The dual complex of Calabi–Yau pairs

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Abstract A log Calabi–Yau pair consists of a proper variety $X$ and a divisor $D$ on it such that $K_X + D$ is numerically trivial. A folklore conjecture predicts that the dual complex of $D$ is homeomorphic to the quotient of a sphere by a finite group. The main result of the paper shows that the fundamental group of the dual complex of $D$ is a quotient of the fundamental group of the smooth locus of $X$, hence its pro-finite completion is finite. This leads to a positive answer in dimension $\leq 4$. We also study the dual complex of degenerations of Calabi–Yau varieties. The key technical result we prove is that, after a volume preserving birational equivalence, the transform of $D$ supports an ample divisor.

A log Calabi–Yau pair, abbreviated as logCY, is a pair $(X, \Delta)$ consisting of a proper variety $X$ and an effective $\mathbb{Q}$-divisor $\Delta$ such that $(X, \Delta)$ is log canonical and $K_X + \Delta$ is $\mathbb{Q}$-linearly equivalent to 0. Any Calabi–Yau variety $X$ can be naturally identified with the log Calabi–Yau pair $(X, 0)$. At the other extreme, if $X$ is a Fano variety and $\Delta \sim_{\mathbb{Q}} -K_X$ is an effective divisor then $(X, \Delta)$ is also logCY (provided that it is log canonical).

Here we are interested in these Fano–type logCYs, especially when $\Delta$ is “large.” Being Fano is not preserved under birational equivalence, thus it is better to define “large” without reference to Fano varieties. There are several...
natural candidates for this notion; we were guided by the concepts of large complex structure limit and maximal unipotent degeneration used in Mirror Symmetry.

**Definition 1** Let \((X, \Delta)\) be a log canonical pair of dimension \(n\) and \(g : (Y, \Delta_Y) \to (X, \Delta)\) a crepant log resolution. That is, \(g_* \Delta_Y = \Delta, K_Y + \Delta_Y \sim_{\mathbb{Q}} g^*(K_X + \Delta)\), \(X\) is smooth and \(\text{Ex}(g) \cup \text{Supp} \Delta_Y\) is a simple normal crossing divisor.

Note that \(\Delta_Y\) is usually not effective but, since \((X, \Delta)\) is log canonical, all divisors appear in \(\Delta_Y\) with coefficient \(\leq 1\). Let \(\Delta_Y^{-1}\) denote the union of all irreducible components of \(\Delta_Y\) whose coefficient equals 1.

The combinatorics of \(\Delta_Y^{-1}\) is encoded in its dual complex, denoted by \(D(\Delta_Y^{-1})\); see Definition 12. By [6] \(D(\Delta_Y^{-1})\) is independent of the choice of \((Y, \Delta_Y)\), up-to PL-homeomorphism. We call this PL-homeomorphism type the dual complex of \((X, \Delta)\) and denote it by \(\mathcal{DMR}(X, \Delta)\).

Note that \(\dim_{\mathbb{R}} \mathcal{DMR}(X, \Delta) \leq \dim X - 1\) since, on a variety of dimension \(n\), at most \(n\) irreducible components of a simple normal crossing divisor meet at a point. We say that \((X, \Delta)\) has maximal intersection if equality holds.

By [29], every finite simplicial complex of dimension \(n - 1\) appears as \(D(X, \Delta)\) for some \(n\)-dimensional simple normal crossing pair \((X, \Delta)\). Thus it is interesting to understand which algebraic restrictions on \((X, \Delta)\) have meaningful topological consequences for \(\mathcal{DMR}(X, \Delta)\). The aim of this paper is to study the dual complex of logCY pairs. The main result is the following.

**Theorem 2** Let \((X, \Delta)\) be a logCY pair and \(\mathcal{DMR}(X, \Delta)\) its dual complex. Assume that \(\dim_{\mathbb{R}} \mathcal{DMR}(X, \Delta) \geq 2\). Then the following hold.

1. \(\mathcal{DMR}(X, \Delta)\) has the same dimension at every point.
2. \(H^i(\mathcal{DMR}(X, \Delta), \mathbb{Q}) = 0\) for \(0 < i < \dim_{\mathbb{R}} \mathcal{DMR}(X, \Delta)\).
3. There is a natural surjection \(\pi_1(X^{\text{sm}}) \to \pi_1(\mathcal{DMR}(X, \Delta))\).
4. The pro-finite completion \(\hat{\pi}_1(\mathcal{DMR}(X, \Delta))\) is finite.
5. The cover \(\mathcal{DMR}(X, \Delta) \to \mathcal{DMR}(X, \Delta)\) corresponding to \(\hat{\pi}_1(\mathcal{DMR}(X, \Delta))\) is the dual complex of a quasi-étale cover \((\tilde{X}, \tilde{\Delta}) \to (X, \Delta)\).

Part (1) is a restatement of earlier results; see [27] or [28, 4.40]. The relationship between the rational homology of the dual complex and the coherent cohomology of \(X\) has been understood for a long time; thus (2) has been known in many cases. Our main contribution is to understand the connection between the fundamental group of the dual complex and the fundamental group of the smooth locus \(X^{\text{sm}} \subset X\). The main conclusion is (3) which in turn implies (5). The finiteness of \(\hat{\pi}_1(X^{\text{sm}})\) follows from [11, 36]; it is conjectured that \(\pi_1(X^{\text{sm}})\) is finite.
If $\dim_{\mathbb{R}} \mathcal{D} \mathcal{M} \mathcal{R}(X, \Delta) = 1$ then $\mathcal{D} \mathcal{M} \mathcal{R}(X, \Delta)$ is either the 1-simplex $[0, 1]$ or $S^1$. In the latter case $\pi_1(\mathcal{D} \mathcal{M} \mathcal{R}(X, \Delta))$ is infinite but usually $X^{\text{sm}}$ has only trivial quasi-étale covers.

On a general logCY pair, $\Delta$ contains divisors with different coefficients, but it turns out that $\mathcal{D} \mathcal{M} \mathcal{R}(X, \Delta)$ is rather special if not all coefficients are equal to 1.

**Proposition 3** Let $(X, \Delta)$ be a logCY pair such that $\Delta$ contains at least one divisor with coefficient $< 1$. Then

1. either $\mathcal{D} \mathcal{M} \mathcal{R}(X, \Delta)$ is contractible
2. or $\dim_{\mathbb{R}} \mathcal{D} \mathcal{M} \mathcal{R}(X, \Delta) \leq \dim_{\mathbb{C}} X - 2$.

It is straightforward to write down examples where $\mathcal{D} \mathcal{M} \mathcal{R}(X, \Delta)$ is contractible or a sphere. More generally, at least some quotients of spheres are easy to get; see Sect. 8. This leads naturally to the following.\(^1\)

**Question 4** Let $(X, \Delta)$ be a logCY pair of dimension $n$. Is $\mathcal{D} \mathcal{M} \mathcal{R}(X, \Delta) \simeq S^{k-1}/G$ for some finite subgroup of $G \subset O_k(\mathbb{R})$ and $k \leq n$. (We use $\simeq$ to denote PL-homeomorphism.)

The group action may have fixed points, thus in general $\mathcal{D} \mathcal{M} \mathcal{R}(X, \Delta)$ is only an orbifold. LogCY varieties also appear as compactifications of character varieties. In this context, Question 4 is studied in [14, 34]. We prove the following in Paragraph 33.

**Proposition 5** The answer to Question 4 is positive if $\dim X \leq 4$ or if $\dim X \leq 5$ and $(X, \Delta)$ is a simple normal crossing pair.

In many questions involving Mori’s program, the 3 and 4 dimensional cases are good indicators of the general situation. However, the proof of Proposition 5 relies on several special low dimensional topological facts, thus it gives only very weak evidence for the general problem. We do not see any good heuristic reason why the answer to Question 4 should be affirmative. From the technical point of view, at least 3 problems remain to be settled.

- Finiteness of the fundamental group of $\mathcal{D} \mathcal{M} \mathcal{R}(X, \Delta)$. We do not know how to prove it but Theorem 2.3 reduces it to the finiteness of $\pi_1(X^{\text{sm}})$.
- Torsion in the integral homology of $\mathcal{D} \mathcal{M} \mathcal{R}(X, \Delta)$. Our methods do not say anything about it.
- Starting in dimension 5 we have to deal with the possibility that $\mathcal{D} \mathcal{M} \mathcal{R}(X, \Delta)$ is singular but $X$ itself has no obvious quasi-étale covers. This seems to us the most likely approach to construct a counter example to Question 4.

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\(^1\) Many people seem to have been aware of this question, among others M. Gross, S. Keel. V. Shokurov, but we could not find any specific mention in the literature.
It is also very unclear which triangulations of a sphere can be realized as dual complexes of logCY pairs. This is quite hard even in dimension 2; see [32] for partial results.

6 (Degenerations of Calabi–Yau varieties) Studying degenerations of Calabi–Yau varieties naturally leads to log Calabi–Yau pairs. Let \(g : Y \rightarrow \mathbb{D}\) be a CY-degeneration over the unit disk \(\mathbb{D}\). That is, \(g\) is proper, \(K_Y \sim_{\mathbb{Q}} 0\) and \((Y, Y_0)\) is dlt where \(Y_0\) is the central fiber. Thus the fiber \(Y_t\) is a Calabi–Yau variety for \(t \neq 0\) and the central fiber is a union of logCY pairs \((X_i, \Delta_i)\) where \(\Delta_i\) is the intersection of \(X_i \subset Y_0\) with the other irreducible components. We are interested in the dual complex of the central fiber \(\mathcal{D}(Y_0)\) especially when the following two conditions are satisfied

1. The general fiber \(Y_t\) is an \(n\)-dimensional Calabi–Yau variety in the strict sense, that is, \(Y_t\) is simply connected and \(h^i(Y_t, \mathcal{O}_{Y_t}) = 0\) for \(0 < i < n\).
2. \(\dim \mathcal{D}(Y_0) = n\). This is a combinatorial version of the large complex structure limit or maximal unipotent degeneration conditions.

**Question 7** Let \(g : Y \rightarrow \mathbb{D}\) be a CY-degeneration of relative dimension \(n\) satisfying the conditions (6.1–2). Is \(\mathcal{D}(Y_0) \simeq \mathbb{S}^n\)?

For \(n = 2\) a positive answer is given by [31]; the general case is proposed in [30]. It is easy to see that \(\mathcal{D}(Y_0)\) is a simply connected rational homology sphere but it is not clear that \(\mathcal{D}(Y_0)\) is a manifold. Using Theorem 2, we prove the following in Paragraph 34.

**Proposition 8** The answer to Question 7 is positive if \(n \leq 3\) or if \(n \leq 4\) and the central fiber is a simple normal crossing divisor.

**1 Volume preserving maps**

For logCY pairs, volume preserving maps, also called crepant birational maps, form the most important subclass of birational equivalences.

**Definition 9** Let \((X_i, \Delta_i)\) be normal pairs. A proper, birational morphism \(g : (X_1, \Delta_1) \rightarrow (X_2, \Delta_2)\) is called crepant if \(g_*(\Delta_1) = \Delta_2\) and \(g^*(K_{X_2} + \Delta_2) \sim_{\mathbb{Q}} K_{X_1} + \Delta_1\). An arbitrary birational map \(g : (X_1, \Delta_1) \dashrightarrow (X_2, \Delta_2)\) between proper pairs is called crepant if it can be factored as

\[
\begin{array}{ccc}
(Y, \Delta_Y) & \rightharpoonup & (X_2, \Delta_2) \\
\downarrow p_1 & \neg & \downarrow p_2 \\
(X_1, \Delta_1) & \dashrightarrow & (X_2, \Delta_2),
\end{array}
\] (9.1)

where the \(p_i\) are proper, birational, crepant morphisms. In characteristic 0 this is equivalent to having a common crepant log resolution. (If the \(X_i\) are...
not proper, the above definition still works and defines proper and crepant birational maps. Note that \( g \) itself is not proper, so the terminology is somewhat confusing.)

