IRREGULARITY OF DISTRIBUTION
IN WASSERSTEIN DISTANCE

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ABSTRACT. We study the non-uniformity of probability measures on the interval and the circle. On the interval, we identify the Wasserstein-$p$ distance with the classical $L^p$-discrepancy. We thereby derive sharp estimates in Wasserstein distances for the irregularity of distribution of sequences on the interval and the circle. Furthermore, we prove an $L^p$-adapted Erdős–Turán inequality.

1. INTRODUCTION

We consider the classical question of irregularity of distribution: if we successively place points in a box, how evenly can we space them? Answers encompass a vast body of theoretical and numerical work. Rather than cite all related literature, we direct the reader to the excellent survey [3] and monograph [1].

In this note, we restrict our attention to one dimension. Given a sequence of points $(x_n)_{n \in \mathbb{N}}$ in the interval $I = [0,1)$ how well can the empirical measures

$$\mu_N := \frac{1}{N} \sum_{n=1}^{N} \delta_{x_n}$$

approximate the uniform distribution $\lambda$? Our answer involves several distances $d$ on the space of probability measures $\mathcal{P}(I)$, but all agree that if the first $N$ points are evenly spaced on $I$,

$$d(\mu_N, \lambda) \sim \frac{C}{N}. \quad (2)$$

However, the truncations $(x_n)_{n=1}^{N}$ of a fixed sequence $(x_n)_{n \in \mathbb{N}}$ cannot all be evenly spaced, so we naturally wonder whether (2) can hold for all $N \in \mathbb{N}$. Indeed, van der Corput conjectured and van Aardenne-Ehrenfest confirmed that such uniform even spacing is impossible [31,30].

These classical results concern distances $d$ based on the discrepancy function

$$D_N(x) := \mu_N([0,x)) - x \quad \text{for } x \in I.$$

For $p \in [1,\infty]$, the $L^p$-discrepancy of a sequence $(x_n)$ at stage $N$ is $\|D_N\|_{L^p}$. Thus the $L^\infty$-discrepancy is simply the Kolmogorov–Smirnov statistic [16].
The following theorem sharply answers van der Corput's conjecture in $L^p$-discrepancy, and unites the work of numerous authors. To state it cleanly, we let

$$\alpha_p := \begin{cases} \frac{1}{p} & \text{if } p \in [1, \infty), \\ 1 & \text{if } p = \infty. \end{cases}$$

**Theorem 1 (25, 27, 12, 17, 32, 6, 5).** For every $p \in [1, \infty]$, there exists a constant $C_p > 0$ such that for any sequence $(x_n)_{n \in \mathbb{N}} \subset I$,

$$\|D_N\|_{L^p} \geq C_p \frac{\log \alpha_p N}{N}$$

holds for infinitely many $N \in \mathbb{N}$. Furthermore, this bound is sharp.

**Remark 1.** Since $D_N$ is an antiderivative of $\mu_N - \lambda$, we in fact have

$$\|D_N\|_{L^p} = \|\mu_N - \lambda\|_{W^{1,p}}. \quad (3)$$

Thus Theorem 1 quantifies the non-uniformity of $\mu_N$ in the negative Sobolev norm $\| \cdot \|_{W^{-1,p}}$, which we define in Section 2.

In this note, we are interested in the distance between $\mu_N$ and $\lambda$ in Wasserstein metrics. These hail from the venerable theory of optimal transport, but their application to irregularity of distribution appears to be recent [28, 4]. We recall that the Wasserstein-$p$ distance on a metric space $X$ measures the $L^p$-weighted cost of transporting one probability measure to another. Precisely, for $p \in [1, \infty]$ and $\mu, \nu \in \mathcal{P}(X)$,

$$W^X_p(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \| d_X \|_{L^p(\pi)} \quad (4)$$

where $d_X : X \times X \to [0, \infty)$ denotes the metric on $X$, and $\Pi(\mu, \nu)$ denotes the set of coupling measures from $\mu$ to $\nu$, i.e. the set of probability measures on $X \times X$ with marginals $\mu$ and $\nu$.

For us, $X$ will be the interval $I = [0, 1)$ or the circle $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. In these one-dimensional spaces, the optimal coupling in (4) assumes a particularly simple form. This allows us to identify the Wasserstein and Sobolev metrics.

**Proposition 2.** Let $X$ be $I$ or $\mathbb{T}$, and let $\lambda$ denote the uniform measure on $X$. Then for all $p \in [1, \infty]$ and $\mu \in \mathcal{P}(X)$,

$$W^X_p(\mu, \lambda) = \|\mu - \lambda\|_{W^{-1,p}(X)} \quad (5)$$

**Remark 2.** This identity extends Kantorovich–Rubinstein duality to $p > 1$, and clarifies a well-known infinitesimal equivalence between $W_2$ and $H^{-1}$ [13, 19, 2]. In this $p = 2$ case, it likewise sharpens a bound of Peyre [21], who investigated the relationship between $W_2$ and $H^{-1}$ in detail. However, these prior results hold in more general settings, while (5) seems restricted to distances from the uniform measure in one dimension. We note that on the interval, the $p = 2$ case of (5) appears in [26, Ex. 64].

With this identity, we can use classical theory to establish irregularity of distribution in Wasserstein metrics on the interval and circle.
Theorem 3. Let $X$ be $I$ or $T$. For every $p \in [1, \infty]$ there exists a constant $C_p > 0$ such that for any sequence $(x_n)_{n \in \mathbb{N}} \subset X$,

$$W_p^X(\mu_N, \lambda) \geq C_p \frac{\log^{\alpha_p} N}{N}$$

holds for infinitely many $N \in \mathbb{N}$. Furthermore, this bound is sharp.

