Group theory aspects of spectral problems on spherical factors

J.S. Dowker

Theory Group,
School of Physics and Astronomy,
The University of Manchester,
Manchester, England

The Ray–Singer isospectral theorem (1971) is applied to a general spectral function for Laplacians of twisted $p$–forms (say) on homogeneous Clifford–Klein factors of the three–sphere. The inducing formulae necessary to express any spectral quantity for any twisting in terms of those for cyclic subgroups of the tetrahedral, octahedral and icosahedral deck groups are detailed. Further, Artin’s theorem allows the McKay correspondence to be obtained. The isospectral theorem is shown to yield a derivation of the Sunada construction which is equivalent to the later one by Pesce.

\footnote{dowker@man.ac.uk}
1. Introduction.

This paper is a product of my ongoing interest in explicit calculations of spectral quantities, such as the Casimir energy and effective action, on spherical factors as examples of manageable manifolds of non-trivial topology and vaguely physical significance. It is always possible, of course, to go for generality, and treat symmetric spaces, however I prefer to concentrate on rather specific examples, in particular on factors of the three-sphere, not only because the techniques are commonly available but also because the factor possibilities are more extensive and interesting.

The situation I wish to address here is a fairly common one and is the same as that considered in [1] namely that of a (complex) field, of some particular space-time character, belonging to a representation of an internal symmetry group, $G$, and defined on a factor, $S^3/\Gamma$ (to be specified later). Space–time could be $T \times S^3/\Gamma$ but it is the spectral problem on the spherical factor that I propose to concentrate on and I henceforth ignore any time dependence.

In [1] we computed the Casimir energy, for various fields, representations and factors. The calculations for $\Gamma$ one of the binary polyhedral groups, $T', O', Y'$, were performed individually. The basic principle that I wish to investigate here is that all computations can be reduced to those for cyclic groups.

I have implemented this earlier, [2], but only for untwisted fields (and homogeneous factors). That it is possible in general follows from Artin’s theorem and an isospectral result contained in Ray and Singer, [3], applied to the analytic torsion, a specific spectral quantity, but valid generally. I explain this in the next section.

2. The setup.

The standard, generic setup, e.g. [3], is a field, $\tilde{\phi}$, defined on the simply connected universal covering space, $\tilde{\mathcal{M}}$, satisfying the periodicity conditions (twisting),

$$\tilde{\phi}(x\gamma) = \tilde{\phi}(x) \rho(\gamma)$$

which projects down to a ‘multivalued field’, $\phi$, on $\mathcal{M} = \tilde{\mathcal{M}}/\Gamma$. The matrix, $\rho$, is a representation of the space group, $\Gamma$, in the internal group, $\mathcal{G}$, i.e. $\rho \in \text{Hom}(\Gamma, \mathcal{G})$ and $\rho(\gamma\gamma') = \rho(\gamma)\rho(\gamma')$. For concreteness I take $\mathcal{G}$ to be $U(N)$ and $\phi$ in the fundamental representation so that $\rho$ is, initially, an $N \times N$ matrix, $|\rho_{ij}|$. Also, I will choose $\phi$ to be a space-time scalar field, or, possibly, a $p$–form. The set-up is a particular case of a more general situation, termed an ‘automorphic’ field theory in
[4]. In the present paper, \( \phi \) is a section of a flat vector bundle which implies, technically, that it is possible to choose frames such that the heat–kernel, for example, on \( \tilde{\mathcal{M}} \) is proportional to the unit matrix in internal (fibre) space.

A (Laplacian) spectral quantity, \( S(\mathcal{M}; \rho) \), is defined to be a function of the spectrum, \( \{\lambda_n(\rho)\} \), of the de Rham Laplacian \(^2\) on \( \mathcal{M} \) for fields satisfying the twisting (1). Examples might be the fully traced heat–kernel and \( \zeta \)–function. For a flat vector bundle these quantities involve only the character of the twisting, \( \rho \), and not the complete representation. To this fact can be traced the computational tractability.

Ray and Singer, [3], prove an ‘isospectral’ theorem,

\[
S(\mathcal{M}_1; \text{Ind} \rho) = S(\mathcal{M}_2; \rho),
\]

where \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) are both covered by \( \tilde{\mathcal{M}} \) and \( \Gamma_2 \subset \Gamma_1 \). They apply it to the analytic torsion, but it holds in general. I give a quick proof in the next section, for completeness and then apply this, rather simple, result to the spherical factors mentioned above with \( \Gamma_1 \) one of \( T' \), \( O' \) or \( Y' \) and \( \Gamma_2 \) a cyclic subgroup (when \( \mathcal{M}_2 \) is a lens space and, therefore, ‘simpler’).

