Abstract

The Move language provides abstractions for programming with digital assets via a mix of value semantics and reference semantics. Ensuring memory safety in programs with references that access a shared, mutable global ledger is difficult, yet essential for the use-cases targeted by Move. The language meets this challenge with a novel memory model and a modular, intraprocedural static reference safety analysis that leverages key properties of the memory. The analysis ensures the absence of memory safety violations in all Move programs (including ones that link against untrusted code) by running as part of a load-time bytecode verification pass similar to the JVM [12] and CLR [15]. We formalize the static analysis and prove that it enjoys three desirable properties: absence of dangling references, referential transparency for immutable references, and absence of memory leaks.

1 Introduction

The Move language [4] provides abstractions for writing safe smart contracts [24, 26] via a mix of value semantics and reference semantics. Programmers can use value semantics and the move operator to transfer ownership of an asset to another user or to a procedure. For example, a procedure signature like fun buy(c: Coin): Car intuitively says: "if you give me ownership of a coin, I will give you ownership of a car". By contrast, references enable programmers to temporarily share a value for reading or writing. The signature fun register(c: &Car, fee: Coin): Registration says "if you show me that you own a car and pay a fee, I will give you a registration for the car".

However, careless mixing of value and reference semantics can lead to memory safety issues. Code like let x = 5; return &x creates a dangling reference to local memory (as in C). In addition, the move operator introduces new kinds of reference errors: let y = move x; (*x ("use-after-move", similar to a null dereference) and let y = &x; let z = move x; (y is a dangling reference to moved memory).

Smart contract languages like Move must prevent memory safety issues by design because they need to support deterministic execution in an adversarial environment.

Deterministic execution. A blockchain is a replicated state machine [11, 21] where replicas are known as validators. Users in the system send transactions (i.e., programs in a language like Move) to a validator that advances the state machine. The validators execute a consensus protocol (e.g., [5, 17]) to agree on the ordering and results of executing transactions. State-machine replication requires determinism; if execution is nondeterministic, validators may not achieve consensus and the system cannot make progress. Since violation of memory safety typically manifests as undefined behavior, the resulting nondeterminism could stall the entire system.

Adversarial environment. Smart contracts deployed on a blockchain store digital assets with real-world value, but must tolerate arbitrary interactions with untrusted code running in the same address space. Even if the deployed code is completely safe on its own, an attacker can intentionally write memory-unsafe code that attempts to break the integrity of the deployed code. Thus, Move must prevent memory safety issues in both the deployed code and the attacker code.

Existing approaches are inadequate. Unfortunately, traditional approaches for ensuring memory safety are not suitable for Move's deployment model. For example, Move cannot rely on a source language with a strong type system (e.g., Rust, OCaml) to prevent memory safety issues because Move bytecode is stored and executed on the ledger. An attacker that wishes to subvert the source-level type system can write and deploy bytecode directly.

Similarly, Move cannot utilize GC for safe memory management. Like other blockchain languages, Move uses gas metering [8] to provide a deterministic metric for execution cost of contracts. The gas cost of a program must closely track its execution time to avoid denial-of-service vulnerabilities [19]. The unpredictable nature of GC does not mix well with the precise accounting required by gas metering.

Contributions. Move must support a rich programming model. An account-based blockchain is a shared, mutable global ledger. A single transaction can mutate an arbitrary number of accounts in the ledger (e.g., by sending funds to each).
Move must expose a programming model that supports mutable references to global state, but without sacrificing reference safety.

Move addresses safe programming of smart contracts by chaining together three new ideas. First, memory management in Move is built atop a forest of tree-shaped values. The roots of this forest are either local variables on the call stack or global memory indexed by a statically-known type and dynamically determined account address. Move allows the creation of references to values and embedded sub-values, but does not allow references to be embedded inside a value. This design ensures that a reference can be canonically represented as a rooted path of fields.

Second, Move imposes the abstraction of a directed and acyclic borrow graph over the forest of values comprising the program memory. The nodes in this borrow graph are abstract locations representing values on the call stack, values indexed by a particular type in global memory, or references stored on the call stack or operand stack. Each edge in the borrow graph is labeled by a relative path indicating an ownership transfer along that path.

Third, Move provides an intraprocedural static analysis to automatically check that a Move bytecode program adheres to the ownership discipline enforced by the borrow graph abstraction. This analysis ensures important properties —no memory leaks, no dangling references, and referential transparency— on Move bytecode programs.

We have implemented our analysis in the Move bytecode verifier which is run whenever new bytecode is loaded into the ANON blockchain[3], in a manner similar to the JVM [12] and CLR [15]. Therefore, runtime memory-safety properties are enforced on all loaded bytecode regardless of provenance. Our implementation is fast; we report numbers in §6.

We have also implemented our analysis in the compiler for the Move source language. The Move language is being used by ANON developers to implement the rules governing the ANON Payment Network. Anecdotal experience indicates that the source borrow checker is a useful tool that helps developers write safe code.

## 2 Language Design for Reference Safety

In this section, we will first give an overview of memory management in the Move language with emphasis on the key feature for facilitating static analysis: tree-shaped values that enable us to represent reference values as structured paths (§2.1).

Next, we present a static analysis that builds on these features to prevent dangling references. We organize our discussion around the three primary challenges our analysis faces: preventing dangling references to memory in the same procedure (§2.2), to memory in different procedures (§2.3), and to global memory (§2.4). The general reference analysis leverages Move’s type-indexed global memory and encapsulation features to enforce a global property with intraprocedural checks.

Each of the analysis sections contain examples of bad Move code that must be rejected. The analysis operates on Move bytecode, but we will write our examples in Move source code for readability. The table below summarizes how source code instructions compile to stack-based bytecode instructions.

| Move Source Code | Stack Instruction |
|------------------|-------------------|
| x = &y            | BorrowLoc(y); StoreLoc(x) |
| x = &y.f          | BorrowField(f, y); StoreLoc(x) |
| x = *y            | CopyLoc(y); ReadRef; StoreLoc(x) |
| *x = y            | MoveLoc(y); BorrowLoc(x); WriteRef |

Although the source language abstracts away some bytecode features such as the operand stack, we have chosen examples that capture the essence of reference issues in the bytecode.

### 2.1 Memory management

Move has three kinds of storage:

1. **Procedure-local variables** Each procedure frame in the call stack has a fixed set of local variables that are uninitialized at the beginning of a procedure. Initialized variables can store values and references.

2. **Shared operand stack** All procedures in the call stack share a single operand stack that can store both values and references. Procedures can use the operand stack both for local computation and to share arguments/return values with other procedures. At the beginning of program execution, the call stack holds one frame and the operand stack is empty. The same conditions must hold for a program to terminate without an error.

3. **Global storage** Move has no global variables, no heap, and cannot access the filesystem or network. However, programs can access persistent data by reading from and writing to shared global storage that persists across program executions. Global storage is organized as a partial map from 16 byte account addresses to record values: (Addr × Type) → Record. § 2.4 explains the design of and motivation for the global storage in more detail.

**Values are Tree-Shaped.** Move has primitive data values, nominally typed records, resizable vectors, and reference values. Move programmers can create references that point into local variables or into global storage, but not into the operand stack. Reference types are tagged with permissions: either mutable (written `&mut`) or immutable (written `&`). References to other references (e.g., `&u64`) are not allowed.

Both records and vectors can store primitive values, and other records/vectors, but not references. Global storage can hold records, but not references. This ensures that non-reference values and the global storage are always tree-shaped.

**References are Structured Paths.** In a byte-addressable memory, a reference value is a dynamically determined index
into the array of memory. This unstructured representation makes it difficult to reason about the relationship between two different reference values—e.g., “can writing through reference i change the memory pointed to by reference j?”

By contrast, Move storage, values, and reference-related instructions are designed to ensure that a reference can be represented as a structured access path [9]. For example: say we have a record value \( f: \{ g: 1 \}, h: [2, 3] \) of type \( T \), where the \( [\) syntax denotes a record and the \( ] \) syntax denotes a vector. If this value is stored in a local variable \( x \), we can represent a reference to the value stored by field \( g \) as the path \( x/f/g \). Similarly, we can represent a reference stored at index 1 of the vector in field \( h \) as \( x/h/1 \). If the same record value is in global storage at account address \( A \), we can represent these paths as \( A/T/f/g \) and \( A/T/h/1 \).

A path is a canonical representation for a particular location in memory.Syntactically distinct paths refer to distinct memory locations. In addition, the structured nature of paths introduces a partial order on reference values. Two reference values either have a prefix relationship (e.g., \( x/f \) is a prefix of \( x/f/g \)), an extension relationship (e.g., \( x/f/g \) is an extension of \( x/f \)), or are incomparable (e.g., \( y/f/g \) and \( x/f/g \)). Writing to a path cannot change the memory named by incomparable paths. As we will see, our reference safety analysis takes advantage of these nice properties to simplify static reasoning about code that uses references.

### 2.2 Preventing Dangling References to Locals

We begin our discussion of the reference safety analysis with a pair of code snippets that exemplify the problems the analysis must prevent. The code comments in the snippet show the abstract state of the analysis, but we will ignore them at first.

The program below creates a reference \( r \) to the \( f \) field of formal parameter \( c \), moves the value stored in \( c \) into \( x \) and then reads \( r \). The `move` assignment works by assigning the value bound to \( c \) to \( x \) and then “clearing” \( c \) by assigning it to \( \bot \).

```rust
fun dangle_after_move(c: Coin) {
    let r = &c.f; // c⇒r
    let x = move c; // ⊥
    let y = *r; // read from dangling ref!
}
```

The ability (and in some cases, requirement) to move values instead of copying them is a key feature of the Move language—it prevents “double spending” of monetary values like \( c \). However, the move causes the reference value stored in \( r \) to become dangling.

Similarly, if the programmer creates a reference to a value that is overwritten via a destructive update, a dangling reference may result. The snippet below creates a vector \( v \) of size 1, acquires a reference \( ref \) into index 0 of the vector, and then reassigns \( v \) to an empty vector via the write to \( v_{ref} \). As a result, the write \( *ref \) accesses a dangling reference.

```rust
let v = Vector::singleton(5);
```

### Ownership and Borrowing

Our analysis enforces a programming discipline based on ownership to prevent the problems above. A location in memory (either a local variable, stack location, or global key) of type \( T \) is the owner of the value it stores. A value of type \( T \) can only be moved or (if \( T \) is a mutable reference) written via its owning location. However, a value stored in a local or global can be borrowed from its owner by creating a reference to it (e.g., \( &v \)) or extending an existing reference (e.g., \( &ref. f \)). The analysis will not allow the value to be moved or written until all borrows have ended (i.e., the reference values produced by the borrows have been destroyed). This discipline has a natural recursive structure: if \( y \) borrows from \( x \) and then \( z \) subsequently borrows from \( y, x \) does not regain ownership until both the \( y \) and the \( z \) borrows have ended.

As we will show in §4, a program that follows these rules cannot create a dangling reference by writing path \( p \) when a reference path that is a strict extension of \( p \) exists elsewhere or moving a variable \( x \) when a reference path that is an extension of \( x \) exists elsewhere. As a side benefit, this discipline ensures referential transparency for immutable references.

### Borrow Analysis Mechanics

The key piece of analysis state is a borrow graph where nodes represent references or values and a directed edge \( A \xrightarrow{p} B \) means “path \( A/p \) is borrowed by \( B \)”. Here, \( A/p \) is an abstraction of the path representation described in §2.1 extended with some additional components: \( e \) for a direct borrow of a local variable and \( * \) for all suffixes of a path. The analysis adds a borrow edge when a reference value is created and eliminates borrow edges when a reference value is destroyed (e.g., popped off the stack or consumed by an instruction).

Returning to our examples above, each instruction is annotated with the borrow graph after it executes. A \( ⊥ \) indicates that the borrow checker would reject the program after processing the instruction. The first program is straightforward: the \( &c.f \) instruction creates a borrow of \( c \), then the analysis rejects the program at \( move \ c \) because there is a borrow edge rooted in \( c \).

In the second program, the borrow edge \( v \xrightarrow{v_{ref}} \) represents that the local \( v_{ref} \) holds a direct reference to the vector in \( v \). The next instruction `let ref = &mut v[0]` generates the edge \( v \xrightarrow{ref} \). This is our first encounter with abstraction in the analysis: the \( * \) means that the analysis does not know which offsets of \( v \) have been borrowed by \( ref \). Finally, the analysis chooses to reject the write to \( v_{ref} \) because it borrows from \( v \) and the \( * \) edge represents an outstanding borrow on \( v \).
2.3 Borrow Discipline Across Procedures

Move procedures can accept reference arguments and return references, which introduces new ways to create dangling references. However, we can extend the borrow discipline and its corresponding analysis to prevent these as well.

First, we consider the problem of returning dangling references. Each procedure below returns a reference value to its caller by pushing the return value on the operand stack (we write $S_i$ for the $i$th operand stack slot) and then executing the Ret instruction (not shown). The first two procedures in the snippet below return dangling references to local variables of the procedure, but the second two return safe references to memory that will outlive the procedure.

```plaintext
fun ret_local(): &u64 ( let x = 7; &x /= x ⇒ $S_0$ */ ) †
fun ret_param(x: u64): &u64 ( &x /= x ⇒ $S_0$ */ ) †
fun ret_ref_param(x: &u64): &u64 ( x /= 0 */ ) †
fun ret_borrowed_param(s: &S): &u64 ( &s.f /= s/s ⇒ $S_0$/* ) †
```

Each procedure shows the borrow graph before the procedure returns. The analysis will not allow a return value to have any borrow of a local variable (ret_local) or a formal with a non-reference type (ret_param). The ret_ref_param procedure is safe because it has no borrows—it returns a copy of a reference parameter. Finally, ret_borrowed_param has a borrow of formal s, but this is ok because s is a reference parameter rather than a local. The analysis has access to a procedure’s type information, and it also tracks the type of each stack location in a separate abstract domain.

Procedure Calls Require Ownership. Lastly, we consider the problem of handling procedure calls. Our static analysis is modular, so it must soundly summarize the effects of a procedure call with no information other than the callee’s type signature. A reference value returned by a callee is conservatively assumed to be borrowed from all of the procedure’s reference arguments, with extra care to avoid conflating immutable and mutable references (see the Call rule in §4 for details).

A subtle consequence of the borrow discipline described in the previous section is that every non-reference value pushed on the operand stack has no outstanding borrows. Since procedures pass arguments to their callees on the operand stack, this ensures that a caller cannot retain references to a value passed to a callee. With this possibility out of the way, the only danger that remains is a dangling reference caused by a reference value written in a callee. To prevent this, the analysis enforces a single rule: an argument with a mutable reference type must not have any outstanding borrows.

The two examples below show unsafe call sites. In both cases, the arguments are evaluated left to right and pushed on the operand stack before the Call instruction. The borrow graphs are at the program point before this instruction. The analysis rejects each example because stack slot $S_1$ is passed as an argument, but is borrowed by $S_0$.

```plaintext
a(&mut x, &mut x) // x ⇒ $S_1$, $S_1$ ⇒ $S_0$ †
b(&mut y, &mut y) // y ⇒ $S_1$, $S_1$ ⇒ $S_0$ †
```

In addition to preventing dangling references, this strategy for handling calls ensures a very useful property: a mutable reference parameter cannot alias with any other parameter! Eliminating aliased mutable data allows Move programmers to write procedures without defensive checks for aliasing and greatly facilitates precise and scalable static program verification in the Move Prover [6].

2.4 Dangling References to Global Memory

We conclude our informal presentation of the analysis by considering the thorny problem of allowing mutable access to global memory while preventing dangling references. At first blush, this might seem impossible to do with a modular analysis—the whole point of global memory is that you can access it from anywhere! However, Move’s global memory instructions are carefully designed to enable local reasoning about the safety of global memory accesses. The two analysis extensions described in this section build on this design to prevent dangling references to global memory using the existing borrow analysis machinery.

Global Memory Indexed By Encapsulated Types. Move’s global memory is a partial map indexed by a pair of a statically chosen record type $T$ and a dynamically chosen account address $a$. A value stored at key $(T, a)$ is a record of type $T$. Record types are declared in modules consisting of type and procedure declarations. The following table summarizes the global state operations available in Move.

| Operation          | Description                                      |
|--------------------|--------------------------------------------------|
| `move_to<T>(a, T)` | Publish $T$ at address $a$                       |
| `move_from<T>(a)`  | Remove $T$ from $a$                              |
| `borrow_global<T>(a): &mut T` | Get mutable ref to $T$ at $a$                     |

Programmers can publish a value of type $T$ to an address in global state, remove the value of type $T$ stored at an address, and acquire references to a value already published in global state. Modules encapsulate access to their declared types; in particular, the global state access operations can only be used on a type declared inside the current module.

