MASA’S AND CERTAIN TYPE I 
CLOSED FACES OF $C^*$-ALGEBRAS

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*Dedicated to the memory of George W. Mackey*

**Abstract.** Let $A$ be a separable $C^*$-algebra and $A^{**}$ its enveloping $W^*$-algebra. A result of Akemann, Anderson, and Pedersen states that if $\{p_n\}$ is a sequence of mutually orthogonal, minimal projections in $A^{**}$ such that $\sum_k p_n$ is closed, $\forall k$, then there is a MASA $B$ in $A$ such that each $\varphi_n|B$ is pure and has a unique state extension to $A$, where $\varphi_n$ is the pure state of $A$ supported by $p_n$. We generalize this result in two ways: We prove that $B$ can be required to contain an approximate identity of $A$, and we show that the countable discrete space which underlies the result cited can be replaced by a general totally disconnected space. We consider two special kinds of type I closed faces, both related to the above, atomic closed faces and closed faces with nearly closed extreme boundary. One specific question is whether an atomic closed face always has an “isolated point”. We give a counterexample for this and also show that the answer is yes if the atomic face has nearly closed extreme boundary. We prove a complement to Glimm’s theorem on type I $C^*$-algebras which arises from the theory of type I closed faces. One of our examples is a type I closed face which is isomorphic to a closed face of every non-type I separable $C^*$-algebra and which is not isomorphic to a closed face of any type I $C^*$-algebra.

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0. Introduction.

This paper was inspired by the paper [5] of C. Akemann, J. Anderson, and G. Pedersen. Much of the terminology used in this section is explained in later sections. To explain the connection with [5], we begin with:

**Proposition 0.1.** Let $A$ be a $C^*$-algebra and $(p_n)$ a sequence of mutually orthogonal, minimal (rank one) projections in $A^{**}$. Let $p = \sum_{1}^{\infty} p_n$, and let $\varphi_n$ be the pure state supported by $p_n$. Then either of the following hypotheses implies that $p$ is closed:

(i) ([5, 2.7(1)⇒(2)]). There is a strictly positive element $e$ such that each $\varphi_n$ is definite on $e$ and $\varphi_n(e) \to 0$.

(ii) ([12, Lemma 3]). There is a strictly positive element $e$ such that $\sum_{1}^{\infty} \varphi_n(e) < \infty$.

In circumstances similar to 0.1, [5] proves the existence of a MASA $B$ such that each $\varphi_n|_B$ has the unique extension property. The hypotheses require that $A$ be non-unital. It is known (see [6, §4]) that a non-unital $C^*$-algebra $A$ may have MASAs which do not hereditarily generate $A$, or equivalently which do not contain an approximate identity of $A$. If the MASA constructed in [5] does not hereditarily generate $A$, the situation is intuitively unsatisfactory. (See the first paragraph of [5, §2].)

To investigate strengthening the result of [5], consider $\tilde{A}$, the result of adjoining an identity to $A$, and the pure state $\varphi_\infty$ defined by $\varphi_\infty(\lambda 1_{\tilde{A}} + a) = \lambda$. The existence of a MASA $B$ of $A$ such that each $\varphi_n|_B$, $1 \leq n < \infty$, has the unique extension property and such that $B$ hereditarily generates $A$ is equivalent to the existence of a MASA $B_1$ of $\tilde{A}$ such that each $\varphi_n|_{B_1}$, $1 \leq n \leq \infty$, has the unique extension property ($B = B_1 \cap A$, $B_1 = \tilde{B}$).

Now the hypotheses of [5] imply that $\sum_{n \in I} p_n$ is closed for every subset $I$ of $\mathbb{N}$. Thus $\{p_n : 1 \leq n < \infty\}$ has properties analogous to those of the discrete topological space $\mathbb{N}$. But when $p_\infty$, the support projection of $\varphi_\infty$, is added to the set, the new set resembles the non-discrete space $\mathbb{N} \cup \{\infty\}$. Thus we seek a generalization of the MASA result of [5] based on a class of topological spaces which includes $\mathbb{N} \cup \{\infty\}$. We accomplish this in Corollary 2.4: Let $A$ be a separable $C^*$-algebra and $X$ a totally disconnected, second countable, locally compact Hausdorff space. Assume that for each $x$ in $X$, $p_x$ is a minimal projection in $A^{**}$, with associated pure state $\varphi_x$, such that the $p_x$’s are mutually orthogonal and for each closed (compact) subset $S$ of $X$, $\sum_{x \in S} p_x$ is the atomic part of a closed (compact) projection, $p_S$, in $A^{**}$. Then there is a MASA $B$ of $A$ such that $B$ hereditarily generates $A$ and each $\varphi_x|_B$ has the unique extension property. Moreover, each $p_S$ is in $B^{**}$.

If $X$ is general, the hypotheses of the above result may seem rather stringent. Partly in order to justify the generality of the result, we attempt to investigate the circumstances in which the hypotheses will be satisfied. A first observation is that every element of $C_0(X)$ (respectively, $C_b(X)$) gives rise to an element of $p_XA^{**}p_X$ which is strongly $q$-continuous (respectively $q$-continuous) on $p_X$. (The concept “$q$-continuous on $p$” was defined in [7]. In [11], “strongly $q$-continuous
on $p^*$ was defined and \textquoteleft\textquoteleft Tietze extension theorems\textquoteright\ for both kinds of relative $q$-continuity were given.) Thus in Section 3 we give some basic results and examples on the subject of how many relatively $q$-continuous elements are supported by a given closed projection.

We also focus on a more specific question suggested by the theory of scattered $C^*$-algebras: Suppose that $p$ is an atomic closed projection in $A^{**}$ and that $pA^*p$ is norm separable. Is there a minimal projection $p_0$ such that $p_0 \leq p$ and $p - p_0$ is closed? Such a $p_0$ would give an \textquoteleft\textquoteleft isolated point\textquoteright\ of the closed face $F(p)$ supported by $p$. This question is related to the special case of 2.4 where $X$ is countable. Clearly, if we seek to prove that certain conditions imply the hypotheses of 2.4, then we must be able to prove that these conditions imply a positive answer to our isolated point question. Note also that when $X$ in 2.4 is countable, then $p_X$ is atomic, and the words \textquoteleft\textquoteleft the atomic part of\textquoteright\ can be omitted.

We give a counterexample for the isolated point question in Section 3, but we also give a positive result which has the following hypothesis (nearly closed extreme boundary):

\begin{equation}
(P(A) \cap F(p))^- \subset \{0\} \cup [t, 1]P(A) \quad \text{for some } t \in (0, 1].
\end{equation}

Here $P(A)$ is the pure state space of $A$, and (NCEB) holds in particular if the set of extreme points of $F(p)$ is (weak$^*$) closed. Lest (NCEB) seem unnatural or excessively strong, we point out a connection with [5, §4]. Circumstances not covered by 0.1 are actually considered in [5]. Suppose $\{\varphi_n : 1 \leq n < \infty\}$ is a collection of mutually orthogonal pure states such that $\varphi_n \overset{w^*}{\to} 0$ and each equivalence class is finite. With the additional assumption that there is a uniform bound on the size of the equivalence classes, the authors of [5] show in §4 that the needed conditions ([5, 2.7(2)]) are satisfied. Without the uniform boundedness hypothesis, we can show easily that [5, 2.7(2)] is equivalent to (NCEB).

Our positive result, which is in Section 4, is roughly that if $p$ is a closed projection satisfying (NCEB), then equivalence of pure states gives a proper closed map from $[P(A) \cap F(p)]^- \setminus \{0\}$ onto a locally compact Hausdorff space $X$. If $pA^*p$ is norm separable, then $X$ is countable and hence scattered. In general, $X$ need not even be totally disconnected, of course.

There are some technicalities involving direct integral theory required in order to prove that closed subsets of $X$ give rise to closed projections. This is what leads us to the study in Section 5 of type I closed faces, where the face $F(p)$ is called type I if $pA^{**}p$ is a type I $W^*$-algebra. Obviously every atomic face is type I, and also $F(p)$ is type I when $p$ (is closed and) satisfies (NCEB), at least if $A$ is separable. Our results on type I closed faces are only rudimentary, and we think the concept is worthy of further study.

Partly because theorems are not always discovered in logical order, our efforts to expand on the results of [5] have led us in several directions. The different parts of this paper, though closely related, do not mesh perfectly. In Section 7 we attempt to exhibit the formal relationships among the previous sections. The earlier sections can in large part be read independently of one another, except that Section 6 is
a continuation of Section 4 relying on Section 5. The promised complement to Glimm’s theorem is Proposition 5.11.

A preliminary preprint of this paper was circulated several years ago. Some results overlapping with Section 3 have been independently found by E. Kirchberg (cf. [22, Lemma 2.3]).

1. Preliminaries

A will always denote a \( C^* \)-algebra and \( A^{**} \) its enveloping \( W^* \)-algebra. For \( h \) in \( A_{sa}^{**} \) and \( F \) a Borel set in \( \mathbb{R} \), \( E_F(h) \) denotes the spectral projection of \( h \) for \( F \). For many of our proofs \( A \) must be separable, but we rarely require that \( A \) be unital. \( S(A) \) is the state space of \( A \), \( P(A) \) the pure state space, and \( Q(A) \) the quasi-state space (the set of positive functionals of norm at most 1). If \( p \) is a projection in \( A^{**} \), \( F(p) = \{ \varphi \in Q(A) : \varphi(1-p) = 0 \} \), the norm closed face of \( Q(A) \) supported by \( p \). (Elements of \( A^* \) are regarded as functionals on \( A^{**} \) without notice.) Topological terminology regarding \( A^* \) refers to the weak* topology unless the contrary is explicitly indicated. A projection \( p \) in \( A^{**} \) is called open ([1]) if it is the support projection of a hereditary \( C^* \)-subalgebra of \( A \) and closed if \( 1-p \) is open. Effros proved in [15, Theorem 4.8] (cf. [25, 3.10.7]) that \( A \) is a “closed face of \( S \)”, where \( S \) be unital.

Lemma 1.1. Let \( (p_n) \) be a sequence of mutually orthogonal minimal projections in \( A^{**} \) and \( p = \sum_1^\infty p_n \). If, \( \forall a \in A \), \( \pi^{**}(p)\pi(a)\pi^{**}(p) \) is a compact operator on \( H_\pi \), where \( \pi \) is the reduced atomic representation of \( A \), then \( p \) is closed.

The proof of 1.1 and the fact that it implies 0.1(ii) is identical to the proof of Lemma 3 in [12]. Lemma 1.1 implies 0.1(i) because in that case \( \pi^{**}(p)\pi(e)\pi^{**}(p) \) is a diagonal operator whose matrix elements approach zero. (If \( A \) is \( \sigma \)-unital, it is enough to verify the compactness for a strictly positive element of \( A \) as shown in [12].) Lemma 1.1 also applies under the Standing Assumptions of [5, §4], since then \( \pi^{**}(p)\pi(a)\pi^{**}(p) \) is a block-diagonal operator with bounded block size - in particular it is a \((2N+1)\)-diagonal operator.

Despite this, we offer the following new proof of 0.1(i), which may be instructive:
The hypothesis that $\varphi_n$ is definite on $e$ is equivalent to $p_n e = e p_n$. Thus if $\lambda_n = \varphi_n(e)$, then $p_n \leq E_{\{\lambda_n\}}(e)$. Let $\epsilon_k = \sup\{\lambda_n : n > k\}$ and $q_k = \sum_{1}^{k} p_n \lor E_{[0, \epsilon_k]}(e)$. Since $[0, \epsilon_k]$ is a closed set, $E_{[0, \epsilon_k]}(e)$ is closed, and thus [1, Theorem II.7] implies that $q_k$ is closed. Since $E_{[0]}(e) = 0$, $p = \wedge_{1}^{\infty} q_k$, and [1, Proposition II.5] implies $p$ is closed.

**Theorem 1.2.** Let $(p_n)$ be a sequence of mutually orthogonal minimal projections in $A^{**}$ and $p = \sum_{1}^{\infty} p_n$. Then the following are equivalent:

(i) Every subprojection of $p$ in $A^{**}$ is closed.

(ii) $\sum_{n \in I} p_n$ is closed for each subset $I$ of $\mathbb{N}$.

(iii) $\sum_{k} p_n$ is closed, $\forall k$ (cf. [5, 2.7(2)]).

(iv) $\pi^{**}(p) \pi(A) \pi^{**}(p) \subset \mathcal{K}(H_{\pi})$, where $\pi$ is the reduced atomic representation of $A$.

**Proof.** (i)⇒(ii)⇒(iii) is obvious.

(iii)⇒(iv): Let $q_k = \sum_{k} p_n$. Then $F(q_k)$ is a closed subset of $Q(A)$ and $\bigcap_{1}^{\infty} F(q_k) = \{0\}$. By this and [5, 2.3], $(q_k)$ approaches infinity in the sense of [5]. By definition, $\|aq_k\| \to 0$, $\forall a \in A$. Therefore $\pi(a) \pi^{**}(p - q_k) \to \pi(a) \pi^{**}(p)$ in norm. Since $\pi^{**}(p - q_k)$ is a finite rank operator, this implies $\pi(a) \pi^{**}(p)$, and a fortiori $\pi^{**}(p) \pi(a) \pi^{**}(p)$, is compact.

(iv)⇒(i): Assume $p' \in A^{**}$ and $p' \leq p$. Then $\pi^{**}(p') \pi(A) \pi^{**}(p') \subset \mathcal{K}(H_{\pi})$. Clearly there are mutually orthogonal minimal projections $p'_n$ such that $p' = \sum_{1}^{\infty} p'_n$. Thus $p'$ is closed by 1.1.

Perhaps it should also be mentioned that if $I$ in 1.2(ii) is finite, then $\sum_{n \in I} p_n$ is finite rank and hence compact ([1, Corollary II.8]).

If $\varphi$ is in $P(A)$ and $B$ is a $C^{*}$-subalgebra of $A$, we say that $\varphi | B$ has the unique extension property (UEP) if $\varphi | B \in P(B)$ and $\varphi$ is the only element of $S(A)$ which extends $\varphi | B$. The next proposition is probably not original (see [5, p. 267]).

**Proposition 1.3.** Assume $p$ is minimal projection in $A^{**}$ and $\varphi$ is the associated pure state. If $B$ is a $C^{*}$-subalgebra of $A$, then $\varphi | B$ has (UEP) if and only if $p$ is in $B^{**}$.

**Proof.** Of course $B^{**}$ is identified with the weak* closure of $B$ in $A^{**}$. First assume (UEP) and let $q$ be the support projection of $\varphi | B$, so that $q$ is a minimal projection in $B^{**}$. If $\psi$ is in $S(A)$ and $\psi(q) = 1$, then $\psi | B$ is in $F(q) \cap S(B)$, and hence $\psi | B = \varphi | B$. By (UEP), $\psi = \varphi$. Thus we have shown that $F(q)$, computed in $A^{*}$, is one dimensional, and this clearly implies $q = p$.

Conversely, assume $p \in B^{**}$. Since $p$ is minimal in $A^{**}$, it is clearly minimal in $B^{**}$. Since $\varphi | B(p) = 1$ and $\|\varphi | B\| \leq \|\varphi\| = 1$, $\varphi | B$ is a state supported by $p$. Therefore $\varphi | B \in P(B)$. If $\psi \in S(A)$ and $\psi | B = \varphi | B$, then $\psi$ and $\varphi$ agree also on $B^{**}$. Thus $\psi(p) = \varphi(p) = 1$, $\psi$ is supported by $p$, and hence $\psi = \varphi$.

Recall the condition (NCEB), which was defined in Section 0 for any projection $p$ in $A^{**}$. It is also convenient to have a name for the special case of (NCEB) where
$t = 1$.

\[(CEB) \quad [P(A) \cap F(p)]^- \subset \{0\} \cup P(A).\]

The phrase “closed extreme boundary”, is accurate only when $p$ is closed, but the main uses of (NCEB) and (CEB) are for projections known \textit{a priori} to be closed.

**Theorem 1.4.** Let $(p_n)$ be a sequence of mutually orthogonal minimal projections in $A^{**}$, $(\varphi_n)$ the associated sequence of pure states, and $p = \sum_1^\infty p_n$. If the equivalence classes of \{\varphi_n\} are finite and $\varphi_n \xrightarrow{w^*} 0$, then the following are equivalent:

(i) $\sum_k^\infty p_n$ is closed, $\forall k$.

(ii) $p$ satisfies (NCEB).

(iii) $p$ satisfies (CEB).

(iv) $[F(p) \cap P(A)]^- \subset \{0\} \cup [t, 1]S(A)$ for some $t$ in $(0, 1]$.

**Proof.** Let $\Gamma_1, \Gamma_2, \ldots$ be the equivalence classes of \{\varphi_n\}, and let $q_i = \sum_{\varphi_n \in \Gamma_i} p_n$. Thus each $q_i$ is a finite rank, and hence compact, projection in $A^{**}$.

