Algebraization of difference eigenvalue equations related to $U_q(sl_2)$

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**Abstract**

A class of second order difference (discrete) operators with a partial algebraization of the spectrum is introduced. The eigenfunctions of the algebraized part of the spectrum are polynomials (discrete polynomials). Such difference operators can be constructed by means of $U_q(sl_2)$, the quantum deformation of the $sl_2$ algebra. The roots of polynomials determine the spectrum and obey the Bethe Ansatz equations. A particular case of difference equations for $q$-hypergeometric and Askey-Wilson polynomials is discussed. Applications to the problem of Bloch electrons in magnetic field are outlined.
1 Introduction

In this paper we attempt to describe difference equations of second order in one variable,

\[ a(z)\Psi(q^2 z) + d(z)\Psi(q^{-2} z) + v(z)\Psi(z) = E\Psi(z), \tag{1} \]

having polynomial eigenfunctions of the form

\[ \Psi(z) = \prod_{m=1}^{N} (z - z_m). \tag{2} \]

Here \( q \) is a parameter and the polynomials are parametrized by their roots \( z_m \). We call it algebraization of the spectrum.

Difference equations are closely related to the discrete equations. The discrete equation

\[ a_n\psi_{n+1} + d_n\psi_{n-1} + v_n\psi_n = E\psi_n. \tag{3} \]

may be obtained from the difference equation (1) by setting \( z = q^{2n} \):

\[ \Psi(q^{2n}) = \psi_n, \quad a(q^{2n}) = a_n, \quad d(q^{2n}) = d_n, \quad v(q^{2n}) = v_n. \tag{4} \]

Therefore, if the difference equation has a polynomial solution, then one can find a solution of the corresponding discrete equation as a polynomial on a discrete support:

\[ \psi_n = \prod_{m=1}^{N} (q^{2n} - z_m). \tag{5} \]

Generally only a part of the spectrum of difference and discrete operators can be algebraized. However, in some cases the entire spectrum is algebraic. The important examples of this type are periodic discrete equations. They appear if \( q \) is a root of unity, i.e., \( q^{2Q} = 1 \), where \( Q \) is an integer. Then we obtain equations with periodic coefficients

\[ a_n = a_{n+Q}, \quad d_n = d_{n+Q}, \quad v_n = v_{n+Q} \]

and the periodicity condition

\[ \psi_n = \psi_{n+Q}. \tag{6} \]

Some of the equations (1-3) have important physical applications. In particular, the method of algebraization has been applied (1-3) to the problem of
Bloch electrons in magnetic field on a lattice (the Azbel-Hofstadter problem).

Discrete equations with periodic coefficients are richer than the difference equation with $|q| = 1$. One can impose the quasiperiodic boundary condition

$$\psi_n = e^{ikn} \psi_{n+Q},$$

where $k$ is the Bloch momentum. In this case the spectrum generally has $Q$ bands. The periodic boundary condition (1) describes the crosssection of the band spectrum at $k = 0$. The quasiperiodic equations ($k \neq 0$) have no difference analog. We do not consider them in this paper.

The class of algebraized operators may be extended by the "gauge" transformation $\Psi(z) \rightarrow U(z)\Psi(z)$, where $U(z)$ may not be a polynomial. Then $a(z) \rightarrow U(q^2z)^{-1} a(z) U(z)$, $d(z) \rightarrow U(q^{-2}z)^{-1} d(z) U(z)$, $v(z) \rightarrow v(z)$. In particular, one can always choose the Jacobi operator (1,3) to be symmetric: $a(z) = d(q^2z)$.

In Appendix A we show that the difference equation (1) has polynomial solutions if $a(z)$, $d(z)$, $v(z)$ are certain Laurent polynomials of order 2. They are determined by 7 parameters (not including $q$) and may be computed directly. However, the algebraization of difference equations has an intimate relation with the representation theory. The relation between representations of the $sl_2$ Lie algebra and second order differential equations having polynomial solutions is known (see e.g. [1, 2, 3, 4] for early works, [5, 6, 7] for recent systematic treatment and [8] for a review). In this paper we show that difference equations having polynomial solutions can be classified according to representations of the quantum algebra $U_q(sl_2)$, the $q$-deformation of $sl_2$.

To construct linear operators with partially algebraized spectrum we employ the following strategy. Consider the algebra $\mathcal{D}$ of difference operators acting in the space of complex functions. Let $\mathcal{A} \subset \mathcal{D}$ be its subalgebra such that $\mathcal{A}$ has a finite-dimensional irreducible representation. This means that elements of $\mathcal{A}$ leave invariant a finite-dimensional functional subspace. Some of the eigenfunctions of operators from $\mathcal{A}$ must then belong to this subspace. Let us choose the invariant functional subspace to be the linear space $Pol_n$ of polynomials of degree at most $n$. Our task, then, will be to realize a finite dimensional representation of the algebra $\mathcal{A}$ in this space.

The basis of the space $Pol_n$ may be chosen as monomials $z^k$, $k = 0, 1, 2, \ldots, n$. In this basis, the algebra $\mathcal{A}$ is naturally decomposed into the
raising (lowering) parts $A_+$ ($A_-$) and diagonal operators $A_0$ (Cartan subalgebra) with the property $A_+ A_0 \subset A_+$, $A_- A_0 \subset A_-$. The space $Pol_n$ is invariant if and only if $A_+ z^n = A_- z^0 = 0$. This means that the representation of $A$ must have highest and lowest weights. Then diagonalizable operators from $A$ must have $n + 1$ different polynomial eigenfunctions. When the subalgebra $A$ and its representation are given we want to select the difference operators of second order. In this case roots of the polynomial eigenfunctions satisfy Bethe equations.

This strategy has been used for construction and classification of differential operators having polynomial eigenfunctions [4, 5, 6, 7, 8, 10, 11, 12] (a different approach was suggested in [9]). Let us use here the same notation $D$ for the algebra of differential operators. Then the subalgebra $A \subset D$ is a factor of $U(sl_2)$ (the universal enveloping of $sl_2$) over its center. The algebra $sl_2$ can be realized by first order differential operators,

$$S_3 = z \frac{d}{dz} - j, \quad S_+ = z (2j - z \frac{d}{dz}), \quad S_- = \frac{d}{dz}. \quad (8)$$

In the invariant subspace $Pol_{2j}$ the representation (8) has highest and lowest weights $\pm j$ (integer or half-integer spin of the representation). This realization provides the embedding of $A$ into the algebra of differential operators $D$. More precisely, $A$ in this approach is identified with the factorial algebra $U(sl_2)/((S^2 - j(j + 1))$ over the ideal generated by $S^2 - j(j + 1)$, where $S^2$ is the Casimir (central) element.

Then the Hamiltonian of the Euler top,

$$H = \sum \alpha_{ij} S_i S_j + \sum \beta_i S_i, \quad (9)$$

i.e., a general bilinear form in the $sl_2$ generators gives a family of second order differential operators having in general $2j + 1$ independent polynomial eigenfunctions.

Similar arguments may be applied to the quantum algebra $U_q(sl_2)$. In this case one may identify $A$ with a certain factor of $U_q(sl_2)$ over its center. Its highest and lowest weight representations ($q$-analog of (8)) are known and presented in Sect.2. In this paper we show that homogeneous bilinear and linear forms in generators of $U_q(sl_2)$ ($q$-deformations of the Euler top (8)) provide the difference equations of interest (Sect.3).
In addition to $U_q(sl_2)$ some other deformations of the $sl_2$ algebra are known (Appendix B). They also can be used to construct difference equations with polynomial solutions (we note that a deformation not equivalent to $U_q(sl_2)$, which generates a subalgebra of $U_q(sl_2)$, has been used in the paper [13] in attempt to construct algebraized difference operators (see Appendix B)). Among different deformations the quantum algebra $U_q(sl_2)$ plays a special role. It indicates a relation of difference algebraized operators and classical discrete integrable systems (in fact the former are Lax operators for a class of nonlinear integrable equations). In this paper we do not use the coproduct structure of the quantum algebra. However, we anticipate an upgrading of the present approach for quantum integrable systems.

Recently we have shown that the spectrum of the Harper equation (also known as the discrete Mathieu or "almost Mathieu" equation)

$$\psi_{n+1} + \psi_{n-1} + 2 \cos(k + n\Phi)\psi_n = E\psi_n$$

(10)

at $\cos(kQ) = -1$, where $Q$ is the denominator of the parameter $\Phi = 2\pi P/Q$, is algebraized [2]:

$$\psi_n = e^{-i\Phi n(n-1)} \prod_{m=1}^{Q-1} (e^{i\Phi_n} - z_m).$$

The roots $z_m$ obey the Bethe Ansatz like equations

$$z_l^2 = e^{i\pi P} \prod_{m=1, m \neq l}^{Q-1} \frac{e^{i\Phi} z_l - z_m}{z_l - e^{i\Phi} z_m}, \quad l = 1, ..., N$$

(11)

and determine the spectrum [2]:

$$E = (1 - e^{i\Phi}) \sum_{m=1}^{Q-1} z_m.$$ 

The Harper equation describes the Bloch particle on a square lattice in magnetic field (Azbel-Hofstadter problem). In this paper we write the Bethe Ansatz equations for a general class of difference operators having polynomial eigenfunctions. In addition, we present a solution for the Bloch particle on the triangular lattice in Sect.4.

