Some remarks regarding $l$-elements defined in algebras obtained by the Cayley-Dickson process

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Abstract. In this paper, we define a special class of elements in the algebras obtained by the Cayley-Dickson process, called $l$-elements. We find conditions such that these elements to be invertible. These conditions can be very useful for finding new identities, identities which can help us in the study of the properties of these algebras.

Key Words: quaternion algebras; octonion algebras; Cayley-Dickson algebras; special sequences.

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1. Introduction

In the last years, many papers were devoted to the study of some special sequences or special elements, especially in their connections with particular cases of algebras obtained by the Cayley-Dickson (quaternions, octonions). Some of them are: [Ba, Pr; 09], [Ca; 16], [Ci, Ip; 16], [Fa, Pl; 07(1)], [Fa, Pl; 07(2)], [Fl; 09], [Fl, Sa; 15(1)], [Fl, St; 09], [Ha, 15], [Mo; 17], [Si,18].

Since the algebras obtained by the Cayley-Dickson process, denoted $A_l$, are poor in properties when their dimension increase, losing commutativity, associativity and alternativity, in this paper, we will find elements with supplementary properties, namely a special set of invertible elements in $A_l$, which can help us in the study of these algebras. Algebras obtained by the Cayley-Dickson process will be presented in the following.

Let $K$ be a commutative field with $\text{char}(K) \neq 2$. Let $A$ be a finite dimensional unitary algebra over a field $K$. We define a scalar involution

$$\overline{\cdot}: A \to A, a \to \overline{a},$$

to be a linear map with the following properties

$$\overline{ab} = \overline{b} \overline{a}, \overline{a} = a,$$

and

$$a + \overline{a}, a\overline{a} \in K \cdot 1 \text{ for all } a, b \in A.$$
An element $\overline{a} \in A$ is called the **conjugate** of the element $a \in A$, the linear form
$$t : A \to K, \ t (a) = a + \overline{a}$$
and the quadratic form
$$n : A \to K, \ n (a) = a\overline{a}$$
are called the **trace** and the **norm** of the element $a$.

We consider $\gamma \in K$ a fixed non-zero element. We define the following algebra multiplication on the vector space $A \oplus A$:
$$(a_1, a_2) (b_1, b_2) = (a_1b_1 + \gamma b_2a_2, a_2b_1 + b_2a_1).$$  \hfill (1.1.)

The obtained algebra structure over $A \oplus A$, denoted by $(A, \gamma)$, is called the **algebra obtained from** $A$ **by the Cayley-Dickson process**. We have $\dim (A, \gamma) = 2 \dim A$.

Let $x \in (A, \gamma), x = (a_1, a_2)$. The map
$$\overline{\cdot} : (A, \gamma) \to (A, \gamma), \ x \to \overline{x} = (\overline{a}_1, -a_2),$$
is a scalar involution of the algebra $(A, \gamma)$, extending the involution $\overline{\cdot}$ of the algebra $A$. We take
$$t (x) = t(a_1)$$
and
$$n (x) = n(a_1) - \gamma n(a_2)$$
to be the **trace** and the **norm** of the element $x \in (A, \gamma)$.

If we consider $A = K$ and we apply this process $t$ times, $t \geq 1$, we obtain an algebra over $K$, denoted by
$$A_t = \left( \frac{\gamma_1, \ldots, \gamma_t}{K} \right).$$  \hfill (1.2.)

Using induction, in this algebra, the set $\{1, e_2, \ldots, e_n\}, n = 2^t$, generates a basis with the properties:
$$e_i^2 = \gamma_i 1, \ i \in K, \gamma_i \neq 0, \ i = 2, \ldots, n$$  \hfill (1.3.)
and
$$e_i e_j = -e_j e_i = \beta_{ij} e_k, \ \beta_{ij} \in K, \ \beta_{ij} \neq 0, i \neq j, i, j = 2, \ldots n, \quad (1.4.)$$
$\beta_{ij}$ and $e_k$ being uniquely determined by $e_i$ and $e_j$. We remark that if $x \in A_t$ and $n (x) \neq 0$, then $x$ is an invertible element in $A_t$.

