THE MOMENT OF INSTABILITY FOR INTERNAL SOLITARY WAVES

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ABSTRACT. In this note, we define a moment of instability $m(c)$ for internal solitary waves in continuously stratified fluids, which seems not to have been done before. To underline the suitability of the proposed $m(c)$, we identify the relation $m''(c) = 0$ as a formal Fredholm condition, and we show that $m''(c)$ displays a definite sign for small-amplitude waves.

1. INTRODUCTION

Internal solitary waves (ISWs) are ecologically important since they are involved in mixing mechanisms and energy transport in lakes and oceans [2, 5, 14, 13]. In this context, a widely used mathematical model consists of the 2D Euler equations for incompressible, inviscid fluids with non-constant density. This model is given by the equations

\begin{align}
\rho_t &= -u\rho_x - v\rho_y, \\
u_t &= -u\rho_x - v\rho_y - \frac{p_x}{\rho}, \\
v_t &= -u\rho_x - v\rho_y - \frac{p_y}{\rho} - g,
\end{align}

complemented by the incompressibility constraint

\begin{equation}
0 = u_x + v_y,
\end{equation}

the boundary conditions

\begin{equation}
v(t, x, 0) = 0 \quad \text{and} \quad v(t, x, 1) = 0,
\end{equation}

and the far-field conditions

\begin{equation}
(\rho, u, v, p)(t, \pm\infty, y) = (\bar{\rho}(y), 0, 0, \bar{p}(y)), \quad 0 \leq y \leq 1.
\end{equation}

In (1.1), density $\rho$, velocity $(u, v)$, and pressure $p$ are functions of time $t$, horizontal position $x \in \mathbb{R}$ and vertical position $y \in [0, 1]$, and the constant $g$
denotes acceleration due to gravity. The far-field \((\bar{\rho}(y), 0, 0, \bar{\rho}(y))\), itself an \(x\)- and \(t\)-independent solution of (1.1a)-(1.1e) with

\[
\bar{\rho} : [0, 1] \to (0, \infty) \text{ differentiable with } \bar{\rho}'(y) < 0, \ 0 \leq y \leq 1,
\]

and \(\bar{\rho}(y) = -g \int_0^y \bar{\rho}(\eta) \, d\eta\), is called the \textit{quiescent state}. Travelling wave solutions

\[
(\rho, u, v, p)(t, x, y) = (\hat{\rho}, \hat{u}, \hat{v}, \hat{p})(x - ct, y), \quad \text{with some } c > 0,
\]

of (1.1) are called \textit{internal solitary waves} (ISWs) of speed \(c\); we refer to [8, 12, 7] for mathematical results on their existence.

In order to study the stability of ISWs, one could start from the linearization of (1.1) about a given ISW, as done by the author in [10, 11] to find an Evans-function approach to stability.

A different general approach to investigate the stability of solitary waves is based on the moment of instability, see e. g. [6], and references therein. Here, we want to establish the moment-of-instability (MOI) route to stability of ISWs.

2. Definition of a moment of instability \(m(c)\) for ISWs

According to [3], the Euler equations (1.1) for stratified fluids possess a Hamiltonian formulation. In terms of the density \(\rho\), the vorticity-like quantity \(\sigma\), and the associated streamfunction \(\psi\), defined as the solution of

\[
\sigma = -\nabla \cdot (\rho \nabla \psi), \quad \text{with } \psi|_{y=0,1} = 0,
\]

this can be formulated as

\[
\partial_t \begin{pmatrix} \rho \\ \sigma \end{pmatrix} = \mathcal{J}(\rho, \sigma) \left( \tilde{H} - c \tilde{I} \right)'(\rho, \sigma)
\]

in the co-moving frame \(t, \tilde{x} = x - ct, y\) with writing \(x\) instead of \(\tilde{x}\), where the Hamiltonian \(\tilde{H} - c \tilde{I}\) is composed of the energy functional

\[
\tilde{H}(\rho, \sigma) = \int_{\mathbb{R}} \int_0^1 \frac{1}{2} \rho |\nabla \psi|^2 + g y (\rho - \bar{\rho}) \, dy \, dx
\]

and the momentum functional

\[
\tilde{I}(\rho, \sigma) = \int_{\mathbb{R}} \int_0^1 y \sigma \, dy \, dx,
\]

and \(\mathcal{J} = \mathcal{J}(\rho, \sigma)\) denotes the (state-dependent!) skew-symmetric operator

\[
\mathcal{J}(\rho, \sigma) = \begin{pmatrix} 0 & -\rho_x \partial_y + \rho_y \partial_x \\ -\rho_x \partial_y + \rho_y \partial_x & -\sigma_x \partial_y + \sigma_y \partial_x \end{pmatrix}.
\]

