Left non-degenerate set-theoretic solutions of the Yang-Baxter equation and dynamical extensions of q-cycle sets✩

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Abstract

A first aim of this paper is to give sufficient conditions on left non-degenerate bijective set-theoretic solutions of the Yang-Baxter equation so that they are non-degenerate. In particular, we extend the results on involutive solutions obtained by Rump in [36] and answer in a positive way to a question posed by Cedó, Jespers, and Verwimp [18, Question 4.2]. Moreover, we develop a theory of extensions for left non-degenerate set-theoretic solutions of the Yang-Baxter equation that allows one to construct new families of set-theoretic solutions.

Keywords: q-cycle set, cycle set, set-theoretic solution, Yang-Baxter equation, brace, skew brace

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1. Introduction

A set-theoretic solution of the Yang-Baxter equation, or shortly a solution, is a pair \((X, r)\) where \(X\) is a non-empty set and \(r\) is a map from \(X \times X\) into itself such that

\[
r_1 r_2 r_1 = r_2 r_1 r_2,
\]

where \(r_1 := r \times id_X\) and \(r_2 := id_X \times r\). Let \(\lambda_x : X \to X\) and \(\rho_y : X \to X\) be maps such that

\[
r(x, y) = (\lambda_x(y), \rho_y(x))
\]

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for all $x, y \in X$. A set-theoretic solution of the Yang-Baxter equation $(X, r)$ is said to be a left [right] non-degenerate if $\lambda_x \in \text{Sym}(X)$ [$\rho_x \in \text{Sym}(X)$], for every $x \in X$, and non-degenerate if it is left and right non-degenerate.

Drinfeld’s paper [21] stimulated much interest in this subject. In recent years, after the seminal papers by Gateva-Ivanova and Van den Bergh [25] and Etingof, Schedler and Soloviev [22] the involutive solutions $r$, i.e., $r^2 = id_{X \times X}$, have received a lot of attention.

To study involutive solutions, many theory involving a lot of algebraic structures has developed. Several examples of involutive solutions have obtained by using groups, racks, and quandles [23, 24, 15, 2, 17]. In 2005, Rump introduced cycle sets, non-associative algebraic structures that allow one to find involutive left non-degenerate solutions. A set $X$ endowed of an operation $\cdot$ is said to be a cycle set if the left multiplication $\sigma_x : X \rightarrow X, y \mapsto x \cdot y$ is invertible, for every $x \in X$, and the relation

$$(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z)$$

is satisfied, for all $x, y, z \in X$. By cycle sets several families of involutive solutions were determined and several interesting results were obtained (see, for example, [20, 6, 8, 10, 42, 9, 5]).

In 2000, Lu, Yan, and Zhu [34] and Soloviev [41] started the study of non-degenerate solutions that are not necessarily involutive. To obtain new families of bijective non-degenerate solutions, in 2015 Guarnieri and Vendramin [26] introduced the algebraic structure of skew brace, a generalisation of the braces introduced by Rump in [37]. Some works where this structure is studied are [11, 13, 16, 40, 19, 29], just to name a few. As skew braces are the analogue version of braces for non-involutive non-degenerate solutions, $q$-cycle sets, introduced recently by Rump [39], are the analogue version of cycle sets for left non-degenerate solutions that are not necessarily involutive. Recall that a set $X$ with two binary operations $\cdot$ and $: \cdot$ is a $q$-cycle set if $\sigma_x : X \rightarrow X, y \mapsto x \cdot y$ is invertible, for every $x \in X$, and

$$(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z)$$

$$(x : y) : (x : z) = (y : x) : (y : z)$$

hold, for all $x, y, z \in X$. If $(X, \cdot, :)$ is a $q$-cycle set, then the pair $(X, r)$, where $r(x, y) := (\sigma_x^{-1}(y), \sigma_x^{-1}(y) : x)$, for all $x, y \in X$, is a left non-degenerate solution.

Conversely, if $(X, r)$ is a left non-degenerate solution, then the operations $\cdot$ and $:\cdot$ given by $x \cdot y := \lambda_x^{-1}(y)$ and $x : y := \rho_{\lambda_x^{-1}}(y)$, for all $x, y \in X$, give rise to a $q$-cycle set. Thanks to this correspondence, one can move from left non-degenerate solutions to $q$-cycle sets.
In the first part of the paper, we focus on non-degeneracy of bijective solutions. In particular, using q-cycle sets, we show that any finite left non-degenerate bijective solutions is right non-degenerate, giving a positive answer to [18, Question 4.2]. In this way, we also extend the corresponding result for finite involutive solutions provided by Rump in [36, Theorem 2] in terms of cycle sets, and by Jespers and Okniński in [30, Corollary 2.3] in terms of monoids of I-type. In addition, using the result on finite left non-degenerate solutions, we give sufficient conditions to find a class non-degenerate solutions that include properly the finite left non-degenerate ones.

In Section 4, we introduce an equivalence relation for a q-cycle set which we call retraction, in analogy to the retraction of cycle sets, that is compatible with respect to the two operations. Since the retraction of a non-degenerate q-cycle set is again a non-degenerate q-cycle set, we obtain an alternative proof of [28, Theorem 3.3] for non-degenerate solutions $(X, r)$ with the additional property of $r$ bijective. Clearly, we include the result showed in [33, Lemma 8.4].

The final goal of this paper is to develop a theory of extensions of q-cycle sets. Following the ideas of Vendramin for cycle sets [42] and of Nelson and Watterberg for biracks [35], we introduce a suitable notion of dynamical extension of q-cycle sets. Even if dynamical extensions are often hard to find, we introduce several examples of dynamical extensions that are relatively easy to compute. Moreover, we introduce several families of dynamical extensions that provide non-degenerate solutions that are different from those obtained by skew braces. As an application, we construct a semidirect product of q-cycle sets, which is a generalization of the semidirect product of cycle sets introduced by Rump [38] and, referring to [4, Problem 4.15], gives rise to a general definition of semidirect product of biquandles.

2. Basic results and examples

To study non-degenerate solutions, Rump recently introduced in [39] the notion of q-cycle set. Recall that a set $X$ together with two binary operations $\cdot$ and $:$ is a q-cycle set if the function $\sigma_x : X \to X, y \mapsto x \cdot y$ is bijective, for every $x \in X$, and

\begin{align*}
(x \cdot y) \cdot (x \cdot z) &= (y : x) \cdot (y \cdot z) \\
(x : y) : (x : z) &= (y : x) : (y : z) \\
(x \cdot y) : (x \cdot z) &= (y : x) \cdot (y : z)
\end{align*}

hold for all $x, y, z \in X$. Hereinafter, we denote by $q$ and $q'$ the squaring maps related to $\cdot$ and $:$, i.e., the maps given by

$$q(x) := x \cdot x \quad \text{and} \quad q'(x) := x : x,$$
for every $x \in X$.
A q-cycle set $(X, \cdot, :)$ is said to be regular if the function $\delta_x : X \rightarrow X, y \mapsto x : y$ is bijective, for every $x \in X$, and non-degenerate if it is regular and $q$ and $q'$ are bijective. At first sight, the notion of non-degeneracy introduced by Rump seems different, but [39, Corollary 2] ensures that the two definitions are equivalent. The left non-degenerate solution $(X, r)$ provided by a q-cycle set $X$ is given by

$$r(x, y) := (\sigma_x^{-1}(y), \sigma_x^{-1}(y) : x),$$

for all $x, y \in X$, is a left-non degenerate solution. Conversely, if $(X, r)$ is a left non-degenerate solution and $r(x, y) = (\lambda_x(y), \rho_y(x))$, for all $x, y \in X$, then the operations $\cdot$ and $:$ given by

$$x \cdot y := \lambda_x^{-1}(y) \quad x : y := \rho_{\lambda_y^{-1}(y)}(y)$$

for all $x, y \in X$, give rise to a q-cycle set, see [39, Proposition 1]. As one would expect, non-degenerate q-cycle sets corresponds to non-degenerate bijective solutions.

Q-cycle sets allow us to construct a lot of families of solutions obtained in several recent papers. To show this, in the rest of this section we collect several examples of q-cycle sets and highlight some connections with other algebraic structures.

At first, note that q-cycle sets with $x \cdot y = x : y$, for all $x, y \in X$, correspond to cycle sets. Clearly, in this case, [1], (2), and (3) are reduced to

$$(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z).$$

These examples of q-cycle sets determine the “celebrated” left non-degenerate solutions that are also involutive, that in particular are maps $r : X \times X \rightarrow X \times X$ of the form

$$r(x, y) = (\sigma_x^{-1}(y), \sigma_x^{-1}(y) \cdot x),$$

for all $x, y \in X$.

