The shifted Wald distribution represents a non-censored Wiener diffusion model of choice response times: Evidence from simulations and a Go/No-go task

Robert Miller, Stefan Scherbaum, Thomas Goschke, Sören Enge

Institute of Psychology, Technische Universität Dresden, Germany

Author note

This work was partly supported by the German Research Foundation (SFB 940/1 2015). We thank Andrew Heathcote, Dominik Wabersich, Dóra Erbé-Matzke, Matthias Gondan, and Monika Fleischhauer for their most valuable support and comments on the manuscript. Correspondence concerning this article should be addressed to R. Miller, Technische Universität Dresden, Institute of Psychology, Zellescher Weg 19, 01069 Dresden, Germany (e-mail: robert.miller@tu-dresden.de).

Abstract

Cognitive psychologists commonly strive to infer processes/constructs from chronometric data. In this regard, diffusion modeling of response times (RTs) from correct and erroneous responses using the Wiener distribution has become popular because it provides a set of interpretable parameters. An important precondition to estimate these parameters, however, is a sufficient number of RTs from erroneous responses.

In the present article, we show by simulation that the parameters of the Wiener distribution can even be recovered from tasks yielding very high or even perfect response accuracies using the shifted Wald distribution. Specifically, we argue that error RTs can be considered as correct RTs that have undergone informative censoring. Such censoring can be accounted for by techniques of survival analyses that should also be applied whenever task trials are experimentally terminated prior to response execution. We illustrate our reasoning by fitting the Wiener and censored shifted Wald distribution to RTs from 101 participants who completed a Go/No-go task. In accordance with our simulations, Wiener and shifted Wald modeling yielded the same parameter estimates when the number of erroneous responses was predicted to be low. Moreover, the modeling of error RTs as censored correct RTs improved the parameter plausibility with narrow response time windows. Finally, we investigate the relation between the number of RTs and the precision of parameter estimates from Wiener and censored shifted Wald modeling, and provide R code for the convenient application of the outlined analytical procedures.

Keywords: response time, diffusion model, informative censoring, competing risks, Go/No-go task
1. Introduction

Due to the availability of convenient tools, the Wiener diffusion model (Ratcliff, 1978) and its derivatives have become increasingly popular for the simultaneous analysis of response rate and response time (RT) data from two-alternative forced choice (2AFC) tasks (for review see Voss, Nagler & Lerche, 2013). The attractiveness of the diffusion model arises from the fact that its parameters have distinct psychological interpretations, if the evidence accumulation in the course of each trial can be represented as a noisy drift towards either of two decision boundaries. These boundaries are assumed to characterize voluntarily adjustable evidence thresholds whose transgression by the uncontrollable diffusion process initiates the execution of the respective response (Voss et al., 2013; Wagenmakers, 2009). The resulting RTs are determined by the four parameters of the Wiener distribution. These parameters represent (α) the distance between the two decision boundaries, (β) the relative response tendency towards the correct response at the beginning of the trial, (δ) the drift rate towards the correct response, and (θ) the non-decision time, that is, the time that is not related to evidence accumulation but residual processes. Moreover, the complete diffusion model assumes that β, δ, and θ can vary across all trials of a 2AFC task (Ratcliff & Rouder, 1998), that is, the actual RT distribution is the sum of many different Wiener distributions (Vandekerckhove et al., 2011).

In fact, there is good evidence showing that the diffusion model fits the empirical data well, but from a practical point of view, its parameters are only identified (i.e., precisely estimable) if a sufficient number of RTs for both response alternatives is available (Wagenmakers, van der Maas, & Grasman, 2007). Unfortunately, this precondition is either barely met in tasks that yield extremely low error rates (context interference tasks like Stroop or Simon tasks; Pratte, Rouder, Morey, & Feng, 2010), or even impossible to be met in tasks yielding no RTs for erroneous responses at all (response inhibition tasks like the Go/No-go task; Verbruggen & Logan, 2008). In consequence the Wiener diffusion model is hardly applicable to these tasks unless the number of trials is tremendously increased.\footnote{Nonetheless, the Wiener diffusion model can be used to analyze such data if one is willing to constrain the starting point of the diffusion process according to some a priori expectation (e.g. β = α/2 for ambiguous response tendencies; see Wagenmakers et al., 2007).}

In such situations, psychologists often substantiate their hypotheses by the separate analyses of response rates from both response alternatives, but only analyze RTs from the correct responses. As the characterization of correct response RTs by measures of central tendency entails a substantial loss of information, the utilization of complex distributions (e.g., the exponentially-modified Gaussian) has strongly been encouraged to describe RTs (Balota & Yap, 2011). However, Matzke and Wagenmakers (2009) found that the parameters of such distributions are not in strict conformity with the parameters of the complete diffusion model, thereby complicating specific interpretations. This finding even generalized to the shifted Wald (also known as shifted inverse Gaussian) distribution, which is generated by diffusion towards only one decision boundary as compared to the Wiener distribution arising from two decision boundaries.

Indeed, the conceptualization of the complete 2AFC diffusion model as many overlapping Wiener distributions suggests that any RT distribution, which is generated by many diffusion processes towards one decision boundary (i.e., the correct response), should only enable similar interpretations if the contribution of the other decision boundary (i.e. the erroneous response) to the RT distribution is negligible (cf. Jones &
Dzhafarov, 2014). This condition seems to be met for the above-mentioned and many other tasks (see Gondan & Fimm, under review).

Proceeding from this reasoning, the present article investigates the modeling of correct response RTs using the shifted Wald distribution. As we will show, the parameters of shifted Wald distributions actually correspond to the parameters of the Wiener distribution whenever the accuracy of the correct response alternative is high and/or the RTs of the erroneous responses are additionally incorporated as censored data in the shifted Wald distribution of correct response RTs. The utility of (censored) shifted Wald modeling will be demonstrated with empirical data that has been obtained from a Go/No-go task. Finally, we discuss our observations with regard to the psychological interpretability of shifted Wald parameters in 2AFC tasks that are often designed to yield increased error rates by a restriction of response time windows.

