Deformation Quantization of a Certain Type of Open Systems

Florian Becher, Nikolai Neumaier, Stefan Waldmann

Fakultät für Mathematik und Physik
Albert-Ludwigs-Universität Freiburg
Physikalisches Institut
Hermann Herder Straße 3
D 79104 Freiburg
Germany

September 2009

Abstract

We give an approach to open quantum systems based on formal deformation quantization. It is shown that classical open systems of a certain type can be systematically quantized into quantum open systems preserving the complete positivity of the open time evolution. The usual example of linearly coupled harmonic oscillators is discussed.

Contents

1 Introduction 2
2 Classical Open Dynamical Systems 2
3 Deformation Quantization of Open Hamiltonian Systems 7
4 Linearly Coupled Harmonic Oscillators I: Generalities 12
5 Linearly Coupled Harmonic Oscillators II: Examples 17

*Florian.Becher@physik.uni-freiburg.de
†Nikolai.Neumaier@physik.uni-freiburg.de
§Stefan.Waldmann@physik.uni-freiburg.de
1 Introduction

Attempts at the quantization of open systems, especially dissipative systems, have been made for quite some time. Examples can, among many others, be found in [7, 11, 20]. In particular, some approaches to the deformation quantization of genuinely dissipative systems have been conducted, see [13,14]. So far, it seems that no successful attempt has been made at a mathematically consistent systematic quantization of open systems originating from coupled systems.

We chose the framework of deformation quantization. The central object of deformation quantization [3] is the algebra of observables. States are regarded as a derived concept in the sense of normalized positive linear functionals on the algebra of observables in the classical as well as in the quantum case. The star products used to deform the classical algebra of observables in this process are meant to be Hermitian star products. The existence of such star products on the smooth functions of Poisson manifolds has been proven by [18]. For the special case of symplectic manifolds the existence has been proven earlier by [12,16,19], see also the textbook [24] for additional references.

In the manner of speaking of [6], we get an open system (classical and quantum mechanical) by constructing a microscopic model and non-selectively integrating the degrees of freedom of the environment.

As a first step, we give a consistent and general definition of what a classical and quantum open Hamiltonian system in the sense of deformation quantization should be relying on the notion of completely positive evolutions in both cases. As main result we prove that every classical open Hamiltonian system can be deformation quantized preserving complete positivity of the evolution map. A by-product of independent interest is the result that for every Hermitian star product on a Poisson manifold there is a completely positive map into the undeformed algebra of formal series of smooth functions deforming the identity map. Our general formalism is exemplified for two coupled harmonic oscillators.

This article is organized in the following way: In Section 2 a notion of classical open dynamical systems in general and the notion of a classical open Hamiltonian system used for deformation quantization in particular are defined. In Section 3 we will briefly recall the notions of a Hermitian star product and the quantum time evolution with regard to a Hermitian star product. We prove in Theorem 3.5 that for every Hermitian star product one has a completely positive map deforming the identity into the formal series of smooth functions with respect to the undeformed product. This turns out to be the main tool to show Theorem 3.13: every classical open Hamiltonian system can be deformation quantized. In Section 4 as an illustration, we give the standard example of the total time evolution of two one-dimensional linearly coupled harmonic oscillators in the setting of deformation quantization. Section 5 contains the open time evolutions of a coupled harmonic oscillator with respect to states on the bath oscillator corresponding to deformed initial values and to KMS states.

2 Classical Open Dynamical Systems

There are many ways to specify the notion of open dynamical systems. A fairly general approach is obtained as follows: We start with a subsystem whose pure states are described by a smooth manifold $S$ and a bath which is described analogously by a smooth manifold $B$. The combined total system has the Cartesian product $S \times B$ as space of pure states.

An open dynamical system is now a time evolution of (pure) states in $S \times B$ where we only look at the $S$-part “ignoring” the $B$-part. More precisely, this is obtained as follows:

On the total system we specify an ordinary dynamical system, i.e. a vector field $X \in \Gamma^\infty(T(S \times B))$ with flow $\Psi_t : S \times B \rightarrow S \times B$. For simplicity, we may assume that the flow $\Psi_t$ is complete,
otherwise we have to restrict to certain neighbourhoods in \( S \times B \) and finite times in the usual way. With this assumption, \( \Psi_t \) is a smooth one-parameter group of diffeomorphisms of \( S \times B \) with
\[
\frac{d}{dt} \Psi_t = X \circ \Psi_t \quad \text{for all } t \in \mathbb{R}.
\]
(2.1)

Next we consider the canonical projection maps
\[
S \leftarrow S \times B \overset{pr_B}{\rightarrow} B,
\]
(2.2)
which allow to decompose the tangent bundle \( T(S \times B) \) into
\[
T(S \times B) = pr^\#_S TS \oplus pr^\#_B TB,
\]
(2.3)
where \( pr^\#_S TS \) and \( pr^\#_B TB \) denote the pull-backs of the tangent bundles of \( S \) and \( B \), respectively.

Clearly, the map \( pr_S \) forgets the degrees of freedom of the bath and thus corresponds precisely to the idea that we want to ignore the \( B \)-part. However, for the time evolution of \( S \) we still have to specify an initial condition for the bath as well. For the moment, we restrict ourselves to pure states and allow for mixed states later on. Thus let \( x_B \in B \) be a point whence we have the embedding
\[
t_{x_B} : S \ni x_S \mapsto (x_S, x_B) \in S \times B,
\]
(2.4)
which is clearly a diffeomorphism onto its image such that \( pr_S \circ t_{x_B} = id_S \) and \( pr_B \circ t_{x_B} = x_B \) is the constant map.

**Definition 2.1 (Open time evolution, pure case)** For any \( x_B \in B \) the open time evolution \( \Phi^{x_B}_t : S \rightarrow S \) with respect to the total time evolution \( \Psi_t \) of \( S \times B \) and the pure state \( x_B \) of the bath is given by
\[
\Phi^{x_B}_t = pr^\#_S TS \oplus pr^\#_B TB,
\]
(2.5)

Of course, we have to justify this definition and examine some consequences as well as properties of \( \Phi^{x_B}_t \). First of all, the map
\[
\Phi^{x_B} : \mathbb{R} \times S \ni (t, x_S) \mapsto \Phi^{x_B}_t(x_S) \in S
\]
(2.6)
is clearly smooth. However, it does not have the usual properties of an ordinary time evolution: For a fixed time \( t \) the map \( \Phi^{x_B}_t \) needs not to be a diffeomorphism, not even for small times. We only have the following “evolution property” which easily follows from the one-parameter group property of \( \Psi_t \):

**Proposition 2.2** For the open time evolution we have
\[
\Phi^{x_B}_0 = id_S \quad \text{and} \quad \Phi^{pr_B(\Psi_t(x_S, x_B))}_s \circ \Phi^{x_B}_t(x_S) = \Phi^{x_B}_{s+t}(x_S)
\]
(2.7)
for all \( x_S \in S, x_B \in B, \) and \( s, t \in \mathbb{R} \).

**Example 2.3** Let \( S = \mathbb{R} = B \) and consider the time evolution
\[
\Psi_t(x_S, x_B) = (x_S \cos(\nu t) - x_B \sin(\nu t), x_S \sin(\nu t) + x_B \cos(\nu t))
\]
(2.8)
on \( S \times B \).
i.) The simplest case is obtained for $\nu \in \mathbb{R}$ being a non-zero constant. Then the open time evolution for $x_B \in B$ is given by

$$\Phi_t^{x_B}(x_S) = x_S \cos(\nu t) - x_B \sin(\nu t)$$

which is a diffeomorphism for small $t$ but the constant map for $\nu t \in \frac{\pi}{2} + \pi \mathbb{Z}$.

ii.) We can also consider the case where $\nu$ is a function on $S \times B$ depending only on the radius, e.g. $\nu(x_S, x_B) = x_S^2 + x_B^2$. Then $\Psi_t$ is still a one-parameter group of diffeomorphisms and the flow lines are still concentric circles around $(0, 0)$. However, the points in $S \times B$ spin faster the further away from $(0, 0)$ they are. Now the open time evolution is

$$\Phi_t^{x_B}(x_S) = x_S \cos((x_S^2 + x_B^2)t) - x_B \sin((x_S^2 + x_B^2)t).$$

E.g. for $x_B = 0$ this gives $\Phi_t^0(x_S) = x_S \cos(x_S^2t)$ which yields

$$\Phi_t^0 \left( \frac{\pi}{\sqrt{2t}} \right) = 0$$

for all $t > 0$. Since also $\Phi_t^0(0) = 0$ for all $t$ we see that $\Phi_t^0$ cannot be a diffeomorphism, even for arbitrarily small time $t > 0$.

From the example we conclude that the open time evolution $\Phi_t^{x_B}$ in general is not a solution to a probably time-dependent differential equation on $S$ alone, i.e. in general there is no time-dependent vector field $X_t \in \Gamma^\infty(TS)$ with

$$\frac{d}{dt}\Phi_t^{x_B} = X_t \circ \Phi_t^{x_B}.$$  

Nevertheless, this situation of a time-dependent vector field is a particular case of an open time evolution as the next example shows:

**Example 2.4** Let $X_t \in \Gamma^\infty(TS)$ be a smooth time-dependent vector field on $S$ and let $\overrightarrow{X} \in \Gamma^\infty(T(S \times \mathbb{R}))$ be the corresponding time-independent vector field

$$\overrightarrow{X}(x_S, t) = \left( X_t(x_S), \frac{\partial}{\partial t} \bigg|_{t=0} \right),$$

where we use the splitting of $T(S \times \mathbb{R})$ and the canonical constant vector field on the “bath” $B = \mathbb{R}$. For simplicity, we assume that $\overrightarrow{X}$ has a complete flow $\Psi_t$. Then the open time evolution for initial condition $x_B = 0$ of the bath is

$$\Phi_t^0(x_S) = \text{pr}_S(\Psi_t(x_S, 0)).$$

But this is precisely the time evolution of the time-dependent vector field $X_t$, i.e. we have

$$\frac{d}{dt}\Phi_t^0 = X_t \circ \Phi_t^0,$$

as an easy and well-known computation shows. Thus the ordinary time evolution of a time-dependent vector field can be viewed as a particular case of an open time evolution in the sense of Definition 2.1.
In view of the yet to be found quantization of open dynamical systems we consider now the effect of an open time evolution on the functions $C^\infty(S)$ as these will play the role of the observables later. The following statement is obvious:

**Proposition 2.5** Let $x_B \in B$. Then $(\Phi_t^{x_B})^* : C^\infty(S) \rightarrow C^\infty(S)$ is a $^*$-homomorphism for every $t \in \mathbb{R}$ and we have

$$$(2.16) (\Phi_t^{x_B})^* = (\text{id} \otimes \delta_{x_B}) \circ \Psi_t^* \circ \text{pr}_S^*.$$$

Here $\delta_{x_B} : C^\infty(S) \rightarrow C$ denotes the $\delta$-functional at $x_B$, i.e. the evaluation of a function at the point $x_B$. Moreover, $\text{id} \otimes \delta_{x_B}$ is the induced map

$$$(2.17) \text{id} \otimes \delta_{x_B} : C^\infty(S) \otimes C^\infty(B) = C^\infty(S \times B) \rightarrow C^\infty(S),$$$

where $\otimes$ denotes the completed projective tensor product. Note that the involved Fréchet spaces are nuclear anyway.