In [39] Rump showed some connections between q-cycle sets and other algebraic structures. In particular, he observed that skew braces correspond to particular q-cycle sets for which the squaring maps $q$ and $q'$ coincide (see [39, Corollary 2]). In this context, we note that an analogue result follows for left semi-braces, algebraic structures that are a generalisation of skew braces. We specify that, in this paper, for a left semi-brace we mean the structure introduced in [12], named left cancellative left semi-brace in [31]. Specifically, we say that a set $B$ with two operations $+ \text{ and } \circ$ is a left semi-brace if $(B, +)$ is a left cancellative semigroup, $(B, \circ)$ is a group, and

$$a \circ (b + c) = a \circ b + a \circ (a^- + c)$$
holds, for all \(a, b, c \in B\), where \(a^-\) is the inverse of \(a\) with respect to \(\circ\). If \((B, +, \circ)\) is a left semi-brace, by [12, Theorem 9] we have that the map \(r_B : B \times B \to B \times B\) defined by

\[
r_B(a, b) = \left(a \circ (a^- + b), (a^- + b) \circ b\right),
\]

for all \(a, b \in B\), is a left non-degenerate solution. Hence, the q-cycle set associated to \(B\) is the structure \((B, \cdot, :)\) where \(\cdot\) and \(:\) are given by

\[
\begin{align*}
a \cdot b &= \lambda_{a^-}(b) = a^- \circ (a + b) \\
a : b &= (\rho_a(b^-))^- = a^- \circ (b + a),
\end{align*}
\]

for all \(a, b \in B\). Therefore, it is now easy to see that the squaring maps \(q\) and \(q'\) coincide also for q-cycle sets obtained by left semi-braces that are not skew-braces. In the following example, we show a family of concrete examples of q-cycle sets determined by left semi-braces.

**Example 1.** Let \((B, \circ)\) be a group, \(f\) an idempotent endomorphism of \((B, \circ)\), and \((B, +, \circ)\) the left semi-brace in [12, Example 10] where the sum is given by

\[
a + b := b \circ f(a),
\]

for all \(a, b \in B\). Since the left non-degenerate solution associated to \((B, +, \circ)\) is the map \(r_B : B : B \times B \to B \times B\) given by

\[
r(a, b) = \left(a \circ b \circ f(a)^{-1}, f(a)\right),
\]

we obtain that \((B, \cdot, :)\) is a q-cycle with \(\cdot\) and \(:\) defined by

\[
\begin{align*}
a \cdot b := a^{-1} \circ b \circ f(a) \\
a : b := f(b),
\end{align*}
\]

for all \(a, b \in B\). Clearly, if \(f \neq id_B\), then \((B, \cdot, :)\) is not regular.

More in general, let us observe that the hypothesis of idempotency on \(f\) is unnecessary to have a structure of q-cycle set on the group \((B, \circ)\). Obviously, \((B, \cdot, :)\) is regular if and only if the map \(f\) is bijective.

However, in general, the squaring maps \(q\) and \(q'\) do not coincide, as one can see in Examples 2 and Examples 3 below.

**Examples 2.**

1) Let \(k \in \mathbb{N}\), \(B := \mathbb{Z}/m\mathbb{Z}\), \(\cdot\) and \(:\) the binary operations on \(B\) given by \(x \cdot y := y + k\) and \(x : y = y\), for all \(x, y \in B\). Then, \((B, \cdot, :)\) is a regular q-cycle set and the solution associated to \((B, \cdot, :)\) is the map \(r : B \times B \to B \times B\) given by

\[
r(x, y) = (y - k, x),
\]

for all \(x, y \in B\).
2) Let \( B := \mathbb{Z}/3\mathbb{Z} \) and \( \alpha \) the element of \( \text{Sym}(B) \) given by \( \alpha := (0 \ 1) \). Moreover, let \( \cdot \) and \( : \) the binary operations on \( B \) given by \( x \cdot y := \alpha(y) \) if \( x \in \{0,1\} \) and \( x : y := y \) otherwise and \( x : y := y \) if \( x \in \{0,1\} \) and \( x : y := \alpha(y) \) otherwise. Then, \((B,\cdot,:)\) is a regular q-cycle set and the solution \( r \) associated to \((B,\cdot,:)\) is defined by

\[
r(x,y) = \begin{cases} 
(\alpha(y), x) & \text{if } x \in \{0,1\} \\
(y, 2) & \text{if } x = 2 
\end{cases},
\]

for all \( x, y \in B \). It is a routine computation to verify that \( r \) satisfies \( r^4 = \text{id}_{B \times B} \).

The previous examples are all regular q-cycle sets. We conclude the section by showing examples of q-cycle sets that are not regular.

**Examples 3.**

1) Let \( X \) be a set, \( \cdot \) and \( : \) two operations on \( X \) given by

\[
x \cdot y := y \quad x : y := k
\]

for all \( x, y \in X \), where \( k \in X \). Then, it is easy to check that the structure \((X,\cdot,:}\) is a q-cycle set. Clearly, if \(|X| > 1\), then \((X,\cdot,:)\) is not regular. Moreover, the left non-degenerate solution associated to \((X,\cdot,:)\) is the map \( r : X \times X \to X \times X \) given by

\[
r(x,y) = (y,k),
\]

for all \( x, y \in X \). In particular, let us observe that \( r \) satisfies the relation \( r^3 = r^2 \).

2) Let \( X \) be a left quasi-normal semigroup, i.e., \( X \) is a semigroup such that \( x(yz) = (xz)y \), for all \( x, y, z \in X \). Then, it is straightforward to check that \( X \) endowed by the operations \( \cdot \) and \( : \) defined by

\[
x \cdot y := y \quad x : y := yx,
\]

for all \( x, y \in X \), is a q-cycle set. Clearly, \((X,\cdot,:)\) in general is not regular. Note that the solution associated to \( X \) is given by

\[
r(x,y) = (y,xy),
\]

for all \( x, y \in X \), and it coincides with that provided in [14, Examples 6.2]. In particular, we have that \( r \) satisfies the property \( r^5 = r^3 \).
3. Non-degeneracy of q-cycle sets

This section focuses on non-degeneracy of q-cycle sets. Rump in [36, Theorem 2] and Jespers and Okniński in [30, Corollary 2.3], using different languages, showed that every finite involutive left non-degenerate solution is non-degenerate. Recently, Cedó, Jespers and Verwimp asked if the same result is true for every finite bijective left non-degenerate solution.

**Question.** [18, Question 4.2] Is every finite left non-degenerate solution non-degenerate?

A first aim is to use the theory of q-cycle sets to answer in the positive sense. Moreover, we give sufficient conditions to find a class of non-degenerate solutions that include properly the finite left non-degenerate ones.

Initially, we provide the “q-version” of the proof of [36, Theorem 2] provided recently by Jedlicka, Pilitowska, and Zamojska-Dzienio in [28, Proposition 4.7]. Hereinafter, for any regular q-cycle set $X$, we denote by $G(X)$ the subgroup of $\text{Sym}(X)$ given by

$$G(X) := \langle \sigma_x | x \in X \rangle \cup \{\delta_x | x \in X\}$$

and we call it the permutation group associated to $X$.

**Lemma 4.** Let $(X, \cdot, :)$ be a regular q-cycle set such that the associated permutation group $G(X)$ is finite. Then the squaring maps $q$ and $q'$ are surjective.

**Proof.** At first, note that thanks to (2), we have

$$(\sigma_x^{-1}(y) : x) : (\sigma_x^{-1}(y) : x) = y : (x : x)$$

for every $x, y \in X$. Now, since $G(X)$ is finite, there exists a natural number $m$ such that $\delta_x^m(z) = z$, for every $z \in X$. If $m \in \{1, 2\}$ then $q'$ is surjective by [4]. If $m > 2$ and $z \in X$, then

$$z = \delta_z^m(z)$$

$$= \delta_z^{m-2}(z : (z : z))$$

$$= \delta_z^{m-2}(q'(\sigma_z^{-1}(z) : z)))$$

$$= \delta_z^{m-3}(z : q'(\sigma_z^{-1}(z) : z)))$$

and applying repeatedly [3], we obtain that $q'(v) = z$, for some $v \in X$, hence $q'$ is surjective. In a similar way, one can show that $q$ is surjective, therefore the thesis follows.

**Theorem 5.** Let $(X, \cdot, :)$ be a finite regular q-cycle set. Then $X$ is non-degenerate.
Proof. If $X$ is finite then so $G(X)$ is, hence by the previous lemma $q$ and $q'$ are surjective. Again, since $X$ is finite, $q$ and $q'$ are also injective, hence the thesis.

As a consequence of the previous theorem, we obtain the following result that extends the analogous one for finite involutive solutions showed by Rump in [31, Theorem 2] and by Jespers and Okniński in [30, Corollary 2.3].

Corollary 6. Let $(X, r)$ be a finite bijective left non-degenerate solution. Then $(X, r)$ is non-degenerate.

Proof. It follows by the previous theorem and [39, Proposition 1].

