Photoemission with Chemical Potential from QCD Gravity Dual

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Abstract

We consider a $D4 - D8 - \overline{D8}$ brane construction which gives rise to a large $N$ QCD at sufficiently small energies. Using the gravity dual of this system, we study chiral phase transition at finite chemical potential and temperature and find a line of first order phase transitions in the phase plane. We compute the spectral function and the photon emission rate. The trace of the spectral function is monotonic at vanishing chemical potential, but develops some interesting features as the value of the chemical potential is increased.

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1. Introduction

There are indications that heavy ion collisions at RHIC produce strongly coupled quark gluon plasma (sQGP) (see [1] for a recent review and a list of references). String theory gives rise to a number of gauge/gravity dual pairs where strongly coupled gauge theory is related to gravity. Hence, one may expect gauge/gravity duality (at finite temperature) to be particularly useful for studying the qualitative properties of sQGP. Moreover, one may hope that gauge theory intuition and direct connection with experiment will bring new insights about gravity itself.

A particularly attractive model where the low energy degrees of freedom are just that of large $N_c$ QCD arises from the collection of $N_c$ D4 and $N_f$ $D8-\overline{D8}$ branes [2]. (For some work in this direction see [3-13].) In this model one of the spatial directions along the D4 branes is compactified on a circle of radius $R_4$, which sets the scale of the glueball masses and confinement-deconfinement transition in the theory. Another important parameter in the theory is asymptotic separation of the D8 and $\overline{D8}$ branes, denoted by $L$. It sets the scale of the meson masses and determines the temperature of chiral phase transition in the theory [4]. In the regime where $L \ll R_4$ the two sets of phenomena happen at widely separated scales [3].

In this paper we restrict the analysis to this regime and extend the results of [4] to nonzero chemical potential $\mu$, dual to baryon number. We find the line of first order phase transitions in the $T-\mu$ plane. We also compute the spectral function and the photoemission spectrum in the quark-gluon plasma and find an intriguing picture. While at small chemical potential the spectral function is featureless, at larger values of $\mu$ it exhibits a characteristic peak at intermediate frequencies. We also observe that the photoemission spectrum is redshifted as we increase the value of $\mu$. The nature of these phenomena is not completely clear to us, and it remains to be seen whether these are generic features or they are associated with the particular string model [4].

The rest of the paper is organized as follows. In the next Section we review the near-horizon geometry of the D4 branes at finite temperature and zero chemical potential. As usual, in the limit $g_s N_c \gg 1$, $N_c \to \infty$, $N_f = finite$, one can use the DBI action of $D8$ branes to study the physics of fundamental matter. At sufficiently high temperature

\footnote{The emission of photons associated with the weakly gauged $U(1)_R$ symmetry of $\mathcal{N} = 4$ SYM has been previously calculated in [14]. As is well known, in this case nonzero chemical potential dual to the $U(1)_R$ charge destabilizes the theory.}
or chemical potential the thermodynamically preferred configuration is the one where $D8$ branes fall into the horizon. In this situation baryonic $U(1)$ is no longer broken and the corresponding current can be coupled to a photon via the term of the form $eJ_\mu A^\mu$. The number of photons emitted per unit volume per unit time is given by

$$d\Gamma = \frac{d^3k}{(2\pi)^3} \frac{e^2}{2|k|} \eta^{\mu\nu} C_{\mu\nu}^<(|k^\mu|=|k|)$$

where

$$C_{\mu\nu}^<(|k|) \equiv \int d^4X e^{-iKX} \langle J_\mu(0)J_\nu(X) \rangle = n_b(k^0) \chi_{\mu\nu}(K)$$

In eq. (1.2) $n_b(k^0) = 1/(e^{\beta k^0} - 1)$ and $\chi_{\mu\nu}(K)$ is the spectral function, proportional to the imaginary part of the retarded current-current correlation function,

$$\chi_{\mu\nu}(K) = -2ImC_{\mu\nu}^{ret}(K)$$

which we compute using Lorenzian AdS/CFT prescription of [15].

In Section 3 we explain how the phase transition takes place at finite chemical potential. We generalize the photoemission computation for the case of nonvanishing chemical potential in Section 4. Surprisingly, we find that the spectral function exhibits a bump whose physical significance is not clear to us. We discuss our results in Section 5. Asymptotic behavior of the spectral function is computed in appendix.

### 2. Photoemission at zero chemical potential

We will consider the near-horizon geometry of the D4-branes in the limit studied in [3], where the direction transverse to the D8 branes is non-compact. The near-horizon geometry of the $D4$-branes at finite temperature is:

$$ds^2 = \left(\frac{U}{R}\right)^\frac{3}{2} \left(dx_i dx^i + f(U) dt^2 + (dx^4)^2\right) + \left(\frac{U}{R}\right)^{-\frac{3}{2}} \left(\frac{dU^2}{f(U)} + U^2 d\Omega_4^2\right)$$

where $f(U) = 1 - U_T^3/U^3$. Here $t$ is the Euclidean time, $i = 1, \ldots, 3$; $U$ and $\Omega_4$ label the radial and angular directions of $(x^5, \ldots, x^9)$. The parameter $R$ is given by

$$R^3 = \pi g_s N_c l_s^3 = \pi \lambda$$
where in the last equality and in the rest of the paper we set $\alpha' = 1$. The fourbrane geometry also has a non-trivial dilaton and RR-four form