Remark 3. To our knowledge, irregularity results on the circle have not appeared previously in the literature. The distinction between $I$ and $T$ may seem trivial, but there exist sequences on $I$ that are asymptotically more evenly distributed when viewed on $T$. Indeed, we shall identify $W_2^T(\mu_N, \lambda)$ with the classical diaphony $[35, 36]$. When $(x_n)$ is the van der Corput sequence $[32]$, the asymptotic difference between $W_2^T(\mu_N, \lambda)$ and $W_2^I(\mu_N, \lambda)$ follows from $[24]$.

Remark 4. The lower bounds in Theorem 3 answer several open questions posed by Steinerberger and Brown in $[28, 4]$.

We close with an explicit upper bound on non-uniformity. The Hausdorff–Young inequality bounds the Sobolev distance $\|\mu - \lambda\|_{W^{-1,p}(T)}$ via the Fourier coefficients $\hat{\mu}$ when $p \in [2, \infty)$. However, we might hope that non-uniformity to scale $\frac{1}{n}$ could be detected by the first $n$ Fourier coefficients. When $p = \infty$, this is a consequence of the classical Erdős–Turán inequality. We develop an $L^p$ variant for all $p \in [2, \infty]$.

Proposition 4. There exists universal $C > 0$ such that for all $p \in [2, \infty]$, $n \in \mathbb{N}$, and $\mu \in P(T)$,

$$\|\mu - \lambda\|_{W^{-1,p}(T)} \leq C \frac{n}{n} + C \left( \sum_{k=1}^{n-1} \left| \frac{\hat{\mu}(k)}{k^q} \right| \right)^{\frac{1}{q}},$$

where $q \in [1,2]$ is the Hölder-conjugate of $p$. Furthermore, let $\bar{D}$ denote the mean of the discrepancy function of $\mu$. Then on the interval,

$$\|\mu - \lambda\|_{W^{-1,p}(I)} \leq C \frac{n}{n} + C \left( \sum_{k=1}^{n-1} \left| \frac{\hat{\mu}(k)}{k^q} + |\bar{D}|^q \right| \right)^{\frac{1}{q}}.$$  

Remark 5. In $[28]$, Steinerberger introduced an Erdős–Turán inequality for the Wasserstein-1 distance. The $p = \infty$ case of (6) shows that in fact the same bound holds for the Wasserstein-$\infty$ distance. As we shall argue in Section 4, the $p = \infty$ case is equivalent to the classical Erdős–Turán inequality.

We prove Proposition 2 in Section 2 and use it to establish Theorem 3 in Section 3. We prove Proposition 4 in Section 4 and apply it to the equidistribution of quadratic residues in finite fields.
ACKNOWLEDGEMENTS

We warmly thank Stefan Steinerberger and Andrea Ottolini for their suggestions and encouragement. This work was supported by the Fannie and John Hertz Foundation and by NSF grant DGE-1656518.

2. WASSERSTEIN AND SOBOLEV CONCUR

We define the homogeneous negative Sobolev norm by duality. For \( q \in [1, \infty] \) and \( X = I \) or \( \mathbb{T} \), let
\[
\| f \|_{\dot{W}^{-1,q}(X)} := \| f' \|_{L^q(X)}
\]
denote the homogeneous Sobolev seminorm of a measurable function \( f : X \to \mathbb{R} \) with weak derivative \( f' \). For Hölder-conjugate \((p,q)\), we define
\[
\| \mu - \lambda \|_{\dot{W}^{-1,p}(X)} := \sup \left\{ \int_X f(\mu - \lambda) \bigg| \| f \|_{\dot{W}^{-1,q}(X)} \leq 1 \right\}.
\]

Note that the seminorm \( \| \cdot \|_{\dot{W}^{-1,q}(X)} \) is invariant under constant shifts of \( f \). Such shifts leave the above expression unchanged, because \( \mu - \lambda \) is orthogonal to the constant function.

We first relate our Sobolev norm to the \( L^p \)-discrepancy.

Lemma 5. Fix \( p \in [1, \infty] \) and \( \mu \in \mathcal{P}(X) \). Let \( F \) be an antiderivative of \( \mu \). Then
\[
\| \mu - \lambda \|_{\dot{W}^{-1,p}(I)} = \| F - F(0) \|_{L^p(I)} \tag{8}
\]
and
\[
\| \mu - \lambda \|_{\dot{W}^{-1,p}(\mathbb{T})} = \inf_{y \in \mathbb{R}} \| F - y \|_{L^p(\mathbb{T})}. \tag{9}
\]

Proof. On \( I = [0,1) \), we have
\[
\int_I f(\mu - \lambda) = \int_I f \, dF = - \int_I F \, df + Ff \bigg|_0^1.
\]
Since \( \mu - \lambda \) is mean zero, \( F(0) = F(1) \) and
\[
\int_I f(\mu - \lambda) = - \int_I [F - F(0)] f'.
\]
Taking the supremum over all \( f' \in L^q(I) \), standard \( L^p \)-duality yields \((8)\).