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1 A more general algebraization in terms of meromorphic functions on algebraic curves rather than polynomials has been introduced by Faddeev and Kashaev [14] for quasiperiodic Harper’s equation for an arbitrary $k$. 
In Sect. 2 we give main formulas related to $U_q(sl_2)$ and its representations. In Sect. 3 we consider the spectral problems for difference operators appearing as linear and bilinear forms in generators of $U_q(sl_2)$. They are difference (discrete) analogs of the algebraic forms of known differential equations having polynomial solutions [8, 9, 11, 12, 15]. Besides, we point out a class of difference spectral problems solvable in terms of $q$-deformed classical orthogonal polynomials with discrete measure (big $q$-Jacobi polynomials). Possible applications to the problem of Bloch particle in magnetic field are outlined in Sect. 4. In Sect. 5 the continuum ("classical") limit is discussed. A related class of difference equations with solutions of the form of symmetric Laurent polynomials is treated in Sect. 6. The solutions of a particular case of these equations are Askey-Wilson polynomials.

### 2 The quantum algebra $U_q(sl_2)$ and difference operators

The algebra $U_q(sl_2)$ (a $q$-deformation of the universal enveloping of the $sl_2$) is generated by the elements $A, B, C, D$, with the commutation relations \[ AB = qBA, \quad BD = qDB, \]
\[ DC = qCD, \quad CA = qAC, \]
\[ AD = 1, \quad [B, C] = \frac{A^2 - D^2}{q - q^{-1}}. \] (12)

The deformation parameter $q$ may be considered as a formal variable. In the classical limit $q \to 1$, the quantum algebra turns into the universal enveloping of $sl_2$: $(A - D)/(q - q^{-1}) \to S_3, B \to S_+, C \to S_-.$

The central element of this algebra is a $q$-analog of the Casimir operator

\[ \Omega = \frac{q^{-1}A^2 + qD^2}{(q - q^{-1})^2} + BC. \] (13)

As $q \to 1$, $\Omega - 2(q - q^{-1})^{-2}$ tends to $\vec{S}^2 + 1/4$. 

6
The commutation relations (12) are simply another way to write the intertwining relation for the \( L \)-operator:

\[
R(u/v)(L(u) \otimes 1)(1 \otimes L(v)) = (1 \otimes L(v))(L(u) \otimes 1)R(u/v)
\]

with the trigonometric \( R \)-matrix

\[
R(u) = \frac{1}{2}(q + 1)(u - q^{-1}u^{-1}) + \frac{1}{2}(q - 1)(u + q^{-1}u^{-1})\sigma_3 \otimes \sigma_3 +
\]

\[
+ (q - q^{-1})(\sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+)
\]

(15)

satisfying the Yang-Baxter relation (\( \sigma_j \) are Pauli matrices; \( \sigma_\pm = (\sigma_1 \pm i\sigma_2)/2 \)).

Generators \( A, B, C, D \) are matrix elements of the \( L \)-operator

\[
L(u) = \begin{bmatrix}
\frac{ukA-u^{-1}k^{-1}D}{q-q^{-1}} & C \\
B & \frac{ukD-u^{-1}k^{-1}A}{q-q^{-1}}
\end{bmatrix}.
\]

Here \( u \) is the spectral parameter and \( k \) is an additional parameter ("rapidity" at the site). Note that the \( R \)-matrix is the \( L \)-operator in the spin 1/2 representation. It is given by the same matrix (15) for \( k = q^{1/2} \) with elements: \( A = q^{1/2}\sigma_3, D = q^{-1/2}\sigma_3, B = \sigma_+, C = \sigma_- \).

Irreducible finite dimensional representations of dimension \( 2j + 1 \) can be expressed in the weight basis, where \( A \) and \( D \) are diagonal matrices: \( A = \text{diag}(q^j, ..., q^{-j}) \). An integer or halfinteger \( j \) is spin of the representation. The value of the Casimir operator (13) in this representation is

\[
\Omega_j = \frac{q^{2j+1} + q^{-2j-1}}{(q - q^{-1})^2}.
\]

These representations can be realized \([21]\) by difference operators acting in the linear space of polynomials \( F(z) \) of degree \( 2j \). Let us introduce "shift" operators \( T_+ \) and \( T_-: T_\pm F(z) = F(q^{\pm 1}z), T_+T_- = 1 \),

\[
T_\pm z = q^{\pm 1}zT_\pm, \quad T_\pm z^{-1} = q^{\mp 1}z^{-1}T_\pm.
\]

Then

\[
A = q^{-j}T_+, \quad D = q^jT_-, \\
B = z(q - q^{-1})^{-1} \left( q^{2j}T_- - q^{-2j}T_+ \right) \\
C = -z^{-1}(q - q^{-1})^{-1} (T_- - T_+).
\]

(19)
Then the lowest weight vector is $F_0(z) = 1$ whereas $F_{2j}(z) = z^{2j}$ is the highest weight vector: $CF_0 = 0$, $BF_{2j} = 0$. The realization (19) is a smooth $q$-deformation of the representation of the $sl_2$ algebra by first order differential operators (8). The Casimir operator (13) in this realization is equal to the $c$-number $\Omega_j$ (17). This means that (19) actually gives a representation of the factor algebra $U_q(sl_2)/(\Omega - \Omega_j)$ over the two-sided ideal generated by $\Omega - \Omega_j$.

If $q$ is a root of unity there is, in addition, three parametric family of finite dimensional representations having, in general, no lowest and no highest weight [20]. Sometimes they are called cyclic (or unrestricted) representations [21]. They correspond to discrete quasiperiodic equations (7) which has no direct difference analog.

3 Second order difference operators related to $U_q(sl_2)$

We call a difference operator in a variable $z$ any linear combination of the form $\sum f_i(z) T_i$, where $f_i(z)$ are rational functions of $z$ and the sum is finite. Such operators form the algebra $\mathcal{D}$ mentioned in the Introduction. The general form of a difference operator of second order then is (cf. (1))

$$\Delta = f_1(z) T_+^s + f_2(z) + f_3(z) T_-^s,$$

where $s$ is an integer.

Second order difference operators appear as linear ($s = 1$) or bilinear ($s = 2$) forms in $U_q(sl_2)$ generators. In fact operators coming from linear forms are equivalent to some bilinear forms due to isomorphism of the algebras generated by $A, q^{1/2}BA, q^{1/2}CD, D$ and $A, B, C, D$. Nevertheless, it is convenient to consider them separately. We begin with linear forms.

3.1 Linear forms

Consider a linear form in the quantum algebra generators

$$L = aA + dD + (q - q^{-1})(bB + cC), \quad (20)$$
where \( a, b, c, d \) are parameters. The diagonalization of this operator in 

\[ 2j + 1 \)-dimensional representations of \( U_q(sl_2) \) leads to the difference equation

\[
(cz^{-1} + aq^{-j} - bq^{-2j}z)\Psi(qz) + (-cz^{-1} + dq^j + bq^{2j}z)\Psi(q^{-1}z) = E\Psi(z). \tag{21}
\]

This equation has \( 2j + 1 \) polynomial solutions. Let us parametrize a polynomial by its roots (2). Plugging (2) in (21) and dividing both sides by \( \Psi(z) \) we get

\[
a(z) \prod_{m=1}^{N} \frac{q^2z - z_m}{z - z_m} + d(z) \prod_{m=1}^{N} \frac{q^{-2}z - z_m}{z - z_m} = E \tag{22}
\]

where we have denoted

\[
a(z) = cz^{-1} + aq^{-j} - bq^{-2j}z, \tag{23}
\]

\[
d(z) = -cz^{-1} + dq^j + bq^{2j}z. \tag{24}
\]

Assume that \( c \neq 0 \). Then two different cases are possible: (1) \( b \neq 0, c \neq 0 \) (the case \( b = c = 0 \) is trivial).