To explain the bookkeeping, I remark, again, that the representation \( \rho_1 \) is \( N \)–dimensional and that an element of \( \text{Hom}(\Gamma_1, \mathcal{G}) \) can be specified by populating the \( N \times N \) matrix with sufficient and suitable irreps of \( \Gamma_1 \), which are known. For example, there are five non–trivial elements of \( \text{Hom}(Y', U(4)) \), corresponding to the reps \( 4, 4_s, 2_s \oplus 2_s, 2'_s \oplus 2'_s \) and \( 2_s \oplus 2'_s \) of \( Y' \). Therefore the elements corresponding to the individual irreps of \( \Gamma_1 \) are building blocks from which any required \( \text{Hom} \) can be assembled. These are the particular objects I seek.

If \( \Gamma_2 \) is a cyclic subgroup, \( \text{Ind} \rho = \text{Ind} \omega \) (where \( \omega \) is a root of unity) and is (equivalent to) a direct sum of irreps of \( \Gamma_1 \). Putting this into (2), the direct sum becomes an algebraic one\(^3\) and so, turning it around, the spectral quantity building block, \( S(\mathcal{M}_1; A) \) (\( A \) is an irrep of \( \Gamma_1 \)), can be obtained as a linear combination of the \( S(\mathcal{M}_2; \omega) \), the algebraic sufficiency of the cyclic quantities being guaranteed by Artin’s theorem. It is often possible to obtain these latter in closed form.

It is my intention in this paper just to detail these linear combinations. For simplicity, I consider only one–sided (homogeneous) factors of the three–sphere.

\(^2\) The results of this paper are actually valid for any natural operator.

\(^3\) Using the assumed additivity, \( S(\mathcal{M}; \rho \oplus \rho') = S(\mathcal{M}; \rho) + S(\mathcal{M}; \rho') \), which holds, e.g., for the \( \zeta \)–function.
3. Isospectrality.

The eigenvalues of the Laplacian on $\tilde{M}/\Gamma$ are the same as those on the covering manifold, $\tilde{M}$, only the degeneracies might differ and the isospectrality, (2), is really just a statement about these.

As is well known in quantum mechanics, (Landau and Lifshitz, [5]), and many other areas, the eigenspace spanned by the eigenvectors with a particular eigenvalue, $\lambda$, on $\tilde{M}$, forms the carrier space of a rep of any symmetry group that $\tilde{M}$ might possess, induced on the eigenvectors by the action of this group on $\tilde{M}$.

Barring accidents, this rep is an irrep of the biggest symmetry group of $\tilde{M}$ which, in the case of $S^3$, is $O(4)$, although this plays no role in the following. The dimension, $d_\lambda$, of this irrep, $\tilde{E}_\lambda$, is the degeneracy of the corresponding energy level, still referring to the Laplacian on $\tilde{M}$ and ignoring any internal degrees of freedom.

For a chain of subgroups, $\Gamma_1 \supset \Gamma_2 \supset \ldots$, each such rep is subduced from the previous one. (As in crystal field theory, I am thinking of the groups, $\Gamma_i$, as finite ones.) The dimension of all these reps is $d_\lambda$,

$$d_\lambda = \dim \tilde{E}_\lambda = \dim \text{Sub} \tilde{E}_\lambda = \dim \text{Sub Sub} \tilde{E}_\lambda = \ldots .$$

For ease, denote the rep of $\Gamma_1$ by $E_\lambda \equiv \text{Sub} \tilde{E}_\lambda$ and assume that, on $M_2$, there is a field twisted by (1) with $\rho$ now an irrep, $B$. The rep $\text{Sub} E_\lambda$ (of $\Gamma_2$) decomposes according to all the possible irreps the frequency of $B$ being the dimension of the twisted eigenspace on $M_2$, i.e. the twisted degeneracy on $M_2$,

$$d^M_\lambda (B) = \langle B \mid \text{Sub} E_\lambda \rangle_{\Gamma_2} .$$

The total degeneracy give the dimension relation,

$$d_\lambda = \sum_i d^M_\lambda (B_i) ,$$

summed over all irreps, $B_i$, in $E_\lambda$. There is a similar formula for every subgroup.

If $\rho$ is not irreducible, the degeneracy is the intertwining number,

$$d^M_\lambda (\rho) = \langle \rho \mid \text{Sub} E_\lambda \rangle_{\Gamma_2} ,$$

by linearity of characters. The standard formula for the bracket is given in the next section.
As noted by Ray and Singer, [3], Frobenius reciprocity allows one to write this as,

\[ \langle \rho \mid \text{Sub} E_\lambda \rangle_{\Gamma_2} = \langle \text{Ind} \rho \mid E_\lambda \rangle_{\Gamma_1}, \]

which is recognised as the degeneracy of the \( \lambda \) eigenspace on \( \mathcal{M}_1 \) for fields twisted by \( \text{Ind} \rho \) and so,

\[ d^{\mathcal{M}_2}_\lambda(\rho) = d^{\mathcal{M}_1}_\lambda(\text{Ind} \rho). \]

This is the required result and leads to (2).

This analysis is a purely representation-theoretic matter, like the Sunada approach to isospectrality, [6], and its generalisations by Pesce, [7], [8], and Sutton, [9], and I will now give a derivation of this based on (2) which is essentially the same as Pesce’s, [10].

