Renormalized SO(5) symmetry in ladders with next-nearest-neighbor hopping

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We study the occurrence of SO(5) symmetry in the low-energy sector of two-chain Hubbard-like systems by analyzing the flow of the running couplings (g-ology) under renormalization group in the weak-interaction limit. It is shown that SO(5) is asymptotically restored for low energies for rather general parameters of the bare Hamiltonian. This holds also with inclusion of a next-nearest-neighbor hopping which explicitly breaks particle-hole symmetry provided one accounts for a different single-particle weight for the quasiparticles of the two bands of the system. The physical significance of this renormalized SO(5) symmetry is discussed.

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Recently, it was shown that both the two-dimensional (2-D) Hubbard and the t–J model enjoy an approximate SO(5) symmetry [1,2], unifying antiferromagnetism (AF) with d-wave superconductivity (dSC) [3]. This symmetry principle gives a definite microscopic description of the AF → dSC transition as the chemical potential is varied. From the SO(5) multiplet structure verified by exact cluster diagonalization, one can see how the SO(5) superspin vector is rotated from AF to dSC direction and show that at the critical chemical potential, the energy barrier ∆E between AF and dSC states is an order of magnitude smaller (∼ J/10) than the exchange coupling J, i.e., the natural parameter in the model. This finding is clearly of importance: while it is well-established that both t–J and Hubbard models reproduce very successfully the “high” and “medium”-energy physics of order U and J ∼ t²/U of the cuprates, the low-energy content of order of the superconducting gap has so far eluded theoretical investigations. The variance ∆E of the multiplet splitting is a well-defined measure of “how-good” the SO(5) symmetry is realized in the bare model considered for a given system size. It seems then a natural question to ask, whether this deviation from exact SO(5) symmetry remains small or even vanishes if one goes to the infinite-volume limit and to lower energies, i.e. under renormalization-group (RG) flow.

In this Letter, we demonstrate that the low-energy regime of rather general Hubbard-type models including finite-ranged interactions (provided they are weak) between one and two dimensions, i.e. ladders, is indeed dominated by the scaling towards an SO(5) -invariant model. This is a remarkable result, since it is the first model, which is non-SO(5) invariant at the bare starting (microscopic) level, where the existence of SO(5) symmetry is proven for low energies [3]. We first show that this holds for the particle-hole- (ph) symmetric case with nearest-neighbor hopping only [3]. In addition, we consider the effect of a next-nearest-neighbor (intrachain) hopping t₂ which produces an explicit breaking of the ph symmetry. We demonstrate that also in this case the system becomes SO(5) symmetric for low energies (at least up to order t₂) provided one considers a renormalized SO(5) transformation [3], which takes into account a different renormalization of the single-particle weight of the bonding and antibonding bands. This result is of importance because of two points: (i) it sheds light on the effect of the longer-range hoppings in general on the fate of SO(5) symmetry. This issue is also under intensive discussion in the case of 2-D systems [2,8,9]: t₂ is known to strongly affect AF correlations and the Fermi-surface topology in the cuprates. (ii) In addition, the renormalized SO(5) representation introduced here for the first time, is likely to be realized in a significantly larger class of physical systems allowing, for example, for asymmetries in the AF- and dSC- phases, such as different transition temperatures.

Specifically, we consider two coupled chains in the band representation with total “low-energy” action S = S₀ + Sᵣ, where the non-interacting part S₀ can be written at a given point τ in the RG flow as

\[ S₀ = \sum_{k,\sigma} C_{k\sigma}^⁺ Z^{-1}_{k,\tau} [iω - ν V_{k,\tau} k] C_{k\sigma}. \]

