Ordinary Complex Differential Equations with Applications in Science and Engineering

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Abstract

In this work, we spotted the light on one of the really important concepts and turned it to a mathematical branch instead of separate equations studied individually in different specializations of science. The existence and uniqueness of solutions for complex differential equations have been proved with many mathematical generalized tools. Homotopy Perturbation Method has been used and implemented as a method for solving linear complex differential equations with which is the first time such a method used to solve an equation in the complex plane. An analytical method of solutions has been investigated deeply with complex series solutions have generalized also using Laurent series expansions. Some really important application in engineering and complex geometry has been implemented with deep understanding, such as Airfoil application for airplane and spaceships wings and the fans of jet engines, and the Schwarz-Christoffel transformation.
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Chapter 1

Background

This chapter introduces the background needed in the preceding chapters and for that purpose, we are dependent on many references research papers. For complex analysis, we based on the concepts in [53], [30], [31], [9] and [29], where almost all what we needed about complex analysis where there. In terms of functional analysis we took the main concepts and theories the well known references mentioned in the following [46], [8], [12], [47], [14], [5], [2], [22], [13] [26]. In terms of differential equation and its basic concepts we based on the most recent book written in this scope of science, some of them are listed here ([40]), ([15]), ([18]), ([39]), ([24]).

1.1 Literature Review

Complex analysis and differential equations have been playing a major significant role in modern physics and engineering modulations and its applications especially in quantum mechanics, fluid mechanics, nuclear engineering and complex geometry. Most of the mathematical models use differentiation to form and calculation the rate of change involved in that particular model, and its concept so clear and has no mysteries for the none mathematics researchers, but when we turn out attention to the differentiation of the functions in the complex domain the will be complicated a and needs to be categorized and ordered in a well known unique title which it's already has been in this work that none of the others is alike. There is a lot of work and researches has been done about differential equations in the real line and also there is much work that is considered as a complex analysis has been achieved, but as a differential equations with complex dependent and independent variables never been showed as a specific branch by it one self. First work came as illustration for
the theoretical concept and properties by ([43]), this book was just a skim for the idea of ordinary differential equations in the complex domain with no methods of solution and no applications and any practical notification. After that ([44]) was just expansion and extension of [43] who touched on the same concepts and theorems but with deeper theoretical taste. As we tracking the development of the The wheel of evolution in terms of analytical solution of complex differential equations no such work has been mentioned to before, but in 2006 there is some tries of solving some specific and restricted parameters complex differential equations numerically to fined an approximations for the actual solution in certain domain, which is done mostly by physicists and engineers ([41]) and ([42]). In 20013 a collection of methods for some numbers of complex differential equations in a circular complex domain has appeared to the show ([50]). Quantum physics has been always the perfect field for complex analysis and complex differential operators, in 2013 ([3]) studied some Numerical study of quantized vortex interaction for some types of equations on bounded domains. In 2015 a really extraordinary aspect has been revealed by ([21]) when they studied the oscillation of radical solutions of some types of complex differential equations. Signal processing researcher took in 2017 a very good offered to study the Complex-valued differential operator in his paper ([17], [7]) to investigate multicomponent signal separation. Finally in 2017 a quantum physicist gave fantastic relation of the linear complex differential equations and Schrödinger equations in his work, ([27]). In this work we instructing a full classification and mathematically codified of complex deferential equations with deeply investigated theoretical reasoning and methods of solutions.
1.2 Outline of Thesis

In chapter one a wide background information and concepts have been deeply reviewed and represented in a particular way that makes it in a harmony with the main target of the whole work. The chapter is classified to a number of sections according to the needs of the main work. An introduction to functional analysis, complex analysis, and differential equations. In section one we had a background about complex analysis with subsections listed as complex functions, power series, Cauchy integrals, and analytical continuation. In subsection two we skimmed the concept of differential equations. Section three took over the functional analysis with subsections initiated with metric spaces, and went on to linear transformations, fixed point theorem, and functional inequalities. In chapter two the theoretical concepts have been serially generalized for complex field and proofed. The start was the existence and uniqueness of complex solution by using a several of theories and then the singularities in its both types: fixed and movable, analytical continuation, Painleve’s determination theorem. Chapter three contains all the unique work that has never been mentioned as the way it is in this work. Methods of solution take place in this chapter. We investigated and created methods of analytical solutions for many popular types of linear complex differential equations. And then we generalized Frobenius method. In chapter four we revealed the importance of the work this research all about, and how the modern engineering and geometry involve complex differential equations of the studied kind. Application in mathematical mapping is Schwarz-Christoffel Transformation that maps the real line on to a n’the polygon in the complex plane. Airfoil shape where the design of the airplane wing as, which is called Joukowski Transformation.

1.3 Complex Analysis

Complex analysis plays a major role in the modern application in physics and engineering, so in this section, we are showing some essential concepts as a background for the related work in the rest of the thesis. We started with the definition of the function, which is the main brick in our building and then we illustrate the important core theories for the modern complex analysis and its interpretation in terms of functional analysis matter.
1.3.1 Complex Functions

Definition 1.3.1. The complex variable \( z = x + iy \) is represented geometrically by the points of \( \mathbb{R}^2 \), the euclidean plane, so that to \( z_0 = x_0 + iy_0 \) in \( \mathbb{C} \) corresponds the point \((x_0, y_0)\) of \( \mathbb{R}^2 \). We have

\[
z = r(\cos(\theta) + i\sin(\theta)) \equiv re^{i\theta}
\]

(1.1)

where

\[
r = (x^2 + y^2)^{1/2}, \quad \theta = \tan^{-1} \frac{y}{x}
\]

(1.2)

We write \( x = Re(z) \) (read “\( x \) is the real part of \( z \)”), \( y = Im(z) \) (read “\( y \) is the imaginary part of \( z \)”). Furthermore, \( r = |z| \) is the absolute value of \( z \), and \( \theta = \arg(z) \) is the argument of \( z \), which is determined only up to multiples of \( 2\pi \).

Property 1.3.1. For every complex number \( z \in \mathbb{C} \) and for every real number \( a \in \mathbb{R} \) we have

\[
|z_1 + z_2| \leq |z_1| + |z_2|, \quad |az| = |a||z|, \quad |z_1z_2| = |z_1||z_2|
\]

(1.3)

Definition 1.3.2. Let \( z_0 = x_0 + iy_0 \). And hence \( |z - z_0| = (x - x_0)^2 + (y - y_0)^2 \) is the absolute distance between the two points \( z = x + iy \) and \( z_0 = x_0 + iy_0 \), the points \( z = x + iy \) will be the points that satisfy the following,

\[
|z - z_0| < \rho \quad \rho > 0,
\]

(1.4)

located on the circle of radius \( \rho \) and has the center \( z_0 \) See Figure 1.2

Example 1.1 (Circles In Complex Plan).
1.3 Complex Analysis

Figure 1.2 Circle in Complex Plan

(1) $|z| = 1$ is a circle of 1 unit radius and center $0 + 0i$.

(2) By rearrange $|z - 1 + 3i| = 5$ to make it as $|z - (1 - 3i)| = 5$, we can notice from (1.4) that the first equation represents a circle of 5 units radius and has a center at the complex point $z_0 = 1 - 3i$.

**Definition 1.3.3.**
The points $z$ which satisfy the following inequality $|z - z_0| \leq \rho$ has two possibilities, this point can be on the rim of the circle $|z - z_0| = \rho$ or can be located inside the circle. We always say that any set of complex points which satisfy the inequality $|z - z_0| \leq \rho$ is a disk on the complex plain of radius $\rho$ and has the center at $z_0$. But the complex points $z$ that defined by the restricted inequality $|z - z_0| < \rho$ located inside the circle and not on its rim, a circle of radius $\rho$ and has a center at the point $z_0$. This set of complex points is named a **neighborhood** of $z_0$. Accidentally, we are going to need using a **neighborhood** of $z_0$ that also means that $z_0$ be inside the neighborhood. Such a neighborhood is satisfied by the inequality $0 < |z - z_0| < \rho$ and is named a **deleted neighbor** of $z_0$. For instance, $|z| < 1$ means a neighborhood in complex plane of the origin, when $0 < |z| < 1$ means a deleted neighborhood in complex plane of the origin too; $|z - 3 + 4i| < 0.01$ means a neighborhood in complex domain of $3 - 4i$, when the inequality $0 < |z - 3 + 4i| < 0.01$ represents a deleted neighborhood in the complex plane of $3 - 4i$.

**Definition 1.3.4.**
A point in complex plane $z_0$ is said to be an interior point of $S$ a set in the complex plane if there are some neighborhood of the point $z_0$ which located entirely inside $S$. And if every point $z$ of the complex set $S$ is an interior point, then $S$ will be called an open set. See Figure 1.3.
**Definition 1.3.5.** A set $B$ is a closed set if its complement is an open set.

For instance, the following inequality $Re(z) > 1$ refers to a right half-plane, which is an open complex set. All complex elements $z = x + iy$ for that $x > 1$ are contained in this set. If we take, for instance, $z_0 = 1.1 + 2i$, then the neighborhood of $z_0$ is located entirely in the complex set that represented by $|z - (1.1 + 2i)| < 0.05$. See Figure 1.4a. The complex set $S$ of points in the complex plane satisfied by $Re(z) \geq 1$ is not open set because of every neighborhood of a point located on the line $x = 1$ has to possess points in the set $S$ and has points not contained $S$. See Figure 1.4b.

**Definition 1.3.6.** A complex function is a mapping $f$ that its domain and its co-domain are sets contained entirely in $\mathbb{C}$.

**Definition 1.3.7.** Let the complex function $f$ is defined in a deleted neighborhood of $z_0$ and let that $L$ is a number in the complex plain. The limit of the function $f$ as $z$
1.3 Complex Analysis

goes to $z_0$ exists and equals to $L$, and we write it as $\lim_{z \to z_0} f(z) = L$, if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(z) - L| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.

**Definition 1.3.8.** Let $f(z) = u(x, y) + iv(x, y)$, $z_0 = x_0 + iy_0$, and $L = u_0 + iv_0$. Then $\lim f(z) = L$ iff

$$\lim_{(x, y) \to (x_0, y_0)} u(x, y) = u_0 \quad \text{and} \quad \lim_{(x, y) \to (x_0, y_0)} v(x, y) = v_0$$

**Definition 1.3.9.** A complex function $f$ is called continuous function at a complex point $z_0$ if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

**Definition 1.3.10.** Let the complex function $f$ is defined in a neighborhood of a point $z_0$. The derivative of the function $f$ at $z_0$, denoted by $f'(z_0)$, is

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

**Definition 1.3.11.** A complex function $w = f(z)$ is said to be analytic at a point $z_0$ if $f$ is differentiable at $z_0$ and at every point in some neighborhood of $z_0$.

**Definition 1.3.12.** Suppose $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point $z = x + iy$. Then at $z$ the first-order partial derivatives of $u$ and $v$ exist and satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

### 1.3.2 Power Series

Power series served as the foundation of the function theory developed by Weierstrass. He did not invent them, but he perfected their theory. Let there be given a sequence $\{a_n\}$ of complex numbers; form the series

**Definition 1.3.13.** If $\alpha$ is a complex number and $z \neq 0$, then the complex power $z^\alpha$ is defined to be:

$$e^{\alpha \ln z}$$

**Definition 1.3.14.** A sequence $\{z_n\}$ is a function whose domain is the set of Positive integers and whose range is a subset of the complex numbers $\mathbb{C}$. 

Example 1.2 (Example of Complex Sequence). For example the sequence \( \{z_n\} = \{1 + i^n\} \) where \( n \) is an integer number, then
\[
\{z_n\} = \{1 + i, 0, 1 - i, 2 + i, \ldots\}
\]

![Figure 1.5 Example of a Complex Sequence in The Complex Plane](image)

Example 1.3 (A Convergent Complex Sequence). The sequence \( \left\{ \frac{p + 1}{n} \right\} \) is convergent since \( \lim_{n \to \infty} \frac{p + 1}{n} = 0 \). As we see from
\[
-1, -i, \frac{1}{3}, \frac{i}{4}, -\frac{1}{5}, \ldots
\]
And in figure 1.6, the terms of the sequence, marked by colored dots, spiral in toward the point \( z = 0 \) as \( n \) increases.

![Figure 1.6 A Convergent Complex Sequence in The Complex Plane](image)
Definition 1.3.15. A geometric series is any series of the form
\[ \sum_{k=1}^{k} a z^{k-1} = a + az + az^2 + \ldots + az^{n-1} + \ldots \]
where the \( n \)'th term of the sequence of partial sums is
\[ S_n = a + az + az^2 + \ldots + az^{n-1} \]

Definition 1.3.16. An infinite series \( \sum_{k=1}^{\infty} z_k \) is said to be absolutely convergent if \( \sum_{k=1}^{\infty} |z_k| \) converges. An infinite series \( \sum_{k=1}^{\infty} z_k \) is said to be conditionally convergent if it converges but \( \sum_{k=1}^{\infty} |z_k| \) diverges.

Theorem 1.3.1. Suppose \( f \) be analytic function within a domain \( D \) and suppose \( z_0 \) be a point in \( D \). Then \( f \) has the following series form
\[ f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \]
valid for the largest circle \( C \) centered at \( z_0 \) and radius \( R \) that located entirely inside \( D \). See Figure 1.7

Figure 1.7 The Contour C
**Theorem 1.3.2 (Laurent’s Theorem).** Let $f$ be analytical within the annular domain $D$ defined by $r < |z - z_0| < R$. Then $f$ has the following series representation

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z - z_0)^k$$

valid for $r < |z - z_0| < R$. The coefficients $a_k$ are given by

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s - z_0)^{k+1}} ds, \quad k = 0, \pm 1, \pm 2, \ldots,$$

Where $C$ is a simple closed curve that lies entirely within $D$ and has $z_0$ in its interior. See Figure 1.8a.

![Figures for some contours](image)

Figure 1.8 Graphs for some contours

Now we put the main theories we are going to use in late work

**Theorem 1.3.3.** Let the series

$$\sum_{n=0}^{\infty} a_n z^n$$

Suppose that there is a $z_0 \neq 0$ such that the sequence $\{|a_n z_0^n|\}$ is bounded. Then the series (1.5) is absolutely convergent for $|z| < |z_0|$, uniformly for $|z| \leq |z_0| - \delta$, $\delta > 0$.

This means that the positive reals fall into two classes:

1- $K_1$ contains those numbers $r$ for which $|a_n| r^n$ is a bounded sequence.
2. $K_2$ contains the remaining positive numbers.

There is a number $R$ which is the supremum of the numbers in class $K_1$ (equivalently, the infimum of the numbers in $K_2$), and now it is seen that the series converges absolutely for $|z| < R$ and diverges for $|z| > R$. For $R$ we have the expression

$$\frac{1}{R} = \lim_{n \to \infty} \sup |a_n|^{\frac{1}{n}}$$

(1.6)

The sequence $\{|a_n|^{\frac{1}{n}}\}$ may have a single limit, in which case we have $\frac{1}{R}$. This was the case considered by Cauchy. The sequence may, however, have more than one limit point, even infinitely many. In any case there is a largest limit point, and this is the superior limit. This $R$ is the radius of convergence of the power series. It can take any value in $[0, \infty]$.

If $R > 0$, the sum of the series is evidently a continuous function of $z$ in the circle of convergence $|z| < R$, say $f(z)$. Moreover, we can differentiate term by term to obtain the power series

$$f_1(z) = \sum_{n=1}^{\infty} na_n z^{n-1}$$

(1.7)

with the same circle of convergence. Furthermore, for $|z| \leq R_0 < R$, $|z + h| \leq R_0$ we find that

$$\left| \frac{1}{h} [f(z + h) - f(z)] - f_1(z) \right| \leq \frac{1}{2} |h| \sum_{n=2}^{\infty} n(n-1)|a_n| R_0^{n-2},$$

(1.8)

which goes to zero with $|h|$. It follows that $f(z)$ is differentiable and $f'(z) = f_1(z)$. It follows also that the sum of the power series is a holomorphic function of $z$. Using the same technique, one sees that $f(z)$ has derivatives of all orders for $|z| < R$. The function $f(z)$ can be expanded in a Taylor series about any point $a$ in the circle of convergence, and the resulting series, a power series in $z - a$, has a radius of convergence $R_a$

$$R - |a| \leq R_a \leq R + |a|$$

(1.9)

Weierstrass obtained this by elementary expansions and rearrangements. We have

$$z^n = (z - a + a)^n = \sum_{k=0}^{n} \binom{n}{k} (z - a)^k a^{n-k}$$

(1.10)

Substitution of this in (1.5) gives a double series,
This series is absolutely convergent for $|z - a| + |a| < R$. Now, an absolutely convergent double series can be summed by columns as well as by rows and, in fact, by any process that is exhaustive. Here (1.11) is the sum by rows. Summing by columns, we get a power series in $z - a$, where the coefficient of $(z - a)^k$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (z - a)^k$$

so that

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (z - a)^k,$$

and this representation is guaranteed to hold for $|z - a| < R - |a|$. Now this is a power series with a radius of convergence $R_u$ which may very well exceed $R - |a|$. In the circle of convergence, $|z - a| < R_u$, (1.10) defines a analytic function of $z$; this function coincides with $f(z)$ in the lens-shaped region of intersection of the two disks, where they are defined, and the union of the two disks is the domain of definition of a single holomorphic function which is represented by (1.5) in one disk and by (1.13) in the other. We can repeat this process for all points $a$ with $|a| < R$. We obtain a family of power series (1.13) which represent the same holomorphic function in their domains of convergence. The union of all the disks is the domain of definition that can be reached by direct rearrangements. There is at least one point on the circle of convergence $|z| = R$ which stays outside of all disks $|z - a| < R_u$ with $|a| < R$. This is a singular point, and every power series admits of at least one singular point on its circle of convergence. It is possible for all points on the boundary to be singular. This will happen iff $R_u = R - |a|$ for all $a$ with $|a| < R$. We then say that $z - R$ is the natural boundary of $f(z)$. An extreme example of this phenomenon is furnished by the series

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^{2n}$$

with $R = 1$. Here the series, as well as all all the derived series, converge absolutely on $|z| = 1$. The unit circle nevertheless is the natural boundary. Such natural boundaries are typical for so-called lacunary series, where there are long and widening gaps in the expansion.
1.3 Complex Analysis

1.3.3 Cauchy Integrals

Consider a function \( z \to f(z) \), analytic in a simply connected domain \( D \). Set

\[
f(z) = U(z) + V(z)
\]

and define the integral

\[
\int_{z_1}^{z_2} F(z)dz = \int_{z_1}^{z_2} [Udx - Vdy] + i \int_{z_1}^{z_2} [Udy + Vdx]
\]

where the path of integration is a curve joining \( z_1 \) with \( z_2 \) in \( D \) and the integrals are line integrals in the sense of the calculus. Since \( U \) and \( V \) satisfy the Cauchy-Riemann equations (1.6), Cauchy claimed that the integral is independent of the path joining \( z_1 \), and \( z_2 \), or, in other words, the integral along a closed contour \( C \) is 0:

\[
\int_C f(z)dz = 0,
\]

It is required that the curve \( C \) have an arc length (=be rectifiable) which requires a representation

\[
z = z(t) \quad 0 \leq t \leq L
\]

where \( L \) is the length of \( C \) and \( t \to z(t) \) is a continuous function of bounded variation. The integral then becomes a so-called Riemann-Stieltjes integral:

\[
\int_0^L f[z(t)]dz(t)
\]

which is the limit of Riemann-Stieltjes sums,

\[
\sum_{j=0}^{n} f[z(t_j)][z(t_j) - z(t_{j-1})]
\]

The limit exists for any continuous integrand \( f \) and integrator \( z \) of bounded variation. In this setting one proves the theorem for \( f = 1 \) and \( f = z \). Then one observes that:

i. \( C \) may be approximated arbitrarily closely by a closed polygon.

ii. a polygon may be triangulated.

iii. the theorem is proved for a small triangle.

iv. hence is true for a polygon and an arbitrary rectifiable curve.
The integral is additive with respect to the path and linear with respect to the integrand. If $D$ is not simply connected, the integral along $C$ equals a sum of integrals around the holes that are inside $C$. The Cauchy integral is an exceedingly powerful tool. If $z \to f(z)$ is analytic in a simply connected domain $D$ and on its rectifiable boundary $C$, then

$$
\frac{1}{2\pi i} \int_C \frac{f(t)dt}{(t-z)} = \begin{cases} 
0 & z \in \text{Ext}(C) \\
f(z) & z \in \text{Int}(C) \cup C
\end{cases}
$$

We can differentiate under the sign of integration as often as we please, and the formal $n$'th derivative represents $f^n(z)$:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(t)dt}{(t-z)^{n+1}}$$

Thus an analytic function has derivatives of all orders, a property proved for power series in Section (1.3.2) It has been observed that the property of defining an analytic function of $z$ inside $C$ resides, not in the factor $f(t)$ in the integral, but in the Cauchy kernel $\frac{1}{t-z}$, for we can replace $f(t)$ by any continuous function $F(t)$ without losing authenticity of the integral, which, however, may not be zero outside of $C$ in this case. It is the Cauchy kernel which is an analytic function of $z$ as long as $z$ is kept away from the contour of integration $C$. We can expand the kernel in powers of $z$ or of $\frac{1}{z}$, multiply by $f(t)$, and integrate term by term, as is usually permitted by uniform convergence of the series. The linearity in $f$ often implies continuity with respect to $f$. Among the many results obtainable by such considerations we list

**Theorem 1.3.4.** If $z \to f(z)$ is analytic in $D_f$, if the disk $D = \{z : |z-a| < R\}$ lies in $D_f$, then $f(z)$ can be expanded using Taylor Series

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (z-a)^k$$

the series being absolutely convergent in the disk $D$. 

**Proof.** We have

$$\frac{1}{t-z} = \frac{1}{(t-a)-(z-a)} = \sum_{k=0}^{\infty} \frac{(z-a)^k}{(t-a)^{k+1}}$$

which converges uniformly in $z$ and $t$ if $|z-a| \leq R-\delta$, $|t-a| = R$. Multiplication by $f(t)$ and termwise integration yields (1.23) in view of (1.22), where $z$ is replaced by $a$. 

\hfill \Box
Theorem 1.3.5. If \( z \to f(z) \) is analytic in an annulus.

\[
0 \leq r_1 < |z - a| < r_2 \leq \infty \tag{1.25}
\]

then

\[
f(z) = \sum_{n=-\infty}^{\infty} a_n (z - a)^n \\
a_n = \frac{1}{2\pi i} \int_C \frac{f(t)\,dt}{(t - a)^{n+1}} \tag{1.26}
\]

where \( C : |t - a| = r, r_1 < r < r_2 \).

We are going to not give the proof but mention that in addition to (1.24) there is needed all \( z \) which satisfies

\[
\frac{1}{t - z} = -\sum_{n=0}^{\infty} \frac{(t - a)^n}{(z - a)^{n+1}} \tag{1.27}
\]

which is valid for \( |z - a| > |t - a| \). The case in which \( z = a \) is an isolated singularity is particularly important. Here \( r_1 = 0 \), and the negative powers in (1.26) constitute the principal part of the singularity. There are three different possibilities.

i. No negative powers. We define \( f(a) = a_0 \); the singularity is removable.

ii. A finite number of negative powers, \( a_n = 0 \) for \( n < -m \) but \( a_{-m} \neq 0 \). This is a pole of order \( m \), and \( (z - a)^m f(z) \) is analytic at \( z = a \).

iii. Infinitely many negative powers. Here \( z = a \) is an essential singularity. In any neighborhood of \( z = a \) the function \( z \to f(z) \) assumes any preassigned value \( c \) infinitely often with at most two exceptions (theorem of Emile Picard).

The property of being analytic may be said to be hard to acquire, but once acquired it persists. It can be expected to survive a passage to the limit. The simplest case is

**Theorem 1.3.6.** Suppose that \( \{f_n(z)\} \) is a sequence of analytic functions in a domain \( D \), which converges uniformly to a function \( f(z) \) in \( D \); then \( f(z) \) is analytic.

The functions analytic in \( D \) form a normed algebra under the sup norm \( \|f\| = \sup_{z \in D} |f(z)| \). Convergence in the norm is uniform convergence in the ordinary sense and the algebra is complete, so the theorem follows. This case is almost trivial, but we can greatly weaken the assumptions by using induced convergence. Here is an example. Instead of assuming convergence in a domain, we can assume it in a subset from which it spreads to the whole domain.
Theorem 1.3.7. Let \( \{f_n(z)\} \) be a sequence of functions analytic in a domain \( D \). Let \( C \) be a simple, closed, rectifiable oriented curve which, together with its interior, lies in \( D \). Suppose that the sequence \( \{f_n(t)\} \) converges uniformly with respect to \( t \) on \( C \). Then there exists a function \( z \to f(z) \) analytic in the interior of \( C \) such that \( f_n(z) \) converges to \( f(z) \) uniformly in the interior of \( C \). Moreover, if \( S \) is any subset of the interior of \( C \) having a positive distance from \( C \), and if \( p \) is any positive integer, then the sequence \( \{f_n^{(p)}(z)\} \) converges uniformly to \( f^{(p)}(z) \) in \( S \).

We will not prove this theorem, but we call attention to the fact that the principle of the maximum implies that a Cauchy sequence \( \{f_n(t)\} \) on \( C \) is also a Cauchy sequence inside and on a zero of \( f(z) \) is by definition a point where the function is zero. It is of order \( m \) if the Taylor expansion starts with the term \( a_m(z-a)^m \), \( a_m \neq 0 \). The zeros of an analytic function can not have a cluster point in the interior of a domain where \( f(z) \) is analytic except when the function is identically zero. If \( z = a \) is a limit point of zeros of \( f(z) \), then in the Taylor expansion

\[
f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + ...
\]

the constant term \( a_0 = f(a) \) is zero by the continuity of the function. But then

\[
f_1(z) = a_1 + a_2(z-a) + a_3(z-a)^2 + ...
\]

also has infinitely many zeros with \( z = a \) as a limit point. This forces \( a_1 \) to be zero and soon; all coefficients are zero, and \( f(z) \) is identically zero. This implies that, if two functions \( f(z) \) and \( g(z) \) analytic in a domain \( D \) coincide for infinitely many values of \( z \) with a limit point in \( D \), they are identical in all of \( D \), for \( h(z) = f(z) - g(z) \) has infinitely many zeros and is thus identically zero. This is known as the identity theorem. Instead of zeros we may of course consider any other fixed value of the function. We see that limit points of zeros or of a value \( c \) are singular points of the function. The calculus of residues occupied a central position in Cauchy’s work. Suppose that \( z = a \) is an isolated singular point of a function \( z \to f(z) \) in the neighborhood of which \( f(z) \) is single valued. There is then an associated Laurent expansion (1.26). The coefficient \( a_{-1} \) is the residue of \( f(z) \) at \( z = a \). The reason for the name is that \( a_{-1} \), is all that is left when we form

\[
\frac{1}{2\pi i} \int_C f(t) \, dt
\]

where \( C \) is a small circle, \( |t-a| = r \), for we can substitute the Laurent series and integrate termwise since the series is uniformly convergent on the circle. We have
\[ \int_C (t-a)^{-n-1} dt = r^{-n} i \int_0^{2\pi} e^{(-ni\theta)} d\theta \] (1.28)

and the integral is zero unless \( n = 0 \), when it equals \( 2\pi i \). This gives the residue theorem.

**Theorem 1.3.8.** If \( f(z) \) is analytic in a simply connected domain \( D \) except for isolated singularities at \( z = s_1, s_2, \ldots, s_n \), then

\[ \int_C f(t) dt = 2\pi i \sum_{j=1}^n r_j \] (1.29)

where \( r_j \) is the residue of \( f(z) \) at \( z = s_j \), and \( C \) is the boundary of \( D \) supposed to be rectifiable.

This is so because the integral along \( C \) equals the sum of the integrals around the small circles surrounding the singularities \( s_j \). In all these formulas the integrals are taken in the positive sense. An important consequence is

**Theorem 1.3.9.** If \( f(z) \) is analytic inside and on \( C \) except for poles, then

\[ \frac{1}{2\pi i} \int_C \frac{f'(t)}{f(t)} dt = Z_f - P_f \] (1.30)

where \( Z_f \) is the number of zeros, and \( P_f \), the number of poles inside \( C \).

**Proof.** It is assumed that neither zeros nor poles are located on \( C \). The integrand is then analytic inside and on \( C \) except for simple poles at the zeros and poles of \( f(z) \). At a zero the residue equals the multiplicity of the zero, whereas at a pole the residue is the negative of the multiplicity of the pole. The conclusion then follows from Theorem 1.3.8

The integral in (1.30) can be evaluated directly since the integrand is the derivative of

\[ \log(f(t)) = \log|f(t)| + i\arg(f(t)) \]

Here the real part returns to its original value when \( z \) returns to the starting point after having described \( C \) once in the positive sense. The imaginary part, however, does not necessarily return to its initial value but will differ from it by a multiple of \( 2\pi \). From Theorem 1.3.9 we then get the so-called principle of the argument:

**Theorem 1.3.10.** Under the assumptions of Theorem 1.3.9 the increase in the argument of \( f(z) \) after \( C \) has been described once in the positive sense is \( 2\pi(Z_f - P_f) \).
A useful addition to Theorem is given by

**Theorem 1.3.11.** Under the assumptions of Theorems 1.3.9 and Theorem 1.3.10, suppose in addition that \( g(z) \) is analytic inside and on \( C \). Then

\[
\frac{1}{2\pi i} \oint_C g(t) \frac{f'(t)}{f(t)} \, dt = \sum g(a_j) - \sum g(b_k)
\]

(1.31)

where the summation is extended over the zeros and poles \( b_k \) of \( f(z) \), and each summand is repeated as often as the multiplicity of the zero or pole requires.