$$e^\Phi = g_s \left( \frac{U}{R} \right)^{\frac{3}{4}}, \quad F_4 = dC_3 = \frac{2\pi N_c}{V_S} \epsilon_4 \quad (2.3)$$

where $V_S$ and $\epsilon_4$ are the volume and the volume form of the unit 4-sphere. Finite temperature implies that $t$ is a periodic variable,

$$t \sim t + \beta \quad (2.4)$$

On the other hand, in order for (2.1) to describe a non-singular space, $t$ must satisfy

$$t = t + \frac{4\pi R^{3/2}}{3U_T^{1/2}} \quad (2.5)$$

Hence, the temperature is related to the minimal value of $U$, denoted by $U_T$, as $T = 3U_T^{1/2}/4\pi R^{3/2}$.

The induced metric on the D8 branes takes the following form (we assume that the temperature is sufficiently high, so the chiral symmetry is restored [4])

$$ds^2 = \left( \frac{U}{R} \right)^{3/2} \left( \delta_{ij} dx^i dx^j - f(U) dt^2 \right) + \left( \frac{U}{R} \right)^{-3/2} \frac{dU^2}{f(U)} + R^{3/2} U^{1/2} d\Omega_4^2 \quad (2.6)$$

In order to find the photon emission rate we need to find current-current correlators. According to gauge/gravity duality the current-current correlators are computed by analyzing the linearized perturbations of the dual gauge field, which in our case is the electromagnetic field living on the D8 brane. In the following we compute photoemission due to a single brane. The total result contains an additional factor of $2N_f$.

At non-zero temperature, rotation plus the gauge invariance implies that the correlator has the form

$$C^{ret}_{\mu\nu}(k) = P^{T}_{\mu\nu}(k) \Pi^T(k) + P^{L}_{\mu\nu}(k) \Pi^L(k) \quad (2.7)$$

where the non-zero components of the $P^{T}_{\mu\nu}(k)$ are

$$P^{T}_{ij}(k) = \delta_{ij} - \frac{k_i k_j}{k^2} \quad (2.8)$$

and

$$P^{L}_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} - P^{T}_{\mu\nu}(k) \quad (2.9)$$
Substituting in (1.3) we find that the trace of the spectral function is

\[ \chi_{\mu}^{\mu} = -4Im \Pi^T(k_{\mu}) - 2Im \Pi^L(k_{\mu}) \]  

(2.10)

In this paper we are interested in computing the rate of production of real photons, hence we will only consider lightlike momenta. In this case \( \Pi^L(k_{\mu}) \) vanishes (otherwise the correlator (2.7) would be singular on the lightcone) and only \( \Pi^T(k_{\mu}) \) will contribute to the trace of the spectral function.

Now we turn on electromagnetic field on the brane. We will be interested in field configurations which do not depend on the coordinates on \( S^4 \), we also set \( A_M \) along the \( S^4 \) to zero. The DBI action is

\[ S = -g_sT_8V_SR^3 \int d^4x \int dUUe^{-\Phi}\sqrt{-\det(g_{AB} + 2\pi F_{AB})} \]  

(2.11)

where \( (A, B = 0, \cdots, 3, U) \), \( T_8 = 1/(2\pi)^8g_s \) is the tension of the brane and \( V_S = 8\pi^2/3 \) is the volume of the unit four-sphere. Expanding it in the second order in the field strength we find

\[ S = -(2\pi^2)T_8V_SR^{3/2} \int d^4x \int dUU^{5/2} \left[ -\frac{1}{f(U)} \left( \frac{R}{U} \right)^3 \vec{E}^2 - F_{0U}^2 + \left( \frac{R}{U} \right)^3 \sum_{i<j} F_{ij}^2 + f(U) \sum_i F_{iU}^2 \right] \]  

(2.12)

It is convenient to make a change of variables

\[ y = \frac{2R^{3/2}}{\sqrt{U}}, \quad U = \frac{4R^3}{y^2} \]  

(2.13)

which brings (2.12) to the form

\[ S = -\frac{64}{3}\pi^4 T_8 R^6 \int d^4x \int dy y^{-2} \left[ -\frac{1}{f(y)} \vec{E}^2 - F_{0y}^2 + \sum_{i<j} F_{ij}^2 + f(y) \sum_i F_{iy}^2 \right] \]  

(2.14)

where

\[ f(y) = 1 - \frac{y^6}{y_T^6}, \quad y_T = \frac{3}{2\pi T} \]  

(2.15)

The space is restricted to lie in the region \( y \in (0, y_T) \), whose upper limit corresponds to the black hole horizon. It is convenient to perform Fourier transform in the space-time

\[ A_C(x^\mu, y) = \int \frac{d^4k}{(2\pi)^4} e^{i k_{\mu} x^\mu} A_C(k_{\mu}, y). \]  

(2.16)
Choosing \( k_\mu = (-\omega, 0, 0, q) \), we find the following e.o.m. satisfied by the transverse electric field

\[
E''_\perp + \left( \frac{f'}{f} - \frac{2}{y} \right) E'_\perp + \frac{\omega^2 - q^2 f}{f^2} E_\perp = 0
\]

