Parameter estimation for the mixed sub-fractional Ornstein-Uhlenbeck process

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Abstract

This paper deals with the problem of estimating the drift parameter in the mixed sub-fractional Ornstein-Uhlenbeck process when a continuous record of observations is available. Based on the fundamental martingale of the mixed sub-fractional Brownian motion and its corresponding Skorohod integral, we propose the maximum likelihood estimator and discuss the obstruction of obtaining its asymptotic properties. Alternatively, the least squares estimator is proposed and its asymptotic behaviors are studied. To overcome the difficulty of simulation, a simulation friendly estimator is provided and its asymptotic properties are presented. Simulation studies show that the proposed ergodic-type estimator performs well.

Key Words: sub-fractional Brownian motion, fundamental martingale, Skorohod integral, parameter estimation

1 Introduction

The Ornstein-Uhlenbeck process has been extensively applied in various fields, as diverse as economics, finance, high technology, biology, physics, chemistry, medicine and environmental studies. In fact, the standard Ornstein-Uhlenbeck models, including the diffusion models based on the Brownian motion and the jump-diffusion models driven by Lévy processes, provide good service in cases where the data demonstrate the Markovian property and the lack of memory. However, over the past few decades, numerous empirical studies have found that the phenomenon of long-range dependence may observe in data of hydrology, geophysics, climatology and telecommunication, economics and finance. In the continuous time situation, the best known and widely used stochastic process that exhibits long-range dependence is of course the fractional Brownian motion (fBm), which describes the degree of dependence by the Hurst parameter. Consequently, fBm is the usual candidate to capture some properties of “Real-world” data and the applications of the fractional Ornstein-Uhlenbeck process (FOUP) have recently experienced intensive development (see, for example, [8, 7]). The development of the application for fBm and FOUP naturally led to the statistical
inference for stochastic models driven by fBm. Consequently, the parameter estimation problems
for stochastic models driven by fBm and FOUP have been of great interest in the past decade, and
beside being a challenging theoretical problem. Some surveys and complete literatures related to
the parametric and other inference procedures for stochastic models driven by fBm could be found
in [16, 12].

Although fBm has been applied in various scientific areas, many authors have proposed to use
some more general fractional Gaussian processes, such as sub-fractional Brownian motion (sfBm),
bi-fractional Brownian motion and weighted fractional Brownian motion. In contrast to the exten-
sive studies on fBm, there has been only a little systematic investigation on the statistical inference
of other fractional Gaussian processes. The papers of [14, 9] considered the least squares estimator
(LSE) for the mean reversion speed parameter in the sub-fractional Ornstein-Uhlenbeck processes.
Recently, the paper of [6] has used the tool of fundamental martingale, the asymptotical properties
of the eigenvalues and eigenfunctions of the fractional covariance operators to study the maximum
likelihood estimator of the drift parameter for the mixed fractional Ornstein-Uhlenbeck process. It
should be pointed out that, to the best of our knowledge, there is no statistical inference of the
mixed sub-fractional Ornstein-Uhlenbeck process (msfOUP). The difficulty is that the properties
of ergodicity can not been applied when the increment of the sfBm is not stationary. This paper
will fill in the gaps in this area. using the least squares method, we consider the problem of esti-
mating the mean reversion speed for msfOUP for all $H \in (1/2, 1)$ based on continuous observation.
Both the strong consistency and the asymptotic distributions are established for LSE based on the
technics inspired from [10, 11].

The rest of this paper is organized as follows. Section 2 contains some preliminaries about
msfBm. Section 3 discusses the maximum likelihood estimator of the drift parameter for msfOUP.
The LSE and its asymptotic properties is provided. For the sake of simulating, a simulation friendly
estimator is also proposed. for the sake of completeness, the LSE for msfOUP in the non-ergodic
case is discussed in Section 4. Section 5 is devoted to presenting Monte Carlo studies on the finite
sample properties of the practical estimator. Some technical proofs are collected in the Appendix.

2 Preliminaries

In this section, we first describe some basic facts on the msfBm. Then, we introduce the fundamen-
tal martingale of the msfBm. After that, Malliavin derivative and adjoint operator with respect
to msfBm are proposed. Finally, we define stochastic integral (Skorohod integral and path-wise
integral) with respect to the msfBm using the tool of fundamental martingale.

2.1 msfBm

The sfBm arises from occupation time fluctuations of branching particle systems with Poisson initial
condition and has properties analogous to those of fBm (self-similarity, long-range dependence,
Hölder paths). However, in comparison with fBm, sfBm has nonstationary increments and the
increments over non overlapping intervals are more weakly correlated and their covariance decays
polynomially as a higher rate in comparison with fBm (for this reason, in [2] it is called sfBm). It’s
well-known that sfBm with index $H \in (0, 1)$ is a mean zero Gaussian process $S^H = \{S^H_t, t \geq 0\}$
with $S_0^H = 0$ and the covariance

$$R_H(s, t) = \mathbb{E}(S^H_t S^H_s) = t^{2H} + s^{2H} - \frac{1}{2} (|t - s|^{2H} + |t + s|^{2H}), s, t \in [0, T].$$
where the stochastic process $S^H = (S^H_t)_{0 \leq t \leq T}$ is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

For $H = 1/2$, $S^H$ coincides with the standard Brownian motion. In fact, $S^H$ is neither a semimartingale nor a Markov process unless $H = 1/2$. For $H > 1/2$, we can see that

$$\mathbb{E}S^H_t S^H_s = \int_0^t \int_0^s K_H(r,u)drdu, \quad 0 \leq s, t \leq T$$

with

$$K_H(s,t) = \frac{\partial^2}{\partial s \partial t} R_H(s,t) = H(2H-1) \left(|s-t|^{2H-2} - (s+t)^{2H-2} \right).$$

(1)

Let us mention that the existence of sub-fBm $S^H_t$ is obvious. For any $H \in (0,1)$, we consider the process

$$\frac{1}{\sqrt{2}} (B^H_t + B^H_{-t}), \quad H \in (0,1), \quad 0 \leq t \leq T$$

(2)

where $B^H = (B^H_t)_{-\infty < t < \infty}$ is fBm with Hurst index $H$ on the whole real line.

Now, it is easy to see that the covariance of the process (2) is precisely $R_H(s,t)$. It is worth to emphasize that the properties mentioned here make sfBm a possible candidate for models which involve long-range dependence, self-similarity and non-stationary. Therefore, sfBm has been used to capture the price fluctuations of the financial asset (see, for example, [13]). However, the paper [22] has stated that the Black-Scholes model driven by an sfBm allows arbitrage opportunity. Following the idea of [4], we introduce the msfBm, $\xi = (\xi_t, 0 \leq t \leq T)$ which is defined by

$$\xi_t = W_t + S^H_t,$$

(3)

where $W = (W_t, 0 \leq t \leq T)$ is a standard Brownian motion and $S^H = (S^H_t, 0 \leq t \leq T)$ is an independent sub-fractional Brownian motion.

As stated in [2, 20, 19], sfBm can be presented by

$$S^H_t = c_H \int_0^t n_H(s,t)dB_s, \quad 0 \leq t \leq T,$$

where $(B_t)_{0 \leq t \leq T}$ is a standard Brownian motion and the kernel $n_H(s,t)$ is

$$n_H(s,t) = s^{3/2-H} \int_s^t (x^2 - s^2)^{H-3/2}dx1_{(0,t)}(s)$$

and

$$c_H^2 = \frac{2^{2-2H} \Gamma(1+2H) \sin(\pi H)}{\Gamma^2(H-1/2)}.$$

From this kernel let us define the operator $\Psi$ for $f : [0,t] \to \mathbb{R}$

$$(\Psi f)(s,t) = \int_s^t f(r)(r^2 - s^2)^{H-3/2}dr, \quad 0 \leq s \leq t \leq T.$$  

(4)

Let us define the space

$$\Lambda^{H-1/2}_t := \left\{f : [0,t] \to \mathbb{R} \mid \int_0^t (s^{3/2-H} (\Psi f)(s,t))^2ds < \infty \right\}.$$
with the scalar product
\[
\langle f, g \rangle_{\mathcal{H}^{H-1/2}} := c_{H}^{2} \int_{0}^{t} s^{3-2H}(\Psi f)(s,t)(\Psi g)(s,t).
\] (5)

This expression can be rewritten as
\[
\langle f, g \rangle_{\mathcal{H}^{H-1/2}} = \int_{0}^{t} \int_{0}^{t} f(u)g(v)K_{H}(u,v)du dv = \mathbb{E} \int_{0}^{t} f(s) dS_{s}^{H} \int_{0}^{t} g(s) dS_{s}^{H}.
\]

where \( K_{H}(u,v) \) is defined in (4). It is not difficult to check that \( L^{2}([0,T]) \subset \mathcal{H}^{H-1/2} \).

Let \( \mathcal{F} = (\mathcal{F}_{t}^{\xi}) \) and \( \mathcal{F} = (\mathcal{F}_{t}) \), \( t \in [0,T] \) be the nature filtrations of \( \xi \) and \( (W,S^{H}) \) respectively, then as presented in [3] we have the following result.

