Raising and Lowering operators of spin-weighted spheroidal harmonics

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In this paper we generalize the spin-raising and lowering operators of spin-weighted spherical harmonics to linear-in-$\gamma$ spin-weighted spheroidal harmonics where $\gamma$ is an additional parameter present in the second order ordinary differential equation governing these harmonics. One can then generalize these operators to higher powers in $\gamma$. Constructing these operators required calculating the $\ell$, $s$- and $m$-raising and lowering operators (and various combinations of them) of spin-weighted spherical harmonics which have been calculated and shown explicitly in this paper.

I. INTRODUCTION

Spin-weighted spheroidal harmonics, $sS^\gamma_{\ell,m}$, arise naturally in any analysis of the angular dependence of propagating fields on rotating, Kerr black hole space-time background, and are most studied in the differential equations governing linear electromagnetic and gravitational perturbations. When the spin, $s$, of the propagating field is zero, these angular eigenfunctions become the oblate (scalar) spheroidal harmonics. When the black hole is spherically symmetric, the full angular eigenfunctions are the spin-weighted spherical harmonics, well known in other areas of Physics. In this work we will focus, for the first time, on describing the operators for raising and lowering the spin index of the spin-weighted spheroidal harmonics. We will do this to first order in the parameter perturbing away from the spin-weighted spherical harmonics, and lay the groundwork for extending the procedure to higher order. For simplicity, we will generally assume an unwritten factor of $e^{imφ}$ throughout, and shall concentrate primarily on the $θ$-dependence, since the azimuthal eigen-equation is rather trivial.

Spin-weighted spheroidal harmonics satisfy the angular part of Teukolsky’s master equation:

\begin{equation}
\hat{\gamma}_s Y_{\ell,m} = \frac{1}{\sin θ} \frac{d}{dθ} \left( \sin θ \frac{dz}{dθ} \right) \left( m^2 + s^2 + 2ms \cos θ \right) \frac{\gamma^2 \cos^2 θ + 2sg\cos θ - sE^\gamma_{\ell,m}}{\sin^2 θ} = 0 \tag{1}
\end{equation}

where $s$ is the spin weight of the harmonic, and $sE^\gamma_{\ell,m}$ is the eigenvalue which, in the limit $\gamma \rightarrow 0$, is $ℓ(ℓ+1)$. As with any 2nd order differential equation, Eq. (1) has two linearly independent solutions, one of which, $sS^\gamma_{\ell,m}$ is generally used for describing scalar, (massless) neutrino, electromagnetic and gravitational perturbations. In the limit $\gamma \rightarrow 0$, these harmonics are the spin-weighted spherical harmonics, $sY_{\ell,m}$. In the limit $s \rightarrow 0$, $sS^\gamma_{\ell,m}$ are the ordinary spheroidal harmonics, $S^\gamma_{\ell,m}$, and $sY_{\ell,m}$ are the ordinary spherical harmonics, $Y_{\ell,m}$. The $sY_{\ell,m}$ appear as a solution to the equation

\begin{equation}
\frac{1}{\sin θ} \frac{d}{dθ} \left( \sin θ \frac{dz}{dθ} \right) - \left( m^2 + s^2 + 2ms \cos θ \right) \frac{\gamma^2 \cos^2 θ + 2sg\cos θ - sE^\gamma_{\ell,m}}{\sin^2 θ} = 0. \tag{2}
\end{equation}

To build $sY_{\ell,m}$, one repeatedly applies spin raising and lowering operators on ordinary spherical harmonics, separately computing eigenfunctions with positive and negative values of spin-weight:

\begin{equation}
\begin{aligned}
sY_{\ell,m} &= \bar{\partial}_s \bar{\partial}_{s-2} \cdots \bar{\partial}_{s-|s|} Y_{\ell,m}, \\
-|s|Y_{\ell,m} &= \bar{\partial}_{s+|s|} \bar{\partial}_{s+2} \cdots \bar{\partial}_{s+|s|} Y_{\ell,m}, \tag{3}
\end{aligned}
\end{equation}

where

\begin{align*}
\bar{\partial}_s &= \frac{\partial_0 - m \csc θ - s \cot θ}{\sqrt{(ℓ-s)(ℓ+s+1)}}, \\
\bar{\partial}_{s+|s|} &= \frac{\partial_0 + m \csc θ + s \cot θ}{\sqrt{(ℓ+s)(ℓ-s+1)}}.
\end{align*}

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\[ oY_{\ell,m} = \sqrt{\frac{(2\ell + 1)(\ell - m)!}{4\pi(\ell + m)!}} \rho^m_e(\cos\theta)e^{im\phi}. \]  

(4)

Here the \( \partial_\alpha \) is the raising operator, \( \partial_\gamma \) is the lowering operator, and \( P^m_\ell \) are the associated Legendre functions. \( \partial_\alpha \) and \( \partial_\gamma \) are given a subscript here to show which spin-weighted quantities they act on. For each \( s \), the \( sY_{\ell,m} \) are complete and orthogonal functions on the 2-sphere, and are related to the Wigner D-rotation matrices by

\[ sY_{\ell,m}(\theta, \phi) = \sqrt{\frac{2\ell + 1}{4\pi}} D^\ell_{s\pm m}(\theta, \phi, 0). \]  

(5)

