Quantum optical reconstruction scheme using weak values

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(Dated: November 9, 2012)

A quantum state contains the maximal amount of information available for a given quantum system. In this paper we use weak-value expressions to reconstruct quantum states of continuous-variable systems in the quantum optical domain. The role played by postselecting measured data will be particularly emphasized in the proposed setup, which is based on an interferometer just using simple homodyne detection.

PACS numbers: 03.65.Ta, 03.65.Wj

I. INTRODUCTION

Since the launch of the idea and formalism by Aharonov, Albert, and Vaidman [1, 2] a considerable amount of research has focused on weak measurements. Basic questions behind the concept have been clarified in a series of papers [3–8], and it was possible to extend the original proposal in various directions [9–14]. It is fascinating to see how the weak-value formalism touches very different fields like phase-space physics [15, 16], quantum trajectories [17, 18], contextuality [19], and quantum cloning [20].

In parallel to these theoretical developments the experimental realization [21] of weak measurements has recently found a rich territory of applications [22–24]. Even the most fundamental debates about the realism of quantum observables [25–27] and the role of trajectories in double-slit experiments [28, 29] can now be attacked with the assistance of weak measurements carried out in the laboratory.

In the present paper we concentrate on yet another crosslink: Lately quantum state reconstruction [30, 31] using weak measurements has become a topic of great interest [32–36]. Several works analyze the aspect of a weak interaction [37–39] in non-standard quantum state estimation. Obviously there is a special role played by postselection in creating weak values of observables. This thought is also pursued in Ref. [32] where a quantum wave function has been reconstructed using the special form of weak values. Later this procedure was generalized for mixed quantum states [40], drawing connections between the structure of weak values and certain phase-space distributions [41, 42]. A similar connection was also established in Refs. [43] and [44] and later used to build yet another reconstruction scheme [36, 45].

In this article we first recall the special form of weak values used to derive reconstruction relations. Our focus then lies in establishing a direct connection between a quantum optical implementation and these reconstruction relations. The main contribution of this work therefore consists of working out a realizable quantum optical scheme that fully exploits the form of weak values to reconstruct a continuous-variable state of a single mode of the quantized electromagnetic field.

In Sec. II we will recapitulate the formalism of weak measurements. Section III provides the reconstruction relations for quantum states of continuous-variable systems. In Sec. IV we show how to realize the method with a quantum optical setup. Finally, in Sec. V we simulate the optical experiment for a special quantum state of light and discuss to which class of states the reconstruction scheme can be applied.

II. WEAK MEASUREMENTS

In this section we will briefly recall the concepts of weak measurements [1] to introduce our notation and to make the paper self-consistent. An appropriate scaling allows us to choose dimensionless quantities everywhere.

A weak measurement can be divided into four successive steps as depicted in Fig. 1. In the first step the

![FIG. 1. Basic setup of a weak measurement which can be applied to reconstruct the unknown state \(|\psi\rangle\) of a system \(S\) with the help of a meter \(M\) initially prepared in a specific state \(|\phi\rangle\). After system and meter have been separately defined in the first step, they interact weakly in the second step. This leaves \(S\) and \(M\) in the joint state \(|\Psi\rangle\). A suited projective measurement with outcome \(b_j\) in the third step, the postselected expectation value \(E^{(j)}[\hat{\delta}_M]\) of a convenient meter observable \(\hat{\delta}_M\) can be obtained in the fourth step. We demonstrate in Sec. III that the probability distribution of finding the eigenvalue \(b_j\) and the expectation value \(E^{(j)}[\hat{\delta}_M]\) are sufficient to reconstruct the state \(|\psi\rangle\).]
system $S$ is given in the unknown state $|\psi\rangle$ and the meter $M$ is prepared in the state $|\phi\rangle$ which we assume to be the Gaussian

$$|\phi\rangle = \frac{1}{\pi^{1/4}} \int_{-\infty}^{\infty} dx \ e^{-\frac{x^2}{2}} |x\rangle$$

in the scaled position $x$ with vanishing first moments. In the second step system and meter interact. At this point the interaction will be simply modeled by the von Neumann Hamiltonian $[47]$

$$\hat{H} = g\delta(t - t_0)\hat{A}_S\hat{p}_M,$$

where $g$ is the interaction strength and $\hat{A}_S$ represents a certain observable of the system which is coupled to the momentum operator $\hat{p}_M$ of the meter. The interaction happens instantaneously at time $t_0$ and is weak, i.e. $g$ is small. In Sec. [48] all these model-like assumptions are shown to be implementable in a quantum optical setup for state reconstruction.