Now, we use the previous result on finite solutions to find a larger class of non-degenerate solutions. At first, we have to introduce a preliminary lemma. We recall that a subset $Y$ of a $q$-cycle set $X$ is said to be $G(X)$-invariant if $g(y) \in Y$, for all $g \in G(X)$ and $y \in Y$.

Lemma 7. Let $X$ be a regular $q$-cycle set and $Y$ a $G(X)$-invariant subset of $X$. Then, $Y$ is a regular $q$-cycle set with respect to the operations induced by $X$. Moreover, if $X$ is non-degenerate then so $Y$ is.

Proof. Is is a straightforward calculation.

Theorem 8. Let $X$ be a regular $q$-cycle set such that the orbits of $X$ with respect to the action of $G(X)$ have finite size. Then, $X$ is non-degenerate.

Proof. We have to show that the squaring maps $q$ and $q'$ are bijective. Let $x$ be an element of $X$ and $Y$ the orbit of $X$ such that $x \in Y$. By the previous lemma and the hypothesis, $Y$ is a finite regular $q$-cycle set and, by Theorem 5 it is non-degenerate. Therefore, there exist $x_1, x_2 \in Y$ such that $q(x_1) = x$ and $q'(x_2) = x$. Hence, we have that the squaring maps of $X$ are surjective. Now, suppose that $y, z, t$ are elements of $X$ such that $q(y) = q(z) = t$ and let $T$ be the orbit of $X$ such that $t \in T$. Then, we have that $y = \sigma_y^{-1}(t) \in T$ and $z = \sigma_z^{-1}(t) \in T$. Since $T$ is non-degenerate, we obtain that $y = z$, hence the squaring map $q$ of $X$ is injective. In a similar way, one can show that the squaring map $q'$ of $X$ is injective.

We conclude the section by translating the previous result in terms of solutions. Note that, if $X$ is a regular $q$-cycle set and $(X, r)$ the associated solution, where $r(x, y) := (\lambda_x(y), \rho_y(x))$, for all $x, y \in X$, then the associated permutation group $G(X)$ coincides with the subgroup of $\text{Sym}(X)$ generated by the set $\{\lambda_x | x \in X\} \cup \{\eta_x | x \in X\}$, where $\eta_x(y) = \rho_{\lambda_x^{-1}(x)}(y)$, for all $x, y \in X$. 

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Corollary 9. Let \((X, r)\) be a bijective left non-degenerate solution such that the orbits of \(X\) respect to the action of \(G(X)\) have finite size. Then \((X, r)\) is non-degenerate.

Proof. It follows by the previous theorem and [39, Proposition 1].

4. Retraction of regular q-cycle sets

In this section we introduce the “q-version” of the retract relation introduced by Rump in [36] for cycle sets. Namely, we introduce a suitable notion of retraction that is a congruence of q-cycle sets, i.e., it is a congruence with respect to the two operations \(\cdot\) and \(:\). Furthermore, we show that, for the specific class of non-degenerate q-cycle sets, the quotient of a q-cycle set by its retraction is still a q-cycle set.

Definition 1. Let \((X, \cdot, :)\) be a regular q-cycle set and \(\sim\) the relation on \(X\) given by

\[x \sim y :\iff \sigma_x = \sigma_y\text{ and }\delta_x = \delta_y\]

for every \(x, y \in X\). Then \(\sim\) will be called the retract relation.

In analogy to [36, Lemma 2], we show that \(\sim\) is always a congruence of a q-cycle set.

Proposition 10. Let \((X, \cdot, :)\) be a regular q-cycle set. Then, the retraction \(\sim\) is a congruence of \((X, \cdot, :)\).

Proof. Let \(x, y, z, t\) be such that \(x \sim y\) and \(z \sim t\). Then, using (1),

\[(x \cdot z) \cdot (x \cdot k) = (y \cdot z) \cdot (y \cdot k) = (z : y) \cdot (z \cdot k) = (t : y) \cdot (t \cdot k) = (y \cdot t) \cdot (y \cdot k),\]

therefore \(\sigma_{x \cdot z} = \sigma_{y \cdot t}\). Moreover, using (2),

\[(x : z) \cdot (x \cdot k) = (y : z) \cdot (y \cdot k) = (z \cdot y) \cdot (z \cdot k) = (t \cdot y) \cdot (t \cdot k) = (y : t) \cdot (y \cdot k),\]

therefore \(\sigma_{x : z} = \sigma_{y : t}\). Similarly, one can show that \(\delta_{x \cdot z} = \delta_{y \cdot t}\) and \(\delta_{x : z} = \delta_{y : t}\), hence the thesis.

To prove the main result of this section we make use of the extension on the free group \(F(X)\) of a q-cycle set \(X\). Initially, we recall the following proposition contained in [39].

Proposition 11. [39, Theorem 1] Let \(X\) be a non-degenerate regular q-cycle set. Then \(X\) can be extended to a q-cycle set on the free group \((F(X), \circ)\) such that \(1 \cdot a = 1 : a = a\) and
1) \((a \circ b) \cdot c = a \cdot (b \cdot c)\)
2) \((a \circ b) : c = a : (b : c)\)
3) \(a \cdot (b \circ c) = ((c : a) \cdot b) \circ (a \cdot c)\)
4) \(a : (b \circ c) = ((c \cdot a) : b) \circ (a : c)\),

are satisfied, for all \(a, b, c \in F(X)\). Furthermore, in \(F(X)\) we have that
\[
x \cdot y^{-1} = (\delta y^{-1}(x) \cdot y)^{-1} \quad \text{and} \quad x : y^{-1} = (\sigma_y^{-1}(x) : y)^{-1}
\]
hold, for all \(x, y \in X\).

As a consequence of the previous proposition we have the following result.

**Lemma 12.** Let \(X\) be a non-degenerate regular q-cycle set. Then, the following hold:

1) \(a \cdot x \in X\), for all \(a \in F(X)\) and \(x \in X\);
2) \(a : x \in X\), for all \(a \in F(X)\) and \(x \in X\);
3) for all \(x, x_1, \ldots, x_n \in X\) and \(\epsilon, \epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}\), there exist \(t_1, \ldots, t_n \in X\) and \(\eta, \eta_1, \ldots, \eta_n \in \{1, -1\}\) such that
   \[
   \sigma_x \cdot (x_1^{\epsilon_1} \circ \cdots \circ x_n^{\epsilon_n}) = t_1^{\eta_1} \circ \cdots \circ t_n^{\eta_n};
   \]
4) for all \(x, x_1, \ldots, x_n \in X\) and \(\epsilon, \epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}\), there exist \(t_1, \ldots, t_n \in X\) and \(\eta, \eta_1, \ldots, \eta_n, \epsilon \in \{1, -1\}\) such that
   \[
   \delta_x \cdot (x_1^{\epsilon_1} \circ \cdots \circ x_n^{\epsilon_n}) = t_1^{\eta_1} \circ \cdots \circ t_n^{\eta_n}.
   \]

**Proof.**
1) If \(x, y \in X\), by 1) in Proposition 11, it follows that
   \[
   \sigma_{x^{-1}} \cdot \sigma_x (y) = x^{-1} \cdot (x \cdot y) = (x^{-1} \circ x) \cdot y = 1 \cdot y = y
   \]
   and, analogously, \(\sigma_x \cdot \sigma_{x^{-1}} (y) = y\), i.e., \(\sigma_{x^{-1}} = \sigma_x^{-1}\). Thus, if \(a \in F(X)\), there exist \(x_1, \ldots, x_n \in X\) and \(\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}\) such that \(a = x_1^{\epsilon_1} \circ \cdots \circ x_n^{\epsilon_n}\), hence, by 1) in Proposition 11, it holds that
   \[
   a \cdot x = \sigma_{x_1^{\epsilon_1}} \cdots \sigma_{x_n^{\epsilon_n}} (x) = \sigma_{x_1^{\epsilon_1}} \cdots \sigma_{x_n^{\epsilon_n}} (x) \in X.
   \]
2) The proof is similar to 1).
3) We proceed by induction on \(n \in \mathbb{N}\). Let \(n = 1, x, x_1 \in X, \epsilon, \epsilon_1 \in \{-1, 1\}\). Let us note that if \(\epsilon_1 = 1\), then the thesis follows by 1). If \(\epsilon_1 = -1\), since by Proposition 11 \(a \cdot 1 = 1\), for every \(a \in F(X)\), by 3) in Proposition 11 it follows that \(1 = x'^{-1} \cdot (x_1 \circ x_1^{-1}) = ((x_1^{-1} : x') \cdot x_1) \circ (x' : x_1^{-1})\) and so
   \[
   \sigma_{x'} (x_1^{-1}) = ((x_1^{-1} : x') \cdot x_1)^{-1}
   \]
   \]
with \((x_1^{-1} : x^t) \cdot x_1 \in X\) by 1). Now, suppose the thesis holds for a natural number \(n\) and let \(x_1, \ldots, x_n \in X\), \(\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}\). Then, by 3) in Proposition 11, we have that
\[
\sigma_{x^t} (x_1^{-1} \circ \cdots \circ x_n^{-1}) = \left( \left( x_1^\epsilon \circ \cdots \circ x_n^\epsilon \right) \circ \left( x_1^{-1} \circ \cdots \circ x_n^{-1} \right) \right) \circ \left( x^t \cdot x_n^{-1} \right),
\]
therefore, by induction hypothesis, the thesis follows.

