Gauge and parametrization dependencies of the one-loop counterterms in the Einstein gravity

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Abstract

The parametrization and gauge dependencies of the one-loop counterterms on the mass-shell in the Einstein gravity are investigated. The physical meaning of the loop calculation results on the mass shell and the parametrization dependence of the renormgroup functions in the nonrenormalizable theories are discussed.

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1 Introduction

The construction of quantum gravity is one of the fundamental problems in the modern theoretical physics. The coincidence of the Einstein gravity with all experimental results is very good [1]; however, this theory is not renormalizable. Einstein’s gravity is the finite theory at the one-loop level in the absence of both matter fields and cosmological constant [4]. But this theory diverges in the two-loop order [3], [4]. The interaction of the gravity with the matter fields gives rise to nonrenormalizable theories yet at the one loop level [3], [4]. Attempts to improve the renormalizability of this theory by adding the terms quadratic in the curvature tensor or corresponding matter fields failed. In the first case, the obtained theory is renormalizable but it is not unitary because the ghosts and tachyons are present in the spectrum of the theory [7] - [11]. The second case led to the discovery of supergravity [12], [13]. Due to the presence of the local supersymmetry, supergravity is two-loop finite. But, the divergent terms are present in the three-loop order [14], [15]. Recently, the superstring [16] and the canonical approach [17] to quantum gravity have been proposed as a sensible theory of quantum gravity.

Most of the calculations confirming the perturbation nonrenormalizability of quantum gravity have been made by the background field method [18] - [22]. This method was suggested to obtain covariant results of the loop calculations. In the background field method, all dynamical fields $\varphi^j$ are expanded with respect to background values, according to

$$\varphi^j = \varphi^j_b + \phi^j_q,$$

and only the quantum fields $\phi^j_q$ are integrated over in the path integral. The background fields $\varphi^j_b$ are effectively external sources. For the one-particle irreducible diagrams there is a difference between the normal field theory and the background field method insofar as the gauge-fixing term may introduce additional vertices. B.DeWitt has proved that these additional vertices do not influence the $S$-matrix and the $S$-matrix in the formalism of the background field method is equivalent to the conventional $S$-matrix [18], [19]. This proof has later been extended in a lot of papers [20] - [26]. Hence, the counterterms on the mass-shell in the background field method must be independent of the gauge-fixing parameters and the reparametrization of the quantum fields. These statements are called the DeWitt-Kallosh theorem [18], [19], [21] and equivalence theorem [27] - [30] respectively.

The equivalence theorem states, that the $S$-matrix of the renormalizable theory is independent of the following change of variables:

$$\varphi^j \rightarrow \varphi^j = \varphi^j + \left(\varphi^2\right)^j + \left(\varphi^3\right)^j + \ldots$$

In the case of the quantum gravity, this statement is divided into two parts:

1. It is well known that there is considerable freedom in what one considers to be gravitational fields. We can consider the arbitrary tensor density $\tilde{g}_{\mu\nu} = g_{\mu\nu}(-g)^r$ or $\tilde{g}_{\mu\nu} = g_{\mu\nu}(-g)^s$ as gravitational variables. In accordance with the equivalence theorem the loop counterterms on the mass-shell must be independent of the choices of gravitational variables.
2. The loop counterterms on the mass-shell are independent of the redefinition of quantum fields of the form

\[ h_{\mu\nu} \rightarrow h_{\mu\nu} + k \left( h^2 \right)_{\mu\nu} + k^2 \left( h^3 \right)_{\mu\nu} + \ldots \]

This redefinition must influence only the higher loop results off the mass-shell.

By means of the corresponding choice of gravitational variables or the corresponding quantum field redefinition, one can considerably reduce the number and the type of interaction vertices. For example, if we consider \( g_{\mu\nu} \) as a gravitational variable, the number of three-point interactions in the Einstein gravity is equal to 13 [3]; if the tensor density \( g^{\mu\nu} \sqrt{-g} \) is selected as a dynamical variable, the number of three-point interactions is equal to six [31]; combining both methods reduces the number of three-point interactions to two [4]. However, the Einstein gravity is not a renormalizable theory. Thus, the equivalence theorem may not be fulfilled. A systematic search in the present context has never been undertaken in quantum gravity. The dependence of the nonlinear renormalizable quantum field theories on the choice of the parametrization have been investigated in the paper [32].