A postselective measurement is then performed in the third and fourth step. Postselection begins by making a projective measurement of an observable $\hat{B}_S$ of the system. We obtain one of its eigenvalues $b_j$ with probability

$$p(b_j) = \langle \Psi | (|b_j\rangle_S \langle b_j| \otimes \hat{\sigma}_M) | \Psi \rangle,$$}

where $|\Psi\rangle \equiv \exp[-i\hat{A}_S\hat{p}_M]|\psi\rangle|\phi\rangle$ denotes the joint state of system and meter after the interaction.

The fourth and final step consists of measuring a suitable observable $\hat{\sigma}_M$ of the meter, given that we have found the eigenvalue $b_j$ before. The corresponding postselected expectation value

$$E^{(j)}[\hat{\sigma}_M] = \frac{1}{p(b_j)} \langle \Psi | (|b_j\rangle_S \langle b_j| \otimes \hat{\sigma}_M) | \Psi \rangle$$

can now be approximated by keeping all terms up to linear order in the coupling $g$. With the notation $\{\}$ for the commutator and $\{\}$ for the anticommutator, this yields $[8]$

$$E^{(j)}[\hat{\sigma}_M] = ig\text{Re}[A^{(j)}_w]|\phi\rangle\langle \hat{p}_M, \hat{\sigma}_M||\phi\rangle + + g\text{Im}[A^{(j)}_w]|\langle \hat{p}_M, \hat{\sigma}_M||\phi\rangle,$$}

where we have used the vanishing first moments of $|\phi\rangle$, Eq. [11], and the definition of the weak value $[1]$

$$A^{(j)}_w = \frac{\langle b_j| \hat{A}_S|\psi\rangle}{\langle b_j|\psi\rangle}. $$

The use of weak measurements for quantum state reconstruction $[32, 33, 36, 49]$ relies on the special form of this definition. Notably, it resembles expressions as they are used in interferometric state reconstruction algorithms $[48]$. Before using this special form of the definition of the weak value to derive reconstruction relations in Sec. [11] we finally look at the case where the observable $\hat{\sigma}_M$ to be measured in step 4 is the scaled position $\hat{x}_M$ of the meter. Using the relations

$$\langle \phi | \hat{p}_M, \hat{x}_M | \phi \rangle = 0 \quad \text{and} \quad [\hat{p}_M, \hat{x}_M] = -i$$

we can simplify the expectation value Eq. [5] and arrive at

$$E^{(j)}[\hat{x}_M] = g\text{Re} \left[ A^{(j)}_w \right].$$

So, for this special choice of meter observables, the real part of the weak value $A^{(j)}_w$ is a measurable quantity.

III. RECONSTRUCTION RELATIONS

In this section we will present the very heart of our reconstruction scheme. It turns out that the real part of a weak value as in Eq. [8] and the postselection probability Eq. [3] can be used to reconstruct an unknown state $|\psi\rangle$ of the system. We will focus on the case leading to reconstruction relations for the momentum representation

$$\langle P|\psi\rangle \equiv \psi(P) \equiv |\psi(P)|e^{i\varphi(P)}.$$}

Note that this means to reconstruct the modulus $|\psi(P)|$ and the phase $\varphi(P)$. This also means that the relations which we will present are not directly applicable to the case of a mixed state.

