Solving linear systems over idempotent semifields through \(\textit{LU}\)-factorization

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Abstract
In this paper, we introduce and analyze a new \(\textit{LU}\)-factorization technique for square matrices over idempotent semifields. In particular, more emphasis is put on “max-plus” algebra here but the work is extended to other idempotent semifields as well. We first determine the conditions under which a square matrix has \(\textit{LU}\) factors. Next, using this technique, we propose a method for solving square linear systems of equations whose system matrices are \(\textit{LU}\)-factorizable. We also give conditions for an \(\textit{LU}\)-factorizable system to have solutions. This work is an extension of similar techniques over fields. Maple\(^\circledR\) procedures for this \(\textit{LU}\)-factorization are also included.

Keywords  Semiring · Idempotent semifield · \(\textit{LU}\)-factorization · Linear system of equations

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1 Introduction

Linear algebra provides significantly powerful problem-solving tools. Many problems in the mathematical sciences consist of systems of linear equations in numerous unknowns. Nonlinear and complicated problems often can be treated by linearization and turned into linear problems that are the “best possible linear approximation”. There are some well-known symbolic and numeric methods and results for the analysis and solution of linear systems of equations over fields and rings. In this work, we propose and analyze a method for solving linear systems of equations over idempotent semifields based on our new \(\textit{LU}\)-factorization technique. We show that this new interpretation of the \(\textit{LU}\) method can be used for a wider range of square matrices and better resembles the well-known factorization method in traditional linear algebra. Note that idempotent semifields are a special class of semirings. In [2, theorem 2.1] and [10, theorem 2] some methods are presented for solving linear systems over idempotent semifields.

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proposed \textit{LU}-factorization technique in this paper is a computational solution method. This work is at the intersection of numerical linear algebra, and pure mathematics. To this end, Maple\textsuperscript{®} procedures are included in the “Appendix”.

Semirings are a generalization of rings and lattices. These algebraic structures are similar to rings where division and subtraction are not needed or can not be defined. Semirings have applications in various areas of mathematics and engineering such as formal language, computer science, optimization theory, control theory, etc. (see e.g. [3, 6, 11, 12, 17]).

The first definition of a semiring was given by Vandiver [16] in 1934. A semiring \((S, \oplus, \otimes, 0, 1)\) is an algebraic structure in which \((S, \oplus)\) is a commutative monoid with an identity element 0 and \((S, \otimes)\) is a monoid with an identity element 1, connected by ring-like distributivity. The additive identity 0 is multiplicatively absorbing, and \(0 \neq 1\).

Some articles have touched on techniques for \textit{LU} decomposition over tropical semirings (see [7, 8, 15]). In this work, we present a new \textit{LU}-factorization technique for a matrix using its entries. This approach enables us to solve systems of linear equations over idempotent semifields applying our \textit{LU} factors. Note that similar to the traditional linear algebra, we set the main diagonal entries of \(L\) to 1 for simplicity. We also show that the solution through this method is maximal. A maximal solution is determined with respect to the defined total order on the idempotent semifield. We consider the linear system of equations, \(AX = b\), where \(A = (a_{ij})\) is a square matrix, \(b = (b_i)\) is a column vector and \(X\) is an unknown vector over an idempotent semifield. A semifield is a commutative semiring such that every nonzero element of it is invertible. We say a matrix \(A\) has an \textit{LU}-factorization if there exist a lower triangular matrix \(L\) and an upper triangular matrix \(U\) such that \(A = LU\).

It is noteworthy that in [15], using a different approach and structure for \textit{LU} decomposition, Tan shows that a square matrix \(A\) over a commutative semifield has an \textit{LU}-factorization if and only if every leading principle submatrix of \(A\) is invertible. Moreover, Cuning-hame-Green proves in [4] that a square matrix \(A\) over an idempotent semifield is invertible if and only if every row and every column of it contains exactly one nonzero element. The same proposition has been proved for certain semifields, namely Boolean algebra and tropical semifields, in [13] and [1], respectively. This means that from Tan’s perspective and in his proposed structure in [15], only diagonal matrices are \textit{LU}-factorizable in semifields and a fortiori in idempotent semifields. In this work, we propose a somewhat different structure for \textit{LU} decomposition which is more aligned with the version from traditional linear algebra. As it turns out, using this method, one can look for and possibly find \textit{LU} factors for square matrices that are not necessarily diagonal.

In traditional linear algebra, \textit{LU} decomposition is the matrix form of Gaussian elimination. Since our goal here is to take advantage of the decomposition technique for solving a system of linear equations, we define a new \textit{LU}-factorization that can be used for an arbitrary non-diagonal matrix, \(A\), in “max-plus algebra”. We give criteria for the existence of a lower triangular matrix, \(L_A\), and an upper triangular matrix, \(U_A\), such that \(A = L_AU_A\). We call \(L_A\) and \(U_A\) the \textit{LU} factors of \(A\). See Sect. 3 for more details.

In Sect. 4, assuming that the matrix \(A\) can be written in an \textit{LU} form, we use the factorization to solve the system of linear equations \(AX = b\) for \(X\). In fact, \(AX = b\) may be rewritten as \(LUX = b\). To find \(X\), we must first solve the system \(LZ = b\) for \(Z\), where \(UX = Z\). Once \(Z\) is found, we solve the system \(UX = Z\) for \(X\). We present some theorems on when \(LZ = b\) and \(UX = Z\) have solutions.

Section 5 concerns the extension of the proposed \textit{LU}-decomposition and system solution from max-plus algebra to arbitrary idempotent semifields.
In the “Appendix” of this paper, we give some Maple procedures as follows. Tables 1 and 2 are subroutines for finding the determinant of a square matrix, and its permutation matrix, respectively. Table 3 gives a code for calculating matrix multiplications. Tables 5 and 6 consist of programs for solving the systems \( LX = b \) and \( UX = b \), respectively. These procedures are all written for max-plus algebra.

We now proceed with some basic definitions and theorems in Sect. 2.

2 Definitions and preliminaries

In this section, we go over some definitions and preliminary notions. For convenience, we use \( \mathbb{N} \), and \( n \) to denote the set of all positive integers, and the set \( \{1, 2, \ldots, n\} \) for \( n \in \mathbb{N} \), respectively.

Definition 1 (See [5]) A semiring \((S, \oplus, \otimes, 0, 1)\) is an algebraic system consisting of a nonempty set \( S \) with two binary operations, addition and multiplication, such that the following conditions hold:

1. \((S, \oplus)\) is a commutative monoid with identity element 0;
2. \((S, \otimes)\) is a monoid with identity element 1;
3. Multiplication distributes over addition from either side, that is \( a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c) \) and \( (b \oplus c) \otimes a = (b \otimes a) \oplus (c \otimes a) \) for all \( a, b, c \in S \);
4. The neutral element of \( S \) with respect to addition is an absorbing element, that is \( a \otimes 0 = 0 = 0 \otimes a \) for all \( a \in S \);
5. \( 1 \neq 0 \).

A semiring is called commutative if \( a \otimes b = b \otimes a \) for all \( a, b \in S \).

Definition 2 Let \( S \) be a commutative semiring. An element \( a \in S \) is multiplicatively cancellative if \( a \otimes b = a \otimes c \) implies \( b = c \). A semiring \( S \) is called multiplicatively cancellative if every element of \( S \) is multiplicatively cancellative.

Definition 3 Let \( S \) be a semiring. An element \( a \in S \) is additively idempotent if and only if \( a \oplus a = a \). A semiring \( S \) is called additively idempotent if every element of \( S \) is additively idempotent.

Definition 4 A semiring \( S \) is called zerosumfree if \( a \oplus b = 0 \) implies that \( a = 0 = b \), for any \( a, b \in S \).

Definition 5 A commutative semiring \((S, \oplus, \otimes, 0, 1)\) is called a semifield if every nonzero element of \( S \) is multiplicatively invertible.

