Hidden Symmetries in the 6-Vertex Model of Statistical Physics

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Dedicated to L.D. Faddeev on his 60th birthday

Abstract

The transfer matrix of the 6-vertex model of two-dimensional statistical physics commutes with many (more complicated) transfer matrices, but these latter, generally, do not commute between each other. The studying of their action in the eigenspaces of the 6-vertex model transfer matrix becomes possible due to a “multiplicative property” of the vacuum curves of \( \mathcal{L} \)-operators from which transfer matrices are built. This approach allowed, in particular, to discover for the first time the fact that the dimensions of abovementioned eigenspaces must be multiples of (big enough) degrees of the number 2.

Since the discovery in 1931 of the famous Bethe ansatz \([1]\) for the eigenvectors of one-dimensional quantum Heisenberg magnetic model Hamiltonian, a lot of papers were devoted to studying the properties and generalizations of that ansatz. Now Bethe ansatz is usually considered in the framework of quantum inverse problem method \([2]\) which has united in a natural way the main achievements of one-dimensional quantum field theory and two-dimensional statistical physics, and linked the ideas in these fields to the exactly solvable nonlinear equations of classical mathematical physics (“soliton equations”). Progress was achieved not only in the classification of eigenvectors in “thermodynamic limit” (infinite length of the chain), but also for the chain of finite length (see e.g. \([3]\), where classification of eigenvectors is presented for an isotropic magnetic—“XXX model”. Nevertheless, now still, in this paper author’s opinion, there are many undiscovered mysteries in the Bethe ansatz.

There is, however, a case in which full solution of the eigenvector problem is not difficult—the case when the model can be reduced to “free fermions”. For 6-vertex model studied in this paper (and for the “XXZ magnetic model” connected with it) this means that the “coupling constant” \( \eta \) equals \( \pi/4 \). Baxter \([4]\) surmised that the next simplest cases will appear when \( \eta \) equals other rational multiples of \( \pi \). Peculiar properties of such \( \eta \) values became clearly seen when exactly for those values new solutions of the Yang–Baxter equation were
constructed—the $L$-operators associated with 6-vertex model $R$-operators 
(see also formulae (6–11) below). Next, the author of this paper applied to 
those $L$-operators the idea of vacuum curve. Vacuum curves (see about 
them) of such $L$-operators turned out to have a very simple form 
This together with their multiplication properties (see Subsection below) allowed 
the author to discover for the first time that the spectrum of 6-vertex model 
transfer matrix (and XXZ model Hamiltonian) is highly degenerate—the multi-
plecies of degeneracy grow, roughly speaking, as $2^{\text{const}}N$ with $N \rightarrow \infty$, where 
$N$ is the chain length. The present paper concludes the series of two papers (the 
first one was ) where the author’s approach to the solutions of Yang–Baxter 
equation associated with algebraic curves of genus $g > 1$, and to the problem of 
hidden symmetries of 6-vertex model transfer matrix, is described.

1 The group of matrices of $L$-operator vacuum 
curve coefficients

1.1
Consider the Yang–Baxter equation

$$R(\lambda - \mu) L(\lambda) L(\mu) = L(\mu) L(\lambda) R(\lambda - \mu), \quad (1)$$

where $R(\lambda - \mu)$ is the $R$-matrix of 6-vertex model of two-dimensional statistical 
physics. There exists, firstly, the following solution of $(1)$:

$$L(\lambda) = \begin{pmatrix}
\sin(\lambda + \eta) & 0 & \sin(\lambda - \eta) & 0 \\
0 & \sin(\lambda + \eta) & 0 & \sin(\lambda - \eta) \\
\sin(\lambda + \eta) & 0 & \sin(\lambda - \eta) & 0 \\
0 & \sin(\lambda + \eta) & 0 & \sin(\lambda - \eta)
\end{pmatrix}. \quad (2)$$

A number of other solutions of $(1)$ can be constructed through the multiplication 
procedure of the abovementioned ones by the following operations: a) construction of 
(inhomogeneous) monodromy matrices

$$L(\lambda) = \begin{pmatrix} a_0 & 0 \\ 0 & d_0 \end{pmatrix} \quad \text{and} \quad L(\lambda) = \begin{pmatrix} 0 & b_0 \\ c_0 & 0 \end{pmatrix}. \quad (3)$$

In case of a “generic” parameter $\eta$, all the known solutions of $(1)$ are obtained 
from the abovementioned ones by the following operations: a) construction of 
(inhomogeneous) monodromy matrices

$$L(\lambda) = \prod_{i=1}^{M} L^{(i)}(\lambda_i + \lambda), \quad (4)$$
where $L^{(i)}(\lambda)$ are arbitrary solutions of (1), $\lambda_i$ are constants; b) taking a direct sum in quantum spaces (with the same auxiliary space)

$$L(\lambda) = \bigoplus_{i=1}^{K} L^{(i)}(\lambda_i + \lambda),$$

and c) restriction to an invariant subspace in the quantum space (if such subspace exists; to be exact, the operator $L$ is restricted to the tensor product of auxiliary space by the invariant subspace of quantum space) or taking the corresponding factor operator.

