A Rank Revealing Randomized Singular Value Decomposition (R³SVD) Algorithm for Low-rank Matrix Approximations

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Abstract— In this paper, we present a Rank Revealing Randomized Singular Value Decomposition (R³SVD) algorithm to incrementally construct a low-rank approximation of a potentially large matrix while adaptively estimating the appropriate rank that can capture most of the actions of the matrix. Starting from a low-rank approximation with an initial guessed rank, R³SVD adopts an orthogonal Gaussian sampling approach to obtain the dominant subspace within the leftover space, which is used to add up to the existing low-rank approximation. Orthogonal Gaussian sampling is repeated until an appropriate low-rank approximation with satisfactory accuracy, measured by the overall energy percentage of the original matrix, is obtained. While being a fast algorithm, R³SVD is also a memory-aware algorithm where the computational process can be decomposed into a series of sampling tasks that use constant amount of memory. Numerical examples in image compression and matrix completion are used to demonstrate the effectiveness of R³SVD in low-rank approximation.

Index Terms— Randomized Singular Value Decomposition, Rank Revealing Algorithm, Low-rank Matrix Approximation, Orthogonal Gaussian Sampling, Memory-aware Algorithm

I. INTRODUCTION

Considering an \( m \times n \) matrix \( A \) with rank \( r \), the optimal \( k \)-rank (\( k \leq r \)) approximation \( A_k \) of matrix \( A \) yields minimum approximation error among all possible \( m \times n \) matrices of rank \( k \) [1], i.e.,

\[
\|A - A_k\|_F = \min_{\text{rank}(X) = k} \|A - X\|_F^2.
\]

Within controllable approximation error, a good low-rank approximation of a large matrix can reduce storage requirement and accelerate matrix operations such as matrix-vector or matrix-matrix multiplications. If \( A \) is a matrix representing data affinity in a large dataset, low-rank approximation algorithms can be used for dimensionality reduction or noise elimination. As a result, constructing appropriate low-rank approximations of large matrices plays a central role in many data analytic applications [2, 3, 4, 5, 6], such as principle component analysis, compressed sensing, data compression, signal processing, machine learning, and matrix completion.

The optimal \( k \)-rank approximation \( A_k \) can be straightforwardly obtained by computing full Singular Value Decomposition (SVD) and then truncating it by selecting the top \( k \) dominant singular values and their corresponding singular vectors such that

\[
A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T,
\]

where \( k \leq r \), \( \sigma_1, \sigma_2, \ldots, \sigma_k \) are the singular values of \( A \) in non-increasing order, and \( u_1, \ldots, u_k \) and \( v_1, \ldots, v_k \) are the corresponding left and right singular vectors, respectively. Here, by tuning the value of \( k \), the low-rank matrix approximation error measured by Frobenius norm can be controlled by

\[
\|A - A_k\|_F^2 = \sum_{l=k+1}^{r} \sigma_l^2.
\]

However, numerically computing the full SVD of a matrix when both \( m \) and \( n \) are large is often computationally costly as well as memory intensive. As an efficient alternative, randomized algorithms to approximate SVD have attracted great interest recently and become competitive for computing rapid low-rank approximations of large matrices [3, 7, 8, 9]. Instead of passing over the large matrix in full SVD, the randomized SVD algorithms focus on efficiently sampling the important matrix elements. Many sampling strategies, including uniform column/row sampling (with or without replacement) [10, 11], diagonal sampling or column-norm sampling [12], sampling with \( k \)-means clustering [13], and Gaussian sampling [14], have been proposed. As a result, compared to full SVD, randomized SVD methods are memory efficient and can often obtain good low-rank approximations in a significantly faster way.

Nevertheless, most of these randomized SVD algorithms require the rank value \( k \) to be given as an input parameter in advance. In fact, in many practical applications, \( k \) is unknown beforehand but is of great importance to the accuracy of the solutions. In general, underestimating \( k \) can introduce unacceptable large error in the low-rank approximation while
overestimating $k$ can lead to unnecessary computational and memory costs.

In this paper, we present a Rank-Revealing Randomized SVD (R$^3$SVD) algorithm whose goal is to incrementally construct a low-rank approximation while estimating the appropriate $k$ value in an adaptive manner. The fundamental idea behind R$^3$SVD is importance sampling – a new form of Gaussian sampling based on orthogonal projection is derived to obtain the dominant subspace orthogonal to the existing low-rank approximation, which is used to add up to existing low-rank approximation. Moreover, R$^3$SVD is a memory-aware algorithm due to the fact that its computation can be tailored into subsequent tasks that can fit in constant amount of memory. We use several application examples, including image compression and matrix completion, to demonstrate the effectiveness of R$^3$SVD.

The rest of the paper is organized as follows. We review the randomized SVD algorithm based on Gaussian Sampling for low-rank approximation in Section II. In Section III, we describe our R$^3$SVD algorithm and justify its properties. Numerical examples are presented in Section IV. Finally, Section V summarizes our conclusions and future research directions.