On \( \mathbb{T} \) there are no boundary terms, so \( \int_\mathbb{T} f(\mu - \lambda) = - \int_\mathbb{T} Ff' \). However, a function in \( L^q(\mathbb{T}) \) is a derivative precisely when it has mean zero. We thus take the supremum over all \( f' \in L^q_0(\mathbb{T}) \), the mean-zero subspace of \( L^q(\mathbb{T}) \). By subspace-quotient duality,
\[
\| \mu - \lambda \|_{\dot{W}^{-1,p}(\mathbb{T})} = \| F \|_{L^p(\mathbb{T})/\mathbb{R}}.
\]
Now \((9)\) follows from the definition of the quotient norm. \( \Box \)
In particular, Lemma 5 implies (3). We now turn to Wasserstein metrics. Before proving Proposition 2, we note that the \( p = 1 \) case simply restates Kantorovich–Rubinstein duality for our measures [13]. Indeed, \( \|f\|_{W^{1,\infty}(X)} = \text{Lip}(f) \), so we can write (5) as

\[
W^X_{1}(\mu, \lambda) = \sup \left\{ \int_X f(\mu - \lambda) \left| \text{Lip}(f) \leq 1 \right. \right\}.
\]

Proof of Proposition 2. We handle the interval first, so fix \( \mu \in \mathcal{P}(I) \) and \( p \in [1, \infty) \). It is well-known that the optimal transport map from \( \mu \) to \( \lambda \) is monotone, i.e. preserves the order of the mass [33, §2.2]. Let \( F_{\mu} \) denote the unique left-continuous antiderivative of \( \mu \) such that \( F_{\mu}(0) = 0 \), so that \( F_{\mu}(x) = \mu([0,x)) \) for \( x \in I \). Let \( F_{\mu}^{-1} \) denote the left-continuous pseudo-inverse of \( F_{\mu} \):

\[
F_{\mu}^{-1}(u) := \inf\{x \in I \mid F_{\mu}(x) \geq u\}.
\]

Then for each \( u \in (0, 1] \), the optimal transport plan from \( \mu \) to \( \lambda \) moves mass at position \( F_{\mu}^{-1}(u) \) to position \( u \). Thus

\[
W^I_{p}(\mu, \lambda)^p = \int_I |F_{\mu}^{-1}(u) - u|^p du.
\]

We now appeal to a curious identity from single-variable calculus, which we state in more general terms in anticipation of the circle case.

Lemma 6. Let \( h: \mathbb{R} \to \mathbb{R} \) be left-continuous and non-decreasing such that \( h - \text{id} \) is 1-periodic. Let \( h^{-1} \) denote its left-continuous pseudo-inverse. Then for continuous \( \phi: \mathbb{R} \to \mathbb{R} \),

\[
\int_I \phi(h(x) - x) \, dx = \int_I \phi(u - h^{-1}(u)) \, du.
\]

Proof. We can approximate \( h \) and \( h^{-1} \) pointwise almost-everywhere by \( h_n \) and \( h_n^{-1} \), respectively, for some sequence of \( C^1 \) diffeomorphisms \( (h_n) \). We may thus assume that \( h \) is itself a \( C^1 \) diffeomorphism.

Let \( I_1 \) and \( I_2 \) denote the left- and right-hand sides of (10), respectively. With the change of variables \( u = h(x) \), we have

\[
I_2 = \int_{h^{-1}(I)} \phi(h(x) - x)h'(x) \, dx.
\]

Now \( h - \text{id} \) is 1-periodic, so \( h^{-1}(I) \) is an interval of length 1. Since \( h' \) is also 1-periodic, we can shift the region of integration in (11) to \( I \) without changing the value of the integral. If \( \psi \) denotes an anti-derivative of \( \phi \), we find

\[
I_2 - I_1 = \int_I \phi(h(x) - x)[h'(x) - 1] \, dx = \int_I d[\psi(h - \text{id})] = \psi(h - \text{id})|_0^1 = 0.
\]

The final equality follows from 1-periodicity. □
On the interval, we can restrict the domain of $h$ to $I$. Taking $h = F^{-1}_\mu$ and $\phi(z) = |z|^p$, we find

$$W^I_p(\mu, \lambda)^p = \int_I |F^{-1}_\mu(u) - u|^p \, du = \int_I |F_\mu(x) - x|^p \, dx.$$ 

Now $F := F_\mu - \text{id}$ is an antiderivative of $\mu - \lambda$ with $F(0) = 0$, so by Lemma 5

$$W^I_p(\mu, \lambda) = \|F\|_{L^p(I)} = \|\mu - \lambda\|_{W^{-1, p}(I)}.$$

We next modify the argument to work on the torus. Given $\mu \in \mathcal{P}(\mathbb{T})$, we can lift $\mu$ and $\lambda$ to 1-periodic measures on the universal cover $\mathbb{R}$. These measures have infinite mass on $\mathbb{R}$, so we search for transport plans which are optimal with respect to any local modification. We still expect the monotone transport plans to be locally optimal, but these are no longer unique. Indeed, there is ambiguity in where we begin “filling-in” the mass of $\mu$ with that of $\lambda$. That is, to transport $\lambda$ to $\mu$ we could first move mass at 0 to position $y$, and then fill-in monotonically around that starting point. We call this the $y$-monotonic transport plan.

In [7], Delon, Salomon, and Sobolevski confirm that the locally optimal transport plans are precisely the $y$-monotonic plans. Furthermore, an optimal plan on $\mathbb{T}$ lifts to a locally optimal plan on $\mathbb{R}$. Thus an optimal transport plan on $\mathbb{T}$ is the restriction of a $y$-monotonic plan to $I$ for some $y \in \mathbb{R}$.

