DISTINCT ZEROS OF THE RIEMANN ZETA-FUNCTION

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Abstract. In this paper, we prove that there are more than 66.036% of zeros of the Riemann zeta-function are distinct.

1. Introduction

Let $\zeta(s)$ be the Riemann zeta-function, where $s = \sigma + it$. It is defined for $\sigma > 1$ by

$$\zeta(s) = \sum_{n\geq 1} n^{-s}. $$

The Riemann-von Mangoldt formula [10] states that $N(T)$, the number of non-trivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with $0 < \gamma \leq T$, satisfies

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} + \frac{7}{8} + S(T) + O\left(\frac{1}{T}\right),$$

$$S(T) = \frac{1}{\pi} \arg \zeta\left(\frac{1}{2} + iT\right) \ll \log T.$$ 

It is generally believed that all the zeros of $\zeta(s)$ are simple (also distinct), which is known as the Simple Zero Conjecture. We define the number of simple zeros and the number of distinct zeros for the Riemann zeta-function as follows

$$N_d(T) = ||\rho = \beta + i\gamma : 0 < \gamma \leq T, \zeta(\rho) = 0||,$$

$$N_s(T) = ||\rho = \beta + i\gamma : 0 < \gamma \leq T, \zeta(\rho) = 0, \zeta'(\rho) \neq 0||.$$ 

The Simple Zero Conjecture means $N_d(T) = N_s(T) = N(T)$.

Due to Levinson’s method, it is known that more than two-fifths of the zeros are simple (see [2, 4]). In 1995, Farmer [6] introduced a combination method, which is based on proportions of simple zeros of $\xi^{(n)}(s)$, $n \geq 0$, and proved that at least 63.9% of the zeros of the Riemann zeta-function are distinct.

In this paper, by the method introduced in our work [11], we prove that there are more than 66.03% of zeros of the Riemann zeta-function are distinct.

Theorem 1. For $T$ sufficiently large, we have

$$N_d(T) \geq 0.66036N(T).$$

In this paper we always assume that $T$ is a large parameter and $L = \log T$. The number of additional zeros of a analytic function $f$ caused by multiplicity means the number of zeros of $f$ counted according to multiplicity minus one.

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We note that if \( \rho \) is a non-simple zero of \( \zeta(s) \), it must be a zero of
\[
G(s) = \zeta(s)\psi_1(s) + \zeta'(s)\psi_2(s)
\]
with multiplicity reduced by at most one, here \( \psi_1(s) \) and \( \psi_2(s) \) can be any analytic function. Thus the number of additional zeros of \( \zeta(s) \) caused by multiplicity in any region is not more than the number of zeros of \( G(s) \) in the same region. We partition the whole plane into the left part \( \text{Re}(s) < 1/2 \) and the right part \( \text{Re}(s) \geq 1/2 \). Then we will evaluate the number of additional zeros of \( \zeta(s) \) caused by multiplicity in each side with different \( G(s) \).

To the left side, we choose \( G(s) = \xi'(s) \), where
\[
\xi(s) = H(s)\zeta(s)
\]
with
\[
H(s) = 1/2s(s - 1)\pi^{-s/2}\Gamma(s/2).
\]
It is known that for any integer \( n \geq 0 \) the number of zeros for \( \xi^{(n)}(s) \) with \( 0 < t < T \) is \( N(T) + O(\log T) \) (to see [3, 8]). The functional equation for \( \zeta(s) \) says that
\[
(1)
\]
\[
(2)
\]
\[
(3)
\]
Differentiating both side of the above formula in \( s \) we have
\[
\xi'(s) = -\xi'(1 - s).
\]
Hence if \( \rho \) is a zero of \( \xi'(s) \), so is \( 1 - \rho \). Let \( N_{\xi',c}(T) \) be the number of zeros of \( \xi'(1/2 + it) \) with \( 0 < t < T \). Then it follows from the symmetry of zeros of \( \xi'(s) \) that the number of additional zeros of \( \zeta(s) \) in the left side is not more than
\[
\frac{1}{2}(N(T) - N_{\xi',c}(T)) + O(\log T).
\]
To the right side, we choose \( G(s) \) as in (1) with
\[
(4)
\]
\[
\psi_1(s) = \sum_{n \leq y} \frac{\mu(n)}{n^{s+1/L}}P_1\left(\frac{\log y/n}{\log y}\right),
\]
\[
\psi_2(s) = \frac{1}{L} \sum_{n \leq y} \frac{\mu(n)}{n^{s+1/L}}P_2\left(\frac{\log y/n}{\log y}\right),
\]
where \( \mu \) is the Mobius function, \( y = T^\theta \) with \( 0 < \theta < 4/7 \). Here \( P_1, P_2 \) are polynomials with \( P_1(0) = P_2(0), P_1(1) = 1 \) which will be specified later (to see [2, 5, 9] for any more choice of \( \psi_1 \)).