A finite-dimensional algebra $A$ is a **division** algebra if and only if $A$ does not contain zero divisors. For other details regarding algebras obtained by the Cayley-Dickson process, the reader is referred to [Sc:66].
For $t = 2$, we obtain the generalized quaternion algebras, denoted $\mathbb{H}(\gamma_1, \gamma_2)$, and for $t = 3$, we obtain the generalized octonion algebras, denoted $\mathbb{O}(\gamma_1, \gamma_2, \gamma_3)$.

For example, for $x \in \mathbb{H}(\gamma_1, \gamma_2)$, $x = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3$, its norm is

$$n(x) = x\overline{x} = x_0^2 - \gamma_1 x_1^2 - \gamma_2 x_2^2 + \gamma_1 \gamma_2 x_3^2 \in K.$$  \hspace{1cm} (1.5.)

If, for $x \in \mathcal{A}$, where $\mathcal{A} = \{ \mathbb{H}(\gamma_1, \gamma_2), \mathbb{O}(\gamma_1, \gamma_2, \gamma_3) \}$, the relation $n(x) = 0$ implies $x = 0$, then these algebras are division algebras. A quaternion and an octonion non-division algebras are called split algebras. Using the above notations, we remark that the quaternion algebra $\mathbb{H}(-1,-1) = \left(\frac{\sqrt{-1}, -1}{\mathbb{R}}\right)$ and the octonion algebra $\mathbb{O}(-1,-1,-1) = \left(\frac{\sqrt{-1}, -1, -1}{\mathbb{R}}\right)$ are division algebras. If the field $K$ is a finite field, then the quaternion and octonion algebras are always split algebras.

An algebra $A \in \mathcal{A}$ such that $n(xy) = n(x)n(y)$, for all $x, y \in A$, is called a composition algebra. For $t \geq 4$, the algebras $A_t$ are not composition algebras.

A unitary algebra $A \neq K$ such that we have $x^2 + \alpha_x x + \beta_x = 0$, for each $x \in A$, with $\alpha_x, \beta_x \in K$, is called a quadratic algebra. An algebra $A$ with a scalar involution is quadratic. From here, we get that the algebras obtained by the Cayley-Dickson process are quadratic.

An algebra $A$ is called alternative if $x^2 y = x(xy)$ and $xy^2 = (xy)y$, for all $x, y \in A$, flexible if $x(yx) = (xy)x = xyx$, for all $x, y \in A$ and power associative if the subalgebra $\langle x \rangle$ of $A$, generated by any element $x \in A$, is associative.

Algebras $A_t$ are noncommutative, for $t \geq 2$, nonassociative, for $t \geq 3$, non-alternative, for $t \geq 4$ and, in general, non division algebras. They are power associative and flexible for all $t$.

2. $l$-numbers and some of their properties

In [Ho; 61] and [Fl, Sa; 17], were defined and studied the $(a, b, x_0, x_1)$ numbers. If we consider $l$ a nonzero natural number, for $a = l, b = 1, x_0 = 0, x_1 = 1$, in [Sa; 18] was considered the following sequence, where

$$p_n = l \cdot p_{n-1} + p_{n-2}, \ n \geq 2, p_0 = 0, p_1 = 1.$$  \hspace{1cm} (2.1.)

Since these numbers are $(l, 1, 0, 1)$ numbers, we will call them $l$-numbers. For $l = 1$, it is obtained the Fibonacci numbers and for $l = 2$, it is obtained the Pell numbers.