Let us first consider the general case (1). Assume that \( \Psi(z) \) is nondegenerate, i.e., all \( z_m \)'s are different. The l.h.s. of (22) is a meromorphic function, whereas the r.h.s. is a constant. To make them equal we must cancel all the singularities of the l.h.s. They appear at singular points of \( a(z) \), and \( d(z) \) (simple poles at \( z = 0 \) and \( z = \infty \)) and at \( z = z_m \). The singular part at \( z = 0 \) vanishes automatically (note that no one of \( z_m \)'s can be equal to 0). The residue at infinity is equal to \( bq^{2j-N}(1 - q^{2N-4j}) \). Its vanishing determines the degree of the polynomial: \( N = 2j \). Comparing the constant terms in the both sides of (22) one finds the energy spectrum:

\[
E = aq^j + dq^{-j} - b(q - q^{-1}) \sum_{m=1}^{2j} z_m. \tag{25}
\]

Annihilation of poles at \( z = z_m \) gives the following Bethe-ansatz equations:

\[
\frac{bq^{2j}z_l^2 - aq^{-j}z_l - c}{bq^{-2j}z_l^2 - aq^{-j}z_l - c} = -\prod_{m=1, m \neq l}^{2j} \frac{qz_l - z_m}{z_l - qz_m}. \tag{26}
\]

In the case (2) the operator \( L \) is triangular in the basis of monomials. It includes only \( A, D \) and \( C \) (the generators of the Borel subalgebra). As a
result the coefficients of $\Psi(z)$ can be recursively determined. Therefore there is only one polynomial solution for each degree $N = 1, ..., 2j$. One can see it from (22). Its l.h.s. is regular at $z = \infty$ hence there is no restriction on degree of the polynomials:

$$\frac{dq^j z_l - c}{aq^{-j} z_l + c} = q^N \prod_{m=1, m \neq l}^N \frac{q z_l - qz_m}{z_l - qz_m}. \quad (27)$$

The energy spectrum is given by the simple formula

$$E_N = aq^{-j} + dq^{-N}. \quad (28)$$

The Bethe equations (27) must have exactly one solution (modulo permutation of the roots). In section 3.2 we discuss a relation of the solutions with $q$-orthogonal polynomials.

### 3.2 Bilinear forms

Consider a general bilinear form in $U_q(sl_2)$ generators:

$$G = aA^2 + dD^2 + (q - q^{-1})(c_2CA + b_2BD + b_3BA + c_3CD) + (q - q^{-1})^2(b_1B^2 + c_1C^2), \quad (29)$$

where $a, d, c_i, b_i$ ($i = 1, 2, 3$) are arbitrary parameters.

In the representation of spin $j$ it is a difference operator:

$$G\Psi(z) = a(z)\Psi(q^2z) + d(z)\Psi(q^{-2}z) - v(z)\Psi(z), \quad (30)$$

where

$$a(z) = b_1q^{-4j+1}z^2 - b_3q^{-3j}z + aq^{-2j} + c_2q^{-j}z^{-1} + c_1q^{-1}z^{-2}, \quad (31)$$

$$d(z) = b_1q^{4j-1}z^2 + b_2q^{3j}z + dq^{2j} - c_3q^jz^{-1} + c_1qz^{-2}, \quad (32)$$

$$v(z) = (q + q^{-1})(b_1z^2 + c_1z^{-2}) + (c_2q^{-j} - c_3q^j)z^{-1} + (b_2q^{-j} - b_3q^j)z. \quad (33)$$

The difference operator given by the linear form (21) is a particular case of the bilinear form

$$aA^2 + dD^2 + (q - q^{-1})(q^jCA + q^jBD + q^{-j}BA + q^{-j}CD),$$
where $q$ is to be changed to $q^2$.

Let us plug (2) in (30) and divide both sides by $\Psi(z)$. We get

$$a(z) \prod_{m=1}^{N} \frac{q^2 z - z_m}{z - z_m} + d(z) \prod_{m=1}^{N} \frac{q^{-2} z - z_m}{z - z_m} - v(z) = E. \quad (34)$$

Suppose that at least one of the coefficients $c_1, c_2, c_3$ is nonzero, then there are three main different cases:

(i) at least one of $b_1, b_2, b_3$ is nonzero and both $a(z)$ and $d(z)$ (see (31), (32)) are nonzero (this is the case of generic position; the other two can be considered as exceptional cases);

(ii) All $b$’s are zero: the quadratic form (29) includes only $A, D$ and $C$ (generators of the Borel subalgebra of $U_q(sl_2)$). In this case the eigenfunctions are big $q$-Jacobi polynomials [23]. They include all $q$-deformed classical orthogonal polynomials with discrete measure [23];

(iii) One of the functions $a(z)$ or $d(z)$ is identically zero. In this case the second-order difference equation (30) reduces to a first-order equation.

Let us consider the case (i) first. The l.h.s. of (34) is a meromorphic function, whereas the r.h.s. is a constant. To make them equal we must cancel all the singularities of the l.h.s. They appear at singular points of $a(z), d(z)$ and $v(z)$ (double and simple poles at $z = 0$ and $z = \infty$) and at $z = z_m$ (again, we consider non-degenerate case when all of them are simple poles). Note that the case when at least one of $z_m$’s is zero (i.e., coincides with the pole of $a(z)$ and $d(z)$) needs special consideration. At the moment we assume that $z_m \neq 0$. The singular part at $z = 0$ vanishes automatically. Vanishing of the singular part at $z = \infty$,

$$b_1(q^{2N-4j+1} + q^{-2N+4j-1} - q - q^{-1})z^2 + b_2(q^{-2N+3j} - q^{-j})z +$$

$$b_3(q^j - q^{-2N-3j})z + z(q - q^{-1})b_1(q^{2N-4j} - q^{-2N+4j}) \sum_{m=1}^{N} z_m, \quad (35)$$

determines degree of the polynomial: $N = 2j$. If $b_1 = 0$ and $b_2/b_3 = -q^{2M}$ for an integer $M > -j$, $N = j + M$ is also possible. Below we consider only the generic case $N = 2j$.

Comparing the constant terms in the both sides of (34) we find the energy spectrum:

$$E = b_1(q - q^{-1})(q^2 - q^{-2}) \sum_{n<m}^{2j} z_n z_m -$$
\[-(q - q^{-1})(b_2 q^{-j+1} + b_3 q^{j-1}) \sum_{m=1}^{2j} z_m + a q^{2j} + dq^{-2j} \]  

Finally, annihilation of poles at \( z = z_m \) gives the following Bethe-Ansatz equations

\[
\frac{d(z_l)}{a(z_l)} = q^{4j} \prod_{m=1, m \neq l}^{2j} \frac{q^2 z_l - z_m}{z_l - q^2 z_m}, \quad l = 1, \ldots, 2j.  
\]  

The Bethe equations is a system of \( 2j \) algebraic equations. It must have exactly \( 2j + 1 \) solutions corresponding to different eigenfunctions. In the case (i) all of them are polynomials of one and the same degree \( 2j \).

Note that \( d(z)/a(z) \) is a rational function having at most 4 zeros and 4 poles. Therefore (37) looks like a system of Bethe equations for a \( XXZ \)-spin chain on at most 4 sites with different spins at the sites. The reader familiar with the algebraic Bethe ansatz [24, 25] should notice that eq.(34) is the Baxter identity for eigenvalues of the transfer matrix \( t(z) \) for this system

\[
\Psi(z) t(z) = a(z) \Psi(q^2 z) + d(z) \Psi(q^{-2} z),  
\]  

In this context the eq.(38) determines \( t(z) \) and \( \Psi(z) \) provided that \( a(z) \) and \( d(z) \) are known functions.

In the special case \( q^{2j+1} = 1 \) the difference operator has finite dimension \( 2j + 1 \). Therefore polynomial eigenfunctions and the Bethe Ansatz cover all the spectrum. There are no more solutions other than polynomials.

### 3.3 Triangular and first order operators

#### 3.3.1 \( q \)-Hypergeometric equations

The difference operators that include only elements \( A, D \) and \( C \) (case (ii)) lead to \( q \)-hypergeometric equations. These operators preserve not only the space \( \text{Pol}_{2j} \) for some \( j \), but all the spaces \( \text{Pol}_n \) for any \( n \geq 0 \). Indeed, they are lower triangular in the basis of monomials \( z^k \). The l.h.s. of (34) is now regular at \( z = \infty \), so there is no restriction on degree of the polynomials. The Bethe equations are valid for any \( N < 2j + 1 \):

\[
\frac{d(z_l)}{a(z_l)} = q^{2N} \prod_{m=1, m \neq l}^{N} \frac{q^2 z_l - z_m}{z_l - q^2 z_m}, \quad l = 1, \ldots, N.  
\]
Triangularity leads to a very simple structure of the spectrum of $G$:

$$E_N = aq^{2N-2j} + dq^{2j-2N}, \quad N = 0, ..., 2j. \quad (40)$$

A general operator of this kind has the same spectrum as the diagonal operator $aA^2 + dD^2$; it does not depend on the coefficients $c_1, c_2, c_3$.