The Sunada construction uses three (finite) groups \( \Gamma, \Gamma_1 \) and \( \Gamma_2 \), with \( \Gamma_1 \) and \( \Gamma_2 \) subgroups of \( \Gamma \), and all taken as symmetry groups of some covering manifold \( \tilde{\mathcal{M}} \). Define, as above,

\[ \mathcal{M} = \tilde{\mathcal{M}}/\Gamma, \quad \mathcal{M}_1 = \tilde{\mathcal{M}}/\Gamma_1, \quad \mathcal{M}_2 = \tilde{\mathcal{M}}/\Gamma_2, \]

and apply (2) to the two pairs, \( (\mathcal{M}, \mathcal{M}_1) \) and \( (\mathcal{M}, \mathcal{M}_2) \) to give

\[ S(\mathcal{M}; \text{Ind} \rho_1) = S(\mathcal{M}_1; \rho_1) \]
\[ S(\mathcal{M}; \text{Ind} \rho_2) = S(\mathcal{M}_2; \rho_2) \]

so that, if \( \text{Ind} \rho_1 \) and \( \text{Ind} \rho_2 \) are \( \Gamma \)-equivalent, one obtains the ‘isospectral’ relation,

\[ S(\mathcal{M}_1; \rho_1) = S(\mathcal{M}_2; \rho_2), \]

generalising Sunada’s original theorem, to which it reduces when \( \rho_1 \) and \( \rho_2 \) are the trivial reps making the equivalence condition somewhat restrictive, but more significant in that the vector bundles are ‘the same’.

---

4 The workers in this field seem to be unaware of the relevance of the Ray–Singer theorem.
4. Inducing representations.

There is a standard technique for computing representations of a group, $G$, from those of a subgroup, $H$, which is especially easy if $H$ is abelian. Given the character tables, routine algebra will produce the answer.

Some basic, textbook facts and organising notation are necessary. I denote the cyclic subgroups of the binary polyhedral groups generically by $\mathbb{Z}_q$. The non–trivial irreps of $\mathbb{Z}_q$ are generated by the $r$–powers of a primitive $q$–th root of unity where $1 \leq r \leq q - 1$. The reps induced by these irreps are denoted initially by $\text{Ind}(\omega^r_q)$, where $\omega_q = e^{2\pi i/q}$. (It is not necessary to consider subgroups, if any, of $\mathbb{Z}_q$, independently, since inducing is transitive.\[^5\])

The necessary values of the order, $q$, are contained in the Threlfall–Coxeter presentation of the binary polyhedral groups, $\langle l, m, n \rangle$

\[
\langle l, m, n \rangle: R^l = S^m = T^n = RST, \quad l = 2, \ m = 3, \ n = 3, 4, 5.
\]

These imply that $(RST)^2 = E = \text{id}$, e.g. Coxeter and Moser, [12], §6.5 so the orders, $q$, are $2l, 2m$ and $2n$.

Because the $\mathbb{Z}_q$ meet all the conjugacy classes of $\langle l, m, n \rangle$ they are sufficient, by Artin’s theorem, to induce all its irreps.

The sufficiency of the irreps of the $\mathbb{Z}_q$ can also be checked in the following way. The number of irreps of $\langle l, m, n \rangle$ is the same as that of the conjugacy classes and these comprise the three, ‘non–trivial’ types,

\[
[R^j]: \quad 1 \leq j \leq l - 1
\]
\[
[S^j]: \quad 1 \leq j \leq m - 1
\]
\[
[T^j]: \quad 1 \leq j \leq n - 1,
\]

(noting that $[T^j] = [T^{2n-j}] = [T^{-j}]$, etc.\[^6\]) plus the two trivial classes, $[E]$ and $[RST] = [E]$, which each contain just a single element. This counting is, self–evidently, exactly that of the effectively independent irreps of the cyclic subgroups, generated by $R, S$ and $T$. Expanding on this a little; the elements (classes), $T^j$ and $T^{2n-j}$ of the subgroup, $\approx \mathbb{Z}_{2m}$, generated by $T$, lie in the same class $[T^j]$ of

\[^5\] Mackey, [11], calls this ‘inducing in stages’. Sometimes the phrase ‘inducing through’ principle is used.

\[^6\] The tetrahedral case, $\langle 2, 3, 3 \rangle$, is more involved in that the two larger class sets are cross–linked, e.g. $[S] = [T^{-1}]$. I give some details in Appendix A, plus some other information.
The non-trivial irreps of \( \mathbb{Z}_{2n} \), \( T \to \omega_{2n}^l \) and \( T \to \omega_{2n}^{2n-r} \) therefore give the same induced character, and so are not distinct, in this regard. The counting of the distinct non-trivial irreps is thus the same as that of the non-trivial classes, (6). The trivial reps \( T \to \omega_{2n}^0 (\equiv 1) \) and \( T \to \omega_{2n}^n (\equiv \overline{1}) \) therefore give the same induced character, and so are not distinct, in this regard. The counting of the distinct non-trivial irreps is thus the same as that of the non-trivial classes, (6). The trivial reps \( T \to \omega_{2n}^0 (\equiv 1) \) and \( T \to \omega_{2n}^n (\equiv \overline{1}) \) are actually common to all cyclic subgroups and contribute \( 1 + 1 \) to the total number of distinct irreps of the complete set, \( \mathbb{Z}_{2l}, \mathbb{Z}_{2m} \) and \( \mathbb{Z}_{2n} \), which is \( 1 + 1 + l - 1 + m - 1 + n - 1 \), the usual value. This confirms Artin’s theorem in this case.