Unlike the way \( sY_{\ell,m} \) are calculated using raising and lowering operators, spin-weighted spheroidal harmonics are usually calculated as a sum over \( sY_{\ell,m} \) or as a sum over Jacobi polynomials. In this paper we work on generalizing \( \partial_\alpha \) and \( \partial_\gamma \) to operators that raise and lower the spin-weight of spheroidal harmonics, \( sS^\ell_{\gamma,m} \). That is, \( z = sS^\ell_{\gamma,m} \) and \( y = s\mp 1S^\ell_{\gamma,m} \) are solutions to two different differential versions of Eq. (11), one being \( s^\gamma_{\ell,m} z = 0 \) and the other being \( s\mp 1S^\ell_{\gamma,m} y = 0 \), and we will find a relation of the form:

\[ y = \alpha z + \beta \partial_\gamma z, \]  

(6)

to linear order in \( \gamma \), between the solutions \( (y \text{ and } z) \) of these equations.

The paper is organized as follows. In Section II, we summarize Whiting’s earlier work on finding relations between solutions of two differential equations. In Section III, we use this work to calculate the different \( \ell \)-, \( s \)- and \( m \)-raising and lowering relations of \( sY_{\ell,m} \). In Section IV, we build the linear-in-\( \gamma \) \( s \)-raising and lowering operators for \( sS^\ell_{\gamma,m} \).

II. EARLIER WORK ON RELATING SOLUTIONS OF TWO DIFFERENTIAL EQUATIONS

Relations of the general form which we seek have been studied previously by one of us and were extensively used in \( \partial_\alpha \) to show mode stability for the perturbations being discussed here. We now give a brief, and slightly more general, introduction, while more complete details can be found in this paper. Thus, we suppose that \( y(x) \) and \( z(x) \) satisfy

\[ y'' + py' + qy = 0 \quad \text{and} \quad z'' + pz' + Qz = 0, \]  

(7)

in which \( ' = d/dx \), and seek conditions that \( \alpha \) and \( \beta \) must satisfy in order that

\[ y = \alpha z + \beta z', \]  

(8)

should hold. More specifically, since each of Eq. (7) is second order, two linearly independent solutions exist, say \( (y_1, y_2) \) and \( (z_1, z_2) \) respectively, and we will actually demand that the mapping (8) applies more fully, so that:

\[ y_1 = \alpha z_1 + \beta z_1', \]  

\[ y_2 = \alpha z_2 + \beta z_2'. \]  

(9)

That is, every solution for \( y_1 \) and \( y_2 \) will map to a solution for \( z_1 \) and \( z_2 \). Defining the relevant Wronskians by:

\[ W(y_1, y_2) \equiv W_y = y_1'y_2 - y_1y_2' = C_ye^{-\int p dx}, \]  

\[ W(z_1, z_2) \equiv W_z = z_1'z_2 - z_1z_2' = C_ze^{-\int p dx}, \]  

(10)

where \( C_y \) and \( C_z \) are constants, we can invert Eqs (8) to find \( \alpha \) and \( \beta \):

\[ \alpha = \frac{1}{W_z} (y_1 z'_2 - y_2 z'_1), \]  

\[ \beta = -\frac{1}{W_z} (y_1 z_2 - y_2 z_1). \]  

(11)

Clearly, \( \alpha \) and \( \beta \) are determined entirely by the solutions they map between. Differentiating (8) once and using Eq. (7) for \( z \) we find:

\[ y' = (\alpha' - \beta Q)z + (\alpha + \beta' - \beta P)z'. \]  

(12)

Eqs (8) and (12) together can be inverted to give \( z \) and \( z' \) in terms of \( y \) and \( y' \). For this we will also need to define:

\[ k = (\alpha + \beta' - \beta P)\alpha - (\alpha' - \beta Q) = \frac{W_y}{W_z}. \]  

(13)

Then

\[ z = \frac{1}{k} ((\alpha + \beta')y - \beta y'), \]  

\[ z' = \frac{1}{k} ((\alpha' - \beta Q)y + \alpha y'). \]  

(14)

Further differentiating Eq. (12), and using both Eqs (13), we can deduce:

\[ 0 = (\alpha'' + p\alpha' + (q - Q)\alpha - 2Q\beta' - Q'\beta - (p - P)Q\beta) z + (2\alpha' + (p - P)\alpha + \beta'' + (p - 2P)\beta' + (q - Q)\beta - P'\beta - (p - P)P\beta) z', \]  

(15)

in which each coefficient must separately be zero because of Eqs (9). Thus:

\[ \alpha'' + p\alpha' + (q - Q)\alpha - 2Q\beta' - Q'\beta - (p - P)Q\beta = 0 \]  

\[ 2\alpha' + (p - P)\alpha + \beta'' + (p - 2P)\beta' + (q - Q)\beta - P'\beta - (p - P)P\beta = 0. \]  

(16)

With the appropriate combination of these, we can now show constructively that:

\[ k' = (P - p)k, \]  

(17)

as already follows from Eqs (10) and (13) above. In the application we have in mind, \( P = p \), so that \( k = \text{const.} \)
We could also check the integrability of Eqs (14) which, with Eq. (17) and some algebra, yields the second of Eqs (16).