The situation is much more interesting if the parameter $\eta$ in (2) is commensurable with $\pi$. Let

$$\frac{\eta}{\pi} = \frac{m}{n},$$

with relatively prime $m$ and $n$. Then to the solutions of (1) one must add the $L$-matrices

$$L(\lambda) = \left( \begin{array}{cc} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{array} \right),$$

where $A(\lambda), \ldots D(\lambda)$ of size $n \times n$ are of the form (with zeros at the blank spaces):

$$A(\lambda) = a \begin{pmatrix} \sin(\lambda + \rho + (n-1)\eta) \\ \vdots \\ \sin(\lambda + \rho + (1-n)\eta) \end{pmatrix},$$

$$B(\lambda) = \begin{pmatrix} 0 \\ b_{21} \\ \vdots \\ b_{n,n-1} \\ 0 \end{pmatrix},$$

$$C(\lambda) = \begin{pmatrix} 0 \\ c_{12} \\ \vdots \\ c_{n-1,n} \\ 0 \end{pmatrix},$$

$$D(\lambda) = d \begin{pmatrix} \sin(\lambda + \sigma + (1-n)\eta) \\ \vdots \\ \sin(\lambda + \sigma + (n-1)\eta) \end{pmatrix}.$$
Here $a, d, \rho, \sigma$ and all entries in matrices $B(\lambda)$ and $C(\lambda)$ are constants, with the following relation satisfied (subtraction in indices is understood mod $n$):

$$b_{k,k-1}c_{k-1,k} = \Delta + \frac{ad}{2} \cos(\rho - \sigma + 2\eta(n - 2k)),$$

(11)

$k = 1, \ldots, n$; $\Delta$ is a constant, too.

$L$-matrices of the form (6–11) are interesting because they have no generating vector in their quantum space (see [7]), i.e. no vector annihilated by $C(\lambda)$ for all $\lambda$. Instead, they have vacuum vectors in the sense of [8]. This leads to the important role of the vacuum curve $\Gamma_L(\lambda)$ of operator $L(\lambda)$—an algebraic curve in $C^2$ given by equation

$$\det(uA(\lambda) + B(\lambda) - uC(\lambda) - vD(\lambda)) = 0.$$

The explicit form of $\Gamma_L(\lambda)$ for all cases we are interested in has been calculated in [9], see also [10] (all the results of [9, 10] that are of interest to us here are easily carried over to the general case, with no restriction $C(\lambda) = B(\lambda)^T$ of [9, 10] on the operators (6–11)). The easiest case is that of the odd $n$, so we will assume this oddness up to Subsection 1.5.

**Theorem 1.1** [9, 10]. The vacuum curve $\Gamma_L(\lambda)$ of an $L$-operator of the form (6–11) is given by equation

$$v^n = \frac{\alpha(\lambda)u^n + \beta(\lambda)}{\gamma(\lambda)u^n + \delta(\lambda)},$$

(12)

where $\alpha(\lambda) = \det A(\lambda), \ldots, \delta(\lambda) = \det D(\lambda)$.

Let us associate with an $L$-matrix of the form (6–11) a matrix

$$M_L(\lambda) = \begin{pmatrix} \alpha(\lambda) & \beta(\lambda) \\ \gamma(\lambda) & \delta(\lambda) \end{pmatrix}. $$

(13)

It is natural to regard $M_L(\lambda)$ as determined up to a meromorphic scalar factor $g(\lambda)$. Below in this paper we always assume that $\det M_L(\lambda) \neq 0$.

**Theorem 1.2** The vacuum curve $\Gamma_L(\lambda)$ of the monodromy matrix (4) composed of $L$-matrices of the form (6–11) has the form

$$\left(v^n - \frac{\alpha(\lambda)u^n + \beta(\lambda)}{\gamma(\lambda)u^n + \delta(\lambda)}\right)^K = 0,$$

where $K$ is a positive integer, and

$$\left(\begin{array}{cc} \alpha(\lambda) & \beta(\lambda) \\ \gamma(\lambda) & \delta(\lambda) \end{array}\right) = \prod_{i=1}^{M} M_{\lambda_i}(\lambda_i + \lambda).$$

(14)
Proof follows easily from Theorem 1.1 and the results of papers [8, 9, 10].

