Fully constrained Majorana neutrino mass matrices using $\Sigma (72 \times 3)$

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Abstract In 2002, two neutrino mixing ansatze having trimaximally mixed middle ($\nu_2$) columns, namely tri-chi-maximal mixing ($T\chi M$) and tri-phi-maximal mixing ($T\phi M$), were proposed. In 2012, it was shown that $T\chi M$ with $\chi = \pm \frac{\pi}{16}$ as well as $T\phi M$ with $\phi = \pm \frac{\pi}{16}$ leads to the solution, $\sin^2 \theta_{13} = \frac{2}{3} \sin^2 \frac{\pi}{16}$, consistent with the latest measurements of the reactor mixing angle, $\theta_{13}$. To obtain $T\chi M(\chi = \pm \frac{\pi}{16})$ and $T\phi M(\phi = \pm \frac{\pi}{16})$, the type I see-saw framework with fully constrained Majorana neutrino mass matrices was utilised. These mass matrices also resulted in the neutrino mass ratios, $m_1 : m_2 : m_3 = \frac{(2+\sqrt{2})}{1+\sqrt{2}(2+\sqrt{2})} : 1 : \frac{(2+\sqrt{2})}{-1+\sqrt{2}(2+\sqrt{2})}$.

In this paper we construct a flavour model based on the discrete group $\Sigma (72 \times 3)$ and obtain the aforementioned results. A Majorana neutrino mass matrix (a symmetric $3 \times 3$ matrix with six complex degrees of freedom) is conveniently mapped into a flavon field transforming as the complex six-dimensional representation of $\Sigma (72 \times 3)$. Specific vacuum alignments of the flavons are used to arrive at the desired mass matrices.

1 Introduction

The neutrino mixing information is encapsulated in the unitary PMNS mixing matrix which, in the standard PDG parameterisation [1], is given by

$$U_{PMNS} = \begin{pmatrix}
  c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
  -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\
  s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13}
\end{pmatrix} \times \text{diag}(1, e^{\frac{i\pi\phi}{16}}, e^{\frac{i\pi\alpha}{16}}),$$

where $s_{ij} = \sin \theta_{ij}, c_{ij} = \cos \theta_{ij}$. The three mixing angles $\theta_{12}$ (solar angle), $\theta_{23}$ (atmospheric angle) and $\theta_{13}$ (reactor angle) along with the CP-violating complex phases (the Dirac phase, $\delta$, and the two Majorana phases, $\alpha_{21}$ and $\alpha_{31}$) parameterise $U_{PMNS}$. In comparison to the small mixing angles observed in the quark sector, the neutrino mixing angles are found to be relatively large [2]:

$$\sin^2 \theta_{12} = 0.271 \rightarrow 0.345, \quad (2)$$

$$\sin^2 \theta_{23} = 0.385 \rightarrow 0.635, \quad (3)$$

$$\sin^2 \theta_{13} = 0.01934 \rightarrow 0.02392. \quad (4)$$

The values of the complex phases are unknown at present. Besides measuring the mixing angles, the neutrino oscillation experiments also proved that neutrinos are massive particles. These experiments measure the mass-squared differences of the neutrinos and currently their values are known to be [2],

$$|\Delta m^2_{21}| = 70.3 \rightarrow 80.9 \text{ meV}^2, \quad (5)$$

$$|\Delta m^2_{31}| = 2407 \rightarrow 2643 \text{ meV}^2. \quad (6)$$

Several mixing ansatze with a trimaximally mixed second column for $U_{PMNS}$, i.e. $|U_{e2}| = |U_{e3}| = |U_{\nu_2}| = \frac{1}{\sqrt{2}}$, were proposed during the early 2000s [3–7]. Here we briefly revisit two of those, the tri-chi-maximal mixing ($T\chi M$) and the tri-phi-maximal mixing ($T\phi M$),$^1$ which are relevant to

$^1$ $TM_i$ ($TM'$) has been proposed [8,9] as a nomenclature to denote the mixing matrices that preserve various columns (rows) of the tri-
Table 1: The standard PDG observables \( \theta_{13}, \theta_{12}, \theta_{23} \) and \( \delta \) in terms of the parameters \( \chi \) and \( \phi \). Note that the range of \( \chi \) as well as \( \phi \) is \(-\frac{\pi}{2} \) to \( +\frac{\pi}{2} \). In \( T\chi M \) (T\psi M), the parameter \( \chi \) (\( \phi \)) being in the first and the fourth quadrant correspond to \( \delta \) equal to \( +\frac{\pi}{2} \) (0) and \(-\frac{\pi}{2} \) (\( \pi \)), respectively.

|           | \( \sin^2 \theta_{13} \) | \( \sin^2 \theta_{12} \) | \( \sin^2 \theta_{23} \) | \( \delta \) |
|-----------|-----------------|-----------------|-----------------|-----------|
| \( T\chi M \) | \( \frac{2}{3} \sin^2 \chi \) | \( \frac{1}{(3-2\sin^2 \chi)} \) | \( \frac{1}{2} \) | \( \pm \frac{\pi}{2} \) |
| \( T\psi M \) | \( \frac{2}{3} \sin^2 \phi \) | \( \frac{1}{(3-2\sin^2 \phi)} \) | \( \frac{2\sin^2 \left( \frac{\pi}{2} + \phi \right)}{(3-2\sin^2 \phi)} \) | 0, \( \pi \) |

our model. They can be conveniently parameterised [5] as follows:

\[
U_{T\chi M} = \begin{pmatrix}
\frac{\sqrt{2}}{\sqrt{3}} \cos \chi & \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{3} \sin \chi \\
\frac{\cos \chi}{\sqrt{6}} - i \frac{\sin \chi}{\sqrt{3}} & \frac{\cos \chi}{\sqrt{6}} + i \frac{\sin \chi}{\sqrt{3}} & \frac{\cos \chi}{\sqrt{6}} - i \frac{\sin \chi}{\sqrt{3}} \\
\frac{\cos \chi}{\sqrt{6}} + i \frac{\sin \chi}{\sqrt{3}} & \frac{\cos \chi}{\sqrt{6}} - i \frac{\sin \chi}{\sqrt{3}} & \frac{\cos \chi}{\sqrt{6}} + i \frac{\sin \chi}{\sqrt{3}}
\end{pmatrix},
\]

(7)

\[
U_{T\psi M} = \begin{pmatrix}
\frac{\sqrt{2}}{3} \cos \phi & \frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{3} \sin \phi \\
\frac{\cos \phi}{\sqrt{6}} - i \frac{\sin \phi}{\sqrt{3}} & \frac{\cos \phi}{\sqrt{6}} + i \frac{\sin \phi}{\sqrt{3}} & \frac{\cos \phi}{\sqrt{6}} - i \frac{\sin \phi}{\sqrt{3}} \\
\frac{\cos \phi}{\sqrt{6}} + i \frac{\sin \phi}{\sqrt{3}} & \frac{\cos \phi}{\sqrt{6}} - i \frac{\sin \phi}{\sqrt{3}} & \frac{\cos \phi}{\sqrt{6}} + i \frac{\sin \phi}{\sqrt{3}}
\end{pmatrix}.
\]

(8)

Both \( T\chi M \) and \( T\psi M \) have one free parameter each (\( \chi \) and \( \phi \)) which directly corresponds to the reactor mixing angle, \( \theta_{13} \), through the \( U_{33} \) elements of the mixing matrices. The three mixing angles and the Dirac CP phase obtained by relating Eq. (1) with Eqs. (7), (8) are shown in Table 1.

In \( T\chi M \), since \( \delta = \pm \frac{\pi}{2} \), CP violation is maximal for a given set of mixing angles. The Jarlskog CP-violating invariant [10–14] in the context of \( T\chi M \) [5] is given by

\[
J = \frac{\sin 2 \chi}{6 \sqrt{3}}.
\]

(9)

On the other hand, \( T\psi M \) is CP conserving, i.e. \( \delta = 0, \pi \), and thus \( J = 0 \).

Since the reactor angle was discovered to be non-zero at the Daya Bay reactor experiment in 2012 [15], there has been a resurgence of interest [16–27] in \( T\chi M \) and \( T\psi M \) and their equivalent forms. For any \( CP \)-conserving (\( \delta = 0, \pi \)) mixing matrix with non-zero \( \theta_{13} \) and trimaximally mixed \( v_2 \) column, we can have an equivalent parameterisation realised using the \( T\psi M \) matrix. Here the “equivalence” is with respect to the neutrino oscillation experiments. The oscillation scenario is completely determined by the three mixing angles and the Dirac phase (Majorana phases are not observable in neutrino oscillations), i.e. we have a total of four degrees of freedom in the mixing matrix. If we assume \( CP \) conservation and also maximal mixing [4]. By this notation, both \( T\chi M \) and \( T\psi M \) fall under the category of \( TM_2 \). To be more specific, \( TM_2 \), which breaks \( CP \) maximally, is \( T\chi M \) and \( TM_2 \), which conserves \( CP \) is \( T\psi M \).

\[
\text{assume that the } v_2 \text{ column is trimaximally mixed, then there is only one degree of freedom left. It is exactly this degree of freedom which is parameterised using } \phi \text{ in } T\psi M \text{ mixing. Similarly any mixing matrix with } \delta = \pm \frac{\pi}{2}, \theta_{13} \neq 0 \text{ and trimaximal } v_2 \text{ column is equivalent to } T\chi M \text{ mixing.}
\]

In 2012 [22], shortly after the discovery of the non-zero reactor mixing angle, it was shown that \( T\chi M_{(\chi=\pm \frac{\pi}{2})} \) as well as \( T\psi M_{(\phi=\pm \frac{\pi}{2})} \) results in a reactor mixing angle,

\[
\sin^2 \theta_{13} = \frac{2}{3} \sin^2 \frac{\pi}{16} = 0.025,
\]

(10)

consistent with the experimental data. The model was constructed in the Type-1 see-saw framework [28–31]. Four cases of Majorana mass matrices were discussed:

\[
M_{Maj} \propto \begin{pmatrix}
2 - \sqrt{2} & 0 & 1 - \sqrt{2} \\
0 & 1 & 0 \\
1 - \sqrt{2} & 0 & 1 + \sqrt{2}
\end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 - \sqrt{2} \\
0 & 1 & 0 \\
1 - \sqrt{2} & 0 & 1 + \sqrt{2}
\end{pmatrix}
\]

(11)

\[
M_{Maj} \propto \begin{pmatrix}
2 - \sqrt{2} & 0 & 1 - \sqrt{2} \\
0 & 1 & 0 \\
1 - \sqrt{2} & 0 & 1 + \sqrt{2}
\end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 - \sqrt{2} \\
0 & 1 & 0 \\
1 - \sqrt{2} & 0 & 1 + \sqrt{2}
\end{pmatrix}
\]

(12)

where \( M_{Maj} \) is the coupling among the right-handed neutrino fields, i.e. \( (v_R)^T M_{Maj} v_R \). In Ref. [22], the mixing matrix was modelled in the form

\[
U_{PMNS} = \frac{1}{\sqrt{3}} \begin{pmatrix}
1 & 1 & 1 \\
1 & \omega & \bar{\omega} \\
1 & \bar{\omega} & \omega
\end{pmatrix} U_v ,
\]

with \( \omega = e^{i \frac{2\pi}{3}} \) and \( \bar{\omega} = e^{-i \frac{2\pi}{3}} \),

(13)

in which the 3 × 3 trimaximal contribution came from the charged-lepton sector. \( U_v \), on the other hand, was the contribution from the neutrino sector. The four \( U_v \)s vis à vis the four Majorana neutrino mass matrices given in Eqs. (11) and (12), gave rise to \( T\chi M_{(\chi=\pm \frac{\pi}{2})} \) and \( T\psi M_{(\phi=\pm \frac{\pi}{2})} \), respectively. All the four mass matrices, Eqs. (11), (12), have the eigenvalues \( \frac{1 + \sqrt{2(2 + \sqrt{2})}}{2 + \sqrt{2}} \), 1 and \( \frac{1 + \sqrt{2(2 + \sqrt{2})}}{2 + \sqrt{2}} \). Due to the see-saw mechanism, the neutrino masses become inversely proportional to the eigenvalues of the Majorana mass matrices, resulting in the neutrino mass ratios
Using these ratios and the experimentally measured mass-squared differences, the light neutrino mass was predicted to be around 25 meV. In this paper we use the discrete group $\Sigma(72 \times 3)$ to construct a flavon model that essentially reproduces the above results. Unlike in Ref. [22] where the neutrino mass matrix was decomposed into a symmetric product of two matrices, here a single sextet representation of the flavour group is used to build the neutrino mass matrix. A brief discussion of the group $\Sigma(72 \times 3)$ and its representations is provided in Sect. 2 of this paper. Appendix A contains more details such as the tensor product expansions of its various irreducible representations (irreps) and the corresponding Clebsch–Gordan (C–G) coefficients. In Sect. 3, we describe the model with its fermion and flavon field content in relation to these irreps. Besides the aforementioned sextet flavon, we also introduce triplet flavons in the model to build the charged-lepton mass matrix. The flavons are assigned specific vacuum expectation values (VEVs) to obtain the required neutrino and charged-lepton mass matrices. A detailed description of how the charged-lepton mass matrix attains its hierarchical structure is deferred to Appendix B. In Sect. 4, we obtain the phenomenological predictions and compare them with the current experimental data along with the possibility of further validation from future experiments. Finally, the results are summarised in Sect. 5. The construction of suitable flavon potentials which generate the set of VEVs used in our model is demonstrated in Appendix C.

