Boundary correlation functions of the six and nineteen vertex models with domain wall boundary conditions

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Abstract

Boundary correlation functions of the six and nineteen vertex models on an $N \times N$ lattice with domain wall boundary conditions are studied. The general expression of the boundary correlation functions is obtained for the six vertex model by use of the quantum inverse scattering method. For the nineteen vertex model, the boundary correlation functions are shown to be expressed in terms of those for the six vertex model.

Keywords: integrable vertex model, quantum inverse scattering method, correlation function

1 Introduction

The six vertex model is one of the most fundamental exactly solved models in statistical physics [1, 2, 3, 4]. Not only the periodic boundary condition but also the domain wall boundary condition is an interesting boundary condition. For example, the partition function is deeply related to the norm [5] and the scalar product [6] of the XXZ chain. The determinant formula of the partition function [7, 8] lead Slavnov [6] to obtain a compact representation of the scalar product, which plays a fundamental role in calculating correlation functions of the XXZ chain [9, 10, 11, 12]. The determinant formula also led to a deep advance in enumerative combinatorics [13, 14, 15]. For example, it was used to give a concise proof of the numbers of the alternating sign matrices for a given size. Recently, the correspondences between the partition function and the Schur polynomial [16] and KP $\tau$ function [17] have been revealed. The determinant representations of partition functions have been extended to other models such as the higher spin vertex models [18], Felderhof models [19] and so on. The domain wall boundary conditions are also interesting from the physical point of view since it exhibits phase separation phenomena [20, 21, 22, 23, 24].

The calculation of correlation functions are also interesting in the domain wall boundary condition itself. Several kinds of them such as the boundary one point functions, two point functions, boundary polarization [25, 26, 27, 28] and the emptiness formation probability [29] have been calculated.

In this paper, we calculate boundary correlation functions for the six and nineteen vertex models on an $N \times N$ lattice with domain wall boundary condition. For the six vertex model, we use the quantum inverse scattering method, and apply the approach of [29] to build and solve two recursive relations for the boundary correlation functions, providing for them a general expression. The boundary correlation functions we consider includes the boundary polarization and boundary emptiness formation probability (EFP) as special cases.

Next, by use of fusion, we show that the boundary correlation functions for the nineteen vertex (Fateev-Zamołodchikov [30]) model can be reduced to those for the six vertex model. In particular, the EFP of length $s$ for the nineteen vertex model reduces to that of length $2s$ for the six vertex model.
The outline of this paper is as follows. In the next section, we define the six vertex model with domain wall boundary condition. The general expression for the boundary correlation functions is obtained in section 3. In section 4, the boundary correlation functions of the nineteen vertex model are considered by use of of fusion. The emptiness formation probability for the nineteen vertex model is expressed in the determinant form in the homogeneous limit in section 5.

2 Six vertex model

The six vertex model is a model in statistical mechanics, whose local states are associated with edges of a square lattice, which can take two values. The Boltzmann weights are assigned to its vertices, and each weight is determined by the configuration around a vertex. What plays the fundamental role is the $R$-matrix

$$R(\alpha, \nu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{\sh(\lambda - \nu)}{\sh(\lambda + \nu)} & 0 & \frac{\sh(\lambda - \nu)}{\sh(\lambda + \nu)} & 0 \\ \frac{\sh(\lambda - \nu)}{\sh(\lambda + \nu)} & \frac{\sh(\lambda - \nu)}{\sh(\lambda + \nu)} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (1)$$

which satisfies the Yang-Baxter equation

$$R_{12}(\lambda, \nu) R_{13}(\lambda, \mu) R_{23}(\nu, \mu) = R_{23}(\nu, \mu) R_{13}(\lambda, \mu) R_{12}(\lambda, \nu). \quad (2)$$

We consider the six vertex model on a $N \times N$ lattice depicted in Figure. The spins are aligned all up at the bottom and right boundaries, and all down at the top and left boundaries. At the intersection of the $\alpha$-th row (from the bottom) and the $k$-th column (from the left), we associate the statistical weight

$$\mathcal{L}_{ak}(\lambda_\alpha, \nu_k) = \sh(\lambda_\alpha - \nu_k + \eta/2) R_{ak}(\lambda_\alpha - \eta/2, \nu_k)$$

$$= \begin{pmatrix} a(\lambda_\alpha, \nu_k) & 0 & 0 & 0 \\ 0 & b(\lambda_\alpha, \nu_k) & c & 0 \\ 0 & c & b(\lambda_\alpha, \nu_k) & 0 \\ 0 & 0 & 0 & a(\lambda_\alpha, \nu_k) \end{pmatrix}, \quad (3)$$

where

$$a(\lambda, \nu) = \sh(\lambda - \nu + \eta/2), \ b(\lambda, \nu) = \sh(\lambda - \nu - \eta/2), \ c = \sh \eta. \quad (4)$$

We refer to the $\alpha$-th row as the auxiliary space $V_\alpha$, and the $k$-th column as the quantum space $H_k$. Let us denote $\{\lambda\} = \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$, $\{\nu\} = \{\nu_1, \nu_2, \ldots, \nu_N\}$, and the basis (dual basis) of the spin-1/2 representation as $|+\rangle, |-\rangle \ ((|+\rangle, |-\rangle))$.

The partition function of the six vertex model, which is the summation of products of statistical weights over all possible configurations can be formally represented as

$$Z_N(\{\lambda\}, \{\nu\}) = a\langle +||q\langle -|| \prod_{\alpha, k=1}^N \mathcal{L}_{ak}(\lambda_\alpha, \nu_k)||-\rangle_q||+\rangle_q, \quad (5)$$

where $||+\rangle = \otimes_{k=1}^N |+\rangle_k, ||-\rangle = \otimes_{k=1}^N |-\rangle_k, \langle +|| = \otimes_{k=1}^N \langle +||_k, \langle -|| = \otimes_{k=1}^N \langle -||_k$, and we distinguish the spins on the quantum and auxiliary spaces by the subscripts "q" and "a". The partition function has the following determinant form

$$Z_N(\{\lambda\}, \{\nu\}) = \frac{\prod_{\alpha=1}^N \prod_{k=1}^N a(\lambda_\alpha, \nu_k)b(\lambda_\alpha, \nu_k) \det M(\{\lambda\}, \{\nu\})} {\prod_{1 \leq \alpha < \beta \leq N} d(\lambda_\alpha, \lambda_\beta) \prod_{1 \leq j < k \leq N} d(\nu_j, \nu_k).} \quad (6)$$
where
\[ d(\lambda, \nu) = \text{sh}(\lambda - \nu), \quad M_{\alpha k} = \varphi(\lambda_\alpha, \nu_k), \quad \varphi(\lambda, \nu) = \frac{c}{a(\lambda, \nu)b(\lambda, \nu)}. \] (7)