The main requirement is the natural linear equivalence

\[
p_1^*(K_{X_1} + \Delta_1) \sim Q p_2^*(K_{X_2} + \Delta_2).
\]

A proper, crepant birational map \( g : (X_1, \Delta_1) \dasharrow (X_2, \Delta_2) \) between log canonical pairs is called thrifty if there are closed subsets \( Z_i \subset X_i \) of codimension \( \geq 1 \) that do not contain any of the log canonical centers of the \((X_i, \Delta_i)\) such that \( g \) restricts to an isomorphism \( X_1 \setminus Z_1 \cong X_2 \setminus Z_2 \).

If \((X, \Delta)\) is logCY then a global section of \( \mathcal{O}_X(mK_X + m\Delta) \) determines a natural volume form on \( X \setminus \text{Supp} \Delta \), up to scalar. Then a map \( g : (X_1, \Delta_1) \dasharrow (X_2, \Delta_2) \) is crepant birational iff it is volume preserving, up to a scalar.

It is important to note that in (9.1) usually one can not choose \( Y \) such that \( \Delta_Y \) is effective, not even if we allow \( Y \) to be singular. However, as we see next, such a choice of \( Y \) is possible if we allow the \( p_i \) to be rational contractions.

**Definition 10** Let \( X_i \) be normal, proper varieties. A birational map \( g : X_1 \dasharrow X_2 \) is called a contraction if \( g^{-1} : X_2 \dasharrow X_1 \) does not have any exceptional divisors. Observe that \( g \) is an isomorphism in codimension 1 iff both \( g \) and \( g^{-1} \) are contractions. (Note that a birational morphism is always a contraction. For a birational map there does not seem to be a generally accepted notion of contraction; the above definition is natural and quite useful.)

Let \( g : (X_1, \Delta_1) \dasharrow (X_2, \Delta_2) \) be a birational contraction that is crepant. Then \( g_* (\Delta_1) = \Delta_2 \) and \( g^*(K_{X_2} + \Delta_2) \sim Q K_{X_1} + \Delta_1 \). The converse is usually not true. (For example, a flop is crepant birational but a flip is not.) However, if the \((X_i, \Delta_i)\) are logCY then the linear equivalence (9.2) is automatic and the converse holds.

The following lemma gives several useful ways of factoring a crepant birational map. In the logCY case, a detailed understanding of volume preserving maps is given in [5], which also suggested to us (11.4).

**Lemma 11** Given two proper, log canonical pairs \((X_i, \Delta_i)\), the following are equivalent.

1. There is a crepant birational map \( g : (X_1, \Delta_1) \dasharrow (X_2, \Delta_2) \).
2. There is a log canonical pair \((Y, \Delta_Y)\) and crepant, birational contraction maps

\[
(X_1, \Delta_1) \dasharrow_{p_1} (Y, \Delta_Y) \dasharrow_{p_2} (X_2, \Delta_2).
\]

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(3) There is a pair \((Y, \Delta_Y)\) and crepant, birational contraction maps
\[
(X_1, \Delta_1) \xrightarrow{\ p_1^1\ } (Y, \Delta_Y) \xrightarrow{\ p_2^2\ } (X_2, \Delta_2),
\]
where \((Y, \Delta_Y)\) is \(\mathbb{Q}\)-factorial, dlt and \(p_1\) is a morphism.

(4) There are \(\mathbb{Q}\)-factorial, dlt pairs \((X'_i, \Delta'_i)\) and crepant, birational maps
\[
(X_1, \Delta_1) \xleftarrow{\ p_1^1\ } (X'_1, \Delta'_1) \xrightarrow{\ \phi\ } (X'_2, \Delta'_2) \xrightarrow{\ p_2^2\ } (X_2, \Delta_2),
\]
where the \(p_i\) are morphisms, \(\phi\) is crepant birational, thrifty and an isomorphism in codimension 1.

Proof It is clear that each assertion implies the previous one.

In order to see (1) \(\Rightarrow\) (3), let \(\phi : (X_1, \Delta_1) \longrightarrow (X_2, \Delta_2)\) be a crepant, birational map and \(E_j \subset X_2\) the exceptional divisors of \(\phi^{-1}\). The discrepancy \(a(E_j, X_1, \Delta_1)\) equals minus the coefficient of \(E_j\) in \(\Delta_2\), thus it is \(\leq 0\).

By \([28, \text{1.38}]\) there is a \(\mathbb{Q}\)-factorial, dlt, crepant modification \(p_1 : (Y, \Delta_Y) \rightarrow (X_1, \Delta_1)\) that extracts precisely the \(E_j\) and possibly some other divisors with discrepancy \(-1\). Then \(p_2 := \phi \circ p_1^{-1}\) is a crepant, birational contraction map.

The proof of (4) is similar. We take a common log resolution \(\pi_i : (Y', \Delta') \rightarrow (X_i, \Delta_i)\) and write \(\Delta' = \Delta'' + \Delta_R\) where \(\text{Supp} \Delta''\) consists of the set of divisors on \(Y'\) that are in one of the following three sets: birational transforms of \(\Delta_i\), divisors that are exceptional for exactly one of the \(\pi_i\) or divisors with coefficient 1 in \(\Delta'\). Let \(F\) be the sum of all other \(\pi_i\)-exceptional divisors.

As in \([28, \text{1.35}]\), for \(i = 1, 2\) run the \((Y', \Delta'' + (1 - \epsilon)F)\text{-MMP over } X_i\) for \(0 < \epsilon \ll 1\) to get \(p_i : (X'_i, \Delta'_i) \rightarrow (X_i, \Delta_i)\). If \(R\) is an extremal ray that we contract then \((R \cdot (K + \Delta'' + (1 - \epsilon)F)) < 0\). Since \(K + \Delta'\) is numerically \(\pi_i\)-trivial, this is equivalent to
\[
(R \cdot (K + \Delta'' + (1 - \epsilon)F - K - \Delta')) = (R \cdot ((1 - \epsilon)F - \Delta_R)) < 0.
\]

By our choice of \(F\), \((1 - \epsilon)F - \Delta_R\) is effective and its support is \(F\). Thus the extremal rays contracted are always contained in \(F\) and the MMP contracts all the divisors contained in \(F\). Note also that \((Y', \Delta'')\) and \((Y', \Delta'' + (1 - \epsilon)F)\) have the same lc centers and they are not contained in \(F\). Thus \(Y' \longrightarrow X'_i\) is a local isomorphism at all the lc centers of \((Y', \Delta')\) hence the induced birational map \((X'_1, \Delta'_1) \longrightarrow (X'_2, \Delta'_2)\) is thrifty and an isomorphism in codimension 1.

\(\square\

**Definition 12** (Dual complex) Let \(E\) be a simple normal crossing variety over a field \(k\) with irreducible components \(\{E_i : i \in I\}\). (Our main interest is in the case \(k = \mathbb{C}\) but sometimes it is convenient to allow \(k\) to be arbitrary.)
A stratum of $E$ is any irreducible component $F \subset \cap_{i \in J} E_i$ for some $J \subset I$.

The dual complex of $E$, denoted by $D(E)$, is a CW-complex whose vertices are labeled by the irreducible components of $E$ and for every stratum $F \subset \cap_{i \in J} E_i$ we attach a $(|J|-1)$-dimensional cell. Note that for any $j \in J$ there is a unique irreducible component of $\cap_{i \in J \setminus \{j\}} E_i$ that contains $F$; this specifies the attaching map. (The dual complex is a regular $\Delta$-complex in the terminology of [15].)

It is very important for us that crepant birational equivalence does not change the dual complex.

**Theorem 13** [6] The dual complexes of proper, log canonical, crepant birational pairs are PL-homeomorphic to each other.

Using Theorem 13 our aim is to study the dual complex $\mathcal{DMR}(X, \Delta)$ of logCY pairs in 2 steps. First we show that $(X, \Delta)$ is crepant birational to a “Fano–type” logCY pair. There are several natural ways to define “Fano–type.” For our current purposes the best seems to be to assume that $\Delta=1$ supports a big and semi-ample divisor.

The second step is the study of $\mathcal{DMR}(X, \Delta)$ in the Fano–type cases.

**Definition 14** Let $X$ be a variety and $D$ an effective divisor. We say that $D$ fully supports a divisor that is ample (or big, or semi-ample or ...) if there is an effective divisor $H$ that is ample (or big, or semi-ample or ...) such that $\text{Supp } H = \text{Supp } D$.

Example 63 shows the important difference between supporting a big and semi-ample divisor and fully supporting a big and semi-ample divisor.

15 (Dlt singularities) Divisorial log terminal singularities form a very convenient class to work with; see [22, 2.37] or [28, 2.8]. The precise definition is important for the proof of Theorem 49, but for most everything else all one needs to know is that dlt pairs behave very much like simple normal crossing pairs.

If $(X, \Delta)$ is dlt then, by [6, Thm. 3], $\mathcal{DMR}(X, \Delta)$ can be computed directly from $\Delta=1$ as in Definition 12. In particular, $\dim X = n$ and $(X, \Delta)$ has maximal intersection iff there are $n$ divisors $D_1, \ldots, D_n \subset \Delta=1$ such that $D_1 \cap \cdots \cap D_n$ is non-empty (hence necessarily 0-dimensional).

If $(X, \Delta)$ is dlt then for every stratum $W \subset X$ there is a natural $\mathbb{Q}$-divisor $\text{Diff}^* W \Delta$ such that $(W, \text{Diff}^* W \Delta)$ is dlt and $K_W + \text{Diff}^* W \Delta \sim_{\mathbb{Q}} (K_X + \Delta)|_W$; see [28, Sect. 4.1] for the definition and basic properties. Thus $(W, \text{Diff}^* W \Delta)$ is also logCY.

In general $\text{Diff}^* W \Delta$ is somewhat complicated to determine, but if all divisors in $\Delta$ have coefficient 1 and $K_X + \Delta$ is Cartier then the same holds for $\text{Diff}^* W \Delta$ and $\text{Diff}^* W \Delta$ consists of the intersections of $W$ with those irreducible components of $\Delta$ that do not contain $W$.  

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This implies that for every irreducible divisor $D \subset \Delta = 1$, the link of $[D] \in \mathcal{DMR}(X, \Delta)$ is PL-homeomorphic to $\mathcal{DMR}(D, \text{Diff}^*_D \Delta)$. Thus the local structure of $\mathcal{DMR}(X, \Delta)$ is determined by the lower dimensional logCY pairs $(D, \text{Diff}^*_D \Delta)$.

## 2 Reduction steps

We discuss various steps that simplify $(X, \Delta)$ without changing the dual complex or changing it in simple ways. The final conclusion is that one should focus on logCY pairs $(X, \Delta)$ such that

- $(X, \Delta)$ is dlt, $\mathbb{Q}$-factorial,
- $X$ is rationally connected,
- $\Delta = \Delta = 1$,
- $\Delta$ fully supports a big, semi-ample divisor and
- $K_X + \Delta \sim 0$.

16 (Disconnected case) By [27] or [28, 4.37], if $(X, \Delta)$ is logCY and the dual complex $\mathcal{DMR}(X, \Delta)$ is disconnected then $(X, \Delta)$ is crepant birational to a product $(Y, \Delta_Y) \times (\mathbb{P}^1, [0] + [\infty])$ where $(Y, \Delta_Y)$ is klt. In particular $\mathcal{DMR}(X, \Delta) \simeq S^0$. Thus in the sequel we need to deal only with the connected case.

17 (Index 1 cover) Assume that $\Delta = \Delta = 1$ and let $m$ be the smallest natural number such that $m(K_X + \Delta) \sim 0$. Correspondingly there is a degree $m$ quasi-étale (that is, étale in codimension 1) cover $(\tilde{X}, \tilde{\Delta}) \to (X, \Delta)$ and $\mathcal{DMR}(X, \Delta) \simeq \mathcal{DMR}(\tilde{X}, \tilde{\Delta})/\mathbb{Z}_m$.