II. RANDOMIZED ALGORITHM WITH GAUSSIAN SAMPLING

Our R$^3$SVD algorithm is based on the randomized SVD algorithm with Gaussian sampling proposed by Halko et al. [14, 15], although it can be straightforwardly extended to other randomized SVD algorithms with different matrix sampling strategies. In this paper, R$^3$SVD is referred to the randomized SVD algorithm with Gaussian sampling. The basic idea of RSVQ is to use Gaussian vectors to construct a small condensed subspace from the range of $A$, whose dominant actions could be quickly estimated from this small subspace with relatively low computation cost while yielding high confidence. The procedure of R$^3$SVD is described in Algorithm 1.

Algorithm 1: Randomized SVD (RSVD) Algorithm with Gaussian Sampling

Input: $A \in \mathbb{R}^{m \times n}$, a target matrix rank $k \in \mathbb{N}$, and an oversampling parameter $p \in \mathbb{N}$ satisfying $k + p \leq \min(m, n)$.
Output: Low rank approximation $U_k \in \mathbb{R}^{m \times k}$, $\Sigma_k \in \mathbb{R}^{k \times k}$, and $V_L \in \mathbb{R}^{n \times k}$.

Construct an $n \times (k + p)$ Gaussian random matrix $\Omega$

$Y = A\Omega$

Compute an orthogonal basis $Q = qr(Y)$

$B = Q^TA$

$[U_B, \Sigma_B, V_B] = svd(B)$

Update $U_{old} = QU_B$

$U_k = U_{old}(1:k, 1:k), \Sigma_k = \Sigma_B(1:k, 1:k)$, and $V_L = V_B(:, 1:k)$

Given a desired rank $k$ and an oversampling parameter $p$ (typically a small constant), RSVQ constructs an $n \times (k + p)$ Gaussian random matrix block $\Omega$, whose elements are normally distributed. $\Omega$ condenses a large matrix $A$ into a “tall-and-skinny,” dense block matrix $Y$ by $Y = A\Omega$. $Y$ captures the most important actions of $A$ and a basis $Q$ is derived by decomposing $Y$. $Q$ is designed to approximate the left singular vectors of $A$ by minimizing $||Q\Omega^TA - A||_F^2$. Then, $Q$ is applied back to $A$ to obtain a “short-and-wide” block matrix $B = Q^TA$. Calculation of SVD on $B$ yields an approximate Singular Value Decomposition of $A$. The result $U_k\Sigma_kV_L^T$ forms a $k$-rank matrix approximation to $A$.

Compared to full SVD directly operating on the $m \times n$ matrix $A$, which is rather computational costly when both $m$ and $n$ are large, the major operations in RSVQ are carried out on the block matrices instead. These block matrix operations include matrix-block matrix multiplications as well as QR and SVD decompositions on the block matrices. Specifically, matrix-block matrix multiplications take $O((k + p)^2(m + n))$ floating-point operations, where $T_{mult}$ denotes the computational cost of a matrix-vector multiplication. For a large matrix $A$ where $m, n >> k + p$, the computational cost of matrix-block matrix multiplications dominates those of block QR or SVD decomposition operations, which requires $O((k + p)^2(m + n))$ floating-point operations. RSVQ needs to store the intermediate matrices, such as $\Omega$, $Y$, $Q$, and $B$, and thus its space complexity is $O(2(m + n)(k + p))$. As a result, RSVQ is usually more efficient than the full SVD algorithms in terms of computational and memory cost, but with a tradeoff of accuracy.

The desired rank $k$ is a required input parameter in the randomized SVD algorithms. However, in many practical applications, the value of $k$ is unknown beforehand and needs to be appropriately estimated. In the literature, two strategies have been proposed to estimate the appropriate value of $k$. One strategy is based on preprocessing, which intends to obtain an appropriate $k$ value before carrying out RSVQ. For instance, Voronin and Martinsson [16] proposed two algorithms, Autorank I and Autorank II, to evaluate a basis $Q$ for a range space that captures the most actions of matrix $A$. Autorank I is based on overestimation by using a very large value $k$ at the beginning and then selecting dominant information from the resulting pool of singular values/vectors. Although Autorank I is often able to obtain good low-rank approximations, largely overestimated $k$ will result in significant computational cost increase, because the computational cost of decomposing the tall-and-skinny or short-and-wide block matrices in RSVQ grows rapidly with $O(k^2)$ and is no longer negligible. At the same time, the memory requirement of Autorank I increases in the order of $k$ too. Instead of overestimating $k$, Autorank II gradually samples the range of $A$ guided by error $||(Q^TA - A)||_F^2$ in order to obtain a good estimation of $k$. Similar to Autorank II, the Adaptive Randomized Range Finder algorithm [14] employs the incremental sampling approach with a probabilistic error estimator based on the relation between the rank $k$ with respect to the theoretical error bound to predict a reasonable basis $Q$ with a reasonable value of $k$. However, this
III. RANK REVEALING RANDOMIZED SVD (R’SVD) ALGORITHM

