ULTRA-DISCRETIZATION OF THE $D^{(3)}_4$-GEOMETRIC CRYSTALS TO THE $G^{(1)}_2$-PERFECT CRYSTALS

MANA Igarashi, KAILASH C. MISRA, AND TOSHIKI NAKASHIMA

ABSTRACT. Let $\mathfrak{g}$ be an affine Lie algebra and $\mathfrak{g}^L$ be its Langlands dual. It is conjectured in [15] that $\mathfrak{g}$ has a positive geometric crystal whose ultra-discretization is isomorphic to the limit of certain coherent family of perfect crystals for $\mathfrak{g}^L$. We prove that the ultra-discretization of the positive geometric crystal for $\mathfrak{g} = D^{(3)}_4$ given in [6] is isomorphic to the limit of the coherent family of perfect crystals for $\mathfrak{g}^L = G^{(1)}_2$ constructed in [21].

1. Introduction

Let $A = (a_{ij})_{i,j \in I}, I = \{0, 1, \ldots, n\}$ be an affine Cartan matrix and $(A, \{\alpha_i\}_{i \in I}, \{\alpha_i^\vee\}_{i \in I})$ be a given Cartan datum. Let $\mathfrak{g} = \mathfrak{g}(A)$ denote the associated affine Lie algebra [16] and $U_q(\mathfrak{g})$ denote the corresponding quantum affine algebra. Let $P = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \cdots \oplus \mathbb{Z}\Lambda_n \oplus \mathbb{Z}\delta$ and $P^\vee = \mathbb{Z}\alpha_0^\vee \oplus \mathbb{Z}\alpha_1^\vee \oplus \cdots \oplus \mathbb{Z}\alpha_n^\vee \oplus \mathbb{Z}d$ denote the affine weight lattice and the dual affine weight lattice respectively. For a dominant weight $\lambda \in P^+ = \{\mu \in P \mid \mu(h_i) \geq 0 \text{ for all } i \in I\}$ of level $l = \lambda(c) (c = \text{canonical central element})$, Kashiwara defined the crystal base $(L(\lambda), B(\lambda))$ [10] for the integrable highest weight $U_q(\mathfrak{g})$-module $V(\lambda)$. The crystal $B(\lambda)$ is the $q = 0$ limit of the canonical basis [20] or the global crystal basis [11]. It has many interesting combinatorial properties. To give explicit realization of the crystal $B(\lambda)$, the notion of affine crystal and perfect crystal has been introduced in [7]. In particular, it is shown in [7] that the affine crystal $B(\lambda)$ for the level $l \in \mathbb{Z}_{\geq 0}$ integrable highest weight $U_q(\mathfrak{g})$-module $V(\lambda)$ can be realized as the semi-infinite tensor product $\cdots \otimes B_l \otimes B_0 \otimes B_0$, where $B_0$ is a perfect crystal of level $l$. This is known as the path realization. Subsequently it is noticed in [9] that one needs a coherent family of perfect crystals $\{B_l\}_{l \geq 1}$ in order to give a path realization of the Verma module $M(\lambda)$ (or $U_q^{-}(\mathfrak{g})$). In particular, the crystal $B(\infty)$ of $U_q(\mathfrak{g})$ can be realized as the semi-infinite tensor product $\cdots \otimes B_\infty \otimes B_\infty \otimes B_\infty$ where $B_\infty$ is the limit of the coherent family of perfect crystals $\{B_l\}_{l \geq 1}$ (see [9]). At least one coherent family $\{B_l\}_{l \geq 1}$ of perfect crystals and its limit is known for $\mathfrak{g} = A^{(1)}_n, B^{(1)}_n, C^{(1)}_n, D^{(1)}_n, A^{(2)}_{2n-1}, A^{(2)}_{2n}, D^{(2)}_{n+1}, D^{(3)}_4, G^{(1)}_2$ (see [8, 9, 22, 13, 21]).

A perfect crystal is indeed a crystal for certain finite dimensional module called Kirillov-Reshetikhin module (KR-module for short) of the quantum affine algebra $U_q(\mathfrak{g})$ [15, 4, 5]. The KR-modules are parametrized by two integers $(i, l)$, where $i \in I \setminus \{0\}$ and $l$ any positive integer. Let $\{c_{i,l}\}_{i \in I \setminus \{0\}}$ be the set of level 0 fundamental weights [12]. Hatayama et al [4, 5] conjectured that any KR-module valid.
$W(\varpi_i)$ admit a crystal base $B_{i,l}$ in the sense of Kashiwara and furthermore $B_{i,l}$ is perfect if $l$ is a multiple of $c_i^\vee := \max(1/\alpha_i^\vee, 1/\alpha_i^{\vee, -1})$. This conjecture has been proved recently for quantum affine algebras $U_q(\mathfrak{g})$ of classical types ([26], [2], [3]). When \( \{B_{i,l}\}_{l \geq 1} \) is a coherent family of perfect crystals we denote its limit by $B_\infty(\varpi_i)$ (or just $B_\infty$ if there is no confusion).

On the other hand the notion of geometric crystal is introduced in [1] as a geometric analog to Kashiwara’s crystal (or algebraic crystal) [10]. In fact, geometric crystal is defined in [11] for reductive algebraic groups and is extended to general Kac-Moody groups in [22]. For a given Cartan datum $(A, \{\alpha_i\}_{i \in I}, \{\alpha_i^{\vee}\}_{i \in I})$, the geometric crystal is defined as a quadruple $\mathcal{V}(\mathfrak{g}) = (X, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\xi_i\}_{i \in I})$, where $X$ is an algebraic variety, $e_i : \mathbb{C}^\times \times X \to X$ are rational $\mathbb{C}^\times$-actions and $\gamma_i, \xi_i : X \to \mathbb{C} (i \in I)$ are rational functions satisfying certain conditions (see Definition 2.1). Geometric crystals have many properties similar to algebraic crystals. For instance, the product of two geometric crystals admits the structure of a geometric crystal if they are induced from unipotent crystals (see [1]). A geometric crystal is said to be a positive geometric crystal if it admits a positive structure (genometric crystal if they are induced from unipotent crystals (see [1]). A geometric crystal is said to be a positive geometric crystal if it admits a positive structure (see Definition 2.5). A remarkable relation between positive geometric crystals and algebraic crystals is the ultra-discretization functor $UD$ between them (see Section 2.4). Applying this functor, positive rational functions are transferred to piecewise linear functions by the simple correspondence:

$$x \times y \mapsto x + y, \quad \frac{x}{y} \mapsto x - y, \quad x + y \mapsto \max\{x, y\}.$$

Let $G$ denote the affine Kac-Moody group associated with the affine Lie algebra $\mathfrak{g}$. Let $B^\pm$ be fixed Borel subgroups and $T$ the maximal torus of $G$ such that $B^+ \cap B^- = T$. Set $y_i(c) := \exp(cf_i)$, and let $\alpha_i^{\vee}(c) \in T$ be the image of $c \in \mathbb{C}^\times$ by the group morphism $\mathbb{C}^\times \to T$ induced by the simple coroot $\alpha_i^{\vee}$. We set $Y_i(c) := y_i(c^{-1})\alpha_i^{\vee}(c) = \alpha_i^{\vee}(c)y_i(c)$. Let $W$ (resp. $\tilde{W}$) be the Weyl group (resp. the extended Weyl group) associated with $\mathfrak{g}$. The Schubert cell $X_w := BwB/B (w = s_{i_1} \cdots s_{i_k} \in W)$ is birationally isomorphic to the variety

$$B^-_i := \{Y_{i_1}(x_1) \cdots Y_{i_k}(x_k) \mid x_1, \cdots, x_k \in \mathbb{C}^\times \} \subset B^-,$$

and $X_w$ has a natural geometric crystal structure, where $i = i_1, \cdots, i_k$ is a reduced word for $w$. ([11], [22]).

Let $W(\varpi_i)$ be the KR-module (also called the fundamental representation) of $U_q(\mathfrak{g})$ with $\varpi_i$ as an extremal weight ([22]). Let us denote its specialization at $q = 1$ by the same notation $W(\varpi_i)$. It is a finite-dimensional $\mathfrak{g}$-module (not necessarily irreducible). Let $\mathbb{P}(\varpi_i)$ be the projective space $(W(\varpi_i) \setminus \{0\})/\mathbb{C}^\times$. For any $i \in I$ the translation $t(c_i^{\vee})\varpi_i$ belongs to $\tilde{W}$ (see [15]). For a subset $J$ of $I$, let us denote by $\mathfrak{g}_J$ the subalgebra of $\mathfrak{g}$ generated by $\{e_i, f_i\}_{i \in J}$. For an integral weight $\mu$, define $I(\mu) := \{j \in I \mid (\alpha_j^{\vee}, \mu) \geq 0\}$. We recall the following conjecture stated in [14].

**Conjecture 1.1** ([14]). For any $i \in I \setminus \{0\}$ there exist a unique variety $X$ endowed with a positive $\mathfrak{g}$-geometric crystal structure and a rational mapping $\pi : X \to \mathbb{P}(\varpi_i)$ satisfying the following property:

(i) for an arbitrary extremal vector $u \in W(\varpi_i)_\mu$, writing the translation $t(c_i^{\vee})\mu$ as $\tau u \in \tilde{W}$ with a Dynkin diagram automorphism $\tau$ and $w = s_{i_1} \cdots s_{i_k}$, there exists a birational mapping $\xi : B^-_{i_1, \cdots, i_k} \to X$ such that $\xi$ is a morphism of $\mathfrak{g}_{I(\mu)}$-geometric crystals and that the composition $\pi \circ \xi : B^-_{i_1, \cdots, i_k}$...
The ultradiscretization of the geometric crystals

\[ \mathbb{P}(\pi_i) \text{ coincides with } Y_i(x_1) \cdots Y_i(x_k) \mapsto Y_i(x_1) \cdots Y_i(x_k) \pi, \]
where \( \pi \) is the line including \( u \).

(ii) the ultradiscretization (see Sect. 2) of \( X \) is isomorphic to the crystal \( B_\infty = B_\infty(\pi_i) \) of the Langlands dual \( g^L \).

In [13], it has been shown that this conjecture is true for \( i = 1 \) and \( g = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, A_{2n}^{(2)}, D_{n+1}^{(2)}. \) In [24], a positive geometric crystal for \( g = G_2^{(1)} \) and \( i = 1 \) has been constructed and it is shown in [25] that the ultradiscretization of this positive geometric crystal is isomorphic to the limit of the coherent family of perfect crystals for \( g^L = D_4^{(3)} \) given in [13].

More recently, two of the authors have constructed a positive geometric crystal for \( g = D_4^{(3)}, i = 1 \) in [6]. In this paper we describe the structure of the crystal obtained by the ultradiscretization of the geometric crystal \( V(g) \) constructed in [6] and then prove that it is isomorphic to the limit \( B_\infty \) of the coherent family of perfect crystals for its Langlands dual \( g^L \) constructed in [21]. This proves Conjecture 4.5 in [6].

This paper is organized as follows. In Section 2, we recall necessary definitions and facts about geometric crystals. In Section 3, we review needed facts about affine crystals and perfect crystals. We recall from [21] the coherent family of perfect crystals for \( g = G_2^{(1)} \) and its limit in Section 4. In Sections 5, we review the positive geometric crystal \( V(g) \) for \( g = D_4^{(3)} \) constructed in [6]. In Section 6, we state and prove our main result (Theorem 7.1).