2. On the correspondence of diffusion processes with one or two decision boundaries

To increase the clarity of the following sections, we want to illustrate our reasoning by means of a typical Go/No-go task, which is commonly employed to assess response inhibition processes of cognitive control (Verbruggen & Logan, 2008). Characteristically, each trial of a Go/No-go task requires participants to press a certain button in response to a prepotent Go-signal (that is drawn from a larger set of stimuli, e.g. letters), but to omit this button press whenever a No-go-signal is encountered (e.g., if the letter ‘X’ is presented). In such a one-choice task, omitted button presses in Go-trials (i.e., erroneous responses) are not supposed to exist, because any participant should be able to correctly detect the Go-signal with unlimited time and after sufficient evidence has accumulated. For reasons of feasibility, however, Go-trials are terminated at a certain point in time, which will necessarily yield a small number of erroneous responses because the accumulated evidence may have been insufficient for response execution prior to trial termination. Alternatively, Go-trials may also yield imperfect response accuracies because participants make the irreversible decision to omit the button press at a certain point in time (Gomez, Ratcliff & Perea, 2007). Thus Go-trials do not necessarily require only one choice but could as well represent a special type of a 2AFC task, where the RTs of erroneous responses are unobservable (e.g., Trueblood, Endres, Busemeyer & Finn, 2011). This process-ambiguity makes the Go/No-go task a practical example to elaborate the correspondence between parameters of diffusion processes that involve only one or two decision boundaries.

Considering the case of two decision boundaries, the RT distribution of the correct responses and of the (unobservable) erroneous responses in Go-trials corresponds to the Wiener distribution, whose density function is provided in Eq. 1 (Navarro & Fuss, 2009; see also Gondan, Blurton & Kesselmeier, 2014).

\[
\text{Eq. (1)}
\]

\[
\text{Wiener}(t|\alpha, \beta, \delta, \theta) = \frac{1}{\alpha^2} \exp\left[-\alpha\beta\delta - \frac{1}{2} \delta^2(t - \theta)\right] \sum_{k=1}^{\infty} k \exp\left[-\frac{k^2\pi^2(t - \theta)}{2\alpha^2}\right] \sin(k\pi\theta)
\]

As can be seen in Figure 1, the Wiener distribution determined by noisy evidence accumulation at mean drift rate \(\delta\), until either the upper decision boundary for the button press \((y_1)\) is reached, or that for omitting the button press \((y_2)\) is “accidentally” transcended due to noise. Notably, the latter also results in the termination of any further evidence accumulation. Hence erroneous response RTs cannot contribute to
forming the RT distribution of correct responses (i.e. the density of first-passage times at $\gamma_1$).

Figure 1. Illustration of a diffusion process with one or two decision boundaries. If two decision boundaries existed ($\gamma_1$ and $\gamma_2$), any evidence accumulation (solid black and grey paths) would stop upon transgression of either boundary and the respective response will be executed (RT$_{correct}$ or RT$_{error}$). If only the upper boundary existed ($\gamma_1$), evidence accumulation would not stop upon any transgression of the lower boundary ($\gamma_2$) but would continue (dotted grey path) until the process was experimentally terminated at a certain point in time (exp. censored RT$_{correct}$) or $\gamma_1$ was finally reached (RT$_{correct}$). The former case is a typical example for right censoring, where only the interval of the true RT$_{correct}$ would be known [ranging from exp. censored RT$_{correct}$ to $\infty$]. Analogously, the erroneous response RT$_{error}$ is essentially a random censored RT$_{correct}$ [ranging from RT$_{error}$ to $\infty$]. Proceeding from this reasoning, any RT drawn from the Wiener distribution of correct responses (black) is an RT drawn from the shifted Wald distribution of correct responses (grey) that has undergone informative censoring.

The same paths of accumulated evidence could arise upon the existence of only one decision boundary, but in contrast to the diffusion process with two boundaries, evidence accumulation would not stop when the lower decision boundary for omitting the button press ($\gamma_2$) was reached. Instead, the drift would continue towards the upper decision boundary ($\gamma_1$) until the correct response was finally executed at some delayed point in time (see dotted grey line in Figure 1). The resulting distribution of correct response RTs in Go-trials is characterized by the density function of the shifted Wald distribution (Eq. 2; Heathcote, 2004), which has a more pronounced skew than implied by the Wiener distribution of correct response RTs. This is because an erroneous transgression of $\gamma_2$ would not result in the termination of the diffusion process at the expense of a larger amount of prolonged but correct responses.
Eq. (2)

\[ s_{Wald}(t|\gamma, \delta, \theta) = \frac{\gamma}{\sqrt{2\pi(t-\theta)}} \exp \left[ -\frac{(\gamma - \delta t + \delta \theta)^2}{2(t-\theta)} \right] \]

As mentioned above, any trial of a Go/No-go task needs to be terminated at a certain, experimentally controlled point in time (e.g., after 1.5 sec) which will also entail erroneous responses (i.e., omitted button presses) if \( \gamma_1 \) is not reached by then. Under such conditions, neither the Wiener, nor the shifted Wald distribution can account for correct response RTs anymore unless all of these erroneous response RTs are considered as unobservable correct response RTs that have been censored at the time of trial termination (cf. Ulrich & Miller, 1991).