To determine which $y$ is optimal on $\mathbb{T}$, we compute the transport cost $C_p[y]$ in $I$ under the $y$-monotonic plan. With $F_\mu$ defined as before, let $F^y_\mu := F_\mu(\cdot - y)$. Then the cost is

$$C_p[y]^p = \int_I |(F^y_\mu)^{-1}(u) - u|^p \, du$$

and [7] shows that

$$W^\mathbb{T}_p(\mu, \lambda) = \inf_{y \in \mathbb{R}} C_p[y].$$

Applying Lemma 3 with $h = F^y_\mu$ and $\phi(z) = |z|^p$, we find

$$W^\mathbb{T}_p(\mu, \lambda)^p = \inf_{y \in \mathbb{R}} \int_I |(F^y_\mu)^{-1}(u) - u|^p \, du = \inf_{y \in \mathbb{R}} \int_I |F^y_\mu(x) - x|^p \, dx.$$ 

Shifting by $y$, periodicity and Lemma 5 imply

$$W^\mathbb{T}_p(\mu, \lambda) = \inf_{y \in \mathbb{R}} \|F - y\|_{L^p(\mathbb{T})} = \|\mu - \lambda\|_{W^{-1, p}(\mathbb{T})}.$$

We have thus verified (5) on $I$ and $\mathbb{T}$ for all $p \in [1, \infty)$. Taking $p \to \infty$, we automatically obtain (5) at the endpoint $p = \infty$. Nevertheless, we offer a self-contained proof in this case.

Assume $\mu \neq \lambda$. For $A \subset X$ and $\varepsilon > 0$, let

$$A^\varepsilon := \{x \in X \mid \text{dist}(x, A) \leq \varepsilon\}$$

denote the $\varepsilon$-neighborhood of $A$. As a consequence of Strassen’s theorem [11] [29],

$$W^X_\infty(\mu, \lambda) = \inf \{\varepsilon \geq 0 \mid \mu(A) \leq \lambda(A^\varepsilon) \text{ for all Borel } A \subset X\}. \quad (12)$$
Let \( \varepsilon_0 := \|\mu - \lambda\|_{\mathcal{W}^{-1,\infty}(X)} > 0 \), and fix a Borel set \( A \subset X \). Let

\[
A^* = \{ x \in A^{\varepsilon_0} : \text{dist}(x, \partial A^{\varepsilon_0}) \geq \varepsilon_0 \}.
\]

Clearly, \( A \subset A^* \). Also, \( A^{\varepsilon_0} \) and \( A^* \) are unions of closed intervals, and each point in \( \partial A^* \) corresponds to an \( \varepsilon_0 \)-wide buffer of \( A^{\varepsilon_0} \) around \( A^* \). (Note that on \( I \) we use the subspace topology, so \( \{0,1\} \) are never boundary points of closed sets.) Thus

\[
\mu(A^*) - \lambda(A^*) = \int_X 1_{A^*}(\mu - \lambda) \leq \|1_{A^*}\|_{TV} \|\mu - \lambda\|_{\mathcal{W}^{-1,\infty}(X)} = |\partial A^*| \varepsilon_0
\]

and

\[
\mu(A) \leq \mu(A^*) \leq |\partial A^*| \varepsilon_0 + \lambda(A^*) = \lambda(A^{\varepsilon_0} \setminus A^*) + \lambda(A^*) = \lambda(A^{\varepsilon_0}).
\]

By \( \text{(12)} \), we have \( W^X_{\varepsilon_0}(\mu, \lambda) \leq \varepsilon_0 = \|\mu - \lambda\|_{\mathcal{W}^{-1,\infty}(X)} \).

For the other direction, fix \( \varepsilon \in (0, \varepsilon_0) \) and let \( \delta := \varepsilon_0 - \varepsilon \). First consider \( X = \mathbb{T} \), and let \( F \) denote a left-continuous anti-derivative of \( \mu - \lambda \). By Lemma \( \text{[5]} \)

\[
\|\mu - \lambda\|_{\mathcal{W}^{-1,\infty}(\mathbb{T})} = \inf_{y \in \mathbb{R}} \|F - y\|_{L^\infty(\mathbb{T})}.
\]

Evidently, this infimum is attained at \( y = \frac{\sup F + \inf F}{2} \). Thus

\[
\varepsilon_0 = \|\mu - \lambda\|_{\mathcal{W}^{-1,\infty}(\mathbb{T})} = \frac{\sup F - \inf F}{2}.
\]

Take \( x_\pm \in \mathbb{T} \) such that

\[
F(x_+) > \sup F - \delta \quad \text{and} \quad F(x_-) < \inf F + \delta.
\]

After a rotation, we may assume that \( 0 \leq x_- \leq x_+ < 1 \) in our representation \( \mathbb{T} = \mathbb{R}/\mathbb{Z} \). Let \( J = [x_-, x_+] \). Then

\[
\mu(J) - \lambda(J) = F(x_+) - F(x_-) > \sup F - \inf F - 2\delta = 2\varepsilon_0 - 2\delta = 2\varepsilon.
\]

Hence

\[
\mu(J) > \lambda(J) + 2\varepsilon \geq \lambda(J^\varepsilon).
\]

Since such an interval exists for all \( \varepsilon < \varepsilon_0 \), \( \text{(12)} \) implies

\[
W^\mathbb{T}_{\varepsilon_0}(\mu, \lambda) \geq \varepsilon_0 = \|\mu - \lambda\|_{\mathcal{W}^{-1,\infty}(\mathbb{T})}.
\]

When \( X = I \), we let \( F \) denote the left-continuous antiderivative such that \( F(0) = 0 \), so that \( \|\mu - \lambda\|_{\mathcal{W}^{-1,\infty}(I)} = \|F\|_{L^\infty(I)} \). If we take \( J = [0, x_+] \) or \([x_-, 1)\), manipulations much as above yield

\[
\mu(J) > \lambda(J) + \varepsilon \geq \lambda(J^\varepsilon),
\]

and the conclusion follows as before. \( \square \)
3. Irregularity of distribution in one dimension

We now pivot to the irregularity of distribution of sequence \( s \). Since we have identified the \( L^p \)-discrepancy with the Sobolev and Wasserstein distances, Theorem 3 largely follows from the classical result s in Theorem 1. The principal remaining obstacle is the set of lower bounds in Theorem 3 on the circle. To prove these lower bounds, we follow the harmonic analysis approach of Roth and Halász [25, 12]. Bilyk lucidly presents this method in [3], which we follow closely.