2. Geometric crystals

In this section, we review Kac-Moody groups and geometric crystals following [27], [19], [11].

2.1. Kac-Moody algebras and Kac-Moody groups. Fix a symmetrizable generalized Cartan matrix \( A = (a_{ij})_{i,j \in I} \) with a finite index set \( I \). Let \( (t, \{ \alpha_i \}_{i \in I}, \{ \alpha_i^\vee \}_{i \in I}) \) be the associated root data, where \( t \) is a vector space over \( \mathbb{C} \) and \( \{ \alpha_i \}_{i \in I} \subset t^* \) and \( \{ \alpha_i^\vee \}_{i \in I} \subset t \) are linearly independent satisfying \( \alpha_j(\alpha_i^\vee) = \delta_{ij} \).

The Kac-Moody Lie algebra \( g = g(A) \) associated with \( A \) is the Lie algebra over \( \mathbb{C} \) generated by \( t \), the Chevalley generators \( e_i \) and \( f_i \) (\( i \in I \)) with the usual defining relations [17, 24]. There is the root space decomposition \( g = \bigoplus_{\alpha \in t^*} g_\alpha \). Denote the set of roots by \( \Delta := \{ \alpha \in t^*| \alpha \neq 0, g_\alpha \neq 0 \} \). Set \( Q = \sum_i \mathbb{Z} \alpha_i, \quad Q^+ = \sum_i \mathbb{Z}_{\geq 0} \alpha_i, \quad Q^\vee := \sum_i \mathbb{Z} \alpha_i^\vee \) and \( \Delta^+ := \Delta \cap Q^+ \). An element of \( \Delta^+ \) is called a positive root. Let \( P \subset t^* \) be a weight lattice such that \( \mathbb{C} \otimes P = t^* \), whose element is called a weight.

Define simple reflections \( s_i \in \text{Aut}(t) \) (\( i \in I \)) by \( s_i(h) := h - \alpha_i(h) \alpha_i^\vee \), which generate the Weyl group \( W \). It induces the action of \( W \) on \( t^* \) by \( s_i(\lambda) := \lambda - \lambda(\alpha_i^\vee) \alpha_i \). Set \( \Delta^\vee := \{ w(\alpha)|w \in W, i \in I \} \), whose element is called a real root.

Let \( g^L \) be the derived Lie algebra of \( g \) and let \( G \) be the Kac-Moody group associated with \( g^L \) (27). Let \( U_\alpha := \exp g_\alpha \) (\( \alpha \in \Delta^\vee \)) be the one-parameter subgroup of \( G \). The group \( G \) is generated by \( U_\alpha \) (\( \alpha \in \Delta^\vee \)). Let \( U^\pm \) be the subgroup generated by \( U_{\pm \alpha} \) (\( \alpha \in \Delta^\vee = \Delta^\vee \cap Q^+ \)), i.e., \( U^\pm := \langle U_{\pm \alpha}|\alpha \in \Delta^\vee \rangle \).

For any \( i \in I \), there exists a unique homomorphism \( \phi_i : SL_2(\mathbb{C}) \to G \) such that
\[
\phi_i \left( \begin{array}{cc} c & 0 \\ 0 & c^{-1} \end{array} \right) = e^{\alpha_i^\vee}, \quad \phi_i \left( \begin{array}{cc} 1 & t \\ 0 & 1 \end{array} \right) = \exp(t \epsilon_i), \quad \phi_i \left( \begin{array}{cc} 1 & 0 \\ t & 1 \end{array} \right) = \exp(t f_i).
\]
where $c \in \mathbb{C}^\times$ and $t \in \mathbb{C}$. Set $\alpha_i^\vee(c) := e^{\alpha_i^\vee}$, $x_i(t) := \exp(t e_i)$, $y_i(t) := \exp(t f_i)$, $G_i := \phi_i(SL_2(\mathbb{C}))$, $T_i := \phi_i(\{ \text{diag}(c,c^{-1}) | c \in \mathbb{C}^\times \})$ and $N_i := N_G(T_i)$. Let $T$ (resp. $N$) be the subgroup of $G$ with the Lie algebra $t$ (resp. generated by the $N_i$'s), which is called a maximal torus in $G$, and let $B^\pm = U^\pm T$ be the Borel subgroup of $G$. We have the isomorphism $\phi : W \simarrow N/T$ defined by $\phi(s_i) = N_iT/T$. An element $\pi_i := x_i(-1)y_i(1)x_i(-1) = \phi_i \left( \begin{pmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{pmatrix} \right)$ is in $N_G(T)$, which is a representative of $s_i \in W = N_G(T)/T$.

2.2. Geometric crystals. Let $X$ be an ind-variety, $\gamma_i : X \to \mathbb{C}$ and $\varepsilon_i : X \to \mathbb{C}$ $(i \in I)$ rational functions on $X$, and $e_i : \mathbb{C}^\times \times X \to X ((c,x) \mapsto e_i(c)(x))$ a rational $\mathbb{C}^\times$-action.

**Definition 2.1.** A quadruple $(X, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$ is a $G$ (or $g$)-geometric crystal if

(i) $\{1\} \times X \subset \text{dom}(e_i)$ for any $i \in I$.

(ii) $\gamma_j(e_i^j(x)) = e^{\alpha_{ij}^\vee}\gamma_j(x)$.

(iii) $e_i$'s satisfy the following relations.

$$e_i^a c_i = e_j^c e_i^b, \quad e_j^d c_i = e_i^e c_j^d = e_j^e c_i^d, \quad e_i^e c_i^d e_j^e = e_j^e c_i^d e_j^e c_i^d, \quad e_i^e c_i^d e_j^e = e_j^e c_i^d e_i^e c_j^d e_i^e$$

if $a_{ij} = a_{ji} = 0$,

$$e_i^a c_i = e_j^c e_i^b, \quad e_j^d c_i = e_i^e c_j^d = e_i^e c_j^d e_i^e c_j^d e_i^e$$

if $a_{ij} = a_{ji} = -1$,

$$e_i^a c_i^d e_j^e = e_j^e c_i^d e_j^e c_i^d, \quad e_i^e c_i^d e_j^e = e_j^e c_i^d e_i^e c_j^d e_i^e$$

if $a_{ij} = -3, a_{ji} = -1$,

(iv) $\varepsilon_i(e_i^j(x)) = c^{-1}\varepsilon_i(x)$ and $\varepsilon_i(e_i^j(x)) = \varepsilon_i(x)$ if $a_{ij} = a_{ji} = 0$.

The condition (iv) is slightly modified from the one in [6, 21, 23].

Let $W$ be the Weyl group associated with $g$. Define $R(w)$ for $w \in W$ by

$$R(w) := \{(i_1, i_2, \cdots, i_l) \in I^l | w = s_{i_1} s_{i_2} \cdots s_{i_l}\},$$

where $l$ is the length of $w$. Then $R(w)$ is the set of reduced words of $w$. For a word $i = (i_1, \cdots, i_l) \in R(w)$ ($w \in W$), set $\alpha^{(l)} := s_{i_1} \cdots s_{i_{j+1}}(\alpha_{i_j})$ $(1 \leq j \leq l)$ and

$$e_i : T \times X \to X \quad (t, x) \mapsto e_i^j(x) := e^{\alpha^{(l)}(t)} e_{i_1}^{\alpha^{(2)}(t)} \cdots e_{i_l}^{\alpha^{(l)}(t)}(x).$$

Note that the condition (iii) above is equivalent to the following: $e_i = e_i'$ for any $w \in W$, i.e. $i \in R(w)$.

2.3. Geometric crystal on Schubert cell. Let $w \in W$ be a Weyl group element and take a reduced expression $w = s_{i_1} \cdots s_{i_l}$. Let $X := G/B$ be the flag variety, which is an ind-variety and $X_w \subset X$ the Schubert cell associated with $w$, which has a natural geometric crystal structure (11, 22). For $i := (i_1, \cdots, i_k)$, set

$$(2.1) \quad B^-_i := \{ Y_i(c_1, \cdots, c_k) := Y_{i_1}(c_1) \cdots Y_{i_k}(c_k) | c_1, \cdots, c_k \in \mathbb{C}^\times \} \subset B^-,$$

where $Y_i(c) := y_i(c)\alpha_i^\vee(c)$. This has a geometric crystal structure (22) isomorphic to $X_w$. The explicit forms of the action $e_i^c$, the rational function $\varepsilon_i$ and $\gamma_i$ on $B^-_i$.
are given by
\[ e_i^j(Y_1(c_1, \cdots, c_k)) = Y_1(C_1, \cdots, C_k), \]
where
\[ \sum_{1 \leq m \leq j, i_m \neq 1} c_1^{a_{i_1} \cdot \cdots \cdot a_{i_{m-1}} \cdot c_m} + \sum_{j < m \leq k, i_m \neq 1} c_1^{a_{i_1} \cdot \cdots \cdot a_{i_{m-1}} \cdot c_m} + \sum_{1 \leq m \leq k, i_m = 1} \frac{1}{c_1^{a_{i_1} \cdot \cdots \cdot a_{i_{m-1}} \cdot c_m}}. \]

(2.3) \[ \epsilon_i(Y_1(c_1, \cdots, c_k)) = \sum_{1 \leq m \leq k, i_m = 1} \frac{1}{c_1^{a_{i_1} \cdot \cdots \cdot a_{i_{m-1}} \cdot c_m}}, \]

(2.4) \[ \gamma_i(Y_1(c_1, \cdots, c_k)) = c_1^{a_{i_1} \cdot \cdots \cdot a_{i_k}}. \]

2.4. Positive structure, Ultra-discretizations and Tropicalizations. Let us recall the notions of positive structure, ultra-discretization and tropicalization.

The setting below is same as in [15]. Let \( T = (\mathbb{C}^*)^l \) be an algebraic torus over \( \mathbb{C} \) and \( X^*(T) := \text{Hom}(T, \mathbb{C}^*) \cong \mathbb{Z}^l \) (resp. \( X_*(T) := \text{Hom}(\mathbb{C}^*, T) \cong \mathbb{Z}^l \)) be the lattice of characters (resp. co-characters) of \( T \). Set \( R := \mathbb{C}(c) \) and define
\[ v : R \setminus \{0\} \rightarrow \mathbb{Z}, \quad f(c) \mapsto \text{deg}(f(c)), \]
where \( \text{deg} \) is the degree of poles at \( c = \infty \). Here note that for \( f_1, f_2 \in R \setminus \{0\} \), we have
\[ v(f_1 f_2) = v(f_1) + v(f_2), \quad v \left( \frac{f_1}{f_2} \right) = v(f_1) - v(f_2). \]

A non-zero rational function on an algebraic torus \( T \) is called positive if it can be written as \( g/h \) where \( g \) and \( h \) are a positive linear combination of characters of \( T \).

Definition 2.2. Let \( f : T \rightarrow T' \) be a rational morphism between two algebraic tori \( T \) and \( T' \). We say that \( f \) is positive, if \( \eta \circ f \) is positive for any character \( \eta : T' \rightarrow \mathbb{C} \).

Denote by \( \text{Mor}^+(T, T') \) the set of positive rational morphisms from \( T \) to \( T' \).

Lemma 2.3 ([1]). For any \( f \in \text{Mor}^+(T_1, T_2) \) and \( g \in \text{Mor}^+(T_2, T_3) \), the composition \( g \circ f \) is well-defined and belongs to \( \text{Mor}^+(T_1, T_3) \).

By Lemma 2.3 we can define a category \( T_+ \) whose objects are algebraic tori over \( \mathbb{C} \) and arrows are positive rational morphisms.