Here, C_{k\sigma} (C_{k\sigma}^⁺) are Grassmann variables associated with the destruction (creation) of a fermion, and σ is the spin. k = (iω, kₓ, ν)} is a shorthand notation for the Matsubara frequency iω, and the momentum perpendicular (kₓ = (0, π)) and parallel (kᵧ) to the chain direction. The latter momentum is measured relative to the Fermi point νkₓkᵧ, associated with the “band” kₓ with Fermi velocity V_{k,τ}, and ν = ± 1 refers to right- and left-moving fermions, respectively. This action is restricted to modes with |kₓ| < Λ with Λ = 1/6e−τ. This weak-coupling RG method has previously been applied to obtain the phase diagram of the two-chain Hubbard model [2]. In order to study the occurrence of SO(5) symmetry in Hubbard-like models, it is convenient to rewrite the interaction part Sᵣ of the action in terms of SO(5) -invariant and SO(5) -breaking terms [10]. Defining the SO(5) spinor as in Ref. [11],

\[ \Psi_{k} \equiv \left\{ C_{k\uparrow}, C_{k\downarrow}, -\cos kₓ C_{k\uparrow}^⁺, -\cos kₓ C_{k\downarrow}^⁺ \right\}^T, \]

where k stands for \{-iω, -kₓ + π, -kᵧ, ν\}, the interacting action Sᵣ can be shown to be expressible in terms of the 4 × 4 charge-rotation Dirac matrix Γ¹⁵ [11].
\[ S_I = \frac{1}{2\beta N} \sum_{k_1 \ldots k_4} g_0(\cdots) \Psi_{k_1}^\dagger \left( 1 + a(\cdots) \Gamma^{15} \right) \Psi_{k_2} \times \Psi_{k_3}^\dagger \left( 1 + b(\cdots) \Gamma^{15} \right) \Psi_{k_4} + (k_1, k_3) \leftrightarrow (k_2, k_4). \] (3)

Here, \((\cdots)\) represent the sets of variables on which the couplings \(g_0, a, b\) depend. As usual, each coupling can be considered as dependent only on the Fermi momenta closest to where the corresponding process takes place, i.e., the \((\cdots)\) are labeled by \((\nu_1, k_{1\perp}; \nu_2, k_{2\perp}; \nu_3, k_{3\perp}; \nu_4, k_{4\perp})\). Moreover, \(\sum\) denotes a sum with conservation of frequency and lattice momentum.

The SO(5) -symmetric part \(S_I^{(0)}\) of \(S_I\) is given by Eq. (3) with \(a(\cdots) = b(\cdots) = 0\). For a general SO(5) -invariant action \(S_I^{(0)}\) it can be shown that one can restrict oneself to the seven independent couplings \(g_0, g_0^{(1);0-\pi}, g_0^{(0);0-\pi}, g_0^{(1);11;0-\pi}, g_0^{(1);11;0-\pi}, g_0^{(2);0-\pi}, g_0^{(1);11;0-\pi}, g_0^{(1);11;0-\pi}\), defined in analogy with the g-ology formalism [14] [15]. The SO(5) -breaking term \(S_I^{(1)}\) of the interacting action with ph symmetry can be demonstrated to be the term proportional to \(g_0(\cdots) a(\cdots) b(\cdots)\) in (3) and thus we define the corresponding SO(5) -breaking couplings as \(g_1(\cdots) \equiv g_0(\cdots) a(\cdots) b(\cdots)\). In this case, we can restrict ourselves to only 5 independent couplings, namely, \(g_1^{(4)}, g_1^{(2)}, g_1^{(1)}, g_1^{(1);11;0-\pi}, g_1^{(1);11;0-\pi}\). At half-filling and with \(t_2 = 0\), the Hamiltonian is ph symmetric, since the velocities of the two bands \(V_{k_2} = 0\) and \(V_{k_\perp} = 0\) are equal. Since the ph-breaking terms in (3) are proportional to \(\Gamma_{15}\), one can set in the ph-symmetric case \(a(\cdots) = b(\cdots)\), and consider the RG flow of the couplings \(g_0(\cdots) = g_1(\cdots)\).

