A moment approach to analytic time-dependent solutions of the Fokker-Planck equation with additive and multiplicative noise

Hideo Hasegawa

Department of Physics, Tokyo Gakugei University, Koganei, Tokyo 184-8501, Japan

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Abstract

An efficient method is presented as a means of an approximate, analytic time-dependent solution of the Fokker-Planck equation (FPE) for the Langevin model subjected to additive and multiplicative noise. We have assumed that the dynamical probability distribution function has the same structure as the exact stationary one and that its parameters are expressed in terms of first and second moments, whose equations of motion are determined by the FPE. Model calculations have shown that dynamical distributions in response to applied signal and force calculated by our moment method are in good agreement with those obtained by the partial difference equation method. As an application of our method, we present the time-dependent Fisher information for the inverse-gamma distribution which is realized in the FPE including multiplicative noise only.

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\footnote{E-mail address: hideohasegawa@goo.jp}
1 INTRODUCTION

The Langvin model is a very important model to describe the diffusion behavior in non-equilibrium systems, and it has been widely applied to various phenomena in physics, chemistry and biology. The Langvin model is usually transformed to the Fokker-Planck equation (FPE) which deals with the probability distribution function (PDF) of a state variable [1]. It is generally not possible to obtain analytic solutions of the second-order partial equations. Indeed, exact analytical solutions of the FPE are known for only a few cases. In most cases, approximate solutions are obtained by using analytic or numerical methods. Typical analytic methods are an appropriate change of variables, eigenfunction expansion, perturbation expansion, path integral, Green’s function, moment method, and the continued-fraction method [1]. When no analytic solutions are available, numerical methods such as finite-difference and finite-element methods have been employed.

For some Langvin models subjected to additive noise only, exact solutions have been obtained. For the linear Langvin model, the exact dynamical solution is expressed by the Gaussian distribution with time-dependent mean and variance of a state variable. For the FPE including a nonlinear diffusion term, some authors have obtained exact dynamical solutions [2, 3, 4]. The generalized FPEs in which time dependences are introduced in drift and diffusion terms have been investigated [4]-[7].

When multiplicative noise is incorporated to the Langvin model, the problem becomes much difficult [8]. For the linear Langvin model subjected to additive and multiplicative noise, the exact stationary solution is available, and it has been considerably discussed in connection with the non-Gaussian PDF in the nonextensive statistics [9]-[13]. An exact dynamical solution for the linear Langvin model subjected to multiplicative noise only is obtained in Ref. [15] although it does not represent the stationary solution. Approximate dynamical solutions of the linear and nonlinear FPEs subjected to multiplicative noise have been discussed with some sophisticated methods such as the polynomial expansion of the logarithmic PDF [16], the linearizing transformation [17] and the direct quadrature method for moment solution [18].

Numerical methods are powerful approaches when exact dynamical solutions are not available. Analytical solutions are, however, indispensable in some subjects. A typical example is a calculation of the time-dependent Fisher information which is expressed by the derivatives of the dynamical PDF with respect to its parameters. In a recent paper [19], we calculated the Fisher information in a typical nonextensive system described by
the linear Langevin model subjected to additive and multiplicative noise. We developed an analytic dynamical approach to the FPE combined with the $q$-moment method in which moments are evaluated over the escort probability distribution $[10]$. The dynamical PDFs calculated by our moment method are shown to be in good agreement with those obtained by the partial difference equation method (PDEM) $[19]$. By using the calculated time-dependent PDF, we discussed the dynamical properties of the Fisher information $[19]$. It is the purpose of the present study to extend such an analytical approach so as to be applied to a wide class of Langevin model with the use of the conventional (normal) moment method instead of the $q$-moment method.

The paper is organized as follows. In Sec. 2, we discuss the adopted Langevin model and moment method to obtain the dynamical PDF. The developed method has been applied to the three Langevin models. We present some numerical calculations of the time-dependent PDF in response to an applied signal and force. Section 3 is devoted to conclusion and discussion on the dynamics of Fisher information of the inverse-gamma distribution.

2 METHOD AND RESULT

2.1 Fokker-Planck equation

We have adopted the Langevin model subjected to cross-correlated additive ($\xi$) and multiplicative noise ($\eta$) given by

$$\frac{dx}{dt} = F(x) + G(x)\eta(t) + \xi(t) + I(t).$$  \hspace{1cm} (1)

Here $F(x)$ and $G(x)$ are arbitrary functions of $x$, $I(t)$ stands for an external input, and $\eta(t)$ and $\xi(t)$ express zero-mean Gaussian white noises with correlations given by

$$\langle \eta(t) \eta(t') \rangle = \alpha^2 \delta(t - t'),$$  \hspace{1cm} (2)

$$\langle \xi(t) \xi(t') \rangle = \beta^2 \delta(t - t'),$$  \hspace{1cm} (3)

$$\langle \eta(t) \xi(t') \rangle = \epsilon \alpha \beta \delta(t - t'),$$  \hspace{1cm} (4)

where $\alpha$ and $\beta$ denote the strengths of multiplicative and additive noise, respectively, and $\epsilon$ the degree of the cross-correlation between the two noise.