Finally we note that the operators in Eq. (7) can be written as:

$$\partial_xx + p\partial_x + q = \left(\partial_x - \frac{\alpha}{\beta} + P\right)\left(\partial_x + \frac{\alpha}{\beta}\right) + \frac{k}{\beta^2},$$

$$\partial_xx + P\partial_x + Q = \left(\partial_x - \frac{\alpha}{\beta} + P\right)\left(\partial_x + \frac{\alpha}{\beta}\right) + \frac{k}{\beta^2},$$

in which the first order operators are effectively intertwined.

III. SPIN-WEIGHTED SPHERICAL HARMONICS

Let us denote spin-weighted spherical harmonics of type-1 and type-2 by \(sY_{\ell,m}\) and \(sX_{\ell,m}\), being two linearly independent solutions of Eq. (2), where, in the notation of section \(\text{[11]}\)

\[ z_1 = sY_{\ell,m}, \quad \text{and} \quad z_2 = sX_{\ell,m}. \]

(19)

To build the harmonics of non-zero spin, we begin with spin-weight zero ordinary spherical harmonics (suppressing \(e^{im\phi}\)):

\[ 0Y_{\ell,m} = \sqrt{\frac{(2\ell + 1)(\ell - m)!}{4\pi(\ell + m)!}} P^m_\ell(\cos \theta), \]

\[ 0X_{\ell,m} = \sqrt{\frac{(2\ell + 1)(\ell + m)!}{4\pi(\ell + m)!}} Q^m_\ell(\cos \theta), \]

(20)

and apply further \(s\)-raising and \(s\)-lowering operators to generate arbitrary spin-weighted spherical harmonics:

\[ \bar{\partial}_s = -\frac{(\partial_\theta - m \csc \theta - s \cot \theta)}{\sqrt{(\ell - s)(\ell + s + 1)}}, \]

\[ \partial_s = \frac{(\partial_\theta + m \csc \theta + s \cot \theta)}{\sqrt{(\ell + s)(\ell - s + 1)}}. \]

(21)

Therefore,

\[ sY_{\ell,m} = \partial_{s-1}\partial_{s-2} \ldots \partial_0 0Y_{\ell,m}, \]

\[ -|s|Y_{\ell,m} = \bar{\partial}_{-s+1}\bar{\partial}_{-s+2} \ldots \bar{\partial}_{-1} \partial_0 0Y_{\ell,m}, \]

(22)

and same holds for \(sX_{\ell,m}\).

From Section \(\text{[11]}\) we know that to find relations between solutions of equations

\[ z'' + Pz' + Qz = 0, \quad y'' + py' + qy = 0 \]

(23)

we need to calculate

\[ \beta = \frac{z_1 y_2 - z_2 y_1}{W(z_1, z_2)}, \quad \text{and} \quad \alpha = \frac{-z_1^* y_2 + z_2^* y_1}{W(z_1, z_2)}, \]

(24)

where \(W(z_1, z_2)\) is defined in \(\text{[11]}\). One then has

\[ y_i = \beta z_i' + \alpha z_i. \]

(25)

Finding \(\beta\)'s and \(\alpha\)'s for various relations, we get

\[ s_{+1}Y_{\ell,m} = \frac{-1}{\sqrt{(\ell - s)(\ell + s + 1)}} \times \left(\partial_\theta - \frac{m}{\sin \theta} - s \cot \theta\right) sY_{\ell,m} \]

(26)

\[ s_{-1}Y_{\ell,m} = \frac{1}{\sqrt{(\ell + s)(\ell - s + 1)}} \times \left(\partial_\theta + \frac{m}{\sin \theta} + s \cot \theta\right) sY_{\ell,m} \]

(27)

\[ sY_{\ell+1,m} = \sqrt{\frac{2\ell + 3}{2\ell + 1}} \sqrt{\frac{(\ell + 1)^2 - m^2}{(\ell + 1)^2 - s^2}} \times \left(\partial_\theta - \frac{ms}{(\ell + 1) \sin \theta} + (\ell + 1) \cot \theta\right) sY_{\ell,m} \]

(28)

\[ sY_{\ell-1,m} = \sqrt{\frac{2\ell - 1}{2\ell + 1}} \frac{-\ell \sin \theta}{\sqrt{(\ell^2 - m^2)(\ell^2 - s^2)}} \times \left(\partial_\theta - \frac{ms}{\ell \sin \theta} - m \cot \theta\right) sY_{\ell,m} \]

(29)

\[ sY_{\ell,m+1} = \sqrt{\frac{\ell - m}{\ell + m + 1}} \times \left(\partial_\theta - \frac{s}{\sin \theta} - m \cot \theta\right) sY_{\ell,m} \]

(30)

\[ sY_{\ell,m-1} = \sqrt{\frac{\ell + m}{\ell - m + 1}} \times \left(\partial_\theta + \frac{s}{\sin \theta} + m \cot \theta\right) sY_{\ell,m} \]

(31)

The above six relations are equivalent to Gauss’s relations for contiguous functions of the hypergeometric function, \(\, _2F_1(a, b; c; z)\). By equating \(\partial_\theta sY_{\ell,m}\) of two different relations, one can get various numbers of recurrence relations. For example, by equating \(\partial_\theta sY_{\ell,m}\) in Eqs (26) and (27), one gets a relations between \(sY_{\ell,m}, \, s_{+1}Y_{\ell,m}\) and \(sY_{\ell-1,m}\). By repeated application of this procedure, various relations can be formed between different \(s_{i+j,m+k}\) (where \(i, j, k\) are integers).

