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The parameter section

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Contents
Bistability: a common feature in some ‘aggregates’ of logistic maps

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Abstract. As it was argued by Anderson [Science 177, 393 (1972)], the ‘reductionist’ hypothesis does not by any means imply a ‘constructionist’ one. Hence, in general, the behavior of large and complex aggregates of elementary components can not be understood nor extrapolated from the properties of a few components. Following this insight, we have simulated different ‘aggregates’ of logistic maps according to a particular coupling scheme. All these aggregates show a similar pattern of dynamical properties, concretely a bistable behavior, that is also found in a network of many units of the same type, independently of the number of components and of the interconnection topology. A qualitative relationship with brain-like systems is suggested.

Keywords: Bistability, coupled logistic oscillators, neural networks

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INTRODUCTION

One of the most challenging scientific problems today is to understand how the millions of neurons of our brain give rise to the emergent property of thinking [1]. Different aspects of neurocomputation take contact on this problem: how brain stores information and how brain processes it to take decisions or to create new information. Other universal properties of this system are more evident. One of them is the existence of a regular daily behavior: the sleep-wake cycle [2, 3]. The internal circadian rhythm is closely synchronized with the cycle of sun light. Roughly speaking and depending on the particular species, the brain is awake during the day and it is slept during the night, or vice versa. All mammals and birds sleep. There is no a well established law relating the size of the animal with the daily time it spends sleeping, but, in general, large animals tend to sleep less than small animals. Hence, at first sight, the bistable sleep-wake behavior seems not depend on the precise architecture of the brain nor on its size. If we represent the brain as a complex network this property would mean that this possible bistability should not depend on the topology (structure) nor on the number of nodes (size) of the network.

So, on one side, it has been recently argued in [4] that the distribution of functional connections \( p(k) \) in the human brain, where \( p(k) \) represents the probability of finding an element with \( k \) connections to other elements of the network, follows the same distribution of a scale-free network [5]. Thus, in that work [4], the human brain is divided in \( 34 \times 64 \times 64 \) sites (called voxels) and the magnetic resonance activity of all the voxels is recorded. Let us observe at this point that if the human brain has around \( 10^{11} \) neurons, then a voxel has around \( 10^5 \) neurons. The calculation of the correlation matrix among the set of the voxels activities shows a power law behavior, \( p(k) \sim k^{-\gamma} \), with \( \gamma \) around 2. This finding means that there are regions in the brain that participate in a large number of tasks while most of the regions are only involved in a tiny fraction of the brain’s activities.

On the other side, it has been shown by Kuhn et al. [6] the nonlinear processing of synaptic inputs in cortical neurons. They studied the response of a model neuron with a simultaneous increase of excitation and inhibition. They found that the firing rate of the model neuron first increases, reaches a maximum, and then decreases at higher input rates. Functionally, this means that the firing rate, commonly assumed to be the carrier of information in the brain, is a non-monotonic function of balanced input. These findings do not depend on details of the model and, hence, are relevant to cells of other cortical areas as well.

Putting together all these facts, we arrive to the central question that we want to bring to the reader: are we able to reproduce the bistability in a complex network independently of the topology and of the number of nodes? The answer is ‘yes’ (Fig. 1). What kind of local dynamics and coupling among nodes must be implemented in order to get this behavior? In the next section, we give a possible strategy for the coupling and the local dynamics which should be implemented in a few or many units network in order to find bistable behavior. An example of this implementation
Figure 1. Question: Is it possible to implement some kind of coupling and nonlinear dynamics in each node of a complex network in order to get bistability? Answer: Yes.

is given in [7]. Here we center our attention in the bistability present in the case of a few coupled functional units. Four different discrete models in two and three dimensions are collected. In view of the results, we want to suggest with all these examples the possibility of ‘constructivism’ in the world of complex systems.

Models of a Few Coupled Functional Units

General model

Our approach consider the so called functional unit, i.e. a neuron or group of neurons (voxels), as a discrete nonlinear oscillator with two possible states: active (meaning one type of activity) or not (meaning other type of activity). Hence, in this naive vision of the brain as a networked system, if \( x_i^n \), with \( 0 < x_i^n < 1 \), represents a measurement of the \( ith \) functional unit activity at time \( n \), it can be reasonable to take the most elemental local nonlinearity, for instance, a logistic evolution [8], which presents a quadratic term, as a first toy-model for the local neuronal activity:

\[
x_{i,n+1} = \bar{p}_i x_i^n (1 - x_i^n).
\]  

(1)

Figure 2. Discrete nonlinear model for the local evolution of a functional unit.