To begin with, we have evaluated the RG equations for the \(g_0(\cdots) = g_1(\cdots)\) couplings [14] at one loop by using the standard g-ology procedure (cf. Refs. [3] [4]), including the interband unklapp processes. As already shown for the two-chain case [4], the system always flows to strong coupling, i.e., the \(g_i\)’s diverge at a value of \(\tau = \tau_c \propto 1/g(\tau = 0), g(\tau)\) being the scale of the interaction (proportional to the maximum of all \(g_i(\tau)\)). This signals an instability towards some gapped state. Nevertheless, the striking new result is that even in a non-SO(5) -invariant system, like, e.g., the Hubbard model, the SO(5) -invariant couplings \(g_0(\cdots) = g_1(\cdots)\) dominate with respect to the symmetry-breaking couplings \(g_1(\cdots)\), when approaching \(\tau_c\). This can be seen from the ratio of the maxima of these two types of couplings, \(g_1^{(\text{max})}(\tau)/g_0^{(\text{max})}(\tau)\) going to zero, as shown in the inset of Fig. 1. Here, \(g_i^{(\text{max})}(\tau)\) is defined as the largest absolute value, and thus the scale of the couplings of a given type \(i (i = 0, 1, \text{ph})\) at a given \(\tau\). This result implies that the low-energy modes of the system can be described by an effective SO(5) -symmetric action, at least for sufficiently small \(g(\tau = 0)\) [3] [4] [5]. In fact, we have verified that this occurs for very general values of the Hamiltonian, including longer-ranged interactions.

A next-nearest-neighbor hopping \(t_2\) breaks ph symmetry explicitly and requires the introduction of a ph breaking interaction \(S_I^{(\text{ph})}\). Here, \(a(\cdots) \neq -b(\cdots)\) in Eq. (3), and thus we need extra couplings \(g_{\text{ph}}(\cdots)\), which we have defined as \(g_{\text{ph}}(\cdots) = g_0(\cdots) (a(\cdots) + b(\cdots))/2\). In this case, one can show that the couplings can be restricted to \(g_{\text{ph}}^{(4)}, g_{\text{ph}}^{(2)}, g_{\text{ph}}^{(1)}\) [17] [18]. The initial \((\tau = 0)\) source of ph-symmetry breaking for \(t_2 \neq 0\) stems from the non-interacting part of the action \(S_0\), due to the difference of the Fermi velocities \(\Delta v_0\) of the two bands. In the following, we will show that SO(5) symmetry is restored (at least up to \(\mathcal{O}(t_2^2)\)), at low energies, even in the presence of this ph- (and thus SO(5) -) breaking term.

FIG. 1. RG flow of the SO(5) -breaking terms \(3g_1^{(\text{max})}g_0^{(\text{max})}(\tau)/\gamma^{(\text{max})}(\tau)\) (full), \(3g_{\text{ph}}^{(\text{max})}g_0^{(\text{max})}(\tau)/(\gamma^{(\text{max})}(\tau))\) (dashed), \(\Delta V_r/\Delta V_{\tau=0}\) (dotted), and \(\Delta Z_{\tau=0}/\gamma^{(\text{max})}(\tau)\) (dash-dotted), as function of \(\tau = -\log (\Lambda/\Lambda_0)\) for \(U = 1, t = t_1 = 0, t_2 = 0.5\) and half filling. Here, \(\Delta V_r = V_{\text{ph}} - V_{\text{r}},\) and \(\Delta Z_{\tau=0} = (\langle Z_{\text{ph}}^{(\tau=0)}(1/\gamma^{(\text{max})}(\tau=0))\rangle - 1) / (V_{\text{r}}/V_{\text{ph}} = 1).\) The inset shows \(3g_1^{(\text{max})}g_0^{(\text{max})}(\tau)\) vs \(\tau U\) for \(t_2 = 0\).

