ON THE $L^2$ STABILITY OF SHOCK WAVES FOR FINITE-ENTROPY SOLUTIONS OF BURGERS

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ABSTRACT. We prove $L^2$ stability estimates for entropic shocks among weak, possibly non-entropic, solutions of scalar conservation laws $\partial_t u + \partial_x f(u) = 0$ with strictly convex flux function $f$. This generalizes previous results by Leger and Vasseur, who proved $L^2$ stability among entropy solutions. Our main result, the estimate

$$\int \mathbb{R} \left| u(t, \cdot) - u_0^{\text{shock}}(\cdot - x(t)) \right|^2 dx \leq \int \mathbb{R} \left| u_0 - u_0^{\text{shock}} \right|^2 + C \mu([0,t] \times \mathbb{R}),$$

for some Lipschitz shift $x(t)$, includes an error term accounting for the positive part of the entropy production measure $\mu = \partial_t (u^2/2) + \partial_x q(u)$, where $q'(u) = uf'(u)$. Stability estimates in this general non-entropic setting are of interest in connection with large deviation principles for the hydrodynamic limit of asymmetric interacting particle systems. Our proof adapts the scheme devised by Leger and Vasseur, where one constructs a shift $x(t)$ which allows to bound from above the time-derivative of the left-hand side. The main difference lies in the fact that our solution $u(t, \cdot)$ may present a non-entropic shock at $x = x(t)$ and new bounds are needed in that situation.

1. INTRODUCTION

We consider bounded weak (not necessarily entropy) solutions of Burgers’ equation

$$\partial_t u + \partial_x \frac{u^2}{2} = 0,$$

or more generally a scalar conservation law

$$(1.1) \quad \partial_t u + \partial_x f(u) = 0, \quad t \geq 0, \quad x \in \mathbb{R},$$

with uniformly convex flux $f'' \geq \alpha > 0$. Let us recall that for any entropy-flux pair $(\eta, q)$ i.e. $\eta'' \geq 0$ and $q' = \eta f'$, the corresponding entropy production of a bounded weak solution $u$ is the distribution

$$(1.2) \quad \mu_\eta = \partial_t \eta(u) + \partial_x q(u).$$

In the special case $\eta(t) = t^2/2$ we will drop the subscript $\eta$ and simply write

$$(1.3) \quad \mu = \partial_t \frac{u^2}{2} + \partial_x q(u), \quad q(v) = \int_0^v t f'(t) dt.$$

For smooth solutions the entropy production $\mu_\eta$ is always zero, but smooth long-time solutions do not exist in general. Entropy solutions are weak solutions whose entropy production is nonpositive, i.e. $\mu_\eta \leq 0$ for all convex entropies $\eta$. Kružkov introduced this concept in [9] and showed that for any bounded initial condition $u_0(x)$ there exists a unique entropy solution.

Date: November 10, 2020.
Finite-entropy solutions. Here in contrast we consider weak solutions whose entropy productions do not necessarily have a sign. Such solutions are not uniquely determined by their initial conditions: they will in general deviate from the unique entropy solution, and the present work addresses the question of estimating this deviation. Our motivation comes from the study of large deviation principles for the hydrodynamic limit of asymmetric interacting particle systems [6, 20], where it is crucial to control how much a general weak solution can deviate from the entropy solution.

To give more details about this issue, we focus on a continuum variant introduced in [16, 2]. There, the question boils down to describing the variational convergence (Γ-convergence) of functionals of the form

\[ E_\varepsilon(u) = \frac{1}{\varepsilon} \int \left[ \varepsilon \partial_x u - \partial_x^{-1}(\partial_t u + \partial_x f(u)) \right]^2 dx dt, \]

in the regime \( \varepsilon \to 0^+ \). The same problem is considered in [17] (with the motivation of providing a variational point of view on the vanishing viscosity method). Limits \( u = \lim u_\varepsilon \) of sequences of bounded energy \( E_\varepsilon(u_\varepsilon) \leq C \) are weak solutions of (1.1), but not necessarily entropy solutions. They belong to the wider class that we will call here finite-entropy solutions: bounded weak solutions of (1.1) such that

(1.4) \( \mu_\eta \) is a Radon measure for all convex \( \eta \),

where \( \mu_\eta \) is the entropy production defined in (1.2). The conjectured limiting energy \( E_0(u) \) is the negative part \( \mu_-([0,T] \times \mathbb{R}) \) of their entropy production, but a proof of this fact is still lacking.

Specifically, the missing part is the upper bound: given a finite-entropy solution \( u \), can one construct an approximating sequence \( u_\varepsilon \to u \) in \( L^1 \) such that \( \lim \sup E_\varepsilon(u_\varepsilon) \leq E_0(u) ? \)

Very similar questions arise in relation with micromagnetics models (the so-called Aviles-Giga energy), we refer to the introduction of [10] for more details. What makes this question hard is the lack of fine knowledge on finite-entropy solutions. Unlike entropy solutions, they are not necessarily of bounded variation (BV). Only very recently E. Marconi [14, 15] proved that their entropy production is a one-dimensional rectifiable measure. This rectifiability result is a remarkable achievement, but it seems that solving the upper bound problem requires other new ideas.

To the best of our knowledge, only two upper bound constructions are available in the literature, with restrictive assumptions on the finite-entropy solution \( u \). The first construction in [17] requires \( u \) to be BV, and the approximating sequence is obtained by mollifying \( u \) and using the fine properties of BV functions. The second construction in [2] is based on approximation by vanishing viscosity, which converges in open regions where the entropy production is \( \leq 0 \). If regions of negative and positive entropy production are not “well separated” this construction breaks down, for want of a good estimate on the distance between \( u \) and entropy solutions when the entropy production changes sign.

