The Uses of Covariant Formalism for Analytical Computation of Feynman Diagrams with Massive Fermions

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Abstract

The bilinear combination of Dirac spinors \( u(p_1, n_1)\bar{u}(p_2, n_2) \) is expressed in terms of Lorentz vectors in an explicit covariant form. The fact that the obtained expression involves only one auxiliary vector makes it very convenient for analytical computations with REDUCE (or FORM) package in the helicity formalism. The other advantage of the proposed formulas is that they apply to massive fermions as well as to massless fermions. The proposed approach is employed for the computation of one-loop Feynman diagrams and it is demonstrated that it considerably reduces the time of computations.

1 Introduction

Over many years, expressions of the type

\[ w(p_1, n_1)\bar{w}(p_2, n_2) = S + V_\mu \gamma^\mu + T_{\mu\nu} \sigma^{\mu\nu} + A_\mu \gamma^\mu \gamma^5 + P \gamma^5 \]

(1)

have received considerable study [1]–[6]. They have been extensively used both in calculations of multiparticle helicity amplitudes and in the analysis of polarization phenomena. The method of computation of multiparticle helicity amplitudes (hereafter referred to as the spinor formalism) is based on the representation of the polarization vectors of gauge bosons in terms of spinors. It has been successfully applied to the computations of tree amplitudes of the reactions involving massless particles [2] and has also been extended to the massive case [7, 8]. However, a computation of a loop diagram with this method presents a challenge.

Another approach is to express the relevant products of spinors in terms of Lorentz vectors (the method of calculations with the use of such expressions is referred to as the helicity formalism). However, the expressions for \( S, V_\mu, etc. \) obtained in [3, 4] are very cumbersome and involve singularities that hinder employing these expressions in computations. One can use a particular basis to avoid the singularities; in this case, there emerges an additional auxiliary vector [5, 6].

In this work, we discuss properties of these auxiliary vectors and demonstrate that an expression of these vectors in terms of momenta simplifies the computations and makes it possible to employ the helicity formalism for an analytical computation of the helicity amplitudes in the one-loop approximation.

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2 Basic Formulas

In order to compute a complete set of the helicity amplitudes of some process involving 2 fermions, it is sufficient to consider the case when the momenta \( p_1 \) and \( p_2 \) and the spin vectors \( \pm N_1 \) and \( \pm N_2 \) of these fermions lie in the one 2-plane; in this case, polarization vectors of the fermions can be expressed in terms of the momenta by the formulas

\[
N_1 = \frac{1}{\Delta} \left( \frac{p_1 \cdot p_2}{m_1} p_1 - m_1 p_2 \right), \quad N_2 = \frac{1}{\Delta} \left( -m_2 p_1 + \frac{p_1 \cdot p_2}{m_2} p_2 \right),
\]

where \( m_1^2 = p_1^2 \), \( m_2^2 = p_2^2 \), and \( \Delta = \sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2} \). This assertion stems from the fact that an arbitrary polarization state of a fermion is a quantum-mechanical superposition of the states \( |N\rangle \) and \( |-N\rangle \), where \( |N\rangle \) is the state with the spin vector directed along some vector \( N \) and \( |-N\rangle \) is the state with the spin vector directed oppositely to \( N \). For this reason, we restrict our attention to the basis composed of the fermion states \( |p_1, N_1\rangle, |p_1, -N_1\rangle, |p_2, N_2\rangle, \) and \( |p_2, -N_2\rangle \) (or, in another notation, \( u(p_1, N_1), u(p_1, -N_1), u(p_2, N_2), u(p_2, -N_2) \)), where the vectors \( N_1 \) and \( N_2 \) are defined by (2); this basis is named ‘diagonal spin basis’ \([5, 6]\).