The DeWitt-Kallosh theorem asserts that the loop counterterms on the mass-shell calculated by means of the background field method are independent of the gauge-fixing term. This statement has been verified in many papers [33] - [36]. It turns out that the proof of this theorem is valid only for renormalizable theories (such as Yang-Mills theory, QED, QCD). For nonrenormalizable theories (such as gravity) the proof of the DeWitt-Kallosh theorem is formal. For example, it has recently been was shown that the one-loop counterterms of the Einstein gravity on the mass-shell depend on the gauge fixing terms [38]. Moreover, we can choose a gauge so that the Einstein gravity interacting with the matter fields will be renormalizable at the one-loop level.

Because of the complexity of the gauge suggested in papers [38], one needs to create a new algorithm for obtaining covariant one-loop results. It will be very nice to verify the gauge dependencies of the one-loop counterterms in a simpler gauge by using the standard well defined algorithm [39] - [41].

The main goal of the present paper is to investigate the influence of the field parametrization and the gauge fixing term on the one-loop counterterms of the Einstein gravity on the mass-shell.

We use the following notation:

\[ c = \hbar = 1; \quad \mu, \nu = 0, 1, 2, 3; \quad k^2 = 16\pi G, \quad (g) = \det(g_{\mu\nu}), \quad \varepsilon = \frac{4 - d}{2} \]

\[ R^\sigma_{\lambda\mu\nu} = \partial_\mu \Gamma^\sigma_{\lambda\nu} - \partial_\nu \Gamma^\sigma_{\lambda\mu} + \Gamma^\sigma_{\alpha\mu} \Gamma^\alpha_{\lambda\nu} - \Gamma^\sigma_{\alpha\nu} \Gamma^\alpha_{\lambda\mu}, \quad R_{\mu\nu} = R^\sigma_{\mu\sigma\nu}, \quad R = R_{\mu\nu} g^{\mu\nu} \]

where \( \Gamma^\sigma_{\mu\nu} \) is the Riemann connection.

Objects marked by the tilde \( \tilde{\cdot} \) are the tensor densities. The other are the tensors.

### 2 One-loop counterterms

One considers the Einstein gravity with the cosmological constant. The action of the theory is
\[
S_{gr} = -\frac{1}{k^2} \int d^4x \sqrt{-g} (R - 2\Lambda) \quad (1)
\]

For calculating the one-loop counterterms we use the background field method and the Schwinger-DeWitt technique. In the gauge theories, the renormalization procedure may violate the gauge invariance at the quantum level, thus destroying the renormalizability of the theory. Therefore, one is bound to apply an invariant renormalization. This can be achieved by applying an invariant regularization and using the minimal subtraction scheme \[12, 13\]. It has been proved that the dimensional regularization \[14 - 17\] is an invariant regularization preserving all the symmetries of the classical action which do not depend explicitly on the space-time dimension \[13, 18, 19\]. It has been shown \[50\] that in general renormalizable and nonrenormalizable theories the background field formalism requires using an invariant renormalization procedure to obtain valid results. A noninvariant regularization or renormalization may break an implicit correlation between different diagrams, which is essential as one formally expands the action in the background and quantum fields. We will use the invariant regularization (dimensional renormalization and minimal subtraction scheme) in our calculations.

When using the invariant renormalization the one-loop correction to the usual effective action is

\[
\Gamma^{(1)} = i \frac{1}{2} \left( \ln \det \Delta_{ab} - 2 \ln \det \Delta_{FP} \right) \quad (2)
\]

where 
\[
\Delta_{FP} \text{ is the Faddeev-Popov ghost operator;}
\]
\[
\Delta_{ab} = \frac{\delta^2 S(\phi)}{\delta \phi^a \delta \phi^b} + P^j_a(\phi) P^b_j(\phi) \quad (3)
\]

and \(P^j_a(\phi)\) is a gauge fixing term.

The divergence part of the one-loop effective action obtained by means of the heat kernel method is

\[
\Gamma^{(1)} = -\frac{1}{32\pi^2\varepsilon} \int d^4x \sqrt{-g}(B_4(\Delta_{ab}) - 2B_4(\Delta_{FP})) \quad (4)
\]

where \(B_4\) is the second coefficient of the spectral expansion of the corresponding differential operator. For the operator

\[
\Delta_{ij} = -\left( \nabla^2 1_{ij} + 2S^\sigma_{ij} \nabla_\sigma + X_{ij} \right) \quad (5)
\]

\(B_4\) is equal to

\[
B_4(\Delta) = Tr \left( \frac{1}{180} \left( R_{\mu\nu\sigma\lambda} - R_{\mu\nu}^2 \right) + \frac{1}{2} \left( Z + \frac{R}{6} \right) + \frac{1}{12} Y_{\mu\nu} Y^{\mu\nu} \right) \quad (6)
\]

where
\[
Z = X - \nabla_\lambda S^\lambda - S_\lambda S^\lambda \\
Y_{\mu\nu} = \nabla_\mu S_\nu - \nabla_\nu S_\mu + S_\mu S_\nu - S_\nu S_\mu + [\nabla_\mu, \nabla_\nu] 1 
\]