4) The proof is similar to 3).

Now, let us introduce the following preliminary lemmas.

**Lemma 13.** Let \(X\) be a non-degenerate regular \(q\)-cycle set and \(x, y \in X\). Then, the following statements are equivalent

1) \(x \cdot z = y \cdot z\) and \(x : z = y : z\), for every \(z \in X\);

2) \(x \cdot z = y \cdot z\) and \(x : z = y : z\), for every \(z \in F(X)\).

**Proof.** Since every element of \(F(X)\) can be written as \(x_1^\epsilon \circ \cdots \circ x_n^\epsilon\) for some \(x_1, \ldots, x_n \in X\) and \(\epsilon_1, \ldots, \epsilon_n \in \{1, -1\}\) we prove “1) \(\Rightarrow\) 2)” by induction on \(n\).

Hence, suppose that \(x : t = y : t\) and \(x : t = y : t\), for every \(t \in X\).

If \(n = 1\) and \(z = x_1^\epsilon\) for some \(x_1 \in X\), \(\epsilon_1 \in \{1, -1\}\) we have two cases: if \(\epsilon_1 = 1\), then \(x : z = y : z\) by the hypothesis, if \(\epsilon_1 = -1\), by equality (2) and the hypothesis we have that
\[
x : (x_1 : x_1) = y : (x_1 : x_1) \Rightarrow \]
\[
\Rightarrow (x_1 \cdot \sigma_{x_1}^{-1}(x)) : (x_1 : x_1) = (x_1 \cdot \sigma_{x_1}^{-1}(y)) : (x_1 : x_1) \]
\[
\Rightarrow (\sigma_{x_1}^{-1}(x) \cdot x_1) : (\sigma_{x_1}^{-1}(x) : x_1) = (\sigma_{x_1}^{-1}(y) : x_1) : (\sigma_{x_1}^{-1}(y) : x_1) \]
\[
\Rightarrow q'(\sigma_{x_1}^{-1}(x) : x_1) = q'(\sigma_{x_1}^{-1}(y) : x_1) \]
\[
\Rightarrow \sigma_{x_1}^{-1}(x) : x_1 = \sigma_{x_1}^{-1}(y) : x_1 \]

and, since \(F(X)\) is a group, it follows that \((\sigma_{x_1}^{-1}(x) : x_1)^{-1} = (\sigma_{x_1}^{-1}(y) : x_1)^{-1}\) and hence, by Proposition 11 \(x : x_1^{-1} = y : x_1^{-1}\). Similarly, one can show that \(x : x_1^{-1} = y : x_1^{-1}\).

Now, suppose the thesis for a natural number \(n\) and let \(z \colonequals x_1^\epsilon \circ \cdots \circ x_n^{-1} \in F(X)\), where \(x_1, \ldots, x_n \in X\) and \(\epsilon_1, \ldots, \epsilon_n \in \{1, -1\}\). Then, by 4) in Proposition 11 we have that
\[
x : (x_1^\epsilon \circ \cdots \circ x_n^{-1}) = \left( \left( x_1^\epsilon \circ \cdots \circ x_n^{-1} \right) \circ \left( x : x_n^{-1} \right) \right) \circ \left( x : x_n^{-1} \right) \]
\[
= \left( \left( x_1^\epsilon \circ \cdots \circ x_n^{-1} \right) \circ \left( \delta_{x_n^{-1}}^{-1} \left( x_1^\epsilon \circ \cdots \circ x_n^{-1} \right) \right) \right) \circ \left( x : x_n^{-1} \right) \]
\[
= \left( \left( x : x_n^{-1} \right) \circ \left( \delta_{x_n^{-1}}^{-1} \left( x_1^\epsilon \circ \cdots \circ x_n^{-1} \right) \right) \right) \circ \left( x : x_n^{-1} \right) \]

and similarly
\[
y : (x_1^\epsilon \circ \cdots \circ x_n^{-1}) = \left( \left( y : x_n^{-1} \right) \circ \left( \delta_{x_n^{-1}}^{-1} \left( x_1^\epsilon \circ \cdots \circ x_n^{-1} \right) \right) \right) \circ \left( y : x_n^{-1} \right) .
\]
Note that, by Lemma 12, there exist \( t_1, ..., t_n \in X \) and \( \eta_1, ..., \eta_n \in \{1, -1\} \) such that \( \delta_{x_{n+1}}^{-1} (x_1^{t_1} \circ ... \circ x_n^{t_n}) = t_1^{\eta_1} \circ ... \circ t_n^{\eta_n} \), and this fact, together with the inductive hypothesis, implies that
\[
x : \delta_{x_{n+1}}^{-1}(x_1^{t_1} \circ ... \circ x_n^{t_n}) = y : \delta_{x_{n+1}}^{-1}(x_1^{t_1} \circ ... \circ x_n^{t_n})).
\]

Moreover, again by inductive hypothesis, it follows that \( x : x_{n+1}^{t_n} = y : x_{n+1}^{t_n} \). Therefore, we showed that \( x : z = y : z \). In a similar way, one can show that \( x \cdot z = y \cdot z \), hence “1) \( \Rightarrow \) 2)” The converse implication is trivial. \( \square \)

Let \( X \) be a regular non-degenerate q-cycle set and \( F(X) \) its extension to the free group on \( X \). In [39, Section 4] Rump define the socle of \( F(X) \), denoted by \( Soc(F(X)) \), as the set
\[
Soc(F(X)) := \{ a \mid a \in F(X), a \cdot b = a : b = b \ \forall \ b \in F(X) \}
\]
and he showed that \( Soc(F(X)) \) is a subgroup of \( F(X) \) and that the factor \( F(X)/Soc(F(X)) \) is again a non-degenerate q-cycle set. From this fact the following lemma follows.

**Lemma 14.** Let \( X \) be a non-degenerate q-cycle set and \( F(X) \) its extension to the free group on \( X \). Then, \( Ret(F(X)) \) is a regular non-degenerate q-cycle set.

**Proof.** It is easy to see that \( x \sim y \) if and only if \( x \circ y^{-1} \in Soc(F(X)) \) for every \( x, y \in F(X) \), hence the result follows by the previous remark. \( \square \)

**Theorem 15.** Let \( X \) be a regular q-cycle set and \( Ret(X) \) its retraction. Then, \( X \) is non-degenerate if and only if \( Ret(X) \) is a non-degenerate q-cycle set.

**Proof.** To avoid confusion, for any element \( x \in X \) we denote by \( \bar{x} \) its equivalence class with respect to the retract relation on \( X \) and by \( \tilde{x} \) its equivalence class with respect to the retract relation on \( F(X) \).

Suppose that \( X \) is non-degenerate. Then, the function
\[
G : X \rightarrow Ret(F(X)), \ x \mapsto \tilde{x}
\]
is a homomorphism of q-cycle sets. Moreover, by Lemma 13 we have that \( G(x) = G(y) \) if and only if \( \bar{x} = \bar{y} \), hence \( Ret(X) \cong G(X) \) as algebraic structures. If \( z \in X \) then
\[
\bar{z} \cdot \bar{x} = \bar{z} \cdot \bar{x}, \quad \bar{z} : \bar{x} = \bar{z} : \bar{x}
\]
\[
\bar{z^{-1}} \cdot \bar{x} = \bar{z^{-1}} \cdot x = \bar{\sigma_{z^{-1}}(x)}
\]
and
\[
\bar{z^{-1}} : \bar{x} = \bar{z^{-1}} : x = \bar{\delta_{z^{-1}}(x)}
\]

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and these equalities imply that $G(X)$ is a $G(\text{Ret}(F(X)))$-invariant subset of $\text{Ret}(F(X))$, hence, by Lemma 7 it is a non-degenerate q-cycle set. Therefore, $\text{Ret}(X)$ is a non-degenerate q-cycle set.

