Self - Interaction Force for the Particle in the Cone Space - Time

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Abstract
The force acting on the charged particle moving along an arbitrary trajectory near the straight cosmic string is calculated. This interaction leads to the scattering of particles by the cosmic string. The scattering cross section is considered.

1 Introduction
Cosmic strings may have resulted from phase transitions in the early Universe [1,2]. Stability of these formations is insured by the appropriate non-zero topological charge (winding number). From the well-known solutions of field equations describing gravitating cosmic strings [3-7] one can see that the energy density is located in a small threadlike region of space, which is why one may approximate them by infinitely thin curves. In this case the cosmic strings has no Newtonian potential [8] and the space-time is flat.

At the same time the non-local display of cosmic strings exists. The cosmic string produces double images through gravitational lensing [8-10]. If a resting particle has electric charge then there will be a repulsive force between

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the string and the particle [11]. A charged particle radiates electromagnetic waves even if it moves uniformly near the cosmic string [12]. There is also the Aharonov-Bohm interaction of cosmic strings with matter [13,19].

In this article we analyse in more detail the force of self-interaction of a charged particle near a straight cosmic string. We obtain this force for the arbitrary trajectory of the particle. In the simplest case of a particle at rest we receive a result which differs from that obtained in Ref.11.

2 Solution of Maxwell’s equations

The spacetime of a straight cosmic string is described by the metric [8]

\[ ds^2 = dt^2 - dz^2 - d\rho^2 - b^2 \rho^2 d\varphi^2 ; \quad b \leq 1 . \] (1)

Spacetime with metric (1) is the product of two dimensional pseudo-Euclidian subspace \((t, z)\) and two dimensional Euclidian cone space \((\rho, \varphi)\), which allows us the Maxwell’s equations

\[ F_{ik} = -\frac{4\pi}{c} j^i = -\frac{4\pi e}{\sqrt{-g}} \int d\tau \delta^{(4)}(x - x(\tau)) u^i(\tau) ; \quad F_{(ik;t)} = 0 , \] (2)

in the Lorentz gauge split into two systems of equations

\[ (\partial_t^2 - \Delta_{(s)}) A^{(s)} = \frac{4\pi}{c} j^{(s)} , \] (3)

where

\[ \Delta_{(s)} = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \left( \frac{\partial}{b \partial \varphi} + i s \right)^2 + \frac{\partial^2}{\partial z^2} , \] (4)

\[ s = 0, 1 ; \quad A^{(1)} = A_\rho + \frac{i}{b \rho} A_\varphi ; \quad A^{(0)} = (A_t, A_z) . \]

Taking the Fourier transformation

\[ A(\omega) = \int_{-\infty}^{+\infty} dt A(t) e^{-i\omega t} , \]
we obtain the following equations

\[ (\Delta_{(s)} + \omega^2) A^{(s)}(r, \omega) = \frac{4\pi}{c} j^{(s)}(r, \omega) . \] (5)
The retarded Green function satisfying the expression
\[(\Delta_{(s)} + \omega^2)G_{(s)}^{(\text{ret})}(\mathbf{r}, \mathbf{r}'; \omega) = -\delta(\rho - \rho')\delta(\varphi - \varphi')\delta(z - z')/b\rho , \quad (6)\]
has been calculated in Ref.12.

This function has the following form
\[G_{(s)}^{(\text{ret})}(\mathbf{r}, \mathbf{r}'; \omega) = -\delta(\rho - \rho')\delta(\varphi - \varphi')\delta(z - z')/b\rho , \quad (6)\]

Hereafter we set \(\Delta z = z - z', \Delta \varphi = \varphi - \varphi', \nu = 1/b \geq 1\); and \(J_p(x)\) - the Bessel function. The Green function (7) can be found by expanding over the full set of eigenfunctions of the operator \(\Delta_{(s)}\).

