MANIFOLDS WHICH ARE IVANOV-PETROVA OR $k$-STANILOV

P. GILKEY, S. NIKČEVIĆ, AND V. VIDEV

Abstract. We present some examples of curvature homogeneous pseudo-Riemannian manifolds which are $k$-spacelike Jordan Stanilov.

1. Introduction

In considering the spectral geometry of the Riemann curvature tensor, one studies when a certain natural operator associated to the curvature has constant Jordan normal form on the natural domain of definition. In this brief note, we consider two such operators – the skew-symmetric curvature operator $R$ and a higher order generalization $\Theta$. We begin by recalling some basic definitions.

1.1. The algebraic context. Let $\mathcal{V} := (V, g, R)$ be a model space where $V$ is a finite dimensional real vector space which is equipped with a non-degenerate inner product $g$ of signature $(p,q)$, and where $R$ is an algebraic curvature tensor on $V$, i.e. $R \in \otimes^4 V^*$ satisfies the usual curvature symmetries:

\begin{align}
R(x, y, z, w) &= R(z, w, x, y) = -R(y, x, z, w), \\
R(x, y, z, w) + R(y, z, x, w) + R(z, x, y, w) &= 0.
\end{align}

Let $R(x, y)$ be the associated curvature operator; it is characterized by the identity:

$R(x, y, z, w) = g(R(x, y)z, w)$.

Let $\tilde{\text{Gr}}_{k,\pm}(V, g)$ (resp. $\text{Gr}_{k,\pm}(V, g)$) be the Grassmannians of oriented (resp. unoriented) spacelike (+) and timelike (−) $k$-planes in $V$. Let $\{e_1, e_2\}$ be an oriented orthonormal basis for $\pi \in \tilde{\text{Gr}}_{k,\pm}(V, g)$. The skew-symmetric curvature operator

$R(\pi) := R(e_1, e_2)$

was introduced by Stanilov in 1990 – see the discussion in Ivanova and Stanilov [11]; it is independent of the particular oriented orthonormal basis chosen for $\pi$. One says $\mathcal{V}$ has constant spacelike (resp. timelike) rank $r$ if $\text{Rank}\{R(\pi)\} = r$ for any oriented spacelike (resp. timelike) 2 plane $\pi$ of $V$. One says $\mathcal{V}$ is spacelike (resp. timelike) Jordan Ivanov-Petrova if the Jordan normal form of $R$ is constant on $\tilde{\text{Gr}}_{2,\pm}(V, g)$. Clearly if $\mathcal{V}$ is spacelike (resp. timelike) Jordan Ivanov-Petrova, then $\mathcal{V}$ has constant spacelike (resp. timelike) rank.

There is a higher order analogue which was introduced Stanilov [14, 15]. If $\{e_1, ..., e_k\}$ is an orthonormal basis for $\pi \in \text{Gr}_{k,\pm}(M, g)$, then the higher order curvature operator is defined by setting:

$\Theta(\pi) := \sum_{i,j} R(e_i, e_j) R(e_i, e_j)$.

This self-adjoint operator is, similarly, independent of the particular orthonormal basis chosen for $\pi$. One says $\mathcal{V}$ is $k$–spacelike (resp. $k$–timelike) Jordan Stanilov if

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the Jordan normal form of \( \Theta \) is constant on \( Gr_{k,\pm}(V,g) \); see also [17] for further details. Up to a suitable normalizing factor,

\[
\Theta(\pi) = \int_{\xi \in Gr_2(\pi)} R(\xi)^2 d\xi
\]

so the higher order curvature operator can be regarded as an average of the square skew-symmetric curvature operator. It is necessary to square \( R(\cdot) \) changes sign if the orientation of \( \pi \) is reversed.

Note that one could in fact define \( R \) (resp. \( \Theta \)) for any non-degenerate oriented 2 plane (resp. non-degenerate unoriented \( k \) plane); we shall restrict ourselves to the spacelike and the timelike planes in the interests of simplicity.

1.2. The geometric context. Let \((M,g)\) be a pseudo-Riemannian manifold of signature \((p,q)\) and dimension \(m = p + q\). Let \( R \) be the Riemann curvature of the Levi-Civita connection. We say that \((M,g)\) is spacelike (resp. timelike) Jordan Ivanov-Petrova if \( T_P := (T_PM, g_P, R_P) \) is spacelike (resp. timelike) Jordan Ivanov-Petrova for every point \( P \) of \( M \). Similarly, we say that \((M,g)\) is \( k \)-spacelike (resp. \( k \)-timelike) Jordan Stanilov if \( T_P \) is \( k \)-spacelike (resp. \( k \)-timelike) Jordan Stanilov for every point \( P \) of \( M \). In both contexts, note that the Jordan normal form is allowed to vary with the point in question.

