Tensionless Strings and Killing(-Yano) Tensors

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Abstract

We construct invariants for bosonic and spinning tensionless (null) strings in backgrounds that carry Killing tensors or Killing-Yano tensors of mixed type. This is facilitated by the close relation of these strings to point particles. We apply the construction to the Minkowski and to the Kerr-Newman backgrounds.

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1 Introduction

In this note we point out that the construction of invariants based on the equation of motion for particles can be straightforwardly applied to tensionless strings\(^1\). This is because such strings can be viewed as collections of massless particles subject to certain constraints.

Killing tensors, Killing-Yano forms and Killing-Yano tensors have a rich history of applications, e.g., to separation of variables in gravity, for finding symmetries of various differential operators [1], [2], to G-structures [3, 4], and for finding geometric invariants\(^2\). Good general references are, e.g., [7], [8] and [9]. For a very recent application, see [10]. For the applications here, see [11] and [12].

Likewise there is by now a large literature on tensionless (null) strings. They were first introduced in [13] and independently rediscovered in a different guise in [14]. Relevant references here are [15], [16] and [17]. In addition, the conformal string [18] and a general discussion of various limits of branes and other gravitational systems [19] may be consulted for a more general picture.

\(^1\)B. Carter once urged U.L. to use the nomenclature “null strings” rather than tensionless strings, presumably thinking of, e.g., cosmic strings with higher-dimensional windings where there are solutions with zero effective tension in the Minkowski part [25]. However, since for us they arise in relativistic strings in the limit that the tension goes to zero, “tensionless” is informative. In fact when supersymmetry is involved, it is not always obvious that the world sheet is null. Here we will use both descriptions interchangeably.

\(^2\)For recent discussions of conserved currents in this context see, e.g., [5], [6].
We first discuss the Killing tensor construction in the bosonic particle case in Sec. 2, then its extension to the tensionless bosonic string in Sec. 3. The tensionless spinning string and its relation to spinning particles is given in Sec. 4. There the Killing tensors are generalized Killing-Yano ones with both symmetric and antisymmetric sets of indices.

2 Background

Given a spacetime $\mathcal{M}$ that admits a symmetric rank $n$ Killing tensor

$$\nabla_{(\mu}K_{\mu_1...\mu_n)} = 0 ,$$

there is a quantity which is conserved along the geodesics of the geometry

$$K = K_{\mu_1...\mu_n} p^{\mu_1} ... p^{\mu_n} ,$$

with

$$p^{\mu} = \partial_\tau X^{\mu}(\tau) := \dot{X}^{\mu} .$$

So one has

$$\frac{D}{d\tau} p^{\mu} = 0$$

with $\frac{D}{d\tau}$ the directional derivative along the geodesic. Conservation follows from

$$\frac{D}{d\tau} K = p^{\mu} \nabla_\mu K = (p^{\mu} \nabla_\mu K_{\mu_1...\mu_n}) p^{\mu_1} ... p^{\mu_n} + K_{\mu_1...\mu_n} \frac{D}{d\tau} (p^{\mu_1} ... p^{\mu_n}) = 0 ,$$

where the first term vanishes due to the defining property of the Killing tensor (2.1) and the second due to (2.4). Note that this is a purely geometric construction assuming the existence of a Killing tensor and geodesics.

When we consider the relativistic massless particle action

$$S_1 = \frac{1}{2} \int d\tau \dot{X}^{\mu} G_{\mu\nu}(X) \dot{X}^{\nu} ,$$

the equations of motion describe geodesics and the above construction yields a quantity $K$ which is conserved on the motion.

If we try to extend the construction to the bosonic string, we run into difficulties since the equations of motion are not amenable to the same treatment. In [7] the bosonic and the spinning strings are nevertheless treated in a similar way by imposing gauge conditions and constraints that effectively reduce them to particles. Here we instead turn to their tensionless limits which are particle like by construction.

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3See, e.g., [20].
3 The tensionless bosonic string

Tensionless strings (null strings) can be thought of as the high energy limit of ordinary strings. Here we investigate conserved currents for that limit.

