Self-Contained Graphs

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Abstract

A self-contained graph is an infinite graph which is isomorphic to one of its proper induced subgraphs. In this paper, these graphs are studied by presenting some examples and defining some of their sub-structures such as removable subgraphs and the foundation. Meanwhile, we try to find out under what conditions on two arbitrary removable subgraphs, their union is also a removable subgraph. We finally introduce coverings and some other areas for future investigations.

Keywords: self-contained graph, removable subgraph, the foundation, torsion subgraph, graph alternative conjecture.

1. Introduction

Although it is impossible for finite graphs, there are infinite graphs which are isomorphic to one of their proper subgraphs. Infinite empty graphs (which are graphs with no edges), rays, infinite stars and complete infinite graphs are trivial examples, but there are more interesting ones. In this paper, our intention is to study properties and structures of self-contained graphs which are isomorphic to one of their proper induced subgraphs.

The problem of considering an isomorphism of a graph to one of its proper subgraphs has its origin at least to 1970s, for example, see [6]. But self-contained graphs have only recently fascinated mathematicians by the so-called “Graph alternative conjecture” which has started in [2] where Bonato and Tardif studied twins of infinite graphs under the phrase “mutually embeddable graphs”; two non-isomorphic graphs $G$ and $H$ are called “strong twins” if $G$ is isomorphic to a proper induced subgraph of $H$ and $H$ is also isomorphic to a proper induced subgraph of $G$. They asked a question that if $G$ and

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H are twins, then do G and H belong to an infinite family of twins? Few years later, they extended their study of twins in [3] where they noted that if an infinite graph has a strong twin, then it is isomorphic to one of its proper induced subgraphs, i.e., in our phrase, every graph that has a strong twin is also self-contained. They also conjectured that every infinite tree has either infinitely many tree-twins or none. They called it “the tree alternative conjecture” and proved it for all rayless trees [3].

In 2009, Tyomkyn proved that the tree alternative conjecture is true for all rooted trees [9]. Moreover, using Schmidt’s method (which has been introduced in English by Halin in [7]), Bonato et al. proved that (i) a rayless graph has either infinitely many twins or none, and (ii) a connected rayless graph has either infinitely many connected twins or none [1].

Unfortunately, there are no further developments to the graph alternative conjecture. One obstacle in this regard is that self-contained graphs are not being studied yet. Here in this paper, we start studying this kind of graphs by presenting some examples and recognising some of their internal structures such as removable subgraphs, the foundation, torsion of a removable subgraph, etc. We then try to find out under what conditions on two removable subgraph of a self-contained graph, their union is also a removable subgraph. Solving this could be a big step toward solving the general case of the graph alternative conjecture, because it may enable us to inductively construct infinitely many twins for a graph that we know it already has a twin.

To read this paper, besides a general knowledge of graph theory and infinite sets, only few definitions of infinite graph theory is needed, all of which can be found in Section 8 of [5].

To simplify, we use the notation ∅ for the null graph, the unique graph that has no vertices. Furthermore, we use the notations ⊆ and ⊂ for the subgraph relations, and $G \setminus H$ for the induced subgraph $G[V(G) \setminus V(H)]$ where $H$ is itself an induced subgraph of $G$.

2. Self-contained graphs: notations, structures and examples

Although there may be interesting examples of disconnected self-contained graphs and most of our results are valid for them, in this paper, with an exception of Theorem 2.5, we are interested in connected graphs. By theorem 8.2.1 of [5], every connected infinite graph contains a ray or a vertex of infinite degree (an infinite star), so, every connected
infinite graph contains a proper self-contained subgraph (not necessarily an induced one). Moreover, a self-contained graph can be a digraph or a multigraph, but we limit our attention to simple graphs.

Let $G$ be a self-contained graph. A non-empty proper subgraph $H$ of $G$ is a removable subgraph of $G$ if $G \setminus H \cong G$. Then we write $H \in \text{Rem}(G)$ and by $\text{Iso}_G(H)$ we mean the set of all isomorphisms $f : G \rightarrow G \setminus H$. It can be easily seen (at the sight of Proposition 2.3) that $|\text{Rem}(G)| \geq \aleph_0$. The linkage of $H$, $\text{link}(H)$, is the edges joining $H$ to $G \setminus H$, i.e., $\text{link}(H) = E(G) \setminus (E(H) \cup E(G \setminus H))$.

