1. Introduction

1.1. Statements. In [Gu1] we showed that vector bundles on affine toric varieties are trivial. An easy generalization of this result to the stable situation yields the equality $K_0(R) = K_0(R[M])$ for every regular ring $R$ and every seminormal monoid $M$ (see [Gu2], also [Sw, Corollary 1.4]). The key case is the class of finitely generated normal monoids which, up to isomorphism, can be described as additive submonoids of $\mathbb{Z}^r$ of the type $C \cap \mathbb{Z}^r$ for some finite-polyhedral convex rational cone $C \subset \mathbb{R}^r$, $r \in \mathbb{N}$.

The direct higher $K$-theoretic analogue of this fact would be the equalities $K_i(R) = K_i(R[M])$, $i > 0$. But this is harder. In fact, Srinivas has shown [Sr] that the equality fails even for the simplest singular toric cone $\mathbb{C}[X^2, XY, Y^2]$ already for $i = 1$. In [Gu4], using different arguments, we showed that $K_1(R) \neq K_1(R[M])$ for essentially all finitely generated submonoids of $\mathbb{Z}^r$, not isomorphic to $\mathbb{Z}^r$.

As a plausible substitute for the equalities $K_i(R) = K_i(R[M])$, we raised in [Gu2] the following question. Assume $c$ is a natural number $\geq 2$ and $M \subset \mathbb{Z}^r$ is a submonoid without non-trivial units. One has the homothety $c \cdot : M \to M$, $m \mapsto cm$. It induces the ring homomorphism $R[{-c}] : R[M] \to R[M]$ and, hence, the group homomorphism $K_i(R[M]) \to K_i(R[M])$ which we denote by $c_*$ (for fixed $i$). Now consider an element $x \in K_i(R[M])$. Is it true that $(c^j)_*(x) \in K_i(R)$ for $j \gg 0$? Speaking loosely: is the multiplicative action of $\mathbb{N}$ on $K_i(R[M])$ nilpotent?

The conjectural positive answer to this question is strong enough to contain Quillen’s fundamental result on $K$-homotopy invariance of regular rings (see below) and the aforementioned equality $K_0(R) = K_0(R[M])$ for arbitrary seminormal monoid $M$ (Proposition 3.5(a,b)). Moreover, we explain how the positive answer yields a similar behavior of arbitrary equivariant closed subsets with respect to the embedded torus (Proposition 3.5(c)), which should be thought of as a higher version of Vorst’s results [Vo2]. We also show that all of this generalizes to not necessarily affine toric varieties (Proposition 4.7).

Such a positive answer for $i = 1$ is obtained in [Gu2] and for $i = 2$ in [Msh]. In [Gu3] we showed that the answer is again ‘yes’ for all higher $K$-groups when the cone $\mathbb{R}_+ M \subset \mathbb{R}^r$ is simplicial, i.e., spanned by linearly independent vectors. (See also Theorem 6.4.) Hence the following conjecture (about a slightly stronger nilpotence property), which contains all the previous results in a uniform way:

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**Conjecture 1.1.** Let $M$ be arbitrary commutative, cancellative, torsion free monoid without non-trivial units, $R$ be a regular ring and $i > 0$. Then for every sequence $c = (c_1, c_2, \ldots)$ of natural numbers $\geq 2$ and every element $x \in K_i(R[M])$ one has $(c_1 \cdots c_j)_*(x) \in K_i(R)$ for $j \gg 0$ (depending on $x$).

It follows from the geometric approach, developed in [Gu1] (and adapted here to higher $K$-groups), that this conjecture admits the following reformulation.

Let $C \subset \mathbb{R}^r$ be a finite-polyhedral rational cone and $H \subset \mathbb{R}^r$ be a rational hyperplane, cutting $C$ into two non-degenerate subcones $C = C' \cup C''$ in such a way that all extremal rays of $C$, except one, belong to $C''$ and the distinguished extremal ray belongs to $C''$. Then Conjecture 1.1 is equivalent to the claim that for an element $x \in K_i(R[C \cap \mathbb{Z}^r])$ and a sequence $c$ of natural numbers as above $(c_1 \cdots c_j)_*(x) \in \text{Im} \left( K_i(R[C' \cap \mathbb{Z}^r]) \to K_i(R[C \cap \mathbb{Z}^r]) \right)$ for $j \gg 0$ (Lemma 5.2). This version of Conjecture 1.1 could loosely be called “pyramidal descent”.

The first main result of this paper “almost” accomplishes the pyramidal descent when the coefficient ring is a field (before [Gu1]).

The ring $\Lambda$ is the ring of endomorphisms of certain rank $2$ vector bundle on certain quasiprojective variety. Its construction involves higher $K$-theoretic analogues of essentially all steps in [Gu1], with use of results from [Gu2]. This becomes possible in combination with Suslin-Wodzicki’s solution to the excision problem and the Mayer-Vietoris sequence for singular varieties due to Thomason.

If the action of the big Witt vectors on $K_i(A, A^+)$ for graded not necessarily commutative $k$-algebras $A = A_0 \oplus A_1 \oplus \cdots$ satisfies a very natural condition when char $k = 0$ (see Question 10.2) then one can also eliminate the distinguished monomial in $\Lambda$. This would complete the proof of Conjecture 1.1 in $0$ characteristic.

Goodwillie’s theorem on the relative theories for nilpotent ideals yields a partial result on Question 10.2 in the arithmetic case which, in combination with Theorem 9.3 and Keating’s results on congruence-triangular matrices, leads to the second main result of the paper (Theorem 11.1). Namely, we present first non-simplicial examples – actually, an explicit infinite series of pairwise different combinatorial types of polyhedral cones, verifying Conjecture 1.1 (when $R$ is a number field) for all higher $K$-groups simultaneously. These are the cones whose compact cross sections are pyramids or bipyramids in an iterative way. The simplest nonsimplicial example of such toric varieties is the cone over the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$, the triviality of vector bundles on which had been a challenge in the 80s [L][Sw, §13] (before [Gu1]).

### 1.2. $K$-homotopy invariance

Let $R$ be a ring and let $\mathbb{N}$ act nilpotently on $K_i(R[\mathbb{Z}_+^r])$, $r \in \mathbb{N}$. For a natural number $c$ the endomorphism $R[-c] : R[\mathbb{Z}_+^r] \to R[\mathbb{Z}_+^r]$ makes $R[\mathbb{Z}_+^r]$ a free module of rank $c^r$ over itself. Thus, using transfer maps,
we see that for any element \( x \in K_i(R[\mathbb{Z}_+^s]) \) there exists a natural number \( j \) such that \( c^j \cdot x \in K_i(R) \). Since the same is true for another natural number \( c' \), coprime to \( c \), we get \( x \in K_i(R) \).

That there is no way to use the same trick for general finitely generated monoids is explained by the following fact [BGu]: for a finitely generated submonoid \( M \subset \mathbb{Z}^r \) without non-trivial units, a ring \( R \), and a natural number \( c > 1 \) the endomorphism \( R[-c] : R[M] \to R[M] \) makes \( R[M] \) an \( R[M] \)-module of finite projective dimension if and only if \( M \cong \mathbb{Z}_+^s \) for some \( s \in \mathbb{Z}_+ \).

1.3. Contents. The paper is organized as follows. In §2–§6 we make preliminary reductions and develop convex geometry for monoids (with emphasis on polarized monoids). In §7 we introduce the crucial variety \( X \) and in §8 we make a reduction in the study of its \( K \)-theory. The first main result is proved in §9, while §10 provides a link with the actions of Witt vectors. The second main result is proved in §11.

1.4. Preliminaries. All the considered monoids \( M \) are assumed to be commutative, cancellative and torsion free. In other words, we assume that the natural homomorphisms \( M \to \text{gp}(M) \to \mathbb{Q} \otimes \text{gp}(M) \) are injective, where \( \text{gp}(M) \) refers to the universal group associated to \( M \) (the Grothendieck group, or the group of differences of \( M \)). Equivalently, our monoids can be thought of as additive submonoids of real spaces (\( M \to \mathbb{R} \otimes \text{gp}(M) \)). This enables us to use polyhedral geometry in their study.

When we treat monoids separately the monoid operation is written additively. In monoid rings we switch to the multiplicative notation.

For a monoid \( M \) its group of units (i. e. the maximal subgroup of \( M \)) is denoted by \( U(M) \). We put rank \( M = \dim_{\mathbb{Q}} \mathbb{Q} \otimes \text{gp}(M) \).

For a subset \( W \) of an Euclidean space \( \text{conv}(W) \) refers to the convex hull of \( W \).

The rings below, unless specified otherwise, are assumed to be commutative and with unit. A free rank \( n \) module, upon the choice of a basis, will be thought of as the module of \( n \)-rows. In particular, its elements are multiplied by \( n \times n \)-matrices from the right, while the homomorphisms are written on the left.

\( c = (c_1, c_2, \ldots) \) will always denote arbitrary sequence of natural numbers \( \geq 2 \).

Finally, \( \mathbb{N} = \{1, 2, \ldots\} \), \( \mathbb{Z}_+ = \{0, 1, 2, \ldots\} \), \( \mathbb{Z}_- = \{0, -1, -2, \ldots\} \) and \( \mathbb{R}_+ \) refers to the nonnegative part of \( \mathbb{R} \).

Our general references to toric geometry are [F][O]. For a survey of the previous results on \( K \)-theory of toric varieties see [Gu5]

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2. Geometry of monoids, normality and seminormality

Here we recall the relationship between elementary convex geometry and monoids as developed in [Gu1][Gu2]. A pure algebraic alternative is found in [Sw, §4,§5].
A monoid \( M \) is called \textit{seminormal} if the following implication holds
\[
(m \in \text{gp}(M), \ 2m \in M, \ 3m \in M) \Rightarrow (m \in M).
\]
A monoid \( M \) is called \textit{normal} if \( m \in M \) whenever \( m \in \text{gp}(M) \) and \( cm \in M \) for some \( c \in \mathbb{N} \).

It is a well known result on monoid domains that for a domain \( R \) both the normality and seminormality conditions for \( R[M] \) are equivalent to the corresponding conditions on \( R \) and \( M \) simultaneously. (A domain \( \Lambda \) is called seminormal if the cancellative and torsion free multiplicative monoid \( \Lambda/U(\Lambda) \) is such.)

Given a finitely generated monoid \( M \) with trivial \( U(M) \). By fixing an embedding \( M \to \text{gp}(M) \to \mathbb{R} \otimes \text{gp}(M) \) we can identify \( M \) with an additive submonoid of the Euclidean space \( \mathbb{R}^{\text{rank} M} \). Clearly, \( M \) gets identified with an additive submonoid of \( \mathbb{Q}^{\text{rank} M} \). We let \( C(M) \) denote the cone \( M \) spans in \( \mathbb{R}^{\text{rank} M} \), i. e. \( C(M) = \mathbb{R}_+ M \). Then \( C(M) \) is a finite polyhedral, rational, strictly convex (sometimes called ‘pointed’) cone – a classical observation in toric geometry. An element \( x \in M \) will be called \textit{extremal} if it lies on an edge (1-dimensional face) of \( C(M) \). It is known (see, for instance, [Sw, Theorem 4.5]) that there exists a rational codimension 1 affine subspace \( H \subset \mathbb{R}^{\text{rank} M} \setminus \{0\} \) such that \( C(M) \) is spanned over \( O \in \mathbb{R}^{\text{rank} M} \) by \( \Phi(M) = H \cap C(M) \) – a finite rational convex polytope. Of course, \( \Phi(M) \) is only defined up to projective transformation. To any submonoid \( M' \subset \mathbb{Q}_+ M \) we associate the convex subset \( \Phi(M') = \mathbb{R}_+ M' \cap H \subset \Phi(M) \). It is spanned by rational points. In general, when we write \( \Phi(L) \) for a monoid \( L \) it is assumed that \( L \subset \mathbb{Q}_+ M \) for some finitely generated monoid \( L \) with trivial \( U(M) \).

A monoid extension \( M_1 \subset M_2 \) is called \textit{integral} if a positive multiple of any element \( m \in M_2 \) belongs to \( M_1 \). If \( M_1 \subset M_2 \) is an integral monoid extension and \( M_1 \) is finitely generated, having no nontrivial units, then \( M_2 \) spans the same finite polyhedral pointed cone in \( \mathbb{R}^{\text{rank} M_1} = \mathbb{R}^{\text{rank} M_2} \). It is also clear that the integrality of the extension \( M_1 \subset M_2 \) is equivalent to the inclusion \( M_2 \subset \mathbb{Q}_+ M_1 = \mathbb{Q}^{\text{rank} M_1} \cap C(M_1) \). The Gordan Lemma asserts that

\textbf{Lemma 2.1.} A finite rank monoid \( M \) is finitely generated if and only if \( \text{gp}(M) \) is finitely generated and the cone \( \mathbb{R}_+ M \subset \mathbb{R}^{\text{rank} M} \) is finite polyhedral rational.

Here we do not assume that \( U(M) = 0 \), i. e. the cone \( \mathbb{R}_+ M \) may not be pointed. A proof in the case when \( U(M) = 0 \) is given in [Sw, Theorem 4.4] and the general case reduces to this case by considering a suitable polyhedral partition of \( \mathbb{R}_+ M \).

For a monoid \( M \) its \textit{normalization} is defined by
\[
n(M) = \{m \in \text{gp}(M) | \exists c \in \mathbb{N} \ \ c_m m \in M\}
\]
and the \textit{seminormalization} is defined by
\[
\text{sn}(M) = \{m \in \text{gp}(M) | \exists 2m, 3m \in M\}.
\]
They satisfy the obvious universality conditions. Observe that \( n(M) = C(M) \cap \text{gp}(M) \) for a finite rank monoid \( M \).

\textbf{Lemma 2.2.} Both normalization and seminormalization preserve finite generation. Moreover, any normal monoid is a filtered union of finitely generated normal
monoids and a seminormal monoid is a filtered union of finitely generated seminormal monoids.

In fact, the group of differences is a filtered union of its finitely generated subgroups. This reduces the problem to the case of finitely generated group of differences. Then we approximate the cone of our monoid by finite rational polyhedral subcones and Lemma 2.1 applies.

For a finite polyhedral pointed cone $C \subset \mathbb{R}^r$, $r \in \mathbb{N}$ we denote by $\text{int}(C)$ the relative interior of $C$. Similarly, for a finite convex polytope $\Phi$ its relative interior is denoted by $\text{int}(\Phi)$. Assume $M$ is a finitely generated monoid with trivial $U(M)$. Then $\text{int}(M)$ refers to the interior ideal $\text{int}(C(M)) \cap M \subset M$ ("ideal" here means $\text{int}(M) + M \subset \text{int}(M)$) and $M_*$ refers to the interior submonoid $\text{int}(M) \cup \{0\} \subset M$. A pure algebraic definition is given in [Sw, §5]: $M_* = \{ m \in M \mid \forall n \in M \exists c \in \mathbb{N} \ c m = n \in M \}$.

More generally, if $M$ is a monoid such that the cone $C(M) = \mathbb{R}_+ M \subset \mathbb{R}^{\text{rank}M}$ is strictly convex finite polyhedral, we will use the notations $\text{int}(M) = \text{int}(C(M)) \cap M \subset M$ and $M_* = \text{int}(M) \cup \{0\}$.

One more notation. Assume $M$ is a monoid and $W \subset \Phi(M)$ is a convex subset. Then we put $M(W) = \mathbb{R}_+ W \cap M$. (Here $\mathbb{R}_+ W$ is the cone with vertex at the origin and spanned by $W$.) In particular, $M_* = M(\text{int}(\Phi(M)))$.

**Lemma 2.3.** Let $M$ be a monoid whose cone $C(M)$ is finite polyhedral and pointed (hence $U(M) = 0$).

(a) $\text{sn}(M_*) = \text{sn}(M)_* = \text{n}(M_*) = \text{n}(M)_*$. In particular, if $M$ is seminormal then $M_*$ is normal.

(b) Assume $M$ is finitely generated and normal and $W \subset \Phi(M)$ is a convex subset such that $\text{dim} W = \text{dim} \Phi(M)$ (i.e. $\text{dim} W = \text{rank}(M) - 1$). Then $M(W)$ contains a free basis of $\text{gp}(M)$. Equivalently, there is a rational simplex $\Delta \subset W$ such that $M(\Delta) \approx \mathbb{Z}_{\text{rank}M}^+$ and $\text{gp}(M(\Delta)) = \text{gp}(M)$.

(c) Assume $W \subset \Phi(M)$ is a convex subset such that $\text{dim} W = \text{dim} \Phi(M)$. Then $\text{gp}(M(W)) = \text{gp}(M)$.

(d) A finitely generated monoid $M$ with trivial $U(M)$ can be embedded into $\mathbb{Z}_{\text{rank}M}^+$ in such a way that $M \subset \mathbb{Z}_{\text{rank}M}^+$ induces the equality $\text{gp}(M) = \mathbb{Z}_{\text{rank}M}^+$.

**Proof.** (a) is proved in [Gu1, Proposition 1.6]. An alternative proof (in the finite generation case) is given in [Sw, Lemma 6.5].

To see (b), fix arbitrarily a unimodular element $m \in \text{gp}(M)$ such that $\mathbb{R}_+ m$ meets $W$ in its relative interior. Since $M$ is normal we have $m \in M$. Complete $m$ to a free basis $\{ m, m_2, \ldots, m_{\text{rank}M} \}$ of $\text{gp}(M)$. If a natural number $c$ is big enough then all the rays $\mathbb{R}_+(m_2 + cm), \ldots, \mathbb{R}_+(m_{\text{rank}M} + cm)$ intersect $W$ in its interior. This is so because the radial directions of the elements $m_2 + cm, \ldots, m_{\text{rank}M} + cm \in \mathbb{R}^{\text{rank}M}$ approximate the direction of $m$ as $c \to \infty$. Therefore, for $c$ big enough we get the desired basis $\{ m, m_2 + cm, \ldots, m_{\text{rank}M} + cm \} \subset \text{gp}(M)$.