The FPE is expressed by $[20, 21, 22]$

$$\frac{\partial}{\partial t} p(x,t) = -\frac{\partial}{\partial x} \left( \left[F(x) + I + \left(\frac{\phi}{2}\right) \left[\alpha^2 G(x)G'(x) + \epsilon \alpha \beta G'(x) \right] \right] p(x,t) \right).$$

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where \( G'(x) = dG(x)/dx \), and \( \phi = 0 \) and \( 1 \) in the Ito and Stratonovich representations, respectively. Although we have adopted the Langevin model for a single variable in this study, it is straightforward to extend it to the coupled Langevin model with the use of the mean-field approximation \[19\].

For \( I(t) = I \), the stationary PDF of \( p(x) \) is expressed by \[14\]

\[
\ln p(x) = X(x) + Y(x) - \left(1 - \frac{\phi}{2}\right) \ln \left(\frac{1}{2}[\alpha^2 G(x)^2 + 2\epsilon\alpha\beta G(x) + \beta^2]\right),
\]

with

\[
X(x) = 2 \int dx \left[ \frac{F(x)}{\alpha^2 G(x)^2 + 2\epsilon\alpha\beta G(x) + \beta^2} \right],
\]

\[
Y(x) = 2 \int dx \left[ \frac{I}{\alpha^2 G(x)^2 + 2\epsilon\alpha\beta G(x) + \beta^2} \right].
\]

### 2.2 Equations of motion for the moments

An equation of motion for the \( n \)th moment is given by

\[
\frac{\partial \langle x^n \rangle}{\partial t} = \int \frac{\partial p(x,t)}{\partial t} x^n dx,
\]

\[
= n \left( \langle x^{n-1}F(x) \rangle + \langle x^{n-1}I(t) \rangle + \frac{\phi}{2} \left[ \alpha^2 \langle x^{n-1}G(x)G'(x) \rangle + \epsilon\alpha\beta \langle x^{n-1}G'(x) \rangle \right] \right)
\]

\[
+ \frac{n(n-1)}{2} \left[ \alpha^2 \langle x^{n-2}G(x)^2 \rangle + 2\epsilon\alpha\beta \langle x^{n-2}G(x) \rangle + \beta^2 \langle x^{n-2} \rangle \right],
\]

where suitable boundary conditions are adopted. For \( n = 1, 2 \), we obtain

\[
\frac{\partial \langle x \rangle}{\partial t} = \langle F(x) \rangle + \langle I(t) \rangle + \frac{\phi}{2} \left[ \alpha^2 \langle G(x)G'(x) \rangle + \epsilon\alpha\beta \langle G'(x) \rangle \right],
\]

\[
\frac{\partial \langle x^2 \rangle}{\partial t} = 2\langle xF(x) \rangle + 2\langle xI(t) \rangle + \phi \left[ \alpha^2 \langle xG(x)G'(x) \rangle + \epsilon\alpha\beta \langle xG'(x) \rangle \right]
\]

\[
+ \alpha^2 \langle G(x)^2 \rangle + 2\epsilon\alpha\beta \langle G(x) \rangle + \beta^2.
\]

Expanding \( x \) as \( x = \mu + \delta x \) and retaining up to \( O((\delta x)^2) \), we obtain equations of motion for the average \( \mu \) \([= \langle x \rangle]\) and variance \( \sigma^2 \) \([= \langle x^2 \rangle - \langle x \rangle^2] \) given by \[13\]

\[
\frac{d\mu}{dt} = f_0 + f_2\sigma^2 + \phi \left( \alpha^2 g_0 g_1 + 3(g_1 g_2 + g_0 g_3)\sigma^2 \right) + \epsilon\alpha\beta(g_1 + 3g_3\sigma^2) + I(t),
\]

\[
\frac{d\sigma^2}{dt} = 2f_1\sigma^2 + (\phi + 1)(g_1^2 + 2g_0 g_2)\alpha^2 \sigma^2 + 2\epsilon\alpha\beta(\phi + 1)g_2 \sigma^2
\]

\[
+ \alpha^2 g_0^2 + 2\epsilon\alpha\beta g_0 + \beta^2,
\]

where \( f_\ell = (1/\ell !) \partial^\ell F(\mu)/\partial x^\ell \) and \( g_\ell = (1/\ell !) \partial^\ell G(\mu)/\partial x^\ell \).
2.3 Model A

2.3.1 Stationary distribution

Our dynamical moment approach will be applied to the three Langevin models A, B and C, which will be separately discussed in Secs. 2.3, 2.4 and 2.5, respectively.

First we consider the model A in which $F(x)$ and $G(x)$ are given by

\begin{align*}
F(x) &= -\lambda x, \\
G(x) &= x,
\end{align*}

with $\epsilon = 0.0$ (i.e., without the cross-correlation), where $\lambda$ expresses the relaxation rate. The model A has been adopted as a microscopic model for nonextensive systems [11]-[13].

From Eq. (5), the FPE in the Stratonovich representation is given by

\begin{align*}
\frac{\partial}{\partial t} p(x,t) &= \frac{\partial}{\partial x} \left[ \lambda x - I(t) \right] p(x,t) + \left( \frac{\beta^2}{2} \right) \frac{\partial^2}{\partial x^2} p(x,t) \\
&\quad + \left( \frac{\alpha^2}{2} \right) \frac{\partial}{\partial x} \left[ x \frac{\partial}{\partial x} \{xp(x,t)\} \right].
\end{align*}