Let \( f : T \rightarrow T' \) be a positive rational morphism of algebraic tori \( T \) and \( T' \). We define a map \( \hat{f} : X_*(T) \rightarrow X_*(T') \) by
\[ \langle \eta, \hat{f}(\xi) \rangle = v(\eta \circ f \circ \xi), \]
where \( \eta \in X^*(T') \) and \( \xi \in X_*(T) \).

Lemma 2.4 ([1]). For any algebraic tori \( T_1, T_2, T_3 \), and positive rational morphisms \( f \in \text{Mor}^+(T_1, T_2) \), \( g \in \text{Mor}^+(T_2, T_3) \), we have \( g \circ f = \hat{g} \circ \hat{f} \).

Let \( \text{Set} \) denote the category of sets with the morphisms being set maps. By the above lemma, we obtain a functor:
\[ UD : \begin{array}{ccc}
T_+ & \rightarrow & \text{Set} \\
T & \rightarrow & X_*(T) \\
(f : T \rightarrow T') & \mapsto & (\hat{f} : X_*(T) \rightarrow X_*(T'))
\end{array} \]
Definition 2.5 ([1]). Let $\chi = (X, \{e_i\}_{i \in I}, \{\omega_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$ be a geometric crystal, $T'$ an algebraic torus and $\theta : T' \to X$ a birational isomorphism. The isomorphism $\theta$ is called positive structure on $\chi$ if it satisfies

(i) for any $i \in I$ the rational functions $\gamma_i \circ \theta : T' \to \mathbb{C}$ and $\varepsilon_i \circ \theta : T' \to \mathbb{C}$ are positive.

(ii) For any $i \in I$, the rational morphism $e_{i, \theta} : \mathbb{C}^\times \times T' \to T'$ defined by $e_{i, \theta}(c, t) := \theta^{-1} \circ e_i^c \circ \theta(t)$ is positive.

Let $\theta : T \to X$ be a positive structure on a geometric crystal $\chi = (X, \{e_i\}_{i \in I}, \{\omega_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$. Applying the functor $UD$ to positive rational morphisms $e_{i, \theta} : \mathbb{C}^\times \times T' \to T'$ and $\gamma \circ \theta : T' \to T$ (the notations are as above), we obtain

\[
\begin{align*}
\tilde{e}_i &:= UD(e_{i, \theta}) : \mathbb{Z} \times X_\ast(T) \to X_\ast(T), \\
\omega_i &:= UD(\gamma_i \circ \theta) : X_\ast(T') \to \mathbb{Z}, \\
\varepsilon_i &:= UD(e_{i, \theta}) : X_\ast(T') \to \mathbb{Z}.
\end{align*}
\]

Now, for given positive structure $\theta : T' \to X$ on a geometric crystal $\chi = (X, \{e_i\}_{i \in I}, \{\omega_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$ with a free pre-crystal structure (see [1, 2.2]) and denote it by $UD_{\theta, T'}(\chi)$. We have the following theorem:

Theorem 2.6 ([1][22]). For any geometric crystal $\chi = (X, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$ and positive structure $\theta : T' \to X$, the associated pre-crystal $UD_{\theta, T'}(\chi) = (X_\ast(T'), \{e_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$ is a crystal (see [1, 2.2]).

Now, let $GC^+$ be a category whose object is a triplet $\langle \chi, T', \theta \rangle$ where $\chi = (X, \{e_i\}, \{\gamma_i\}, \{\varepsilon_i\})$ is a geometric crystal and $\theta : T' \to X$ is a positive structure on $\chi$, and morphism $f : (\chi_1, T'_1, \theta_1) \to (\chi_2, T'_2, \theta_2)$ is given by a morphism $\varphi : X_1 \to X_2$ ($\chi_i = (X_i, \cdot \cdot \cdot)$) such that

\[f := \varphi^{-1} \circ \theta_2 \circ \theta : T'_1 \to T'_2,
\]

is a positive rational morphism. Let $CR$ be a category of crystals. Then by the theorem above, we have

Corollary 2.7. The map $UD = UD_{\theta, T'}$ defined above is a functor

\[
UD : GC^+ \to CR,
\]

\[(\chi, T', \theta) \mapsto X_\ast(T'), \quad (f : (\chi_1, T'_1, \theta_1) \to (\chi_2, T'_2, \theta_2)) \mapsto (\tilde{f} : X_\ast(T'_1) \to X_\ast(T'_2)).
\]

We call the functor $UD$ “ultra-discretization” as [22, 23] instead of “tropicalization” as in [1]. And for a crystal $B$, if there exists a geometric crystal $\chi$ and a positive structure $\theta : T' \to X$ on $\chi$ such that $UD(\chi, T', \theta) \cong B$ as crystals, we call an object $\langle \chi, T', \theta \rangle$ in $GC^+$ a tropicalization of $B$, where it is not known that this correspondence is a functor.

3. Limit of perfect crystals

We review limit of perfect crystals following [9]. (See also [7, 9]).
3.1. Crystals. First we review the theory of crystals, which is the notion obtained by abstracting the combinatorial properties of crystal bases. Let \((A, \{\alpha_i\}_{i \in I}, \{\alpha_i^\vee\}_{i \in I})\) be a Cartan data.

**Definition 3.1.** A crystal \(B\) is a set endowed with the following maps:

- \(\wt : B \rightarrow P,\)
- \(\varepsilon_i : B \rightarrow \mathbb{Z} \sqcup \{-\infty\}, \varphi_i : B \rightarrow \mathbb{Z} \sqcup \{-\infty\}\) for \(i \in I,\)
- \(\tilde{e}_i : B \sqcup \{0\} \rightarrow B \sqcup \{0\}, \tilde{f}_i : B \sqcup \{0\} \rightarrow B \sqcup \{0\}\) for \(i \in I,\)
- \(\varepsilon_i(0) = \tilde{f}_i(0) = 0.\)

Those maps satisfy the following axioms: for all \(b, b_1, b_2 \in B,\) we have

\[
\varphi_i(b) = \varepsilon_i(b) + \langle \alpha_i^\vee, \wt(b) \rangle,
\]

\[
\wt(\tilde{e}_i b) = \wt(b) + \alpha_i \quad \text{if} \quad \tilde{e}_i b \in B,
\]

\[
\wt(\tilde{f}_i b) = \wt(b) - \alpha_i \quad \text{if} \quad \tilde{f}_i b \in B,
\]

\[
\tilde{e}_ib_2 = b_1 \iff \tilde{f}_ib_1 = b_2 \quad (b_1, b_2 \in B),
\]

\[
\varepsilon_i(b) = -\infty \implies \tilde{e}_ib = \tilde{f}_ib = 0.
\]

The following tensor product structure is one of the most crucial properties of crystals.

**Theorem 3.2.** Let \(B_1\) and \(B_2\) be crystals. Set \(B_1 \otimes B_2 := \{b_1 \otimes b_2; b_j \in B_j (j = 1, 2)\}.\) Then we have

(i) \(B_1 \otimes B_2\) is a crystal.

(ii) For \(b_1 \in B_1\) and \(b_2 \in B_2,\) we have

\[
\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} 
\tilde{f}_ib_1 \otimes b_2 & \text{if} \ \varphi_i(b_1) > \varepsilon_i(b_2), \\
 b_1 \otimes \tilde{f}_ib_2 & \text{if} \ \varphi_i(b_1) \leq \varepsilon_i(b_2).
\end{cases}
\]

\[
\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} 
 b_1 \otimes \tilde{e}_ib_2 & \text{if} \ \varphi_i(b_1) < \varepsilon_i(b_2), \\
 \tilde{e}_ib_1 \otimes b_2 & \text{if} \ \varphi_i(b_1) \geq \varepsilon_i(b_2).
\end{cases}
\]

**Definition 3.3.** Let \(B_1\) and \(B_2\) be crystals. A strict morphism of crystals \(\psi : B_1 \rightarrow B_2\) is a map \(\psi : B_1 \sqcup \{0\} \rightarrow B_2 \sqcup \{0\}\) satisfying: \(\psi(0) = 0, \psi(B_1) \subset B_2,\)

\(\psi\) commutes with all \(\tilde{e}_i\) and \(\tilde{f}_i\) and

\[
\wt(\psi(b)) = \wt(b), \quad \varepsilon_i(\psi(b)) = \varepsilon_i(b), \quad \varphi_i(\psi(b)) = \varphi_i(b) \quad \text{for any} \quad b \in B_1.
\]

In particular, a bijective strict morphism is called an isomorphism of crystals.

**Example 3.4.** If \((L, B)\) is a crystal base, then \(B\) is a crystal. Hence, for the crystal base \((L(\infty), B(\infty))\) of the nilpotent subalgebra \(U_q^-(g)\) of the quantum algebra \(U_q(g), B(\infty)\) is a crystal.

**Example 3.5.** For \(\lambda \in P,\) set \(T_\lambda := \{t_\lambda\}.\) We define a crystal structure on \(T_\lambda\) by

\[
\tilde{e}_i(t_\lambda) = \tilde{f}_i(t_\lambda) = 0, \quad \varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty, \quad \wt(t_\lambda) = \lambda.
\]

**Definition 3.6.** For a crystal \(B,\) a colored oriented graph structure is associated with \(B\) by

\[
b_1 \xrightarrow{i} b_2 \iff \tilde{f}_ib_1 = b_2.
\]

We call this graph a crystal graph of \(B.\)
3.2. **Affine weights.** Let \( \mathfrak{g} \) be an affine Lie algebra. The sets \( t, \{\alpha_i\}_{i \in I} \) and \( \{\alpha_i^\vee\}_{i \in I} \) be as in [21]. We take \( \dim t = 2I + 1 \). Let \( \delta \in Q_+ \) be the unique element satisfying \( \{\lambda \in Q | (\alpha_i^\vee, \lambda) = 0 \text{ for any } i \in I\} = Z\delta \) and \( c \in \mathfrak{g} \) be the canonical central element satisfying \( \{h \in Q^\vee | (h, \alpha_i) = 0 \text{ for any } i \in I\} = Zc \). We write ([16, 6.1])

\[
c = \sum a_i^\vee \alpha_i^\vee, \quad \delta = \sum a_i\alpha_i.
\]

Let \((\ , \ )\) be the non-degenerate \( W \)-invariant symmetric bilinear form on \( t^* \) normalized by \((\delta, \lambda) = (\epsilon, \lambda) \) for \( \lambda \in t^* \). Let us set \( t_0^\vee := t^*/C\delta \) and let \( \text{cl} : t^* \to t_0^\vee \) be the canonical projection. Here we have \( t_0^\vee \cong \oplus_i (C\alpha_i^\vee)^* \). Set \( t_0 \equiv \{\lambda \in t^* | (\epsilon, \lambda) = 0\}, (t_0)_{\ast} := \text{cl}(t_0) \). Since \((\delta, \delta) = 0\), we have a positive-definite symmetric form on \( t_0^\vee \) induced by the one on \( t^* \). Let \( \Lambda_i \in t_0^\vee \) be a classical weight such that \( \langle \alpha_i^\vee, \Lambda_j \rangle = \delta_{i,j} \), which is called a fundamental weight. We choose \( P \) so that \( P_{cl} := \text{cl}(P) \) coincides with \( \oplus_{i \in I} Z\Lambda_i \) and we call \( P_{cl} \) a classical weight lattice.