The decision to include a type in these operations simplifies static reasoning about aliasing of locations in global memory. Two global access operations involving keys $(T_1, a_1)$ and $(T_2, a_2)$ can only touch the same memory if $T_1 = T_2$ and $a_1 = a_2$. Combined with type encapsulation, this means that global accesses in distinct modules touch distinct memory by construction. Without this property, a local reference safety analysis would not be practical.

Abstracting Dynamic Global Accesses. Each instruction that accesses global memory indexes it using an address value chosen at runtime. This means that code like:
fun address_aliasing(a1: address, a2: address) acquires T {
    let t_ref = borrow_global<T>(a1); // T⇒t_ref
    let t = move_from<T>(a2); // T⇒t
    let t_ref = ... // accessing a dangling ref
}

may create a dangling reference if a1 and a2 are bound to the same address value. Similarly, performing a move_to<T>(a, T while a reference created by a borrow_global<T>(a) is still active creates a dangling reference.

Rather than attempting to reason about address equality, our analysis conservatively assumes that all global accesses indexed by type T touch the same address. This decision suggests a simple extension to the borrow graph: a T node that abstracts all concrete cells in global memory keyed by type T. In the example above, the analysis introduces a borrow from T at the borrow_global instruction and rejects the program at move_from<T> because there is an active borrow on T. This is exactly how the analysis deals with programs like let x = &y; let z = move y only with T in place of x and global access instructions in the place of local ones.

Global References Cannot Be Returned. The scheme described so far prevents dangling references to a global memory cell of type T inside the module that declares T. Only one issue remains: leaking a reference to a global cell of type T outside of the module which T is declared:

module M1 {
    fun f(a: address): &mut T acquires T { borrow_global<T>(a) }
    fun g(a: address): T acquires T { move_from<T>(a) }
}

module M2 {
    fun bad(a: address) {
        let ref = M1::f(a);
    }
}

let t = M1::g(a); // ref now dangling!
}

The possibility of this leakage undermines our efforts to modularize reasoning about reference invalidation. Thus, our analysis prevents it by banning returns of references to global memory. This happens implicitly by following the return discipline described in §2.3: only borrow references parameter may remain on the stack when a procedure returns, and a global is not a reference parameter.

3 Move Operational Semantics

In this section, we formalize the operational semantics of a subset of the Move language chosen to illustrate the key challenges of reference safety.

Partial functions and lists. We use partial functions to represent record values and for mappings that are parts of semantic states. Lists are used for sequences of field accesses and for the stack component of semantic states.

Following common convention, if f : A → B is a partial function, then dom(f) is its domain and img(f) is its range. We write f[a⇒b] for the function that is the same as f on every argument except a, and which maps a to b. Similarly, f \ a is the partial function equivalent to f except that it is undefined at a.

We write [] for the empty list and e :: l for the result of placing e at the front of list l. Similarly, l :: l' is the list with e appended to l and, by slight overloading of notation, l :: l' is the concatenation of lists l and l'.

Types. Let PrimType be the set of primitive types, including Bool (of Boolean values), Int (of integers), and Addr (of account addresses). Let F be a fixed, finite set of field names. The set ValType of value types is the least set satisfying:

1. PrimType ⊆ ValType; and
2. if f1 ... fn ∈ F are pair-wise distinct and t1 ... tn ∈ ValType, then \{(f1, t1), ..., (fn, tn)\} ∈ ValType.

Let ImmRefType = {Imm(t) | t ∈ ValType}, MutRefType = {Mut(t) | t ∈ ValType}, RefType = ImmRefType ∪ MutRefType, and T = ValType ∪ RefType.

Values. We define the set of values used in computation inductively. Let PrimVal be the set of primitive data values. The set Val of values is the least set satisfying:

1. PrimVal ⊆ Val; and
2. if f1 ... fn ∈ F are pair-wise distinct and v1 ... vn ∈ Val, then \{(f1, v1), ..., (fn, vn)\} ∈ Val.

The judgment v : t, indicating that value v has type t, is defined in the natural way.

Paths. A path is a possibly empty list of field names, representing a sequence of field selections. A path represents a way to start at any location in memory and follow a sequence of field selections to reach a value. It is helpful to visualize a value as a labeled tree whose edges are labeled by field names. A primitive value is a tree consisting of a single leaf. The tree associated with a record value consists of a node and a subtree
for each record component. If $r$ is a record value, then for each $(f,o) \in r$, the edge from $r$ to the subtree for $o$ is labeled by $f$.

Two useful operations on values are (1) the subtree $v[p]$ of $v$ located at path $p$, and (2) the tree $v[p := v']$ obtained by replacing the subtree at path $p$ with tree $v'$. Using the association between values and trees, $v[p]$ is the subtree reached by path $p$ from root $v$, and $v[p := v']$ is the tree with the subtree $v[p]$ replaced with $v'$.

**Concrete states.** The set $Loc$ is an uninterpreted set of locations; the memory of a program is a partial map from $Loc$ to $Val$. A reference $r$ is a pair ref$(c,p)$, where $c \in Loc$ and $p$ is a path. A concrete state $s$ is a tuple $(P,S,M)$ where: (1) Call stack $P$ is a list of frames. Each frame is a triple comprising a procedure $p$, a program counter $t$, and a local store $L$ that maps variables to locations or references. (2) Operand stack $S$ is a list of values and references. (3) Memory $M$ maps locations to values.

A program $\mathcal{P}$ maps each procedure $p$ to a tuple $\mathcal{P}[p]$ which contains a nonempty sequence of bytecodes $\mathcal{P}[p].C$, and a sequence of input/output types $\mathcal{P}[p].I$ and $\mathcal{P}[p].O$.

**Semantics.** Figure 1 shows the formal operational semantics. The first four rules capture the execution of a program $\mathcal{P}$ as a transition relation $\mathcal{P} \vdash (P,S,M) \rightarrow (P',S',M')$. Each of these rules looks up the instruction $\mathcal{P}[p].C[t]$ pointed to by the top frame on the call stack, executes the instruction, and updates the state accordingly. Call establishes a new frame at the top of the call stack. Ret tears down the frame at the top of the call stack and increments the program counter for the new topmost frame. Branch$(t_1,t_2)$ consumes a Boolean value from the top of the operand stack and updates the program counter to $t_1$ if the value is true and to $t_2$ otherwise. The Execute rule is a wrapper for executing the remaining instructions.

The MoveLoc, CopyLoc, and StoreLoc instructions move or copy values between local variables and the operand stack; BorrowLoc, BorrowField, and ReadRef to operate on reference values stored on the operand stack. The FreezeRef rule converts a mutable reference into an immutable one. This instruction is a no-op in the concrete semantics, but performs several important checks in the analysis (see Section 4). Pack and Unpack to create and destroy record values by binding values on the stack to fields or pushing values bound to fields onto the stack (respectively). Pop destroys a value on the operand stack. This is simple in the concrete semantics, but the analysis must perform careful bookkeeping when popping references. The generic Op instruction represents arithmetic/bitwise operations that use the operand stack.

The correctness of executions of $\mathcal{P}$ requires that only instructions in the valid bytecode range are accessed and that each instruction operates on the correct number of values of appropriate type producing the appropriate result and updating the state without violating reference safety. The first requirement is easily handled by a syntactic check that the bytecode sequence of each procedure is nonempty, branch targets are legal indices, and the last instruction is either Branch or Ret. The second requirement is the subject of the next section.
4 Move Borrow Checker

In this section, we present the guarantees offered by the Move borrow checker on the subset of the language formalized in §3. Our formalization requires that there is a type annotation on each bytecode of each procedure. Our analysis will check that the program is well-typed, i.e., all type annotations are consistent with each other. The annotations are derived automatically via a simple fixpoint based on abstract interpretation; this fixpoint is described informally in §5. The annotations on a well-typed program allow us to define an abstraction function 

\[ \text{Abs} \] from a concrete state \( (P,S,M) \) to an abstract state \( \hat{P} \hat{S} \hat{B} \) that replaces each concrete value or reference in \( P(S) \) with its type in \( \hat{P} \hat{S} \), dropping the memory \( M \) entirely replacing it with a borrow graph \( B \) that captures the borrow relationships among values and references in \( P \) and references in \( S \). Finally, we show that every concrete state \( s \) reachable by program \( \mathcal{P} \) is connected to Abs(\( s \)) by an invariant that is sufficient to prove type safety, absence of leaks, and absence of dangling references.

In the abstract semantics used to formalize static checking, a path may optionally end with the distinguished symbol \( * \), representing an unknown (possibly empty) sequence of additional field selections. A path is extensible if it ends in \( * \) and fixed otherwise. We include both kinds of paths in the definitions of path operations below. We write \( p \leq q \) if path \( p \) is a prefix of path \( q \). We write \( p : = q \) for path concatenation which is ordinary concatenation of \( p \) and \( q \) if \( p \) is fixed and \( q \) otherwise.

Borrow graph. An abstract location is either \( \Pi(x,y) \) for some non-negative integers \( x \) and \( y \) or \( \Omega(x) \) for some non-negative integer \( x \). Let \( ALoc \) be the set of all abstract locations. The abstract location \( \Pi(x,y) \) represents the contents of the local variable \( y \) in the call stack frame at position \( x \). The abstract location \( \Omega(x) \) represents the contents of the operand stack at position \( x \). In both cases, we count up from the bottom of the stack. An abstract location can be looked up in a concrete state to return either a reference (both call stack and operand stack), a value (only operand stack), or a concrete location in \( Loc \) (only call stack).

For abstract locations \( x,y \in ALoc \) and path \( p \), the assertion \( Borrow(x,p,y) \), called a borrow edge, is interpreted in a concrete state \( s \) to indicate that \( x.p \) is borrowed by \( y \). If \( p \) is fixed, the reference in \( s \) at position \( y \) has jurisdiction over \( m.p \), where \( m \) is the reference or concrete location at position \( x \) in \( s \). If \( p \) is extensible, then \( x.q \) is borrowed by \( y \) for some extension \( q \) of \( p \). In our static analysis, we therefore assume (to preserve soundness) that \( y \) has jurisdiction over each \( x.q \) for any extension \( q \) of the path \( p \).

A borrow graph \( B \) is a collection of borrow edges. An edge \( Borrow(x,y) \) is subsumed by edge \( Borrow(x,q,y) \) if either \( p = q \) or \( q = r + p \) and \( r \leq p \). If \( G \) and \( H \) are borrow graphs, we write \( G \subseteq H \), if every edge \( Borrow(x,p,y) \) in \( G \) is subsumed by some edge \( Borrow(x,q,y) \) in \( H \). Note that this definition allows additional edges in \( H \) that are not implied by edges in \( G \). Semantically, if \( G \subseteq H \), then \( G \) imposes every restriction on concrete execution that is expressed by \( H \), and possibly additional restrictions. As a result, we will see that every state that satisfies \( G \) also satisfies \( H \).

Abstract state. An abstract state \( \hat{s} \) is a tuple \( \hat{P} \hat{S} \hat{B} \) with three components. The first component \( \hat{P} \), an abstract call stack that matches the concrete call stack, is a list of triples, each comprising a procedure name \( \rho \), a program counter \( \ell \), and a partial map \( \hat{L} \) from variables to \( Type \). The stack \( \hat{P} \) defines a set of stack positions \( \mathcal{P}(\hat{P}) = \{ \Pi(x,y) \mid 0 \leq y < len(\hat{P}) \land y \in \hat{P}[x] \} \), with \( \Pi(x,y) \) indicating the \( y \)-th local variable in the \( x \)-th call frame of the stack. The second component \( \hat{S} \), an abstract operand stack that matches the concrete operand stack, is a list of types. The stack \( \hat{S} \) defines a set of meaningful stack positions \( \mathcal{S}(\hat{S}) = \{ \Omega(x) \mid 0 \leq y < len(\hat{S}) \} \). Treating the abstract state \( \hat{s} \) as a partial map from positions \( dom(\hat{s}) = \mathcal{P}(\hat{P}) \cup \mathcal{S}(\hat{S}) \) to types, we let \( \hat{s}(\Pi(x,y)) \) be the type \( \hat{P}[x][y] \hat{L}[\hat{S}] \hat{S} \) and \( \hat{s}(\Omega(x)) \) be the type \( \hat{S}[\hat{S}] \). The third component \( B \) of the abstract state is a borrow graph with edges connecting nodes from \( dom(\hat{s}) \).

Local abstract state. Since abstract states represent type information, abstract execution of an imperative program instruction is a form of type propagation. The rules for type propagation of local instructions (§4.1) operate over a local abstract state \( \hat{L} \hat{S} \hat{B} \). The first component \( \hat{L} \) maps the variables in the local store of the top frame of the call stack to their types. The second component \( \hat{S} \) contains the types of the values and references for the portion of the operand stack visible to the execution of the procedure in the top frame. This component is empty at the starting bytecode at position 0 and grows and shrinks as values or references are pushed on and popped off it. The third component \( B \) contains the borrow edges for locations in \( \hat{L} \) and \( \hat{S} \).

Let \( ls = \langle \hat{L}, \hat{S}, B \rangle \) be an abstract local state. We define \( dom(ls) = \{ \Pi(i,0) \mid i \in dom(\hat{L}) \} \cup \{ \Omega(i) \mid i \in 0..len(\hat{S}) \} \). The locations in \( dom(ls) \) are defined using the offset 0 for the position of its frame and the bottom of its operand stack. Later, when we define the abstract state corresponding to a concrete state, we will show how local abstract states of each frame on the call stack, can be stitched together by adjusting their locations with respect to an appropriate offset.

We call \( \Pi(i,0) \mid i \in 0..len(\mathcal{P}(\rho)[\ell]) \) the input locations of \( \rho \). The abstract state local is \( well-formed \) for \( \rho \) if (1) every input location of \( \rho \) is in \( dom(ls) \), (2) \( \hat{L} \hat{S} \hat{B} \) is a directed acyclic graph, and (3) for all \( Borrow(x,y) \in \hat{L} \hat{S} \hat{B} \), we have \( x,y \in dom(ls) \), and \( y \) is not an input location of \( \rho \). The propagation on local abstract states (§4.1) uses judgments of the form \( \rho, op : \langle \hat{L}, \hat{S}, B \rangle \rightarrow \langle \hat{L}', \hat{S}', B' \rangle \) indicating that procedure \( \rho \) executing instruction \( op \) from \( \langle \hat{L}, \hat{S}, B \rangle \) results in \( \langle \hat{L}', \hat{S}', B' \rangle \). These
rules are designed to ensure that if \( \langle \bar{L}, \bar{S}, B \rangle \) is well-formed for \( \rho \), then \( \langle \bar{L}', \bar{S}', B' \rangle \) is also well-formed for \( \rho \).

**Well-typed programs.** In addition to input types \( \cP[\rho].I \) and output types \( \cP[\rho].O \), our analysis requires that each procedure \( \rho \) contains an abstract local state \( \cP[\rho].T[i] \) for each offset \( i \in 0..\text{len}(\cP[\rho].C) \) that is well-formed for \( \rho \). Let \( \text{Next}(\rho)[i] \) be the (empty, singleton, or doubleton) set of program counters to which control transfer is possible after executing instruction at position \( i \).

\[
\text{Next}(\rho)[i] = \{ \}, \quad \text{if } \cP[\rho].C[i] = \text{Ret} \{ t_1, t_2 \}, \quad \text{if } \cP[\rho].C[i] = \text{Branch}(t_1, t_2) \{ t+1 \}, \quad \text{otherwise}
\]

We write \( \langle \bar{L}, \bar{S}, B \rangle \subseteq \langle \bar{L}', \bar{S}', B' \rangle \) if \( \bar{L} = \bar{L}', \bar{S} = \bar{S}', \) and \( B \subseteq B' \). Thus, every concrete local state represented by \( \langle \bar{L}, \bar{S}, B \rangle \) can also be represented by \( \langle \bar{L}', \bar{S}', B' \rangle \). A program \( \cP \) is well-typed if for all \( \rho \in \text{dom}(\cP) \):

1. \( (\cP[\rho].I[],[],\{\}) = \cP[\rho].T[0] \).
2. For all \( i \in \text{dom}(\cP[\rho].C) \), there exists \( \langle \bar{L}, \bar{S}, B \rangle \) such that
   (a) \( \rho, \cP[\rho].C[i] + \cP[\rho].T[i] \rightarrow \langle \bar{L}, \bar{S}, B \rangle \), and
   (b) \( \langle \bar{L}, \bar{S}, B \rangle \subseteq \cP[\rho].T[j] \) for all \( j \in \text{Next}(\rho)[i] \).

