Interplay of classical and “quantum” capacitance in a one dimensional array of Josephson junctions

Pedro Ribeiro

CFIF, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais, 1049-001 Lisboa, Portugal
Max Planck Institute for the Physics of Complex Systems - Nöthnitzer Str. 38, 01187 Dresden, Germany

Antonio M. García-García

University of Cambridge, Cavendish Laboratory, JJ Thomson Avenue, Cambridge, CB3 0HE, UK

and

CFIF, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais, 1049-001 Lisboa, Portugal

Even in the absence of Coulomb interactions phase fluctuations induced by quantum size effects become increasingly important in superconducting nano-structures as the mean level spacing becomes comparable with the bulk superconducting gap. Here we study the role of these fluctuations, termed “quantum capacitance”; in the phase diagram of a one-dimensional (1D) ring of ultrasmall Josephson junctions (JJ) at zero temperature by using path integral techniques. Our analysis also includes dissipation due to quasiparticle tunneling and Coulomb interactions through a finite mutual and self capacitance. The resulting phase diagram has several interesting features: A finite quantum capacitance can stabilize superconductivity even in the limit of only a finite mutual-capacitance energy which classically leads to breaking of phase coherence. In the case of vanishing charging effects, relevant in cold atom settings where Coulomb interactions are absent, we show analytically that superfluidity is robust to small quantum finite-size fluctuations and identify the minimum grain size for phase coherence to exist in the array. We have also found that the renormalization group results are in some cases very sensitive to relatively small changes of the instanton fugacity. For instance, a certain combination of capacitances could lead to a non-monotonic dependence of the superconductor-insulator transition on the Josephson coupling.

PACS numbers: 74.20.Fg, 75.10.Jm, 71.10.Li, 73.21.La

The Josephson’s effect reveals the central role played by the phase of the order parameter in superconductivity. It has been exploited in a broad spectrum of research problems and applications: from the study of the pseudogap phase in high Tc materials, fluctuations above Tc and cold atom physics and quantum computing. Of special interest is the study of an array of superconducting grains separated by thin tunnel junctions, usually referred to as Josephson junctions (JJ). The physical properties of JJ arrays are very sensitive to the grain dimensionality, the presence of Coulomb interactions and dissipation (see also the review). Usually it is assumed that each single grain is sufficiently large so that the amplitude of the order parameter, the superconducting gap, is well described by the bulk Bardeen-Cooper-Schiffer (BCS) theory. Moreover it is also commonly assumed that a simple capacitance model is sufficient to account for Coulomb interactions. The phase of each grain is therefore the only effective degree of freedom of the JJ array.

Within this general theoretical framework a broad consensus has emerged on the main features of JJ arrays: for long 1d arrays at zero temperature with negligible dissipation, the existence of long range order depends on the nature of the capacitance interactions. For situations in which only self-capacitance is important superconductivity persists for sufficiently small charging effects provided that the Josephson coupling is strong enough. Despite spatial global long-range order a state of zero resistance will strictly occur only in the case in which the super current is induced by threading a flux in a ring-shaped JJ array. A current in a long but finite linear JJ array will eventually induce a resistance through for sufficiently strong Josephson coupling it is hard to measure it as its typical time scale can be much longer that the experimental observation time. At any finite temperature the resistivity is always finite as a consequence of the unbinding of phase anti-phase slips.

In the opposite limit in which only mutual-capacitance is considered, even small charging effects induce a superconductor-insulator transition. The combined effect of the two types of charging effects, considered in [19,20], can also lead to global long-range order. On a single junction, dissipation by quasiparticle tunneling only renormalizes the value of the capacitance. However dissipation caused by a ohmic resistance induces long range correlations between phase slips and anti phase slips that restore superconductivity provided that the normal resistance is smaller than the quantum one. In order to illustrate the profound impact of dissipation it is worth noting that a state of zero resistance in a 1D JJ array can in some cases coexist with an order parameter whose spatial correlation functions are short-ranged.

The closely related problem of a quantum nanowire was addressed by employing instanton techniques to model phase tunneling and then mapping the resulting effective model onto a 1+1d Coulomb gas where one of the dimensions is imaginary time. For an infinite wire

The Josephson effect reveals the central role played by the phase of the order parameter in superconductivity. It has been exploited in a broad spectrum of research problems and applications: from the study of the pseudogap phase in high Tc materials, fluctuations above Tc and cold atom physics and quantum computing. Of special interest is the study of an array of superconducting grains separated by thin tunnel junctions, usually referred to as Josephson junctions (JJ). The physical properties of JJ arrays are very sensitive to the grain dimensionality, the presence of Coulomb interactions and dissipation (see also the review). Usually it is assumed that each single grain is sufficiently large so that the amplitude of the order parameter, the superconducting gap, is well described by the bulk Bardeen-Cooper-Schiffer (BCS) theory. Moreover it is also commonly assumed that a simple capacitance model is sufficient to account for Coulomb interactions. The phase of each grain is therefore the only effective degree of freedom of the JJ array.

Within this general theoretical framework a broad consensus has emerged on the main features of JJ arrays: for long 1d arrays at zero temperature with negligible dissipation, the existence of long range order depends on the nature of the capacitance interactions. For situations in which only self-capacitance is important superconductivity persists for sufficiently small charging effects provided that the Josephson coupling is strong enough. Despite spatial global long-range order a state of zero resistance will strictly occur only in the case in which the super current is induced by threading a flux in a ring-shaped JJ array. A current in a long but finite linear JJ array will eventually induce a resistance through for sufficiently strong Josephson coupling it is hard to measure it as its typical time scale can be much longer that the experimental observation time. At any finite temperature the resistivity is always finite as a consequence of the unbinding of phase anti-phase slips.

In the opposite limit in which only mutual-capacitance is considered, even small charging effects induce a superconductor-insulator transition. The combined effect of the two types of charging effects, considered in [19,20], can also lead to global long-range order. On a single junction, dissipation by quasiparticle tunneling only renormalizes the value of the capacitance. However dissipation caused by a ohmic resistance induces long range correlations between phase slips and anti phase slips that restore superconductivity provided that the normal resistance is smaller than the quantum one. In order to illustrate the profound impact of dissipation it is worth noting that a state of zero resistance in a 1D JJ array can in some cases coexist with an order parameter whose spatial correlation functions are short-ranged.

The closely related problem of a quantum nanowire was addressed by employing instanton techniques to model phase tunneling and then mapping the resulting effective model onto a 1+1d Coulomb gas where one of the dimensions is imaginary time. For an infinite wire
in the zero temperature limit a superconductor-insulator Berezinsky-Kosterlitz-Thouless (BKT) transition occurs as a function of the system parameters. The role of vortices in 1+1d is played by phase slips which correspond to configurations for which the amplitude of the order parameter vanishes and the phase receives a $2\pi$ boost. By contrast at finite temperature – a similar argument holds for finite length – the time dimension is compactified so, in the absence of dissipation, the Coulomb gas analogy breaks down since for long separations phase and anti-phase slips become uncorrelated. As a consequence phase coherence is lost and the resistance is always finite\cite{10,11,19,20}.

As was mentioned previously all these results assume that the amplitude of the order parameter of each grain, which enters in the definition of the Josephson coupling energy, is not affected by any deviations from the bulk limit and that the phase dynamics is induced only by classical charging effects. Although these assumptions are in many cases sound there are situations in which corrections are expected.