Though Proposition 2.5 is a trivial reformulation of the definition of $\Phi_t^{x_B}$ it gives a new point of view: to this end, recall that a linear functional $\omega_0 : C^\infty(M) \rightarrow C$ is called positive if $\omega_0(f) \geq 0$ for all functions $f \in C^\infty(M)$. Similarly, we can define a positive functional on matrix-valued functions $M_n(C^\infty(M))$. Having the notion of positive linear functionals we can define positive algebra elements by setting that $f \in C^\infty(M)$ is positive if $\omega_0(f) \geq 0$ for all positive linear functionals $\omega_0$. Then it is a true but slightly non-trivial fact that $f$ is positive iff $f(p) \geq 0$ for all points $p \in M$. The same holds for matrix-valued functions: a function $F \in M_n(C^\infty(M))$ is positive iff $F(p)$ is a positive semi-definite matrix for all $p \in M$. Note that in our approach, this is not a definition but a consequence of the more algebraic definition. Finally, a linear map $\phi : C^\infty(M) \rightarrow C^\infty(N)$ is called positive if it maps positive functions to positive functions. More important is the notion of a completely positive map: $\phi$ is called completely positive if all the canonical extensions $\phi : M_n(C^\infty(M)) \rightarrow M_n(C^\infty(N))$ are positive maps for $n \in \mathbb{N}$. Clearly, this is the standard definition valid for every $^*$-algebra over the complex numbers $C$, see e.g. [22] for a detailed exposition and [8, App. B] for a discussion of the case of smooth functions.

Now we come back to our particular situation: while $\Phi_t^*$ and $\text{pr}_S^*$ are canonically given $^*$-homomorphisms of the $^*$-algebras of smooth functions and hence completely positive maps, the map $\text{id} \otimes \delta_{x_B}$ can also be interpreted as a positive (and in fact completely positive) map which coincides with a $^*$-homomorphism $^*$-

"by accident". In particular, we can replace the positive functional $\delta_{x_B}$ by any, not necessarily pure state $\omega_0$ of $C^\infty(B)$, that is, a positive linear normalized functional $\omega_0 : C^\infty(B) \rightarrow C$. This yields the following, more general definition of an open time evolution:

**Definition 2.6 (Open time evolution, general case)** For any state $\omega_0 : C^\infty(B) \rightarrow C$ of the bath, the open time evolution of $S$ with respect to the total time evolution $\Psi_t$ and the state $\omega_0$ is given by

$$$(2.18) (\Phi_t^{\omega_0})^* = (\text{id} \otimes \omega_0) \circ \Psi_t^* \circ \text{pr}_S^*.$$$

**Remark 2.7** Any positive functional $\omega_0 : C^\infty(B) \rightarrow C$ is actually a positive Borel measure with compact support, see e.g. [8, App. B]: for continuous functions this is the famous Riesz Representation Theorem, see e.g. [21, Thm. 2.14], which can be shown to extend to the smooth setting. Therefore, any state $\omega_0 : C^\infty(B) \rightarrow C$ is automatically continuous with respect to the smooth topology. Thus the map $\text{id} \otimes \omega_0$ extends to the completed tensor product making the above expression in (2.18) well-defined.
The notation \((\Phi^\omega_t)^*\) is of course only symbolic as there is clearly no longer an underlying map of manifolds. With this definition we shifted the focus to the observable algebra rather than the underlying geometry.

**Proposition 2.8** For any state \(\omega_0\) of the bath, the open time evolution \((\Phi^\omega_t)^* : C^\infty(S) \rightarrow C^\infty(S)\) is a completely positive map.

**Proof:** Since \(\Psi_t^*\) and \(\text{pr}_B^*\) are \(*\)-homomorphisms we only have to show that \(\text{id} \hat{\otimes} \omega_0\) is a completely positive map from \(C^\infty(S \times B)\) to \(C^\infty(S)\). Thus let \(F \in M_n(C^\infty(S \times B))\) be given and let \(x_s \in S\). Then we have the embedding \(\iota_{x_s} : B \rightarrow S \times B\) whence

\[
\delta_{x_s} \circ (\text{id} \hat{\otimes} \omega_0) = \delta_{x_s} \hat{\otimes} \omega_0 = \omega_0 \circ (\delta_{x_s} \hat{\otimes} \text{id}) = \omega_0 \circ \iota_{x_s}^*.
\]

Since \(\iota_{x_s}^*\) is a \(*\)-homomorphism, the composition \(\omega_0 \circ \iota_{x_s}^*\) is still a positive functional and hence a completely positive map. Thus, applied to \(F^* F\), we get a positive semi-definite matrix \(\omega_0 \circ \iota_{x_s}^* (F^* F) = \delta_{x_s} \circ (\text{id} \hat{\otimes} \omega_0)(F^* F)\). Since this is true for every point \(x_s \in S\), we have a positive element \((\text{id} \hat{\otimes} \omega_0)(F^* F) \in M_n(C^\infty(S))\) proving the claim. \(\blacksquare\)

**Remark 2.9** Since any positive functional \(\omega_0 : C^\infty(B) \rightarrow \mathbb{C}\) is actually a positive Borel measure with compact support, the map \(\text{id} \hat{\otimes} \omega_0\) indeed means to integrate over the bath degrees of freedom with respect to a measure specified by \(\omega_0\).

**Remark 2.10** Note also that in the case of a \(\delta\)-functional instead of an arbitrary state \(\omega_0\), the open time evolution actually is a \(*\)-homomorphism, in contrast to the case of arbitrary states. However, in general, \((\Phi^\omega_t)^*\) is just a completely positive map without any further nice algebraic features.

While up to now we have considered arbitrary dynamical systems, we shall pass to more specific ones: we assume to have a Hamiltonian dynamics on the total space of the system and the bath. In more detail, we choose the rather general setting of Poisson geometry to formulate Hamiltonian dynamics. This framework contains in particular any symplectic phase space such as coadjoint orbits, cotangent bundles or Kähler manifolds. However, also the dual of a Lie algebra is a (linear) Poisson manifold which is important when dealing with symmetries, see e.g. [24, Chap. 3 & Chap. 4] for an introduction.

Thus, let the state space of the system \((S, \pi_S)\) and the one of the bath \((B, \pi_B)\) be in addition Poisson manifolds with Poisson structures \(\pi_S\) and \(\pi_B\). On the total system \(S \times B\) we choose the product Poisson structure

\[
\pi = \text{pr}_S^* \pi_S + \text{pr}_B^* \pi_B.
\]

This means that for functions \(f_S, g_S \in C^\infty(S)\) and \(f_B, g_B \in C^\infty(B)\) the factorizing functions \(f = f_S \otimes f_B\) and \(g = g_S \otimes g_B\) have the Poisson bracket

\[
\{f, g\} = \{f_S, g_S\}_S \otimes f_B g_B + f_S g_S \otimes \{f_B, g_B\}_B.
\]

The dynamics of the total system is given by the Hamiltonian vector field \(X_H \in \Gamma^\infty(T(S \times B))\) with respect to the total Hamiltonian \(H \in C^\infty(S \times B)\). Recall that the Hamiltonian vector field is defined by \(X_H = \{\cdot, H\}\). In typical situations, the total Hamiltonian contains three parts: we have the Hamiltonian \(H_S \in C^\infty(S)\) of the system alone, the Hamiltonian \(H_B \in C^\infty(B)\) of the bath alone, and an interaction Hamiltonian \(H_I \in C^\infty(S \times B)\) such that the total Hamiltonian is

\[
H = \text{pr}_S^* H_S + \text{pr}_B^* H_B + H_I.
\]
Then the total Hamiltonian time evolution is the flow \( \Phi_t : S \times B \rightarrow S \times B \) of \( X_H \) which we assume to be complete for simplicity and analogously to Definition 2.6 the open Hamiltonian time evolution with respect to a given state of the bath is defined as follows:

**Definition 2.11 (Classical open Hamiltonian time evolution)** The classical open Hamiltonian time evolution of the system \( S \) with respect to a total Hamiltonian time evolution \( \Phi_t \) of \( S \times B \) and a given state \( \omega_0 \) of the bath is defined as follows:

\[
(\Phi_t^{\omega_0})^* : C^\infty(S) \rightarrow C^\infty(S)
\]

according to Definition 2.6.

**Remark 2.12** Again, unless we have special circumstances, the open Hamiltonian time evolution is just a completely positive map without any further algebraic features. In particular, there is no reason that \((\Phi_t^{\omega_0})^*\) should preserve Poisson brackets, even for \( \omega_0 = \delta_{x_B} \) being a pure state.