Let \( D \) be the closed rectangle with vertices \( 1/2 + it_0, 3 + it_0, 1/2 + iT, 3 + iT \). Here \( t_0 \leq 2 \) is a given positive constant that is not a ordinate of any zero of \( G(s) \) in the region \( 0 < \text{Re}(s) < 3, \text{Im}(s) > 0 \). Let \( N_G(D) \) denote the zeros of \( G(s) \) in \( D \), including zeros on the left boundary.

Since the number of additional zeros of \( \zeta(s) \) caused by multiplicity is not more than the number of zeros of \( \xi'(s) \) in the left side and not more than \( N_G(D) \) in the right side, we may have the following formula about the number of distinct zeros of \( \zeta(s) \)
\[
(5)
\]
\[
N_d(T) \geq \frac{1}{2}N(T) + \frac{1}{2}N_{\xi',c}(T) - N_G(D).
\]
It is therefore important to give an upper bound for $N_G(D)$ and a lower bound for $N_{\epsilon,\sigma}(T)$. We will obtain a an upper bound for $N_G(D)$ in section 2 and a lower bound better than the known result now for $N_{\epsilon,\sigma}(T)$ in section 3.

2. Upper bound for $N_G(D)$

An upper bound for $N_G(D)$ can be found in a familiar way by applying Littlewood’s formula (to see §9.9 of [10]). Let

$$\sigma_0 = 1/2 - R/L,$$

where $R$ is a constant to be specified precisely later. Let $D_1$ be the closed rectangle with vertices $\sigma_0 + it_0, 3 + it_0, \sigma_0 + iT, 3 + iT$. Suppose $G(3 + it) \neq 0$. Determine $\arg G(\sigma + iT)$ by continuation left from $3 + iT$ and $\arg G(\sigma + it_0)$ by continuation left from $3 + iTt_0$. If a zero is reached on the upper edge, use $\lim G(\sigma + iT + i\epsilon)$ as $\epsilon \to +0$. Make horizontal cuts in $D_1$ from the left side to the zeros of $G$ in $D_1$. Applying the Littlewood’s formula, we have

$$\int_{t_0}^{T} \log |G(\sigma_0 + it)| dt - \int_{t_0}^{T} \log |G(3 + it)| dt$$

$$+ \int_{\sigma_0}^{3} \arg G(\sigma + iT) d\sigma - \int_{\sigma_0}^{3} \arg G(\sigma + it_0) d\sigma$$

$$= 2\pi \sum_{\rho \in D_1} \text{dist}(\rho),$$

(6)

where $\text{dist}(\rho)$ is the distance of $\rho$ from the left side of $D_1$.

Recall the definition of $\psi_i$ for $i = 1, 2$. A direct calculation shows that $\psi_i(s) \ll T$ for $\Re(s) > 0$. Hence $G(s) \ll T^2$ for $\Re(s) > 0$. Then using Jensen’s theorem in a familiar way as in §9.4 of [10], we have

$$\int_{\sigma_0}^{3} \arg G(\sigma + iT) d\sigma = O(\log T).$$

(7)

For $t_0$ is a given constant, it is easy to see

$$\int_{\sigma_0}^{3} \arg G(\sigma + it_0) d\sigma = O(1).$$

(8)

By a direct calculation we can see

$$\zeta'(3 + it) \psi_i(3 + it) \ll O(1/\log T).$$

Hence we have from (11) that

$$\int_{t_0}^{T} \log |G(3 + it)| dt = \int_{t_0}^{T} \log |\zeta(3 + it)\psi_i(3 + it)| dt + O(T/\log T).$$