The construction of the induced reps is standard. Possibly it is easiest to use Frobenius reciprocity which states that the number of times an irrep, \( A \), of \( G \) occurs in the rep induced from one, \( B \), of \( H \) is the same as the number of times \( B \) is contained is the rep of \( H \) subduced from \( A \). There is a standard formula for this frequency as the scalar product (on \( H \)) of the corresponding characters

\[
n(A, \text{Ind } B) = \langle A \ | \ \text{Ind } B \rangle_G = \langle \text{Sub } A \ | \ B \rangle_H = \frac{1}{|H|} \sum_h \chi_{\text{Sub } A}(h) \chi^B(h).
\]

Here, \( H \) is \( \mathbb{Z}_q \) generated by \( R, S, T \) in turn, and calculation produces the following inductions.

**For the tetrahedral case:**

\[
\begin{align*}
T & \leftrightarrow S^{-1} & R \\
0 \uparrow & = 1 + 3 & = 1 + 1' + 1'' + 3 \\
1 \uparrow & = 2''_s + 2_s & = 2_s + 2'_s + 2''_s \\
2 \uparrow & = 1'' + 3 & = 3 + 3 \\
3 \uparrow & = 2'_s + 2''_s & - \\
4 \uparrow & = 1' + 3 & - \\
5 \uparrow & = 2_s + 2'_s & -.
\end{align*}
\]

**For the octahedral case:**

\[
\begin{align*}
T & & S & & R \\
0 \uparrow & = 1 + 2 + 3 & = 1 + 1' + 3 + 3' & = 1 + 2 + 3 + 2 \times 3' \\
1 \uparrow & = 2_s + 4_s & = 2_s + 2'_s + 4_s & = 2_s + 2'_s + 2 \times 4_s \\
2 \uparrow & = 3 + 3' & = 2 + 3 + 3' & = 1' + 2 + 2 \times 3 + 3' \\
3 \uparrow & = 2'_s + 4_s & = 4_s + 4_s & - \\
4 \uparrow & = 1' + 2 + 3' & - & -.
\end{align*}
\]
For the icosahedral case:

|   | $T$ | $S$ | $R$ |
|---|-----|-----|-----|
| 0↑ | $1 + 3 + 3' + 5$ | $1 + 3 + 3' + 2 \times 4 + 5$ | $1 + 3 + 3' + 2 \times 4 + 3 \times 5$ |
| 1↑ | $2_s + 4_s + 6_s$ | $3s + 4s + 6s + 2 \times 6_s$ | $2s + 2's + 2 \times 4_s + 3 \times 6_s$ |
| 2↑ | $3' + 4 + 5$ | $3 + 3' + 4 + 2 \times 5$ | $2 \times (3 + 3' + 4 + 5)$ |
| 3↑ | $2'_s + 4_s + 6_s$ | $2 \times (4_s + 6_s)$ | $-$ |
| 4↑ | $3 + 4 + 5$ | $-$ | $-$ |
| 5↑ | $6_s + 6_s$ | $-$ | $-$ |

The irreps of the groups, $T'$, $O'$, $Y'$, are labelled by their dimension, distinguished by dashes and the spinor, double-valued ones have a suffix ‘$s$’ and the column entries cease when repetitions begin. The notation now is that $r\uparrow$ refers to the rep induced by the cyclic irrep generated by $\omega^r$, $\omega$ being a relevant primitive root of unity; say $\omega_{2n}$ for $T$, $\omega_6$ for $S$ and $\omega_4$ for $R$. To specify the particular generator, if required, I write $3\uparrow T$ etc. often leaving the group implicit. $0\uparrow$ is sometimes referred to as the principal induced rep. (See, e.g. Lomont, [13]) with special properties. For example, it contains the trivial rep exactly once.

There are numerous checks of these results. For example one can induce to $O'$ from $Z_6$ via $T'$. For example, $(2 \uparrow S) \uparrow = (1'' + 3) \uparrow T'_r = 2 + 3 + 3'|O'$ where, for convenience, I have used the induction results from $T'$ to $O'$ listed in Stekolschchik, [14] p.178. Further, adding the complete columns gives the regular representation and corresponds to inducing from the trivial rep of $H = \{E\}$. As is well known, this is a consequence of Frobenius reciprocity.

The previous counting shows that not all these relations are independent, as can be confirmed visually.