By using Eqs. (6)-(8), we obtain the stationary PDF given by

\begin{equation}
p(x) = \left( \frac{1}{Z} \right) \exp \left[ 2c \tan^{-1} (ax) \right] \frac{(1 + a^2 x^2)^{b}}{b},
\end{equation}

with

\begin{align*}
a &= \frac{\alpha}{\beta}, \\
b &= \frac{(2\lambda + \alpha^2)}{2\alpha^2}, \\
c &= \frac{I}{\alpha \beta}, \\
Z &= \sqrt{\pi} \Gamma(b) \Gamma(b - \frac{1}{2}) \left| a \Gamma(b + i c) \right|^2.
\end{align*}

By using Eq. (18), we obtain the mean and variance in the stationary state given by

\begin{align*}
\mu &= \frac{c}{a(b-1)} = \frac{2I}{(2\lambda - \alpha^2)}, \\
\sigma^2 &= \frac{\left( (b-1)^2 + c^2 \right)}{a^2(b-1)^2(2b-3)} = \frac{(\alpha^2 \mu^2 + \beta^2)}{2(\lambda - \alpha^2)}.
\end{align*}

Depending on the model parameters, the stationary PDF given by Eq. (18) may reproduce various PDFs such as the Gaussian, $q$-Gaussian, Cauchy and inverse-gamma PDFs [19].
2.3.2 Dynamical distribution

It is worthwhile to remind the dynamical solution of the FPE given by Eq. (17) in the
limit of $\alpha = 0$ (i.e., additive noise only), for which the time-dependent solution is given
by

$$p(x, t) = \frac{1}{\sqrt{2\pi} \sigma(t)^2} e^{-[x-\mu(t)]^2/2\sigma(t)^2},$$

(25)

with $\mu(t)$ and $\sigma(t)^2$ satisfying equations of motion given by

$$\frac{d\mu(t)}{dt} = -\lambda \mu(t) + I(t),$$

(26)

$$\frac{d\sigma(t)^2}{dt} = -2\lambda \sigma(t)^2 + \beta^2.$$  

(27)

In order to derive the dynamical solution of the FPE for $\alpha \neq 0$ given by Eq. (17),
we adopt the moment approach with the following steps:

1) We assume that dynamical PDF has the same structure as the stationary one, as given
by

$$p(x, t) = \left( \frac{1}{Z(t)} \right) \frac{\exp[2c(t) \tan^{-1}\{a(t)x\}]}{[1 + a(t)^2x^2]^{b(t)}},$$

(28)

with

$$Z(t) = \frac{\sqrt{\pi} \Gamma[b(t)] \Gamma[b(t) - \frac{1}{2}]}{a(t) \Gamma[b(t) + ic(t)]^2},$$

(29)

2) With the assumption (1), we first tried to derive equations of motion for the parameters
of $a(t)$, $b(t)$ and $c(t)$, by using the FPE after Refs. [2, 3, 4]. Unfortunately, it did not
work because functional forms in the left and right sides of the FPE become different.

Then we tried to express the parameters in terms of importance quantities of $\mu(t)$ and
$\sigma(t)^2$ such as to be consistent with the relations for the stationary state given by Eqs.
(23) and (24). Because the number of parameters (three) is larger than two for $\mu(t)$ and
$\sigma(t)^2$, the parameters of $a(t)$, $b(t)$ and $c(t)$ cannot be uniquely expressed in terms of $\mu(t)$
and $\sigma(t)^2$ from Eqs. (23) and (24). If the first three moments in the stationary state are
available, it is possible to uniquely express $a(t)$, $b(t)$ and $c(t)$ in terms of them, though
such a calculation is laborious.

In order to overcome the above problem, we have imposed an additional condition
that expressions for the parameters yield the consistent result in the two limiting cases
of $\alpha \to 0$ and $\beta \to 0$. After several tries, we have decided that $b(t)$ and $c(t)$ in Eqs. (28)
and (29) are expressed as

\[ b(t) = \left[ 1 + a^2 \{ \mu(t)^2 + 3\sigma(t)^2 \} \right] \frac{1}{2a^2\sigma(t)^2}, \]

\[ c(t) = \left[ 1 + a^2 \{ \mu(t)^2 + \sigma(t)^2 \} \right] \frac{\mu(t)}{\sigma(t)^2}, \]

with the time-independent \( a \,(= \alpha/\beta) \) given by Eq. (19). The relations given by Eqs. (30) and (31) are consistent with Eqs. (23) and (24) for the stationary state and they satisfy the above-mentioned limiting conditions, as will be shown shortly.

(3) Equations of motion for \( \mu(t) \) and \( \sigma(t)^2 \) in Eqs. (30) and (31) are obtained from Eqs. (13)-(16), as given by

\[ \frac{d\mu(t)}{dt} = -\lambda \mu(t) + I(t) + \alpha^2 \mu(t)^2, \]

\[ \frac{d\sigma(t)^2}{dt} = -2\lambda \sigma(t)^2 + 2\alpha^2 \sigma(t)^2 + \alpha^2 \mu(t)^2 + \beta^2. \]

Thus the dynamical solution of the FPE given by Eq. (17) is expressed by Eqs. (28)-(33).

In the following, we will show that the relations given by Eqs. (30) and (31) lead to results consistent in the two limiting cases of \( \alpha \to 0 \) and \( \beta \to 0 \).