18 (Rational connectedness) Let $(X, \Delta)$ be a logCY pair. If $X$ is not rationally connected then there is an MRC-fibration $g : X \to Z$ onto a non-uniruled variety by [25, IV.5.2] and [10]. Let $(X_k(Z), \Delta_k(Z))$ denote the generic fiber.

We see in Proposition 19 that every log canonical center dominates $Z$. Thus, by Lemma 29,

$$\mathcal{DMR}(X, \Delta) \simeq \mathcal{DMR}(X_k(Z), \Delta_k(Z)).$$

Thus it is enough to consider those logCY pairs whose underlying variety is rationally connected.

**Proposition 19** Let $(X, \Delta)$ be a dlt logCY pair and $g : X \to Z$ a dominant map to a non-uniruled variety $Z$. Then every irreducible component $D \subset \Delta = 1$ dominates $Z$.

**Proof** We may assume that $Z$ is smooth and projective. Let $C \subset X$ be a general complete intersection curve, in particular $g$ is defined along $C$. Then $(C \cdot (K_X + \Delta)) = 0$ and $(C \cdot D) > 0$. 

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If $D \subset \Delta=\Delta_1$ does not dominate $Z$ then we reach a contradiction by proving that $(C \cdot (K_X + \Delta)) > 0$. By blowing up $Z$ we may assume that $D$ dominates a divisor $D_Z \subset Z$; see [28, 2.22].

Let $Y$ be the normalization of the closure of the graph of $g$. Choose $\Delta_Y$ such that the first projection $(Y, \Delta_Y) \to (X, \Delta)$ is crepant. Since $g$ is defined along $C$ we may identify $C$ with its preimage in $Y$ and so $(C \cdot (K_Y + \Delta_Y)) = 0$.

We apply the canonical class formula (20.4) to the second projection $\pi_2 : Y \to Z$ and write $K_Y + \Delta_Y \sim_Q \pi_2^* (K_Z + J + B)$.

Note that $(C \cdot \pi_2^* K_Z) = (\pi_2^* C) \cdot K_Z \geq 0$ since $Z$ is not uniruled [33] and $(C \cdot \pi_2^* J) = (\pi_2^* C) \cdot J \geq 0$ by (20.5). Finally, (20.8) shows that a divisor $W \subset Z$ appears in $B$ with positive (resp. nonnegative) coefficient if there is divisor $W_Y \subset Y$ dominating $Y$ that appears in $\Delta_Y$ with positive (resp. nonnegative) coefficient. Thus $g(C)$ is disjoint from the non-effective part of $B$ and intersects $D_Z$ nontrivially. Furthermore, $D_Z$ appears in $B$ with positive coefficient. Thus $(C \cdot \pi_2^* B) = (g(C) \cdot B) > 0$. Adding these together we get that $(C \cdot (K_Y + \Delta_Y)) > 0$, a contradiction. $\square$

20 (Kodaira–type canonical class formula) The original formula for elliptic surfaces was further developed by [9,19]. The following is a somewhat simplified version of the form given in [26, 8.5.1].

Let $(Y, \Delta)$ be a log canonical pair where $\Delta$ is not assumed effective. Let $p : Y \to Z$ be a proper morphism to a normal variety $Z$ with geometrically connected generic fiber $X_k(Z)$. Assume that

1. $\Delta_k(Z)$ is effective,
2. $(X_k(Z), \Delta_k(Z))$ is logCY and
3. $K_Y + \Delta$ is $\mathbb{Q}$-linearly equivalent to the pull-back of some $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor from $Z$. (This seems like a strong restriction but it is easy to achieve by changing $\Delta$.)

Let $Z^0 \subset Z$ be the largest open set such that $p$ is flat over $Z^0$ with logCY fibers and set $Y^0 := p^{-1}(Z^0)$. Then one can write

$$K_Y + \Delta \sim_{\mathbb{Q}} p^*(K_Z + J + B)$$

(20.4)

where $J$ and $B$ have the following properties.

1. $J$ is a $\mathbb{Q}$-linear equivalence class, called the modular part. It depends only on the generic fiber $(X_k(Z), \Delta_k(Z))$ and it is the push-forward of a nef class by some birational morphism $Z' \to Z$. In particular, $J$ is pseudo-effective.
2. $B$ is a $\mathbb{Q}$-divisor, called the boundary part. It is supported on $Z \setminus Z^0$.
3. Let $D \subset Z \setminus Z^0$ be an irreducible divisor. Then

$$\text{coeff}_B D = \sup_E \left\{ 1 - \frac{1 + a(Y, \Delta, E)}{\text{mult}_E p^* D} \right\}$$
where the supremum is taken over all divisors over $Y$ that dominate $D$.

This implies the following.

(8) If $D$ is dominated by a divisor $E$ such that $a(Y, \Delta, E) < 0$ (resp. $\leq 0$) then $\text{coeff}_B D > 0$ (resp. $\geq 0$).

(9) If $\Delta$ is effective then so is $B$.

(10) If $(Y, \Delta)$ is lc then $\text{coeff}_B D \leq 1$ for every $D$ and $\text{coeff}_B D = 1$ iff $D$ is dominated by a divisor $E$ such that $a(Y, \Delta, E) = -1$.

The strongest reduction assertion is the following, which is a weak form of our main technical result; see Corollary 58 for the general case.

**Theorem 21** Let $(X, \Delta)$ be a logCY pair. Then there is a volume preserving birational map $(X, \Delta) \to (X_s, \Delta_s)$ such that

1. in the maximal intersection case $\Delta_s = 1$ fully supports a big and semi-ample divisor and
2. in general there is a morphism $q_s : X_s \to Z$ with generic fiber $(X_k(Z), \Delta_k(Z))$ such that
   a. $\mathcal{DMR}(X_k(Z), \Delta_k(Z)) \simeq \mathcal{DMR}(X, \Delta)$ and
   b. $\Delta_k(Z)$ fully supports a big and semi-ample divisor.

22 (Fractional part of $\Delta$) Assume that $\Delta < 1 \neq 0$. Then, by [1,18], the $(X, \Delta_{\leq 1})$-MMP terminates with a Fano-contraction $q : X_r \to Z$. Note that $\mathcal{DMR}(X, \Delta) \simeq \mathcal{DMR}(X_r, \Delta_r)$ by Theorem 13.

If Proposition 24 applies then $\mathcal{DMR}(X_r, \Delta_r^{\leq 1})$ is collapsible to a point and it remains to compare $\mathcal{DMR}(X_r, \Delta_r)$ and $\mathcal{DMR}(X_r, \Delta_r^{=1})$. In general this seems rather difficult and it can happen that $\mathcal{DMR}(X_r, \Delta_r^{=1})$ is empty yet $\mathcal{DMR}(X_r, \Delta_r)$ is not. There is, however, one case when the two are closely related.

Assume that $\Delta^{=1}$ fully supports a big and semi-ample divisor over $Z$. Then $\Delta_m^{=1}$ fully supports a big and mobile divisor over $Z$ for any dlt modification $h : (X_m, \Delta_m) \to (X_r, \Delta_r)$. This implies that $\text{Supp} \Delta_m^{=1}$ dominates $Z$ and

$$\text{Supp} \Delta_m^{=1} = h^{-1}(\text{Supp} \Delta_r^{=1}).$$

Thus [6, Thm. 3] proves that $\mathcal{DMR}(X_m, \Delta_m) \simeq \mathcal{DMR}(X_r, \Delta_r)$ collapses to $\mathcal{DMR}(X_r, \Delta_r^{=1})$. (We do not known whether they are PL-homeomorphic or not.)

Thus, in this case, $\mathcal{DMR}(X, \Delta) \simeq \mathcal{DMR}(X_m, \Delta_m)$ is collapsible to a point.

Combining with Theorem 21, we have proved the following strengthening of Proposition 3.
Corollary 23 Let \((X, \Delta)\) be a logCY pair such that \(\text{DMR}(X, \Delta)\) is not collapsible to a point. Then the model \((X_s, \Delta_s)\) obtained in Theorem 21 also satisfies \(\Delta_s = \Delta_s^{=1}\) in the maximal intersection case and \(\Delta_k(Z) = \Delta_k^{=1}(Z)\) in general.

Let \((Y, \Delta)\) be a dlt pair and \(D \subset \Delta^{=1}\) an irreducible divisor such that \(D \cap W\) is irreducible and non-empty for every log canonical center \(W\). Then \(\text{DMR}(X, \Delta)\) is the cone over \(\text{DMR}(D, \text{Diff}^*_D \Delta)\). If \(q : (Y, \Delta) \rightarrow Z\) is a Fano contraction and \(D\) dominates \(Z\) then it has irreducible and non-empty intersection with every log canonical center; this is a special case of [28, Thm. 4.40]. Applying these repeatedly, we obtain the following.

Proposition 24 Let \((Y, \Delta)\) be a dlt pair and \(q : (Y, \Delta) \rightarrow Z\) a Fano contraction. Assume that \(\text{Supp} \Delta^{=1}\) dominates \(Z\). Then there is a unique smallest lc center \(W \subset Y\) dominating \(Z\) and

\[
\text{DMR}(X, \Delta) = \sigma^r * \text{DMR}(W, \text{Diff}^*_W \Delta),
\]

the join of a simplex \(\sigma^r\) of dimension \(r = \text{codim}_Y W - 1\) and of \(\text{DMR}(W, \text{Diff}^*_W \Delta)\).

In particular, \(\text{DMR}(X, \Delta)\) is contractible, even collapsible.

Note also that the simplex \(\sigma^r\) is PL-homeomorphic to \(S^r / (\tau)\) where \(\tau\) is a reflection on a hyperplane. Thus, as in Example 64, if \(\text{DMR}(W, \text{Diff}^*_W \Delta)\) is the quotient of a sphere then so is \(\text{DMR}(Y, \Delta)\).

3 Basic results on the dual complex

We need two results that connect the topology of \(\mathcal{D}(E)\) and the algebraic geometry of \(E\). The following homological lemma is essentially proved in [13, pp. 68–72]; see also [7, pp. 26–27] and [28, 3.63]. The fundamental group result is rather straightforward; [20, Lem. 25].

Lemma 25 Let \(E = \bigcup_{i \in I} E_i\) be a proper, simple normal crossing variety over \(\mathbb{C}\). Then there are natural injections

\[
H^r(\mathcal{D}(E), \mathbb{C}) \hookrightarrow H^r(E, \mathcal{O}_E).
\]  

Furthermore, if \(H^r(\bigcap_{i \in J} E_i, \mathcal{O}_{\bigcap_{i \in J} E_i}) = 0\) for every \(r > 0\) and every \(J \subset I\) then (25.1) is an isomorphism.

Lemma 26 Let \(E = \bigcup_{i \in I} E_i\) be a connected, simple normal crossing variety over \(\mathbb{C}\). Then there is a natural surjection

\[
\pi_1(E) \twoheadrightarrow \pi_1(\mathcal{D}(E)).
\]
Furthermore, if the $E_i$ are simply connected then (26.1) is an isomorphism.

In some cases one can describe a dual complex using a fibration and finite group actions as in Theorem 21.2.

27 Let $T$ be a simplicial complex and $G$ a finite group acting on $T$. Let $B(T)$ denote the barycentric subdivision of $T$. If $C \in B(T)$ is a simplex and $g \in G$ such that $g(C) = C$ then $g$ also fixes every vertex of $C$. Thus the $G$ action on $B(T)$ naturally extends to a simplicial $G$-action on the topological realization $|B(T)|$.

The quotient is a regular complex denoted by $B(T)/G$. There is a natural map $|B(T)| \to |B(T)/G|$ whose fibers are exactly the $G$-orbits on $|B(T)|$.

Such branched covering spaces are discussed in [8]; we need the following properties.

1. There are natural isomorphisms $H^i(|T|/G, \mathbb{Q}) \cong H^i(|T|, \mathbb{Q})^G$ for every $i$.
2. There is an exact sequence

$$\pi_1(|T|) \to \pi_1(|T|/G) \to G/\langle\text{stabilizers of points}\rangle.$$ 

In particular, if $\pi_1(|T|)$ is finite then so is $\pi_1(|T|/G)$.