Let $X$ be maps from the world sheet $\Sigma$ to the target space $\mathcal{M}$. As described in, e.g., [17] the tensionless limit of the bosonic string may be described by the action

$$S_2 = \frac{1}{2} \int d^2 x V^a V^b \gamma_{ab} := \frac{1}{2} \int d^2 x \partial X^\mu G_{\mu\nu}(X) \partial X^\nu ,$$

(3.1)

where $V^a$ is a covariant world sheet density vector of weight one half, $\partial := V^a \partial_a$ and $\gamma$ is the induced metric

$$\gamma_{ab} = \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) .$$

(3.2)

The field equation for $V^a$ is

$$\delta V^a : V^b \gamma_{ab} = 0 , \quad \Rightarrow \det \gamma_{ab} = 0 .$$

(3.3)

The variation of $X^\mu$ gives

$$\delta X^\mu : \int d^2 x (\partial(\delta X^\mu)G_{\mu\nu}\partial X^\nu + \frac{1}{2} \partial X^\rho G_{\rho\nu,\mu} \partial X^\nu \delta X^\mu) = 0$$

$$\Rightarrow \int d^2 x \tilde{\nabla}(\delta X^\mu G_{\mu\nu}\partial X^\nu)$$

$$- \int d^2 x \delta X^\mu (G_{\mu\nu,\rho} \partial X^\rho \partial X^\nu + G_{\mu\nu} \tilde{\nabla} \partial X^\nu - \frac{1}{2} \partial X^\rho G_{\rho\nu,\mu} \partial X^\nu) = 0 ,$$

(3.4)

where

$$\tilde{\nabla} = V^a \tilde{\nabla}_a = \partial + \tilde{\Gamma}$$

(3.5)

is a world sheet covariant derivative [17]. It satisfies

$$\tilde{\nabla}_a V^a = 0$$

(3.6)

which determines the contracted world sheet connection to obey

$$V^a \tilde{\Gamma}_a^b = -2 \partial_a V^a .$$

(3.7)

Using (3.7) the first integral term in (3.4) becomes a total derivative

$$\int d^2 x \partial_a (V^a \delta X^\mu G_{\mu\nu}\partial X^\nu)$$

(3.8)

and may be discarded. The remaining terms in (3.4) then imply

$$\tilde{\nabla}\partial X^\mu + \partial X^\nu \Gamma^\mu_{\nu\rho} \partial X^\rho = 0 ,$$

(3.9)

where $\Gamma$ (without the tilde) denotes the Levi-Civita connection on $\mathcal{M}$.

The equations (3.3) and (3.9) mean that the world sheet spanned by the tensionless string is a null surface and that the string behaves classically as a collection of massless particles, one at each $\sigma$ position, constrained to move transversally to the direction of the string. This is clarified in appendix C.
### 3.1 Invariances of the action

We look for an invariance of the action modulo the equations (3.9). The field equations are derived assuming variations that vanish on the boundary. Suppose that $\delta X^\mu = K^\mu(X)$ leaves the action invariant but does not vanish on the boundary of the variation. Modulo the field equations, we are then left with

$$\delta S^2 = -\int d^2 x \tilde{\nabla} \left( \delta X^\mu G_{\mu\nu}(X) \partial X^\nu \right)$$

$$= \int d^2 x \left( \tilde{\nabla} K^\nu \partial X^\nu + K^\nu \tilde{\nabla} X^\nu \right)$$

$$= \int d^2 x \left( \partial_\mu K^\nu \partial X^\mu \partial X^\nu - K^\nu \Gamma^\rho_{\mu\nu} \partial X^\mu \partial X^\nu \right)$$

$$= \int d^2 x \left( \nabla_{(\mu} K_{\nu)} \partial X^\mu \partial X^\nu \right) , \tag{3.10}$$

where we used the equation for $\tilde{\nabla} \partial X$ from (3.9). The expression for $\delta S^2$ is seen to vanish when $K$ is a Killing vector. There is a conserved quantity for each Killing vector of the background geometry. In fact, the weaker condition that $K$ is a conformal Killing vector

$$\nabla_{(\mu} K_{\nu)} = \lambda G_{\mu\nu} \tag{3.11}$$

is sufficient in view of (3.3).

It is interesting that the above argument can be extended to include higher rank Killing tensors. To show this, we first note that the action (3.1) is equivalent to the following phase space action

$$S_3 = \int d^2 x \left( -\frac{1}{2} p^\mu p^{\mu} + p^\mu \partial X^\mu \right) . \tag{3.12}$$

The field equations are now

$$p^\mu = \partial X^\mu ,$$

$$p_\mu \partial_a X^\mu = 0 ,$$

$$\nabla p^\mu = \tilde{\nabla} p^\mu + p^\rho \Gamma^\nu_{\rho\mu} p^\nu = 0 , \tag{3.13}$$

where the first equation shows that $p_\mu$ is the “momentum”

$$p_\mu = \frac{\partial L}{\partial (\partial X^\mu)} \tag{3.14}$$

for the Lagrangian in (3.1). Note that the equations (3.13) imply that $p^2 = 0$ on shell.