The first condition states that initially the operand stack and borrow graph are empty. The second condition expresses that the restrictions imposed after executing an instruction continue to the next instruction.

**Abstraction function.** While the rules mention only abstract local states, they also define abstract execution on (full) abstract states. We define an abstraction function from a concrete state to an abstract state. As a concrete execution steps through concrete states, its abstract execution steps through the corresponding abstract states.

We define the abstract state \( \langle \bar{P}, \bar{S}, B \rangle \) corresponding to a concrete state \( \langle P, S, M \rangle \) of a well-typed program \( P \) by looking only at \( P \) and the annotations on \( P \). The component \( \bar{P} \) is obtained by processing each frame \( (\rho, \ell, L) \) in \( P \) separately, replacing \( L \) with \( \cP[\rho].T[\ell].\bar{L} \), and concatenating the results:

\[
\bar{A}(P) = \text{match } P \\
| [] \rightarrow [] \\
| (\rho, \ell, \_):P \rightarrow \text{let } x = \cP[\rho].T[\ell].\bar{L} \text{ in } (\rho, \ell, x):\bar{A}(P') \bar{P} = \bar{A}(P)
\]

The definition of \( \bar{S} \) is not a straightforward concatenation of the operand stacks \( \cP[\rho].T[i] \) for each position \( i \in \text{dom}(P) \) because the operand stack for each frame other than the topmost contains the arguments for the callee. These arguments are removed from the operand stack as a result of the call; consequently, they must be removed prior to concatenation as well. We first derive \( \cP[\rho].\overline{T}[i] \) which removes the callee arguments from the operand stack if instruction \( i \in \rho \) is a call.

\[
\text{if } \cP[\rho].C[i] = \text{Call}(\rho') \\
\quad \text{let } x = \cP[\rho'].I \\
\quad \langle \bar{L}, \text{rev}(x):\bar{S}, B \rangle = \cP[\rho].T[i] \\
\quad R = ((\Omega(\text{len}(\bar{S}) + i), 1, (i, i)) | i \in 0..\text{len}(x)) \text{ in } \\
\quad \langle \bar{L}, \text{rename}(B, R) \rangle
\]

This derivation uses the operation \( \text{rename}(B, R) \) which renames the nodes of edges in \( B \) according to the bijection \( R \). Here \( R \) renames the positions on the operand stack corresponding to callee arguments to appropriate positions in the next frame with offset 1. We now define \( \bar{P} \) by concatenating operand stacks obtained by looking up \( \cP[\rho].T \) for the top frame and \( \cP[\rho].\overline{T} \) for all other frames:

\[
\bar{S}(\bar{P}) = \text{match } \bar{P} \\
| [] \rightarrow [] \\
| (\rho, \ell, \_):P \rightarrow \text{let } B = \text{if } n = \text{len}(P) \text{ then } \cP[\rho].T[\ell].B \text{ else } \cP[\rho].\overline{T}[\ell].B, \\
\quad R = ((\Pi(x,y), \Pi(x+\text{len}(AS(P')))) \cup \{ (\Omega(x), \Omega(x+\text{len}(AS(P'))) | \Omega(x) \} \text{ in } \\
\quad \text{rename}(B, R) \cup \bar{A}(P', n)) \text{ for all other frames: }
\]

The last component \( B \) is defined similarly to \( \bar{S} \), by looking up the borrow graph for each frame, renaming it appropriately to account for the offset of the frame and its corresponding operand stack, and taking the union of all such renamed borrow graphs. The top frame is looked up in \( \cP[\rho].\overline{T} \) but all other frames are looked up in \( \cP[\rho].\overline{T} \).

\[
\bar{A}(P, n) = \text{match } P \\
| [] \rightarrow [] \\
| (\rho, \ell, \_):P \rightarrow \text{let } B = \text{if } n = \text{len}(P) \text{ then } \cP[\rho].T[\ell].B \text{ else } \cP[\rho].\overline{T}[\ell].B, \\
\quad R = ((\Pi(x,y), \Pi(x+\text{len}(AS(P')))) \cup \{ (\Omega(x), \Omega(x+\text{len}(AS(P'))) | \Omega(x) \} \text{ in } \\
\quad \text{rename}(B, R) \cup \bar{A}(P', n)) \text{ for all other frames: }
\]

The constraints on the annotations of a well-typed program, explained earlier, allow us to prove that \( B \) is acyclic, an important property that we leverage in the proof of reference safety. Finally, we get \( \bar{S}(\bar{P}, S, M) = \bar{S}(\bar{P}, S, B) \) where \( P, S, \) and \( B \) are defined as above.

We use the abstraction function to prove critical invariants about executions of well-typed programs. These invariants establish useful properties—type agreement, no memory leaks, no dangling references, and referential transparency. We state the invariants as a predicate \( \text{Inv}(s, s) \) over a concrete state \( s \) and an abstract state \( s \). We use type propagation on local abstract states (§4.1) to prove the following theorem:

**Theorem 1.** Let program \( \cP \) be well-typed. If \( s \) is a concrete state with Inv\( s, \cP(s) \) and \( \cP + s \rightarrow s' \), then Inv\( s', \cP(s') \).

A proof sketch for this theorem is available in the supplemental material. A corollary is that if \( \cP \) starts execution in a concrete state \( s_0 \) such that Inv\( s_0, \cP(s_0) \) holds, then
\(\text{Inv}(s, \text{Abs}(s))\) holds for all states reachable from \(s_0\). Any initial state \(s_0\) of \(\mathcal{P}\) is of the form \(((\rho, 0, \mathcal{X}[,])\{\})\) representing the beginning of a transaction that invokes \(\rho\) with inputs \(\mathcal{X}\) comprising only values (no references), empty operand stack, and empty memory. It is easy to see that \(\text{Inv}(s_0, \text{Abs}(s_0))\) holds if \(\mathcal{P}\) is well-typed. We present \(\text{Inv}\) as the conjunction of four predicates, \(\text{InvA}, \text{InvB}, \text{InvC}, \text{InvD}\), described below.

Type Agreement. Concrete state \(s = \langle P, S, M \rangle\) and abstract state \(\hat{s} = \langle \hat{P}, \hat{S}, B \rangle\) are shape-matching if (1) \(\text{len}(P) = \text{len}(\hat{P})\), (2) \(\text{len}(S) = \text{len}(\hat{S})\), (3) for all \(i \in \text{dom}(P)\), we have \(P[i].\rho = \hat{P}[i].\rho, P[i].t = \hat{P}[i].t\), and \(\text{dom}(P[i].L) = \text{dom}(\hat{P}[i].\hat{L})\). Intuitively, shape-matching states have the same call stack height, the same operand stack height, and agreement between corresponding procedure names, program counters, and set of local variables in each stack frame. Shape-matching states \(s = \langle P, S, M \rangle\) and \(\hat{s} = \langle \hat{P}, \hat{S}, B \rangle\) are further type-matching if for all positions \(n\) in the identical sets of call stack positions \(\text{Pos}(P) = \text{Pos}(\hat{P})\) or in the identical sets of operand stack positions \(\text{Pos}(S) = \text{Pos}(\hat{S})\), we have \(s(n) = \hat{s}(n)\).

\(\text{InvA}(s, \hat{s})\) : \(s\) and \(\hat{s}\) are shape-matching and type-matching.

No Memory Leaks. The following invariant indicates that (1) every local variable on the call stack of \(s\) contains a different location, and (2) locations are not leaked, i.e., \(s.M\) does not contain any location not present in a local variable.

\(\text{InvB}(s, \hat{s})\) : \(s\) and \(\hat{s}\) are shape-matching and type-matching.

No Dangling References. For shape-matching \(s = \langle P, S, M \rangle\) and \(\hat{s} = \langle \hat{P}, \hat{S}, B \rangle\), a borrow edge \(\text{Borrow}(m, p, n)\) in \(B\) is realized in \(s\) if the path \(p\) leads from \(s(m)\) to \(s(n)\), optionally involving additional field selections if \(p\) ends in \. More precisely, we say this graph edge is realized if either

1. \(s(m) = c, s(n) = \text{ref} \langle c, q' \rangle\), and path \(p\) matches \(q'\), or
2. \(s(m) = \text{ref} \langle c, q \rangle, s(n) = \text{ref} \langle c, q' \rangle\), and \(q.p\) matches \(q'\).

Note that the two conditions express the same basic relationship if we identify \(c\) and \(\text{ref} \langle c, e \rangle\). The following invariant allows us to conclude that every reference is rooted in a memory location present in some local variable on the call stack.

\(\text{InvC}(s, \hat{s})\) :

1. \(\hat{s}.B\) is acyclic.
2. For all \(n \in \text{dom}(\hat{s})\) such that \(\hat{s}(n) \in \text{ValType}\), there is no borrow edge in \(\hat{s}.B\) coming into \(n\).
3. For all \(n \in \text{dom}(\hat{s})\) such that \(\hat{s}(n) \in \text{RefType}\), there is a borrow edge in \(\hat{s}.B\) coming into \(n\) that is realized in \(s\).

Referential Transparency. We write \(\text{ref} \langle c, p \rangle \leq \text{ref} \langle d, q \rangle\) if \(c = d\) and \(p \leq q\). We extend \(\leq\) so that \(c \leq \text{ref} \langle c, p \rangle\) for any \(p\). The following invariant indicates that the absence of borrow edges out of an abstract location containing a value or a mutable reference guarantees that mutation via that abstract location, either of the stored value or the value pointed to by the mutable reference, will not invalidate any live reference.

\(\text{InvD}(s, \hat{s})\) : For any distinct \(m, n \in \text{dom}(\hat{s})\) such that \(\hat{s}(n) \in \text{RefType}\) and \(s(m) \leq s(n)\), one of the following hold:

1. \(\hat{s}(m) \in \text{ImmRefType}\) and \(\hat{s}(n) \in \text{ImmRefType}\).
2. \(\hat{s}(m) \notin \text{ImmRefType}\) and there is a path in \(\hat{s}.B\) from \(m\) to \(n\) comprising realized edges in \(s\).
3. \(s(m) = s(n)\) and there is a path in \(\hat{s}.B\) from \(n\) to \(m\) comprising realized edges in \(s\).

4.1 Propagating Local Abstract States

Having explained the overall structure of our soundness argument, we now provide intuition for type propagation on local abstract states. The rule for operation \(op\) derives a judgment of the form \(\rho, op + \langle L, S, B \rangle \rightarrow \langle L', S', B' \rangle\) if certain conditions are satisfied. These conditions include availability of appropriately-typed values in \(L\) or \(S\) and absence of certain edges in the borrow graph \(B\). The state transformation adds or removes a variable-to-type binding in \(L\), pushes or pops types in \(S\), and adds or removes edges in \(B\). The rules for \text{MoveLoc} and \text{StoreLoc} also prevent an input of procedure \(\rho\) from being moved or overwritten to enable accurate tracking of transitive borrow relationships across a procedure call.

The rule for \text{MoveLoc}(x) moves the type of variable \(x\) to the top of operand stack. The rule checks that \(x\) is available in \(L\) and there are no outgoing borrow edges from \(\Pi(0, x)\), the abstract location of \(x\), in case \(x\) is a value. The rule also renames the old position of the moved value to its new position.

The rule for \text{Pop} pops the top of the operand stack and eliminates the local location corresponding to it using a new operation \text{elim}. The expression \(\text{elim}(B, u)\) creates a new borrow graph by eliminating location \(u\) in \(B\) as follows: (1) For every edge \(\text{Borrow}(a, p, u)\) coming into \(a\) and edge \(\text{Borrow}(a, q, b)\) going out of \(a\), add an edge \(\text{Borrow}(a, p :: q, b)\). (2) Delete all edges coming into and going out of \(u\). The definition of \(\text{elim}(B, u)\) ensures that all transitive borrow relationships going through the reference at the top of the operand stack are maintained even when the top is popped.

The two rules for \text{StoreLoc}(x) use a combination of the techniques introduced for handling \(\text{MoveLoc}(x)\) and \text{Pop}. If \(x\) is available in the local store and is a reference type, \(\Pi(0, x)\) is eliminated in the borrow graph. If \(x\) is available in the local store and is a value type, then it is checked that there are no borrow edges going out of \(\Pi(0, x)\). In both cases, the position for the previous top of stack is renamed to \(\Pi(0, x)\) since it is being moved into variable \(x\).

The rule for \text{BorrowLoc}(x) uses the operation \text{factor}. The expression \(\text{factor}(B, u, v)\), where \(u\) may but \(v\) may not have borrow edges incident in \(B\), creates a new borrow graph by redirecting edges going out of \(u\) to go out of \(v\) and adding a new edge \(\text{Borrow}(u, t, o)\). This operation ensures that borrow edges from \(u\) are propagated to \(v\).

The first rule for \text{BorrowField}(f, x) addresses the case when the source reference is mutable and creates a mutable borrow from it. This rule uses a variation of factor named
\[x \notin \text{dom}(\mathcal{P}[\rho], I) \quad \text{Borrow}(\Pi(0, x), ...) \not\in B \quad B' = \text{rename}(B, \{(\Pi(0, x), \Omega(len(\hat{S})))\})\]

\[
\rho.\text{MoveLoc}(x) + (\{\hat{S}, B\} \rightarrow (\hat{L} \setminus x, \hat{L}(x) :: \hat{S}, B')
\]

\[
\rho.\text{Popr}(\{\text{lab}, \hat{S}, B\} \rightarrow (\hat{L}, \hat{S}, B')
\]

\[
x \notin \text{dom}(\mathcal{P}[\rho], I) \quad \hat{L}(x) \in \text{ValType} \quad B' = \text{rename}(\text{elim}(B, \Pi(0, x), \{(\Omega(len(\hat{S})), \Pi(0, x))\}))
\]

\[
\rho.\text{StoreLoc}(x) + (\{\hat{L}, t :: \hat{S}, B\} \rightarrow (\hat{L}[x \mapsto t], \hat{S}, B')
\]

\[
x \notin \text{dom}(\mathcal{P}[\rho], I) \quad x \notin \text{dom}(\hat{L}) \land \hat{L}(x) \in \text{ValType} \quad \text{Borrow}(\Pi(0, x), ...) \not\in B \quad B' = \text{rename}(B, \{(\Omega(len(\hat{S})), \Pi(0, x))\})
\]

\[
\rho.\text{StoreLoc}(x) + (\{\hat{L}, t :: \hat{S}, B\} \rightarrow (\hat{L}[x \mapsto t], \hat{S}, B')
\]

\[
x \in \text{dom}(\hat{L}) \quad \hat{L}(x) = t \quad t \in \text{ValType} \quad B' = \text{factor}(B, \Pi(0, x), \Omega(len(\hat{S})))
\]

\[
\rho.\text{BorrowLoc}(x) + (\{\hat{L}, \hat{S}, B\} \rightarrow (\hat{L}, \text{Mut}(t) :: \hat{S}, B')
\]

\[
x \in \text{dom}(\hat{L}) \quad \hat{L}(x) = \text{Mut}((f, t), ...)
\]

\[
\rho.\text{BorrowField}(f, x) + (\{\hat{L}, \hat{S}, B\} \rightarrow (\hat{L}, \text{Mut}(t) :: \hat{S}, B')
\]

\[
x \in \text{dom}(\hat{L}) \quad \hat{L}(x) = \text{Imm}((f, t), ...)
\]

\[
\rho.\text{BorrowField}(f, x) + (\{\hat{L}, \hat{S}, B\} \rightarrow (\hat{L}, \text{Imm}(t) :: \hat{S}, B')
\]

\[
x \in \text{dom}(\hat{L}) \quad \hat{L}(x) = \text{Mut}((f, t), ...)
\]

\[B' = \text{factor}(B, \Pi(0, x), \Omega(len(\hat{S})))
\]

\[
\rho.\text{CopyLoc}(x) + (\{\hat{L}, \hat{S}, B\} \rightarrow (\hat{L}, \hat{L}(x) :: \hat{S}, B')
\]

