The new form of the equation of state for dark energy fluid and accelerating universe

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We suggest to generalize the dark energy equation of state (EoS) by introduction the relaxation equation for pressure which is equivalent to consideration of the inhomogeneous EoS cosmic fluid which often appears as the effective model from strings/brane-worlds. As another, more wide generalization we discuss the inhomogeneous EoS which contains derivatives of pressure. For several explicit examples motivated by the analogy with classical mechanics the accelerating FRW cosmology is constructed. It turns out to be the asymptotically de Sitter or oscillating universe with possible transition from deceleration to acceleration phase. The coupling of dark energy with matter in accelerating FRW universe is considered, it is shown to be consistent with constrained (or inhomogeneous) EoS.

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\textbf{I. INTRODUCTION}

The number of attempts is aimed to the resolution of dark energy problem (for recent review, see \cite{1,2,3}) which is considered as the most fundamental one in modern cosmology. Among the different descriptions of late-time universe the easiest one is phenomenological approach where it is assumed that universe is filled with mysterious cosmic fluid of some sort. One can mention imperfect fluids \cite{4}, general equation of state (EoS) fluid where pressure is some (power law) function of energy-density \cite{5}, fluids with inhomogeneous equation of state \cite{6}, where EoS with time-dependent bulk viscosity is the particular case \cite{7}, coupled fluids \cite{8,9}, etc. The EoS fluid description may be even equivalent to modified gravity approach as in \cite{10}.

As it has been recently discussed in \cite{11} it is not easy to construct the dark energy model which describes the universe acceleration and on the same time keep untouched the radiation/matter dominated epochs with subsequent transition from deceleration to acceleration. In order to minimize the dark energy effect at intermediate epoch one may speculate about sudden appearance of dark energy around the deceleration-acceleration transition point. In other words, one may suppose that EoS of DE fluid is of the form $p = \theta(t - t_d)w\rho$ where $t_d$ is transition time and $w$ is DE EoS parameter. Before transition point, DE plays a role of usual dust which changes EoS by some unknown scenario. In the similar way, one can generalize other cosmic fluids with more complicated EoSs. This introduces the idea of structure/form changing EoS in different epochs. The simplest example of such cosmic fluid is oscillating dark energy \cite{12,13}. Finally, the reason why cosmic fluid still escapes of direct observations could be that it has completely unexpected properties, for instance, in EoS picture.

In the present letter the new form of dark energy EoS is considered. As the first step, we introduce the relaxation equation for pressure (the analog of energy conservation law for energy-density). It is then shown that such constrained EoS is equivalent to usual but inhomogeneous EoS which is known to be the effective description for brane-worlds or modified gravity \cite{10}. The generalized inhomogeneous EoS which contains time derivatives of pressure is introduced. The number of examples for such EoS cosmic fluids is presented and the corresponding FRW cosmologies are described. It is shown that cosmic speed-up in the examples under consideration corresponds to the asymptotically de Sitter or the oscillating universe where accelerating/decelerating epochs repeat with possibility to cross the phantom barrier or to make transition from deceleration to acceleration. In all cases, dark energy EoS parameter is close to $-1$, being within the observational bounds to it. It is demonstrated that the inclusion of matter may be consistent with constrained EoS.
II. CONSTRANDED EQUATION OF STATE

Let us discuss the possible modification of the equation of state in such a way that it would change its structure/form during the universe evolution. Consider the balance equation for the energy (energy conservation law)

$$\dot{\rho} + 3H(\rho + P) = 0.$$  \hspace{1cm} (1)

It can be represented as a relaxation equation

$$\tau = -\frac{1}{\tau} (\Psi - \Psi_0),$$  \hspace{1cm} (2)

where $\Psi$ coincides with $ρ$, relaxation time $\tau$ is $τ = \frac{1}{\Psi_0}$ and stationary (or equilibrium) value of $ρ$ is $ρ_0 \equiv \Psi_0 = -P$. To formulate consistently this equation we need, as usual, the equation of state (EoS). The standard barotropic EoS is $P = P(ρ)$, providing the equation for $ρ$ only:

$$\frac{1}{3H} \dot{ρ} = -(ρ + P(ρ)).$$  \hspace{1cm} (3)

Let us conjecture now that cosmological fluid is described by different EoS at different epochs. In other words, to describe the transition from one epoch in cosmological evolution to another we try to introduce the transition from one EoS to another, or in simplest form to modify EoS to permit the presence of pressure derivatives.