Let us discuss the asymptotics of this equation in the two regimes: \( y \to y_T \) and \( y \to 0 \). In the near horizon limit the equation is solvable and \( E_\perp \) is

\[
E_\perp = (y_T - y)^{\pm i w/2}
\]

where \( w = \omega y_T / 3 = \omega / 2\pi T \). To compute the retarded correlators we should impose the incoming wave boundary condition, which fixes the sign in the exponent of the (2.18) to be negative. Now let us discuss the opposite limit \( y \to 0 \). In this regime there are again two solutions which have the following asymptotics

\[
F_I = 1 + a_3 y^3 / y_T^3 + \cdots \\
F_{II} = y^3 / y_T^3 + \cdots
\]

Setting \( a_3 = 0 \) uniquely determines \( F_I \). The solution of (2.17) with the incoming wave boundary condition has the form

\[
E_\perp = A F_I + B F_{II},
\]

where \( B \) is proportional to \( A \).

The boundary action for \( E_\perp \) can be determined from

\[
\frac{\delta S}{\delta E_\perp(y = 0, k_\mu)}_{EOM} = \lim_{y \to 0} \frac{\delta S}{\delta E'_\perp(y, k_\mu)} = -\frac{128 \pi^4 T^6 R^{16}}{3 \omega^2} \lim_{y \to 0} y^{-2} f(y) E'_\perp.
\]

Using (2.20) and varying this equation with respect to \( E_\perp \), we find

\[
\frac{\delta^2 S}{\delta A_\perp(0, k_\mu) \delta A_\perp(0, -k_\mu)} = \omega^2 \frac{\delta^2 S}{\delta E_\perp(0, k_\mu) \delta E_\perp(0, -k_\mu)} = -128 \pi^4 T^6 y_T^{-3} \frac{B}{A}
\]

Hence

\[
\text{Im}(\Pi^T(k_\mu)) = -\frac{4\pi}{27} (\lambda T) T^2 N_c \text{Im} \left[ \frac{B}{A} \right]
\]

and the trace of the spectral function for lightlike momenta

\[
\chi^\mu_\mu = -4 \text{Im} \Pi^T(q = \omega) = \frac{16 \pi}{27} \lambda T^3 N_c \text{Im} \left[ \frac{B}{A} \right]
\]

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To determine the spectral function we solve the equation (2.17) numerically with the incoming wave boundary condition and determine $\frac{B}{A}$ as a function of $\omega$ (we set $q = \omega$). The result is depicted on Fig. 1.

For better numerical convergence near horizon it is convenient to rewrite the equation (2.17) in terms of slow varying mode defined as

$$E_\perp = (y_T - y)^{-\frac{2\pi}{\omega}} \varphi(y).$$  \hfill (2.25)

Substituting in (2.17) we find that $\varphi$ satisfies

$$\varphi''(y) + \left[ \frac{iw}{y_T - y} - \frac{6y^5}{y_T^6 - y^6} - \frac{2}{y} \right] \varphi'(y) + \left[ \left(-\frac{w^2}{4} + \frac{iw}{2}\right) \frac{1}{(y_T - y)^2} - \left(\frac{6y^5}{y_T^6 - y^6} + \frac{2}{y}\right) \frac{iw}{2} \frac{1}{y_T - y} + \frac{9w^2y^6y_T^4}{(y_T^6 - y^6)^2} \right] \varphi(y) = 0 \hfill (2.26)$$

As expected the exponents of this equation near $y = y_T$ are 0 and $iw$. The solution with zero exponent corresponds to $E_\perp$ with incoming wave boundary condition.

Fig 1. $3\chi^\mu_\mu(\omega)/(\omega y_T)$ in the units of $\frac{16\pi}{27} \lambda T^3 N_c$ as a function of $\omega y_T$.

Next we would like to discuss the asymptotic behavior of $\chi^\mu_\mu$ at high and low temperatures. We start by discussing the low temperature regime $\omega y_T \to \infty$. It turns out that in this regime we can solve (2.17) analytically using generalized WKB approximation. We perform this calculation in Appendix A and find

$$\chi^\mu_\mu = \frac{16\pi}{27}(\lambda T)^2 N_c I_m \left[ \frac{B}{A} \right] \simeq \frac{16\pi}{81} \lambda T^3 N_c \sin \left( \frac{3\pi}{8} \right) \frac{\Gamma \left( \frac{5}{8} \right)}{\Gamma \left( \frac{11}{8} \right)} \left( \frac{\omega y_T}{8} \right)^\frac{3}{8} \hfill (2.27)$$
This approximation works very well at high frequencies as can be seen from Fig. 2. In principle one can improve this result by including higher order terms in the expansion in powers of $\omega^{-1}$.