**Lemma 2.1.** Consider the mixed sub-fractional Brownian motion \( \xi = (\xi_{t})_{0 \leq t \leq T} \) defined in (3) and let \( \eta \) be a random variable, such that the pair \( (\eta,\xi_{t}) \) forms a Gaussian process, then
\[
\mathbb{E}(\eta | \mathcal{F}_{t}^{\xi}) = \mathbb{E}\eta + \int_{0}^{t} h(s,t)d\xi_{s},
\]
with a unique function \( h(\cdot,t) \in L^{2}([0,t]) \subset \Lambda_{t}^{H-1/2} \).

### 2.2 Stochastic integral with respect to msfBm

Let \( \mathcal{F} = (\mathcal{F}_{t}^{\xi}) \) and \( \mathcal{F} = (\mathcal{F}_{t}) \), \( t \in [0,T] \) be the nature filtrations of \( \xi \) and \( (W,S^{H}) \) respectively. We now construct a process \( M = (M_{t})_{0 \leq t \leq T} \), which is the conditional expectation of \( W \) with respect to \( \mathcal{F}_{t}^{\xi} \)
\[
M_{t} = \mathbb{E}(W_{t} | \mathcal{F}_{t}^{\xi}), \quad t \in [0,T] \tag{6}
\]

Since \( W \) is an \( \mathcal{F} \)-martingale and \( \mathcal{F}_{t}^{\xi} \subseteq \mathcal{F}_{t} \), the process \( M \) is an \( \mathcal{F}_{t}^{\xi} \)-martingale and we have the following result.

**Theorem 2.2.** The martingale \( M \) admits the following representation:
\[
M_{t} = \int_{0}^{t} g(s,t)d\xi_{s}, \quad (M)_{t} = \int_{0}^{t} g(s,t)ds = \int_{0}^{t} g^{2}(s,s)ds, \quad t \geq 0, \tag{7}
\]

where \( g(s,t) \) solves the following integral equation:
\[
g(s,t) + \int_{0}^{t} g(r,t)\kappa(r,s)dr = 1, \quad \kappa(s,t) = H(2H - 1)(|t-s|^{2H-2} - |t+s|^{2H-2}). \tag{8}
\]

On the other hand, we have the innovation representation
\[
\xi_{t} = \int_{0}^{t} G(s,t)dM_{s}, \quad t \in [0,T], \tag{9}
\]
where
\[
G(s,t) := 1 - \frac{1}{g(s,s)} \int_{0}^{t} R(\tau,s)d\tau, \quad 0 \leq s \leq t \leq T. \tag{10}
\]


with
\[ R(s, t) := \frac{\partial}{\partial t} g(s, t), \quad s \neq t \] (11)
and \( \dot{g}(s, t) := \frac{\partial}{\partial t} g(s, t) \).

Using (7) and (9), we have \( F_{\xi t} = F_{Mt}, P - a.s. \) for all \( t \in [0, T] \). We called \( M \) the fundamental martingale of the process \( \xi \). Let us mention that in the equation (11), the function \( g(s, t) \) also satisfies (8) without the restriction of \( s \leq t \).

Proof. With Lemma 2.1 and the normal correlation theorem, for any test function \( h \in L^2([0, t]) \), we have
\[
E \left( W_t - \int_0^t g(s, t) d\xi_s \right) \left( \int_0^t h(s) d\xi_s \right) = 0.
\]
It is easy to check the equation (8). At the same time
\[
\langle M \rangle_t = EM_t^2 = EW_tM_t = EW_t \int_0^t g(s, t) d\xi_s = \int_0^t g(s, t) ds.
\]
With lemma 3.2 of [3], we have
\[
\int_0^t g(s, t) ds = \int_0^t g^2(s, s) ds.
\]
On the other hand, we can also deduce the expression (9) with the same method of Theorem 5.1 in [3]. \( \square \)

### 2.3 Malliavin derivative and adjoint operator with respect to msfBm

Fixe a time interval \([0, T]\), we denote by \( \mathcal{E} \) the set of step function on \([0, T]\). Let \( \mathcal{H} \) be the Hilbert space defined as the closure of \( \mathcal{E} \) with respect to the scalar product
\[
\langle 1_{[0, t]}, 1_{[0, s]} \rangle_{\mathcal{H}} = R_H(t, s),
\]
where \( R_H(t, s) \) is the covariance of \( S_H^t \) and \( S_H^s \). The mapping \( 1_{[0, t]} \rightarrow \xi_t \) can be extended to an isometry between \( \mathcal{H} \) and the Gaussian space \( \mathcal{H}_1 \) associated with \( \xi \). We denote this isometry by \( \varphi \rightarrow \xi(\varphi) \). Here, \( \xi(\varphi) \) is an isonormal Gaussian process associated with the Hilbert space \( \mathcal{H} \), which was introduced by [16] (see Definition 1.1.1 in [16]). For any pair step functions \( f, g \in \mathcal{H} \), we have
\[
\langle f, g \rangle_{\mathcal{H}} = \int_0^T f(t)g(t) dt + \alpha_H \int_0^T \int_0^T f(t)g(s) \left( |t-s|^{2H-2} + (t+s)^{2H-2} \right) dsdt,
\]
where \( \alpha_H = H(2H-1) \).

Following the same steps as Section 1.2 in [16], let \( \mathcal{S} \) be the class of smooth random variables and \( F \in \mathcal{S} \). Then we have
\[
F = f(\xi(h_1), \ldots, \xi(h_n)),
\]
where \( f \in C_b^\infty(\mathbb{R}^n), h_1, \ldots, h_n \in \mathcal{H} \) for \( n \geq 1 \). The Malliavin derivative satisfies the following chain rule, which is provided by the following definition.
Definition 2.3. The derivative of a smooth random variable \( F \) of the form \( (13) \) is the \( \mathcal{H} \)-valued random variable and can be given by

\[
D_\xi F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\xi(h_1), \ldots, \xi(h_n)) h_i. \tag{14}
\]

Let the space \( D_{1,2}^{\xi,2} \) be the closure of the class of smooth random variables \( S \) with respect to the norm

\[
||F||_{1,2} = \left( \mathbf{E}(|F|^2) + \mathbf{E}||D_\xi F||_{\mathcal{H}}^2 \right)^{1/2}.
\]

Consequently, we obtain that the space \( D_{1,2}^{\xi,2} \) is a Hilbert space.

Definition 2.4. Let \( \delta_\xi \) be the adjoint of the operator \( D_\xi \). Then \( \delta_\xi \) is an unbounded operator on \( L^2(\Omega; \mathcal{H}) \) with values in \( L^2(\Omega) \) such that

- For any \( F \in D_{1,2}^{\xi,2} \), the domain of \( \delta_\xi \), denoted by \( \text{Dom}(\delta_\xi) \), is the set of \( \mathcal{H} \)-valued square integrable random variables \( u \in L^2(\Omega; \mathcal{H}) \) such that

  \[
  |\mathbf{E}(\langle D_\xi F, u \rangle_\mathcal{H})| \leq c||F||_2.
  \]

  where \( c \) is a constant depending on \( u \).

- If \( u \) belongs to \( \text{Dom}(\delta_\xi) \), then \( \delta_\xi(u) \) is the element of \( L^2(\Omega) \) characterized by

  \[
  \mathbf{E}(F\delta_\xi(u)) = \mathbf{E}(\langle D_\xi F, u \rangle_\mathcal{H}).
  \]

Now, let \( u \in \text{Dom}(\delta_\xi) \). Then we define the Skorohod integral with respect to the mfBm \( \xi \) by

\[
\delta_\xi(u) = \int_0^T u(t)\delta \xi_t.
\]

Remark 2.5. For the deterministic function \( \psi(t) \in \mathcal{H} \), it is not hard to check that \( \psi(t) \in \text{Dom}(\delta_\xi) \), with the same proof as the stochastic calculus with respect to the standard Brownian motion in [1], since \( \delta_\xi(\psi) \) is the Riemann-Stieltjes integral.

Let \( B = (B_t, 0 \leq t \leq T) \) be the standard Brownian motion with the same filtration of the process \( \xi \). We are interested in the following three question:

1. What is the relationship of the malliavin derivative \( D_\xi \) and \( D_B \)?

2. What is the relationship of the adjoint operator \( \delta_\xi \) and \( \delta_B \) ?

3. What is the operator of \( G^* \) such as the \( K^*_H \) presented in Chapter 5 of [16]?
Let us take a function $\psi(t) \in \mathcal{H}$, which would be derivable. Then, we have
\[
\int_0^T \psi(t) d\xi_t = \psi(T)\xi_T - \int_0^T \psi'(s)\xi_s ds = \psi(T)\xi_T - \int_0^T \psi'(s) \left[ \int_0^s G(\tau, s) dM_\tau \right] ds
\]
\[
= \int_0^T \psi(T)G(\tau, T) dM_\tau - \int_0^T \left[ \int_\tau^T G(\tau, s)\psi'(s) ds \right] dM_\tau
\]
\[
= \int_0^T \left[ \psi(T)G(\tau, T) - \int_\tau^T G(\tau, s)\psi'(s) ds \right] dM_\tau
\]
\[
= \int_0^T \int_\tau^T \psi(t) \frac{\partial G}{\partial t}(\tau, t) dt + \psi(\tau)G(\tau, \tau) dM_\tau.
\]

We define the Wiener process $B = (B_t, 0 \leq t \leq T)$ with the same filtration of $\xi$. Consequently, we have
\[
\int_0^T \psi(t) d\xi_t = \int_0^T \left[ \int_\tau^T g(\tau, t) \frac{\partial G}{\partial t}(\tau, t) dt + \psi(\tau)G(\tau, \tau) \right] dB_\tau.
\]

Let us define the operator $G^*$ from $\mathcal{H}$ to the complete subspace $L^2[0, T]$ as follows
\[
(G^* \psi)(\tau) = \int_\tau^T \frac{\partial G}{\partial t}(\tau, t) \psi(t) dt + \psi(\tau).
\]

Using the operator above, we can define the divergence-type integral for mfBm.