3.3. **Definitions of perfect crystal and its limit.** Let \( \mathfrak{g} \) be an affine Lie algebra, \( P_{cl} \) be a classical weight lattice as above and set \( (P_{cl})_l^+ := \{\lambda \in P_{cl} | (c, \lambda) = \lambda, \langle \alpha_i, \lambda \rangle \geq 0 \} \) \( (l \in \mathbb{Z}_{>0}) \).

**Definition 3.7.** A crystal \( B \) is a perfect crystal of level \( l \) if

(i) \( B \otimes B \) is connected as a crystal graph.

(ii) There exists \( \lambda_0 \in P_{cl} \) such that

\[
\text{wt}(B) \subset \lambda_0 + \sum_{i \neq 0} \mathbb{Z}_{\leq 0} \text{cl}(\alpha_i), \quad \sharp B_{\lambda_0} = 1
\]

(iii) There exists a finite-dimensional \( U'_l(\mathfrak{g}) \)-module \( V \) with a crystal pseudo-base \( B_{ps} \) such that \( B \cong B_{ps}/\pm 1 \)

(iv) The maps \( \varepsilon, \varphi : B_{l}^{\min} := \{b \in B | (c, \varepsilon(b)) = l\} \to (P_{cl}^+)_{l} \) are bijective, where \( \varepsilon(b) := \sum_i \varepsilon_i(b)\Lambda_i \) and \( \varphi(b) := \sum_i \varphi_i(b)\Lambda_i \).

Let \( \{B_l\}_{l \geq 1} \) be a family of perfect crystals of level \( l \) and set \( I := \{(l, b) | l > 0, b \in B_{l}^{\min}\} \).

**Definition 3.8.** A crystal \( B_{\infty} \) with an element \( b_{\infty} \) is called a limit of \( \{B_l\}_{l \geq 1} \)

(i) \( \text{wt}(b_{\infty}) = \varepsilon(b_{\infty}) = \varphi(b_{\infty}) = 0 \).

(ii) For any \( (l, b) \in I \), there exists an embedding of crystals:

\[
f_{(l,b)} : \quad T_{\varepsilon(b)} \otimes B_l \otimes T_{-\varphi(b)} \hookrightarrow B_{\infty} \quad t_{\varepsilon(b)} \otimes b \otimes t_{-\varphi(b)} \mapsto b_{\infty}
\]

(iii) \( B_{\infty} = \bigcup_{(l,b) \in I} \text{Im} f_{(l,b)} \).

As for the crystal \( T_{\lambda} \), see Example 3.5. If a limit exists for a family \( \{B_l\} \), we say that \( \{B_l\} \) is a coherent family of perfect crystals.

The following is one of the most important properties of limit of perfect crystals.

**Proposition 3.9.** Let \( B(\infty) \) be the crystal as in Example 3.4. Then we have the following isomorphism of crystals:

\[
B(\infty) \otimes B_{\infty} \cong B(\infty).
\]
4. Perfect Crystals of type $G_2^{(1)}$

In this section, we review the family of perfect crystals of type $G_2^{(1)}$ and its limit $[21]$. We fix the data for $G_2^{(1)}$. Let $\{\alpha_0, \alpha_1, \alpha_2\}$, $\{\alpha_0^\vee, \alpha_1^\vee, \alpha_2^\vee\}$ and $\{\Lambda_0, \Lambda_1, \Lambda_2\}$ be the set of simple roots, simple coroots and fundamental weights, respectively. The Cartan matrix $A = (a_{ij})_{i,j=0,1,2}$ is given by

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix},$$

and its Dynkin diagram is as follows.

\[ \begin{array}{ccc} \circ_0 & \rightarrow & \circ_2 \\
\end{array} \]

The standard null root $\delta$ and the canonical central element $c$ are given by

$$\delta = \alpha_0 + 2\alpha_1 + 3\alpha_2 \quad \text{and} \quad c = \alpha_0^\vee + 2\alpha_1^\vee + \alpha_2^\vee,$$

where $\alpha_0 = 2\Lambda_0 - \Lambda_1 + \delta, \quad \alpha_1 = -\Lambda_0 + 2\Lambda_1 - 3\Lambda_2, \quad \alpha_2 = -\Lambda_1 + 2\Lambda_2$.

For a positive integer $l$ we introduce $G_2^{(1)}$-crystals $B_l$ and $B_\infty$ as

$$B_l = \left\{ b = (b_1, b_2, b_3, \bar{b}_3, \bar{b}_2, \bar{b}_1) \in (\mathbb{Z}_{\geq 0}/3)^6 \left| \begin{array}{l}
3b_3 \equiv 3\bar{b}_3 \pmod{2}, \\
\sum_{i=1,2}(b_i + \bar{b}_i) + \frac{b_3 + \bar{b}_3}{2} \leq l \\
b_1, b_1, b_2 - b_3, \bar{b}_3 - \bar{b}_2 \in \mathbb{Z}
\end{array} \right. \right\},$$

$$B_\infty = \left\{ b = (b_1, b_2, b_3, \bar{b}_3, \bar{b}_2, \bar{b}_1) \in (\mathbb{Z}/3)^6 \left| \begin{array}{l}
3b_3 \equiv 3\bar{b}_3 \pmod{2}, \\
b_1, b_1, b_2 - b_3, \bar{b}_3 - \bar{b}_2 \in \mathbb{Z}
\end{array} \right. \right\}.$$

Now we describe the explicit crystal structures of $B_l$ and $B_\infty$. Indeed, most of them coincide with each other except for $\varepsilon_0$ and $\varphi_0$. In the rest of this section, we use the following convention: $(x)_+ = \max(x, 0)$. For $b = (b_1, b_2, b_3, \bar{b}_3, \bar{b}_2, \bar{b}_1)$ we denote

\[ s(b) = b_1 + b_2 + \frac{b_3 + \bar{b}_3}{2} + \bar{b}_2 + \bar{b}_1, \]

and

\[ z_1 = \bar{b}_1 - b_1, \quad z_2 = \bar{b}_2 - b_3, \quad z_3 = b_3 - b_2, \quad z_4 = (\bar{b}_3 - b_3)/2. \]

Now we define conditions $(E_1)$-$(E_6)$ and $(F_1)$-$(F_6)$ as follows.

\[ (E_i) \quad (1 \leq i \leq 6) \quad \text{is defined from} \quad (F_i) \quad \text{by replacing} \quad > \quad \text{(resp.} \leq \text{)} \quad \text{with} \quad \geq \quad \text{(resp.} <). \]

We also define

\[ A = (0, z_1, z_1 + z_2, z_1 + z_2 + 3z_4, z_1 + z_2 + z_3 + 3z_4, 2z_1 + z_2 + z_3 + 3z_4). \]

Then for $b = (b_1, b_2, b_3, \bar{b}_3, \bar{b}_2, \bar{b}_1) \in B_l$ or $B_\infty$, $\bar{c}_i b, \bar{f}_i b, \varepsilon_i(b), \varphi_i(b), i = 0, 1, 2$ are given as follows.
\[\hat{e}_{0b} = \begin{cases} (b_1 - 1, \ldots) & \text{if } (E_1), \\
(b_1 - 1, b_1 - 1, \ldots, \tilde{b}_1 + 1) & \text{if } (E_2), \\
(b_1 - 2, \ldots, b_1, \tilde{b}_1 + 1, \ldots) & \text{if } (E_3) \text{ and } z_4 = -\frac{1}{3}, \\
(b_1 - 2, \ldots, b_1, \tilde{b}_1 + 1, \ldots) & \text{if } (E_3) \text{ and } z_4 = -\frac{2}{3}, \\
(b_1 - 1, \ldots, b_1, \tilde{b}_1 + 1, \ldots) & \text{if } (E_5), \\
(b_1 + 1) & \text{if } (E_6), \end{cases}\]

\[\hat{f}_{0b} = \begin{cases} (b_1 + 1, \ldots) & \text{if } (F_1), \\
(b_1 + 1, b_1 - 1, \ldots) & \text{if } (F_2), \\
(b_1 + 1, b_1 - 1, \ldots) & \text{if } (F_3), \\
(b_1 + 1, b_1 - 1, \ldots) & \text{if } (F_4) \text{ and } z_4 = \frac{1}{3}, \\
(b_1 + 1, b_1 - 1, \ldots) & \text{if } (F_4) \text{ and } z_4 = \frac{2}{3}, \\
(b_1 + 1, \ldots) & \text{if } (F_5), \\
(b_1 + 1, \ldots) & \text{if } (F_6), \end{cases}\]

\[\hat{e}_{1b} = \begin{cases} (\ldots, \tilde{b}_2 + 1, \tilde{b}_1 - 1) & \text{if } \tilde{b}_2 - \tilde{b}_3 \geq (b_2 - b_3)_+, \\
(b_1 + 1, \tilde{b}_2 + 1, \ldots) & \text{if } \tilde{b}_2 - \tilde{b}_3 < 0 \leq b_2 - b_3, \\
(b_1 + 1, \tilde{b}_2 - 1, \ldots) & \text{if } (\tilde{b}_2 - \tilde{b}_3)_+ < b_2 - b_3, \end{cases}\]

\[\hat{f}_{1b} = \begin{cases} (b_1 - 1, b_2 + 1, \ldots) & \text{if } (b_2 - b_3)_+ \leq b_2 - b_3, \\
(b_1 - 1, b_2 + 1, \ldots) & \text{if } (b_2 - b_3)_+ \leq b_2 - b_3, \\
(b_1 - 1, b_2 + 1, \ldots) & \text{if } (b_2 - b_3)_+ \leq b_2 - b_3, \end{cases}\]

\[\hat{e}_{2b} = \begin{cases} (\ldots, \tilde{b}_3 + \frac{2}{3}, \tilde{b}_2 - \frac{1}{3}, \ldots) & \text{if } \tilde{b}_3 \geq b_3, \\
(\ldots, \tilde{b}_2 + \frac{1}{3}, \tilde{b}_3 - \frac{2}{3}, \ldots) & \text{if } \tilde{b}_3 < b_3, \end{cases}\]

\[\hat{f}_{2b} = \begin{cases} (\ldots, b_2 - \frac{1}{3}, b_3 + \frac{2}{3}, \ldots) & \text{if } \tilde{b}_3 \leq b_3, \\
(\ldots, b_3 - \frac{1}{3}, b_2 + \frac{2}{3}, \ldots) & \text{if } \tilde{b}_3 > b_3. \end{cases}\]

\[\varepsilon_1(b) = \tilde{b}_1 + (\tilde{b}_3 - \tilde{b}_2 + (b_2 - b_3)_+)_+, \quad \varepsilon_2(b) = b_1 + (b_3 - b_2 + (\tilde{b}_2 - \tilde{b}_3)_+)_+, \quad \varphi_1(b) = b_1 + (b_3 - b_2 + (\tilde{b}_2 - \tilde{b}_3)_+)_+, \quad \varphi_2(b) = 3b_2 + \frac{3}{2}(\tilde{b}_3 - b_3)_+, \]

\[\varepsilon_0(b) = \begin{cases} l - s(b) + \max A - (2z_1 + z_2 + z_3 + 3z_4) & b \in B_1, \\
-ls(b) + \max A - (2z_1 + z_2 + z_3 + 3z_4) & b \in B_{\infty}. \end{cases}\]

\[\varphi_0(b) = \begin{cases} l - s(b) + \max A & b \in B_1, \\
-ls(b) + \max A & b \in B_{\infty}. \end{cases}\]

For \( b \in B_j \) if \( \hat{e}_{ib} \) or \( \hat{f}_{ib} \) does not belong to \( B_j \), namely, if \( b_j \) or \( \tilde{b}_j \) for some \( j \) becomes negative or \( s(b) \) exceeds \( l \), we understand it to be 0.

