SHUBIN TYPE FOURIER INTEGRAL OPERATORS AND EVOLUTION EQUATIONS

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ABSTRACT. We study the Cauchy problem for an evolution equation of Schrödinger type. The Hamiltonian is the Weyl quantization of a real homogeneous quadratic form with a pseudodifferential perturbation of negative order from Shubin’s class. We prove that the propagator is a Fourier integral operator of Shubin type of order zero. Using results for such operators and corresponding Lagrangian distributions, we study the propagator and the solution, and derive phase space estimates for them.

1. Introduction

In this article we study the propagator and solution to the Cauchy problem

\[
\begin{aligned}
\partial_t u(t,x) + i(q^w(x,D) + p^w(x,D))u(t,x) &= 0, \quad t > 0, \quad x \in \mathbb{R}^d, \\
u(0,\cdot) &= u_0 \in \mathcal{S}'(\mathbb{R}^d),
\end{aligned}
\]

where \(q^w(x,D)\) is the Weyl quantization of a real homogeneous quadratic form on \(T^*\mathbb{R}^d\) and \(p^w(x,D)\) is a pseudodifferential perturbation operator of negative order of Shubin type. Particular examples of interest are perturbations to the free Schrödinger equation and the quantum harmonic oscillator.

The Shubin class \(\Gamma^m, m \in \mathbb{R}\), introduced in [27], is defined as the space of all functions \(a \in C^\infty(\mathbb{R}^{2d})\) that satisfy estimates of the form

\[
|\partial_\alpha^a \partial_\xi^\beta a(x,\xi)| \lesssim (1 + |x| + |\xi|)^{m-|\alpha+\beta|}, \quad (x,\xi) \in \mathbb{R}^{2d}, \quad \alpha, \beta \in \mathbb{N}^d.
\]

Differently from the Hörmander symbols, the elements of \(\Gamma^m\) exhibit a symmetric behavior in the decay with respect to \(x\) and \(\xi\). A relevant example in the Shubin classes is the symbol \(a(x,\xi) = |x|^2 + |\xi|^2 \in \Gamma^2\) for the harmonic oscillator operator.

The theory of pseudodifferential operators with symbols in the Shubin classes has been developed in [27] and widely applied to the study of several classes of partial differential equations, see e.g. [1–5,14,15,18,21,23,25,26,28]. Helffer and Robert [14,15] introduced Fourier integral operators (FIOs) with Shubin type amplitudes and phase functions that are generalized quadratic. The main applications concern spectral theory and the construction of a parametrix for the Cauchy problem associated to the harmonic oscillator. Similar oscillatory integrals have been considered by Asada and Fujiwara [1]. We mention also an extension of this theory to more general classes of amplitudes obtained by replacing the weight function \((1 + |x| + |\xi|)\) by multi-quasi-elliptic weights, see [2].

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Recently [6, 7] we considered a class of FIOs with Shubin amplitudes and quadratic phase functions. We proved phase space estimates for an FBI type transform of their Schwartz kernels, see [28] for similar estimates in a particular case. We also proved that every operator in our class can be written as the composition of a metaplectic operator and a pseudodifferential operator with Shubin symbol. As a byproduct of the analysis we obtained a new notion of Lagrangian distributions in the Shubin framework which generalize the properties of the kernels of the FIOs.

In the present paper we apply these results to the study of the Cauchy problem (CP). We assume that the perturbative term has a symbol \( p \in \Gamma^{-\delta} \) for \( \delta > 0 \). It is well known that in case of a vanishing perturbation, \( p = 0 \), the solution operator to the system (CP) is a metaplectic operator \( \mu(\chi_t) \) associated with the Hamiltonian flow \( \chi_t \) of \( q \), see e.g. [10]. In [8] it was proved under less regularity assumptions on \( p \) that the system (CP) admits a propagator from a class of generalized metaplectic operators, see also [9, 30]. These have the form \( \mu(\chi_t) a^w_t(x, D) \), where \( a_t \) belongs to a modulation space of Sjöstrand type, the elements of which are not necessarily smooth. In this paper we make a stronger assumption and obtain a stronger conclusion. Using ideas from [14, 15], we prove that the propagator is an operator of the form \( \mu(\chi_t) a^w_t(x, D) \) for \( a_t \in \Gamma^0 \).

The upshot of this is as follows. Generalized metaplectic operators of the latter form may be represented as FIOs for which we have developed a calculus [7]. This opens up the possibility to study (CP) with tools from microlocal analysis. In particular we study the singularities of solutions to (CP) in detail, proving propagation results for Lagrangian type singularities and phase space estimates. We stress the fact that our results apply but are not limited to Schrödinger type equations since we do not need to assume ellipticity of the principal term \( q^w(x, D) \).

The paper is organized as follows. In Section 2 we recall the technical tools for our analysis, in particular aspects of pseudodifferential quantization, metaplectic and symplectic analysis, Shubin type FIOs, and properties of an FBI type phase space transform which is a fundamental tool. In Section 3 we construct a parametrix to (CP) and prove that the propagator is a Shubin type FIO. Finally in Section 4 we study the singularities of propagators and solutions to (CP) and deduce phase space estimates for them.

2. Preliminaries on microlocal analysis in Shubin’s class

**Basic notation.** The gradient operator with respect to \( x \in \mathbb{R}^d \) is denoted \( \nabla_x \). The symbols \( \mathcal{S}(\mathbb{R}^d) \) and \( \mathcal{S}'(\mathbb{R}^d) \) denote the Schwartz space of rapidly decaying smooth functions and the tempered distributions, respectively. We write \( (f, g) \) for the sesquilinear pairing, conjugate linear in the second argument, between a distribution \( f \) and a test function \( g \), as well as the \( L^2 \) scalar product if \( f, g \in L^2(\mathbb{R}^d) \). The linear pairing of a distribution \( f \) and a test function \( g \) is written \( \langle f, g \rangle \).

The symbols \( T_{x_0} u(x) = u(x - x_0) \) and \( M_{\xi} u(x) = e^{i\langle x, \xi \rangle} u(x) \), where \( \langle \cdot, \cdot \rangle \) denotes the inner product on \( \mathbb{R}^d \), are used for translation by \( x_0 \in \mathbb{R}^d \) and modulation by \( \xi \in \mathbb{R}^d \), respectively, applied to functions or distributions. For \( x \in \mathbb{R}^d \) we use \( \langle x \rangle := \sqrt{1 + |x|^2} \), and Peetre’s inequality is

\[
\langle x + y \rangle^s \leq C_s \langle x \rangle^s \langle y \rangle^{|s|}, \quad x, y \in \mathbb{R}^d, \quad C_s > 0, \quad s \in \mathbb{R}.
\]
The notation \( f(x) \preceq g(x) \) means \( f(x) \leq Cg(x) \) for some \( C > 0 \) for all \( x \) in the domain of \( f \) and of \( g \).

The Fourier transform for \( f \in \mathcal{S}(\mathbb{R}^d) \) is normalized as
\[
\hat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x)e^{-i(x,\xi)} \, dx
\]
which makes it unitary on \( L^2(\mathbb{R}^d) \). We write \( dx = (2\pi)^{-d} \, dx \) for the dual Lebesgue measure, denote by \( M_{d_1 \times d_2}(\mathbb{R}) \) the space of \( d_1 \times d_2 \) matrices with real entries, and by \( \text{GL}(d, \mathbb{R}) \subseteq M_{d \times d}(\mathbb{R}) \) the group of invertible matrices. The orthogonal projection on a linear subspace \( Y \subseteq \mathbb{R}^d \) is denoted \( \pi_Y \).

An integral transform of FBI type. The following integral transform has been used extensively in \([6,7]\) and is used also in this article. For more information see \([6]\).