(iv)$\Rightarrow$(i): For this it is clearly permissible to simplify the notation by assuming $k = 1$. Thus we need to show that $p$, which is $\sum_i q_i$, is closed. According to Proposition 4.2 of [5], for this it is sufficient to show that $(q_i)$ approaches infinity. Let $U$ be a convex neighborhood of $0$ in $A^*$. We need to find $i_0$ such that $F(q_i) \subset U$ for $i \geq i_0$. By the Krein-Milman theorem, it is sufficient to show $F(q_i) \cap P(A) \subset U$ for $i \geq i_0$. If this is false we can find nets $(\psi_j)$ and $(i_j)$ such that $\psi_j \in F(q_{i_j}) \cap P(A)$, $i_j \to \infty$, $\psi_j \to \psi$, and $\psi \neq 0$.

Let $\pi$ be the reduced atomic representation of $A$ and $H_j$ the range of $\pi^{**}(q_{i_j})$. Thus dim $H_j = |\Gamma_j|$. If dim $H_j = 1$ for arbitrarily large $j$, then $\psi_j = \varphi_n$ for $\varphi_n \in \Gamma_{i_j}$; and we already know $\varphi_n \to 0$. Thus we may assume dim $H_j = 2$, $\forall j$. Then we can find unit vectors $v_j, v_j', v_j''$ in $H_j$ such that $(v_j', v_j'') = 0$, the pure states $(\pi(\cdot)v_j', v_j')$ and $(\pi(\cdot)v_j'', v_j'')$ are in $\Gamma_{i_j}$, and $\psi_j = (\pi(\cdot)u_j, u_j)$. Choose a unit vector $v_j$ in span{$v_j', v_j''$} such that $(v_j, u_j) = 0$, and let $\theta_j = (\pi(\cdot)v_j, v_j)$. Then $\theta_j \to 0$. This follows from Lemma 4.1 of [5], with the $N$ of [5] being 2, or it can be proved directly using an argument similar to the one below. Let $f_j = (\pi(\cdot)u_j, v_j)$, which is an element of $A^*$. By the Schwarz inequality, $|f_j(a)| \leq ||\pi(a^*)v_j|| = \theta_j(aa^*)^{1/2}$. Therefore $f_j \to 0$. Then if $w_j = ru_j + sv_j$, with $|r|^2 + |s|^2 = 1$, and $\rho_j = (\pi(\cdot)w_j, w_j)$, we see that $\rho_j \in F(p) \cap P(A)$, and $\rho_j \to |r|^2 \psi$. We can choose $r, s$ such that $0 < |r|^2 ||\psi|| < t$, in contradiction to (iv).

(i)$\Rightarrow$(iii): Suppose $\psi_j \in F(p) \cap P(A)$ and $\psi_j \to \psi$. Then for each $j$ there is $i_j$ such that $\psi_j \in F(q_{i_j})$. If $i_j \to \infty$, then by passing to a subnet we may assume $i_j = i$, $\forall j$. Then it is easy to see that $\psi \in F(q_i) \cap P(A)$. (Each $\psi_j$ is a vector state coming from $H_j$ and the unit sphere of $H_j$ is norm compact.) If $i_j \to \infty$, then for each $k$, $\psi_j \in F(\sum_k^\infty p_n)$ for sufficiently large $j$. By (i), $\psi \in F(\sum_k^\infty p_n)$. Since $\bigwedge_{k=1}^\infty \sum_k^\infty p_n = 0$, $\psi = 0$.

(iii)$\Rightarrow$(ii)$\Rightarrow$(iv) is obvious.

2. Existence of MASA’s
Lemma 2.1. Let $A$ be a $C^*$-algebra and $\tilde{A}$ the result of adjoining a new identity to $A$ (i.e., $\tilde{A} \cong A \oplus \mathbb{C}$ if $A$ is already unital). Let $\varphi_\infty$ in $P(A)$ be defined by $\varphi_\infty(A1_{\tilde{A}} + a) = \lambda$. Assume $B_1$ is a unital $C^*$-subalgebra of $\tilde{A}$ such that $\varphi_\infty|_{B_1}$ has (UEP), and let $B = B_1 \cap A$. Then $B$ hereditarily generates $A$ and $B^{**} = B_1^{**} \cap A^{**}$.

Proof. That $B^{**} = B_1^{**} \cap A^{**}$ follows, for example, from general Banach space theory and the fact that $B_1/B$ is finite dimensional. Now let $p_\infty$ be the support projection of $\varphi_\infty$. Then $p_\infty \in B_1^{**}$ by 1.3. Since $1_{\tilde{A}} \in B_1^{**}$, $1_{\tilde{A}} - p_\infty$ is also in $B_1^{**}$, and of course $1_{\tilde{A}} - p_\infty$ is the identity of $A^{**}$. Thus $1_{\tilde{A}} - p_\infty \in B^{**}$, and this implies that $B$ hereditarily generates $A$.

Lemma 2.2. Let $A$ be a separable unital $C^*$-algebra, $p$ a closed projection in $A^{**}$, and $q$ an open projection in $A^{**}$ such that $q \geq p$. Let $B = \text{her}(q)$, the hereditary $C^*$-subalgebra of $A$ supported by $q$, and let $U$ be a neighborhood of $F(p) \cap S(A)$ in $S(A)$. Then there is a closed projection $p'$ in $B^{**}$ such that $p'p = 0$ and $\varphi(p') = 0$ implies $\varphi \in U$ for $\varphi$ in $S(B)$.

Proof. As usual, we identify $B^{**}$ with $qA^{**}q$ and $S(B)$ with $\{ \varphi \in S(A) : \varphi(q) = 1 \}$. (The weak$^*$ topologies of $A^*$ and $B^*$ agree on $S(B)$.) By Akemann’s Urysohn lemma, [2, Theorem 1.1], there is $a$ in $A_{sa}$ such that $p \leq a \leq q$. Then $a \in B$. Let $C = \text{her}(q-p)$, and let $e$ be a strictly positive element of $C$. Let $b = a - aea$. Then, by an argument of Akemann [3, 1.1], $E_{\{1\}}(b) = p$. Let $p'_n = E_{(-\infty, 1-n^{-1}]}(b)$, where the spectral projection is computed in $B^{**}$. We claim that for $n$ sufficiently large the choice $p' = p'_n$ suffices. If not, for each $n$ there is $\varphi_n$ in $S(B)$ such that $\varphi_n(p'_n) = 0$ and $\varphi_n \notin U$. Then $\varphi_n$ is supported by $q - p'_n = E_{[1-n^{-1}, 1]}(b) \leq E_{[1-n^{-1}, 1]}(b)$. Let $\varphi$ be a cluster point of $(\varphi_n)$ in $S(A)$. Then since each $E_{[1-n^{-1}, 1]}(b)$ is closed in $A^{**}$, $\varphi$ is supported by $\bigwedge_{n=1}^{\infty} E_{[1-n^{-1}, 1]}(b) = E_{\{1\}}(b) = p$. Therefore $\varphi \in F(p) \cap S(A)$, a contradiction since $\varphi_n \notin U$.

Theorem 2.3. Let $A$ be a separable $C^*$-algebra and $X$ a second countable, totally disconnected, locally compact Hausdorff space. Assume that for each $x$ in $X$, $p_x$ is an atomic projection in $A^{**}$, the $p_x$’s are mutually orthogonal, and for every closed (compact) subset $S$ of $X$ there is a closed (compact) projection $p_S$ such that $z_{at}p_S = \sum_{x \in S} p_x$. Then there is a MASA $B$ in $A$ such that $B$ hereditarily generates $A$ and each $p_S$ is in $B^{**}$.

Proof. First we reduce to the case $A$ unital, $X$ compact. To do this, let $\tilde{A}$ be the result of adjoining a new identity to $A$, and let $\tilde{X} = X \cup \{ \infty \}$ be the one point compactification. If we let $p_\infty$ be as in 2.1, all hypotheses of the theorem are satisfied for $\tilde{X}$, $\tilde{A}$. (If $S$ is a compact subset of $X$, then $p_S$ is compact in $A^{**}$ and hence closed in $\tilde{A}^{**}$. Any other closed subset of $\tilde{X}$ is $S \cup \{ \infty \}$ for some closed subset $S$ of $X$. The fact that $p_S$ is closed in $A^{**}$ implies that $p_S + p_\infty$ is closed in $\tilde{A}^{**}$. Since $\tilde{A}$ is unital, “closed” and “compact” mean the same for projections in $\tilde{A}^{**}$.) If $B_1$ satisfies the conclusion of the theorem for $\tilde{A}$, $\tilde{X}$, then by 2.1, $B_1 \cap A$ satisfies the conclusions of the theorem for $A$, $X$. Thus from now on we assume $A$ unital and $X$ compact.
Let $C$ be the usual middle-thirds Cantor set in $[0, 1]$. Then there is a one-to-one continuous function $f : X \rightarrow C$. We will let $\alpha$ and $\beta$ denote finite strings of $+$’s and $-$’s, and $|\alpha|$ denote the length of $\alpha$. Let $C_+, C_- \subseteq X$ be the right and left halves of $C$, $C_{++}, C_{+-}$ the right and left halves of $C_+$, etc. Let $p_\alpha = p_{f^{-1}(C_\alpha)}$, and let $F_\alpha = S(A) \cap F(p_\alpha)$. Note that $p_\alpha + p_{\alpha^-} = 0$ and $p_\alpha = p_{\alpha^+} + p_{\alpha^-}$. This follows, for example, from the theory of universally measurable elements of $A^{**}$, [25, 4.3] and the fact that the relations are satisfied by the atomic parts of the projections. Let $e$ be a strictly positive element of her$(1 - p_X)$. We are going to construct recursively $b_\alpha$ in $A_+$ and an open projection $q_\alpha$ in $A^{**}$ such that:

1. $b_\alpha b_{\alpha^-} = 0$
2. $p_\alpha \leq q_\alpha \leq E_{\{1\}}(b_{\alpha^\pm})$
3. $b_\alpha^+, b_{\alpha^-} \in \text{her}(q_\alpha)$. (Thus $b_\alpha b_{\alpha^\pm} = b_{\alpha^\pm}$.)
4. If $\varphi$ in $S(A)$ is supported by $E_{\{1\}}(b_{\alpha^\pm})$, then $\varphi(e) < |\alpha|^{-1}2^{-|\alpha|}$.

Fix non-negative functions $g_+, g_- \in C([-1, 1])$ such that $g_+ = 1$ on $[\frac{3}{4}, 1]$, $g_+$ is supported on $[\frac{1}{4}, 1]$, $g_- = 1$ on $[-1, -\frac{3}{4}]$, and $g_-$ is supported on $[-1, -\frac{1}{4}]$.

Step 1, $|\alpha| = 1$. Then 3 and 4 are vacuous. Choose $a$ in $A_{sa}$ such that $-1 < a \leq 1$, $p_- \leq E_{\{\alpha\}}(a)$, and $p_+ \leq E_{\{1\}}(a)$. This is easily accomplished by [2, Theorem 1.1] and the continuous functional calculus. Let $b_{\pm} = g_{\pm}(a)$, $q_+ = E_{\{\cdot, 1\}}(a)$, and $q_- = E_{[-1, -\frac{3}{4}]}(a)$.

Step $k$, $|\alpha| = k > 1$. We construct $b_{\beta^\pm}, q_{\beta^\pm}$ for each $\beta$ with $|\beta| = k - 1$, assuming of course that $b_\beta, q_\beta$ have already been constructed. Apply 2.2 to find a closed projection $p'$ in her$(q_\beta)^{**}$ such that $p' p_\beta = 0$ and if $\varphi$ in $S(A)$ is supported by $q_\beta$ and $\varphi(p') = 0$, then $\varphi(e) < |\beta|^{-2} |\beta'|$. Next choose $a$ in her$(q_\beta)$ such that $-1 < a \leq 1$, $p' \leq E_{\{\alpha\}}(a)$, and $p_{\beta^\pm} \leq E_{\{\alpha\}}(a)$. The existence of $a$ can be deduced from [11, 34.3], but it is more elementary to apply Akemann’s Urysohn lemma for her$(q_\beta)$ twice to obtain $a_1$ and $a_2$ with $p_{\beta^+} \leq a_1 \leq 1 - (p' + p_{\beta^-})$ and $p_{\beta^-} \leq a_2 \leq 1 - (p' + p_{\beta^+})$. Then let $a = a_1 a_2$. Then let $q_{\beta^+} = E_{\{\cdot, 1\}}(a)$, $q_{\beta^-} = E_{[-1, -\frac{3}{4}]}(a)$, and $b_{\beta^\pm} = g_{\pm}(a)$.

Now $\{b_\alpha\}$ is commutative, since for $\alpha \neq \alpha'$ either $b_\alpha b_{\alpha'} = 0$, $b_\alpha b_{\alpha'} = b_{\alpha'}$, or $b_\alpha b_{\alpha'} = b_\alpha$. Let $B$ be any MASA containing all $b_\alpha$’s. If $p_\alpha = E_{\{1\}}(b_\alpha)$, then $p_\alpha' \in B^{**}$. Note that $p_{\alpha_1} p_{\alpha_2}' = 0$ if $|\alpha_1| = |\alpha_2|$ and $\alpha_1 \neq \alpha_2$ and that $p_\alpha' \geq p_\alpha$.

We show that $p_X \in B^{**}$ by proving $p_X = \bigcup_{n=1}^{\infty} p_\alpha'$. Clearly the latter is at least $p_X$. Suppose $\varphi \in S(A) \cap F(\bigcup p_\alpha')$. Let $\varphi_\alpha = p_\alpha' \varphi p_\alpha'$. Then $\sum_{|\alpha|=n} \|\varphi_\alpha\| = 1$, $\varphi_\alpha(e) < (n - 1)^{-1} 2^{-(n-1)} \|\varphi_\alpha\|$, by 4, and $\varphi \leq 2^n \sum_{|\alpha|=n} \varphi_\alpha$. Therefore $\varphi(e) < 2(n - 1)^{-1}$. If the above is true for all $n$, then $\varphi(e) = 0$ and hence $\varphi \in F(p_X)$.

Finally, to show that every $p_\beta$ is in $B^{**}$, note that every closed subset of $C$ is the intersection of a sequence of clopen sets and every clopen set is the union of finitely many $C_\alpha$’s. Thus it is sufficient to show that each $p_\alpha$ is in $B^{**}$. We do this by showing that $p_\alpha = p_X \wedge p_\alpha'$. This follows from $p_\alpha' \geq p_\alpha$, $p_\alpha' p_\beta = 0$ if $|\alpha| = |\beta|$ and $\alpha \neq \beta$, and $p_X = \sum_{|\beta|=|\alpha|} p_\beta$. 
Corollary 2.4. Assume the hypotheses of 2.3 and also that each $p_x$ is a minimal projection in $A^{**}$. Let $\varphi_x$ be the pure state supported by $p_x$. Then if $B$ is the MASA of 2.3, $\varphi_x|B$ has the unique extension property, $\forall x \in X$.

Proof. Combine 2.3 and 1.3, and note that $p_x = p_S$ for $S = \{x\}$.

Remark 2.5. Since the construction of the MASA in 2.3 requires only the $p_S$'s, we could start with a more general, but also more abstract, setup, an assignment $S \mapsto S$, $\forall S$. Theorem 3.1. If $\varphi$ satisfies $\varphi(a) = 0$ for $a \in B_{\sigma}$, then $\varphi$ is $\varphi(x) = 0$ for $x \in X$. We will show that $\varphi(x) = 0$ for $x \in X$. Another alternative formulation, using relative $q$-continuity, appears below in 7.1 (see also 7.5). The hypotheses actually used in 2.3 and 2.4 imply a stronger relationship between the structure of $F(p_X)$ and the space $X$.

3. Relative $q$-continuity

Let $p$ be a closed projection in $A^{**}$ and $h$ an element of $pA^{**}p$. Then $h$ is called $q$-continuous on $p$ ([7]) if $E_F(h)$ is closed for every closed subset $F$ of $\mathbb{R}$, where the spectral projection is computed in $pA^{**}p$, and $h$ is called strongly $q$-continuous on $p$ ([11]) if in addition, $E_F(h)$ is compact when $F$ is closed and $0 \notin F$. It was shown in [11, 3.43] that $h$ is strongly $q$-continuous on $p$ if and only if $h = pa$ for some $a$ in $A_{sa}$, such that $pa = ap$, and if $A$ is $\sigma$-unital, then $h$ is $q$-continuous on $p$ if and only if $h = px$ for some $x$ in $M(A)_{sa}$ such that $px = xp$.

It was neglected in [11] to give any serious examples or discussion of how extensive is the set of relatively $q$-continuous elements. For general $h$ in $pA^{**}p$ let us say that $h$ is $q$-continuous or strongly $q$-continuous on $p$ if both $\text{Re} h$ and $\text{Im} h$ are. Let $\text{SQC}(p) = \{h \in pA^{**}p : h$ is strongly $q$-continuous on $p\}$, and let $\text{QC}(p) = \{h \in pA^{**}p : h$ is $q$-continuous on $p\}$. By [11, 3.45], $\text{SQC}(p)$ is a $C^*$-algebra, and if $A$ is $\sigma$-unital, $\text{QC}(p)$ is also a $C^*$-algebra. We say that $p$ satisfies (MSQC) (many strongly $q$-continuous elements) if $\text{SQC}(p)$ is $\sigma$-weakly dense in $pA^{**}p$ and $p$ satisfies (MQC) if $\text{QC}(p)$ is $\sigma$-weakly dense in $pA^{**}p$. The von Neumann and Kaplansky density theorems give many equivalent formulations of (MSQC), and also (MQC) if $A$ is $\sigma$-unital. As for the other extreme, we always have $Cp \subset QC(p)$ and $0 \in QC(p)$. We will show that $QC(p)$ and $SQC(p)$ need not be any bigger. Of course, $QC(p) = SQC(p)$ if and only if $p$ is compact.

