\[ 6k[ \alpha p + \beta a ] + a^2[3\alpha W + a\beta \frac{\partial W}{\partial p}] = 0. \]  

(4.13)

The solution of this system, i.e. the existence of a Noether symmetry, gives \( \alpha, \beta \) and \( W(p) \). It is satisfied for

\[ \alpha = \alpha(a), \quad \beta(a, p) = \beta_0 a^s p, \]  

(4.14)

where \( s \) is a parameter and \( \beta_0 \) is an integration constant. In particular,

\[ s = 0 \rightarrow \alpha(a) = -\frac{\beta_0}{3} a, \quad \beta(p) = \beta_0 p, \quad W(p) = W_0 p, \quad k = 0, \]  

(4.15)

\[ s = -2 \rightarrow \alpha(a) = -\frac{\beta_0}{a} a, \quad \beta(a, p) = \beta_0 \frac{p}{a^2}, \quad W(p) = W_1 p^3, \quad \forall \ k, \]  

(4.16)

where \( W_0 \) and \( W_1 \) are constants. As above, the new set of variables \( Q^j = Q^j(q^i) \) adapted to the foliation induced by \( X \) are given by the system (2.23). Let us discuss separately the solutions (4.13) and (4.16).

**A. The case \( s = 0 \)**

The induced change of variables \( Q \equiv \{a, p\} \rightarrow \tilde{Q} \equiv \{w, z\} \) can be

\[ w(a, p) = a^3 p, \quad z(p) = \ln p. \]  

(4.17)

Lagrangian (4.6) becomes

\[ \tilde{L}(w, \dot{w}, \dot{z}) = \dot{z} \dot{w} - 2w \dot{z}^2 + \frac{\dot{w}^2}{w} - 3W_0 w. \]  

(4.18)

and, obviously, \( z \) is the cyclic variable. The conjugate momenta are

\[ \pi_z \equiv \frac{\partial \tilde{L}}{\partial \dot{z}} = \dot{w} - 4\dot{z} = \Sigma_0, \]  

(4.19)
\[ \pi_w \equiv \frac{\partial \tilde{L}}{\partial \dot{w}} = \dot{z} + \frac{2\dot{w}}{w}. \]  
(4.20)

and the Hamiltonian is

\[ \mathcal{H}(w, \pi_w, \pi_z) = \pi_w \pi_z - \frac{\pi_z^2}{w} + 2w\pi_w^2 + 6W_0 w. \]  
(4.21)

By canonical quantization, reduced dynamics is given by

\[ \left[ \partial_z^2 - 2w^2 \partial_w^2 - w \partial_w \partial_z + 6W_0 w^2 \right] |\Psi\rangle = 0, \]  
(4.22)

\[ -i \partial_z |\Psi\rangle = \Sigma_0 |\Psi\rangle. \]  
(4.23)

However, we have done simple factor ordering considerations in the WDW equation (4.22). Immediately, the wave function has an oscillatory factor, being

\[ |\Psi\rangle \sim e^{i\Sigma_0 z} |\chi(w)\rangle. \]  
(4.24)

The function \(|\chi\rangle\) satisfies the Bessel differential equation

\[ \left[ w^2 \partial_w^2 + i \frac{\Sigma_0}{2} w \partial_w + \left( \frac{\Sigma_0^2}{2} - 3W_0 w^2 \right) \right] \chi(w) = 0, \]  
(4.25)

whose solutions are linear combinations of Bessel functions \(Z_\nu(w)\)

\[ \chi(w) = w^{1/2 - i\Sigma_0/4} Z_\nu(\lambda w), \]  
(4.26)

where

\[ \nu = \pm \frac{1}{4} \sqrt{4 - 9\Sigma_0^2 - i4\Sigma_0}, \quad \lambda = \pm 9 \sqrt{\frac{W_0}{2}}. \]  
(4.27)

The oscillatory regime for this component depends on the reality of \(\nu\) and \(\lambda\). The wave function of the universe, from Noether symmetry (4.15) is then

\[ \Psi(z, w) \sim e^{i\Sigma_0[z-(1/4)\ln w]} w^{1/2} Z_\nu(\lambda w). \]  
(4.28)

For large \(w\), the Bessel functions have an exponential behaviour [20], so that the wave function (4.28) can be written as

\[ \Psi \sim e^{i [\Sigma_0 z - (\Sigma_0/4) \ln w \pm \lambda w]}. \]  
(4.29)