**Proposition 2.1.** Let $G$ be a self-contained graph and $G^c$ be its complement. Then $G^c$ is a self-contained graph. Moreover, if $H \in \text{Rem}(G)$ then $H^c \in \text{Rem}(G^c)$.

**Proof.** Let $H$ be proper subgraph of $G$ such that $V(H) \neq \emptyset$ and $G \cong G \setminus H$. Then $G^c \cong (G \setminus H)^c = G^c \setminus H^c$ which shows that $G^c$ is a self-contained graph and $H^c$ is a removable subgraph of it. \qed

**Example 2.2.** Let $R$ be the random graph (which is also known as the Rado graph). Then $R$ is a self-contained graph that is also self-complimentary, i.e., $R \cong R^c$. For the random graph and its properties see [4] or Section 8.3 of [5].

The following two propositions are easy to prove. Meanwhile, Proposition 2.4 has been stated for locally finite graphs in [6] (Proposition 11, page 263).

**Proposition 2.3.** Let $G$ be a self-contained graph, $P \in \text{Rem}(G)$ and $Q$ be an induced subgraph of $G \setminus P$. Then $Q \in \text{Rem}(G \setminus P)$ if and only if $P \cup Q \in \text{Rem}(G)$.

**Proposition 2.4.** Let $G$ be a self-contained graph and $H \in \text{Rem}(G)$. Then $G$ contains infinitely many vertex disjoint copies of $H$.

The following theorem characterizes removable subgraphs of disconnected self-contained graphs. It worth mentioning that it is true for all graphs with self-embeddings, not only self-contained graphs.

**Theorem 2.5.** Let $G$ be a disconnected self-contained graph. Then atleast one of the following statements is true:

i. $G$ contains a connected component which is also a self-contained graph.

ii. There exists a connected component of $G$ which is a removable subgraph of it.

iii. For each $H \in \text{Rem}(G)$ and $f \in \text{Iso}_G(H)$ there exist infinitely many connected components of $G$, namely $\{G_z\}_{z \in \mathbb{Z}}$, such that for each $z \in \mathbb{Z}$ we have $H$ and $f(H)$ have non-empty intersections with $G_z$. 

Proof. Let the first statement be false about $G$ and $H \in \text{Rem}(G)$. Then there are three possibilities:

Case 1. $H$ contains a connected component $P$ of $G$ as a proper induced subgraph. Then by Proposition 2.4, $G$ contains infinitely many copies of $P$. Suppose $R = \{P_0 = P, P_1, P_2, \ldots\}$ be a subfamily of copies of $P$ in $G$ and define $f : G \rightarrow G \setminus P$ so that it moves $P_i$ to $P_{i+1}$ for each $i = 0, 1, 2, \ldots$ and fixes all other vertices outside $R$. Since $f$ is an isomorphism, $P$ is a removable subgraph of $G$.

Case 2. $H$ does not contain a connected component of $G$, but it is made of parts of finitely many connected components of $G$, namely $G_1, G_2, \ldots, G_k$. If $f \in \text{Iso}_G(H)$, then $f(G_i)$ must be a connected component of $G \setminus H$, for $i = 1, 2, \ldots, k$. So, one of the two possibilities may occur:

Sub-case 2.1. There is $1 \leq i \leq k$ such that $f^n(H \cap G_i) \not\subseteq G_j$ for all $j = 1, 2, \ldots, k$ and $n \in \mathbb{N}$. Then $G_i, f^1(G_i), f^2(G_i), \ldots$ are infinitely many isomorphic copies of the connected components $G_i$, and hence the second statement is true.

Sub-case 2.2. For all $1 \leq i \leq k$ and $n \in \mathbb{N}$ we have $f^n(H \cap G_i) \subset G_j$ for some $j = 1, 2, \ldots, k$. Then $G_i$ must contain one of $f^1(G_i), f^2(G_i), \ldots$, which means that $G_i$ is a self-contained component of $G$, a contradiction to our assumption.