\[
x \in \text{dom}(\hat{L}) \quad \hat{L}(x) = \text{Imm}((f, t), ...)
\]

\[t = \text{Imm}(t') \land t = \text{Mut}(t')
\]

\[
\forall n. \quad \text{Borrow}(\Omega(len(\hat{S})), ...) \not\in B \rightarrow (\{\hat{L}, \hat{S}[n]\} \in \text{ImmRefType}
\]

\[
\rho.\text{FreezeRef}(\{\hat{L}, \hat{S}, B\} \rightarrow (\hat{L}, \text{Imm}(t') :: \hat{S}, B)
\]

\[
\forall n. \quad \text{Borrow}(\Omega(len(\hat{S})), ...) \not\in B \rightarrow (\{\hat{L}, \hat{S}[n]\} \in \text{ImmRefType}
\]

\[
\rho.\text{ReadRef}(\{\hat{L}, \hat{S}, B\} \rightarrow (\hat{L}, \text{Mut}(t') :: \hat{S}, B')
\]

\[
\begin{align*}
\bar{t} &= \{t_1, ..., t_n\} \\
\{t_1, ..., t_n\} &= \{((f_i, t_i) | 1 \leq i \leq n) \\
\bar{t} &= \{t_1, ..., t_n\} \\
\{t_1, ..., t_n\} &= \{((f_i, t_i) | 1 \leq i \leq n) \\
\text{Op}: \bar{t} &= \bar{t}
\end{align*}
\]

\[
\rho.\text{Pack}(t) + (\bar{L}, \bar{t} :: \hat{S}, B) \rightarrow (\bar{L}, \bar{t} :: \hat{S}, B)
\]

\[
\rho.\text{Unpack}(\bar{L}, \bar{t} :: \hat{S}, B) \rightarrow (\bar{L}, \bar{t} :: \hat{S}, B)
\]

\[
\rho.\text{Op}(\bar{L}, \bar{t} :: \hat{S}, B) \rightarrow (\bar{L}, \bar{t} :: \hat{S}, B)
\]

\[
\begin{align*}
is &= 0, \text{len}(\mathcal{P}[\rho'], I) \quad \text{os} = 0, \text{len}(\mathcal{P}[\rho'], O) \\
\forall i \in \text{is}. \quad \mathcal{P}[\rho', I[i] \in \text{MutRefType} \Rightarrow \text{Borrow}(\Omega(len(\hat{S}) + i), ...) \not\in B \\
B_1 &= \text{rename}(B, \{i \in ((\Omega(len(\hat{S}) + i), \Pi(1, i))\})
\end{align*}
\]

\[
B_2 = \text{extend}(B_1, \{\Pi(1, i) | i \in \text{is} \land \mathcal{P}[\rho', I[i] \in \text{MutRefType}, \Omega(len(\hat{S}) + i) | i \in \text{os} \land \mathcal{P}[\rho', O[i] \in \text{RefType})
\]

\[
B_3 = \text{extend}(B_2, \{\Pi(1, i) | i \in \text{is} \land \mathcal{P}[\rho', I[i] \in \text{ImmRefType}, \Omega(len(\hat{S}) + i) | i \in \text{os} \land \mathcal{P}[\rho', O[i] \in \text{ImmRefType})
\]

\[
B' = \text{eval}(B_3, \{\Pi(1, i) | i \in \text{is})
\]

\[
\rho.\text{Call}(\rho') + (\{\hat{L}, \text{rev}(\mathcal{P}[\rho'], I) :: \hat{S}, B\} \rightarrow (\hat{L}, \text{rev}(\mathcal{P}[\rho'], I) :: \hat{S}, B')
\]

\[
\hat{L} = \mathcal{P}[\rho], I \quad \forall x \in \text{dom}(\hat{L}). \hat{L}(x) \in \text{ValType} \Rightarrow \text{Borrow}(\Pi(0, x), ...) \not\in B
\]

\[
\hat{S} = \text{rev}(\mathcal{P}[\rho], O) \quad \forall x \in \text{dom}(\hat{S}). \hat{S}[i] \in \text{MutRefType} \Rightarrow \text{Borrow}(\Omega(i), ...) \not\in B
\]

\[
t = \text{Bool}
\]

\[
\rho.\text{Branch}(t_1, t_2, t) + (\hat{L}, \hat{S}, B) \rightarrow (\hat{L}, \hat{S}, B)
\]
factor, a partial operation with similar inputs as factor. This operation succeeds if there is no edge labeled \( e \) coming out of \( u \) in \( B \), converting each edge of the form \( \text{Borrow}(u, f \circ p, a) \) to \( \text{Borrow}(u, p, a) \) and adding the edge \( \text{Borrow}(u, f, a) \). This rule ensures that any borrows from variable \( x \) along the field \( f \) are instead borrows from the new reference pushed on the operand stack which is itself borrowed from \( x \).

The second rule for \( \text{BorrowField}(f, x) \) addresses the case when the source reference is immutable and creates an immutable borrow from it. This rule simply adds a borrow edge labeled \( f \) between the source reference and the new reference.

The rule for \( \text{CopyLoc}(x) \) is similar to \( \text{BorrowLoc}(x) \) in case the variable \( x \) being copied is a reference. The rule for \( \text{WriteRef} \) checks that the target reference does not have any borrow edges coming out of it. The rules for \( \text{FreezeRef} \) and \( \text{ReadRef} \) both check that the reference operand at the top of the stack is freezable, i.e., all borrowed references from it are immutable. If a mutable reference is freezable, it is safe to convert it into an immutable reference. For space reasons, we skip over the rules for \( \text{Pack}, \text{Unpack}, \) and \( \text{Op} \) which do not perform any reference-related operations.

The rule for \( \text{Call}(\rho') \) can be understood as a sequence of simple steps. First, it checks if no mutable reference being passed to \( \rho' \) is borrowed. Second, it renames the call arguments present on the operand stack to the corresponding locals in the next frame to simulate the call (see definition of \( B_1 \)). Third, it simulates the return from the call by adding borrow edges from input reference parameters to output references returned by the call (see definition of \( B_2 \) and \( B_3 \)) and eliminating the locals in the callee frame (see definition of \( B' \)). The definitions of \( B_2 \) and \( B_3 \) use the \text{extend} operation. The expression \( \text{extend}(B, ws, vs) \), where \( ws \) and \( vs \) are sets of locations with borrow edges incident on locations in \( ws \) but no borrow edges incident on locations in \( vs \), adds an edge \( \text{Borrow}(u, s, a) \) for every \( u \in ws \) and \( v \in vs \). Furthermore, the definition of \( B' \) uses a generalization of \text{elim} that eliminates a set of locations in the borrow graph one at a time.

The rule for \( \text{Ret} \) checks that no local of value type is borrowed, the contents of the operand stack matches the output signature of \( \rho \), and no output that is a mutable reference is borrowed. This ensures that returned references are valid and the caller’s expectations for borrow relationships are sound.

### 4.2 Global Memory

We have formalized a borrow analysis that operates on a subset of Move with references to local variables on the call stack. In this section, we informally describe how to amend this model to support global storage. The analysis extensions are straightforward additions to the borrow graph domain that do not require changes to the existing rules.

We model global storage as an extra component \( G \) in the concrete program state \( ⟨P, S, M, G⟩ \), where \( G \) maps a type \( t \) ∈ \text{ValType} \) and an address \( a \in \text{Addr} \) to a location \( c \in \text{Loc} \). The extended borrow checker uses the \text{acquires} annotation described in §2.4 to abstract \( G \) in the global access instructions described in Section 2.4. We treat each type \( t \) in the \text{acquires} list of a procedure \( ρ \) as an extra local variable \( l_t \) in the intraprocedural borrow checker rules. The borrow checker ensures that if there is a reference into a value published at \( (t, a) \) for any address \( a \), then there is a path to this reference from \( l_t \) in the borrow graph. This guarantee is achieved by treating \( \text{BorrowGlobal}(t) \) similar to \( \text{BorrowLoc}(x) \), which allows us to handle \( \text{MoveTo}(t) \) and \( \text{MoveFrom}(t) \) much like \( \text{StoreLoc}(x) \) and \( \text{MoveLoc}(x) \) (respectively).

The borrow checker must perform one additional check: at a \( \text{MoveFrom}(t) \), \( \text{BorrowGlobal}(t) \), or a call to a procedure that has \( t \) in its \text{acquires} list, there must be no outgoing edges from \( l_t \) in the borrow graph. Additionally, a separate \text{acquires} static analysis checks that a procedure with any such instructions has an \text{acquires} annotation. Together, these ensure that global reference instructions cannot create a dangling reference to global memory.

### 5 Implementation

We have implemented two versions of the borrow checking algorithm in Rust: the bytecode analysis described above as part of the Move bytecode verifier (1072 lines) and source code variant in the Move compiler (1807 lines). The two implementations share a borrow graph abstract domain library (481 lines). All of these components are open-source.

#### 5.1 Borrow Checker in Move Bytecode Verifier

The bytecode verifier plays the important role of gating the admission of code to the ANON blockchain: a module can only be published if it is first certified by the bytecode verifier. The borrow analysis is a key component of the bytecode verifier, but it relies on several auxiliary analysis passes that we will briefly describe. Each pass analyzes a single module in isolation using the type signatures of its dependencies.

**Control-flow graph construction.** The bytecode of each procedure is converted into a control-flow graph over a collection of basic blocks. Each basic block is a non-empty and contiguous sub-sequence of the bytecode such that control-flow instructions — Branch or Ret — only occur as the last instruction of the basic block. Together, the basic blocks are a partition of the entire bytecode sequence. The construction of the control-flow graph attempts to create maximal basic blocks such that there is no jump into the middle of a block. Simple checks such as non-empty bytecode and ending with a control-flow instruction are also performed in this analysis. The granularity of all subsequent analyses is an entire basic block rather than an individual bytecode instruction.

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1 Withheld for double-blind review
Stack usage analysis. This analysis ensures that the shape-matching property from §4 holds for all programs. It tracks the height of the operand stack in each basic block and checks that the heights are equal at each join point.

Value type analysis. The goal of this analysis is to make sure that a value is used only if has not been moved and that each bytecode instruction is applied to values of appropriate type. Each procedure is analyzed separately exploiting the type annotations on inputs and outputs of called procedures. This analysis infers unmoved locals and the types of values on the operand stack using a straightforward dataflow analysis.

Acquires analysis. This straightforward analysis checks that acquires annotations on procedures (see §4.2) are correct.

Borrow analysis. This analysis is the most complex part of reference safety verification. The borrow checker (§4) requires that each instruction be annotated by an abstract state. However, our analysis computes these annotations using a fixpoint computation based on abstract interpretation. The fixpoint computation is performed locally for each procedure with the local abstract state as its abstract domain. The key new insight enabling this analysis is a suitable join operation for the borrow graph. This join of $G$ and $H$ is defined operationally as follows:

1. Take the union of edges in $G$ and $H$.
2. For each edge $\text{Borrow}(x, p, y)$ in the result, drop the edge if it is subsumed by another edge $\text{Borrow}(x, p', y)$.

It is possible for the resulting graph to have cycles even if $G$ and $H$ are acyclic. The join fails in this case and an error is reported. Thus, when a fixpoint is reached successfully, we are guaranteed that the computed annotations create a well-typed program.

5.2 Borrow Checker in the Move Compiler

The bytecode borrow checker is designed to simply and efficiently reject bad code, not help users diagnose reference issues. However, the Move compiler contains a source-level implementation of the borrow checker that augments the core algorithm with important tracking information used to provide informative error messages.

In addition, the compiler uses a liveness analysis to improve the programming experience in two ways: (1) infer whether a use of a source-level variable should emit a $\text{CopyLoc}(x)$ bytecode instruction (if $x$ is live) or a $\text{MoveLoc}(x)$ instruction (otherwise), and (2) immediately releasing dead references to prevent errors in the stricter bytecode borrow checker.

6 Evaluation

We claim that our analysis is useful, precise, and efficient.

| Project    | B   | Proc | \&Proc (%) | GProc (%) | B/ms |
|------------|-----|------|------------|----------|------|
| Anon       | 6.9K| 327  | 196 (60)   | 151 (46) | 1.2K |
| StarCoin   | 6.2K| 351  | 171 (49)   | 123 (35) | 1.3K |
| dFinance   | 1.2K| 109  | 33 (40)    | 30 (28)  | 1.3K |
| Total      | 14.6K| 787  | 400 (51)   | 304 (39) | 1.2K |

Figure 2. Usage of references in three Move codebases. $B$ and $Proc$ quantify the number of bytecodes and declared procedures for each project. \&Proc lists procedures with a reference in their type signatures and GProc shows procedures that access global storage. B/ms shows the average number of bytecode instructions analyzed per millisecond on a 2.4 GHz Intel Core i9 laptop with 64GB RAM.

7| let x = move c;  
   \text{******} Invalid move of local ‘c’  
6| let r = &c.f;  
   ---- It is still being borrowed by this reference

Figure 3. Error message reported by the Move compiler for the first dangling reference example from §2.2.

Utility. Move is a new language that is not (yet) officially supported outside of the Anon project. However, there are two open-source blockchain projects that use Move by maintaining a fork of the Anon codebase: dFinance$^2$ and StarCoin$^3$. Figure 2 summarizes the use of references in each project. We observe that references passed across procedure boundaries (\&Proc) and references to global storage (GProc) are common—over half of procedure signatures contain a reference, and more than a third touch global storage.

Reference-related mistakes (e.g., null dereferences, dangling references) in intricate, reference-heavy code with mutability are ubiquitous in other languages. Anecdotally, Move is no different here—we frequently made such mistakes while developing Anon. The difference is that Move reports these errors at compile-time with a message that points out the unsafe action and the borrow that precludes it (see Figure 3). This helps programmers internalize the discipline enforced by the checker and write reference-safe code.

Precision. Like any static analysis, the Move borrow checker introduces approximations that may lead it to reject safe programs. Sources of imprecision include abstracting references either returned by procedures or created on different sides of a conditional branch, abstracting values in global storage with their types, and preventing returns of global references.

In our experience with Move, we have only encountered expressivity problems with the last restriction. Issues usually arise when a module $M_1$ wants to give a module $M_2$ the ability to perform arbitrary writes to a global value of type $M_1:\T$.  

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2https://github.com/dfinance/dvm/tree/master/stdlib/modules
3https://github.com/starcoinorg/starcoin/tree/master/vm/stdlib/modules
In these cases, we used workarounds such as exposing field setters/getters for the global or combining the two modules. These workarounds are inconvenient but not fatal—a Move module typically encapsulates its global values to enforce key safety invariants, so the pattern of “sharing” globals between modules is uncommon. By contrast, eliminating references to global storage altogether would break key programming patterns such as increasing the balance field of an Account value in-place without removing it from storage.

**Efficiency.** The final column of Figure 2 quantifies the performance of the analysis using bytecodes analyzed/millisecond as a metric. The results show that the modular, intraprocedural analysis runs at a consistent rate on projects of different size. We note that although our current implementation is single-threaded, it would be easy to parallelize analysis of procedures to further increase the speed of the analysis.

7 Related Work

**Borrow-Based Static Analyses.** Rust [14] also uses a borrow-based static analysis to prevent dangling references. Rust’s analysis provides reference safety at the source level, whereas Move provides this guarantee directly for its executable representation via bytecode verification. The analyses support different language features (e.g., Rust allows references in records, Move allows mutable references to global state) and require different annotations (e.g., Rust has reference lifetime annotations, Move has procedure acquires annotations). We prove that Move’s analysis ensures leak freedom (see §4), but we are not aware of a similar proof for Rust.

There are also differences in the analysis mechanics. There are two descriptions of the Rust borrow checker: one that abstracts reference lifetimes and ensures that the lifetimes of related references are properly nested [2] (formalized in [10, 20]), and another that abstracts the relationship between each value and the set of loans [13] involving the value and prevents accesses to loaned values (formalized in [1, 25]).

Move’s analysis is philosophically similar to the second approach, but differs by using a borrow graph domain that preserves structural information about loans and values. This allows simpler handling of features like reborrowing (the Rust term for creating a copy of a unique reference), which requires Rust formalizations [1, 25] to maintain additional state, but is the same as a normal borrow for Move. On the other hand, Rust’s reference lifetime annotations allow programmers to precisely specify the relationship between input and output parameters, but Move does not allow this.