To illustrate the R’SVD algorithm for a given matrix $A \in \mathbb{R}^{m \times n}$, we first define the energy $E(U_L \Sigma_L V_L^T)$ of a $k$-rank approximation $U_L \Sigma_L V_L^T$ as $E(U_L \Sigma_L V_L^T) = \left\|U_L \Sigma_L V_L^T\right\|_F^2$ while the overall energy of $A$ is $E(A) = \left\|A\right\|_F^2 = \sum_{i=1}^{\infty} \sigma_i^2$. Then, the overall energy percentage occupied by the $k$-rank approximation with respect to that of $A$ becomes

$$\varphi = \frac{\left\|U_L \Sigma_L V_L^T\right\|_F^2}{\left\|A\right\|_F^2} = \frac{\sum_{i=1}^{k} \sigma_i^2}{\sum_{i=1}^{\infty} \sigma_i^2},$$

where $\sigma_i$ denotes the $i$th diagonal element of $\Sigma_L$ and $\sigma_i$’s are the actual singular values of $A$. Measuring the percentage of energy of a low-rank approximation with respect to a large matrix has been popularly used in a variety of applications for dimensionality reduction such as Principle Component Analysis (PCA) [4, 18], ISOMAP learning [19]. Locally Linear Embedding (LLE) [20], and Linear Discriminant Analysis (LDA) [21]. According to the Eckart-Young-Mirsky theorem [22], for a fixed $k$, optimal $k$-rank approximation has the overall energy percentage of $\sum_{i=1}^{k} \sigma_i^2 / \left\|A\right\|_F^2$ of $A$.

The adaptivity of R’SVD is achieved by estimating the overall energy percentage of the low-rank approximation obtained so far. The rationale of R’SVD is to build a low-rank approximation incrementally based on orthogonal Gaussian projection. Initially, a $t$-rank approximation is obtained, where $t$ is an initial guess of $k$ which can be justified according to the memory available. The energy percentage is estimated accordingly. If the energy percentage obtained so far does not satisfy the application requirement, a new $t$-rank approximation is calculated in the subspace orthogonal to the space of the previous low-rank approximation. Then, the new $t$-rank approximation will be added to the previous one to form a $2t$-rank approximation and its corresponding energy percentage is estimated. The above process is repeated until the incrementally built low-rank approximation has secured satisfactory percentage of energy from $A$.

Compared to RSVD, R’SVD incorporates three major changes including orthogonal Gaussian sampling, orthogonalization process, and stopping criteria based on energy percentage estimation. By putting all pieces together, R’SVD is described as follows.

Algorithm 2: Rank Revealing Randomized SVD Algorithm

**Input:** $A \in \mathbb{R}^{m \times n}$, sampling size $t \in \mathbb{N}$ per iteration, oversampling number $p \in \mathbb{N}$, power number $q \in \mathbb{N}$, maximum number of iterations $\max(t) \in \mathbb{N}$, and energy threshold $\tau \in \mathbb{R}$.

**Output:** Low rank approximation $U_L \in \mathbb{R}^{m \times k'}$, $\Sigma_L \in \mathbb{R}^{k' \times k'}$, $V_L \in \mathbb{R}^{n \times k'}$, and estimated rank $k'$.

// initialization
Construct an $n \times (t + p)$ standard Gaussian matrix $\Omega

\gamma_0 = \Omega$ and $V_L = \emptyset$, $U_L = \emptyset$, $\Sigma_L = \emptyset$.

$k' = 0$

for $i = 0:maxit$

\begin{align*}
Y_i &= AG_i \\
Q_i &= qr(Y_i, 0)
\end{align*}

$B_i = Q_i^T A$

\begin{align*}
\left[ U_{B_i}, V_{B_i} \right] &= svd(B_i, 0) \\
U_{B_i} &= Q_i U_{B_i} \\
V_{B_i} &= qr(V_{B_i} - V_L(V_L^T V_{B_i}), 0) & // & \text{orthogonalization process}
\end{align*}

$U_L \leftarrow [U_L, U_B(:, 1:t)]$, $\Sigma_L \leftarrow \begin{pmatrix} \Sigma_L & 0 \\ 0 & \Sigma_{B_i}(1:t, 1:t) \end{pmatrix}$, $V_L \leftarrow [V_L, V_{B_i}(:, 1:t)]$

for $j = 1:t$

$k' = i \times t + j$

$\hat{\varphi}_{k'} = \sum_{i=1}^{k'} \sigma_i^2 / \left\|A\right\|_F^2$ // estimate energy percentage

if $\hat{\varphi}_{k'} \geq \tau$, then stop;
end

$G_{i+1} = G_i - V_L(V_L^T G_i)$ // update Gaussian matrix

end

$[\Sigma_L, \text{Idx}] = \text{sort}(\Sigma_L, \text{descend})$; // sort the approximate singular values

$V_L = V_L(:, \text{Idx})$;

$U_L = U_L(:, \text{Idx})$;

A. Orthogonal Gaussian Sampling

Suppose that $V_L$ is an $n \times t$ matrix composed of $t$ right singular vectors of a low-rank approximation $U_L \Sigma_L V_L^T$, which is supposed to capture most of the energy in $A$. Then, the range space, ran($A^T$), can be divided into two orthogonal
spaces: space \( \text{ran}(V_L) \) spanned by the columns in \( V_L \) and its orthogonal complement \( \text{ran}(V_L) \bot \). Obviously, if \( V_L \) consists of only partial dominant actions of \( A \), the rest dominant information is left over in the space \( \text{ran}(V_L) \bot \).