Definition 6 (See [9]) The semifield \((S, \oplus, \otimes, 0, 1)\) is idempotent if it is an additively idempotent, totally ordered, and radicable semifield. Note that the radicability implies the power \( a^q \) be defined for any \( a \in S \setminus \{0\} \) and \( q \in \mathbb{Q} \) (rational numbers). In particular, for any non-negative integer \( p \) we have

\[
a^0 = 1, \quad a^p = a^{p-1}a, \quad a^{-p} = (a^{-1})^p.
\]
The totally ordered operator “\(\leq_S\)” on an idempotent semifield is compatible with the following partial order:

\[
a \leq_S b \iff a \oplus b = b,
\]

that is induced by additive idempotency. Notice that the last partial order equips addition with an extremal property in the form of the following inequalities

\[
a \leq_S a \oplus b, \quad b \leq_S a \oplus b,
\]

which makes idempotent semifields zerosumfree, since we have \(a \geq_S 0\) for any \(a \in S\).

The total order defined on \(S\) is compatible with the algebraic operations, that is:

\[
(a \leq_S b \& c \leq_S d) \rightarrow a \oplus c \leq_S b \oplus d,
\]

\[
(a \leq_S b \& c \leq_S d) \rightarrow a \otimes c \leq_S b \otimes d,
\]

for any \(a, b, c, d \in S\).

Throughout this article, we consider the notation “\(\geq_S\)” as the converse of the order “\(\leq_S\)”, which is a totally ordered operator satisfying \(a \geq_S b\) if and only if \(b \leq_S a\), for any \(a, b \in S\). Furthermore, we use “\(a <_S b\)” whenever \(a \leq_S b\) and \(a \neq b\). The notation “\(>_S\)” is defined similarly.

**Example 1** Tropical semirings are important cases of idempotent semifields that we consider in this work. Examples of tropical semifields are as follows:

\[
\begin{align*}
\mathbb{R}_{\text{max, +}} &= (\mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0), \\
\mathbb{R}_{\text{min, +}} &= (\mathbb{R} \cup \{+\infty\}, \min, +, +\infty, 0), \\
\mathbb{R}_{\text{max, \times}} &= (\mathbb{R}_+ \cup \{0\}, \max, \times, 0, 1), \\
\mathbb{R}_{\text{min, \times}} &= (\mathbb{R}_+ \cup \{+\infty\}, \min, \times, +\infty, 1),
\end{align*}
\]

where \(\mathbb{R}\) is the set of real numbers, and \(\mathbb{R}_+ = \{x \in \mathbb{R} | x > 0\}\). This work particularly concerns \(\mathbb{R}_{\text{max, +}}\) that is called “max–plus algebra” whose additive and multiplicative identities are \(-\infty\) and 0, respectively. Note further that the multiplication \(a \otimes b^{-1}\) in “max–plus algebra” means that \(a + (-b) = a - b\), where “+”, “−” and \(-b\) denote the usual real numbers addition, subtraction and the typical additively inverse of the element \(b\), respectively. Moreover, the defined total order on “max–plus algebra” is the standard less than or equal relation “\(\leq\)” over \(\mathbb{R}\).

The semifields \(\mathbb{R}_{\text{max, +}}, \mathbb{R}_{\text{min, +}}, \mathbb{R}_{\text{max, \times}}\) and \(\mathbb{R}_{\text{min, \times}}\) are isomorphic to one another. Figure 1 shows the isomorphism maps for these semifields [10, fig. 1].

Let \(S\) be a commutative semiring. We denote the set of all \(m \times n\) matrices over \(S\) by \(M_{m \times n}(S)\). For \(A \in M_{m \times n}(S)\), we denote by \(a_{ij}\) and \(A^T\) the \((i, j)\)-entry of \(A\) and the transpose of \(A\), respectively.

For any \(A = (a_{ij}) \in M_{m \times n}(S), B = (b_{ij}) \in M_{m \times n}(S), C = (c_{ij}) \in M_{n \times l}(S)\) and \(\lambda \in S\), we define:

\[
A + B = (a_{ij} \oplus b_{ij})_{m \times n},
\]

\[
AC = \left( \bigoplus_{k=1}^{n} a_{ik} \otimes c_{kj} \right)_{m \times l}.
\]
and
\[ \lambda A = (\lambda \otimes a_{ij})_{m \times n}. \]

It is easy to verify that \( M_n(S) := M_{m \times n}(S) \) forms a semiring with respect to the matrix addition and the matrix multiplication whose additive and multiplicative identities are the matrices \( 0 \) (the matrix of semiring zeros) and \( I_n \) (the matrix with semiring ones on the diagonal and zeros elsewhere), respectively. Take matrices \( A, B \in M_n(S) \). We say that \( A \leq B \) if \( a_{ij} \leq b_{ij} \) for every \( i, j \in \mathbb{N} \). Furthermore, \( A \) is called invertible if \( AC = I_n = CA \) for some \( C \in M_n(S) \).

Let \( A \in M_n(S) \), \( S_n \) be the symmetric group of degree \( n \geq 2 \), and \( A_n \) be the alternating group on \( n \) such that
\[ A_n = \{ \sigma | \sigma \in S_n \text{ and } \sigma \text{ is an even permutation} \}. \]

The positive determinant, \( \det^+(A) \), and negative determinant, \( \det^-(A) \), of \( A \) are
\[ \det^+(A) = \bigoplus_{\sigma \in A_n} \bigotimes_{i=1}^{n} a_{\sigma(i)}, \]
and
\[ \det^-(A) = \bigoplus_{\sigma \in S_n \backslash A_n} \bigotimes_{i=1}^{n} a_{\sigma(i)}. \]

**Definition 7** Let \( S \) be a commutative semiring and \( A \in M_n(S) \). The determinant of \( A \), denoted by \( \det(A) \), is defined by
\[ \det(A) = \det^+(A) \oplus \det^-(A). \]

In particular, for \( S = \mathbb{R}_{\max,+} \), we have \( \det(A) = \max(\det^+(A), \det^-(A)) \). Note that, this definition agrees with that in [14].

Let \( (S, \oplus, \otimes, 0, 1) \) be an idempotent semifield, \( A \in M_n(S) \), \( b \in S^n \) be a column vector and \( X = (x_i)_{i=1}^{n} \) be an unknown vector over \( S \). Then the \( i \)-th equation of the linear system \( AX = b \) is
\[ \bigoplus_{j=1}^{n} (a_{ij} \otimes x_j) = (a_{i1} \otimes x_1) \oplus (a_{i2} \otimes x_2) \oplus \cdots \oplus (a_{in} \otimes x_n) = b_i. \]

Especially, the \( i \)-th equation of the linear system over \( \mathbb{R}_{\text{max,+}} \) is

\[ \max(a_{i1} + x_1, a_{i2} + x_2, \ldots, a_{in} + x_n) = b_i. \]

**Definition 8** Let \( S \) be an idempotent semifield and \( b \in S^n \). Then \( b \) is called a regular vector if it has no zero element.

Without loss of generality, we can assume that \( b \) is regular in the system \( AX = b \). Otherwise, let \( b_i = 0 \) for some \( i \in n \). Then in the \( i \)-th equation of the system, we have \( a_{ij} \otimes x_j = 0 \) for any \( j \in n \), since \( S \) is zerosumfree. As such, \( x_j = 0 \) if \( a_{ij} \neq 0 \). Consequently, the \( i \)-th equation can be removed from the system together with every column \( A_j \) where \( a_{ij} \neq 0 \), and the corresponding \( x_j \) can be set to 0.

**Definition 9** A solution \( X^* \) of the system \( AX = b \) is called maximal if \( X \leq X^* \) for any solution \( X \).

**Theorem 1** (See [4]) Let \( S \) be a semifield and \( A \in M_n(S) \). Then \( A \) is invertible if and only if every row and every column of \( A \) contains exactly one nonzero element.