Case 3. $H$ is made of parts of infinitely many connected components of $G$, and so there must be a countable subfamily of them that could be named $\{G_z\}_{z \in \mathbb{Z}}$. If $f \in \text{Iso}_G(H)$, then $f(G_z)$ must be a connected component of $G \setminus H$, for each $z \in \mathbb{Z}$. So, one of the two possibilities may occur:

Sub-case 3.1. There is $z \in \mathbb{Z}$ such that $f^n(H \cap G_z) \not\subseteq G_t$ for all $t \in \mathbb{Z}$ and $n \in \mathbb{N}$. Then $G_z, f^1(G_z), f^2(G_z), \ldots$ are infinitely many isomorphic copies of the connected components $G_z$ which are also connected components of $G$, and hence the second statement is true again.

Sub-case 3.2. For all $z \in \mathbb{Z}$ and $n \in \mathbb{N}$ we have $f^n(H \cap G_z) \subset G_t$ for some $t \in \mathbb{Z}$. The case $t = z$ is absurd since it means that $G_z$ is self-contained, so we suppose $t \neq z$. Then, for each $z \in \mathbb{Z}$ we have $H$ and $f(H)$ have non-empty intersections with $G_z$, which means that the third statement is true.

Since these cases cover all possibilities and in each case the second or the third statement is true, we are done with the proof. 

Self-contained graphs may have substructures that are not removable. Let us see an example first:

**Example 2.6.** Let $G$ be the graph with vertex set $V = \{0, 1, 2, \ldots\}$ and its edges defined as follows: 0 is adjacent to all other vertices while other vertices are only adjacent to 0 and their consecutive vertices. Consequently, vertex degrees of 0, 1 and $i$ for $i \geq 2$ are
∞, 2 and 3, respectively. It is obvious that $G$ is a self contained graph since there is an isomorphism $G \cong G \setminus \{1\}$. The vertex 0 in Example 2.6 is a vertex of no removable subgraph of $G$.

Let $G$ be a self-contained graph. A subgraph $H$ of $G$ is called an asset to $G$ if the intersection of $V(H)$ and vertex set of every removable subgraph of $G$ is empty. The union of all assets of a self-contained graph $G$, is an asset to $G$ which is called the foundation of $G$ and shown by $\text{Fnd}(G)$. It is easy to verify that

$$\text{Fnd}(G) = \bigcap_{H \in \text{Rem}(G)} G \setminus H.$$ 

**Example 2.7.** We have seen in Example 2.6 that $\text{Fnd}(G)$ can be a finite induced subgraph of $G$. Here are two examples of self-contained graphs, one of which has an infinite foundation and the other has a null one.

(a) Let $G$ be the graph whose vertex set consists of three disjoint copies of natural numbers: $A_n = \{a_{n1}, a_{n2}, \ldots\}$, for $n = 1, 2, 3$. The edges of $G$ is of the following three kinds:

i. $\{a_{1j}a_{1(j+1)}\} \in E(G)$ for all $j \in \mathbb{N}$,

ii. $\{a_{11}a_{2j}\} \in E(G)$ for all $j \in \mathbb{N}$,

iii. $\{a_{14}a_{3j}\} \in E(G)$ for all $j \in \mathbb{N}$.

Then $G$ is a self-contained graph and $V(\text{Fnd}(G)) = A_1$. Hence, $\text{Fnd}(G)$ is a ray.

(b) The $\mathbb{N} \times \mathbb{N}$ grid, the graph on $\mathbb{N}^2$, in which two vertices $(m, n)$ and $(m', n')$ are adjacent if and only if $|m - m'| + |n - n'| = 1$, is a self-contained graph whose foundation is the null graph.

As we have seen in Example 2.7 (b), the foundation of a self-contained graph may have no vertices. A more simple example is a ray, whose foundation is also the null graph. Meanwhile, when $G$ is a self-contained graph with non-empty foundation, every isomorphism of $G$ onto one of its proper induced subgraphs, $H$, maps the foundation of $G$ onto the foundation of $H$. Although $\text{Fnd}(H)$ is again a proper induced subgraph of $G$, it can be different from $\text{Fnd}(G)$ (see Example 3.3). By the way, we have the following proposition.

**Proposition 2.8.** Let $G$ be a self-contained graph and $H \in \text{Rem}(G)$. Then we have $\text{Fnd}(G) \subseteq \text{Fnd}(G \setminus H)$. 