**Low-Level Enforcement of Memory Safety.** Move ensures reference safety for its executable representation without trusting a compiler using the approach of the JVM [7, 12] and CLR [15]: lightweight analysis run in a bytecode verifier. Other approaches to certifying low-level memory safety include typed assembly [16], proof-carrying code [18], and capability machines, specialized hardware with a memory-safe instruction set [22, 23, 27]. Each of these approaches has merits in its targeted application domains. We chose a bytecode language with a co-designed verifier for Move to accommodate mutable, persistent global storage, gas metering, resource types[4], and other features required for Move’s use-cases.

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Additional Material: The Move Borrow Checker

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We prove soundness of the borrow checker (Theorem 1) by showing that the inductive invariant between concrete and abstract states is preserved by execution of any well-typed program. This short addendum to the submitted paper describes the proof and gives the key elements of the technical argument.

The proof of Theorem 1 is decomposed into smaller lemmas. Recall that a concrete state contains the procedure call stack, procedure name, and a program counter that identifies the next instruction. The first lemma shows that if we execute any instruction next from any concrete state, the invariant is preserved between the new concrete state and an abstract state based on the borrow checker rules. This state must be in the \( \subseteq \) relation with the next program annotation. We next use another lemma relating \( \subseteq \) and the invariant to prove the theorem.

We begin with a simpler helper lemma that states an important and useful fact connecting borrow graphs and reference types.

**Lemma 0.** Suppose \( \hat{s} = \text{Abs}(s) = (\hat{L}, \hat{S}, \hat{B}) \) and \( \text{Inv}(s, \hat{s}) \). If \( m, n \in \text{dom}(\hat{s}) \) are distinct, \( \hat{s}(n) \in \text{MutRefType} \), and there is a path in \( B \) realizable in \( s \) from \( m \) to \( n \), then \( \hat{s}(m) \notin \text{ImmRefType} \).

**Proof of Lemma 0.** If there is a path in \( B \) realizable in \( s \) from \( m \) to \( n \), then \( s(m) \leq s(n) \). We know that \( \hat{s}(n) \notin \text{ImmRefType} \). We also know that since the borrow graph in \( \hat{s} \) is acyclic, there is no path in it from \( n \) to \( m \). The definition of referential transparency part of \( \text{Inv} \) allows us to conclude that \( \hat{s}(m) \notin \text{ImmRefType} \). \( \square \)

We now introduce helpful notation. Recall that all byte-code instructions other than call and return operate on a local state \( (L, S, M) \), where \( L \) is a local store from the top frame of a concrete state \( ((\rho, t, L)::P, S, M) \). A local instruction changes local states \( (L, S, M) \rightarrow (L', S', M') \) resulting in a state change

\[ ((\rho, t, L)::P, S, M) \rightarrow ((\rho, t+1, L')::P, S', M') \]

when extended to full states.

In our static analysis, a local instruction defines an associated transition \( (\hat{L}_1, \hat{S}_1, B_1) \rightarrow (\hat{L}_2, \hat{S}_2, B_2) \) of abstract local states, expressed by the borrow checker rules. To extend a local abstract state transition to full states, we view that transition as occurring on the top of any additional abstract call stack. However, unlike the concrete semantics, the abstract semantics are formulated using only a top fragment of the operand stack and a local borrow graph that must be integrated into the full abstract state.

Suppose \( s = ((\rho, t, L)::\_\_\_\_\_) \) has a nonempty call stack. We define \( \text{Base}(s) = ((\rho, t, i)::<AP, A\Sigma, AB(P, len(P)+1)) \).

\( \text{Base}(s) \) is an abstract state with a hole. It is useful to view the abstract state \( \hat{s} = \text{Abs}(s) \) in the form

\[ \hat{s} = \text{Base}(s) \uplus \{\hat{L}, \hat{S}, \hat{B}\} \]

where \( P \{\rho, T[t] = (\hat{L}, \hat{S}, \hat{B}) \}. \) The operation \( \uplus \) in \( \text{Base}(s) \uplus \{\hat{L}, \hat{S}, \hat{B}\} \) fills the hole \( \square \) with \( \hat{L} \), appends \( \hat{s} \) to the operand stack of \( \text{Base}(s) \), and adds the borrow edges in \( \text{rename}(B, R) \) to the borrow graph of \( \text{Base}(s) \), where \( R = \{((0, y), ((\text{len}(P), y)) | y \in \text{dom}(\hat{L})) \cup (\{\Omega(x), \Omega(x + \text{len}(\text{AS}(P))) | x \in 0..\text{len}(\hat{S})\). \)

We will often find it useful in our proof to construct the abstract state \( \text{Base}(s) \uplus \text{ls}' \) where \( \text{ls}' \) is different from \( P \{\rho, T[t] \}. \) The following lemma indicates a key property of \( \text{Base} \) and \( \uplus \).

**Lemma 1.** If

- \( \text{Inv}(s', \text{Base}(s') \uplus \{\hat{L}_1, \hat{S}_1, B_1\}) \)
- \( \{\hat{L}_1, \hat{S}_1, B_1\} \subseteq \{\hat{L}_2, \hat{S}_2, B_2\} \)

then \( \text{Inv}(s', \text{Base}(s') \uplus \{\hat{L}_2, \hat{S}_2, B_2\}) \).

**Proof of Lemma 1.** Let \( \hat{s}_1 = \text{Base}(s') \uplus \{\hat{L}_1, \hat{S}_1, B_1\} \) and \( \hat{s}_2 = \text{Base}(s') \uplus \{\hat{L}_2, \hat{S}_2, B_2\} \). The definition of \( \subseteq \) indicates that if \( \{\hat{L}_1, \hat{S}_1, B_1\} \subseteq \{\hat{L}_2, \hat{S}_2, B_2\} \) then \( \hat{L}_1 = \hat{L}_2 \) and \( \hat{S}_1 = \hat{S}_2 \) and each edge in \( B_1 \) is subsumed by some edge in \( B_2 \). Thus, the call stacks and operand stacks of \( \hat{s}_1 \) and \( \hat{s}_2 \) are identical and each edge in the global borrow graph of \( \hat{s}_1 \) is subsumed by some edge in the global borrow graph of \( \hat{s}_2 \). The preservation of \( \text{Inv} \) follows from the definition of subsumption of a borrow edge. \( \square \)

To aid our proof, we define an auxiliary transition relation \( P + s \rightarrow s' \). We view a transition \( P + s \rightarrow s' \) as happening in two steps. In the first step, the instruction pointed to by the program counter at the top of the call stack in \( s \) is executed to change parts of the state to yield \( s^- \) without adjusting the program counter to point to the next instruction. In the second step, the program counter in \( s^- \) is adjusted to yield \( s' \) and move to the next instruction. We write \( P + s \rightarrow s' \) to denote the first part of this two-step process. The concrete states \( s^- \) and \( s' \) coincide in the two cases when the call stack in \( s^- \) is empty or \( s' \) is obtained from \( s \) by executing a call instruction; in these two cases, \( P + s \rightarrow s' \) is a single step.

The following two lemmas together state that the transformation of the abstract state when a local instruction is executed preserves \( \text{Inv} \). The proof of Lemma 2 is provided later in this document.

**Lemma 2.** If

- \( s = ((\rho, t, L)::\_\_\_\_) \)
- \( \text{Inv}(s, \text{Abs}(s)) \)
- \( P + s \rightarrow s' \) by a local instruction
Theorem 1. Let program $\mathcal{P}$ be well-typed. If $s$ is a concrete state with $Inv(s, Abs(s))$ and $\mathcal{P} + s \rightarrow s'$, then $Inv(s', Base(s') \uparrow (\hat{L}, \hat{S}, \hat{B}))$.

Proof of Theorem 1. Let program $\mathcal{P}$ be well-typed. Let $s$ be a concrete state with $Inv(s, Abs(s))$ and $\mathcal{P} + s \rightarrow s'$. If the transition is due to $\textbf{Emp}$, the call stack and operand stack in $s$'s both empty. Therefore, $Abs(s') = \{\}, \{\}, \{\}$ and $Inv(s', Base(s'))$ holds trivially.

In all cases other than $\textbf{Emp}$, the call stack in $s$'s nonempty. Let $s \rightarrow s'$ and let the top frame of the call stack of $s'$ be $(\rho, t, \ldots)$. We use Lemma 3 to find $(\hat{L}, \hat{S}, \hat{B}) \supseteq \mathcal{P}[\rho]. T[t]$ such that $Inv(s', Base(s') \uparrow (\hat{L}, \hat{S}, \hat{B}))$. We use Lemma 1 to conclude that $Inv(s', Base(s') \uparrow (\hat{L}, \hat{S}, \hat{B}))$. By replacing the program counter in the top frame of the call stack of $s'$, we get $s'$. By replacing the program counter in the top frame of the call stack of $Base(s') \uparrow (\hat{L}, \hat{S}, \hat{B})$, we get $Abs(s')$. Therefore, we get $Inv(s', Base(s'))$.

Proof of Lemma 2. Let $\hat{s} = Abs(s) = (\rho, \hat{L}, \hat{S}, \hat{B})$. Note that $\hat{s}$ has the same procedure name and program counter in the top of the call stack as $s$. We note that $\hat{s} = Base(s) \uparrow \mathcal{P}[\rho]. T[t]$. Let $\mathcal{P}[\rho]. T[t] = (\hat{L}_0, \hat{S}_0, \hat{B}_0)$. Let $s' = Base(s') \uparrow (\hat{L}, \hat{S}, \hat{B})$. We must show $Inv(s', \hat{s})$. The proof proceeds by case analysis according to the local instruction that is executed to reach $s'$. There are four parts of the invariant to check for each instruction: type agreement, no memory leaks, no dangling references, and referential transparency. The first two conditions are straightforward for all local instructions and are therefore omitted from this presentation of the proof.

BorrowLoc($x$). Suppose $\mathcal{P}[\rho]. C(t) = \text{BorrowLoc}(x)$ with $L(x) = c$. This instruction pushes a reference to $c$ on top of the operand stack. The operational semantics give us

$$s' = ((\rho, t, L): P, \text{ref}(c), \{\}, \{\})$$

The rule for BorrowLoc($x$) shows $\hat{L}_0(x) = t$ for $t \in \text{ValType}$, $L = L_0, S = \text{Mut}(t) \uparrow \hat{S}_0$, and $B = \text{factor}(B_0, \Pi(0, x), \Omega(\text{len} \hat{S}_0))$. Thus, $B$ is derived from $B_0$ by

1. adding the edge $\text{Borrow}(\Pi(0, x), c, \Omega(\text{len} \hat{S}_0))$, and
2. replacing each edge $\text{Borrow}(\Pi(0, x), p, n)$ by the edge $\text{Borrow}(\Omega(\text{len} \hat{S}_0), p, n)$.

No dangling references. The only new position in $Abs(s')$ is the new top of the operand stack containing a mutable reference. The new borrow edge $\text{Borrow}(\Pi(0, x), c, \Omega(\text{len} \hat{S}))$ incident on this position is realized. Also, any realized borrow edge leaving $\Pi(0, x)$ that was moved to leave from $\Omega(\text{len} \hat{S})$ continues to be realized edge. Therefore the property holds for $s'$ and $s'$.

Referential transparency. We need to check all $m, n \in \text{dom}(s')$ such that $\hat{s}'(n) \in \text{RefType}$ and $s'(m) \leq s'(n)$. There are three cases depending on whether one of $m$ or $n$ is the new location $\Omega(\text{len} \hat{S})$.

1. $m \neq \Omega(\text{len} \hat{S})$ and $n \neq \Omega(\text{len} \hat{S})$. This case works because the factor operation is such that any realizable path that went through $\Pi(0, x)$ now goes through both $\Pi(0, x)$ and $\Omega(\text{len} \hat{S})$.
2. $m = \Omega(\text{len} \hat{S})$, then $s(\Pi(0, x)) < s(n)$. There is a realizable path in $s$ from $\Pi(0, x)$ to $n$ by induction. This path can be transformed into a realizable path in $s'$ that goes through $\Omega(\text{len} \hat{S})$.
3. $n = \Omega(\text{len} \hat{S})$, then either $m = \Pi(0, x)$ or $s(m) = s'(n)$. In the first case, the required path is simply the new $e$-edge. In the second case, there is a realizable path in $s$ from $\Pi(0, x)$ to $n$ by induction. This path can be transformed into a realizable path in $s'$ that goes through $\Omega(\text{len} \hat{S})$.

BorrowField($f, x$). Suppose $\mathcal{P}[\rho]. C(t) = \text{BorrowField}(f, x)$ with $L(x) = \text{ref}(c, \rho)$. There are two cases, based on whether $\hat{L}_0(x) = \text{Mut}((f, t, \ldots))$ or $\hat{L}_0(x) = \text{Imm}((f, t, \ldots))$. In both cases, the instruction pushes a reference $\text{ref}(c, \rho : f)$ derived from $L(x)$ on top of the operand stack. The operational semantics gives us

$$s' = ((\rho, t, L): P, \text{ref}(c, \rho : f), \{\}, \{\})$$

However, the static analysis differs in the two cases. We prove each case separately.

Case 1. Suppose $\hat{L}_0(x) = \text{Mut}((f, t, \ldots))$ is a mutable reference. The corresponding borrow-checker rule shows $L = L_0, \hat{S} = \text{Mut}(t) \uparrow \hat{S}_0$, and $B = \text{factor}(B_0, \Pi(0, x), \Omega(\text{len} \hat{S}))$. This
borrow graph operation fails and the bytecode program is rejected if there is any edge labeled e or * coming out of \( \Pi(0,x) \) in \( B_0 \). We therefore assume there is no such edge in \( B_0 \) and derive \( B \) from \( B_0 \) by

- Adding the edge \( \text{Borrow}(\Pi(0,x), f, \Omega(\text{len}(\hat{S}))) \), and
- Replacing each edge \( \text{Borrow}(\Pi(0,x), f :: p, n) \) by the edge \( \text{Borrow}(\Omega(\text{len}(\hat{S})), p, n) \).

No dangling references. This proof works exactly like the corresponding proof for BorrowLoc(x).

Referential transparency. We need to check all \( m, n \in \text{dom}(s') \) such that \( s'(n) \in \text{RefType} \) and \( s'(m) \leq s'(n) \). There are three cases depending on whether one of \( m \) or \( n \) is the new location \( \Omega(\text{len}(\hat{S})) \).

If \( m \neq \Omega(\text{len}(\hat{S})) \) and \( n \neq \Omega(\text{len}(\hat{S})) \), then the proof works exactly like the corresponding proof for BorrowLoc(x).

If \( m = \Omega(\text{len}(\hat{S})) \), then \( s(\Pi(0,x)) < s(n). \) There is a realizable path in \( s \) from \( \Pi(0,x) \) to \( n \) by induction. Since there is no edge labeled \( e \) or \( * \) out of \( \Pi(0,x) \), this path must follow one of the factored edges. Therefore, this path can be transformed to go through \( \Omega(\text{len}(\hat{S})) \) in \( s' \).

If \( n = \Omega(\text{len}(\hat{S})) \), then \( s(m) < s(\Pi(0,x)) \) or \( s(\Pi(0,x)) = s(m) \) or \( s(\Pi(0,x)) < s(m) \). In the first case, there is a realizable path in \( s \) from \( m \) to \( \Pi(0,x) \) by induction. We can extend this path to \( n \) by using the realizable edge from \( \Pi(0,x) \) to \( \Omega(\text{len}(\hat{S})) \). In the second case, there must be a realizable path either from \( m \) to \( \Pi(0,x) \) or from \( \Pi(0,x) \) to \( m \) by induction. Since there is no edge labeled \( e \) or \( * \) out of \( \Pi(0,x) \), the realizable path must be from \( m \) to \( \Pi(0,x) \). We can extend this path to \( n \) by using the realizable edge from \( \Pi(0,x) \) to \( \Omega(\text{len}(\hat{S})) \).

In the third case, \( s(m) = s'(m) = s'(n) \) and there is a realizable path in \( s \) from \( \Pi(0,x) \) to \( m \) by induction. Since there is no edge labeled \( e \) or \( * \) out of \( \Pi(0,x) \), this path must follow one of the factored edges. Therefore, this path can be transformed to go through \( \Omega(\text{len}(\hat{S})) \) in \( s' \).

Case 2. Suppose \( L_0(x) = \text{Imm}(\{f, \ldots\}) \) is an immutable reference. The corresponding borrow-checker rule shows \( L = L_0, \hat{S} = \text{Imm}(\Pi(0,x), f, \Omega(\text{len}(\hat{S}))) \).