**Definition 2.1.** Let \( u \in \mathcal{S}'(\mathbb{R}^d) \) and let \( g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\} \). The transform \( u \mapsto T_gu \) is defined by
\[
T_g u(x,\xi) = (2\pi)^{-d/2}(u, T_x M_\xi g), \quad x, \xi \in \mathbb{R}^d.
\]

If \( u \in \mathcal{S}(\mathbb{R}^d) \) then \( T_g u \in \mathcal{S}(\mathbb{R}^{2d}) \) by \([13]\) Theorem 11.2.5. The adjoint \( T^*_g \) is defined by \((T^*_g U, f) = (U, T_g f)\) for \( U \in \mathcal{S}(\mathbb{R}^{2d}) \) and \( f \in \mathcal{S}(\mathbb{R}^d) \). When \( U \) is a polynomially bounded measurable function we write
\[
T^*_g U(y) = (2\pi)^{-d/2} \int_{\mathbb{R}^{2d}} U(x,\xi) T_x M_\xi g(y) \, dx \, d\xi,
\]
where the integral is defined weakly so that \((T^*_g U, f) = (U, T_g f)_{L^2}\) for \( f \in \mathcal{S}(\mathbb{R}^d) \).

**Proposition 2.2.** \([13]\) Theorem 11.2.3 Let \( u \in \mathcal{S}'(\mathbb{R}^d) \) and let \( g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\} \). Then \( T_g u \in C^\infty(\mathbb{R}^{2d}) \) and there exists \( N \in \mathbb{N} \) such that
\[
|T_g u(x,\xi)| \lesssim \langle (x,\xi) \rangle^N, \quad (x,\xi) \in \mathbb{R}^{2d}.
\]
We have \( u \in \mathcal{S}(\mathbb{R}^d) \) if and only if for any \( N \geq 0 \)
\[
|T_g u(x,\xi)| \lesssim \langle (x,\xi) \rangle^{-N}, \quad (x,\xi) \in \mathbb{R}^{2d}.
\]

The transform \( T_g \) is related to the short-time Fourier transform \([13]\)
\[
V_g u(x,\xi) = (2\pi)^{-d/2}(u, M_\xi T_x g), \quad x, \xi \in \mathbb{R}^d,
\]
viz. \( T_g u(x,\xi) = e^{i(x,\xi)} V_g u(x,\xi) \). If \( g, h \in \mathcal{S}(\mathbb{R}^d) \) then
\[
T^*_g T_h u = (h, g) u, \quad u \in \mathcal{S}'(\mathbb{R}^d),
\]
and thus \( \|g\|_{L^2}^{-2} T^*_g T_g u = u \) for \( u \in \mathcal{S}'(\mathbb{R}^d) \) and \( g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\} \), cf. \([13]\).

Finally we recall the definition of the Gabor wave front set which describes global singularities of tempered distributions in phase space, cf. \([15,25,26]\).

**Definition 2.3.** If \( u \in \mathcal{S}'(\mathbb{R}^d) \) and \( g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\} \) then \( z_0 \in T^* \mathbb{R}^d \setminus \{0\} \) satisfies \( z_0 \notin \text{WF}(u) \) if there exists an open cone \( V \subseteq T^* \mathbb{R}^d \setminus \{0\} \) containing \( z_0 \), such that for any \( N \in \mathbb{N} \) there exists \( C_{V,g,N} > 0 \) such that \( |T_g u(z)| \leq C_{V,g,N} |z|^{-N} \) when \( z \in V \).

The Gabor wave front set is hence a closed conic subset of \( T^* \mathbb{R}^d \setminus \{0\} \). If \( u \in \mathcal{S}(\mathbb{R}^d) \) then \( \text{WF}(u) = \emptyset \) if and only if \( u \in \mathcal{S}(\mathbb{R}^d) \) \([15]\) Proposition 2.4].
**Weyl pseudodifferential operators.** We use pseudodifferential operators in the Weyl calculus with Shubin amplitudes \([21, 27]\). Recall that \(a \in C^\infty(\mathbb{R}^{N_1} \times \mathbb{R}^{N_2})\) is a Shubin amplitude of order \(m \in \mathbb{R}\), denoted \(a \in \Gamma^m(\mathbb{R}^{N_1} \times \mathbb{R}^{N_2})\), if it satisfies the estimates
\[
|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \lesssim \langle (x, \xi) \rangle^{m-|\alpha+\beta|}, \quad (\alpha, \beta) \in \mathbb{N}^{N_1} \times \mathbb{N}^{N_2}, \quad (x, \xi) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}.
\]

We write \(\Gamma^m = \Gamma^m(\mathbb{R}^{2d})\) and observe that \(\bigcap_{m \in \mathbb{R}} \Gamma^m = \mathcal{S}'(\mathbb{R}^{2d})\). The space \(\Gamma^m\) is a Fréchet space with respect to the seminorms that are the best constants hidden in \((2.1)\).

To a Shubin amplitude \(a \in \Gamma^m\) one associates its pseudodifferential Weyl quantization, which is the operator \(a^w(x, D)\) with Schwartz kernel
\[
K_a(x, y) = \int_{\mathbb{R}^d} e^{i(x-y, \xi)} a((x+y)/2, \xi) \, d\xi \in \mathcal{S}'(\mathbb{R}^{2d})
\]
interpreted as an oscillatory integral. Then \(a^w(x, D)\) is a continuous operator on \(\mathcal{S}(\mathbb{R}^d)\) that extends uniquely to a continuous operator on \(\mathcal{S}'(\mathbb{R}^d)\). If \(a \in \mathcal{S}'(\mathbb{R}^{2d})\) then \(a^w(x, D) : \mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)\) is continuous when \(\mathcal{S}'(\mathbb{R}^d)\) is equipped with its strong topology. Conversely, any continuous linear operator from \(\mathcal{S}'(\mathbb{R}^d)\), endowed with the strong topology, to \(\mathcal{S}'(\mathbb{R}^d)\) may be represented as \(a^w(x, D)\) for some \(a \in \mathcal{S}'(\mathbb{R}^{2d})\) \([29]\).

For \(a \in \mathcal{S}'(\mathbb{R}^{2d})\) and \(f, g \in \mathcal{S}(\mathbb{R}^d)\) we have
\[
(a^w(x, D)f, g) = (2\pi)^{-d/2}(a, W(g, f))
\]
where \(W(g, f)\) is the Wigner distribution \([10, 13]\).

\[
W(g, f)(x, \xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} g(x+y/2)f(x-y/2) e^{-i(y, \xi)} \, dy \in \mathcal{S}'(\mathbb{R}^d).
\]

The Weyl product \(a \# b : \Gamma^m_1 \times \Gamma^m_2 \rightarrow \Gamma^{m_1+m_2}\) is the continuous product (cf. \([27]\)) on the symbol level corresponding to composition of operators
\[
(a \# b)^w(x, D) = a^w(x, D)b^w(x, D).
\]

There is a scale of Sobolev spaces \(Q^s(\mathbb{R}^d), s \in \mathbb{R}\), defined by
\[
Q^s(\mathbb{R}^d) = \{ u \in \mathcal{S}'(\mathbb{R}^d) : v_u(x, D)u \in L^2(\mathbb{R}^d) \},
\]
where \(v_u(x, \xi) = \langle (x, \xi) \rangle^s\), which is adapted to the Shubin calculus. We have (cf. \([27]\), Corollary 25.2)
\[
\mathcal{S}(\mathbb{R}^d) = \bigcap_{s \in \mathbb{R}} Q^s(\mathbb{R}^d), \quad \mathcal{S}'(\mathbb{R}^d) = \bigcup_{s \in \mathbb{R}} Q^s(\mathbb{R}^d).
\]

The Weyl quantization of \(\Gamma^m\) yields continuous maps
\[
a^w(x, D) : Q^s(\mathbb{R}^d) \rightarrow Q^{s-m}(\mathbb{R}^d), \quad s \in \mathbb{R},
\]
and the \(Q^s \rightarrow Q^{s-m}\) operator norm of \(a^w(x, D)\) can be estimated by a finite linear combination of seminorms of \(a \in \Gamma^m\).