(7)

In the general case the dynamical variable in some metrical theory of gravity is the tensor density \( \tilde{g}_{\mu\nu} = g_{\mu\nu}(-g)^r \) or \( \tilde{g}^{\mu\nu} = g^{\mu\nu}(-g)^s \), where \( r \) and \( s \) are the numbers satisfying the conditions

\[
det \left| \frac{\partial \tilde{g}_{\mu\nu}}{\partial g_{\alpha\beta}} \right| \neq 0 
\]

or

\[
det \left| \frac{\partial \tilde{g}^{\mu\nu}}{\partial g^{\alpha\beta}} \right| \neq 0 
\]

(8) (9)

In accordance with the background field method we rewrite the dynamical field as

\[
\tilde{g}_{\mu\nu} = \tilde{g}_{\mu\nu} + k \tilde{h}_{\mu\nu} 
\]

(10)

where \( \tilde{g}_{\mu\nu} \) is the classical part satisfying the following equation

\[
\frac{\delta S_{gr}}{\delta \tilde{g}_{\mu\nu}} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} \frac{2r + 1}{4r + 1} + \Lambda g_{\alpha\beta} \frac{1}{4r + 1} = 0 
\]

(11)

Having solved this equation we obtain

\[
R_{\mu\nu} = \Lambda g_{\mu\nu} 
\]

(12)

Some functions of \( g_{\mu\nu} \) and their expansion in powers of the quantum field are given below

\[
(-g) = (-\tilde{g})^\frac{1}{t} \\
g_{\mu\nu} = \tilde{g}_{\mu\nu}(-\tilde{g})^{-\frac{r}{t}} \\
g^{\mu\nu} = \tilde{g}^{\mu\nu}(-\tilde{g})^{\frac{s}{t}} \\
\Gamma^\sigma_{\mu\nu} = \frac{1}{2} \tilde{g}^{\sigma\lambda} \left( -\partial_\lambda \tilde{g}_{\mu\nu} + \partial_\mu \tilde{g}_{\lambda\nu} + \partial_\nu \tilde{g}_{\mu\lambda} \right) - \frac{r}{2t} \tilde{g}^{\alpha\beta} \partial_\lambda \tilde{g}_{\alpha\beta} \left( \delta^\sigma_\mu \delta^\lambda_\nu + \delta^\sigma_\nu \delta^\lambda_\mu - \tilde{g}^{\sigma\lambda} \tilde{g}_{\mu\nu} \right) \\
(-g)^m = (-g)^m \left( 1 + k \frac{m}{t} h + \frac{k^2}{2} \left( \frac{m^2}{t^2} h^2 - \frac{m}{t} h_{\alpha\beta} h^{\alpha\beta} \right) + O(k^3) \right) \\
g^{\mu\nu} = g^{\mu\nu} + k \left( \frac{r}{t} h g^{\mu\nu} - h^{\mu\nu} \right) + k^2 h^{\mu\alpha} h^{\nu}_\alpha \\
+ \ k^2 \left( \frac{r^2}{2t^2} h^2 - \frac{r}{2t} h_{\alpha\beta} h^{\alpha\beta} \right) g^{\mu\nu} - k^2 \frac{r}{t} h h^{\mu\nu} + O(k^3) \\
g_{\mu\nu} = g_{\mu\nu} - k \left( \frac{r}{t} h g_{\mu\nu} - h_{\mu\nu} \right) + k^2 \left( \frac{r^2}{2t^2} h^2 + \frac{r}{2t} h_{\alpha\beta} h^{\alpha\beta} \right) g_{\mu\nu} - k^2 \frac{r}{t} h h_{\mu\nu} + O(k^3)
\]
\begin{align}
\Gamma_{\mu\nu} &= \Gamma^\sigma_{\mu\nu} + \frac{k}{2} g^{\sigma\lambda} \left( \nabla_\mu h_{\nu\lambda} + \nabla_\nu h_{\mu\lambda} - \nabla_\lambda h_{\mu\nu} \right) \\
&\quad - \frac{k}{2} \left( \frac{r}{t} \right) g^{\alpha\beta} \nabla_\lambda h_{\alpha\beta} \left( \delta_\mu^\delta h_\nu^\lambda + \delta_\nu^\delta h_\mu^\lambda - g^{\sigma\lambda} g_{\mu\nu} \right) \\
&\quad - \frac{k^2}{2} h^{\lambda\sigma} \left( - \nabla_\lambda h_{\mu\nu} + \nabla_\mu h_{\nu\lambda} + \nabla_\nu h_{\lambda\mu} \right) \\
&\quad + \frac{k^2 r}{2} h^{\alpha\beta} \nabla_\lambda h_{\alpha\beta} \left( \delta_\mu^\delta h_\nu^\lambda + \delta_\nu^\delta h_\mu^\lambda - g^{\sigma\lambda} g_{\mu\nu} \right) + O(k^3) \\
\text{where} \\
h &= h_{\alpha\beta} g^{\alpha\beta} \\
t &\equiv 4r + 1 \neq 0
\end{align}