By identifying the exponential factor of (4.29) with \(S_0\), we can recover the conserved momenta \(\pi_z, \pi_w\) and select classical trajectories. Going back to the old variables, we get the cosmological solutions

\[ a(t) = a_0 e^{(\lambda/6)t} \exp \left\{ -\frac{z_1}{3} e^{-(2\lambda/3)t} \right\}, \]  
(4.30)

\[ p(t) = p_0 e^{(\lambda/6)t} \exp \left\{ z_1 e^{-(2\lambda/3)t} \right\}, \]  
(4.31)

where \(a_0, p_0\) and \(z_1\) are integration constants. It is clear that \(\lambda\) plays the role of a cosmological constant and inflationary behaviour is asymptotically recovered.
B. The case $s = -2$

The new variables adapted to the foliation for the solution $w(a, p) = ap, \quad z(a) = a^2$. (4.32)

and Lagrangian $L$ assumes the form

$$\tilde{L}(w, \dot{w}, \dot{z}) = 3\dot{z}\dot{w} - 6kw - W_1w^3,$$ (4.33)

The conjugate momenta are

$$\pi_z = \frac{\partial \tilde{L}}{\partial \dot{z}} = 3\dot{w} = \Sigma_1,$$ (4.34)

$$\pi_w = \frac{\partial \tilde{L}}{\partial \dot{w}} = 3\dot{z}.$$ (4.35)

The Hamiltonian is given by

$$\mathcal{H}(w, \pi_w, \pi_z) = \frac{1}{3}\pi_z\pi_w + 6kw + W_1w^3.$$ (4.36)

Going over the same steps as above, the wave function of the universe is given by

$$\Psi(z, w) \sim e^{[\Sigma_1z + 9kw^2 + (3W_1/4)w^4]},$$ (4.37)

and the classical cosmological solutions are

$$a(t) = \pm \sqrt{h(t)}, \quad p(t) = \pm c_1 + \frac{(\Sigma_1/3) t}{\sqrt{h(t)}},$$ (4.38)

where

$$h(t) = \left(\frac{W_1\Sigma_1^3}{36}\right)t^4 + \left(\frac{W_1w_1\Sigma_1}{6}\right)t^3 + \left(k\Sigma_1 + \frac{W_1w_1^2\Sigma_1}{2}\right)t^2 + w_1(6k + W_1w_1^2)t + z_2.$$ (4.39)

$w_1, \ z_1 \text{ and } z_2$ are integration constants. Immediately we see that, for large $t$

$$a(t) \sim t^2, \quad p(t) \sim \frac{1}{t},$$ (4.40)

which is a power–law inflationary behaviour.
V. HIGHER THAN FOURTH–ORDER GRAVITY COSMOLOGIES

Minisuperspaces which are suitable for the above analysis can be found for higher than fourth–order theories of gravity as

\[ \mathcal{A} = \int d^4x \sqrt{-g} f(R, \Box R). \tag{5.1} \]

In this case, the configuration space is \( Q = \{a, R, \Box R\} \) considering \( \Box R \) as an independent degree of freedom \([17,18,21]\). The FRW point–like Lagrangian is formally

\[ \mathcal{L} = \mathcal{L}(a, \dot{a}, R, \dot{R}, \Box R, (\Box R)) \tag{5.2} \]

and the constraints

\[ R = -6 \left[ \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right], \tag{5.3} \]

\[ \Box R = \dddot{R} + 3 \frac{\dot{a}}{a} \dot{R} \tag{5.4} \]

holds. Using the above Lagrange multiplier approach, we get the Helmholz point–like Lagrangian

\[ \mathcal{L} = 6aa^2p + 6a^2a\dot{p} - 6kap - a^3\dot{h}q - a^3W(p, q), \tag{5.5} \]

where

\[ p = \frac{\partial f}{\partial R}, \quad q = \frac{\partial f}{\partial \Box R}, \tag{5.6} \]

\[ W(p, q) = h(p)p + g(q)q - f, \tag{5.7} \]

and

\[ h(p) = R, \quad g(q) = \Box R, \quad f = f(R, \Box R). \tag{5.8} \]