No dangling references. This proof works exactly like the corresponding proof for BorrowLoc(x).

Referential transparency. We need to check all \( m, n \in \text{dom}(s') \) such that \( s'(n) \in \text{RefType} \) and \( s'(m) \leq s'(n) \). There are three cases depending on whether one of \( m \) or \( n \) is the new location \( \Omega(\text{len}(\hat{S})) \).

If \( m \neq \Omega(\text{len}(\hat{S})) \) and \( n \neq \Omega(\text{len}(\hat{S})) \), then the proof works exactly like the corresponding proof for BorrowLoc(x).

If \( m = \Omega(\text{len}(\hat{S})) \), then \( s(\Pi(0,x)) < s(n). \) We get \( s(n) \in \text{ImmRefType} \) by induction.

If \( n = \Omega(\text{len}(\hat{S})) \), then \( s(m) < s(\Pi(0,x)) \) or \( s(\Pi(0,x)) = s(m) \) or \( s(\Pi(0,x)) < s(m) \). In the first case, by induction (instantiating \( m \) with \( m \) and \( n \) with \( \Pi(0,x) \)), either \( s(m) \in \text{ImmRefType} \) or \( s(m) \notin \text{ImmRefType} \) and there is a realizable path in \( s \) from \( m \) to \( \Pi(0,x) \) which can be extended to a realizable path in \( s' \) to \( n \). In the second case, by induction (instantiating \( m \) with \( \Pi(0,x) \) and \( n \) with \( m \)), either \( s(m) \in \text{ImmRefType} \) or \( s(m) \notin \text{ImmRefType} \) and there is a realizable path in \( s \) from \( m \) to \( \Pi(0,x) \) which can be extended to a realizable path in \( s' \) to \( n \). In the third case, we get \( s(m) \in \text{ImmRefType} \) by induction.

MoveLoc \( (x) \). Vertices in the borrow graph are renamed from the old position of the moved value to its new position. The new position does not appear as the source or target of any borrow edge in the borrow graph. Therefore, the new borrow graph is isomorphic to the old borrow graph. Properties (No dangling references) and Referential transparency) follow from the fact that these properties held before this instruction was executed.

Pop. The analysis pops the top of the abstract operand stack and eliminates the location corresponding to it using the \textit{elim} operation on borrow graphs. This operation eliminates location \( u \) in \( B_0 \) as follows: (1) For every edge \( \text{Borrow}(a,p,u) \) coming into \( u \) and edge \( \text{Borrow}(a,q,b) \) going out of \( u \), add an edge \( \text{Borrow}(a,p :: q,b) \). (2) Delete all edges coming into and going out of \( u \). This ensures that all transitive borrow relationships going through the reference that has been popped off the operand stack are maintained, preserving (No dangling references) and (Referential transparency) as a result.

StoreLoc \( (x) \). The proof is a combination of the proof for MoveLoc \( (x) \) and \textit{Pop}.

CopyLoc \( (x) \). The proof is similar to \textit{BorrowLoc} \( (x) \) in case the variable \( x \) being copied is a reference. Otherwise, there is no change in the borrow graph and the proof is straightforward.

WriteRef. The proof is similar to \textit{Pop}.

FreezeRef. The operational semantics gives us \( s' = s \). If the reference on top of the abstract operand stack in \textit{Abs}(s) is immutable, the local abstract state does not change and the proof is done. Otherwise, the freezeable mutable reference at the top of the abstract operand stack is converted into an immutable reference.

No dangling references. This proof is straightforward since there is no change in the borrow graph.

Referential transparency. We need to check all \( m, n \in \text{dom}(s') \) such that \( s'(n) \in \text{RefType} \) and \( s'(m) \leq s'(n) \). There are three cases depending on whether one of \( m \) or \( n \) is the location \( \Omega(\text{len}(\hat{S}) - 1) \).

If \( m \neq \Omega(\text{len}(\hat{S}) - 1) \) and \( n \neq \Omega(\text{len}(\hat{S}) - 1) \), then the proof is straightforward.

Suppose \( m = \Omega(\text{len}(\hat{S}) - 1) \). By induction, either (1) there is a path realizable in \( s \) from \( m \) to \( n \), or (2) \( s(m) = s(n) \) and there is a path realizable in \( s \) from \( n \) to \( m \). In the first case, the freezability of \( m \) and Lemma 0 indicate that \( s(n) \in \text{ImmRefType} \). Since \( s(n) = s'(n) \), we are done. In the second case, we can reuse the path since \( s = s' \).
Suppose $n = \Omega(len(\hat{s}) - 1)$. By induction, either (1) $\hat{s}(m) \notin ImmRefType$ and there is a path realizable in $s$ from $m$ to $n$, or (2) $s(m) = s(n)$ and there is a path realizable in $s$ from $n$ to $m$. In the first case, we can reuse the path since $s = s'$ and $\hat{s}(m) = \hat{s}'(m)$. In the second case, the freezability of $n$ and Lemma 0 indicate that $\hat{s}(m) \in ImmRefType$. Since $\hat{s}(m) = \hat{s}'(m)$, we are done. 

ReadRef. The proof is similar to Pop.

We omit discussion of Pack, Unpack, and Op since these instructions do not perform any reference-related operations. □
Revised Material: The Move Borrow Checker

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Revised Operational Semantics

Concrete states. A concrete state $s$ is a pair $⟨P,M⟩$ where:
1. Call stack $P$ is a list of frames. Each frame comprises a procedure $ρ$, a program counter $ℓ$, a local store $L$ and a local operand stack $S$. A local store $L$ maps variables to locations or references and a local operand stack $S$ is a list of values and references. (2) Memory $M$ maps locations to values.

Parameter passing and return. Before calling a procedure, the procedure that is going to make the call places all the procedure parameters on the top of the operand stack, in the correct order. When the Call command, is executed these parameter values are removed from the local operand stack of the caller and placed on the local operand stack of the new call-stack frame for the new procedure. When that procedure returns, the return value is similarly transferred from the returning procedure’s operand stack to the operand stack of the procedure that called it.

Revised Borrow Checker

Positions. In specifying the types of local variables, the type of entries on the operand stack, and relationships between locations and references stored in them, we use two forms of position, written $Π(x,y)$ and $Ω(x,y)$ for non-negative integers $x$ and $y$.

For concrete state $s = ⟨P,M⟩$, the position $Π(x,y)$ refers to the $y$-th local variable in the $x$-th call frame of the stack and $Ω(x,y)$ refers to the $y$-th operand-stack position in the $x$-th call frame of the stack.

We let $dom(P)$ be the set of local variable and operand-stack positions that are used in $P$, i.e., $Π(x,y)$ for $x < y$ less than the total number of call frames in $P$ and $y < len(P)$ less than the number of local variables in the xth frame, counting from the bottom of the call stack, or $Ω(x,y)$ with $x < y$ less than the total number of call frames in $P$ and $y$ less than the height of the operand in the xth frame. If $s = ⟨P,M⟩$, then $dom(s) = dom(P)$.

Abstract state. An abstract state $ś$ is represented by an abstract call stack $P = ˆF_0, ..., ˆF_n$ where each abstract call-stack frame $F_i = (ρ,ℓ,L,S,B)$ comprises a procedure name $ρ$, a program counter $ℓ$, a partial map $L$ from variables to $Type$, a list $S$ of types, and a borrow graph $B$. Absolute positions in the local variable map $L$ and operand-stack types $S$ are defined in the same ways as for a concrete call-stack. We write $dom( ˆP)$ for the set of local variable and operand-stack positions in $P$.

A borrow graph $B$ is a labeled directed graph whose nodes are positions; each edge is labeled by a path. We write $Borrow( x,p,y)$ for the borrow edge from node $x$ to node $y$ with label $p$. If $B$ is the borrow graph of a frame $F_i$, then positions $Π(x,y)$ and $Ω(x,y)$ are interpreted as relative positions that are relative to the height $i$ of this frame in the call stack. Specifically, a relative position $Π(x,y)$ in borrow graph $B$ of frame at height $i$ refers to absolute position $Π(x+i,y)$, and similarly for $Ω(x,y)$.

An abstract call stack $P = ˆF_0, ..., ˆF_n$ defines a borrow graph $BG(P)$ whose nodes are absolute positions. The graph $B = BG(P)$ has edge $Borrow(Π(x+i,y),ρ,Π(x+i,y)) ∈ B$ for each $Borrow(Π(x,y),ρ,Π(y,u)) ∈ B_i$ at stack level $i$, and similarly for all other combinations of $Π(x,y)$ and $Ω(x,y)$ used as relative positions. An abstract call stack $P$ is considered well-formed only if every absolute position produced in this way is in $dom(P)$.

States as partial maps. Concrete and abstract states each induce a partial function from call-stack positions to values or types. Specifically, if $s$ is a concrete state, then for any call-stack position in $dom(s)$, we write $ś(Π(x,y))$ for the value of the $y$-th local variable in the $x$-th call frame of the stack and $ś(Ω(x,y))$ for the value in $y$-th operand-stack position in the $x$-th call frame of the stack. We similarly treat any abstract state $ś$ as a partial map from call-stack positions $dom(ś)$ to types, with $ś(Π(x,y))$ the type given to the $y$-th local variable in the $x$-th call frame of the stack and $ś(Ω(x,y))$ the type in $y$-th operand-stack position in the $x$-th call frame of the stack.

Type and shape matching. Concrete state $s = ⟨P,M⟩$ and abstract state $ś$ are shape-matching if $dom(s) = dom(ś)$, and for all $i ≤ len(P)$, we have $P[i].ρ = ˆP[i].ρ$ and $P[i].ℓ = ˆP[i].ℓ$. Shape-matching states $s$ and $ś$ are further type-matching if for all $i ∈ dom(s) = dom(ś)$ in the identical sets of positions, we have $ś(i) = s(i)$.

Borrow graph intuition. A borrow graph in an abstract state expresses borrow relationships between references in any corresponding concrete state. If concrete state $s$ and abstract state $ś$ are shape-matching and type-matching, with positions $x,y ∈ dom(s) = dom(ś)$, then an edge $Borrow( x,p,y)$ in $ś$ indicates that in $s$, the reference $x.p$ is borrowed by $y$. If $p$ is fixed, the reference in $s$ at position $y$ has jurisdiction over $m.p$, where $m$ is the reference or concrete location at position $x$ in the $s$. If $p$ is extensible, then $x.q$ is borrowed by $y$ for some extension $q$ of $p$.

Realized edge. For shape-matching $s = ⟨P,M⟩$ and $ś$, a borrow edge $Borrow( m.p,n)$ in $BG(ś)$ is realized in $s$ if the path $p$ leads from $s(m)$ to $s(n)$, optionally involving additional field selections if $p$ ends in $.*$. More precisely, we say this graph edge is realized if either
1. $s(m) = c, s(n) = ref( c.q′)$, and path $p$ matches $q′$, or
2. $s(m) = ref( c.q), s(n) = ref( c.q′)$, and $q.p$ matches $q′$.

Note that the two conditions express the same basic relationship if we identify $c$ and ref $⟨c.c⟩$. 
We write \( \text{ref} \ (c,p) \leq \text{ref} \ (d,q) \) if \( c = d \) and \( p \leq q \). We extend \( \leq \) so that \( c \leq \text{ref} \ (c,p) \) for any \( p \).

**Borrow graph subsumption.** An edge \( \text{Borrow}(x,p,y) \) is subsumed by edge \( \text{Borrow}(x,q,y) \) if either \( p = q \) or \( q = r+1 \) and \( r \leq p \). If \( G \) and \( H \) are borrow graphs, we write \( G \subseteq H \), if every edge \( \text{Borrow}(x,p,y) \) in \( G \) is subsumed by some edge \( \text{Borrow}(x,q,y) \) in \( H \). Note that this definition allows additional edges in \( H \) that are not implied by edges in \( G \). Semantically, if \( G \subseteq H \), then \( G \) imposes every restriction on concrete execution that is expressed by \( H \), and possibly additional restrictions. As a result, we will see that every state that satisfies \( G \) also satisfies \( H \).

**Parameter passing and return.** Before a procedure call, the bytecode program is assumed to contain instructions that put the parameters for the call on the top of the operand stack. As part of the Call command, these parameter values are removed from the local operand stack of the caller and placed on the local operand stack of the callee. For this reason, procedure execution begins with stack entries whose types are the declared types of the formal parameters; we manage this by setting types in the abstract state accordingly.

When the procedure returns, the return value is transferred from the returning procedure’s operand stack to the operand stack of the procedure that called it. This is similarly managed in the static analysis by giving the top value in the callee stack the type of the procedure return value.

We use the notation \( \mathcal{P} [\rho, I] \) for the list of parameter types of \( \rho \) and \( \mathcal{P} [\rho, O] \) for the return type, both as specified in the procedure declaration.

**Abstract semantics of instructions (= Type propagation).** The borrow checker analyzes each procedure independently, in a manner that is characterized by a set of rules for propagating type and borrow relationships through individual instructions. These rules are applied successively to each instruction in the procedure, beginning with an abstract state characterizing the local variables and parameters passed into the procedure when it is first called. The data structure used to characterize the static information is an extended abstract call-stack frame, as defined above, although the procedure name and program counter do not affect the semantics of an instruction.

The rules for type propagation of local instructions (§0.1) operate over a local abstract state \( \langle \hat{L}, \hat{S}, B \rangle \), where \( \hat{L} \) maps local variables to types, \( \hat{S} \) is a list of types for the values and references in the operand stack, and \( B \) is a borrow graph over positions in \( \hat{L} \) and \( \hat{S} \). We write \( \hat{L}, \hat{S} \) for the borrow-graph component of the local state, and similarly \( \hat{L}, \hat{S} \) for the operand stack.

**consistent notation**

An abstract local state \( \hat{L}, \hat{S} \) is well-formed for \( \rho \) if (1) every input location of \( \rho \) is in \( \text{dom} (\hat{L}) \), (2) \( \hat{L}, \hat{S} \) is a directed acyclic graph, and (3) for all \( \text{Borrow}(x, y) \) in \( \hat{L}, \hat{S} \), we have \( x, y \in \text{dom} (\hat{L}) \), and \( y \) is not an input location of \( \rho \).

The propagation on local abstract states (§0.1) uses judgments of the form \( \rho, \text{op} \rightarrow \langle \hat{L}, \hat{S}, B \rangle \rightarrow \langle \hat{L}', \hat{S}', B' \rangle \) indicating that procedure \( \rho \) executing instruction \( \text{op} \) from \( \langle \hat{L}, \hat{S}, B \rangle \) results in \( \langle \hat{L}', \hat{S}', B' \rangle \). These rules are designed to ensure that if \( \langle \hat{L}, \hat{S}, B \rangle \) is well-formed for \( \rho \), then \( \langle \hat{L}', \hat{S}', B' \rangle \) is also well-formed for \( \rho \).

Extending to the partial ordering \( \subseteq \) on borrow graphs to abstract states, we write \( \langle \hat{L}, \hat{S}, B \rangle \subseteq \langle \hat{L}', \hat{S}', B' \rangle \) if \( \hat{L} \subseteq \hat{L}' \), \( \hat{S} \subseteq \hat{S}' \), and \( B \subseteq B' \). It will follow from the properties of \( \subseteq \) on borrow graphs that every concrete local state meeting the conditions imposed by \( \langle \hat{L}, \hat{S}, B \rangle \) also meets those imposed by \( \langle \hat{L}', \hat{S}', B' \rangle \).

**Well-typed programs.** In proving the soundness of the borrow checker, we assume the program is annotated with an appropriate abstract state for entry to each instruction. Specifically, for each procedure \( \rho \), we assume there is a given abstract local state \( \mathcal{P} [\rho], T[i] \) for each offset \( i \in 0..\text{len}(\mathcal{P} [\rho], C) \). To be well-typed, each abstract state must be well-formed for the procedure in which it occurs and must satisfy two additional conditions, one involving the initial input conditions and the other involving control flow.

