Viscous Aubry-Mather theory
and the Vlasov equation

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Abstract

The Vlasov equation models a group of particles moving under a potential $V$; moreover, each particle exerts a force, of potential $W$, on the other ones. We shall suppose that these particles move on the $p$-dimensional torus $\mathbb{T}^p$ and that the interaction potential $W$ is smooth. We are going to perturb this equation by a Brownian motion on $\mathbb{T}^p$; adapting to the viscous case methods of Gangbo, Nguyen, Tudorascu and Gomes, we study the existence of periodic solutions and the asymptotics of the Hopf-Lax semigroup.

Introduction

The Vlasov equation models the motion of a group of particles under the action of a time-dependent potential $V$ and a mutual interaction $W$. For definiteness, we shall suppose that the particles move on the torus $\mathbb{T}^p := \mathbb{R}^p / \mathbb{Z}^p$; we put on the position and velocity space $\mathbb{T}^p \times \mathbb{R}^p$ the coordinates $(x,v)$ and we suppose that, at time $t$, the particles are distributed on $\mathbb{T}^p \times \mathbb{R}^p$ according to a probability measure $f_t$. Then, the Vlasov equation has the form

$$\partial_t f_t + \langle v, \partial_x f_t \rangle = \text{div}_v (f_t \partial_x P_t) \quad (VL)_\infty$$

where

$$P(t,x) = V(t,x) + \int_{\mathbb{T}^p \times \mathbb{R}^p} W(x-x')df_t(x',v')$$

and $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^p$. Since [7], one looks for weak solutions of $(VL)_\infty$; in other words, given an initial distribution $f_0$, one looks for a continuous curve of probability measures $f_t$ satisfying

$$\int_{\mathbb{T}^p \times \mathbb{R}^p} \phi(0,x,v)df_0(x,v) + \int_0^{+\infty} \int_{\mathbb{T}^p \times \mathbb{R}^p} [\partial_t \phi(t,x,v) - \langle v, \partial_x \phi(t,x,v) \rangle + \langle \partial_x P(t,x), \partial_v \phi(t,x,v) \rangle]df_t(x,v) = 0$$

for all $\phi \in C_0^\infty([0,+\infty) \times \mathbb{T}^p \times \mathbb{R}^p)$.

Our hypotheses on $V$ and $W$ are the following:

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1) $V \in C(T, C^3(T^p))$, and
2) $W \in C^3(T^p)$. Thus, $W$ lifts to a $C^3$ function on $R^p$, $Z^p$-periodic; we shall also suppose that $W(x) = W(-x)$ and that $W(0) = 0$.

A recent idea (see [11], [12]) is to view $(VL)_\infty$ as a Lagrangian system in the space of measures; indeed, it is possible to define what it means for a curve $\mu_t$ of probability measures on $T^p$ to minimize the Lagrangian action

$$\int_{t_0}^{t_1} \left[ \frac{1}{2} ||\dot{\mu}_t||^2 + \int_{T^p} V(t, x)d\mu_t + \frac{1}{2} \int_{T^p \times T^p} W(x - x')d(\mu_t \times \mu_t)(x, x') \right] dt. \tag{1}$$

The advantages are that one can use the tools of Lagrangian dynamics (Aubry-Mather theory, Hamilton-Jacobi equations, minimal characteristics, etc...) albeit in the difficult "differential manifold" of probability measures.

In this paper, we are going to adapt to the viscous case an older approach: following [7], we jury-rig a fixed point theorem to the viscous Mather theory of [13]. Let us briefly outline what we are doing in the case of periodic orbits.

We let $\psi(t, x)$ be a continuous family of densities, periodic in time; in other words, we ask that

- d1) $\psi \in C(T \times T^p)$
- d2) $\psi \geq 0$
- d3) $\int_{T^p} \psi(t, x)dx = 1$ for all $t \in T$.

Let us define

$$P_\psi(t, x) = V(t, x) + \int_{T^p} W(x - x')\psi(t, x')dx'$$

and, for $c \in R^p$, let us set

$$\mathcal{L}_{c, \psi}: T \times T^p \times R^p \to R, \quad \mathcal{L}_{c, \psi}(t, x, \dot{x}) = \frac{1}{2}||\dot{x}||^2 - \langle c, \dot{x} \rangle - P_\psi(t, x)$$

$$H_\psi: T \times T^p \times R^p \to R, \quad H_\psi(t, x, p) = \frac{1}{2}||p||^2 + P_\psi(t, x).$$

We have the following.

**Theorem 1.** Let $c \in R^p$ and let $\beta > 0$. Then, there is a couple of functions

$$\rho_\beta \in C^1(T \times T^p) \cap C(T, C^2(T^p)), \quad u_\beta \in C^1(T, C^1(T^p)) \cap C(T, C^3(T^p))$$

and $\bar{H}_\beta(c) \in R$ such that $\rho_\beta$ satisfies points d1)-d3) above and

$$\frac{1}{2\beta} \Delta u_\beta + \partial_t u_\beta - H_{\rho_\beta}(t, x, c - \partial_x u_\beta) + \bar{H}_\beta(c) = 0, \quad (HJ)_{\rho_\beta, per}$$

$$\frac{1}{2\beta} \Delta \rho_\beta - \text{div}[\rho_\beta \cdot (c - \partial_x u_\beta)] - \partial_x \rho_\beta = 0. \quad (FP)_{c-\partial_x u_\beta, per}$$

Moreover, among the triples $(\rho_\beta, u_\beta, \bar{H}_\beta(c))$ which solve $(HJ)_{\rho_\beta, per} - (FP)_{c-\partial_x u_\beta, per}$, there is one which minimizes

$$\int_{T \times T^p} \mathcal{L}_{c, \frac{1}{\beta} \rho_\beta}(t, x, \partial_x u_\beta)\rho_\beta(t, x)dtdx. \tag{2}$$
Thus, our "characteristics" are the solutions of a Fokker-Planck equation bringing mass forward in time; the drift of this equation, or the optimal trajectory, is determined by a Hamilton-Jacobi equation, backward in time. This is quite typical for this kind of problems: see for instance equation (5.40) of [17] or theorem 3.9 of [12].

We briefly sketch the proof of theorem 1; the complete details are in section 1 below. First of all, we fix \( \psi \) satisfying points \( d_1 \)-\( d_3 \) above; then we find, as in [13], a couple \( (u_\psi, H_\psi(c)) \) which solves \((HJ)_{\psi,per} \). By [13], the number \( \bar{\rho} \) of a minimum in (2) will follow from the fact that the fixed points of \((HJ)_{\psi,per} \) are unique and \( u_\psi \) is unique up to an additive constant. To \( c - \partial_x u_\psi \) is associated a stochastic flow, whose stationary Fokker-Planck equation is \((FP)_{c-\partial_x u_\psi,per} \); again by [13], \((FP)_{c-\partial_x u_\psi,per} \) has a unique periodic solution \( \rho_\psi \) satisfying \( d_1 \)-\( d_3 \). In other words, we have a map \( \psi \to \rho_\psi \) bringing densities to densities; we shall find a fixed point \( \rho_\beta \) of this map by the Schauder fixed point theorem. We shall see that \((u_{\rho_\beta}, \rho_\beta, H_\rho_\beta(c)) \) solves \((HJ)_{\rho_\beta,per} - (FP)_{c-\partial_x u_{\rho_\beta},per} \) practically by definition; the existence of a minimum in (2) will follow from the fact that the fixed points of \( \rho_\beta \) are a compact set.

In section 2, we study the Hopf-Lax semigroup. We denote by \( \mathcal{M}_1(T^p) \) the space of Borel probability measures on \( T^p \) with the 1-Wasserstein distance (see section 1 for a definition); we shall prove the following.

**Theorem 2.** Let \( U: \mathcal{M}_1(T^p) \to \mathbb{R} \) be of the form
\[
U(\mu) = \int_{T^p} f d\mu
\]
for some \( f \in C^3(T^p) \). Let \( \mu \in \mathcal{M}_1(T^p) \) and let \( m \in \mathbb{N} \). Then, there are \( R_\beta \in C([-m, 0], \mathcal{M}_1(T^p)) \) with density \( \rho_\beta \in C^1([-m, 0] \times T^p) \cap C([-m, 0] \times C^2(T^p)) \) and \( u_\beta \in C^1([-m, 0], C^1(T^p)) \cap C([-m, 0], C^3(T^p)) \) such that \( u_\beta \) solves
\[
\begin{align*}
\frac{1}{2} \Delta u_\beta + \partial_t u_\beta - H_\rho_\beta(t, x, c - \partial_x u_\beta) &= 0, \quad t < 0 \\
u_\beta(0, x) &= f \quad \forall x \in T^p
\end{align*}
\]
and \( R_\beta \) together with its density \( \rho_\beta \) solve
\[
\begin{align*}
\frac{1}{2} \Delta \rho_\beta - \text{div} [\rho_\beta \cdot (c - \partial_x u_\beta)] - \partial_t \rho_\beta &= 0, \quad t > -m \\
R_\beta(-m) &= \mu. 
\end{align*}
\]
Among the solutions \((u_\beta, \rho_\beta)\) of \((HJ)_{\beta,f} - (FP)_{c-\partial_x u_{\rho_\beta}, \mu}\), there is one which minimizes
\[
\int_{-m}^0 dt \int_{T^p} \mathcal{L}_{c, \frac{1}{2}} \rho_\beta(t, x, c - \partial_x u_\beta) \rho_\beta dx + U(\rho_\beta(0)).
\]
We call such a minimum \((\Lambda^m_\mu U)(\mu)\).

Since minimizing over fixed points is uncomfortable, one could ask whether this restriction can be removed, getting a problem more similar to (1).

**Theorem 3.** 1) Let \( U \) and \((\Lambda^m_\mu U)(\mu)\) be as in theorem 2. Then,
\[
(\Lambda^m_\mu U)(\mu) = \min_Y E_w \left\{ \int_{-m}^0 \mathcal{L}_{c, \frac{1}{2}}(t, X, Y) dt \right\} + U(\rho(0))
\]

3
where \( \rho \) solves \((FP)_{-m,Y,\mu}\), \( E_w \) denotes expectation with respect to the Brownian motion \( w \), \( X \) solves the stochastic differential equation

\[
\begin{aligned}
dX(-m, s, x) &= Y(s, X(-m, s, x))dt + dw(s) \quad s \geq -m \\
X(-m, -m, x) &= X_\mu
\end{aligned}
\]

\((SDE)_{-m,Y,\mu}\)

for a random variable \( X_\mu \) of law \( \mu \), independent on \( w(s) \) for \( s \geq -m \). The minimum is taken over all Lipschitz vector fields \( Y \).

2) Any minimal \( Y \) satisfies \( Y = c - \partial_x u \), where \( u \) solves \((HJ)_{\rho,f}\).

In other words, \((HJ)_{\rho,f} - (FP)_{-m,c-\partial_x u,\mu}\) are the Euler-Lagrange equations of the functional \((3)\), exactly as in the zero-viscosity situation (we refer again to [12], theorem 3.9.) We also note a quirk of the notation: in the Hamilton-Jacobi equation we have \( H_{\rho,\beta} \), while in \((3)\) we have \( L_{c,\frac{1}{2} \rho} \); again, we share this factor two with the zero-viscosity situation and we shall see the reason for it in the proof of lemma 2.5 below.

Theorems like theorem 3 are common in the theory of mean field games (see for instance [4]). In the language of mean field games, we are saying that each particle tries to minimize unilaterally the cost

\[
\min_Y E_w \left\{ \int_{-m}^{0} L_{c,\rho}(t, X, Y)dt + f(X(0)) \right\}
\]

where \( X \) solves \((SDE)_{-m,Y,\delta_{x_0}}\) and \( \rho \) is the distribution of the other particles; this is the reason of equation \((HJ)_{\rho,f}\) in theorem 2. The result of the independent efforts of all the particles (or the Nash equilibrium, as it is called) is that the whole community minimizes \((3)\).

Let \( U : M_1(T^p) \to \mathbb{R} \) be bounded; theorem 3 prompts us to define

\[
(\Psi^m_U)(\mu) = \inf_Y \left\{ E_w \int_{-m}^{0} L_{c,\rho}(t, X, Y)dt + f(X(0)) \right\}
\]

\((4)\)

where the infimum is taken over all Lipschitz vector fields \( Y \); the density \( \rho \) satisfies \((FP)_{-m,Y,\mu}\) and \( \rho \) is the distribution of the other particles. Naturally, if \( U \) is linear as in theorem 3, then \( \Psi^m_U = \Lambda^m_U \).

We shall see in proposition 2.10 below that \( \Psi^m_U \) has the semigroup property: \( \Psi^{m+n}_U = \Psi^n_U \circ \Psi^m_U \).

Theorem 3 tells us that the infimum in \((4)\) is a minimum when \( U \) is a linear function on measures as in theorem 2; we don’t know whether this is true when \( U \) is in some more reasonable class, for instance continuous or Lipschitz. We don’t even know whether, for \( U \) continuous, \( \Psi^1_U \) is continuous; however, when \( U \) is linear as in theorem 2, we can prove that \( \Psi^m_U \) is Lipschitz, uniformly in \( m \). This allows us to find, for a suitable \( \lambda \in \mathbb{R} \), Lipschitz fixed points of the operator \( \Psi_{c,\lambda} \) defined by

\[
\Psi_{c,\lambda} : U \to \Psi^1_U + \lambda.
\]

**Theorem 4.** There is a unique \( \lambda \in \mathbb{R} \) for which \( \Psi_{c,\lambda} \) has a fixed point \( \hat{U} \) in \( C(M_1(T^p), \mathbb{R}) \). In other words, for any \( \mu \in M_1(T^p) \), there is a Lipschitz vector field \( \hat{Y} \) such that

\[
(\Psi_{c,\lambda} \hat{U})(\mu) = E_w \left\{ \int_{-m}^{0} L_{c,\rho}(t, \hat{X}, \hat{Y})dt \right\} + \hat{U}(\hat{\rho}(0)),
\]

\((5)\)
where $X$ solves $(SDE)_{-1,\bar{Y},\mu}$ and $\tilde{\rho}$ solves $(FP)_{-1,\bar{Y},\mu}$.

The function $\hat{U}$ is Lipschitz for the 1-Wasserstein distance; by (5), the infimum in the definition of (4) of $\Psi^{1}_{c,\lambda}\hat{U}$ is a minimum.

The proof of this theorem is similar to the corresponding statement in Aubry-Mather theory. Indeed, using an approximation with finitely many particles, we shall prove that, for a suitable $\lambda \in \mathbb{R}$, the sequence $(\Lambda^{n}_{c,\lambda})(0)$ of continuous functions on the compact space $\mathcal{M}_{1}(T^{p})$ is equibounded and equilipschitz; by Ascoli-Arzelà, it has a subsequence converging to a limit $\hat{U}$; we shall prove that $\hat{U}$ is a fixed point of $\Lambda^{c,\lambda}$.

§1

Periodic orbits

In this section, we are going to prove theorem 1. We begin with a study of $(HJ)^{\psi,\text{per}}$; we follow the approach of [13] but, for completeness’ sake, we reprove several results of this paper using, as in [2], the Feynman-Kac formula.

Definitions.

- We group in a set $\text{Den}$ the functions on $T \times T^{p}$ which satisfy points d1)-d3) in the introduction. Clearly, the set $\text{Den}$ is closed in $C(T \times T^{p})$.

- We define $\mathcal{M}_{1}(T^{p})$ as the space of all Borel probability measures on $T^{p}$; if $\mu_{1}, \mu_{2} \in \mathcal{M}_{1}(T^{p})$, we define the 1-Wasserstein distance between them as

$$d_{1}(\mu_{1}, \mu_{2}) = \min \{ \int_{T^{p} \times T^{p}} |x - x'|_{T^{p}} \text{d}\gamma(x, x') \}$$

where $|x - x'|_{T^{p}}$ is the distance on the flat torus $T^{p}$. The minimum is taken over all the Borel probability measures $\gamma$ on $T^{p} \times T^{p}$ whose first and second marginals are, respectively, $\mu_{1}$ and $\mu_{2}$. It is standard (see for instance section 7.1 of [17]) that $d_{1}$ turns $\mathcal{M}_{1}(T^{p})$ into a complete metric space, and induces the weak* topology.

We note that, if $\psi \in \text{Den}$ and $\mathcal{L}^{p}$ denotes the Lebesgue measure on $T^{p}$, then the function $: t \rightarrow \psi(t, \cdot)\mathcal{L}^{p}$ belongs to $C(T, \mathcal{M}_{1}(T^{p}))$.

- We extend the definition of $P_{\psi}$ we gave in the introduction: for $\psi \in C(\mathbb{R}, \mathcal{M}_{1}(T^{p}))$ we set

$$P_{\psi}(t, x) = V(t, x) + \int_{T^{p}} W(x - x')\text{d}\psi(t, x')$$

\text{(1.1)}

**Lemma 1.1.** There is $C_{1} > 0$, independent on $\psi \in C(\mathbb{R}, \mathcal{M}_{1}(T^{p}))$, such that the function $P_{\psi}(t, x)$ defined in (1.1) satisfies

$$\|P_{\psi}\|_{C(\mathbb{R}, C^{3}(T^{p}))} \leq C_{1}.$$

\text{(1.2)}
Proof. We recall that, by definition,

\[ \|P_\psi\|_{C(R, C^3(T^p))} = \sup_{t \in R} \|P_\psi(t, \cdot)\|_{C^3(T^p)} \]

where, as usual,

\[ \|f\|_{C^3(T^p)} = \|f\|_{C^0(T^p)} + \|D_x f\|_{C^0(T^p)} + \|D_x^2 f\|_{C^0(T^p)} + \|D_x^3 f\|_{C^0(T^p)}. \]

By our hypotheses on \( V \) and \( W \), we have that

\[ \|V\|_{C(R, C^3(T^p))} + \|W\|_{C^3(T^p)} = C_1 < +\infty. \quad (1.3) \]

For \( 0 \leq j \leq 3 \), differentiation under the integral sign implies that

\[ D^j x P_\psi(t, x) = D^j x V(t, x) + \int_{T^p} D^j x W(x - x') d\psi(t, x'). \]

Since \( t \to \psi(t, \cdot) \) is continuous from \( R \) to the weak* topology, the formula above implies that \( P_\psi \) is in \( C(R, C^3(T^p)) \). Since \( \psi(t, \cdot) \) is a probability measure and the \( C^3 \) norm is convex, (1.2) follows from the last formula and (1.3).

From now on, we shall fix \( \psi \in Den \); the functions \( P_\psi \), \( L_{c, \psi} \) and \( H_\psi \) are defined as in the introduction. Following [2], we note that, if \((u, A)\) solves

\[
\begin{cases}
\frac{1}{2\beta} \Delta u + \partial_t u - H_\psi(t, x, c - \partial_x u) + A = 0 & \forall t \in R \\
u(t, \cdot) = u(t + 1, \cdot) & \forall t \in R
\end{cases}
\]

and is periodic in space (i.e., it quotients to a continuous function on \( T^p \)), then the couple \((v, A) = (e^{-\beta u}, A)\) is a solution, periodic in space, of the "twisted" Schrödinger equation

\[
\begin{cases}
\partial_t v + e^{-\beta(c, x)} \left[ \frac{1}{2\beta} \Delta + \beta P_\psi(t, x) - \beta A \right] (e^{\beta(c, x)} v) = 0 & \forall t \in R \\
v(t, \cdot) = v(t + 1, \cdot) & \forall t \in R.
\end{cases}
\]

Vice-versa, the logarithm of a positive solution of \((TS)_{\psi, per}\) solves \((HJ)_{\psi, per}\). Thus, solving \((HJ)_{\psi, per}\) reduces to solving \((TS)_{\psi, per}\); that’s what we are going to do next.