The following is one of the main results in [21]:
Theorem 4.1 \cite{21}.  
(i) The $G_2^{(1)}$-crystal $B_l$ is a perfect crystal of level $l$.
(ii) The family of the perfect crystals $\{B_l\}_{l \geq 1}$ forms a coherent family and the crystal $B_\infty$ is its limit with the vector $b_\infty = (0,0,0,0,0)$.

As was shown in \cite{21}, the minimal elements are given

$$(B_l)_{\text{min}} = \{(\alpha, \beta, \beta, \beta, \beta, \alpha) \mid \alpha \in \mathbb{Z}_{\geq 0}, \beta \in (\mathbb{Z}_{\geq 0})/3, 2\alpha + 3\beta \leq l\}.$$  

Let $J = \{(l, b) \mid l \in \mathbb{Z}_{\geq 1}, b \in (B_l)_{\text{min}}\}$ and the maps $\varepsilon, \varphi : (B_l)_{\text{min}} \rightarrow (P^+_1)_l$ be as in Sect. 3. Then we have $\text{wt}b_\infty = 0$ and $\varepsilon_i(b_\infty) = \varphi_i(b_\infty) = 0$ for $i = 0,1,2$.

For $(l, b_0) \in J$, since $\varepsilon(b_0) = \varphi(b_0)$, one can set $\lambda = \varepsilon(b_0) = \varphi(b_0)$. For $b = (b_1, b_2, b_3, b_4, b_5, b_6) \in B_l$ we define a map

$$f_{(l,b_0)} : T_\lambda \otimes B_l \otimes B_{-\lambda} \rightarrow B_\infty$$

by

$$f_{(l,b_0)}(t_\lambda \otimes b \otimes t_{-\lambda}) = b' = (\nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6)$$

where $b_0 = (\alpha, \beta, \beta, \beta, \beta, \alpha)$, and

$$\nu_1 = b_1 - \alpha, \quad \nu_5 = b_1 - \alpha,$$

$$\nu_2 = b_2 - \beta, \quad \nu_3 = b_3 - \beta$$

Finally, we obtain $B_\infty = \bigcup_{(l,b) \in J} \text{Im} f_{(l,b)}$

5. Affine Geometric Crystal $\mathcal{V}_1(D_4^{(3)})$

5.1. Fundamental representation $W(\varpi_1)$ for $D_4^{(3)}$. Let $c = \sum_i a_i^J a_i^Y$ be the canonical central element in an affine Lie algebra $g$ (see \cite{16} 6.1), $\{A_i \mid i \in I\}$ the set of fundamental weight as in the previous section and $\varpi_1 := \Lambda_1 - a_1^J \Lambda_0$ the (level 0)fundamental weight. Let $W(\varpi_1)$ be the fundamental representation of $U_q'(g)$ associated with $\varpi_1$ (\cite{12}).

By \cite{12} Theorem 5.17, $W(\varpi_1)$ is a finite-dimensional irreducible integrable $U_q'(g)$-module and has a global basis with a simple crystal. Thus, we can consider the specialization $q = 1$ and obtain the finite-dimensional $g$-module $W(\varpi_1)$, which we call a fundamental representation of $g$ and use the same notation as above.

We shall present the explicit form of $W(\varpi_1)$ for $g = D_4^{(3)}$.

5.2. $W(\varpi_1)$ for $D_4^{(3)}$. The Cartan matrix $A = (a_{i,j})_{i,j=0,1,2}$ of type $D_4^{(3)}$ is:

$$A = \begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & -3 \\
0 & -1 & 2
\end{pmatrix}.$$  

Then the simple roots are

$$\alpha_0 = 2\Lambda_0 - \Lambda_1 + \delta, \quad \alpha_1 = -\Lambda_0 + 2\Lambda_1 - \Lambda_2, \quad \alpha_2 = -3\Lambda_1 + 2\Lambda_2,$$

and the Dynkin diagram is:

```
0——1——2
```

The $D_4^{(3)}$-module $W(\varpi_1)$ is an 8-dimensional module with the basis,

$$\{v_1, v_2, v_3, v_0, \emptyset, v_4, v_6, v_7, v_8\}.$$
The explicit form of $W(\varpi_1)$ is given in [14].

$$\mathrm{wt}(v_1) = \Lambda_1 - 2\Delta_0, \quad \mathrm{wt}(v_2) = -\Lambda_0 - \Lambda_1 + \Lambda_2, \quad \mathrm{wt}(v_3) = -\Lambda_0 + 2\Lambda_1 - \Lambda_2,$$

$$\mathrm{wt}(v_i) = -\mathrm{wt}(v_i) \quad (i = 1, \ldots, 3), \quad \mathrm{wt}(v_0) = \mathrm{wt}(0) = 0.$$

The actions of $e_i$ and $f_i$ on these basis vectors are given as follows:

$$f_0(v_0, v_\gamma, v_\gamma, v_\gamma, 0) = \left(v_1, v_2, v_3, 0 + \frac{1}{2}v_0, \frac{3}{2}v_1\right),$$

$$f_1(v_1, v_3, v_0, v_\gamma, 0) = (v_2, v_0, 2v_\gamma, v_\gamma),$$

$$f_2(v_2, v_\gamma) = (v_3, v_\gamma),$$

$$e_0(v_1, v_2, v_3, 0, 0) = \left(0 + \frac{1}{2}v_0, v_\gamma, v_\gamma, v_\gamma, \frac{3}{2}v_1\right),$$

$$e_1(v_2, v_0, v_\gamma, v_\gamma) = (v_1, 2v_3, v_0, v_\gamma),$$

$$e_2(v_3, v_\gamma) = (v_2, v_\gamma),$$

where we give non-trivial actions only.

5.3. **Affine Geometric Crystal $V_1(D_4^{(3)})$ in $W(\varpi_1)$**. Let us review the construction of the affine geometric crystal $V_1(D_4^{(3)})$ in $W(\varpi_1)$ following [6].

For $\xi \in (t_i)_{\mathbb{A}}$, let $t(\xi)$ be the translation as in [12 Sect 4] and $\tau_i$ as in [13], indeed, $\tau_i := \max(1, \frac{2}{(t_i, t_i)}) \varpi_i$. Then we have

$$t(\varpi_1) = s_{01}s_{21}s_{21}s_{21} =: w_1,$$

$$t(\mathrm{wt}(\varpi_1)) = s_{01}s_{21}s_{21}s_{01} =: w_2,$$

Associated with these Weyl group elements $w_1$ and $w_2$, we define algebraic varieties $V_1 = V_1(D_4^{(3)})$ and $V_2 = V_2(D_4^{(3)}) \subset W(\varpi_1)$ respectively:

$$V_1 := \{ V_1(x) := Y_0(x_0)V_1(x_1)V_2(x_2)V_1(x_3)V_2(x_4)V_1(x_5)v_1 \mid x_i \in \mathbb{C}^*, (0 \leq i \leq 5) \},$$

$$V_2 := \{ V_2(y) := Y_2(y_2)V_1(y_1)V_2(y_4)V_1(y_3)V_0(y_0)V_1(y_5)v_\gamma \mid y_i \in \mathbb{C}^*, (0 \leq i \leq 5) \}.$$

Owing to the explicit forms of $f_i$’s on $W(\varpi_1)$ as above, we have $f_0^3 = 0, f_1^3 = 0$ and $f_2^2 = 0$ and then

$$Y_i(c) = (1 + \frac{f_i}{c} + \frac{f_i^2}{2c^2})\alpha_i^\vee(c) \quad (i = 0, 1), \quad Y_2(c) = (1 + \frac{f_2}{c})\alpha_2^\vee(c).$$

We get explicit forms of $V_1(x) \in V_1$ and $V_2(y) \in V_2$ as in [24]:

$$V_1(x) = \sum_{1 \leq i \leq 3} (X_i v_i + X_i v_\gamma) + X_0 v_0 + X_0 \varnothing,$$

$$V_2(y) = \sum_{1 \leq i \leq 3} (Y_i v_i + Y_i v_\gamma) + Y_0 v_0 + Y_0 \varnothing,$$

where the rational functions $X_i$’s and $Y_i$’s are all positive in $(x_0, \ldots, x_5)$ and $(y_0, \ldots, y_5)$ respectively (as for their explicit forms, see [6]) and for any $x$ there exist a unique rational function $a(x)$ and $y$ such that $V_2(y) = a(x)V_1(x)$. Using this result, we get the positive birational isomorphism $\varpi : V_1 \rightarrow V_2$ $(V_1(x) \mapsto V_2(y))$ and we know that its inverse $\varpi^{-1}$ is also positive. The actions of $e_0^\vee$ on $V_2(y)$ (respectively $\gamma_0(V_2(y))$ and $e_0(V_2(y))$) are induced from the ones on $Y_2(y_2)V_1(y_1)V_2(y_4)V_1(y_3)V_0(y_0)V_1(y_5)$ as an element of the geometric crystal $V_2$. We define the action $e_0^\vee$ on $V_1(x)$ by

$$e_0^\vee V_1(x) = \varpi^{-1} \circ e_0^\vee \circ \varpi(V_1(x)).$$
We also define $\gamma_0(V_1(x))$ and $\varepsilon_0(V_1(x))$ by

\begin{equation}
\gamma_0(V_1(x)) = \gamma_0(\sigma(V_1(x))), \quad \varepsilon_0(V_1(x)) := \varepsilon_0(\sigma(V_1(x))).
\end{equation}

**Theorem 5.1** ([6]). Together with (5.1), (5.2) on $V_1$, we obtain a positive affine geometric crystal $\chi := \{V_1, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I}\} (I = \{0, 1, 2\})$, whose explicit form is as follows: first we have $e^i_1, \gamma_i$ and $\varepsilon_i$ for $i = 1, 2$ from the formula (2.2), (2.3) and (2.4).

$$e^i_1(V_1(x)) = V_1(x_0, C_1 x_1, x_2, C_3 x_3, x_4, C_5 x_5), \quad e^2_1(V_1(x)) = V_1(x_0, x_1, C_2 x_2, x_3, C_4 x_4, x_5),$$

where

$$C_1 = \frac{x_0 x_1 + x_0 x_2 + x_0 x_3}{x_0 x_1 + x_0 x_2 + x_0 x_3}, \quad C_2 = \frac{x_0 x_3 + x_0 x_4 + x_0 x_5}{x_0 x_3 + x_0 x_4 + x_0 x_5}, \quad C_3 = \frac{x_0 x_4 + x_0 x_5 + x_0 x_6}{x_0 x_4 + x_0 x_5 + x_0 x_6}, \quad C_5 = \frac{x_0 x_2 + x_0 x_3 + x_0 x_4 + x_0 x_5}{x_0 x_2 + x_0 x_3 + x_0 x_4 + x_0 x_5},$$

$$\varepsilon_1(V_1(x)) = \frac{x_0 + x_0 x_2 + x_0 x_4}{x_1 + x_1^2 x_3 + x_1 x_2 x_4 x_5}, \quad \varepsilon_2(V_1(x)) = \frac{x_1^3 + x_1^3 x_3^3}{x_2 + x_2^2 x_4}, \quad \gamma_1(V_1(x)) = \frac{x_0^2 + x_1^2 x_3 x_5^5}{x_0 x_2 x_4}, \quad \gamma_2(V_1(x)) = \frac{x_0^2 + x_1^2 x_3 x_5^5}{x_1^2 + x_3 x_5^5}.$$
Indeed, from the explicit form of $i$

We simplify this by using the following lemma:

use the same notations

of type

\((\mathbb{C}^* )^6 \rightarrow V_1 (x \mapsto V_1 (x)).\) Then by Corollary 2.4 we obtain the ultra-discretization $UD(\chi, T', \theta)$, which is a Kashiwara’s crystal. Now we show that the conjecture in [6] is correct and it turns out to be the following theorem.