The Hamiltonian formulation (2.1), however, does not directly yield a variational principle due to the non-invertibility of \(\mathcal{J}\). Concretely, as a stationary solution of (2.1) an ISW \((\rho^c, \sigma^c)\) satisfies

\[
0 = \mathcal{J}(\rho^c, \sigma^c) \left( \tilde{H} - c \tilde{I} \right)'(\rho^c, \sigma^c)
\]
but, as a little calculation reveals (see, e. g., [3, p. 35]),

\[
(\tilde{H} - c\tilde{I})' (\rho^c, \sigma^c) = \left( gy - \frac{1}{\psi^c} |\nabla \psi^c|^2 \right) \neq 0,
\]

i. e., \((\rho^c, \sigma^c)\) is not a critical point of \(\tilde{H} - c\tilde{I}\)!

This issue can be overcome by modifying \(\tilde{H} - c\tilde{I}\) without spoiling the Hamiltonian structure. In fact, taking the quantities

\[
\Delta H(\rho, \sigma) := -\int_0^1 \int g \left\{ \int_{\tilde{\rho}(y)} \bar{\rho}^{-1}(y) \ d\bar{\rho} \right\} \sigma \ dy \ dx,
\]

\[
\Delta I(\rho, \sigma) := -\int_0^1 \int \bar{\rho}^{-1}(\rho) \sigma \ dy \ dx,
\]

it is easily verified that

\[
J(\rho, \sigma) (\Delta H - c\Delta I)' (\rho, \sigma) = 0 \quad \text{and} \quad (\tilde{H} - c\tilde{I})' (\rho^c, \sigma^c) = 0
\]

with \(H := \tilde{H} + \Delta H\) and \(I = \tilde{I} + \Delta I\). Therefore, replacing \(\tilde{H} - c\tilde{I}\) with

\[
H - cI \equiv \int_0^1 \frac{1}{2} \rho |\nabla \psi|^2 + \int g \left\{ y - \tilde{\bar{\rho}}^{-1}(y) \right\} \ dy \ dx
\]

\[
- c \int_0^1 \int \left\{ y - \tilde{\bar{\rho}}^{-1}(y) \right\} \sigma \ dy \ dx,
\]

results in a modified Hamiltonian formulation such that ISWs are, indeed, critical points of the Hamiltonian. This was already noticed by [1, 6] but, as far as the author is aware, has not been used in connection with the stability of ISWs. For background material on so-called Casimir functionals, for which \(\Delta H - c\Delta I\) is an example, their systematic derivation and their use in hydrodynamic contexts, see [1] and references therein.

Now, we are in a position to define the moment of instability for ISWs in the usual way:

\[
m(c) := (H - cI) (\rho^c, \sigma^c).
\]

Since \((H - cI)' (\rho^c, \sigma^c) = 0\) by construction, we immediately have the usual relation

\[
m''(c) \equiv \frac{d^2}{dc^2} (H - cI) (\rho^c, \sigma^c) = -\frac{d}{dc} I(\rho^c, \sigma^c).
\]

In the rest of the paper, we study this \(m(c)\). In Sec. 2 we show that \(m''(c) < 0\) for ISWs of sufficiently small amplitude. In Sec. 3 we show in a quite general situation, which covers ours, that the condition \(m''(c) = 0\) can be read as a formal Fredholm condition.

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3. Proving $m''(c) < 0$ for small ISWs

For small waves\footnote{We assume here the genericity condition $\int_0^1 \bar{\rho}(y) \varphi_0^3(y) \, dy \neq 0$ which is necessary for the validity of the approximate expressions; cf. \cite{9}.} we have \cite{4, 7}

\[ c = c_0 + \varepsilon^2, \]

\[ \psi^c(x, y) = \varepsilon^2 A(\varepsilon x) \varphi_0(y) + O(\varepsilon^4), \]

\[ \rho^c(x, y) = \bar{\rho}(y) - \frac{1}{c_0} \varepsilon^2 A(\varepsilon x) \bar{\rho}'(y) \varphi_0(y) + O(\varepsilon^4), \]

where

\[ A''(X) = -\frac{1}{s} A(X) - \frac{r}{s} A(X)^2 \quad \text{and} \quad (\bar{\rho}(y) \varphi_0^3(y))^' = \frac{9}{c_0} \bar{\rho}'(y) \varphi_0(y). \]

With these expressions at hand, it is straightforward to evaluate $m''(c)$.