These results are obtained on the basis of two complementary RG calculations. Calculation (i) considers the RG flow of the self-energy parameters \(V_{k_\perp} = Z_{k_\perp}^{-1}\) at two loops, and of the coupling parameters \(g_i(\tau)\) at one loop, taking the \(\tau\) dependence of all the parameters at each RG step fully into account. This first calculation (i), although not rigorously controlled (see below), is motivated by the fact that we are interested in studying the RG flow of the self-energy, which is the leading symmetry-breaking term when \(t_2 \neq 0\) is included [13]. In a second calculation (ii), we will show how our main results about the renormalized SO(5) symmetry obtained within this first procedure can be achieved also in an alternative, more controlled way, where we consider only the renormalization of the \(g_i\). Nevertheless, the first calculation (i) is instructive, in order to provide a physical interpretation for the single-particle renormalization factors \(Z\) as discussed the conclusions. Indeed, in procedure (i) the \(Z\) factors, and thus the renormalized SO(5) transformation, derive naturally from the RG flow, while in (ii) they are introduced right at the outset.

In calculation (i), the relevant part of the renormalized action has the form of Eq. (3) with \(\tau\)-dependent Fermi velocities and single-particle weights. The flow of these parameters is shown in Fig. 1. As the bare \((\tau = 0)\)
Hamiltonian, we take the half-filled Hubbard ladder with isotropic intrachain and interchain hoppings $t = t_u = 1$, next-nearest-neighbor hopping $t_2 = 0.5$ (corresponding to $\Delta V_{\tau=0} \approx 1.9$), and $U = 1$ \cite{21}. Actually, we have verified that the results we are discussing below are rather general and hold also in the presence of anisotropy $t_\perp \neq t$ and nearest-neighbor interactions ($\gtrsim - U$). The $t_2 = 0$ case \cite{3} discussed above, is plotted in the inset for comparison.

For the $t_2 \neq 0$ case, $\Delta V_{\tau}$ (dotted line) flows to zero, but $\Delta Z^{-1}_{\tau}$ (dash-dotted line, initially zero) scales to unity. Therefore, the initial asymmetry between the bands due to the different Fermi velocities is transferred into a difference in the single-particle weights $Z$, such that for large $\tau$, $Z^{-1}_{\tau} \to \sqrt{V_{\tau=0}/V_{\tau=n}}$. In order to restore the coefficient of the $i\omega$ term in \cite{3} to unity, the standard procedure \cite{13} is to reabsorb this renormalization into the definition of new Grassmann variables $\bar{C}_{k\sigma}$ and to set $\sqrt{\frac{Z^{-1}_{\tau}}{Z^{-1}_{\tau=0}}} \bar{C}_{k\sigma} = \tilde{C}_{k\sigma}$. This standard procedure is dictated by the requirement to identify the canonical Fermi operators with correct anticommutation relations, as will be discussed at the end.

![FIG. 2. $\tilde{SO}(5)$ -breaking couplings $3\tilde{g}_{\parallel}^{(max)}/\tilde{g}_{0}$ (full) and $3\tilde{g}_{\perp}^{(max)}/\tilde{g}_{0}$ (dashed) as a function of $\tau$. (a) shows the results of the RG procedure (i) and (b) shows the results of (ii) as defined in the text. The parameters are as in Fig. 1.](image)

