Sequent Calculi and Interpolation for Non-Normal Logics

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Abstract

G3-style Sequent calculi for the logics in the cube of non-normal modal logics and for their deontic extensions are introduced. For each of the calculi considered, we prove that weakening and contraction are height-preserving admissible, and we give a syntactic proof of the admissibility of cut. This implies that the subformula property holds for them and that they are decidable. These calculi are shown to be equivalent to the axiomatic ones and, therefore, they are sound and complete with respect to neighbourhood semantics. Finally, we give a Maehara-style proof of Craig’s interpolation theorem for most of the logics considered.

Keywords: Non-normal logics, sequent calculi, structural proof theory, interpolation, decidability.

1 Introduction

For many interpretations of the modal operators – e.g., for deontic, epistemic, game-theoretic, and high-probability interpretations – it is necessary to adopt logics that are weaker than the normal modal ones; e.g., deontic paradoxes, see [14], are one of the main motivations for adopting a non-normal deontic logic. Non-normal logics, see [2] for naming conventions, are quite well understood from a semantic point of view where they are studied mostly by means of neighbourhood semantics [7, 20]. Nevertheless, until recent years their proof theory has been rather limited since it was mostly confined to Hilbert-style axiomatic systems. This situation seems to be rather critical since it is difficult to build countermodels in neighbourhood semantics and it is difficult to find derivations in axiomatic systems. When the aim is to find derivations and to analyse their structural properties, sequent calculi are to be preferred to axiomatic systems. Recently different kinds of sequent calculi for non-normal logics have been proposed: Gentzen-style calculi [8, 9, 10, 18]; labelled calculi based on translations into normal modal logics [5, 19]; labelled calculi based on the internalization of neighbourhood semantics [15]; and, finally, linear nested sequents [11].

This paper, which extends the results presented in [18], concentrates on Gentzen-style calculi since they are better suited than their extensions to give (computationally optimal) decision procedures and constructive proofs of interpolation theorems. The existing Gentzen-style calculi for non-normal logics, see Section 5, either cover only the so-called monotone non-normal logics [8], or do not allow to eliminate all the structural rules of inference [9, 10] and, therefore, it is not possible to use them to determine whether a given formula is derivable or not by means of a root-first proof search procedure. Furthermore, although there are rules of inference that capture both deontic axioms $D^\circ := \Box A \supset \Diamond A$ and $D^\perp := \neg \square \perp [9, 23]$, to our knowledge there is no rule that captures satisfactorily only one of them (when they are not interderivable). Finally, despite the existence of semantic proofs of Craig’s interpolation theorem for non-normal logics, see e.g. [7], for most of them there is no constructive proof of this result. This paper studies cut- and contraction-free G3-style sequent calculi for all the logics in the cube of non-normal modalities and for their extension with the deontic axioms $D^\circ$ and $D^\perp$. The calculi we present have the subformula property, allow for a straightforward decision procedure by a terminating root-first proof search, and in most cases they allow us to give a constructive proof of Craig’s interpolation theorem.

We proceed as follows: Section 2 summarizes the basic notions of axiomatic systems and of neighbourhood semantics for non-normal logics. Section 3 presents G3-style sequent calculi for
These logics for then showing that weakening and contraction are height-preserving admissible and that cut is (syntactically) admissible. As consequences of the admissibility of the structural rules, we have that they allow for a terminating proof-search and that they are equivalent to the axiomatic systems. Section 4 gives a Maehara-style constructive proof of Craig’s interpolation theorem for the logics not containing rule LR-C nor L-D°C (see Table 4). Finally, Section 5 considers related works.

2 Non-normal Logics

2.1 Axiomatic Systems

We introduce, following [2], the basic notions of non-normal modal logics that will be used later on. Given a countable set of propositional variables \( \{p_n : n \in \mathbb{N}\} \), the formulas of the modal language \( L \) are generated by:

\[
A ::= p_n \mid \bot \mid A \land A \mid A \lor A \mid A \supset A \mid \Box A .
\] (1)

We remark that \( \bot \) is a 0-ary logical symbol, this will be extremely important in the proof of Craig’s interpolation theorem. As usual \( \neg A \) is a shorthand for \( A \supset \bot \), \( \top \) for \( \bot \supset \bot \), \( A \leftrightarrow B \) for \( (A \supset B) \land (B \supset A) \), and \( \Diamond A \) for \( \neg \Box \neg A \). We follow the usual conventions for parentheses.

Let \( L \) be the logic containing all \( L \)-instances of propositional tautologies as axioms, and the modus ponens (MP) as inference rule. The minimal non-normal modal logic \( E \) is the logic \( L \) plus the rule \( RE \) of Table 2. We will consider all the logics that are obtained by extending \( E \) with some set of axioms from Table 1. We will denote the logics accordingly to the axioms that define them, e.g. \( EC \) is the logic \( E \oplus C \), and \( EMD^\perp \) is \( E \oplus M \oplus D^\perp \). By \( X \) we denote any of these logics and we write \( X \vdash A \) whenever \( A \) is a theorem of \( X \). We will call modal the logics containing neither \( D^\perp \) nor \( D^\Diamond \), and deontic the logics containing at least one of them. Observe that we have followed the usual naming conventions for the modal axioms, but we have introduced new conventions for the deontic ones: the deontic axiom \( D^\perp \) is usually called \( CON \).
and $D^\Box$ is usually called $D$, cf. [1]. Finally, we will use $XD$ for a deontic logic extending $X$ that has both $D^\perp$ and $D^\Box$ as theorems.

It is also possible to give an equivalent rule-bases axiomatization of some of these logics, see [2]. In particular, the logic $EM$, also called $M$, can be axiomatized as $L$ plus the rule $RM$ of Table 2. The logic $EMC$, also called $R$, can be axiomatized as $L$ plus the rule $RR$ of Table 2. Finally, the logic $EMCN$, i.e. the smallest normal modal logic $K$, can be axiomatized as $L$ plus the rule $RK$ of Table 2. These rule-based axiomatizations will be useful later on since they ease the proof of the equivalence between axiomatic systems and sequent calculi (Theorem 3.9).

The following proposition states the well-known relations between the theorems of non-normal modal logics, for a proof the reader is referred to [2].

**Proposition 2.1.** For any formula $A \in \mathcal{L}$ we have that $E \vdash A$ implies $M \vdash A$; $M \vdash A$ implies $R \vdash A$; $R \vdash A$ implies $K \vdash A$. Analogously for the logics containing axiom $N$ and/or axiom $C$.

Axiom $D^\perp$ is $K$-equivalent with the schema $D^\Box := \neg(\Box A \land \Box \neg A)$, but the correctness of $D^\Box$ has been a big issue in the literature on deontic logic. This fact urges us to study logics weaker than $KD$ with respect to which $D^\perp$ and $D^\Box$ are no more equivalent [2]. The deontic formulas $D^\perp$ and $D^\Box$ have the following relations in the logics we are considering.

**Proposition 2.2.** $D^\perp$ and $D^\Box$ are independent in $E$; $D^\perp$ is derivable from $D^\Box$ in non-normal logics containing at least one of the axioms $M$ and $N$; $D^\Box$ is derivable from $D^\perp$ in non-normal logics containing axiom $C$.

In Figure 1 the reader finds the lattice of non-normal modal logics, see [2, p. 237], and in Figure 2 the lattice of non-normal deontic logics.

### 2.2 Semantics

The most widely known semantics for non-normal logics is neighbourhood semantics. We sketch its main tenets following [2], where neighbourhood models are called minimal models.
Definition 2.3. A *neighbourhood model* is a triple $\mathcal{M} := \langle W, N, P \rangle$, where $W$ is a non-empty set of possible worlds; $N : W \rightarrow 2^W$ is a neighbourhood function that associates to each possible world $w$ a set $N(w)$ of subsets of $W$; and $P$ gives a truth value to each propositional variable at each world.

The definition of truth of a formula $A$ at a world $w$ of a neighbourhood model $\mathcal{M}$ – $\models^\mathcal{M} A$ – is the standard one for the classical connectives with the addition of

$$\models^\mathcal{M} \Box A \iff \exists w : \models^\mathcal{M} A,$$  \hspace{1cm} (2)

where $\models^\mathcal{M} A$ is the truth set of $A$ – i.e., $\models^\mathcal{M} A = \{ w : \models^\mathcal{M} A \}$. We say that a formula $A$ is *valid* in a class $\mathcal{C}$ of neighbourhood models iff it is globally true in every world of every $\mathcal{M} \in \mathcal{C}$.

In order to give soundness and completeness results for non-normal modal and deontic logics with respect to (classes of) neighbourhood models, we introduce the following definition.

Definition 2.4. Let $\mathcal{M} := \langle W, N, P \rangle$ be a neighbourhood model, $X, Y \in 2^W$, and $w \in W$; we say that:

- $\mathcal{M}$ is *supplemented* whenever if $X \cap Y \in N(w)$ then $X \in N(w)$ and $Y \in N(w)$;
- $\mathcal{M}$ is *closed under finite intersection* whenever , if $X \in N(w)$ and $Y \in N(w)$ then $X \cap Y \in N(w)$;
- $\mathcal{M}$ contains the *unit* whenever $W \in N(w)$;
- $\mathcal{M}$ is *non-blind* whenever if $X \in N(w)$ then $X \neq \emptyset$;
- $\mathcal{M}$ is *complement-free* whenever if $X \in N(w)$, then $W - X \notin N(w)$.

Proposition 2.5. We have the following correspondence results between $\mathcal{L}$-formulas and the properties of the neighbourhood function defined above:
• Axiom $M$ corresponds to supplementation;
• Axiom $C$ corresponds to closure under finite intersection;
• Axiom $N$ corresponds to containment of the unit;
• Axiom $D_{\bot}$ corresponds to non-blindness;
• Axiom $D_{\circ}$ corresponds to complement-freeness.

**Theorem 2.6.** $E$ is sound and complete with respect to the class of all neighbourhood models.
Any logic $X$ which is obtained by extending $E$ with some axioms from Table 1 is sound and complete with respect to the class of all neighbourhood models which satisfies all the properties corresponding to the axioms of $X$.