We expand the action (1) in powers of the quantum field and pick out the terms quadratic in the quantum fields

\begin{align}
L_{\text{eff}} &= \left( \frac{1}{4} \nabla_\sigma h_{\alpha\beta} \nabla_\lambda h^{\alpha\beta} - \frac{1}{2} \nabla_\mu h_{\nu\sigma} \nabla_\lambda h^{\mu\nu} + \frac{1}{2} \frac{2r + 1}{t} \nabla_\mu h_\nu h^{\mu\nu} \\
&\quad - \frac{1}{2} \frac{6r^2 + 4r + 1}{2t^2} \nabla_\sigma h_\lambda h_\nu h^{\sigma\lambda} - \frac{1}{2} h^{\mu\nu} X_{\mu\nu\alpha\beta} h_{\alpha\beta} \right) \sqrt{-g} \\
\text{where} \\
X_{\mu\nu\alpha\beta} &= R_{\mu\alpha\beta} g_{\nu\beta} + R_{\mu\nu\alpha\beta} - 2\rho R_{\mu\nu} g_{\alpha\beta} + \rho^2 R g_{\mu\nu} g_{\alpha\beta} \\
&\quad - \rho R_{\mu\alpha} g_{\nu\beta} + \frac{\Lambda}{t} g_{\mu\alpha} g_{\nu\beta} - \frac{\Lambda}{2t^2} g_{\mu\nu} g_{\alpha\beta} \\
\rho &= \frac{t + 1}{4t}
\end{align}

The effective Lagrangian (16) is invariant under the general coordinate transformation

\begin{align}
x^\mu \rightarrow' x^\mu &= x^\mu + k \xi^\mu(x) \\
\tilde{h}_{\mu\nu}(x) \rightarrow' \tilde{h}_{\mu\nu}(x) &= \tilde{h}_{\mu\nu}(x) - k \partial_\mu \xi^\alpha \tilde{g}_{\alpha\nu}(x) - k \partial_\nu \xi^\alpha \tilde{g}_{\mu\alpha}(x) \\
&\quad - k \xi^\alpha \partial_\alpha \tilde{g}_{\mu\nu}(x) - 2rk \partial_\alpha \xi^\alpha \tilde{g}_{\mu\nu}(x) + O(k^2)
\end{align}

Now we investigate the parametrization dependence of the one-loop counterterms. To use the standard method of calculation (5) and (6), we fix the gauge invariance by the following condition:

\begin{align}
F_\mu &= \nabla_\nu \tilde{h}^{\nu}_{\mu} - \rho \nabla_\mu \tilde{h} \\
L_{gf} &= \frac{1}{2} F_\mu F_\nu g^{\mu\nu} (-g)^{\frac{1-n}{2}}
\end{align}
The ghost action obtained in the standard way is

$$L_{gh} = \bar{\psi}^{\mu} \left( g_{\mu\nu} \nabla^{2} + R_{\mu\nu} \right) \psi^{\nu} \sqrt{-g}$$

(22)

The one-loop counterterms off the mass-shell are

$$\Delta \Gamma^{(1)}_{\infty} = -\frac{1}{32\pi^2\varepsilon} \int d^4x \sqrt{-g} \left( \Lambda^2 \left( 8 + 2t^2 - 8t + \frac{18}{t^2} \right) \right. \right.$$ 

\[+ \Lambda R \left( -\frac{4}{3} - t^2 + \frac{8}{3} t - \frac{9}{t^2} \right) + R_{\mu\nu} R^{\mu\nu} \left( -\frac{3}{10} + 2t - t^2 \right) \]

\[+ \frac{53}{45} \left( R_{\mu\nu\alpha\lambda} R^{\mu\nu\alpha\lambda} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right) \]

\[+ \frac{R^2}{60} \left( -\frac{49}{8} + \frac{3t^2}{8} - \frac{2t}{3} + \frac{9}{8t^2} \right) \]