The simplest way is to introduce the relaxation equation for pressure

$$\tau \dot{P} + P = f(ρ, a(t)).$$  \hspace{1cm} (4)

When $τ = 0$ and $f(ρ, a(t)) = P(ρ)$ we recover the standard EoS. Such an equation may be considered as some (dynamical) constraint to usual EoS. Of course, the physical sense of such equation (unlike to energy conservation law) is not clear at the moment although some explanations are given in the next section. In daily life, however, there could occur similar phenomena where the time change of the pressure depends on the density. For example, consider the water. There is a pressure in the steam, which is the gas of water. When the density increases, the molecules of water make drops of water, like fog. The pressure of the drops could be neglected. At high density, the total pressure could decrease. The equation (4) seems to express such a process. Then if dark energy consists of particles or some objects with internal structure, there may occur the phase transition like that between steam and water drop. At the point of phase transition, since the system becomes unstable, the pressure may be governed by a equation like (4).

We prefer to measure the relaxation time $τ$ in terms of Hubble function $H$, i.e., consider $τH = ξ = const$. In this case it is convenient to use a new variable $x = \frac{a(t)}{a(0)}$. In terms of $x$ the expression $τ \dot{P}$ simplifies as

$$τ \dot{P} = ξ x^2 \frac{dP}{dx}$$  \hspace{1cm} (5)

and one obtains the pair of relaxation type equations for $ρ$ and $P$

$$\frac{1}{3} x^3 \frac{dρ}{dx} + ρ = -P,$$  \hspace{1cm} (6)

$$ξ x^2 \frac{dP}{dx} + P = f(ρ, x).$$  \hspace{1cm} (7)

Extrating $P$ from the first equation and inserting it to the second one we obtain the second order, master equation for the energy density $ρ$

$$x^2 \frac{d^2 ρ}{dx^2} + x \frac{dρ}{dx} \left(4 + \frac{1}{ξ} \right) + 3 ξ [ρ + f(ρ, x)] = 0$$  \hspace{1cm} (8)

This is new, dynamical equation to energy-density which is compatible with energy conservation law.

As the explicit example, let the function $f$ be of the form (of course, more complicated choices may be considered)

$$f(ρ, x) = -ρ + γ(x)(ρ - ρ_c),$$  \hspace{1cm} (9)

where $ρ_c = const$ is some critical value of the energy density, and $γ(x) = γ_0 + α x^m$. When $ρ_c = 0$ and $α = 0$ we recover the standard linear EoS

$$P = (γ_0 - 1)ρ.$$  \hspace{1cm} (10)

The equation for $ρ$ can be reduced to the Bessel equation and the solution is of the form

$$ρ = ρ_c + x^\sigma \left[C_1 J_\nu(\hat{a} x^\lambda) + C_2 J_{-\nu}(\hat{a} x^\lambda)\right].$$  \hspace{1cm} (11)

Here

$$σ = -2 - \frac{m}{2}, \hspace{1cm} λ = \frac{m}{2}, \hspace{1cm} \hat{a} = α^{m/2}.$$  \hspace{1cm} (12)

Using (9), we find

$$P = -ρ_c - \frac{m}{2} + 1 \right) x^\sigma \left[C_1 J_\nu(\hat{a} x^\lambda) + C_2 J_{-\nu}(\hat{a} x^\lambda)\right]$$

$$- \frac{λ \hat{a}}{6} x^{\sigma+\lambda} \left[C_1 (J_{\nu-1}(\hat{a} x^\lambda) - J_{-\nu+1}(\hat{a} x^\lambda))$$

$$+ C_2 (J_{-\nu-1}(\hat{a} x^\lambda) - J_{\nu+1}(\hat{a} x^\lambda))\right].$$  \hspace{1cm} (13)
When $x$ is large, $\rho$ behaves as an oscillating function
\[
\rho \sim \rho_c + \sqrt{\frac{2}{\pi a}} x^{\sigma + \lambda/2} \left\{ C_1 \cos \left( \hat{a} x^\lambda - \frac{2\nu + 1}{4} \right) + C_2 \cos \left( \hat{a} x^\lambda - \frac{2\nu + 1}{4} \right) \right\}.
\]
(14)