**Theorem 6.1.** The crystal $UD(\chi, T', \theta)$ as above is isomorphic to the crystal $B_{\infty}$ of type $G_2^{(1)}$ as in Sect. [3]

In order to show the theorem, we shall see the explicit crystal structure on $\mathcal{X} := UD(\chi, T', \theta)$. Note that $UD(\chi) = \mathbb{Z}^6$ as a set . Here as for variables in $\mathcal{X}$, we use the same notations $c, x_0, x_1, \ldots, x_5$ as for $\chi$.

For $x = (x_0, x_1, \ldots, x_5) \in \mathcal{X}$, it follows from the results in the previous section that the functions $\text{wt}_i$ and $\varepsilon_i (i = 0, 1, 2)$ are given as:

\[
\begin{align*}
\text{wt}_0(x) &= 2x_0 - x_1 - x_3 - x_5, \quad \text{wt}_1(x) = 2(x_1 + x_3 + x_5) - x_0 - x_2 - x_4, \\
\text{wt}_2(x) &= 2(x_2 + x_4) - 3(x_1 - x_3 - x_5).
\end{align*}
\]

Set

\[
\begin{align*}
\alpha &:= 2x_0 + x_2 + x_3, \quad \beta := x_1 + x_2 + 2x_3 + x_5, \quad \gamma := x_0 + x_1 + 3x_3, \\
\delta &:= x_0 + x_2 + 2x_3, \quad \epsilon := x_0 + x_1 + x_2 + x_4, \\
\phi &:= x_0 + x_1 + x_2 + x_3 + x_5.
\end{align*}
\]

Then we have

\[
\begin{align*}
\varepsilon_0(x) &= \max(\alpha, \beta, \gamma, \delta, \epsilon, \phi) - (3x_0 + x_2 + x_3), \\
\varepsilon_1(x) &= \max(x_0 - x_1, x_0 + x_2 - 2x_1 - x_3, x_0 + x_2 + x_4 - 2x_1 - 2x_3 - x_5), \\
\varepsilon_2(x) &= \max(3x_1 - x_2, 3x_1 + 3x_3 - 2x_2 - x_4).
\end{align*}
\]

Indeed, from the explicit form of $G$ as in the previous section we have

\[
UD(G)|_{c=-1} = \max(-3 + 3\alpha, -2 + 2\alpha + \delta, -2 + 2\alpha + \gamma, -1 + \alpha + 2\delta, -1 + \alpha + \gamma + \delta, -3\delta, -1 + \alpha + 2\gamma, \gamma + 2\delta, 2\gamma + \delta, 3\gamma, -3 + 2\alpha + \epsilon, -2 + \alpha + \delta + \epsilon, -1 + \alpha + \gamma + \epsilon, -1 + 2\delta + \epsilon, 2\gamma + \epsilon, -3 + 2\alpha + 2\epsilon, -2 + \delta + 2\epsilon, 2\gamma, -3 + 3\epsilon, -3 + 2\alpha + \phi, -2 + \alpha + \delta + \phi, -2 + \alpha + \gamma + \phi, -1 + 2\delta + \phi, -1 + \gamma + \delta + \phi + \beta, 2\beta + \delta, -1 + 2\gamma + \phi, \beta + \gamma + \delta, -3 + \alpha + \epsilon + \phi, -2 + \alpha + \beta + \phi, -1 + \gamma + \epsilon + \phi, -1 + \alpha + \gamma + \epsilon, -1 + \beta + \delta + \epsilon, -1 + \beta + \gamma + \phi, -1 + \alpha + 2\beta, -1 + \beta + \gamma + \phi, 2\beta + \delta, 2\beta + \gamma, -3 + \epsilon + 2\phi, -2 + \delta + \epsilon + \phi, -1 + \beta + \epsilon + \phi), -1 + 2\beta + \epsilon, -3 + 3\phi, -2 + \beta + 2\phi, -1 + 2\delta + \phi, 3\beta).
\]

We simplify this by using the following lemma:
Lemma 6.2. For \( m_1, \cdots, m_k \in \mathbb{R} \) and \( t_1, \cdots, t_k \in \mathbb{R}_{\geq 0} \) such that \( t_1 + \cdots + t_k = 1 \), we have

\[
\max \left( m_1, \cdots, m_k, \sum_{i=1}^{k} t_i m_i \right) = \max(m_1, \cdots, m_k)
\]

Since we have

\[
\begin{align*}
-2 + 2\alpha + \delta &= \frac{2(-3 + 3\alpha) + 3\delta}{3}, \quad -2 + 2\alpha + \gamma = \frac{2(-3 + 3\alpha) + 3\gamma}{3}, \\
-1 + \alpha + 2\delta &= \frac{2 \cdot 3\delta + (-3 + 3\alpha)}{3}, \quad -1 + \alpha + \delta = \frac{(-3 + 3\alpha) + 3\gamma + 3\delta}{3}, \\
-1 + \alpha + 2\gamma &= \frac{(-3 + 3\alpha) + 2 \cdot 3\gamma}{3}, \quad \gamma + 2\delta = \frac{2 \cdot 3\delta + 3\gamma}{3}, \quad \text{etc.}
\end{align*}
\]

by this lemma we get

\[
 UD(G)|_{c=-1} = \max(-3 + 3\alpha, 3\beta, 3\gamma, 3\delta, -3 + 3\epsilon, -3 + 3\phi, -1 + \alpha + \gamma + \epsilon, \gamma + \delta + \epsilon, \\
\gamma + 2\epsilon, 2\gamma + \epsilon, -1 + \gamma + \epsilon + \phi, \beta + \gamma + \epsilon).
\]

Next, we describe the actions of \( \hat{f}_i \) (\( i = 0, 1, 2 \)). Set \( \Xi_j := UD(G_j)|_{c=-1} \) (\( j = 1, \cdots, 5 \)). Then we have

\[
\begin{align*}
\Xi_1 &= \max(-1 + x_0 - x_1, x_0 + x_2 - 2x_1 - x_3, x_0 + x_2 + x_4 - 2x_1 - 2x_3 - x_5) \\
&\quad -\max(x_0 - x_1, x_0 + x_2 - 2x_1 - x_3, x_0 + x_2 + x_4 - 2x_1 - 2x_3 - x_5), \\
\Xi_3 &= \max(-1 + x_0 - x_1, -1 + x_0 + x_2 - 2x_1 - x_3, x_0 + x_2 + x_4 - 2x_1 - 2x_3 - x_5) \\
&\quad -\max(-1 + x_0 - x_1, x_0 + x_2 - 2x_1 - x_3, x_0 + x_2 + x_4 - 2x_1 - 2x_3 - x_5), \\
\Xi_5 &= \max(-1 + x_0 - x_1, -1 + x_0 + x_2 - 2x_1 - x_3, -1 + x_0 + x_2 + x_4 - 2x_1 - 2x_3 - x_5) \\
&\quad -\max(-1 + x_0 - x_1, -1 + x_0 + x_2 - 2x_1 - x_3, x_0 + x_2 + x_4 - 2x_1 - 2x_3 - x_5), \\
\Xi_2 &= \max(-1 + 3x_1 - x_2, 3x_1 + 3x_3 - 2x_2 - x_4) - \max(3x_1 - x_2, 3x_1 + 3x_3 - 2x_2 - x_4), \\
\Xi_4 &= \max(-1 + 3x_1 - x_2, -1 + 3x_1 + 3x_3 - 2x_2 - x_4) \\
&\quad -\max(-1 + 3x_1 - x_2, 3x_1 + 3x_3 - 2x_2 - x_4).
\end{align*}
\]

Therefore, for \( x \in \mathcal{X} \) we have

\[
\begin{align*}
\hat{f}_1(x) &= (x_0, x_1 + \Xi_1, x_2, x_3 + \Xi_3, x_4, x_5 + \Xi_5), \\
\hat{f}_2(x) &= (x_0, x_1, x_2 + \Xi_2, x_3, x_4 + \Xi_4, x_5).
\end{align*}
\]
We obtain the action $\tilde{c}_i$ ($i = 1, 2$) by setting $c = 1$ in $UD(C_i)$. Finally, we describe the action of $\tilde{f}_0$. Set

$$
\Psi_0 := \max(-2 + \alpha, \beta, -1 + \gamma, -1 + \delta, -1 + \epsilon, -1 + \phi) - \max(\alpha, \beta, \gamma, \delta, \epsilon, \phi) + 1,
$$

$$
\Psi_1 := \max(-1 + \alpha, \beta, -1 + \gamma, \delta, -1 + \epsilon, -1 + \phi) - \max(\alpha, \beta, \gamma, \delta, \epsilon, \phi) + 1,
$$

$$
\Psi_2 := \max(-3 + 3\alpha, 3\beta, 3\gamma, 3\delta, -3 + 3\epsilon, -3 + 3\phi, -1 + \alpha + \gamma + \epsilon, \gamma + \delta + \epsilon, \gamma + 2\epsilon, 2\gamma + \epsilon, -1 + \gamma + \epsilon + \phi, \beta + \gamma + \epsilon) - 3\max(\alpha, \beta, \gamma, \delta, \epsilon, \phi) + 3,
$$

$$
\Psi_3 := \max(-2 + \alpha, \beta, -1 + \gamma, -1 + \delta, -1 + \epsilon, -1 + \phi) + \max(-1 + \alpha, \beta, \gamma, \delta, -1 + \epsilon, -1 + \phi) - \max(\alpha, \beta, \gamma, \delta, \epsilon, \phi) + 2,
$$

$$
\Psi_4 := 3\max(-2 + \alpha, \beta, -1 + \gamma, -1 + \delta, -1 + \epsilon, -1 + \phi) - \max(-3 + 3\alpha, 3\beta, 3\gamma, 3\delta, -3 + 3\epsilon, -3 + 3\phi, -1 + \alpha + \gamma + \epsilon, \gamma + \delta + \epsilon, \gamma + 2\epsilon, 2\gamma + \epsilon, -1 + \gamma + \epsilon + \phi, \beta + \gamma + \epsilon) + 3,
$$

$$
\Psi_5 := \max(-2 + \alpha, \beta, -1 + \gamma, -1 + \delta, -1 + \epsilon, -1 + \phi) - \max(1 + \alpha, \beta, \gamma, \delta, \epsilon, -1 + \phi) + 1,
$$

where $\alpha, \beta, \cdots, \phi$ are as in (6.1). Therefore, by the explicit form of $e_0^2$ as in the previous section, we have

$$
\tilde{f}_0(x) = (x_0 + \Psi_0, x_1 + \Psi_1, x_2 + \Psi_2, x_3 + \Psi_3, x_4 + \Psi_4, x_5 + \Psi_5).
$$

We have the explicit form of $\tilde{e}_0$ by setting $c = 1$ in $UD(C_i)$. Now, let us show the theorem.