The quotient construction naturally arises for families of simple normal crossing varieties.

**Lemma 28** $E = \bigcup_{i \in I} E_i$ be a simple normal crossing variety and $q : E \to Z$ a morphism such that every stratum of $E$ dominates $Z$. Let $z \in Z$ be a general point and $E_z$ the fiber over $z$. Then $E_z$ is a simple normal crossing variety and there is a finite group $G$ acting on $\mathcal{D}(E_z)$ such that $\mathcal{D}(E) = \mathcal{D}(E_z)/G$.

**Proof** By shrinking $Z$ we may assume that $q$ is smooth on every stratum of $E$. Then $z \mapsto \mathcal{D}(E_z)$ defines a locally trivial fiber bundle. Since $\mathcal{D}(E_z)$ is a finite simplicial complex, the monodromy of the fiber bundle is a finite group $G$ and $\mathcal{D}(E) = \mathcal{D}(E_z)/G$. $\square$

Algebraically minded readers may prefer to think of Lemma 28 as a combination of the next two claims.

**Lemma 29** $E = \bigcup_{i \in I} E_i$ be a simple normal crossing variety over a field $k$ and $q : E \to Z$ a morphism. Then the generic fiber $E_k(Z)$ is a simple normal crossing variety over the function field $k(Z)$ and $\mathcal{D}(E_k(Z))$ is a subcomplex of $\mathcal{D}(E)$. Furthermore, if every stratum dominates $Z$ then $\mathcal{D}(E_k(Z)) = \mathcal{D}(E)$.

**Lemma 30** Let $K/k$ be a Galois extension with Galois group $G$. Let $E_k$ be a simple normal crossing variety over $k$. Then $G$ acts on $\mathcal{D}(E_K)$ and $\mathcal{D}(E_k) = \mathcal{D}(E_K)/G$. 
4 Homology of the dual complex

The following proves (2.2).

**Proposition 31** Let \((X, \Delta)\) be a logCY pair. Then

\[
H^i(\mathcal{D\mathcal{M\mathcal{R}}}(X, \Delta), \mathbb{Q}) = 0 \quad \text{for} \quad 0 < i < \dim \mathcal{D\mathcal{M\mathcal{R}}}(X, \Delta).
\]

**Proof** Assume first that \(X\) is rationally connected, \(\Delta = \Delta^{=1}\) and \(K_X + \Delta^{=1} \sim 0\). Set \(n := \dim X\). Then \(H^i(X, \mathcal{O}_X) = 0\) for \(i > 0\) by [4,23] and

\[
H^i(X, \mathcal{O}_X(-\Delta^{=1})) = H^i(X, \mathcal{O}_X(K_X)) = 0 \quad \text{for} \quad i < n
\]

by Serre duality. The long cohomology sequence of the exact sequence

\[
0 \to \mathcal{O}_X(-\Delta^{=1}) \to \mathcal{O}_X \to \mathcal{O}_{\Delta^{=1}} \to 0
\]

now implies that \(H^i(\Delta^{=1}, \mathcal{O}_{\Delta^{=1}}) = 0\) for \(0 < i < n - 1\). Thus \(H^i(\mathcal{D\mathcal{M\mathcal{R}}}(X, \Delta), \mathbb{Q}) = 0\) for \(0 < i < n - 1\) by Lemma 25.

We try to use a similar argument in general; the problem is that, in (31.1), instead of \(\mathcal{O}_X(K_X)\) we have \(\mathcal{O}_X(-\Delta^{=1})\) and \(-\Delta^{=1}\) is \(\mathbb{Q}\)-linearly equivalent to \(K_X + \Delta^{<1}\). The presence of the fractional part \(\Delta^{<1}\) and the \(\mathbb{Q}\)-linear (as opposed to linear) equivalence both cause problems.

Assume next that \((X, \Delta)\) is dlt and \(\Delta^{=1}\) fully supports a big and semi-ample divisor \(M\). Note that

\[
0 \sim_{\mathbb{Q}} K_X + \epsilon M + (\Delta - \epsilon M) \quad \text{and} \quad \Delta^{=1} \sim \epsilon M + (\Delta^{=1} - \epsilon M),
\]

thus by vanishing we see that

\[
H^i(X, \mathcal{O}_X) = 0 \quad \text{for} \quad i > 0 \quad \text{and} \quad H^i(X, \mathcal{O}_X(-\Delta^{=1})) = 0 \quad \text{for} \quad i < n.
\]

Using (31.1) these imply that \(H^i(\Delta^{=1}, \mathcal{O}_{\Delta^{=1}}) = 0\) for \(0 < i < n - 1\).

Finally, by Theorem 13, we are free to replace \((X, \Delta)\) with any other logCY pair that is crepant birational to it. We first apply Theorem 21, then Lemmas 28 and 27 to reduce to the already established case when \(\Delta^{=1}\) fully supports a big and semi-ample divisor.

Next we study the top cohomology of the structure sheaf and of the dual complex.

32 (The top cohomology of logCY pairs) Let \((X, \Delta)\) be a dlt, logCY pair of dimension \(n\). Then \(H^n(X, \mathcal{O}_X) = 0\) save when \(\Delta = 0\) and \(K_X \sim 0\).
Let $D \subset \Delta^{=1}$ be an irreducible component such that $H^{n-1}(D, \mathcal{O}_D) \neq 0$. As we noted above, then $\text{Diff}_D^* \Delta = 0$, thus $D$ is a connected component of $\text{Supp} \Delta$. As we noted in Paragraph 16, there are 2 possibilities. Either $\Delta^{=1} = D$ or $\Delta^{=1} = D + D'$ has 2 irreducible components which are crepant birational to each other.

Let $(Y, \Delta_Y)$ be a dlt pair. Set $X := \Delta^{=1}_Y$ and assume that $(C \cdot (K_Y + \Delta_Y)) = 0$ for every curve $C \subset X$. We can then view $(X, \Delta := \text{Diff}_X^* \Delta_Y)$ as a reducible logCY pair. A non-embedded definition of such pairs, called semi-dlt pairs, is given in [28, Sect. 5.4]. The precise definition is not important for now, we will only use the case when $X = \Delta^{=1}_Y$ as above.

Using these observations inductively, we get the following.

**Claim 32.1.** Let $(X, \Delta) = \bigcup_i (X_i, \Delta_i)$ be a connected, semi-dlt, logCY pair of dimension $n$. Assume that it has a stratum $W \subset X$ of dimension $r$ such that $H^r(W, \mathcal{O}_W) \neq 0$. Then

1. all strata have dimension $\geq r$,
2. the $r$-dimensional strata are crepant birational to each other and
3. $\text{dim} \mathcal{D}(X) = n - r - 1$.

Let $X = \bigcup_{i \in I} X_i$ be a simple normal crossing variety. The cohomology of $\mathcal{O}_X$ is computed by a spectral sequence whose $E_1$ terms are

$$E_1^{pq} = H^q(X_p, \mathcal{O}_{X_p}) \quad \text{where} \quad X_p := \bigsqcap_{J \subset I, |J| = p+1} \bigcap_{i \in J} X_i.$$  \hspace{1cm} (32.2)

In the bottom row $q = 0$ we find the complex that computes the cohomology of $\mathcal{D}(X)$. Note also that if $H^{\dim W}(W, \mathcal{O}_W) = 0$ holds for every positive dimensional stratum then the only term that contributes to $H^n(X, \mathcal{O}_X)$ is $E_2^{n0} = H^n(\mathcal{D}(X), \mathbb{C})$. We have thus proved the following.

**Claim 32.3.** Let $(X, \Delta) = \bigcup_i (X_i, \Delta_i)$ be a connected, semi-dlt, logCY pair of dimension $n$ such that $H^n(X, \mathcal{O}_X) \neq 0$. Then

1. either $H^n(\mathcal{D}(X), \mathbb{Q}) = \mathbb{Q}$, 
2. or $\text{dim} \mathcal{D}(X) < n$.

Next we prove Proposition 5.

**33 (Dimension induction)** Let $(X, \Delta)$ be a dlt logCY pair of dimension $n$ such that $K_X + \Delta \sim 0$. As we noted in Paragraph 15, the local structure of $\mathcal{D}\mathcal{M}\mathcal{R}(X, \Delta)$ is determined by the lower dimensional logCY pairs.

Assume first that $(X, \Delta)$ has maximal intersection. Using Proposition 31 and Paragraph 32 we see that $H^{n-1}(\mathcal{D}\mathcal{M}\mathcal{R}(X, \Delta), \mathbb{Q}) = \mathbb{Q}$ hence $\mathcal{D}\mathcal{M}\mathcal{R}(X, \Delta)$ is a rational homology sphere. In low dimensions we obtain complete answers to Question 4.
If \( n = 1 \) then \( X = \mathbb{P}^1 \) and \( DMR(X, \Delta) \simeq S^0 \).

If \( n = 2 \) then \( X \) is a rational surface and \( DMR(X, \Delta) \simeq S^1 \).

If \( n = 3 \) then \( DMR(X, \Delta) \) is a 2-manifold that is a rational homology sphere. Thus \( DMR(X, \Delta) \simeq S^2 \).

If \( n = 4 \) then \( DMR(X, \Delta) \) is a 3-manifold that is a rational homology sphere. The fundamental group of a 3-manifold is residually finite [16], thus (2.4) implies that the fundamental group itself is finite. By (2.5) the universal cover is a simply connected homology sphere, thus \( DMR(X, \Delta) \simeq S^3 \).

(This uses the Poincaré conjecture.) Note however that we do not claim that \( DMR(X, \Delta) \) is the sphere \( S^3 \).

Thus, starting with \( n = 5 \) we do not claim that \( DMR(X, \Delta) \) is a manifold.

We can, however, do better if \((X, \Delta)\) is a simple normal crossing pair. In this case Theorem 36 implies that \( DMR(X, \Delta) \) and its links are simply connected. Thus we get that \( DMR(X, \Delta) \) is homeomorphic to a sphere if \( \dim X \leq 5 \). (Conjecturally, PL-homeomorphic to a sphere.)

If \( n = 6 \) then \( DMR(X, \Delta) \) is a 5-manifold that is a simply connected rational homology sphere. Our results say nothing about the torsion group \( H_2(DMR(X, \Delta), \mathbb{Z}) \).

Finally consider the case when \((X, \Delta)\) does not have maximal intersection. A similar induction shows that, for \( \dim DMR(X, \Delta) \leq 3 \), the universal cover is a simply connected manifold, possibly with boundary. Thus \( DMR(X, \Delta) \) is either a sphere or a ball.

34 Let \( g : Y \to \mathbb{D} \) be a CY-degeneration of relative dimension \( n \) satisfying the conditions (6.1–2).

If \( X_i \subset Y_0 \) is an irreducible component and \( D_i \subset X_i \) is the intersection of \( X_i \) with the other components then \((X_i, D_i)\) is a logCY pair and \( DMR(X_i, D_i) \) is PL-homeomorphic to the link of \([X_i] \in D(Y_0)\). Thus if \( n \leq 3 \) or if \( n \leq 5 \) and the central fiber is a simple normal crossing divisor then \( D(Y_0) \) is a manifold using the results of Paragraph 33.

Since \( Y_0 \) has Du Bois singularities (cf. [28, Chap. 6]) we see that \( h^i(Y_0, \mathcal{O}_{Y_0}) = h^i(Y_t, \mathcal{O}_{Y_t}) = 0 \) for \( 0 < i < n \), thus \( D(Y_0) \) is a rational homology sphere.

Finally, there are natural surjections

\[
\pi_1(Y_0) \to \pi_1(Y) \cong \pi_1(Y_0) \to \pi_1(D(Y_0)).
\]

We assumed that \( Y_t \) is simply connected for \( t \neq 0 \), hence \( \pi_1(Y_0) \) is a quotient of \( \pi_1(\mathbb{D}\setminus\{0\}) \cong \mathbb{Z} \). Since \( Y_0 \) is reduced, \( Y \to \mathbb{D} \) has a section, thus \( \pi_1(\mathbb{D}\setminus\{0\}) \) gets killed in \( \pi_1(Y_0) \).