**Definition 2.6.** Let $H > 1/2$ and $u$ be a stochastic process $u(\omega) : [0, T] \to \mathcal{H}$ such that $G^* u$ is Skorohod integral with respect to the standard Brownian motion $B(t)$. Then we define the extended Wiener integral of $u$ with respect to the mfBm $\xi$ as
\[
\xi(u) := \int_0^T (G^* u)(\tau) d\delta B_\tau.
\]

It is easy to check that $\text{Dom}(\delta_\xi) = (G^*)^{-1}(\text{Dom}(\delta_B))$ and for $u \in \text{Dom}(\delta_\xi)$ the Itô-Skorohod integral $\delta_\xi(u)$ coincides with the divergence-type integral $\xi(u)$ defined in (15). At the same time, we have the following result.

**Lemma 2.7.** For any $F \in \mathcal{D} B^1,2 = \mathcal{D} \xi^1,2$, we have
\[
G^* D_\xi F = D_B F
\]
where $D_B$ denotes the derivative operator with respect to the standard Brownian motion $B$ and $\mathcal{D} B^1,2$ the corresponding Sobolev space.
2.4 Path-wise integral

Let \((u_t)_{t \in [0, T]}\) be a process with integrable trajectories. The symmetric integral of \(u\) with respect to \(\xi\) is defined as

\[
\int_0^T u(s) \circ d\xi_s = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^T u(s) [\xi(s + \epsilon) - \xi(s - \epsilon)] \, ds
\]

provided that the limit exists in probability.

**Lemma 2.8.** Suppose that \((u_t)_{t \in [0, T]}\) is a stochastic process in \(D^{1,2}_\xi\),

\[
\int_0^T \int_0^T |D^\xi u(t)| \left( |t - s|^{2H-2} + (t + s)^{2H-2} \right) \, ds \, dt < \infty, \text{ a.s} \tag{16}
\]

and

\[
\int_0^T |D^\xi u(t)| \, dt < \infty, \text{ a.s} \tag{17}
\]

where \(D^\xi u(t)\) means \(D^\xi_s u(t)\) when \(s = t\). Then the symmetric integral exists and the following relation holds:

\[
\int_0^T u(t) \circ d\xi_t = \delta \xi (u) + H(2H-1) \int_0^T \int_0^T D^\xi u(t) \left( |t - s|^{2H-2} + (t + s)^{2H-2} \right) \, ds \, dt + \frac{1}{2} \int_0^T D^\xi u(t) \, dt. \tag{18}
\]

**Proof.** Let

\[ u^\epsilon_t = \frac{1}{2\epsilon} \int_{t-\epsilon}^{t+\epsilon} u(s) \, ds. \]

Using the integration by part formula, we have

\[ \delta \xi F u = F \delta \xi (u) - \langle D^\xi F, u \rangle_H, \]

for \(F \in D^{1,2}_\xi\) and \(u \in Dom(\delta \xi)\). When \(u\) satisfied the conditions \([16]\) and \([17]\), we have

\[
\int_0^T u(s) \frac{\xi_{s+\epsilon} - \xi_{s-\epsilon}}{2\epsilon} \, ds = \int_0^T u(s) \frac{1}{2\epsilon} \int_{s-\epsilon}^{s+\epsilon} d\xi_a \, ds
\]

\[
= \int_0^T \delta \xi \left( u(s) \frac{1}{2\epsilon} 1_{[s-\epsilon, s+\epsilon]} \right) \, ds + \frac{1}{2\epsilon} \int_0^T \langle D^\xi u(s), 1_{[s-\epsilon, s+\epsilon]} \rangle_H \, ds
\]

\[
= \delta \xi (u^\epsilon) + \frac{1}{2\epsilon} \int_0^T \langle D^\xi u(s), 1_{[s-\epsilon, s+\epsilon]} \rangle_H \, ds.
\]

The equation \([18]\) can be easily obtained by taking the limit \(\epsilon \to 0\) on both sides of the above equation. \qed
3 Estimating the drift parameter for msfOUP

The msfOUP $X = (X_t)_{0 \leq t \leq T}$ is determined by the following stochastic differential equation

$$dX_t = -\vartheta X_t dt + d\xi_t, \ 0 \leq t \leq T. \quad (19)$$

Here the parameter $\vartheta$ is unknown and to be estimated with the observed process $X$. We will present first of all the Maximum Likelihood Estimator and the difficulties to deal with this estimator and then we will study the Least square estimator.

**Remark 3.1.** The Cox-Ingersoll-Ross process driven by msfBm with symmetric path-wise integral has the relationship with the O-U process. Let

$$d\tilde{X}_t = a\tilde{X}_t dt + \sqrt{\tilde{X}_t} \circ d\xi_t, \ a > 0, \ X_0 > 0$$

and $\tau$ be the first moment of $\tilde{X}$ reaching zero by the latter. Then, the process $\tilde{X} = (\tilde{X}, t \geq 0)$ can be represented by

$$\tilde{X}_t = \tilde{Y}_t^2 1_{t \leq \tau}$$

where $\tilde{Y} = (\tilde{Y}_t, t \geq 0)$ is the msfOUP. In this situation, we can observe the process $\tilde{X}_t$ and estimate the parameter $a$ with the maximum likelihood estimation.

### 3.1 Maximum likelihood estimator

With the fundamental martingale of the msfBm, it is easy to get the maximum likelihood estimator of $\vartheta$. Integrating kernel $g(s, t)$ with respect to $X$ gives a semimartingale

$$Z_t := \int_0^t g(s, t) dX_s = -\vartheta \int_0^t Q_s d\langle M \rangle_s + M_t \quad (20)$$

where

$$Q_t := \frac{d}{d\langle M \rangle_t} \int_0^t g(s, t) X_s ds \quad (21)$$

and the filtration generated by $X$ and $Z$ coincide.

By the Girsanov theorem the Likelihood function $L(\vartheta, X_T)$ can be written as

$$L(\vartheta, X_T) = \exp \left( -\vartheta \int_0^T Q_t dZ_t - \frac{\vartheta^2}{2} \int_0^T Q_t^2 d\langle M \rangle_t \right).$$

Consequently, the MLE is given by

$$\hat{\vartheta}_T = -\frac{\int_0^T Q_t dZ_t}{\int_0^T Q_t^2 d\langle M \rangle_t} \quad (22)$$

and the estimation error will be

$$\hat{\vartheta}_T - \vartheta = -\frac{\int_0^T Q_t dM_t}{\int_0^T Q_t^2 d\langle M \rangle_t} \quad (23)$$
When \( g(s,t) \) has no explicit solution, it is not easy to obtain the asymptotic law for \( \hat{\vartheta}_T \) even we use the method of the mixed fractional case in [5] and [6]. On the other hand, for large \( T \), to simulate the \( g(t,T) \) when \( T \) fixed, we will divided \( T \) with small interval with distance of \( 1/10 \). Then the computer will calculate the inverse of a large dimensional matrix but not sparse matrix, which will be so slow. In order to avoid these troubles, we try to study the LSE.

### 3.2 Least square estimator

First of all, the solution of (19) can be written as

\[
X_t = \int_0^t e^{-\vartheta(t-s)}d\xi_s, \quad 0 \leq t \leq T, \quad X_0 = 0.
\]

(24)

The least squares estimator aims to minimize

\[
\int_0^T |\dot{X}_t + \vartheta X_t|^2 dt = \int_0^T \dot{X}_t dt + 2\vartheta \int_0^T X_t dX_t + \vartheta^2 \int_0^T X_t^2 dt.
\]

As stated in [10], this is a quadratic function of \( \vartheta \) although \( \int_0^T \dot{X}_t dt \) does not exist. The minimum is achieved when

\[
\hat{\vartheta}_T = -\frac{\int_0^T X_t dX_t}{\int_0^T X_t^2 dt} = \vartheta - \frac{T}{\int_0^T X_t^2 dt},
\]

(25)

where the stochastic integral \( \int_0^T X_t d\xi_t \) is interpreted as the Skorohod integral and it will be denoted as \( \int_0^T X_t d\xi_t \).