Introducing the monodromy matrix
\[ T_\alpha(\lambda_\alpha, \{\nu\}) = L_{\alpha N}(\lambda_\alpha, \nu_N) \cdots L_{\alpha 1}(\lambda_\alpha, \nu_1) = \begin{pmatrix} A(\lambda_\alpha, \{\nu\}) & B(\lambda_\alpha, \{\nu\}) \\ C(\lambda_\alpha, \{\nu\}) & D(\lambda_\alpha, \{\nu\}) \end{pmatrix}, \] (8)

the partition function can be represented as
\[ Z_N(\lambda_\alpha, \{\nu\}) = q^{\langle ||B(\lambda_\alpha, \{\nu\}) \cdots B(\lambda_\alpha, \{\nu\})|| \rangle} \] (9)

From the Yang-Baxter equation, one has
\[ R_{\alpha \beta}(\mu - \lambda)T_\alpha(\mu, \{\nu\})T_\beta(\lambda, \{\nu\}) = T_\beta(\lambda, \{\nu\})T_\alpha(\mu, \{\nu\})R_{\alpha \beta}(\mu - \lambda). \] (10)

From (10), one has
\[ A(\lambda, \{\nu\})B(\mu, \{\nu\}) = f(\lambda, \mu)B(\mu, \{\nu\})A(\lambda, \{\nu\}) + g(\mu, \lambda)A(\lambda, \{\nu\})B(\mu, \{\nu\}), \] (11)
\[ B(\lambda, \{\nu\})A(\mu, \{\nu\}) = f(\lambda, \mu)A(\mu, \{\nu\})B(\lambda, \{\nu\}) + g(\mu, \lambda)B(\lambda, \{\nu\})A(\mu, \{\nu\}), \] (12)
\[ B(\lambda, \{\nu\})B(\mu, \{\nu\}) = B(\mu, \{\nu\})B(\lambda, \{\nu\}), \] (13)

where
\[ f(\mu, \lambda) = \frac{\text{sh}(\lambda - \mu + \eta)}{\text{sh}(\lambda - \mu)}, \quad g(\mu, \lambda) = \frac{\text{sh} \eta}{\text{sh}(\lambda - \mu)}, \] (14)

for example.
3 Boundary correlation functions of the six vertex model

In this section, we consider the following boundary correlation functions

\[ F_N^{(r, \epsilon_1, \cdots, \epsilon_s)}(\{\lambda\}, \{\nu\}) = \frac{\tilde{F}_N^{(r, \epsilon_1, \cdots, \epsilon_s)}(\{\lambda\}, \{\nu\})}{Z_N(\{\lambda\}, \{\nu\})}, \]

\[ \tilde{F}_N^{(r, \epsilon_1, \cdots, \epsilon_s)}(\{\lambda\}, \{\nu\}) = a(+||q(-|| \prod_{a=r+1}^{N} \prod_{k=1}^{N} \mathcal{L}_{\alpha k}(\lambda_a, \nu_k) \prod_{k=1}^{s} \pi_{\epsilon_k}^+ \prod_{a=1}^{N} \mathcal{L}_{\alpha k}(\lambda_a, \nu_k)||-||_{q}), \]

where \( \pi_{\epsilon_k}^+ = |+\rangle_{kk}^\dagger \langle +| \) and \( \pi_{\epsilon_k}^- = |-\rangle_{kk}^\dagger \langle -| \) is a projection onto the up and down spin respectively. Some special cases of this general boundary correlation function reduces to the ones previously considered \[25, 26, 27, 28, 29\]. We calculate the boundary correlation functions by use of the quantum inverse scattering method, applying the approach of \[29\].

![Figure 2: An example of boundary correlation function.](image)

First, note that (16) can be expressed as

\[ \tilde{F}_N^{(r, \epsilon_1, \cdots, \epsilon_s)}(\{\lambda\}, \{\nu\}) = q(-||B(\lambda_N, \{\nu\}) \cdots B(\lambda_{r+1}, \{\nu\}) \prod_{k=1}^{s} \pi_{\epsilon_k}^+ B(\lambda_r, \{\nu\}) \cdots B(\lambda_1, \{\nu\})||+||_{q}), \]

in the quantum inverse scattering language.

We introduce the following two-site model \[10\] in order to obtain recursive relations between boundary correlation functions of different lattice sizes.

\[ T(\lambda, \{\nu\}) = T_2(\lambda, \{\nu\}\{\nu_1\}) T_1(\lambda, \nu_2) \]

\[ T_2(\lambda, \{\nu\}\{\nu_1\}) = \mathcal{L}_{\alpha N}(\lambda, \nu_N) \cdots \mathcal{L}_{\alpha 2}(\lambda, \nu_2) \]

\[ = \left( \begin{array}{cc} A_2(\lambda, \{\nu\}\{\nu_1\}) & B_2(\lambda, \{\nu\}\{\nu_1\}) \\ C_2(\lambda, \{\nu\}\{\nu_1\}) & D_2(\lambda, \{\nu\}\{\nu_1\}) \end{array} \right), \]

