Blow up and global existence for the periodic Phan-Thein-Tanner model

Yuhui Chen\textsuperscript{a}, Wei Luo\textsuperscript{b,\textasteriskcentered}, Zheng-an Yao\textsuperscript{b}

\textsuperscript{a}School of Aeronautics and Astronautic, Sun Yat-sen University, Guangzhou, 510275, CHINA
\textsuperscript{b}School of Mathematics, Sun Yat-sen University, Guangzhou, 510275, CHINA

Abstract

In this paper, we mainly investigate the Cauchy problem for the periodic Phan-Thein-Tanner (PTT) model. This model is derived from network theory for the polymeric fluid. We prove that the strong solutions of PTT model will blow up in finite time if the trace of initial stress tensor $\text{tr} \tau_0(x)$ is negative. This is a very different with the other viscoelastic model. On the other hand, we obtain the global existence result with small initial data when $\text{tr} \tau_0(x) \geq c_0 > 0$ for some $c_0$. Moreover, we study about the large time behaviour.

2010 AMS Classification: 35A01, 35B45, 35Q35, 76A05, 76D03.

Keywords: The Phan-Thein-Tanner Model; Blow Up; Global Existence.

1. Introduction

In this paper, we consider the initial value problem for the following periodic incompressible Phan-Thein-Tanner (PTT) model:\cite{20,19}:

\[
\begin{aligned}
& u_t + u \cdot \nabla u - \mu \Delta u + \nabla p = \mu_1 \text{div} \tau, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{T}^3, \\
& \tau_t + u \cdot \nabla \tau + (a + b \text{tr} \tau)\tau + Q(\tau, \nabla u) = \mu_2 D(u), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{T}^3, \\
& \text{div} u = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{T}^3, \\
& u|_{t=0} = u_0(x), \quad \tau|_{t=0} = \tau_0(x), \quad x \in \mathbb{T}^3.
\end{aligned}
\]

(PTT)

Here $u$ stands for the velocity and $p$ is the scalar pressure of fluid, $\tau$ is the stress tensor. $D(u)$ is the symmetric part of $\nabla u$, that is

\[D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T).\]
\( Q(\tau, \nabla u) \) is a given bilinear form

\[
Q(\tau, \nabla u) = \tau \Omega(u) - \Omega(u)\tau + \lambda (D(u)\tau + \tau D(u)),
\]

where \( \Omega(u) \) is the skew-symmetric part of \( \nabla u \), namely

\[
\Omega(u) = \frac{1}{2}(\nabla u - (\nabla u)^T).
\]

\( \mu > 0 \) is the viscosity coefficient and \( \mu_1 \) is the elastic coefficient. \( a \) and \( \mu_2 \) are associated to the Debroah number \( De = \frac{\mu_2}{\mu_1} \), which indicates the relation between the characteristic flow time and elastic time. \( \lambda \in [-1,1] \) is a physical parameter. In particular, we call the system co-rotational case when \( \lambda = 0 \). \( b \geq 0 \) is a constant relate to the rate of creation or destruction for the polymeric network junctions.

If \( b = 0 \), the system \( \text{(PTT)} \) reduce to the famous Oldroyd-B model (See \[17\]) which has been studied widely. Let us review some mathematical results for the related Oldroyd type model. C. Guillopé and J. C. Saut \[9, 10\] proved the existence of local strong solutions and the global existence of one dimensional shear flows. In \[7\], E. Fernández-Cara, F. Guillén and R. Ortega studied the local well-posedness in Sobolev spaces. J. Chemin and N. Masmoudi \[4\] proved the local well-posedness in critical Besov spaces and give a low bound for the lifespan. In the co-rotational case, P. L. Lions and N. Masmoudi \[14\] proved the global existence of weak solutions. In \[13\], F. Lin, C. Liu and P. Zhang proved that if the initial data is a small perturbation around equilibrium, then the strong solution is global in time. The similar results were obtained in several papers by virtue of different methods, see Z. Lei and Y. Zhou \[12\], Z. Lei, C. Liu and Y. Zhou \[11\], T. Zhang and D. Fang \[21\], Y. Zhu \[22\]. D. Fang, M. Hieber and R. Zi proved the global existence of strong solutions with a class of large data \[5, 6\]. For the Oldroyd-B model, the global existence of strong solutions in two dimension without small conditions is still an open problem.

In this paper, we focus on the PTT model \( b \neq 0 \). To our knowledge, there are a lot of numerical results about the PTT model (See \[18, 16, 15, 2, 8\]). However, there is no any well-posedness results about the PTT model. The nonlinear term \( (\text{tr} \tau)\tau \) in the PTT model will leads to some interesting phenomenon that is quiet different between the Oldroyd-B model. By virtue of the characteristic method, we prove that the strong solution of \( \text{(PTT)} \) will blow up in finite time when the initial data \( \text{tr} \tau_0 < 0 \). This phenomenon can not be founded in other viscoelastic model.

On the other hand, when \( \text{tr} \tau_0 \) has a positive low bound \( c_0 \), we can prove the global existence of strong solution with small initial data. The idea is inspired by the method applied in \[22\]. The main different is to deal with the nonlinear term \( (\text{tr} \tau)\tau \). In \[22\], the author observe that the linearized
system of the Oldroyd-B model has the form
\[
\begin{cases}
\quad u_t - \Delta u = P \text{ div } \tau, \\
\quad \tau_t = D(u),
\end{cases}
\]
where $P$ is the Leray project operator. Then, one can see that both $u$ and $P \text{ div } \tau$ satisfy the following wave equation with damping term:
\[
W_{tt} - \Delta W_t - \frac{1}{2} \Delta W = 0.
\]
Based on the above dissipative structure of $u$ and $P \text{ div } \tau$, the author in [22] proves the global existence of the strong solution for the Oldroyd-B model with small initial data. However, from the linearized system, we cannot obtain any dissipation for $\tau$. Thus the nonlinear term $(\text{tr } \tau) \tau$ is a difficult term even though the initial data is small. In order to deal with this difficult term, we use the Lagrange coordinate to write the equation of $\text{tr } \tau$, and then we can obtain some decay estimate of the quantity $\text{tr } \tau$. In the later, we will see that the estimate of $\text{tr } \tau$ is depend on some power of $\frac{1}{c_0}$ and the initial data. Since the initial data is small, it follows that $\frac{1}{c_0}$ is very large. The key point in this paper is to control the power of $\frac{1}{c_0}$ such that $\text{tr } \tau$ is still small for any time. Combining the decay estimate of $\text{tr } \tau$ with the dissipative structure for the linearized system of (PTT), we can prove the global existence.

Throughout of the paper, we assume that $a = \lambda = 0$ and $b = \mu = \mu_1 = \mu_2 = 1$. There is no derivative in the additional term $(\text{tr } \tau) \tau$ in (PTT), the proof of local well-posedness for (PTT) is similar to the Oldroyd-B model (See [7, 13]) and we omit the detail here.

Our main results can be stated as follow:

**Theorem 1.1.** Let $(u_0, \tau_0) \in H^2(\mathbb{T}^3)$, assume that $\text{tr } \tau_0(x_0) < 0$ for some $x_0 \in \mathbb{T}^3$, then the corresponding solution to the system (PTT) blows up in finite time. More precisely, there exists a time $T$ with $0 < T \leq -\frac{1}{\text{tr } \tau_0(x_0)}$ such that
\[
\lim_{t \to T^-} \text{tr } \tau(t, q(t, x_0)) = -\infty. \tag{1.1}
\]

**Theorem 1.2.** Suppose that $\text{div } u_0 = 0$, $\int_{\mathbb{T}^3} u_0 dx = 0$, $(\tau_0)_{ij} = (\tau_0)_{ji}$, and the initial data $(u_0, \tau_0) \in H^2(\mathbb{T}^3)$. There exists $\epsilon_0, \tilde{\epsilon}_0$ such that if
\[
0 < \delta_0^2 := \|(u_0, \tau_0)\|_{H^2(\mathbb{T}^3)}^2 \leq \epsilon_0,
\]
\[
0 < c_0^{-2} \| \nabla^2 \text{tr } \tau_0\|_{L^2(\mathbb{T}^3)}^2 \leq \tilde{\epsilon}_0, \quad \text{tr } \tau_0(x) \geq c_0 := \frac{1}{2} \delta_0,
\]
then the problem (PTT) admits a unique global solution $(u(t), \tau(t))$ satisfying that for all $t \geq 0$:
\[
\|u(t)\|_{H^2(\mathbb{T}^3)}^2 + \|\tau(t)\|_{H^2(\mathbb{T}^3)}^2 + \int_0^t \left( \|\nabla u(s)\|_{H^2(\mathbb{T}^3)}^2 + \|P \text{ div } \tau(s)\|_{H^1(\mathbb{T}^3)}^2 + \|\nabla \text{tr } \tau(s)\|_{H^1}^2 \right) ds \leq C \delta_0^2,
\]
where $C > 0$ is a positive constant independent of $t$. Moreover, we have
\[
\|u\|_{H^2(T^3)} + \|\operatorname{P div} \tau\|_{L^2(T^3)} \leq C(\|u_0\|_{H^2(T^3)}, \|\tau_0\|_{H^2(T^3)})(1 + t)^{-\frac{3}{2} + \frac{\epsilon}{2}},
\] (1.2)
where $\epsilon > 0$ is a small constant.

**Remark 1.3.** By virtue of Sobolev's embedding, one can see that
\[
c_0 \leq \operatorname{tr} \tau_0 \leq 3\|\tau_0\|_{L^\infty} \leq 3\|\tau_0\|_{H^2} = 3\delta_0,
\]
it follows that $c_0 = k\delta_0$ for some $k \leq 3$. For simplicity, we choose $k = \frac{1}{2}$ to prove our main result.