Since $\sigma - \lambda/2 = -2 - (3/4)m - 1/\xi$, if we naturally assume that $m$ and $\xi$ should be positive, the second term damps with oscillation. Then $\rho$ goes to a constant $\rho \to \rho_c$. On the other hand, for large $x$, $P$ behaves as
\[
P \sim -\rho_c - \frac{\lambda \hat{a}}{3} \sqrt{\frac{2}{\pi a}} x^{\sigma + \lambda/2} \left\{ C_1 \cos \left( \hat{a} x^\lambda - \frac{2\nu - 1}{4} \right) + C_2 \cos \left( \hat{a} x^\lambda - \frac{2\nu + 1}{4} \right) \right\}.
\]
(15)

Since $\sigma + \lambda/2 = -2 - m/4 - 1/\xi$, when $m$ and $\xi$ are positive, the second term damps with oscillation again and $P$ goes to a constant $P \to -\rho_c$. Then the effective EoS parameter $w \equiv P/\rho$ goes to $-1$, which corresponds to a cosmological constant.

The Bessel function is quasi-oscillating and we obtain an infinite number of epochs, in which $\rho$, $P$, $H$ and $a$ are also quasi-oscillating. In other words we have an infinite number of points in which the deceleration replaces the acceleration and vice-versa.

The presence of $\rho_c$ can guarantee that $\rho$ is positive, thus, $H^2$ is also positive. Nevertheless, $P$ can change its sign, and this phenomenon can mimic the dark energy effect.

When $\rho_c = 0$, $\alpha = 0$ the equation becomes of the Euler type, and the solution is also very simple.

III. THE RELATION WITH STANDARD EQUATION OF STATE.

We now consider the relation with the standard EoS. Let us start with the scale factor dependent EoS:
\[
P = g(\rho, a).
\]
(16)

Then we have
\[
e^{-t/H} \frac{d}{dt} \left( e^{t/H} P \right) = \frac{1}{\tau} P + \dot{P} = \frac{1}{\tau} g(\rho, a) + \frac{\partial g(\rho, a)}{\partial \rho} \dot{\rho} + \frac{\partial g(\rho, a)}{\partial a} a H.
\]
(17)

By using the conservation law (11), one can rewrite (16) in a form similar to (4):
\[
\tau \dot{P} + P = g(\rho, a) + \tau H \left( -3 \frac{\partial g(\rho, a)}{\partial \rho} (\rho + g(\rho, a)) + \frac{\partial g(\rho, a)}{\partial a} a \right).
\]
(18)

When other contributions to the energy density can be neglected, the first FRW equation looks as
\[
\frac{3}{\kappa^2} H^2 = \rho.
\]
(19)

Then Eq. (18) can be rewritten as
\[
\tau \dot{P} + P = g(\rho, a) + \tau \kappa \sqrt{\frac{\rho}{3}} \left( -3 \frac{\partial g(\rho, a)}{\partial \rho} (\rho + g(\rho, a)) + \frac{\partial g(\rho, a)}{a} \right).
\]
(20)

By comparing (20) with (4), we may identify
\[
f(\rho, a) = g(\rho, a) + \tau \kappa \sqrt{\frac{\rho}{3}} \left( -3 \frac{\partial g(\rho, a)}{\partial \rho} (\rho + g(\rho, a)) + \frac{\partial g(\rho, a)}{a} \right).
\]
(21)

This shows the relation between standard (generally speaking, inhomogeneous EoS) and relaxation equation for pressure.