In the high temperature regime we can solve the equation (2.17) perturbatively. The leading behavior is computed appendix A and yields

$$\chi_\mu = \frac{16\pi}{27} (\lambda T)^2 N_c \text{Im} \left[ \frac{B}{A} \right] \simeq \frac{16\pi}{81} \lambda T^3 N_c \omega y_T.$$  \hspace{1cm} (2.28)

3. Chiral phase transition at finite chemical potential

In this section we Wick rotate time coordinate to study the chiral phase transition at finite chemical potential. Turning on chemical potential is equivalent to turning on non-trivial, purely imaginary $A_0^{(0)}(U)$. Then the average 4d charge density is given by the variation of the action w.r.t. $A_0^{(0)}(\infty)$

$$\rho = iT \frac{\delta S}{\delta A_0^{(0)}(U = \infty)} |_{EOM} = iT \lim_{U \to \infty} \frac{\delta S}{\delta A_0^{(0)}(U)}$$  \hspace{1cm} (3.1)

To find the action we allow non-trivial dependence $x^4 \equiv \tau$ on the radial coordinate $U$ and compute the induced metric on the world volume of the D-brane. The resulting action is

$$S = \frac{T_8 V_S R^{3/2}}{T} \int d^3x \int dU U^{5/2} \sqrt{1 + 4\pi^2 F_0^{(0)} + (U/R)^3 f(U) \tau'^2}$$  \hspace{1cm} (3.2)

\[2\] We use $A_U = 0$ gauge.
It is convenient to replace $A_0^{(0)} \rightarrow iA_0^{(0)}$, so that $\mu = A_0^{(0)}(\infty) + \text{const}$ (we will determine this constant below). Then the equations of motion which follow from (3.2) are

$$\frac{d}{dU} \left[ \frac{2\pi U^{5/2}A_0''^{(0)}}{\sqrt{1 - (2\pi A_0''^{(0)})^2 + \left(\frac{U}{\pi}\right)^3 f(U)\tau'^2}} \right] = 0$$

and

$$\frac{d}{dU} \left[ \frac{U^{11/2}f(U)\tau'}{\sqrt{1 - (2\pi A_0''^{(0)})^2 + \left(\frac{U}{\pi}\right)^3 f(U)\tau'^2}} \right] = 0$$

(3.3)

which should be solved with boundary conditions $\tau(\infty) = \pm L/2$ and $A_0^{(0)}(\tau = \pm L/2) = \mu + \text{const}$. As in the case of zero chemical potential there are two types of solutions: $\tau' = 0$, which has $U(N_f) \times U(N_f)$ chiral symmetry and the U shaped solution, which breaks the symmetry down to $U(N_f)$. The phase in which chiral symmetry is broken does not have free charges, since the quarks form $q\bar{q}$ pairs which are neutral (the baryons are infinitely heavy in $N_c \rightarrow \infty$ limit). Hence we expect that this phase should be unaffected by non-zero chemical potential. On the other hand the phase in which the chiral symmetry is restored has free charges and should be sensitive to non-zero chemical potential. We come to the same conclusion by analyzing the solutions of (3.3). Indeed it is easy to see that for the U shaped solution (3.3) imply that $A_0^{(0)}$ is a monotonic function of $\tau$, hence in order to satisfy the boundary conditions one has to have $A_0^{(0)} = 0$. The situation is very similar to the Hawking-Page transition in AdS space in the presence of chemical potential. In this case the thermal AdS solution does not allow non-zero charge density, while the AdS black hole does [16]. The reason for this behavior from the CFT side is the same as in our situation: the AdS solution corresponds to the confined phase in which we do not have free charges.

For $\tau' = 0$ the action (3.2) takes the following form

$$S = \frac{T_8V_8R^{3/2}}{T} \int d^3 x \int dU U^{5/2} \sqrt{1 - 4\pi^2 F_{0U}^{(0)2}}$$

(3.4)

which yield e.o.m

$$\frac{2\pi U^{5/2}A_0''^{(0)}}{\sqrt{1 - 4\pi^2(A_0''^{(0)})^2}} = C$$

(3.5)

3 The equations (3.3) were obtained in [1], but their $A_0^{(0)}$ for the U shaped solution is not a continuous function at $\tau = 0$.  

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To fix the constant in the relation between $\mu$ and $A_0^{(0)}(\infty)$ we require $4d$ charge density to vanish for $\mu = 0$. This implies

$$\mu = A_0^{(0)}(\infty) - A_0^{(0)}(U_T)$$

(3.6)

Setting $A_0^{(0)}(U_T) = 0$, the chemical potential is

$$\mu = A_0^{(0)}(\infty) = \frac{1}{2\pi} \int_{U_T}^{\infty} dU \sqrt{\frac{C^2}{C^2 + U^5} - \frac{U_T}{C^{2/5}} \, \frac{\Gamma(\frac{3}{10})\Gamma(\frac{6}{5})}{\sqrt{\pi}} \, \frac{2}{\Gamma(2/5)} \, 2F_1 \left( \frac{1}{5}, \frac{1}{2}, \frac{6}{5}, -\frac{U_T^5}{C^2} \right)}$$

(3.7)

The thermal expectation value of $4d$ charge density is related to $C$ as follows

$$\rho = 2\pi T_8 V_S R^{3/2} C$$

(3.8)

The relation between the chemical potential and the charge density simplifies considerably for $C \gg U_T^{5/2}$ and $C \ll U_T^{5/2}$. For the former we find

$$\rho \simeq \sqrt{2} \frac{\Gamma(\frac{3}{10})\Gamma(\frac{6}{5})}{12\pi^2} \left( \frac{\mu^5}{\sqrt{\pi}} \right)^{-5/2} N_c \lambda^{-1/2} \mu^{5/2}$$

(3.9)

while for the latter

$$\rho \simeq N_c \left[ \frac{4\pi}{27} (\lambda T) T^2 \mu + \frac{35}{27} \lambda T \mu^3 \right].$$

(3.10)

Rewriting the conditions of reliability for (3.9) and (3.10) in terms of $\mu$, $\lambda$ and $T$, we find that (3.9) is reliable for $\mu \gg (\lambda T) T$, while (3.10) for $\mu \ll (\lambda T) T$. It is instructive to compare these relations with corresponding relations for $4d$ free fermions. In this case the charge density is

$$\rho_{\text{free}} = \frac{N_c}{6\pi^2} \left[ \mu \pi^2 T^2 + \mu^3 \right]$$

(3.11)

We see that while small $\mu$ behavior in our system bears some similarity with the free fermion case, the large $\mu$ behavior is completely different.