In the Riemannian setting \((p = 0)\), the Jordan normal form is determined by the eigenvalue structure and, as every \( k \) plane is spacelike, we shall drop the qualifiers ‘spacelike’ and ‘Jordan’. This is not true in the higher signature context which is why we focus on the Jordan normal form, i.e. the conjugacy class, instead of only on the eigenvalue structure.

1.3. Ivanov-Petrova tensors and manifolds. One has the following result, which is due to Gilkey, Leahy, and Sadofsky [4] and Gilkey [2] in the Riemannian setting \((p = 0)\), which was generalized by Zhang [18, 19] to the Lorentzian \((p = 1)\) setting, and which was extended by Stavrov [16] to the higher signature setting:

**Theorem 1.1.** Let \( V \) be a model space with constant spacelike rank \( r \).

1. If \( p = 0 \) and if \( q \neq 3, 4, 7 \), then \( r = 2 \).
2. If \( q \geq 11 \), if \( 1 \leq p \leq (q - 6)/4 \), and if the set \( \{ q, q + 1, \ldots, q + p \} \) does not contain a power of \( 2 \), then \( r = 2 \).

This result is important as one has the following classification result [4, 8]. Let \( g_V \) be a metric on a finite dimensional real vector space \( V \). If \( \phi \) is a self-adjoint linear map of \( V \), then we define an algebraic curvature tensor \( R_\phi \) on \( V \) by setting:

\[
R_\phi(x, y)z := g_V(\phi y, z)\phi x - g_V(\phi x, z)\phi y.
\]

**Theorem 1.2.** Let \( q \geq 5 \). The following assertions are equivalent:

1. The model space \( V \) is spacelike rank 2 Jordan Ivanov-Petrova.
2. There exists \( C \neq 0 \) and a self-adjoint map \( \phi \) of \( V \) so \( R = cR_\phi \) where one of the following 3 conditions on \( \phi \) holds:
   a. \( \phi \) is an isometry of \((V, g)\), i.e. \( g(\phi x, \phi y) = g(x, y) \forall x, y \in V \).
   b. \( \phi \) is a para-isometry of \((V, g)\), i.e. \( g(\phi x, \phi y) = -g(x, y) \forall x, y \in V \).
   c. \( \phi^2 = 0 \) and \( \ker \phi \) contains no spacelike vectors.

In the metric setting, one has [4, 9, 10]:

**Theorem 1.3.** Let \((M, g)\) be a connected spacelike Jordan Ivanov-Petrova pseudo-Riemannian manifold of signature \((p,q)\). Assume either \((p,q) = (0,4)\) or that \( q \geq 5 \). Assume that \( R(x) \) is not nilpotent for at least one spacelike 2 plane in \( TM \) and that \( T_P \) has spacelike rank 2 for all \( P \in M \). Let \( R = cR_\phi \) be as in Theorem 1.2 where \( \phi = \phi(P) \). Then \( \phi \) is an isometry, \( \phi^2 = \text{Id} \), and one of the following 2 cases holds:

1. \( \phi = \pm \text{Id} \) and \((M,g)\) has constant sectional curvature.
Theorem 1.4. Let \((M, g)\) be a connected spacelike Jordan Ivanov-Petrova pseudo-Riemannian manifold of signature \((p, q)\). Assume that \(M, g\) is not nilpotent for at least one spacelike 2-plane in \(TM\) and that \(T_P\) has spacelike rank 2 for all \(P \in M\). Then

1. \((M, g)\) is \(k\)-spacelike Jordan Stanilov for any \(2 \leq k \leq q\).
2. \((M, g)\) is \(k\)-timelike Jordan Stanilov for any \(2 \leq k \leq p\).

We will also establish the following partial converse in the Riemannian setting.

Theorem 1.5. Let \((M, g)\) be a connected Riemannian manifold of dimension \(m\) where \(m \neq 3, 7\). If \((M, g)\) is 2-Stanilov, then \((M, g)\) is Ivanov-Petrov-a with constant spacelike rank 2.

Theorem 1.6. Assume \(\text{Rank}(H) \geq 2\). Then \((M_f, g_f)\) is:

1. spacelike Jordan Ivanov-Petrov-a if and only if \(\text{det}(H)\) is never zero.
2. timelike Jordan Ivanov-Petrov-a if and only if \(\text{det}(H)\) is never zero.
3. \(k\)-spacelike and \(k\)-timelike Jordan Stanilov for \(2 \leq k \leq p\) for any \(f\).