(a) \( \alpha \to 0 \) case

In the limit of \( \alpha \to 0 \) (i.e., additive noise only), \( p(x, t) \) given by Eq. (28) reduces to

\[ p(x, t) \propto e^{-a(t)^2 b(t) x^2 + 2a(t) c(t) x} \]

\[ \to e^{-[x-\mu(t)]^2/2\sigma(t)^2}, \]

because Eqs. (30) and (31) with \( a \to 0.0 \) yield

\[ a(t)^2 b(t) = \left[ 1 + a^2 \{ \mu(t)^2 + 3\sigma(t)^2 \} \right] \frac{1}{2\sigma(t)^2} \to \frac{1}{2\sigma(t)^2}, \]

\[ 2a(t) c(t) = \left[ 1 + a^2 \{ \mu(t)^2 + \sigma(t)^2 \} \right] \frac{\mu(t)}{\sigma(t)^2} \to \frac{\mu(t)}{\sigma(t)^2}. \]

Equation (35) agrees with the Gaussian distribution given by Eq. (24).

(b) \( \beta \to 0 \) case

In the opposite limit of \( \beta = 0.0 \) (i.e., multiplicative noise only), the stationary PDF given by Eqs. (18) and (22) with \( I > 0 \) leads to the inverse-gamma distribution expressed by

\[ p(x) = \frac{\kappa^{\delta-1}}{\Gamma[\delta-1]} x^{-\delta} e^{-\kappa/x} \Theta(x), \]

\[ (38) \]
where
\[ \delta = 2b, \quad (39) \]
\[ \kappa = \frac{2c}{a}, \quad (40) \]

Here \( \Gamma(x) \) denotes the gamma function and \( \Theta(t) \) the Heaviside function: \( \Theta(t) = 1 \) for \( t > 0 \) and zero otherwise. From Eq. (38), we obtain the average and variance in the stationary state given by
\[ \mu = \frac{\kappa}{(\delta - 2)}, \quad (41) \]
\[ \sigma^2 = \frac{\kappa^2}{(\delta - 2)(\delta - 3)}, \quad (42) \]

from which \( \delta \) and \( \kappa \) are expressed in terms of \( \mu \) and \( \sigma^2 \) as
\[ \delta = \frac{\mu^2 + 3\sigma^2}{\sigma^2}, \quad (43) \]
\[ \kappa = \frac{\mu^2 + \sigma^2}{\sigma^2} \mu. \quad (44) \]

On the contrary, the dynamical PDF given by Eq. (28) in the limit of \( \beta \to 0.0 \) (and \( I > 0 \)) reduces to
\[ p(x,t) \propto e^{-2c(t)/a(t)} \Theta(x) \to e^{-\kappa(t)/x} \Theta(x), \quad (45) \]

because Eqs. (30) and (31) with \( \beta \to 0.0 \) \( (a \to \infty) \) lead to
\[ \delta(t) = 2b = [1 + a^2\{\mu(t)^2 + 3\sigma(t)^2\}] \to \frac{\mu(t)^2 + 3\sigma(t)^2}{\sigma(t)^2}, \quad (46) \]
\[ \kappa(t) = \frac{2c(t)}{a(t)} = \frac{[1 + a^2\{\mu(t)^2 + \sigma(t)^2\}]\mu(t)}{a^2\sigma(t)^2} \to \frac{\mu(t)^2 + \sigma(t)^2}{\sigma(t)^2} \mu(t). \quad (47) \]
Equations (45), (46) and (47) agree with Eqs. (38), (43) and (44), respectively.

Thus the expressions given by Eqs. (30) and (31) yield the consistent result covering the two limits of \( \alpha \to 0.0 \) and \( \beta \to 0.0 \).

### 2.4 Model B

#### 2.4.1 Stationary distribution

Next we consider the model B in which \( F(x) \) and \( G(x) \) are given by Eqs. (15) and (16) with \( \epsilon \neq 0 \). From Eqs. (15)-(18), the stationary PDF is given by
\[ p(x) = \left( \frac{1}{Z} \right) \frac{\exp[2c\tan^{-1}\{a(x + f)\}]}{[1 + a^2(x + f)^2]^b}, \quad (48) \]
with

\[ a = \frac{\alpha}{\beta \sqrt{1 - \epsilon^2}}, \]
\[ b = \frac{2\lambda + \alpha^2}{2\alpha^2}, \quad (49) \]
\[ c = \frac{(I + \lambda f)}{\alpha \beta \sqrt{1 - \epsilon^2}}, \quad (50) \]
\[ f = \frac{\epsilon \beta}{\alpha}, \quad (51) \]
\[ Z = \sqrt{\pi} \frac{\Gamma(b) \Gamma(b - \frac{1}{2})}{a |\Gamma(b + ic)|^2}. \quad (52) \]

Equations (48) and (53) yield the average and variance in the stationary state given by

\[ \mu = \frac{c}{a(b - 1)} - f = \frac{(2I + \epsilon \alpha \beta)}{(2\lambda - \alpha^2)}, \quad (54) \]
\[ \sigma^2 = \frac{[(b - 1)^2 + c^2]}{a^2(2b - 3)} = \frac{\alpha^2 \mu^2 + 2\epsilon \alpha \beta \mu + \beta^2}{2(\lambda - \alpha^2)}. \quad (55) \]