2 The group $\Sigma(72 \times 3)$ and its representations

Discrete groups have been used extensively in the description of flavour symmetries. Historically, the study of discrete groups can be traced back to the study of symmetries of geometrical objects. Tetrahedron, cube, octahedron, dodecahedron and icosahedron, which are the famous Platonic solids, were known to the ancient Greeks. These objects are the only regular polyhedra with congruent regular polygonal faces. Interestingly, the symmetry groups of the platonic solids are the most studied in the context of flavour symmetries too - $A_4$ (tetrahedron), $S_4$ (cube and its dual octahedron) and $A_5$ (dodecahedron and its dual icosahedron). These polyhedra live in the three-dimensional Euclidean space. In the context of flavour physics, it might be rewarding to study similar polyhedra that live in three-dimensional complex Hilbert space. In fact, five such complex polyhedra that correspond to the five Platonic solids exist as shown by Coxeter [32]. They are $3\{3\}3\{3\}3$, $2\{3\}2\{4\}p$, $p\{4\}2\{3\}2$, $2\{4\}3\{3\}3$, $3\{3\}3\{4\}2$ where we have used the generalised schlafli symbols [32] to represent the polyhedra. The polyhedron $3\{3\}3\{3\}3$ known as the Hessian polyhedron can be thought of as the tetrahedron in the complex space. Its full symmetry group has 648 elements and is called $\Sigma(216 \times 3)$. Like the other discrete groups relevant in flavour symmetry, $\Sigma(216 \times 3)$ is also a subgroup of the continuous group $U(3)$. The principal series of $\Sigma(216 \times 3)$ [33] is given by

$$\{e\} \triangleleft \Delta(27) \triangleleft \Delta(54) \triangleleft \Sigma(72 \times 3) \triangleleft \Sigma(216 \times 3).$$

Our flavour symmetry group, $\Sigma(72 \times 3)$, is the maximal normal subgroup of $\Sigma(216 \times 3)$. So we get $\Sigma(216 \times 3)/\Sigma(72 \times 3) = Z_3$. Various details as regards the properties of the group $\Sigma(72 \times 3)$ and its representations can be found in Refs. [33–37]. Note that $\Sigma(72 \times 3)$ is quite distinct from $\Sigma(216)$, which is defined using the relation $\Sigma(216 \times 3)/Z_3 = \Sigma(216)$. In other words, $\Sigma(216 \times 3)$ forms the triple cover of $\Sigma(216)$.

$\Sigma(216 \times 3)$ as well as $\Sigma(216)$ is sometimes referred to as the Hessian group. In terms of the GAP [38,39] nomenclature, we have $\Sigma(216 \times 3) \equiv \text{SmallGroup}(648,532)$, $\Sigma(72 \times 3) \equiv \text{SmallGroup}(216,88)$ and $\Sigma(216) \equiv \text{SmallGroup}(216,153)$.

We find that, in the context of flavour physics and model building, $\Sigma(72 \times 3)$ has an appealing feature: it is the smallest group containing a complex three-dimensional representation whose tensor product with itself results in a complex six-dimensional representation, i.e.

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \bar{\mathbf{3}}.$$  

(16)

With a suitably chosen basis for $\mathbf{6}$ we get

$$\mathbf{6} \equiv \begin{pmatrix} a_1 b_1 \\ a_2 b_2 \\ a_3 b_3 \end{pmatrix}, \quad \bar{\mathbf{3}} \equiv \begin{pmatrix} 1 \sqrt{2} (a_2 b_3 - a_3 b_2) \\ 1 \sqrt{2} (a_3 b_1 + a_1 b_3) \\ 1 \sqrt{2} (a_1 b_2 + a_2 b_1) \end{pmatrix}$$

(17)

where $(a_1, a_2, a_3)^T$ and $(b_1, b_2, b_3)^T$ represent the triplets appearing in the LHS of Eq. (16). All the symmetric components of the tensor product together form the representation $\mathbf{6}$ and the antisymmetric components form $\bar{\mathbf{3}}$. For the $SU(3)$ group it is well known that the tensor product of two $\mathbf{3}$s gives rise to a symmetric $\mathbf{6}$ and an antisymmetric $\bar{\mathbf{3}}$. $\Sigma(72 \times 3)$ being a subgroup of $SU(3)$, of course, has its $\mathbf{6}$ and $\bar{\mathbf{3}}$ embedded in the $\mathbf{6}$ and $\bar{\mathbf{3}}$ of $SU(3)$.

\[\text{We studied the comprehensive list of finite subgroups of } U(3) \text{ provided in Ref. [40] and determined that } \Sigma(72 \times 3) \text{ is the smallest group having this feature.}\]
In general, the $\nu$ or $d$ which transforms as a $6$ handed neutrino Weyl spinors. We may couple leads to a conjugate sextet, $\bar{\nu}_v$, which transforms as a $\bar{6}$, a suitably chosen vacuum expectation value (VEV) for the flavon field.

Consider the complex conjugation of Eq. (16), i.e. $\bar{\mathbf{3}} \otimes \mathbf{3} = \mathbf{6} \oplus \mathbf{3}$. Let the right-handed neutrinos form a triplet, $\nu_R = (\nu_{R1}, \nu_{R2}, \nu_{R3})^T$, which transforms as a $\mathbf{3}$. Symmetric (and also Lorentz invariant) combination of two such triplets leads to a conjugate sextet, $\bar{\nu}_v$, which transforms as a $\bar{6}$, a suitably chosen vacuum expectation value (VEV) for the flavon field.

To describe the representation theory of $\Sigma(72 \times 3)$ we largely follow Ref. [33]. $\Sigma(72 \times 3)$ can be constructed using four generators, namely $C$, $E$, $V$ and $X$ [33]. For the three-dimensional representation, we have

$$C \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E \equiv \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad V \equiv -i \sqrt{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad X \equiv -i \sqrt{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$  (21)

The characters of the irreducible representations of $\Sigma(72 \times 3)$ are given in Table 2. Tensor product expansions of various representations relevant to our model are given in Appendix A. There we also provide the corresponding C–G coefficients and the generator matrices.

### 3 The model

In this paper we construct our model in the Standard Model framework with the addition of heavy right-handed neutrinos. Through the type I see-saw mechanism, light Majorana neutrinos are produced. The fermion and flavon content of the model, together with the representations to which they belong, are given in Table 3. In addition to $\Sigma(72 \times 3)$,
we have introduced a flavour group $C_4 = \{1, -1, i, -i\}$ for obtaining the observed mass hierarchy for the charged leptons. The Standard Model Higgs field is assigned to the trivial (singlet) representation of the flavour groups.

For the charged leptons, we obtain the mass term

$$H = \sqrt{\frac{2\Lambda_{\beta\alpha}}{A^2}} H + \mathcal{H}.T. \quad (22)$$

where $H$ is the Standard Model Higgs, $A$ is the cut-off scale, $y_T$ and $y_{\mu}$ are the coupling constants for the $\tau$-sector and the $\mu$-sector, respectively. $\Lambda_{\beta\alpha}$ is the conjugate triplet obtained from $\phi_{\beta}$ and $\phi_{\alpha}$, constructed in the same way as the second part of Eq. (17),

$$\Lambda_{\beta\alpha} \equiv \left( \sqrt{2} (\phi_{\beta2} \phi_{\alpha3} - \phi_{\beta3} \phi_{\alpha2}), \sqrt{2} (\phi_{\beta3} \phi_{\alpha1} - \phi_{\beta1} \phi_{\alpha3}), \sqrt{2} (\phi_{\beta1} \phi_{\alpha2} - \phi_{\beta2} \phi_{\alpha1}) \right) \quad (23)$$

where $\phi_{\alpha} = (\phi_{\alpha1}, \phi_{\alpha2}, \phi_{\alpha3})^T$ and $\phi_{\beta} = (\phi_{\beta1}, \phi_{\beta2}, \phi_{\beta3})^T$.

$L^\dagger \tau_R$ transforms as $3 \times i$ under the flavour group, $\Sigma(72 \times 3) \times C_4$. The flavon $\tilde{\phi}_{\beta}$ transforms as $\bar{3} \times -i$ and hence it couples to $L^\dagger \tau_R$ as shown in Eq. (22). No other coupling involving $\tau_R, \mu_R$ or $e_R$ with either $\tilde{\phi}_{\beta}$ or $\bar{\phi}_{\alpha}$ is allowed, given the $C_4$ assignments in Table 3. However, $L^\dagger \mu_R$ and $\Lambda_{\beta\alpha}$, which transform as $3 \times 1$ and $\bar{3} \times 1$, respectively, can couple, Eq. (22). Note that $\Lambda_{\beta\alpha}$ is a second-order product of $\phi_{\beta}$ and $\phi_{\alpha}$ and it is antisymmetric. No other second-order product transforming as $\bar{3}$ exists, since the antisymmetric product of $\phi_{\beta}$ with itself or $\phi_{\alpha}$ with itself vanishes. $\mathcal{H}.T.$ represents all the higher-order terms, i.e. the terms consisting of higher-order products of the flavons, coupling to $e_R, \mu_R$ and $\tau_R$. It can be shown that, for obtaining a flavon term coupling to the $e_R$, we require at least quartic order.\(^4\)

The VEV of the Higgs, $(0, \bar{h}_o)$, breaks the weak gauge symmetry. For the flavons $\bar{\phi}_{\alpha}$ and $\tilde{\phi}_{\beta}$, we assign the vacuum alignments\(^3\)

Table 3 The flavour structure of the model. The three families of the left-handed-weak-isospin lepton doublets form the triplet $L$ and the three right-handed heavy neutrinos form the triplet $v_R$. The flavons $\bar{\phi}_{\alpha}$, $\tilde{\phi}_{\beta}$ and $\xi$, are scalar fields and are gauge invariants. On the other hand, they transform non-trivially under the flavour groups

| Fermions | $e_R$ | $\mu_R$ | $\tau_R$ | $L$ | $v_R$ | $\phi_{\alpha}$ | $\phi_{\beta}$ | $\xi$ |
|----------|-------|---------|---------|-----|-------|----------------|----------------|-----|
| $\Sigma (72 \times 3)$ | 1     | 1       | 1       | 3   | $\bar{3}$ | 3               | 3              | 6   |
| $C_4$    | $-1$  | 1       | $i$     | $i$ | $-i$  | $i$            | 1              | 1   |

