A novel quantitative inverse scattering scheme using interior resonant modes

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Abstract

This paper is devoted to a novel quantitative imaging scheme of identifying impenetrable obstacles in time-harmonic acoustic scattering from the associated far-field data. The proposed method consists of two phases. In the first phase, we determine the interior eigenvalues of the underlying unknown obstacle from the far-field data via the indicating behavior of the linear sampling method. Then we further determine the associated interior eigenfunctions by solving a constrained optimization problem, again only involving the far-field data. In the second phase, we propose a novel iteration scheme of Newton’s type to identify the boundary surface of the obstacle. By using the interior eigenfunctions determined in the first phase, we can avoid computing any direct scattering problem at each Newton’s iteration. The proposed method is particularly valuable for recovering a sound-hard obstacle, where the Newton’s formula involves the geometric quantities of the unknown boundary surface in a natural way. We provide rigorous theoretical justifications of the proposed method. Numerical experiments in both 2D and 3D are conducted, which confirm the promising features of the proposed imaging scheme. In particular, it can produce quantitative reconstructions of high accuracy in a very efficient manner.

Keywords: inverse scattering problem, sound-hard obstacle, interior resonant modes, linear sampling method, Newton-type method

(Some figures may appear in colour only in the online journal)
1. Introduction

In this article, we are concerned with the inverse acoustic scattering problem of reconstructing an impenetrable obstacle by the associated far-field measurement. To begin with, we present the mathematical setup of the inverse scattering problem for our study.

Let \( k \in \mathbb{R}_+ \) be the wavenumber of a time-harmonic wave and \( D \subset \mathbb{R}^m, m = 2, 3 \) be a bounded domain with a Lipschitz-boundary \( \partial D \) and a connected complement \( \mathbb{R}^m \setminus D \), which signifies the target obstacle in our study. We take the incident field \( u' \) to be a time-harmonic plane wave of the form

\[
u'(x, d, k) = e^{ikx \cdot d}, \quad x \in \mathbb{R}^m, \]

where \( i := \sqrt{-1} \) is the imaginary unit, \( d \in \mathbb{S}^{m-1} \) is the direction of propagation and \( \mathbb{S}^{m-1} := \{ x \in \mathbb{R}^m : |x| = 1 \} \) is the unit circle/sphere in \( \mathbb{R}^m \). Let \( u' \) and \( u := u' + u' \) signify the scattered and total wave fields, respectively. The forward scattering problem is described by the following Helmholtz system:

\[
\begin{align*}
\Delta u + k^2 u &= 0 & \text{in } \mathbb{R}^m \setminus D, \\
B(u) &= 0 & \text{on } \partial D, \\
\lim_{r \to \infty} r^{\frac{m-1}{2}} \left( \frac{\partial u}{\partial r} - iku' \right) &= 0, & r = |x|, \\
\end{align*}
\]

where the last limit is known as the Sommerfeld radiation condition which holds uniformly in \( \hat{x} := x/|x| \in \mathbb{S}^{m-1} \) and characterizes the outgoing nature of the scattered field. In (1.1), \( B(u) = u \) or \( B(u) = \partial u/\partial v \), respectively, signify the physical scenarios that the obstacle \( D \) is sound-soft or sound-hard. Here and also in what follows, \( \nu \in \mathbb{S}^{m-1} \) denotes the exterior unit normal vector to \( \partial D \). The well-posedness of the scattering system (1.1) can be conveniently found in [26]. There exists a unique solution \( u \in H^1_{\text{loc}}(\mathbb{R}^m \setminus D) \), and it admits the following asymptotic expansion [11]:

\[
u'(x, d, k) = \frac{e^{i|\hat{x}|}}{|x|^{\frac{m-1}{2}}} \left\{ u^\infty(\hat{x}, d, k) + O \left( \frac{1}{|x|} \right) \right\} \quad \text{as } |x| \to \infty,
\]

which holds uniformly for all directions \( \hat{x} \in \mathbb{S}^{m-1} \). The complex-valued function \( u^\infty \) defined on the unit circle/sphere \( \mathbb{S}^{m-1} \) is known as the far-field pattern of the scattered field \( u' \), which encodes the information of the scattering obstacle \( D \). The inverse scattering problem of our concern is to recover \( D \) by knowledge of \( u^\infty \); that is,

\[
\mathcal{F}(D) = u^\infty(\hat{x}, d, k), \quad (\hat{x}, d, k) \in \mathbb{S}^{m-1} \times \mathbb{S}^{m-1} \times V,
\]

where \( V \) is an open interval in \( \mathbb{R}_+ \), and \( \mathcal{F} \) is an abstract operator defined by (1.1). Though the forward scattering problem (1.1) is linear, it can be straightforwardly verified that the inverse problem (1.2) is nonlinear.

The inverse scattering problem (1.2) is prototypical model of fundamental importance for many scientific and industrial applications including radar/sonar, medical imaging, geophysical exploration and nondestructive testing. There are rich results in the literature for (1.2), both theoretically and numerically. It is known that one has the unique identifiability for (1.2), namely the correspondence between \( u^\infty \) and \( D \) is one-to-one. Many numerical reconstruction algorithms have been developed in solving (1.2) as well. In fact, it is impossible for us to list and discuss all of the existing numerical studies for (1.2) in the literature. In what follows, we only discuss a few selective ones to motivate our current study. Roughly speaking, the existing numerical approaches can be classified into two categories: quantitative ones and qualitative.
Quantitative approaches employ Newton’s linearization and/or optimization strategies to tackle (1.2) directly. We refer the reader to the Newton’s iterative method [11, 15, 37], the decomposition method [11, 14, 17] and the recursive linearization method [2, 3] for the relevant results. The other class of approaches resorts to establishing a criterion to distinguish the interior and exterior of the obstacle $D$, and hence can qualitatively reconstruct the boundary of $D$. These include the linear sampling method (LSM) [5, 7, 10, 23], the factorization method [18], the direct sampling method [13, 22, 29] and the probe method [32, 33].

In this article, we develop a novel quantitative reconstruction scheme for (1.2). A key ingredient of our method is to connect the exterior scattering problem (1.1) to its interior counterpart:

$$\begin{align*}
\Delta v + k^2 v &= 0 \quad \text{in } D, \\
B(v) &= 0 \quad \text{on } \partial D,
\end{align*}$$

where $v \in H^1(D)$. Equation (1.3) is the classical Dirichlet/Neumann Laplacian eigenvalue problem and $v$ represents an interior resonant mode. It has been well understood, say e.g. there exist infinitely many discrete eigenvalues accumulating only at $\infty$, and the eigenspace associated with each eigenvalue is finite-dimensional. It turns out that the exterior problem (1.1) and the interior problem (1.3) are dual to each other, in particular in the sense that one can read off the spectral system of (1.3) from the far-field data of (1.1). This is done by employing a certain indicating behavior of the LSM and by solving a constrained optimization problem, both only involving the far-field data in a simple manner, which form the first phase of the proposed method. In the second phase, we derive an iteration formula of Newton’s type which can be used to quantitatively reconstruct the boundary of the obstacle. By utilizing the interior eigenfunctions determined in the first phase, the shape derivative involved in the Newton’s iteration can be calculated in a very efficient way without the need to solve any forward scattering problem. To highlight the novel contributions of the current study, we present three remarks as follows.