To understand the details of the chiral symmetry restoration we need to compare the grand potentials of the two phases

$$\delta \Omega = \Omega_{\text{st}} - \Omega_U = T_8 V_S R^{3/2} \left[ \int_{U_T}^{U_0} dU \, \frac{U^5}{\sqrt{U^5 + C^5}} + \right.$$  

$$\left. \int_{U_0}^{\infty} dUU^{5/2} \left( \frac{U^{5/2}}{\sqrt{U^5 + C^5}} - \sqrt{1 + \frac{f(U_0)U_0^8}{f(U)U^8 - f(U_0)U_0^8}} \right) \right]$$

(3.12)
where $U_0$ is the value of $U$ at the tip of the U shaped solution. Let us first discuss the $LT \to 0$ case. Solving numerically the equation $\delta \Omega = 0$ we find the critical value for $C$
\[
\frac{U_0}{C^{2/5}} \simeq 1.83
\]  
Using (3.3) to express $U_0$ in terms of the asymptotical separation $L$ and substituting into (3.7) we find
\[
\mu_c \simeq \frac{8R^3L^{-2}}{1.83} \frac{\Gamma(3/10)\Gamma(6/5)}{\sqrt{\pi}} \left[ \frac{\Gamma(9/16)}{\Gamma(6/5)} \right]^2 \simeq 0.22L^{-2}\lambda
\]  
The phase diagram at non-zero $T$ is given by
\[
\mu_c(T) = \phi(LT)\lambda L^{-2},
\]  
where $\phi(x)$ is monotonically decreasing function, which can be determined from (3.12).

4. Photoemission at finite chemical potential

Now we would like to compute the photoemission in the presence of non-zero chemical potential. As before we turn on electromagnetic field on the brane and expand (2.11) in the second order in the field strength. Note that $\text{det}(g_{AB} + 2\pi F_{AB})$ is an even function of $F$, hence
\[
\text{det}(g_{AB} + 2\pi F_{AB}) = \text{det}g_{AB}(1 + (F^2 - \text{terms}) + (F^4 - \text{terms}))
\]  
We set $A_0 = A_0^{(0)} + A_0$, $A_B = A_B$ for $B \neq 0$, $F_{0U} = F_{0U}^{(0)} + F_{0U}$ and $F_{AB} = F_{AB}$ for $AB \neq 0U$. In addition to the DBI action there is also contribution coming from the CS term
\[
S_{CS} = i\mu_9 \int \text{Tr} [\exp(2\pi F_2) \wedge C_3] = i\mu_9 \frac{(2\pi)^3}{3!} \int F \wedge F \wedge F \wedge C_3
\]  
where $\mu_9 = 1/(2\pi)^8 g_s$. Integrating this action over the internal $S^4$ we find
\[
S_{CS} = i\frac{N_c}{24\pi^2} \int A \wedge F \wedge F
\]  
Using the expansion (4.1) we find
\[
S = -2\pi^2 T_8 V_S R^{3/2} \int d^4x \int \frac{dU}{\sqrt{U^5 + C^2}} \left[ -\frac{U^5 + C^2}{f(U)} \left( \frac{R}{U} \right)^3 \bar{E}^2 - \frac{(U^5 + C^2)^2}{U^5} F_{0U}^2 + \left( \frac{R}{U} \right)^3 U^5 \sum_{i<j} F_{ij}^2 + (U^5 + C^2) f(U) \sum_i F_{iU}^2 \right] + S_{CS}
\]  
\[\text{Note that there is a term in action linear in } F_{0U}. \text{ This term enforces the equations of motion for } F_{0U}^{(0)} \text{ hence it is omitted in the expression for the action.} \]
The action in coordinates (2.13) is
\[
S = -\frac{64}{3}\pi^4 T_8 R^6 \int d^4x \int \frac{dy y^{-2}}{\sqrt{1 + \tilde{C}^2 y^{10}/y_T^{10}}} \left[ -\frac{1 + \tilde{C}^2 y^{10}/y_T^{10}}{f(y)} E^2 - (1 + \tilde{C}^2 y^{10}/y_T^{10})^2 F_{0y}^2 + \sum_{i<j} F_{ij}^2 + (1 + \tilde{C}^2 y^{10}/y_T^{10}) f(y) \sum_i F_{iy}^2 \right] + S_{CS}
\]
(4.5)
where
\[
\tilde{C}^2 = \frac{C^2 y_T^{10}}{2^{10} R^{15}} \sim \frac{\rho^2}{(\lambda T)^4 N_c^2 T^6}.
\]
(4.6)
The variation of the Chern-Simons term is
\[
\delta S = i \frac{N_c}{8 \pi^2} \int \delta A \wedge F^{(0)} \wedge F
\]
(4.7)
Taking into account that
\[
\frac{dA_0^{(0)}}{dy} = -8 R^3 \frac{\tilde{C} y^2 y_T^{-5}}{2\pi \sqrt{1 + \tilde{C}^2 y^{10}/y_T^{10}}}
\]
(4.8)
we find that the equations of motion for the transverse modes are
\[
E_1'' + \left( \frac{f'}{f} + \frac{3}{y} - \frac{5}{y} \left( 1 + \frac{1}{1 + \tilde{C}^2 y^{10}/y_T^{10}} \right) \right) E_1' + \frac{\omega^2 - \frac{q^2}{(1 + \tilde{C}^2 y^{10}/y_T^{10}) f}}{f^2} E_1 + \frac{3}{(1 + \tilde{C}^2 y^{10}/y_T^{10}) f} q \tilde{C} y^4 y_T^{-5} E_2 = 0
\]
\[
E_2'' + \left( \frac{f'}{f} + \frac{3}{y} - \frac{5}{y} \left( 1 + \frac{1}{1 + \tilde{C}^2 y^{10}/y_T^{10}} \right) \right) E_2' + \frac{\omega^2 - \frac{q^2}{(1 + \tilde{C}^2 y^{10}/y_T^{10}) f}}{f^2} E_2 - \frac{3}{(1 + \tilde{C}^2 y^{10}/y_T^{10}) f} q \tilde{C} y^4 y_T^{-5} E_1 = 0
\]
(4.9)
Note that the asymptotics of this equation as \( y \to y_T \) and \( y \to 0 \) are the same as for (2.17).
The solution with boundary conditions (2.18) for \( E_\perp = E_1 \) and \( E_2|_{y=y_T} = E_2'|_{y=y_T} = 0 \) has a form
\[
E_1 = A_1 F_I + B_1 F_{II}
\]
(4.10)
\[
E_2 = A_2 F_I + B_2 F_{II}
\]
(4.11)
where the coefficients $A_i, B_i$ can be computed numerically. By rotational invariance, boundary conditions $E_{\perp} = E_2$ and $E_1|_{y=yT} = E'_1|_{y=yT} = 0$ give rise to the solution