(9)

Since for $\sigma > 1$

$$\log \zeta(s) = - \sum_{n \geq 1} \Lambda(n) \frac{n^s \log n}{n^s},$$

(3)
it follows taking the real part that

$$(10) \quad \int_{t_0}^{T} \log |\zeta(3 + it)| dt \ll 1.$$ 

For the entire function $\psi_i(s)$, it is easy to see, for $\sigma \geq 3$,

$$|\psi_i(s) - 1| \leq \frac{1}{2^\sigma} + \frac{1}{3^\sigma} + \int_{3}^{\infty} \frac{v}{\nu^\sigma} \leq \frac{1}{2^\sigma} + \frac{5}{2} \frac{1}{3^\sigma} < 2^{1-\sigma}.$$ 

Therefore, $\log \psi_i(s)$ is analytic for $\sigma \geq 3$. Integrating on the contour $\sigma + iT$, $3 \leq \sigma < \infty$; $3 + it$, $t_0 \leq t \leq T$; $\sigma + iT$, $3 \leq \sigma < \infty$ gives

$$(11) \quad \int_{t_0}^{T} \log |\psi_i(3 + it)| dt \leq 8 \int_{3}^{\infty} \frac{d\sigma}{2^\sigma} = O(1).$$

Substituting (10), (11) into (9) we have

$$(12) \quad \int_{t_0}^{T} \log |G(3 + it)| dt \ll T / \log T.$$ 

Then using (7), (8), (12) in (6), we have

$$\int_{t_0}^{T} \log |G(\sigma_0 + it)| dt = 2\pi \sum_{\rho \in D} \text{dist}(\rho).$$

Since all the zeros of $G$ in closed rectangle $D$ are at least distance $1/2 - \sigma_0$ from $\sigma = \sigma_0$, it follows that

$$(13) \quad 2\pi(1/2 - \sigma_0)N_G(D) \leq \int_{t_0}^{T} \log |G(\sigma_0 + it)| dt + O(T / \log T).$$

Using the concavity of the logarithm,

$$\int_{t_0}^{T} \log |G(\sigma_0 + it)| dt = \frac{1}{2} \int_{t_0}^{T} \log |G(\sigma_0 + it)|^2 dt$$

$$\leq \frac{1}{2} T \log \left( \frac{1}{T} \int_{t_0}^{T} |G(\sigma_0 + it)|^2 dt \right).$$

Substituting $\sigma_0 = 1/2 - R/L$, then we have from (13) that

$$(14) \quad N_G(D) \leq \frac{T L}{4\pi R} \log \left( \frac{1}{T} \int_{t_0}^{T} |G(\sigma_0 + it)|^2 dt \right).$$

Hence an upper bound for $N_G(D)$ may be obtained by evaluating the mean value integral of $G$ on the $\sigma_0$-line.

To evaluate the mean value integral of $G$ on the $\sigma_0$-line, we need the following two Lemmas.

**Lemma 2.** Suppose that $\delta > 0$ and $\Delta = T^{1-\delta}$. Then

$$\frac{1}{T} \int_{t_0}^{T} |G(\sigma_0 + it)|^2 dt = \frac{1}{\Delta \pi^{1/2}} \int_{-\infty}^{\infty} e^{-(t-w)^2/\Delta} |G(\sigma_0 + it)|^2 dt + o_\delta(1)$$

uniformly for $T \leq w \leq 2T$. 

This lemma follows exactly as in Section 3 of Balasubramanian, Conrey, and Heath-Brown [11].