These imply that \( \pi_1(D(Y_0)) \) is trivial and so \( D(Y_0) \) is a simply connected rational homology sphere. For \( n \leq 4 \) this implies that it is homeomorphic to
a sphere (conjecturally PL-homeomorphic to a sphere). This completes the proof of Proposition 8.

5 Fundamental groups of logCY pairs

In this section we study various fundamental groups associated to a logCY pair. It is easy to see that usually $X$ itself is simply connected. It is much more interesting to understand the fundamental group of the smooth locus $\pi_1(X_{\text{sm}})$ and the fundamental group of the dual complex. Note that while the latter is a crepant-birational invariant, the fundamental group of the smooth locus is not; see Example 62.

35 (General set-up) Let $X \subset \mathbb{CP}^N$ be a normal, projective variety and $D \subset X$ a divisor. Fix a smooth metric on $\mathbb{CP}^N$ and for $0 < \delta \ll 1$ let $D_\delta \subset X$ denote the $\delta$-neighborhood of $D$. Then $D$ is a deformation retract of $D_\delta$, hence $\pi_1(D_\delta) \cong \pi_1(D)$. If $(X, D)$ is snc or dlt then we can form the dual complex $\mathcal{D}(D)$ and we have a surjection

$$\pi_1(D_\delta) \cong \pi_1(D) \twoheadrightarrow \pi_1(\mathcal{D}(D)).$$

If $Z \subset X \setminus D_\delta$ is any closed subset then there is a natural map $L_D : \pi_1(D_\delta) \to \pi_1(X \setminus Z)$. Very little is known about this map in general but if $D$ is ample and $\dim X \geq 3$ then it is an isomorphism by the Lefschetz hyperplane theorem for large enough finite $Z \subset \text{Sing } X$. Thus we have the following inclusions

$$X_{\text{sm}} \hookrightarrow X \setminus Z \hookleftarrow D_\delta \hookleftarrow D$$

and these induce maps on the fundamental groups

$$\pi_1(X_{\text{sm}}) \to \pi_1(X \setminus Z) \overset{L_D}{\twoheadleftarrow} \pi_1(D_\delta) \cong \pi_1(D) \to \pi_1(\mathcal{D}(D)). \quad (35.1)$$

Thus if $L_D$ is an isomorphism then we obtain surjections

$$\pi_1(X_{\text{sm}}) \to \pi_1(X \setminus Z) \to \pi_1(D) \to \pi_1(\mathcal{D}(D)). \quad (35.2)$$

We would like to apply this to $(X, \Delta = 1)$ for a logCY pair $(X, \Delta)$. Typically $\Delta = 1$ is quite negative but Theorem 49 shows that ampleness can be achieved for a suitable crepant birational model $(\bar{X}, \bar{\Delta})$. Unfortunately $(\bar{X}, \bar{\Delta})$ is not dlt. We thus have to find a dlt model $(X_s, \Delta_s)$ where $\Delta_s^{-1}$ is close enough to being ample that the above arguments apply. An extra complication is that $\pi_1(X_{\text{sm}})$ is not a birational invariant. At the end we prove (35.2) for $D = \Delta = 1$.

If (35.2) holds then any finite covering space $\mathcal{D}(D) \to \mathcal{D}(D)$ lifts to a finite, étale cover of $X \setminus Z$, hence to a finite, possibly ramified cover $p : \bar{X} \to X$ that
is étale along $D$. If every prime divisor $E \subset X$ has non-empty intersection with $D$ then $p$ is quasi-étale. If $(X, D)$ is dlt, logCY and $p$ is quasi-étale then $(\tilde{X}, \tilde{D} := p^{-1}(D))$ is also logCY and $\mathcal{D}(\tilde{D}) = \mathcal{D}(D)$.

This gives a realization of covering spaces of $\mathcal{D}(D)$ as dual complexes of quasi-étale covers of $(X, D)$. Thus (35.2) for $D = \Delta^=1$ implies (2.5).

The main result of this section is the following restatement of (2.3).

**Theorem 36** Let $(X, \Delta)$ be a dlt logCY pair such that $\dim \mathcal{DMR}(X, \Delta) \geq 2$. Then there are surjections

$$\pi_1(X^{\text{sm}}) \twoheadrightarrow \pi_1(\Delta^=1) \twoheadrightarrow \pi_1(\mathcal{DMR}(X, \Delta)).$$

We start with the equality of the last two groups.

**Lemma 37** Let $(X, \Delta)$ be a dlt logCY pair that has maximal intersection. Then $\pi_1(\Delta^=1) \cong \pi_1(\mathcal{DMR}(X, \Delta))$.

**Proof** By [28, 4.40] the irreducible components of $\Delta^=1$ are also dlt logCY pairs that have maximal intersection. Thus Proposition 19 implies that they are rationally connected. A proper, rationally connected variety is simply connected by [4,23], thus Lemma 26 shows that

$$\pi_1(\Delta^=1) \cong \pi_1(\mathcal{D}(\Delta^=1)) \cong \pi_1(\mathcal{DMR}(X, \Delta)).$$

$\square$

We next prove a variant of Theorem 36.

**Proposition 38** Let $(X, \Delta)$ be a dlt logCY pair of dimension $\geq 3$. Assume that $\Delta^=1$ fully supports a big and semi-ample divisor and every prime divisor $E \subset X$ has non-empty intersection with $\Delta^=1$. Then there are surjections

$$\pi_1(X^{\text{sm}}) \twoheadrightarrow \pi_1(\Delta^=1) \twoheadrightarrow \pi_1(\mathcal{DMR}(X, \Delta)).$$

**Proof** By assumption there is a birational morphism $f : X \to Y$ and an ample divisor $H$ on $Y$ such that $\text{Supp} \ f^{-1}(H) = \text{Supp} \ \Delta^=1$. Corollary 40 shows that $\pi_1(X^{\text{sm}}) \twoheadrightarrow \pi_1(\Delta^=1)$ is surjective while $\pi_1(\Delta^=1) \twoheadrightarrow \pi_1(\mathcal{DMR}(X, \Delta))$ is surjective by Lemma 26.

$\square$

The key ingredient of the above proof is the following immediate consequence of [12, Part II, Thm. 1.1]. As in Paragraph 35, $H_\delta \supset H$ denotes a neighborhood of $H$ (in the Euclidean topology) such that $H$ is a deformation retract of $H_\delta$.  

$\square$ Springer
Proposition 39 Let $U$ be a smooth (possibly non-proper) variety, $f : U \to W$ a birational morphism and $H$ an ample divisor on $W$. Assume that $\dim f^{-1}(w) \leq \dim U - 2$ for every $w \in W \setminus H$.

Then there is a natural isomorphism

$$\pi_1(f^{-1}H) \cong \pi_1(U).$$

Corollary 40 Let $f : X \to Y$ be a birational morphism between normal, projective varieties. Let $H$ be an ample divisor on $Y$ such that every prime divisor $E \subset X$ has non-empty intersection with $f^{-1}(H)$. Then there is a natural surjection

$$\pi_1(X^\text{sm}) \twoheadrightarrow \pi_1(f^{-1}(H)).$$

Proof Set $H^\text{sm}_\delta := f^{-1}H \cap X^\text{sm}$. By Proposition 39, there is an isomorphism

$$\pi_1(H^\text{sm}_\delta) \cong \pi_1(X^\text{sm})$$

and the natural injection $H^\text{sm}_\delta \hookrightarrow f^{-1}(H_\delta)$ induces a surjective morphism

$$\pi_1(H^\text{sm}_\delta) \twoheadrightarrow \pi_1(f^{-1}(H_\delta)).$$

Finally we can choose $H_\delta$ such that $f^{-1}(H_\delta)$ retracts to $f^{-1}(H)$ hence $\pi_1(f^{-1}(H_\delta)) \cong \pi_1(f^{-1}(H))$. Combining these we get a surjection

$$\pi_1(X^\text{sm}) \twoheadrightarrow \pi_1(f^{-1}(H)).$$

Lemma 41 Let $f : X \dashrightarrow Y$ be a rational contraction between normal, proper varieties. Then there is a natural surjection

$$\pi_1(Y^\text{sm}) \twoheadrightarrow \pi_1(X^\text{sm}).$$

Proof $f^{-1}$ gives a birational morphism $Y^\text{sm} \setminus \text{Ex}(f^{-1}) \hookrightarrow X^\text{sm}$ which gives a surjection

$$\pi_1(Y^\text{sm} \setminus \text{Ex}(f^{-1})) \twoheadrightarrow \pi_1(X^\text{sm}).$$

Since $f$ is a rational contraction, $\text{Ex}(f^{-1})$ has complex codimension $\geq 2$ thus $\pi_1(Y^\text{sm} \setminus \text{Ex}(f^{-1})) = \pi_1(Y^\text{sm}).$
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42 (Proof of Theorem 36) The plan is to use Theorem 49 to find a crepant birational logCY pair \((\bar{X}, \bar{\Delta})\) such that \(\bar{\Delta} = 1\) fully supports an ample divisor \(H\) and then use Proposition 38 on a dlt modification of \((\bar{X}, \bar{\Delta})\). This almost works and there are only two problems: \(\pi_1(X_{s_{\text{sm}}})\) is not a birational invariant and we have to deal with the relative case. In order to deal with the first problem we have to work with a different model to which Proposition 38 does not apply directly. Nonetheless we follow this path and use [6, Thm. 3] to go around the difficulties.

We use Corollary 58 to obtain crepant birational maps

\[
\psi := g^{-1} \circ \phi : (X, \Delta) \to (\bar{X}, \bar{\Delta})
\]

satisfying the properties (58.1–5). By Lemma 41 we have a natural surjection \(\pi_1(X_{\text{sm}}) \to \pi_1(X_{s_{\text{sm}}})\) and \(\text{DMR}(X, \Delta) \cong \text{DMR}(X_s, \Delta_s)\). Using [6, Thm. 3] we obtain that \(D(g^{-1}(H))\) is homotopy equivalent to \(\text{DMR}(X_s, \Delta_s)\), hence

\[
\pi_1(D(g^{-1}(H))) \cong \pi_1(\text{DMR}(X_s, \Delta_s)) \cong \pi_1(\text{DMR}(X, \Delta)).
\]

Thus it is enough to prove that there is a natural surjection

\[
\pi_1(X_{s_{\text{sm}}}) \to \pi_1(g^{-1}(H)).
\]

Next we look at the morphism \(q_s : X_s \to Z\) in Corollary 58. If \(Z\) is a point (this is always the case if \((X, \Delta)\) has maximal intersection) the surjectivity of (42.3) is implied by Corollary 40.

If \(Z\) is positive dimensional, let \(Z^* \subset Z\) be an open subset such that both \(q_s : X_s \to Z\) and its restriction \(q_H : g^{-1}(H) \to Z\) become topological fiber bundles \(q^*_s : X^*_s \to Z^*\) and \(q^*_H : g^{-1}(H)^* \to Z^*\). For both of these, the fundamental group surjects onto the fundamental group of \(Z^*\) and the kernel is the image of the fundamental group of the fiber. We have already established a surjection between the fundamental groups of the respective fibers, thus we have a surjection

\[
\pi_1(X^*_s) \to \pi_1(g^{-1}(H)^*),
\]

which in turn gives a surjection

\[
\pi_1(X^*_s) \to \pi_1(g^{-1}(H)).
\]

Equivalently, every étale cover of \(g^{-1}(H)\) comes from an étale cover of a suitable Zariski open neighborhood \(X_s \supset U \supset g^{-1}(H)\).
Now we can use (58.4) to obtain that $X_s \setminus U$ has codimension $\geq 2$, thus, by the purity of branch loci, (42.5) extends to a surjection
\[
\pi_1(X_s^{\text{sm}}) \twoheadrightarrow \pi_1(g^{-1}(H)). \tag{42.6}
\]
Putting all of these together we have a chain of natural surjections
\[
\pi_1(X^{\text{sm}}) \twoheadrightarrow \pi_1(X_s^{\text{sm}}) \twoheadrightarrow \pi_1(g^{-1}(H)) \\
\twoheadrightarrow \pi_1(D(g^{-1}(H))) \cong \pi_1(\mathcal{DMR}(X_s, \Delta_s)).
\]

The arguments of Paragraph 35 show the following variant of (2.5).