Finally, as an example of Eq. (18) we show:

\[ \partial_\theta \cot \theta \partial_\theta - \frac{m^2 + s^2 + 2ms \cos \theta}{\sin^2 \theta} + (\ell + 1) \]

(32)

\[ = \left(\partial_\theta + \frac{m}{\sin \theta} + (s + 1) \cot \theta\right) \left(\partial_\theta - \frac{m}{\sin \theta} - s \cot \theta\right) \]

(33)
\[ \left( \partial_\theta - \frac{m}{\sin \theta} - (s-1) \cot \theta \right) \left( \partial_\theta + \frac{m}{\sin \theta} + s \cot \theta \right) + (\ell + s)(\ell - s + 1) \]  
\[ = \left( \partial_\theta - \frac{ms}{(\ell + 1)\sin \theta - \ell \cot \theta} \right) \times \left( \partial_\theta + \frac{ms}{(\ell + 1)\sin \theta + (\ell + 1) \cot \theta} \right) + \frac{(\ell + 1)^2 - m^2}{(\ell + 1)^2 - s^2} \]  
\[ \frac{(\ell^2 - m^2)}{(\ell^2 - s^2)} \]  

(34)

IV. SPIN-WEIGHTED SPHEROIDAL HARMONICS

Let us denote the spin-weighted spheroidal harmonics of type-1 and type-2 by \( s^\gamma_{l,m} \) and \( T^\gamma_{l,m} \), being two linearly independent solutions of Eq. (1). To linear-order-in-\( \gamma \), expressions for \( s^\gamma_{l,m} \) and \( T^\gamma_{l,m} \) have been given by Press and Teukolsky. We write:

\[ s^\gamma_{l,m} = sY_{l,m} + \gamma b^m_{l+1} sY_{l+1,m} + \gamma b^m_{l-1} sY_{l-1,m} + O(\gamma^2) \]
\[ sT^\gamma_{l,m} = sX_{l,m} + \gamma b^m_{l+1} sX_{l+1,m} + \gamma b^m_{l-1} sX_{l-1,m} + O(\gamma^2) \]

(39)

Let’s first look at the Wronskian to \( O(\gamma) \),

\[ W(s^\gamma_{l,m}, s^\gamma_{l+1,m}) = \]
\[ (sY_{l,m} sY'_{l,m} - sX_{l,m} sY'_{l,m}) + \gamma b^m_{l+1} \left( - sY'_{l+1,m} sX_{l+1,m} + sX'_{l+1,m} sY_{l+1,m} \right) + \gamma b^m_{l-1} \left( - sY'_{l-1,m} sX_{l-1,m} + sX'_{l-1,m} sY_{l-1,m} \right) + O(\gamma^2) \]

(40)

In the above equation, the expression in the first parenthesis is \( sW_{l,m} \), the expression in the second parenthesis is \( s^\gamma_{l+1,m} \rightarrow sY_{l+1,m} \), the expression in the third parenthesis is \( \gamma s^\gamma_{l+1,m} \rightarrow sY'_{l+1,m} \), the expression in the fourth parenthesis is \( \gamma s^\gamma_{l-1,m} \rightarrow sY'_{l-1,m} \), and the expression in the fifth parenthesis is \( \gamma s^\gamma_{l-1,m} \rightarrow sY'_{l-1,m} \), all of which are known. The analytical expression of \( \beta 's \) and \( \alpha 's \) are

\[ s^\gamma_{l+1,m} \]
\[ s^\gamma_{l-1,m} \]

Using these together, we get

\[ W(s^\gamma_{l,m}, s^\gamma_{l+1,m}) = -\frac{2\ell + 1}{4\pi \sin \theta} \left( 1 + \frac{2ms^2\gamma}{\ell^2(\ell + 1)^2} \right) + O(\gamma^2) \]

(48)

We now wish to find the \( s \)-raising operator of \( s^\gamma_{l,m} \) to linear-order-in-\( \gamma \). We first start with finding the \( \beta \) and \( \alpha \) in the following relation

\[ \gamma = \beta z' + \alpha z + O(\gamma^2) \]

(49)

where \( y = s^\gamma_{l,m} \rightarrow s^\gamma_{l,m} \), and \( z = s^\gamma_{l,m} \rightarrow s^\gamma_{l,m} \). Let’s first have a look at \( \beta \),

\[ \beta = \frac{s^\gamma_{l,m} s^\gamma_{l+1,m} - s^\gamma_{l,m} s^\gamma_{l+1,m}}{W(s^\gamma_{l,m}, s^\gamma_{l+1,m})} \]
\[ = -\frac{4\pi \sin \theta}{2\ell + 1} \left( 1 - \frac{2ms^2\gamma}{\ell^2(\ell + 1)^2} \right) \times \left( s^\gamma_{l,m} s^\gamma_{l+1,m} - s^\gamma_{l,m} s^\gamma_{l+1,m} \right) + O(\gamma^2) \]

(50)
To calculate the above, we need
\[
\left( sS^\gamma_{\ell,m} + sT^\gamma_{\ell,m} + s^2S^\gamma_{\ell,m} \right) = (sY_{\ell,m} + sX_{\ell,m} + s^2Y_{\ell,m}) + \gamma b_{\ell+1} \left( sY_{\ell+1,m} + sX_{\ell+1,m} + s^2Y_{\ell+1,m} \right) + \gamma b_{\ell-1} \left( sY_{\ell-1,m} + sX_{\ell-1,m} + s^2Y_{\ell-1,m} \right) + \gamma b_{\ell-1} \left( sY_{\ell+1,m} + sX_{\ell+1,m} + s^2Y_{\ell+1,m} \right).
\] (51)