The claim (c) follows similarly, using the following observation: for any element $m \in \text{int}(M(W))$ and $n \in M$ there exists a natural number $c$ such that $n + cm \in M(W)$, yielding the inclusion $n = (n + cm) - cm \in \text{gp}(M(W))$. 
(d) is proved in [Gu3, Theorem 1.3.2]. Winfried Bruns suggested the following alternative short proof based on (b). By Lemma 2.2 we can additionally assume that $M$ is normal. Consider the dual cone

$$C(M)^* = \{ \xi \in (\mathbb{R} \otimes \text{gp}(M))^* \mid \forall x \in C(M) \, \xi(x) \geq 0 \}$$

in the dual space $(\mathbb{R} \otimes \text{gp}(M))^* = \text{Hom}_\mathbb{R}(\mathbb{R} \otimes \text{gp}(M)), \mathbb{R})$. It is again a pointed convex polyhedral cone of the same dimension $\dim C(M)$. By (b) there is a free monoid $F^* \subset C(M)^*$ such that $\text{gp}(F^*) = \text{Hom}_\mathbb{Z}(\text{gp}(M), \mathbb{Z})$. Then we have the induced embedding $M \subset F^{**}$ where $F^{**} = C(F^*)^* \cap \text{gp}(M) \approx \mathbb{Z}^{\text{rank} M}$.

\[ \square \]

**Lemma 2.4.** Assume $M$ is a finitely generated normal monoid with trivial $U(M)$ and let $E \subset C(M)$ be an edge. Assume $t$ is the (unique) generator of $E \cap M \approx \mathbb{Z}_+$. Then $\mathbb{Z}_+(-t) + M \approx \mathbb{Z} \times N$ for some finitely generated normal monoid $N$ for which $\text{rank } N = \text{rank } M - 1$ and $U(N) = 0$.

(See, for instance, [Gu1, Theorem 1.8][Sw, Lemma 8.7].)

**Corollary 2.5.** Let $M$ and $t$ be as in Lemma 2.4. Then the submonoid $(\mathbb{Z}_+(-t) + M) \setminus \mathbb{N}(-t) \subset \mathbb{Z}_+(-t) + M$ is a filtered union of monoids of the type $\mathbb{Z}t \times N_c$, $c \in \mathbb{N}$ where the $N_c$ are finitely generated normal monoids and $U(N_c) = 0$.

**Proof.** Fix any representation $\mathbb{Z}_+(-t) + M \approx \mathbb{Z} \times N$. By Lemma 2.3(d) there is a basis $\{b_1, \ldots, b_{\text{rank} N}\} \subset N$ of the group $\text{gp}(N)$ such that $\Phi(N) \subset \text{conv}\{b_1, \ldots, b_{\text{rank} N}\}$. For a natural number $c$ consider the monoid

$$N_c = (\mathbb{Z}(b_1 - ct) + \cdots + \mathbb{Z}(b_{\text{rank} N} - ct)) \cap (\mathbb{Z}_+(-t) + M),$$

the intersection being considered in $\text{gp}(M)$. Using the fact that the radial directions of the elements $b_1 - ct, \ldots, b_{\text{rank} N} - ct \in \mathbb{R}^{\text{rank} M}$ approximate that of $-t$ when $c \to \infty$ we obtain the desired filtered union representation

$$\bigcup_{c \in \mathbb{N}} (\mathbb{Z}t + N_c) = (\mathbb{Z}_+(-t) + M) \setminus \mathbb{N}(-t).$$

\[ \square \]

Below, when the monoid operation will be written multiplicatively, we will use the notation $t^{-1} M$ for $\mathbb{Z}_+(-t) + M$.

3. **Excision**

Recall that a non-unital not necessarily commutative ring $I$ satisfies excision in algebraic $K$-theory if the natural homomorphisms $K_i(I,I) \to K_i(A,I)$, $i \in \mathbb{Z}$ are isomorphisms for any ring $A$, containing $I$ as a two-sided ideal. Here $\tilde{I} = \mathbb{Z} \ltimes I$ refers to the universal ring obtained by adjoining unit (for details see [SuW]). Actually we only need to require the excision for positive indices $i$ because for $i \leq 0$ it is always satisfied [Ba, Theorem XII.8.3]. For such a non-unital ring $I$ and a ring $A$, containing $I$ as a two-sided ideal, we have the long exact sequence

$$\cdots \to K_{i+1}(A/I) \to K_i(I) \to K_i(A) \to K_i(A/I) \to \cdots$$

$(i \in \mathbb{Z})$, where $K_*(I) = K_*(\tilde{I}, I)$. 

Theorem 3.1 (Suslin-Wodzicki [SuW]). A non-unital (not necessarily commutative) ring $I$ satisfies excision if for any finite system $a_1, \ldots, a_n \in I$ there exist elements $b_1, \ldots, b_n, u, v \in I$ such that $a_i = b_iuv$ for $i \in [1, n]$ and the left annihilators of $u$ and $uv$ coincide.

In this section we show that "interior ideals" of certain monoid rings satisfy excision in $K$-theory.

For a monoid $M$ we denote by $M^c$ the inductive limit of the diagram

$$M \overset{c_1}{\to} M \overset{c_2}{\to} \cdots.$$ 

We will think of $M^c$ as the filtered union of the ascending chain

$$\frac{M}{c_1} \subset \frac{M}{c_1c_2} \subset \cdots,$$

the union being considered in $\mathbb{Q} \otimes \text{gp}(M)$.

Lemma 3.2. $M^c$ is seminormal.

Proof. Assume $2m, 3m \in M^c$ for some $m \in \text{gp}(M^c)$. Since 2 and 3 generate the additive monoid $\mathbb{Z}_+ \setminus \{1\}$ there exists $j \in \mathbb{N}$ such that

$$c_{j+1}m \in \frac{M}{c_1 \cdots c_j}.$$

But then

$$m \in \frac{M}{c_1 \cdots c_j c_{j+1}} \subset M^c.$$

□

Lemma 3.3. Let $R$ be a ring and $M$ be a finitely generated monoid with trivial $U(M)$. Then the ideal $J = R\text{int}(M^c) \subset R[M^c]$ satisfies excision in $K$-theory.

Proof. Assume $a_1, \ldots, a_n \in J$ for some $n \in \mathbb{N}$. Fix any element $m \in \text{int}(M)$. There exists $j \in \mathbb{N}$ such that the element

$$m_j = m^{(c_1 \cdots c_j)^{-1}} \in M^{(c_1 \cdots c_j)^{-1}}$$

satisfies the condition (in the multiplicative notation):

$$a_1(m_j)^{-1}, \ldots, a_n(m_j)^{-1} \in R[\text{int}(C(M)) \cap \text{gp}(M^c)].$$

By Lemmas 2.3(a) and 3.2 we have $\text{int}(C(M)) \cap \text{gp}(M^c) = \text{int}(M^c)$. Now Theorem 3.1 applies because of the equalities in $\text{int}(M^c)$:

$$m_j = (m_j)^{c_{j+1}} \cdot ((m_j)^{c_{j+1}})^{-c_{j+1}^{-1}}.$$

□

Based on Lemma 3.3 we can make the first reduction in Conjecture 1.1.

Lemma 3.4. To prove Conjecture 1.1 for a regular coefficient ring $R$ it suffices to show that $K_i(R) = K_i(R[M^c])$ $(i \in \mathbb{N})$ for all finitely generated normal monoids $M$ with trivial groups of units.

(Clearly, $(M^c)_* = (M_*)^c$ and, therefore, we can use the notation $M^c_*$.)
Proof. Since any monoid is a filtered union of finitely generated monoids and $K$-functors commute with filtered inductive limits Conjecture 1.1 reduces to the case of finitely generated monoids. We want to show that $K_i(R) = K_i(R[L^\ell])$ assuming the equalities as in the lemma, where $L$ is any finitely generated monoid with trivial $U(L)$. By Lemmas 2.3(a) and 3.3 (the latter being applied to the extension $R \text{int}(L^\ell) \subset R[L^\ell]$) one has $K_i(R[L^\ell], R \text{int}(L^\ell)) = 0$. By Lemma 3.3 the extension $R \text{int}(L^\ell) \subset R[L^\ell]$ yields the natural isomorphisms $K_i(R[L^\ell]) = K_i(R[L^\ell]/R \text{int}(L^\ell))$. Put $A_{-1} = R[L^\ell]$ and $A_0 = R[L^\ell]/R \text{int}(L^\ell)$.

Let $F \subset \Phi(L)$ be any facet (codimension 1 face). Then the ideal $R \text{int}(L^\ell(F)) \subset R[L^\ell(F)]$ is an ideal of the bigger ring $A_0$ as well. Using the same arguments as above we arrive at the equalities $K_i(A_0) = K_i(A_1)$ where $A_1 = A_0/R \text{int}(L^\ell(F))$. Next we do the same reduction with respect to another facet of the polytope $\Phi(L)$, and so on. By the same token we get a sequence of rings $A_0, A_1, \ldots, A_s$ such that

$$A_k = A_{k-1}/R \text{int}(L^\ell(F_k)) \quad \text{and} \quad K_i(A_{k-1}) = K_i(A_k)$$

for the corresponding enumeration of the facets $F_k \subset \Phi(L)$, $k \in [1, p]$ where $F_1 = F$.

Thereafter we handle the codimension 2 faces of $\Phi(L)$, and so on. Finally, by annihilating all non-trivial monomials, we shall descend to the coefficient ring $R$. (When we treat 0-dimensional faces, i. e. the vertices of $\Phi(L)$, we follow the convention – an interior of a point is the point itself.) Finally, we obtain a sequence of rings $A_{-1}, A_0, \ldots, A_q = R (q \geq p)$ such that $K_i(A_{-1}) = K_i(A_0) = \cdots = K_i(A_q) = K_i(R)$. \hfill $\Box$

Proposition 3.5. Let $R$ be a ring and $M$ a be finitely generated monoid with trivial $U(M)$.

(a) If $\mathbb{N}$ acts nilpotently on $SK_0(R[M])$ then $SK_0(R) = SK_0(R[M])$

(b) If $\mathbb{N}$ acts nilpotently on $K_0(R[M])$, $R$ is a domain and both $R$ and $M$ are seminormal then $K_0(R) = K_0(R[M])$.

(c) If $I \subset M$ is a proper ideal (i.e. $I \neq M$ and $IM \subset I$) and $\mathbb{N}$ acts nilpotently on $K_i(R[M])$ ($i \in \mathbb{N}$) then $\mathbb{N}$ acts nilpotently also on $K_i(R[M]/I)$.

(The action of $\mathbb{N}$ on $SK_0(R[M])$ and $K_i(R[M]/I)$ and the corresponding nilpotence is understood in the obvious sense.)

Actually one can drop the finite generation condition on $M$ but we skip this. The arguments below follow closely [Gu3, §3.3]. We give a shortened version, in which Lemma 2.3 is used implicitly several times.

Proof. (a) It follows easily from the results in [1] that for a ring extension $A \subset A[b] = B$, such that $b^2, b^3 \in A$ the homomorphism $SK_0(A) \to SK_0(B)$ is injective.

Let $c$ be a natural number. Consider the endomorphism $-c : M \to M$. We fix a basis $\{m_1, \ldots, m_r\}$ of gp(M) which is a subset of int($M$). Then $R[\text{Im}(-c) + \mathbb{Z}_+m_1]$ is a free $R[\text{Im}(-c)]$-module of rank $c$. In particular, one easily concludes (using transfer maps for $K_0$) that any element $x \in SK_0(R[M])$, which is mapped to zero in $SK_0(R[\text{Im}(-c) + \mathbb{Z}_+m_1])$, is of $c$-torsion. By the infinite iteration, involving periodically all the elements $m_1, \ldots, m_r$, we conclude that every element of

$$\ker (SK_0(R[M]) \to SK_0(R[M + (\mathbb{Z}_+m_1 + \cdots + \mathbb{Z}_+m_r)^c]))$$,

The action of $\mathbb{N}$ on $SK_0(R[M])$ and $K_i(R[M]/I)$ and the corresponding nilpotence is understood in the obvious sense.
where $c' = (c, c, \ldots)$, is annihilated by some positive power of $c$.

In view of the equality $s_n([M + (\mathbb{Z}_+ m_1 + \cdots + \mathbb{Z}_+ m_r) c']_*) = M_c'$ and the aforementioned corollary of Ischebeck's results, the 'interior excision' arguments (as in the proof of Lemma 3.4) for $SK_0$-groups show the implication

$$(\mathbb{N} \text{ acts nilpotently on } SK_0(R[M])) \Rightarrow (SK_0(R[M])/SK_0(R) \text{ is of } c\text{-torsion}).$$

We are done because the same is true for another natural number $c'$, coprime to $c$.

(b) Since for seminormal monoids $M$ it is well known that $\text{Pic}(R) = \text{Pic}(R[M])$ provided $R$ is a seminormal domain, the claim follows from (a).

(c) One only needs to observe that the limit ring

$$\lim_{j \to \infty} (R[M]/RI \xrightarrow{R[-s]/RI} R[M]/RI)$$

is a "face" ring of the type we had in the proof of Lemma 3.3. This is so because the limit contains no non-zero nilpotent elements, corresponding to monomials from $M$. Therefore, the similar arguments as in that proof apply. \hfill \Box

4. Localization

A Karoubi square is a commutative square of (not necessarily commutative) rings of the type

$$\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
S^{-1}A & \rightarrow & S^{-1}B
\end{array}$$

where $S$ is a central multiplicative subset of $A$ which is regular on $A$ and $B$ and such that $A/sA \to B/sB$ is an isomorphism for every $s \in S$ [Sw, §2]. In other words we require that $A \to B$ is an analytic isomorphism along $S$ [Vo1][W1]. The basic fact on Karoubi squares is that it implies the long exact sequence

$$\cdots \to K_i(A) \to K_i(S^{-1}A) \oplus K_i(B) \to K_i(S^{-1}B) \to K_{i-1}(A) \to \cdots$$

([Vo1].) This is shown by comparing the two localization sequences corresponding to $A \to S^{-1}A$ and $B \to S^{-1}B$, with use of the equivalence $H_S(A) \approx H_S(B)$ due to Karoubi [Ka] (see also [W1, §1]).

We will need the following lemma, based on an observation of Swan [Sw, §10].

Assume $R = (R, \mu, k)$ is a local ring and $M$ is a monoid, not necessarily finitely generated, for which $\Phi(M)$ is defined. Let $W \subset \Phi(M)$ be a convex open subset for which $\text{dim } W = \text{dim } \Phi(M)$. Put $\mathfrak{m} = \mu + R(M(W) \setminus \{1\}) (\in \text{max } R[M(W)])$.

**Lemma 4.1.** Let $A \subset B$ be two (not necessarily commutative) $R[M(W)]$-algebras ($R[M(W)]$ is in their centers) such that $M(W)^{-1}A = M(W)^{-1}B$. Then the commutative square of rings

$$\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
A_{\mathfrak{m}} & \rightarrow & B_{\mathfrak{m}}
\end{array}$$

is a Karoubi square.
Proof. Consider a maximal ideal \( \mathfrak{M} \in \text{max } R[M(W)] \). We claim that there are only two possibilities: either \( \mathfrak{N} = \mathfrak{M} \) or \( \mathfrak{N} \cap M(W) = \emptyset \). In fact, if \( m \in \mathfrak{N} \cap M(W) \) then for any \( n \in M(W) \setminus \{1\} \) there exists a natural number \( c \) such that \( n^c m^{-1} \in M(W) \) (as in the proof of Lemma 2.3(c)). Therefore, \( n^c \in \mathfrak{N} \) which yields \( n \in \mathfrak{N} \). Since the only maximal ideal of \( R[M(W)] \) containing \( M(W) \setminus \{1\} \) is \( \mathfrak{N} \) the claim follows.

Pick \( s \in S = R[M(W)] \setminus \mathfrak{M} \). We want to show that \( A/(s) \to B/(s) \) is an isomorphism. This can be checked locally on max \( R[M(W)] \). At \( \mathfrak{M} \) both the source and the target localize to 0. If \( \mathfrak{N} \in \text{max } R[M(W)] \) is another maximal ideal then we know that \( M(W) \subset S \). Therefore, \( (A/(s))_{\mathfrak{M}} \to (B/(s))_{\mathfrak{M}} \) is a further localization of the identity mapping \( M(W)^{-1}A/(s) \to M(W)^{-1}B/(s) \).

One more localization result.

**Lemma 4.2.** To prove Conjecture 1.1 it suffices to show that \( K_i(R) = K_i(R[M_\mathfrak{N}]) \) (\( i \in \mathbb{N} \)) for every local regular ring \( R \) and every finitely generated normal monoid \( M \) with trivial \( U(M) \).

The proof below is based on the following local-global patching for \( K \)-groups due to Vorst [Vo1]. It is a higher (stable) analogue of Quillen’s original local-global patching for projective modules [Q2].

**Theorem 4.3.** Let \( A \) be a ring and \( x \in K_i(A[T]) \) (\( T \) a variable). Then \( x \in K_i(A) \) iff \( x_\mu \in K_i(A_\mu) \) for any \( \mu \in \text{max } A \).

**Proof of Lemma 4.2.** By Lemma 3.4 all we need is to show that the coefficient ring in the equalities \( K_i(R) = K_i(R[M_\mathfrak{N}]) \) admits globalization.

**Step 1.** Assume \( R \) is any regular ring and \( M \) is as in the lemma. Assume further we have shown that \( K_i(\Lambda) = K_i(\Lambda[L_\mathfrak{N}]) \) for any regular local ring \( \Lambda \) and any finitely generated normal monoid \( L \) with trivial \( U(L) \). Then by Lemma 3.4 we have \( K_i(R_\mu) = K_i(R_\mu[R[M_\mathfrak{N}^\times \mathbb{Z}_+]]) \) for all \( \mu \in \text{max } R \). Fix an element \( x \in K_i(R[M_\mathfrak{N}^\times \mathbb{Z}_+]) \). There are elements \( a_1, \ldots, a_p \in R \) and a natural number \( j \in \mathbb{N} \) such that \( (c_1 \cdots c_j)_* x_{a_u} \in K_i(R_{a_u}) \), \( u \in [1, p] \) and \( a_1 R + \cdots + a_p R = R \), where \( c_\bullet \) refers to the endomorphism of the group \( K_i(R[M_\mathfrak{N}^\times \mathbb{Z}_+]) \) induced by \( -^c : \mathbb{Z}_+ \to \mathbb{Z}_+ \).

It follows that \( (c_1 \cdots c_j)_* x_{\mathfrak{N}} \in K_i(R[M_\mathfrak{N}^\times \mathfrak{N}]) \) for any \( \mathfrak{N} \in \text{max } R[M_\mathfrak{N}] \). By Theorem 4.3 we conclude \( (c_1 \cdots c_j)_* x \in K_i(R[M_\mathfrak{N}]) \). Next we derive the equality

\[
(*) \quad K_i(R[M_\mathfrak{N}]) = K_i(R[M_\mathfrak{N}^\times \mathbb{Z}_+]).
\]

Pick an element \( y \in K_i(R[M_\mathfrak{N}^\times \mathbb{Z}_+]) \). It admits a representation \( y = (c_1 \cdots c_{j_1})_* (z) \) for some \( j_1 \in \mathbb{N} \) and \( z \in K_i(R[M_\mathfrak{N}^\times \mathbb{Z}_+]) \). By the remarks above, applied to the sequence \( c' = (c_{j_1+1}, c_{j_1+2}, \ldots) \), there exists an index \( j_2 \geq j_1 \) such that

\[
(c_{j_1+1} \cdots c_{j_2})_* (z) \in K_i(R[M_\mathfrak{N}^\times]),
\]

which implies \((*)\).