**Lemma 3.** Let $a, b \in \mathbb{C}$ with $a, b \ll 1$, and put $\alpha = a/L$, $\beta = b/L$ where $L = \log T$. Let $s_0 = 1/2 + iw$ with $T \leq w \leq 2T$. Suppose that $\delta > 0$, $\Delta = T^{1-\delta}$ and that $y = T^\theta$ with $0 < \theta < 4/7$. For $i, j = 1, 2$, let

$$g(a, b, w, P_1, P_2) = \frac{1}{i\Delta \pi^{1/2}} \int_{(1/2)} e^{(s-s_0)^2\Delta^{-2}} \zeta(s + \alpha)\zeta(1 - s + \beta)\psi_1(s - R/L)\psi_2(1 - s - R/L)ds,$$

where $(1/2)$ denotes the straight line path from $1/2 - i\infty$ to $1/2 + i\infty$. Then

$$g(a, b, w, P_1, P_2) = \frac{\Sigma(a, b, P_i, P_j)}{\theta(a + b)} - e^{-a - b} \int_{(1/2)} e^{(s-s_0)^2\Delta^{-2}} G(s - R/L)G(1 - s - R/L)ds + o_\delta(1).$$

This lemma is the Lemma 2 of Conrey [4].

We now evaluate the mean value integral of $G$ on the $\sigma_0$-line. From Lemma 2 we have

$$\frac{1}{T} \int_{t_0}^{T} |G(\sigma_0 + it)|^2 dt = \frac{1}{\Delta \pi^{1/2}} \int_{-\infty}^{\infty} e^{-(t-w)^2\Delta^{-2}} |G(\sigma_0 + it)|^2 dt + O_\delta(1)$$

$$= \frac{1}{i\Delta \pi^{1/2}} \int_{(1/2)} e^{(s-s_0)^2\Delta^{-2}} G(s - R/L)G(1 - s - R/L)ds + o_\delta(1).$$

Then recalling the definition of $G$ in (11) we have

$$\frac{1}{T} \int_{t_0}^{T} |G(\sigma_0 + it)|^2 dt = \frac{1}{i\Delta \pi^{1/2}} \int_{(1/2)} e^{(s-s_0)^2\Delta^{-2}} \zeta\psi_1(s - R/L)\zeta\psi_1(1 - s - R/L)ds$$

$$+ \frac{1}{i\Delta \pi^{1/2}} \int_{(1/2)} e^{(s-s_0)^2\Delta^{-2}} \zeta\psi_1(s - R/L)\zeta'\psi_2(1 - s - R/L)ds$$

$$+ \frac{1}{i\Delta \pi^{1/2}} \int_{(1/2)} e^{(s-s_0)^2\Delta^{-2}} \zeta'\psi_2(s - R/L)\zeta\psi_1(1 - s - R/L)ds$$

$$+ \frac{1}{i\Delta \pi^{1/2}} \int_{(1/2)} e^{(s-s_0)^2\Delta^{-2}} \zeta'\psi_2(s - R/L)\zeta'\psi_2(1 - s - R/L)ds + o_\delta(1).$$

We may evaluate every item in the above formula by using Lemma 3. Then

$$\frac{1}{T} \int_{t_0}^{T} |G(\sigma_0 + it)|^2 dt = g(a, b, w, P_1, P_1)\big|_{a = b = -R} \quad + \partial_\gamma g(a, b, w, P_1, P_1)\big|_{a = b = -R}$$

$$+ \partial_\delta g(a, b, w, P_2, P_1)\big|_{a = b = -R} \quad + \partial_\delta \partial_\gamma g(a, b, w, P_2, P_1)\big|_{a = b = -R} + o_\delta(1).$$

(17)
Taking $\theta = 4/7 - \epsilon, R = 1.023,$
\begin{align*}
P_1(x) &= x - 0.064x(1 - x) + 0.112x^2(1 - x), \\
P_2(x) &= 1.305x - 0.276x^2 - 0.025x^3,
\end{align*}
and making $\epsilon \to 0$ in (17), we get from (14) that
\[N_{G}(D) \leq 0.27442N(T).\]

3. Zeros of $\xi'(s)$ on the critical line (1/2-line)

The fact that a positive proportion of zeros for $\xi'(s)$ are on the critical line was first proved by Levinson. Using the method introduced in his work [7], Levinson [8] proved that at least 71% zeros of $\xi'(s)$ are on the critical line.

In 1983, Conrey [3] made a careful study of zeros of $\xi^{(n)}(s)$ on the critical line and proved that at least 81.37% zeros of $\xi'(s)$ are on the critical line, and later, in his work [4], he successfully proved that the mollifier with length $\theta = 4/7 - \epsilon$ is available. It is obvious that this result can be used in [3]. Using the mollifier with length $\theta = 4/7 - \epsilon$ in [3], one may obtain that at least 82.402% zeros of $\xi'(s)$ are on the critical line.