Introducing a phase space function

$$K^\mu(X, p) = K^\mu(X) + K^{\mu\nu}(X) p_\nu + K^{\mu\nu\rho}(X) p_\nu p_\rho + \ldots , \tag{3.15}$$
we may repeat the derivation that led to (3.10): Assume that $\delta X^\mu = K^\mu$ is a variation that does not vanish on the boundary but leaves (3.12) invariant. The variation of (3.12) leads to terms that vanish due to the equations (3.13) and leaves us with

$$
\delta S_3 = \int d^2 x \tilde{\nabla} (p_\mu K^\mu) .
$$

Using the field equations (3.13), it is easy to see that this vanishes term by term when the expansion coefficients in (3.15) are Killing tensors (2.1) or conformal Killing tensors:

$$
\nabla_{(\mu} K_{\mu_1...\mu_n)} = G_{(\mu_1...\mu_n)} K^{\lambda}_{\mu_2...\mu_n}) .
$$

At this point we may invoke a version of Noether’s theorem to conclude that the following is annihilated by $\tilde{\nabla}$ on-shell

$$
J = \frac{\partial L}{\partial (\partial X^\mu)} K^\mu = G_{\mu\nu} \partial X^\nu K^\mu = p_\mu K^\mu \Rightarrow \tilde{\nabla} J = 0 .
$$

Unlike the usual Noether procedure, where time is the only variable, this does not mean that $J$ is constant since it is only shown to be independent of one combination of the world sheet coordinates. To proceed, we write

$$
0 = \tilde{\nabla} J = V^a \tilde{\nabla}_a J = \tilde{\nabla}_a (V^a J) := \tilde{\nabla}_a J^a = \tilde{\nabla}_\tau J^\tau + \tilde{\nabla}_\sigma J^\sigma ,
$$

which states that $J^a$ is (world sheet) divergence free. Here we have used $\tilde{\nabla}_a V^a = 0$ and denoted the world sheet coordinates by $(\tau, \sigma)$. Integrating over $\sigma$ and using the divergence theorem, we have that

$$
\frac{d}{d\tau} \int d\sigma J^\tau = \int d\sigma \tilde{\nabla}_\tau J^\tau = - \int d\sigma \tilde{\nabla}_\sigma J^\sigma = 0 ,
$$

where the last equality is also obvious in a world sheet diffeomorphism gauge\(^4\) where $(V^\tau, V^\sigma) = (v, 0)$ with $v$ a constant.

The interpretation as particles suggests that we can construct invariants directly along the same lines as for particles using Killing tensors and momenta. Since the derivation runs parallel to the discussion about invariances of the Lagrangian, we relegate this to Appendix A.

So we have shown that a bosonic tensionless string, subject to the relations (3.3), (3.9), in a geometry that allows Killing tensors, has invariants that are $\sigma$ integrals of the expression in the expansion of $p_\mu K^\mu$ using (3.15).

Minkowski space has the maximal number of Killing vectors possible. The 15 (conformal) Killing vectors of the $D = 4$ Minkowski space are:

**Translations $P_\mu$:** \( \eta^{\mu\nu} e_\nu \),

**Rotations in the $\mu\nu$-plane $L_{\mu\nu}$:** \( X_{[\mu} e_{\nu]} \),

**Dilatations $S$:** \( X^\mu e_\mu \),

**Special Conformal $K_\mu$:** \( 2X_\mu X^\nu e_\nu - X \cdot X e_\mu \).

\(^4\)The available gauges for tensionless strings in this formulation are discussed in [17].
The corresponding generators are listed in the left column and \( e_\mu \) denote the coordinate basis vectors.