### 3 Deformation Quantization of Open Hamiltonian Systems

In this section we will establish the deformation quantized version of the open Hamiltonian time evolution. To this end, we recall that a *formal star product* on a Poisson manifold \((M, \pi)\) is an associative \( \mathbb{C}[[\hbar]] \)-bilinear multiplication

\[
f \star g = \sum_{r=0}^{\infty} \hbar^r C_r(f, g)
\]

for \( f, g \in C^\infty(M)[[\hbar]] \) such that \( C_0(f, g) = fg \) is the undeformed commutative product, \( C_1(f, g) - C_1(g, f) = i\{f, g\} \) with the Poisson bracket \(\{\cdot, \cdot\}\), \(1 \star f = f = f \star 1\) for the constant function 1, and all \(C_r\) are bidifferential operators \([3]\), see also \([24]\) for a pedagogical introduction. The reason that we chose formal star products where a priori no convergence in \(\hbar\) is controlled, is that for this situation we have the powerful existence and classification theorems of deformation quantization at hand. Physically, of course, one would like to have convergence or at least some asymptotic statements. In many examples this is possible but we shall not enter this rather technical issue here any further.

In the sequel, the case where the star product \(\star\) is *Hermitian* will be important, i.e.

\[
\overline{f \star g} = g \star \overline{f}
\]

for all \( f, g \in C^\infty(M)[[\hbar]] \) where \( \overline{\hbar} = \hbar \) is treated as a real quantity. This \(\star\)-involution will be necessary to have the honest interpretation of the algebra \((C^\infty(M)[[\hbar]], \star)\) as observable algebra of the quantum system corresponding to the classical system.

Having the observable algebra, it is natural to define the states in the same way as classically: we use positive linear functionals. Now however, we have to specify first what a *positive formal series* should be. Here we can rely on the following definition. A non-zero real formal power series \( a = \sum_{r=r_0}^{\infty} \hbar^r a_r \in \mathbb{R}[[\hbar]] \) is called positive if its lowest non-zero component is positive, \( a_{r_0} > 0 \). In this case we write \( a \geq 0 \). This is a good definition for many reasons: if we view formal series as arising from asymptotic expansions then this is what remains from a positive function. More algebraically, \( \mathbb{R}[[\hbar]] \) becomes an *ordered ring* by this definition, hence we can rely on the rich and well-developed theory of *-algebras over ordered rings, see e.g. \([9, 23]\) for an overview and \([24, \text{Chap. 7}]\) for an introduction and further references.
For star product algebras we can proceed analogously to the classical case and define a \( \mathbb{C}[[\hbar]] \)-linear functional \( \omega : C^\infty(M)[[\hbar]] \to \mathbb{C}[[\hbar]] \) to be positive if
\[
\omega(\mathcal{F} \ast f) \geq 0 \tag{3.3}
\]
for all \( f \in C^\infty(M)[[\hbar]] \). It can be shown that it suffices to check (3.3) for \( f \in C^\infty(M) \) without higher orders of \( \hbar \). Analogously, we define positive linear functionals for matrix-valued functions \( F \in M_n(C^\infty(M)[[\hbar]]) \) where the star product is extended to matrices in the usual way. Having positive functionals we define \( f \in C^\infty(M)[[\hbar]] \) or \( F \in M_n(C^\infty(M)[[\hbar]]) \) to be a positive algebra element if
\[
\omega(f) \geq 0 \quad \text{and} \quad \Omega(F) \geq 0 \tag{3.4}
\]
for all positive functionals \( \omega \) and \( \Omega \), respectively. Finally, a \( \mathbb{C}[[\hbar]] \)-linear map \( \phi : C^\infty(M)[[\hbar]] \to C^\infty(N)[[\hbar]] \) between two star product algebras on possibly different underlying manifolds is called positive if \( \phi \) maps positive elements to positive elements. Equivalently, \( \phi \) is called positive if \( \omega \circ \phi \) is a positive functional on \( C^\infty(M)[[\hbar]] \) for all positive functionals \( \omega \) on \( C^\infty(N)[[\hbar]] \). The map \( \phi \) is called completely positive if this is also true for arbitrary matrix-valued functions, i.e. if \( \phi^{(n)} : M_n(C^\infty(M)[[\hbar]]) \to M_n(C^\infty(N)[[\hbar]]) \) is positive for all \( n \in \mathbb{N} \). Note that even though these definitions are in complete analogy to the classical situation, it is nevertheless crucial to have a good notion of positive formal power series in \( \mathbb{R}[[\hbar]] \).

**Remark 3.1** It is clear that the above concepts generalize immediately to \(*\)-algebras \( A \) over a ring \( \mathbb{C} = \mathbb{R}(i) \) where \( \mathbb{R} \) is an ordered ring and \( i \) is a square root of \( -1 \). Even though many of the following considerations generalize to this algebraic framework as well, we shall focus on the more particular situation of star products.

**Remark 3.2** In the following, completely positive maps will play a crucial role. It is easy to see that positive functionals are in fact completely positive maps. Also \(*\)-homomorphisms are completely positive. Moreover, the composition of completely positive maps as well as convex combinations of completely positive maps are again completely positive. Finally, less evident but nevertheless true is the fact that the algebraic tensor product of completely positive maps is again completely positive. In general, this last statement is wrong for positive maps.

To describe the positive \( \mathbb{C}[[\hbar]] \)-linear functionals of \( (C^\infty(M)[[\hbar]], \ast) \) one first notes that \( \omega \) is necessarily of the form
\[
\omega = \sum_{r=0}^{\infty} h^r \omega_r \quad \text{with linear maps} \quad \omega_r : C^\infty(M) \to \mathbb{C}. \tag{3.5}
\]
Then the positivity \( \omega(\mathcal{F} \ast f) \geq 0 \) in the sense of formal power series immediately implies that \( \omega_0(\mathcal{F} f) \geq 0 \) classically, i.e. \( \omega_0 \) is a positive \( \mathbb{C} \)-linear functional. This raises the question whether every classical state \( \omega_0 \) can be “quantized” into a state \( \omega \) with respect to the star product. In other words, we ask whether every classical state is the classical limit of some quantum state. Physically, this is absolutely necessary as quantum theory is believed to be the more fundamental description of nature. Fortunately, we can rely on the following theorem [10], even for the case of matrices. But first we give a definition which shall simplify the further considerations.

**Definition 3.3 (Square preserving map)** A \( \mathbb{C}[[\hbar]] \)-linear map \( S = id + \sum_{r=1}^{\infty} h^r S_r : C^\infty(M)[[\hbar]] \to C^\infty(M)[[\hbar]] \) with differential operators \( S_r, \ S(1) = 1, \) and \( S(\mathcal{F}) = \mathcal{S}(f) \) is called preserving
squares with respect to \(\star\), if there are formal series of differential operators \(D_{r,I} : C^\infty(M)[[\hbar]] \rightarrow C^\infty(M)[[\hbar]]\) for \(r \in \mathbb{N}_0\) and \(I\) running over a finite range (possibly depending on \(r\)) such that

\[
S(f \star g) = \sum_{r=0}^{\infty} \hbar^r \sum_I D_{r,I}(f)D_{r,I}(g)
\]

for all \(f, g \in C^\infty(M)[[\hbar]]\).

**Remark 3.4** It is fairly simple to see that a map preserving squares according to Definition 3.3 is in fact a completely positive map from the quantized algebra \((C^\infty(M)[[\hbar]], \star)\) to the classical algebra \((C^\infty(M)[[\hbar]], \cdot)\) with the undeformed product.

**Theorem 3.5** Given a Hermitian star product \(\star\), there exists a globally defined map \(S\) preserving squares with respect to \(\star\).

**Proof:** By [10] we know that for a Hermitian star product \(\star\) on an open subset \(U \subseteq \mathbb{R}^n\) there exists a map preserving squares with respect to \(\star\), denoted by

\[
S(f \star g) = \sum_{r=0}^{\infty} \hbar^r \sum_I D_{r,I}(f)D_{r,I}(g).
\]

For the Poisson manifold \(M\) with star product \(\star\) we choose a finite atlas. Note that we can always find an atlas consisting of \(\dim(M) + 1\) not necessarily connected charts. Denote the domains of the charts by \(U_\alpha \subseteq M\). Next we choose a corresponding subordinate finite quadratic partition of unity \(\chi_\alpha \in C^\infty(M)\), i.e. \(\text{supp} \ \chi_\alpha \subseteq U_\alpha\) and \(\sum_\alpha \chi_\alpha \chi_\alpha = 1\). Now let \(S_\alpha\) be the locally available maps preserving squares with respect to \(\star|_{U_\alpha}\) with corresponding locally defined differential operators \(D_{r,I,\alpha}\). Then we set

\[
S(f) = \sum_\alpha \chi_\alpha \chi_\alpha S_\alpha (f|_{U_\alpha})\quad (\star)
\]

Clearly, this gives a globally well-defined formal series of differential operators with \(S(f) = S(f)\) and \(S(1) = 1\). Moreover, since the star product is bidifferential, we have \((f \star g)|_{U_\alpha} = f|_{U_\alpha} \star g|_{U_\alpha}\) and hence we can apply \((\star)\) to obtain

\[
S(f \star g) = \sum_{r=0}^{\infty} \hbar^r \sum_{I,\alpha} \chi_\alpha D_{r,I,\alpha}(f)\chi_\alpha D_{r,I,\alpha}(g)\quad (\star)
\]

**Remark 3.6** Recently, a \(C^*\)-algebraic version of this theorem was obtained for particular strict deformation quantizations in [17].

The proof of Theorem 3.5 immediately leads to the following consequence.

**Corollary 3.7** For every Hermitian star product \(\star\) on a Poisson manifold there exists an equivalent star product \(\star'\) with the property that every classically positive linear functional \(\omega_0\) is also positive with respect to \(\star'\).

**Proof:** This is now easy, as we take a map \(S\) preserving squares with respect to \(\star\). Then the star product \(f \star' g = S(S^{-1}(f) \star S^{-1}(g))\) is easily shown to do the job.
Remark 3.8 Rephrasing the result from [10] in terms of Theorem 3.5 says that every classical positive linear functional $\omega_0$ can be deformed into a positive linear functional with respect to a Hermitian star product. Indeed, $\omega_0 \circ S$ will be such a deformation, even universal for all $\omega_0$ once $S$ is specified. In general, correction terms in higher orders of $\hbar$ are necessary to obtain positivity. Moreover, they are by far not unique and neither is the map $S$. This is of course to be expected, both from a physical and mathematical point of view. Finally, note that each term $\omega_0 \circ S_\tau$ is continuous in the smooth topology, since the classical functional $\omega_0$ is continuous and the differential operators $S_\tau$ are as well.