The zeros and poles of \( f(z) \) are still simple poles of the integrand, and at a zero of \( f \) of multiplicity \( \mu_j \), the residue is \( \mu_j g(a_j) \), and similarly at the poles. Cauchy’s formula (1.3.10) invites some comments. Replace \( z \) by \( z_0 \), and let the path of integration be the circle \( t = z_0 + re^{i\theta} \), where \( \theta \) goes from zero to \( 2\pi \). The result is

\[
f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) \, d\theta
\]

(1.32)

The right-hand side is the mean value in the sense of the integral calculus over the interval \((0, 2\pi)\) of the integrand. This is a basic property of analytic functions but is shared with harmonic (logarithmic) potential functions. From (1.32) we also get

\[
|f(z_0)| \leq \max |f(z)| \quad \text{for} \quad |z - z_0| = r
\]

(1.33)

In this relation the inequality normally holds, equality can hold if and only if \( f(z) \) equals its maximum for all \( z \), to start with, all \( z \) on the circle and ultimately all \( z \) in the plane, and \( f(z) = Me^{ia} \), a , where \( M \) is the maximum and \( a \) is real, fixed. This is a form of the principle of the maximum. The principle asserts that the absolute value of an analytic function \( f(z) \) cannot have a local maximum unless it is a constant. If in 3D-space we plot the surface

\[
u = |f(x + iy)|^2
\]

(1.34)

where \( f \) is not a constant, if \( f(x_0 + iy_0) = f(z_0) \neq 0 \), there are paths on the surface leading from \( z = z_0 \), \( u = u_0 \) along which \( u \) is strictly increasing, and also paths along which \( u \) is strictly decreasing. The latter type of path naturally is missing if \( u_0 = 0 \). We formulate a form of the principle which is sufficient for our purposes.

**Theorem 1.3.12.** If \( f(z) \) is analytic inside and on the rectifiable curve \( C \), and if \( M(f,C) \) is the maximum of \( |f(z)| \) on \( C \), then for all \( z \) inside \( C \)

\[
|f(z)| \leq M(f,C)
\]

(1.35)
Proof. The use of Cauchy's integral below is due to Edmund Landau, who made profound contributions to function theory and analytic number theory. We have

$$[f(z)]^k = \frac{1}{2\pi i} \int_C \frac{[f(t)]^k}{t-z} \, dt$$

whence

$$|f(z)|^k \leq [2\pi d(z,C)]^{-1} L[M(f,C)]^k$$

Here $d(z,C)$ is the distance of $C$ from the point $z$ in its interior, and $L$ is the length of $C$. We extract the $k$'th root and pass to the limit with $k$ to obtain (1.35)

1.3.4 Analytic Continuation

Analytic continuation is a concept introduced by Weierstrass and basic for his attack on function theory. A function $z \rightarrow f(z)$ is defined originally by a power series, say,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with a radius of convergence $R$. If $R = \infty$ and the function is entire, there is no continuation problem. Also, $R = 0$ is out (we disregard the possibility that the series may be summable by some method or other). If $0 < R < \infty$, there is a continuation problem. We saw in Section 1.3.2 that $f(z)$ admits of expansions in a Taylor series,

$$f(z; a) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(z-a)^k$$

obtained by direct rearrangement of (1.37) after setting $z = a + (z-a)$ and expanding. The series $f(z; a)$ converges for $|z-a| < R_a$, where $R - |a| \leq R_a \leq R + |a|$. If $R_a = R - |a|$, the point of contact of $|z-a| = R_a$ with $|z| = R$ is a singular point of $f(z)$ and analytic continuation in the direction $\arg(z) = \arg(a)$ is not possible. On the other hand, if $R - |a| < R_a$, the disk $|z-a| < R_a$ is partly outside $|z| < R$, and in the lens-shaped overhang $f(z; a)$ defines an analytic continuation of $f(z)$. This process is repeated for all $a$, with $|a| < R$. If for all such $a$‘s we get $R_a = R - |a|$, no analytic continuation is possible and $|z| = R$ is the natural boundary of $f(z)$.

If, on the other hand, $R_a > R - |a|$ for some values of $a$, the union of the disks $|z-a| < R_a$ is a simply connected domain $D_1$, in which our function is defined by one of the series $f(z; a)$ with $|a| < R$. Moreover, if a point $z_0 \in D_1$, it belongs to infinitely
many disks $|z-a| < R_a$ and the corresponding series $f(z;a)$ all assign the same value to $f(z)$ at $z = z_0$. We now repeat the process for points at in $D_1$ with $|a_1| \geq R$. This gives a set of power series $f(z;a_1)$ obtained by double rearrangements: of the original series at $z = a$ with $|a| < R$, and of the series $f(z;a)$ at $z = a_1$. This gives a definition of $f$ in the union of all the disks. Since this set may be self-overlapping, we are no longer assured of the consistency of the definition but have to keep account of the steps involved.

Suppose that we can find a sequence of points $a_0, a_1, ..., a_n$ such that power series in terms of $z - a_1$ are obtained by repeated rearrangements of (1.37). Suppose that

$$|a_0| < R, \quad |a_1 - a_0| < R_{a_0}, \quad |a_n - a_{n-1}| < R_{a_{n-1}}$$

(1.39)

Here

$$f_1(z) = \sum a_{j,k}(z-a)^k_j, \quad |z-a| < R_j$$

There is then defined a branch of $f(z)$ at $z = a_n$ obtained by analytic continuation of $f(z)$ using the intermediary points $a_0, a_1, ..., a_n$. According to Weierstrass, the totality of such series constitutes an analytic function $f(z)$. To these regular elements of $f$ are further adjoined the following:

1. Polar elements $(z-a)^{-m} \sum_{j=0}^{\infty} b_j (z-a)^j$, one for each pole.

2. Algebraic elements $(z-a)^{-k} \sum_{j=0}^{\infty} c_j (z-a)^{jk}$ for algebraic branch points.

3. Elements at infinity. These may be regular: $\sum_{j=0}^{\infty} d_j z^{-j}$, polar: $z^m \sum_{j=0}^{\infty} d_j z^{-j}$

algebraic: $z^k \sum_{j=0}^{\infty} d_j z^{-jk}$.

Essential singular points are not considered as belonging to the domain of definition of the function and contribute no functional elements. Logarithmic singularities also are noncontributing.

At a given point $z = a$ there are normally infinitely many elements, regular and/or singular, we have the distinct elements of $f(z)$ at $z = a$ form a countable set. This we see as follows. An element at $z = a$ is the end product of a chain of rearrangements. We can order these first according to the number of steps involved, say $n$. For each $n$ we have rearrangements at points $a_0, a_1, ..., a_n$. Without loss of generality we may assume that the $a$’s used have rational coordinates (these numbers are dense in $\mathbb{C}$). Now the points with rational coordinates form a denumerable set. Thus we are dealing with a countable number of countable sets, and such a set is itself countable. Hence the different determinations at $z = a$ form either a finite set or a countable
1.3 Complex Analysis

one. Leaving these general considerations, we turn to the principle of permanence of functional equations, which is a very important fact in the theory of CDE's. It is hard to find a definition of the term functional equation, and the mathematicians who deal with the subject normally exclude differential and integral equations from consideration, so we cannot expect much help from that quarter. Before going any further we have to prove the double series theorem of Weierstrass, which will be needed here and later in this thesis.

**Theorem 1.3.13.** Suppose that the functions \( z \to f_n(z) \), \( n = 0, 1, 2, \ldots \), are analytic in a disk \(|z - a| < R\) and that the series

\[
\sum_{n=0}^{\infty} f_n(z) \equiv f(z)
\]  

(1.40)

converges uniformly in \(|z - a| \leq \rho\) for each \( \rho \) with \( 0 < \rho < 1 \). Suppose also that

\[
f_n(z) = \sum_{k=0}^{\infty} a_{k,n}(z-a)^k
\]  

(1.41)

Then each of the series

\[
\sum_{n=0}^{\infty} a_{k,n} \equiv A_k, \quad k = 0, 1, 2, \ldots
\]  

(1.42)

converges and, for \(|z - a| < R\),

\[
f(z) = \sum_{k=0}^{\infty} A_k(z-a)^k
\]  

(1.43)

**Proof.** This follows from Theorem 1.3.7, which implies that the sum of the series (1.40) is an analytic function \( f(z) \), that the series may be differentiated term by term \( p \) times, and that the resulting sum of \( p \)'th derivatives is the \( p \)'th derivative of the sum \( f(z) \), where \( p \) is any integer. In particular, we have convergence for \( z = a \) and this implies the assertions.

The name double series theorem refers to the fact that under the stated conditions the order of summation in the series

\[
\sum_{n=0}^{\infty} \left[ \sum_{k=0}^{\infty} a_{k,n}(z-a)^k \right]
\]  

(1.44)
may be interchanged. We will prove a special form of the principle of permanence of functional equations, which is sufficient for most of our needs and is connected with our work on functional inequalities Section. There, was found that, if \( X \) is a partially ordered metric space with elements \( f \), if \( T \) is a mapping of \( X \) into itself, and if \( T \) or one of its powers is a contraction with fixed point \( g \), then the elements of \( X \) which satisfy the inequality

\[
f \leq T[f]
\]

also satisfy

\[
f \leq g, \quad \text{where} \quad g = T[g],
\]

this is a functional equation and obeys the principle of permanence if the data are given an analytical form. We consider a function \( T(z,w) \) of two complex variables given by the series

\[
T(z,w) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk} z^j w^k, \quad a_{00} = 0
\]

(1.47)

It is assumed that there exist values of \( z \) and \( w \), different from \((0,0)\), for which the series is absolutely convergent, say \( z = a, w = b \). These numbers \( a \) and \( b \) may be assumed to be positive. The reader should observe that a double power series does not have a unique radius of convergence, though there exist pairs of associated radius: if you lower one, you can increase the other. Our assumption implies that the terms of the series are bounded for \( z = a, w = b \),

\[
|a_{jk}| a^j b^k \leq M,
\]

(1.48)

by an obvious generalization of Theorem 1.5 this implies that the series (1.47) is absolutely convergent for \(|z| < a, |w| < b\).

Next, suppose that we have found a function

\[
z \to f(z) \equiv \sum_{m=1}^{\infty} c_m z^m
\]

(1.49)

such that

(1) there exist a pair of numbers \( s < a, t < b \) and (1.49) is absolutely convergent for \(|z| \leq s\) and \(|f(z)| \leq t\) for such values.
We have then the following principle of permanence of functional equations:

**Theorem 1.3.14.** If $T$ satisfies the conditions just stated, and if the solution $f(z)$ as well as the composite function $F(z) = T[z, f(z)]$ can be continued along the same path from the origin, then all along the path (1.50) holds; for instance, the continuation of the solution is the solution of the continuation of the equation.

**Proof.** We assume that the continuation involves a chain of disks $D_0, D_1, \ldots, D_n$, where $D_0 = \{z; |z| < s\}$ and the center $a_j$ of $D_j$ lies in $D_{j-1}$. In $D_j$ we have functional elements $f_j(z)$ and $F_j(z) = T[z, f(z)]$ represented by convergent power series in $z - a_j$. Now, in $D_0$ we have $f(z) \equiv F(z)$, and this equality holds at $z = a_1$, and in some neighborhood thereof. But the identity theorem then requires that $f_1(z) = F_1(z)$ in $D_1$, in particular, at $z = a_2$ so that $f_2(z) = F_2(z)$, and so on.

**Notation 1.1.** notice that this is not an existence proof. It is by no means clear that our assumptions are strong enough to guarantee the existence of a solution. All that is claimed is that, if we have found an analytic solution and we can continue it analytically together with the right member of the equation, then it remains a solution. It should also be observed that (1.46) is a rather special case of a functional equation and that the law of permanence holds in much more general situations.

**Definition 1.3.17.** Let $w = f(z)$ be a complex mapping defined in a domain $D$ and let $z_0$ be a point in $D$. Then we say that $w = f(z)$ is conformal at $z_0$ if for every pair of smooth oriented curves $\Gamma$ and $C$ in $D$ intersecting at $z_0$ the angle between $\Gamma$ and $C$ at $z_0$ is equal to the angle between the image curves $\Gamma'$ and $C'$ at $f(z_0)$ in both magnitude and sense.

## 1.4 Differential Equations

**Definition 1.4.1.** The differential equation is a relation between one or more independence variables in a certain field and a dependent variable and its derivative with respect to one or more independence variables, and it's denoted by $\text{DE}$.

**Definition 1.4.2.** The solution of $\text{DE}$ is any well defined function in a particular interval that satisfies the $\text{DE}$ and has constants as much as the order of
Theorem 1.4.1. Suppose \( a_n(x), a_{n-1}(x), \ldots, a_1(x), a_0(x) \) and \( G(x) \) are continuous functions on some interval \( I \) and suppose \( a_n(x) \neq 0 \) for every \( x \) in that mentioned interval. If \( x = x_0 \) is any point in that interval, then a solution \( y(x) \) of the initial-value problem is existed on the interval and it is unique.

Theorem 1.4.2. Consider the IVB

\[
\frac{df}{dx} = f(x, y), \quad y(x_0) = x_0
\]

(1.51)

1.5 Functional Analysis

Functional analysis has been already a revolutionary branch of mathematics where the most intractable problems could not have been solved without it. Quantum mechanics has no meaning mathematically without Hilbert spaces for example when He formed a wave of real and imaginary state of the electron wave as vector. Some essential concepts are introduced here as background for the concepts mentioned in this thesis.

1.5.1 Metric Spaces

A metric space is one in which there is defined a notion of distance subject to the following conditions:
(1) For any pair of points \( x \) and \( y \) of a set \( A \), a number \( d(x, y) \geq 0 \) is defined and called the distance from \( x \) to \( y \) such that \( d(x, y) = 0 \) iff \( x = y \).

(2) \( d(x, y) = d(y, x) \).

(3) For any \( z \) in \( X \) we have \( d(x, y) \leq d(x, z) + d(z, y) \).

We say that a linear vector space is normed if the following conditions hold:

(1) For each \( x \in X \) there is assigned a number \( \| x \| \leq 0 \) such that \( \| x \| = 0 \) iff \( x = 0 \).

(2) \( \| \alpha x \| = |\alpha| \| x \| \) for each \( \alpha \) in the scalar field.

(3) \( \| x + y \| \leq \| x \| + \| y \| \).

A normed linear vector space becomes a metric space by setting

\[
d(x, y) = \| x - y \| \tag{1.52}
\]

In a metric space we can do analysis since the fundamental operation of analysis, that of finding limits of a sequence, becomes meaningful. If \( x_n \) is a sequence in the metric space \( X \), we say that \( n \) converges to \( x_0 \) and

\[
x_0 = \lim_{x \to \infty} x_0 \quad \lim d(x_0, x_0) = 0 \quad \text{as} \quad x \to \infty \tag{1.53}
\]

We say that \( x_n \) is a **Cauchy sequence** if, given any \( \epsilon > 0 \), there exists an \( N \) such that

\[
d(x_m, x_n) < \epsilon \quad \text{for} \quad m, n > N \tag{1.54}
\]

If (1.5) holds, it follows that \( x_n \) is a Cauchy sequence, but the converse is not necessarily true, for there may be gaps in the space. A metric space \( X \) is said to be complete if all Cauchy sequences converge to elements of the space. Euclidian spaces are complete, and so are various function spaces that will be encountered in the following. The space \( \mathbb{Q} \) of rational numbers is not complete. Various notions of real analysis are meaningful in complete metric spaces, such as the concepts of closure, open set, closed set, and \( \epsilon \)-neighborhood. The Bolzano-Weierstrass theorem need not be valid in a complete metric space, that is, there may be bounded infinite point sets without a limit point. Incidentally, “bounded” means that the set can be enclosed in a “sphere” \( d(x, 0) < R \). The topological diameter \( d(A) \) of a subset of \( X \) is the least upper bound of the distances \( d(x, y) \) for \( x \) and \( y \) in \( A \).
1.5.2 Linear Transformation From \( \mathbb{C}^n \) to \( \mathbb{C}^n \)

The simplest of all linear transformations are those which map \( \mathbb{C}^n \) into itself. If \( T \) is such a transformation, then \( T \) is uniquely determined by linearity and its effect on the basis of \( \mathbb{C}^n \). Any set of \( n \) linearly independent vectors would serve as a basis, but we may just as well use the unit vectors

\[
e_j = (\delta_{jk}) \tag{1.55}
\]

where \( \delta_{jk} \) is the Kronecker delta, that is, the vector whose \( j \)'th component is one, all others being zero. This gives

\[
x = x_1 e_1 + x_2 e_2 + \ldots + x_n e_n \tag{1.56}
\]

if \( x = (x_1, x_2, \ldots, x_n) \) in the coordinate system defined by the vectors \( e_j \). Now \( T \) takes vectors into vectors, so there are \( n^2 \) complex numbers \( a_{jk} \) such that

\[
T[e_k] = a_{1k}e_1 + a_{2k}e_2 + \ldots + a_{nk}e_n, \quad k = 1, 2, \ldots, n \tag{1.57}
\]

The linearity of \( T \) then gives

\[
T[x] = T \left( \sum_{k=1}^{n} x_k e_k \right) = \sum_{k=1}^{n} x_k T[e_k]
\]

or

\[
T[x] = \sum_{j=1}^{n} \left( \sum_{k=1}^{n} a_{jk} x_k \right) e_j \equiv y \tag{1.58}
\]

from which we can read off the components of the vector \( y \). The quadratic array

\[
A = \begin{bmatrix}
    a_{11} & a_{12} & \ldots & a_{1n} \\
    a_{21} & a_{22} & \ldots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & \ldots & a_{nn}
\end{bmatrix} \tag{1.59}
\]

is known as a (matrix) more precisely, the matrix of the transformation \( T \) with respect to the chosen basis. We can now write \( T \) symbolically as

\[
y = T[x] = A x \tag{1.60}
\]
where the last member may be considered as the product of the matrix $A$ with the column vector $x$, the result being the column vector $y$. We have to decide when the mapping defined by $T$ is one to one. Here the condition $T[x] = 0$ implies that $x = 0$ now takes the form that the homogeneous system

$$\sum a_{jk}x_k = 0 \quad j = 1, 2, \ldots, n$$  \hspace{1cm} (1.61)

must have the unique solution

$$x_1 = x_2 = \ldots = x_n = 0$$

This will happen as long as

$$\det(A) \neq 0$$  \hspace{1cm} (1.62)

In this case the mapping is also onto, since for a given vector $y$ we can solve the system

$$\sum_{k=1}^{n} a_{jk}x_k = y_j, \quad j = 1, 2, \ldots, n$$  \hspace{1cm} (1.63)

uniquely for $x = (x_1, x_2, \ldots, x_n)$. It follows that $T$ has a unique inverse, also an element of $E[C^n]$, a linear bounded transformation of $C^n$ into itself. With this transformation goes a matrix $A^{-1}$ which we refer to as the inverse of $A$. The fact that its elements may be computed from (1.4.9) shows that the element in the place $(j,k)$ is $A_{kj}\Delta$, where $A_{jk}$ is the cofactor of $a_{ik}$ in the determinant $\Delta = \det(A)$.

We can define algebraic operations and a norm in the set $\mathfrak{R}$ of $n - b$ matrices in terms of which the set becomes a *Banach algebra*. This follows from the fact that there is a one to one correspondence between the linear transformations $T$ in $E(C^n)$ and their matrices. Then to $T_1 + T_2$, $\alpha T$, and $T_1 T_2$ correspond

$$\begin{align*}
(a_{jk} + b_{jk}) &\equiv A + B \hspace{1cm} (1.64) \\
(\alpha a_{jk}) &\equiv \alpha A \hspace{1cm} (1.65) \\
\left(\sum_{m=1}^{n} a_{jm}b_{mk}\right) &\equiv AB \hspace{1cm} (1.66)
\end{align*}$$

A number of different but equivalent norms may be defined for $\mathfrak{R}_n$. A suitable one for analysis is

$$||A|| = \max_j \sum_{k=1}^{n} |a_{jk}|$$  \hspace{1cm} (1.67)
We have then
\[ \|AB\| \leq \|A\|\|B\| \]

Since \( \mathbb{R}^n \) is complete in the normed metric, it follows that \( \mathbb{R}^n \) is a B-algebra. We have seen that \( A \) has an inverse iff \( \det(A) \neq 0 \). If this is the case, \( A \) is said to be regular, otherwise, singular. Together with the given matrix \( A \) we consider the family of matrices

\[ \lambda \xi - A \]

where \( \lambda \) runs through the complex field \( \mathbb{C} \) and \( \xi = (\delta_{jk}) \) the \( n \times n \) unit matrix. These matrices are normally regular, but there exist \( n \) values of \( \lambda \) for which \( \lambda \xi - A \) is singular: the \( n \) roots of the characteristic equation of \( A \)

\[ \det(\lambda \xi - A) = 0 \quad (1.68) \]

The roots \( \lambda_1, \lambda_2, \ldots, \lambda_n \) form the spectrum \( \sigma(A) \) of \( A \). They are known as characteristic values, latent roots, or eigenvalues. For these values of \( \lambda \) one can find vectors \( x_k \) in \( \mathbb{C}^n \) of norm 1 such that

\[ A x_k = \lambda_k x_k \quad (1.69) \]

The characteristic vectors \( x_k \) are linearly independent and may be chosen so that they form an orthogonal system; in this case the inner product for \( x = x_k, y = x_m, k \neq m \). This holds even if the Equation (1.68) has multiple roots. A matrix \( A \) is singular iff zero belongs to the spectrum.

### 1.5.3 Fixed Point Theorem

A continuous map of the unit ball in \( \mathbb{R}^n \) into itself must necessarily leave at least one point invariant. Such a point is known as a fixed point, and an assertion about the existence of fixed points is known as a fixed point theorem. We will prove some theorems of this nature.

**Theorem 1.5.1.** If \( T \) is a contraction defined on a complete metric space \( X \), then there is one and only one fixed point.

**Proof.** The triangle inequality plays a basic role here. We start with an arbitrary point \( x_1 \in X \) and form its successive transforms under \( T \):

\[ x_{n+1} = T(x_n) \quad n = 1, 2, \ldots \quad (1.70) \]
These elements form a Cauchy sequence, and $X$ being a complete metric space, $x_0 = \lim x_n$ exists and is to be proved to be a fixed point, in fact, the only such point. Now it is sufficient to prove that, $\forall \varepsilon > 0$, there is an $N$ such that

$$d(x_n, x_{n+p}) < \varepsilon \quad n > N, \quad p = 1, 2, \ldots$$

To this end we note that by the triangle inequality the left member does not exceed

$$d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+p-1}, x_{n+p})$$

Now, using the contraction hypothesis, we see that

$$d(n, m+1) \leq kd(x_{m-1}, x_m) \leq \ldots \leq k^{n-1}d(x_1, x_2)$$

and thus

$$d(x_n, x_{n+p}) \leq (k^{n-1} + k^n + \ldots + k^{n+p-1})d(x_1, x_2)$$

$$\leq \frac{k^{n-1}}{1-k}d(x_1, x_2)$$

This expression can be made as small as we want by choosing $n$ large enough. Hence $\{x_n\}$ is a Cauchy sequence regardless of the choice of $x_1$, and the limit $x_0$ exists. Since $x_{n+1} = T(x_n)$

we conclude that

$$x_0 = \lim x_{n+1} = \lim T(x_n) = T[\lim x_n] = T(x_0)$$

where we have used the continuity of $T$. It is seen that $x_0$ is indeed a fixed point. Suppose that $y_0$ is a fixed point. Then

$$d(x_0, y_0) = d[T(x_0), T(y_0)] \leq kd(x_0, y_0)$$

Since $k < 1$, this implies that $d(x_0, y_0) = 0$ or $y_0 = x_0$, so there is one and only one fixed point.

The restriction to contraction operators is a drawback, but it can occasionally be avoided by observing the following.

**Corollary 1.5.1.** There is a unique fixed point if some power of $T$ is a contraction.
Proof. If $T^m$ is a contraction, then there exists a fixed point $x_0$ such that. We have also

$$T^m(x_0) = x_0$$

for any choice of $x_1$. Here we set $x_1 = T(x_0)$ and find that

$$(T^m)^n[T(x_0)] = T[(T^m)^n(x_0)] = T(x_0)$$

When $n$ becomes infinite, the first member tends to 0, so we have $T(x_0) = x_0$ or $T$ admits $x_0$ as a fixed point. Since any fixed point of $T$ is a fixed point of $T^m$ and the latter has a unique fixed point, it follows that $x_0$ is the unique fixed point of $T$. \[\square\]

**Theorem 1.5.2.** Let $X$ be a B-space. Let $z_0$ be a given element of $X$, and let $S$ belong to $E(X)$ and be such that

$$\sum_0^\infty \|S^n\| < \infty$$

(1.71)

Then the transformation

$$T(x) = z_0 + S[x]$$

(1.72)

has a unique fixed point $x_0$ given by

$$x_0 = z_0 + \sum_1^\infty S^n[z_0]$$

(1.73)

It is enough to observe that the series converges in norm and $S$ can be applied termwise to the series and shows that $T(x_0) = x_0$.

Consider, in particular, Volterra’s equation

$$f(t) = g(t) + \int_0^t K(s,t)f(s)ds,$$  

(1.74)

here, the kernel $K(s,t)$ and $g(t)$ are known and $f(t)$ is to be found. We consider the particular case in which the kernel is a function of $s$ alone.

**Theorem 1.5.3.** Suppose that $g(t) \in C[0,a]$ and $K(s) \in L(0,a)$. Then the equation

$$f(t) = g(t) + \int_0^t K(s)f(s)ds$$

(1.75)
has a unique solution in \( C[0,a] \), namely,

\[
f(t) = g(t) + \int_{0}^{t} K(s) \exp \left[ \int_{s}^{t} K(u) du \right] g(s) ds
\]  

(1.76)

**Corollary 1.5.2.** For \( K(t) = K \) the equation

\[
f(t) = g(t) + K \int_{0}^{t} f(s) ds
\]  

(1.77)

has the unique solution

\[
f(t) = g(t) + K \int_{0}^{t} \exp[K(t-s)]g(s) ds
\]  

(1.78)

### 1.5.4 Functional Inequalities

This is a field of increasing importance. We shall consider inequalities of the form

\[
f(t) \leq T[f](t)
\]  

(1.79)

We consider a complete metric space \( X \), the elements of which are mappings from some interval \([a,b]\) in \( \mathbb{R}^1 \). Here \( T \) is a mapping of \( X \) into itself, and the problem is to discuss the inequality. Can it be satisfied by elements \( f \) of \( X \)? Is it trivial in the sense that it is satisfied by all \( f \)'s in \( X \)? If neither of the above is true, characterize the elements of \( X \) for which (1.79) holds. Is it so restrictive that it holds for one and only one \( f \)? It can be seen that there are a number of pertinent questions. The discussion in fixed point theorem leads to several functional inequalities which are categorical or determinative in the sense that there is a single element of the space under consideration which satisfies the inequality. If in (1.75) and (1.77) we assume \( g \) to be identically zero, then \( f \) is identically zero. This suggests

**Theorem 1.5.4.** Let \( X \) be the positive cone of \( C[0,a] \), \( 0 < a < \infty \). Let \( K(t) \in L(0,a) \), continuous and non-negative on the half-open interval \((0,a]\). If \( f \in X \) and if for \( 0 \leq t \leq a \)

\[
f(t) \leq \int_{0}^{t} K(s) f(s) ds
\]  

(1.80)

then \( f \) is identically zero.

**Proof.** This functional inequality is very important for the theory of differential equations (DE's) since it underlies uniqueness proofs based on a Lipschitz or, more
generally, a Carathéodory condition. We shall obtain the theorem as a consequence of more general theorems, but in view of its importance it is desirable to give a direct short proof.

Set

\[ F(t) = \int_0^t K(s)f(s)ds \]  

(1.81)

This is an element of \( X \) and \( F(0) = 0 \). Furthermore, for \( 0 < t \)

\[ F'(t) = K(t)f(t) \leq K(t)F(t), \]

so that (1.80) implies that

\[ F'(t) - K(t)F(t) \leq 0 \]

(1.82)

This we multiply by the positive function \( e^{\int_0^t K(u)du} \), result is an exact derivative so that

\[ \frac{d}{dt} \left\{ F(t)e^{-\int_0^t K(u)du} \right\} \leq 0 \]

Since \( F(0) = 0 \), this shows that \( F(t) \leq 0 \) for \( 0 < t \leq a \). But we already know that \( F(t) \geq 0 \). To satisfy both inequalities we must have \( F(t) = 0 \). This implies that \( f(t) \equiv 0 \), as asserted.