First, determining the interior eigenvalues from the associated far-field data via the LSM has been studied in [6, 20, 24, 30]. In [24], the determination of the interior Dirichlet Laplacian eigenfunctions has been further investigated. In fact, it is proposed in [24] that the Dirichlet eigenfunctions can be used to recover $\partial D$ (in a qualitative manner). This is natural since the Dirichlet eigenfunctions vanish on $\partial D$. However, such an idea cannot be extended to the sound-hard case since identifying the place where the Neumann data vanish requires knowing the normal vector of $\partial D$ which is equivalent to knowing $\partial D$. It is our aim of developing an imaging scheme that can recover sound-hard obstacles by making use of the interior resonant modes. It turns out that the scheme we develop can not only recover sound-hard obstacles but can also yield highly-accurate quantitative reconstructions. The key idea is to establish an iteration formula of Newton’s type by using the homogeneous Neumann condition on $\partial D$, which can then be used for identifying the unknown $\partial D$. A salient feature of this Newton’s iteration approach is that it is essentially derivative-free in the sense that the involved shape derivatives can be explicitly calculated by using the Neumann eigenfunctions that have been already determined, and there is no need to calculate any direct scattering problem. Moreover, the idea can be directly extended to the sound-soft case to produce quantitative reconstructions that outperform the qualitative reconstructions in [24]. However, it can be seen even at this point that the sound-hard case is technically more challenging than the sound-soft case. Hence, we shall mainly stick our study to the sound-hard case and only remark the sound-soft case throughout the rest of the paper.
Second, the far-field data used in (1.2) is significantly overdetermined. In fact, most of the existing methods in the literature, in particular those qualitative ones, make use of far-field data corresponding to a fixed $k$. Moreover, it is widely conjectured that one can determine an obstacle by at most a few or even a single far-field measurements, namely a few $k$ and $d$ or even fixed $k$ and $d$ in (1.2); see e.g. [8, 9, 25, 28, 34] for related theoretical uniqueness and stability results and [22] for related numerical reconstructions if a-priori geometrical knowledge is available on $D$. The overdetermination in (1.2), especially on $k$, is mainly needed to compute the interior eigenvalues located inside $V$, which is the prerequisite to the eigenfunction determination. As mentioned in the first remark, the eigenfunctions are critical to avoid computing shape derivatives for the Newton’s iteration in our method. Hence, the overdetermined data are an unobjectionable cost for achieving both high accuracy and high efficiency. It is interesting to note that all of the aforementioned qualitative methods inevitably involve shape derivatives and hence the calculation of a large amount of forward scattering problems, and moreover suffer from the local minima issue. In fact, to our best knowledge, no 3D quantitative reconstruction of a sound-hard obstacle was ever conducted in the literature due to the highly complicated computational nature. Nevertheless, we refer interested readers to [16], which explores a more intricate electromagnetic inverse scattering problem involving the use of a regularized iterative Newton scheme. In section 4, we present both 2D and 3D reconstructions and it can be seen that our method is computationally straightforward. Moreover, it can be seen that our method is robust and insensitive to the initial guess, and this is physically justifiable as the interior resonant modes carry the geometrical information of $D$ in a sensible way (though implicitly in the sound-hard case). It is also interesting to note the similarity shared by our method and the machine learning approaches for inverse obstacle problems. In [12, 36], machine learning approaches were developed that can yield a highly-accurate reconstruction of a target obstacle by using only a few far-field measurements. However, a large amount of data as well as computations are needed to train the neural networks therein, and hence are computationally more costly.

Third, we would like to mention two practical scenarios where our method might find applications to corroborate our viewpoint in the above remark. In many practical applications, say e.g. photo-acoustic tomography, time-dependent data are collected which can yield multiple frequency data as needed in (1.2) via temporal Fourier transform [27]. The other application is the bionic approach of generating human body shape [21], where overdetermined data are not a practical drawback, but highly accurate reconstructions are needed. We shall consider the application of our method in those practical setups in our future work.

The rest of the paper is organized as follows. Section 2 introduces the LSM to determine the interior eigenvalues from the multi-frequency far-field data. Then by the Herglotz wave approximation, we present an efficient optimization algorithm to reconstruct the interior eigenfunctions. In section 3, a novel Newton iterative formula is proposed to reconstruct the sound-hard obstacle via the use of the interior eigenfunctions. Numerical experiments are conducted in section 4 to verify the promising features of our method.

### 2. Determination of Neumann eigenvalues and eigenfunctions

In this section, we aim to determine the interior eigenvalues and eigenfunctions to (1.3) from the far-field data in (1.2) corresponding to the unknown $D$. As discussed in the previous section, we shall mainly focus on the sound-hard case, namely $B(u) = \partial u / \partial \nu$ and only remark the extension to the sound-soft case.
To begin with, we introduce the LSM for reconstructing the Neumann eigenvalues. The LSM is a widely used method to recover the shape of the scatterer without a priori information of the boundary condition of the scatterer. The basic idea of the LSM is choosing an approximate indicator function and distinguishing whether the sampling point is inside or outside the scatterer. To that end, we present the indicator function of the LSM. Define the test function by
\[ \Phi^\infty(\hat{x}, z, k) = e^{-ik\cdot z}, \quad \hat{x} \in \mathbb{S}^{m-1}, \]
where \( z \in \mathbb{R}^m \) denotes the sampling point. The key ingredient of the LSM is to find the kernel \( g_z \in L^2(\mathbb{S}^{m-1}) \) as a solution to the following integral equation
\[ (F_k g)(\hat{x}) = \Phi^\infty(\hat{x}, z, k), \]  
(2.1)
where \( F_k \) is the far field operator from \( L^2(\mathbb{S}^{m-1}) \) to \( L^2(\mathbb{S}^{m-1}) \) and it is defined by
\[ (F_k g)(\hat{x}) := \int_{\mathbb{S}^{m-1}} u^\infty(\hat{x}, d, k) g(d) \, ds(d), \quad \hat{x} \in \mathbb{S}^{m-1}. \]  
(2.2)

We note that the equation (2.1) is ill-posed not only due to the discontinuous dependence on the data, but more importantly due to the fact that the equation in general does not have a solution [6]. In what follows, we adopt Tikhonov regularization to solve the Fredholm integral equation (2.1). Due to the existence of noisy data in practice, we suppose that \( F^\delta_k \) is the corresponding operator to the noisy measurements \( u^\infty, \delta \) and then instead seek the unique minimizer \( g^\delta_z \in L^2(\mathbb{S}^{m-1}) \) of the following Tikhonov functional
\[ \|F^\delta_k g - \Phi^\infty(\cdot, z, k)\|_{L^2(\mathbb{S}^{m-1})}^2 + \eta \|g\|_{L^2(\mathbb{S}^{m-1})}^2, \]  
(2.3)
where \( \eta \) is a regularization parameter and it satisfies \( \eta(\delta) \to 0 \) as \( \delta \to 0 \).

Next, we discuss how to recover the Neumann eigenvalues by using the LSM. According to the rigorous justification of the rationale behind the LSM (see [1, 30]), one can use the following lemma to distinguish whether \( k^2 \) is a Neumann eigenvalue or not.

**Lemma 2.1.** Define the Herglotz wave function by
\[ v_{g,k}(x) := \int_{\mathbb{S}^{m-1}} e^{ikx \cdot d} g(d) \, ds(d), \quad x \in \mathbb{R}^m, \]  
(2.4)
where \( g \in L^2(\mathbb{S}^{m-1}) \) is referred to as the Herglotz kernel. Suppose that \( g^\delta \) is the unique minimizer of the Tikhonov functional (2.3). Then, for almost every \( z \in D, \|v_{g^\delta,k}\|_{H^1(D)} \) is bounded as \( \delta \to 0 \) if and only if \( k^2 \) is not a Neumann eigenvalue.

**Remark 2.1.** Due to the fact that the scatterer \( D \) is unknown, it is impossible to identify the Neumann eigenvalues based on the behavior of \( \|v_{g^\delta,k}\|_{H^1(D)} \). By lemma 2.1, we refer the reader to [30] for a strategy to determine the Neumann eigenvalues based on the property of the Herglotz kernel \( g^\delta \). The main idea is, for a point \( z \in D \), the norm \( \|g^\delta_k\|_{L^2(\mathbb{S}^{m-1})} \) against the wavenumber \( k \) is relatively large when \( k^2 \) is a Neumann eigenvalue, but relatively small when \( k^2 \) is not a Neumann eigenvalue.