## 3 Sequent Calculi

We introduce sequent calculi for non-normal logics that extend the multiset-based sequent calculus $G3cp$ [16, 17, 22] for classical propositional logic – see Table 3 – by adding some modal rule from Table 4. In particular, we consider the modal sequent calculi given in Table 5, which will be shown to capture the modal logics of Figure 1, and their deontic extensions given in Table 6, which will be shown to capture all deontic logics of Figure 2. The rules $L-D_{\bot}$, $L-D_{\circ}$, $L-D_{\circ C}$ and $L-D^{*}$ are called **modal** even though the label **deontic** would have been more appropriate.\(^1\) We adopt the following notational conventions: we use $G3X$ to denote a generic calculus from either Table 5 or Table 6, and we use $G3Y(Z)$ to denote both $G3Y$ and $GRYZ$.

For an introduction to $G3cp$ and the relevant notions, the reader is referred to [16, Chapter 3]. We sketch here the main notions that will be used in this paper. A **derivation** of a sequent $\Gamma \Rightarrow \Delta$ – where $\Gamma$ and $\Delta$ are finite, possibly empty, multisets of formulas, and where if $\Pi$ is the (possibly empty) multiset $A_1, \ldots, A_m$ then $\square \Pi$ is the (possibly empty) multiset $\square A_1, \ldots, \square A_m$ – in $G3X$ is a tree of sequents having $\Gamma \Rightarrow \Delta$ as root, initial sequents or instances of rule $L_{\bot}$ as leaves, and all edges obtained by applications of rules of $G3X$. In the rules in Tables 3 and 4, the multisets $\Gamma$ and $\Delta$ are called **contexts**, the other formulas occurring in the conclusion (premiss(es), resp.) are called **principal** (active). In a sequent the **antecedent** (**succedent**) is the multiset occurring to the left (right) of the sequent arrow $\Rightarrow$. As for $G3cp$, a sequent $\Gamma \Rightarrow \Delta$ has the following **denotational interpretation**: the conjunction of the formulas in $\Gamma$ implies the disjunction of the formulas in $\Delta$. As measures for inductive proofs we use the weight of a formula and the height of a derivation. The **weight** of a formula $A$, $w(A)$, is defined inductively as follows: $w(\bot) = w(p_i) = 0$; $w(\square A) = w(A) + 1$; $w(A \circ B) = w(A) + w(B) + 1$ (where $\circ$ is one of the binary connectives $\land, \lor, \supset$). The **height** of a derivation is the length of its longest branch minus one. A rule of inference is said to be (**height-preserving**) admissible in $G3X$ if, whenever its premises are derivable in $G3X$, then also its conclusion is derivable (with at most the same derivation height) in $G3X$.

### 3.1 Structural rules of inference

We are now going to prove that the calculi $G3X$ have the same good structural properties of $G3cp$: weakening and contraction are height-preserving admissible in $G3X$ and cut is admissible in $G3X$. All proofs are extension of those for $G3cp$, see [16, Chapter 3]; in most cases, the modal rules have to be treated differently from the propositional ones because of the presence of

\(^{1}\) Observe that it is possible to consider rule $L-D_{\bot}$ ($L-D_{\circ}$) as a particular case of rule $L-D^{*}$ ($L-D_{\circ C}$). We have chosen to take it as an independent rule for the sake of simplicity.
Table 3: The sequent calculus G3cp

Initial sequents:
\[ p_n, \Gamma \Rightarrow \Delta, p_n \]

Propositional rules:
\[ A, B, \Gamma \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B \quad \Gamma \Rightarrow \Delta, A \land B \quad \Gamma \Rightarrow \Delta, A \lor B \quad \Gamma \Rightarrow \Delta, A \rightarrow B \]

Table 4: Modal rules

\[ \square A, \Gamma \Rightarrow \square \Delta, \square B \]
\[ \Diamond A, \Gamma \Rightarrow \Diamond \Delta, \Diamond B \]
\[ A \Rightarrow B \quad B \Rightarrow A \]

Table 5: Modal sequent calculi (✓ = primitive, ⋆ = admissible, − = neither)

| G3E | G3EN | G3M | G3MN | G3C | G3CN | G3R | G3K |
|-----|------|-----|------|-----|------|-----|-----|
| LR-E | ✓ | ✓ | ⋆ | ⋆ | ⋆ | ⋆ | ⋆ |
| LR-M | ✓ | ✓ | ⋆ | ⋆ | ⋆ | ⋆ | ⋆ |
| LR-C | ✓ | ✓ | ⋆ | ⋆ | ⋆ | ⋆ | ⋆ |
| LR-R | ✓ | ✓ | ⋆ | ⋆ | ⋆ | ⋆ | ⋆ |
| LR-K | ✓ | ✓ | ⋆ | ⋆ | ⋆ | ⋆ | ⋆ |
| R-N | ✓ | ✓ | ⋆ | ⋆ | ⋆ | ⋆ | ⋆ |

Table 6: Deontic sequent calculi (✓ = primitive, ⋆ = admissible, − = neither)

| G3E(N)D | G3ED | G3E(N)D | G3M(N)D | G3M(N)D | G3CD | G3C(N)D | GARD | GAKD |
|---------|-----|--------|--------|--------|------|--------|------|------|
| L-D-1  | ✓ | − | − | ✓ | ✓ | ✓ | − | ⋆ | ⋆ | ⋆ |
| L-D₀  | − | ✓ | ✓ | − | − | − | − | ⋆ | ⋆ | ⋆ | ⋆ |
| L-Dⁿ  | − | − | − | − | − | − | ✓ | ⋆ | ⋆ | ⋆ | ⋆ |
| L-D⁺  | − | − | − | − | − | − | ✓ | ✓ | ✓ | ✓ | ✓ |
empty contexts in the premiss(es) of the modal ones. We adopt the following notational convention: given a derivation tree \( D_k \), the derivation tree of the \( n \)-th leftmost premiss of its last step is denoted by \( D_{kn} \). We begin by showing that the restriction to atomic initial sequents, which is needed to have the propositional rules invertible, is not limitative in that initial sequents with arbitrary principal formula are derivable in \( \text{G3X} \).

**Proposition 3.1.** Every instance of \( A, \Gamma = \Rightarrow \Delta, A \) is derivable in \( \text{G3X} \).

**Proof.** By induction on the weight of \( A \). If \( w(A) = 0 \) – i.e., \( A \) is atomic or \( \bot \) – then we have an instance of an initial sequent or of a conclusion of \( L \bot \) and there is nothing to prove. If \( w(A) \geq 1 \), we argue by cases according to the construction of \( A \). In each case we apply, root-first, the appropriate rule(s) in order to obtain sequents where some proper subformula of \( A \) occurs both in the antecedent and in the succedent. The claim then holds by the inductive hypothesis (IH). To wit, if \( A \equiv \Box B \) and we are in \( \text{G3M(ND)} \), we have:

\[
\begin{align*}
\Box B \Rightarrow B & \quad \text{IH} \\
\Box B, \Gamma \Rightarrow \Delta, \Box B & \quad \text{LR-M}
\end{align*}
\]

**Theorem 3.2.** The left and right rules of weakening are height-preserving admissible in \( \text{G3X} \)

\[
\begin{align*}
\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} & \quad \text{LW} \\
\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A} & \quad \text{RW}
\end{align*}
\]

**Proof.** The proof is a straightforward induction on the height of the derivation \( D \) of \( \Gamma \Rightarrow \Delta \). If the last step of \( D \) is by a propositional rule, we have to apply the same rule to the weakened premiss(es), which are derivable by IH, see [16, Thm. 2.3.4]. If it is by a modal one, we proceed by adding \( A \) to the appropriate context of the conclusion of that instance of a modal rule. To illustrate, if the last rule is \( \text{LR-E} \), we prove that \( \text{LW} \) is height-preserving admissible by transforming

\[
\begin{align*}
\vdots D_1 & \quad \vdots D_2 \\
B \Rightarrow C & \quad C \Rightarrow B \\
\Box B, \Gamma \Rightarrow \Delta, \Box C & \quad \text{LR-E}
\end{align*}
\]

into

\[
\begin{align*}
\vdots D_1 & \quad \vdots D_2 \\
B \Rightarrow C & \quad C \Rightarrow B \\
\Box B, A, \Gamma \Rightarrow \Delta, \Box C & \quad \text{LR-E}
\end{align*}
\]

Before considering contraction, we recall some facts that will be useful later on.

**Lemma 3.3.** In \( \text{G3X} \) it holds that:

1. The rules \( \frac{\Gamma \Rightarrow \Delta, \bot}{\Gamma \Rightarrow \Delta} \quad \text{R}_{\bot} \) and \( \frac{\top, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \quad \text{L}_{\top} \) are height-preserving admissible.

2. The rules \( \frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta} \quad \text{L}\neg \) and \( \frac{\neg A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A} \quad \text{R}\neg \) are admissible.

**Proof.** 1. An induction on the height of the derivation of the premiss (no rule has \( \bot \) principal in the succedent or \( \top \) principal in the antecedent).
2. By a root-first proof search, using the admissibility of the right rule of weakening for $R^\neg$.

**Lemma 3.4.** All propositional rules are height-preserving invertible in $G3X$, that is the derivability of (a possible instance of) a conclusion of a propositional rule entails the derivability, with at most same derivation height, of its premiss(es).

**Proof.** Same as for $G3cp$, see [16, Thm. 3.1.1].

**Theorem 3.5.** The left and right rules of contraction are height-preserving admissible in $G3X$

\[
\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}
\]

**Proof.** The proof is by simultaneous induction on the height of the derivation $D$ of the premiss for left and right contraction. The base case is straightforward. For the inductive steps, we have different strategies according to whether the last step in $D$ is by a propositional or by a modal rule. If the last step in $D$ is by a propositional rule, we have two subcases: if the contraction formula is not principal in that step, we apply the inductive hypothesis and then the rule. Else we start by using the height-preserving invertibility – Lemma 3.4 – of that rule, and then we apply the inductive hypothesis and the rule see [16, Thm. 3.2.2] for details.