In his seminal paper [25], Roth observed a crucial equivalence between irregularity problems on \( I \) and \( I^2 \). Roth proved that point distributions on \( I^2 \) are irregular for all \( N \in \mathbb{N} \), and showed that this implies irregularity in one dimension for infinitely many \( N \). We adapt this approach to distributions on the circle.

Given a finite point set \( \{ x_n \}_{n=1}^N \subset \mathbb{T} \times I \), let \( \mu \) denote its empirical measure as in (1). We define the two-dimensional discrepancy function \( D : \mathbb{T} \times I \to \mathbb{R} \) by

\[
D(x, y) := \mu([0, x) \times [0, y)) - xy.
\]

By Proposition 2 and Lemma 5, we wish to control the one-dimensional discrepancy \( D_N \) in the quotient space \( L^p(\mathbb{T})/\mathbb{R}1 \). We must therefore measure \( D \) in a space that is compatible with these quotients on horizontal slices. Let \( V^p := 1 \otimes L^p(I) \) denote the closed subspace of \( L^p(\mathbb{T} \times I) \) consisting of functions that are independent of the first variable. We will control \( D \) in the quotient space \( L^p(\mathbb{T} \times I)/V^p \). The stratified structure of this space permits us to descend to the quotient \( L^p(\mathbb{T})/\mathbb{R}1 \) on horizontal slices.

**Lemma 7.** There exists a universal constant \( C > 0 \) such that for any \( N \in \mathbb{N}, p \in [1, \infty], \) and point set \( \{ x_n \}_{n=1}^N \subset \mathbb{T} \times I \), the discrepancy function \( D \) of the point set satisfies

\[
\|D\|_{L^p(\mathbb{T} \times I)/V^p} \geq C \frac{\log^p N}{N}. \tag{14}
\]

Before proving the lemma, we use it to establish our main theorem.

**Proof of Theorem 3.** The identities (3) and (5) imply

\[
\|D_N\|_{L^p(I)} = W^p_I(\mu_N, \lambda).
\]

Thus on the interval, Theorem 3 simply restates the classical results in Theorem 1. Furthermore, any transport plan on \( I \) can be interpreted as a plan on \( \mathbb{T} \), so \( W^p_\mathbb{T} \leq W^p_I \). It follows that the upper bounds in Theorem 1 on \( I \) also serve as upper bounds in Theorem 3 on \( \mathbb{T} \). See also [25] for an upper bound proven specifically for the Wasserstein-2 distance on \( \mathbb{T} \). It remains only to prove the lower bounds on \( \mathbb{T} \).

Fix \( p \in [1, \infty] \) and a sequence \( (x_n)_{n \in \mathbb{N}} \subset \mathbb{T} \). For each \( m \in \mathbb{N} \), we form an \( m \)-point set \( \{ x_n \}_{n=1}^m \subset \mathbb{T} \times I \) by

\[
x_n^m := \left( x_n, \frac{n-1}{m} \right).
\]
That is, we embed the first $m$ terms of $(x_n)$ in the cylinder by interpreting the “time” $\frac{n-1}{m}$ as a second coordinate. Let $D^m$ denote the two-dimensional discrepancy function associated to this $m$-point set by (13). By Lemma 7
\[ \|D^m\|_{L^p(T \times I)} \geq C \frac{\log^{\alpha_p} m}{m}. \]
That is,
\[ \inf_{g \in L^p(I)} \|D^m - 1 \otimes g\|_{L^p(T \times I)} \geq C \frac{\log^{\alpha_p} m}{m}. \]
By Fubini, there exists $t_m \in I$ such that
\[ \inf_{y \in \mathbb{R}} \|D^m(\cdot, t_m) - y\|_{L^p(T)} \geq C \frac{\log^{\alpha_p} m}{m}. \]
Now $t_m$ is nearly the time corresponding to the index $N_m := \lceil mt_m \rceil$. Writing $t_m = N_m - \delta$ with $\delta \in [0, \frac{1}{m})$, we have
\[ D^m(x, t_m) = \frac{N_m}{m} D_{N_m}(x) - \delta x \quad \text{for all } x \in T, \]
where $D_{N_m}$ is the one-dimensional discrepancy of $(x_n)$ at stage $N_m$. Therefore
\[ \|D_{N_m}\|_{L^p(T) / \mathbb{R}^1} \geq \frac{m}{N_m} \left( \|D^m(\cdot, t_m)\|_{L^p(T) / \mathbb{R}^1} - \delta \|\text{id}\|_{L^p(T)} \right) \]
\[ \geq \frac{C}{N_m} \left( \log^{\alpha_p} m - C' \right). \]
Once $m$ is sufficiently large, we find
\[ \|D_{N_m}\|_{L^p(T) / \mathbb{R}^1} \geq \frac{C}{N_m} \frac{\log^{\alpha_p} m}{m} \geq C \frac{\log^{\alpha_p} N_m}{N_m}, \quad \text{(15)} \]
where we have allowed the constant $C > 0$ to change from line to line. The first inequality in (15) cannot hold as $m \to \infty$ if the sequence $(N_m)_{m \in \mathbb{N}}$ remains bounded, so it must hold for infinitely many $N_m$. □

Remark 6. In fact, the $p = 2$ case of Theorem 3 follows from classical results. Indeed, in this case Proposition 2 and Parseval imply
\[ W^T_{2}(\mu_N, \lambda) = \|\mu_N - \lambda\|_{H^{-1}(\mathbb{T})} = \frac{1}{2\pi} \left( \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{|\hat{\mu}_N(k)|^2}{k^2} \right)^{\frac{1}{2}}. \quad \text{(16)} \]
The final expression is the diaphony of Zinterhof (up to a factor of $2\pi$). In [23], Pro˘ınov announced that the diaphony exceeds $C \frac{\sqrt{\log N}}{N}$ infinitely often, confirming Theorem 1 in the case $p = 2$. (We were, however, unable to locate a published proof.)