In this spirit, the only estimate [10] we are aware of is not homogeneous:

\[ \int_{[0,1]^t \times [-1,1]^x} |u - u^{\text{ent}}|^4 \leq C \mu_+([0,2]^t \times [-2,2]^x) \gamma \quad \text{for some } \gamma \in (0,1), \]

where \( u^{\text{ent}} \) is the entropy solution with initial data \( u_0 \) and \(|u| \leq 1\). If one applies (a rescaled version of) this estimate in small regions where \( \mu_+ \) is small, after summing over all regions the right-hand side may become very large because of the small exponent \( \gamma \). As a consequence, this estimate cannot be used to remove the main restriction (that the
regions where the entropy production has a constant sign must be well separated) in the approximation scheme of [2]. One would rather need an estimate that is homogeneous, hence amenable to summing rescaled applications of it.

In this work we prove a first step towards such estimate, beginning with the distance of $u$ to entropic shocks: if a solution $u$ starts close to a shock and $\mu_+$ is small, then $u$ remains close to a shock, and this is quantified via a homogeneous estimate. More precisely, our main result takes the form of an $L^2$ stability estimate for entropic shocks. For entropy solutions this question was addressed in [12, 13] using relative entropy methods. Here we generalize their methods to solutions whose entropy production does not necessarily have a sign. Loosely stated, we prove (Theorem 1.1)

\[ \int |u(t) - \text{shock}|^2 \, dx \leq \int |u(0) - \text{shock}|^2 \, dx + C \int_0^t \int \mu_+ (dt, dx), \]

where the shock at time $t$ is a shift of the initial shock.

**Strong and very strong traces.** As in [13, 7, 8], in order to implement the relative entropy method we need to assume that $u$ has traces on Lipschitz curves, in a strong enough sense. From [21], it is known that finite-entropy solutions have traces which are reached strongly in $L^1$. We call this the strong trace property, precisely defined as follows. A bounded function $u: [0, T] \times \mathbb{R} \to \mathbb{R}$ satisfies the strong trace property if for any Lipschitz path $x: [0, T] \to \mathbb{R}$ there exist traces $t \mapsto u(t, x(t) \pm)$ on each side of $x(t)$, such that

\[(1.5) \quad \text{ess lim}_{y \to 0^+} \int_0^T |u(t, x(t) \pm y) - u(t, x(t) \pm)| \, dt = 0.\]

In [21] entropy solutions are considered, but the proof there uses only a kinetic formulation which is also valid for finite-entropy solutions [4]. The results of [21] also include traces along constant time lines, implying that (for an a.e. representative)

\[(1.6) \quad [0, T] \ni t \mapsto u(t, \cdot) \in L^1_{loc} \quad \text{is continuous},\]

whenever $u$ is a finite-entropy solution of (1.1).

Unfortunately the strong trace property turns out not to be enough for our purposes, and as in [13, 7, 8] we will in fact require an even stronger property. We say that a bounded function $u: [0, T] \times \mathbb{R} \to \mathbb{R}$ satisfies the very strong trace property if for any Lipschitz path $x: [0, T] \to \mathbb{R}$ there exist traces $t \mapsto u(t, x(t) \pm)$ such that (for an a.e. representative of $u$)

\[(1.7) \quad \text{ess lim}_{y \to 0^+} u(t, x(t) \pm y) = u(t, x(t) \pm) \quad \text{for a.e. } t \in [0, T].\]

By dominated convergence the very strong trace property does imply the strong trace property. Functions $u \in BV([0, T] \times \mathbb{R})$ satisfy the very strong trace property, but it is not known whether finite-entropy solutions satisfy it.

**Stability of shocks in $L^2$ for finite-entropy solutions.** We are now ready to state our main result, on the $L^2$ stability of an entropic shock wave $u^{\text{shock}}$ with initial datum

\[(1.8) \quad u_0^{\text{shock}}(x) = u_t 1_{x < \ell} + u_r 1_{x > \ell}, \quad u_t > u_r,\]

that is, $u^{\text{shock}}(t, x) = u_0^{\text{shock}}(x - \sigma t)$, with shock speed $\sigma = (f(u_r) - f(u_\ell))/(u_r - u_\ell)$.

**Theorem 1.1.** Let $f: \mathbb{R} \to \mathbb{R}$ be such that $f'' \geq \alpha > 0$. Let $u: [0, T] \times \mathbb{R} \to \mathbb{R}$ be a bounded finite-entropy solution (1.4) of

\[ \partial_t u + \partial_x f(u) = 0, \quad u(0, x) = u_0(x). \]
Assume that \( u \) satisfies the very strong trace property \( (1.7) \). Let \( u_{\text{shock}} \) be the entropic shock wave with initial datum \( u_0^{\text{shock}} \) \( (1.8) \), and set \( M = \sup_I f'' \) and \( S = \sup_I |f'| \), where \( I = [\min(u_\ell, \inf u), \max(u_\ell, \sup u)] \).

There exists a Lipschitz path \( h: [0, T] \to \mathbb{R} \) such that \( h(0) = 0 \) and

\[
(1.9) \quad \int_{-R}^{R} |u(t, x) - u_{\text{shock}}(t, x - h(t))|^2 \, dx \leq \int_{-R-tS}^{R+tS} \left| u_0 - u_0^{\text{shock}} \right|^2 \, dx \\
+ C \frac{M^3}{\alpha^3} \mu_+([0, t] \times [-R - tS, R + tS]),
\]

for all \( t \in [0, T] \), all \( R > 0 \) and some absolute constant \( C > 0 \), where \( \mu \) is the entropy production \( (1.3) \) associated with \( \eta(t) = t^2/2 \).