Let \( S \) be a commutative semiring, \( A \in M_n(S) \) and \( V, W \subseteq n \). We denote by \( A[V|W] \) the matrix with row indices indexed by indices in \( V \), in an increasing order, and column indices indexed similarly by indices in \( W \). The matrix \( A[V|W] \) is called a submatrix of \( A \). Particularly, \( A[i|i] \) is called a leading principle submatrix of \( A \) for any \( i \in n \).

**Theorem 2** (See [15]) Let \( (S, \oplus, \otimes, 0, 1) \) be a commutative semiring. Then for any \( A \in M_n(S) \), the following statements are equivalent.

1. All the leading principle submatrices of \( A \) are invertible.
2. \( A \) has an \( LU \)-factorization where \( L \) is an invertible lower triangular matrix with 1’s on its main diagonal, and \( U \) is an invertible upper triangular matrix with \( u_{ii} \in U(S) \) where \( U(S) \) denotes the set of all multiplicatively invertible elements of \( S \).

Let \( S \) be an idempotent semifield and \( A \in M_n(S) \). Suppose that \( A \) has an \( LU \)-factorization as mentioned in Theorem 2. Then all the leading principle submatrices of \( A \) are invertible. Moreover, per Theorem 1, we can obtain \( a_{ii} \neq 0 \) for any \( i \in n \). According to this interpretation, \( A \) has an \( LU \)-factorization if and only if it is a diagonal matrix with nonzero diagonal elements.

In what follows, through a different lens, we interpret and devise an \( LU \)-factorization method for certain square matrices which are not necessarily diagonal, and use it to solve linear systems of equations.

Note that our main focus in the following two sections is on the “\( \text{max} - \text{plus algebra} \)”. 

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3 LU-factorization

Throughout this section, we consider $S = \mathbb{R}_{\text{max},+}$. Take an arbitrary matrix $A = (a_{ij}) \in M_n(S)$. Let two matrices $L_A$ and $U_A$ be as follows

$$L_A = (l_{ij}); \quad l_{ij} = \begin{cases} a_{ij} - a_{jj} & \text{if } i \geq j, \\ -\infty & \text{otherwise}, \end{cases}$$  \hspace{1cm} (1)$$

and,

$$U_A = (u_{ij}); \quad u_{ij} = \begin{cases} a_{ij} & \text{if } i \leq j, \\ -\infty & \text{otherwise}. \end{cases}$$  \hspace{1cm} (2)$$

We say a matrix $A \in M_n(S)$ has an LU-factorization if $A = L_A U_A$. For simplicity, we show $L_A$ just by $L$ and $U_A$ just by $U$ from this point on. Clearly, $L$ and $U$ are lower and upper triangular matrices, respectively.

Let $\sigma \in S_n$ be a permutation. Then $P_\sigma$ is the permutation matrix corresponding to $\sigma$. It is a square matrix obtained from the same size identity matrix through a permutation of the rows. It is clear that every row and every column of a permutation matrix contains exactly one nonzero element, therefore it is invertible according to Theorem 1. As a result, the system of equations $AX = b$ and $P_\sigma AX = P_\sigma b$ have the same solutions.

Note that without loss of generality, we can assume that

$$\det(A) = a_{11} + \cdots + a_{nn};$$

otherwise, there exists a permutation matrix $P_\sigma$ such that

$$\det(P_\sigma A) = (P_\sigma A)_{11} + \cdots + (P_\sigma A)_{nn}.$$

In the following theorem, we present the necessary and sufficient conditions for the existence of the presented LU factors of an arbitrary square matrix $A$.

**Theorem 3** Let $A \in M_n(S)$ such that $\det(A) = a_{11} + \cdots + a_{nn}$ and the matrices $L$ and $U$ be defined by (1) and (2), respectively. Then $A = LU$ if and only if for any $1 < i, j \leq n$ and $i \neq j$,

$$a_{ij} = \max_{k=1}^{r}(\det(A[\{k, i\} | \{k, j\}]) - a_{kk}),$$

where $r = \min\{i, j\} - 1$.

**Proof** Suppose that the conditions (3) hold for any $1 < i, j \leq n$ and $i \neq j$. Then it suffices to show that for all $i, j \in \mathbb{Z}^+$, $a_{ij} = (LU)_{ij}$. To this end, we consider the following cases:

1. Let $i = 1$. Then for all $j \in \mathbb{Z}^+$. 
\[(LU)_{ij} = \bigoplus_{k=1}^{n}(l_{ik} \otimes u_{kj})
\]

\[= \max_{k=1}^{n}(l_{ik} + u_{kj})
\]

\[= l_{i1} + u_{ij}
\]

\[= a_{ij},
\]

given that for \(k > 1, l_{ik} = -\infty\).

2. Similarly, for \(j = 1\), we have \((LU)_{11} = a_{11}\) for all \(i \in \mathbb{n}\) since \(u_{k1} = -\infty\) for all \(k > 1\).

3. Now assume that \(i = j\) for all \(i, j \in \mathbb{n}\). Then

\[(LU)_{ii} = \bigoplus_{k=1}^{n}(l_{ik} \otimes u_{ki})
\]

\[= \max_{k=1}^{n}(l_{ik} + u_{ki})
\]

\[= \max_{k=1}^{n}(a_{ik} - a_{kk} + a_{ki})
\]

\[= a_{ii}.
\]

The equality (4) holds since for all \(k > i, u_{ki} = l_{ik} = -\infty\). Furthermore, with respect to any permutation \(\sigma = (ik)\), we have

\[a_{1\sigma(1)} \otimes \cdots \otimes a_{n\sigma(n)} \leq \det(A)\]

i.e.,

\[a_{ik} + a_{ki} + \sum_{j=1, j \neq k, i}^{n} a_{jj} \leq a_{ii} + a_{kk} + \sum_{j=1, j \neq k, i}^{n} a_{jj}.
\]

Since \(S\) is a multiplicatively cancellative semifield, we get \(a_{ik} + a_{ki} \leq a_{ii} + a_{kk}\) which yields the equality (5).

4. For other entries \((i,j > 1; i \neq j)\), we have

\[(LU)_{ij} = \bigoplus_{k=1}^{n}(l_{ik} \otimes u_{kj})
\]

\[= \max_{k=1}^{\min\{i,j\}}(l_{ik} + u_{kj})
\]

\[= \max_{k=1}^{\min\{i,j\}}(a_{ik} - a_{kk} + a_{kj})
\]
\[
= \max_{k=1}^{\min\{i,j\}-1} \left( a_{ik} - a_{kk} + a_{kj}, a_{ij} \right)
\]
\[
= \max_{k=1}^{\min\{i,j\}-1} \left( \max_{j=1}^{i,j} (a_{ik} + a_{kj} - a_{kk}, a_{ij} + a_{kk} - a_{kk}) \right)
\]
\[
= \max_{k=1}^{\min\{i,j\}-1} \left( \max_{j=1}^{i,j} (a_{ik} + a_{kj}, a_{ij} + a_{kk} - a_{kk}) \right)
\]
\[
= \max_{k=1}^{\min\{i,j\}-1} \left( \det(A[\{k,i\} \mid \{k,j\}]) - a_{kk} \right)
\]
\[
= a_{ij}.
\]

The equality (6) holds, because

\[
(\forall k)(k > i, k > j \Rightarrow l_{ik} = -\infty, u_{kj} = -\infty).
\]

As a result, for any \( k > \min\{i,j\} \), \( l_{ik} + u_{kj} = -\infty \). The equality (7) holds, because \( k = \min\{i,j\} \), which means \( k = i \) or \( k = j \), implies \( a_{ik} - a_{kk} + a_{kj} = a_{ij} \). Conversely, assume that \( A \) has the presented \( LU \)-factorization, that is \( a_{ij} = (LU)_{ij} \). Then by (1) and (2) the proof is complete.

\[\Box\]

**Example 2** Consider \( A \in M_4(S) \) as follows

\[
A = \begin{bmatrix}
7 & -1 & 3 & 0 \\
4 & 5 & 1 & -2 \\
1 & -6 & 2 & -5 \\
-2 & -9 & -5 & 0
\end{bmatrix},
\]

\( A \) has the following \( LU \) factors:

\[
L = \begin{bmatrix}
0 & -\infty & -\infty & -\infty \\
-3 & 0 & -\infty & -\infty \\
-6 & -11 & 0 & -\infty \\
-9 & -14 & -7 & 0
\end{bmatrix},
\]

\( U = \begin{bmatrix}
7 & -1 & 3 & 0 \\
-\infty & 5 & 1 & -2 \\
-\infty & -\infty & 2 & -5 \\
-\infty & -\infty & -\infty & 0
\end{bmatrix}. \)

Note that a Maple procedure for calculating the \( LU \) factors of a square matrix from Theorem 3 is given in Table 4.