Conversely, suppose that $\text{Ret}(X)$ is a non-degenerate q-cycle set. We have to show that the maps $q$ and $q'$ of $X$ are bijective. Let $x, y \in X$ such that $x \cdot x = y \cdot y$. Then, $\tilde{x} \cdot \tilde{x} = \tilde{y} \cdot \tilde{y}$ and, by the non-degeneracy of $\text{Ret}(X)$, we have that $\tilde{x} = \tilde{y}$. Hence, $y \cdot x = x \cdot x = y \cdot y$, therefore $x = y$ and the injectivity of $q$ follows. Moreover, if $x \in X$, there exists a unique $\tilde{y} \in \text{Ret}(X)$ such that $\tilde{y} \cdot \tilde{y} = \tilde{x}$. Since $\tilde{x} = \tilde{y} \cdot \sigma_y^{-1}(x)$, we have that $\sigma_y^{-1}(x) = \tilde{y}$, and hence

$$x = \sigma_y(\sigma_y^{-1}(x)) = \sigma_{\sigma_y^{-1}(x)}(\sigma_y^{-1}(x)) = q(\sigma_y^{-1}(x))$$

therefore $q$ is bijective. In the same way, one can show that $q'$ is bijective.  

If $X$ is a non-degenerate q-cycle set and $(X, r)$ is the associated solution, it is easy to see that $x \sim y$ if and only if $\lambda_x = \lambda_y$ and $\rho_x = \rho_y$. Indeed, if $x, y \in X$,

$$x \sim y \iff \sigma_x = \sigma_y \text{ and } \delta_x = \delta_y$$

$$\iff \sigma_x^{-1} = \sigma_y^{-1} \text{ and } \delta_x q' = \delta_y q'$$

$$\iff \sigma_x^{-1} = \sigma_y^{-1} \text{ and } q'(\sigma_x^{-1}(x) : z) = q'(\sigma_y^{-1}(y) : z) \forall z \in X$$

$$\iff \lambda_x = \lambda_y \text{ and } \rho_x = \rho_y.$$  

Hence, the previous theorem give us an alternative proof of [28, Theorem 3.3], in terms of solutions $(X, r)$ having bijective map $r$. Moreover, it shows a kind of converse for left non-degenerate bijective solutions: indeed, if $X$ retracts to a non-degenerate solution then so $X$ is.

**Corollary 16.** Let $(X, r)$ be a bijective left non-degenerate solution of the Yang-Baxter equation and $\text{Ret}(X)$ the retraction of the associated q-cycle set. If $(X, r)$ is non-degenerate then, for every $\tilde{x} \in \text{Ret}(X)$, we can define the function $\tilde{\lambda}_{\tilde{x}}$ by

$$\tilde{\lambda}_{\tilde{x}} : \text{Ret}(X) \longrightarrow \text{Ret}(X), \ \tilde{y} \mapsto \lambda_x(y)$$

for every $\tilde{y} \in \text{Ret}(X)$. Moreover, the pair $(\text{Ret}(X), \tilde{r})$, where $\tilde{r}$ is given by

$$\tilde{r}(\tilde{x}, \tilde{y}) := (\lambda_x(y), \lambda_x(y) : x)$$

is a bijective non-degenerate solution of the Yang-Baxter equation.

Conversely, suppose that the function $\tilde{\lambda}_{\tilde{x}}$ is well defined for every $\tilde{x} \in \text{Ret}(X)$ and that the pair $(\text{Ret}(X), \tilde{r})$ defined as above is a non-degenerate bijective solution. Then, $(X, r)$ is non-degenerate.
5. Dynamical extensions of q-cycle sets

Inspired by the extensions of cycle sets and racks, introduced by Vendramin in [12] and Andruskiwitsch and Graña in [1], and the dynamical cocycles of biracks [31], introduced by Nelson and Watterberg, in this section we develop a theory of dynamical extensions of q-cycle sets.

**Theorem 17.** Let \((X, \cdot, :)\) be a q-cycle set, \(S\) a set, \(\alpha : X \times X \times S \rightarrow \text{Sym}(S)\) and \(\alpha' : X \times X \times S \rightarrow S^S\) maps such that

\[
\alpha_{(x,y),(x,z)}(\alpha_{(y,z)}(s,t), \alpha_{(x,z)}(s,u)) = \alpha_{(y,x),(y,z)}(\alpha'_{(y,x)}(t,s), \alpha_{(y,z)}(t,u)) \tag{6}
\]

\[
\alpha'_{(x,y),(x,z)}(\alpha'_{(x,y)}(s,t), \alpha_{(x,z)}(s,u)) = \alpha'_{(y,x),(y,z)}(\alpha_{(y,x)}(t,s), \alpha'_{(y,z)}(t,u)) \tag{7}
\]

\[
\alpha'_{(x,y),(x,z)}(\alpha_{(x,y)}(s,t), \alpha_{(x,z)}(s,u)) = \alpha_{(y,x),(y,z)}(\alpha'_{(y,x)}(t,s), \alpha'_{(y,z)}(t,u)) \tag{8}
\]

hold, for all \(x, y, z \in X, s, t, u \in S\). Then, the triple \((X \times S, \cdot, :)\) where

\[
(x, s) \cdot (y, t) := (x \cdot y, \alpha_{(x,y)}(s,t))
\]

\[
(x, s) : (y, t) := (x : y, \alpha'_{(x,y)}(s,t)),
\]

for all \(x, y \in X\) and \(s, t \in S\), is a q-cycle set. Moreover, \((X \times S, \cdot, :)\) is regular if and only if \(\alpha'_{(x,y)}(s, -) \in \text{Sym}(S)\), for all \(x, y \in X\) and \(s \in S\).

**Proof.** Since

\[
((x, s) \cdot (y, t)) \cdot ((x, s) \cdot (z, u)) = ((x \cdot y) \cdot (x \cdot z), \alpha_{(x,y),(x,z)}(\alpha_{(x,y)}(s,t), \alpha_{(x,z)}(s,u)))
\]

and

\[
((y, t) : (x, s)) \cdot ((y, t) : (z, u)) = ((y : x) \cdot (y : z), \alpha_{(y,x),(y,z)}(\alpha'_{(y,x)}(t,s), \alpha_{(y,z)}(t,u))),
\]

for all \(x, y, z \in X, s, t, u \in S\), we have that (6) is equivalent to (1). In the same way one can show that (7) and (8) are equivalent to (2) and (3). The rest of the proof is a straightforward calculation. \(\Box\)

**Definition 2.** We call the pair \((\alpha, \alpha')\) dynamical pair and we call the q-cycle set \(X \times_{\alpha, \alpha'} S := (X \times S, \cdot, :)\) dynamical extension of \(X\) by \(S\).

Clearly, every dynamical extension of a cycle set \([12]\) is a dynamical extension of a q-cycle set for which \(\alpha = \alpha'\).

If \(X\) and \(S\) are q-cycle sets and

\[
\alpha_{(x,y)}(s,t) = \alpha'_{(x,y)}(s,t) := t
\]

for all \(x, y \in X, s, t \in S\), \(X \times_{\alpha, \alpha'} S\), is a dynamical extension which we call trivial dynamical extension of \(X\) by \(S\). The goal of the next section is to provide further examples of dynamical extensions.
Definition 3. Let $X$ and $Y$ be q-cycle sets. A homomorphism $p : X \rightarrow Y$ is called covering map if it is surjective and all the fibers

$$p^{-1}(y) := \{ x | x \in X, \; p(x) = y \}$$

have the same cardinality. Moreover, a q-cycle set $X$ is simple if, for every covering map $p : X \rightarrow Y$, we have that $|Y| = 1$ or $p$ is an isomorphism.

Examples 18.

1) Every q-cycle set having a prime number of elements is simple.

2) Let $X := \{0, 1, 2, 3\}$ be the q-cycle set given by

$$\sigma_0 := (1 \ 3) \quad \sigma_1 := (0 \ 3) \quad \sigma_2 := (0 \ 1 \ 3) \quad \sigma_3 := (0 \ 1)$$

$$\delta_0 = \delta_1 = \delta_3 := \text{id}_X \quad \delta_2 := (0 \ 1 \ 3).$$

Suppose that $p : X \rightarrow Y$ is a covering map. Hence, $Y$ has necessarily two elements. Moreover, since $p(0) \cdot p(0) = p(0) : p(0) = p(0)$, $Y$ is the q-cycle set given by $\sigma_y = \delta_y = \text{id}_Y$, for every $y \in Y$. This fact implies that $X$ has a subset $I$ of 2 elements such that $\sigma_x(I) = \delta_x(I) = I$, for every $x \in X$, but this is a contradiction. Then, $X$ is simple.

The following theorem, that is a “q-version” of [42, Theorem 2.12], states that every non-simple q-cycle set can be found as a dynamical extension of a smaller q-cycle set.

Theorem 19. Let $X$ and $Y$ be q-cycle sets and suppose that $p : Y \rightarrow X$ is a covering map. Then, there exist a set $S$ and a dynamical pair $(\alpha, \alpha')$ such that $Y \cong X \times_{\alpha, \alpha'} S$.