The term “censoring” originated in the field of survival analysis and denotes missing data problems, in which only the interval of the unobserved time-to-event data is known (Miller, 1998). Such situations arise whenever the event of interest (i.e., the response to the Go-trial) will eventually have occurred outside the period of data collection (i.e., after the time-out of the Go-trial). In the following sections, we adapt this concept of right-censored RTs for the shifted Wald modeling of data, which have been sampled from the Wiener distribution (i.e., they are actually generated by a diffusion process with two decision boundaries). As can be derived from Figure 1, the major conceptual difference to an experimentally controlled time of trial termination (exp. censored RT_{correct}) is that the censoring point is not constant across all trials but is determined by the RT of the respective erroneous response (RT_{error} = random censored RT_{correct}). If such censoring information were appropriately incorporated into the modeling procedure or the data were not censored at all (i.e., no erroneous responses are present), the parameters of the shifted Wald distribution are supposed to exactly correspond to the parameters of the Wiener distribution. Under these circumstances, the shifted Wald distribution can be adopted for obtaining psychologically meaningful RT parameters of many 2AFC tasks.

Table 1. The relation between shifted Wald and Wiener parameters, and their psychological interpretations.

| Shifted Wald parameter | Wiener parameter configuration | Psychological interpretation |
|------------------------|-------------------------------|-----------------------------|
| \( \delta \)           | \( \delta \)                   | Speed of evidence accumulation for the correct response alternative |
| \( \gamma \)           | \( \alpha^*(1-\beta) \)       | Speed-accuracy tradeoff with reference to the starting point of the decision process |
| \( \theta \)           | \( \theta \)                   | Time that is not related to the decision process (e.g., sensory encoding or execution of motor responses) |

Note. In contrast to the diffusion process outlined in the previous section (upper boundary = \( \gamma_1 \), lower boundary = \( \gamma_2 \), starting point = 0), the boundaries of the Wiener distribution are defined by \( \alpha \) (upper boundary) and 0 (lower boundary), with the starting point of the diffusion process at \( \beta \) (i.e., the relative initial response tendency towards the correct response alternative). Translating this parameterization, the decision boundaries of the correct and erroneous response are defined as \( \gamma_1 = \alpha^*(1-\beta) \) and \( \gamma_2 = \alpha^*\beta \), respectively.
Proceeding from the outlined correspondence of diffusion processes with one or two decision boundaries, we performed a simulation study showing how the parameters obtained by both, regular and censored shifted Wald modeling relate to different parameter configurations of the Wiener distribution. All shifted Wald parameters and their corresponding Wiener parameter configurations are listed in Table 1. In a first step, we generated artificial RTs from a large set of Wiener parameters covering previous findings (Matzke & Wagenmakers, 2009). These data were subsequently submitted to (censored) shifted Wald modeling using the maximum likelihood method. Finally, the resulting parameter estimates were predicted by the parameters of the data-generating Wiener distributions using locally weighted scatterplot smoothing (Cleveland & Devlin, 1988). Technical details about the Monte-Carlo simulations and the R code for implementing the discussed shifted Wald models of partially censored RTs are provided in the Appendices A and B.

3.1. Trial-invariant diffusion parameters

First, we investigated the case where all randomness in the RT distributions is attributable to the diffusion process, implying Wiener distributions with no parameter variability across trials. These simulations’ results are presented in Figure 2.

Regarding the parameter recovery by conventional shifted Wald modeling (light grey lines), the simulations revealed pronounced linear relations between the corresponding shifted Wald and Wiener parameters when the proportion of correct responses amounted to more than ~95%. However, our results also confirmed a growing divergence of Wiener and shifted Wald parameters as the proportion of correct responses decreased due to liberal boundary separations (α) and/or slow drift rates (δ). Specifically, the latter also led to biased estimates of boundary separations and non-decision times (θ). Likewise, response tendencies (β) toward the erroneous response alternative entailed a growing discrepancy between parameter estimates (data not shown).

In accordance with our reasoning, censored shifted Wald modeling (dark grey lines) was well able to improve the recovery of α and δ, but the bias of α and θ resulting from a slow drift of the data-generating diffusion process could not be adjusted for. The failure of censored shifted Wald modeling to fully recover Wiener parameters at increasing error rates is not surprising, because any censoring induced by erroneous responses will be informative for a delayed execution of the correct responses (see Figure 1) unless the processes governing correct and erroneous responses were independent (cf. Jones & Dzhafarov, 2014). Technically speaking, the Markov property of the diffusion process ensures that any correct RT that has undergone censoring at a lower decision boundary γ2 (< γ1) will necessarily have a response tendency β = 0 implying γ = α for all erroneous responses (Gondan & Finm, under review). Such informative censoring is well known to introduce bias when the likelihood function of non-informatively censored distributions is maximized (e.g., Siannis et al., 2005).

In order to resolve this issue, we implemented another variant of censored shifted Wald modeling that considered the probability of error commission as a competing risk for correct responses (see Putter et al., 2007). Specifically, we modeled both correct RTs and error RTs using two censored shifted Wald distributions. Both RT distributions featured different decision boundaries (γ1 and γ2, respectively), which depended on a shared response tendency parameter and the distance in between them (see Table 1). Moreover, the distribution of error RTs was constrained to arise from the negative drift...
parameter of the distribution of correct RTs (i.e. $\delta_1 = \delta$ and $\delta_2 = -\delta$). Thus, these two distributions mimicked the Wiener model as two connected evidence accumulators. As can be seen in Figure 2 (black lines), the competing risks approach resulted in an almost perfect recovery of the Wiener parameters when error rates were not negligible, with small bias only occurring when estimating $\alpha$ at extremely slow drift rates $\delta$. By contrast, the recovery of all Wiener parameters was extraordinarily insensitive to manipulations of $\alpha$ and $\theta$. However, we need to emphasize, that the accuracy of the competing risks approach came at the cost of the same limitations as the Wiener model of 2AFC tasks, that is, any response tendency $\beta$ that is not identified by the data will necessarily entail an unidentifiable boundary separation $\alpha$ (or decision boundaries $\gamma_1$ and $\gamma_2$) unless the former is fixed to some a-priori assigned value. However, many tasks feature an equal attractiveness of both response alternatives (i.e., a response tendency of $\beta = \alpha/2$, see Wagenmakers et al., 2007), which may relax this potential issue to some degree.

