Manifold Topology, Observables and Gauge Group

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Abstract

The relation between manifold topology, observables and gauge group is clarified on the basis of the classification of the representations of the algebra of observables associated to positions and displacements on the manifold. The guiding, physically motivated, principles are i) locality, i.e. the generating role of the algebras localized in small, topological trivial, regions, ii) diffeomorphism covariance, which guarantees the intrinsic character of the analysis, iii) the exclusion of additional local degrees of freedom with respect to the Schroedinger representation. The locally normal representations of the resulting observable algebra are classified by unitary representations of the fundamental group of the manifold, which actually generate an observable, “topological”, subalgebra. The result is confronted with the standard approach based on the introduction of the universal covering $\tilde{M}$ of $M$ and on the decomposition of $L^2(\tilde{M})$ according to the spectrum of the fundamental group, which plays the role of a gauge group. It is shown that in this way one obtains all the representations of the observables iff the fundamental group is amenable. The implications on the observability of the Permutation Group in Particle Statistics are discussed.

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1 Introduction

For the analysis of quantum mechanical systems with a non-trivial manifold $\mathcal{M}$ as configuration space, the rôle of the topology of $\mathcal{M}$ has been investigated by many authors [1], [2], [3], [4], [5], [6], [7]. The involved fundamental issues include the control of the representations of the Diffeomorphisms Group of the manifold [8], the identification of the observable algebras and the classification of their representations. Similar problems arise for the formulation of Local Quantum (Field) Theory on space-time manifolds [9].

Historically, the main approaches to Quantum Mechanics (QM) on manifolds, the Functional Integral formulation [10, 4] and Geometric Quantization (GQ) of symplectic manifolds [11, 12, 13, 14], produced substantially equivalent results. Apart from the possible presence of an “internal” space, such methods lead in fact to Schroedinger QM in $L^2(\tilde{\mathcal{M}})$, $\tilde{\mathcal{M}}$ the universal covering space of $\mathcal{M}$, with $\pi_1(\mathcal{M})$ playing the rôle of a gauge group. The centre of the representation of the gauge group in $L^2(\tilde{\mathcal{M}})$ is observable and reduces $L^2(\tilde{\mathcal{M}})$ to inequivalent Quantum Mechanical “sectors”, with a multiplicity given by the action of the gauge group.

The result can be seen as a generalization of Dirac treatment of Identical Particles [15], which identifies the observable algebra through the invariance under the Permutation Group, which therefore plays the role of a gauge group, and classifies its representations by those of the gauge group. In three space dimensions, the relation with Dirac treatment of identical particles is indeed very close, since the Permutation Group coincides with the fundamental group of the configuration manifold of identical particles [16].

In view of their implications on basic issues, however, the above treatments leave open substantial questions. In fact, they rely on “quantization” prescriptions, associating a quantum system to a “corresponding” classical one, without an independent examination the physical basis and interpretation of the adopted approach. The situation is therefore quite different with respect to the case of QM in $\mathbb{R}^d$, where the approach based on Weyl algebras provides a clear and unique result, through the Von Neumann uniqueness theorem.

In particular, the question arises whether an observable role is associated to the entire group $\pi_1(\mathcal{M})$, or only to its center, as indicated by Dirac analysis. To this purpose, clearly, the identification of the QM observables on the basis of physical principles becomes the decisive issue, also because no comparable phenomena appear for classical particles.

The need of a description in terms of observables and states has been emphasized by Landsman [6, 16]. His approach is in fact based on the construction of an observable algebra, which is obtained by “Deformation Quantization” on
\( \mathcal{M} \), taking a quotient with respect to the action (on \( \mathcal{M} \)) of \( \pi_1(M) \), which therefore plays the role of a gauge group. In Landsman’s analysis, the same group also appears as a unitary group in the observable algebra, and in such a role it provides the classification of its representations. Thus, on one side Landsman’s treatment follows the approach and confirms the conclusions of the previous literature: the fundamental group of the manifold plays the role of a gauge group, which identifies the observable algebra starting from \( \mathcal{M} \); on the other, Landsman’s classification of the irreducible representations of the observable algebra does not involve the gauge group, but rather an isomorphic observable group. Clearly, two questions arise:

First, since \( \mathcal{M} \) is taken as a starting point in all the above approaches, one may ask whether the above results depend on such a choice, which does not seem to have a direct physical interpretation. In any case, as argued above, a direct foundation on physical principles is lacking.

Second, if \( \mathcal{M} \) needs not to appear in the formulation, the classification of the representations in terms of \( \pi_1(M) \) as a gauge group is in question. Moreover, even if the different formulations can be compared, the classification in terms of \( \pi_1(M) \) as a gauge group and as an observable group are not a priori related and to which extent they may coincide is an open problem.

An answer to the first question has been given in ref. \[7\], which avoids the introduction of \( \mathcal{M} \) and directly identifies the observable algebra as generated by coordinates and momentum variables associated to vector fields on \( M \). The construction only employs the manifold \( M \), the identification of the observables does not make any reference to \( \mathcal{M} \) and no action of \( \pi_1(M) \) as a gauge group appears. The irreducible representations of the observable algebra are then classified by a representations of \( \pi_1(M) \), which appears within each representation space. By irreducibility of the observables in each sector, such a representation of \( \pi_1(M) \) is automatically given by strong limits of observables.

The purpose of this note is twofold: first, to construct the observable algebra on the basis of clear physical principles, holding independently of the topology of the manifold. The identification of the observable algebra will be performed purely in terms of a collection of local algebras, localized in “small”, topologically trivial, regions. This also allows for a purely local analysis of the degrees of freedom which may appear, as a consequence of diffeomorphism invariance, in addition to Schroedinger QM. The local and global effects are therefore separated completely, and the role of the topology precisely appears when the local (essentially Schroedinger) descriptions are glued together.

The second aim is to confront the resulting representations, classified by an observable \( \pi_1(M) \), with those obtained by the Dirac gauge group method.
Outline of the strategy and main results

The Observable Algebra for the description of a quantum particle on a $d$-dimensional manifold $\mathcal{M}$ is constructed according to the following physically motivated principles and with the following results:

1. **The local structure**
   a) *Locality*: The observable algebra is generated by a collection of local algebras associated to (topologically trivial) “small” regions $\mathcal{O}$, technically defined as regions homeomorphic to a subdisk of an open disk (homeomorphism to an open disk would not be enough to exclude, e.g., the entire space $\mathbb{R}^d$).
   
   b) *Diffeomorphism Covariance*: The identification of the observables is exclusively linked to the manifold structure. Locally, it amounts to independence of coordinates, as well as of a metric, special transformation groups etc.; globally, it provides the intrinsic geometric links between the observable algebras localized in different regions, just as Poincaré covariance does in relativistic quantum field theory.

   c) *Positions and "local Movements" on the manifold*: The local algebras are generated by position and “local trasportation” variables, the latter identified with the unitary groups $G(\mathcal{O})$ describing displacements along (all, localized) vector fields with support in $\mathcal{O}$.

   The local algebras are therefore assumed to be generated by $C^0$ functions $\alpha(x)$ with support in $\mathcal{O}$, as position variables, and by unitaries $U(g)$ representing the diffeomorphisms groups $G(\mathcal{O})$. They are identified with the Crossed Products $\Pi(\mathcal{O}) \equiv C^0(\mathcal{O}) \times G(\mathcal{O})$, defined by the relation $U(g) \alpha(x) U(g)^* = \alpha(g^{-1}x)$, representing the action of diffeomorphisms $g \in G(\mathcal{O})$ on functions on $\mathcal{O}$.

   The algebras $\Pi(\mathcal{O})$ generate $\Pi(\mathcal{M}) \equiv C^0(\mathcal{M}) \times G_L(\mathcal{M})$, as global observable algebra, with $G_L(\mathcal{M})$ the group generated by all the $G(\mathcal{O})$ (see Section 2.3). No independent global variables are therefore introduced and the topology of $\mathcal{M}$ only affects the result of algebraic operations on local observables.

2. **The Local One (Spinless) Particle Interpretation**

   As a consequence of diffeomorphism invariance, the local algebras admit many inequivalent representations (also describing many particle systems); hence, for a one-particle description, a selection is needed to avoid, at the local level, the presence of additional degrees of freedom, with respect to those appearing in Schroedinger QM. Spin and internal degrees of freedom, excluded by the above requirement, may be added by an explicit extension of the observable algebra.
The representation of the local algebras can be taken in spaces of the form
\[ \mathcal{H}(O) = L^2(O, d\mu) \times K(O), \]
(see Sect.2 for the exact characterization), with the association of unitary "cocycles" \( V_g(x) \) in \( K(O) \), \( g \in \mathcal{G}(O), \ x \in O \), to local displacements.

Up to unitary equivalence, such cocycles are characterized by those representing the subgroup \( \mathcal{G}(O, x) \subset \mathcal{G}(O) \) which leaves a point \( x \in O \) stable; the unitary class of their representations is independent from the point and the region.

Such cocycles, describing modifications of local displacements by operations which leave the point invariant, have the physical interpretation of describing "internal degrees of freedom". Their triviality, which excludes such degrees of freedom, is required by a Local One Particle Interpretation and turns out to be equivalent to the Schrödinger representation of the local algebras \( \Pi(O) \), up to a multiplicity (Locally Schrödinger condition).

The exclusion of internal degrees of freedom is actually a general question for representations of observable algebra which include momenta describing "movements" besides those associated to translations. In fact, it appears already in the case of one spinless quantum particle in \( \mathbb{R}^d \) with momenta indexed by the generators of the euclidean group, where non-trivial cocycles are associated to rotations and describe spin; more generally, the same problem appears for the representation of the stability group of a point in the case of the algebra constructed from a group \( G \) and associated to the manifold \( G/H \), \( H \) a subgroup of \( G \) [18].

As a result, on the basis of the above principles, we obtain a complete identification of the local observable algebras and of their representations; they simply describe Schrödinger QM, up to a multiplicity, in diffeomorphism covariant variables.

The control of QM of a particle on a manifold is then reduced to the classification of the representations of the global algebra \( \Pi(M) \equiv C^0(M) \times \mathcal{G}_L(M) \) which reduce to a multiple of the Schrödinger representation when restricted to \( \Pi(O) \), for all \( O \).

Equivalently, in the spirit of Local Quantum Theory [19], the local observable algebras can be identified with the weak closures \( \mathcal{A}(O) \) of \( \Pi(O) \) in (any) LS representations of \( \Pi(M) \). The global observable algebra for one particle on \( M \) is then identified with the algebra \( \mathcal{A}(M) \) generated by them in the sum of such representations. LS representations of \( \Pi(M) \) coincide with locally normal representations of \( \mathcal{A}(M) \).
3. **Classification of Locally Schroedinger (LS) representations**

By the above results, the classification of the LS representations of $\Pi(M)$ is reduced to the analysis of the cocycles $V_{g_n \ldots g_1}(x)$ associated to products of localized diffeomorphisms $g_i \in \mathcal{G}(\mathcal{O}_1)$. They are shown to depend only on the topological equivalence class of the path from $x$ to $g_n \ldots g_1 x$ given by $g_n \ldots g_1$, and to define a unitary representation of the path groupoid, which is classified, up to unitary equivalence, by its restriction to closed paths with (any) fixed base point, i.e., by a representation of $\pi_1(M)$.