$$E_2 = A_1 F_I + B_1 F_{II}$$

$$E_1 = -A_2 F_I - B_2 F_{II}$$

To compute $\chi_\mu^\mu$, we need to take the linear combination of the solutions such that the leading term in $E_2$ (or $E_1$) near the boundary vanishes, and then repeat (2.21)-(2.23).

The spectral function for lightlike momenta is

$$\chi_\mu^\mu = -4 Im \Pi^T(q = \omega) = \frac{16\pi}{27} (\lambda T) T^2 N_c Im \left[ \frac{A_1 B_1 + A_2 B_2}{A_1^2 + A_2^2} \right]$$

Solving (4.9) with incoming wave boundary condition numerically we find the spectral function in the case of non-zero chemical potential. The result is depicted on Fig 3.

![Fig 3.](image)

Fig 3. $3\chi_\mu^\mu(\omega)/(\omega y_T)$ in the units of $\frac{16\pi}{27} \lambda T^3 N_c$ as a function of $\omega y_T$ for $\tilde{C} = 200$. The result is somewhat surprising: while the spectral function for zero chemical potential is featureless, we see that at non-zero chemical potential things change and we get a second maximum. The nature of this peak is not clear to us.

The leading behavior of the spectral function at high temperature is computed in appendix A

$$\chi_\mu^\mu = \frac{16\pi}{27} \lambda T^3 N_c Im \left[ \frac{A_1 B_1 + A_2 B_2}{A_1^2 + A_2^2} \right] \approx \frac{16\pi}{27} (\lambda T) T^2 N_c \omega y_T \sqrt{1 + \tilde{C}^2}.$$
5. Discussion

In this paper we extended the results of [4] to the case of nonvanishing chemical potential. We found a line of first order phase transitions in the $\mu - T$ plane which intersects the axes at ($\mu = 0, T = 0.15L^{-1}$) and ($\mu_c = 0.22L^{-2}\lambda, T = 0$). Note that the value of $\mu_c$ is of the order of constituent quark mass in the phase with broken chiral symmetry.5

We also studied photoemission at finite temperature and chemical potential. At small chemical potential the spectral function is featureless while at larger values of $\mu$ it exhibits a characteristic peak at intermediate frequencies.

Using (1.1) we rewrite the photon emission rate as function of the frequency

$$\frac{d\Gamma}{d\omega} = \frac{e^2}{4\pi} \frac{\omega \chi_{\mu}^4}{\omega/T - 1}$$  \hspace{1cm} (5.1)

Then for the zero chemical potential we plot the spectrum on Fig 4. using the results of section 2. For comparison we also plot the black body radiation spectrum on the same graph

$$\frac{d\Gamma_p}{d\omega} \propto \frac{\omega^2}{e^{\omega/T} - 1}$$  \hspace{1cm} (5.2)

Fig 4. $d\Gamma/d\omega$ in the units of $\frac{4e^2}{81}(\lambda T)T^3N_c$ as a function of $\omega/T$—red curve and the black body radiation spectrum (in arbitrary units)—blue curve.