**Theorem 1.5.5.** Let \( X \) be the subspace of \( C^+[0,a] \), the elements of which satisfy \( f(0) = 0 \), \( \lim_{h \to 0} f(h)/h = 0 \). Then, if \( f \in X \), and if

\[ f(t) \leq \int_0^t f(s)\frac{ds}{s} \]

then

\[ f(t) = 0 \]  

(1.83)

**Theorem 1.5.6.** Let \( X \) be a complete metric space which is partially ordered in such a manner that if \( x_n \) is an increasing sequence in \( X \), so that \( x_n \leq x_{n+1} \) for all \( n \), and if \( \lim_{n \to \infty} x_n = x_0 \) exists in the sense of the metric, then \( x_n \leq x_0 \) for all \( n \). Let \( T \) be an order-preserving mapping of \( X \) into \( X \) such that \( T^m \) is a contraction for some \( m \). Let \( f_0 \) be the unique fixed point of \( T \). Then

\[ f \leq T[f] \quad \text{implies} \quad f \leq f_0 \]  

(1.84)
**Proof.** We say that \( T \) is order-preserving if for \( f_1, f_2 \in X \)

\[
f_1 \leq f_2 \quad \text{implies} \quad T[f_1] \leq T[f_2]
\]  

(1.85)

Suppose that \( f \in X_0 \), the subset of \( X \) for which the inequality is meaningful. Since \( X_0 \) contains \( f_0 \) at least, it is not void. Then

\[
f \leq T[f] \leq T^2[f] \leq \ldots \leq T^n[f] \leq \ldots
\]

Now \( T^n[f] \) tends to the limit \( f_0 \) as \( n \to \infty \) for

\[
\lim_{k \to \infty} T^{km}[T^j(f)] = f_0 \quad j = 0, 1, \ldots, m - 1
\]

since \( T^m \) is a contraction and the limit is the same for all elements of \( X \). It follows that the increasing sequence \( \{T^n[f]\} \) converges to \( f_0 \). Since order is preserved under the limit operation, we have \( f \leq f_0 \) and the theorem is proved

\[\Box\]

If \( T[f] \) is defined by (1.81), \( T \) is order-preserving since the kernel \( K(s) \) is non-negative and the space \( X \) is linear. Here \( T \) usually does not define a contraction, but all powers \( T^m \) with a sufficiently large \( m \) are contractions for

\[
T^n[f](t) = \frac{1}{(n-1)!} \int_0^t K(s) \left[ \int_s^t K(u)du \right]^{n-1} f(s)ds
\]

(1.86)

the norm of which goes rapidly to zero as \( n \) becomes infinite. This provides another proof for Theorem 1.6.1. We can also apply the *Volterra* fixed point theorem to functional inequalities.

**Theorem 1.5.7.** Let \( X \) be a partially ordered B-space such that the positive cone \( X^+ \) is a closed point set. Let \( S \) be a linear bounded positive transformation on \( X \) to \( X \) and such that

\[
\sum_{n=1}^{\infty} \|S^n\| < \infty
\]

(1.87)

Let \( g \) be a given element of \( X^+ \), and \( f_0 \) the unique fixed point of

\[
T[f] = g + S[f]
\]

(1.88)

Then

\[
f \leq g + S[f] \quad \text{implies} \quad f \leq f_0
\]

(1.89)
Background

Proof. That $S$ is positive means that it maps $X^+$ into itself. All the powers of $S$ are then also positive and $S$ is order-preserving. The existence of a unique fixed point follows from Theorem 1.5.2 We have $\lim_{n \to \infty} T^n[f] = f_0$ for any $f$ in particular, for an $f$ satisfying the first inequality under (1.89). Now we have

$$T^n[f] \leq g + S[g] + S^2[g] + \ldots + S^n[g] + S^n[f]$$  \hspace{1cm} (1.90)

This is an increasing sequence which goes to the limit $f_0$ as $n$ becomes infinite. This, combined with $f \leq T[f]$ and the order-preserving properties of limits, leads to the desired result.

We state a couple of applications of this theorem which are of special importance to the theory of DE's.

**Theorem 1.5.8.** Let $K \in C^+(a, b) \cap L(a, b)$, and let $g$ and $f$ belong to $C^+[a, b]$. Suppose that for all $t$ in $[a, b]$

$$f(t) \leq g(t) + \int_a^t K(s)f(s)ds$$  \hspace{1cm} (1.91)

Then

$$f(t) \leq g(t) + \int_a^t K(s)\exp\left[\int_s^t K(u)du\right]g(s)ds$$  \hspace{1cm} (1.92)

We see that if $g(f) \equiv 0$; then $f(t) = 0$ and Theorem 1.79 is again obtained. It is worth while stating the case $K(t) \equiv K$ as a separate result.

**Theorem 1.5.9.** If $K(t) = K$ and $f$ and $g$ are in $C^+[a, b]$, then

$$f(t) \leq g(t) + K\int_a^t f(s)ds$$  \hspace{1cm} (1.93)

implies that

$$f(t) \leq g(t) + K\int_a^t \exp[K(t-s)]g(s)ds$$  \hspace{1cm} (1.94)

If $g(t)$ is also a constant, then

$$f(t) \leq g + K\int_a^t f(s)ds$$  \hspace{1cm} (1.95)

implies that

$$f(t) \leq g\exp[K(t-a)]$$  \hspace{1cm} (1.96)

**Notation 1.2.** The inequalities listed under Theorems 1.5.8 and 1.5.9 are known as Gronwall's lemma.
Chapter 2

Theory of Ordinary Complex Differential Equations

In this chapter, we study and investigate and modify the theories of existences and uniqueness of the solution of complex differential equations. This kind of work has no enough references as much as the other parts of the research. We discussed the concepts based on ([43]), ([44]), ([16]), ([6]). This chapter studies the differential equation when the input and output variables are in complex domain, The pivot of the chapter will be the matter of the existence an uniqueness of the solution of the differential equation in the complex plane that will be called in this work the complex differential equation or CDE as stand for it.

2.1 Definitions

**Definition 2.1.1.** A complex differential equation is an equation contains derivatives for an analytic complex function with respect to one or more independence variables

\[ w = F(z_1, z_2, \ldots, z_n), \]

where \(z_1, z_2, \ldots, z_n\) are complex variables. The general form of complex differential equations,

\[ F(z, w, w', w'', \ldots, w^{(n)}) = 0, \]

it will be called CDE as stands for it. \( w = F(z_1, z_2, \ldots, z_n) \) will be the solution for the CDE.
**Definition 2.1.2.** The order of the complex differential equation is the highest derivative in the complex differential equation.

**Definition 2.1.3.** The degree of complex differential equation is the power of the highest derivative in the complex differential equation.

**Notation 2.1.** The complex differential equation has two types: ordinary complex differential equations and partial complex differential equations.

1. **Ordinary complex differential equations (OCDE’s):** have one dependence and one independence complex variables.

2. **Partial complex differential equations (OCDE’s):** have more than one independence complex variable.

**Notation 2.2.** In this work we will study the ordinary complex differential equations.

**Definition 2.1.4.** The general solution to the OCDE is an analytic function

\[ w = F(z), \]

which satisfies the OCDE.

**Definition 2.1.5.** The particular solution is a specified general solution with a particular value for the constant \( C \).

**Definition 2.1.6.** A solution that does not come from the general solution is called the singular solution.

**Note:** Now we have a very important two questions;

1. Is there a solution for any CDE?

2. Is that solution unique?

### 2.2 Existence and Uniqueness

In this section an answer for the last two questions will be investigated. Let’s study at first the ordinary complex differential equation of the first order.

Consider the OCDE:

\[ w' = F(z,w), \]
and suppose \((z,w) \mapsto f(z,w)\) is analytic in a neighborhood of \((z_0,w_0)\). So is there a function of the variable \(z\) let say \(w = w(z, z_0, w_0)\) such that it is analytic in some neighborhood of \(z = z_0\) and for all \(z\) in the neighborhood with \(w_0 = (z_0, z_0, w_0)\) such that

\[
w'(z; z_0; w_0) = f(z, w(z; z_0; w_0)),
\]

Cauchy showed the existence and uniqueness of such a solution. To this end, he developed some original ways. For the real case, he worked out a step-by-step method using linear approximations. And he used series expansions and the majorant method. So, for the a solution in the complex plane we will show some methods as analog to the real solution.

### 2.2.1 Fixed Point Method

We are concerned with the ODE equation of first order and degree one,

\[
w' = F(z, w)
\]

where \((z,w) \mapsto F(z,w)\) is analytic in the dicylinder

\[
D : |z - z_0| \leq a, \quad |w - w_0| \leq b,
\]

It is required to find a function \(w(z; z_0, w_0)\), analytic in some disk \(|z - z_0| < r \leq a\), such that

\[
w'(z, z_0, w_0) = F(z, w(z, z_0, w_0)) \quad w(z_0; z_0, w_0) = w_0,
\]

To apply Theorem 1.5.1 we have to exhibit a complete metric space \(X\) consisting of functions \(z \mapsto g(z)\) and define an operator \(T\) which maps \(X\) into itself and is a strict contraction. Also, \(X\) has to be chosen so that the existing unique fixed point is the desired solution, as a first step, we integrate (2.1) by the integral equation

\[
f(z) = w_0 + \int_{z_0}^{z} F(s, f(s))ds,
\]

If this equation has a unique solution \(f\), then \(f(z)\) also satisfies (2.1) including the initial condition. This suggests defining an operator \(T\) by

\[
T[g](z) = w_0 + \int_{z_0}^{z} F[s, g(s)]ds.
\]
where $T$ operates on a space to be defined. We notice that $F$ satisfies two conditions that used in the following, namely,

$$|F(z,w)| < M,$$  
(2.3)

$$|F(z,u) - F(z,v)| < K|u - v|,$$  
(2.4)

for suitably chosen constants $K$ and $M$ and $(z,w), (z,u),$ and $(z,v)$ in $D$. Since $D$ is closed, $F$ is certainly bounded in $D$ and so is $F_w(z,w)$, the partial of $F$ with respect to $w$. The Lipschitz condition (2.3) is implied by the boundedness of the partial derivative. We can now state and prove the following theorem.

**Theorem 2.2.1.** Under the stated assumptions on $F$, in the disk $D_0 : |z - z_0| < r$, where

$$r < \min \left( \frac{a}{M}, \frac{b}{K}, \frac{1}{K} \right),$$  
(2.5)

(2.1) has a unique analytic solution satisfying the initial-value condition $w(z_0; z_0, w_0) = w_0$.

**Proof.** Let $X$ be the set of all functions $z \to g(z)$, analytic and bounded in $D_0$ so that

$$g(z_0) = w_0, \quad \text{and} \quad |g - w_0| \leq b,$$

where

$$\|g - w_0\| = \sup_{z \in D_0} |g(z) - w_0|.$$

Under these assumptions the composite function $z \to F[z, g(z)]$ exists and is analytic in $D_0$, and its norm is at most $M$. The holomorphism follows from an applications of the series Theorem (1.3.13) plus analytic continuation. This implies that $z \to T[g](z)$ is also analytic (the integral from $z_0$ to $z$ of a analytic function is analytic); it takes the value $w_0$ at $z = z_0$ and

$$\|T[g] - w_0\| \leq Mr \leq b,$$

by (2.5). Hence $T$ maps $X$ into itself. The contraction property follows from the Lipschitz condition, for we have
\[ |T[g](z) - T[h](z)| = \int_{z_0}^{z} |F[s, g(s)] - F[s, h(s)]| \, ds \leq K \int_{z_0}^{z} |g(s) - h(s)| \, ds \leq Kr \| g - h \|, \]

Since \( Kr < 1 \) by (2.5), it is seen that

\[ \| T[g] - T[h] \| \leq Kr \| g - h \| = K \| g - h \|, \]

where \( k = Kr < 1 \). Also, \( X \) is a metric space under the sup norm; it is complete, for if a sequence \( \{g_n\} \) of \( X \) is Cauchy, then \( \lim g_n(z) = g(z) \) exists uniformly in \( D_0 \). Furthermore, \( g \) is analytic in \( D_0 \), \( g(z_0) = w_0 \), and \( \|g - w_0\| \leq b \). Thus \( g \) in \( X \) and \( X \) is complete. All the assumptions of Theorem 1.5.1 are satisfied, and we conclude that \( X \) has a unique fixed point under \( T \). This function \( z \rightarrow f(z) \) satisfies (2.2) and hence is our required solution \( w(z; z_0, w_0) \) of (2.1), by the proof of Theorem 1.5.1 we could start with any element \( g \) of \( X \) and obtain the invariant element \( f \) as

\[ f = \lim T^n[g], \]

the convergence of this sequence is comparatively slow. Convergence as the geometric series \( \sum_{i=1}^{\infty} K^n \) is all that this method gives. The method of successive approximations of Picard gives convergence as an exponential series and also dispenses with the obnoxious condition \( Kr < 1 \). \( \square \)

### 2.2.2 The Method of Successive Approximation

Consider

\[ w' = F(z, w) \quad w(z_0) = w_0 \] (2.6)

Here \((z, w) \rightarrow F(z, w)\) is analytic in the closed dicylinder

\[ D : |z - z_0| \leq a, \quad |w - w_0| \leq b, \]

and, with \( 1 \leq K \),

\[ |F(z, w)| \leq M, \]

\[ |F(z, u) - F(z, v)| \leq K|u - v|, \]
for \((z,w), (z,u),\) and \((z,v)\) in \(D\). We introduce a disk,

\[ D_0 : |z - z_0| \leq r, \quad \text{with} \quad r < \min \left\{ a, \frac{b}{M} \right\}, \]

Note that the condition \(rK < 1\) is no longer imposed. Then (2.6) has a unique analytic solution in \(D_0\).

**Theorem 2.2.2.** Under the stated conditions (2.6) has a unique analytic solution in \(D_0\).

**Proof.** The desired solution satisfy the following

\[ w(z) = w_0 + \int_{z_0}^{z} F(s, w(s)) ds, \]

We now define a sequence of approximations \(w_n(z)\) recursively by

\[ w(z_0) = w_0 \quad w_n(z) = w_0 + \int_{z_0}^{z} F(s, w_{n-1}(s)) ds \quad n > 0, \]

It should be shown that the definitions make sense for \(z\) in \(D_0\). This involves showing that the approximations exist as analytic functions in \(D_0\) and that \(|w_n(z) - w_0| \leq b\).

Suppose that this has been achieved for \(n < m\). The implication is that \(w_{m-1}(z)\) exists in \(D_0\), where it is analytic and satisfies \(|w_{m-1}(z) - w_0| \leq b\). This in turn implies that the composite function \(z \rightarrow F(z, w_{m-1}(z))\) exists and is analytic in \(D_0\). It follows that the integral exists and is a analytic function of \(z\) in \(D_0\), furthermore,

\[ \left| \int_{z_0}^{z} f(s, w_{m-1}(s)) \right| \leq M|z - z_0| \leq Mr < b, \]

so that \(w_m(z)\) is analytic in \(D_0\) and \(|w_m(z) - w_0| \leq b\). Thus the approximations exist for all \(n\), are analytic in \(D_0\), and take the value \(w_0\) at \(z = z_0\), convergence must be proved. This follows from the Lipschitz condition, which has not been used so far.

We have

\[ |w_1(z) - w_0| \leq M|z - z_0| \leq KM|z - z_0| \]

so that

\[ |w_2(z) - w_1(z)| = \left| \int_{z_0}^{z} \{ F(s, w_1(s)) - F(s, w_0) \} ds \right| \]

We have
and
\[ |w_2(z) - w_1(z)| \leq K \left| \int_{z_0}^z |w_1(s) - w_0| |ds| \right| \leq K^2 M \left| \int_{z_0}^z |s - z_0| |ds| \right| = K^2 M \frac{1}{2} |z - z_0|^2 \]

to evaluate the integral, set \( s = z_0 + (z - z_0)t \), where \( t \in [0, 1] \), this suggests taking as induction hypothesis the assumption
\[ |w_k(z) - w_{k-1}| \leq M \frac{K^k}{k!} |z - z_0|^k, \quad (2.7) \]
which holds for \( k = 2 \), giving
\[ |w_{k+1}(z) - w_k(z)| \leq K \]
Hence,
\[ |w_{k+1}(z) - w_k(z)| = \left| \int_{z_0}^z \{ F(s, w_k(s)) - F(s, w_{k-1}(s)) \} ds \right| \leq K \left| \int_{z_0}^z |w_k(s) - w_{k-1}(s)| |ds| \right| \leq \frac{K^{k+1}}{k!} \left| \int_{z_0}^z |s - z_0|^k |ds| \right| = M \frac{K^{k+1}}{(k+1)!} |z - z_0|^{k+1} \]
Thus (2.7) holds for all \( k \), and it follows that the series
\[ w(z) = w_0 + \sum_{n=1}^{\infty} [w_n(z) - w_{n-1}(z)], \]
converges absolutely and uniformly in \( D_0 \). This implies also that \( w(z) \) is analytic, at least in \( Int(D_0) \), and continuous on the boundary. Furthermore,
\[ \lim w_n(z) = w(z), \quad \lim F(z, w_{n-1}(z)) = F(z, w(z)) \]
and finally
as desired. This is clearly a solution of (2.6). It remains to show that the solution is unique. This may be proved in a number of different ways by invoking the Lipschitz condition again. Suppose that \( f(z) \) is a solution; we may assume that it is also analytic in \( D_0 \). Again we have

\[
f(z) = z_0 + \int_{z_0}^{z} f(s, f(s)) \, ds
\]

From (2.7) we obtain

\[
f(z) - w_n(z) = \int_{z_0}^{z} \{ F(s, f(s)) - F(z, w_{n-1}) \} \, ds,
\]

whence

\[
|f(z) - w_n(z)| \leq K \left| \int_{z_0}^{z} [f(s) - w_{n-1}(s)] \, ds \right|
\leq K^2 \left| \int_{z_0}^{z} [f(t) - w_{n-2}(t)] \, dt \right|
\leq K^2 \left| \int_{z_0}^{z} |s - z_0| |f(s) - w_{n-2}(s)| \, ds \right|.
\]

Repeated use of the same device finally gives

\[
|f(z) - w_n(z)| \leq K^k \left( \frac{K}{(n-1)!} \right) \left| \int_{z_0}^{z} |s - z_0|^{n-1} |f(s) - w_0| \, ds \right|
\]

Here, we may assume that \(|f(z) - w_n(z)| \leq b\), so that

\[
|f(z) - w_n(z)| \leq \frac{(Kr)^n}{n!} b,
\]

and this approach zero as \( n \) becomes infinite. Hence \( f(z) = w(z) \) and the solution is unique.

We can get another uniqueness proof from Theorem 1.5.4 If We have two solutions, \( w(z) \) and \( f(z) \), their difference satisfies

\[
f(z) - w(z) = \int_{z_0}^{z} \{ F(s, f(s)) - F(s, w(s)) \} \, ds,
\]
and if \(|f(z) - w(z)| = g(z)|\) then
\[
g(z) \leq k \left| \int_{z_0}^{z} g(s) \, ds \right|
\]

Here, we set, with \(\theta = \arg(z - z_0)\),
\[
s = z_0 + te^{i\theta}, \quad z = z_0 + ue^{i\theta}, \quad g(s) = h(t), \quad g(z) = h(u)
\]
and obtain the following inequality
\[
h(u) \leq K \int_{0}^{u} h(t) \, dt,
\]
Here \(h(t)\) is continuous and non-negative. But by Theorem 1.5.4 the only solution of this inequality in \(C^+[0,a]\) is \(h(t) = 0\). This gives \(g(z) = 0\) and \(f(z) = w(z)\).

### 2.2.3 Majorants and Majorant Method

Consider the ODE
\[
w'(z) = F(z, w), \quad w(z_0) = w_0 \tag{2.9}
\]
where \(F(z, w)\) is analytic in dicylinder defined by
\[
D : |z - z_0| \leq a, \quad |w - w_0| \leq b,
\]

**Definition 2.2.1.** Let
\[
f(z) = \sum_{n=0}^{\infty} c_n z^n \tag{2.10}
\]
be a power series with a positive radius of convergence, and let
\[
g(z) = \sum_{n=0}^{\infty} C_n z^n \tag{2.11}
\]
be a power series with non-negative coefficients and the radius of convergence \(R\). we say that \(g(z)\) is a majorant of \(f(z)\) and symbolized by
\[
f(z) \ll g(z),
\]
if
\[ |c_n| \leq C_n \quad \forall n, \]

This clearly implies that the radius of convergence of (2.10) is at least \( R \). The relation \( \ll \) is transitive.

\[ f \ll g \quad \text{and} \quad g \ll h \quad \text{imply} \quad f \ll h, \]

**Lemma 2.2.1.** If \( m \) is a positive integer and \( f \ll g \), then

\[ [f(z)]^m \ll [g(z)]^m, \]

Furthermore,

\[ f^m(z) \ll g^m(z), \]

\[ \int_0^z f(t)dt \ll \int_0^z g(t)dt, \]

Repeated integration also preserve the majorant relation.

**Lemma 2.2.2.** Let

\[ h(w) = \sum_{n=0}^{\infty} h_n w^n, \quad H(w) = \sum_{n=0}^{\infty} H_n w^n \quad \text{with} \quad h(w) \ll H(w), \]

the series is being convergent for \( |w| \leq R_0 \). Suppose \( f \ll g \), \( f(0) = g(0) = 0 \), and \( |g(z)| \leq R_0 \) for \( |z| \leq R \). Then

\[ h[f(z)] \ll H[g(z)] \]

**Proof.** Suppose that \( f_i \) and \( g_i \), for \( j = 1, 2, \ldots, m \) are power series, the \( g_i \)'s being absolutely convergent for \( |z| \leq R \), and suppose that

\[ f_i \ll g_i, \quad j = 1, 2, \ldots, m. \]

Let \( a_1, a_2, \ldots, a_m \) be arbitrary numbers, and \( |a_j| \leq A_j \). Then

\[ a_1 f_1 + a_2 f_2 + \ldots + a_m f_m \ll A_1 g_1 + A_2 g_2 + \ldots + A_m g_m \]

Since the right member is power series in \( z \) convergent for \( |z| \leq R \), the same holds for the left member.
We apply this to case
\[ f_j(z) = [f(z)]^j, \quad g_j(z) = [g(z)]^j, \quad a_j = h_j, \quad A_j = H_j, \]
Then we have for each \( m \)
\[ h_1 f(z) + h_2 [f(z)]^2 + \ldots + h_m [f(z)]^m \leq H_1 g(z) + H_2 [g(z)]^2 + \ldots + H_m [g(z)]^m \tag{2.12} \]
Here the right-hand member is the \( m \)th partial sum of the series \( \sum_{n=1}^{\infty} H_n w^n \) with \( w \) replaced by \( g(z) \).
In the disk \( |z| < R - \delta \) we fined an \( \epsilon > 0 \) such that \( |g(z)| < R_0 - \epsilon \). Since the series \( \sum_{n=1}^{\infty} H_n (R_0 - \epsilon)^n \) converges, the series \( \sum_{n=1}^{\infty} H_n |g(z)|^n \) converges uniformly in \( |z| < R_0 - \delta \) to analytic function \( L(z) \) with non-negative coefficient in its Maclaurin expansion.
The series
\[ K(z) = \sum_{n=1}^{\infty} h_n [f(z)]^n, \]
is also convergent for \( |z| < R_0 \), and majorant relation (2.12), which holds for a partial sums, holds also in the limit so that \( K(z) \leq L(z) \), as asserted.

The majorant concept extends to functions of several variables. Suppose, in particular, that
\[ F(z, w) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} c_{jk} z^j w^k \tag{2.13} \]
is analytic function of \((z, w)\) in the dicylinder
\[ D : |z| \leq a, \ |w| \leq b \]
Suppose that
\[ G(z, w) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} C_{jk} z^j w^k \tag{2.14} \]
is also analytic in \( D \). We say that \( G \) is a majorant of \( F \):
\[ F(z, w) \ll G(z, w) \quad \text{if} \quad |c_{jk}| \leq C_{jk}, \quad \forall j, k \tag{2.15} \]
We consider the OCDE(2.9), were for simplicity we take $z_0 = w_0 = 0$. Here the fundamental fact is that a *majorant relation* for the right-hand sides implies the corresponding majorant relations.

**Theorem 2.2.3.** Let $F(z, w)$ be defined by the series (2.13), and let $G(z, w)$ be a majorant of $F(z, w)$ defined by (2.14) and (2.15). Suppose that

$$W'(z) = G[z, W(z)], \quad W(0) = 0 \tag{2.16}$$

has solution

$$W(z) = \sum_{j=1}^{\infty} C_j z^j \tag{2.17}$$

which convergence for $|z| < r$.

Let

$$w(z) = \sum_{j=1}^{\infty} c_j z^j \tag{2.18}$$

be a formal solution of (2.9). Then

$$w(z) \ll W(z),$$

and the series (2.17) is absolutely convergent for $|z| < r$ and is the unique solution of (2.9)

**Proof.** The derivation of (2.18) involves a number of operations on power series which are legitimate if and only if the series are absolutely convergent. The series are "formal" if the required operations have been performed without regard to legitimate, the justification being given a *posteriori* when it is found that the series are indeed absolutely convergent.

We have to construct the composite series

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} c_{jk} z^j \left[ \sum_{p=1}^{\infty} c_{p} z^p \right]^k,$$

Here we start by forming the $k$th powers by using Cauchy's product formula (valid for absolutely convergent series and leading to absolutely convergent series). The $k$'th series is multiplied by $c_{jk} z^j$. and the result is summed for $j$ and $k$. We then rearrange the result as a power series in $z$. This can be justified with the aid of the double series.
2.2 Existence and Uniqueness

provided that we have absolute convergence. In the result the coefficient of \( z^n \) will be a polynomial in the \( c_{jk} \)'s and the \( c_p \)'s, say

\[ M(c_{jk}; c_p), \]

Here the big question is, what \( c_{jk} \)'s occur, and what \( c_p \)? To the first question we can reply that a necessary condition for \( c_{jk} \) to occur in \( M_n \) is that \( j + k \leq n \). This is gratifying, for it means that \( c_{n+1} \) depends only on a finite number of the known coefficients \( c_{jk} \), at most \( \frac{1}{2}(n+1)(n+2) \) in number. Furthermore, the only \( c_p \)'s that can occur are \( c_1, c_2, \ldots, c_n \), all of which have been determined by the time the \( (n+1) \)th coefficient is considered. This means that the coefficients \( c_p \), can be determined successively and uniquely. The numerical constants which enter when the \( k \)th powers are formed are positive integers, a fact which is also important. The series (2.18) is a formal solution if

\[ (n+1)c_{n+1} = M_n(c_{jk}; c_p) \]  

for all \( n \). The first three coefficients in (2.19) are

\[ c_1 = c_{00} \]
\[ c_2 = \frac{1}{2}(c_{10} + c_{01}c_{00}) \]  
\[ c_3 = \frac{1}{2}[c_{20} + c_{11}c_{00} + c_{02}(c_{00})^2 + \frac{1}{2}c_{01}c_{10} + \frac{1}{2}(c_{01})^2c_{00}]. \]

This means that all coefficients \( c_p \) are ultimately determined in terms of the coefficients of \( F(z, w) \). If the same procedure is applied to the majorant equation (2.16), we get exactly the same formulas for determining the \( C_n \)'s, provided that we replace the lower case letters in \( M_n \) by the corresponding capitals. Then, it is clear that

\[ |c_n| \leq C_n \quad \forall n, \]

so \( W(z) \) is a majorant for \( w(z) \) provided that the series (2.17) has a disk of convergence. If we know that this is the case, it follows that (2.18) has a radius of convergence at least as large as that of \( W(z) \). And now all the operations performed to get the coefficients are justified, and the formal series is an actual solution. The coefficients \( c_n \) are uniquely determined so that (2.18) is the only solution which is analytic in some neighborhood of \( z = 0 \). There is still the possibility of the existence of a non-analytic
solution. If this would be given by a series in terms of fractional powers of \( z \), either the series itself or some derivative thereof would become infinite as \( z \to 0 \). This cannot be a real solution of an equation of type (2.9), for a solution of such an equation must possess derivatives of all orders continuous at \( z = 0 \). Non-analytic solutions can also be excluded by one of the uniqueness theorems of Section 2.2.2.

### 2.3 Singularities

In previous section, our equation of the study

\[
\begin{align*}
  w'(z) &= F(z, w(z)), \\
  w(z_0) &= w_0
\end{align*}
\]

led to the conclusion that there is a unique analytic solution whenever the mapping \((z, w) \to F(z, w)\) is analytic at \((z_0, w_0)\). In this section we will study what happens at a point where this assumption does not hold. Such a point is a singularity at least for the solution which approaches \(w_0\) when \(z \to z_0\). But there are other possibilities. Solutions may become infinite as \(z \to z_0\) or may not tend to any limit, finite or infinite. Usually the point at infinity is a fixed singularity.

#### 2.3.1 Fixed and Movable Singularities

Singularities of CDE’s are of two types: fixed singularity and movable singularity. Fixed singularities are external in the sense that they are given position and nature by the DE, and at least the position should be obtainable in a finite number of steps without knowledge of the solutions. Movable singularities, on the other hand, depend on internal parameters, and normally such a point can be put anywhere in the complex plane by manipulating an internal parameter. Shift the parameter and the singularity moves. Thus the equation

\[
w' = w^2 - \pi \cot^2(\pi z),
\]

has fixed singularities at the integers and at infinity. The finite singularities are simple poles for some solutions, but the general solution has transcendental branch points where infinitely many branches are permuted. The movable singularities are simple poles and may be placed anywhere in the complex plane.

The nature of a movable singularity will depend on the nature of the equation. Thus the equations
are satisfied by
\[ w = \log \left( \frac{1}{z - c} \right) \quad \text{and} \quad v = \exp \left( \frac{1}{z - c} \right), \]
respectively. In the first case there is a movable logarithmic branch point; in the second, a movable essential singularity.

In the following discussion such cases will be excluded. In the first order case (but not for systems) this can be achieved by assuming that \( F(z, w) \) is a rational function of \( w \) with coefficients which are algebraic functions of \( z \). This restriction makes the number of fixed singularities finite and permits only algebraic movable singularities. During most of our work we will make the further restriction that the coefficients are single valued; this implies that they may be assumed to be polynomials. We consider equations of the form

\[ w' = \frac{P(z, w)}{Q(z, w)} \quad (2.21) \]

With
\[ P(z, w) = \sum_{j=0}^{p} A_j(z)w^j, \quad Q(z, w) = \sum_{k=0}^{q} B_k(z)w^k, \]
where the \( A' \)s and \( B' \)s are algebraic functions of \( z \).