### 2.4.2 Dynamical distribution

With the use of the procedure mentioned for the model A in Sec. 2.3, the dynamical PDF \( p(x, t) \) of the model B is assumed to be given by Eq. (48) but with \( b \) and \( c \) replaced by

\[ b(t) = \frac{1 + a^2\{[\mu(t) + f]^2 + 3\sigma(t)^2]\}}{2a^2\sigma(t)^2}, \quad (56) \]
\[ c(t) = \frac{1 + a^2\{[\mu(t) + f]^2 + \sigma(t)^2]\}[,\mu(t) + f]}{2a\sigma(t)^2}, \quad (57) \]

which agree with Eqs. (54) and (55) in the stationary state. Equations of motion for \( \mu(t) \) and \( \sigma(t)^2 \) in Eqs. (56) and (57) are given by

\[ \frac{d\mu(t)}{dt} = -\lambda \mu(t) + I(t) + \frac{\alpha^2 \mu(t)}{2} + \frac{\epsilon \alpha \beta}{2}, \quad (58) \]
\[ \frac{d\sigma(t)^2}{dt} = -2\lambda \sigma(t)^2 + 2\alpha^2 \sigma(t)^2 + \alpha^2 \mu(t)^2 + 2\epsilon \alpha \beta \mu(t) + \beta^2, \quad (59) \]

which are derived from Eqs. (13)-(16).

It is noted that in the limits of \( \alpha \to 0.0 \) and \( \beta \to 0.0 \), the cross-correlation between additive and multiplicative noise does not work, and the result for the model B reduces to that for the model A. Thus the moment method with the use of Eqs. (56) and (57) leads to the results consistent in the limits of \( \alpha \to 0.0 \) and \( \beta \to 0.0 \), where \( p(x, t) \) becomes the Gaussian and inverse-gamma distributions, respectively.
2.5 Model C

2.5.1 Stationary distribution

Now we consider the model C in which \( F(x) \) and \( G(x) \) are given by

\[
F(x) = -\lambda(x + s), \quad (60)
\]

\[
G(x) = \sqrt{x^2 + 2sx + r^2}, \quad (61)
\]

with \( \epsilon = 0.0 \) where \( \lambda \) expresses the relaxation rate, and \( r \) and \( s \) are parameters. We assume that \( \epsilon = 0.0 \) because we cannot obtain the analytic stationary PDF for \( \epsilon \neq 0.0 \). The model C with \( r = s = 0.0 \) is nothing but the model A. From Eqs. (6)- (8), the stationary PDF for the model C is given by

\[
p(x) \propto \left[ 1 - \frac{\alpha^2}{D}(x + s)^2 \right]^{-b} e^{Y(x)}, \quad (62)
\]

where

\[
Y(x) = \begin{cases} 
\frac{I}{\alpha \sqrt{D}} \ln \left| \frac{x + s - \sqrt{D}}{x + s + \sqrt{D}} \right|, & \text{for } D > 0 \\
\frac{2I}{\alpha \sqrt{-D}} \tan^{-1} \left( \frac{\alpha(x + s)}{\sqrt{-D}} \right), & \text{for } D < 0 \\
-\frac{2I}{\alpha^2 (x + s)}, & \text{for } D = 0
\end{cases} \quad (63-65)
\]

with

\[
D = \alpha^2(s^2 - r^2) - \beta^2, \quad (66)
\]

\[
b = \frac{(2\lambda + \alpha^2)}{2\alpha^2}. \quad (67)
\]

When we consider the case of \( D < 0 \), the stationary PDF is rewritten as

\[
p(x) = \left( \frac{1}{Z} \right) \frac{\exp[2c \tan^{-1}(a(x + s))]}{[1 + a^2(x + s)^2]^b}, \quad (68)
\]

with

\[
a = \frac{\alpha}{\sqrt{\beta^2 + \alpha^2(r^2 - s^2)}}, \quad (69)
\]

\[
c = \frac{I}{\alpha \sqrt{\beta^2 + \alpha^2(r^2 - s^2)}}, \quad (70)
\]

\[
Z = \sqrt{\pi} \Gamma(b) \Gamma(b - \frac{1}{2}) \frac{\Gamma(b + ic)}{a \Gamma(b + ic)^2}. \quad (71)
\]
From Eq. (68), we obtain $\mu$ and $\sigma^2$ in the stationary state expressed by

$$
\mu = \frac{c}{a(b-1)} - s = \frac{2I}{(2\lambda - \alpha^2)} - s, \quad (72)
$$

and

$$
\sigma^2 = \frac{[(b-1)^2 + c^2]}{a^2(b-1)(2b-3)} = \frac{[\alpha^2(\mu^2 + 2s\mu + r^2) + \beta^2]}{2(\lambda - \alpha^2)}. \quad (73)
$$

### 2.5.2 Dynamical distribution

In order to obtain the dynamical PDF for the model C, we adopt the same procedure as those for the models A and B. We assume that the dynamical solution $p(x,t)$ of the model C is given by Eq. (68) but with $b$ and $c$ replaced by

$$
b(t) = \frac{1 + a^2[(\mu(t) + s)^2 + 3\sigma(t)^2]}{2a^2\sigma(t)^2}, \quad (74)
$$

and

$$
c(t) = \frac{1 + a^2[(\mu(t) + s)^2 + \sigma(t)^2][\mu(t) + s]}{2a\sigma(t)^2}, \quad (75)
$$

which agree with Eqs. (72) and (73) in the stationary state. Equations of motion for $\mu(t)$ and $\sigma(t)^2$ in Eqs. (74) and (75) are given by

$$
\frac{d\mu(t)}{dt} = -\left(\lambda - \frac{\alpha^2}{2}\right)[\mu(t) + s] + I(t), \quad (76)
$$

and

$$
\frac{d\sigma(t)^2}{dt} = -2(\lambda - \alpha^2)\sigma(t)^2 + \alpha^2[\mu(t)^2 + 2s\mu(t) + r^2] + \beta^2, \quad (77)
$$

which are derived from Eqs. (13), (14), (60) and (61). It is easy to see that the moment method with the use of Eqs. (74) and (75) lead to the results consistent in the limits of $\alpha \to 0.0$ and $\beta \to 0.0$, where $p(x,t)$ becomes the Gaussian and inverse-gamma distributions, respectively.