5 The constituent quark mass is given by the mass of the string which goes from the tip of D8 brane all the way down to the horizon. In this system it is much larger then the meson mass.
We observe that at zero chemical potential the spectrum of radiation is very similar to the Planckian spectrum. As we turn on the chemical potential things start to change. The spectrum becomes more and more red-shifted as we increase the chemical potential.

The natural question is whether something similar happens in other models. One particularly simple way of incorporating fundamental matter involves adding D7 branes to the stack of D3 branes [17]. The resulting theory is four dimensional, but the presence of bosonic fields charged under the $U(1)_V$ does not allow turning on nonzero chemical potential. Similar problems persist for the D4-D6 system recently studied in [18].

The model we consider does not have charged bosonic fields. On the other hand one must bear in mind that it involves four dimensional fundamental matter and five dimensional gauge field and only becomes four dimensional QCD for energies much smaller then $1/R_4$. This is the reason for the appearance of the factors $(\lambda T)$ in the expressions for the spectral functions in this paper. (In comparison, such factors are absent in the four-dimensional model of [14].) It would be nice to better understand the physical origin of the features of the photoemission spectrum in strongly coupled gauge theories at nonvanishing chemical potential.

**Acknowledgements:**

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6 One may still induce nonzero charge density by condensing the corresponding bosonic fields.

7 The value of $\mu$ in this system is bounded from above by the scalar mass, while we observed that the nontrivial features in the spectral density appear at very large values of $\mu$. 

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Appendix A. Asymptotics for the spectral function

In this appendix we find the asymptotic behavior of the spectral function at high and low temperatures. We start by discussing the low temperature regime \(\omega y_T \to \infty\). It turns out that in this regime we can solve (2.17) analytically using generalized WKB approximation. We will follow the approach of [20]. First we introduce new variable

\[ \tilde{E}_\perp \equiv \sqrt{1 - y^6} E_\perp \]  

(A.1)
to eliminate the first derivative term in (2.17)

\[ \tilde{E}_\perp'' + \left[ \frac{\omega^2 y^6}{(1 - y^6)^2} - \frac{2y^{12} - 13y^6 + 2}{y^2(1 - y^6)^2} \right] \tilde{E}_\perp = 0 \]  

(A.2)

In the \(\omega \to \infty\) regime the first term in the brackets dominates almost everywhere on the segment \(y \in (0, 1)\). As we approach \(y = 0\) point the second term starts contributing. Keeping only the most singular contribution from the second term we obtain the approximate equation

\[ \tilde{E}_\perp'' + \left[ \frac{\omega^2 y^6}{(1 - y^6)^2} - \frac{2}{y^2} \right] \tilde{E}_\perp = 0 \]  

(A.3)

Rewriting (A.3) in terms of new variables

\[ z^6 \equiv \left(\frac{dy}{dz}\right)^2 \frac{y^6}{(1 - y^6)^2} \]  

\[ W \equiv \frac{\tilde{E}_\perp}{\sqrt{y}} \]  

(A.4)

we obtain

\[ \frac{d^2W}{dz^2} + \left[ \omega^2 z^6 - \frac{2}{z^2} \right] W = 0. \]  

(A.5)

This equation can be solved analytically in terms of Bessel functions

\[ W(z) = \mathcal{A} \Gamma \left( \frac{5}{8} \right) \left( \frac{\omega}{8} \right)^{\frac{3}{8}} \sqrt{z} J_{-\frac{3}{8}} (\omega z^4/4) + \mathcal{B} \Gamma \left( \frac{11}{8} \right) \left( \frac{\omega}{8} \right)^{-\frac{3}{8}} \sqrt{z} J_{\frac{3}{8}} (\omega z^4/4) \]  

(A.6)

\[ ^8 \text{Here and below we rescaled } y \to y/y_T \text{ and } \omega \to \omega y_T \text{ for simplicity.} \]
As we will see shortly the first term corresponds to the $F_I$ in (2.20), while the second term corresponds to $F_{II}$. Hence we identified the integration constants with $A$ and $B$. Then the approximate solution to (2.17) is

$$E_\perp(y) = \frac{z(y)^{3/2}}{y^{3/2}} - yW(z(y)).$$

(A.7)

To find the spectral function we will need the behavior of $z$ as $y \to 1$ and $y \to 0$. Using (A.4) we find

$$z^{4/4} = \frac{1}{2\sqrt{3}} \left( \arctan \left( \frac{1+2y}{\sqrt{3}} \right) - \arctan \left( \frac{2y-1}{\sqrt{3}} \right) \right) - \frac{1}{6} \log(1-y) - \frac{1}{6} \log(1+y) + \frac{1}{12} \log(1-y+y^2) + \frac{1}{12} \log(1+y+y^2) - \frac{\pi}{6 \sqrt{3}}$$

(A.8)

Hence

$$z = y + \cdots, \quad \text{for } y \to 0$$

$$z^4 = -\frac{2}{3} \log(1-y) + \cdots, \quad \text{for } y \to 1$$

(A.9)

Using the asymptotic behavior of the Bessel functions

$$J_\alpha(x) \simeq \frac{1}{\Gamma(\alpha+1)} \left( \frac{x}{2} \right)^\alpha \quad \text{for } x \to 0$$

