Spatial geometry of the rotating disk and its non-rotating counterpart

Klaus Kassner

I. Institut für Theoretische Physik,
Otto-von-Guericke-Universität Magdeburg, Germany

(Dated: 23 February 2012)

A general relativistic description of a disk rotating at constant angular velocity is given. It is argued that conceptually this direct approach poses fewer problems than the special relativistic one. For observers on the disk, the geometry of their proper space is hyperbolic. This has interesting consequences concerning their interpretation of the geometry of a non-rotating disk having the same radius. The influence of clock synchronization on spatial measurements is discussed.

PACS numbers: 03.30.+p; 04.20.-q; 04.20.Cv
Keywords: General relativity, rotating disk, space-time splitting, clock synchronization

I. INTRODUCTION

In his famous 1916 paper introducing general relativity,1 Einstein invoked a rotating frame of reference to give an example of how non-Euclidean geometry may arise in relativistic physics. He considered an inertial frame and a circle, the center of which was at rest in the frame, plus a second frame rotating about the symmetry axis perpendicular to the circle. For symmetry reasons, the geometric figure is a circle in both frames. Measuring its circumference and diameter in the usual way, i.e., by laying out rods along both lines, the ratio of the number of rods needed in the inertial frame would approach π using sufficiently short rods. The same kind of measurement would produce a number exceeding π in the rotating frame, because due to Lorentz contraction more rods would be needed along the circumference but not along the diameter. Contraction arises only parallel to the direction of motion, so radially arranged rods would not suffer from it. The conclusion would then be that the geometry of the circle is non-Euclidean and hyperbolic in the rotating frame.

In later discussions, the circle was replaced by a solid disk (Ref. 2 is an example). This should be quite in the spirit of Einstein, whose basic premise was that spatial geometry is determined by the physical properties of bodies serving as measuring devices, bodies that are approximately rigid, within the limitations set by his theory. The circle then becomes the circumference of the disk. In order to have a physical realization of an “inertial” circle as well, one may imagine a second non-rotating disk under the first, with exactly the same radius. If the two disks are sufficiently close to each other, the outline of their circumferences will essentially be a single spatial curve, and the question is then, how observers in different states of motion will assess the length of this curve.

When publishing these ideas, Einstein had already thought hard for a number of years about the generalization of special relativity, so one might be inclined to believe that he got his introductory example right. And indeed, many researchers later confirmed and extended his results.2–5

However, surprisingly there are to this day controversies about the geometry of and the physics on, a rotating disk. The subject is still alive, as is attested by the existence of a whole book6 devoted to contributions by both opponents and defenders of Einstein’s results. Some authors even consider special relativity to be inapplicable to rotating frames.6,7 Since not all of the divergences can possibly be reconciled, the reader should be warned that there must be errors in several of these contributions. Statements range from the geometry of the rotating disk being hyperbolic, corresponding to a relationship $L > 2\pi R$ between its circumference $L$ and its radius $R$, to the disk remaining Euclidean (6) ($L = 2\pi R$) and the idea that an observer traveling around a circle at constant speed should assess its geometry to be elliptic (5) ($L < 2\pi R$).

The dispute is all the more surprising, as the assumption would seem reasonable that a correct application of general relativity should resolve the issue beyond any doubt. To be sure, because gravity is not involved, a complete description of the physics on a rotating disk can be given within special relativity. Yet this appears to be one of the rare cases where a direct application of general relativity renders the problem simpler than arguing within the special theory. Applicability of the general theory cannot be denied, whereas the – mistaken – idea still seems to be lurking in some minds that special relativity does not work in accelerated frames.

Furthermore, general relativity, being firmly geometry-based, offers certain conceptual advantages, improving clarity. Consider for example the proposition by Semon et al.6,7 that an observer moving around a circle, measuring each length element of its circumference to be Lorentz contracted, will find a total length smaller than $2\pi R$, implying an elliptic geometry. Such an idea seems to make sense within the framework of the special theory, where accelerated motion has an absolute character. The general theory, however, tells us that acceleration is indistinguishable from the effect of an appropriate gravitational field,6,7 so any observer may consider herself at rest. Beyond that, if one cherishes the notion of a spatial geometry that may depend on the state of motion of an observer, which is what consideration of the rotating and static disks suggests, then it seems obvious that in associating a space with an observer, that space must not move with respect to the observer (rather than just might not move). Otherwise we would have to consider an infinite (three-parameter) family of spaces associated with each observer, which would severely exacerbate the uniqueness problems that we will see arise anyway.

Hence, what we mean by a space associated with a particular observer is one, in which the latter is at rest. But what could such an observer learn about the space at large by locally measuring length elements of a curve moving past herself? While the moving curve certainly contains information about distant pieces of space from the past, we must admit that this does not help a lot, since we know from general relativity that geometries may be time dependent, so the information we get by measuring the curve element at the current moment may already be outdated. Special relativity even suggests that this is precisely what happens. At any instant in time, there is a comoving inertial observer having
II. TWO PARADOXES

A. Bell’s spaceship paradox

The thread-between-spaceships paradox was not actually invented by Bell. He just immensely contributed to its popularization.

Imagine two spaceships moving with exactly the same acceleration program in a given inertial system $S$ along a straight line, which we may choose as the $x$ axis. Obviously their distance within $S$ must remain constant by definition. Suppose a thread connects the two spaceships. The length of the thread at the beginning of the flight is exactly equal to the distance of the two spacecraft. Beyond that, the thread is not assumed strong enough to withstand the thrust of the rocket engines, should there be a conflict about the distance at which the thread tries to keep the spaceships and the distance their acceleration dictates. The question then is this: will the thread break or will it not?

The situation is depicted in Fig. 1 displaying the world lines of the two spaceships in a Minkowski diagram, with the coordinates in $S$ given as $x$ and $ct$. The shaded area is a section of the world sheet of the thread.

![Illustration of Bell’s spaceship paradox. Dash-dotted lines denote world lines of light.](image)

At first sight, one might wonder why the question of a breaking thread should arise at all. In any event, the distance of the spaceships remains constant, so why should there be a problem? Well, the thread is set in motion, so it ought to move along a straight line that makes the same angle with the world line of the trailing craft, we know that we simply have to draw a straight line, which we may choose as the $x$ axis. Obviously their distance within $S$ must remain constant by definition. Suppose a thread connects the two spaceships. The length of the thread at the beginning of the flight is exactly equal to the distance of the two spacecraft. Beyond that, the thread is not assumed strong enough to withstand the thrust of the rocket engines, should there be a conflict about the distance at which the thread tries to keep the spaceships and the distance their acceleration dictates. The question then is this: will the thread break or will it not?