**Remark 2.1.** ([Sa; 18]). Let $(p_n)_{n \geq 0}$ be the sequence previously defined. Then, the following relations are true:

1) $$p_n^2 + p_{n+1}^2 = p_{2n+1}, \ (\forall) \ n \in \mathbb{N}.$$
2) For \( \alpha = \frac{1 + \sqrt{5}}{2} \) and \( \beta = \frac{1 - \sqrt{5}}{2} \), we obtain that

\[
\begin{align*}
p_n &= \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^n - \beta^n}{\sqrt{l^2 + 4}}, \quad (\forall) n \in \mathbb{N},
\end{align*}
\]

called the Binet’s formula for the sequence \((p_n)_{n\geq 0}\).

**Proposition 2.2.** With the above notations, the following relations hold:
1) If \( d \mid n \), then \( p_d \mid p_n \).
2) \( p_m + p_n = p_{m+n+1} + p_{m-1}p_n \).

**Proof.**
1) We use the Binet formula from the above remark. If \( d \mid n, n = dc \),

\[
p_n = \frac{\alpha^{dc} - \beta^{dc}}{\sqrt{l^2 + 4}} = \frac{(\alpha^d)^c - (\beta^d)^c}{\sqrt{l^2 + 4}} = \frac{(\alpha^d - \beta^d)M}{\sqrt{l^2 + 4}},
\]

therefore \( p_d \mid p_n \).

2) We use induction after \( n \). For \( n = 0 \), we have \( p_m = p_m \). Supposing affirmation true for each \( k \leq n \), we will prove for \( n + 1 \). We have

\[
P_{m+n+1} = lP_{m+n} + P_{m+n-1} = l (P_{m}P_{n+1} + P_{m-1}p_n) +
+ P_{m}P_{n+1} + P_{m-1}p_n = P_{m} (lP_{n+1} + p_n) + P_{m-1} (lp_{n} + p_{n-1}) =
= P_{m}P_{n+2} + P_{m-1}p_{n+1}.
\]

**Proposition 2.3.** Let \((p_n)_{n\geq 0}\) be the \( l \)-sequence defined in (2.1). Then, the following relations are true:

i) \( p_n + p_{n+4} = \left(l^2 + 2\right) \cdot p_{n+2}; \quad (2.2.) \)

ii) \( p_n + p_{n+8} = \left(l^2 + 2\right)^2 - 2 \cdot p_{n+4}; \quad (2.3.) \)

For \( k \geq 3 \), we have

\[
\begin{align*}
p_n + p_{n+2^k} &= \left[ \left( \left( \left( l^2 + 2 \right)^2 - 2 \right)^2 - 2 \right)^2 - 2 \right] \cdot p_{n+2^{k-1}}. \quad (2.4.)
\end{align*}
\]

**Proof.**
1) With the above notations, we remark that \( \alpha^2 + \frac{1}{\alpha^2} = \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = l^2 + 2 \). Applying Binet’s formula for \( l \)-sequence, we have:

\[
\begin{align*}
p_n + p_{n+4} &= \frac{\alpha^n - \beta^n}{\alpha - \beta} + \frac{\alpha^{n+4} - \beta^{n+4}}{\alpha - \beta} =
\end{align*}
\]

\[
\begin{align*}
&= \frac{\alpha^{n+2} \cdot (\alpha^2 + \frac{1}{\alpha^2}) - \beta^{n+2} \cdot (\beta^2 + \frac{1}{\beta^2})}{\alpha - \beta} =
\end{align*}
\]
\[
(l^2 + 2) \cdot \frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta} = (l^2 + 2) p_{n+2}.
\]

ii) Using i), we obtain:

\[
p_n + p_{n+8} = (p_n + p_{n+4}) + (p_{n+4} + p_{n+8}) - 2 \cdot p_{n+4} =
= (l^2 + 2) \cdot p_{n+2} + (l^2 + 2) \cdot p_{n+6} - 2 \cdot p_{n+4} =
= (l^2 + 2) \cdot (p_{n+2} + p_{n+6}) - 2 \cdot p_{n+4} =
= [((l^2 + 2)^2 - 2)^2 - 2] \cdot p_{n+4}.
\]

iii) We use induction after \( k \in \mathbb{N}, k \geq 3 \), to prove the following statement:

\[
P(k) : p_n + p_{n+2^k} = \left( \left( \left( \left( (l^2 + 2)^2 - 2 \right)^2 - 2 \right)^2 - 2 \right)^2 - 2 \right)^{k-2 \text{ times of } -2} p_{n+2^{k-1}}.
\]