Next, we proceed to recover the corresponding Neumann eigenfunctions. We first show that the Herglotz wave function can be used to approximate the solution of the Helmholtz equation by the following lemma.

**Lemma 2.2 ([35]).** Let \( D \subset \mathbb{R}^m \) be a bounded domain of class \( C^{\alpha,1} \), \( \alpha \in \mathbb{N} \cup \{0\} \) with a connected complement \( \mathbb{R}^m \backslash D \). Let \( H \) be the space of all Herglotz wave functions of the form (2.4).
Define, respectively,
\[ \mathcal{H}(D) := \{ u|_D : u \in \mathcal{H} \}, \]
and
\[ \mathcal{U}(D) := \{ u \in C^\infty(D) : \Delta u + k^2 u = 0 \text{ in } D \}. \]
Then \( \mathcal{H}(D) \) is dense in \( \mathcal{U}(D) \cap H^{\alpha+1}(D) \) with respect to the \( H^{\alpha+1}(D) \)-norm.

The following theorem plays an important role to determine the Neumann eigenfunctions.

**Theorem 2.1.** Assume that \( D \) is of class \( C^{0,1} \) when \( m = 2 \), and of class \( C^{1,1} \) when \( m = 3 \), and \( \mathbb{R}^m \backslash \overline{D} \) is connected. Suppose that \( k^2 \in \mathbb{R}_+ \) is a Neumann eigenvalue of \( -\Delta \) in \( D \) and \( u_k \) is an associated normalized eigenfunction, i.e. \( \|u_k\|_{H^1(D)} = 1 \). For any sufficiently small \( \varepsilon \in \mathbb{R}_+ \), there exists \( g_\varepsilon \in L^2(S^{m-1}) \) such that
\[
\|F_k g_\varepsilon\|_{L^2(S^{m-1})} \lesssim \varepsilon \quad \text{and} \quad \|v_{\varepsilon,k}\|_{H^1(D)} = 1, \tag{2.5}
\]
where \( F_k \) is the far field operator defined by (2.2) and \( v_{\varepsilon,k} \) is the Herglotz wave function defined by (2.4) with the kernel \( g_\varepsilon \). Here and also in what follows, \( a \lesssim b \) stand for \( a \leq Cb \) with a constant \( C > 0 \) depending only on \( D \) and \( k \).

On the other hand, if \( k^2 \in \mathbb{R}_+ \) is a Neumann eigenvalue and \( g_\varepsilon \) satisfies (2.5), then the Herglotz wave \( v_{\varepsilon,k} \) is an approximation to a Neumann eigenfunction \( u_k \) associated with the Neumann eigenvalue \( k^2 \) in the \( H^1(D) \)-norm, namely,
\[
\|v_{\varepsilon,k} - u_k\|_{H^1(D)} \lesssim \varepsilon.
\]

**Proof.** Let \( u_k \) be a normalized Neumann eigenfunction associated with the Neumann eigenvalue \( k^2 \), then \( u_k \in H^1(D) \) is a solution to the Neumann eigenvalue problem
\[
\Delta u_k + k^2 u_k = 0 \quad \text{in } D, \quad \frac{\partial u_k}{\partial \nu} = 0 \quad \text{on } \partial D.
\]
According to the denseness in lemma 2.2, for any \( 0 < \varepsilon < 1/2 \), there exists \( \tilde{g}_\varepsilon \in L^2(S^{m-1}) \) such that
\[
\|v_{\varepsilon,k} - u_k\|_{H^1(D)} < \varepsilon, \tag{2.6}
\]
where \( v_{\varepsilon,k} \) is the Herglotz wave function with the kernel \( \tilde{g}_\varepsilon \). By the triangle inequality, one can find that
\[
\|v_{\varepsilon,k}\|_{H^1(D)} \leq \|v_{\varepsilon,k} - u_k\|_{H^1(D)} + \|u_k\|_{H^1(D)} < \varepsilon + \|u_k\|_{H^1(D)}.
\]
Similarly, we have
\[
\|v_{\varepsilon,k}\|_{H^1(D)} \geq \|u_k\|_{H^1(D)} - \|v_{\varepsilon,k} - u_k\|_{H^1(D)} > \|u_k\|_{H^1(D)} - \varepsilon.
\]
Since \( u_k \in H^1(D) \) is a normalized eigenfunction, using the last two equations, one can obtain that
\[
\|v_{\varepsilon,k}\|_{H^1(D)} \in (1 - \varepsilon, 1 + \varepsilon) \subset \left( \frac{1}{2}, \frac{3}{2} \right).
\]
Letting
\[
g_\varepsilon := \frac{\tilde{g}_\varepsilon}{\|v_{\varepsilon,k}\|_{H^1(D)}},
\]
one can find that the Herglotz wave function \( v_{g, k} \) with the kernel \( g \) satisfies
\[
\| v_{g, k} \|_{H^s(D)} = 1. \tag{2.8}
\]
From (2.6)–(2.8), we can deduce that
\[
\| v_{g, k} - u_k \|_{H^s(D)} \leq \left\| v_{g, k} - v_{g, k} \|_{H^s(D)} + \| v_{g, k} - u_k \|_{H^s(D)} \right\| + \| v_{g, k} - u_k \|_{H^s(D)}
\]
\[
< \| v_{g, k} \|_{H^s(D)} [1 - \| v_{g, k} \|_{H^s(D)}] + \epsilon
\]
\[
< 2\epsilon.
\]
By the trace theorem, we can derive that
\[
\left\| \frac{\partial v_{g, k}}{\partial \nu} - \frac{\partial u_k}{\partial \nu} \right\|_{H^{-1/2}(\partial D)} \lesssim \epsilon. \tag{2.9}
\]
Noting that \( \frac{\partial u_k}{\partial \nu} = 0 \) on \( \partial D \), we clearly have from (2.9) that
\[
\left\| \frac{\partial v_{g, k}}{\partial \nu} \right\|_{H^{-1/2}(\partial D)} \lesssim \epsilon. \tag{2.10}
\]
It is directly verified that \( F_k g \) is the far-field pattern of the exterior scattering problem (1.1) associated with the incident field \( v_{g, k} \). Hence, based on the well-posedness of the forward scattering problem (1.1), together with (2.10), one can conclude that
\[
\| F_k g \|_{L^2(\mathbb{S}^{n-1})} \lesssim \epsilon, \tag{2.11}
\]
which proves the first part of the theorem.