Bell insisted that it is real enough to make the thread break and that in teaching relativity, this point should be driven home with the students. In a comoving frame, the thread should keep its proper length, if it is unstressed. This means it must be Lorentz contracted in $S$. Therefore, if its length in $S$ is kept fixed artificially, it will experience Lorentz contraction as “real”. It is considered an effect resulting from different points of view of different observers, not a change of the object.

Bell insisted that it is real enough to make the thread break and that in teaching relativity, this point should be driven home with the students. In a comoving frame, the thread should keep its proper length, if it is unstressed. This means it must be Lorentz contracted in $S$. Therefore, if its length in $S$ is kept fixed artificially, it will experience Lorentz contraction as “real”. It is considered an effect resulting from different points of view of different observers, not a change of the object.

Can this be rationalized from the point of view of an observer comoving with an end point of the thread? Consider the trailing spaceship in the picture. Its instantaneous inertial system $S'$ is described by the axes $x'$ and $ct'$. To obtain the axis $x'$ describing simultaneous events from the point of view of the trailing craft, we know that we simply have to draw a straight line that makes the same angle with the world line of a right-moving light beam as the $ct'$ axis — this guarantees...
that the world line of light is a bisection of the time and space axes and the second postulate of special relativity is satisfied.

If the spacemanship inhabitant wishes to determine the length of the thread, he has to measure the extension of its world sheet along a line of simultaneity, i.e., along the \( x' \) axis. This length is given by the spatial distance of the events \( A \) and \( C \) of the diagram. Because the length units along the \( x \) and \( x' \) axes are not normally equal in Minkowski diagrams, we cannot easily read a quantitative relationship off the figure. But we see that \( C \) is later on the world line of the leading spacemanship than \( B \), where it had the same speed as the trailing craft in \( A \). So the leading spacemanship has a larger velocity in \( C \), an event which in the \( x' \prime \) frame is simultaneous with \( A \). This means that the observer at \( A \) finds the leading spacemanship to increase its distance from his (and to have done so from the beginning of the trip). Therefore, the thread has become longer and will break as soon as its yield limit is reached.

B. Ehrenfest’s paradox

Ehrenfest’s paradox\(^{25} \) has to do with the question whether a disk can be set into rotation via Born rigid motion\(^{28} \). From the very beginnings of special relativity, it was clear that the theory does not permit the existence of rigid bodies in the sense of classical mechanics, because accelerating such a body would mean transmission of a signal instantaneously from one of its ends to the other, which is incompatible with relativistic kinematics.

However, rigid motion in the Born sense seemed feasible. A body moves rigidly, if in a given inertial system each of its volume elements is contracted in the direction of motion by the Lorentz factor corresponding to its instantaneous velocity. Hence, its dimensions in its own successive rest-frames are preserved, implying, as Rindler emphasizes\(^{27} \), that this definition is frame-independent. When a body is set in motion this way, it will not experience any internal stresses (if it had none before being moved). Translational Born rigid acceleration is indeed possible\(^{26} \).

Ehrenfest considered a rotating cylinder\(^{25} \). Assuming the rotation to be rigid, he found two contradictory conclusions. Radial line elements of the cylinder would not be Lorentz contracted, being perpendicular to the local motion, so the radius \( R \) of the cylinder would be unchanged. Hence, its circumference would be \( L = 2\pi R \). On the other hand, the circumference of the cylinder would have to be Lorentz contracted, because line elements along the rim are aligned with the direction of local motion. Hence, the circumference would be \( L < 2\pi R \). We have two incompatible results.

A solution of Ehrenfest’s paradox follows from the fact that there is no Born rigid accelerated rotation.

In order to see this, let us make use of what we have learned from Bell’s spacemanship paradox. Suppose we wish to increase the rotation speed of our cylinder by applying accelerating forces to elements of its rim. Figure 2 gives a visualization. The cross section of the cylinder surface is divided into \( N \) equal segments. To accelerate segment \( i \), forces have to be applied at its endpoints numbered \( i-1 \) and \( i \). These accelerations all have to be equal due to the rotational symmetry of the problem. But then each segment is in a similar situation as the thread in Bell’s spacemanship paradox. Its ends are subject to the same acceleration program (as viewed by an observer \( C \) at the central axis of the cylinder), so the segment will be longitudinally stressed, contrary to the assumption of Born rigid motion.

\[
\begin{align*}
x &= \gamma (x' + vt) , & t &= \gamma \left(t' + \frac{x'}{c^2}v\right),
\end{align*}
\]

with \( \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \). This tells us that, given \( t_{i-1}' = t_i' \), \( t_i \) will be larger than \( t_{i-1} \) by \( \gamma \frac{v^2}{c^2} (x_i' - x_{i-1}' \prime) \), the difference being proportional to the length \( x_i' - x_{i-1}' \prime \) of the segment. All that is important is that by repeating the argument for all segments, we find that we must have \( t_0 < t_1 < \ldots < t_i < \ldots < t_{N-1} < t_N \). But point number 0 and point number \( N \) are identical. Hence we have \( t_0 < t_N = t_0 \), a clear and evident contradiction.

Ununiformly accelerated rotation is possible, accelerated rigid rotation is not.

Note that a disk rotating at constant angular velocity does conform with the definition of Born rigid motion, if we assume Einstein’s result to be correct. In fact, the notion of Lorentz contraction refers to the comparison of lengths of the same object in two different frames, not to that of lengths before and after acceleration\(^{26} \). According to Einstein, the length of the circumference of the rotating disk is \( L = 2\pi R \) in the non-rotating frame and \( L' = \gamma L \) in the rotating one, so the circumference is reduced by precisely the factor \( 1/\gamma \) in the frame with respect to which it moves\(^{22} \).

III. REALIZABILITY

A typical argument against the discussed arrangement of two disks is that the material properties of disks make it impossible to keep the rotating disk flat or to keep the two disks
on top of each other. In a discussion, a student once argued that the top disk would have to spin so fast that it would fly away. The argument is invalid for the simple reason that we do not need a disk that is spinning extremely fast. Relativity is not only valid at large speeds! Its predictions hold at any speed, we just have to measure more precisely to detect and quantify effects at smaller speeds.

Consider a steel disk of 1 m diameter and a mass of 1000 kg (this corresponds to about 16 cm height). Since the transversal speed of sound in steel is about 3200 m/s, we expect its elastic yield limit not to be approached before parts of it are rotating at this speed. To be safe, let us rotate it at 500 rps, meaning that its circumference moves at about 1500 m/s, i.e., $v/c \approx 5 \times 10^{-6}$. Special relativistic effects are $O((v/c)^2)$, so we need a precision in time measurement of 1 part in $4 \times 10^{15}$. With current atomic clocks, precisions that are better than this by a factor of $2.5 \times 10^4$ are achievable. Chip-scale atomic clocks were mounted at NIST already in 2004 (http://tf.nist.gov/afm/smallclock/CSAC.html), albeit with a precision of $10^{-15}$ only. This would not yet suffice, but is only off by a factor of 4 (while big atomic clocks achieve $10^{-13}$).