It is appropriate to illustrate a brief abstract of the theory of algebraic functions. We will need some of the properties of such functions for the discussion of the fixed singularities of (2.21). For each algebraic function \( z \to A(z) \) there is a unique irreducible polynomial in two variables:

\[ V(z, u) = P_0(z)u^n + P_1(z)u^{n-1} + \ldots + P_{n-1}(z)u + P_n(z), \]

such that \( V[z, A(z)] \equiv 0 \). Here the \( p' \)s are polynomials in \( z \) without a common factor. Conversely, every such polynomial \( V(z, u) \) defines an algebraic function. The zeros of \( A(z) \) are the zeros of \( p_n(z) \); the infinitudes, the zeros of \( p_0(z) \). The point at infinity may be a singular point of \( A(z) \). Additional singularities may be furnished by the zeros of the discriminant equation

\[ D_0(z) = 0. \]
If \( z = z_0 \) is a root of this equation, then the equation

\[
V(z_0, u) = 0,
\]

has multiple roots and \( z = z_0 \) is an algebraic branch point of \( A(z) \). Here \( D_0(z) \) is the resultant of eliminating \( u \) between the two equations \( V(z, u) = 0 \) and \( V_u(z, u) = 0 \). Since the number of multiple roots is at most equal to the total number of roots, which is \( n \), we see that the number of singular points of \( A(z) \) is finite and is at most \( n + 1 + \text{deg}(p_0) \).

It should be noted that the algebraic functions of one complex variable form a field over \( C \): if \( A_1 \) and \( A_2 \) are algebraic functions, then so are their sum, product, and quotients.

We can start an enumeration of the possible fixed singular points of the CDE (2.21). The first subset of such points is formed by the singularities of the \( A \)'s and the \( B \)'s, say,

\[
S_2 : s_1, s_2, \ldots, s_u, \infty,
\]

This is a finite set.

The second subset \( S_2 \) is the set of points \( t_j \), if any, such that \( Q(t_j, w) \) identically zero, say

\[
S_2 : t_1, t_2, \ldots, t_v
\]

At \( z = t_j \) all the algebraic functions \( B_0, B_1, \ldots, B_q \) are zero, and this means that the corresponding polynomials \( V_j(z, 0) \) are zero, where \( V_j(z, u) \) is the defining polynomial for \( B_j(z) \). Now the common roots of a number of polynomials can be found, for instance by the method of Bezout. We can affirm, however, that the number of common roots cannot exceed \( \min_j(m_j) \), where \( m_j \) is the degree of \( V_j(z, 0) \).

The third subset,

\[
S_3 : e_1, e_2, \ldots, e_\lambda,
\]

is formed by the roots of another discriminant \( D_3(z) \), the vanishing of which expresses the existence of numbers \( w_0 \) such that

\[
P(e_j, w_0) = 0, \quad Q(e_j, w_0) = 0,
\]

so that \( P/Q \) takes on the indeterminate form \( 0/0 \) at \( (z_0, w_0) \). Now the condition for \( P(z, w) = 0, Q(z, w) = 0 \) to have a common root is the vanishing of a determinant with
2.3 Singularities

$p + q$ rows and columns, the entries being the $p + q + 2$ coefficients $A_j$ and $B_k$ with the rows filled out with zeros. Now $D_3(z)$ is an algebraic function of $z$ obtained by expanding the determinant, and as an algebraic function it satisfies a polynomial equation $V(z, u) = 0$. Thus the $e$'s are the roots of the polynomial $V(z, 0)$. Hence the set $S_3$ is also finite. In studying the infinitudes of the solutions of (2.21), we use the transformation $w = \frac{1}{v}$, which transforms (2.21) into

$$v' = \frac{v^2 P(z, \frac{1}{v})}{Q(z, \frac{1}{v})} = \frac{P_1(z, v)}{Q_1(z, v)},$$

where $P_1$, and $Q_1$, are polynomials in $v$ with the $A_j$'s and $B_k$'s as coefficients. The fixed singularities of this equation are also fixed singularities of (2.21). The sets $S_1$, and $S_2$ are unchanged, but the result of eliminating $v$ between $P_1 = 0$ and $Q_1 = 0$ is not necessarily the same as that of eliminating $w$ between $P = 0$ and $Q = 0$, since now the common roots of $P_1(z, 0) = 0$ and $Q_1(z, 0) = 0$ are relevant. Thus we have possibly an additional set of fixed singularities,

$$S_4 : i_1, i_2, \ldots, i_n.$$  \hfill (2.22)

The set (2.22) is a finite one, and the union of the four sets $S_1 - S_4$ is a finite set $S$. This is the set of potential fixed singularities of (2.21).

A point of the set $S$ need not be a singularity of all or any of the solutions, as is illustrated by the following CDE's:

$$u' = 1 + z^2 u - z^2 u^2, \quad w' = \frac{\mu w}{z},$$

In both cases $z = 0$ as a singularity of the coefficients is a potential singularity. The first equation has the particular solution $u = z$, which is analytic at $z = 0$. The general solution has a branch point of order 1 there \textit{(order 1 means two branches)}. The second equation has the general solution $Cz^\mu$, which is analytic at $z = 0$ if and only if $\mu$ is a positive integer. There is a pole if $\mu$ is a negative integer, an algebraic branch point if $\mu$ is a rational number, and a transcendental critical point in all other cases. Here the nature of the singularity depends essentially on the algebraic nature of the external parameter $\mu$.

$$P[f(z), g(z)] \equiv 0$$
In particular, if \( f(z) \) is elliptic so is \( f'(z) \), and the two have the same periods so that there exists a polynomial CDE

\[
P(w, w') = \sum c_{jk}(w)^j(w')^k = 0 \tag{2.23}
\]

which is satisfied by \( f(z) \) and, more generally, by \( f(z + a) \), where \( a \) is an arbitrary constant since (2.23) is autonomous, does not contain \( z \) explicitly. Similar equations of the second or higher orders also hold for elliptic functions.

### 2.3.2 Analytic Continuation

We consider again the IVB

\[
w' = F(z, w) = \frac{P(z, w)}{Q(z, w)}, \quad w(z_0) = w_0 \tag{2.24}
\]

with the unique solution \( w(z; z_0, w_0) \). Here \( P \) and \( Q \) are polynomials in \( w \) of degree \( p \) and \( q \), respectively, and the coefficients are algebraic functions of \( z \). Furthermore, \((z, w) \to F(z, w)\) is analytic at \((z_0, w_0)\). In particular, \( z_0 \) does not belong to the set \( S \) of fixed singularities. Starting at \( z = z_0 \), we continue \( w(z; z_0, w_0) \) analytically along a path \( C \). We continue \( F[z, w(z; z_0, w_0)] \) along the same path. This means that \( z \) has to avoid the singularities of the coefficients \( A_j \) and \( B_k \), as well as points where \( Q(z, w) \) could vanish. If \( w(z; z_0, w_0) \) admits of analytic continuation, so do its powers of finite order; both \( P[z, w(z; z_0, w_0)] \) and \( Q[z, w(z; z_0, w_0)] \) can be continued along any path along which \( w[z; z_0, w_0] \) can be continued, but their quotient may possibly impose some conditions on the path \( C \). At any rate, under the stated assumptions the law of permanence of functional equations (Theorem 1.3.14) permits us to conclude that the analytic continuation of the solution \( w(z; z_0, w_0) \) satisfies the CDE all along \( C \). Moreover, if \( z = z^* \) is a point on \( C \) and if \( w(z^*; z_0, w_0) = w^* \), there exists a disk, its center at \( z^* \), in which \( w(z^*, z_0, w_0) \) coincides with the local solution \( w(z; z^*, w^*) \). We now suppose that \( w(z; z_0, w_0) \) has been continued analytically along \( C \) from \( z = z_0 \) to \( z = z_1 \), and ask what will happen to the solution as \( z \to z_1 \). There are various possibilities as follows.

1. \( w(z; z_0, w_0) \to w_1 \), and \( F(z, w) \) is analytic at \((z_1, w_1)\)
2. \( F(z, w) \) is not analytic at \((z_1, w_1)\), but \( [F(z, w)]^{-1} \) is analytic there and \( Q(z_1, w) \neq 0 \).
3. As in possibility 2, but \( Q(z_1, w) \equiv 0 \).
4 $P(z, w)$ and $Q(z, w)$ are analytic at $(z_1, w_1)$, but both are zero there.

5 At least one of the functions $P$ and $Q$ is not analytic at $(z_1 w_1)$.

6 As $z \to z_1, w \to \infty$

7 $w(z; z_0, w_0)$ does not tend to any limit, finite or infinite, as $z \to z_1$

We list here the results to be proved in the next theorems. In case:

1 The point $z = z_1$ is not a singularity.

2 The point is a movable algebraic branch point.

3 -5 and 7 The point $z = z_1$ belongs to the set $S$ of fixed singular points.

6 The point is a singularity which may be fixed or movable.

Let us start with case 1.

[Case: 1]

**Theorem 2.3.1. [Painleve Theorem]** If analytic continuation of a given solution of (2.24) leads to a point $z_1$ and a function value $w = w_1$ and if $F(z, w)$ is analytic at $(z_1, w_1)$, then the local solution $w(z; z_1, w_1)$ gives the analytic continuation of the given solution in the disk where $w(z; z_1, w_1)$ is analytic.

**Proof.** In this case a local solution $w(z; z^*, w^*)$ exists for all $z^*$ on $C$, including the point $z = z_1$, where we can take $w^* = w_1$. Now $w(z; z_1, w_1)$ is analytic in some disk $|z - z_1| < r$, and this disk contains the end of $C$. If $z^* \in C$ and $|z_1 - z^*| < r$, then at $z = z^*$ we have two solutions, $w(z; z^*, w^*)$ and $w(z; z_1, w_1)$. They are identical provided that they take on the same value at $z = z^*$. In this case $w(z; z_1, w_1)$ is the analytic continuation of $w(z; z^*, w^*)$ and vice versa. Now along $C$ the radius of convergence of the power series representing $w(z; z^*, w^*)$ is a continuous, positive function of $z^*$ bounded away from zero. If we observe that $F(z, w)$ is analytic at each point $(z^*, w^*)$ for $z^*$ on $C$ and $w^* = w(z^*, z_0, w_0)$, except possibly at the endpoint, where $w^* = w_1$. We may assume that $C$ is rectifiable and that

$$C : z = z^*(t), \quad 0 \leq t \leq L,$$

where $L$ is the length of $C$. This is a compact set in the plane. At each point $z^*(t)$ on $C$ there are corresponding quantities $a(t), b(t)$, and $M(t)$. These quantities refer to the
local expansion of \( F(z,w) \) in powers of \( z - z^*(t) \) and \( w - w^*(t) \) and may be assumed to vary continuously with \( t \). Then asserts that the local solution \( w[z; z^*(t), w^*(f)] \) has an expansion which converges at least for

\[
|z - z^*(t)| < r(t), \quad 0 \leq t \leq L, \tag{2.25}
\]

where

\[
r(t) = a(t) \left( 1 - \exp \left[ -\frac{b(t)}{2a(t)M(t)} \right] \right),
\]

Now we can cover \( C \) by a set of disks defined by (2.25), and since \( C \) is compact the Heine-Borel theorem ensures the existence of a finite sub-covering corresponding to

\[
t_0 = 0 < t_1 < t_2 < \ldots < t_n < L,
\]

We may assume that the sub-covering is so dense that the center of the \((j + 1)\)th disk lies in the \( j \)th one. This implies that the \( n \)th disk contains \( z_1 \), so that \( w[z, z^*(t_n), w^*(t_n)] \) is analytic at \( z = z_1 \). Since this solution is the result of \( n \) successive rearrangements (See 1.39) of the solution \( w(z; z_0, w_0) \) at the points \( z^*(t_j), j = 1, 2, ..., n \), its value at \( z = z_1 \) is \( w \), the postulated limit of the solution for analytic continuation along \( C \). Thus we have two solutions of the CDE at \( z = z_1 \), with the same value \( w_1 \) there. Hence they are identical, and the theorem is proved. \( \square \)

[Case: 2]

**Theorem 2.3.2.** If \( w(z; z_0, w_0) \to w_1 \), as \( z \to z_1 \), if \( P \) and \( Q \) are analytic at \( (z_1, w_1) \), if \( P(z_1, w_1) \neq 0 \), and if \( Q(z_1, w) \) has a zero of exact multiplicity \( k \) at \( w = w_1 \), then \( z = z_1 \), is an algebraic branch point of order \( k \) for \( w(z; z_0, w_0) \).

**Proof.** The assumptions evidently imply that \( F(z, w) \) is not analytic at \( (z_1, w_1) \), while \( [F(z, w)]^{-1} = G(z, w) \) is analytic and not zero at all points \( (z_1, w) \). We consider the solution of the IVB,

\[
\frac{dz}{dw} = G(z, w), \quad z(w_1) = z_1 \tag{2.26}
\]

This equation has a unique solution by the existence and uniqueness theorems, say,

\[
z(w; w_1, z_1) = z_1 + \sum_{n=1}^{\infty} c_n (w - w_1)^n \tag{2.27}
\]
Since $G(z_1, w)$ has a multiple zero at $w = w_1$ a certain number of the coefficients $c_n$ are zero. To make this precise, suppose that $G(z, w)$ is presented to us in the form

$$G(z, w) = \sum_{n=0}^{\infty} g_n(w)(z - z_1)^n,$$

where the functions $w \to g_n(w)$ are analytic in one and the same disk $|w - w_1| < r$ for $z = z_1$ this reduces to $g_0(w)$, so that

$$G(z_1, w) = g_0(w) = \sum g_{m0}(w - w_1)^m,$$

Now by assumption $Q(z_1, w)$ has a zero of order $k$ at $w = w_1$, and, since $P(z_1, w_1) \neq 0$, this implies the same property for $G(z_1, w)$. It follows that

$$g_{00} = g_{10} = \ldots = g_{(k-1)0} = 0, \quad g_{k0} \neq 0,$$

We have now from (2.26)

$$\sum_{m=1}^{\infty} mc_m(w - w_1)^{m-1} = \sum_{n=0}^{\infty} g_n(w) \left[ \sum_{m=0}^{\infty} c_m(w - w_1)^m \right]^n \tag{2.28}$$

Suppose that in the expansion (2.27) the series starts with the power $(w - w_1)^p$. This means that in (2.28) we have on the left $(w - w)^{p-1}$ as the lowest power, while on the right $(w - w_1)^k$ furnishes the lowest power. Hence

$$p = k + 1, \quad (k+1)c_{k+1} = g_{k0} \neq 0,$$

Thus, we have

$$z - z_1 = (w - w_1)^{k+1} \sum_{m=0}^{\infty} c_{k+1+m}(w - w_1)^m,$$

Since the power series does not vanish for $w = w_1$ we can extract $(k+1)$th root on both sides and obtain

$$(z - z_1)^{1/(k+1)} = (w - w_1) \sum_{m=0}^{\infty} d_{k+1+m}(w - w_1)^m,$$

Here $d_{k+1} = (c_{k+1})^{1/(k+1)} \neq 0$, and the power series has a positive radius of convergence by the double series theorem of Weierstrass.

We set
and want to solve the equation

\[ s = t \sum_{m=0}^{\infty} d_{k+1+m} t^m, \]

for \( t \), the assumptions of which are satisfied. Hence

\[ t = \sum_{j=1}^{\infty} b_j s^j, \]

with a positive radius of convergence. Reverting to the old variables, we get

\[ w - w_1 = \sum_{j=1}^{\infty} b_j (z-z_1)^{\frac{1}{k+1}}, \]

This means that the analytic continuation of \( w(z; z_0, w_0) \) along \( C \) has led to a branch point of order \( k \) of the solution where \( k + 1 \) branches are permuted cyclically.

Here a question arises: In what sense is this a movable singularity, or would it have to rate as a fixed one? To elucidate this point let us examine the conditions for this type of singularity to occur. Give \( z \) a value \( z_0 \), not belonging to the set \( S \) of fixed singularities, and examine the equation

\[ Q(z^0, w) = \sum_{k=0}^{q} B_k(z^0) w^k = 0 \quad (2.29) \]

This is an algebraic equation for \( w \) of degree \( q \) unless \( B_q(z^0) = 0 \). Normally there are \( q \) roots \( w_1^0, w_2^0, ..., w_q^0 \). If \( w_0^0 \) is one of these roots, the event \( w(z) \to w_0^0 \) as \( z \to z_0^0 \) gives rise to a branch point for \( w(z) \) at \( z = z_0 \), and normally the branch point is of order 1 with two permuting branches. The roots of (2.29) are continuous (even analytic) functions of \( z_0 \), so a small change in \( z_0 \) imposes a small change in \( w_0 \). This means that there is a solution \( w^*(z) \) close to \( w(z) \) which has a branch point \( z_0^* \) near to \( z_0 \), and normally this branch point is also of order 1. In this sense we can speak of movable branch points.

We consider cases 3-5.

The point \( z_0 \) is a fixed singularity in each of the three cases, and almost anything and everything can happen at such a point. For case 6, here we set \( w = \frac{1}{r} \). The transformed equation is
\[ v' = -v^2 \frac{P(z, \frac{1}{v})}{Q(z, \frac{1}{v})}, \quad v(z) \to 0 \quad \text{as} \quad z \to z_1 \quad (2.30) \]

Again, we assume that \( z_1 \notin S \). There are two distinct subcases according to whether \( q + 2 - p > 0 \) or \( \leq 0 \).

**Case 6:1.** if \( q + 2 > p \) (2.30) becomes

\[ v' = -v^{q+2-p} \sum A_j(z) v^{p-j} - \sum B_k(z) v^{q-k} \equiv \frac{P_1(z, v)}{Q_1(z, v)}. \]

Here \( P_1(0, 0) = 0 \) and \( Q_1(z_1, 0) = B_q(z_1) \), which is \( \neq 0 \) since \( z_1 \) is not a fixed singularity. For the same reason all the coefficients \( A_j \) and \( B_k \) are analytic at \( z = z_1 \). Since it follows that the last member of (2.30) is analytic at \( (z_1, 0) \), (2.30) has a unique solution which takes the value 0 at \( z = z_1 \). By inspection one sees that \( v(z) \equiv 0 \) is that solution, and this means that case 6:1 cannot occur.

**Case 6:2.** Suppose first that \( q + 2 = p \). Then (2.30) takes the form

\[ v' = G(z, v), \quad \text{where} \quad G(z_1, 0) = -\frac{A_p(z_1)}{B_q(z_1)}, \]

Now, \( G(z, w) \) is analytic at \( (z_1, 0) \), so there is a unique solution which vanishes at \( z = z_1 \). If \( A_p(z_1) \neq 0 \), this is a simple zero; but if \( A_p(z) \) has a zero of multiplicity \( m \) at \( z = z_1 \), then \( v(z; z_1, 0) \) has a zero of multiplicity \( m + 1 \) at \( z = z_1 \). This means a pole of multiplicity \( m + 1 \) for \( w(z; z_0, w_0) \). Next, suppose that \( q + 2 < p \). Then we replace (2.30) by

\[ \frac{dz}{dv} = v^{p-q-2} H(v, z), \quad z(0) = z_1 \quad (2.31) \]

with obvious notation. Here we may assume \( H(v, z) \) to be analytic at \((0, z_1)\) and different from zero. In this case \( A_p(z_1) \neq 0 \) since \( z_1 \) does not belong to \( S \), and this guarantees for \( H \) to analytic.

Now, \( H(0, z_1) = 0 \) if and only iff \( B_q(z_1) = 0 \), so our assumption implies that \( B_q(z_1) \neq 0 \). If this assumption is not satisfied, the following argument has to be modified along the lines used in the proof of Theorem 2.3.2 Under the stated assumption, (2.31) has a unique solution of the form

\[ z - z_1 = v^{p-q-1}(c_0 + c_1 v + \ldots), \quad c_0 \neq 0, \]

Here we extract the \((p - q - 1)\)th root and invert the result to obtain
\[ v(z) = \sum_{j=1}^{\infty} d_j (z - z_1)^{\frac{j}{p - q - 1}} \quad d_1 \neq 0, \]

Now, \( w(z) \) is the reciprocal of this, so that
\[ w(z) = (z - z_1)^{\frac{1}{p - q - 1}} \sum_{j=0}^{\infty} e_j (z - z_1)^{\frac{j}{p - q - 1}} \] (2.32)

where \( e_0 \neq 0 \) and the series has a positive radius of convergence. If \( B_q(z_1) \) should be zero and, more generally, \( H(v, z_1) \) has a zero of order \( k \), then in (2.32) \( p - q - 1 \) should be replaced by \( p - q - 1 + k \), as is seen by imitating the argument used in the proof of Theorem 2.3.2 Thus we have proved

**Theorem 2.3.3.** Equation (2.24) has no movable infinitudes if \( p < q + 2 \). There are movable poles if \( p = q + 2 \). The poles are simple unless accidentally \( A_p(z) \) vanishes at the point in question. If \( p > q + 2 \), there are movable branch points where the solution becomes infinite. Normally \( p - q - 1 \) branches are permuted at such a point.
Chapter 3

Analytical Solution of Ordinary CDE

In this chapter we made a method of solution for the complex ordinary complex differential equations. This method has never been mentioned before in such a way as organized way, but we based on the ordering of the real differential equations in some of the most known references.

3.1 First-Order Linear Complex Differential Equations

The first order ordinary complex differential equations is written in the general form

$$\frac{dw}{dz} + P(z)w = Q(z)$$  \hspace{1cm} (3.1)

where $P(z)$ and $Q(z)$ are analytic functions on their domain, so, the solution will be gained by several steps. Let assume that there is an analytical complex valued function $v(z)$, when we multiply the 3.1 by $v(z)$ we get the following form

$$v(z)\frac{dw}{dz} + v(z)P(z)w = v(z)Q(z),$$  \hspace{1cm} (3.2)

now we have to know that we have already assumed that $v(z)$ satisfied...

$$\frac{d}{dz}[v(z)w(z)] = v(z)Q(z),$$  \hspace{1cm} (3.3)

so that, by integration both sides of 3.2 we get

$$v(z)w(z) = \int v(z)Q(z)dz.,$$  \hspace{1cm} (3.4)
and

\[ w = \frac{1}{v(z)} \int v(z)Q(z)\,dz, \quad (3.5) \]

Now we need to find \( v(z) \) for every equation 3.1 we could get, because of the equality

\[ \frac{d}{dz}(vw) = v\frac{dw}{dz} + vPw, \quad (3.6) \]

hold, the we fine the missing part

\[ v\frac{dw}{dz} + w\frac{dv}{dz} = v\frac{dw}{dz} + vPw \Rightarrow vw' = vPw \quad (3.7) \]

we can see easily that

\[ v = e^{\int P(z)} \quad (3.8) \]

**Example 3.1.**
Consider the following first order ordinary CDE

\[ w' + w = z, \]

we have

\[ P(z) = 1, \quad Q(z) = z, \quad v = e^{\int dz} \]

and

\[ w = \frac{1}{e^z} \int ze^z\,dz \quad \Rightarrow \quad w = \frac{1}{e^z}[e^z(z-1) + C] \quad (3.9) \]

so the general solution has the following form

\[ w = Ce^{-z} + z - 1 \quad (3.10) \]

**Example 3.2.**
Let the first order ordinary CDE

\[ (1 + 3i)w' + zw = z^2, \]

First we put the equation in the standard form

\[ w' + \frac{z}{1+3i}w = \frac{z^2}{1+3i}, \]
3.2 N’th-Order Linear Complex Differential Equations

Now, we find the integration factor $v(z)$

$$v(z) = e^{\int \frac{1}{1+3i} dz} \Rightarrow v(z) = e^{\frac{z^2}{2(1+3i)}},$$

then, substitute $V(z)$ in the following

$$w(z) = \frac{1}{v(z)} \int v(z) q(z) dz,$$

We get

$$w(z) = e^{-\frac{z^2}{2(1+3i)}} \int e^{\frac{z^2}{2(1+3i)}} \frac{z^2}{1+3i} dz,$$

And after simplifying the general solution will be

$$w(z) = \frac{1}{1+3i} e^{\frac{z^2}{2(1+3i)}} \int z^2 e^{\frac{z^2}{2}} dz,$$

Then

$$w(z) = \frac{1}{1+3i} e^{\frac{z^2}{2(1+3i)}} \left[\frac{ze^{\frac{z^2}{2}}}{2} - \frac{1}{4} \sqrt{\pi} \text{erfi}(z) + C\right].$$

3.2 N’th-Order Linear Complex Differential Equations

The general form of n’th order linear complex differential equations is

$$\sum_{i=0}^{n} f_i(z) w^{(n)}(z) = G(z),$$

where $f_i(z) i = 1, 2, 3, 4, \ldots$, are complex functions or polynomials or constants. We are going to call it n’th OLCDE, There are two types of such equations:

1- Homogeneous n’th OLCDE when $G(z) = 0$

2- Non-homogeneous n’th OLCDE when $G(z) \neq 0$
3.2.1 The Homogeneous N’th Order Linear CDE with Constant Coefficients

The standard form of the N’th order linear complex differential equations is:

\[ \sum_{i=0}^{n} \alpha_i w^{(i)} = G(z), \]  

(3.11)

where \( \alpha_i \) are complex constants and \( G(z) = 0 \)

**Example 3.3.**
The Homogeneous LCDE

\[ w'' - w = 0, \]

Let assume that the solution is \( w = e^{rz} \) so we will fined \( w'' \), and substitute \( w \) and \( w'' \) in the main equation

\[ w' = re^{rz} \text{ and } w'' = r^2 e^{rz} \]

\[ r^2 e^{rz} - e^{rz} = 0 \quad \Rightarrow \quad r^2 - 1 = 0 \quad \Rightarrow \quad r = \pm r \]

(3.12)

from the 3.12 we can see the CDE has two solutions, and from the linearity property which says every linear compilation of two solution is also a solution, so that The general solution will be

\[ w(z) = c_1 e^{-z} + c_2 e^{z}, \]

**Example 3.4.**
Consider the LCDE

\[ w'' - (8 + 6i)w = 0, \]

we start with the equation

\[ r^2 - (8 + 6i) = 0 \quad \Rightarrow \quad r^2 = 8 + 6i \quad \Rightarrow \quad r = \sqrt{8 + 6i}, \]

Now we have to find the value of \( r \) let assume that \( x + yi \) is the value of \( r \), then we can put the following relation

\[ (x + yi)^2 = 8 + 6i, \]
where \( x, y \) are in \( \mathbb{R} \)

\[
x^2 - y^2 + xyi = 8 + 6i,
\]
from this equality we get two equations

\[
x^2 - y^2 = 8, \quad (3.13)
\]
and

\[
xy = 6, \quad (3.14)
\]
Then from Equation (3.13) and (3.14) we can see that

\[
r = \pm (3 + i),
\]
So the general solution will be

\[
w(z) = c_1 e^{(3+i)z} + c_2 e^{-(3+i)z},
\]

**Example 3.5.**
Let the OLCDE equation

\[
w''' - w = 0,
\]
we can see easily that the characteristic equation is \( r^3 - 1 = 0 \) so

\[
(r - 1)(r^2 + r + 1) = 0,
\]
we clearly get that \( r = 1 \) or \( r^2 + r + 1 = 0 \), for the second part we use the quadratic formula

\[
r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},
\]
\[
r = \frac{-1 \pm \sqrt{3}i}{2},
\]
the general solution will be

\[
w(z) = c_1 e^z + c_2 e^{-\frac{1}{2} + \frac{\sqrt{3}}{2}i} + c_3 e^{-\frac{1}{2} - \frac{\sqrt{3}}{2}i},
\]

**Example 3.6.**
Consider the equation
\[ w'' - 4w' + 4w = 0, \]

the characteristic equation \( r^2 - 4r + 4 = 0 \) \( \Rightarrow (r - 2)^2 = 0 \) then \( r = 2 \) which is repeated root, so the general solution will be

\[ w(z) = c_1e^{2z} + c_2ze^{2z}, \]

Example 3.7.
Consider the problem

\[ w^{(4)} + 2w'' + w = 0, \quad (3.15) \]

the characteristic equation is \( m^4 + 2m^2 + 4 = 0 \) and the roots of its simplified

\[ (m^2 + 1)^2 = 0 \]

is \( m = \pm i \) so the general solution is

\[ w(z) = c_1e^{iz} + c_2e^{-iz} + c_3ze^{iz} + c_4ze^{-iz}, \]

3.3 Reduction Of The Order

In the last section we find the general solution of a homogeneous linear complex differential equation

\[ \sum_{i=0}^{n} a_iw^{(i)}(z) = 0, \quad (3.16) \]

is a linear combination \( w = c_1w_1 + c_2w_2 \), where \( w_1 \) and \( w_2 \) are solutions that form a linear independent set on domain \( D \). We will examine a method for determining these solutions when the coefficients of the differential equation in (3.16) are constants. This method, which is a straightforward exercise in algebra, breaks down in a few cases and yields only a single solution \( w_1 \) of the CDE. It turns out that we can construct a second solution \( w_2 \) of a homogeneous equation (3.16) (even when the coefficients in (3.16) are variable) provided that we know a nontrivial solution \( w_1 \) of the CDE. The basic idea described in this section is that equation (3.16) can be reduced to a linear first-order CDE by means of a substitution involving the known solution \( w_1 \). A second solution \( w_2 \) of (3.16) is apparent after this first-order differential equation is solved.
3.4 Reduction Of Order Method

Suppose that \( w_1 \) denotes a nontrivial solution of (3.16) and that \( w - 1 \) is defined on an domain \( D \). We seek a second solution \( w_2 \) so that the set consisting of \( w_1 \) and \( w_2 \) is linearly independent on \( D \). Recall from Linear Algebra that if \( w_1 \) and \( w_2 \) are linear independent and when divide on by the other \( \frac{w_2}{w_1} \) is non-constant on \( D \) — that is, \( \frac{w_2(z)}{w_1(z)} = u(z) \) or \( w_2(z) = u(z)w_1(z) \). The function \( u(z) \) will be formed by solving \( w_2(z) = u(z)w_1(z) \) and the given linear complex differential equation. This method is named reduction of order due to we have to solve a linear first-order complex differential equation to get \( u \).