In sufficiently small grains close to the critical temperature it is well documented that homogeneous path integral configurations different from the mean field prediction, the so called static paths, contribute significantly to the specific heat and other thermodynamical corrections, the so called static paths, contribute significantly to the specific heat and other thermodynamical contributions. For single nano-grains at intermediate temperatures it has been shown recently\cite{8} that, even in the limit of vanishing Coulomb interactions, deviations from mean-field predictions occur due to the non trivial interplay of thermal and quantum fluctuations induced by finite size effects. Experimentally it is also well established\cite{9} that substantial deviations from mean-field predictions occur in isolated nano-grains. Indeed it has recently been reported\cite{10,25} that quantum size effects enhance the superconducting gap of single isolated Sn nanograins with respect to the bulk limit. It is therefore of interest to understand in more detail the role of these finite size effects in arrays of ultrasmall JJ where the mean level energy spacing of single grains is smaller, but comparable, to the superconducting gap. This paper is a step in this direction. We study the stability of phase coherence in arrays of 1D JJ at zero temperature. Our formalism includes the above quantum fluctuations induced by size effects, charging effects and dissipation by quasiparticle tunneling. Starting from a microscopic Hamiltonian for a 1D JJ ring-shaped array of nanograins at zero temperature, we map the problem onto a Sine Gordon Hamiltonian where we identify the region of parameters in which long-range order persists in the presence of phase fluctuations. In the limit of vanishing charging energy, relevant for cold atom experiments, we find the minimum size for which the JJ array can be superfluid as a function of the wire resistance in the normal state. We also show that quantum fluctuations induced by finite size effects can in principle stabilize superconductivity in the limit of a negligible self-capacitance energy but a finite mutual capacitance energy. We have also identified a region parameters in which it is observed a non-monotonic dependence of the superconductor-insulator transition on the Josephson coupling.\cite{8}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1}
\caption{Sketch a closed ring of Josephson junctions pierced by a total flux $\Phi$.}
\end{figure}

\section{I. THE MODEL}

We consider the system sketched in Fig.1, consisting of an array of $L$ superconducting grains with periodic boundary conditions and a total magnetic flux $\Phi$ passing through it, that can be modeled by the Hamiltonian:

$$H = \sum_{r=1}^{L} H_r^{\text{BCS}} + H_r^{\text{SC}} + H_r^{\text{MC}} + H_{r,r+1}^{T}.$$ \hfill (1)

Each isolated superconducting grain is described by the BCS term,\cite{20,29}

$$H_r^{\text{BCS}} = \sum_{\alpha, \sigma} \epsilon_{\alpha, r} c_{\alpha, \sigma, r}^{\dagger} c_{\alpha, \sigma, r}$$ \hfill (2)

$$- g_r \delta_r \left( \sum_{\alpha} c_{\alpha, 1, r}^{\dagger} c_{\alpha, -1, r}^{\dagger} \right) \left( \sum_{\alpha'} c_{\alpha', -1, r} c_{\alpha', 1, r} \right),$$

accounting for the effective attractive electron-electron interactions in the region where the grain size is much smaller then the bulk superconducting coherence length. $\alpha, -\alpha$ label single-particle states related by time reversal symmetry with energies $\epsilon_{\alpha} = \epsilon_{-\alpha}, \sigma = \pm 1$ is the spin label and $\delta_r$ and $g_r$ are, respectively, the mean level spacing (inversely proportional to the grain volume) and the dimensionless coupling constant of grain $r$. We further assume the presence of self and mutual capacitive terms of the form

$$H_r^{S} = \frac{1}{2C_r^{S}} \left( \hat{N}_r - N_r^{S} \right)^2,$$ \hfill (3)

$$H_{r,r+1}^{M} = \frac{1}{2C_r^{M}} \left( \hat{N}_r - \hat{N}_{r+1} - N_r^{M} \right)^2,$$ \hfill (4)

accounting for the repulsive Coulomb interaction within each grain and between electrons in neighboring grains. $\hat{N}_r = \sum_{\alpha, \sigma} c_{\alpha, \sigma, r}^{\dagger} c_{\alpha, \sigma, r}$ is the total number of electrons, $C_r^{S}$ is the self-capacitance the of grain $r$ and $C_r^{M}$ the
mutual capacitance between nearest neighbor grains $r$ and $r+1$. The constants $N^{S}_r$ and $N^{M}_r$ can be adjusted by applying suitable gate voltages. Finally, the hopping of electrons between grains is captured by the term

$$H^{T}_{r,r+1} = \sum_{\alpha \sigma \alpha'} T^{\alpha \alpha'}_{r,r+1} c_{\alpha \sigma r}^\dagger c_{\alpha' \sigma r+1} + \text{h.c.}, \quad (5)$$

where the hybridization matrix $T^{\alpha \alpha'}_{r,r+1} \propto \int \psi_{\alpha \sigma r}(x) \psi_{\alpha' \sigma r+1}(x) \, dx$ is proportional to the overlap of the single-particle wave functions of two neighboring grains. In the regime of interest here - small grain sizes with respect to the bulk coherence length - the simplifying assumption that the hybridization is energy independent $T^{\alpha \alpha'}_{r,r+1} = t_{r,r+1}$ can safely be used and thus $H^{T}_{r,r+1}$ simplifies to

$$H^{T}_{r,r+1} = t_{r,r+1} \sum_{\sigma} \left( \sum_{\alpha} c_{\alpha \sigma r}^\dagger \right) \left( \sum_{\alpha} c_{\alpha \sigma r+1} \right) + \text{h.c.}, \quad (6)$$

with $\Phi = \sum_{r} \arg t_{r,r+1}$ the total flux passing through the ring.

II. FINITE SIZE CORRECTIONS TO THE ACTION OF A JOSEPHSON JUNCTION’S ARRAY

A. Partition function in the path integral formalism

In this section we write the partition function $Z = \text{Tr} \left[ e^{-\beta H} \right]$ in the path-integral form and identify the finite size corrections to the action. This is done by inserting $L$ complex-valued Hubbard-Stratonovich fields (HSF) $\Delta_r$ to decouple the BCS term in the superconducting channel, $L$ real valued HSF $V^{S}_r$, conjugate to the number of particles on each grain, and $L$ real valued HSF $V^{M}_r$, conjugate to the difference of the number of particles in neighboring grains. Using the notation $\Psi = (c_{a,1,1}, c_{a,-1,1}, c_{a,1,2}, c_{a,-1,2}, \ldots)^T$, the partition function reads $Z = \int Dc \, D\Delta \, DV \, e^{-S}$, with the action

$$S = -\Psi^\dagger G^{-1} \Psi + \int_0^\beta d\tau \sum_r \left[ \frac{1}{g_r} \Delta_r^\dagger \Delta_r \right] - \frac{C^{S}_r}{2} \left(V^{S}_r\right)^2 - \frac{C^{M}_r}{2} \left(V^{M}_r\right)^2 + i N^{S}_r V^{S}_r + i N^{M}_r V^{M}_r \right], \quad (7)$$

where the full Green’s function is given by

$$G^{-1} = \begin{pmatrix} G^{-1}_{11} & T_{21}^\dagger \\ T_{21} & G^{-1}_{22} \end{pmatrix}, \quad (8)$$

and

$$G^{-1}_{r,r+1} = \begin{pmatrix} -\partial_\tau - \xi_{a,r}(\tau) & \Delta_r(\tau) \\ \Delta_r^\dagger(\tau) & -\partial_\tau + \xi_{a,r}(\tau) \end{pmatrix}, \quad (9)$$

is the inverse of the electronic propagators restricted to grain $r$. Here we defined $\xi_{a,r}(\tau) = \xi_{a,r} - i V^{S}_r(\tau) - i V^{M}_r(\tau)$ and the hybridization matrix $T_{r,r+1} = \begin{pmatrix} t_{r+1,r} & 0 \\ 0 & -t_{r+1,r} \end{pmatrix}$.

Integrating out $\Psi$ yields the action

$$S = -\text{Tr} \ln [-G^{-1}] + \int_0^\beta d\tau \sum_r \left[ \frac{1}{g_r} \Delta_r^\dagger \Delta_r \\ \frac{C^{S}_r}{2} \left(V^{S}_r\right)^2 + i N^{S}_r V^{S}_r + \frac{C^{M}_r}{2} \left(V^{M}_r\right)^2 + i N^{M}_r V^{M}_r \right], \quad (10)$$

solely in terms of the HSF.