We note that if not following the way in [3] but using the Levinson’s method generalized by Conrey in [4], we may obtain a better result on this problem. From the functional equation (3) it is easy to see that $\xi^{(n)}(s)$ is real for $s = 1/2 + it$ when $n$ is even and is purely imaginary when $n$ is odd. Let $\delta \neq 0$ be real, $g_n, n \geq 1$, be complex numbers with $g_n$ real if $n$ is even and $g_n$ purely imaginary if $n$ is odd. Now define
\[
\eta(s) = (1 - \delta)\xi(s) + \delta \xi'(s)L^{-1} + \sum_{n=1}^{N} g_n \xi^{(n)}(s)L^{-n}
\]
for some fixed $N$. Then, for $s = 1/2 + it$,
\[
\delta \xi'(s) = \text{Im}\eta(s),
\]
so that $\xi'(s) = 0$ on $\sigma = 1/2$ if and only if $\text{Im}\eta(s) = 0$. Observe that for every change of $\pi$ in the argument of $\eta(s)$ it must be the case that $\text{Im}\eta(s)$ has at least one zero. Hence it follows that
\[N_{\eta, \epsilon}(T) \geq \frac{1}{\pi} \Delta \arg\eta(s),\]
where $\Delta \arg$ stands for the variation of the argument as $s$ runs over the critical line from $1/2 + it_0$ to $1/2 + iT$ passing the zeros of $\eta(s)$ from the east side.