In addition the drift \( h \) is controlled by

\[
(1.10) \quad c \frac{\alpha}{M^2} (u_\ell - u_r) \int_0^t h'(\tau)^2 \, d\tau \leq \int_{-2St}^{2St} (u_0 - u_0^{\text{shock}})^2 \, dx \\
+ \frac{M^3}{\alpha^3} \mu_+([0, t] \times [-2St, 2St])
\]

for some absolute constant \( c > 0 \) and all \( t \in [0, T] \).

Remark 1.2. We were not able to remove the very strong trace assumption from this statement. In the proof it is used at one single point, Lemma 3.3 where we construct the drift function \( h(t) \) as a solution of a differential inclusion involving the traces of \( u \) along \( h(t) \). Other places where traces are needed require only the strong trace property \( (1.5) \), satisfied by finite-entropy solutions.

Remark 1.3. The necessity of introducing a drift \( h(t) \), and the near-optimality of estimate \( (1.10) \) when \( \mu_+ = 0 \) and \( u_\ell = -u_r = 1 \), are proved in [23] Proposition 1.2).

To prove Theorem 1.1 we adapt the relative entropy arguments used in [12, 13, 8] (see also [18, 7, 19, 23]). The relative entropy method was introduced in [3, 5] to study the \( L^2 \) stability of smooth solutions among entropy solutions, and later refined in [12, 13] to obtain the \( L^2 \) stability (up to a drift) of shock waves. This method is also relevant in the study of hydrodynamic limits for fluid equations [22]. The basic idea is that for any constant \( v_0 \), one has an identity of the form

\[
\frac{1}{2} \partial_t (u - v_0)^2 = \mu - \partial_x q(u; v_0).
\]

Stability of the constant state \( v_0 \) when \( \mu \leq 0 \) then follows by integrating over \( x \in \mathbb{R} \), provided \( q(u; v_0) \) is nice enough (e.g. has compact support). In the case of finite-entropy solutions, one also has to take into account the contribution of \( \mu_+ \). But when studying the stability of a shock, one integrates \( \partial_t (u - u_\ell)^2 \) and \( \partial_t (u - u_r)^2 \) on two complementary half-lines, and boundary terms appear at the junction.

The crucial remark used in [12, 13, 8] is that, if the initial shock is shifted by a well-chosen length \( x(t) \), then the boundary terms combine into a nonpositive contribution. There are two cases to consider, depending on whether or not \( u(t, \cdot) \) jumps at \( x(t) \). At times \( t \) where it does not jump, the situation is the same for entropy or finite-entropy solutions, and the ideas of [12, 13, 8] apply also in our case. But at times \( t \) where it does jump, an entropy solution can only make a negative jump, while a finite-entropy solution can also make a positive jump. More precisely, denoting by \( (u_-, u_+) \) the values of the jump of \( u \), it
is shown in [12, 13] that the dissipation rate $D(u_-, u_+; u_\ell, u_r)$ coming from the boundary terms satisfies

$$ D(u_-, u_+; u_\ell, u_r) \leq 0 \quad \text{whenever } u_- \geq u_+ \text{ and } u_\ell \geq u_r. $$

To include finite-entropy solutions, we have to consider also what happens when $u_- < u_+$. One cannot expect the dissipation rate $D$ to remain $\leq 0$, but what we do show (see Proposition 2.1) is that its positive part is controlled by the entropy cost of the jump, in other words by $\mu_+$. This crucial observation enables us to adapt the techniques of [12, 13, 8] to our situation and to prove the stability estimate (1.9). In fact we prove a sharper upper bound on $D$, thanks to which the control (1.10) on the drift $h(t)$ can then be obtained as in [8].

The article is organized as follows. In section 2 we prove the new bound on the dissipation rate $D$ appearing in the relative entropy method. In section 3 we recall and adapt the arguments of [12, 13] and conclude with the proof of Theorem 1.1.

Acknowledgements. X.L. is partially supported by ANR project ANR-18-CE40-0023 and COOPINTER project IEA-297303.

2. Upper bound on the dissipation rate $D$

We start by setting some notations. We denote by $\eta$, $q$ the entropy-flux pair given by

$$ \eta(x) = \frac{x^2}{2}, \quad q(x) = \int_0^x \eta' f', $$

and by $\eta(\cdot|\cdot), q(\cdot; \cdot)$ the corresponding relative entropy-flux pair

$$ \eta(x|a) = \eta(x) - \eta(a) - \eta'(a)(x-a) = \frac{(x-a)^2}{2}, \\
q(x;a) = q(x) - q(a) - \eta'(a)(f(x) - f(a)). $$

The propagation speed of a shock $(u_-, u_+)$ is constrained by the Rankine-Hugoniot condition:

$$ \sigma(u_-, u_+) = \frac{f(u_+) - f(u_-)}{u_+ - u_-}. $$

Given two shocks $(u_-, u_+)$ and $(u_\ell, u_r)$ we define the dissipation rate

$$ (2.1) \quad D(u_-, u_+; u_\ell, u_r) := q(u_+; u_r) - q(u_-; u_\ell) - \sigma(u_-, u_+)(\eta(u_+|u_r) - \eta(u_-|u_\ell)). $$

As explained in the introduction, this corresponds to the boundary terms which arise when calculating

$$ \partial_t \int \eta(u(t,x)|u_0^{\text{shock}}(x-x(t))) \, dx, $$

at times $t$ where $u(t, \cdot)$ has a jump $(u_-, u_+)$ at $x = x(t)$.

