Fully Quantum Arbitrarily Varying Channels: Random Coding Capacity and Capacity Dichotomy

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Abstract—we consider a model of communication via a fully quantum jammer channel with quantum jammer, quantum sender and quantum receiver, which we dub quantum arbitrarily varying channel (QAVC). Restricting to finite dimensional user and jammer systems, we show, using permutation symmetry and a de Finetti reduction, how the random coding capacity (classical and quantum) of the QAVC is reduced to the capacity of a naturally associated compound channel, which is obtained by restricting the jammer to i.i.d. input states. Furthermore, we demonstrate that the shared randomness required is at most logarithmic in the block length, using a random matrix tail bound. This implies a dichotomy theorem: either the classical capacity of the QAVC is zero, and then also the quantum capacity is zero, or each capacity equals its random coding variant.

I. FULLY QUANTUM AVC AND RANDOM CODES

We consider a simple, fully quantum model of arbitrarily varying channel (QAVC). Namely, we have three agents, Alice (sender), Bob (receiver) and Jamie (jammer), each controlling a quantum system \( A \), \( B \), and \( J \), respectively. The channel is simply a completely positive and trace preserving (cptp) map \( \mathcal{N} : \mathcal{L}(A \otimes J) \rightarrow \mathcal{L}(B) \), and we assume it to be memoryless on blocks of length \( \ell \), i.e. \( \mathcal{N}^{\otimes \ell} : \mathcal{L}(A^\ell \otimes J^\ell) \rightarrow \mathcal{L}(B^\ell) \), with \( A^\ell = A \otimes \cdots \otimes A \) (\( \ell \) times), etc. However, crucially, neither Alice’s nor Jamie’s input states need to be tensor product or even separable states. We shall assume throughout that all three Hilbert spaces \( A \), \( B \) and \( J \) have finite dimension, \( |A|, |B|, |J| < \infty \). The previously introduced AVC model of Ahlswede and Blinovsky [5], and more generally Ahlswede et al. [4], is obtained by channels \( \mathcal{N} \) that first dephase the input \( J \) in a fixed basis, so that the choices of the jammer are effectively reduced to basis states \( |ij\rangle \) of \( J \) and their convex combinations. Note that this generalises the classical AVC, which is simply a channel with input alphabet \( X \times S \) and output alphabet \( Y \), given by transition probabilities \( N(y|x,s) \), and such a channel can always be interpreted as a cptp map. This model has been considered in [15], [20], however in those works principally from the point of view that Jamie is helping Alice and Bob, passively, by providing a suitable input state to \( J \). Contrary to the classical AVC and the AVC considered in [5], [4], where the jammer effectively always selects a tensor product channel between Alice and Bob, the fact that we allow general quantum inputs on \( J^\ell \), including entangled states, permits Jamie to induce non-classical correlations between the different channel systems. These correlations, as was observed in [19], [20], are not only highly nontrivial, but can also have a profound impact on the communication capacity of the channel between Alice and Bob. In the present context, however, Jamie is fundamentally an adversary.

Define a (deterministic) classical code for \( \mathcal{N} \) of block length \( \ell \) as a collection \( \mathcal{C} = \{ (\rho_m, D_m) \colon m = 1, \ldots, M \} \) of states \( \rho_m \in \mathcal{S}(A^\ell) \) and POVM elements \( D_m \) not acting on \( B^\ell \), such that \( \sum_{m=1}^M D_m = I \). Its rate is defined as \( \frac{1}{\ell} \log M \), the number of bits encoded per channel use. Its error probability is defined as the average over uniformly distributed messages and with respect to a state \( \sigma \) on \( J^\ell \):

\[
P_{\text{err}}(C, \sigma) := \frac{1}{M} \sum_{m=1}^M \text{Tr} \left( \mathcal{N}^{\otimes \ell} (\rho_m \otimes \sigma) \right) (I - D_m).
\]

For the transmission of quantum information, define a (deterministic) quantum code for \( \mathcal{N} \) of block length \( \ell \) as a pair \( Q = (\mathcal{E}, \mathcal{D}) \) of cptp maps \( \mathcal{E} : \mathcal{L}(C^\ell) \rightarrow \mathcal{L}(A^\ell) \) and \( \mathcal{D} : \mathcal{L}(B^\ell) \rightarrow \mathcal{L}(C^\ell) \). Its rate is \( \frac{1}{\ell} \log L \), the number of qubits encoded per channel use, and the error is quantified, with respect to a state \( \sigma \) on \( J^\ell \), as the “infidelity”

\[
\hat{F}(Q, \sigma) := 1 - \text{Tr} \left( \mathcal{I} \otimes \mathcal{D} \circ \mathcal{N}^{\otimes \ell} \circ \mathcal{E} \right) \Phi_Q \cdot \Phi_{\sigma},
\]

with the maximally entangled state \( \Phi_Q = \frac{1}{\sqrt{L}} \sum_{ij} |ii\rangle \langle jj| \). Here, we have introduced the channels \( N_\lambda : \mathcal{L}(A) \rightarrow \mathcal{L}(B) \) defined by fixing the jammer’s state to \( \sigma \), \( N_\lambda(\rho) := N(\rho \otimes \sigma) \).