To estimate the change in argument of $\eta(s)$ on the critical line, we let $\eta(s) = H(s)V(s)$, where $H(s)$ is defined in (2) and
\[
V(s) = (1 - \delta)\xi(s) + \frac{\delta}{L} \left( \frac{H'}{H}(s)\xi(s) + \xi'(s) \right) + \sum_{n=1}^{N} \frac{g_n}{L^n} \sum_{k=0}^{n} \binom{n}{k} \frac{H^{(n-k)}(s)}{H(s)} \xi^{(k)}(s).
\]
By the Stirling formula, for $|t| \geq 2$, we have
\[
\arg H(1/2 + it) = \frac{t}{2} \log \frac{|t|}{2\pi e} + O(1),
\]
and
\[ \frac{H^{(n)}}{H}(s) = \left( \frac{1}{2} \log \frac{s}{2\pi} \right)^n (1 + O(1/|t|)) \]
for \( t \geq 10, \) \( 0 < \sigma < A_1, \) here \( A_1 \) can be any positive constant. (For a proof of these formulas, see Lemma 1 of [3].) Hence we may have
\[ (19) \quad \Delta \arg \eta(1/2 + it)|_{t_0}^T = T \log \frac{T}{2\pi e} + \Delta \arg V(1/2 + it)|_{t_0}^T + O(T) \]
and denote \( V(s) \) by
\[ V(s) = \left\{ \left( 1 - \delta + \delta \left( 1 + \frac{d}{L ds} \right) Q_0 \left( \frac{\log s}{2\pi} + 1 + \frac{d}{L ds} \right) \right) \zeta(s) \right\}(1 + O(1/|t|)) \]
with
\[ Q_0(x) = 1 + \sum_{n=1}^{N} \frac{\mu(n)}{\delta x} x^{n-1}. \]
As in (4), we use the mollifier
\[ \psi(s) = \sum_{n \leq y} \frac{\mu(n)}{n^{1+\theta/\delta} L P(\log y/n)} (\log y/n), \]
where \( \mu \) is the Mobius function, \( y = T^{\theta} \) with \( 0 < \theta < 4/7. \) Here \( P \) is a polynomial with \( P(0) = 0, \) \( P(1) = 1 \) which will be specified later. By the Cauchy’s argument principle, it is not difficult to see
\[ \left| \Delta \arg V(1/2 + it)|_{t_0}^T \right| = 2\pi N_{\xi}(D)(1 + o(1)). \]
Here \( N_{\xi}(D) \) denote the number of \( \xi(s) \) in the closed rectangle \( D \) defined before. Then, if \( Q_0(1/2) = 2, \) by applying Jenson’s theorem and Littlewood’s formula as in section [2] we can show that
\[ (21) \quad |\Delta \arg V(1/2 + it)|_{t_0}^T | \leq \frac{L}{R} \int_{t_0}^T \log |V\psi(\sigma_0 + it)| dt(1 + o(1)), \]
where
\[ \sigma_0 = 1/2 - R/L, \]
\( R \ll 1 \) is a positive real number. Here, the reason for requiring the condition \( Q_0(1/2) = 2 \) is to ensure the integration \( \int_{t_0}^T \log |V\psi(3 + it)| dt \) caused by using Littlewood’s formula is \( O(T/L). \) To evaluate the integral in the right of (21) we use the following useful approximation to \( V(s), \)
\[ U(s) = \left( 1 - \delta + \delta \left( 1 + \frac{2d}{L ds} \right) Q \left( -\frac{1}{L ds} \right) \right) \zeta(s), \]
where
\[ Q(x) = \frac{1}{2} Q_0(1/2 - x). \]
If restrict \( Q(x) \) be real polynomial, then the restriction of \( g_n \) and the condition that \( Q_0(1/2) = 2 \) are equivalent to \( Q'(x) = Q'(1 - x) \) and \( Q(0) = 1. \) It is easy to see that the error caused
by the substitution of $V$ with $U$ can be absorbed by the error term in (21). Hence we have from (18)-(23) that
\[
N_{\xi',\varepsilon}(T) \geq N(T) - \frac{L}{\pi R} \int_0^T \log |U\psi(\sigma_0 + it)| dt (1 + o(1)) + O(L).
\]
Recall that $\sigma_0 = 1/2 - R/L$. From the concavity of the logarithm we have
\[
(23) \quad N_{\xi',\varepsilon}(T) \geq N(T) - \frac{TL}{2\pi R} \log \left( \frac{1}{T} \int_0^T |U\psi(\sigma_0 + it)|^2 dt \right).
\]
By Lemma 2 and Lemma 3 we have
\[
\frac{1}{T} \int_0^T |U\psi(\sigma_0 + it)|^2 \sim \left(1 - \delta + \delta(1 + 2\delta_a)Q(-\delta_a)\right)\left(1 - \delta + \delta(1 + 2\delta_b)Q(-\delta_b)\right)
\times \left(\frac{g(b, a) - e^{-a-b}g(-a, -b)}{\theta(a + b)}\right)_{a = b = -R},
\]
where
\[
g(a, b) = \int_0^1 \left( P'(t) + a\theta P(t) \right) \left( P'(t) + b\theta P(t) \right) dt.
\]
Let $\theta = 4/7 - \epsilon$, $R = 1.104$, $\delta = 0.869$. Taking
\[
P(x) = x - 0.274 x(1 - x) - 0.334 x^2(1 - x) + 0.005 x^3(1 - x),
\]
\[
Q(x) = 1 - 0.609 x - 0.572 (x^2/2 - x^3/3) - 4.895 (x^3/3 - x^4/2 + x^5/5)
\]
into (24) and making $\epsilon \to 0$, then we have by (23) that
\[
N_{\xi',\varepsilon}(T) \geq 0.86957N(T).
\]
It is easy to see that the way in this section can also give better numerical results about the proportion of zeros of $\xi^{(n)}(s)$, $n \geq 2$ on the critical line, however one may find that this way is useless when consider simple zeros of $\xi^{(n)}(s)$, $n \geq 1$ on the critical line.

4. COMPLETION OF THE PROOF

We have obtained that
\[
N_C(D) \leq 0.27442N(T)
\]
in section 2 and
\[
N_{\xi',\varepsilon}(T) \geq 0.86957N(T).
\]
in section 3, then by (5) we have
\[
N_d(T) \geq \frac{1}{2}N(T) + \frac{1}{2}N_{\xi',\varepsilon}(T) - N_C(D)
\geq \left(\frac{1}{2} + 0.434785 - 0.27442\right)N(T) > 0.66036N(T),
\]
Hence we have proved Theorem 1.
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