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Proof. Suppose \( v \in V(Fnd(G)) \) but \( v \notin V(Fnd(G\setminus H)) \). Hence, there is \( P \in \text{Rem}(G\setminus H) \) such that \( v \in V(P) \). Now, by Proposition 2.3 we have \( H \cup P \in \text{Rem}(G) \) and \( v \in V(H \cup P) \), which is a contradiction since \( v \) is a vertex of the foundation of \( G \). \( \square \)

We end this section with following theorem and its corollaries. Meanwhile, if \( P = x_0 x_2 \ldots x_k \) is a path from \( x_0 \) to \( x_k \), then \( x_i P x_j \) represents the subpath \( x_i x_{i+1} \ldots x_j \) of \( P \).

**Theorem 2.9.** Let \( G \) be a connected self-contained graph. If \( Fnd(G) = \emptyset \) then \( G \) contains a ray.

**Proof.** We inductively construct a ray in \( G \). In each step, we add a finite path to a path we have already chosen and make sure that vertex sets of these two paths have only a singleton as their intersection; the last vertex of the first one coincides with the first vertex of the one we want to choose. Let \( v_i \in V(G) \) and, put \( P_0 \) be the path consisting of the single vertex \( v_i \). Then since \( Fnd(G) = \emptyset \), there is \( H_1 \in \text{Rem}(G) \) with isomorphism \( f_i \in \text{Iso}_G(H_1) \) such that \( P_0 \subseteq H_1 \). Moreover, put \( H_0 = \emptyset \).

Now, suppose we are already decided about \( v_i, H_i, f_i \) and \( P_{i-1} \) for \( i = 1, 2, \ldots \). Since \( G \setminus H_{i-1} \) is still connected, there is a finite path \( P^*_i \) in \( G \setminus H_{i-1} \) with endpoints \( v_i \) and \( f_i(v_i) \). Suppose \( P^*_i = x_1 x_2 \ldots x_k \) for some natural \( k \geq 2 \). So, \( x_1 = v_i \) and \( x_k = f_i(v_i) \). Since \( H_i \) and \( f_i(H_i) \) are vertex disjoint and \( x_1 \in V(H_i) \) and \( x_k \in V(f_i(H_i)) \), it can be inferred that there is \( 1 < j \leq k \) such that \( x_1 P^*_i x_{j-1} \subseteq H_i \) but \( x_j \notin V(H_i) \). So, the edge \( x_{j-1} x_j \) must be an edge of \( \text{link}(H_i) \).

On the other hand, since \( Fnd(G \setminus H_i) = \emptyset \), there is \( H^*_i \in \text{Rem}(G \setminus H_i) \) such that \( x_j \in V(H^*_i) \). By Proposition 2.3, \( H_{i+1} = H_i \cup H^*_i \) is a removable subgraph of \( G \), and hence, there is an isomorphism \( f_{i+1} \in \text{Iso}_G(H_{i+1}) \). Now, we put \( v_{i+1} = x_j \) and \( P_i = P_{i-1} \cup v_i P^*_i v_{i+1} \).

It remains to show that \( P_{i-1} \) and \( v_i P^*_i v_{i+1} \) are edge-disjoint. This actually is simple, because if \( P_{i-1} = v_1 y_2 \ldots y_{t-1} v_i \), then \( v_1 P_{i-1} y_{t-1} \subseteq H_{i-1} \) but \( v_i P^*_i v_{i+1} \subseteq G \setminus H_{i-1} \).

Hence, by the above induction, the induced subgraph of \( P = \bigcup_{i \in \mathbb{N}\cup\{0\}} P_i \) in \( G \) is a ray and we are done with the proof. \( \square \)

**Corollary 2.10.** Let \( G \) be connected rayless self-contained graph. Then \( Fnd(G) \neq \emptyset \).

**Corollary 2.11.** Let \( G \) be self-contained graph such that \( G \setminus Fnd(G) \) is connected. Then, \( G \) contains a ray.

3. **Torsion of a removable subgraph**

In this section, we try to find an answer to the following question: let \( P \) and \( Q \) be removable subgraphs of a self-contained graph \( G \). Under what conditions on \( P \) and \( Q \), we have \( P \cup Q \) is another removable graph of \( G \)? Before answering to this, we show that there are self-contained graphs that for some removable subgraphs of them like \( H \),
the foundation does not remain invariant after removal of $H$. This fact uncover another substructure of self-contained graphs which has an impact on the answer to the our question.