If the last step in $D$ is by a modal rule, we have two subcases: either (the last step is by one of $LR-C$, $LR-R$, $LR-K$, $L-D^\circ$, $L-D^{\circ \circ}$ and $L-D^*$ and) both occurrences of the contraction formula $A$ of $LC$ are principal in the last step, else one or no instance of the contraction formula $A$ is principal in the last step and the other(s) is(are) introduced in the appropriate context of its conclusion. In the fist subcase, we apply the inductive hypothesis to the premiss and then the rule. An interesting example is when the last step in $D$ is by $L-D^\circ$. We transform

\[
\vdash D_1 \quad \vdash D_2 \\
B, B \Rightarrow \quad \Rightarrow B, B \\
\square B, \Box B, \Gamma \Rightarrow \Delta \\
\square B, \Gamma \Rightarrow \Delta \\
\frac{IH(D_1)} {IH(D_2)}
\]

where $IH(D_1)$ is obtained by applying the inductive hypothesis for the left rule of contraction to $D_1$ and $IH(D_2)$ is obtained by applying the inductive hypothesis for the right rule of contraction to $D_2$.

In the second subcase, we apply an instance of the same modal rule which introduces one less occurrence of $A$ in the appropriate context of the conclusion. Let’s consider $RC$. If the last step is by $LR-M$ and no instance of $A$ is principal in the last rule, we transform

\[
\vdash D_1 \\
B \Rightarrow C \\
\square B, \Gamma' \Rightarrow \Delta', A, A, \Box C \\
\square B, \Gamma' \Rightarrow \Delta', A, \Box C \\
\frac{LR-M}{RC} \\
\frac{LR-M}{RC}
\]

\[
\vdash D_1 \\
B \Rightarrow C \\
\square B, \Gamma' \Rightarrow \Delta', A, \Box C \\
\square B, \Gamma' \Rightarrow \Delta', A, \Box C \\
\frac{LR-M}{RC} \\
\frac{LR-M}{RC}
\]
Theorem 3.6. The rule of cut is admissible in G3X

\[
\frac{\Gamma \rightarrow \Delta, D}{\Gamma, \Pi \rightarrow \Delta, \Sigma} \quad \text{Cut}
\]

Proof. We consider an uppermost application of Cut and we show that either it is eliminable, or it can be permuted upward in the derivation until we reach sequents where that instance of Cut is eliminable. The proofs, one for each calculus, are by induction on the weight of the cut formula \(D\) with a sub-induction on the sum of the heights of the derivations of the two premisses (cut-height for short.). As in [16, Thm. 3.2.3], the proof can be organized in 5 exhaustive cases:

1. The left premiss is an initial sequent or a conclusion of \(L\);
2. The right premiss is an initial sequent or a conclusion of \(L\);
3. The cut formula in not principal in the left premiss;
4. The cut formula is principal in the left premiss only;
5. The cut formula is principal in both premisses.

- **Cases (1) and (2).** Same as for G3cp, see [16, Thm. 3.2.3] for the details.

- **Case (3).** We have many subcases according to the last rule applied in the derivation of the left premiss \((D_1)\). For the propositional rules, we refer the reader to [16, Thm. 3.2.3], where it is given a procedure that allows to reduce the cut-height. If the last rule applied in \(D_1\) is a modal one, we can transform the derivation into a cut-free one because the conclusion of Cut is derivable directly by ending \(D_1\) with the appropriate instance of the same modal rule. We present explicitly only LR-E and L-D⊥, all other transformations being similar.

**LR-E:** If the left premiss is by rule LR-E, we transform

\[
\frac{\vdash D_{11} \quad \vdash D_{12}}{A \Rightarrow B \quad B \Rightarrow A} \quad \text{LR-E} \quad \frac{\vdash D_2}{\square A, \Gamma' \Rightarrow \Delta', \square B, \Sigma} \quad \text{Cut} \quad \frac{\vdash D_{11} \quad \vdash D_{12}}{A \Rightarrow B \quad B \Rightarrow A} \quad \text{LR-E}
\]

**L-D⊥:** If the left premiss is by rule L-D⊥, we transform

\[
\frac{\vdash D_{11} \quad \vdash D_2}{A \Rightarrow \square A, \Gamma' \Rightarrow \Delta, D} \quad \text{L-D⊥} \quad \frac{\vdash D_{11} \quad \vdash D_2}{D, \Pi \Rightarrow \Delta, \Sigma} \quad \text{Cut} \quad \frac{\vdash D_{11} \quad \vdash D_{12}}{A \Rightarrow \square A, \Gamma' \Rightarrow \Delta, \Sigma} \quad \text{L-D⊥}
\]

- **Case (4)** If the cut formula \(D\) is principal in the left premiss only, the procedure is analogous to that for case (3). If the right premiss has been derived by a propositional rule, see [16, Thm. 3.2.3]. If it has been derived by a modal rule, we can once again obtain the conclusion of Cut by an appropriate instance of the same rule. The details are left to the reader.

- **Case (5)** If the cut formula \(D\) is principal in both premisses, we have cases according to the principal operator of \(D\); in each case we have a procedure that allows to reduce the weight
of the cut formula, possibly increasing the cut-height. For the propositional cases, which are the same for all the logics considered here, see [16, Thm. 3.2.3].

If $D \equiv \Box C$, we consider the different logics one by one, without repeating the common cases.

- **G3E(ND).** Both premisses are by rule $LR-E$, we have

  \[
  \frac{\vdash D_{11}, D_{12} \vdash D_{21}, D_{22} \quad A \Rightarrow C \quad C \Rightarrow A \quad LR-E \quad \Box A, \Gamma' \Rightarrow \Delta, \Box C \quad \Box C, \Pi \Rightarrow \Sigma', \Box B}{\vdash D_{21}, \Delta, \Sigma', \Box B \Rightarrow C \quad LR-E \quad \Box A, \Gamma', \Pi \Rightarrow \Delta, \Sigma', \Box B}
  \]

  and we transform it into the following derivation that has two cuts with cut formulas of lesser weight, which are admissible by IH.

  \[
  \frac{\vdash D_{11}, D_{21} \vdash D_{22}, D_{12} \quad A \Rightarrow C \quad C \Rightarrow B \quad LR-E \quad \Box A, \Gamma' \Rightarrow \Delta, \Box C \quad \Box C, \Pi \Rightarrow \Sigma', \Box A}{\vdash D_{21}, \Delta, \Sigma', \Box B \Rightarrow C \quad LR-E \quad \Box A, \Gamma', \Pi \Rightarrow \Delta, \Sigma', \Box B}
  \]

- **G3EN(D).** Left premiss by $R-N$ and right one by $LR-E$. We transform

  \[
  \frac{\vdash D_{11}, D_{21} \vdash D_{22} \quad \Box A, \Gamma' \Rightarrow \Delta, \Box C \quad C \Rightarrow A \Rightarrow C \quad LR-E \quad \Box C, \Pi \Rightarrow \Sigma', \Box A}{\vdash D_{22}, \Delta, \Sigma', \Box A \Rightarrow \Delta, \Sigma \quad LR-E \quad \Box A, \Gamma', \Pi \Rightarrow \Delta, \Sigma}
  \]

- **G3E(N)$D^\perp$.** Left premiss by $LR-E$, and right one by $L-D^\perp$. We transform

  \[
  \frac{\vdash D_{11}, D_{12} \vdash D_{21}, D_{22} \quad A \Rightarrow C \Rightarrow A \quad LR-E \quad \Box A, \Gamma' \Rightarrow \Delta, \Box C \quad \Box C, \Pi \Rightarrow \Sigma}{\vdash D_{21}, \Delta, \Sigma \Rightarrow A \quad L-D^\perp \quad \Box A, \Gamma', \Pi \Rightarrow \Delta, \Sigma}
  \]

- **G3E(N)$D^\circ$.** Left premiss is by $LR-E$, and right one by $L-D^\circ$. We transform ($|\Xi| \leq 1$)

  \[
  \frac{\vdash D_{11}, D_{12} \vdash D_{21}, D_{22} \quad A \Rightarrow C \Rightarrow A \quad LR-E \quad \Box A, \Gamma' \Rightarrow \Delta, \Box C \quad \Box C, \Xi, \Pi' \Rightarrow \Sigma}{\vdash D_{22}, \Delta, \Sigma \Rightarrow \Xi \quad L-D^\circ \quad \Box A, \Gamma', \Xi, \Pi' \Rightarrow \Delta, \Sigma}
  \]

- **G3E(N)$D$.** Left premiss by $LR-E$ and right one by $L-D^\perp$ or $L-D^\circ$. Same as above.

- **G3END^\perp$.** Left premiss by $R-N$ and right one by $L-D^\perp$. We transform

  \[
  \frac{\vdash D_{11} \vdash D_{21} \quad \Box A, \Gamma' \Rightarrow \Delta, \Box C \quad C \Rightarrow \Sigma}{\vdash D_{21}, \Delta, \Sigma \Rightarrow C \quad L-D^\perp \quad \Box A, \Gamma', \Pi \Rightarrow \Delta, \Sigma}
  \]

  and

  \[
  \frac{\vdash D_{11} \vdash D_{21} \quad \Box A, \Gamma' \Rightarrow \Delta, \Box C \quad C \Rightarrow \Sigma}{\vdash D_{21}, \Delta, \Sigma \Rightarrow C \quad L-D^\perp \quad \Box A, \Gamma', \Pi \Rightarrow \Delta, \Sigma}
  \]
- **G3END.** Left premiss by R-N and right one by L-D\(^{\circ}\). We transform \(|\Xi| \leq 1\)

\[
\frac{\vdash D_{11}}{\Gamma \Rightarrow \Delta, \Box C}
\]

\[
\frac{\vdash D_{21}}{C, \Xi \Rightarrow C, \Xi}
\]

\[
\vdash D_{22}
\]

\[
\vdash D_{11}, \vdash D_{21}
\]

\[
\text{L-D}^{\circ}
\]

\[
\text{Cut}
\]

\[
\Xi \Rightarrow \Sigma
\]

\[
\text{Cut}
\]

\[
\square \Xi, \Gamma, \Pi' \Rightarrow \Delta, \Sigma
\]

\[
\text{(*)}
\]

where \((*)\) is an instance of \(L-D^{\perp}\) if \(|\Xi| = 1\), else \((|\Xi| = 0\) and it is some instances of \(LW\) and \(RW\).