**Remark 1.4.** The conditions $c_0^{-2}\|\nabla^2 \operatorname{tr} \tau_0\|_{L^2(T^3)}^2 \leq \tilde{\epsilon}_0$ and $\operatorname{tr} \tau_0 \geq c_0$ are feasible. For example, choose a periodic stress tensor $\tau_0$ such that
\[
\operatorname{tr} \tau_0 = \frac{3}{2}c_0 + \frac{1}{4}c_0^2\epsilon_0 \sin(x_1) \sin(x_2) \sin(x_3).
\]
As $c_0 > 0$ is a small constant, one can easily check that
\[
0 < c_0^{-2}\|\nabla^2 \operatorname{tr} \tau_0\|_{L^2(T^3)}^2 \leq \epsilon_0, \quad \operatorname{tr} \tau_0(x) \geq c_0.
\]

**Remark 1.5.** If $x \in \mathbb{R}^3$ and $\tau \in H^2(\mathbb{R}^3)$, we see that $\lim_{|x| \to \infty} \tau = 0$. Thus, we can't suppose that $\operatorname{tr} \tau_0 \geq c_0$ for some positive constant $c_0$. The Theorem 1.2 is false in this case. The global wellposedness of (PTT) in the whole space is also an interesting problem.

**Remark 1.6.** For the periodic Navier-Stokes equation, if $\int_{T^3} u_0 = 0$, one can obtain the $H^2$-norm decay rate of velocity $u$ is $e^{-ct}$. However, for the system (PTT), there is no any decay for the stress tensor $\tau$, and we only obtain an algebra decay in Theorem 1.2.

The remainder of the paper is organized as follows. In Section 2 we give some key lemmas which will be used in the sequel. In Section 3, we prove the blow up phenomenon of the system (PTT). Section 4 is devoted to study about the global existence and decay rate of the strong solution with small initial data.

# 2. Preliminary

We recall a few estimates for the flow of a smooth vector field.

**Lemma 2.1.** Let $v$ be a smooth time-dependent vector field with bounded first order space derivatives. Let $q_t$ satisfy
\[
q_t(x) = x + \int_0^t v(s, q_s(x)) ds.
\]
Lemma 2.2. For any smooth tensor \([\tau_{ij}]_{3 \times 3}\) and three dimensional vector \(u\), it always holds that

\[
\begin{align*}
\mathbb{P} \text{div}(u \cdot \nabla \tau) &= \mathbb{P}(u \cdot \nabla \text{div} \tau) + \mathbb{P}(\nabla u \cdot \nabla \tau) - \mathbb{P}(\nabla u \cdot \nabla \Delta^{-1} \text{div} \tau), \\
\mathbb{P} \text{div}((\text{tr} \tau) \mathbb{P} \text{div} \tau) &= \mathbb{P}((\text{tr} \tau) \mathbb{P} \text{div} \tau) + \mathbb{P}(\tau \cdot \nabla (\text{tr} \tau)) - \mathbb{P}(\nabla (\text{tr} \tau) \Delta^{-1} \text{div} \tau),
\end{align*}
\]

where

\[
(\nabla u \cdot \nabla \tau)_i = \sum_j \partial_j u \cdot \nabla \tau_{ij}, \quad (\nabla u \cdot \nabla \Delta^{-1} \text{div} \tau)_i = \partial_i u \cdot \nabla \Delta^{-1} \text{div} \tau.
\]

Proof. The first equality has been proved in [22]. We only deal with the second equality. Using direct computation we have

\[
\mathbb{P} \text{div}((\text{tr} \tau) \mathbb{P} \text{div} \tau) = \mathbb{P}(\tau \cdot \nabla (\text{tr} \tau) + (\text{tr} \tau) \text{div} \tau) = \mathbb{P}(\tau \cdot \nabla (\text{tr} \tau)) + \mathbb{P}((\text{tr} \tau) \text{div} \tau).
\]

Denote \(\mathbb{P}^\perp = \Delta^{-1} \nabla \text{div} \), we compute as follows

\[
\begin{align*}
\mathbb{P}((\text{tr} \tau) \mathbb{P} \text{div} \tau) &= \mathbb{P}((\text{tr} \tau) \mathbb{P} \text{div} \tau) + \mathbb{P}((\text{tr} \tau) \Delta^{-1} \nabla \text{div} \tau) \\
&= \mathbb{P}((\text{tr} \tau) \mathbb{P} \text{div} \tau) + \mathbb{P}\nabla((\text{tr} \tau) \Delta^{-1} \text{div} \tau) - \mathbb{P}(\nabla (\text{tr} \tau) \Delta^{-1} \text{div} \tau) \\
&= \mathbb{P}((\text{tr} \tau) \mathbb{P} \text{div} \tau) - \mathbb{P}(\nabla (\text{tr} \tau) \Delta^{-1} \text{div} \tau).
\end{align*}
\]

Hence we proof the Lemma. \(\square\)

By a directly calculation, the following technical lemma holds true.

Lemma 2.3. Let \(r > 0, c_0 > 0\), one has

\[
\int_0^t e^{-(t-s)}(1+c_0s)^{-r}ds \lesssim e^{-\frac{c_0}{2}} \int_0^t (1+c_0s)^{-r}ds \lesssim \begin{cases} c_0^{-1} e^{-\frac{c_0}{2}}, & \text{for } r > 1, \\ c_0^{-1} e^{-\frac{c_0}{2}} (1+c_0t)^r, & \text{for } r = 1, \\ c_0^{-1} e^{-\frac{c_0}{2}} (1+c_0t)^{1-r}, & \text{for } r < 1, \end{cases}
\]

and

\[
\int_0^t e^{-(t-s)}(1+c_0s)^{-r}ds \lesssim (1+c_0t)^{-r} \int_0^t e^{-(t-s)}ds \lesssim (1+c_0t)^{-r},
\]

where \(\epsilon > 0\) is a small but fixed constant.
Notation. Since all function spaces in through out the paper are over $\mathbb{T}^3$, for simplicity, we drop $\mathbb{T}^3$ in the notation of function spaces if there is no ambiguity. $A \lesssim B$ stands for $A \leq CB$ for some constant $C > 0$ independent of $A$ and $B$.

3. Proof of the Theorem 1.1

In this section, we are going to prove that the solution will blow up in finite time.

Proof. Suppose that $T^*$ is the maximal existence time for the solution $(u, \tau)$ with the initial data $u_0$ and $\tau_0$. Applying operator $\text{tr}$ to the second equation of (PTT), and using the fact that $\text{tr} D(u) = \text{div} u = 0$ and $\text{tr} Q(u, \tau) = 0$, we get

$$\text{tr} \tau_t + u \cdot \nabla \text{tr} \tau + (\text{tr} \tau)^2 = 0.$$  \hfill (3.1)

Let us consider the trajectory equation,

$$\frac{d}{dt} q(t, x) = u(t, q(t, x)), \quad q(0, x) = x.$$  

Thanks to (3.1), we have

$$\frac{d}{dt} (\text{tr} \tau)(t, q(t, x)) = -(\text{tr} \tau)^2(t, q(t, x)),$$

which yields that

$$\text{tr} \tau(t, q(t, x)) = \frac{\text{tr} \tau_0(x)}{1 + \text{tr} \tau_0(x)t}, \quad \forall t \in [0, T],$$  \hfill (3.2)

where $0 < T < T^*$. Since $\text{tr} \tau_0(x_0) < 0$, it follows that $\text{tr} \tau(t, q(t, x_0)) \to -\infty$ as $t \to -\frac{1}{\text{tr} \tau_0(x_0)}$. Hence the maximal existence time $T^* \leq -\frac{1}{\text{tr} \tau_0(x_0)}$.

Remark 3.1. Thanks to

$$\lim_{t \to T^*} \text{tr} \tau(t, q(t, x_0)) = -\infty,$$

we have $\lim_{t \to T^*} \|\text{tr} \tau\|_{L^\infty} = \infty$. Using the Sobolev embedding, we see that

$$\lim_{t \to T^*} \|\tau\|_{H^2} \geq \lim_{t \to T^*} C\|\tau\|_{L^\infty} \geq \lim_{t \to T^*} C\|\text{tr} \tau\|_{L^\infty} = \infty.$$

Thus, our theorem implies that the $H^2$-norm of $\tau$ will blow up in finite time. However, we do not know whether the velocity will blow up or not before $T^*$.

Remark 3.2. If $a \neq 0$ and $b > 0$, by the same token we have

$$\frac{d}{dt} (\text{tr} \tau)(t, q(t, x)) = -b(\text{tr} \tau)^2(t, q(t, x)) - a \text{tr} \tau(t, q(t, x)) \leq -\frac{b}{2}(\text{tr} \tau)^2(t, q(t, x)) + \frac{a^2}{2b}.$$  

From the above estimate, we can also obtain a blow up result when the initial data satisfy that $\text{tr} \tau_0(x_0) < -\frac{|a|}{b}$. 

Remark 3.3. Indeed we can obtain the blow up rate. Since
\[
\frac{d}{dt}(\tau(t)) = -(\tau(t))^2,
\]
we have \(\tau(t) \leq \tau_0\) for any \(t\) small initial data and \(\tau\) small.

4. Proof of the Theorem 1.2

Now we turn our attention to prove the global existence and decay rate of the strong solution with small initial data and \(\tau(x) \geq c_0 > 0\). First, we give the some basic energies and time-weighted energies as follows,
\[
\mathcal{E}(t) = \|u_0\|_{L^2(\mathbb{T}^3)}^2 + \|\tau_0\|_{L^2(\mathbb{T}^3)}^2, \quad \bar{\mathcal{E}}(t) = c_0^{-2} \|\nabla^2 \tau_0\|_{L^2(\mathbb{T}^3)}^2,
\]
\[
\mathcal{E}_1(t) = \sup_{0 \leq s \leq t} (\|u(s)\|_{L^2(\mathbb{T}^3)}^2 + \|\tau(s)\|_{L^2(\mathbb{T}^3)}^2) + \int_0^t \|\nabla u(s)\|_{L^2(\mathbb{T}^3)}^2 ds + \int_0^t \|\nabla P \tau(s)\|_{L^2(\mathbb{T}^3)}^2 ds,
\]
\[
\mathcal{E}_2(t) = \sup_{0 \leq s \leq t} (1 + c_0 s)^{3-\epsilon} (\|\nabla^2 u(s)\|_{L^2(\mathbb{T}^3)}^2 + \|\nabla P \tau(s)\|_{L^2(\mathbb{T}^3)}^2),
\]
\[
\mathcal{E}_3(t) = \sup_{0 \leq s \leq t} (1 + s)^{3-\epsilon} (\|\nabla^2 u(s)\|_{L^2(\mathbb{T}^3)}^2 + \|\nabla P \tau(s)\|_{L^2(\mathbb{T}^3)}^2),
\]
\[
\mathcal{E}_4(t) = c_0^{-1} \int_0^t (1 + c_0 s)^{3-\epsilon} \|\nabla \tau(s)\|_{L^2(\mathbb{T}^3)}^2 ds,
\]
\[
\mathcal{E}_5(t) = c_0^{-1} \int_0^t (1 + c_0 s)^{3-\epsilon} \|\nabla^2 \tau(s)\|_{L^2(\mathbb{T}^3)}^2 ds,
\]
for any \(\epsilon > 0\) is a fixed but small constant, and \(P = I - \triangle^{-1} \nabla \div\) is the Leray projection operator.