Note that we use the language of “deterministic” code, although in quantum information this is indistinguishable from stochastic encoders; it is meant to differentiate from “random” codes, which use shared correlation: A random classical [quantum] code for \( \mathcal{N} \) of block length \( \ell \) consists of a random variable \( \lambda \) with a well-defined distribution and a family of deterministic codes \( C_\lambda [Q_\lambda] \). The error probability if \( (C_\lambda) \), always with respect to a state \( \sigma \) on \( J^\ell \), is simply the expectation over \( \lambda \), i.e. \( \mathbb{E}_\lambda P_{\text{err}}(C_\lambda, \sigma) \). The error of the random quantum code is similarly \( \mathbb{E}_\lambda \hat{F}(Q_\lambda, \sigma) \).

The operational interpretation of the random code model is that Alice and Bob share knowledge of the random variable \( \lambda \), and use \( C_\lambda \) accordingly, but that Jamie is ignorant of it. This shared randomness is thus a valuable resource, which for random codes is considered freely available, whose amount, however, we would like to control at the same time.
The capacities associated to these code concepts are defined as usual, as the maximum achievable rate as block length goes to infinity and the error goes to zero:

\[ C_{\text{det}}(N) := \lim_{\ell \to \infty} \sup_{\sigma} \frac{1}{\ell} \log M \text{ s.t. sup } P_{e}^ {\ell}(C, \sigma) \to 0, \]

\[ C_{\text{rand}}(N) := \lim_{\ell \to \infty} \sup_{\sigma} \frac{1}{\ell} \log L \text{ s.t. sup } \mathbb{E}_{\lambda} P_{e}^ {\ell}(C, \sigma) \to 0, \]

\[ Q_{\text{det}}(N) := \lim_{\ell \to \infty} \sup_{\sigma} \frac{1}{\ell} \log L \text{ s.t. sup } \tilde{F}(Q, \sigma) \to 0, \]

\[ Q_{\text{rand}}(N) := \lim_{\ell \to \infty} \sup_{\sigma} \frac{1}{\ell} \log L \text{ s.t. sup } \mathbb{E}_{\lambda} \tilde{F}(Q, \sigma) \to 0. \]

If in the above error maximisations Jamie is restricted to tensor power states \( \sigma \otimes \ell \), the QAVC model becomes a compound channel: \( N_{\ell}^{\otimes \ell} = (N_{\sigma})^{\otimes \ell} \), \( \sigma \in \mathcal{S}(J) \). Its classical and quantum capacities are denoted \( C((N_{\sigma})_{\sigma}) \) and \( Q((N_{\sigma})_{\sigma}) \), respectively.

II. RANDOM CODING CAPACITIES: FROM QAVC TO ITS COMPOUND CHANNEL

By definition, (see also [7], [8] and [11])

\[ C_{\text{det}}(N) \leq C_{\text{rand}}(N) \leq C((N_{\sigma})_{\sigma}), \]

and

\[ Q_{\text{det}}(N) \leq Q_{\text{rand}}(N) \leq Q((N_{\sigma})_{\sigma}). \]

(1)

Here, we show that for the random capacity, the rightmost inequalities are identities, by proving bounds in the opposite direction. For the quantum capacity, this was done in [19] Appendix A]. To present the argument, define the permutation operator \( U^\pi \) acting on the tensor power \( A^\pi \) as permuting the subsystems, for a permutation \( \pi \in S_{\ell} \): \n
\[ U^\pi = U^\alpha \rho^\alpha U^\dagger. \]

which extends uniquely by linearity. This is a unitary representation of the symmetric group, which is defined for any Hilbert space. The quantum channel obtained by the conjugation action of \( U^\pi \) is denoted \( U_{\pi}(\alpha) = U^\pi \rho^\alpha (U^\dagger)^\pi \).

**Proposition 1** Let \( Q = (E, D) \) be a quantum code for the compound channel \( \{N_{\sigma}\}_{\sigma \in \mathcal{S}(J)} \) at block length \( \ell \) of size \( L \) and with fidelity \( 1 - \epsilon \), i.e. for all \( \sigma \in \mathcal{S}(J) \)

\[ \tilde{F}(Q, \sigma^{\otimes \ell}) = 1 - \text{Tr}

\[ \left((\text{id} \otimes D \otimes \sigma^{\otimes \ell} \otimes \varepsilon^{\ell})\Phi_{L}\right) \Phi_{L} \leq \epsilon. \]

Then, the random quantum code \( (Q_{\pi})_{\pi \in \mathcal{S}(J)} \) with uniformly distributed random permutation \( \pi \) of \([\ell]\), defined by

\[ Q_{\pi} = (U_{\pi} \circ E, D \circ U_{\pi} \circ \varepsilon), \]

has infidelity \( \mathbb{E}_{\pi} \tilde{F}(Q_{\pi}, \sigma^{\otimes \ell}) \leq \epsilon' \leq \epsilon(\ell + 1)^{1/2} \) for the QAVC \( N \).