Additionally, if $F$ denotes the antiderivative of $\mu - \lambda$ with $F(0) = 0$ and $\bar{F} := \int_I F$, Proposition 2 and Lemma 5 imply
\[ W^T_{2}(\mu_N, \lambda)^2 = W^T_{2}(\mu_N, \lambda)^2 + \bar{F}^2. \]
This is simply a reinterpretation of a formula of Koksma [15, 14]. It precisely quantifies the additional transport cost incurred when the movement of mass across the point $0 \in \mathbb{T}$ is forbidden.

Proof of Lemma 7. Fix $N \in \mathbb{N}$, the point set $\{x_n\}_{n=1}^N \subset \mathbb{T} \times I$, and its discrepancy function $\mathcal{D}$. In [3], Bilyk condenses ideas of Roth and Halász to prove

$$\|\mathcal{D}\|_{L^p(\mathbb{T} \times I)} \geq C \frac{\log^a(p) N}{N}$$

for any $p \in [1, \infty]$. (Note that our discrepancy function differs from that in [3] by a factor of $N^{-1}$. ) This suffices for irregularity on the interval. On the torus, however, we must measure $\mathcal{D}$ in the smaller quotient norm of $L^p / V^p$. We claim that Bilyk actually proves the stronger bound (14).

When $p = 2$, Bilyk follows Roth and proves (17) via duality [3, §2]. He uses Haar wavelets to construct an explicit test function $F$ on $\mathbb{T} \times I$ such that

$$\|F\|_{L^2(\mathbb{T} \times I)} \asymp \sqrt{\log N} \quad \text{and} \quad \langle \mathcal{D}, F \rangle_{L^2(\mathbb{T} \times I)} \gtrsim \frac{\log N}{N}.$$  

The estimate (17) then follows from Cauchy-Schwarz. To control $\mathcal{D}$ in $L^2 / V^2$, we must handle coset representatives of the form $\mathcal{D} - 1 \otimes g$ for $g \in L^2(I)$. Fortuitously, the Haar wavelets comprising $F$ are mean-zero along every horizontal slice. It follows that

$$\langle F, 1 \otimes g \rangle = 0$$

for all $g \in L^p(I)$. Therefore

$$\|\mathcal{D} - 1 \otimes g\|_{L^2} \geq \frac{\langle \mathcal{D} - 1 \otimes g, F \rangle}{\|F\|_2} = \frac{\langle \mathcal{D}, F \rangle}{\|F\|_2} \gtrsim \frac{\sqrt{\log N}}{N}. $$

Since this holds uniformly for $g \in L^2(I)$, we obtain

$$\|\mathcal{D}\|_{L^2 / V^2} \gtrsim \frac{\sqrt{\log N}}{N},$$

i.e. the $p = 2$ case of Lemma 7.

When $p \in (1, \infty)$, Bilyk employs Littlewood–Paley theory [3, §3] to show that $\|F\|_q \asymp \sqrt{\log N}$, where $q$ denotes the Hölder exponent dual to $p$. We thus obtain Lemma 7 for $p \in (1, \infty)$ in an identical manner. When $p = 1$ or $\infty$, Halász had the insight of using “Riesz products” as test functions [12]. The construction is slightly more elaborate, but in [3, §4] we nonetheless arrive at test functions in the span of the Haar basis which are mean-zero on horizontal slices. These test functions are thus also immune to the quotient space complication, and imply the $p = 1$ and $\infty$ cases of Lemma 7. \hfill $\square$
4. An $L^p$ Erdős–Turán inequality

The classical Erdős–Turán inequality \[8, 9\] controls another notion of discrepancy on the circle:
$$\text{disc}(\mu, \lambda) := \sup_{J \subset \mathbb{T}} |\mu(J) - \lambda(J)|,$$
where the supremum ranges over all subintervals $J \subset \mathbb{T}$. It states:
$$\text{disc}(\mu, \lambda) \leq \frac{C}{n} + C \sum_{k=1}^{n-1} \frac{|\hat{\mu}(k)|}{k} \quad \text{for all } n \in \mathbb{N}. \tag{18}$$

In fact, this form of discrepancy is cleanly related to those we’ve already considered:
$$\text{disc}(\mu, \lambda) = 2 \|\mu - \lambda\|_{W^{-1,\infty}(\mathbb{T})} = 2W_{\infty}^p(\mu, \lambda). \tag{19}$$

After all, if $F$ denotes a left-continuous antiderivative of $\mu - \lambda$, the regularity of $\mu$ implies
$$\sup_{J \subset \mathbb{T}} |\mu(J) - \lambda(J)| = \sup_{s \in [t, t+\delta]} (s - t) - \mu([t, s]) \leq \delta.$$

By (19), the $p = \infty$ case of Proposition 4 is equivalent to (18).