**Step 2.** By Lemma 2.3(d) we know that \( M \) can be embedded into a free monoid. As a result, there is a \( \mathbb{Z}_+ \)-grading \( R[M] = R \oplus R_1 \oplus \cdots \) for which \( M \) consists of homogenous elements. It follows that there is a \( \mathbb{Z}_+ \) grading

\[
R[M_\mathfrak{N}] = \bigoplus_{\mathbb{Z}_+} R_\alpha, \quad R_0 = R.
\]
By Lemma 3.5 in [Gu2] (a straightforward generalization from polynomial algebras to monoid algebras of s. c. Weibel homotopy trick; see below) for any functor $\mathcal{F}$ from the category of rings to that of abelian groups the following implications holds

$$ (\mathcal{F}(R[M_i^*]) = \mathcal{F}(R[M_i^*][Z_+^i])) \Rightarrow (\mathcal{F}(R) = \mathcal{F}(R[M_i^*])) $$

By (⋆) we are done.

Recall, that the just mentioned ‘Weibel homotopy trick’ is the following assertion: for any functor $\mathcal{F}$ from rings to abelian groups and any graded ring $A = A_0 \oplus A_1 \oplus \cdots$ the following implication holds

$$ \mathcal{F}(A) = \mathcal{F}(A[\mathbb{Z}_+]) \Rightarrow \mathcal{F}(A_0) = \mathcal{F}(A). $$

In the proof of Theorem 9.3 below we will make an essential use of the Mayer-Vietoris sequence for singular varieties due to Thomason [TT, Theorem 8.1]:

**Theorem 4.4.** Let $U$, $V$ be two quasi-compact open subschemes of a quasi-separated scheme. Then there is a natural long exact Mayer-Vietoris sequence

$$ \cdots \to K_i(U \cup V) \to K_i(U) \oplus K_i(V) \to K_i(U \cap V) \to K_{i-1}(U \cup V) \to \cdots $$

**Remark 4.5.** Actually, the $K$-groups mentioned here are those of Waldhausen, associated to the appropriate categories of perfect complexes on the given schemes. However, these $K$-groups coincide with Quillen’s $K$-groups for quasiprojective (in particular, affine) schemes over a commutative ring [TT, §3]. In particular, this is so for schemes of the type $X$ from §7 (see Remark 7.1(a)). Accordingly, all the $K$-groups that appear in this paper are a priori assumed to be those of Quillen [Q1].

**Lemma 4.6.** For a ring $R$ and a monic polynomial $f \in R[T]$ the natural homomorphism $K_i(R[T]) \to K_i(R[T, f^{-1}])$ is injective.

For $i = 0, 1$ this is proved in [MuP, §1]. We use similar ideas.

**Proof.** First we observe that for any ring $B$ and its two comaximal elements $b_1, b_2$ Theorem 4.4 implies the long exact sequence

$$ \cdots \to K_i(B) \to K_i(B_{b_1}) \oplus K_i(B_{b_2}) \to K_i(B_{b_1b_2}) \to K_{i-1}(B) \to \cdots $$

To prove the lemma it is enough to show that $K_i(R[T]) \to K_i(R[T, T^{-1}f^{-1}])$ is injective.

Let $f = T^n + a_{n-1}T^{n-1} + \cdots + a_0$. We write $f = gY^n$, where $Y = T^{-1}$ and $g = 1 + a_{n-1}Y + \cdots + a_0Y^n$. We have the equality $R[T, T^{-1}f^{-1}] = R[Y, Y^{-1}g^{-1}]$. Since $Y$ and $g$ are comaximal in $R[Y]$ the observation above yields the exact sequence

$$ K_i(R[Y]) \to K_i(R[Y^{\pm 1}]) \oplus K_i(R[Y, g^{-1}]) \to K_i(R[Y, Y^{-1}g^{-1}]). $$

By the Fundamental Theorem [Gra] we have the embeddings

$$ K_i(R[T]) \to K_i(R[T^{\pm 1}]), \quad K_i(R[Y]) \to K_i(R[Y^{\pm 1}]) = K_i(R[T^{\pm 1}]) $$

with $K_i(R)$ the intersection of their images. Thus, the exact sequence above implies

$$ \ker(K_i(R[T]) \to K_i(R[T, T^{-1}f^{-1}])) = \ker(K_i(R) \to K_i(R[T, T^{-1}f^{-1}])). $$
Now the proof is completed by Lemma 1.1 in [MuP] which says that for any functor $\mathfrak{F}$ with transfers from rings to abelian groups the homomorphisms of the type $\mathfrak{F}(R) \to \mathfrak{F}(R[T, h^{-1}])$, $h$ monic, are always injective. \qed

Next we show that Conjecture 1.1 admits a natural generalization to not necessarily affine (normal) toric varieties.

Let $X$ be a (normal) toric variety over a regular ring $R$. The multiplicative actions of $\mathbb{N}$ on the affine toric varieties in the standard open covering are compatible. In particular, we have the actions $\mathbb{N} \times X \to X$ and $\mathbb{N} \times (X \times \mathbb{A}^1_R) \to X \times \mathbb{A}^1_R$, where we mean the trivial action on $\mathbb{A}^1_R$. Denote by $K_i(X)^c$ and $K_i(X \times \mathbb{A}^1_R)^c$ the inductive limits of the corresponding $K_i$-groups with respect to the successive endomorphisms, given by $c$.

**Proposition 4.7.** Conjecture 1.1 is equivalent to the equalities $K_i(X \times \mathbb{A}^1_R)^c = K_i(X)^c$ for all quasiprojective toric varieties $X$.

**Proof.** If $X$ is affine then $X = Y \times T$ for some affine toric variety $Y$, whose polyhedral cone is strictly convex, and some torus $T$. Using the Fundamental Theorem, the desired equality is easily derived in the affine case from Conjecture 1.1.

Next we reduce the general case to the affine case as follows. Let $X$ be given by a fan $\mathcal{F}$. Pick a maximal cone $\sigma \in \mathcal{F}$ and let $\mathcal{F} - \sigma$ denote the fan determined by the remaining maximal cones. Denote by $U$ and $V$ the open toric subvarieties of $X$, corresponding to $\sigma$ and $\mathcal{F} - \sigma$. Theorem 4.4 yields the long exact sequence

$$
\cdots \to K_{i+1}(U \cap V) \to K_i(X) \to K_i(U) \oplus K_i(V) \to K_i(U \cap V) \to \cdots
$$

By passing to the inductive limits we get the exact sequence (in the self explanatory notation)

$$
\cdots \to K_{i+1}(U \cap V)^c \to K_i(X)^c \to K_i(U)^c \oplus K_i(V)^c \to K_i(U \cap V)^c \to \cdots .
$$

$X \times \mathbb{A}^1_R$ is again a toric variety. Moreover, $U \times \mathbb{A}^1_R$ and $V \times \mathbb{A}^1_R$ are open toric subvarieties covering $X \times \mathbb{A}^1_R$ so that the underlying fans are products of the corresponding fans by the same ray. Writing up the corresponding Mayer-Vietoris sequences one obtains $K_*(X)^c = K_*(X \times \mathbb{A}^1_R)^c$, provided the same equality holds for $U$ and $V$. Iterating the process we will produce toric varieties whose fans contain less and less top dimensional cones. But for single cones we are already done.

Conversely, using the same homotopy trick as in Step 2 in the proof Theorem 4.3 one can show that the equalities $K_i(X \times \mathbb{A}^1_R)^c = K_i(X)^c$ for affine toric varieties $X$, whose cones are pointed, are in fact equivalent to Conjecture 1.1. \qed

5. **Pyramidal Extensions**

**Definition 5.1.** A monoid extension $M \subset N$ is called pyramidal if $M$ and $N$ are finitely generated normal monoids without nontrivial units, $\text{gp}(M) = \text{gp}(N)$, and there is a representation of the form $\Phi(N) = \Phi(M) \cup \delta$ for some rational pyramid $\delta$ so that $\dim \Phi(N) = \dim \Phi(M) = \dim \delta$ and the polytope $\Phi(M)$ meets $\delta$ in its base.

Recall that a polytope is called pyramidal if it is a convex hull of its facet and a vertex not in this facet. The equivalent reformulation of Conjecture 1.1 given in the lemma below will be called pyramidal descent.
**Lemma 5.2.** Conjecture 1.1 is equivalent to the claim that for every pyramidal extension $M \subset N$, a local regular ring $R$, an element $x \in K_i(R[N_s])$ ($i \in \mathbb{N}$) and a sequence $c = (c_1, \ldots)$ the inclusion $(c_1 \cdots c_j)_*(x) \in \text{Im} \left( K_i(R[M_s]) \to K_i(R[N_s]) \right)$ holds for $j \gg 0$.

**Proof.** That the conjecture implies the claim is clear. Next observe that the mentioned inclusions are equivalent to the surjectivity of the homomorphisms $K_i(R[M_s]) \to K_i(R[N_s])$ for all pyramidal extensions $M \subset N$ and all sequences $c$.

Let $N$ be a finitely generated normal monoid with trivial $U(N)$. Consider any sequence $N = N_1, N_2, N_3, \ldots$ of normal submonoids of $N$, satisfying the condition – for any $k \in \mathbb{N}$ either $N_{k+1} \subset N_k$ is a pyramidal extension or $N_k \subset N_{k+1}$. (Observe that $\text{gp}(N_k) = \text{gp}(N)$, $k \in \mathbb{N}$.) Then our condition yields surjectivity of the homomorphisms $K_i(R[N_{k,s}]) \to K_i(R[N_3])$. Such a sequence of monoids will be called **admissible**.

According to Lemma 2.8 in [Gu1] for any rational interior point $\xi \in \Phi(N)$ and its any neighborhood $\xi \in U \subset \Phi(N)$ there is an admissible sequence of monoids $N, N_2, N_3, \ldots$ such that $\Phi(N_k) \subset U$ for all sufficiently big indices $k$.

By Lemma 2.3 there exists a rational simplex $\Delta \subset \text{int}(\Phi(N))$ such that $N(\Delta)$ is a free submonoid of $N$. These observations altogether show that (assuming the pyramidal descent) the homomorphism $K_i(R[N(\Delta)_s]) \to K_i(R[N_3])$ is surjective. But then $K_i(R[N(\Delta)_s]) \to K_i(R[N_3])$ is a surjection as well. Since $N(\Delta)_s$ is a filtered union of free monoids Quillen’s theorem implies $K_i(R[N_3]) = K_i(R)$. By Lemma 4.2 we are done. \qed

### 6. Approximations by polarized monoids

The results of this section are refined versions of the results from §1.3 in [Gu2].

**Definition 6.1.** A polarized monoid $N$ is a triple $(t, \Gamma, N)$ where $N$ is a finitely generated normal monoid with trivial $U(N)$, $\Gamma \subset \Phi(N)$ is a rational polytope and $t \in N$ is a non-zero element – the **pole**, such that the following hold:

1. $C(N) = \mathbb{R}_+ t + \mathbb{R}_+ \Gamma$,
2. $N \cap (\mathbb{R}_+ t + \mathbb{R}_+ F) \approx \mathbb{Z}_+ \times N(F)$ for any facet $F \subset \Gamma$.

In particular, in a polarized monoid $(t, \Gamma, N)$ one has that $\mathbb{R}_+ t$ is an edge of $C(N)$, $t$ is the generator of $N \cap \mathbb{R}_+ t \approx \mathbb{Z}_+$, and $t \notin \mathbb{R}F$ (the hyperplane spanned by $F$) for any facet $F \subset \Gamma$.

In a polarized monoid $(t, \Gamma, N)$ a facet $F \subset \Gamma$ is called **positive** if $t$ and $\text{int}(C(N))$ lie on the same side relative to the hyperplane $\mathbb{R}F \subset \mathbb{R} \otimes \text{gp}(M)$, otherwise $F$ is called **negative**.

In what follows we will use the notation $N^- = \mathbb{Z}_+(-t) + N(\Gamma) \subset \text{gp}(N)$. Clearly, for a polarized monoid $(t, \Gamma, N)$ the triple $(-t, \Gamma, N^-)$ is also a polarized monoid. It will be called the **antipode** of $(t, \Gamma, N)$.

The main approximation result for polarized monoids is as follows.

**Theorem 6.2.** Let $M \subset N$ be a pyramidal extension of monoids. Then for any natural number $s$ there exist systems of polarized monoids $(t_s, \Gamma_{1a}, N_{1a})$, $(t_a, \Gamma_{2a}, N_{2a})$, \ldots, $(t_a, \Gamma_{sa}, N_{sa})$, $a \in \mathbb{N}$ such that the following hold:
Lemma 6.3. Let $L$ be a monoid such that $\Phi(L)$ is a simplex. Then $L_\ast$ is a filtered union of free monoids.

The monoids of this type are usually called simplicial in the toric literature. Notice, that monoids of the type $L_\ast$ for $L$ simplicial are in general not filtered unions of free monoids, an example $-(\mathbb{Z}_+(2,0)+\mathbb{Z}_+(1,1)+\mathbb{Z}_+(0,2))^3$ where $3 = (3,3,\ldots)$.

Lemma 6.3 is exactly Approximation Theorem A from [Gu2] in the special case when $c$ is a constant sequence. The proof of the general claim makes no difference.

First let us show how Lemma 6.3 proves Conjecture 1.1 in the simplicial case.

Theorem 6.4. Conjecture 1.1 is true for arbitrary simplicial monoid $M$.

Proof. Assume $R$ is a regular ring. By Lemma 3.3 $R\text{int}(M_\ast)$ satisfies excision. On the other hand the exact sequence

$$0 \to R\text{int}(M_\ast) \to R[M_\ast] \to R[M_\ast]/R\text{int}(M_\ast) \to 0$$

and Lemma 6.3 show that $K_i(R\text{int}(M_\ast)) = 0$ for $i \in \mathbb{Z}$. (Here we use the $K$-homotopy invariance of $R$.) Since all faces of $\Phi(M)$ are also simplicial the same arguments as in the proof of Lemma 3.4 provide the desired process of “annihilating” the interior monoids. \hfill \Box

Proof of Theorem 6.2. Fix an index $j \in \mathbb{N}$ and finite subsets $W \subset \text{int}(\Phi(N))$, $W' \subset \text{int}(\Phi(M))$. We need to show that there are polarized monoids $(t,\Gamma_1,N_1),\ldots,(t,\Gamma_s,N_s)$ for which the conditions (a) and (b) are satisfied and, moreover, $W \subset \Phi(N_1)$, $W' \subset \Gamma_1$ and

$$G_j := \frac{\text{gp}(N)}{c_1\cdots c_j} \subset \text{gp}(N_1).$$

Let $v$ be the vertex of $\Phi(N)$ not in $\Phi(M)$. We can choose a rational point $v' \in \text{int}(\Phi(N)) \setminus \Phi(M)$ close to $v$ so that it does not belong to any of the hyperplanes $\mathbb{R}F(\mathbb{R} \otimes \text{gp}(N))$, where $F$ runs through the facets of $\Phi(M)$.

Consider any rational simplex $\Delta$ in the affine hull of $\Phi(N)$ which has one vertex at $v'$ and contains $W$ in its interior (such exists). We have the normal monoid

$$L = \{ n \in \text{gp}(N), \ n \neq 0 \mid \mathbb{R}^*_+n \cap \Delta \neq \emptyset \} \cup \{0\}.$$

By Lemma 2.1 $L$ is a finitely generated normal monoid and by Lemma 2.3(c) $\text{gp}(N) = \text{gp}(L)$. By Lemma 6.3 there exists a free submonoid $L_0 \subset L_\ast$ such that $G_j \subset \text{gp}(L_0)$,
\( W \subset \text{int}(\Phi(L_0)) \) and the simplex \( \Phi(L_0) \) has a vertex \( v_0 \) close to \( v' \), not in the affine span of any of the facets \( F \subset \Phi(M) \).

Put \( \Delta_0 = \Phi(L_0) \) and let \( \delta_0 \subset \Delta_0 \) be the facet opposite to \( v_0 \). The free generator of \( L_0(\{v_0\}) \approx \mathbb{Z}_+ \) will be denoted by \( \tau \).

Consider the polar projection \( \pi : \Phi(M) \to \text{Aff}(\delta_0) \) into the affine hull of \( \delta_0 \) with respect to the pole \( v_0 \). Then \( \dim \pi(F) = \dim F = \dim \delta_0 \) for any facet \( F \subset \Phi(M) \).

By Lemma 2.3(c) for any facet \( F \subset \Phi(M) \) there is a free basis
\[
\{l_{\pi(F),1}, \ldots, l_{\pi(F),\text{rank }N-1} \subset L(\pi(F))\}
\]
of the group \( \text{gp} \left( L(\delta_0) \right) \approx \mathbb{Z}^{\text{rank }N-1} \). Fix such bases arbitrarily. We can also fix a natural number \( j' > j \) such that
\[
\tau_k = \frac{\tau}{c_{j'}c_{j'+1} \cdots c_{j'+k}} \in N^*_e, \quad k \in \mathbb{N}.
\]
For a natural number \( k \) and a facet \( F \subset \Phi(M) \) we have the systems
\[
\{\tau_k, l_{\pi(F),1} + a_1 \tau_k, \ldots, l_{\pi(F),\text{rank }N-1} + a_{\text{rank }N-1} \tau_k\} \subset N^*_e
\]
where \( a_1, \ldots, a_{\text{rank }N-1} \in \mathbb{Z}_+ \). The free monoids \( N(F, k, a_1, \ldots, a_{\text{rank }N-1}) \) they generate all have the same groups of differences, provided \( k \) is fixed. Denote by \( G_k \) this common group of differences.

We let \( \delta(F, k, a_1, \ldots, a_{\text{rank }N-1}) \subset \Phi(N(F, k, a_1, \ldots, a_{\text{rank }N-1})) \) denote the facet opposite to \( v_0 \).