In i) we proved \( P(2) \), in ii) we proved \( P(3) \). Supposing that \( P(k) \) is true, it is easy to prove \( P(k+1) \), in the same way as we passed from \( P(2) \) to \( P(3) \). □

Remark 2.4. We will denote

\[
M_k = \left[ \left( \left( \left( (l^2 + 2)^2 - 2 \right)^2 - 2 \right)^2 - 2 \right)^2 - 2 \right]^{k-2 \text{ times of } -2}, k \geq 3. \tag{2.5.}
\]

We remark that \( M_{k+1} = M_k^2 - 2 \). We put \( M_2 = l^2 + 2 \). It results, \( M_k > 0 \), for all \( k \in \mathbb{N}, k \geq 2 \).

3. \( l \)-elements in algebras obtained by the Cayley-Dickson process

With the above notations, we will denote

\[
S_t = M_2 M_3 M_4 \ldots M_{t-2} M_{t-1} M_t, t \geq 2.
\]

We will put \( S_1 = 1 \).

Let \( A_t = \left( \frac{2}{K-1} \right) \) be an algebra obtained by the Cayley-Dickson process and \( n \in \mathbb{N} \). We define the \( n \)-th \( l \)-element \( P_n \in A_t \) to be an element of the form

\[
P_n = p_n \cdot 1 + p_{n+1} \cdot e_1 + p_{n+2} \cdot e_2 + p_{n+3} \cdot e_3 + \ldots + p_{n+2^{t-1}} \cdot e_{t-1}. \tag{3.1.}
\]

Proposition 3.1. Let \( v \) be a real number, \( q \) be a prime number, \( K \in \{ \mathbb{Q}, \mathbb{R}, \mathbb{Z}_q \} \), and \( A_t = \left( \frac{2}{K-1} \right) \) be an algebra obtained by the Cayley-Dickson process of dimension \( r = 2^t, t \geq 2 \), and
$P_n = p_n \cdot 1 + p_{n+1} \cdot e_1 + p_{n+2} \cdot e_2 + p_{n+3} \cdot e_3 + \ldots + p_{n+2^l-1} e_{l-1},$

be an $l$–element. Therefore, the following statements are true:

1) We have

$$n(P_n) = S_{t-1}(p_{2n+2^{l-1}-1} - vp_{2n+2^{l-1}-1 -2^{l-1}}).$$

(3.2.)

2) If $v \in \mathbb{R} - (-1,1)$, the $n$-th $l$–element $P_n$ is invertible in $A_t = \left(\frac{-1}{v}, \frac{-1}{v-1}\right)$, for all $n \in \mathbb{N}$.

3) If $v = -1$, we have

$$n(P_n) = S_t p_{2n+2^{l-1}-1}.$$  

(3.3.)

Proof. 1) Using Remark 2.1 and Proposition 2.3, we compute the norm. In this case, we obtain:

$$n(P_n) = p_n^2 + p_{n+1}^2 + p_{n+2}^2 + \ldots + p_{n+2^{l-1}-1}^2 - v(p_{2n+2^{l-1}} + \ldots + p_{n+2^{l-2}} + p_{n+2^{l-1}}) =$$

$$= p_{2n+1} + p_{2n+5} + p_{2n+9} + \ldots - v(p_{2n+2^{l-1}} + \ldots + p_{n+2^{l-1}-3}) =$$

$$= (l^2 + 2)p_{2n+3} + (l^2 + 2)p_{2n+11} + \ldots - v(... + (l^2 + 2)p_{n+2^{l-2}}) =$$