Our goal in this section is to compare the dissipation rate $D$ with the entropy cost of the jump $(u_-, u_+)$, given by

$$ (2.2) \quad E(u_-, u_+) = q(u_+) - q(u_-) - \sigma(u_-, u_+)(\eta(u_+) - \eta(u_-)). $$
This formula corresponds to the fact that, if a solution $u$ has a jump $(u_-(t), u_+(t))$ along a curve $x(t)$, and is smooth everywhere else, then by the BV chain rule (see e.g. [11 § 3.10]) the entropy production $\mu$ is given by

$$\mu(A) = \int 1_{(t,x(t))\in A} E(u_-(t), u_+(t)) \, dt.$$  

The main result of this section is the following.  

**Proposition 2.1.** For $u_\ell \geq u_r$ and any $u_\pm \in \mathbb{R}$ we have

$$D(u_-, u_+; u_\ell, u_r) \leq C_1 \frac{M^3}{\alpha^3} \max(E(u_-, u_+), 0)$$

$$- C_2 \alpha (u_\ell - u_-) \left[ (u_\ell - u_-)^2 + (u_r - u_+)^2 \right],$$

for some absolute constants $C_1, C_2 > 0$ and $0 < \alpha \leq M$ such that $\alpha \leq f'' \leq M$ on the convex hull of \{u_-, u_+, u_\ell, u_r\}.

**Proof of Proposition 2.1.** The case $u_- \geq u_+$ can be inferred from the arguments in [8 Lemma 4.1], only the case $u_- < u_+$ is really new. For the reader’s convenience we include a proof in both cases. Only the values of $f$ on the convex hull of \{u_-, u_+, u_\ell, u_r\} play a role in this inequality, so we assume without loss of generality that $\alpha \leq f'' \leq M$ on $\mathbb{R}$.

**Case 1:** $u_+ \geq u_-$.  

We split $D$ as

$$D(u_-, u_+; u_\ell, u_r) = E(u_-, u_+) + F_-(u_-, u_+; u_\ell, u_r) + D(u_-, u_+; u_\ell, u_r)$$

$$= E(u_-, u_+) + F_-(u_-, u_+; u_\ell, u_r) + D(u_+, u_+; u_\ell, u_r)$$

where

$$F_-(u_-, u_+; u_\ell, u_r) = D(u_-, u_+; u_\ell, u_r) - E(u_-, u_+) - D(u_-, u_-; u_\ell, u_r)$$

$$= \eta'(u_r) \left[ f(u_-) - f(u_+) + \sigma(u_-, u_+)u_+ - f'(u_-)u_+ \right]$$

$$+ (\sigma(u_-, u_+) - f'(u_-)) \left[ \eta(u_-) - \eta(u_\ell) - u_\ell \eta'(u_\ell) \right]$$

$$F_+(u_-, u_+; u_\ell, u_r) = D(u_+, u_+; u_\ell, u_r) - E(u_+, u_+) - D(u_+, u_-; u_\ell, u_r)$$

$$= \eta'(u_\ell) \left[ f(u_-) - f(u_+) - \sigma(u_-, u_+)u_- + f'(u_+)u_+ \right]$$

$$+ (\sigma(u_-, u_+) - f'(u_+)) \left[ \eta(u_-) - \eta(u_\ell) - u_\ell \eta'(u_\ell) \right]$$

We also define

$$\Delta := u_+ - u_- \geq 0, \quad A_\pm := u_\ell - u_\pm \geq B_\pm := u_r - u_\pm,$$

and start by remarking that

$$E(u_-, u_+) = \frac{\beta}{12} \Delta^3$$

for some $\beta \in [\alpha, M]$,

$$F_\pm(u_-, u_+; u_\ell, u_r) = \frac{\gamma_\pm}{4} \Delta (A_\pm^2 - B_\pm^2)$$

for some $\gamma_\pm \in [\alpha, M]$,

$$D(u_\pm, u_\pm; u_\ell, u_r) \leq -\frac{\alpha}{6} (A_\pm^2 - B_\pm^3).$$

**Proof of (2.4).** Recalling that $\eta(t) = t^2/2$ and $q'(t) = tf'(t)$ we have
\[
E(u_-, u_+) = q(u_+) - q(u_-) - \sigma(u_-, u_+)(\eta(u_+) - \eta(u_-)) \\
= \int_{u_-}^{u_+} t f'(t) \, dt - \frac{u_+ + u_-}{2} \int_{u_-}^{u_+} f'(t) \, dt \\
= \frac{1}{4} (u_+ - u_-)^2 \int_{-1}^{1} s f' \left( \frac{u_+ + u_-}{2} + s \frac{u_+ - u_-}{2} \right) \, ds \\
= \frac{1}{4} (u_+ - u_-)^2 \int_{-1}^{1} s \left[ f' \left( \frac{u_+ + u_-}{2} + s \frac{u_+ - u_-}{2} \right) - f' \left( \frac{u_+ + u_-}{2} \right) \right] \, ds \\
= \frac{1}{8} (u_+ - u_-)^3 \int_{0}^{1} \int_{-1}^{1} s^2 f'' \left( \frac{u_+ + u_-}{2} + st \frac{u_+ - u_-}{2} \right) \, ds \, dt.
\]

This last expression implies (2.4) since \( \alpha \leq f'' \leq M \) and \( \int_{0}^{1} \int_{-1}^{1} s^2 \, ds \, dt = 2/3 \).