We apply the unitary transformation

$$U = \text{diag} \left\{ e^{i \frac{1}{2} \phi_1(\tau)}, e^{-i \frac{1}{2} \phi_1(\tau)}, e^{i \frac{1}{2} \phi_2(\tau)}, e^{-i \frac{1}{2} \phi_2(\tau)}, \ldots \right\}$$

with $\phi_r(\tau) = \phi_r(\tau + \beta) + 2 \pi n_{\phi_r}$ ($n_{\phi_r} \in \mathbb{Z}$) to the electronic propagator $G^{-1}$ in order to render its off-diagonal anomalous elements $\Delta_r(\tau) = s_r(\tau) e^{i \phi_r(\tau)}$, where $s_r(\tau), \phi_r(\tau) \in \mathbb{R}$. Note that for odd $n_{\phi}$, one has that $\text{Tr}_f \left[ G^{-1} \right] = \text{Tr}_b \left[ U^\dagger G^{-1} U \right]$, where $\text{Tr}_f$ denotes the trace over anti-periodic functions (fermionic) and $\text{Tr}_b$ the trace over periodic functions (bosonic). For a generic $n_{\phi}$ we will denote $\text{Tr}_{n_{\phi}} = \text{Tr}_f$ for $n_{\phi}$ even and $\text{Tr}_{n_{\phi}} = \text{Tr}_b$ for $n_{\phi}$ odd. Whenever we have two such indices we will use $\text{Tr}_{n_{\phi_1} n_{\phi_2}}$ for the time periodicity in indices 1 and 2. Note however that this complication is only formal as we will be interested in the low temperature properties of this action where the distinction between even and odd $n_{\phi}$’s can be safely ignored. After this transformation we get

$$\hat{G}^{-1} = U^\dagger G^{-1} U = \begin{pmatrix} \hat{G}^{-1}_{11} & \hat{T}_{21}^\dagger \\ \hat{T}_{21} & \hat{G}^{-1}_{22} \end{pmatrix}, \quad (11)$$

with

$$\hat{G}^{-1} = -1 \times \begin{pmatrix} \partial_\tau + \xi_{a,r}(\tau) + i \frac{1}{2} \partial_\tau \phi_r(\tau) & -s_r(\tau) \\ -s_r(\tau) & \partial_\tau - \xi_{a,r}(\tau) - i \frac{1}{2} \partial_\tau \phi_r(\tau) \end{pmatrix}, \quad (12)$$

and

$$\hat{T}_{r+1,r} = \begin{pmatrix} t_{r+1,r} e^{i \frac{1}{2} [\phi_{r+1}(\tau)-\phi_r(\tau)]} & 0 \\ 0 & -\tilde{t}_{r+1,r} e^{-i \frac{1}{2} [\phi_{r+1}(\tau)-\phi_r(\tau)]} \end{pmatrix}. \quad (13)$$
Moreover, assuming the hopping amplitude to be small, we may develop the $\text{Tr} \ln [-\bar{G}^{-1}]$ term to second order in $|t_{r+1,r}|$ and obtain the action

\[
S[s,\phi,V] = \sum_{r} \left\{ \int_{0}^{\beta} d\tau \left[ \frac{1}{g_{r}\delta_{r}} \dot{s}_{r}^{\dagger} s_{r} \right. \right. \\
+ \left. \left. \frac{C_{r}^{S}}{2} (V_{r}^{S})^{2} + iN_{r}^{S} V_{r}^{S} + \frac{C_{r}^{M}}{2} (V_{r}^{M})^{2} + iN_{r}^{M} V_{r}^{M} \right] \\
- \text{Tr}_{n_{s}} \ln [-\bar{G}_{r}] \\
+ \text{Tr}_{n_{s},n_{s+1}} \left[ \bar{G}_{r+1} \bar{T}_{r+1,i} \bar{G}_{r} \bar{T}_{r,r+1} \right] \right\}. \tag{14}
\]

B. Leading behavior in $\delta$

The action above Eq. (14) is suitable for a saddle-point expansion in both $s$ and $V$ fields since the action for each grain is an extensive quantity in the number of electrons within that grain ($N_{r} \approx E_{D}/\delta_{r}$). Notice however that the saddle-point equations cannot be explicitly evaluated as $\bar{G}^{-1}$ depends on $\phi_{r}(\tau)$. We proceed by noting that $\partial_{\tau} \phi_{r}(\tau)$ is small, as the phase varies smoothly as a function of $\tau$ for sufficiently low temperatures. Formally we set $V_{r}^{S}(\tau) = V_{r,0}^{S} + \delta V_{r}^{S}(\tau)$, $V_{r}^{M}(\tau) = V_{r,0}^{M} + \delta V_{r}^{M}(\tau)$ and $s_{r}(\tau) = s_{r,0} + \delta s_{r}(\tau)$ where the subscript 0 denotes the static component (constant in $\tau$) of the different quantities and the fluctuation around the static value, to be considered at quadratic order, are denoted by $\delta V_{r}^{S}$, $\delta V_{r}^{M}$ and $\delta s_{r}$. Physically, $s_{r,0}$ is the amplitude of the condensate on grain $i$ and the terms $iV_{r,0}^{S}$, $iV_{r,0}^{M}$ in $\mathbb{R}$ leads to a renormalization of the chemical potential: $\tilde{\varepsilon}_{\alpha,r} = \varepsilon_{\alpha,r} - iV_{r,0}^{S} - iV_{r,0}^{M} + iV_{r,1,0}^{M}$.

For equally spaced levels and a particle-hole-symmetric single-particle density of states the tunneling term can be simplified at low temperatures\cite{33}

\[
\text{Tr} \left[ \bar{G}_{r+1} \bar{T}_{r+1,i} \bar{G}_{r} \bar{T}_{r,r+1} \right] \approx \frac{C_{r}^{J}}{8} \int d\tau \left\{ \partial_{\tau} \left[ \phi_{r+1}(\tau) - \phi_{r}(\tau) \right] \right\}^{2} \\
- I_{r}^{F} \int d\tau \cos \left[ \phi_{r+1}(\tau) - \phi_{r}(\tau) + \phi_{r}^{c} \right] \tag{15}
\]

where $\phi_{r}^{c}$ is the phase of the hopping term $t_{r+1,r} = |t_{r+1,r}| e^{i\phi_{r}^{c}}$, $C_{r}^{J}$ is the quasi-particle induced capacitance and $I_{r}^{F}$ is the junction’s critical current between grains $r$ and $r + 1$, given respectively by\cite{33}

\[
C_{r}^{J} = 2 \frac{4 |t_{r+1,r}|^{2}}{\delta_{r}\delta_{r+1}} \times \int_{\varepsilon_{r,0}}^{\infty} d\nu_{1} \int_{\varepsilon_{r+1,0}}^{\infty} d\nu_{2} \frac{\nu_{1}\nu_{2}}{\left( \nu_{2}^{2} - 2 \nu_{1} \nu_{2} \right) \left( \nu_{2}^{2} - s_{r+1,0}^{2} \right)} \tag{15}
\]

and

\[
I_{r}^{F} = \frac{8 |t_{r+1,r}|^{2}}{\delta_{r}\delta_{r+1}} \times \int_{s_{r,0}}^{\infty} d\nu_{1} \int_{s_{r+1,0}}^{\infty} d\nu_{2} \frac{s_{r,0}s_{r+1,0}}{\left( \nu_{2}^{2} - 2 \nu_{1} \nu_{2} \right) \left( \nu_{2}^{2} - s_{r+1,0}^{2} \right)} \tag{16}
\]

Note that for $s_{r,0} = s_{r+1,0} = s_{0}$ these expressions simplify to $C_{r}^{J} = C_{J} = \frac{2}{N} \sum_{s_{r}} R_{N}$ and $I_{r}^{F} = I_{c} = \frac{\pi}{2} \frac{\delta s_{r}}{2}$. with $R_{N} = \left( \frac{4\tilde{\varepsilon}_{\alpha,r}^{2}}{\delta_{s}^{2}} \right)^{-1}$ the normal state resistance of the junction.