- **G3M(ND).** Both premisses are by rule LR-M, we transform

\[
\frac{\vdash D_{11}}{A \Rightarrow C}
\]

\[
\vdash D_{21}
\]

\[
\vdash D_{21}
\]

\[
\vdash D_{11}, \vdash D_{21}
\]

\[
\text{LR-M}
\]

\[
\text{Cut}
\]

\[
\square A, \Gamma', \Pi \Rightarrow \Delta, \Sigma', \Box B
\]

\[
\text{LR-M}
\]

- **G3MN(D).** Left premiss by R-N and right one by LR-M. Similar to the case with left premiss by R-N and right one by LR-E.

- **G3M(N)D\(^{\perp}\) and G3M(ND).** Left premiss is by LR-M, and right one by either \(L-D^{\perp}\) or \(L-D^{\circ}\). Similar to the analogous cases with left premiss by LR-E.

- **G3MND\(^{\perp}\) and G3MND.** The cases with left premiss by R-N and right one by a deontic rule have already been considered.

- **G3C(ND).** Both premisses are by rule LR-C. Let us agree to use \(\Lambda\) to denote the non-empty multiset \(A_1, \ldots, A_n, \) and \(\Xi\) for the (possibly empty) multiset \(B_2, \ldots B_m\). We transform

\[
\Lambda \Rightarrow C, C \Rightarrow A_1, \ldots, C \Rightarrow A_n, C, \Xi \Rightarrow E, E \Rightarrow C, E \Rightarrow B_m
\]

\[
\vdash D_{11}, \vdash D_{21}, \vdash D_{22}, \vdash D_{11}, \vdash D_{21}
\]

\[
\text{LR-C}
\]

\[
\text{Cut}
\]

\[
\square \Lambda, \Gamma' \Rightarrow \Delta, \Box C
\]

\[
\square A, \Gamma', \Pi' \Rightarrow \Delta, \Sigma', E
\]

is transformed into a derivation with \(n + 1\) cuts on formulas of lesser weight.

- **G3CN(D).** Left premiss by R-N and right premiss by LR-C. We have

\[
\frac{\vdash D_{11}}{\Gamma \Rightarrow \Delta, \Box C}
\]

\[
\vdash D_{21}
\]

\[
\vdash D_{21}
\]

\[
\vdash D_{11}, \vdash D_{21}
\]

\[
\text{LR-C}
\]

\[
\text{Cut}
\]

\[
\Gamma, \Box A_1, \ldots, \Box A_n, \Pi' \Rightarrow \Delta, \Sigma', \Box B
\]

where \(A_1, \ldots, A_n\) (and thus also \(\Box A_1, \ldots, \Box A_n\)) may or may not be the empty multiset. If \(A_1, \ldots, A_n\) is not empty, we transform it into the following derivation having one cut with cut
formula of lesser weigh

\[
\vdash_{\mathcal{D}_{11}} C, A_1, \ldots, A_n \Rightarrow B
\]

\[
\vdash_{\mathcal{D}_{21}} B \Rightarrow A_1 \quad \vdash_{\mathcal{D}_{A_n}} \quad C \Rightarrow A_n
\]

\[
\text{Cut}
\]

\[
\Gamma, \Box A_1, \ldots, \Box A_n, \Pi' \Rightarrow \Delta, \Sigma', \Box B
\]

If, instead, \(A_1, \ldots, A_n\) is empty, we transform it into

\[
\vdash_{\mathcal{D}_{11}} C \quad \vdash_{\mathcal{D}_{21}} B
\]

\[
\text{Cut}
\]

\[
\Gamma, \Pi' \Rightarrow \Delta, \Sigma', \Box B
\]

- **G3CD\(^c\).** Left premise by \(LR-C\) and right premise by \(L-D^c\). We transform (we assume \(\Xi = A_1, \ldots, A_k\), \(\Theta = C, B_2, \ldots, B_m\) and \(\Lambda = D_1, \ldots, D_n\))

\[
\vdash_{\mathcal{D}_{11}} \Xi \Rightarrow C \quad \vdash_{\mathcal{D}_{A_i}} C \Rightarrow A_i \in \Xi
\]

\[
\vdash_{\mathcal{D}_{21}} \Theta, \Lambda \Rightarrow \{B_i, D_j \Rightarrow: B_i \in \Theta \text{ and } D_j \in \Lambda\}
\]

\[
\text{Cut}
\]

\[
\Box \Xi, \Box B_1, \ldots, \Box B_m, \Box \Lambda, \Box \Gamma', \Pi' \Rightarrow \Delta, \Sigma
\]

into the following derivation having \(1 + (k \times n)\) cuts on formulas of lesser weight.

- **G3C(N)D.** Left premise by \(LR-C\) and right one by \(L-D^*\). It is straightforward to transform the derivation into another one having one cut with cut formula of lesser weight.

- **G3R(D).** Both premises are by rule \(LR-R\), we transform

\[
\vdash_{\mathcal{D}_{11}} C \quad \vdash_{\mathcal{D}_{21}} C \Rightarrow B
\]

\[
\text{Cut}
\]

\[
\Gamma', \Pi' \Rightarrow \Delta, \Sigma', \Box B
\]

\[
\text{Cut}
\]

\[
\Delta, \Sigma', \Box B
\]

- **G3RD\(^*\).** Left premise is by \(LR-R\), and right one by \(L-D^*\), we transform

\[
\vdash_{\mathcal{D}_{11}} C \quad \vdash_{\mathcal{D}_{21}} C \Rightarrow B
\]

\[
\text{Cut}
\]

\[
\Box A, \Box \Xi, \Box \Gamma', \Box \Psi, \Pi' \Rightarrow \Delta, \Sigma
\]

- **G3K(D).** All subcases are similar to the respective ones for **G3R(D).**
3.2 Decision procedure for G3X

As a corollary of the admissibility of the structural rules, it holds that each calculus \( \text{G3X} \) has the (strong) subformula property since in all rules in Tables 3 and 4 each active formula is a proper subformula of a principal formula and no formula disappears in moving from premise(s) to conclusion. As usual, this gives us a syntactic proof of consistency and it gives us an effective method to decide the derivability of a sequent in \( \text{G3X} \): we start from the desired sequent \( \Gamma \Rightarrow \Delta \) and we construct, root-first, all possible \( \text{G3X} \)-derivation trees until either we find a tree where each leaf is an initial sequent or a conclusion of \( L \bot \) – we have found a \( \text{G3X} \)-derivation of \( \Gamma \Rightarrow \Delta \) – or we have checked all possible \( \text{G3X} \)-derivations and we have found none – \( \Gamma \Rightarrow \Delta \) is not \( \text{G3X} \)-derivable. Observe that, given that the modal rules are not invertible, in the root-first decision procedure we may need backtracking when they are applied (or we can apply all possible instances of a modal rule in parallel). More in details, to decide whether a sequent \( \Gamma \Rightarrow \Delta \) is derivable in \( \text{G3X} \), we apply the following recursive procedure (we assume that at each node is associated the list of the instances of modal rules that can be applied to it and have not been applied yet).

**Definition 3.7 (G3X-Decision tree).**

1. We write the one node sequent \( \Gamma \Rightarrow \Delta \);
2. If the tree constructed at stage \( n \) is a \( \text{G3X} \)-derivation the procedure ends; else we consider each leaf of the tree constructed at stage \( n \) and
   - If some instance of a propositional rule is applicable root-first, we apply it; else
   - If, for some \( m > k \geq 1 \), \( m \) instances of some modal \( \text{G3X} \)-rules are applicable and \( m - k \) of those instances have been already applied, we apply root-first the \( m - k + 1 \)-instance of a modal rule; else
   - We go back in that branch until we reach a node where \( m \) instance of some modal rule where applicable and only \( m - k \) (for \( m > k \geq 1 \)) have been applied in the previous stages, we apply the \( m - k + 1 \)-instance; else
3. The procedure ends and we conclude that \( \Gamma \Rightarrow \Delta \) is not \( \text{G3X} \)-derivable.

By inspecting the rules in Tables 3 and 4 it is easy to acknowledge that the procedure defined above terminates. Let the weight of a sequent be the sum of the weights of the formulas occurring in it. If at step \( n \) of a \( \text{G3X} \)-decision tree for \( \Gamma \Rightarrow \Delta \) we have \( \Pi \Rightarrow \Sigma \) in a leaf, then if at step \( n + 1 \) we are either in subcase \( n + 1 \) or \( n + 2 \), we obtain a finite number of leaves of strictly lesser weight. If we are in subcase \( n + 1 \), we backtrack in that branch until we reach a node \( \Pi' \Rightarrow \Sigma' \) where \( j \) instances of modal rules still have to be applied. We apply one of these instances, thus obtaining a leaf of weight strictly lesser than \( \Pi' \Rightarrow \Sigma' \) and we know we may backtrack to the node \( \Pi' \Rightarrow \Sigma' \) at most other \( j - 1 \) times. Hence, at each (non-final) step we introduce sequents of lesser weight and, possibly, we diminish the number of times that we have to backtrack to a given node. After a finite amount of steps, we will reach a tree that either is a \( \text{G3X} \)-derivation of \( \Gamma \Rightarrow \Delta \), or that is such that all its leaves have weight 0 and where we cannot backtrack anymore, hence \( \Gamma \Rightarrow \Delta \) is not \( \text{G3X} \)-derivable.