### 2.6 Model calculations

We will present some numerical calculations in this subsection. In order to examine the validity of the moment approach, we have employed the partial difference equation derived from Eq. (5) with $\phi = 1$, as given by

$$
p(x, t + v) = p(x,t) + \left(-F' + \frac{\alpha^2}{2}[(G')^2 + GG^{(2)}] + \frac{\epsilon\alpha\beta}{2}G'^{(2)}\right)v p(x,t)
$$

$$
+ \left[-F - I(t) + \frac{3\alpha^2}{2}GG' + \frac{3\epsilon\alpha\beta}{2}G'\right] \left(\frac{v}{2u}\right)[p(x+u) - p(x-u)]
$$

$$
+ \left(\frac{\alpha^2}{2}G^2 + \epsilon\alpha\beta G + \frac{\beta^2}{2}\right) \left(\frac{v}{u^2}\right)[p(x+u,t) + p(x-u,t) - 2p(x,t)]. \quad (78)
$$
where \( u \) and \( v \) denote incremental steps of \( x \) and \( t \), respectively. We impose the boundary condition:

\[
p(x, t) = 0, \quad \text{for } |x| \geq x_m
\]  

(79)

with \( x_m = 5 \), and the initial condition of \( p(x, 0) = p_0(x) \) where \( p_0(x) \) is the stationary PDF. We have chosen parameters of \( u = 0.05 \) and \( v = 0.0001 \) such as to satisfy the condition: \( (\alpha^2 x_m^2 v/2u^2) < 1/2 \), which is required for stable, convergent solutions of the PDEM.

First we apply a pulse input signal given by

\[
I(t) = \Delta I \Theta(t - 2)\Theta(5 - t) + I_b,
\]

(80)

with \( \Delta I = 0.5 \) and \( I_b = 0.0 \) to the model B with \( \lambda = 1.0, \alpha = 0.5, \beta = 0.5 \) and  in the introduced correlation of \( \epsilon = 0.5 \). By an applied pulse at \( 2.0 \leq t < 6.0 \), \( \mu(t) \) and \( \sigma(t)^2 \) are increased, and the position of \( p(x, t) \) moves rightward with slightly distorted shapes. After an input pulse diminishes at \( t \geq 6.0 \), the PDF gradually restores to its original stationary shape.

Next we apply the pulse input given by Eq. (80) with \( \Delta I = 0.5 \) and \( I_b = 0.0 \) to the model C with \( \lambda = 1.0, \alpha = 0.5, \beta = 0.5, r = 0.5 \) and 2. Figure 3 shows the time-dependent PDF at various \( t \) in response to an applied input. Results of the moment method calculated with the use of Eqs. (68), (71), (74)-(77) are shown by solid curves while those of the PDEM with Eq. (78) are expressed by dashed curves. In the stationary state at \( t < 2.0 \), we obtain \( \mu(t) = -0.2 \) and \( \sigma(t)^2 = 0.2 \). By an applied pulse at \( 2.0 \leq t < 6.0 \), \( \mu(t) \) and \( \sigma(t)^2 \) are increased and the position of \( p(x, t) \) moves rightward with slightly changed shapes.
given by
\[ \lambda(t) = \Delta \lambda \Theta(t - 2)\Theta(6 - t) + \lambda_b, \quad \text{(81)} \]
with $\Delta \lambda = 1.0$ and $\lambda_b = 1.0$ to the model C with $\alpha = 0.5$, $\beta = 0.0$, $r = 0.5$, $s = 0.2$ and $I = 0.0$. Equation (81) stands for an application of an external force of $-\Delta \lambda x$ at $2.0 \leq t < 6.0$. The time dependent $p(x,t)$ is plotted in Fig. 5, where solid and dashed curves denote the results of the moment method and PDEM, respectively. Figure 6 shows the time dependences of $\mu(t)$ and $\sigma(t)^2$. When an external force is applied at $2.0 \leq t < 6.0$, the width of the PDF is reduced and $\sigma(t)^2$ is decreased while $\mu(t)$ has no changes.

It is noted in Figs. 1-6 that results of $p(x,t)$, $\mu(t)$ and $\sigma(t)^2$ calculated by the moment method are in good agreement with those obtained by the PDEM.