**Proposition 2** Let \( C = \{(\rho_{m}, D_{m}) : m = 1, \ldots, M\} \) be a code of block length \( \ell \) for the compound channel \( \{N_{\sigma}\}_{\sigma \in \mathcal{S}(J)} \) with error probability \( \epsilon \), i.e. for all \( \sigma \in \mathcal{S}(J) \)

\[ P_{e}(C, \sigma^{\otimes \ell}) = \frac{1}{M} \sum_{m=1}^{M} \text{Tr}

\[ \left(N^{\otimes \ell}_{\sigma} (\rho_{m})(\mathbb{I} - D_{m})\right) \leq \epsilon. \]

Then, the random code \( (C_{\pi})_{\pi \in \mathcal{S}(J)} \) with a uniformly distributed random permutation \( \pi \) of \([\ell]\), defined by

\[ C_{\pi} := \{(U^{\pi} \rho_{m} U^{\dagger}, U^{\pi} D_{m} U^{\dagger}) : m = 1, \ldots, M\}, \]

has error probability \( \epsilon' \leq \epsilon(\ell + 1)^{1/2} \) for the QAVC \( N \).

**Proof** We only prove Proposition 2 since Proposition 1 has been argued in [19] Appendix A], with analogous proofs. For an arbitrary state \( \zeta \) on \( J^{\ell} \), the error probability of the random code \( (C_{\pi})_{\pi \in \mathcal{S}(J)} \) can be written as

\[ \mathbb{E}_{\pi} P_{e}(C_{\pi}, \zeta) \]

\[ = \frac{1}{M} \sum_{m=1}^{M} \mathbb{E}_{\pi} \text{Tr}

\[ \left(U^{\pi}(N^{\otimes \ell}(U^{\pi} \rho_{m} U^{\dagger}, \zeta)) U^{\pi}(\mathbb{I} - D_{m})\right) \]

\[ = \frac{1}{M} \sum_{m=1}^{M} \text{Tr}

\[ \left(N^{\otimes \ell}(\rho_{m}, \mathbb{E}_{\pi} U^{\pi} \zeta U^{\pi}(\mathbb{I} - D_{m}))\right), \]

(2)

where in the last line we have exploited the \( \mathcal{S}_{\ell} \)-covariance of the tensor product channel \( N^{\otimes \ell} \). The crucial feature of the last expression is that it shows that the error probability that the jammer can achieve with \( \zeta \) is the same as that of the state \( \zeta' = \mathbb{E}_{\pi} U^{\pi} \zeta U^{\pi} = \frac{1}{\ell!} \sum_{\pi \in \mathcal{S}_{\ell}} U^{\pi} \zeta U^{\pi} \).

This is, by its construction, a permutation-symmetric state, and we can apply the de Finetti reduction from [13]:

\[ \zeta' \leq (\ell + 1)^{1/2} \int_{\sigma \in \mathcal{S}(J)} \mu(d\sigma) \sigma^{\otimes \ell} :=: (\ell + 1)^{1/2} \mathcal{F}, \]

with a uniform probability measure \( \mu \) on the states of \( J \), whose detailed structure is given in [13], but which is not going to be important for us.

Indeed, inserting this into the last line of eq. (2), and using complete positivity of \( N \), we obtain the upper bound

\[ \mathbb{E}_{\pi} P_{e}(C_{\pi}, \zeta) \leq \frac{1}{M} \sum_{m=1}^{M} \text{Tr}

\[ \left(N^{\otimes \ell}(\rho_{m}, \mathcal{F}) (\mathbb{I} - D_{m})\right) \]

\[ \leq (\ell + 1)^{1/2} \frac{1}{M} \sum_{m=1}^{M} \text{Tr}

\[ \left(N^{\otimes \ell}(\rho_{m}, \mathcal{F}) (\mathbb{I} - D_{m})\right), \]

\[ = (\ell + 1)^{1/2} \int_{\sigma \in \mathcal{S}(J)} \mu(d\sigma) \frac{1}{M} \sum_{m=1}^{M} \text{Tr}

\[ \left(N^{\otimes \ell}(\rho_{m}, \sigma^{\otimes \ell})\right) (\mathbb{I} - D_{m})\]

\[ = (\ell + 1)^{1/2} \int_{\sigma \in \mathcal{S}(J)} \mu(d\sigma) P_{e}(C, \sigma^{\otimes \ell}) \leq (\ell + 1)^{1/2} \epsilon, \]

where in the last step we have used the assumption that for every jammer state of the form \( \sigma^{\otimes \ell} \), the error probability is bounded by \( \epsilon \).