To prove Proposition 4, we introduce a lemma in the mode of Ganelius \[10\]. For a real function $V \in L^\infty(\mathbb{T})$, define the one-sided modulus of continuity
$$\omega(\delta; V) := \sup_{s \in [t, t+\delta]} |V(s) - V(t)|.$$

Lemma 8. There exists a universal $C > 0$ such that for all $p \in [2, \infty]$, $n \in \mathbb{N}$, and $V \in L^\infty(\mathbb{T})$,
$$\|V\|_{L^p(\mathbb{T})} \leq C \omega \left( \frac{C}{n}; V \right) + C \left( \sum_{k=0}^{n-1} |\hat{V}(k)|^q \right)^{\frac{1}{q}},$$
where $q$ is the Hölder-conjugate of $p$.

Proposition 4 follows easily from Lemma 8.

Proof of Proposition 4. Let $F$ denote the mean-zero antiderivative of $\lambda - \mu$ (opposite our earlier choice). By Lemma 5, $\|\mu - \lambda\|_{W^{-1,p}(\mathbb{T})} \leq \|F\|_{L^p(\mathbb{T})}$. Now note that
$$\omega(\delta; F) = \sup_{s \in [t, t+\delta]} [(s - t) - \mu([t, s])] \leq \delta.$$

In particular, when $\delta = \frac{C}{n}$, Lemma 8 implies
$$\|F\|_{L^p(\mathbb{T})} \leq \frac{C}{n} + C \left( \sum_{k=0}^{n-1} |\hat{F}(k)|^q \right)^{\frac{1}{q}}, \tag{20}$$
where we have allowed $C$ to change from line to line. Now $\hat{F}(0) = 0$ because $F$ is mean-zero, and $\hat{F}(k) = \frac{\hat{\mu}(k)}{2\pi ik}$ for $k \geq 1$. Thus (20) implies (3).

Now consider the interval. The discrepancy function $D(x) := \mu([0, x)) - x$ is a left-continuous antiderivative of $\mu - \lambda$ with $D(0) = 0$. By Lemma 5, $\|\mu - \lambda\|_{\bar{W}^{-1,p}(I)} = \|D\|_{L^p(I)}$. Now $D(0) = \int_I D =: D$. Using $D$ in the place of $F$ in (20) and retaining the $k = 0$ term in the sum, we obtain (7). \qed

To prove the lemma, we follow the approach of Ganelius [10].

Proof of Lemma 8. Fix $n \in \mathbb{N}$ and $V \in L^\infty(\mathbb{T})$. We first assume $p \in [2, \infty)$. Let
\[
\Psi_n(t) := \frac{\sin^2(\pi nt)}{n \sin^2(\pi t)}
\]
denote the Fejér kernel. Then $\Psi_n$ has mass 1 concentrated at scale $\frac{1}{n}$ around $t = 0$. Precisely, there exists $C_0 > 0$ such that for all $\delta \in (0, 1)$,
\[
\int_{-\delta}^{\delta} \Psi_n(t) \, dt \geq 1 - \frac{C_0}{n\delta}.
\]
Fix $\delta := \frac{8C_0}{n}$, so that
\[
\int_{\mathbb{T}\setminus[-\delta,\delta]} \Psi_n \leq \frac{1}{8}.
\]
Let $\omega := \omega(2\delta; V)$. Define the subsets
\[
\Omega^\pm := \{ t \in \mathbb{T} \mid \pm V(t) > \omega \}.
\]
By the definition of $\omega$, the translated sets $\Omega^+ - \delta$ and $\Omega^- + \delta$ are disjoint.

Next, let $\Psi_n^1 := 1_{[-\delta,\delta]} \Psi_n$ and $\Psi_n^2 := \Psi_n - \Psi_n^1$. By the generalized AM-GM inequality, $|a + b|^p \leq 2^{p-1}(|a|^p + |b|^p)$ for all $a, b \in \mathbb{R}$. After substitution and rearrangement, this implies
\[
|a - b|^p \geq 2^{1-p} |a|^p - |b|^p \quad \text{for all } a, b \in \mathbb{R}. \tag{21}
\]
In particular,
\[
\int_{\mathbb{T}} |\Psi_n^1 \ast V|^p \geq 2^{1-p} \int_{\mathbb{T}} |\Psi_n^1 \ast V|^p - \int_{\mathbb{T}} |\Psi_n^2 \ast V|^p. \tag{22}
\]
We control the second term with Young’s inequality:
\[
\|\Psi_n^2 \ast V\|_{L^p}^p \leq \|\Psi_n^2\|_{L^1}^p \|V\|_{L^p}^p \leq 2^{-3p} \|V\|_{L^p}^p.
\]
For the first term in (22), write
\[
\int_{\mathbb{T}} |\Psi_n^1 \ast V|^p \geq \int_{\Omega^+ - \delta} |\Psi_n^1 \ast V|^p + \int_{\Omega^- + \delta} |\Psi_n^1 \ast V|^p.
\]
Now if $t \in \Omega^+$, the definition of $\omega$ implies
\[
\Psi_n^1 \ast V(t - \delta) = \int_{-\delta}^{\delta} \Psi_n(s) V(t - \delta - s) \, ds \geq [V(t) - \omega] \int_{-\delta}^{\delta} \Psi_n(s) \, ds \geq \frac{1}{2} [V(t) - \omega].
\]
Similarly, if \( t \in \Omega^- \), we have
\[
|\Psi_n^1 \ast V(t + \delta)| \geq \frac{1}{2} |V(t)| - \omega|.
\]
Thus by (21) again,
\[
\int_{\Omega^+ \mp \delta} |\Psi_n^1 \ast V|^p \geq 2^{-p} \int_{\Omega^\pm} (|V| - \omega)^p \geq 2^{-p} \int_{\Omega^\pm} (2^{1-p} |V|^p - \omega^p).
\]
On the other hand, \( |V| \leq \omega \) on \( T \setminus (\Omega^+ \cup \Omega^-) \), so
\[
\int_{T \setminus (\Omega^+ \cup \Omega^-)} |V|^p \leq \omega^p.
\]
Collecting these results, (22) implies
\[
\int_{T} |\Psi_n \ast V|^p \geq 2^{2-3p} \int_{T} |V|^p - 2^{2-2p} \omega^p - 2^{2-3p} \int_{T} |V|^p \geq 2^{1-3p} \int_{T} |V|^p - 2^{2-2p} \omega^p.
\]
Rearranging, we find
\[
\int_{T} |V|^p \leq 2^{2p} \omega^p + 2^{3p} \int_{T} |\Psi_n \ast V|^p.
\]
Taking \( \frac{1}{p} \) powers, this becomes
\[
\|V\|_{L^p} \leq 4 \omega + 8 \|\Psi_n \ast V\|_{L^p}.
\]
(23)
The same holds for \( p = \infty \) by taking limits, or by trivially adjusting the above manipulations. In fact, (23) holds for all \( p \geq 1 \).