R\text{SV}D is designed to incrementally add up a low-rank approximation. Therefore, R\text{SV}D needs to sample the space \( \text{ran}(V_L) \bot \) orthogonal to \( V_L \) to extract the left-over dominant information of \( A \). Here, we construct a sampling matrix \( G \) such as

\[
G = (I - P_V)\Omega
\]

where \( P_V = V_L V_L^T \) is an orthogonal projection onto the space \( \text{ran}(V_L) \). \( \Omega \) is a standard Gaussian matrix, and \( I \) is the identity matrix. Theorem 1 shows that \( G \) is a Gaussian matrix orthogonal to \( \text{ran}(V_L) \).

**Theorem 1.** Assuming that \( V_L \) is an \( n \times t \) non-empty matrix with orthonormal columns, then

1) \( G \) is orthogonal to \( V_L \); and
2) elements in \( G \) are normally distributed.

**Proof.** 1) Since \( V_L \) is an \( n \times t \) matrix with orthonormal columns, \( P_V \) can be derived as \( P_V = V_L V_L^T \). Obviously, \( V_L^T (I - P_V) V_L \) is an orthogonal projection onto the \( \text{ran}(V_L) \bot \), which is the orthogonal complement of space \( \text{ran}(V_L) \). We can denote an \( n \times (n-t) \) matrix \( \tilde{V} = (\tilde{v}_{ij}) \) as a basis of the space \( \text{ran}(V_L) \bot \) and then \( I - P_V = \tilde{V} \tilde{V}^T \). Then, each element \( g_{ij} \) in \( G \) can be expressed as

\[
g_{ij} = \sum_{s=1}^{n-t} \left( \sum_{h=1}^{t} \tilde{v}_{hs} \tilde{v}_{sh} \right) \omega_{sj}
\]

where \( \omega_{sj} \) denotes an element of \( \Omega \) in row \( s \) of column \( j \).

Since element \( \omega_{sj} \) ’s are independent standard normal distributed variables, the characteristic function \( \Phi_{g_{ij}}(x) \) can be obtained as

\[
\Phi_{g_{ij}}(x) = \prod_{s=1}^{n-t} \prod_{h=1}^{t} e^{-\frac{1}{2}(\tilde{v}_{hs} \tilde{v}_{sh} x)^2} = \prod_{h=1}^{t} \prod_{i=1}^{n-t} e^{-\frac{1}{2}(\tilde{v}_{ih} \tilde{v}_{sh} x)^2} = \prod_{i=1}^{n-t} e^{-\frac{1}{2}\tilde{v}_{ih}^2 (\tilde{v}_{sh} x)^2}.
\]

As the columns of \( \tilde{V} \) are orthonormal such that \( (\sum_{s=1}^{n-t} \tilde{v}_{sh}^2) = 1 \), we have

\[
\Phi_{g_{ij}}(x) = e^{-\frac{1}{2}\tilde{v}_{ih}^2 x^2}.
\]

Since the characteristic function uniquely determines the probability distribution of a random variable \([23]\), it suffices to show that \( g_{ij} \) is normally distributed with expected value zero and variance \( \delta_{ij}^2 = \sum_{h=1}^{t} \tilde{v}_{ih}^2 \), i.e., \( g_{ij} \sim N(0, \delta_{ij}^2) \).

To avoid resampling of the space \( \text{ran}(V_L) \), the product of matrix \( AG \) in R\text{SV}D focuses on revealing the dominant information from the space \( \text{ran}(V_L) \bot \) orthogonal to \( \text{ran}(V_L) \).

Since the number of dominant singular values is unknown in advance, R\text{SV}D generates a series of Gaussian matrices \( G_1, G_2, \ldots \) to iteratively explore the orthogonal subspace of the obtained low-rank approximation until a satisfactory low rank approximation is obtained.

In the situation when the singular spectrum of matrix \( A \) decays slowly, Gaussian matrix can be applied to a power multiplication of matrix \( A \) to improve the approximation accuracy. This is referred to as the power scheme suggested in \([14]\). In R\text{SV}D, the power scheme with Gaussian matrices \( G_i \) is applied to \((A(I - P_V)A^T)^q A\) where \( q \) is a power scalar. This power scheme is able to refine the sampled space \( V_L \) orthogonal to space \( \text{ran}(V_L) \). Algorithm 3 shows the R\text{SV}D algorithm with power scheme.