In this way, the non-interacting part of the (renormalized) action will again be symmetric under exchange of the two bands (and thus $SO(5)$ symmetric in the new fields). This transformation, however, also affects the interaction part, and one should consistently redefine the renormalized $SO(5)$ spinor in \cite{3} to $\bar{\Psi}_k$, whereby the $C_{k\sigma}$ are again replaced with the $\tilde{C}_{k\sigma}$. The couplings defined in this way are of course different from the original ones and we will distinguish them with a tilde, i.e. $\bar{g}^{(i)}_i \to \tilde{g}^{(i)}_i$. The remarkable result is that the transformation which makes the non-interacting part of the action $SO(5)$ -symmetric also restores $SO(5)$ in the interacting part. This is demonstrated in Fig. 2a, which plots the ratio of the $\tilde{g}^{(max)}_i$ as a function of the flow parameter $\tau$. We note that the non-$SO(5)$ couplings $\tilde{g}^{(max)}_i$ and $\tilde{g}^{(max)}_{\text{ph}}$ all flow to zero (relative to the $\tilde{g}^{(max)}_0$). Thus, at large $\tau$, $SO(5)$ symmetry is restored for low energies in the $a \rightarrow -$ basis. However, at the energy scale where $\Delta V_{\tau}$ starts to decrease and $\Delta Z^{-1}_{\tau}$ starts to become finite ($\tau \sim 7$ in Fig. 1), the renormalized couplings can be shown to become large and the weak-coupling expansion is no longer controlled, as anticipated.

To support this physically appealing yet uncontrolled calculation, we verify, in terms of a controlled RG calculation \cite{17} (i.e., at one loop), that $SO(5)$ symmetry is indeed recovered at least up to $O(t_2^2)$. This alternative derivation clarifies why the single-particle weights renormalize proportionally to $(\sqrt{V_{\tau=0}})^{-1}$, as obtained asymptotically in the two-loop calculation. In this RG procedure (ii), we start from the action $S_0 + S_I$ and carry out the transformation $\bar{T}$ on the Grassmann variables right at the outset, where $\bar{T}$ is defined as $\bar{T}C_{i\omega,k_i,k_\perp} = 1/\sqrt{u_{k_i}} \tilde{C}_{i\omega',k_i',k_\perp}$ with $i\omega' = i\omega/u_{k_i}$ and $k_i' = k_i + u_{k_i}$, and $u_{k_i} = \sqrt{V_{k_i}}$ \cite{12}. Such a transformation, which is always possible with Grassmann variables, is motivated by our first calculation (i). By changing the sum over $i\omega$ and $k_i$ into a sum over $i\omega'$ and $k_i'$ separately for each band, the non-interacting part of the action again recovers its explicit $SO(5)$ symmetry. Furthermore, by defining a new $SO(5)$ spinor $\bar{\Psi}_k$ in terms of the $\bar{C}$, we obtain new couplings $\tilde{g}_0^{(i)}, \tilde{g}_\perp^{(i)},$ and $\tilde{g}_{\text{ph}}^{(i)}$, as in step (i). In Fig. 2b, we show the corresponding RG flow of the ratios of the $\tilde{g}^{(max)}_i$. With increasing RG parameter $\tau$ the ph-symmetry breaking term $\tilde{g}_{\text{ph}}^{(max)}/\tilde{g}_0^{(max)}$ vanishes (full line), while the $SO(5)$ -breaking term $\tilde{g}_\perp^{(max)}/\tilde{g}_0^{(max)}$ goes to a finite but rather small value $\tilde{g}_\perp (\tau = 0) \sim 0$ (dashed line). The $SO(5)$ symmetry thus is recovered up to a very high degree of precision for low energies $\tilde{g}_0$ \cite{3}. In contrast with this, the results of procedure (ii) is controlled for small $g(\tau = 0) \sim 0$. Our two-loop calculation (i) further suggests that even this $SO(5)$ -breaking term of order $t_2^2$ might be removed by the self-energy renormalization.

The renormalized $SO(5)$ symmetry introduced here, and the related renormalization of the single-particle weights $Z_{k_i,\tau}$, can be understood in terms of a simplified scheme, which renormalizes the Hamiltonian, by restricting the Hilbert space to a subspace with energy $\omega$ smaller than a certain cutoff $\omega_0 \propto \Delta_0 \exp(-\tau)$. In the restricted subspace, the total integrated spectral density (which we identify with $Z_{k_i,\tau}$) will be less than one. Since the spectral sum rule identifies $Z_{k_i,\tau}$ with the anticommutator of the Fermi operators $C_k$, the canonical Fermi operators with anticommutator equal to 1 in this subspace are the transformed field operators $\tilde{C}_k$ introduced above \cite{22}.