(Proof of Theorem 6.1) Define the map

$$
\Omega: (x_0, \cdots, x_5) \mapsto (b_1, b_2, b_3, b_4, b_5, b_6),
$$

by

$$
b_1 = x_5, \quad b_2 = \frac{1}{3} x_4 - x_5, \quad b_3 = x_3 - \frac{2}{3} x_4, \quad b_4 = \frac{2}{3} x_2 - x_3, \quad b_5 = x_1 - \frac{1}{3} x_2, \quad b_6 = x_0 - x_1,
$$

and $\Omega^{-1}$ is given by

$$
x_0 = b_1 + b_2 + \frac{b_3 + b_4}{2} + \frac{b_5 + b_6}{2}, \quad x_1 = b_1 + b_2 + \frac{b_3 + b_4}{2} + \frac{b_5 + b_6}{2},
$$

$$
x_2 = 3b_1 + 3b_2 + \frac{3(b_3 + b_4)}{2}, \quad x_3 = 2b_1 + 2b_2 + b_3, \quad x_4 = 3b_1 + 3b_2, \quad x_5 = b_1,
$$

which means that $\Omega$ is bijective. Here note that $\frac{3(b_3 + b_4)}{2} \in \mathbb{Z}$ by the definition of $B_\infty$ as in Sect.4. We shall show that $\Omega$ is commutative with actions of $\tilde{f}_i$ and preserves the functions $\text{wt}_i$ and $\varepsilon_i$, that is,

$$
\tilde{f}_i(\Omega(x)) = \Omega(\tilde{f}_i x), \quad \text{wt}_i(\Omega(x)) = \text{wt}_i(x), \quad \varepsilon_i(\Omega(x)) = \varepsilon_i(x) \quad (i = 0, 1, 2).
$$

Indeed, the commutativity $\tilde{c}_i(\Omega(x)) = \Omega(\tilde{c}_i x)$ is shown by a similar way. First, let us check $\text{wt}_i$: Set $b = \Omega(x)$ and let $(z_1, z_2, z_3, z_4)$ be as in (4.2). By the explicit
forms of $\omega_i$ on $X$ and $B_\infty$, we have

\[
\begin{align*}
\omega_0(\Omega(x)) &= \omega_0(\Omega(x)) - \omega_0(\Omega(x)) = 2z_1 + z_2 + z_3 + 3z_4 \\
&= 2(b_1 - b_1) + (b_2 - \bar{b}_2) + (b_3 - b_3) + \frac{3}{2}(\bar{b}_3 - b_3) = 2(b_1 - b_1) + \bar{b}_2 - b_2 + \frac{\bar{b}_3 - b_3}{2} \\
&= 2x_0 - x_1 - x_3 - x_5 = \omega_0(x),
\end{align*}
\]

\[
\begin{align*}
\omega_1(\Omega(x)) &= \omega_1(\Omega(x)) - \omega_1(\Omega(x)) \\
&= b_1 + (b_3 - b_2 + (\bar{b}_3 - b_3) +) - (\bar{b}_1 + (\bar{b}_3 - \bar{b}_2 + (b_2 - b_3) +) +) \\
&= b_1 - b_1 - \bar{b}_2 + b_3 - b_3 = 2(x_1 + x_3 + x_5) - x_0 - x_2 - x_4 = \omega_1(x),
\end{align*}
\]

\[
\begin{align*}
\omega_2(\Omega(x)) &= \omega_2(\Omega(x)) - \omega_2(\Omega(x)) = 3b_2 + \frac{3}{2}(b_3 - b_2) + \frac{3}{2}(b_3 - b_3) + \\
&= 3b_2 - 3\bar{b}_2 + \frac{3}{2}(b_3 - b_3) = 2(x_2 + x_4) - 3(x_1 + x_3 + x_5) = \omega_2(x).
\end{align*}
\]

Next, we shall check $\varepsilon_i$:

\[
\begin{align*}
\varepsilon_1(\Omega(x)) &= \bar{b}_1 + (b_3 - \bar{b}_3 + (b_2 - b_3) +) + \\
&= \max(b_1, b_1 + b_3 - b_2, b_1 + b_3 - b_2 + b_2 - b_3) \\
&= \max(x_0 - x_1, x_0 - 2x_1 + x_2 - x_3, x_0 - 2x_1 + x_2 - 2x_3 + x_4 - x_5) = \varepsilon_1(x),
\end{align*}
\]

\[
\begin{align*}
\varepsilon_2(\Omega(x)) &= 3b_2 + \frac{3}{2}(b_3 - b_2) + \frac{3}{2}(b_3 - \bar{b}_3) \\
&= \max(3x_1 - x_2, 3x_1 - 2x_2 + 3x_3 - x_4) = \varepsilon_2(x).
\end{align*}
\]

Here let us see $\varepsilon_0$:

\[
\begin{align*}
\varepsilon_0(\Omega(x)) &= \varepsilon_0(\Omega(x)) \\
&= b_1 + (b_3 - \bar{b}_3 + (b_2 - b_3) +) + \\
&= \max(b_1, b_1 - b_2, b_1 - b_3 + b_2 - b_3) \\
&= \max(x_0 - x_1, x_0 - 2x_1 + x_2 - x_3, x_0 - 2x_1 + x_2 - 2x_3 + x_4 - x_5) = \varepsilon_0(\Omega(x)).
\end{align*}
\]

On the other hand, we have

\[
\begin{align*}
\varepsilon_0(x) &= -(3x_0 + x_2 + x_3) + \max(\alpha, \beta, \gamma, \delta, \varepsilon, \phi).
\end{align*}
\]

which shows $\varepsilon_0(\Omega(x)) = \varepsilon_0(x)$.

Let us show $f_i(\Omega(x)) = \Omega(f_i(x)) (x \in X, i = 0, 1, 2$). As for $f_1$, set

\[
A = x_0 - x_1, B = x_0 + x_2 - 2x_1 - x_3, C = x_0 + x_2 + x_4 - 2x_1 - 2x_3 - x_5.
\]

Then we obtain $\Xi_1 = \max(A - 1, B, C) - \max(A, B, C)$, $\Xi_3 = \max(A - 1, B - 1, C) - \max(A - 1, B, C)$, $\Xi_5 = \max(A - 1, B - 1, C - 1) - \max(A - 1, B - 1, C)$. Therefore, we have

\[
\begin{align*}
\Xi_1 &= -1, \quad \Xi_3 = 0, \quad \Xi_5 = 0, \quad \text{if } A > B, C \\
\Xi_1 &= 0, \quad \Xi_3 = -1, \quad \Xi_5 = 0, \quad \text{if } A \leq B > C \\
\Xi_1 &= 0, \quad \Xi_3 = 0, \quad \Xi_5 = -1, \quad \text{if } A, B \leq C,
\end{align*}
\]
which implies
\[ \tilde{f}_1(x) = \begin{cases} 
(x_0, x_1 - 1, x_2, \ldots, x_5) & \text{if } A > B, C \\
(x_0, \ldots, x_3 - 1, x_4, x_5) & \text{if } A \leq B > C \\
(x_0, \ldots, x_4, x_5 - 1) & \text{if } A, B \leq C 
\end{cases} \]

Since \( A = b_1, B = b_1 + b_3 - b_2 \) and \( C = b_1 + b_3 - b_2 + b_2 - b_3 \), we get \((b = \Omega(x))\)

\[ \Omega(\tilde{f}_1(x)) = \begin{cases} 
(\ldots, b_2 - 1, b_1 + 1) & \text{if } b_2 - b_3 > (b_2 - b_3)_+, \\
(\ldots, b_3 - 1, b_3 + 1, \ldots) & \text{if } b_2 - b_3 \leq 0 < b_3 - b_2, \\
(b_1 - 1, b_2 + 1, \ldots) & \text{if } (b_2 - b_3)_+ \leq b_2 - b_3, 
\end{cases} \]

which is the same as the action of \( \tilde{f}_1 \) on \( b = \Omega(x) \) as in Sect.4. Hence, we have \( \Omega(\tilde{f}_1(x)) = \tilde{f}_1(\Omega(x)) \).

Let us see \( \Omega(\tilde{f}_2(x)) = \tilde{f}_2(\Omega(x)) \). Set

\[ L = 3x_1 - x_2, \quad M := 3x_1 + 3x_3 - 2x_2 - x_4. \]

Then \( \Xi_2 = \max(-1 + L, M) - \max(L, M) \) and \( \Xi_4 = \max(-1 + L, -1 + M) - \max(-1 + L, M) \). Thus, one has

\[ \Xi_2 = -1, \quad \Xi_4 = 0 \text{ if } L > M, \]
\[ \Xi_2 = 0, \quad \Xi_4 = -1 \text{ if } L \leq M, \]

which means
\[ \tilde{f}_2(x) = \begin{cases} 
(x_0, x_1, x_2 - 1, x_3, x_4, x_5) & \text{if } L > M, \\
(x_0, x_1, x_2, x_3, x_4 - 1, x_5) & \text{if } L \leq M. 
\end{cases} \]

Since \( L - M = x_2 - 3x_3 + x_4 = \frac{3(b_3 - b_1)}{2} \), one gets
\[ \Omega(\tilde{f}_2(x)) = \begin{cases} 
(\ldots, b_3 - \frac{2}{3}, b_2 + \frac{1}{3}, \ldots) & \text{if } b_3 > b_2, \\
(\ldots, b_2 - \frac{1}{3}, b_3 + \frac{2}{3}, \ldots) & \text{if } b_3 \leq b_2. 
\end{cases} \]

where \( b = \Omega(x) \). This action coincides with the one of \( \tilde{f}_2 \) on \( b \in B_\infty \) as in Sect.4. Therefore, we get \( \Omega(\tilde{f}_2(x)) = \tilde{f}_2(\Omega(x)) \).

Finally, we shall check \( \tilde{f}_0(\Omega(x)) = \Omega(\tilde{f}_0(x)) \). For the purpose, we shall estimate the values \( \Psi_0, \cdots, \Psi_5 \) explicitly.

First, the following cases are investigated:

\begin{enumerate}
\item[(f1)] \( \beta \geq \gamma, \delta, \epsilon, \phi, \ \phi \geq \alpha, \ \delta \geq \alpha \)
\item[(f2)] \( \beta < \delta \geq \alpha, \epsilon, \ \alpha > \phi, \ \beta \geq \phi \)
\item[(f3)] \( \beta, \delta < \gamma \geq \alpha, \epsilon, \phi \)
\item[(f4)] \( \beta, \delta < \epsilon \geq \alpha, \phi, \ \epsilon = \gamma + 1 \)
\item[(f4')] \( \beta, \delta < \epsilon \geq \alpha, \phi, \ \epsilon = \gamma + 2 \)
\item[(f4'')] \( \beta, \delta < \epsilon \geq \alpha, \phi, \ \epsilon > \gamma + 2 \)
\item[(f5)] \( \beta, \gamma, \epsilon < \phi \geq \alpha, \ \alpha > \delta, \ \beta \geq \delta \)
\item[(f6)] \( \alpha > \gamma, \delta, \epsilon, \phi, \ \delta, \phi > \beta. \)
\end{enumerate}

It is easy to see that each of these conditions are equivalent to the conditions \((F_1)\)-\((F_6)\) in Sect.4, more precisely, we have \((f1) \leftrightarrow (F_1)\) \((i = 1, 2, 3, 5, 6)\), \((f4) \leftrightarrow (F_4)\) and \(z_4 = \frac{1}{3}, (f4') \leftrightarrow (F_4)\) and \(z_4 = \frac{2}{3}\) and \((f4'') \leftrightarrow (F_4)\) and \(z_4 \neq \frac{1}{3}, \frac{2}{3}\), and that \((f1)-(f6)\) cover all cases and they have no intersection.
Let us show \((f1) \iff (F1)\): the condition \((f1)\) means \(\beta - \gamma = -(z_1 + z_2) \geq 0, \beta - \delta = -z_1 \geq 0, \beta - \epsilon = -(z_1 + z_2 + 3z_4) \geq 0\) and \(\beta - \phi = -(z_1 + z_2 + z_3 + 3z_4) \geq 0\), which is equivalent to the condition \(z_1 + z_2 \leq 0, z_1 \leq 0, z_1 + z_2 + 3z_4 \leq 0\) and \(z_1 + z_2 + z_3 + 3z_4 \leq 0\). (Note that \(\phi - \alpha = \beta - \delta - \alpha = \beta - \phi\)) This is just the condition \((F1)\). Other cases \(i = 2, 3, 5, 6\) are shown similarly. Next, let us see the cases \((f4)\), \((f4')\) and \((f4'')\). Indeed,

\[
\epsilon - \gamma = x_2 - 3x_3 + x_4 = \frac{3}{2}(\overline{b}_3 - b_3) = 3z_4.
\]

Thus, we can easily get that \((f4) \iff (F4)\) and \(z_4 = \frac{1}{3}\), \((f4') \iff (F4)\) and \(z_4 = \frac{2}{3}\), and \((f4'') \iff (F4)\) and \(z_4 \neq \frac{1}{3}, \frac{2}{3}\).

