Coordinate Geometric Approach to Spherometer

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Abstract

The spherometer used for measuring radius of curvature of spherical surfaces is explicitly based on a geometric relation unique to circles and spheres. We present an alternate approach using coordinate geometry, which reproduces the well-known result for the spherometer and also leads to a scheme to study aspherical surfaces. We shall also briefly describe some of the modified spherometers.

Keywords and phrases: Spherometer, Aspherical Surfaces, Coordinate Representation, Cylindrometer, Ring-Spherometer, Quadricimeter

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1 Introduction

The focal length of spherical lenses is governed by the Lens-Maker’s Equation, which contains the radii of both surfaces of the lens [1]. Spherometers are precision instruments, which were designed by the opticians of the early-nineteenth century (or even earlier) to deduce the radius of curvature of spherical surfaces, as the name suggests [2]. The initial design of a three-legged base supporting a central micrometer screw, was quickly adopted as the basic standard and remains in use to this day [3]-[9]. The error arising
from the deviations from the tripod design have been studied [10]. The need
to handle aspherical surfaces (non-spherical surfaces) has resulted in mod-
ifications of the traditional spherometers [11]. One of the most prominent
modifications resulted in the invention of the cylindrometer, a dual device,
which can additionally measure the radius of curvature of a right circular
cylinder [12]-[14]. The working of both the spherometer and the cylindrome-
ter (also known as the Cylindro-Spherometer and Sphero-Cylindrometer) are
based on a geometric relation unique to circles and spheres, dating back to
the time of Euclid. In this article, we present an alternate derivation of
the spherometer formula using coordinate geometry, which reproduces the
familiar result for the spherometer. This approach, using the powerful tech-
niques of coordinate geometry is suitable to a generalization of the traditional
spherometer to devices, which can characterize aspherical surfaces [15]. We
shall first review the traditional derivation of the spherometer formula and
then describe the coordinate geometric approach. An Appendix is dedicated
to the variants and modifications of the common spherometer.

2 Spherometer: the Traditional Approach

The commonly available spherometers consist of a tripod framework supported on three fixed legs of equal lengths. The tips of the three legs lie on the corners of an equilateral triangle of side $L$. An accurately cut micrometer-screw passes through the nut fixed at the centroid of the equilateral triangle. The micrometer-screw is parallel to the three fixed legs. A large circular disc with typically a hundred divisions is attached to the top of the micrometer-screw. A small millimeter scale is vertically attached to the tripod framework (parallel to the axis of the micrometer-screw). The two scales together (working on the principle of the screw gauge) provide the relative height of the micrometer-screw (known as *sagitta*) with respect to the tips of the fixed legs. When the spherometer is placed on a spherical surface, the underlying geometry is a circle with the equilateral triangle inscribed in it. The great circle (whose radius is $R$) touches the tip of the micrometer-screw of height $h$, measured relative to the plane containing the tips of the three fixed legs, as seen in Fig.-2. Applying the Pythagorean theorem in Fig.-2, we have $R^2 = r^2 + (R - h)^2$, leading to

$$R(h) = \frac{r^2}{2h} + \frac{h}{2},$$

$$h(r) = R - \sqrt{R^2 - r^2} = \frac{r^2}{R + \sqrt{R^2 - r^2}}. \quad (1)$$

The radius, $r$ of the circumcircle is related to the side, $L$ of the equilateral triangle by the relation $r = L/\sqrt{3}$. So, the radius of the sphere is given by

$$R(h) = \frac{L^2}{6h} + \frac{h}{2}. \quad (2)$$

3 Coordinate Geometric Approach to the Spherometer

A sphere is uniquely determined by specifying four points on it [16]-[18]. This is precisely the situation in a spherometer: the three points are the fixed tips
of the tripod and the fourth point is the movable tip of the micrometer-
screw. It is possible to have a coordinate representation of the spherometer
as follows. The tips of the three fixed legs (lying on the corners of the
equilateral triangle of side $L$) and the tip of the movable micrometer-screw
(lying at the centroid of the equilateral triangle), on a plane surface can be
chosen to lie completely in the $X$-$Y$ plane, without any loss of generality.
The tip of the micrometer-screw moves parallel to the $Z$-axis. In this choice
of representation, when the spherometer is placed on a plane surface, the
$z$-coordinates of all the four points are identically zero. We call this as
the ground state of the spherometer. When the spherometer is placed on a
spherical surface, only the $z$-coordinate of the micrometer-screw changes to
$h$ and the remaining three points remain unchanged.