Theorem 3.1. If $p$ is a closed projection in $A^{**}$, then the following are equivalent:

1. $p$ satisfies (MSQC).
2. $pAp = SQC(p)$.
3. $pAp$ is an algebra.
4. $pA^p$ is a Jordan algebra.
5. $F(p)$ is isomorphic to the quasi-state space of a $C^*$-algebra.

Remarks. If $F_1$ and $F_2$ are closed faces of $C^*$-algebras, we say they are isomorphic if there is a 0-preserving affine isomorphism which is also a (weak) homeomorphism. An intrinsic characterization of $pA^p$ was observed in [11] (a portion of the proof of 3.5 for which no originality was claimed): $pA^p$ is the set of continuous affine functionals vanishing at 0 on $F(p)$. With help of [15] one can find intrinsic characterizations of $QC(p)$ and $SQC(p)$. One of the consequences of [7, 4.4, 4.5] is that $pA^{**}p$ is the bidual of the Banach space $pA^p$. In [14] we will give an intrinsic characterization of $pM(A)p$. Thus many questions concerning a closed face of a $C^*$-algebra $A$ can be treated intrinsically, without knowing what $A$ is.

The $C^*$-algebra of 5 is determined only up to Jordan $*$-isomorphism.

Proof. 1 $\Rightarrow$ 2: Since $SQC(p) \subset pA^p$ and $pA^{**}p$ is the bidual of $pA^p$, $SQC(p)$ is dense in $pA^p$ in the weak Banach space topology. Therefore $SQC(p)$ is norm dense in $pA^p$. But $SQC(p)$ is norm closed (since it is a $C^*$-algebra, for example).

2 $\Rightarrow$ 3 $\Rightarrow$ 4: Obvious.

4 $\Rightarrow$ 1: Let $a \in A_{sa}$. Then $papap \in pA^p$. Let $(e_n)_{n \in D}$ be an approximate identity of her$(1 - p)$. Then $pa(1 - e_n)ap \to papap$. By Dini’s theorem for continuous functions on $F(p)$, this convergence is uniform. Thus $\|pa(1 - e_n - p)ap\| \to 0$, $\|(1 - e_n - p)^{1/2}ap\| \to 0$, and $\|(1 - e_n - p)ap\| \to 0$. It follows that $(1 - p)ap \in A^p$, since $A^p$ is closed by an argument similar to [7, 4.4]. If $(1 - p)ap = xp$ for $x$ in $A$, then $pap = 0$. Therefore $x \in L + R$, where $L = \{b \in A : bp = 0\}$ and $R = L^* = \{b \in A : pb = 0\}$, (proof of [7, 4.4]).

3 $\Rightarrow$ 2 $\Rightarrow$ 4 is obvious from previous remarks and is also essentially included in the proof of [7, 4.5].

That 5 implies 5 is obvious from previous remarks and the fact ([4, Theorem III.3]) that 2 is true when $p = 1$.

Theorem 3.2. Let $A$ be a $\sigma$-unital $C^*$-algebra, $p$ a closed projection in $A^{**}$, and let $B = SQC(p)$. If $B$ is non-degenerately embedded in $pA^{**}p$, then $M(B)$ is naturally isomorphic to $QC(p)$.

Remarks. 1. When $B = pA^p$ (i.e., when the conditions of 3.1 hold), this result was partly proved in [7, 4.5].

2. It follows from 3.2 that if $SQC(p)$ is non-degenerate in $pA^{**}p$ and if $p$ does not satisfy (MSQC), then $p$ does not satisfy (MQC). This is so because $M(B) \subset B''$.

Proof. Let $A^{**}$ be represented on $H$ via the universal representation of $A$. The non-degeneracy hypothesis means that $B$ is non-degenerately represented on $pH$. Therefore $M(B)$ is isomorphic to the idealizer of $B$ in $B(pH)$. It follows that if $F$ is a closed subset of $\mathbb{R}$ and $h$ is in $M(B)_{sa}$ then there is a hereditary $C^*$-subalgebra $B_0$ of $B$ such that any approximate identity of $B_0$ converges to $p - E_F(h)$, where
the spectral projection is computed in $B(pH)$. Let $\overline{B} = \{ a \in A : ap = pa \}$, and let $\overline{B_0}$ be the inverse image of $B_0$ in $\overline{B}$. If $q$ is the limit in $B(H)$ of an approximate identity of $\overline{B_0}$, then $q$ is an open projection in $A^{**}$, $qp = pq$, and $qp = p - E_F(h)$. Thus $E_F(h)$ is $p \land (1 - q)$, a closed projection in $A^{**}$, and $h$ is in $QC(p)$.

Conversely, if $x \in QC(p)$ and $b \in B$, then $x = p\overline{x}$ and $b = p\overline{b}$ where $\overline{x} \in M(A)$, $\overline{xp} = p\overline{x}$, and $\overline{b} \in \overline{B}$. Then $xb = p\overline{x}\overline{b} \in B$ and $bx = p\overline{b}\overline{x} \in B$. Thus $x \in M(B)$.

Remark. The σ-unitality was used only in the second part of the proof.

**Theorem 3.3.** If $A$ in 3.1 is σ-unital, then the following conditions are equivalent to 1-5 of 3.1:

6. $pAp \subseteq QC(p)$.

7. $pM(A)p = QC(p)$.

**Proof.** That $2 \Rightarrow 6$ is obvious.

$6 \Rightarrow 3$: Let $x$ be in $pAp$ and let $a$ be in $A$. Write $x = p\overline{x}$ where $\overline{x} \in M(A)$ and $\overline{xp} = p\overline{x}$. Then $xap = p\overline{x}ap = p^2\overline{xp}p \in pAp$.

That $7$ implies $6$ is obvious.

$2 \Rightarrow 7$: Clearly we have the non-degeneracy required for 3.2. Let $x$ be in $pAp$ and let $y$ be in $M(A)$. Write $x = p\overline{x}$ where $\overline{x} \in A$ and $p\overline{x} = \overline{xp}$. Then $xyp = p(\overline{yx})p \in pAp$, and $pyxp = p(y\overline{x})p \in pAp$. Thus, in the notation of 3.2, $pyp \in M(B)$, and hence $pyp \in QC(p)$.

**Example 3.4.** In this example $p$ is closed, infinite rank, abelian, and atomic, and $pA^*p$ is norm separable. Also $SQC(p) = \{0\}$ but $p$ satisfies $(MQC)$. In particular, $p$ is a counterexample for the question raised in Section 0 about isolated points.

In fact, if $p_0$ is a minimal projection, $p_0 \leq p$, and $p - p_0$ is closed, then obviously $p_0 \in SQC(p)$.

Let $A = C([0,1]) \otimes K$. Here $K$ is the algebra of compact operators on a separable infinite dimensional Hilbert space $H$, $\{e_1,e_2,\ldots\}$ is an orthonormal basis of $H$, and $P_n$ is the projection on span$\{e_1,\ldots,e_n\}$. A criterion for weak semicontinuity from [11, §5.G] will be used to describe closed projections in $A^{**}$. A closed projection is given by a projection-valued function $P : [0,1] \rightarrow B(H)$ such that if $h$ is any weak cluster point of $P(y)$ as $y \rightarrow x$, then $h \leq P(x)$. More precisely, $P$ describes the atomic part of a closed projection $p$, and $P$ determines $p$ since a closed projection is determined by its atomic part. (In our case $p$ will equal its atomic part.) We will construct a countable subset $S$ of $[0,1]$ and unit vectors $v(x)$ for each $x$ in $S$. For $x$ in $S$, $P(x)$ is the rank one projection on $\mathbb{C}v(x)$, and for $x$ not in $S$, $P(x) = 0$.

The following trivial lemma is stated for purposes of reference:

**3.4.1.** Let $\{x_i\}$ be a sequence of distinct points in $[0,1]$ and let $D$ be a countable subset of $[0,1]$. Then there are distinct points $y_{ij}$ in $[0,1] \setminus (\{x_i\} \cup D)$ such that $|y_{ij} - x_i| \leq \frac{1}{2^{(i+j)}}$.

We will take $S = \bigcup_0^\infty S_n$, a disjoint union, where $S_n$ and $v|S_n$ will be constructed recursively so that $\|P_nv(x)\| \leq n^{-\frac{1}{2}}$ for $x$ in $S_n$.

Step 0: Take $S_0 = \{\frac{1}{2}\}$, $v(\frac{1}{2}) = e_1$. 


Step 1: Take \( S_1 = \{ x_i \} \) where the \( x_i \)'s are distinct, \( x_i \neq \frac{1}{2} \), and \( x_i \to \frac{1}{2} \) as \( i \to \infty \). Let \( v(x_i) = 2^{-\frac{i}{2}}e_1 + 2^{-\frac{i}{2}}e_{i+1} \) for \( i = 1, 2, \ldots \).

\[ : \]

Step \( n \) \( (n > 1, \text{ step } n-1 \text{ already completed}) \): Write \( S_{n-1} = \{ x_1, x_2, \ldots \} \).

Choose \( y_{ij} \)'s as in 3.4.1 with \( D = \bigcup_0^{n-2} S_k \). Let \( S_n = \{ y_{ij} : i, j = 1, 2, \ldots \} \)

and \( v(y_{ij}) = n^{-\frac{1}{2}}v(x_i) + (1-n^{-1})^\nu w_{ij} \), where \( w_{ij} \) is a unit vector such that \( (w_{ij}, v(x_i)) = 0 \) and \( P_{i+j+n}w_{ij} = 0 \).

The first step in the proof is to show that we get a closed projection. Thus we may assume, after passing to a subsequence, that

\[ \forall \epsilon > 0 \quad \exists n_0 \quad \text{such that} \quad \| v(x_i) - \epsilon \| < \epsilon \quad \forall i > n_0. \]

We must show \( h \leq P(t) \). We have no difficulty if \( P(t_r) = 0 \). Thus we may assume, after passing to a subsequence, that \( t_r \in S_n(r) \). If \( n(r) \to \infty \), then since

\[ \| P_n(t_r)P(t_r)P_n(r) \| \leq n(r)^{-1}, \]

we must have \( h = 0 \). Thus, after again passing to a subsequence, we may assume \( n(r) = n, \forall r \). Now it is easy to see by induction that \( \bigcup_0^{n} S_k \) is closed. In fact, every cluster point of \( S_n \) is in \( S_{n-1} = \bigcup_0^{n-1} S_k \). The proof that \( h \leq P(t) \) will be left to the reader in the cases \( n = 0, n = 1 \). If \( n > 1 \), write \( t_r = y_{i(r), j(r)} \) in the notation of step \( n \). If \( i(r) + j(r) \to \infty \), we may assume, after passing to a subsequence, that \( t_r = t, \forall r \), a trivial case. If \( i(r) + j(r) \to \infty \), then \( t_r \in S_m \) for some \( m < n \). We use induction on \( n - m \). First suppose \( i(r) \to \infty \).

Then, passing to a subsequence, we assume \( i(r) = r, \forall r \). Then \( t = x_i \), and the construction shows that \( h = n^{-1}P(x_i) \). If \( i(r) \to \infty \), let \( t_r = x_{i(r)} \). Then \( t_r \to t \), and \( [v(t_r) - n^{-1}v(t')] \to h' \). This shows that \( h = n^{-1}h' \), where \( P(t_r) \to h' \).

Since \( h' \leq P(t) \) by induction, we conclude that \( h \leq P(t) \), as desired.

Now since \( A \) is separable, every state in \( F(p) \) is the resultant of a probability measure on \( F(p) \cap P(A) \). Since \( F(p) \cap P(A) \) is countable, the integral is a Bochner integral and thus the resultant is an atomic state. This shows that \( p \) is atomic, as claimed. Also, \( pA^*p \) is norm separable, being isometrically isomorphic to \( \ell^1(S) \).

That \( p \) is abelian, in other words \( pA^{**}p \) is abelian, is now obvious (cf [10]).

Now if \( h \) is in \( pA^{**}p \), \( h \) is determined by a function \( \lambda \) in \( \ell^\infty(S) \), where \( h(x) = \lambda(x)P(x), \ x \in S \). If \( h \) is in \( SQC(p) \), then \( h = p\tilde{h} \), where \( \tilde{h} \in A \) and \( p\tilde{h} = \tilde{h}p \).

In particular, \( \tilde{h}(\cdot) \) is a norm continuous function from \( [0,1] \) to \( K \). If \( x \in S \), there is a sequence \( (x_n) \) in \( S \) such that \( x_n \to x \) and \( P(x_n) \to tP(x) \), where \( 0 < t < 1 \). Since

\[ P(\cdot)\tilde{h}(\cdot)P(\cdot) = \lambda(\cdot)P(\cdot), \]

we conclude that \( \lambda(x_n) \to t\lambda(x) \). Since also \( h^2 \in \overline{SQC(p)} \), we also have \( \lambda(x_n)^2 \to t\lambda(x)^2 \). This implies \( \lambda(x) = 0 \). Since \( x \) is arbitrary, \( h = 0 \).

(The only property of \( h \) actually used is that \( h, h^2 \in pAp \).)

Finally we note that any continuous function on \( [0,1] \) gives rise to an element \( \tilde{h} \) of the center of \( M(A) \). Thus \( p\tilde{h} \in QC(p) \). It is easy to see that such elements of \( QC(p) \) generate \( pA^{**}p \) as a \( W^* \)-algebra, and hence \( p \) satisfies \( (MQC) \).

**Example 3.5.** By modifying the previous example, we can obtain either of the following:

(a) a compact projection \( \hat{p} \) such that \( QC(\hat{p}) = \mathbb{C}\hat{p} \)

(b) a closed projection \( p_1 \) such that \( SQC(p_1) = \{ 0 \} \) and \( QC(p_1) = \mathbb{C}p_1 \).

In both cases we will still have \( p \) infinite rank, abelian, and atomic and \( pA^*p \)
norm separable, and of course \( A \) will still be separable.

(a) Let \( A \) and \( p \) be as in 3.4, and consider \( \tilde{A} \) and \( \tilde{p} = p + p_\infty \). \( \tilde{A}^{**} \) is identified with \( A^{**} \oplus \mathbb{C} \) and \( p_\infty \) has its usual meaning, so that \( p_\infty = 0 \oplus 1 \) in \( A^{**} \oplus \mathbb{C} \) and \( \tilde{p} = p \oplus 1 \). Then \( \tilde{p} \) is closed, and hence compact, in \( \tilde{A}^{**} \). Suppose \( x = \lambda \tilde{A} + a, \lambda \in \mathbb{C}, a \in A, \) and \( x\tilde{p} = \tilde{p}x \). Then \( ap = pa \), and hence by 3.4, \( ap = 0 \). It follows easily that \( \tilde{p}x = \lambda \tilde{p} \). Therefore \( QC(\tilde{p}) = \mathbb{C}\tilde{p} \).

(b) We will use a \( C^* \)-algebra \( A_1 \) which is a maximal hereditary \( C^* \)-subalgebra of \( \tilde{A} \). Let \( p_0 \) be the minimal projection in \( \tilde{A}^{**} \) (actually in \( A^{**} \)) corresponding to the projection \( P(1/2) \) in the notation of 3.4 (\( p_0 \) corresponds to the pure state \( \varphi_0 \) where \( \varphi_0(a) = (a(1/2)e_1, e_1) \)). Then \( p_0 \leq p \leq \tilde{p} \). Let \( A_1 = \text{her}(1 - p_0) \) and \( p_1 = \tilde{p} - p_0 \) in \( A_1^{**} \). Then \( A_1^{**} \) is identified with \( (1 - p_0)\tilde{A}^{**}(1 - p_0) \). It is easy to see that \( p_1 \) is closed in \( A_1^{**} \): The complement projection to \( p_1 \) in \( A_1^{**} \) is \( 1 - p_0 - p_1 = 1 - \tilde{p} \), and this supports a hereditary \( C^* \)-subalgebra of \( \tilde{A} \) which happens to be contained in \( A_1 \) also. If \( x \in A_1 \) and \( x\tilde{p}_1 = p_1x \), then \( x \) is also in \( \tilde{A} \) and \( x\tilde{p} = \tilde{p}x \). Thus by (a), \( \tilde{p}x = \lambda \tilde{p} \) and hence \( p_1x = \lambda p_1 \). But \( x \in A_1 \) implies \( x\tilde{p}_0 = p_0x = 0 \). Since \( \tilde{p}x = \lambda \tilde{p} \) implies \( p_0x = \lambda p_0, \lambda = 0 \). Therefore \( SQC(p_1) = \{0\} \).