Example 3.8.

Since 3.17 is a solution for the CDE \( y'' - y = 0 \)

\[
 w_1 = e^z, \tag{3.17}
\]

we are going to use reduction of order to find a second solution \( w_2 \).

If \( w = u(z)w_1(z) = u(z)e^z \), then the Product Rule gives

\[
 w' = ue^z + e^z u', \quad w'' = ue^z + 2e^z u' + e^z u'',
\]

and so

\[
 w'' - w = e^z(u'' + 2u') = 0,
\]

Since \( e^z \neq 0 \) then \( u'' + 2u = 0 \). If we substitute \( s = u' \) in the linear second-order complex equation in \( u \) we have \( s' + 2s = 0 \) which is a linear first-order complex equation in \( s \). And by using the integrating factor \( e^{2s} \) we will have \( \frac{d}{dz}[e^{2z}s] = 0 \). After the integration we get \( s = c_1e^{-2z} \) or \( u' = c_1e^{-2z} \). Integrating one more time and get \( u = -\frac{1}{2}c_1e^{-2z} + c_2 \).

Thus

\[
 w = u(z)e^z = -\frac{c_1}{2}e^{-z} + c_2e^z, \tag{3.18}
\]

By picking \( c_2 = 0 \) and \( c_1 = -2 \), we obtain the desired second solution, \( w_2 = e^{-z} \).

Because \( W(e^z, e^{-z}) \neq 0 \) for every \( z \), the solutions are linearly independent on some \( D \).

Since we have shown that \( w_1 = e^z \) and \( w_2 = e^{-z} \) are linearly independent solutions of a linear second-order equation, the expression in (3.18) is actually the general solution of \( w'' - w = 0 \) on \( \mathbb{C} \).
3.5 The General Case

Suppose we divide the equation by $a_1(z)$ to put the equation $a_1(z)w'' + a_2(z)w' + a_3(z) = 0$ in the standard form

$$w'' + P(z)w + Q(z) = 0,$$  \hspace{1cm} (3.19)

where $P(z)$ and $Q(z)$ are continuous functions on a domain $D$. And suppose that $w_1(z)$ is an existed solution of (3.19) on $D$ and that $w_1(z) \neq 0$ for every $z$ in the domain. If we let $w = u(z)y_1(z)$, it follows that

$$w' = uw'_1 + w_1u', \quad w'' = uw''_1 + 2w'u' + w_1u'',$$

$$w'' + Pw' + Qw = u[w''_1 + Pw'_1 + Qw_1] + w_1u'' + (2w_1 + Pw_1)u' = 0,$$

This implies, we should have

$$w'' + Pw' + Qw = \text{ or } w_1u'' + (2w_1 + Pw_1)u' = 0,$$  \hspace{1cm} (3.20)

when we let $s = u'$. Notice that the last equation in (3.20) is both linear and separable. And with separating the complex variables and integrating both sides, we obtain

$$\frac{ds}{s} + 2\frac{s'}{s^2}dz + Pdz = 0,$$

Then

$$sw_1^2 = c_1e^{-\int Pdz},$$

We are going to fined the solution to the last equation for $s$, use $w = u'$, and then, integrate both sides again:

$$u = c_1 \int \frac{e^{-\int Pdz}}{w_1^2} + c_2,$$

Afer choosing $c_1 = 1$ and $c_2 = 0$, we definitely get from $w = u(z)y_1(z)$ that a second solution for the complex equation (3.19) will be

$$w_2y_1(z) \int \frac{e^{-\int P(z)dz}}{w_1(z)},$$  \hspace{1cm} (3.21)
It will make a very good revision of differentiation to make sure that the complex function \( w_2(z) \) showed in (3.21) go with the complex equation (3.19) and \( w_1 \) and \( w_2 \) are linearly independent on any domain on which \( w_1(z) \) is not vanish.

**Example 3.9.**

The function \( w_1 = z^2 \) is a solution of

\[
z^2 w'' - 3zw' + 4y = 0,
\]

From the standard form of the equation,

\[
w'' - \frac{3}{z} w' + \frac{4}{z^2} w = 0,
\]

we find from 3.21

\[
w_2 = z^2 \int \frac{e^{3z} \frac{d}{dz}}{e^3} \frac{dz}{z} = z^2 \int \frac{dz}{z} = z^2 \ln(z),
\]

The general solution on the domain of \( \ln(z) \) is given by \( w = c_1 w_1 + c_2 w_2 \) that is,

\[
w = c_1 z^2 + c_2 z^2 \ln(z),
\]

### 3.5.1 The Non-Homogeneous N’th Order Linear Complex Differential Equations

To find the solution of the part \( G(z) \) we should first we find the solution for the homogeneous part and combine it with the non-homogeneous part as linear combination. And to do this we are going to do **Undetermined Coefficients** method.

### 3.6 Undetermined Coefficients

We can denote to the homogeneous part by \( L(z) \) as a linear combination of \( w \) derivatives and the constant coefficients, so, the non-homogeneous linear ordinary complex differential equation with constant coefficients can be represented as

\[
L(z) = G(z),
\]
and the function $G(z)$ consists of finite sums and products of constants, polynomials, exponential functions $e^{rz}$, sines and cosines. In this method we assume the particular solution $w_p(z)$ for the non-homogeneous part is a combination of from the table below but with undetermined coefficients, and then we substitute into the main ODE and get the coefficients.

**Table 3.1 Trial Particular Solutions**

| $g(z)$ | The From $w_p(z)$ |
|--------|-------------------|
| 1 (any constant) | $A$ |
| 2 $5z + 7 + 3i$ | $Az + B$ |
| 3 $(3 + i)z^2 - 2$ | $Az^2 + Bz + C$ |
| 4 $iz^3 - z + 1 + i$ | $Az^3 + Bz^2 + Cz + E$ |
| 5 $(1 - i)\sin(2 - i)z$ | $A\cos((2 - i)z) + B\sin((2 - i)z)$ |
| 6 $\cos(3iz)$ | $A\cos(3iz) + B\sin(3iz)$ |
| 7 $e^{iz}$ | $Ae^{iz}$ |
| 8 $(9z - (2 + 5i))e^{-iz}$ | $(Az + B)e^{-iz}$ |
| 9 $e^{iz}e^{-z}$ | $(Az^2 + Bz + C)e^{-z}$ |
| 10 $e^{iz}\sin((3 + 2i)z)$ | $Ae^{iz}\cos((3 + 2i)z) + Be^{iz}\sin((3 + 2i)z)$ |
| 11 $5iz^2\sin(4z)$ | $(Az^2 + Bz + C)\cos(4z)(Ez^2 + Fz + G)\sin(4z)$ |
| 12 $ze^{iz}\cos(4z)$ | $(Az + B)e^{iz}\cos(4z) + (Cz + E)e^{iz}\sin(4z)$ |

**Example 3.10.**  
Consider the problem

$$w'' - 2w' - 3w = 1 - z^2,$$

first we find the solution for the homogeneous $w'' - 2w' - 3w = 0$ by the characteristic equation $m^2 - 2m - 3 = 0 \Rightarrow m = 3, m = -1$

the solution would be $w_h = c_1e^{3z} + c_2e^{-z}$.

To solve the non-homogeneous part we assume the solution is

$$w_p = Az^2 + Bz + C, \quad (3.22)$$

$$w_p' = 2Az + b, \quad (3.23)$$

$$w_p'' = 2A \quad (3.24)$$
Now we substitute 3.22 and 3.23 and 3.24 in 3.10 and we will get $A = -\frac{1}{3}$, $B = -\frac{4}{9}$, and $C = \frac{5}{27}$ so the general solution will be $w = w_h + w_p$.

$w(z) = c_1 e^{3z} + c_2 e^{-z} \left( -\frac{1}{3}z^2 - \frac{4}{9}z + \frac{5}{27} \right)$.

**Example 3.11.**
Consider the LCDE $w'' - w' = 2\sin(z)$.

Let's take the homogeneous part $w' - w' = 0$ so the characterization equation is $m^2 - m = 0 \Rightarrow m(m - 1) = 0$ $m = 0$ or $m = 1$ so that $w_h = c_1 + c_2 e^z$ to find the particular solution we can not assume that $w = A\sin(z)$ because we get two values for $A$ at the same time so we had to take $w = A\cos(z) + B\sin(z)$.

$w' = -A\sin(z) + B\cos(z)$ \quad $w'' = -A\cos(z) - B\sin(z)$,

and when we substituted the derivatives and solving for $A$ ans $B$ we get $A = -1$ and $B = 0$ so

$w_p = \cos(z) - \sin(z)$,

and the general solution is $w = w_h + w_p$

$w = c_1 + c_2 e^z + \cos(z) - \sin(z)$.

**Example 3.12.**
Consider the problem

$w'' - 3w' + 2w = 5e^z$.

We can easily see that $W_h = c_1 e^z + c_2 e^{2z}$ and the particular part should be of the form $w_p = Ae^z$ and we substitute $w'_p = Ae^z$ and $w''_p = Ae^z$ in the problem to find that $A = -5$ and the general solution will be

$w = c_1 e^z + c_2 e^{2z} - 5e^z$.

**Example 3.13.**
Consider the problem

$w'' - 6iw' - 9w = e^{(1+i)z}$,
we can see the character equation is \( m^2 - 6i - 9 = 0 \Rightarrow m^2 - 6i + i^29 = 0 \Rightarrow (m - 3i)^2 = 0 \) so that \( m = 3i \) then \( w_h = c_1 e^{3iz} + c_2 ze^{3iz} \), Then the non-homogeneous solution is neither \( Ae^{(1+i)z} \) nor \( Aze^{(1+i)z} \) so that we will take \( w_p(z) = Az^2 e^{(1+i)z} \).

\[
\begin{align*}
\frac{d^2}{dz^2}w_p(z) &= A(1 + i)z^2 e^{(1+i)z} + 2Aze^{(1+i)z}, \\
\frac{dp}{dz}w_p(z) &= A(1 + i)z^2 e^{(1+i)z} + 2Aze^{(1+i)z} + 2(1 + i)z e^{(1+i)z} + 2A e^{(1+i)z},
\end{align*}
\]

by substituting \( w_p' \) and \( w_p'' \) in the main CDE and equalizing the similar terms we get \( A = \frac{1}{2} \) so that \( w_p = \frac{1}{2}z^2 e^{(1+i)z} \) and the general solution would be

\[
w(z) = c_1 e^{3iz} + c_2 ze^{3iz} + \frac{1}{2}z^2 e^{(1+i)z},
\]

### 3.7 The Method of Variation of Parameters

We could see that undetermined coefficients method has weaknesses which limited its applications in physics and engineering: The OCDE should have constant coefficients, and the function \( G(z) \) must be of the type listed in table.

In this section, we are studying a method that doesn’t have such restrictions, named (Variation of Parameters).

#### 3.7.1 Linear First-Order CDE Revisited

In section 3.1 we saw that the general solution for the class of equations which has the form

\[
\frac{dw}{dz} + P(z)w = f(z)
\]

where \( P(z) \) and \( f(z) \) are analytic functions in its domain \( D \), using the integration factor method, the general solution of 3.25 on \( D \) was found to be

\[
w = c_1 e^{-\int P(z)dz} + e^{-\int P(z)dz} \int e^{p(z)dz}f(z)dz,
\]

Now we are going to find the general solution for 3.25 using (Variation of Parameters) method. by taking the homogeneous part
\[
\frac{dw}{dz} + P(z)w = 0,
\] (3.26)

It is clear that \( w_1 = e^{-\int p(z)dz} \) is the solution for the homogeneous part 3.26, and from the linearity property of DE solutions we conclude that \( w_1 = c_1 e^{\int p(z)dz} \), now we will use the new method to find the solution for the non-homogeneous part \( f(z) \).

The variation of parameters method consists of finding a particular solution of 3.26 of 3.25 of the form \( w_p = u_1(z)w_1(z) \), in other words we have changed the constant \( c_1 \) by a function \( u_1(z) \).

Substituting \( w_p = u_1w_1 \) into 3.25 and using the Product Rule gives

\[
\frac{d}{dz}[u_1w_1] + P(z)u_1w_1 = f(z),
\]

\[
u_1 \frac{dw_1}{dz} + w_1 \frac{du_1}{dz} + P(z)u_1w_1 = f(z),
\]

\[0 \text{ because of 3.26}\]

\[
u_1 \frac{dw_1}{dz} + P(z)w_1 + w_1 \frac{du_1}{dz} = f(z),
\]

so

\[w_1 \frac{du_1}{dz} = f(z),\]

By separating variables and integrating, we find \( w_1 \):

\[dw_1 = \frac{f(z)}{w_1(z)}dz \quad \text{yields} \quad u_1 = \int \frac{f(z)}{w_1(z)}dz,\]

Hence the sought-after particular solution is

\[w_p = u_1w_1 = w_1 \int \frac{f(z)}{w_1(z)}dz,\]

from the fact that \( w_1 = e^{-\int p(z)dz} \) we see that the last result is:

\[w_p = e^{-\int p(z)dz} \int e^{p(z)dz}f(z)dz,\]
### 3.7.2 Linear Second-Order Ordinary Complex Differential Equations

Consider the case of linear second order complex equations

\[ a_2(z)w'' + a_1(z)w' + a_0(z)w = G(z) \]  

(3.27)

we will start this method by putting 3.27 in the standard form

\[ w'' + P(z)w' + Q(z)w = G(z) \]  

(3.28)

by dividing by the leading coefficient \( a_2(z) \). In (3.28) we suppose that coefficient functions \( P(z), Q(z), \) and \( G(z) \) are Analytic on specific common domain \( D \). As we have noticed in Section 3.6, it is not very hard to get the complementary complex solution \( w_h = c_1 w_1(z) + c_2 W_2(z) \), the general desired solution of the associated homogeneous complex equation of (3.27), when the coefficients are known constants. Analogous to the preceding discussion in (3.7.1), we now ask: Can the parameters \( c_1 \) and \( c_2 \) in \( w_h \) can be replaced with functions \( u_1 \) and \( u_2 \), or "variable parameters", so that

\[ w_p = u_1(z)w_1(z) + u_2(z)w_2(z) + u_2w_2' + w_2u_2' \]  

(3.29)

is a particular solution of (3.28)? To answer this question we substitute (3.29) into (3.28). Using the Product Rule to differentiate \( W_p \) twice, we get

\[ w'_p = u_1w'_1 + w_2u'_2 \]

\[ w''_p = u_1w''_1 + w'_1u'_1 + w_2u''_2 + w'_2u'_2 + w_2u'_2 + u'_1w'_1 + u'_2w'_2 + w_2u'_2 + u'_2w'_2 \]  

(3.30)

Substituting (3.30) and the foregoing derivatives into (3.28) and grouping terms yields

\[ w''_p + Pw'_p + Qw_p = u_1 \left( w''_p + Pw'_1 + Qw_1 \right) + u_2 \left( w''_2 + Pw'_2 + Qw_2 \right) \]

\[ + w_1u''_1 + u'_1w'_1 + w_2u''_2 + P[w_1u'_1 + w_2u'_2] + w'_1u'_1 + w'_2u'_2 \]

\[ = \frac{d}{dz}[w_1u'_1] + \frac{d}{dz}[w_2u'_2] + P[w_1u'_1 + w_2u'_2] + w'_1u'_1 + w'_2u'_2 \]

\[ = \frac{d}{dz}[w_1u'_1 + w_2u'_2] + P[w_1u'_1 + w_2u'_2] + w'_1u'_1 + w'_2u'_2 = G(z) \]  

(3.31)
Because we are looking for two unknown complex functions $u_1$ and $u_2$, for that we are in need of two equations. We can easily get these two equations by strictly making a further assumption that the functions $u_1$ and $u_2$ satisfy $w_1u'_1 + w_2u'_2 = 0$. This assumption does not come out of the last equation but is prompted by the first two terms in (3.31), since if we demand that $w_1u'_1 + w_2u'_2 = 0$, then (3.31) reduces to $w_1u'_1 + w_2u'_2 = G(z)$. We now have what we were looking for which are our desired two equations, from these equations and their derivatives $u_1'$ and $u_2'$. By and using Cramer’s Rule, the solution the system will be easy to fined,

$$w_1u'_1 + w_2u'_2 = 0$$
$$w'_1 + w'_2 = G(z)$$

first we should find the determinants and the applying the following,

$$u'_1 = \frac{W_1}{W} = -\frac{w_2G(z)}{W}$$
$$u'_2 = \frac{W_2}{W} = \frac{w_1G(z)}{W}$$

where

$$W = \begin{vmatrix} w_1 & w_2 \\ w'_1 & w'_2 \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & w_2 \\ G(z) & w'_2 \end{vmatrix}, \quad W_2 = \begin{vmatrix} w_1 & 0 \\ w'_1 & G(z) \end{vmatrix}$$

The functions $u_1$ and $u_2$ are found by integrating the results in (3.32). The determinant $W$ is calculated as we where finding the Wronskian of $w_1$ and $w_2$. By linear independence of $w_1$ and $w_2$ on $D$, we know that $W(w_1(z),w_2(z)) \neq 0$ for every $z$ in the domain $D$.

**Example 3.14.**
Consider the CDE

$$w'' - 4w' + 4w = (z + 1)e^{2z}$$

From the auxiliary equation $m^2 - 4m - 4 = (m - 2)^2 = 0$ we have $w_h = c_1e^{2z} + c_2ze^{2z}$. With the identifications $w_1 = e^{2z}$ and $w_2 = ze^{2z}$, we next compute the Wronskian:

$$W(e^{2z},ze^{2z}) = W = \begin{vmatrix} e^{2z} & ze^{2z} \\ 2e^{2z} & 2ze^{2z} + e^{2z} \end{vmatrix}$$

Since the given differential equation is already in form (3.28) (that is, the coefficient of $w''$ is 1), we identify $G(z) = (z + 1)e^{2z}$. From (3.33) we get
Analytical Solution of Ordinary CDE

\[
W_1 = \begin{vmatrix}
0 & ze^{2z} \\
(z + 1)e^{2z} & 2ze^{2z} + e^{2z}
\end{vmatrix} = -(z + 1)ze^{4z}, \quad W_2 = \begin{vmatrix}
e^{2z} & 0 \\
2e^{2z} & (z + 1)e^{2z}
\end{vmatrix},
\]

and so from (3.32)

\[
u_1' = -\frac{(z + 1)ze^{4z}}{e^{4z}} = -z^2 - z, \quad u_2' = \frac{(z + 1)e^{4z}}{e^{4z}} = z + 1,
\]

It follows that \(u_1 = -\frac{1}{3}z^3 - \frac{1}{2}z^2, \quad u_2 = \frac{1}{2}z^2 + z\).

Hence

\[
w_p = \left(-\frac{1}{3}z^3 - \frac{1}{2}z^2\right)e^{2z} + \left(\frac{1}{2}z^2 + z\right)ze^{2z} = \frac{1}{6}z^3e^{2z} + \frac{1}{2}z^2e^{2z},
\]

and

\[
w = w_h + w_p = c_1e^{2z} + c_2ze^{2z} + \frac{1}{6}z^3e^{2z} + \frac{1}{2}z^2e^{2z},
\]

**Example 3.15.**

Consider the CDE \(4w'' + 36w = \csc[(3 - i)z]\)

We first of all, we should make the complex equation in the standard formation (3.28) by dividing by 4

\[
w'' + 9w = \frac{1}{4}\csc[(3 - i)z],
\]

The solution for the homogeneous part is \(w_h = c_1e^{3iz} + c_2e^{-3iz}\) so \(w_1 = e^{3iz}\) and \(w_2 = e^{-3iz}\)

\[
W(e^{3iz}, e^{-3iz}) = \begin{vmatrix}
e^{3iz} & e^{-3iz} \\
3ie^{3iz} & -3ie^{-3iz}
\end{vmatrix} = 6i,
\]

\[
W_1 = \begin{vmatrix}
0 & ze^{-3iz} \\
\frac{1}{4}\csc[(3 - i)z] & -3ie^{-3iz}
\end{vmatrix} = -\frac{1}{4}e^{-3iz}\csc[(3 - i)z],
\]

\[
W_2 = \begin{vmatrix}
e^{3iz} & 0 \\
3ie^{3iz} & \frac{1}{4}\csc[(3 - i)z]
\end{vmatrix} = \frac{1}{4}e^{-3iz}\csc[(3 - i)z],
\]

\[
u_1' = \frac{1}{24i}e^{-3iz}\csc[(3 - i)z],
\]

\[
u_1' = -\frac{1}{24i}e^{-3iz}\csc[(3 - i)z],
\]
so that,
\[ u_1 = \frac{e^{3iz} \left[ \log(\cos \left( \frac{1}{2} (-3 + i)z \right)) - \log(-\sin \left( \frac{1}{2} (-3 + i)z \right)) \right]}{24i(-3 + i)}, \]
\[ u_2 = -\frac{e^{3iz} \left[ \log(\cos \left( \frac{1}{2} (-3 + i)z \right)) - \log(-\sin \left( \frac{1}{2} (-3 + i)z \right)) \right]}{24i(-3 + i)}, \]

then

\[ w = w_h + w_p = c_1 e^{3iz} + c_2 e^{-3iz} + \left( \frac{e^{3iz} \left[ \log(\cos \left( \frac{1}{2} (-3 + i)z \right)) - \log(-\sin \left( \frac{1}{2} (-3 + i)z \right)) \right]}{24i(-3 + i)} \right) e^{3iz} + \left( -\frac{e^{3iz} \left[ \log(\cos \left( \frac{1}{2} (-3 + i)z \right)) - \log(-\sin \left( \frac{1}{2} (-3 + i)z \right)) \right]}{24i(-3 + i)} \right) e^{-3iz} \]

3.8 Higher-Order Ordinary Complex Differential Equations

The method that we have already studied in the last section about the non-homogeneous second-order complex differential equations can be generalized to linear \( n \)'th-order complex differential equations that have been put into the standard form

\[ w^n + P_{n-1}(z)w^{(n-1)} + \ldots + P_1(z)w' + P_0(z)w = G(z) \quad (3.34) \]

If \( w_h = c_1 w_1 + c_2 w_2, \ldots, c_n w_n \) is the complementary function for (3.34), then a particular solution is

\[ w_p = u_1(z)w_1(z) + u_2(z)w_2(z) + \ldots + u_n(z)w_n(z), \]

where \( u_k', k = 1, 2, \ldots \) are determined by the \( n \) equations

\[
\begin{align*}
  w_1 u_1' & + w_2 u_2' + \cdots + w_n u_n' = 0 \\
  w_1 u_1'' & + w_2 u_2'' + \cdots + w_n u_n'' = 0 \\
  & \vdots & \vdots & \vdots \quad \vdots \\
  w_1^{(n-1)} u_1' & + w_2^{(n-1)} u_2' + \cdots + w_n^{(n-1)} u_n' = G(z)
\end{align*}
\]

and the \( u \)'s are found with

\[ u_k' = \frac{W_k}{W} \quad k = 1, 2, \ldots, \]
Example 3.16.

Consider the equation,

\[ w''' + w' = \tan(z - 2i), \]

the complementary solution will be \( m^3 + m = 0 \Rightarrow m(m^2 - 1) = 0 \)
so,

\[ w_h = c_1 + c_2e^{iz} + c_3e^{-iz}, \]

Now, \( w_1 = 1, w_2 = e^{iz} \) and \( w_3 = e^{-iz} \) then

\[
W = \begin{vmatrix}
1 & e^{-i} & e^i \\
0 & -ie^{-i} & ie^i \\
0 & e^i & -e^{-i}
\end{vmatrix} = 2i, \]

\[
W_1 = \begin{vmatrix}
0 & e^{-i} & e^i \\
0 & -ie^{-i} & ie^i \\
tan(z-2i) & -e^{-i} & -e^i
\end{vmatrix} = 2i\tan(z-2i),
\]

\[
W_2 = \begin{vmatrix}
1 & 0 & e^i \\
0 & 0 & ie^i \\
0 & \tan(z-2i) & -e^i
\end{vmatrix} = -ie^i\tan(z-2i),
\]

\[
W_3 = \begin{vmatrix}
1 & e^{-i} & 0 \\
0 & -ie^{-i} & 0 \\
e^{-i} & \tan(z-2i)
\end{vmatrix} = -ie^{-i}\tan(z-2i),
\]

\[
u'_1 = \frac{W_1}{W} = \tan(z-2i),
\]

\[
u'_2 = \frac{W_2}{W} = \frac{1}{2}e^i\tan(z-2i),
\]

\[
u'_3 = \frac{W_3}{W} = \frac{1}{2}e^{-i}\tan(z-2i),
\]

then, by integrating the factors we get

\[ u_1 = -\log[\cos(2i - z)], \]

\[ u_2 = -\frac{1}{2}e^i\log[\cos(2i - z)], \]
3.9 Cauchy-Euler Ordinary Complex Differential Equation

\[ u_3 = -\frac{1}{2} e^{-i \log(2i)} \],

so, the general solution is:

\[ w = w_h + w_p = c_1 + c_2 e^{iz} + c_3 e^{-iz} + \log(\cos(z - 2i)) \left\{ \frac{1}{2} (e^i + e^{-i} - 1) \right\}, \]

### 3.9 Cauchy-Euler Ordinary Complex Differential Equation

The same connected simplicity with which we could gain the explicit solutions of higher-order linear complex differential equations with constant coefficients in the last sections, in general, we will do the same to higher-order linear complex equations with variable coefficients. We are going to see in Section 3.10 when a linear CDE has variable coefficients, the best that we can usually expect is to find an approximated solution using a Laurent series. The types of the complex differential equation which we consider in this section is a kind exception to this rule; it is a linear equation with variable coefficients whose general solution can always be represented in terms of powers of \( z \), sines, cosines, and logarithmic complex functions. And, its methods of solution is quite similar to those for constant-coefficient equations in that an assistant equation must be solved.

A linear complex differential equation of the form

\[
a_n z^n \frac{d^n w}{dz^n} + a_{n-1} z^{n-1} \frac{d^{n-1} w}{dz^{n-1}} + \ldots + a_1 z \frac{dw}{dz} + a_0 w = G(z),
\]

where the coefficients \( a_n, a_{n-1}, \ldots, a_0 \) are constants, is known as a Cauchy-Euler equation. The observable characteristic of this type of equation is that the degree \( k = n, n-1, \ldots, 1, 0 \) of the coefficients \( z^k \) matches the order \( k \) of differentiation \( d^k w/dz^k \): we start the discussion with a detailed examination of the forms of the general solutions of the homogeneous second-order equation

\[
a z^2 \frac{d^2 w}{dz^2} + b z \frac{dw}{dz} + c w = 0,
\]

The solution of higher-order equations follows analogously. Also, we can solve the non-homogeneous equation \( a z^2 w'' + b z w' + c w = G(z) \) by variation of parameters, once we have determined the complementary function \( w_h \).
Notation 3.1. The coefficient $az^2$ of $w$ is zero at $z = 0$. Hence to guarantee that the fundamental results of the existence of a unique solution are applicable to the Cauchy-Euler equation, we focus our attention on finding the general solutions defined on the domain:

3.9.1 Method of Solution

We try a solution of the form $w = z^m$, where $m$ is to be determined. Analogous to what happened when we substituted $e^{mz}$ into a linear equation with constant coefficients, when we substitute $x^m$, each term of a Cauchy-Euler equation becomes a polynomial in $m$ times $z^m$, since

$$a_k z^k \frac{d^k w}{dz^k} = a_k z^k m(m-1)(m-2) \ldots (m-k+1)z^{m-k} = a_k m(m-1)(m-2) \ldots (m-k+1)z^m$$

(3.36)

For example, when we substitute $w = z^m$, the second-order equation becomes

$$az^2 \frac{d^2 w}{dz^2} + bw \frac{dw}{dz} + cw = am((m-1) + bm + c)z^m,$$

Thus $w = z^m$ is a solution of the complex differential equation whenever $m$ is a solution of the auxiliary equation

$$am(m-1) + bm + c = 0 \quad or \quad am^2 - (b-a)m + c = 0 \quad (3.37)$$

There are three different cases to be considered, depending on whether the roots of this quadratic equation are real and distinct, real and equal, or complex. In the last case the roots appear as a conjugate pair.