3.3 Equivalence with the axiomatic systems

It is now time to show that the sequent calculi introduced are equivalent to the non-normal logics of Sect. 2. We write \( \text{G3X} \vdash \Gamma \Rightarrow \Delta \) if the sequent \( \Gamma \Rightarrow \Delta \) is derivable in \( \text{G3X} \), and we say that a formula \( A \) is derivable in \( \text{G3X} \) whenever \( \text{G3X} \vdash \Rightarrow A \). We begin by proving the following
Lemma 3.8. All the axioms of the axiomatic system \( X \) are derivable in \( G3X \).

Proof. A straightforward root-first application of the rules of the appropriate sequent calculus, possibly using Prop. 3.1. As an example, we show that the deontic axiom \( D \Perp \) is derivable by means of rule \( L-D \Perp \) and that axiom \( C \) is derivable by means of \( LR-C \).

\[
\begin{align*}
A, B & \Rightarrow A & \text{3.1} \\
A, B & \Rightarrow B & \text{3.1} \\
A, B & \Rightarrow A \land B & \text{3.1} \\
A & \Rightarrow A \land B & \text{R\land} \\
A & \Rightarrow A & \text{L\land} \\
A & \Rightarrow B & \text{L\land} \\
A, B & \Rightarrow B & \text{LR-C} \\
\end{align*}
\]

Next we prove the equivalence of the sequent calculi for non-normal logics with the corresponding axiomatic systems in the sense that whenever a sequent \( \Gamma \Rightarrow \Delta \) is derivable in \( G3X \), its characteristic formula \( \land \Gamma \supset \lor \Delta \) is derivable in \( X \), where the empty antecedent stands for \( \top \) and the empty succedent for \( \bot \). As a consequence each calculus is sound and complete with respect to the appropriate class of neighbourhood models, cf. Sect. 2.2.

Theorem 3.9. Derivability in the sequent system \( G3X \) and in the axiomatic system \( X \) are equivalent, i.e.

\[
G3X \vdash \Gamma \Rightarrow \Delta \iff X \vdash \land \Gamma \supset \lor \Delta
\]

Proof. To prove the right-to-left implication, we argue by induction of the height of the axiomatic derivation in \( X \). The base case is covered by Lemma 3.8. For the inductive steps, the case of MP follows by the admissibility of Cut and the invertibility of rule \( R \supset \). If the last step is by \( RE \), then \( \Gamma = \emptyset \) and \( \Delta = \Box C \leftrightarrow \Box D \). We know that (in \( X \)) we have derived \( \Box C \leftrightarrow \Box D \) from \( C \leftrightarrow D \). Remember that \( C \leftrightarrow D \) is defined as \( (C \supset D) \land (D \supset C) \). Thus we assume, by inductive hypothesis (IH), that \( G3ED \vdash \Rightarrow C \supset D \) and \( G3ED \vdash \Rightarrow D \supset C \), and we proceed as follows

\[
\begin{align*}
C \Rightarrow D & \text{3.4} \\
C & \Rightarrow D \text{3.4} \\
\Box C & \Rightarrow \Box D \text{ R\supset} \\
\Box C & \Rightarrow \Box D \text{ R\supset} \\
(\Box C \supset \Box D) \land (\Box D \supset \Box C) & \text{LR-E} \\
\end{align*}
\]

For the converse implication, we assume \( G3X \vdash \Gamma \Rightarrow \Delta \), and show, by induction on the height of the derivation in sequent calculus, that \( X \vdash \land \Gamma \supset \lor \Delta \). If the derivation has height 0, we have an initial sequent – so \( \Gamma \land \Delta \neq \emptyset \) or an instance on \( L \bot \) – thus \( \bot \in \Gamma \). In both cases the claim holds. If the height is \( n + 1 \), we consider the last rule applied in the derivation. If it is a propositional one, the proof is straightforward. If it is a modal rule, we argue by cases.

If the last step of a derivation in \( G3(n) \) is by rule \( LR-E \), we have derived \( \Box C, \Gamma' \Rightarrow \Delta', \Box D \) from \( C \Rightarrow D \) and \( D \Rightarrow C \). By IH, \( ED \vdash C \leftrightarrow D \), thus \( ED \vdash \Box C \leftrightarrow \Box D \). By some propositional steps we conclude \( ED \vdash (\Box C \land \Gamma') \supset (\lor \Delta' \lor \Box D) \). The cases of \( LR-M, LR-R, \) and \( LR-K \) can be treated in a similar manner (thanks, respectively, to the rule \( RM, RR, RK \) from Table 2).
If we are in \textbf{G3C(ND)}, suppose the last step is the following instance of LR-C:
\[
\frac{C_1, \ldots, C_k \implies D \quad D \implies C_1 \quad \ldots \quad D \implies C_k}{\Box C_1, \ldots, \Box C_k, \Gamma' \implies \Delta', \Box D} \text{ LR-C}
\]

By IH, we have that \(\mathbf{C(ND)} \vdash D \supset C_i\) for all \(1 \leq i \leq k\), and, by propositional reasoning, we have that \(\mathbf{C(ND)} \vdash D \supset C_1 \land \cdots \land C_k\). We also know, by IH, that \(\mathbf{C(ND)} \vdash C_1 \land \cdots \land C_k \supset D\). By applying \text{RE} to these two theorems we get that
\[
\mathbf{C(ND)} \vdash \Box (C_1 \land \cdots \land C_k) \supset \Box D
\]
(3)

By using axiom \(C\) and some propositional steps, we know that
\[
\mathbf{C(ND)} \vdash C_1 \land \cdots \land C_k \supset \Box (C_1 \land \cdots \land C_k)
\]
(4)

By applying transitivity to (4) and (3) and some propositional steps, we conclude that
\[
\mathbf{C(ND)} \vdash (\Box C_1 \land \cdots \land \Box C_k \land \bigwedge \Gamma') \supset (\bigvee \Delta' \lor \Box D)
\]

Let’s now consider rule \(L-D^\perp\). Suppose we are in \textbf{G3xD^\perp} and we have derived \(\Box C, \Gamma' \implies \Delta\) from \(C \implies \perp\). By IH, \(\text{xD}^\perp \vdash C \supset \perp\), and we know that \(\text{xD}^\perp \vdash \perp \supset C\). Thus by \text{RE} (or \text{RM}, or \text{RR}), we get \(\text{xD}^\perp \vdash \Box C \supset \perp\). Now, by contrapositively it and then applying a \(M\) with the axiom \(D^\perp\), we get that \(\text{xD}^\perp \vdash \neg \Box C\). By some easy propositional steps we conclude \(\text{xD}^\perp \vdash (\Box C \land \bigwedge \Gamma') \supset \bigvee \Delta\). The case \(R-N\) is similar.

Let’s consider rules \(L-D^\circ\). Suppose we are in \textbf{G3ED}^\circ and we have derived \(\Box A, \Box B, \Gamma' \implies \Delta\) from the premisses \(A, B \implies \perp\) and \(\perp \implies A, B\). By induction we get that \(\text{ED}^\circ \vdash A \land B \supset \perp\) and \(\text{ED}^\circ \vdash A \lor B\). Hence, \(\text{ED}^\circ \vdash B \supset \neg A\) and \(\text{ED}^\circ \vdash \neg A \supset B\). By applying \text{RE} we get that
\[
\text{ED}^\circ \vdash \Box B \supset \neg \Box A
\]
which, thanks to axiom \(D^\circ\), entails that
\[
\text{ED}^\circ \vdash \Box B \supset \neg \Box A
\]
By some propositional steps we conclude
\[
\text{ED}^\circ \vdash \Box A \land \Box B \land \bigwedge \Gamma' \supset \bigvee \Delta
\]
Notice that we can assume, w.l.o.g., that instances of rule \(L-D^\circ\) always have two principal formulas since otherwise the calculus would prove the empty sequent (for the same reason we will assume that neither \(\Pi\) nor \(\Sigma\) is empty in instances of rule \(L-D^\circ\)).

Let’s consider rule \(L-D^\circ_{\text{c}}\). Suppose we have a \textbf{G3CD}^\circ\)-derivation whose last step is:
\[
\frac{\Pi, \Sigma \implies \{ \implies A, B : A \in \Pi \text{ and } B \in \Sigma \}}{\Box \Pi, \Box \Sigma, \Gamma' \implies \Delta'}
\]

By induction and by some easy propositional steps we know that \(\text{ECD}^\circ \vdash \bigwedge \Pi \leftrightarrow \neg \bigwedge \Sigma\). By rule \text{RE} we derive \(\text{ECD}^\circ \vdash \Box \bigwedge \Pi \supset \Box \neg \bigwedge \Sigma\), which, thanks to axiom \(D^\circ\), entails that \(\text{ECD}^\circ \vdash \Box \bigwedge \Pi \supset \neg \Box \bigwedge \Sigma\). By contrapositively it and then applying two (generalized) instances of axiom \(C\) we obtain \(\text{ECD}^\circ \vdash \bigwedge \Pi \supset \neg \bigwedge \Sigma\). By some easy propositional steps we conclude that \(\text{ECD}^\circ \vdash \bigwedge \bigwedge \Pi \land \Box \bigwedge \Sigma \land \bigwedge \Gamma' \supset \bigvee \Delta\).

The admissibility of \(L-D^\circ\) in \textbf{EC(N)D, RD, and KD} is similar to that of \text{LR-C}, save that in (3) we replace \(\Box D\) with \(\perp\) and then we use theorem \(D^\perp\) to transform it into \(\perp\).

By combining this with the results in Theorem 2.6 we have the following result.

\textbf{Corollary 3.10.} The calculus \textbf{G3X} is sound and complete with respect to the class of all neighbourhood models for \(X\).
4 Craig’s Interpolation Theorem

In this section we give a constructive proof of Craig’s interpolation theorem by means of the well-known Maehara’s technique, see [12, 13], for each modal or deontic logic \( X \) which does not have \( C \) as axiom.