With these approximations the action reads

\[
S[s,\phi,V] = S_{0} + \int d\tau \sum_{r} \left\{ \Omega_{r} \delta s_{r}^{2}(\tau) \right. \\
+ \left. \frac{C_{r}^{S}}{2} \delta V_{r}^{S}(\tau) \right\}^{2} + \frac{C_{r}^{M}}{2} \delta V_{r}^{M}(\tau) \right\}^{2} + \frac{1}{2} C_{r}^{\delta} \delta s_{r}^{2}(\tau) \\
- i \left( \langle N_{0,r} \rangle \right) \partial_{\tau} \phi_{r}(\tau) + \frac{C_{J}^{I}}{8} \left( \partial_{\tau} \left[ \phi_{r+1}(\tau) - \phi_{r}(\tau) \right] \right)^{2} \\
- \frac{I_{r}^{F}}{2} \cos \left[ \phi_{r+1}(\tau) - \phi_{r}(\tau) + \phi_{r}^{c} \right] \tag{17}
\]

where

\[
S_{0} = \sum_{i} \left\{ \text{Tr} \ln [-\bar{G}^{-1}_{r}i] \\
+ \beta \left[ \frac{1}{g_{r}\delta_{r}} s_{r,0}^{2} + \frac{C_{r}^{S}}{2} (V_{r,0}^{S})^{2} + iN_{r}^{S} V_{r,0}^{S} \right. \left. + \frac{C_{r}^{M}}{2} (V_{r,0}^{M})^{2} + iN_{r}^{M} V_{r,0}^{M} \right] \right. \tag{18}
\]

only depend on the static saddle-point values, $\Omega_{r} = \frac{1}{g_{r}\delta_{r}}$, $\xi_{r} = \sqrt{\varepsilon_{r}^{2} + \varepsilon_{0}^{2}}$, $\langle N_{0,r} \rangle = \frac{1}{2} \sum_{\alpha} \left( 1 - \frac{\varepsilon_{\alpha,r}}{\xi_{r}} \right)$.

\[
\bar{G}_{r,0}^{-1}(i\omega_{n}) = \left( \begin{array}{cc}
\varepsilon_{n} - \varepsilon_{\alpha,r} & s_{r,0} \\
\varepsilon_{n} + \varepsilon_{\alpha,r} & i\omega_{n} \end{array} \right) \tag{19}
\]

and where we also define

\[
\varphi_{r}(\tau) = \delta V_{r}^{S}(\tau) + \delta V_{r}^{M}(\tau) - \delta V_{r-1}^{M}(\tau) - \frac{1}{2} \partial_{\tau} \phi_{r}(\tau) \tag{20}
\]

and the finite size induced self-capacitance $C_{\delta,r} = \frac{\frac{4}{N}}{\varepsilon_{r}}$.

Eq. (17) is now suitable to a static-path treatment once the fluctuations are integrated out. Here, as we are only interested in the phase dynamics at low temperatures we set the static components to their mean-field values and integrate out the gapped fluctuations both in the $s$ and $V$ fields. In the limit $\beta_{0} \gg 1$ the final action in terms of the phase degrees of freedom and assuming translational invariance in the couplings $C_{r}^{J} = C_{S}$,
\[ C^M_r = C_M, \quad C_{\delta,r} = C_\delta, \] is given by,

\[ S = \frac{1}{8} \int d\tau \sum_{r,r'} \partial_r \phi_r (\tau) [C_R]_{r,r'} \partial_r \phi_{r'} (\tau) + i \frac{1}{2} \sum_r \langle N_{0,r} \rangle \int d\tau \partial_r \phi_r (\tau) - \frac{I_c}{2} \sum_r \int d\tau \cos \left( \phi_{r+1} (\tau) - \phi_r (\tau) + \phi_r^t \right) \tag{21} \]

where \( C_R = \frac{1}{C_S - C_M^2 \Delta_1^2} - C_J \Delta_1^2 \) is the capacitance matrix, with \( \Delta_1 \) the discrete derivative: \( (\Delta_1 \phi)_r = \phi_r - \phi_{r-1} \), \( \phi_r^t = \arg t_{r,r+1} \) is the phase of the hopping term, \( \langle N_{0,r} \rangle = \frac{1}{2} \sum_{\alpha} \left( 1 - \frac{\xi_{\alpha,r}}{\xi_{\alpha,r}} \right) \) is the average number of electrons in grain \( r \) and

\[ \hat{C}_S = \left( \frac{1}{C_S} + \frac{\delta}{2} \right)^{-1} \tag{22} \]

is the grain self-capacitance renormalized by quantum finite size effects. Note that on the lattice \( \sum_j (\Delta_1 \phi)_j (\Delta_1 \phi')_j = - \sum j \phi_j (\Delta_1 \Delta_1 \phi')_j \), with \( (\Delta_1 \phi)_r = \phi_{r+1} - \phi_r \), for sake of simplicity we use the notation \( \Delta_1^2 \) to denote the lattice Laplacian \( \Delta_1 \Delta_1 \).

**Equation 21** is the central result of this section, it contains the effective low energy theory for a junction at \( t \), including charging effects, quasiparticle dissipation and for the first time quantum fluctuations induced by finite size effects \( C_\delta \). The Berry phase term - second term of Eq. \( 21 \) - ensures that, in the ground-state (i.e. for \( T = 0 \)), the average number of electrons on each grain is even. In the following we assume that this condition is fulfilled and drop this term.

Note that for a set of isolated finite-size grains with \( I_c = C_J = C_M^{-1} = 0 \) no superconducting phase ensues as the action in Eq. \( 21 \) reduces to \( \frac{1}{8} \int d\tau \partial_r \phi_r (\tau) \) with the phase stiffness \( \varphi = \frac{\Delta_1}{4} \) controlling the exponential time decay of the order parameter correlation function \( \bar{\Psi} (\tau) = \sum_\alpha c_{r,\alpha} (\tau) c_{r,\alpha} (\tau) \): \( \langle \bar{\Psi} (\tau) \bar{\Psi} (\tau') \rangle \propto \delta_{\tau,\tau'} e^{-}\frac{|r_{\tau-1}|}{2\varphi} \).

**III. SUPERCONDUCTING TRANSITION**

**A. Hamiltonian Formulation**

In this section we analyze the action given by Eq. \( 21 \), without the Berry phase term \( \int d\tau \partial_r \phi_r (\tau) \) as we assume an even number of particles in each grain. The calculation is carried out by first mapping this equation onto an equivalent Coulomb gas model. The Coulomb gas is subsequently transformed into a Sine-Gordon action for which a perturbative RG treatment can be effectively performed.

First we provide a description of the model in terms of the effective low energy Hamiltonian for the phase degrees of freedom in order to make contact with previous works where this effective description is taken as the starting point of the calculation. The initial step is the discretization of the imaginary time in Eq. \( 21 \): \( \tau = \Delta \tau \bar{\tau} \) (with \( \bar{\tau} = 1, \ldots, N \) and \( N \Delta \tau = \beta \) ). Using the identity

\[ \lim_{\Delta \tau \rightarrow 0} \sum_{n=n_1, \ldots, n_N} e^{-\Delta \tau \frac{n}{A^{-1}} + \Delta \tau \beta n} = \left( \frac{2\pi}{\det (A) \Delta \tau} \right)^N e^{-\frac{\Delta \tau \beta}{2} A b b} \tag{23} \]

the partition function can be rewritten as \( Z = \int D\phi \sum_n e^{-i S[\phi, n]}, \) with

\[ S[\phi, n] = \sum_{\bar{\tau}, r, r'} 2 \Delta \tau n (\bar{\tau}, r) [C_R^{-1}]_{r,r'} n (\bar{\tau}, r') - \sum_{\bar{\tau}, r} i n (\bar{\tau}, r) [\phi (\bar{\tau} + 1, r) - \phi (\bar{\tau}, r)] - \frac{I_c}{2} \sum_{\bar{\tau}, r} \Delta \tau \cos \left( \phi (\bar{\tau}, r + 1) - \phi (\bar{\tau}, r) + \phi_r^t \right) \tag{24} \]

In this form, Eq. \( 24 \) can readily be interpreted as the Trotter-sliced action coming from the Hamiltonian

\[ H = \sum_{r,r'} 2 \bar{\eta}_r [C_R^{-1}]_{r,r'} \bar{\eta}_{r'} - \frac{I_c}{2} \sum_{r,r'} \left( \bar{\phi}_{r+1} - \bar{\phi}_r + \phi_r^t \right) \tag{25} \]

where \( \bar{\eta}_r = (-i \partial_{\phi_r}) \), the variable conjugated to \( \bar{\phi}_r \), is the number of Cooper-pairs in grain \( r \).