Then the solution of equation (3) maybe presented in the following form
\[A_{(s)}^{(s)}(\mathbf{r}, t) = -2e^{\int_{\infty}^{-\infty} d\tau \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{i\omega(t-t(\tau))+ik(z-z(\tau))}} \times \sum_{n=\infty}^{+\infty} \frac{1}{2\pi b} e^{i n \Delta \varphi} \int_{0}^{+\infty} \frac{\lambda d\lambda}{(\omega - i 0)^2 - k^2 - \lambda^2} J_{|n\nu+q|}(\lambda \rho) J_{|n\nu+q|}(\lambda \rho') , \quad (8)\]

where \(u^{(1)}(\tau) = u_{p}(\tau) + i\nu u_{\varphi}(\tau)/\rho(\tau)\), \(u^{(0)}(\tau) = (u_{t}(\tau), u_{z}(\tau))\).

Integrating successively over the \(\omega, k, \lambda\) and, where possible, summing the series we obtain (see appendix):
\[A_{(s)}^{(s)} = e^{\nu \int_{-\infty}^{\infty} d\tau u^{(s)}(\tau)\theta(t - t(\tau))} \cdot \frac{\theta(q)}{\rho(\tau)\sqrt{4q|q - 1|}} \times \{\theta(1 - q)Q(s) + \theta(q - 1)Q_{\nu}(s)\} \quad (9)\]

where
\[q = [(t - t(\tau))^2 - (z - z(\tau))^2 - (\rho - \rho(\tau))^2]/4\rho(\tau) , \beta = 2q - 1 + \sqrt{4q(q - 1)} , \]

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\( Q(s) = \sum_{n=-\infty}^{\infty} [e^{is \arccos(1-2q)} \delta(\Delta \varphi(\tau) + 2\pi n + \nu \arccos(1-2q)) + e^{-is \arccos(1-2q)} \delta(\Delta \varphi(\tau) + 2\pi n - \nu \arccos(1-2q))] , \quad (10) \)

\[ Q_\nu(s) = -(-1)^s \frac{\sin(\pi \nu)}{\pi} \{ \cosh(s \ln \beta)[\cosh(\nu \ln \beta) \cos(\Delta \varphi(\tau)) - \cos(\pi \nu)] - i \sinh(s \ln \beta) \sinh(\nu \ln \beta) \sin(\Delta \varphi(\tau)) \}/\{ \cosh(\nu \ln \beta) - \cos(\Delta \varphi(\tau) + \pi \nu) \} . \]

From (9)-(11) one can see that the electromagnetic field arising from the particle consists of two parts. The first, singular term, describes the propagation of radiation along the isotropic geodesics; this is because the condition of vanishing of the arguments of \( \delta \) - functions is equivalent to zero value of the square of the geodesic interval between the points of observations and position of charge. The sum of \( \delta \) - functions occurs due to the fact that in the cone spacetime any two events may be connected by several geodesic lines. The smaller the parameter \( b \), the more of these geodesic lines exist. Indeed, when \( b \) is small, the cone is sharper and looks like a cylinder locally, but cylindric space has an infinite number of geodesics connecting two points. If there are several isotropic geodesics connecting a point of observation and a position of charge, or if there are several of these, then all of them must be taken into account.

From expression (10) one may extract the restrictions on the possible values of integer \( n \). Since the function \( \arccos x \) changes from 0 to \( \pi \), we infer the following expression:

\[-\nu \pi \leq \Delta \varphi + 2\pi n \leq \nu \pi . \quad (11)\]

This condition coincides with that from Ref. 14.

The second non-local term vanishes when \( \nu \) is integer. We emphasize that a particle moving uniformly in spacetime (1) does not radiate the electromagnetic wave when the analogy condition is satisfied. Inequality \( q \geq O \) shows that the field at the point of observation is due to the motion of a particle into the cone of past events.
3 Self-interaction force

In order to obtain this force we use the traditional method with the help of which the Dirac - Lorentz force has been calculated [15]. Previously, this method has been used to obtain the gravitational - induced self - interaction force acting on the charged particle situated in the field of strong plane gravitational wave [16].