**Algorithm 3: Rank Revealing Randomized SVD Algorithm with Power Scheme**

**Input:** \( A \in \mathbb{R}^{m \times n} \), sampling size \( t \in \mathbb{N} \) per iteration, oversampling number \( p \in \mathbb{N} \), power number \( q \in \mathbb{N} \), maximum number of iterations \( max_t \in \mathbb{N} \), and energy threshold \( \tau \in \mathbb{R} \).

**Output:** Low rank approximation \( U_L \in \mathbb{R}^{m \times k'} \), \( \Sigma_L \in \mathbb{R}^{k' \times k'} \), \( V_L \in \mathbb{R}^{n \times k'} \), and estimated rank \( k' \).

// initialization

Construct an \( n \times (t + p) \) standard Gaussian matrix \( \Omega \)

\[
\Omega = \mathcal{O} \quad \text{and} \quad V_L = \emptyset, U_L = \emptyset, \Sigma_L = \emptyset \quad k' = 0
\]

for \( i = 0: \text{maxit} \)

\[
y_i = A G_i \quad Q_i = q r(Y_i, 0)
\]

for \( j = 1: q \)

\[
y_i = A^T Q_i \quad y_i = y_i - V_L (V_L^T y_i) \quad Q_i = q r(Y_i, 0) \quad Y_i = A Q_i \quad Y_i = y_i - V_L (V_L^T y_i) \quad Q_i = q r(Y_i, 0)
\]

end

\[
B_i = Q_i^T A \quad \left[ U_{B_i}, \Sigma_{B_i}, V_{B_i} \right] = \text{svd}(B_i, 0) \quad U_{B_i} = Q_i U_{B_i} \quad V_{B_i} = q r(V_{B_i} - V_L (V_L^T V_{B_i}), 0) \quad \text{// orthogonalization process}
\]

\[
U_L \leftarrow [U_L, U_{B_i}(:, 1: t)] \quad \Sigma_L \leftarrow \begin{bmatrix} \Sigma_L & 0 \\ 0 & \Sigma_{B_i}(1: t, 1: t) \end{bmatrix} \quad V_L \leftarrow [V_L, V_{B_i}(:, 1: t)]
\]

for \( j = 1: t \)

\[
k' = i \times t + j \quad \tilde{\varphi}_{k'} = \frac{\psi_{k'}^2}{||A||^2} \quad \text{// estimate energy percentage}
\]

if \( \tilde{\varphi}_{k'} \geq \tau \), then stop;

end

\[
G_{i+1} = G_i - V_L (V_L^T G_i) \quad \text{// update Gaussian matrix}
\]
end

$[\Sigma_L, \text{Idx}] = \text{sort}(\Sigma_L, \text{`descend'})$; // sort the approximate singular values

$V_L = V_L (:, \text{Idx})$;

$U_L = U_L (:, \text{Idx})$;

\[ V_L = [V_1, V_2, ..., V_t] \] denote a matrix containing the approximate right singular vectors obtained in R^2SVD after the $i$th iteration step. Then, the singular vectors in $V_{i+1}$ must be orthogonal to $V_L$. However, the inherent numerical errors may cause loss of orthogonality between $V_{i+1}$ and $V_L$. To ensure the orthogonality property, we generate $V_{i+1}$ by employing an orthogonalization process to remove the components of $V_{Bj}$ that are not orthogonal to the previous right singular vectors in $V_L$ such that $V_{i+1} = qr(V_{Bj} - V_L(V_L^T V_{Bj}), 0)$.

Proposition 2 indicates that the resulting matrix $V_{i+1}$ generated at the $(i+1)$th iteration step in R^2SVD is orthogonal to $V_L$.

**Proposition 2.** $V_L^T V_{i+1} = 0$ holds.

**Proof.** Given $Z_{i+1} = V_{Bj} - V_L(V_L^T V_{Bj})$, we can get $V_L^T Z_{i+1} = V_L^T (V_{Bj} - V_L(V_L^T V_{Bj})) = 0$.

Since $V_{i+1}$ is a basis of $\text{ran}(Z_{i+1})$, $V_L^T V_{i+1} = 0$ holds.

Based on Proposition 2, the orthogonality property of the resulting left singular vectors $U_L$ is proved in Proposition 3.

**Proposition 3.** $U_L^T U_{i+1} = 0$ holds.

**Proof.** Denoting the QR decomposition of $Y_{i+1}$ by $Y_{i+1} = Q_{i+1} R_{i+1}$. We can have

\[
Q_{j+1}^T Q_{j+1} = Q_j^T Y_{i+1} R_{i+1}^{-1}
\]

\[
= Q_j^T A G_{i+1} R_{i+1}^{-1}
\]

\[
= Q_j^T A (I - P_{V_j}) \Omega R_{i+1}^{-1}
\]

\[
= B_j (I - P_{V_j}) \Omega R_{i+1}^{-1}
\]

\[
= U_{Bj}^T \Sigma_b V_{Bj}^T (I - P_{V_j}) \Omega R_{i+1}^{-1}
\]

where $V_L = [V_1, V_2, ..., V_t]$. Let $V^- = [V_1, V_2, ..., V_{j-1}]$ and $V^+ = [V_{j+1}, ..., V_t]$ for $j \leq i$. According to Proposition 2 that the columns in $V_L$ are orthogonal to each other, $(I - P_{V_j})$ can be expressed as $(I - P_{V_j}) = (I - P_{V^-}) (I - P_{V^+})$.