Let's look at \( \alpha \),
\[
\alpha = \frac{-sS^\gamma_{\ell,m} + sT^\gamma_{\ell,m} + s^2S^\gamma_{\ell,m}}{W(sS^\gamma_{\ell,m}, sT^\gamma_{\ell,m})} = \frac{4\pi \sin \theta}{2\ell+1} \left( 1 - \frac{2m^2\gamma}{\ell^2(\ell+1)^2} \right) \left( -sS^\gamma_{\ell,m} + sT^\gamma_{\ell,m} + s^2S^\gamma_{\ell,m} \right) + O(\gamma^2).
\] (52)

To calculate the above, we need
\[
\left( -sS^\gamma_{\ell,m} + sT^\gamma_{\ell,m} + s^2S^\gamma_{\ell,m} \right) = \left( -sY_{\ell,m} + sX_{\ell,m} + s^2Y_{\ell,m} \right) + \gamma b_{\ell+1} \left( -sY_{\ell+1,m} + sX_{\ell+1,m} + s^2Y_{\ell+1,m} \right) + \gamma b_{\ell-1} \left( -sY_{\ell-1,m} + sX_{\ell-1,m} + s^2Y_{\ell-1,m} \right) + \gamma b_{\ell-1} \left( -sY_{\ell+1,m} + sX_{\ell+1,m} + s^2Y_{\ell+1,m} \right).
\] (53)

In Eqs (51), the expression in the first set of parenthesis are the \( \beta s\ell,m \) and \( \alpha s\ell,m \) of \( sY_{\ell,m} \rightarrow s+1Y_{\ell,m} \), the expression in the second set of parenthesis are the \( \beta s\ell+1,m \) and \( \alpha s\ell+1,m \) of \( sY_{\ell+1,m} \rightarrow s+1Y_{\ell+1,m} \), the expression in the third set of parenthesis are the \( \beta s\ell-1,m \) and \( \alpha s\ell-1,m \) of \( sY_{\ell-1,m} \rightarrow s+1Y_{\ell-1,m} \), the expression in the fourth set of parenthesis are the \( \beta s\ell,m \) and \( \alpha s\ell,m \) of \( sY_{\ell,m} \rightarrow s+1Y_{\ell+1,m} \), and the expression in the fifth set of parenthesis are the \( \beta s\ell,m \) and \( \alpha s\ell,m \) of \( sY_{\ell,m} \rightarrow s+1Y_{\ell-1,m} \). To get the \( \beta \)'s and \( \alpha \)'s of the last four expression, we will need the following relations,
\[ \beta_{s,\ell,m} \rightarrow (s+1,\ell-1,m) = \frac{\sqrt{2\ell - 1}}{2\ell + 1} \times \frac{-(\ell + s)(m + \ell \cos \theta)}{\sqrt{(\ell^2 - m^2)(\ell^2 - s^2)(\ell + s)(\ell - s - 1)}}. \]

\[ \alpha_{s,\ell,m} \rightarrow (s+1,\ell,m) = \frac{1}{\sqrt{(\ell^2 - m^2)(\ell^2 - s^2)(\ell + s)(\ell + s + 1)}} \times \left[ \frac{m}{\sin \theta} + \left( s + \frac{1}{2} \right) \cot \theta \right]. \]

\[ \alpha_{s,\ell+1,m} \rightarrow (s+1,\ell,m) = \frac{\sqrt{2\ell + 1}}{2\ell + 3} \times \frac{-\sqrt{\ell + s + 1} \times \left[ (s^2 + m^2)(s^2 + 1) + \ell(m^2 + s^2 + 2) + 1 \right] \csc \theta \times \left[ (s + 1)[m + (\ell + 1) \cos \theta] + (\ell + 1)^2 \sin \theta \right]}{(\ell + s + 1)[m + (\ell + 1) \cos \theta]} \times \frac{\ell}{2(-m + \ell \cos \theta)} \times \frac{1}{2(-m + \ell \cos \theta)} \times \frac{\ell^2 \cos 2\theta}{2(-m + \ell \cos \theta)} \times \frac{\ell^2}{2(-m + \ell \cos \theta)}. \]

\[ s_{\ell,m} = (s^2_{\ell,m} \theta + s\alpha_{\ell,m}) s S_{\ell,m} + O(\gamma^2) \quad (56) \]

\[ \beta_{s,\ell,m} = \frac{-1}{\sqrt{(\ell - s)(\ell + s + 1)}} \times \frac{m(\ell^2 + \ell + 2s + 1)\gamma}{\ell(\ell + 1)\sqrt{(\ell - s)(\ell + s + 1)}} - \frac{(2s + 1) \cos \gamma}{\ell(\ell + 1)\sqrt{(\ell - s)(\ell + s + 1)}} \quad (57) \]

\[ \alpha_{s,\ell,m} = \frac{m \csc \theta + s \cot \theta}{\sqrt{(\ell - s)(\ell + s + 1)}} + \frac{m \left[ (\ell + 1) + 3\ell s(\ell + 1) + 2s^2 + s \right] \cot \gamma}{\ell^2(\ell + 1)^2 \sqrt{(\ell - s)(\ell + s + 1)}} + \frac{[s(\ell + 1)(2s + 1) + m^2(\ell^2 + \ell + 2s + 1)] \csc \gamma}{\ell^2(\ell + 1)^2 \sqrt{(\ell - s)(\ell + s + 1)}} \quad (58) \]