\[ T_1(\lambda, \nu_1) = \mathcal{L}_{\alpha 1}(\lambda, \nu_1). \]
Applying
\[ 1(+|B(\lambda, \{\nu\})|+)_1 = b(\lambda, \nu_1)B_2(\lambda, \{\nu\} \setminus \nu_1), \]
\[ 1(-|B(\lambda, \{\nu\})|+)_1 = cA_2(\lambda, \{\nu\} \setminus \nu_1), \]
\[ 1(+|B(\lambda, \{\nu\})|)_1 = 0, \]
\[ 1(-|B(\lambda, \{\nu\})|)_1 = a(\lambda, \nu_1)B_2(\lambda, \{\nu\} \setminus \nu_1), \]
(21)
iteratively, one has
\[
1(+|B(\lambda_n, \{\nu\}) \cdots B(\lambda_1, \{\nu\})|+)_1 = \prod_{j=1}^{n} b(\lambda_j, \nu_1)B_2(\lambda_n, \{\nu\} \setminus \nu_1) \cdots B_2(\lambda_1, \{\nu\} \setminus \nu_1),
\]
(22)
\[
1(-|B(\lambda_n, \{\nu\}) \cdots B(\lambda_1, \{\nu\})|)_1 = \prod_{j=1}^{n} a(\lambda_j, \nu_1)B_2(\lambda_n, \{\nu\} \setminus \nu_1) \cdots B_2(\lambda_1, \{\nu\} \setminus \nu_1),
\]
(23)
\[
1(-|B(\lambda_n, \{\nu\}) \cdots B(\lambda_1, \{\nu\})|+)_1 = \sum_{\alpha=1}^{n} \prod_{\beta=\alpha+1}^{\alpha-1} b(\lambda_\beta, \nu_1)B_2(\lambda_n, \{\nu\} \setminus \nu_1) \cdots B_2(\lambda_\alpha+1, \{\nu\} \setminus \nu_1)
\times A_2(\lambda_\alpha, \{\nu\} \setminus \nu_1)B_2(\lambda_\alpha-1, \{\nu\} \setminus \nu_1) \cdots B_2(\lambda_1, \{\nu\} \setminus \nu_1).
\]
(24)
Combining (24),
\[
A_2(\lambda, \{\nu\} \setminus \nu_1)B_2(\mu, \{\nu\} \setminus \nu_1) = f(\lambda, \mu)B_2(\mu, \{\nu\} \setminus \nu_1)A_2(\lambda, \{\nu\} \setminus \nu_1)
+ g(\mu, \lambda)B_2(\lambda, \{\nu\} \setminus \nu_1)A_2(\mu, \{\nu\} \setminus \nu_1),
\]
(25)
and
\[
A_2(\lambda, \{\nu\} \setminus \nu_1) \otimes_{k=2}^{N} a(\lambda, \nu_k) \otimes_{k=2}^{N} \langle + \rangle_k,
\]
(26)
we get
\[
1(-|B(\lambda_r, \{\nu\}) \cdots B(\lambda_1, \{\nu\})|+)_1
= \sum_{\alpha=1}^{r} \prod_{\beta=\alpha+1}^{\alpha-1} b(\lambda_\beta, \nu_1) \prod_{\beta=1}^{\alpha} f(\lambda_\alpha, \lambda_\beta) \prod_{k=2}^{N} a(\lambda_\alpha, \nu_k) \prod_{k=1}^{r} B_2(\lambda_k, \{\nu\} \setminus \nu_1) \otimes_{k=2}^{N} \langle + \rangle_k.
\]
(27)
In the same way as (27), we can also show the following relation
\[
\langle -|B(\lambda_r, \{\nu\}) \cdots B(\lambda_{r+1}, \{\nu\})|+ \rangle_1
= \sum_{\alpha=r+1}^{N} \prod_{\beta=\alpha+1}^{\alpha-1} b(\lambda_\beta, \nu_1) \prod_{\beta=1}^{\alpha} f(\lambda_\alpha, \lambda_\beta) \prod_{k=2}^{N} b(\lambda_\alpha, \nu_k) \otimes_{k=2}^{N} \langle - \rangle_k \prod_{k=2}^{N} B_2(\lambda_k, \{\nu\} \setminus \nu_1),
\]
(28)
utilizing (24),
\[
B_2(\lambda, \{\nu\} \setminus \nu_1)A_2(\mu, \{\nu\} \setminus \nu_1) = f(\lambda, \mu)A_2(\mu, \{\nu\} \setminus \nu_1)B_2(\lambda, \{\nu\} \setminus \nu_1)
+ g(\mu, \lambda)A_2(\lambda, \{\nu\} \setminus \nu_1)B_2(\mu, \{\nu\} \setminus \nu_1),
\]
(29)
and
\[
\otimes_{k=2}^{N} \langle - \rangle A_2(\lambda, \{\nu\} \setminus \nu_1) = \prod_{k=2}^{N} b(\lambda, \nu_k) \otimes_{k=2}^{N} \langle - \rangle.
\]
(30)
From (23) and (27), one can derive one recursive relation for the boundary correlation functions between different lattice sizes [29]

\[ F_N^{(r-s, \ldots, s)}(\{\lambda\}, \{\nu\}) = \prod_{\beta=r+1}^{N} f(\lambda_{s}, \lambda_{\beta}) \prod_{k=1}^{N} a(\lambda_{s}, \nu_{k}) \]

\[ \times F_{N-1}^{(r-s, \ldots, s)}(\{\lambda\} \setminus \{\lambda_{s}\}, \{\nu\} \setminus \{\nu_{k}\}). \]  

(31)

We can obtain another recursive relation from (22) and (28)

\[ F_N^{(r+\ldots, s)}(\{\lambda\}, \{\nu\}) = \prod_{\beta=r+1}^{N} b(\lambda_{s}, \nu_{j}) \prod_{k=1}^{N} c(\lambda_{s}, \nu_{k}) \]

\[ \times F_{N-1}^{(r+\ldots, s)}(\{\lambda\} \setminus \{\lambda_{s}\}, \{\nu\} \setminus \{\nu_{k}\}). \]  

(32)

Solving these two recursive relations (31) and (32), one obtains the general expression for the boundary correlation functions as

\[ F_N^{(r-s, \ldots, s)}(\{\lambda\}, \{\nu\}) = \frac{1}{\det M(\{\lambda\}, \{\nu\})} \prod_{j=1}^{s} \prod_{\beta=r+1}^{N} \frac{b(\lambda_{s}, \nu_{j})}{\prod_{\alpha=1}^{N} a(\lambda_{s}, \nu_{j})} \sum_{\alpha=1}^{N} \sum_{\alpha \neq \alpha_{1}, \ldots, \alpha_{s-1}} \cdots \sum_{\alpha_{s-1} \neq \alpha_{s-1}} \frac{1}{H^{s}_{\alpha_{s-1} \cdots \alpha_{s}}} \]

\[ \times (-1)^{\sum_{k=1}^{s} \chi(\alpha_{k}, \alpha_{s}) + \sum_{k=1}^{s} (a_{k} - 1 - r(e_{k} + 1)/2) + \sum_{k=1}^{s} (e_{k} + 1)(N-k)/2} \]

\[ \times \prod_{j=1}^{s} H^{s}_{\alpha_{s}}(\lambda_{\alpha_{s}}) \prod_{1 \leq j < k \leq s} E^{s+}_{\alpha_{j}}(\lambda_{\alpha_{j}}, \lambda_{\alpha_{k}}, \nu_{j}, \nu_{k}) \det M(\{\lambda\} \setminus \{\lambda_{s}\}, \{\nu\} \setminus \{\nu_{j}, \nu_{k}\}), \]  