Since $V_j = qr(V_{Bj} - V^- (V^-^T V_{Bj}), 0)$, it follows that

\[(I - P_{V^-}) V_{Bj} = V_j R_j.
\]

Therefore,

\[
V_{Bj}^T (I - P_{V^-}) = V_{Bj}^T (I - P_{V^-}) (I - P_{V^+})
\]

\[= R_j^T V_j^T (I - P_{V_j}) (I - P_{V^+})
\]

\[= 0.
\]

and thus $Q_{j+1}^T = 0$, for $j \leq i$. In conclusion,

\[
U_L^T U_{i+1} = [U_1, U_2, ..., U_t]^T U_{i+1}
\]

The orthogonalization process requires $O((2ti + 1)(t + p)n)$ operations to ensure the orthogonality properties of singular vectors obtained in the previous iterations. Moreover, by taking advantage of the orthogonality between $V_{i+1}$ and $V_L$, the next Gaussian matrices $G_{i+1}$ can be fast generated using the following short recursive formula,

\[
G_{i+1} = \left( I - \sum_{j=1}^{i} P_{V_j} \right) \Omega = G_i - P_{V_j} G_i,
\]

where $P_{V_j}$ is the orthogonal projection onto the space spanned by $V_j$, such that $P_{V_j} = V_j V_j^T$, and $\Omega$ is a standard Gaussian matrix. Since $G_{i+1}$ is generated directly from $G_i$, the orthogonal Gaussian sampling takes only $O((2t + 1)(t + p)n)$ operations.

**C. Energy Estimation and Stopping Criteria**

The incremental low-rank approximation buildup process in R^2SVD will be terminated when sufficient percentage of energy of $A$ is secured. The energy percentage threshold $\tau$ is typically specified by the applications, which often ranges from 80% to 99%.

Let $U_L = [U_1, U_2, ..., U_t]$ denote a matrix of the approximate left singular vectors. The actual energy percentage of the low-rank approximation obtained at the $i$th iteration step can be evaluated based on

\[
\varphi = \frac{\|U_L U_L^T A\|_F^2}{\|A\|_F^2}.
\]

However, to avoid costly calculation of $\|U_L U_L^T A\|_F^2$ at each iteration, in this paper, we adopt the following measure $\hat{\varphi}_k$, to quickly estimate the energy percentage of the obtained low-rank approximation. Here

\[
\hat{\varphi}_k = \sum_{i=1}^{k} \sigma_i'^2 / \|A\|_F^2,
\]

where $\sigma_i'$ denotes the $i$th approximate singular value in R^2SVD. Proposition 4 shows that the estimated energy $\hat{\varphi}_k$ is equivalent to the actual energy $\varphi$. Therefore, it guarantees that the low-rank approximation obtained by R^2SVD satisfies the accuracy requirement of the applications.

**Proposition 4.** $\hat{\varphi}_k = \varphi$

**Proof.** Since the columns in $U_L$ are orthogonal, we have

\[
\|U_L U_L^T A\|_F^2 = \sum_{j=1}^{i} \|U_j U_j^T A\|_F^2
\]

\[
= \sum_{j=1}^{i} \|U_j^T A\|_F^2
\]
\[
\begin{align*}
\sum_{j=1}^{l} \left\| U_{B_j}^T Q_j^T A \right\|^2_F &= \sum_{j=1}^{l} \left\| U_{B_j}^T U_{B_j} \Sigma_{B_j} V_{B_j}^T \right\|^2_F \\
&= \sum_{j=1}^{l} \left\| \Sigma_{B_j} \right\|^2_F \\
&= \sum_{i=1}^{k_r} \sigma_i'^2
\end{align*}
\]

Hence,
\[
\tilde{\phi}_{k_r} = \frac{\sum_{i=1}^{k_r} \sigma_i'^2}{\| A \|^2_F} = \phi' = \frac{\| U_L U_L^T A \|^2_F}{\| A \|^2_F}.
\]

It is important to note the approximate singular values \( \sigma_i' \)'s are available during the calculation of SVD on \( B_1 \), where \( B_1 = Q_1^T A \). Therefore, the energy percentage can be evaluated at (almost) no cost.