**ACKNOWLEDGMENTS**

This work was supported in part by the European Research Council under the European Union’s Seventh Framework Programme (FP7/2007-2013)/ERC grant agreement no. 304978 to the University of Southampton, and NSF Grants PHY 1205906 and PHY 1314529 to the University of Florida. Hospitality for BFW at the University of Southampton at an early stage in this work.
is gratefully acknowledged. Support from the CNRS through the IAP, where part of this work was carried out, is also acknowledged, as well as support from the French state funds managed by the ANR within the Investissements d’Avenir programme under Grant No. ANR-11-IDEX-0004-02.

Appendix A: Building $sT_{\ell,m}^\gamma$ for a special cases

For cases such as $m = \ell$, $s = -\ell$ or $m = \ell = s$, one cannot simply use the general form,

$$sT_{\ell,m}^\gamma = sX_{\ell,m} + \gamma s b_{\ell,\ell+1}^m sX_{\ell+1,m} + \gamma s b_{\ell,\ell-1}^m sX_{\ell-1,m}, \quad (A1)$$

with $s b_{\ell,\ell-1}^m = 0$ as when building $sS_{\ell,m}^\gamma$ for these cases. Instead, it then becomes imperative to carefully look at the analyticity of the Clebsch-Gordan coefficients in $s b_{\ell,\ell-1}^m$, the coefficients of $\tilde{\sigma}^s Q_{\ell}^{(m)}$, and at times even multiplying them with extra coefficients so that $sT_{\ell,m}^\gamma$ for such cases satisfy Eq. (1).

In general cases, it is straightforward to use

$$s b_{\ell,m}^m = 2s \left( \frac{2\ell + 1}{2\ell - 1} \right)^{1/2} \times \frac{\langle \ell, m; 1, 0|\ell', m \rangle \langle \ell, -s; 1, 0|\ell', -s \rangle}{\ell(\ell + 1) - \ell'(\ell' + 1)}, \quad (A2)$$

$$sX_{\ell,m}(\theta) = \sqrt{\frac{(2\ell + 1)(\ell - m)!}{4\pi(\ell + m)!}} \tilde{\sigma}_s Q_{\ell}^{(m)}(\cos \theta) \quad (A3)$$

where $\langle | \rangle$ are the Clebsch-Gordan coefficients which can be calculated using

$$\langle j_1, m_1; j_2, m_2| j, m \rangle = \sqrt{\frac{(j_1 + j_2 - j)!}{(j_1 - j_1 - j)!}\times\frac{(j + j_1 - j_1)!}{(j + j_1 + j_1)!}} \times \sum_{k=-\infty}^{\infty} (-1)^k \sqrt{\frac{(j_1 + m_1)!}{(j_1 - m_1)!}}\times\frac{(j_2 + m_2)!}{(j_2 - m_2)!}\times\frac{(j + m)!}{(j + m)!}\times\frac{(j - m)!}{(j - m)!}\times\frac{(j - j_1 - m_2 + k)!}{(j - j_1 - m_2 + k)!}. \quad (A4)$$

Now, we study the following special cases.

1. $m = \ell$ and $|s| \neq \ell$

In this case, there is a cancellation in the last term of the right-hand-side of Eq. (A1). The $s b_{\ell,\ell-1}^m$ includes $\langle \ell, m; 1, 0|\ell - 1, m \rangle$ which go to zero as

$$\frac{(2m - 1)}{\sqrt{2(2m + 1)}}(\ell - m) + O(\ell - m)^{3/2}. \quad (A5)$$

And $\sqrt{\ell - m)!}$ in the numerator of the square-root of Eq. (A3) blows up as $\frac{1}{\ell - m}$ which cancels the one above. Using these and after empirically finding the coefficient $c_\ell$, we have

$$sT_{\ell,m}^\gamma = sX_{\ell,m} + \gamma s b_{\ell,\ell+1}^m sX_{\ell+1,m} + \gamma c_\ell s b_{\ell,\ell-1}^m sX_{\ell-1,m} + O(\gamma^2) \quad (A6)$$

And $\sqrt{\ell - m)!}$ in the denominator of the square-root of Eq. (A3) goes to zero as $\sqrt{\ell - m)}$. $sQ_{\ell,m}$ for $m = -\mu$ and $\ell = -\ell$ (where $\mu > 0$) blows up to leading

2. $m = -\ell$ and $|s| \neq \ell$

The $s b_{\ell,\ell-1}^m$ includes $\langle \ell, m; 1, 0|\ell - 1, m \rangle$ which goes to zero as

$$\frac{2}{1 - 2m} \sqrt{\ell + m) + O(\ell + m)^{3/2}}. \quad (A8)$$

And $\sqrt{\ell - m)!}$ in the denominator of the square-root of Eq. (A3) goes to zero as $\sqrt{\ell + m)}$. $sQ_{\ell,m}$ for $m = -\mu$ and $\ell = -\ell$ (where $\mu > 0$) blows up to leading
order as
\[ (-1)^s \sqrt{\frac{2\ell+1}{2\ell-1}} \frac{\tan^s(\theta/2) Q_{\ell-1}^{(m)}}{(\mu+1)! (\mu-s-1)!} \tan^s(\theta/2) Q_{\ell-1}^{(m)} \cos(\theta) \]
(A9)