Another argument holds that general relativistic effects might mask or even overwhelm special relativistic ones, due to the mass of the disk. These effects arising from the stress-energy tensor of the system (and including the effects of mechanical stresses in the disk) may be simply estimated for our small-mass system, they are on the order of $\Phi/c^2$, where $\Phi$ is the Newtonian gravitational potential. The absolute value of the potential of our disk is certainly smaller than that of a sphere of the same mass having as diameter the smallest extension of the disk, i.e., 16 cm. We find $|\Phi|/c^2 < 10^{-23}$ which is below the discussed measuring accuracy, so separability of the effects of acceleration and true space-time curvature would not be an issue in our example.

Observers on the non-rotating disk are inertial. In inertial frames, Euclidean geometry holds, so for these observers both disks will have a circumference of $2\pi R$. Why does Lorentz contraction not reduce the circumference of the rotating disk? Of course, on setting the disk in rotation, the material elements in its periphery will try to Lorentz contract. But they cannot do so arbitrarily as they are connected to other material elements. What will happen instead is that the rim of the disk will develop tensile tangential stresses. As discussed in the last section, the situation is quite similar to the one in Bell’s spaceship paradox.

An alternative way of viewing this is to say that the material of the disk does Lorentz contract but that it is in addition elastically strained in a way that compensates the contraction. Incidentally, the radius of the disk will tend to shrink due to the tensile tangential stresses in the disk, a tendency that may be partially compensated or even overcompensated by centrifugal forces. So in order to have two disks precisely of the same radius, the one that is rotating may have to be taken of different size before starting to spin it. It is also possible to make a stress-free rotating disk by spinning a mould filled with molten steel and cooling it to perform solidification while the whole arrangement is rotating. A similar argument has been given by Rindler. The resulting steel disk will experience compressive tangential stresses along its rim when it is not rotating anymore.

It should be noted that details of how the rotating disk might be realized are irrelevant for an understanding of Einstein’s thought experiment. All that is important is that we have a disk spinning at constant angular velocity and a non-rotating disk that has the same radius as the rotating one.

IV. METRIC DESCRIPTION

In general relativity, the geometry of space-time is described by a metric tensor $(g_{\mu\nu})$, generalizing the Minkowski metric $(\eta_{\mu\nu}) = \text{diag}(-1,1,1,1)$. Moreover, the general relativistic metric may vary as a function of space and time. The line element $ds$ of four-dimensional space-time is given by $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, where Einstein’s summation convention has been used (the subscripts and superscripts running from 0 to 3). All local geometric properties of space-time can be deduced from $(g_{\mu\nu})$.

Points on our disks may be specified in terms of cylindrical coordinates, in which the Minkowski line element of the inertial observer $C$ at the center is rewritten as

$$ds^2 = -c^2dt^2 + dr^2 + r^2d\varphi^2 + dz^2.$$  

We might suppress the summand $dz^2$, as $dz = 0$ on the disks, but we keep it to remind ourselves of the four-dimensional nature of space-time. The non-rotating disk would then be given by $0 \leq r \leq R$, $0 \leq \varphi < 2\pi$, $z = -\Delta z$, and $t$ varies through its lifetime. Each material point on the disk would have fixed coordinates $r$, $\varphi$, and $z$. The rotating disk would be described in terms of the coordinates $0 \leq r \leq R$, $0 \leq \varphi < 2\pi$, $z = 0$ and $t$ varying through its lifetime. Its material points would have $r$ and $z$ fixed and $\varphi = \omega t + \varphi_p$ mod $2\pi$, with $\omega$ the angular velocity of the disk, assumed constant, and $\varphi_p$ the angular position of the point at time $t = 0$. The rim velocity is $v = \omega R$ and of course $v < c$, i.e., for given $\omega$, $R$ cannot be larger than $c/\omega$.

Now we wish to describe the rotating disk from the point of view of comoving observers, meaning that material points have fixed coordinates. An important question then is how to choose the time coordinate – after all, the proper times of observers sitting at different radial coordinates of the disk run at different rates due to different time dilation factors. As it turns out, there is no way to use this local proper time as a global time coordinate on the disk for any useful time interval. However, one of the nicer features of general relativity is that the choice of time coordinate is (with minor restrictions) as arbitrary as the choice of spatial coordinates. This freedom of choice comes at a price – velocities change on a redefinition of time, so the postulate of the constancy of the speed of light does not hold for coordinate velocities. Light propagation is instead described by $ds^2 = 0$.

A convenient choice of the time coordinate is to just keep the time $t$ of the central observer $C$. So we take as new coordinates

$$\hat{t} = t, \quad \hat{r} = r, \quad \hat{\varphi} = \varphi - \omega t \text{ mod } 2\pi, \quad \hat{z} = z$$  

which transforms the line element (2) into

$$ds^2 = -\left(1 - \frac{\omega^2r^2}{c^2}\right)c^2d\hat{t}^2 + 2\frac{\omega r^2}{c}c d\hat{t}d\hat{\varphi} + dr^2 + r^2d\hat{\varphi}^2 + d\hat{z}^2.$$  

(4)
The proper time interval \(d\tau\) for an observer sitting at a point with fixed \(\hat{r}, \hat{\varphi}, \hat{z}\) can be read off this equation using \(ds^2 = -c^2 dt^2\) and setting \(d\hat{r} = d\hat{\varphi} = d\hat{z} = 0\):

\[
d\tau = \sqrt{1 - \frac{\omega^2 \vec{r}^2}{c^2}} \, dt,
\]

as expected from time dilation for a velocity \(\omega \vec{r}\).

At this point, two things may be noted. First, the metric describes a frame that is not time-orthogonal. The components of the metric tensor are scalar products of base vectors spanning the four-space under consideration. Therefore, if the metric contains nondiagonal terms (here summing up to the prefactor of the \(d\hat{r} d\hat{\varphi}\) term), the corresponding pair of base vectors is not orthogonal. In our case, we have a mixing of the time and angular coordinates. Second, the metric is obtained from the Minkowski metric \(\bar{\omega}\) by a mere coordinate transformation. Hence, it cannot describe a space-time with a different geometry. Coordinates are just a way to label space-time points, they do not modify nor create the geometry. This means that the space-time of the rotating disk is just the Minkowski space-time, it is flat, there is no curvature. The situation changes when one is dealing with true gravitating bodies – they curve space-time.