All unitary representations of $\pi_1(M)$ are shown to define admissible cocycles, so that the correspondence between LS representations of $\Pi(M)$ and unitary representations of $\pi_1(M)$ is one to one. If $M$ is simply connected, the analogue of Von Neumann uniqueness theorem holds: the (locally normal) representation of $\mathcal{A}(M)$ is unique (and Schroedinger), up to a multiplicity.

The above representations of $\pi_1(M)$, multiplied by projections over small regions $\mathcal{O}$, are shown to describe observables corresponding to the transport of the particle, starting in the region $\mathcal{O}$, along the corresponding loops. Moreover, for compact $M$, $\mathcal{A}(M)$ contains a subalgebra isomorphic to the group algebra of $\pi_1(M)$. The classification of the representations is therefore always given by an observable representation of $\pi_1(M)$.

4. **The gauge and observable realizations of $\pi_1(M)$**

A comparison with the Dirac gauge group approach [15] is provided by the analysis of the Schroedinger representation of $\mathcal{A}(M)$ in $L^2(\hat{M})$. Identifying $\hat{M}$ with a space of pairs $x \in M, \gamma \in \pi_1(M)$, one has

$$L^2(\hat{M}) \sim L^2(M) \times L^2(\pi_1(M)).$$

The usual “gauge” action of $\pi_1(M)$ in $\hat{M}$ is given by

$$\psi(x, \gamma) \rightarrow \psi(x, \gamma \circ \delta),$$

i.e., by its right regular representation in $L^2(\pi_1(M))$. It clearly commutes with the position observables and also with the unitaries $U(g)$, which act as in the Schroedinger representation in $L^2(M)$, combined with the left regular representation of $\pi_1(M)$ in $L^2(\pi_1(M))$ (see eq. 3.7).

It follows that, in the representation of $\mathcal{A}(M)$ in $L^2(\hat{M})$, two commuting unitary representations of $\pi_1(M)$ appear, respectively as a gauge group and as an observable group. Being given by (a multiple of) the left and right regular representations, the two representations are unitarily equivalent; they generate Von Neumann algebras which are the commutant one of the other in $L^2(\pi_1(M))$, so that their centres coincide and give rise to the same reduction.
of $L^2(\tilde{M})$. Therefore, the classifications of the representations of $\mathcal{A}(M)$ in $L^2(\tilde{M})$ given by the observable and by the gauge group coincide.

As a result, only the completeness of the Dirac approach is in question, i.e. whether all the (locally normal) irreducible representations of $\mathcal{A}(M)$ appear in the reduction of $L^2(\tilde{M})$. Equivalently, whether all the irreducible representations of $\pi_1(M)$ appear in the (possibly integral) reduction of its regular representation. The answer to this question is known and rather simple: this happens if and only if the group is amenable.

If $\pi_1(M)$ is not amenable, the role of its regular representation and of $L^2(\tilde{M})$ is completely different, since even the identity representation of $\pi_1(M)$, corresponding to the ordinary Schroedinger representation of the observables, is not present in the reduction. Proposition 3.3 below shows that in this case all the representations of $\mathcal{A}(M)$ can still be obtained in suitable Hilbert spaces of functions on $\tilde{M}$ (with non $L^2$ scalar products), but the role of the gauge group is in general lost.

5. Implications for Identical Particles

Particularly significant is the case of $N$ identical particles, which can be described as the quantum system associated to the $N$-particle manifold $M_S$, defined by identifying configurations obtained by permuting the particles and excluding the set $\Delta$ of coincident points [17].

In the case of $N$ particles in Euclidean space of dimension greater than two, the fundamental group of $M_S$ is the permutation group $S_N$ and $M_S$ may be identified with $\mathbb{R}^{Nd} \setminus \Delta$. From the above results it follows that

i) since $S_N$ is finite (and therefore amenable), the Dirac representation of the observable algebra in $L^2(\mathbb{R}^{Nd})$ ($\Delta$ being here irrelevant) contains all its irreducible representations, classified by the representations of $S_N$ in $L^2(\mathbb{R}^{Nd})$ as a gauge group.

ii) an observable unitary representations of the Permutation Group is present in $L^2(\mathbb{R}^{Nd})$, unitarily equivalent to the gauge representation. In each reduction space, the gauge and observable representations are complex conjugates.

The observable representation acts (see Sect. 3) by physical operations which shift the position of each particle (as a gauge invariant variable, independent of particle labels), along paths which interchange the position of the particles. Apart from the abelian case, $N = 2$, the (gauge invariant) operators implementing such actions have nothing to do with gauge transformations, which in fact act on the particle labels and are not observable.

Clearly, the presence of the observable representation of $S_N$ allows for a direct physical (and topological) reinterpretation of the Dirac classification, in terms of unitaries describing permutations as physical operations.
2 The observable algebra and its Locally Schrödinger representations

2.1 The algebra generated by local coordinates and local displacements

In the standard case of a quantum mechanical system with $\mathbb{R}^d$ as configuration space, the observables are usually generated by the Cartesian coordinates in $\mathbb{R}^d$ and the associated momenta; Schrödinger Quantum Mechanics then arises as the unique (regular) representation of the Weyl algebra, generated by the exponentials of such variables. On the other hand, when the configurations are described by a $C^\infty$ connected ($d$-dimensional) manifold $\mathcal{M}$ no global coordinate system is available and, even locally, there are no distinguished coordinate systems, nor intrinsic finite dimensional transformation groups.

Following the general philosophy of Local Quantum Theory [19], we start by considering “local” open regions $\mathcal{O}$, topologically trivial and topologically localizable, in the sense that they are proper subsets of larger similar regions. In the following, $\mathcal{M}$ will indicate any $C^\infty$ connected manifold.

**Definition 2.1** An open region $\mathcal{O} \subset \mathcal{M}$ is called small if there is a diffeomorphism of a neighbourhood $\mathcal{O}'$ of $\mathcal{O}$ taking $\mathcal{O}'$ to an open ball, $|x| < 1$, $x \in \mathbb{R}^d$, and $\mathcal{O}$ to the open ball $|x| < 1/2$.

Hereafter, $\mathcal{O}$ will always denote a small region. Clearly, each point of $\mathcal{M}$ has a small open neighbourhood $\mathcal{O}$ and a denumerable set of such regions covers $\mathcal{M}$.

Our guiding principle for the construction of the local observable algebras is to recognize as fundamental the covariance under diffeomorphisms, linking the observables to the intrinsic geometry of $\mathcal{M}$. For each $\mathcal{O}$ we introduce:

i) as “position observables” the $C^\infty$ (complex valued) functions $\alpha(x)$, $x \in \mathcal{M}$, with compact support in $\mathcal{O}$, generating, with the Sup norm and together with a common identity $1$, a diffeomorphism invariant $C^*$ algebra $C(\mathcal{O})$; then, $C(\mathcal{O})$ is isomorphic to the algebra generated by the continuous functions on $\mathcal{O}$, vanishing at its boundary, and by the constant functions.

ii) as “local generalized momenta”, the vector fields $v \in L(\mathcal{O})$, $L(\mathcal{O})$ the Lie algebra of $C^\infty$ vector fields with compact support in $\mathcal{O}$, by compactness of their support, they integrate to one parameter groups $g_{\lambda v}$, $\lambda \in \mathbb{R}$, generating diffeomorphism groups $G(\mathcal{O})$, with a common identity $e$ and $G(\mathcal{O}_1) \subset G(\mathcal{O}_2)$ for $\mathcal{O}_1 \subset \mathcal{O}_2$. In physical terms, $G(\mathcal{O})$ describes displacements localized in $\mathcal{O}$, acting on (any) configuration of the system.
The vector space of the pairs \(\{\alpha, g\}, \alpha \in C(O), g \in \mathcal{G}(O)\), with the operations \(\{\alpha_1, g_1\} \{\alpha_2, g_2\} = \{\alpha_1 \alpha_2^g, g_1 g_2\}\), \(\alpha_2^g(x) \equiv \alpha_2(g^{-1}x)\) \(\{\alpha, g\}^* = \{\alpha, g^{-1}\}\), defines the **crossed product \(C^*\) algebra** \(\Pi(O) \equiv C(O) \times \mathcal{G}(O)\).

In the construction of the crossed product, \(\mathcal{G}(O)\) is taken as a topological group with the discrete topology and \(\Pi(O)\) is generated by the finite sums \(\sum_i \alpha_i g_i, \alpha_i \in C(O), g_i \in \mathcal{G}(O)\), with norm \(\sum_i \text{Supp}_x|\alpha_i(x)|\), see, e.g., [20]. In the following, for simplicity, we adopt the notations: \(U(g) \equiv \{1, g\}, \alpha(x) \equiv \{\alpha, e\}\); then the basic crossed product algebraic relations read

\[
U(g)^{-1} = U(g)^*, \quad U(g)\alpha(x)U(g)^* = \alpha(g^{-1}x), \quad \text{Supp}\, \alpha \subset O, \quad g \in \mathcal{G}(O).
\]  

(2.1)

The local “position” algebras \(C(O)\) satisfy isotony, \(C(O_1) \subset C(O_2)\) for \(O_1 \subset O_2\) and generate, in the Sup norm on \(M\) (1 being identified with the function 1 on \(M\)), the \(C^*\) algebra \(C(M)\) of continuous functions on \(M\), if \(M\) is compact, and on its one-point compactification \(\hat{M} = M \cup \{x_\infty\}\) (the Gelfand spectrum of \(C(M)\)) otherwise. In the latter case, the diffeomorphisms of \(\mathcal{G}(O)\) extend to diffeomorphisms of \(\hat{M}\), with \(g x_\infty = x_\infty; \alpha(x_\infty) \equiv 0\) for \(\alpha(x)\) in \(C^\infty(O)\), for all \(O\).

The local groups \(\mathcal{G}(O)\) coincide with the connected component of the identity of the diffeomorphism group of \(O\) (see [3]). They obviously satisfy isotony and generate a group \(\mathcal{G}_L(M)\), uniquely associated to \(M\), defined by the formal products of their elements modulo the group relations holding in each region \(O\). Its elements are therefore strings \(g_1 \ldots g_n\), modulo the equivalence relation defined by any sequence of replacements of a substring of elements localized in some \(O\) by another string with the same product in \(\mathcal{G}(O)\). \(\mathcal{G}_L(M)\) acts on \(M\) by diffeomorphisms (depending in fact only on the equivalence class of \(g_1 \ldots g_n\), which will be denoted by the same symbol. \(\mathcal{G}_L(M)\) formalizes, in the construction of the observable algebra, the principle that **finite sequences of local operations still define physical operations and are only constrained by the validity of all local relations**.