Theorem 1.2 prompts one to associate to a monodromy matrix $\mathcal{L}(\lambda)$ as well the matrix (14) of its coefficients. In this case, the condition

$$(u, v) \in \Gamma_{\mathcal{L}}(\lambda) \Rightarrow v^n = \frac{\alpha(\lambda)u^n + \beta(\lambda)}{\gamma(\lambda)u^n + \delta(\lambda)}$$

holds, while we are paying no attention to the fact that $\Gamma_{\mathcal{L}}$ may consist of several identical components. Let us, then, associate to the 6-vertex model $\mathcal{L}$-matrix (2) and its multiplied versions the identity matrix $M_{\mathcal{L}}(\lambda)$, and to the matrices (3) $M_{\mathcal{L}}(\lambda) = \begin{pmatrix} a^n_0 & 0 \\ 0 & d^n_0 \end{pmatrix}$ and $M_{\mathcal{L}}(\lambda) = \begin{pmatrix} 0 & b^n_0 \\ c^n_0 & 0 \end{pmatrix}$ respectively. Now allow ourselves to include in a monodromy matrix (4) the $\mathcal{L}$-matrices mentioned in this paragraph as well as $\mathcal{L}$-matrices (6–11). Using the results from [9, 10] we find that to such a monodromy matrix the matrix

$$\begin{pmatrix} \alpha(\lambda) & \beta(\lambda) \\ \gamma(\lambda) & \delta(\lambda) \end{pmatrix}$$

obtained from relation (14) is associated in the sense of (15), as before.

Note that matrices $M_{\mathcal{L}}(\lambda)$ are periodic with period $\pi/n$.

1.2

In papers [9, 10] an involution $\mathcal{L}(\lambda) \mapsto \hat{\mathcal{L}}(\lambda)$ has been introduced that maps an $\mathcal{L}$-matrix of the form (6–11) into such a matrix $\hat{\mathcal{L}}(\lambda)$ that the vacuum curve of the monodromy matrix $\mathcal{L}(\lambda) \hat{\mathcal{L}}(\lambda)$ has an identity matrix of coefficients $M_{\hat{\mathcal{L}}}(\lambda)$. Here we will slightly change the definition of this involution (without changing the vacuum curve of $\mathcal{L}(\lambda)$) and extend it to other $\mathcal{L}(\lambda)$ as follows: for any $\mathcal{L}$-operator (21), with the only condition on $A(\lambda), \ldots, D(\lambda)$ that they satisfy the 6-vertex model commutation relations, introduce $\hat{\mathcal{L}}(\lambda)$ by the formula

$$\hat{\mathcal{L}}(\lambda) = \begin{pmatrix} D(\lambda)^T & -B(\lambda)^T \\ -C(\lambda)^T & A(\lambda)^T \end{pmatrix}.$$  

1.3

Now let us change the roles of quantum and auxiliary spaces of monodromy matrices $\mathcal{L}(\lambda)$ introduced in the end of Subsection 1.1 and consider for a given $\mathcal{L}(\lambda)$ an inhomogeneous transfer matrix

$$T(\lambda) = \text{Tr} \prod_{i=1}^{N} \mathcal{L}(\mu_i + \lambda),$$
\(\mu_i\) being fixed numbers, which acts in a \(2^N\)-dimensional linear space \(H\), the tensor product of auxiliary, from the viewpoint of equation (1), spaces. It is shown in the papers 9, 10 that if an identity matrix \(M_L(\lambda)\) corresponds in the sense of (15) to a matrix \(L(\lambda)\) then \(T(\lambda)\) commutes with the analogous transfer matrix built up of any other matrix \(L(\lambda)\). Guided by this fact, let us study the action of transfer matrices of the form (17) in a space \(H_w \subset H\), an eigenspace for all the transfer matrices corresponding to \(L(\lambda)\)'s with the identity matrix \(M_L(\lambda) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\).

**Theorem 1.3** Let monodromy matrices \(L_1(\lambda)\) and \(L_2(\lambda)\) have the same vacuum curve \(\Gamma(\lambda)\) (for all \(\lambda\)). Let the restrictions to \(H_w\) of transfer matrices corresponding to them according to (17) be non-degenerate in \(\lambda = 0\):

\[
\det T_1(0)\bigg|_{H_w} \neq 0, \quad \det T_2(0)\bigg|_{H_w} \neq 0.
\]

Let, finally, exist a monodromy matrix \(L_3(\lambda)\) such that \(L_1(\lambda)L_3(\lambda)\) has an identity matrix of vacuum curve coefficients and the transfer matrix built up of \(L_3(\lambda)\) also satisfies

\[
\det T_3(0)\bigg|_{H_w} \neq 0.
\]

Then the equality

\[
T_1(0)\bigg|_{H_w} = h T_2(0)\bigg|_{H_w}
\]

holds, with \(h\) a numeric factor.

**Proof.** According to the definition of \(H_w\) and assumptions of the theorem, we have

\[
T_1(\lambda)T_3(\lambda)\bigg|_{H_w} = h_1(\lambda), \quad T_2(\lambda)T_3(\lambda)\bigg|_{H_w} = h_2(\lambda),
\]

where \(h_1(\lambda), h_2(\lambda)\) are functions such that \(h_1(0) \neq 0, h_2(0) \neq 0\). Putting \(h = h_2(0)/h_1(0)\), we come to (18). The theorem is proved.