We start by regarding a weak measurement of the position observable $\hat{A}_S = \hat{X}_S$ and a projective measurement of the conjugate variable, the momentum $\hat{B}_S = \hat{P}_S$, in the postselection step. If we insert these special choices into the definition of the weak value, Eq. [10], we get

$$X^{(P)}_w \equiv \frac{\langle P|\hat{X}_S|\psi\rangle}{\langle P|\psi\rangle} = i\frac{d\varphi(P)}{\psi(P)},$$

where we have also used the momentum representation of the position observable $\hat{X}_S$. Finally, rewriting this with the polar decomposition Eq. [10] results in

$$X^{(P)}_w = i \frac{d\varphi(P)}{|\psi(P)|} - \frac{d\varphi(P)}{dP}.$$}

An analogous relation was first mentioned in the context of Bohmian mechanics and its connection to weak values $[13, 18, 49]$. We will utilize it to derive a reconstruction relation which then turns out to be particularly suited for the quantum optical implementation in the next section. By taking the real part of Eq. [11] we find in particular that

$$\text{Re} \left[ X^{(P)}_w \right] = -\frac{1}{g} E^{(P)}[\hat{x}_M].$$}

If we now again use the special choice of observables $\hat{A}_S = \hat{X}_S$ and $\hat{B}_S = \hat{P}_S$ in Eq. [8] and combine the result with Eq. [12], we arrive at

$$\frac{d\varphi(P)}{dP} = -\frac{1}{g} E^{(P)}[\hat{x}_M].$$
This leads directly to the reconstruction relation for the phase,

\[ \varphi(P) = -\frac{1}{\mathcal{g}} \int_{0}^{P} dP' \mathbb{E}(P') [\hat{x}_M], \]

(14)
in which the lower integral bound is arbitrary, since any global phase of the state \(|\psi\rangle\) cannot be detected.

To complete the state reconstruction we still need the absolute value \(|\psi(P)|\) of the system state in momentum representation. Hence we calculate the probability Eq. 3 of finding the eigenvalue \(P\) in the projective measurement on the system. When we again retain only terms up to linear order in \(g\), we arrive at

\[ p(P) = \langle \Psi | (|P\rangle S \langle P| \otimes \hat{1}_M) |\Psi\rangle = |\psi(P)|^2, \]

(15)
keeping in mind that the first moments of the meter state \(|\phi\rangle\), Eq. 11, vanish. Thus the reconstruction relation for the absolute value reads

\[ |\psi(P)| = \sqrt{p(P)}. \]

(16)
Note that the information on the modulus \(|\psi(P)|\) just comes from simple projective measurements of momentum performed in step 3 of the scheme shown in Fig. 1. The corresponding information on the phase \(\varphi(P)\), according to Eq. (14), can be extracted only from a set of postselected data, i.e., a continuum of expectation values ordered with respect to the momentum of the system.

IV. QUANTUM OPTICAL IMPLEMENTATION

The reconstruction relations introduced in the previous section rely on the special form of the interaction Hamiltonian Eq. 2. So it is a crucial question to ask where in nature the momentum observable of one subsystem, here \(\hat{P}_M\) of the meter \(M\), is coupled to the position observable of another subsystem, here \(\hat{X}_S\) of the unknown system \(S\). In the next section we recall a straightforward optical realization for which the derived reconstruction relations hold. For another optical version with a very different aim, see Ref. 50.

A. Beam splitter interaction

In linear quantum optics 51 a beam splitter simply couples two modes of light; see Fig. 2. One mode represents the system \(S\) prepared in the state \(|\psi\rangle\) and the second mode will be the meter in state \(|\phi\rangle\). The joint state after the beam splitter reads 30, 51

\[ |\Psi\rangle = \exp \left[ -i \theta \left( \hat{X}_S \hat{P}_M - \hat{P}_S \hat{x}_M \right) \right] |\psi\rangle |\phi\rangle, \]

(17)
where \(\theta\) is the parameter that determines reflection and transmission of the device 52. Here the observables

\[ \hat{X}_S = \frac{1}{\sqrt{2}} \left( \hat{a}_S + \hat{a}_S^\dagger \right) \] and \[ \hat{P}_S = \frac{1}{i \sqrt{2}} \left( \hat{a}_S - \hat{a}_S^\dagger \right) \]

(18)
are the position and momentum quadrature observables 51 of the system mode, which are defined by the usual creation and annihilation operators \(\hat{a}_S^\dagger\) and \(\hat{a}_S\). Analogous relations hold for the quadratures \(\hat{x}_M\) and \(\hat{p}_M\) of the meter mode. Hence we see that the beam splitter interaction closely resembles the interaction modeled by the Hamiltonian of Eq. 2. The additional term, determined by the product \(\hat{P}_S \hat{x}_M\), will actually not affect the postselected expectation value Eq. 8.