Let \(s \geq 2\). In Section 3 we exhibit family of manifolds of signature \((2s, s)\) which are spacelike Jordan Ivanov-Petrov-a but not timelike Jordan Ivanov-Petrov-a. Thus the notions spacelike and timelike are distinct. This family provides the first example of spacelike Ivanov-Petrov-a manifolds of spacelike rank 4. The manifolds will be \(k\)-spacelike Jordan Stanilov for all admissible \(k\); they will be \(k\)-timelike Jordan Stanilov only for \(k = 2s\).

Let \((\vec{u}, \vec{v})\) be coordinates on \(\mathbb{R}^{2s}\) where \(\vec{u} = (u_1, \ldots, u_s), \vec{v} = (t_1, \ldots, t_s)\), and \(\vec{v} = (u_1, \ldots, u_s)\). We define a metric of signature \((2s, s)\) on \(M_{2s} := \mathbb{R}^{2s}\) by setting:

\[
\begin{align*}
g_{3s}(\partial_{i_1}^u, \partial_{i_2}^v) &= -2\delta_{ij} \sum_{1 \leq k \leq s} u_k t_k, \\
g_{3s}(\partial_{i_1}^v, \partial_{i_2}^v) &= g_{3s}(\partial_{i_1}^v, \partial_{i_2}^v) = 0, \\
g_{3s}(\partial_{i_1}^u, \partial_{i_2}^u) &= g_{3s}(\partial_{i_1}^u, \partial_{i_2}^u) = \delta_{ij}, \\
g_{3s}(\partial_{i_1}^v, \partial_{i_2}^u) &= g_{3s}(\partial_{i_1}^v, \partial_{i_2}^u) = 0. \\
\end{align*}
\]

One says that \((M, g)\) is curvature homogeneous if there exists a model \(\mathcal{V}\) and isomorphisms \(\psi_P : T_P \rightarrow \mathcal{V}\) for all \(P \in M\); see [12, 13] for further details. These manifolds were first introduced in [6] to provide examples of curvature homogeneous spacelike Oxserman manifolds which are not locally homogeneous and where the Jacobi operator was nilpotent of order 3.

Theorem 1.7. The manifolds \((M_{3s}, g_{3s})\) are...
(1) Spacelike rank 4 Jordan Ivanov-Petrova.
(2) Not timelike Jordan Ivanov-Petrova.
(3) $k$-Spacelike Jordan Stanilov for $2 \leq k \leq s$.
(4) $k$-Timelike Jordan Stanilov if and only if $k = 2s$.

Here is a brief guide to this paper. In Section 2, we establish Theorems 1.4 and 1.6. In Section 3, we review results of [5] to sketch the proof of Theorem 1.6. Section 4 comprises the body of this paper. We first determine the curvature tensor of the metric in question. Then we show the space is curvature homogeneous and determine the model space. We complete the proof of Theorem 1.7 by establishing the corresponding assertions for the model space.

2. Relationships between the Stanilov and Ivanov-Petrova condition

Proof of Theorem 1.4 Let $(M, g)$ be a connected spacelike Jordan Ivanov-Petrova pseudo-Riemannian manifold of signature $(p, q)$ where $q \geq 5$ or where $(p, q) = (0, 4)$. Assume that $R(\pi)$ is not nilpotent for at least one spacelike 2 plane in $TM$ and that $T_P$ has spacelike rank 2 for all $P \in M$. We may then use Theorem 1.3 to see $R = cR_0$ where $c^2 = 1$ and $\phi$ is an isometry. If $\{e_1, e_2\}$ is an orthonormal basis for an oriented spacelike 2 plane $\pi$, then $R(e_1, e_2)x = c(g(\phi e_2, x)\phi e_1 - g(\phi e_1, x)\phi e_2)$.

Let $\rho_\pi$ be orthogonal projection on $\pi$. By Display (2.a),

$$R(\pi) : \phi e_2 \to c\phi e_1, \quad R(\pi) : \phi e_1 \to -c\phi e_2, \quad R(\pi) : x \to 0,$$

(2.a)

$$R(\pi)^2 : \phi e_2 \to -c^2\phi e_2, \quad R(\pi)^2 : \phi e_1 \to -c^2\phi e_1, \quad R(\pi)^2 : x \to 0.$$

(2.b)

Let $\pi_k \in \mathrm{Gr}_{k,+}(M, g)$, then we can use Equation (2.b) to obtain

$$\Theta(\pi_k) = -(k - 1)c^2\rho_\pi.$$

This shows $\Theta(\pi_k)$ has constant Jordan normal form and hence $(M, g)$ is k-spacelike Jordan Stanilov. The argument that $(M, g)$ is k-timelike Jordan Stanilov is essentially the same modulo an appropriate change of signs and thus is omitted. □