Proof. Since all the fibers $p^{-1}(x)$ are equipotent, there exist a set $S$ and a bijection $f_x : p^{-1}(x) \rightarrow S$. Let $\alpha : X \times X \times S \rightarrow \text{Sym}(S)$ and $\alpha' : X \times X \times S \rightarrow S^S$ given by

$$\alpha(x,z)(s,t) := f_{x,z}(f_x^{-1}(s) \cdot f_z^{-1}(t))$$

and

$$\alpha'(x,z)(s,t) := f_{x,z}(f_x^{-1}(s) \cdot f_z^{-1}(t))$$

for all $x, z \in X, s, t \in S$. Then,

$$\alpha(x,y,z)(\alpha(x,y)(r,s), \alpha(x,z)(r,t)) =$$

$$= f_{(x,y)}(x,z)(f_{x,y}^{-1}(\alpha(x,y)(r,s)) \cdot f_{x,z}^{-1}(\alpha(x,z)(r,t)))$$

$$= f_{(x,y)}(x,z)(f_{x,y}(f_x^{-1}(r) \cdot f_y^{-1}(s)) \cdot f_{x,z}(f_x^{-1}(r) \cdot f_z^{-1}(t)))$$

$$= f_{(x,y)}(x,z)((f_x^{-1}(r) \cdot f_y^{-1}(s)) \cdot (f_x^{-1}(r) \cdot f_z^{-1}(t)))$$

$$= f_{(y,z)}(y,z)((f_y^{-1}(s) : f_z^{-1}(t)) \cdot (f_y^{-1}(s) : f_z^{-1}(t)))$$

$$= f_{(x,y)}(x,z)(f_{y,z}^{-1}(f_{y,x}(f_y^{-1}(s)) : f_x^{-1}(r)) \cdot f_{y,z}(f_y^{-1}(s) : f_z^{-1}(t)))$$

$$= \alpha(x,y,z)(\alpha'(y,z)(s,r), \alpha(y,z)(s,t)).$$

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for all $x, y, z \in X$ and $r, s, t \in S$, hence (6) holds. In the same way, one can check that the pair $(\alpha, \alpha')$ verifies (7) and (8), hence, by Theorem 17, $X \times_{\alpha, \alpha'} S$ is a q-cycle set. The map $\phi : Y \to X \times_{\alpha, \alpha'} S, y \mapsto (p(y), f_{p(y)}(y))$ is a bijective homomorphism. Indeed,

$$\phi(y \cdot z) = (p(y \cdot z), f_{p(y \cdot z)}(y \cdot z))$$

$$= (p(y) \cdot p(z), f_{p(y \cdot z)}(f_{p(y)}(y)) \cdot f_{p(z)}(f_{p(z)}(z))))$$

$$= (p(y) \cdot p(z), \alpha_{(p(y), p(z))}(f_{p(y)}(y), f_{p(z)}(z)))$$

$$= (p(y), f_{p(y)}(y)) \cdot (p(z), f_{p(z)}(z))$$

$$= \phi(y) \cdot \phi(z),$$

for all $y, z \in Y$. In a similar way one can show that $\phi(y : z) = \phi(y) : \phi(z)$, for all $y, z \in Y$. By the surjectivity of $p$ and the bijectivity of the maps $f_y$ the bijectivity of $\phi$ also follows, hence the thesis.

6. Examples and constructions

In this section, we collect several both known and new examples of dynamical extension of q-cycle sets. Finally, we introduce a new construction of q-cycle sets, the semidirect product.

Some dynamical cocycles of biracks founded in \[4, 35, 27\] provide dynamical extensions of q-cycle sets (one can see this fact by a long but easy calculation). Using non-degenerate cycle sets, in \[42, 7, 3\] several examples of dynamical extensions were obtained.

We start by giving an example of a dynamical extension of a degenerate cycle set that is similar to those obtained in \[7\] and \[3\].

**Example 20.** Let $X$ be the cycle set on $\mathbb{Z}$ given by $x \cdot y := y - \min\{0, x\}$, for all $x, y \in X$ (see \[36, Example 1\]), and let $S$ be the Klein group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Let $\alpha : X \times X \times S \to \text{Sym}(S)$ be the function given by

$$\alpha_{(i,j)}((a, b), (c, d)) := \begin{cases} (c, d - (a - c)) & \text{if } i = j, \\ (c - b, d) & \text{if } i \neq j \end{cases}$$

for all $i, j \in X, (a, b), (c, d) \in S$, and set $\alpha' := \alpha$. Then, $X \times_{\alpha, \alpha'} S$ is a dynamical extension of $X$ by $S$.

By an easy calculation one can see that the previous example is an irretractable cycle set. Moreover, it is degenerate: indeed, $q(-2, 0, 0)$ is equal to $q(-1, 0, 0)$. Recently, Cedó, Jespers and Verwimp \[18\] studied the links between non-degeneracy and irretractability of cycle sets (note that they used the language of set-theoretic solutions). As mentioned in \[18, p. 12\], they did not
know whether there exist examples of irretractable degenerate cycle sets: in this direction, the previous example answers in the positive sense.

Inspired by the dynamical extensions provided in [3], in the following example we give a family of dynamical extensions that determine new q-cycle sets that are not cycle sets.

**Example 21.** Let $X$ be the cycle set given by $x \cdot y = y$ for every $x, y \in X$, and $G$ an abelian group. Let $\alpha, \alpha' : X \times X \times (G \times G) \rightarrow \text{Sym}(G \times G)$ be the maps given by

$$\alpha(x, y)((s_1, s_2), (t_1, t_2)) := \begin{cases} (t_1 + t_2 - s_2, t_2) & \text{if } x = y \\ (t_1, t_2 + s_1) & \text{if } x \neq y \end{cases}$$

and

$$\alpha'(x, y)((s_1, s_2), (t_1, t_2)) := \begin{cases} (t_1 - t_2 + s_2, t_2) & \text{if } x = y \\ (t_1, t_2 + s_1) & \text{if } x \neq y \end{cases}$$

for all $x, y \in X, s_1, s_2, t_1, t_2 \in G$. Then $(\alpha, \alpha')$ is a dynamical pair and hence $X \times_{\alpha, \alpha'} (G \times G)$ is a q-cycle set.

**Example 22.** Let $B$ be a group, $f$ an endomorphism of $B$ and $(B, \cdot, :)$ the q-cycle in Example II where $x \cdot y = x^{-1}yf(x)$ and $x : y = f(y)$, for all $x, y \in B$. Let $S = B$, $\alpha : B \times B \times S \rightarrow \text{Sym}(S)$ and $\alpha' : B \times B \times S \rightarrow S^S$ be the maps defined by

$$\alpha(x, y)(s, t) := x^{-1}t$$
$$\alpha'(x, y)(s, t) := f(t),$$

for all $x, y, s, t \in B$. Then, it is a routine computation to verify that $(\alpha, \alpha')$ is a dynamical pair. Thus, the dynamical extension $B \times_{\alpha, \alpha'} S$ of $B$ by $S$ is such that

$$(x, s) \cdot (y, t) = (x^{-1}yf(x), x^{-1}t)$$
$$(x, s) : (y, t) = (f(y), f(t)),$$

for all $x, y, s, t \in B$. Let us observe that the q-cycle set $B \times_{\alpha, \alpha'} S$ is regular if and only if $f$ is bijective. Moreover, the left non-degenerate solution associated to $B \times_{\alpha, \alpha'} S$ is given by

$$r((x, s), (y, t)) = ( (xyf(x)^{-1}, xt), (f(x), f(s)) ),$$

for all $x, y, s, t \in B$.

**Example 23.** Let $X$ be a left quasi-normal semigroup and $(X, \cdot, :)$ the q-cycle set in 2) of Examples III where $x \cdot y = y$ and $x : y = yx$, for all $x, y \in X$. Let
$S = X, \alpha : X \times X \times S \to \text{Sym}(S)$ and $\alpha' : X \times X \times S \to S^S$ be the maps defined by

$$\alpha_{(x,y)}(s,t) = t$$
$$\alpha'_{(x,y)}(s,t) = tx,$$

for all $x, y, s, t \in X$. Then, $(\alpha, \alpha')$ is a dynamical pair. In fact, clearly (6) is satisfied. Moreover, if $x, y, s, t \in X$, we have that

$$\alpha'_{(x,y)}(s,uy) = \alpha'_{(y,x)}(\alpha(y,x)(t,s), \alpha'_{(y,z)}(t,u))$$

and

$$\alpha'_{(x,y)}(s,uy) = \alpha'_{(y,x)}(\alpha(y,x)(t,s), \alpha'_{(y,z)}(t,u)),$$

i.e., (7) and (8) hold. Therefore, the dynamical extension $X \times_{\alpha,\alpha'} S$ of $X$ by $S$ is such that

$$(x,s) \cdot (y,t) = (y, t)$$
$$(x,s) : (y,t) = (yx, tx),$$

for all $x,y,s,t \in X$. Note that $X \times_{\alpha,\alpha'} S$ in general is not regular. Moreover, the left non-degenerate solution associated to the q-cycle set $X \times_{\alpha,\alpha'} S$ is given by

$$r ((x,s), (y,t)) = ((y,t), (xy, sy)).$$

Let us observe that this solution satisfies the property $r^5 = r^3$, just like for the solution associated to the q-cycle set $(X, \cdot, \cdot)$. The following example is the analogue of the abelian extension of cycle sets studied by Lebed and Vendramin [32].