B. Implementation

We can now specify the calculations in Sec. III using the beam splitter interaction given by Eq. 11. The state \(|\psi\rangle\) of the system mode is still arbitrary; the special state of the meter mode is given in Eq. 14, where \(|\phi\rangle\) is now a position quadrature eigenstate. Thus Eq. 14 is just the \(x\) representation of the vacuum state \(|\phi\rangle = |0\rangle\).

The first step of the postselection is now a projective measurement of the momentum quadrature observable \(\hat{B}_S = \hat{P}_S\), as depicted in Fig. 2. The probability \(p(P)\) of finding a certain eigenvalue \(P\) can then be calculated based on Eqs. (14) and (17). When we neglect all terms of order higher than linear in \(\theta\) and keep in mind that the first moments of \(|0\rangle\) vanish, we will get the result

\[ p(P) = |\psi(P)|^2 \]

in complete analogy to Eq. (13).

Now we can continue with the expectation value of the position quadrature \(\hat{x}_M\), also neglecting all terms of order higher than linear in \(\theta\), which results in

\[ \mathbb{E}(P)[\hat{x}_M] = \theta \left\{ \text{Re} \left[ \hat{X}_S(P) \right] - \text{Im} \left[ \hat{P}_S(P) \right] \right\}. \]

(19)
Furthermore we find

$$\text{Im} \left[ P_w^{(P)} \right] = \text{Im} \left[ \frac{\langle P | \hat{P}_S | \psi \rangle}{\langle P | \psi \rangle} \right] = 0, \quad (20)$$

since $\hat{P}_S$ is a Hermitian operator. Thus the expectation value Eq. (19) simplifies and is now equivalent to Eq. (5) when we replace the interaction constant $g$ by the beam splitter parameter $\theta$. The additional term $\hat{P}_S \hat{x}_M$ in the beam splitter transformation Eq. (17), from which the imaginary part of the weak value $P_w^{(P)}$ originates, has no influence on this expectation value. Hence a beam splitter interaction with a small parameter $\theta$ and a specifically chosen postselection measurement can be used to obtain the real part of the position quadrature weak value $X_w^{(P)}$. Based on this information and the knowledge about the probability $p(P)$ of finding an eigenvalue $P$ in the postselection measurement, the unknown state $|\psi\rangle$ of the system mode $S$ can be reconstructed using exactly the reconstruction relations derived in Eqs. (14) and (16).

Note that the reconstruction relations for phase, Eq. (14), and modulus, Eq. (16), of a momentum representation rely on the special choice of the weakly measured observable $\hat{A}_S = \hat{X}_S$ and the postselection observable $\hat{B}_S = \hat{P}_S$ as well as on the form of the interaction, Eq. (17). A different choice of observables and interaction may thus lead to reconstruction relations suitable for representations in terms of other degrees of freedom. This suggests that the general idea behind this reconstruction scheme might be applicable to a broader range of physical systems, not only light modes coupled by a beam splitter.

V. SIMULATION

To see if the reconstruction scheme presented in the previous paragraphs is really feasible, we will numerically simulate the setup shown in Fig. 2 and investigate the various influences of relevant parameters.

A. Influence of $\theta$

We will first focus on the errors introduced by neglecting higher orders of the beam splitter parameter $\theta$. Moreover, it is then interesting to see how a statistical error in the measured observables due to a finite number of measurement runs influences the reconstruction quality. To give a first example, we choose the state

$$|\psi\rangle = N(|\alpha = 1\rangle + |\beta = 2 e^{i\pi/4}\rangle) \quad (21)$$

of the system mode $S$, where $N$ is a normalization constant and $|\alpha\rangle$ and $|\beta\rangle$ are coherent states of light. Note that this state in fact contains nontrivial phase and modulus dependencies which allow us to simulate essential features of the reconstruction.