We shall derive the a priori estimates of \(\mathcal{E}_1(t), \mathcal{E}_2(t), \mathcal{E}_3(t), \mathcal{E}_4(t)\) and \(\mathcal{E}_5(t)\) respectively.

4.1. The estimates of \(\mathcal{E}_1(t)\)

Applying the operator \(\nabla^k\) \((k = 0, 1, 2)\) to the \((PTT)\) system and by virtue of the standard energy estimate, we have
\[
\frac{1}{2} \frac{d}{dt} \left( \|u\|_{H^2}^2 + \|\tau\|_{H^2}^2 \right) + \|\nabla u\|_{H^2}^2 = -2 \sum_{k=0}^2 \int_{\mathbb{T}^3} \nabla^k (u \cdot \nabla u) \nabla^k u dx - \sum_{k=0}^2 \int_{\mathbb{T}^3} \nabla^k (u \cdot \nabla \tau + (\tau \cdot \nabla) \tau + Q(\tau, \nabla u)) \nabla^k \tau dx
\]
\[
+ 2 \sum_{k=0}^2 \int_{\mathbb{T}^3} (\nabla^k \div \nabla^k u + \nabla^k D(u) \nabla^k \tau) dx = I_1 + I_2 + I_3.
\]
For the first term \( I_1 \), notice that \( \text{div} \ u = 0 \), by virtue of Hölder’s inequality and Sobolev’s embedding, we obtain that
\[
\int_0^t I_1(s) ds = - \sum_{k=0}^{2} \int_0^t \int_{\mathbb{T}^3} \nabla^k (u) \cdot \nabla u \nabla^k u(s) dx ds
\]
\[
\lesssim \int_0^t \left( \| \nabla u(s) \|_{L^2} \| \nabla u(s) \|_{L^3} \| \nabla^2 u(s) \|_{L^6} + \| \nabla u(s) \|_{L^3} \| \nabla^2 u(s) \|_{L^2} \right) ds
\]
\[
\lesssim \sup_{0 \leq s \leq t} \| u(s) \|_{H^{3/2}} \int_0^t \| \nabla u(s) \|_{H^{3/2}}^2 ds \lesssim E_1^2(t).
\]

Because of \( \int_{\mathbb{T}^3} u_0(x) dx = 0 \), we deduce from the first equation of [PTT] that \( \int_{\mathbb{T}^3} u(t, x) dx = 0 \). For the term \( I_2 \), by virtue of Poincaré’s inequality, we get that
\[
\int_0^t I_2(s) ds = - \sum_{k=0}^{2} \int_0^t \int_{\mathbb{T}^3} \nabla^k (u) \cdot \nabla \tau + (\text{tr} \ \tau) \tau + Q(\tau, \nabla u)) \nabla^k \tau(s) dx ds
\]
\[
\lesssim \int_0^t \left\{ \| \nabla^3 u(s) \|_{L^2} + \| \nabla^4 u(s) \|_{L^2} \right\} ds
\]
\[
\lesssim \sup_{0 \leq s \leq t} \| \nabla \tau(s) \|_{H^{3/2}} \left( \int_0^t (1 + c_0 s)^{-3 + \epsilon} ds \right)^{\frac{1}{2}}
\]
\[
\times \left( \int_0^t (1 + c_0 s)^{3 - \epsilon} \left( \| \nabla^3 u(s) \|_{L^2} + \| \nabla^4 u(s) \|_{L^2} \right) \right)^{\frac{1}{2}}
\]
\[
\lesssim c_0^{-\frac{1}{2}} E_1(t) E_2^2(t) + E_1(t) E_4^2(t) E_5^2(t) + E_1(t) E_4^2(t) E_5^2(t).
\]

For the last term \( I_3 \), using integration by parts and the symmetry \( \tau_{ij} = \tau_{ji} \), we have
\[
I_3 = \sum_{k=0}^{2} \int_{\mathbb{T}^3} \nabla^k \text{div} \ \tau \nabla^k u + \nabla^k D(u) \nabla^k \tau dx = 0.
\]

Integrating (4.7) with time, according to above estimates, we deduce that
\[
E_{11} = \sup_{0 \leq s \leq t} \left( \| u(s) \|_{H^2}^2 + \| \tau(s) \|_{H^2}^2 \right) + \int_0^t \| \nabla u(s) \|_{H^2}^2 ds
\]
\[
\lesssim E_0(t) + c_0^{-\frac{1}{2}} \left( E_1^2(t) + E_2^2(t) \right) + E_1(t) E_4^2(t) E_5^2(t) + E_1(t) E_4^2(t) E_5^2(t).
\]

Operating \( \mathbb{P} \) on the first equation of the [PTT] system, we have the following equation
\[
u_t + \mathbb{P}(u \cdot \nabla u) - \Delta u = \mathbb{P} \text{div} \ \tau.
\]

Applying \( \nabla^k \ (k = 0, 1) \) to the above equation, taking inner product with \( \nabla^k \mathbb{P} \text{div} \ \tau \), then we obtain
that
\[ \| \mathbb{P} \text{div} \tau \|_{H^1}^2 = \sum_{k=0,1} \int_{\mathbb{T}^3} \nabla^k u_t \nabla^k \mathbb{P} \text{div} \tau dx + \sum_{k=0,1} \int_{\mathbb{T}^3} \nabla^k \mathbb{P} (u \cdot \nabla u) \nabla^k \mathbb{P} \text{div} \tau dx \]
\[ - \sum_{k=0,1} \int_{\mathbb{T}^3} \nabla^k \Delta u \nabla^k \mathbb{P} \text{div} \tau dx = J_1 + J_2 + J_3. \tag{4.9} \]

For the first term \( J_1 \), using integration by parts, we rewrite this term into the following form
\[ J_1 = \sum_{k=0,1} \frac{d}{dt} \left\{ \int_{\mathbb{T}^3} \nabla^k u \nabla^k \mathbb{P} \text{div} \tau dx \right\} - \sum_{k=0,1} \int_{\mathbb{T}^3} \nabla^k u \nabla^k \mathbb{P} \text{div} \tau_1 dx = J_{11} + J_{12}, \]

where
\[ \int_0^t J_{11}(s) ds \lesssim \int_0^t \frac{d}{ds} \left\{ \| u(s) \|_{H^1} \| \tau(s) \|_{H^2} \right\} ds \lesssim \mathcal{E}(0) + E_{11}(t). \]

Operating \( \mathbb{P} \text{div} \) on the second equation of the \( \{\mathbb{P} \mathbb{T} \mathbb{U}\} \) system, we have the following equation
\[ \mathbb{P} \text{div} \tau_1 + \mathbb{P} \text{div}(u \cdot \nabla \tau + (\text{tr} \tau) \tau + Q(\tau, \nabla u)) = \frac{1}{2} \Delta u, \]

which leads to
\[ \int_0^t J_{12}(s) ds = \sum_{k=0,1} \int_0^t \int_{\mathbb{T}^3} (-1)^k \Delta^k u (\mathbb{P} \text{div}(u \cdot \nabla \tau + (\text{tr} \tau) \tau + Q(\tau, \nabla u))) - \frac{1}{2} \Delta u(s) dx ds \]
\[ \lesssim \int_0^t \left\{ \| \nabla u \|_{L^2}^2 + \| u(s) \|_{L^\infty} \| u \|_{L^2} \| \nabla \mathbb{P} \text{div} \tau \|_{L^2} + \| u(s) \|_{L^\infty} \| \nabla u \|_{L^2} \| \tau \|_{H^1} \right. \]
\[ + \| u(s) \|_{L^\infty} \| \text{tr} \tau \|_{L^2} \| \mathbb{P} \text{div} \tau \|_{L^2} \| u \|_{L^\infty} \| \nabla \text{tr} \tau \|_{L^2} \| \tau \|_{L^2} \]
\[ + \left. \| \nabla u \|_{L^2} \| \nabla \text{tr} \tau \|_{L^2} \| \mathbb{P} \text{div} \tau \|_{L^2} \| u \|_{L^\infty} \| \nabla \tau \|_{L^2} \right\} \right\} ds \]
\[ \lesssim \int_0^t \| \nabla u \|_{H^1}^2 ds + \sup_{0 \leq s \leq t} (\| u(s) \|_{H^2} + \| \tau(s) \|_{H^2}) \]
\[ \times \int_0^t (\| \nabla u \|_{H^1}^2 + \| \mathbb{P} \text{div} \tau \|_{H^1}^2 + \| \nabla u \|_{H^2}^2 \| \text{tr} \tau \|_{L^2}^2) ds \]
\[ \lesssim \mathcal{E}_{11} + \mathcal{E}_{11}^2 (t) + c_0 \mathcal{E}_1(t) \mathcal{E}_{11}^2 (t). \]

Thus, we get that
\[ \int_0^t J_1(s) ds \lesssim \mathcal{E}(0) + \mathcal{E}_{11}^2 (t) + c_0 \mathcal{E}_1(t) \mathcal{E}_{11}^2 (t). \]