\[ \square \]

To apply this, we need compound channel codes with error decaying faster than any polynomial. This is no problem, as there are several constructions giving even exponentially small error for rates arbitrarily close to the compound channel capacity, both for classical [7], [21] and quantum codes [8].
Corollary 3 Let $N$ be a QAVC. Its classical random coding capacity is given by
\[
C_{\text{rand}}(N) = C([N_\sigma]_\sigma) = \lim_{\ell \to \infty} \frac{1}{\ell} \max_{\{p_z, \omega_z\}} \inf I(X : B^\ell),
\]
where $I(X : B^\ell) = S(\sum_z p_z \omega_z) - \sum_z p_z S(\omega_z)$ is the Holevo information of the ensemble $\{p_z, \omega_z = N^{\otimes \ell}(\rho_z \otimes \sigma)\}$ \cite{7,21}.

Similarly, its quantum random coding capacity is
\[
Q_{\text{rand}}(N) = Q([N_\sigma]_\sigma) = \lim_{\ell \to \infty} \frac{1}{\ell} \max_{\{p_z, \omega_z\}} \inf I(I(R) : B^\ell),
\]
where $I(I(R) : B^\ell) = S(\Omega^{B^\ell}) - S(\Omega)$ is the coherent information of the state $\Omega = (\text{id} \otimes N_\sigma)(\Phi^{RA_\ell} \otimes \sigma)$ \cite{22}.

### III. Capacity Dichotomy: Elimination of Correlation from Random Codes

For classical AVCs or AVQCs with classical channel, the observations of Ahlswede \cite{22} show that the random coding capacity can always be attained using at most $O(\log \ell)$ bits of shared randomness. This is done by i.i.d. sampling the shared random variable $\lambda$, thus approximating, for each channel state $\sigma$, $E_{\lambda}P_{\text{err}}(C_\lambda, \sigma)$ by an empirical mean over $n$ realisations of $\lambda$, except with probability exponentially small in $n$. Then, the union bound can be used because the jammer has “only” exponential in $\ell$ many choices. On the face of it, this strategy looks little promising for QAVCs: the jammer’s choices form a continuum, and even if we realise that we can discretise $S(J^\ell)$, any net of states is exponentially large in the dimension $\ell$, i.e. doubly exponentially large in $\ell$, resulting in a naive bound of $O(\ell)$ for the shared randomness required. However, the linearity of the quantum formalism comes to our rescue.

**Observation 4** From the point of view of the jammer, the error probability of a classical code is an observable, $P_{\text{err}}(C, \sigma) = \text{Tr} \sigma E$, with a POVM element $E = E(C)$ depending in a systematic way on the code. Likewise, the infidelity of a quantum code can be written $\tilde{F}(Q, \sigma) = \text{Tr} \sigma G$ for a POVM element $G = G(Q)$.

**Proof** Indeed, using the Heisenberg picture (adjoint map) $N^*$,
\[
P_{\text{err}}(C, \sigma) = \frac{1}{M} \sum_{m=1}^M \text{Tr} (N^{\otimes \ell}(\rho_m \otimes \sigma))(1 - D_m)
= \text{Tr} \sigma \left[ \frac{1}{M} \sum_{m=1}^M \text{Tr} N^{\otimes \ell}(\rho_m \otimes 1)(1 - D_m) \right],
\]
so that $E = \frac{1}{M} \sum_{m=1}^M \text{Tr} N^{\otimes \ell}(\rho_m \otimes 1)(1 - D_m)$, which is manifestly a POVM element, i.e. $0 \leq E \leq 1$.

Lkewise, for the infidelity,
\[
\tilde{F}(Q, \sigma) = \text{Tr} (\text{id} \otimes \mathcal{D} \otimes N^{\otimes \ell} \otimes E)(\Phi_L \otimes \sigma)(1 - \Phi_L)
= \text{Tr} (\Phi_L \otimes \sigma) \cdot (\text{id} \otimes \mathcal{E}^* \otimes N^{\otimes \ell} \otimes \mathcal{D}^*)(1 - \Phi_L)
\]

with $G = \text{Tr}_{AA'}(\Phi_L \otimes 1)(\text{id} \otimes \mathcal{E}^* \otimes N^{\otimes \ell} \otimes \mathcal{D}^*)(1 - \Phi_L)$.