Now the minisuperspace is three–dimensional but, again, the Noether symmetries can be recovered. Cases of physical interest \([21]\) are

\[ f(R, \Box R) = F_0 R + F_1 R^2 + F_2 R \Box R, \tag{5.9} \]

\[ f(R, \Box R) = F_0 R + F_1 \sqrt{R \Box R}, \tag{5.10} \]

discussed in details in \([22]\). Also here the existence of the symmetry selects the form of the model and allows to reduce the dynamics. Once it is identified, we can perform the change of variables induced by foliation using Eqs. \((2.23)\), if a symmetry is present, or Eqs. \((2.23)\), is two symmetries are present. In both cases,
\[ \mathcal{Q} \equiv \{ a, R, \Box R \} \rightarrow \tilde{\mathcal{Q}} \equiv \{ z, u, w \}, \]  

(5.11)

where one or two variables are cyclic in Lagrangian (5.5). Taking into account, for example, the case (5.10), we get

\[ \tilde{\mathcal{L}} = 3[w\dot{w}^2 - kw] - F_1 \left[ 3w\dot{w}^2 u + 3w^2 \dot{w} u + \frac{w^3 \dddot{u}}{2u^2} - 3kwu \right], \]  

(5.12)

where we assume \( F_0 = -1/2 \), the standard Einstein coupling, \( z \) is the cyclic variable and

\[ z = R, \quad u = \sqrt{\frac{\Box R}{R}}, \quad w = a. \]  

(5.13)

The conserved quantity is

\[ \Sigma_0 = \frac{w^3 \dot{u}}{2u^2}. \]  

(5.14)

Using the canonical procedure of quantization and deriving the WDW equation from (5.12), the wave function of the universe is

\[ |\Psi > \sim e^{i\Sigma_0 z}|\chi(u) > |\Theta(w) >, \]  

(5.15)

where \( \chi(u) \) and \( \Theta(w) \) are combinations of Bessel functions. The oscillatory subset of the solution is evident. In the semi–classical limit, using the conserved momentum (5.14), we obtain the cosmological behaviours

\[ a(t) = a_0 t, \quad a(t) = a_0 t^{1/2}, \quad a(t) = a_0 e^{\kappa t}, \]  

(5.16)

depending on the choice of boundary conditions.

**VI. DISCUSSION AND CONCLUSIONS**

In this paper, we discussed the connection of Noether symmetries for minisuperspace cosmological models to the recovering of classical solutions. If the wave function of the universe is related to the probability to get a given classical cosmology, the existence of such symmetries tell us when the WDW wave function of the universe has oscillatory behaviours connected to the recovering of correlations between coordinates and conjugate canonical momenta \( [9] \). In this sense, the Hartle criterion to get correlations capable of selecting classical universes works.

Some remarks are necessary at this point. First of all, we have to stress that the wave function is *only* related to the probability to get a certain behaviour but it is not the probability amplitude since, till now, quantum cosmology is not a unitary theory. Furthermore, the Hartle criterion works in the context of an Everett–type interpretation of quantum cosmology \( [23,24] \) which assumes the ideas that the universe branches into a large number of copies of itself whenever a measurement is made. This point of view is called *Many Worlds* interpretation of quantum cosmology. Such an interpretation is just one way of thinking
and gives a formulation of quantum mechanics designed to deal with correlations internal to individual, isolated systems. The Hartle criterion gives an operative interpretation of such correlations. In particular, if the wave function is strongly peaked in some region of configuration space, we predict that we will observe the correlations which characterize that region. On the other hand, if the wave function is smooth in some region, we predict that correlations which characterize that region are precluded to the observations.

If the wave function is neither peaked nor smooth, no predictions are possible from observations. In other words, we can read the correlations of some region of minisuperspace as causal connections. However, the validity of minisuperspace approximation is often not completely accepted and it is still matter of debate [25].

As we said above, the analogy with standard quantum mechanics is straightforward. By considering the case in which the individual system consists of a large number of identical subsystems, one can derive from the above interpretation, the usual probabilistic interpretation of quantum mechanics for the subsystems [1,9].

What we proposed in this paper is a criterion by which the Hartle point of view can be recovered without arbitrariness. If a Noether symmetry (or more than one) is present for a given minisuperspace model, then oscillatory subsets of the wave function of the universe are found. Viceversa, oscillatory parts of the wave function can be always connected to conserved momenta and then to Noether symmetries.

From a general point of view, this is the same philosophy of many branches of physics: Finding symmetries allows to solve dynamics, gives the main features of systems and simplify the interpretation of results.

However the above scheme should be enlarged to more general classes of minisuperspaces in order to seek for its application to the full field theory (i.e. to the infinite–dimensional superspace). Only in this sense, one could claim for the validity of the approach to the full semiclassical limit of quantum cosmology.
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