\(^3\) Refer to Appendix B for an analysis of the higher order products of $\phi_{\alpha}$ and $\phi_{\beta}$.\(^4\) Refer to Appendix C for the details of the flavon potential that leads to these VEVs.

\[ (\bar{\phi}_{\alpha}) = V^\dagger (1, 0, 0)^T m, \quad (\tilde{\phi}_{\beta}) = V^\dagger (0, 0, 1)^T m \quad (24) \]

where $V$ is one of the generators of $\Sigma(72 \times 3)$ given in Eq. (21) and is proportional to the $3 \times 3$ trimaximal matrix. The constant $m$ has dimensions of mass. Substituting these vacuum alignments in Eq. (22) leads to the following charged-lepton mass term:

\[ \left( \begin{array}{c} e_L^\dagger \\ \mu_L^\dagger \\ \tau_L^\dagger \end{array} \right) V^\dagger \left( \begin{array}{ccc} \mathcal{O}(\epsilon^4) & \mathcal{O}(\epsilon^4) & 0 \\ 0 & y_m^\mu h_o \epsilon^2 + \mathcal{O}(\epsilon^4) & 0 \\ \mathcal{O}(\epsilon^4) & \mathcal{O}(\epsilon^4) & y_m^\tau h_o \epsilon + \mathcal{O}(\epsilon^3) \end{array} \right) \left( \begin{array}{c} e_R \\ \mu_R \\ \tau_R \end{array} \right) \quad (25) \]

where $\epsilon = \frac{m}{\Lambda}$. The matrix elements, $\mathcal{O}(\epsilon^3)$ and $\mathcal{O}(\epsilon^4)$, are of the order of $\epsilon^3$ and $\epsilon^4$, respectively. They are the result of the higher-order terms in Eq. (22) containing cubic and quartic flavon products.\(^3\) The mass matrix shown in Eq. (25) is approximately diagonalised\(^5\) by left multiplying it with $V$. It is apparent that the charged-lepton masses, i.e. the eigenvalues of the mass matrix, are in the ratio $\mathcal{O}(\epsilon) : \mathcal{O}(\epsilon^2) : \mathcal{O}(\epsilon^4)$. This is consistent with the experimentally observed mass hierarchy, $\left( \frac{m_\mu}{m_\tau} \right) \approx \left( \frac{m_\mu}{m_\mu} \right)^2$.

Now, we write the Dirac mass term for the neutrinos:

\[ 2y_{\nu} L^\dagger v_R \tilde{H}, \quad (26) \]

where $\tilde{H}$ is the conjugate Higgs and $y_{\nu}$ is the coupling constant. With the help of Eq. (20), we also write the Majorana mass term for the neutrinos:

\[ y_m \bar{\nu}^T \xi, \quad (27) \]

where $y_m$ is the coupling constant. Let $(\xi)$ be the VEV acquired by the sextet flavon $\xi$, and let $(\bar{\xi})$ be the corresponding $3 \times 3$ symmetric matrix of the form given in Eq. (20). Combining the mass terms, Eqs. (26) and (27), and using the VEVs of the Higgs and the flavon, we obtain the Dirac–Majorana mass matrix:

\[ M = \left( \begin{array}{cc} 0 & y_m h_o I \\ y_m h_o I & y_m (\bar{\xi}) \end{array} \right) \quad (28) \]

The $6 \times 6$ mass matrix $M$, forms the coupling

\[ M_{ij} v_i v_j \quad \text{with} \quad v = \left( \begin{array}{c} v_L^3 \\ v_R^3 \end{array} \right) \quad (29) \]

where $v_L^3 = (v_\alpha, v_\mu, v_\tau)^T$ are the left-handed neutrino flavour eigenstates.

\(^5\) The effect of higher-order elements on diagonalisation is discussed in Sect. 4.
Since $y_{u}h_{u}$ is at the electroweak scale and $y_{m}(\xi)$ is at the high energy flavon scale (> $10^{10}$ GeV), small neutrino masses are generated through the see-saw mechanism. The resulting effective see-saw mass matrix is of the form

$$M_{ss} = -(y_{u}h_{u})^{2} (y_{m}(\xi))^{-1}. \quad (30)$$

From Eq. (30), it is clear that the see-saw mechanism makes the light neutrino masses inversely proportional to the eigenvalues of the matrix $\langle \xi \rangle$. We now proceed to construct the four cases of the mass matrices, Eqs. (11), (12), all of which result in the neutrino mass ratios, Eq. (14). To achieve this we choose suitable vacuum alignments$^{6}$ for the sextet flavon $\chi$.

3.1 $T\chi M_{(\chi=+\frac{\pi}{16})}$

Here we assign the vacuum alignment

$$\langle \xi \rangle = (2 - \sqrt{2}, 1, 0, 0, 1, 0) \text{ } \text{ } \text{ } \text{ } \text{ } m. \quad (31)$$

Using the symmetric matrix form of the sextet given in Eq. (20), we obtain

$$\langle \xi \rangle = \left( \begin{array}{ccc} 2 - \sqrt{2} & 0 & \frac{1}{\sqrt{2}} \\
0 & 1 & 0 \\
\frac{1}{\sqrt{2}} & 0 & 0 \end{array} \right) \text{ } \text{ } \text{ } \text{ } \text{ } m. \quad (32)$$

Diagonalising the corresponding effective see-saw mass matrix $M_{ss}$, Eq. (30), we get

$$U_{\nu}^{\dagger} M_{ss} U_{\nu} = \frac{(y_{u}h_{u})^{2}}{y_{m}m} \text{ \ \ \text{Diag} \left( \frac{(2+\sqrt{2})}{1+\sqrt{2}(2+\sqrt{2})}, \frac{1}{\sqrt{2}}, \frac{(2+\sqrt{2})}{1+\sqrt{2}(2+\sqrt{2})} \right)} \quad (33)$$

leading to the neutrino mass ratios, Eq. (14). The unitary matrix $U_{\nu}$ is given by

$$U_{\nu} = i \left( \begin{array}{ccc} \cos \left( \frac{5\pi}{16} \right) & 0 & i \sin \left( \frac{5\pi}{16} \right) \\
0 & 1 & 0 \\
\sin \left( \frac{5\pi}{16} \right) & i \cos \left( \frac{5\pi}{16} \right) \end{array} \right). \quad (34)$$

The product of the contribution from the charged-lepton sector i.e. $V$ from Eqs. (25), (21) and the contribution from the neutrino sector i.e. $U_{\nu}$ from Eqs. (33), (34) results in the $T\chi M_{(\chi=+\frac{\pi}{16})}$ mixing:

$$U_{PMNS} = V U_{\nu} = \left( \begin{array}{ccc} 1 & 0 & 0 \\
0 & \omega & 0 \\
0 & 0 & \tilde{\omega} \end{array} \right) \times \left( \begin{array}{ccc} \frac{\sqrt{2}}{2} \cos \chi & \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2} \sin \chi \\
-\cos \chi & -i \sin \chi & \frac{\sqrt{2}}{2} \cos \chi - i \sin \chi \\
\cos \chi & -i \sin \chi & \frac{\sqrt{2}}{2} \cos \chi - \sin \chi \cos \chi \end{array} \right) \times \left( \begin{array}{ccc} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & i \end{array} \right) \quad (35)$$

with $\chi = +\frac{\pi}{16}$.

3.2 $T\phi M_{(\phi=+\frac{\pi}{16})}$

Here we assign the vacuum alignment

$$\langle \xi \rangle = \left( 0, 1, (2 - \sqrt{2}), 0, 1, 0 \right) \text{ } \text{ } \text{ } \text{ } \text{ } m \quad (36)$$

resulting in the symmetric matrix

$$\langle \xi \rangle = \left( \begin{array}{ccc} 0 & 0 & \frac{1}{\sqrt{2}} \\
0 & 1 & 0 \\
\frac{1}{\sqrt{2}} & 0 & (2 - \sqrt{2}) \end{array} \right) m. \quad (37)$$

In this case, the diagonalising matrix is

$$U_{\nu} = i \left( \begin{array}{ccc} \cos \left( \frac{5\pi}{16} \right) & 0 & i \sin \left( \frac{5\pi}{16} \right) \\
0 & 1 & 0 \\
\sin \left( \frac{5\pi}{16} \right) & -i \cos \left( \frac{5\pi}{16} \right) \end{array} \right) \quad (38)$$

and the corresponding mixing matrix is

$$U_{PMNS} = V U_{\nu} = \left( \begin{array}{ccc} 1 & 0 & 0 \\
0 & \omega & 0 \\
0 & 0 & \tilde{\omega} \end{array} \right) \times \left( \begin{array}{ccc} \frac{\sqrt{2}}{2} \cos \chi & \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2} \sin \chi \\
-\cos \chi & -i \sin \chi & \frac{\sqrt{2}}{2} \cos \chi - i \sin \chi \\
\cos \chi & -i \sin \chi & \frac{\sqrt{2}}{2} \cos \chi - \sin \chi \cos \chi \end{array} \right) \times \left( \begin{array}{ccc} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -i \end{array} \right) \quad (39)$$

with $\chi = -\frac{\pi}{16}$.

3.3 $T\phi M_{(\phi=+\frac{\pi}{16})}$

Here we assign the vacuum alignment

$$\langle \xi \rangle = \left( (i + \frac{1-i}{\sqrt{2}}), 1, -i + \frac{1+i}{\sqrt{2}}, 0, (\sqrt{2}-1), 0 \right) \text{ } \text{ } \text{ } \text{ } \text{ } m \quad (40)$$

$^{6}$ Refer to Appendix C for the details of the flavon potentials that lead to these VEVs.
In this case, the diagonalising matrix is
\[
\langle \xi \rangle = \begin{pmatrix} i + \frac{1+i}{\sqrt{2}} & 0 & 1 - \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ 1 - \frac{1}{\sqrt{2}} & 0 & i + \frac{1+i}{\sqrt{2}} \end{pmatrix} m.
\]

In this case, the diagonalising matrix is
\[
U_\nu = i \begin{pmatrix} \frac{1}{\sqrt{2}} e^{-i \frac{\pi}{16}} & 0 & \frac{1}{\sqrt{2}} e^{i \frac{\pi}{16}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} e^{i \frac{\pi}{16}} & 0 & \frac{1}{\sqrt{2}} e^{-i \frac{\pi}{16}} \end{pmatrix}
\]
and the corresponding mixing matrix is
\[
U_{PMNS} = VU_\nu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \bar{\omega} \end{pmatrix}
\times \begin{pmatrix} \sqrt{\frac{2}{3} \cos \phi} & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3} \sin \phi} \\ -\cos \phi & -\sin \phi & \frac{1}{\sqrt{3} \bar{\omega}} \\ -\sin \phi & \cos \phi & \frac{1}{\sqrt{3} \omega} \end{pmatrix}
\times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -i \\ 0 & 1 & 0 \end{pmatrix}
\]
with \( \phi = -\frac{\pi}{16} \).