Obviously, for a random classical code $(C_\lambda)$, the expected error probability is
\[
E_\lambda P_{\text{err}}(C_\lambda, \sigma) = \text{Tr} \sigma (E_\lambda G_\lambda),
\]
with the POVM elements $E_\lambda = E(C_\lambda)$ associated to each code $C_\lambda$. Likewise for a random quantum code.

For a random classical code $(C_\lambda)$, the jammer’s goal is to maximise the error probability, choosing $\sigma$ in the worst possible way. But from the present perspective that the error probability is an observable for Jamie, it is clear that $\sup_\sigma E_\lambda P_{\text{err}}(C_\lambda, \sigma)$ is simply the maximum eigenvalue of $E$. \hfill $\square$

We say, following general convention, that a random classical or quantum code $(C_\lambda)$ or $(Q_\lambda)$ has error $\epsilon$ (without reference to any specific state of the jammer) if
\[
\sup_\sigma E_\lambda P_{\text{err}}(C_\lambda, \sigma) \leq \epsilon \quad \text{or} \quad \sup_\sigma E_\lambda \tilde{F}(Q_\lambda, \sigma) \leq \epsilon,
\]
respectively. By the above discussion is equivalent to
\[
E_\lambda E \leq \epsilon 1 \quad \text{or} \quad E_\lambda G \leq \epsilon 1,
\]
in the sense of the operator order. This is an extremely useful way of characterising that the random code has a given error.

Our goal now is to select a “small” number of $\lambda$’s, say $\lambda_1, \ldots, \lambda_n$, such that
\[
\frac{1}{n} \sum_{\mu=1}^n E_{\lambda_\mu} \leq (\epsilon + \delta) 1,
\]
ensuring that the random code $(C_{\lambda_\mu})_{\mu=1}^n$, with uniformly distributed $\nu \in [n]$, has error probability $\epsilon + \delta$. This is precisely the situation for which the matrix tail bounds in \cite{23} were developed. Indeed, quoting \cite{23} Thm. 19, for i.i.d. $\nu_\mu \sim P_\lambda$,
\[
\Pr \left\{ \frac{1}{n} \sum_{\mu=1}^n E_{\lambda_\mu} \leq (\epsilon + \delta) 1 \right\} \leq |J|^{\ell_1} \cdot \exp(-nD(\epsilon + \delta)[\nu]),
\]
with the binary relative entropy $D(u||v) = u \ln \frac{u}{v} + (1 - u) \ln \frac{1 - u}{1 - v}$, which can be lower bounded by Pinsker’s inequality, $D(u||v) \geq 2(u - v)^2$. Note that both the logarithm (ln) and the exponential (exp) are understood to base $e$.

Thus, for $n > \frac{1}{2\epsilon^2}(\ln |J|)\ell$, the right hand probability bound above is less than 1, so that there exist $\lambda_1, \ldots, \lambda_n$ with \cite{4}.

The number of bits needed to be shared between Alice and Bob to achieve this, is $\log n$, which we may choose to be $\leq \log \ell - 2 \log \delta + \log \ln |J|$, which is not zero, but has zero rate as $\ell \to \infty$. Exactly the same argument applies to a random quantum code $(Q_\lambda)$. We record this as a quotable statement.

**Proposition 5** Let $(C_\lambda : \lambda \in \Lambda)$ be a random classical code of block length $\ell$ for the QAVC $N : A \otimes J \to B$, with error probability $\epsilon$. Then for $\delta > 0$, there exist $\lambda_1, \ldots, \lambda_n \in \Lambda$, with $n \leq 1 + \frac{1}{2\epsilon^2}(\ln |J|)\ell$, such that the random code $(C_{\lambda_\mu} : \nu \in R [n])$ has error probability $\leq \epsilon + \delta$. 
For a random quantum code $\{Q_\lambda : \lambda \in \Lambda\}$, with infidelity $\epsilon$, we similarly have that the random code $\{Q_\lambda^{\nu} : \nu \in R [n]\}$ has infidelity $\leq \epsilon + \delta$. \hfill $\square$

Remark We have discussed here from the beginning the version of the capacity with average probability of error (and arbitrary encodings). Following Ahlswede [2] and the generalisation of his method above, investing another $O(\log \ell)$ bits of shared randomness, or losing $O(\log \ell)$ bits from the code, we can convert any code with error $\epsilon$ into one with maximum error $\leq 2\epsilon$. We omit the details of this argument, as it is exactly as in [2].