(33)

where \( S_{N}^{N} = \{1, \ldots, r\}, S_{\alpha}^{N,r} = \{r+1, \ldots, N\}, e(\lambda, \nu) = sh(\lambda - \nu + \eta) \),

\[ H^{s}_{\alpha}(\lambda) = \prod_{\beta=r+1}^{N} e(\lambda_{\alpha}, \lambda_{\beta}) \prod_{k=1}^{N} b(\lambda_{s}, \nu_{k}) \]

\[ H^{s}_{\alpha}(\lambda) = \prod_{\beta=r+1}^{N} e(\lambda_{\alpha}, \lambda_{\beta}) \prod_{k=1}^{N} a(\lambda_{s}, \nu_{k}) \]

\[ E^{s+}_{\alpha_{j}}(\lambda_{\alpha_{j}}, \lambda_{\alpha_{k}}, \nu_{j}, \nu_{k}) = \frac{a(\lambda_{\alpha_{j}}, \nu_{k})}{d(\lambda_{\alpha_{j}}, \lambda_{\alpha_{k}})} \]

\[ E^{s+}_{\alpha_{j}}(\lambda_{\alpha_{j}}, \lambda_{\alpha_{k}}) = \frac{b(\lambda_{\alpha_{j}}, \nu_{j})}{d(\lambda_{\alpha_{j}}, \lambda_{\alpha_{k}})} \]

\[ E^{s+}_{\alpha_{j}}(\lambda_{\alpha_{j}}, \lambda_{\alpha_{k}}) = \frac{c(\lambda_{\alpha_{j}}, \nu_{k})}{d(\lambda_{\alpha_{j}}, \lambda_{\alpha_{k}})} \]

and \( \chi(\beta, \alpha) = 1 \) for \( \beta > \alpha \) and 0 otherwise. The proof is given in the Appendix. As a special case \((\epsilon_{j} = -j, j = 1, \ldots, s)\), the boundary correlation function reduces to the emptiness formation probability [29] (cf. [10]), which gives the probability of finding a sequence of all spins down of length \( s \) from the left boundary.
4 Nineteen vertex model

In this section, we consider the nineteen (spin-1 or Fateev-Zamolodchikov) vertex model. The nineteen vertex model can be constructed from the gauge transformed spin-1/2 $R$-matrix

$$R^+_{jk}(\lambda, \nu) = \phi_j(\lambda)\phi_k(\nu)R_{jk}(\lambda, \nu)\phi_j^{-1}(\lambda)\phi_k^{-1}(\nu),$$

(40)

where $\phi(\lambda) = \text{diag}(1, e^\lambda)$ and the projection operator

$$P = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{e^n}{2ch^2\eta} & 0 & 0 \\
0 & \frac{1}{2ch^2\eta} & \frac{1}{2ch^2\eta} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},$$

(41)

The basis (dual basis) of the spin-1 representation $\{|+\rangle \otimes |-\rangle, \{-\rangle \otimes |+\rangle\}$ is given in terms of basis of spin-1/2 representation as $|1\rangle = |+\rangle \otimes |+\rangle$, $|0\rangle = (1 + e^{-2\eta}(-1/2(|+\rangle \otimes |-\rangle + e^{-\eta}(-\rangle \otimes |+\rangle), |1\rangle = |+\rangle \otimes |+\rangle$, $|0\rangle = |+\rangle \otimes |+\rangle$. The gauge transformed spin-1 $R$-matrix can be constructed as $[31, 32, 33]$

$$R^+_{jk}(z, w) = P_{2K-1,2K}P_{2J,2J-1}R^+_{2J,2J-1}(z + \eta, w)R^+_{2J,2J-1}(w + \eta, z + \eta) \times R^+_{2J-1,2J-1}(z, w)R^+_{2J-1,2J-1}(w, z + \eta)P_{2J,2J-1}P_{2K-1,2K}.$$

(42)

The symmetric spin-1 $R$-matrix can be obtained from $R^+_{12}(z, w)$ by gauging out factors as

$$R^+_{12}(z, w) = \Phi^{-1}_{,j}(z)\Phi^{-1}_{,k}(w)R^+_{12}(z, w)\Phi_{,j}(z)\Phi_{,k}(w),$$

(43)

where $\Phi(z) = \text{diag}(1, e^z, e^{2z})$.

For the nineteen vertex model on a $N \times N$ lattice with domain wall boundary condition, all spins are aligned $+1$ at the bottom and right boundaries, and $-1$ at the top and left boundaries. At the intersection of the $\alpha$-th row (from the bottom) and the $k$-th column (from the left), the statistical weight $L^\alpha_k(z, w_k) = R^\alpha_k(z, w_k)$ is associated. We also set $L^\alpha_k(z, w_k) = R^\alpha_k(z, w_k)$, $L^\alpha_k(z, w_k) = R^\alpha_k(z, w_k)$, $L^\alpha_k(z, w_k) = R^\alpha_k(z, w_k)$ for later convenience.

We consider the boundary correlation functions for this nineteen vertex model

$$F^1_N(r, \delta_1, \cdots, \delta_s)(\{z\}, \{w\}) = \frac{F^1_N(r, \delta_1, \cdots, \delta_s)(\{z\}, \{w\})}{Z^+_N(\{z\}, \{w\})},$$

(44)

$$Z^+_N(\{z\}, \{w\}) = a(1||-1|| \prod_{\alpha=1}^N L^\alpha_k(z, w_k)|| - 1)_q, \tag{45}$$

$$\tilde{F}^1_N(r, \delta_1, \cdots, \delta_s)(\{z\}, \{w\}) = a(1||-1|| \prod_{\alpha=r+1}^N \prod_{k=1}^N L^\alpha_k(z, w_k) \prod_{k=1}^N \pi_k^\alpha \prod_{\alpha=1}^N \prod_k L^\alpha_k(z, w_k)|| - 1)_q, \tag{46}$$

where $\pi_k^\alpha = |\delta_k\rangle_kk\langle \delta_k|$, $\delta_k = 1, 0, -1$ and $||1\rangle = \otimes_{k=1}^N |1\rangle_k$, $||-1\rangle = \otimes_{k=1}^N |1\rangle_k$, $\langle 1\rangle = \otimes_{k=1}^N \langle 1\rangle_k$, $\langle -1\rangle = \otimes_{k=1}^N \langle -1\rangle_k$.