The coefficient \( c_{\ell} \) is found empirically to be
\[ (-1)^\ell \sqrt{\frac{2\ell+1}{2\ell-1}} \sqrt{\ell(\ell+m)} \] (11)
where
\[ c_{\ell} = (-1)^\ell \sqrt{\frac{2\ell+1}{2\ell-1}}, \] (11)
\[ s b_{\ell,\ell-1} = 2s \left( \frac{2\ell+1}{2\ell-1} \right)^{1/2} \]
\[ \times -\frac{2}{\ell(\ell+1) - \ell(\ell-1)}, \] (12)
\[ s \tilde{X}_{\ell-1,m} = (-1)^\ell \sqrt{\frac{(2\ell+1)(\ell - m - 1)!}{4\ell(\ell+1)(\ell-m-1)!}} \]
\[ \times \left( \frac{2s-2}{2s} \right) \frac{(-s)(\ell-s; 1,0|\ell-1, -s)!}{\ell(\ell+1) - \ell(\ell-1)}, \] (13)
where appropriate powers of \( \sqrt{\ell + m} \) in each of \( c_{\ell}, s b_{\ell,\ell-1} \) and \( s \tilde{X}_{\ell-1,m} \) cancel each other to give a finite contribution.

3. \( |m| \neq \ell \) and \( s = -\ell \)

The \( s b_{\ell,\ell-1} \) includes \( \langle \ell, -s; 1,0|\ell-1, -s \rangle \) which, in the limit \( \ell \to -s \), goes to zero as
\[ -\sqrt{\frac{2}{1+2s}} \frac{(\ell-s)}{(\ell-s)^{3/2}} + O(\ell-s)^{3/2}, \] (14)
and \( s Q_{\ell-1,m}(\theta) \) blows up as
\[ \frac{(-1)^s}{2s} \sqrt{\frac{2s-2}{2s}} \left( s \frac{(s+m-1)!}{(s-1)!} \right) \frac{\csc^{s+m} \left( \frac{\theta}{2} \right) \sec^{s-m} \left( \frac{\theta}{2} \right)}{(\ell+s)}, \] (15)
Factors of \( \sqrt{\ell-s} \) in the above two equations cancel each other and \( s T_{\ell,m} \) is then given by
\[ s T_{\ell,m} = s X_{\ell,m} + \gamma s b_{\ell,\ell+1,m} X_{\ell+1,m} + \gamma c_{\ell} s \tilde{b}_{\ell,\ell-1,m} X_{\ell-1,m} + O(\gamma^2) \] (16)
where
\[ \tilde{b}_{\ell,\ell-1} = 2s \left( \frac{2\ell+1}{2\ell-1} \right)^{1/2} \]
\[ \times \left( \frac{2s}{2s+1} \right) \frac{(\ell,m; 1,0|\ell-1, -s)!}{\ell(\ell+1) - \ell(\ell-1)}, \] (17)
\[ s \tilde{X}_{\ell-1,m} = \left( \frac{2\ell+1}{2\ell-1} \right)(\ell-1-m)! \left( \frac{(-1)^m}{2s} \right) \]
\[ \times \left( \frac{2s-2}{2s} \right) \frac{(s+m-1)!}{(s-1)!} \times \csc^{s+m} \left( \frac{\theta}{2} \right) \sec^{s-m} \left( \frac{\theta}{2} \right), \] (18)

4. \( |m| \neq \ell \) and \( s = \ell \)

The \( s b_{\ell,\ell-1} \) includes \( \langle \ell, -s; 1,0|\ell-1, -s \rangle \) which, in the limit \( \ell \to -s \), goes to zero as
\[ -\frac{(2s+1)}{2(\ell+s)} \frac{(\ell+s)}{(\ell-s)^{3/2}} + O(\ell-s)^{3/2}, \] (19)
and \( s Q_{\ell-1,m}(\theta) \) blows up as
\[ \frac{(-1)^s}{2s} \sqrt{\frac{2s-2}{2s}} \left( s \frac{(s+m-1)!}{(s-1)!} \right) \]
\[ \times \csc^{s+m} \left( \frac{\theta}{2} \right) \sec^{s-m} \left( \frac{\theta}{2} \right) \]
\[ \sqrt{\ell+s}, \] (20)
The \( \sqrt{\ell+s} \) in the above two equations cancel each other and \( s T_{\ell,m} \) is then given by
\[ s T_{\ell,m} = s X_{\ell,m} + \gamma s b_{\ell,\ell+1,m} X_{\ell+1,m} + \gamma c_{\ell} s \tilde{b}_{\ell,\ell-1,m} X_{\ell-1,m} + O(\gamma^2) \] (21)
where
\[ c_{\ell} = -\frac{2}{2\ell-1} \]
\[ s b_{\ell,\ell-1} = 2s \left( \frac{2\ell+1}{2\ell-1} \right)^{1/2} \]
\[ \times \left( \frac{2s+1}{2s} \right) \frac{(\ell,m; 1,0|\ell-1, -s)!}{\ell(\ell+1) - \ell(\ell-1)}, \] (22)
\[ s \tilde{X}_{\ell-1,m} = \left( \frac{2\ell+1}{2\ell-1} \right)(\ell-1-m)! \left( \frac{(-1)^m}{2s} \right) \]
\[ \times \left( \frac{2s-2}{2s} \right) \frac{(s+m-1)!}{(s-1)!} \times \csc^{s+m} \left( \frac{\theta}{2} \right) \sec^{s-m} \left( \frac{\theta}{2} \right) \]
\[ \sqrt{\ell+s}, \] (23)
\[ \times \csc^{s-m}(\theta/2) \sec^{s+m}(\theta/2). \] (A23)