4. Spectral consequences

The transition to spectral quantities, $S(\tilde{M}/\Gamma; \rho)$, converts the decompositions into algebraic equations which can be solved for the $S(\tilde{M}/\Gamma; A) \equiv S(A)$, where $A$ is an irrep, in terms of $S(\tilde{M}/\Gamma; r\uparrow \gamma)$. These, from (2), equal the lens space quantities $S(\tilde{M}/Z_q; r) \equiv S(r; \gamma)$, where $q$ is the order of the generator $\gamma = R, S, T$. I recall that $r$ labels the twisting of the $U(1)$ bundle on the lens space.

The spinor and non-spinor reps separate and in the case of $(2, 3, 5)$ elimination
yields,
\[ S \begin{pmatrix} 2_s \ \\ 2'_s \ \\ 4_s' \ \\ 6_s' \end{pmatrix} = \begin{pmatrix} 0 & -1 & -1/2 & 1 \\ -1 & 0 & -1/2 & 1 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1/2 & 0 \end{pmatrix} S \begin{pmatrix} 1; T \\ 3; T \\ 5; T \\ 1; S \end{pmatrix} \] (7)
and
\[ S \begin{pmatrix} 1 \\ 3 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & -1 & -1/2 \\ 0 & 0 & -1 & 0 & 1/2 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1/2 \end{pmatrix} S \begin{pmatrix} 0; T \\ 2; T \\ 2; S \\ 2; R \end{pmatrix}. \] (8)

For \( \langle 2, 3, 4 \rangle \),
\[ S \begin{pmatrix} 2_s \\ 2'_s \\ 4_s \\ 4_s' \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1/2 \\ -1 & 1 & 0 \end{pmatrix} S \begin{pmatrix} 1; T \\ 1; S \\ 3; S \end{pmatrix} \] (9)
and
\[ S \begin{pmatrix} 1 \ \\ 1' \ \\ 2 \ \\ 3 \ \\ 3' \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1/2 & -1 & -1/2 \\ 0 & 0 & 1/2 & -1 & 1/2 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1/2 & 0 & 1/2 \\ 0 & 1 & 1/2 & 0 & -1/2 \end{pmatrix} S \begin{pmatrix} 0; T \\ 2; T \\ 2; S \\ 2; R \end{pmatrix}, \] (10)

while for \( \langle 2, 3, 3 \rangle \),
\[ S \begin{pmatrix} 2_s \\ 2'_s \\ 2''_s \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix} S \begin{pmatrix} 1; T \\ 3; T \\ 5; T \end{pmatrix} \] (11)
and
\[ S \begin{pmatrix} 1 \ \\ 1' \ \\ 3 \ \\ 1'' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 0 & 1 & -1/2 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 0 & 1/2 \end{pmatrix} S \begin{pmatrix} 0; T \\ 2; T \\ 4; T \\ 2; R \end{pmatrix}. \] (12)

As an example of a consistency check, one of many, the decomposition, \( 1 \uparrow R = 2_s + 2'_s + 2 \times 4_s \) in the octahedral case (not used in the derivation of (9)) becomes \( \mathcal{S}(R; 1) = \mathcal{S}(S; 1) + \mathcal{S}(S; 3)/2 \). This relates, as an illustration, the analytic torsions on \( \mathbb{Z}_4 \) and \( \mathbb{Z}_6 \) lens spaces and, using Ray’s formula, numerically is \( 2 = 1 \times \sqrt{4} \). (The additive \( S \) is the logarithm of the torsion.)
5. Use of the \( \mathbb{Z}_2 \) subgroup.

While the three cyclic subgroups generated by \( R, S \) and \( T \) are sufficient, additional use of the \( \mathbb{Z}_2 \) subgroup generated by the central element, \( \mathcal{E} = RST \), provides a more symmetrical formulation.

Inducing gives,

\[
\begin{align*}
T' & \uparrow = 1 + 1' + 1'' + 3 \times 3 = 1 + 1' + 2 \times 2 + 3 \times (3 + 3') = 1 + 3 \times (3 + 3') + 4 \times 4 + 5 \times 5' \\
1 \uparrow & = 2 \times (2_s + 2'_s + 2''_s) = \quad 2 \times (2_s + 2'_s + 2 \times 4_s) = 2 \times (2_s + 2'_s + 2 \times 4_s + 3 \times 6_s)
\end{align*}
\]

which illustrates nicely the regular rep result, \( 0_{\{E \uparrow} = 0_{\{E,\mathcal{E}\uparrow} + 1_{\{E,\mathcal{E}\uparrow} \), mentioned before.

Furthermore, the value of \( S \) evaluated for the trivial bundle, \( S(1) \), can be expressed purely in terms of untwisted lens space values. For all factors of \( S^3 \), apart from lens spaces themselves, it is easily established from the above listings that

\[
S(1) = \frac{1}{2} \left( S(0; T) + S(0; S) + S(0; R) - S(0; RST) \right). \tag{14}
\]

This relation was derived in [2] using a geometric, cyclic decomposition of the traced (untwisted) heat–kernel (or, equivalently, the \( \zeta \)–function) on orbifold factors of the two–sphere which was obtained earlier in [15]. The information used is, of course, contained in the symmetry groups. An application to analytic torsion was made in [16] (see also Tsuchiya, [17]) and to Casimir energies in [2].