We use the description of \(Q^s\) in terms of localization operators \([21\) Proposition 1.7.12\]. Let \(\psi_0 = \pi^{-d/4} e^{-|x|^2/2}, x \in \mathbb{R}^d\). A localization operator \(A_a\) with symbol \(a \in \mathcal{S}'(\mathbb{R}^{2d})\) is defined by
\[
(A_a u, f) = (a T_{\psi_0} u, T_{\overline{\psi_0}} f), \quad u, f \in \mathcal{S}(\mathbb{R}^d).
\]
In terms of the localization operator $A_{u} := A_{u_{I}}$, the space $Q'(\mathbb{R}^{d})$ is the Hilbert modulation space of all $u \in \mathcal{S}'(\mathbb{R}^{d})$ such that $A_{u} u \in L^{2}(\mathbb{R}^{d})$, equipped with the norm $\|u\|_{Q'} = \|A_{u} u\|_{L^{2}}$.

It is possible to express localization operators as pseudodifferential operators (cf. [21 Section 1.7.2]) writing $A_{u} = b^{w}(x, D)$ where

$$b = \pi^{-d} e^{-|\cdot|^{2}} a.$$  

**Metaplectic operators.** We view $T^{*} \mathbb{R}^{d} \cong \mathbb{R}^{d} \times \mathbb{R}^{d}$ as a symplectic vector space equipped with the canonical symplectic form

$$\sigma((x, \xi), (x', \xi')) = \langle x', \xi \rangle - \langle x, \xi' \rangle, \quad (x, \xi), (x', \xi') \in T^{*} \mathbb{R}^{d}.$$  

The real symplectic group $\text{Sp}(d, \mathbb{R}) \subseteq \text{GL}(2d, \mathbb{R})$ is the set of matrices that leaves $\sigma$ invariant. A special symplectic matrix is the metaplectic group $\text{Mp}(d, \mathbb{R})$. We view $\text{Mp}(d, \mathbb{R})$ equipped with the canonical symplectic form

$$\sigma((x, \xi), (x', \xi')) = \langle x', \xi \rangle - \langle x, \xi' \rangle, \quad (x, \xi), (x', \xi') \in T^{*} \mathbb{R}^{d}.$$  

The two-to-one projection $\pi : \text{Mp}(d, \mathbb{R}) \to \text{Sp}(d, \mathbb{R})$ is a homomorphism modulo sign (cf. [10][17]). The operators $\mu(\chi)$ are homeomorphisms on $\mathcal{S}$ and on $\mathcal{S}'$, and are called metaplectic operators.

The metaplectic representation is the mapping $\text{Sp}(d, \mathbb{R}) \ni \chi \mapsto \mu(\chi)$ which is a homomorphism modulo sign

$$\mu(\chi_{1})\mu(\chi_{2}) = \pm \mu(\chi_{1}\chi_{2}), \quad \chi_{1}, \chi_{2} \in \text{Sp}(d, \mathbb{R}).$$

Two ways to overcome the sign ambiguity are to pass to “double-valued maps”, or to a representation of the so called 2-fold covering group of $\text{Sp}(d, \mathbb{R})$. The latter group is called the metaplectic group $\text{Mp}(d, \mathbb{R})$. The two-to-one projection $\pi : \text{Mp}(d, \mathbb{R}) \to \text{Sp}(d, \mathbb{R})$ is $\mu(\chi) \mapsto \chi$ whose kernel is $\{\pm 1\}$. The sign ambiguity may be fixed (hence it is possible to choose a section of $\pi$) along a continuous path $\mathbb{R} \ni t \mapsto \chi_{t} \in \text{Sp}(d, \mathbb{R})$. This involves the so called Maslov factor [20].

**Fourier integral operators with Shubin amplitudes.** In [7] we have introduced a class of Fourier integral operators (FIOs) with quadratic phase functions and Shubin amplitudes. The space of Shubin type FIOs of order $m \in \mathbb{R}$ associated with $\chi \in \text{Sp}(d, \mathbb{R})$, denoted $\mathcal{F}^{m}(\chi)$, consists of those operators $\mathcal{F}$ whose kernels admit oscillatory integral representations of the form

$$K_{a, \varphi}(x, y) = \int_{\mathbb{R}^{2d}} e^{i \varphi(x, y, \theta)} a(x, y, \theta) \, d\theta, \quad (x, y) \in \mathbb{R}^{2d},$$

where $a \in \Gamma^{m}(\mathbb{R}^{2d} \times \mathbb{R}^{N})$. The phase function $\varphi$ is a real quadratic form on $\mathbb{R}^{2d+N}$ which parametrizes the twisted graph Lagrangian

$$\Lambda'_{\chi} = \{(x, y, \xi, -\eta) \in T^{*} \mathbb{R}^{2d} : (x, \xi) = \chi(y, \eta)\} \subseteq T^{*} \mathbb{R}^{2d}$$

corresponding to $\chi \in \text{Sp}(d, \mathbb{R})$. (Cf. [7] Definitions 3.5 and 4.1.) We will not use this representation here but merely recall the following result [7 Theorem 4.15], see
The closure generates a strongly continuous group $\mathbb{R}$ operators on $L^2_s$ equipped with the domain $q$ its Weyl quantization $\mu$. First discuss the complex phase factor of $\mu$. $p_3$. The free evolution.

We first discuss the solution operator (propagator) in the (CP) $\mu K \in [7, \text{Proposition 4.16 and Corollary 5.4}]$. Suppose $j = 1$ $\mu$ metaplectic operator. The factorization is uniquely determined by the order of arrangement and choice of phase factor of $\mu(\chi)$. In particular $\mathcal{F}^m(I)$, where $I \in \text{GL}(2d, \mathbb{R})$ is the identity matrix, is the space of pseudodifferential operators with Shubin amplitudes of order $m \in \mathbb{R}$. A kernel of the form $K_{1,\varphi}$, i.e. trivial amplitude, corresponds to the operator $C_\varphi \mu(\chi)$ where $C_\varphi \in \mathbb{C} \setminus 0$ (cf. $[7][9][21]$). A fundamental result for FIOs is the following composition theorem.

**Theorem 2.5.** [4, Proposition 4.10] Let $\chi_j \in \text{Sp}(d, \mathbb{R})$ and suppose $\mathcal{H}_j \in \mathcal{F}^m(\chi_j)$, for $j = 1, 2$. Then $\mathcal{H}_1 \mathcal{H}_2 \in \mathcal{F}^{m_1+m_2}(\chi_1\chi_2)$.

We state the mapping properties of FIOs with respect to the Shubin–Sobolev spaces $Q^s$ and the Gabor wave front set respectively.

**Proposition 2.6.** [4, Proposition 4.16 and Corollary 5.4] Suppose $\chi \in \text{Sp}(d, \mathbb{R})$ and $\mathcal{H} \in \mathcal{F}^m(\chi)$. Then $\mathcal{H} : Q^s(\mathbb{R}^d) \to Q^{s-m}(\mathbb{R}^d)$ is continuous for all $s \in \mathbb{R}$. For all $u \in \mathcal{F}'(\mathbb{R}^d)$ we have $\text{WF}(\mathcal{H} u) \subseteq \chi \text{WF}(u)$.

### 3. Application to evolution equations

Consider the initial value Cauchy problem associated with a real homogeneous quadratic form $q \in \Gamma^2$ defined by $q(x, \xi) = \langle (x, \xi), Q(x, \xi) \rangle$ where $(x, \xi) \in \mathbb{R}^{2d}$ and $Q \in M_{2d \times 2d}(\mathbb{R})$ is symmetric, and a negative order perturbation $p \in \Gamma^{-\delta}$ where $\delta > 0$.

\[
\begin{aligned}
&\partial_t u(t, x) + i(q^w(x, D) + p^w(x, D))u(t, x) = 0, \quad t > 0, \quad x \in \mathbb{R}^d, \\
u(0, \cdot) = u_0 \in \mathcal{S}(\mathbb{R}^d).
\end{aligned}
\]

**3.1. The free evolution.** We first discuss the solution operator (propagator) in the unperturbed case $p = 0$. In this case the propagator is given by a metaplectic operator, given for $t \in \mathbb{R}$ by

\[
e^{-itq^w(x, D)} = \mu(\chi_t)
\]

where $\chi_t = e^{2itF} \in \text{Sp}(d, \mathbb{R})$ and $F = \mathcal{F}Q \in M_{2d \times 2d}(\mathbb{R})$ (see e.g. [7][8][10][23]). Note that the complex phase factor of $\mu(\chi_t)$ is fixed uniquely by the criterion $\mu(\chi_t)|_{t=0} = \text{id}$. We first discuss $\mu(\chi_t)$ as a group on $L^2(\mathbb{R}^d)$, then on $Q^s(\mathbb{R}^d)$.