After this discussion of states we also need a notion of time evolution for star product algebras. Here we can rely on the following facts. For a given Hamiltonian $H \in C^\infty(M)[[\hbar]]$, where we might even allow for some $\hbar$-dependent correction terms, we consider the Heisenberg equation

$$ \frac{d}{dt} f(t) = \frac{i}{\hbar}[H,f(t)]_\star $$

for $f(t) \in C^\infty(M)[[\hbar]]$. Note that the right-hand side is a well-defined formal power series since the commutator vanishes in zeroth order. For simplicity, we assume that the Hamiltonian vector field corresponding to the zeroth order $H_0$ of $H$ has a complete flow $\Phi_t$. In this case, one can show that (3.7) has a solution for all times with the following properties: There exists a formal series of time-dependent differential operators $T_t = \text{id} + \sum_{r=1}^\infty \hbar^r T_t^{(r)}$ on $M$ such that

$$ A_t = \Phi_t^* \circ T_t : C^\infty(M)[[\hbar]] \rightarrow C^\infty(M)[[\hbar]] $$

(3.8)

is a one-parameter group of automorphisms of $\star$ with $f(t) = A_t f$ being the unique solution of (3.7) with initial condition $f(0) = f$. Moreover, $A_t$ commutes with the commutator $[H,\cdot]_\star$ and we have conservation of energy $A_t H = H$ as usual. Finally, if $\star$ is a Hermitian star product and $H = \overline{H}$ a real Hamiltonian then $A_t$ is even a $\star$-automorphism for each $t$. For details on this quantized version of the classical time evolution we refer to [24, Sect. 6.3.4] and references therein.

After this preparatory discussion we come back to our original situation of a coupled total system $S \times B$. As we already have a nice separation of the total Poisson structure into the Poisson structure of the system and the one of the bath, we shall require the same feature also for the quantization. Thus, we assume to have Hermitian star products $\star_S$ on $S$ and $\star_B$ on $B$, respectively. Then this immediately induces a Hermitian star product $\star = \star_S \otimes \star_B$ on $S \times B$ in such a way that

$$ (C^\infty(S)[[\hbar]], \star_S) \xrightarrow{\text{pr}_S^*} (C^\infty(S \times B)[[\hbar]], \star) \xrightarrow{\text{pr}_B^*} (C^\infty(B)[[\hbar]], \star_B) $$

(3.9)

are both $\star$-homomorphisms of the involved star products. On factorizing functions we have

$$ f \star g = (f_S \star_S g_S) \otimes (f_B \star_B g_B), $$

(3.10)

where $f = f_S \otimes f_B$ and $g = g_S \otimes g_B$ for $f_S, g_S \in C^\infty(S)[[\hbar]]$ and $f_B, g_B \in C^\infty(B)[[\hbar]]$. Clearly, (2.21) becomes the first order limit of (3.10) in the commutators.

Remark 3.9 It will be crucial for our approach that the algebraic structure of the observables is a priori given and will stay untouched. The physical interpretation is, that whatever the time evolution will be, the way how certain quantities, the observables, are measured is independent of any sort of dynamics but a purely kinematical property of the physical system. Thus our star products $\star, \star_S$, and $\star_B$ will be given once and for all and not changed by the open time evolution. Note that this is not the only possibility to deal with open systems: in [14] the star product itself was modified in order to describe a damped harmonic oscillator.
It is now rather obvious what a good definition of a quantized open Hamiltonian time evolution in deformation quantization should be:

**Definition 3.10 (Quantized open Hamiltonian time evolution)** Let $H \in C^\infty(S \times B)[[\hbar]]$ be a Hamiltonian with complete time evolution $A_t$ and let $\omega : C^\infty(B)[[\hbar]] \to \mathbb{C}[[\hbar]]$ be a positive $\mathbb{C}[[\hbar]]$-linear functional. Then the quantized open Hamiltonian time evolution of $S$ with respect to $\omega$ is

$$A_t^\omega = (id \otimes \omega) \circ A_t \circ \text{pr}^*_B : C^\infty(S)[[\hbar]] \to C^\infty(S)[[\hbar]]. \tag{3.11}$$

**Remark 3.11** The above completed tensor product is understood order by order in $\hbar$. Thus we have to require that $\omega = \sum_{r=0}^{\infty} \hbar^r \omega_r$ is continuous in each order of $\hbar$, i.e. each $\omega_r$ is a continuous linear functional with respect to the smooth topology. In view of Theorem 3.5 and Remark 3.8 this seems to be a very reasonable assumption.

**Remark 3.12** Putting Theorem 3.5 Remark 3.8 the existence of Hermitian star products in [18], and the existence of the quantum time evolution of Equation (3.6) together it is easy to see that any classical open Hamiltonian time evolution can be quantized into a quantized open Hamiltonian time evolution. Conversely, the classical limit of any quantized open Hamiltonian time evolution is a classical open Hamiltonian time evolution for the classical limit of the Hamiltonian and with respect to the classical limit of the quantum state by construction as $A_t^\omega = (\Phi_t^{(0)})^* + \mathcal{O}(\hbar)$.

In view of Definition 3.10 it is tempting to believe that the quantized open Hamiltonian time evolution $A_t^\omega$ is completely positive. Indeed, if we would have used the algebraic tensor product in (3.11) instead of the completed one $\hat{\otimes}$ in every order of $\hbar$, then this would be a trivial statement: the algebraic tensor product of the completely positive maps id and $\omega$ is again completely positive, and so is the composition with the completely positive $^*\text{-homomorphisms}$ $A_t$ and $\text{pr}^*_B$. However, the crucial point is that the Fréchet topology of the smooth functions and the $\hbar$-adic topology originating from the ring ordering are not very well compatible. In fact, it is not clear whether the completed tensor product is completely positive or not. Note that this is rather different from the $C^*$-algebraic case where the completed projective tensor product of completely positive maps is always completely positive. From that point of view, the following principal result on the quantized open Hamiltonian time evolution is non-trivial:

**Theorem 3.13** Let $\omega$ be a positive $\mathbb{C}[[\hbar]]$-linear functional on $(C^\infty(B)[[\hbar]], \ast_B)$ of the form

$$\omega = \omega_0 \circ S \tag{3.12}$$

with $S$ preserving squares with respect to $\ast_B$. Then any quantized open Hamiltonian time evolution with respect to $\omega$ is completely positive.

**Proof:** As $\text{pr}^*_B$ and $A_t$ are $^*\text{-homomorphisms}$, the only thing left to show is that $id \otimes \omega$ is completely positive. We extend $S$ to matrices as usual. For $F \in M_n(C^\infty(S \times B)[[\hbar]])$ we have

$$t^*_x(B)((id \otimes S)(F^* \ast F)) = \sum_{r=0}^{\infty} \hbar^r \sum_l t^*_x(B)(D_{r,l}(F))^* \ast_S t^*_x(B)(D_{r,l}(F)),$$

since the restriction to $x_B \in B$ commutes with the pointwise products in (3.6). Now let $\mu : C^\infty(S)[[\hbar]] \to \mathbb{C}[[\hbar]]$ be a positive $\mathbb{C}[[\hbar]]$-linear functional with respect to $\ast_S$. Then for every $x_B$

$$t^*_x(B)((\mu \otimes S)(F^* \ast F)) = \sum_{r=0}^{\infty} \hbar^r \sum_l \mu(t^*_x(B)(D_{r,l}(F))^* \ast_S t^*_x(B)(D_{r,l}(F))) \in M_n(\mathbb{C})[[\hbar]]$$

11
is positive. So if \( \omega_0 \) is classically positive, we conclude that \( \omega_0 \circ (\mu \otimes S)(F^* \star F) \geq 0 \). But 
\[ \omega_0 \circ (\mu \otimes S) = \mu \circ (\text{id} \otimes \omega_0 \circ S) = \mu \circ (\text{id} \otimes \omega). \]
Thus, \( \mu \circ (\text{id} \otimes \omega)(F^* \star F) \) is positive for all positive functionals \( \mu \). This implies that \( (\text{id} \otimes \omega)(F^* \star F) \) is a positive algebra element for all matrices \( F \) and hence \( \text{id} \otimes \omega \) is a completely positive map as claimed. \( \blacksquare \)

Remark 3.14 The assertion of Theorem 3.13 is actually true for more quantum states than the ones of type (3.12); we will see examples later on in Proposition 5.21. We also note that a possible failure of the complete positivity of \( A_t \) should be seen as an artifact of the rather fine (and not too physical) \( h \)-adic topology of formal power series in \( h \). One would expect reasonable behaviour as soon as one enters a convergent regime like strict deformation quantization.

Remark 3.15 In general, the quantized open Hamiltonian time evolution \( A_t \) is no \(*\)-automorphism of \( (C^\infty(S)[[h]], \star_s) \). Furthermore, a close look at Equation (3.11) shows that usually \( A_s \circ A_t \neq A_{s+t} \) as expected from a microscopic system.

Remark 3.16 Using the notions of super manifolds and star products on super symplectic manifolds according to [4,15] one can easily extend our formalism to this framework. This way, one can incorporate spin systems.