1.4

The matrices \(M_{L}(\lambda)\) of vacuum curve coefficients of monodromy matrices \(L(\lambda)\) introduced in Subsection 1.3 and determined up to equivalence

\[
M_{L}(\lambda) \sim g(\lambda)M_{L}(\lambda), \quad g(\lambda) \neq 0,
\]

form a group which we will denote \(\mathcal{G}\). The composition law in that group is consistent with the composition of \(L\)-matrices (in the sense of making monodromy matrices, as in 11), with \(M_{L}(\lambda)\) being the inverse for \(M_{L}(\lambda)\) (Subsection 1.2).

Define now for the subspace \(H_w \subset H\) introduced in Subsection 1.3 a subgroup \(\mathcal{G}_w \subset \mathcal{G}\) that acts projectively in \(H_w\) in a natural way. Namely, \(\mathcal{G}_w\)
consists of the matrices \( M_{\mathcal{L}}(\lambda) \) for those \( \mathcal{L}(\lambda) \) for which \( \det T(0)\big|_{H_w} \neq 0 \) and, as in Theorem 1.3, a \( \mathcal{L}_3(\lambda) \) exists such that \( M_{\mathcal{L}L_3}(\lambda) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) (equalities of this kind are understood, of course, to within a scalar factor) and \( \det T_3(0) \neq 0 \).

Then the action of \( \mathcal{G}_w \) is given by the formula

\[
M_{\mathcal{L}} \mapsto T(0)\big|_{H_w},
\]

which is well-defined according to Theorem 1.3.

1.5

Thus, in this section a homomorphism was constructed from the semigroup of monodromy matrices (with making of them “larger” monodromy matrices as composition law) to a group of meromorphic \( 2 \times 2 \)-matrices depending on \( \lambda \) trigonometrically and determined up to a meromorphic scalar factor. This homomorphism can be in a sense inverted (Subsection 1.4, formula (19)). The usefulness of this homomorphism will be shown in the next section.

The constructions of this section can be extended to the case of even \( n = 2p \) using ideas of [9, 10]. In particular, when constructing monodromy matrices (4) one should use, instead of \( \mathcal{L} \)-matrices (6–11), the matrices \( \mathcal{L}_+(\lambda) \) introduced in [9, 10].

2 Degeneracies in the spectrum of the 6-vertex model transfer matrix

2.1

Let \( \eta = m\pi/n \), as in Section 1, with \( m \) and \( n \) relatively prime integers. For simplicity, let us again, up to Subsection 2.7, assume that \( n \) is odd. Denote as

\[
\mathcal{L}_0(\lambda) = \begin{pmatrix} \mathcal{A}_0(\lambda) & \mathcal{B}_0(\lambda) \\ \mathcal{C}_0(\lambda) & \mathcal{D}_0(\lambda) \end{pmatrix}
\]

the \((n-1)\)th symmetric degree of the 6-vertex model \( \mathcal{L} \)-operator. To be exact, \( \mathcal{L}_0(\lambda) \) is such an \( \mathcal{L} \)-operator that \( \mathcal{A}_0(\lambda), \ldots, \mathcal{D}_0(\lambda) \) act in a linear space of dimension \( n \) and possess a generating vector \( \Omega \) with properties

\[
\mathcal{C}_0(\lambda)\Omega \equiv 0, \quad \mathcal{A}_0(\lambda)\Omega = \sin \lambda \cdot \Omega, \quad \mathcal{D}_0(\lambda)\Omega = \sin(\lambda + 2\eta) \cdot \Omega,
\]

obtained from (7–10) when \( a = d = 1 \), \( \rho = \sigma = (1 - n)\eta \), \( b_{1n} = c_{n1} = 0 \).

Let

\[
T_0(\lambda) = \operatorname{Tr} \prod_{i=1}^N \mathcal{L}_0(\mu_i + \lambda)
\]
be an inhomogeneous transfer matrix acting in the space $H$—the tensor product
of $N$ two-dimensional spaces (as in Subsection 1.3).

Recall that $H_w$ denotes a common eigenspace of all transfer matrices

$$T(\lambda) = \text{Tr} \prod_{i=1}^{N} \mathcal{L}(\mu_i + \lambda) \quad (20)$$

with the identity (strictly speaking, scalar) matrix of vacuum curve coefficients

$$M_L(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

in particular, of transfer matrix $T_0(\lambda)$. The aim of this section is to prove the
following theorem.