In the following proposition we use the notation $\text{Fin}(H)$ for the set $\{a \in V(H) : \deg_G(a) < \infty\}$ where $H$ is an induced subgraph of $G$.

**Proposition 3.1.** Let $G$ be a self-contained graph, $v \in V(\text{Fnd}(G))$ and

$$|\text{Fin}(\text{Fnd}(G))| < \infty.$$ 

If $\deg_G(v) < \infty$ then $N_G(v) \subseteq V(\text{Fnd}(G))$.

**Proof.** If $N_G(v) \not\subseteq V(\text{Fnd}(G))$, then $v$ is adjacent to one vertex $b$ which is not a vertex of the foundation. Hence, $b$ is a vertex of a removable subgraph $H$ whose elimination from $G$ reduces the degree of $v$. Now let $f : G \rightarrow G \setminus H$ be an isomorphism. By Proposition 2.8, $f$ induces an injection on $\text{Fnd}(G)$. Hence, $v, f(v), f^2(v), ...$ are all distinct vertices of $\text{Fnd}(G) \subseteq f(\text{Fnd}(G)) \subseteq f^2(\text{Fnd}(G)) ...$ respectively. Therefore, for each $n \in \mathbb{N}$, $f^n(\text{Fnd}(G))$ is isomorphic to $\text{Fnd}(G)$ and contains $n$ distinct finite degree vertices. So, by common induction, $\text{Fnd}(G)$ contains infinitely many finite-degree vertices, contradiction to our assumption. \qed

**Corollary 3.2.** Let $G$ be a connected self-contained graph and $\text{Fnd}(G) \neq \emptyset$. Then, if $\text{Fnd}(G)$ is a finite graph, then it contains a vertex $v$ which has an infinite degree in $G$.

**Example 3.3.** The condition $|\text{Fin}(\text{Fnd}(G))| < \infty$ in Proposition 3.1 is not redundant. Let $A = \{a_1, a_2, \ldots\}$, $B = \{b_1, b_2, \ldots\}$ and $C = \{c_0, c_{-1}, c_{-2} \ldots\}$ be three sets equipotent to natural numbers disjoint from $\mathbb{Z}$ and from each other, and suppose that $G$ is a graph with vertex set $V(G) = \mathbb{Z} \cup A \cup B \cup C$ and edge set of the form of union of the following four sets:

i \quad \{\{v_1v_2\} : v_1, v_2 \in \mathbb{Z} \text{ and } |v_1 - v_2| = 1\}

ii \quad \{\{a_jj\} : j \in \mathbb{N}\}

iii \quad \{\{b_jj\} : j \in \mathbb{N}\}

iv \quad \{\{c_jj\} : j \in \mathbb{Z} \text{ and } j \leq 0\}.

Then $G$ is a self-contained graph since $P = \{a_1\}$ and $Q = \{b_1\}$ are two removable subgraph of $G$. Moreover, $\text{Fnd}(G) = \mathbb{Z} \cup C$ and for every $j > 0$, $j$ is a finite-degree vertex of the foundation of $G$ which has a neighbor out of $\text{Fnd}(G)$.  

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Example 3.3 reveals another important substructure of self-contained graphs. Let $G$ be a self-contained graph and $H \in \text{Rem}(G)$. A vertex $v$ of $G$ is called a twisted vertex for $H$ if there exists $P \in \text{Rem}(G)$ such that $v \in V(P)$ and $v \in \text{Fnd}(G \setminus H)$. The subgraph induced by all twisted vertices for $H$ is called the torsion of $H$ and is denoted by $\text{Tor}_G(H)$. Meanwhile, when $\text{Tor}_G(H) = \emptyset$ we say $H$ is a torsion-free removable subgraph of $G$. Moreover, we say $G$ is torsion-free self-contained graph if all removable subgraphs of it are torsion-free.

Torsion of a removable subgraph can be the null graph, as it is in all examples prior to Example 3.3. But in Example 3.3, if we consider $H = \{a_1\}$ as a removable subgraph of $G$, we have $\text{Tor}_G(H) = \{b_1\}$. By the way, it is obvious from previous definition that for a self-contained graph $G$, if $\text{Fnd}(G) = \emptyset$, then for all removable subgraph $H$ of $G$ we have $\text{Tor}_G(H) = \emptyset$, i.e. $G$ is torsion-free.