Proof of Theorem 1.5 Let $(M, g)$ be a Riemannian manifold which is 2-Stanilov. Let $\{\lambda_i(\pi)\}$ be the eigenvalues of $R(\pi)$ for $\pi \in \overline{\mathrm{Gr}}_{2,+}(T_P M)$. Then $\{\lambda_i^2(\pi)\}$ are the eigenvalues of $\Theta(\pi)$. Since these eigenvalues are independent of $\pi$, since $\overline{\mathrm{Gr}}_{2,+}(T_P M)$ is connected, and since the eigenvalues vary continuously, we may conclude that $\Theta(\pi)$ also has constant eigenvalues. Since the Jordan normal form is determined by the eigenvalue structure in the positive definite setting, we can conclude that $(M, g)$ is Ivanov-Petrova. □

3. Stanilov manifolds of neutral signature $(p, p)$

The following manifolds were first introduced in [5] and we follow the discussion there to see the metric $g_f$ of Equation (1.6) is a hypersurface metric. Let $\{U_1, ..., U_p, V_1, ..., V_p, W\}$ be a basis for $\mathbb{R}^{2p+1}$ where $p \geq 2$. Introduce a non-degenerate inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{2p+1}$ by defining:

$$\langle U_i, V_j \rangle = \langle V_j, U_i \rangle = \delta_{ij}, \quad \langle W, W \rangle = 1,$$

$$\langle U_i, U_j \rangle = \langle V_i, V_j \rangle = \langle U_i, W \rangle = \langle V_i, W \rangle = \langle W, U_i \rangle = \langle W, V_i \rangle = 0.$$

Introduce coordinates $(x, y)$ on $\mathbb{R}^p$ where $x = (x_1, ..., x_p)$ and $y = (y_1, ..., y_p)$. Let $f = f(x)$ be a smooth real valued function on $O \subset \mathbb{R}^p$. Define an embedding of $M_f := O \times \mathbb{R}^p$ in $\mathbb{R}^{2p+1}$ by setting

$$\Psi_f(x, y) = \sum_{1 \leq i \leq s} (x_i U_i + y_i V_i) + f(x)W.$$
Let $g_f$ be the induced hypersurface pseudo-Riemannian metric on $M_f$:

$$g_f(\partial^r_i, \partial^r_j) = \partial^r_i f \cdot \partial^r_j f, \quad g_f(\partial^p_i, \partial^p_j) = g_f(\partial^p_i, \partial^p_j) = \delta_{ij}, \quad g_f(\partial^p_i, \partial^p_j) = 0. $$

Let $H_{ij} := \partial^r_i \partial^r_j f$ be the Hessian, let $L$ be the second fundamental form, and let $S$ be the shape operator:

$$(3.a) \quad L(\partial^r_i, \partial^r_j) = H_{ij}, \quad L(\partial^p_i, \partial^p_j) = L(\partial^p_j, \partial^p_i) = 0, \quad L(\partial^p_i, \partial^p_j) = 0,$$

$$g_f(S(\cdot, \cdot) = L(\cdot, \cdot), \quad S(\partial^r_i) = \sum_j H_{ij} \partial^r_j, \quad S(\partial^p_i) = 0. $$

We then have

$$(3.b) \quad R(Z_1, Z_2, Z_3, Z_4) = L(Z_1, Z_4)L(Z_2, Z_3) - L(Z_1, Z_3)L(Z_2, Z_4),$$

$$R(Z_1, Z_2)Z_3 = g_f(S(Z_2), Z_3)S(Z_1) - g_f(S(Z_1), Z_3)S(Z_2).$$

Let $\ell := \text{Rank}(H) = \dim \text{Range}(S)$; by assumption $2 \leq \ell \leq p$.

**Proof of theorem** Let $\mathcal{X} := \text{Span}\{\partial^r_i\}$ and $\mathcal{Y} := \text{Span}\{\partial^p_i\}$. If $Z_1$ and $Z_2$ are arbitrary tangent vectors, then we may use Displays (3.a) and (3.b) to see that

$$R(Z_1, Z_2)\mathcal{X} \subset \mathcal{Y} \quad \text{and} \quad R(Z_1, Z_2)\mathcal{Y} = 0.$$

Consequently $R(Z_1, Z_2)^2 = 0$ and thus $\Theta(\pi) = 0$ for any spacelike or timelike $k$-plane $\pi$. Thus, trivially, $(M_f, g_f)$ is $k$-spacelike and $k$-timelike Jordan Ivanov-Petrov for any admissible $k$.