3.4 T\phi M_{\phi=-\frac{\pi}{16}}

Here we assign the vacuum alignment
\[
\langle \xi \rangle = \begin{pmatrix} -i + \frac{1+i}{\sqrt{2}} & 1, i + \frac{1-i}{\sqrt{2}} & 0, (\sqrt{2} - 1), 0 \end{pmatrix} \rangle m
\]
resulting in the symmetric matrix
\[
\langle \xi \rangle = \begin{pmatrix} 0 & 1 - \frac{1}{\sqrt{2}} \\ 1 - \frac{1}{\sqrt{2}} & 0 & i + \frac{1+i}{\sqrt{2}} \end{pmatrix}
\]
In this case, the diagonalising matrix is
\[
U_\nu = i \begin{pmatrix} \frac{1}{\sqrt{2}} e^{-i \frac{\pi}{16}} & 0 & \frac{1}{\sqrt{2}} e^{i \frac{\pi}{16}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} e^{i \frac{\pi}{16}} & 0 & \frac{1}{\sqrt{2}} e^{-i \frac{\pi}{16}} \end{pmatrix}
\]
and the corresponding mixing matrix is
\[
U_{PMNS} = VU_\nu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \bar{\omega} \end{pmatrix}
\times \begin{pmatrix} \sqrt{\frac{2}{3} \cos \phi} & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3} \sin \phi} \\ -\cos \phi & -\sin \phi & \frac{1}{\sqrt{3} \bar{\omega}} \\ -\sin \phi & \cos \phi & \frac{1}{\sqrt{3} \omega} \end{pmatrix}
\times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -i \\ 0 & 1 & 0 \end{pmatrix}
\]
with \( \phi = -\frac{\pi}{16} \).

As stated earlier, the four cases, Eqs. (32), (37), (41), (45), result in the same neutrino mass ratios, Eq. (14).

Symmetries of the VEVs of the sextet flavons

A careful inspection of the Majorana matrices, Eqs. (11), (12), reveals several symmetries which could be attributed to the underlying symmetries of the VEVs of the sextet flavons, Eqs. (31), (36), (40), (44). The VEVs, Eqs. (31), (36), (and thus the mass matrices, Eqs. (11)) are composed of real numbers implying they remain invariant under complex conjugation. Therefore, they do not contribute to CP violation. In our model, \( U_{PMNS} = VU_\nu \) where \( V \) originates from the charged-lepton mass matrix, Eq. (25). Since \( V \) is maximally CP-violating (\( \delta = \frac{\pi}{2} \)), the resulting leptonic mixing, \( VU_\nu \), is also maximally CP-violating (\( T\chi M \)). Note that \( UT\chi M \), Eq. (7), is symmetric under the conjugation and the exchange of \( \mu \) and \( \tau \) rows. This generalised CP symmetry under the combined operations of \( \mu-\tau \) exchange and complex conjugation is referred to as \( \mu-\tau \) reflection symmetry in previous publications [5,16,41–43]. The conjugation symmetry in the neutrino VEVs together with maximal CP violation from the charged-lepton sector produces the \( \mu-\tau \) reflection symmetry of \( U_{PMNS} \).

Consider the exchange of the first and the third rows as well as the columns of the mass matrix, Eq. (20). This is equivalent to the exchange of the first and the third elements and the fourth and the sixth elements of the sextet flavon, Eq. (19). In \( \Sigma (72 \times 3) \), this exchange can be realised using the group transformation by the unitary matrix \( E.V.V \),

\[
E.V.V \equiv \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]
with \( E \) and \( V \) given in Eqs. (21), (67). By the group transformation we imply left and right multiplication of the mass matrix using the \( 3 \times 3 \) unitary matrix and its transpose or
equivalently left multiplication of the sextet flavon using the $6 \times 6$ unitary matrix. The mass matrices, Eqs. (12), and the corresponding flavon VEVs, Eqs. (40), (44), are invariant under the transformation by $E.V.V$ together with the conjugation. The VEVs break $\Sigma(72 \times 3)$ almost completely except for $E.V.V$ with conjugation which remains as their residual symmetry. The resulting mixing matrix, $U_{PMNS} = V U_\nu$, is $T \phi M$, which is real and $CP$ conserving. $E.V.V$-conjugation symmetry in the neutrino VEVs together with maximal $CP$ violation from the charged-lepton sector produces the $CP$ symmetry of $U_{PMNS}$.

Both $T \chi M$ and $T \phi M$ have a trimaximal second column. This feature of the mixing matrix was linked to the “magic” symmetry of the mass matrix [42,44–46]. In our model, the charged-leptonic contribution, $V$, is trimaximal. Because of the vanishing of the fourth and the sixth elements of the sextet VEVs, Eqs. (31), (36), (40), (44) which correspond to the off-diagonal zeros present in the mass matrices, Eqs. (11), (12), the trimaximality of $V$ carries over to $U_{PMNS} = V U_\nu$.

Consider the unitary matrix,

$$A \equiv \text{diag}(-1, 1, -1).$$  

(49)

Group transformation by $A$ is the multiplication of the off-diagonal Majorana matrix elements by $-1$. Invariance under $A$, implies these elements vanish and ensures trimaximality. In the literature, small groups like the Klein group [16,47–51] are often used to implement symmetries like the generalised $CP$ and the trimaximality as the residual symmetries of the mass matrix. However, $A$ is not a group member of $\Sigma(72 \times 3)$. In our model, the vanishing mass matrix elements arise as a consequence of the specific choice of the flavon potential, Eq. (142), rather than the result of a residual symmetry under $\Sigma(72 \times 3)$.

The presence of a simple set of numbers in the VEVs (and the mass matrices) is suggestive of additional symmetry transformations (like the one generated by $A$, Eq. (49)) which are not a part of $\Sigma(72 \times 3)$. The present model only serves as a template for constructing any fully constrained Majorana mass matrix using $\Sigma(72 \times 3)$. We impose additional symmetries on the mass matrix by using flavon potentials with a carefully chosen set of parameters, Table 8. Realising these symmetries naturally by incorporating more group transformations along with $\Sigma(72 \times 3)$ in an expanded flavour group requires further investigation.

\[\frac{1}{3 - 2 \sin^2 \left( \frac{\pi}{16} \right)} = 0.342.\]  

(53)

\[\sin^2 \theta_{12} = \frac{1}{3 - 2 \sin^2 \left( \frac{\pi}{16} \right)}\]

8. Besides in Ref. [22], this value was predicted in the context of $\Delta(6\alpha^2)$ symmetry group in Ref. [24] and later obtained in Ref. [25].

9. Whether this correction has a reducing or enhancing effect on $\sin^2 \theta_{13}$, is determined by the relative phase between the $(U_{PMNS})_{13}$ and the correction, which in turn is determined by the phases of the elements in the mass matrix, Eq. (25).

\[\text{For a range of values of the mass matrix elements, we have numerically verified that a reducing effect can be achieved.}\]
This is within 3σ errors of the experimental values, although there is a small tension towards the upper limit. For the atmospheric angle, $T\chi M$ predicts maximal mixing:

$$\sin^2 \theta_{23} = \frac{1}{2}$$

(54)

which is also within 3σ errors. The NuFIT data as well as other global fits [52,53] are showing a preference for non-maximal atmospheric mixing. As a result there has been a lot of interest in the problem of octant degeneracy of $\theta_{23}$ [54–61]. $T\phi M$ predicts this non-maximal scenario of atmospheric mixing. $T\phi M(\phi=\frac{\pi}{16})$ and $T\phi M(\phi=-\frac{\pi}{16})$ correspond to the first and the second octant solutions, respectively. Using the formula for $\theta_{23}$ given in Table 1, we get

$$T\phi M(\phi=\frac{\pi}{16}) : \sin^2 \theta_{23} = \frac{2 \sin^2 \left(\frac{\pi}{16} + \frac{\pi}{16}\right)}{3 - 2 \sin^2 \left(\frac{\pi}{16}\right)}$$

$$= 0.387,$$

(55)

$$T\phi M(\phi=-\frac{\pi}{16}) : \sin^2 \theta_{23} = \frac{2 \sin^2 \left(\frac{\pi}{16} - \frac{\pi}{16}\right)}{3 - 2 \sin^2 \left(\frac{\pi}{16}\right)}$$

$$= 0.613.$$

(56)

The Dirac $CP$ phase, $\delta$, has not been measured yet. The discovery that the reactor mixing angle is not very small has raised the possibility of a relatively earlier measurement of $\delta$ [62–64]. $T\chi M$ having $\delta = \pm \frac{\pi}{2}$ should lead to large observable $CP$-violating effects. Substituting $\chi = \pm \frac{\pi}{16}$ in Eq. (9), our model gives

$$J = \pm \frac{\sin \frac{\pi}{16}}{6\sqrt{3}}$$

$$= \pm 0.0368.$$  

(57)

which is about 40% of the maximum value of the theoretical range, $-\frac{1}{6\sqrt{3}} \leq J \leq +\frac{1}{6\sqrt{3}}$. On the other hand, $T\phi M$, with $\delta = 0$, $\pi$ and $J = 0$, is $CP$ conserving.

The neutrino mixing angles are fully determined by the model, Eqs. (10), (53), (54), (55), (56). Hence, we simply compared the individual mixing angles with the experimental data in the earlier part of this section. Regarding the neutrino masses, the model predicts their ratios, Eq. (14). To compare this result with the experimental data, which gives the mass-squared differences, Eqs. (5), (6), we utilise a $\chi^2$ analysis,

$$\chi^2 = \sum_{x = \Delta m^2_{21}, \Delta m^2_{31}} \left(\frac{x_{\text{model}} - x_{\text{expt}}}{\sigma x_{\text{expt}}}\right)^2.$$  

(58)

We report that the predicted neutrino mass ratios are consistent with the experimental mass-squared differences. Using the $\chi^2$ analysis we obtain,

$$\sum m_i < 183 \text{ meV.}$$

(60)
Our predictions Eqs. (59), give a sum

$$\sum_i m_i = 107.6^{+0.71}_{-0.65} \text{ meV}$$ (61)

which is not far below the current cosmological limit. Improvements in the cosmological bounds from Planck data are expected. Future ground-based CMB polarisation experiments such as Polarbear-2 [74] and Square Kilometer Array-2 [75], could lower the cosmological limit to below 100 meV and could also determine the mass hierarchy. Such results may support or rule out our model.

Neutrinoless double beta decay experiments seek to determine the nature of the neutrinos as Majorana or not. These may support or rule out our model. and could also determine the mass hierarchy. Such results

Neutrinoless double beta decay experiments seek to determine the nature of the neutrinos as Majorana or not. These may support or rule out our model. and could also determine the mass hierarchy. Such results

The most stringent upper bounds on the value of $|m_{\beta\beta}|$ have been set by Heidelberg–Moscow [77,78], Cuoricino [79], NEMO3 [80], EXO200 [81] and GERDA [82] experiments. Combining their results leads to the bounds of the order of a few hundreds of meV [83]. New experiments such as CUORE [84], SuperNEMO [85] and GERDA-2 [86] will improve the measurements on $|m_{\beta\beta}|$ to a few tens of meV and thus may support or rule out our model.

Note that both $+i$ and $-i$ appearing in the diagonal phase matrices in Eqs. (35), (39), (43), (47) correspond to $\alpha_{31} - 2\delta = \pi$.

Renormalisation effects on the observables

The see-saw mechanism requires the existence of a heavy Majorana mass term coupling the right-handed neutrinos together. Our model, combined with the observed neutrino mass-squared differences, predicts that the neutrino masses are a few tens of meVs. This places the see-saw scale (also the flavon scale) at around $10^{12}$ GeV. As such, this is the scale at which the fully constrained mass matrices, as proposed in our model, are generated. In order to accurately compare the model with the observed masses and mixing parameters, it is necessary to calculate its renormalisation group (RG) evolution from the high energy scale down to the electroweak scale.