For the sake of completeness let us identify the solutions of (30) in the case (ii) with known $q$-orthogonal polynomials. Consider the big $q$-Jacobi polynomials $P^{(\alpha,\beta)}_n(z; \gamma, \delta; q^2)$ (where we have used the standard notation from the textbook [23]; to shorter the notation we denote them as $P_n(z)$). They can be expressed in terms $q$-hypergeometric series:

$$P^{(\alpha,\beta)}_n(z; \gamma, \delta; q^2) = \phi_2 \left[ \begin{array}{c} q^{-2n}, q^{2\alpha+2\beta+2n+2}, q^{2\alpha+2\gamma-1}z \\ q^{2\alpha+2}, -q^{2\alpha+2\delta-\gamma-1} \\ q^2, q^2 \end{array} \right] (41)$$

and obey the difference equation [23]

$$(q^{2\alpha-2\beta+2} + (q^{2\alpha\delta} - q^{2\beta\gamma})z^{-1} - \gamma\delta q^{-2}z^{-2})P_n(q^2z) + (1 + (\delta - \gamma)z^{-1} - \gamma\delta z^{-2})P_n(q^{-2}z) - (1 + q^{2\alpha}\delta - (1 + q^{2\beta})\gamma)z^{-1} - \gamma\delta(1 + q^{-2})z^{-2}]P_n(z) = (q^{-2n} + q^{2n+2\alpha+2\beta+2})P_n(z). \quad (42)$$

This equation is equivalent to (30) provided $d \neq 0$, $b_1 = b_2 = b_3 = 0$, and $\gamma - \delta = c_3d^{-1}q^{-j}$, $-\gamma\delta = c_1d^{-1}q^{-1-2j}$, $q^{2\alpha\delta} - q^{2\beta\gamma} = c_2d^{-1}q^{-3j}$, $q^{2\alpha+2\beta} = ad^{-1}q^{2-4j}$.

Note that (41) is formally symmetric with respect to the change $n \leftrightarrow -\alpha - \beta - n - 1$. This symmetry becomes important if $m \equiv \alpha + \beta + n + 1$ is a negative integer. Then the series (41) truncates at $n$-th term if $n < |m|$, but if $|m| < n$ it truncates at $|m|$-th term. One has $\alpha + \beta = \log_{q^2}(a/d) - 2j - 1$. In particular, if $a = d$ the series (41) is symmetric with respect to $n \leftrightarrow 2j - n$.

In other words, for integer $j$ it gives only $j + 1$ different eigenfunctions (they are polynomials of degrees 0, 1, ..., $j$). It can be easily seen that in this case the original operator contains Jordan cells and, therefore, is not completely diagonalizable (i.e., it has less than $2j + 1$ different eigenfunctions). In this case the formula (41) still gives all the eigenfunctions.

At $\delta = 0$, the big $q$-Jacobi polynomials $P^{(\alpha,\beta)}_n(z; \gamma, 0; q)$ reduce (up to an $n$-independent factor) to the little $q$-Jacobi polynomials $P^{(\beta,\alpha)}_n(z\gamma^{-1}, q)$. 

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They provide eigenfunctions of the linear form (20) at $b = 0$. In this case $q^\beta = -1$, $q^\alpha = -ad^{-1}q^{-1-2j}$, $\gamma = cd^{-1}q^{-j}$. It is known that the little $q$-Jacobi polynomials can be expressed through the $q$-hypergeometric function $\phi_1$:

$$ p_n^{(\beta, \alpha)}(z\gamma^{-1}, q) = \phi_1 \left[ \begin{array}{c} q^{-n}, q^{\alpha+\beta+n+1} \\ q^{\beta+1} \\ q^{\gamma^{-1}z} \end{array} \right]. \tag{43} $$

Again, if the operator is not completely diagonalizable, (43) gives less than $2j + 1$ different eigenfunctions.

The following comment is in order. In contrast to the linear forms (27) the system of algebraic equations (39) generally has more than one solution for a given $N$. However, only one of them corresponds to an eigenfunction. The point is that the Bethe equations have been derived under assumption that all roots are nonzero. In fact one or more zero roots are not forbidden (for instance, $q$-Legendre polynomials of odd degree are odd functions). However, they change the residues at $z = 0$ of the first two terms in (34), so this case needs a special consideration. To illustrate this let us give an example.

**Example.** Consider the spectral problem for the following quadratic forms:

$$ G_1 = A^2 - D^2 + (q - q^{-1})(q^{j-1}CA + BD - BA - q^{1-j}CD), $$

$$ G_2 = q^{-j}A^2 + q^jD^2 + (q - q^{-1})(q^{-j}CA + q^jCD) + (q - q^{-1})^2C^2. $$

Though $G_1$ corresponds to the case (i) and $G_2$ to the case (ii) they lead to identical Bethe equations. For instance, at $j = 1/2$ the equation is

$$ \frac{qz_1^2 - q^{1/2}z_1 + q^{1/2}}{q^{-1}z_1^2 + q^{-1/2}z_1 + q^{-1/2}} = q. $$

It has two roots: $z_1' = 0$ and $z_1'' = 2(q^{1/2} - q^{-1/2})^{-1}$. Both of them give an eigenfunction of $G_1$, while for $G_2 z_1'$ is an artifact.)

### 3.3.2 First order operators

At last, we consider the case (iii). When, say, $d(z) = 0$ the operator is

$$ G_0 = aA^2 + (q - q^{-1})(c_2C + b_3B)A. \tag{44} $$

These operators appear in harmonic analysis on the quantum group [26], [27].
The Bethe equations (37) are simplified as
\[
a(z_l) \prod_{k=1,k \neq l}^{2j} (q^2 z_l - z_k) = 0,
\]
so eigenfunctions and the spectrum can be easily found. Let \( z_{\pm} \) be the two roots of the quadratic equation \( a(z) = 0 \), then the \( m \)-th eigenfunction \( (m = 0, 1, \ldots, 2j) \) and the spectrum are
\[
\Psi_m(z) = \prod_{k=0}^{2j-1-m} (z - z_+ q^{-2k}) \prod_{l=0}^{m-1} (z - z_- q^{-2l}),
\]
\[
E_m = b_3 q^{-j} (z_- q^{2j-2m} + z_+ q^{2m-2j}).
\]

3.3.3 Algebraic structure unifying triangular and first order operators

The similarity of the spectra of operators of types (ii) and (iii) is not an accident. It turns out that these operators form a simple quadratic algebra. Consider two operators of the type (iii)
\[
H_1 = \mu_1 BA + \mu_2 CA + \mu_3 A^2,
\]
\[
H_2 = \nu_1 DB + \nu_2 DC + \nu_3 D^2
\]
with arbitrary coefficients \( \mu_i \) and \( \nu_i \) and a general operator of the type (ii)
\[
H_3 = (q^2 - q^{-2}) \left( \frac{\mu_1 \nu_2 A^2 + \mu_2 \nu_1 D^2}{(q - q^{-1})^2} - q \mu_2 \nu_2 C^2 - \mu_2 \nu_3 DC - \mu_3 \nu_2 CA \right).
\]
It is straightforward to check that these operators form the closed algebra \footnote{A similar algebra has been considered by Granovskii and Zhedanov in [28], a particular case \( h_1 = h_2 = h_3 = 0 \) has been discussed in [30], [32].}
\[
q^{-1} H_i H_j - q H_j H_i = g_{ik} H_k + h_k
\]
where \( \{ijk\} \) stands for any cyclic permutation of \( \{123\} \). The structure constants \( g_i \) and \( h_i \) may be expressed in terms of coefficients \( \mu_i \) and \( \nu_i \) and the value of the Casimir operator \( \Omega \) (13).
The algebra (48) allows one to find the spectrum of $H_1$, $H_2$ or $H_3$ in an algebraic way [25]. Suppose one knows an eigenvector of $H_1$. Then, some linear combination of $H_i$ is a creation operator: it creates a new eigenvector by acting to the known one. As a result the spectrum of $H_1$ is found to be a trigonometric function of the number of level.

4 Difference periodic equations and the group of magnetic translations

Difference periodic equations and discrete equations with periodic coefficients appear when $q$ is a root of unity:

$$q = \exp(i\pi P/Q),$$

(49)

where $P$ and $Q$ are coprime integers. Some of them have important applications in physics.

Let us briefly recall the problem of Bloch particles in magnetic field (Azbel-Hofstadter problem). Consider a particle on a two-dimensional square lattice. The Schrödinger equation has the form

$$\sum_{\mu} t_{\mu} e^{iA_{\mu}(\vec{n})} \psi(\vec{n} + \vec{\mu}) = E \psi(\vec{n}),$$

(50)

where $\vec{n}$ is a lattice site, $\vec{\mu}$ is a lattice vector and $t_{\mu} = t_{-\mu}$ is a hopping amplitude. The gauge potential $A_{\mu}(\vec{n}) = -A_{-\mu}(\vec{n} - \vec{\mu})$ describes a homogenous magnetic field with flux $\Phi = 2\pi P/Q$ per plaquette of the square lattice:

$$\prod_{\text{plaquette}} e^{iA_{\mu}(\vec{n})} = e^{i\Phi} \equiv q^2.$$

In this context the parameter $q$ has a clear interpretation: it is a flux per half of the plaquette. The problem may be expressed in terms of magnetic translations [3]:

$$T_{\mu} \psi(\vec{n}) = e^{iA_{\mu}(\vec{n})} \psi(\vec{n} + \vec{\mu}).$$
Let \( x \) and \( y \) be unit vectors along \( x \) and \( y \) directions on the square lattice and \( \vec{n} = (n_x, n_y) \). They generate the algebra of magnetic translations

\[
T_{\vec{n}}T_{\vec{m}} = q^{n_ym_x-n_xm_y}T_{\vec{n}+\vec{m}}, \quad T_{\vec{n}}^{-1} = T_{-\vec{n}},
\]

\[
TyT_x = q^2T_xTy, \quad TyT_{-x} = q^{-2}T_{-x}Ty.
\]

The operators \( T_{\vec{n}}^Q \) are central elements. As an algebra over its center the algebra of magnetic translations is the finite Heisenberg-Weyl algebra.