4 Linearly Coupled Harmonic Oscillators I: Generalities

As an example, consider the well-known linear coupling of two one-dimensional harmonic oscillators. We shall describe a one-dimensional harmonic oscillator as a Hamiltonian system \( (M, \pi, H) \), given by \( M = T^*\mathbb{R}_q \simeq \mathbb{R}_q^2 \), with Hamiltonian \( H(q, p) = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}q^2 \), where \( m, \omega \in \mathbb{R}^+ \). The Poisson bracket is then determined by

\[
\{q, p\} = 1, \quad \{q, q\} = 0 = \{p, p\}.
\]

Now let us take \( S = M = B \). The Hamiltonian system \( (S \times B, \pi, H) \) describing the linearly coupled identical harmonic oscillators is then given by the smooth manifold \( S \times B \simeq \mathbb{R}_{q_1, p_1} \times \mathbb{R}_{q_2, p_2} \) with the corresponding Poisson bracket as given by Equation (2.21). In the following, we shall use the same symbols \( q_1, p_1, q_2, p_2 \) for the coordinate functions on \( S, B, \) and \( S \times B \), respectively, in order to simplify our notation. In the same spirit, we simply write \( H_1 = H_{q_1} + H_{q_2} + H_{q_3} \) for the total Hamiltonian without the explicit use of \( p_1 \) and \( p_2 \). For the linearly coupled harmonic oscillators the interaction term is given by \( H_1 = \frac{\kappa}{2}(q_1 - q_2)^2 \), with \( \kappa \in \mathbb{R}^+ \) being the coupling constant. Using the new and still global coordinate functions

\[
q_1 = \frac{1}{\sqrt{2}}(q_1 + q_2), \quad p_1 = \frac{1}{\sqrt{2}}(p_1 + p_2), \quad q_2 = \frac{1}{\sqrt{2}}(q_1 - q_2), \quad p_2 = \frac{1}{\sqrt{2}}(p_1 - p_2),
\]

we can bring the total Hamiltonian to normal form and find the well-known expression

\[
H = \frac{1}{2m}(p_1^2 + p_2^2 + \frac{m\omega^2}{2}q_1^2 + \frac{m\omega^2}{2}q_2^2) \quad \text{with} \quad \nu^2 = \nu_1^2 + \frac{2\kappa}{m}. \tag{4.1}
\]

The classical time evolution \( \Phi_t \) is known to be a linear map for all \( t \) which we can express in matrix form as

\[
\Phi_t = \frac{1}{2}\begin{pmatrix}
\cos(\nu t) + \cos(\nu_1 t) & \sin(\nu t) + \sin(\nu_1 t) & \cos(\nu_2 t) - \cos(\nu_1 t) & \sin(\nu t) - \sin(\nu_1 t) \\
-m(\nu \sin(\nu t) + \nu_1 \sin(\nu_1 t)) & \cos(\nu t) + \cos(\nu_1 t) & -m(\nu \sin(\nu_2 t) - \nu \sin(\nu_1 t)) & \sin(\nu t) + \sin(\nu_1 t) \\
\cos(\nu t) - \cos(\nu_1 t) & \sin(\nu t) - \sin(\nu_1 t) & \cos(\nu_2 t) + \cos(\nu_1 t) & \sin(\nu t) + \sin(\nu_1 t) \\
-m(\nu \sin(\nu t) - \nu_1 \sin(\nu_1 t)) & \cos(\nu t) - \cos(\nu_1 t) & -m(\nu \sin(\nu_2 t) + \nu_1 \sin(\nu_1 t)) & \cos(\nu t) + \cos(\nu_1 t)
\end{pmatrix}
\]
with respect to the global linear coordinates \( q_S, p_S, q_B, p_B \). Thus, the open time evolution \( \Phi^ω_t \) of the open subsystem with regard to the state \( ω_0 \) of the bath takes the form

\[
(\Phi^ω_t)^* \begin{pmatrix} q_S \\ p_S \end{pmatrix} = \frac{1}{2} \begin{pmatrix}
\cos(\nu t) + \cos(\nu_κ t) & \sin(\nu t) + \sin(\nu_κ t) \\
-m(\nu \sin(\nu t) + \nu_κ \sin(\nu_κ t)) & \cos(\nu t) + \cos(\nu_κ t)
\end{pmatrix} \begin{pmatrix} q_S \\ p_S \end{pmatrix} \\
+ \frac{1}{2} \begin{pmatrix}
\omega_0(q_B) (\cos(\nu t) - \cos(\nu_κ t)) + \omega_0(p_B) (\sin(\nu t) - \sin(\nu_κ t)) \\
-\omega_0(q_B)m(\nu \sin(\nu t) - \nu_κ \sin(\nu_κ t)) + \omega_0(p_B)(\cos(\nu t) - \cos(\nu_κ t))
\end{pmatrix}.
\]

(4.2)

Analogously to the classical case we shall use the normal coordinates in order to simplify the computation of the quantum time evolution of the total system. Moreover, it will be advantageous to combine the real \( q_1, p_1, q_2, \) and \( p_2 \) into complex coordinates which will play the role of annihilation and creation “operators” later on. We set

\[
z_k = \sqrt{\frac{mν_k}{2}} q_k + i \sqrt{\frac{mν_k}{2}} p_k, \\
\overline{z}_k = \sqrt{\frac{mν_k}{2}} q_k - i \sqrt{\frac{mν_k}{2}} p_k
\]

and hence

\[
q_k = \frac{1}{\sqrt{mν_k}} (z_k + \overline{z}_k), \\
p_k = \sqrt{\frac{mν_k}{2}} (z_k - \overline{z}_k)
\]

(4.3)

for \( k = 1, 2 \) and \( ν_l = ν, ν_2 = ν_κ \). With respect to these global coordinate functions on \( M \) the total Hamiltonian can be written as \( H = \frac{ν}{2} z_1^2 + \frac{16}{νν_κ} z_2^2 \). For the Poisson brackets one obtains \( \{z_k, z_l\} = 0 = \{z_k, \overline{z}_l\} \) and \( \{z_k, \overline{z}_l\} = \frac{2}{\sqrt{mν_k}} \delta_{kl} \) for all \( k, l = 1, 2 \). The Hamiltonian for the system will take a slightly more complicated form, namely

\[
H_S = \frac{ν}{4} z_1^2 + \frac{ν^2}{16ν_κ} (z_2 + \overline{z}_2)^2 - \frac{ν_κ}{16} (z_2 - \overline{z}_2)^2 + \frac{ν^2}{8\sqrt{νν_κ}} (z_1 + \overline{z}_1)(z_2 + \overline{z}_2) - \sqrt{\frac{νν_κ}{8}} (z_1 - \overline{z}_1)(z_2 - \overline{z}_2).
\]

(4.4)

On the other hand, we will also need “factorizing” complex coordinates with respect to the original Darboux coordinates on the system \( S \) and the bath \( B \). Hence we set

\[
z_S = \sqrt{mν q_S + i \sqrt{mν} p_S}, \quad \overline{z}_S = \sqrt{mν q_S - i \sqrt{mν} p_S}, \quad \text{and} \quad z_B = \sqrt{mν q_B + i \sqrt{mν} p_B}, \quad \overline{z}_B = \sqrt{mν q_B - i \sqrt{mν} p_B}.
\]

(4.5)

In these coordinates, the Hamiltonians of the system and the bath are given by \( H_S = \frac{ν}{2} z_S^2 \) and \( H_B = \frac{ν}{2} z_B^2 \). The interaction term now reads \( H_I = \frac{κ}{mν_S} (z_S + \overline{z}_S - z_B - \overline{z}_B)^2 \). Again, for the Poisson brackets one finds \( \{z_k, z_l\} = 0 = \{z_k, \overline{z}_l\} \) and \( \{z_k, \overline{z}_l\} \) enters into the Poisson brackets only for \( k, l = s, b \).

After these preparations, we can specify the star product on the total algebra of observables. We take the Weyl-Moyal star product on the total system \( S \times B \) defined by

\[
f \starWeyl g = \sum_{r=0}^{∞} \sum_{l=0}^{∞} \left( \frac{-i}{2} \right)^{r-l} \frac{r!}{l!(r-l)!} \frac{h^r}{i^{l_1} \cdots i^{l_r}} \sum_{i_1, \ldots, i_r=1}^{2} \frac{\partial^r f}{\partial q_{i_1} \cdots \partial q_{i_r} \partial p_{i_{r+1}} \cdots \partial p_{i_r}} \frac{\partial^r g}{\partial p_{i_1} \cdots \partial p_{i_r} \partial q_{i_{r+1}} \cdots \partial q_{i_r}}
\]

(4.6)

for \( f, g \in C^∞(S \times B)[[h]], \) see e.g. [3] and [24, Chap. 5].

**Remark 4.1** The Weyl-Moyal star product on a flat symplectic phase space \( \mathbb{R}^{2n} \) is uniquely determined by the requirement of invariance under the affine symplectic group. Under the usual quantization map into differential operators it corresponds to the total symmetrization, see e.g. [24, Chap. 5] for a detailed discussion. We also note that \( \starWeyl = \starWeyl_S \otimes \starWeyl_B \) as required by our general framework.
While the Weyl-Moyal star product is the most natural one with respect to phase space symmetries, it has certain technical disadvantages: when dealing with harmonic oscillators, for technical reasons it will be more convenient to employ a Wick star product. Such a Wick star product is no longer unique, but depends on the choice of a compatible linear complex structure on the phase space which is nothing but the choice of a harmonic oscillator. Therefore, we will have different Wick star products adapted to the various harmonic oscillators on hand: either with or without the coupling. In detail, one passes from the Weyl-Moyal star product to the Wick star product by means of an equivalence transformation explicitly given by

$$S = \exp (\hbar \Delta) \quad \text{with} \quad \Delta = \sum_{k=1}^{2} \frac{\partial^2}{\partial z_k \partial \bar{z}_k}.$$  \hfill (4.7)

Then the Wick star product $\star_{\text{Wick}}$ is defined by

$$f \star_{\text{Wick}} g = S(S^{-1}(f) \star_{\text{Weyl}} S^{-1}(g)).$$  \hfill (4.8)

for $f, g \in C^\infty(S \times B)[[\hbar]]$. Alternatively, we ignore the coupling term and use the complex coordinates $z_S, \bar{z}_S$ for the system and $z_B, \bar{z}_B$ for the bath. This gives the two equivalence transformations

$$S_S = \exp \left( \hbar \frac{\partial^2}{\partial z_S \partial \bar{z}_S} \right) \quad \text{and} \quad S_B = \exp \left( \hbar \frac{\partial^2}{\partial z_B \partial \bar{z}_B} \right),$$  \hfill (4.9)

acting on functions on S and B, respectively. Analogously to (4.8) we get Wick star products $\star_{\text{Wick}}^S$ and $\star_{\text{Wick}}^B$ for the system and the bath, respectively. Since we ignored the coupling terms in the definition of the latter two Wick star products, we have

$$S \neq S_S \otimes S_B \quad \text{and} \quad \star_{\text{Wick}} \neq \star_{\text{Wick}}^S \otimes \star_{\text{Wick}}^B.$$  \hfill (4.10)

The total time evolution with respect to $\star$ and $H$ can actually be calculated in a much easier way than by solving the corresponding evolution equation (3.7): we first compute the time evolution with respect to the Wick star product $\star_{\text{Wick}}$, which turns out to be simple, and then transform the time evolved observables back using $S$.