Proposition [5] allows us to assess the leftmost inequalities in the capacity order from eq. [1]. Because the randomness needed is so little, it can be generated by a channel code losing no rate. Hence, in a certain sense, they are also identities, except in the somewhat singular case the deterministic classical capacity vanishes:

**Corollary 6** The classical capacity of a QAVC $N$ is either 0 or, if it is positive, it equals the random coding capacity:

$$\begin{align*}
C_{\text{der}}(N) &= \begin{cases} 
C_{\text{rand}}(N) & \text{if } C_{\text{der}}(N) > 0, \\
0 & \text{otherwise}.
\end{cases}
\end{align*}$$

Similarly, for the quantum capacity:

$$Q_{\text{der}}(N) = \begin{cases} Q_{\text{rand}}(N) & \text{if } C_{\text{der}}(N) > 0, \\
0 & \text{otherwise}. \hfill \square
\end{cases}$$

**IV. Discussion and Outlook**

We have shown that in a fully quantum jammer channel model (QAVC), the random coding capacity, for both quantum and classical transmission, can be reduced to the capacity of a corresponding compound channel; furthermore, by extension of the “elimination of correlation” technique, that the shared randomness required has zero rate; thus implying dichotomy theorems for the deterministic classical and quantum capacities. Since the derandomisation leaves so little randomness, we can apply the results also to say something about the identification capacity of QAVCs: Either the ID-capacity vanishes, or it equals the random coding capacity $C_{\text{rand}}(N)$.

Our work leaves two important open questions: First, to give necessary and sufficient conditions for vanishing classical capacity. For classical AVCs this is the co-called “symmetrising ability” [17], [14]. But what is the analogue of this condition for quantum channels?

Second, both parts of our reasoning relied on the finite dimensionality of the jammer system $J$. It is not so clear how to deal with infinite dimension of $J$, on the other hand. A priori we have a problem already in Proposition [2] since the de Finetti reduction has an upper bound depending on the dimension $|J|$. However, one can prove the random coding capacity theorem directly from first principles, without recourse to de Finetti reductions.

Then, we have the problem again in the derandomisation step, which requires bounded $|J|$ to apply the matrix tail bound. We need some kind of quantum net argument to be able to go to a finite dimensional subspace $J' < J$ that somehow approximates the relevant features of $N$ up to error $\eta$ and block length $\ell$. Classically, the finiteness of the alphabet of channel states is irrelevant, as long as we have finite sender and receiver alphabets. The reason is that for each block length $\ell$ we can choose a subset of channel states of size polynomial in $\ell$, corresponding to an $\frac{1}{2}\ell$-net of channels realised by the jammer, for any fixed $\eta > 0$. Indeed, for the QAVC with classical jammer, which may be described by a state set $S$, the following statements are easily obtained by standard methods.

**Lemma 7** For every $\eta > 0$, there exists a set $S' \subset S$ of cardinality $|S'| \leq \left(\frac{10|A|^2}{\eta}\right)^{2|A|^2|B|^2}$, with the property that for every $s \in S$ there is an $s' \in S'$ with $\frac{1}{2}\|N_s - N_{s'}\|_o \leq \eta$, where the norm is the diamond norm (aka completely bounded trace norm) on channels [11], [22].

By applying this lemma with $\frac{1}{2}\eta$, the “telescoping trick” and the triangle inequality to bound $\frac{1}{2}\|N_s^{\ell} - N_{s'}^{\ell}\|_o$ for $s^{\ell} \in S^{\ell}$ and $s^{\ell'} \in S'^{\ell}$, we obtain:

**Lemma 8** For every $\eta > 0$ and integer $\ell$, there exists a subset $S' \subset S$ of cardinality $|S'| \leq \left(\frac{10|A|^2}{\eta}\right)^{2|A|^2|B|^2}$, such that

$$\sup_{s^{\ell} \in S^{\ell}} \mathbb{E}_\lambda P_{\text{err}}(C, s^{\ell}) \leq \sup_{s^{\ell} \in S^{\ell}} \mathbb{E}_\lambda P_{\text{err}}(C, s^{\ell'}) \leq \sup_{s^{\ell} \in S^{\ell}} \mathbb{E}_\lambda P_{\text{err}}(C, s^{\ell'})$$

for any random code $\{C_\lambda : \lambda \in \Lambda\}$. Similar for the infidelity of random quantum codes. \hfill $\square$

Since we need to entangle both the $A$’s and the $J$’s, it seems that the most natural approach is to answer the following question.

**Question 9** Let $N : \mathcal{L}(A \otimes J) \to \mathcal{L}(B)$ be a cptp map with finite dimensional $A$ and $B$, and $\eta > 0$. It is possible to find a subspace $J' \subset J$ of dimension bounded by some polynomial in $\eta^{-1}$, with the following property?

For every Hilbert space $K$ and state $\sigma$ on $J \otimes K$, there exists another state $\sigma'$ on $J' \otimes K$ such that $\frac{1}{2}\|N_\sigma - N_{\sigma'}\|_o \leq \eta$.