For \( \phi \in C(T^p) \) and \( A \in R \) we consider the evolution (or involution, since it goes backward in time) problem

\[
\begin{cases}
\partial_t v + e^{-\beta(c, x)} \left[ \frac{1}{2\beta} \Delta + \beta P_\psi(t, x) - \beta A \right] (e^{\beta(c, x)} v) = 0, & t \leq 0 \\
v(0, x) = \phi(x).
\end{cases}
\]

If \( t \leq 0 \), we can use the Feynman-Kac formula (see for instance [6]) and write the unique solution of \((TS)_{\psi, \phi}\) as

\[ v(t, x) = (L_{(\psi, A, t)} \phi)(x), \quad t \leq 0. \]
where
\[
(L_{\psi,A,t}\phi)(x) = e^{-\beta(c,x)} \cdot E_w \left[ e^{\int_0^\beta [P_{\psi} (\tau \frac{1}{\sqrt{\beta}} w(\tau) + x) - A]d\tau} e^{\beta (c \frac{1}{\sqrt{\beta}} w(0) + x)} \phi \left( \frac{1}{\sqrt{\beta}} w(0) + x \right) \right]. \tag{1.5}
\]
In the formula above, \( w \) is a Brownian motion on \([t, +\infty)\) with \( w(t) = 0 \), and \( E_w \) is the expectation with respect to the Wiener measure.

We shall see in lemma 1.4 below that there is a bijection between the positive eigenfunctions of \( L_{(\psi,0,-1)} \) and the positive solutions of \((TS)_{\beta,\text{per}}\); now, we prove that such eigenfunctions exist.

**Lemma 1.2.** 1) (Existence) there is \((v, B) \in C(T^p) \times \mathbb{R}\) such that
\[
\begin{cases}
L_{(\psi,0,-1)}v = Bv \\
v > 0 \\
B > 0.
\end{cases} \tag{1.6}_\psi
\]

2) (Uniqueness) Let \((v_1, B_1)\) and \((v_2, B_2)\) solve \((1.6)_\psi\); then, \( B_1 = B_2 \) and \( v_1 = \alpha v_2 \) for some \( \alpha > 0 \). In particular, there is a unique couple \((v_\psi, B_\psi)\) which satisfies \((1.6)_\psi\) and such that
\[
\int_{T^p} v_\psi(x)dx = 1. \tag{1.7}
\]

**Proof.** We recall from papers (see also chapter XVI of [3] for G. Birkhoff’s original exposition) a few facts about the Perron-Frobenius theorem. Let us denote by \( C_+ \subset C(T^p) \) the cone of strictly positive, continuous functions. We forego the easy proof that \( L_{(\psi,0,-1)} \) brings \( C_+ \) into itself.

Let \( v_1, v_2 \in C_+ \); we say that \( v_1 \) and \( v_2 \) are equivalent, or \( v_1 \simeq v_2 \), if \( v_1 = tv_2 \) for some \( t > 0 \). Given \( v_1, v_2 \in C_+ \), we define
\[
\alpha(v_1, v_2) = \sup \{ t > 0 : v_2 - tv_1 \in C_+ \}
\]
and
\[
\theta(v_1, v_2) = - \log [\alpha(v_1, v_2)\alpha(v_2, v_1)].
\]
It turns out ([16]) that \((C_+, \theta)\) is a complete metric space. We refer again to [16] or [3] for the proof that
\[
\theta(L_{(\psi,0,-1)}v_1, L_{(\psi,0,-1)}v_2) \leq (1 - e^{-D})\theta(v_1, v_2)
\]
where
\[
D = \sup_{v_1, v_2 \in C_+} \theta(L_{(\psi,0,-1)}v_1, L_{(\psi,0,-1)}v_2).
\]
As a consequence, points 1) and 2) follow from the contraction mapping theorem if we prove that \( D < +\infty \). Actually, we are going to show that \( D \) is bounded from above independently of \( \psi \in Den \); equivalently, the Lipschitz constant of \( L_{(\psi,0,-1)} \) does not depend on \( \psi \). We shall need this fact in the next lemma.

Let \( v_1, v_2 \in C_+ \). Recalling the definition of \( \theta \), we see that
\[
\theta(v_1, v_2) \leq \log \left( \frac{\max v_2}{\min v_1} \frac{\max v_1}{\min v_2} \right), \tag{1.8}
\]
Thus, \( D < +\infty \) follows if we prove that there is \( C_3 > 0 \) such that
\[
\frac{\max L_{(\psi,0,-1)^{\nu}}}{\min L_{(\psi,0,-1)^{\nu}}} \leq C_3
\] (1.9)
for all \( \nu \in C_+ \); since the term on the left is homogeneous of degree zero in \( \nu \), we can suppose that \( \nu \) satisfies (1.7).

We prove (1.9); in the following, \( C_i \) always denotes a constant independent on \( \nu \) and \( \psi \). By (1.7) and the fact that \( \nu > 0 \), we have that
\[
(L_{(\psi,0,-1)^{\nu}})(x) \geq e^{-\beta(c,x)} e^{\beta \max P_\nu} \frac{1}{(2\pi)^p} \int_{\mathbb{R}^p} e^{\beta(c,\sqrt[\nu]{z+x})} v(\frac{1}{\sqrt[\nu]{z+x}}) e^{-\frac{|z|^2}{2}} dz.
\]
Setting \( \frac{1}{\sqrt[\nu]{z}} = y \) and simplifying \( e^{-\beta(c,x)} \) outside the integral with \( e^{\beta(c,x)} \) inside, we get the first inequality below
\[
(L_{(\psi,0,-1)^{\nu}})(x) \geq e^{\beta \max P_\nu} \frac{(\beta)}{2\pi} \int_{\mathbb{R}^p} e^{\beta(c,y)} v(x+y) e^{-\frac{y^2}{2}} dy \geq e^{-\beta C_1} \frac{(\beta)}{2\pi} \int_{[0,1]^p} e^{\beta(c,y)} v(x+y) e^{-\frac{y^2}{2}} dy.
\]
The second inequality above comes from lemma 1.1 and the fact that \( \psi \), which belongs to \( C_+ \), is positive. By lemma 1.1, the constant \( C_1 \) does not depend on \( \psi \in Den \).

We assert that
\[
(L_{(\psi,0,-1)^{\nu}})(x) \geq e^{-\beta C_1} \frac{(\beta)}{2\pi} \min_{\nu \in [0,1]^p} \left[ e^{\beta(c,y)} e^{-\frac{y^2}{2}} \right] \int_{[0,1]^p} v(x+y) dy = C_5
\] (1.10)
for a constant \( C_5 > 0 \) independent on \( \psi \) and \( \nu \). Indeed, the inequality follows since \( \nu \) is positive; since \( \nu \) is periodic and satisfies (1.7), the integral above is 1, and the equality follows.

For the estimate from above, we get from (1.5) that
\[
(L_{(\psi,0,-1)^{\nu}})(x) \leq e^{-\beta(c,x)} e^{\beta \max P_\nu} \frac{1}{(2\pi)^p} \int_{\mathbb{R}^p} e^{\beta(c,\sqrt[\nu]{z+x})} v(\frac{1}{\sqrt[\nu]{z+x}}) e^{-\frac{|z|^2}{2}} dz.
\]
We simplify \( e^{-\beta(c,x)} \) outside the integral with \( e^{\beta(c,x)} \) inside; now lemma 1.1 gives us the first inequality below; the equality follows from the change of variables \( \frac{1}{\sqrt[\nu]{z}} = y \).
\[
(L_{(\psi,0,-1)^{\nu}})(x) \leq \frac{e^{\beta C_1}}{(2\pi)^p} \int_{\mathbb{R}^p} e^{\beta(c,\sqrt[\nu]{z+x})} v(\frac{1}{\sqrt[\nu]{z+x}}) e^{-\frac{|z|^2}{2}} dz = e^{\beta C_1} \frac{(\beta)}{2\pi} \int_{\mathbb{R}^p} e^{\beta(c,y)} v(x+y) e^{-\frac{y^2}{2}} dy.
\]
Since \( v \) is positive periodic, and by (1.7) integrates to 1 on the unit cube, we get the first inequality below.
\[
(L_{(\psi,0,-1)^{\nu}})(x) \leq e^{\beta C_1} \frac{(\beta)}{2\pi} \sum_{k \in \mathbb{Z}_+} \max_{y \in [0,1]^p} \left[ e^{\beta(c,y)} e^{-\frac{y^2}{2}} \right] \leq \]
\[ C_6 \sum_{k \in \mathbb{Z}^p} e^{\beta \overline{|\nabla| \sqrt{p} e^{-\beta (|k| - \sqrt{p})^2}}. \]

Since the sum in the last formula is finite, we get that
\[
(L_{(\psi, 0, -1)} v)(x) \leq C_7 \quad \forall x \in T^p
\]
for a constant \( C_7 > 0 \) independent on \( \psi \) and \( v \). Now (1.9) follows from the last formula and (1.10); we have seen that (1.9), by the contraction mapping theorem, implies points 1) and 2) of the thesis.

---

**Lemma 1.3.** Let \( \psi \in Den \) and let \((v, B)\) be as in the last lemma. Then, \( v, B \in C^3(T^p) \) and the following two points hold.

1) (Uniform estimates) There is \( C_8 > 0 \), independent on \( \psi \in Den \), such that
   
i) \[ ||v||_{C^3(T^p)} \leq C_8, \]
   
   ii) \[ \frac{1}{C_8} \leq v(x) \leq C_8 \quad \forall x \in T^p \]
   
   iii) \[ \frac{1}{C_8} \leq B \leq C_8. \]

2) (Continuous dependence) The function
\[
K: Den \to C^3(T^p) \times \mathbb{R}, \quad K: \psi \to (v, B)
\]
is continuous.

**Proof.** We prove point 1). Since \( v \) satisfies (1.7), by (1.10) and (1.11) there is \( C_8 > 1 \) such that
\[
\frac{1}{C_8} \leq \min L_{(\psi, 0, -1)} v \leq \max L_{(\psi, 0, -1)} v \leq C_8.
\]
Since we also have that \( L_{(\psi, 0, -1)} v = B v \), we get that
\[
\frac{1}{B} \cdot \frac{1}{C_8} \leq v \leq \frac{1}{B} C_8.
\]
Integrating on \( T^p \) and using (1.7), we get that
\[
\frac{1}{B} \cdot \frac{1}{C_8} \leq \int_{T^p} v \, dx = 1 \leq \frac{1}{B} \cdot C_8
\]
from which iii) of point 1) follows.

From point iii) and (1.12), possibly increasing \( C_8 \), we get point ii). We show i).
We would like to differentiate under the integral sign in (1.5); we cannot do this immediately, because we only know that the final condition \( \phi \) (which in our case is \( v_\psi \)) is in \( C^0 \). Let \( E_{(0,z)} \) denote the expectation of the Brownian bridge with \( w(-1) = 0 \) and \( w(0) = z \); by (1.5) we get that, for \( v \in \mathcal{C}_+ \),

\[
(L_{(\psi,0,-1)}v)(x) = e^{-\beta(c,x)} \frac{1}{\sqrt{(2\pi)^p}} \int_{\mathbb{R}^p} e^{\frac{-|y|^2}{2}} e^{\beta(c,y) E_{(0,0)}(y,0)} E_{(0,0)}(y,0) \left[ e^{\int_0^t \beta P_\psi(\tau, \sqrt{\beta}) w(\tau + x) d\tau} \right] dy.
\]

Setting \( \frac{1}{\sqrt{\beta}} z + x = y \), we get that

\[
(L_{(\psi,0,-1)}v)(x) = \sqrt{\left( \frac{\beta}{2\pi} \right)^p} e^{-\beta(c,x)} \int_{\mathbb{R}^p} e^{-\frac{\beta}{2}|y-x|^2} e^{\beta(c,y)} E_{(0,0)}(y,0) \left[ e^{\int_0^t \beta P_\psi(\tau, \sqrt{\beta}) w(\tau + x) d\tau} \right] dy.
\]

We recall from [14] that, if \( \tilde{w} \) is a Brownian bridge with \( \tilde{w}(-1) = \tilde{w}(0) = 0 \), then \( w(t) := \sqrt{\beta/(y-x)} (t+1) + \tilde{w}(t) \) is a Brownian bridge with \( w(-1) = 0, w(0) = \sqrt{\beta/(y-x)} \). This and the last formula imply that

\[
(L_{(\psi,0,-1)}v)(x) = \sqrt{\left( \frac{\beta}{2\pi} \right)^p} e^{-\beta(c,x)} \int_{\mathbb{R}^p} e^{-\frac{\beta}{2}|y-x|^2} e^{\beta(c,y)} E_{(0,0)}(y,0) \left[ e^{\int_0^t \beta P_\psi(\tau, \sqrt{\beta}) w(\tau + x) d\tau} \right] dy.
\]

The formula above allows us to differentiate under the integral sign, even if \( v \) is only \( C^0 \); using lemma 1.1, we easily get

\[
||L_{(\psi,0,-1)}v||_{C^3(\mathbb{T}^p)} \leq C_9 ||v||_{C^3(\mathbb{T}^p)}
\]

for a constant \( C_9 \) independent of \( \psi \). By \( ii) \), we get that

\[
||L_{(\psi,0,-1)}v||_{C^3(\mathbb{T}^p)} \leq C_8 \cdot C_9.
\]

Since \( L_{(\psi,0,-1)}v = B_\psi v_\psi \), formula \( i) \) now follows from \( iii) \).

We prove point 2); in the first three steps below, we show a weaker result, namely that the map \( \psi \to v_\psi \) is continuous from \( \text{Den} \) to \( C^0(\mathbb{T}^p) \); this will follow from the theorem of contractions depending on a parameter applied to the map

\[
\Xi: (\text{Den}, ||\cdot||_{\sup}) \times (\mathcal{C}_+, \theta) \to (\mathcal{C}_+, \theta), \quad \Xi: (\psi, v) \to L_{(\psi,0,-1)}v.
\]

**Step 1.** We begin to observe that \( \theta \) and the sup norm induce equivalent topologies on the subset \( \mathcal{A} \) of the functions of \( \mathcal{C}_+ \) which satisfy (1.7). Indeed, (1.8) proves that the \( C^0 \) topology is stronger; for the opposite inclusion, let \( \theta(v_n, v) \to 0 \) and let \( v_n, v \) satisfy (1.7). Since \( \theta(v_n, v) \to 0 \), we have that, for any \( \epsilon > 0 \) and \( n \) large enough,

\[
\frac{1 - \epsilon}{\alpha(v_n, v)} \leq \alpha(v, v_n) \leq \frac{1 + \epsilon}{\alpha(v_n, v)}.
\]
The definition of $\alpha$ implies the first two inequalities below; the last one follows by the first inequality above.

\[
\alpha(v, v_n) v \leq v_n \leq \frac{1}{\alpha(v_n, v)} v \leq \frac{\alpha(v, v_n)}{1 - \epsilon} v.
\]

Since $v$ and $v_n$ satisfy (1.7), if we integrate the formula above on $T^p$, we get that $\alpha(v_n, v) \rightarrow 1$ and that $\alpha(v, v_n) \rightarrow 1$; since $\min v > 0$, again from the formula above we get that $v_n \rightarrow v$ uniformly.

**Step 2.** Let $v \in C_+$ be fixed; we assert that the map : $\psi \rightarrow \Xi(\psi, v)$ is continuous from the $\norm{\cdot}_{\sup}$ to the $\theta$ topology. Indeed, we saw in step 1 that, on $C_+$, the $C^0$ topology is stronger than the $\theta$ topology; thus, it suffices to prove that $\Xi(\cdot, v) : (Den, \norm{\cdot}_{\sup}) \rightarrow (C_+, \norm{\cdot}_{\sup})$ is continuous. The proof of this, which ends the proof of the assertion, follows by applying the theorem of continuity under the integral sign to (1.5), and we forego it.

**Step 3.** We assert that the map : $\psi \rightarrow v_\psi$ is continuous from $(Den, \norm{\cdot}_{\sup})$ to $(A, \norm{\cdot}_{\sup})$; by step 1, it suffices to prove that it is continuous from $(Den, \norm{\cdot}_{\sup})$ to $(A, \theta)$. We have seen in the proof of lemma 1.2 that $\psi \rightarrow \Xi(\psi, v)$ is a contraction for the $\theta$-topology, whose Lipschitz constant does not depend on $\psi$. Since $\psi \rightarrow \Xi(\psi, v)$ is continuous by step 2, we can apply the theorem of contractions depending on a parameter, and get that the map : $\psi \rightarrow v_\psi$ is continuous from $(Den, \norm{\cdot}_{\sup})$ to $(C_+, \theta)$, as we wanted.

**Step 4.** We assert that the map : $\psi \rightarrow B_\psi$ is continuous from $Den$ to $\mathbb{R}$. Since $L_{(\psi,0,1)}v_\psi = B_\psi v_\psi$, it suffices to prove that both maps : $\psi \rightarrow v_\psi$ and : $\psi \rightarrow L_{(\psi,0,1)}v_\psi$ are continuous from $Den$ to $C^0(T^p)$. The first fact has been proven in step 3; we prove that $\psi \rightarrow L_{(\psi,0,1)}v_\psi$ is continuous. Indeed,

\[
\|L_{(\psi',0,1)}v_\psi' - L_{(\psi,0,1)}v_\psi\|_{\sup} \leq \|L_{(\psi',0,1)}(v_\psi' - v_\psi)\|_{\sup} + \|(L_{(\psi',0,1)} - L_{(\psi,0,1)})v_\psi\|_{\sup}.
\]

Now the assertion follows from the fact that (with the sup norm in all spaces) $\psi \rightarrow v_\psi$ is continuous, that $\psi \rightarrow L_{(\psi,0,1)}v$ is continuous, and that $v \rightarrow L_{(\psi,0,1)}v$ is uniformly Lipschitz by (1.13).

**End of the proof of point 2.** For $\phi \in C_+$, we get from (1.5) that

\[
L_{(\psi, r, -1)}\phi = e^{-\beta r} L_{(\psi,0, -1)}\phi.
\]

(1.14)

Setting $A_\psi = \frac{1}{\beta} \log B_\psi$, the formula above implies that

\[
L_{(\psi, A_\psi, -1)}v_\psi = v_\psi.
\]

(1.15)

The same proof which yielded (1.13) also yields that there is $C_{10} > 0$ such that, if $A$ and $A'$ satisfy the estimate of point 1), iii) of this lemma, then

\[
\|L_{(\psi, A, -1)}v - L_{(\psi', A', -1)}v\|_{C^3(T^p)} \leq C_{10}(\|\psi - \psi'\|_{C^0(T^p)} + |A - A'|) \cdot \|v\|_{C^0(T^p)}.
\]

(1.16)

Thus,

\[
\|v_\psi - v_\psi'\|_{C^3(T^p)} = \|L_{(\psi, A_\psi, -1)}v_\psi - L_{(\psi', A_\psi', -1)}v_\psi'\|_{C^3(T^p)} \leq \\
\|L_{(\psi, A_\psi, -1)}(v_\psi - v_\psi')\|_{C^3(T^p)} + \|(L_{(\psi, A_\psi, -1)} - L_{(\psi', A_\psi', -1)})v_\psi'\|_{C^3(T^p)} \leq \\
C_9 \|v_\psi - v_\psi'\|_{C^0(T^p)} + C_{10}(\|\psi - \psi'\|_{C^0(T^p)} + |A_\psi - A_\psi'|) \cdot \|v_\psi'\|_{C^0(T^p)}
\]

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where the the equality comes from (1.15) and the last inequality comes from (1.13) and (1.16). Since the map \( \psi \to v_\psi \) is continuous from \( \text{Den} \) to the \( C^0 \) topology by step 3, and \( \psi \to A_\psi \) is continuous too (because \( \psi \to B_\psi \) is continuous by step 4 and point 1), \( iii \) of this lemma holds), point 2) follows.