It is an elementary geometric observation that if \( k \) is sufficiently big then the rational points \( \mathbb{R}_+ (l_{\pi(F),i} + a_i \tau_k) \cap \Phi(N), \quad i \in [1, \text{rank }N-1] \) move “sufficiently slowly” towards the point \( v_0 \) when the \( a_i \) run through \( \mathbb{Z}_+ \). Moreover, all the time these rational points remain correspondingly in the segments \( \mathbb{R}_+ l_{\pi(F),i} \cap \Phi(N), \mathbb{R}_+ \tau_k \cap \Phi(N) \), \( i \in [1, \text{rank }N-1] \). In particular, we can choose \( k \) big enough and \( a_{F,1}, \ldots, a_{F,\text{rank }L-1} \in \mathbb{Z}_+ \) in such a way that the affine hulls of the simplices \( \delta(F, k, a_{F,1}, \ldots, a_{F,\text{rank }L-1}) \) bound a (convex) subpolytope \( \Gamma_1 \subset \text{int}(\Phi(M)) \) with the same number of facets as \( \Phi(M) \) such that \( W' \subset \Gamma_1 \), while the convex hull of \( \Gamma_1 \) and \( v_0 \) contains \( W \). Moreover, by approximating the facets of \( \Phi(M) \) with a sufficient precision, we can also achieve that \( v_0 \) is not in the affine hull of any of the facets of \( \Gamma_1 \).

It is an easy exercise to show that the triple \( (\tau_k, \Gamma_1, N_1) \) is a polarized monoid, where \( N_1 = \mathbb{R}_+ \text{conv}(\Gamma_1, v_0) \cap G_k \). For instance, the normality of \( N_1 \) follows from the facts that \( \Phi(N_k) \) is a union of the pyramids with vertex at \( v_0 \) and bases the positive facets of \( \Gamma_1 \), that these pyramids define normal submonoids of \( N_1 \) (they are free extensions of the normal monoids corresponding to these facets), and that their groups of differences are all the same.

Moreover, \( \Gamma_1 \) can even be chosen in such a way that there are \( 2s - 1 \) polytopes \( \Gamma_2, \ldots, \Gamma_{2s} \), obtained in the same way as \( \Gamma_1 \), for which the following hold
- \( \Gamma_1 \subset \text{int}(\Gamma_2) \), \( \ldots, \Gamma_{2s-1} \subset \text{int}(\Gamma_{2s}) \) and \( \Gamma_{2s} \subset \text{int}(\Phi(M)) \),
- \( \Gamma_2, \ldots, \Gamma_{2s} \) possess all the aforementioned properties of \( \Gamma_1 \).

In fact, one first “refines” the triple \( (\tau_k, \Gamma_1, N_1) \) by passing to \( \tau_{k'} \) with \( k' > k \), leaving the polytope \( \Gamma_1 \) untouched. The monoid \( N_1 \) is then changed by the monoid \( N_1' \) which is generated by \( \tau_{k'} \) and the elements of \( N_1 \) living on the positive facets.
of $\Gamma_1$. The point is that this new $N'_1$ is again a polarized monoid and the addition of $\pm \tau k'$ to the elements of $N'_1(\Gamma_1) \setminus \{0\}$ has sufficiently small effect on the radial directions (provided $k'$ is big enough). In particular, we can "blow up" the polytope $\Gamma_1$ suitably in any prescribed number of steps, remaining inside \text{int}(\Phi(M)).

It is immediate from Definition 6.1 that the $s$-tuple of polarized monoids 
\[(\tau_k, \Gamma_1, N_1), (\tau_k, \Gamma_3, N_3), \ldots, (\tau_k, \Gamma_{2s-1}, N_{2s-1})\]
satisfies the desired conditions.

Actually, we have even achieved $\text{gp}(N_1) = \text{gp}(N_3) = \cdots = \text{gp}(N_{2s-1})$. \hfill $\Box$

7. The scheme $X$

From now on we will mainly work over a coefficient field $k$, not necessarily algebraically closed.

Let $(t, \Gamma, N)$ denote a polarized monoid and $(t^{-1}, \Gamma, N^{-})$ be its antipode (in the multiplicative notation). The non-affine scheme $X$ is defined by gluing $\text{Spec}(k[N])$ and $\text{Spec}(k[N^{-}])$ along their open subscheme $\text{Spec}(k[t^{-1}N])$. (Here the equality $t^{-1}N = tN^{-}$ is used.)

**Remark 7.1.** (a) In other words $X$ is a toric variety over $k$ whose fan consists of 2 maximal equidimensional cones, sharing a facet. Namely, we mean the cones $C(N)^{\ast}, C(N^{-})^{\ast} \subset (\mathbb{R}^{\text{rank} N})^{\ast}$, where $-^{\ast}$ refers to "dual". We also have the equality $C(N)^{\ast} \cup C(N^{-})^{\ast} = C(N(\Gamma))^{\ast}$. In particular, $X$ is quasiprojective over $k$. In fact, the fan of $X$ admits a completion to a projective fan (the one defining a projective toric variety) – an elementary geometric observation. Therefore, $X$ is an open subscheme of a projective toric variety (see [O, Ch. 2]).

(b) Formal similarities of $X$ with a projective line allow us to use in §8 some of Quillen’s ideas from his computation of the $K$-theory of a projective line [Q1, §8.3]. However, it is not until §9 that we use peculiar properties of polarized monoids.

The category of sheaves of locally free $O_X$-modules will be denoted by $\mathcal{P}(X)$. Since $X$ is connected any object of $\mathcal{P}(X)$ has a constant rank.

The objects of $\mathcal{P}(X)$ can be thought of as triples $(P, P^{-}, \Theta)$ where $P \in \mathbb{P}(k[N])$, $P^{-} \in \mathbb{P}(k[N^{-}])$, and $\Theta : t^{-1}P \to tP^{-}$ is an isomorphism of $k[t^{-1}N]$-modules. In this terminology a morphism $(P_1, P_{1}^{-}, \Theta_1) \to (P_2, P_{2}^{-}, \Theta_2)$ is just a pair of morphisms $f^{+} : P_1 \to P_2$ and $f^{-} : P_{1}^{-} \to P_{2}^{-}$ correspondingly in the categories $\mathbb{P}(k[N])$ and $\mathbb{P}(k[N^{-}])$, such that $f^{-} \circ \Theta_1 = \Theta_2 \circ f$.

Given an object $(P, P^{-}, \Theta) \in \mathcal{P}(X)$. By [Gu1] $P$ ($P^{-}$) is free $k[N]$-module ($k[N^{-}]$-module). By fixing bases in $P$ and $P^{-}$ we can associate to $\Theta$ a matrix $\vartheta \in GL_{\text{rank} P}(k[t^{-1}N])$. We will say that $\Theta$ is represented by $\vartheta$.

The following is obvious.

**Lemma 7.2.** Assume $(P_1, P_{1}^{-}, \Theta_1), (P_2, P_{2}^{-}, \Theta_2) \in \mathcal{P}(X)$.

(a) If $\Theta_1$ is represented by a matrix $\vartheta_1$ then the set of matrices representing $\Theta_1$ is exactly $\{r\tau \vartheta_1 \sigma \mid r \in GL_{\text{rank} P}(k[N]), \sigma \in GL_{\text{rank} P}(k[N^{-]})\}$.

(b) $(P_1, P_{1}^{-}, \Theta_1) \approx (P_2, P_{2}^{-}, \Theta_2)$ if and only if $\text{rank} P_1 = \text{rank} P_2$ and for some (equivalently, for all) matrices $\vartheta_1$ and $\vartheta_2$, representing correspondingly $\Theta_1$ and $\Theta_2$, there exist $\tau$ and $\sigma$ as above such that $r\tau \vartheta_1 \sigma = \vartheta_2$. 

We have the natural augmentation $k[t^{-1}N] \to k[t^\pm 1]$, defined by $t^{-1}N \setminus U(t^{-1}N) \to 0$. For a matrix $\beta$, defined over $k[t^{-1}N]$, its image under this augmentation will be denoted by $\beta(0)$.

**Definition 7.3.** $\mathcal{P}(X)^0 \subset \mathcal{P}(X)$ is the full subcategory of the objects $(P, P^-, \Theta)$ such that $\Theta$ is represented by a matrix $\vartheta$ satisfying the condition $[\vartheta(0)^{-1} \cdot \vartheta] = 0 \in K_1(k[t^{-1}N])$.

Any natural number $c$ gives rise to an endomorphism of $X$ which on the affine charts $\text{Spec}(k[N]), \text{Spec}(k[N^-]) \subset \text{Spec}(k[t^{-1}N])$ is given by $k[-c]$. The induced endomorphism of $K_i(X)$ ($i \in \mathbb{N}$) will be denoted by $c_*$.

**Lemma 7.4.** $\mathcal{P}(X)^0$ is closed under extensions in $\mathcal{P}(X)$ and for an element $z \in K_i(X)$ we have $(c_1 \cdots c_j)_*(z) \in \text{Im}(K_i(\mathcal{P}(X)^0) \to K_i(X))$ provided $j \gg 0$.

In the proof we will use the main result of [Gu2] (the stable version):

**Theorem 7.5.** $SK_i(k[L^c]) = 0$ for any monoid $L$.

The stronger result $SL_n(k[L^c]) = E_n(k[L^c])$ for $n \geq 3$ is proved in [Gu2, Theorem 2.1] when $c_1 = c_2 = \cdots$. But the same claim remains true for arbitrary $c$. In fact, the only place in [Gu2] that uses the sequence $(c, c, \ldots)$ is Approximation Theorem B there. But Theorem 6.2 is a refined version for arbitrary $c$.

**Proof of Lemma 7.4.** The closedness under extensions of $\mathcal{P}(X)^0$ follows from the following observations. Assume $(P_1, P_1^-, \Theta_1), (P_2, P_2^-, \Theta_2) \in \mathcal{P}(X)^0$ and $\vartheta_1 \in GL_{\text{rank} P_1}(k[t^{-1}N])$ and $\vartheta_2 \in GL_{\text{rank} P_2}(k[t^{-1}N])$ represent correspondingly $\Theta_1$ and $\Theta_2$. Then for an exact sequence in $\mathcal{P}(X)$ of the type

$$0 \to (P_1, P_1^-, \Theta_1) \to (P_3, P_3^-, \Theta_3) \to (P_2, P_2^-, \Theta_2) \to 0$$

there is a representation of $\Theta_3$ by a matrix of the form

$$\vartheta_3 = \begin{pmatrix} \vartheta_1 & \vartheta' \\ 0 & \vartheta_2 \end{pmatrix} \in GL_{\text{rank} P_1 + \text{rank} P_2}(k[t^{-1}N])$$

where $\vartheta'$ is a $(\text{rank} P_1) \times (\text{rank} P_2)$-matrix over $k[t^{-1}N]$. It is clear that $[\vartheta_3(0)^{-1} \vartheta_3] = 0 \in K_1(k[t^{-1}N])$ provided $[\vartheta_1(0)^{-1} \vartheta_1] = 0$ and $[\vartheta_2(0)^{-1} \vartheta_2] = 0$ in $K_1(k[t^{-1}N])$.

Let a continuous map $\zeta : S^{i+1} \to BQ(P(X))$ represent $z$. $(S^{i+1}$ is $(i + 1)$-sphere.) There are finitely many simplices in the simplicial complex $BQ(P(X))$ which intersect $\text{Im}(\zeta)$. Let $(P_\lambda, P_\lambda^-, \Theta_\lambda), \lambda \in \Lambda$ be the vertices of these simplices and the matrices $\vartheta_\lambda \in GL_{\text{rank} P_\lambda}(k([t^{-1}N]))$ represent the $\Theta_\lambda$. Since $U(k[t^{-1}N]) = U(k[t^\pm 1])$, Theorem 7.5 implies $[(c_1 \cdots c_j)_* (\vartheta_\lambda(0)^{-1} \cdot \vartheta_\lambda)] = 0$ in $K_1(k[t^{-1}N]), \lambda \in \Lambda$, provided $j \gg 0$. (Here $c_#$ refers to the corresponding endomorphism of $GL(k[t^{-1}N])$.)

**8. Polarization descent**

In this section we use the same notation as in §7.

For a natural number $r$ we denote by $\mathcal{P}(X)(r)^0 \subset \mathcal{P}(X)^0$ the full subcategory of the objects whose ranks are nonnegative multiples of $r$. It is clear that $\mathcal{P}(X)(r)^0$ is closed under extensions in $\mathcal{P}(X)^0$. 
Definition 8.1. Assume \( r \in \mathbb{N} \) and \( a, b \in \mathbb{Z} \cup \{ -\infty, +\infty \} \), \( a \leq b \). Then
\[
\mathcal{P}(X)(r)^0_{a,b} \subset \mathcal{P}(X)(r)^0
\]
is the full subcategory of the objects \((P, P^{-}, \Theta)\) such that \( \Theta \) can be represented by a matrix \( \vartheta \in GL_{\text{rank} P}(k[t^{-1}N]) \) satisfying the conditions:
\[
(1) \quad \vartheta(0) = \begin{pmatrix} t^{u_1} & 0 & \cdots & 0 \\ 0 & t^{u_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t^{u_{\text{rank} P}} \end{pmatrix}
\]
for some integral numbers \( a \leq u_1, \ldots, u_{\text{rank} P} \leq b \),
\[
(2) \quad [\vartheta(0)^{-1} \vartheta] = 0 \in K_1(k[t^{-1}N]).
\]
The polarization of an object of \( \mathcal{P}(X)(r)^0_{-\infty, +\infty} \) is by definition the smallest segment \([a, b]\) such that this object is in \( \mathcal{P}(X)(r)^0_{a,b} \). The polarization is well defined because \( (\mathcal{O}_{\mathcal{P}^1}(u_1) \oplus \cdots \oplus \mathcal{O}_{\mathcal{P}^1}(u_n) \approx \mathcal{O}_{\mathcal{P}^1}(v_1) \oplus \cdots \oplus \mathcal{O}_{\mathcal{P}^1}(v_m)) \Rightarrow (n = m \text{ and the } u \text{ coincide with the } v \text{ up to permutation}) \). Here \( \mathbb{P}^1 = \mathbb{P}^1_k \) denotes the projective line over \( k \) and \( \mathcal{O}_{\mathcal{P}^1}(u) = (k[t], k[t^{-1}], \text{multiplication by } t^{-u}) \).

Lemma 8.2. \( \mathcal{P}(X)(r)^0_{0,1} \) is closed under extensions in \( \mathcal{P}(X)(r)^0 \) and the embedding \( \mathcal{P}(X)(r)^0_{0,1} \subset \mathcal{P}(X)(r)^0 \) induces isomorphisms on \( K \)-groups.

Proof. We will only treat the case \( r = 1 \). The general case makes no difference.

Step 1. Here we prove that \( \mathcal{P}(X)^0_{-\infty, +\infty} = \mathcal{P}(X)^0 \).

Consider an object \((P, P^{-}, \Theta) \in \mathcal{P}(X)^0 \). By Definition 7.3 \( \Theta \) can be represented by a matrix \( \vartheta \in GL_{\text{rank} P}(k[t^{-1}N]) \) such that \( [\vartheta(0)^{-1} \vartheta] = 0 \in K_1(k[t^{-1}N]) \). By Grothendieck’s theorem [Gro] on vector bundles on projective lines (proved by elementary methods for arbitrary fields in [HMa]) there are \( \sigma \in GL_{\text{rank} P}(k[t]) \) and \( \tau \in GL_{\text{rank} P}(k[t^{-1}]) \) such that
\[
\sigma \vartheta(0) \tau = (\sigma \vartheta \tau)(0) = \begin{pmatrix} t^{u_1} & 0 & \cdots & 0 \\ 0 & t^{u_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t^{u_{\text{rank} P}} \end{pmatrix}
\]
for some \( u_1, \ldots, u_{\text{rank} P} \in \mathbb{Z} \). We have
\[
((\sigma \vartheta \tau)(0)^{-1}(\sigma \vartheta) = \tau^{-1} \vartheta(0)^{-1} \vartheta = 0 \in K_1(k[t^{-1}N]).
\]
By Lemma 7.2 we are done.

Step 2. We prove that \( \mathcal{P}(X)^0_{a,b} \) is closed under extensions in \( \mathcal{P}(X)^0 \) for arbitrary \( a, b \in \mathbb{Z} \cup \{ \pm \infty \} \), \( a \leq b \).

By the previous step the reduction modulo \( k(t^{-1}N) \cap \mathbb{Z} \subset k[t^{-1}N] \) shows that it is enough to prove the following.
Claim. Assume we are given a commutative diagram of free $k[t^\pm 1]$-modules

$$
\begin{array}{ccc}
0 & \rightarrow & k[t^\pm 1]^m \\
\downarrow \alpha & & \downarrow \beta \\
0 & \rightarrow & k[t^\pm 1]^{m+n}
\end{array}
\quad
\begin{array}{ccc}
f^+ & : & k[t^\pm 1]^{m+n} \\
\downarrow \gamma & & \downarrow \\
g^+ & : & k[t^\pm 1]^n \rightarrow 0
\end{array}
\quad
\begin{array}{ccc}
0 & \rightarrow & k[t^\pm 1]^n \\
\downarrow \gamma & & \downarrow \\
0 & \rightarrow & k[t^\pm 1]^{m+n}
\end{array}
$$

with exact rows, where $f^+$ and $g^+$ are defined over $k[t]$ and $f^-$ and $g^-$ are defined over $k[t^{-1}]$. Assume, further, $\alpha$, $\beta$ and $\gamma$ are correspondingly by matrices of the type

$$
\left( \begin{array}{cccc}
t^u_1 & 0 & \ldots & 0 \\
0 & t^u_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & t^u_m
\end{array} \right),
\left( \begin{array}{cccc}
t^v_1 & 0 & \ldots & 0 \\
0 & t^v_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & t^v_m+n
\end{array} \right)
$$

Then $(u_1, \ldots, u_m, w_1, \ldots, w_n) \Rightarrow (v_1, \ldots, v_{m+n})$.

We introduce the following notation:

(i) the matrices of $f^+$ and $f^-$ (w.r.t. the standard bases) are correspondingly $(f^+_{ij})_i$, $i \in [1, m]$, $j \in [1, m+n]$, $f^+_{ij} \in k[t]$ and $(f^-_{ij})_i$, $i \in [1, m]$, $j \in [1, m+n]$, $f^-_{ij} \in k[t^{-1}]$,

(ii) the matrices of $g^+$ and $g^-$ are correspondingly $(g^+_{ij})_j$, $i \in [1, m+n]$, $j \in [1, m+n]$, $g^+_{ij} \in k[t]$ and $(g^-_{ij})_j$, $i \in [1, m+n]$, $l \in [1, n]$, $g^-_{ji} \in k[t^{-1}]$.