Here, $N_\sigma$ and $N_{\sigma'}$ are channels from $A$ to $B \otimes K$, defined by inserting the respective state into the jammer register:

$$N_\sigma(\rho) := (N \otimes \text{id}_K)(\rho \otimes \sigma), \quad N_{\sigma'}(\rho) := (N \otimes \text{id}_K)(\rho \otimes \sigma').$$

We can reduce this to the more elementary question of approximating the output of the “Choi channel” $\Gamma : \mathcal{L}(J) \to \mathcal{L}(C)$, with $C = A \otimes B$, defined by $\Gamma(\sigma) = (\text{id}_A \otimes N)(\Phi^{AA' \otimes \sigma})$, mapping each $\sigma$ to the Choi state of the channel $N_\sigma$: Namely, the question is whether for every Hilbert space $K$...
and state $\sigma$ on $J \otimes K$, does there exist a state $\sigma'$ on $J' \otimes K$ such that
\[
\frac{1}{2} \| (\Gamma \otimes id_1)(\sigma - \sigma') \|_1 \leq \eta := \eta / |A|^2.
\]

We now show that a positive answer to Question 9 with deviation $\frac{2}{3}$, could be used to replace the $\ell$ steps of each full dimensional approximation. In this way, we would be able to find, for every state $\sigma$ on $J'$, another state $\sigma''$ on $J''$, with
\[
\frac{1}{2} \| (\Lambda^{\otimes \ell})_{\sigma''} - (\Lambda^{\otimes \ell})_{\sigma'} \|_1 \leq \eta.
\]

(5)

**Proof** Set $\sigma^{(0)} := \sigma$; we shall define a sequence of approximants $\sigma^{(i)}$ on $J^{\otimes i} \otimes J^{\otimes \ell-i}$ (i = 1, . . . , $\ell$), as follows:

To obtain $\sigma^{(1)}$, we apply Question 9 with $K = J^{\otimes \ell-1}$ (the last $\ell-1$ of the $J$-systems) to obtain
\[
\frac{1}{2} \| \Lambda^{\otimes \ell}_{\sigma^{(1)}} - \Lambda^{\otimes \ell}_{\sigma} \|_1 \leq \eta / \ell,
\]
where the notation $\Lambda^{\otimes \ell}_{\sigma^{(1)}} := id_{\otimes \ell-1} \otimes \Lambda \otimes id_{\otimes \ell-1}$ indicates application of the channel to the i-th system in $J^{\ell-1}$. Proceeding inductively, assume that we already have constructed a state $\sigma^{(i)}$ on $J^{\otimes i} \otimes J^{\otimes \ell-i}$, Question 9 applied to $K = J^{\otimes i-1} \otimes J^{\otimes \ell-i}$ (i.e. all the $J^{\ell-i}$ systems and the last $\ell-i$ of the $J$'s) gives us a state $\sigma^{(i+1)}$ on $J^{\otimes i+1} \otimes J^{\otimes \ell-i-1}$ such that
\[
\frac{1}{2} \| \Lambda^{\otimes \ell}_{\sigma^{(i+1)}} - \Lambda^{\otimes \ell}_{\sigma^{(i)}} \|_1 \leq \eta / \ell.
\]

Since the diamond norm is contractive under composition with ctp maps, we obtain for all i = 1, . . . , $\ell$ that
\[
\frac{1}{2} \| (\Lambda^{\otimes \ell})_{\sigma^{(i+1)}} - (\Lambda^{\otimes \ell})_{\sigma^{(i)}} \|_1 \leq \eta / \ell,
\]
and via the triangle inequality we arrive at eq. (5), by letting $\sigma' := \sigma^{(\ell)}$ and recalling $\sigma = \sigma^{(0)}$. \(\square\)

This would mean that any behaviour that the jammer can effect by choosing states on $J'$, can be approximated up to $\pm \eta$ (on block length $\ell$) by choices from $J''$, analogously to Lemma 8 which actually provides a positive answer to Question 9 in the case of a classical jammer. Since $|J'|$ is bounded polynomially in $\ell$, we could apply now Proposition 5 and incur an additional term of $O(\log \ell)$ in the shared randomness required, in particular it will still be of zero rate.

A third complex of questions concerns the extension of the present results to other quantum channel capacities. This is easy along the above lines for cases like the entanglement-assisted capacity (cf. [11], [16]), but challenging for others, such as the private capacity [12], [15]. This is interesting because the error criterion (of decodability and privacy) does not seem to correspond to an observable on the jammer system. We leave this and the other open problems for future investigation.

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**REFERENCES**

1. D. Aharonov, A. Kitaev and N. Nisan, “Quantum circuits with mixed states”, in: Proc. 30th STOC, pp. 20-30, ACM Press, 1998; arXiv:quant-ph/9806029.

2. R. Ahlswede, “Elimination of correlation in random codes for arbitrarily varying channels”, Z. Wahrschein. Verw. Gebiete 44(2):159-175 (1978).