In the next lemma, we show how the fixed points of \( L_{\psi,0,-1} \) induce solutions of \( (TS)_\psi,\text{per} \).

**Lemma 1.4.** 1) (Existence) Given \( \psi \in \text{Den} \), we can find \( A \in \mathbf{R} \) and \( \hat{v} \in C(T, C^1(T^p)) \cap C^1(T, C^1(T^p)) \) such that \( \hat{v} > 0 \) and \( (\hat{v}, A) \) solves \( (TS)_\psi,\text{per} \).

2) (Uniqueness) Let us suppose that \( (\hat{v}, A) \) and \( (\hat{v}^1, A^1) \) are two solutions of \( (TS)_\psi,\text{per} \) with \( \hat{v} > 0 \) and \( \hat{v}^1 > 0 \). Then, \( A = A_1 \) and \( \hat{v} = \lambda \hat{v}^1 \) for some \( \lambda > 0 \).

3) (Estimates) Let us call \( (\hat{v}_\psi, A_\psi) \) the solution of \( (TS)_\psi,\text{per} \) such that \( \hat{v}_\psi > 0 \) and \( \hat{v}_\psi(0, \cdot) \) satisfies (1.7). Then, there is a constant \( C_{10} > 0 \), independent on \( \psi \in \text{Den} \), such that

\[
|A_\psi| + ||\hat{v}_\psi||_{C(T, C^1(T^p))} + ||\hat{v}_\psi||_{C^1(T, C^1(T^p))} \leq C_{10} \tag{1.17}
\]

and

\[
\frac{1}{C_{10}} \leq \hat{v}_\psi(t, x) \leq C_{10} \quad \forall (t, x) \in T \times T^p. \tag{1.18}
\]

4) (Continuous dependence) Let \( (\hat{v}_\psi, A_\psi) \) be as in point 3), and let us consider the map \( I: \psi \to (\hat{v}_\psi, A_\psi) \).

Then, \( I \) is continuous from \( \text{Den} \) to \( [C(T, C^3(T^p)) \cap C^1(T, C^1(T^p))] \times \mathbf{R} \).

**Proof.** As in the proof of lemma 1.3, we set \( A_\psi = \frac{1}{\beta} \log B_\psi \). For \( t \leq 0 \), we set

\[
\hat{v}_\psi(t, x) = (L_{\psi, A_\psi, t}) v_\psi(x). \tag{1.19}
\]

By (1.15), we get that \( \hat{v}_\psi(-1, x) = \hat{v}(0, x) \); in other words, \( \hat{v}_\psi \) quotients on \( T \times T^p \); equivalently, it satisfies the second formula of \( (TS)_\psi,\text{per} \).

Let us prove (1.18) for the function \( \hat{v}_\psi \) defined by (1.19); we prove the inequality on the left, since the one on the right is analogous.

The first equality below is (1.19). Since \( \hat{v}_\psi \) is periodic, we can suppose that \( t \in [-1, 0] \); now (1.5), implies the first inequality below; the second inequality follows from lemma 1.1 and the fact that \( t \in [-1, 0] \); the third one comes from point 1), \( ii \) and \( iii \) of lemma 1.3.

\[
\hat{v}_\psi(t, x) = (L_{\psi, A_\psi, t}) v_\psi(x) \geq e^{-A_\psi} e^{\min(t) P_\psi} (\min v_\psi) E_w \left( e^{\beta(c, \sqrt{\mathbf{v}}(0))} \right) = e^{-A_\psi} e^{\min(t) P_\psi} (\min v_\psi) \frac{1}{\sqrt{(2\pi |t|)^p}} \int_{\mathbf{R}^p} e^{-\frac{x^2}{2|t|}} e^{\beta(c, x)} dx \geq e^{-A_\psi} C_9 \min v_\psi \geq C_9 \frac{C_9}{C_8}.
\]

This yields the inequality on the left of (1.18).
We prove (1.17). We begin to note that the estimate on \( A_\psi \) follows by point 1), \( iii) \) of lemma 1.3, and by the fact that \( A_\psi = \frac{1}{\beta} \log B_\psi \).

We end the proof of (1.17) with the estimates on the derivatives. Let \( \tilde{w} \) be the Brownian bridge with \( \tilde{w}(-1) = 0 = \tilde{w}(0) \) and let \( \tilde{E}_{(0,0)} \) denote its expectation; for \( t < 0 \), let \( w \) be the Brownian bridge with \( w(t) = 0 = w(0) \) and let \( E_{(0,0)} \) denote its expectation; we recall that
\[
w(s) = \frac{1}{\sqrt{|t|}} \tilde{w} \left( \frac{s}{|t|} \right).
\]
This yields the second inequality below, while (1.19) yields the first one; the third one comes from the change of variables \( s = \frac{t}{|t|} \).

\[
\hat{v}_\psi(t,x) = \left( \frac{\beta}{2\pi|t|} \right)^{\frac{z}{2}} e^{-\beta \langle c,x \rangle}.
\]
\[
\int_{\mathbb{R}^p} e^{-\frac{|z|^2}{2\pi|t|}} e^{\beta \langle c, \frac{z}{\sqrt{|t|}} + x \rangle} v_\psi \left( \frac{1}{\sqrt{\beta}} z + x \right) E_{(0,0)} \left[ e^{\int_{t}^{0} \beta P_\psi(\tau,x + \frac{1}{\sqrt{|t|}} \tilde{w}(\tau) + \frac{z}{\sqrt{|t|}}) d\tau} \right] dz = \left( \frac{\beta}{2\pi|t|} \right)^{\frac{z}{2}} e^{-\beta \langle c,x \rangle}.
\]
\[
\int_{\mathbb{R}^p} e^{-\frac{|z|^2}{2\pi|t|}} e^{\beta \langle c, \frac{z}{\sqrt{|t|}} + x \rangle} v_\psi \left( \frac{1}{\sqrt{\beta}} z + x \right) \tilde{E}_{(0,0)} \left[ e^{\int_{t}^{0} \beta P_\psi(\tau,x + \frac{1}{\sqrt{|t|}} \tilde{w}(\tau) + \frac{z}{\sqrt{|t|}}) d\tau} \right] dz = \left( \frac{\beta}{2\pi|t|} \right)^{\frac{z}{2}} e^{-\beta \langle c,x \rangle}.
\]
\[
\int_{\mathbb{R}^p} e^{-\frac{|z|^2}{2\pi|t|}} e^{\beta \langle c, \frac{z}{\sqrt{|t|}} + x \rangle} v_\psi \left( \frac{1}{\sqrt{\beta}} z + x \right) \tilde{E}_{(0,0)} \left[ e^{\int_{t}^{0} \beta P_\psi(\tau,x + \frac{1}{\sqrt{|t|}} \tilde{w}(\tau) + \frac{z}{\sqrt{|t|}}) d\tau} \right] dz = \left( \frac{\beta}{2\pi|t|} \right)^{\frac{z}{2}} e^{-\beta \langle c,x \rangle}.
\]

By point 1), \( i) \) of lemma 1.3, we can differentiate under the integral sign and get that
\[
||\hat{v}_\psi||_{C([-2,-1],C^3(T^p))} + ||\hat{v}_\psi||_{C^1([-2,-1],C^3(T^p))} \leq C_{10}.
\]
Since \( \hat{v}_\psi \) is periodic in time, (1.17) follows.

By theorem 9.1 and proposition 6.6 of [6], the Feynman-Kac formula holds for the unbounded final condition \( e^{\beta \langle c,x \rangle} v_\psi \); this, (1.19) and (1.5) imply that \( \hat{v}_\psi \) satisfies the first formula of \((TS)_{\psi,\text{per}} \) for \( t < 0 \); since it is periodic in \( t \), it satisfies it for all times. Moreover, \( \hat{v}_\psi > 0 \) because, by (1.19) and (1.5), it is an integral, with a positive weight, of the positive \( v_\psi \). This ends the proof of point 1).

We have just seen that (1.19) gives a bijection between the periodic, positive solutions of \((TS)_{\psi,\text{per}} \) and the positive eigenfunctions of \( L_{(\psi,0,-1)} \); since the latter are unique up to a multiplicative constant by point 2) of lemma 1.2, we get that the former too are unique up to a multiplicative constant; this proves point 2).

We prove point 4). To prove that the map \( \psi \to A_\psi \) is continuous, it suffices to note that \( A_\psi = \frac{1}{\beta} \log B_\psi \), that the map \( \psi \to B_\psi \) is continuous by point 2) of lemma 1.3, and that \( B_\psi \) is bounded away from zero and infinity by point 1), \( iii) \) of the same lemma.

By point 2) of lemma 1.3, we know that \( \psi \to v_\psi \) is continuous from \( Den \) to \( C^3(T^p) \); this and (1.19) easily imply that \( \psi \to \hat{v}_\psi \) is continuous from \( Den \) to \( C(T, C^3(T^p)) \cap C^1(T, C^1(T^p)) \).
Lemma 1.5. \( 1 \) (Existence and uniqueness) There is a unique couple \( \hat{H}_\psi(c) \in \mathbb{R} \) and \( u_\psi \in C(T, C^3(T^p)) \cap C^1(T, C^1(T^p)) \) which solves \((HJ)_\psi,\text{per}\) and satisfies
\[
\int_{T^p} u_\psi(0, x) \, dx = 0. \tag{1.21}
\]

2) (Estimates) There is \( C_{11} > 0 \), independent on \( \psi \in \text{Den} \), such that, if \( u_\psi \) is as in point 1), then
\[
|\bar{H}_\psi(c)| + ||u_\psi||_{C(T, C^3(T^p))} + ||u_\psi||_{C^1(T, C^1(T^p))} \leq C_{11}. \tag{1.22}
\]

3) (Continuous dependence) The couple \( (u_\psi, \bar{H}_\psi(c)) \) depends continuously on \( \psi \): if \( \psi_n \to \psi \) in \( \text{Den} \), then \( \bar{H}_{\psi_n}(c) \to \bar{H}_\psi(c) \) in \( \mathbb{R} \) and \( u_{\psi_n} \to u_\psi \) in \( C(T, C^3(T^p)) \cap C^1(T, C^1(T^p)) \).

Proof. By lemma 1.4, there is a unique couple
\[
(\hat{v}_\psi, A_\psi) \in [C(T, C^3(T^p)) \cap C^1(T, C^1(T^p))] \times \mathbb{R}
\]
which solves \((TS)_{\psi,\text{per}}\) and such that \( \hat{v}_\psi(0, \cdot) \) is positive and satisfies (1.7). We have seen at the beginning of this section that, for any \( \lambda > 0 \), the couple
\[
(u_\psi, \bar{H}_\psi(c)) := (-\frac{1}{\beta} \log(\lambda \hat{v}_\psi), -\frac{1}{\beta} A_\psi) \tag{1.23}
\]
solves \((HJ)_{\psi,\text{per}}\); vice-versa, if \( u_\psi \) solves \((HJ)_{\psi,\text{per}}\), then its exponential solves \((TS)_{\psi,\text{per}}\). Thus, if we define \( u_\psi \) as above, for the unique \( \lambda \) for which (1.21) holds, we have existence. Now point 2) of lemma 1.4 implies that all positive solutions of \((TS)_{\psi,\text{per}}\) are of the form \((\lambda \hat{v}_\psi, A_\psi)\); since we have just seen that there is a bijection between the solutions of \((HJ)_{\psi,\text{per}}\) and the positive solutions of \((TS)_{\psi,\text{per}}\), we get uniqueness.

Formula (1.22) follows from (1.23); indeed, the derivatives of the logarithm of \( \hat{v}_\psi \) are bounded by (1.17) and (1.18). In an analogous way, point 3) follows from point 4) of lemma 1.4.

Let the Lagrangian \( L_{c,\psi} \) be as in the introduction, and let \( u_\psi \) be as in lemma 1.5. It is well-known ([9]) that \( u_\psi \) satisfies, for \( t \leq 0 \),
\[
u(t, x) = \min_{Y} E_{\nu} \left\{ \int_{t}^{0} L_{c,\psi}(s, z(s), Y(s, z(s))) \, ds + u_\psi(0, z(0)) \right\}
\]
where \( z \) solves the stochastic differential equation
\[
z(s) = Y(s, z(s)) \, ds + \frac{1}{\sqrt{\beta}} \, dw(s) \quad s \geq t \quad (SDE)_{t, Y, \delta s}
z(t) = x
\]
and \( Y(t,z) \) varies among the vector fields continuous in \( t \) and Lipschitz in \( z \). We have denoted by \( E_{w} \) the expectation with respect to the Wiener measure. From [9], we get that the minimal \( Y_{\psi} \) is given by

\[
Y_{\psi}(t,x) = c - \partial_{x}u_{\psi}(t,x).
\]

By (1.22), there is \( C_{12} > 0 \) such that, for any \( \psi \in Den, \)

\[
||Y_{\psi}||_{C(T,C^{2}(T^{p}))} + ||Y_{\psi}||_{C^{2}(T,C(T^{p}))} \leq C_{12}.
\] (1.24)

**Definition.** We group in a set \( Vect \) all the vector fields \( Y : T \times T^{p} \to \mathbb{R}^{p} \) which satisfy (1.24). The distance on \( Vect \) is given by the norm of (1.24).

We would like to consider the law of the stochastic differential equation above when the initial condition is distributed according to a measure \( \mu \). One way to do this is to call \( \rho_{x_{0}} \) the solution of \((FP)_{t,Y,\delta_{x_{0}}} \) and to set

\[
\rho(s,x_{0}) = \int_{T^{p}} \rho_{x_{0}}(s,x)d\mu(x_{0}).
\]

Another one, which yields the same law, is to suppose that the Brownian motion is on a probability space \( \Omega \) on which there is a random variable \( M \) independent on \( w(s) \) for \( s \geq t \) and with law \( \mu \); we consider the solution \( z \) of the stochastic differential equation above with initial condition \( M \) and we say that \( z \) solves \((SDE)_{t,Y,\mu} \).

Let \( Y \in Vect \); by [13], there is \( \mu \in C(T,M_{1}(T^{p})) \) which is invariant by the stochastic differential equation; in other words, there is a measure \( \mu_{0} \) such that, if \( \mu_{t} \) is the measure induced by a solution \( z \) of \((SDE)_{0,Y,\mu_{0}} \) for \( t \geq 0 \), then \( \mu_{0} = \mu_{1} \). Equivalently, we are saying that there is a weak solution \( \mu \) of \((FP)_{Y,per} \). We sketch a proof of this fact: the map which brings the measure \( \mu_{0} \) into \( \mu_{1} \), the solution of the Fokker-Planck equation at time 1, has a fixed point by the Schauder theorem.

We shall use the following classical uniqueness result ([7], proposition 1, [1], theorem 4.1, [17], theorem 5.34) to prove that \( \mu \) has a smooth density \( \rho_{\mu} \).

**Lemma 1.6.** Let \( Y \in Vect \). For \( i = 1,2 \), let the map \( \nu^{i} : [0, +\infty) \to M_{1}(T^{p}) \) be continuous and let it be a weak solution of the Fokker-Planck equation, i. e.

\[
\int_{T^{p}} \phi(0,x)d\nu_{0}^{i}(x) + \int_{0}^{+\infty} dt \int_{T^{p}} \left[ \partial_{t}\phi + \frac{1}{2\beta} \Delta \phi + \langle Y, \partial_{x}\phi \rangle \right] d\nu_{t}^{i} = 0
\] (1.25)

for all \( \phi \in C_{c}^{1}([0, +\infty) \times T^{p}) \cap C([0, +\infty), C^{2}(T^{p})) \). Let \( \nu_{0}^{1} = \nu_{0}^{2} \). Then, \( \nu_{t}^{1} = \nu_{t}^{2} \) for all \( t \geq 0 \).

**Proof.** We begin to note that \( \mu_{t} = \nu_{t}^{2} - \nu_{t}^{1} \) satisfies

\[
\int_{0}^{+\infty} dt \int_{T^{p}} \left[ \partial_{t}\phi + \frac{1}{2\beta} \Delta \phi + \langle Y, \partial_{x}\phi \rangle \right] d\mu_{t} = 0 \quad \forall \phi \in C_{c}^{1}([0, +\infty) \times T^{p}) \cap C([0, +\infty), C^{2}(T^{p})).
\] (1.26)
We have to prove that \( \mu_t = 0 \) for all \( t \geq 0 \).

We define the operator \( A_Y \) as
\[
A_Y \phi = \frac{1}{2\beta} \Delta \phi + \langle Y, \partial_x \phi \rangle.
\]
Let \( \gamma \in C^1_c([0, +\infty) \times T^p) \) and let \( t \) be so large that \( \text{supp}\gamma \subset [0, t) \times T^p \). The heat equation with time reversed and final condition in \( t \)
\[
\begin{align*}
\partial_s \phi + A_Y \phi &= \gamma \quad s < t \\
\phi(t, x) &= 0 \quad \forall x \in T^p
\end{align*}
\]
has a unique solution \( \phi \). We set
\[
\psi(s, x) = \begin{cases} 
\phi(s, x) & s \leq t \\
0 & s > t
\end{cases}
\]
and we see that \( \psi \in C^1_c([0, +\infty) \times T^p) \cap C([0, +\infty), C^2(T^p)) \). Indeed, \( \psi \) is \( C^1 \) in \( t \) and \( C^2 \) in \( x \) on \( s < t \) by theorem 9 of chapter 1 of [10]; it is obviously \( C^2 \) on \( s > t \); it is \( C^2 \) also in a neighbourhood of \( s = t \), because, by the uniqueness of the Cauchy problem for the equation \( \partial_s \phi + A_Y \phi = \gamma \), and the fact that \( \gamma(s, x) = 0 \) for \( s \in [t - \epsilon, t] \) we have that \( \phi(s, x) = 0 \) for \( s \geq t - \epsilon \). We use \( \psi \) as a test function in (1.26), getting the second equality below.
\[
0 = \int_0^{+\infty} ds \int_{T^p} [\partial_s \psi + A_Y \psi] d\mu_s(x) = \int_0^{+\infty} ds \int_{T^p} \gamma(s, x) d\mu_s(x).
\]
Since the formula above holds for all \( \gamma \in C^1_c([0, +\infty) \times T^p) \), we get the thesis.

\[
\| \mu_t \|_{C^1(\mathbb{T} \times T^p)} + \| \mu_t \|_{C(\mathbb{T}, C^2(T^p))} \leq C_{13}.
\]

**Lemma 1.7.** Let \( Y \in \text{Vect} \). By [13], there is \( \mu \in C(\mathbb{T}, \mathcal{M}_1(T^p)) \) which solves \( (FP)_{Y, per} \) in the weak sense. Then, the following holds.

1) The measure \( \mu \) has density \( \rho_Y \in \text{Den} \).
2) The measure \( \mu \) is unique.
3) There is \( C_{13} > 0 \), independent on \( Y \in \text{Vect} \), such that
\[
\| \rho_Y \|_{C^1(\mathbb{T} \times T^p)} + \| \rho_Y \|_{C(\mathbb{T}, C^2(T^p))} \leq C_{13}.
\]
4) If \( Y_n \in \text{Vect} \) for all \( n \), if \( Y \in \text{Vect} \) and \( Y_n \rightarrow Y \) in \( C(\mathbb{T} \times T^p) \), then \( \rho_{Y_n} \rightarrow \rho_Y \) in \( C(\mathbb{T} \times T^p) \).