**Theorem 4.1** (Craig’s interpolation theorem). Let \( A \supset B \) be a theorem of a logic \( X \) which does not have \( C \) as (non-eliminable) axiom. It holds that

- there is a formula \( I \), which contains propositional variables common to \( A \) and \( B \) only, such that both \( A \supset I \) and \( I \supset B \) are theorems of \( X \).

In order to prove this theorem, we use the following notions

**Definition 4.2.** A partition of a sequent \( \Gamma \Rightarrow \Delta \) is any pair of sequents
\( (\Gamma _1 \Rightarrow \Delta _1 \parallel \Gamma _2 \Rightarrow \Delta _2) \) such that \( \Gamma _1, \Gamma _2 = \Gamma \) and \( \Delta _1, \Delta _2 = \Delta \).

A \( G3X \)-interpolant of a partition \( (\Gamma _1 \Rightarrow \Delta _1 \parallel \Gamma _2 \Rightarrow \Delta _2) \) is any formula \( I \) such that:

1. all propositional variables in \( I \) are in \( (\Gamma _1 \cup \Delta _1) \cap (\Gamma _2 \cup \Delta _2) \),
2. \( G3X \vdash \Gamma _1 \Rightarrow \Delta _1, I \) and \( G3X \vdash I, \Gamma _2 \Rightarrow \Delta _2 \).

If \( I \) is a \( G3X \)-interpolant of the partition \( (\Gamma _1 \Rightarrow \Delta _1 \parallel \Gamma _2 \Rightarrow \Delta _2) \), we write
\[ (G3X \vdash) \langle \Gamma _1 \Rightarrow \Delta _1 \parallel \Gamma _2 \Rightarrow \Delta _2 \rangle \]

where one or more of the multisets \( \Gamma _1, \Gamma _2, \Delta _1, \Delta _2 \) may be empty. When the set of propositional variables in \( (\Gamma _1 \cup \Delta _1) \cap (\Gamma _2 \cup \Delta _2) \) is empty the \( X \)-interpolant has to be constructed from \( \bot \) (and \( \top \)). The proof of Theorem 4.1 is by the following lemma, originally due to Maehara [12].

**Lemma 4.3** (Maehara’s lemma). If \( G3X \vdash \Gamma \Rightarrow \Delta \) and \( LR-C \) (and \( L-D^{\ominus C} \)) is not a primitive rules of \( G3X \), every partition of \( \Gamma \Rightarrow \Delta \) has a \( G3X \)-interpolant.

*Proof.* The proofs is by induction on the height of the derivation \( D \) of \( \Gamma \Rightarrow \Delta \). Basically we have to show that each partition of an initial sequent (or of a conclusion of a \( 0 \)-premiss rule) has a \( G3X \)-interpolant and that for any rule of \( G3X \), we have an effective procedure that outputs a \( G3X \)-interpolant for any partition of its conclusion from the interpolant(s) of its premiss(es). The proof is modular and, hence, we can consider the modal rules directly without having to consider the different modal calculi.

For the base case of initial sequents with \( p \) principal formula, we have four possible partitions, whose interpolants are:

\[
\begin{align*}
(1) \langle p, \Gamma _1 \Rightarrow \Delta _1 \parallel p, \Gamma _2 \Rightarrow \Delta _2 \rangle & \quad (2) \langle p, \Gamma _1 \Rightarrow \Delta _1 \parallel p, \Gamma _2 \Rightarrow \Delta _2 \rangle \\
(3) \langle \Gamma _1 \Rightarrow \Delta _1 \parallel p, \Gamma _2 \Rightarrow \Delta _2 \rangle & \quad (4) \langle \Gamma _1 \Rightarrow \Delta _1 \parallel p, \Gamma _2 \Rightarrow \Delta _2 \rangle \\
\end{align*}
\]

and for the base case of rule \( L \bot \), we have:

\[
\begin{align*}
(1) \langle \bot, \Gamma _1 \Rightarrow \Delta _1 \parallel \Gamma _2 \Rightarrow \Delta _2 \rangle & \quad (2) \langle \Gamma _1 \Rightarrow \Delta _1 \parallel \bot, \Gamma _2 \Rightarrow \Delta _2 \rangle \\
\end{align*}
\]

For the proof of (some of) the propositional cases the reader is referred to [22, pp. 117-118], thus we have only to prove that all the modal rules of Table 4 (save for \( LR-C \) and \( L-D^{\ominus C} \)) behave as desired.

- **LR-E** If the last rule applied in \( D \) is
we have four kinds of partitions of the conclusion:

(1) \( \langle A \Rightarrow B \mid \mid \Rightarrow \rangle \quad \langle B \Rightarrow A \mid \mid \Rightarrow \rangle \)

(2) \( \langle A \Rightarrow B \mid \mid \Rightarrow \rangle \quad \langle B \Rightarrow A \mid \mid \Rightarrow \rangle \)

(3) \( \langle A \Rightarrow B \mid \mid \Rightarrow \rangle \quad \langle B \Rightarrow A \mid \mid \Rightarrow \rangle \)

In each case we have to choose a suitable partition of the premiss (s) that allows us to construct a \textbf{G3E(ND)}-interpolant for the partition under consideration.

In case (1) we have

\[
\begin{array}{c}
\langle A \Rightarrow B \mid \mid \Rightarrow \rangle \quad \langle B \Rightarrow A \mid \mid \Rightarrow \rangle \\
\langle \square A, \Gamma' \Rightarrow \Delta_1, \square B \mid \mid \Gamma_2 \Rightarrow \Delta_2 \rangle
\end{array}
\]

To wit, by IH there is some \( C (D) \) that is a \textbf{G3E(ND)}-interpolant of the partition of the left (right) premiss. Thus both \( C \) and \( D \) contains only propositional variables common to \( A \) and \( B \); and (i) \( \vdash A \Rightarrow B, C \) (ii) \( \vdash C \Rightarrow (iii) \vdash B \Rightarrow A, D \) and (iv) \( \vdash D \Rightarrow . \) Since the common language of the partitions of the premisses is empty i.e. \( (\Gamma_1 \cup \Delta_1) \cap (\Gamma_2 \cup \Delta_2) = \emptyset \), no propositional variable can occur in \( C \) nor in \( D \), therefore \( C \equiv \bot \) and \( D \equiv \bot \). Here is a proof that \( \bot \) is a \textbf{G3E(ND)}-interpolant of the partition under consideration:

\[
\begin{array}{c}
\langle A \Rightarrow B, \bot \mid \Rightarrow \rangle \\
\langle B \Rightarrow A, \bot \mid \Rightarrow \rangle
\end{array}
\]

In case (2) we have

\[
\begin{array}{c}
\langle A \Rightarrow \mid \mid \Rightarrow B \rangle \\
\langle B \Rightarrow \mid \mid \Rightarrow A \rangle
\end{array}
\]

The proof goes as follows. By IH it holds that some \( C \) and \( D \) are \textbf{G3E(ND)}-interpolants of the given partitions of the premisses. Thus (i) \( \vdash A \Rightarrow C \) (ii) \( \vdash C \Rightarrow B \) (iii) \( \vdash B \Rightarrow D \) and (iv) \( \vdash D \Rightarrow A \). Here is a proof that \( \square C \) is a \textbf{G3E(ND)}-interpolant of the partition under consideration (observe that \( \square D \) works equally well):

\[
\begin{array}{c}
\langle C \Rightarrow B \rangle \\
\langle C \Rightarrow D \rangle
\end{array}
\]

In case (3) we have

\[
\begin{array}{c}
\langle A \Rightarrow D \rangle \\
\langle B \Rightarrow C \rangle
\end{array}
\]

In case (4) we have

\[
\begin{array}{c}
\langle C \Rightarrow A \rangle \\
\langle D \Rightarrow B \rangle
\end{array}
\]
By IH, there are $C$ and $D$ that are $G3E(ND)$-interpolants of the partitions of the premisses. Thus (i) $\vdash B, C$ (ii) $\vdash C, A \implies$ (iii) $\vdash A, D$ and (iv) $\vdash D, B \implies$. We prove that $\Diamond D$ is a $G3E(ND)$-interpolant of the (given partition of the) conclusion as follows:

$$
\begin{array}{c}
\begin{array}{c}
\text{By IH, there are } C \text{ and } D \text{ of the partitions of the conclusion (and of the appropriate partition of the premiss). The proofs are parallel to those for LR-E.}
\end{array}
\end{array}
$$

In case (4) we have

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(1) \( \langle \Box A, \Box \Pi_1, \Gamma'_1 \Rightarrow \Delta'_1, \Box B \mid \Box \Pi_2, \Gamma'_2 \Rightarrow \Delta_2 \rangle \)

(2) \( \langle \Box A, \Box \Pi_1, \Gamma'_1 \Rightarrow \Delta_1 \mid \Box \Pi_2, \Gamma'_2 \Rightarrow \Delta'_2, \Box B \rangle \)

(3) \( \langle \Box \Pi_1, \Gamma'_1 \Rightarrow \Delta'_1, \Box B \mid \Box A, \Box \Pi_2, \Gamma'_2 \Rightarrow \Delta_2 \rangle \)

(4) \( \langle \Box \Pi_1, \Gamma'_1 \Rightarrow \Delta_1 \mid \Box A, \Box \Pi_2, \Gamma'_2 \Rightarrow \Delta'_2, \Box B \rangle \)

In case (1) we have two subcases according to whether \( \Pi_2 \) is empty or not. If it is not empty we have

\[
\begin{array}{c}
\langle A, \Pi_1 \Rightarrow B \mid \Pi_2 \Rightarrow \rangle \\
\langle \Box A, \Box \Pi_1, \Gamma'_1 \Rightarrow \Delta'_1, \Box B \mid \Box \Pi_2, \Gamma'_2 \Rightarrow \Delta_2 \rangle
\end{array}
\]