The commutative diagram above implies

$$f^+_{ij} t^{v_j} = f^-_{ij} t^{u_i} \text{ for all } i, j \quad (8.2)_1$$

and

$$g^+_{ji} t^{u_i} = g^-_{ji} t^{v_j} \text{ for all } j, l. \quad (8.2)_2$$

Fix any index $j \in [1, m+n]$. If there exists $i \in [1, m]$ such that $f^+_{ij} \neq 0$ then $(8.2)_1$ implies $u_i \geq v_j$. Now consider the case when $f^+_{ij} = f^-_{ij} = 0$ for all $i \in [1, m]$. Then our diagram splits as follows

$$
\begin{array}{ccc}
0 & \rightarrow & k[t^\pm 1]^m \\
\downarrow \alpha & & \downarrow \\
0 & \rightarrow & k[t^\pm 1]^{m+n}
\end{array}
\quad
\begin{array}{ccc}
f^+ & : & k[t^\pm 1]^{m+n} \\
\downarrow \beta & & \downarrow \\
g^+ \oplus g^- & : & k[t^\pm 1]^{m+n} \rightarrow 0
\end{array}
\quad
\begin{array}{ccc}
0 & \rightarrow & k[t^\pm 1]^{m+n} \\
\downarrow \gamma & & \downarrow \\
0 & \rightarrow & k[t^\pm 1]^n
\end{array}
$$

where

$$0 \rightarrow k[t^\pm 1]^m \xrightarrow{f^+} k[t^\pm 1]^{m+n-1} \oplus k[t^\pm 1] \xrightarrow{g^+_1 \oplus g^-_1} k[t^\pm 1]^{n-1} \oplus k[t^\pm 1] \rightarrow 0$$

are the exact sequences of free $k[t^\pm 1]$-modules, obtained by deleting the $j$-th direct summand in $k[t^\pm 1]^{m+n}$, and the mappings $g^+_1, g^-_1 : R[t^\pm 1] \rightarrow k[t^\pm 1]$ establish an isomorphism between the $O_{P1}$-sheaves $O(-v_j)$ and $O(-w_l)$ for some $l \in [1, n]$. But in this case $v_j = w_l$. Therefore, in all cases we have $v_j \leq \min_{i,l}(u_i, w_l)$.

Using $(8.2)_2$ and dual arguments we get $v_j \geq \min_{i,l}(u_i, w_l)$. 
Step 3. By Step 1 we have the filtered union representation

\[ \mathcal{P}(X)^0 = \bigcup_{a=0}^{\infty} \mathcal{P}(X)^0_{a, +\infty}. \]

Therefore, if we knew that the embeddings \( \mathcal{P}(X)^0_{a, +\infty} \subset \mathcal{P}(X)^0_{a-1, +\infty}, a \leq 0 \), induce isomorphisms on \( K \)-groups we could conclude that the embedding \( \mathcal{P}(X)^0_{0, +\infty} \subset \mathcal{P}(X)^0 \) also induces isomorphisms on \( K \)-groups.

Fix a nonpositive integral number \( a \) and consider the two exact functors

\[ F_1, F_2 : \mathcal{P}(X)^0_{a-1, +\infty} \to \mathcal{P}(X)^0_{a, +\infty} \]

given correspondingly by

\[ F_1((P, P^{-}, \Theta)) = (P, P^{-}, t\Theta) \oplus (P, P^{-}, t\Theta), \quad F_1((f^+, f^-)) = (f^+, f^-) \]

and

\[ F_2((P, P^{-}, \Theta)) = (P, P^{-}, t^2\Theta), \quad F_1((f^+, f^-)) = (f^+, f^-). \]

The exact "Koszul sequence" (where we use matrix theoretical notation)

\[
\begin{array}{cccc}
0 & \to & k[t^{-1}N]^n & \to \kern-15pt \xrightarrow{(t^{-1})^\otimes n} \kern-15pt k[t^{-1}N]^n \oplus k[t^{-1}N]^n & \to \kern-15pt \xrightarrow{(1)^\otimes n} \kern-15pt k[t^{-1}N]^n & \to 0 \\
& & \downarrow t^2\Theta & \downarrow t\Theta \oplus t\Theta & & \downarrow \Theta \\
0 & \to & k[t^{-1}N]^n & \to \kern-15pt \xrightarrow{(1-t^{-1})^\otimes n} \kern-15pt k[t^{-1}N]^n \oplus k[t^{-1}N]^n & \to \kern-15pt \xrightarrow{(1^{-1})^\otimes n} \kern-15pt k[t^{-1}N]^n & \to 0 \\
\end{array}
\]

implies the exact sequence of exact functors \( 0 \to \iota \circ F_2 \to \iota \circ F_1 \to 1 \to 0 \), where \( \iota : \mathcal{P}(X)^0_{a, +\infty} \to \mathcal{P}(X)^0_{a-1, +\infty} \) is the identity embedding and \( 1 \) is the identity endofunctor of \( \mathcal{P}(X)^0_{a-1, +\infty} \). By \([Q1, \S3]\) we get \( \iota_* \circ ((F_1)_* - (F_2)_*) = \iota_* \), for the corresponding \( K \)-group homomorphisms.

It remains to show that \(( (F_1)_* - (F_2)_* ) \circ \iota_* = 1_* \), the right hand side denoting the identity endomorphism of the corresponding \( K \)-group of \( \mathcal{P}(X)^0_{a, +\infty} \). But this equality is derived by literally the same arguments once we observe the exact "Koszul sequence" \( 0 \to F_2 \circ \iota \to F_1 \circ \iota \to 1 \to 0 \) of endofunctors of \( \mathcal{P}(X)^0_{a, +\infty} \).

Step 4. Because of the filtered union representation

\[ \mathcal{P}(X)^0_{0, +\infty} = \bigcup_{b=1}^{+\infty} \mathcal{P}(X)^0_{0, b} \]

the previous step shows that we only need that the embeddings \( \mathcal{P}(X)^0_{0, b} \subset \mathcal{P}(X)^0_{0, b+1}, b \geq 1 \), induce isomorphisms on \( K \)-groups.

Assume \( b \in [1, +\infty) \). Consider an object \(( P, P^{-}, \Theta ) \in \mathcal{P}(X)^0_{0, b+1} \) and fix a matrix \( \vartheta \) satisfying the conditions in Definition 8.1 (with respect to \( a = 0 \) and \( b = b + 1 \)). We let \( u_1, \ldots, u_n \in [0, b + 1] \) be the corresponding numbers, where \( n = \text{rank} P \).

Without loss of generality we can assume \( u_1, \ldots, u_l = b + 1 \) and \( u_{l+1}, \ldots, u_n \in [0, b] \) for some \( l \in [0, n] \) (none of the values \( l = 0 \) and \( l = n \) being excluded).
Consider the matrix 
\[
\rho = \begin{pmatrix} t1_t & 0 \\ 0 & 1_{n-t} \end{pmatrix}.
\]

We have the following (non-functorial) commutative diagram with exact rows
\[
\begin{array}{ccc}
0 \to k[t^{-1}N]^n & \xrightarrow{(\rho-1)} & k[t^{-1}N]^n \oplus k[t^{-1}N]^n \\
& \xrightarrow{\phi} & k[t^{-1}N]^n \to 0 \\
0 \to k[t^{-1}N]^n & \xrightarrow{(1, -\rho^{-1})} & k[t^{-1}N]^n \oplus k[t^{-1}N]^n \\
& \xrightarrow{\rho^{-1} \varrho \rho^{-1}} & k[t^{-1}N]^n \to 0 \\
\end{array}
\]
where \( 1 = 1_{n \times n} \) and the matrices at the vertical rows refer to the corresponding maps.

Observe that the matrices at the second and third vertical rows satisfy the conditions (1) and (2) in Definition 8.1 so that we get objects of polarization \([0, b]\)
(one uses that \(\rho(0) = \rho\)). Therefore, any object \((P, P^-, \Theta) \in \mathcal{P}(X)^0_{0,b+1}\) admits a “co-resolution”
\[
0 \to (P, P^-, \Theta) \to (P_1, P_1^-, \Theta_1) \to (P_2, P_2^-, \Theta_2) \to 0
\]
whose second and third terms belong to \(\mathcal{P}(X)^0_{0,b}\). In other words, for the dual categories
\[
\mathcal{P}^b = (\mathcal{P}(X)^0_{0,b})^{op} \quad \text{and} \quad \mathcal{P}^{b+1} = (\mathcal{P}(X)^0_{0,b+1})^{op}
\]
any object \(p \in \mathcal{P}^{b+1}\) admits a resolution \(0 \to p_2 \to p_1 \to p \to 0\) where \(p_1, p_2 \in \mathcal{P}^b\).
Since \(K\)-theories of dual exact categories are the same it suffices to show that the embedding \(\mathcal{P}^b \subset \mathcal{P}^{b+1}\) induces isomorphisms on \(K\)-groups. In view of what has been said above we see that Resolution Theorem [Q1, §4] applies once we show the implication: if \(0 \to p_2 \to p_1 \to p_0 \to 0\) is exact in \(\mathcal{P}^{b+1}\) and \(p_1 \in \mathcal{P}^b\) then \(p_2 \in \mathcal{P}^b\).

Returning to the original categories, we claim that for any exact sequence
\[
0 \to (P_0, P_0^-, \Theta_0) \to (P_1, P_1^-, \Theta_1) \to (P_2, P_2^-, \Theta_2) \to 0
\]
in \(\mathcal{P}(X)^0_{0,b+1}\) the following implication holds
\[
((P_1, P_1^-, \Theta_1) \in \mathcal{P}(X)^0_{0,b}) \Rightarrow ((P_2, P_2^-, \Theta_2) \in \mathcal{P}(X)^0_{0,b}).
\]

Introducing the notation as in Claim in Step 2 this implication rewrites as follows:
\[
(v_1, \ldots, v_{m+n} < b+1) \Rightarrow (w_1, \ldots, w_n < b+1).
\]
First observe that for any \(h \in [1, n]\) there exists \(j_h \in [1, m + n]\) such that \(g_{j_h}^+ \neq 0\). In fact, assuming to the contrary that such \(j_h\) does not exist, the mapping \(g^+\) (and, therefore, \(g^-\)) would not be surjective, which is excluded. By (8.2)_2 we have
\[
g_{j_h}^+ t^{v_h} = g_{j_h}^- t^{v_h}, \quad h \in [1, n].
\]
In particular, \(w_h \leq v_{j_h}\). This completes the proof. \(\square\)
Let $\mathcal{P}(\mathbb{P}^1_k)_{0,1}$ denote the full subcategory of $\mathcal{P}(\mathbb{P}^1_k)$ whose objects are $\mathcal{O}^a \oplus \mathcal{O}(-1)^b$, $a, b \in \mathbb{Z}_+$. It is clear from Claim in Step 2 in the proof above that $\mathcal{P}(\mathbb{P}^1_k)_{0,1}$ is an exact category. For $\mathcal{E}, \mathcal{F} \in \mathcal{P}(\mathbb{P}^1_k)_{0,1}$ the product $\mathcal{F}^* \otimes \mathcal{E}$ is of type $\mathcal{O}(-1)^a \oplus \mathcal{O}^b \oplus \mathcal{O}(1)^c$. Hence $0 = H^1(\mathbb{P}^1_k, \mathcal{F}^* \otimes \mathcal{E}) = \text{Ext}(\mathcal{F}, \mathcal{E})$ ([Ha, Ch.3 §6]), and we get

**Lemma 8.3.** Any short exact sequence in $\mathcal{P}(\mathbb{P}^1_k)_{0,1}$ splits.

9. Almost pyramidal descent

The peculiar property of polarized monoids we need is Lemma 9.1 below, yielding Claim A and Claim B in the proof of Theorem 9.3. We present it in the general form involving arbitrary commutative rings.

First a word on notation. Let $(t, \Gamma, L)$ be a polarized monoid. If $R$ is a local ring and $\mu \subset R$ is its maximal ideal we let $\mathfrak{m}(\mu) \in \max R[L(\Gamma)]$ denote the maximal ideal generated by $\mu$ and $L(\Gamma) \setminus \{1\}$. Also, for a natural number $r$ we put

- $G_r = E_r(R[t^{-1}L]_{\mathfrak{m}}) \cap GL_r(R[t^{-1}L]_{\mathfrak{m}}, \mathfrak{m}R[t^{-1}L]_{\mathfrak{m}})$,
- $G_r^+ = E_r(R[L]_{\mathfrak{m}}) \cap GL_r(R[L]_{\mathfrak{m}}, \mathfrak{m}R[L]_{\mathfrak{m}})$,
- $G_r^- = E_r(R[L^{-1}]_{\mathfrak{m}}) \cap GL_r(R[L^{-1}]_{\mathfrak{m}}, \mathfrak{m}R[L^{-1}]_{\mathfrak{m}})$.

The subset $\{g^+g^- : g^+ \in G_r^+, \ g^- \in G_r^-\} \subset G_r$ will be denoted by $G_r^+G_r^-.

**Lemma 9.1.** $G_r = G_r^+G_r^-$ for any local ring $R$ and any natural number $r \geq 3$.

**Remark 9.2.** Lemma 9.1 is a polarized version of the pivotal technical fact in [Su] (Proposition 5.6). It is proved in [Gn2, Proposition 2.14, Step 4]. A generalization of Lemma 9.1 to a relative case $I \subset R$ is obtained in [Sch].

Let $k$ be a field. Put

$$\Lambda = \Lambda_{(t, \Gamma, L)} = \{(\varphi_{uv}) \in M_{2 \times 2}(k[L])\},$$

where

- $\varphi_{11}, \varphi_{22} \in k[L(\Gamma)]$,
- $\varphi_{12} \in k[L(\Gamma)] \cap t^{-1}k[L(\Gamma)]$,
- $\varphi_{21} \in k[L(\Gamma)] + tk[L(\Gamma)]$.

Consider the subring

$$\Lambda' = \Lambda'_{(t, \Gamma, L)} = \{(\varphi_{uv}) : \varphi_{21}(0) \in k\} \subset \Lambda.$$

(As usual, $\varphi(0)$ is the image of $\varphi$ under the augmentation $k[L] \rightarrow k[t]$.) Then $\Lambda$ differs from $\Lambda'$ in a single monomial – the pole $t$ (in the 21 position).

**Theorem 9.3.** Let $k$ be a field, $M \subset N$ be a pyramidal extension of monoids and $z \in K_i(k[N_s])$ ($i \in \mathbb{N}$). Assume Conjecture 1.1 holds for all monoids of rank $< \text{rank } N$ and all coefficient fields. Then for every sequence $c = (c_1, c_2, \ldots)$ there exists a polarized monoid $(t, \Gamma, L)$ such that:

(a) $L \subset N_s$ and $\Gamma \subset \text{int}(\Phi(M))$,
(b) $\Lambda'_{(t, \Gamma, L)} \subset M_{2 \times 2}(k[M_s])$,
(c) $(c_1 \cdots c_j)_s(z) \in \text{Im } (K_i(\Lambda_{(t, \Gamma, L)}) \rightarrow K_i(M_{2 \times 2}(k[N_s])) = K_i(k[N_s]))$ for $j \gg 0$. 

Proof. Step 1. Let \( z(0) \in K_i(k) \) denote the image of \( z \) under the homomorphism \( K_i(k[N_\ast]) \to K_i(k) \), induced by \( N_\ast \setminus \{1\} \to 0 \in k \). Changing \( z \) by \( z - z(0) \) we can without loss of generality assume \( z(0) = 0 \).

Denote by \( x \) the image of \( z \) in \( K_i(k[N_\ast]) \). Then \( x(0) = 0 \) w.r.t. the natural augmentation \( k[N_\ast] \to k \).

Step 2. By Theorem 6.2 (and continuity of algebraic \( K \)-functors) there are six polarized monoids \( (t, \Gamma_1, N_1), \ldots, (t, \Gamma_6, N_6) \) such that

(i) \( N_1 \subset \cdots \subset N_6 \subset N_\ast \) and \( \Gamma_1 \subset \cdots \subset \Gamma_6 \subset \text{int}(\Phi(M)) \),

(ii) \( t^z(N_1(\Gamma) \setminus \{1\}) \subset \text{int}(N_{l+1}(\Gamma_{l+1})) \), for \( l = 1, 5 \).

(iii) \( x \in \text{Im} \left( K_i(k[N_1]) \to K_i(k[N_6]) \right) \).

We will assume that \( x \) is the image of \( x_1 \in K_i(k[N_1]) \).

Next we show that one can also assume

(iv) \( x_1 \in \text{Ker} \left( K_i(k[N_1]) \to K_i(k[t^{-1}N_1]) \right) \).

Since \( N_1, \ldots, N_6 \) are finitely generated monoids there is an index \( j \in \mathbb{N} \) such that

\[
\begin{align*}
N_1 & \subset c_j \cdots c_{j'}, \quad \ldots, \quad N_6 & \subset c_j \cdots c_{j'} \subset N_\ast \\
\end{align*}
\]

for any natural number \( j' \geq j \). Denote these six polarized monoids respectively by \( N_{1j'}, \ldots, N_{6j'} \). They are naturally isomorphic to \( N_1, \ldots, N_6 \). Clearly, the same conditions (i)–(iii) are satisfied for them. In particular, if we show that \( x_1 \in \text{Ker} \left( K_i(k[N_1]) \to K_i(k[(t^{-1}N_1)^\ell]) \right) \), where \( \ell = (c_j, c_{j+1}, \ldots) \), then we achieve the validity of all the conditions (i)–(iv) for the isomorphic system of monoids \( N_{1j'}, \ldots, N_{6j'} \) with \( j' \gg 0 \).

By Corollary 2.5 there is an intermediate monoid \( N_1 \subset \mathbb{Z}_\ast t \times N_0 \subset t^{-1}N_1 \), where \( N_0 \subset t^{-1}N_1 \) is a normal submonoid for which \( U(N_0) \) is trivial. In view of the inclusions \( k[N_1] \subset k[\mathbb{Z}_\ast t \times N_0^\ell] = k[N_0^\ell][t] \subset k[(t^{-1}N_1)^\ell] \) it suffices to show that \( x_1 \in \text{Ker} \left( K_i(k[N_1]) \to K_i(k[N_0^\ell][t]) \right) \). By Lemma 4.6 this is equivalent to the inclusion \( x_1 \in \text{Ker} \left( K_i(k[N_1]) \to K_i(k(t)[N_0^\ell]) \right) \). The hypothesis of the theorem implies that the augmentation \( k(t)[N_0^\ell] \to k(t), N_0^\ell \setminus \{1\} \to 0 \) induces the isomorphism \( K_i(k(t)[N_0^\ell]) = K_i(k(t)) \). Now the composite mapping \( k[N_1] \to k(t)[N_0^\ell] \to k(t) \) factors through \( k[t] \). Finally, \( K_i(k[t]) = K_i(k) \) and by Step 1 we are done.