Since $R(\pi)^2 = 0$ for any oriented spacelike (resp. timelike) 2-plane, the Jordan normal form of $R(\pi)$ is determined by $\text{Rank}(R(\pi))$. Let $\{Z_1, Z_2\}$ be an orthonormal basis for $\pi$. Expand $Z_\mu = X_\mu + Y_\mu$ for $X_\mu \in \mathcal{X}$ and $Y_\mu \in \mathcal{Y}$. Then $R(\pi) = R(X_1, X_2)$ as well. Note that $\text{Rank}(S) = \text{Rank}(S)|_{\mathcal{X}}$. Since $\pi$ is spacelike (resp. timelike), $X_1$ and $X_2$ are linearly independent. We have

$$\text{Rank}(R(\pi)) = \begin{cases} 0 \text{ if } S(X_1) \text{ and } S(X_2) \text{ are linearly dependent,} \\ 2 \text{ if } S(X_1) \text{ and } S(X_2) \text{ are linearly independent.} \end{cases}$$

If $\text{Rank}(S) = p$, then $\{S(X_1), S(X_2)\}$ is a linearly independent set and thus $\text{Rank}(R(\pi)) = 2$; consequently $(M_f, g_f)$ is spacelike (resp. timelike) Jordan Ivanov-Petrov. On the other hand, if $2 \leq \text{Rank}(S) \leq p - 1$, then we may choose spacelike (resp. timelike) 2 planes $\pi_1$ and $\pi_2$ so $\text{Rank}(R(\pi_1)) = 2$ and $\text{Rank}(R(\pi_2)) = 0$ and thus $(M_f, g_f)$ is not spacelike (resp. timelike) Jordan Ivanov-Petrov. \hfill \square

4. $k$-Stanilov manifolds in signature $(2s, s)$

4.1. The curvature tensor of the manifolds $(M_{3s}, g_{3s})$. We adopt the notation of Display (1.d). We begin our study of the manifold $(M_{3s}, g_{3s})$ by showing:

**Lemma 4.1.** Let $R_{3s}$ be the curvature tensor of the pseudo-Riemannian manifold $(M_{3s}, g_{3s})$ defined in Display (1.d). Then the non-zero entries in $R_{3s}$ are, up to the usual $\mathbb{Z}_2$ symmetries of Equation (1.d), given by:

$$R_{3s}(\partial^r_i, \partial^r_j, \partial^u_k) = |u|^2 \quad \text{and} \quad R_{3s}(\partial^u_i, \partial^u_j, \partial^r_k) = 1. $$

**Proof.** Let $i \neq j$. The non-zero Christoffel symbols of the second kind are given by:

$$g_{3s}(\nabla_\partial^r_i \partial^r_j, \partial^u_k) = -u_i, \quad g_{3s}(\nabla_\partial^u_i \partial^r_j, \partial^r_k) = -u_i, \quad g_{3s}(\nabla_\partial^u_i \partial^u_j, \partial^u_k) = -u_k,$$

$$g_{3s}(\nabla_\partial^r_i \partial^u_j, \partial^r_k) = u_j, \quad g_{3s}(\nabla_\partial^u_i \partial^r_j, \partial^u_k) = u_j, \quad g_{3s}(\nabla_\partial^u_i \partial^u_j, \partial^r_k) = -u_j.$$
We may then raise indices to see the non-zero covariant derivatives are given by:
\[
\nabla_{\partial_i} \partial_i^u = -t_i \partial_i^u + \sum_{k \neq i} t_k \partial_k^u - \sum_{1 \leq k \leq s} u_k \partial_k^u,
\]
\[
\nabla_{\partial_i} \partial_j^u = -t_j \partial_i^u - t_i \partial_j^u,
\]
\[
\nabla_{\partial_i} \partial_i^u = \nabla_{\partial_i} \partial_j^u = -u_i \partial_i^u, \text{ and}
\]
\[
\nabla_{\partial_i} \partial_j^u = -u_j \partial_i^u.
\]

We have \(\nabla \partial_i^u = 0\). Thus if at least one \(z_\mu \in \{\partial_i^u\}\), \(R_{3s}(z_1, z_2, z_3, z_4) = 0\). Similarly, if at least two of the \(z_\mu\) belong to \(\{\partial_i^u\}\), then \(R_{3s}(z_1, z_2, z_3, z_4) = 0\). Furthermore \(R_{3s}(\partial_i^u, \partial_j^u, \partial_k^u, *) = 0\) if the indices \(\{i, j, k\}\) are distinct. Finally,
\[
\nabla_{\partial_i} \nabla_{\partial_j} \partial_i^u = -\partial_i^u + |u|^2 \partial_i^u \text{ and } \nabla_{\partial_i} \nabla_{\partial_j} \partial_i^u = 0.
\]

The Lemma now follows. \(\square\)