$$= (l^2 + 2)\left[(l^2 + 2)^2 \cdot 2p_{2n+7} + \ldots - v(... + (l^2 + 2)^2 \cdot 2)p_{2n+2^{l-1}-9}\right] =$$

$$= M_2 M_3[M_4 p_{2n+15} + \ldots - v(... + M_4 p_{n+2^{l-1}-17})] =$$

$$= M_2 M_3 M_4 ... M_{t-2}[M_{t-1} p_{2n+2^{l-1}-1} - v M_{t-1} p_{2n+2^{l-1}-1 - 2^{l-1}}] =$$

$$= M_2 M_3 M_4 ... M_{t-2} M_{t-1} (p_{2n+2^{l-1}-1} - vp_{2n+2^{l-1}-1 -2^{l-1}}).$$

2) Indeed, since $2n + 2^{l-1} - 1 < 2n + 2^{l+1} - 1 - 2^{l-1}$, we have that $p_{2n+2^{l-1}-1 -2^{l-1}} > p_{2n+2^{l-1}-1}$, since $(p_n)_{n \geq 0}$, is an increasing sequence. From here, we get that $n(P_n) \neq 0$, even if $v$ is positive or negative. Therefore $P_n$ is an invertible element in $A_t = \left(\frac{-1}{v}, \frac{-1}{v-1}\right)$.

3) Taking $v = -1$ in relation (3.2), we get

$$n(P_n) = M_2 M_3 M_4 ... M_{t-2} M_{t-1} M_t(p_{2n+2^{l-1}-1} + p_{2n+2^{l-1}-1 -2^{l-1}}) =$$

$$= M_2 M_3 M_4 ... M_{t-2} M_{t-1} M_t p_{2n+2^{l-1}}.$$

□

An idempotent element in a unitary ring, with $0 \neq 1$, is an element $x$, $x \neq 0$ and $x \neq 1$, such that $x^2 = x$.

**Theorem 3.2.** With the above notations, let $q$ be a prime number and let $d$ be the first positive integer such that $q \mid p_d$. 


1) Therefore \( q \mid p_n \) if and only if \( d \mid n \).
2) If \( \gcd(q, S_t) = 1, \ t \geq 2, \) and \( d \) is an even number, then all \( l \)-elements \( P_n, n \geq d, \) from the algebra \( A_t = \left( \frac{-1}{Z_q} \right) \) are invertible.
3) If \( t \in \{2, 3\}, \ \gcd(q, S_t) = 1, \ t \geq 2, \) and \( d \) is an odd number such that \( d \mid (2^t - 1) \), then there are no idempotent \( l \)-elements in the algebra \( A_t = \left( \frac{-1}{Z_q} \right) \).

**Proof.**

1) Supposing that \( q \mid p_n \), let \( n = dc + r, 0 \leq r < d \), therefore

\[
p_n = p_{dc+r} = p_r p_{dc+1} + p_{r-1} p_{dc},
\]

from Proposition 2.2. Since \( q \mid p_d \), we have that \( q \mid p_{dc} \). Using that \( q \mid p_n \), it results \( q \mid p_{dc+1} \). If \( q \mid p_r \), we have a contradiction with the choice of the number \( d \). If \( q \mid p_{dc+1} \), we have \( q \mid p_{dc-1} \) and so on. Then we obtain that \( q \mid p_0 \), false. Therefore, \( r = 0 \) and \( d \mid n \). Conversely, assuming that \( d \mid n \), we have \( n = dc, c \) a natural number. From here, we have \( p_d \mid p_n \), then \( q \mid p_n \).

2) Indeed, since \( n(P_n) = S_t p_{2n+2^t-1} \equiv 0 \pmod{q} \), we obtain \( p_{2n+2^t-1} \equiv 0 \pmod{q} \). It results \( d/(2n + 2^t - 1) \), which is false.