(A.10)

we confirm the identifications made in (A.6). To find the spectral function we should look at the solution (A.7) in the opposite regime $y \to 0$ ($z \to \infty$) and impose the incoming wave boundary conditions

$$E_\perp \simeq (1-y)^{-i\omega/6} \simeq e^{i\omega^4/4}$$

(A.11)

Using the asymptotic form of Bessel functions at large value of the argument

$$J_\alpha(x) \simeq \sqrt{\frac{2}{\pi x}} \cos \left( x - \frac{\alpha\pi}{2} - \frac{\pi}{4} \right)$$

(A.12)

we find

$$\Im \left[ \frac{B}{A} \right] = \sin \left( \frac{3\pi}{8} \right) \frac{\Gamma \left( \frac{3}{8} \right)}{\Gamma \left( \frac{1}{8} \right)} \left( \frac{\omega}{8} \right)^{\frac{3}{4}}$$

(A.13)

The case of the finite chemical potential can be treated similarly, although the calculation is more involved and we will not pursue it here.

Now let us discuss the high temperature case $\omega \to 0$. In this regime we can solve equations (2.17) and the corresponding equation for the finite chemical potential (4.9) perturbatively in $\omega = 3w$

$$E_\perp(y) = (1-y)^{-i\omega} \phi(y) = (1-y)^{-i\omega} (F^{(0)}(y) + wF^{(1)}(y) + \cdots)$$

(A.14)
We start by discussing the zero chemical potential case. Then $\varphi(y)$ satisfies equation (2.26). In the $w^0$ order the equation has two solutions one of which is constant while the other is logarithmically divergent at the horizon. The solution with incoming wave boundary conditions correspond to the constant solution which we set to one. Substituting (A.14) into (2.26) we find that $F^{(1)}(y)$ at $w$ order satisfies

$$F^{(1)}(y)'' - \left[ \frac{6y^5}{1-y^6} + \frac{2}{y} \right] F^{(1)}(y)' + \frac{i}{2} \left[ \frac{1}{(1-y)^2} - \left( \frac{6y^5}{1-y^6} + \frac{2}{y} \right) \frac{1}{1-y} \right] = 0$$  \hspace{1cm} (A.15)

Solving (A.15) we find

$$F^{(1)} = \frac{i}{12} \left( (5-2iC_1) \log(1-y) - (1+2iC_1) \log \left( \frac{1+y+y^2}{1+y^3} \right) \right)$$  \hspace{1cm} (A.16)

The coefficient in front of the first term should be zero to satisfy the boundary condition. Using this explicit solution we find that near horizon $E_\perp$ is

$$E_\perp(y) = 1 + iwy^3 + \cdots$$  \hspace{1cm} (A.17)

Hence to the leading order in $w$

$$Im \left[ \frac{B}{A} \right] \simeq w.$$  \hspace{1cm} (A.18)

In the case of finite chemical potential we have

$$E_1 = (1-y)^{-\frac{i\varphi}{y}} \varphi_1(y) = (1-y)^{-\frac{i\varphi}{y}} (F^{(0)}_1(y) + wF^{(1)}_1(y) + \cdots)$$
$$E_2 = (1-y)^{-\frac{i\varphi}{y}} \varphi_2(y) = (1-y)^{-\frac{i\varphi}{y}} (F^{(0)}_2(y) + wF^{(1)}_2(y) + \cdots)$$  \hspace{1cm} (A.19)

The incoming wave boundary conditions for $E_1$ and $E_2 |_{y=y_T} = E_1' |_{y=y_T} = 0$ imply that

$$F^{(0)}_1(y) = 1 \quad and \quad F^{(0)}_2(y) = 0$$  \hspace{1cm} (A.20)

Then in the next to leading order the two equations in (4.9) decouple and we find that $F^{(1)}_1$ satisfies the following equation

$$F^{(1)}_1(y)'' - \left[ \frac{6y^5}{1-y^6} - \frac{3}{y} + \frac{5}{y(1+C^2y^{10})} \right] F^{(1)}_1(y)' + \frac{i}{2} \left[ \frac{1}{(1-y)^2} - \left( \frac{6y^5}{1-y^6} - \frac{3}{y} + \frac{5}{y(1+C^2y^{10})} \right) \frac{1}{1-y} \right] = 0$$  \hspace{1cm} (A.21)
$F_2^{(1)}$ satisfies similar equation, which we will not need at this order. Solving (A.21) we find

$$F_1^{(1)}(y) = C_1 \int_0^y \frac{y^2}{(1 - y^6)\sqrt{1 + \tilde{C}^2 y^{10}}} dy + i \log(1 - y) \quad (A.22)$$

The integral above is logarithmically divergent near $y = 1$. Imposing the regularity condition we find $C_1$. Near $y = 0$ we can again approximate the integral by dropping the square root in the denominator. Hence we find

$$E_1(y) = 1 + iw\sqrt{1 + \tilde{C}^2 y^3} + \cdots, \quad (A.23)$$
$$E_2(y) = wF_2^{(1)}(y) + \cdots$$

and

$$Im \left[ \frac{A_1 B_1 + A_2 B_2}{A_1^2 + A_2^2} \right] = Im \left[ \frac{B_1}{A_1} \right] \simeq w\sqrt{1 + \tilde{C}^2}. \quad (A.24)$$
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