(23)

On the mass-shell we have

$$\Delta \Gamma^{(1)}_{\infty} = -\frac{1}{32\pi^2\varepsilon} \int d^4x \sqrt{-g} \left( \frac{53}{45} R^2_{\mu\nu\sigma\lambda} \right. \right.$$ 

\[\left. - \frac{58}{5} \Lambda^2 \right) \] 

(24)

This result coincides with the result obtained in the paper [51]. The one-loop counterterms on the mass-shell calculated in the gauge (20) in the Einstein gravity are independent of the choice of parametrization of the gravitational field. The case \( \tilde{g}^{\mu\nu} = g^{\mu\nu}(-g)^s \) can be considered analogously in this way and does not give essentially new results.

Now, we change the gauge fixing term and investigate the gauge and parametrization dependencies of the one-loop counterterms on the mass-shell. The most general gauge linear in the quantum field is

$$F_{\mu} = \alpha \nabla_{\nu} \tilde{h}_{\mu}^{\nu} + \beta \nabla_{\mu} \tilde{h} + T_{\mu\nu\rho\sigma} \tilde{h}^{\rho\sigma} + S_{\mu}^{\nu\sigma\lambda} \nabla_{\nu} \tilde{h}_{\sigma\lambda}$$

(25)

where

\( \alpha \) and \( \beta \) are the arbitrary constants;

\( T_{\mu\alpha\beta} \) and \( S_{\mu}^{\nu\alpha\beta} \) are some tensors depending on the background field \( g_{\mu\nu} \), functions of \( g_{\mu\nu} \) (such as \( R_{\alpha\beta\gamma}^{\sigma}, R_{\mu\nu}, R \)) and the covariant derivatives \( \nabla_{\sigma} \). Expression (25) being the most general gauge for the gravity, is defined by the following conditions:

1. Lorentz covariance

2. the number of derivatives with respect to the quantum fields is smaller than or equal to one

3. linear in the quantum field

Using a gauge of this type, one can simplify the calculations of counterterms in some models.

In the previous papers, the one-loop counterterms for the Einstein gravity were calculated in the following gauges:
1. $r = 0; T_{\mu\alpha\beta} = S_\mu^{\nu\alpha\beta} = 0$ \[34, 35\]

The one-loop counterterms off the mass-shell depend on the parameters $\alpha$ and $\beta$. On the mass the one-loop counterterms coincide with the result (24).

2. $r = 0; \alpha = \beta = 0; S_\mu^{\nu\alpha\beta} = 0$ \[37\]

The calculations were made by means of the diagrams technique. The metric was expanded around the flat background. It is impossible to write the results of calculations in the covariant way.

3. $r = 0; T_{\sigma\mu\nu} = 0$ \[38\]

To calculate the one-loop counterterms in the covariant way, one needs to create a new algorithm for the calculations. The results on the mass-shell depend on the $S_\mu^{\nu\alpha\beta}$

We consider the case $r \neq 0$, $\alpha = 1$, $\beta = -\rho, S_\mu^{\nu\alpha\beta} = 0, T_{\sigma\mu\nu} \neq 0$

This choice of parameters allows us to use the standard algorithm for the one-loop calculations. We will use this gauge for investigation of gauge and parametrization dependencies of the one-loop counterterms on the mass-shell. The gauge involved is the following

$$F_\mu = \nabla_\nu \tilde{h}_\mu - \rho \nabla_\mu \tilde{h} + T_{\mu\nu\sigma} \tilde{h}^{\nu\sigma}$$ \[26\]

The arbitrary tensor $U_{\sigma\mu\nu}$ can be decomposed into its irreducible parts:

$$U_{\sigma\mu\nu} = A_\sigma g_{\mu\nu} + B_\mu g_{\sigma\nu} + C_\nu g_{\mu\sigma} + \frac{1}{6} \varepsilon_{\sigma\mu\nu\lambda} \tilde{U}^\lambda + \bar{U}_{\sigma\mu\nu}$$ \[27\]

where $\tilde{U}^\lambda$ is the axial part defined by

$$\tilde{U}^\lambda = \varepsilon^{\lambda\mu\rho\nu} U_{\sigma\mu\nu}$$ \[28\]

and $A_\sigma, B_\mu, and C_\nu$, are the vector fields defined by

$$A_\sigma \equiv \frac{1}{18} \left( 5U_{\sigma\lambda}^\lambda - U_{\sigma\lambda}^\lambda - U_{\sigma\lambda}^\lambda \right)$$ \[29\]