**Theorem 2.1** Let $w_0(\lambda)$ be an eigenvalue of transfer matrix $T_0(\lambda)$ in $H_w$. If
there are $K_w$ mutually different mod $\pi/n$ zeroes $\lambda = \nu_1, \ldots, \nu_{K_w}$ among the
simple zeroes of the function $w_0(\lambda)$ (multiple zeroes are not taken into account here) then $\dim H_w$ is divisible by $2^{K_w}.$

2.2

Let

$$\mathcal{L}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \quad (21)$$

be an $\mathcal{L}$-operator of the form $[3 \ 11]$, i.e. $A(\lambda), \ldots, D(\lambda)$ act in an $n$-dimensional
space while the generating vector $\Omega$ may not exist. Let $T(\lambda)$ be given by for-

mula (20) and $\hat{T}(\lambda)$ be composed in the same way from the $\mathcal{L}$-operator $[13].$

$$\hat{\mathcal{L}}(\lambda) = \begin{pmatrix} D(\lambda)^T & -B(\lambda)^T \\ -C(\lambda)^T & A(\lambda)^T \end{pmatrix}.$$ 

**Lemma 2.1** In the previous paragraph notations,

$$T(\lambda)\hat{T}(\lambda) = \text{const} \cdot T_0(\lambda - \phi_1) T_0(\lambda - \phi_2), \quad (22)$$

where $\phi_1$ and $\phi_2$ are zeroes of the function $\det M_L(\lambda)$—the determinant of the
vacuum curve coefficient matrix of the operator $\mathcal{L}(\lambda)$.

**Proof.** The statement that the formula (22) is valid with some $\phi_1$ and $\phi_2$
is a reformulation of lemmas 5 and 6 of [9] (see also [11]), and, according to
the proof of the second of those lemmas, $\phi_1$ and $\phi_2$ are zeroes of $\det \mathcal{L}(\lambda)$ (of
multiplicity $n$). The fact that $\phi_1$ and $\phi_2$ are zeroes of $\det M_L(\lambda)$ (generally, of
multiplicity one), easily follows from the explicit form of $\mathcal{L}(\lambda)$ and the definition
of $M_L(\lambda)$ (see Theorem [1] and formula [13]). The lemma is proved.
2.3

Let, besides the mentioned in Theorem 2.1 zeroes \( \lambda = \nu_1, \ldots, \nu_K \) in \( H_w \) zeroes of multiplicity \( \geq 2 \): \( \lambda = \nu_{(K_w+1)} \), \( \ldots \), \( \nu_{M_w} \mod \pi \). Let \( T(\lambda) \) be a transfer matrix corresponding according to (20) to such an operator (monodromy matrix) \( \mathcal{L}(\lambda) \) whose vacuum curve coefficient matrix \( M_{\mathcal{L}}(\lambda) \) is degenerate in the points \( \lambda = \phi_1, \ldots, \phi_q \mod \pi/n \): \( \det M_{\mathcal{L}}(\phi_i) = 0, \ 1 \leq i \leq q \).

Recall that \( M_{\mathcal{L}}(\lambda) \) is defined up to a meromorphic scalar factor \( g(\lambda) \). We can thus assume that in each point \( \phi_i \) the entries of matrix \( M_{\mathcal{L}}(\lambda) \) are finite and not all equal to zero. The fact that zeroes of \( \det M_{\mathcal{L}}(\lambda) \) are situated with period \( \pi/n \) follows from periodicity of \( M_{\mathcal{L}}(\lambda) \), see a remark in the end of Subsection 1.1.

Lemma 2.2 If \( \mathcal{L}(\lambda) \) and \( T(\lambda) \) described above are such that for all \( i, j, \ 1 \leq i \leq q, \ 1 \leq j \leq M_w \),

\[ \phi_i + \nu_j \neq 0 \mod \pi/n, \]

then there exists an \( \mathcal{L} \)-operator \( \tilde{\mathcal{L}}(\lambda) \) with the same vacuum curve

\[ M_{\mathcal{L}}(\lambda) = M_{\mathcal{L}}(\lambda) \]

such that the transfer matrix \( \tilde{T}(\lambda) \) corresponding to it according to (20) is non-degenerate in \( H_w \) for \( \lambda = 0 \).

Proof. Let \( \mathcal{L}_1(\lambda) \) be an \( \mathcal{L} \)-operator of the type described in the beginning of Subsection 2.2, and let us choose it so that zeroes of \( \det M_{\mathcal{L}_1}(\lambda) \) be exactly in the points \( \phi_1 \) and \( \phi_2 \) and the relation

\[ \ker M_{\mathcal{L}_1}(\phi_i) = \ker M_{\mathcal{L}}(\phi_i), \ i = 1, 2 \]

be valid (this can always be done, with changing, if necessary, the numbering of points \( \phi_i, \ 1 \leq i \leq q \)). From (24) it follows that there exists a decomposition

\[ M_{\mathcal{L}}(\lambda) = M_{\mathcal{L}_1}(\lambda)M_{\mathcal{L}_1}(\lambda), \]

where \( \det M_{\mathcal{L}_1}(\lambda) \) has by 2 zeroes mod \( \pi/n \) less than \( \det M_{\mathcal{L}}(\lambda) \). Proceeding further this way, we get

\[ M_{\mathcal{L}}(\lambda) = M_0M_{\mathcal{L}_1}(\lambda)\ldots M_{\mathcal{L}_1}(\lambda), \]