Thus we consider $q^w(x, D)$ as an unbounded operator in $L^2(\mathbb{R}^d)$. Since $q$ is real-valued its Weyl quantization $q^w(x, D)$ is a symmetric operator. The closure of $-iq^w(x, D)$ equipped with the domain $\mathcal{S}$ equals its maximal realization, denoted $M_q$ [19 pp. 425–26]. The closure generates a strongly continuous group $\mathbb{R} \ni t \mapsto e^{-itq^w(x, D)}$ of unitary operators on $L^2$. The group gives the unique solution $\mu(\chi_t)u_0$ to (CP) in the space $C([0, \infty), L^2) \cap C^1((0, \infty), L^2)$.
for $u_0 \in D(M_q) \subseteq L^2$, see [22] Theorem 4.1.3].

Next we fix $s \in \mathbb{R}$ and consider $q^u(x, D)$ as an unbounded operator on $Q^s(\mathbb{R}^d)$. In this case $\mu(\chi_t)$ is in general no longer unitary but we still have the following result.

**Proposition 3.1.** For $s \in \mathbb{R}$ the group $\mathbb{R} \ni t \mapsto \mu(\chi_t)$ is a strongly continuous group of operators on $Q^s(\mathbb{R}^d)$ whose generator is a closed extension of $-iq^u(x, D)$.

**Proof.** By [12] Prop. 400, $\mu(\chi_t)$ is for fixed $t \in \mathbb{R}$ a homeomorphism on $Q^s$. First we prove a uniform bound for $\|\mu(\chi_t)\| \leq (Q^s)$ over $-T \leq t \leq T$ where $T > 0$.

By (3.2) we have $A_s = a^{u^s}(x, D)$ where $a(z) = \pi^{-d}(e^{-|z|^2} * v_s)(z)$ with $z \in \mathbb{R}^{2d}$. This implies $a \in \Gamma^s$ and $a$ is elliptic by [20] Proposition 2.3]. Since $\|\mu(\chi_t)f\|_{L^2} = \|f\|_{L^2}$ for all $t \in \mathbb{R}$ and $f \in L^2$ we obtain for $u \in Q^s$ using (3.2)

$$\|\mu(\chi_t)u\|_{Q^s} = \|A_s\mu(\chi_t)u\|_{L^2}$$

$$= \|\mu(\chi_t)^{-1}a^u(x, D)\mu(\chi_t)u\|_{L^2}$$

$$= \|a \circ \chi_t^{-1}a^u(x, D)\|_{L^2}$$

Indeed, by [21] Proposition 1.7.12] the inverse of $A_s$ exists and $A_s^{-1} = b^u(x, D)$ where $b \in \Gamma^{-s}$. We have

$$|\partial^2 (a \circ \chi_t)(z)| \leq C_\alpha e^{2t|\alpha|} \|F^{(|\alpha| + 2|\alpha|)}(z)|^{s-|\alpha|}, \ z \in \mathbb{R}^{2d}, \ \alpha \in \mathbb{N}^{2d}. $$

The set of symbols $a \circ \chi_t \in \Gamma^s$ is thus uniformly bounded over $t \in [-T, T]$. Thus $(a \circ \chi_t)\#b \in \Gamma^0$ is uniformly bounded over $t \in [-T, T]$.

By the Calderón–Vaillancourt theorem (see e.g. [10] Theorem 2.73] $\|a \circ \chi_t^s(x, D)A_s^{-1}\|_{L^2(\mathbb{R}^d)} < \infty$ uniformly over $t \in [-T, T]$. We have shown

$$\sup_{|t| \leq T} \|\mu(\chi_t)\| \leq (Q^s) < \infty, \ s \in \mathbb{R}. $$

Next let $f \in \mathcal{S}$ and write as above

$$A_s(\mu(\chi_t) - I)f = \mu(\chi_t)(a \circ \chi_t)u(x, D)f - a^u(x, D)f$$

$$= \mu(\chi_t)(a \circ \chi_t - a)u(x, D)f + (\mu(\chi_t) - I)a^u(x, D)f.$$ 

Since $\mu(\chi_t)$ is unitary on $L^2$ we obtain for $|t| \leq 1$

$$\|\mu(\chi_t) - I\|_{L^2} = \|A_s\mu(\chi_t) - I\|_{L^2}$$

$$\leq \|\mu(\chi_t)(a \circ \chi_t - a)u(x, D)f\|_{L^2} + \|\mu(\chi_t) - I)a^u(x, D)f\|_{L^2}$$

$$= \|a \circ \chi_t - a\|^u(x, D)f\|_{L^2} + \|\mu(\chi_t) - I)a^u(x, D)f\|_{L^2}.$$ 

The fact that $a \circ \chi_t - \chi_t \to 0$ in $\Gamma^{s+\nu}$ as $t \to 0$ provided $\nu > 0$ (cf. [17] Proposition 18.1.2] implies

$$\lim_{t \to 0} \frac{\|a \circ \chi_t - a\|^u(x, D)f\|_{L^2}}{\|f\|_{Q^{s+\nu}}} = 0.$$ 

Combining with the known strong continuity of $\mu(\chi_t)$ on $L^2$ we have shown

$$\lim_{t \to 0} \|\mu(\chi_t) - I\|_{Q^s} = 0, \ f \in \mathcal{S}(\mathbb{R}^d).$$
Finally, combining (3.3) with the fact that $\mathcal{S}$ is dense in $Q^s$ we can use [11, Proposition I.5.3] to conclude that $\mu(\chi_t)$ is a strongly continuous group on $Q^s$. □

The equality (3.1) thus holds on $Q^s$. The operators $\mu(\chi_t)$ are not necessarily unitary if $s \neq 0$. Due to (2.4) we may allow $u_0 \in \mathcal{S}'(\mathbb{R}^d)$. In fact for some $s \in \mathbb{R}$ we then have $u_0 \in Q^{s+2} \subseteq D(q^w(x,D))$, where $q^w(x,D)$ is considered an unbounded operator in $Q^s$.

By abuse of notation we use $q^w(x,D)$ to also denote a closed extension as an operator in $Q^s$. Again [22, Theorem 4.1.3] implies that $\mu(\chi_t)u_0$ is the unique solution to (CP) in $C([0,\infty),Q^s) \cap C^1((0,\infty),Q^s)$. We summarize:

**Proposition 3.2.** For $s \in \mathbb{R}$ the equation (CP) with $p = 0$ is solved uniquely by the strongly continuous group of operators $e^{-itq^w(x,D)} = \mu(\chi_t)$ on $Q^s(\mathbb{R}^d)$, and for each $t \in \mathbb{R}$ it is an FIO in $\mathcal{S}'(\chi_t)$. We have for $u_0 \in Q^{s+2}$ the unique solution

$$e^{-itq^w(x,D)}u_0 \in C([0,\infty),Q^s) \cap C^1((0,\infty),Q^s).$$

### 3.2. Construction of a parametrix to the perturbed equation.

We will now consider (CP) with a nonzero perturbation $p \in \Gamma^{-\delta}$. As a first step we note that the perturbation operator is bounded $p^w(x,D) : Q^s(\mathbb{R}^d) \rightarrow Q^{s+\delta}(\mathbb{R}^d)$ and compact $p^w(x,D) : Q^s(\mathbb{R}^d) \rightarrow Q^s(\mathbb{R}^d)$ [27, Proposition 25.4]. Perturbation theory (see e.g. [8, [11, Theorems III.1.3 and III.1.10]) gives the following conclusion.