**Proof of (2.5).** We have

\[
F_- = \frac{1}{2} (u_\ell - u_r)(u_\ell + u_r - 2u_-) \frac{1}{u_+ - u_-} \int_{u_-}^{u_+} (f'(t) - f'(u_-)) \, dt \\
= \frac{1}{2} (A_- - B_-)(A_- + B_-) \int_{0}^{1} f' \left( u_- + s(u_+ - u_-) \right) - f' \left( u_- \right) \, ds \\
= \frac{1}{2} \Delta(A_-^2 - B_-^2) \int_{0}^{1} \int_{0}^{1} t \, f'' \left( u_- + st(u_+ - u_-) \right) \, ds \, dt,
\]

which gives (2.5) for \( F_- \) since \( \alpha \leq f'' \leq M \) and \( \int_{0}^{1} \int_{0}^{1} t \, ds \, dt = 1/2 \). Similarly

\[
F_+ = -\frac{1}{2} \Delta(A_+^2 - B_+^2) \int_{0}^{1} \int_{0}^{1} t \, f'' \left( u_+ - st(u_+ - u_-) \right) \, ds \, dt
\]

which gives (2.5) for \( F_+ \).

**Proof of (2.6).** We have

\[
D(u, u; u_\ell, u_r) = \int_{u_\ell}^{u_r} t f'(t) \, dt - u_\ell \int_{u_\ell}^{u_r} f'(t) \, dt - u_r \int_{u_r}^{u} f'(t) \, dt \\
+ \frac{1}{2} (u_\ell - u_r)(u_\ell + u_r - 2u) f'(u) \\
= \int_{u}^{u_\ell} (t - u_\ell) f'(t) \, dt + \int_{u}^{u} (t - u_r) f'(t) \, dt \\
+ \left( \int_{u}^{u_\ell} (t - u_\ell) + \int_{u_r}^{u} (t - u_r) \right) f'(u) \\
= \int_{u}^{u_\ell} (t - u_\ell)(f'(t) - f'(u)) \, dt + \int_{u_r}^{u} (t - u_r)(f'(t) - f'(u)) \, dt \\
= \int_{u}^{u_\ell} (t - u_\ell)(t - u) \int_{0}^{1} f''(u + s(t - u)) \, ds \, dt \\
+ \int_{u}^{u} (t - u_r)(t - u) \int_{0}^{1} f''(u + s(t - u)) \, ds \, dt.
\]
If \( u \in [u_r, u_\ell] \) we see that \( f'' \geq \alpha \) implies
\[
D(u, u; u_\ell, u_r) \leq -\alpha \left( \int_u^{u_\ell} (u_\ell - t)(t - u) \, dt + \int_u^{u_r} (t - u_r)(u - t) \, dt \right)
= -\frac{\alpha}{6} (A^3 - B^3),
\]
where \( A = u_\ell - u \) and \( B = u_r - u \). If \( u \leq u_r \) we rewrite the above as
\[
D(u, u; u_\ell, u_r) = -\int_u^{u_\ell} (u_\ell - t)(t - u) \int_0^1 f''(u + s(t - u)) \, ds \, dt
- \int_u^{u_r} (u - t)(t - u_r) \int_0^1 f''(u + s(t - u)) \, ds \, dt,
\]
and if \( u \geq u_\ell \) as
\[
D(u, u; u_\ell, u_r) = -\int_u^{u_\ell} (u_\ell - t)(u - t) \int_0^1 f''(u + s(t - u)) \, ds \, dt
- \int_u^{u_r} (t - u_r)(u - t) \int_0^1 f''(u + s(t - u)) \, ds \, dt,
\]
and in both cases we deduce again that (2.6) is valid.

Combining (2.3) with (2.4)-(2.6) we obtain
\[
D(u_-, u_+; u_\ell, u_r) \leq \frac{\beta}{12} \Delta^3 + \frac{\gamma}{4} \Delta (A^2_\pm - B^2_\pm) - \frac{\alpha}{6} (A^3_\pm - B^3_\pm).
\]
Since \( \Delta \geq 0 \), for any \( A \geq B \) and \( \gamma, \lambda > 0 \), by Young’s inequality \( ab \leq \frac{1}{8} a^3 + \frac{3}{8} b^3 \) \((a,b \geq 0)\) we have
\[
\frac{\gamma}{4} \Delta |A^2 - B^2| \leq \frac{\gamma}{12} \Delta^3 + \frac{3}{8} \lambda \frac{3}{2} |A^2 - B^2|^\frac{3}{2}.
\]
From
\[
|A^2 - B^2|^3 = (A - B)^2 |A^2 - B^2| (A + B)^2 \leq 2(A - B)^2 (A^2 + B^2)^2
\leq 8(A - B)^2 (A^2 + AB + B^2)^2 = 8(A^3 - B^3)^2,
\]
we deduce
\[
\frac{\gamma}{4} \Delta |A^2 - B^2| \leq \frac{\gamma}{12} \Delta^3 + \sqrt{2} \frac{\gamma}{3} \lambda \frac{3}{2} (A^3 - B^3),
\]
and we see that choosing \( \lambda^3 = \frac{a^2}{32 \gamma^2} \) leads to
\[
\frac{\gamma}{4} \Delta |A^2 - B^2| \leq \frac{8 \gamma}{3} \lambda^3 \Delta^3 + \frac{a}{12} (A^3 - B^3).
\]
Plugging this into (2.7) yields
\[
D(u_-, u_+; u_\ell, u_r) \leq \left( 1 + \frac{32 \gamma}{\beta \alpha^2} \right) \frac{\beta}{12} \Delta^3 - \frac{\alpha}{12} (A^3_\pm - B^3_\pm).
\]
Recalling (2.4)-(2.5) this implies
\[
D(u_-, u_+; u_\ell, u_r) \leq CM^3 \frac{\alpha^3}{\lambda^3} E(u_-, u_+) - \frac{\alpha}{12} (A^3_\pm - B^3_\pm),
\]
for $C = 33$. Remarking that
\[ A^3 - B^3 = (A - B)(A^2 + AB + B^2) \geq (A - B)\frac{A^2 + B^2}{2}, \]
we deduce
\[ D(u_-, u_+; u_\ell, u_\rho) \leq C\frac{M^3}{\alpha^3}E(u_-, u_+) \]
\[ -\frac{\alpha}{24}(u_\ell - u_\rho)[(u_\ell - u_-)^2 + (u_\rho - u_+)^2], \]
which proves Proposition 2.1 when $u_+ \geq u_-$. 