### 3 CONCLUSION AND DISCUSSION

The moment approach to the FPE discussed in preceding Sec. 2 may be applied to various Langevin models provided analytic expressions for the PDF and for the first- and second-order moments in the stationary state are available. For example, when $F(x)$ and $G(x)$ are given by
\[ F(x) = -\lambda x|x|^{r-1}, \quad \text{and} \quad G(x) = x|x|^{s-1}, \quad \text{for} \quad r \geq 0, \quad s \geq 0, \]
the stationary PDF with $\epsilon = 0.0$ is given by
\[ p(x) \propto (\alpha^2|x|^{2s} + \beta^2)^{-1/2} \exp[X(x) + Y(x)], \quad \text{(84)} \]
with
\[ X(x) = -\left( \frac{2\lambda|x|^{r+1}}{\beta^2(r+1)} \right) F \left( 1, \frac{r+1}{2s}, \frac{r+1}{2s} + 1; -\frac{\alpha^2|x|^{2s}}{\beta^2} \right), \quad \text{(85)} \]
\[ Y(x) = \left( \frac{2I|x|}{\beta^2} \right) F \left( 1, \frac{1}{2s}, \frac{1}{2s} + 1; -\frac{\alpha^2|x|^{2s}}{\beta^2} \right), \quad \text{(86)} \]
where $F(a,b,c;z)$ denotes the hypergeometric function. Equations of motion for $\mu$ and $\sigma^2$ are given by
\[ \frac{d\mu}{dt} = -\lambda \mu |\mu|^{r-1} + I - \left( \frac{\lambda}{2} \right) r(r-1)\mu |\mu|^{r-3}\sigma^2 \]
\[ + \left( \frac{\alpha^2}{2} \right) [s\mu|\mu|^{2s-2} + s(s-1)(2s-1)\mu|\mu|^{2s-4}\sigma^2], \quad \text{(87)} \]
\[ \frac{d\sigma^2}{dt} = -2\lambda r|\mu|^{r-1}\sigma^2 + 2s(2s-1)\alpha^2|\mu|^{2s-2}\sigma^2 + \alpha^2|\mu|^{2s} + \beta^2. \quad \text{(88)} \]
If analytic expressions for stationary values of $\mu$ and $\sigma^2$ are obtainable from Eqs. (87) and (88), we may apply our moment method to the FPE given by Eqs. (5), (82) and (83) with the following steps: (1) adopting the stationary PDF given by Eqs. (84)-(86), and (2) expressing its parameters in terms of the time-dependent $\mu(t)$ and $\sigma(t)^2$ in an appropriate way, as mentioned for models A, B and C.

As an application of our method, we have calculated the Fisher information for the dynamical inverse-gamma distribution, which is realized for $\beta = 0.0$ in the model A [Eq. (45)],

$$p(x, t) = \frac{\kappa^{\delta(t)-1}}{\Gamma[\delta(t) - 1]} x^{-\delta(t)} e^{-\kappa(t)/x} \Theta(x), \quad (89)$$

with the time-dependent $\delta(t)$ and $\kappa(t)$ given by Eqs. (46) and (47). With the use of Eq. (89), the Fisher information matrix given by

$$g_{ij} = \left\langle \left( \frac{\partial \ln p(x)}{\partial \theta_i} \right) \left( \frac{\partial \ln p(x)}{\partial \theta_j} \right) \right\rangle, \quad (90)$$

is expressed by

$$g_{\delta\delta} = \psi'[\delta(t) - 1], \quad (91)$$

$$g_{\kappa\kappa} = \frac{[\delta(t) - 1]^2}{\kappa(t)^2}, \quad (92)$$

$$g_{\kappa\delta} = \frac{[\delta(t) - 1]}{\kappa(t)} \{ \psi[\delta(t) - 1] - \psi[\delta(t)] \}, \quad (93)$$

where $\psi(x)$ and $\psi'(x)$ are di- and tri-gamma functions, respectively.

Figure 7 shows the time-dependent inverse-gamma distribution $p(x, t)$ when an input pulse given by Eq. (80) with $\Delta I = 0.3$ and $I_0 = 0.2$ is applied to the model A with $\lambda = 1.0$, $\alpha = 0.5$ and $\beta = 0.0$. Solid and dashed curves show the results of the moment method and the PDEM, respectively. The time dependences of $\mu(t)$ and $\sigma(t)^2$ and of Fisher information are plotted in Fig. 8(a) and (b), respectively, where solid (dashed) curves express the result of the moment method (PDEM). By an applied pulse at $2.0 \leq t < 6.0$, $\mu(t)$ and $\sigma(t)^2$ are increased and the position of the PDF moves rightward with an increased width. An applied pulse increases $g_{\kappa\kappa}$ while it decreases $g_{\delta\kappa}$. An interesting behavior is observed in $g_{\delta\delta}$ which is decreased at $t = 2.0$ when a pulse is applied, but afterward it seems to gradually reduce to the stationary value. A similar behavior is realized in $g_{\delta\delta}$ also at $t \geq 6.0$ when the applied pulse is off.

In our previous paper [19], we applied the $q$-moment approach to the model A, deriving equations similar to Eqs. (30)-(33) in the (normal) moment approach. There are some
differences between the $q$- and normal-moment approaches. The stationary variance $\sigma_q^2$ in the $q$-moment approach evaluated over the escort distribution is stable for $0 \leq \alpha^2/\lambda < \infty$ whereas $\sigma^2$ in the normal-moment approach is stable for $0 \leq \alpha^2/\lambda < 1.0$ [Eqs. (24), (55) or (73)]. Although the time dependences of $\mu_q(t)$ and $\sigma(t)_q^2$ calculated by the $q$-moment approach are similar to those of $\mu(t)$ and $\sigma(t)^2$ for a small $\alpha$, the difference between them becomes significant for a large $\alpha$ (see Fig. 13 in Ref. [19]). These differences yield the quantitative difference in $p(x,t)$ calculated by the normal- and $q$-moment methods, although both the methods lead to qualitatively similar results. We note that in the limit of $\alpha = 0.0$ (i.e., additive noise only), the dynamical solution given by the $q$- or normal-moment method reduces to the Gaussian solution given by Eqs. (25)-(27). Thus the $q$- or normal-moment approach is a generalization of the Gaussian solution to the FPE with $\alpha \neq 0.0$ given by Eq. (5).