![Figure 1: Momenta \( p_1, p_2 \) and vectors of spin \( N_1, N_2 \) lie in the one 2-plane](image)

In this basis, the combination of Dirac spinors \( u(p_1, N_1)\bar{u}(p_2, N_2) \), where \( p_1 \) and \( p_2 \) are the momenta and \( N_1 \) and \( N_2 \) are the polarization vectors, has the form \([9]\)

\[
u\bar{u}(\pm, \pm) = \left( j_1 \frac{1 \pm \gamma^5}{2} - j_2 \frac{1 \mp \gamma^5}{2} + m_1 \hat{k}_2 - m_2 \hat{k}_1 \right) \frac{\bar{\omega}_\pm}{\sqrt{2}},
\]

(3)
where \( u\bar{u}(\pm, \mp) = (j_1 k_1 + m_1) \hat{k}_2 \frac{1 \pm \gamma^5}{2} + (j_2 k_2 + m_2) \hat{k}_1 \frac{1 \pm \gamma^5}{2}, \) \((4)\) 

and the light-like vectors \( k_1, k_2, \) and \( \omega_\pm \) are given by 
\[
 k_1 = \frac{1}{2\Delta} \left( j_1 p_1 - \frac{m_1}{m_2} j_3 p_2 \right), \quad k_2 = \frac{1}{2\Delta} \left( -\frac{m_2}{m_1} j_2 p_1 + j_1 p_2 \right), \quad k_1 \cdot k_2 = \frac{1}{2}, \quad (6) 
\]

\[
 \omega_\mp = -\frac{1}{\sqrt{2\Delta}} \left( \hat{k}_1 \hat{q} \hat{k}_2 \frac{1 \pm \gamma^5}{2} + \hat{k}_2 \hat{q} \hat{k}_1 \frac{1 \mp \gamma^5}{2} \right), \quad (7)
\]

and \( q \) is an arbitrary vector such that \( \Delta_3 \neq 0^2 \). To simplify computations, one should choose vector \( q \) such that either \( q \cdot q = 0 \) or \( q \cdot k_1 = q \cdot k_2 = 0 \). In spite of the presence of an arbitrary vector \( q \) in (7), \( \omega_+ \) and \( \omega_- \) are almost independent of \( q \): they only acquire a phase as the vector \( q \) varies.

The explicit expressions (7) for \( \omega_\pm \) allow to compute the complete set of the helicity amplitudes using only momenta of the particles, that is, without resort to specific polarization vectors \([2]\) or ‘fundamental’ spinors \([7]\). Such computation is possible due to the arbitrariness in a choice of the ‘gauge’ vector \( q \). To explain this in more detail, let us consider a diagram involving a fermion line of the type \( \mathcal{F} = \bar{u}(p_2, n_2)\mathcal{O}u(p_1, n_1) \), where \( \mathcal{O} \) is something like \( T \) in formula (10) below. Since \( \mathcal{F} = \text{Tr} \mathcal{O}u(p_1, n_1)\bar{u}(p_2, n_2) \), it can be evaluated with the formulas (3)–(7); this being so, the momentum of any particle other than a fermion of momentum \( p_1 \) or \( p_2 \) can be inserted in (7) for vector \( q \).

The only exception from this rule is the decay \( f_1 \to f_2 \gamma \), where \( f_1 \) and \( f_2 \) are heavy and light fermions and \( \gamma \) is the photon (or gauge boson). In this case, there is no momentum independent of the fermion momenta and, therefore, vector \( q \) with the required properties does not exist. In the analysis of such reactions, the vectors \( \omega_\pm \) should be identified with the polarization vectors of the photon.

### 3 An Example of Computations

Our approach is based on the representation of spinors in terms of vectors, its application to loop computations is straightforward. Therein lies the difference between the proposed approach and that by Ballesterro \textit{et al.} \([7]\), in which all relevant vectors should be expressed in terms of spinors. We illustrate the proposed method by computing the imaginary part of the one-loop diagram for the decay \( K(p_K) \to \mu(k)\nu(k')\gamma(q) \) (see Fig. 2). The kinematical variables here are defined as follows:

\[
 x = \frac{2p_K \cdot q}{M_K^2}, \quad y = \frac{2p_K \cdot k}{M_K^2}, \quad \rho = \frac{m_2^2}{M_K^2}, \quad (9)
\]

\[
 \lambda = \frac{1 - y + \rho}{x}, \quad \zeta = (1 - \lambda)(1 - x) - \rho.
\]

\footnote{Here and below, \( \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \) and \( \varepsilon^{0123} = +1 \). In this case, \( \text{Tr}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta \gamma^5 = -4i\varepsilon^{\mu\nu\alpha\beta} \) and the operator \( g(l, a) \) in the \textit{REDUCE} package (symbol \( l \) marks the fermion line) should be identified with \( -\gamma^5 \).}
where \( m_\ell \) is the lepton mass and \( M_K \) is the kaon mass.