It presents only one stable state for each \( \bar{p}_i \). Then, there is no bistability in the basic component of our models. For \( \bar{p}_i < 1 \), the dynamics dissipates to zero, \( x_i^n = 0 \), then it can represent the functional unit with no activity. For \( 1 < \bar{p}_i < 4 \), the dynamics is non null and it would represent an active functional unit. This local transition is controlled by the parameter \( \bar{p}_i \). The functional dependence of this local coupling on the neighbor states is essential in order to get a good brain-like behavior (i.e., as far as the bistability of the sleep-wake cycle is concerned) of the network. As a first approach, we can take \( \bar{p}_i \) as a linear function depending on the actual mean value, \( X_i^n \), of the neighboring signal activity and expanding the interval \((1, 4)\) in the form:

\[
\bar{p}_i = p_i (3X_i^n + 1), \quad \text{(excitation coupling)}
\]

or

\[
\bar{p}_i = p_i (-3X_i^n + 4), \quad \text{(inhibition coupling)}
\]

(2)  

(3)

with

\[
X_i^n = \frac{1}{N_i} \sum_{j=1}^{N_i} x_{j,n}.
\]  

(4)

\( N_i \) is the number of neighbors of the \( ith \) functional unit, and \( p_i \), which gives us an idea of the interaction of the functional unit with its first-neighbor functional units, is the control parameter. This parameter runs in the range \( 0 < p_i < p_{\text{max}} \), where \( p_{\text{max}} \geq 1 \). When \( p_i = p \) for all \( i \), the dynamical behavior of these networks with the excitation type coupling [7] presents an attractive global null configuration that has been identified as the turned off state of the network. Also they show a completely synchronized non-null stable configuration that represents the turned on state of the network. Moreover, a robust bistability between these two perfect synchronized states is found in that particular model (see [7] for more details). For different models with a few coupled functional units we sketch in the
Models of two functional units

Let us start with the simplest case of two interconnected \((x_n, y_n)\) functional units. Three different combinations of couplings are possible: \textit{(excitation, excitation)}, \textit{(excitation, inhibition)} and \textit{(inhibition, inhibition)}.

\[
\begin{align*}
x_{n+1} &= p (3y_n + 1)x_n(1-x_n), \\
y_{n+1} &= p (3x_n + 1)y_n(1-y_n).
\end{align*}
\]

The first two cases of coupling present bistability in different regions of the parameter space. The third case also shows bistability in a very narrow interval of parameter space but it requires a very fine inspection \[\text{(12)}\] that we shelve for a further work.

\textit{Model with mutual excitation}

The dynamics of the \textit{(excitation, excitation)} case \[\text{(9)}\] is given by the coupled equations:

\[
\begin{align*}
x_{n+1} &= p (3y_n + 1)x_n(1-x_n), \\
y_{n+1} &= p (3x_n + 1)y_n(1-y_n).
\end{align*}
\]

The regions of the parameter space (Fig. 4) where we can find bistability are:

- For \(0.75 < p < 0.86\), the synchronized state, \(x_0 = (\bar{x}, \bar{x}) = P_4\), with \(\bar{x} = \frac{1}{3}\{1 + (4 - \frac{3}{p})^{\frac{1}{2}}\}\), which arises from a saddle-node bifurcation for the critical value \(p = 0.75\), is a stable turned on state. This state coexists with the turned off state \(x_0 = 0\). The system presents now bistability and depending on the initial conditions, the final state can be \(x_0\) or \(x_+\). Switching on the system from \(x_0\) requires a level of noise in both functional units sufficient to render the activity on the basin of attraction of \(x_+\). On the contrary, switching off the two functional units network can be done, for instance, by making zero the activity of one functional unit, or by doing the coupling \(p\) lower than 0.75.

- For \(0.86 < p < 0.95\), the active state of the network is now a period-2 oscillation, namely the period-2 cycle \(P_5, P_6\) in Fig. 4. This new dynamical state bifurcates from \(x_+\) for \(p = 0.86\). A smaller noise is necessary to activate the system from \(x_0\). Making zero the activity of one functional unit continues to be a good strategy to turn off the network.

- For \(0.95 < p < 1\), the active state acquires a new frequency and presents quasiperiodicity (the invariant closed curves of Fig. 4). It is still possible to switch off the network by putting to zero one of the functional units.

\textbf{FIGURE 3.} Two functional coupled units.