IV. GENERALIZED INHOMOGENEOUS EQUATION OF STATE

As it was indicated above, there is a possibility that the EoS contains $\dot{P}$ or even higher time derivatives of pressure. More generally, the EoS could depend on $H$ or $\dot{H}$ (inhomogeneous EoS) like
\[
U(\rho, P, \dot{P}, H, \dot{H}) = 0.
\]
(22)

Note that many effective dark energy models like brane-worlds, modified gravity and string compactifications have such a form (for very recent example compatible with observational data, see [14] and references therein). As particular example, one may consider
\[
U(\rho, P, \dot{P}, H, \dot{H}) = \dot{P} + \frac{\dot{H}}{H} (\rho + P) + \frac{W(\rho)}{3H}.
\]
(23)
Here \( W(\rho) \) is a proper function of the energy density \( \rho \). Using the energy conservation law (11), one gets
\[
\dot{\rho} = W(\rho) .
\] (24)

If \( \rho \) is regarded as a coordinate, Eq. (24) has a form of Newtonian equation of motion of the classical particle with the “force” \( W \). For example, if \( W \) is a constant \( W = w_0 \), we find \( \rho \) behaves as a coordinate of the massive particle in the uniform gravity:
\[
\rho = \frac{w_0}{2} (t - t_0)^2 + c_0 .
\] (25)
Here \( t_0 \) and \( c_0 \) are constants of the integration. As an another example, we may consider the case of the harmonic oscillator:
\[
W(\rho) = -\omega^2 (\rho - \rho_0) .
\] (26)

Then an oscillating energy density follows [12, 13]:
\[
\rho = \rho_0 + A \sin (\omega t + \alpha) .
\] (27)

If other contributions to the energy density may be neglected, by using the first FRW equation (19), we find the behavior of the Hubble rate, for
\[
H = \frac{\kappa}{\sqrt{3}} \sqrt{\frac{w_0}{2} (t - t_0)^2 + c_0} .
\] (28)

As \( \dot{H} < 0 \) when \( t < t_0 \), and \( \dot{H} > 0 \) when \( t > t_0 \), there is a transition from non-phantom era to phantom one at \( t = t_0 \). For (27), we have oscillating \( H \):
\[
H = \frac{\kappa}{\sqrt{3}} \sqrt{\rho_0 + A \sin (\omega t + \alpha)} .
\] (29)

When we neglect the other contributions to the energy density and pressure, we also have
\[
-\frac{2}{\kappa^2} \dot{H} = \rho + p .
\] (30)

Combining (30) with (19), one may define the effective EoS parameter \( w_{\text{eff}} \) by
\[
w_{\text{eff}} = -1 - \frac{2\dot{H}}{3H^2} .
\] (31)

Hence, for (28)
\[
w_{\text{eff}} = -1 - \frac{t - t_0}{\sqrt{3\kappa (\frac{w_0}{2} (t - t_0)^2 + c_0)^{3/2}}} ,
\] (32)
which surely crosses \( w_{\text{eff}} = -1 \) when \( t = t_0 \). On the other hand, for (29), one gets
\[
w_{\text{eff}} = -1 - \frac{2A \omega \cos (\omega t + \alpha)}{\sqrt{3\kappa (\rho_0 + A \sin (\omega t + \alpha))^{3/2}}} ,
\] (33)
which oscillates around \( w_{\text{eff}} = -1 \) as in [13].

As an another example, we consider the EoS
\[
\dot{P} - 3H (\rho + P) = U(H) .
\] (34)

Here \( U(H) \) is a proper function of the Hubble rate \( H \). Then by using (11), one arrives at
\[
\dot{\rho} + \dot{P} = U(H) .
\] (35)

In a simplest case, \( U(H) = 0 \), it follows
\[
\rho + P = c \ (c : \text{constant}) .
\] (36)

When the other contributions to the energy density and pressure are neglected, because of (30), we find \( \dot{H} \) is constant and
\[
\dot{H} = -\omega^2 \frac{\kappa^2 c}{2} .
\] (37)

As an another case, we may consider
\[
U(H) = \frac{2\omega^2}{\kappa^2} H .
\] (38)
Here \( \omega \) is a constant. Then combining (30), (31), and (38), we find
\[
\dot{H} = -\omega^2 H ,
\] (39)
which is the equation typical for the harmonic oscillator in classical mechanics. Hence, the oscillating Hubble rate is obtained
\[
H = H_0 \sin (\omega t + \alpha) .
\] (40)
Here \( H_0 \) and \( \alpha \) are constants of the integration. Thus, we demonstrated that inhomogeneous generalized EoS (linear in the pressure derivative) leads to the interesting accelerating (often oscillating) late-time universe.