The following two propositions express some properties of torsions of removable subgraphs:

**Proposition 3.4.** Let $G$ be a self-contained graph, $P, Q \in \text{Rem}(G)$ and $P \subseteq Q$. Then $\text{Tor}_G(P) \subseteq \text{Tor}_G(Q)$.

**Proof.** If $P = Q$, there is nothing to prove. Hence, suppose $Q \setminus P \neq \emptyset$. Then $Q \setminus P \in \text{Rem}(G \setminus P)$. Now, if $a \in V(\text{Tor}_G(P))$, then $a \in V(\text{Fnd}(G \setminus P)) \subseteq V(\text{Fnd}((G \setminus P) \setminus (Q \setminus P)))$ which means that $a \in \text{Fnd}(G \setminus Q)$. We have $a \notin Q$ since $a \notin P$ and $a \notin Q \setminus P$ because $Q \setminus P \in \text{Rem}(G \setminus P)$ cannot contain a vertex of the foundation of $G \setminus P$. Therefore, $a$ is a vertex of a removable subgraph of $G$ (since $a \in V(\text{Tor}_G(P))$ which is also a vertex of the foundation of $G \setminus Q$, and consequently, $a \in V(\text{Tor}_G(Q))$.

**Proposition 3.5.** Let $G$ be a self-contained graph, $P, Q \in \text{Rem}(G)$ and $Q \cap \text{Tor}_G(P) \neq \emptyset$. Then $P \cap \text{Tor}_G(Q) \neq \emptyset$.

**Proof.** First, we prove that there does not exist an $H \in \text{Rem}(G \setminus Q)$ such that $P \subseteq H$. If on the contrary $P \subseteq H \in \text{Rem}(G \setminus Q)$ then $Q \cup H \in \text{Rem}(G)$ and by Proposition 3.4 we have $\text{Tor}_G(P) \subseteq \text{Tor}_G(Q \cup H)$; which is a contradiction since $Q \cap \text{Tor}_G(P) \neq \emptyset$ but no vertex of $\text{Tor}_G(Q \cup H)$ is a vertex of $Q$.

Therefore, since no removable subgraph of $G \setminus Q$ contains all vertices of $P$, there exist an $a \in V(P)$ which a vertex of $\text{Fnd}(G \setminus Q)$. So, $P \cap \text{Tor}_G(Q) \neq \emptyset$.

**Theorem 3.6.** Let $G$ be a self-contained graph, $H \in \text{Rem}(G)$ and $\text{Tor}_G(H) \neq \emptyset$. Then $\text{Fnd}(G)$ is a self-contained graph which has a removable subgraph $P$ isomorphic to $\text{Tor}_G(H)$.

**Proof.** Let $f : G \rightarrow G \setminus H$ be an isomorphism. Hence, we have $\text{Fnd}(G) \cong f(\text{Fnd}(G)) = G[V(\text{Fnd}(G)) \cup V(\text{Tor}_G(H))]$, which means that $f(\text{Fnd}(G))$ is a self-contained graph and
\[ \text{Tor}_G(H) \in \text{Rem}\left( f\left( \text{Fnd}(G) \right) \right). \] Therefore, Fnd(G) is isomorphic to a self-contained graph that has a removable subgraph \( P = f^{-1}(\text{Tor}_G(H)) \) which is isomorphic to Tor\(_G\)(H). \qed

Even when \( G \) is a torsion-free self-contained graph and \( P, Q \in \text{Rem}(G) \), we cannot say for sure that \( P \cup Q \in \text{Rem}(G) \), because they may have non-empty intersection.

**Example 3.7.** Let \( G \) be the infinite graph with the vertex set \( \mathbb{N} \cup \{0\} \) and its edges are of the following:

- 1 is only adjacent to 0, but 0 is adjacent to all natural \( n \equiv \pm 1 \pmod{3} \),

- If \( n \) is a natural number that \( n \equiv 0 \pmod{3} \), then \( n \) is adjacent to \( n - 1 \) and \( n + 1 \).

Then \( G \) is a torsion-free self-contained graph which has \( P = G[\{2, 3, 4\}] \), \( Q = G[\{1, 2, 3\}] \) and \( R = G[\{1, 5, 6\}] \) as some of its removable subgraphs. While \( P \cup R \) is a removable subgraph of \( G \), the other two union subgraphs, \( P \cup Q \) and \( Q \cup R \), are not.