In addition to (14), it is readily found that the same combination evaluated for the first (spinor) twisting, yields the result, valid for \( T', O' \) and \( Y' \),

\[
S(2_s) = S(1; R) + S(1; S) + S(1; T) - S(1; RST). \tag{15}
\]

(For the tetrahedral case, \( S(1; S) \) equals \( S(5; T) \).)

In similar vein, I find some other universal relations,

\[
\begin{align*}
S(3) & = S(1) + S(2; R) + S(2; S) + S(2; T) - S(2; RST) \\
S(4_s) & = S(2_s) + S(3; R) + S(3; S) + S(3; T) - S(3; RST)
\end{align*}
\]

and also, just for \( Y' \),

\[
\begin{align*}
S(3') & = S(1) + S(4; R) + S(4; S) + S(4; T) - S(4; RST) \\
S(6_s) & = S(4_s) + S(5; R) + S(5; S) + S(5; T) - S(5; RST)
\end{align*}
\]

with trivial equalities, \( S(i + 4; R) = S(i; R) = S(4 - i; R), S(2; RST) = S(0; RST) \), etc.

\textsuperscript{7} For the tetrahedral case, \( S(4_s) \) is zero.
6. The McKay correspondence

Making the cyclic twisting, \( r \), correspond to \( j \) in (6), gives a two–to–one correspondence between the irreps of the three cyclic subgroups\(^8\) and the conjugacy classes, and thence the irreps of \( \langle l, m, n \rangle \). In fact one can go further and link up with the McKay correspondence in the following fashion.

Represent, in the usual cyclotomic way, the inducing cyclic irrep generators, \( \omega^r \), by points on three distinct unit circles, best pictured as great circles on a two–sphere, intersecting at the common, trivial rep points, \( \mathbf{1} \) and \( \overline{\mathbf{1}} \), represented by the north and south poles. Then, for each circle, identify a semicircle with its reflection under a \( \mathbb{Z}_2 \) orbifold action with \( \mathbf{1} \) and \( \overline{\mathbf{1}} \) as fixed points. The three resulting semicircles give a graph with these points as two three–nodes connected by three arcs with \( l - 1 \), \( m - 1 \) and \( n - 1 \) two–nodes each. This is a compactification of the extended Dynkin diagram for \( \tilde{E}_{n+3} \) obtained by linking the two shorter arms to the ‘affine’ node, \( \mathbf{1} \). This seems, to me, a more symmetrical arrangement. The possibility of adding the identity element node to the end of each branch of the Dynkin diagram is noted by Rossmann, [18]. Identifying these nodes goes a little further and is in keeping with the geometrical interpretation where both \( \{E\} \) and \( \{\overline{E}\} \) correspond to rotations through \( 2\pi \) and is the reason I refer to \( \{\overline{E}\} \) as a trivial class and to \( \overline{\mathbf{1}} \) as a trivial rep.

This is not the standard form of the McKay correspondence, which usually labels the nodes by the equivalence classes of the irreps of \( \langle l, m, n \rangle \), but Artin’s theorem demonstrates they are effectively the same. Furthermore, the construction of the previous paragraph obviously applies with the inducing cyclic irreps replaced by the conjugacy classes of \( \langle l, m, n \rangle \), \( [R^j] \) etc. This yields the dual of the standard correspondence.

The class version of the McKay correspondence has been encountered before by Ito and Reid, [19], and discussed in more detail from an algebraic geometry perspective by Brylinski, [20], whose Thm.4.1, gives the rules for constructing a graph which turns out to be a Dynkin diagram. The role of the quaternion representation is played by a ‘special’ class, corresponding to an end vertex of the graph. Whether two vertices (classes) are graphically connected depends on relations between representatives of these two classes and the special one.

Suter, [21], contains suitably labelled Dynkin diagrams and other useful information.

\(^8\) I do not distinguish between the irreps generated by different primitive roots of unity.
7. Concluding remarks.

I have given formulae, eqns. (7) to (12), that enable any spectral quantity for a flat, twisted vector bundle over tetrahedral, octahedral and icosahedral space to be found from the corresponding quantity on lens spaces with various twistings. Applications will be dealt with elsewhere. For example, the results of Cisneros–Molina, [22], on the $\eta$–invariant of twisted Dirac operators can be obtained in a more direct fashion.

Appendix A. The tetrahedral classes.