**Proof.** Classical results about PDE’s (see lemma 2.3 below for more details) imply that there is a density \( \rho_{x_0} \), smooth on \( (0, +\infty) \times T^p \), which solves
\[
\begin{align*}
\rho_{x_0}(t, \cdot) &\rightarrow \delta_{x_0} \quad \text{as} \quad t \rightarrow 0 \\
\frac{1}{2\beta} \Delta \rho_{x_0} - \text{div}[\rho_{x_0} \cdot Y] - \partial_t \rho_{x_0} &= 0 \\
\rho_{x_0}(t, \cdot \! \cdot \! \cdot) \mathcal{L}^p &\rightarrow \delta_{x_0}
\end{align*}
\]
where \( \mathcal{L}^p \) denotes the Lebesgue measure on \( T^p \). It is standard that, for \( t > 0 \), \( \rho_{x_0}(t, \cdot) \) satisfies properties d2) and d3) of the introduction, and that
\[
\| \rho_{x_0} \|_{C^1([1, 2] \times T^p)} + \| \rho_{x_0} \|_{C([1, 2], C^2(T^p))} \leq C_{13}
\] (1.27)
Lemma 1.8. There is a continuous map $\text{Den}$ is continuous; by point 3), it has image in $\text{Den}$. Again from point 3), we get that $\rho_Y$ is a classical solution of $(FP)_{Y,\text{per}}$; since by [13] there is only one of them, we get point 2).

We prove point 4). Let $Y_n \to Y$ in $C(\mathbf{T} \times \mathbf{T}^p)$, and let $\rho_{Y_n}$ and $\rho_Y$ solve $(FP)_{Y_n,\text{per}}$ and $(FP)_{Y,\text{per}}$ respectively. We have just proved that $\rho_{Y_n}$ satisfies point 3) of the thesis; thus, we can apply Ascoli-Arzelà and get that, up to subsequences, $\rho_{Y_n} \to \rho$ in $C(\mathbf{T} \times \mathbf{T}^p)$. Taking limits in (1.25) we see that $\rho$ is a weak, periodic solution of $(FP)_{Y,\text{per}}$; by the uniqueness of point 2), we get that $\rho = \rho_Y$. Thus, any subsequence of $\rho_{Y_n}$ has a sub-subsequence converging to $\rho_Y$ in $C(\mathbf{T} \times \mathbf{T}^p)$; by a well-known principle, this implies that $\rho_{Y_n} \to \rho_Y$ in $C(\mathbf{T} \times \mathbf{T}^p)$.

\\\n
**Definition.** Let $C_{13}$ be as in lemma 1.7. We group in a set $\rho \in \text{Den}^{reg}$ the elements of $\text{Den}$ which belong to $\text{Lip}(\mathbf{T} \times \mathbf{T}^p)$ and such that $||\rho||_{\text{Lip}(\mathbf{T} \times \mathbf{T}^p)} \leq C_{13}$. By point 3) of lemma 1.7, if $Y \in \text{Vect}$, then $\rho_Y \in \text{Den}^{reg}$.

**Lemma 1.8.** There is a continuous map $\Phi: \text{Den} \to \text{Den}$ whose fixed points $\rho_\beta$ induce solutions $(u_{\rho_\beta}, \rho_\beta, \bar{H}_{\rho_\beta}(c))$ of $(HJ)_{\rho_\beta,\text{per}} - (FP)c - \partial_x u_{\rho_\beta,\text{per}}$. Moreover, $\Phi(\text{Den}) \subset \text{Den}^{reg}$.

**Proof.** We define the map $\Phi$ by composition. By lemma 1.5 and formula (1.24), we know that there is a map

$$\Phi_1: \text{Den} \to \text{Vect} \times \mathbb{R}, \quad \Phi_1: \psi \to (c - \partial_x u_{\psi}, \bar{H}_{\psi}(c)).$$

This map is continuous by point 3) of lemma 1.5.

Let $\rho_Y$ be as in point 1) of lemma 1.7; by point 4) of this lemma, the map

$$\Phi_2: \text{Vect} \times \mathbb{R} \to \text{Den}, \quad \Phi_2: (Y, \lambda) \to \rho_Y$$

is continuous; by point 3), it has image in $\text{Den}^{reg}$. Thus, the map $\Phi = \Phi_2 \circ \Phi_1$ is continuous from $\text{Den}$ to $\text{Den}$, and has image in $\text{Den}^{reg}$, as we wanted.

Let now $\rho_\beta$ be a fixed point of $\Phi$; we recall that $\Phi_1(\rho_\beta) = (c - \partial_x u_{\rho_\beta}, \bar{H}_{\rho_\beta}(c))$, with $(u_{\rho_\beta}, \bar{H}_{\rho_\beta}(c))$ which satisfies $(HJ)_{\rho_\beta,\text{per}}$ and (1.21). Moreover,

$$\rho_\beta = \Phi_2 \circ \Phi_1(\rho_\beta) = \Phi_2(c - \partial_x u_{\rho_\beta}, \bar{H}_{\rho_\beta}(c)).$$

for a constant $C_{13} > 0$ which depends only on the $C^1$ norm of $Y$; as a consequence, $C_{13}$ is the same for all $Y \in \text{Vect}$ and $x_0 \in \mathbf{T}^p$ (again, we refer the reader to lemma 2.3 below).
Proof of theorem 1. We begin to show that there are couples \((u_\beta, \rho_\beta)\) which satisfy \((HJ)_{\rho_\beta, \text{per}} - (FP)_{c-\partial_x u_{\rho_\beta}, \text{per}}\). By lemma 1.8, this follows if we show that \(\Phi\) has fixed points. But this is true by Schauder’s fixed point theorem: indeed, by lemma 1.8, \(\Phi\) is a continuous map from \(Den\) to itself which preserves the compact, convex set \(Den^{reg}\).

Let us now call \(S\) the set of the triples \((u, \rho, H)\) such that \(\rho \in Den\) is a weak solution of \((FP)_{c-\partial_x u, \text{per}}\) and \((u, H)\) is a classical solution of \((HJ)_{\rho, \text{per}}\). Let \((u^n, \rho^n, H^n) \in S\) be such that

\[
\int_{T \times T} L_c \|\rho^n(t, x, \partial_x u^n)\|_{L^2} dt dx \to \inf_{(u, \rho, H) \in S} \int_{T \times T} L_c \|\rho(t, x, \partial_x u)\|_{L^2} dt dx.
\]

(1.29)

By lemma 1.1, \(L_c\|\rho\|_{L^2}\) is bounded from below independently on \(\rho\); as a consequence, the inf in the right hand side of (1.29) is finite. Note that, if \(\rho^n \in Den\), lemma 1.5 implies that \(c-\partial_x u^n \in \text{Vect}\); since \(\rho^n\) is a fixed point, we get by lemma 1.7 that \(\rho^n \in Den^{reg}\); since \(Den^{reg}\) is compact in \(Den\), we can suppose that, up to subsequences,

\(\rho^n \to \bar{\rho} \quad \text{in} \quad Den\).

By point 3) of lemma 1.5, this implies that

\((u^n, H^n) \to (\bar{u}, \bar{H}) \quad \text{in} \quad [C(T, C^1(T^p))) \cap C^1(T, C^1(T^p))] \times \mathbb{R},\)

with \((\bar{u}, \bar{H})\) solving \((HJ)_{\bar{\rho}, \text{per}}\). This and point 4) of lemma 1.7 yield that \(\bar{\rho} = \rho_{c-\partial_x \bar{u}}\) solves \((FP)_{c-\partial_x \bar{u}, \text{per}}\) and satisfies the estimate of point 3) of that lemma. In other words, \((\bar{u}, \bar{\rho}, \bar{H}) \in S\); now (1.29) and the last three formulas easily imply that

\[
\int_{T \times T} L_c \|\bar{\rho}(t, x, \partial_x \bar{u})\|_{L^2} dt dx = \inf_{(u, \rho, H) \in S} \int_{T \times T} L_c \|\rho(t, x, \partial_x u)\|_{L^2} dt dx
\]

yielding the thesis.

§2

The evolution equation

In this section, we shall prove theorems 2 and 3. We begin with some notation.

We recall that the map

\(: (\mu, \nu) \to d_1(\mu, \nu)\)

is convex, i.e.

\[d_1((1-\lambda)\nu_1 + \lambda \mu_1, (1-\lambda)\nu_2 + \lambda \mu_2) \leq (1-\lambda)d_1(\nu_1, \mu_1) + \lambda d_1(\nu_2, \mu_2).\]
Indeed, the dual formulation
\[
d_1(\mu, \nu) = \sup \left\{ \int_{T^p} f d(\mu - \nu) : f \in Lip_1(T^p) \right\}
\]
implies that \(d_1\) is the supremum of a family of linear functions. Since the functions \(f\) in the dual formulation belong to \(Lip_1(T^p)\) and \(T^p\) has diameter \(\sqrt{p}\), we can as well suppose that \(||f||_\infty \leq \frac{1}{\sqrt{p}}\); as a consequence,
\[
d_1(\mu, \nu) \leq \sqrt{p}||\mu - \nu||_{tot},
\]
(2.1)
where \(||\cdot||_{tot}\) denotes total variation.

**Definition.** We are going to denote by the norm symbol the distance on \(C([-m, 0], M_1(T^p))\), which is no norm at all: if \(R_1, R_2 \in C([-m, 0], M_1(T^p))\), then we set
\[
||R_1 - R_2||_{C([-m, 0], M_1(T^p))} = \sup_{t \in [-m, 0]} d_1(R_1(t), R_2(t)).
\]

Though this is no norm, it is convex thanks to the convexity of \(d_1\):
\[
||(1 - \lambda)R_1 + \lambda R_2 - (1 - \lambda)\tilde{R}_1 - \lambda \tilde{R}_2||_{C([-m, 0], M_1(T^p))} \leq (1 - \lambda)||R_1 - \tilde{R}_1||_{C([-m, 0], M_1(T^p))} + \lambda||R_2 - \tilde{R}_2||_{C([-m, 0], M_1(T^p))}.
\]
(2.2)

**Definition.** For \(\mu \in M_1(T^p)\) and \(m \in \mathbb{N}\), we group in a set \(Den_m(\mu)\) all the maps \(R \in C([-m, 0], M_1(T^p))\) such that \(R(-m) = \mu\). This space inherits the distance of \(C([-m, 0], M_1(T^p))\).

**Lemma 2.1.** Let \(f \in C^3(T^p)\) and let \(H^Z(t, q, p) = \frac{1}{2}||p||^2 + Z(t, q)\), with \(Z \in C([-m, 0], C^3(T^p))\).

1) Then, there is a unique solution \(u^Z\) of
\[
\begin{cases}
\frac{1}{2\beta} \Delta u^Z + \partial_t u^Z - H^Z(t, x, c - \partial_x u^Z) = 0, & t \in [-m, 0] \\
u^Z(0, x) = f & \forall x \in T^p.
\end{cases}
\]
(\(HJ\))^Z

2) There is \(C_{13} > 0\), only depending on \(||f||_{C^3(T^p)}, ||Z||_{C([-m, 0], C^3(T^p))}\) and \(m\), such that
\[
||u^Z||_{C^1([-m, 0], C^1(T^p))} + ||u^Z||_{C([-m, 0], C^3(T^p))} \leq C_{13}.
\]

3) The map
\[
: Z \rightarrow u^Z
\]
is continuous from \(C([-m, 0], C^3(T^p))\) to \(C([-m, 0], C^3(T^p)) \cap C^1([-m, 0], C^1(T^p))\).

**Proof.** We know that the twisted Schroedinger equation with potential \(Z\) and final condition \(e^{-\beta f} \in C^3(T^p)\) has a unique solution \(v^Z\), which can be represented by the Feynman-Kac formula (1.20) with \(e^{-\beta f}\).
in stead of $v_\mu$ and $Z$ in stead of $P_\mu$. Since $e^{-\beta f} > 0$, we get that $v^Z > 0$ too. We saw in section 1 that $u^Z = -\frac{1}{\beta} \log v^Z$ solves $(HJ)^Z$, and point 1) follows.

Points 2) and 3) follow as in section 1 if we prove that

$$||\partial_x u^Z||_{C^1([-m,0],C^1(T^p))} + ||v^Z||_{C([-m,0],C^3(T^p))} \leq 14$$

and that the map $: Z \to (v^\mu, \partial_x u^\mu)$ is continuous. Since $e^{-\beta f}$, the final condition of the Schroedinger equation, is of class $C^3$, this is a standard result; for instance, differentiation under the integral sign in (1.20) gives the estimate on $||v^Z||_{C([-m,0],C^3(T^p))}$; from this and the fact that $v^Z$ solves the Schroedinger equation, we get the estimate on $||v^Z||_{C([-m,0],C^3(T^p))}$.

Recalling lemma 1.1, we get this immediate consequence.

**Corollary 2.2.** 1) Let $f \in C^3(T^p)$, let $\mu \in M_1(T^p)$ and let $R \in Den^m(\mu)$. Then, there is a unique solution $u_R$ of

$$\frac{1}{2\beta} \Delta u_R + \partial_t u_R - H_R(t, x, c - \partial_x u_R) = 0, \quad t \in [-m, 0]$$

$$u_R(0, x) = f \quad \forall x \in T^p.$$  

2) There is $C_{14} = C_{14}(m) > 0$, independent of $\mu \in M_1(T^p)$ and on $R \in Den^m(\mu)$, such that

$$||u_R||_{C^1([-m,0],C^1(T^p))} + ||u_R||_{C([-m,0],C^3(T^p))} \leq 14.$$

3) The map

$$: R \to u_R$$

is continuous from $Den^m(\mu)$ to $C([-m,0],C^3(T^p)) \cap C^4([-m,0],C^1(T^p))$.

**Definition.** By point 2) of corollary 2.2, there is $C_{15} > 0$ such that, setting $Y = c - \partial_x u_R$, we have

$$||Y||_{C^1([-m,0],C^1(T^p))} + ||Y||_{C([-m,0],C^2(T^p))} \leq 15$$

with $C_{15}$ independent on $R \in C([-m,0],M_1(T^p))$. We group in a set $Vect^m$ all the vector fields $Y$ on $[-m,0] \times T^p$ which satisfy the estimate above. The distance on $Vect^m$ is the one induced by the norm above.

**Lemma 2.3.** Let $Y \in Vect^m$, and let $\mu \in M_1(T^p)$. Then, the following holds.

1) There is a unique $R_Y \in C([-m,0],M_1(T^p))$ which solves $(FP)_{-m,Y,\mu}$ in the weak sense.

2) For $t \in (-m,0]$, $R_Y(t)$ has density $\rho_Y$. There are $C_{16}, C_{17} : (-m,0) \to [0, +\infty)$, independent on $Y \in Vect^m$ and on $\mu \in M_1(T^p)$, such that

a) $C_{17}(T) \to 0$ as $T \to -m$, $C_{16}$ and $C_{17}$ are bounded on $(-m + \epsilon,0]$ for all $\epsilon \in (0,m)$ and
b) For $T \in (-m, 0]$, we have
\[
\begin{align*}
\|\rho_Y\|_{C^{1}((T,0) \times \mathbf{T}^{p})} + \|\rho_Y\|_{C^{2}((T,0),C^{2}((T,0) \times \mathbf{T}^{p})} & \leq C_{16}(T) \\
d_{1}(R_{Y}(T), \mu) & \leq C_{17}(T)
\end{align*}
\]
(2.3)
where $d_1$ denotes the 1-Wasserstein distance.

**Proof.** The uniqueness of point 1) comes from lemma 1.6; for the existence, we begin to recall from PDE theory (see for instance chapter 1 of [10]) that, for $x_0 \in \mathbf{T}^{p}$, $(FP)_{-m,Y,\delta_{x_0}}$ has a solution $R_{x_0}$ with density $\rho_{x_0}$. Always from [10], the function $\rho_{x_0}$ satisfies the first formula of (2.3) for a constant $C_{16}(T)$ which depends neither on $x_0 \in \mathbf{T}^{p}$ nor on the particular element $Y \in Vect_{m}$. Moreover, as $T \to -m$, we get from [10] that, if $g \in C(\mathbf{T}^{p})$, then
\[
\int_{\mathbf{T}^{p}} g(x) dR_{x_0}(T) \to g(x_0)
\]
uniformly in $x_0 \in \mathbf{T}^{p}$; since $d_1$ induces the weak* topology and $\mathbf{T}^{p}$ is compact, we have that $d_{1}(R_{x_0}(T), \delta_{x_0}) \leq C_{17}(T)$, for a constant $C_{17}(T)$ which depends neither on $x_0 \in \mathbf{T}^{p}$ nor on $Y \in Vect_{m}$, and such that $C_{17}(T) \to 0$ as $T \to -m$. In other words, $\rho_{x_0}$ satisfies (2.3) for two uniform constants $C_{16}(T)$, $C_{17}(T)$, depending neither on $x_0$ nor on $Y \in Vect_{m}$.

Now we set
\[
\rho_Y(t, x) = \int_{\mathbf{T}^{p}} \rho_{x_0}(t, x) d\mu(x_0).
\]
(2.4)
Clearly, $\rho_Y$ is a solution of $(FP)_{-m,Y,\mu}$, and this ends the proof of point 1).

We have seen that $\rho_{x_0}$ satisfies the first formula of (2.3); since norms are convex, (2.4) implies that $\rho_Y$ too satisfies this formula. Now $\rho_{x_0}$ satisfies $d_{1}(R_{x_0}(T), \delta_{x_0}) \leq C_{17}(T)$, and the map
\[
: (\mu, \nu) \to d_{1}(\mu, \nu)
\]
is convex; it follows again by (2.4) that $\rho_Y$ too satisfies the second formula of (2.3).

\[
\\
\]
**Definition.** We define $Den_{m}^{reg}(\mu)$ as the subset of the elements $R \in Den_{m}(\mu)$ which, for $t \in (-m,0]$, have a density $\rho$ with respect to the Lebesgue measure. Moreover, we ask that $R$ and $\rho$ satisfy
\[
\begin{align*}
\|\rho\|_{Lip([T,0] \times \mathbf{T}^{p})} & \leq C_{16}(T), \quad \forall T \in (-m,0] \\
d_{1}(R(T), \mu) & \leq C_{17}(T), \quad \forall T \in (-m,0]
\end{align*}
\]
(2.5)
where $C_{16}(T)$ and $C_{17}(T)$ are the same two constants of (2.3). By lemma 2.3, if $Y \in Vect$, $\mu \in \mathcal{M}_{1}(\mathbf{T}^{p})$ and $R_{Y}$ solves $(FP)_{-m,Y,\mu}$ in the weak sense, then $R_{Y} \in Den_{m}^{reg}(\mu)$.

**Lemma 2.4.** $Den_{m}^{reg}(\mu)$ is compact in $Den_{m}(\mu)$ for the $C([-m,0],\mathcal{M}_{1}(\mathbf{T}^{p}))$ topology.

**Proof.** Let $R_{n} \in Den_{m}^{reg}(\mu)$ have density $\rho_{n}$ for $n \in \mathbf{N}$. We must show that it has a subsequence converging in $Den_{m}(\mu)$.
Since $\rho_n$ satisfies the first formula of (2.5), Ascoli-Arzelà implies that, up to subsequences, $\rho_n \to \rho$ in $C^0_{\text{loc}}((-m,0] \times T^p)$; clearly, $\rho$ satisfies the first formula of (2.5). Denoting by $L^p$ the Lebesgue measure on $T^p$, we set $R(t) = \rho(t)L^p$ and we see that, for any fixed $T \in (-m,0]$, 

$$d_1(R_n(T), R(T)) \leq \sqrt{p}||R_n(T) - R(T)||_{L^1(T^p)} = \sqrt{p}||\rho_n(T) - \rho(T)||_{L^1(T^p)} \to 0 \quad \text{as} \quad n \to +\infty$$

where the first inequality comes from (2.1) and the limit from the fact that $\rho_n \to \rho$ in $C^0_{\text{loc}}((-m,0] \times T^p)$.

Since $R_n$ satisfies the second formula of (2.5), we have that

$$d_1(R_n(T), \mu) \leq C_{17}(T), \quad \forall T \in (-m,0], \quad \forall n \geq 1.$$ 

The last two formulas imply that $R$ satisfies the second formula of (2.5).