$$B_\sigma \equiv \frac{1}{18} \left( -U_{\sigma\lambda}^\lambda + 5U_{\sigma\lambda}^\lambda - U_{\sigma\lambda}^\lambda \right)$$ \[30\]

$$C_\sigma \equiv \frac{1}{18} \left( -U_{\sigma\lambda}^\lambda - U_{\sigma\lambda}^\lambda + 5U_{\sigma\lambda}^\lambda \right)$$ \[31\]

and $\bar{U}_{\sigma\mu\nu}$ is the traceless part satisfying the following conditions:

$$\bar{U}^{\nu}_{\mu\nu} = \bar{U}^{\nu}_{\nu\mu} = \bar{U}^{\mu}_{\nu\mu} \equiv 0 \quad (32)$$

$$\bar{U}_{\sigma\mu\nu} + \bar{U}_{\nu\sigma\mu} + \bar{U}_{\mu\sigma\nu} = 0$$ \[33\]

The tensor $T_{\sigma\mu\nu}$ presented in the gauge \[24\] satisfies the condition
\[ T_{\sigma\mu\nu} = T_{\sigma\nu\mu} \tag{34} \]

Then, the decomposition of \( T_{\sigma\mu\nu} \) can be written in the following way:

\[ T_{\sigma\mu\nu} = T_\sigma g_{\mu\nu} + C_\nu g_{\mu\sigma} + C_\mu g_{\nu\sigma} + T_{\sigma(\mu\nu)} \tag{35} \]

The number of counterterms off the mass shell including \( T^4 \) and \( RT^2 \), where \( T^4 \) and \( RT^2 \) are the symbolic notation for contractions of the tensor \( T_{\sigma\mu\nu} \) or the curvature tensor and tensor \( T_{\sigma\mu\nu} \), respectively, are about 150. The calculation of these counterterms is very cumbersome. To reduce the number of possible counterterms and to facilitate the calculations, we consider the three particular cases of the gauge (26).

1. \( T_{\sigma\mu\nu} \) is an arbitrary tensor satisfying two conditions
   - \( T_{\sigma\mu\nu} \) is the symmetrical tensor: \( T_{\sigma\mu\nu} = T_{(\sigma\mu\nu)} \)
   - \( T_{\sigma\mu\nu} \) is the traceless tensor: \( T_{(\sigma\mu\nu)} g^{\mu\nu} = 0 \)

2. \( T_{\sigma\mu\nu} = T_\sigma g_{\mu\nu} \)
   where \( T_\sigma \) is an arbitrary vector.

3. \( T_{\sigma\mu\nu} = C_\mu g_{\sigma\nu} + C_\nu g_{\sigma\mu} \) where \( C_\sigma \) is an arbitrary vector.

In the first case, the gauge fixing term is

\[ F_\mu = \nabla_\nu \tilde{h}^\nu_\mu - \rho \nabla_\mu \tilde{h} + T_{(\mu\alpha\beta)} \tilde{h}^{\alpha\beta} \tag{36} \]

The ghost action is

\[ L_{gh} = \overline{\psi} \left( g_{\mu\nu} \nabla^2 + 2 T^\sigma_{\mu\nu} \nabla_{\sigma} + R_{\mu\nu} \right) c' \sqrt{-g} \tag{37} \]

In the second case, the gauge fixing term and the ghost action are:

\[ F_\mu = \nabla_\nu \tilde{h}^\nu_\mu - \rho \nabla_\mu \tilde{h} + T_\mu \tilde{h} \tag{38} \]

\[ L_{gh} = \overline{\psi} \left( g_{\mu\nu} \nabla^2 + 2 T_\mu \nabla_\nu + R_{\mu\nu} \right) c' \sqrt{-g} \tag{39} \]

In the third case, the gauge fixing term and the ghost action are:

\[ F_\mu = \nabla_\nu \tilde{h}^\nu_\mu - \rho \nabla_\mu \tilde{h} + 2 C_\nu \tilde{h}^\nu_\mu \tag{40} \]

\[ L_{gh} = \overline{\psi} \left( g_{\mu\nu} \nabla^2 + 2 C_\nu \nabla_\mu + 2 g_{\mu\nu} C_\sigma \nabla_{\sigma} + 4 r C_\mu \nabla_\nu + R_{\mu\nu} \right) c' \sqrt{-g} \tag{41} \]

The results of the one-loop calculation on the mass-shell coincide with the standard results (24). Hence the one-loop counterterms of the Einstein gravity on the mass-shell do not depend on the choice of the tensor \( T_{\sigma\mu\nu} \).
3 The physical meaning of the results of the loop calculation