Let $s \in \mathbb{R}$. The solution to (CP) for $u_0 \in Q^{s+2}(\mathbb{R}^d)$ is $T_tu_0$ where

$$T_t = \mu(\chi_t)C_t, \quad t \geq 0.$$  

Here $C_t$ is a strongly continuous semigroup of operators on $Q^s(\mathbb{R}^d)$ with operator norm estimate

$$\|C_t\|_{\mathcal{B}(Q^s)} \leq Me^{t(\omega + M\|p^w(x,D)\|_{\mathcal{B}(Q^s)})}, \quad t \geq 0,$$

where $M \geq 1$ and $\omega \geq 0$, and

$$C_t = \text{id} + \sum_{n=1}^{\infty} (-i)^n \int_0^t \cdots \int_0^{t_{n-1}} \int_0^{t_1} P_{t_n} \cdots P_{t_1} \, dt_1 \cdots dt_n$$

with convergence in the $\mathcal{B}(Q^s)$ norm. Here $P_t = p^w(x,D) = (p \circ \chi_t)^w(x,D)$. The integrals are Bochner integrals of operator-valued functions. The propagator (3.4) is a strongly continuous semigroup of operators on $Q^s$.

By [13, Corollary 11.2.6 and Lemma 11.3.3] the $Q^s$-norms for $s \geq 0$ are a family of seminorms for $\mathcal{S}'(\mathbb{R}^d)$. Thus $C_t : \mathcal{S} \rightarrow \mathcal{S}$ is continuous.

We show that $C_t$ is a pseudodifferential operator. First we use results in [8] to prove that $C_t$ has a pseudodifferential operator symbol in a space larger than $\Gamma^0$. By [16, Remark 2.18] we have

$$\Gamma^{-\delta} \subseteq \Gamma^0_0 = \bigcap_{s \geq 0} M_{1\otimes v_s}^{\infty,1}(\mathbb{R}^{2d})$$

where $M_{1\otimes v_s}^{\infty,1}$ denotes a Sjöstrand modulation space [13] with the weight $v_s(z), z \in \mathbb{R}^{2d}$, and where $\Gamma^0_0$ denotes the space of smooth symbols whose derivatives are in $L^\infty$. From [8, Theorem 4.1] it follows that $C_t = c^w_t(x,D)$ where

$$c_t \in \bigcap_{s \geq 0} M_{1\otimes v_s}^{\infty,1} = \Gamma^0_0, \quad t \geq 0.$$
By duality $c^\nu_t(x,D)$ extends uniquely to a continuous operator on $\mathcal{S}'(\mathbb{R}^d)$.

The outcome of this argument is that the propagator (3.11) is of the form

$$T_t = \mu(\chi_t)c^\nu_t(x,D), \quad t \geq 0.$$  

If $u_0 \in \mathcal{S}'(\mathbb{R}^d)$ then $u_0 \in Q^{s+2}$ for some $s \in \mathbb{R}$. Again, by [22, Theorem 4.1.3], $T_t u_0$ is the unique solution to (CP) in $C([0, \infty), Q^s) \cap C^1((0, \infty), Q^s)$.

Our objective is to improve (3.10) into

$$c_t \in \Gamma^0, \quad t \geq 0,$$

which implies that the propagator $T_t$ is an FIO of order zero.

The strategy to prove (3.8) is as follows. We first construct an FIO parametrix $\{\mathcal{K}_t\}_{t \geq 0}$ to the equation (CP), that is a family of maps $\{\mathcal{K}_t\}_{t \geq 0}$, where $\mathcal{K}_t \in \mathcal{S}^0(\chi_t)$ for $t \geq 0$, which satisfies

$$\begin{align*}
\partial_t \mathcal{K}_t u_0 + i(q^\nu(x,D) + p^\nu(x,D))\mathcal{K}_t u_0 &= g(t), \quad t > 0, \\
\mathcal{K}_0 u_0 &= u_0, \quad u_0 \in \mathcal{S}'(\mathbb{R}^d),
\end{align*}$$

where $g \in C([0, \infty), \mathcal{S}'(\mathbb{R}^d))$. We then prove that $\mathcal{K}_t - T_t = \mathcal{R}_t$ is regularizing, which implies that $T_t = \mathcal{K}_t - \mathcal{R}_t \in \mathcal{S}^0(\chi_t)$ is an FIO.

Thus we start by proving the following result.

**Theorem 3.3.** The Cauchy problem (CP) admits an FIO parametrix $\mathcal{K}_t \in \mathcal{S}^0(\chi_t)$ for $t \geq 0$ such that $\mathcal{K}_0 = I$.

The proof is carried out in several steps.

**Lemma 3.4.** Let $T > 0$ and $n \geq 1$. The family of Weyl symbols

$$p_{t_1} \# \cdots \# p_{t_n} \in \Gamma^{-\delta n}, \quad t_j \in [0, T], \quad 1 \leq j \leq n,$$

is uniformly bounded in $\Gamma^{-\delta n}$, and

$$[0, T]^n \ni (t_1, t_2, \ldots, t_n) \mapsto p_{t_1} \# \cdots \# p_{t_n} \in \Gamma^{-(\delta - \nu)n}$$

is continuous for any $\nu > 0$.

**Proof.** We have

$$\left| \partial^\alpha p_t(z) \right| \leq C_n e^{2\|F\|((\delta + 2|\alpha|)}|z|^{-\delta - |\alpha|}, \quad z \in \mathbb{R}^d, \quad \alpha \in \mathbb{N}^d,$$

which proves that $p_t \in \Gamma^{-\delta}$ uniformly over $t \in [0, T]$. The estimates (3.10) also show that

$$p_t \in C([0, T], \Gamma^{-\delta + \nu})$$

if $\nu > 0$. The result is thus a consequence of the continuity of the Weyl product on the spaces $\Gamma^n$ (see e.g. [21, Theorem 1.2.16]).

Fix $T > 0$. For $t \in [0, T]$ we set $b_{t,0} = 1$, and for $n \geq 1$

$$b_{t,n} = (-i)^n \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} p_{t_1} \# \cdots \# p_{t_n} dt_{n-1} \cdots dt_1.$$
In particular, \( b_{t,n} = 0 \) for \( n \geq 1 \). By Lemma 3.4, \( b_{t,n} \) makes sense as an integral and \( b_{t,n} \in \Gamma^{-\delta n} \). From (2.3) it follows that integration commutes with the Weyl product so that

\[
b_{t,n}^{(n)}(x,D) = (-i)^n \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} P_{t_1} \cdots P_{t_n} dt_n \cdots dt_1, \quad n \geq 0.
\]

Using (3.10), the recursion

\[
b_{t,n} = -i \int_0^t \partial_t p_t \# b_{t,n-1} dt, \quad n \geq 1,
\]

and induction, one shows

\[
b_{t,n} = C([0,T],\Gamma^{-\delta n}), \quad n \geq 0.
\]

Hence

\[
|\partial_z b_{t,n}(z)| \leq C_{\alpha}(z)^{-\delta n - |\alpha|}, \quad \alpha \in \mathbb{N}^{2d}, \quad 0 \leq t \leq T.
\]

We also have for \( n \geq 1 \)

\[
|\partial_z^\alpha \partial_t b_{t,n}(z)| \leq C_{\alpha}(z)^{-\delta n - |\alpha|}, \quad \alpha \in \mathbb{N}^{2d}, \quad 0 \leq t \leq T.
\]

In fact (3.13) gives for \( n \geq 1 \) and \( t > 0 \)

\[
\partial_t b_{t,n} = -i p_t \# b_{t,n-1}.
\]

Hence (3.11) and (3.14) show that

\[
\partial_t b_{t,n} \in C([0,T],\Gamma^{m_0 + \nu})
\]

provided \( \nu > 0 \). (Note that the continuity extends to include the end points of the interval \([0,T]\).) Lemma 3.4 and (3.15) imply that

\[
\partial_t b_{t,n} \in \Gamma^{-\delta n}
\]

uniformly over \( t \in [0,T] \). This proves (3.16). By differentiating (3.17) and using

\[
\partial_t (p_t(z)) = \langle (\nabla p)(\chi_t z), 2F\chi_t z \rangle \in \Gamma^{-\delta}
\]

we also obtain

\[
\partial_t^2 b_{t,n} \in \Gamma^{-\delta n}
\]

uniformly over \( t \in [0,T] \) for all \( n \geq 1 \).

We need the following technical result to sum the \( b_{t,n} \) asymptotically.