It remains to prove that $R_n \to R$ in $C([-m,0], M_1(T^p))$; it suffices to note that, for $\delta \in (0,m)$,

$$\sup_{t \in [-m,0]} d_1(R_n(t), R(t)) \leq \sup_{t \in [-m,-m+\delta]} [d_1(R_n(t), \mu) + d_1(\mu, R(t))] + \sup_{t \in [-m+\delta,0]} d_1(R_n(t), R(t)) \leq 2C_{17}(-m+\delta) + \sqrt{p} \sup_{t \in [-m+\delta,0]} ||\rho_n(t) - \rho(t)||_{L^1(T^p)}$$

where the last inequality comes from the second formula of (2.5) and from (2.1). Since $C_{17}(T) \to 0$ as $T \searrow -m$, we can fix $\delta > 0$ so that the first term on the right is smaller than $\epsilon$; having thus fixed $\delta$, we take $n$ so large that, by convergence in $C^0_{\text{loc}}((-m,0] \times T^p)$, the second term on the right is smaller than $\epsilon$, and we are done.

\]

We only sketch the proof of the next lemma, since it is identical to point 4) of lemma 1.7.

**Lemma 2.5.** Given $\epsilon > 0$, we can find $\delta > 0$ with the following property. Let $Y, \bar{Y} \in Vect_m$ and let $\mu \in M_1(T^p)$; let $R_{\bar{Y}}$ and $R_Y$ satisfy $(FP)_{-m, Y, \mu}$ and $(FP)_{-m, Y, \mu}$ respectively. Let $||\bar{Y} - Y||_{C([-m,0], T^p)} \leq \delta$. Then, $||R_{\bar{Y}} - R_Y||_{C([-m,0], M_1(T^p))} \leq \epsilon$.

**Proof.** Let $\{Y_n\}_{n \geq 1}$, $\{\bar{Y}_n\}_{n \geq 1}$ be two sequences in $Vect_m$ and let $\{\mu_n\}_{n \geq 1} \subset M_1(T^p)$. We suppose that $||\bar{Y}_n - Y_n||_{C([-m,0], T^p)} \to 0$; we let $R_{\bar{Y}_n}$ solve $(FP)_{-m, \bar{Y}_n, \mu_n}$ and $R_{Y_n}$ solve $(FP)_{-m, Y_n, \mu_n}$; we have to prove that

$$||R_{\bar{Y}_n} - R_{Y_n}||_{C([-m,0], M_1(T^p))} \to 0.$$ 

Let us suppose by contradiction that this does not hold; then there is $\epsilon > 0$ and a subsequence (which we denote by the same index) such that

$$||R_{\bar{Y}_n} - R_{Y_n}||_{C([-m,0], M_1(T^p))} > \epsilon \quad \forall n.$$ 

Since $\bar{Y}_n, Y_n \in Vect_m$ and $||\bar{Y}_n - Y_n||_{C([-m,0], T^p)} \to 0$, by Ascoli-Arzelà up to taking subsequences we can suppose that $\bar{Y}_n, Y_n \to Y$ in $C([-m,0] \times T^p)$; we can also suppose that $\mu_n \to \mu$. To reach a contradiction
with the formula above, it suffices to show that $R_{Y_n}$ and $R_{\bar{Y}_n}$ both converge to $R_Y$; since the proof for $R_{Y_n}$ is analogous, we prove convergence for $R_{Y_n}$.

We note that $\{R_{Y_n}\}$ is contained in $Den^\text{reg}_m(\mu_n)$ by lemma 2.3; thus, by lemma 2.4, it has a subsequence converging to a limit $R$. Since $R_{Y_n}$ is a weak solution of $(FP)_{-m,Y_n,\mu_n}$, we easily get that $R$ is a weak solution of $(FP)_{-m,Y,\mu}$; by lemma 1.6, $R = R_Y$. In other words, every subsequence of $R_{Y_n}$ has a sub-subsequence converging to $R_Y$; this implies that $R_{Y_n}$ converges to $R_Y$, and we are done.

\\

**Proof of theorem 2.** For $Q \in Den_m(\mu)$, let $u_Q$ be as in corollary 2.2; for $Y \in Vect_m$, let $R_Y = \rho_Y L^p$ be as in lemma 2.3. The two maps

$$\begin{align*}
: Den_m(\mu) &\to Vect_m, \\
: Q &\to c - \partial_x u_Q
\end{align*}$$

and

$$\begin{align*}
: Vect_m &\to Den_m(\mu), \\
: Y &\to R_Y
\end{align*}$$

are both continuous: the first one, by point 3) of of corollary 2.2, the second one by lemma 2.5. Let us call $\Phi$ their composition:

$$\Phi: Den_m(\mu) \to Den_m(\mu), \quad \Phi: Q \to R_{c - \partial_x u_Q}.$$ 

Being the composition of two continuous functions, $\Phi$ is continuous; moreover, by point 2) of lemma 2.3, it has image in $Den^\text{reg}_m(\mu)$; this latter set is clearly convex, and it is compact in $Den_m(\mu)$ by lemma 2.4. Thus, we have that

$$\Phi: Den^\text{reg}_m(\mu) \to Den^\text{reg}_m(\mu).$$

We apply the Schauder fixed point theorem and we get that $\Phi$ has a fixed point in $Den^\text{reg}_m(\mu)$. With the same argument as in the proof of theorem 1, we see that, if $R$ is a fixed point of $\Phi$, then $(u_R, R)$ solves $(HJ)_{R,f} - (FP)_{-m,c - \partial_x u_R,\mu}$. This yields existence.

We continue as in the proof of theorem 1. Let us call $S$ the set of the couples $(u, R)$ where $u$ is a classical solution of $(HJ)_{R,f}$ and $R \in Den_m(\mu)$ is a weak solution of $(FP)_{-m,c - \partial_x u,\mu}$.

Let us consider a sequence $(u_n, R_n) \in S$ such that, denoting by $\rho_n$ the density of $R_n$,

$$\int_{-m}^0 dt \left( \int_{T^p} L_{\epsilon, \frac{1}{2} R_n}(t, x, c - \partial_x u_n)\rho_n dt dx \right) \to \inf_{(u, R) \in S} \int_{-m}^0 dt \left( \int_{T^p} L_{\epsilon, \frac{1}{2} R}(t, x, c - \partial_x u)\rho dt dx \right).$$

Whatever is $R_n \in Den_m(\mu)$, $u_n$ satisfies the estimates of point 2) of corollary 2.2; in particular, $c - \partial_x u_n \in Vect_m$. Since $R_n$ satisfies $(FP)_{-m,c - \partial_x u_n,\mu}$ lemma 2.3 implies that $R_n \in Den^\text{reg}_m(\mu)$; by lemma 2.4, up to subsequences we can suppose that $R_n \to \tilde{R}$, with $\tilde{R} \in Den^\text{reg}_m(\mu)$. By point 3) of corollary 2.2, we get that $u_n \to \tilde{u}$ in $C^1([-m,0], C^4(T^p)) \cap C([-m,0], C^4(T^p))$, and that $\tilde{u}$ solves $(HJ)_{\tilde{R},f}$. Thus, $(\tilde{u}, \tilde{R}) \in S$; now, the formula above easily implies that $(\tilde{u}, \tilde{R})$ is minimal in $S$. 

\\

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We turn to the proof of theorem 3; our route will pass through an approximation with a finite number of particles.

Definitions. Let us define the Lagrangian for one particle as

\[ L_c: T \times T^p \times \mathbb{R}^p \to \mathbb{R}, \quad L_c(t, x, y) = \frac{1}{2} |y|^2 - \langle c, y \rangle - V(t, x). \]

The Lagrangian for \( n \) particles, each of mass \( \frac{1}{n} \), is

\[ L^n_c: T \times (T^p)^n \times (\mathbb{R}^p)^n \to \mathbb{R} \]

\[ L^n_c(t, (x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \frac{1}{n} \sum_{i=1}^{n} L_c(t, x_i, y_i) + \frac{1}{2n^2} \sum_{i,j=1}^{n} W(x_i - x_j). \]

Let \( U \) be as in the statement of theorem 2. For any given \( z = (z_1, \ldots, z_n) \in (T^p)^n \), we define

\[ U^n(-m, z) = \inf_{u_1, \ldots, u_n} \left\{ \int_{-m}^{0} L^n_u(s, X^n(-m, s, z), Y^n(s, X^n(-m, s, z)))ds + U(R^n(-m, 0)) \right\}. \tag{2.6} \]

The infimum above is over all vector fields \( Y^n(s, x) = (Y^n_1(s, x_1), Y^n_2(s, x_2), \ldots, Y^n_n(s, x_n)) \) continuous in \( s \) and Lipschitz in \( x \); each component of the function

\[ X^n(-m, s, z) = (X^n_1(-m, s, z_1), X^n_2(-m, s, z_2), \ldots, X^n_n(-m, s, z_n)) \in (\mathbb{R}^p)^n \]

solves the stochastic differential equation on \( \mathbb{R}^p \)

\[ \begin{cases} 
\diff X^n_i(-m, s, z_i) = Y^n_i(s, X^n_i(-m, s, z_i)) \diff t + \diff w_i(s) & s \geq -m, \quad i \in \{1, \ldots, n\} \\
X^n_i(-m, -m, z_i) = z_i. 
\end{cases} \tag{SDE} \]

In the formula above, each \( w_i \) is a standard Brownian motion on \( \mathbb{R}^p \); the \( w_i \) are independent and \( E_{u_1, \ldots, u_n} \) denotes the expectation with respect to the product of the Wiener measures. It remains to define \( R^n(-m, 0) \); to do this, we let \( \rho^n_i(-m, s, x) \) be the density on \( T^p \) which solves \( (FP)_{-m, Y^n_i, \delta_{z_i}} \) and we set, for \( t \in [-m, 0] \),

\[ \rho^n(-m, t, x) = \frac{1}{n} \sum_{i=1}^{n} \rho^n_i(-m, t, x), \quad R^n(-m, t) = \rho^n(-m, t, x)\mathcal{L}^p. \tag{2.7} \]

We note that we are not considering the most general vector field \( Y \) on \( (T^p)^n \). On the contrary, we assign to each particle \( x_i \in T^p \) a control \( Y_i \) which depends only on \( x_i \), and not on the positions of the other particles; these, however, interact with \( x_i \) via the potential \( W \). We have chosen this particular problem because we want \( U_n(-m, z) \) to converge, as \( n \to +\infty \), to \( \Lambda^n U \); we recall that, in the definition of \( \Lambda^n U \), there is a control \( Y \) which depends on the single particle in \( T^p \).

Lemma 2.6. Let us suppose that \( U \) is as in the statement of theorem 1 and let \( U^n(-m, z) \) be defined as in (2.6). Then for any fixed \( n \in \mathbb{N} \), the infimum in (2.6) is a minimum.
Proof. Let \( \{Y^{n,k}\}_{k \geq 1} \) be a minimizing sequence. We are going to show that we can build another minimizing sequence, say \( \{\tilde{Y}^{n,k}\}_{k \geq 1} \), which is Lipschitz in \((t,x)\) uniformly in \(k\). Once we know this, the lemma follows by Ascoli-Arzelà.

For the vector field \( Y^{n,k} \), let us define \( \rho_{i}^{n,k} \) and \( \rho^{n,k} \) as in (2.7); we set

\[
L_{c,Y^{n,k},i}^{n}[\cdot,0] \times \mathbb{T}^{p} \times \mathbb{R}^{p} \rightarrow \mathbb{R},
\]

\[
L_{c,Y^{n,k},i}^{n}(s,x,\dot{x}) = L_{c}(s,x,\dot{x}) - \frac{1}{n} \sum_{j \neq i} \int_{\mathbb{T}^{p}} W(x - y) \rho_{j}^{n,k}(-m,s,y)dy.
\]  

Note that, in contrast with \( L_{c}^{n} \), a factor \( \frac{1}{n} \) in the interaction sum is missing. We know from lemma 1.1 that the potential in \( L_{c,Y^{n,k},i}^{n} \) satisfies a uniform \( C^{3} \) estimate. By [9], for \((t,x) \in [-m,0] \times \mathbb{T}^{p} \), there is \( \tilde{Y}_{i}^{n,k} \) on which the minimum below is attained

\[
u_{i}^{n,k}(t,x) := \min_{u^{n,k}_{i}} \left\{ \int_{t}^{0} L_{c,Y^{n,k},i}(s,X,Y)ds + f(X(t,0,x)) \right\}
\]  

with \( X(t,s,x) \) which solves \((SDE)_{t,Y,\delta_{i}}\); the minimum is taken over all the Lipschitz vector fields \( Y \). Always by [9], \( \tilde{Y}_{i}^{n,k} = c - \partial_{x}u_{i}^{n,k}(t,x) \) and \( u_{i}^{n,k} \) solves the Hamilton-Jacobi equation for the Lagrangian \( L_{c,Y^{n,k},i}^{n} \) and final condition \( f \). By lemma 2.1,

\[
||\nu_{i}^{n,k}||_{C^{1}([-m,0],C^{3}(\mathbb{T}^{p}))} + ||u_{i}^{n,k}||_{C^{1}([-m,0],C^{3}(\mathbb{T}^{p}))}
\]

is bounded in terms of the \( C^{3} \) norm of the potential of \( L_{c,Y^{n,k},i}^{n} \). By lemma 1.1, the latter depends neither on \( n \) nor on \( k \); thus, \( \tilde{Y}_{i}^{n,k} \) belongs to \( \text{Vect}_{m} \); in particular, it is Lipschitz uniformly in \( n \) and \( k \).

In the following, whenever we have a drift, say \( Y_{i}^{B} \), we shall denote by \( X_{i}^{B}(t,s,x_{i}) \) the solution of \((SDE)_{t,Y,\delta_{i}}\); we shall set \( X^{B} = (X_{1}^{B}, \ldots, X_{n}^{B}) \) and \( z = (z_{1}, \ldots, z_{n}) \).

We are going to isolate the first particle and show that the mean action decreases if we substitute \( Y_{1}^{n,k} \) with the smoother \( \tilde{Y}_{1}^{n,k} \) defined above. Since the interaction potential is even and satisfies \( W(0) = 0 \), we get the first equality below: since the Brownian motions \((w_{1}, \ldots, w_{n})\) are independent, we get the second one.

\[
E_{w_{1},\ldots,w_{n}} \left\{ \int_{-m}^{0} L_{c}^{n}(s,X_{1}^{n,k}(-m,s,z),Y_{n}^{n,k}(s,X_{n}^{n,k}(-m,s,z)))ds + U(R^{n}(-m,0)) \right\} = \]

\[
E_{w_{1},\ldots,w_{n}} \left\{ \frac{1}{n} \sum_{j \neq 1} \int_{-m}^{0} L_{c}(s,X_{1}^{n,k}(-m,s,z_{j}),Y_{n}^{n,k}(s,X_{n}^{n,k}(-m,s,z_{j})))ds + \right\} + \]

\[
\frac{1}{2n} \sum_{i,j \neq 1} \int_{-m}^{0} W(X_{i}^{n,k}(-m,s,z_{i}) - X_{j}^{n,k}(-m,s,z_{j}))ds + \frac{1}{n} \sum_{j \neq 1} f(X_{j}(-m,0,z_{j})) \right\} + \]

\[
E_{w_{1},\ldots,w_{n}} \left\{ \frac{1}{n} \int_{-m}^{0} [L_{c}(s,X_{1}^{n,k}(-m,s,z_{1}),Y_{1}^{n,k}(s,X_{1}^{n,k}(-m,s,z_{1}))) + \right\} + \]

\[
\frac{1}{n} \sum_{j \neq 1} \int_{-m}^{0} W(X_{1}^{n,k}(-m,s,z_{1}) - X_{j}^{n,k}(-m,s,z_{j}))ds + \frac{1}{n} f(X_{1}(-m,0,z_{1})) \right\} = \]

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If we consider \( \tilde{Y}_{n,k} \) instead of \( Y_{1,n,k}, \ldots, Y_{n,k} \), we see that the terms \( a1 \) and \( a2 \) in the formula above remain the same, while, by our choice of \( \tilde{Y}_{n,k} \), \( a3 \) gets smaller. After applying this procedure to each coordinate, we get a sequence \( \tilde{Y}_{n,k} = (\tilde{Y}_{1,n,k}, \ldots, \tilde{Y}_{n,k}) \) which satisfies the following two properties.

\[
\begin{align*}
E_{w_1, \ldots, w_n} \left\{ \int_{-m}^{0} L_c(s, X_{j}^{n,k}(-m, s, z_j)) ds + f(X_j(-m, 0, z_j)) \right\} \\
E_{w_1, \ldots, w_n} \left\{ \int_{-m}^{0} L_c(s, \tilde{X}_{j}^{n,k}(-m, s, z_j)) ds + f(X_j(-m, 0, z_j)) \right\}
\end{align*}
\]

If we consider \( \tilde{Y}_{n,k} \) instead of \( Y_{1,n,k}, \ldots, Y_{n,k} \), we see that the terms \( a1 \) and \( a2 \) in the formula above remain the same, while, by our choice of \( \tilde{Y}_{n,k} \), \( a3 \) gets smaller. After applying this procedure to each coordinate, we get a sequence \( \tilde{Y}_{n,k} = (\tilde{Y}_{1,n,k}, \ldots, \tilde{Y}_{n,k}) \) which satisfies the following two properties.

1. \( E_{w_1, \ldots, w_n} \left\{ \int_{-m}^{0} L_c(s, X_{i}^{n,k}(-m, s, x)) ds + f(X_i(-m, 0, x)) \right\} \leq E_{w_1, \ldots, w_n} \left\{ \int_{-m}^{0} L_c(s, \tilde{X}_{i}^{n,k}(-m, s, x)) ds + f(X_i(-m, 0, x)) \right\} \)

2. \( \tilde{Y}_{n,k} \in (Vect_m)^n \). In particular, \( \tilde{Y}_{n,k} \in Vect_m \) for all \( i, n \) and \( k \); as a consequence, we can apply point 2) of lemma 2.3, getting that \( \{\rho_{i,n,k}^n\}_{n,k} \subset Den_m^{reg}(\delta_{z_i}) \) for all \( i \). By lemma 2.4, we find a subsequence, which we denote by the same index, such that

\[
(\rho_{1,n,k}^n L^p, \ldots, \rho_{n,n,k}^n L^p) \rightarrow (\rho_{1}^n L^p, \ldots, \rho_{n}^n L^p) \quad \text{in} \quad Den_m(\delta_{z_1}) \times \ldots \times Den_m(\delta_{z_n}).
\]

Thus, for each \( i \),

\[
\frac{1}{n} \sum_{j \neq i} \int_{T^p} W(x - y)\rho_{j}^{n,k}(-m, s, y) dy \rightarrow \frac{1}{n} \sum_{j \neq i} \int_{T^p} W(x - y)\rho_{j}^{n}(-m, s, y) dy
\]

in \( C([-m, 0], C^1(T^p)) \). By point 3) of lemma 2.1, this implies that

\[
(\tilde{Y}_{1,n,k}, \ldots, \tilde{Y}_{n,n,k}) \rightarrow (\tilde{Y}_1^n, \ldots, \tilde{Y}_n^n) \quad \text{in} \quad C^1([-m, 0], C(T^p)) \cap C([-m, 0], C^2(T^p))
\]

and that each \( \tilde{Y}_i^n \) is minimal for \( L_{c,Y_i}^n \). By the last formula and lemma 2.5, we get that \( \rho_i^n \) solves \( (FP)_{-m,\tilde{Y}_i^n,\delta_{z_i}} \). The last three formulas imply that

\[
E_{w_1, \ldots, w_n} \left\{ \int_{-m}^{0} L_c(s, \tilde{X}_{j}^{n,k}(-m, s, z_j)) ds + U(R^n(-m, 0, z)) \right\} = \\
\lim_{n \rightarrow +\infty} E_{w_1, \ldots, w_n} \left\{ \int_{-m}^{0} L_c(s, \tilde{X}_{j}^{n,k}(-m, s, z_j)) ds + U(R^n(-m, 0, z)) \right\}.
\]
Moreover, the function $U$ is a Nash equilibrium (\cite{4}). Note one fact about the value function $Y$:

Let $Y = \sup_{i} \left( \int_{t}^{s} L_{c,Y,i}(s, X(s, t), X(t, 0, x)) ds + f(X(t, 0, x)) \right)$

where $X$ solves $(SDE)_{t,x},Y,\delta_{s}$ and the minimum is taken among all Lipschitz vector fields $Y$ on $[-m,0] \times T^{p}$. Then, for each $i$ we have that $Y_{i}^{n}(t, x) = c - \partial_{x}u_{i}^{n}(t, x)$.