We therefore associate to \(M\) the **crossed product \(C^*\) algebra** \(\Pi(M) \equiv C(M) \times \mathcal{G}_L(M)\), defined as above, with eqs. (2.1) satisfied for all \(\alpha \in C(M)\) and \(g \in \mathcal{G}_L(M)\). \(\Pi(M)\) is generated by the local algebras \(\Pi(O)\) and is invariant under \(\text{Diff}(M)\), the entire diffeomorphism group of \(M\), since both \(C(M)\) and \(\mathcal{G}_L(M)\) are invariant.

As we shall see, diffeomorphism invariance makes such algebras, already for regions \(O\), much richer than, e.g., Weyl algebras in \(\mathbb{R}^d\); in fact, they admit many inequivalent representations, with different physical interpretation. This happens because additional independent “momentum” variables appear, as a consequence of diffeomorphism invariance.
2.2 Regular representations and their cocycles

In order to classify the physically relevant representations of $\Pi(\mathcal{M})$, we start by characterizing, under physically motivated conditions, the representations of its local subalgebras $\Pi(\mathcal{O})$, which generate it.

First, in order to ensure the existence of local momenta, we consider representations of $\Pi(\mathcal{M})$ in separable Hilbert spaces, satisfying, for each $\mathcal{O}$, strong continuity in $\lambda$ of the local groups $U(g_{\lambda v})$, $v \in \mathcal{L}(\mathcal{O})$.

Then, the representations of $\Pi(\mathcal{M})$, as well as of $\Pi(\mathcal{O})$, are characterized by the following Proposition, which applies to a generic manifold; for economy of notation, it is stated for $\Pi(\mathcal{M})$.

**Proposition 2.2** A representation $\pi$ of $\Pi(\mathcal{M})$ in a separable Hilbert space, with $U(g_{\lambda v})$ strongly continuous in $\lambda$ (regular representation) is unitarily equivalent to one in

$$\mathcal{H}_{\pi} = L^2(\mathcal{M}, d\mu) \times K \oplus K_\infty,$$

(2.2)

$K$ and $K_\infty$ separable Hilbert spaces, $d\mu$ equivalent to the Lebesgue measure on $\mathcal{M}$ (in any system of local coordinates).

Identifying the elements of $\mathcal{H}_{\pi}$ with $L^2$ functions $\psi(x)$, $x \in \mathcal{M}$, taking values in $K$ for $x \in \mathcal{M}$ and in $K_\infty$ for $x = x_\infty$, the action of the representatives of the elements of $\Pi(\mathcal{M})$ is given by:

$$\pi(\alpha)\psi(x) = \alpha(x)\psi(x), \quad \pi(U(g)) = C_g V_g;$$

(2.3)

$$C_g \psi(x) \equiv \psi(g^{-1} x) J_g(x)^{1/2}, \quad V_g \psi(x) = V_g(x)\psi(x),$$

(2.4)

$$J_g(x) \equiv [d\mu(g^{-1} x)/d\mu(x)], \quad V_g(x) \text{ a family of unitary operators, in $K$ for $x \in \mathcal{M}$ and in $K_\infty$ for $x = x_\infty$, weakly measurable in $x$, satisfying}$$

$$C_h^{-1} V_g(x) C_h = V_g(hx),$$

(2.5)

$$V_g(hx)V_h(x) = V_{gh}(x),$$

(2.6)

a.e. in $x \in \mathcal{M}$.

Two (regular) representations $\pi_1$, $\pi_2$ of $\Pi(\mathcal{M})$ in $L^2(\mathcal{M}, d\mu) \times K_i \oplus K_{\infty,i}$, $i = 1, 2$, are unitarily equivalent iff there exists a weakly measurable family of isometric operators $S(x)$ from $K_i$ to $K_2$ for $x \in \mathcal{M}$, and from $K_{\infty,1}$ to $K_{\infty,2}$ for $x = x_\infty$, such that

$$S(gx) V_g^{(1)}(x) S(x)^{-1} = V_g^{(2)}(x).$$

(2.7)
The proof is essentially the same as in [7], Lemma 3, only strong continuity of \( U(g_\lambda v) \) for \( v \in \mathcal{L}(\mathcal{O}) \) being required; the irreducibility condition is replaced by separability of \( \mathcal{H}_\pi \). The compactification of \( \mathcal{M} \) arises as the Gelfand spectrum of \( C(\mathcal{M}) \) and gives rise in general to a representation of \( \Pi(\mathcal{M}) \), in \( K_\infty \), which assigns the null value to all functions with compact support and reduces to a unitary representation of \( \mathcal{G}_L(\mathcal{M}) \) by \( V_g(x_\infty) \).

Thus, up to unitary equivalence, the regular representations of the local algebras \( \Pi(\mathcal{O}) \) are given in \( L^2(\mathcal{O}, d\mu) \times K(\mathcal{O}) \oplus K_\infty(\mathcal{O}) \) by eqs. (2.3), (2.4), with \( \alpha \in C(\mathcal{O}), g \in \mathcal{G}(\mathcal{O}) \). Given a regular representation of \( \Pi(\mathcal{M}) \), by Proposition 2.2, the corresponding representation spaces are \( K(\mathcal{O}) = K \) for all \( \mathcal{O} \) and

\[
K_\infty(\mathcal{O}) = L^2(\mathcal{M} \setminus \mathcal{O}, d\mu) \times K \oplus K_\infty.
\]

By Proposition 2.2, the classification of the representations of \( \Pi(\mathcal{M}) \) reduces to that of the unitary operators \( V_g(x) \). For \( V_g(x) \equiv 1 \) and \( K \) one dimensional, the representation in \( L^2(\mathcal{M}, d\mu) \) defines the Schrödinger representation \( \pi_S(\Pi(\mathcal{M})) \).

Eq. (2.6) becomes a group cocycle relation if the excluded zero measure subset can be taken independent of \( g \) and \( h \). In fact, in this case, the operators \( V_g(x) \) are well defined as maps from \( g \in \mathcal{G}_L(\mathcal{M}) \) to the group \( \mathcal{U} \) of unitaries in \( K \) depending on \( x \in \mathcal{M} \setminus A, A \) of zero measure; eq. (2.6) is then the cocycle relation associated to the map

\[
g \to \varphi(g), \quad \text{Aut}(\mathcal{U}) \ni \varphi(g) : V(x) \to V(gx).
\]

**Definition 2.3** A representation \( \pi \) of \( \Pi(\mathcal{M}) \) will be called *cocycle-regular* if it is unitarily equivalent to a regular representation where eqs. (2.6) define a group cocycle relation, for almost all \( x \) in \( \mathcal{M} \).

In a cocycle-regular representation of \( \Pi(\mathcal{M}) \) the operators \( V_g(x) \) provide a representation of the stability groups

\[
\mathcal{G}(\mathcal{O}, x) \equiv \{ g \in \mathcal{G}(\mathcal{O}) : gx = x \}, \quad x \in \mathcal{O}
\]

in \( K \), for almost all \( x \in \mathcal{M} \).

For different \( \mathcal{O} \subset \mathcal{M}, x \in \mathcal{O} \), the groups \( \mathcal{G}(\mathcal{O}) \) are isomorphic, and the same applies to \( \mathcal{G}(\mathcal{O}, x) \). In fact, it is enough to consider \( \mathcal{O}_1, \mathcal{O}_2 \) disjoint, \( x_i \in \mathcal{O}_i \); then, there exists a region \( \mathcal{O} \supset (\mathcal{O}_1 \cup \mathcal{O}_2) \) and a diffeomorphism \( g \in \mathcal{G}(\mathcal{O}) \) transforming \( \mathcal{O}_1 \) in \( \mathcal{O}_2 \) and \( x_1 \) in \( x_2 \), constructed, e.g., by contracting \( \mathcal{O}_1 \) to sufficiently small balls around \( x_i \), interpolated by local translations in a cylinder around a path from \( x_1 \) to \( x_2 \).

The analysis of local cocycles allows for a characterization of the representations of the local algebras.
Proposition 2.4 A cocycle-regular representation $\pi$ of $\Pi(O)$, by Proposition 2.2 in $L^2(O, d\mu) \times K(O) \oplus K_\infty(O)$, defines unitary representations $R(O, x)$ of $G(O, x)$ in $K(O)$, for almost all $x \in O$, all unitarily equivalent and a unitary representation $R_\infty$ of $G(O)$ in $K_\infty(O)$. The corresponding unitary equivalence classes determine the cocycles $V_g(x)$, and therefore $\pi(\Pi(O))$, up to unitary equivalence.

Conversely, any pair of unitary representations of $G(O, x)$ and $G(O)$, respectively $R, R_\infty$, in spaces $K, K_\infty$ (strongly continuous in the parameters of one-dimensional subgroups) determines a cocycle-regular representation $\pi_{R, R_\infty}$ of $\Pi(O)$, in $L^2(O, d\mu) \times K \oplus K_\infty$.

Proof. By definition of small regions, there exist $O' \supset O \sim \{y \in \mathbb{R}^d, |y| < 1\}$ and a subgroup $T$ of $G(O')$ acting in $O$ as translations, $\tau(a)$, sending $y$ to $y + a$, for $y, y + a \in O$. Given $x_0 \in O$, for all $x$ in $O$ there is a (unique) translation, $\tau(x - x_0)$ sending $x_0$ to $x$. For all $g \in G(O)$, two applications of eq. (2.6) give

$$V_{\tau(gx-x_0)}^{-1}(x_0) V_g(x) V_{\tau(x-x_0)}(x_0) = V_{\tau(x_0-gx)} g \tau(x-x_0)(x_0),$$  \hspace{1cm} (2.8)

a.e. in $x, x_0$. The operators in the r.h.s. are indexed by elements of $G(O, x_0)$ and, by eq. (2.6), give a unitary representation of it. By the equivalence criterion, eq. (2.7), taking $S(x) = V_{\tau(x-x_0)}^{-1}(x_0)$, the representation of $\Pi(O)$ is unitarily equivalent to that given by

$$V_g^{T,x_0}(x) \equiv R^{x_0}(\tau(x_0 - gx) g \tau(x - x_0)), \hspace{1cm} (2.9)$$

with

$$R^{x_0}(h) \equiv V_h(x_0) \hspace{1cm} (2.10)$$

a unitary representation of $G(O, x_0)$ in $K(O)$. Clearly, $V_h^{T,x_0}(x_0) = R^{x_0}(h)$. For $x = x_\infty$, by the cocycle equation and the invariance of $x_\infty$,

$$R_\infty(g) \equiv V_g(x_\infty) \hspace{1cm} (2.11)$$

gives a unitary representation of $G(O)$ in $K_\infty(O)$. For different $x_0 \in O$ the groups $G(O, x_0)$ are isomorphic and their representations $R^{x_0}$ are unitarily equivalent.