The Killing vectors \( V^\mu \) from (3.24) give the following invariants \( Q \) (3.20):

\[
Q^P_\nu = \int d\sigma p_\mu \delta^\nu_\mu = \int d\sigma p_\nu , \tag{3.25}
\]

\[
Q^L_{\mu\nu} = \int d\sigma p_\mu X_\nu , \tag{3.26}
\]

\[
Q^S = \int d\sigma p_\mu X^\mu , \tag{3.27}
\]

\[
Q^K_\mu = \int d\sigma (2(p \cdot X) X_\mu - X \cdot X p_\mu) . \tag{3.28}
\]

In the gauge \( V^a = (v, 0) \), the first equation in (3.3) becomes (C.4)

\[
\dot{X}^2 = \dot{X} X' = 0 , \tag{3.29}
\]

and, on shell, (3.25) becomes

\[
Q^P_\nu = P_\nu := \int d\sigma \dot{p}_\nu = v \int d\sigma \dot{X}_\nu , \tag{3.30}
\]

where

\[
\dot{p}_\nu = \frac{\partial L}{\partial X^\nu} \tag{3.31}
\]

is the usual momentum. The relations (3.29)–(3.31) are precisely the coordinate choices made in [13] where the total momentum (3.30) is shown to be invariant. Similarly, (3.26) represents angular momentum in this gauge. All the relations (3.25)–(3.28) are direct extensions of those for massless particles.

All the higher Minkowski space Killing tensors are reducible, i.e., sums of products of Killing vectors, except the metric. However the integrand involving the metric vanishes on shell (for all geometries). We therefore turn to the more interesting example provided by the Kerr-Newman metric in \( D = 4 \): The metric and the vector potential are given by

\[
ds^2 = - \left( \frac{\Delta - a^2 \sin^2 \theta}{\Sigma} \right) dt^2 - \frac{2a \sin^2 \theta \left( r^2 + a^2 - \Delta \right)}{\Sigma} dt d\phi + \left( \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}{\Sigma} \right) \sin^2 \theta d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 , \tag{3.32}
\]

\[
A_a \, dx^a = - \frac{q_T}{\Sigma} \left( dt - a \sin^2 \theta \, d\phi \right) , \tag{3.33}
\]

where

\[
\Sigma = r^2 + a^2 \cos^2 \theta \quad \text{and} \quad \Delta = r^2 + a^2 + q^2 - 2Mr . \tag{3.34}
\]
The nontrivial Killing tensor for the Kerr-Newman metric has components
\[
K_{tt} = \frac{(a^3 \cos 2\theta + a^3 + 2r^2)^2 \left(r^2 \sin^2 \theta + \Delta \cos^2 \theta\right)}{4\Sigma^3},
\]
\[
K_{t\phi} = -\frac{a \sin^2 \theta \left(a^2 \cos 2\theta + a^2 + 2r^2\right)^2 \left(a^2 r^2 + a^2 \Delta \cos^2 \theta + r^4\right)}{4\Sigma^3},
\]
\[
K_{rr} = -\frac{a^2 \cos^2 \theta \left(a^2 \cos 2\theta + a^2 + 2r^2\right)^2 \left(a^2 r^2 + a^2 \Delta \cos^2 \theta + r^4\right)}{4\Sigma},
\]
\[
K_{t\phi} = \frac{\sin^2 \theta \left(a^2 \cos 2\theta + a^2 + 2r^2\right) \left(a^4 \Delta \sin^2 \theta \cos^2 \theta + r^2 \left(a^2 + r^2\right)^2\right)}{4\Sigma^3}.
\]

One has \(p^\mu = \partial(t, r, \theta, \phi)^\mu\) and thus a \(\tau\) invariant (A.1) is given by \(c(\sigma) = K_{\mu \nu} p^\mu p^\nu\), which reduces to the classical “Carter constant” \([21]\) at each \(\sigma\) when \(V^a = (v, 0)\). We thus find an invariant which is the integral of these
\[
C = \int d\sigma c(\sigma) .
\]  

4 The tensionless spinning string

In this section we extend the construction of Sec. 3 to spinning strings in a flat background.

The tensionless limit of the spinning string with an \(O(N)\) symmetry introduced in \([16]\) has the Lagrangian
\[
2\mathcal{L} = (V^a \partial_a X^\mu + i\lambda^\mu \chi_i)(V^b \partial_b X_\mu + i\lambda_\mu \chi_i) + i\lambda^\mu V^a \partial_a \lambda_\mu + A_{ij} \lambda^\mu \chi_i \lambda^\mu \chi_j .
\]

Here \(\lambda\) and \(\chi\) are spinorial\(^5\) world sheet densities and \(A\) is antisymmetric in the \(O(N)\) indices \(i, j\). Gauge fixing the world sheet diffeomorphisms displays the model as a set of spinning HPPT particles \([22]\), \([23]\). For our purpose we take \(N = 1\) and thus consider
\[
2\mathcal{L}_C = (\partial X^\mu + i\lambda^\mu \chi)(\partial X_\mu + i\lambda_\mu \chi) + i\lambda^\mu \partial \lambda_\mu ,
\]
where we used the definition \(\partial = V^a \partial_a\). One feature of the complete set of field equations is that, on shell,
\[
(\partial X^\mu + i\lambda^\mu \chi)(\partial X_\mu + i\lambda_\mu \chi) = 0 .
\]

