Quantum conditional query complexity

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Abstract

We define and study a new type of quantum oracle, the quantum conditional oracle, which provides oracle access to the conditional probabilities associated with an underlying distribution. Amongst other properties, we (a) obtain speed-ups over the best known quantum algorithms for identity testing, equivalence testing and uniformity testing of probability distributions; (b) study the power of these oracles for testing properties of boolean functions, and obtain an algorithm for checking whether an $n$-input $m$-output boolean function is balanced or $\epsilon$-far from balanced; and (c) give a sub-linear algorithm, requiring $\tilde{O}(n^{3/4}/\epsilon)$ queries, for testing whether an $n$-dimensional quantum state is maximally mixed or not.

1 Introduction

One of the fundamental challenges in statistics is to infer information about properties of large datasets as efficiently as possible. This is becoming increasingly important as we collect progressively more data about our world and our lives. Often one would like to determine a certain property of the collected data while having no physical ability to access all of it. This can be formalised as the task of property testing: determining whether an object has a certain property, or is ‘far’ from having that property, ideally minimising the number of inspections of it. There has been an explosive growth in recent years in this field [7, 19, 20], and particularly in the sub-field of distribution testing, in which one seeks to learn information about a data set by drawing samples from an associated probability distribution.

The classical conditional sampling oracle (COND) [2, 10, 12] grants access to a distribution $D$ such that one can draw samples not only from $D$, but also from $D_S$, the conditional distribution of $D$ restricted to an arbitrary subset $S$ of the domain. Such oracle access reveals a separation between the classical query complexity of identity testing (i.e. whether an unknown distribution $D$ is the same as some known distribution $D^*$), which takes a constant number of queries, and equivalence testing (i.e. whether two unknown distributions $D_1$ and $D_2$ are the same), which requires $\Omega(\sqrt{\log \log N})$ queries, where $N$ is the size of the domain [2]. In this paper we introduce a natural quantum version of the COND oracle (see Definition 2.4 below) and study its computational power.

More specifically, we will consider the PCOND (pairwise-COND) oracle, which only accepts query subsets $S$ of cardinality 2 or $N$, and introduce the PQCOND (pairwise-QCOND) oracle. While being rather restricted in comparison to the full COND and QCOND oracles, they nevertheless offer significant advantages over the standard sampling oracles.
1.1 Results

Quantum algorithms for property testing problems. We study the following property testing tasks for classical probability distributions and present efficient algorithms for their solution using our PQCOND oracle. We compare our results with previously known bounds for the standard quantum sampling oracle QSAMP and the classical PCOND oracle.

1. **Uniformity Test**: Given a distribution $D$ and a promise that $D$ is either the uniform distribution $A$ or $|D - A| \geq \epsilon$, where $| \cdot |$ is the $L_1$-norm, decide which of the options holds.

2. **Known-distribution Test**: Given a fixed distribution $D^*$ and a promise that either $D = D^*$ or $|D - D^*| \geq \epsilon$, decide which of the options holds.

3. **Unknown-distribution Test**: Given two distributions $D_1$ and $D_2$ and a promise that either $D_1 = D_2$ or $|D_1 - D_2| \geq \epsilon$, decide which of the options holds.

4. **Distance from uniformity**: Given a distribution $D$ and the uniform distribution $A$, estimate $\hat{d} = |D - A|$. The query complexities for the above problems are listed in Table 1, with our new results given in the last column. The notation $\tilde{O}(\cdot)$ denotes $O(f(N, \epsilon) \log^k f(N, \epsilon))$ for some $k$, i.e. logarithmic factors are hidden.

| Task                     | Standard quantum oracle (QSAMP) | PCOND oracle [10] | PQCOND oracle [this work] |
|--------------------------|---------------------------------|------------------|---------------------------|
| Uniformity Test          | $O\left(\frac{N^{1/2}}{\epsilon^2}\right)$ [9] | $\tilde{O}\left(\frac{1}{\epsilon^2}\right)$ | $\tilde{O}\left(\frac{1}{\epsilon}\right)$ |
| Known-distribution Test  | $\tilde{O}\left(\frac{N^{1/3}}{\epsilon^2}\right)$ [13] | $\tilde{O}\left[\left(\frac{\log N}{\epsilon}\right)^4\right]$ | $\tilde{O}\left[\left(\frac{\log N}{\epsilon}\right)^3\right]$ |
| Unknown-distribution Test| $O\left(\frac{N^{1/2}}{\epsilon^2}\right)$ [9] | $\tilde{O}\left[\left(\frac{\log^2 N}{\epsilon}\right)^3\right]$ | $\tilde{O}\left[\left(\frac{\log^2 N}{\epsilon}\right)^2\right]$ |
| Distance from uniformity | $O\left(\frac{N^{1/2}}{\epsilon^2}\right)$ [9] | $\tilde{O}\left(\frac{1}{\epsilon^{20}}\right)$ | $\tilde{O}\left(\frac{1}{\epsilon^{13}}\right)$ |

Table 1: Query complexity for property testing problems using three different access models: the standard quantum oracle (QSAMP), the PCOND oracle, and our PQCOND oracle.

Testing properties of boolean functions. A slight modification of the PQCOND oracle will allow for the testing of properties of boolean functions.

Given $f : \{0,1\}^n \rightarrow \{0,1\}^m$ with $n \geq m$, define $F_i := |\{x \in \{0,1\}^n : f(x) = i\}|/2^n$ for $i \in \{0,1\}^m$. The function $f$ is promised to be either:

- a balanced function, i.e. $F_i = \frac{1}{2^m}$ $\forall i \in \{0,1\}^m$; or

- $\epsilon$-far from balanced, i.e. $\sum_{i \in \{0,1\}^m} |F_i - \frac{1}{2^m}| \geq \epsilon$.

Provided we have PQCOND access to $f$, we present a quantum algorithm that decides which of these is the case using $\tilde{O}(1/\epsilon)$ queries.

Quantum spectrum testing. We consider a quantum cloud-based computation scenario in which one or more small, personal quantum computers query a central quantum data hub $Q$ to deduce properties of a dataset.
Suppose this hub has access to an \( n \)-dimensional mixed state \( \rho \) (in the form of a full classical description, or simply through having access to a large number of copies of \( \rho \)), and a personal quantum computer \( \mathcal{P} \) wishes to determine properties of \( \rho \). Each query from \( \mathcal{P} \) is effected as follows:

1. \( \mathcal{P} \) prepares a state of three registers: the first is classical and describes a basis \( \mathcal{B} = \{ |b_i \rangle \}_{i \in [n]} \); the second and third are quantum, prepared in a state of \( \mathcal{P} \)'s choosing. \( \mathcal{P} \) sends the three registers to \( \mathcal{Q} \).

2. Given these registers, \( \mathcal{Q} \) provides \( \text{PQCOND} \) access to the distribution \( D^{(n, \mathcal{B})}(i) = \text{Tr}(\rho |b_i \rangle \langle b_i|) = \langle b_i | \rho | b_i \rangle \), with the quantum registers being the input and output registers for the \( \text{PQCOND} \) query. \( \mathcal{Q} \) finally returns the quantum registers to \( \mathcal{P} \).

We consider the problem of testing whether or not \( \rho \) is the maximally mixed state. More formally, it is promised that one of the following holds:

- \( \| \rho - \mathbb{I} / n \|_1 = 0 \), i.e. \( \rho \) is the maximally mixed state; or
- \( \| \rho - \mathbb{I} / n \|_1 \geq \epsilon \), i.e. \( \rho \) is \( \epsilon \)-far from the maximally mixed state,

where \( \| \cdot \|_1 \) is the trace norm\(^1\). The task for \( \mathcal{P} \) is to decide which is the case.

We present a quantum algorithm to decide the above problem that uses \( \tilde{O}(n^{3/4} / \epsilon) \) \( \text{PQCOND} \) queries.

This problem has also been studied in a setting where \( \mathcal{P} \) has access to an unlimited number of copies of the state \( \rho \)[24], and an optimal algorithm was presented that used \( \tilde{O}(n / \epsilon^2) \) copies of the state.

### 1.2 Motivation

The conditional access model is versatile and well-suited to a wide range of practical applications, a few of which are mentioned below.

**Lottery machine.** A gravity pick lottery machine works as follows: \( N \) balls, numbered 1, \ldots, \( N \), are dropped into a spinning machine, and after a few moments a ball is released. One might wish to determine whether or not such a machine is fair, i.e. whether or not a ball is released uniformly at random. A distribution testing algorithm would correctly decide between the following options (assuming that one is guaranteed to be true) with high probability:

- The lottery machine is fair and outputs \( i \) with probability \( 1 / N \);
- The lottery machine is \( \epsilon \)-far from uniform.

In this example, access to a \( \text{COND} \) oracle is equivalent to being able to choose which balls are allowed into the spinner. Classically, it is known that \( \Theta(N^{1/2} / \epsilon^4) \) queries [5] to the \( \text{SAMP} \) oracle are required to determine whether or not a distribution generated by such a lottery machine is uniform. However, given access to the corresponding quantum oracle, \( \text{QSAMP} \), only \( O(N^{1/3} / \epsilon^{4/3}) \) queries are required [9]. Using the \( \text{PQCOND} \) oracle we are able to achieve this with \( \tilde{O}(1 / \epsilon) \) queries.