For the second term \( J_2 \) and the last term \( J_3 \), we can directly compute
\[ \int_0^t J_2(s) ds = \sum_{k=0,1} \int_0^t \int_{\mathbb{T}^3} \nabla^k \mathbb{P} (u \cdot \nabla u) \nabla^k \mathbb{P} \text{div} \tau(s) dx ds \]
\[ \lesssim \int_0^t (\| u(s) \|_{L^\infty} \| \nabla u \|_{H^1} + \| \nabla u(s) \|_{L^\infty} \| \nabla u \|_{L^2}) \| \mathbb{P} \text{div} \tau \|_{H^1} ds \]
\[ \lesssim \sup_{0 \leq s \leq t} \| u(s) \|_{H^2} \left( \int_0^t \| \nabla u(s) \|_{H^2}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t \| \mathbb{P} \text{div} \tau(s) \|_{H^1}^2 ds \right)^{\frac{1}{2}} \lesssim \mathcal{E}_{11}^2 (t), \]
and
\[
\int_0^t J_3(s) ds = - \sum_{k=0, 1} \int_0^t \int_{\mathbb{T}^3} \nabla^k \Delta u \nabla^k P \text{ div } \tau(s) dx ds \lesssim E_1^\frac{1}{11}(t) E_{12}^\frac{1}{12}(t).
\]

Integrating (4.9) with time, combining the above estimates, yields that
\[
E_{12}(t) = \int_0^t \|P \text{ div } \tau(s)\|_{H^1}^2 ds \lesssim E(0) + E_{11} + E_{11}^\frac{1}{1} + c_0^\frac{1}{2} E_1(t) E_{42}^\frac{1}{2}(t).
\]  

(4.10)

Adding up $C_1 \times (4.8)$ and (4.10), choosing $C_1 > 0$ fixed but large enough, then we obtain that
\[
E_1(t) \lesssim E(0) + c_0^\frac{1}{2} E_1(t) E_4^\frac{1}{4}(t) + c_0^\frac{1}{2} \left( E_1^\frac{1}{2}(t) + E_2^\frac{1}{2}(t) + E_3^\frac{1}{3}(t) E_5^\frac{1}{5}(t) + E_1(t) E_5^\frac{1}{5}(t). \right)
\]  

(4.11)

### 4.2. The estimates of $E_2(t)$

Taking $\nabla^2$ on the first equation of the system (4.12), and adding the time weight $(1 + c_0 t)^{3-\epsilon}$, then we have

\[
\frac{1}{2} \frac{d}{dt} \left( (1 + c_0 t)^{3-\epsilon} (\| \nabla^2 u \|^2_{L^2} + 2 \| \nabla \text{ div } \tau \|^2_{L^2}) \right) + (1 + c_0 t)^{3-\epsilon} \| \nabla^3 u \|^2_{L^2}
\]

\[
= \frac{(3-\epsilon)c_0}{2} (1 + c_0 t)^{2-\epsilon} \| \nabla^2 u \|^2_{L^2} + 2 \| \nabla \text{ div } \tau \|^2_{L^2}
\]

\[
- (1 + c_0 t)^{3-\epsilon} \int_{\mathbb{T}^3} \nabla^2 (u \cdot \nabla u) \nabla^2 u dx
\]

\[
- 2 (1 + c_0 t)^{3-\epsilon} \int_{\mathbb{T}^3} \nabla \text{ div } (u \cdot \nabla \tau + (\text{tr } \tau) \tau + Q(\tau, \nabla u)) \nabla \text{ div } \tau dx
\]

\[
+ (1 + c_0 t)^{3-\epsilon} \int_{\mathbb{T}^3} (\nabla^2 \tau \nabla^2 u + \nabla \Delta u \nabla \text{ div } \tau) dx = K_1 + K_2 + K_3 + K_4.
\]  

(4.13)

For the first term $K_1$ and the second term $K_2$, by virtue of Poincare’s inequality, we can directly derive

\[
\int_0^t K_1(s) ds = \int_0^t (3 - \epsilon)c_0 (1 + c_0 s)^{2-\epsilon} (\| \nabla^2 u(s) \|^2_{L^2} + 2 \| \nabla \text{ div } \tau(s) \|^2_{L^2}) ds
\]

\[
\lesssim c_0 \int_0^t (1 + c_0 s)^{3-\epsilon} (\| \nabla^3 u(s) \|^2_{L^2} + \| \nabla \text{ div } \tau(s) \|^2_{L^2}) ds \lesssim c_0 E_2(t),
\]

and

\[
\int_0^t K_2(s) ds = - \int_0^t \int_{\mathbb{T}^3} (1 + c_0 s)^{3-\epsilon} \nabla^2 (u \cdot \nabla u) \nabla^2 u(s) dx ds
\]

\[
\lesssim \int_0^t (1 + c_0 s)^{3-\epsilon} \| u(s) \|_{H^2} \| \nabla^3 u(s) \|^2_{L^2} ds \lesssim E_1^\frac{1}{12}(t) E_2(t).
\]
We turn to deal with the wildest term $K_3$. Applying Lemma 2.2 Hölder’s inequality and Sobolev’s embedding, then we deduce that

\[
\int_0^t K_3(s) ds = -2 \int_0^t \int_{\mathcal{T}_3} (1 + c_0 s)^{3-\varepsilon} \nabla P \cdot \nabla \tau P \, dx \, ds
\]

\[
\leq \int_0^t (1 + c_0 s)^{3-\varepsilon} \left( \| \nabla u(s) \|_{L^\infty} + \| \nabla \tau(s) \|_{L^\infty} \right) \| \nabla^2 \tau(s) \|_{L^2} \left( \| \nabla P \cdot \nabla \tau(s) \|_{L^s} + \| \nabla \tau(s) \|_{H^s} \right) ds
\]

\[
\leq \sup_{0 \leq s \leq t} \| \tau(s) \|_{H^s} \int_0^t (1 + c_0 s)^{3-\varepsilon} \| \nabla^3 u(s) \|_{L^{s+1}} \| \nabla P \cdot \nabla \tau(s) \|_{L^s} + \| \nabla \tau(s) \|_{H^s} \| \nabla P \cdot \nabla \tau(s) \|_{L^s} ds
\]

\[
\leq \mathcal{E}_2(t) + \mathcal{E}_3(t) \left( \mathcal{E}_1(t) + \mathcal{E}_5(t) \right).
\]

Taking advantage of integration by parts and div $u = 0$, we can compute

\[
K_4 = (1 + c_0 s)^{3-\varepsilon} \int_{\mathcal{T}_3} (\nabla^2 \tau \nabla^2 u + \nabla \Delta u \nabla P \cdot \nabla \tau) dx = 0.
\]

Integrating (4.13) with time, combing the above estimates, yields that

\[
\mathcal{E}_{21}(t) = \sup_{0 \leq s \leq t} (1 + c_0 s)^{3-\varepsilon} \| \nabla^3 u(s) \|_{L^s} + \| \nabla P \cdot \nabla \tau(s) \|_{L^s} + \int_0^t (1 + c_0 s)^{3-\varepsilon} \| \nabla^3 u(s) \|_{L^s} ds
\]

\[
\leq \mathcal{E}(0) + c_0 \mathcal{E}_2(t) + \mathcal{E}_3(t) \left( \mathcal{E}_1(t) + \mathcal{E}_5(t) \right).
\]

Operating $\nabla P$ on the first equation of the (PTT) system, we have the following equation

\[
\nabla u_t + \nabla P (u \cdot \nabla u) - \nabla \Delta u = \nabla P \cdot \nabla \tau.
\]

Taking $L^2$ inner product with $\nabla P \cdot \nabla \tau$, adding the time weight $(1 + c_0 t)^{3-\varepsilon}$, then we obtain that

\[
(1 + c_0 t)^{3-\varepsilon} \| \nabla P \cdot \nabla \tau \|_{L^2}
\]

\[
= (1 + c_0 t)^{3-\varepsilon} \int_{\mathcal{T}_3} \nabla u_t \nabla P \cdot \nabla \tau dx + (1 + c_0 t)^{3-\varepsilon} \int_{\mathcal{T}_3} \nabla P (u \cdot \nabla u) \nabla P \cdot \nabla \tau dx
\]

\[
- (1 + c_0 t)^{3-\varepsilon} \int_{\mathcal{T}_3} \nabla \Delta u \nabla P \cdot \nabla \tau dx = L_1 + L_2 + L_3.
\]

For the first term $L_1$, integrating by parts, we see that

\[
L_1 = \frac{d}{dt} \left( (1 + c_0 t)^{3-\varepsilon} \int_{\mathcal{T}_3} \nabla u \nabla P \cdot \nabla \tau dx \right) - (3 - \varepsilon) c_0 (1 + c_0 t)^{2-\varepsilon} \int_{\mathcal{T}_3} \nabla u \nabla P \cdot \nabla \tau dx
\]

\[
- (1 + c_0 t)^{3-\varepsilon} \int_{\mathcal{T}_3} \nabla u \nabla P \cdot \nabla \tau dx = L_{11} + L_{12} + L_{13},
\]

where

\[
\int_0^t L_{11}(s) ds \leq \int_0^t \frac{d}{ds} \left( (1 + c_0 s)^{3-\varepsilon} \| \nabla u(s) \|_{L^2} \| \nabla P \cdot \nabla \tau(s) \|_{L^2} \right) ds \leq \mathcal{E}(0) + \mathcal{E}_{21}(t),
\]
and

\[
\int_0^t L_{12}(s) ds \lesssim c_0 \left( \int_0^t (1 + c_0 s)^{3-\varepsilon} \|
abla^3 u(s) \|^2_{L^2} ds \right)^{\frac{1}{2}} \left( \int_0^t (1 + c_0 s)^{3-\varepsilon} \| \nabla \mathbb{P} \text{ div } \tau(s) \|^2_{L^2} ds \right)^{\frac{1}{2}} \lesssim c_0 \mathcal{E}_{21}^\frac{1}{2}(t) \mathcal{E}_{22}^\frac{1}{2}(t).
\]

Operating \( \mathbb{P} \text{ div} \) on the second equation of the system (P111), we have the following equation

\[
\mathbb{P} \text{ div } \tau + \mathbb{P} \text{ div}(u \cdot \nabla \tau + (\tau \cdot \nabla) \tau + Q(\tau, \nabla u)) = \frac{1}{2} \triangle u,
\]

from which we deduce that

\[
\int_0^t L_{13}(s) ds = - \int_0^t \int_{\mathbb{T}^3} (1 + c_0 s)^{3-\varepsilon} \triangle u \left( \mathbb{P} \text{ div}(u \cdot \nabla \tau + (\tau \cdot \nabla) \tau + Q(\tau, \nabla u)) - \frac{1}{2} \triangle u \right) dx ds.
\]