We will need a phase space formulation and define
\[
p_\mu = \frac{\partial \mathcal{L}_C}{\partial (\partial X^\mu)} = \partial X_\mu + i\lambda_\mu \chi .
\]

\(^5\)Spinors are just Grassmann numbers in this formulation.
Using this we find the equivalent \((X, p, \lambda)\) phase space Lagrangian

\[
\mathcal{L}_P = -\frac{1}{2}p^2 + p^\mu (\partial X_\mu + i\lambda_\mu \chi) + \frac{i}{2} \lambda^\mu \tilde{\nabla} \lambda_\mu ,
\]

where we suppressed the Minkowski metric in the contractions. The corresponding action is invariant under local supersymmetry transformations given in Appendix B. The equations of motion are

\[
\begin{align*}
\delta X^\mu &: \tilde{\nabla} p_\mu = 0 \\
\delta p_\mu &: p_\mu = \partial X_\mu + i\lambda_\mu \chi \\
\delta \lambda^\mu &: p_\mu \chi + \tilde{\nabla} \lambda_\mu = 0 \\
\delta V^a &: p_\mu \partial_a X^\mu + \frac{i}{2} \lambda_\mu \partial_a \lambda^\mu = 0 \\
\delta \chi &: p_\mu \chi = 0 .
\end{align*}
\]

On shell we have (4.3)

\[
p_\mu p^\mu = 0 .
\]

In the partial gauge choice \((V^0, V^1) = (e^{-1/2}, 0)\), the Lagrangian (4.5) becomes

\[
\mathcal{L} = -\frac{e}{2} \dot{p}^2 + \dot{p}^\mu (\dot{X}_\mu + i\dot{\lambda}_\mu \dot{\chi}) + \frac{i}{2} \dot{\lambda}^\mu \dot{\lambda}_\mu .
\]

after the redefinitions

\[
p^\mu = e^{1/2} \dot{p}^\mu , \quad \lambda^\mu = e^{1/4} \dot{\lambda}^\mu , \quad \chi = e^{-3/4} \dot{\chi} .
\]

As a result of this exercise, we see that an alternative formulation of the spinning null string is given by the Lagrangian (4.8), but with all fields depending on the additional coordinate \(\sigma\) and a constraint that corresponds to integrating out \(V^a\) in (4.5). The latter reads

\[
p_\mu \partial_a X^\mu + \frac{i}{2} \lambda_\mu \partial_a \lambda^\mu = e^{1/2} \left( \dot{p}_\mu \partial_a X^\mu + \frac{i}{2} \dot{\lambda}_\mu \partial_a \lambda^\mu \right) = 0 .
\]

We have thus recovered in first order form the fact that the tensionless spinning string in a particular gauge is equivalent to a bunch of spinning particles, one at each \(\sigma\), moving under the orthogonality constraint (4.10) (see Appendix C). We now note that the Lagrangian (4.8) is formally equivalent to that of a spinning particle [23] and we want to use this analogy to construct invariants for the tensionless spinning string based on the Lagrangian (4.8), mimicking the construction for the spinning particle [22], [12].

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6The covariant derivative \(\tilde{\nabla}\) on the density \(\lambda\) reduces to \(\partial\) in the Lagrangian due to the Grassmann property of \(\lambda\).
4.1 Invariants for the spinning particle

In this subsection we review the analysis of [12]. The Lagrangian (4.8) is taken to represent the spinning particle, so now there are only $\tau$ dependent fields. There is a natural symplectic form $\omega$ associated with this Lagrangian:

$$\omega = dX^\mu \wedge dp_\mu - \frac{i}{2} d\lambda^\mu \wedge d\lambda_\mu . \tag{4.11}$$

The corresponding Poisson bracket is

$$\{ F, G \} = \frac{\partial F}{\partial X^\mu} \frac{\partial G}{\partial p_\mu} - \frac{\partial F}{\partial p_\mu} \frac{\partial G}{\partial X^\mu} + i(-1)^f \frac{\partial F}{\partial \lambda^\mu} \frac{\partial G}{\partial \lambda_\mu} = (-1)^{fg+1} \{ G, F \} . \tag{4.12}$$