In these terms the Hamiltonian of the problem is

\[
H = \sum_{\vec{\mu}} t_{\vec{\mu}}T_{\vec{\mu}}.
\]

The quantum algebra \( U_q(sl_2) \) may be expressed through the Heisenberg-Weyl generators, i.e., in terms of magnetic translations \( \mathbb{H} \). There are many different ways to do it (see e.g. \[33\]). One of them (for odd \( P \)) is

\[
A^2 = T_{x+y}, \quad D^2 = T_{-x-y}, \quad CA = iq^{1/2}(q - q^{-1})^{-1}(T_{-x} + T_y),
\]

\[
BD = iq^{1/2}(q - q^{-1})^{-1}(T_x + T_{-y}), \quad CD = iq^{-1/2}(q - q^{-1})^{-1}(T_{-x} + T_{-2x-y}),
\]

\[
BA = iq^{-1/2}(q - q^{-1})^{-1}(T_x + T_{2x+y}),
\]

\[
B^2 = -(q - q^{-1})^{-2}(T_{x-y} + T_{3x+y} + (q + q^{-1})T_{2x}),
\]

\[
C^2 = -(q - q^{-1})^{-2}(T_{x+y} + T_{-3x-y} + (q + q^{-1})T_{-2x}).
\]

The central element \([13]\) in this representation is

\[
\Omega = -2(q - q^{-1})^{-2}.
\]

Comparing with \([17]\) we see that this value corresponds to the representation \([19]\) with \( j = (Q - 1)/2 \) and dimension \( Q \). The same value \([24]\) corresponds to a 2-parametric family of the \( Q \)-dimensional cyclic representations \([20]\).

Plugging \((53)\) into \((29)\) we obtain

\[
G = aT_{x+y} + dT_{-x-y} - b_1T_{x-y} - c_1T_{-x+y} - b_1T_{3x+y} - c_1T_{-3x-y} +
+ iq^{1/2}(c_2T_y + b_2T_{-y}) + iq^{1/2}(b_2T_x + c_2T_{-x}) + iq^{-1/2}(b_3T_x + c_3T_{-x}) +
+ iq^{-1/2}(b_3T_{2x+y} + c_3T_{-2x-y}) - (q + q^{-1})(c_1T_{-2x} + b_1T_{2x}).
\]

\((55)\)
The operator $G$ is hermitian if $d = \bar{a}$, $b_1 = \bar{c}_1$, $-qb_2 = \bar{c}_2$, $b_3 = -q\bar{c}_3$. It describes a particle in a magnetic field on the square lattice with a hopping along $x - y$ and $x + y$ diagonals, along $x, y$ and $2x$ bonds and along $2x + y$ and $3x + y$ diagonals.

Although the Hamiltonians (55) contains many parameters, the number of interesting problems is limited since the hopping amplitudes $t_{\vec{\mu}}$ must not depend on the magnetic flux (i.e., on $q$). Then only few cases remain. Among them, there is a triangular lattice: $b_1 = c_1 = b_3 = c_3 = 0, c_2 = b_2 = \pm iq^{-1/2}$, $a = d = t \neq 0$ (the sign depends on parity of $(P - 1)/2$: it is "+" for $(P - 1)/2$ even and "−" otherwise),

$$H = t(T_{x+y} + T_{x-y}) + T_x + T_{-x} + T_y + T_{-y}. \quad (56)$$

Here the triangular lattice is treated as a square lattice with $x + y$ diagonals. The square lattice may be obtained at $t = 0$. This case has been considered in [1].

In terms of generators of $U_q(sl_2)$, the Hamiltonian (56) is

$$H = t(A^2 + D^2) \pm iq^{-1/2}(q - q^{-1})(CA + BD) \quad (57)$$

and under the representation [13] we obtain the spectral problem

$$(z^{-1} - tq)\Psi(q^2z) + (zq^{-2} - tq^{-1})\Psi(q^{-2}z) - (z + z^{-1})\Psi(z) = E\Psi(z) \quad (58)$$

How does this equations appears from the original problem? Let us choose a specific gauge to implement the flux $\Phi/2$ per elementary triangle:

$$A_x = -\Phi n_x, \quad A_y = \Phi n_x, \quad A_{x+y} = \Phi/2. \quad (59)$$

Then, the Schrödinger equation acquires the form

$$
e^{-i\Phi n_x}\psi(n_x + 1, n_y) + e^{i\Phi(n_x - 1)}\psi(n_x - 1, n_y)$$

$$+ e^{i\Phi n_x}\psi(n_x, n_y + 1) + e^{-i\Phi n_x}\psi(n_x, n_y - 1) + t(e^{i\Phi/2}\psi(n_x + 1, n_y + 1) + e^{-i\Phi/2}\psi(n_x - 1, n_y - 1) = E\psi(n_x, n_y). \quad (60)$$

Solutions are the Bloch waves:

$$\psi(n_x, n_y) = e^{ik_x n_x + ik_y n_y}\psi_{n_x, n_y},$$

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where \( \vec{k} = (k_x, k_y) \) is the Bloch momentum and \( \psi_n \) is \( Q \)-periodic discrete function. In these terms (60) turns into the following version of the Harper equation:

\[
( e^{i k_x - i \Phi_n} + t e^{i(k_x + k_y) + i \Phi/2} ) \psi_{n+1} + ( e^{-i k_x + i \Phi_n - i \Phi} + t e^{-i(k_x + k_y) - i \Phi/2} ) \psi_{n-1} + 2 \cos(n \Phi + k_y) \psi_n = E \psi_n.
\]

The spectrum of the problem has \( Q \) bands \( E = E_i(k_x, k_y), \ i = 1, \ldots, Q \). For \( k_x = 0, k_y = \pi \) ("midband points" of the spectrum) Eq. (61) becomes

\[
(q^{-2n} - tq) \psi_{n+1} + (q^{2n-2} - tq^{-1}) \psi_{n-1} - (q^{2n} + q^{-2n}) \psi_n = E \psi_n \quad (62)
\]

with the periodic boundary condition (6). This discrete equation is equivalent to the difference equation (58). Applying the results of Sect.3 to (57), (58) we obtain the Bethe equations

\[
\frac{z_l(z_l - t q)}{1 + t z_l} = \prod_{m=1}^{Q-1} \frac{q^2 z_l - z_m}{z_l - q^2 z_m} \quad (63)
\]

and the energies of the mid points of each band:

\[
E = (q^2 - 1) \sum_{m=1}^{Q-1} z_m - t(q + q^{-1}). \quad (64)
\]

Harper’s equation (61) for an arbitrary Bloch momentum \( \vec{k} \) may be obtained from (57) by using cyclic representations [20]. These representations generally has no highest or lowest weight. The wave functions are not polynomials in this case. Nevertheless the Bethe Ansatz a kind of algebraization in terms of meromorphic functions on higher genus algebraic curves is still possible. The method which generalizes the algebraic Bethe Ansatz has been developed in Ref.[33] and applied to the chiral Potts model. Recently, Faddeev and Kashaev [14] used this method for the Harper equation on the square lattice to obtain the Bethe Ansatz equations for arbitrary \( \vec{k} \).

Simplicity of the spectrum (10) of triangular operators considered in the Sect.3 has a clear interpretation in terms of magnetic translations. In this case the Hamiltonian allows hopping only to the South-West, North-East and North-West directions on the lattice. Therefore, no closed loop trajectory is possible. As a result, the particle does not feel the magnetic field and the spectrum remains free.
5 Continuum limit. Differential equations

Difference equations of Sect.3 may have different continuum limits. Let us consider the simplest one, when we assume that $(\Psi(q^2z) - \Psi(q^{-2}z))/(q - q^{-1})$ has a regular limit as $q \to 1$. Let us set $q = e^{\bar{h}}$ and also assume that $q^j = 1 + \mathcal{O}(\bar{h})$, so $j$ is a fixed integer or half-integer number (let us note that this condition is not valid for the periodic difference equations considered in the Sect.4, where $q^{2j+1} = \pm 1$). Then

$$A = 1 + \hbar S_3 + \frac{1}{2} \hbar^2 S_3^2 + \mathcal{O}(\hbar^3), \quad D = 1 - \hbar S_3 + \frac{1}{2} \hbar^2 S_3^2 + \mathcal{O}(\hbar^3),$$

$$B = S_+ + \mathcal{O}(\hbar^2), \quad C = S_- + \mathcal{O}(\hbar^2)$$

as $\hbar \to 0$. Here $S_\pm, S_3$ are generators of $U(sl_2)$ realized as differential operators (8).