It is well known that in quantum gravity the results of the loop calculations off the mass shell calculated by means of the background field method depend on the gauge fixing term and the choice of the parametrization of quantum fields. For example, one considers the result (23) and using the standard arguments calculates some renormgroup functions. One considers only first two counterterms (\(\Lambda^2\) and \(\Lambda R\)). The expression

\[
f d^4 x \sqrt{-g} \left( R_{\mu\nu\sigma\lambda} R^{\mu\nu\sigma\lambda} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right)
\]

is proportional to the topological number of space-time, the so called Euler number. Hence, this expression is some number. It can be assumed that other structures appearing in one-loop counter terms (\(R^2_{\mu\nu}\) and \(R^2\)) are comparably small in concrete physical applications. This situation corresponds to the low energy limit. From the renormalization group analysis \([52], [53]\) it is well known that the terms with higher derivatives play the essential role only in the high energy limit. But in the low energy limit the essential role belongs to the terms with two derivatives. In this way, in the low energy limit we consider only the \(\Lambda^2\) and \(R\Lambda\) terms. Then, under this consideration the theory is renormalizable. At the one loop level, one needs to renormalize the cosmological constant \(\Lambda\) and the gravitational constant \(k^2\). The cosmological constant can be represented in the following form:

\[
\Lambda = \frac{\lambda}{k^2}
\]

where \(\lambda\) is the dimensionless constant. Then, from expression (24) one gets the renormalization group equations

\[
\beta_\lambda = \mu^2 \frac{\partial \lambda}{\partial \mu^2} = -\frac{\lambda^2}{32\pi^2} \left( (t - 2)^2 + \frac{9}{t^2} \right)
\]

\[
\gamma_{k^2} = \mu^2 \frac{\partial k^2}{\partial \mu^2} = -\frac{\lambda}{32\pi^2} k^2 \left( t^2 - \frac{8}{3} t + \frac{4}{3} + \frac{9}{t^2} \right)
\]

where \(\mu^2\) is the renormalization point mass and \(\gamma\) is the anomalous dimension of the gravitational constant \(k^2\). We see that asymptotical freedom for the cosmological constant \(\lambda\) is preserved for an arbitrary choice of the field parametrization. But anomalous dimension of the gravitational constant \(k^2\) drastically depends on the parametrization. In general, it is possible to find such a parametrization that the anomalous dimension will be equal to zero. The parametrization dependence of the renormalization group functions, such as the \(\beta\)-function and anomalous dimension, have the same treatment as the gauge and scheme dependencies of these functions in the ordinary quantum field theory.

In general, in the nonrenormalizable quantum gravity all numerical coefficients of the counterterms calculated by means of the background field method off the mass-shell depend on the choice of the gauge and parametrization. The standard choice of \(g_{\mu\nu}\) and \(\Phi_{mat}\) as dynamical variables, where \(g_{\mu\nu}\) and \(\Phi_{mat}\) are the metric and material fields, respectively, is simply a particular choice of a possible parametrization. The
loop counterterms off the mass shell obtained by means of these variables are also parametrization dependent.

The dependence of the one-loop counterterms off the mass-shell on the method of calculation have been discussed also in paper [54]. It has been shown that the Einstein gravity in the first-order formalism corresponds to some choice of the parametrization of the metric field.

In this situation the question arises: what is the physical parametrization? Quite recently Fujikawa has suggested a very beautiful way to define the physical parametrization [55]. The true dynamical variables are defined from the anomaly-free condition on the BRST-transformation connected with the general coordinate transformations. This prescription must be fulfilled for each variable separately. This condition means that the dynamical variables are some tensor densities \( \tilde{\varphi} \) obtained from the initial fields \( \varphi \) by multiplication by corresponding degree of \( (-g)^{\frac{N-4}{4}} \). For example, the physical dynamical variables in the quantum gravity must be \( g_{\mu\nu}(-g)^{\frac{N-4}{4}} \) or \( g^{\mu\nu}(-g)^{\frac{N+4}{4}} \) where \( N \) is the space-time dimension. All material fields must be replaced by some tensor density fields. These results can be obtained from the functional integral approach [56] without the BRST-symmetry. However, the gauge dependence of the results of the loop calculation is present even in this physical parametrization.

An other way to obtain physical results is the use of the gauge and parametrization independent Vilkovisky-DeWitt effective action instead of the ordinary effective action [57] - [60]. But the calculations of the loop correction to the Vilkovisky-DeWitt effective action are very cumbersome because of the nonlocal terms in its definition. Moreover, the gauge and parametrization invariance has been proved only for the renormalizable theories. In the nonrenormalizable theories the Vilkovisky-DeWitt effective action can give rise to gauge or parametrization dependent results off the mass shell. The connection between ordinary and Vilkovisky-DeWitt effective actions in the Einstein gravity have been discussed in paper [61].