**Lemma 3.2.** For \( H > 1/2 \), we have

\[
\hat{\vartheta}_T = -\frac{X_T^2}{2\int_0^T X_t^2 dt} + \alpha H \int_0^T \int_0^t \exp(-\vartheta(t-s)) \left((t-s)^{2H-2} + (t+s)^{2H-2}\right) dsdt + \frac{T}{2\int_0^T X_t^2 dt}.
\]

(26)

**Proof.** In the equation (25) we denote the \( \int_0^T X_t d\xi_t \) as the Skorohod integral and with (18) and (24) we have

\[
\int_0^T X_t d\xi_t = \int_0^T X_t \circ d\xi_t - \alpha H \int_0^T \int_0^t \exp(-\vartheta(t-s)) \left((t-s)^{2H-2} + (t+s)^{2H-2}\right) dsdt - \frac{T}{2}
\]

\[
= \frac{X_T^2}{2} + \vartheta \int_0^T X_t^2 dt - \alpha H \int_0^T \int_0^t \exp(-\vartheta(t-s)) \left((t-s)^{2H-2} + (t+s)^{2H-2}\right) dsdt - \frac{T}{2}
\]

which achieves the proof. \( \square \)

**Theorem 3.3.** The LSE \( \hat{\vartheta}_T \) defined in (26) converge to \( \vartheta \) in probability as \( T \to \infty \). Thus \( \forall \varepsilon > 0 \)

\[
\lim_{T \to \infty} P \left( |\hat{\vartheta}_T - \vartheta| > \varepsilon \right) = 0.
\]

The asymptotical laws of the LSE defined in (25) depends on the the Hurst parameter \( H \) and we have the following results.
Theorem 3.4. For $H \in (1/2, 3/4)$ we have

$$\sqrt{T} \left( \hat{\vartheta}_T - \vartheta \right) \xrightarrow{\mathcal{L}} \mathcal{N} \left( 0, \sigma_H^2 \right),$$

(27)

where $\sigma_H = \sqrt{\frac{\vartheta^{1-4H} H^2 (4H-1) \Gamma(2H) + \frac{\Gamma(2H) \Gamma(3-4H) \Gamma(4H-1)}{\Gamma(2-2H)}}{\vartheta^{-2H} HT(2H) + \frac{1}{2H}}}$.

For $H = 3/4$, the LSE is also asymptotically normal with the convergence rate $\frac{\sqrt{T}}{\sqrt{\log(T)}}$, that is

$$\frac{\sqrt{T}}{\sqrt{\log(T)}} \left( \hat{\vartheta}_T - \vartheta \right) \xrightarrow{\mathcal{L}} \mathcal{N} \left( 0, \frac{9}{4\vartheta^2} \left( \frac{3\sqrt{\pi} \vartheta^{-3/2}}{4} + \frac{1}{2} \right)^2 \right).$$

(28)

For $H > 3/4$, we have

$$T^{2-2H} \left( \hat{\vartheta}_T - \vartheta \right) \xrightarrow{\mathcal{L}} - \frac{\vartheta^{-1} R_1}{\vartheta^{-2H} HT(2H) + \frac{1}{2H}}.$$ 

(29)

where $R_1$ is the Rosenblatt random variables defined in Theorem 5.2 of [11].

3.3 A practical estimator

Though we have obtained some desired asymptotical properties of LSE, $\hat{\vartheta}_T$ depends on the unknown parameter $\vartheta$, the parameter we want to estimate. Thus, it is impossible to do the simulation. Fortunately, using (40), we have

$$\frac{1}{T} \int_0^T X_t^2 dt \xrightarrow{P} \frac{1}{2\vartheta} + H \vartheta^{-2H} \Gamma(2H).$$

Let us define a function $p(\vartheta) = \frac{1}{2\vartheta} + H \vartheta^{-2H} \Gamma(2H)$. Then a practical estimator $\tilde{\vartheta}_T$ can be defined by

$$\tilde{\vartheta}_T = p^{-1} \left( \frac{1}{T} \int_0^T X_t^2 \right).$$

(30)

Obvious this simulation friendly estimator converges to $\vartheta$ in probability. Moreover, we can obtain the asymptotical normality of $\tilde{\vartheta}_T$ with the Delta method. For the sake of saving space, we only present the case of $H \in (1/2, 3/4)$ here and the other two cases ($H = 3/4$ and $H \in (3/4, 1)$) can also be obtained by the same method.

Theorem 3.5. As $T \to \infty$, when $H \in (1/2, 3/4)$

$$\sqrt{T} \left( \tilde{\vartheta}_T - \vartheta \right) \xrightarrow{\mathcal{L}} \mathcal{N} \left( 0, \frac{\sigma_H^2 (HT(2H) \vartheta^{1-2H} + \frac{1}{2})}{\vartheta^2} \right),$$

where $\sigma_H = \sqrt{\frac{\vartheta^{1-4H} H^2 (4H-1) \Gamma(2H) + \frac{\Gamma(2H) \Gamma(3-4H) \Gamma(4H-1)}{\Gamma(2-2H)}}{\vartheta^{-2H} HT(2H) + \frac{1}{2H}}}$.
Proof. From equation (26) we have

\[
\frac{1}{T} \int_0^T X_t^2 dt - p(\vartheta) = \frac{1}{T} H(2H - 1) \int_0^T \int_0^t \exp(-\vartheta(t-s))((t-s)^{2H-2} + (t+s)^{2H-2}) ds dt \\
- \frac{X_T^2}{\vartheta_T} + \frac{1}{2} \frac{1}{\vartheta_T} - \left( \frac{1}{2}\vartheta + H\vartheta^{-2H}\Gamma(2H) \right) \\
\overset{a.s.}{\rightarrow} \left( \frac{1}{2} + H\vartheta^{1-2H}\Gamma(2H) \right) \left( \frac{1}{\vartheta_T} - \frac{1}{\vartheta} \right) + o\left( \frac{1}{T^{1-\alpha}} \right)
\]

where \( \alpha \) is a small real number and the last convergence comes from the property of \( X_T^2 \). Using the Delta method three times, we can achieve this proof. \( \square \)

Remark 3.6. Here for the simulation, we have to use the function of \( p^{-1}(\vartheta) \) but this is not an explicit function, the numerical result of the inverse function will be applied in MATLAB.

Remark 3.7. Since the simulation friendly estimator, \( \hat{\vartheta}_T \), does not contain any stochastic integral and hence it is simpler to simulate. Motivated by Eq. (5.1) in [11], we choose to work with the formula (30) by replacing the Riemann integral in the denominator by its corresponding approximate Riemann sums in discrete integer time. Specifically, we define,

\[
\hat{\vartheta}_N = p^{-1} \left( \frac{1}{N} \sum_{i=1}^N X_{id} \right), \tag{31}
\]

where \( d > 0 \) the sampling interval and the process \( X_t \) is observed at discrete-time instants \( t_i = id, \ i = 1, 2, \ldots, N. \)

Remark 3.8. Let \( N \to \infty, d \to 0 \) and \( H \in \left( \frac{1}{2}, 1 \right) \). Borrowing the idea of [11] and using Theorem 3.5, we can prove the strong consistency and the asymptotic laws for the practical estimator \( \hat{\vartheta}_n \) for under some mild conditions.

4 Non ergodic case

Let us recall that for msfOUP of [19] will be obviously a non-ergodic process when \( \vartheta < 0 \). The paper of [9] has considered the general Gaussian case for the non-ergodic Ornstein-Uhlenbeck process and interpreted the stochastic integral \( \int_0^T X_t dX_t \) as the Young integral. Following the idea of [9], we will also interpret the stochastic integral \( \int_0^T X_t dX_t \) as the Young integral. Moreover, we will provide the intuition why we choose the Young integral and propose the asymptotic properties of LSE for msfOUP in non-ergodic process. We first introduce the following result.

Lemma 4.1. For \( H > 1/2 \) and \( \vartheta < 0 \), we have

\[
\lim_{T \to \infty} \frac{H(2H - 1) \int_0^T \int_0^t \exp(-\vartheta(t-s))((t-s)^{2H-2} + (t+s)^{2H-2}) ds dt}{\int_0^T X_t^2 dt} = 0
\]

and

\[
\lim_{T \to \infty} \frac{T}{\int_0^T X_t^2 dt} = 0.
\]
Proof. Although for $0 \leq t \leq T$, $c > 0$ and $\gamma > 0$, the condition of the msfBm $\xi_t : E\xi_t^2 \leq ct^\gamma$ is not satisfied, we can divide the process, $\xi_t$, into two parts: one is on the interval $[0, 1]$ and the other is on interval $[1, \infty]$. For any interval, this condition is satisfied and the proof for Lemma 2.1 in [9] can be achieved by these two parts. Thus we obtain

$$\lim_{T \to \infty} e^{2\theta T} \int_0^T X_t^2 dt = -\frac{\vartheta}{2} Z_\infty^2,$$

where $Z_t := \int_0^t e^{\vartheta s} \xi_s ds$, $t \geq 0$ and $Z_t \to Z_\infty$ almost surely and in $L^2(\Omega)$.

Moreover, when $\vartheta < 0$, it is easy to check that

$$\lim_{T \to \infty} e^{2\theta T} = 0 \quad (33)$$

and

$$\lim_{T \to \infty} e^{2\theta T} \int_0^T \int_0^t \exp(-\vartheta(t-s))(t-s)^{2H-2} + (t+s)^{2H-2})ds dt = 0 \quad (34).$$

Combining (32), (33) with (34), we obtain the desired convergence. Thus we complete the proof.

In fact, using the Skorohod integral, we can see that LSE of $\vartheta$ are the same in both cases of $\vartheta$ and $\vartheta < 0$. Thus, form (26), we have

$$\dot{\vartheta}_T = -\frac{X_T^2}{2 \int_0^T X_t^2 dt} + \frac{H(2H-1) \int_0^T \int_0^t \exp(-\vartheta(t-s))(t-s)^{2H-2} + (t+s)^{2H-2})ds dt}{\int_0^T X_t^2 dt} + \frac{T}{2 \int_0^T X_t^2 dt}.$$

From Lemma 4.1 and the expression of $\dot{\vartheta}_T$, we can see that the first term of $\dot{\vartheta}_T$, which is identical to LSE based on the Young integral, plays a dominant role asymptotically. Hence, we will also interpret the stochastic integral $\int_0^T X_t dX_t$ in LSE as the Young integral. Thus, LSE can be written as

$$\dot{\vartheta}_T = -\frac{X_T^2}{2 \int_0^T X_t^2 dt}.$$

(35)

Following similar steps as [9], we can obtain the asymptotic consistency and asymptotic law of $\dot{\vartheta}_T$.