**Example 24.** Let $X$ be a cycle set, $A$ an abelian group, $a, b, a', b' \in A$ and $f, f' : X \times X \to A$ given by

$$f(x,y) := \begin{cases} a & \text{if } x = y \\ b & \text{if } x \neq y \end{cases}$$

and

$$f'(x,y) := \begin{cases} a' & \text{if } x = y \\ b' & \text{if } x \neq y \end{cases}$$
for all $x, y \in X$. Define on $X \times A$ the function $\alpha, \alpha' : X \times X \times A \rightarrow \text{Sym}(A)$ given by

$$\alpha_{(x,y)}(s,t) = t + f(x,y)$$

and

$$\alpha'_{(x,y)}(s,t) = t + f'(x,y),$$

for all $x, y \in X$. Then, $(\alpha, \alpha')$ is a dynamical pair and hence $X \times_{\alpha, \alpha'} A$ is a $q$-cycle set. In particular, we have that

$$(x, s) \cdot (y, t) = (x \cdot y, t + f(x, y))$$

$$(x, s) : (y, t) = (x \cdot y, t + f'(x, y)),$$

for all $x, y \in X$.

A particular family of dynamical extensions allow us to define a kind of semidirect product of $q$-cycle sets. Since this construction is similar to the ones of other algebraic structures, for the convenience of the reader we write the two operations $\cdot$ and $:$ of the $q$-cycle set without using the dynamical pair.

**Proposition 25.** Let $X, S$ be $q$-cycle sets, $\theta : X \rightarrow \text{Aut}(S), x \mapsto \theta_x$ such that

$$\theta_{x \cdot y} \theta_x = \theta_{y \cdot x} \theta_y,$$

for all $x, y \in X$. Define on $X \times S$ the operations $\cdot$ and $:$ by

$$(x, s) \cdot (y, t) := (x \cdot y, \theta_{x \cdot y}(s) \cdot \theta_{y \cdot x}(t))$$

$$(x, s) : (y, t) := (x : y, \theta_{x : y}(s) : \theta_{y : x}(t))$$

for all $x, y \in X$, $s, t \in S$. Then, $(X \times S, \cdot, :)$ is a $q$-cycle set which we call the semidirect product of $X$ and $S$.

**Proof.** Let $(x, s), (y, t), (z, u) \in X \times S$. Then

$$(x, s) \cdot (y, t) = (x \cdot y, \theta_{x \cdot y}(s) \cdot \theta_{y \cdot x}(t))$$

$$= (x \cdot y, \theta_{x \cdot y}(s) \cdot \theta_{y \cdot x}(t) \cdot \theta_{x \cdot z}(s) \cdot \theta_{z \cdot x}(u))$$

$$= ((x \cdot y) \cdot (x \cdot z), \theta_{(x \cdot y) \cdot (x \cdot z)}(\theta_{x \cdot y}(s) \cdot \theta_{y \cdot x}(t)) \cdot \theta_{(x \cdot z) : (x \cdot y)}(\theta_{x \cdot z}(s) \cdot \theta_{z \cdot x}(u)))$$

and, thanks to the hypothesis, we obtain that

$$(x, y) \cdot (x, y) = \theta_{x \cdot y}(s) \cdot \theta_{y \cdot x}(t) \cdot \theta_{x \cdot z}(s) \cdot \theta_{z \cdot x}(u)$$

(9)

and

$$(y, z) \cdot (y, z) = \theta_{y \cdot z}(t) \cdot \theta_{z \cdot y}(u)$$

(10)
Indeed,
\[
\theta(x,y)(x,z)(\theta_x(y(s) \cdot \theta_y(x(t)) \cdot \theta_{x,z}(s) \cdot \theta_{x,z}(u)) = (\theta_x(y(s)) \cdot \theta_{x,z}(t)) \cdot \theta_{x,z}(s) \cdot \theta_{x,z}(u))
\]
\[
= (\theta_x(y(s)) \cdot \theta_{x,z}(t)) \cdot \theta_{x,z}(s) \cdot \theta_{x,z}(u))
\]
\[
= (\theta_x(y(s)) \cdot \theta_{x,z}(t)) \cdot \theta_{x,z}(s) \cdot \theta_{x,z}(u))
\]
\[
= (\theta_x(y(s)) \cdot \theta_{x,z}(t)) \cdot \theta_{x,z}(s) \cdot \theta_{x,z}(u))
\]

i.e., \[\text{(9)}\] holds. On the other hand, we have that
\[
\theta_{y,z}(t) \cdot \theta_{y,z}(s) = (\theta_{y,z}(t) \cdot \theta_{y,z}(s))
\]
and, by the hypothesis, it follows that
\[
\theta_{y,z}(t) \cdot \theta_{y,z}(s) = (\theta_{y,z}(t) \cdot \theta_{y,z}(s))
\]
\[
\theta_{y,z}(t) \cdot \theta_{y,z}(s) = (\theta_{y,z}(t) \cdot \theta_{y,z}(s))
\]
\[
\theta_{y,z}(t) \cdot \theta_{y,z}(s) = (\theta_{y,z}(t) \cdot \theta_{y,z}(s))
\]

i.e., \[\text{(10)}\] holds. Therefore, since we showed that the first members of \[\text{(9)}\] and \[\text{(10)}\] coincide and \(X\) is a \(\sigma\)-cycle set, we have that equality \[\text{(11)}\] holds. In a similar way, one can check that \[\text{(2)}\] and \[\text{(3)}\] hold, hence it remains to show that \(\sigma_{(x,s)}\) is bijective for every \((x,s) \in X \times S\). Suppose that \((x,s) \cdot (y,t) = (x,s) \cdot (z,u)\) for some \((x,s),(y,t),(z,u) \in X \times S\). Then,
\[
(x, y, \theta_{x,y}(s) \cdot \theta_{y,z}(t)) = (x, z, \theta_{x,z}(s) \cdot \theta_{x,z}(u))
\]
and, since \(X\) and \(S\) are \(\sigma\)-cycle sets, it follows that \(y = z\) and \(\theta_{y,z}(t) = \theta_{y,z}(u)\). Therefore, from the bijectivity of \(\theta_{y,z}\), we have that \(t = u\), hence \(\sigma_{(x,s)}\) is injective. Finally, we have that \((x,s) \cdot (\sigma_{x}^{-1}(y), \theta_{y}(s)) = (y,t)\), hence the thesis.

By semidirect product of \(\sigma\)-cycle sets we are able to construct further dynamical extensions.

**Example 26.** Let \(X := \{0,1,2\}\) be the \(\sigma\)-cycle set given by \(x \cdot y := y\), for all \(x,y \in X\), and
\[
x : 0 = x : 1 := 0 \quad x : 2 = 2,
\]
for every \(x \in X\). Moreover, let \(Y := \{0,1,2\}\) be the \(\sigma\)-cycle set given by
\[
x \cdot y = x : y = y,
\]
for all \(x,y \in Y\), and \(\theta : X \rightarrow \text{Aut}(Y)\) given by \(\theta(0) = \theta(1) := (0\ 1)\) and \(\theta(2) := id_Y\). Then, the semidirect product \((X \times Y, \cdot)\) is a \(\sigma\)-cycle set of order 9. In particular, note that it is not regular.
Note that, if $X$ and $S$ are cycle sets, the semidirect product (as q-cycle sets) of $X$ and $S$ coincides with the semidirect product of cycle sets developed by Rump in [38]. Moreover, since biquandles in [4] can be viewed as particular q-cycle sets, referring to Problem 4.15 in [4] itself, the previous proposition gives rise to a general definition of semidirect product of biquandles.