3. R. Ahlswede and A. Winter, “Strong Converse for Identification via Quantum Channels”, IEEE Trans. Inf. Theory 48(3):569-579 (2002); arXiv:quant-ph/0101212.

4. R. Ahlswede, I. Bjelaković, H. Boche and J. Nötzel, “Quantum Capacity under Adversarial Quantum Noise: Arbitrarily Varying Quantum Channels”, Commun. Math. Phys. 317:103-156 (2013); arXiv:quant-ph/1010.0418.

5. R. Ahlswede and V. Blinovsky, “Classical Capacity of Classical-Quantum Arbitrarily Varying Channels”, IEEE Trans. Inf. Theory 55(2):526-533 (2007).

6. G. Aubrun and S. Szarek, Alice and Bob Meet Banach – The Interface of Asymptotic Geometric Analysis and Quantum Information Theory, Amer. Math. Soc. Math. Surveys & Monographs, vol. 223, Providence RI, 2017.

7. I. Bjelaković and H. Boche, “Classical capacities of compound and averaged quantum channels”, IEEE Trans. Inf. Theory 55(7):3360-3374 (2009); arXiv:quant-ph/0710.3027.

8. I. Bjelaković, H. Boche and J. Nötzel, “Entanglement Transmission and Generation under Channel Uncertainty: Universal Quantum Channel Coding”, Commun. Math. Phys. 292(1):55-97 (2008). Erratum: Commun. Math. Phys. 317(1):98-102 (2013).

9. I. Bjelaković, H. Boche, G. Janßen and J. Nötzel, “ Arbitrarily varying and compound classical-quantum channels and a note on quantum zero-error capacities”, in: Information Theory, Combinatorics, and Search Theory: In Memory of Rudolf Ahlswede (H. Aydinian, F. Cacalese, C. Deppe, eds.), LNCS 7777, pp. 247-283, Springer Verlag, Berlin Heidelberg, 2013; arXiv:quant-ph/1209.6352v2.

10. V. Blinovsky and M. Cai, “Quantum-Quantum Arbitrarily Varying Wiretap Channel”, Information Theory, Combinatorics, and Search Theory: In Memory of Rudolf Ahlswede (H. Aydinian, F. Cacalese, C. Deppe, eds.), LNCS 7777, pp. 234-246, 2013; arXiv:cs.IT/1208.1151.

11. H. Boche, G. Janßen and S. Kaltendalder, “Entanglement-assisted classical capacities of compound and arbitrarily varying quantum channels”, arXiv:quant-ph/1606.09314 (2016).

12. M. Cai, A. Winter and R. W. Yeung, “Quantum Privacy and Quantum Wiretap Channels”, Problems Inf. Transm. 40(4):318-336 (2004).

13. M. Christandl, R. König and R. Renner, “Postselection Technique for Quantum Channels with Applications to Quantum Cryptography”, Phys. Rev. Lett. 102:020504 (2009); arXiv:quant-ph/0809.3019.

14. I. Csiszár and P. Narayan, “ The capacity of the arbitrarily varying channel revisited: positivity, constraints”, IEEE Trans. Inf. Theory 34(2):181-193 (1988).

15. I. Devetak, “ The private classical information capacity and quantum information capacity of a quantum channel”, IEEE Trans. Inf. Theory 51(1):44-55 (2005); arXiv:quant-ph/0304127.

16. R. Duan, S. Severini and A. Winter, “ On Zero-Error Communication via Quantum Channels in the Presence of Noisefeedback”, IEEE Trans. Inf. Theory 62(9):5260-5277 (2016); arXiv:quant-ph/1502.02987.

17. T. Ericson, “ Exponential error bounds for random codes in the arbitrarily varying channel”, IEEE Trans. Inf. Theory 31(1):42-48 (1985).

18. P. Hayden, D. Leung, P. W. Shor and A. Winter, “ Randomising Quantum States: Constructions and Applications”, Commun. Math. Phys. 250:371-394 (2004); arXiv:quant-ph/0307104.

19. S. Karumanchi, S. Mancini, A. Winter and D. Yang, “Quantum Channel Capacities with Passive Environment Assistance” IEEE Trans. Inf. Theory 62(4):1733-1747 (2016); arXiv:quant-ph/1407.8160v2.

20. S. Karumanchi, S. Mancini, A. Winter and D. Yang, “Classical capacities of quantum channels with environment assistance”, Problems Inf. Transm. 52(3):214-238 (2016); arXiv:quant-ph/1602.02036v2.

21. M. Mosonyi, “Coding Theorems for Compound Problems via Quantum Rényi Divergences”, IEEE Trans. Inf. Theory 61(6):2997-3012 (2015); arXiv:quant-ph/1310.7525.

22. J. Watrous, "Semidefinite Programs for Completely Bounded Norms", Theory of Computing 5(1):217-238 (2009); arXiv:quant-ph/0901.4709.