Step 3. We let \( X_1 \) denote the non-affine toric variety corresponding to \( N_1 \) in the sense of \( \S 7 \). By Theorem 4.4 (and Remark 4.5) we have the long exact sequence

\[
\cdots \to K_i(X_1) \to K_i(k[N_1]) \oplus K_i(k[N_1^-]) \to K_i(k[t^{-1}N_1]) \to \cdots
\]

Hence by (iv) in Step 2 there exists \( z_1 \in K_i(X_1) \) mapping to \( x_1 \).
For any natural number \( r \) we have the commutative diagram of exact categories and exact functors

\[
\begin{array}{cccc}
P(X_1) & \xrightarrow{\Delta} & P(X_1) \times \cdots \times P(X_1) & \xrightarrow{\oplus} & P(X_1) \\
\downarrow & & \downarrow & & \downarrow \\
P(k[N_1]) & \xrightarrow{\Delta} & P(k[N_1]) \times \cdots \times P(k[N_1]) & \xrightarrow{\oplus} & P(k[N_1])
\end{array}
\]

where \( \Delta \) refers to the diagonal embeddings and the number of factors is \( r \). Passing to \( K \)-groups we get the commutative diagram

\[
\begin{array}{cccc}
K_i(P(X_1)(r)) & & \\
\downarrow & & \downarrow \\
K_i(X_1) & \xrightarrow{r} & K_i(X_1) \\
\downarrow & & \downarrow \\
K_i(k[N_1]) & \xrightarrow{r} & K_i(k[N_1])
\end{array}
\]

Therefore, for any natural number \( r \) there is an element \( z_r \in K_i(P(X_1)(r)) \) mapping to \( rx_1 \) under the homomorphism \( K_i(P(X_1)(r)) \to K_i(k[N_1]) \).

**Step 4.** As in Step 2 there exists an index \( j \in \mathbb{N} \) such that

\[
N_{1j'} = \frac{N_1}{c_j \cdots c_{j'}} \cdots , N_{6j'} = \frac{N_6}{c_j \cdots c_{j'}} \subset N_s^e
\]

for any natural number \( j' \geq j \).

By Lemma 7.4 for any natural number \( r \) there exists \( j_r > j \) such that the element \( (c_j \cdots c_{j'})_*(z_r) \) is in \( \text{Im} \ (K_i(P(X_1)(r)'^0) \to K_i(P(X_1)(r))) \). In particular, by Step 3 we have \( rx \in \text{Im} \ (K_i(P(X_{1j'})(r)'0) \to K_i(k[N_{1j'}])) \), where the mapping between the \( K \)-groups is induced by the composite exact functor

\[
P(X_{1j'})(r)'0 \to P(X_{1j'})(r) \to P(k[N_{1j'}]) \to P(k[N_s^e]).
\]

Here \( X_{1j'} \) is the scheme associated with \( N_{1j'} \). By Lemma 8.2

\[
rx_{2n} \in \text{Im} \ (K_i(P(X_{1j'})(r)'0,1) \to K_i(k[N_s^e])_2),
\]
where the mapping between the $K$-groups is induced by the ‘upper route’ in the commutative diagram of exact categories and exact functors

\[
\begin{array}{c}
P(X_{1,j})(r) \longrightarrow P(k[N_{1,j}]) \longrightarrow P(k[N_4^e]) \\
P(X_{1,j})(r)_{0,1} \supset \quad P(k[N_4^e]_{\mathfrak{M}}) \quad \quad (9.3)_3 \\
P((X_{1,j},\mathfrak{N}_{1,j}))(r) \longrightarrow P((X_{2,j},\mathfrak{N}_{2,j}))(r) \longrightarrow P((X_{4,j},\mathfrak{N}_{4,j}))
\end{array}
\]

That the ‘lower route’ in $(9.3)_3$ is in fact possible follows from $(9.3)_1$ and the condition (i) in Step 2. Here we have used the following notation:

\[
\mathfrak{M} = k(M_1^e \setminus \{1\}) \in \text{max } k[M_1^e],
\]

\[
\mathfrak{N}_{1,j} = k(N_{1,j},(\Gamma_1) \setminus \{1\}) \in \text{max } k[N_{1,j},(\Gamma_1)],
\]

and the scheme $(X_{1,j},\mathfrak{N}_{1,j})$ is defined by the push-out diagram

\[
\begin{array}{c}
\text{Spec}(k[N_{1,j},\mathfrak{N}_{1,j}]) \longrightarrow (X_{1,j})_{\mathfrak{N}_{1,j}} \\
\text{Spec}(k[k^{-1}N_{1,j},\mathfrak{N}_{1,j}]) \longrightarrow \text{Spec}(k[N_{1,j}^{-1},\mathfrak{N}_{1,j}])
\end{array}
\]

while $P(X_{1,j})(r)$ refers to the full subcategory of those locally free coherent sheaves on $X_{1,j}$ whose ranks are non-negative multiples of $r$. The other members of the diagram are defined similarly with respect to the corresponding polarized monoids, and the exact functors indicated are the ones induced by the appropriate scalar extensions, restrictions of vector bundles to open subschemes and category embeddings.

**Step 5.** Fix arbitrarily a natural number $r \geq \text{rank } N + 2$. We are going to study the lower route in the diagram $(9.3)_3$.

The monoids $N_{1,j},\ldots,N_{6,j}$ are naturally isomorphic to $N_1,\ldots,N_6$ so that they satisfy the obvious analogues of the conditions (i)–(iv) in Step 2. In order to simplify the notation we will below omit the subindex $j_r$. Thus, the mentioned lower route looks as follows

\[
\begin{array}{c}
P(X_1)(r)_{0,1} \supset \quad P(k[N_4^e]_{\mathfrak{M}}) \\
P((X_1,\mathfrak{N}_1))(r) \longrightarrow P((X_2,\mathfrak{N}_2))(r) \longrightarrow P((X_4,\mathfrak{N}_4))
\end{array}
\]

By Step 4 there is an element $\zeta_r \in K_1(P(X_1)(r)_{0,1})$ mapping to $r x_{\mathfrak{M}}$. Clearly, $\zeta_r = (c_j \cdots c_j)_*(z_r)$ where $z_r$ is the same as in Step 3.
Choose a nonzero object \((P, P^-, \Theta) \in \mathcal{P}(X_1)(r)_{0,1}\). By Lemma 7.2 and Definition 8.1 this object up to isomorphism has the form

\[ P = k[N_1]^n, \ P^- = k[N_1^-]^n, \ \vartheta(0)^{-1} \vartheta \in E(k[t^{-1}N_1]), \ \vartheta(0) = \begin{pmatrix} t^{u_1} & 0 & \ldots & 0 \\ 0 & t^{u_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & t^{u_n} \end{pmatrix} \]

for some \(u_1, \ldots, u_n \in \{0,1\}\), where \(\vartheta\) is the matrix of \(\Theta\) in the standard basis of \(k[t^{-1}N_1]^n\).

Since \(\text{dim}(k[t^{-1}N_1]) = \text{rank } t^{-1}N_1 = \text{rank } N \leq r - 2 \leq n - 2\) the injective stabilization estimate for \(K_1 \text{ [Ba, Ch.5, §4]} \text{ [Va]} \implies \vartheta(0)^{-1} \vartheta \in E_n(k[t^{-1}N_1]).\) It is also clear that \(\vartheta(0)^{-1} \vartheta \in GL_n(k[t^{-1}N_1], \mathfrak{N}_1 k[t^{-1}N_1]_1)\). Therefore, by Lemma 9.1 there exist \(\gamma^+ \in GL_n(k[N_1]_{\mathfrak{N}_1}, \mathfrak{N}_1 k[N_1]_{\mathfrak{N}_1})\) and \(\gamma^- \in GL_n(k[N_1^-]_{\mathfrak{N}_1}, \mathfrak{N}_1 k[N_1^-]_{\mathfrak{N}_1})\) such that \(\vartheta(0)^{-1} \vartheta = \gamma^+ \gamma^-\), the equality being considered in \(GL_n(k[t^{-1}N_1]_{\mathfrak{N}_1})\). We have \(\vartheta = (\vartheta(0)^{-1} \vartheta(0)^{-1}) \vartheta(0)^{-1} \gamma^-\), where by the condition (ii) in Step 2 (namely, \(t^\pm(N_1(\Gamma_1) \setminus \{1\}) \subset \text{int}(N_2(\Gamma_2))\)) the inclusion \(\vartheta(0)^{-1} \vartheta(0)^{-1} \gamma^- \in GL_n(k[N]_{\mathfrak{N}_1})\) holds. Therefore, by Lemma 7.2 we arrive at Claim A below.

First one notation. For \(h = 2, 4\) we put

\[ \mathcal{O}_{X_h} = (k[N_h]_{\mathfrak{N}_h}, k[N_h^-]_{\mathfrak{N}_h}, 1_{t^{-1}k[N_h]_{\mathfrak{N}_h}}) \]

and

\[ \mathcal{O}_{X_h}(-1) = (k[N_h]_{\mathfrak{N}_h}, k[N_h^-]_{\mathfrak{N}_h}, t \cdot 1_{t^{-1}k[N_h]_{\mathfrak{N}_h}}) \].

**Claim A.** Any object of \(\mathcal{P}(X_1)(r)_{0,1}\) is mapped to an object of \(\mathcal{P}((X_2)_{\mathfrak{N}_3})(r)\) which up to isomorphism (within \(\mathcal{P}((X_2)_{\mathfrak{N}_3})\)) is of the type \(\mathcal{O}_{X_2}^a \oplus \mathcal{O}_{X_2}(-1)^b\) for some \(a, b \in \mathbb{Z}_+\).

**Step 6.** Consider any morphism in \(\mathcal{P}((X_2)_{\mathfrak{N}_3})\) of the form

\[ (f^+, f^-) : \mathcal{O}_{X_2}^a \oplus \mathcal{O}_{X_2}(-1)^b \to \mathcal{O}_{X_2}^c \oplus \mathcal{O}_{X_2}(-1)^d.\]

We let \((f^i_{ij})^+\) and \((f^i_{ij})^-\) denote the matrices of \(f^+\) and \(f^-\) in the corresponding standard bases. Then we see that there are systems \(u_1, \ldots, u_{a+b} \in \{0,1\}\) and \(v_1, \ldots, u_{c+d} \in \{0,1\}\) such that

\[ f^i_{ij} t^{u_j} = f^i_{ij} t^{u_j}, \quad i \in [1, a+b], \quad j \in [1, c+d]. \]

(Compare with the equations (8.2) and (8.2)2.) By the condition (ii) in Step 2 (namely, \(t^\pm(N_2(\Gamma_2) \setminus \{1\}) \subset \text{int}(N_3(\Gamma_3))\)) we get the inclusions

\[ f^i_{ij} \in k[N_3(\Gamma_3)]_{\mathfrak{N}_3} + k \cdot t \quad \text{and} \quad f^i_{ij} \in k[N_3(\Gamma_3)]_{\mathfrak{N}_3} + k \cdot t^{-1} \quad \text{(9.3)} \]

for all \(i\) and \(j\).

**Claim B.** Any short exact sequence in \(\mathcal{P}((X_2)_{\mathfrak{N}_3})\) of the type

\[ 0 \to \mathcal{O}_{X_2}^a \oplus \mathcal{O}_{X_2}(-1)^b \to \mathcal{O}_{X_2}^a \oplus \mathcal{O}_{X_2}(-1)^b \xrightarrow{f} \mathcal{O}_{X_2}^{a'} \oplus \mathcal{O}_{X_2}(-1)^{b'} \to 0 \]

is mapped to a split short exact sequence in \(\mathcal{P}((X_4)_{\mathfrak{N}_4})\).
By Lemma 8.3 any such a sequence reduces modulo \( \mathcal{A}_2 \) to a split sequence in \( \mathcal{P}(\mathbb{P}^1_{0,1}) \). Let \( \pi : \mathcal{P}(X_2)_{\mathfrak{S}_2} \to \mathcal{P}(\mathbb{P}^1_{0,1}) \) denote the exact functor, obtained by reduction modulo \( \mathfrak{S}_2 \), and let \( g \) be a monomorphism that splits \( \pi(f) \).

We have the scalar extension functor \( \iota : \mathcal{P}(\mathbb{P}^1_k) \to \mathcal{P}(X_2)_{\mathfrak{S}_2} \). The composite \( \pi \circ \iota \) is isomorphic to the identity functor on \( \mathcal{P}(\mathbb{P}^1_k) \). We can assume

\[ \pi(f \circ \iota(g)) = \mathbf{1}_{\mathcal{O}^{x \in \mathcal{O}(1)}_{x_1}}. \]

Claim B is proved once we show that the composite \( f \circ \iota(g) \) is mapped to an automorphism of \( \mathcal{O}_{X_1}^a + \mathcal{O}_{X_1}(-1)^b \) under the functor \( \mathcal{P}(X_2)_{\mathfrak{S}_2} \to \mathcal{P}(X_4)_{\mathfrak{S}_4} \).

It is easily observed that the positive and negative components of \( \iota(g) \) have correspondingly entries from \( k + k \cdot t \) and \( k + k \cdot t^{-1} \). But it then follows from the condition (ii) in Step 2 (namely, \( t^{+1}(N_3(\Gamma_3) \setminus \{1\}) \subset \text{int}(N_4(\Gamma_4)) \)), the inclusion (9.3.4) and the equality (9.3.5) that both positive and negative components of \( f \circ \iota(g) \) are defined over the local ring \( k[N_4(\Gamma_4)]_{\mathfrak{S}_4} = k[N_4^{-}(\Gamma_4)]_{\mathfrak{S}_4} \), that is the corresponding matrices w.r.t. the standard bases have entries from this local ring. Since these matrices are invertible modulo \( \mathfrak{S}_4 \) they are invertible themselves.

**Step 7.** Let \( \mathcal{E} \subset \mathcal{P}(X_4)_{\mathfrak{S}_4} \) denote the full subcategory whose objects are \( \mathcal{O}_{X_4}^a + \mathcal{O}_{X_4}(-1)^b \), \( a, b \in \mathbb{Z}_+ \) (in the selfexplanatory notation). This is an additive category. We equip it with an exact structure with respect to the class of split short sequences — any additive category carries such an exact structure, as shown by the Yoneda embedding \((* \mapsto \text{Hom}(-, *))\) in the category of contravariant additive functors with values in abelian groups. By the previous steps we have the natural commutative triangle of exact categories and exact functors

\[ \begin{array}{ccc}
\mathcal{P}(X_1)(r)_{0,1} & \longrightarrow & \mathcal{P}(X_4)_{\mathfrak{S}_4} \\
\mathcal{E} \end{array} \]

Consider a morphism \( f = (f^+, f^-) : \mathcal{O}_{X_4}^a + \mathcal{O}_{X_4}(-1)^b \to \mathcal{O}_{X_4}^a + \mathcal{O}_{X_4}(-1)^b \). For the corresponding matrices we have the equality

\[
\begin{pmatrix}
f_{a_1 \times a_2}^+ & t f_{a_2 \times b_2}^+ \\
f_{b_1 \times a_2}^+ & t f_{b_1 \times b_2}^+
\end{pmatrix} = \begin{pmatrix}
f_{a_1 \times a_2}^- & f_{a_2 \times b_2}^- \\
t f_{b_1 \times a_2} & t f_{b_1 \times b_2}
\end{pmatrix}.
\]

Since \( N_4 \) is polarized (and \( U(k[N_4(\Gamma_4)]_{\mathfrak{S}_4}) = k^* + \mathfrak{M}_4 k[N_4(\Gamma_4)]_{\mathfrak{S}_4} \)) we get:

- the entries of \( f_{a_1 \times a_2}^+ \) and \( f_{b_1 \times b_2}^+ \) belong to \( k[N_4(\Gamma_4)]_{\mathfrak{S}_4} \),
- the entries of \( f_{a_1 \times b_2}^+ \) belong to \( \mathfrak{M}_4 k[N_4(\Gamma_4)]_{\mathfrak{S}_4} \cap t^{-1} \mathfrak{M}_4 k[N_4(\Gamma_4)]_{\mathfrak{S}_4} \),
- the entries of \( f_{b_1 \times a_2}^+ \) belong to \( k + \mathfrak{M}_4 k[N_4(\Gamma_4)]_{\mathfrak{S}_4} + t \mathfrak{M}_4 k[N_4(\Gamma_4)]_{\mathfrak{S}_4} + k \cdot t \).

Conversely, any matrix over \( k[N_4]_{\mathfrak{S}_4} \) of size \( (a_1 + b_1) \times (a_2 + b_2) \), whose entries satisfy the three conditions above, is a positive component of a unique morphism in \( \mathcal{E} \).

Consider the subring \( \Lambda_0 = \{ (\varphi_{uv}) \} \subset \mathbb{M}_{2 \times 2}(k[N_4]_{\mathfrak{S}_4}) \) where

- \( \varphi_{11}, \varphi_{22} \in k[N_4(\Gamma_4)]_{\mathfrak{S}_4} \),
- \( \varphi_{12} \in \mathfrak{M}_4 k[N_4(\Gamma_4)]_{\mathfrak{S}_4} \cap t^{-1} \mathfrak{M}_4 k[N_4(\Gamma_4)]_{\mathfrak{S}_4} \),
- \( \varphi_{21} \in k + \mathfrak{M}_4 k[N_4(\Gamma_4)]_{\mathfrak{S}_4} + t \mathfrak{M}_4 k[N_4(\Gamma_4)]_{\mathfrak{S}_4} + k \cdot t \).
Consider the full subcategory $\mathcal{E}_{\Lambda_0} \subset \mathcal{E}$ of objects of the type $O_{X_4}^* \oplus O(-1)^*_X$. By the mentioned description of morphisms in $\mathcal{E}$ we have the natural equivalence of categories $\mathcal{E}_{\Lambda_0} \approx \mathcal{F}(\Lambda_0)$ – the category of left free $\Lambda_0$-modules (after thinking of elements of $\Lambda_0$ as the corresponding endomorphisms of $O_{X_4} \oplus O(-1)^*_X$).

Clearly, $\mathcal{E}_{\Lambda_0}$ is a cofinal subcategory of the exact category $\mathcal{E}$ in which all exact sequences split by definition. Therefore, by [Gra, §The plus construction] we have $K_i(\mathcal{E}_{\Lambda_0}) = K_i(\mathcal{E})$. (Recall $i \geq 1$; for the Grothendieck groups we have $K_0(\mathcal{E}) = \mathbb{Z} \oplus \mathbb{Z}$ and $K_0(\mathcal{E}_{\Lambda_0}) = \mathbb{Z}$.)

We have the exact functor $\mathcal{P}(\mathcal{E}_{\Lambda_0}) \to \mathcal{P}(k[N_5^t])$ given by picking up the “positive” part in the objects and morphisms in $\mathcal{E}_{\Lambda_0}$. In other words, we mean the composite functor $\mathcal{E}_{\Lambda_0} \subset \mathcal{E} \to \mathcal{P}((X_4)_{S_4}) \to \mathcal{P}(k[N_4]) \to \mathcal{P}(R[N_5^t])$, where the third functor is given by restriction to an open subscheme and the fourth functor corresponds to a scalar extension.