**Theorem 4.2.** Let $\vartheta < 0$ and $H > 1/2$. As $T \to \infty$, the estimator in (35) is strong consistency and asymptotical Cauchy

$$e^{-\vartheta T}(\dot{\vartheta}_T - \vartheta) \xrightarrow{L} -\frac{2}{\vartheta} C(1),$$

where $C(1)$ is a standard Cauchy distribution with the probability density function $\frac{1}{\pi(1+x^2)}$.

**Proof.** The proof for this theorem is almost the same as in [9] which only needs to divide the independent part of sfBm and the standard Brownian motion. \qed
5 Simulation study

In this section, we study the finite sample properties of the proposed practical estimator, ̃θₚ when $H < 3/4$. Several experiments are designed via data simulated from msfOUP with different parameter values for $H$ and $θ$ and different values for $d$ and $T$. The main difficulty is to simulate a path from a sfBm. To the best of our knowledge, only one method has been proposed in the literature, which is based on the random walk (see e.g., [15]). Notice that the simulation method proposed by [15] is not stable because of its weak convergence. To improve the convergence rate, we simulate sfBm using (2). For this purpose, we first generate fBm using Paxson’s algorithm (see, [18]). Then, for fixed $n$, we take a sequence of $X(0)$ and different values for $H$.

$$X_n = B^H_n - B^H_{n-1}, X_{-n} = B^H_{-n+1} - B^H_{-n}, n = 1, 2, 3, \ldots.$$  

After that, we obtain the vector $Y = (X_{-n}, \ldots, X_{-1}, X_1, \ldots, X_n)$, which is a standard fractional Gaussian noise with 2$n$ dimension. Thus, we have

$$B^H_1 = Y_{n+1}, B^H_2 = Y_{n+1} + Y_{n+2}, \ldots, B^H_n = Y_{n+1} + \ldots + Y_{2n}$$

and

$$B^H_1 = -Y_n, B^H_2 = -(Y_n + Y_{n-1}), B^H_n = -(Y_n + Y_{n-1} + \ldots + Y_1).$$

Finally, we obtain a sequence of sfBm

$$S^H_1 = \frac{1}{\sqrt{2}}(B^H_1 + B^H_{-1}), \ldots, S^H_n = \frac{1}{\sqrt{2}}(B^H_n + B^H_{-n}). \quad (36)$$

**Proposition 5.1.** For $p, q \in \mathbb{Z}_+$, the covariance of $S^H_p$ and $S^H_q$ are

$$\mathbb{E}S^H_p S^H_q = p^{2H} + q^{2H} - \frac{1}{2}(|p - q|^{2H} + |p + q|^{2H}).$$

**Proof.** We will use the mathematical induction to prove this conclusion. Without loss of generality, we suppose $p > q$. A standard calculation shows that

$$\mathbb{E}S^H_{p-1} S^H_q = (p - 1)^{2H} + q^{2H} - \frac{1}{2}(|p - q - 1|^{2H} + |p + q - 1|^{2H})$$

then

$$\mathbb{E}S^H_p S^H_q = \mathbb{E}S^H_{p-1} S^H_q + \mathbb{E}(Y_{n+p} - Y_{n-(p-1)})S^H_q$$

$$= \sum_{i=1}^{q} \rho(p - i) - \sum_{i=0}^{q-1} \rho(p + i)$$

where $\rho(|k|)$ is the auto-covariance of the fractional Gaussian noise.

Consequently, by a simple calculation, we have

$$\mathbb{E}S^H_p S^H_q = \frac{1}{2}(p^{2H} + (p - q - 1)^{2H} - (p - 1)^{2H} - (p - q)^{2H} - \frac{1}{2}((p + q)^{2H} - (p + q - 1)^{2H} + (p + q - 1)^{2H})$$

$$+ (p - 1)^{2H} + q^{2H} - \frac{1}{2}(|p - q - 1|^{2H} + |p + q - 1|^{2H})$$

$$= p^{2H} + q^{2H} - \frac{1}{2}(|p - q|^{2H} + |p + q|^{2H}).$$

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Using once more the mathematical induction for $S^H_1$ and $S^H_p$, we achieve the desired result.

Now, we can provide an algorithm for the estimating the drift parameter for msfOUP by Monte Carlo simulation:

(i) Set the sampling size $N$ and the time span $T$ and obtain the sampling interval by $d = T/N$;

(ii) Set the values of two variables $H$ and $\vartheta$;

(iii) Generate fBm $B^H_{2T}$ based on Paxson’s method (see [18]), with the sampling interval $d$ and $2N$ points. Thus, we have $0 = t_0 < t_1 < t_2 < \cdots < t_{2N} = T$, $t_i - t_{i-1} = d$;

(iv) consider the sequence of $Y$ with $2N$ dimension

\[
Y_1 = B^H_{t_1}, Y_2 = B^H_{t_2} - B^H_{t_1}, \ldots, Y_i = B^H_{t_i} - B^H_{t_{i-1}}, \ldots, Y_{2N} = B^H_{2N} - B^H_{t_{2N-1}};
\]

(v) Construct a new real line fBm denoted $\tilde{B}^H$:

\[
\tilde{B}^H_{t_i} = \sum_{j=1}^{i} Y_{N+j}, \ i = 1, 2, \ldots, N, \quad \tilde{B}^H_{-t_i} = -\sum_{j=1}^{i} Y_{N-(j-1)}, \ i = 1, 2, \ldots, N;
\]

(vi) Use $S^H_{id} = \frac{1}{\sqrt{2}}(\tilde{B}^H_{t_i} + \tilde{B}^H_{-t_i})$ to obtain sfBm;

(vii) Set $X_0 = 0$ and simulate the observations $X_d, \ldots, X_Nd$ for different values of $H$ and $\vartheta$.

Here, we approximate the msfOUP by the Euler scheme:

\[
X_{(i+1)d} = X_{id} - \vartheta dX_{id} + \left(S^H_{(i+1)d} - S^H_{id}\right) + \left(W_{(i+1)d} - W_{id}\right), \quad i = 0, \ldots, N. \quad (37)
\]

For each case, we simulate $l = 10000$ paths.

(viii) Obtain the practical estimator of (31), by solving the equation

\[
\frac{1}{N} \sum_{i=1}^{N} X^2_{id} = \vartheta^{-2H} H \Gamma(2H) + \frac{1}{2\vartheta}, \quad \text{numerically.}
\]

Now, setting $\vartheta = 0.3$, $T = 16$, $d = 1/2^8$ and $X_0 = 0$, we simulate some paths of msfOUP with different Hurst parameters ($H = 0.52, 0.62, 0.72$). The simulation paths reflex the main property of msfOUP: a large value of $H$ corresponds to a smoother sample path. In other words, for smaller values of $H$, the sample paths of a msfOUP fluctuate more wildly.
In what follows, for some fixed sampling intervals $d = 1/12$ (e.g., data collected by monthly observations) and $d = 1/250$ (e.g., data collected by daily observations), we carry out a simulation study proposed above. Then, we obtain the practical estimator $\tilde{\vartheta}_n$ using some generating datasets with different sampling size $N$ and different time span $T$. For each case, replications involving $l = 10000$ samples are simulated from the true model. The following table reports the mean, the median and standard deviation (S.Dev.) of the practical type estimator proposed by (31) for different sample sizes and different time span, where the true values denote the parameter values used in the Monte Carlo simulation.

| True value | 0.1000 | 0.5000 | 1.0000 | 2.0000 | 0.1000 | 0.5000 | 1.0000 | 2.0000 |
|------------|--------|--------|--------|--------|--------|--------|--------|--------|
| $T = 10$   |        |        |        |        |        |        |        |        |
| Mean       | 0.1461 | 0.6582 | 1.2895 | 2.4880 | 0.1223 | 0.5461 | 0.9743 | 2.3435 |
| Median     | 0.1464 | 0.6783 | 1.3058 | 2.3888 | 0.1173 | 0.5341 | 0.9439 | 2.3639 |
| S.Dev.     | 0.7544 | 0.8502 | 1.0040 | 0.7787 | 0.7014 | 0.8316 | 0.9814 | 0.7225 |
| $T = 20$   |        |        |        |        |        |        |        |        |
| Mean       | 0.1286 | 0.5835 | 1.1310 | 2.3596 | 0.1143 | 0.5149 | 1.0282 | 2.0672 |
| Median     | 0.1324 | 0.5288 | 1.1559 | 2.3092 | 0.1277 | 0.5122 | 1.0448 | 2.0862 |
| S.Dev.     | 0.2358 | 0.3152 | 0.3910 | 0.5065 | 0.3678 | 0.4927 | 0.6155 | 0.5093 |
Table 2 Estimation results with the Hurst parameter $H = 0.65$

| True value | 0.1000 | 0.5000 | 1.0000 | 2.0000 | 0.1000 | 0.5000 | 1.0000 | 2.0000 |
|------------|--------|--------|--------|--------|--------|--------|--------|--------|
| $T = 10$   |        |        |        |        |        |        |        |        |
| Mean       | 0.1492 | 0.6582 | 1.3698 | 2.6292 | 0.1114 | 0.5739 | 1.1608 | 2.2122 |
| Median     | 0.1542 | 0.6783 | 1.3015 | 2.6721 | 0.1131 | 0.5647 | 1.1532 | 2.2314 |
| S.Dev.     | 0.7461 | 0.8502 | 0.6455 | 0.8072 | 0.7182 | 0.7754 | 0.5010 | 0.5717 |
| $T = 20$   |        |        |        |        |        |        |        |        |
| Mean       | 0.1885 | 0.5844 | 1.1397 | 2.3824 | 0.1025 | 0.5153 | 1.0611 | 1.9846 |
| Median     | 0.1304 | 0.5888 | 1.1435 | 2.4303 | 0.1044 | 0.5088 | 1.0739 | 2.0764 |
| S.Dev.     | 0.2412 | 0.3258 | 0.4069 | 0.5299 | 0.0991 | 0.1172 | 0.1285 | 0.1445 |

From numerical computations, we can see that the practical type estimator proposed in this paper performs well for the Hurst parameters $H > \frac{1}{2}$. As is expected, the simulated mean of these estimators converges to the true value rapidly and the simulated standard deviation decreases to zero with a slight positive bias as the sampling interval tends to zero and the time span goes to infinite.