We use the Mathematica package, REAP [87], to numerically study the RG evolution of the masses and the mixing observables. The Mathematica code for calculating the RG evolution relevant to the model is given below:

```mathematica
Needs["REAP`RGE" ];
RGEAdd["SM"];
RGESetInitial[10^12,
RGE[Theta][12] -> 33.79 Degree,
RGE[Theta][23] -> 45.00 Degree,
RGE[Theta][13] -> 9.165 Degree,
RGE[Delta] -> 90.00 Degree,
RGEMLightest -> 0.03055,
RGE[CapitalDelta][m2sol] -> 0.000119,
RGE[CapitalDelta][m2atm] -> 0.0037490,
RGEY[Nu] -> {[0. , 0. , 0. , 0. , 0. , 0. ]};
RGEY[Nu], RGESolution[100,RGEY[Nu]]
```

In the above code, the initial values of the mixing observables and the masses are set at $10^{12}$ GeV. The mixing observables are chosen such that they correspond to $\chi_{M_{\text{flav}}}$ and $\chi_{M_{\text{flav}}}$. We set the masses to be 30.55, 32.33 and 69.24 meV. These specific values are chosen such that they are consistent with Eq. (14) (at $10^{12}$ GeV) and give the best fit to the observed mass-squared differences when renormalised to the electroweak scale (100 GeV). MSNParameters in the code gives the renormalised parameters at 100 GeV as its output and thus we get $\theta_{12} = 33.78^\circ$, $\theta_{23} = 45.00^\circ$, $\theta_{13} = 9.165^\circ$, $\delta = 90.00^\circ$, $m_1 = 24.63$ meV, $m_2 = 26.07$ meV, $m_3 = 56.16$ meV. From these values we conclude that, under the conditions of our model, renormalisation has virtually no

10 Note that both $+i$ and $-i$ appearing in the diagonal phase matrices in Eqs. (35), (39), (43), (47) correspond to $\alpha_{31} - 2\delta = \pi$.

11 These masses correspond to $\chi_{\min}^2 = 1.23$ as calculated using Eq. (58).
effect on the mixing parameters. On the other hand, it affects our predictions for the masses, Eqs. (59), (61), (65), by a few percentage points.

Analysis of RG equations [87–92] show that, even though the neutrino masses (the fermion masses in general) evolve appreciably, their ratios evolve slowly. This behaviour is sometimes referred to as “universal scaling”. For our model, the light neutrino masses evolve by around 20%, while their ratios by less than 1%. This ensures that the mass ratios, Eq. (14), theorised at the high energy scale remain practically valid at the electroweak scale as well.

5 Summary

In this paper we utilise the group $\Sigma(72 \times 3)$ to construct fully constrained Majorana mass matrices for the neutrinos. These mass matrices reproduce the results obtained in Ref. [22] i.e., $T \chi M(\chi = \pm \frac{\pi}{16})$ and $T \phi M(\phi = \pm \frac{\pi}{16})$ mixings along with the neutrino mass ratios, Eq. (14). The mixing observables as well as the neutrino mass ratios are shown to be consistent with the experimental data. $T \chi M(\chi = \pm \frac{\pi}{16})$ and $T \phi M(\phi = \pm \frac{\pi}{16})$ predict the Dirac CP-violating effect to be maximal (at fixed $\theta_{13}$) and null, respectively. Using the neutrino mass ratios and the light neutrino masses evolve by around 20%, while their ratios evolve slowly. This behaviour is sometimes referred to as “universal scaling”. For our model, the neutrino mass ratios, Eq. (14), theorised at the high energy scale remain practically valid at the electroweak scale as well.

Appendix A: Irreps of $\Sigma(72 \times 3)$ and their tensor product expansions

(i) $3 \otimes 3 = 6 \oplus \bar{3}$

The generator matrices for the triplet representation are provided in Eq. (21). We define the basis for the sextet representation using Eqs. (17). The resulting generator matrices are

$C \equiv \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega \end{pmatrix}$,

$E \equiv \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$.

$V = -\frac{1}{3} \begin{pmatrix} 1 & 1 & 1 & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1 & \bar{\omega} & \bar{\omega} & \sqrt{2} \bar{\omega} & \sqrt{2} \bar{\omega} & \sqrt{2} \bar{\omega} \\ 1 & \omega & \bar{\omega} & \sqrt{2} \omega & \sqrt{2} \omega & \sqrt{2} \omega \\ \sqrt{2} & \sqrt{2} & \sqrt{2} & -1 & -1 & -1 \\ \sqrt{2} \bar{\omega} & \sqrt{2} \bar{\omega} & \sqrt{2} \bar{\omega} & -1 & -\omega & -\omega \\ \sqrt{2} \omega & \sqrt{2} \omega & \sqrt{2} \omega & -1 & -\omega & -\omega \end{pmatrix}$.

$X = -\frac{1}{3} \begin{pmatrix} 1 & 1 & \omega & \sqrt{2} \omega & \sqrt{2} \omega & \sqrt{2} \omega \\ 1 & \bar{\omega} & \bar{\omega} & \sqrt{2} \bar{\omega} & \sqrt{2} \bar{\omega} & \sqrt{2} \bar{\omega} \\ \bar{\omega} & 1 & \bar{\omega} & \sqrt{2} \bar{\omega} & \sqrt{2} \bar{\omega} & \sqrt{2} \bar{\omega} \\ \sqrt{2} \omega & \sqrt{2} \omega & \sqrt{2} \omega & -1 & -1 & -\omega \\ \sqrt{2} \omega & \sqrt{2} \omega & \sqrt{2} \omega & -1 & -\bar{\omega} & -\bar{\omega} \\ \sqrt{2} \bar{\omega} & \sqrt{2} \bar{\omega} & \sqrt{2} \bar{\omega} & -1 & -\bar{\omega} & -\bar{\omega} \end{pmatrix}$.

(ii) $3 \otimes \bar{3} = 1 \oplus 8$

With $(a_1, a_2, a_3)^T$ and $(\bar{b}_1, \bar{b}_2, \bar{b}_3)^T$ transforming as $3$ and $\bar{3}$, the tensor product expansion, Eq. (68), is given by

With (a1, a2, a3)^T and (b1, b2, b3)^T transforming as 3 and \bar{3}, the tensor product expansion, Eq. (68), is given by
$1 \equiv \frac{1}{\sqrt{3}} (a_1 \tilde{b}_1 + a_2 \tilde{b}_2 + a_3 \tilde{b}_3)$,

\[
(\frac{1}{\sqrt{6}} a_1 \tilde{b}_1 - \frac{\sqrt{2}}{\sqrt{3}} a_2 \tilde{b}_2 + \frac{1}{\sqrt{6}} a_3 \tilde{b}_3, \frac{1}{\sqrt{2}} (a_1 \tilde{b}_1 - a_3 \tilde{b}_3), \frac{1}{\sqrt{2}} (a_2 \tilde{b}_2 + a_3 \tilde{b}_3), \frac{1}{\sqrt{2}} (a_1 \tilde{b}_1 + a_2 \tilde{b}_2), \frac{1}{\sqrt{2}} (a_3 \tilde{b}_3 - a_2 \tilde{b}_2), \frac{1}{\sqrt{2}} (a_3 \tilde{b}_3 - a_1 \tilde{b}_1)) \quad (69)
\]

In this basis, the generator matrices of the octet representation are

$C \equiv \frac{1}{2}
\begin{pmatrix}
2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & -\sqrt{3} & 0 \\
0 & 0 & -1 & 0 & 0 & -\sqrt{3} & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & -\sqrt{3} \\
0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}
$

$E \equiv \frac{1}{2}
\begin{pmatrix}
-1 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\
-\sqrt{3} & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$

$V \equiv \frac{1}{6}
\begin{pmatrix}
0 & 0 & \sqrt{3} & \sqrt{3} & \sqrt{3} & 3 & \sqrt{3} & 3 \\
0 & 0 & 3 & 3 & 3 & 3 & -\sqrt{3} & -\sqrt{3} & -\sqrt{3} \\
\sqrt{3} & 3 & 4 & -2 & -2 & 0 & 0 & 0 & 0 \\
0 & \sqrt{3} & -2 & 1 & 1 & 2 & \sqrt{3} & -\sqrt{3} & -\sqrt{3} \\
-3 & \sqrt{3} & 0 & 2 \sqrt{3} & -2 \sqrt{3} & 0 & 0 & 0 & 0 \\
-3 & \sqrt{3} & 0 & -\sqrt{3} & \sqrt{3} & 0 & 0 & 0 & 0 \\
-3 & \sqrt{3} & 0 & -\sqrt{3} & \sqrt{3} & 0 & 0 & 0 & 0
\end{pmatrix}
$

$X \equiv \frac{1}{6}
\begin{pmatrix}
0 & 0 & -2 \sqrt{3} & \sqrt{3} & \sqrt{3} & 0 & -3 & 3 \\
0 & 0 & 0 & -3 & 3 & 2 \sqrt{3} & -\sqrt{3} & -\sqrt{3} & -\sqrt{3} \\
\sqrt{3} & -3 & 1 & 1 & 1 & 4 & -\sqrt{3} & \sqrt{3} & 0 \\
0 & -2 \sqrt{3} & 0 & -2 & -2 & 0 & 2 \sqrt{3} & \sqrt{3} & \sqrt{3} \\
\sqrt{3} & 3 & -2 & -2 & -2 & 1 & -2 \sqrt{3} & 0 & -\sqrt{3} \\
3 & \sqrt{3} & -\sqrt{3} & \sqrt{3} & \sqrt{3} & 0 & 3 & 3 & 0 \\
0 & -2 \sqrt{3} & -2 \sqrt{3} & 0 & -\sqrt{3} & 0 & 0 & 0 & 0 \\
-3 & \sqrt{3} & 0 & 2 \sqrt{3} & \sqrt{3} & 0 & 0 & 0 & -3
\end{pmatrix}
$

We define the basis for the doublet representation in such a way that $6$ is simply the Kronecker product of $2$ and $\bar{3}$, i.e.

$6 \equiv \begin{pmatrix} a_1 \tilde{b}_1 \\
a_1 b_2 \\
a_1 b_3 \\
a_2 b_1 \\
a_2 b_2 \\
a_2 b_3 \end{pmatrix}
$

where $(a_1, a_2)^T$ and $(\tilde{b}_1, \tilde{b}_2, \tilde{b}_3)^T$ represent $2$ and $\bar{3}$, respectively. In such a basis, the generator matrices for the doublet are

$C \equiv \begin{pmatrix} 1 & 0 \\
0 & 1 \end{pmatrix}, \quad E \equiv \begin{pmatrix} 1 & 0 \\
0 & 1 \end{pmatrix}, \quad V \equiv \frac{i}{\sqrt{3}} \begin{pmatrix} 1 & \sqrt{2} \\
\sqrt{2} & -1 \end{pmatrix}, \quad X \equiv \frac{i}{\sqrt{3}} \begin{pmatrix} 1 & \sqrt{2} \omega \\
\sqrt{2} \omega & -1 \end{pmatrix}
$

$2 \otimes 2 = 1 \oplus 1^{(0,1)} \oplus 1^{(1,0)} \oplus 1^{(1,1)} \quad (74)
$

The singlets $1^{(p,q)}$ transform as

$C \equiv 1, \quad E \equiv 1, \quad V \equiv (-1)^p, \quad X \equiv (-1)^q \quad (75)
$

In terms of the tensor product expansion, Eq. (74), these singlets are given by

$1 \equiv a^T u b, \quad 1^{(0,1)} \equiv a^T u_1 b, \quad 1^{(1,0)} \equiv a^T u_\omega b, \quad 1^{(1,1)} \equiv a^T u_{\bar{\omega}} b $ (76)

where $a$ and $b$ represent the doublets in Eq. (74) and $u, u_1$, $u_\omega$ and $u_{\bar{\omega}}$ are unitary matrices,

$u = ia_2, \quad u_1 = u V, \quad u_\omega = u X, \quad u_{\bar{\omega}} = -u \bar{X} \quad (77)$

with $a_2$ being the second Pauli matrix and $V, X$ being the generators of the doublet representation, Eq (73).