This can be shown as follows. By IH, there is a \( \text{G3R(D*)-interpolant} \) \( C \) of the chosen partition of the premiss. Thus (i) \( \vdash A, \Pi_1 \Rightarrow B, C \) and (ii) \( \vdash C, \Pi_2 \Rightarrow \), and we have the following proofs

\[
\begin{array}{c}
\frac{\Box A, \Box \Pi_1 \Rightarrow B, C \quad (i)}{-C, \Box A, \Box \Pi_1 \Rightarrow B \quad L^\rightarrow} \\
\frac{\Box A, \Box \Pi_1, \Gamma'_1 \Rightarrow \Delta'_1, \Box B \quad \Box \Pi_2 \Rightarrow -C \quad \text{LR-R}}{\Box \Pi_1, \Gamma'_1 \Rightarrow \Delta'_1, \Box B, -C \Rightarrow \Delta_2 \quad R^\rightarrow}
\end{array}
\]

When \( \Pi_2 \) (and \( \Box \Pi_2 \)) is empty we cannot proceed as above since we cannot apply \( \text{LR-R} \) in the right derivation. But in this case we know that the \( \text{G3R(D*)-interpolant} \) of the premiss is \( \perp \) since the common language is empty. Hence we have

\[
\begin{array}{c}
\langle A, \Pi_1 \Rightarrow B \mid \Rightarrow \rangle \\
\langle \Box A, \Box \Pi_1, \Gamma'_1 \Rightarrow \Delta'_1, \Box B \mid \Gamma'_2 \Rightarrow \Delta_2 \rangle
\end{array}
\]

Cases (2) and (3) are similar to the corresponding cases for rule \( \text{LR-E} \):

\[
\begin{array}{c}
\langle A, \Pi_1 \Rightarrow C \mid \Pi_2 \Rightarrow B \rangle \\
\langle \Box A, \Box \Pi_1, \Gamma'_1 \Rightarrow \Delta_1 \mid \Box \Pi_2, \Gamma'_2 \Rightarrow \Delta_2, \Box B \rangle
\end{array}
\]

\[
\begin{array}{c}
\langle \Pi_1 \Rightarrow B \mid C, \Pi_2 \Rightarrow \rangle \\
\langle \Box \Pi_1, \Gamma'_1 \Rightarrow \Delta_1 \mid \Box B, \Box C, \Box \Pi_2, \Gamma'_2 \Rightarrow \Delta_2 \rangle
\end{array}
\]

In case (4) we have two subcases according to whether \( \Pi_1 \) is empty or not:

\[
\begin{array}{c}
\langle \Rightarrow \mid \Rightarrow, A, \Pi_2 \Rightarrow B \rangle \\
\langle \Gamma'_1 \Rightarrow \Delta_1 \mid \Box A, \Box \Pi_2, \Gamma'_2 \Rightarrow \Delta'_2, \Box B \rangle
\end{array}
\]

\[
\begin{array}{c}
\langle \Pi_1 \Rightarrow \Rightarrow, C, \Pi_2 \Rightarrow B \rangle \\
\langle \Box \Pi_1, \Gamma'_1 \Rightarrow \Delta_1 \mid \Box A, \Box \Pi_2, \Gamma'_2 \Rightarrow \Delta'_2, \Box B \rangle
\end{array}
\]

The proofs are similar to those of case (1).

- **LR-K** If the last rule applied in \( \mathcal{D} \) is

\[
\begin{array}{c}
\Pi \Rightarrow B \\
\Box \Pi, \Gamma \Rightarrow \Delta, \Box B
\end{array}
\]
we give directly the $G3K(D)$-interpolants of the two possible partitions of the conclusion:

| $\langle \Pi_1 \Rightarrow C \parallel \Pi_2 \Rightarrow B \rangle$ | $\langle \Pi_1 \Rightarrow B \parallel \Pi_2 \Rightarrow \parallel \rangle$ |
|---------------------------------------------------------------|
| $\langle \Box \Pi_1, \Gamma' \Rightarrow \Delta_1 \parallel C \parallel \Box \Pi_2, \Gamma' \Rightarrow \Delta_2 \parallel \Box B \rangle$ | $\langle \Box \Pi_1, \Gamma' \Rightarrow \Delta_1' \parallel C \parallel \Box \Pi_2, \Gamma' \Rightarrow \Delta_2 \parallel \Box B \rangle$ |

The proofs are, respectively, parallel to those for cases (2) and (3) of $LR-E$.

- **L-$D^\perp$** If the last rule applied in $D$ is
  \[ A \Rightarrow \Box A, \Gamma \Rightarrow \Delta \]
  we have two kinds of partitions of the conclusion, whose $G3XD^\perp$-interpolants are, respectively:

| $\langle A \Rightarrow \parallel \parallel \parallel \rangle$ | $\langle \parallel \parallel \parallel \parallel \parallel \rangle$ |
|---------------------------------------------------------------|
| $\langle \Box A, \Gamma' \Rightarrow \Delta_1 \parallel \Gamma_2 \Rightarrow \Delta_2 \rangle$ | $\langle \Gamma_1 \Rightarrow \Delta_1 \parallel \Box A, \Gamma_2 \Rightarrow \Delta_2 \rangle$ |

- **L-$D^\Diamond$** If the last rule applied in $D$ is
  \[ A, B \Rightarrow \Box A, \Box B, \Gamma \Rightarrow \Delta \]
  we have three kinds of partitions of the conclusion:

  (1) $\langle \Box A, \Box B, \Gamma' \Rightarrow \Delta_1 \parallel \Gamma_2 \Rightarrow \Delta_2 \rangle$
  
  (2) $\langle \Gamma_1 \Rightarrow \Delta_1 \parallel \Box A, \Box B, \Gamma_2 \Rightarrow \Delta_2 \rangle$
  
  (3) $\langle \Box A, \Gamma' \Rightarrow \Delta_1 \parallel \Box B, \Gamma_2 \Rightarrow \Delta_2 \rangle$

In cases (1) and (2) we have, respectively (the proofs are left to the reader):

| $\langle A, B \Rightarrow \parallel \parallel \parallel \rangle$ | $\langle \parallel \parallel \parallel \parallel \parallel \parallel \rangle$ | $\langle \parallel \parallel \parallel \parallel \parallel \parallel \parallel \rangle$ |
|---------------------------------------------------------------|
| $\langle \Box A, \Box B, \Gamma' \Rightarrow \Delta_1 \parallel \Gamma_2 \Rightarrow \Delta_2 \rangle$ | $\langle \Gamma_1 \Rightarrow \Delta_1 \parallel \Box A, \Box B, \Gamma_2 \Rightarrow \Delta_2 \rangle$ | $\langle \Box A, \Box B, \Gamma_2 \Rightarrow \Delta_2 \rangle$ |

Finally, in case (3) we have:

| $\langle A \Rightarrow \parallel C \parallel B \Rightarrow \parallel \rangle$ | $\langle \parallel \parallel A \Rightarrow \parallel \parallel \parallel \parallel \rangle$ |
|---------------------------------------------------------------|
| $\langle \Box A, \Gamma' \Rightarrow \Delta_1 \parallel \Box C \parallel \Box D \parallel \Gamma' \Rightarrow \Delta_2 \rangle$ | $\langle \Box A, \Gamma_1 \Rightarrow \Delta_1 \parallel \Box C \parallel \Box D \parallel \Gamma' \Rightarrow \Delta_2 \rangle$ |

To wit, by IH, we can assume that $C$ is an interpolant of the partition of the left premiss and $D$ of the right one, hence we have the following $G3YD^\circ$-derivation ($Y \in \{E, M\}$):

\[
\begin{array}{c}
D \Rightarrow B \quad \text{IH} \\
C, D \Rightarrow \quad \text{Cut} \\
C, B \Rightarrow \quad \text{IH} \\
A \Rightarrow C \quad \text{IH} \\
\hline
A \Rightarrow C \quad \text{IH} \\
\hline
D \Rightarrow A \quad \text{IH} \\
\hline
C, D \Rightarrow \quad \text{Cut} \\
\hline
\hline
\end{array}
\]
Another analogous derivations (where at the step by $L-D^\circ$ we have weakened the conclusion with $\Box B, \Gamma_2', \Delta_2$ instead of with $\Box A, \Gamma_1', \Delta_1$) shows that $G3YD^\circ \vdash \Box C \land \Box D, \Box B, \Gamma_2 \Rightarrow \Delta_2$. It is also immediate to notice that $\Box C \land \Box B$ satisfies the language condition for being a $G3YD^\circ$-interpolant of the conclusion since, by IH, we know that each propositional variable occurring in $C \cup B$ occurs in $A \cap B$.

- **L-D** If the last rule applied in $D$ is

$$
\begin{array}{c}
\Pi \Rightarrow \\
\square \Pi, \Gamma \Rightarrow \Delta \\
\end{array}
$$

we have the following kind of partition

$$\{ \square \Pi_1, \Gamma_1' \Rightarrow \Delta_1 \mid \square \Pi_2, \Gamma_2' \Rightarrow \Delta_2 \}$$

If $\Pi_1$ is not empty we have:

$$
\begin{array}{c}
\square \Pi_1, \Gamma_1' \Rightarrow \Delta_1, \square C \\
\square \Pi_2, \Gamma_2' \Rightarrow \Delta_2
\end{array}
\stackrel{L-D^\ast}{\Rightarrow}
\begin{array}{c}
\{ \Pi_1 \Rightarrow C \mid \Pi_2 \Rightarrow \} \\
L-D^\ast
\end{array}
$$

By IH, there is some $C$ that is an interpolant of the premiss. It holds that $\vdash \Pi_1 \Rightarrow C$ and $\vdash C, \Pi_2 \Rightarrow$. We show that $\Box C$ is a $G3YD$-interpolant ($\mathbf{Y} \in \{R,K\}$) of the partition of the conclusion as follows:

$$
\begin{array}{c}
\square \Pi_1 \Rightarrow C \\
\Pi_1 \Rightarrow C
\end{array}
\stackrel{L-R^\ast}{\Rightarrow}
\begin{array}{c}
\square \Pi_2, \Gamma_2' \Rightarrow \Delta_2 \\
\square \Pi_2, \Gamma_2' \Rightarrow \Delta_2
\end{array}
\stackrel{L-D^\ast}{\Rightarrow}
\begin{array}{c}
\Pi_1 \Rightarrow C \\
\Pi_2 \Rightarrow C
\end{array}
$$

If, instead, $\Pi_1$ is empty we have:

$$
\begin{array}{c}
\Gamma_1 \Rightarrow \Delta_1, \Box C \\
\square \Pi_2, \Gamma_2' \Rightarrow \Delta_2
\end{array}
\stackrel{L-D^\ast}{\Rightarrow}
\begin{array}{c}
\{ \Rightarrow \mid \Pi_2 \Rightarrow \} \\
L-D^\ast
\end{array}
$$

To wit, by IH we know there is a $G3YD$-interpolant $C$ of the partition of the premiss. Thus $\vdash \Rightarrow C$ and $\vdash C, \Pi_2 \Rightarrow$. Since the common language is empty, $C \equiv \top$. We know that $G3YD \vdash \Gamma_1 \Rightarrow \Delta_1, \Box \top$ (since rule $L-D^\ast$ makes $\Rightarrow \Box \top$ derivable) and $G3YD \vdash \Box \top, \Box \Pi_2, \Gamma_2' \Rightarrow \Delta_2$ (by applying $L-\top$ and $L-D^\ast$ to the derivable (by IH) sequent $\top, \Pi_2 \Rightarrow$).