where \( M_0 \) is a constant matrix, namely \( M_0 = \begin{pmatrix} a_0 & 0 \\ 0 & d_0 \end{pmatrix} \) or \( \begin{pmatrix} 0 & b_0 \\ c_0 & 0 \end{pmatrix} \), while \( \mathcal{L}_2(\lambda), \ldots, \mathcal{L}_{q/2}(\lambda) \) are \( \mathcal{L} \)-operators of the same type as \( \mathcal{L}_1(\lambda) \) (the total number of zeroes \( \phi_i \), counted with regard to their multiplicities, is of course always even).
It follows from Lemma 2.1 that the transfer matrices $T_1(\lambda), \ldots, T_{q/2}(\lambda)$ built of operators $L_1(\lambda), \ldots, L_{q/2}(\lambda)$ can be degenerate in $\lambda = 0$ only if (23) is violated. The same applies, therefore, to the transfer matrix $\hat{T}(\lambda)$ constructed from the $L$-operator

$$\hat{\mathcal{L}} = \mathcal{L}_0\mathcal{L}_{q/2}(\lambda) \ldots \mathcal{L}_1(\lambda),$$

where $\mathcal{L}_0$, of course, corresponds to the matrix $M_0$. Lemma 2.2 is proved.

2.4

Let now $T'(\lambda)$ and $T''(\lambda)$ be two transfer matrices constructed according to formula (20) from operators $L'(\lambda)$ and $L''(\lambda)$ such that $\det M_{L'}(\lambda) = 0$ in the points $\phi_1', \ldots, \phi_{q'}'$, and $\det M_{L''}(\lambda) = 0$ in the points $\phi_1'', \ldots, \phi_{q''}''$. Let the conditions

$$\phi_i' + \nu_j \neq 0 \pmod{\pi/n},$$
$$\phi_i'' + \nu_j \neq 0 \pmod{\pi/n},$$

where, as before, $\nu_j, 1 \leq j \leq M_w$, are zeroes of $T_0(\lambda)|_{H_w}$, be valid for all $i, j$, except $i = j = 1$, i.e.

$$\phi_1' + \nu_1 = 0,$$
$$\phi_1'' = \phi_1''.''

**Lemma 2.3** If the operators $L'(\lambda)$ and $L''(\lambda)$ from the previous paragraph are such that

$$\text{Ker } M_{L'}(-\nu_1) = \text{Ker } M_{L''}(-\nu_1), \quad (25)$$

then

$$\text{Ker } T_0'(0)|_{H_w} = \text{Ker } T_0''(0)|_{H_w}. \quad (26)$$

**Proof.** One can find in the same way as in the proof of Lemma 2.1 that

$$T'(\lambda)\hat{T}'(\lambda) = \text{const } \prod_{i=1}^{q'} T_0(\lambda - \phi_i').$$

Thus, $T'(\lambda)\hat{T}'(\lambda)|_{H_w}$ is a scalar operator having a simple zero in $\lambda = 0$, whence

$$\text{Ker } T'(0)|_{H_w} = \text{Im } \hat{T}'(0)|_{H_w}. \quad (27)$$

From (25) and the fact that $M_{L'}(\lambda)$ is proportional to $(M_{L'}(\lambda))^{-1}$ it follows that $M_{L''}(\lambda)M_{L'}(\lambda)$ is non-degenerate in $\lambda = -\nu_1$ (to within a scalar factor!). Thus, $\mathcal{L}(\lambda) = \mathcal{L''}(\lambda)\hat{\mathcal{L}}'(\lambda)$ satisfies the conditions of Lemma 2.2, from which
it follows that at $\lambda \to 0$ $T''(\lambda)\hat{T}'(\lambda)\big|_{H_w}$ is proportional to a non-degenerate operator (the operator $\hat{T}(0)$ in Lemma 2.2 notations). Thus,

$$\text{Ker} T''(\lambda)\big|_{H_w} = \text{Im} \hat{T}'(0)\big|_{H_w} \quad (28)$$

Comparing (27) with (28), we come to (26). The lemma is proved.

2.5

Now in this subsection let the operator

$$L(\lambda) = \left( \begin{array}{cc} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{array} \right)$$

and the transfer matrix $T(\lambda)$ constructed according to (20) have the following properties:

a) $A(\lambda), \ldots, D(\lambda)$ act in an $n$-dimensional space, with no generating vector $\Omega$ possessing the property $C\Omega \equiv 0$;

b) the vacuum curve coefficient matrix has the form

$$M_L(\lambda) = \left( \begin{array}{cc} \alpha(\lambda) & \beta(\lambda) \\ \beta(\lambda) & \alpha(\lambda) \end{array} \right);$$

c) $T(\lambda)\hat{T}(\lambda) = \text{const} \cdot T_0(\lambda - \phi_1)T_0(\lambda - \phi_2)$,

with all the sums

$$\phi_i + \nu_j, \quad i = 1, 2; \quad 1 \leq j \leq M_w$$

pairwise different (recall that $\nu_j$ are zeroes of the scalar operator $T_0(\lambda)\big|_{H_w}$, and the first $K_w$ of them are the simple ones).