Next, we prove that the Herglotz wave \( v_{g, k} \) is an approximation to the Neumann eigenfunction \( u_k \). Let \( v_{g, k}, u_{g, k}^s \) and \( u_{g, k} \) be, respectively, the incident, scattered, and total wave fields. It is clear that one has
\[
\begin{cases}
\Delta v_{g, k} + k^2 v_{g, k} = 0 & \text{in } D, \\
\frac{\partial v_{g, k}}{\partial \nu} = -\frac{\partial u_{g, k}^s}{\partial \nu} & \text{on } \partial D.
\end{cases} \tag{2.12}
\]
As we mentioned before, \( F_k g \) is the far-field pattern of \( u_{g, k}^s \). According to (2.5) and the quantitative Rellich theorem (see [4] as well as remark 2.2 in what follows), there is
\[
\left\| \frac{\partial u_{g, k}}{\partial \nu} \right\|_{L^2(\partial D)} \leq \psi(\epsilon), \tag{2.13}
\]
where \( \psi(\epsilon) := C_1 (\ln(\ln(C_0 \epsilon^{-1}))^{-1/2} \) is a real-valued function of double logarithmic type with positive constants \( C_0 \) and \( C_1 \), and it satisfies \( \psi(\epsilon) \to 0 \) as \( \epsilon \to 0^+ \). Here, both constants \( C_0 \) and \( C_1 \) depend only on \( D \) and \( k \). The equations (2.12) and (2.13) imply that
\[
\left\| \frac{\partial v_{g, k}}{\partial \nu} \right\|_{L^2(\partial D)} \leq \psi(\epsilon). \tag{2.14}
\]
Next, we consider the boundary value problem
\[
\begin{cases}
\Delta u + k^2 u = 0 & \text{in } D, \\
\frac{\partial u}{\partial \nu} = \frac{\partial v_{g, k}}{\partial \nu} & \text{on } \partial D.
\end{cases} \tag{2.15}
\]
We proceed to show that (2.15) is uniquely solvable in $H^1(D)/\mathcal{V}$, where $\mathcal{V}$ signifies the finite-dimensional eigenspace associated with the Neumann eigenvalue $k^2$ for $-\Delta$, and $H^1(D)/\mathcal{V}$ is the quotient space. To that end, we make use of the Fredholm theory for elliptic partial differential equations (cf theorem 4.10 in [31]), and first need to verify the compatibility condition is fulfilled. In fact, noting the homogeneous adjoint problem to (2.15) is exactly the Neumann eigenvalue problem. Letting $v \in H^1(D)$ be a Neumann eigenfunction, one can verify directly that

$$\int_{\partial D} v \cdot \frac{\partial v_{g, k}}{\partial \nu} = \int_D (\Delta v_{g, k} \cdot v - v_{g, k} \cdot \Delta v) = 0.$$ 

Hence, the compatibility condition is fulfilled and (2.15) is uniquely solvable in $H^1(\Omega)/\mathcal{V}$. Clearly, $v_{g, k}$ is a particular solution to (2.15), and we have from [31, theorem 4.10] that

$$\|v_{g, k} + \mathcal{V}\|_{H^1(D)/\mathcal{V}} \leq \tilde{C} \left\| \frac{\partial v_{g, k}}{\partial \nu} \right\|_{H^{-1/2}(\partial D)} \leq \tilde{C} \left\| \frac{\partial v_{g, k}}{\partial \nu} \right\|_{L^2(\partial D)} \leq \tilde{C} \psi(\varepsilon),$$

(2.16)

where $\tilde{C}$ depending only on $D$ and $k$. By (2.16), we readily see that there exists a Neumann eigenfunction $u_k$ such that

$$\|v_{g, k} - (-w_k)\|_{H^1(D)} \leq 2\tilde{C} \psi(\varepsilon).$$

(2.17)

Let $u_k := -w_k$, it is clear that $u_k$ is also a Neumann eigenfunction. Furthermore, the last inequality can be rewritten as

$$\|v_{g, k} - u_k\|_{H^1(D)} \leq 2\tilde{C} \psi(\varepsilon),$$

where $\psi(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. Hence, $v_{g, k}$ is an approximation to the Neumann eigenfunction $u_k$.

The proof is complete.

**Remark 2.2.** In the proof of theorem 2.1, we make use of the so-called quantitative Rellich theorem which states that if the far-field is smaller than $\varepsilon$ (i.e. $\|F_k g\|_{L^2(\partial D \setminus D)} \leq \varepsilon$ in our case), then the scattered field (i.e. $u_{s, k'}$ in our case) is also small up to the boundary of the scatterer in the sense of (2.13), where $\psi(\varepsilon)$ is the stability function in [4]. In fact, in [4], the quantitative Rellich theorem is established for medium scattering. But as long as the scattered field is Hölder continuous up the boundary of the scatterer, the result in [4] can be straightforwardly extended to the case of obstacle scattering. In theorem 2.1, the regularity assumptions on $\partial D$ guarantees that the scattered field is indeed Hölder continuous up to $\partial D$. In fact, let us consider (1.1) to ease the exposition. In two dimensions, $\partial u/\partial \nu|_{\partial D} \in L^2(\partial D)$. It follows from regularity results for elliptic problems in Lipschitz domains [31, theorem 4.24] that $u|_{\partial D} \in H^1(\partial D)$, and hence from [31, theorem 6.12] and the accompanying discussion that $u \in H^{3/2}_{loc}(\mathbb{R}^2 \setminus D)$. Then by the standard Sobolev embedding theorem, one can directly verify that $u$ is Hölder continuous up to $\partial D$ and therefore $u' = u - u'$ is also Hölder continuous up to $\partial D$. In three dimensions, since $\partial D \in C^{1,1}$, one can make use of the [31, theorem 4.18] to show that $u \in H^{1/2}_{loc}(\mathbb{R}^3 \setminus D)$. Again by the Sobolev embedding theorem, one can see that $u$ and hence $u'$ are Hölder continuous up to $\partial D$. We believe the regularity assumption in three dimensions can be relaxed to be purely Lipschitz as that in two dimensions. However, this is not the focus of the current study. We also refer to [34] for a different approach of quantitatively continuing the far-field data to the boundary of the scatterer.
Remark 2.3. Theorem 2.1 presents an approach for reconstructing the Neumann eigenfunctions from far-field data. This method relies on the fundamental assumption that \( k^2 \) is the Neumann eigenvalue and the measurements are the corresponding far-field data. In essence, the theorem states that if \( g_z \) satisfies the constrained inequality (2.5), then we can approximate the eigenfunction \( u_k \) by using the Herglotz wave function \( v_{r,k} \) with the kernel \( g_z \).

Based on theorem 2.1 and supposing that \( k^2 \) is a Neumann eigenvalue to \( D \) and \( F_k \) is the corresponding far-field operator to the measured far-field data, we can see that the following optimization problem:

\[
\min_{g \in L^2(\mathbb{S}^{m-1})} \| F_k g \|_{L^2(\mathbb{S}^{m-1})} \quad \text{s.t.} \quad \| v_{r,k} \|_{L^2(D)} = 1
\]  

has a solution \( g \in L^2(\mathbb{S}^{m-1}) \). It is worth mentioning that we replace the \( H_1 \)-norm in (2.5) by the \( L_2 \)-norm in (2.18) because \( v_{r,k} \) is an entire solution to the Helmholtz equation and these two norms are equivalent. Moreover, the constraint in (2.18) is to ensure that \( g \in L^2(\mathbb{S}^{m-1}) \) is a non-trivial solution. Since \( D \) is unknown, it is unpractical to solve the optimization problem (2.18) with the constraint term \( \| v_{r,k} \|_{L^2(D)} = 1 \). Thus, we consider the following optimization problem instead:

\[
\min_{g \in L^2(\mathbb{S}^{m-1})} \| F_k g \|_{L^2(\mathbb{S}^{m-1})} \quad \text{s.t.} \quad \| v_{r,k} \|_{L^2(D)} = 1,
\]

where \( B \) is bounded domain such that \( D \subset B \). In practice, one can simply choose \( B \) to be a large ball containing \( D \). Finally, the corresponding eigenfunction \( u_k \) can be approximated by the Herglotz wave function \( v_{r,k} \) with the solved kernel \( g \). In summarizing our discussion above, we can formulate the following scheme (scheme 1) to determine the Neumann eigenvalues and the corresponding eigenfunctions. It is emphasized that all the results in this section hold equally for the sound-soft case.

**Scheme 1. Determination of Neumann eigenvalues and eigenfunctions.**

- **Step 1** Collect a family of far-field data \( u^{\alpha,\delta}(\hat{x}, d, k) \) for \((\hat{x}, d, k) \in \mathbb{S}^{m-1} \times \mathbb{S}^{m-1} \times V \), where \( V \) is an open interval in \( \mathbb{R}_+ \).
- **Step 2** Pick a point \( z \in D \) (a priori information) and for each \( k \in V \), find the minimizer \( g_{z,k}^\delta \) of (2.3).
- **Step 3** Plot \( \| g_{z,k}^\delta \|_{L^2(\mathbb{S}^{m-1})} \) against \( k \in V \) and find the Neumann eigenvalues from the peaks of the graph.
- **Step 4** For each determined Neumann eigenvalue, solve the optimization problem (2.19) and obtain the Herglotz kernel function \( g_k \) by using the gradient total least square method as proposed in [24].
- **Step 5** With the computed Herglotz kernel function \( g_k \), then the corresponding eigenfunction \( u_k \) can be approximated by the Herglotz wave function according to the definition (2.4).