3.9.2 Distinct Roots

Let $m_1$ and $m_2$ denote the roots of (3.36) such that $m_1 \neq m_2$. Then $w_1 = z^{m_1}$ and $w_2 = z^{m_2}$ form a fundamental set of solutions. Hence the general solution is

$$w = c_1 z^{m_1} + c_2 z^{m_2},$$

Example 3.17.
Consider the CDE
\[ z^2w'' - 2zw' - 4w = 0 \]

Rather than just memorizing Equation (3.37), it is preferable to assume \( w = z^m \) as the solution a few times to understand the origin and the difference between this new form of the auxiliary equation and that obtained in Section 3.6. Differentiate twice,

\[
w' = z^{m-1}, \quad w'' = m(m-1)z^{m-2},
\]

and substitute back into the differential equation:

\[
z^2w'' - 2zw' - 4w = m(m-1)z^2z^{m-2} - 2mzz^{m-1} - 4z^m
\]

\[
= z^m(m(m-1) - 2m - 4) = z^m(m^2 - 3m - 4) = 0
\]

if \( m^2 - 3m - 4 = 0 \) Now \( (m+1)(m-4) = 0 \) implies \( m_1 = -1, m_2 = 4 \), so

\[ w = c_1x^{-1} + c_2x^4, \]

### 3.9.3 Repeated Roots

If the roots of (3.37) are repeated (that is, \( m_1 = m_2 \)), then we obtain only one solution -namely, \( w = z^{m_1} \). When the roots of the quadratic equation \( am^2 + (b - a)m + c = 0 \) are equal, the discriminant of the coefficients is necessarily zero. It follows from the quadratic formula that the root must be \( m_1 - (b - a)/2a \). Now we can construct a second solution \( w_2 \), using (3.21) of Section 3.4 We first write the Cauchy-Euler equation in the standard form

\[ w'' + \frac{b}{az}w' + \frac{c}{az^2}, \]

and make the identifications \( P(x) = b/az \) and \( \int (b/ax)dz = (b/a)\ln(z) \). Thus

\[
w_2 = z^{m_1} \int \frac{e^{-(b/a)\ln(z)}}{x^{2m_1}}dz
\]

\[
= z^{m_1} \int \frac{dz}{z}
\]

\[
= z^{m_1} \ln(z)
\]
The general solution will be then
\[ w = c_1 z^{m_1} + c_2 z^{m_1} \ln(z), \]

**Example 3.18.**
Consider the problem
\[ (1-i)w'' - 3iw' + (3+4i)w = 0, \]
by the auxiliary equation \( am^2 + (b-a)m + c = 0 \) we get \( a = 1-i, \ b-a = -(1+2i), \)
and \( c = 3 + 4i \)
so,
\[ m = \frac{-(1+2i) \pm \sqrt{(1+2i)^2 - 4(1-i)(3+4i)}}{2(1-i)}, \]
we get from this equation \( m_1 = \frac{15}{2} + 7i, \ m_2 = 8 - \frac{17}{2}i \)
so the general solution will be
\[ w = c_1 z^{\frac{15}{2} + 7i} + c_2 z^{8 - \frac{17}{2}i}, \]

**Notation 3.2.** For higher-order equations, if \( m_1 \) is a root of multiplicity \( k \), then it can be shown that
\[ z^{m_1}, z^{m_1} \ln(z), z^{m_1} [\ln(z)]^2, \ldots, z^{k-1} [\ln(z)]^{k-1}, \]

**Notation 3.3 (Non-homogeneous Equations).** The method of undetermined coefficient described in Sections 3.6 does not carry over, in general, to non-homogeneous linear complex differential equations with variable coefficients. Consequently, in our next example the method of variation of parameters is employed.

**Example 3.19.**
Consider the problem
\[ z^2 w'' - 3zw' + 3w = (2+i)z^4 e^{3iz}, \]
Since the equation is non-homogeneous, we first solve the associated homogeneous equation. From the auxiliary equation \((m - 1)(m - 3) = 0\) we find that \( w_h = c_1 z + c_2 z^3 \). Now before using variation of parameters to find a particular solution \( w_p = u_1 w_1 + u_2 w_2 \), recall that the formulas \( u_1' = W_1 / W \) and \( u_2' = W_2 / W \), where \( W_1, W_2, \) and \( W \) are the determinants, were derived under the assumption that the complex differential equation has been put into the standard form \( w'' + P(z)w' + Q(z)w = G(z) \).
Therefore we divide the given equation by $z^2$, and from
\[ w'' + \frac{3}{z} w' + \frac{3}{z^2} w = (2 + i)z^2 e^{3iz}, \]
we make the identification $G(z) = (2 + i)z^2 e^{3iz}$. Now with $w_1 = z$, $w_2 = z^3$ and
\[
W = \begin{vmatrix} z & z^3 \\ 1 & 3z^2 \end{vmatrix} = 2z^2, \quad W_1 = \begin{vmatrix} 0 & z^3 \\ (2 + i)z^2 e^{3iz} & 3z^2 \end{vmatrix} = -(2 + i)z^5 e^{3iz},
\]
\[
W_2 = \begin{vmatrix} z & 0 \\ 1 & (2 + i)z^2 e^{3iz} \end{vmatrix} = (2 + i)z^3 e^{3iz},
\]
we found
\[
u_1' = -\frac{(2 + i)z^5 e^{3iz}}{2z^2} = -\frac{1}{2} (2 + i)z^3 e^{2iz},
\]
\[
u_2' = \frac{(2 + i)z^3 e^{3iz}}{2z^2} = \frac{1}{2} (2 + i)ze^{3iz},
\]
then
\[
u_1 = -\frac{(2 + i)}{54} e^{3iz}(-9iz^3 + 9z^2 + 6iz - 2)
\]
\[
u_1 = \frac{(2 + i)}{18} z^{3i}(1 - 3iz)
\]
so
\[
w = w_h + w_p, \quad w = c_1 z + c_2 z^3 + \left(-\frac{(2 + i)}{54} e^{3iz}(-9iz^3 + 9z^2 + 6iz - 2)\right)z + \left(\frac{(2 + i)}{18} z^{3i}(1 - 3iz)\right)z^3,
\]

### 3.10 Series Solutions Of Linear OCD Equations

In Section 3.2.1 we saw that solving a homogeneous linear OCDE with constant coefficients was essentially a problem in algebra. By finding the roots of the auxiliary equation, we could write a general solution of the CDE as a linear combination of the elementary complex functions $e^{\alpha z}$, $z^k e^{\alpha z}$, and so no. But as was pointed out in the introduction to Section 3.9, most linear higher-order CDEs with variable coefficients
cannot be solved in terms of elementary complex functions. A usual course of action for equations of this sort is to assume a solution in the form of an infinite series and proceed in a manner similar to the method of undetermined coefficient (Section 3.6).

3.10.1 Solutions of OCDE About Ordinary Points

In this section we turn to the more important problem of finding power series solutions of linear second-order complex differential equations. More to the point, we are going to find solutions of linear second-order equations in the form of Laurent series whose center is a complex number that is an ordinary point of the CDE.

Definition 3.10.1 (Ordinary and Singular Points). Consider the second order CDE

\[ a_2(z)w'' + a_1(z)w' + a_0(z)w = 0, \quad (3.38) \]

and by dividing 3.38 by \( a_2(z) \) we get

\[ w'' + P(z)w' + Q(z)w \quad (3.39) \]

A point \( z = z_0 \) is said to be an ordinary point of the differential equation (3.38) if both coefficients \( P(x) \) and \( Q(x) \) in the standard form (3.39) are analytic at \( z_0 \). A point that is not an ordinary point of (3.38) is said to be a singular point of the CDE.

Notation 3.4. When \( a_2(z) \), \( a_1(z) \), and \( a_2(z) \) are polynomials a point \( z = z_0 \) is an ordinary point of 3.38 if \( a_2(z) \neq 0 \), whereas \( z = z_0 \) is a singular point of 3.38 if \( a_2(z) = 0 \).

Theorem 3.10.1 (Existence of Power Series Solutions). If \( z = z_0 \) is an ordinary point of the differential equation (3.38), we can always find two linearly independent solutions in the form of a Laurent series centered at \( z_0 \), that is,

\[ w = \sum_{n=-\infty}^{\infty} C_n(z-z_0)^n, \]

A power series solution converges at least on some annular domain defined by \( r \leq |z-z_0| < R \).

where the \( C_n \) and \( z_0 \) are constants, defined by a line integral which is a generalization of Cauchy’s integral formula:

\[ c_n = \frac{1}{2\pi i} \oint_{y} \frac{f(z)dz}{(z-z_0)^{n+1}}, \quad n = 1, 2, 3, 4, ..., \]
where $\gamma$ in any closed simple path in the annual domain and $f(z)$ is an analytic function.

A solution of the form $w = \sum_{-\infty}^{\infty} c_n (z-z_0)^n$ is said to be a solution about the ordinary point $z_0$.

**Example 3.20.**
Consider the CDE

$$w'' + zw = 0,$$  \hspace{1cm} (3.40)

Since there are no singular points, Theorem 3.10.1 guarantees two power series solutions centered at $z = 0$ that converge for $|z| < \infty$. Substituting $w = \sum_{-\infty}^{\infty} c_n z^n$ and the second derivative $w'' = \sum_{-\infty}^{\infty} n(n-1) c_n z^{n-2} + \sum_{-\infty}^{\infty} n(n-1) c_n z^{n-2}$ into 3.40.

$$w'' + zw = \sum_{-\infty}^{\infty} n(n-1) c_n z^{n-2} + \sum_{-\infty}^{\infty} n(n-1) c_n z^{n-2} + z \sum_{-\infty}^{\infty} c_n z^n = 0$$

$$= \sum_{-\infty}^{\infty} n(n-1) c_n z^{n-2} + \sum_{n=2}^{\infty} n(n-1) c_n z^{n-2} + \sum_{-\infty}^{\infty} c_n z^{n+1} = 0$$

$$= -3 \sum_{-\infty}^{\infty} (n+2)(n+1) c_{n+2} z^n + \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} z^n + \sum_{-\infty}^{\infty} c_{n-1} z^n = 0$$

$$= -3 \sum_{-\infty}^{\infty} (n+2)(n+1) c_{n+2} z^n + \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} z^n + \sum_{-\infty}^{\infty} c_{n-1} z^n = 0$$

$$= \sum_{n=0}^{\infty} [(n+2)(n+1) c_{n+2} + c_{n-1}] z^n + \sum_{n=0}^{\infty} \frac{1}{z} + c_{-3} \frac{1}{z^2}$$

we can see that $c_{-2} = 0$ and $c_{-3} = 0$ and ,

$$(n+2)(n+1) c_{n+2} + c_{n-1} \rightarrow c_{n+2} = -\frac{c_{n-1}}{(n+2)(n+1)},$$
so we can see clearly that $c_{-i} = 0$ for $i = 1, 2, 3, 4, \ldots$, now we see the right side of the series.

at this point, we can write the general solution as,

$$w = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \ldots,$$

and after substitution we get

$$w = c_0 + c_1 z + 0 - \frac{c_0}{2.3} z^3 - \frac{c_1}{3.4} z^4 + 0 + \frac{c_0}{2.3.5.6} z^6 + \frac{c_1}{3.4.6.7} z^7 + 0 - \frac{c_0}{2.3.5.6.8.9} z^9 \ldots$$
after grouping the terms containing \( c_0 \) and containing \( c_1 \) we obtain,

\[
w = c_0 w_1(z) + c_1 w_2(z),
\]

where

\[
w_1(z) = 1 - \frac{1}{2.3} z^3 + \frac{1}{2.3.5.6} z^6 - \frac{1}{2.3.5.6.8.9} z^9 + \ldots = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{2.3 \ldots (3n-1)(3n)} z^{3n}
\]

\[
w_2(z) = z - \frac{1}{3.4} z^4 + \frac{1}{3.4.6.7} z^7 - \frac{1}{3.4.6.7.9.10} z^{10} + \ldots = z + \sum_{n=1}^{\infty} \frac{(-1)^n}{3.4 \ldots (3n)(3n+1)} z^{3n+1}
\]

**Example 3.21.**

Consider the CDE,

\[
(z^2 + 1)w'' + zw' - w = 0,
\]

we can see that the singular points are \( z = \pm i \), so we can find the series solution centered on \( 0 \), and for which the annular domain is \( 0 < |z| < 1 \) then we can get rid of the negative index to the Laurent series to be

\[
w = \sum_{n=0}^{\infty}
\]

\[
(z^2 + 1) \sum_{n=2}^{\infty} n(n-1)c_n z^{n-2} + z \sum_{n=1}^{\infty} nc_n z^{n-1} - \sum_{n=0}^{\infty} c_n z^n
\]

\[
= \sum_{n=2}^{\infty} n(n-1)c_n z^n + \sum_{n=2}^{\infty} n(n-1)c_n z^{n-2} + \sum_{n=1}^{\infty} nc_n z^n - \sum_{n=0}^{\infty} c_n z^n
\]

\[
= 2c_2 - c_0 + 6c_3 z + \sum_{n=2}^{\infty} [n(n-1)c_n + (n+2)(n+1)c_{n+2} + nc_n - c_n] z^n
\]

\[
= 2c_2 - c_0 + 6c_3 z + \sum_{n=2}^{\infty} [(n+1)(n-1)c_n + (n+2)(n+1)c_{n+2}] z^n
\]

from the inequality we conclude that

\[
c_2 = \frac{1}{2} c_0
\]

\[
c_3 = 0
\]

\[
c_{n+2} = \frac{1-n}{n+2} c_n, \quad n = 2, 3, 4, ...
\]
Substituting $n = 2, 3, 4, 5, \ldots$ into the last formula gives

\[
c_4 = -\frac{1}{4}c_2c_2 = -\frac{1}{2.3}c_0 = -\frac{1}{232}c_0
\]

\[
c_5 = -\frac{2}{5}c_3 = 0
\]

\[
c_6 = -\frac{3}{6}c_4 = \frac{1}{2.4.6}c_0 = \frac{1.3}{2^43!}c_0
\]

\[
c_7 = -\frac{4}{7}c_5 = 0
\]

\[
c_8 = \frac{5}{8}c_6 = -\frac{3.5}{2.4.6.8}c_0 = -\frac{1.3.5}{2^44!}c_0
\]

\[
c_9 = -\frac{6}{9}c_7 = 0
\]

\[
c_{10} = -\frac{7}{10}c_8 = \frac{3.5.7}{2.4.6.8.10}c_0 = \frac{1.3.5.7}{2^5}c_0
\]

and so on, Therefore

\[
w = c_0 + c_1z + c_2z^2 + c_3z^3 + c_4z^4 + c_5z^5 + \ldots,
\]

\[
w = c_0 + c_1z + c_2z^2 + c_3z^3 + c_4z^4 + c_5z^5 + \ldots
\]

\[
= C_0 \left[ 1 + \frac{1}{2}z^2 - \frac{1}{2^22!}z^4 + \frac{1.3}{2^33!}z^6 - \frac{1.3.5}{2^44!}z^8 + \frac{1.3.5.7}{2^55!}z^{10} \right] + c_1z
\]

\[
= c_0w_1(z) + c_1w_2(z)
\]

The solutions are the polynomial $w_2(z) = z$ and the power series,

\[
w_1(z) = 1 + \frac{1}{2}z^2 + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{1.3.5\ldots(2n-3)}{2^nn!}z^n \quad |z| < 1,
\]

**Example 3.22.**

Consider the CDE,

\[w'' + \cos(z)w = 0,\]

We see that $z = 0$ is an ordinary point of the equation because, as we have already seen, $\cos(z)$ is analytic at that point. Using the Maclaurin series for $\cos(z)$ along with the usual assumption $w = \sum_{n=0}^{\infty} c_nz^n$ we get
\[ w'' + \cos(z)w = \sum_{n=2}^{\infty} n(n-1)c_n z^{n-2} + \left( 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots \right) \sum_{n=0}^{\infty} c_n z^n \]

\[ = 2c_2 + 6c_3z + 12c_4z^2 + 20c_5z^3 + \ldots + \left( 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots \right) \left( c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots \right) \]

\[ = 2c_2 + c_0 + (6c_3 + c_1)z + \left( 12c_4 + c_2 - \frac{1}{2}c_0 \right) z^2 + \left( 20c_5 + c_3 - \frac{1}{2}c_1 \right) z^3 + \ldots = 0 \]

it follows that,

\[ 2c_2 + c_0 = 0 \quad 6c_3 + c_1 = 0 \quad 12c_4 + c_2 - \frac{1}{2}c_0 = 0 \quad 20c_5 + c_3 - \frac{1}{2}c_1 = 0, \]

and so on, this gives \( c_2 = -\frac{1}{2}c_0, c_3 = -\frac{1}{6}c_1, c_4 = \frac{1}{12}, c_5 = \frac{1}{30}c_1, \ldots \) by grouping terms we get to the general solution \( w = c_0 w_1(z) + c_1 w_2(z), \) where

\[ w_1(z) = 1 - \frac{1}{2}z^2 + \frac{1}{12}z^2 - \ldots \quad w_2(z) = x - \frac{1}{6}z^3 + \frac{1}{30}z^5 - \ldots, \]

### 3.10.2 Solutions of OCDE About Singular Points

The two different complex differential equations,

\[ w'' + zw = 0 \quad \text{and} \quad zw'' + w = 0, \]

are similar only in that they are both examples of simple linear second-order CDEs with variable coefficients. That is all they have in common. Since \( z = 0 \) is an ordinary point of \( w'' + zw = 0, \) we saw in Section 3.10.1 that there was no problem in finding two distinct power series solutions centered at that point. In contrast, because \( z = 0 \) is a singular point of \( zw'' + w = 0, \) finding two infinite series solutions of the equation about that point becomes a more difficult task. The solution method that is discussed in this section does not always yield two infinite series solutions. When only one solution is found, we can use the formula given in (3.21) of Section 3.3 to find a second solution

A singular point \( z_0 \) of a linear differential equation

\[ a_2(z)w'' + a_1(z)w' + a_0(z)w = 0, \quad (3.41) \]
is further classified as either regular or irregular. The classification again depends on the functions $P$ and $Q$ in the standard form

$$w'' + P(z)w' + Q(z)w = 0, \quad (3.42)$$

**Definition 3.10.2.** A singular point $z = z_0$ is said to be a regular singular point of the complex differential equation (3.41) if the functions $p(z) = (z - z_0)P(z)$ and $q(z) = (z - z_0)^2Q(z)$ are both analytic at $z_0$. A singular point that is not regular is said to be an irregular singular point of the equation.

### 3.11 Generalized Frobenius Method

To solve a complex differential equation (3.41) about a regular singular point, we generalize the following theorem due to the eminent German mathematician Ferdinand Georg Frobenius (1849–1917)

**Theorem 3.11.1.** If $z = z_0$ is a regular singular point of the differential equation (3.41), then there exists at least one solution of the form,

$$w = (z - z_0)^s \sum_{-\infty}^{\infty} c_n (z - z_0)^n = \sum_{-\infty}^{\infty} c_n (z - z_0)^{n+s},$$

where the number $r$ is a constant to be determined. The series will converge at least on some annular domain $r < |z - z_0| < R$.

**Example 3.23.** Consider the problem

$$3zw'' + w' - w = 0, \quad (3.43)$$

Because $z = 0$ is a regular singular point of the differential equations we try to find a solution of form $w = \sum_{-\infty}^{\infty} c_n z^{n+r}$. Now,

$$w' = \sum_{-\infty}^{\infty} (n+r)c_n z^{n+r-1} \quad \text{and} \quad w'' = \sum_{-\infty}^{\infty} (n+r)(n+r-1)c_n z^{n+r-2},$$

Since the only singular point is only $z = 0$ so the left side of Laurent series will be vanished
3zw'' + w' - y = 3 \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n z^{n+r-1} + \sum_{n=0}^{\infty} (n+r)c_n z^{n+r} - \sum_{n=0}^{\infty} c_n z^{n+r}

= \sum_{n=0}^{\infty} (n+r)(3n+3r-2)c_n z^{n+r-1} - \sum_{n=0}^{\infty} c_n z^{n+r}

= z^r \left[ r(3r-2)c_0 z^{-1} + \sum_{n=1}^{\infty} (n+r)(3n+3r-2)c_n z^{n-1} - \sum_{n=0}^{\infty} c_n z^n \right]

= z^r \left[ r(3r-2)c_0 z^{-1} + \sum_{n=0}^{\infty} [(k+r+1)(3k+3r+1)c_{k+1} - c_k] z^k \right] = 0

which implies that,

\[ r(3r-2) = 0, \]

and,

\[ (k+r+1)(3k+3r+1)c_{k+1} - c_k = 0, \quad k = 0, 1, 2, \ldots, \]

when we take \( c_0 = 0 \) nothing will be gained then we take

\[ r(3r-2) = 0, \quad (3.44) \]

and

\[ c_{k+1} = \frac{c_k}{(k+r+1)(3k+3r+1)} \quad k = 0, 1, 2, \ldots, \quad (3.45) \]

when we substitute in (3.45), the two values of \( r \) that satisfy the quadratic equation (3.44), \( r_1 = \frac{2}{3} \) and \( r_2 = 0 \), give two recurrence relations:

\[ r_1 = \frac{2}{3}, \quad c_{k+1} = \frac{c_k}{(3k+5)(k+1)} \quad k = 0, 1, 2, \ldots, \quad (3.46) \]

\[ r_2 = 0, \quad c_{k+1} = \frac{c_k}{(k+1)(3k+1)} \quad k = 0, 1, 2, \ldots, \quad (3.47) \]
from (3.46) we find

\[ c_1 = \frac{c_0}{5.1} \]
\[ c_2 = \frac{c_1}{8.2} = \frac{c_0}{2!5.8} \]
\[ c_3 = \frac{c_2}{11.3} = \frac{c_0}{3!5.8.11} \]
\[ c_4 = \frac{c_3}{14.4} = \frac{c_0}{4!5.8.11.14} \]
\[ \vdots \]
\[ c_n = \frac{c_0}{n!5.8.11\ldots(3n+2)} \]

and from (3.47) we find

\[ c_1 = \frac{c_0}{1.1} \]
\[ c_2 = \frac{c_1}{2.4} = \frac{c_0}{2!1.4} \]
\[ c_3 = \frac{c_2}{3.7} = \frac{c_0}{3!1.4.7} \]
\[ c_4 = \frac{c_3}{4.10} = \frac{c_0}{4!1.4.7.10} \]
\[ \vdots \]
\[ c_n = \frac{c_0}{n!1.4.7\ldots(3n-2)} \]

Here we encounter something that did not happen when we obtained solutions about an ordinary point; we have what looks to be two different sets of coefficients, but each set contains the same multiple \( c_0 \). If we omit this term, the series solutions are

\[
w_1(z) = z^2 \left[ 1 + \sum_{n=1}^{\infty} \frac{1}{n!5.8.11\ldots(3n+2)} z^n \right], \tag{3.48}
\]

\[
w_2(z) = z^0 \left[ 1 + \sum_{n=1}^{\infty} \frac{1}{n!1.4.7\ldots(3n-2)} z^n \right], \tag{3.49}
\]

By the ratio test it can be demonstrated that both (3.46) and (3.47) converge for all values of \( z \)—that is, \( |z| < \infty \). Also, it should be apparent from the form of these solutions that neither series is a constant multiple of the other, and therefore \( w_1(z) \) and \( w_2(z) \) are linearly independent on the entire \( z \)-axis. Hence by the superposition
principle, $w = c_1w_1 + c_2w_2$ is another solution of (3.43). On any domain that does not contain the $z = 0$, this linear combination represents the general solution of the differential complex equation.
Chapter 4

Homotopy Perturbation Method for Solving Linear Complex Differential Equations

The most important mathematical models for physical phenomena is the differential equation. The motion of objects, Fluid and heat flow, bending and cracking of materials, vibrations, chemical reactions and nuclear reactions are all modeled by systems of differential equations. Moreover, Numerous mathematical models in science and engineering are expressed in terms of unknown quantities and their derivatives. Many applications of differential equations (DEs), particularly ODEs of different orders, can be found in the mathematical modeling of real life problems(Mechee et al. (2014)). The homotopy perturbation method (HPM) is an efficient technique to find the approximate solutions for ordinary and partial differential equations which describe different fields of science, physical phenomena, engineering, mechanics, and so on. HPM was proposed by Ji-Huan He in 1999 for solving linear and nonlinear differential equations and integral equations. Many researchers used HPM to approximate the solutions of differential equations and integral equations([48], [25]). Complex differential equations and their solutions play a major role in science and engineering. When a mathematical model is formulated for a physical problem, it is often represented by complex differential equations that cannot be solved explicitly by analytic techniques. In this case, it is often necessary to move to approximation and numerical methods to find solutions of such equations. For instance, the vibrations of a one-mass system with two degrees of freedom are mostly described using the differential equation with a complex dependent variable which is usually linear complex differential equation. The solution of the differential equation clarifies the linear
phenomena which occur in the system. The modern physics has some applications that using differential equations, ordinary or partial, in the complex domain. Recently, we have studied a wide class of complex differential equations (CDEs), when the input and output or mostly just the dependent variables are in the complex domain, which is used as mathematical models in many physically scientific fields and applied science. The approximated solutions of this class of differential equations have studied using modded homotopy perturbation method (HPM). This study will reveal the significance and consolidation of the most powerful fields in mathematics, complex analysis and differential equations. CDEs have been used in some applications in engineering and physics. For example, a time-harmonic form of Maxwell’s equations and Schrodinger equation are examples of CPDEs and the six Painleve equations (PI-PVI), Riccati equation and Schwarzian equation, all are examples of CODEs. The most important mathematical models for physical phenomena is the differential equation. Motion of objects, Fluid and heat flow, bending and cracking of materials, vibrations, chemical reactions and nuclear reactions are all modeled by systems of differential equations. Moreover, Numerous mathematical models in science and engineering are expressed in terms of unknown quantities and their derivatives. Many applications of differential equations (DEs), particularly ODEs of different orders, can be found in the mathematical modeling of real life problems ([35], [36]). The homotopy perturbation method (HPM) is an efficient technique to find the approximate solutions for ordinary and partial differential equations which describe different fields of science, physical phenomena, engineering, mechanics, and so on. HPM was proposed by Ji-Huan He in 1999 for solving linear and nonlinear differential equations and integral equations. Many researchers used HPM to approximate the solutions of differential equations and integral equations ([49], [25] & [34]). Complex tools are widely used in mathematical physics. The solution of mathematical model of physical problem is often made simpler through the use of complex analysis. Another particularly important application of complex numbers is in the quantum mechanics where they play a central role representing the state, or wave function, of a quantum system. The modern physics has some applications using differential equations, ordinary or partial, in complex domain. Recently, we have studied a wide class of complex differential equations (CDEs), when the input and output variables are in complex domain, which is used as mathematical models in many physically significant fields and applied science. The approximated solutions of this class of differential equations have studied using modded homotopy perturbation method (HPM). This study will reveal the significance and consolidation of the most powerful
fields in mathematics, complex analysis and differential equations. CDEs have been used in some applications in engineering and physics. For example, Time-Harmonic form of Maxwell’s equations and Schrodinger equation are examples of CPDEs and the six Painleve equations (PI-PVI), Riccati equation and Schwarzian equation are a examples of CODEs. The objective of this paper is the studying of the approximated solutions of CODEs using HPM.

4.1 Preliminary

High-Order Linear Complex Differential Equations

To present a review of the homotopy perturbation method for solving the linear CDEs, we consider the following CDE:

\[
\sum_{k=0}^{n} f_k(z)w^{(k)}(z) = f(z), \quad z, w(z) \in C, \quad (4.1)
\]

with initial conditions

\[
w^{k}(z_0) = \alpha_k, \quad (4.2)
\]

For all \( k = 0, 1, 2, \ldots, n - 1 \), where \( \alpha_k \in C \).

The implementation of proposed modified HPM for solving Linear CDEs (4.1) according the following algorithm: Homotopy Perturbation Method (HPM)

4.2 Analysis of The Method

4.2.1 Homotopy Perturbation Method (HPM)

In this section, we present a brief description of the HPM, to illustrate the basic ideas of the homotopy perturbation method, we consider the following differential equation ([37], [10], [4] & [1]):

\[
A(w) - f(z) = 0, \quad z \in \Omega, \quad (4.3)
\]
with boundary conditions:

\[ B(w, \partial w) = 0, \quad z \in \partial \Omega, \quad (4.4) \]

where \( A \) is a general differential operator, \( B \) is a boundary operator, \( f(z) \) a known analytic function and \( \partial \Omega \) is the boundary of the domain \( \Omega \). The operator \( A \) can be generally divided into two parts of \( L \) and \( N \) where \( L \) is linear part, while \( N \) is the nonlinear part in the DE, Therefore Eq. \((4.3)\) can be rewritten as follows \([19]\) :

\[ L(w) + N(w) - f(z) = 0. \quad (4.5) \]

By using homotopy technique, One can construct a homotopy

\[ V(z, p) : \Omega \times [0,1] \rightarrow C \]

which satisfies:

\[ H(v, p) = (1-p)[L(v) - L(w_0)] + p[L(v) + N(v) - f(z)] = 0, \quad (4.6) \]

or

\[ H(v, p) = L(v) - L(w_0) + p L(w_0 + p[N(v) - f(z)]) = 0, \quad (4.7) \]

where \( p \in [0,1], \quad z \in \Omega \) & \( p \) is called homotopy parameter and \( u_0 \) is an initial approximation for the solution of equation \((4.3)\) which satisfies the boundary conditions obviously, Using equation \((4.6)\) or \((4.7)\), we have the following equation:

\[ H(v, 0) = L(v) - L(w_0) = 0 \quad (4.8) \]

and

\[ H(v, 1) = L(v) + N(v) - f(z) = 0. \quad (4.9) \]

Assume that the solution of \((4.6)\) or \((4.7)\) can be expressed as a series in \( p \) as follows:

\[ V = v_0 + pv_1 + p^2v_2 + p^3v_3 + \cdots = \sum_{i=0}^{\infty} p^i v_i \quad (4.10) \]

set \( p \rightarrow 1 \) results in the approximate solution of \((4.3)\).
4.2 Analysis of The Method

Consequently,

\[ w(z) = \lim_{p \to 1} V = v_0 + v_2 + v_3 + \cdots = \sum_{i=0}^{\infty} v_i \quad (4.11) \]

It is worth to note that the major advantage of He’s homotopy perturbation method is that the perturbation equation can be freely constructed in many ways and approximation can also be freely selected. Analysis of HPM for Solving

4.2.2 Analysis of The Proposed Method

Firstly, we start with the initial approximation \( w_0(z) = g(z) \).