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### 4 Solving a linear system of equations

In this section, we first discuss and analyze a system of the form \( LX = b \) with a lower triangular matrix, \( L \), followed by a system of the form \( UX = b \) with an upper triangular matrix, \( U \). The combination of these results gives the solutions of the system \( AX = b \) if the \( LU \) factors of \( A \) exist. Note that solving triangular systems over semifields is discussed in [4] and [10]. Our results here are more consistent with our interpretation of the \( LU \)-factorization and better lend themselves to an algorithmic approach. The Maple\textsuperscript{®} procedures in the “Appendix” are the implementations of this idea.
4.1 L-system

Here, we study the solution of the lower triangular system $LX = b$ where $L \in M_n(S)$ and $b \in S^n$ is a regular vector. The $i$-th equation of this system is
\[
\max(l_{i1} + x_1, l_{i2} + x_2, \ldots, l_{ii} + x_i, -\infty) = b_i.
\]

**Theorem 4** Let $LX = b$ be a linear system of equations with a lower triangular matrix $L \in M_n(S)$ and a regular vector $b \in S^n$. Then the system $LX = b$ has the maximal solution $X^* = (b_i - l_{ii})_{i=1}^n$ if $l_{ik} - l_{kk} \leq b_i - b_k$ for any $2 \leq i \leq n$ and $1 \leq k \leq i - 1$. Moreover, if all the inequalities $l_{ik} - l_{kk} \leq b_i - b_k$ are proper, then the maximal solution $X^*$ of the system $LX = b$ is unique.

**Proof** The proof is through induction on $i$. For $i = 2$ ($k = 1$), the second equation of the system $LX = b$ in the form
\[
\max(l_{21} + x_1, l_{22} + x_2) = b_2
\]
implies that $x_2 \leq b_2 - l_{22}$, since $l_{21} - l_{11} \leq b_2 - b_1$ and $x_1 = b_1 - l_{11}$. We also show that the statement is true for $i = 3$ ($k = 1, 2$). Since the inequalities $l_{31} - l_{11} \leq b_3 - b_1$ and $l_{32} - l_{22} \leq b_3 - b_2$ hold, replacing for $x_1$, and $x_2$ in the third equation, $\max(l_{31} + x_1, l_{32} + x_2, l_{33} + x_3) = b_3$, yields $x_3 \leq b_3 - l_{33}$.

Suppose that the statements are true for all $i \leq m - 1$, i.e., $x_i = b_i - l_{ii}$ and $x_i \leq b_i - l_{ii}$, for any $2 \leq i \leq m - 1$. Now, let $i = m$ ($k = 1, \ldots, m - 1$). Then $l_{ik} - l_{kk} \leq b_i - b_k$, and by the induction hypothesis, $x_i \leq b_i - l_{ii}$ for any $2 \leq i \leq m - 1$. As such, in the $m$-th equation of the system we get $x_m \leq b_m - l_{mm}$. Hence, the system $LX = b$ has the maximal solution $X^* = (b_i - l_{ii})_{i=1}^n$. Clearly, if $l_{ik} - l_{kk} < b_i - b_k$ for any $2 \leq i \leq n$ and $1 \leq k \leq i - 1$, then the maximal solution $X^*$ is unique. \hspace{1cm} \square

**Remark 1** The inequalities in Theorem 4 give a sufficient condition for the existence of the solutions of $LX = b$, but it is not a necessary condition.

For example, let $L \in M_3(S)$ be a lower triangular matrix and $LX = b$ has solutions $X \leq (b_i - l_{ii})_{i=1}^3$. Then the equation $\max(l_{31} + x_1, l_{32} + x_2, l_{33} + x_3) = b_3$ implies $l_{32} + x_2 \leq b_3$. However, considering $x_2 \leq b_2 - l_{22}$, we can not necessarily conclude that $l_{32} - l_{22} \leq b_3 - b_2$.

Importantly, the contrapositive of Theorem 4 gives a necessary condition for when the system $LX = b$ has no solution. It states that if the system $LX = b$ has no solution, then there must exist some $2 \leq i \leq n$ and some $1 \leq k \leq i - 1$ such that $l_{ik} - l_{kk} > b_i - b_k$.

We now present a step by step method for solving a lower triangular linear system, $LX = b$, and finding the components $x_k$, $1 \leq k \leq n$, of $X$ if the system has any solutions. A Maple® code for this technique is given in Table 5.
4.2 A descriptive method for solving $LX = b$

Let $x_k$ and $b_i$ be the $k$-th entries ($1 \leq k \leq n$) of the unknown vector $X$ and the constant vector $b$, respectively, of the lower triangular system $LX = b$.

- **Step 1** From the first row of the system ($i = 1$), we have $x_1 = b_1 - l_{11}$.

- **Step 2** We now check the feasibility of the next rows, $2 \leq i \leq n$, for $k = 1$.

  - **Case 1** If for some $i$, $l_{i1} - l_{11} > b_i - b_1$, then $l_{i1} + x_i = l_{11} + b_1 - l_{11} > b_i$ which means the $i$-th row and therefore the system has no solution and the process is terminated without yielding any solution.

  - **Case 2** If for all $i$, $l_{i1} - l_{11} \leq b_i - b_1$, then $l_{i1} + x_i \leq b_i$. In particular for $i = 2$, we have $l_{21} + x_1 \leq b_2$ and the second row yields $x_2 \leq b_2 - l_{22}$. This takes us to the next step.

- **Step 3** We now check the feasibility of the rows, $k + 1 \leq i \leq n$, for each $2 \leq k \leq n - 1$ and exactly in that order.

  - **Case 1** If for some $i$, $l_{ik} - l_{kk} > b_i - b_k$ or $b_k - l_{kk} > b_i - l_{ik}$, then given already that $x_k \leq b_k - l_{kk}$, we end up with one of the following cases:

    (i) if $x_k = b_k - l_{kk}$, then $l_{ik} + x_k > b_i$ which means the $i$-th row and therefore the system has no solution and the process is terminated without yielding any solution,

    (ii) else if $x_k < b_k - l_{kk}$, then we set $x_k = b_i - l_{ik}$. In particular and in order to attain the maximal solution, we can actually set $x_k = \min_{i \in I} \{b_i - l_{ik}\}$, where $I \subseteq \{k + 1, \ldots, n\}$ is the set of all $i$ such that $l_{ik} - l_{kk} > b_i - b_k$. Next, we replace $k$ with $k + 1$ and repeat this step as long as $k \leq n - 2$. If $k = n - 1$, we end up with $x_n \leq b_n - l_{nn}$ and the system has solutions, so we stop here.

  - **Case 2** If for all $i$, $l_{ik} - l_{kk} \leq b_i - b_k$, then $l_{ik} + x_k \leq b_i$. In particular for $i = k + 1$, we have $l_{(k+1)k} - l_{kk} \leq b_{(k+1)} - b_k$ which implies $l_{(k+1)k} + x_k \leq l_{(k+1)k} + b_k - l_{kk} \leq b_{(k+1)}$. Note that already $x_k \leq b_k - l_{kk}$. Now the $(k + 1)$-st row of the system, $\max(l_{(k+1)1} + x_1, l_{(k+1)2} + x_2, \ldots, l_{(k+1)(k+1)} + x_{(k+1)}) = b_{(k+1)}$, gives $x_{(k+1)} \leq b_{(k+1)} - l_{(k+1)(k+1)}$. At this point, we should return to the beginning of this step as long as $k \leq n - 2$. If $k = n - 1$, then $x_n \leq b_n - l_{nn}$ and the system has solutions, so we stop here.