\textbf{FIGURE 4.} Bistability in 2 functional units with excitation type coupling.
Model with excitation + inhibition

The dynamics of the \((excitation, inhibition)\) case \([10]\) is given by the coupled equations:

\[
\begin{align*}
x_{n+1} &= p (3y_n + 1)x_n(1 - x_n), \quad (7) \\
y_{n+1} &= p (-3x_n + 4)y_n(1 - y_n). \quad (8)
\end{align*}
\]

The regions of the parameter space (Fig. 5) where we can find bistability are:

- For \(1.051 < p < 1.0851\), a stable period three cycle \((Q_1, Q_2, Q_3)\) appears in the system. It coexists with the fixed point \(P_1\). When \(p\) is increased, a period-doubling cascade takes place and generates successive cycles of higher periods \(3 \times 2^n\). The system presents bistability. Depending on the initial conditions, both populations \((x_n, y_n)\) oscillate in a periodic orbit or, alternatively, settle down in the fixed point. The borders between the two basins are complex.

- For \(1.0851 < p < 1.0997\), an aperiodic dynamics is possible. The period-doubling cascade has finally given birth to an order three cyclic chaotic band(s) \((A_{31}, A_{32}, A_{33})\). The system can now present an irregular oscillation besides the stable equilibrium with final fixed populations. The two basins are now fractal.

![FIGURE 5. Bistability in 2 functional units with excitation+inhibition type coupling.](image)

Models of three functional units

Following the strategy given by relation \(2\) several models with three functional units can be established. We have studied in detail two of them \([11]\) and their bistable behavior is reported here.

Model with local mutual excitation

Let us start with the case of three alternatively interconnected \((x_n, y_n, z_n)\) functional units under a mutual excitation scheme.

\[
\begin{align*}
x_{n+1} &= p (3y_n + 1)x_n(1 - x_n), \quad (9) \\
y_{n+1} &= p (3z_n + 1)y_n(1 - y_n), \quad (10) \\
z_{n+1} &= p (3x_n + 1)z_n(1 - z_n). \quad (11)
\end{align*}
\]

![FIGURE 6. Three alternatively coupled functional units under the excitation scheme.](image)
The regions of the parameter space where we have found bistability are:

- For $0.93310 < p < 0.95334$, a big invariant closed curve (ICC) $C_1$ coexists with a period-3 orbit that bifurcates, first to an order-3 cyclic ICC (Fig. 7), and finally to an order-3 weakly chaotic ring (WCR) before disappearing.
- For $0.98418 < p < 0.98763$, the ICC $C_1$ coexists with another ICC $C_2$ (see Ref. [11]) that becomes chaotic, by following a period doubling cascade of tori, before disappearing.
- For $1.00360 < p < 1.00402$, the ICC $C_1$ coexists with a high period orbit that gives rise to an ICC $C_3$. This ICC also becomes a chaotic band (see Ref. [11]) by following a period doubling cascade of tori before disappearing.

**Model with global mutual excitation**

We expose now the case of three globally interconnected $(x_n, y_n, z_n)$ functional units under a mutual excitation scheme.

\[
\begin{align*}
x_{n+1} &= p \left( x_n + y_n + z_n + 1 \right) x_n \left( 1 - x_n \right), \\
y_{n+1} &= p \left( x_n + y_n + z_n + 1 \right) y_n \left( 1 - y_n \right), \\
z_{n+1} &= p \left( x_n + y_n + z_n + 1 \right) z_n \left( 1 - z_n \right).
\end{align*}
\]

**Figure 8.** Three globally coupled functional units under the excitation scheme.

Then the dynamics of the system is given by the coupled equations:
For the whole range of the parameter, $0 < p < 1.17$, bistability is present in this system:

- Firstly, two order-2 cyclic ICC coexist before becoming two order-2 cyclic chaotic attractors by contact bifurcations of heteroclinic type. Finally the two chaotic attractors become a single one before disappearing.

**CONCLUSIONS**

One of the more challenging problems in nonlinear science is the goal of understanding the properties of neuronal circuits [13]. Synchrony and bistability are two important dynamical behaviors found in those circuits. In this work, different coupling schemes for networks with local logistic dynamics are proposed. It is observed that these types of couplings generate a global bistability between two different dynamical states. This property seems to be topology and size independent. This is a direct consequence of the local mean-field multiplicative coupling among the first-neighbors. If a formal and naive relationship is established between these two states and the sleep-wake states of a brain, respectively, one would be tempted to assert that these types of couplings in a network, regardless of its simplicity, give us a good qualitative model for explaining that specific bistability. Following this insight, different low-dimensional systems with logistic components coupled under these schemes have been presented. The regions where the dynamics shows bistability have been identified. Other low and high dimensional models merit a similar detailed inspection in the future. This study could put in evidence the possibility of ‘constructivism’ in the world of complex systems.

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