FIG. 3. (Color online) Error introduced by just keeping terms up to linear order in $\theta$ in the weak-value expressions. Depicted are the corresponding Wigner functions: $W_e$ of the exact state, Eq. (21), $W_1$ of the reconstructed state $|\psi_{\text{rec}}(\theta = 0.05)\rangle$ and $W_2$ of the reconstructed state $|\psi_{\text{rec}}(\theta = 0.05)\rangle$. The reconstruction error $\delta$, Eq. (22), quantifies the total error.
First we have to decide how small $\theta$ shall be chosen to justify neglecting all terms of order higher than linear in $\theta$. Therefore we calculate the exact expectation value, Eq. (4), and the exact probability, Eq. (3), by using the full beam splitter transformation, Eq. (17), applied to the system state $|\psi\rangle$, Eq. (21), and the meter state $|\phi\rangle = |0\rangle$. Next we insert the results of these calculations into the reconstruction relations given in Eqs. (14) and (16). As these relations have been derived disregarding all terms of order higher than linear in $\theta$, they might be rough approximations, unless $\theta$ is small enough. To quantify the error, we compute the reconstruction error

$$\delta \equiv 1 - \left| \langle \psi | \psi_{\text{rec}}(\theta) \rangle \right|^2$$

between the reconstructed state $|\psi_{\text{rec}}(\theta)\rangle$ and the exact state $|\psi\rangle$.

In Fig. 3 we compare Wigner functions $|\psi\rangle$ and $|\phi\rangle$ of the reconstructed states to the Wigner function of the exact state and we also show the corresponding reconstruction error, Eq. (22). It is clearly visible that the choice $\theta = 0.05$ for the beam splitter results in a small error introduced by retaining only terms of order up to linear in $\theta$. Moreover, we also note that even for a 50:50 beam splitter [cf. Fig. 3(c)], one can still reconstruct some features of the state $|\psi\rangle$, Eq. (21), by using the proposed scheme. However, negative parts of the Wigner functions are very sensitive to a badly chosen parameter $\theta$.

Remarkably for a 50:50 beam splitter, i.e., $\theta = \pi/4$, the setup depicted in Fig. 2 is nothing else than an eight-port interferometer (see, for example, [51]). In this case the reconstruction error $\delta$ is large for our way of processing the measured data. Yet the Husimi-Kano $Q$ function [58, 59] may be obtained directly as the joint probability distribution of such an eight-port interferometer [60]. However, it is quite complicated to extract the underlying quantum state from a $Q$ function. On the other hand, if $\theta$ is small we can directly obtain the absolute value and the phase of an unknown state by using the simple relations Eqs. (14) and (16). At this point it is interesting to see how in two different measurement regimes two very different ways of processing measured data each lead to the full state information.

### B. Statistical error

Up to now we have dealt with exact expectation values and full probability distributions, which in a real experiment are accessible only in the limit of an infinite number $N$ of measurement runs. To account for the influence of the statistical error due to a finite $N$, we also perform a Monte Carlo simulation for the optical setup shown in Fig. 2 with the state of Eq. (21). For this simulation, as for any real experiment, we furthermore have to choose a certain binning $\Delta P$ [61] of the momentum quadrature values $P$. Hence we obtain approximated probability distributions for the $P$ quadratures and approximated values for the postselected expectation values Eq. (19). These estimates for probability and expectation value are then inserted into the reconstruction relations Eqs. (14) and (16).

Figures 4 and 5 represent two typical measurements for a properly chosen beam splitter parameter $\theta = 0.05$. The number of measurement runs is $N = 10^4$ and $N = 10^5$, respectively. We show the pure measurement data, namely, approximated probabilities and approximated expectation values, as well as the resulting reconstructed phase. The steps in the probabilities and expectation values as well as the edges in the reconstructed phases are an artifact of the finite bin width $\Delta P$. As expected, increasing the number of measurement runs $N$ improves the reconstruction quality.