To evidence the asymptotic laws of $\Phi$, we next investigate the asymptotic distributions of $\Phi$. Thus, we focus on the distributions of the following statistics:

$$
\Phi(N, H, \vartheta, d, X) = \frac{\vartheta \sqrt{N d}}{\sigma_H \left( H \Gamma(2H) \vartheta^{1-2H} + \frac{1}{2} \right)} \left( \hat{\vartheta}_N - \vartheta \right). \tag{38}
$$

Here, the chosen parameters are $\vartheta = 0.1$, $H = 0.618$ and we take $T = 16$ and $h = \frac{1}{250}$. We perform 10,000 Monte Carlo simulations of the sample paths generated by the process of (37). The results are presented in the following Figure and Table 3.
Table 3. The comparisons of statistical properties between \( \Phi(N, H, \vartheta, d, X) \) and \( N(0,1) \).

| Statistics    | Mean     | Median  | Standard Deviation | Skewness | Kurtosis |
|---------------|----------|---------|--------------------|----------|----------|
| \( N(0,1) \) | 0        | 0       | 1                  | 0        | 3        |
| \( \Phi(N, H, \vartheta, d, X) \) | 0.0003419 | 0.0716  | 0.000011648        | 0.0246   | 4.5534   |

The histogram indicates that the normal approximation of the distribution of the statistic \( \Phi(N, H, \vartheta, d, X) \) is reasonable even when sampling size \( N \) is not so large. From Table 3, we can see that the empirical mean, standard deviation, skewness and kurtosis are close to their asymptotic counterparts, which confirms our theoretical analysis: the convergence of the distribution of \( \Phi(N, H, \vartheta, d, X) \) is fast. Thus, the density plot of the simulation results is close to the kernel of the limiting distribution of \( \Phi(N, H, \vartheta, d, X) \) proposed by (38) when \( H = 0.618 \). For \( H > \frac{3}{4} \), the limiting distribution, known as Rosenblatt distribution, is not known to have a closed form. Readers who are interested in the density plot of Rosenblatt random variable are referred to [21] and the references therein.

6 Appendix

6.1 Proof of Main Theorem

In this part we will prove the main results of Theorem 3.3 and Theorem 3.4. First of all, let us introduce the following Lemmas.

Lemma 6.1. Let \( S_t^H \) be a sub-fractional Brownian motion. Then, we have

\[
E \left[ \int_s^T e^{-\vartheta(t-s)} dS^H_t \int_t^T e^{-\vartheta(\eta-t)} dS^H_\eta \right] \leq C_{\vartheta, H} |t-s|^{2H-2}
\]

and

\[
E \left[ \int_t^s e^{-\vartheta(t-u)} dS^H_u \int_0^s e^{-\vartheta(s-v)} dS^H_v \right] \leq C_{\vartheta, H} |t-s|^{2H-2}.
\]

Proof. For fixed real number \( t, s \geq 0 \), we have \( |t+s|^{2H-2} \leq |t-s|^{2H-2} \). Then with the method of the web only Lemma 5.4 of [10], we have the conclusions immediately. \( \square \)

Lemma 6.2. For \( H > 1/2 \), we have

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T E \left( \int_0^s e^{-\vartheta(s-u)} dB^H_u \int_s^T e^{-\vartheta(v-s)} dB^H_v \right) ds = 0. \quad (39)
\]
Proof. We calculate the expectation

\[ \mathbb{E} \left[ \int_0^s e^{-\vartheta(s-u)} dB_u \int_s^T e^{-\vartheta(v-s)} dB_v \right] = \mathbb{E} \left[ \int_0^T e^{-\vartheta(s-u)} dB_u \int_s^T e^{-\vartheta(v-s)} dB_v \right] \\
= \int_0^T \int_0^s e^{-\vartheta(s-u)} e^{-\vartheta(v-s)} (v-u)^{2H-2} du dv \\
= \int_0^T \int_0^s e^{-\vartheta(v-u)} (v-u)^{2H-2} du dv \\
= \int_z \int_0^z T^{2H} (e^{-\vartheta(y-x)})^T (y-x)^{2H-2} dy dx. \]

Using the fact \(0 < y - x < 1\) and the L'Hôpital's rule, we can easily verify (39).

6.1.1 Proof of Theorem 3.3

The result in [10] gives the strong consistency for the LSE from the ergodicity. Since the increment of the sfBm is not stationary, we can not use the ergodicity to prove the consistency of \(\hat{\vartheta}_T\). Now, a standard calculation yields

\[ \lim_{T \to \infty} \frac{H(2H-1)}{T} \int_0^T \int_0^t \exp(-\vartheta(t-s))(t-s)^{2H-2} ds dt \\
= \lim_{T \to \infty} \frac{H(2H-1)}{T} \int_0^T \int_0^t u^{2H-2} e^{-\vartheta u} du dt = \vartheta^{1-2H} H \Gamma(2H). \]

On the other hand, a straightforward calculation shows that

\[ \int_0^T \int_0^t \exp(-\vartheta(t-s))(t+s)^{2H-2} ds = \int_0^T \int_0^t \exp(\vartheta(s+t) - 2\vartheta t)(t+s)^{2H-2} ds dt \\
= \int_0^T \exp(-2\vartheta t) \int_0^{2t} \exp(\vartheta u) u^{2H-2} du dt. \]

With the L'Hôpital's rule, we have

\[ \lim_{T \to \infty} \frac{1}{T} \int_0^T \exp(-2\vartheta t) \int_0^{2t} \exp(\vartheta u) u^{2H-2} du dt = 0. \]

Now we only need to prove that

\[ \frac{1}{T} \int_0^T X_t^2 dt \rightarrow P \frac{1}{2\vartheta} + H\vartheta^{-2H} \Gamma(2H). \]  

(40)

For \(0 \leq t \leq T\), let \(W_t\) be a standard Brownian motion and \(X_t = X_t^{(1)} + X_t^{(2)}\). Then, we have

\[ dX_t^{(1)} = -\vartheta X_t dt + dW_t, \quad 0 \leq t \leq T \]
and
\[ dX_t^{(2)} = -\vartheta X_t dt + dS_t^H, \quad 0 \leq t \leq T. \]

With the property of ergodicity, we have
\[ \frac{1}{T} \int_0^T (X_t^{(1)})^2 dt \xrightarrow{a.s.} \frac{1}{2\vartheta}. \]

By a simple calculation, we have
\[
E \left( \frac{1}{T} \int_0^T (X_t^{(2)})^2 dt \right)^2 = \frac{1}{T^2} \int_0^T \int_0^T E \left( \left( X_t^{(2)} \right)^2 \left( X_s^{(2)} \right)^2 \right) dtds \\
= \frac{1}{T^2} \int_0^T \int_0^T E \left( X_t^{(2)} \right)^2 E \left( X_s^{(2)} \right)^2 dtds \\
+ \frac{2}{T^2} \int_0^T \int_0^T E \left( X_t^{(2)} X_s^{(2)} \right)^2 dtds.
\]

From [9], we have
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T E \left( X_t^{(2)} \right)^2 = H \vartheta - 2H \Gamma(2H).
\]

Consequently, we have
\[
\lim_{T \to \infty} \frac{1}{T^2} \int_0^T \int_0^T E \left( X_t^{(2)} \right)^2 E \left( X_s^{(2)} \right)^2 dtds = (H \vartheta - 2H \Gamma(2H))^2.
\]  

(41)

Lemma [6.1] yields
\[
\lim_{T \to \infty} \frac{2}{T^2} \int_0^T \int_0^T E \left( X_t^{(2)} X_s^{(2)} \right)^2 = 0.
\]  

(42)

Combining (41), (42), the independence of \(X_t^{(1)}\) and \(X_t^{(2)}\), and Chebyshev’s Inequality, we can obtain
\[
\frac{1}{T} \int_0^T \left( X_t^{(2)} \right)^2 P \rightarrow H \vartheta - 2H \Gamma(2H),
\]

which complete the proof.

6.1.2 Proof of Theorem 3.4

**Step 1:** We shall use Malliavin calculus and the fourth moment theorem (see, for example, Theorem 4 in [17]) to prove (27).