Conversely, for given $x_0, T$, by eqs. (2.9) (2.11), any pair of (strongly continuous) representations $R, R_\infty$ of $G(O, x_0), G(O)$ in separable Hilbert spaces $K$, $K_\infty$ define operators $V_g^{T,x_0}(x)$ satisfying eqs. (2.6) and, by Proposition 2.2, a cocycle-regular representation $\pi_{R, R_\infty}$ of $\Pi(O)$. By eqs. (2.8), (2.9), the representations of $\Pi(O)$ given by $R, R_\infty$, are unitarily equivalent for different $T, x_0$. 
Examples of non-trivial $R$ are given by representations $\rho_n$ of $\Pi(\mathcal{O})$ in $L^2(\mathcal{O} \times \mathcal{O} \times \ldots \times \mathcal{O}, d\mu(x_1) \ldots d\mu(x_n)) \equiv L^2(\mathcal{O}^n)$ defined by

$$\rho_n(\alpha(x))\psi(x_1, \ldots, x_n) = \alpha(x_1) \psi(x_1, \ldots, x_n),$$

$$\rho_n(U(g))\psi(x_1, \ldots x_n) = \psi(g^{-1}x_1, \ldots, g^{-1}x_n) \Pi(g) J_g(x_i)^{1/2}. $$

Here, $K = L^2(\mathcal{O}^{n-1})$ and $\mathcal{G}(\mathcal{O}, x)$ is represented there by the unitary change of variables $\Pi_{i=2}^n C_g(x_i)$. In these examples, $K_\infty = 0$.

### 2.3 Locally Schroedinger representations

The representations of $\Pi(\mathcal{O})$ of the above examples describe states of $n$ particles in $\mathcal{O}$, the position of the first particle being described by $\alpha(x)$ and the other particles by the variables $U(g)$. Thus, already for local algebras and independently of the topology of $\mathcal{M}$, additional conditions are needed in order to select the representations of $\Pi(\mathcal{M})$ with a local one-particle interpretation.

Given a cocycle-regular representation $\pi$ of $\Pi(\mathcal{M})$, the unitary equivalence class of the representations $R(\mathcal{O}, x)$, $R_\infty(\mathcal{O})$ do not depend on $\mathcal{O}$, by the argument before Proposition 2.4. A non-trivial $R(\mathcal{O}, x)$ describes an additional action of $U(g)$, $g \in \mathcal{G}(\mathcal{O})$, besides the change of variables $C_g$, i.e., additional localized degrees of freedom with respect to the Schroedinger positions and momenta in $\mathcal{O}$. Similarly, a non trivial $R_\infty(\mathcal{O})$ describes, in $K_\infty(\mathcal{O})$, an action of $\mathcal{G}(\mathcal{O})$ on states localized outside $\mathcal{O}$ and must be excluded if $\pi(\mathcal{G}(\mathcal{O}))$ has to act locally.

The same conclusion is obtained considering that the regions $\mathcal{O}$ are diffeomorphic to $\mathbb{R}^d$, so that the representations of $\Pi(\mathcal{O})$ should be compared (they cannot be distinguished in a diffeomorphism invariant way) with those arising in a diffeomorphism invariant formulation of QM of a particle in $\mathbb{R}^d$.

In fact, the condition $R(\mathbb{R}^d, x) = R_\infty(\mathbb{R}^d) = I$ identifies the Schroedinger representation $\pi_S(\Pi(\mathbb{R}^d))$, up to unitary equivalence, multiplicity and the addition of a trivial representation ("states at infinity"); $R(\mathcal{O}, x) = R_\infty(\mathcal{O}) = I$ identifies the Schroedinger representation $\pi_S(\Pi(\mathcal{O}))$ in $L^2(\mathcal{O}, d\mu)$, apart from unitary equivalence, multiplicity and the addition of a trivial representation, given by $\pi(\alpha(x)) = 0$, $\pi(U(g)) = 1$, always allowed for the description of states localized in $\mathcal{M} \setminus \mathcal{O}$.

Thus, we are led to consider representations of $\Pi(\mathcal{M})$ with $R(\mathcal{O}, x) = I$ and $R_\infty(\mathcal{O}) = I$, for all $\mathcal{O} \subset \mathcal{M}$. Since $R_\infty(\mathcal{O}) = I$ also implies $R_\infty = I$, i.e., a trivial representation of $\Pi(\mathcal{M})$ in $K_\infty$, we set in the following $K_\infty = 0$ and therefore, by Proposition 2.2, consider representations of $\Pi(\mathcal{M})$ in $L^2(\mathcal{M}, d\mu) \times K$. 
By the above analysis, such conditions fix the representations of the local algebras: by Propositions 2.2, 2.4, each of them is unitarily equivalent, in $L^2(M, d\mu) \times K$, to a representation with $V_g(x) = I$ for all $g \in G(O)$, i.e., to the restriction of $\pi_S(\Pi(M))$ to $\Pi(O)$, hereafter called the Schrödinger representation of $\Pi(O)$ in $L^2(M, d\mu)$, with a multiplicity given by $K$.

**Definition 2.5** A representation $\pi$ of $\Pi(M)$, with $K_\infty = 0$, is Locally Schrödinger (LS) if it is cocycle-regular and, for all $O \subset M$, $R(O, x)$, $R_\infty(O)$ are the identity; equivalently, if the representations of $\Pi(O)$ are unitarily equivalent to their Schrödinger representation in $L^2(M, d\mu)$, apart from a (common, at most denumerable) multiplicity.

The LS condition is a non-trivial restriction already at the algebraic level, since $\pi_S(\Pi(O))$ is not a faithful representation of $\Pi(O)$. In fact, e.g., for $U(g), \alpha \in \Pi(O)$, if $gx = x$, $\forall x \in \text{Supp} \alpha(x)$, $\pi_S((U(g) - 1)\alpha = 0)$ whereas, in $\Pi(O)$, $||(U(g) - 1)\alpha|| = 2\sup |\alpha(x)|$.

**Remark.** The LS condition is also equivalent to the local validity of the Lie-Rinehart relations of the generators of $\text{Diff}(M)$ as a module on $C_0^\infty(M)$ [7], allowing to express them in terms of $d$ independent momenta:

$$T_{\sum_i \alpha_i v_i} = \frac{1}{2} \sum_i (\alpha_i T_{v_i} + T_{v_i} \alpha_i), \quad \forall \alpha_i \in C_0^\infty(O), \ \forall v_i \in \mathcal{L}(O). \quad (2.12)$$

Eqs. (2.12) hold in fact in the Schrödinger representation and the converse follows from the proof of Theorem 3.5 in [7].

By the LS condition, for any region $O$, one may take, as the observable algebra in $O$, the Von Neumann closure $\mathcal{A}(O)$ of $\pi_S(\Pi(O))$. As operator algebras in the space of the sum, $\rho(\Pi(M))$, of the LS representations of $\Pi(M)$, the Von Neumann algebras $\mathcal{A}(O)$ generate a $C^*$ algebra $\mathcal{A}(M)$, which can be taken as the (one-particle) observable algebra associated to the entire manifold. $\mathcal{A}(M)$ contains $\rho(\Pi(M))$ as a weakly dense subalgebra, so that representations of $\mathcal{A}(M)$ define representations of $\Pi(M)$ and unitarily equivalent LS representations of $\Pi(M)$ define unitarily equivalent representations of $\mathcal{A}(M)$. Therefore, the LS representations of $\Pi(M)$ coincide with the locally normal ([19], p.131) representations of $\mathcal{A}(M)$.

By irreducibility of the Schrödinger representation of $\Pi(O)$ in $L^2(O, d\mu)$, the algebras $\mathcal{A}(O)$ are isomorphic to the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators in a separable Hilbert space. However, the Schrödinger condition is purely local and the immersion of the local algebras in $\mathcal{A}(M)$ gives rise to a non trivial “bundle” structure, associated to the topology of $M$, which plays the essential role in the classification of the representations of $\mathcal{A}(M)$.
2.4 Classification of the LS representations by the fundamental group

By Proposition 2.2 and Definition 2.5, a LS representation of $\Pi(\mathcal{M})$ is given by a collection of representations of the local algebras $\Pi(O), O \subset \mathcal{M}$, all in $L^2(\mathcal{M}, d\mu) \times K$ and determined by unitary local intertwiners with the Schroedinger representation. The associated cocycles are locally trivial and our aim is to characterize the cocycles $V_g(x), g \in G_L(\mathcal{M})$, (eq.(2.3)) associated to products of localized diffeomorphisms; by Proposition 2.2, they identify the representation of $\Pi(\mathcal{M})$. The analysis will consist of the following steps.

a) Local Schroedinger intertwiners

The following Proposition characterizes the local cocycles for a LS representation of $\Pi(\mathcal{M})$.

**Proposition 2.6** A regular representation $\pi$ of $\Pi(\mathcal{M})$, with $K_\infty = 0$, is LS iff for (one and therefore) all $O$, there exist (unique) weakly measurable unitary operators in $K$, $W_O(x,y), x,y \in O$ such that, $\forall g \in G(O)$,

$$V_g(x) = W_O(gx,x), \quad \forall x \in O; \quad V_g(x) = 1, \quad \forall x \notin O. \quad (2.13)$$

They satisfy

$$W_O(y,x)W_O(z,x) = W_O(y,z), \quad W_O(y,x) = W_O(x,y)^{-1}, \quad (2.14)$$

$$W_{O_1}(y,x) = W_{O_2}(y,x), \quad \forall x,y \in O_1, \quad O_1 \subset O_2. \quad (2.15)$$

**Proof.** By eq. (2.7) the quasi-equivalence of the representation of $\Pi(O)$ to its Schroedinger representation in $L^2(\mathcal{M}, d\mu) \times K$ implies the existence of weakly measurable unitary operators $W_O(x)$ in $K$, such that, $\forall g \in G(O)$,

$$V_g(x) = W_O(gx)x)W_O(x)^{-1}, \quad \forall x \in O; \quad V_g(x) = 1, \quad \forall x \notin O. \quad (2.16)$$

Then,

$$W_O(y,x) \equiv W_O(y)W_O(x)^{-1} \quad (2.17)$$

is well defined for all $x,y \in O$ and weakly measurable, since $W_O(x)$ is measurable in $x$. Eqs. (2.13), (2.14), (2.15) immediately follow from the definitions. Conversely, eqs. (2.13), (2.14) imply eqs. (2.16), with $W_O(x) \equiv W_O(x,x_0)$ (for any choice of $x_0 \in O$, apart from a zero measure subset). Therefore, the representation of $\Pi(O)$, in $L^2(\mathcal{M}, d\mu) \times K$, is unitarily equivalent to the Schroedinger representation in the same space, with multiplicity given by $K$. 
b) **Homotopy classes of products of local intertwiners**

As a next step, on the basis of eqs. (2.13) (2.14) (2.15), we show that, for \( g = g_n \ldots g_1, g_i \in G(O_i) \), \( V_g(x) \) is given by a string of local factors \( W_{O_i}(x_{i+1}, x_i) \), \( x_{i+1} = g_i x_i \); the resulting operators will be shown to be invariant under small displacements of the intermediate points \( x_i \) and therefore indexed by the homotopy class of the corresponding path.