Now recall that
\[
\hat{\Psi}_n(k) = \left(1 - \frac{|k|}{n}\right) 1_{[-n,n]}(k) \leq 1_{[1-n,n-1]}(k)
\]
When \( p \geq 2 \), the Hausdorff–Young inequality implies
\[
\|\Psi_n \ast V\|_{L^p} \leq \left(\sum_{k=1-n}^{n-1} |\hat{V}(k)|^q\right)^{\frac{1}{q}}.
\]
Since \( V \) is real, we obtain
\[
\|V\|_{L^p} \leq C \omega \left(\frac{C}{n} V\right) + C \left(\sum_{k=0}^{n-1} |\hat{V}(k)|^q\right)^{\frac{1}{q}},
\]
perhaps after increasing \( C \). \( \square \)
4.1. **An application in number theory.** As an example, we use Propositions 4 and 1 to study the distribution of quadratic residues in finite fields of prime order. Given a prime \( p \), we consider the set

\[
Q_p := \left\{ \left\{ \frac{m^2}{p} \right\} \mid 1 \leq m \leq p \right\} \subset \mathbb{T},
\]

where \( \{ \cdot \} \) denotes the fractional part. We measure the equidistribution of \( Q_p \) in Wasserstein-\( r \) distances. We mimic the approach of Steinerberger [28], who considered \( r \in \{2, \infty\} \). We extend his results to the intermediate exponents \( r \in (2, \infty) \).

Let \( \mu_p \) denote the empirical measure of \( Q_p \) as in (1). To apply Proposition 4, we must control the Gauss sum

\[
\hat{\mu}_p(k) = \frac{1}{p} \sum_{m=1}^{p} e^{-2\pi i m^2/p}.
\]

This is a classical object in analytic number theory. Gauss showed:

\[
|\hat{\mu}_p(k)| = \begin{cases} 
1 & \text{if } p | k, \\
\frac{1}{\sqrt{p}} & \text{if } p \nmid k. 
\end{cases} \tag{24}
\]

Now let \((r, s)\) be Hölder-conjugates with \( r \in [2, \infty] \). By Proposition 4 with \( n = p \), we have

\[
W_T^*(\mu_p, \lambda) \leq C_p + C \left( \sum_{k=1}^{p-1} |\hat{\mu}_p(k)|^s \right)^{\frac{1}{s}} \leq C_p + C \left( \frac{1}{\sqrt{p}} \sum_{k=1}^{p-1} k^{-s} \right)^{\frac{1}{s}}.
\]

Thus when \( r \in [2, \infty) \), there exists a constant \( C_r > 0 \) such that

\[
W_T^*(\mu_p, \lambda) \leq \frac{C_r}{\sqrt{p}} \tag{25}
\]

In fact, this holds for all \( r \in [1, \infty) \) by Hölder’s inequality. However, when \( r = \infty \) we lose a logarithmic factor:

\[
W_T^*(\mu_p, \lambda) \leq \frac{C_\infty \log p}{\sqrt{p}}. \tag{26}
\]

In light of (19), we have thus recovered a well-known result of Pólya and Vinogradov [22, 34].

Establishing lower bounds seems more challenging, but we can handle the case \( p = 2 \) explicitly. Indeed, (16) and (24) imply

\[
W_T^*(\mu_p, \lambda)^2 = \frac{1}{4\pi^2} \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{|\hat{\mu}_p(k)|^2}{k^2} = \frac{1}{4\pi^2} \left( \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{k^2} + \frac{1}{p} \sum_{k \in \mathbb{Z} \setminus \mathbb{Z} \setminus \{0\}} \frac{1}{k^2} \right).
\]

Rearranging these sums, we obtain

\[
W_T^*(\mu_p, \lambda) = \sqrt{\frac{p^2 + p - 1}{12p^3}} \geq \frac{1}{\sqrt{12p}}.
\]
It follows that (25) is tight up to the value of $C_r$ for all $r \in [2, \infty)$. However, assuming the generalized Riemann hypothesis, Montgomery and Vaughan \cite{MR1738163} showed that (26) can be improved to

$$W^T_\infty(\mu_p, \lambda) \leq C \log \log p \sqrt{p}.$$ 

A construction of Paley \cite{MR102006} suggests this is optimal up to the value of $C$.

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