At the beginning it is necessary to receive in the integral form the tensor of electromagnetic field $F_{ik} = \partial_i A_k - \partial_k A_i$ at the point of observation. The force acting on the charge $e$, moving with velocity $u^k$ at the point of observation has the following form

$$F_i = eF_{ik}u^k.$$ (12)

Next, we situate the point of observation on the world line of charge:

$$x^i = x^i(\tau_1) ; \quad u^k = u^k(\tau_1).$$

Apparently, the electromagnetic field at the point situated on the trajectory of the particle is created by the particle situated at this point. Therefore we may expand the integrand in a power series over $(\tau - \tau_1)$. The first, divergent term proportional to the $du^i/ds$ is removed via renormalization of particle mass. The next term, proportional to the $d^2u^i/ds^2$ is the Dirac - Lorentz force. The other terms of the expansion are equal to zero.

Let us apply this procedure to our case. Electromagnetic field and self-interaction force are split into two parts in accordance with decomposition in (9). The first part leads to the Dirac - Lorentz force. Indeed, the two points at the infinitely short distance are connected by the only geodesic line. By $\Delta \varphi \to 0$ and $\nu < 2$ (this case corresponds to the real physical situation [8]) from condition (12) it follows that $n = 0$.

The second term, being the gravitational-induced self-interaction force, is regular on the charge. Therefore, for simplicity we calculate only the electromagnetic field potential at the location of particle $x^i = x^i(\tau_1)$. They have the following form

$$A_{2k}(x^i(\tau_1)) = e\nu \frac{\sin(\pi \nu)}{\pi} \int_{-\infty}^{+\infty} d\tau \frac{\theta(\Delta t)\theta(q-1)N_k}{\rho(\tau_1)\rho(\tau)\sqrt{4q(q-1)}},$$ (13)

where

$$N_4 = u_4(\tau)H_1(0),$$

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\[ N_z = u_z(\tau)H_1(0), \]
\[ N_\rho = u_\rho(\tau)H_1(1) + u_\phi(\tau)H_2(1)/b\rho(\tau), \]
\[ N_\phi = u_\phi(\tau)H_1(1)\rho(\tau)/\rho(\tau) - u_\rho(\tau)H_2(1)b\rho(\tau_1), \]  \(14\)

and \( H_1 \) and \( H_2 \) are determined via the \(11\)

\[ Q_\nu(s) = \frac{\sin(\pi\nu)}{\pi}\{H_1(s) - iH_2(s)\}. \]  \(15\)

Let us consider the simple case of the resting particle with the following trajectory of motion

\[ t(\tau) = \tau, \quad z(\tau) = z_0, \quad \rho(\tau) = \rho_0, \quad \phi(\tau) = \phi_0. \]  \(16\)

At first, we calculate \( A_{2k} \) at the arbitrary point with coordinates \((t, z, \rho, \phi)\).

Substituting the trajectory \(17\) in \(14\)-(\(16\)) and defining a new variable \( y = \ln \beta(\tau) \) we obtain

\[ A_{24} = -e\frac{\nu\sin(\pi\nu)}{\pi} \int_0^{+\infty} \frac{dy}{\sqrt{\Delta z^2 + \Delta \rho^2 + 4\rho_0 \cosh^2(y/2)\cosh(\nu y)}} \times \frac{\cosh(\nu y) \cos(\Delta \phi) - \cos(\pi\nu)}{[\cosh(\nu y) - \cos(\Delta \phi + \pi\nu)][\cosh(\nu y) - \cos(\Delta \phi - \pi\nu)]}, \]

\[ A_{2\alpha} = 0. \]  \(17\)

At the location of the charge the field \(18\) has form

\[ A_4 = L\frac{e}{\rho_0}, \quad A_\alpha = 0, \]

\[ L(\nu) = -\frac{\nu\sin(\pi\nu)}{2\pi} \int_0^{+\infty} \frac{dy}{\cosh(y/2)} \frac{1}{\cosh(\nu y) - \cos(\pi\nu)}. \]