**Proposition 3.8.** Let \( G \) be a self-contained graph and \( P, Q \in \text{Rem}(G) \) such that \( P \cap Q = \emptyset \). Then \( P \cup Q \in \text{Rem}(G) \) if and only if there exists an isomorphism \( f : G \to G \setminus P \) such that \( f^{-1}(Q) \in \text{Rem}(G) \). In particular, if there is an isomorphism \( f : G \to G \setminus P \) such that \( f(Q) = Q \), then \( P \cup Q \in \text{Rem}(G) \).

**Proof.** If \( f^{-1}(Q) \in \text{Rem}(G) \), then \( Q \in \text{Rem}(G \setminus P) \) and thus by Proposition 2.3 we have \( P \cup Q \in \text{Rem}(G) \). Meanwhile, when \( P \cup Q \in \text{Rem}(G) \), we must have \( Q \in \text{Rem}(G \setminus P) \) and hence there is an isomorphism \( f : G \to G \setminus P \) such that \( f^{-1}(Q) \in \text{Rem}(G) \). \qed

Now, one may think that for a self-contained graph \( G \) and \( P, Q \in \text{Rem}(G) \), if \( Q \cap (P \cup \text{Tor}_G(P)) = \emptyset \) then \( P \cup Q \in \text{Rem}(G) \). As we show in the following example, sometimes it is not true.

**Example 3.9.** Let \( R \) be a graph consisting of infinitely many disjoint copies of the graph \( G \) we have introduced in Example 3.3, i.e. \( G_1, G_2, \ldots \) and a vertex \( a \) which is adjacent to all other vertices of double rays of copies of \( G \). Then \( R \) is a self-contained graph since every \( G_i \) is removable subgraph of \( R \) for \( i = 1, 2, \ldots \). Moreover, if \( P_i \) and \( Q_i \) are copies of removable subgraphs of \( G_i \), we have introduced in Example 3.3, then they are also removable subgraphs of \( R \). But we have \( \text{Tor}_{G_i}(P_i) = Q_i \) while \( \text{Tor}_R(P_i) = \emptyset \). So, although \( Q_i \cap (P_i \cup \text{Tor}_R(P_i)) = \emptyset \), we have \( P_i \cup Q_i \notin \text{Rem}(R) \).

Let \( G \) be a self-contained graph and \( P \in \text{Rem}(G) \). A vertex \( v \) of \( G \) is a **curly vertex to \( P \)** if there exists a self-contained removable subgraph \( Q \in \text{Rem}(G) \) such that \( P \in \text{Rem}(Q) \) and \( v \in V(\text{Tor}_Q(P)) \). The set of all curly vertices to \( P \) is shown by \( \text{Curl}_G(P) \).

We end this section with the following conjecture:

**Conjecture 3.10.** Let \( G \) be a self-contained graph and \( P, Q \in \text{Rem}(G) \). If \( Q \cap (P \cup \text{Tor}_G(P) \cup \text{Curl}_G(P)) = \emptyset \) then \( Q \in \text{Rem}(G \setminus P) \).
4. Areas for future investigations

In this section, we consider some questions that are arose naturally about self-contained graphs. Roughly speaking, a connected self-contained graph $G$, is a graph constructed from its foundation and iterated copies of its removable subgraphs and their torsions. The main question is how precise this property can be expressed.

Let $G$ be a connected self-contained graph, $H \in \text{Rem}(G)$. We show the induced subgraph $G[\bigcup_{f \in \text{Iso}(H)} f^i(H \cup \text{Tor}_G(H))_{i=0}^\infty]$ by the notation $\varphi_G(H)$. When $H, P \in \text{Rem}(G)$, we say $H$ and $P$ are congruent and write $H \cong P$ if $\varphi_G(H) = \varphi_G(P)$.