That space-time is flat does not mean that space has to be flat, as can be easily seen from appropriate analogs in Euclidean space. The surface of a sphere is clearly curved, it has nonvanishing Gaussian curvature. Still the sphere is part of a flat Euclidean space, so a flat space can accommodate a curved space of smaller dimensionality. Just as the surface of a sphere may be obtained by setting one of the spherical coordinates of three-space, the radial one, equal to a constant, we may construct a hypersurface of four-space by setting the time variable equal to a constant – this hypersurface will correspond to a spatial slice of four-space. Doing so in \(\bar{\omega}\), we obtain the spatial line element given by

\[
d\ell^2 = d\tau^2 + r^2 d\hat{\varphi}^2 + d\hat{z}^2,
\]

which is the standard form of the line element of Euclidean geometry in cylindrical coordinates. Hence, the three-curvature defined by the hyperplane of simultaneity \(t = \text{const.}\) is flat. This is not too surprising, given that the time coordinate \(t\) is identical to \(t\), the time of an inertial observer \((C)\).

However, the splitting of space-time into \((1+3)\)-dimensional submanifolds describing time and space is not unique, so the resulting three-dimensional geometry depends on the chosen time coordinate, which corresponds to a particular synchronization procedure. The time coordinate \(t\) does not conform with Einstein synchronization \(\omega\) for the local clocks of disk observers. This is obvious for clocks at different radii corresponding to different rates of proper time, but it is also true for clocks having the same radial coordinate, as we shall see below. In general, there is a great variety of possible synchronization choices, essential restrictions being only that the simultaneity relation so established be symmetric and transitive and that simultaneous events remain space-like. Any space-like foliation of space-time (if one exists) may serve to define three-dimensional sets of simultaneous events and thus give a prescription for the synchronization of (non-standard) clocks. For example, it is quite permissible to define a new time coordinate on the disk via \(\bar{t} = \bar{t} - \alpha \vec{r}\), where \(\alpha\) is some constant satisfying \(a < 1/c\). Rewriting the metric in terms of this new time coordinate and the old spatial coordinates (i.e., letting \(\bar{r} = \hat{r}, \bar{\varphi} = \hat{\varphi}, \bar{z} = \hat{z}\) and setting \(d\bar{t} = 0\), we obtain for the spatial line element

\[
d\ell^2 = \left[1 - \frac{a^2 (c^2 - \omega^2 \bar{r}^2)}{c^2}\right] d\tau^2 + 2\omega \bar{r}^2 a d\bar{\varphi} d\bar{\varphi} + \bar{r}^2 d\bar{\varphi}^2 + d\bar{z}^2.
\]

To assess what kind of geometry is described by this spatial metric, we calculate the measured radius \(\bar{R}\) and circumference \(\bar{L}\) of a circle with coordinate radius \(R\):

\[
\bar{R} = \int_{d\bar{\varphi} = d\bar{z} = 0} \sqrt{1 - \frac{a^2 (c^2 - \omega^2 \bar{r}^2)}{c^2}} \, d\bar{r} < R,
\]

\[
\bar{L} = \int_{d\bar{r} = d\bar{z} = 0} \sqrt{2} \, Rd\bar{\varphi} = 2\pi R,
\]

from which we conclude that \(\bar{L}/\bar{R} > 2\pi\), hence the spatial geometry connected with this global time coordinate is non-Euclidean and hyperbolic. Note that the time coordinate \(\bar{t}\) has no particular significance, it was just introduced to demonstrate that the spatial geometry may depend on the choice of clock synchronization.

Klauber insists that “regardless of how one believes time should be defined on the disk” the geometry of the latter must be Riemann flat, because any finite object traveling a geodesic path in the plane of the disk surface will not experience any tidal stresses. From general relativity, we expect objects moving along geodesics (of space-time) to experience tidal forces, if and only if space-time is curved. Therefore Klauber’s observation implies that space-time must be flat. This is what we assumed all along. The rotating disk is not supposed to have a sufficiently large mass for gravity to play a role. But that does not tell us anything about space. In fact, general relativity informs us that objects moving along geodesics will not experience any tidal stresses in a flat space-time, regardless of how it is divided into \((1+3)\) submanifolds. These tidal forces will not care at all about spatial curvature.

Neither this hyperbolic geometry nor the aforementioned Euclidean one correspond to what an observer at rest on the rotating disk will measure using standard clocks and standard rulers. Standard clocks run at the rate of her proper time and the proper length of a standard ruler is given by \(\sqrt{c^2 - \omega^2 \bar{r}^2}\). For an observer sitting at a point \((\bar{r}, \bar{\varphi}, \bar{z})\) and setting \(d\bar{t} = 0\), we obtain the measured radius \(\bar{R}\) and circumference \(\bar{L}\) of a circle with coordinate radius \(R\):

\[
\bar{R} = \int_{d\bar{\varphi} = d\bar{z} = 0} \sqrt{1 - \frac{a^2 (c^2 - \omega^2 \bar{r}^2)}{c^2}} \, d\bar{r} < R,
\]

\[
\bar{L} = \int_{d\bar{r} = d\bar{z} = 0} \sqrt{2} \, Rd\bar{\varphi} = 2\pi R,
\]

from which we conclude that \(\bar{L}/\bar{R} > 2\pi\), hence the spatial geometry connected with this global time coordinate is non-Euclidean and hyperbolic. Note that the time coordinate \(\bar{t}\) has no particular significance, it was just introduced to demonstrate that the spatial geometry may depend on the choice of clock synchronization.

Klauber insists that “regardless of how one believes time should be defined on the disk” the geometry of the latter must be Riemann flat, because any finite object traveling a geodesic path in the plane of the disk surface will not experience any tidal stresses. From general relativity, we expect objects moving along geodesics (of space-time) to experience tidal forces, if and only if space-time is curved. Therefore Klauber’s observation implies that space-time must be flat. This is what we assumed all along. The rotating disk is not supposed to have a sufficiently large mass for gravity to play a role. But that does not tell us anything about space. In fact, general relativity informs us that objects moving along geodesics will not experience any tidal stresses in a flat space-time, regardless of how it is divided into \((1+3)\) submanifolds. These tidal forces will not care at all about spatial curvature.