\[
\mathcal{I}(\rho^c, \sigma^c) = \frac{1}{c} \int_{\mathbb{R}} \int_0^1 \rho^c |\nabla \psi^c|^2 \, dy \, dx \\
= \frac{1}{c} \int_{\mathbb{R}} \int_0^1 \left( \bar{\rho}(y) - \frac{1}{c_0} \varepsilon^2 A(\varepsilon x) \bar{\rho}'(y) \varphi_0(y) + O(\varepsilon^4) \right) \\
\times \left( (\varepsilon^3 A'(\varepsilon x) \varphi_0(y))^2 + (\varepsilon^2 A(\varepsilon x) \varphi_0^3(y))^2 + O(\varepsilon^5) \right) \, dy \, dx \\
= \frac{\varepsilon^4}{c_0} \int_{\mathbb{R}} \int_0^1 \bar{\rho}(y) A(\varepsilon x)^2 \varphi_0'(y)^2 \, dy \, dx + O(\varepsilon^5) \\
= \frac{\varepsilon^4}{c_0} \int_{\mathbb{R}} A(\varepsilon x)^2 \, dx \int_0^1 \bar{\rho}(y) \varphi_0'(y)^2 \, dy + O(\varepsilon^5) \\
= \varepsilon^3 \left( \frac{1}{c_0} \int_{\mathbb{R}} A(X)^2 \, dX \int_0^1 \bar{\rho}(y) \varphi_0'(y)^2 \, dy + O(\varepsilon^5) \right) \\
= K (c - c_0)^{\frac{3}{2}} + O \left( (c - c_0)^{\frac{5}{2}} \right)
\]

with the finite, positive constant

\[ K := \frac{1}{c_0} \int_{\mathbb{R}} A(X)^2 \, dX \int_0^1 \bar{\rho}(y) \varphi_0'(y)^2 \, dy > 0. \]

Hence, we derive that

\[ m''(c) = -\frac{d}{dc} \mathcal{I}[\rho^c, \sigma^c] = -\frac{3}{2} K (c - c_0)^{\frac{3}{2}} + O \left( (c - c_0)^{\frac{5}{2}} \right) < 0 \]

holds for $0 \leq c - c_0 \ll 1$, i.e., for sufficiently small waves.

4. Characterizing $m''(c) = 0$ as a Fredholm condition

To simplify the notation, we write $\phi = (\rho^c, \sigma^c)$ for the ISW in the following. In the situation above, differentiating the profile equation

\[ (\mathcal{H} - c \mathcal{I})' (\phi) = 0 \]

with respect to the position yields

\[ (\mathcal{H} - c \mathcal{I})'' (\phi) \frac{\partial \phi}{\partial x} = 0, \]
while differentiating it with respect to the speed results in

\begin{equation}
(H - cI)''(\phi) \frac{\partial \phi}{\partial c} = I'(\phi).
\end{equation}

Eqs. (4.2), (4.3) give

\begin{equation}
J(H - cI)''(\phi) \frac{\partial \phi}{\partial x} = 0
\end{equation}
and

\begin{equation}
J(H - cI)''(\phi) \frac{\partial \phi}{\partial c} = JI'(\phi).
\end{equation}

As

\begin{equation}
JI'(\phi) = -\frac{\partial \phi}{\partial x},
\end{equation}
eqs. (4.4) and (4.5) state that 0 is an at least double eigenvalue for

\[ \dot{u} = J(H - cI)'(u), \]

with $\frac{\partial \phi}{\partial x}$ as an eigenfunction and $\frac{\partial \phi}{\partial c}$ as a first-order generalized eigenfunction.

Now, a second-order generalized eigenfunction $\psi$ would solve

\begin{equation}
J(H - cI)''(\phi)\psi = \frac{\partial \phi}{\partial c}.
\end{equation}

According to the Fredholm alternative, eq. (4.7) has a non-trivial solution if and only if its right hand side $\frac{\partial \phi}{\partial c}$ is orthogonal to the solution $\chi$ of the adjoint homogeneous equation

\begin{equation}
0 = (J(H - cI)''(\phi))^*\chi = -((H - cI)''(\phi)J)\chi.
\end{equation}

As (4.2) and (4.6) imply

\begin{equation}
0 = -((H - cI)''(\phi)J)I'(\phi) \quad \text{and thus} \quad \chi = I'(\phi),
\end{equation}
the existence of $\psi \neq 0$ consequently is equivalent to

\begin{equation}
0 = \frac{d}{dc} I'(\phi) = \left< I'(\phi), \frac{\partial \phi}{\partial c} \right>,
\end{equation}
i. e., vanishing of the moment of instability.

Remarks. (i) The above argument slightly varies the one given by Zumbrun in [16] (Sec. 1, between the statements of Corollary 1.4 and Remark 1.5).
(ii) This argument literally applies to the situation of Grillakis et. al. [6] by changing to their notation

\[ E = H, Q = I, J = J, \phi = (\rho^c, \sigma^c), T'(0) = \partial_x. \]

Hence, it can be applied to various contexts that fall into this class.
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