Now \( A_1 \) can be regarded as the set of all norm continuous functions \( f : [0, 1] \to \tilde{K} \) such that \( f(1/2)P(1/2) = P(1/2)f(1/2) = 0 \) and the image of \( f \) in \( \tilde{K}/K \) is constant. Since \( 1/2 \) is not an isolated point of \( [0, 1] \), \( M(A_1) \) can be regarded as a set of functions \( g : [0, 1] \setminus \{1/2\} \to \tilde{K} \) (cf. [7, Theorem 3.3] and note that \( \tilde{K} \) is unital). The requirements on \( g \) are:

(i) \( g \) is norm continuous and bounded.
(ii) \( \lim_{t \to 1/2} (1 - P(1/2))g(t)(1 - P(1/2)) \) exists in norm.
(iii) \( \lim_{t \to 1/2} \|P(1/2)g(t)(1 - P(1/2))\| = \lim_{t \to 1/2} \|g(t)(1 - P(1/2))P(1/2)\| = 0 \).
(iv) If we write \( g(t) = \lambda(t)1_{\tilde{K}} + x(t), \lambda(t) \in \mathbb{C}, x(t) \in K \), then \( \lambda(\cdot) \) is a constant.

To see these, the main thing to note is that the constant function \( 1 - P(1/2) \) is in \( A_1 \).

Now assume \( g \), as above, commutes with \( p_1 \). Then \( x(t) \) commutes with \( P(t) \) for all \( t \) in \( S \setminus \{1/2\} \), in the notation of 3.4. Just as in 3.4, this implies \( P(t)x(t) = 0 \) for \( t \) in \( S \setminus \{1/2\} \); i.e., \( x\tilde{p}_0 = 0 \) and \( p_1G = \lambda p_1 \). Thus \( QC(p_1) = \mathbb{C}p_1 \).

**Example 3.6.** Here we show, by a simpler example, how badly Theorem 3.2 can fail when the non-degeneracy hypothesis is eliminated. By [7, Theorem 2.7], if \( B \) is a non-unital separable \( C^* \)-algebra, then \( M(B) \) is non-separable. Thus if \( A \) is separable and \( SQC(p) \) is non-unital (in particular non-trivial), and if the conclusion of 3.2 is true, then \( QC(p) \) is much larger than \( SQC(p) \). In this example, \( SQC(p) \) is (infinite dimensional and) non-unital and \( QC(p) = SQC(p) + \mathbb{C}p \).

Let \( A = c \otimes K \). Thus \( A^{**} \) can be identified with the set of bounded collections \( \{h_n : 1 \leq n \leq \infty \}, h_n \in B(H) \). Let \( v_n = 2^{-1/2}e_1 + 2^{-1/2}e_{n+1}, n < \infty, v_\infty = e_1 \), let \( p_n \) be the projection with range \( \mathbb{C}v_n \), and let \( p = \{p_n\} \) in \( A^{**} \). Then \( p \) is closed since \( p_n \overset{w}{\to} \frac{1}{2}p_\infty \), and clearly \( p \) is abelian. Any element of \( pA^{**}p \) is given by \( h_n = \lambda_n p_n, 1 \leq n \leq \infty, \{\lambda_n\} \) bounded. An easy argument, which is part of 3.4, shows that if
$h \in SQC(p)$ then $\lambda_\infty = 0$ and $\lambda_n \to 0$ as $n \to \infty$. Conversely, any such $h$ is in $SQC(p)$; in fact $h \in A \cap pA^{**}$. Thus $SQC(p) \cong c_0$. Next we show that $h \in QC(p)$ implies $\lambda_n \to \lambda_\infty$. If this is false for $h = h^*$, then there is a closed subset $F$ of $\mathbb{R}$ such that $\lambda_\infty \notin F$ and $\lambda_n \in F$ for infinitely many $n$. If $q = E_F(h)$, then $q_\infty = 0$ and $q_n = p_n$ for infinitely many $n$. Since $p_n \to \frac{1}{2}p_\infty \neq 0$, $q$ is not closed and $h$ is not $q$-continuous on $p$. Thus $QC(p) \cong c$ and $QC(p)/SQC(p)$ is one dimensional.

4. Closed faces with (NCEB).

If $\hat{A}$ is the spectrum of $A$ and $p$ is a projection in $A^{**}$, we will denote by $X$ the set of all $[\pi]$ in $\hat{A}$ such that $\pi^{**}(p) \neq 0$. For $[\pi]$ in $X$ let $p[\pi]$ be the atomic projection in $A^{**}$ corresponding to $\pi^{**}(p)$. Thus $z_{at}p = \sum_{x \in X} p_x$. If $p$ is closed, or even universally measurable, then $p$ is determined by the $p_x$'s. If $\varphi$ and $\psi$ are in $(0, \infty)P(A)$, we will say that $\varphi$ and $\psi$ are equivalent, and write $\varphi \sim \psi$, if the pure states $\frac{\varphi}{\|\varphi\|}$ and $\frac{\psi}{\|\psi\|}$ are equivalent.

The proof of the next theorem and some of the other geometric arguments in this paper were inspired by Glimm [16].

**Theorem 4.1.** If $p$ is a projection in $A^{**}$ and if $p$ satisfies (NCEB), then $p_x$ is finite rank, $\forall x \in X$.

**Proof.** Let $\pi$ be an irreducible representation belonging to $x$, and let $H_\pi$ be the range of $\pi^{**}(p)$. If the conclusion is false, there is an infinite orthonormal sequence, $\{e_1, e_2, \ldots\}$, in $H_\pi$. Choose $t > 0$ such that $[P(A) \cap F(p)]^- \subset [t, 1]P(A) \cup \{0\}$ and choose $s$ such that $0 < s < t$. Let $v_n = s^{1/2}e_i + (1 - s)^{1/2}e_n$, $n > 2$, where $i$ is 1 or 2. Define $\varphi_n, \psi_n$ in $P(A) \cap F(p)$ by $\varphi_n(a) = (\pi(a)v_n, v_n)$, $\psi_n(a) = (\pi(a)e_n, e_n)$. Let $\theta$ be any cluster point of $(\psi_n)$ in $Q(A)$. Since $e_n \xrightarrow{w} 0$, $(\pi(a)e_n, e_n) \to 0$, $\forall a \in A$. Therefore $s\psi_i + (1 - s)\theta$ is a cluster point of $(\varphi_n)$. If $\theta = 0$, we have a contradiction to (NCEB), since $0 < s < t$. Therefore $\theta \in [t, 1]P(A)$. Since we must also have $s\psi_i + (1 - s)\theta \in [t, 1]P(A)$, it follows that $\theta = r_1\psi_i$ for some $r_1 \geq t > 0$. We have shown that $\theta = r_1\psi_1$ and $\theta = r_2\psi_2$, a contradiction.

For the rest of this section we assume that $p$ is closed and satisfies (NCEB). Let $\tilde{X} = [P(A) \cap F(p)]^- \setminus \{0\}$. Then $\tilde{X} \subset F(p) \cap [t, 1]P(A)$ and $\tilde{X}$ is locally compact, since $\tilde{X} \cup \{0\}$ is closed. We identify $X$ with the set of equivalence classes in $\tilde{X}$ via $f : \tilde{X} \to X$, where $f(\varphi) = [\pi_\varphi]$. Give $X$ the quotient topology arising from $f$.

**Lemma 4.2.** $f$ is a closed map.

**Proof.** The main point is to show the following: If $(\varphi_i)_{i \in D}$ and $(\psi_i)_{i \in D}$ are nets in $\tilde{X}$ such that $\varphi_i \sim \psi_i$, $\varphi_i \to \varphi$, and $\psi_i \to \psi$, then either $\varphi = \psi = 0$ or $\varphi, \psi \in \tilde{X}$ and $\varphi \sim \psi$. Assume this is false and consider first the case $\varphi = 0$, $\psi \in \tilde{X}$. Let $\pi$ be the reduced atomic representation of $A$, $H = H_\pi$, and choose vectors $u_i, v_i$ in $\pi^{**}(p)H$ of norm at most 1 such that $\varphi_i = (\pi(\cdot)u_i, v_i)$, $\psi_i = (\pi(\cdot)v_i, v_i)$. If $g_i(a) = (\pi(a)u_i, v_i)$, then $|g_i(a)| \leq \|\pi(a)u_i\| = \varphi_i(a^*a)^{1/2} \to 0$. Therefore $g_i \to 0$. Now choose $r_i$ in $\mathbb{R}$ such that $\|w_i\| = 1$, where $w_i = r_iu_i + \left(\frac{1}{2}\right)^{1/2}v_i$. Since
\[ \|u_i\|^2 \geq t, \{r_i\} \text{ is bounded. Let } \theta_i = (\pi(\cdot)w_i,w_i). \text{ It follows from the above that } \theta_i \in F(p) \cap P(A) \text{ and } \theta_i \to \frac{1}{t} \psi. \text{ Since } 0 < \|\frac{1}{t} \psi\| < t, \text{ this contradicts (NCEB)}. \]

Next assume \( \varphi, \psi \in \tilde{X} \) and \( \varphi \not\sim \psi \). Then there are invariant subspaces \( H_1 \) and \( H_2 \) of \( H \), corresponding to inequivalent irreducible representations, and non-zero vectors \( u \in H_1, v \in H_2 \) such that \( \varphi = (\pi(\cdot)u,u) \) and \( \psi = (\pi(\cdot)v,v) \). Let \( u_i, v_i, \) and \( g_i \) be as above with the extra condition that \( Re(u_i,v_i) \geq 0 \). Passing to a subnet, we may assume \( g_i \to g, g \in A^* \). Since \( |g_i(a)| \leq \varphi_i(a^*a)^{1/2}, \forall a \in A \), then \( |g(a)| \leq \varphi(a^*a)^{1/2} \). From the Hahn-Banach and Riesz-Fisher theorems we see that \( g = (\pi(\cdot)u,u') \) for some \( u' \in H \). Clearly, we may assume \( u' \in H_1 \). Similarly, \( |g_i(a)| \leq \psi_i(aa^*)^{1/2} \), and hence \( |g(a)| \leq \psi(aa^*)^{1/2} \). Therefore \( g = (\pi(\cdot)v',v) \) for some \( v' \in H_2 \). It follows that \( g = 0 \) ([20]). Now choose \( r_i \in \mathbb{R}^+ \) such that \( \|w_i\| = 1 \), where \( w_i = r_i(u_i + v_i) \). Since \( 2t \leq \|u_i + v_i\|^2 \leq 4 \), \( \{r_i\} \) is bounded and bounded away from 0. If \( \theta_i = (\pi(\cdot)w_i,w_i) \), then \( \theta_i \in F(p) \cap P(A) \) and every cluster point of \( (\theta_i) \) is of the form \( r^2(\varphi + \psi) \) for some cluster point \( r \) of \( (r_i) \). Since this last functional is not a multiple of a pure state, this contradicts (NCEB).

To complete the proof of the lemma, we have to show that the saturation of a closed set is closed. Suppose \( Y \) is a closed subset of \( \tilde{X} \), \( \varphi_i \in f^{-1}(f(Y)) \), and \( \varphi_i \to \varphi \) in \( \tilde{X} \). Choose \( \psi_i \) in \( Y \) such that \( \varphi_i \sim \psi_i \). Passing to a subnet if necessary, we may assume \( \psi_i \to \psi \). By what has already been proved \( \psi \in \tilde{X} \) and \( \psi \sim \varphi \). Since \( Y \) is closed, \( \psi \in Y \) and hence \( \varphi \in f^{-1}(f(Y)) \). Thus \( f^{-1}(f(Y)) \) is closed (relative to \( \tilde{X} \)).

**Theorem 4.3.** \( X \) is a locally compact Hausdorff space and \( f \) is a proper map from \( \tilde{X} \) to \( X \).

*Proof.* The fibers of \( f \), i.e., the sets \( f^{-1}(\{x\}), x \in X \), are compact (even norm compact) by 4.1. This, 4.2, and the fact that \( \tilde{X} \) is locally compact Hausdorff imply that \( X \) is locally compact Hausdorff, by standard point set topology. Any closed map with compact fibers is proper; i.e., the inverse image of a compact set is compact.

**Remarks.** 1. It follows from 4.3, or it could be deduced directly from the proof of 4.2, that the saturation of a compact subset of \( \tilde{X} \) is compact.

2. The topology of \( X \) is stronger than, and in general unequal to, the relative topology that \( X \) inherits from the usual hull-kernel topology of \( \hat{A} \). In fact, using [5] and 1.4, we can easily construct a closed projection satisfying (NCEB) and even (CEB) such that \( X \) is a countably infinite discrete space and the image of \( X \) in \( \text{prin} A \) consists of one point. Thus the relative topology is trivial on \( X \).

**Lemma 4.4.** Assume \( p \) is an atomic closed projection satisfying (NCEB) and that \( pA^*p \) is norm separable. Then for every closed subset \( S \) of \( X \), \( \sum_{x \in S} p_x \) is a closed projection.

*Proof.* Since \( pA^*p \) has a linear subspace isometric to \( \ell^1(X) \), \( X \) must be countable. Let \( p_S = \sum_{x \in S} p_x \). Then every element of \( F(p_S) \) is the resultant of a probability measure supported by \( [F(p_S) \cap P(A)] \cup \{0\} \), and \( a \) fortiori supported by \( f^{-1}(S) \cup \{0\} \). Since \( f^{-1}(S) \cup \{0\} \) is compact, every element of \( F(p_S^-) \) is the resultant of a probability measure on \( f^{-1}(S) \cup \{0\} \). Since \( f^{-1}(S) \) is the disjoint union of
countably many fibers of \( f \), since each of these fibers is contained in \( F(p_x) \) for some \( x \) in \( S \), and since each \( p_x \) is finite rank and hence closed, it is easy to see that any such resultant is in \( F(p_S) \). Thus \( F(p_S) \) is closed and hence \( p_S \) is closed.

**Corollary 4.5.** Under the same assumptions, if \( p \neq 0 \), there is a minimal projection \( p_0 \) such that \( p_0 \leq p \) and \( p - p_0 \) is closed. Also for every non-zero closed subprojection \( p' \) of \( p \), there is a minimal projection \( p_0 \) such that \( p_0 \leq p' \) and \( p' - p_0 \) is closed.

**Proof.** Since \( X \) is countable and locally compact Hausdorff, the Baire category theorem implies that \( X \) has an isolated point \( x_0 \). Let \( p_0 \) be any minimal subprojection of \( p_{x_0} \). Then \( p - p_0 = (p_{x_0} - p_0) + p_{X \setminus \{x_0\}} \), the sum of two orthogonal closed projections. Therefore \( p - p_0 \) is closed ([1, Theorem II.7]).

**Remarks.** 1. In Section 6 we will generalize 4.4 and 4.5 by dropping the requirement that \( pA^*p \) be norm separable, but we will add the assumption that \( A \) is separable. We are not sure what technical assumptions are really needed.

2. Corollary 4.5 and Examples 3.4 and 3.5(a) constitute our results on the “isolated point” question raised in Section 0. The second sentence of 4.5 is closely analogous to the definition of a scattered topological space and less closely analogous to the definition of scattered \( C^* \)-algebras. Obviously we have not found a necessary and sufficient condition for this to hold. Example 3.4 shows that we cannot replace (NCEB) by the weaker condition \( F(p) \cap P(A)^- \subset [0,1]P(A) \), and 3.5(a) shows we cannot weaken (NCEB) to \( F(p) \cap P(A)^- \subset \{0\} \cup [t,1]S(A) \). Any closed face of a scattered \( C^* \)-algebra satisfies the conclusion of 4.5 but not necessarily the hypothesis. Example 5.12 below, whose primary purpose is something else, is a closed face satisfying the conclusion of 4.5 (the proof of this is in 7.9), but not (NCEB), and which is not isomorphic to a closed face of any scattered \( C^* \)-algebra.

We now consider the geometry of \( F(p) \) in more detail.

**Theorem 4.6.** If \( p \) is a closed projection satisfying (NCEB) and if \( (x_i)_{i \in D} \) is a net in \( X \) converging to \( x \), then there is a subnet \( (x_j)_{j \in I} \) such that one of the following holds:

1. We have rank \( p_{x_j} = k \leq n = \text{rank } p_x, \forall j \); and there are orthonormal bases \( \{e_1^j, \ldots, e_k^j\} \) of range \( \pi_j^*(p_{x_j}) \) and \( \{e_1, \ldots, e_n\} \) of range \( \pi^*(p_x) \) and an \( n \times k \) matrix \( T \) such that \( tI_k \leq T^*T \leq I_k \) and \( \forall z \in C^k, \varphi_j(z) = \varphi(w) \), where \( \pi_j \) and \( \pi \) are irreducible representations belonging to \( x_j \) and \( x \), \( v_j = \sum_1^k z_m e_m^j, v = \sum_1^n w_m e_m, w = Tz, \varphi_j(z) = (\pi_j(\cdot)v_j, v_j) \), and \( \varphi(w) = (\pi(\cdot)v, v) \).

2. There is \( \varphi \) in \( P(A) \cap F(p_x) \) such that every cluster point of \( (\varphi_j) \) is a multiple of \( \varphi \), \( \forall \varphi_j \in P(A) \cap F(p_{x_j}) \).