In the following example, we intend to solve a lower triangular linear system through the above-described method.

**Example 3** Consider the following system:

$$
\begin{bmatrix}
3 & -\infty & -\infty & -\infty \\
-5 & 4 & -\infty & -\infty \\
6 & 18 & -2 & -\infty \\
1 & 14 & -6 & 3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= 
\begin{bmatrix}
6 \\
-2 \\
10 \\
5
\end{bmatrix}.
$$

It is clear that $x_1 = 3$ and $l_{11} - l_{11} \leq b_i - b_1$ for any $2 \leq i \leq 4$. In particular for $i = 2$, we have $l_{21} + x_1 = b_2$ and the second row of the system implies $x_2 \leq b_2 - l_{22}$ (i.e., $x_2 \leq -6$). We now move to step 3 for $k = 2$. It is easy to check $l_{i2} - l_{22} > b_i - b_2$ for $i = 3, 4$. As such, we consider $x_2 = \min\{b_2 - l_{32}, b_4 - l_{42}\} = -9$, because $x_2$ is not necessarily equal to $b_2 - l_{22}$. Next, we replace $x_1$ and $x_2$, obtained from the previous steps, in the third row.
which implies \( l_{31} + x_1, l_{32} + x_2 < b_3 \) and consequently \( x_3 = b_3 - l_{33} \). We repeat this step for \( k = 3 \). The inequality \( l_{43} - l_{33} > b_4 - b_3 \) yields \( l_{43} + x_3 > b_4 \) and therefore the system has no solution.

**Proposition 1**  The system \( LX = b \) has no solution if \( l_{11} - l_{11} > b_i - b_1 \), for some \( 2 \leq i \leq n \).

**Proof**  If \( l_{11} - l_{11} > b_i - b_1 \), then \( l_{11} + b_1 - l_{11} > b_i \). Since \( x_1 = b_1 - l_{11}, l_{11} + x_1 > b_i \). This means the system has no solutions. \( \square \)

### 4.3 U-system

In this section, we study the solution of the upper triangular system \( UX = b \) with \( U \in M_n(S) \) and \( b \in S^n \) that is a regular vector. Note that we can rotate an upper triangular matrix and turn it into a lower triangular matrix through a clockwise 180-degree rotation. As such, the \( U \)-system \( UX = b \) becomes an \( L \)-system \( LX' = b' \) with the following matrix:

\[
\begin{bmatrix}
 u_{nn} & -\infty & \cdots & -\infty \\
 u_{(n-1)n} & u_{(n-1)(n-1)} & \cdots & -\infty \\
 \vdots & \vdots & \ddots & \vdots \\
 u_{1n} & u_{1(n-1)} & \cdots & u_{11}
\end{bmatrix},
\]

where \( l_{ij} = u_{(n-i+1)(n-j+1)}, x'_i = x_{(n-i+1)} \) and \( b'_i = b_{(n-i+1)} \), for every \( 1 \leq i, j \leq n \) and \( j \leq i \).

**Theorem 5**  Let \( UX = b \) be a linear system of equations with an upper triangular matrix \( U \in M_n(S) \) and a regular vector \( b \in S^n \). Then the system \( UX = b \) has the maximal solution \( X^* = (b_j - u_{jj})_{j=1}^{n} \) if \( u_{(n-i)k} - u_{kk} \leq b_{(n-i)} - b_k \) for any \( 1 \leq i \leq n - 1 \) and \( n - i + 1 \leq k \leq n \). Moreover, if the inequalities \( u_{(n-i)k} - u_{kk} \leq b_{(n-i)} - b_k \) are proper, then the maximal solution \( X^* \) of the U-System is unique.

**Proof**  We first convert the upper triangular system \( UX = b \) into a lower triangular system \( LX' = b' \) as explained above. We can now rewrite \( u_{(n-i)k} - u_{kk} \leq b_{(n-i)} - b_k \) as

\[
l_{(i+1)(i+1-k)} - l_{(n+1-k)(n+1-k)} \leq b_{(i+1)} - b_{(n+1-k)},
\]

Consequently, without loss of generality, we have \( l_{ss} - l_{tt} \leq b_s - b_t \) for every \( 2 \leq s \leq n \), \( 1 \leq t \leq s - 1 \). Hence by Theorem 4, the system \( LX' = b' \) has solutions \( X' = (x'_i)_{i=1}^{n} \) where \( x'_i \leq b'_i - l_{ii} \) which implies \( x_{(n-i+1)} \leq b_{(n-i+1)} - u_{(n-1)(n-i+1)} \). Thus, \( UX = b \) has the maximal solution \( X^* = (b_j - u_{jj})_{j=1}^{n} \). Note further that the maximal solution is unique if \( u_{(n-i)k} - u_{kk} < b_{(n-i)} - b_k \) for any \( 1 \leq i \leq n - 1 \) and \( n - i + 1 \leq k \leq n \). \( \square \)

We can devise a step by step method for solving the upper triangular system, \( UX = b \), similar to the one given in Sect. 4.2 for \( LX = b \). As explained above, we can fairly easily turn an upper triangular system into a lower triangular one. A Maple\textsuperscript{®} procedure based on the step by step method for solving the \( UX = b \) system is given in Table 6.
4.4 LU-system

The LU-factorization method is well-known for solving systems of linear equations. In this section, we find the solutions of the system $AX = b$ for any $A \in M_n(S)$ and a regular vector $b \in S^n$, where $A$ has LU factors.

**Theorem 6** Let $A \in M_n(S)$ and suppose $A$ has LU factors given by (1) and (2), respectively. The system $AX = b$ has the maximal solution $X^* = (b_i - a_{ii})_{i=1}^n$ if $a_{ik} - a_{kk} \leq b_i - b_k$ and $a_{(n-j)l} - a_{ll} \leq b_{(n-j)} - b_l$ for any $2 \leq i \leq n$, $1 \leq k \leq i - 1$, $1 \leq j \leq n - 1$, and $n - j + 1 \leq l \leq n$. Furthermore, the system $AX = b$ has the unique solution $X^* = (b_i - a_{ii})_{i=1}^n$ if all the inequalities $a_{ik} - a_{kk} \leq b_i - b_k$ and $a_{(n-j)l} - a_{ll} \leq b_{(n-j)} - b_l$ are proper.

**Proof** Let the matrix $A$ have LU factors. Then the system $AX = b$ may be rewritten as $L(UX) = b$. To obtain $X$, we must first decompose $A$ and then solve the system $LZ = b$ for $Z$, where $UX = Z$. Once $Z$ is found, we solve the system $UX = Z$ for $X$.