The coefficients of $G$ (29) may also depend on $\bar{h}$. Suppose they are regular at $\bar{h} = 0$. Up to the first order in $\bar{h}$ one has:

$$a = a^{(0)} + \hbar a^{(1)}, \quad d = d^{(0)} + \hbar d^{(1)}, \quad b_i = b_i^{(0)} + \hbar b_i^{(1)}, \quad c_i = c_i^{(0)} + \hbar c_i^{(1)}, \quad i = 1, 2, 3.$$ 

In order to get a non-trivial limit one should put $a^{(0)} = d^{(0)}, \quad c_2^{(0)} = -c_3^{(0)}, \quad b_2^{(0)} = -b_3^{(0)}$. Then we obtain a general bilinear form in the $sl_2$ generators (the quantum Euler top (9)). Components of the matrix $\alpha_{ij}$ and the vector $\beta_i$ are $\alpha_{33} = a^{(0)}, \quad \alpha_{++} = 2b_2^{(0)}, \quad \alpha_{--} = 2c_1^{(0)}, \quad \alpha_{-,3} = 2c_2^{(0)}, \quad \alpha_{+3} = -2b_2^{(0)}, \quad \alpha_{+-} = \alpha_{3,-} = 0, \quad \beta_3 = a^{(1)} - d^{(1)}, \quad \beta_- = c_2^{(1)} + c_3^{(1)}, \quad \beta_+ = b_2^{(1)} + b_3^{(1)}$. Clearly, a global $SL(2)$-rotation leaves the spectrum invariant. Therefore, in the case of generic position, one may diagonalize the matrix $\alpha_{ij}$ and reduce the number of parameters to 4. Indeed, there are 3 momenta of inertia (eigenvalues of $\alpha_{ij}$) and direction of the ”magnetic field” $\vec{\beta}$ (parametrized e.g. by 2 Euler angles). However, the Casimir element $\vec{S}^2$ contributes only to a c-number term, so we are left with 4 parameters. In terms of differential operators the adjoint action of the group amounts to linear fractional transformation. We do not know $q$-analog of this transformation in the discrete case. An explicit form of the differential equation which corresponds to the quantum Euler top in the representation (8) is

$$Q_4(z) \frac{d^2}{dz^2} + (Q_2(z) - (j - \frac{1}{2})Q'_2(z)) \frac{d}{dz} + \left(\frac{1}{3} j (j - \frac{1}{2})Q''_2(z) - j Q'_2(z)\right), \quad (65)$$

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where \( Q_k(z) \) are general polynomials in \( z \) of degree at most \( k \). These are the only differential operators of second order that have polynomial eigenfunctions \(^{[12]}\) (see also \(^{[10]}\)).

The quasiclassical version of Bethe equations \(^{(37)}\) describes roots of the polynomial eigenfunctions and algebraized part of the spectrum:

\[
Q_2(z_l) - (j - 1/2)Q_4(z) = -2 \sum_{m=1, m \neq l}^{2j} \frac{1}{z_l - z_m},
\]

\[
E = 4b_1 \sum_{n<m}^{2j} z_n z_m + (b_2^{(1)} + b_3^{(1)}) \sum_{m=1}^{2j} z_m + \text{const}.
\]

Diagonalization of \(^{(65)}\) in terms of Bethe equations has been proposed in ref.\(^{[9]}\).

If \( Q_4(z) \) has 4 simple roots, the spectral problem for \(^{(65)}\) is equivalent to the Heun equation \(^{[34]}\) (the algebraic forms of Lame and Mathieu equations are degenerate cases of the Heun equation). We, therefore, may call the difference equation \(^{(65)}\) as \( q \)-Heun (Mathieu, Lame) algebraic equations.

There is also degenerate (triangular) case. The case (ii) corresponds to the hypergeometric differential operator

\[
Q_2(z) \frac{d^2}{dz^2} + (Q_1(z) - (j - 1/2)Q_2(z)) \frac{d}{dz}.
\]

Its eigenfunctions are classical orthogonal polynomials. The classical limit of operators of the case (iii) of Sect.3 are first order differential operators.

At last, let us discuss the continuum limit of the linear form \(^{(20)}\). Adopting the similar notation for \( \hbar \)-dependent coefficients \( (a = a^{(0)} + \hbar a^{(1)}, \text{etc}) \) as \( \hbar \to 0 \) and putting \( a^{(0)} = d^{(0)}, b^{(0)} = c^{(0)} = 0 \) we obtain the operator:

\[
L = 2\hbar^2 (2a^{(0)} S_3^2 + \sum \beta_i S_i),
\]

where \( \beta_3 = a^{(1)} - d^{(1)}, \beta_+ = b^{(1)}, \beta_- = c^{(1)} \). As a differential operator it is:

\[
z^2 \frac{d^2}{dz^2} + (-\mu z^2 + (1 - 2j + \nu)z + \lambda) \frac{d}{dz} + 2j\mu z,
\]

where \( \mu = b^{(1)}/(2a^{(0)}), \nu = (a^{(1)} - d^{(1)})/(2a^{(0)}), \lambda = c^{(1)}/(2a^{(0)}) \). After the change of variable \( z = e^{-x} \) and a ”gauge” transformation eliminating the
term with first derivative, this operator acquires the form $\frac{d^2}{dx^2} - V(x)$ with

$$V(x) = \frac{1}{4}(\mu^2 e^{-2x} + \lambda^2 e^{2x}) - \frac{1}{2}(\mu(2j + 1 + \nu)e^{-x} + \lambda(2j + 1 - \nu)e^x).$$

If $c^{(1)} = 0$ or $b^{(1)} = 0$ it is the Morse potential.

6 Difference equations solvable in symmetric Laurent and Askey-Wilson polynomials

In this section we describe a related family of difference operators with eigenfunctions given by symmetric Laurent polynomials in one variable $y$, i.e., invariant under the inversion $y \to y^{-1}$. Differential operators of this kind may be obtained from (65) and (68) by the change of variable $z = (y + y^{-1})/2$ (that is the change $z = \cos \theta$, $e^{i\theta} = y$ which turns Legendre polynomials into spherical harmonics). In the case of difference equations, there is no direct $q$-analog of ”changes of variables”. We must use another operator algebra.

The proper operator algebra has been obtained in the paper [35] as a certain degenerate case of the Sklyanin algebra [20]:

$$\tilde{D} \tilde{C} = q \tilde{C} \tilde{D}, \quad \tilde{C} \tilde{A} = q \tilde{A} \tilde{C},$$

$$\tilde{A} \tilde{B} - q \tilde{B} \tilde{A} = q \tilde{D} \tilde{B} - \tilde{B} \tilde{D} = -(1/4)(q^2 - q^{-2})(\tilde{D} \tilde{C} - \tilde{C} \tilde{A}),$$

$$\{\tilde{A}, \tilde{D}\} = (1/4)(q - q^{-1})^2 \tilde{C}^2,$$

$$\{\tilde{B}, \tilde{C}\} = \frac{\tilde{A}^2 - \tilde{D}^2}{q - q^{-1}}.$$
It can be realized by the shift operators (18):

\[
\tilde{A} = \frac{q^{-j}}{y - y^{-1}}(yT_+ - y^{-1}T_-), \quad \tilde{D} = \frac{q^j}{y - y^{-1}}(-y^{-1}T_+ + yT_-),
\]

\[
\tilde{B} = \frac{1}{2(q - q^{-1})(y - y^{-1})}(-q^{-2j}y^2 + q)(q^{2j-1}y^{-2} + 1)T_+ + (q^{-2j}y^{-2} + q)(q^{2j-1}y^2 + 1)T_-,
\]

\[
\tilde{C} = 2\frac{(q - q^{-1})(y - y^{-1})}{(q - q^{-1})(y - y^{-1})}(T_+ - T_-).
\]

(71)

The representation space of this algebra is spanned by \(y^k + y^{-k}, k = 0, 1, \ldots, 2j\) (symmetric Laurent polynomials). The standard quantum algebra \(U_q(sl_2)\) can be obtained as a contraction of (70): \(\tilde{C} = \epsilon^2C, \tilde{A} = \epsilon A, \tilde{D} = \epsilon D, \tilde{B} = B, \epsilon \to 0\). There are two central elements:

\[
\tilde{\Omega}_0 = \tilde{A}\tilde{D} + \frac{1}{4q}(q - q^{-1})^2\tilde{C}^2,
\]

\[
\tilde{\Omega}_1 = \frac{q^{-1}\tilde{A}^2 + q\tilde{D}^2}{(q - q^{-1})^2} + \tilde{B}\tilde{C} + \frac{1}{2}(q + q^{-1})\tilde{C}^2.
\]

(72)

(73)

In the continuum limit one obtains \(U(sl_2)\) in the representaion (19), where \(z = (y + y^{-1})/2\). Homogeneous bilinear (and linear) forms in \(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}\) give rise to a class of difference equations partially solvable in symmetric Laurent polynomials.