Neither this hyperbolic geometry nor the aforementioned Euclidean one correspond to what an observer at rest on the rotating disk will measure using standard clocks and standard rulers. Standard clocks run at the rate of her proper time and the proper length of a standard ruler is given by \(c/2\) times the time a light signal will take on a standard clock (which is at rest with respect to the ruler) to travel from one end of the ruler to the other and back. How can we obtain the geometry on the rotating disk as measured by standard rulers? We may employ one of the pillars on which general relativity is founded, the equivalence principle. It states that in a sufficiently small freely falling system of reference the laws of physics have the same form as in the inertial frames of special relativity. Expressed differently, local freely falling frames are inertial frames. In mathematical terms, this means that for any metric describing a piece of space-time (and the gravitational fields arising from its curvature) there are coordinate transformations mapping it locally to a Minkowski metric. The laws of physics in these small space-time patches are known from special relativity. Transforming back to the original metric, we get the form of physical laws in the presence of space-time curvature, i.e., gravitational fields. Transforming to the Minkowski metric locally, we obtain a local decomposition of space-time into the proper time of the freely falling observer and a small platform of three-space, in which standard rulers can be established just as in special relativity. Since the rotating disk is no inertial system, there is no global coordinate transformation mapping the global time to
the proper time of material points of the disk and establishing a common spatial frame for them. However, to find a local transformation mapping the metric to Minkowskian form is easy. In fact, since the squared line element is just a quadratic form, it can be diagonalized by completion of squares, and a diagonal metric with the correct signature can be reduced to the standard (η_{μν}) by appropriate rescaling of the coordinate differentials.

$$d\tau^2 = \frac{d\hat{t}^2}{c^2} + \frac{d\hat{r}^2}{c^2 (1 - \frac{\omega^2 \hat{r}^2}{c^2})} + \frac{d\hat{\phi}^2}{c^2 (1 - \frac{\omega^2 \hat{r}^2}{c^2})} + d\hat{z}^2,$$

where

$$d\hat{\varphi} = \sqrt{1 - \frac{\omega^2 \hat{r}^2}{c^2}} d\hat{\varphi},$$

$$d\hat{z} = d\hat{z}.$$

In the last line of (7), the metric has Minkowskian form, i.e., its spatial part may be interpreted as defining the proper length element:

$$d\ell^2 = d\hat{r}^2 + \frac{1}{1 - \frac{\omega^2 \hat{r}^2}{c^2}} d\hat{\phi}^2 + d\hat{z}^2.$$

A circle with coordinate radius \( \hat{r} = R \) will also have the measured radius \( R \), because for \( d\hat{\varphi} = d\hat{z} = 0 \) we have \( d\ell = d\hat{r} \).

Its circumference will be

$$L = \int_{d\hat{\varphi} = 0}^{2\pi} R d\hat{\varphi} = \frac{2\pi R}{\sqrt{1 - \frac{\omega^2 \hat{r}^2}{c^2}}},$$

Since the speed of rotation of an observer \( M \) at \( \hat{r} = R \) about \( C \) (at \( \hat{\varphi} = 0 \)) is \( v = \omega R \), this satisfies

$$L = L' = 2\pi R \gamma > 2\pi R,$$

which is the result obtained by Einstein and others. Hence, it seems expedient to consider the hyperbolic geometry described by the proper line element (3) the natural geometry of the rotating disk. It should be emphasized that this geometry is not obtainable from a hypersurface of constant time in any synchrony. Instead, it may be visualized as constructed from local patches of space orthogonal to the world lines of material points on the disk. Although the construction to obtain the corresponding proper length element has been known and used correctly by a number of authors, it has been put on a rigorous formal basis only recently via definition of the so-called relative space. In this approach, world lines are used to define equivalence classes constituting the points and local space platforms of relative space, which then becomes the quotient space of the world tube of the disk referred to these classes. This is a precise formulation of our introductory observation that space is only defined by a collection of observers (which are test particles in Ref. [2]). The relative-space approach is claimed to be generalizable to non-stationary and non-symmetric frames of reference whereas in preceding treatments of the rotating disk, the definition of the union of space platforms as a globally valid geometry relied on the fact that the metric is independent of time. The procedure of laying out physical rulers is indefinitely repeatable (always giving the same spatial geometry) only if the metric does not change with time. In this case, it establishes a spatial geometry that is independent of the notion of simultaneity, because the length of a standard ruler at rest with respect to an observer can be ascertained with a single clock without the necessity of synchronizing several clocks.

Let us now turn to the geometry of the non-rotating disk as assessed by observers on the rotating disk. This is interesting, because the non-rotating disk is a moving object according to these observers, and it must somehow be embedded in their hyperbolic space. Observers on the rotating disk at \( \hat{r} = R \) may synchronize their standard clocks according to the Einstein prescription, which means to use the time \( \hat{t} \). Setting \( d\hat{\varphi} = 0 \) in the first equation of (3), we have

$$d\hat{\varphi} = \frac{v R}{c^2} \gamma^2 d\hat{\varphi},$$

and integrating from \( \hat{\varphi} = 0 \) to \( \hat{\varphi} = 2\pi \) along the perimeter, we find \( \Delta t = 2\pi R \gamma^2 \frac{\omega}{c^2} \mu = \gamma^2 \omega \theta' \).

To interpret this result let us imagine a standard clock \( B \) that is Einstein synchronized with some clock \( A \) on the disk edge to be taken around the disk instantaneously (i.e., it ticks off no time, \( d\hat{t} = 0 \)) in the positive angular direction and then be compared with clock \( A \). Because according to the central observer, the time \( \Delta t \) has passed on the return of clock \( B \) and clock \( A \) is running slow by a factor of \( 1/\gamma \) with respect to the central clock, clock \( A \) will have covered an interval \( \Delta t' = \Delta t/\gamma = \mu \theta' \) during \( B \)'s journey. Hence clock \( B \) will lag behind \( A \) by \( \Delta t' \). In reality, clock \( B \) cannot be moved instantaneously, but if we move \( B \) sufficiently slow that time dilation with respect to \( A \) is negligible, then clock \( B \) will have ticked off an interval \( \Delta t' \) less than \( A \) when they meet again. The phenomenon of the time gap \( \Delta t' \) arising with this synchronization method (which is closely related to the kinematic resolution of Ehrenfest's paradox) has been discussed by many authors with a particularly transparent exposition given by Cranor et al. As long as we do not close the space along which we synchronize, Einstein synchronization is however possible without contradiction.

Consider now the circumference of the non-rotating disk given by \( 0 \leq \varphi < 2\pi, r = R \). What angle will it cover on the rotating disk? Since the non-rotating disk is moving with respect to the spinning one, this question makes sense only with the proviso “at a given time”. But then synchronization comes into play. If we assume Einstein synchronization, we have to evaluate the angle at \( d\hat{t} = 0 \), i.e., \( d\hat{\varphi} \) is given by (11). Moreover, we have

$$d\varphi = d\hat{\varphi} + \omega d\hat{t} = d\hat{\varphi} \left(1 + \frac{\omega^2 \gamma^2}{c^2}\right) = \gamma^2 d\hat{\varphi},$$

hence, when \( \varphi \) runs from 0 to 2\( \pi \), \( \varphi \) will only cover an interval of length \( 2\pi /\gamma^2 \). This is smaller than 2\( \pi \), so Einstein synchronization was legitimate. Hence, we find that the non-rotating disk does not fully cover the rotating one! The time
gap translates into a spatial gap, the circumference of the non-rotating disk is not a closed curve for observers on the rotating disk. Of course, the question immediately arises, what is in the gap? The answer is that there is no discontinuity. Since equal times for corotating observers correspond to times increasing along the direction of rotation for observers on the static disk, the continuation of the latter is just another copy of itself but at later times (and possibly more than one copy, depending on the value of $\gamma$). Discontinuities in the course of events on the non-rotating disk will only seem to arise when observations are compared at equal times on clocks that got out of synchronization, being separated by a full turn around the rotating disk. But this is easily explicable as a problem of those clocks, not one of the static disk.