With

$$H = \frac{1}{2} p^2 , \quad Q = \lambda \cdot p , \tag{4.13}$$

we have

$$\{ X^\mu , p_\nu \} = \delta^\mu_\nu , \quad \{ \lambda^\mu , \lambda^\nu \} = -i\eta^{\mu\nu} \Rightarrow \{ Q, Q \} = -2iH . \tag{4.14}$$

We may express the local supersymmetry of (4.5) using the Poisson brackets:

$$\{ Q, X^\mu \} = -\lambda^\mu$$
$$\{ Q, \lambda^\mu \} = -ip^\mu$$
$$\{ Q, p^\mu \} = 0 . \tag{4.15}$$

In addition, there are the local supergravity transformations

$$\delta e = -\epsilon \chi$$
$$\delta \chi = i\dot{\epsilon} . \tag{4.16}$$

The phase space invariants based on this Lagrangian involve (generalized) superconformal Killing-Yano tensors, as discussed in [12]. Here we recapitulate this in the present setting.

In [12], a generalization of conformal Killing-Yano (CKY) forms to mixed symmetry conformal Killing-Yano tensor (CKYT) $A_{p,q}$ of type $(p, q)$ was introduced. This is a traceless mixed tensor with the Young tableau

$$A_{p,q} \sim \begin{array}{c}
\begin{array}{c}
q \\
\end{array}
\end{array}$$

(4.17)

with $(p + 1)$ boxes in the first column. The differential constraint satisfied by such a CKYT is that, when a derivative is applied to $A_{p,q}$, the traceless tensor corresponding to the Young
tableau with one extra box on the first row has to vanish, i.e.,

\[
\partial A_{p,q} \ni q_{+1} = 0. \tag{4.18}
\]

Such tensors appear naturally in the context of the spinning particle in [11], albeit there in the context of global supersymmetry in curved backgrounds.

A function on phase space, of the form\footnote{C.f. (2.2).}

\[
\mathcal{F} = F(X, \lambda)^{\mu_1 \ldots \mu_q} p_{\mu_1} \cdots p_{\mu_q} \tag{4.19}
\]

can be expanded in the odd variables to give a sum of terms of the following kind

\[
\lambda^p A_{p,q} p^q := \lambda^{\nu_1 \ldots \nu_{p-q}} A_{\nu_1 \ldots \nu_{p-q}} p^{\mu_1 \ldots \mu_q}, \tag{4.20}
\]

where the multi-index \(\lambda\) and \(p\) denote \(p\)-fold and \(q\)-fold products of the odd coordinates and the even momenta, respectively. A world line super-invariant is a function \(F\) of the phase-space variables which is weakly annihilated by \(Q\), \(\{Q, F\} \approx 0\). Since \(\{Q, Q\} \sim H \sim \frac{d}{d\tau}\) such a function will automatically be a constant of the motion modulo in the particle case. If we are given such an invariant function with lowest \(\lambda\) component \(\lambda^p A_{p,q} p^q\), then we may use the invariance condition \(\{Q, \mathcal{F}\} \approx 0\) to determine the rest of the \(\lambda\) components. The result is

\[
\mathcal{F} = \lambda^p A_{p,q} p^q + \alpha(p, q)\lambda^{p+2} dA_{p+2,q-1} p^{q-1} := A + dA, \tag{4.21}
\]

where \(A_{p,q}\) is in the representation (4.17), satisfies the constraint (4.18) and

\[
\alpha(p, q) := i \frac{(-1)^{(p+1)q}}{(1 + p + q)}, \tag{4.22}
\]

\[
(dA_{p+2,q-1})_{\nu_1 \ldots \nu_{p+2}, \mu_1 \ldots \mu_{q-1}} := \partial_{[\nu_1} A_{\nu_2 \ldots \nu_{p+2}, \mu_1 \ldots \mu_{q-1}]}. \tag{4.23}
\]