We show that the above boundary correlation functions can be reduced to those for the six vertex model calculated in the previous section. Instead of directly dealing with (44), we consider

$$F^1_N(r, \delta_1, \cdots, \delta_s)(\{z\}, \{w\}) = \frac{F^1_N(r, \delta_1, \cdots, \delta_s)(\{z\}, \{w\})}{Z^+_N(\{z\}, \{w\})}, \tag{47}$$

$$Z^+_N(\{z\}, \{w\}) = Z^+_N(\{z\}, \{w\})_{L^+L^+}, \tag{48}$$

$$\tilde{F}^1_N(r, \delta_1, \cdots, \delta_s)(\{z\}, \{w\}) = \tilde{F}^1_N(r, \delta_1, \cdots, \delta_s)(\{z\}, \{w\})_{L^+L^+}. \tag{49}$$
We also define $F_{N}^{1/2(r,\cdots,\cdots)}$ and $F_{N}^{1/2(r,\cdots,\cdots)}$ as well, replacing $L^1$ by $L^{1/2}$ and $L^{1/2}$, respectively. From (33), one can see

$$F_{N}^{1+(r,\delta_1,\cdots,\delta_s)}(\{z\},\{w\}) = F_{N}^{1+(r,\delta_1,\cdots,\delta_s)}(\{z\},\{w\}),$$  

(50)

since

$$Z_{N}^{1+}(\{z\},\{w\}) = e^{2\sum_{k=1}^{N} w_{k} - 2 \sum_{\alpha=1}^{N} (z_{\alpha} - \eta/2)} Z_{N}^{1}(\{z\},\{w\}),$$  

(51)

$$\tilde{F}_{N}^{1+(r,\delta_1,\cdots,\delta_s)}(\{z\},\{w\}) = e^{2\sum_{k=1}^{N} w_{k} - 2 \sum_{\alpha=1}^{N} (z_{\alpha} - \eta/2)} \tilde{F}_{N}^{1+(r,\delta_1,\cdots,\delta_s)}(\{z\},\{w\}).$$  

(52)

Thus, we can consider $F_{N}^{1+(r,\delta_1,\cdots,\delta_s)}(\{z\},\{w\})$ instead, which is easier to handle than $F_{N}^{1+(r,\delta_1,\cdots,\delta_s)}(\{z\},\{w\})$ itself.

We reduce $F_{N}^{1+(r,\delta_1,\cdots,\delta_s)}(\{z\},\{w\})$ to the boundary correlation functions of the six vertex model by use of fusion. The following relations are used.

$$P_{2K-1,2K}^{\delta_{+}}(z + \eta, w)R_{2J,2K}^{\delta_{+}}(z, w) = P_{2K-1,2K}^{\delta_{+}}(z + \eta, w)R_{2J,2K}^{\delta_{+}}(z, w),$$  

(53)

$$P_{2J,2J-1}^{\delta_{+}}(z + \eta, w)R_{2J,2J-1}^{\delta_{+}}(z, w) = P_{2J,2J-1}^{\delta_{+}}(z + \eta, w)R_{2J,2J-1}^{\delta_{+}}(z, w),$$  

(54)

$$P_{2k-2k}^{\delta_{+}}(z + \eta, w)R_{2k-2k}^{\delta_{+}}(z, w) = P_{2k-2k}^{\delta_{+}}(z + \eta, w)R_{2k-2k}^{\delta_{+}}(z, w),$$  

(55)

$$\pi_{k}^{\delta_{+}} = \frac{1}{2} \sum_{\epsilon_{k-1},\epsilon_{k} = \pm} C_{\epsilon_{k-1},\epsilon_{k}}^{\delta_{+}} C_{\epsilon_{k-1},\epsilon_{k}}^{\delta_{+}}.$$  

(56)

where $C_{\sigma_{+}}^{\delta_{+}} = C_{\sigma_{-}}^{\delta_{-}} = C_{\sigma_{-}}^{\delta_{+}} = 1$ and 0 otherwise.

First, utilizing (53), (54) and (55), one has

$$F_{N}^{1+(r,\delta_1,\cdots,\delta_s)}(\{z\},\{w\}) = \prod_{k=1}^{N} P_{2k-2k}^{\delta_{+}}(z + \eta, w)R_{2k-2k}^{\delta_{+}}(z, w)$$  

$$\times \prod_{\alpha=r+1}^{N} \tilde{T}_{\alpha}(z_{\alpha},\{w\})(\underbrace{||}_{\alpha}) + \langle \psi, \rangle,$$  

(57)

where $|| = \otimes_{k=1}^{2N}(\alpha_{k},\epsilon_{k})$, $\langle \psi, \rangle = \otimes_{k-1}^{2N}(\alpha_{k},\epsilon_{k})$, $\langle \psi, \rangle = \otimes_{k=1}^{2N}(\alpha_{k},\epsilon_{k})$, $\langle \psi, \rangle = \otimes_{k=1}^{2N}(\alpha_{k},\epsilon_{k})$, and

$$\tilde{T}_{\alpha}(z_{\alpha},\{w\}) = \prod_{k=1}^{N} L_{2\alpha,2k}^{1/2}(z_{\alpha} + \eta, w_{k})L_{2\alpha,2k}^{1/2}(z_{\alpha} + \eta, w_{k})$$  

$$\times \prod_{k=1}^{N} L_{2\alpha,2k-1}^{1/2}(z_{\alpha}, w_{k})L_{2\alpha,2k-1}^{1/2}(z_{\alpha}, w_{k}).$$  

(58)

Applying (59), we have

$$F_{N}^{1+(r,\delta_1,\cdots,\delta_s)}(\{z\},\{w\}) = \prod_{k=1}^{N} P_{2k-2k}^{\delta_{+}}(z + \eta, w)R_{2k-2k}^{\delta_{+}}(z, w)$$  

$$\times \prod_{\alpha=r+1}^{N} \tilde{T}_{\alpha}(z_{\alpha},\{w\})(\underbrace{||}_{\alpha}) + \langle \psi, \rangle,$$  

(59)

$$\times \prod_{\alpha=r+1}^{N} \tilde{T}_{\alpha}(z_{\alpha},\{w\})(\underbrace{||}_{\alpha}) + \langle \psi, \rangle.$$  