**Predicting movie preferences.** Suppose we had a large enough amount of data about two movies, \( A \) and \( B \), in order to access the joint probability distribution \( D \) describing how many people watch these movies on any given day. One would like to find out if people watching movie \( A \) are more likely to watch movie \( B \). More generally, we ask: is \( D \) a product of two independent distributions, or are viewings of movie \( A \) correlated with viewings of movie \( B \)? The distribution testing algorithm can be used to decide between the following options:

\(^1\)For an \((n \times n)\) matrix \( A \), \( \| A \|_1 = \text{Tr} \sqrt{AA^\dagger} = \sum_{i \in [n]} a_i \), where the \( a_i \) are the singular values of \( A \).
• $D$ is independent; i.e. $D$ is a product of two distributions, $D = D^{(A)} \times D^{(B)}$;
• $D$ is $\epsilon$-far from independent; i.e. it is $\epsilon$-far from every product distribution.

Other tests. There is a wide range of other informative property tests, including:
• Checking if two unknown distributions are identical.
• Checking if a distribution is identical to a known reference distribution.
• Estimating the support size of a distribution.
• Estimating the entropy of a distribution.

Many of these have been extensively studied in the classical [6,10,11,13,14,18,22,28] and quantum [9,23] literature, and near-optimal bounds have often been placed on the number of queries required to solve the respective problems.

1.3 Outline

In Section 2 we introduce notation and define our quantum conditional oracles. In Section 3 we prove a lemma that is subsequently used to obtain our main technical tool—the QC \textsc{Compare} function, which efficiently compares conditional probabilities of a distribution. In Section 4 we apply it to obtain new, efficient query complexity bounds for property testing of probability distributions. In Section 5 we test properties of boolean functions, before presenting a quantum spectrum test in Section 6.

2 Preliminaries and Notation

Let $D$ be a probability distribution over a finite set $[N] := \{0, 1, \ldots, N - 1\}$, where $D(i) \geq 0$ is the weight of the element $i \in [N]$. Furthermore, if $S \subseteq [N]$, then $D(S) = \sum_{i \in S} D(i)$ is the weight of the set $S$. If $D(S) > 0$, define $D_S$ to be the conditional distribution, i.e. $D_S(i) := D(i)/D(S)$ if $i \in S$ and $D_S(i) = 0$ if $i \notin S$.

Below, we recall the definitions of the classical and quantum sampling oracles, and subsequently define the classical and quantum conditional sampling oracles.

\textbf{Definition 2.1 (Classical Sampling Oracle [10])}. Given a probability distribution $D$ over $[N]$, we define the classical sampling oracle $\textsc{Samp}_D$ as follows: each time $\textsc{Samp}_D$ is queried, it returns a single $i \in [N]$, where the probability that element $i$ is returned is $D(i)$.

\textbf{Definition 2.2 (Quantum Sampling Oracle [9])}. Given a probability distribution $D$ over $[N]$, let $T \in \mathbb{N}$ be some specified integer, and assume that $D$ can be represented by a mapping $O_D : [T] \to [N]$ such that for any $i \in [N]$, $D(i)$ is proportional to the number of elements in the pre-image of $i$, i.e. $D(i) = |\{t \in [T] : O_D(t) = i\}|/T$. In other words, $O_D$ labels the elements of $[T]$ by $i \in [N]$, and the $D(i)$ are the frequencies of these labels, and are thus all rational with denominator $T$.

Then each query to the quantum sampling oracle $\textsc{Qsamp}_D$ applies the unitary operation $U_D$, described by its action on basis states:

$$U_D |t\rangle |0\rangle = |t\rangle |\beta + O_D(t) \mod N\rangle .$$

In particular,

$$U_D |t\rangle |0\rangle = |t\rangle |O_D(t)\rangle .$$
As an example, note that querying with a uniformly random \( t \in [T] \) in the first register will result in \( i \in [N] \) in the second register with probability \( D(i) \).

**Definition 2.3** (Classical Conditional Sampling Oracle [10]). Given a probability distribution \( D \) over \([N]\) and a set \( S \subseteq [N] \) such that \( D(S) > 0 \), we define the classical conditional sampling oracle \( \text{COND}_D \) as follows: each time \( \text{COND}_D \) is queried with query set \( S \), it returns a single \( i \in [N] \), where the probability that element \( i \) is returned is \( D_S(i) \).

We are now ready to define a new quantum conditional sampling oracle, a quantum version of \( \text{COND}_D \).

**Definition 2.4** (Quantum Conditional Sampling Oracle). Given a probability distribution \( D \) over \([N]\), let \( T \in \mathbb{N} \) be some specified integer, and assume that there exists a mapping \( O_D : \mathcal{P}([N]) \times [T] \rightarrow [N] \), where \( \mathcal{P}([N]) \) is the power set of \([N]\), such that for any \( S \subseteq [N] \) with \( D(S) > 0 \) and any \( i \in [N] \), \( D_S(i) = |\{ t \in [T] : O_D(S, t) = i \}|/T \).

Then each query to the quantum conditional sampling oracle \( \text{QCOND}_D \) applies the unitary operation \( U_D \), defined below.

\( U_D \) acts on 3 registers:
- The first consists of \( N \) qubits, whose computational basis states label the \( 2^N \) possible query sets \( S \);
- The second consists of \( \log T \) qubits that describe an element of \([T]\); and
- The third consists of \( \log N \) qubits to store the output, an element of \([N]\).

The action of the oracle on basis states is
\[
U_D |S\rangle |t\rangle |0\rangle = |S\rangle |t\rangle |\beta\rangle + O_D(A, t) \mod N\rangle .
\]

In particular,
\[
U_D |S\rangle |t\rangle |0\rangle = |S\rangle |t\rangle |O_D(A, t)\rangle .
\]

**Remark:** Note that querying \( \text{QCOND}_D \) with query set \( S = [N] \) is equivalent to a query to \( \text{QSAMP}_D \).

The \( \text{PCOND}_D \) oracle, described in [10], only accepts query subsets \( S \) of cardinality 2 or \( N \). Below we define its quantum analogue, the \( \text{PQCOND}_D \) oracle.

**Definition 2.5** (Pairwise Conditional Sampling Oracle). The \( \text{PQCOND}_D \) oracle is equivalent to the \( \text{QCOND}_D \) oracle, with the added requirement that the query set \( S \) must satisfy \( |S| = 2 \) or \( N \), i.e. the distribution can only be conditioned over pairs of elements or the whole set.

## 3 Efficient comparison of conditional probabilities

In this section we first prove a lemma to improve the dependency on success probability for a general probabilistic algorithm. We subsequently use this result to prove our main technical tool, the \( \text{QCOMPARE} \) algorithm, which compares conditional probabilities of a distribution, and is crucial to our improved property tests.

### 3.1 Improving dependence on success probability

The following lemma, proved in Section A.1, provides a general method for improving the dependence between the number of queries made and the success probability of the algorithm.
Lemma 3.1. Suppose an algorithm $\text{ALG}1(\xi, \epsilon, \delta)$ ($\epsilon > 0, \delta \in (0, 1]$) outputs an (additive) approximation to $f(\xi) \in \mathbb{R}$. More formally, suppose it outputs $\tilde{f}(\xi)$ such that $\mathbb{P}[|\tilde{f}(\xi) - f(\xi)| \leq \epsilon] \geq 1 - \delta$ using $\text{ALG}1(\xi, \epsilon, \delta)$ queries to a classical/quantum oracle, for some function $M$.

Then there exists an algorithm $\text{ALG}2(\xi, \epsilon, \delta)$ that makes $\Theta(M(\xi, \epsilon, \frac{1}{\epsilon} \log(1/\delta)))$ queries to the same oracle and outputs $\tilde{f}(\xi)$ such that $\mathbb{P}[|\tilde{f}(\xi) - f(\xi)| \leq \epsilon] \geq 1 - \delta$, i.e. the dependence of the number of queries on the success probability can be taken to be $\log(1/\delta)$.

Applying this lemma to Theorem 5 of [9] gives an exponential improvement, from $1/\delta$ to $\log(1/\delta)$, in the dependence on the success probability given there. This is summarised in the theorem below.

Theorem 3.2. There exists a quantum algorithm $\text{ADDESTPROB}(D, S, M)$ that takes as input a distribution $D$ over $[N]$, a set $S \subseteq [N]$ and an integer $M$. The algorithm makes exactly $M$ queries to the $\text{QSAMP}_D$ oracle and outputs $\tilde{D}(S)$, an approximation to $D(S)$, such that $\mathbb{P}[|\tilde{D}(S) - D(S)| \leq \epsilon] \geq 1 - \delta$ for all $\epsilon > 0$ and $\delta \in (0, 1]$ satisfying

$$M \geq c \log(1/\delta) \max\left(\frac{\sqrt{D(S)}}{\epsilon}, \frac{1}{\sqrt{\epsilon}}\right),$$

where $c = O(1)$ is some constant.