$6 \otimes 3 = 2 \oplus 8 \oplus 8 \quad (78)$

The C–G coefficients for the above tensor product expansion are given by

$2 \equiv \begin{pmatrix} \frac{1}{\sqrt{3}} a_1 b_1 + \frac{1}{\sqrt{3}} a_2 b_2 + \frac{1}{\sqrt{3}} a_3 b_3 \\
\frac{1}{\sqrt{3}} a_4 b_1 + \frac{1}{\sqrt{3}} a_5 b_2 + \frac{1}{\sqrt{3}} a_6 b_3 \end{pmatrix} \quad (79)$

(iii) $2 \otimes \bar{3} = 6$
where $(a_1, a_2, a_3, a_4, a_5, a_6)^T$ and $(b_1, b_2, b_3)^T$ represent the sextet and the triplet appearing in the LHS of Eq. (78).

\[ (vi) \quad 6 \otimes \tilde{3} = 3 \oplus 3^{(0,1)} \oplus 3^{(1,0)} \oplus 3^{(1,1)} \]  

The representations $3^{(0,1)}$, $3^{(1,0)}$ and $3^{(1,1)}$ are simply the product of the triplet 3 and the singlets $1^{(0,1)}$, $1^{(1,0)}$ and $1^{(1,1)}$, respectively,

\[ 3^{(p,q)} = 3 \otimes 1^{(p,q)} . \]  

The C–G coefficients for the tensor product expansion, Eq. (82), are given by

\[ 3 \equiv \begin{pmatrix} 
-\frac{1}{\sqrt{2}} a_1 b_1 + \frac{1}{\sqrt{2}} a_3 b_3 \\
\frac{1}{\sqrt{2}} a_2 b_2 + \frac{1}{\sqrt{2}} a_4 b_4 \\
\frac{1}{\sqrt{6}} a_3 b_3 - \frac{\sqrt{3}}{2} a_2 b_2 + \frac{\sqrt{3}}{2} a_4 b_4
\end{pmatrix} , \]  

\[ \tilde{6} \equiv \begin{pmatrix} 
-\frac{1}{\sqrt{2}} a_1 b_1 + \frac{1}{\sqrt{2}} a_3 b_3 \\
\frac{1}{\sqrt{2}} a_2 b_2 + \frac{1}{\sqrt{2}} a_4 b_4 \\
\frac{1}{\sqrt{6}} a_3 b_3 + \frac{\sqrt{3}}{2} a_2 b_2 + \frac{\sqrt{3}}{2} a_4 b_4
\end{pmatrix} , \]  

\[ 3^{(0,1)} \equiv \begin{pmatrix} 
\frac{1}{\sqrt{6}} a_1 b_1 + \frac{1}{\sqrt{6}} a_2 b_2 + \frac{1}{\sqrt{6}} a_3 b_3 + \frac{1}{\sqrt{6}} a_4 b_4 + \frac{1}{\sqrt{6}} a_5 b_5 + \frac{1}{\sqrt{6}} a_6 b_6 \\
\frac{1}{\sqrt{6}} a_1 b_1 + \frac{1}{\sqrt{6}} a_2 b_2 + \frac{1}{\sqrt{6}} a_3 b_3 + \frac{1}{\sqrt{6}} a_4 b_4 + \frac{1}{\sqrt{6}} a_5 b_5 + \frac{1}{\sqrt{6}} a_6 b_6 \\
\frac{1}{\sqrt{6}} a_1 b_1 + \frac{1}{\sqrt{6}} a_2 b_2 + \frac{1}{\sqrt{6}} a_3 b_3 + \frac{1}{\sqrt{6}} a_4 b_4 + \frac{1}{\sqrt{6}} a_5 b_5 + \frac{1}{\sqrt{6}} a_6 b_6
\end{pmatrix} , \]  

and

\[ 3^{(1,0)} = \begin{pmatrix} 
\frac{1}{\sqrt{6}} a_1 b_1 + \frac{1}{\sqrt{6}} a_2 b_2 + \frac{1}{\sqrt{6}} a_3 b_3 + \frac{1}{\sqrt{6}} a_4 b_4 + \frac{1}{\sqrt{6}} a_5 b_5 + \frac{1}{\sqrt{6}} a_6 b_6 \\
\frac{1}{\sqrt{6}} a_1 b_1 + \frac{1}{\sqrt{6}} a_2 b_2 + \frac{1}{\sqrt{6}} a_3 b_3 + \frac{1}{\sqrt{6}} a_4 b_4 + \frac{1}{\sqrt{6}} a_5 b_5 + \frac{1}{\sqrt{6}} a_6 b_6 \\
\frac{1}{\sqrt{6}} a_1 b_1 + \frac{1}{\sqrt{6}} a_2 b_2 + \frac{1}{\sqrt{6}} a_3 b_3 + \frac{1}{\sqrt{6}} a_4 b_4 + \frac{1}{\sqrt{6}} a_5 b_5 + \frac{1}{\sqrt{6}} a_6 b_6
\end{pmatrix} , \]

\[ 3^{(1,1)} = \begin{pmatrix} 
\frac{1}{\sqrt{6}} a_1 b_1 + \frac{1}{\sqrt{6}} a_2 b_2 + \frac{1}{\sqrt{6}} a_3 b_3 + \frac{1}{\sqrt{6}} a_4 b_4 + \frac{1}{\sqrt{6}} a_5 b_5 + \frac{1}{\sqrt{6}} a_6 b_6 \\
\frac{1}{\sqrt{6}} a_1 b_1 + \frac{1}{\sqrt{6}} a_2 b_2 + \frac{1}{\sqrt{6}} a_3 b_3 + \frac{1}{\sqrt{6}} a_4 b_4 + \frac{1}{\sqrt{6}} a_5 b_5 + \frac{1}{\sqrt{6}} a_6 b_6 \\
\frac{1}{\sqrt{6}} a_1 b_1 + \frac{1}{\sqrt{6}} a_2 b_2 + \frac{1}{\sqrt{6}} a_3 b_3 + \frac{1}{\sqrt{6}} a_4 b_4 + \frac{1}{\sqrt{6}} a_5 b_5 + \frac{1}{\sqrt{6}} a_6 b_6
\end{pmatrix} . \]

Here the sextet, 6, appears more than once in the symmetric part. So there is no unique way of decomposing the product space into the sum of the irreducible sextets, i.e. the C–G coefficients are not uniquely defined. To solve this problem, we utilise the group $\Sigma(216 \times 3)$, which has $\Sigma(72 \times 3)$ as one of its subgroups. $\Sigma(216 \times 3)$ has three distinct types of sextets [33], 60, 61, 62. The sextet of $\Sigma(72 \times 3)$ can be embedded in any of these three sextets of $\Sigma(216 \times 3)$. The tensor product expansion for two 60s of $\Sigma(216 \times 3)$ is given by

\[ 6^0 \otimes 6^0 = \tilde{60}^0 \oplus 6^1 \oplus \tilde{6^2} \oplus 3^0 \oplus \tilde{3^0} \oplus 9 . \]  

In Eq. (90), the decomposition of the symmetric part into the irreducible sextets is unique. Hence we embed the irreps of $\Sigma(72 \times 3)$ in the irreps of $\Sigma(216 \times 3)$,
where \((a_1, a_2, a_3, a_4, a_5, a_6)^T\) and \((b_1, b_2, b_3, b_4, b_5, b_6)^T\) represent the sextets appearing in the LHS of Eq. (89). In Eqs. (92)–(99) we have used the curly bracket and the square bracket to denote the symmetric sum and the antisymmetric sum, respectively, i.e. \(a_i b_j = a_i b_j + a_j b_i\) and \(a_i b_j = a_i b_j - a_j b_i\).

\((viii) \quad 6 \otimes \bar{6} = 1 \oplus 8 \oplus 1^{(0,1)} \oplus 1^{(1,0)} \oplus 1^{(1,1)} \oplus 8 \oplus 8 \oplus 8\) (100)

We are not listing the C–G coefficients for the above expansion, since they are not used in our model.

**Appendix B: Hierarchical structure of the charged-lepton mass matrix**

The triplet flavons, \(\phi_a\) and \(\phi_b\), transform as \(3 \times -i\) and \(3 \times i\) under \(\Sigma(72 \times 3) \times C_4\). Table 3. \(L^\dagger \tau_R, L^\dagger \mu_R, L^\dagger e_R\) transform as \(3 \times i, 3 \times 1, 3 \times -1\), respectively. Therefore, the flavons and their tensor products which transform as \(3\) under \(\Sigma(72 \times 3)\) and \(-i, 1, -1\) under \(C_4\) couple with \(L^\dagger \tau_R, L^\dagger \mu_R, L^\dagger e_R\), respectively. \(C_4\) is responsible for restricting the allowed couplings and produces the hierarchical structure of the mass matrix. In Sect. 3, we showed that \(\bar{\phi}_\beta\) and \(A_{\beta\mu}\) couple to the \(\tau\) and \(\mu\) sectors. After symmetry breaking, the flavons attain the VEVs \(\langle \phi_a \rangle = V^\dagger (1, 0, 0)^T m\) and \(\langle \phi_b \rangle = V^\dagger (0, 1, 0)^T m\), Eq. (24). The resulting tau and muon masses are of the order of \(\epsilon\) and \(\epsilon^2\), respectively. The term \(\mathcal{H}.\mathcal{T}\). in Eq. (22) contains all the higher-order products of the flavons transforming as \(3\) and \(-i, 1, -1\) coupling to \(\tau, \mu, \epsilon\) sectors. In this appendix, we analyse the cubic and the quartic products which give rise \(O(\epsilon^3)\) and \(O(\epsilon^4)\) mass matrix elements, respectively in Eq. (25). We neglect the products beyond quartic order.

**Cubic products**

\((i) \quad 3 \otimes 3 \otimes 3 = (6 \oplus \bar{3}) \otimes 3 = 2 \oplus 8 \oplus 8 \oplus 1 \oplus 8\) (101)

\((ii) \quad \bar{3} \otimes \bar{3} \otimes 3 = (6 \oplus \bar{3}) \otimes 3 = 2 \oplus 8 \oplus 8 \oplus 1 \oplus 8\) (102)

\((iii) \quad 3 \otimes 3 \otimes 3 = (6 \oplus \bar{3}) \otimes 3 = 3 \oplus \bar{6} \oplus z^{(0,1)} \oplus z^{(1,0)} \oplus z^{(1,1)} \oplus \bar{6} \oplus 3\) (103)

The above expansions, Eqs. (101)–(103), do not contribute to any coupling, since \(\bar{3}\) does not appear in their RHS.

\((iv) \quad \bar{3} \otimes \bar{3} \otimes 3 = (6 \oplus \bar{3}) \otimes 3 = \bar{3} \oplus 6 \oplus z^{(0,1)} \oplus z^{(1,0)} \oplus z^{(1,1)} \oplus \bar{6} \oplus \bar{3}\) (104)

In the above expansion, \(\bar{3}\) appears twice in the RHS. In terms of the components of the triplets, these \(\bar{3}\)s are given by

First \(\bar{3} = \frac{1}{2\sqrt{2}} \left( \bar{a}_1(2\bar{b}_1c_1 + \bar{b}_2c_2 + \bar{b}_3c_3) + \bar{b}_1(\bar{a}_2c_2 + \bar{a}_3c_3), \right.

\(\bar{a}_2(2\bar{b}_2c_2 + \bar{b}_3c_3 + \bar{b}_1c_1) + \bar{b}_2(\bar{a}_3c_3 + \bar{a}_1c_1), \)

\(\bar{a}_3(2\bar{b}_3c_3 + \bar{b}_1c_1 + \bar{b}_2c_2) + \bar{b}_3(\bar{a}_1c_1 + \bar{a}_2c_2) \right)\) (105)
where \((\tilde{a}_1, \tilde{a}_2, \tilde{a}_3)^T, (\tilde{b}_1, \tilde{b}_2, \tilde{b}_3)^T\) and \((c_1, c_2, c_3)^T\) are the \(\tilde{3}, \tilde{3}\) and \(\tilde{3}\) appearing in the LHS of Eq. (104). The product \(\tilde{3} \otimes \tilde{3} \otimes \tilde{3}\) can be obtained in terms of the flavons \(\phi_\alpha\) and \(\phi_\beta\) in several different ways. These are listed in Table 4. For each combination of flavons, we provide the corresponding \(C_4\) representation. Under \(\langle \phi_\alpha \rangle \propto (1, 0, 0)\) and \(\langle \phi_\beta \rangle \propto (0, 0, 1),\) we calculate the vacuum alignments of the cubic \(\tilde{3}s\) given in Eqs. (105) and (106). These are listed in the last two columns of the table.