In fact, given \( g \in \mathcal{G}_L(M) \), \( g = h_N \ldots h_1 \), by eq. (2.10) \( V_g(x) \) is represented by a product of terms of the form \( V_{h_i}(y_i) \). Dropping all the factors with \( h_i y_i = y_i \) (equal to the identity by eq. (2.13)), we are left with a subset \( g_1 \ldots g_n \), \( g_i \in \mathcal{G}(O_i) \), such that

\[
g x = g_n \ldots g_1 x, \quad x_i, x_{i-1} \in O_i \quad x_i \equiv g_i x_{i-1} \neq x_{i-1}, \quad x_0 \equiv x, \quad x_n \equiv g x.
\]

Then, by eq. (2.13), one has

\[
V_g(x) = V_{g_n}(x_{n-1}) \ldots V_{g_1}(x) = W_{O_n}(g x, x_{n-1}) W_{O_{n-1}}(x_{n-1}, x_{n-2}) \ldots W_{O_1}(x_1, x).
\] (2.18)

**Lemma 2.7** Given \( x, y \in \mathcal{M} \), a regular path \( \gamma(y, x) \) starting at \( x \) and ending at \( y \), a partition of \( \gamma \), \( \gamma = \gamma_n(y, x_{n-1}) \circ \ldots \circ \gamma_1(x_1, x) \), open sets \( O_1 \ldots O_n \) with \( O_i \supset \gamma_i \) and unitary operators \( W_{O}(y, x) \) in \( K \) satisfying eqs. (2.14), (2.15), the operators

\[
W(y, x, \gamma) \equiv W_{O_n}(y, x_{n-1}) W_{O_{n-1}}(x_{n-1}, x_{n-2}) \ldots W_{O_1}(x_1, x),
\] (2.19)

only depend on \( x, y \) and on the homotopy class \([\gamma]\) of \( \gamma(y, x) \). They satisfy the composition law

\[
W(y, x, [\gamma]) W(x, z, [\delta]) = W(y, z, [\gamma] \circ [\delta]).
\] (2.20)

For almost all \( x \), the **topological operators** \( W(x, x, [\gamma]) \) provide a unitary representation \( \mathcal{R}_x^x([\gamma]) \) of \( \pi_1(\mathcal{M}) \), all belonging to the same **equivalence class** \([\mathcal{R}]\).

Two systems of (weakly measurable) unitary operators, \( W_1, W_2(y, x, [\gamma]) \), satisfying eq. (2.20), are related by (weakly measurable) unitaries \( S(z) \),

\[
W_2(y, x, [\gamma]) = S(y) W_1(y, x, [\gamma]) S(x)^{-1}
\] (2.21)

iff, for some (and then for all) \( x \in \mathcal{M} \), the corresponding systems for closed paths are unitarily equivalent:

\[
\mathcal{R}_x^x([\gamma]) = S(x) \mathcal{R}_x^x([\gamma]) S^{-1}(x),
\] (2.22)
equivalently, iff the corresponding equivalence classes \([\mathcal{R}_i] \), \( i = 1, 2 \), coincide.
Proof. Given two choices \( \{ \gamma_i, O_i \} \) and \( \{ \gamma'_i, O'_i \} \), a “combined” partition of \( \gamma \) is defined by the sequence \( \{ y_k \} \) obtained by ordering the points \( x_i, x'_j \); each piece \( \gamma''_i(y_i, y_{i-1}) \) of the so obtained partition is contained in some \( O''_i \subset O_k \cap O'_{k'} \) for some pair \( k, k' \). Therefore, by \( \text{eq.}(2.15) \) the operators \( W(y, x, \gamma) \) associated to \( \{ \gamma''_i, O''_i \} \) coincide with both the operators constructed from \( \gamma_i \) and \( \gamma'_i \). Since the sets \( O_i \) can be kept fixed for a small deformation of \( \gamma \), the result only depends on the homotopy class of \( \gamma \). Independence from the partition of \( \gamma \) immediately implies \( \text{eq.}(2.20) \).

By \( \text{eqs.}(2.20) \), for all \( x, y \), fixed a path \( \delta \) from \( y \) to \( x \),

\[ [\gamma] \rightarrow [\delta^{-1} \circ \gamma \circ \delta] \]

is a bijection between (the equivalence classes of) the closed paths with base point \( x \) and those with base point \( y \), and

\[ W(y, y, [\delta^{-1} \circ \gamma \circ \delta]) = W(x, y, [\delta])^{-1} W(x, x, [\gamma]) W(x, y, [\delta]). \quad (2.23) \]

Then, \( \text{eq.}(2.21) \) implies \( \text{eq.}(2.22) \) yielding the unitary equivalence of the representations \( \mathcal{R}^x \) of \( \pi_1(M) \) for different \( x \in M \); conversly, it is easy to check that, given \( x \) and \( S(x) \) satisfying \( \text{eq.}(2.22) \),

\[ S'(y) \equiv W_2(y, x, \gamma) S(x) W_1(y, x, \gamma)^{-1} \quad (2.24) \]

satisfies \( \text{eq.}(2.21) \), for any choice of curves \( \gamma \) starting at \( x \) and ending at \( y \) and it is measurable in \( x \) if the curves depend continuously on \( x \) in an open set \( M_0 \) with complement of zero measure, as in the proof of Proposition 2.4.

c) Classification by the fundamental group

By the above results, in a LS representation \( \pi \) of \( \Pi(M) \), the operators \( \pi(U(g)) \) representing a product of localized diffeomorphisms, \( g = g_n \ldots g_1 \), are given by \( \text{eqs.}(2.3) \), with

\[ V_g(x) = W(gx, x, [\gamma_g]) \quad (2.25) \]

where \( W(gx, x, [\gamma_g]) \) is defined by \( \text{eqs.}(2.18),(2.19) \) and \( \gamma_g \) is the integral curve, from \( x \) to \( gx \), defined by \( g_n \ldots g_1 \).

By \( \text{eq.}(2.7) \) two LS representations \( \pi_i \) of \( \Pi(M) \) are unitarily equivalent iff the corresponding \( W \) operators are related by \( \text{eq.}(2.21) \), and therefore, by Lemma 2.7, iff the associated equivalence classes \( [\mathcal{R}_{\pi_i}] \) of representations of \( \pi_1(M) \) coincide. Hence, the LS representations of \( \Pi(M) \) are classified by the corresponding unitary representations of \( \pi_1(M) \). The following Lemma shows that all the unitary representation of \( \pi_1(M) \) appear in the classification.
Lemma 2.8 Any unitary representation $\mathcal{R}$ of $\pi_1(M)$ in a (separable) Hilbert space $K$ defines a LS representation $\pi_{\mathcal{R}}$ of $\Pi(M)$ in $L^2(M, d\mu) \times K$, with the same equivalence class, $[\mathcal{R}_{\pi_{\mathcal{R}}}] = [\mathcal{R}]$.

Proof. By Proposition 2.2, given $\mathcal{R}$ we have to construct weakly measurable unitary operators $V_{\beta}(x)$ satisfying eq. (2.6).

To this purpose, we fix a point $x_0 \in M$ and associate to each $x \in M$ a path $\delta(x, x_0)$ from $x_0$ to $x$. As in the proof of Proposition 2.4, such paths can be taken continuous in $x$, in the $C^0$ topology of paths, for all $x$ in a set $M_0 \subset M$, with a complement of zero measure. Then, to each $\gamma(y, x)$ from $x$ to $y$, $\forall x, y \in M$, we associate the closed path

$$\beta(\gamma(y, x)) \equiv \delta(x_0, y) \circ \gamma(y, x) \circ \delta(x, x_0),$$

(2.26)

with $\delta(x_0, y) \equiv (y, x_0)^{-1}$. Clearly, the equivalence class $[\beta]$ only depends on $[\gamma]$. Composing two paths, $\gamma(y, x)$, $\gamma(x, z)$, gives rise to the composition of the corresponding images in $\pi_1(M)$:

$$\beta(\gamma(y, x) \circ \gamma(x, z)) = \delta(x_0, y) \circ \gamma(y, x) \circ \delta(x, x_0) \circ \delta(x_0, x) \circ \gamma(x, z) \circ \delta(z, x_0) =$$

$$= \beta(\gamma(y, x)) \circ \beta(\gamma(x, z)).$$

(2.27)

Given a unitary representation $\mathcal{R}$ of $\pi_1(M)$ in $K$, we then define, for all $g \in G_1(M), x \in M$ (omitting for simplicity the equivalence class notation for the paths)

$$V_g(x) = W(gx, x, \gamma_g) \equiv \mathcal{R}(\beta(\gamma_g(gx, x))),$$

(2.28)

with, as above, $\gamma_g(gx, x)$ the integral curve associated to $x$ and $g$. For $gx = x = x_0$, eq. (2.28), identifies $\mathcal{R}([\gamma_g])$ with the topological factors $W(x_0, x_0, \gamma_g)$ representing $\pi_1(M)$ with base point $x_0$.

By the above continuity property of paths, $V_g(x)$ is locally constant in $x$ and $gx$ in $M_0$; it is therefore strongly continuous in the parameters of one-dimensional subgroups of $G_L(M)$ and defines a cocycle: by eqs. (2.28), (2.27), $\forall x \in M$

$$V_g(hx) V_h(x) = \mathcal{R}(\beta(\gamma_g(ghx, hx))) \mathcal{R}(\beta(\gamma_h(hx, x))) =$$

$$= \mathcal{R}(\beta(\gamma_{gh}(ghx, x))) = V_{gh}(x).$$

We have therefore a unitary representation of $\Pi(M)$ in $L^2(M, d\mu) \times K$, which is regular and cocycle-regular, with $\mathcal{R}$ as the associated representation of $\pi_1(M)$. For the proof of the LS property, eqs. (2.13), (2.10), consider $x, z \in O, g \in G(O)$; since $[\beta]$ only depends on $[\gamma]$, using eq. (2.27), $\forall \gamma(gx, z), \gamma(z, x) \subset O$,

$$V_g(x) = \mathcal{R}(\beta([\gamma_g(gx, x)])) = \mathcal{R}(\beta([\gamma(gx, z)])) \mathcal{R}(\beta([\gamma(z, x)])),$$
which is of the form of eq. (2.10) since $R(\beta([\gamma(y,z)]))$ only depend on $y$, for fixed $z$.

Eq. (2.28) also provides a direct relation, in $\pi_R$, between the topological operators associated to closed paths with different base points: by using the existence, for any closed $\gamma$, of a path $\gamma_g$ with the same base point $x$ and $[\gamma_g] = [\gamma]$, one has

$$W(x,x,\gamma) = R(\beta([\gamma])) = W(x_0,x_0,\beta(\gamma)).$$

(2.29)

In conclusion, we have:

**Theorem 2.9** The Locally Schroedinger representations $\pi$ of $\Pi(M)$, equivalently, the locally normal representations of the observable algebra $\mathcal{A}(M)$, are given by eqs. (2.3), (2.25). Up to unitary equivalence, they are classified by the associated unitary equivalence class of representations of $\pi_1(M)$, $[R]$. Any unitary representation $R$ of $\pi_1(M)$ (in a separable space) define a LS representation $\pi_R$ of $\Pi(M)$, equivalently, a locally normal representations of $\mathcal{A}(M)$, with $[R]$ the associated representation of $\pi_1(M)$.