**Proof.** If for one $i$ we had $\bar{Y}_{i}^{n} \neq c - \partial_{x}u_{i}^{n}$, then, isolating particle $i$ as in the last lemma, we could see that the vector field

$$(\bar{Y}_{1}^{n}, \ldots, \bar{Y}_{i-1}^{n}, c - \partial_{x}u_{i}^{n}, \bar{Y}_{i+1}^{n}, \ldots, \bar{Y}_{n}^{n})$$

has a lower Lagrangian action, contradicting the minimality of $\bar{Y}^{n}$.

**Corollary 2.7.** Let $\bar{Y}^{n}(t, x) = (\bar{Y}_{1}^{n}(t, x_{1}), \ldots, \bar{Y}_{n}^{n}(t, x_{n}))$ be minimal in (2.6) and let $L_{c,Y,i}^{n}$ be defined as in (2.8). Let

$$u_{i}^{n}(t, x) = \min_{Y} E_{w} \left\{ \int_{t}^{s} L_{c,Y,i}^{n}(s, X(s, t), X(t, 0, x)) ds + f(X(t, 0, x)) \right\}$$

where $X$ solves $(SDE)_{t,x},Y,\delta_{s}$ and the minimum is taken among all Lipschitz vector fields $Y$ on $[-m,0] \times T^{p}$. Then, for each $i$ we have that $\bar{Y}_{i}^{n}(t, x) = c - \partial_{x}u_{i}^{n}(t, x)$.

**Proof.** If for one $i$ we had $\bar{Y}_{i}^{n} \neq c - \partial_{x}u_{i}^{n}$, then, isolating particle $i$ as in the last lemma, we could see that the vector field

$$(\bar{Y}_{1}^{n}, \ldots, \bar{Y}_{i-1}^{n}, c - \partial_{x}u_{i}^{n}, \bar{Y}_{i+1}^{n}, \ldots, \bar{Y}_{n}^{n})$$

has a lower Lagrangian action, contradicting the minimality of $\bar{Y}^{n}$.

**Lemma 2.8.** Let $\mu \in \mathcal{M}_{1}(T^{p})$ and let us suppose that

$$\frac{1}{n}(\delta_{z_{1}} + \ldots + \delta_{z_{n}}) \to \mu \ \text{in} \ \mathcal{M}_{1}(T^{p}). \ (2.10)$$

Let $Y^{n} = (Y_{1}^{n}, \ldots, Y_{n}^{n})$ be a drift minimal in (2.6); by corollary 2.7, $Y_{i}^{n} = c - \partial_{x}u_{i}^{n}$ for the value function $u_{i}^{n}$ defined in (2.9). Let $\rho^{n}$ be defined as in (2.7).

Then, there is $(u, \rho)$ which satisfies $(HJ)_{\rho,f} - (FP)_{-m,c-\partial_{x}u,\mu}$, and a subsequence $\{n_{k}\}$ such that

$$\rho^{n_{k}} L^{p} \to \rho L^{p} \ \text{in} \ C([-m,0],\mathcal{M}_{1}(T^{p}))$$

$$\sup_{0 \leq i \leq n_{k}} ||u_{i}^{n_{k}} - \partial_{x}u||_{C([-m,0],C^{1}(T^{p}))} + ||u_{i}^{n_{k}} - u||_{C([-m,0],C^{2}(T^{p}))} \to 0 \ (2.11)$$

Moreover, the function $U^{n_{k}}(-m, z_{1}, \ldots, z_{n_{k}})$ defined in (2.6) converges to the function $U(-m, \mu)$ defined by

$$U(-m, \mu) = \int_{[-m,0] \times T^{p}} \mathcal{L}_{c,-\frac{1}{2}}(t, x, c - \partial_{x}u) \rho(t, x) dxdt + U(\rho). \ (2.12)$$

**Proof.** Since $Y_{i}^{n} = c - \partial_{x}u_{i}^{n}$, the third formula of (2.11) follows from the second one; we prove the first two ones.
Step 1. We prove the convergence of the densities.

For \(i, j \in \{1, \ldots, n\}\), we consider the densities

\[
\hat{\rho}^n_i(-m, s, x) = \frac{1}{n-1} \sum_{l \neq i} \rho^n_l(-m, s, x), \quad \hat{\rho}^n_j(-m, s, x) = \frac{1}{n-1} \sum_{l \neq j} \rho^n_l(-m, s, x)
\]

where \(\hat{\rho}^n_i\) is the same as in formula (2.7). Let \(R^n_i = \rho^n_i \mathcal{L}^p\), \(\hat{R}^n_i = \hat{\rho}^n_i \mathcal{L}^p\) and \(R^n = \rho^n \mathcal{L}^p\). Formula (2.1) implies the first inequality below, while the second one follows from the fact that \(\rho^n_i\) and \(\hat{\rho}^n_i\) are probability densities.

\[
d_1(\hat{R}^n_i(-m, s), \hat{R}^n_j(-m, s)) \leq \sqrt{n-1} \left| \frac{1}{n-1} \sum_{l \neq i} \rho^n_l(-m, -s, \cdot) - \frac{1}{n-1} \sum_{l \neq j} \rho^n_l(-m, -s, \cdot) \right|_{L^1(\mathcal{D}p)} = \frac{\sqrt{n-1}}{n-1} \left| \sum_{l \neq i} \rho^n_l(-m, -s, \cdot) - \sum_{l \neq j} \rho^n_l(-m, -s, \cdot) \right|_{L^1(\mathcal{D}p)} \\
\leq \frac{2}{n-1} \forall s \in [-m, 0], \quad \forall i, j \in \{1, \ldots, n\}.
\]

By (2.8),

\[
L^p_{i, Y^n, i} = \frac{1}{2}[|\dot{c}|^2 - \langle c, \dot{x} \rangle - V(t, x) - \frac{n-1}{n} \int_\mathcal{D}p W(x-y)\hat{\rho}^n_i(t, y)dy.
\]

By lemma 1.1, we get the second inequality below.

\[
\left| \left| V(t, x) + \frac{n-1}{n} \int_\mathcal{D}p W(x-y)\hat{\rho}^n_i(t, y)dy \right| \right|_{C((-m, 0), C^1(\mathcal{D}p))} \leq \left| \left| V(t, x) \right| \right|_{C((-m, 0), C^1(\mathcal{D}p))} + \left| \left| \int_\mathcal{D}p W(x-y)\hat{\rho}^n_i(t, y)dy \right| \right|_{C((-m, 0), C^1(\mathcal{D}p))} \leq C_1.
\]

As a result, the value function \(u^n_i\) satisfies point 2) of corollary 2.2; thus, \(Y^n_i \in \text{Vect}_m\) and we can apply lemma 2.3, getting that \(R^n_i \in \text{Den}_{m}^{reg}\). Since this set is convex, (2.13) implies that \(\hat{R}^n_i \in \text{Den}_{m}^{reg}\); by lemma 2.4, we have that \(\text{Den}_{m}^{reg}\) is a compact set; thus, fixing \(i = 1\), there is \(n_k \to +\infty\) such that \(\hat{R}^n_k\) converges to \(R \in \text{Den}_{m}^{reg}\), in particular, \(R\) and its density \(\rho\) satisfy (2.5). This gives convergence only for \(\hat{R}^n_k\); however, from (2.14) we get that

\[
\sup_{i \in \{1, \ldots, n\}} \sup_{s \in [-m, 0]} d_1(\hat{R}^n_k(-m, s), \hat{R}^n_k(-m, s)) \to 0 \quad \text{as} \quad k \to +\infty
\]

which implies that all \(\hat{R}^n_k\) converge to the same limit \(R\). By the same argument of (2.14),

\[
d_1(R^n_k(-m, s), \hat{R}^n_k(-m, s)) \leq \frac{1}{n}.
\]

Thus, (2.15) implies the first formula of (2.11).

Step 2. We prove the convergence of the solutions of Hamilton-Jacobi. We set

\[
\begin{align*}
W^k_i(s, x) &= \frac{n-1}{n} \int_\mathcal{D}p W(x-y)\hat{\rho}^n_i(-m, s, y)dy = \frac{n-1}{n} \int_\mathcal{D}p W(x-y)d\hat{R}^n_k(-m, s)(y) \\
\tilde{W}(s, x) &= \int_\mathcal{D}p W(x-y)\hat{\rho}(-m, s, y)dy = \int_\mathcal{D}p W(x-y)dR(-m, s)(y).
\end{align*}
\]

By (2.15) and the fact that \(d_1\) induces weak* convergence, we get that

\[
\sup_{i \in \{1, \ldots, n\}} ||W^k_i - \tilde{W}||_{C((-m, 0), C^1(\mathcal{D}p))} \to 0 \quad \text{as} \quad k \to +\infty.
\]
Now, \( u^n_{i_k} \) is the value function of \( L_{c,Y^n_{i_k,i}} \), whose potential is \( V(t,x) + W^n_{i_k}(t,x) \); by the last formula, we can apply point 3) of lemma 2.1 and get that \( u^n_{i_k} \) satisfies the limit in the second formula of (2.11), with \( u \) a solution of \( (HJ)^{V+W} \) or, which is the same, of \( (HJ)_{\rho,f} \).

**Step 3.** We prove that the limit density \( \rho \) solves \( (FP)_{-m,c-\partial_x u,\mu} \).

From now on, for ease of notation, we drop the \( n_k \) of the subsequence. We recall that each \( \rho^n_i \) solves \( (FP)_{-m,Y^n_i,\delta_{n_i}} \); by the third formula of (2.11) and lemma 2.5, we get that, if \( \bar{\rho}_i \) is a solution of \( (FP)_{-m,c-\partial_x u,\delta_{n_i}} \), then

\[
\sup_i \| \rho^n_i \mathcal{L}^p - \bar{\rho}_i \mathcal{L}^p \|_{\mathcal{C}([0,1],\mathcal{M}(\mathcal{F}^p))} \to 0 \quad \text{as} \quad n \to +\infty.
\]

Now (2.2) implies the inequality below, and the last formula implies the limit.

\[
\| \rho^n \mathcal{L}^p - \frac{1}{n} \sum_{i=1}^n \rho^n_i \mathcal{L}^p \|_{\mathcal{C}([0,1],\mathcal{M}(\mathcal{F}^p))} \leq \frac{1}{n} \sum_{i=1}^n \| \rho^n_i \mathcal{L}^p - \bar{\rho}_i \mathcal{L}^p \|_{\mathcal{C}([0,1],\mathcal{M}(\mathcal{F}^p))} \to 0.
\]

This means that \( \rho^n \mathcal{L}^p \) and \( \frac{1}{n} \sum_{i=1}^n \rho^n_i \mathcal{L}^p \) have the same limit; we saw in step 1 that \( \rho^n \mathcal{L}^p \) converges to \( \rho \mathcal{L}^p \); thus, to prove that \( \rho \mathcal{L}^p \) solves \( (FP)_{-m,c-\partial_x u,\mu} \), it suffices to prove that the limit of \( \frac{1}{n} \sum_{i=1}^n \rho^n_i \mathcal{L}^p \) solves the same equation. This follows easily, since by definition \( \frac{1}{n} \sum_{i=1}^n \rho^n_i \mathcal{L}^p \) solves the Fokker-Planck equation with drift \( c - \partial_x u \) and initial condition \( \frac{1}{n}(\delta_{z_1} + \ldots + \delta_{z_n}) \), and (2.10) holds.

**Step 4.** We prove the last assertion of the lemma; the equality below comes from (2.6) and the fact that \( Y^n \) is minimal.

\[
U^n(-m,(z_1,\ldots,z_n)) = E_{w_1,\ldots,w_n}\left\{ \int_{-m}^0 L^n_c(s,X^n(-m,s,x),Y^n(s,X^n(-m,s,x)))\,ds \right\} + U(\rho^n(-m,0,\cdot)).
\]

We recall that, by corollary 2.7,

\[
Y^n = (c - \partial_x u^n_1,\ldots,c - \partial_x u^n_n).
\]

Now \( \rho^n \mathcal{L}^p \) is the push-forward of the Wiener measure by \( X^n_i \), and the Brownian motions \( w_i \) are independent. This implies that

\[
U^n(-m,(z_1,\ldots,z_n)) = \frac{1}{n} \sum_{i=1}^n \int_{[-m,0] \times \mathcal{T}^p} L_c(t,x,c - \partial_x u^n_i) \rho^n_i(t,x)\,dt\,dx -
\]

\[
\frac{1}{2n^2} \sum_{i \neq j \in (1,\ldots,n)} \int_{[-m,0] \times \mathcal{T}^p \times \mathcal{T}^p} W(x_i - x_j) \rho^n_i(-m,t,x_i) \rho^n_j(-m,t,x_j)\,dx_i\,dx_j\,dt + \int_{\mathcal{T}^p} f(x) \rho^n(-m,0,x)\,dx.
\]

Using (2.11), we get immediately that

\[
U^n(-m,(z_1,\ldots,z_n)) \to \int_{[-m,0] \times \mathcal{T}^p} L_c(x,c - \partial_x u) \rho(-m,0,x)\,dx + \int_{\mathcal{T}^p} f(x) \rho(-m,0,x)\,dx.
\]
Proof of theorem 3. Let the measure $\mu$, the couple $(u, \rho)$ and the function $U(-m, \mu)$ be as in the last lemma; let the operator $\Lambda^m_{c}$ be as in the introduction, and let $Y = c - \partial_x u$. We are going to prove that
\[
(\Lambda^m_{c} U)(\mu) = U(-m, \mu) = E_w \left\{ \int_{-m}^0 \mathcal{L}_{c, \frac{\rho}{\mu}}(s, X, Y) ds + f(X(-m, 0, \mu)) \right\} = \min_{\tilde{Y}} E_w \left\{ \int_{-m}^0 \mathcal{L}_{c, \frac{\rho}{\mu}}(s, \tilde{X}, \tilde{Y}) ds + f(\tilde{X}(-m, 0, \mu)) \right\}. \tag{2.18}
\]
The functions $\rho$ and $\tilde{\rho}$ in the formula above satisfy $(FP)_{-m,Y,\mu}$ and $(FP)_{-m,Y,\mu}$ respectively, while $X$ and $\tilde{X}$ satisfy $(SDE)_{-m,Y,\mu}$ and $(SDE)_{-m,Y,\mu}$ respectively. The minimum is taken over all Lipschitz vector fields $\tilde{Y}$.

Note that, in principle, $U(-m, \mu)$ could depend on the subsequence $\{n_k\}_{k \geq 1}$ chosen in lemma 2.8; the formula above says that this is not the case. Moreover, it says that any $(u, \rho)$ arising in lemma 2.8 as the limit of a subsequence, minimizes the last expression of (2.18).

The second equality of (2.18) follows from lemma 2.8: it is just another way of writing (2.12). Again by lemma 2.8, $(u, \rho) \in \mathbb{S}$, and thus, by the definition of $(\Lambda^m_{c} U)(\mu)$,
\[
(\Lambda^m_{c} U)(\mu) \leq E_w \left\{ \int_{-m}^0 \mathcal{L}_{c, \frac{\rho}{\mu}}(s, X, Y) ds + f(X(-m, 0, \mu)) \right\} = U(-m, \mu). \tag{2.19}
\]
Now we prove that
\[
U(-m, \mu) \leq \inf_{\tilde{Y}} E_w \left\{ \int_{-m}^0 \mathcal{L}_{c, \frac{\rho}{\mu}}(s, \tilde{X}, \tilde{Y}) ds + f(\tilde{X}(-m, 0, z)) \right\}. \tag{2.20}
\]
To prove this, we consider the n-particle value function $U^n(-m, z_1, \ldots, z_n)$. Let $\tilde{Y}$ be a Lipschitz vector field on $[-m, 0] \times \mathbb{T}$. Let $\tilde{X}^n$ solve $(SDE)_{-m,\tilde{Y},\delta_{z_1}}$; let us suppose that (2.10) holds. Let us set $\tilde{X}^n = (\tilde{X}^n_1, \ldots, \tilde{X}^n_n)$ and $Y^n = (\tilde{Y}, \ldots, \tilde{Y})$. Let $\tilde{\rho}$ solve $(FP)_{-m,\tilde{Y},\mu}$ and let $\tilde{\rho}_i$ solve $(FP)_{-m,\tilde{Y},\delta_{z_1}}$; by linearity, we get that $\frac{1}{n} \sum_{i=1}^n \tilde{\rho}_i(-m, s, x)$ solves $(FP)_{-m,\tilde{Y},\frac{1}{n} \delta_{z_1 + \ldots + z_n}}$. In other words, $\tilde{\rho}$ and $\frac{1}{n} \sum_{i=1}^n \tilde{\rho}_i(-m, s, x)$ solve a Fokker-Planck equation with the same drift, but initial distributions $\mu$ and $\frac{1}{n} \delta_{z_1 + \ldots + z_n}$ respectively; by (2.10), it is standard to see that
\[
\sup_{s \in [-m, 0]} d_1 \left( \frac{1}{n} \sum_{i=1}^n \tilde{\rho}_i(-m, s, x) \mathcal{L} \tilde{\rho}_i, \tilde{\rho}(-m, s, x) \mathcal{L} \right) \rightarrow 0 \quad \text{as} \quad n \rightarrow +\infty. \tag{2.21}
\]
We set $z = (z_1, \ldots, z_n)$ and by (2.6) we get the inequality below; the first equality is the definition of $L^n_c$, the second one comes from the fact that the Brownian motions $w_1, \ldots, w_n$ are independent.
\[
U^n(-m, z) \leq E_{w_1, \ldots, w_n} \left\{ \int_{-m}^0 L^n_c(s, \tilde{X}^n(-m, s, z), \tilde{Y}^n(s, \tilde{X}^n(-m, s, z)) ds + \frac{1}{n} \sum_{i=1}^n f(\tilde{X}_i(-m, 0, z_i)) \right\} = E_{w_1, \ldots, w_n} \left\{ \frac{1}{n} \sum_{i=1}^n \int_{-m}^0 L_c(s, \tilde{X}_i, \tilde{Y}_i) ds - \frac{1}{2n^2} \sum_{i \neq j} \int_{-m}^0 W(\tilde{X}_i - \tilde{X}_j) ds + \frac{1}{n} \sum_{i=1}^n f(\tilde{X}_i(-m, 0, z_i)) \right\}
\]

\[
\frac{1}{n} \sum_{i=1}^{n} \int_{[-m,0] \times T^p} L_c(s, x, \tilde{Y}(s, x)) \hat{\rho}_i(-m, s, x) \, ds \, dx - \\
\frac{1}{2n^2} \sum_{i \neq j}^{n} \int_{[0,m] \times T^p \times T^p} W(x_i - x_j) \hat{\rho}_i(-m, s, x_i) \hat{\rho}_j(-m, s, x_j) \, ds \, dx_i \, dx_j + \frac{1}{n} \sum_{i=1}^{n} \int_{T^p} f(x) \hat{\rho}_i(-m, 0, x) \, dx.
\]

We take limits in the formula above, using the last assertion of lemma 2.8 for the left hand side and (2.21) for the right hand side; we get that

\[
U(-m, \mu) \leq \int_{-m}^{0} L_{c, \tilde{\rho}}(s, x, \tilde{Y}) \rho(t, x) \, dt \, dx + \int_{T^p} f(x) \hat{\rho}(-m, 0, x) \, dx.
\]

Since \( \tilde{Y} \) is an arbitrary Lipschitz vector field, we get that (2.20) holds.