The linear form (20) (with \(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}\) in place of \(A, B, C, D\)) gives the operator

\[
\tilde{a}(y) = \frac{\tilde{a}(q^{-j}y)}{2(y - y^{-1})}T_+ - \frac{\tilde{a}(q^{-j}y^{-1})}{2(y - y^{-1})}T_-.
\]

(74)

where

\[
\tilde{a}(y) = -b(y^2 + y^{-2}) + 2ay - 2dy^{-1} + 4c - b(q + q^{-1}).
\]

(75)

If \(b \neq 0\) and \(c \neq 0\) the algebraic eigenfunctions of (74) are given by

\[
\Psi(y) = y^{-2j} \prod_{l=1}^{2j}(y - y_l)(y - y_l^{-1})
\]

(76)
and $y_l$’s satisfy the Bethe equations

$$\frac{\tilde{a}(q^{-j}y_k^{-1})(y_k - qy_l^{-1})}{\tilde{a}(q^{-j}y_k)(qy_k - y_l^{-1})} = -\prod_{l=1, l\neq k}^{2j} \frac{(qy_k - y_l)(qy_k - y_l^{-1})}{(y_k - qy_l)(y_k - qy_l^{-1})},$$

(77)

with the eigenvalues being expressed through $y_l$’s as follows:

$$E = aq^j + dq^{-j} - \frac{1}{2}b(q - q^{-1})\sum_{l=1}^{2j}(y_l + y_l^{-1})$$

(78)

(compare with (25).

The general bilinear form (29) in $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ gives the difference operator

$$G = A(y)(T^2_+ - 1) + A(y^{-1})(T^2_+ - 1) + W(y),$$

(79)

where

$$A(y) = \frac{\sum_{s=-4}^{4} a_s q^{-js}y^s}{(y - y^{-1})(qy - q^{-1}y^{-1})},$$

(80)

$$W(y) = \frac{1}{4}(q^{2j} - q^{-2j}) (b_1(q^{2j-1} - q^{-2j+1})(y^2 + y^{-2}) +$$

$$+ 2(b_2q^j + b_3q^{-j})(y + y^{-1})),$$

(81)

and

$$a_0 = 4c_1 + \frac{1}{2}b_1(q^2 + q^{-2} + 1), \quad a_1 = 2q(c_2 + \frac{1}{4}(q^{-2}b_2 - (q + q^{-1})b_3)),\quad a_{-1} = -2q^{-1}(c_3 + \frac{1}{4}(q^2b_3 - (q + q^{-1})b_2)), \quad a_2 = qa + \frac{1}{4}q(q + q^{-1})^2b_1,$$

$$a_{-2} = q^{-1}d + \frac{1}{4}q^{-1}(q + q^{-1})^2b_1,$$

$$a_3 = -\frac{1}{2}q b_3, \quad a_{-3} = \frac{1}{2}q^{-1}b_2, \quad a_4 = \frac{1}{4}q^2b_1, \quad a_{-4} = \frac{1}{4}q^{-2}b_1.$$

(82)

In the generic case (at least one of $b_i$’s is not 0) the algebraic eigenfunctions of (79) are given by (76), where $y_l$’s obey the following system of Bethe equations:

$$\frac{\sum_{s=-4}^{4} a_s q^{-js}y_k^s}{\sum_{s=-4}^{4} a_s q^{-js}y_k^s} = \prod_{l=1, l\neq k}^{2j} \frac{(q^2y_k - y_l)(q^2y_ky_l - 1)}{(y_k - q^2y_l)(y_ky_l - q^2)}.$$
The eigenvalues are

\[ E = \frac{1}{4}b_1(q - q^{-1})(q^2 - q^{-2}) \sum_{l<m}^{2j} (y_l + y_{l^{-1}})(y_m + y_{m^{-1}}) - \]

\[ -\frac{1}{2}(q - q^{-1})(b_2q^{-j} + b_3q^{j-1}) \sum_{l=1}^{2j} (y_l + y_{l^{-1}}) + (q^{2j} - q^{-2j})(a - d) - \]

\[ -\frac{1}{4}b_1(q + q^{-1})(q^{2j} - q^{-2j})^2 + \frac{1}{2}jb_1(q - q^{-1})(q^2 - q^{-2}). \] (84)

The "exactly solvable" case \( b_1 = b_2 = b_3 = 0 \) when the operator is triangular is of particular importance. Its eigenfunctions are known as Askey-Wilson polynomials \[22, 23\]. In this case \( W(y) = 0 \) in (79). For references we recall the conventional form of the difference equation for Askey-Wilson polynomials:

\[ A(y)(\Psi_n(q^2y) - \Psi_y(y)) + A(y^{-1})(\Psi_n(q^{-2}y) - \Psi_y(y)) = \]

\[ = (q^{-2n} - 1)(1 - q^{2n-2}w_1w_3w_4)\Psi_n(y) \] (85)

with

\[ A(y) = \frac{\prod_{\alpha=1}^{4}(1 - w_\alpha y)}{(1 - y^2)(1 - q^2y^2)}, \] (86)

where \( w_i \) are independent parameters. The zeros \( y_l, y_{l^{-1}} \) of the Askey-Wilson polynomials satisfy the system of Bethe equations (83). In the notation of eqs. (85), (86) they are

\[ \prod_{\alpha=1}^{4} \frac{y_k - w_\alpha}{w_\alpha y_k - 1} = \prod_{l=1,l\neq k}^{n} \frac{(q^2y_k - y_l)(q^2y_ky_l - 1)}{(y_k - q^2y_l)(y_ky_l - q^2)}. \] (87)

The form of these equations suggests to ask for an interpretation of the Askey-Wilson polynomials in terms of integrable spin chains with boundaries.

Setting \( y = e^{2x}, w_1 = q^\alpha, \) \( w_2 = q^\beta, w_3 = -q^\gamma, w_4 = -q^\delta, \) one readily finds that in the continuum limit the difference operator in the l.h.s. of (85) turns into

\[ \frac{d^2}{dx^2} = \frac{1}{\sinh(2x)}(2(\gamma + \delta - \alpha - \beta) + (2 - \alpha - \beta - \gamma - \delta) \cosh(2x)) \frac{d}{dx}. \] (88)

They are connected with the previous ones in an obvious way; in particular, \( w_1w_2w_3w_4 = q^{2-4j}a/d. \)
Note that in the continuum limit the number of independent parameters is reduced to 2. Eliminating the first derivative term by means of a suitable "gauge" transformation, one arrives at the Schrödinger equation for the generalized Pöschl-Teller potential. The same potential appears from (68) after a suitable change of variable and a "gauge" transformation. The solutions of the Pöschl-Teller equation are Jacobi polynomials.

A particular case of Askey-Wilson polynomials $w_1 = -w_3$, $w_2 = -w_4 = q$ is Rogers-Askey-Ismail (or $q$-Gegenbauer) polynomials [23]. The corresponding spectral problem (85) in this case have a simple multivariable generalization. It is the $q$-analogue (or relativistic extension) of the Calogero-Moser system [36].

The interrelation between big $q$-Jacobi and Askey-Wilson polynomials is two-fold. First, the former can be obtained [26] under a certain scaling limit from the latter, although no parameter is lost in the course of this limit. The two families appear to be equivalent. Furthermore, there exists a similarity transformation (isospectral transformation) connecting the two families (see the end of the next section). One can interpret this transformation as change of variables in the difference equation.

7 Conclusion and discussion

In this paper we have attempted to classify the difference and discrete operators that preserve a space of polynomials. These operators may be represented as bilinear (or linear) forms of the generators of $U_q(sl_2)$ ($q$-analogs of the quantum Euler top). Difference and discrete equations appearing as spectral problems for these operators have polynomial solutions. Roots of polynomials and eigenvalues are given by the Bethe Ansatz equations. Generally polynomial eigenfunctions cover only a part of the spectrum. Sometimes all the spectrum is algebraized. Differential equations having polynomial solutions appear in a certain continuous limit of the difference equations. They have been classified previously in Refs. [10, 9, 11].

Some periodic difference equations with an incommensurate period have a peculiar multifractal spectrum (Harper’s equation [10] is an example). The Bethe Ansatz equations may be helpful for a statistical description of the multifractality.
Algebraized difference equations have an intimate relation with integrable quantum and classical nonlinear systems. Below we discuss a particular aspect of this relation.

The Heun (Mathieu, Lame) operator (65) written in the algebraic form may be transformed to the Schrödinger-type) operator

\[ H = \frac{d^2}{dx^2} + V(x) \]  \hspace{1cm} (89)

by the change of variable

\[ x = \int \frac{dy}{\sqrt{Q_4(y)}} \]  \hspace{1cm} (90)

and a subsequent ”gauge” transformation. This is the transcendental form of the Heun (Mathieu, Lame) operator. The potential \( V(x) \) is a certain elliptic function. In the case of Lame equation it is the Weierstrass function: \( V(x) = n(n+1)\wp(x) \).