In fact, using (25), we have
\[
\sqrt{T} \left( \hat{\vartheta}_T - \vartheta \right) = -\frac{1}{\sqrt{T}} \int_0^T \int_0^T e^{-\vartheta(t-s)} d\xi_s^H d\xi_t^H \\
= \frac{1}{T} \int_0^T X_t^2 dt,
\]

where \(F_T\) is the double stochastic integral
\[
F_T = \frac{1}{2\sqrt{T}} \int_2 \left( e^{-\vartheta(t-s)} \right) = \frac{1}{2\sqrt{T}} \int_0^T \int_0^T e^{-\vartheta(t-s)} d\xi_s^H d\xi_t.
\]  

(43)
By [10], we know that \( \frac{1}{T} \int_0^T X_t^2 \, dt \) converges in probability and in \( L^2 \) as \( T \) tends to infinity to \( \frac{1}{2\theta} + H\theta^{-2H}\Gamma(2H) \). From Theorem 4 of [17], we have to check the following two conditions:

(i). \( \mathbf{E}(F_T^2) \) converges to a constant as \( T \) tends to infinity

\[
\lim_{T \to \infty} \mathbf{E}F_T^2 = \theta^{1-4H}H^2(4H-1) \left( \Gamma(2H)^2 + \frac{\Gamma(2H)\Gamma(3-4H)\Gamma(4H-1)}{\Gamma(2-2H)} \right) + \frac{1}{2\theta}.
\]

(ii). \( \|DF_T\|^2_{H^2} \) converges in \( L^2 \) to a constant as \( T \) tends to infinity.

We first check the condition (i). When \( W_t \) and \( S_t^{H_1} \) are independent, we have \( \mathbf{E}F_T^2 = \mathbf{E}(F_{t,T}^2 + F_{T,t}^2) \).

A standard calculation together with [12] yields

\[
\mathbf{E}F_{2,T}^2 = \frac{\alpha^2_H}{2T} \int_{[0,T]^4} \exp(-\theta|u_2 - s_2| - \theta|u_1 - s_1|)|u_2 - u_1|^{2H-2}|s_2 - s_1|^{2H-2}dS_1^2ds_2du_1du_2
\]

\[
- \frac{\alpha^2_H}{2T} \int_{[0,T]^4} \exp(-\theta|u_2 - s_2| - \theta|u_1 - s_1|)|u_2 - u_1|^{2H-2}|s_2 + s_1|^{2H-2}dS_1^2ds_2du_1du_2
\]

\[
- \frac{\alpha^2_H}{2T} \int_{[0,T]^4} \exp(-\theta|u_2 - s_2| - \theta|u_1 - s_1|)|u_2 + u_1|^{2H-2}|s_2 - s_1|^{2H-2}dS_1^2ds_2du_1du_2
\]

\[
+ \frac{\alpha^2_H}{2T} \int_{[0,T]^4} \exp(-\theta|u_2 - s_2| - \theta|u_1 - s_1|)|u_2 + u_1|^{2H-2}|s_2 + s_1|^{2H-2}dS_1^2ds_2du_1du_2.
\]

From [10], we have

\[
\lim_{T \to \infty} \frac{\alpha^2_H}{2T} \int_{[0,T]^4} \exp(-\theta|u_2 - s_2| - \theta|u_1 - s_1|)|u_2 - u_1|^{2H-2}|s_2 - s_1|^{2H-2}dS_1^2ds_2du_1du_2
\]

\[
= \theta^{1-4H}H^2(4H-1) \left( \Gamma(2H)^2 + \frac{\Gamma(2H)\Gamma(3-4H)\Gamma(4H-1)}{\Gamma(2-2H)} \right). \tag{45}
\]

A simple calculation yields

\[
\lim_{T \to \infty} \mathbf{E}F_{1,T}^2 = \lim_{T \to \infty} \frac{1}{T} \int_0^T \int_0^t e^{-\theta(t-s)}dsdt = \frac{1}{2\theta} \tag{46}
\]

Now, if we can prove

\[
\lim_{T \to \infty} \frac{1}{T} \int_{[0,T]^4} \exp(-\theta|u_2 - s_2| - \theta|u_1 - s_1|)|u_2 - u_1|^{2H-2}|s_2 + s_1|^{2H-2}dS_1^2ds_2du_1du_2 = 0, \tag{47}
\]

then the last three terms of \( \mathbf{E}F_{2,T}^2 \) will tend to zero with the fact \( |s_2 + s_1|^{2H-2} \leq |s_2 - s_1|^{2H-2} \).

Denote

\[
I_T = \frac{1}{T} \int_{[0,T]^4} e^{-\theta|u_2 - s_2| - \theta|u_1 - s_1|} \left( |u_2 - u_1|^{2H-2}|s_2 + s_1|^{2H-2} \right) ds_1^2ds_2du_1du_2.
\]

\[21\]
Using the L'Hôpital's rule, we have
\[
\frac{dI_T}{dT} = \int_{[0,T]^3} e^{-\vartheta(T-s_2) - \vartheta|s_1-u_1|} \left( (T - u_1)^{2H-2} (s_2 + s_1)^{2H-2} \right) ds_1 du_1 du_2.
\]

Let \( T - s_2 = x_1, T - s_1 = x_2, T - u_1 = x_3. \) Ignoring the sign, we have
\[
\frac{dI_T}{dT} = \int_{[0,T]^3} e^{-\vartheta x_1 - \vartheta|x_2-x_3|} \left( x_3^{2H-2} (T - x_1 + T - x_2)^{2H-2} \right) dx_1 dx_2 dx_3
\leq e^{-\vartheta T} \int_{[0,T]^3} e^{\vartheta y_1 - \vartheta|y_2-y_3|} \left( y_3^{2H-2} (y_1 + T - x_2)^{2H-2} \right) dy_1 dx_2 dx_3.
\]

Let \( J_T = \int_{[0,T]^3} e^{\vartheta y_1 - \vartheta|y_2-y_3|} \left( x_3^{2H-2} y_1^{2H-2} \right) dy_1 dx_2 dx_3. \) Then, using the L'Hôpital's rule, we get
\[
\frac{dJ_T}{dT} = e^{-\vartheta T} \int_{[0,T]^2} e^{\vartheta y_1 + \vartheta x_3} (y_1^{2H-2} x_3^{2H-2}) dy_1 dx_3.
\]

On the other hand, we can easily obtain \( \frac{d}{dT} e^{-\vartheta T} = -\vartheta e^{-\vartheta T}. \) Moreover, with the L'Hôpital’s rule, it is easy to check that
\[
\lim_{T \to \infty} J_T e^{\vartheta T} = 0,
\]
which implies the equation (47).

Consequently, with (47) and the fact \(|s_2 + s_1|^{2H-2} \leq |s_2 - s_1|^{2H-2},\) it is easy to see that
\[
\lim_{T \to \infty} \frac{1}{T} \int_{[0,T]^4} \exp \left( -\vartheta |u_2 - s_2| - \vartheta |u_1 - s_1| \right) |u_2 + u_1|^{2H-2} |s_2 + s_1|^{2H-2} ds_1 ds_2 du_1 du_2 = 0. \tag{48}
\]

Combining (45), (46), (47) with (48), we verify condition (i).

Now we will check the condition condition (ii). For \( s \leq T, \) we have
\[
D_s F_T = \frac{X_s}{\sqrt{T}} + \frac{1}{\sqrt{T}} \int_s^T e^{-\vartheta(t-s)} d\xi_t.
\]

From (12) we have
\[
\|D_s F_T\|_H^2 = \frac{1}{T} \int_0^T \left( X_s + \int_s^T e^{-\vartheta(t-s)} d\xi_t \right)^2 \, ds + \frac{H(2H-1)}{T} \int_0^T \int_0^T \left( X_s + \int_s^T e^{-\vartheta(t-s)} d\xi_t \right) \left( X_u + \int_u^T e^{-\vartheta(t-u)} d\xi_t \right) (|u-s|^{2H-2} + |u+s|^{2H-2}) \, du \, ds
\]

We first consider the first term of the above equation. A straightforward calculation shows that
\[
\frac{1}{T} \int_0^T \left( X_s + \int_s^T e^{-\vartheta(t-s)} d\xi_t \right)^2 \, ds = \frac{1}{T} \int_0^T \left( X_s^2 + 2X_s \int_s^T e^{-\vartheta(t-s)} d\xi_t + \left( \int_s^T e^{-\vartheta(t-s)} d\xi_t \right)^2 \right) \, ds
= A_T^{(1)} + A_T^{(2)} + A_T^{(3)}.
\]
From the proof of Theorem 3.3, it is easy to see
\[ A_T^{(1)} = \frac{1}{T} \int_0^T X_s^2 ds \]

converges in \( L^2 \) as \( T \) tends to infinity. On the other hand
\[ A_T^{(3)} = \frac{1}{T} \int_0^T \left( \int_s^T e^{-\vartheta(t-s)} d\xi_t \right)^2 ds = \frac{1}{T} \int_0^T \left( \int_0^u e^{-\vartheta(u-x)} d\xi_x \right)^2 du \]
also converges in \( L^2 \).