**Corollary 43** Let $(X, \Delta)$ be a logCY pair that has maximal intersection. Then every finite degree covering space of $\mathcal{DMR}(X, \Delta)$ is the dual complex of a suitable quasi-étale cover of $(X, \Delta)$.

It is natural to conjecture that, under the assumptions of Theorem 36, $\pi_1(X^{\text{sm}})$—and hence also the other 2 groups—are finite, but we have only the following weaker result.

**Proposition 44** Let $(X, \Delta)$ be a dlt, logCY pair. Then

1. either $\hat{\pi}_1(X^{\text{sm}})$ is finite
2. or there is a quasi-étale cover $(\tilde{X}, \tilde{\Delta}) \to (X, \Delta)$ and a dominant map $q : \tilde{X} \to Z$ onto a non-uniruled variety.

In particular, if $(X, \Delta)$ has maximal intersection then the groups $\hat{\pi}_1(X^{\text{sm}})$ and $\hat{\pi}_1(\mathcal{DMR}(X, \Delta))$ are finite.

The key result we use is the following local variant.

**Theorem 45** [36] Let $(0 \in X)$ be a normal singularity over $\mathbb{C}$ that is potentially dlt. (That is, there is a divisor $\Delta$ such that $(X, \Delta)$ is dlt.) Then there is a Euclidean open neighborhood $0 \in V \subset X$ such that $\hat{\pi}_1(V \setminus \{0\})$ is finite.

A formal argument can be used to go from the local to the global situation.

---

**46 (Local and global fundamental groups)** Let $M$ be a compact simplicial complex and $C \subset M$ a closed subcomplex. Set $N := M \setminus C$ and let $B \subset N$ be a nowhere dense, closed subcomplex. Assume that every $p \in B$ has a contractible neighborhood $U_p$ such that $V_p := U_p \cap (N \setminus B)$ is connected.

By repeatedly applying van Kampen’s theorem we get the following.

**Claim 46.1.** The natural map $\pi_1(N \setminus B) \to \pi_1(N)$ is surjective and its kernel is generated (as a normal subgroup) by the images of the maps $\pi_1(V_p) \to \pi_1(N \setminus B)$. The same holds for the pro-finite completion $\hat{\pi}_1(N \setminus B) \to \hat{\pi}_1(N)$.
Assume now that the \( \hat{\pi}_1(V_p) \) are all finite. Since \( M \) is compact, the images \( \text{im}[\hat{\pi}_1(V_p) \to \hat{\pi}_1(N \setminus B)] \) form finitely many conjugacy classes. Thus there is a finite index normal subgroup \( G \subset \hat{\pi}_1(N \setminus B) \) such that

\[
G \cap \text{im} \left[ \hat{\pi}_1(V_p) \to \hat{\pi}_1(N \setminus B) \right] = \{1\} \quad \forall p \in B. \tag{46.2}
\]

Let \( s_B : (N \setminus B)^{\sim} \to (N \setminus B) \) be the corresponding cover; it extends to a ramified cover \( s : \tilde{N} \to N \).

By construction the maps \( \hat{\pi}_1(\tilde{V}_p) \to \hat{\pi}_1(N \setminus \tilde{B}) \) are all trivial where \( \tilde{V}_p \) denotes the preimage of \( V_p \). Using (46.1) for \( \tilde{B} \subset \tilde{N} \) we obtain the following.

Claim 46.3. With the above notation and assumptions there is a natural isomorphism \( \hat{\pi}_1(\tilde{N} \setminus \tilde{B}) \cong \hat{\pi}_1(\tilde{N}) \). Furthermore, a similar isomorphism holds for every finite degree covering space of \( \tilde{N} \setminus \tilde{B} \).

(1) either \( \hat{\pi}_1(X^{\text{sm}}) \) is finite

(2) or there is a quasi-étale cover \( p : \tilde{X} \to X \) such that \( \hat{\pi}_1(\tilde{X}) = \hat{\pi}_1(\tilde{X}^{\text{sm}}) \) is infinite.

Proof By [17] there is a Whitney stratification

\[
X_0 := X \supset X_2 := \text{Sing} X \supset X_3 \supset \cdots \supset X_{n+1} = \emptyset
\]

such that \( X \) is (locally in the Euclidean topology) a product along each \( X_i \setminus X_{i+1} \) and \( \dim X_i = n - i \); see also [12, Part I. Chap. 1].

We are done if \( \hat{\pi}_1(X \setminus X_2) \) is finite. Otherwise, assume that \( \hat{\pi}_1(X \setminus X_i) \) is infinite for some \( i \). Let \( (0 \in Z_i) \) denote a general transversal slice of \( X \) along \( X_i \setminus X_{i+1} \). Then \( (0 \in Z_i) \) is potentially log terminal, hence, by Theorem 45, every point \( x \in X_i \setminus X_{i+1} \) has a Euclidean open neighborhood \( U_x := U_x \setminus X_i \) is connected and \( \hat{\pi}_1(V_x) \) is finite.

Choose a compactification \( \tilde{X} \supset X \) and apply (46.3) with \( M := \tilde{X}, C := X_{i+1} \cup (\tilde{X} \setminus X) \) and \( B := X_i \) to obtain a quasi-étale cover \( p : \tilde{X} \to X \) such that \( \hat{\pi}_1(\tilde{X} \setminus X_i) \cong \hat{\pi}_1(\tilde{X} \setminus \tilde{X}_{i+1}) \). Thus \( \hat{\pi}_1(\tilde{X} \setminus \tilde{X}_{i+1}) \) is also infinite.

We can now replace \( X \) by \( \tilde{X} \) and apply the above argument with \( i \) replaced by \( i+1 \). After \( n \) steps we get a quasi-étale cover \( \tilde{X}^{(n)} \to X \) such that \( \hat{\pi}_1(\tilde{X}^{(n)}) \) is infinite.

\[\square\]
48 (Proof of Proposition 44) Let \( p : \tilde{X} \to X \) be a quasi-étale cover. Then \( (\tilde{X}, \tilde{\Delta} := p^*\Delta) \) is also a dlt, logCY pair. If there is a dominant map \( q : \tilde{X} \to Z \) onto a non-uniruled variety then we are in the second case. Otherwise, as we noted in Paragraph 18, \( \tilde{X} \) is rationally connected, hence simply connected. Thus \( \hat{\pi}_1(X^\text{sm}) \) is finite by Corollary 47.

6 Construction of Fano models

Our main technical theorem says that every logCY pair \((X, \Delta)\) with maximal intersection is crepant birational to a logCY pair whose boundary is big. For some applications one needs to know what happens without maximal intersection and it is crucial to have very tight control over the exceptional divisors of the crepant birational equivalence.

Theorem 49 Let \((X, \Delta)\) be a \(\mathbb{Q}\)-factorial, dlt, logCY pair. Then there is a crepant birational map \(\phi : (X, \Delta) \to (\overline{X}, \overline{\Delta})\) to a logCY pair and a morphism \(q : \overline{X} \to Z\) such that

1. \(\overline{\Delta}^=\) fully supports a \(q\)-ample divisor,
2. every log canonical center of \((\overline{X}, \overline{\Delta})\) dominates \(Z\),
3. \(\tilde{E} \subset \overline{\Delta}^=\) for every \(\phi^{-1}\)-exceptional divisor \(\tilde{E} \subset \tilde{X}\) and
4. \(\phi^{-1}\) is an isomorphism over \(\tilde{X} \setminus \tilde{\Delta}^=\).

Note that in general \(\tilde{X}\) is neither \(\mathbb{Q}\)-factorial nor dlt, but, by (4), \(\tilde{X} \setminus \tilde{\Delta}^=\) is \(\mathbb{Q}\)-factorial and klt.

50 (Outline of the proof) The proof, inspired by [3], focuses on guaranteeing the properties (49.1–2), then we note that (49.3–4) are also achieved.

Assume that we have \((\tilde{X}, \tilde{\Delta})\) and let \(\overline{H}\) be a \(q\)-ample divisor supported on \(\overline{\Delta}^=\). Then \(- (K_{\tilde{X}} + \tilde{\Delta} - \epsilon \overline{H})\) is \(q\)-ample, thus \(q\) is a Fano contraction for the pair \((\tilde{X}, \tilde{\Delta} - \epsilon \overline{H})\). It is thus reasonable to hope that we can obtain it using MMP for \((X, \Delta - \epsilon H)\). We do not know what \(H\) should be, so we start by pretending that \(\overline{H} = \overline{\Delta}^=\) and \(\epsilon = 1\).

Step 1. Run the \((X, \Delta^< 1)\)-MMP. It ends with a Fano contraction \(g : (X_r, \Delta_{r}^< 1) \to Z_1\) where \(\Delta_{r}^= := - (K_{X_r} + \Delta_{r}^< 1)\) is \(g\)-ample. Thus (49.1) holds but we have no information on (49.2).

Step 2. Apply the canonical bundle formula of Paragraph 20 and write

\[
K_{X_r} + \Delta_r \sim_{\mathbb{Q}} g^* (K_{Z_1} + J_1 + B_1).
\]

A key point is that \((Z_1, J_1 + B_1)\) behaves very much like a logCY pair. (Conjecturally one can choose a divisor \(D_1 \sim_{\mathbb{Q}} J_1\) such that \((Z_1, D_1 + B_1)\) is a logCY pair.) Furthermore, (20.10) essentially says that \(B_1^= \neq 0\) iff \((X_r, \Delta_r)\)
has a log canonical center that does not dominate $Z_1$. (This actually holds only for suitable birational models.)

**Step 3.** Repeat Step 1 for $(Z_1, J_1 + B_1)$ to get $Z_1 \dashrightarrow Z_2$. (In general there are serious technical problems working with pairs $(Z, \Delta)$ where $\Delta$ is not known to be effective, but one can do this much.)

**Step 4.** Show that a suitable birational model of the composite $X \dashrightarrow Z_1 \dashrightarrow Z_2$ satisfies (49.1) and repeat the arguments. At each step the dimension of $Z_i$ decreases, so eventually we stop and then (49.1–2) hold.

**Step 5.** Prove that the properties (49.3–4) also hold.

**Definition 51** Given a logCY pair $(Y, \Delta_Y)$ and a dominant morphism $q : Y \to Z$, a log canonical center $W \subset Y$ is called **vertical** if $q|_W : W \to Z$ is not dominant.

(Proof of Theorem 49) Note that $Z = \bar{X} = X$ works if $(X, \Delta)$ is klt and Paragraph 16 settles the case when $\mathcal{DMR}(X, \Delta)$ is disconnected. Thus we may assume that $\text{Supp} \Delta = \emptyset$ is connected and nonempty.

Next we find a logCY pair $(\tilde{X}, \tilde{\Delta})$ that is crepant birational to $(X, \Delta)$ and a morphism $q : \tilde{X} \to Z$ such that

1. $\tilde{\Delta} = \emptyset$ fully supports a divisor $\tilde{H}$ such that the corresponding map $\psi := \psi_{\tilde{H}} : \tilde{X} \dashrightarrow \text{Proj}_Z \bigoplus_m q_*(O_{\tilde{X}}(m\tilde{H}))$ is a morphism outside the vertical log canonical centers and an isomorphism outside the log canonical centers,

2. the properties (49.3–4) are satisfied and

3. $\dim Z$ is the smallest possible.

(The trivial case $Z = \tilde{X} = X$ satisfies (52.1), (52.3) and (52.4) since any divisor is relatively ample for the identity morphism. Thus such $(\tilde{X}, \tilde{\Delta})$ and $q : \tilde{X} \to Z$ exist.)