### 3. Newton-type method based on the interior resonant modes

In this section, we develop a novel Newton-type method to reconstruct the shape of a sound-hard obstacle based on the interior resonant modes determined in the previous section. Noting that the Neumann eigenfunctions satisfy the following system

\[
\Delta u_k + k^2 u_k = 0,
\]

with the boundary condition \( \partial u_k / \partial \nu = 0 \) on \( \partial D \), we can determine the unknown boundary \( \partial D \) by using the boundary property of Neumann eigenfunctions. From the previous section, it is noted that the Neumann eigenfunction \( u_k \) can be approximated by a Herglotz wave function
v_{g,k}. Hence, we will present an approach to recover the shape of a sound-hard obstacle based on the approximation of the Herglotz wave functions v_{g,k} within Neumann eigenvalues.

Let γ signify the boundary of a bounded domain. We define an operator G that maps the boundary γ onto the trace of \( \partial v_{g,k} / \partial \nu \) on γ, that is,

\[
G : \gamma \rightarrow \left. \frac{\partial v_{g,k}}{\partial \nu} \right|_\gamma.
\]

In terms of the above operator G, we seek the boundary γ such that the Herglotz wave function \( v_{g,k} \) approximately satisfies the Neumann boundary condition:

\[
G(\gamma) \approx 0. \tag{3.2}
\]

Next, we introduce the Newton-type method to recover the boundary of the obstacle. Following the idea of Newton method, we replace the previous nonlinear equation (3.2) by the linearized equation

\[
G(\gamma) + G'(\gamma)h = 0. \tag{3.3}
\]

To improve the approximate boundary γ, we need to solve (3.3) for the shift \( h \), which allows us to obtain a new approximation given by \( \tilde{\gamma} := \gamma + h \). Actually, the key point for solving the last linearized equation is to determine the Fréchet derivative of the operator G. Inspired by the work [19], we seek to improve and present a simpler Fréchet derivative in two and three dimensions, respectively.

For the two-dimensional case, we assume that γ is a closed curve and it is parameterized by

\[
\gamma = \left\{ z(\phi) = \left( z^{(1)}(\phi), z^{(2)}(\phi) \right) \right\} \in \mathbb{R}^2 : \phi \in [0, 2\pi].
\]

Furthermore, the updated parameterization boundary \( \tilde{\gamma} \) is given by

\[
\tilde{\gamma} = \{ z(\phi) + h(\phi) : \phi \in [0, 2\pi] \}.
\]

For the further analysis, we use the notations

\[
z_0 = \left( \frac{\partial z^{(1)}(\phi)}{\partial \phi}, \frac{\partial z^{(2)}(\phi)}{\partial \phi} \right)^\top, \quad z^\perp = \left( \frac{\partial z^{(2)}(\phi)}{\partial \phi}, -\frac{\partial z^{(1)}(\phi)}{\partial \phi} \right)^\top, \quad \nu(z) := \frac{z^\perp}{|z^\perp|}. \tag{3.4}
\]

Here, \( z_0 \) denotes the tangential vector, \( z^\perp \) denotes the exterior normal vector and \( \nu \) denotes the exterior unit normal vector on boundary γ. In what follows, the gradient of a scalar-valued differentiable function \( f(y) \in \mathbb{C}(\mathbb{R}^m) \) is defined by

\[
\nabla_f(y) := \left( \frac{\partial f(y)}{\partial y_1}, \ldots, \frac{\partial f(y)}{\partial y_m} \right)^\top, \quad y = (y_1, \ldots, y_m)^\top, \quad m = 2, 3,
\]

and the gradient of a vector-valued differentiable function \( P(y) \in (\mathbb{C}(\mathbb{R}^m))^m \) is defined by

\[
\nabla_y P(y) := \begin{pmatrix}
\frac{\partial p_1(y)}{\partial y_1} & \ldots & \frac{\partial p_1(y)}{\partial y_m} \\
\vdots & \ddots & \vdots \\
\frac{\partial p_m(y)}{\partial y_1} & \ldots & \frac{\partial p_m(y)}{\partial y_m}
\end{pmatrix}, \quad y = (y_1, \ldots, y_m)^\top, \quad P = (p_1, \ldots, p_m)^\top, \quad m = 2, 3.
\]

In addition, some similar notations of \( z \) are also used to \( h \).

Now we characterize the Fréchet derivative in the following theorem.
**Theorem 3.1.** Let \( z: [0, 2\pi] \to \mathbb{R}^2 \) be a \( C^2 \) function and \( v_{g,k} \) be the Herglotz wave function that approximates a Neumann eigenfunction \( u_k \) associated with the Neumann eigenvalue \( k^2 \). Then the operator \( G: z \mapsto \nu(z) \cdot \nabla_z v_{g,k}(z) \) defined in (3.1) is Fréchet differentiable, i.e.

\[
|G(z + h) - G(z) - G'(z)h| = O(\|h\|^2 C_2), \quad \|h\|_{C^1([0,2\pi])} \to 0,
\]

and its derivative is given by

\[
G'(z)h = \frac{1}{|z_\phi|} (I - \nu\nu^\top) h_\phi^\top \cdot \nabla_z v_{g,k} + \nu \cdot (\nabla_z \nabla_z^\top v_{g,k}) h.
\]

(3.5)

Here the derivative \( z_\phi \) and the exterior unit normal vector \( \nu \) are defined in (3.4), \( I \) is the 2 \times 2 identity matrix and \( h_\phi^\top = (\partial h^{(2)}/\partial \phi, -\partial h^{(1)}/\partial \phi)^\top \).

**Proof.** Recall that

\[
G(z) = \nu(z) \cdot \nabla_z v_{g,k}(z),
\]

then we have the following decomposition

\[
G(z + h) - G(z) = (\nu(z + h) - \nu(z)) \cdot \nabla_z v_{g,k}(z) + \nu(z) \cdot (\nabla_z v_{g,k}(z + h) - \nabla_z v_{g,k}(z)).
\]

(3.6)

Noting that \( \nu = z_\phi^\top/|z_\phi| \), using Taylor’s formula, one can deduce that

\[
\nu(z + h) - \nu(z) = \frac{z_\phi^\top + h_\phi^\top}{|z_\phi^\top + h_\phi^\top|} - \frac{z_\phi^\top}{|z_\phi^\top|}
\]

\[
= \nabla_{z_\phi} \left( \frac{z_\phi^\top}{|z_\phi^\top|} \right) h_\phi^\top + O(|h_\phi|^2)
\]

\[
= \frac{I|z_\phi^\top| - z_\phi^\top |z_\phi^\top|^\top}{|z_\phi^\top|^2} h_\phi^\top + O(|h_\phi|^2)
\]

\[
= \frac{1}{|z_\phi^\top|} (I - \nu\nu^\top) h_\phi^\top + O(|h_\phi|^2), \quad \|h\|_{C^2} \to 0.
\]

Similarly, we can derive that

\[
\nabla_z v_{g,k}(z + h) - \nabla_z v_{g,k}(z) = \nabla_z (\nabla_z v_{g,k}(z))^\top h + O(|h|^2)
\]

\[
= \nabla_z (\nabla_z^\top v_{g,k}(z)) h + O(|h|^2), \quad \|h\|_{C^1} \to 0.
\]