**Step 8.** Consider the two subrings $\Lambda_4 = \Lambda_{(t, \Gamma_4, N_4)}$ and $\Lambda_5 = \Lambda_{(t, \Gamma_5, N_5)}$ of the matrix ring $M_{2 \times 2}(k[N_4^t])$. Put $S = k^* + k(\text{int}(N_5(\Gamma_5))) \subset k[N_5(\text{int}(\Gamma_5))]$. We regard $S$ a central multiplicative set in $M_{2 \times 2}(k[N_5^t])$ through the diagonal embedding.

It is easily observed that $(\Lambda_4)_{S_4} = \Lambda_0$. (For an essentially equivalent equality see [Gu2, Proposition 2.14, Step 5].) The inclusion $\Gamma_4 \subset \text{int}(\Gamma_5)$ implies $\Lambda_0 \subset S^{-1}\Lambda_5$. Therefore, we have the commutative diagram (with obvious morphisms)

\[
\begin{array}{ccc}
\Lambda_5 & \to & M_{2 \times 2}(k[N_5]) \\
\downarrow & & \downarrow \\
\Lambda_4 & \to & M_{2 \times 2}(k[N_4]) \\
\downarrow & & \downarrow \\
\Lambda_0 & \to & M_{2 \times 2}(k[N_4])_{S_4} \\
\downarrow & & \downarrow \\
S^{-1}\Lambda_5 & \to & S^{-1}M_{2 \times 2}(k[N_5])
\end{array}
\] (9.3)7

By Lemmas 2.3 and 4.1 the boundary of (9.3)7, which consists of $k[N_5(\text{int}(\Gamma_5))]$-algebra homomorphisms, is a Karoubi square. Hence the long exact sequence

\[
\cdots \to K_i(\Lambda_5) \to K_i(k[N_5]) \oplus K_i(S^{-1}\Lambda_5) \to K_i(S^{-1}k[N_5]) \to \cdots
\] (9.3)8

By Step 7 we have the natural commutative diagram

\[
\begin{array}{ccc}
K_i(k[N_4]) & \to & K_i(k[N_5]) \\
\downarrow & & \downarrow \\
K_i(\mathcal{P}(X_1)(r)_0^1) & \to & K_i(k[N_5])_{S_4} \\
\downarrow & & \downarrow \\
K_i(\Lambda_0) & \to & K_i(k[N_5^t])
\end{array}
\]
Also, using $\Gamma_5 \subset \text{int}(\Phi(M))$, we have the homomorphisms with the same composite

$$K_i(P(X_1)(r)_{0,1}) \to K_i(k[N_5]) \to K_i(S^{-1}k[N_5]) \to K_i(k[N^*_5])_m.$$  

Therefore, using the element $\zeta \in K_i(P(X_1)(r)_{0,1})$ (see Step 5) and the sequence (9.3)$_8$, we see that $rx \in \text{Im}\,(K_i(A_5) \to K_i(k[N^*_5]))$ for $r \geq \text{rank}\,N + 2$, i.e. $x \in \text{Im}\,(K_i(A_5) \to K_i(k[N^*_5]))$.

**Step 9.** For a natural number $c$ put $\Lambda^c = \Lambda_{(r, \Gamma_5, N^*_5)}$. Clearly, $N^*_5 \subset \Gamma_5 \subset N_*$ for $j \gg 0$. Then Step 8 implies $(c_1 \cdots c_j)_s(z) \in \text{Im}\,(K_i(\Lambda_5^c) \to K_i(k[N^*_5]))$ for $j \gg 0$. It is easily observed that all the polarized monoids $T^c \subset N_5^*$ that have shown up above work equally well for the elements $(c_1 \cdots c_j)_s(z) \in \text{Im}\,(K_i(\Lambda_5^c) \to K_i(k[N^*_5]))$.

By the conditions (i) and (ii) in Step 2 (namely, $\Gamma_6 \subset \text{int}(\Phi(M))$) and $\Lambda^c := K_i(k^[N^*_5]_\Gamma \setminus \{1\}) \subset \text{int}(N_0(\Gamma_6))$ we have the inclusion

$$\Lambda^c_{(r^c_j, \Gamma_5, N^*_5)} \subset M_{2 \times 2}(k[M^*_5]).$$

Therefore, the polarized monoid $(t^{c_1 \cdots c_j}, \Gamma_5, N^*_5)$ is the desired one for $j \gg 0$. \hfill \Box

### 10. The action of Witt vectors

Assume $R$ is a general commutative ring. Stienstra [St] has studied a continuous Witt($R$)-module structure on $NK_i(R)$, $i \in \mathbb{Z}_+$. (Such actions more or less implicitly were previously defined by Bloch [Bl].) Recall the additive group of Witt($R$) is the multiplicative group $1 + TR[[T]]$ and the multiplicative structure is determined by

$$(1 - rT^m) \cdot (1 - sT^n) = (1 - r^{n/d}s^{m/d}T^mT^n)^d, \quad r, s \in R, \quad d = \gcd(m, n).$$

We have the decreasing ideal filtration $I_m(R) = 1 + T^mR[[T]]$. “Continuous” here means that the annihilator of any element $z \in NK_i(R)$ contains $I_m(R)$ for some $m$.

Weibel [W2] has generalized these operations to the graded situation as follows. Assume $A = A_0 \oplus A_1 \oplus \cdots$ is a graded, not necessarily commutative ring and $R \subset A_0$ is a subring in the center of $A$. Then there is a *functorial* continuous Witt($R$)-module structure on $K_i(A, A^+)$. (Here $A^+ = 0 \oplus A_1 \oplus A_2 \oplus \cdots$.)

In the special case $A = A_0[T]$ the action of $1 - rT^n \in \text{Witt}(R)$, $r \in R$ on $NK_i(A_0)$ is the effect of the composite functor

$$\mathbb{P}(A_0[T]) \xrightarrow{t_n} \mathbb{P}(A_0[T]) \xrightarrow{r} \mathbb{P}(A_0[T]) \xrightarrow{(v_n)^*} \mathbb{P}(A_0[T])$$

where $v_n : A_0[T] \to A_0[T]$ is given by $T \mapsto T^n$, $r : A_0[T] \to A_0[T]$ is given by $T \mapsto rT$ and $t_n$ is the scalar restriction through $v_n$. This determines the action of the whole Witt($R$) because any element $\omega(T) \in \text{Witt}(R)$ has a unique convergent expansion $\omega(T) = \prod_{n \geq 1}(1 - r^nT^n)$. Moreover, since for $\omega(T) \in I_m(R)$ the expansion has the form $\omega(T) = \prod_{n \geq m}(1 - r^nT^n)$, we conclude that

$$I_m(R)NK_i(A_0) \subset \text{Im}\,(K_i(A_0[\{T^n\}_{n \geq m}] \to K_i(A_0[T])), \quad m \in \mathbb{N}.$$  

(10.1)

Now assume a natural number $n$ is invertible in $A_0$ and $z \in NK_i(A_0)$ is in the image of $NK_i((v_n)_*)$. Since $n$ is invertible also in Witt($R$) we have $\frac{1}{n}z \in NK_i(A_0)$. On the other hand the composite $NK_i(t_n) \circ NK_i((v_n)_*) : NK_i(A_0[T] \to NK_i(A_0[T])$ is
the multiplication by \( n \). Therefore, \( z \in \text{Im} (NK_i((v_n)_*) \circ NK_i(t_n) \circ NK_i((v_n)_*)) \subseteq \text{Im} (NK_i((v_n)_*) \circ NK_i(t_n)) = (1 - T^n) \ast NK_i(A_0), \) referring to the Witt(\( R \))-action.

In particular, if \( \mathbb{Q} \subseteq A_0 \) we have the equality

\[ I_m(R)NK_i(A_0) = \text{Im} \left( K_i(A_0)[\{T^n\}_{n \geq m}] / K_i(A_0) \to NK_i(A_0) \right), \quad m \in \mathbb{N}. \tag{10} \]

Assume for simplicity \( R = k \) – a characteristic 0 field. Since Witt(\( k \)) \( \cong \Pi_1^{\infty} k \) (ghost map) with the product topology, a continuous Witt(\( k \))-module is just a \( k \)-vector space with a grading by natural numbers. Further, the Weibel map of graded algebras \( w : A \to A[T], \sum_j a_j = \sum_j a_j T^j \) (\( A[T] \) being graded w.r.t. the degrees of \( T \)) induces an embedding of Witt(\( k \))-modules \( K_i(A, A^+) \to NK_i(A) \) – a consequence of the fact that \( w \) splits the non-graded ring homomorphism \( A[T] \to A, T \mapsto 1 \).

Using (10\( _2 \)) (for the ring \( A \)) we obtain the embedding \( K_i(A_{[n]}, A^+_{[n]}) \to I_n(k)NK_i(A) \) where \( A_{[n]} = A_0 \oplus 0 \oplus \cdots \oplus 0 \oplus A_n \oplus A_{n+1} \oplus \cdots \). Summarizing, we get

**Proposition 10.1.** The following hold in the category of \( k \)-vector spaces:

1. (a) \( K_i(A, A^+) = \bigoplus_{j=1}^{\infty} V_j \)
2. (b) \( I_n(k)K_i(A, A^+) = 0 \oplus \cdots \oplus 0 \oplus V_n \oplus V_{n+1} \oplus \cdots, \quad n \in \mathbb{N}, \)
3. (c) \( \text{Im} \left( K_i(A_{[n]}, A^+_{[n]}) \to K_i(A, A^+) \right) \subseteq I_n(k)K_i(A, A^+), \quad n \in \mathbb{N}. \)

We would like to have a weaker version of (10\( _1 \)) in the graded situation as follows.

**Question 10.2.** Let \( k \) be a characteristic 0 field and \( A = A_0 \oplus A_1 \oplus \cdots \) be a graded not necessarily commutative \( k \)-algebra. Assume \( A_1 = A_2 = \cdots = A_{m-1} = 0, \) \( \dim_k(A_m) = 1 \) for some \( m \in \mathbb{N} \), and \( \dim_k(A_j) < \infty \) for all \( j \geq 0 \). Does one have the inclusions \( I_n(k)K_i(A, A^+) \subseteq \text{Im} \left( K_i(A_{[m+1]} \to K_i(A) \right) \) for \( n \gg 0 \)?

The relevance of Question 10.2 is that the positive answer to it would complete the proof of Conjecture 1.1 for fields of characteristic 0.

To see this consider a pyramidal extension of monoids \( M \subseteq N \) and an element \( z \in K_i(k[N_\ast]) \) with the property \( z(0) \in K_i(k) \). Let \( (t, \Gamma, L) \) be as in Theorem 9.3. We know that there is a grading \( k[N_\ast] = k \oplus R_1 \oplus \cdots \) such that \( N_\ast \) consists of homogeneous elements. It is easily observed that we can choose the grading of \( k[N_\ast] \) in such a way that the pole \( t \) is the element of the smallest positive degree within the polarized monoid \( L \). By Proposition 10.1 we have \( c_\ast(z) \in I_c(k)K_i(k[N_\ast], k[N_\ast]^+) \) for \( c \in \mathbb{N} \). Further, we have the induced gradings \( \Lambda_{(t, \Gamma, L)} = \Lambda_0 \oplus \Lambda_1 \oplus \cdots \) and \( \Lambda'_{(t, \Gamma, L)} = \Lambda'_0 \oplus \Lambda'_1 \oplus \cdots \) for these rings are in the same relation as \( A \) and \( A_{[m+1]} \) in Question 10.2. Therefore, the positive answer to this question (and the Morita invariance of the Witt(\( k \)) actions, [W2]) would imply \( (c_1 \cdots c_j)_\ast(z) \in \text{Im} \left( K_i(\Lambda'_{(t, \Gamma, L)} \to K_i(k[N_\ast]) \right) \) for \( j \gg 0 \). But the mentioned map factors through \( K_i(k[M_\ast]) \) and, hence, the pyramidal descent is achieved.

In one situation, which is to some extent dual to the situation considered so far, we have the following partial result on Question 10.2. It will be used in §11 – in a different way but again with essential use of Theorem 9.3 – to prove Conjecture 1.1 for certain class of non-simplicial monoids.
Lemma 10.3. Let $k$ be a number field (i.e., a finite extension of $\mathbb{Q}$) and $A = A_0 \oplus A_1 \oplus \cdots$ be a graded $k$-algebra such that $\dim_k A < \infty$. Then $K_i(A, A^+)$ is a finite dimensional $k$-vector space. In particular, $I_n(k)K_i(A, A^+) = 0$ for $n \gg 0$.

Proof. $A^+$ is a nilpotent ideal. So by Goodwillie’s result [Go] we have the isomorphism of abelian groups $K_i(A, A^+) \approx HC_{i-1}(A, A^+)$, the right hand side being a finite dimensional $\mathbb{Q}$-vector space. □

Remark 10.4. (a) It is very natural to expect that the answer to Question 10.2 is ‘yes’ when $k$ is a number field. This would proof Conjecture 1.1 for toric varieties over such fields. Notice, however, that one needs to modify the argument in Step 1 in the proof of Theorem 9.3 so that the inductive step no longer increases the transcendence degree of the ground field. Here we only mention that this is always possible. For the special class of bipyramidal cones the details are included in Step 1 in the proof of Theorem 11.1 below.

(b) We also expect that $I_n(k)K_i(A, A^+) = 0$ for $n \gg 0$ provided $\text{char } k = 0$ and $\dim_k A < \infty$. This would extend Theorem 11.1 to all characteristic 0 fields.

11. Bipyramidal monoids

Let $P \subset \mathbb{R}^r$ be a finite convex polytope of dimension $\dim P < r$ and $\sigma \subset \mathbb{R}^r$ be a closed segment (a 1-dimensional polytope). Assume $\text{int}(\sigma) \cap \text{int}(P)$ is a point. Then the polytope $Q = \text{conv}(\sigma \cup P)$ will be called a bipyramid over $P$. The class of polytopes $\mathfrak{P}$ is defined recursively by the condition:

- Points are in $\mathfrak{P}$ and if $P \in \mathfrak{P}$ then pyramids and bipyramids over $P$ are in $\mathfrak{P}$.

Every positive dimensional polytope $P \in \mathfrak{P}$ admits a finite, not necessarily unique sequence of polytopes $P_1 \subset P_2 \subset \cdots \subset P_d = P$, $d = \dim P$ such that $P_1$ is a segment and $P_s$ is either a pyramid or a bipyramid over $P_{s-1}$, $s \in [2, \dim P]$. To such a sequence we associate a sequence $\sigma(P)$ of length $d - 1$, consisting of 0s and 1s, as follows: the $s$th member of $\sigma(P)$ is 1 if $P_{d-s+1}$ is a bipyramid over $P_{d-s}$ and 0 otherwise. For instance, simplices are characterized by the condition $\sigma(P) = 0 \ldots 0$, $\sigma(P) = 11$ means that $P$ is an octahedron, $\sigma(P) = 01$ if $P$ is a pyramid over a square etc. The sequence $\sigma(P)$ is uniquely determined by the combinatorial type of $P$. To see this notice that no bipyramid can simultaneously be a pyramid because arbitrary facet of a bipyramid admits at least two vertices not in this facet. In particular, the first member of $\sigma(P)$ is uniquely determined. But then we can apply induction on the length of such sequences because the combinatorial type of the preceding member $P_{d-1}$ is determined by that of $P$ – an easy observation. As a result, we can identify $\sigma(P)$ with the combinatorial type of $P$. Therefore, in dimension $r > 0$ we have $2^{r-1}$ different $\mathfrak{P}$-combinatorial types of which one is simplicial.

We order the set of $\mathfrak{P}$-combinatorial types with respect first to the dimension and then to the lexicographic order of the $\sigma$-sequences.

A finite polyhedral pointed cone $C \subset \mathbb{R}^r$ will be called of type $\mathfrak{P}$ if its polytopal cross section is in $\mathfrak{P}$. A monoid $N$ without nontrivial units will be called of type $\mathfrak{P}$ if
the polytope $\Phi(N)$ is in $\mathfrak{P}$. Obviously, each $\mathfrak{P}$-combinatorial type admits infinitely many nonisomorphic monoids, whose $\Phi$-polytopes are of this combinatorial type.

**Theorem 11.1.** Let $\mathbb{Q} \subset k$ be an algebraic extension of fields, $N$ be a monoid of type $\mathfrak{P}$ and $i \in \mathbb{N}$. Then $N$ acts nilpotently on $K_i(k[N])$. If, in addition, $N$ is normal and finitely generated then $N$ acts nilpotently on $K_i(\text{Proj}(k[N]))$ for arbitrary grading $k[N] = k \oplus R_1 \oplus \cdots$ making the elements of $N$ homogeneous.

Here the nilpotence of the action in the projective case is understood in the sense of Proposition 4.7. Also, we remark that the normality assumption on $N$ in the second half of Theorem 11.1 can actually be dropped.

Among the toric varieties over $k$ covered by this theorem, the cone over the Segre embedding $\mathbb{P}_k^1 \times \mathbb{P}_k^1 \rightarrow \mathbb{P}_k^3$ is the simplest which is not simplicial and

$$\text{Proj}(k[U, V, W, X, Y, Z]/(UX - WZ, UX - VY)) \quad (\deg(U) = \cdots = \deg(Z) = 1)$$

is the simplest projective variety whose standard affine covering involves no simplicial toric variety.

We need a preparation. Let $C \subset \mathbb{R}^r$ be a rational bipyramidal cone and $C = C_1 \cup C_2$ be a decomposition into pyramidal cones. (We call a cone ‘bipyramidal’ or ‘pyramidal’ if its polytopal cross section is such.) Consider a nonzero element $t \in C_2 \cap \mathbb{Z}^r$ such that $\mathbb{R}_+ t$ is the unique extremal ray of $C_2$ not in $C_1 \cap C_2$. Denote by $L$ the monoid $C_1 \cap \mathbb{Z}^r$ and by $l$ the extremal ray of $C_1$ not in $C_1 \cap C_2$.

Consider the subring $\Lambda_{t,C} = \{\varphi_{uv}\} \subset M_{2 \times 2}(k[\mathbb{Z}^r])$ given by

- $\varphi_{11}, \varphi_{22} \in k[L]$,
- $\varphi_{12} \in k[L] \cap t^{-1}k[L]$,
- $\varphi_{21} \in k[L] + tk[L]$.

Assume the following holds

- $\omega = l \cap (- t + (C_1 \cap C_2)) \in \mathbb{Z}^r$.