Figure 2. Recovery of Wiener parameters by different types of shifted Wald analyses. In all simulations, one parameter of the Wiener distribution was systematically varied (see Appendix B). The reference parameter manifestations were $\delta = 2.5$ sec$^{-1}$, $\alpha = 2$, $\beta = 0.5$, and $\theta = 0.55$ sec. Different shadings represent recoveries by conventional shifted Wald modeling (light grey), censored shifted Wald modeling (dark grey), and censored shifted Wald modeling with competing risks (black).
3.2. Diffusion parameter variability across trials

RT modeling from only one response alternative (e.g. the correct response) does not necessarily require variable diffusion parameters because RT variability can just arise from the stochasticity in the process itself (Jones & Dzhafarov, 2014). With two competing response alternatives, however, the Wiener diffusion model can only account for empirical phenomena if parameter variability across different trials is explicitly assumed (Ratcliff & Rouder, 1998). In consequence we complement the simulations in section 3.1 by investigating the sensitivity of (censored) shifted Wald modeling of chronometric data to such parameter variability. The results of these simulations are presented in Figure 3 and essentially replicate the findings of Matzke and Wagenmakers (2009): Conventional and censored shifted Wald analyses are comparably robust to variability in drift rates and (to a lesser extent) response tendencies across trials. However, non-decision time variability may severely bias the recovery of the data-generating Wiener parameters.

Figure 3. Sensitivity of the different shifted Wald analyses to variability of Wiener parameters across trials i. Proceeding from the reference manifestations at $\delta = 2.5$ sec$^{-1}$, $\beta = 0.5$, and $\theta = 0.55$ sec, variability of the respective parameter was systematically introduced according to the common distribution assumptions of the complete Wiener diffusion model (see Jones & Dzhafarov, 2014): $\delta_i \sim \text{Normal}(\delta, \sigma_\delta)$, $\beta_i \sim \text{Uniform}(\beta - \sigma_\beta, \beta + \sigma_\beta)$, and $\theta_i \sim \text{Uniform}(\theta - \sigma_\theta, \theta + \sigma_\theta)$. 
4. Correspondence of shifted Wald and Wiener parameters in a Go/No-go task

So far, we have shown that the shifted Wald and Wiener parameters are actually comparable if (A) either the competition between correct and erroneous responses is adequately modeled (when error rates are moderate), or (B) the number of error RTs is comparably low due to high drift rates, conservative decision boundaries $\gamma$ or response tendencies towards the correct response alternative.\(^2\)

Whenever a cognitive task yields a moderate number of commission errors (i.e. $>5\%$ that do not result from response time out), there will probably be a sufficient number of error RTs to fit the Wiener diffusion model. Thus the competing risks approach to censored shifted Wald modeling is mostly interesting from a theoretical perspective, but of less practical value for dealing with situation (A). Similarly in situation (B), shifted Wald and censored shifted Wald analyses will asymptotically yield the same parameter estimates. Thus one may trade off the additional implementation burden of censored shifted Wald analyses against their prospected gain.

However, the concept of censoring may yield immense advantages if the majority of erroneous responses is caused by narrow response time windows (yielding unobservable error RTs) and not by the diffusion process itself [i.e. situation (B) is disguised as situation (A)]. Such situations are supposed to arise whenever response tendencies towards the correct response alternative are experimentally enforced by response priming, or by prepotent trial types (e.g. a larger frequency of Go-trials as compared to No-go-trials), but the response time window is severely constrained. Proceeding from this reasoning we want to show, that shifted Wald parameters can actually serve as plausible proxies for Wiener parameters when estimated from empirical data that are obtained under such limiting conditions.

4.1. Material and methods

In order to accomplish this goal, we present the results from (conventional and censored) shifted Wald modeling and Wiener modeling of the chronometric data from 101 undergraduate students providing 189 – 198 valid Go-trial RTs in Go/No-go task\(^3\). These Go-trial RTs were collected during two experimental blocks featuring trial timeout times of 450 ms and 350 ms, respectively, which were presented in a counterbalanced order. Proceeding from the assumption that drift rates ($\delta$) and non-decision times ($\theta$) were unlikely to systematically vary across both blocks, an intra-individually constant decision boundary ($\gamma_{450\text{ms}} = \gamma_{350\text{ms}}$) would necessarily entail an increased number of erroneous (timed-out) responses in the 350 ms condition. We hypothesized that such an increased rate of erroneous responses due to trial time-out could be partly compensated if the participants reduced their decision thresholds accordingly ($\gamma_{450\text{ms}} > \gamma_{350\text{ms}}$). Proceeding from this reasoning, decision boundaries were allowed to vary across blocks whereas drift-rates and non-decision times were constrained to equality.

\(^2\) Although, both situations also assume that across-trial variability in response tendencies or non-decision times is largely absent, we will for now disregard this potential issue to reduce complexity.

\(^3\) Each trial of the Go/No-go task required the same response to any presented Go-stimulus (all latin letters except for „X”), but to omit this response whenever the No-go-stimulus („X“) was encountered. The proportion of Go-trials amounted to 80%. Due to the sensitivity of shifted Wald and Wiener modeling to fast responses whenever response tendencies are not allowed to vary across trials, all RTs < 150 ms were discarded prior to data analysis (cf. Matzke & Wagenmakers, 2009).
As Go-trials cannot yield erroneous response RTs because of a decision to omit the response at a certain point in time but only due to trial time-out (see section 2), regular shifted Wald and Wiener modeling could only be performed using the correct response RTs. Thus, we also needed to fix the response tendency parameter of the Wiener distribution to the proportion of Go-trials (i.e., $\beta = 0.8$), which enabled the conversion of boundary separations to decision thresholds ($\gamma = 0.2\alpha$). Upon presence of such pronounced response tendencies, the Wiener distribution posits that almost all erroneous response RTs are due to trial time-out, and thus transitions into the Wald distribution if drift rates and decision boundaries are sufficiently large (see Figure 2).