Substituting the last two equations into (3.6), one has

\[
G(z + h) - G(z) = \frac{1}{|z_\phi|} (I - \nu\nu^\top) h_\phi^\top \cdot \nabla_z v_{g,k} + \nu \cdot (\nabla_z \nabla_z^\top v_{g,k}) h + O(|h|^2), \quad \|h\|_{C^2} \to 0.
\]

Further, using the definition of the Fréchet derivative, it is easy to verify that the operator \( G \) is Fréchet differentiable and its derivative is given by

\[
G'(z)h = \frac{1}{|z_\phi|} (I - \nu\nu^\top) h_\phi^\top \cdot \nabla_z v_{g,k} + \nu \cdot (\nabla_z \nabla_z^\top v_{g,k}) h.
\]

The proof is complete. \( \square \)
Next, we consider the three-dimensional case and let $\gamma$ be a closed surface, which is parameterized by
\[
\gamma = \left\{ z(\theta, \phi) = (z^{(1)}(\theta, \phi), z^{(2)}(\theta, \phi), z^{(3)}(\theta, \phi)) \in \mathbb{R}^3 : (\theta, \phi) \in [0, \pi] \times [0, 2\pi] \right\}.
\]

We define two orthogonal unit tangential vector fields $\tau_1$ and $\tau_2$ on the surface $\gamma$, that is,
\[
\tau_1 = \frac{z_\theta}{|z_\theta|}, \quad \tau_2 = \frac{z_\phi - (z_\theta \cdot z_\phi)z_\theta}{|z_\theta - (z_\theta \cdot z_\phi)z_\theta|},
\]
where $z_\theta = \partial z/\partial \theta$ and $z_\phi = \partial z/\partial \phi$. Therefore, the normal vector can be represented by
\[
\nu(z) := \frac{\tau_1 \times \tau_2}{|\tau_1 \times \tau_2|} = \frac{z_\theta \times z_\phi}{|z_\theta \times z_\phi|}, \quad \nu(z) \equiv z_\nu.
\]

In the following theorem, we characterize the Fréchet derivative of $G$ in three dimensions.

**Theorem 3.2.** Let $z : [0, \pi] \times [0, 2\pi] \rightarrow \mathbb{R}^3$ be a $C^2$ function and $v_{g,k}$ be the Herglotz wave function that approximates some Neumann eigenfunction $u_k$ associated with the Neumann eigenvalue $k^2$. Then the operator $G : z \mapsto \nu(z) \cdot \nabla_z v_{g,k}(z)$ defined in (3.1) is Fréchet differentiable, i.e.
\[
|G(z + h) - G(z) - G'(z)h| = \mathcal{O}(\|h\|^2_{C^2}), \quad \|h\|_{C^2([0,\pi] \times [0,2\pi])} \rightarrow 0,
\]
and its derivative is given by
\[
G'(z)h = \frac{1}{|z_\theta \times z_\phi|} ((I - \nu \nu^\top)(h_\theta \times z_\phi + z_\theta \times h_\phi) \cdot \nabla_z v_{g,k} + \nu \cdot (\nabla_z \nabla_z^\top v_{g,k} h).
\]

Here $I$ is the $3 \times 3$ identity matrix, $h_\theta = \partial h/\partial \theta$, $h_\phi = \partial h/\partial \phi$, $z_\theta = \partial z/\partial \theta$, $z_\phi = \partial z/\partial \phi$ and $\nu$ is defined in (3.7).

**Proof.** Using Taylor’s formula, one can derive that
\[
\nu(z + h) - \nu(z) = \frac{(z + h)_\theta \times (z + h)_\phi}{|(z + h)_\theta \times (z + h)_\phi|} - \frac{z_\theta \times z_\phi}{|z_\theta \times z_\phi|}
\]
\[
= \nabla_{z_\theta} \left( \frac{z_\theta \times z_\phi}{|z_\theta \times z_\phi|} \right) h_\theta + \nabla_{z_\phi} \left( \frac{z_\theta \times z_\phi}{|z_\theta \times z_\phi|} \right) h_\phi + \mathcal{O}(|h_\theta|^2) + \mathcal{O}(|h_\phi|^2)
\]
\[
= \frac{1}{|z_\theta \times z_\phi|} [h_\theta \times z_\phi + z_\theta \times h_\phi - \nu((z_\phi \times \nu) \cdot h_\theta + (\nu \times z_\theta) \cdot h_\phi)]
\]
\[
+ \mathcal{O}(|h_\theta|^2) + \mathcal{O}(|h_\phi|^2), \quad \|h\|_{C^2} \rightarrow 0.
\]

According to the mixed product, we have
\[
(z_\phi \times \nu) \cdot h_\theta = \nu \cdot (h_\theta \times z_\phi), \quad (\nu \times z_\theta) \cdot h_\phi = \nu \cdot (z_\theta \times h_\phi).
\]

Similar to the two dimensional case, by a straightforward calculation, one can deduce that
\[
G'(z)h = \frac{1}{|z_\theta \times z_\phi|} ((I - \nu \nu^\top)(h_\theta \times z_\phi + z_\theta \times h_\phi) \cdot \nabla_z v_{g,k} + \nu \cdot (\nabla_z \nabla_z^\top v_{g,k} h),
\]
which completes the proof. \qed
As the Herglotz wave function is analytic, the gradient and the Hessian Matrix of \( v_{g,k} \) defined in theorems 3.1 and 3.2 can be expressed as

\[
\nabla x v_{g,k}(x) = \Im \int_{\gamma} \, d^2 x \, g(d) \, ds(d),
\]

\[
\nabla x \nabla^\top x v_{g,k}(x) = -k^2 \int_{\gamma} \, d^2 x \, e^{i k x \cdot d} g(d) \, ds(d), \quad x \in \gamma,
\]

(3.8)

where the Herglotz kernel \( g \) is determined by solving the optimization problem (2.19).

**Remark 3.1.** It is noted that the Fréchet derivatives presented in theorems 3.1 and 3.2 are simpler and more intuitionistic compared with the formula defined in [19]. We would like to mention that the gradient and Hessian matrices of \( v_{g,k} \) in (3.8) are computed with respect to \( x \) and they are given in the Cartesian coordinate system. Moreover, based on the Herglotz wave approximation, it is very easy to numerically calculate the Fréchet derivatives since only cheap integrations are involved in the evaluation of the gradient and Hessian matrices of \( v_{g,k} \).

We would like to emphasize that it is non-trivial to show that \( G' \) is injective, namely, the operator \( G' \) may be not invertible. Therefore, we employ the standard Tikhonov regularization scheme with a regularization parameter \( \alpha \) for solving linearized equation (3.3) at each iteration. As discussed above, we propose the following Newton-type scheme.

**Algorithm 1 (Newton-type method).** Assume that \( \alpha > 0 \) is a regularization parameter. Given an initial guess \( z_0 \), the \( n \)th step is to compute \( h \) from the following equation

\[
(\alpha I + \left( G'(z_{n-1}, k) \right)^* G'(z_{n-1}, k)) h = -(G'(z_{n-1}, k))^* G(z_{n-1}, k),
\]

where \( k^2 \) is a Neumann eigenvalue. Further, we update the parameterized boundary by \( z_n = z_{n-1} + h \).

Noting that the Newton iteration usually produces local minima for solving the inverse obstacle problems. To overcome local minimum and extend the convergence range, we propose a multifrequency Newton-type iterative algorithm.

**Algorithm 2 (Multifrequency Newton-type method).** Assume that \( \alpha > 0 \) is a regularization parameter. Given an initial guess \( z_0 \), the \( n \)th step is to compute \( h \) from the following equation

\[
(\alpha I + \left( F'(z_{n-1}) \right)^* F'(z_{n-1})) h = -(F'(z_{n-1}))^* F(z_{n-1}),
\]

where \( F(z_{n-1}) = \left[ G(z_{n-1}, k_1), G(z_{n-1}, k_2), \ldots, G(z_{n-1}, k_\ell) \right]^\top \) and \( k_1, k_2, \ldots, k_\ell \) are \( \ell \) different square roots of Neumann eigenvalues. Further, one can update the parameterized boundary by \( z_n = z_{n-1} + h \).