The coupling between the arms of equal length of the Dynkin diagram of $\langle 3, 3, 2 \rangle$ is expressed by the class equalities, $[S] = [T^{-1}]$ and $[S^2] = [T^{-2}]$ which can be shown by exhibiting the conjugation. For example, $S^{-1} = U^{-1}TU$, where $U$ has to be a group element. In fact $U = T^{-1}RT$ which is proved using the presentation relations, (5). Directly,

$$U^{-1}TU = T^{-1}R^{-1}TTT^{-1}RT = T^{-2}S^{-1}TST^2 = T^4S^5TST^2$$

$$= T^4S^7T = T^4ST = T^3S^2 = S^5 = S^{-1},$$

where I have used the relations $R = ST$, $TST = S^2$ and $T^3 = S^3$.

Ito and Reid, [19], treat the conjugacy relations using a different presentation.

Rossmann, [18] Lemma 2.2, has also considered this coupling using a standard quaternion representation of the generators given, e.g., in Coxeter, [23]. The conjugation (by $i$) stated in [18] appears to be in error. It should be by $j$. The change from $i$ to $j$ corresponds to the conjugation (rotation) by $T$ in the above definition of $U$. See Coxeter, [23],p.75.

This cross–linking means that the picture leading to the (compactified) Dynkin diagram has to be slightly amended for the $\langle 2, 3, 3 \rangle$ case so that the $\mathbb{Z}_2$ action (which is an inversion involution) now identifies a semicircle of one circle with a semicircle of the other. The upshot is that the Dynkin diagram consists of a complete circle for $S$ (or $T$) and a semicircle for $R$.

There is no cross–linking for the octahedral and icosahedral cases. For example one can show that $T = U^{-1}T^{-1}U$ where $U = SRS^{-1}$ by a similar manipulation as above. These conjugacy relations have a geometrical significance.
Appendix B. Induced representations and isopectrality again

There are many treatments of the notion of induced representations, which goes back to Frobenius. Most use the coset decomposition of $G$. I set up the left one,

$$G = g_1H + g_2H + \ldots + g_nH = \bigcup_{i=1}^{n} g_iH, \quad n = |G|/|H|. \quad (16)$$

The $g_i$ can be taken as the representatives of the cosets. If a particular set of representatives is chosen, every group element, $g$, can be written uniquely as $g = g_i h$ for some $g_i$ and $h \in H$.

An unfussy way of proceeding is the following. Consider the basis vectors, $|B, m\rangle$ of a rep, $B$, of the subgroup, $H$, and define the new vectors, $|B, i, m\rangle$ by the object,

$$|B, i, m\rangle = g_i |B, m\rangle, \quad i = 1, \ldots, n, \quad m = 1, \ldots, d, \quad (17)$$

the linear space of which I show to be closed under action by $G$. Consider

$$g |B, i, m\rangle = gg_i |B, m\rangle = g_jh |B, m\rangle = g_j |B, m'\rangle D^{\mathbf{B}}_{m'm}(h) = |B, j, m'\rangle D^{\mathbf{B}}_{m'm}(h), \quad (18)$$

where the left coset decomposition has been used. Hence the vectors, (17), form the basis for the carrier space of a representation of $G$ induced from $B$, of $H$, and denoted $B(H) \uparrow G$ or $B \uparrow$, for short.

One can extract the representation matrices of $B \uparrow$ from (18),

$$D^{\mathbf{B}}_{jm',im}(g) = D^{\mathbf{B}}_{m'm}(h)$$

where $h = g_j^{-1} g g_i$.

In the above manipulation, (18), $g_i$ and $g$ are given, then $g_j$ and $h$ are uniquely determined (given the set of representatives). However, if we write

$$D^{\mathbf{B}}_{jm',im}(g) = D^{\mathbf{B}}_{m'm}(g_j^{-1} g g_i),$$

where now $g_i$, $g$ and $g_j$ are given, we have to ensure that $g_j^{-1} g g_i$ belongs to $H$. This can be achieved by including a ‘Kronecker delta–function’

$$D^{\mathbf{B}}_{jm',im}(g) = \sigma_{ji}(g) D^{\mathbf{B}}_{m'm}(g_j^{-1} g g_i), \quad (19)$$
where
\[ \sigma_{ji}(g) = \begin{cases} 1, & \text{if } g_j^{-1} g g_i \in H \\ 0, & \text{otherwise} \end{cases} \]
or one could write,
\[ D_{jm',im}^B(g) = \hat{D}_{m'm}(g_j^{-1} g g_i), \]
with the same import.

I now give an alternative approach to the isospectrality, \( (2) \). The space of \( p \)-forms on \( M_2 \) twisted by a rep, \( D^B \), can be identified with that of forms, \( \phi \), taking values in \( \mathbb{C}^d \), on the covering manifold, \( \tilde{M} \), satisfying \( (1) \), \( (I \ have \ dropped \ the \ tilde) \),
\[ \phi(x \gamma_2) = \phi(x) D^B(\gamma_2), \quad \gamma_2 \in \Gamma_2. \] (20)
Similarly, the space of \( p \)-forms, \( \hat{\phi} \), on \( M_1 \) twisted by \( \uparrow B \) and taking values in \( \mathbb{C}^{nd} \), is equivalent to that of forms on \( \tilde{M} \) satisfying,
\[ \hat{\phi}(x \gamma) = \hat{\phi}(x) D^B(\gamma), \quad \gamma \in \Gamma_1. \] (21)
where \( D^B \) is defined by \( (19) \).