4.2. Results

Consistent with our prediction, we found that the parameter estimates obtained by regular shifted Wald and Wiener modeling corresponded almost perfectly across the whole sample (all $r$'s $>.99$), with a mean $\delta = 13.26$ sec$^{-1}$, $\theta = 0.03$ sec, and both $\gamma$'s $\geq 3.58$ (see Table 2). As expected, we also found that $\gamma_{450ms}$ was significantly larger than $\gamma_{350ms}$ [$t_{(100)} = 11.94, p < .001$], suggesting that the participants indeed lowered their decision boundaries in response to a faster time-out of Go-trials.

Table 2. Diffusion parameters as estimated from correct and erroneous Go-trial RTs by censored shifted Wald modeling, and from correct Go-trial RTs by shifted Wald/Wiener modeling.

| Parameters | censored shifted Wald modeling | shifted Wald/Wiener modeling |
|------------|--------------------------------|-----------------------------|
|            | $Y_{450ms}$                  | $Y_{350ms}$                  | $Y_{450ms}$                  | $Y_{350ms}$                  |
| Moments    |                                |                             |                             |                             |
| Mean       | 9.37                          | 2.32                        | 2.24                        | 0.09                        |
| SD         | 1.96                          | 1.02                        | 1.00                        | 0.07                        |
| Quantiles  |                                |                             |                             |                             |
| Min        | 4.89                          | 0.72                        | 0.58                        | 0.00                        |
| 25th       | 8.11                          | 1.42                        | 1.47                        | 0.01                        |
| Median     | 9.08                          | 2.25                        | 2.19                        | 0.09                        |
| 75th       | 10.39                         | 3.05                        | 2.91                        | 0.15                        |
| Max        | 15.90                         | 4.77                        | 4.96                        | 0.22                        |
| Correlations |                                |                             |                             |                             |
| (1) $\delta$ | 1                             |                             |                             |                             |
| (2) $\gamma_{450ms}$ | 0.82                          | 1                           |                             |                             |
| (3) $\gamma_{350ms}$ | 0.80                          | 0.98                        | 1                           |                             |
| (4) $\theta$ | -0.54                         | -0.88                       | -0.88                       | 1                           |
| (5) $\delta$ | 0.62                          | 0.29                        | 0.29                        | 0.05                        |
| (6) $\gamma_{450ms}$ | 0.56                          | 0.57                        | 0.55                        | -0.38                       |
| (7) $\gamma_{350ms}$ | 0.56                          | 0.53                        | 0.57                        | -0.38                       |
| (8) $\theta$ | -0.33                         | -0.56                       | -0.58                       | 0.70                        |

Notably, the estimates of drift rate were mostly larger as compared to previous findings using shifted Wald modeling of correct responses RTs in Go/No-go tasks (Trueblood et al., 2011). These accelerated drift rates are presumably attributable to the dominance of fast correct response RTs (i.e., slow correct response RTs are often discarded due to trial time-out), which is supported by comparably large error rates in the 350 ms condition ($M \pm SD: 0.25 \pm 0.14$) and to a smaller extent in the 450 ms condition ($M \pm SD: 0.04 \pm 0.03$).
Note. The upper decision boundaries of the Wiener distribution were calculated as γ = 0.2α. Dark grey cells contain estimates of relative parameter correspondence between both modeling approaches, whereas light grey cells contain stability estimates for the upper decision boundary across both conditions.

As the substantial number of errors in the 350 ms condition likely results in biased parameter estimates due to censoring (see section 3), we tried to account for both, correct response RTs and erroneous response RTs using censored shifted Wald modeling. The results of these analyses are listed in Table 2. As can be seen, the estimated mean drift rate decreased to 9.37 sec⁻¹, which agrees much better with previous findings (see Trueblood et al., 2011). Moreover, the change of the decision boundaries from the 450 ms to the 350 ms time-out condition remained to be significant [t(100) = 3.71, p < .001]. Finally, censored shifted Wald modeling seemed to better identify mean non-decision times as compared to regular shifted Wald or Wiener modeling [P(θ > 10 ms) = 74.2% versus P(θ > 10 ms) = 30.7%].

5. Precision of censored shifted Wald and Wiener modeling

Another important aspect for the utility of censored shifted Wald modeling relates to the number of RTs that are necessary to obtain comparable precise estimates of drift rate (δ), decision boundary (γ), and non-decision time (θ). Therefore we investigated by simulation how the precision of Wiener parameter recovery by censored shifted Wald modeling increases if the number of available RTs grows (see Appendix B). In order to compare these results with other methods to estimate the Wiener parameters, we also investigated the precision of three optimization routines provided by the fast-dm package (Voss & Voss, 2007) and the EZ algorithm (Wagenmakers et al., 2007), which was specifically developed to perform diffusion modeling of sparse chronometric data. Precision was quantified as coefficients of variation (CVs) of the respective parameter estimate [CV(φ) = SD(φ) / Mean(φ)]⁴. We consider a heuristic threshold of CV = 0.1 as indicative of a sufficiently precise parameter estimate.