**V. THE EQUATION OF STATE QUADRATIC ON THE PRESSURE DERIVATIVE**

In this section, as an immediate generalization, the case that the EoS is not linear on \( \dot{P} \) but quadratic is considered.

Let the equation to pressure and its derivatives looks like an energy in the classical mechanics:
\[
E = \frac{1}{2} \dot{p}^2 + V(P) .
\] (41)
Here \( E \) is a constant but as it is an analogue of the energy, it is denoted as \( E \). We should note that \( E \) does not correspond to real energy in universe. This may be also considered as implicit form of EoS.

First example is

\[
V(P) = \dot{a}P , \tag{42}
\]

with a constant \( \dot{a} \). Then by the analogy with the classical mechanics, we find

\[
P = -\frac{1}{2} \dot{a}t^2 + v_0t + p_0 ,
\]

\[
E = \frac{1}{2} v_0^2 + \dot{a}p_0 . \tag{43}
\]

Here \( v_0 \) and \( p_0 \) are constants.

In case that other contributions to the total energy density are large, as in the early universe, the Hubble rate \( H \) could not be so rapidly changed. Then we may assume that the Hubble rate \( H \) could be almost constant \( H = H_0 \). Using (1), one obtains

\[
\rho = \rho_0 e^{-3H_0 t}
\]

\[
= -\frac{\ddot{a}}{2} \left( \frac{2}{27H_0^3} - \frac{2t}{9H_0^2} + \frac{t^2}{3H_0} \right)
\]

\[
- v_0 \left( -\frac{1}{9H_0^2} + \frac{t}{3H_0} \right) + p_0 . \tag{44}
\]

Here \( \rho_0 \) is a constant. The explicit form of (inhomogeneous) EoS may be found combining two above equations.

On the other hand, we may also consider the case that the other contributions to the energy density and pressure are neglected as in late-time or future universe. Then deleting \( \rho \) from (1) and (15), we have

\[
\dot{H} + \frac{3}{2} H^2 + \frac{\kappa^2}{2} P = 0 . \tag{45}
\]

For (15), Eq. (16) admits the solution

\[
H = h_0 t + h_1 , \tag{46}
\]

when

\[
\dot{a} = 3h_0 , \quad v_0 = -3h_0 h_1 , \\
p_0 = -h_0 - h_1^2 . \tag{47}
\]

For (16), the effective EoS parameter \( w_{\text{eff}} \) defined by (11) has the following form:

\[
w_{\text{eff}} = -1 - \frac{2h_0}{3(h_0 t + h_1)^2} , \tag{48}
\]

which goes to \(-1\) when \( t \) goes to infinity. Hence, the emerging universe seems to be the asymptotically de Sitter one.

Second example is

\[
V(P) = \frac{1}{2} \omega^2 P^2 . \tag{49}
\]

Then we have

\[
P = A \sin (\omega t + \alpha) \\
E = A \omega^2 . \tag{50}
\]

where \( A \) and \( \alpha \) are constants. Then in the case that other contribution to the total energy density is large, as in the early universe, the Hubble rate \( H \) could be almost constant \( H = H_0 \), we find

\[
\rho = \rho_0 e^{-3H_0 t}
\]

\[
= -\frac{A}{9H_0^2 + \omega^2} (3H_0 \sin (\omega t + \alpha) + \omega \cos (\omega t + \alpha)) , \tag{51}
\]

with a constant \( \rho_0 \). This corresponds to de Sitter universe.

On the other hand, when other contributions to the total energy density can be neglected, as in the late-time universe, by using (15), one gets

\[
\frac{d^2 a^{3/2}}{dt^2} + \frac{3}{4} \kappa^2 A \sin (\omega t + \alpha) a^{1/2} = 0 . \tag{52}
\]

By defining a new variable \( s \)

\[
s \equiv \omega t + \alpha + \frac{\pi}{2} , \tag{53}
\]

one obtains a kind of Mathieu equation:

\[
0 = \frac{d^2 a^{3/2}}{ds^2} + \frac{3\kappa^2}{4\omega^2} \cos s a^{1/2} , \tag{54}
\]

whose solution is given by

\[
a^{3/2} = \sum_{n=0}^{\infty} c_n \cos(nt) + \sum_{n=1}^{\infty} s_n \sin(nt) . \tag{55}
\]