To show the equivalence of these two spaces, one constructs a one–to–one mapping between them, \( [3] \).

For a form \( \phi \) on \( \tilde{M} \), with values in \( \mathbb{C}^d \) let \( S\phi \) be the form on \( \tilde{M} \) with values in \( \mathbb{C}^{nd} \), defined by
\[ (S\phi)(x) = \sum_{\oplus i} \phi(x \gamma_i) \]
where the \( \gamma_i \) are the representatives of the left cosets of \( \Gamma_2 \) in \( \Gamma_1 \).

The action of \( \Gamma_1 \) on \( S\phi \), is
\[ (S\phi)(x \gamma) = \sum_{\oplus i} \phi(x \gamma \gamma_i) \]
\[ = \sum_{\oplus i} \phi(x \gamma_j \gamma_j^{-1} \gamma_2 \gamma_i) \]
for any \( \gamma_j \). Now let \( j \) range over \( 1 \to n \) and sum over \( j \). If \( \gamma_j^{-1} \gamma_2 \gamma_i \) belongs to \( \Gamma_2 \), then one can apply the action \( (20) \). If \( \gamma_j^{-1} \gamma_2 \gamma_i \) does not belong to \( \Gamma_2 \), one would want the result to be zero. This can be achieved by extending the action to all of \( \Gamma_1 \) by using the \( \hat{D} \), which vanishes for all these other \( \gamma_j \)s. Then
\[ (S\phi)(x \gamma) = \sum_{\oplus i} \sum_j \phi(x \gamma_j) \hat{D}(\gamma_j^{-1} \gamma_2 \gamma_i) \]
\[ = (S\phi)(x) D^B(\gamma), \]
13
and $S\phi$ obeys (21). The map $S$ is therefore into. To show it is also onto, one needs the converse. A form taking values in $C^d$ has the general structure $\hat{\phi} = \sum_{i=1}^{n} \phi_i$ where each $\phi_i$ takes values in $C^d$. If it satisfies (21), then the component $\phi_1$ satisfies (20) and, further, $\hat{\phi} = S\phi_1$. Hence $S\phi$ is everything.

The projection $S$ commutes with the Laplacian since the latter commutes with the action of $\Gamma_1$ and so $S$ preserves eigenspaces.

References.

1. Dowker, J.S. and Jadhav, S. Phys. Rev. D39 (1989) 1196.
2. Dowker, J.S. Class. Quant. Grav. 21 (2004) 4247.
3. Ray, D.B., and Singer, I. Adv. in Math. 7 (1971) 145.
4. Banach, R. and Dowker, J.S. J. Phys. A12 (1979) 2527.
5. Landau, L.D. and Lishitz, E.M. Quantum Mechanics (Pergamon Press, London, 1958).
6. Sunada, T. Ann. of Math. 121 (1985) 169.
7. Pesce, H. Comm. Math. Helv. 71 (1996) 243.
8. Gornet, R., and McGowan, J. J. Comp. and Math. 9 (2006) 270.
9. Sutton, C.J. Equivariant isospectrality and isospectral deformations on spherical orbifolds, [ArXiv:math/0608567].
10. Pesce, H. Contemp. Math 173 (1994) 231.
11. Mackey, G. Induced representations (Benjamin, New York, 1968).
12. Coxeter, H.S.M. and Moser, W.O.J. Generators and relations of finite groups (Springer. Berlin, 1957).
13. Lomont, J.S. Applications of finite groups (Academic Press, New York, 1959).
14. Stekholschkik, R. Notes on Coxeter transformations and the McKay correspondence. (Springer, Berlin, 2008).
15. Chang, P. and Dowker, J.S. Nucl. Phys. B395 (1993) 407.
16. Dowker, J.S. and Chang, Peter Analytic torsion on spherical factors and tessellations, arXiv:math.DG/0904.0744 .
17. Tsuchiya, N. J. Fac.Sci., Tokyo Univ. Sect.1 A, Mathematics 23, 289-295 (1976).
18. Rossman, W. McKay’s correspondence and characters of finite subgroups of $SU(2)$ Progress in Math. Birkhauser (to appear).
19. Ito, Y. and Reid, M. The McKay correspondence for finite subgroups of $SL(3,C)$ Higher dimensional varieties, (Trento 1994), 221-240, (Berlin, de Gruyter 1996).
20. Brylinski, J-L., *A correspondence dual to McKay’s* ArXiv [alg-geom/9612003].
21. Suter, R. *Manusc. Math.* **122** (2007) 1-21.
22. Cisneros–Molina, J.L. *Geom. Dedicata* **84** (2001) 207.
23. Coxeter, H.S.M. *Regular Complex Polytopes*, (Cambridge University Press, Cambridge, 1975).