A multiplicative version Theorem 3.2 follows straightforwardly:

Theorem 3.3. There exists a quantum algorithm $\text{MULESTPROB}(D, S, M)$ that takes as input a distribution $D$ over $[N]$, a set $S \subseteq [N]$ and an integer $M$. The algorithm makes exactly $M$ queries to the $\text{QSAMP}_D$ oracle and outputs $\tilde{D}(S)$, an approximation to $D(S)$, such that $\mathbb{P}[|\tilde{D}(S) - D(S)| \leq \epsilon] \geq 1 - \delta$ for all $\epsilon, \delta \in (0, 1]$ satisfying

$$M \geq \frac{c\log(1/\delta)}{\epsilon \sqrt{D(S)}},$$

where $c = O(1)$ is some constant.

Access to the $\text{QCOND}_D$ oracle effectively gives us access to the oracle $\text{QSAMP}_{DS}$ for any $S \subseteq [N]$, and this allows us to produce stronger versions of Theorems 3.2 and 3.3:

Theorem 3.4. There exists a quantum algorithm $\text{ADDESTPROBQCOND}(D, S, R, M)$ that takes as input a distribution $D$ over $[N]$, a set $S \subseteq [N]$ with $D(S) > 0$, a subset $R \subseteq S$ and an integer $M$. The algorithm makes exactly $M$ queries to the $\text{QCOND}_D$ oracle and outputs $\tilde{D}_S(R)$, an approximation to $D_S(R)$, such that $\mathbb{P}[|\tilde{D}_S(R) - D_S(R)| \leq \epsilon] \geq 1 - \delta$ for all $\epsilon > 0$ and $\delta \in (0, 1]$ satisfying

$$M \geq c \log(1/\delta) \max\left(\frac{\sqrt{D_S(R)}}{\epsilon}, \frac{1}{\sqrt{\epsilon}}\right),$$

where $c = O(1)$ is some constant.

Theorem 3.5. There exists a quantum algorithm $\text{MULESTPROBQCOND}(D, S, R, M)$ that takes as input a distribution $D$ over $[N]$, a set $S \subseteq [N]$ with $D(S) > 0$, a subset $R \subseteq S$ and an integer $M$. The algorithm makes exactly $M$ queries to the $\text{QCOND}_D$ oracle and outputs $\tilde{D}_S(R)$, an approximation to $D_S(R)$, such that $\mathbb{P}[|\tilde{D}_S(R) - D_S(R)| \leq \epsilon] \geq 1 - \delta$ for all $\epsilon, \delta \in (0, 1]$ satisfying

$$M \geq \frac{c\log(1/\delta)}{\epsilon \sqrt{D_S(R)}},$$

where $c = O(1)$ is some constant.
### 3.2 The QCOMPARE algorithm

An important routine used in many classical distribution testing protocols (see \cite{10}) is the COMPARE function, which outputs an estimate of the ratio $r_{X,Y} := D(Y)/D(X)$ of the weights of two disjoint subsets $X, Y \subseteq [N]$ over $D$. As stated in Section 3.1 of \cite{10}, if $X$ and $Y$ are disjoint, $D(X \cup Y) > 0$, and $1/K \leq r_{X,Y} \leq K$ for some integer $K \geq 1$, the algorithm outputs $\tilde{r}_{X,Y} \in [1 - \eta, 1 + \eta]r_{X,Y}$ with probability at least $1 - \delta$ using only $\Theta(K \log(1/\delta)/\eta^2)$ COND$_D$ queries. Surprisingly, the number of queries is independent of $N$, the size of the domain of the distribution.

Here we introduce a procedure called QCOMPARE that uses the COND$_D$ oracle and subsequent quantum operations to perform a similar function to COMPARE, achieving the same success probability and bound on the error with $\Theta(\sqrt{K} \log(1/\delta)/\eta)$ queries.

We now use ADDESTPROBQCOND and MULESTPROBQCOND to create the QCOMPARE procedure.

**Algorithm 1 QCOMPARE**

**Input:** COND access to a probability distribution $D$ over $[N]$, disjoint subsets $X, Y \subseteq [N]$ such that $D(X \cup Y) > 0$, parameters $K \geq 1$, $\eta \in (0, \frac{1}{8K})$, and $\delta \in (0, 1]$.

1. Set $M = \Theta\left(\frac{\sqrt{K} \log(1/\delta)}{\eta}\right)$.
2. Set $\bar{w}_+(X) = $ ADDESTPROBQCOND$(D, X \cup Y, X, M)$.
3. Set $\bar{w}_+(Y) = $ ADDESTPROBQCOND$(D, X \cup Y, Y, M)$.
4. Set $\bar{w}_-(X) = $ MULESTPROBQCOND$(D, X \cup Y, X, M)$.
5. Set $\bar{w}_-(Y) = $ MULESTPROBQCOND$(D, X \cup Y, Y, M)$.
6. Check that $\bar{w}_+(X) \leq \frac{3K}{6K+1} - \frac{\eta}{3}$. If the check fails, return Low and exit.
7. Check that $\bar{w}_+(Y) \leq \frac{3K}{6K+1} - \frac{\eta}{3}$. If the check fails, return High and exit.
8. Return $\tilde{r}_{X,Y} = \frac{\bar{w}_-(Y)}{\bar{w}_-(X)}$.

**Theorem 3.6.** Given the input as described, QCOMPARE (Algorithm 1) outputs Low, High, or a value $\tilde{r}_{X,Y} > 0$, and satisfies the following:

1. If $1/K \leq r_{X,Y} \leq K$, then with probability at least $1 - \delta$ the procedure outputs a value $\tilde{r}_{X,Y} \in [1 - \eta, 1 + \eta]r_{X,Y}$.
2. If $r_{X,Y} > K$ then with probability at least $1 - \delta$ the procedure outputs either High or a value $\tilde{r}_{X,Y} \in [1 - \eta, 1 + \eta]r_{X,Y}$.
3. If $r_{X,Y} < 1/K$ then with probability at least $1 - \delta$ the procedure outputs either Low or a value $\tilde{r}_{X,Y} \in [1 - \eta, 1 + \eta]r_{X,Y}$.

The procedure performs $\Theta\left(\frac{\sqrt{K} \log(1/\delta)}{\eta}\right)$ COND$_D$ queries on the set $X \cup Y$ via use of ADDESTPROBQCOND and MULESTPROBQCOND.

The proof of this theorem is given in Section A.2.

### 4 Property testing of probability distributions

We now apply our results to obtain new algorithms for a number of property testing problems.

**Corollary 4.1.** Let $A^{(N)}$ be the uniform distribution on $[N]$ (i.e. $A^{(N)}(i) = 1/N, i \in [N]$). Given PQCOND access to a probability distribution $D$ over $[N]$, there exists an algorithm that uses $\tilde{O}(1/\epsilon)$ PQCOND$_D$ queries...
and decides with probability at least 2/3 whether

- \(|D - A(N)| = 0 \) (i.e. \(D = A(N)\)), or
- \(|D - A(N)| \geq \epsilon\),

provided that it is guaranteed that one of these is true. Here \(|\cdot|\) is the \(L_1\)-norm\(^2\).

**Proof.** We replace the calls to COMPARE with the corresponding calls to QCOMPARE in Algorithm 4 of [10]. For this method, calls to QCOMPARE only require conditioning over pairs of elements, and hence the PQCOND\(_D\) oracle may be used instead of QCOND\(_D\). \(\square\)

**Remark:** The corresponding classical algorithm (Algorithm 4 in [10]) uses \(\tilde{O}(1/\epsilon^2)\) PCOND\(_D\) queries. The authors also show (Section 4.2 of [10]) that any classical algorithm making COND\(_D\) queries must use \(\Omega(1/\epsilon^2)\) queries to solve this problem with bounded probability. Thus the above quantum algorithm is quadratically more efficient than any classical COND algorithm.

**Corollary 4.2.** Given the full specification of a probability distribution \(D^*\) (i.e. a known distribution) and PQCOND access to a probability distribution \(D\), both over \([N]\), there exists an algorithm that uses \(\tilde{O}\left(\frac{\log^3 N}{\epsilon}\right)\) PQCOND\(_D\) queries and decides with probability at least 2/3 whether

- \(|D - D^*| = 0 \) (i.e. \(D = D^*\)), or
- \(|D - D^*| \geq \epsilon\),

provided that it is guaranteed that one of these is true.

**Proof.** We replace the calls to COMPARE with the corresponding calls to QCOMPARE in Algorithm 5 of [10]. \(\square\)

**Remark:** The corresponding classical algorithm (Algorithm 5 in [10]) uses \(\tilde{O}\left(\frac{\log^4 N}{\epsilon^4}\right)\) PCOND\(_D\) queries.

**Corollary 4.3.** Given PQCOND access to probability distributions \(D^{(1)}\) and \(D^{(2)}\) over \([N]\), there exists an algorithm that decides, with probability at least 2/3, whether

- \(|D^{(1)} - D^{(2)}| = 0 \) (i.e. \(D^{(1)} = D^{(2)}\)), or
- \(|D^{(1)} - D^{(2)}| \geq \epsilon\),

provided that it is guaranteed that one of these is true. The algorithm uses \(\tilde{O}\left(\frac{\log^4 N}{\epsilon^4}\right)\) PQCOND\(_{D^{(1)}}\) and PQCOND\(_{D^{(2)}}\) queries.