Applying the Lemma 2.2, we get that

\[
\int_0^t L_{13}(s) ds \lesssim \int_0^t (1 + c_0 s)^{3-\varepsilon} \left\{ \| \nabla^2 u(s) \|^2_{L^2} + \| \nabla^2 u(s) \|_{L^6} \left[ \| u(s) \|_{L^3} \| \nabla \mathbb{P} \text{ div } \tau(s) \|_{L^2} + \| \nabla u(s) \|_{L^2} \| \nabla \tau(s) \|_{L^3} \right]
+ \| \tau(s) \|_{L^2} \left[ \| \nabla \text{ tr } \tau(s) \|_{L^2} + \| \nabla^2 u(s) \|_{L^2} \right] \right\} ds + \| \nabla^2 u(s) \|_{L^2} \| \text{ tr } \tau(s) \|_{L^\infty} \| \mathbb{P} \text{ div } \tau(s) \|_{L^2},
\]

\[
\lesssim \mathcal{E}_{21}(t) + \mathcal{E}_{1}^\frac{2}{7}(t) \mathcal{E}_{2}(t) + c_0 \mathcal{E}_{1}^\frac{2}{7}(t) \mathcal{E}_{2}^\frac{2}{7}(t) \left( \mathcal{E}_{4}^\frac{2}{7}(t) + \mathcal{E}_{5}^\frac{2}{7}(t) \right).
\]

From the above estimates, we obtain that

\[
\int_0^t L_1(s) ds \lesssim \mathcal{E}(0) + \mathcal{E}_{21}(t) + c_0 \mathcal{E}_{21}^\frac{1}{2}(t) \mathcal{E}_{22}^\frac{1}{2}(t) + \mathcal{E}_{1}^\frac{2}{7}(t) + \mathcal{E}_{2}^\frac{2}{7}(t) + c_0 \mathcal{E}_{1}^\frac{2}{7}(t) \mathcal{E}_{2}^\frac{2}{7}(t) \left( \mathcal{E}_{4}^\frac{2}{7}(t) + \mathcal{E}_{5}^\frac{2}{7}(t) \right).
\]

For the second term \( L_2 \) and the last term \( L_3 \), we can directly derive

\[
\int_0^t L_2(s) ds \lesssim \int_0^t (1 + c_0 s)^{3-\varepsilon} \left\{ \| u \|_{L^\infty} \| \nabla^2 u \|_{L^2} + \| \nabla u \|_{L^\infty} \| \nabla u \|_{L^2} \right\} \| \nabla \mathbb{P} \text{ div } \tau \|_{L^2} ds \lesssim \mathcal{E}_{1}^\frac{2}{7}(t) \mathcal{E}_{2}(t),
\]

and

\[
\int_0^t L_3(s) ds \lesssim \int_0^t (1 + c_0 s)^{3-\varepsilon} \| \nabla^3 u \|_{L^2} \| \nabla \mathbb{P} \text{ div } \tau \|_{L^2} ds \lesssim \mathcal{E}_{21}^\frac{1}{2}(t) \mathcal{E}_{22}^\frac{1}{2}(t).
\]

Integrating (4.15) with time, combining the above estimates, then we have

\[
\mathcal{E}_{22}(t) = \int_0^t (1 + c_0 s)^{3-\varepsilon} \| \nabla \mathbb{P} \text{ div } \tau(s) \|^2_{L^2} ds \lesssim \mathcal{E}(0) + \mathcal{E}_{21}(t) + \mathcal{E}_{1}^\frac{2}{7}(t) + \mathcal{E}_{2}^\frac{2}{7}(t) + c_0 \mathcal{E}_{1}^\frac{2}{7}(t) \mathcal{E}_{2}^\frac{2}{7}(t) \left( \mathcal{E}_{4}^\frac{2}{7}(t) + \mathcal{E}_{5}^\frac{2}{7}(t) \right),
\]

Adding up \( C_2 \times (4.14) \) and (4.16), choosing \( C_2 > 0 \) is fixed but large enough, then we obtain that

\[
\mathcal{E}_2(t) \lesssim \mathcal{E}(0) + c_0 \mathcal{E}_{1}^\frac{2}{7}(t) \mathcal{E}_{2}^\frac{2}{7}(t) \left( \mathcal{E}_{4}^\frac{2}{7}(t) + \mathcal{E}_{5}^\frac{2}{7}(t) \right) + \mathcal{E}_{1}^\frac{2}{7}(t) + \mathcal{E}_{2}^\frac{2}{7}(t).
\]
4.3. The estimates of $E_3(t)$

The main difficult is to estimate $E_3(t)$. Let us consider the initial value problem for the following system:

\[
\begin{aligned}
&u_t - \Delta u - \mathbb{P} \text{div} \tau = -\mathbb{P}(u \cdot \nabla u), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{T}^3, \\
&(\mathbb{P} \text{div} \tau)_t - \frac{1}{2} \Delta u = -\mathbb{P} \text{div}(u \cdot \nabla \tau) + (\text{tr} \tau) \tau + Q(\tau, \nabla u), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{T}^3, \\
&\text{div} u = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{T}^3, \\
&u|_{t=0} = u_0(x), \quad \tau|_{t=0} = \tau_0(x), \quad x \in \mathbb{T}^3.
\end{aligned}
\]  \hspace{1cm} (4.18)

In terms of the semigroup theory, the solution $(u, \mathbb{P} \text{div} \tau)$ of the (4.18) can be expressed for $U = (u, \mathbb{P} \text{div} \tau)^t$ as

\[
U_t(t, x) = BU(t, x) + H(t, x), \quad U(0, x) = U_0(x), \quad t \geq 0,
\]

where $H(t, x) = (H_1(t, x), H_2(t, x))^t = (-\mathbb{P}(u \cdot \nabla u)(t, x), -\mathbb{P} \text{div}(u \cdot \nabla \tau) + (\text{tr} \tau) \tau + Q(\tau, \nabla u))(t, x)^t$.

Applying the Fourier transform to the system (4.18), we have

\[
\hat{U}_t(t, k) = A(k)\hat{U}(t, k) + \hat{H}(t, k), \quad \hat{U}(0, k) = \hat{U}_0(k), \quad t \geq 0,
\]

where $k = (k_1, k_2, k_3) \in \mathbb{Z}^3$, $A(k)$ is defined as

\[
A(k) = \begin{pmatrix}
-|k|^2 I_{3 \times 3} & I_{3 \times 3} \\
-\frac{|k|^2}{2} I_{3 \times 3} & 0
\end{pmatrix}.
\]

The eigenvalues of the matrix $A(k)$ are computed from the determinant

\[
\det(A(k) - \lambda I) = \left(\lambda^2 + |k|^2 \lambda + \left|\frac{k}{2}\right|^3\right) = 0,
\]

which implies

\[
\lambda_1 = \lambda_1(|k|)\text{(triple root)}, \quad \lambda_2 = \lambda_2(|k|)\text{(triple root)}.
\]

Thus, the semigroup $e^{tA(k)}$ is expressed as

\[
e^{tA(k)} = e^{\lambda_1 t} P_1 + e^{\lambda_2 t} P_2,
\]

where the project operators $P_i$ can be computed as

\[
P_i = \prod_{j \neq i} \frac{A(k) - \lambda_j I}{\lambda_i - \lambda_j}.
\]

By a direct computation, we deduce the Fourier transform $\hat{G}(t, k)$ of Green’s function $G(t, x)$ of $e^{tB}$ as

\[
\hat{G}(t, k) = e^{tA(k)} = \left(\begin{pmatrix}
\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} & \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \\
\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{2(\lambda_1 - \lambda_2)} & \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}
\end{pmatrix} I_{3 \times 3} \right) = \left(\begin{pmatrix}
\hat{N}(t, k) \\
\hat{M}(t, k)
\end{pmatrix}
\right).
\]
Then we have the following decomposition for the Fourier transform of \( u, \mathbb{P} \text{ div } \tau \) as

\[
\hat{u}(t, k) = \hat{N}(t, k) \cdot \hat{U}_0(k) + \int_0^t \hat{N}(t - s, k) \cdot \hat{H}(s, k) ds,
\]

\[
\mathbb{P} \text{ div } \tau(t, k) = \hat{M}(t, k) \cdot \hat{U}_0(k) + \int_0^t \hat{M}(t - s, k) \cdot \hat{H}(s, k) ds.
\] (4.19)

We need to verify the approximation of the Fourier transform of Green’s function for all \( k \in \mathbb{Z}^3 \).

If \( |k| \leq 1 \), we see that

\[
\lambda_1 = -\frac{1}{2} + \frac{i}{2}, \quad \lambda_2 = -\frac{1}{2} - \frac{i}{2},
\] (4.20)

which leads to

\[
\frac{\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \lesssim e^{-\frac{t}{2}}, \quad \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} = 2e^{-\frac{t}{2}} \sin(t/2) \lesssim e^{-\frac{t}{2}}.
\]

If \( |k| \geq 2 \), we see that

\[
\lambda_1 = -\frac{1}{2} |k|^2 - \frac{1}{2} \sqrt{(|k|^2 - 1)^2 - 1}, \quad \lambda_2 = -\frac{1}{2} |k|^2 + \frac{1}{2} \sqrt{(|k|^2 - 1)^2 - 1},
\] (4.21)

and we have

\[
\frac{\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \lesssim e^{-\frac{t}{2}}, \quad \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \lesssim \frac{1}{|k|^2} e^{-\frac{t}{2}}.
\]

An easy computation together with the formula of the \( \hat{G}(t, k) \) gives

\[
\hat{N}(t, k) \cdot \hat{U}_0(k) = (\frac{\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}}{\lambda_1 - \lambda_2} - \frac{|k|^2 (e^{\lambda_1 t} - e^{\lambda_2 t})}{\lambda_1 - \lambda_2}) \hat{u}_0(k) + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \mathbb{P} \text{ div } \tau_0(k),
\]

and

\[
\hat{M}(t, k) \cdot \hat{U}_0(k) = -\frac{|k|^2 (e^{\lambda_1 t} - e^{\lambda_2 t})}{2(\lambda_1 - \lambda_2)} \hat{u}_0 + \frac{\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \mathbb{P} \text{ div } \tau_0.
\]