The relation between the CVs of the recovered Wiener parameters and the number of available RTs is depicted in Figure 4. As can be seen, drift rate and non-decision time are quite precise when obtaining maximum likelihood (ML) estimates from at least 50 RTs. Precise estimates of the decision boundary, however, require approximately twice as many RTs. To our surprise, the EZ method performs worse as compared to ML estimates when estimating non-decision times. Moreover and consistent with our expectations (Voss et al., 2013), the Kolmogorov-Smirnov (KS) and the χ² approach do not yield precise estimates when only a moderately sized set of RTs is available. This relative inefficiency of both approaches has been suggested to go along with an increased robustness to outliers as compared to ML or EZ estimates (see Voss et al., 2013) – an observation that is consistent with previous findings reporting better recovery of boundary separations α and drift rates δ by the EZ algorithm as compared to χ² approaches (Van Ravenzwaaij & Oberauer, 2009). Proceeding from our analyses, we recommend to employ censored shifted Wald modeling using ML optimization only when approximately 100 RTs per condition have been obtained from a 2AFC task. However, this amount can probably be lowered upon switching to hierarchical distribution modeling (Farrell & Ludwig, 2008, see also Vandekerckhove et al., 2011).

⁴ CVs are scale-invariant and therefore easily interpretable across different parameter ranges. Moreover, the quantification of precision by means of CVs can account for the heteroscedasticity of parameter estimates that we observed with the recovery of drift rates and decision boundaries in Figure 2.
Figure 4. Relations between the number of RTs and the precision (CV) of the respective parameter estimate from censored shifted Wald modeling, the EZ algorithm (Wagenmakers et al., 2007), and three optimization routines of the fast-dm software package (Voss & Voss, 2007; ML = maximum likelihood, KS = Kolmogorov-Smirnov, $\chi^2$ = chi-square). All simulations are based on the following parameter set of the Wiener distribution: drift rate $\delta = 2.5$ sec$^{-1}$, response tendency $\beta = 0.5$, boundary separation $\alpha = \gamma / (1 - \beta) = 2$, and non-decision time $\theta = 0.55$ sec.
6. Conclusions, limitations and perspectives

In the present report, we have shown that the parameter estimates from shifted Wald modeling will correspond to Wiener parameters if the number of erroneous responses is low (~5%) due to high drift rates, conservative decision boundaries, and a-priori response tendencies towards the correct response alternative. As these conditions seem to be met in many 2AFC tasks (e.g., Gomez et al., 2007; Pratte et al., 2010; Gondan & Fimm, under review), shifted Wald parameters suggest analogous psychological interpretations of chronometric data, which can rarely be obtained by using other distributions (see Matzke & Wagenmakers, 2009; Balota & Yap, 2011). In contrast to modeling RTs using the Wiener distribution, shifted Wald modeling does not require erroneous responses if the probabilities of committing either response are independent. Moreover, approximately 100 correct response RTs suffice for obtaining comparably precise parameter estimates, which renders shifted Wald modeling a convenient and easily applicable tool for the analysis of many 2AFC tasks.

Nonetheless, our simulations also revealed that parameter estimates become biased if drift rates are slow, decision boundaries are liberal, or there is a strong response tendency towards the erroneous response alternative. All of these three sources of bias can potentially restrict the interpretability of shifted Wald parameters unless they are minimized by appropriate modifications of the task’s design (see Wagenmakers, 2009). Strikingly any experimental restriction of time window during which a response can be committed may also foster such biased parameter estimates by promoting a lowering of decision boundaries, and introducing a censoring mechanism that is informative for the complete distribution of correct response RTs and mimics erroneous responses. While bias that results from actual erroneous responses can be largely removed by the modeling of correct and error RTs as competing events that have undergone censoring, one can only partially account for such bias when the number of erroneous responses due to timeout is not negligible (and error RTs are consequently not available). Therefore it may be worth to consider allocating sufficient time for a correct processing of investigated tasks, that is, trials should not to be censored prior to task completion (Ulrich & Miller, 1994). Furthermore, overly liberal decision boundaries can be avoided by instructing the participants to focus on response accuracy (as opposed to speed).

Finally, we need to highlight that only one trial-invariant shifted Wald distribution may not completely account for empirical phenomena in 2AFC tasks, which has been the major reason for introducing across-trial variability of diffusion parameters in the first place (e.g., Ratcliff & Rouder, 1998). In consequence, it is not surprising that shifted Wald modeling will probably fail to recover Wiener parameters if such variability is explicitly assumed\(^5\) (Matzke & Wagenmakers, 2009). This issue was shown to particularly limit the applicability of (censored) shifted Wald modeling when non-decision times (and to a lesser extent response tendencies) vary substantially across trials. In such situations, one could either obtain adjusted estimates of Wald parameters using mixture modeling (Wagenmakers et al., 2008), and/or combine the outlined concept of censored RT modeling with shifted Wald distributions that incorporate such parameter variability across trials (see Logan et al., 2014, for the Wald distribution with across-trial variability in response tendencies). Thus censored shifted Wald modeling could actually provide similarly interpretable parameters as the Wiener diffusion model while bypassing the need to impose more or less arbitrary constraints on unidentifiable parameters.

---

\(^5\) As Wagenmakers and colleagues (2008) aptly remarked “...it is easy to generate data from a complex model and show that the simpler model, nested within the complex model, fails to recover parameters well”.
References

Balota DA, Yap MJ (2011). Moving beyond the mean in studies of mental chronometry: the power of response time distributional analyses. Current Directions in Psychological Science, 20, 160-166.

Cleveland, WS, Devlin SJ (1988). Locally weighted regression: an approach to regression analysis by local fitting. Journal of the American Statistical Association, 83, 596-610.

Farrell S, Ludwig CJH (2008). Bayesian and maximum likelihood estimation of hierarchical response time models. Psychonomic Bulletin & Review, 15, 1209-1217.