**Proof.** We replace the calls to COMPARE with the corresponding calls to QCOMPARE in Algorithm 9 of [10]. As a by-product of this process, the function ESTIMATE-NEIGHBORHOOD (Algorithm 2 in [10]), using \(\tilde{O}\left(\frac{\log(1/\delta)}{\kappa \eta \beta \bar{P} \delta}\right)\) PCOND queries, is replaced by an algorithm QESTIMATE-NEIGHBORHOOD, which uses \(\tilde{O}\left(\frac{\log(1/\delta)}{\kappa \eta \beta \bar{P} \delta}\right)\) PQCOND queries. \(\square\)

**Remark:** This is to be compared with Algorithm 9 in [10], which uses \(\tilde{O}\left(\frac{\log^6 N}{\epsilon^2}\right)\) PCOND\(_{D^{(1)}}\) and PCOND\(_{D^{(2)}}\) queries.

\(^2\)For two distributions \(D_1\) and \(D_2\) over \([N]\), \(|D^{(1)} - D^{(2)}| = \sum_{i \in [N]} |D^{(1)}(i) - D^{(2)}(i)|\).
Corollary 4.4. Given PQCOND access to a probability distribution $D$ over $[N]$, there exists an algorithm that uses $\tilde{O}(1/\epsilon^{13})$ queries and outputs a value $\hat{d}$ such that $|\hat{d} - D - A(N)| = O(\epsilon)$.

Proof. We replace the calls to COMPARE with the corresponding calls to QCOMPARE in Algorithm 11 of [10]. In addition, we trivially replace all queries to the SAMP$_D$ oracle with queries to PQCOND$_D$ with query set $[N]$. As a by-product of this process, the function FIND-REFERENCE (Algorithm 12 in [10]), using $\tilde{O}(1/\kappa^{20})$ PCOND and SAMP queries, is replaced by an algorithm QFIND-REFERENCE, which uses $\tilde{O}(1/\kappa^{13})$ PQCOND queries. \qed

Remark: The corresponding classical algorithm (Algorithm 11 in [10]) uses $\tilde{O}(1/\epsilon^{20})$ queries.

5 Property testing of Boolean functions

The results in Section 4 can be applied to test properties of Boolean functions. One of the more important challenges in field of cryptography is to determine whether or not a given boolean function is ‘balanced’. We give an algorithm to solve this problem with a constant number of PQCOND queries.

Consider a function $f : \{0,1\}^n \rightarrow \{0,1\}^m$, for $n,m \in \mathbb{N}$ with $n \geq m$. If $m = 1$, we might consider the following problem:

Problem 5.1 (Constant-balanced problem). Given $f : \{0,1\}^n \rightarrow \{0,1\}$, decide whether

- $f$ is a balanced function, i.e. $|\{x \in \{0,1\}^n : f(x) = 0\}|/2^n = |\{x \in \{0,1\}^n : f(x) = 1\}|/2^n = \frac{1}{2}$, or
- $f$ is a constant function, i.e. $f(x) = 0 \quad \forall x \in \{0,1\}^n$ or $f(x) = 1 \quad \forall x \in \{0,1\}^n$,

provided that it is guaranteed that $f$ satisfies one of these conditions.

With standard quantum oracle access to $f$, this problem can be solved exactly with one query, through use of the Deutsch-Jozsa algorithm [15,17]. Consider the following extension of this problem:

Problem 5.2. Given $f : \{0,1\}^n \rightarrow \{0,1\}$, write $F_i := |\{x \in \{0,1\}^n : f(x) = i\}|/2^n$. Decide whether

- $f$ is a balanced function, i.e. $F_0 = F_1 = \frac{1}{2}$, or
- $f$ is $\epsilon$-far from balanced, i.e. $|F_0 - \frac{1}{2}| + |F_1 - \frac{1}{2}| = 2|F_0 - \frac{1}{2}| \geq \epsilon$,

provided that it is guaranteed that $f$ satisfies one of these conditions.

This problem can be solved with bounded probability by querying $f$ several times. In addition, it can be solved using the QSAMP oracle. To understand how this works, set $T = 2^n, N = 2, O_D = f$ in Definition 2.2 so that $D(i) = F_i$. Then Theorem 3.2 can be used to estimate $D(0) = F_0$ to error $\epsilon/3$ with probability $1 - \delta$ using $O(\log(1/\delta) / \epsilon)$ queries.

Now we consider an even more general problem:

Problem 5.3. Given $f : \{0,1\}^n \rightarrow \{0,1\}^m$, write $F_i := |\{x \in \{0,1\}^n : f(x) = i\}|/2^n$. Decide whether

- $f$ is a balanced function, i.e. $F_i = \frac{1}{2^m} \forall i \in \{0,1\}^m$, or
- $f$ is $\epsilon$-far from balanced, i.e. $\sum_{i \in \{0,1\}^m} |F_i - \frac{1}{2^m}| \geq \epsilon$,

provided that it is guaranteed that $f$ satisfies one of these conditions.

By allowing PQCOND access to $f$, this can be solved in $\tilde{O}(1/\epsilon)$ queries. In what sense do we allow PQCOND access to $f$? We relate $f$ to a probability distribution by setting $N = 2^n, D_{[N]}(i) = F_i$, and using
the definition of \( D_S(i) \) given at the start of Section 2. The problem is then solved by an application of the algorithm presented in Corollary 4.1.

6 Quantum Spectrum Testing

Recall the quantum cloud-based computation scenario presented in Section 1.1. It is easy to see that for any basis \( \mathcal{B}, D^{1/n}_{[n]} = \mathcal{A}^{(n)} \), where \( \mathcal{A}^{(n)} \) is the uniform distribution over \([n] \). Then for any state \( \rho \),

- if \( \|\rho - 1/n\|_1 = 0 \), then \( D^{\rho B}_{[n]} - \mathcal{A}^{(n)} \) = 0 for any basis \( \mathcal{B} \);
- if \( \|\rho - 1/n\|_1 \geq \epsilon \), perhaps we can choose a basis \( \mathcal{B} \) such that \( D^{\rho B}_{[n]} - \mathcal{A}^{(n)} \geq \nu(\epsilon, n) \), for some function \( \nu \).

Corollary 4.1, with distance parameter \( \nu(\epsilon, n) \), could then be used to distinguish between these two options.

As the first case above is immediate, we henceforth assume that \( \|\rho - 1/n\|_1 \geq \epsilon \). In order to simplify the analysis, we assume that \( n \) is even, let \( \Delta = \rho - 1/n \), and introduce \[ \delta^{(B)} := \left| D^{\rho B}_{[n]} - \mathcal{A}^{(n)} \right| = \sum_{i \in [n]} |\langle b_i | \Delta | b_i \rangle |. \]

Let \( \tilde{\mathcal{B}} = \{ |\tilde{b}_i \rangle \}_{i \in [n]} \) be the eigenbasis of \( \Delta \), and let \( d_i := \langle \tilde{b}_i | \Delta | \tilde{b}_i \rangle, i \in [n] \) be the eigenvalues. Thus, \( \Delta = \sum_{i \in [n]} d_i |\tilde{b}_i \rangle \langle \tilde{b}_i | \). Note that \( \text{Tr} \Delta = \sum_{i \in [n]} d_i = 0 \), and also \( \eta := \|\rho - 1/n\|_1 = \|\Delta\|_1 = \sum_{i \in [n]} |d_i| \geq \epsilon \).

Now suppose we choose a basis \( \mathcal{B} = \{ |b_i \rangle \}_{i \in [n]} \) uniformly at random, i.e. we choose \( W \in \mathcal{U}(n) \) uniformly at random according to the Haar measure, and set \( |b_i \rangle = W |\tilde{b}_i \rangle \). Then

\[ \delta^{(B)} = \sum_{i \in [n]} |\langle b_i | \Delta | b_i \rangle | = \sum_{i \in [n]} \left| \sum_{j \in [n]} |W_{ji}|^2 d_j \right|. \]

The triangle inequality then gives

\[ \delta^{(B)} \geq \left| \sum_{j \in [n]} \left( \sum_{i \in [n]} |W_{ji}|^2 \right) d_j \right| = \left| \sum_{j \in [n]} d_j \right| = 0; \quad \delta^{(B)} \leq \sum_{j \in [n]} \left( \sum_{i \in [n]} |W_{ji}|^2 \right) |d_j| = \eta. \tag{1} \]

Let \( \nu^{(i)}_j = |W_{ji}|^2 \), introduce the vector \( V^{(i)} = (\nu^{(i)}_0, \ldots, \nu^{(i)}_{n-1}) \), and write \( d = (d_0, \ldots, d_{n-1}) \). Then

\[ \delta^{(B)} = \sum_{i \in [n]} |V^{(i)} \cdot d|. \]

We now make use of Sykora’s theorem [27], which states that if \( W \) is chosen uniformly at random according to the Haar measure on \( \mathcal{U}(n) \), then the vector \( V^{(i)} \), for any \( i \), is uniformly distributed over the probability simplex

\[ T_n = \{ (\nu_0, \ldots, \nu_{n-1}) : \nu_i \in [0, 1], \sum_{i \in [n]} \nu_i = 1 \}. \]

Since all of the \( V^{(i)} \)'s have the same distribution, we see that

\[ \mathbb{E} \left( \delta^{(B)} \right) = n \mathbb{E}(|V \cdot d|), \]

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where $V$ is a generic $V^{(i)}$.