Since being congruent is an equivalence relation among removable subgraphs of $G$, it partitions Rem($G$). Now, we can easily prove the following theorem:

**Theorem 4.1.** Let $G$ be a connected self-contained graph. Then there exist a family of its removable subgraph

$$\mathcal{R} = \{H_i : H_i \in \text{Rem}(G), i \in I\}$$

such that

$$G = G[\text{Fnd}(G) \cup \left( \bigcup_{H \in \mathcal{R}} \varphi_G(H) \right)]$$

**Proof.** Let $\overline{H}$ denotes the equivalence class of removable subgraph $H$ of $G$. If we put $\mathcal{R} = \{H_i : H_i \in \text{Rem}(G), i \in I\}$, and $\overline{H_i} \neq \overline{H_j}$ whenever $i \neq j$} be a set of all congruent classes, then Theorem 4.1 must hold since if $v \in V(G)$ such that $v \notin V\left(G[\text{Fnd}(G) \cup \left( \bigcup_{H \in \mathcal{R}} \varphi_G(H) \right)]\right)$, then either $v \in V(\text{Fnd}(G))$ which is impossible, or $v \notin V(\text{Fnd}(G))$ which means that there is $H_j \in \text{Rem}(G)$ such that $v \in V(H_j)$. Suppose $j^* \in I$ such that $\overline{H_{j^*}} = \overline{H_j}$. Then $\varphi_G(H_{j^*}) = \varphi_G(H_j)$ and thus $v \in \varphi_G(H_{j^*})$, which is a contradiction. 

If Theorem 4.1 is true for $G$ and $\mathcal{R}$, then we call $\mathcal{R}$ to be a covering (for removable part) of $G$.

Coverings for self-contained graphs can be very complicated in general, but sometimes they have some straightforward ones. Let $G$ be a self-contained graph and it has $\mathcal{R} = \{M\}$ as its covering. Then Theorem 4.1 has a simple consequence for $G$:

$$G = G[\text{Fnd}(G) \cup \varphi_G(M)]$$

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A removable subgraph, sharing this property with $M$, deserves a good name: a \textit{monomer} for $G$.

All the examples of self-contained graphs we have considered are those which have monomers. So, the following question arises naturally.

\textbf{Question 4.2.} Does all self-contained graphs contain monomers? If not, under what conditions can we say a self-contained graph must contain a monomer?

Another good question arises via the $\mathbb{N} \times \mathbb{N}$ grid of Example 2.7, which is the Cartesian product of two rays. For the definition and properties of the Cartesian product of two graphs see [8]. The following proposition tells us why it is not odd that the $\mathbb{N} \times \mathbb{N}$ grid is self-contained:

\textbf{Proposition 4.3.} Let $G$ be a self-contained graph and $H$ be an arbitrary graph. Then the Cartesian product $G \square H$ is a self-contained graph.

\textit{Proof.} Let $P \in \text{Rem}(G)$. Then $P \square H \in \text{Rem}(G \square H)$. The details are left to the reader. \qed

Now the question is what is the relation of removable subgraphs of the product graph and those of factors or how the foundations of factors transform under the product to the foundation of the product graph? Furthermore, if we know that $G$ has infinitely many (strong) twins or none, what can be said about its product to an arbitrary graph $H$?

\textbf{References}

\textbf{References}

[1] Bonato, A., Bruhn, H., Diestel, R., Spr"ussel, P., \textit{Twins of rayless graphs}, Journal of Combinatorial Theory, Series B, Volume 101, 2011, pp 60–65.

[2] Bonato, A., Tardif, C., \textit{Large Families of Mutually Embeddable Vertex-Transitive Graphs}, J. Graph Theory, Volume 43, 2003, pp 99–106.

[3] Bonato, A., Tardif, C., \textit{Mutually embeddable graphs and the tree alternative conjecture}, Journal of Combinatorial Theory, Series B, Volume 96, 2006, pp 874–880.

[4] Cameron, P. J., \textit{The random graph}, Algorithms and Combinatorics, Volume 14, 1997, pp 333–351.
[5] Diestel, R., *Graph Theory*, 4th edition, Springer, 2010.

[6] Halin, R., *Automorphisms and endomorphisms of infinite locally finite graphs*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg, Volume 39, Issue 1, 1973, pp 251–283.

[7] Halin, R., *The structure of rayless graphs*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg, Volume 68, 1998, pp 225–253.

[8] Hammack, R., *Handbook of product graphs*, 2nd edition, CRC Press, 2011.

[9] Tyomkyn, M., *A proof of the rooted tree alternative tree conjecture*, Discrete Math., Volume 309, 2009, pp 5963–5967.