**Lemma 3.5.** Let \( T > 0 \) and \( a_{t,n} \in C([0,T],\Gamma^{m_n}) \) where \( (m_n)_{n \in \mathbb{N}} \) is a decreasing sequence tending to \( -\infty \) for \( n \to \infty \). Assume that \( \partial_t a_{t,n} \in \Gamma^{m_n} \) uniformly over \( t \in [0,T] \) for every \( n \in \mathbb{N} \). Then there exists a symbol function \( t \mapsto a_t \in C([0,T],\Gamma^{m_0}) \) such that for every \( N \in \mathbb{N} \setminus 0 \) we have

\[
a_t - \sum_{n=0}^N a_{t,n} \in C([0,T],\Gamma^{m_{N+1}}).
\]

We write \( a_t \sim \sum_{n=0}^\infty a_{t,n} \).
The proof is a variant of the proof of [27, Proposition 3.5] and other similar results for symbols depending on a parameter. The essential modification of the standard proof to obtain continuity is to use

\[ |\partial_x^s (a_{t+s,n}(z) - a_{t,n}(z))| \leq |s| \sup_{0 \leq s \leq 1} |\partial_x^s \partial_t a_{t+s,n}(z)|. \]

We omit further details except for the statement that the symbol \( a_t \) is constructed as

\( a_t = \sum_{n=0}^{\infty} \psi_n a_{t,n} \)

where \( \psi_0(z) = 1 \), and for \( n \geq 1 \), \( \psi_n(z) = \psi(z/r_n) \) where \( \psi \in C^\infty(\mathbb{R}^d) \), \( \psi(z) = 0 \) for \( |z| \leq 1 \), \( \psi(z) = 1 \) for \( |z| \geq 2 \), and \( (r_n)_{n \geq 1} \) a sufficiently rapidly increasing sequence of positive reals. This shows that we can extend Lemma 3.5 to take account also of higher derivatives in \( t \) by modifying the constants \( r_n \).

Applying Lemma 3.6 to the sequences \( (b_{r,n})_{n \geq 0} \) defined in (3.12) and \( (\partial b_{r,n})_{n \geq 1} \) simultaneously, and using (3.14), (3.15), (3.16), (3.18) and (3.19) we obtain \( b_t \in C([0,T],\Gamma^0) \cap C^1([0,T],\Gamma^{-\delta}) \) such that

\[ b_t \sim \sum_{n=0}^{\infty} b_{t,n}, \quad \partial_t b_t \sim \sum_{n=1}^{\infty} \partial_t b_{t,n}. \]

Note that \( b_{t,0}(x,D) = I \).

**Lemma 3.6.** For \( T > 0 \) we have

\[ r_t = \partial_t b_t + ip_t \# b_t \in C([0,T], \mathcal{F}(\mathbb{R}^d)). \]

**Proof.** Let \( N \in \mathbb{N} \setminus 0 \). By (3.17) and Lemma 3.5 we have

\[ \partial_t b_t + i \sum_{n=1}^{N} p_{t} \# b_{t,n-1} = \partial_t b_t - \sum_{n=1}^{N} \partial_t b_{t,n} \in C([0,T], \Gamma^{-\delta(N+1)}). \]

On the other hand we also observe that

\[ ip_t \# b_t - i \sum_{n=1}^{N} p_{t} \# b_{t,n-1} = ip_t \# \left( b_t - \sum_{n=0}^{N-1} b_{t,n} \right) \in C([0,T], \Gamma^{-\delta(N+1)+\nu}) \]

again using Lemma 3.5 and (3.11). Since \( N > 0 \) is arbitrary the claim follows.

**Proof of Theorem 3.3.**

Weyl quantization of Lemma 3.6 gives

\[ \partial_t b_t^w(x,D) = -ip_t^w(x,D) b_t^w(x,D) + r_t^w(x,D), \quad t \geq 0. \]

For \( u_0 \in \mathcal{F}'(\mathbb{R}^d) \) we define \( \mathcal{X}_0 u_0 = u_0 \) and for \( t > 0 \)

\[ v_t = \mathcal{X}_t u_0 = \mu(\chi_t) b_t^w(x,D) u_0 \in \mathcal{F}'(\mathbb{R}^d). \]

For \( t > 0 \) we have

\[ i \partial_t v_t = i \partial_t (\mu(\chi_t) b_t^w(x,D) u_0) \]

\[ = q^w(x,D) v_t + \mu(\chi_t) (p_t^w(x,D) b_t^w(x,D) + ir_t^w(x,D)) u_0 \]

\[ = q^w(x,D) v_t + p_t^w(x,D) v_t + i\mu(\chi_t) r_t^w(x,D) u_0. \]
If we set
\[ g(t) = \mu(\chi_t)r_t^w(x, D)u_0 \]
it remains to show \( g \in C([0, \infty), \mathcal{S}(\mathbb{R}^d)) \). Writing
\[
g(t + s) - g(t) = \mu(\chi_{t+s})(r_{t+s}^w(x, D) - r_t^w(x, D)) u_0 + \mu(\chi_t)(\mu(\chi_s) - I) r_t^w(x, D) u_0
\]
the latter claim is a consequence of Proposition 3.1 combined with (3.26) and the fact that by (3.21) \( u_0 \in Q^m \) for some \( m \in \mathbb{R} \). We have thus shown that \( \mathcal{K}_t \) is a parametrix to (CP).

3.3. The propagator to the perturbed equation as an FIO. In order to show (3.8) it remains to connect the parametrix with the solution operator (3.7) (cf. the proof of (14) Proposition 3.1]). Let \( u_0 \in \mathcal{S}'(\mathbb{R}^d) \). Then for some \( s \in \mathbb{R} \) we have \( u_0 \in Q^{s+2}(\mathbb{R}^d) \subseteq D(q^w(x, D) + p^w(x, D)) \) where \( q^w(x, D) + p^w(x, D) \) is considered an unbounded operator in \( Q^s(\mathbb{R}^d) \).

**Lemma 3.7.** If \( T > 0 \) then
\[
(3.22) \quad t \mapsto \mu(\chi_t)r_t^w(x, D)u_0 \in C([0, T], Q^s),
\]
\[
(3.23) \quad t \mapsto v_t \in C([0, T], Q^s), \quad \text{and}
\]
\[
(3.24) \quad t \mapsto \partial_t v_t \in C((0, T), Q^s).
\]

**Proof.** Property (3.22) is a consequence of Proposition 3.1 Lemma 3.6 and (2.5). Lemma 3.7 gives \( b_t \in C([0, T], \Gamma^0) \). Hence property (3.23) follows from Proposition 3.1 and (2.5).

Finally we show (3.24). From (3.21) we obtain for \( t > 0 \)
\[
\partial_t v_t = -iq^w(x, D)\mu(\chi_t)b_t^w(x, D)u_0 - ip^w(x, D)\mu(\chi_t)b_t^w(x, D)u_0 + \mu(\chi_t)r_t^w(x, D)u_0.
\]
In order to prove (3.24) we must show
\[
(3.25) \quad t \mapsto p^w(x, D)\mu(\chi_t)b_t^w(x, D)u_0 \in C([0, T], Q^s),
\]
\[
(3.26) \quad t \mapsto q^w(x, D)\mu(\chi_t)b_t^w(x, D)u_0 \in C((0, T), Q^s).
\]
by virtue of (3.22). Claim (3.25) is a consequence of (3.23), \( p \in \Gamma^\delta \) and (2.5). Finally claim (3.26) is a consequence of \( q \in \Gamma^0 \), (2.5) and Proposition 3.1. \( \square \)

From Lemma 3.7 we may conclude that
\[
t \mapsto \mathcal{K}_t u_0 \in C([0, T], Q^s) \cap C^1((0, T), Q^s).
\]
From Proposition 3.1 combined with \( b_t \in C([0, T], \Gamma^0) \) and (2.5) we obtain \( \mathcal{K}_t u_0 \in Q^{s+2} \subseteq D(q^w(x, D) + p^w(x, D)) \) for \( t > 0 \). Since \([0, T] \ni t \mapsto \mathcal{K}_t u_0 \) solves (3.9) with \( u_0 \in Q^{s+2}(\mathbb{R}^d) \), it is a classical solution according to (22) Definition 4.2.1.