3) Supposing that there are \( l \)-elements \( P_n \in A_t = \left( \frac{-1}{Z_q} \right) \) such that \( P_n^2 = P_n \) in \( A_t \), we obtain \( P_n \cdot (P_n - 1) = 0 \). Since \( t \in \{2, 3\} \), the algebra \( A_t \) is a composition algebra. It results that \( P_n \) and \( P_n - 1 \) are zero divisors in the algebra \( A_t \) and their norm are zero. These are equivalent with the system

\[
\begin{cases}
 n(P_n) = 0 \\
 n(P_n - 1) = 0
\end{cases}
\]

in \( Z_q \). Using Proposition 3.1, we get

\[
\begin{cases}
 S_t p_{2n+2^t-1} \equiv 0 \pmod{q} \\
 (p_n - 1)^2 + p_{n+1}^2 + p_{n+2}^2 + \ldots + p_{n+2^t-1}^2 \equiv 0 \pmod{q}.
\end{cases}
\]

Since \( \gcd(q, S_t) = 1 \), we obtain \( p_{2n+2^t-1} \equiv 0 \pmod{q} \), that means \( q \mid p_{2n+2^t-1} \).

It results, from above, that

\[
2n + 2^t - 1 \equiv 0 \pmod{d}
\]

which is equivalent with

\[
n \equiv 0 \pmod{d}
\]

and

\[
S_t p_{2n+2^t-1} - 2p_n + 1 \equiv 0 \pmod{q}.
\]

From the last relation, we have \( \bar{0} = 2p_n - 1 \) in \( Z_q \). From here, since \( n \equiv 0 \pmod{d} \), we get \( 2p_{\bar{n}} - 1 = q - 1 \), which is false. We obtain that there are no idempotent \( l \)-elements in the algebra \( A_t = \left( \frac{-1}{Z_q} \right) \). □
Remark 3.3. Statement i) from the above Theorem was proved in [Da, Dr; 70] for the Fibonacci sequence, that means for \( l = 1 \).

4. Examples

Example 4.1. For \( t = 2 \), when \( \gamma_1 = -1 \) and \( \gamma_2 = q \) is a prime positive integer, it is known, from ([La; 04]), that the quaternion algebra \( \mathbb{H}_Q (-1, q) \) splits if and only if \( q \equiv 1 \pmod{4} \). We wonder how many invertible \( l \)-elements are in the algebra \( \mathbb{H}_Q (-1, q) \)? For a prime positive integer, \( q \equiv 1 \pmod{4} \), we consider the quaternion algebra \( A_t = \left( \frac{-1, q}{Q} \right) = H_Q (-1, q) \). Let \( P_n \) be the \( n \)-th \( l \)-quaternion. Then, we have \( n (P_n) = p_{2n+1} - q \cdot p_{2n+5} \), therefore the \( n \)-th \( l \)-quaternion \( P_n \) is invertible for all \( n \in \mathbb{N} \). Indeed, using Proposition 3.1, we have:

\[
n (P_n) = p_n^2 + p_{n+1}^2 - q \cdot p_{n+2}^2 - q \cdot p_{n+3}^2 = p_{2n+1} - q \cdot p_{2n+5}.
\]

Remarking that the sequence of \( l \)-numbers is strictly increasing and \( q \geq 1 \), it results that \( p_{2n+1} < q \cdot p_{2n+5} \), for all \( n \in \mathbb{N} \). Therefore \( n (P_n) \neq 0 \), for all \( n \in \mathbb{N} \). We obtain that \( P_n \) is invertible, for all \( n \in \mathbb{N} \).

Example 4.2. Even if the octonion algebra \( \mathbb{O} (-1, -1, -1) = \left( \frac{-1, -1, -1}{R} \right) \) is division algebra, for \( t = 4 \), the real sedenion algebra

\[
\left( \frac{-1, -1, -1, -1}{R} \right),
\]

with the basis \( \{1, e_1, \ldots, e_{15}\} \), is not a division algebra. For example,

\[(e_3 + e_{10}) (e_6 - e_{15}) = 0, (see [Fl; 13]) \].