### 4.2 Application to the tensionless spinning string

To use these results for the tensionless spinning string, we replace the time derivative in the above construction by \(\partial\), make a final gauge fixing \(V^a \to (1, 0)\) and observe that \(\mathcal{F}\) in (4.21) obeys

\[
\tilde{\nabla} \mathcal{F} = 0 \tag{4.24}
\]
on shell, i.e. using the field equations that follow from (4.8). These are

\[ p^2 = 0 \]
\[ \tilde{\nabla} p_\mu = 0 \quad \rightarrow \quad \dot{p}_\mu = 0 \]
\[ \tilde{\nabla} \lambda_\mu + i\chi \lambda_\mu = 0 \quad \rightarrow \quad \dot{\lambda}_\mu = 0 \]
\[ p_\mu = e^{-1} (\partial X_\mu - i\chi \lambda_\mu) \quad \rightarrow \quad p_\mu = \dot{X}_\mu \]
\[ \lambda p = 0 \ , \quad (4.25) \]

where the right hand sides are the fully gauge fixed relations, including \( \chi = 0 \). The same arguments as in the bosonic case in Sec. 3 now ensure that

\[ \frac{d}{d\tau} \int F d\sigma = 0 \quad (4.26) \]

on the motion.

A simple example of these invariants is given by the \( \tau \)-invariant with leading term \( A_{0,1} \):

\[ F = A_{\mu} p^\mu - \frac{i}{2} \lambda^{\nu_1} \lambda^{\nu_2} \partial_{\nu_1} A_{\nu_2} \ . \quad (4.27) \]

It is not difficult to see that \( \tilde{\nabla} F = \dot{F} = 0 \) using (4.25) and \( \partial_{(\mu} A_{\nu)} = 0 \).

When the leading term is instead \( A_{1,1} \) we have

\[ F = \lambda^\nu A_{\nu,\mu} p^\mu + \frac{i}{2} \lambda^{\nu_1} \lambda^{\nu_2} \lambda^{\nu_3} \partial_{[\nu_1} A_{\nu_2,\nu_3]} \ . \]
\[ \Rightarrow \tilde{\nabla} F = i\lambda^\nu \chi A_{\nu,\mu} p^\mu + \lambda^\nu \partial_{\rho} A_{\nu,\mu} (ep^\rho + i\chi \lambda^\rho) p^\mu + \chi \lambda^{\nu_1} \lambda^{\nu_2} \lambda^{\nu_3} \partial_{[\nu_1} A_{\nu_2,\nu_3]} \]
\[ + \frac{i}{3} \lambda^{\nu_1} \lambda^{\nu_2} \lambda^{\nu_3} \partial_{\rho} \partial_{[\nu_1} A_{\nu_2,\nu_3]} (ep^\rho + i\chi \lambda^\rho) \]
\[ \rightarrow \lambda^\nu \partial_{\rho} A_{\nu,\mu} p^\rho p^\mu + \frac{i}{3} \lambda^{\nu_1} \lambda^{\nu_2} \lambda^{\nu_3} \partial_{\rho} \partial_{[\nu_1} A_{\nu_2,\nu_3]} p^\rho = 0 \ , \quad (4.28) \]

where we used \( \partial_{(\mu} A_{\nu,\rho)} = 0 \) and \( \partial_{\nu_1} \partial_{(\mu} A_{\nu_2,\nu_3)} = 0 \) to show that the last line vanishes.

These arguments extend to arbitrary \( F \).

5 Conclusions

In this note we point out how to construct new invariants for tensionless strings, covering the tensionless bosonic (null) string in a curved background and the tensionless spinning (null) string in flat background. Obvious further work would be to consider the spinning tensionless (null) string in curved background and to discuss the zero tension superstring. The latter has the flat super space action [15]

\[ S_{\text{super}} = \int d^2 x V^a V^b \Pi_\alpha^a \Pi_\beta^b \quad (5.1) \]

which is (3.1) with the space time supersymmetrisation

\[ \partial_a X^\mu \rightarrow \Pi_\alpha^\mu = \partial_a X^\mu - i\partial \Gamma^\mu \partial_a \theta \ , \quad (5.2) \]
where $\theta$ is the antisymmetric spinor partner to $X^\mu$. This system should be open to the same analysis as the tensionless bosonic (null) string. However, there is the added complication in curved supergravity backgrounds of defining super Killing-Yano tensors.

The tensionless strings may be viewed as high energy limits of ordinary strings [17] and hence the invariants discussed in this note should be relevant for ordinary strings in that limit [24].

In the zero tension limit the strings lose their world sheet Weyl invariance but gain ambient space time conformal invariance. In the present note, this is partly reflected in allowing for conformal Killing vectors, but in looking for the most general conserved quantities that should be more systematically taken into account. A good setting for that would be the conformal string described in [18].