- **R-N** If the last rule applied in $D$ is

$$
\begin{array}{c}
\square A \Rightarrow \\
\Gamma \Rightarrow \Delta, \Box A
\end{array}
\stackrel{R-N}{\Rightarrow}
\begin{array}{c}
\{ \Rightarrow \mid \Pi_2 \Rightarrow \} \\
L-D^\ast
\end{array}
$$

The interpolants for the two possible partitions are:

| (1) | (2) |
|-----|-----|
| $\{ \vee \mid \Pi_2 \Rightarrow \} \vdash \vee$ | $\{ \vee \mid A \} \vdash \vee$ |
| $\{ \Gamma_1 \Rightarrow \Delta_1, \Box A \mid \Gamma_2 \Rightarrow \Delta_2 \}$ | $\{ \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2, \Box A \}$ |

This completes the proof.
Proof of Theorem 4.1. Assume that $A \supset B$ is a theorem of $X$. By Theorem 3.9 and Lemma 3.4 we have that $G3X \vdash A \Rightarrow B$. By Lemma 4.3 (taking $A$ as $\Gamma_1$ and $B$ as $\Delta_2$ and $\Gamma_2, \Delta_1$ empty) and Theorem 3.9 there exists a formula $I$ that is an interpolant of $A \supset B$ -- i.e. $I$ is such that all propositional variables occurring in $I$, if any, occur in both $A$ and $B$ and such that $A \supset I$ and $I \supset B$ are theorems of $X$. \hfill $\square$

Observe that the proof is constructive in that Lemma 4.3 gives a procedure extract an interpolant for $A \supset B$ from a given derivation of $A \Rightarrow B$. Furthermore the proof is purely proof-theoretic in that it makes no use of model-theoretic notions.

Craig’s theorem is often -- e.g. in [13] for an extension of classical logic -- stated in the following stronger version:

If $A \supset B$ is a theorem of the logic $X$, then

1. If $A$ and $B$ share some propositional variable, there is a formula $I$, which contains propositional variables common to $A$ and $B$ only, such that both $A \supset I$ and $I \supset B$ are theorems of $X$;

2. Else either $\neg A$ or $B$ is a theorem of $X$.

But the second condition doesn’t hold for modal and deontic logics where at least one of $N := \Box \top$ and $D^\perp := \Diamond \top$ is not a theorem. To wit, let $p$ and $q$ be two different propositional variables, it holds that $E \vdash \Box(p \supset p) \supset \Box(q \supset q)$, but neither $\neg \Box(p \supset p)$ nor $\Box(q \supset q)$ is a theorem of $E$ (the same holds in $M$, $R$, and in their deontic extensions). In Fig. 3 we give the construction of an $E$-interpolant of $\Box(p \supset p) \supset \Box(q \supset q)$.

Among the deontic logics considered here, the stronger version of Craig’s theorem holds only for $\text{END}^{\perp(\Diamond)}$, $\text{MND}^{\perp(\Diamond)}$, and $\text{KD}$, as shown by the following

Corollary 4.4. Let $\text{XD}$ be one of $\text{END}^{\perp(\Diamond)}$, $\text{MND}^{\perp(\Diamond)}$, and $\text{KD}$. If $A \supset B$ is a theorem of $\text{XD}$ and $A$ and $B$ share no propositional variable, then either $\neg A$ or $B$ is a theorem of $\text{XD}$.

Proof. Suppose that $\text{XD} \vdash A \supset B$ and that $A$ and $B$ share no propositional variable, then the interpolant $I$ is constructed from $\perp$ and $\top$ by means of classical and deontic operators. Whenever $D^\perp$ and $N$ are theorems of $\text{XD}$, we have that $\Diamond \top \leftrightarrow \top, \Box \top \leftrightarrow \top, \Diamond \bot \leftrightarrow \bot$, and $\Box \bot \leftrightarrow \bot$ are theorems of $\text{XD}$. Hence, the interpolant $I$ is (equivalent to) either $\perp$ or $\top$. In the first case $\text{XD} \vdash \neg A$ and in the second one $\text{XD} \vdash B$. \hfill $\square$

As noted in [3, p. 298], Corollary 4.4 is a Halldén-completeness result. A logic $X$ is Halldén-complete if, for every formulas $A$ and $B$ that share no propositional variable, $X \vdash A \lor B$ if and only if $X \vdash A$ or $X \vdash B$. All the modal and deontic logics considered here, being based on classical logic, are such that $A \supset B$ is equivalent to $\neg A \lor B$. Thus the deontic logics considered in Corollary 4.4 are Halldén-complete, whereas all other non-normal logics for which we have proved interpolation are Halldén-incomplete since, as noted above, they don’t satisfy Corollary 4.4.
5 Related Works and conclusion

We conclude by sketching here the Gentzen-style calculi for non-normal logics presented by Lavendhomme and Lucas[10] and those presented by Indrzejczak [8, 9].

In [10] set-based sequent calculi for the logics in Figure 1 are introduced and a decision procedure based on these calculi is given. The modal rules considered in [10] are like the rules LR-E, LR-M, LR-C, LR-R, and R-N given in Table 4 save that (i) the ones in [10] don’t have contexts in their conclusion and, therefore, weakening has to be taken as a primitive (and non-eliminable) rule of inference; and (ii) sequents are defined as pairs of sets of formulas instead of as pairs of multisets of formulas. Having defined sequent as sets, the rule of contraction is implicitly built into the rules in [10] and cannot be expressed as an independent rule. Hence contraction cannot be eliminated from the calculi in [10]. Moreover [10] shows the admissibility of cut only with respect to (calculi containing) rules analogous to LR-E and LR-M and does not consider rules for the deontic axioms. Finally, the decision procedure for non-normal logics given in [10] is based on a model-theoretic inversion technique so that it is possible to define a root-first procedure that outputs a derivation for all valid sequents and a finite countermodel for all invalid ones. If we set aside the model-theoretic make-up, the decision procedure in [10] is like the one in Definition 3.7 save that backtracking is replaced by the parallel application of all possible instances of modal rules.

In [8, 9] multiset-based sequent calculi for the non-normal logics E(N) and M(N), as well as for their extension with the axioms $D^\Diamond$ (as well as $T$, $4$, $5$, and $B$) are given. The rules LR-E, LR-M, and R-N are defined exactly as in Table 4, but the deontic axiom $D^\Diamond$ is expressed by the following rule:

$$A, B \Rightarrow (\Rightarrow A, B)$$

where the right premiss is present when we are working over LR-E and we don’t want to have $\Rightarrow \neg \Box \bot$ among the derivable sequents, and it has to be omitted when we work over LR-M (or R-N) and/or when we want to have $\Rightarrow \neg \Box \bot$ derivable. In the calculi in [8, 9] weakening and contraction are taken as primitive rules and not as admissible one as we did here. Even if it is easy to show that weakening is eliminable from the calculi in [8, 9], contraction cannot be eliminated because the rule $D-2$ has exactly two principal formulas and, therefore, it is not possible to permute contraction up with respect to instances of rule $D-2$ as we did for $L-D^\Diamond$ in Theorem 3.5. The presence of a non-eliminable rule of contraction makes the elimination of cut more problematic: in most cases we cannot eliminate the cut directly, but we have to consider the rule known as multicut, see [16, p. 88]. This shows how an apparently minor change in the formulation of some rule can have major effects on the structural rules of the calculus.

From the perspective of structural proof analysis, the calculi considered here are better behaved than the ones considered in [10, 8, 9] because here we have proved that weakening and contraction are height-preserving admissible and we have given a syntactic proof of cut elimination for all logics considered. In particular, contraction, which here is height-preserving admissible, cannot be eliminated from the non-normal calculi of [10, 8, 9]. Given that, as it is well known, contraction can be as bad as cut for root-first proof search – we may continue to duplicate some formula forever – we believe this is a substantial improvement.

To our knowledge in the literature there is no other systematic proof-theoretic study of interpolation in non-normal logics. In [3, Chap(s). 3.8 and 6.6] a constructive proof of Craig’s interpolation theorem (and also of some stronger interpolation theorem such as Lyndon’s one) is given for the modal logics K and R, and for some of their extensions, including the deontic ones, but the proof makes use of model-theoretic notions. A proof of interpolation by the
Maehara-technique in the logic KD is given in [23]. For a thorough study of interpolation in modal logics we refer the reader to [4]. A model theoretic proof of interpolation for E is given in [7], and a coalgebraic proof of (uniform) interpolation for M is given in [21]. We have not been able to prove interpolation for calculi containing LR-C (and L-D°C) and, as far as we know, it is still an open problem whether the corresponding non-normal logics have the interpolation property or not.

Thanks. Thanks are due to Tiziano Dalmonte for many helpful suggestions.

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