Gomez P, Ratcliff R, Perea M (2007). A model of the go/no-go task. Journal of Experimental Psychology: General, 136, 389-413.

Gondan M, Blurton SP, Kesselmeier M (2014). Even faster and even more accurate first-passage time densities and distributions for the Wiener diffusion model. Journal of Mathematical Psychology, 60, 20-22.

Gondan M, Fimm B (under review). Less biased response times.

Heathcote A (2004). Fitting Wald and ex-Wald distributions to response time data: an example using functions for the S-PLUS package. Behavior Research Methods, Instruments, & Computers, 36, 678-694.

Jones M, Dzhafarov EN (2014). Unfalsifiability and mutual translatability of major modeling schemes for choice reaction time. Psychological Review, 121, 1-32.

Logan GD, Van Zandt T, Verbruggen F, Wagenmakers EJ (2014). On the ability to inhibit thought and action: General and special theories of an act of control. Psychological Review, 121, 66-95.

Matzke D, Wagenmakers E-J (2009). Psychological interpretation of the ex-Gaussian and shifted Wald parameters: a diffusion model analysis. Psychonomic Bulletin & Review, 16, 798-817.

Miller RG (1998). Survival analysis. 2nd ed. John Wiley & Sons: New York.

Navarro DJ, Fuss IG (2009). Fast and accurate calculations for first-passage times in Wiener diffusion models. Journal of Mathematical Psychology, 53, 222-230.

Pratte MS, Rouder JN, Morey RD, Feng C (2010). Exploring the differences in distributional properties between Stoop and Simon effects using delta plots. Attention, Perception, & Psychophysics, 72, 2013-2025.

Putter H, Fiocco M, Geskus RB (2007). Tutorial in biostatistics: competing risks and multi-state models. Statistics in Medicine, 26, 2389-2430.

R Core Team (2014). R: A language and environment for statistical computing. R Foundation for Statistical Computing: Vienna.

Ratcliff R (1978). A theory of memory retrieval. Psychological Review, 85, 59-108.

Ratcliff R, Rouder JN (1998). Modeling response times for two-choice reaction times. Psychological Science, 9, 347-356.

Siannis F, Copas J, Lu G (2005). Sensitivity analysis for informative censoring in parametric survival models. Biostatistics, 6, 77-91.

Trueblood JS, Endres MJ, Busemeyer JR, Finn PR (2011). Modeling response times in a go/no-go discrimination task. Cognitive Science, 1866-1871.

Ulrich R, Miller J (1994). Effects of truncation on reaction time analysis. Journal of Experimental Psychology: General, 123, 34-80.

Vandekerckhove J, Tuerlinckx F, Lee MD (2011). Hierarchical diffusion models for two-choice response times. Psychological Methods, 16, 44-62.
Verbruggen F, Logan GD (2008). Automatic and controlled response inhibition: associative learning in the go/no-go and stop-signal paradigms. *Journal of Experimental Psychology: General*, 137, 649-672.

Van Ravenzwaaij D, Oberauer K (2009). How to use the diffusion model: Parameter recovery of three methods: EZ, fast-dm, and DMAT. *Journal of Mathematical Psychology*, 53, 463-473.

Voss A, Voss J (2007). Fast-dm: A free program for efficient diffusion model analysis. *Behavior Research Methods*, 39, 767-775.

Voss A, Nagler M, Lerche V (2013). Diffusion models in experimental psychology: a practical introduction. *Experimental Psychology*, 60, 385-402.

Wabersich D, Vandekerckhove J (2014). The RWiener package: an R package providing distribution functions for the Wiener diffusion model. *The R Journal*, 6, 49-56.

Wagenmakers E-J, van der Maas HLJ, Grasman RPPP (2007). An EZ-diffusion model for response time and accuracy. *Psychonomic Bulletin & Review*, 14, 3-22.

Wagenmakers E-J, van der Maas HLJ, Dolan CV, Grasman RPPP (2008). EZ does it! Extensions of the EZ-diffusion model. *Psychonomic Bulletin & Review*, 15, 1229-1235.

Wagenmakers E-J (2009). Methodological and empirical developments for the Ratcliff diffusion model of response times and accuracy. *European Journal of Cognitive Psychology*, 21, 641-671.
Appendix A

Shifted Wald distribution and likelihood function for censored shifted Wald modeling

Henceforth, the density function and the distribution function of the shifted Wald distribution are indicated by the subscripts pdf and cdf, respectively.

The likelihood function of any RT distribution that has undergone non-informative, right censoring (cf. Miller, 1998; Ulrich & Miller, 1994) is defined as the product of the probability density of n manifest data \( x_{1,...,n} \) (e.g. correct RTs) and the survival probability of m censored data \( y_{1,...,m} \) (e.g. error RTs):

\[
L(x_{1,...,n}, y_{1,...,m}) = \prod_{i=1}^{n} sWald_{pdf}(x_i) \prod_{j=1}^{m} [1 - sWald_{cdf}(y_j)]
\]

If the censoring mechanism is informative because of a competing risk to commit another response, the joint likelihood function is defined as the product of the two likelihood functions of both censored RTs distribution (cf. Putter et al., 2007; Miller, 1998):

\[
L(x_{1,...,n}, y_{1,...,m}) = \prod_{i=1}^{n} sWald_{pdf}(x_i) \prod_{j=1}^{m} [1 - sWald_{cdf}(y_j)] \prod_{j=1}^{m} sWald_{pdf}(y_j) \prod_{i=1}^{n} [1 - sWald_{cdf}(x_i)]
\]

While \( sWald_{pdf} \) is given in Eq. 2, \( sWald_{cdf} \) is provided below (Heathcote, 2004):

\[
sWald_{cdf}(t|\gamma, \delta, \theta) = \Phi \left( \frac{\delta t - \delta \theta - \gamma}{\sqrt{t - \theta}} \right) + \exp[2\gamma\delta] \Phi \left( - \frac{\delta t - \delta \theta + \gamma}{\sqrt{t - \theta}} \right)
\]

with \( \Phi \) denoting the distribution function of the unit normal distribution.