Here the coefficients \( c_n \) and \( s_n \) are given by recursively solving the following equations:

\[
c_0 = c , \quad c_1 = 0 \\
- n^2 c_n + \frac{q}{2} (c_{n-1} + c_{n+1}) = 0 \quad (n \geq 1) , \\
s_1 = s , \quad s_2 = -\frac{2}{q} s , \\
- n^2 s_n + \frac{q}{2} (s_{n-1} - s_{n+1}) = 0 \quad (n \geq 2) , \\
q \equiv \frac{3\kappa^2}{4\omega^2} . \tag{56}
\]
Hence, $a$ has a periodicity $1/\omega$. In the expression \( (55) \), $a$ is not always positive. Then physically the regions where $a^{3/2}$ is not negative could be allowed and the points $a = 0$ could correspond to Big Bang/Big Crunch/Big Rip \[ 16 \].

We should note that the expressions of $\rho_0$ in \[ 14 \] and \[ 51 \] are not always positive. Then only the period(s) where $\rho_0$ is positive could be allowed in the real universe.

**VI. COUPLING WITH THE MATTER**

**A. No direct interaction between dark energy and matter**

Let us now include the matter. For simplicity, we consider the matter with constant EoS parameter $w_m$ so that the matter energy density $\rho_m$ is given by

$$\rho = \rho_0 \left( \frac{a(t)}{a(t_0)} \right)^{-3(1+w_m)}.$$  \( (57) \)

In case of \[ 11 \], the total energy density is given by

$$\rho_{\text{tot}} = \rho_c + x^\sigma \left[ C_1 J_\nu(\tilde{a} x^\lambda) + C_2 J_-\nu(\tilde{a} x^\lambda) \right] + \rho_0 x^{-3(1+w_m)}$$  \( (58) \)

and the Hubble rate $H$ is given by

$$H = \kappa \left\{ \frac{1}{3} \left( \rho_c + x^\sigma \left[ C_1 J_\nu(\tilde{a} x^\lambda) + C_2 J_-\nu(\tilde{a} x^\lambda) \right] + \rho_0 x^{-3(1+w_m)} \right) \right\}^{1/2}.$$  \( (59) \)

In future, $x$ becomes large, then the Hubble rate $H$ goes to a constant (with oscillations):

$$H \to \kappa \sqrt{\frac{\rho_0}{3}},$$  \( (60) \)

which tells $w_{\text{eff}} \to -1$. On the other hand, in the early universe, $x$ should be small. Hence, one finds

$$H = \kappa \left\{ \frac{1}{3} \left( \rho_c + x^\sigma \left[ C_1 \left( \frac{\tilde{a} x^\lambda}{2} \right)^\nu + C_2 \left( \frac{\tilde{a} x^\lambda}{2} \right)^{-\nu} \right] + \rho_0 x^{-3(1+w_m)} \right) \right\}^{1/2}.$$  \( (61) \)

If $-3(1+w_m) < \sigma - \lambda$, the contribution from matter becomes dominant and Hubble rate is

$$H = \kappa \left( \frac{\rho_0}{3} \right)^{1/2} x^{-3(1+w_m)/2},$$  \( (62) \)

which gives, as well-known,

$$H \sim \frac{2}{3(1+w_m)} t.$$  \( (63) \)

On the other hand if $\sigma - \lambda \nu < -3(1 + w_m) < 0$, Hubble rate is

$$H = \kappa \left\{ \frac{C_2}{3} \left( \frac{\dot{a}}{2} \right)^{-\nu} \right\}^{1/2} e^{(\sigma - \lambda \nu)/2},$$  \( (64) \)

which gives

$$H = \frac{2}{3 \kappa} \frac{\sigma - \lambda \nu}{t}.$$  \( (65) \)

By comparing \[ 65 \] with \[ 63 \] or \[ 11 \], it follows that the effective EoS parameter is given by

$$w_{\text{eff}} = -1 - \frac{\sigma - \lambda \nu}{3}.$$  \( (66) \)

For the model in \[ 23 \], solving \[ 21 \], we find the $t$-dependence of $\rho$. Then the FRW equation gives

$$\frac{3}{\kappa^2} H^2 = \rho(t) + \rho_0 \left( \frac{t}{a(t_0)} \right)^{-3(1+w_m)}.$$  \( (67) \)