Furthermore, we notice that in general those parts of the reconstructed phases located in the vicinity of $P = 0$ are in better agreement with the exact phase than outer parts situated at larger values of $P$. There are two reasons for this. First, according to Eq. (14), we start the integration of the postselected expectation value at $P = 0$. Therefore, the reconstructed and exact phases agree perfectly at this point. Then we need more and more expectation values to reconstruct the phase for increasing values of $P$. Hence the statistical errors involved in every single expectation value sum up. Second, the statistical error of a single expectation value increases when the corresponding probability of obtaining the outcome $P$ decreases. This is most clearly visible from the outer parts of the estimated expectation values in Figs. 4 and 5 which tend to fluctuate more strongly around the exact value as the probability $p(P)$ gets smaller.

Due to these intricacies, the reconstructed phase is more sensitive to statistical errors than the reconstructed modulus. Even for $N = 10^5$ measurement runs (see Fig. 5) the reconstructed phase does not match the exact value as nicely as does the reconstructed modulus. For $N = 10^4$ (see Fig. 4) this contrast is even more pronounced. As in other reconstruction schemes [30, 31, 48], it is harder to reconstruct the phase than the absolute value of the quantum state.

However, in the scheme discussed here this sensitivity of the phase can be fully traced back to the measured data: The set of postselected expectation values needs to be known with good statistical confidence and on the whole $P$ axis. This also means that quantum states $|\psi\rangle$ with regions in the quadrature distribution $|\psi(P)|^2$ that are nonzero and separated by an intermediate region where the quadrature distribution is equal to zero cannot be reconstructed easily. In these gaps we will never find a postselected expectation value and hence we will lose the phase relation between parts of the quadrature representation $\psi(P)$ separated by these gaps. To a certain degree this can be seen already in Figs. 4 and 5. For some values of $P$ we are missing the corresponding expectation values (see, e.g., $P = -2.75$ in Fig. 5), and hence we lose track of the phase. However, as this happens here in a region where $|\psi(P)|$ stays exponentially small, it is not very problematic and the reconstruction error $\delta$ is still.
FIG. 4. (Color online) Monte Carlo simulated (red, solid) and exact (black, dashed) values of probability $p(P)$, expectation value $E^{(P)}[\hat{x}_M]$, and phase $\varphi(P)$ as a function of the momentum quadrature $P$ for the state in Eq. (21). Depicted are results for $N = 10^4$ measurement runs for a beam splitter parameter $\theta = 0.05$ and a bin width $\Delta P = 0.1$. The reconstruction error is $\delta = 0.08$. One can see that the statistics is good enough to reconstruct the modulus $|\psi(P)| = \sqrt{p(P)}$. However, the postselected expectation value shows considerable deviations. This is even more pronounced in those regions where we have a small probability to find a certain value of the momentum quadrature. As a consequence the reconstructed phase soon differs from the exact value.

FIG. 5. (Color online) Monte Carlo simulated (red, solid) and exact (black, dashed) values of probability $p(P)$, expectation value $E^{(P)}[\hat{x}_M]$, and phase $\varphi(P)$ as a function of the momentum quadrature $P$ for the state in Eq. (21). Depicted are results for $N = 10^5$ measurement runs for a beam splitter parameter $\theta = 0.05$ and a bin width $\Delta P = 0.1$. The reconstruction error is $\delta = 0.008$. In comparison with Fig. 4 the statistics here is much better. The postselected data now show stronger fluctuations only in those regions where the probability $p(P)$ is small. Hence the reconstructed phase deviates considerably only when the modulus has already become exponentially small.
small.

C. Counterexample

An explicit example of a state showing a distinct gap in the probability distribution $|\psi(P)|^2$ is the state

$$|\psi\rangle = N (|\alpha = 2i\rangle - |\beta = -2i\rangle),$$

(23)

where $N$ again marks the normalization constant for a superposition of two coherent states $|\alpha\rangle$ and $|\beta\rangle$. In Fig. 6 we show the corresponding Monte Carlo simulated data in comparison to the exact results. The probability $|\psi(P)|^2$ around the origin is so small that there are values of $P$ which never appear on our meter HDS in all of the $N = 10^5$ simulated measurements. It is exactly those values of $P$ where we can find no estimate for the post-selected expectation value. Therefore, the reconstructed expectation value is not continuous around the origin and we cannot use Eq. (14) to reconstruct the phase on the complete $P$ axis.