(60)
Using (53) and (55), one gets
\[ F_N^{1+ \{r, \delta_1, \cdots, \delta_s\}}(\{z\}, \{w\}) = \sum_{\epsilon_1, \cdots, \epsilon_{2s} = \pm} \prod_{k=1}^{s} C_{\epsilon_{2k-1}, 1-\epsilon_{2k}}^{\delta_k} \times \prod_{k=1}^{s} \frac{\pi_{\epsilon_{2k-1}}}{\pi_{\epsilon_{2k}}} \prod_{\alpha=r+1}^{N} \tilde{T}_\alpha(z_{\alpha}, \{w\}) \]
\[ \times \prod_{k=1}^{s} \frac{\pi_{\epsilon_{2k-1}}}{\pi_{\epsilon_{2k}}} \ni_{\alpha}^{q-z_{\alpha}} \ni_{\alpha}^{-q} \]
\[ = \sum_{\epsilon_1, \cdots, \epsilon_{2s} = \pm} \prod_{k=1}^{s} C_{\epsilon_{2k-1}, 1-\epsilon_{2k}}^{\delta_k} F_N^{1/2+(2r, \epsilon_1, \cdots, \epsilon_{2s})}(\{\tilde{z}\}, \{\tilde{w}\}), \]
(61)
where \( \{\tilde{z}\} = \{z_1, z_1 + \eta, z_2, z_2 + \eta, \cdots, z_N, z_N + \eta\} \), \( \{\tilde{w}\} = \{w_1 + \eta, w_1, w_2 + \eta, w_2, \cdots, w_N + \eta, w_N\} \).

As the simplest case, one has [18] \( F_N^{1/2+(2r, \epsilon_1, \cdots, \epsilon_{2s})}(\{\tilde{z}\}, \{\tilde{w}\}) = \tilde{Z}_N^{1/2+}(\{\tilde{z}\}, \{\tilde{w}\}) \).

Thus we have
\[ F_N^{1+ \{r, \delta_1, \cdots, \delta_s\}}(\{z\}, \{w\}) = \sum_{\epsilon_1, \cdots, \epsilon_{2s} = \pm} \prod_{k=1}^{s} C_{\epsilon_{2k-1}, 1-\epsilon_{2k}}^{\delta_k} F_N^{1/2+(2r, \epsilon_1, \cdots, \epsilon_{2s})}(\{\tilde{z}\}, \{\tilde{w}\}). \]
(63)

From (60), one can see
\[ F_N^{1/2+(2r, \epsilon_1, \cdots, \epsilon_{2s})}(\{\tilde{z}\}, \{\tilde{w}\}) = F_N^{1/2+(2r, \epsilon_1, \cdots, \epsilon_{2s})}(\{\tilde{z}\}, \{\tilde{w}\}), \]
(64)
since
\[ Z_N^{1/2+}(\{\tilde{z}\}, \{\tilde{w}\}) = e^{2(2r-2)} \sum_{z_{\alpha=1}}^{N} z_{\alpha} \sum_{=1}^{N} \eta_{\alpha} \tilde{Z}_N^{1/2+}(\{\tilde{z}\}, \{\tilde{w}\}), \]
\[ \tilde{F}_N^{1/2+(2r, \epsilon_1, \cdots, \epsilon_{2s})}(\{\tilde{z}\}, \{\tilde{w}\}) = e^{2(2r-2)} \sum_{z_{\alpha=1}}^{N} z_{\alpha} \sum_{=1}^{N} \eta_{\alpha} \tilde{F}_N^{1/2+(2r, \epsilon_1, \cdots, \epsilon_{2s})}(\{\tilde{z}\}, \{\tilde{w}\}). \]
(65)

Combining (61), (62) and (63), one finally has
\[ F_N^{1+ \{r, \delta_1, \cdots, \delta_s\}}(\{z\}, \{w\}) = \sum_{\epsilon_1, \cdots, \epsilon_{2s} = \pm} \prod_{k=1}^{s} C_{\epsilon_{2k-1}, 1-\epsilon_{2k}}^{\delta_k} F_N^{1/2+(2r, \epsilon_1, \cdots, \epsilon_{2s})}(\{\tilde{z}\}, \{\tilde{w}\}), \]
(67)

which means that the boundary correlation functions for the nineteen vertex model on an \( N \times N \) lattice with spectral parameters \( \{z\}, \{w\} \) can be reduced to those for the six vertex model on a \( 2N \times 2N \) lattice with spectral parameters \( \{\tilde{z}\}, \{\tilde{w}\} \). Note that \( F_N^{1/2+(2r, \epsilon_1, \cdots, \epsilon_{2s})}(\{\tilde{z}\}, \{\tilde{w}\}) \) is exactly \( F_N^{1/2+(2r, \epsilon_1, \cdots, \epsilon_{2s})}(\{\tilde{z}\}, \{\tilde{w}\}) \) in the previous section since the corresponding Boltzmann weights are different just by an overall factor, which do not affect correlation functions.

5 Homogeneous limit of the emptiness formation probability

Let us consider the homogeneous limit of the emptiness formation probability (EFP) for the nineteen vertex model. As a special case of (67), one has
\[ F_N^{1+ \{r, \delta\}}(\{z\}, \{w\}) = F_N^{2/(2r, \delta)}(\{\tilde{z}\}, \{\tilde{w}\}), \]
(68)
i.e., the EFP of length \( s \) for the nineteen vertex model with spectral parameters \( \{z\}, \{w\} \) reduces to the EFP of length \( 2s \) for the six vertex model with spectral parameters \( \{\tilde{z}\} = \{z_1, z_1 + \eta, z_2, z_2 + \eta, \cdots, z_N, z_N + \eta\} \).
\[ \eta, \cdots, z_N, z_N + \eta \}, \{ \bar{w} \} = \{ w_1 + \eta, w_1, w_2 + \eta, w_2, \cdots, w_N + \eta, w_N \}. \] Let us set \( z_j = z + \xi_j \). One finds that \( F_{2N}^{(2r,(-)^{2s})}(\{ z \}, \{ \bar{w} \}) \) can be expressed in the determinant form as

\[
F_{2N}^{(2r,(-)^{2s})}(\{ z \}, \{ \bar{w} \}) = \frac{X_1 \cdot X_2 X_3}{\det M(\{ z \}, \{ \bar{w} \}) \cdot \det \Psi \cdot \prod_{1 \leq j < k < 2s} e(z + \epsilon_j, z + \epsilon_k)} |_{\epsilon_1 = \cdots = \epsilon_{2s} = 0},
\]