By the definition of $L$, we have $l_{ik} = a_{ik} - a_{kk}$, for any $i, k \in n$ and $i > k$. As such, the inequalities $a_{ik} - a_{kk} \leq b_i - b_k$ for any $2 \leq i \leq n$ and $1 \leq k \leq i - 1$ can be rewritten as the inequalities $l_{ik} - l_{kk} \leq b_i - b_k$. Therefore, due to Theorem 4, the system $LZ = b$ has maximal solution $Z = (b_i - l_{ii})_{i=1}^n$ where $l_{ii} = 0$. Similarly, we can turn the inequalities $a_{(n-j)l} - a_{ll} \leq b_{(n-j)} - b_l$, for any $1 \leq j \leq n - 1$, and $n - j + 1 \leq l \leq n$ into the inequalities $l_{(n-j)l} - l_{ll} \leq b_{(n-j)} - b_l$, since $a_{ij} = a_{ij}$, for any $i, j \in n$ and $i < j$. Then due to Theorem 5, the system $UX = Z$ has maximal solution $X^* = (z_i - a_{ii})_{i=1}^n$. Consequently, the system $AX = b$ has the maximal solution $X^* = (b_i - a_{ii})_{i=1}^n$. Clearly, the obtained maximal solution is unique whenever all the assumed inequalities are proper. \[\square\]

**Example 4** Let $A \in M_4(S)$. Consider the following system $AX = b$:

\[
\begin{bmatrix}
4 & 1 & 4 & 3 \\
-1 & 0 & 1 & 4 \\
3 & 7 & 8 & 1 \\
5 & 2 & 5 & -2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= 
\begin{bmatrix}
3 \\
4 \\
9 \\
4
\end{bmatrix}
\]

Here, $\det(A) = a_{13} + a_{24} + a_{32} + a_{41}$, but there exists a permutation matrix $P_\sigma$ corresponding to the permutation $\sigma = (1324)$:

\[
P_\sigma = \begin{bmatrix}
-\infty & -\infty & -\infty & 0 \\
-\infty & -\infty & 0 & -\infty \\
0 & -\infty & -\infty & -\infty \\
-\infty & 0 & -\infty & -\infty
\end{bmatrix}
\]

such that $P_\sigma A$ has the following LU factors:

\[
L = \begin{bmatrix}
0 & -\infty & -\infty & -\infty \\
-2 & 0 & -\infty & -\infty \\
-1 & -6 & 0 & -\infty \\
-6 & -7 & -3 & 0
\end{bmatrix},
U = \begin{bmatrix}
5 & 2 & 5 & -2 \\
-\infty & 7 & 8 & 1 \\
-\infty & -\infty & 4 & 3 \\
-\infty & -\infty & -\infty & 4
\end{bmatrix}.
\]

We can now use the LU method to solve the system $(P_\sigma A)X = P_\sigma b$.
Due to Theorem 6, we must first solve the system $LZ = P_\sigma b$:

\[
\begin{bmatrix}
0 & -\infty & -\infty & -\infty \\
-2 & 0 & -\infty & -\infty \\
-1 & -6 & 0 & -\infty \\
-6 & -7 & -3 & 0
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2 \\
z_3 \\
z_4
\end{bmatrix}
= \begin{bmatrix}4 \\ 9 \\ 3 \\ 4
\end{bmatrix}.
\]

Since $l_{ik} - l_{kk} \leq b'_i - b'_k$ for any $2 \leq i \leq 4$ and $1 \leq k \leq i - 1$, the maximal solution of the system $LZ = b$ is $(b'_i - l_{ii})_{i=1}^4 = (b'_1)_{i=1}^4$, where $b' = P_\sigma b$:

\[
Z = \begin{bmatrix}4 \\ 9 \\ 3 \\ 4
\end{bmatrix}.
\]

We shall now solve the system $UX = Z$:

\[
\begin{bmatrix}
5 & 2 & 5 & -2 \\
-\infty & 7 & 8 & 1 \\
-\infty & -\infty & 4 & 3 \\
-\infty & -\infty & -\infty & 4
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= \begin{bmatrix}z_1 \\ z_2 \\ z_3 \\ z_4
\end{bmatrix}.
\]

By Theorem 5, the maximal solution is $X^* = (z_1 - u_{ii})_{i=1}^4$ or in fact $X^* = (b'_i - u_{ii})_{i=1}^4$, since $u_{(4-i)k} - u_{kk} \leq b_{(4-i)-k} - b_k$ for any $1 \leq i \leq 3$ and $4 - i + 1 \leq k \leq 4$:

\[
X^* = \begin{bmatrix}-1 \\ 2 \\ -1 \\ 0
\end{bmatrix}.
\]

Note that the systems $AX = b$ and $(P_\sigma A)X = P_\sigma b$ have the same solutions.

**Remark 2** Theorem 6 shows that if the system $(LU)X = b$ does not have any solutions, then either the system $LZ = b$ or the system $UX = Z$ must not have any solutions.

The next example shows that if the system $LZ = b$ has no solution, then neither does the system $AX = b$.

**Example 5** Let $A \in M_4(S)$. Consider the following system $AX = b$:

\[
\begin{bmatrix}
7 & -1 & 3 & 0 \\
4 & 5 & 1 & -2 \\
1 & -6 & 2 & -5 \\
-2 & -9 & -5 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= \begin{bmatrix}5 \\ 2 \\ -1 \\ -9
\end{bmatrix}.
\]
By Theorem 3, $A$ has an $LU$-factorization. It is fairly straightforward to verify that $LZ = b$ has no solution, because $l_{41} > b_4 - b_1$. We have $a_{41} + x_1 > b_4$, since $x_1 = b_1 - a_{11}$ and $l_{41} = a_{41} - a_{11}$ which implies $AX = b$ not have any solutions.

**Theorem 7** Let $A \in M_n(S)$. Suppose $A$ has an $LU$-factorization. If the system $LZ = b$ has no solution, then neither does the system $AX = b$.

**Proof** Assume that $LZ = b$ does not have any solutions. Based on the method described in Sect. 4.2, if the system does not have any solutions, then $l_{ik} - l_{kk} > b_i - b_k$ and $b_i - l_{ik} < z_k \leq b_k - l_{kk}$ for some $2 \leq i \leq n$ and $1 \leq k \leq i - 1$. As such, we have $l_{ik} + z_k > b_i$, where $z_k$ is obtained from the $k$-th equation of the system $UX = Z$. That means

$$
l_{ik} + z_k = l_{ik} + \max(u_{kk} + x_k, \ldots, u_{kn} + x_n) = \max(a_{ik} - a_{kk} + a_{kk} + x_k, \ldots, a_{ik} - a_{kk} + a_{kn} + x_n).
$$

(8)

Since the matrix $A$ has an $LU$-factorization, then Theorem 3 leads to

$$(LU)_{ij} = \max_{r=1}^{\min\{i,j\}-1} \left( \det(A[\{i\} | \{j\}]) - a_{ij} \right) = \max_{r=1}^{\min\{i,j\}-1} \max(a_{ij}, a_{ij} - a_{rn} + a_{rn}) = a_{ij},$$

where $r = \min\{i,j\} - 1$. As such, $a_{ik} - a_{kk} + a_{kj} \leq a_{ij}$, for any $k + 1 \leq j \leq n$, since $1 \leq k \leq i - 1 (k < \min\{i,j\})$. Due to (8), the following inequality is obtained

$$l_{ik} + z_k \leq \max(a_{ik} + x_k, a_{i(k+1)} + x_{k+1}, \ldots, a_{in} + x_n).$$

Moreover, in the $i$-th equation of the system $LZ = b$, we have $l_{ik} + z_k > b_i$. Consequently, in the $i$-th equation of the system $AX = b$, we get $\max(a_{i1} + x_1, \ldots, a_{in} + x_n) > b_i$ which means $AX = b$ has no solution. \qed

**Remark 3** Take the system $AX = b$. Let $A \in M_n(S)$ have an $LU$-factorization and $UX = Z$. If $LZ = b$ has some solution, but $UX = Z$ does not have any solutions, then $AX = b$ has no solution.

### 5 Solving linear systems in other idempotent semifields

Throughout this section, let $S$ be an idempotent semifield and $A = (a_{ij}) \in M_n(S)$ with $\det(A) = a_{11} \otimes \cdots \otimes a_{nn}$. Take a lower triangular matrix $L$ and an upper triangular matrix $U$ over $S$ as follows.

$$L = (l_{ij}); \quad l_{ij} = \begin{cases} a_{ij} \otimes a_{ij}^{-1} & \text{if } i \geq j \\ 0 & \text{otherwise}, \end{cases}$$

and

$$U = (u_{ij}); \quad u_{ij} = \begin{cases} a_{ij} & \text{if } i \leq j \\ 0 & \text{otherwise}. \end{cases}$$
We say \( A \) has an \( LU \)-factorization if \( A = LU \).