The filling of a spatial gap on the rotating disk by a later replica of the non-rotating one is a consequence of the fact that the static disk is really a space-time object and that pieces of it belonging to different times are scrolled onto a single time for observers on the rotating disk. Its material points cover merely part of the spinning disk, if counted only once, viz. between two encounters of a corotating observer with a (straight) line connecting the center of the static disk with a point on its circumference. Space is nonetheless hyperbolic and the missing part of the rotating disk is covered by an additional copy (or more than one) of (part of) the non-rotating disk. The shape of the first copy, obtained by measuring the “material” circumference at every radius along a line of Einstein simultaneity, is shown in Fig. 3.

![FIG. 3. Coverage of the rotating disk (dashed line) by the non-rotating one (solid line). Projection from hyperbolic to Euclidean plane. $v(R) = 0.75c \Rightarrow \gamma^2(v(R)) = 2.29.$](image)

It should be pointed out that the observations of a traveler moving at constant speed around a circle discussed by Semon et al. are completely explicable by these phenomena without the need of invoking elliptic geometry. Evidently, the situation considered is equivalent to an observer $M$ on the rotating disk measuring the length of the rim of the non-rotating disk. A detailed description of the measuring procedure is not given in Ref. 13 but it is obvious that the authors assume Einstein synchronization. Then the relative speed of the rim of the non-rotating disk with respect to the observer is $-v$. The total time $M$ takes to complete a full turn, as observed by $C$, around the circle is $\Delta t_0 = 2\pi R/v$ according to $C$ and, taking time dilation into account, $\Delta t'_0 = 2\pi R/v\gamma = L/v\gamma = L'/\gamma V_0^2 = L''/v$ according to $M$, who will conclude the length to be $L' = L/\gamma < L$. This view can be defended, as we have seen. However, the conclusion that this constitutes a measurement of the circumference of a circle and that therefore the geometry is elliptic fails, due to the fact that the curve is not even closed in $M$’s reference frame, because the perimeter of the non-rotating disk covers only $1/\gamma^2$ of the circumference of the disk, on which $M$ is living. Note that we do have Lorentz contraction of the non-rotating disk here: in its own frame its circumference is $L$, in $M$’s frame (with respect to which it rotates) it is $L/\gamma$, i.e., contracted but it is not a full circle.

The description of the non-rotating disk is remarkably complex, if Einstein synchronized clocks are used on the spinning disk. It becomes much simpler if we admit a different synchronization, discussed by Cranor et al. To realize it, send a light flash from the center of the rotating disk and have all observers on its rim set their standard clocks to the same time on arrival of the light signal. Let us call this method central synchronization. The relationship between increments of “local center-triggered time” $\hat{t}$ and “local Einstein time” $\tilde{t}$ is

$$\tilde{t} = \hat{t} + \frac{\omega R^2}{c^2 \sqrt{1 - \frac{\omega^2 R^2}{c^2}}} \Phi \Rightarrow \hat{t} = \tilde{t} - \gamma \omega d\Phi,$$

i.e., clocks at fixed radius are advanced by a fixed amount of time proportional to $\Phi$ with respect to Einstein synchronization. This amount is just sufficient to close the time gap and to allow clocks to be synchronized around the disk. The second equality of (13) shows that centrally synchronized clocks share the notion of simultaneity with the central observer $C$ ($\tilde{t} = 0$ implies $d\Phi = 0$ and vice versa) and likewise with observers on the non-rotating disk. To determine the length of the circumference of the static disk, $M$ may still use the time $\Delta t'_0$ needed to circle around it – this interval depends only on the rate of her clock, not on its initial setting. But the relative velocity of fixed points on the rim of the non-rotating disk with respect to $M$ is now different. To calculate it, we have

$$d\Phi = 0 (\Rightarrow d\tilde{t} = -\omega \hat{t} dt) \Rightarrow \hat{t} = \tilde{t}.$$  

Hence, $M$ obtains as length of the material circumference of the non-rotating disk $\Delta t'_0 \gamma^2 v = (L/\gamma V_0) \gamma^2 v = \gamma L = L'$. The non-rotating disk now covers the rotating one completely and has the same spatial geometry, its circumference is a closed curve for $M$ and agrees with that of her own disk.

At this point, it may be appropriate to discuss another objection by Klauber against the non-Euclidean nature of the space of the rotating disk. He considers a tape measure around the rim of the disk and states it not to “meet up with itself at the same point in time”. This is of course the time lag discussed in the last few paragraphs and rendering richer the interpretation of measurements of the non-rotating disk – which would be moving with respect to the tape. However, there is no problem for the rotating disk on which the tape is at rest. Klauber’s argument again neglects the difference between space and space-time. While a tape measure may be considered an approximately one-dimensional object in space, it is two-dimensional in space-time, and its world sheet does meet up with itself, not at a point in time, but along a whole world line. For the purpose of measuring the length of the edge, it does not matter at all whether one end of the tape is in place a little earlier than the other. All that matters is that the two marks on the tape the readings of which constitute the act of measurement, have a common piece of world line, along which they can be compared.

V. CONCLUSIONS

As we have seen, general relativity allows several spatial geometries to be associated with the set of observers on the
rotating disk. This is due to the non-uniqueness of the splitting of space-time into space and time, which is closely related to the fact that clock synchronization is conventional\textsuperscript{13,14}, i.e., can be largely done at liberty (as long as contradictions such as the assigning of two different times to the same space-time event are avoided).

Locally, this splitting can be made unique by specifying a world line describing an observer defined to be at rest and an infinitesimal space platform orthogonal to it. If moreover a set of observers whose world lines fill a finite piece of space-time can be regarded as being at rest — operationally this means that light signals sent by one of these observers to another and reflected back always take the same time\textsuperscript{15} — this leads to a geometry that is distinguished by not being dependent on synchronization. In the case of our disk, this is its proper geometry, corresponding to a frame of reference that coincides with its local rest frame everywhere\textsuperscript{16}.

When geometric dimensions of objects are being measured that move with respect to this reference frame, then synchronization dependence is inevitable as was exemplified by two possible descriptions of the non-rotating disk in the frame of the spinning disk. However, no matter what synchronization is chosen, these objects will “fit” into the space defined by the rest frame. The static disk, as observed by inhabitants of the rotating disk, sits in a locally hyperbolic space whether we choose Einstein synchronization, where length measurement of its circumference would suggest otherwise, or central synchronization, where the measured length matches well with its local rest frame everywhere\textsuperscript{16}.