**Case 2:** $u_- > u_+$. 

Following [18] (see also [8] Lemma 3.3) we write $D$ as a combination of integrals of the function
\[ g(t) = f(t) - \sigma(u_-, u_+)t - (f(u_+) - \sigma(u_-, u_+)u_+) \]
\[ = f(t) - \sigma(u_-, u_+)t - (f(u_+) - \sigma(u_-, u_+)u_+), \]
where the second equality follows from the definition of $\sigma$. Specifically we have
\[ D(u_-, u_+; u_\ell, u_\rho) = -\int_{u_\rho}^{u_\ell} g(t) dt + \int_{u_\ell}^{u_-} g(t) dt. \]
As remarked in [18], since $u_+ \leq u_-$ and $u_\rho \leq u_\ell$, this can be written as
\[ (2.8) \quad -D = \int_I |g| + \int_J |g|, \]
where $I$ and $J$ are disjoint intervals such that
\[ I \cup J = ([u_\rho, u_-] \cup [u_\ell, u_\rho]) \setminus ([u_+, u_-] \cap [u_\rho, u_\ell]). \]
We denote by $g_0$ the function $g$ in the special case $f(t) = f_0(t) = t^2/2$, i.e.
\[ g_0(t) = \frac{1}{2}(t - u_+)(t - u_-). \]
Since $g - \alpha g_0$ is a convex function vanishing at $u_+$ and $u_-$, we have
\[ g(t) \geq \alpha g_0(t) \quad \text{for } t \in \mathbb{R} \setminus [u_+, u_-], \]
\[ g(t) \leq \alpha g_0(t) \quad \text{for } t \in [u_+, u_-]. \]
Therefore $|g| \geq \alpha |g_0|$ on $\mathbb{R}$ and one infers from $(2.8)$ that
\[ -D \geq -\alpha D_0, \]
where $D_0$ is the dissipation rate [2.1] for $f(t) = f_0(t) = t^2/2$. To conclude the proof of Proposition 2.1 it suffices to check that
\[ (2.9) \quad -D_0(u_-, u_+; u_\ell, u_\rho) \geq c(u_\ell - u_\rho)[(u_\ell - u_-)^2 + (u_\rho - u_+)^2], \]
for some absolute constant $c > 0$. To this end we compute
\[ \int_{u_\rho}^{u_\ell} g_0 = \frac{1}{6}(u_\rho - u_+)^3 + \frac{1}{4}(u_\rho - u_-)(u_\rho - u_+)^2, \]
\[ \int_{u_-}^{u_\ell} g_0 = \frac{1}{6}(u_\ell - u_-)^3 + \frac{1}{4}(u_- - u_+)(u_\ell - u_-)^2, \]
so that, setting $H = u_\ell - u_-$ and $K = u_r - u_+$, we find

$$-D_0 = \frac{1}{4}(u_- - u_+)(H^2 + K^2) + \frac{1}{6}(H^3 - K^3).$$

This implies

$$-D_0 = \frac{1}{4}(u_\ell - u_r - H + K)(H^2 + K^2) + \frac{1}{6}(H^3 - K^3)$$

As $H - K = u_\ell - u_r - (u_- - u_+) \leq u_\ell - u_r$ we deduce

$$-D_0 \geq \frac{1}{4}(u_\ell - u_r)(H^2 + K^2) - \frac{1}{12}(H - K)^2$$

$$\geq \frac{1}{12}(u_\ell - u_r)(H^2 + K^2),$$

proving (2.9) with $c = 1/12$. □

3. The stability estimate

We follow [12, 13, 18], where $u$ is an entropy solution, and explain how their methods adapt to our more general situation. Recall that our goal is to control the increase of

$$F(t) = \int_{R - St}^{R + St} \eta(u(t, x)|u^\text{shock}_0(x - x(t))) \, dx,$$

for a well-chosen Lipschitz path $x(t)$ and $R \geq St$.

First we establish properties of the traces of $u$ along Lipschitz curves, which require only the strong trace property (and are thus valid for finite-entropy solutions).

**Lemma 3.1.** Let $u$ be a weak bounded solution of (1.1) satisfying the strong trace property (1.5). Let $x : [0, T] \to \mathbb{R}$ be a Lipschitz path, and $u(t, x(t)\pm)$ the traces of $u$ along $x(t)$. Then for almost every $t \in [0, T]$ we have the Rankine-Hugoniot relation

$$f(u(t, x(t)+)) - f(u(t, x(t)-)) = x'(t)(u(t, x(t)+) - u(t, x(t)-)).$$

**Proof of Lemma 3.1.** This is proved in [13] Lemma 6]. We sketch the proof for the reader’s convenience. The Rankine Hugoniot relation (3.2) follows from testing the equation

$$\partial_t u + \partial_x f(u) = 0,$$

against a test function $\chi$ of the form

$$\chi(t, \xi) = \psi(t)(\Phi_\varepsilon(\xi - x(t)) + \Phi_\varepsilon(x(t) - \xi) - 1),$$

where $1_{y < -\varepsilon} \leq \Phi_\varepsilon(y) \leq 1_{y < 0},$

The strong trace property (1.5) ensures convergence, as $\varepsilon \to 0$, to (3.2) tested against $\psi(t)$. □

Next we establish a formula for the variations of quantities of the form

$$t \mapsto \int_{x(t)}^{y(t)} \eta(u(t, x)|v_0) \, dx,$$
for some constant $v_0$.