where

\[
X_1 = \frac{\prod_{j=1}^{2s} \prod_{k=j}^{N} d(w_j + \eta, w_k) \prod_{k=j+1}^{N} d^2(w_j, w_k) \prod_{k=j+1}^{N} d(w_j, w_k + \eta)}{\prod_{j=1}^{2s} \prod_{k=j}^{N} a(z_\beta + \eta, w_j)a^2(z_\beta, w_j)a(z_\beta, w_j + \eta)[\prod_{k=r+1}^{N} b(z_\beta + \eta, w_j)b^2(z_\beta, w_j)b(z_\beta, w_j + \eta)]},
\]

\[
X_2 = \frac{\prod_{j=1}^{2s} \prod_{k=j}^{N} e(z_\beta, z + \epsilon_j)e(z_\beta + \eta, z + \epsilon_j)[\prod_{k=r+1}^{N} d(z_\beta + \eta, z + \epsilon_j)]}{\prod_{j=1}^{2s} \prod_{k=1}^{N} b(z + \epsilon_j, w_k)b(z + \epsilon_j, w_k + \eta)},
\]

\[
X_3 = \sum_{j=1}^{s} \sum_{k=j}^{s} a(z + \epsilon_{2j-1}, w_k) \prod_{k=j+1}^{s} a(z + \epsilon_{2j-1}, w_k + \eta)
\]

\[
\times \prod_{k=2}^{s} b(z + \epsilon_{2k-1}, w_j) \prod_{k=1}^{s-1} b(z + \epsilon_{2k-1}, w_j + \eta)
\]

\[
\times \prod_{k=1}^{s} b(z + \epsilon_{2k}, w_j) \prod_{j=1}^{k-1} b(z + \epsilon_{2k}, w_j + \eta),
\]

and \( \Psi \) is a \( 2N \times 2N \) matrix whose \((j, k)\)-th block matrix element is given by

\[
\left( \begin{array}{cc} \exp(\xi_j \partial_{z_{2k-1}}) & \exp(\xi_j \partial_{z_{2k}}) \\ \exp((\xi_j + \eta) \partial_{z_{2k-1}}) & \exp((\xi_j + \eta) \partial_{z_{2k}}) \end{array} \right),
\]

for \( j = 1, \cdots, N, k = 1, \cdots, s \) and

\[
\left( \begin{array}{cc} \varphi(z_j, w_k + \eta) & \varphi(z_j, w_k) \\ \varphi(z_j + \eta, w_k + \eta) & \varphi(z_j + \eta, w_k) \end{array} \right),
\]

for \( j = 1, \cdots, N, k = s + 1, \cdots, N \).

Now let us take the homogeneous limit by putting \( \xi_j, w_j, j = 1, \cdots, N \) to zero in the order \( w_1 \to 0, \cdots, w_N \to 0, \xi_1 \to 0, \cdots, \xi_N \to 0 \). We have

\[
F_{2N}^{(2r,(-)^{2s})} = Y_1 \cdot \det m \cdot \frac{Y_2 Y_3}{\det \Psi \cdot \prod_{1 \leq j < k < 2s} \sh(\epsilon_j - \epsilon_k + \eta)} |_{\epsilon_1 = \cdots = \epsilon_{2s} = 0},
\]

10
where

\[
Y_1 = \frac{(-1)^{sN-s(s+1)/2} \prod_{j=1}^{s} (N-j)!}{a^{s}(z + \eta, 0)b(N-r)s(z + \eta, 0)a^{2s}(z, 0)b^{2(N-r)}s(z, 0)a^{s}(z - \eta, 0)b(N-r)s(z - \eta, 0)},
\]

\[
Y_2 = \prod_{j=1}^{2s} \frac{\text{sh}'(-\epsilon_j + 2\eta)\text{sh}N(-\epsilon_j + \eta)\text{sh}N-r(-\epsilon_j)}{\text{sh}^N(\epsilon_j + z - \eta/2)\text{sh}^N(\epsilon_j + z - 3\eta/2)},
\]

\[
Y_3 = \prod_{j=1}^{s} \text{sh}^{s-j+1}(z + \epsilon_{2j-1} + \eta/2)\text{sh}^{s-j}(z + \epsilon_{2j-1} - \eta/2)
\times \prod_{j=1}^{s} \text{sh}^{s-j}(z + \epsilon_{2j} + \eta/2)\text{sh}^{s-j}(z + \epsilon_{2j} - \eta/2)
\times \prod_{k=2}^{s} \text{sh}^{k-1}(z + \epsilon_{2k-1} - \eta/2)\text{sh}^{k-1}(z + \epsilon_{2k-1} - 3\eta/2)
\times \prod_{k=1}^{s} \text{sh}^{k-1}(z + \epsilon_{2k} - \eta/2)\text{sh}^{k}(z + \epsilon_{2k} - 3\eta/2),
\]

\[m\] is a \(2N \times 2N\) matrix whose \((j,k)\)-th block matrix element is given by

\[
\begin{pmatrix}
\partial_z^{j+k-2} \varphi(z - \eta, 0) & \partial_z^{j+k-2} \varphi(z, 0) \\
\partial_z^{j+k-2} \varphi(z, 0) & \partial_z^{j+k-2} \varphi(z + \eta, 0)
\end{pmatrix},
\]

for \(j, k = 1, \cdots, N\), and \(\psi\) is a \(2N \times 2N\) matrix whose \((j,k)\)-th block matrix element is given by

\[
\begin{pmatrix}
\partial_{\epsilon_{2k-1}}^{j+k-2} & \partial_{\epsilon_{2k}}^{j+k-2} \\
\exp(\eta \partial_{\epsilon_{2k-1}}) \partial_{\epsilon_{2k-1}}^{j+k-2} & \exp(\eta \partial_{\epsilon_{2k}}) \partial_{\epsilon_{2k}}^{j+k-2}
\end{pmatrix},
\]

for \(j = 1, \cdots, N, k = 1, \cdots, s\) and

\[
\begin{pmatrix}
\partial_z^{j+k-s-2} \varphi(z - \eta, 0) & \partial_z^{j+k-s-2} \varphi(z, 0) \\
\partial_z^{j+k-s-2} \varphi(z, 0) & \partial_z^{j+k-s-2} \varphi(z + \eta, 0)
\end{pmatrix},
\]

for \(j = 1, \cdots, N, k = s + 1, \cdots, N\).