**Proof.** If rank \( p_{x_i} \rightarrow \infty \), we first pick a subnet such that rank \( p_{x_j} = k, \forall j \). If rank \( p_{x_i} \rightarrow \infty \), we must show there is a subnet satisfying 2; and we do this by contradiction. Thus assume there are a subnet \( (x_j) \) and pure states \( \theta_j, \psi_j \) in \( F(p_{x_j}) \) such that \( (\theta_j) \) and \( (\psi_j) \) converge to non-proportional elements of \( F(p_x) \). In the first case choose an arbitrary orthonormal basis \( \{e_1^j, \ldots, e_k^j\} \) of range \( \pi_j^*(p_{x_j}) \). In the second case let \( k = n + 1 \) and choose an orthonormal set \( \{e_1^j, \ldots, e_k^j\} \) in range
\[ \pi^*(p_{x_j}) \] such that \( \theta_j = (\pi_j(\cdot)v_j, v_j) \) and \( \psi_j = (\pi_j(\cdot)v'_j, v'_j) \) with \( v_j, v'_j \) unit vectors in span \( \{e_1^j, \ldots, e_k^j\} \).

In both cases define \( f^j_{\ell m} \) in \( A^* \) by \( f^j_{\ell m} = (\pi_j(\cdot)e^j_m, e^j_\ell) \), \( 1 \leq \ell, m \leq k \). Passing to a subnet, we may assume \( f^j_{\ell m} \to f_{\ell m}, \forall \ell, m \). Since the matrix \([f^j_{\ell m}]\) represents a positive linear functional on \( A \otimes M_k \), the same must be true of the matrix \([f_{\ell m}]\). The GNS representation of a state, the vectors \( \psi_f \) for some representation \( \tilde{\pi} \) of \( A \), and \([f_{\ell m}]\) must be the vector state induced by a vector \((u_1, \ldots, u_k)\) in \( H_{\tilde{\pi}} \oplus \cdots \oplus H_{\tilde{\pi}} \). In other words, \( f_{\ell m} = (\tilde{\pi}(\cdot)u_m, u_\ell) \). Since \( f_{\ell m} \in [t,1][P(A) \cap F(p_x)], \tilde{\pi} \cong \pi \oplus \cdots \oplus \pi \). Thus we may write \( u_\ell = (u_{\ell 1}, \ldots, u_{\ell r}), r \leq k, \) where \( u_{\ell p} \in \text{range} \, \pi^{**}(p_{x_j}) \).

Now \( f_{\ell m} = \sum_1^r(\pi(\cdot)u_{\ell p}, u_{\ell p}) \). Since \( f_{\ell m} \in [t,1][P(A)], \) there must be a non-zero vector \( y_\ell \) in range \( \pi^{**}(p_{x_j}) \) such that \( u_{\ell p} = \lambda_{\ell p} y_\ell \) with \( (\lambda_{\ell p}) \) a non-zero element of \( \mathbb{C}^r \). If \( z \in \mathbb{C}^k \) and \( \varphi_j(z) \) is as above, then \( \varphi_j(z) = \sum \tilde{z}_{\ell m} f^j_{\ell m} z_m \) and hence \( \varphi_j(z) \to \sum \tilde{z}_{\ell f_{\ell m}} z_m = (\tilde{\pi}(\cdot) \sum z_{\ell} u_\ell, \sum z_{\ell} u_\ell) \). Since this functional is a multiple of a pure state, the vectors \( \sum z_{\ell} u_\ell, 1 \leq p \leq r \), must be proportional. Suppose, for example, that \( y_1 \) and \( y_2 \) are linearly independent. Then the choice \( z = (1,1,0, \ldots, 0) \) shows that \( (\lambda_{1.1}) \) and \( (\lambda_{1.2}) \) are proportional. For \( \ell > 2, y_\ell \) cannot be a multiple of both \( y_1 \) and \( y_2 \). Therefore all \( (\lambda_{\ell p}) \) are proportional. Changing notation, we may write \( u_{\ell p} = \lambda_{\ell p} y_\ell \). Then \( \varphi_j(z) \to (\sum_1^r |\lambda_p|^2)(\pi(\cdot) \sum z_{\ell} y_\ell, \sum z_{\ell} y_\ell) \). Now choose any orthonormal basis of range \( \pi^{**}(p_{x_j}) \) and let \( T \) be the matrix of \( z \to (\sum_1^r |\lambda_p|^2)^{1/2} \sum z_{\ell} y_\ell \). Since \( t \|z\|_2^2 \leq \|\lim \varphi_j(z)\| \leq \|z\|_2^2 \), we must have \( tI_k \leq T^*T \leq I_k \). This implies \( k \leq n \) so that 1 is proved. The other alternative is that span \{\( y_\ell \)\} is one dimensional. Then \( \varphi' = (\pi(\cdot)y_1, y_1) \) and \( \varphi = \frac{\varphi'}{\|\varphi\|} \). Since each \( f_{\ell m} \) is proportional to \( \varphi \), \( (\varphi_j(z)) \) converges to a multiple of \( \varphi \), \( \forall z \in \mathbb{C}^k \), and more generally every cluster point of \( (\varphi_j(z_j)) \) is a multiple of \( \varphi \) for any bounded net \( (z_j) \) in \( \mathbb{C}^k \). If \( k = \text{rank} \, p_{x_j} \), this proves 2. In the original second case, rank \( p_{x_j} \to \infty, k = n + 1 \), this proves the contradiction that establishes 2.

We say that a C*-algebra \( A \) satisfies (CEB) or (NCEB) if the closed projection 1 in \( A^{**} \) satisfies (CEB) or (NCEB). In [17, §5] Glimm proved a necessary and sufficient condition for \( A \) to satisfy a property weaker than (NCEB), \( \overline{P(A)} \subset [0,1]P(A) \). His condition is:

(i) \( A \) is CCR,

(ii) \( \widehat{A} \) is Hausdorff, and

(iii) \( \{\pi\} \in \widehat{A} \) and \( \text{dim} \, \pi > 1 \) implies \( \pi \) is regular.

Given (i) and (ii), (iii) can be restated as follows: If \( I \) is the ideal of \( A \) such that \( \widehat{I} = \{[\pi] \in \widehat{A} : \text{dim} \, \pi > 1 \} \), then \( I \) is a continuous trace C*-algebra. (See [27] for the theory of continuous trace C*-algebras.) It is presumably an easy exercise to derive a characterization of C*-algebras satisfying (CEB) or (NCEB) (they are equivalent for C*-algebras) from Glimm’s result. In Corollary 4.7 below we derive such a characterization instead from 4.1-4.6. The purpose is not to put this result on the record, so long after [17]. The purpose is as follows: The class of closed faces of C*-algebras admits more varied behavior than the class of C*-algebras. One illustration of this is the contrast between the facts on the isolated point question
for atomic closed faces of $C^*$-algebras and the facts on scattered $C^*$-algebras ([18], [19]). Another illustration is the contrast between 4.6 and 4.7. (We will show by example that all of the behavior contemplated by 4.6 really occurs.) The exercise of deriving 4.7 from 4.1 to 4.6 gives some insight into why the behavior of closed faces is more varied than that of $C^*$-algebras.

If $A$ is a CCR $C^*$-algebra with Hausdorff spectrum, then $A$ is isomorphic to the set of continuous sections vanishing at $\infty$ of elementary $C^*$-algebras. If $x_0 \in \hat{A}$ and $A(x_0)$ is one dimensional, then there is a continuous section $e(\cdot)$ such that $e(x_0) = 1_A(x_0)$ and $e(x)$ is a projection for $x$ in some neighborhood of $x_0$ ([17]). We will say $A$ is locally unital at $x_0$ if $e(x) = 1_A(x)$ in some neighborhood of $x_0$.

**Corollary 4.7.** The following are equivalent for a $C^*$-algebra $A$:

1. $A$ satisfies (CEB)
2. $A$ satisfies (NCEB)
3. (i) Every irreducible representation of $A$ is finite dimensional, (ii) $\hat{A}$ is Hausdorff,
   (iii) $\forall n > 1, \{[\pi] : \dim \pi = n\}$ is an open subset of $\hat{A}$, and
   (iv) $A$ is locally unital at each $[\pi]$ with $\dim \pi = 1$.

**Remark.** Condition 3(iii) says that the ideal $I$ discussed above is the $c_0$ direct sum of $n$-homogeneous $C^*$-algebras for various values of $n$. Thus the comparison of 3 with Glimm’s condition is clear.

**Proof.** $2 \Rightarrow 3$: (i) follows from 4.1 with $p = 1$. Since $p = 1$, $X = \hat{A}$. Since the map from $P(A)$ to $\hat{A}$ is open for the hull-kernel topology ([17]), the hull-kernel topology is the quotient topology; i.e., our topology on $X$ agrees with the usual one when $p = 1$. Thus (ii) follows from 4.3. Again since the map from $P(A)$ to $\hat{A}$ is open, if $\dim \pi > 1$ and $[\pi_i] \to [\pi]$, then after passing to a subnet, we can find $\varphi_i, \psi_i$ in $P(A)$ associated to $\pi_i$ such that the nets $(\varphi_i)$, $(\psi_i)$ converge to distinct pure states associated to $\pi$. Thus alternative 2 of 4.6 cannot hold, and $\limsup(\dim \pi_i) \leq \dim \pi$. It is always true in a $C^*$-algebra that $\liminf(\dim \pi_i) \geq \dim \pi$ (but for a closed face we can have $\liminf(\text{rank } p_{\pi_i}) < \text{rank } p_{\pi}$). This shows (iii). If $x_0$, $e(\cdot)$ are as above and $A$ is not locally unital at $x_0$, then we can find $(x_i)$ such that $x_i \to x_0$ and $e(x_i) \neq 1$, $\forall i$. Then we can find $\varphi_i$ in $P(A)$ associated to $x_i$ such that $\varphi_i(e) = \frac{1}{2}$. It follows that $\|\varphi\| = \frac{1}{2}$ for any cluster point $\varphi$ of $(\varphi_i)$, in contradiction to (NCEB). This proves (iv).

That 1 implies 2 is obvious, and the proof that 3 implies 1 is left to the reader.

**Examples 4.8.** (a) We can illustrate alternative 1 of 4.6 with $A = c \otimes K$. Choose $k$ and $n$ with $k \leq n$, $t > 0$, and an $n \times k$ matrix $T$ such that $tI_k \leq T^*T \leq I_k$. Let $S = (1 - T^*T)^{1/2}$, a $k \times k$ matrix. Let $p_\infty$ be the projection on $\text{span}\{e_1, \ldots, e_n\}$ and for $j < \infty$ let $p_j$ be the range projection of $\begin{pmatrix} T \\ S \end{pmatrix}$, where the matrix is regarded as a linear isometry from $\mathbb{C}^k$ to $\text{span}\{e_1, \ldots, e_n, e_{n+j}, \ldots, e_{n+j+k-1}\}$. If $p = \{p_j : 1 \leq j \leq \infty\}$, then $p$ is a closed projection in $A^{**}$, $p$ satisfies (NCEB) ((CEB) if
5. Type I Closed Faces and Atomic Closed Faces

Methods can easily be used to construct a subnet and a positive \( \varphi \) such that \( \text{tI} \) von Neumann algebra. Clearly Remark. In 4.6.2 we showed only that every cluster point of \( (\varphi) \) sequences can satisfy 4.6.1, with different choices of \( F \) and \( A \) orthonormal basis \( \{\varphi_j\} \) corresponds to the columns of \( (T_S) \).

If we want a more complicated example, say one where two different subsequences give two different matrices, we can easily modify the above. Choose \( k' \leq n \) and an \( n \times k' \) matrix \( T' \) such that \( TI_k \leq T^*T' \leq I_k' \). Let \( \tilde{p}_{2j-1} \) be the above \( p_j \), and let \( \tilde{p}_{2j} \) be the above \( p_j \) constructed from \( T' \) instead of \( T \).

(b) As a first example for alternative 2 of 4.6, consider \( A_1 = \{(a_n)_{1}^{\infty} : a_n \in \mathcal{K} \text{ and } (a_n) \text{ converges to a scalar in norm}\} \). Then \( A_1^* \) can be identified with the set of bounded collections \( \{h_n : 1 \leq n \leq \infty\} \) such that \( h_n \in B(H) \oplus \mathbb{C} \) for \( n < \infty \) and \( h_\infty \in \mathbb{C} \). Choose any sequence \( (n_j) \) of positive integers and define a closed projection \( p \) in \( A_1^* \) by: \( p = \{p_j\}, p_\infty = 1_\mathcal{K}, \) and \( p_j \) is a rank \( n_j \) projection in \( B(H) \) for \( j < \infty \). It is easy to see that \( p \) satisfies (CEB) and 4.6.2. This easy example shows that there is no restriction on rank \( p_{x_j} \), when 4.6.2 holds, but this is all that it shows.

(c) For more complicated examples, in particular examples where some subsequences satisfy 4.6.1 and others 4.6.2, we can use \( A_2 = A_1 \otimes \mathcal{K} \). Then \( A_2^* \cong A_1^* \otimes B(H) \).

The construction in (a) above can also be used for \( A_2 \). Let \( \tilde{p}_\infty = 1 \otimes p_\infty \) and \( \tilde{p}_j = q_0 \otimes p_j \) for \( j < \infty \), where the \( p_j \)'s are as in (a) and \( q_0 \) is a rank 1 projection in the \( B(H) \)-component of \( \mathcal{K}^{**} \). It is easy to see that \( \tilde{p} \) is closed in \( A_2^* \) and that \( F(\tilde{p}) \) is isomorphic to the closed face \( F(p) \) of \( (c \otimes \mathcal{K})^{**} \).

We can also construct examples of 4.6.2 using \( A_2 \). Let \( T \) be a positive \( k \times k \) matrix such that \( TI_k \leq T^2 \leq I_k \), and let \( u \) be a unit vector in \( \text{span}\{e_1, \ldots, e_n\} \) where \( k \) and \( n \) are arbitrary. Let \( S = (1 - T^2) \frac{1}{2} \) and define a closed projection \( \tilde{p} \) in \( A_2^* \) by: \( \tilde{p} = \{\tilde{p}_j : 1 \leq j \leq \infty\}, \tilde{p}_\infty = 1 \otimes p_\infty \) for \( p_\infty \) the projection on \( \text{span}\{e_1, \ldots, e_n\} \), and \( \tilde{p}_j \) is the range projection of \( (T_S) \) where now \( (T_S) \) sends \( \mathbb{C}^k \) to \( \text{span}\{e_1 \otimes u, e_2 \otimes u, \ldots, e_k \otimes u, e_1 \otimes e_{n+j}, \ldots, e_1 \otimes e_{n+j+k-1}\} \).

Then 4.6.2 holds with \( \varphi \) given by \( \varphi(a) = (a_\infty u, u) \). Also the columns of \( (T_S) \) give an orthonormal basis \( \{e^j_1, \ldots, e^j_k\} \) of range \( \tilde{p}_j \), and, using the notation of 4.6.1, \( \varphi_j(z) \to \|Tz\|\varphi \). It is easy to see that \( \tilde{p} \) satisfies (NCEB).

By using the idea of the second paragraph of (a), we can construct a closed projection such that different subsequences exhibit different behavior. Some subsequences can satisfy 4.6.1, with different choices of \( T \) and \( k \), and some can satisfy 4.6.2 with \( \varphi_j(z) \to \|Tz\|\varphi \) for different choices of \( T, k, \) and \( \varphi \).

Remark. In 4.6.2 we showed only that every cluster point of \( (\varphi_j) \) is a multiple of \( \varphi \) and did not describe which multiples arise. When rank \( p_{x_j} \) is bounded, the same methods can easily be used to construct a subnet and a positive \( k \times k \) matrix \( T \) such that \( TI_k \leq T^2 \leq I_k \) and \( \varphi_j(z) \to \|Tz\|\varphi \).

5. Type I Closed Faces and Atomic Closed Faces

If \( p \) is a projection in \( A^{**} \), we say that \( p \) or \( F(p) \) is type I if \( pA^{**}p \) is a type I von Neumann algebra. Clearly \( p \) is type I if and only if \( c(p) \), the central cover of
Lemma 5.1. Let \( A \) be a separable \( C^* \)-algebra and \( p \) a type I closed projection in \( A^{**} \). Let \( \mu \) be a probability measure on \( F(p) \), and let \( \pi = \int \omega d\mu(\omega) \), the direct integral. Then \( \pi \) is a type I representation.

Proof. Let \( \varphi = \int \omega d\mu(\omega) \), the resultant of \( \mu \). Then \( \varphi \in F(p) \), since \( F(p) \) is closed. Therefore \( \pi_\varphi \) is type I, and \( \pi_\varphi \) is a subrepresentation of \( \pi \). We claim \( \pi \) and \( \pi_\varphi \) have the same central support in \( A^{**} \) (i.e. \( \pi \) is quasi-equivalent to \( \pi_\varphi \)). Therefore \( \pi \) is also type I.