In this situation $\omega$ and $t + \omega$ both belong to $L$. We also have $k[L] \cap t^{-1}k[L] = \omega k[L]$ and $c \cdot t + (c - 1) \cdot \omega = t + (c - 1) \cdot (t + \omega) \in t + L$ for $c \in \mathbb{N}$. It follows that for every natural number $c$ we have the $k$-algebra endomorphism:

$$\tilde{c} : \Lambda_{t,C} \rightarrow \Lambda_{t,C}, \quad (\varphi_{uv}) \mapsto \begin{pmatrix} c_\#(\varphi_{11}) & \omega^{-c+1}c_\#(\varphi_{12}) \\ \omega^{c-1}c_\#(\varphi_{21}) & c_\#(\varphi_{22}) \end{pmatrix}$$

where $c_\# = k[-c]$. In this notation we have the following

**Lemma 11.2.** Let $C' \subset \mathbb{R}^r$ be a polyhedral rational pointed cone such that $C_1 \setminus \{0\} \subset \text{int}(C')$. Then $\text{Im}(\tilde{c}) \subset M_{2 \times 2}(k[(C' \cap \mathbb{Z}^r)]_c)$ for $c \gg 0$.

**Proof.** Since $t + \omega \in \text{int}(C')$ we have $\omega^{-1}t \in \text{int}(C')$ for $c \gg 0$ and this yields the desired inclusion. \hfill $\Box$

Let $N$ be arbitrary finitely generated, normal monoid with trivial $U(N)$ and $C(N) = C' \cup C''$ be a decomposition into two nondegenerate pyramidal cones, sharing a base facet. Put $M = N(C')$. Then $M \subset N$ is a pyramidal extension of monoids. Assume $(t, \Gamma, L)$ is a polarized monoid such that $L \setminus \{0\} \subset \text{int}(N)$, $\Gamma \subset \text{int}(\Phi(M))$, and $t \in C'' \setminus C'$. As usual, $t = (c_1, c_2, \ldots)$ where $c_1, c_2, \ldots \geq 2$. 
Lemma 11.3. There exists a rational bipyramidal cone $C$ with the property $C \setminus \{0\} \subset \text{int}(C(N))$ and admitting a decomposition $C = C_1 \cup C_2$ into pyramidal cones in such a way that the following hold (in the real space $\mathbb{R} \otimes \text{gp}(N)$):

(a) $\Gamma \subset \text{int}(C_1)$, $C_1 \setminus \{0\} \subset \text{int}(C')$, and $L \subset C$,

(b) $C_2 = \mathbb{R}_+ t + (C_1 \cap C_2)$,

(c) $l \cap (t - t + (C_1 \cap C_2)) \in \text{gp}(N)^e$ where $l \subset C_1$ is the extremal ray not in $C_1 \cap C_2$.

Proof. We can gradually approximate $\text{int}(C')$ by appropriate pyramidal rational cones $C_\alpha$, $\alpha \in \mathbb{N}$ such that $C_\alpha \setminus \{0\} \subset \text{int}(C')$. Since $\text{gp}(N)^e \subset \mathbb{R} \otimes \text{gp}(N)$ is a dense subset, we can also keep the condition $l_\alpha \cap (-t + F_\alpha) \in \text{gp}(N)^e$ satisfied, where $F_\alpha \subset C_\alpha$ is the facet ‘close’ to $C' \cap C''$ and $l_\alpha$ is the extremal ray of $C_\alpha$ not in $F_\alpha$. Then $C = C_\alpha + \mathbb{R}_+ t$ is the desired bipyramidal cone for $\alpha \gg 0$ whose desired decomposition into pyramidal cones is $C = C_\alpha \cup (F_\alpha + \mathbb{R}_+ t)$.

We also need several results on triangular matrix rings.

Berrick and Keating showed [BeKe] that for not necessarily commutative rings $A$ and $B$ and an $A - B$ bimodule $U$ the embedding of rings

$$A \oplus B \to \begin{pmatrix} A & 0 \\ U & B \end{pmatrix}$$

induces isomorphisms on $K$-groups. Further, let $R$ be a commutative ring and $I \subset R$ be a projective ideal. Put

$$A = \begin{pmatrix} R & I \\ R & R \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} R/I & 0 \\ R/I & R/I \end{pmatrix}.$$ 

For every natural index $i$ we have the group homomorphisms

- $f : K_i(A) \to K_i(M_{2 \times 2}(R)) = K_i(R)$,
- $g : K_i(A) \to K_i(B)$,
- $h : K_i(B) \to K_i(M_{2 \times 2}(R/I)) = K_i(R/I)$.

Next is the commutative case of Keating’s result.

Theorem 11.4 (Keating [Ke]). For any natural index $i$ we have

(a) $h((a, b)) = a + b$ for $a, b \in K_i(R/I)$. In particular, $\text{Ker}(h) \approx K_i(R/I)$.

(b) $f : K_i(A) \to K_i(R)$ is a split epimorphism.

(c) $g$ restricts to an isomorphism $\text{Ker}(f) \to \text{Ker}(h)$. In particular, $K_i(A) = K_i(R) \oplus K_i(R/I)$.

Proof of Theorem 11.1. There is no loss of generality in assuming that $\mathbb{Q} \subset k$ is a finite extension.

Step 1. First we consider the affine case.

We have well ordered the set of combinatorial types of the cones at the issue and the proof can be carried out by induction with respect to this order. Theorem 6.4 gives the result for monoids $N$ such that $\sigma(\Phi(N)) = 0 \ldots 0$. This includes the case rank $N \leq 2$. 
All faces of a polytope in $\mathfrak{P}$ are in $\mathfrak{P}$. Therefore, by the obvious version of Lemma 3.4 for monoids of type $\mathfrak{P}$ it suffices to show that $N$ acts nilpotently on $K_i(k[N_\ast])$ where $N$ is in addition a finitely generated and normal monoid.

In order to apply Theorem 9.3 we have to resolve one difficulty—the induction hypothesis on rank $N$, used in Step 2 in the proof of Theorem 9.3, increases the transcendence degree of the ground field. Here we develop an alternative, circumventing such an increase for our special cones.

There is no loss of generality in assuming $\Phi(N) = 0 \cdots 01\delta_s \cdots \delta_{d-1}$, for some $s \in [1,d]$ ($d = \text{rank}(N) - 1$). The cone $C(N)$ decomposes into two pyramidal cones $C(N) = C' \cup C''$ whose polytopal cross sections are both of type $0 \cdots 00\delta_s \cdots \delta_{d-1}$. Let $M \subset N$ be the corresponding pyramidal extension so that $C(M) = C'$. By the obvious adaptation of Theorem 6.2 to the monoids of type $\mathfrak{P}$ we can assume that the polarized monoids $(t,\Gamma,N_\alpha)$ are such that the $\Gamma_\alpha$ are pyramids whose bases approximate $C' \cap C''$ and $\sigma(\Gamma_\alpha) = 0 \cdots 00\delta_s \cdots \delta_{d-1}$, notation as in Theorem 6.2. In this situation the monoids $(t^{-1}\alpha N_\alpha) \setminus (t^{-1}\alpha)^N$ are filtered unions of finitely generated normal monoids $N_{\alpha\beta}$ of type $\mathfrak{P}$ such that $\sigma(\Phi(N_{\alpha\beta})) = \sigma(\Gamma_\alpha) = 0 \cdots 00\delta_s \cdots \delta_{d-1} < \sigma(\Phi(N))$. Therefore, for each element $z \in K_i(k[N_\alpha])$, $z(0) = 0$, the condition (iv) in Step 2 in the proof of Theorem 9.3 is achieved by the induction assumption.

All the other arguments in the proof of Theorem 9.3 go through with respect to the system $(t,\Gamma,N_\alpha)$. So we reach the situation when:

- $N$ is a finitely generated normal monoid without nontrivial units,
- $C(N) = C' \cup C''$ is a decomposition into two pyramidal cones sharing a base facet and $\sigma(\Phi(M)) < \sigma(\Phi(N))$ where $M = C' \cap N$,
- $(t,\Gamma,L)$ is a polarized monoid such that $L \subset N_\ast$ and $\Gamma \subset \text{int}(\Phi(M))$,
- $z \in K_i(k[N_\alpha])$, $z(0) = 0$, and $(c_1 \cdots c_j) \ast(z) \in \text{Im} \left(K_i(\Lambda(t,\Gamma,L) \to K_i(k[N_\ast]))\right)$ for $j \gg 0$ (notation as in Theorem 9.3).

By induction assumption it suffices to achieve the pyramidal descent for the extension $M \subset N$, i.e. $(c_1 \cdots c_j) \ast(z) \in \text{Im} \left(K_i(k[M_\ast]) \to k[N_\ast]\right)$ for $j \gg 0$.

**Step 2.** Fix $j_1$ such that $(c_1 \cdots c_{j_1}) \ast(z) \in \text{Im} \left(K_i(\Lambda(t,\Gamma,L) \to K_i(k[N_\ast]))\right)$. Next we choose a bipyramidal cone $C \setminus \{0\} \subset \text{int}(C(N))$ as in Lemma 11.3 with respect to the cone $C'$, the polarized monoid $(t,\Gamma,L)$, and the sequence $c' = (c_{j_1+1},c_{j_1+2},\ldots)$. Put $\omega = l \cap ( - t + (C_1 \cap C_2))$ (notation as in that lemma) and choose $j_2 > j_1$ such that $\omega \in (c_{j_1+1} \cdots c_{j_2})^{-1}\text{gp}(N)$. For simplicity of notation put $\kappa = c_{j_1+1} \cdots c_{j_2}$ and consider the ring $\Lambda_{t,\ast,C}$ as in Lemma 11.2 with respect to the lattice $\text{gp}(N) \approx \mathbb{Z}^{\text{rank } N}$.

We have the diagram of ring embeddings

$$
\xymatrix{ \Lambda_{(t,\ast,(\Gamma,L))} \ar[r] & \Lambda_{t,\ast,C} \\
A \ar[r] & M_{2\times 2}(k[N]) \ar[r] & M_{2\times 2}(k[N_\ast]) \\
k[N] \ar@{^{(}->}[u] & & \Delta 
}
$$
where $N = C \cap \text{gp}(N)$,

$$A = \begin{pmatrix} k[N] & \omega^k k[N] \\ k[N] & k[N] \end{pmatrix} \subseteq M_{2 \times 2}(k[N]), \quad \Delta(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix},$$

and all the other homomorphisms are the identity embeddings. As observed in Step 9 in the proof of Theorem 9.3, $(c_1 \cdots c_j, c)_*(z) \in \text{Im} \left( K_i(\Lambda_{(t, \Gamma, \Lambda^+)} \to (k[N_\star])) \right)$ for any natural number $c$. By choosing $c = c_{j_2+1}c_{j_2+2} \cdots c_j$ for $j > j_2$ we get

$$(c_1 \cdots c_{j_1}, c_{j_1+1} \cdots c_{j_2}c_{j_2+2} \cdots c_j)_*(z) \in \text{Im} \left( K_i(\Lambda_{(t, \Gamma, \Lambda^+)} \to M_{2 \times 2}(k[N_\star])) \right)$$

for all $j > j_2$. We will denote by $z_{c_1 \cdots c_j}$ such preimages in $K_i(\Lambda_{(t, \Gamma, \Lambda^+)})$.

Since $z(0) = 0$ we can assume that $(c_1 \cdots c_j)_*(z)$ has a preimage in $K_i(k[N])/K_i(k)$ for $j \gg 0$ with respect to the right-most arrow in the diagram above.

Fix a grading $k[N_\star] = k \oplus R_1 \oplus \cdots$ making element of $N_\star$ homogenous. It restricts to a grading $k[N] = k \oplus S_1 \oplus \cdots$. The homomorphism $k[N] \xrightarrow{\Delta} A \subseteq M_{2 \times 2}(k[N])$ induces a continuous Witt($k$)-module endomorphism of $K_i(k[N])/K_i(k)$ which is a $k$-vector space (see §10). On the other hand, by Theorem 11.4(a) (and the Berrick-Keating result on triangular matrices) this endomorphism is just the multiplication by 2. Therefore, there is a preimage $z_{c_1 \cdots c_j} \in K_i(k[N])$ of the image of $z_{c_1 \cdots c_j}$ in $K_i(M_{2 \times 2}(k[N])) = K_i(k[N])$ under the homomorphism $K_i(N) \xrightarrow{\Delta^\star} K_i(A) \to K_i(M_{2 \times 2}(k[N]))$.

**Step 3.** We claim that for $j \gg 0$ the preimages $z_{c_1 \cdots c_j} \in K_i(\Lambda_{(t, \Gamma, \Lambda^+)})$ and $z_{c_1 \cdots c_j}^N \in K_i(k[L])$ can be chosen in such a way that they map to the same elements in $K_i(A)$.

By Theorem 11.4 we have $K_i(A) = K_i(k[N]) \oplus K_i(k[N]/k[N])$ and it is enough to show that the elements $z_{c_1 \cdots c_j}$ with $j \gg 0$ can be chosen in such a way that their images belong to $K_i(k[N]) \oplus 0$.

First observe, that we can assume $z_{c_1 \cdots c_j}(0) = 0$. Then, by Proposition 10.1 and the fact that

$$(c_1 \cdots c_j)_*(z) \in \text{Im} \left( K_i(k[N_\star]_{c_1 \cdots c_j}], k[N_\star]_{c_1 \cdots c_j}]^+ \right) \to K_i(k[N_\star], k[N_\star]^+) \right),$$

we can further choose

$$z_{c_1 \cdots c_j} \in I_{c_1 \cdots c_j}(k)K_i(\Lambda_{(t, \Gamma, \Lambda^+)}, \Lambda_{(t, \Gamma, \Lambda^+)})$$

(w.r.t. to the induced gradings). The image of $z_{c_1 \cdots c_j}$ in $K_i(A)$ belongs to the subgroup $I_{c_1 \cdots c_j}(k)K_i(A, A^+) \subseteq K_i(A, A^+)$. Denote this image by $(z', z'')$ for the appropriate elements

$$z' \in I_{c_1 \cdots c_j}(k)K_i(k[N], k[N]^+), \quad z'' \in I_{c_1 \cdots c_j}(k)K_i(k[N]/\omega^k k[N], (k[N]/\omega^k k[N])^+).$$

Recall, by Theorem 11.4 we have the natural identification

$$K_i(k[N]/\omega^k k[N]) = \text{Ker} \left( K_i(B) \to K_i(k[N]/\omega^k k[N]) \right)$$

where

$$B = \begin{pmatrix} k[N]/\omega^k k[N] & 0 \\ k[N]/\omega^k k[N] & k[N]/\omega^k k[N] \end{pmatrix},$$
and moreover, using this identification, \( z'' \) is in the image of the composite map
\[
K_i(\Lambda(t^r,\Gamma,L^s)) \to K_i(\Lambda^{t^r}) \to K_i(A) \to K_i(M_{2 \times 2}(k[N])) \to K_i(k[N]/\omega^* k[N]).
\]
Now the image of \( \Lambda^{t^r} \) in \( B \) is
\[
\left( \bar{\Lambda} \quad 0 \right)
\]
where \( \bar{\Lambda} \) and \( \bar{U} \) denote respectively the images of \( k[L^\times(\Gamma)] \) and \( k[L^\times(\Gamma)] + t^r k[L^\times(\Gamma)] \) in \( k[N]/\omega^* k[N] \). Since \( \mathbb{R}_+ \Gamma \setminus \{0\} \subset \text{int}(C_1) \subset \text{int}(\mathcal{C}) \) we have \( \dim k \bar{\Lambda} < \infty \), the cone \( C_1 \) being as in Lemma 11.3. (Actually, \( \dim k \bar{U} < \infty \) as well.) By Berrick-Keating’s result on triangular matrices and Lemma 10.3 we get the desired vanishing of \( z'' \) for \( j \gg 0 \).

**Step 4.** For any natural number \( c \) we have the following diagram

\[
\begin{array}{cccccc}
K_i(\Lambda^{t^r}) & \to & K_i(A) & \to & K_i(M_{2 \times 2}(k[N^*_c])) \\
\downarrow & & & & & \downarrow \\
K_i(\Lambda^{t^r},C) & \to & K_i(k[N]) & \to & K_i(k[N]) \\
\downarrow & & & & & \downarrow \\
K_i(\Lambda^{t^r}) & \to & K_i(A) & \to & K_i(M_{2 \times 2}(k[N^*_c])) \\
\downarrow & & & & & \downarrow \\
K_i(\Lambda^{t^r},C) & \to & K_i(k[N]) & \to & K_i(k[N])
\end{array}
\]

with the obvious homomorphisms. (The endomorphism \( \tilde{c} : \Lambda^{t^r} \to \Lambda^{t^r} \) extends naturally to an endomorphism of \( A \), which we denote by the same \( \tilde{c} \).) All the squares with continuous arrows commute, while there is no obvious reason why the square with dashed arrows should commute – certainly, it does not commute on the level of the underlying rings. By Step 3 the preimages \( z_{c_1 \ldots c_j} \) and \( z_{c_1 \ldots c_j}^{N} \) agree in \( K_i(A) \) for \( j \gg 0 \). Therefore, by Lemma 11.2 we achieve the desired pyramidal descent by fixing \( j \gg 0 \) and running \( c \) through \( \{c_{j+1} \ldots c_{j+k}\}_{k \gg 0} \). The affine case has been proved.

**Step 5.** For the projective varieties we can use the result in the affine case – the same arguments as in the proof of Proposition 4.7 go through. One only needs to observe that the standard affine covering of \( \text{Proj}(k[N]) \) admits the following description. It consists of the affine toric varieties \( \text{Spec}(k[N_v]) \), defined by the data:
- \( v \) runs through the vertex set of \( \Phi(N) \),
- \( \lambda \) is a natural number such that \( \lambda v \in \text{gp}(N) \),
- \( C_v \subset \mathbb{R} \otimes \text{gp}(N) \) is the \((\dim C - 1)\)-dimensional cone spanned by \( \lambda \Phi(N) \) at \( \lambda v \),
- \( N_v = -\lambda v + (C_v \cap \text{gp}(N)) \).
The cones $C_v$ are all of type $\mathfrak{P}$ provided $N$ is such, equivalently – corner cones of polytopes in $\mathfrak{P}$ are of type $\mathfrak{P}$. Notice that we also need to take care on monomial localizations of $k[N]$ – they also show up in our Mayer-Vietoris sequences. But such localizations are Laurent polynomial extensions of monoid rings whose underlying monoids are of type $\mathfrak{P}$. In particular, we can use the Fundamental Theorem as in the proof of Proposition 4.7 (the affine case). □

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