Thus the gravitational induced self-interaction force has the only component

\[ F^0_\nu = -F_{2\rho} = -e\partial_\rho A_{24} = L\frac{e^2}{\rho_0^2}. \]  \(19\)

When \(\nu - 1\) is small we have the following result:

\[ L \approx \frac{\pi(\nu - 1)}{8} \approx \frac{\pi(1 - b)}{8} = \frac{\pi G\mu}{2c^2}. \]  \(20\)
Next we calculate $A_{1k}$ at the point with coordinates $t = \tau_1, z = z_0, \varphi = \varphi_0, \rho = \rho_1$. Let us consider the $\nu < 2$. Then from condition (12) we have $n = 0$, and

$$A_{14} = \frac{e}{|\rho_0 - \rho_1|}, \quad A_{1\alpha} = 0. \quad (21)$$

Thus in this approach the self-interaction force is different from that calculated in Ref.11. The difference is connected with the coefficient $L$ which may be obtained from results of article [14].

It is possible to calculate self-interaction force for the arbitrary values of $\nu > 1$. In this case the first part of potential $A_{14}$ for the trajectory (17) contains both non-regular part (22) and regular part. Eventually the regular on the particle part of potential has the following form:

$$A_4 = L_0 \frac{e}{\rho_0}, \quad A_\alpha = 0, \quad (22)$$

where

$$L_0(\nu) = \sum_{n=1}^{[\nu/2]} \int_0^1 \frac{dq}{\sqrt{q}} \delta(q - \sin^2(\pi n/\nu)) + L(\nu). \quad (23)$$

The coefficient (24) have the simple form when $\nu$ is integer:

$$L_0(2k + 1) = \sum_{n=1}^{k} |\sin \frac{\pi n}{2k + 1}|^{-1},$$

$$L_0(2k) = \frac{1}{2} + \sum_{n=1}^{k-1} |\sin \frac{\pi n}{2k}|^{-1}. \quad (24)$$

From this expression one can see that the self-interaction force can build up to high values when the $\nu$ increase. Notes, as $\nu \to \infty$, the spacetime of cosmic string tends to the cylindric spacetime [7].

4 Remarks and Conclusion

In this article we investigated the forces acting on the charged particles in the spacetime of a straight cosmic string. Electromagnetic potential is split in two parts (9). The first part is non-regular at the location of the particle and leads to the Dirac-Lorentz force. The second non-local part is regular on the charge and leads to the additional gravitational-induced force.
Electromagnetic potential (14) and, consequently, the self-interaction force, depends on the past history of the charge. In the simple case of the resting particle this force repels the particle from the string.

It must be emphasized that the foregoing interaction leads to the scattering of charged particles by the cosmic string alongside the scattering of matter from cosmic string of radius \( R \) [18] and Aharonov - Bohm scattering of fermions [19]. By virtue of the fact that the interaction between charge and string is the Coulomb repulsion (20) the scattering cross section has the following form

\[
d\sigma_{st} = L_0^2 d\sigma_r = L_0^2 \left( \frac{e^2}{2\varepsilon} \right)^2 \frac{\cos\left(\frac{\theta}{2}\right)}{\sin^3\left(\frac{\theta}{2}\right)} d\theta ,
\]

where \( d\sigma_r \) is Rutherford cross section, \( \theta \) is angle of scattering, \( \varepsilon \) is energy of particle before scattering and \( L_0 \) is given by the equation (24).

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**Appendix**

In order to obtain (9) we must to take into account the following integrals and series: 2.5.25(9), 5.4.12(1) from [15], 2.12.42(16) from [17], and the well-known expression in the theory of distributions

\[
\sum_{n=-\infty}^{+\infty} e^{inx} = 2\pi \sum_{n=-\infty}^{+\infty} \delta(x + 2\pi n) .
\]

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