$L(\lambda)$ with properties a)–c) always exists. It follows from the property b) that $M_L(\lambda)$ and $M_C(\mu)$ commute for all $\lambda, \mu$, so $T(\lambda)$ and $T(\mu)$, and also $T(\lambda)$ and $\hat{T}(\mu)$, commute as well.

Lemma 2.4 For $1 \leq j \leq K_w$ the decomposition takes places

$$H_w = \text{Ker} T(\phi_1 + \nu_j)\big|_{H_w} \oplus \text{Ker} \hat{T}(\phi_1 + \nu_j)\big|_{H_w}.$$ 

Proof. The scalar operator

$$\left( T(\phi_1 + \nu_j + \lambda)\hat{T}(\phi_1 + \nu_j + \lambda) \right)\big|_{H_w}$$
has a simple zero in \( \lambda = 0 \), which can be written in the form

\[
\left( T(\phi_1 + \nu_1)T(\phi_1 + \nu_2) + \lambda \frac{dT(\phi_1 + \nu_2 + \lambda)}{d\lambda} \right)_{\lambda=0}^T(\phi_1 + \nu_2) + \\
\lambda T(\phi_1 + \nu_2) \frac{dT(\phi_1 + \nu_2 + \lambda)}{d\lambda} \bigg|_{\lambda=0} = \text{const} \cdot \lambda + o(\lambda).
\]

From the terms of order zero in \( \lambda \) we get

\[
\dim \text{Ker} T(\phi_1 + \nu_2) \bigg|_{H_w} + \dim \text{Ker} \hat{T}(\phi_1 + \nu_2) \bigg|_{H_w} \geq \dim H_w. \quad (29)
\]

From the terms of the first order in \( \lambda \) we see, taking into account the commutativity of \( T \) and \( dT/d\lambda \) which follows, as was explained above, from the condition b) of this subsection, that there cannot exist a nonzero vector \( \Phi \in H_w \) with properties \( T(\phi_1 + \nu_2)\Phi = 0 \) and \( \hat{T}(\phi_1 + \nu_2)\Phi = 0 \), which means

\[
\text{Ker} T(\phi_1 + \nu_2) \bigg|_{H_w} \cap \text{Ker} \hat{T}(\phi_1 + \nu_2) \bigg|_{H_w} = 0. \quad (30)
\]

Relations (29) and (30) together mean exactly what was required in the lemma, so the proof is complete.

For each subset \( A \subset \{1, \ldots, K_w\} \) of the set of integers from 1 to \( K_w \), let us introduce a subspace \( H(A) \subset H_w \):

\[
H(A) = \bigcap_{i \in A} \text{Ker} T(\phi_1 + \nu_i) \bigg|_{H_w} \bigcap_{j = 1}^{K_w} \text{Ker} \hat{T}(\phi_1 + \nu_j) \bigg|_{H_w}.
\]

**Lemma 2.5** The dimensions if subspaces \( H(A) \) are equal for all \( A \); there is a decomposition

\[
H_w = \bigoplus_A H(A). \quad (31)
\]

**Proof** The decomposition (31) readily follows from Lemma 2.4 and the commutativity of \( T(\lambda), T(\mu), \hat{T}(\lambda'), \hat{T}(\mu') \) for all \( \lambda, \mu, \lambda', \mu' \). To prove the equalness of the dimensions of \( H(A) \), it is sufficient to construct for any pair \( A_1, A_2 \) a non-degenerate operator \( F \) mapping \( H(A_1) \) into \( H(A_2) \). Let e.g. \( A_1 = \{1, \ldots, K_w\} \) and \( A_2 = \{2, \ldots, K_w\} \). Let us construct the operator \( F \) with properties

\[
F \text{Ker} T(\phi_1 + \nu_1) \bigg|_{H_w} = F \text{Ker} \hat{T}(\phi_1 + \nu_1) \bigg|_{H_w}, \\
F \text{Ker} T(\phi_1 + \nu_2) \bigg|_{H_w} = F \text{Ker} T(\phi_1 + \nu_2) \bigg|_{H_w}, \\
\vdots \\
F \text{Ker} T(\phi_1 + \nu_{K_w}) \bigg|_{H_w} = F \text{Ker} T(\phi_1 + \nu_{K_w}) \bigg|_{H_w}.
\]
Applying Lemma 2.3 we find that we can set
\[ F = \tilde{T}(0) \bigg|_{H_w}, \]
where \( \tilde{T}(\lambda) \) is the transfer matrix built of an operator \( \tilde{L}(\lambda) \) with properties

\[
\begin{align*}
M_{\tilde{L}}(-\nu_1) & \text{Ker} M_{\tilde{L}}(\phi_1) = \text{Ker} M_{\tilde{L}}(\phi_1), \\
M_{\tilde{L}}(-\nu_2) & \text{Ker} M_{\tilde{L}}(\phi_1) = \text{Ker} M_{\tilde{L}}(\phi_1), \\
& \cdots \cdots \\
M_{\tilde{L}}(-\nu_{K_w}) & \text{Ker} M_{\tilde{L}}(\phi_1) = \text{Ker} M_{\tilde{L}}(\phi_1).
\end{align*}
\]

Recall that \( M_{\tilde{L}}(\lambda) \) consists of trigonometrical polynomials whose degree depends on \( \tilde{L} \). Choosing this degree big enough, one can satisfy all the conditions (32) together with nondegeneracy of \( \tilde{T}(0) \bigg|_{H_w} \). The lemma is proved.