Secondly, we can construct a homotopy for CDE (4.1) as follow:

\[
H(w(z), p) = (1 - p)(w^{(n)}(z) - w_0^{(n)}(z)) + p\left( \sum_{k=0}^{n} f_k(z)w^{(k)}(z) - f(z) \right) = 0. \quad (4.12)
\]

Thirdly, using Taylor expansion about \( z = 0 \), by substituting the Taylor expansions for the coefficients functions

However,

\[
H(w(t), p) = (1 - p)\left( \sum_{i=0}^{\infty} \sum_{k=0}^{n} p^i w^{(n)}_i(z) - w_0^{(n)}(z) \right) + p\left( \sum_{i=0}^{\infty} \sum_{k=0}^{n} p^i f_k(z)w^{(k)}_i(z) - f(z) \right) = 0. \quad (4.13)
\]

Fourthly, suppose that the solution of Equation (4.13) is in the form

\[
w(z) = w_0(z) + pw_1(z) + p^2w_2(z) + p^3w_3(z) + \ldots \quad (4.14)
\]
Fifthly, collecting terms of the same power of $p$ gives, as shown in the following equations:

\[ p^0: \quad w_0^{(n)}(z) - w_0^{(n)}(z) = 0, \quad (4.15) \]

\[ p^1: \quad w_1^{(k)}(z) + \sum_{k=0}^{n} f_k(z)w_0^{(k)}(z) - f(z) = 0, \quad (4.16) \]

\[ p^2: \quad w_2^{(k)}(z) + \sum_{k=0}^{n} f_k(z)w_1^{(k)}(z) = 0, \quad (4.17) \]

\[ p^3: \quad w_3^{(k)}(z) + \sum_{k=0}^{n} f_k(z)w_2^{(k)}(z) = 0, \quad (4.18) \]

\[ p^4: \quad w_4^{(k)}(z) + \sum_{k=0}^{n} f_k(z)w_3^{(k)}(z) = 0, \quad (4.19) \]

\[ p^5: \quad w_5^{(k)}(z) + \sum_{k=0}^{n} f_k(z)w_4^{(k)}(z) = 0, \quad (4.20) \]

\[ \ldots \]

Hence, for $m = 2, 3, 4, \ldots$ we have,

\[ p^m: \quad w_m^{(k)}(z) + \sum_{k=0}^{n} f_k(z)w_{m-1}^{(k)}(z) = 0. \quad (4.21) \]

\[ \ldots \]

Finally, using the Equations (4.15)-(4.21) with some simplifications, we get the following terms of the solution:

\[ w_0(z) = g(z), \quad (4.22) \]

\[ w_1(z) = -\int \int \cdots \left( \sum_{k=0}^{n} f_k(z)w_0^{(k)}(z) - f(z) \right) dz \cdots dz, \quad (4.23) \]

\[ w_2(z) = -\int \int \cdots \left( \sum_{k=0}^{n} f_k(z)w_1^{(k)}(z) \right) dz \cdots dz, \quad (4.24) \]

\[ w_3(z) = -\int \int \cdots \left( \sum_{k=0}^{n} f_k(z)w_2^{(k)}(z) \right) dz \cdots dz, \quad (4.25) \]

\[ w_4(z) = -\int \int \cdots \left( \sum_{k=0}^{n} f_k(z)w_3^{(k)}(z) \right) dz \cdots dz. \quad (4.26) \]
and

\[
    w_5(z) = -\int \cdots \int (\sum_{k=0}^{n-1} f_k(z)w^{(k)}_4(z))dz \cdots dz, \quad \text{...(4.27)}
\]

Hence, the general term has the following form:

\[
    w_m(z) = -\int \cdots \int (\sum_{k=0}^{n-1} f_k(z)w^{(k)}_{m-1}(z))dz \cdots dz, \quad m = 2, 3, 4, \ldots \quad \text{...(4.28)}
\]

Then the solution of Equation (4.1) is

\[
    w(z) = w_0(z) + w_1(z) + w_2(z) + w_3(z) + w_4(z) + w_5(z) + \ldots \quad \text{...(4.29)}
\]

### 4.3 Implementations

In order to assess the accuracy of the solving generalized linear CDEs by HPM we will introduce some different examples in general and to compare the approximated solution with the exact solutions for these problems, we will consider the following.

**Example 4.1.** Consider the following CDE:

\[
    w'(z) - w(z) = 0, \quad w > 0, z \in C \quad \text{...(4.30)}
\]

subject to the initial condition

\[
    w(0) = 1,
\]

with the exact solution

\[
    w(z) = e^z.
\]

Comparing Equation (4.30) with Equation (4.1), we have \( n = 1, \quad f_0(z) = -1, \quad f_1(z) = 1, \) and \( f(z) = 0 \)

The initial approximation has the form \( w_0(z) = 1 \) substituting Equation (4.30) into
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Equations (4.22)-(4.28), we have

\[ w_1(z) = z \]  
\[ w_2(z) = \frac{z^2}{2!} \]  
\[ w_3(z) = \frac{z^3}{3!} \]  
\[ w_4(z) = \frac{z^4}{4!} \]

Then, the general solution of Equation (4.30) is written as follow:

\[ w(z) = w_0(z) + w_1(z) + w_2(z) + w_3(z) + w_4(z) + \ldots = e^z \]  

Example 4.2. Consider the following CDE:

\[ w''(z) + w(z) = 0, \quad -1 < w > 1, z \in C \]  

subject to the initial condition

\[ w(0) = 0, \quad w'(0) = 1 \]

with the exact solution

\[ w(z) = \sin(z). \]

Comparing Equation (4.36) with Equation (4.1), we have \( n = 2, \quad f_0(z) = 1, \quad f_1(z) = 0, \quad f_2(z) = 1, \) and \( f(z) = 0. \)

The initial approximation has the form \( w_0(z) = z \) substituting Equation (4.36) into
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Equations (4.22)-(4.28), we have

\[ w_1(z) = -\frac{z^3}{3!} \]  \hspace{1cm} (4.37)
\[ w_2(z) = \frac{z^5}{5!} \]  \hspace{1cm} (4.38)
\[ w_3(z) = -\frac{z^7}{7!} \]  \hspace{1cm} (4.39)
\[ w_4(z) = \frac{z^9}{9!} \]  \hspace{1cm} (4.40)
\[ \ldots \]
\[ w_m(z) = (-1)^m \frac{z^{2m+1}}{(2m+1)!} \]  \hspace{1cm} (4.41)

Then, the general solution of Equation (4.36) is written as follow:

\[ w(z) = w_0(z) + w_1(z) + w_2(z) + w_3(z) + w_4(z) + \ldots = \sin(z) \]  \hspace{1cm} (4.42)

**Example 4.3.** Consider the following Linard CDE:

\[ w''(z) - w(z) + z = 0, \quad z \in C \]  \hspace{1cm} (4.43)

subject to the initial condition

\[ w(0) = 2\alpha, \quad w'(0) = \beta \]

Where \( \alpha = a_1 + ib_1, \quad \beta = a_2 + ib_2, \) with the exact solution

\[ w(z) = c_1 e^z + c_2 e^{-z} + z. \]

Comparing Equation (4.43) with Equation (4.1), we have \( n = 2, \quad f_0(z) = -1, \quad f_1(z) = 0, \quad f_2(z) = 1, \) and \( f(z) = z \)

The initial approximation has the form \( w_0(z) = 2\alpha + \beta z \) substituting Equation (4.43)
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into Equations (4.22)-(4.28), we have

\[ w_1(z) = \frac{2\alpha z^2}{2!} + (\beta - 1)\frac{z^3}{3!}, \] (4.44)

\[ w_2(z) = \frac{2\alpha z^4}{4!} + (\beta - 1)\frac{z^5}{5!}, \] (4.45)

\[ w_3(z) = \frac{2\alpha z^6}{6!} + (\beta - 1)\frac{z^7}{7!}, \] (4.46)

\[ w_4(z) = \frac{2\alpha z^8}{8!} + (\beta - 1)\frac{z^9}{9!}, \] (4.47)

\[ \ldots \]

\[ w_m(z) = \frac{2\alpha z^{2m}}{2m!} + (\beta - 1)\frac{z^{2m+1}}{(2m+1)!}, \] (4.48)

Suppose \( \beta = 1 \) then, the general solution of Equation (4.43) is written as follow:

\[ w(z) = w_0(z) + w_1(z) + w_2(z) + w_3(z) + w_4(z) + \ldots \]

\[ = \alpha (e^z + e^{-z}) + z \] (4.49)

**Example 4.4.** Consider the following CDE:

\[ w'''(z) + iw(z) = 0, \quad z \in C \] (4.50)

subject to the initial conditions

\[ w(0) = 1, \quad w'(0) = i, \quad w''(0) = -1 \]

with the exact solution

\[ w(z) = ae^{iz}. \]

Comparing Equation (4.50) with Equation (4.1), we have \( n = 3 \), \( f_0(z) = i \), \( f_1(z) = 0 \), \( f_2(z) = 0 \), \( f_3(z) = 1 \), and \( f(z) = 0 \)

From the initial conditions evince \( z = ix \)

Then, the initial approximation has the form \( w_0(z) = 1 + ix - \frac{x^2}{2!} \), substituting Equation
(4.50) into Equations (4.22)-(4.28), we have

\[ w_1(z) = -\frac{i x^3}{3!} + \frac{x^4}{4!} + \frac{i x^5}{5!}, \]
\[ w_2(z) = -\frac{x^6}{6!} - \frac{i x^7}{7!} + \frac{x^8}{8!}, \]
\[ w_3(z) = \frac{i x^9}{9!} - \frac{x^{10}}{10!} - \frac{i x^{11}}{11!}, \]
\[ w_4(z) = \frac{x^{12}}{12!} + \frac{i x^{13}}{13!} - \frac{x^{14}}{14!}. \]

Then, the general solution of Equation (4.50) is written as follow:

\[ w(z) = w_0(z) + w_1(z) + w_2(z) + w_3(z) + w_4(z) + \ldots \]
\[ = e^{ix} \] (4.55)

**Example 4.5.** Consider the following CDE:

\[ w'''(z) + 3w''(z) - 4w(z) = 0, \quad z \in C \] (4.56)

subject to the initial condition

\[ w(0) = a, \quad w'(0) = b, \quad w''(0) = c \]

Where \( a, \ b, \ c \in C \), with the exact solution

\[ w(z) = c_1 e^{z} + c_2 e^{-2z} + c_3 z e^{-2z}. \]

Comparing Equation (4.56) with Equation (4.1), we have \( n = 2 \), \( f_0(z) = -1 \), \( f_1(z) = 0 \), \( f_2(z) = 1 \), and \( f(z) = z \)

The initial approximation has the form \( w_0(z) = a + bz + c z^2 \) substituting Equation
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(4.56) into Equations (4.22)-(4.28), we have

\[ w_1(z) = 4a \frac{z^3}{3!} + 4b \frac{z^4}{4!} + 4c \frac{z^5}{5!}, \quad (4.57) \]
\[ w_2(z) = 16a \frac{z^6}{6!} + 16b \frac{z^7}{7!} + 16c \frac{z^8}{8!}, \quad (4.58) \]
\[ w_3(z) = 64a \frac{z^9}{9!} + 64b \frac{z^{10}}{10!} + 64c \frac{z^{11}}{11!}, \quad (4.59) \]
\[ w_4(z) = 256a \frac{z^{12}}{12!} + 256b \frac{z^{13}}{13!} + 256c \frac{z^{14}}{14!}, \quad (4.60) \]
\[ \vdots \]
\[ w_m(z) = 4^m (a \frac{z^{3m}}{3m!} + b \frac{z^{3m+1}}{(3m+1)!} + c \frac{z^{3m+2}}{(3m+2)!}) \quad (4.61) \]

Suppose \( \beta = 1 \) then, the general solution of Equation (4.56) is written as follow:

\[ w(z) = w_0(z) + w_1(z) + w_2(z) + w_3(z) + w_4(z) + \ldots \]
\[ = a + bz + c \frac{z^2}{2!} + 4a \frac{z^3}{3!} + 4b \frac{z^4}{4!} + 4c \frac{z^5}{5!} + 16a \frac{z^6}{6!} + 16b \frac{z^7}{7!} + 16c \frac{z^8}{8!} \]
\[ w_3(z) = 64a \frac{z^9}{9!} + 64b \frac{z^{10}}{10!} + 64c \frac{z^{11}}{11!} + 256a \frac{z^{12}}{12!} + 256b \frac{z^{13}}{13!} + 256c \frac{z^{14}}{14!} \]
\[ + \cdots + 4^m (a \frac{z^{3m}}{3m!} + b \frac{z^{3m+1}}{(3m+1)!} + c \frac{z^{3m+2}}{(3m+2)!}) + \ldots \quad (4.62) \]
Chapter 5

Applications

In the previous chapters, we fully constructed a solid foundation of the existence and uniqueness of the general solution of complex differential equations. Analytical methods for solving complex differential equations has never been classified and derived as this work. This work has an extraordinary step up which is some real world applications in physics and engineering fluids. We discussed this concept and its application based on the following researches ([33]), ([32]), ([11]), ([45]), ([38]), ([20]), ([52]), ([51]), ([28]), ([23]).

5.1 Application Of The Schwarz-Christoffel Transformation

This revealing the powerful features and features of complex differential equations, one of the well known complex conformal mapping or as it is usually called a transformation. This transformation makes a mapping between the $x$ axis and the upper half of the complex plane onto a given close polygon and the interior point of that polygon in $w$ plane where both $z$ and $w$ are complex plains. This transformation is used to solve problems in Electronic potential theory and fluid.

5.1.1 Mapping The Real Axis Onto A Polygon

In this part, at first, we’re strutting the most important and the main form of the Schwarz-Christoffel transformation. Suppose that we denote the tangent unit vector of a smooth arc $C$ in the complex plain at the point $z_0$ by $t$ and we say for instance that we denote the tangent unit vector in the corresponding complex plain to $\Gamma$ the
image of $C$ controlled by the transformation $w = f(z)$ is $\tau$. Assume that $f$ is an analytic complex function at $z_0$ and, $f'(z_0) \neq 0$. According to the conformal mapping property (Preservation of Angles),

$$\arg(\tau) = \arg(f'(z_0)) + \arg(t).$$

(5.1)

particularly, if $C$ is a segment of the positive $x$ axis, then $t = 1$ and $\arg(t) = 0$ at each point $z_0 = x$ on $C$. In that case, equation (5.1) becomes

$$\arg(\tau) = \arg(f'(z_0)) + \arg t.$$  

(5.2)

$$\arg(\tau) = \arg(f'(z_0)).$$  

(5.3)

Whenever $f'(z)$ owns a constant argument along the segment of its domain, then $\arg(\tau)$ is also constant. Therefore the image of $C$ by the transformation is $\Lambda$ is also a segment on its domain of a straight line.

We heading now to initially construct the mentioned transformation $w = f(z)$ that maps the whole $x$ axis in a domain onto a polygon of $n$ sides in another domain, where $x_1, x_2, ..., x_{n-1}$, and $\infty$ are the points on that axis in the origin domain which its images are will be the vertices of the polygon in the mapped domain and where

$$x_1 \leq ... \leq x_{n-1}.$$  

The vertices in the polygon are the $n$ points $w_j = f(x_j)$ ($j = 1, 2, ..., n-1$) and $w_n = f(\infty)$. The function $f$ then should be such that $\arg(f'(z))$ moves one constant each time we move one value on $x$ value to another at the points $z = x_j$ Figure (5.1).

If the function $f$ is chosen such that

$$f'(z) = A(z - x_1)^{-k_1}(z - x_2)^{-k_2}...(z - x_{n-1})^{-k_{n-1}},$$  

(5.4)

Note that $A$ is a complex constant value and every $k_j$ is a real constant since it is an exponent the argument of the derivative (the tangent) $f'(z)$ varies in the designated mode as $z$ represents the real axis. This is seen by writing the argument of the derivative (5.4) as
\[ \arg f'(z) = \arg A - k_1 \arg (z - x_1) - k_2 \arg (z - x_2) - \ldots - k_{n-1} \arg (z - x_{n-1}). \] (5.5)

When \( z = x \) and \( x < x_1 \)
\[ \arg (z - x_1) = \arg (z - x_2) = \ldots = \arg (z - x_{n-1}) = \pi \] (5.6)

while \( x_1 < x < x_2 \), the argument \( \arg (zx_1) \) is 0 and each of the other arguments is \( \pi \).

According to Equation (5.5) the \( \arg f'(z) \) increases abruptly by the angle \( k_1 \pi \) as \( z \) moves to the right through the point \( z = x_1 \). It again moves in value by the amount \( k_2 \pi \), as \( z \) passes through the point \( x_2 \). When viewing of Equation (5.2) we will see the unit vector \( \tau \) is constant complex number in direction while \( z \) moves from \( x_j \) to \( x_{j+1} \) the point \( w \) so that moves in that fixed direction along a straight line. The direction of \( \tau \) changes abruptly so that by the angle \( k_j \pi \) at complex domain where the image point \( w_j \) of \( x_j \) takes place as shown in Figure(5.1). Those angles \( k_j \pi \) are the exterior angles of the polygon described by the point \( w \). The exterior angles can be limited to angles between \( -\pi \) and \( \pi \), in which case \( -1 < k_j < 1 \). We assume that all sides of the polygon do not cross each other and the polygon is given a positive or counterclockwise orientation. The sum of the exterior angles of a closed polygon is, then, \( 2\pi \); and the exterior angle at the vertex \( w_n \), which is the image of the point \( z = \infty \), can be written
\[ k_n \pi = 2\pi - (k_1 + k_2 + \ldots + k_{n-1}) \pi. \] (5.7)

Thus the numbers \( k_j \) must necessarily satisfy the conditions
\[ k_1 + k_2 + \ldots + k_{n-1} + k_n = 2, \quad -1 < k_j < 1 \quad (j = 1, 2, \ldots, n). \] (5.8)
Note that $k_n = 0$ if
\[ k_1 + k_2 + \ldots + k_{n-1} = 2 \]  
(5.9)

This means that the direction of $\tau$ does not change at the point $w_n$. So $w_n$ is not a vertex, and the polygon has $n - 1$ sides. The existence of a mapping function $f$ whose derivative is given by equation (5.4) will be established in the next section.

5.2 Schwarz-Christoffel Transformation

In the previous section
\[ f'(z) = A(z-x_1)^{-k_1}(z-x_2)^{-k_2}(z-x_{n-1})^{-k_{n-1}}, \]  
(5.10)

for the derivative of a function that is to map the $x$ axis onto a polygon, let the factors $(z-x_j)^{-k_j}$ ($j = 1, 2, \ldots, n - 1$) represent branches of power functions with branch cuts extending below that axis. To be specific, write
\[ (z-x_j)^{-k_j} = e^{-k_j \log(z-x_j)} = e^{-k_j(\ln|z-x_j| + i\theta_j)}, \]  
(5.11)

and then
\[ (z-x_j)^{-k_j} = |z-x_j|^{-k_j} e^{-ik_j\theta_j} \quad \left( -\frac{\pi}{2} < \theta_j < \frac{3\pi}{2} \right) \]  
(5.12)

where $\theta_j = \arg(z-x_j)$ and $j = 1, 2, \ldots, n$. This makes $f'(z)$ analytic everywhere in the half plane $y > 0$ except at the $n - 1$ branch points $x_j$. If $z_0$ is a point in that region of analyticity, denoted here by $R$, then the function
\[ F(z) = \int_{z_0}^{z} f'(s)ds \]  
(5.13)

is single-valued and analytic throughout the same region, where the path of integration from $z_0$ to $z$ is any contour lying within $R$. Moreover, $F'(z) = f(z)$.

To define the function $F$ at the point $z = x_1$ so that it is continuous there, we note that $(z-x_1)^{k_1}$ is the only factor in Equation (5.10) that is not analytic at $x_1$. Hence if $\phi(z)$ denotes the product of the rest of the factors in that expression, $\phi(z)$ is analytic at the point $x_1$ and is represented throughout an open disk $|z-x_1| < R_1$ by its Taylor series about $x_1$. So we can write
\[ f'(z) = (z-x_1)^{-k_1} \phi(z) \]
\[ = (z-x_1)^{-k_1} \left[ \phi(x_1) + \frac{\phi'(x_1)}{1!}(z-x_1) + \frac{\phi''(x_1)}{2!}(z-x_1) + \ldots \right] \]
or
\[ f'(z) = \phi(x_1)(z-x_1)^{-k_1} + (z-x_1)^{1-k_1} \psi(z), \tag{5.14} \]

where \( \psi \) is analytic and therefore continuous throughout the entire open disk. Since \( 1-k_1 > 0 \), the last term on the right in equation (5.5) thus represents a continuous function of \( z \) throughout the upper half of the disk, where \( \text{Im}z \geq 0 \), if we assign it the value zero at \( z = x_1 \). It follows that the integral
\[
\int_h^z (s-x_1)^{1-k_1} \psi(s) \, ds,
\]
of that last term along a contour from \( h \) to \( z \), where \( h_1 \) and the contour lie in the half disk, is a continuous function of \( z \) at \( z = x_1 \). The integral
\[
\int_h^z (s-x_1)^{-k_1} \, ds = \frac{1}{1-k_1} [(z-x_1)^{1-k_1} - (h-x_1)^{1-k_1}],
\]
along the same path also represents a continuous function of \( z \) at \( x_1 \) if we define the value of the integral there as its limit as \( z \) approaches \( x_1 \) in the half disk. The integral of the function (5.14) along the stated path from \( h \) to \( z \) is, then, continuous at \( z = x_1 \); and the same is true of integral (5.13) since it can be written as an integral along a contour in \( R \) from \( z_0 \) to \( h \) plus the integral from \( h \) to \( z \). The above argument applies at each of the \( n-1 \) points \( x_j \) to make \( F \) continuous throughout the region \( y \geq 0 \).

From equation (5.10), we can show that for a sufficiently large positive number \( R \), a positive constant \( M \) exists such that if \( \text{Im}z > 0 \), then
\[
|f'(z)| < \frac{M}{|z|^{2-k_n}} \quad \text{whenever} \quad |z| > R, \tag{5.15}
\]
Since \( 2-k_n > 1 \), this order property of the integrand in Equation (5.13) grantees the existence of the limit of the integral there as \( z \) goes to infinity; that is, a number \( W_n \) exists such that
\[
\lim_{z \to \infty} F(z) = W_n \quad (\text{Im}z \geq 0) \tag{5.16}
\]
Our mapping function, whose derivative is given by Equation (5.10), can be formulated \( f(z) = F(z) + B \), where \( B \) is a complex constant. The resulting transformation, is the Schwarz Christoffel transformation. According to Equation (5.16), the image \( w_n \) of the point \( z = \infty \) exists and \( w_n = W_n + B \). If \( z \) is an interior point of the upper half plane \( y \geq 0 \) and \( x_0 \) is any point on the \( x \)-axis other than one of the \( x_j \), then the angle from the vector \( t \) at \( x_0 \) up to the line segment joining \( x_0 \) and \( z \) is positive and less than \( \pi \). At the image, \( w_0 \) of \( x_0 \), the corresponding angle from the vector \( \tau \) to the image of the line segment joining \( x_0 \) and \( z \) has that same value. Thus the images of interior points in the half plane lie to the left of the sides of the polygon, taken counterclockwise.

Given a specific polygon \( P \), let us examine the number of constants in the Schwarz Christoffel transformation that must be determined in order to map the \( x \)-axis onto \( P \). For this purpose, we may write \( z_0 = 0 \), \( A = 1 \), and \( B = 0 \) and simply require that the \( x \)-axis be mapped onto some polygon \( P' \) similar to \( P \).

The size and position of \( P' \) can then be adjusted to match those of \( P \) by introducing the appropriate constants \( A \) and \( B \). The numbers \( k_j \) are all determined from the exterior angles at the vertices of \( P \). The \( n - 1 \) constants \( x_j \) remain to be chosen. The image of the \( x \)-axis is some polygon \( P' \) that has the same angles as \( P \). But if \( P' \) is to be similar to \( P \), then \( n - 2 \) connected sides must have a common ratio to the corresponding sides of \( P \); this condition is expressed by means of \( n - 3 \) equations in the \( n - 1 \) real unknowns \( x_j \). Thus two of the numbers \( x_j \), or two relations between them, can be chosen arbitrarily, provided those \( n - 3 \) equations in the remaining \( n - 3 \) unknowns have real-valued solutions. When a finite point \( z = x_n \) on the \( x \)-axis, instead of the point at infinity, represents the point whose image is the vertex \( w_n \), then the Schwarz Christoffel transformation takes the form

\[
 w = A \int_{z_0}^{z} (s - x_1)^{-k_1} (s - x_2)^{-k_2} \ldots (s - x_n)^{-k_n} ds + B 
\]  

where \( k_1 + k_2 + \ldots + k_n = 2 \). The exponents \( k_j \) are determined from the exterior angles of the polygon. But, in this case, there are \( n \) real constants \( x_j \) that must satisfy the \( n - 3 \) equations noted above. Thus three of the numbers \( x_j \), or three conditions on those \( n \) numbers, can be chosen arbitrarily when transformation (5.17) is used to map the \( x \)-axis onto a given polygon.
5.2 Schwarz-Christoffel Transformation

5.2.1 Rectangles and Triangles

The Schwarz Christoffel transformation is written in terms of the points $x_j$ and not in terms of their images, which are the vertices of the polygon. No more than three of those points can be chosen arbitrarily; so, when the given polygon has more than three sides, some of the points $x_j$ must be determined in order to make the given polygon, or any polygon similar to it, be the image of the $x$ axis. The selection of conditions for the determination of those constants that are convenient to use often requires ingenuity. Another limitation in using the transformation is due to the integration that is involved. Often the integral cannot be evaluated in terms of a finite number of elementary functions. In such cases, the solution of problems by means of the transformation can become quite involved. If the polygon in the Figure (5.2) is a triangle with vertices at the points $w_1$, $w_2$, and $w_3$ the transformation can be written:

$$w = A \int_0^z (s-x_1)^{-k_1}(s-x_2)^{-k_2}(s-x_3)^{-k_3}ds + B,$$

(5.18)

where $k_1 + k_2 + k_3 = 2$. In terms of the interior angles $\theta_j$

$$k_j = 1 - \frac{1}{\pi} \theta_j \quad (j = 1, 2, 3),$$

Here we have taken all three points $x_j$ as finite points on the $x$ axis. Arbitrary values can be assigned to each of them. The complex constants $A$ and $B$, which are associated with the size and position of the triangle, can be determined so that the upper half plane is mapped onto the given triangular region.

If we take the vertex $w_3$ as the image of the point at infinity, the transformation becomes

![Figure 5.2 Rectangles and Triangles](image-url)
where arbitrary real values can be assigned to \(x_1\) and \(x_2\). The integrals in equations (5.18) and (5.19) do not represent elementary functions unless the triangle is degenerate with one or two of its vertices at infinity. The integral in equation (5.19) becomes an elliptic integral when the triangle is equilateral or when it is a right triangle with one of its angles equal to either \(\frac{\pi}{3}\) or \(\frac{\pi}{4}\).