**Lemma 3.2.** Let $u$ be a finite-entropy solution of (1.1). Let $y, z : [0, T] \to \mathbb{R}$ be Lipschitz paths, let $0 \leq t_1 < t_2 \leq T$ and assume that

$$y(\tau) < z(\tau) \quad \forall \tau \in (t_1, t_2).$$

For any $v_0 \in \mathbb{R}$, we have

$$\int_{y(t_2)}^{z(t_2)} \eta(u(t_2, \xi)|v_0) \, d\xi - \int_{y(t_1)}^{z(t_1)} \eta(u(t_1, \xi)|v_0) \, d\xi$$

$$= \int 1_{t_1 < \tau < t_2, y(\tau) < \xi < z(\tau)} \mu(d\tau, d\xi)$$

$$+ \int_{t_1}^{t_2} [q(u(\tau, y(\tau)+); v_0) - y'(\tau)\eta(u(\tau, y(\tau)+)|v_0)] \, d\tau$$

$$- \int_{t_1}^{t_2} [q(u(\tau, z(\tau)-); v_0) - z'(\tau)\eta(u(\tau, z(\tau)-)|v_0)] \, d\tau.$$

**Proof of Lemma 3.2.** The proof is essentially the same as e.g. [13, Lemma 6], see also [7, Lemma 2.4]. We sketch it here for the reader’s convenience, the only difference being that we keep the terms involving $\mu$. We may assume without loss of generality that $y < z$ in $[t_1, t_2]$ (otherwise consider instead $[t_1 + \delta, t_2 - \delta]$ and let $\delta \to 0^+$ at the end).

We test the identity

$$\partial_t \eta(u|u_0) + \partial_x q(u|u_0) = \mu,$$

against a test function $\chi$ of the form

$$\chi(\tau, \xi) = \psi_\varepsilon(\tau) (\Phi_\varepsilon(y(\tau) - \xi) - \Phi_\varepsilon(\xi - z(\tau))),$$

where $1_{t_1 + \varepsilon < \tau < t_2 - \varepsilon} \leq \psi_\varepsilon(\tau) \leq 1_{t_1 < \tau < t_2}$, $1_{x < -\varepsilon} \leq \Phi_\varepsilon(x) \leq 1_{x < 0}$.

and obtain (3.3) as $\varepsilon \to 0^+$, thanks to the strong trace property (1.5) and the time-continuity property (1.6).

Using Lemma 3.2 we will obtain a formula for the variations of $F(t)$ (3.1), and thanks to Lemma 3.1 we will see that at any time $t$ where $u(t, \cdot)$ has a jump $(u_-, u_+)$ at $x(t)$, the increase of $F(t)$ is controlled by $\mu$ plus the dissipation rate $D$, which owing to Proposition 2.1 is in turn controlled by $\mu_+$. Note that so far this is valid for any Lipschitz curve $x(t)$.

However, in order to control the increase of $F(t)$ at times $t$ where $u(t, \cdot)$ does not jump at $x(t)$, we will need to constrain $x'(t)$. The next lemma gives us a tool to do so. This is the only place where we need the very strong trace property.

**Lemma 3.3.** Let $V : \mathbb{R} \to \mathbb{R}$ be continuous. Let $u : [0, T] \times \mathbb{R}$ be bounded and satisfy the very strong trace property. Then there exists a Lipschitz path $x : [0, T] \to \mathbb{R}$ such that $x(0) = 0$ and

$$\min (V(u_+(t)), V(u_-(t))) \leq x'(t) \leq \max (V(u_+(t)), V(u_-(t)))$$

for a.e. $t \in [0, T]$, where $u_{\pm}(t) = u(t, x(t) \pm)$ denote the traces of $u$ along $x(t)$, given by the very strong trace property.

**Proof of Lemma 3.3.** This is proved in [13, Proposition 1]. The path $x(t)$ is obtained as a limit of paths $x_k(t)$ solving $x_k'(t) = \tilde{V}(t, x_k(t))$, where $\tilde{V}_k$ is a mollification (with respect to the $x$ variable) of $V \circ u$. □
We are now ready to prove our main result.