In general relativity, a single (typically non-inertial) observer defines a local reference frame at best, because the continuation of his local platform of space (given by a small hypersurface of simultaneity) is not unique. Generally, a physical reference frame is a time-like congruence, a set of world lines of observers or test particles filling space-time densely\textsuperscript{17}. This congruence then defines a single frame of reference for a whole set of observers, contrary to the statement by Nikolić.

Very often, frames of reference are not even specified in general relativity but only coordinate systems. In terms of these, a frame of reference may be considered an equivalence class of coordinate systems\textsuperscript{18} connected by so-called internal transformations, where the new space coordinates may depend only on the old ones but not on time. The general form of such a coordinate transformation is

$$x'^{i} = x^{i}(x^{0}, x^{1}, x^{3}) ,$$

with the side conditions $g_{00} > 0$, $\partial x^{0}/\partial x'^{0} > 0$ (making sure that both $x$ and $x'$ are time coordinates) and $g_{ij}dx^{i}dx^{j} < 0$ ($i, j = 1 \ldots 3$, allowing the interpretation of $x'$ as spatial coordinates). Due to these conditions, not all coordinate systems are admissible as representations of a reference frame. For instance, in the Schwarzschild metric\textsuperscript{27} $g_{00}$ changes sign on crossing the event horizon. So it describes a single reference frame only when restricted to the region outside the horizon. Coordinate frames are thus both more general and less general than physical reference frames. They are more general, because one global coordinate system may comprise more than one frame of reference — the Schwarzschild metric may be used to describe an observer falling across the horizon. They are less general, because the same reference frame is describable by many coordinate systems.

As pointed out by Nikolić\textsuperscript{14} the spatial line element\textsuperscript{9} cannot be used for radii $\hat{r} > R_{\odot} \equiv c/\omega$. But this is not required either, as it is the line element of observers at rest on a rotating disk, and such a disk must have a radius smaller than $R_{\odot}$. While it may be debatable that the rotating coordinates describe a reference frame extending through all of space ($g_{00}$ changes sign at $\hat{r} = R_{\odot}$), they are perfectly applicable beyond $R_{\odot}$. Language is often sloppy and a clear distinction not made in general relativity between coordinate systems and reference frames. Because of time-non-orthogonality, the metric\textsuperscript{4} does not even become singular at $\hat{r} = R_{\odot}$, its eigenvalues all remain nonzero. Moreover, $R_{\odot}$ definitely does not have the properties of a horizon. What happens beyond $R_{\odot}$ is that the requirement $ds^{2} < 0$ for a time-like world line cannot be satisfied with $d\hat{r} = d\phi = d\hat{s} = 0$, meaning that no observers can exist at rest. Inertial frames are dragged in the counterrotating direction. Nikolić mentions the stars moving at superluminal velocity around the Earth with respect to the Earth’s rotating (coordinate) frame. They have to do so in order to avoid being faster than light with respect to a local inertial system. The motion of a star in the the equatorial plane is given by $\hat{\phi} = -\omega t + \varphi_{0}$, and null geodesics describing incoming starlight satisfy

$$ds^{2} = 0 \Rightarrow \left(\frac{d\hat{r}}{dt}\right)^{2} + \hat{r}^{2}\left(\frac{d\phi}{dt}\right)^{2} + 2\hat{r}\frac{d\hat{r}}{dt}\frac{d\phi}{dt} = c^{2} - \omega^{2}\hat{r}^{2} . \quad (15)$$

As long as $\hat{r} > R_{\odot}$, we must have $d\phi/dt < 0$, so there are no constant-$\hat{r}$ light paths. In fact, (15) is solved by $d\hat{r}/dt = -c$, $d\phi/dt = -\omega$, which remains valid for $\hat{r} \leq R_{\odot}$. So the light from a star reaches us along a spiral when viewed from the rotating coordinate frame, and one with typically many turns (4 x 365 turns for a star at $\hat{r} = 4$ lightyears). If the star is at $\hat{r} = R_{\odot}$, then its coordinate speed is $\omega R_{\odot}$, which is larger than $c$ for all stars but the Sun. The radius distance\textsuperscript{29} of the star is $R_{\odot}/\gamma \approx R_{\odot}$. While radial ruler distances seem problematic, in general the ruler distance is not well-defined beyond $R_{\odot}$ in rotating coordinates — there is no way to lay out rods at rest with respect to the coordinate frame. To extend length measurements via rulers, one first would have to continue the definition of the physical reference frame beyond $R_{\odot}$ by specifying appropriately moving observers. The definition of a reference frame given by Rizzi and Ruggierio\textsuperscript{6} does not require the observers to be at rest with respect to each other. Nikolić’s suggestion to use the line element\textsuperscript{6} corresponds to
taking the reference frame of all observers at rest with respect to the inertial observer \( C \) and is a valid approach, but by no means the only option.

According to Nikolić, there is something wrong with the standard solution to Ehrenfest’s paradox as he tries to show by the example of a rotating ring (with radius \( r' \)) in a non-rotating circular gutter (with radius \( r = r' \)).\(^{32}\) He is bothered by the following problem. In keeping with the prevalent view (which is also the one presented here), the proper length of a ring rotating at speed \( v \) would be \( 2πr'γ(v) \). The proper length of the gutter, in which the ring turns, would be only \( 2πr = 2πr' \). From the point of view of the gutter, the ring is Lorentz contracted to a length of \( 2πr' \), so it fits inside the gutter. But from the point of view of the ring, the gutter should be Lorentz contracted, i.e., have a length \( 2πr'γ(v) < 2πrγ(v) \). Yet, since the ring is inside the gutter, the length of the gutter cannot be smaller than that of the ring.

This problem has been completely resolved in the preceding section without giving up the hyperbolic geometry of the space of the ring. As often is the case in discussions of length contraction, the key is that the problem has been (unintentionally) posed in terms that are based on an absolute notion of simultaneity.

Let us unveil the relativistic solution in a somewhat pictorial way. Imagine the rotating ring to be inhabited by observer ants, equipped with clocks, who decide to measure the distance between white marks they found at certain points on the gutter (in fact, there is only a single white mark, but the ants do not know that yet). All the ants have to do is to line up at equal distances around the ring and to find out which ants see a white mark at a predetermined time. The number of ants between two successful observers (plus one) multiplied by the length of an ant’s watched segment is the (approximate) length between the marks.