Since \( \int_{\mathbb{T}^3} u(t, x) dx = 0, \int_{\mathbb{T}^3} \mathbb{P} \text{ div } \tau(t, x) dx = 0, \forall t \geq 0 \) it follows that \( \hat{U}(0) = 0 \). Then we have the \( L^2 \)-decay rate on the derivatives of \((N \ast U_0, M \ast U_0)\) as

\[
\|\nabla^\alpha (N \ast U_0)(t)\|_{L^2(\mathbb{T}^3)}^2 \lesssim \sum_{k \in \mathbb{Z}^3} |\hat{N}(t, k) \cdot \nabla^\alpha \hat{U}_0(k)|^2
\]

\[
\lesssim \sum_{k \in \mathbb{Z}^3/\{0\}} e^{-t} |\nabla^\alpha \hat{u}_0(k)|^2 + \sum_{k \in \mathbb{Z}^3/\{0\}} \frac{e^{-t} |\nabla^\alpha \tau_0(k)|^2}{|k|^2} \]

\[
\lesssim e^{-t} (\|\nabla^\alpha \hat{u}_0\|_{L^2(\mathbb{T}^3)}^2 + \|\nabla^\alpha \tau_0\|_{L^2(\mathbb{T}^3)}^2), \quad \forall \alpha \in \mathbb{N}^3,
\] (4.22)
Applying a similar argument as in the estimate of the derivatives of \((N \ast u_0, M \ast u_0)\), then we can obtain that
\[
\|\nabla^\alpha N(t-s,k) \ast H(s,k)\|_{L^2(T^3)}^2 \lesssim \sum_{k \in \mathbb{Z}^3} e^{-t-s} \left( \|\nabla^\alpha (u \cdot \nabla u)(s,k)\|^2 + \|\nabla^\alpha \mathbb{P} \text{div}(u \cdot \nabla \tau + (\text{tr} \tau) \tau + Q(\tau, \nabla u))(s,k)\|^2 \right)
\]
and
\[
\|\nabla^\alpha M(t-s,k) \ast H(s,k)\|_{L^2(T^3)}^2 \lesssim \sum_{k \in \mathbb{Z}^3} e^{-t-s} \left( \|\nabla^\alpha (u \cdot \nabla u)(s,k)\|^2 + \|\nabla^\alpha \mathbb{P} \text{div}(u \cdot \nabla \tau + (\text{tr} \tau) \tau + Q(\tau, \nabla u))(s,k)\|^2 \right)
\]
where \(\tilde{\alpha} \in \{\alpha, \alpha - 1, \alpha - 2\}\), and
\[
\|\nabla^\alpha N(t-s,k) \ast H(s,k)\|_{L^2(T^3)}^2 \lesssim \sum_{k \in \mathbb{Z}^3} e^{-t-s} \left( \|\nabla^\alpha (u \cdot \nabla u)(s,k)\|^2 + \|\nabla^\alpha \mathbb{P} \text{div}(u \cdot \nabla \tau + (\text{tr} \tau) \tau + Q(\tau, \nabla u))(s,k)\|^2 \right)
\]
\[
\|\nabla^\alpha M(t-s,k) \ast H(s,k)\|_{L^2(T^3)}^2 \lesssim \sum_{k \in \mathbb{Z}^3} e^{-t-s} \left( \|\nabla^\alpha (u \cdot \nabla u)(s,k)\|^2 + \|\nabla^\alpha \mathbb{P} \text{div}(u \cdot \nabla \tau + (\text{tr} \tau) \tau + Q(\tau, \nabla u))(s,k)\|^2 \right)
\]
\[
\lesssim e^{-t-s} \left( \|\nabla^\alpha (u \cdot \nabla u)(s)\|_{L^2}^2 + \|\nabla^\alpha \mathbb{P} \text{div}(u \cdot \nabla \tau + (\text{tr} \tau) \tau + Q(\tau, \nabla u))(s)\|_{L^2}^2 \right), \quad \forall \alpha \in \mathbb{N}^3.
\]
Specially, together with (4.22), (4.23), (4.24) and (4.25), we get that
\[
\|u(t)\|_{L^1} \lesssim e^{-\frac{t}{4}} \mathcal{E}^\frac{1}{2}(0) + \sum_{\alpha = 0}^2 \int_0^t e^{-\frac{s}{4}} \|\nabla^\alpha (u \cdot \nabla u)(s)\|_{L^2} ds + \int_0^t e^{-\frac{s}{4}} \mathbb{P} \text{div}(u \cdot \nabla \tau)(s)\|_{L^2} ds
\]
\[
+ \int_0^t e^{-\frac{s}{4}} \mathbb{P} \text{div}(\text{tr} \tau \tau)(s)\|_{L^2} ds + \int_0^t e^{-\frac{s}{4}} \|\mathbb{P} \text{div} Q(\tau, \nabla u)(s)\|_{L^2} ds
\]
\[
= e^{-\frac{t}{4}} \mathcal{E}^\frac{1}{2}(0) + O_1 + O_2 + O_3 + O_4.
\]
Applying the Lemma 2.3 we verify that
\[
O_1 \lesssim \int_0^t e^{-\frac{s}{4}} \|u(s)\|_{H^2} \|\nabla u(s)\|_{H^2} ds
\]
\[
\lesssim \sup_{0 \leq s \leq t} \|u\|_{H^2} \left( \int_0^t (1 + c_0 t)^{3-\epsilon} \|\nabla^3 u(s)\|_{L^2}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t e^{-(t-s)} (1 + c_0 t)^{-3+\epsilon} ds \right)^{\frac{1}{2}}
\]
\[
\lesssim e^{-\frac{1}{4}}(t) \mathcal{E}^\frac{1}{2}(t) \left( e^{-\frac{t}{4}} + (1 + c_0 t)^{-\frac{1}{2}+\epsilon} \right).
\]
Taking advantage of the Lemma 2.2 we deduce that
\[
O_2 \lesssim \int_0^t e^{-\frac{s}{4}} \left( \|u\|_{L^\infty} \|\nabla^\mathbb{P} \text{div} \tau(s)\|_{L^2} + \|\nabla u\|_{L^6} \|\nabla \tau\|_{L^3} \right) ds \lesssim e^{-\frac{1}{4}}(t) \mathcal{E}^\frac{1}{2}(t) \left( e^{-\frac{t}{4}} + (1 + c_0 t)^{-\frac{1}{2}+\epsilon} \right),
\]
4.4. The estimates of \( E_4(t) \) and \( E_5(t) \)

From (3.2), we have

\[
\text{tr} \, \tau(t, x) = \frac{\text{tr} \, \tau_0(q_t^{-1}(x))}{1 + \text{tr} \, \tau_0(q_t^{-1}(x))t}, \quad \forall t \in [0, T].
\]

which implies that

\[
\nabla \text{tr} \, \tau(t, x) = \frac{(\nabla q_t^{-1}(x) \cdot \nabla) \text{tr} \, \tau_0(q_t^{-1}(x))}{(1 + \text{tr} \, \tau_0(q_t^{-1}(x))t)^2},
\]

and

\[
\nabla^2 \text{tr} \, \tau(t, x) = \frac{\partial_k \partial_l \text{tr} \, \tau_0(q_t^{-1}(x)) \cdot \nabla(q_t^{-1}(x))^k \cdot \nabla(q_t^{-1}(x))^l}{(1 + \text{tr} \, \tau_0(q_t^{-1}(x))t)^2} + \frac{\partial_l \text{tr} \, \tau_0(q_t^{-1}(x)) \cdot \nabla^2(q_t^{-1}(x))^k}{(1 + \text{tr} \, \tau_0(q_t^{-1}(x))t)^2} - \frac{2t \cdot \partial_l \text{tr} \, \tau_0(q_t^{-1}(x)) \cdot \partial_l \text{tr} \, \tau_0(q_t^{-1}(x)) \cdot \nabla(q_t^{-1}(x))^k \cdot \nabla(q_t^{-1}(x))^l}{(1 + \text{tr} \, \tau_0(q_t^{-1}(x))t)^3}.
\]

Taking \( L^2 \)-norm to the above equations and using the fact that \( \text{div} \, u = 0 \), then we deduce that

\[
\| \nabla \text{tr} \, \tau(t, x) \|_{L^2}^2 \lesssim \| \nabla q_t^{-1}(x) \|_{L^2}^2 \int_{\mathbb{T}^3} \frac{|\nabla \text{tr} \, \tau_0(q_t^{-1}(x))|^2}{|1 + \text{tr} \, \tau_0(q_t^{-1}(x))t|^4} \, dx \lesssim \| \nabla q_t^{-1}(x) \|_{L^2}^2 \int_{\mathbb{T}^3} \frac{|\nabla \text{tr} \, \tau_0(x)|^2}{|1 + \text{tr} \, \tau_0(x)t|^4} \, dx \lesssim \| \nabla \text{tr} \, \tau_0 \|^2_{L^2} \| \nabla q_t^{-1}(x) \|_{L^\infty}^2 \int_{\mathbb{T}^3} \frac{|\nabla \text{tr} \, \tau_0(x)|^2}{|1 + \text{tr} \, \tau_0(x)t|^4} \, dx \lesssim \| \nabla \text{tr} \, \tau_0 \|^2_{L^2} \| \nabla q_t^{-1}(x) \|_{L^\infty}^2 \int_{\mathbb{T}^3} \frac{|\nabla \text{tr} \, \tau_0(x)|^2}{|1 + \text{tr} \, \tau_0(x)t|^4} \, dx.
\]
and
\[
\| \nabla^2 \tau(t, x) \|_{L^2}^2 \lesssim \| \nabla q_t^{-1}(x) \|_{L^\infty}^4 \int_{T^3} \frac{|\nabla^2 \tau_0(x)|^2}{|1 + \tau_0(x)|t^4} dx + \| \nabla^2 q_t^{-1}(x) \|_{L^\infty}^2 \int_{T^3} \frac{|\nabla \tau_0(x)|^2}{|1 + \tau_0(x)|t^4} dx
\]
\[
+ 4 \| \nabla q_t^{-1}(x) \|_{L^2}^4 \int_{T^3} t^2 |\nabla \tau_0(x)|^4 \frac{1}{|1 + \tau_0(x)|t^0} dx
\]
\[
\lesssim \left( \| \nabla^2 \tau_0 \|_{L^2}^2 + c_0^{-2} \| \nabla \tau_0 \|_{L^4}^4 \| \nabla q_t^{-1}(x) \|_{L^\infty} \| \nabla q_t^{-1}(x) \|_{L^\infty} \right) \frac{1}{|1 + c_0t|^4} + \| \nabla \tau_0 \|_{L^2}^2 \| \nabla^2 q_t^{-1}(x) \|_{L^\infty}.
\]