The total time evolution $A_t^\text{Wick}$ with respect to the Wick star product is determined by

$$\frac{d}{dt} A_t^\text{Wick} f = \frac{i}{\hbar} [H, A_t^\text{Wick} f]_{\star_{\text{Wick}}} = \{A_t^\text{Wick} f, H\}$$  \hfill (4.11)

for $f \in C^\infty(S \times B)[[\hbar]]$ due to the fact that $H = \frac{\nu}{2} z_1^2 + \frac{\kappa}{2} z_2^2$. It immediately follows that the time evolution is just the classical one, i.e. $A_t^\text{Wick} = \Phi_t^*$, and no higher order correction terms arise. But then it is clear that the time evolution with respect to $\star$ is given by conjugation with $S$ since $SH = H + c$ with a constant $c = \hbar \frac{\nu + \kappa}{2}$. Hence, we have

$$A_t = S^{-1} \circ \Phi_t^* \circ S.$$  \hfill (4.12)

As a consequence we immediately obtain the following result for the open time evolution with respect to the Weyl-Moyal star product:

**Proposition 4.2** The deformed time evolution of the open subsystem with respect to the functional $\omega$ is given by

$$A_t^\omega = (\text{id} \otimes \omega) \circ S^{-1} \circ \Phi_t^* \circ S \circ \text{pr}_S^*.$$  \hfill (4.13)
Remark 4.3 The *-automorphism $A_t$ obviously restricts to the polynomials $\text{Pol}(S \times B)[\hbar]$. Thus, being only interested in polynomial observables leads to a convergent formulation of the deformed time evolution of the open harmonic oscillator if the quantized state $\omega$ used to reduce the total dynamics gives a finite order in $\hbar$ for every polynomial on the bath. This will be the case for the deformed $\delta$-functionals as well as for the KMS functionals in Section 5. Thence, here we recover the usual quantum mechanical formulation including the convergence in $\hbar$.

To further illustrate the above situation we compute the open time evolution of some specific observables of the system. Here we still allow for a general state $\rho$.

As a first step we calculate the total quantum time evolutions of the total system for $q_s$ and $p_s$. To do so, we will have to evaluate the chain of maps (4.13) applied to these observables. First we note that

$$S q_s = q_s = S^{-1} q_s \quad \text{and} \quad S p_s = p_s = S^{-1} p_s. \quad (4.14)$$

Then the classical time evolution is linear whence applying the transformation $S^{-1}$ again does not give additional terms. We conclude that

$$A_t q_s = \Phi_t^* q_s \quad \text{and} \quad A_t p_s = \Phi_t^* p_s. \quad (4.15)$$

For the Hamiltonian $H_s$ of the system the calculation is slightly more complicated: First we note that applying $S$ yields an additional constant, namely

$$S H_s = H_s + \frac{\hbar}{4} \left( \nu + \nu_\kappa - \frac{\kappa}{m \nu_\kappa} \right). \quad (4.16)$$

Now the total classical time evolution of $H_s$ is quite complicated and can be computed most easily from $\Phi_t^* z_1 = \exp(-i \nu t) z_1$ and $\Phi_t^* z_2 = \exp(-i \nu_\kappa t) z_2$ and (4.3). The remarkable fact is now that

$$\Delta \Phi_t^* H_s = \Delta H_s = \frac{1}{4} \left( \nu + \nu_\kappa - \frac{\kappa}{m \nu_\kappa} \right) \quad (4.17)$$

for all $t$. Thus applying $S^{-1}$ to $\Phi_t^* H_s$ gives $\Phi_t^* H_s$ minus the same constant as we obtained in (4.16). We conclude that also for the Weyl star product

$$A_t H_s = \Phi_t^* H_s. \quad (4.18)$$

Replacing the complex coordinates and their (simple) time evolution by the original real coordinates we get the explicit total classical and hence also quantum time evolutions for $q_s$, $p_s$, and $H_s$

$$A_t q_s = \Phi_t^* q_s = \frac{1}{2} \left( \cos(\nu t) + \cos(\nu_\kappa t) \right) q_s + \left( \frac{\sin(\nu t)}{2 \nu} + \frac{\sin(\nu_\kappa t)}{2 \nu_\kappa} \right) p_s$$

$$\quad + \frac{1}{2} \left( \cos(\nu t) - \cos(\nu_\kappa t) \right) q_B + \left( \frac{\sin(\nu t)}{2 \nu} - \frac{\sin(\nu_\kappa t)}{2 \nu_\kappa} \right) p_B, \quad (4.19)$$

$$A_t p_s = \Phi_t^* p_s = -\frac{m}{2} \left( \nu \sin(\nu t) + \nu_\kappa \sin(\nu_\kappa t) \right) q_s + \frac{1}{2} \left( \cos(\nu t) + \cos(\nu_\kappa t) \right) p_s$$

$$\quad - \frac{m}{2} \left( \nu \cos(\nu t) - \nu_\kappa \cos(\nu_\kappa t) \right) q_B + \frac{1}{2} \left( \cos(\nu t) - \cos(\nu_\kappa t) \right) p_B, \quad (4.20)$$

and

$$A_t H_s = \Phi_t^* H_s = \left( \frac{m}{8} (\nu \sin(\nu t) + \nu_\kappa \sin(\nu_\kappa t))^2 + \frac{m^2 \nu^2}{8} (\cos(\nu t) + \cos(\nu_\kappa t))^2 \right) q_s^2$$

15
+ \left( \frac{1}{8\sin} (\cos(\nu t) + \cos(\nu_r t))^2 \right. \\
+ \left. \frac{m\nu^2}{8} \left( \frac{\sin(\nu t)}{\nu} + \frac{\sin(\nu_r t)}{\nu_r} \right)^2 \right) \frac{p_s^2}{2} \\
+ \left( \frac{m}{8} (\nu \sin(\nu t) - \nu_r \sin(\nu_r t))^2 \right. \\
+ \left. \frac{m\nu^2}{8} (\cos(\nu t) - \cos(\nu_r t))^2 \right) \frac{q_s^2}{2} \\
+ \left( \frac{1}{8\sin} (\cos(\nu t) - \cos(\nu_r t))^2 \right. \\
+ \left. \frac{m\nu^2}{8} \left( \frac{\sin(\nu t)}{\nu} - \frac{\sin(\nu_r t)}{\nu_r} \right)^2 \right) \frac{p_r^2}{2} \\
+ \left( -\frac{1}{4} \nu (\sin(\nu t) + \nu_r \sin(\nu_r t)) \right. \\
+ \left. \frac{m\nu^2}{4} \left( \frac{\sin(\nu t)}{\nu} + \frac{\sin(\nu_r t)}{\nu_r} \right) \right) \frac{q_s q_p}{2} \\
+ \left( -\frac{1}{4} \nu (\sin(\nu t) + \nu_r \sin(\nu_r t)) \right. \\
+ \left. \frac{m\nu^2}{4} \left( \frac{\sin(\nu t)}{\nu} + \frac{\sin(\nu_r t)}{\nu_r} \right) \right) \frac{q_s q_p}{2} \\
+ \left( -\frac{1}{4} \nu (\sin(\nu t) + \nu_r \sin(\nu_r t)) \right. \\
+ \left. \frac{m\nu^2}{4} \left( \frac{\sin(\nu t)}{\nu} + \frac{\sin(\nu_r t)}{\nu_r} \right) \right) \frac{q_s p_r}{2} \\
+ \left( -\frac{1}{4} \nu (\sin(\nu t) + \nu_r \sin(\nu_r t)) \right. \\
+ \left. \frac{m\nu^2}{4} \left( \frac{\sin(\nu t)}{\nu} + \frac{\sin(\nu_r t)}{\nu_r} \right) \right) \frac{q_s p_r}{2} \\
+ \left( -\frac{1}{4} \nu (\sin(\nu t) + \nu_r \sin(\nu_r t)) \right. \\
+ \left. \frac{m\nu^2}{4} \left( \frac{\sin(\nu t)}{\nu} + \frac{\sin(\nu_r t)}{\nu_r} \right) \right) \frac{q_s q_p}{2}.

The reason for transforming the time evolved observables back to the Darboux coordinate functions \( q_s, p_s, q_p, \) and \( p_p \) is not just an addiction to extensive exercise: It is in these variables where we can apply the final map \( \hat{\Theta} \otimes \omega \) needed for the open time evolution, where \( \omega \) is a state of the bath with respect to \( \hat{\Theta}_b \). The procedure is very simple: we will have to replace all bath variables by their expectation values with respect to \( \omega \), i.e. \( q_b \) is to be replaced by \( \omega(q_b) \), \( q_b p_b \) replaced by \( \omega(p_b p_b) \) et cetera. We will not write down the explicit formulas as these are now obtained from (4.19), (4.20), and (4.21) just by copying.

Remark 4.4 Note that for these observables, the open time evolutions in the classical and quantum regime only differ by the (possibly) different expectation values with respect to \( \omega \) and its classical limit \( \omega_0 \). In general, we have to expect additional quantum corrections from the total time evolution as well.
5 Linearly Coupled Harmonic Oscillators II: Examples

The first example of a state for the bath is a deformation of the $\delta$-functional. Thus, fix a point $(q_{B0}, p_{B0})$ in the bath and consider $\delta_{(q_{B0}, p_{B0})}$. For the Weyl-Moyal star product this will no longer be a positive functional, see e.g. [24, Sect. 7.1.3]. However, for the Wick star product $\star_{W}^{\text{Wick}}$ on the bath the $\delta$-functional will be positive without corrections. Thus using the equivalence transformation $S_{B}$ we obtain a positive functional $\delta_{(q_{B0}, p_{B0})} \circ S_{B}$ with respect to the Weyl-Moyal star product. Note that the equivalence transformation $S_{B}$ is precisely a map preserving squares with respect to the Weyl-Moyal star product which is evident from the explicit formula for $\star_{W}^{\text{Wick}}$. In fact, this was the first example of a map preserving squares which is also heavily used in the proofs in [10]. More physically speaking, $\delta_{(q_{B0}, p_{B0})} \circ S_{B}$ corresponds to a coherent state localized around the point $(q_{B0}, p_{B0})$.