We employ the Cutkosky rules \cite{10} to replace cut propagators by the \( \delta \) functions. Thus we obtain the expression for the imaginary part of the amplitude,

\[
\mathcal{M}'' = \frac{\alpha F}{2\pi} G_\nu e V_{us}^* \int dr \frac{\delta(r^2 - m_\ell^2)\delta ((k+q-r)^2)}{Z(r \cdot q, r \cdot k)} \bar{u}(k')(1 + \gamma^5)T(r, k', q, \epsilon) v(k),
\]

where \( T = 8(\hat{k} + \hat{q})(1 - \gamma^5)(m_\ell - \hat{q} - \hat{k})\gamma^\nu(m_\ell - \hat{q} - \hat{r})\hat{\epsilon}(m_\ell - \hat{r})\gamma^\nu, \) \( Z \) is the product of the remaining (uncut) propagators, \( r \) is the loop momentum, spinors \( u \) and \( v \) describe the neutrino and muon, and \( \hat{\epsilon} = \epsilon_\mu(\pm)\gamma^\mu \). Here

\[
\epsilon_\mu(\pm) = \frac{\sqrt{2}}{2M_K \sqrt{1 - x - \rho}} \left( -x\lambda k_\mu + x(1 - \lambda)k'_\mu - (1 - \rho - x)q_\mu \mp \frac{2i}{M_K^2} \varepsilon_{k'q\mu} \right).
\]

is the polarization vector of the photon. The factor \( \frac{\alpha F}{2\pi} G_\nu e V_{us}^* \) in formula (10) should be thought of as an effective coupling constant.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Diagram giving a contribution to the amplitude of the decay \( K \to \mu \nu \gamma \).}
\end{figure}

Our method is based on computation of the helicity amplitudes.\(^3\) Making use of the formulas

\[
v_\mu(k, -N)\bar{u}_\nu(k') = \frac{\hat{k} - m_\ell \hat{k}'}{2M_K \sqrt{1 - x - \rho}} (1 + \gamma^5),
\]

\[
v_\mu(k, N)\bar{u}_\nu(k') = \frac{M_K^2(1-x-\rho) - m_\ell k'}{2M_K \sqrt{1 - x - \rho}} \hat{\omega}_-(1 + \gamma^5),
\]

\(^3\)Here there are four helicity amplitudes: the muon helicity (we select the reference frame comoving with the center of mass of the muon and neutrino) takes one of the two values \( \pm 1/2 \); photon helicity in any reference frame is \( \pm 1 \).
\[ \dot{\omega} = -\frac{\sqrt{2}}{2M_K^2x\sqrt{\lambda\zeta}} \left( \hat{k}\hat{k}'(1 + \gamma^5) + \hat{k}'\hat{k}(1 - \gamma^5) - \frac{2M_K^2\rho x \lambda}{1 - x - \rho} \hat{k}' \right) \] (13)

and
\[ N_\nu = \frac{(1 - x - \rho)k_\nu - 2\rho k_\nu'}{m_\ell(1 - x - \rho)}, \] (14)

we express the quantity \( \tilde{u}(k')T(r, k, k', q, \epsilon)v(k) \) (which is nothing but the respective helicity amplitude) in the integrand in (10) in terms of the scalar products of \( k, k', q, \) and \( r \) for each fixed polarization state of the muon and photon and only then perform integration with respect to \( r \). The two integrations are trivial due to the \( \delta \) functions, the integrand takes the form of a rational function \( F = F(x, \lambda, \rho, v, \cos \phi) \):

\[ \int dr \delta(r^2 - m_\ell^2) \delta ((k + q - r)^2) F(r) = \frac{x(1 - \lambda)}{8\tau} \int_{-1}^{1} dv \int_{0}^{2\pi} d\phi F, \] (15)

where
\[ v = \frac{4\tau}{x^2(1 - \lambda)^2} \frac{q \cdot r}{M_K} - \frac{2\rho}{x(1 - \lambda)} - 1, \] (16)