The \( l \)-element \( P_1 = p_1 + \sum_{i=1}^{15} p_{i+1} \cdot e_i 
\]

has the norm \( n (P_1) = M_2 M_3 M_4 p_{17} \neq 0 \), therefore \( P_1 \) is an invertible element.

Let \( q \) be a prime positive integer, \( q \geq 3 \), and \( (\mathbb{Z}_q, +, \cdot) \) be the finite field. It is known that the quaternion algebra \( \mathbb{H}_{\mathbb{Z}_q} (-1, -1) \) splits. Therefore, it has proper zero divisors. In the paper [Sa;17], using many properties of Fibonacci and Lucas numbers, we determined Fibonacci quaternions and generalized Fibonacci-Lucas quaternions which are invertible, respectively zero divisors elements in some quaternion algebras \( \mathbb{H}_{\mathbb{Z}_q} (-1, -1) \). Here, for \( q = 3 \), \( q = 5 \), respectively \( q = 7 \), we get the invertible elements, respectively zero divisors elements and idempotent elements in the quaternion algebras \( \mathbb{H}_{\mathbb{Z}_3} (-1, -1) \), \( \mathbb{H}_{\mathbb{Z}_5} (-1, -1) \), \( \mathbb{H}_{\mathbb{Z}_7} (-1, -1) \). A quaternion from \( \mathbb{H}_{\mathbb{Z}_q} (-1, -1) \) is a zero divisor if and only if its norm is zero.
Example 4.3. Let $q \geq 3$ be a prime positive integer, let $H_{Z_q}(-1,-1)$ be the quaternion algebra and let $P_n$ be the $n$-th generalized $l$–quaternion. Then, from Proposition 3.1, taking $t = 2$, we have
\[ n(P_n) = (l^2 + 2)p_{2n+3}. \]

Example 4.4. All Pell quaternions in the quaternion algebra $H_{Z_3}(-1,-1)$ are zero divisors. Indeed, we apply Proposition 3.1, that means $l = 2$, and we get $n(P_n) = \tilde{0}$ in $Z_3$, for all $n \in \mathbb{N}$.

Example 4.5. Let $(p_n)_{n \geq 0}$ be the Pell sequence. Then, we have: $p_n \mid 5$ if and only if $n \equiv 0 \pmod{3}$. Indeed, $d = 3$ is the first number such that $p_d \mid 5$ and we apply Theorem 3.2.

Example 4.6. A Pell quaternion is a zero divisor in the quaternion algebra $H_{Z_3}(-1,-1)$ if and only if $n \equiv 0 \pmod{3}$. Indeed, this statement results from Theorem 3.2.

Example 4.7. There are no idempotent Pell quaternions in the quaternion algebra $H_{Z_5}(-1,-1)$. Indeed, it is a particular case of Theorem 3.2.

Example 4.8. Let $(p_n)_{n \geq 0}$ be the Pell sequence. Then, we have: $p_n \mid 7$ if and only if $n \equiv 0 \pmod{6}$. Obviously, $d = 6$ is the first number such that $p_d \mid 7$. Therefore, we apply Theorem 3.2.

Example 4.9. There are no Pell quaternion zero divisors in the quaternion algebra $H_{Z_7}(-1,-1)$. Therefore all Pell quaternions $P_n$ from the quaternion algebra $H_{Z_7}(-1,-1)$ are invertible. It is a consequence of the Theorem 3.2.

Conclusions. In this paper, we found elements, namely $l$–elements, with supplementary properties which can help us in the study of algebras $A_t$ obtained by the Cayley-Dickson process. In Proposition 3.1 and Theorem 3.2, we found conditions such that these elements are invertible. These conditions can be very useful in solving equations in algebras $A_t$ or to find new identities, identities which can help us in the study of the properties of these algebras.

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