Finally, we would like to remark the extension to the sound-soft case. Indeed, by modifying the definition of the operator \( G \) in (3.1) via replacing \( \partial u/\partial n \) by \( u \), all the results in this section can be readily extended to the sound-soft case. To that end, it suffices to note that the Fréchet derivative for a sound-soft obstacle is given by

\[
G'(z) h = \nabla_z v_{g,k} \cdot h,
\]

and it holds for both two and three dimensions.
Table 1. The list of major parameters used in the numerics.

| Parameters                           | 2D    | 3D    |
|--------------------------------------|-------|-------|
| $M$: number of observation directions | 64    | 500   |
| $N$: number of incident directions   | 64    | 450   |
| $N_z$: number of terms in Fourier expansion | 20    | 8     |
| $\delta$: noise level of measured data | 1%    | 1%    |
| $\alpha$: regularization parameter  | $10^{-5}$ | $10^{-4}$ |
| $\|h\|$: stop criterion            | $10^{-5}$ | $10^{-4}$ |

4. Numerical experiments

In this section, several numerical experiments are presented to verify the effectiveness and efficiency of the proposed methods. All the numerical results are conducted for recovering sound-hard obstacles, which present more challenges than the sound-soft case. To avoid inverse crime, the artificial far-field data are calculated by the finite element method (FEM), which is written as

\[ f_u^\infty(\hat{x}_i, d_j; k_s), \quad i = 1, 2, \ldots, M, \quad j = 1, 2, \ldots, N, \quad s = 1, 2, \ldots, L. \]

Here $\hat{x}_i$ denotes the observation direction, $d_j$ denotes the incident direction and $k_s$ denotes the wavenumber. The observation and incident directions are chosen as the equidistantly distributed points on the unit circle (2D) or unit sphere (3D). Moreover, the wavenumbers are also equidistantly distributed in the open interval $V \subset \mathbb{R}_+$. To test the stability of the method, for any fixed wavenumber, we add some random noise to the measurement matrix:

\[ U^\delta = U + \delta \|U\| \frac{R_1 + R_2}{\|R_1 + R_2\|} \]

where $\delta > 0$ is the noise level, and $R_1$ and $R_2$ are two uniform random matrices that range from $-1$ to $1$. To facilitate the analysis and comparison of the reconstructions, we list the major parameters for the two- and three-dimensional reconstructions in table 1.

4.1. Recover the eigenvalues and eigenfunctions

In this part, we shall devote to recovering the eigenvalues and the corresponding eigenfunctions from the measured far field data. Here we consider a pear-shaped domain in two dimension, which is parameterized as

\[ x(\phi) = (2 + 0.3 \cos 3\phi)(\cos \phi, \sin \phi), \quad \phi \in [0, 2\pi]. \]  

(4.1)

We set $M = 64$, $N = 64$ and $L = 2001$, that is, the artificial far-field data are obtain at 64 observation directions, 64 incident directions and 2001 equally distributed wavenumbers in the interval $[1.2, 3.2]$.

To begin with, we introduce the LSM to determine the eigenvalues. According to scheme 1, we plot the indicator function $\|g_{\zeta,k}^S\|_{L^2(S^2)}$ for $k \in [1.2, 3.2]$ in figure 1, where the interior test point is given by $z = (1, 1)$. In figure 1(a), the dashed red lines denote the location of square root of eigenvalues computed by the FEM and the solid blue line denotes the value of $\|g_{\zeta,k}^S\|_{L^2(S^2)}$. 


As expected, one can observe that the indicator function (solid blue line) has clear spikes near the locations of the real eigenvalues (dashed red lines). Thus, we can pick up the eigenvalues via the locations of the spikes. Moreover, we present the plot of the indicator function \( \| \mathbf{g}_{\delta z}^k \|_{L^2(\Omega')} \) with 1% noise in figure 1(b). Although the indicator function exhibits oscillating phenomenon, the eigenvalues can still be picked up from the locations of the spikes. In order to exhibit the accuracy of the recovery results quantitatively, we also present the eigenvalues computed by the FEM in table 2. One can observe that the LSM is valid to pick up the eigenvalues. There are two main reasons for utilizing a substantial amount of wavenumbers. Firstly, a finer discretization provides clearer spikes at the eigenvalues, which improves the accuracy of the solution. Secondly, the eigenfunctions are highly sensitive to changes in the eigenvalues, and hence it is crucial to obtain more precise eigenvalues to accurately reconstruct the corresponding eigenfunctions. Actually, [30] offers a guidance for designing a suitable discretization scheme for the wavenumber band.

Next, we aim to recover the corresponding eigenfunctions from the far field data associated with Neumann eigenvalues. Here, we shall use the gradient total least square method.
Table 2. The first seven square root of real eigenvalues of the square domain. FEM: computed the exact domain by the finite element method; LSM: recovered from the far-field data by the linear sampling method.

| Index of eigenvalue | 1     | 2     | 3     | 4     | 5     | 6     | 7     |
|---------------------|-------|-------|-------|-------|-------|-------|-------|
| FEM:                | 1.559 | 1.709 | 2.072 | 2.329 | 2.394 | 2.873 | 3.006 |
| LSM with no noise:  | 1.559 | 1.708 | 2.071 | 2.329 | 2.394 | 2.873 | 3.005 |
| LSM with 1% noise:  | 1.562 | 1.708 | 2.073 | 2.328 | 2.395 | 2.873 | 3.004 |

Figure 2. Contour plots of the exact and reconstructed eigenfunctions with different eigenvalues. The top row: the eigenfunctions recovered by using the finite element method; the bottom row: the Herglotz wave recovered by using the gradient total least square method.

as proposed in [24]. To reduce the oscillations, we add a penalty term to the optimization problem (2.19), that is,

$$
\min_{g \in L^2(B^{n-1})} \| F_g \|_{L^2(B^{n-1})} + \beta \| \nabla g \|_{L^2(B^{n-1})} \quad \text{s.t.} \quad \| v_{g,k} \|_{L^2(B)} = 1,
$$

where $\beta > 0$ denotes the regularization parameter. In the numerical part, the regularization parameter is chosen as $\beta = 10^{-2}$ and the radius of domain $B$ is given by $r = 3$. To test the stability, we add extra 1% noise to the far field data associated with Neumann eigenvalues. For comparison, we use the FEM to compute the interior Neumann eigenvalue problem (1.3) and obtain the eigenfunctions $u_k$ with different eigenvalues, see the top row of figure 2. The bottom row of figure 2 present the recovered Herglotz wave equations with recovered Neumann eigenvalues by using the gradient total least square method. Here the dashed white lines denote
the exact pear-shaped domain. One can observe that the recovered Herglotz wave functions are very close to the exact Neumann eigenfunctions inside the obstacle.

4.2. Reconstruct shapes

In this section, we provide several two- and three-dimensional numerical examples to verify the efficiency of algorithms 1 and 2.

4.2.1. 2D reconstructions.