The following lemma allows us to relate a lower bound on $\mathbb{E}(\delta^{(B)})$ to a lower bound on $\mathbb{P}[\delta^{(B)} \geq \lambda]$, for some $\lambda$.

**Lemma 6.1.**

$$\mathbb{P}\left[\delta^{(B)} \geq \lambda\right] \geq \frac{1}{\eta} \left(\mathbb{E}\left(\delta^{(B)}\right) - \lambda^2\right)$$

**Proof.** Let $p = p(\mu)$ be the probability density function for $\delta^{(B)}$. As noted in eq. (1), $0 \leq \delta^{(B)} \leq \eta$. Thus, for $\lambda \in [0, \eta]$ we can write

$$\mathbb{E}\left(\delta^{(B)}\right) = \int_0^\eta \mu p(\mu) d\mu = \int_0^\lambda \mu p(\mu) d\mu + \int_\lambda^\eta \mu p(\mu) d\mu \leq \int_0^\lambda \lambda \cdot 1 d\mu + \int_\lambda^\eta \eta p(\mu) d\mu = \lambda^2 + \eta \mathbb{P}\left[\delta^{(B)} \geq \lambda\right].$$

Rearranging the inequality gives the result.  

**Remark:** One might consider using Chebyshev’s inequality [21] to place a bound on $\mathbb{P}[\delta^{(B)} \geq \lambda]$. The above lemma achieves a tighter bound, however, which is necessary for the remainder of this section.

We now write $\mathbb{E}(|V \cdot d|)$ as an integral over the probability simplex $T_n$. We have

$$\mathbb{E}(f(V)) = \int_{T_n} f(V) dV := (n-1)! \int_{v_0=0}^1 \cdots \int_{v_{n-1}=0}^1 \delta(1 - \sum_{i \in [n]} v_i) f(V) dv_0 \cdots dv_{n-1}$$

where $dV = (n-1)! \delta(1 - \sum_{i \in [n]} v_i) dv_0 \cdots dv_{n-1}$ is the normalised measure on $T_n$, defined so that $\mathbb{E}(1) = 1$.

Now note that the integral expression for $\mathbb{E}(|V \cdot d|) = \mathbb{E}(|v_0d_0 + \cdots v_{n-1}d_{n-1}|)$ is completely symmetric in the $v_i$’s (and hence in the $d_i$’s). Thus, if $\sigma$ is a permutation on $[n]$, we have that

$$\mathbb{E}(|v_0d_0 + \cdots v_{n-1}d_{n-1}|) = \mathbb{E}(|v_0d_{\sigma(0)} + \cdots v_{n-1}d_{\sigma(n-1)}|).$$
Using this observation, we can write

\[
\mathbb{E}(|v_0d_0 + \cdots v_{n-1}d_{n-1}|)
= \frac{1}{n}\left[\mathbb{E}(|v_0d_{c(0)} + \cdots v_{n-1}d_{c(n-1)}|) + \mathbb{E}(|v_0d_{c(1)} + \cdots v_{n-1}d_{c(0)}|) + \cdots + \mathbb{E}(|v_0d_{c(n-1)} + \cdots v_{n-1}d_{c(n-2)}|)\right]
\]

\[
= \frac{1}{n}\left[\mathbb{E}(|v_0d_{c(0)} + \cdots v_{n-1}d_{c(n-1)}|) + \mathbb{E}(|v_0d_{c(1)} - \cdots v_{n-1}d_{c(0)}|) + \cdots + \mathbb{E}(|v_0d_{c(n-1)} - \cdots v_{n-1}d_{c(n-2)}|)\right]
\]

\[
\geq \frac{1}{n}\mathbb{E}\left[|v_0(d_{c(0)} - d_{c(1)} + \cdots - d_{c(n-1)}) + v_1(d_{c(1)} - d_{c(2)} + \cdots - d_{c(0)}) + \cdots + v_{n-1}(d_{c(n-1)} - d_{c(0)} + \cdots - d_{c(n-2)})|\right]
\]

\[
= \frac{1}{n}\mathbb{E}\left[|d_{c(0)} - d_{c(1)} + d_{c(2)} + \cdots - d_{c(n-1)}| \mathbb{E}(|v_0 - v_1 + v_2 - \cdots - v_{n-1}|),\right)
\]

where in eq. (2) minus signs are added inside every other expectation (note that \(n\) is even), and eq. (3) is derived using the triangle inequality.

Since \(\sigma\) was an arbitrary permutation, we can instead write

\[
\mathbb{E}(|V \cdot d|) \geq \frac{1}{n}\left[\max_{c \in \text{Sym}([n])} |d_{c(0)} - d_{c(1)} + d_{c(2)} - \cdots - d_{c(n-1)}|\right] \mathbb{E}(|v_0 - v_1 + v_2 - \cdots - v_{n-1}|),
\]

where \(\text{Sym}([n])\) symmetric group on \([n]\), and hence

\[
\mathbb{E}\left(\delta^{(B)}\right) \geq M^{(d)}E_n,
\]

where

\[
M^{(d)} := \max_{c \in \text{Sym}([n])} |d_{c(0)} - d_{c(1)} + d_{c(2)} - \cdots - d_{c(n-1)}|, \quad (4)
\]

\[
E_n := \mathbb{E}(\|v_0 - v_1 + v_2 - \cdots - v_{n-1}\|), \quad (5)
\]

Evaluation of \(M^{(d)}\) and \(E_n\) is carried out in Sections B.1 and B.2, where we find that \(M^{(d)} \geq \frac{1}{2}\eta\) and \(E_n \geq \frac{1}{2\sqrt{n}}\). Hence

\[
\mathbb{E}\left(\delta^{(B)}\right) \geq \frac{\eta}{4\sqrt{n}}.
\]

Use of Lemma 6.1 immediately tells us that

\[
\mathbb{P}\left[\delta^{(B)} \geq \lambda\right] \geq \frac{1}{4\sqrt{n}} - \frac{\lambda^2}{\eta^2}.
\]

Setting \(\lambda = \frac{\min(1,\epsilon)}{\sqrt{8n^{1/4}}}\) and recalling that \(\epsilon \leq \eta\) gives

\[
\mathbb{P}\left[\delta^{(B)} \geq \frac{\min(1,\epsilon)}{\sqrt{8n^{1/4}}}\right] \geq \frac{1}{4\sqrt{n}} - \frac{\min(1,\epsilon)^2}{8\eta\sqrt{n}} \geq \frac{1}{4\sqrt{n}} - \frac{1}{8\sqrt{n}} = \frac{1}{8\sqrt{n}}.
\]

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Suppose we repeat this test $k$ times, choosing different bases $B_1, \ldots, B_k$ uniformly at random according to the Haar measure on $U(n)$. We call $B$ ‘good’ if $\delta(B) \geq \frac{\min(1, \epsilon)}{\sqrt{8n^{3/4}}}$. Let $K(k)$ represent the event that at least one of $B_1, \ldots, B_k$ is ‘good’. Then

$$\mathbb{P}[K(k)] \geq 1 - \left(1 - \frac{1}{8\sqrt{n}}\right)^k.$$  

Setting $k = 32\sqrt{n}$ gives

$$\mathbb{P}[K(32\sqrt{n})] \geq 1 - \frac{1}{e^4} \geq \frac{49}{50}.$$

### 6.1 Executing the algorithm

The algorithm given in Corollary 4.1 succeeds with probability at least $\frac{2}{3}$. Suppose we run the algorithm $l$ times in total. Then by using a Chernoff bound (eq. (1) in [10]), it follows that

- if the distributions are ‘equal’, $\mathbb{P} [\text{algorithm outputs } \text{Equal} \geq \frac{1}{2}l \text{ times}] \geq 1 - e^{-l/18}$;
- if the distributions are ‘far’, $\mathbb{P} [\text{algorithm outputs } \text{Far} \geq \frac{1}{2}l \text{ times}] \geq 1 - e^{-l/18}$.

The full algorithm has been set out below.

**Algorithm 2 MaximallyMixedStateTest($\rho$)**

**Input:** PQCOND access to a probability distribution $D^{(\rho, B)}_{[n]}$ over $[n]$ for any $B$, as described in Section 6, and parameter $\epsilon$. Set $l = 128 \log n$.

1. Choose $k = 32\sqrt{n}$ bases $B_1, \ldots, B_k$ uniformly at random.
2. For each $j = 1, \ldots, k$, run the algorithm given in Corollary 4.1 on the distribution $D^{(\rho, B_j)}_{[n]}$ $l$ times, returning $u_j = 1$ if at least $\frac{l}{2}$ of the runs return Far, and $u_j = 0$ otherwise.
3. If any $u_j$ is equal to 1, output Far, otherwise output Equal.