**B. Partition function of the Coulomb gas**

We follow the procedure of [30] to re-write the action of a Josephson junction array in terms of the partition function of a classical Coulomb gas. Using the Villain decomposition of the cosine term

\[ e^{z \cos (\theta)} \simeq I_0 (z) \sum_{m=-\infty}^{\infty} e^{-\frac{1}{2} \mu (z) m^2} e^{i m \theta} \tag{26} \]

with \( I_0 (z) \) a modified Bessel function of the first kind, valid for both, large and small \( z \) respectively with

\[ \mu (z) = \begin{cases} \frac{-2}{z} \ln (z/2) & \text{for } z \ll 1 \\ z^{-1} & \text{for } z \gg 1 \end{cases} \tag{27} \]
Eq. (24) can be written as

\[ S[\phi,n] = \sum_{\tilde{\tau},r,r'} 2\Delta \tau n_0(\tilde{\tau},r) \left[ C^{-1}_R \right]_{rr'} n_0(\tilde{\tau},r') + \sum_{\tilde{\tau},r} \left\{ \frac{1}{2}\mu \left( \frac{I_0\Delta \tau}{2} \right) n_1^2(\tilde{\tau},r) - i\phi^\dagger_1 n_1(\tilde{\tau},r) - i\phi(\tilde{\tau},r) \times \times [n_1(\tilde{\tau}, r - 1) - n_1(\tilde{\tau},r) + n_0(\tilde{\tau} - 1, r) - n_0(\tilde{\tau},r)] \right\} \]

(28)

where we relabel \( n \rightarrow n_0 \) in Eq. (24) and \( m \rightarrow n_1 \) in Eq. (26) in order to interpret \( n_\mu(\tilde{\tau},r) \) as an integer field living on links of a square lattice - an integer-valued one-form on the square lattice - with \( n_0 \) corresponding to time-like and \( n_1 \) to space-like links.

Integrating out the \( \phi \) field yields the divergence-free constraint

\[ \partial a \equiv \Delta a n_1 + \Delta_0 n_0 = 0, \quad (29) \]

where \( \Delta_0 f(\tilde{\tau},r) = f(\tilde{\tau},r) - f(\tilde{\tau} - 1, r) \) is the discrete derivative along the time direction. Locally such constraint can be satisfied by writing \( n \) as the rotational of an integer valued field living on the centers of plaquettes - an integer-valued lattice 2-form - \( n = \partial a \) or in components: \( n_0 = -\Delta_1 a_{01}, n_1 = \Delta_0 a_{01}, \) where the subscript of \( a \) denotes that this field lives on spacial-temporal plaquettes. The operator \( \partial \) can be seen as the lattice exterior coderivative. Globally, the most generic solution of the constraint in Eq. (29) includes a non-trivial divergence-free field that cannot be written as a rotational. On a torus, such general solution can be decomposed as \( n = \partial a + \sum_\alpha c_\alpha b_\alpha^\alpha \). More explicitly,

\[ n_0(\tilde{\tau},r) = -\Delta_1 a_{01}(\tilde{\tau},r) + \sum_{\alpha=0,1} c_\alpha b_\alpha^0(\tilde{\tau},r) \quad (30) \]

\[ n_1(\tilde{\tau},r) = \Delta_0 a_{01}(\tilde{\tau},r) + \sum_{\alpha=0,1} c_\alpha b_\alpha^1(\tilde{\tau},r) \quad (31) \]

where \( b^0 \) and \( b^1 \) (with \( \partial b^\alpha = 0 \)) are integer-valued 1-forms on the lattice that cannot be written as a rotational. They are chosen, see Fig. 2, to have a minimum flux along time and space directions respectively: \( \sum_{\tilde{\tau},r} b^0_\mu(\tilde{\tau},r) = N \delta_{0\mu}, \sum_{\tilde{\tau},r} b^1_\mu(\tilde{\tau},r) = L \delta_{0\mu} \). \( c_{\alpha=0,1} \) are integer-valued coefficients labeling different topological sectors. Note that in the infinite volume limit, i.e. zero temperature and \( L \rightarrow \infty \), the \( b^0 \) terms can be dropped in the solution as the space becomes topologically trivial. Later on we will drop the \( b^0 \) contribution as we are interested in the zero temperature limit.

In terms of the \( a \) field and the integers \( c_0 \) and \( c_1 \), the partition function is given by the unconstrained sum \( Z = \sum_{a,c} e^{-S[a,c]} \) with

\[ S[a,c] = \sum_{\tilde{\tau},r,r'} 2\Delta \tau \left[ \Delta_1 a(\tilde{\tau},r) - \sum_\alpha c_\alpha b_\alpha^0(\tilde{\tau},r) \right] \times \left[ C^{-1}_R \right]_{rr'} \left[ \Delta_1 a(\tilde{\tau},r') - \sum_\alpha c_\alpha b_\alpha^0(\tilde{\tau},r') \right] - ic_1 \Phi + \frac{1}{2} \sum_{\tilde{\tau},r} \mu \left( \frac{I_0\Delta \tau}{2} \right) \left| \Delta_0 a(\tilde{\tau},r) + c_1 b_1^0(\tilde{\tau},r) \right|^2 \quad (32) \]

where the total flux \( \Phi = \sum_\tau \phi^\dagger_\tau \).

Using the Poisson summation formula \( \sum_m f(a) = \sum_m \int d\psi f(\psi) e^{2\pi im\psi} \) to improve the convergence of the sum over Eq. (33) and integrating over \( \psi \) yields

\[ Z = \sum_{m,c} \delta \sum_{m=0} e^{-\sigma[c,m]} e^{ic_1 \Phi} \quad (33) \]

where the sum over \( m \) is restricted such that the so-called neutrality condition \( \sum_{r \tilde{r}} m(\tilde{r},r) = 0 \) is fulfilled and

\[ S[c,m] = \left( \frac{2\pi i}{2} \right)^2 \sum_{\tilde{\tau},r,r'} m(\tilde{\tau},r) G(\tilde{\tau} - \tilde{\tau}', r - r') m(\tilde{\tau}', r') - 2\pi i \sum_\alpha c_\alpha \sum_{\tilde{\tau},r} m(\tilde{\tau},r) \left( \partial^{-1} b_\alpha^0 \right)(\tilde{\tau},r). \quad (34) \]

with \( (\partial^{-1} b_\alpha^0)_{01} = (\Delta_0^2 + \Delta_1^2)^{-1} (\Delta_1 b_\alpha^0 - \Delta_0 b_\alpha^1) \) the inverse of the \( \partial \) operator defined in Eq. (29). The last term in Eq. (34) for \( b^1 \) can be simplified to

\[ \sum_{\tilde{\tau},r} m(\tilde{\tau},r) \left( \partial^{-1} b^1 \right)(\tilde{\tau},r) = \sum_j \left[ (\Delta_0^2 + \Delta_1^2)^{-1} \Delta_0 m \right](0,r) \cdot \]

The Green’s function is given by

\[ \left( \begin{array}{c} 1 \\ 1 - C_M^{-1} \tilde{C}_S \Delta_1^2 \end{array} \right)^{-1} \left\{ \begin{array}{c} 1 - \tilde{C}_S^{-1} C_J \Delta_1^2 \\ \mu \left( \frac{I_0\Delta \tau}{2} \right) \Delta_0^2 \end{array} \right\} \quad (35) \]
In summary, after integrating over the $\psi$ field that represents small phase fluctuations, the action in Eq. (44) is given solely in terms of topological excitations, $m$, that can be interpreted as an instanton field representing a phase slip. The corrections due to non-vanishing values of $C_{S}^{-1}C_{I}$ and $C_{M}^{-1}C_{S}$ do not change the nature of the long-range interaction between the phase slips, as they multiply higher powers of the discrete Laplacian. Nonetheless they appear in Eq. (34) in inequivalent ways, further we will see this translates to different contributions to the monopole energy to create monopole pairs.