The products transforming as \(i\) under \(C_4\) cannot couple to any of the right-handed charged leptons. On the other hand \(\phi_\alpha \phi_\beta \phi_\alpha \phi_\beta\) and \(\phi_\beta \phi_\beta \phi_\alpha \phi_\alpha\), which transform as \(-i\), couple with \(r_T\). From the table, it is clear that these products lead to non-vanishing elements in the third position only. The cubic products provide \(O(e^3)\) contributions to the mass matrix. The aforementioned position corresponds to the position of the \(O(e^3)\) element in the mass matrix, Eq. (25).

### Quartic products

(i) \(3 \otimes 3 \otimes 3 \otimes 3\)

\[
\begin{align*}
&= (6 \otimes 3) \otimes (6 \otimes 3) \\
&= 6 \oplus 6 \oplus 6 \oplus 6 \oplus 3^{(0,1)} \oplus 3^{(1,0)} \oplus 3^{(1,1)} \\
&\quad \oplus 3 \oplus 6 \oplus 3^{(0,1)} \oplus 3^{(1,0)} \oplus 3^{(1,1)} \\
&\quad \oplus 3 \oplus 6 \oplus 3^{(0,1)} \oplus 3^{(1,0)} \oplus 3^{(1,1)} \\
&\quad \oplus 6 \oplus 3 \\
&= (6 \otimes \tilde{3}) \otimes (6 \otimes \tilde{3})
\end{align*}
\]

(ii) \(\tilde{3} \otimes \tilde{3} \otimes \tilde{3}\)

\[
\begin{align*}
&= (6 \otimes \tilde{3}) \otimes (6 \otimes \tilde{3})
\end{align*}
\]

Table 5 Quartic products of \(\phi_\alpha\) and \(\phi_\beta\) of the form \(\tilde{3} \otimes \tilde{3} \otimes \tilde{3} \otimes \tilde{3}\) leading to \(\tilde{3}\)

| \(\phi_\alpha \phi_\alpha \phi_\beta \phi_\alpha\) | 1 | (0, 0, 0) | (0, 0, 0) |
| \(\phi_\alpha \phi_\alpha \phi_\beta \phi_\beta\) | -1 | (0, 0, 1) | (0, 0, 0) |
| \(\phi_\alpha \phi_\beta \phi_\beta \phi_\beta\) | 1 | (0, 0, 0) | (0, 0, 0) |
| \(\phi_\beta \phi_\beta \phi_\beta \phi_\beta\) | 1 | (0, 0, 0) | (0, 0, 0) |

\[
\begin{align*}
&= 1 \oplus 8 \oplus 1^{(0,1)} \oplus 1^{(1,0)} \oplus 1^{(1,1)} \oplus 8 \oplus 8 \\
&\oplus 2 \oplus 8 \oplus 8 \oplus 2 \oplus 8 \oplus 8 \\
&\oplus 1 \oplus 8
\end{align*}
\]

(iii) \(\tilde{3} \otimes \tilde{3} \otimes \tilde{3} \otimes \tilde{3}\)

\[
\begin{align*}
&= (6 \otimes 3) \otimes (3 \otimes 3) \\
&= (\tilde{3} \otimes 6 \otimes 3) \oplus \tilde{3}^{(0,1)} \oplus \tilde{3}^{(1,0)} \oplus \tilde{3}^{(1,1)} \\
&\oplus 6 \otimes 6 \otimes 3 \oplus 6 \otimes 3^{(0,1)} \oplus 6 \otimes 3^{(1,0)} \oplus 6 \otimes 3^{(1,1)} \\
&\oplus 3 \otimes 6 \otimes 3 \oplus 3 \otimes 3^{(0,1)} \oplus 3 \otimes 3^{(1,0)} \oplus 3 \otimes 3^{(1,1)} \\
&\oplus 6 \otimes 3 \\
&= (6 \otimes \tilde{3}) \otimes (3 \otimes \tilde{3})
\end{align*}
\]

(iv) \(\tilde{3} \otimes \tilde{3} \otimes \tilde{3} \otimes \tilde{3}\)

\[
\begin{align*}
&= (6 \otimes \tilde{3}) \otimes (3 \otimes \tilde{3})
\end{align*}
\]

This tensor product corresponds to the conjugation of Eq. (107). The conjugate expansion will have four \(\tilde{3}s\) in the RHS. All the products of \(\phi_\alpha\) and \(\phi_\beta\) in the form of \(3 \otimes 3 \otimes 3 \otimes 3\) are listed in Table 6, along with their respective \(C_4\) representations. Under \(\langle \phi_\alpha \rangle \propto (1, 0, 0)\) and \(\langle \phi_\beta \rangle \propto (0, 0, 1),\) we calculate the vacuum alignments of the above-mentioned four \(\tilde{3}s\) and list them in the table.

For the sake of brevity, we do not provide the explicit expressions of these quartic \(\tilde{3}s\). However, it is straightforward to obtain them, as was the case for the cubic \(\tilde{3}s\), Eqs. (105, 106).

We do not provide the expressions of these \(\tilde{3}s\) also.
An invariant term (singlet) can be constructed using the tensor product expansion of a 3 and a 3, Eq. (68). This expansion is valid for both $\Sigma(72 \times 3)$ and $SU(3)$. It is well known that the singlet constructed from a triplet and its conjugate is the square of the norm of the triplet, e.g. for the flavon $\phi_\alpha = (\phi_1, \phi_2, \phi_3)^T$, we have $|\phi_\alpha|^2 = \phi_1^2 + \phi_1 \phi_2 + \phi_1 \phi_3$. Next we take the symmetric part of the tensor product of two 3s to obtain a 6, similar to Eq. (18),

$$S_\alpha = \begin{pmatrix} 
\phi_1^2 \\
\phi_2^2 \\
\phi_3^2 \\
\sqrt{2}\phi_1 \phi_2 \\
\sqrt{2}\phi_1 \phi_3 \\
\sqrt{2}\phi_2 \phi_3 
\end{pmatrix}.$$ (112)

With this sextet, we may construct a singlet by combining it with its conjugate, i.e. $\tilde{S}_\alpha$. It can be shown that $\tilde{S}_\alpha S_\alpha = |\phi_\alpha|^4$. Therefore, without loss of generality we may choose,

$$T_{\phi_\alpha} = (|\phi_\alpha|^2 - m^2)^2$$ (113)

as the potential term having a minimum at $(1, 0, 0)^T m$. $T_{\phi_\alpha}$ is invariant not only under $\Sigma(72 \times 3)$ but also under the continuous symmetry, $SU(3)$, and hence the minima of the potential are not discrete. This issue can be tackled in two different ways. We may add higher-order non-renormalisable terms which break $SU(3)$ to $\Sigma(72 \times 3)$. Or we may introduce extra flavons whose sole purpose is to break $SU(3)$ while maintaining renormalisability. In this paper we choose the later approach.

To achieve $SU(3)$ breaking we introduce a flavon $\eta_\alpha = (\eta_\alpha^u, \eta_\alpha^d)^T$, which transforms as a doublet under $\Sigma(72 \times 3)$, Eqs. (73). For $\eta_\alpha$, we construct the following potential:

$$T_{\eta_\alpha} = (|\eta_\alpha|^2 - m^2)^2 + Re^2(\eta_\alpha^T u_1 \eta_\alpha) + Re^2(\bar{\omega} \eta_\alpha^T u_\omega \eta_\alpha),$$ (114)

where $Re^2$ denotes the square of the real part. Since the terms $\eta_\alpha^T u_1 \eta_\alpha$, $\eta_\alpha^T u_\omega \eta_\alpha$ and $\eta_\alpha^T u_\omega \eta_\alpha$ transform as $I^{(0,1)}$, $I^{(1,0)}$ and $I^{(1,1)}$, respectively, Eqs. (76), their squares are invariants. Therefore it is evident that $Re^2(\eta_\alpha^T u_1 \eta_\alpha)$, $Re^2(\bar{\omega} \eta_\alpha^T u_\omega \eta_\alpha)$ and $Re^2(\bar{\omega} \eta_\alpha^T u_\omega \eta_\alpha)$ are also invariants. In terms of the components of $\eta_\alpha$, the invariants in Eq. (114) are given by

$$Re^2(\eta_\alpha^T u_1 \eta_\alpha) = Re^2\left(\frac{\sqrt{2}}{\sqrt{3}}(\eta_\alpha^1 \eta_\alpha^2 - \sqrt{2}\eta_\alpha^1 \eta_\alpha^2 - i\eta_\alpha^2)^2\right),$$ (116)

$$Re^2(\bar{\omega} \eta_\alpha^T u_\omega \eta_\alpha) = Re^2\left(\bar{\omega} \frac{\sqrt{2}}{\sqrt{3}}(\eta_\alpha^1 \eta_\alpha^2 - \sqrt{2}\eta_\alpha^1 \eta_\alpha^2 + i\eta_\alpha^2)^2\right),$$ (117)

| $C_4$ | First 3 | Second 3 | Third 3 | Fourth 3 |
|-----------------|--------|--------|--------|--------|
| $\phi_1 \phi_2 \phi_3 \phi_4$ | -1 | (0, 0, 0) | (0, 0, 0) | (0, 0, 0) |
| $\phi_1 \phi_2 \phi_3 \phi_4$ | 1 | (0, 0, 0) | (0, -1/2, 0) | (0, 0, 0) |
| $\phi_1 \phi_2 \phi_3 \phi_4$ | 1 | (0, 0, 0) | (0, 0, -1/2) | (0, 0, 0) |
| $\phi_1 \phi_2 \phi_3 \phi_4$ | -1 | (0, 0, 0) | (0, 0, 0) | (0, 0, 0) |
| $\phi_1 \phi_2 \phi_3 \phi_4$ | -1 | (0, 0, 0) | (0, 0, 0) | (0, 0, 0) |
| $\phi_1 \phi_2 \phi_3 \phi_4$ | -1 | (0, 0, 0) | (0, 0, 0) | (0, 0, 0) |

(0, 0, 1), the four 3s attain specific alignments which we have calculated and provided in the table.

The quartic products in Tables 5, 6 with the $C_4$ representations 1 and $-1$ couple to $\mu_R$ and $e_R$, respectively. The VEVs with $C_4 = 1$ have non-zero elements in the first, the second and the third positions while the VEVs with $C_4 = -1$ have non-zero elements in the first and the third positions only. The quartic products provide $O(\epsilon^4)$ contributions to the mass matrix. The aforementioned positions correspond to the positions of the $O(\epsilon^4)$ elements in the mass matrix, Eq. (25).

### Appendix C: Flavon potentials

Here we discuss the flavon potentials that lead to the vacuum alignments assumed in our model. It should be noted that even though our construction results in the required VEVs, we are not doing an exhaustive analysis of the most general flavon potentials involving all the possible invariant terms. However, the content we include is sufficient to realise our VEVs.