In conclusion, we have shown that the effective low-energy action (or Hamiltonian) of a ladder with weak interaction is asymptotically $SO(5)$ symmetric \cite{3}. With the inclusion of a next-nearest-neighbor hopping $t_2$ the
action is invariant under a generalized $\tilde{SO}(5)$ transformation \(\tilde{C}\), which performs a “stretched” $SO(5)$ rotation of the order parameters. Physically, this $SO(5)$ symmetry may be present in the low-energy sector of a larger and more generic class of physical systems than the ordinary $SO(5)$. Moreover, since this stretched rotation does not conserve the norm of the superspin (order-parameter) vector $\nu$, a renormalized $SO(5)$ theory can possibly admit asymmetries between the antiferromagnetic and superconducting phases, like for example the difference in $T_c$'s $[\nu, k_{\perp 1}; \nu_2, k_{\perp 2}]|_{\nu_3, k_{\perp 3}; \nu_4, k_{\perp 4}}$.

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[5] The results of our RG calculations for $t_2 = 0$ imply that the low-energy approximate behavior of any correlation function $RF(C_{\omega, k_{\perp 1}, k_{\perp 2}})$ (functional of the fermion fields) is identical with its $SO(5)$-transformed one $\tilde{R}F(\tilde{C}_{\omega, k_{\perp 1}, k_{\perp 2}})$ up to relative corrections vanishing for small $g(\tau = 0)$. For example, the ratio of the spin and of the charge gap $\Delta_s/\Delta_e \rightarrow 1$ for $g(\tau = 0) \rightarrow 0$. In the PH-broken case, this holds for correlations functions transformed under $\tilde{R}$ up to corrections of order $t_2^2$. A transformation on a correlation function is defined naturally, when the transformation on the fermion operators is given. The $SO(5)$ transformation $\tilde{R}$ has the usual form $\tilde{R} C \equiv e^{4i\lnC} e^{-4i\ln L}$ with $L$ an $SO(5)$ generator $[\nu, \nu_2, \nu_3, \nu_4, \nu_5]$.

[6] For the ph-symmetric case, see also H.-H. Lin, L. Balents, and M. P. A. Fisher, Phys. Rev. B 58, 1794 (1998).

[7] Throughout the paper, we denote by $\tilde{SO}(5)$ the renormalized $SO(5)$ symmetry, whenever the corresponding transformation $\tilde{R}$ is expressed in terms of the original fermion fields $C$. The transformation $\tilde{R}$ is obtained by first performing the $\tilde{T}$ transformation (described in the text), then the $SO(5)$ transformation on the new fields $\tilde{C}$, and then transforming back to the original fermion fields, i. e., $\tilde{R} = \tilde{T}^{-1} R T$. To illustrate what invariance under $\tilde{R}$ means, let’s take the non-interacting case. The Greens function of the band $k_{\perp 1} = 0$ is $G(\omega, k_{\perp 1}, k_{\perp 1} = 0) = \rho/(\omega - V_0 k_{\perp 1})$. We choose a particular $SO(5)$ transformation for $\tilde{R}$, namely, the exchange of the two bands. We thus have $\tilde{R} G(\omega, k_{\perp 1}, 0) = T^{-1} R G(\omega, \sqrt{V_0}, k_{\perp 1} \sqrt{V_0}, 0) = T^{-1} R (\omega) G(\omega, k_{\perp 1} \sqrt{V_0}, k_{\perp 1} \sqrt{V_0}, 0) = (V_0/\sqrt{V_0}, k_{\perp 1} \sqrt{V_0}) \equiv \pi = (1/\omega - V_0 k_{\perp 1})$. Our RG calculation additionally shows that the invariance under $\tilde{R}$ holds (asymptotically) also for the interacting case, which is a non-trivial result.

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