Applying the Lemma 2.1, then we get
\[
\int_0^t (1 + c_0s)^{-\epsilon} \| \nabla \tau(s) \|_{L^2}^2 ds
\]
\[
\lesssim c_0^{-1} \| \nabla \tau_0 \|_{L^2}^2 \exp \left( \int_0^t \| \nabla u(s) \|_{L^\infty} ds \right)
\]
\[
\lesssim c_0^{-1} \| \nabla \tau_0 \|_{L^2}^2 \exp \left\{ \left( \int_0^t (1 + c_0s)^{-3-\epsilon} \| \nabla^3 u(s) \|_{L^2}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t (1 + c_0s)^{-3-\epsilon} ds \right)^{\frac{1}{2}} \right\}
\]
\[
\lesssim c_0^{-1} \| \nabla \tau_0 \|_{L^2}^2 \exp \left( c_0^{-\frac{1}{2}} \mathcal{E}_2(t) \right),
\]
which implies
\[
\mathcal{E}_4(t) \lesssim c_0^{-2} \mathcal{E}(0) \exp \left( c_0^{-\frac{1}{2}} \mathcal{E}_2(t) \right). \tag{4.29}
\]

Applying the Lemma 2.1 again, then we get
\[
\int_0^t (1 + c_0s)^{-3-\epsilon} \| \nabla^2 \tau(s) \|_{L^2}^2 ds
\]
\[
\lesssim c_0^{-1} \left( \| \nabla^2 \tau_0 \|_{L^2}^2 + c_0^{-2} \| \nabla \tau_0 \|_{L^2} \| \nabla^2 \tau_0 \|_{L^2} + c_0^{-1} \| \nabla \tau_0 \|_{L^2} \| \nabla q_t^{-1}(x) \|_{L^\infty} \| \nabla q_t^{-1}(x) \|_{L^\infty} \right)
\]
\[
\lesssim c_0^{-1} \left( \| \nabla^2 \tau_0 \|_{L^2}^2 + c_0^{-2} \| \nabla \tau_0 \|_{L^2} \| \nabla \tau_0 \|_{L^2} \| \nabla^2 \tau_0 \|_{L^2} \right) \exp \left( \int_0^t \| \nabla u(s) \|_{L^\infty} ds \right)
\]
\[
+ c_0^{-1} \| \nabla \tau_0 \|_{L^2} \exp \left( \int_0^t \| \nabla u(s) \|_{L^\infty} ds \right) \int_0^t \| \nabla^2 u(s) \|_{L^\infty} \exp \left( \int_0^s \| \nabla u(s') \|_{L^\infty} ds' \right) ds
\]
\[
\lesssim c_0^{-1} \left( \| \nabla^2 \tau_0 \|_{L^2}^2 + c_0^{-2} \| \nabla \tau_0 \|_{L^2} \| \nabla^2 \tau_0 \|_{L^2} \right) \exp \left( c_0^{-\frac{1}{2}} \mathcal{E}_2(t) \right)
\]
\[
+ c_0^{-1} \| \nabla \tau_0 \|_{L^2} \exp \left( c_0^{-\frac{1}{2}} \mathcal{E}_2(t) \right) \int_0^t \| \nabla^2 u(s) \|_{L^\infty} ds.
\]

Now we estimate the \( \int_0^t \| \nabla^2 u(s) \|_{L^\infty} ds \). For this purpose, we rewrite the first equation of the system \( \text{P}^TT\text{I} \) as following:
\[
u_t - \Delta u = \mathbb{P} \text{ div } \tau - \mathbb{P}(u \cdot \nabla u),
\]
then we have
\[
u(t, x) = e^{-t \Delta} u_0(x) + \int_0^t e^{-(t-s) \Delta} (\mathbb{P} \text{ div } \tau - \mathbb{P}(u \cdot \nabla u))(s) ds.
\]
By standard estimates about the Heat equation, we obtain that
\[ \| \nabla^2 e^{-t \triangle} u_0(x) \|_{L^\infty} \lesssim \sum_{k \in \mathbb{Z}^d \setminus \{0\}} e^{-|k|^2 t} |k|^2 |\widehat{u}_0(k)| \lesssim e^{-t} \| u_0 \|_{L^2}. \]

Using the (4.27), then we have
\[
\begin{align*}
\| \int_0^t \nabla^2 e^{-(t-s) \triangle} (P \div \tau) (s) ds \|_{L^\infty} & \lesssim \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \int_0^t e^{-|k|^2 (t-s)} |k|^2 |P \div \tau (s, k)| ds \\
& \lesssim \int_0^t e^{-(t-s)} \| P \div \tau (s) \|_{L^2} ds \\
& \lesssim \mathcal{E}_4(0) \int_0^t e^{-(t-s)} e^{-\frac{s}{2}} ds + c_0^{-\frac{1}{2}} \left( \mathcal{E}_4^2(t) \mathcal{E}_2^2(t) + c_0^2 \mathcal{E}_4^2(t) \mathcal{E}_2^2(t) \right) \int_0^t e^{-(t-s)} e^{\frac{s}{2}} ds \\
& \quad + \left( \mathcal{E}_4^2(t) \mathcal{E}_2^2(t) + c_0^2 \mathcal{E}_4^2(t) \mathcal{E}_2^2(t) \right) \int_0^t e^{-(t-s)} (1 + c_0 s)^{-\frac{3}{2} + \frac{s}{2}} ds \\
& \lesssim e^{-\frac{s}{2}} \mathcal{E}_4^2(0) + c_0^{-\frac{1}{2}} \left( \mathcal{E}_4^2(t) \mathcal{E}_2^2(t) + c_0^2 \mathcal{E}_4^2(t) \mathcal{E}_2^2(t) \right) e^{-\frac{s}{2}} + \left( \mathcal{E}_4^2(t) \mathcal{E}_4^2(t) + c_0^2 \mathcal{E}_4^2(t) \mathcal{E}_4^2(t) \right) (1 + c_0 t)^{-\frac{3}{2} + \frac{s}{2}},
\end{align*}
\]

and
\[
\begin{align*}
\| \int_0^t \nabla^2 e^{-(t-s) \triangle} (u \cdot \nabla u) (s) ds \|_{L^\infty} & \lesssim \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \int_0^t e^{-|k|^2 (t-s)} |k|^2 |P(u \cdot \nabla u) (s, k)| ds \\
& \lesssim \int_0^t e^{-(t-s)} \| u \cdot \nabla u (s) \|_{L^2} ds \\
& \lesssim \mathcal{E}_4^2(t) \mathcal{E}_2^2(t) \left( \int_0^t e^{-2(t-s)} (1 + c_0 s)^{-3+\epsilon} ds \right)^{\frac{s}{2}} \\
& \lesssim c_0^{-\frac{1}{2}} \mathcal{E}_4^2(t) \mathcal{E}_2^2(t) e^{-\frac{s}{2}} + \mathcal{E}_4^2(t) \mathcal{E}_2^2(t) (1 + c_0 s)^{-\frac{3}{2}+\frac{s}{2}}.
\end{align*}
\]

Together with the above estimates, we have
\[
\| \nabla^2 u(t) \|_{L^\infty} \lesssim e^{-\frac{s}{2}} \mathcal{E}_4^2(0) + c_0^{-\frac{1}{2}} \left( \mathcal{E}_4^2(t) \mathcal{E}_2^2(t) + c_0^2 \mathcal{E}_4^2(t) \mathcal{E}_2^2(t) \right) e^{-\frac{s}{2}} + \left( \mathcal{E}_4^2(t) \mathcal{E}_4^2(t) + c_0^2 \mathcal{E}_4^2(t) \mathcal{E}_4^2(t) \right) (1 + c_0 t)^{-\frac{3}{2} + \frac{s}{2}},
\]

integrating with time, then we get that
\[
\int_0^t \| \nabla^2 u(s) \|_{L^\infty} ds \lesssim \mathcal{E}_4^2(0) + c_0^{-\frac{1}{2}} \mathcal{E}_4^2(t) \mathcal{E}_2^2(t) + c_0^{-\frac{1}{2}} \mathcal{E}_4^2(t) \mathcal{E}_4^2(t). \tag{4.30}
\]

Finally, we have
\[
\begin{align*}
\int_0^t (1 + cs)^{3-\epsilon} \| \nabla^2 \tau (s) \|_{L^2}^2 ds \\
& \lesssim c_0^{-1} \left( \| \nabla^4 \tau_0 \|_{L^2}^2 + c_0^{-2} \| \nabla \tau_0 \|_{L^2} \| \nabla^2 \tau_0 \|_{L^2} \right) \exp \left( c_0^{-\frac{1}{2}} \mathcal{E}_4^2(t) \right) \\
& \quad + c_0^{-1} \| \nabla \tau_0 \|_{L^2} \exp \left( c_0^{-\frac{1}{2}} \mathcal{E}_4^2(t) \right) \left( \mathcal{E}_4^2(0) + c_0^{-1} \mathcal{E}_4^2(t) \mathcal{E}_4^2(t) + c_0^{-\frac{1}{2}} \mathcal{E}_4^2(t) \mathcal{E}_4^2(t) \right).
\end{align*}
\]

Combining with the above estimates yields that
\[
\begin{align*}
\mathcal{E}_5(t) & \lesssim (\mathcal{E}(0) + c_0^{-1} \mathcal{E}_4^2(0) \mathcal{E}_4^2(0)) \exp \left( c_0^{-\frac{1}{2}} \mathcal{E}_4^2(t) \right) + c_0^2 \mathcal{E}(0) \exp \left( c_0^{-\frac{1}{2}} \mathcal{E}_4^2(t) \right) \\
& \quad \times \left( \mathcal{E}_4^2(0) + c_0^{-1} \mathcal{E}_4^2(t) \mathcal{E}_4^2(t) + c_0^{-\frac{1}{2}} \mathcal{E}_4^2(t) \mathcal{E}_4^2(t) \right). \tag{4.31}
\end{align*}
\]
4.5. Proof of the main theorem