The proof of Theorem 2.1 comes now to its end with an observation that the number of subspaces \( H(A) \) equals \( 2^{K_w} \).

2.6

Let us apply the obtained results to calculating the degeneracy multiplicity of the 6-vertex model transfer matrix eigenvalue corresponding to the “naked vacuum”, i.e. the eigenvector
\[ \left( 1 \atop 0 \right) \otimes \cdots \otimes \left( 1 \atop 0 \right). \]
Let us assume that the chain length \( N \) is a multiple of \( n \). A simple calculation shows that in this case \( K_w = N/n \). Hence, the degeneracy multiplicity is divisible by \( 2^{N/n} \).

2.7

Thus, the results of Section 2 have been applied to calculating the degeneracy multiplicities of the 6-vertex model transfer matrix spectrum. These multiplicities turned out to be divisible by high (as it is seen from the example in Subsection 2.6) degrees of the number 2.

In the case of even \( n = 2p \), one can perform all the reasoning in much the same way as above. Some necessary complications follow from the paper [9] (or [10]). In particular, the transfer matrices must be constructed using the operator \( \mathcal{L}_4(\lambda) \) (or \( \mathcal{L}(\lambda) \)) instead of \( \mathcal{L}(\lambda) \). Theorem 2.1 remains valid for \( n = 2p \) if one changes mod \( \pi/n \) to mod \( \pi/p \) in its formulation.
References

[1] Bethe H. Zur Theorie der Metalle. I. Eigenwerte und Eigenfunktionen Atomkette // Zeitschrift für Physik.—1931. B. 71, Hefte 3–4. S. 205–226.

[2] Takhtajan L.A., Faddeev L.D. Quantum Inverse Problem Method and the XYZ Heisenberg Model // Uspekhi Mat. Nauk—1979. V. 34, issue 5(209).—P. 13–63. (In Russian)

[3] Takhtajan L.A., Faddeev L.D. The Spectrum and Excitation Scattering in the Homogeneous Isotropic Heisenberg Magnetic // Zapiski Nauch. Seminarov LOMI—1981. V. 109.—P. 134–179. (In Russian)

[4] Baxter R. Exactly Solved Models in Statistical Mechanics. N.Y. etc.: Academic Press, 1982.

[5] Kulish P.P., Sklyanin E.K. Quantum spectral transform method. Recent developments // Lect. Notes in Phys.—1982. V. 151. P. 61–119.

[6] Kulish P.P., Reshetikhin N.Yu., Sklyanin E.K. Yang—Baxter equations and representation theory // Lett. Math. Phys.—1981. V. 5, No. 5. P. 393–403.

[7] Izergin A.G., Korepin V.E. Lattice version of quantum field theory models in two dimensions // Nucl. Phys.—1982.—V. B205[FS5], No. 3.—P. 401–413.

[8] Krichever I.M. Baxter Equations and the Algebraic Geometry // Funkc. Analiz.—1981. V. 15, issue. 2.—P. 22–35. (In Russian)

[9] Korepanov I.G. Vacuum Curves of L-Operators Connected with the 6-Vertex Model, and Construction of R-Operators / Chelyabinsk Polytechnical Institute.—Chelyabinsk, 1986.—40 pages.—Deposited in the VINITI April 2, 1986, No. 2271-V86. (In Russian)

[10] Korepanov I.G. Vacuum Curves of L-Operators Connected with the 6-Vertex Model // Algebra i Analiz.—1994.—V. 6. issue 2.—P. 176–194. (In Russian)

[11] Korepanov I.G. Hidden Symmetries in the 6-Vertex Model / Chelyabinsk Polytechnical Institute.—Chelyabinsk, 1987.—12 pages.—Deposited in the VINITI February 27, 1987, No. 1472-V87. (In Russian)

[12] Korepanov I.G. On the Spectrum of the 6-Vertex Model Transfer Matrix / Chelyabinsk Polytechnical Institute.—Chelyabinsk, 1987.—14 pages.—Deposited in the VINITI May 7, 1987, No. 3268-V87. (In Russian)

[13] Korepanov I.G. Applications of Algebro-Geometrical Constructions to the Triangle and Tetrahedron Equations. Ph.D. Thesis. Leningrad.: LOMI, 1990. (In Russian)
[14] Korepanov I.G. *The Vacuum Vector Method in the Theory of Yang–Baxter Equation*. In the book: *Applied Problems of Matematical Analisys: Collection of scientific works / Chelyabinsk Polytechnical Institute.*—Chelyabinsk, 1986.—P. 39–48. (In Russian)