C. Flux quantization

To understand how the flux piercing the ring gets quantized in the superconducting phase, where the density of instantons (phase slips) vanishes, let us examine the partition function given in Eq. (33). For simplicity let us first take the zero temperature limit in order to ignore the $b^{0}$ field. The flux $\Phi$ is imposed to the system assuming that the magnetic field far from the ring is constant and perpendicular to the $z$ axes in Fig. (1). A complete description of the system array+field should include the dynamics of the electromagnetic field as well. However this is too involved and not really needed here, the only thing that is required is to remember that the spacial distribution of the electromagnetic field (and thus the flux piercing the ring) is itself determined by an action containing the electromagnetic contribution plus the coupling of the electromagnetic field to the instanton configurations given by the last term of Eq. (34).

Performing the summation over $c_{1}$ in Eq. (33) one observes that the partition function of a system with flux $\Phi$ can be written as

$$Z = \sum_{m,c_{1}} \delta_{\sum_{m=0}} \delta_{2\pi} (\Phi - \Phi_{m}^{1})$$

$$\times e^{- \frac{i2\pi m}{\sum_{r}(\frac{\Delta}{\Delta_{0}+\Delta_{1}})_{m}(0,r)}}$$

$$\times m(\bar{\tau},r) \cdot m(\bar{\tau},r') \cdot m(\bar{\tau},r') (36)$$

where $\delta_{2\pi}$ is the $2\pi$-periodic delta function and $\Phi_{m}^{1} = 2\pi \sum_{r} \left[ \frac{\Delta_{0}}{\Delta_{0}+\Delta_{1}} \right] (0,r) \in \mathbb{R}$. To the action of the free electromagnetic action one should thus add the monopole contribution $F[\Phi] = -\ln Z$. Directly from Eq. (36) one can observe that if the density of phase-slips vanishes (i.e. $\sum_{r} m^{2}(\bar{\tau},r) = 0$) then $\Phi_{m}^{1} = 0$ and thus $\Phi$ has to be quantized in multiples of $2\pi$. When phase-slips proliferate, $\Phi_{m}^{1}$ is a fraction of $2\pi$, for a generic configuration of instantons $m$, the summation over all $m$ configurations allows for a continuum value of $\Phi$.

D. Superconducting-Insulating Transition

Having understood how the flux gets quantized once instantons are suppressed, let us neglect the topological terms (i.e. set $c_{0,1} = 0$ in Eq. (33)), in order to study the superconducting-insulating transition. A simple way of addressing this question is to map the problem to the Sine-Gordon model. The main result we report in this section is that the superconducting insulating phase transition is Kosterlitz-Thouless like, even in the presence of a finite $C_{M}$ and $C_{I}$. This extends the results of Ref. [23], where the case $C_{M} \neq 0$, $C_{I} = 0$ is considered. Nonetheless $C_{M}$ and $C_{I}$ renormalize the instanton-core energy in rather different ways. By studying how this energy gets renormalized we obtain the behavior of the superconducting-insulating transition line as a function of $I_{c},C_{S},C_{M}$ and $C_{I}$. We note that $C_{S}$ also includes a term $\propto 1/\delta$ coming from quantum fluctuations induced by finite size effects that so far had not investigated in the literature.

The first step to get the Sine-Gordon action is to regularize the instanton interaction kernel at the origin $G(\bar{\tau},j) \rightarrow G(\bar{\tau},j) - G(0,0)$ in Eq. (34) by making use of the neutrality condition. After this procedure the asymptotic $\bar{\tau},j \rightarrow \infty$ form of the instanton (anti) instanton interaction is given by

$$(2\pi)^{2} [G(\bar{\tau},r) - G(0,0)] \simeq$$

$$\tilde{G}(\bar{\tau} - \bar{\tau}',r - r') - \nu,$$ (37)

where

$$\tilde{G}(\bar{\tau},r) = -2\pi \sqrt{I_{c}C_{S}/8 \ln \left( \sqrt{\tau^{2}/\lambda^{2} + \tau'^{2}} \right)}$$ (38)

is the long-range instanton interaction potential and

$$\nu = \sqrt{I_{c}C_{S}/8 \kappa} \left( \lambda, \tilde{C}_{S}/C_{M}, C_{I}/\tilde{C}_{S} \right)$$ (39)

is the instanton-core energy. Choosing the regulator $\Delta \tau \approx \sqrt{\tilde{C}_{S}/2I_{c}}$ such that $I_{c}\Delta \tau/2 \gg 1$ and $\Delta \tau \ll 2\tilde{C}_{S}$, we observe by Eq. (27) that $\mu \left( \frac{I_{c}\Delta \tau}{2} \right) \simeq \frac{2}{2\Delta \tau}$. The anisotropy between time and space directions $\lambda = \sqrt{\tilde{C}_{S}/2I_{c}}(\Delta \tau)^{2}$ is thus of order 1. $\kappa$ depends on all ratios $\lambda, \tilde{C}_{S}/C_{M}$ and $C_{I}/\tilde{C}_{S}$, however it is mildly varying as a function of $\lambda$ around $\lambda = 1$. In the following we take the $\lambda = 1$ prescription for our numerical analysis.

The function $\kappa$ can be computed numerically by subtracting the asymptotic behavior $\tilde{G}(\bar{\tau},r)$ to the right hand site of Eq. (37) and numerically integrating the resulting expression. After a careful analysis of the numerical data to ensure that the asymptotic values are well reproduced we obtained the results of Fig. (3)-(a).

Using the neutrality condition once more, the action acquires the Coulomb (lattice) gas form

$$S[m] \simeq \sum_{r} \sum_{\bar{\tau}} \sum_{\bar{\tau}' \neq \bar{\tau}} m(\bar{\tau},r) \tilde{G}(\bar{\tau} - \bar{\tau}',r - r') m(\bar{\tau}',r')$$

$$+ \nu \sum_{\bar{\tau}} \sum_{r} \left[ m(\bar{\tau},r) \right]^{2}$$ (40)
The (lattice) Sine-Gordon model can be obtained by inserting a Hubbard-Stratonovich field and using the identity given in Eq. (26):

\[ Z = \sum_{m} \delta \sum_{m=0} e^{-S[m]} \]

\[ \propto \int D\psi e^{-\frac{1}{2}G^{-1}\psi + \nu \sum_{x} \cos(\psi_{x})} \]  

(41)

with \( \mu(u) = \nu \) given by Eq. (27). Note that in this mapping the neutrality condition is assured by the fact that \( G^{-1}(\omega = 0, k = 0) = 0 \). The usual (continuum) Sine-Gordon action, that maintains the universal properties of the lattice model, is obtained taking the continuum limit by formally introducing a regularizing lattice constant \( a \) and taking the limit \( a \to 0 \). In the continuous form the inverse of the kernel \( G \) can be straightforwardly identified:

\[ \frac{1}{\pi}(1/\omega_{1}^{2} + \lambda\partial_{x}^{2}) \ln \left( \sqrt{\pi^{2} + x_{1}^{2}} \right) = \delta(x_{0}) \delta(x_{1}) \]

After a rescaling of the axes in the \( x_{0} \) direction, one obtains the continuum Sine-Gordon action

\[ S = -\frac{1}{2} \int d^{2}x \left[ g(\nabla\psi)^{2} - \lambda a^{2}u \cos(\psi) \right] \]  

(42)

with \( g = \frac{1}{(2\pi)^{2}} \sqrt{\frac{8}{I_{c}C_{S}}} \). This model has a phase transition for \( g = g_{c} \), that can be estimated by a perturbative renormalization group procedure to first order in \( \epsilon \) [38]

\[ g_{c} = \frac{1}{8\pi} - y_{1} u + O(u^{2}) \]  

(43)

where \( y_{1} \simeq 1/8 \) and \( u(0) \simeq -2 \ln(u/2) \).