The integration of these equations provides the objective functions for (censored) shifted Wald modeling that can be optimized using common model fitting routines. Below, we provide functions for R statistical software (R Core Team, 2014) to estimate range-constrained shifted Wald parameters from manifest correct response RTs and censored correct response RTs (i.e., RTs from erroneous responses) using the Nelder-Mead algorithm:

```r
#pdf of sWald distribution
dswald <- function(x,d,a,t0,b){
  x <- x - t0; (a*(1-b)) / sqrt(2*pi*x^3) * exp(-(a*(1-b)-d*x)^2 / (2*x))
}

#cdf of sWald distribution
pswald <- function(x,d,a,t0,b) {
  x <- x - t0; pnorm((d*x-(a*(1-b)))/sqrt(x)) + exp(2*(a*(1-b))*d)*pnorm(-
(d*x+(a*(1-b)))/sqrt(x))
}
```
#negative loglikelihood (objective) functions for sWald modeling

#sWald without censoring
fobjswald <- function(x, rt_hit, b){
  x <- exp(x)
  -sum(log(dswald(rt_hit, x[1], x[2], x[3], b)))
}

#sWald with non-informative censoring
fobjcswald <- function(x, rt_hit, rt_miss, b){
  x <- exp(x)
  -sum(log(dswald(rt_hit, x[1], x[2], x[3], b))) -sum(log(1 - pswald(rt_miss, x[1], x[2], x[3], b)))
}

#sWald with informative censoring
fobjcswaldi <- function(x, rt_hit, rt_miss, b){
  x <- exp(x)
  -sum(log(dswald(rt_hit, x[1], x[2], x[3], b))) -sum(log(1 - pswald(rt_miss, x[1], x[2], x[3], b))) -sum(log(1 - pswald(rt_miss, -x[1], x[2], x[3], 1-b))) -sum(log(1 - pswald(rt_hit, -x[1], x[2], x[3], 1-b)))
}

#initial value function for sWald modeling (method of moments)
swstart <- function(x, p = 0.9) {
  t0 <- p*min(x); x <- x-t0; d <- sqrt(mean(x)/var(x)); a <- d*mean(x); c(d,a,t0)
}

#function to estimate sWald parameters from (un)censored RTs
swfit <- function(RT_hit, RT_miss, b=.5, scal=1000, inform=F){
  if(length(RT_miss) == 0){
    rt_hit <- RT_hit/scal
    temp <- optim(log(swstart(rt_hit)), fobjswald, rt_hit=rt_hit, b=b)
  }else{
    rt_hit <- RT_hit/scal
    rt_miss <- RT_miss/scal
    if(inform == F){
      start <- log(swstart(c(rt_hit,rt_miss)))
      temp <- optim(start, fobjcswald, rt_hit=rt_hit, rt_miss=rt_miss, b=b)
    }elseif(inform == T){
      start <- log(swstart(c(rt_hit,rt_miss)))
      temp <- optim(start, fobjcswaldi, rt_hit=rt_hit, rt_miss=rt_miss, b=b)
    }
  }
  return(list(param=exp(temp$par), loglik=-temp$value))
}
Appendix B

Monte-Carlo simulations of Wiener parameter recovery and its sensitivity and precision

All simulations were conducted using R 3.0.1 (R Core Team, 2014), the RWiener package (Wabersich & Vandekerckhove, 2014), and fast-dm (Voss & Voss, 2007). To investigate the recovery of the Wiener parameters by (censored) shifted Wald modeling, each parameter was separately varied across a range that was reported in previous studies employing the diffusion model (see Matzke & Wagenmakers, 2009, and Table 2).

Table B.1. Wiener parameters that were used for the simulations presented in Figure 2. All parameters varied in between their minima and maxima with 1,000 equally spaced steps.

| Wiener parameter          | Reference manifestation | Minimum | Maximum |
|---------------------------|-------------------------|---------|---------|
| drift rate $\delta$      | 2.50                    | 0.00    | 5.86    |
| boundary separation $\alpha$ | 2.00                    | 0.56    | 3.93    |
| non-decision time $\theta$ | 0.55                    | 0.21    | 0.94    |

Thereafter, each of the resulting parameter configurations was used to generate 1,000 random first-passage times of the respective Wiener diffusion process with two decision boundaries. The first-passage times of the upper decision boundary (i.e., the RTs of correct responses) were used to maximize the pdf-part of likelihood function for censored shifted Wald modeling. The first-passage times of the lower decision boundary (i.e., the RTs of erroneous responses) were considered as censored correct response RTs, whose likelihood was maximized with the survival function of the shifted Wald distribution. The parameter set maximizing the respective likelihood function was saved. The congruencies of these recovered parameters and the manipulated Wiener parameters are visualized in the Figures 2 and 3. The whole procedure was implemented for two different response tendency scenarios, that is, $\beta = 0.5$, reflecting an ambiguous response tendency in 2AFC tasks (as in Stroop tasks), and $\beta = 0.8$, reflecting the presence of a predominant response (as in Go/No-go tasks).

Analogously, the precision of censored shifted Wald modeling was evaluated by performing simulations. In contrast to the above outlined procedure, all Wiener parameters were held constant at their reference manifestation (see Table 2). However, the number of randomly generated first-passage times was systematically varied on the log-scale with 2,500 equally spaced steps, starting at 25 RTs and stopping at 250 RTs. Subsequently, the parameters’ conditional variances were predicted by the number of RTs using smoothing splines. Thereafter, these variances were used to calculate the respective CV($\phi$) visualized in Figure 4.