**Definition 4.2.** Let \(\{U_1, ..., U_s, T_1, ..., T_s, V_1, ..., V_s\}\) be a basis for \(\mathbb{R}^{3s}\) where \(s \geq 2\). Let \(\mathcal{V}_{3s} := (\mathbb{R}^{3s}, g_{3s}, R_{3s})\) where the non-zero entries of the metric \(g_{3s}\) and of the algebraic curvature tensor \(R_{3s}\), up to the usual \(Z_2\) symmetries, are
\[
g_{3s}(U_i, U_i) = g_{3s}(V_i, U_i) = 1, \quad g_{3s}(T_i, T_i) = -1, \quad \text{and} \quad R_{3s}(U_i, U_j, T_i) = 1 \quad \text{for} \quad i \neq j.
\]

Set \(Z^\pm := U_i \pm \frac{1}{2} V_i\). Then \(\text{Span}\{Z_i^+\}\) is a maximal spacelike subspace of \(\mathbb{R}^{3s}\) and \(\text{Span}\{T_i, Z_i^-\}\) is the complementary maximal timelike subspace. Thus \(\mathbb{R}^{3s}\) has signature \((2s, s)\). A basis \(\mathcal{B} = \{\tilde{U}_1, ..., \tilde{U}_s, \tilde{T}_1, ..., \tilde{T}_s, \tilde{V}_1, ..., \tilde{V}_s\}\) for \(\mathbb{R}^{3s}\) is said to be normalized if the relations given above in Display 4.4 hold for \(\mathcal{B}\).

**Lemma 4.3.** \((M_{3s}, g_{3s})\) is curvature homogeneous with model space \(\mathcal{V}_{3s}\).

**Proof.** Fix \(P \in M_{3s}\). Let constants \(\varepsilon_i\) and \(\varrho_i\) be given. We define a new basis for \(T_PM\) by setting:
\[
U_i := \partial_i^u + \varepsilon_i \partial_i^v + \varrho_i \partial_i^w, \quad T_i := \partial_i^f + \varepsilon_i \partial_i^v, \quad \text{and} \quad V_i := \partial_i^v.
\]

Let \(i \neq j\). Since \(g_{3s}(U_i, T_i) = \varepsilon_i - \varepsilon_i = 0\), the possibly non-zero entries of \(g_{3s}\) and \(R_{3s}\) are, up to the usual \(Z_2\) symmetries, given by
\[
g_{3s}(U_i, U_i) = g_{3s}(U_i, V_i) = 1, \quad g_{3s}(T_i, T_i) = -1, \quad g_{3s}(U_i, V_i) = 1, \quad R_{3s}(U_i, U_j, T_i) = 1, \quad \text{and} \quad R_{3s}(U_i, U_j, U_j, U_i) = |u|^2 + 2 \varepsilon_i + 2 \varepsilon_j.
\]

We set
\[
\varepsilon_i := -\frac{1}{4} |u|^2 \quad \text{and} \quad \varrho_i := \frac{1}{2} \varepsilon_i^2 - \frac{1}{4} g_{3s}(\partial_i^u, \partial_i^u).
\]

This ensures that \(g_{3s}(U_i, U_i) = 0\) and \(R_{3s}(U_i, U_j, U_j, U_i) = 0\) and establishes the existence of a basis with the normalizations of Definition 4.2. \(\square\)

Lemma 4.3 shows that the manifold \((M_{3s}, g_{3s})\) is curvature homogeneous; the work of [7] shows it is not locally homogeneous. We shall prove Theorem 1.2 by establishing the corresponding assertions for the model space \(\mathcal{V}_{3s}\).

### 4.2. The skew-symmetric curvature operator

Theorem 1.2a(1,2) will follow from the following result concerning the model space \(\mathcal{V}_{3s}\).

**Lemma 4.4.** Let \(R_{3s}\) be the skew-symmetric curvature operator defined by \(R_{3s}\).

1. If \(\pi\) is an oriented spacelike 2 plane, then \(\text{Rank}\{R_{3s}(\pi)\} = 4\), \(\text{Rank}\{R_{3s}(\pi)\}^2 = 2\), and \(R_{3s}(\pi)^3 = 0\).
2. The model space \(\mathcal{V}_{3s}\) is spacelike rank 4 Jordan Ivanov-Petrova.
3. The model space \(\mathcal{V}_{3s}\) is not timelike Jordan Ivanov-Petrova.
4.3. Definite bilinear form \( \tilde{g} \)

We complete the proof of Theorem 1.7 by showing:

\[
\xi : U_i \to \sum_j \xi_{ij} U_j, \quad \xi : T_i \to \sum_j \xi_{ij} T_j, \quad \text{and} \quad \xi : V_i \to \sum_j \xi_{ij} V_j.
\]