Substituting this values in Eq. (43) one obtains the phase transition condition

\[ \sqrt{\frac{8}{I_{c}C_{S}}} = \frac{\pi}{2} \left[ 1 - Ae^{-\frac{1}{2}V_{0}^{\epsilon}/\epsilon^{2}} \right] \]  

(44)

where \( A = 16\pi y_{1}\lambda \).
Eq. (44) predicts the form of the Kosterlitz-Thouless transition line as a function of $\tilde{C}_S/C_M$, $C_1/\tilde{C}_S$ and the non-universal constant $A$. We have now all the ingredients to discuss the phase diagram of the 1D JJ array.

IV. DISCUSSION

The phase diagram as a function of $\tilde{C}_S/C_M$ for $C_1 = 0$ is depicted in Fig. (3)-(b). As was expected the stability of the superconducting phase is reduced upon increasing the ratio $\tilde{C}_S/C_M$, in agreement with Ref. [27] where a perturbative analysis around $\tilde{C}_S/C_M = 0$ was performed. For $\tilde{C}_S/C_M \to \infty$ it is well known that the system is always in the insulating phase independently of the value of $\sqrt{I_c\tilde{C}_S}/8$. The expression Eq. (44) interpolates between this two regimes. It predicts a critical value $\tilde{C}_S/C_M \approx 0.375$ above which the system is always in the insulating phase in the $\sqrt{I_c\tilde{C}_S}/8 \to \infty$ limit. For this critical ratio $\kappa$ vanishes and becomes negative ($\kappa < 0$) for larger values of $\tilde{C}_S/C_M$ which, for sufficiently large $\sqrt{I_c\tilde{C}_S}/8$, renders the system insulating due to the proliferation of phase slips. The non-linearity of the relation Eq. (44) induces a striking feature in the transition line for $A$ smaller than unity: superconductivity is predicted to have a re-entrant behavior. Here, upon increasing $\sqrt{I_c\tilde{C}_S}/8$, the system passes from insulator to superconductor and again to insulator. This is a rather contra-intuitive behavior as one would naively expect that an increase of the Josephson energy (proportional to $L_c$) always enhances superconductivity. It would be very interesting to search for experimental signatures of this phenomena. However we must also note that $A$ is a non-universal constant that depends on various factors including the accuracy to which the instanton fugacity is computed, the exact choice of $\Delta \gamma$ and the system parameters. At present we cannot rule out that in the range of plausible parameters for realistic materials $A \geq 1$ and this non-monotonicity is not observed. Another potential limitation of our results is that, since Eq. (44) is only valid for small values of $u$, the obtained transition lines are only qualitatively correct.

As is observed in Fig. (3)-(c), the presence of a finite $C_1$, in the limit $C_M \to \infty$, increases the stability of the superconducting phase. Even away from this limit, a finite $C_1$ makes more robust the superconducting phase. In Fig. (3)-(d) it is depicted the full phase diagram as a function of $\sqrt{I_c\tilde{C}_S}/8$, $\tilde{C}_S/C_M$ and $C_1/\tilde{C}_S$ for different values of the non-universal constant $A$. Another striking feature of the phase diagram, besides the re-entrant behavior mentioned previously, is the fact that, even for a relatively large ratio $\tilde{C}_S/C_M$ which brings the system deep into the insulating phase, a fairly small value of $C_1/\tilde{C}_S$ can restore superconductivity.

There are also intriguing features related to the interplay between quantum capacitance and charging effects. For instance in the limit in which the charging energy is only due to a finite mutual capacitance there is no global superconductivity as phase fluctuations in each grain are independent. However the inclusion of “quantum” capacitance $C_\delta$, induced by quantum size effects not related to Coulomb interactions, changes this picture qualitatively. From Eq. (44) it is clear that a finite $C_\delta$ might stabilize superconductivity in a certain range of parameters even if the self-capacitance energy is zero. Therefore a finite “quantum” capacitance, which occur in all systems no matter the nature of the interactions, can help restore long-range order in some cases.

V. APPLICATION TO COLD ATOM PHYSICS

\[ \delta_c = \begin{cases} A = 0.0, & \text{A = 0.8,} \\ A = 0.1, & \text{A = 1.2} \end{cases} \]

\[ \frac{\delta_c}{\Delta}, \frac{R_q}{R_N} \]

Figure 4: Phase diagram as a function of $\delta_c/\Delta$ and $R_q/R_N$ plotted for different values of the non-universal constant $A$. Below the curves the system is superconducting and above it behaves as an insulator.

In this section we investigate the fate of superconductivity in an array in which Coulomb interactions are absent in the limit in which the grain mean level spacing $\delta$ becomes comparable to the bulk gap. For that purpose we study the interplay between the Josephson coupling, the quantum capacitance $C_\delta \sim 2/\delta$, and the quasiparticle dissipation $C_M$. This question can be easily addressed by solving Eq. (44) in the limit of negligible charging energies. This is not of academic interest as it is possible to study experimentally 1D JJ arrays in a cold atom setting with no Coulomb interactions at all. Moreover in cold atom physics many parameters such the tunneling rate, directly related to $R_N$, and the gap $\Delta_0$ can be controlled with great precision so an experimental verification seems feasible.

For sufficiently small grains it is broadly expected that superconductivity will not survive unless the grains are strongly coupled so that the effective granularity of the array is heavily suppressed. Likewise we expect to have global superconductivity for large grains where quantum fluctuations are negligible. Therefore for a given value of the normal resistance $R_N$ there must exist a minimum grain size for which phase coherence can occur despite a finite “quantum” capacitance $C_\delta$. According to Eq. (44),
the best case scenario for the array to stay superconducting corresponds to the limit of infinite fugacity (or $A = 0$) which sets the following lower bound on the grain mean level spacing $\delta_c \approx \frac{\pi L}{4} = \frac{\pi^2 \Delta_0 R_q}{32 N^2}$ from which it is possible to estimate the minimum grain size. For metallic superconductors the above estimation results in a minimum grains size is of order $L \sim 5\text{nm}$ though important variations are expected depending on the material. A finite fugacity is expected to weaken the superfluid state and therefore to decrease $\delta_c$. The evolution of $\delta_c$ as a function of $R_q/R_N$ for different values of $C_J$ and the non-universal parameter $A$, depicted in Fig. 4, agree with this prediction. Note that no re-entrant behavior is observed as there is no charging energy related to a mutual capacitance. Finally we note that our calculation is only valid for $\delta/\Delta_0 \ll 1$ so, from the above expression for $\delta_c$, it is clear that phase coherence is attainable even in the region $R_N \sim R_q$ where the contact among grains is weak and only induces a small smoothing of the spectral density.