Proof of Theorem 1.1 We apply Lemma 3.3 to the function $V = f'$ and obtain a Lipschitz path $x: [0, T] \to \mathbb{R}$ such that
\[ x'(t) = f'(u(t)) \quad \text{for a.e. } t \in [0, T] \setminus J, \]
where, denoting by $u_{\pm}(t) = u(t, x(t) \pm)$ the traces of $u$ along $x(t)$, we let $J$ be the “jump set”
\[ J = \{ t \in [0, T]: u_-(t) \neq u_+(t) \}, \]
and write $u(t) = u_-(t) = u_+(t)$ for $t \in [0, T] \setminus J$. For $t \in J$, Lemma 3.1 ensures that $x'(t) = \sigma(u_-(t), u_+(t))$, and since $\sigma(u, u) = f'(u)$ we deduce that
\[ x'(t) = \sigma(u_-(t), u_+(t)) \quad \text{for a.e. } t \in [0, T]. \]
Let $R > 0$ and $F(t)$ defined by (3.1) for all $t \leq R/S$. We assume without loss of generality that $R \geq ST$ (otherwise replace $T$ by $R/S$). Consider the time
\[ t_0 = \sup \{ t \in [0, T]: -R + St < x(t) < R - St \}. \]
By definition of $S = \sup_{t} |f'|$ we know that $|x'| \leq S$ and deduce
\[ -R + St < x(t) < R - St \quad \forall t \in [0, t_0], \]
and $x(t) \in (-\infty, -R + St) \cup [R - St, \infty) \quad \forall t \geq t_0.$
For $t \in [0, t_0]$ we have
\[ F(t) = \int_{-R+St}^{x(t)} \eta(u(t, x)|u_\ell) \, dx + \int_{x(t)}^{R-St} \eta(u(x, t)|u_\ell) \, dx. \]
We apply the variation formula (3.3) to compute $F(t) - F(0)$. Note that the identity
\[ \frac{q(u; v)}{\eta(u|v)} = \frac{2}{(u - v)^2} \int_{v}^{u} (t - v)f'(s) \, dt = 2 \int_{0}^{1} s f'(us + v(1 - s)) \, ds. \]
and the definition of $S$ ensure that
\[ |q(u; v)| \leq S \eta(u|v) \quad \forall u, v \in I = [\min(u_\ell, \inf u), \max(u_\ell, \sup u)]. \]
As a consequence, whenever $y'(t) = S$ or $z'(t) = -S$ the corresponding term in the right-hand side of (3.3) gives a nonpositive contribution, and we deduce
\[ F(t) - F(0) \leq \mu_+(B^S_{R,t}) + \int_{0}^{t} \left[ q(u_+(\tau); u_\ell) - q(u_-(\tau); u_\ell) \right] d\tau, \]
where
\[ B^S_{R,t} = \{ (\tau, \xi): 0 < \tau < t, -R + \tau S < \xi < R - \tau S \}. \]
Recalling (3.4) that $x' = \sigma(u_-, u_+)$ a.e. in $[0, T]$, we recognize the dissipation rate $D$ (2.1) and rewrite the above as
\[ F(t) - F(0) \leq \mu_+(B^S_{R,t}) + \int_{0}^{t} D(u_-(\tau), u_+(\tau); u_\ell, u_\ell) \, d\tau. \]
Since
\[\sigma(u_-, u_+) - \sigma(u_0, u_0) = \int_0^1 \left[ f'(tu_- + (1-t)u_+) - f'(tu_0 + (1-t)u_0) \right] dt,\]
and \(f'\) is \(M\)-Lipschitz on \(I\) we have
\[|\sigma(u_-, u_+) - \sigma(u_0, u_0)| \leq \frac{M}{2} (|u_-| - u_0 + |u_0 - u_+|),\]
so using the upper bound on \(D\) provided by Proposition 2.1 we deduce from (3.5) that
\[F(t) - F(0) \leq \mu_+(B_{R,t}^s) + C_1 \frac{M^3}{\alpha^3} \int_0^t \max(E(\tau), 0) d\tau - C_2(u_\ell - u_r) \frac{\alpha}{M^2} \int_0^t (x'(\tau) - \sigma)^2 d\tau,\]
where \(E(\tau)\) is the entropy cost of the jump \((u_-(\tau), u_+(\tau))\) (2.2). From the characterization [11, 4] of the one-dimensional part of \(\mu\) we have
\[\mu_+[(\tau, x(\tau))] = \max(E(\tau), 0) d\tau,\]
and obtain
\[(3.6) \quad F(t) \leq F(0) + C_1 \frac{M^3}{\alpha^3} \mu_+(B_{R,t}^s) - C_2(u_\ell - u_r) \frac{\alpha}{M^2} \int_0^t (x'(\tau) - \sigma)^2 d\tau, \quad \forall t \in [0, t_*].\]
Now for \(t \in [t_*, T]\) we have
\[F(t) = \int_{-R+St}^{R-St} \eta(u(x, t)|v_0) dx,\]
where \(v_0 = u_0\) or \(u_\ell\). Therefore, applying (3.3) and remarking again that the terms involving \(y'(t) = S\) and \(z'(t) = -S\) give nonpositive contributions, we obtain
\[F(t) - F(t_*) \leq \mu_+(B_{R,t}^s \setminus B_{R,t_*}^s) \quad \forall t \in [t_*, T].\]
Combining this with the estimate obtained in \([0, t_*]\) and recalling the definition (3.1) of \(F(t)\) we deduce that
\[
\int_{-R+St}^{R-St} \left| u(x, t) - u_0^{shock}(x - x(t)) \right|^2 dx \leq \int_{-R}^{R} \left| u_0 - u_0^{shock} \right|^2 dx + C \frac{M^3}{\alpha^3} \mu_+(B_{R,t}^s) \quad \forall t \in [0, T].
\]
Provided we set \(h(t) = x(t) - \sigma t\) and replace \(R\) by \(R + St\), this implies our main result (1.9) since \(B_{R+tSt}^s \subset [0, t] \times [-R-St, R-St].\)

To prove estimate (1.10) on \(h(t)\), simply remark that (3.6) readily implies that
\[
c \frac{\alpha}{M^3} (u_\ell - u_r) \int_0^t h'(\tau)^2 d\tau \leq \int_{-R}^{R} (u_0 - u_0^{shock})^2 dx + M^3 \frac{\alpha^3}{\mu_+(B_{R,t}^s)}
\]
for some absolute constant \(c > 0\), provided \(-R + St < x(t) < R - St\). Since we know that \(|x(t)| \leq St\) we may choose \(R = 2St\), and deduce (1.10). \(\square\)
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