**Corollary 2.10** The commutant and the centre of $\pi_R(\mathcal{A}(M))''$, in the representation space $L^2(M,d\mu) \times K$, are given by the commutant $R'$ and the centre $R' \cap R''$ of the representation $R$ of $\pi_1(M)$ in $K$:

$$\pi_R(\mathcal{A}(M))' = I \times R',$$

(2.30)

$$\pi_R(\mathcal{A}(M))' \cap \pi_R(\mathcal{A}(M))'' = I \times (R' \cap R'').$$

(2.31)

**Corollary 2.11** (Analogue of Von Neumann uniqueness theorem) If $M$ is simply connected, the locally normal representation of $\mathcal{A}(M)$ is unique, up to multiplicity and coincides, up to unitary equivalence, with the Schroedinger representation in $L^2(M,d\mu)$.

**Proof of Corollary 2.10.** A (bounded) operator $A$ in $L^2(M,d\mu) \times K$ commuting with all the multiplication operator $\pi_R(\alpha(x))$ is a multiplication operator $A(x)$ in $K$. By eq. (2.26) with $x = x_0$, for all $y \in M$, $\beta(\delta(y,x_0))$ is the identity. Taking a diffeomorphism $g$, sending $x_0$ into $y$, such that $[\gamma_g(y,x_0)] = [\delta(y,x_0)]$, the vanishing of the commutator of $A(x)$ with all the operators $\pi_R(U(g))$ implies, by eq. (2.28), that $A(x)$ is constant, a.e. in $x$, i.e., $A = I \times A_K$.

Since $A$ commutes with $\pi_R(U(g))$, $A_K$ commutes with $V_g(x)$ and therefore, by eq. (2.28), it commutes with the representation $R$ of $\pi_1(M)$ in $K$. Conversely,
by eq. (2.28), operators $I \times B$, with $B \in \mathcal{R}'$, commute with $\pi_R(\mathcal{A}(\mathcal{M}))$. This immediately implies

$$\pi_R(\mathcal{A}(\mathcal{M})))'' = \mathcal{B}(L^2(\mathcal{M}, d\mu)) \times \mathcal{R}''$$

and eq. (2.31) follows.

### 2.5 Topological observables

Theorem 2.9 and Corollary 2.10 classify the representations of $\mathcal{A}(\mathcal{M})$ in terms of weak limits of local observables, given by the centre of the associated representation $\mathcal{R}$ of $\pi_1(\mathcal{M})$. They leave open the question of whether the operators of $\mathcal{R}$, and those of $\mathcal{R}' \cap \mathcal{R}''$, already belong to the “quasi local” observable algebra $\mathcal{A}(\mathcal{M})$.

**Theorem 2.12** For all $x \in \mathcal{M}$ there exists $\mathcal{O} \ni x$ such that, for all $\mathcal{R}$, $\pi_R(\mathcal{A}(\mathcal{M}))$ contains the topological operators $W(x, x, [\gamma])_P$, eq. (2.19), with $P$ the projection on $L^2(\mathcal{O}, d\mu) \times \mathcal{K}$; actually they belong to (the representation of) the algebra generated by a finite number of algebras $\mathcal{A}(\mathcal{O}_i)$, $\mathcal{O}_i$ contained in a neighbourhood of $\gamma$.

Furthermore, if $\mathcal{M}$ is compact

i) $\mathcal{A}(\mathcal{M})$ contains a “topological” $C^*$-subalgebra $\mathcal{T}(\mathcal{M})$, isomorphic to the group algebra of $\pi_1(\mathcal{M})$ (generated in norm by the sum of its unitary representations). For all $\mathcal{R}$, $\pi_R(\mathcal{T}(\mathcal{M}))$ is generated by the operators $W(x_0, x_0, [\gamma])$ representing $\pi_1(\mathcal{M})$ in $\mathcal{K}$ and the representations of $\mathcal{T}(\mathcal{M})$ classify the (locally normal) representations of $\mathcal{A}(\mathcal{M})$.

ii) if $\pi_1(\mathcal{M})$ is finite, $\mathcal{T}(\mathcal{M})$ is a Von Neumann algebra and its centre $\mathcal{Z}$ classifies the (locally normal) representations of $\mathcal{A}(\mathcal{M})$.

iii) if $\pi_1(\mathcal{M})$ abelian, the spectrum of $\mathcal{T}(\mathcal{M})$ labels the factorial (locally normal) representations of $\mathcal{A}(\mathcal{M})$.

**Proof.** The topological factors $W(x, x, [\gamma])$ can be represented in terms of localized observables as follows. Given $g = g_n \ldots g_1$, $g_i$ localized in $\mathcal{O}_i$, with $gx = x$ for some $x \in \mathcal{O}_1$, by eqs. (2.25), (2.3) one has, $\forall y \in \mathcal{O}_1$

$$W(gy, y, [\gamma_g]) = V_g(y) = C^{-1}_g \pi(U(g_n) \ldots U(g_1)), \quad (2.32)$$

with $\gamma_g(gy, y) \subset \cup_i \mathcal{O}_i$; moreover, for all $y \in \mathcal{O} \subset \mathcal{O}_1$, such that $g(\mathcal{O}) \subset \mathcal{O}_1$, by eqs. (2.19), (2.14), one has

$$W(gy, y, [\gamma_g]) = W_{\mathcal{O}_1}(gy, x) W(x, x, [\gamma_g(x, x)]) W_{\mathcal{O}_1}(x, y). \quad (2.33)$$
In a representation $\pi_R$, one can choose $O_1$ such that, by eq. (2.28), the operators $W_{O_1}$ in eq. (2.33) are the identity, and therefore

$$W(x, x, [\gamma g]) P_O = C^{-1}_g \pi_R(U(g_n) \ldots U(g_1)) P_O,$$

with $P_O$ the projection on $L^2(O, d\mu) \times K$. $\pi_R(U(g_n) \ldots U(g_1)) P_O$ maps $L^2(O, d\mu) \times K$ into $L^2(g(O), d\mu) \times K$. A projector $P_{g(O)}$ can therefore be inserted in the r.h.s. of eq. (2.34) and $C^{-1}_g P_{g(O)}$, as an operator in $L^2(O_1, d\mu)$, belongs to the (weakly closed) algebra $A(O_1)$, by irreducibility of the Schroedinger representation of $A(O_1)$ in $L^2(O_1, d\mu)$. By the argument before eq. (2.29), eq. (2.34) applies to all $[\gamma] \in \pi_1(M)$, with almost any $x$ as base point, and therefore gives the unitary representations of $\pi_1(M)$, characterizing the representations of $A(M)$ directly in terms of observables localized along the paths $\gamma$.

i) Since $\gamma \rightarrow \beta(\gamma)$ has an inverse

$$\gamma(\beta) = \delta(x, x_0) \beta \delta(x_0, x),$$

by eq. (2.29) one has

$$R(\beta) = W(x, x, \gamma(\beta))$$

with $\beta$ any closed path with base point $x_0$ and $x$ almost any point in $M$. Since, as above, there is a $\gamma_g$ in the equivalence class of $\gamma(\beta)$, eqs. (2.35), (2.34) imply

$$R(\beta) P_O = W(x, x, [\gamma_g]) P_O$$

for all closed paths $\beta$ with base point $x_0$, almost all $x \in M$, $\gamma_g \in [\gamma(\beta)]$. If $M$ is compact, it may be covered by a finity family $\{O_{\alpha}\}$ and therefore by disjoint measurable sets $O_{\alpha} \subset O_{\alpha}$; multiplying eq. (2.36) on the right by $P_{O_{\alpha}}$ and summing over $\alpha$, one gets

$$R(\beta) = \sum_{\alpha} W(x_\alpha, x_\alpha, [\beta]) P_{O_{\alpha}} \equiv \pi_R(A(\{x_\alpha, g_{\alpha}\}))$$

with $A(\{x_\alpha, g_{\alpha}\}) \in A(M)$, by eq. (2.34).

Since this holds for any $R$ and all LS representations of $\Pi(M)$ are unitarily equivalent to a $\pi_R$, the observable algebra $A(M)$, which has been defined in the sum of the LS representations of $\Pi(M)$, contains a $C^*$-subalgebra $T(M)$ isomorphic to the group algebra of $\pi_1(M)$. By eq. (2.29), the generators of $T(M)$ are represented in $\pi_R$ by $W(x_0, x_0, [\beta])$. By Theorem 2.9, the representations of $T(M)$ classify the (locally normal) representations of $A(M)$.

ii) If $\pi_1(M)$ is finite, its representations are finite dimensional, apart from multiplicities, and therefore norm and weak closures coincide.
iii) If $\pi_1(\mathcal{M})$ is abelian, the Von Neumann algebra generated by the representations of $\mathcal{T}(\mathcal{M})$ is the algebra of Borel functions (with the weak topology defined by Borel measures) on its spectrum, which therefore indexes the factorial representations.

If the diffeomorphism $\tilde{g}$ defined by $g = g_n \ldots g_1$ belongs to the connected component of the identity of the diffeomorphism group of $\mathcal{O}_1$, then $C_g = \pi_\mathcal{R}(U(\tilde{g}))$ and the representation of the topological factors, eq.(2.34), only involves the local algebras $\mathcal{A}(\mathcal{O}_i)$. If $\mathcal{M}$ is not orientable, this is not in general the case, and $C_g$ only belongs to the weakly closed algebra $\mathcal{A}(\mathcal{O}_1)$ [21].

The classification in terms of a locally Schroedinger description and unitary representations of $\pi_1(\mathcal{M})$ was also obtained in [7], for the crossed product algebra $C(\mathcal{M}) \times \text{Diff}(\mathcal{M})^c$, Diff$(\mathcal{M})^c$ the universal covering group of Diff$(\mathcal{M})$, under a “Lie-Reinhardt” condition, which, for the local algebras, is equivalent to the LS property. However, the starting point is different in the two approaches: $C(\mathcal{M}) \times \text{Diff}(\mathcal{M})^c$ includes from the beginning the operations of transport on closed paths as independent variables.

On the contrary, in the present approach, the observable algebra is generated by the local algebras $\mathcal{A}(\mathcal{O})$, associated to topologically trivial regions, homeomorphic to $\mathbb{R}^d$, and therefore assumed to be represented as in ordinary Schroedinger QM, apart from unitary equivalence and multiplicity. Topological effects arise from the collection of local Schroedinger Quantum Mechanical descriptions. They appear in products of local observables and are characterized by the representation of operations of physical transport along non-trivial closed paths, constructed as a sequence of strictly localized operations.

The result for $C(\mathcal{M}) \times \text{Diff}(\mathcal{M})^c$ actually follows from the present analysis through the extension of eq.(2.25) to the representatives $U(g_{\lambda v})$ of all the one-parameter subgroups of Diff$(\mathcal{M})^c$, which follows, as in [7], Proposition 4.4., from the identification of the corresponding generators.