**Theorem 8** Let \( A \in M_n(S) \) and \( n \) be a positive integer such that \( \det(A) = a_{11} \otimes \cdots \otimes a_{nn} \) and matrices \( L \) and \( U \) be defined as above. Then \( A = LU \) if and only if for any \( 1 < i, j \leq n \) and \( i \neq j \),

\[
a_{ij} = \bigoplus_{k=1}^{r}(\det(A\{ \{ k, i \} \mid \{ k, j \} \}) \otimes a_{kk}^{-1}),
\]

where \( r = \min\{i, j\} - 1. \)

**Proof** Note that \( S \) is additively idempotent and the total order \( \preceq_S \) on \( S \) is compatible with addition and multiplication. Moreover, the total order \( \preceq_S \) induces a partial order on \( S \), which is stated in Sect. 2. Considering these properties, the proof is similar to that of Theorem 3. \( \square \)

**Theorem 9** Let \( LX = b \) be a linear system of equations with a lower triangular matrix \( L \in M_n(S) \) and a regular vector \( b \in S^n \). Then the system \( LX = b \) has the maximal solution \( X^* = (b_i \otimes l_{ii}^{-1})_{i=1}^{n} \) if \( l_{ik} \otimes l_{kk}^{-1} \preceq_S b_i \otimes b_k^{-1} \), for any \( 2 \leq i \leq n \) and \( \leq k \leq i - 1 \). Moreover, if all the inequalities are proper, then the maximal solution \( X^* \) of the system is unique.

**Proof** The proof is by induction on \( i \). For \( i = 2 (k = 1) \), since \( l_{21} \otimes l_{11}^{-1} \preceq_S b_2 \otimes b_1^{-1} \) and \( x_1 = b_1 \otimes l_{11}^{-1} \), we have \( l_{21} \otimes x_1 \preceq_S b_2 \). As such, the second equation of the system, \( (l_{21} \otimes x_1) \otimes (l_{22} \otimes x_2) = b_2 \), implies \( x_2 \preceq_S b_2 \otimes l_{22}^{-1}. \)

Suppose that for \( i \leq m - 1 \), the statements are true , i.e. \( x_i = b_i \otimes l_{ii}^{-1} \) and \( x_i \preceq_S b_i \otimes l_{ii}^{-1}, \) for any \( 2 \leq i \leq m - 1 \). Let \( i = m, (k = 1, \ldots , m - 1) \). Then the proof is similar to that of Theorem 4. Hence, the system \( LX = b \) has the maximal solution \( X^* = (b_i \otimes l_{ii}^{-1}). \)

**Remark 4** Similarly, we can extend Theorems 5, 6 and 7 to other idempotent semifields as follows.

1. Let \( UX = b \) be a linear system of equations with an upper triangular matrix \( U \in M_n(S) \) and a regular vector \( b \in S^n \). Then the system \( UX = b \) has the maximal solution \( X^* = (u_{i} \otimes u_{ii}^{-1})_{i=1}^{n} \) if \( u_{(n-i)k} \otimes u_{kk}^{-1} \preceq_S u_{i} \otimes b_i^{-1} \), for any \( 1 \leq i \leq n - 1, n - i + 1 \leq k \leq n \). Moreover, if the inequalities \( u_{(n-i)k} \otimes u_{kk}^{-1} \preceq_S u_{i} \otimes b_i^{-1} \) are proper, then the maximal solution \( X^* \) of the U-System \( UX = b \) is unique.

2. Let \( A \in M_n(S) \) and suppose \( A \) has \( LU \) factors given by (1) and (2), respectively. The system \( AX = b \) has the maximal solution \( X^* = (a_{i} \otimes a_{ii}^{-1})_{i=1}^{n} \) if \( a_{ik} \otimes a_{kk}^{-1} \preceq_S b_i \otimes b_k^{-1}, \) and \( a_{(n-j)k} \otimes a_{kk}^{-1} \preceq_S b_{(n-j)} \otimes b_k^{-1}, \) for every \( 2 \leq i \leq n, \quad 1 \leq k \leq i - 1, \quad 1 \leq j \leq n - 1, \quad n - j + 1 \leq l \leq n \). Furthermore, the system \( AX = b \) has the unique solution \( X^* = (a_{i} \otimes a_{ii}^{-1})_{i=1}^{n} \) if the inequalities \( a_{ik} \otimes a_{kk}^{-1} \preceq_S b_i \otimes b_k^{-1} \) and \( a_{(n-j)k} \otimes a_{kk}^{-1} \preceq_S b_{(n-j)} \otimes b_k^{-1} \) are proper.

3. Let \( A \in M_n(S) \). Suppose \( A \) has an \( LU \)-factorization. If the system \( LZ = b \) has no solution, then neither does the system \( AX = b \).

Moreover, for solving lower and upper triangular systems of equations over idempotent semifields, we can devise methods similar to the one given in Sect. 4.2 for max-plus algebra.
Example 6 Let $A \in M_4(S)$ where $S = \mathbb{R}_{\min}$. Consider the following system $AX = b$:

$$
\begin{bmatrix}
1 & 6 & 9 & 8 \\
6 & 2 & 7 & 5 \\
9 & 7 & 1 & 7 \\
8 & 5 & 6 & 3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= 
\begin{bmatrix}
4 \\
6 \\
1 \\
6
\end{bmatrix},
$$

where $\det(A) = a_{11} \times a_{22} \times a_{33} \times a_{44} = 6$. We use the presented $LU$-method to solve this system. Due to the extension of the Theorem 6 in Remark 4, we must first solve the system $LZ = b$:

$$
\begin{bmatrix}
1 & +\infty & +\infty & +\infty \\
6 & 1 & +\infty & +\infty \\
9 & 7/2 & 1 & +\infty \\
8 & 5/2 & 6 & 1
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2 \\
z_3 \\
z_4
\end{bmatrix}
= 
\begin{bmatrix}
4 \\
6 \\
1 \\
6
\end{bmatrix}.
$$

Since $l_{ik} \otimes f_{kk}^{-1} \leq_S b_{i} \otimes b_{ii}^{-1}$ for any $2 \leq i \leq 4$ and $1 \leq k \leq i - 1$, the maximal solution of the system $LZ = b$ is $(b_{i} \otimes b_{ii}^{-1})_{i=1}^{d} = (b_{i})_{i=1}^{d}$:

$$
Z = 
\begin{bmatrix}
4 \\
6 \\
1 \\
6
\end{bmatrix}.
$$

We shall now solve the system $UX = Z$:

$$
\begin{bmatrix}
1 & 6 & 9 & 8 \\
+\infty & 2 & 7 & 5 \\
+\infty & +\infty & 1 & 7 \\
+\infty & +\infty & +\infty & 3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= 
\begin{bmatrix}
z_1 \\
z_2 \\
z_3 \\
z_4
\end{bmatrix}.
$$

By the extension of the Theorem 5 in Remark 4, the maximal solution is $X^* = (z_{i} \otimes u_{ii}^{-1})_{i=1}^{d}$ or in fact $X^* = (b_{i} \otimes u_{ii}^{-1})_{i=1}^{d}$, since $u_{(4-i)k} \otimes u_{kk}^{-1} \leq_S b_{4-i} \otimes b_{kk}^{-1}$ for any $1 \leq i \leq 3$ and $4 - i + 1 \leq k \leq 4$:

$$
X^* = 
\begin{bmatrix}
4 \\
3 \\
2
\end{bmatrix}.
$$

It should be noted that $a \leq_S b$ means that $a \geq b$ for any $a, b \in \mathbb{R}_{\min}$, where “$\geq$” is the standard greater than or equal relation over $\mathbb{R}_{+}$.