Assembling the pieces gives the following conclusion. The map \( t \mapsto (\mathcal{K}_t - T_t)u_0 \) solves the equation
\[
\begin{cases}
\partial_t (\mathcal{K}_t - T_t)u_0 + i(q^w(x, D) + p^w(x, D))(\mathcal{K}_t - T_t)u_0 = \mu(\chi_t)r_t^w(x, D)u_0, & t > 0, \\
(\mathcal{K}_0 - T_0)u_0 = 0
\end{cases}
\]
and
\[
t \mapsto (\mathcal{K}_t - T_t)u_0 \in C([0, T], Q^s) \cap C^1((0, T), Q^s).
\]
Combining (3.22) in Lemma 3.7 with [22 Corollary 4.2.2] gives, invoking (3.7),
\[ \mathcal{R}_t u_0 := (\mathcal{K}_t - T_t) u_0 = \int_0^t T_{t-s} \mu(\chi_s) r^w_s(x, D) u_0 ds \]
\[ = \mu(\chi_t) \int_0^t \mu(\chi_{s-t}) c^w_{t-s}(x, D) \mu(\chi_s) r^w_s(x, D) u_0 ds. \]

**Lemma 3.8.** For $t > 0$ the kernel of the operator $\mu(\chi_t) \mathcal{R}_t$ belongs to $\mathcal{S}(\mathbb{R}^{2d})$.

**Proof.** By [29, Eq. (50.17) p. 525 and Theorem 51.6] the conclusion follows if we can show that the operator
\[ \int_0^t \mu(\chi_{s-t}) c^w_{t-s}(x, D) \mu(\chi_s) r^w_s(x, D) ds \]
is continuous $\mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ when $\mathcal{S}'(\mathbb{R}^d)$ is equipped with its strong topology [24 Section V.7]. By Proposition 3.1 and (3.5), intersected over $s > 0$, it suffices to show that $r^w_t(x, D) : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is continuous uniformly over $t \in [0, T]$, when $\mathcal{S}'(\mathbb{R}^d)$ is equipped with the strong topology.

Let $K_t \in \mathcal{S}(\mathbb{R}^{2d})$ denote the kernel of $r^w_t(x, D)$. According to (2.2) $K_t$ and $r_t$ are related by the composition of a linear change of variables and partial inverse Fourier transform. These are continuous operators on $\mathcal{S}(\mathbb{R}^{2d})$ and thus Lemma 3.6 implies $K_t \in C([0, T], \mathcal{S}(\mathbb{R}^{2d}))$.

We use the seminorms on $g \in \mathcal{S}(\mathbb{R}^d)$
\[ ||g||_{\alpha, \beta} = \sup_{x \in \mathbb{R}^d} |x^\alpha \partial^\beta g(x)|, \quad \alpha, \beta \in \mathbb{N}^d. \]

Let $u \in \mathcal{S}'(\mathbb{R}^d)$ and $\alpha, \beta \in \mathbb{N}^d$. We have
\[ ||r^w_t(x, D) u||_{\alpha, \beta} = \sup_{x \in \mathbb{R}^d} |\langle x^\alpha \partial^\beta K_t(x, \cdot), u \rangle| \]
\[ = \sup_{g \in B} ||g(u)|| \]
where
\[ B = \{ x^\alpha \partial^\beta K_t(x, \cdot) \in \mathcal{S}(\mathbb{R}^d), x \in \mathbb{R}^d \} \subseteq \mathcal{S}(\mathbb{R}^d). \]

Let $\gamma, \kappa \in \mathbb{N}^d$ be arbitrary. We have
\[ \sup_{g \in B} ||g||_{\gamma, \kappa} = \sup_{x, y \in \mathbb{R}^d} |x^\alpha y^\gamma \partial^\beta \partial^\delta K_t(x, y)| = ||K_t||_{(\alpha, \gamma), (\beta, \kappa)} < \infty \]
where the latter seminorm bound is uniform over $t \in [0, T]$. Thus $B \subseteq \mathcal{S}(\mathbb{R}^d)$ is a bounded set which implies that $u \mapsto \sup_{g \in B} ||g(u)||$ is a seminorm on $\mathcal{S}'(\mathbb{R}^d)$ endowed with the strong topology.

Since $\alpha, \beta \in \mathbb{N}^d$ are arbitrary, (3.27) combined with [24 Theorem V.2] thus prove the claim that $r^w_t(x, D) : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is continuous uniformly over $t \in [0, T]$, when $\mathcal{S}'(\mathbb{R}^d)$ is equipped with the strong topology. \hfill \□

Combining Lemma 3.8 with the fact that an operator has kernel in $\mathcal{S}(\mathbb{R}^{2d})$ if and only if its Weyl symbol belongs to the same space, we obtain
\[ \mathcal{R}_t = \mu(\chi_t) a^w_t(x, D) \]
where \( a_t \in \mathcal{S}^m(\mathbb{R}^d) \) and \( t \geq 0 \). This gives finally
\[
T_t = \mathcal{H}_t - \mathcal{R}_t = \mu(\chi_t)\left(b_t^m(x, D) - a_t^m(x, D)\right), \quad t \geq 0,
\]
which, in view of (3.7), implies that \( a_t = b_t - a_t \in \Gamma^0 \) which is the sought improvement to (3.6). Thus (3.8) has at long last been proved.

This means that we have finally obtained our main result.

**Theorem 3.9.** The Cauchy problem \([\text{CP}]\) has an FIO propagator \( T_t \in \mathcal{S}^0(\chi_t) \) for \( t \geq 0 \).

### 4. Singularities of the solutions

In this section we will discuss propagation of singularities under equation \([\text{CP}]\). Theorem 3.9 and Proposition 4.1 give the following result.

**Proposition 4.1.** Let \( T_t \in \mathcal{S}^0(e^{2tF}) \) be the propagator to Cauchy problem \([\text{CP}]\). If \( u_0 \in \mathcal{S}'(\mathbb{R}^d) \) then for \( t \geq 0 \)
\[
\text{WF}(T_t u_0) \subseteq e^{2tF} \text{WF}(u_0).
\]

A more refined concept of singularities are (Shubin) \( \Gamma \)-Lagrangian distributions \([7]\) which we now explain. A Lagrangian linear subspace \( \Lambda \subseteq T^*\mathbb{R}^d \) is a space of dimension \( d \) on which the restriction of \( \sigma \) vanishes. If \( \Lambda \) is Lagrangian then so is \( \chi \Lambda \) for each \( \chi \in \text{Sp}(d, \mathbb{R}) \). An example of a Lagrangian is \( \Lambda_0 = \mathbb{R}^d \times \{0\} \), and all other Lagrangians may be obtained by application of elements \( \chi \in \text{Sp}(d, \mathbb{R}) \) to \( \Lambda_0 \) \([7]\).

A Lagrangian \( \Lambda \subseteq T^*\mathbb{R}^d \) may be parametrized in the form
\[
\Lambda = \{(X, AX + Z) \in T^*\mathbb{R}^d, \ X \in Y, \ Z \in Y^\perp\} \subseteq T^*\mathbb{R}^d
\]
where \( Y \subseteq \mathbb{R}^d \) is a linear subspace and \( A \in M_{d \times d}(\mathbb{R}) \) is a symmetric matrix that leaves \( Y \) invariant, see \([23]\). It then automatically leaves \( Y^\perp \) invariant so can be written
\[
A = A_Y + A_{Y^\perp}
\]
where \( A_Y = \pi_Y A \pi_Y \) and \( A_{Y^\perp} = \pi_{Y^\perp} A \pi_{Y^\perp} \).

Note that the subspace \( Y \subseteq \mathbb{R}^d \) is uniquely determined by \( \Lambda \), but the matrix \( A \) is not. In fact \( A_Y \) is uniquely determined, but \( A_{Y^\perp} \) can be any matrix such that \( Y \subseteq \text{Ker} A_{Y^\perp} \) and \( A_{Y^\perp} \) leaves \( Y^\perp \) invariant.