3 The observable and the gauge action of $\pi_1(\mathcal{M})$

3.1 The Dirac approach to QM on manifolds

In his treatment of $N$ identical particles, Dirac proposed a solution of two problems: the explicit identification of the observables and the classification of their representations. In terms of operator algebras, his strategy was:

i) to consider the algebra of $N$ distinguishable particles, which may be taken as the Weyl algebra for $3N$ degrees of freedom $\mathcal{A}_W(3N)$, or as the algebra
\( \mathcal{B}(L^2(\mathbb{R}^{3N})) \) of all bounded operators in \( L^2(\mathbb{R}^{3N}) \), and to define the observable algebra \( \mathcal{A}_S(N) \) of \( N \) identical particles as the subalgebra invariant under the permutation group \( S_N \), which thus plays the role of a gauge group.

ii) to obtain representations of \( \mathcal{A}_S(N) \) by the decomposition of the unique (by the Von Neumann theorem) representation of \( \mathcal{A}_W(N) \), or of the unique normal (irreducible) representation of \( \mathcal{B}(L^2(\mathbb{R}^{3N})) \). Both are defined in \( L^2(\mathbb{R}^{3N}) \) and the decomposition is given by their commutant, generated by the gauge representation of \( S_N \) in \( L^2(\mathbb{R}^{3N}) \).

Since the configuration manifold \( \mathcal{M}_S \) of \( N \) identical particles in three dimensions (introduced above) has \( S_N \) as fundamental group and \( \mathbb{R}^{3N} \) as universal covering manifold, Dirac strategy may be translated as a recepee for Quantum Mechanics on a manifold \( \mathcal{M} \), consisting in the following steps:

i) the configuration manifold \( \mathcal{M} \) is replaced by its universal cover, \( \tilde{\mathcal{M}} \);
ii) the observable algebra \( \mathcal{A}(\mathcal{M}) \) is identified as the subalgebra of \( \mathcal{B}(L^2(\tilde{\mathcal{M}}, d\tilde{\mu})) \) (\( d\tilde{\mu} \) locally in the equivalence class of the Lebesgue measure) invariant under the action of \( \pi_1(\mathcal{M}) \) in \( L^2(\tilde{\mathcal{M}}) \);
iii) the representations of \( \mathcal{A}(\mathcal{M}) \) are obtained by the decomposition of \( L^2(\tilde{\mathcal{M}}) \) according to the representations of \( \pi_1(\mathcal{M}) \).

Such an approach had a substantial influence on the treatments of QM on manifolds; it provides a natural framework for the introduction of \( \tilde{\mathcal{M}} \), also suggested by the Aharonov-Bohm phenomenon and naturally associated to the Functional Integral approach, and for the associated role of \( \pi_1(\mathcal{M}) \) as a gauge group. The question then arises about the physical justification of the choices underlying the Dirac strategy and about the completeness of the classification of the representations of the observable algebra obtained in that way. A related question is whether, in that approach, the role of the fundamental group is only that of gauge transformations.

The aim of this section is to discuss the relation between Dirac strategy and the treatment of QM on manifolds discussed in Section 2, which only relies on physically motivated principles. The essential results are the following:

A. If \( \pi_1(\mathcal{M}) \) is \textit{amenable} (i.e., it admits an invariant mean, which is always the case for finite and abelian groups), then:

i) a representations of \( \pi_1(\mathcal{M}) \) by \textit{observable operators} is always present in \( L^2(\tilde{\mathcal{M}}) \), unitarily equivalent to its representation as a gauge group;
ii) the gauge and the observable topological classification of the representations of \( \mathcal{A}(\mathcal{M}) \) in \( L^2(\mathcal{M}) \) are the same. The observable and gauge representations of \( \pi_1(\mathcal{M}) \) associated to irreducible representations of \( \mathcal{A}(\mathcal{M}) \) coincide, apart from a complex conjugation;
iii) the irreducible (locally Schroedinger) representations of \( \mathcal{A}(\mathcal{M}) \) are \textit{all} con-
tained in the reduction of $L^2(\tilde{\mathcal{M}})$.

B. If $\pi_1(\mathcal{M})$ is not amenable, results i) and ii) still hold, but the reduction of $L^2(\tilde{\mathcal{M}})$ does not contain all the irreducible (locally Schroedinger) representations of $\mathcal{A}(\mathcal{M})$, and in fact even its ordinary Schroedinger representation is not obtained in the reduction.

In the derivation of the above result, the central role is played by the following facts:

i) the gauge representation of $\pi_1(\mathcal{M})$ in $L^2(\tilde{\mathcal{M}})$ is a multiple of its right regular representation;

ii) consequently, corresponding left regular representations can be constructed and are observable;

iii) completeness of the regular representation, i.e. the presence of all the irreducible representations in its (possibly integral) decomposition is equivalent [22] to amenability of $\pi_1(\mathcal{M})$.

Non amenable groups may appear for manifolds describing relevant physical situations, e.g. in the case of a plane with $n$ holes, $n > 1$, where the fundamental group is freely generated by $n$ elements and therefore non amenable.

For non amenable groups, even if the role of $L^2(\tilde{\mathcal{M}})$ and of the gauge group are lost, a modification of Dirac strategy still applies: all the irreducible LS representations of $\mathcal{A}(\mathcal{M})$ can still be obtained by the action of coordinates and vector fields of $\mathcal{M}$ on Hilbert spaces of wavefunctions on $\tilde{\mathcal{M}}$, with suitable (non $L^2$) scalar products.

### 3.2 Equivalence of the Dirac approach for amenable $\pi_1$

To prove the above results, we have to confront the “extended Schroedinger” representation of $\mathcal{A}(\mathcal{M})$ in $L^2(\tilde{\mathcal{M}})$ arising in the Dirac strategy with the classification given by Theorem 2.9.

a) The standard representation of $\tilde{\mathcal{M}}$

A standard representation of $\tilde{\mathcal{M}}$, together with the gauge action of $\pi_1(\mathcal{M})$ on it, is obtained as follows:

Denoting, as before, by $\gamma(y, x)$ the continuous paths in $\mathcal{M}$ from $x$ to $y$, given $x_0 \in \mathcal{M}$, $\tilde{\mathcal{M}}$ can be identified as the space of pairs

$$\tilde{\mathcal{M}} = \{ (x, [\gamma]) ; x \in \mathcal{M}, \gamma = \gamma(x, x_0) \}. \quad (3.1)$$
with manifold structure given by the basis of open sets

\[ \tilde{O}(x, \gamma) \equiv \{(y, [\gamma_y]); y \in O_x \subset M, [\gamma_y] = [\gamma(y, x) \circ \gamma(x, x_0)], \gamma(y, x) \subset O_x\}, \tag{3.2} \]

indexed by and homeomorphic to the small neighbourhoods \( O_x \) of \( x \). The resulting space covers \( M \) and is simply connected.

\( \pi_1(M) \) acts on \( \tilde{M} \) by its right action on the paths \( \gamma(x, x_0) \),

\[ r(\eta) : (x, [\gamma]) \mapsto (x, \gamma \circ \eta^{-1}), \tag{3.3} \]

\( \eta \in \pi_1(M) \) with base point \( x_0 \), and clearly

\[ M = \tilde{M}/r(\pi_1(M)). \tag{3.4} \]

\( r(\pi_1(M)) \) has therefore the role of a gauge group, associated to the redundant description given by \( \tilde{M} \). No identification of \( M \) with a subset of \( \tilde{M} \) is assumed at this point, nor is it implied by eqs.\( (3.3),(3.4) \).

b) The standard representation of \( \mathcal{A}(M) \) in \( L^2(\tilde{M}) \)

A standard representation of \( \mathcal{A}(M) \) in \( L^2(\tilde{M}) \) is provided by multiplication of pairs \( (x, [\gamma]) \) by functions of \( x \) and by the action of localized diffeomorphisms of \( M \) on \( \tilde{M} \). In fact, for all \( O \), diffeomorphisms of \( M \) localized in \( O \), \( g_{xv} \), define diffeomorphisms of \( \tilde{M} \), by

\[ g_{xv}(x, [\gamma]) \equiv (g_{xv}x, [\gamma_{g}(gx, x) \circ \gamma]), \]

with \( \gamma_{g}(gx, x) \) the integral curve of \( v \), as above (the homotopy class of the curve in the r.h.s. only depending on \([\gamma])\).

On \( \tilde{M} \) we consider measures \( d\tilde{\mu} \), locally in the class of the Lebesgue measure on the disks to which the \( \tilde{O}_x \) are homeomorphic. \( \mathcal{A}(M) \) is then represented in \( L^2(\tilde{M}, d\tilde{\mu}) \) by its “extended Schroedinger” representation \( \tilde{\pi}_S \), with action given by equations of the form of eqs.\( (2.3) \), with \( K \) one-dimensional and \( V_g = 1 \)

\[ \tilde{\pi}_S(\alpha\psi)(x, [\gamma]) = \alpha(x)\psi(x, [\gamma]) \tag{3.5} \]

\[ \tilde{\pi}_S(U(g_{xv})\psi)(x, [\gamma]) = \psi(g^{-1}x, [\gamma_{g}(g^{-1}x, x) \circ \gamma]) J_g(x, [\gamma])^{1/2}. \tag{3.6} \]

As above, \( J_g(\xi) \equiv [d\tilde{\mu}(g^{-1}\xi)/d\tilde{\mu}(\xi)] \), \( \xi \equiv (x, [\gamma]) \).

Eqs.\( (3.5),(3.6) \) reproduce the representation adopted in the Dirac approach; actually, if \( \mathcal{A}(M) \) is identified with the Dirac “enlarged algebra”, one of the Dirac steps, i.e. the choice of its Schroedinger representation, is forced by the uniqueness result for simply connected manifolds, Corollary 2.11.
The (Dirac) gauge representation of $\pi_1(\mathcal{M})$ is given by its right action in $\tilde{\mathcal{M}}$, which defines a unitary representation

$$
\psi(x, [\gamma]) \mapsto \psi(r(\eta^{-1})(x, [\gamma])) J^n(x, [\gamma])^{1/2}
$$

in $L^2(\tilde{\mathcal{M}}, d\tilde{\mu})$, commuting with $\tilde{\pi}_S(\mathcal{A}(\mathcal{M}))$ by eqs.$(3.5),(3.6)$, with $J^n$ a Jacobian factor as in eq.$(3.6)$.

If $\pi_1(\mathcal{M})$ is finite, since localized diffeomorphisms of $\mathcal{M}$ act in finite union of small regions in $\tilde{\mathcal{M}}$, $\tilde{\pi}_S(\mathcal{A}(\mathcal{M}))$ is contained in $\pi_S(\mathcal{A}(\tilde{\mathcal{M}}))$ and coincides with its subalgebra invariant under the gauge representation of $\pi_1(\mathcal{M})$.

c) Equivalence of $\tilde{\pi}_S(\mathcal{A}(\mathcal{M}))$ to a representations in $L^2(\mathcal{M}) \times l^2(\pi_1(\mathcal{M}))$

In order to confront $\tilde{\pi}_S(\mathcal{A}(\mathcal{M}))$ with the classification in Theorem 2.9, it is convenient to convert it to the form given by Proposition 2.2. Introducing $\delta(x, x_0)$ and $\beta(\gamma)$ as in the proof of Lemma 2.8, eq. (2.26),

$$
\beta(\gamma(x, x_0)) \equiv \delta(x_0, x) \circ \gamma(x, x_0)
$$

$[\beta]$ depends only on $[\gamma(x, x_0)]$ and belongs to $\pi_1(\mathcal{M})$, with $x_0$ as base point. The correspondence is invertible and $\tilde{\mathcal{M}}$ can therefore be represented as

$$
\tilde{\mathcal{M}} = \{ \tilde{x} = (x, [\beta]); x \in \mathcal{M}, [\beta] \in \pi_1(\mathcal{M}) \} = \mathcal{M} \times \pi_1(\mathcal{M}), \quad (3.7)
$$

In this representation, $\tilde{\mathcal{M}}$ consists of copies of $\mathcal{M}$, indexed by $\pi_1(\mathcal{M})$ and permuted by its action, eq.$(3.3)$. Clearly, also the left action of $\pi_1(\mathcal{M})$,

$$
l(\eta) : (x, [\beta]) \mapsto (x, [\eta \circ \beta]), \quad (3.8)
$$

acts by permutation of the copies of $\mathcal{M}$ in $\tilde{\mathcal{M}}$ and commutes with the right action. Contrary to the right action, the left action depends on the above construction, indexed by the family of paths $\delta(x, x_0)$, equivalently, by the corresponding family of embeddings of $\mathcal{M}$ in $\tilde{\mathcal{M}}$.