Proof. In this subsection, we will combine the above a priori estimates of $E_1(t)$, $E_2(t)$, $E_3(t)$, $E_4(t)$, $E_5(t)$ together and give the proof of the Theorem 1.2. For any $t \in [0, T]$, we have

$$\Lambda_1(t) = E_1(t) + E_2(t) \leq C^* E(0) + C \epsilon_0 \left( E_1^\frac{\epsilon}{2}(t) + E_2^\frac{\epsilon}{2}(t) \right) + C \epsilon_0 \left( E_1(t) + E_2(t) \right) \left( E_3^\frac{\epsilon}{2}(t) + E_4^\frac{\epsilon}{2}(t) \right),$$

(4.32)

$$+ C E_1(t) \left( E_4^\frac{\epsilon}{2}(t) + E_5^\frac{\epsilon}{2}(t) \right).$$

(4.33)

$$\Lambda_2(t) = E_4(t) \leq \epsilon_0^2 \exp \left( C \epsilon_0 \right),$$

(4.34)

$$\Lambda_3(t) = E_5(t) \leq \left( E_0 + c_0^{-1} E_1^\frac{\epsilon}{2}(0) \right) \exp \left( c_0^{-\frac{\epsilon}{2}} E_2(t) \right) + c_0^{-2} E(0) \exp \left( c_0^{-\frac{\epsilon}{2}} E_2(t) \right) \times \left( E_2^\frac{\epsilon}{2}(0) + c_0^{-1} E_1^\frac{\epsilon}{2}(t) E_2^\frac{\epsilon}{2}(t) + c_0^{-\frac{\epsilon}{2}} E_1^\frac{\epsilon}{2}(t) E_2^\frac{\epsilon}{2}(t) \right).$$

Due to the local existence theory, there exists a positive time $T$ such that

$$\Lambda_1(t) \leq 2 C^* \delta_0^2, \quad \Lambda_2(t) \leq C^*, \quad \Lambda_3(t) \leq C^* \left( \epsilon_0 + \delta_0^\frac{\epsilon}{2} \right), \quad \forall t \in [0, T].$$

(4.35)

Let $T^*$ be the maximal time for what (4.35) holds. Since $E(0) \leq \delta_0^2 \leq \epsilon_0$, $\hat{E}(0) \leq \epsilon_0$, $c_0 = \frac{1}{2} \delta_0$, then it follows that for any $t \in [0, T^*)$

$$c_0^{-\frac{\epsilon}{2}} \left( E_1^\frac{\epsilon}{2}(t) + E_2^\frac{\epsilon}{2}(t) \right) \leq \delta_0^\frac{\epsilon}{2},$$

$$c_0^{-\frac{\epsilon}{2}} \left( E_1(t) + E_2(t) \right) \left( E_3^\frac{\epsilon}{2}(t) + E_4^\frac{\epsilon}{2}(t) \right) \leq c_0^{-\frac{\epsilon}{2}} \left( E_1(t) + E_2(t) \right),$$

$$E_1(t) \left( E_4^\frac{\epsilon}{2}(t) + E_5^\frac{\epsilon}{2}(t) \right) \leq \left( \epsilon_0 + \delta_0^\frac{\epsilon}{2} \right)^\frac{\epsilon}{2} E_1(t),$$

$$\left( E_0 + c_0^{-1} E_1^\frac{\epsilon}{2}(0) \right) \exp \left( c_0^{-\frac{\epsilon}{2}} E_2(t) \right) \leq \epsilon_0, \quad E_2^\frac{\epsilon}{2}(0) + c_0^{-1} E_1^\frac{\epsilon}{2}(t) E_2^\frac{\epsilon}{2}(t) + c_0^{-\frac{\epsilon}{2}} E_1^\frac{\epsilon}{2}(t) E_2^\frac{\epsilon}{2}(t) \leq \delta_0^\frac{\epsilon}{2}.$$

Under the setting of initial data, there exists small enough numbers $\epsilon_0, \tilde{\epsilon}_0$ such that $E(0) \leq \delta_0^2 \leq \epsilon_0$, $\hat{E}(0) \leq \tilde{\epsilon}_0$. By virtue of (4.32), (4.33), (4.34), and the smallness assumption on $\delta_0$ and $\epsilon_0$, we get that

$$\Lambda_1(t) \leq C^* \delta_0^2 + C \delta_0^\frac{\epsilon}{2} + C \left( \epsilon_0 + \delta_0^\frac{\epsilon}{2} \right) \Lambda_1(t) < 2 C^* \delta_0^2,$$

$$\Lambda_2(t) \leq C \Lambda_1(t) \leq C^* \Lambda_1(t) \leq C^* \left( \epsilon_0 + \delta_0^\frac{\epsilon}{2} \right).$$

By standard continuity argument and total energy (4.32), (4.33), (4.34), we can show that $T^* = \infty$ provided that $\delta_0$ and $\epsilon_0$ is small enough. Moreover, we deduce the time decay estimate for the velocity $u$ and the quantity $P \text{div} \tau$ from (4.32) that

$$\|u\|_{H^2} + \|P \text{div} \tau\|_{L^2} \lesssim (1 + t)^{-\frac{\epsilon}{2} + \frac{\epsilon}{2}}.$$

Hence, we finish the proof of the Theorem 1.2.

□
Acknowledgements. This work was partially supported by NSF of China (Grants No.11701586, No.11671407, No.11431015 and No.11801586) and the Central Universities (Grants No. 18lgpy66).

References

[1] H. Bahouri, J.-Y. Chemin, and R. Danchin. *Fourier analysis and nonlinear partial differential equations*, volume 343 of *Grundlehren der Mathematischen Wissenschaften*. Springer, Heidelberg, 2011.

[2] O. Bautista, S. Sánchez, J. C. Arcos, and F. Méndez. Lubrication theory for electro-osmotic flow in a slit microchannel with the Phan-Thien and Tanner model. *J. Fluid Mech.*, 722:496–532, 2013.

[3] R. B. Bird, R. C. Armstrong, and O. Hassager. *Dynamics of Polymeric Liquids*, volume 1. Wiley, New York, 1977.

[4] J.-Y. Chemin and N. Masmoudi. About lifespan of regular solutions of equations related to viscoelastic fluids. *SIAM J. Math. Anal.*, 33(1):84–112, 2001.

[5] D. Fang, M. Hieber, and R. Zi. Global existence results for Oldroyd-B fluids in exterior domains: the case of non-small coupling parameters. *Math. Ann.*, 357(2):687–709, 2013.

[6] D. Fang and R. Zi. Global solutions to the Oldroyd-B model with a class of large initial data. *SIAM J. Math. Anal.*, 48(2):1054–1084, 2016.

[7] E. Fernández-Cara, F. Guillén, and R. R. Ortega. Some theoretical results concerning non-Newtonian fluids of the Oldroyd kind. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 26(1):1–29, 1998.

[8] I. E. Garduño, H. R. Tamaddon-Jahromi, K. Walters, and M. F. Webster. The interpretation of a long-standing rheological flow problem using computational rheology and a PTT constitutive model. *J. Non-Newton. Fluid Mech.*, 233:27–36, 2016.

[9] C. Guillopé and J.-C. Saut. Existence results for the flow of viscoelastic fluids with a differential constitutive law. *Nonlinear Anal.*, 15(9):849–869, 1990.

[10] C. Guillopé and J.-C. Saut. Global existence and one-dimensional nonlinear stability of shearing motions of viscoelastic fluids of Oldroyd type. *RAIRO Modél. Math. Anal. Numér.*, 24(3):369–401, 1990.
[11] Z. Lei, C. Liu, and Y. Zhou. Global solutions for incompressible viscoelastic fluids. *Arch. Ration. Mech. Anal.*, 188(3):371–398, 2008.

[12] Z. Lei and Y. Zhou. Global existence of classical solutions for the two-dimensional Oldroyd model via the incompressible limit. *SIAM J. Math. Anal.*, 37(3):797–814, 2005.

[13] F.-H. Lin, C. Liu, and P. Zhang. On hydrodynamics of viscoelastic fluids. *Comm. Pure Appl. Math.*, 58(11):1437–1471, 2005.

[14] P. L. Lions and N. Masmoudi. Global solutions for some Oldroyd models of non-Newtonian flows. *Chinese Ann. Math. Ser. B*, 21(2):131–146, 2000.

[15] Y. Mu, G. Zhao, A. Chen, and X. Wu. Modeling and simulation of three-dimensional extrusion swelling of viscoelastic fluids with PTT, Giesekus and FENE-P constitutive models. *Internat. J. Numer. Methods Fluids*, 72(8):846–863, 2013.

[16] Y. Mu, G. Zhao, X. Wu, and J. Zhai. Modeling and simulation of three-dimensional planar contraction flow of viscoelastic fluids with PTT, Giesekus and FENE-P constitutive models. *Appl. Math. Comput.*, 218(17):8429–8443, 2012.

[17] J. G. Oldroyd. Non-Newtonian effects in steady motion of some idealized elastico-viscous liquids. *Proc. Roy. Soc. London. Ser. A*, 245:278–297, 1958.

[18] P. J. Oliveira and F. T. Pinho. Analytical solution for fully developed channel and pipe flow of Phan-Thien–Tanner fluids. *J. Fluid Mech.*, 387:271–280, 1999.

[19] N. Phan-Thien. A nonlinear network viscoelastic model. *Journal of Rheology*, 22(3):259–283, 1978.

[20] N. Phan-Thien and R. I. Tanner. A new constitutive equation derived from network theory. *Journal of Non-Newtonian Fluid Mechanics*, 2(4):353 – 365, 1977.

[21] T. Zhang and D. Fang. Global existence of strong solution for equations related to the incompressible viscoelastic fluids in the critical $L^p$ framework. *SIAM J. Math. Anal.*, 44(4):2266–2288, 2012.

[22] Y. Zhu. Global small solutions of 3D incompressible Oldroyd-B model without damping mechanism. *J. Funct. Anal.*, 274(7):2039–2060, 2018.