D. Complexity Analysis

As discussed above, at each iteration, \( R^3 \text{SVD} \) carries out orthogonal Gaussian sampling to compute a new \( t \)-rank approximation of the leftover subspace orthogonal to the low-rank approximation obtained so far. Suppose that \( R^3 \text{SVD} \) uses \( s \) iterations to achieve a satisfactory low-rank approximation with \( k' \approx ts \) as the result rank and assume that the computational cost of matrix-block matrix multiplications dominates those of QR and SVD decompositions on the block matrices. The computational cost of \( R^3 \text{SVD} \) is
\[
O(2(k' + sp)T_{\text{mult}}).
\]

In the case that matrix \( A \) is sparse and both \( m \) and \( n \) are large, we can obtain the time complexity with simpler terms. In particular, as \( T_{\text{mult}} \approx cm \), where \( c \) is sparsity ratio, the time complexity of \( R^3 \text{SVD} \) can be expressed as \( O(k'^2 \min(m, n)) \).

In additional to the storage of matrix \( A \), the major computations of \( R^3 \text{SVD} \) are carried out on a series of block matrices with \((t + p)\) columns or rows. Therefore, where \( t < k \), \( R^3 \text{SVD} \) takes a constant space complexity of \( O(2(m + n)(t + p)) \), which is lower than that of \( R^3 \text{SVD} \) and \( A \) multiplication in restarting \( R^3 \text{SVD} \). It is also important to notice that the algorithms based on the strategy of estimating \( k \) before \( R^3 \text{SVD} \), including Autorank II, Adaptive Randomized Range Finder, and Randomized Blocked Algorithm require more computational time as well as the memory than restarting \( R^3 \text{SVD} \). The Adaptive Randomized Range Finder uses a probabilistic error estimator, which leads to a highly overestimated rank \((641)\).

### IV. NUMERICAL RESULTS

In this section, we use several numerical examples to demonstrate the effectiveness of \( R^3 \text{SVD} \) for low-rank approximation in image compression and matrix completion.

A. Comparisons with RSVD

We compare the performance of \( R^3 \text{SVD} \), full SVD, Autorank II, restarting RSVD, Adaptive Randomized Range Finder algorithm, and Randomized Blocked algorithm in constructing low rank approximations to compress a 7671 \( \times \) 7680 NASA synthesis image chosen from the Mars Exploration Rover mission [24]. The energy percentage threshold \( \tau \) is set to 99%.

Both \( R^3 \text{SVD} \) and RSVD start with an initial guess \( t = 15 \) of the target rank and \( p = 5 \) extra oversampling vectors. The power scheme is not applied, such that \( q = 0 \). In restarting RSVD, as the approximate singular values are available during each RSVD trial, the energy estimation introduced in Section III is used. If the guessed rank turns out to be insufficient to obtain a low rank approximation with satisfactory accuracy, the restarting approach repeats the RSVD computation with a gradually increasing rank \( \Delta t = 15 \). Table I compares the computational performance of full SVD, Autorank II algorithm, Adaptive Randomized Range Finder algorithm, Randomized Blocked Algorithm, \( R^3 \text{SVD} \), and restarting RSVD in terms of rank, computational time, maximum memory usage, and energy percentage of the obtained low rank approximation. The optimal low-rank approximation (rank 46) to obtain 99% energy of the original matrix can be obtained by carrying out full SVD, which takes over 760 seconds on a Dell Precision-M6500 laptop (Intel CoreTM i5CPU, 2.67GHz, 4GBRAM). Restarting RSVD reduces the computational time to 13.77 seconds with a low-rank approximation of rank 79. Compared to restarting RSVD, \( R^3 \text{SVD} \) further reduces both the computational time to 4.54 (32.97%) and rank to 62 (78.48%). This is because \( R^3 \text{SVD} \) carries out important sampling based on the approximate right singular vectors in \( V_L \), which is computed by multiplying \( A \) twice per iteration. The power iteration allows more precise estimation of the dominant actions than a single iteration of \( A \) multiplication in restarting RSVD. It is also important to notice that the algorithms based on the strategy of estimating \( k \) before \( R^3 \text{SVD} \), including Autorank II, Adaptive Randomized Range Finder, and Randomized Blocked Algorithm require more computational time as well as the memory than restarting RSVD and \( R^3 \text{SVD} \). The Adaptive Randomized Range Finder uses a probabilistic error estimator, which leads to a highly overestimated rank (641).