The topological space of \( \Gamma \)-Lagrangian distributions of order \( m \in \mathbb{R} \) with respect to a Lagrangian \( \Lambda \subseteq T^*\mathbb{R}^d \) is a subspace denoted \( I^m_{\Gamma}(\mathbb{R}^d, \Lambda) \subseteq \mathcal{S}'(\mathbb{R}^d) \) \([7]\) Definition 6.3]. It can be defined as follows (cf. \([7]\) Corollary 6.12]).

**Definition 4.2.** A distribution \( u \in \mathcal{S}'(\mathbb{R}^d) \) satisfies \( u \in I^m_{\Gamma}(\mathbb{R}^d, \Lambda) \) if there exist \( \chi \in \text{Sp}(d, \mathbb{R}) \) that maps \( \chi : \mathbb{R}^d \times \{0\} \to \Lambda \) isomorphically, and \( a \in \Gamma^m(\mathbb{R}^d) \) such that \( u = \mu(\chi)a \).

By \([7]\) Proposition 6.7] we have \( \text{WF}(u) \subseteq \Lambda \) when \( u \in I^m_{\Gamma}(\mathbb{R}^d, \Lambda) \). Kernels of FIOs are \( \Gamma \)-Lagrangian distributions associated with the twisted graph Lagrangian \((2.10)\) of \( \chi \) in \( T^*\mathbb{R}^{2d} \) \([7]\) Theorem 7.2]. We have the following result on the action of FIOs on \( \Gamma \)-Lagrangian distributions \([7]\) Theorem 6.11].
Theorem 4.3. Suppose $\chi \in \text{Sp}(d, \mathbb{R})$, $\mathcal{K} \in \mathcal{S}^m(\chi)$ and let $\Lambda \subseteq T^*\mathbb{R}^d$ be a Lagrangian. Then

$\mathcal{K} : I^m_t(\mathbb{R}^d, \Lambda) \to I^{m+\delta}_{t}(\mathbb{R}^d, \chi\Lambda)$

is continuous.

As a consequence we obtain the following result which can be seen as a refinement of Proposition 4.1.

Theorem 4.4. Let $T_t \in \mathcal{S}^0(e^{2tF})$ be the propagator to Cauchy problem (CP), let $\Lambda \subseteq T^*\mathbb{R}^d$ be a Lagrangian and let $u_0 \in I^m_t(\mathbb{R}^d, \Lambda)$. Then for all $t \geq 0$

$T_t u_0 \in I^m_t(\mathbb{R}^d, e^{2tF} \Lambda)$.

4.1. Phase space estimates on the solutions. In this section we derive phase space estimates for the propagator and solutions to (CP). The estimates for the propagator will be relative to the underlying twisted graph Lagrangian \((2.10)\) of a symplectic matrix $\chi$. We have the following characterization of the kernels of FIOs (cf. [7, Theorem 5.2]) and \([28]\).

Definition 4.5. We define the following phase factor adjusted versions of the transform $T_\theta$ where $g \in \mathcal{S}(\mathbb{R}^d) \setminus 0$.

1. If $\chi \in \text{Sp}(d, \mathbb{R})$ and $u \in \mathcal{S}(\mathbb{R}^{2d})$ then

$T^\chi_{g@g} u(z, \zeta) = e^{-\frac{i}{2}(\langle z, \zeta \rangle + \sigma(\langle z, \zeta \rangle, \langle z, \zeta \rangle))} T^g_{g@g} u(z, \zeta), \quad (z, \zeta) \in T^*\mathbb{R}^{2d}$.

2. If $\Lambda \subseteq T^*\mathbb{R}^d$ is a Lagrangian parametrized by $Y \subseteq \mathbb{R}^d$ and $A \in M_{d \times d}(\mathbb{R})$ as in \((4.1)\) and $u \in \mathcal{S}(\mathbb{R}^d)$ then

$T^\Lambda_g u(x, \xi) = e^{-i(\langle x, \xi \rangle + \frac{1}{2}(x, Ax))} T_g u(x, \xi), \quad (x, \xi) \in T^*\mathbb{R}^d$.

We have the following characterization of the kernels of FIOs (cf. [7, Theorem 5.2] and \([28]\)).

Proposition 4.6. Let $K \in \mathcal{S}'(\mathbb{R}^{2d})$, $\chi \in \text{Sp}(d, \mathbb{R})$ and $g \in \mathcal{S}(\mathbb{R}^d) \setminus 0$. Then $K$ is the kernel of an FIO in $\mathcal{S}^m(\chi)$ if and only if the estimates

$|L_1 \cdots L_k T^\chi_{g@g} K(z, \zeta)| \lesssim (1 + \text{dist}(z, \zeta, \Lambda^\prime - \chi))^{m-k} (1 + \text{dist}(z, \zeta, \Lambda^\prime))^{-N}$,

hold for all $k, N \in \mathbb{N}$, where $(z, \zeta) \in T^*\mathbb{R}^{2d}$ and $L_j = \langle a_j, \nabla_{z, \zeta} \rangle$ with $a_j \in \Lambda^\prime$ for $j = 1, 2, \ldots, k$.

We also have the following phase space characterization of Lagrangian distributions [7, Proposition 6.14].

Proposition 4.7. Let $\Lambda \subseteq T^*\mathbb{R}^d$ be a Lagrangian parametrized by $Y \subseteq \mathbb{R}^d$ and $A \in M_{d \times d}(\mathbb{R})$ as in \((4.1)\) and let $V \subseteq T^*\mathbb{R}^d$ be a subspace transversal to $\Lambda$. A distribution $u \in \mathcal{S}'(\mathbb{R}^d)$ satisfies $u \in I^m_t(\mathbb{R}^d, \Lambda)$ if and only if for any $g \in \mathcal{S}(\mathbb{R}^d) \setminus 0$ and for any $k, N \in \mathbb{N}$ we have

$|L_1 \cdots L_k T^\Lambda_g u(x, \xi)| \lesssim (1 + \text{dist}(x, \xi, V))^{m-k} (1 + \text{dist}(x, \xi, \Lambda))^{-N}$,

with $(x, \xi) \in T^*\mathbb{R}^d$, where $L_j = \langle b_j, \nabla_{x, \xi} \rangle$ are first order differential operators with $b_j \in \Lambda$, $j = 1, \ldots, k$. 
Applying Propositions 4.6 and 4.7 to Proposition 4.1 and Theorem 4.4 respectively, we obtain the following phase space estimates for the propagator $T_t$ and the solution to (CP), see also [8, 28] for related results in different symbol classes. Note that our regularity assumptions allow for a precise estimate also of the derivatives of the propagator and solutions.

**Theorem 4.8.** Suppose $g \in \mathcal{S}(\mathbb{R}^d) \setminus 0$ and set $\chi_t = e^{2tF}$.

1. The kernel $K_t$ of the propagator $T_t$ to (CP) satisfies the estimates for $t \geq 0$

$$|L_1 \cdots L_k T^N_{g \otimes g} K_t(z, \zeta)| \lesssim (1 + \text{dist}((z, \zeta), \Lambda'_{\chi_t}))^{-N},$$

for all $k, N \in \mathbb{N}$, where $L_j = \langle a_j, \nabla_{z,\zeta} \rangle$ with $a_j \in \Lambda'_{\chi_t}$ for $j = 1, 2, \ldots, k$.

2. Suppose $\Lambda \subseteq T^*\mathbb{R}^d$ is a Lagrangian and set $\Lambda_t = \chi_t \Lambda$. If $u_0 \in I^m_{\text{reg}}(\mathbb{R}^d, \Lambda)$ then the solution $T_t u_0$ to (CP) satisfies for $t \geq 0$ and $k, N \in \mathbb{N}$

$$|L_1 \cdots L_k T^N_{g}(T_t u_0)(x, \xi)| \lesssim (1 + \text{dist}((x, \xi), \Lambda_t))^{-N},$$

where $L_j = \langle b_j, \nabla_{x,\xi} \rangle$ are first order differential operators with $b_j \in \Lambda_t$, $j = 1, \ldots, k$ and $V_t$ is a subspace transversal to $\Lambda_t$.

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