References

[1] N. Andruskiewitsch, M. Graña, From racks to pointed Hopf algebras, Adv. Math. 178 (2) (2003) 177–243. URL https://doi.org/10.1016/S0001-8708(03)00018-3

[2] D. Bachiller, Solutions of the Yang-Baxter equation associated to skew left braces, with applications to racks, J. Knot Theory Ramifications 27 (8) (2018) 1850055, 36. URL https://doi.org/10.1142/S0218216518500554

[3] D. Bachiller, F. Cedó, E. Jespers, J. Okniński, A family of retractable square-free solutions of the Yang-Baxter equation, Forum Math. 29 (6) (2017) 1291–1306. URL https://doi.org/10.1515/forum-2015-0240

[4] V. Bardakov, T. Nasybullov, M. Singh, General constructions of biquandles and their symmetries, Preprint. URL https://arxiv.org/abs/1908.08301

[5] M. Bonatto, M. Kinyon, D. Stanovský, P. Vojtechovský, Involutive latin solutions of the Yang-Baxter equation, Preprint. URL https://arxiv.org/pdf/1910.02148.pdf

[6] M. Castelli, F. Catino, M. M. Miccoli, G. Pinto, Dynamical extensions of quasi-linear left cycle sets and the Yang-Baxter equation, J. Alg. Appl. 18 (11) (2019) 1950220. URL https://doi.org/10.1142/S0219498819502207

[7] M. Castelli, F. Catino, G. Pinto, A new family of set-theoretic solutions of the Yang-Baxter equation, Comm. Algebra 46 (4) (2017) 1622–1629. URL http://dx.doi.org/10.1080/00927987.2017.1350700

[8] M. Castelli, F. Catino, G. Pinto, About a question of Gateva-Ivanova and Cameron on square-free set-theoretic solutions of the Yang-Baxter equation, Comm. Algebra, In press. URL https://doi.org/10.1080/00927987.2020.1713328
[9] M. Castelli, F. Catino, G. Pinto, Indecomposable involutive set-theoretic solutions of the Yang-Baxter equation, J. Pure Appl. Algebra 220 (10) (2019) 4477–4493. URL https://doi.org/10.1016/j.jpaa.2019.01.017

[10] M. Castelli, G. Pinto, W. Rump, On the indecomposable involutive set-theoretic solutions of the Yang-Baxter equation of prime-power size, Comm. Algebra, In press. URL https://doi.org/10.1080/00927872.2019.1710163

[11] F. Catino, I. Colazzo, P. Stefanelli, On regular subgroups of the affine group, Bull. Aust. Math. Soc. 91 (1) (2015) 76–85. URL http://dx.doi.org/10.1017/S000497271400077X

[12] F. Catino, I. Colazzo, P. Stefanelli, Semi-braces and the Yang-Baxter equation, J. Algebra 483 (2017) 163–187. URL https://doi.org/10.1016/j.jalgebra.2017.03.035

[13] F. Catino, I. Colazzo, P. Stefanelli, Skew left braces with non-trivial annihilator, J. Algebra Appl. 18 (2) (2019) 1950033, 23. URL https://doi.org/10.1142/S0219498819500336

[14] F. Catino, M. Mazzotta, P. Stefanelli, Set-theoretical solutions of the Yang-Baxter and pentagon equations on semigroups, Accepted on Semigroup Forum. URL https://arxiv.org/abs/1910.05393

[15] F. Cedó, E. Jespers, Á. Del Río, Involutive Yang-Baxter groups, Transactions of the American Mathematical Society 362 (5) (2010) 2541–2558. URL https://doi.org/10.1090/S0002-9947-09-04927-7

[16] F. Cedó, E. Jespers, J. Okniński, Braces and the Yang-Baxter equation, Comm. Math. Phys. 327 (1) (2014) 101–116. URL https://doi.org/10.1007/s00220-014-1935-y

[17] F. Cedó, E. Jespers, J. Okniński, Set-theoretic solutions of the Yang-Baxter equation, associated quadratic algebras and the minimality condition, Rev. Mat. Complut. (2020). URL https://doi.org/10.1007/s13163-019-00347-6

[18] F. Cedó, E. Jespers, C. Verwimp, Structure monoids of set-theoretic solutions of the Yang-Baxter equation, Preprint. URL https://arxiv.org/abs/1912.09710

[19] F. Cedó, A. Smoktunowicz, L. Vendramin, Skew left braces of nilpotent type, Proc. Lond. Math. Soc. (3) 118 (6) (2019) 1367–1392. URL https://doi.org/10.1112/plms.12209
[20] P. Dehornoy, Set-theoretic solutions of the Yang-Baxter equation, RC-calculus, and Garside germs, Adv. Math. 282 (2015) 93–127.
URL https://doi.org/10.1142/S0219498815500012

[21] V. G. Drinfel’d, On some unsolved problems in quantum group theory, in: Quantum groups (Leningrad, 1990), vol. 1510 of Lecture Notes in Math., Springer, Berlin, 1992, pp. 1–8.
URL https://doi.org/10.1007/BFb0101175

[22] P. Etingof, T. Schedler, A. Soloviev, Set-theoretical solutions to the Quantum Yang-Baxter equation, Duke Math. J. 100 (2) (1999) 169–209.
URL http://doi.org/10.1215/S0012-7094-99-10007-X

[23] T. Gateva-Ivanova, Set-theoretic solutions of the Yang-Baxter equation, braces and symmetric groups, Adv. Math. 338 (2018) 649–701.
URL https://doi.org/10.1016/j.aim.2018.09.005

[24] T. Gateva-Ivanova, S. Majid, Matched pairs approach to set theoretic solutions of the Yang–Baxter equation, J. Algebra 319 (4) (2008) 1462–1529.
URL https://doi.org/10.1016/j.jalgebra.2007.10.035

[25] T. Gateva-Ivanova, M. Van den Bergh, Semigroups of I-Type, J. Algebra 206 (1) (1998) 97–112.
URL https://doi.org/10.1006/jabr.1997.7399

[26] L. Guarnieri, L. Vendramin, Skew braces and the Yang-Baxter equation, Math. Comp. 86 (307) (2017) 2519–2534.
URL https://doi.org/10.1090/mcom/3161

[27] E. Horvat, Constructing biquandles, Preprint.
URL https://arxiv.org/abs/1810.03027

[28] P. Jedlicka, A. Pilitowska, A. Zamojska-Dzienio, The retraction relation for biracks, J. Pure Appl. Algebra 223 (8) (2019) 3594–3610.
URL https://doi.org/10.1016/j.jpaa.2018.11.020

[29] E. Jespers, L. Kubat, A. Van Antwerpen, L. Vendramin, Factorizations of skew braces, Math. Ann. 375 (3-4) (2019) 1649–1663.
URL https://doi.org/10.1007/s00208-019-01909-1

[30] E. Jespers, J. Okniński, Monoids and groups of $I$-type, Algebr. Represent. Theory 8 (5) (2005) 709–729.
URL https://doi.org/10.1007/s10468-005-0342-7

[31] E. Jespers, A. Van Antwerpen, Left semi-braces and solutions of the Yang-Baxter equation, Forum Math. 31 (1) (2019) 241–263.
URL https://doi.org/10.1515/forum-2018-0059
[32] V. Lebed, L. Vendramin, Homology of left non-degenerate set-theoretic solutions to the Yang-Baxter equation, Adv. Math. 304 (2017) 1219–1261. URL https://doi.org/10.1142/80218196716500570

[33] V. Lebed, L. Vendramin, On structure groups of set-theoretic solutions to the Yang-Baxter equation, Proc. Edinb. Math. Soc. (2) 62 (3) (2019) 683–717. URL https://doi.org/10.1017/s0013091518000548

[34] J.-H. Lu, M. Yan, Y.-C. Zhu, On the set-theoretical Yang-Baxter equation, Duke Math. J. 104 (1) (2000) 1–18. URL http://dx.doi.org/10.1215/S0012-7094-00-10411-5

[35] S. Nelson, E. Watterberg, Birack dynamical cocycles and homomorphism invariants, J. Algebra Appl. 12 (8) (2013) 1350049, 14. URL https://doi.org/10.1142/S0219498813500497

[36] W. Rump, A decomposition theorem for square-free unitary solutions of the quantum Yang-Baxter equation, Adv. Math. 193 (2005) 40–55. URL https://doi.org/10.1016/j.jalgebra.2006.03.040

[37] W. Rump, Braces, radical rings, and the quantum Yang-Baxter equation, J. Algebra 307 (1) (2007) 153–170. URL https://doi.org/10.1016/j.jalgebra.2006.03.040

[38] W. Rump, Semidirect products in algebraic logic and solutions of the quantum Yang-Baxter equation, J. Algebra Appl. 7 (4) (2008) 471–490. URL https://doi.org/10.1142/S0219498808002904

[39] W. Rump, A covering theory for non-involutive set-theoretic solutions to the Yang-Baxter equation, J. Algebra 520 (2019) 136–170. URL https://doi.org/10.1016/j.jalgebra.2018.11.007

[40] A. Smoktunowicz, L. Vendramin, On skew braces (with an appendix by N. Byott and L. Vendramin), J. Comb. Algebra 2 (1) (2018) 47–86. URL https://doi.org/10.4171/JCA/2-1-3

[41] A. Soloviev, Non-unitary set-theoretical solutions to the quantum Yang-Baxter equation, Math. Res. Lett. 7 (5-6) (2000) 577–596. URL https://doi.org/10.4310/MRL.2000.v7.n5.a4

[42] L. Vendramin, Extensions of set-theoretic solutions of the Yang-Baxter equation and a conjecture of Gateva-Ivanova, J. Pure Appl. Algebra 220 (2016) 2064–2076. URL https://doi.org/10.1142/S1005386716600183