However, as shown in Fig. 6 we can integrate the reconstructed expectation values in those regions where $|\psi(P)|^2$ is large enough and reconstruct a phase. To this end we have to choose two different starting points of the integration Eq. (14), each lying within one of these regions. This means we explicitly pick a certain “global” phase in each of the two regions. Hence we introduce an arbitrary relative phase shift between the two regions.

Now let us compare this to the exact result also shown in Fig. 6. The exact value of the phase $\varphi(P)$ jumps by $\pi$ at the origin and is otherwise constant. The two parts of the reconstructed phase show this constant behavior in the corresponding regions where the influence of the statistical error is small, i.e., where $|\psi(P)|^2$ is reasonably large. The difference between these two constant values is exactly the relative phase shift introduced by simply choosing a “global” phase in each of the two regions separately. Therefore, without any additional measurements we cannot determine this relative phase shift. Consequently for this example reconstruction fails. Fortunately, gaps in the probability distribution are visible from the measured data and therefore indicate whether we can rely on the reconstructed state or not.

A possible workaround is the reconstruction of a unitarily transformed state, not showing any gaps. One example would be to measure suitable rotated quadratures on the homodyne detectors. This is equivalent to implementing a unitary transformation rotating the Wigner function of the input state. But this procedure will not dispose of gaps in a state with a rotationally invariant Wigner function, such as, for example, a Fock state. So there is no general recipe to find a unitary transformation that is easily implementable and removes the gaps.

FIG. 6. (Color online) Monte Carlo simulated (red, solid) and exact (black, dashed) values of probability $p(P)$, expectation value $E(P)\{\hat{x}_M\}$, and phase $\varphi(P)$ as a function of the momentum quadrature $P$ for the state in Eq. (23). Depicted are results for $N = 10^5$ measurement runs for a beam splitter parameter $\theta = 0.05$ and a bin width $\Delta P = 0.1$. The small probability at the origin causes gaps in the simulated expectation values. Hence we lose track of the phase at the origin and can only guess the relative phase between the two regions left and right of the origin. Therefore direct reconstruction of this state fails.
VI. CONCLUSION

In this paper we have shown how to use weak measurements to reconstruct a continuous-variable state in quantum optics. The modulus of a suitably chosen representation can be reconstructed from projective measurements on the system itself, while the corresponding phase has to be extracted from a complete set of post-selected expectation values measured on a weakly coupled meter device. An appealing facet of the presented reconstruction scheme is the simplicity of the reconstruction relations which turn measured data into quantum state information. Moreover, this measurement concept can be realized with basic elements of linear optics and homodyne detection. However, the sorting required for the postselection of data poses additional problems. The reconstruction of the phase is not straightforward for states with gaps in the probability distribution which determines the sorting. A generalization of the presented scheme to mixed states is desirable, but based on the presented reconstruction relations not straightforward.

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In more detail this means that a reflection coefficient \( \sin \theta \) quantifies how strongly the meter mode couples to the system mode and vice versa. Hence for \( \theta \ll 1 \) we obtain a beam splitter with very low reflection, which corresponds to the weak coupling regime. On the other hand the value \( \theta = \pi/4 \) marks a 50:50 beam splitter with balanced reflection and transmission.

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[61] The bin size we choose throughout this article is an empirical value that seems to be appropriate for the performed simulations. As the transformation of simulated data invokes only a square root, Eq. (16), and an integration, Eq. (14), the results are quite stable to a variation of the bin size. Of course for a finite amount of measurement runs the bin size has to unite two opposing goals: (i) a small bin size allows us to perfectly map the shape of the probability distribution whereas (ii) it should be chosen as large as possible to increase the number of counts per bin, thus minimizing the statistical error. Therefore, an optimization is an appealing challenge which is closely connected to the methods of M. Bellini, A. S. Coelho, S. N. Filippov, V. I. Man’ko, and A. Zavatta, Phys. Rev. A 85, 052129 (2012).