To see the claimed quasi-equivalence, let \( \nu_\omega \) be the cyclic vector in \( H_\omega \) produced by the GNS construction, and let \( \nu = \int \omega d\mu(\omega) \), a vector in \( H_\pi \). Then \( (\pi(a)\nu, \nu) = \varphi(a) \). For every \( \mu \)-measurable subset \( S \) of \( F(p) \) (\( \mu \) is a Borel measure, and “\( \mu \)-measurable” means measurable with respect to the completion of \( \mu \)) there is a projection \( P_S \) in \( \pi(A)' \) such that the corresponding subrepresentation of \( \pi \) is \( \int_S \pi_\omega d\mu(\omega) \). It is easy to see that \( H_\pi \) is the smallest closed invariant subspace containing \( P_S \nu \) for all such \( S \). Moreover the cyclic subrepresentation of \( \pi \) generated by \( P_S \nu \) is equivalent to a subrepresentation of \( \pi_\varphi \). These remarks complete the proof.

The main fact needed from direct integral theory is something that the author learned from G. W. Mackey and is expressed as a lemma. For the ideas in the proof see [23], pages 112-117, and [24], especially page 159. The basic point is that the direct integral decomposition into irreducibles of a type I representation is almost unique.

Lemma 5.2 (Mackey). Let \( A \) be a separable \( C^* \)-algebra, let \( \pi' \) and \( \pi'' \) be measurable fields of irreducible representations of \( A \) defined over standard measure spaces \( S' \) and \( S'' \), and let \( \pi' = \int_{S'} \pi'_s d\mu'(s) \), \( \pi'' = \int_{S''} \pi''_s d\mu''(s) \). Assume that \( \pi'_s \) is inequivalent to \( \pi''_s \), \( \forall s' \in S', \forall s'' \in S'' \) and that \( \pi' \) and \( \pi'' \) are type I representations. Then \( \pi' \) and \( \pi'' \) are disjoint (i.e., their central supports in \( A^{**} \) are orthogonal).

Lemma 5.3. Let \( A \) be a separable \( C^* \)-algebra and \( p \) a type I closed projection in \( A^{**} \). Assume \( \mu \) and \( \nu \) are positive finite measures on \( F(p) \cap P(A) \) such that \( \int \omega d\mu(\omega) = \int \omega d\nu(\omega) \). Let \( E \) be a saturated Borel subset (or, more generally, a saturated \( (\mu + \nu) \)-measurable subset) of \( F(p) \cap P(A) \). Then \( \int_E \omega d\mu(\omega) = \int_E \omega d\nu(\omega) \) and in particular \( \mu(E) = \nu(E) \).

Proof. Let \( \varphi = \int \omega d\mu(\omega) = \int \omega d\nu(\omega) \), \( \varphi' = \int \omega d\mu(\omega) \), and \( \varphi'' = \int \omega d\nu(\omega) \). As in the proof of 5.1, there are vectors \( \nu' \) in \( H_\pi' \) and \( \nu'' \) in \( H_\pi'' \) which induce the functional \( \varphi \). Thus \( \nu'' \) is a partial isometry. Let \( P' \) and \( P'' \) be the projections in \( \pi'(A)' \) and \( \pi''(A)' \) defined from \( E \). Thus \( \mu(E) = (P'\nu', \nu') \) and \( \nu(E) = (P''\nu'', \nu'') \). By 5.2 and 5.1, \( (1-P'')(U-P') = P'E(1-
$P'_E = 0$. Therefore $P'_E v'$ is in the initial space of $U$ and $UP'_E v' = P''_E v''$. The conclusion follows.

**Lemma 5.4.** Let $A$ be a separable $C^*-$algebra, $p$ a type I closed projection in $A^{**}$, and $E$ a saturated Borel subset of $F(p) \cap P(A)$. Then there is a projection $p_E$ in $A^{**}$ such that $p_E \leq p$ and $F(p_E)$ is the set of resultants of probability measures on $E \cup \{0\}$.

**Proof.** Let $F_1$ be the set of resultants of probability measures on $E \cup \{0\}$. We claim that $F_1$ is a norm closed sub-face of $F(p)$. The result then follows from [15, Theorem 4.4 and p. 396] (cf. [25, 3.6.11]).

To see the claim, note that by Choquet theory every element of $F(p)$ is the resultant of a probability measure on $[F(p) \cap P(A)] \cup \{0\}$. Let $E' = [F(p) \cap P(A)] \setminus E$. Then $F_1 = \{ \int \omega d\mu(\omega) : \mu(E') = 0 \}$. Suppose $\varphi_i = \int \omega d\mu_i(\omega)$, $i = 1, 2$, and $t\varphi_1 + (1-t)\varphi_2 \in F_1$, $0 < t < 1$. By 5.3, $t\mu_1(E') + (1-t)\mu_2(E') = 0$. Therefore $\mu_1(E') = \mu_2(E') = 0$, and $\varphi_1, \varphi_2 \in F_1$. Thus $F_1$ is a face.

To see that $F_1$ is norm closed, assume $\varphi = \int \omega d\mu(\omega)$ where $\mu(E') = \delta > 0$. We claim that dist$(\varphi, F_1) \geq \delta$. Suppose $\psi = \int \omega d\nu(\omega)$ where $\nu(E') = 0$ and $\|\varphi - \psi\| = r$. Then $\varphi - \psi = \lambda_1 - \lambda_2$ where $\lambda_1, \lambda_2 \geq 0$ and $\|\lambda_1\| + \|\lambda_2\| = r$. If $\lambda_i = \int \omega d\mu_i(\omega)$, for positive measures $\mu_1$, $\mu_2$ on $F(p) \cap P(A)$, then $\mu + \mu_2$ and $\nu + \mu_1$ have the same resultant. Therefore by 5.3, $\mu(E') + \mu_2(E') = \nu(E') + \mu_1(E')$. Therefore $\mu(E') \leq \mu_1(E') \leq r$. This proves the claim and completes the proof of the lemma.

**Remarks.** Although the conclusion of 5.4 has what we need, more is true. Also $F(p_E) \cap S(A)$ is a split face of $F(p) \cap S(A)$, the complement being $F(p_{E'}) \cap S(A)$. This means that $p_E$ and $p_{E'}$ are centrally disjoint projections and $p_E + p_{E'} = p$. Also $p_E$ satisfies the barycenter formula. (The barycenter formula is discussed below before 5.13). A related statement is that $F(p_E)$ is closed under resultants. The hypotheses of 5.4 could be weakened. We could assume that $p$ satisfies the barycenter formula instead of that $p$ is closed, and we could assume $E$ universally measurable instead of Borel.

**Lemma 5.5.** Let $A$ be a separable $C^*-$algebra and $p$ a closed projection in $A^{**}$. If $\pi^{**}(p)$ has finite rank for every irreducible representation $\pi$ of $A$, then $p$ is type I.

**Proof.** Let $\pi = \int^{\oplus} \pi_s d\mu(s)$ be a standard direct integral, where each $\pi_s$ is irreducible. Since $p$ is closed, $\pi^{**}(p) = \int^{\oplus} \pi^{**}_{s}(p)d\mu(s)$, where $\pi^{**}_{s}(p)$ is a Borel operator field. Therefore rank $(\pi^{**}(p))$ is a Borel function, and by hypothesis it is everywhere finite-valued.

From the above it follows that any representation of $A$ in a separable Hilbert space can be written as a direct sum, $\pi = \bigoplus_{n=0}^{\infty} \pi_n$, such that $\pi_n = \int^{\oplus} \pi_s d\mu(s)$ and rank $(\pi^{**}_{s}(p)) = n, \forall s \in S_n$. It was shown by A. Amitsur and J. Levitzki in [9] that there is a non-commutative polynomial $G_n$ of $2n$ variables such that $G_n$ vanishes on $M_{2n}^{2n}$ but not on $M_{n+1}^{2n}$ (cf. [21, Lemma 2], where a weaker but adequate result is proved). Also if $G_n$ vanishes on $M^{2n}$ for a $W^*-$algebra $M$, then $M$ is a direct
sum of type $I_k$ algebras for $k \leq n$. Clearly $G_n$ vanishes on $[\pi_n^{**}(p)\pi_n(A)\pi_n^{**}(p)]^{2n}$, $n > 0$, and hence by strong continuity $G_n$ vanishes on $[\pi_n^{**}(pA^{**}p)]^{2n}$. Therefore $\pi_n^{**}(pA^{**}p)$ is type I, $\forall n$. (For $n = 0$, $\pi_0^{**}(p) = 0$). If $z_n$ is the central support of $\pi_n$ in $A^{**}$, and $z(\pi)$ is the central support of $\pi$, then $z(\pi) = \sup_n z_n$. Since we have shown that $z_n pA^{**}p$ is type I, $\forall n$, then $z(\pi)pA^{**}p$ is type I. Since $\sup\{z(\pi): \pi$ as above$\} = 1$, $pA^{**}p$ is type I.

**Corollary 5.6.** If $A$ is a separable $C^*$-algebra, $p$ is a closed projection in $A^{**}$, and if $p$ satisfies (NCEB), then $p$ is type I.

**Proof.** Combine 4.1 and 5.5.

We have already defined the concept of an atomic projection in $A^{**}$. We say that $p$ is strongly atomic if $p$ is atomic and $pA^*p$ is norm separable. If $A$ is separable the separability of $pA^*p$ can be rephrased: There are only countably many points $[\pi]$ in $A$ such that $\pi^{**}(p) \neq 0$.

**Question 5.7.** If $A$ is separable, is every closed atomic projection in $A^{**}$ strongly atomic?

If $p$ is closed and atomic and if $\mu$ is a probability measure on $F(p) \cap P(A)$, then $\int \omega d\mu(\omega)$ is in $F(p)$ and hence is an atomic state. If $A$ is separable, it follows from 5.3, for example, that $\mu$ is supported by the union of countably many equivalence classes. If $p$ is not strongly atomic, this means that there are uncountably many equivalence classes in $F(p) \cap P(A)$ but every finite measure is concentrated on the union of countably many. It follows that the relation of equivalence of pure states is not countably separated on $F(p) \cap P(A)$. (If it were countably separated, the quotient Borel space would be an uncountable analytic Borel space ([24, Theorem 5.1]) and hence would support a continuous measure. This measure could be lifted to $F(p) \cap P(A)$ by the von Neumann selection lemma.) In particular $A$ is not type I. Also $p$ does not satisfy (NCEB), since the space $X$ of Section 4 is second countable and hence countably separated when $A$ is separable. This reasoning suggests the following:

**Question 5.8.** If $A$ is a separable $C^*$-algebra and $p$ is a type I closed projection in $A^{**}$, is equivalence of pure states countably separated on $F(p) \cap P(A)$?

Obviously 5.8 is analogous to Mackey’s conjecture ([23, p. 85] or [24, p. 163]), which was proved by Glimm in [17]. Of course [17] proved much more than Mackey’s conjecture. We do not know whether there is a structure theorem for type I closed faces of similar power to Glimm’s theorem. Because the variety of closed faces of $C^*$-algebras is so great, there is not enough evidence to support a conjecture on any of these questions.

If the answer to 5.8 is yes for a particular $p$, then a standard form for elements of $F(p) \cap S(A)$ can be established. Let $X$ be the set of equivalence classes of $F(p) \cap P(A)$, an analytic Borel space which is in one-one correspondence with a subset $\{[\pi_x]: x \in X\}$ of $\hat{A}$. Then an element $\varphi$ of $F(p) \cap S(A)$ is determined by a probability measure $\mu$ on $X$ and a measurable function $f: X \to S(A)$ such that $f(x)$ is supported by $\pi_x^{**}(p)$. In fact $\varphi$ is the resultant of a probability measure
\( \mathcal{P} \) on \( F(p) \cap P(A) \). Even though \( \mathcal{P} \) is not unique, 5.3 implies its pushforward to \( X \) is unique. The function \( f \) is obtained by writing \( \mathcal{P} = \int_X \nu_x d\mu(x) \), where \( \nu_x \) is supported on the equivalence class \( x \), and \( f(x) = \int \omega d\nu_x(\omega) \). It can be shown that \( f \) is unique modulo null sets. Thus, under the hypotheses given, the Choquet decomposition of \( \varphi \) is almost unique in a sense roughly analogous to Mackey’s result that the direct integral decomposition of a type I representation into irreducibles is almost unique.

There is a converse question to 5.7 which we can answer. The proof is valid even for \( A \) nonseparable.

**Proposition 5.9.** If \( A \) is a \( C^* \)-algebra and \( p \) is a closed projection in \( A^{**} \) such that \( z_{at} p A^* p \) is norm separable, then \( p \) is atomic and hence strongly atomic.

**Proof.** There is an increasing sequence \( (p_n) \) of finite rank projections such that \( p_n \to z_{at} p \). By 4.5.12 or 4.5.15 of [25], \( z_{at} p \) is universally measurable. Since \((1 - z_{at}) p\) is a universally measurable operator whose atomic part is 0, \((1 - z_{at}) p = 0 \) ([25, 4.3.15]).

The following lemma, or the ideas in its proof, might be useful in connection with questions 5.7, 5.8. It will also be used to prove a complement to Glimm’s theorem.

**Lemma 5.10.** If \( p \) is an atomic projection in \( A^{**} \) such that \( p A^* p \) is norm separable, then \( F(p) \cap P(A) \) is an \( F_\sigma \) set relative to \( P(A) \).

**Proof.** The lemma can be rephrased more concretely: Let \( \pi : A \to B(H) \) be an irreducible representation, let \( H_0 \) be a separable closed subspace of \( H \), and let \( P_0 = \{(\pi(\cdot)v, v) : v \) is a unit vector in \( H_0 \} \). Then \( P_0 \) is an \( F_\sigma \) set relative to \( P(A) \).

The proof is similar to that of 4.1. Let \( H_1, H_2, \ldots \) be an increasing sequence of finite dimensional subspaces such that \( H_n = (\cup_1^n H_n)^\perp \), and let \( p_n \) be the projection on \( H_n \). Let \( V_n = \{v \in H_0 : \|v\| = 1 \text{ and } \|p_n v\| \geq \frac{1}{2}\} \) and \( P_n = \{(\pi(\cdot)v, v) : v \in V_n\} \). Then \( P_0 = \cup_1^\infty P_n \), and we will show \( P_n \) closed relative to \( P(A) \). Suppose \( v_i \in V_n \), \( \varphi_i = (\pi(\cdot)v_i, v_i) \), and the net \((\varphi_i)\) converges to a pure state \( \varphi \). Passing to a subnet if necessary, we may assume \( v_i \to v \) for some \( v \) in \( H_0 \). Clearly \( \|v\| \leq 1 \) and \( \|p_n v\| \geq \frac{1}{2} \). Then \( v_i = u_i + w_i \), where \( u_i \to v \) in norm, \( w_i \to 0 \), and \( (u_i, w_i) = 0 \). Therefore \( (\pi(a)u_i, w_i) \to 0, \forall a \in A \). Passing to a further subnet, we may assume \((\pi(\cdot)w_i, v_i)\) converges to some \( \psi \) in \( Q(A) \). Then \( \varphi = (\pi(\cdot)v, v) + \psi \). Since \( \varphi \) is pure, \( \psi \) must be proportional to \((\pi(\cdot)v, v) \). Therefore \( \varphi = (\pi(\cdot)v_1, v_1) \) where \( v_1 = v/\|v\| \). Since \( \|p_n v_1\| \geq \|p_n v\|, \varphi \in P_n \).

**Proposition 5.11.** If \( A \) is a separable \( C^* \)-algebra and \( \pi : A \to B(H) \) is an irreducible representation such that \( \pi(A) \not\subseteq K(H) \), then there are uncountably many inequivalent irreducible representations of \( A \) with the same kernel as \( \pi \).

**Remark.** Glimm’s theorem implies that there are uncountably many irreducibles with the same kernel, but so far as we know, it was not previously known that that kernel can be taken to be the same as the kernel of the given \( \pi \).

**Proof.** By replacing \( A \) with its quotient by the kernel of \( \pi \), we may reduce to the case \( \pi \) faithful. Assume that \( A \) has only countably many faithful irreducible
representations. Since \( \hat{A} \) is second countable, there is a countable set \( \{I_n\} \) of non-zero (closed, two-sided) ideals such that every non-faithful representation of \( A \) vanishes on some \( I_n \). Then since \( \pi \) is a dense point in \( \hat{A} \), the hull of \( I_n \) has empty interior in \( \hat{A} \). Let \( F_n = \{ \varphi \in P(A) : \varphi|I_n = 0 \} \). Since the map from \( P(A) \) to \( \hat{A} \) is open, we have that \( F_n \) is a closed nowhere dense set relative to \( P(A) \). It now follows from the Baire category theorem, applied to \( P(A) \), and 5.10 that there is a faithful irreducible representation \( \pi' \) whose associated pure states have non-empty interior in \( P(A) \). From the openness of the map from \( P(A) \) to \( \hat{A} \), we conclude that \( \hat{A} \) has an open point, whence \( A \) has an ideal \( K \), necessarily essential, such that \( \hat{K} \) has only one point. The proof is concluded by showing \( K \cong \mathcal{K}(H) \), and this can be done in at least two ways. There is a simple way to prove that every separable \( C^* \)-algebra whose spectrum is a single point must be elementary (i.e., the affirmative answer to Naimark’s question in the separable case), or one can apply Glimm’s theorem to \( K \).

The following example demolishes one naive conjecture with regard to the structure of type I closed faces.