Finally, By a straight but redundant calculation, we can also have
\[
\mathbb{E}(A_T^{(2)}) = \frac{1}{T^2} \int_0^T \int_0^T \mathbb{E} \left( X_s X_u \left( \int_s^T e^{-\vartheta(t-s)} d\xi_t \right) \left( \int_u^T e^{-\vartheta(t-u)} d\xi_t \right) \right) dsdu
\]
\[
= \frac{1}{T^2} \int_0^T \int_0^T \mathbb{E} X_s X_u \mathbb{E} \left( \int_s^T e^{-\vartheta(t-s)} d\xi_t \right) \left( \int_u^T e^{-\vartheta(t-u)} d\xi_t \right) dsdu
\]
\[
+ \frac{1}{T^2} \int_0^T \int_0^T \mathbb{E} \left( X_s \int_s^T e^{-\vartheta(t-s)} d\xi_t \right) \mathbb{E} \left( X_u \int_u^T e^{-\vartheta(t-u)} d\xi_t \right) dsdu
\]
\[
+ \frac{1}{T^2} \int_0^T \int_0^T \mathbb{E} \left( X_s \int_u^T e^{-\vartheta(t-u)} d\xi_t \right) \mathbb{E} \left( X_u \int_s^T e^{-\vartheta(t-s)} d\xi_t \right) dsdu.
\]

With the independence of the \( W_t \) and \( S_t^H \) in the msfBm motion, the convergence to 0 for the standard Brownian motion case in the proof of Theorem 3.4 of [10] and Lemma 6.1, we can get
\[
\lim_{T \to \infty} \frac{1}{T^2} \int_0^T \int_0^T \mathbb{E} X_s X_u \mathbb{E} \left( \int_s^T e^{-\vartheta(t-s)} d\xi_t \right) \left( \int_u^T e^{-\vartheta(t-u)} d\xi_t \right) dsdu = 0.
\]

Combining the conclusion of the \( A_T^{(2)} \) of standard Brownian motion case in [10], Lemma 6.2, and the fact \( (s+t)^{2H-2} \leq |s-t|^{2H-2} \) for \( s, t > 0 \), we have
\[
\lim_{T \to \infty} \frac{1}{T^2} \int_0^T \int_0^T \mathbb{E} \left( X_s \int_s^T e^{-\vartheta(t-s)} d\xi_t \right) \mathbb{E} \left( X_u \int_u^T e^{-\vartheta(t-u)} d\xi_t \right) dsdu = 0.
\]

For the third term in [10], using the same method, we have
\[
\lim_{T \to \infty} \frac{1}{T^2} \int_0^T \int_0^T \mathbb{E} \left( X_s e^{-\vartheta(t-s)} d\xi_t \right) \mathbb{E} \left( X_u e^{-\vartheta(t-u)} d\xi_t \right) dsdu = 0.
\]

Now let us look at the third term of \( \| D_s F_T \|_{L^2}^2 \). A standard calculation yields
\[
C_T = \frac{H(2H-1)}{T} \int_0^T \int_0^T \left( X_s + \int_s^T e^{-\vartheta(t-s)} d\xi_t \right) \left( X_u + \int_u^T e^{-\vartheta(t-u)} d\xi_t \right)
\]
\[
\left( |u-s|^{2H-2} + |u+s|^{2H-2} \right) duds
\]
\[
= \frac{H(2H-1)}{T} \left( C_T^{(1)} + 2C_T^{(2)} + C_T^{(3)} \right),
\]
where
\[ C_T^{(1)} = \int_0^T \int_0^T X_s X_u \left( |u - s|^{2H-2} + |u + s|^{2H-2} \right) duds, \]
\[ C_T^{(2)} = \int_0^T \int_0^T \left( X_u \int_s^T e^{-\vartheta(t-s)} d\xi_t \right) \left( |u - s|^{2H-2} + |u + s|^{2H-2} \right) duds, \]
and
\[ C_T^{(3)} = \int_0^T \int_0^T \left( \int_s^T e^{-\vartheta(t-s)} d\xi_t \right) \left( \int_u^T e^{-\vartheta(t-u)} d\xi_t \right) \left( |u - s|^{2H-2} + |u + s|^{2H-2} \right) duds. \]

We want to prove that
\[ \lim_{T \to \infty} \mathbb{E} \left[ (C_T - \mathbb{E} C_T)^2 \right] = 0. \] (50)

Since \( X_t \) is Gaussian we can write
\[
\mathbb{E} \left( |C_T^{(1)} - \mathbb{E} C_T^{(1)}|^2 \right) 
= 2 \int_{[0, T]^4} \mathbb{E}(X_s X_t) \mathbb{E}(X_u X_v) \left( |u - s|^{2H-2} + (u + s)^{2H-2} \right) \left( |v - t|^{2H-2} + (v + t)^{2H-2} \right) dudvdsdt 
\leq 8 \int_{[0, T]^4} \mathbb{E}(X_s X_t) \mathbb{E}(X_u X_v) \left( |u - s|^{2H-2} + |v - t|^{2H-2} \right) dudvdsdt.
\]

With the same method of Theorem 3.4 in [10], Lemma 6.1 and the independence of the \( W_t \) and \( S^H_t \) in msfBm, we have
\[ \lim_{T \to \infty} \frac{1}{T^2} \mathbb{E} \left( |C_T^{(1)} - \mathbb{E} C_T^{(1)}|^2 \right) = 0. \] (51)

In the same way we can prove that
\[ \lim_{T \to \infty} \frac{1}{T^2} \mathbb{E} \left( |C_T^{(i)} - \mathbb{E} C_T^{(i)}|^2 \right) = 0, \quad i = 2, 3. \] (52)

Combining (51) with (52), we obtain (50). It is not difficult to check that \( \lim_{T \to \infty} \mathbb{E} C_T \) exists. Finally, we obtain that \( C_T \) converges in \( L^2 \) to a constant. Thus, condition (ii) satisfies.

**Step 2:** Case \( H = 3/4 \). From (25) we have
\[
\frac{\sqrt{T}}{\sqrt{\log(T)}} (\bar{\vartheta}_T - \vartheta) = -\frac{F_T}{\sqrt{\log(T)}} - \frac{1}{T} \int_0^T X_t^2 dt,
\]
where \( F_T \) is defined by (44).

We still use the fourth moment theorem (see, for example, Theorem 4 in [17]) and check two conditions of **Step 1**. Using same calculations of **Step 1**, we can show that
\[
\lim_{T \to \infty} \frac{1}{T \log(T)} \int_{[0, T]^4} e^{-\vartheta|s_2 - u_2| - \vartheta|s_1 - u_1||s_2 - s_1|^{2H-2}(u_2 + u_1)^{2H-2} du_1 du_2 ds_1 ds_2 = 0
\]

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and
\[
\lim_{T \to \infty} \frac{1}{T \log(T)} \int_{[0,T]^4} e^{-\vartheta|s_2-u_2|-\vartheta|s_1-u_1|}(s_2 + s_1)^{2H-2}(u_2 + u_1)^{2H-2}du_1du_2ds_1ds_2 = 0.
\]

On the other hand, a straightforward calculation shows that
\[
\lim_{T \to \infty} E \left( \frac{1}{2\sqrt{T \log T}} \int_0^T \int_0^T e^{-\vartheta|t-s|}dW_t dW_s \right)^2 = 0.
\]

Then, we have
\[
\lim_{T \to \infty} E \left( \frac{F_T}{\sqrt{\log(T)}} \right)^2 = \frac{H^2(2H-1)^2}{2T \log(T)} \int_{[0,T]^4} e^{-\vartheta|s_2-u_2|-\vartheta|s_1-u_1|}(s_2 - s_1)^{2H-2}(u_2 - u_1)^{2H-2}du_1du_2ds_1ds_2
\]
\[
= \frac{9}{4\vartheta^2},
\]
where the equality comes from Lemma 6.6 in [11].

Thus, condition (i) and condition (ii) are obvious when we add a term of \( \frac{1}{\sqrt{\log T}} \) and \( T^{8H-6} = 1 \) with \( H = \frac{3}{4} \).

**Step 3:** In this step we will prove the theorem when \( 3/4 < H < 1 \). From (25), we have
\[
T^{2-2H} (\bar{\vartheta} - \vartheta) = -\frac{T^{1-2H}}{2} \int_0^T \int_0^T e^{-\vartheta|t-s|}d\xi_t d\xi_s
\]
\[
\rightarrow \frac{1}{T} \int_0^T X_t^2 dt.
\]

Let us mention that the condition (ii) in Step 1 will not be satisfied when \( H > 3/4 \). Fortunately, we still have the following convergence:
\[
\lim_{T \to \infty} T^{3-4H} \frac{1}{T} \int_{[0,T]^4} e^{-\vartheta|s_2-u_2|-\vartheta|s_1-u_1|}(s_2 - s_1)^{2H-2}(u_2 + u_1)^{2H-2}du_1du_2ds_1ds_2 = 0
\]

and
\[
\lim_{T \to \infty} T^{3-4H} \frac{1}{T} \int_{[0,T]^4} e^{-\vartheta|s_2-u_2|-\vartheta|s_1-u_1|}(s_2 + s_1)^{2H-2}(u_2 + u_1)^{2H-2}du_1du_2ds_1ds_2 = 0.
\]

With the similarity of the process \( \xi \) and Lemma 6.6 in [11], we have
\[
T^{1-2H} \int_0^T \int_0^T e^{-\vartheta|t-s|}d\xi_t d\xi_s \rightarrow 2\bar{\vartheta}^{-1} R_1
\]
which achieves the proof.
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