**Initial state condition.** Because \( \text{jm: fix this} \), we require that \( \langle \mathcal{P} [\rho], \mathcal{I}, \{\} \rangle = \mathcal{P} [\rho], T[0] \), meaning … (on procedure entry, the operand stack and borrow graph are empty; possibly not correct because of parameter passing on the operand stack)

Second, the abstract state annotations must reflect the control flow of the program. To state this, we let \( \text{Next}(\rho)[i] \) be the set of program counters to which control transfer is possible after executing instruction at position \( i \), specifically

\[
\text{Next}(\rho)[i] = \{i\}, \quad \text{if } \mathcal{P} [\rho], C[i] = \text{Ret}(\ell_1, \ell_2), \quad \text{if } \mathcal{P} [\rho], C[i] = \text{Branch}(\ell_1, \ell_2)
\]

\[
\{i+1\}, \quad \text{otherwise}
\]

**Control flow condition.** For all for all \( i \in \text{dom} (\mathcal{P} [\rho], C) \), there must be an abstract state with \( \rho, \mathcal{P} [\rho], C[i] \rightarrow \mathcal{P} [\rho], T[i] \rightarrow \langle \hat{L}, \hat{S}, B \rangle \) according to the propagation rules and \( \langle \hat{L}, \hat{S}, B \rangle \subseteq \mathcal{P} [\rho], T[j] \) for all \( j \in \text{Next}(\rho)[i] \). In other words, any conditions expressed in the abstract state propagation rules must be met and the abstract state produced by applying the propagation rules must be \( \subseteq \) the annotation of any next program state. We do not require equality because there may be more than one control path for reaching the next instruction and the borrow checking algorithm may therefore select some abstract state that is \( \subseteq \) all of them.

A program \( \mathcal{P} \) is well-typed if the initial state condition and the control flow condition above are met for all \( \rho \in \text{dom} (\mathcal{P}) \).

**State abstraction function.** We can combine program annotations, which give a local abstract state for each program point, with a concrete state that has a stack of program points, to produce an abstract state. This abstraction function has the property that as a concrete execution steps through concrete states, abstract execution steps through the corresponding abstract states.
Given any concrete state \( s = (P, M) \) for a well-typed program \( P \), we define the abstract state \( \text{Abs}(P) = P \) by induction on the height of \( P \).

If the concrete call-stack \( P = [] \) is the empty list, then \( \text{Abs}(P) = [] \) is empty.

If the concrete call-stack is \( (\rho, t, \ell, S):P \), the corresponding abstract call-stack is 
\[
(\rho, t, \hat{\ell}, \hat{S}, B) : \text{Abs}(P)
\]
where \( \langle \hat{L}, \hat{S}, B \rangle : P \langle \rho_0 \rangle : T \langle \ell_0 \rangle \) is the type annotation for program point \( t \) in the procedure \( \rho \). For \( s = (P, M) \), we let \( \text{Abs}(s) = \text{Abs}(P) \).

**Borrow Checker Invariants**

We use the abstraction function to prove critical invariants about executions of well-typed programs. These invariants establish four properties: type agreement, no memory leaks, no dangling references, and referential transparency. We combine these invariants as a predicate \( \text{Inv}(s, \hat{s}) \) over a concrete state \( s \) and an abstract state \( \hat{s} \).

**Theorem 1.** Let program \( P \) be well-typed. If \( s \) is a concrete state with \( \text{Inv}(s, \text{Abs}(s)) \) and \( P + s \to s' \), then \( \text{Inv}(s', \text{Abs}(s')) \).

A corollary is that if \( P \) starts execution in a concrete state \( s_0 \) such that \( \text{Inv}(s_0, \text{Abs}(s_0)) \) holds, then \( \text{Inv}(s, \text{Abs}(s)) \) holds for all states reachable from \( s_0 \). Any initial state \( s_0 \) of \( P \) is of the form \( \langle (\rho, 0, \overline{T}), [], \{\} \rangle \) representing the beginning of a transaction that invokes \( \rho \) with inputs \( \overline{T} \) comprising only values (no references), empty operand stack, and empty memory. It is easy to see that \( \text{Inv}(s_0, \text{Abs}(s_0)) \) holds if \( P \) is well-typed.

We define \( \text{Inv} \) as the conjunction of four predicates, \( \text{InvA}, \text{InvB}, \text{InvC}, \text{InvD} \), described below.

**Type Agreement.**
\( \text{InvA}(s, \hat{s}) : s \) and \( \hat{s} \) are shape-matching and type-matching.

**No Memory Leaks.** The following invariant indicates that (1) every local variable on the call stack of \( s \) contains a different location, and (2) locations are not leaked, i.e., \( s . M \) does not contain any location not present in a local variable.

\( \text{InvB}(s, \hat{s}) : \) The relation \( \{(n, s(n)) \mid n \in \text{dom}(\hat{s}) \land s(n) \in \text{Loc}\} \) is a bijection from its domain to \( \text{dom}(s . M) \).

\( \text{jm}: \) Since \( \text{dom}(\hat{s}) = \text{dom}(s) \), it looks like this can be stated as a property of \( s \) without referring to \( \hat{s} \). Does this property depend on the borrow checker?

**No Dangling References.** The following invariant allows us to conclude that every reference is rooted in a memory location present in some local variable on the call stack.

\( \text{InvC}(s, \hat{s}) : \)
1. \( \text{BG}(\hat{s}) \) is acyclic.
2. For all \( n \in \text{dom}(\hat{s}) \) such that \( \hat{s}(n) \in \text{ValType} \), there is no borrow edge in \( \text{BG}(\hat{s}) \) coming into \( n \).

3. For all \( n \in \text{dom}(\hat{s}) \) such that \( \hat{s}(n) \in \text{RefType} \), there is a borrow edge in \( \text{BG}(\hat{s}) \) coming into \( n \) that is realized in \( s \).

**Referential Transparency.** The following invariant indicates that the absence of borrow edges out of an abstract location containing a value or a mutable reference guarantees that mutation via that abstract location, either of the stored value or the value pointed to by the mutable reference, will not invalidate any live reference.

\( \text{InvD}(s, \hat{s}) : \) For any distinct \( m, n \in \text{dom}(\hat{s}) \) such that \( \hat{s}(n) \in \text{RefType} \) and \( s(m) \leq s(n) \), one of the following hold:
1. \( \hat{s}(m) \in \text{ImmRefType} \) and \( \hat{s}(n) \in \text{ImmRefType} \).
2. \( \hat{s}(m) \notin \text{ImmRefType} \) and there is a path in \( \text{BG}(\hat{s}) \) from \( m \) to \( n \) comprising realized edges in \( s \).
3. \( s(m) = s(n) \) and there is a path in \( \text{BG}(\hat{s}) \) from \( n \) to \( m \) comprising realized edges in \( s \).

The following lemma states an important and useful fact connecting borrow graphs and reference types.

**Lemma 0.** Suppose \( \hat{s} = \text{Abs}(s) \) and \( \text{InvD}(s, \hat{s}) \). If \( m, n \in \text{dom}(\hat{s}) \) are distinct, \( \hat{s}(n) \in \text{MutRefType} \), and there is a path in \( \text{BG}(\hat{s}) \) realizable in \( s \) from \( m \) to \( n \), then \( \hat{s}(m) \notin \text{ImmRefType} \).

**Proof of Lemma 0.** If there is a path in \( B = \text{BG}(\hat{s}) \) realizable in \( s \) from \( m \) to \( n \), then \( s(m) \leq s(n) \). We know that \( \hat{s}(n) \notin \text{ImmRefType} \). We also know that since the borrow graph in \( \hat{s} \) is acyclic, there is no path in it from \( n \) to \( m \). The definition of referential transparency property expressed in \( \text{InvD} \) allows us to conclude that \( \hat{s}(m) \notin \text{ImmRefType} \).

**0.1 Propagating Local Abstract States**

Having explained the overall structure of our soundness argument, we now provide intuition for type propagation on local abstract states. The rule for operation \( op \) derives a judgment of the form \( \rho, op \langle \hat{L}, \hat{S}, B \rangle \to \langle \hat{L}', \hat{S}', B' \rangle \) if certain conditions are satisfied. These conditions include availability of appropriately-typed values in \( \hat{L} \) or \( \hat{S} \) and absence of certain edges in the borrow graph \( B \). The state transformation adds or removes a variable-to-type binding in \( \hat{L} \), pushes or pops types in \( \hat{S} \), and adds or removes edges in \( B \). The rules for \( \text{MoveLoc} \) and \( \text{StoreLoc} \) also prevent an input of procedure \( p \) from being moved or overwritten to enable accurate tracking of transitive borrow relationships across a procedure call.

**Operations on borrow graphs.** There are six operations on borrow graphs and two queries.

1. The operation \( \text{rename}(B, u, v) \) replaces every edge \( \text{Borrow}(u, p, w) \in B \) with \( \text{Borrow}(v, p, w) \), for all \( w \). This is used when position \( u \) is eliminated and \( v \) is created, such as when a local variable is eliminated and its value pushed onto the operand stack. In the case the second argument is a set, \( \text{rename}(B, \{(u_i, v_i) \mid i \in is\}) \) replaces each \( \text{Borrow}(u_i, p, w) \) with \( \text{Borrow}(u_i, p, w) \), for all \( w \).
(2) The operation \( \text{factor}(B, u, v) \) creates a new borrow graph by replacing each edge \( \text{Borrow}(u, p, w) \) going out of \( u \) by \( \text{Borrow}(u, p, w) \) going out of \( v \) and adding a new edge \( \text{Borrow}(u, f, v) \). The node \( u \) may have borrow edges incident in \( B \) but \( v \) may not.

(3) The operation \( \text{factor}_f \) is similar to \( \text{factor} \). It is a partial operation and may fail. If there is no edge labeled \( f \) or \( * \) coming out of \( u \) in \( B \), then \( \text{factor}_f(B, u, v) \) succeeds and replaces each edge of the form \( \text{Borrow}(u, f :: p, a) \) by \( \text{Borrow}(u, p, a) \). The operation also adds an edge \( \text{Borrow}(u, f, v) \).

(4) The operation \( \text{Borrow}(u, p, o) \) adds the edge \( \text{Borrow}(u, p, o) \) if it is not already present.

(5) The operation \( \text{extend}(B, u, v) \) is applicable to sets \( u \) and \( v \) of positions, where there may be borrow edges incident on locations in \( u \) but no borrow edges incident on locations in \( v \). The result adds an edge \( \text{Borrow}(u, v, x) \) to \( B \) for every \( u \in u \) and \( v \in v \).

(6) The operation \( \text{elim}(B, u) \) creates a new borrow graph by eliminating location \( u \) in \( B \) as follows: (1) For every edge \( \text{Borrow}(a, p, u) \) coming into \( u \) and edge \( \text{Borrow}(a, q, b) \) going out of \( u \), add an edge \( \text{Borrow}(a, p, q) \). (2) Delete all edges coming into and going out of \( u \).

(7) The predicate \( \text{Unborrowed}(B, u) \) is true if there are no outgoing borrow edges from \( u \) in \( B \).

(8) The set \( \text{Borrowed}(B, u) \) is the set of all \( v \) borrowed from \( u \) in \( B \). \text{jm: Is this set of } v \text{ with } \text{Borrow}(u, p, a) \text{ for all } p? \text{ Or just } \? \text{ Or any path?} \text{ USED IN "The rules for FreezeRef and ReadRef both check that the reference operand at the top of the stack is freezable, i.e., all borrowed references from it are immutable."}

**Rules for instructions.** The rule for \( \text{MoveLoc}(x) \) moves the type of variable \( x \) to the top of operand stack. The rule checks that \( x \) is available in \( L \) and there are no outgoing borrow edges from \( \Pi(0, x) \), the abstract location of \( x \), in case \( x \) is a value. The rule also renames the old position of the moved value in the borrow graph to its new position.

The rule for \( \text{Pop} \) pops the top of the operand stack and eliminates the location corresponding to it using a new operation \( \text{elim} \). \text{jm: The definition of } \text{elim}(B, u) \text{ ensures that all transitive borrow relationships going through the reference at the top of the operand stack are maintained even when the top is popped.}

The two rules for \( \text{StoreLoc}(x) \) use a combination of the techniques introduced for handling \( \text{MoveLoc}(x) \) and \( \text{Pop} \). If \( x \) is available in the local store and is a reference type, \( \Pi(0, x) \) is eliminated in the borrow graph. If \( x \) is available in the local store and is a value type, then it is checked that there are no borrow edges going out of \( \Pi(0, x) \). In both cases, the position for the previous top of stack is renamed to \( \Pi(0, x) \) since it is being moved into variable \( x \).

The rule for \( \text{BorrowLoc}(x) \) uses the operation \( \text{factor} \). \text{jm: explain This operation ensures that borrow from } u \text{ are propagated to } v.\text{jm: Add explanation of these now?}

The first rule for \( \text{BorrowField}(f, x) \) addresses the case when the source reference is mutable and creates a mutable borrow from it. This rule uses \( \text{factor} \), which succeeds if there is no edge labeled \( f \) or \( * \) coming out of \( u \) in \( B \), converting each edge of the form \( \text{Borrow}(u, f :: p, a) \) to \( \text{Borrow}(u, p, a) \) and adding the edge \( \text{Borrow}(u, f, a) \). This rule ensures that any borrow from variable \( x \) along the field \( f \) are instead borrow from the new reference pushed on the operand stack which is itself borrowed from \( x \).

The second rule for \( \text{BorrowField}(f, x) \) addresses the case when the source reference is immutable and creates an immutable borrow from it. This rule simply adds a borrow edge labeled \( f \) between the source reference and the new reference.

The rule for \( \text{CopyLoc}(x) \) is similar to \( \text{BorrowLoc}(x) \) in case the variable \( x \) being copied is a reference. The rule for \( \text{WriteRef} \) checks that the target reference does not have any borrow edges coming out of it. The rules for \( \text{FreezeRef} \) and \( \text{ReadRef} \) both check that the reference operand at the top of the stack is freezable, i.e., all borrowed references from it are immutable. If a mutable reference is freezeable, it is safe to convert it into an immutable reference.

For space reasons, we skip over the rules for \( \text{Pack} \), \( \text{Unpack} \), and \( \text{Op} \) which do not perform any reference-related operations. \text{jm: Add explanation of these now?}

The rule for \( \text{Call}(\rho) \) can be understood as a sequence of simple steps. First, it checks that no mutable reference being passed to \( \rho \) is borrowed. Second, it renames the call arguments present on the operand stack to the corresponding locals in the next frame to simulate the call (see definition of \( B_1 \)). Third, it simulates the return from the call by adding borrow edges from input reference parameters to output references returned by the call (see definition of \( B_2 \) and \( B_3 \)) and eliminating the locals in the callee frame (see definition of \( B' \)). The definitions of \( B_2 \) and \( B_3 \) use the \( \text{extend} \) operation and applies \( \text{elim} \) to a set of locations to eliminate each of them from the borrow graph.

The rule for \( \text{Ret} \) checks that no local of value type is borrowed, the contents of the operand stack matches the output signature of \( \rho \), and no output that is a mutable reference is borrowed. This ensures that returned references are valid and the caller’s expectations for borrow relationships are sound.

**Soundness proof.** We prove soundness of the borrow checker (Theorem 1) by showing that the inductive invariant between concrete and abstract states is preserved by execution of any well-typed program.

The proof of Theorem 1 is decomposed into several lemmas. \text{jm: fix this:} The first lemma shows that if we execute any instruction next from any concrete state, the invariant is preserved between the new concrete state and an abstract
The preservation of references, and referential transparency. The first two conditions are identical on this position is realized. Also, any realized borrow checker rules. This state must be in the $\subseteq$ relation with with the next program annotation. We next use another lemma relating $\subseteq$ and the invariant to prove the theorem.

Lemma 1. Let $s = (P,M)$ be a concrete state with non-empty call stack $\mathcal{P} = \langle p.t,L_1,S_1 \rangle :: P_0$. Let $\hat{s} = \text{Abs}(s) = \langle p.t,L_1,S_1,B_1 \rangle :: \text{Abs}(P_0)$ be the corresponding abstract state. If

- $\text{Inv}(\hat{s},\hat{s})$
- $L_1.S_1.B_1 \subseteq \langle L_2.S_2.B_2 \rangle$

then $\text{Inv}(s,\langle p.t,L_2,S_2.B_2 \rangle :: \text{Abs}(P_0))$.