and the azimuth angle \( \phi \) specifies the direction of the projection of the vector \( r \) onto the plane orthogonal to the 4-vectors \( k \) and \( q \). The function \( F(x, \lambda, \rho, M_K/M_\pi, v, \cos \phi) \) is a polynomial in \( \cos \phi \), therefore, integration with respect to \( \phi \) is trivial; this being so, the integrand in the integral with respect to \( v \) has the form \( A(v) + B/(Cv + D) \), where \( A(v) \) is a polynomial in \( v \) and \( B, C, D \) are independent of \( v \). The computation of the diagram in Fig. 2 is made with the \textit{REDUCE} package [11]. This diagram is calculated exactly, no approximation is used [12]. A computation in the helicity formalism is by an order of magnitude faster than a computation with the traditional method.

Now we present the contribution of the diagram in Fig. 2 to the imaginary part of the helicity amplitudes as it was calculated using the above procedure:

%%% The variables used below are defined as follows:
let \( w = 1 - \lambda \mbd; \)
let \( b^2 = x \lambda \mbd; \)
let \( c^2 = x(1 - \lambda) \mbd; \)
let \( az^2 = 1 - \rho - x \mbd; \)
let \( d^2 = c^2az^2 - \rho b^2 \mbd; \)

%%% Results of computations:
\[
\begin{align*}
\text{GammaMinusMuMinus} & := \frac{(16 \rho^2 m_e^4 w^4 x^2 (w - 1) + 16 \rho^2 m_K^4 w^4 x^2 (w - 1) + 4 \rho m_e m_K^4 w^2 x^4 w x^2 (w^2 - 1) + 8 \rho m_K^4 w^2 x^2 (w^2 - x - 1))}{(\rho a z b \ast d + a z b \ast d w x)}$
\end{align*}
\]

\[
\begin{align*}
\text{GammaMinusMuPlus} & := \frac{(16 \rho^2 m_K^4 w^4 x^2 (w - 1) + 4 \rho m_K^4 w^2 x^2 (w - 1))}{(\rho a z + a z w x)}$
\end{align*}
\]
\[ \text{GammaPlusMuMinus} := (16\rho^3m_eM_K^4w^2x( - w + 1) + 8\rho^2m_eM_K^4w^2x( - 2w^2x - w^2 + 2w + 3x - 2) + 4\rho M_K^4w^2x^3( - w^3x + 9w^2x + 6w + 8x - 6) + 4meM_K^4w^3x^4( - w^2x + w + x - 1)) / (\rho az + azw^2x) + \ln(w^2x/\rho + 1) \cdot (8\rho b^2dM_K^4)/az \]

\[ \text{GammaPlusMuPlus} := (8\rho^2M_K^5w^2x( w + 1) + 4\rho M_K^5w^2x^2( - w^2x + 6x - 4) + 4M_K^5w^3x^3(x - 1)) / (\rho az + azw^2x) + \ln(w^2x/\rho + 1) \cdot ( - 8\rho^2M_K^5x + 8\rho M_K^5w^2x( - x + 1))/az, \]

where \( \lambda = 1 - \lambda \), \( \rho = \rho \), \( m_e = m_e \), \( M_K = M_K \), the variable \( \text{GammaPlusMuMinus} \) is the imaginary part of the amplitude for the photon helicity +1 and the muon helicity −1/2 (in the reference frame comoving with the center of mass of the lepton pair) etc. Here we avoid cumbersome denominators by expressing all vectors in terms of the light-like vectors \( q \).

\[ k_1 = \frac{1}{M_K \sqrt{1 - x - \rho}} \left( k - \frac{\rho}{1 - x - \rho} k' \right), \quad \text{and} \quad k_2 = \frac{k'}{M_K \sqrt{1 - x - \rho}} \]  

from the outset.