5. \( m = \ell \) and \( s = \ell \)

This case is a hybrid form of the \( s = \ell \) and the \( m = \ell \) cases. The two Clebsch-Gordan coefficients in \( \tilde{s}_\ell^{\ell, m} \), and the denominator in the square-root of the coefficient of \( Q_{\ell - 1,m} \) have the same form as the \( s = \ell \) and \( m = \ell \) cases. The coefficient \( c_\ell \) is also same as the one in the \( m = \ell \) case. \( sQ_{\ell - 1,m} \) blows up as

\[ (-1)^s \sqrt{(2s - 1)!} \csc^{2s}(\theta/2) \] (A24)

which cancels the Clebsch-Gordan coefficient containing \( s \). We then have

\[ sT_{\ell, m}^\gamma = sX_{\ell, m} + \gamma s\tilde{b}^m_{\ell, \ell - 1} sX_{\ell + 1, m} \]

where

\[ \gamma c_\ell s\tilde{b}^m_{\ell, \ell - 1} s\tilde{X}_{\ell - 1, m} \] (A25)

6. \( m = \ell \) and \( s = -\ell \)

This case is a hybrid form of the \( s = -\ell \) and the \( m = \ell \) cases. The two Clebsch-Gordan coefficients in \( b^m_{\ell, \ell - 1} \), and the denominator in the square-root of the coefficient of \( sQ_{\ell - 1,m} \) have the same form as the \( s = -\ell \) and \( m = \ell \) cases. \( sQ_{\ell - 1,m} \) blows up as

\[ (-1)^s \sqrt{(2s - 1)!} \sec^{2s}(\theta/2) \] (A27)

which cancels the Clebsch-Gordan coefficient containing \( s \). We then have

\[ sT_{\ell, m}^\gamma = sX_{\ell, m} + \gamma s\tilde{b}^m_{\ell, \ell - 1} sX_{\ell + 1, m} \]

where

\[ \gamma c_\ell s\tilde{b}^m_{\ell, \ell - 1} s\tilde{X}_{\ell - 1, m} \] (A28)

7. \( m = -\ell \) and \( s = \ell \)

This case is a hybrid of the \( m = -\ell \) and the \( s = \ell \) cases. The two Clebsch-Gordan coefficients in \( b^m_{\ell, \ell - 1} \) go to zero as \( \epsilon^{1/2} \) each, and the \( \sqrt{(\ell + 1 - m)!} \) in the expression of \( sX_{\ell - 1,m} \) goes to zero as \( \epsilon^{1/2} \). \( sQ_{\ell - 1,m} \) blows up as \( \epsilon^{-2} \) and we empirically find that the coefficient \( s\gamma c_\ell \) is \((-1)^s \sqrt{(2\ell - 1)!} \epsilon^{1/2} \) canceling the extra \( \epsilon^{-1/2} \) left in \( sQ_{\ell - 1,m} \). Here \( \epsilon \) is the small factor \( (\ell + m) \) or \( (\ell - m) \). We then have

\[ sT_{\ell, m}^\gamma = sX_{\ell, m} + \gamma s\tilde{b}^m_{\ell, \ell - 1} sX_{\ell + 1, m} \]

where

\[ \gamma c_\ell = (-1)^s \sqrt{(2s - 1)!}, \]

\[ \tilde{s}_\ell^{\ell, m} = 2s \left( \frac{2\ell + 1}{2\ell - 1} \right)^{1/2} \left( \frac{2m - 1}{\ell(1 + m) - (\ell - 1)} \right)^{-\sqrt{\frac{2(2m - 1)}{2\ell + s}}}, \]

\[ \tilde{s}_\ell^{\ell, m} = \frac{\sqrt{2\ell - 1}}{4\ell(1 + m) - 2s - 1} \frac{(2s - 1)!}{\csc^{2s}(\theta/2)}. \] (A31)

8. \( m = -\ell \) and \( s = -\ell \)

Identical cancellations take place as in the previous case. We have

\[ sT_{\ell, m}^\gamma = sX_{\ell, m} + \gamma s\tilde{b}^m_{\ell, \ell - 1} sX_{\ell + 1, m} \]

where

\[ \gamma c_\ell = (-1)^{s+1} \frac{2\sqrt{(2\ell - 1)!}}{(2\ell - 1)}, \]

\[ \tilde{s}_\ell^{\ell, m} = 2s \left( \frac{2\ell + 1}{2\ell - 1} \right)^{1/2} \left( \frac{-\sqrt{\frac{2(2m - 1)}{2\ell + s}}}{\sqrt{1 - 2m} \ell(1 + m) - 2s - 1} \right), \]

\[ \tilde{s}_\ell^{\ell, m} = \sqrt{2\ell - 1} \frac{(2s - 1)!}{\csc^{2s}(\theta/2)} \] (A33)

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