Note that property (1) is clearly invariant under birational maps that are isomorphisms outside the vertical log canonical centers (we will have many such maps during the proof). We then have to go from this property to $q$-ampleness at the end using Lemma 56.

We are done if the property (49.2) also holds, thus assume that there is a vertical log canonical center. Next we first improve $\tilde{X}$, then $Z$ and finally obtain a contradiction by showing that $\dim Z$ is not the smallest possible.

By assumption (49.4) $\tilde{X} \setminus \tilde{\Delta} = \emptyset$ is $\mathbb{Q}$-factorial and dlt, thus we can choose a $\mathbb{Q}$-factorial dlt modification $\pi : (\tilde{X}_1, \tilde{\Delta}_1) \to (\tilde{X}, \tilde{\Delta})$ that is an isomorphism over $\tilde{X} \setminus \tilde{\Delta} = \emptyset$. Then $\pi^*H$ also satisfies property (1). Thus we may replace $(\tilde{X}, \tilde{\Delta})$ by $(\tilde{X}_1, \tilde{\Delta}_1)$ and assume from now on that $(\tilde{X}, \tilde{\Delta})$ is $\mathbb{Q}$-factorial and dlt.
Next write $\tilde{\Delta} = \Gamma^h + \Gamma^v$ as the sum of its horizontal and vertical parts. By Lemma 57, after extracting some divisors we may also assume that $\Gamma^v \neq 0$ and $(\tilde{X}, \tilde{\Delta} < 1 + \Gamma^h)$ has no vertical log canonical centers. [This is the only step that introduces divisors that appear in (49.3).]

For the 3rd step, we run a $(K\tilde{X} + \tilde{\Delta} < 1 + \Gamma^h)$-MMP using [18, Thm. 1.1]. After replacing $\tilde{X}$ with the resulting minimal model and $Z$ by the corresponding canonical model, we may also assume that $K\tilde{X} + \tilde{\Delta} < 1 + \Gamma^h$ is the pull-back of a $\mathbb{Q}$-divisor from $Z$. Lemma 55 guarantees that (49.4) still holds. We can now apply the canonical bundle formula of Paragraph 20 and write

$$K\tilde{X} + \tilde{\Delta} < 1 + \Gamma^h \sim_{\mathbb{Q}} q^*(K_Z + J + B^h) \quad \text{and} \quad K\tilde{X} + \tilde{\Delta} < 1 + \Gamma^h + \Gamma^v \sim_{\mathbb{Q}} q^*(K_Z + J + B).$$

Note that the $J$-parts are the same by (20.5) and the $\mathbb{Q}$-linear equivalence $\Gamma^v \sim_{\mathbb{Q}} q^*(B - B^h)$ implies that $\Gamma^v = q^*(B - B^h)$. Furthermore, using (20.7–10) we see that $B - B^h$ equals the reduced part $B = 1$. Thus $\Gamma^v = q^*(B = 1)$.

So far we have focused on optimizing the choice of $\tilde{X}$; next we consider $Z$. As we change $Z$, we have to keep changing $\tilde{X}$ to ensure that we have a morphism $\tilde{X} \to Z$.

As explained in Paragraph 53, we can run a $(Z, J + B^h)$-MMP to get $(\tilde{Z}, \tilde{J} + \tilde{B}^h)$ and $q_Z : \tilde{Z} \to W$. Using Lemma 54 we see that, by passing to a suitable crepant birational model of $(\tilde{X}, \tilde{\Delta} < 1 + \Gamma^h + \Gamma^v)$ we may assume that we have a morphism $\tilde{q} : \tilde{X} \to \tilde{Z}$ and $\Gamma^v = q^*(B = 1)$ remains true. Furthermore, $Z \to \tilde{Z}$ is an isomorphism outside $\tilde{Z}\setminus B = 1$ by Lemma 55 and so (49.4) still holds by Lemma 54.

We aim to use

$$(\tilde{X}, \tilde{\Delta} = \tilde{\Delta} < 1 + \tilde{\Gamma}^h + \tilde{\Gamma}^v) \quad \text{and} \quad q_Z \circ \tilde{q} : \tilde{X} \to \tilde{Z} \to W$$

to contradict the minimality of dim $Z$. The question is whether property (1) is satisfied or not. The problem is that there are log canonical centers that are vertical for $q$ but not vertical for $q_Z \circ \tilde{q}$.

We know that $\tilde{B} = 1$ supports a $q_Z$-ample divisor $H_Z$ and $\tilde{\Delta} = 1$ supports a divisor $\tilde{H}$ whose restriction over $\tilde{Z}\setminus \tilde{B} = 1$ is the pull back of a relatively ample divisor under a birational morphism. Since $\tilde{q}^{-1}(\tilde{B} = 1) = \text{Supp} \tilde{\Gamma}^v \subset \text{Supp} \tilde{\Delta} = 1$, we see that $\tilde{H}, \tilde{q}^*H_Z$ are both supported on $\tilde{\Delta} = 1$. If $\tilde{H}_X$ is $\tilde{q}$-ample then $\tilde{H}_X + m\tilde{q}^*H_Z$ is $q_Z \circ \tilde{q}$-ample for $m \gg 1$ and we are done.

Finally, we aim to use Lemma 56 for $A := \tilde{H}$ to pass to another crepant birational model

$$(\tilde{X}, \tilde{\Delta}) \quad \text{and} \quad \tilde{X} \xrightarrow{\tilde{q}} \tilde{Z} \xrightarrow{q_Z} W$$
where $\tilde{H}_X$ is $\bar{q}$-ample. Note that we can not apply Lemma 56 directly to $(\tilde{X}, \tilde{\Delta})$ since $\tilde{\Gamma}^v$ contributes lc centers that do not dominate $\tilde{Z}$. However, $\tilde{\Gamma}^v$ is the pull-back of $\tilde{B}^{=1}$, thus numerically $\bar{q}$-trivial. Therefore we can apply Lemma 56 to $(\tilde{X}, \tilde{\Delta} \prec_1 + \tilde{\Gamma}^h)$ to get $(\tilde{X}, \tilde{\Delta}) \to \tilde{Z}$ as needed.

By property (1), $\psi_H$ is an isomorphism over $\tilde{X} \backslash \tilde{\Delta}^{=1}$ thus (49.4) still holds for the final $(\tilde{X}, \tilde{\Delta})$.

We have used several lemmas during the previous proof.

53 (MMP for $(Z, J + B)$) In general very little is known about MMP for a pair $(Z, \Theta)$ where $\Theta$ is not effective but this can be done in the special case when $K_Z + \Theta$ is not pseudo-effective and $(Z, \Theta)$ is a limit of klt pairs.

To make this precise, fix an ample divisor $H$ and assume that there is an effective $\mathbb{Q}$-divisor $\Delta_\epsilon \sim_\mathbb{Q} \Theta + \epsilon H$ such that $(Z, \Delta_\epsilon)$ is klt and $K_Z + \Delta_\epsilon$ is still not pseudo-effective. Then every step of the $(K_Z + \Delta_\epsilon)$-MMP with scaling of $H$ is also a step of the $(K_Z + \Theta)$-MMP with scaling of $H$ and the program ends with a Fano contraction $q : (Z_m, \Theta_m + \epsilon H_m) \to W$ such that $-(K_{Z_m} + \Theta_m)$ is $q$-ample.

We remark that $Z$ does not have to be $\mathbb{Q}$-factorial. Given a birational contraction $g_i : (Z_i, \Theta_i) \to W_i$, let $Z_i+1$ be the canonical modification of $(W_i, (g_i)_*\Theta_i)$; the latter exist by [1]. The termination of this process also follows from the finiteness of models as in [1, Thm. E].

The following is essentially proved in [2, 18].

Lemma 54 Let $(X, \Delta)$ be a $\mathbb{Q}$-factorial, quasi-projective, dlt pair with a projective morphism $f : X \to Z$ such that $f_*\mathcal{O}_X = \mathcal{O}_Z$. Assume that $K_X + \Delta \sim_\mathbb{Q} f^*N$ for a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $N$ on $Z$. Let $g : Z \to W$ be a birational morphism such that $g_*\mathcal{O}_Z = \mathcal{O}_W$ and assume that $R(Z/W, N) := \oplus_m g_*\mathcal{O}_Z(mN)$ is a finitely generated algebra over $W$. Set $Z^+ := \text{Proj}_W R(Z/W, N)$. Then there is a model $X^+$ of $(X, \Delta)$ over $W$, such that

1. $f$ extends to a morphism $f^+ : X^+ \to Z^+$ and
2. $X \dasharrow X^+$ is an isomorphism over the locus where $Z \dasharrow W$ is an isomorphism.

Proof It follows from [18, 2.12] that $(X, \Delta)$ has a good minimal model $X^+$ over $W$. So $X^+$ admits a morphism to $Z^+$. Furthermore, by [18, 2.9], we know that $X^+$ can be obtained by running a suitable $(K_X + \Delta)$-MMP over $W$. Thus all steps are isomorphisms over the locus where $Z \dasharrow W$ is an isomorphism. \hfill $\Box$

Lemma 55 Let $g : (X, \Delta_1 + \Delta_2) \to Z$ be a relative logCY where the $\Delta_i$ are effective and $\Delta_2$ is $\mathbb{Q}$-Cartier. Let $\phi : X \dasharrow X'$ be a rational map obtained by running a $(K_X + \Delta_1)$-MMP. Then $\phi^{-1}$ is an isomorphism on $X' \backslash \text{Supp} \Delta_2'$.
Proof. It is enough to check this when $\phi$ is a single step of the MMP. So assume that $\phi$ corresponds to the extremal ray $R \subset NE(X/Z)$. Let $g : X \rightarrow W$ denote the corresponding contraction and set

$$X' := \text{Proj}_W \sum_{m \geq 0} \mathcal{O}_W ([mK_W + mg_*\Delta_1]) = \text{Proj}_W \sum_{m \geq 0} \mathcal{O}_W ([m \cdot mg_*\Delta_2]).$$

If a curve $C' \subset X'$ is contracted by $g'$ then $(C' \cdot \Delta'_2) < 0$, thus $\text{Ex}(g') \subset \text{Supp} \Delta'_2$. Therefore $g'$ is an isomorphism on $X' \setminus \text{Supp} \Delta'_2$.

Pick $w \in W$. If $g'$ is a local isomorphism over $w$ but $g$ is not then there is a $g$-exceptional divisor $E$ such that $w \in g(E)$. Let $C \subset X$ be an irreducible curve such that $[C] \in R$. Then $(C \cdot (K_X + \Delta_1)) < 0$ hence $(C \cdot \Delta_2) > 0$. In particular $E \subset \text{Supp} \Delta_2$. An effective exceptional divisor can not be relatively nef, thus $w \in \text{Supp}(g_*\Delta_2)$. Therefore $\phi^{-1}$ is an isomorphism on $X' \setminus \text{Supp} \Delta'_2$.

**Lemma 56** Let $g : (X, \Delta) \rightarrow Z$ be a relative log CY and $A$ a $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor on $X$. Let $Z^0 \subset Z$ be an open subset such that

1. the restriction of $|A|$ to $X_0 := g^{-1}(Z_0)$ is base point free and big,
2. no lc center of $(X, \Delta)$ is contained in $g^{-1}(Z \setminus Z^0)$.

Then there is a relative log CY $g' : (X', \Delta') \rightarrow Z$ and a crepant birational contraction map $\phi : (X, \Delta) \dasharrow (X', \Delta')$ such that $\phi_*A$ is $q'$-ample.