**Note added**

The imaginary part of the above diagram is needed for the determination of the transverse component of the muon spin, which may provide an indicator of CP violation. It should be noted that, using the relations similar to (3) and (4), the average value of the transverse component of the muon spin in the reaction under consideration can be expressed in terms of the helicity amplitudes as follows:

\[ \xi = \frac{1}{N^2} \left( M_+ M_- - M_- M_+ - M_- M_+ + M_+ M_- - M_+ M_- + M_+ M_- \right), \]  

where \( N \) is the normalization factor, \( N^2 = \sum_{i,j=\pm} |M_{ij}|^2 \), and \( M_{r,s} \) and \( M''_{r,s} \) are the real and imaginary parts of the helicity amplitudes, \( M_{r,s} = M_{r,s}^r + iM_{r,s}^i \) (indices \( r, s = \pm \) mark the photon and muon helicities).

**Appendix. Properties of the Vectors \( \omega^\pm_{\mu} \).**

Any ordered pair of real light-like vectors \( k_1 \) and \( k_2 \): \( k_1 \cdot k_2 = \frac{1}{2} \) determines the other pair of the complex zero-norm vectors \( \omega_+ \) and \( \omega_- \), which are complex-conjugated to each other and satisfy the relations

\[ \omega_+ \cdot k_1 = \omega_+ \cdot k_2 = 0, \quad \omega_- \cdot k_1 = \omega_- \cdot k_2 = 0, \quad \omega_- \cdot \omega_+ = -1. \]  

\footnote{It measures a half of the muon polarization}
Now, let the tensors
\[ k_1^\mu \omega_+^\nu - \omega_+^\mu k_1^\nu \quad \text{and} \quad k_2^\mu \omega_-^\nu - \omega_-^\mu k_2^\nu \]
be self-dual:
\[ \hat{k}_1 \omega_+^\nu (1 - \gamma^5) = \hat{k}_2 \omega_-^\nu (1 - \gamma^5) = 0, \quad (20) \]
whereas the tensors
\[ k_2^\mu \omega_+^\nu - \omega_+^\mu k_2^\nu \quad \text{and} \quad k_1^\mu \omega_-^\nu - \omega_-^\mu k_1^\nu \]
be anti-self-dual:
\[ \hat{k}_2 \omega_+^\nu (1 + \gamma^5) = \hat{k}_1 \omega_-^\nu (1 + \gamma^5) = 0. \quad (21) \]
The vectors \( \omega_+ \) and \( \omega_- \) are determined by the pair \( k_1, k_2 \) and the requirements (19) and (20)–(21) up to a phase.

- The vectors \( \omega_+ \) (\( \omega_- \)) can be interpreted as the polarization vectors of the photon of momentum \( k_1 \) and positive (negative) helicity (in the gauge \( k_2 \cdot A = 0 \)).

The vectors \( \omega_\pm \) can be represented in the explicit form\(^5\) \cite{6, 9}
\[ \omega_\pm^\mu = \frac{1}{\sqrt{2\Delta_3}} (k_1 \cdot k_2 q^\mu - q \cdot k_2 k_1^\mu - q \cdot k_1 k_2^\mu \pm i \varepsilon_{\mu \nu \alpha \beta \gamma} k_1^\nu k_2^\alpha k_3^\beta k_4^\gamma), \quad (22) \]
where \( \Delta_3 \) is defined in (8).

It should also be mentioned that, in the framework of the Cartan theory of spinors \cite{13}, a Dirac spinor in 4-dimensional space can be interpreted as an isotropic 2-plane in 5-dimensional pseudo-Euclidean space (4 dimensions in it are associated with the Minkowski space-time and the 5th dimension is associated with \( \gamma^5 \) giving the signature \((+,-,-,-,-)\)). This being so, such plane for the spinor\(^6\) \( u(p,n) \) is nothing but the linear span of the zero-norm 5-vectors \((1,n)\) and \((0, \omega_+)\), where the 4-vector \( \omega_+ \) is associated with the vectors
\[ k_1 = \frac{1}{2} \left( \frac{p}{m} + n \right) \quad \text{and} \quad k_2 = \frac{1}{2} \left( \frac{p}{m} - n \right) \]
according to the above procedure.

References

\footnote{Note that \( \gamma^5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \) and \( \varepsilon^{0123} = +1 \).
\footnote{Solution of the Dirac equation with 4-momentum \( p \) \((p^2 = m^2)\) and vector of spin \( n \) \((n^2 = -1)\).}

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