For the two-dimensional case, we choose an approximation of the boundary with the form

\[ z(\phi) = \{ r(\phi) (\cos \phi, \sin \phi) : \phi \in [0, 2\pi] \}, \]

where \( r \in C^2([0, 2\pi]) \) is the radial function and it is given by trigonometric polynomials of order less than or equal \( N_z \in \mathbb{N} \), i.e.

\[ r(\phi) = a_0 + \sum_{j=1}^{N_z} (a_j \cos j \phi + b_j \sin j \phi), \quad \phi \in [0, 2\pi]. \]

Here \( a_0, a_1, a_2, \ldots, a_{N_z} \) and \( b_1, b_2, \ldots, b_{N_z} \) are unknown Fourier coefficients. In what follows, the order is chosen as \( N_z = 20 \) and the stop criterion is set to be \( \| h \|_{L^2([0,2\pi])} < 10^{-5} \), defined in equation (3.3). Moreover, the regularized parameter \( \alpha \) is given by \( \alpha = 10^{-5} \). In what follows, the solid black lines denote the exact sound-hard obstacle, the dotted grey lines denote the initial guess and the red dashed lines denote the recovered shapes.

In the first example, we consider the pear-shaped domain as shown in (4.1). Since the initial value plays an important role for the Newton iterative method, we test the proposed algorithm 1 with different initial guesses. Let the initial shape be a circle centered at the origin with a radius of \( R \). Figure 3 presents the reconstructions of the pear-shaped obstacle with different initial guesses. It is noted that the stopping criteria is achieved between 15 and 20 iterations for this example. Figures 3(a) and (b) respectively shows the recovery results for the minimum and maximum radius with square root of Neumann eigenvalue \( k_1 = 1.562 \). Correspondingly, figures 3(c) and (d) respectively shows the reconstructed results for the minimum and maximum radius with square root of Neumann eigenvalue \( k_2 = 1.708 \). Through the numerical experiments, we find that for each Neumann eigenvalue there exists an interval of the radius such that the Newton iteration is convergent. Moreover, we test the multifrequency Newton-type method for recovering the shape. Figure 4 presents the reconstructed shapes via algorithm 2, where \( R = 1 \) is the minimum radius and \( R = 3.2 \) is the maximum radius. Here, we use four different Neumann eigenvalues. Comparing figures 3 and 4, one can observe that the multifrequency Newton-type approach has larger convergence range than single-frequency Newton-type approach. In addition, we present a numerical example for reconstructing the sound-soft obstacle in figure 5, where the square root of the Dirichlet eigenvalue is given by \( k = 3.2784 \). It is clear to see that the proposed imaging approach based on the interior resonant modes can be extended to recover the sound-soft obstacle.

In the second example, we consider a concave case and the scatterer is given by a kite-shaped domain, which is parameterized by

\[ x(\phi) = (\cos \phi + 0.65 \cos 2\phi - 0.65, 1.5 \sin \phi), \quad \phi \in [0, 2\pi]. \]

Here 3001 equally distributed wavenumbers in the interval [1, 4] are used to determine the eigenvalues by using the LSM. We obtain eight square root of eigenvalues, i.e. \( k_1 = 1.1136 \), \( k_2 = 1.4494 \), \( k_3 = 2.2629 \), \( k_4 = 2.3044 \), \( k_5 = 2.9178 \), \( k_6 = 3.1950 \), \( k_7 = 3.5283 \)
Figure 3. Reconstructions of the pear-shaped obstacle with different initial guesses $R$.

Figure 4. Reconstructions of the pear-shaped obstacle by using algorithm 2 with different initial guesses $R$. 
Figure 5. Reconstructions of the sound-soft pear-shaped obstacle by using algorithm 1 with different initial guesses $R$.

Figure 6. Reconstructions of the kite-shaped scatterer by using algorithm 2. (a)–(c) Reconstructions with three eigenvalues; (d) reconstructions with six eigenvalues.
and $k_8 = 3.7477$. Figure 6 presents reconstructions of the kite-shaped scatterer by using the multifrequency Newton-type method, i.e. algorithm 2. The top row of figure 6 shows the recovery results with the first three eigenvalues, where the initial shape is a circle centered at $(0,0)$ with radius $R = 2$. The bottom row of figure 6 shows the recovery results with the first six eigenvalues, where the initial guess is a circle centered at $(0,0)$ with radius $R = 1$. To test the stability of the proposed approach, 1% and 5% noise are respectively added to the measured far-field data. From figures 6(a) and (b), it is clear to see that the accuracy is reduced as the noise increases. Furthermore, according to figures 6(c) and (d), one can find that the accuracy is improved as the number of frequencies increases.

4.2.2. 3D reconstructions. \textcolor{black}{In this part, we consider a more challenging case for reconstructing the three-dimensional scatterer. For the three-dimensional case, we choose an approximation of the surface boundary with the form}

$$z(\theta, \phi) = \{r(\theta, \phi) (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) : \theta \in [0, \pi], \phi \in [0, 2\pi]\},$$

where $r \in \mathbb{C}^2([0, \pi] \times [0, 2\pi])$ is the radial function and it is given by spherical harmonics with order less than or equal $N_z \in \mathbb{N}$, i.e.

$$r(\theta, \phi) = \sum_{\ell=0}^{N_z} \sum_{s=-\ell}^{\ell} (a_{\ell}^s \sin s\phi + b_{\ell}^s \cos s\phi) P_{\ell}^s(\cos \theta).$$

In what follows, the regularized parameter $\alpha$ is given by $\alpha = 10^{-4}$, the stop criterion is set to be $\|h\|_{L^2([0,\pi] \times [0,2\pi])} < 10^{-4}$ and the order is chosen as $N_z = 8$. In addition, the measurement directions are chosen to be 500 pseudo-uniformly distributed measured points on the unit sphere. The incident directions are chosen to be 15 $\times$ 30 uniform rectangular mesh of $[0, \pi] \times [0, 2\pi]$. Due to the heavy computational cost in determining the eigenvalue, we use the FEM to calculate the eigenvalue in the following three-dimensional examples.

We first consider an ellipsoid domain in three dimensions (see figure 7(a)), which is parameterized by

$$x(t, \tau) = (3 \sin t \cos \tau, 3 \sin t \sin \tau, 6 \cos t), \quad t \in [0, \pi], \tau \in [0, 2\pi].$$

Here 1% noise is added to perturb the far field data and we use algorithm 1 to recover the shape of the obstacle. Figures 7(b) and (c) show the reconstructed shapes with different initial guesses, where the square root of eigenvalue is chosen as $k = 1.4513$. From the surface plots, it can be seen that the reconstructions are very close to the exact obstacle. Figure 7(d) shows the relationship between the initial guess $R$ and iteration number $N_i$. One can find that the proposed method converges rapidly, requiring fewer than ten iterations, when the initial value of $R$ falls within the interval $(3.5, 5)$.

Next, we are devoted to the identification of a concave scatter, where the domain is parameterized by

$$x(t, \tau) = (1.5 \sin t \cos \tau, 1.5 \sin t \sin \tau, 0.2 - \cos t - 0.65 \cos 2\tau),$$

for $t \in [0, \pi]$ and $\tau \in [0, 2\pi]$. Actually, it is difficult to determine the boundary of the obstacle by using single frequency data. Therefore, the multifrequency Newton-type method is used to recover the shape with two Neumann eigenvalues ($k_1 = 2.4514$ and $k_2 = 3.2630$). To exhibit
the iterative process, we plot recovery results with different iteration numbers \( N_i \) in figure 8. One can observe that the proposed approach demonstrates better imaging performance as the number of iterations increases.

**Remark 4.1.** In the numerical part, the parameterized boundary \( \gamma \) is given by the so-called star-shaped curve (in \( \mathbb{R}^2 \)) or surface (in \( \mathbb{R}^3 \)). The properties of star-shaped domain are used in two aspects. Firstly, it is convenient to calculate the derivative vector and norm vector by using the star-shaped domain. Secondly, it could significantly reduce the computational cost in the numerical reconstructions since only the first few Fourier coefficients are needed to recover.
Figure 8. Surface plots of the exact and reconstructed results with different iteration numbers $N_t$. 
Data availability statement

The data cannot be made publicly available upon publication because no suitable repository exists for hosting data in this field of study. The data that support the findings of this study are available upon reasonable request from the authors.

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