Proof: If $\langle L_1.S_1.B_1 \rangle \subseteq \langle L_2.S_2.B_2 \rangle$. Then there exists an abstract state $s' = (P',\ell',L'$.S') such that

- $\text{Inv}(s',\langle L'.S'.B' \rangle :: \text{Abs}(P''))$
- $\langle L'.S'.B' \rangle \subseteq \mathcal{P}[\ell'[\ell']].$

Proof of Lemma 1. If $\langle L_1.S_1.B_1 \rangle \subseteq \langle L_2.S_2.B_2 \rangle$ then by definition of $\subseteq$ it follows that $L_1 = L_2$, $S_1 = S_2$, and each edge in $B_1$ is subsumed by some edge in $B_2$. Thus, the call stacks and operand stacks of $s_1$ and $s_2$ are identical and each edge in the global borrow graph of $s_1$ is subsumed by some edge in the global borrow graph of $s_2$. The preservation of $\text{Inv}$ follows from the definition of subsumption of a borrow edge.

Lemma 2. Let $s = (P,M)$ be a concrete state with call stack $\mathcal{P} = \langle p.t,L_1,S_1 \rangle :: P_0$. Let $\hat{s} = \text{Abs}(s) = \langle p.t,L_1,S_1,B_1 \rangle :: \text{Abs}(P_0)$ be the corresponding abstract state. If

- $\text{Inv}(\hat{s},\hat{s})$
- $\mathcal{P} \vdash s \rightarrow s'$ by a local instruction
- $\rho,\mathcal{P}[\rho].C[\ell] + \mathcal{P}[\rho].T[\ell] \rightarrow \langle L.S.B \rangle$

then $\text{Inv}(s,\langle p.t,L_2,S_2.B_2 \rangle :: \text{Abs}(P_0))$.

Proof of Lemma 2. Let $\hat{s} = \text{Abs}(s) = \langle p.t,L.\ell \rangle :: \text{Abs}(P_0)$. Note that $\hat{s}$ has the same procedure name and program counter in the top of the call stack as $s$. We note that $\hat{s} = \text{Base}(s) \uplus \mathcal{P}[\rho].T[\ell]$. Let $\mathcal{P}[\rho].T[\ell] = \langle L_0, \hat{S}_0, B_0 \rangle$. Let $s' = \text{Base}(s) \uplus \langle \hat{L}.S.B \rangle$. We must show $\text{Inv}(s',\hat{s})$. The proof proceeds by case analysis according to the local instruction that is executed to reach $s'$. There are four parts of the invariant to check for each instruction: type agreement, no memory leaks, no dangling references, and referential transparency. The first two conditions are straightforward for all local instructions and are therefore omitted from this presentation of the proof.

$\text{BorrowLoc}(x)$. Suppose $\mathcal{P}[\rho].C[\ell] = \text{BorrowLoc}(x)$ with $L(x) = c$. This instruction pushes a reference to $c$ on top of the operand stack. The operational semantics give us

- $s' = (\langle p.t,\ell \rangle :: \mathcal{P}[\rho].T[\ell], c :: \text{Abs}(\hat{S}_0))$.

The rule for $\text{BorrowLoc}(x)$ shows $L_0(\ell) = \ell$ for $\ell \in \text{ValType}$, $L = L_0, \hat{S} = \text{Mut}(\ell) :: \text{Abs}(B_0, B_0, \text{factor} (B_0, 0, 0, \Omega (\text{len}(\hat{S}))))$. Thus, $B$ is derived from $B_0$ by

1. adding the edge $\text{Borrow}(\Pi (0, 0, x), \ell, \Omega (\text{len}(\hat{S})))$,
2. replacing each edge $\text{Borrow}(\Pi (0, 0, x), p, n)$ by the edge $\text{Borrow}(\Omega (\text{len}(\hat{S})), p, n)$.

$\text{No dangling references}$. The only new position in $\text{Abs}(s')$ is the new top of the operand stack containing a mutable reference. The new borrow edge $\text{Borrow}(\Pi (0, 0, x), \ell, \Omega (\text{len}(\hat{S})))$ incident on this position is realized. Also, any realized borrow edge leaving $\Pi (0, x)$ that was moved to leave from $\Omega (\text{len}(\hat{S}))$ continues to be a realized edge. Therefore the property holds for $s'$ and $s$.

Referential transparency. We need to check all $m, n \in \text{dom}(s')$ such that $s'(n) \in \text{RefType}$ and $s'(m) \leq s'(n)$. There are three cases depending on whether one of $m$ or $n$ is the new location $\Omega (\text{len}(\hat{S}))$.

Let $m \neq \Omega (\text{len}(\hat{S}))$ and $n \neq \Omega (\text{len}(\hat{S}))$. This case works because the factor operation is such that any realizable path
that went through \( \Pi(0, x) \) now goes through both \( \Pi(0, x) \) and \( \Omega(\text{len}(\hat{S})) \).

If \( m = \Omega(\text{len}(\hat{S})) \), then \( s(\Pi(0, x)) < s(n) \). There is a realizable path in \( s \) from \( \Pi(0, x) \) to \( n \) by induction. This path can be transformed into a realizable path in \( s' \) that goes through \( \Omega(\text{len}(\hat{S})) \).

If \( n = \Omega(\text{len}(\hat{S})) \), then either \( m = \Pi(0, x) \) or \( s(m) = s'(m) = s'(n) \). In the first case, the required path is simply the new edge. In the second case, there is a realizable path in \( s \) from \( \Pi(0, x) \) to \( m \) by induction. This path can be transformed into a realizable path in \( s' \) that goes through \( \Omega(\text{len}(\hat{S})) \).

BorrowField\((f, x)\). Suppose \( P[p].C(t) = \text{BorrowField}(f, x) \) with \( L(x) = \text{ref}(c.p) \). There are two cases, based on whether \( L_0(x) = \text{Mut}((f, t, \ldots)) \) or \( L_0(x) = \text{Imm}((f, t, \ldots)) \). In both cases, the instruction pushes a reference \( \text{ref}(c.p :: f) \) derived from \( L(x) \) on top of the operand stack. The operational semantics gives us:

\[
\hat{s} = (\langle p.f.t.l.p : \text{ref}(c.p :: f) : S, M \rangle,
\]

However, the static analysis differs in the two cases. We can examine each case separately.

**Case 1.** Suppose \( L_0(x) = \text{Mut}((f, t, \ldots)) \) is a mutable reference. The corresponding borrow-checker rule shows \( L = L_0, \hat{S} = \text{Mut}(t) :: S_0, \) and \( B = \text{factor}(B_0, \Pi(0, x), \Omega(\text{len}(\hat{S}))) \). This borrow graph operation fails and the bytecode program is rejected if there is any edge labeled \( e \) or \( * \) coming out of \( \Pi(0, x) \) in \( B_0 \). We therefore assume there is no such edge in \( B_0 \) and derive \( B \) from \( B_0 \) by:

- Adding the edge \( \text{Borrow}(\Pi(0, x), f, \Omega(\text{len}(\hat{S}))) \), and
- Replacing each edge \( \text{Borrow}(\Pi(0, x), f :: p.n) \) by the edge \( \text{Borrow}(\Omega(\text{len}(\hat{S})), p.n) \).

No dangling references. This proof works exactly like the corresponding proof for BorrowLoc\((x)\).

Referential transparency. We need to check all \( m, n \in \text{dom}(\hat{s}') \) such that \( \hat{s}'(n) \in \text{RefType} \) and \( \hat{s}'(m) \leq \hat{s}'(n) \). There are three cases depending on whether one of \( m \) or \( n \) is the new location \( \Omega(\text{len}(\hat{S})) \).

If \( m \neq \Omega(\text{len}(\hat{S})) \) and \( n \neq \Omega(\text{len}(\hat{S})) \), then the proof works exactly like the corresponding proof for BorrowLoc\((x)\).

If \( m = \Omega(\text{len}(\hat{S})) \), then \( s(\Pi(0, x)) < s(n) \). There is a realizable path in \( s \) from \( \Pi(0, x) \) to \( n \) by induction. Since there is no edge labeled \( e \) or \( * \) out of \( \Pi(0, x) \), this path must follow one of the factored edges. Therefore, this path can be transformed to go through \( \Omega(\text{len}(\hat{S})) \) in \( s' \).

If \( n = \Omega(\text{len}(\hat{S})) \), then \( s(m) < s(\Pi(0, x)) \) or \( s(\Pi(0, x)) = s(m) < s(\Pi(0, x)) < s(m) \). In the first case, there is a realizable path in \( s \) from \( m \) to \( \Pi(0, x) \) by induction. We can extend this path to \( n \) by using the realizable edge from \( \Pi(0, x) \) to \( \Omega(\text{len}(\hat{S})) \).

In the third case, \( s(m) = s'(m) = s'(n) \) and there is a realizable path in \( s \) from \( \Pi(0, x) \) to \( m \) by induction. Since there is no edge labeled \( e \) or \( * \) out of \( \Pi(0, x) \), this path must follow one of the factored edges. Therefore, this path can be transformed to go through \( \Omega(\text{len}(\hat{S})) \) in \( s' \).

**Case 2.** Suppose \( \hat{L}_0(x) = \text{Imm}((f, t, \ldots)) \) is an immutable reference. The corresponding borrow-checker rule shows \( L = L_0, \hat{S} = \text{Imm}(t) :: S_0, \) and \( B = B_0 \cup \{ \text{Borrow}(\Pi(0, x), f, \Omega(\text{len}(\hat{S})) \} \).

No dangling references. This proof works exactly like the corresponding proof for BorrowLoc\((x)\).

Referential transparency. We need to check all \( m, n \in \text{dom}(\hat{s}') \) such that \( \hat{s}'(n) \in \text{RefType} \) and \( \hat{s}'(m) \leq \hat{s}'(n) \). There are three cases depending on whether one of \( m \) or \( n \) is the new location \( \Omega(\text{len}(\hat{S})) \).

If \( m \neq \Omega(\text{len}(\hat{S})) \) and \( n \neq \Omega(\text{len}(\hat{S})) \), then the proof works exactly like the corresponding proof for BorrowLoc\((x)\).

If \( m = \Omega(\text{len}(\hat{S})) \), then \( s(\Pi(0, x)) < s(n) \). We get \( s(n) \in \text{ImmRefType} \) by induction.

If \( n = \Omega(\text{len}(\hat{S})) \), then \( s(m) < s(\Pi(0, x)) \) or \( s(m) = s(\Pi(0, x)) \) or \( s(\Pi(0, x)) < s(m) \). In the first case, by induction (instantiating \( m \) with \( m \) and \( n \) with \( \Pi(0, x) \)), either \( s(m) \in \text{ImmRefType} \) or \( s(m) \notin \text{ImmRefType} \) and there is a realizable path in \( s \) from \( m \) to \( \Pi(0, x) \) which can be extended to a realizable path in \( s' \) to \( n \). In the second case, by induction (instantiating \( m \) with \( \Pi(0, x) \) and \( n \) with \( m \)), either \( s(m) \in \text{ImmRefType} \) or \( s(m) \notin \text{ImmRefType} \) and there is a realizable path in \( s \) from \( m \) to \( \Pi(0, x) \) which can be extended to a realizable path in \( s' \) to \( n \).

In the third case, we get \( s(m) \in \text{ImmRefType} \) by induction.

**MoveLoc\((x)\).** Vertices in the borrow graph are renamed from the old position of the moved value to its new position. The new position does not appear as the source or target of any borrow edge in the borrow graph. Therefore, the new borrow graph is isomorphic to the old borrow graph. Properties (No dangling references) and Referential transparency) follow from the fact that these properties held before this instruction was executed.

**Pop.** The analysis pops the top of the abstract operand stack and eliminates the location corresponding to it using the **elim** operation on borrow graphs. This operation eliminates location \( u \) in \( B_0 \) as follows: (1) For every edge \( \text{Borrow}(a, p, u) \) coming into \( u \) and edge \( \text{Borrow}(u, q, b) \) going out of \( u \), add an edge \( \text{Borrow}(a, p :: q, b) \). (2) Delete all edges coming into and going out of \( u \). This ensures that all transitive borrow relationships going through the reference that has been popped off the operand stack are maintained, preserving (No dangling references) and (Referential transparency) as a result.

**StoreLoc\((x)\).** The proof is a combination of the proof for **MoveLoc\((x)\)** and **Pop.**

**CopyLoc\((x)\).** The proof is similar to **BorrowLoc\((x)\)** in case the variable \( x \) being copied is a reference. Otherwise,
there is no change in the borrow graph and the proof is straightforward.

WriteRef. The proof is similar to Pop.

FreezeRef. The operational semantics gives us \( s' = s \). If the reference on top of the abstract operand stack in \( \text{Abs}(s) \) is immutable, the local abstract state does not change and the proof is done. Otherwise, the freezeable mutable reference at the top of the abstract operand stack is converted into an immutable reference.

No dangling references. This proof is straightforward since there is no change in the borrow graph.

Referential transparency. We need to check all \( m,n \in \text{dom}(s') \) such that \( \dot{s}'(n) \in \text{RefType} \) and \( s'(m) \leq s'(n) \). There are three cases depending on whether one of \( m \) or \( n \) is the location \( \Omega(\text{len}(\hat{S}) - 1) \).

If \( m \neq \Omega(\text{len}(\hat{S}) - 1) \) and \( n \neq \Omega(\text{len}(\hat{S}) - 1) \), then the proof is straightforward.

Suppose \( m = \Omega(\text{len}(\hat{S}) - 1) \). By induction, either (1) there is a path realizable in \( s \) from \( m \) to \( n \), or (2) \( s(m) = s(n) \) and there is a path realizable in \( s \) from \( n \) to \( m \). In the first case, the freezeability of \( m \) and Lemma 0 indicate that \( \dot{s}(n) \in \text{ImmRefType} \). Since \( \dot{s}(n) = \dot{s}'(n) \), we are done. In the second case, we can reuse the path since \( s = s' \).

Suppose \( n = \Omega(\text{len}(\hat{S}) - 1) \). By induction, either (1) \( \dot{s}(m) \notin \text{ImmRefType} \) and there is a path realizable in \( s \) from \( m \) to \( n \), or (2) \( s(m) = s(n) \) and there is a path realizable in \( s \) from \( n \) to \( m \). In the first case, we can reuse the path since \( s = s' \) and \( \dot{s}(m) = \dot{s}'(m) \). In the second case, the freezeability of \( n \) and Lemma 0 indicate that \( \dot{s}(m) \in \text{ImmRefType} \). Since \( \dot{s}(m) = \dot{s}'(m) \), we are done.

ReadRef. The proof is similar to Pop.

We omit discussion of Pack, Unpack, and Op since these instructions do not perform any reference-related operations.

\[ \square \]

**Old version, for comparison**

To aid our proof, we define an auxiliary transition relation \( \mathcal{P} \vdash s \rightarrow s' \). We view a transition \( \mathcal{P} \vdash s \rightarrow s' \) as happening in two steps. In the first step, the instruction pointed to by the program counter at the top of the call stack in \( s \) is executed to change parts of the state to yield \( s^- \) without adjusting the program counter to point to the next instruction. In the second step, the program counter in \( s^- \) is adjusted to yield \( s' \) and move to the next instruction. We write \( \mathcal{P} \vdash s \rightarrow s^- \) to denote the first part of this two-step process. The concrete states \( s^- \) and \( s' \) coincide in the two cases when the call stack in \( s' \) is empty or \( s' \) is obtained from \( s \) by executing a call instruction; in these two cases, \( \mathcal{P} \vdash s \rightarrow s' \) is a single step.

The following two lemmas together state that the transformation of the abstract state when a local instruction is executed preserves \( \text{Inv} \). The proof of Lemma 2 is provided later in this document.

**Lemma 2-old.** If

- \( s = (\rho, \ell, \cdot, \cdot) \)
- \( \text{Inv}(s, \text{Abs}(s)) \)
- \( \mathcal{P} \vdash s \rightarrow s' \) by a local instruction
- \( \rho, \mathcal{P}[\rho], C[t] \vdash \mathcal{P}[\rho], T[t] \rightarrow (\hat{L}, \hat{S}, B) \)

then \( \text{Inv}(s', \text{Base}(s) \uparrow (\hat{L}, \hat{S}, B)) \).

**Lemma 3-old.** If

- \( \text{Inv}(s, \text{Abs}(s)) \)
- \( \mathcal{P} \vdash s \rightarrow s' \)
- \( \mathcal{P} \vdash (\rho, \ell, \cdot, \cdot) \)

then there exists \( (\hat{L}, \hat{S}, B) \) such that

- \( \text{Inv}(s', \text{Base}(s') \uparrow (\hat{L}, \hat{S}, B)) \)
- \( (\hat{L}, \hat{S}, B) \subseteq \mathcal{P}[\rho], T[t] \).