The analysis of this algorithm is separated into two cases:

- $\|\rho - 1/n\|_1 = 0$: The probability that a particular $u_j$ is equal to 1 in Step 2 is less than $e^{-l/18}$. Thus, the probability of the algorithm failing is, by the union bound,$^3$ at most $32\sqrt{n} \cdot e^{-l/18} \leq \frac{1}{2}$, and hence the algorithm outputs Equal with probability at least $\frac{3}{4}$.

- $\|\rho - 1/n\|_1 \geq \epsilon$: Suppose that $B_j$ is ‘good’. Then with probability at least $1 - e^{l/18} \geq \frac{99}{100}$, we get $u_j = 1$, and the algorithm will output Far in Step 3. The probability that one of $B_1, \ldots, B_k$ is ‘good’ is at least $\frac{49}{50}$, and hence the probability that the entire algorithm outputs Far is at least $0.97 \geq \frac{3}{4}$.

Each run of the algorithm given in Corollary 4.1 requires $\tilde{O}(n^{1/4}/\epsilon)$ PQCOND queries if $\epsilon \leq 1$, and hence in total Algorithm 2 requires

$$\tilde{O} \left( kl \frac{n^{1/4}}{\epsilon} \right) = \tilde{O} \left( \frac{n^{3/4}}{\epsilon} \right)$$

PQCOND queries.

---

$^3$For a countable set of events $A_1, A_2, \ldots$, we have that $\mathbb{P} [\bigcup_i A_i] \leq \sum_i \mathbb{P}[A_i]$.
7 Discussion

Quantum conditional oracles give us new insights into the kinds of information that are useful for testing properties of distributions. In many circumstances such oracles serve as natural models for accessing information. In addition, they are able to demonstrate separations in query complexity between a number of problems, thereby providing interesting new perspectives on information without trivialising the set-up. We now mention some open questions.

Group testing and pattern matching are further important areas to which our notion of a quantum conditional oracle could be applied. The structure of questions commonly considered there suggest that use of \( \text{PQCOND} \) would decrease the query complexity dramatically for many practically relevant problems compared to the best known quantum and classical algorithms \([1,3,8,16]\).

In our algorithms, we have made particular use of the \( \text{PQCOND} \) oracle, the quantum analogue of the \( \text{PCOND} \) oracle. It is noted in \([10]\) that the unrestricted \( \text{COND} \) oracle offers significant advantages over the \( \text{PCOND} \) oracle for many problems, and it is possible that similar improvements could be achieved for some quantum algorithms through use of the unrestricted \( \text{QCOND} \) oracle.

The algorithm that we present for quantum spectrum testing (Algorithm 2) chooses several bases \( B_1, \ldots, B_k \) independently and uniformly at random. It remains open, however, whether or not a more adaptive approach to choosing bases will yield an algorithm requiring fewer queries.

Our definition of the spectrum testing problem in Section 6 made use of the trace norm, \( \| \cdot \|_1 \). One might wonder how the query complexity would be affected if the problem were defined with a different norm, such as the operator norm\(^4\), \( \| \cdot \|_\infty \). Numerical simulations and limited analysis suggest that the probability of picking a ‘good’ basis \( B \) tends to 1 as \( n \to \infty \), and hence that the number of queries required to distinguish between the two options would be independent of \( n \). We leave the proof of this conjecture as an open question.

Appendix

A Efficient comparison of conditional probabilities

A.1 Proof of Lemma 3.1

We first state the procedure for \( \text{ALG2}(\xi, \epsilon, \delta) \) (\( \epsilon > 0, \delta \in (0,1) \)).

1. Run \( \text{ALG1}(\xi, \epsilon, \frac{1}{10}) \) \( m \) times, where \( m = \Theta(\log(1/\delta)) \) (and such that \( m \) is even). Denote the outputs as \( \tilde{f}_1, \ldots, \tilde{f}_m \), labelled such that \( \tilde{f}_1 \leq \cdots \leq \tilde{f}_m \).

2. Output \( \tilde{f}_{m/2} \).

Consider \( \text{ALG1}(\xi, \epsilon, \frac{1}{10}) \), and let \( E_1 \) be the event that \( |\tilde{f}(\xi) - f(\xi)| \leq \epsilon \), which is equivalent to the event that \( \tilde{f}(\xi) \in [f(\xi) - \epsilon, f(\xi) + \epsilon] \). Then we have that \( \mathbb{P}[E_1] \geq \frac{9}{10} \).

Let \( Y \) be a random variable that takes the value 1 if \( E_1 \) occurs during a run of \( \text{ALG1}(\xi, \epsilon, \frac{1}{10}) \), and 0 otherwise. Let \( Y_1, \ldots, Y_m \sim Y \) be i.i.d. random variables. Let \( E_2 \) be the event that at least \( \frac{8}{10} m \) of the \( Y_i \) output 1 (i.e. the event that \( E_1 \) occurs at least \( \frac{8}{10} m \) times).

Using a Chernoff bound (here we use eq. (1) in \([10]\)), it is easy to see that \( \mathbb{P}[E_2] \geq 1 - \exp(-\frac{1}{15}m) \).

\(^4\)For an \((n \times n)\) matrix \( A, \|A\|_\infty = \max_{i \leq \nu} a_i \), where the \( a_i \) are the singular values of \( A \).
Setting \( m = \Theta(\log(1/\delta)) \) and rounding \( m \) up to the nearest multiple of 2 then gives \( \Pr[E_2] \geq 1 - \delta \).

Thus, we see that, with probability at least \( 1 - \delta \), Step 1 results in \( \tilde{f}_1 \leq \cdots \leq \tilde{f}_m \) such that \( |\tilde{f}_i - f(\xi)| \leq \epsilon \) for at least \( \frac{8}{10}m \) values of \( i \in \{1, \ldots, m\} \). Henceforth we assume that \( E_2 \) occurs. Now consider \( \tilde{f}_{m/2} \).

Suppose \( \tilde{f}_{m/2} \not\in [f(\xi) - \epsilon, f(\xi) + \epsilon] \). Then one of the two following statements must hold:

- \( \tilde{f}_{m/2} < f(\xi) - \epsilon \). Since \( \tilde{f}_1 \leq \cdots \leq \tilde{f}_{m/2} \), we have that \( \tilde{f}_1, \ldots, \tilde{f}_{m/2} \not\in [f(\xi) - \epsilon, f(\xi) + \epsilon] \), which contradicts the statement of \( E_2 \);

- \( \tilde{f}_{m/2} > f(\xi) + \epsilon \). Since \( \tilde{f}_{m/2} \leq \cdots \leq \tilde{f}_m \), we have that \( \tilde{f}_{m/2}, \ldots, f_m \not\in [f(\xi) - \epsilon, f(\xi) + \epsilon] \), which contradicts the statement of \( E_2 \).

Hence we conclude that \( \tilde{f}_{m/2} \in [f(\xi) - \epsilon, f(\xi) + \epsilon] \).

\[ \square \]

Remark: It is worth noting that the method used in the above proof could also apply to different kinds of algorithms, and not just the specific algorithm \textsc{Alg1}.

### A.2 Proof of Theorem 3.6

We prove this case-by-case. We define the shorthand \( w(X) := D_{X \cup Y}(X) = D(X) / D(X \cup Y) \), \( w(Y) := D_{X \cup Y}(Y) = D(Y) / D(X \cup Y) \) and note that \( r_{X,Y} = w(Y) / w(X) \). In addition, since \( w(X) + w(Y) = 1 \), it is straightforward to show the following inequalities for a constant \( T \geq 1 \):

\[
\begin{align*}
    r_{X,Y} \geq \frac{1}{T} & \implies w(X) \leq \frac{T}{T+1}, \quad w(Y) \geq \frac{1}{T+1} \\
    r_{X,Y} \leq \frac{1}{T} & \implies w(X) \geq \frac{T}{T+1}, \quad w(Y) \leq \frac{1}{T+1} \\
    r_{X,Y} \geq T & \implies w(X) \leq \frac{1}{T+1}, \quad w(Y) \geq \frac{T}{T+1} \\
    r_{X,Y} \leq T & \implies w(X) \geq \frac{1}{T+1}, \quad w(Y) \leq \frac{T}{T+1}
\end{align*}
\]

(6)

The strict versions of these inequalities also hold true.

1. \( \frac{1}{K} \leq r_{X,Y} \leq K \)

   In this case we wish our algorithm to output \( \hat{r}_{X,Y} \in [1 - \eta, 1 + \eta]r_{X,Y} \).