For this particular state we note that for the observables at most linear in $q_{B}$ and $p_{B}$ the operator $S_{B}$ does not have a non-trivial effect. Moreover, for the quadratic terms $q_{B}^{2}$, $p_{B}^{2}$, and $q_{B} p_{B}$ the operator $S_{B}$ only gives a correction term in first order of $\hbar$. Explicitly, we obtain

$$S_{B} q_{B} = q_{B}, \quad S_{B} p_{B} = p_{B}, \quad S_{B} (q_{B} p_{B}) = q_{B} p_{B},$$

(5.1)

$$S_{B} q_{B}^{2} = q_{B}^{2} + \frac{\hbar}{2} \frac{1}{m \nu}, \quad \text{and} \quad S_{B} p_{B}^{2} = p_{B}^{2} + \hbar \frac{m \nu}{2}.$$  

(5.2)

From these computations we see that the open time evolutions with respect to $\delta_{(q_{B0}, p_{B0})} \circ S_{B}$ are given by

$$A_{t}^{\delta_{(q_{B0}, p_{B0})} \circ S_{B}} (q_{S}) = \left( \Phi_{t}^{\delta_{(q_{B0}, p_{B0})}} \right)^{*} q_{S},$$

(5.3)

$$A_{t}^{\delta_{(q_{B0}, p_{B0})} \circ S_{B}} (p_{S}) = \left( \Phi_{t}^{\delta_{(q_{B0}, p_{B0})}} \right)^{*} p_{S},$$

(5.4)

$$A_{t}^{\delta_{(q_{B0}, p_{B0})} \circ S_{B}} H_{S} = \left( \Phi_{t}^{\delta_{(q_{B0}, p_{B0})}} \right)^{*} H_{S} + \frac{\hbar}{16} \left( \frac{1}{\nu} (\nu \sin(\nu t) - \nu \kappa \sin(\nu \kappa t)) + \frac{\nu}{\kappa} \sin(\nu \kappa t) \right)^{2} + 2 \nu (\cos(\nu t) - \cos(\nu \kappa t)).$$

(5.5)

**Remark 5.1** The classical open time evolutions of $q_{S}$, $p_{S}$, and $H_{S}$ in (5.3), (5.4), and (5.5) are obtained by replacing the functions $q_{B}$, $p_{B}$, and their powers in the Equations (4.19), (4.20), and (4.21) by their values at $q_{B0}$ and $p_{B0}$.

**Remark 5.2** The deformation of the $\delta$-functional necessary in order to ensure complete positivity leads to non-classical components of the open time evolution.

Next we will study quantized states fulfilling a formal KMS condition, corresponding to “thermal equilibrium states” of the bath.

To this end, we first recall that for every symplectic star product $\star$ for $C^{\infty}(M)[[\hbar]]$ there is a unique trace functional

$$\text{tr} : C^{\infty}_{0}(M)[[\hbar]] \rightarrow \mathbb{C}[[\hbar]],$$

(5.6)
Depending on the Hamiltonian \( H \), the Hamiltonian used permits a normalization by rendering the integrations in \([1, 2]\) in the context of deformation quantization, leads to the following result: up to we choose a map \( \mu \).

**Proof:**

We choose a map \( \mu \) such that

\[
\mu_{KMS}(f) = \text{tr}(\text{Exp}(-\beta H) \star f)
\]

for \( f \in C^\infty_0(M)[[\hbar]] \), where \( \text{Exp}(\beta H) \in C^\infty_0(M)[[\hbar]] \) is obtained by

\[
\frac{d}{d\beta} \text{Exp}(\beta H) = H \star \text{Exp}(\beta H)
\]

with initial condition \( \text{Exp}(0) = 1 \). The classical limit of \( \text{Exp}(H) \) is the ordinary exponential \( \exp(H_0) \).

The KMS condition for inverse temperature \( \beta \) and Hamiltonian \( H \) for a \( \mathbb{C}[[\hbar]] \)-linear functional as formulated in \([1, 2]\) in the context of deformation quantization, leads to the following result: up to normalization the KMS functional is uniquely determined and explicitly given by

\[
\mu_{KMS}(f) = \text{tr}(\text{Exp}(-\beta H) \star f)
\]

for \( f \in C^\infty_0(M)[[\hbar]] \), see \([5]\) for the proof and \([24, \text{Sect. 7.1.4}]\) for more details on KMS functionals. In particular, we note that \((5.9)\) is a positive functional.

**Remark 5.3** Depending on the Hamiltonian \( H \), \( \mu_{KMS} \) may or may not be normalizable. Whenever the Hamiltonian used permits a normalization by rendering the integrations in \( \mu_{KMS}(1) \) well-defined, we will denote \( \frac{1}{\mu_{KMS}(1)} \mu_{KMS} \) by \( \omega_{KMS} \) and call it a KMS state.

Before entering the particular example again, we note that in the symplectic case the open quantum time evolution with respect to a KMS functional is necessarily completely positive. This will follow at once from this proposition:

**Proposition 5.4** Let the system be an arbitrary Poisson manifold and let the bath be symplectic. Given the KMS functional \( \mu_{KMS} \) with respect to an arbitrary \( H_B \in C^\infty(B)[[\hbar]] \) and inverse temperature \( \beta \), the map \( \text{id} \otimes \mu_{KMS} : (C^\infty_0(S \times B)[[\hbar]], \star) \to (C^\infty_0(S)[[\hbar]], \star_s) \) is completely positive.

**Proof:** We choose a map \( S \) for the bath whose existence is guaranteed by Theorem 3.13. In the proof of Theorem 3.13 we have seen that for every positive \( \mathbb{C}[[\hbar]] \)-linear functional \( \mu : M_n(C^\infty_0(S)[[\hbar]]) \to \mathbb{C}[[\hbar]] \) the combined map

\[
\mu \otimes S : (M_n(C^\infty_0(S \times B)[[\hbar]]), \star) \to (C^\infty_0(B)[[\hbar]], \cdot)
\]

is positive. It follows that for \( F \in M_n(C^\infty_0(S \times B)[[\hbar]]) \) the function \( (\mu \otimes S)(F \star F) \) is at every point \( x_B \in B \) either a formal series with positive lowest order term or zero. To avoid trivialities, assume that \( (\mu \otimes S)(F \star F) \) is not identically zero. Let \( r_0 \) be the minimal exponent with \( (\mu \otimes S)(F \star F) = \hbar^{r_0}a_{r_0} + \cdots \) and \( a_{r_0} \geq 0 \) not identically zero. By continuity, there is an open subset \( U \subseteq B \) with \( a_{r_0}(x_B) > 0 \) for \( x_B \in U \). But this implies that \( (\mu_{KMS} \circ S^{-1}) \circ (\mu \otimes S)(F \star F) = \hbar^{r_0}b_{r_0} + \cdots \) with \( b_{r_0} > 0 \) since the zeroth order of \( S \) is the identity and the zeroth order of \( \mu_{KMS} \) is the integration over all of \( B \). Since \( \mu \) is arbitrary and using

\[
(\mu_{KMS} \circ S^{-1}) \circ (\mu \otimes S) = \mu \circ (\text{id} \otimes \mu_{KMS}),
\]

i.e. \( \text{tr}(f \star g) = \text{tr}(g \star f) \). Choosing the normalization of \( \text{tr} \) appropriately one obtains a positive trace, see e.g. \([24, \text{Sect. 6.3.5}]\) for a detailed discussion and references. For the Weyl-Moyal star product, the trace is known to be

\[
\text{tr}(f) = \int_{\mathbb{R}^n} f(x) \, d^{2n}x,
\]

i.e. the integration with respect to the Liouville volume. In fact, it can be shown that in the symplectic case the lowest order of \( \text{tr} \) is necessarily of this form: it is just the integration over the whole manifold with respect to the Liouville volume.

The second ingredient we need is the \(*\)-exponential \( \text{Exp} \), as introduced in \([3]\). Instead of defining the exponential function by means of the series, the following approach favoured in \([5]\), see also \([24, \text{Sect. 6.3.1}]\), will be used. For \( H \in C^\infty(M)[[\hbar]] \) one defines \( \text{Exp}(\beta H) \in C^\infty(M)[[\hbar]] \) to be the unique solution of the differential equation

\[
\frac{d}{d\beta} \text{Exp}(\beta H) = H \star \text{Exp}(\beta H)
\]

see \([5]\) for the proof and \([24, \text{Sect. 7.1.4}]\) for more details on KMS functionals. In particular, we note that \((5.9)\) is a positive functional.
this shows that \((\text{id} \otimes \mu_{\text{KMS}})(F^* \star F)\) is a positive algebra element in \(M_n(C^\infty(S)[[\hbar]])\) with respect to \(\star_\hbar\).

Back to our specific example, we consider the harmonic oscillator as the Hamiltonian \(H_\hbar \in C^\infty(\mathbb{R}^2)\) and the Weyl-Moyal star product \(\star_\hbar\) as before. In this case, the star exponential of \(H_\hbar\) has been computed explicitly by [3]. One has

\[
\text{Exp}(-\beta H_\hbar) = \frac{1}{\cosh \left( \frac{\hbar \beta \nu}{2} \right) \cosh \left( \frac{\hbar \beta \nu}{2} \right)} \exp \left( -\frac{2H_\hbar}{\hbar \nu} \tanh \left( \frac{\hbar \beta \nu}{2} \right) \right)
\]

for \(\beta > 0\) and \(\nu > 0\), which is a well-defined formal power series in \(\hbar\). Note that in [3] the exponential \(\text{Exp}(\frac{1}{\hbar} H)\) requires a convergent setting due to the \(\hbar\) in the denominator. In our case, the situation is much simpler. In fact, differentiating \((5.10)\) with respect to \(\beta\) gives the defining differential equation \((5.8)\) right away.

As in the textbooks on statistical mechanics, we can now calculate the partition function \(Z\) as the normalization factor of the KMS state on the bath by formally calculating Gaussian integrals.