6 Concluding remarks

In this paper, we extended the $LU$-factorization technique to idempotent semifields. We stated the criteria under which a matrix can have $LU$ factors, and when it does, what the factors look like. Importantly, we used the results in solving linear systems of equations when the solution exists. One can use these $LU$ factors in the design of numerical
algorithms in idempotent semifields. Other important properties of these \(LU\) factors can also be studied especially relative to well-known classic results from linear algebra.

**Appendix**

See Tables 1, 2, 3, 4, 5 and 6.

**Table 1** Finding the determinant of a square matrix in max-plus

```
MaxPlusDet := proc (A::Matrix)
    local i, j, s, n, detA, ind, K, V;
    description "This program finds the determinant of a square matrix in max-plus."
    Use LinearAlgebra in
    n := ColumnDimension(A);
    V := Matrix(n);
    ind := Vector(n);
    if n = 1 then
        V := A[1, 1];
        detA := V;
        ind[1] := 1
    elif n = 2 then
        V[1, 1] := A[1, 1] + A[2, 2];
        V[1, 2] := A[1, 2] + A[2, 1];
        V[2, 1] := A[1, 2] + A[2, 1];
        V[2, 2] := A[1, 1] + A[2, 2];
        detA := max(V);
        for s to 2 do
            K := V[s, 1 .. 2];
            ind[s] := max[index](K)
        end do;
    else
        for i to n do
            for j to n do
                V[i, j] := A[i, j] + op(1, MaxPlusDet(A[[i .. i-1, i+1 .. n], [j .. j-1, j+1 .. n]]));
            end do;
            detA := max(V);
            K := V[i, 1 .. n];
            ind[i] := max[index](K);
        end do;
    end if;
end use;
[detA, ind, V]
end proc:
```

**Table 1** Finding the determinant of a square matrix in max-plus
Table 2  Finding the permutation matrix of a square matrix in max-plus

\begin{verbatim}
Proc := proc (A::Matrix)
local i, n, I, V, d, L, j, L1, P;
description "This program finds the permutation matrix based on the determinant of a square matrix in max-plus.";
use LinearAlgebra in
n := ColumnDimension(A);
d := op(1, MaxPlusDet(A));
P := Matrix(1 .. n, 1 .. n, (-1)*Float(infinity));
V := op(3, MaxPlusDet(A));
L := [];
L1 := [];
for i to n do
  for j to n do
    if V(i, j) = d then
      if j in L then
        L1 := [op(L1), j]
      else
        L := [op(L), j];
        break
      end if;
    end if;
  end do;
P[j, i] := 0
end do;
end use;
P
end proc;
\end{verbatim}

Table 3  Calculation of matrix multiplication in max-plus

\begin{verbatim}
Matmul := proc (A::Matrix, B::Matrix)
local i, j, m, n, p, q, C, L;
description "This program finds the multiplication of two matrices in max-plus.";
use LinearAlgebra in
m := RowDimension(A);
n := ColumnDimension(A);
p := RowDimension(B);
q := ColumnDimension(B);
C := Matrix(m, q);
if n <> p then
  print('Impossible');
  break
else
  for i to m do
    for j to q do
      L := [seq(A[i, k]+B[k, j], k = 1 .. n)];
      C[i, j] := max(L)
    end do;
  end if;
end use;
C
end proc;
\end{verbatim}
Table 4  Calculating the LU factors of a square matrix in max-plus

\[
\begin{align*}
\text{maxLU} & := \text{proc (A::Matrix)} \\
& \text{local i, j, m, n, k, h, V, L, U, P, B, s, r;}
\end{align*}
\]

Use LinearAlgebra in

\[
\begin{align*}
n & := \text{ColumnDimension}(A); \\
L & := \text{Matrix}(1 .. n, 1 .. n, (-1)\ast\text{Float}(\text{infinity})); \\
U & := \text{Matrix}(1 .. n, 1 .. n, (-1)\ast\text{Float}(\text{infinity})); \\
P & := \text{Pmat}(A); \\
B & := \text{Matmul}(P, A);
\end{align*}
\]

for h to n do

\[
\begin{align*}
L[h, 1] & := B[h, 1] - B[1, 1]; \\
U[1, h] & := B[1, h]; \\
U[h, h] & := B[h, h]; \\
L[h, h] & := 0;
\end{align*}
\]

end do;

for i from 2 to n do

\[
\begin{align*}
\text{for j from 2 to n do}
& \text{if i <> j then}
& \quad r := \text{min}(i, j) - 1;
& \quad V := \text{Vector}(r);
& \quad \text{for k to r do}
& \quad \quad V[k] := \text{max}(B[i, k] + B[k, j], B[i, j] + B[k, k]) - B[k, k];
& \quad \text{end do;}
& \quad s[i, j] := \text{max}(V);
& \quad \text{if s[i, j] = B[i, j] then}
& \quad \quad \text{if j < i then}
& \quad \quad \quad L[i, j] := B[i, j] - B[j, j];
& \quad \quad \quad \text{elif i < j then}
& \quad \quad \quad \quad U[i, j] := B[i, j];
& \quad \quad \quad \text{end if;}
& \quad \quad \text{else}
& \quad \quad \quad \text{print('No solution');}
& \quad \quad \quad \text{break}
& \quad \quad \text{end if;}
& \quad \text{end if;}
& \text{end do;}
\end{align*}
\]

end if;

end use:

\[
\begin{align*}
[P, L, U]
\end{align*}
\]

end proc:
Table 5  Solving the system $LX = b$ in max-plus

```maple
maxLsystem:=proc(L::Matrix, b::Vector)
local i, k, n, x, c, V;
description "This program solves a lower triangular system in max-plus."
use LinearAlgebra in
n:= ColumnDimension(L);
x := Vector(n);
c[1] := 1; \#(Equality Flag)
x[1] := b[1]-L[1, 1];
for i from 2 to n do
  for k to i-1 do
    V:= Vector(i-1);
    if x[k]<b[i]-L[i, k] then
      V[k]:= 1;
x[i]:= b[i]-L[i,i];
    elseif x[k]= b[i]-L[i, k] then
      x[i]:= b[i]-L[i,i];
      print('x'[i]<= b[i]- L[i, i]);
    elseif x[k]> b[i]-L[i, k] then
      if c[k]= 1 then
        print('no solution');
        break
      end if;
    end if;
    if max(V)= 1 then
      c[i]:= 1;
    else
      c[i]:=0;
    end if:
  end do:
end use:
x
end proc:
```
Table 6 Solving the system $UX = b$ in max-plus

```plaintext
maxUsystem:=proc(U::Matrix, b::Vector)
local i, k, n, x, c, V;
description "This program solves an upper triangular system in max-plus."
use LinearAlgebra in
n:= ColumnDimension(U);
x := Vector(n);
c[n] := 1; \#(Equality Flag)
x[n] := b[n]-U[n, n];
for i from 2 to n do
  for k to i-1 do
    V:= Vector(i-1);
    if x[n+1- k]<b[n+1- i]-U[n+1- i, n+1- k] then
      V[n+1- k]:= 1;
      x[n+1- i]:= b[n+1- i]-U[n+1-i, n+1- i];
    elif x[n+1- k]= b[n+1- i]-U[n+1- i, n+1- k] then
      x[n+1- i]:= b[n+1- i]-U[n+1- i, n+1- i];
      print('x'[n+1- i]<= b[n+1- i]- U[n+1- i, n+1- i]);
    elif x[n+1- k]> b[n+1- i]-U[n+1- i, n+1- k] then
      if c[n+1- k]= 1 then
        print('no solution');
        break
      elif c[n+1- k]=0 then
        x[n+1- k]:= b[n+1- i]-U[n+1- i, n+1- k];
        x[n+1- i]:= b[n+1- i]- U[n+1- i, n+1- i];
        V[n+1- k]:= 1;
        print('x'[n+1- k]<= b[n+1- i]- U[n+1- i, n+1- k]);
      end if;
    end if;
  end do:
  if max(V)= 1 then
    c[n+1-i] := 1;
  else
    c[n+1-i] := 0;
  end if:
end do:
end use:
x
end proc:
```

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