| TABLE I: PERFORMANCE COMPARISON OF R³SVD, FULL SVD, AUTORANK II, RESTARTING RSVD, ADAPTIVE RANDOMIZED RANGE FINDER, AND THE RANDOMIZED BLOCKED ALGORITHM |
|-----------------|-------------|-------------------|-----------------|
| Rank | Computational Time (second) | Maximum Memory Usage (bytes) | Energy Percentage Achieved |
|-----------------|-------------|-------------------|-----------------|
| Full SVD | 46 | 760.55 | 1.41 \( \times \) 10\(^7\) | 99.024% |
| Autorank II Algorithm [16] | 105 | 32.66 | 2.03 \( \times \) 10\(^7\) | 99.184% |
| Adaptive Randomized Range Finder Algorithm [14] | 641 | 55.65 | 1.19 \( \times \) 10\(^8\) | 99.999% |
| Randomized Blocked Algorithm [17] | 105 | 20.90 | 4.80 \( \times \) 10\(^8\) | 99.184% |
| Restarting RSVD | 79 | 13.77 | 1.60 \( \times \) 10\(^7\) | 99.000% |
| R³SVD | 62 | 4.54 | 4.91 \( \times \) 10\(^8\) | 99.006% |
Another important advantage of R$^3$SVD is that R$^3$SVD maintains constant memory usage in the computational process. Fig. 1 shows the memory usages in R$^3$SVD and restarting RSVD as the guessed rank gradually increases. One can find that for a larger guessed rank, restating RSVD requires more memory because of decomposing block matrices with more columns or rows. In contrast, the decomposition operations in R$^3$SVD are carried out on block matrices with fixed $(t + p)$ number of columns or rows. As a result, the memory usage does not increase as the guessed rank increases during R$^3$SVD. As shown in Table I, the memory usage in R$^3$SVD is significantly less than those of the other algorithms.

Fig. 2 presents the compressed images in R$^3$SVD, where Fig. 2(a) is the original image and Figs. 2(b) to 2(d) illustrate the adaptive compressed images with increasing ranks. With the resulting 55-rank low-rank approximation, a compressed image with 99.18% energy of the original image is obtained.

One advantage of R$^3$SVD is that the computational process can be tailored into a series of sampling tasks that can fit into the available memory in a computer via adjusting the sampling size parameter $t$. Fig. 3 compares the memory usage in R$^3$SVD with $t = 20, 15, 10, and 5$. One can find that a smaller sample size in R$^3$SVD yields proportionally less consumption of memory but without significantly affecting the rank in the obtained low-rank approximation. The resulting ranks are 63, 62, 61, and 60, respectively. Therefore, calculating sampling size parameter $t$ according to the available memory in a computer can lead to the best computational performance of R$^3$SVD.

B. Application in Matrix Completion

R$^3$SVD can be effectively applied to applications of matrix completion, whose goal is to recover the missing (unknown) entries of an incomplete matrix [6, 25, 26, 27]. Matrix completion algorithms have been widely used in many applications, including machine learning [28, 29], computer vision [30], and image/video processing [31]. Low rank matrix approximation is a core component in many matrix completion algorithms. The computational efficiency of constructing high-quality low rank approximation is essential
to the performance of these matrix completion algorithms.

We modify the Singular Value Thresholding (SVT) algorithm [6] by replacing the underlying Lanczos algorithm with our R$^3$SVD algorithm to compute dominant singular values and vectors at each SVT iteration. Fig. 4 shows a $1024 \times 1024$ aerial image chosen from the USC-SIPI Image Database [32] as well as 10% of the pixels uniformly sampled from the image (the background is set to grey to highlight these samples). As shown in Table II, the modified SVT algorithm obtains the completed image with similar recovery error and rank as that of the original SVT. In comparison, replacing Lanczos algorithm with R$^3$SVD significantly reduces the overall computational time in SVT. This is due to the fact that the Lanczos bidiagonalization algorithm with partial reorthogonalization used in original SVT has computational complexity of $O(\min(m,n)^2k)$ [33, 34] while R$^3$SVD offers a faster way with computational complexity of $O(\min(m,n)k^2)$ in contrast. As a result, the modified SVT method using R$^3$SVD achieves about 1.69 times speedup over the original SVT method using Lanczos algorithm.

V. CONCLUSIONS

We present an R$^3$SVD algorithm based on orthogonal sampling to gradually build up a low rank approximation of a given matrix to satisfy application specific accuracy. A random matrix based on the orthogonal complement operator is derived to enable R$^3$SVD to concentrate on sampling the orthogonal subspace of the existing low-rank approximation. Compared to the algorithms based on preprocessing strategy by estimating the appropriate rank $k$ before RSVD, R$^3$SVD is more efficient in terms of both computation time and memory while providing a better rank estimation. Moreover, as a memory-aware algorithm, R$^3$SVD is particular favorable for many real-life applications running in limited computer memory. The effectiveness of R$^3$SVD has been demonstrated in numerical applications including image compression and matrix completion.

The importance sampling approach proposed in this paper can also be used for other randomized algorithms by sampling the most important subspaces toward the solutions. The R$^3$SVD algorithm described in this paper is based on Gaussian sampling [14, 15], which can also be extended to other randomized SVD strategies such as column or row sampling [10, 11].

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**TABLE II**

THE COMPLETED IMAGES USING THE ORIGINAL SVT ALGORITHM AND THE MODIFIED SVT ALGORITHM USING R$^3$SVD

| Completed Image $X$ | Elapsed Time (seconds) | Rank | $\frac{\|A - X\|^2_F}{\|A\|^2_F}$ |
|--------------------|------------------------|------|----------------------------------|
| Original SVT       | 3457.122               | 190  | $6.772 \times 10^{-2}$           |
| Modified SVT using R$^3$SVD | 2045.295               | 189  | $6.772 \times 10^{-2}$           |

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