Correspondence between solutions of the algebraic and transcendental forms of the equations is peculiar. The potential \( V(x) \) is a periodic function, so the spectrum of the operator (89) has band structure. Some polynomial eigenfunctions of the algebraic form (65) describe edges of the bands, others correspond to unnormalizable eigenfunctions \([10],[12]\). Both operators (89) and (65) may have also other eigenfunctions which do not correspond to each other.

An important property of the potential of the Lame equation in the transcendental form is its direct generalization to the many body case:

\[ -\sum_i \partial_i^2 + \sum_{i>j} n(n+1)\wp(x_i - x_j). \]

This is the elliptic version of the celebrated Calogero-Moser model \([37]\). Than several questions arise:

(i) Is there a many-body version of the Lame equation in the algebraic form? If yes, it must be a new quantum integrable theory, equivalent to the Calogero-Moser model, and having an explicit quantum group symmetry;

(ii) Do the general Heun (Mathieu) operator in algebraic (65) or transcendental (89) form have a many-body generalization?

\[ ^5 \text{For its explicit form see [1], [3].} \]
The same questions apply to the $q$-deformed equations:

(iii) The operator (30) is the $q$-deformation of the algebraic form (65) of the Heun operator. What is it a $q$-analog of its transcendental form (89) and the transformation (90)?

(iv) The $q$-deformation of the Calogero-Moser hamiltonian is known. It is the Macdonald operator [38]. What is the $q$-deformed many-body version of the general Heun operator? What is the algebraic form of this many-body theory?

Here we may give a partial answer to the question (iii). We suggest that the difference equation for the Askey-Wilson polynomials [23] (see eq.(85) ) is the $q$-deformation of the trigonometric limit of the Lame operator in the transcendental form. Under some a similarity transformation it becomes equivalent to the triangular operator of the case (ii) of Sect.3. The explicit form of this transformation is quite complicated. Its existence follows from the representation theory of the quadratic algebra [18] developed in the paper [23]. We plan to address these questions elsewhere.

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Appendix A

Let us show that linear difference operators of the form \((1)\) have \((n + 1)\)-dimensional invariant subspace of polynomials \(Pol_n\) iff the coefficients \(a(z), d(z), v(z)\) are certain Laurent polynomials of order 2 with 7 independent parameters. We assume that \(n > 1\). Let us denote \(f_1(z) = a(z), f_2(z) = v(z), f_3(z) = d(z),\) and consider the decomposition of the coefficients

\[ f_i(z) = f_i^{(0)} + f_i^{(+)}(z) + f_i^{(-)}(z), \]

into a constant \(f_i^{(0)}\), strictly positive \(f_i^{(+)}\) and negative \(f_i^{(-)}\) degrees of \(z\).

Similarly, the operator \(\Delta = \Delta^{(0)} + \Delta^{(+)} + \Delta^{(-)}\), where

\[ \Delta \equiv f_1 T_+ + f_2 + f_3 T_- \]

is decomposed into a raising (lowering) parts \(\Delta^{(+)}(\Delta^{(-)}\) and a diagonal operator \(\Delta^{(0)}\). Clearly, \(\Delta\) leaves \(Pol_n\) invariant if and only if both \(\Delta^{(+)}\) and \(\Delta^{(-)}\) do so separately. Consider, say, \(\Delta^{(+)}\). To preserve the space \(Pol_n\) it must annihilate \(z^n\) and transform each vector \(z^k\) \((k < n)\) to a linear combination of \(z^{k+1}, z^{k+2},...,z^n\). This gives a number of conditions, the first three of them are

\[
q^n f_1^{(+)}(z) + f_2^{(+)}(z) + q^{-n} f_3^{(+)}(z) = 0, \\
q^{n-1} f_1^{(+)}(z) + f_2^{(+)}(z) + q^{-n+1} f_3^{(+)}(z) = A_1^{(+)} z, \tag{A1} \\
q^{n-2} f_1^{(+)}(z) + f_2^{(+)}(z) + q^{-n+2} f_3^{(+)}(z) = A_2^{(+)} z^2 + A_3^{(+)} z,
\]

where \(A_i^{(+)}\) are arbitrary coefficients. This is a system of linear equations for \(f_i^{(+)}\) with non-zero determinant (it is equal to \((q - q^{-1})(q + q^{-1} - 2)\)). Therefore, \(f_i^{(+)}(z)\) are uniquely determined from \((A1)\) to be linear combinations of \(z\) and \(z^2\) with 3 independent parameters \(A_i^{(+)}\). All the other conditions are then automatically satisfied. Similar arguments applied to \(\Delta^{(-)}\) allow one to find \(f_i^{(-)}(z)\) as linear combinations of \(z^{-1}\) and \(z^{-2}\) parametrized by 3 independent constants \(A_i^{(-)}\).

The total number of independent parameters is 9 \((A_i^{(\pm)}, f_i^{(0)}; i = 1, 2, 3)\), i.e., the linear space of 2-nd order difference operators preserving the space.
of polynomials is 9-dimensional. Two of these parameters correspond to
the constant term and the common factor in (18), so there are 7 essential
parameters (not including \( q \)).

Calculating \( f_i^{(\pm)}(z) \) explicitly, one finds that \( \Delta \) (18) coincides (after obvious changes \( n \to 2j, q \to q^2 \)) with the operator (30) obtained by means of
\( U_q(sl_2) \).

Appendix B

B1. There are other weight representations of \( U_q(sl_2) \) by the "shift" operators \( T_\pm \) (see (18)). One of them is

\[
A = q^{-j}T_+, \quad D = q^jT_-
\]

\[
B = (q - q^{-1})^{-1}z(-q^{-j}T_+ + q^j - q^{-j-1} + q^{j-1}T_-), \quad (B1)
\]

\[
C = (q - q^{-1})^{-1}z^{-1}(q^{-j}T_+ + q(q^j - q^{-j-1}) - q^{j+1}T_-).
\]

The representation space again consists of polynomials of degree \( 2j \) and the
value of the Casimir operator is the same as in (17). The classical limit of
(B1) is the same as that of (19). Bearing in mind applications to 2-nd order
difference equations, (19) is more convenient than (B1) because any homo-
genous bilinear form in \( A, B, C, D \) realized as in (19) becomes a 2-nd order
difference operator, whereas a general bilinear form in the representation
(B1) gives an operator of fourth order.

Another representation is realized in the space of Taylor series in \( z \) [39, 40]:

\[
A = q^jT_-, \quad D = q^{-j}T_+
\]

\[
B = (q - q^{-1})^{-2}z^{-1}(q^{-2j-1}(1 - T_+^2) + q^{2j+1}(1 - T_-^2)), \quad (B2)
\]

\[
C = z.
\]

There is an invariant subspace spanned by \( z^k \) with \( k > 2j \). The induced
representation in the (finite-dimensional) factorspace is equivalent to the spin
j representation. The representation (B2) (in a slightly different form) was used in [39] for discretizing the Schrödinger equation with $x^{-2}$-potential.

**B2.** Algebraized difference equations may be obtained from any weight representations of $q$-deformations of $sl_2$ other than $U_q(sl_2)$ (12).

One of them is generated by $J_\pm, J_0$ [41]:

$$q^{-1}J_+J_0 - qJ_0J_+ = -J_+, \quad qJ_-J_0 - q^{-1}J_0J_- = J_-,$$

$$q^{-2}J_+J_- - q^2J_-J_+ = (q + q^{-1})J_0.$$  

This algebra covers a part of $U_q(sl_2)$:

$$J_+ = BD = (q - q^{-1})^{-1}q^j z(q^{2j}T^2 - q^{-2j}),$$

$$J_- = CD = (q - q^{-1})^{-1}q^j z^{-1}(1 - T^2),$$

$$J_0 = (q - q^{-1})^{-1}(1 - wD^2) = (q - q^{-1})^{-1} \left(1 - \frac{q^{4j+1} + q^{-1}}{q + q^{-1}}T^2\right),$$

(B4)

where

$$w = (q + q^{-1})^{-1}(q - q^{-1})^2 \Omega.$$  

Another "half" of the algebra may be obtained by the authomorphism: $A \rightarrow D, D \rightarrow A, B \rightarrow C, C \rightarrow B$.

This algebra has been used in the papers [13, 10, 42] to construct difference equations having polynomial solutions. A class of difference equations obtained from this algebra is of the form $G\Psi = \varepsilon T^2\Psi$, where $G$ is the invertible Jacobi operator (29). Generally these equations do not correspond to any hermitian operator.

All other deformations of $sl_2$ have a similar embedding into $U_q(sl_2)$. More information about different $q$-deformed algebras and interrelations between them may be found in [43]. Some particular examples were studied in [30, 34, 32]. Another algebra of difference operators which preserves the space of symmetric Laurent polynomials is considered in Sect.6.
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