For the case \[ 25 \], when $t$ is large enough, the second term in the r.h.s. of \[ 67 \] could be neglected and we will obtain \[ 28 \]. If $\rho_0$ in \[ 25 \] is small enough, when $t \sim t_0$, the second term in \[ 67 \] could be dominant and we may obtain \[ 63 \]. For the case \[ 29 \], in the early universe, where $a$ is small, the second term in \[ 67 \] could be dominant and one obtains \[ 63 \], again. Especially for the dust $w_m = 0$, we find $H \sim 2t/3$, that is, $a \sim t^2$. In the late time universe, the first term could be dominant and one gets \[ 29 \].

Three years WMAP data are recently analyzed in Ref. \[ 16 \], which shows that the combined analysis of WMAP with supernova Legacy survey (SNLS) constrains the dark energy equation of state $w_{DE}$ pushing it towards the cosmological constant. The marginalized best fit values of the equation of state parameter at 68% confidence level are given by $-1.14 \leq w_{DE} \leq -0.93$. In case of a prior that universe is flat, the combined data gives $-1.06 \leq w_{DE} \leq -0.90$.

In our models, as shown in \[ 10 \], \[ 14 \], \[ 25 \], and \[ 29 \], the effective EoS parameter is $w_{\text{eff}} \sim -1$ and there is no contradiction with the above WMAP data. We should also note that when matter is coupled, we find $w_{\text{eff}} \sim w_m$ in the early universe, as in \[ 29 \]. Thus when $w_m < -1/3$, there should occur the transition from deceleration to acceleration.
B. Dark energy interacting with matter

Generally speaking, the matter interacts with the dark energy. In such a case, the total energy density $\rho_{\text{tot}}$ consists of the contributions from the dark energy and the matter: $\rho_{\text{tot}} = \rho + \rho_m$. If we define, however, the matter energy density $\rho_m$ properly, we can also define the matter pressure $p_m$ and the dark energy pressure $p$ by

$$ p_m \equiv -\rho_m + \frac{\dot{\rho_m}}{3H}, \quad P \equiv P_{\text{tot}} - P_m. \quad (68) $$

Here $P_{\text{tot}}$ is the total pressure. Hence, the matter and dark energy satisfy the energy conservation laws separately,

$$ \dot{\rho} + 3H (\rho + P) = 0, \quad \dot{\rho} + 3H (\rho + P) = 0. \quad (69) $$

In case, however, that the EoS parameter $w_m$ for the matter is almost constant, one may write the conservation law as

$$ \dot{\rho_m} + 3H (1 + w_m) \rho_m = Q, \quad (70) $$

and therefore for the dark energy

$$ \dot{\rho} + 3H (\rho + P) = -Q, \quad (71) $$

so that the total energy density and the pressure satisfy the conservation law:

$$ \dot{\rho}_{\text{tot}} + 3H (\rho_{\text{tot}} + P_{\text{tot}}) \rho_m = 0. \quad (72) $$

In (70), $Q$ expresses the shift from the constant EoS parameter case.

As an example, we consider the case that $Q$ is given by a function $q = q(a)$ as

$$ Q = H a q'(a) \rho_m. \quad (73) $$

Combining (73) with (70), one gets

$$ \rho_m = \rho_{m0} a^{-3(1 + w_m)} e^{q(a)}. \quad (74) $$

Here $\rho_{m0}$ is a constant of the integration. Hence, the conservation law (11) is modified, through (71) as

$$ \dot{\rho} + 3H (\rho + P) = -\rho_{m0} H a^{-2(3 + w_m)} q'(a) e^{q(a)}, \quad (75) $$

and (11) is also modified as

$$ 1 \frac{dx}{3} \frac{dp}{dx} + \rho = -P - S(x). \quad (76) $$

Here

$$ S(x) \equiv -\frac{\rho_{m0}}{3} (a(t_0)x)^{-2(3 + w_m)} q'(a(t_0)x) e^{q(a(t_0)x)}. \quad (77) $$