   From eq. (6), we immediately have that

   \[
   \frac{1}{K+1} \leq w(X), w(Y) \leq \frac{K}{K+1}.
   \]

   (7)

   Steps 2 and 3 use \textsc{AdjEstProbCond} to estimate \( w(X) \) and \( w(Y) \) to within additive error \( \eta/3 \) with probability at least \( 1 - \delta/4 \). As stated in Theorem 3.4, this requires

   \[
   \Theta \left( \max \left( \frac{\sqrt{w(X)}}{\eta}, \frac{1}{\sqrt{\eta}} \right) \log(1/\delta) \right) = \Theta \left( \frac{\log(1/\delta)}{\eta} \right)
   \]

   queries to \textsc{QCondD}, where the equality is due to the fact that \( w(X) \leq 1 \), and thus \( M \) (defined in Algorithm 1) queries suffice.
Step 4 uses \textsc{MULESTPROBQCOND} to estimate \(w(X)\) to within multiplicative error \(\eta/3\) with probability at least \(1 - \delta/4\). From Theorem 3.5, we clearly require

\[
\Theta \left( \frac{\log(1/\delta)}{\eta \sqrt{w(Y)}} \right) = \Theta \left( \frac{\sqrt{K} \log(1/\delta)}{\eta} \right)
\]

queries to \textsc{QCOND}_D in order to achieve these, where the equality is due to eq. (7), and thus \(M\) queries suffice. Step 5 requires the same number of queries.

With a combined probability of at least \(1 - \delta\), Steps 2–5 all pass, and produce the following values:

- \(\tilde{w}_+^X \in [w(X) - \eta/3, w(X) + \eta/3]\),
- \(\tilde{w}_+^Y \in [w(Y) - \eta/3, w(Y) + \eta/3]\),
- \(\tilde{w}_\times^X \in [1 - \eta/3, 1 + \eta/3]w(X)\),
- \(\tilde{w}_\times^Y \in [1 - \eta/3, 1 + \eta/3]w(Y)\).

From eq. (7), we see that

\[
\tilde{w}_+^X, \tilde{w}_+^Y \leq \frac{K}{K+1} + \frac{\eta}{3} < \frac{3K}{3K+1} - \frac{\eta}{3},
\]

where the final inequality is due to the algorithm’s requirement that \(\eta < \frac{1}{8K}\).

Thus, the checks in Steps 6 and 7 pass, and Step 8 gives us

\[
\tilde{r}_{X,Y} \in [1 - \eta, 1 + \eta]r_{X,Y}.
\]

2. This is split into 4 sub-cases.

a) \(K < r_{X,Y}\)
   i) \(3K < r_{X,Y}\)

   \textit{In this case we wish our algorithm to output High.}

   From eq. (6) we have that

   \[
   w(X) < \frac{1}{3K+1}, \quad w(Y) > \frac{3K}{3K+1}.
   \]

   As in Case 1, Steps 2 and 3 allow us to gain

   - \(\tilde{w}_+^X \in [w(X) - \eta/3, w(X) + \eta/3]\),
   - \(\tilde{w}_+^Y \in [w(Y) - \eta/3, w(Y) + \eta/3]\),

   with combined probability at least \(1 - \delta/2\). (We henceforth assume that we have gained such values.)

   Using eq. (8) it is easy to show that \(\tilde{w}_+^X < \frac{3K}{3K+1} - \frac{\eta}{3}\) and that \(\tilde{w}_+^Y > \frac{3K}{3K+1} - \frac{\eta}{3}\). Hence the check in Step 6 passes, but the check in Step 7 fails, and the algorithm outputs High and exits.
ii) $K < r_{X,Y} \leq 3K$

In this case we wish our algorithm to either output High or output $\tilde{r}_{X,Y} \in [1 - \eta, 1 + \eta] r_{X,Y}$.

From eq. (6), we have that

$$\frac{1}{3K + 1} \leq w(X) < \frac{1}{K + 1} \left( \frac{K}{3K + 1} < \right) \frac{K}{1 + K} < w(Y) \leq \frac{3K}{3K + 1}.$$  \hfill (9)

Thus, with $\Theta(\sqrt{K} \log(1/\delta) / \eta)$ queries, as in Case 1, we gain

$$\tilde{w}_+(X) \in [w(X) - \eta/3, w(X) + \eta/3],$$

$$\tilde{w}_-(Y) \in [w(Y) - \eta/3, w(Y) + \eta/3],$$

$$\tilde{w}_+(X) \in [1 - \eta/3, 1 + \eta/3] w(X),$$

$$\tilde{w}_-(Y) \in [1 - \eta/3, 1 + \eta/3] w(Y),$$

with combined probability at least $1 - \delta$. (We henceforth assume that we have gained such values.)

Using eq. (9), we see that $\tilde{w}_+(X) < \frac{3K}{3K + 1} - \frac{\eta}{3}$, and thus Step 6 will pass.

Assuming the check in Step 7 passes, Step 8 will output $\tilde{r}_{X,Y} \in [1 - \eta, 1 + \eta] r_{X,Y}$.

However, given the upper bound for $w(Y)$ in eq. (9), it is possible to have $\tilde{w}_+(Y) > \frac{3K}{3K + 1} - \frac{\eta}{3}$, causing the check in Step 7 to fail and the algorithm to output High.

b) $r_{X,Y} < 1/K$

i) $r_{X,Y} < 1/(3K)$

This is equivalent to the condition that $3K < r_{Y,X}$, and thus follows the same argument as Case 2(a)ii, with $X$ and $Y$ interchanged and an output of Low instead of High.

ii) $1/(3K) \leq r_{X,Y} < 1/K$

This is equivalent to the condition that $K < r_{Y,X} \leq 3K$, and thus follows the same argument as Case 2(a)iii, with $X$ and $Y$ interchanged and an output of Low instead of High.

\hfill \square

B Quantum Spectrum Testing

B.1 Evaluating $M^{(d)}$

This section provides a lower bound for the quantity $M^{(d)}$, as defined in eq. (4).

Let $D^+$ be the set of non-negative $d_i$’s, labelled such that $d_0^+ \geq d_1^+ \geq \cdots$, and similarly let $D^-$ be the set of negative $d_i$’s, labelled such that $d_0^- \leq d_1^- \leq \cdots$. w.l.o.g. suppose $|D^-| \geq |D^+|$.

Let $|D^+| = \frac{n}{2} - k$, where $k \leq \frac{n}{2}$. Thus $|D^-| = \frac{n}{2} - k$. Note that $\sum_i d_i^+ = -\sum_i d_i^- = \frac{1}{2} n$.

We now define $\sigma$ so that the following statements are true:

- $d_i(1) = d_0^+, d_i(3) = d_1^+, \ldots, d_i(\pi - 1) = d_{\pi - 1}^-$.
• \( d_{\nu(0)} = d_{\nu 0}^+, \ d_{\nu(2)} = d_{\nu 1}^+, \ldots, d_{\nu(n-2k-2)} = d_{\nu n-2k-1}^+; \)

• \( d_{\nu(n-2k)}, d_{\nu(n-2k+2)}, \ldots, d_{\nu(n-2)} \) can be filled with the remaining members of \( D^- \).

Then

• \( d_{\nu(0)} + d_{\nu(2)} + \cdots + d_{\nu(n-2k-2)} = \frac{1}{2} \eta; \)

• \( d_{\nu 0}^-, \ldots, d_{\nu 2-1}^- \leq d_{\nu 2}^- \equiv -d_{\nu(1)} - d_{\nu(3)} - \cdots - d_{\nu(n-1)} \geq -\frac{n}{2} d_{\nu 2}^-; \)

• \( d_{\nu 2}^-, \ldots, d_{\nu n-k+1}^- \geq d_{\nu n-k-1}^- \equiv d_{\nu(n-2k)} + d_{\nu(n-2k+2)} + \cdots + d_{\nu(n-2)} \geq k d_{\nu 2}^- \).

Hence

\[
|d_{\nu(0)} - d_{\nu(1)} - d_{\nu(2)} - \cdots - d_{\nu(n)}| \geq \left| \frac{1}{2} \eta + \left( k - \frac{n}{2} \right) d_{\nu 2}^- \right| \geq \frac{1}{2} \eta,
\]

where the final inequality follows since \( k \leq \frac{n}{2} \) and \( d_{\nu 2}^- < 0 \).

Thus \( M^{(d)} \geq \frac{1}{2} \eta. \)

**B.2 Evaluating \( E_n \)**

**This section provides a lower bound for the quantity \( E_n \), as defined in eq. (5).**

To evaluate \( E_n \) we will use the Hermite-Genocchi Theorem (Theorem 3.3 in [4]), which relates integrals over the probability simplex to associated divided differences.

The divided difference of \( n \) points \((x_0, f(x_0)), \ldots, (x_{n-1}, f(x_{n-1}))\) is defined by

\[
f[x_0, \ldots, x_{n-1}] := \sum_{j \in [n]} \frac{f(x_j)}{\prod_{k \neq j} (x_j - x_k)}, \tag{10}
\]

where limits are taken if any of the \( x_j \) are equal. It can be shown that for repeated points (see Exercise 4.6.6 in [26])

\[
f[\underbrace{x_0, \ldots, x_0}_{(r_0+1) \text{ times}}, \underbrace{x_1, \ldots, x_1}_{(r_1+1) \text{ times}}, x_2, \ldots, x_{n-1}] = \frac{1}{r_0! r_1!} \frac{\partial^{r_0+r_1}}{\partial x_0^{r_0} \partial x_1^{r_1}} f[x_0, x_1, x_2, \ldots, x_{n-1}], \tag{11}
\]

where \( x_0, \ldots, x_{n-1} \in \mathbb{R} \) are distinct.