Let \( \pi \) be an oriented spacelike 2 plane. By applying a symmetry of the form given in Equation (4.11), we may suppose that \( \pi = \operatorname{Span}\{X_1, X_2\} \) where

\[
X_1 = U_1 + \sum_{1 \leq i \leq s} \{b_{1i} T_i + c_{1i} V_i\} \quad \text{and} \quad X_2 = U_2 + \sum_{1 \leq i \leq s} \{b_{2i} T_i + c_{2i} V_i\}.
\]

Let \( c := g_{3s}(X_1, X_1)g_{3s}(X_2, X_2) - g_{3s}(X_1, X_2)^2 > 0 \) and let \( E := R_{3s}(X_1, X_2) \). We then have \( R_{3s}(\pi) = \frac{1}{\sqrt{c}} E \). Let \( i \geq 3 \). There exist real numbers \( \varepsilon_{ij} = \varepsilon_{ij}(b,c) \) and \( \quad \varepsilon_{ij} = \rho_{ij}(b,c) \), which play no role in the subsequent development, so that

\[
\Xi : U_i \to T_2 + \sum_{1 \leq k \leq s} \varepsilon_{1k} V_k, \quad \Xi : T_i \to -V_2, \quad \Xi : V_i \to 0,
\]

(4.5)

\[
\Xi : U_2 \to -T_1 + \sum_{1 \leq k \leq s} \varepsilon_{2k} V_k, \quad \Xi : T_1 \to V_1, \quad \Xi : V_1 \to 0,
\]

\[
\Xi : U_1 \to \varepsilon_{1i} V_1 + \varepsilon_{2i} V_2, \quad \Xi : T_i \to 0, \quad \Xi : V_i \to 0.
\]

Assertion (1) now follows; Assertion (2) follows from Assertion (1). Let

\[
\pi_1 := \operatorname{Span}\{T_1, T_2\} \quad \text{and} \quad \pi_2 := \operatorname{Span}\{Z_1^-, Z_2^-\}
\]

be timelike 2 planes with \( R_{3s}(\pi_1) = 0 \) and \( R_{3s}(\pi_2) \neq 0 \). Assertion (3) follows. \( \square \)

4.3. The higher order curvature operator of \( V_{3s} \).

Define a positive semi-definite bilinear form \( \tilde{g} \) on \( \mathbb{R}^{3s} \) by setting

\[
\tilde{g}(U_i, U_j) = \delta_{ij}, \quad \tilde{g}(U_i, V_j) = \tilde{g}(V_j, U_i) = 0, \quad \tilde{g}(U_i, T_j) = \tilde{g}(T_j, U_i) = 0, \quad \tilde{g}(T_i, T_j) = 0, \quad \tilde{g}(T_i, V_j) = \tilde{g}(V_j, T_i) = 0, \quad \tilde{g}(V_i, V_j) = 0.
\]

This inner product is invariant under the action of \( O(s) \) described in Equation (4.14). If \( \pi \) is a linear subspace of \( \mathbb{R}^{3s} \), set

\[
\ell(\pi) := \operatorname{Rank}\{\tilde{g}|_{\pi}\}.
\]

We complete the proof of Theorem 1.7 by showing:

**Lemma 4.5.** Let \( \Theta_{3s} \) be the higher order curvature operator defined by \( V_{3s} \).

1. If \( \pi \) is a spacelike \( k \)-plane, then \( \operatorname{Rank}\{\Theta_{3s}(\pi)\} = k \) and \( \Theta_{3s}(\pi)^2 = 0 \).

2. If \( 2 \leq k \leq s \), then \( \Theta_{3s} \) is \( k \)-spacelike Jordan Stanilov.

3. Let \( \pi \) be a timelike 2 plane. If \( \ell(\pi) \geq 2 \), then \( \operatorname{Rank}\{\Theta_{3s}(\pi)\} = \ell(\pi) \) and \( \Theta_{3s}(\pi)^2 = 0 \). If \( \ell(\pi) \leq 1 \), then \( \Theta_{3s}(\pi) = 0 \).

4. \( V_{3s} \) is \( k \)-timelike Jordan Stanilov if and only if \( k = 2s \).