The general equation of a sphere has four independent constants and is

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0.$$  \(3\)

The radius of this sphere is $R = \sqrt{u^2 + v^2 + w^2 - d}$ and its centre is at
$(-u, -v, -w)$. The four constants in the above equation are determined
uniquely from the four points lying on the sphere. It is to be noted that the
four points can not be coplanar. Using the three points of the tips of the fixed legs (inbuilt in the construction of the spherometer and the choice of the coordinate system), we obtain $d = 0$, $u = (-1/2)L$, and $v = (-\sqrt{3}/6)L$. The coordinates of the micrometer-screw lead to $w = (L^2/6h) - h/2$. Substituting the values of these four constants, the radius of the sphere is readily obtained

$$R = \sqrt{u^2 + v^2 + w^2 - d} = \frac{L^2}{6h} + \frac{h}{2}.$$ (4)

Thus, we reproduce the familiar result using coordinate geometry. The choice of the coordinate representation ensures that the equation of the sphere is dependent only on one measurable parameter, namely the height $h$ of the screw. The central feature of the coordinate geometric approach is that it does not use the underlying geometric relation explicitly, as is customary in the traditional approach. Moreover, it is suited to incorporate the changes in the geometry of the spherometer. For instance, during construction or during operation (particularly, when the spherometer is large and heavy) the device may deviate from the equilateral triangle geometry. In which a case one only need to revise the ground state of the spherometer (which can be done even during the measurements) and then solve for the equation of the sphere. Another advantage of using the coordinate geometric approach is
that it offers a scheme to generalize the traditional spherometer to devices for studying aspherical surfaces.

4 Concluding Remarks

In the language of coordinate geometry, a sphere is uniquely determined by specifying four points on it. In the spherometer the four points are the tips of the tripod and the movable tip of the micrometer-screw. A right circular cylinder is uniquely determined by specifying five points on it; consequently the cylindrometer is a five point device. The tips of four fixed legs on a square base and the movable tip of the micrometer-screw at the centre account for the five points. Other surfaces such as cylindrical lenses or paraboloidal reflectors arise naturally in optics. The study of elliptic and hyperbolic mirrors dates back to the time of Greeks [19, 20] and the medieval Arabs [21, 22]. This has necessitated the study of a general class of surfaces known as the quadratic surfaces. In the context of optics the quadratic surfaces have been classified by enumerating the Hamiltonian orbits [23]-[25]. A quadratic surface is described by the general second-order equation, which has nine independent constants [16]-[18]. The nine points (no four points being coplanar) uniquely determine the quadratic surface. So, we require a nine point device to identify any quadratic surface. The four-point coordinate representation of a spherometer, has been generalized to a nine-point device, the quadricmeter, which generates the equation of the quadratic surfaces in terms of measurable laboratory parameters [26]. The quadricmeter is the instrument devised to identify, distinguish and measure the various characteristics (axis, foci, latera recta, directrix, etc.,) completely characterizing the quadratic surfaces. The characterization is done using the standard techniques of coordinate geometry. One may use MS EXCEL [27]-[31] or a versatile symbolic package such as the MATHEMATICA incorporating the graphic environment [32, 33]. In both the spherometer and the cylindrometer, one assumes the surface to be either spherical or cylindrical respectively. In the case of the quadricmeter, there are no such assumptions. The name quadricmeter originates from the word quadrics used for quadratic surfaces and was preferred over conicoidmeter.
Appendix: Modifications of the Common Spherometer

Some of the widely used modifications of the common spherometer are:

**Ball Spherometer:**
Accuracy of measurement requires the legs of the spherometer to have sharp tips, which can damage the surfaces under study. Blunt tips may save the surface but compromise on the accuracy. This situation has been overcome in the ball spherometer. The tips of the legs are replaced by ball-bearings of radius $r_0$, modifying the basic relation to

$$R(h) = \frac{r^2}{2h} + \frac{h}{2} \pm r_0,$$

where the positive and negative signs are for the convex and concave surfaces respectively.

**Ring-Spherometer:**
One of the widely used variations of the tripod spherometer is the ring-spherometer, which has a continuous ring of radius $r$ all the way around. The radius, $R$ of the spherical surface is

$$R(h) = \frac{r^2}{2h} + \frac{h}{2}.$$

Ring-spherometers measure up to the edge of a surface providing an average curvature. The ring needs to be fairly sharp at the edge or the ring will measure differently for concave and convex surfaces. This can be corrected by using the internal and external radii of the ring for convex and concave surfaces respectively.

**Cylindrometer:**
The spherometer can be easily modified into a cylindrometer (also known as the Cylindro-Spherometer and Sphero-Cylindrometer) by replacing the tripod of the spherometer with a square framework supported on four fixed legs of equal lengths. The tips of the four legs lie on the corners of a square of side $L$. The cylindrometer enables the measurement of the radius of a right
circular cylinder in addition to the radius of spherical surfaces [12]-[14]. The corresponding expressions are

\[ R_{\text{cylinder}} = \frac{L^2}{8h} + \frac{h}{2}, \quad (A.3) \]
\[ R_{\text{sphere}} = \frac{L^2}{4h} + \frac{h}{2}. \quad (A.4) \]

**Aspherical Surfaces:**
To some extent the aspherical surfaces can be analyzed using the common spherometer. In many situations, the aspheric surfaces can be approximated as conic sections of revolution. Incorporating the eccentricity, \( e \) of the of the conic, Eq. (1) modifies to

\[ h(r) = \frac{r^2}{R + \sqrt{R^2 - (1 - e^2)r^2}}. \quad (A.5) \]

Different conics have different eccentricities (for sphere \( e = 0 \); for prolate ellipsoid \( 0 < e < 1 \), for oblate ellipsoid \( e < 0 \), for paraboloid, \( e = 1 \), and for
hyperboloid $e > 1$). A series expansion of Eq. (A.5) is used to calculate the aspheric departure [1]. A pair of readings obtained from two ring-spherometers of different sizes, close to the vertex can also be used to characterize such surfaces.

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