**Example 5.12.** If \( A \) is any non-type I separable \( C^* \)-algebra, then \( A \) has a type I closed face \( F(p) \) (\( p \) is even compact) such that \( F(p) \) is not isomorphic to a closed face of any type I \( C^* \)-algebra.

If \( A \) is not unital, we consider \( A^{**} \) as a subalgebra of \( \hat{A}^{**} \) and construct \( p \) as a projection in \( A^{**} \) closed in \( \hat{A}^{**} \), so that \( p \) will be compact. Since \( A \) is not type I, there is an irreducible representation \( \pi \) such that \( \pi(A) \not\subset \mathcal{K}(H\pi) \). For the natural extension of \( \pi \) to \( \hat{A} \), also denoted \( \pi \), we also have \( \pi(\hat{A}) \not\subset \mathcal{K}(H\pi) \). Let \( v_0 \) be a unit vector in \( H\pi \), \( \varphi_0 = (\pi(\cdot)v_0, v_0) \) and \( p_0 \) the support projection in \( A^{**} \) of \( \varphi_0 \).

By a result of Glimm [16, Theorem 2], there is a sequence \( \{v_n\} \) of unit vectors in \( H\pi \) such that \( v_n \xrightarrow{w} 0 \) and \( (\pi(\cdot)v_n, v_n) \to \varphi_0 \) in \( \hat{A}^* \). By using the Gram-Schmidt process, we can find a subsequence \( \{v_{n_i}\} \) and an orthonormal sequence \( \{w_{n_i}\} \) such that \( (w_{n_i}, v_0) = 0 \) and \( \|w_{n_i} - v_{n_i}\| \to 0 \). Let \( \varphi_{i} = (\pi(\cdot)w_{n_i}, w_{n_i}) \).

Since \( p_0 \) is a minimal projection in \( A^{**} \), it is closed in \( \hat{A}^{**} \). Let \( B \) be the hereditary \( C^* \)-subalgebra of \( \hat{A} \) supported by \( 1 - p_0 \), and let \( e \) be a strictly positive element of \( B \). Since \( \varphi_i \to \varphi_0 \) in \( \hat{A}^* \) and \( \varphi_0|B = 0 \), \( \varphi_i(e) \to 0 \). Passing to a subsequence, we may assume \( \sum \varphi_i(e) < \infty \). Let \( p_i \) be the support projection of \( \varphi_i \). \( p_i \) is in \( B^{**} \cap A^{**} \), considered as a subalgebra of \( \hat{A}^{**} \). By 0.1(ii), \( \sum p_i \) is closed in \( B^{**} \). Thus if \( p = \sum p_i \), \( p \) is closed in \( \hat{A}^{**} \). Since \( p \in A^{**} \), \( p \) is a compact projection in \( A^{**} \). Since \( pA^{**}p \cong B(H_0) \) where \( H_0 = \text{span}\{v_0, w_{n_1}, w_{n_2}, \ldots\} \), \( p \) is a type I projection.

Suppose \( F(p) \) were isomorphic to a closed face, \( F(p') \), of a type I \( C^* \)-algebra \( A' \). Since \( p'(A')^{**}p' \) can be identified with the space of bounded affine functionals vanishing at 0 on \( F(p') \), \( p'(A')^{**}p' \) is Jordan *-isomorphic to \( pA^{**}p \). Therefore \( p'(A')^{**}p' \) is a type I factor, and \( p' \) is associated with a single irreducible representation, \( \pi' \), of \( A' \). Since \( A' \) is type I, \( \pi'(A') \subset \mathcal{K}(H_{\pi'}) \). Let \( \varphi_i' \), \( i \geq 0 \), be the element of \( F(p') \) corresponding to \( \varphi_i \). Then \( \varphi_i' \to \varphi_0' \) in \( A^{**} \). This contradicts the facts that \( \{\varphi_i'\} \) arises from an orthonormal sequence of vectors in \( H_{\pi'} \) and \( \pi'(A') \subset \mathcal{K}(H_{\pi'}) \).
We think it is fairly obvious from the proof of 0.1(ii) ([12, Lemma 3]), that the faces $F(p)$ constructed above are all isomorphic. In Section 7 we will determine the structure of $pAp$, and this will be our formal proof of this fact.

Finally, we want to generalize 5.5 for use in connection with a remark in Section 7. If $h \in A^{**}$, we say that $h$ satisfies the barycenter formula if, when regarded as a function on $Q(A)$, $h$ is measurable with respect to (the completion of) any regular Borel measure and $\varphi(h) = \int h(\omega)d\mu(\omega)$ whenever $\mu$ is a regular Borel measure on $Q(A)$ and $\varphi$ is the resultant of $\mu$. If $A$ is separable, it is sufficient to verify the formula for measures supported on $P(A)$. Also when $A$ is separable, the barycenter formula is equivalent to: $\pi^{**}(h)$ is a measurable field of operators and $\pi^{**}(h) = \int \pi^{**}(h)d\mu(s)$, whenever $\pi = \int \pi d\mu(s)$, a standard direct integral; and again it is sufficient to verify this in the special case where each $\pi_s$ is irreducible. Thus for $A$ separable the set of elements of $A^{**}$ satisfying the barycenter formula is a $C^*$–algebra closed under weak sequential convergence. This $C^*$–algebra is at least as large as $\{h : \text{Re } h, \text{Im } h \text{ are universally measurable}\}$ and appears to be a good thing to use, though the monotone sequential closure of $A$ (discussed in [25, §4.5]) would do for our purposes. For $A$ non-separable, we know of nothing more general than the space of universally measurable operators ([26]).

**Theorem 5.13.** If $A$ is a separable $C^*$–algebra, $p$ is a projection in $A^{**}$ satisfying the barycenter formula, and if $\pi^{**}(p)\pi(A)\pi^{**}(p) \subset K(H_\pi)$ for every irreducible representation $\pi$ of $A$, then $p$ is type I.

*Proof.* First note that the proof of 5.5 is equally valid if $p$ satisfies the barycenter formula instead of being closed. Let $e$ be a strictly positive element of $A$. Then for $\varepsilon > 0$, $E_{[\varepsilon, \infty)}(pep)$ satisfies the barycenter formula and $\pi^{**}(E_{[\varepsilon, \infty)}(pep))$ has finite rank for $\pi$ irreducible. Therefore each $p_n$ is type I where $p_n = E_{[n^{-1}, \infty)}(pep)$. Since $p_n \not
rightarrow p$, $p$ is type I.

### 6. More on Closed Faces with (NCEB) for $A$ Separable.

The notations $X, p_x, \tilde{X}$, and $f$ have the same meanings as in Section 4.

**Theorem 6.1.** If $A$ is a separable $C^*$–algebra, $p$ is a closed projection in $A^{**}$, and if $p$ satisfies (NCEB), then $\sum_{x \in S} p_x$ is the atomic part of a closed projection $p_S$ for every closed subset $S$ of $X$. Also $p$ satisfies (CEB) if and only if $p_S$ is compact for $S$ compact.

*Proof.* By 5.6, $p$ is type I. Let $\tilde{S} = f^{-1}(S)$, a closed subset of $\tilde{X}$, let $E = \tilde{S}\cap P(A) = \tilde{S} \cap S(A)$, a saturated subset of $F(p) \cap P(A)$, and let $p_S$ be the projection called $p_E$ in 5.4. By 5.4, $F(p_S) = \{\omega d\mu(\omega) : \mu \text{ is a probability measure on } E \cup \{0\} = \{\int \omega d\mu(\omega) : \mu \text{ is a probability measure on } \tilde{S} \cup \{0\}\}$. Since $\tilde{S} \cup \{0\}$ is a compact subset of $A^*$, $F(p_S)$ is closed, and hence $p_S$ is closed by [15, Theorem 4.8]. By 5.4, $F(p_S) \cap P(A) = E$, and this implies that the atomic part of $p_S$ is $\sum_{x \in S} p_x$.

If $p$ satisfies (CEB) and $S$ is compact, then $E = \tilde{S}$, and $\tilde{S}$ is compact by 4.3. Thus $F(p_S) \cap S(A) = \{\omega d\mu(\omega) : \mu \text{ is a probability measure on } \tilde{S}\}$, a closed subset of $A^*$. Therefore $p_S$ is compact. Conversely, if $S$ compact implies $p_S$ compact and
if \( \varphi_n \to t \varphi \), for \( \varphi_n, \varphi \) in \( F(p) \cap P(A) \), then there is a compact set \( S \) such that \( f(\varphi_n) \in S \) for \( n \) sufficiently large. Since \( F(p_S) \cap S(A) \) is closed, it follows that \( t = 1 \).

**Remarks.**

1. If \( p \) satisfies only (NCEB) and \( S \) is compact, then \( p_S \) is nearly relatively compact in the sense of [13].

2. The hypothesis of 4.4 included the assumption that \( p \) be strongly atomic, though this term was not used. Theorem 6.1 shows that this assumption can be dropped if \( A \) is separable. Also the discussion after 5.7 shows that if \( A \) is separable, \( p \) is closed and atomic, and \( p \) satisfies (NCEB), then \( p \) is strongly atomic. Thus for \( A \) separable, the hypothesis of 4.5 can be weakened by replacing strongly atomic with atomic.

3. In view of the remarks after 5.4, it is not hard to calculate the facial topology on the extreme boundary of \( F(p) \), when \( p \) is closed and satisfies (CEB). Its \( T_0 \)-ification is the compact Hausdorff space \( X \cup \{\infty\} \). If \( p \) satisfies only (NCEB), we still see that the closed split faces of \( F(p) \) containing \( 0 \) are in one-to-one correspondence with the closed subsets of \( X \).

### 7. Some Relationships among Prior Sections and Concluding Remarks.

Each of the three main parts of this paper (Sections 2, 3, and 4-6) studies a different generalization of the situation considered in [5] (1.4 and 7.2 below are used to justify this statement). Sections 3-6 were motivated by our desire to investigate the circumstances in which 2.4 applies, but the detailed discussion below makes it clear that we have not solved this problem – if it can be called a “problem”. There is a broader “problem” to which all three parts of this paper are relevant: Study the structure of those closed faces of \( C^\ast \)-algebras which are closely modeled on locally compact Hausdorff spaces. We now discuss the relationships among the prior sections.

First we consider the relationship between Sections 2 and 3. The next result and the remarks following show that if we were willing to use the theory of relative \( q \)-continuity in the construction of MASA’s, it would have been sufficient to prove the special case of 2.4 in which the projection \( p_X \) is central and abelian. However, so far as we know, this special case is no easier.

**Theorem 7.1.** Let \( A \) be a separable \( C^\ast \)-algebra and \( p \) a closed projection in \( A^{**} \). Suppose \( B \) is a commutative \( C^\ast \)-subalgebra of \( SQC(p) \) which is non-degenerately embedded in \( pA^{**}p \). If \( \hat{B} \) is totally disconnected, then there is a commutative \( C^\ast \)-subalgebra \( C \) of \( A \) such that \( C \) contains an approximate identity of \( A \), \( p \in C^{**} \), and \( pC = B \).

**Proof.** Let \( \overline{B} = \{ a \in A : ap = pa \text{ and } pa \in B \} \). Then \( \text{her}(1 - p) \) is an ideal of \( \overline{B} \) and \( \overline{B}/\text{her}(1 - p) \cong B \). We can apply 2.4 (or 2.3) with \( \overline{B} \) playing the role of \( A \) and with \( X = \hat{B} \). For \( x \in X \), \( p_x \) is the support projection in \( \overline{B}^{**} \) of the pure state of \( \overline{B} \) given by \( x \). Let \( C \) be the MASA of \( \overline{B} \) given by 2.4. Since \( B \) is non-degenerate in \( pA^{**}p \), \( \overline{B} \) hereditarily generates \( A \). Since \( C \) hereditarily generates \( \overline{B} \) by 2.4, \( C \) also hereditarily generates \( A \). One way to deduce from 2.4 that \( pC \), which is the
image of \( C \) in \( \overline{B}/\text{her}(1 - p) \), is all of \( B \) is to quote the classical Stone-Weierstrass theorem.

Suppose \( p \) is a closed projection in \( A^{**} \) such that \( SQC(p) \) is non-degenerate in \( pA^{**}p \) (cf. 3.2) and that \( B \) is a MASA in \( SQC(p) \) which hereditarily generates \( SQC(p) \). If \( A \) is separable and \( \hat{B} \) is totally disconnected, then 7.1 gives a commutative algebra \( C \) (which could be assumed a MASA in \( A \)). For each \( x \) in \( \hat{B} \) we have a pure state \( \varphi_x \) of \( B \) (or \( C \)) which is supported by a minimal projection \( p_x \) in \( B^{**} \), and it follows from \( pC = B \) and \( p \in C^{**} \) that also \( p_x \in C^{**} \). If \( p_x \) is minimal in \( A^{**} \), then \( \varphi_x \) satisfies (UEP) relative to the inclusion of \( C \) in \( A \). If \( pAp \) is an algebra (cf 3.1 and 7.2 below), then we need only start with a MASA \( B \) in \( pAp \) which hereditarily generates \( pAp \) and such that each pure state of \( B \) satisfies (UEP) relative to \( pAp \). It was pointed out in Section 0 that under the hypotheses of 2.4 every element of \( C_0(X) \) gives an element of \( SQC(p_X) \). It can be shown that \( C_0(X) \) is thus embedded as a MASA in \( SQC(p_X) \) and that \( C_0(X) \) is nondegenerate in \( p_X A^{**}p_X \). Thus the above discussion applies.

**Proposition 7.2.** Conditions (i)-(iv) of 1.2 imply that \( pAp \) is an algebra, and \( p \) satisfies:

\[
(G) \quad [P(A) \cap F(p)]^- \subset [0, 1]P(A).
\]

**Proof.** The reduced atomic representation, \( \pi \), of \( A \) is faithful on \( pA^{**}p \). Moreover, \( \pi^{**}(pA^{**}p) \cap K(H_\pi) \) is a \( C^* \)-algebra, and by 1.2(iv), \( \pi^{**}(pAp) \) is contained in this algebra. We show equality. Let \( h \) be an element of \( pA^{**}p \) such that \( \pi^{**}(h) \) is compact. It is sufficient to show \( h \in SQC(p) \). If \( F \) is a closed subset of \( \mathbb{R} \), then 1.2(i) implies that \( E_F(h) \), computed in \( pA^{**}p \), is closed. (In fact we don’t need \( F \) closed for this.) If \( 0 \notin F \) (and \( F \) is closed), then \( \pi^{**}(E_F(h)) \) is a finite rank operator on \( H_x \), and by [1, Corollary II.8] this implies \( E_F(h) \) is compact. Thus \( h \in SQC(p) \). Then (G) follows from [17, §5] for example.

The same reasoning shows \( pA^{**}p = QC(p) \).

Next we consider the relationship between Sections 2 and 4. By 6.1 if \( A \) is separable and \( p \) is a closed projection in \( A^{**} \) satisfying (CEB) then we have the hypotheses of 2.3 (except for total disconnectedness of \( X \)). By 4.1 each \( p_x \) is of finite rank. For 2.4, we would want each \( p_x \) to be of rank 1. This happens for the \( p_x \)'s of Section 4 if and only if \( p \) is abelian. If \( p \) is not abelian, it might be possible to write \( p_x = p_{x,1} + \cdots + p_{x,n_x} \) where the \( p_{x,i} \)'s are minimal and \( \{p_{x,i}\} \) satisfies the hypotheses of 2.4 with \( X \) replaced by some space \( \overline{X} \). Then \( \overline{X} \) would map onto \( X \) by a closed continuous map with finite fibers. However, Example 7.6(a) below shows that this is not always possible.

Conversely, suppose the hypotheses of 2.4 are satisfied. By 1.4, if \( X \) is countable and discrete and each equivalence class of \( \{\varphi_x : x \in X\} \) is finite, then \( p_X \) satisfies (CEB). If \( p_X \) is abelian, we can deduce (7.3 below) that \( p_X \) satisfies (CEB) even for \( X \) not discrete; but it is fairly obvious (cf 7.6(b) below) that in general \( p_X \) need not satisfy (NCEB) or even (G).
In retrospect it seems that (G) is worthy of more study in the present context despite the fact, as pointed out in the remark following 4.5, that it does not imply a positive answer to the isolated point question. One reason is mentioned in 7.2 above. However, the conclusion of 1.4 is definitely false if we drop the hypothesis of finite equivalence classes. (This follows from 4.1.) It may be that (G) is part of the hypothesis of a nice result. Also, even though, by Example 3.4, (G) does not imply that $F(p)$ is associated with a locally compact Hausdorff space, we do not know whether (G) implies that $F(p)$ is associated with a Hausdorff space. We will show below that (G) does imply that $p$ is type I.

One could also consider weaker conditions than (G):

1. $[P(A) \cap F(p)]^- \subset \{\text{type I factorial quasi-states}\}$

2. $[P(A) \cap F(p)]^- \subset z_{at}A^*$.

(2) is suggested by the theory of perfect $C^*$-algebras ([8]).