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A Particle-like invariants

Consider the quantity

$$\mathbb{K} = K_{\mu_1...\mu_n} p^{\mu_1} ... p^{\mu_n}, \quad (A.1)$$

where $\mathbb{K}$ is a function of $X$ (not a density) and $K$ is a Killing tensor of rank $n$ satisfying (2.1) or a conformal Killing tensor (3.17). Then the following world sheet covariant derivative of $\mathbb{K}$ vanishes:

$$\tilde{\nabla} \mathbb{K} = \partial K_{\mu_1...\mu_n} p^{\mu_1} ... p^{\mu_n} + n K_{\mu_1...\mu_n} (\tilde{\nabla} p^{\mu_1}) ... p^{\mu_n}$$

$$= \nabla_\mu K_{\mu_1...\mu_n} \partial X^\mu p^{\mu_1} ... p^{\mu_n} + n \Gamma^\lambda_{\mu_1...\mu_n} K_{\mu_2...\mu_n,\lambda} \partial X^\mu p^{\mu_1} ... p^{\mu_n}$$

$$- n K_{\mu_1...\mu_n} (p^{\sigma} \Gamma^\mu_{\sigma p} p^{\rho}) p^{\mu_2} ... p^{\mu_n} = 0, \quad (A.2)$$

where we have used the field equation (3.13) and the definition of covariant derivative in getting to the second line. The first term vanishes due to (3.13) and (2.1) or (3.17), the remaining two cancel after using (3.13) once more.

Note that, as before, this does not mean that $\mathbb{K}$ is constant since it is only shown to be independent of one combination of the world sheet coordinates. If we again go to a world sheet diffeomorphism gauge where $V^\alpha = (V^\tau, 0)$, the $\tau$ derivative of $\mathbb{K}$ vanishes but not the $\sigma$ derivative. However, if we again use\(^9\) $\tilde{\nabla}_\alpha V^\alpha = 0$, we may write

$$0 = \tilde{\nabla} \mathbb{K} = V^\alpha \tilde{\nabla}_\alpha \mathbb{K} = \tilde{\nabla}_\alpha (V^\alpha \mathbb{K}) , \quad (A.3)$$

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\(^9\)This is the condition allowing partial integration found in [17].
so that the construction in Sec. 3.1 applies

\begin{equation}
\partial_\tau (V^\tau K) + \partial_\sigma (V^\sigma K) = 0 \, .
\end{equation}

Treating this as the conservation of the current $V^\mu K$, we integrate over $\sigma$:

\begin{equation}
\int \partial_\tau (V^\tau K)d\sigma = \frac{d}{d\tau} \int (V^\tau K)d\sigma = -\int \partial_\sigma (V^\sigma K)d\sigma = 0 \, .
\end{equation}

The last equality follows after specifying the boundary of the $\sigma$ integral, or by choosing the gauge $V^a = (V^\tau, 0)$.

**B  SUSY transformations**

Here are the local supersymmetry transformations that leave the action for (4.5) invariant:

\begin{align*}
\delta X^\mu &= i\epsilon \lambda^\mu \\
\delta p_\mu &= i\epsilon \tilde{\nabla} \lambda^\mu \\
\delta \lambda^\mu &= -\epsilon (\partial X^\mu + \frac{i}{2} \lambda^\mu \chi) \\
\delta V^a &= iV^a (\epsilon \chi) \\
\delta \chi &= \tilde{\nabla} \epsilon .
\end{align*}

**C  The tensionless string as a collection of massless particles**

As an alternative to the covariant treatment presented above, we may from the outset choose a world sheet diffeomorphism gauge

\begin{equation}
V^a = (v, 0) \, ,
\end{equation}

with $v$ a constant, corresponding to the conformal gauge in the tensile theory. We are then looking at the action

\begin{equation}
\int d^2x \ v^2 \hat{X}^\mu G_{\mu\nu}(X) \hat{X}^\nu \, ,
\end{equation}

which leads to the geodesic equation

\begin{equation}
\ddot{\hat{X}}^\mu + \Gamma^\mu_{\rho\nu} \hat{X}^\rho \hat{X}^\nu = 0 \, ,
\end{equation}

subject to the constraints from $V^a$ variation\(^{10}\) (3.3)

\begin{equation}
\hat{X}^2 = \dot{\hat{X}} \hat{X}' = 0 \, .
\end{equation}

This brings out the physical interpretation of the bosonic null string as a collection of massless particles moving along null geodesics orthogonal to the $\sigma$ direction of the string.

\(^{10}\)C.f. the vanishing of the energy momentum tensor for the tensile string.
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