To summarize, in nonrenormalizable theories the results of the loop calculations off the mass shell within the background field method are physically meaningless. For the results of the loop background field method calculations on the mass shell in some nonrenormalizable theory are be physically meaningful, one needs to prove or verify the validity of the DeWitt-Kallosh theorem and equivalence theorem for this theory.

4 Conclusion

The background field formalism is a powerful tool for the loop calculations. Its validity is based on the statement that the S-matrix in the formalism of the background field method is equivalent to the conventional S-matrix. The consequence of this equivalence is the gauge and parametrization independence of the loop counterterms on the mass shell calculated in the background field method. For nonrenormalizable theories, such as Einstein gravity the proof of this statement is formal. In this way the question arises about the physical meaning of the loop results calculated by the background field method in the nonrenormalizable theories. Can we obtain some physical quantities or some physical information from these calculations? If the DeWitt-Kallosh theorem and
equivalence theorem are fulfilled in some nonrenormalizable theory, then it is possible to obtain some physical information from the results of the loop calculations on the mass shell. Therefore, one needs to verify the validity of the DeWitt-Kallosh theorem and equivalence theorem for each nonrenormalizable theory.

In this paper the gauge and parametrization dependencies of the one-loop counterterms of the Einstein gravity were verified. The gauge (26) and arbitrary parametrization were considered. It turns out that on the mass shell the one-loop counterterms do not depend on the considered gauge and parametrization. However, as has been shown in papers [38], the one loop counterterms on the mass shell in the most general gauge (25) depend on the gauge parameter. Hence, the DeWitt-Kallosh theorem is not valid in this gauge.

What is the reason?. Maybe one needs to modify the statement of the DeWitt-Kallosh theorem for nonrenormalizable gauge theories?. For example, we can say that in the nonrenormalizable gauge theories the DeWitt-Kallosh theorem is valid only in the physical, so called Landau-DeWitt gauge, defined as

\[ f_\alpha = R_{a\beta}(\varphi)\phi^\beta \]  

(45)

where \( \varphi^\alpha \) and \( \phi^\alpha \) are the background and quantum fields, respectively, and \( R_{i\alpha}(\varphi) \) are the generators of the gauge transformations. For the quantum gravity, the Landau-DeWitt gauge is defined by

\[ f_\mu = \nabla_\nu h^\nu_\mu + \beta \nabla_\mu h \]  

(46)

\[ L_{gf} = \frac{1}{2\alpha} f_\mu f^\mu \]  

(47)

where \( \alpha \) and \( \beta \) are arbitrary numbers. In papers [34], [35] it has been shown that the one-loop counterterms on the mass shell do not depend on the gauge parameters \( \alpha \) and \( \beta \). Then, we suggested that in the Landau-DeWitt gauge the effective action would be connected with the S-matrix. Hence, the results obtained in the Landau-DeWitt gauge on the mass-shell have the physical meaning. Then, in the gauge distinct from Landau-DeWitt gauge the ordinary effective action on the mass-shells does not imply physical quantities and one needs to define some reduction method to obtain physical quantities from the usual effective action in a nonphysical gauge.

To verify this statement one needs to calculate the gauge dependence of the one-loop counterterms on the mass shell in the gauge distinct from the Landau-DeWitt gauge. The gauge (26) and gauge (25) satisfy this condition. These gauges are equivalent. Then, the results of the loop calculations in these gauges must have the same physical meaning. In the gauge (26) the one-loop counterterms on the mass shell depend on the gauge fixing term and, as consequence, are meaningless. Hence, the results of the loop calculations in the gauge (26) do not have physical meaning as well. But the results of the loop calculations in the gauge (24) on the mass-shell coincide with the results of the loop calculations in the standard gauge (4). Then, the results of calculations by the loop background field method in the Einstein gravity in an arbitrary gauge do not have the physical meaning. We cannot obtain some physical information from these
calculations. In this way the results of the loop calculations do not give information about renormalizability of the theory.

It is possible that in arbitrary nonrenormalizable theories the ordinary effective action (and maybe the Vilkovisky-DeWitt effective action) on the mass-shell does not give any physical quantities at all. The validity of the DeWitt-Kallosh and equivalence theorem in particular gauges does not contradict this statement. This is simply a fortunate event. To obtained physical information in nonrenormalizable theories, one needs to define the physical quantities and to calculate loop corrections only to these physical quantities.

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