VI. SIZE DEPENDENCE OF CLASSICAL AND “QUANTUM” CAPACITANCE

As the grain size decreases both classical and quantum capacitance play a more important role in the description of the array. Naively one might think that for sufficiently small grains charging effects are in general less important than quantum capacitance effects since the former $E_c \propto 1/L^2$ but the latter is proportional to $\delta \propto 1/L^3$. However we note the capacitance and the mean level spacing depends on completely different parameters, the former on the dielectric constant of the material and the details of the substrate while the latter on the Fermi energy and the effective electronic mass. As a result it is plausible that, even if the area scaling holds, both contributions might still be similar for grain sizes $L \sim 10\text{nm}$. This is consistent with the experimental results $\delta^{\text{Pb}}$ for Pb superconducting islands where it was possible to reproduce the expected classical scaling of the capacitance with the area only for relatively large grains. Indeed in a Si(111) substrate the charging energy and the mean level spacing of a $L \sim 7\text{nm}$ grain with $C \approx 40aF$ can be comparable. Therefore quantum fluctuations, not related to charging effects, must be taken into account in any quantitative theoretical model of superconducting nanograins.

VII. CONCLUSIONS

We have investigated the robustness of superconductivity in a 1D JJ array of nanograins at zero temperature. We go beyond the standard theoretical treatment of this problem by including quantum fluctuations, not related to Coulomb interactions, induced by finite size effects, referred to as “quantum capacitance”. By using path integral techniques we have studied the phase diagram of this system including also charging effects and quasiparticle dissipation. We have treated the model analytically by mapping it onto a 1+1D Coulomb gas and then to a sine Gordon model which is known to undergo a Kosterlitz-Thouless transition. For sufficiently large grains long range order is always robust to small self-capacitance charging effects. However the combined effect of a vanishing self-capacitance energy and a finite mutual capacitance energy leads to breaking of phase coherence. We have shown that even in this limit superconductivity is stabilized by a quantum capacitance. In systems with vanishing charging effects, relevant in cold atom experiments, we have shown that long range order persists up to normal resistances comparable to the quantum one. We have also identified the minimum grain size for global superconductivity to occur in this limit. We have found that the phase diagram resulting from the renormalization group analysis is to some extent sensitive to specific details of the model embodied in a non-universal prefactor of the fugacity. As an example, for certain capacitance configurations, small changes in the pre-factor of the fugacity can lead to rather contrain- tuitive results such as a transition from superconductor to insulator by increasing the Josephson coupling.

Acknowledgments

AMG acknowledges financial support from PTDC/FIS/111348/2009, a Marie Curie International Reintegration Grant PIRG07-GA-2010-26817 and EPSRC grant EP/I004637/1.

1 B. D. Josephson, Phys. Lett. 1, 251 (1962); B. D. Josephson, Rev. Mod. Phys. 46, 251 (1974).
2 R. M. Bradley and S. Doniach, Phys. Rev. B 30, 1138 (1984).
3 V. Ambegaokar, U. Eckern, G. Schon, Phys. Rev. Lett. 48, 1745 (1982).
4 K. A. Matveev, A. I. Larkin, and L. I. Glazman, Phys. Rev. Lett. 89, 096802 (2002).
5 I. Giaever, Phys. Rev. Lett. 5, 147 (1960).
6 N. Bergeal, et al., Nature Phys. 4, 608 (2008).
7 D. J. Scalapino, Phys. Rev. Lett. 24, 1052 (1970).
8 S. Bose, et al., Nature Mat. 9, 550 (2010); I. Brihuega, et al., Phys. Rev. B 84, 104525 (2011).
9 P. Ribeiro and A. M. Garcia-Garcia, Phys. Rev. Lett. 108, 097004 (2012).
10 S.E. Korshunov, Sov. Phys. JETP 68, 609 (1989); G. Rastelli, I. M. Pop, W. Guichard, and F.W.J. Hekking, arXiv:1201.0539; G. Refael, E. Demler, Y. Oreg, and D. S. Fisher Phys. Rev. B 75, 014522 (2007).
11 R. C. Jaklevic, J. Lambe, A. H. Silver, and J. E. Mercereau,
Phys. Rev. Lett. 12, 159 (1964).
12 K. Senapati, M. G. Blamire, Z. H. Barber, Nature Mat. 10, 849 (2011).
13 Y. Nakamura, Y. A. Pashkin and J. S. Tsai, Nature 398, 786 (1999); V. Bouchiat et al., Physica Scripta T76 165 (1998).
14 A. D. Caldeira and A. J. Leggett, Phys. Rev. Lett. 48, 1571 (1982).
15 M. P. A. Fisher, Phys. Rev. B, 36 1917 (1987).
16 A. Kampf and G. Schoen, Phys. Rev. B, 36 3651 (1987).
17 S. Chakravarty, G. L. Ingold, S. Kivelson and A. Luther, Phys. Rev. Lett. 56, 2303 (1986).
18 H. C. Fu, A. Seidel, J. Clarke, and D-H. Lee Phys. Rev. Lett. 96, 157005 (2006); H. P. Buchler, V. B. Geshkenbein, and G. Blatter, Phys. Rev. Lett. 92, 067007 (2004).
19 A. D. Zaikin, D. S. Golubev, A. van Otterlo, and G. T. Zimanyi Phys. Rev. Lett. 78, 1552 (1997).
20 J. S. Langer and V. Ambegaokar, Phys. Rev. 164, 498 (1967); D. McCumber and B. Halperin, Phys. Rev. B 1, 1054 (1970).
21 I. S. Beloborodov, A. V. Lopatin, V. M. Vinokur, and K. B. Efetov, Rev. Mod. Phys. 79, 469 (2007).
22 S. Eley, S. Gopalakrishnan, P. M. Goldbart, N. Mason Nature Physics 8, 59 (2012).
23 P. A. Bobbert, R. Fazio, and G. Schön, A. D. Zaikin, Phys. Rev. B 45, 2294 (1992).
24 S. V. Panyukov and A. D. Zaikin, J. Low Temp. Phys. 75, 361 (1989).
25 S. Katsumoto, J. Low Temp. Phys. 98, 287 (1995); G. Schön and A.D. Zaikin Phys. Rep. 198, 237 (1990); R. Fazio and H. van der Zant, Phys. Rep. 355, 235 (2001).
26 B. Muhlschlegel, D. J. Scalapino, and R. Denton, Phys. Rev. B 6, 1767 (1972).
27 D. C. Ralph, C. T. Black, and M. Tinkham, Phys. Rev. Lett. 74, 3241 (1995).
28 F. S. Cataliotti et al., Science 293 843 (2001).
29 A. M. Garcia-Garcia et al., Phys. Rev. Lett. 100, 187001 (2008).
30 R. Parmenter, Phys. Rev. 166, 392 (1968); C. J. Thompson and J. M. Blatt, Phys. Lett. 5, 6 (1963).
31 C. Brun, K. H. Müller, L-P. Hong, F. Patthey, C. Flindt, W.-D. Schneider, Phys. Rev. Lett. 108, 126802 (2012).
32 J. M. Kosterlitz, D. J. Thouless, J. Phys. C: Solid State Phys. 6, 1181 (1973).
33 K. Matveev and A. Larkin, Phys. Rev. Lett. 74, 3749 (1997).
34 V. Ambegaokar and A. Baratoff, Phys. Rev. Lett. 10, 486 (1963).
35 U. Eckern, G. Schön, and V. Ambegaokar, Phys. Rev. B 30, 6419 (1984).
36 J. José, L. Kadanoff, S. Kirkpatrick, and D. Nelson, Phys. Rev. B 16, 1217 (1977).
37 M.-S. Choi, et al., Phys. Rev. B 57, R716 (2005).
38 S. E. Korshunov, Europhys. Lett. 9, 107 (1989); S. E. Korshunov, Zh. Eksp. Teor. Fiz. 95, 1058 (1989); P. A. Bobbert, R. Fazio, G. Schon, and G. T. Zimanyi, Phys. Rev. B41, 4009 (1990).
39 D. J. Amit, Y. Y. Goldschmidt, G. Grinstein, J. Phys. A: Math. Gen. 13, 585 (1980)