Now, the Hermite-Genocchi Theorem states that

\[
f[x_0, \ldots, x_{n-1}] = \frac{1}{(n-1)!} \int_{T_n} f^{(n-1)}(v_0 x_0 + \cdots v_{n-1} x_{n-1}) \ dV,
\]

where we recall that \( dV = (n-1)! \delta(1 - \sum_{i \in [n]} v_i) \ d v_0 \cdots d v_{n-1}. \)

In order to evaluate \( E_n \), we set \( f^{(n-1)}(\xi) = (n-1)! |\xi|. \) Thus

\[
f(\xi) = \begin{cases} \frac{1}{n!} \xi^n & \xi \geq 0 \\ -\frac{1}{n!} \xi^n & \xi < 0 \end{cases}
\]

and \( E_n = f[1, -1, 1, -1, \ldots, 1, -1] \).

Let \( m = \frac{n}{2} - 1 \) (i.e. \( n = 2m + 2 \)). Then by eq. (11) we have that

\[
E_{2m+2} = \frac{1}{m!^2} \left. \partial_0^m \partial_1^m f[x_0, x_1] \right|_{x_0 = -1, x_1 = 1},
\]

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where we have used the notation \( \partial_i \equiv \frac{\partial}{\partial x_i} \).

In the neighbourhood of \( x_0 = -1, x_1 = 1 \), we have (by eq. (10))

\[
f[x_0, x_1] = -\frac{1}{2m + 2} \frac{x_0^{2m+2} + x_1^{2m+2}}{x_0 - x_1},
\]

and thus

\[
E_{2m+2} = -\frac{1}{2m + 2 m!} A|_{x_0=-1, x_1=1},
\]

where

\[
A = \partial_0^m \partial_1^m \left( \frac{x_0^{2m+2} + x_1^{2m+2}}{x_0 - x_1} \right).
\]

We see that

\[
A = \partial_1^m \partial_0^m \left( \frac{x_0^{2m+2}}{x_0 - x_1} \right) - \partial_0^m \partial_1^m \left( \frac{x_1^{2m+2}}{x_1 - x_0} \right)
\]

\[
= \partial_1^m \partial_0^m \left( \frac{x_0^{2m+2}}{x_0 - x_1} \right) - \text{(same term with } x_0 \text{ and } x_1 \text{ interchanged).}
\]

We use the Leibniz product rule\(^5\) to deduce that

\[
\partial_0^m \left( x_0^n \left( \frac{1}{x_0 - x_1} \right) \right) = \sum_{k=0}^{m} \binom{m}{k} \left[ \frac{(2m + 2)!}{(2m + 2 - k)!} x_0^{2m+2-k} \right] \left[ \frac{(-1)^{m-k}}{(x_0 - x_1)^{m+1-k}} (m-k)! \right],
\]

and hence that the first term in eq. (13) is

\[
\partial_1^m \partial_0^m \left( x_0^n \left( \frac{1}{x_0 - x_1} \right) \right)
\]

\[
= \sum_{k=0}^{m} \binom{m}{k} \left[ \frac{(2m + 2)!}{(2m + 2 - k)!} x_0^{2m+2-k} \right] \left[ \frac{(-1)^{m-k}}{(x_0 - x_1)^{2m+1-k}} (2m-k)! \right]
\]

\[
= (2m + 2)! (-1)^m \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^{m-k}}{(2m + 2 - k)!} \frac{x_0^{2m+2-k}}{(x_0 - x_1)^{2m+1-k}}
\]

\[
= (2m + 2)! (-1)^m (x_0 - x_1) \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^{m-k}}{(2m + 2 - k)!} \frac{x_0^{2m+2-k}}{(2m + 1 - k)(x_0 - x_1)^{2m+2-k}}.
\]

Substituting this into eq. (13) and setting \( x_0 = -1, x_1 = 1 \) gives

\[
A|_{x_0=-1, x_1=1} = -4(2m + 2)! (-1)^m \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^{m-k}}{(2m + 2 - k)(2m + 1 - k)} \left( \frac{1}{2} \right)^{2m+2-k}.
\]

Now set

\[
B = (-1)^m \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^{m-k}}{(2m + 2 - k)(2m + 1 - k)} 
\]

so that

\[
A|_{x_0=-1, x_1=1} = -4(2m + 2)! B|_{\gamma = \frac{1}{2}}.
\]

\(^5\) \((\mu v)^{m} = \sum_{k=0}^{m} \binom{m}{k} u^{(k)} v^{(m-k)}\)
Next, note that
\[ \frac{\partial^2 B}{\partial \gamma^2} = (-1)^m \sum_{k=0}^{m} \binom{m}{k} (-1)^k \gamma^{2m-k} = \gamma^m \sum_{k=0}^{m} \binom{m}{k} (-\gamma)^{m-k} = \gamma^m (1 - \gamma)^m, \]

and thus
\[ B|_{\gamma = \frac{1}{2}} = \int_{z=0}^{1} \int_{\alpha=0}^{z} \alpha^m (1 - \alpha)^m \, d\alpha \, dz + C \]
\[ = \int_{z=0}^{1} B_z(m + 1, m + 1) \, dz + C, \]

where \( B_z(p, q) = \int_{0}^{z} \alpha^{p-1} (1 - \alpha)^{q-1} \, d\alpha \) is the incomplete Beta function. By setting \( m = 0 \) it is easy to deduce that \( C = 0 \).

Now, the indefinite integral of the incomplete Beta function is
\[ \int B_z(p, q) \, dz = zB_z(p, q) - B_z(p + 1, q), \]

and hence we deduce that
\[ B|_{\gamma = \frac{1}{2}} = \frac{1}{2} B_{\frac{1}{2}}(m + 1, m + 1) - B_{\frac{1}{2}}(m + 2, m + 1) \]
\[ = \int_{0}^{\frac{1}{2}} \alpha^m (1 - \alpha)^m \, d\alpha - \int_{0}^{\frac{1}{2}} \alpha^{m+1} (1 - \alpha)^m \, d\alpha \]
\[ = \frac{1}{2} \left[ \int_{0}^{\frac{1}{2}} \alpha^m (1 - \alpha)^m \left( \frac{1 - 2\alpha}{1 - \alpha} \right) \, d\alpha \right] \]
\[ = \frac{1}{2} \int_{0}^{\frac{1}{2}} \left( \alpha^m (1 - \alpha)^{m+1} - \alpha^{m+1} (1 - \alpha)^m \right) \, d\alpha \]
\[ = \frac{1}{2(m + 1)} \int_{0}^{\frac{1}{2}} \frac{d(\alpha^{m+1} (1 - \alpha)^m)}{d\alpha} \, d\alpha \]
\[ = \frac{1}{2(m + 1)} \left[ \alpha^{m+1} (1 - \alpha)^m \right]_{0}^{1/2} \]
\[ = \frac{1}{2^{2m+3}(m + 1)}. \]

Substituting this into eq. (14) and subsequently into eq. (12), we get
\[ E_{2m+2} = -\frac{1}{2m+2} \frac{1}{m!} \cdot 4(2m + 2)! \cdot \frac{1}{2^{2m+3}(m + 1)} \]
\[ = \frac{(2m + 1)!}{2^{2m+1}m!(m + 1)} \]
\[ = \frac{2m + 1}{m + 1} \cdot \frac{(2m)!}{m!^2} \cdot \frac{1}{2^{2m+1}}. \]  

(15)

Stirling’s formula [25] tells us that for \( m \geq 1 \)
\[ \sqrt{2\pi}m^{m+\frac{1}{2}}e^{-m} < m! < \sqrt{2\pi}m^{m+\frac{1}{2}}e^{-m}e^{\frac{1}{12}}, \]
and thus
\[
\frac{(2m)!}{m!^2} > \frac{\sqrt{2\pi (2m)^{2m+\frac{1}{2}} e^{-2m}}}{2\pi m^{2m+1} e^{-2m} e^\frac{1}{2}} = \frac{2^m e^{-\frac{1}{2}}}{\sqrt{m\pi}}.
\]
Since \(\frac{2m+1}{m+1} \geq \frac{3}{2}\), eq. (15) tells us that
\[
E_{2m+2} > \frac{3e^{-\frac{1}{2}}}{4\sqrt{\pi}} \cdot \frac{1}{\sqrt{m}}.
\]
Replacing \(m\) with \(\frac{n}{2} - 1\), we deduce that
\[
E_n > \frac{3e^{-\frac{1}{2}}}{4\sqrt{\pi}} \cdot \frac{1}{\sqrt{\frac{n}{2} - 1}} > \frac{3e^{-\frac{1}{2}}}{4\sqrt{\pi}} \cdot \frac{1}{\sqrt{\frac{n}{2}}} = \frac{3e^{-\frac{1}{2}}}{2\sqrt{2\pi}} \cdot \frac{1}{\sqrt{n}} > \frac{1}{2\sqrt{n}}.
\]
(16)
The case when \(m = 0\) is easily dealt with through direct calculation using eq. (15), giving \(E_2 = \frac{1}{2}\). Hence we conclude that eq. (16) holds for all positive, even \(n\).

Remark: Using the asymptotic form of Stirling’s formula, it can be shown that \(E_n \sim \sqrt{\frac{2}{\pi \sqrt{n}}}\) for large \(n\).

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