The triplet flavons: $\phi_\alpha$, $\phi_\beta$

First we consider the triplet flavons $\phi_\alpha$ and $\phi_\beta$. Our target is to obtain the VEVs, $\langle \phi_\alpha \rangle = V(1, 0, 0)^T m$ and $\langle \phi_\beta \rangle = V(0, 0, 1)^T m$, Eqs. (24). The flavons $\phi_\alpha$ and $\phi_\beta$ transform as 3, Table 3. The $3 \times 3$ maximal matrix $V$, Eqs. (21), is one of the generators of 3. Therefore if the potentials of $\phi_\alpha$ and $\phi_\beta$ have minima at $(1, 0, 0)^T m$ and $(0, 0, 1)^T m$, then they have minima at $(1, 0, 0)^T m$ and $V(0, 0, 1)^T m$ as well. The $3 \times 3$ cyclic matrix $E$, Eqs. (21), is another generator of 3. Therefore, if the potential has a minimum at $(1, 0, 0)^T m$, then it has a minimum at $(0, 0, 1)^T m$ also. So, for obtaining the target VEVs, all we need to do is to construct a potential with a minimum at $(1, 0, 0)^T m$.
\[ \text{Re}^2(\omega \eta_\alpha^T u_{\alpha} \eta_\alpha) = \text{Re}^2 \left( \frac{i \sqrt{3}}{\sqrt{3}} (\omega \eta_\alpha^2 - \sqrt{2} \eta_\alpha \eta_\beta - \omega \eta_\beta^2) \right). \]  

(118)

If we assign

\[ (\eta_\alpha) = (1, 0)^T m, \]

(119)

it is evident that each of these invariants vanishes. Therefore the potential, Eq. (114), attains its minimum value of zero at \( \eta_\alpha = (1, 0)^T m \) (and also at the states generated by the discrete transformations on \( (1, 0)^T m \)). Note that the first term, \( \langle \eta_\alpha^2 - m^2 \rangle^2 \), is \( SU(2) \) invariant. The other three terms break the continuous \( SU(2) \) group and all its \( U(1) \) subgroups so that only discrete \( SU(2) \) invariants generated by Eqs. (73) are present in the potential.

The Kronecker product of \( \eta_\alpha(2) \) and \( \phi_\alpha(3) \), calculated using Eq. (72), leads to a sextet \( (6) \),

\[ K_\alpha = \begin{pmatrix} \eta_\alpha \phi_\alpha \\ \eta_\alpha \phi_\beta \\ \eta_\alpha \phi_\gamma \\ \eta_\gamma \phi_\alpha \\ \eta_\gamma \phi_\beta \\ \eta_\beta \phi_\gamma \end{pmatrix}. \]

(120)

We utilise \( S_\alpha \), Eq. (112), and \( K_\alpha \), Eq. (120), to couple together the flavons \( \eta_\alpha, \phi_\alpha \) and their conjugates and thus we construct

\[ T_{\phi_\alpha} = (S_\alpha - K_\alpha)^T (S_\alpha - K_\alpha) \]

(121)

as an invariant.\(^{15}\) If we assign

\[ (\phi_\alpha) = (1, 0, 0)^T m, \]

(122)

its symmetric product, \( S_\alpha \), becomes \((1, 0, 0, 0, 0, 0)^T m^2 \). The Kronecker product, \( K_\alpha \), of \( (\eta) = (1, 0)^T m \) and \( (\phi_\alpha) = (1, 0, 0)^T m \) also becomes \((1, 0, 0, 0, 0, 0)^T m^2 \). Therefore, \( T_{\phi_\alpha} \) vanishes (which is its minimum value) at assigned VEVs, Eqs. (119), (122).

Combining Eqs. (113), (114), (121), we obtain the following renormalisable potential term for the flavon \( \phi_\alpha \):

\[ T_{\phi_\alpha} + T_{\eta_\alpha} + T_{\phi_\alpha \eta_\alpha} \]

(123)

which is \( \Sigma(72 \times 3) \) invariant and at the same time devoid of continuous symmetries. A similar potential can be constructed for the flavon \( \phi_\beta \) also,

\[ T_{\phi_\beta} + T_{\eta_\beta} + T_{\phi_\beta \eta_\beta}, \]

(124)

\(^{15}\) Under the group \( C_4 \), Table 3, \( \phi_\alpha \) belongs to \(-i\). Hence \( \eta_\alpha \) needs to transform as \( i \) to ensure the invariance of Eq. (121).

\(^{16}\) Under the group \( C_4 \), Table 3, \( \phi_\beta \) belongs to \( i \). Hence \( \eta_\beta \) needs to transform as \(-i\) to ensure the invariance of \( T_{\phi_\beta \eta_\beta} \) in Eq. (124).
Our first step is to write the potential terms for $\eta$, $\phi_a$ and $\phi_b$, similar to Eq. (126),
\begin{align}
T_{\phi_a} + T_{\phi_b} + T_{\eta} + T_{\phi_a\eta} + T_{\phi_b\eta} + T_{\phi_a\phi_b}.
\end{align}
(127)

The individual invariant terms in Eq. (127) are
\begin{align}
T_{\phi_a} &= (|\phi_a|^2 - m^2)^2, \\
T_{\phi_b} &= (|\phi_b|^2 - m^2)^2, \\
T_{\eta} &= (|\eta|^2 - m^2)^2 \\
&+ \text{Re}^2(\eta^T u_1 \eta) + \text{Re}^2(\bar{\omega} \eta^T u_\omega \eta) + \text{Re}^2(\omega \eta^T u_\omega \eta), \\
T_{\phi_a\eta} &= (\tilde{S}_a - \tilde{K}_a)^T (S_a - K_a), \\
T_{\phi_b\eta} &= (\tilde{S}_b - \tilde{K}_b)^T (S_b - K_b), \\
T_{\phi_a\phi_b} &= |\phi_a^\dagger \phi_b|^2,
\end{align}
(128)–(133)

where $S_a$, $K_a$ and $S_b$, $K_b$ are defined similar to $S_a$, $K_a$ in Eqs. (112), (120) having $\phi_a$, $\eta_a$ replaced with $\phi_b$, $\eta$ and $\phi_b$, $\eta$, respectively. As described earlier, it is straightforward to show that each term in Eqs. (128)–(133) vanishes, if we assign the following VEVs:
\begin{align}
\langle \eta \rangle &= (1, 0, 0)^T m, \\
\langle \phi_a \rangle &= (1, 0, 0)^T m, \\
\langle \phi_b \rangle &= (0, 0, 1)^T m.
\end{align}
(134)–(136)

$S_a$ and $S_b$ are the sextets constructed from $\phi_a$ and $\phi_b$, respectively. We may also construct a sextet combining $\phi_a$ and $\phi_b$ together,
\begin{align}
S_{ab} &= \left(\begin{array}{c}
\phi_{a1}\phi_{b1} \\
\phi_{a2}\phi_{b2} \\
\phi_{a3}\phi_{b3}
\end{array}\right) \\
&\left(\begin{array}{c}
\frac{1}{\sqrt{2}} (\phi_{a2}\phi_{b3} + \phi_{a3}\phi_{b2}) \\
\frac{1}{\sqrt{2}} (\phi_{a3}\phi_{b1} + \phi_{a1}\phi_{b3}) \\
\frac{1}{\sqrt{2}} (\phi_{a1}\phi_{b2} + \phi_{a2}\phi_{b1})
\end{array}\right).
\end{align}
(137)

Using the VEVs, Eqs. (135), (136), we obtain
\begin{align}
\langle S_a \rangle &= (1, 0, 0, 0, 0, 0)^T m^2, \\
\langle S_b \rangle &= (0, 0, 1, 0, 0, 0)^T m^2, \\
\langle S_{ab} \rangle &= (0, 0, 0, 0, \frac{1}{\sqrt{2}} 0)^T m^2.
\end{align}
(138)–(140)

We use the sextet $\xi = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6)$ to construct the Majorana neutrino mass matrix, Eq. (20). In the VEV of $\xi$, if any two among the three elements $\xi_4$, $\xi_5$ and $\xi_6$ become zero, then two off-diagonal elements in the mass matrix vanishes. $U_\nu$ effectively becomes a $2 \times 2$ unitary matrix and $U_{PMNS} = VU_\nu$, attains one trimaximal column. In all the four VEVs, Eqs. (31), (36), (40), (44), we can see that $\xi_4 = 0$ and $\xi_6 = 0$ leading to the trimaximal second column, i.e. $|U_{e2}| = |U_{\mu 2}| = |U_{\tau 2}| = \frac{1}{\sqrt{3}}$. Both $T_\chi M$ and $T_\phi M$ belong to the larger class of mixing schemes in which one neutrino is trimaximally mixed [7]. As the first step in constructing the potential for $\xi$, we consider the tensor product of two $\xi$s, Eq. (89), and obtain a sextet,
\begin{align}
\bar{X} &= \left(\begin{array}{c}
\frac{1}{\sqrt{3}} \xi_1 (2 \xi_3) - \frac{1}{\sqrt{3}} \xi_4 \xi_6 \\
\frac{1}{\sqrt{3}} \xi_2 (3 \xi_1) - \frac{1}{\sqrt{3}} \xi_5 \xi_6 \\
\frac{1}{\sqrt{3}} \xi_3 (1 \xi_2) - \frac{1}{\sqrt{3}} \xi_6 \xi_6 \\
\frac{1}{\sqrt{3}} \xi_4 (5 \xi_6) - \frac{1}{\sqrt{3}} \xi_1 (1 \xi_4) \\
\frac{1}{\sqrt{3}} \xi_5 (6 \xi_4) - \frac{1}{\sqrt{3}} \xi_2 (2 \xi_5) \\
\frac{1}{\sqrt{3}} \xi_6 (4 \xi_5) - \frac{1}{\sqrt{3}} \xi_3 (3 \xi_6)
\end{array}\right),
\end{align}
(141)
as shown in Eq. (92). $\bar{X}$ transforms as a $\bar{6}$. Note that, when $\xi_4 = 0$ and $\xi_6 = 0$, the fourth and sixth elements of $\bar{X}$ also vanishes.

If the second column of $U_{PMNS}$ is trimaximally mixed, then the VEV, $\langle \xi \rangle$, as well as the resulting $\langle \bar{X} \rangle$ have non-zero elements only in the first, second, third and fifth positions. As shown in Eqs. (138), (139), (140), $\langle S_a \rangle$, $\langle S_b \rangle$ and $\langle S_{ab} \rangle$ have non-zero elements only in the first, third and fifth position, respectively. Therefore, a linear combination of $\langle \xi \rangle$, $\langle X_\chi \rangle$, $\langle S_a \rangle$, $\langle S_b \rangle$ and $\langle S_{ab} \rangle$ can be constructed which fully vanishes. With this information in hand, we construct the potential term,
\begin{align}
T_\xi &= (m \xi + \tilde{c}_1 \bar{X} + \tilde{c}_2 \tilde{S}_a + \tilde{c}_3 \tilde{S}_b + \tilde{c}_4 \tilde{S}_{ab})^T \\
&\times (m \xi + c_1 X + c_2 S_a + c_3 S_b + c_4 S_{ab}),
\end{align}
(142)

where $c_1$, $c_2$, $c_3$ and $c_4$ are constants. $T_\xi$ couples the sextet flavon, $\xi$ with the triplet flavons, $\phi_a$ and $\phi_b$. Any neutrino mass matrix which leads to a trimaximally mixed column can be obtained using a potential of the form, Eq. (142). The values of the constants resulting in the four VEVs, Eqs. (31), (36), (40), (44), are given in Table 8.

Using an appropriate choice of the constants, $c_1$, $c_2$, $c_3$, and $c_4$, we may obtain any mixing scheme within the constraint of a trimaximal column. It can be shown that, having the symmetry of $c_2$ and $c_3$ being real (invariant under complex conjugation) leads to $T_\chi M$. In the case of $T_\phi M$, the symmetry is the simultaneous conjugation and interchange of $c_2$ and $c_3$. Additionally, the fact that $c_1$, $c_2$, $c_3$ and $c_4$ are related by simple ratios points to the presence of more symmetries, the study of which is beyond the scope of this paper.

With the help of first- and second-order partial derivatives of a given potential, its minima can be calculated, as was done in previous work, e.g. in Ref. [93]. Using such a procedure,
along with numerical analysis, we have verified that every potential discussed here has a discrete set of minima and that the quoted VEVs are included among those minima in each case.

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