Under the condition \((f1)\) \((\iff (F1))\), we have

\[
\Psi_0 = \Psi_1 = \Psi_5 = 1, \Psi_2 = \Psi_4 = 3, \quad \Psi_3 = 2,
\]

which means \(\tilde{f}_0(x) = (x_0 + 1, x_1 + 1, x_2 + 3, x_3 + 2, x_4 + 3, x_5 + 1)\). Thus, we have

\[
\Omega(\tilde{f}_0(x)) = (b_1 + 1, b_2, \cdots, \overline{b}_1),
\]

which coincides with the action of \(\tilde{f}_0\) under \((F2)\) in Sect.4. Similarly, we have

\[
(f2) \quad \implies (\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5) = (0, 1, 3, 1, 0, 0) \implies \tilde{f}_0(x) = (x_0, x_1 + 1, x_2 + 3, x_3 + 1, x_4, x_5), \implies \Omega(\tilde{f}_0(x)) = (b_1, b_2, b_3 + 1, \overline{b}_3 + 1, \overline{b}_2, \overline{b}_1 - 1),
\]

which coincides with the action of \(\tilde{f}_0\) under \((F2)\) in Sect.4.

\[
(f3) \quad \implies (\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5) = (0, 0, 3, 2, 0, 0) \implies f_0(x) = (x_0, x_1 + 1, x_2 + 3, x_3 + 2, x_4, x_5), \implies \Omega(f_0(x)) = (b_1, b_2, b_3 + 2, \overline{b}_3, \overline{b}_2 - 1, \overline{b}_1),
\]

which coincides with the action of \(\tilde{f}_0\) under \((F3)\) in Sect.4.

\[
(f4) \quad \implies (\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5) = (0, 0, 2, 2, 1, 0) \implies f_0(x) = (x_0, x_1 + 1, x_2 + 2, x_3 + 2, x_4 + 1, x_5), \implies \Omega(f_0(x)) = (b_1, b_2 + \frac{1}{3}, b_3 + \frac{4}{3}, \overline{b}_3 - \frac{2}{3}, \overline{b}_2 - \frac{2}{3}, \overline{b}_1),
\]

which coincides with the action of \(\tilde{f}_0\) under \((F4)\) and \(z_4 = \frac{1}{3}\) in Sect.4.

\[
(f4') \quad \implies (\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5) = (0, 0, 1, 2, 2, 0) \implies f_0(x) = (x_0, x_1 + 1, x_2 + 2, x_3 + 2, x_4 + 2, x_5), \implies \Omega(f_0(x)) = (b_1, b_2 + \frac{5}{3}, b_3 + \frac{4}{3}, \overline{b}_3 - \frac{4}{3}, \overline{b}_2 - \frac{1}{3}, \overline{b}_1),
\]

which coincides with the action of \(\tilde{f}_0\) under \((F4)\) and \(z_4 = \frac{2}{3}\) in Sect.4.

\[
(f4'') \quad \implies (\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5) = (0, 0, 0, 2, 3, 0) \implies f_0(x) = (x_0, x_1 + 1, x_2 + 2, x_3 + 3, x_5), \implies \Omega(f_0(x)) = (b_1, b_2 + 1, b_3, \overline{b}_3 - 2, \overline{b}_2, \overline{b}_1),
\]

which coincides with the action of \(\tilde{f}_0\) under \((F4)\) and \(z_4 \neq \frac{1}{3}, \frac{2}{3}\) in Sect.4.

\[
(f5) \quad \implies (\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5) = (0, 0, 0, 1, 3, 1) \implies f_0(x) = (x_0, x_1, x_2, x_3 + 1, x_4 + 3, x_5 + 1), \implies \Omega(f_0(x)) = (b_1 + 1, b_2, b_3 - 1, \overline{b}_3 - 1, \overline{b}_2, \overline{b}_1),
\]
which coincides with the action of $\tilde{f}_0$ under $(F_5)$ in Sect.4.

\[(\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5) = (-1, 0, 0, 0, 0, 0)\]

\[\Rightarrow \tilde{f}_0(x) = (x_0 - 1, x_1, x_2, x_3, x_4, x_5),\]

\[\Rightarrow \Omega(\tilde{f}_0(x)) = (b_1, b_2, b_3, b_4, b_5, b_6 - 1),\]

which coincides with the action of $\tilde{f}_0$ under $(F_0)$ in Sect.4. Now, we have $\Omega(\tilde{f}_0(x)) = \tilde{f}_0(\Omega(x))$. Therefore, the proof of Theorem 6.1 has been completed.

\[\qed\]

\section*{References}

[1] Berenstein A. and Kazhdan D., Geometric crystals and Unipotent crystals, GAFA 2000(Tel Aviv,1999), Geom Funct.Anal.2000, Special Volume, Part I, 188–236.

[2] Fourier G., Okado M., Schilling A., Kirillov-Reshetikhin crystals for nonexceptional types, Adv. Math. 222 (3) (2009), 1080–1116.

[3] Fourier G., Okado M., Schilling A., Perfectness of Kirillov-Reshetikhin crystals for nonexceptional types, Contemporary Mathematics 506, (2010), 127-143.

[4] Hatayama G., Kuniba A., Okado M., Takagi T. and Yamada Y., Remarks on fermionic formula, Contemp. Math. 248 (1999), 243–291.

[5] Hatayama G., Kuniba A., Okado M., Takagi T. and Tsoboi Z., Paths, crystals and fermionic formulae, in “MathPhys Odyssey 2001-Integrable Models and Beyond In Honor of Barry M.McCoy”, Edited by M.Kashiwara and T.Miwa, Birkhäuser (2002), 205–272.

[6] Igarashi M. and Nakashima T., Affine Geometric Crystal of type $D_4^{(3)}$, Contemporary Mathematics 506, (2010), 215-226.

[7] Kang S-J., Kashiwara M., Misra K.C., Miwa T., Nakashima T. and Nakayashiki A., Affine crystals and vertex models, Int.J.Mod.Phys..A7 Suppl.1A (1992), 449–484.

[8] Kang S-J., Kashiwara M., Misra K.C., Miwa T., Nakashima T. and Nakayashiki A., Perfect crystals of quantum affine Lie algebras, Duke Math. J., 68(3), (1992), 499-607.

[9] Kang S-J., Kashiwara M. and Misra K.C., Crystal bases of Verma modules for quantum affine Lie algebras, Compositio Mathematica 92 (1994), 299–345.

[10] Kashiwara M., Crystallizing the $q$-anologue of universal enveloping algebras, Commun. Math. Phys., 133 (1990), 249–260.

[11] Kashiwara M., On crystal bases of the $q$-anologue of universal enveloping algebras, Duke Math. J., 63 (1991), 465–516.

[12] Kashiwara M., On level-zero representation of quantized affine algebras, Duke Math. J., 112 (2002), 499-525.

[13] Kashiwara M., Level zero fundamental representations over quantized affine algebras and Demazure modules. Publ. Res. Inst. Math. Sci. 41 (2005), no. 1, 223–250.

[14] Kashiwara M., Misra K., Okado M. and Yamada D., Perfect crystals for $U_q(D_4^{(3)})$, Journal of Algebra, 317, no.1, (2007), 392-423.

[15] Kashiwara M., Nakashima T. and Okado M., Affine geometric crystals and limit of perfect crystals, math.QA/0512657 (to appear in Trans.Amer.Math.Soc.).

[16] Kac V.G., Infinite dimensional Lie algebras, Cambridge Univ. Press, 3rd edition (1990).

[17] Kac V.G. and Peterson D.H., Defining relations of certain infinite-dimensional groups; in “Arithmetic and Geometry”(Artin M.,Tate J.,eds), 141–166, Birkhäuser, Boston-Basel-Stuttgart, (1983).

[18] Kirillov A.N. and Reshetikhin N., Representations of Yangians and multiplicity of occurrence of the irreducible components of the tensor product of representations of simple Lie algebras, J. Sov. Math. 52 (1990), 3156–3164.

[19] Kumar S., Kac-Moody groups, their Flag varieties and Representation Theory, Progress in Mathematics 204, Birkhauser Boston, 2002.

[20] Lusztig G., Canonical bases arising from quantized enveloping algebras, J. Amer. Math. Soc. 3 (1990), 447–498.

[21] Misra K.C., Mohamad M., and Okado M., Zero action on perfect crystals for $U_q(G_2^{(1)})$, SIGMA, (submitted), 2009.

[22] Nakashima T., Geometric crystals on Schubert varieties, Journal of Geometry and Physics, 53 (2), 197–225, (2005).
[23] Nakashima T., Geometric crystals on unipotent groups and generalized Young tableaux, Journal of Algebra, 293, No.1, 65–88, (2005).
[24] Nakashima T., Affine Geometric Crystal of type $G_2^{(1)}$, Contemporary Mathematics, 442, 179–192, Amer.Math.Soc., Providence, RI, (2007).
[25] Nakashima T., Ultra-discretization of the $G_2^{(1)}$-Geometric Crystals to the $D_4^{(3)}$-Perfect Crystals, to appear in Proceedings of International Conference in Nagoya 2006, arXiv:0712.3894.
[26] Okado M., Schilling A., Existence of Kirillov-Reshetikhin crystals for nonexceptional types, Represent. Theory 12 (2008), 186–207.
[27] Peterson D.H., and Kac V.G., Infinite flag varieties and conjugacy theorems, Proc. Nat. Acad. Sci. USA, 80, 1778–1782, (1983).
[28] Yamane S., Perfect Crystals of $U_q(G_2^{(1)})$, J. Algebra 210, no.2, 440–486, (1998).

Department of Mathematics, Sophia University, Kioicho 7-1, Chiyoda-ku, Tokyo 102-8554, Japan
E-mail address: mana-i@hoffman.cc.sophia.ac.jp

Department of Mathematics, North Carolina State University, Raleigh, NC 27695-8205, USA
E-mail address: misra@math.ncsu.edu

Department of Mathematics, Sophia University, Kioicho 7-1, Chiyoda-ku, Tokyo 102-8554, Japan
E-mail address: toshiki@math.sophia.ac.jp