In summary, by using the second-order moment method, we have discussed the analytic time-dependent solution of the FPE which includes additive and multiplicative noise as well as external perturbations. It has been demonstrated that dynamical PDFs calculated by the moment approach are in good agreement with those obtained by the PDEM. Our moment method has some disadvantages. The variance $\sigma^2$ diverges at $\alpha^2/\lambda \geq 1.0$, for which our method cannot be applied. If an applied perturbation induces a large $\mu(t)$ and/or $\sigma(t)^2$, our method leads to poor results which are not in good agreement to those calculated by the PDEM. These are inherent in the moment approximation in which each moment is required to be small. Despite these disadvantages, however, our moment method has following advantages: (i) obtained dynamical solutions are compatible with the exact stationary solutions in the Langevin models A, B and C, (ii) it is useful for various subjects in which analytical dynamical PDFs are indispensable (e.g., Ref. [19]), and (iii) the second-order moment approach is more tractable than sophisticated methods [16]-[18] for the FPE subjected to multiplicative noise. As for the item (iii), it is possible to take account of contributions from higher-order moments than second-order ones with the use of Eq. (10), though actual calculations become tedious.

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Figure 1: (Color online) The time-dependent probability distribution $p(x, t)$ of the model B with $\lambda = 1.0$, $\alpha = 0.5$, $\beta = 0.5$ and $\epsilon = 0.5$, in response to an applied pulse given by Eq. (80) with $\Delta I = 0.5$ and $I_b = 0.0$: solid and dashed curves denote results calculated by the moment method and the PDEM, respectively, curves being consecutively shifted downward by 0.25 for a clarity of the figure.

Figure 2: (Color online) The time dependence of $\mu(t)$ and $\sigma(t)^2$ of the model B with $\lambda = 1.0$, $\alpha = 0.5$, $\beta = 0.5$ and $\epsilon = 0.5$, calculated by the moment method with Eqs. (76) and (77) (solid curves) and by the PDEM (dashed curves) in response to an applied pulse $I(t)$ given by Eq. (80) with $\Delta I = 0.5$ and $I_b = 0.0$ (the chain curve).

Figure 3: (Color online) The time-dependent probability distribution $p(x, t)$ of the model C with $\lambda = 1.0$, $\alpha = 0.5$, $\beta = 0.5$, $r = 0.5$ and $s = 0.2$, in response to an applied pulse given by Eq. (80) with $\Delta I = 0.5$ and $I_b = 0.0$: solid and dashed curves denote results calculated by the moment method and the PDEM, respectively, curves being consecutively shifted downward by 0.25 for a clarity of the figure.

Figure 4: (Color online) The time dependence of $\mu(t)$ and $\sigma(t)^2$ of the model C with $\lambda = 1.0$, $\alpha = 0.5$, $\beta = 0.5$, $r = 0.5$ and $s = 0.2$, calculated by the moment method with Eqs. (76) and (77) (solid curves) and by the PDEM (dashed curves), in response to an applied pulse $I(t)$ given by Eq. (80) with $\Delta I = 0.5$ and $I_b = 0.0$ (the chain curve).

Figure 5: (Color online) The time-dependent probability distribution $p(x, t)$ of the model C with $I = 0.0$, $\alpha = 0.5$, $\beta = 0.0$, $r = 0.5$ and $s = 0.2$, when the time-dependent relaxation rate $\lambda(t)$ given by Eq. (81) is applied: solid and dashed curves denote results calculated by the moment method and the PDEM, respectively, curves being consecutively shifted downward by 0.25 for a clarity of the figure.

Figure 6: (Color online) The time dependence of $\mu(t)$ and $\sigma(t)^2$ of the model C with $\lambda = 1.0$, $\alpha = 0.5$, $\beta = 0.5$, $r = 0.5$ and $s = 0.2$, calculated by the moment method with Eqs. (76) and (77) (solid curves) and by the PDEM (dashed curves) when the time-dependent relaxation rate $\lambda(t)$ given by Eq. (81) is applied (the chain curve), $\lambda$ being divided by a factor of five.
Figure 7: (Color online) The time-dependence of the inverse-gamma distribution $p(x, t)$ of the model A with $\lambda = 1.0$, $\alpha = 0.5$ and $\beta = 0.0$ in response to an applied pulse given by Eq. (80) with $\Delta I = 0.3$ and $I_b = 0.2$: solid and dashed curves denote results calculated by the moment method and the PDEM, respectively. Curves are consecutively shifted downward by 1.0 for a clarity of the figure.

Figure 8: (Color online) (a) The time dependence of $\mu(t)$ and $\sigma(t)^2$ of the model A with $\lambda = 1.0$, $\alpha = 0.5$ and $\beta = 0.0$ calculated by the moment method (solid curves) and PDEM (dashed curves) in response to an applied pulse given by Eq. (80) with $\Delta I = 0.3$ and $I_b = 0.2$ (the chain curve), $\sigma(t)^2$ being multiplied by a factor of five. (b) The time dependence of the Fisher information matrix calculated by Eqs. (91)-(93): $g_{\delta\delta}$ (the solid curve), $g_{\kappa\kappa}$ (the dashed curve) and $g_{\delta\kappa}$ (the chain curve), $g_{\delta\delta}$ and $g_{\kappa\kappa}$ being multiplied by factors of 10 and 1/10, respectively (see text).
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