### 6 Conclusion

In this paper, we considered correlation functions for the six and nineteen vertex models on an \(N \times N\) lattice with domain wall boundary conditions. For the six vertex model, we derived the general expression for the boundary correlation function by solving two recursive relations obtained by the quantum inverse scattering method. The result includes the boundary one point functions and emptiness formation probability as special cases.

For the nineteen vertex (Fateev-Zamolodchikov) model, by use of fusion, we have shown that the boundary correlation functions reduce to those for the six vertex model. In particular, the emptiness formation probability of length \(s\) for the nineteen vertex model on an \(N \times N\) lattice reduces to that of length \(2s\) for the six vertex model on a \(2N \times 2N\) lattice with appropriate spectral parameters.

The correlation functions "off the boundary" can in principle be expressed as a linear sum of the boundary correlation functions obtained in this paper. However, since this means we need to sum over intermediate spin states, the expression gets complicated. Simplifying the expression of the correlation functions "off the boundary" is an important problem to be considered in the future.

Another interesting problem is to extend the analysis to other models or boundary conditions such as higher rank models, Felderhof model, reflecting end, etc.
Appendix

We prove (39) by induction. We show the expression holds for lattice size \( N \) and \( \epsilon_1 = - \) from the recursive relation (31). One can similarly show for \( \epsilon_1 = + \) from (32). Suppose (33) holds for lattice size \( N - 1 \). For \( \alpha_1 \in S^{N-r}_r \), one has

\[
\frac{1}{\det M(\{\lambda\} \setminus \{\alpha_1\}, \{\nu\})} \prod_{k=j+1}^{N} d(\nu_j, \nu_k) \prod_{\beta=1}^{N} a(\lambda_{\beta}, \nu_{\beta}) \prod_{\beta=r+1}^{N} b(\lambda_{\beta}, \nu_{\beta}) \sum_{\alpha_2 \in S^{N-r}_r} \cdots \sum_{\alpha_s \neq \alpha_1, \ldots, \alpha_{s-1}} (-1)^{\sum_{i<j} \chi(\alpha_i, \alpha_j) + \sum_{i=1}^s (\alpha_i - 1 - r (\epsilon_1 + 1)/2) + \sum_{k=1}^s (\epsilon_1 + 1)(N-k)/2} \times \prod_{j=2}^{s} H_{j}^{(\alpha_{j})} (\lambda_{\alpha_j}, \lambda_{\alpha_j}, \nu_j, \nu_k) \prod_{2 \leq j < k \leq s} E_{j \lambda}^{(\nu_j, \nu_k)} (\lambda_{\alpha_j}, \lambda_{\alpha_j}, \nu_j, \nu_k) \end{equation}

where

\[
m^{+, -(\lambda_{\alpha_j}, \lambda_{\alpha_j}, \nu_j)} = \frac{a(\lambda_{\alpha_1}, \nu_1)}{d(\lambda_{\alpha_1}, \lambda_{\alpha_1})}, \quad m^{-, -(\lambda_{\alpha_j}, \lambda_{\alpha_j}, \nu_j)} = \frac{b(\lambda_{\alpha_1}, \nu_1)}{e(\lambda_{\alpha_1}, \lambda_{\alpha_1})}.
\]

We also have the following recursive relation (7) for the partition function

\[
\frac{Z_N - (\{\lambda\} \setminus \{\alpha_1\}, \{\nu\})}{Z_N (\{\lambda\}, \{\nu\})} = \frac{(-1)^{\alpha - 1}}{a(\lambda_{\alpha_1}, \nu_1)} \prod_{\beta=1}^{N} d(\lambda_{\beta}, \lambda_{\beta}) \prod_{k=2}^{N} a(\lambda_{\beta}, \nu_k) b(\lambda_{\beta}, \nu_k) \times \frac{\det M(\{\lambda\} \setminus \{\alpha_1\}, \{\nu\})}{\det M(\{\lambda\}, \{\nu\})}.
\]

Combining (31), (A.1) and (A.3), one has

\[
\frac{1}{\det M(\{\lambda\}, \{\nu\})} \prod_{j=1}^{s} \prod_{\beta=1}^{N} d(\nu_j, \nu_k) \prod_{\beta=1}^{N} a(\lambda_{\beta}, \nu_{\beta}) \prod_{\beta=r+1}^{N} b(\lambda_{\beta}, \nu_{\beta}) \sum_{\alpha_2 \in S^{N-r}_r} \cdots \sum_{\alpha_s \neq \alpha_1, \ldots, \alpha_{s-1}} (-1)^{\sum_{i<j} \chi(\alpha_i, \alpha_j) + \sum_{i=1}^s (\alpha_i - 1 - r (\epsilon_1 + 1)/2) + \sum_{k=1}^s (\epsilon_1 + 1)(N-k)/2} \times \prod_{j=2}^{s} H_{j}^{(\alpha_{j})} (\lambda_{\alpha_j}, \lambda_{\alpha_j}, \nu_j, \nu_k) \prod_{2 \leq j < k \leq s} E_{j \lambda}^{(\nu_j, \nu_k)} (\lambda_{\alpha_j}, \lambda_{\alpha_j}, \nu_j, \nu_k) \end{equation}

\[
eq \epsilon \prod_{\beta=1}^{N} f(\lambda_{\beta}, \lambda_{\beta}) \prod_{\beta=1}^{N} d(\lambda_{\beta}, \lambda_{\alpha_1}) \prod_{\beta=1}^{N} b(\lambda_{\alpha_1}, \nu_k) (\lambda_{\alpha_j}, \lambda_{\alpha_j}, \nu_j) \prod_{j=2}^{s} a(\lambda_{\alpha_1}, \nu_j) m^{+, -(\lambda_{\alpha_j}, \lambda_{\alpha_j}, \nu_j)} b(\lambda_{\alpha_1}, \nu_k) \prod_{\beta=1}^{N} d(\lambda_{\beta}, \lambda_{\alpha_1}) \prod_{\beta=1}^{N} b(\lambda_{\alpha_1}, \nu_k) \cdot H_{j}^{(\lambda_{\alpha_j})} (\lambda_{\alpha_j}, \lambda_{\alpha_j}, \nu_j, \nu_k) = H_{j \lambda}^{(\alpha_{j})} (\lambda_{\alpha_j}, \lambda_{\alpha_j}, \nu_j, \nu_k).
\]

one can see that (A.4) is exactly the expression (39) for \( \epsilon_1 = - \).
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