Let now \((\bar{u}, \bar{\rho}) \in S\) be minimal in the definition of \((A^u_m U)(\mu)\); setting \(\tilde{Y} = e - \partial_x \bar{u}, \) (2.20) implies the inequality below, while the equality comes from our choice of \(\tilde{Y}\).

\[
U(-m, \mu) \leq E_{w} \left\{ \int_{-m}^{0} L_{c, \tilde{\rho}}(s, x, \tilde{Y}) ds + f(\tilde{X}(-m, 0, z)) \right\} = (A^u_m U)(\mu).
\]

This yields the inequality opposite to (2.19). In other words, we have proven the first equality of (2.18); the second one, as we have seen, is lemma 2.8. As for the third one, it suffices to prove the inequality opposite to (2.20), which we do presently.

Let \((z_1, \ldots, z_n)\) satisfy (2.10), let \(Y^n = (Y^n_1, \ldots, Y^n_n)\) be minimal in (2.6), and let us set \(\bar{Y}^n = Y^n_1\). Let \(\tilde{\rho}^n\) satisfy \((FP)_{-m, \bar{Y}^n, \mu, \cdot}\). By (2.11) and (2.17), we obtain that there is \(\gamma_n \to 0\) such that

\[
E_{w} \left\{ \int_{-m}^{0} L_{c, \tilde{\rho}^n}(s, \tilde{X}^n, \bar{Y}^n) ds + f(\tilde{X}^n(-m, 0, \mu)) \right\} \leq \\
E_{w_{1, \ldots, w_n}} \left\{ \int_{-m}^{0} \frac{1}{n} \sum_{i=1}^{n} L_c(t, X_i^n, Y_i^n) + \frac{1}{2n^2} \sum_{i \neq j}^{n} W(X_i^n - X_j^n) \right\} ds + \frac{1}{n} \sum_{i=1}^{n} f(X_i^n(-m, 0, \mu)) + \gamma_n.
\]

Since the limit of the term on the right is \(U(-m, \mu)\) by lemma 2.8, we get that

\[
\inf_{\tilde{Y}} \left\{ \int_{-m}^{0} L_{c, \tilde{\rho}}(s, \tilde{X}, \tilde{Y}) ds + f(\tilde{X}^n(-m, 0, \mu)) \right\} \leq U(-m, \mu)
\]
yielding the inequality opposite to (2.20).

\[
\Box
\]

We need the following lemma to prove the semigroup property.

**Lemma 2.9.** Let \(Y_1\) be a Lipschitz vector field on \([- (n + m), -n] \times T^p\), and let \(Y_2\) be a Lipschitz vector field on \([-n, 0] \times T^p\). Then, for all \(\epsilon, \delta \in (0, 1)\), there is a Lipschitz vector field \(Y\) which coincides with \(Y_1\) when \(t \in [- (n + m), -n]\), and with \(Y_2\) when \(t \in [-n + \delta, 0]\). Moreover, \(Y\) satisfies

\[
E_{w} \left\{ \int_{-(n+m)}^{0} L_{c, \tilde{\rho}^n}(s, X(-(n + m), s, \mu), Y(s, X(-(n + m), s, \mu))) ds \right\} \leq 
\]

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Thus, it suffices to prove that

\[ E_w \left\{ \int_{-(n+m)}^{-n} \mathcal{L}_{e^{\frac{1}{2} \rho_Y}} (s, X_1(-(n+m), s, \mu), Y_1(s, X_1(-(n+m), s, \mu))ds \right\} +
\]

\[ E_w \left\{ \int_{-n}^{0} \mathcal{L}_{e^{\frac{1}{2} \rho_Y}} (s, X_2(-n, s, \rho_Y(-n)), Y_2(s, X_2(-n, s, \rho_Y(-n)))ds \right\} + \epsilon. \quad (2.22) \]

In the formula above, \( X_1 \) solves \( (SDE)_{-(n+m)}, \nu, \mu \), \( X_2 \) solves \( (SDE)_{-m,\nu,\rho_Y} \), \( X \) solves \( (SDE)_{-(n+m), Y, \mu} \) and \( \rho_1, \rho_2, \rho_Y \) are the densities of the laws of \( X_1, X_2 \) and \( X \) respectively.

Moreover, we can require that

\[ |U(\rho_Y(0)L^\rho) - U(\rho_Y(0)L^\rho)| \leq \epsilon. \quad (2.23) \]

**Proof.** Let \( \delta \in (0, \delta) \); it is always possible to find a Lipschitz vector field \( Y \) coinciding with \( Y_1 \) on \([- (n + m), -n] \times \mathbb{T}^p \) and with \( Y_2 \) on \([-n + \delta, 0] \times \mathbb{T}^p \), and such that \( ||Y||_{\infty} \) is bounded uniformly in \( \delta \); we forego the easy proof of this fact.

We note that \( X = X_1 \) when \( s \in [-(n+m),-n] \), since both functions solve the same stochastic differential equation; as a consequence,

\[ E_w \left\{ \int_{-(n+m)}^{-n} \mathcal{L}_{e^{\frac{1}{2} \rho_Y}} (s, X_1(-(n+m), s, \mu), Y(s, X_1(-(n+m), s, \mu))ds \right\} =
\]

\[ E_w \left\{ \int_{-n}^{-n} \mathcal{L}_{e^{\frac{1}{2} \rho_Y}} (s, X_1(-(n+m), s, \mu), Y(s, X_1(-(n+m), s, \mu))ds \right\}. \]

Thus, it suffices to prove that

\[ E_w \left\{ \int_{-n}^{0} \mathcal{L}_{e^{\frac{1}{2} \rho_Y}} (s, X_2(-n, s, \rho_Y(-n)), Y_2(s, X_2(-n, s, \rho_Y(-n)))ds \right\} \leq
\]

\[ E_w \left\{ \int_{-n}^{0} \mathcal{L}_{e^{\frac{1}{2} \rho_Y}} (s, X_2(-n, s, \rho_Y(-n)), Y_2(s, X_2(-n, s, \rho_Y(-n)))ds \right\} + \epsilon. \quad (2.24) \]

To prove this, we recall that \( X \) and \( X_2 \) solve two stochastic differential equations with drift \( Y \) and \( Y_2 \) respectively; this means that, for \( s \geq -n \) and any trajectory \( w \) of the Brownian motion, we have that

\[ X(-n, s, \rho_Y(-n))(w) = X(-n, -n, \rho_Y(-n))(w) + \int_{-n}^{s} Y(\tau, X_2(-n, \tau, \rho_Y(-n))(w))d\tau + w(s) - w(-n) \quad (2.25) \]

and

\[ X_2(-n, s, \rho_Y(-n))(w) = X_2(-n, -n, \rho_Y(-n))(w) + \int_{-n}^{s} Y_2(\tau, X_2(-n, \tau, \rho_Y(-n))(w))d\tau + w(s) - w(-n). \quad (2.26) \]

Since \( X_2(-n, -n, \rho_Y(-n)) \) and \( X(-n, -n, \rho_Y(-n)) \) have the same law \( \rho_Y(-n) \), we can as well suppose that

\[ X_2(-n, -n, \rho_Y(-n))(w) = X(-n, -n, \rho_Y(-n))(w) \quad (2.27) \]
for all realizations $w$ of the Brownian motion. Subtracting (2.26) from (2.25) and using the formula above, we get the inequality below; the equality is the definition of the function $a$.

$$|X(-n, s, \rho_{Y_1}(-n))(w) - X_2(-n, s, \rho_{Y_1}(-n))(w)| \leq$$

$$\int_{-n}^{s} |Y(\tau, X(-n, \tau, \rho_{Y_1}(-n))(w)) - Y_2(\tau, X_2(-n, \tau, \rho_{Y_1}(-n))(w))|d\tau =$$

$$\int_{-n}^{s} a(\tau, X(-n, \tau, \rho_{Y_1}(-n))(w), X_2(-n, \tau, \rho_{Y_1}(-n))(w))|d\tau$$

Since $Y$ and $Y_2$ are bounded uniformly in $\bar{\delta}$, we get that $|a| \leq M$ if $\tau \in [-n, 0]$ for a constant $M$ independent on $\bar{\delta}$; since $Y$ coincides with the Lipschitz $Y_2$ on $[-n + \bar{\delta}, 0]$, we get that, for $\tau \geq -n + \bar{\delta}$,

$$|a(\tau, x, y)| \leq K|x - y|$$

for a constant $K$ independent on $\bar{\delta}$. From the last two formulas, we get that

$$|X(-n, s, \rho_{Y_1}(-n))(w) - X_2(-n, s, \rho_{Y_1}(-n))(w)| \leq$$

$$\int_{-n}^{-n + \bar{\delta}} M|\tau| + \int_{-n + \bar{\delta}}^{s} K|X(-n, \tau, \rho_{Y_1}(-n))(w) - X_2(-n, \tau, \rho_{Y_1}(-n))(w)|d\tau.$$

Using the Gronwall lemma and (2.27), we get that there is a function $\gamma(\bar{\delta})$, tending to zero as $\bar{\delta}$ tends to zero, such that

$$|X(-n, s, \rho_{Y_1}(-n))(w) - \tilde{X}(-n, s, \rho_{Y_1}(-n))(w)| \leq \gamma(\bar{\delta})$$

for all realizations $w$ of the Brownian motion. From this, (2.24) follows easily.

On the other hand, it is easy to see that the formula above implies that, as $\bar{\delta} \to 0$, $\rho_Y(0)\mathcal{L}^p$ converges weak* to $\rho_{Y_2}(0)\mathcal{L}^p$. Since $U$ is Lipschitz for the 1-Wasserstein distance, (2.23) follows.

\\

**Proposition 2.10.** 1) The map $\Psi^m_c$ defined in the introduction has the semigroup property, i.e. for $n, m \geq 0$ and $U \in C(\mathcal{M}_1(\mathbb{T}^p), \mathbb{R})$,

$$\Psi^{n+m}_c U = \Psi^n_c \circ \Psi^m_c U.$$  

2) If $U \leq V \in C(\mathcal{M}_1(\mathbb{T}^p), \mathbb{R})$, then $\Psi^m_c U \leq \Psi^m_c V.$

3) For all $a \in \mathbb{R}$ and $U \in C(\mathcal{M}_1(\mathbb{T}^p), \mathbb{R})$, $\Psi^m_c (U + a) = (\Psi^m_c U) + a$.

**Proof.** Properties 2) and 3) follow in a standard way from the definition of $\Psi^m_c$; we prove 1).

Let $\mu \in \mathcal{M}_1$; by the definition of $\Psi^{n+m}_c U$ as an infimum, for any $\epsilon > 0$ we can find a Lipschitz vector field $Y$ such that

$$\Psi^{n+m}_c U(\mu) \geq E_w \left\{ \int_{-(n+m)}^{0} \mathcal{L}_{c, \frac{1}{2} \rho_Y}(s, X(-(n + m), s, \mu), Y(s, X(-(n + m), s, \mu)))ds \right\} + U(\rho_Y(0)\mathcal{L}^p) - \epsilon$$

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where $X$ solves $(SDE)_{-(n+m),Y,\mu}$ and $\rho_Y$ is, as usual, the solution of the Fokker-Planck equation with initial condition $\mu$. By the Chapman-Kolmogorov formula, the formula above implies the first inequality below.

$$
(\Psi^{n+m}_cU)(\mu) \geq E_w \left\{ \int_{-(n+m)}^{n} \mathcal{L}_{c,\frac{1}{2}\rho_Y} (s, X(-(n+m), s, \mu), Y(s, X(-(n+m), s, \mu)))ds \right\} + 
E_w \left\{ \int_{-n}^{0} \mathcal{L}_{c,\frac{1}{2}\rho_Y} (s, X(-n, s, \rho_Y(-n)), Y(s, X(-n, s, \rho_Y(-n))))ds \right\} + U(\rho_Y(0)\mathcal{L}^p) - \epsilon \geq 
E_w \left\{ \int_{-(n+m)}^{-n} \mathcal{L}_{c,\frac{1}{2}\rho_Y} (s, X(-(n+m), s, \mu), Y(s, X(-(n+m), s, \mu)))ds \right\} + (\Psi^n_cU)(\rho_Y(-n)\mathcal{L}^p) - \epsilon \geq 
(\Psi^n_c \circ \Psi^n_c U)(\mu) - \epsilon.
$$

The second and third inequalities above come from the definition of $\Psi^n_cU$ and $\Psi^n_c \circ \Psi^n_c U$ as infima. Since $\epsilon$ is arbitrary, this means that

$$
(\Psi^{n+m}_cU)(\mu) \geq (\Psi^n_c \circ \Psi^n_c U)(\mu).\quad (2.28)
$$

We prove the opposite inequality. By the definition of $\Psi^n_c \circ \Psi^n_c(U)$, we can find a Lipschitz vector field $Y_1$ such that

$$
(\Psi^n_c \circ \Psi^n_c U)(\mu) \geq E_w \left\{ \int_{-(m+n)}^{-(n+m)} \mathcal{L}_{c,\frac{1}{2}\rho_{Y_1}} (s, X_1(-(n+m), s, \mu), Y_1(s, X_1(-(n+m), s, \mu)))ds \right\} + 
\Psi^n_c(U)(\rho_{Y_1}(-n)\mathcal{L}^p) - \epsilon.
$$

By the definition of $\Psi^n_cU$, we can find another Lipschitz vector field $Y_2$ such that

$$
\Psi^n_cU(\rho_{Y_1}(-n)) \geq E_w \left\{ \int_{-n}^{0} \mathcal{L}_{c,\frac{1}{2}\rho_{Y_2}} (s, X_2(-n+m), s, \rho_{Y_1}(-n)), Y_2(s, X_2(-n+m), s, \rho_{Y_1}(-n)))ds \right\} + 
U(\rho_{Y_2}(0)\mathcal{L}^p) - \epsilon.
$$

Let $\epsilon, \delta > 0$; by lemma 2.9, we can find a Lipschitz vector field $Y$ equal to $Y_1$ on $[-(n+m), -n] \times \mathbb{T}^p$ and to $Y_2$ on $[-n + \delta, 0] \times \mathbb{T}^p$, such that (2.22) and (2.23) holds. The first inequality below comes from the definition of $\Psi^{n+m}_cU$ as a infimum; the second one from (2.22) and (2.23); the third and fourth ones come from (2.30) and (2.29) respectively.

$$
(\Psi^{n+m}_cU)(\mu) \leq E_w \left\{ \int_{-(n+m)}^{0} \mathcal{L}_{c,\frac{1}{2}\rho_Y} (s, X(-(n+m), s, \mu), Y(s, X(-(n+m), s, \mu)))ds \right\} + U(\rho_Y(0)\mathcal{L}^p) \leq 
E_w \left\{ \int_{-(n+m)}^{-n} \mathcal{L}_{c,\frac{1}{2}\rho_{Y_1}} (s, X_1(-(n+m), s, \mu), Y_1(s, X_1(-(n+m), s, \mu)))ds \right\} + 
E_w \left\{ \int_{-n}^{0} \mathcal{L}_{c,\frac{1}{2}\rho_{Y_2}} (s, X_2(-n, s, \rho_{Y_1}(-n)), Y_2(s, X_2(-n, s, \rho_{Y_1}(-n)))ds \right\} + U(\rho_{Y_2}(0)\mathcal{L}^p) + 2\epsilon \leq 
E_w \left\{ \int_{-(n+m)}^{-n} \mathcal{L}_{c,\frac{1}{2}\rho_{Y_1}} (s, X_1(-(n+m), s, \mu), Y_1(s, X_1(-(n+m), s, \mu)))ds \right\} + (\Psi^n_cU)(\rho_{Y_1}(-n)\mathcal{L}^p) + 3\epsilon \leq 
$$
$(\Psi^n_c \circ \Psi^m_c U)(\mu) + 4\epsilon$.

Since $\epsilon > 0$ is arbitrary, we get the inequality opposite to (2.28), and thus the thesis.

§3

Fixed points

As in [8] and in [15], the following proposition is essential in proving theorem 4.

**Proposition 3.1.** Let $U$ be linear as in theorem 2. Then, there is $L > 0$, independent on $n$, such that $\Lambda^n_c U = \Psi^n_c U$ is $L$-Lipschitz for the Wasserstein distance $d_1$.

To prove this proposition, we shall need two lemmas.

**Lemma 3.2.** Let $R \in C([-m,0], M_1(T^p))$ and let $u$ solve $(HJ)_{R,f}$. Then, there is $C > 0$, independent both of $m \in \mathbb{N}$ and of $R \in C([-m,0], M_1(T^p))$, such that $||\partial_x u(t, \cdot)||_{C^1([-m,0], C(T^p))} + ||\partial_x u(t, \cdot)||_{C^1([-m,0], C^2(T^p))} \leq C$.

**Proof.** We have seen in section 1 that, if $u$ is a solution of $(HJ)_{R,f}$, $v = e^{-\beta a}$ and $a \in \mathbb{R}$, then $e^{-\beta a} v = e^{-\beta (u+a)}$ is a solution of $(TS)_{R,e^{-\beta (f+a)}}$ with $A = 0$. Let $a_k \in \mathbb{R}$ such that $e^{-\beta a_k} v(-k, \cdot) = e^{-\beta (u+a_k)}$ satisfies (1.7) for $k = 0, 1, 2, \ldots$. By the Feynman-Kac formula, for $k \geq 0$,

$$e^{-\beta a_k} v(-k - 1, \cdot) = L_{(\psi, 0, -1)}(e^{-\beta a_k} v(-k, \cdot)). \quad (3.1)$$

Since $e^{-\beta a_k} v(-k, \cdot)$ satisfies (1.7), formulas (1.10) and (1.11) hold and we get that, for $k \geq 0$,

$$\frac{1}{C_1} \leq e^{-\beta a_k} v(-k - 1, x) \leq C_1 \quad \forall x \in T^p.$$

We consider (3.1) with $e^{-\beta a_k} v(-k - 1, x)$ on the right hand side and differentiate under the integral sign; proceeding as in lemma 1.3, and using the last formula, we get that, for $k \geq 0$,

$$||e^{-\beta a_k} v(-k - 2, \cdot)||_{C^3(T^p)} \leq C_2.$$

As in lemma 1.4, this implies that there is $C_3 > 0$, independent on $k \geq 0$ (it depends only on $C_1$ and $C_2$) such that, for $k \geq 0$,

if $t \in [-k + 3, -(k + 2)]$, then $\frac{1}{C_3} \leq e^{-\beta a_k} v(t, \cdot) \leq C_3$ and

$$||e^{-\beta a_k} v(t, \cdot)||_{C^1([-k+3, -(k+2)], C^3(T^p))} + ||e^{-\beta a_k} v(t, \cdot)||_{C^1([-k+3, -(k+2)], C^1(T^p))} \leq C_3.$$
By our definition of $v$,

$$
\text{for } t \in [-{(k+3)}, -(k+2)], \quad u = -\frac{1}{\beta} \log(e^{-\beta ax v}) - a_k.
$$

From the two formulas above, we get that

$$
||\partial_x u||_{C([-m,-2], C^3(T^p))} + ||\partial_x u||_{C([-m,-2], C(T^p))} \leq C \quad \text{for } t \leq -2.
$$

It remains to bound $\partial_x u(t,x)$ when $t \in [-2,0]$; since $f \in C^3(T^p)$, this follows by differentiation under the integral sign in (1.20), and we are done.

\[\]

We recall some notation: in the following $U^n(-m,z)$ will be the minimum in (2.6); moreover, given $z = (z_1, \ldots, z_n) \in (R^p)^n$, we set

$$
|z|_1 = |z_1| + \ldots + |z_n|
$$

and we define $z' \in (R^p)^{n-1}$ by $z = (z_1, \ldots, z_n) = (z_1, z')$.