**Proposition 5.5** The normalization factor \(\mu_{\text{KMS}}(1)\) is explicitly given by

\[
\mu_{\text{KMS}}(1) = 2\pi \hbar \frac{\exp \left( -\frac{\hbar \beta \nu}{2} \right)}{1 - \exp \left( -\hbar \beta \nu \right)} \in \mathbb{R}[[\hbar]].
\]

The partition function is the formal Laurent series

\[
Z = \frac{\exp \left( -\frac{\hbar \beta \nu}{2} \right)}{1 - \exp \left( -\hbar \beta \nu \right)} \in \mathbb{R}((\hbar)).
\]

The crucial point is that \(\mu_{\text{KMS}}(1)\) has a well-defined classical limit while \(Z\) has a simple pole at \(\hbar = 0\). Therefore, we can use this normalization factor to obtain the well-defined KMS state

\[
\omega_{\text{KMS}}(f) = \frac{1}{2\pi \hbar Z} \int \text{Exp}(-\beta H_\hbar) \star_\hbar f \, dq_\hbar \, dp_\hbar
\]

for \(f \in C^\infty(B)[[\hbar]]\) such that the integral \((5.13)\) is convergent order by order in \(\hbar\). Note that the inverse of \(2\pi \hbar Z\) is again a well-defined formal power series.

As for the \(\delta\)-functional, we shall now compute the open quantum time evolution of the observables \(q_\hbar, p_\hbar,\) and \(H_\hbar\) also with respect to the KMS state \(\omega_{\text{KMS}}\). To this end, we need the expectation values of \(q_\hbar, p_\hbar, q_\hbar^2, p_\hbar^2,\) and \(q_\hbar p_\hbar\) in order to evaluate \((4.19), (4.20),\) and \((4.21)\).

**Lemma 5.6** One has the following expectation values

\[
\omega_{\text{KMS}}(q_\hbar) = \omega_{\text{KMS}}(p_\hbar) = \omega_{\text{KMS}}(q_\hbar p_\hbar) = 0,
\]

\[
\omega_{\text{KMS}}(q_\hbar^2) = \frac{3\hbar}{2m \nu \tanh \left( \frac{\hbar \beta \nu}{2} \right)}, \quad \text{and} \quad \omega_{\text{KMS}}(p_\hbar^2) = \frac{3m \nu \hbar}{2 \tanh \left( \frac{\hbar \beta \nu}{2} \right)}.
\]

which are formal power series in \(\mathbb{C}[[\hbar]]\).

**Proof:** This is of course textbook knowledge. Nevertheless, we sketch the computation in order to illustrate the star product formalism used. The first observation is that the trace functional \(\text{tr}\) for the Weyl-Moyal star product has the remarkable feature

\[
\text{tr}(f \star g) = \text{tr}(fg),
\]
see e.g. [24, Ex. 6.3.33]. Strictly speaking, one of the functions has to have compact support. However, if one is the Gaussian $\text{Exp}(-\beta H_0)$ then the rapid decay allows to perform the integrations by parts also for observables like polynomials. Thus we can use this feature to simplify $\omega_{\text{KMS}}(f)$ considerably for the above observables. Since $\text{Exp}(-\beta H_0)$ is just a Gaussian we are left with the well-known computation of some Gaussian integrals.

Using these expectation values, we can apply the general formulas (4.19), (4.20), and (4.21) and substitute there the observables $q_0$, $p_0$, $q_0p_0$, and $p_0^2$ by their expectation values with respect to $\omega_{\text{KMS}}$. This then gives the open time evolutions of $q_s$, $p_s$ and $H_s$. Remarkably, many terms disappear thanks to the vanishing of (5.14). In detail, we have

\[
A_t^{\text{KMS}} q_s = \frac{1}{2} \left( \cos(\nu t) + \cos(\nu_\kappa t) \right) q_s + \frac{1}{2} \left( \frac{\sin(\nu t)}{m\nu} + \frac{\sin(\nu_\kappa t)}{m\nu_\kappa} \right) p_s,
\]

(5.16)

\[
A_t^{\text{KMS}} p_s = -\frac{m}{2} \left( \nu \sin(\nu t) + \nu_\kappa \sin(\nu_\kappa t) \right) q_s + \frac{1}{2} \left( \cos(\nu t) + \cos(\nu_\kappa t) \right) p_s,
\]

(5.17)

\[
A_t^{\text{KMS}} H_s = \left( \frac{m}{8} \left( \nu \sin(\nu t) + \nu_\kappa \sin(\nu_\kappa t) \right)^2 + \frac{mv^2}{8} \left( \cos(\nu t) + \cos(\nu_\kappa t) \right)^2 \right) q_s^2
\]

\[+ \left( \frac{1}{8m} \left( \sin(\nu t) + \sin(\nu_\kappa t) \right)^2 + \frac{mv^2}{8} \left( \frac{\sin(\nu t)}{m\nu} + \frac{\sin(\nu_\kappa t)}{m\nu_\kappa} \right)^2 \right) p_s^2
\]

\[+ \left( -\frac{1}{4} \left( \nu \sin(\nu t) + \nu_\kappa \sin(\nu_\kappa t) \right) \left( \cos(\nu t) + \cos(\nu_\kappa t) \right) \right)
\]

\[+ \frac{mv^2}{4} \left( \frac{\sin(\nu t)}{m\nu} + \frac{\sin(\nu_\kappa t)}{m\nu_\kappa} \right) \left( \cos(\nu t) + \cos(\nu_\kappa t) \right) \right) q_s p_s
\]

\[+ \left( \frac{1}{\nu} \left( \nu \sin(\nu t) - \nu_\kappa \sin(\nu_\kappa t) \right)^2 + 2\nu \left( \nu t - \cos(\nu_\kappa t) \right)^2
\]

\[+ \nu \left( \sin(\nu t) - \frac{\nu}{\nu_\kappa} \sin(\nu_\kappa t) \right)^2 \right) \frac{3\hbar}{16 \tanh \left( \frac{\hbar\nu}{2} \right)}.
\]

(5.18)

References

[1] Basart, H., Flato, M., Lichnerowicz, A., Sternheimer, D.: Deformation Theory applied to Quantization and Statistical Mechanics. Lett. Math. Phys. 8 (1984), 483–494.

[2] Basart, H., Lichnerowicz, A.: Conformal Symplectic Geometry, Deformations, Rigidity and Geometrical (KMS) Conditions. Lett. Math. Phys. 10 (1985), 167–177.

[3] Bayen, F., Flato, M., Frønsdal, C., Lichnerowicz, A., Sternheimer, D.: Deformation Theory and Quantization. Ann. Phys. 111 (1978), 61–151.

[4] Bordemann, M.: The deformation quantization of certain super-Poisson brackets and BRST cohomology. In: Dito, G., Sternheimer, D. (Eds.): Conférence Moshé Flato 1999. Quantization, Deformations, and Symmetries. Mathematical Physics Studies no. 22, 45–68. Kluwer Academic Publishers, Dordrecht, Boston, London, 2000.

[5] Bordemann, M., Römer, H., Waldmann, S.: A Remark on Formal KMS States in Deformation Quantization. Lett. Math. Phys. 45 (1998), 49–61.

[6] Breuer, H. P., Petruccione, F.: Concepts and Methods in the Theory of Open Quantum Systems. In: Benatti, F., Floreanini, R. (Eds.): Irreversible Quantum Dynamics, vol. 622 in Lecture Notes in Physics, 65–79. Springer-Verlag, Berlin, 2003. [quant-ph/0302047].

[7] Brittin, W. E.: A Note on the Quantization of Dissipative Systems. Physical Review 77.3 (1950), 396–397.

[8] Bursztyn, H., Waldmann, S.: Algebraic Rieffel Induction, Formal Morita Equivalence and Applications to Deformation Quantization. J. Geom. Phys. 37 (2001), 307–364.
[9] Bursztyn, H., Waldmann, S.: \emph{Completely positive inner products and strong Morita equivalence}. Pacific J. Math. \textbf{222} (2005), 201–236.

[10] Bursztyn, H., Waldmann, S.: \emph{Hermitian star products are completely positive deformations}. Lett. Math. Phys. \textbf{72} (2005), 143–152.

[11] Dekker, H.: \emph{On the Quantization of Dissipative Systems in the Lagrange-Hamilton Formalism}. Zeitschrift für Physik B \textbf{21} (1975), 295–300.

[12] DeWilde, M., Lecomte, P. B. A.: \emph{Existence of Star-Products and of Formal Deformations of the Poisson Lie Algebra of Arbitrary Symplectic Manifolds}. Lett. Math. Phys. \textbf{7} (1983), 487–496.

[13] Dito, G., Léandre, R.: \emph{Stochastic Moyal product on the Wiener space}. J. Math. Phys. \textbf{48} (2007), 023509.

[14] Dito, G., Turrubiates, F. J.: \emph{The damped harmonic oscillator in deformation quantization}. Phys.Lett. \textbf{A352} (2006), 309–316.

[15] Eckel, R.: \emph{Quantisierung von Supermannigfaltigkeiten à la Fedosov}. PhD thesis, Fakultät für Physik, Albert-Ludwigs-Universität, Freiburg, September 2000.

[16] Fedosov, B. V.: \emph{Quantization and the Index}. Sov. Phys. Dokl. \textbf{31}.11 (1986), 877–878.

[17] Kaschek, D., Neumaier, N., Waldmann, S.: \emph{Complete Positivity of Rieffel’s Deformation Quantization}. J. Noncommut. Geom. \textbf{3} (2009), 361–375.

[18] Kontsevich, M.: \emph{Deformation Quantization of Poisson manifolds}. Lett. Math. Phys. \textbf{66} (2003), 157–216.

[19] Omori, H., Maeda, Y., Yoshioka, A.: \emph{Weyl Manifolds and Deformation Quantization}. Adv. Math. \textbf{85} (1991), 224–255.

[20] Razavy, M.: \emph{On the Quantization of Dissipative Systems}. Zeitschrift für Physik B \textbf{26} (1977), 201–206.

[21] Rudin, W.: \emph{Real and Complex Analysis}. McGraw-Hill Book Company, New York, 3. edition, 1987.

[22] Schmüdgen, K.: \emph{Unbounded Operator Algebras and Representation Theory}, vol. 37 in \emph{Operator Theory: Advances and Applications}. Birkhäuser Verlag, Basel, Boston, Berlin, 1990.

[23] Waldmann, S.: \emph{States and Representation Theory in Deformation Quantization}. Rev. Math. Phys. \textbf{17} (2005), 15–75.

[24] Waldmann, S.: \emph{Poisson-Geometrie und Deformationsquantisierung. Eine Einführung}. Springer-Verlag, Heidelberg, Berlin, New York, 2007.