**Proof.** Fix \( 2 \leq k \leq s \). Let indices \( \alpha \) and \( \beta \) range from 1 through \( k \). Let \( \pi \) be a spacelike \( k \)-plane in \( \mathbb{R}^{3s} \). We diagonalize the quadratic form \( \tilde{g}|_{\pi} \) with respect to the positive definite quadratic form \( g|_{\pi} \) to choose an orthonormal basis \( \{X_\alpha\} \) for \( \pi \) so

\[
\tilde{g}(X_\alpha, X_\beta) = a_{\alpha\beta} a_{\alpha\beta} a_{\alpha\beta} \text{ where } a_{\alpha} > 0.
\]

By replacing \( \pi \) by \( \xi \cdot \pi \) for an appropriately chosen symmetry \( \xi \) in \( O(s) \), we may assume without loss of generality that

\[
X_\alpha = a_{\alpha} U_\alpha + \sum_{1 \leq i \leq s} \{b_{\alpha i} T_i + c_{\alpha i} V_i\} \quad \text{for} \quad 1 \leq \alpha \leq k,
\]

where the real numbers \( b_{\alpha i} \) and \( c_{\alpha i} \) play no role in the subsequent discussion. Let \( \alpha \neq \beta \). Let \( 1 \leq i \leq s \). We use Equation (4.5) to see that:

\[
R_{3s}(X_\alpha, X_\beta)^2 U_i = \begin{cases} a_{\alpha i}^2 a_{\beta i}^2 V_i & \text{if } i = \alpha, \beta, \\ 0 & \text{otherwise}. \end{cases}
\]

\[
R_{3s}(X_\alpha, X_\beta)^2 T_i = 0, \quad \text{and} \quad R_{3s}(X_\alpha, X_\beta)^2 V_i = 0.
\]
Remark: Let \( \vec u \) follow. Let \( V \) be a timelike 2-plane. We apply exactly the same diagonalization argument to see that, after replacing \( \pi \) for generic members of the family. These manifolds are not locally homogeneous on \( M \).

It was shown in [7] that these spaces are curvature homogeneous with model space \( M \) on \( O(\times k) \).

Since \( \Theta(3s) \) is not \( k \)-timelike Jordan Stanilov. If \( 2 \leq k \leq 2s-1 \), then set:

\[
\pi_1 := \begin{cases} 
\text{Span}\{T_1, ..., T_k\} & \text{if } 2 \leq k \leq s, \\
\text{Span}\{T_1, ..., T_s, Z_1, ..., Z_{k-s}\} & \text{if } s < k < 2s,
\end{cases}
\]

\[
\pi_2 := \begin{cases} 
\text{Span}\{T_1, ..., T_{k-2}, Z_1, Z_2\} & \text{if } 2 \leq k \leq s, \\
\text{Span}\{T_1, ..., T_s-1, Z_1, ..., Z_{k+1-s}\} & \text{if } s < k < 2s.
\end{cases}
\]

Then

\[
\text{Rank } \Theta(3s)(\pi_1) = \begin{cases} 
0 & \text{if } 2 \leq k \leq s+1, \\
k-s & \text{if } s+1 \leq k \leq 2s,
\end{cases}
\]

\[
\neq \text{Rank } \Theta(3s)(\pi_2) = \begin{cases} 
2 & \text{if } 2 \leq k \leq s+1, \\
k+1-s & \text{if } s+1 \leq k \leq 2s.
\end{cases}
\]

This shows \( \mathcal{V}_{3s} \) is not \( k \)-timelike Jordan Stanilov.

4.4. Remark: One can generalize the pseudo-Riemannian manifold \( (M_{3s}, g_{3s}) \) as follows. Let \( \vec u := (u_1, ..., u_s) \), \( \vec t := (t_1, ..., t_s) \), and \( \vec v := (v_1, ..., v_s) \) give coordinates \((\vec u, \vec t, \vec v)\) on \( \mathbb{R}^{3s} \) for \( s \geq 2 \). Let \( F(\vec u) := f_1(u_1) + ... + f_s(u_s) \) be a smooth function on an open subset \( \mathcal{O} \subset \mathbb{R}^s \). Define a pseudo-Riemannian metric \( g_F \) of signature \((2s, s)\) on \( M_F := \mathcal{O} \times \mathbb{R}^{2s} \) whose non-zero components are given by:

\[
g_F(\partial^u_i, \partial^u_j) = -2F(\vec u) - 2 \sum_i u_it_i, \\
g_F(\partial^u_i, \partial^F_j) = g_F(\partial^F_i, \partial^u_j) = 1, \\
g_F(\partial^F_i, \partial^F_j) = -1.
\]

It was shown in [7] that these spaces are curvature homogeneous with model space \( \mathcal{V}_{3s} \) and thus arguments given above show that the conclusions of Theorem 4.7 apply to all of the manifolds in this family. These manifolds are not locally homogeneous for generic members of the family.
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PG: Mathematics Department, University of Oregon, Eugene Or 97403 USA.
Email: gilkey@darkwing.uoregon.edu

SN: Mathematical Institute, Sanu, Knez Mihailova 35, p.p. 367, 11001 Belgrade, Yugoslavia. Email: stanan@mi.sanu.ac.yu

VV: Mathematics Department, Thracian University, University Campus, 6000 Stara Zagora, Bulgaria. Email: videv@uni-sz.bg