Proof. Let $\pi : \tilde{X} \rightarrow X$ be a $\mathbb{Q}$-factorial dlt modification of $(X, \Delta)$ with $K_{\tilde{X}} + \tilde{\Delta} = \pi^*(K_X + \Delta)$. Let $D \in \pi^*|A|$ a general member. Then $D$ does not contain any of the log canonical centers of $(\tilde{X}, \tilde{\Delta})$, hence $(\tilde{X}, \tilde{\Delta} + \epsilon D)$ is also dlt for $0 < \epsilon \ll 1$. By [18] it has a relative canonical model $(X', \Delta' + \epsilon D')$. Then $D'$ is $g'$-ample since $K_{X'} + \Delta'$ is numerically $g'$-trivial and $D' \sim_g \phi_*A$. ☐

**Lemma 57** Let $(Y, \Delta)$ be a $\mathbb{Q}$-factorial dlt pair and $\mathcal{C} \subset \mathcal{DMR}(Y, \Delta)$ a closed subcomplex that contains all the vertices. Let $W_C \subset Y$ be the union of the strata corresponding to those simplices that are not contained in $C$.

Then there is a $\mathbb{Q}$-factorial dlt pair $(Y_C, \Delta_C)$ and a proper, crepant, birational morphism $g : (Y_C, \Delta_C) \rightarrow (Y, \Delta)$ such that

1. $g$ is an isomorphism over $Y \setminus W_C$,
2. all the exceptional divisors have discrepancy $-1$ and
3. $\mathcal{DMR}(Y_C, g_*^{-1}\Delta) = \mathcal{C}$.

Proof. Assume first that $(Y, \Delta)$ is a simple normal crossing pair. Then the process is straightforward. First we blow up the strata corresponding the maximal dimensional simplices not contained in $C$, then the strata corresponding to codimension 1 simplices not contained in $C$ and so on.
In general, first we take a thrifty resolution $(Y_1, \Delta_1) \to (Y, \Delta)$ as in [28, Sect. 2.5] where $\Delta_1$ is the birational transform of $\Delta$. Then $(Y_1, \Delta_1)$ is a simple normal crossing pair, thus we can apply the previous argument to get $(Y_2, \Delta_2) \to (Y_1, \Delta_1) \to (Y, \Delta)$ where $\Delta_2$ is the birational transform of $\Delta_1$. Finally we take a minimal model for $(Y_2, \Delta_2 + E)$, where $E$ is the exceptional divisor of $Y_2 \to Y$. As in [28, 1.35], this removes the exceptional divisors whose discrepancy is $> -1$.  

In general the pair $(\bar{X}, \bar{\Delta})$ in Theorem 49 is neither unique nor dlt and for some applications we need a carefully chosen dlt modification. Start with a projective logCY pair $(X, \Delta)$. After extracting divisors with discrepancy $-1$ we may assume that $(X, \Delta)$ is dlt and $\mathbb{Q}$-factorial. Then apply Theorem 49 to get $(\bar{X}, \bar{\Delta})$ and $q : \bar{X} \to Z$. Finally let $g : (X_s, \Delta_s) \to (\bar{X}, \bar{\Delta})$ be a $\mathbb{Q}$-factorial, dlt, crepant modification that extracts precisely all $\phi$-exceptional divisors and possibly some other divisors with discrepancy $-1$; see [28, 1.38]. Then $g^{-1} \circ \phi$ is a crepant birational map and we obtain the following.

**Corollary 58** Every logCY pair $(X, \Delta)$ is crepant birational to a $\mathbb{Q}$-factorial, dlt, logCY pair $(X_s, \Delta_s)$ with crepant birational maps

$$
\psi := g^{-1} \circ \phi : (X, \Delta) \dashrightarrow (\bar{X}, \bar{\Delta}) \xleftarrow{g} (X_s, \Delta_s)
$$

such that there is a morphism $q_s : X_s \to Z$ with the following properties.

1. Every log canonical center of $(X_s, \Delta_s)$ dominates $Z$,
2. $\Delta_s = 1 \subset g^{-1}(\bar{\Delta} = 1)$,
3. $g^{-1}(\bar{\Delta} = 1)$ fully supports a $q_s$-big and $q_s$-semi-ample divisor,
4. every prime divisor $E_s \subset X_s$ has non-empty intersection with $g^{-1}(\bar{\Delta} = 1)$.
5. $\psi^{-1}$ is a crepant, birational contraction and $E_s \subset \Delta_s = 1$ for every $\psi^{-1}$-exceptional divisor $E_s \subset X_s$.

## 7 The boundary of crepant birational pairs

So far we have studied the dual complex of logCY pairs $(X, \Delta)$. Some of the invariance results have their counterparts for $[\Delta]$ as well.

**Theorem 59** Let $g : (X_1, \Delta_1) \dashrightarrow (X_2, \Delta_2)$ be a proper, crepant birational map of dlt log pairs as in (9). Assume that either the $\Delta_i$ are effective or the $(X_i, \Delta_i)$ are simple normal crossing pairs. Set $D_i := \Delta_i = 1$. Then

1. $H^i(D_1, \mathcal{O}_{D_1}) \cong H^i(D_2, \mathcal{O}_{D_2})$ for every $i$ and
2. $\pi_1(D_1) \cong \pi_1(D_2)$. 

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Proof In the simple normal crossing case \( g \) can be factored as a composite of smooth blow-ups and blow-downs whose centers have simple normal crossings with \( \text{Supp} \Delta \). For any one such blow-up the claimed isomorphisms (59.1–2) are clear.

In order to reduce to this case let \((X, \Delta)\) be a dlt log pair with \( \Delta \) effective; thus \([\Delta] = \Delta^{=1} \). Let \( p : (Y, \Delta_Y) \to (X, \Delta) \) a thrifty resolution as in [28, Sect. 2.5]. Set \( D := p^* \Delta^{=1} \). Consider the exact sequence

\[
0 \to \mathcal{O}_Y(-D) \to \mathcal{O}_Y \to \mathcal{O}_D \to 0.
\]

Note that \( R^i p_* \mathcal{O}_Y = 0 \) for \( i > 0 \) since \( X \) has rational singularities and [28, 2.87] implies that \( R^i p_* \mathcal{O}_Y(-D) = 0 \) for \( i > 0 \) and \( p_* \mathcal{O}_D = \mathcal{O}_{[\Delta]} \). Therefore \( H^i(D, \mathcal{O}_D) \cong H^i([\Delta], \mathcal{O}_{[\Delta]}) \).

It remains to establish that (59.2) holds for \( p \). For this it is enough to prove that every fiber of \( g := p|_D : D \to [\Delta] \) is simply connected. If \([\Delta] \) is normal, this follows from [24, 7.5] in the algebraic case and [35] in general. The proof of [24, 7.5] works in the non-normal case; most likely the same applies to [35] but we have not checked the details. One can, however, go around this problem as follows.

Fix a point \( x \in [\Delta] \) and let \( S_1, \ldots, S_r \subset X \) be the irreducible components of \([\Delta] \) passing through \( x \) with birational transforms \( D_1, \ldots, D_r \subset Y \).

We use induction on \( r \) and on the dimension. If \( r = 1 \) then, as we noted, \( g^{-1}(x) \) is simply connected. For \( r > 1 \), let \( h \) denote the restriction of \( g \) to \( D_r \) and \( g_{r-1} \) the restriction of \( g \) to \( D_1 \cup \cdots \cup D_{r-1} \). Both \( h^{-1}(x) \) and \( g_{r-1}^{-1}(x) \) are simply connected by induction. Their intersection is isomorphic to the fiber we get from the lower dimensional thrifty resolution

\[
(D_r, \text{Diff}^*_{D_r} \Delta_Y) \to (S_r, \text{Diff}^*_{S_r} \Delta),
\]

hence also simply connected. Thus \( g^{-1}(x) \) is also simply connected by van Kampen’s theorem. \( \square \)

8 Examples

Example 60 Start with \( X_1 = \mathbb{P}^1 \) and \( \Delta = (0:1)+(1:0) \). Let \( \tau_1 \) be the involution \((x:y) \mapsto (y:x)\). Set \( (X_n, \Delta_n) := (X_1, \Delta_1)^n \), \( \tau_n \) the involution \((\tau_1, \ldots, \tau_1)\) and \( (Y_n, \Delta_n^Y) := (X_n, \Delta_n)/\tau_n \).

Then \( \mathcal{D}M\mathcal{R}(X_n, \Delta_n) \) is the boundary of the cube of dimension \( n \), thus PL-homeomorphic to \( \mathbb{S}^{n-1} \) while \( \mathcal{D}M\mathcal{R}(Y_n, \Delta_n^Y) \cong \mathbb{R}^{n-1} \).

Example 61 Let \( G \) be a group of order \( m \). It acts on \( \mathbb{P}^{m-1} \) by permuting the coordinates. Pick a general hyperplane and move it around by \( G \) to get a logCY
The dual complex of Calabi–Yau pairs

pair \((\mathbb{P}^{m-1}, \Delta_m)\). The dual complex is the boundary of the \((m - 1)\)-simplex, thus PL-homeomorphic to \(S^{m-2}\).

Next we take the quotient by \(G\). The boundary consists of 1 divisor with complicated self-intersections. It is thus better to take the barycentric subdivision first and then take the quotient. See [6, Rem. 10] on how to do this with blow-ups to obtain a dlt pair \((X_m, \Delta_m)\). The resulting quotient need not be dlt; such examples led to the introduction of quotient-dlt pairs in [6, Sect. 5].

The dual complex of the quotient is \(S^{m-2}/G\). Note that usually \(G\) has fixed points on \(S^{m-2}\). The only exception occurs when \(G\) is cyclic of prime order \(m = p\). In these cases \(\pi_1(D\text{MR}(X, \Delta)) \cong \mathbb{Z}/p\).

Example 62 Let \(A\) be an Abelian surface and \(X = A/(\pm)\) the corresponding Kummer surface with minimal resolution \(X'\). Then \(X\) and \(X'\) are crepant-birational. Note that \(X'\) is smooth and simply connected while \(\pi_1(X_{sm})\) is infinite.

There are similar examples involving only rational surfaces. Start with \(\mathbb{P}^1 \times \mathbb{P}^1\) and an involution \(\tau\) with 4 isolated fixed points. Set \(X = (\mathbb{P}^1 \times \mathbb{P}^1)/\langle \tau \rangle\) with minimal resolution \(X'\). In this example \(X'\) is smooth and simply connected while \(\pi_1(X_{sm}) \cong \mathbb{Z}/2\).

Example 63 Let \(Y\) be a smooth CY and \(L\) a very ample line bundle. Set \(X := \mathbb{P}_Y(L + O_Y)\) and \(D, D' \subset X\) the 2 sections. Then \((X, D + D')\) is a logCY.

Choose indexing such that \(O_X(D)|_D \cong L\). Then \(D\) is big and semi-ample. Thus \(D + D'\) supports—but not fully supports—a big and semi-ample divisor and yet \(X\) is not rationally connected.

The linear system \(|D|\) is base point free; let \(D_1, \ldots, D_n\) be general members. Set \(\Delta := \frac{1}{n}(D_1 + \cdots + D_n) + D'\). Then \((X, \Delta)\) is a log CY with a morphism \(X \to Y\) to a CY but, for \(n \geq 3\), \((X, \Delta) \to Y\) is not a product, not even locally analytically at the generic point.

This contrasts with the product theorem of [21] for Calabi–Yau varieties without boundary divisors.

Example 64 Let \((X_i, \Delta_i)\) be two logCY pairs. The product

\[(X_1, \Delta_1) \times (X_2, \Delta_2) := (X_1 \times X_2, X_1 \times \Delta_2 + \Delta_1 \times X_2)\]

is also logCY and

\[\text{D} \text{M} \text{R} ((X_1, \Delta_1) \times (X_2, \Delta_2)) \cong \text{D} \text{M} \text{R} (X_1, \Delta_1) \ast \text{D} \text{M} \text{R} (X_2, \Delta_2)\]

where \(\ast\) denotes the join. Thus if \(\text{D} \text{M} \text{R} (X_i, \Delta_i) \cong S^{n_i}/G_i\) then

\[\text{D} \text{M} \text{R} ((X_1, \Delta_1) \times (X_2, \Delta_2)) \cong S^{n_1 + n_2 + 1}/G_1 \times G_2.\]
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