Obviously, the result will depend on how the ants have synchronized their clocks. Suppose they use Einstein synchronization. Then there is necessarily a desynchronization gap \( Δt'_0 \) between the clocks of one pair of ants, and they have to make sure to choose their measuring instant such that this gap does not appear inside the length interval to be determined. If there are \( N \) ants, we can number them so that the gap is between ant no. 1 and its immediate neighbour no. \( N \) (numbering is in the sense of rotation). Then the gap will not appear between the two ants seeing the mark, if measurement is taken to be in the direction of increasing ant number and the instant of measurement is chosen to be slightly before the first white mark passes ant no. 1 (the ants can estimate this time by observing several revolutions before making the actual measurement). Let us further assume that the gutter is slowly and uniformly (in its own frame) heated and that all ants continually measure and log its local temperature. Temperature measurements of all ants will be compared for the point in time when the length measurement is made.

What are the ants’ observations? First, they note that the measured temperature of the gutter is not uniform. It increases in the sense of rotation, because what is the same time for the ants becomes increasingly later for the gutter. Second, they find white marks at distances \( 2πr'/γ^2 \). That is, there are two white marks at least, observed by two different ants, and they have different temperatures.\(^{33}\) While it is not immediately obvious to the ants that this is actually the same white mark, they may note, if the gutter is not perfectly regular, that the scratch pattern near the white marks is always the same. Hence, they will conclude that the length of the gutter between two white marks is smaller than the length of the ring but that there is a suspiciously similar piece of gutter following, different only by a higher temperature.

So the result is that the gutter manages to contain the ring although it is shorter. This is due to the fact that length elements of the gutter at a fixed ring time correspond to length elements at different gutter times. Aligning them does not produce a closed curve in a space-time diagram but a helix. If we could disregard that Einstein synchronization of clocks around the full ring (and further) leads to contradictions (we would have two ants at the same segment with different clock readings but claiming to have synchronized clocks), then the gutter might contain an object of arbitrary length!\(^{47}\)

The situation is far simpler, if the ants use central synchronization for their clocks (employing either of the three methods described in Ref.\(^{43}\)). Then what is simultaneous for the gutter is simultaneous for the ants. Hence, the gutter has the same temperature everywhere at the moment of measurement. Moreover, only a single ant will find the white mark in its segment. The length of the gutter is \( 2πr'γ \), i.e., identical to that of the ring.

Note that certain well-known features of special relativity such as mutual time dilation, mutual length contraction, reciprocity of relative velocities (if inertial system \( S' \) moves at velocity \( v \) in \( S \), \( S \) will move at \( −v \) in \( S' \)) and isotropy of inertial systems all depend on Einstein synchronization. A detailed discussion of these aspects that do not seem to be well-known is beyond the scope of this paper. Einstein synchronization is the most favored method of defining simultaneity in inertial systems, among other reasons, because it leads to symmetry between different inertial systems and makes them isotropic. Also the second postulate of special relativity is valid in this synchronization as a statement on one-way velocities of light. With arbitrary synchronizations, it remains true only referring to the round-trip velocity of light along closed curves.\(^{35} \)

Clearly, in a rotating system, different synchronizations may be more favorable. If we use central synchronization on the ring, there is no Lorentz contraction of the gutter, but the ring is still Lorentz contracted in the frame of the gutter. So the mutuality of length contraction gets lost.

---

1. A. Einstein, “Die Grundlage der allgemeinen Relativitätstheorie,” Ann. Phys. \textbf{354}, 769–822 (1916), English Translation in \textit{The Principle of Relativity} (Methuen, 1923, reprinted by Dover Publications, New York, 1952), pp. 109 – 164.
2. C. W. Berenda, “The Problem of the Rotating Disk,” Phys. Rev. \textbf{62}, 280–290 (1942)
3. Ø. Grøn, “Relativistic description of a rotating disk,” Am. J. Phys. \textbf{43}, 869–876 (1975)
4. T. A. Weber, “A note on rotating coordinates in relativity,” Am. J. Phys. \textbf{65}, 486–487 (1997)
5. G. Rizzi and A. Tartaglia, “Speed of Light on Rotating Platforms,” Found. Phys. \textbf{28}, 1663–1683 (1998)
6. G. Rizzi and A. Tartaglia, “On local and global measurements of the speed of light on rotating platforms,” Found. Phys. Lett. \textbf{12}, 179–186 (1999)
Similarly, the right amount of Lorentz contraction would also be present, if the circumference was $2\pi R/\gamma$ in the non-rotating frame and $2\pi R$ in the rotating one. But then we would have the paradoxical situation described by Ehrenfest, unless we assumed space to be positively curved in an inertial frame, for which there is no reason.

Length measurements can be reduced to time measurements without loss of accuracy using light signals, because the speed of light is known exactly by definition.

Since general relativity encompasses special relativity, all special relativistic effects are strictly speaking also general relativistic ones. What is meant, is however clear: general relativistic effects as opposed to special relativistic ones are effects that are due to true gravitational sources, effects described by the field equations of general relativity that cannot be derived from special relativity.

For more realism, we might give the disk a finite thickness, letting $z$ vary between $-\Delta z_1$ and $-\Delta z_2$.

One can formally rewrite the metric in terms of the local coordinate differentials should be total differentials. Hence we are using standard rulers and clocks at $t = 0$. Then set $B$ to the time reading on $A$ plus half the time span elapsed on $B$ between emission and reception of the signal.

However, if an extended body is not allowed to move along a geodesic of space-time, then it may experience tidal forces. An object kept at a fixed position will be subject to centrifugal forces known to produce tidal stresses.

Defining the spatial line element by a slice of constant $z$ we get:

$$\sqrt{g_{rr}} dr^2 + \sqrt{g_{zz}} dz^2$$

There is an alternative way of determining the proper spatial line element via the calculation of the time a light signal will need to travel to a close-by point and back.

While this is physically well-motivated, the calculation is slightly more complicated. Moreover, its use of the mathematical statement of the equivalence principle is less direct.

For a local transformation, other than a global one, we do not have integrability conditions due to the fact that coordinate differentials should be total differentials.

There is an alternative way of determining the proper spatial line element via the calculation of the time a light signal will need to travel to a close-by point and back. While this is physically well-motivated, the calculation is slightly more complicated. Moreover, its use of the mathematical statement of the equivalence principle is less direct.

For a local transformation, other than a global one, we do not have integrability conditions due to the fact that coordinate differentials should be total differentials.

There is an alternative way of determining the proper spatial line element via the calculation of the time a light signal will need to travel to a close-by point and back. While this is physically well-motivated, the calculation is slightly more complicated. Moreover, its use of the mathematical statement of the equivalence principle is less direct.

For a local transformation, other than a global one, we do not have integrability conditions due to the fact that coordinate differentials should be total differentials.

There is an alternative way of determining the proper spatial line element via the calculation of the time a light signal will need to travel to a close-by point and back. While this is physically well-motivated, the calculation is slightly more complicated. Moreover, its use of the mathematical statement of the equivalence principle is less direct.