Note that Eq. (5) is also modified: it now contains the inhomogeneous terms:

$$ x^2 \frac{d^2 \rho}{dx^2} + x \frac{d\rho}{dx} \left( 4 + \frac{1}{\xi} \right) + \frac{3}{\xi} [\rho + f(\rho, x)] $$

$$ = 3x \frac{dS(x)}{dx} + \frac{3}{\xi} S(x) $$

$$ = 3x^{1-1/\xi} \frac{d}{dx} \left( x^{1/\xi} S(x) \right). \quad (78) $$

Let a (special) solution of (5) be $\rho = \rho_s(x)$. Then in case of (19) with $\gamma(x) = \gamma_0 + \alpha x^m$, the general solution corresponding to (11) is given by

$$ \rho = \rho_s(x) + x^\sigma \left[ C_1 J_\nu (\hat{a} x^\lambda) + C_2 J_{-\nu} (\hat{a} x^\lambda) \right]. \quad (79) $$

where $\rho_s$ should be included in $\rho_s(x)$. It is also noted that the initial conditions are relevant to determine $C_1$ and $C_2$ but irrelevant for $\rho_s(x)$. As an example, we can find

$$ \rho_s(x) = \rho_e + \rho_0 x^\eta, \quad (80) $$

with constants $\rho_0$ and $\eta$ when

$$ e^{q(a)} = e^{q_0} - \frac{\rho_0}{\rho_{m0}} \left[ \left( \eta^2 - 3\eta/4 + \eta/\xi + 3\gamma_0/\xi \right) a_0^{-\eta} a_0^{\eta + 3(1 + w_m)} \right. $$

$$ \left. + \frac{3\alpha/\xi}{(m + \eta + 1/\xi) (m + \eta + 3(1 + w_m))} \right] $$

$$ - \frac{3s_0 a_0^{\nu} x^{-1/\xi + 3(1 + w_m)}}{\rho_{m0} (\nu - 1/\xi + 3(1 + w_m))}, \quad (81) $$

which gives

$$ S = \frac{\rho_0}{3} \left[ \left( \eta^2 - 3\eta/4 + \eta/\xi + 3\gamma_0/\xi \right) x^\eta \right. $$

$$ + \frac{3\alpha x^m + \eta}{(m + \eta + 1/\xi) \xi} \right] + s_0 x^{-1/\xi}. \quad (82) $$

In (81) and (82), $q_0, s_0$ are constants and $a_0 \equiv a(t_0)$. In case that

$$ \eta + 3(1 + w_m), m + \eta + 3(1 + w_m), -1/\xi + 3(1 + w_m) < 0, \quad (83) $$

we find $e^{q(a)} \to e^{q_0}$ when $a$ becomes large, that is, in the late time universe. Thus, $\rho_m \to \rho_{m0} e^{q_0} a^{-2(3 + w_m)}$. Furthermore if $\eta < 0$, we find $\rho_s \to \rho_0$, that is, $H$ goes to a constant, which may lead to the asymptotically deSitter space. Clearly, for more complicated coupling $Q$, more sophisticated accelerating cosmology may be constructed.
VII. DISCUSSION

In summary, we discussed the constrained EoS for cosmic fluid where the relaxation equation for pressure is introduced. It is shown that such EoS is equivalent to usual inhomogeneous EoS which contains scale factor dependent terms. Subsequently, the generalized inhomogeneous EoS with time derivatives of pressure is presented. For the number of explicit examples, the accelerating dark energy cosmology as follows from such EoS cosmic fluid is constructed. It turns out to be the asymptotically de Sitter universe or oscillating universe with long accelerating phase and transition from deceleration to acceleration. The consistent coupling of such constrained EoS dark fluid with matter is discussed. It is shown that emerging FRW cosmology may be consistent with three years WMAP data.

Of course, there are many ways to generalize the EoS for cosmic fluid and to investigate the corresponding impact of such generalization to dark cosmos. The physics behind such generalization remains to be quite obscure (as dark energy itself and its sudden appearance). At best, this may be considered as some phenomenological approximation. Nevertheless, having in mind, that most of modern attempts to understand dark energy including strings/M-theory, brane-worlds, modified gravity, etc lead to effective description in terms of cosmic fluid with unusual form of EoS, it turns out to be extremely powerful approach. From another side, the reconstruction of the cosmic fluid EoS may be done for any given cosmology compatible with observational data which may finally select the true dark energy theory.

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