With regard to the relationship between Sections 3 and 4, we note that a closed projection $p$ satisfying (CEB) need not satisfy (MSQC) (cf. 7.6(c) below). However, it follows from 6.1 that $p$ does satisfy the hypothesis of 3.2 (A separable). Also if $p$ is closed and abelian and satisfies (CEB), then $p$ does satisfy (MSQC). (It follows from 5.3 that $F(p)$ is isomorphic to the set of probability measures on $\tilde{X} \cup \{0\}$. Is there a less technical proof?) It may be that there are other useful concepts on the extensiveness of $SQC(p)$.

At the end of this section we return to Example 5.12, partly to show that it does satisfy the conclusion of 4.5. A complete theory for closed faces of $C^*$-algebras analogous to the theory of scattered $C^*$-algebras might have to be quite complicated.

**Proposition 7.3.** Assume the hypotheses of 2.4 and also that $p_X$ is abelian. Then $p_X$ satisfies (CEB).

**Remark.** The hypothesis that $p_X$ is abelian can be rephrased more concretely: The $\varphi_x$’s are mutually inequivalent. ([10]).

**Proof.** Since $p_X$ is abelian, $P(A) \cap F(p_X) = \{\varphi_x : x \in X\}$. Suppose $\varphi_{x_i} \to \psi$ in $Q(A)$. Passing to a subnet, we may assume $x_i \to x$ in $X$ or $x_i \to \infty$. If $x_i \to x$, let $\{S_j\}$ be a set of compact neighborhoods of $x$ such that $\bigcap_j S_j = \{x\}$. Since $p_{S_j}$ is compact and $\varphi_{x_i} \in F(P_{S_j})$ for $i$ large, we conclude that $\|\psi\| = 1$ and $\psi \in \bigcap_j F(p_{S_j}) = F(\bigwedge_j p_{S_j})$. Since a closed projection is determined by its atomic part, $\bigwedge_j p_{S_j} = p_x$, and hence $\psi = \varphi_x$. If $x_i \to \infty$, let $\{U_j\}$ be a set of relatively compact open subsets of $X$ such that $\bigcup_j U_j = X$, and let $S_j = X \setminus U_j$. Since $x_i$ is eventually in $S_j$ and since $p_{S_j}$ is closed, we see that $\psi \in \bigcap_j F(P_{S_j}) = F(\bigwedge_j p_{S_j}) = \{0\}$.

It would be desirable if the hypotheses in 2.4 that certain projections are atomic parts of closed projections could be stated entirely in terms of pure states (or
equivalently, minimal projections). This can be done in a situation of intermediate generality. Consider the following conditions for a projection $p$ in $A^{**}$:

\begin{enumerate}
  \item[(3)] \exists t \in (0, 1] such that $[P(A) \cap F(p)]^- \subset \{0\} \cup [t, 1][P(A) \cap F(p)]$.
  \item[(4)] $\{0\} \cup [P(A) \cap F(p)]$ is closed.
  \item[(5)] $[P(A) \cap F(p)]$ is closed.
\end{enumerate}

Conditions (3) and (4) are strengthenings of (NCEB) and (CEB) respectively, and are equivalent to (NCEB) and (CEB) if $p$ is closed. But we are interested in the case where $p$ is atomic. If $p$ is the atomic part of a closed projection $q$, then $q$ satisfies (NCEB) or (CEB) if and only if $p$ satisfies (3) or (4). If $p$ is atomic and satisfies (3) or (4), is $p$ necessarily the atomic part of a closed projection? We can prove this if $p$ is strongly atomic, in which case $p$ itself is closed. (In general let $C$ be the closed convex hull of $\{0\} \cup [P(A) \cap F(p)]$. If $q$ exists, then $F(q) = C$. The tricky thing is to prove that $C$ is a face of $Q(A)$.)

**Lemma 7.4.** If $p$ is a strongly atomic projection satisfying (3), then $p$ is closed.

**Proof.** Let $X = \{0\} \cup [P(A) \cap F(p)]^-$ and let $C$ be the closed convex hull of $X$. Then every element of $C$ is the resultant of a probability measure on $X$. Since $X$ is norm separable, the resultant is actually a Bochner integral. Hence $C \subset F(p)$. The reverse inclusion follows easily from the structure of atomic von Neumann algebras.

**Remark.** The same argument works if (3) is replaced by a similar modification of (2).

**Corollary 7.5.** Let $X$ be a locally compact Hausdorff space with only countably many points, and let $\{p_x : x \in X\}$ be a family of mutually orthogonal minimal projections in $A^{**}$. If $\sum_{x \in S} p_x$ satisfies (3) whenever $S$ is a closed subset of $X$ and (5) when $S$ is compact, and if $A$ is separable, then we have the hypotheses and conclusions of Corollary 2.4.

**Remarks.** 1. Let $\varphi_x$ be the pure state supported by $p_x$. If $p = \sum_{x \in X} p_x$ is abelian, i.e. if the $\varphi_x$’s are mutually inequivalent, then the hypotheses on $\sum_{x \in S} p_x$ can be stated more concretely:

\begin{enumerate}
  \item[(6)] If $x_n \to x$, then $\varphi_{x_n} \to \varphi_x$, and if $x_n \to \infty$, then $\varphi_{x_n} \to 0$.
\end{enumerate}

We can actually replace the hypotheses on $\sum_{x \in S} p_x$ by (6) if we assume only that the equivalence classes have bounded finite cardinality. (Note that they have to be finite by 4.1 if $p$ satisfies (3).) The proof of this uses Akemann’s result in [1] that the supremum of finitely many mutually commuting closed projections is closed.

2. Even when $X$ is uncountable, the hypotheses on $\sum_{x \in S} p_x$ in 2.4 can be modified somewhat: If $\sum_{x \in X} p_x$ is the atomic part of a closed projection, and
if $\sum_{x \in S} p_x$ satisfies (3) for $S$ closed and (5) for $S$ compact, then we have the hypotheses of 2.4. We will not provide a complete proof of this because it would be rather technical and it is not clear that the result is a big improvement on 2.4. The main lemma is the following:

Let $p$ be a closed projection satisfying (NCEB) and $q$ a subprojection of $z_{st}p$. If $A$ is separable and $q$ satisfies (3), then $q$ is the atomic part of a closed projection.

The proof of this uses 5.6, the other results of Section 5 (in particular the discussion following 5.8), and the von Neumann selection lemma.

**Examples 7.6.** (a) Let $A = c \otimes K$ and define a closed projection $p$ in $A^{**}$ by letting $p_\infty$ be the projection on $\text{span}\{e_1, e_2\}$ and $p_n$ the projection on

$$\begin{cases}
\mathbb{C}e_1, & n = 3k + 1 \\
\mathbb{C}e_2, & n = 3k + 2 \\
\mathbb{C}(2^{-\frac{k}{2}}e_1 + 2^{-\frac{k}{2}}e_2), & n = 3k
\end{cases}$$

It is easy to see that $p$ satisfies (CEB). Let $\varphi_n$ be the pure state of $A$ supported by $p_n$, $n < \infty$, and suppose $B$ is a MASA of $A$ such that each $\varphi_n|_B$ satisfies (UEP). Thus each $p_n$ is in $B^{**}$. If $b \in B$, then $e_1$ is an eigenvector of each $b_{3k+1}$ and hence $e_1$ is an eigenvector of $b_\infty$. Similarly $e_2$ and $2^{-\frac{1}{2}}e_1 + 2^{-\frac{1}{2}}e_2$ are eigenvectors of $b_\infty$. Therefore all three eigenvalues are the same and $b_\infty p_\infty = \lambda p_\infty$. It follows that $p_\infty$ is a minimal projection of $B^{**}$. Thus no matter how we write $p_\infty = p' + p''$, with $p'$ and $p''$ rank one projections, we cannot achieve the conclusion of 2.4, let alone the hypotheses.

(b) First note that if $p$ is the projection of 5.12, then we have the hypotheses of 2.4 with $X = \mathbb{N} \cup \{\infty\}$ and $p = p_X$. Since all of the $\varphi_n$’s, $1 \leq n \leq \infty$, are equivalent, it is easy to see that the non-pure state $\frac{1}{2}\varphi_1 + \frac{1}{2}\varphi_\infty$ is in $[P(A) \cap F(p)]^-$ (cf [16]), so that $p$ does not satisfy (G).

It is better to give an example where the equivalence classes of $\{\varphi_x : x \in X\}$ are finite, since by 4.1 there is no hope of (NCEB) without this finiteness. A standard example suffices for this. Let $A = \{a \in c \otimes M_2 : a_\infty \text{ is diagonal}\}$. Let $B = \{a \in A : a_n \text{ is diagonal, } \forall n\}$. Then $B$ is a MASA in $A$, and we let $X = \widehat{B}$, the disjoint union of two copies of $\mathbb{N} \cup \{\infty\}$. It is clear that for $x$ in $X$ the pure state $\psi_x$ of $B$ satisfies (UEP); and if $p_x$ is the support projection of $\psi_x$, we have the hypothesis of 2.4 with $p_X = 1_A$. Since $\widehat{A}$ is not Hausdorff, it follows from [17, Thm. 6] that $p_X$ does not satisfy (G). Of course, this is also easy to see explicitly.

It is possible to give a similar example in which $\widehat{A}$ is Hausdorff, but a different condition of [17, Theorem 6] is violated. Let $A = \{a \in c \otimes M_2 \otimes M_2 : a_\infty \in M_2 \otimes I_2\}$. If $B = \{a \in A : a_n \in D_2 \otimes D_2, n < \infty; a_\infty \in D_2 \otimes I_2\}$, where $D_2 = \{d \in M_2 : d \text{ is diagonal}\}$, then $B$ is a MASA in $A$ and we can proceed similarly to the above. Again $X$ is the disjoint union of two copies of $\mathbb{N} \cup \{\infty\}$ (arising more naturally as $\mathbb{N} \cup \mathbb{N} \cup \{\infty\}$).

(c) Consider one of the examples of alternative 2 of 4.6 constructed in 4.8(c). Let $k = n = 2$ and $T = 1$. We then get a projection $p$ satisfying (CEB) where the space $X$ of Section 4 is $\mathbb{N} \cup \{\infty\}$, rank $p_n = 2, 1 \leq n \leq \infty$, and there is a single
element \( \varphi \) of \( F(p_\infty) \cap P(A) \) such that every sequence \( (\psi_n) \) with \( \psi_n \) in \( F(p_n) \cap P(A) \) converges to \( \varphi \). In this case we can write \( p_n = p_{n,1} + p_{n,2} \) so that the hypotheses of 2.4 are satisfied. All we have to do is take \( p_{\infty,1} \) to be the support projection of \( \varphi \). \( \bar{X} \) will be homeomorphic to \( \mathbb{N} \cup \{ \infty \} \), but it arises as the disjoint union of \( \mathbb{N} \cup \mathbb{N} \cup \{ \infty \} \) with an isolated point. This example does not satisfy (MSQC). One way to see this is to note that the saturation of an open subset of \( F(p) \cap P(A) \) need not be open, and hence \( F(p) \) is not isomorphic to the quasi-state space of a \( C^* \)-algebra. Explicitly, any element \( h \) of \( SQC(p) \) (or \( QC(p) \)) must have \( \varphi \) definite on \( h_\infty \).

Lemma 7.7. If \( A \) is a \( C^* \)-algebra, \( p \) is a projection in \( A^{**} \), and \( p \) satisfies (G), then \( \pi^{**}(pAp) \subset \mathcal{K}(H_\pi) \) for every irreducible representation \( \pi \) of \( A \).

Proof. Part of the proof of 4.1 applies: If \( (e_n) \) is an orthonormal sequence in the range of \( \pi^{**}(p) \) and \( \psi_n = (\pi(\cdot)e_n,e_n) \), we can conclude that \( \psi_n \to 0 \). If \( E_{\{e,\infty\}}(\pi^{**}(pap)) \) has infinite rank for some \( a \in A_+ \) and \( \epsilon > 0 \), then we can obtain a contradiction by taking the \( e_n \)'s in the range of \( E_{\{e,\infty\}}(\pi^{**}(pap)) \).

Corollary 7.8. If \( A \) is a separable \( C^* \)-algebra, \( p \) is a projection in \( A^{**} \) satisfying the barycenter formula (in particular if \( p \) is closed), and \( p \) satisfies (G), then \( p \) is type I.

Proof. Apply 5.13.

7.9. Continuation of Example 5.12.

(a) A subprojection \( p' \) of \( p \) is closed if and only if \( p' \) has finite rank or \( p' \geq p_0 \).

Proof. If \( p' \) has finite rank, then \( p' \) is closed by [1]. If \( p' \geq p_0 \), then \( p' - p_0 \) is closed in \( B^{**} \) by 0.1(ii). Therefore \( p' \) is closed in \( \tilde{A}^{**} \) and a fortiori in \( A^{**} \).

If \( p' \) has infinite rank, then the range of \( p' \) contains an infinite dimensional subspace \( H' \) of \( \text{span}\{w_{n_1}, w_{n_2}, \ldots\} = \text{range}(p - p_0) \). (This last is a codimension 1 subspace of \( H_0 = \text{range} \, p \).) Let \( (u_n) \) be a sequence of unit vectors in \( H' \) such that \( u_n \xrightarrow{w} 0 \) and \( \psi_n = (\pi(\cdot)u_n, u_n) \). Then \( \psi_n|_B \to 0 \), since \( \pi^{**}(pBp) \subset \mathcal{K}(H_\pi) \). Since \( p \) is compact, it follows that \( \psi_n \to \varphi_0 \). If \( p' \) is closed, this implies \( \varphi_0 \in F(p') \) and hence \( p' \geq p_0 \).

(b) For any non-zero closed subprojection \( p' \) of \( p \) there is a minimal projection \( p_1 \) such that \( p_1 \leq p' \) and \( p' - p_1 \) is closed.

Proof. If \( p' \) has infinite rank, then \( p' \geq p_0 \). We can find a minimal projection \( p_1 \) such that \( p_1 \leq p' - p_0 \). Then \( p_0 \leq p' - p_1 \) so that \( p' - p_1 \) is closed. If \( p' \) has finite rank then \( p' - p_1 \) is closed for any choice of \( p_1 \).

(c) If \( pA^{**}p \) is identified with \( B(H_0) \), then

\[
    pAp = \{ x \in B(H_0) : x - \varphi_0(x)I_{H_0} \in \mathcal{K}(H_0) \} \\
    = \{ x \in B(H_0) : x - (x_{v_0}, v_0)I_{H_0} \in \mathcal{K}(H_0) \}.
\]

Proof. Let \( H_1 = H_0 \ominus Cv_0 \). By construction and the proof of 7.2, applied to \( B \) and \( p - p_0 \), \( pBp = \mathcal{K}(H_1) \). Note that since \( p \) is compact, \( pAp = p\tilde{A}p \). To show that \( pAp \) is contained in the set indicated, it is enough to show \( pap \) is compact when
\(a \in \tilde{A}\) and \(\phi_0(a) = 0\). By [25, 3.13.6], \(a = l + r\), where \(l \in \tilde{A}B\) and \(r \in B\tilde{A}\). Since \(x\) is compact if and only if \(x^*x\) is compact, \(Bp \subset \mathcal{K}(H_\pi)\); and similarly \(pB \subset \mathcal{K}(H_\pi)\). Therefore \(pap \in \mathcal{K}(H_0)\).

For the reverse inclusion, since \(p \in pAp\), \(\mathcal{K}(H_1) \subset pAp\), and \(pAp\) is self-adjoint, it is sufficient to show that \(pAp\) contains every rank 1 operator \(x\) of the form \(v \rightarrow (v, v_0)v_1\) for \(v_1 \in H_1\). By the Kadison transitivity theorem ([20]) there is \(a\) in \(A\) such that \(av_0 = v_1\) and \(a^*v_0 = 0\). By the above, since \((av_0, v_0) = 0\), \(pap\) is compact. Hence \((pap - x) \in \mathcal{K}(H_1) = pBp\). This implies that \(x\) is in \(p\tilde{A}p\).

Since by [7, 4.4], the bidual of \(pAp\) is \(pA^{**}p\), and since the predual of a \(W^*\)-algebra is unique, it follows from (c) that the dual space of \(pAp\) is \(\mathcal{T}(H_0)\), the set of trace class operators on \(H_0\). A concrete statement of this reads:

If \(v_0\) is a unit vector in the (separable, infinite dimensional) Hilbert space \(H_0\) and

\[
A_{v_0} = \{x \in B(H_0) : x - (xv_0, v_0)I_{H_0} \text{ is compact}\},
\]

then the dual space of \(A_{v_0}\) is naturally isometrically isomorphic to \(\mathcal{T}(H_0)\). In particular, for \(T \in \mathcal{T}(H_0)\), \(\|T\|_1 = \sup\{|Tr(Tx)| : x \in A_{v_0}, \|x\| \leq 1\}\).

It is amusing to give a direct proof of this statement (which removes the parenthetical part of the hypothesis). The main step is to prove the second sentence.

Finally, we note that 5.12 gives another example of how the behavior of closed faces of \(C^*\)-algebras differs from that of \(C^*\)-algebras. If \(\pi\) is an irreducible representation of a \(C^*\)-algebra \(A\), then \(\pi(A) \cap \mathcal{K}(H_\pi)\) is either 0 or \(\mathcal{K}(H_\pi)\). The analogous statement for Example 5.12 (replacing \(A\) by \(pAp\)) is false.
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