**Lemma 3.3.** Let $U$ be as in theorem 2 and let $U^n(-m,z)$ be defined as in (2.6). Then, there is a constant $C > 0$ such that, for all positive integers $n$ and $m$, we have

$$
|U^n(-m,z) - U^n(-m,\tilde{z})| \leq \frac{C}{n} |z - \tilde{z}|_1.
$$

**Proof.** It suffices to prove that, for $i = 1, \ldots, n$, the function

$$
: z_i \to U^n(-m,z_1,\ldots, z_{i-1},z_{i+1},\ldots, z_n)
$$

is $\frac{C}{n}$-Lipschitz and that the constant $C$ does not depend neither on $m \in \mathbb{N}$ nor on $(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n) \in (T^p)^{n-1}$. We shall prove this for $i = 1$; from this the general case follows, since $U$ is a symmetric function of $(z_1, \ldots, z_n)$.

We write the function $U^n(-m,(z_1, z'))$ as in lemma 2.6, isolating particle $z_1$:

$$
U^n(-m,(z_1, z')) = E_{w_1,\ldots, w_n} \left\{ \frac{1}{n} \sum_{j \neq 1} \int_{-m}^{0} L_c(s,X_j(-m,s,z_j),Y_j(s,X_j(-m,s,z_j)))ds + \right.
$$

$$
\left. \frac{1}{2n^2} \sum_{j \neq 1} \int_{-m}^{0} W(X_j(-m,s,z_1) - X_j(-m,s,z_j))ds + \frac{1}{n} \sum_{j \neq 1} f(X_j(-m,0,z_j)) \right\} +
$$

$$
\frac{1}{n} E_{w_1} \left\{ \int_{-m}^{0} L_{c,Y,1}^n(s,X_1(-m,s,z_1),Y_1(s,X_1(-m,s,z_1)))ds + f(X_1(-m,0,z_1)) \right\}
$$

where the vector field $Y = (Y_1, \ldots, Y_n)$ is minimal. By the definition of $U^n(-m,(\tilde{z}_1, z'))$ as a minimum, we get that

$$
U^n(-m,(\tilde{z}_1, z')) \leq E_{w_1,\ldots, w_n} \left\{ \frac{1}{n} \sum_{j \neq 1} \int_{-m}^{0} L_c(s,X_j(-m,s,z_j),Y_j(s,X_j(-m,s,z_j)))ds + \right.
$$

$$
\left. \frac{1}{2n^2} \sum_{j \neq 1} \int_{-m}^{0} W(X_j(-m,s,\tilde{z}_1) - X_j(-m,s,z_1))ds + \frac{1}{n} \sum_{j \neq 1} f(X_j(-m,0,\tilde{z}_1)) \right\} + 
$$

$$
\frac{1}{n} E_{w_1} \left\{ \int_{-m}^{0} L_{c,Y,1}^n(s,X_1(-m,s,\tilde{z}_1),Y_1(s,X_1(-m,s,\tilde{z}_1)))ds + f(X_1(-m,0,\tilde{z}_1)) \right\}
$$

(3.2)
\[
\frac{1}{2n^2} \sum_{i,j \neq 1} \int_{-m}^{0} W(X_i(-m, s, z_i) - X_j(-m, s, z_j))ds + \frac{1}{n} \sum_{j \neq 1} f(X_j(-m, 0, z_j)) + \\
\frac{1}{n} E_{w, \epsilon} \left\{ \int_{-m}^{0} L_{c,Y,1}^n(s, X_i(-m, s, \tilde{z}_i), Y_i(s, X_i(-m, s, \tilde{z}_i)))ds + f(X_i(-m, 0, \tilde{z}_i)) \right\}.
\]  

(3.3)

The term with \(E_{w_1, \ldots, w_n}\) is identical in (3.2) and (3.3); defining \(u_1^n\) as in corollary 2.7, we see that the term with \(E_{w_1}\) is equal to the function \(u_1^n(-m, z_1)\) in (3.2), and to \(u_1^n(-m, \tilde{z}_1)\) in (3.3); subtracting (3.2) from (3.3), we get that

\[
U^n(-m, (\tilde{z}_1, z')) - U^n(-m, (z_1, z')) \leq \frac{1}{n} [u_1^n(-m, \tilde{z}_1) - u_1^n(-m, z_1)] \leq \frac{C}{n} |\tilde{z}_1 - z_1|
\]

where the last inequality comes from lemma 3.2. Exchanging the roles of \(z_1\) and \(\tilde{z}_1\), we get that the function \(z_1 \rightarrow U^n(-m, (z_1, z'))\) is \(\frac{C}{n}\)-Lipschitz; we saw at the beginning of the proof that this implies the thesis.

\\

**Proof of proposition 3.1.** By lemma 2.8, we know that

\[
\text{if } \frac{1}{n}(\delta_{z_1} + \ldots + \delta_{z_n}) \rightarrow \mu \text{ then } U^n(-m, z_1, \ldots, z_n) \rightarrow U(-m, \mu).
\]

We saw in (2.18) that \(U(-m, \mu) = (\Lambda_{c, \mu})^n U(\mu)\). Thus it suffices to show that

\[
|U^n(-m, (x_1, \ldots, x_n)) - U^n(-m, (y_1, \ldots, y_n))| \leq L d_1 \left( \frac{1}{n}(\delta_{x_1} + \ldots + \delta_{x_n}), \frac{1}{n}(\delta_{y_1} + \ldots + \delta_{y_n}) \right).
\]

It is standard ([5]) that

\[
d_1\left(\frac{1}{n}(\delta_{x_1} + \ldots + \delta_{x_n}), \frac{1}{n}(\delta_{y_1} + \ldots + \delta_{y_n})\right) = \min_{\sigma} \frac{1}{n} \sum_{i=1}^{n} |x_i - y_{\sigma(i)}|
\]

(3.7)

where the minimum is taken over all the permutations \(\sigma\) of \([1, \ldots, n]\). In terms of transport, when we are connecting two \(n\)-uples of deltas, there is not just a minimal transfer plan, but a minimal transfer map.

Since

\[
U^n(-m, (y_1, \ldots, y_n)) = U^n(-m, (y_{\sigma(1)}, \ldots, y_{\sigma(n)})�,)
\]

we have to prove that, for \(\sigma\) minimal in (3.7),

\[
|U^n(-m, (x_1, \ldots, x_n)) - U^n(-m, (y_{\sigma(1)}, \ldots, y_{\sigma(n)}))| \leq C \frac{1}{n} \sum_{i=1}^{n} |x_i - y_{\sigma(i)}|.
\]

But this is an immediate consequence of lemma 3.3.

\\

When \(U\) is linear, we define \(\Lambda_{c, \lambda} U = \Lambda_c U + \lambda\); thus, in case of a linear \(U\), we have that \(\Lambda_{c, \lambda} U = \Psi_{c, \lambda} U\) for the operator \(\Psi_{c, \lambda}\) defined in the introduction. In the next lemma, we stick to the \(\Lambda_{c, \lambda}\) notation.
Lemma 3.4. Let the operator $\Lambda_{c,\lambda}$ be defined as in the introduction and let $U = 0$. Then, there is a unique $\lambda \in \mathbb{R}$ such that

$$\hat{U}(\mu) = \liminf_{n \to +\infty} (\Lambda_{c,\lambda}^n 0)(\mu)$$

is finite for all $\mu \in \mathcal{M}_1$. Moreover, $\hat{U}$ is $L$-Lipschitz for the constant $L$ of proposition 3.1.

Proof. Clearly, there is at most one $\lambda \in \mathbb{R}$ for which the lim inf above is finite; let us prove that it exists. This means finding $\lambda \in \mathbb{R}$ such that, for all $\mu \in \mathcal{M}_1(T^p)$,

$$-\infty < \liminf_{n \to +\infty} (\Lambda_{c,\lambda}^n 0)(\mu) < +\infty. \quad (3.8)$$

Note that the formula above implies that $\hat{U}$ is finite; it is $L$-Lipschitz because it is the lim inf of $L$-Lipschitz functions.

By proposition 3.1, $\Lambda_{c,0}^n 0$ is $L$-Lipschitz for all $n \in \mathbb{N}$; since $\mathcal{M}_1(T^p)$ is a compact metric space, we can find $M > 0$ such that

$$\max \Lambda_{c,0}^n 0 - \min \Lambda_{c,0}^n 0 \leq M \quad \forall n \geq 1. \quad (3.9)$$

Possibly taking a larger $M$, we can suppose that

$$||\Lambda_{c,0}^1 0||_{\sup} \leq M.$$ 

By point 2) of proposition 2.10, this implies the first inequality below; the equality follows by point 1), and the second inequality by point 3) of the same proposition.

$$(\Lambda_{c,0}^2 0)(\mu) = (\Lambda_{c,0}^1 (\Lambda_{c,0}^1 0))(\mu) \leq \Lambda_{c,0}^1 (0 + M) \leq 2M.$$ 

Exchanging signs, this implies that

$$||\Lambda_{c,0}^2||_{\sup} \leq 2M.$$ 

Iterating, we get

$$||\Lambda_{c,0}^n||_{\sup} \leq nM \quad \forall n \geq 1. \quad (3.10)$$

We set

$$a_n = \min_{\mu} (\Lambda_{c,0}^n 0)(\mu) \quad \text{and} \quad -\lambda = \liminf_{n \to +\infty} \frac{a_n}{n}. \quad (3.11)$$

From (3.10), it follows that $\lambda \in [-M, M]$. We assert that $\lambda$ satisfies (3.8). We prove the inequality on the left of (3.8), since the one on the right is analogous; actually, we are going to prove that, for all $\mu \in \mathcal{M}_1(T^p)$ and all $n \in \mathbb{N}$,

$$(\Lambda_{c,\lambda}^n 0)(\mu) > -10M.$$ 

Indeed, let us suppose by contradiction that, for some $m \in \mathbb{N}$ and $\bar{\mu} \in \mathcal{M}_1(T^p)$, we have

$$(\Lambda_{c,\lambda}^m 0)(\bar{\mu}) \leq -10M.$$ 

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By (3.9), this implies that, for all \( \mu \in M_1(T^p) \),
\[
(\Lambda^{m}_{c,\lambda}(\mu)) \leq -9M. \tag{3.12}
\]
Let \( \mu \in M_1(T^p) \); the first inequality below comes from (3.12) and points 1) and 2) of proposition 2.10, the equality from point 3) of the same proposition, the last inequality from (3.12).
\[
(\Lambda^{2m}_{c,\lambda}(\mu)) \leq [\Lambda^{m}_{c,\lambda}(-9M)](\mu) = -9M + (\Lambda^{m}_{c,\lambda}(\mu)) \leq -18M.
\]

Proceeding by induction, we find that
\[
(\Lambda^{km}_{c,\lambda}(\mu)) \leq -9kM \quad \forall \mu \in M_1(T^p).
\]
Since
\[
(\Lambda^{km}_{c,\lambda}(\mu)) = (\Lambda^{km}_{c,\lambda}(\mu)) + km\lambda,
\]
we get that
\[
(\Lambda^{km}_{c,\lambda}(\mu)) \leq -9kM - (km)\lambda \quad \forall \mu \in M_1(T^p).
\]
By the definition of \( \lambda \) in (3.11), this implies that \( -\lambda \leq -\lambda - \frac{9M}{m} \); this contradiction proves (3.8) and thus the lemma.

Proof of theorem 4. Let \( \hat{U} \) be as in the last lemma; since \( \hat{U} \) may not be linear, we switch to the \( \Psi^{1}_{c,\lambda} \) notation. We have to prove that \( \hat{U} \) is a fixed point of \( \Psi^{1}_{c,\lambda} \) and that a minimizing vector field exists.

Let \( \mu \in M_1(T^p) \) and \( \epsilon > 0 \); by the definition of \( \Psi^{1}_{c,\lambda} \hat{U} \), we can find a Lipschitz vector field \( \bar{Y} \) for which
\[
E_w \left\{ \int_0^1 \mathcal{L}_{\mathcal{C}^{1}_{\mathcal{P}}}(s, X, Y)ds \right\} + \hat{U}(\bar{\rho}(1)\mathcal{L}^p) + \lambda \leq (\Psi^{1}_{c,\lambda} \hat{U})(\mu) + \epsilon. \tag{3.13}
\]
To use this formula, we are going to express \( \hat{U}(\bar{\rho}(1)\mathcal{L}^p) \) by the limit of lemma 3.4.

Let \( n \in \mathbb{N} \); by theorem 3, applied to \( f = 0 \) with an obvious translation in time, we can find \( Y_n \) be such that
\[
(\Psi^{n}_{c,\lambda}0)(\mu) = \min_Y E_w \left\{ \int_1^{n+1} \mathcal{L}_{\mathcal{C}^{1}_{\mathcal{P}}}(s, X, Y)ds \right\} = E_w \left\{ \int_1^{n+1} \mathcal{L}_{\mathcal{C}^{1}_{\mathcal{P}}}(s, X_n, Y_n) \right\}
\]
where \( \rho_n \) stays for \( \rho_{Y_n} \) and \( \rho \) for \( \rho_Y \); the initial time for (SDE) and (FP) is 1. Let \( \tilde{Y} \) be equal to \( \bar{Y} \) on \([0, 1] \times T^p\), to \( Y_n \) on \([1 + \delta, n + 1] \times T^p\) and a Lipschitz connection in between. By lemma 2.9, we can choose the Lipschitz connection in such a way that
\[
E_w \left\{ \int_0^1 \mathcal{L}_{\mathcal{C}^{1}_{\mathcal{P}}}(s, \bar{X}(0, s, \mu), \bar{Y}(s, \bar{X}(0, s, \mu)))ds \right\} + \\
E_w \left\{ \int_1^{n+1} \mathcal{L}_{\mathcal{C}^{1}_{\mathcal{P}}}(s, X_n(1, s, \mu), Y_n(s, X_n(1, s, \rho_Y(1))))ds \right\} \geq
\]

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\[ E_w \left\{ \int_0^{n+1} \mathcal{L}_{c, \hat{w}}(s, \bar{X}(0, s, \mu), \bar{Y}(s, \bar{X}(0, s, \mu))) ds \right\} \geq \epsilon. \]  \tag{3.14}

Now (3.13) and the definition of \( \hat{U} \) imply the first inequality below, (3.14) the second one while the third one follows from the definition of \( \Psi^{n+1}_{c, \lambda} 0 \). The equality at the end follows by the definition of \( \hat{U} \).

\[ (\Psi^1_{c, \lambda} \hat{U})(\mu) \geq E_w \left\{ \int_0^1 \mathcal{L}_{c, \hat{w}}(s, \bar{X}(0, s, \mu), \bar{Y}) ds + \lambda \right\} + \liminf_{n \to +\infty} E_w \left\{ \int_0^{n+1} \mathcal{L}_{c, \hat{w}}(s, X_n(0, s, \mu), Y_n) ds + \lambda n \right\} - \epsilon \geq \liminf_{n \to +\infty} E_w \left\{ \int_0^{n+1} \mathcal{L}_{c, \hat{w}}(s, \bar{X}(0, s, \mu), \bar{Y}) ds + \lambda(n+1) \right\} - 2\epsilon \geq \liminf_{n \to +\infty} (\Psi^{n+1}_{c, \lambda} 0)(\mu) - 2\epsilon = \hat{U}(\mu) - 2\epsilon. \]

Since \( \epsilon > 0 \) is arbitrary, we get that

\[ (\Psi^1_{c, \lambda} \hat{U})(\mu) \geq \hat{U}(\mu) \quad \forall \mu \in M_1(T^p). \]  \tag{3.15}

On the other hand, let \( \mu \in M_1(T^p) \) and let \( Y_n \) minimize in the definition of \( (\Psi^{n+1}_{c, \lambda} 0)(\mu) \). Then, by the definition of \( \hat{U} \), we get the equality below.

\[ \hat{U}(\mu) = \liminf_{n \to +\infty} E_w \left\{ \int_0^{n+1} \mathcal{L}_{c, \hat{w}}(s, X_n(0, s, \mu), Y_n) ds + \lambda(n+1) \right\} \]

where \( \rho_n \) stays for \( \rho_{Y_n} \). Let \( \{n_h\} \) be a subsequence on which the \( \liminf \) is attained. By lemma 3.2, \( Y_{n_h} \) is uniformly Lipschitz. Thus, we can apply Ascoli-Arzelà and, after further refining \( \{n_h\} \), we can suppose that \( Y_{n_h} \) converges to a vector field \( \bar{Y} \) in the \( C^0 \) topology. The formula above yields the first equality below, while the second one follows from the fact that \( Y_{n_h} \) converges uniformly to \( \bar{Y} \); the first inequality follows from the definition of \( \hat{U}(\bar{\rho}(1)L^p) \), the second one from the definition of \( \Psi^1_{c, 0} \).

\[ \hat{U}(\mu) = \liminf_{h \to +\infty} E_w \left\{ \int_0^1 \mathcal{L}_{c, \hat{w}}(s, X_{n_h}(0, s, \mu), Y_{n_h}) ds + \lambda \right\} + \liminf_{h \to +\infty} E_w \left\{ \int_1^{n_h+1} \mathcal{L}_{c, \hat{w}}(s, X_{n_h}(1, 0, \rho_{n_h}(1)), Y_{n_h}) ds + \lambda n_h \right\} \geq \liminf_{h \to +\infty} E_w \left\{ \int_0^1 \mathcal{L}_{c, \hat{w}}(s, \bar{X}, \bar{Y}) ds + \lambda \right\} + \hat{U}(\bar{\rho}(1)L^p) \geq \Psi^1_{c, 0} \hat{U}(\mu) \quad \forall \mu \in M_1(T^p) \]

where \( \bar{\rho} \) stays for \( \rho_{\bar{Y}} \). This proves the inequality opposite to (3.15). Thus, \( \hat{U} = \Psi^1_{c, 0} \hat{U} \); by the last formula, this implies that \( \bar{Y} \) satisfies (5).

It remains to prove that the constant \( \lambda \) is unique. Let \( \Psi^1_{c, \lambda_1} \hat{U}_1 = \hat{U}_1 \) and \( \Psi^2_{c, \lambda_2} \hat{U}_2 = \hat{U}_2 \). Let us suppose by contradiction that \( \lambda_1 < \lambda_2 \). Since \( \hat{U}_i \) is a continuous fixed point, we can suppose that \( \|\hat{U}_i\|_{\sup} \leq M \) for
\(i = 1, 2\); as a consequence, \(\hat{U}_2 \geq \hat{U}_1 - 2M\). By proposition 2.10, the first inequality below follows; the second equality follows from the fact that \(\hat{U}_1\) is a fixed point.

\[
\Psi_{c,\lambda_2}^n \hat{U}_2 \geq \Psi_{c,\lambda_2}^n \hat{U}_1 - 2M \geq \Psi_{c,\lambda_1}^n \hat{U}_1 + n(\lambda_2 - \lambda_1) - 2M = \\
\hat{U}_1 + n(\lambda_2 - \lambda_1) - 2M \geq \hat{U}_2 + n(\lambda_2 - \lambda_1) - 4M.
\]

For \(n\) large enough, the last formula contradicts the fact that \(\hat{U}_2\) is a fixed point.

\[
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\]
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