Actually, as it is clear by a transformation to the first representation of $\tilde{\mathcal{M}}$, eq.$(3.1)$, the above left action of $\pi_1(\mathcal{M})$ only depends upon the family of isomorphisms between the fundamental groups with different base points $x \in \mathcal{M}$ given by the paths $\delta(x, x_0)$. It is immediate to verify that such isomorphisms are unique (i.e. $\delta(x, x_0)$ independent) iff $\pi_1(\mathcal{M})$ is abelian; in this case, the right and left actions coincide, up to an inversion.

Since, apart from a set of zero measure, all points of $\tilde{\mathcal{M}}$ have a neighbourhood with $[\beta]$ fixed, the measure on $\tilde{\mathcal{M}}$ can be chosen, in the same measure class, as $d\tilde{\mu}(x, [\beta]) = d\mu(x)$, independent of $[\beta]$. This provides the unitary equivalence

$$
L^2(\tilde{\mathcal{M}}, d\tilde{\mu}) \sim L^2(\mathcal{M}, d\mu) \times l^2(\pi_1(\mathcal{M})). \quad (3.9)
$$
In this space, the representations $\tilde{\pi}_S(\mathcal{A}(M))$ takes the form, for simplicity without change in notation,

$$\tilde{\pi}_S(\alpha)\psi(x, [\beta]) = \alpha(x)\psi(x, [\beta])$$  \hspace{1cm} (3.10)

$$\tilde{\pi}_S(U(g_{\lambda v}))\psi(x, [\beta]) = \psi(g^{-1} x, [\theta^{-1}(g, x) \circ \beta])J_g(x)^{1/2},$$  \hspace{1cm} (3.11)

with

$$\theta^{-1}(g, x) \equiv \delta(x_0, g^{-1} x) \circ \gamma_g(g^{-1} x, x) \circ \delta(x, x_0) \in \pi_1(M).$$  \hspace{1cm} (3.12)

d) The left and right regular representation of $\pi_1(M)$ in $L^2(\tilde{M})$

Eq. (3.11) gives the action of $U(g)$ in terms of the left regular representation $R_l$ of $\pi_1(M)$ in $l^2(\pi_1(M))$,

$$(I \times R_l(\theta))\psi(x, [\beta]) = \psi(l(\theta^{-1})(x, [\beta]))$$  \hspace{1cm} (3.13)

On the other hand, the right action of $\pi_1(M)$ in $\tilde{M}$ defines the right regular representation $R_r$ of $\pi_1(M)$ in $l^2(\pi_1(M))$ and a unitary representation $I \times R_r(\eta)$ in $L^2(\tilde{M}, d\mu)$,

$$(I \times R_r(\eta))\psi(x, [\beta]) = \psi(r(\eta^{-1})(x, [\beta])),$$  \hspace{1cm} (3.14)

which, by eqs. (3.10), (3.11), commutes with $\tilde{\pi}_S(\mathcal{A}(M))$.

We recall that the left and right regular representations $R_l, R_r$ of a discrete group $G$ in $l^2(G)$ (with basis $e_g, g \in G$) are unitarily equivalent and generate isomorphic Von Neumann algebras $\mathcal{N}_l, \mathcal{N}_r$, which are the commutant one of the other, $\mathcal{N}_l' = \mathcal{N}_r', \mathcal{N}_r' = \mathcal{N}_l$. Therefore, their centres coincide and give the central decomposition of both $R_l$ and $R_r$ in $l^2(G)$. Moreover, for any central projection $P$, the left and right representations in the corresponding space $P l^2(G)$ have $P e_1$, 1 the identity in $G$, as a cyclic vector and are defined by complex conjugate matrix elements

$$(P e_1, R_l(g) P e_1) = (P e_1, R_r(g) P e_1).$$

Keeping the same notation for $G = \pi_1(M)$, it follows from Theorem 2.9 and eqs. (3.11), (3.13) that $\mathcal{N}_l \cap \mathcal{N}_l'$ is the centre of the Von Neumann closure of the observable algebra, $\tilde{\pi}_S(\mathcal{A}(M))''$. Since, by the irreducibility of the Schroedinger representation, the algebra generated by $\tilde{\pi}_S(\alpha)$ and $\tilde{\pi}_S(U(g_{\lambda v}))$, with $\theta(g, x) = 1$ a.e. in $x$, is weakly dense in $L^2(M)$,

$$\tilde{\pi}_S(\mathcal{A}(M))' = \mathcal{N}_r.$$

We have therefore:
Theorem 3.1  In the representation $\tilde{\pi}_S(\mathcal{A}(\mathcal{M}))$ in

$$L^2(\tilde{\mathcal{M}}, d\tilde{\mu}) \sim L^2(\mathcal{M}, d\mu) \times l^2(\pi_1(\mathcal{M}))$$

the observable factors classifying the representations of $\mathcal{A}(\mathcal{M})$ (Theorem 2.9) are given (a.e.) by the left regular representation of $\pi_1(\mathcal{M})$ in $l^2(\pi_1(\mathcal{M}))$, eq. (3.13).

$\pi_1(\mathcal{M})$ also acts as a gauge group by its right regular representation, eq. (3.14), which generates the commutant of $\tilde{\pi}_S(\mathcal{A}(\mathcal{M}))$.

The observable and gauge representation of $\pi_1(\mathcal{M})$ are unitarily equivalent. The centres of the Von Neumann algebras generated by them coincide and give the same reduction of $\tilde{\pi}_S(\mathcal{A}(\mathcal{M}))$.

e) Completeness of the Dirac approach

If (and only if) $\pi_1(\mathcal{M})$ is amenable, its regular representation contains all its irreducible representations (in the weak, integral decomposition, sense if it is infinite). In its reduction, as recalled above, central projectors give rise to left (observable) and right (gauge) representations $R$ and $\bar{R}$, defined by complex conjugate matrix elements on a cyclic vector. This gives

Theorem 3.2 If $\pi_1(\mathcal{M})$ is amenable, the central decomposition of $\tilde{\pi}_S(\mathcal{A}(\mathcal{M}))$ in $L^2(\tilde{\mathcal{M}}, d\tilde{\mu})$ is given by

$$L^2(\tilde{\mathcal{M}}) = \sum_i L^2(\mathcal{M}) \times K^o_i \times K^g_i,$$

the sum ranging over all the irreducible representations $R_i$ of $\pi_1(\mathcal{M})$. $\pi_1(\mathcal{M})$ acts as an observable group in $K^o_i$, with representation $R_i$, and as a gauge group in $K^g_i$, with a complex conjugate representation $\bar{R}_i$. The sum is replaced by an integral if $\pi_1(\mathcal{M})$ is infinite, with all the irreducible representations appearing in the support of the corresponding measure.

If $\pi_1(\mathcal{M})$ is not amenable, Eqs. (3.10), (3.11) still allow for the construction of all the irreducible LS representations of $\mathcal{A}(\mathcal{M})$ on suitable, non $L^2$ Hilbert spaces of wavefunctions on $\tilde{\mathcal{M}}$.

In fact, given an irreducible unitary representation $\mathcal{R}$ of $\pi_1(\mathcal{M})$ in a (automatically separable) Hilbert space $K$ and a non-zero vector $v \in K$ (cyclic by irreducibility), a scalar product on the vector space $V$ of finite linear combinations $\sum_g \lambda_g e_g, e_g \in \pi_1(\mathcal{M})$ is defined by

$$(e_g, e_h)_\mathcal{R} \equiv (\mathcal{R}(g)v, \mathcal{R}(h)v)_K$$

(3.16)
The left representation \( \rho_l \) of \( \pi_1(M) \) in \( V \), \( \rho_l(g)e_g \equiv e_{g\circ h} \), preserves the above scalar product since

\[
(\rho_l(g)e_h, \rho_l(g)e_m)_\mathcal{R} = (\mathcal{R}(g \circ h) v, \mathcal{R}(g \circ m) v)_K = (\mathcal{R}(h) v, \mathcal{R}(m) v)_K = (e_h, e_m)_\mathcal{R}.
\]

It therefore extends to the Hilbert completion \( \mathcal{V}_\mathcal{R} \) of \( V \), where it is unitarily equivalent to \( \mathcal{R} \), the two representations being given by the same expectations on the cyclic vectors \( e_1, v \).

Then, on the completion \( \mathcal{H}_\mathcal{R} \) of the space of complex functions \( \psi(x, [\beta]) \), on \( \tilde{M} \sim M \times \pi_1(M) \), with finite support in the second argument and scalar product

\[
(\psi(x, [\beta]), \chi(x, [\beta])) \equiv \int (\overline{\psi(x, [\beta])}, \chi(x, [\beta]))_\mathcal{R} d\mu(x) \quad (3.17)
\]

eqs.(3.10), (3.11), (3.12) define a representation of \( \mathcal{A}(M) \) having \( \mathcal{R} \) as associated representation of \( \pi_1(M) \). By Theorem 2.9, we have therefore:

**Proposition 3.3** All the irreducible, locally normal, representations of \( \mathcal{A}(M) \) are unitarily equivalent to representations, defined by eqs. (3.10), (3.11), (3.12), on Hilbert spaces of functions on \( \tilde{M} \), with scalar product defined by eq. (3.17).

If the corresponding representation of \( \pi_1(M) \) is finite dimensional, the scalar product can be chosen so that the gauge group is represented in the resulting spaces. In fact, in this case, eq.(3.10) can be substituted by

\[
(e_g, e_h)_\mathcal{R} \equiv Tr_K(\mathcal{R}(e_g)^{-1}\mathcal{R}(e_h)). \quad (3.18)
\]

The right action of \( \pi_1(M) \),

\[
\rho_r(e_g)e_h \equiv e_{h \circ g^{-1}},
\]

then defines a unitary representation of \( \pi_1(M) \) in \( \tilde{V}_\mathcal{R} \), and therefore in \( \mathcal{H}_\mathcal{R} \), commuting with the representation of \( \mathcal{A}(M) \) and complex conjugate to \( \mathcal{R} \).
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