**Example 5.1.** Let’s take for example the equilateral triangle, \(K_1 = K_2 = K_3 = \frac{2}{3}\). It is convenient to write \(x_1 = -1\), \(x_2 = 1\), and \(x_3 = \infty\) ant by using Equation (5.19), with \(z_0 = 1\), \(A = 1\), and \(B = 0\). The transformation will be

\[
w = A \int_{z_0}^{z} (s - x_1)^{-k_1} (s - x_2)^{-k_2} ds + B, \tag{5.19}
\]

The image of the point \(z = 1\) is clearly \(w = 0\); that is, \(w_2 = 0\). If \(z = -1\) in this integral, one can write \(s = x\), where \(-1 < x < 1\). Then

\[
x + 1 > 0 \quad \text{arg}(x + 1) = 0, \tag{5.20}
\]

while

\[
|x - 1| = 1 - x \quad \text{and} \quad \text{arg}(x - 1) = \pi,
\]

then

\[
w = \int_{1}^{-1} (x + 1) \frac{2}{3} (1 - x) \frac{2}{3} e^{-\frac{2\pi}{3}i} \tag{5.21}
\]

\[
e^{\frac{2\pi}{3}} \int_{0}^{1} \frac{2dx}{(1 - x^2)^\frac{3}{2}}
\]

when \(z = -1\). With the substitution \(x = \sqrt{t}\), the last integral here reduces to a special case of the one used in defining the beta function. Let \(b\) denote its value, which is positive:

\[
b = \int_{0}^{1} \frac{2dx}{(1 - x^2)^\frac{3}{2}} = \int_{0}^{1} t^{-\frac{1}{2}} (1 - t)^{-\frac{3}{2}} dt = B\left(\frac{1}{2}, \frac{1}{3}\right),
\]

The vertex \(w_1\) is, therefore, the point Figure (5.3).
\[ w_1 = b e^{\frac{2\pi i}{3}}, \]

The vertex \( w_3 \) is on the positive \( u \) axis because

\[ w_3 = \int_1^\infty (x + 1)^{-\frac{1}{3}}(x - 1)^{-\frac{2}{3}} dx = \int_1^\infty \frac{dx}{(x^2 - 1)^{\frac{1}{3}}}, \]

But the value of \( w_3 \) is also represented by integral (5.20) when \( z \) tends to infinity along the negative \( x \) axis; that is,

\[ w_3 = \int_1^{-1} (|x + 1||x - 1|)^{-\frac{1}{3}} e^{-\frac{2\pi i}{3}} dx \]

\[ + \int_{-1}^\infty (|x + 1||x - 1|)^{-\frac{3}{3}} e^{-\frac{4\pi i}{3}} \]

In view of the first of expressions (5.1) for \( w_1 \), then,

\[ w_3 = w_1 + e^{\left(\frac{2\pi i}{3}\right)} \int_{-1}^{-\infty} (|x + 1||x - 1|)^{-\frac{2}{3}} dx \]

\[ = b e^{\frac{2\pi i}{3}} + e^{\left(-\frac{2\pi i}{3}\right)} \int_1^\infty \frac{dx}{(x^2 - 1)^{\frac{1}{3}}} \]

or

\[ w_3 = b e^{\frac{2\pi i}{3}} + w_3 e^{-\frac{2\pi i}{3}}, \]

Solving for \( w_3 \), we find that

\[ w_3 = b, \]

We have thus verified that the image of the \( x \) axis is the equilateral triangle of side \( b \) shown in Figure (5.3). We can also see that
\[ w = \frac{b}{2} e^{\frac{z_0}{2}} \quad z = 0, \]

When the polygon is a rectangle, each \( k_j = \frac{1}{2} \). If we choose \( \pm 1 \) and \( \pm a \) as the points \( x_j \) whose images are the vertices and write

\[ g(z) = (z + a)^{-\frac{1}{2}}(z + 1)^{-\frac{1}{2}}(z - 1)^{-\frac{1}{2}}(z - 1)^{-\frac{1}{2}} \]  \quad (5.22)

where \( 0 \leq \text{arg}(z - x_j) \leq \pi \), the Schwarz–Christoffel transformation becomes

\[ w = -\int_0^z g(s) ds, \]  \quad (5.23)

except for a transformation \( W = Aw + B \) to adjust the size and position of the rectangle. Integral (5.23) is a constant times the elliptic integral

\[ \int_0^z (1 - s^2)^{-\frac{1}{2}}(1 - k^2 s^2)^{-\frac{1}{2}} ds \quad \left(k = \frac{1}{a}\right), \]

but the form (5.22) of the integrand indicates more clearly the appropriate branches of the power functions involved.

### 5.2.2 The Flow of a Fluid in a Channel with an Abrupt Change in Its Breadth

the use of the Schwarz–Christoffel transformation, let us find the complex potential for the flow of a fluid in a channel with an abrupt change in its breadth (Figure 5.4). We take our unit of length such that the breadth of the wide part of the channel is \( \pi \) units; then \( h\pi \), where \( 0 < h < 1 \), represents the breadth of the narrow part. Let the real constant \( V_0 \) denote the velocity of the fluid far from the offset in the wide part; that is,

\[ \lim_{u \to -\infty} V = V_0, \]

where the complex variable \( V \) represents the velocity vector. The rate of flow per unit depth through the channel, or the strength of the source on the left and of the sink on the right, is then

\[ Q = \pi V_0 \]  \quad (5.24)
The cross section of the channel can be considered as the limiting case of the quadrilateral with the vertices \( w_1, w_2, w_3, \) and \( w_4 \) shown in Figure 5.4 as the first and last of these vertices are moved infinitely far to the left and right, respectively. In the limit, the exterior angles become

\[
\begin{align*}
    k_1 \pi &= \pi \\
    k_2 \pi &= \frac{\pi}{2} \\
    k_3 \pi &= -\frac{\pi}{2} \\
    k_4 \pi &= \pi,
\end{align*}
\]

To simplify the determination of the constants \( A \) and \( x_2 \) here, we proceed at once to the complex potential of the flow. The source of the flow in the channel infinitely far to the left corresponds to an equal source at \( z = 0 \). The entire boundary of the cross section of the channel is the image of the \( x \) axis. In view of equation (5.24), then, the function

\[
F = V_0 \log z = V_0 \ln r + iV_0 \theta
\]

is the potential for the flow in the upper half of the \( z \) plane, with the required source at the origin. Here the stream function is \( \psi = V_0 \theta \). It increases in value from 0 to \( V_0 \pi \) over each semicircle \( z = Re^{i\theta} \) \((0 \leq \theta \leq \pi)\) as \( \theta \) varies from 0 to \( \pi \).

The complex conjugate of the velocity \( V \) in the \( w \) plane can be written

\[
V(w) = \frac{dF}{dw} = \frac{dF}{dz} \frac{dz}{dw}
\]

Thus, by referring to equations (5.24) and (5.26), we can see that

\[
V_0 = \frac{V_0}{A} \sqrt{x_2},
\]
At the limiting position of $w_4$, which corresponds to $z = \infty$, let the real number $V_4$ denote the velocity. Now it seems plausible that as a vertical line segment spanning the narrow part of the channel is moved infinitely far to the right, $V$ approaches $V_4$ at each point on that segment. We could establish this conjecture as a fact by first finding $w$ as the function of $z$ from equation (5.26); but, to shorten our discussion, we assume that this is true, Then, since the flow is steady,

$$\pi h V_4 = \pi V_0 = Q,$$

or $V_4 = \frac{V_0}{h}$. Letting $z$ tend to infinity in equation (5.27), we find that

$$V_0 = \frac{V_0}{h} = A,$$

thus

$$A = h, \quad x_2 = h^2,$$

and

$$V(w) = \frac{V_0}{h} \left( \frac{z - h^2}{z - 1} \right)^{\frac{1}{2}} \quad (5.28)$$

From equation (5.28), we know that the magnitude $|V|$ of the velocity becomes infinite at the corner $w_3$ of the offset since it is the image of the point $z = 1$. Also, the corner $w_2$ is a stagnation point, a point where $V = \theta$. Hence, along the boundary of the channel, the fluid pressure is greatest at $w_2$ and least at $w_3$. To write the relation between the potential and the variable $w$, we must integrate equation (5.25), which can now be written

$$\frac{dw}{dz} = \frac{h}{z} \left( \frac{z - 1}{z - h^2} \right)^{\frac{1}{2}} \quad (5.29)$$

we substitute by $w = \sum_{i=0}^{\infty} p^i w_i$ and $w' = \sum_{i=0}^{\infty} p^i w'_i$ to get,

$$H(w, p) = (1 - p)(w'_0 + p w'_1 + p^2 w'_2 + ... - w'_0) + p(w'_0 + p w'_1 + p^2 w'_2 + ... - \frac{h}{z} \left( \frac{z - 1}{z - h^2} \right)^{\frac{1}{2}}) = 0,$$
5.2 Schwarz-Christoffel Transformation

\[ p^0 : w'_0 - w'_0 = 0 \]
\[ p^1 : w'_1 + w'_0 - \frac{h}{z} \left( \frac{z-1}{z-h^2} \right)^{\frac{1}{2}} = 0, \]
\[ p^2 : w'_2 - w'_1 + w'_1 = 0, \]
\[ p^3 : w'_3 - w'_2 + w'_2 = 0, \]
\[ \vdots \]
\[ w_1 = \int \left( \frac{h}{z} \left( \frac{z-1}{z-h^2} \right)^{\frac{1}{2}} - w'_0 \right) dz \]
\[ w = w_0 + w_1 \]

By substituting a new variable \( s \), where

\[ \frac{z-h^2}{z-1} = s^2, \]

one can show that equation (5.29) reduces to

\[ \frac{dw}{dz} = 2h \left( \frac{1}{1-s^2} - \frac{1}{h^2 - s^2} \right). \]

We are going to solve this equation by HMT method, let \( w(a) = b \) is the IV.

\[ H(w, p) = (1 - p)(w' - w'_0) + p \left( w' - \frac{h}{z} \left( \frac{z-1}{z-h^2} \right)^{\frac{1}{2}} \right) = 0 \]

Hence

\[ w = h \log \frac{1+s}{1-s} - \log \frac{h+s}{h-s} \]  \hspace{1cm} (5.30)

The constant of integration here is zero because when \( z = h^2 \), the quantity \( s \) is zero and so, therefore, is \( w \). In terms of \( s \), the potential \( F \) of equation (5.26) becomes

\[ F = V_0 \log \frac{h^2 - s^2}{1 - s^2}, \]

consequently,
\[ s^2 = \frac{e^{F_{0}} - h^2}{e^{F_{0}} - 1} \]  

(5.31)

By substituting \( s \) from this equation into equation (5.30), we obtain an implicit relation that defines the potential \( F \) as a function of \( w \).

### 5.2.3 Electronic Potential About An Edge of A Conducting Plate

Two parallel conducting plates of infinite extent are kept at the electrostatic potential \( V = 0 \), and a parallel semi-infinite plate, placed midway between them, is kept at the potential \( V = 1 \). The coordinate system and the unit of length are chosen so that the plates lie in the planes \( v = 0 \), \( v = \pi \), and \( v = \frac{\pi}{2} \) (Figure 5.5). Let us determine the potential function \( V(u,v) \) in the region between those plates.

![Figure 5.5 Two Conducting Plates](image)

The cross section of that region in the \( uv \) plane has the limiting form of the quadrilateral bounded by the dashed lines in Figure 5.5 as the points \( w_1 \) and \( w_3 \) move out to the right and \( w_4 \) to the left. In applying the Schwarz-Christoffel transformation here, we let the point \( x_4 \), corresponding to the vertex \( w_4 \), be the point at infinity. We choose the points \( x_1 = -1 \), \( x_3 = 1 \) and leave \( x_2 \) to be determined. The limiting values of the exterior angles of the quadrilateral are

\[ k_1 \pi = \pi \quad k_2 \pi = -\pi \quad k_3 \pi = k_4 \pi = \pi, \]

Thus

\[ \frac{dw}{dz} = A(z + 1)^{-1}(z - x_2)(z - 1)^{-1} = A \left( \frac{z - x_2}{z^2 - 1} \right) = \frac{A}{2} \left( \frac{1 + x_2}{z + 1} + \frac{1 - x_2}{z - 1} \right), \]

\[ H(w, p) = (1 - p)(w' - w'_0) + p(w' - A(\frac{z - x_2}{z^2 - 1})) = 0 \]

we substitute by \( w = \sum_{i=0}^{\infty} p_i w_i \) and \( w' = \sum_{i=0}^{\infty} p'_i w'_i \) to get,
\[ H(w, p) = (1 - p)(w_0' + pw_1' + p^2w_2' + ... - w_0') + p(w_0' + pw_1' + p^2w_2' + ...) - A \left( \frac{z - x_2}{z^2 - 1} \right) = 0, \]

\[
\begin{align*}
ccp^0 & : w_0' - w_0' = 0 \\
p^1 & : w_1' + w_0' - A \left( \frac{z - x_2}{z^2 - 1} \right) = 0, \\
p^2 & : w_2' - w_1' + w_1' = 0, \\
p^3 & : w_3' - w_2' + w_2' = 0, \\
& \vdots \\
w_1 & = \int (A \left( \frac{z - x_2}{z^2 - 1} \right) - w_0')dz \\
w & = w_0 + w_1
\end{align*}
\]

then

\[ w = \int (A \left( \frac{z - x_2}{z^2 - 1} \right)) dz \]

and so the transformation of the upper half of the \( z \) plane into the divided strip in the \( w \) plane has the form

\[ w = \frac{A_1}{2} \left[ (1 + x_2) \log(z + 1) + (1 - x_2) \log(z - 1) \right] + B. \]  \hspace{1cm} (5.32)

Let \( A_1, A_2 \) and \( B_1, B_2 \) denote the real and imaginary parts of the constants \( A \) and \( B \). When \( z = x \), the point \( w \) lies on the boundary of the divided strip; and, according to equation (5.32),

\[ u + iv = \frac{A_1 + iA_2}{2} \left\{ (1 + x_2)[|x + 1| + i\arg(x + 1)] + (1 - x^2)[|x - 1| + i\arg(x - 1)] \right\} + B_1 + iB_2 \]  \hspace{1cm} (5.33)

To determine the constants here, we first note that the limiting position of the line segment joining the points \( w_1 \) and \( w_4 \) is the \( u \) axis. That segment is the image of the part of the \( x \) axis to the left of the point \( x_1 = -1 \); this is because the line segment
joining $w_3$ and $w_4$ is the image of the part of the $x$ axis to the right of $x_3 = 1$, and the other two sides of the quadrilateral are the images of the remaining two segments of the $x$ axis. Hence when $v = 0$ and $u$ tends to infinity through positive values, the corresponding point $x$ approaches the point $z = -1$ from the left. Thus

$$\arg(x + 1) = \pi \quad \arg(x - 1) = \pi,$$

and $\ln|x + 1|$ tends to $\infty$. Also, since $-1 < x_2 < 1$, the real part of the quantity inside the braces in equation (5.33) tends to $-\infty$. Since $v = 0$, it readily follows that $A_2 = 0$; for, otherwise, the imaginary part on the right would become infinite. By equating imaginary parts on the two sides, we now see that

$$\frac{A_1}{2} [(1 + x_2)\pi + (1 - x_2)\pi] + B_2;$$

Hence

$$-\pi A_1 = B_2, \quad A_2 = 0. \quad (5.34)$$

The limiting position of the line segment joining the points $w_1$ and $w_2$ is the half line $v = \frac{\pi}{2} \ (u \geq 0)$. Points on that half line are images of the points $z = x$, where $-1 < x \leq x_2$; consequently,

$$\arg(x + 1) = 0, \quad \arg(x - 1) = \pi,$$

Identifying the imaginary parts on the two sides of equation (5.33), we thus arrive at the relation

$$\frac{\pi}{2} = \frac{A_1}{2} (1 - x_2)\pi + B_2 \quad (5.35)$$

Finally, the limiting positions of the points on the line segment joining $w_3$ to $w_4$ are the points $u + \pi i$, which are the images of the points $x$ when $x > 1$. By identifying, for those points, the imaginary parts in equation (5.33), we find that

$$\pi = B_2,$$

Then, in view of equations (5.34) and (5.35),

$$A_1 = -1, \quad x_2 = 0,$$
Thus $x = 0$ is the point whose image is the vertex $w = \frac{\pi i}{2}$; and, upon substituting these values into equation (5.33) and identifying real parts, we see that $B_1 = 0$. Transformation (5.32) now becomes

$$w = -\frac{1}{2} \left[ \log(z + 1) + \log(z - 1) \right] + \pi i,$$  \hspace{1cm} (5.36)

or

$$z^2 = 1 + e^{-2w}.$$  \hspace{1cm} (5.37)

Under this transformation, the required harmonic function $V(u, v)$ becomes a harmonic function of $x$ and $y$ in the half plane $y > 0$; and the boundary conditions indicated in Figure (5.6) are satisfied. Note that $x_2 = 0$ now. The harmonic function in that half plane which assumes those values on the boundary is the imaginary component of the analytic function

$$\frac{1}{\pi} = \log \frac{z-1}{z+1} = \frac{1}{\pi} \ln \frac{r_1}{r_2} + \frac{i}{\pi} (\theta_1 - \theta_2),$$

where $\theta_1$ and $\theta_2$ range from 0 to $\pi$. Writing the tangents of these angles as functions of $x$ and $y$ and simplifying, we find that

$$\tan \pi V = \tan(\theta_1 - \theta_2) = \frac{2y}{x^2 + y^2 - 1}.$$ \hspace{1cm} (5.38)

Equation (5.37) furnishes expressions for $x^2 + y^2$ and $x^2 - y^2$ in terms of $u$ and $v$. Then, from Equation (5.38), we find that the relation between the potential $V$ and the coordinates $u$ and $v$ can be written

$$\tan \pi V = \frac{1}{s} \sqrt{e^{-4u} - s^2}.$$ \hspace{1cm} (5.39)

where
\[ s = -1 + \sqrt{1 + 2e^{-2u} \cos(2\nu) + e^{-4u}}, \]

### 5.3 Application Of Joukowski Mapping: Airfoil

An airfoil is the distinguished shape of a plane wing, the blade of a jet engine or rotor and many other blades, Figure (5.10). An object that has shape with airfoil design is moving into a particular fluid produces a kind of force called aerodynamic force. The parts of such a force are should be perpendicular to where the fluid oriented and has been named the lift force. The parts that are parallel to the orientation of where the flow is going is called drag force. The airfoil of wings of the planes that fly at less or equal to sound speed have a characteristic distinguished shape with a nice rounded leading front strong edge, and a very sharp trailing back well designed edge, mostly with an asymmetric significant calculated curvature of the upper cover and the lower boundary ready for high pressure surface. Foils forms of the same job that is created with tasks related to water fluid are named generally hydrofoils. The lift force on an airfoil is mainly and clearly the sequence of the angle of the strong fluid (air) attack it has and shapes it is made as. When turned at a proper angle, the position of the airfoil will change and diverts the incoming air to the upper surface, and this state is going to produce a kind of force on the upper boundary for the airfoil in the direction of the flow of the fluid that is on the other side to the noted deflection. This type of force is called aerodynamic force and it can be made normally of two parts: lift force and drag force. Most of the shapes that have foil forms require making an angle with the real line of fluid which attacks larger than zero to produce the mentioned lift force, but airfoils that are cambered can make lift force at zero angles of fluid attack.

![Figure 5.7 Airfoil Cross-section](image-url)
The most of the fixed-wings are created in a shape with the airfoil based cross cut. Airfoils are widely used and it is very important in propellers in jet engines and compressors. Any buddy now has his own angle of fluid attack in a particular fluid field, like for example a flat plate, a big high building for facing the flow, a bridge deck is going to produce an aerodynamic lift force making with the flow 90 degree angle. Airfoils are the most effective lifting molds, it can make more lift force with less drag force. A lift force and drag force that made the flowing wind is achieved in wind tunnel testing is shown Figure (5.7). The curve is an airfoil that has a positive camber, so some lifting force is created at zero fluid attack angles. By increasing the fluid attack angle, lift rises in an approximately linear function named List curve slope. At near 18 degrees angle the airfoil design stalls and the lift force falls off suddenly away from that. The decline in lift force can be illustrated by the effect of the upper cover of the wing. The distance that will be between the upper and lower covers of the wing turns the airfoil’s powerful shape to be a lift force source, it decreases its efficient camber that adjusts all the flow field of the fluid so as to decrease the circulation of the fluid around the wing cut section and the lift point. The more distance between the covers means a thicker layer and also causes a large rise in pressure in the drag force so that all the amount of the drag force that rises sharply approaching and preceding the stall back point.

Airfoil form is a very significant aspect of aerodynamics. Different airfoils designs serve various flight projects in practical world. Asymmetric airfoils designs what do not have an identical upper and lower covers can create lift at zero angles of fluid attack, But a symmetric airfoil is what do have a identical upper and lower covers may better readjust and put the wing in a specific angle to give the wing some lift force. In the area of the small wings -ailerons- and approaching a wingtip a symmetric design shape of airfoil can be used to extend the of what the angle of fluid attack can produce to bypass rotation–stall. Therefore a large amount of measures for the expected angles can be used without limitations of the layers. Normal use of airfoils designs have a rounded smooth leading front edge, which is generally inconsiderate to the fluid attack slope from real axis. The cross view of the wing is not absolutely circular, although: radius of the curved front end is extended before the wing reaches the greatest thickness it could to slightly reduce the opportunity of any frame layer that distinguish. This makes the wing stronger in notable state and shifts or moves the point of the highest distance between the upper and lower covers back from the front edge of the wing. The airfoils of planes that fly with the normal speed at or lower than the sound speed are extra more angular in shape and can have a very
sharpened leading front strong edge and it is very sensitive to the angle of fluid attack of the wing. A super-critical airfoil design have its highest thickness property near to the leading front edge of the wing to have plenty of lengths to gently shock the very high-speed flow, the flow that flows in more than the sound speed back to normal speeds or speeds under the sound barrier. The advanced shape that modern aircraft wings have and it can own various airfoil sections along the whole the span of the wing for each one optimized for the needed requirements of each solid part of the wing. Adjustable high-lift devices, can easily flaps support, that is suited and modified to airfoils on nearly every modern and new aircraft design. A trailing very back edge cover in the wing acts likewise the front to the aileron; although, it is presented to a specific special design to an aileron can be retreated somewhat into the wing if not in use. See figure (5.8)
5.3 Application Of Joukowski Mapping: Airfoil

5.3.1 Airfoil Terminology

Consider a general airfoil:

The various terms that we will define in the section below are badly related to the airfoils design in Figure (5.10):

- The upper part of the surface or the upper cover is usually faced with very high velocity so that a notable reduction in the static pressure will be there.

- The lower part surface or the lower cover has a relatively and desired the huge increase in static pressure more than the upper part of the wing. The pressure difference between these two very important surfaces surfaces provides the lift force created for a presented airfoil.

The geometrical aspects of the airfoil design shape are illustrated with many kinds of terms:
• The leading front edge of the wing is the location where the beginning of the airfoil is, that has the greatest curving (smallest radius) and it is the most important part of the wing from a dynamic perspective.

• The trailing back (rear) edge of the wing is determined likewise where the location of smallest curving be at the very back of the airfoil design in the wing.

• The chord is a very important line placed through the line joining the leading front and trailing rear edges, some times we call it the chord distance or just chord in other times, \( c \), is the value of the distance of the chord line.

The design of the shape of the airfoil is very well defined and related to the next highly important parameters:

• The mean camber line or the mean line: it is the locus of the intermediate points between the upper cover and lower cover and Its important shape depends on the how the wing look like along the chord;

• The distance between the two covers of the airfoil changes slightly along the chord. It can be held in on of these two methods:

  • Thickness vertical to the found camber line.
  • Thickness vertical to the found chord line.

Some very significant parameters used to represent the formation of an airfoil’s design shape are its camber and crucial thickness. Assuredly, relevant thoughts used to explain the airfoil’s important role when it is running through a fluid field:

• The center aerodynamic is where the chordwise length is highly concerning and the wanted pitching time that is free of any the lift force coefficient and the attack fluid angle.

• The pressure center is the very important in chord-wise where it is placed about where the pitching moment is zero.

5.3.1.1 Thin Airfoil Theory

Small airfoil idea is a very important and simple that relates wing angle to the attack fluid to give it a calculated lift force for not compressed air. This idea considers the amount of flow around the airfoil of the wing as a cut 2D view of the whole wing
that flow throughout a thin airfoil design. It can be created as marking an airfoil of zero thickness and infinite wingspan.

Thin airfoil approach was particularly important in its time because it contributed a sound theoretical basis for the next significant characteristics of airfoils in two-dimensional flow:

1. On an airfoil that has an identical covers the center of pressure on the cover and the center of the aerodynamic lies precisely one the quarter of the chord after the leading front significant edge.

2. On an airfoil that is firmly cambered the center point of the aerodynamic field situated precisely one-quarter of the chord after the significant leading front edge.

3. The amount of the lift scale coefficient varies according to the angle of fluid attack line that is $2\pi$ units per radian.

As a consequence of the item in (3) the amount of lift scale coefficient of a section for a symmetric wing airfoil of infinite wingspan is:

$$c_l = 2\pi \alpha,$$

where $c_l$ is the scale lift coefficient for a specified section on the wing

$\alpha$ is the angle of where the fluid attacks the front end of the wing in radians measured with very relative to the chord line.

(The above expression is also applicable to a cambered airfoil where $\alpha$ is the angle of attack measured relative to the zero-lift line instead of the chord line.)

Also as a consequence of the term in (3), the scale lift coefficient of a specified section is a clear cambered airfoil of infinite wingspan is:

$$c_l = c_{l_0} + 2\pi \alpha,$$

where $c_{l_0}$ is the lift scale coefficient in a particular section and the angle of fluid attacks the wing will be definitely equal to zero.

Thin airfoil idea or concept does not account for the stall of the wings airfoil in the projects that usually occurs at an angle of fluid attack between $10^\circ$ and $15^\circ$ for typical and normal airfoils.
This is what airfoil means in usual, next we’re investigating the role of complex differential equations in such applications.

5.3.1.2 The Flow about Joukowski Airfoil

The following graphs clarify how an outer vortex able to rise the amount of the lift force on the airfoil. Figure (5.11) shows the airfoil flying at speed $U$ inside an no compressed density $\rho$ air.

![Figure 5.11 Airfoil and Its Flow without otter Vortex](image)

To satisfy the Kutta-condition at the sharp trailing edge, a circulation $\Gamma$ should be created around the airfoil, what gives increasing to lift of $\rho U/\Gamma$ per unit span of the wing. When the vortex of the circulation $\kappa$ is put above the airfoil, it makes a reversed fluid motion on the upper surface, so that a stagnation point will appear there if the same circulation was maintained around the airfoil. The flow pattern is graphed in Figure (5.9).

![Figure 5.12 Airfoil with Outer Vortex and The Same Circulation](image)

Though, to fade the velocity that is definitely finite placed at the ending of the cross-section of the wing that named as the sharp edge, another circulation $\Delta \Gamma$ must be made to move the point which called the point of stagnation back to the wing trailing
Figure 5.13 Flow around the airfoil and Kutta condition with outer external vortex at (the sharp trailing edge).

e, so Kutta-condition is satisfied again. A vortex that has a circulation $-\Delta \Gamma$ is placed in the back of the airfoil at the trailing edge as in Figure (5.13).

For that, the wing has then possessed a lift of $\rho(\Delta \Gamma + \kappa)$ unit per span. The dependence of the risen lift on the strength and location of the vortex is one of the critical concerns of this study, when the stable status for a given airfoil specifications has been gained, we determine the stationary equilibrium places of the external vortex and calculate the related lift. Furthermore, the stability of the vortex at every equilibrium positions it has is examined. Our investigation is meant to be for a steady two-dimensional flow past an airfoil, under this assumption that the fluid is not viscid and is not compressed. The vortex-induced drag cannot be calculated using this model.

### 5.3.2 The Flow Around Joukowski Airfoil

In regard to Figure (5.14), we consider the transformations

$$z = 2(\zeta + \frac{1}{\zeta}),$$  \hspace{1cm} (5.40)

$$\zeta = \zeta' + \delta^i \mu,$$ \hspace{1cm} (5.41)

where $z = x + iy$ and $\zeta = \xi + i\eta$ and $\zeta' = \xi' + i\eta'$ are complex variables. These transformations map the radius a circle that has zero as a center of the $\zeta$ plane into a Joukowski airfoil in physical $z$ plane, whose chord is generally greater than 2 but is equal to 2 when the airfoil has the flat plate form. As a particular case, the point $\zeta = 1$ is mapped into the sharp trailing edge of the airfoil. The shape design of the airfoil is maintained by changing the values of parameters $\delta$ and $\mu$. For instance, the
Figure 5.14 Transformation of the flow about a circle in the \( \zeta \) plane into that about an airfoil in the \( z \) plane.

generally \textit{cambered} airfoil will be a symmetric airfoil as \( \mu = \pi \), it will be a circular arc as \( \mu = \frac{\pi}{2} \), and it will be a flat plate if \( \delta = 0 \). The radius \( a \) of the circle and the angle \( \theta \) shown in Figure (5.14) can be expressed in terms of these two parameters as:

\[
a = \sqrt{1 - 2\delta \cos(\mu) + \delta^2}, \tag{5.42}
\]

\[
\beta = \tan^{-1}\left(\frac{\delta \sin \mu}{1 - \delta \cos \mu}\right). \tag{5.43}
\]

Under the same functions (trans.) between \( \zeta \) and \( z \) domains, a uniform flow that has speed \( U = \frac{1}{2} \) creating an angle \( \alpha \) with the real line maps uniform flow of speed \( U = 1 \) without changing its direction, and the vortex which has circulation known as \( \kappa \) at \( \zeta_0 \) that maps into the same strength vortex that located at \( z_0 \), where

\[
\zeta_0 = \zeta'_0 + \delta e^{i\mu}, \tag{5.44}
\]

\[
\zeta'_0 = \rho_0 e^{i\theta_0}. \tag{5.45}
\]

Let \( w = \phi + \psi \) be a complex function that represents the potential flow function, in a way \( \phi \) will be the potential function of the velocity and \( \psi \) expresses the function of the stream. The potential function is first will be expressed in \( \zeta' \) domain, That has the supports from the origin flow, the vortex in \( \zeta \) domain, the image of it in the circle, and the circulation around the circle center.
\[ w = \frac{1}{2} e^{-i\alpha} \left( \xi' + \frac{a^2}{\xi'} e^{2i\alpha} \right) + i \frac{\kappa}{2\pi} \left[ \log(\xi' - \xi'_0) - \log(\xi' - \frac{a^2}{\xi'_0}) \right] + i (\Gamma + \kappa) \log(\xi'), \tag{5.46} \]

Then, the velocity function (complex) of the flow that is around the airfoil where it has been derived from Equation (5.46)

\[
\frac{dw}{dz} = \frac{dw}{d\zeta'} \frac{d\zeta'}{\zeta} = \frac{dw}{d\zeta} \tag{5.47}
\]

where

\[
\frac{dw}{d\zeta'} = \frac{1}{2} e^{-i\alpha} - \frac{1}{2} \left( \frac{a}{\zeta'} \right)^2 e^{i\alpha} + \frac{i\kappa}{2\pi} \left[ \frac{1}{\zeta' - \xi'_0} - \frac{1}{\zeta' - \frac{a^2}{\xi'_0}} \right] + \frac{i}{2\pi} (\Gamma + \kappa) \tag{5.48}
\]

\[
\frac{dz}{d\zeta} = \frac{\zeta^2 - 1}{2\zeta^2} \tag{5.49}
\]

Our mathematical issue now begins as follows. For a given airfoil (with specified \( \delta \) and \( \mu \)) at a given angle of attack \( \alpha \), we will determine the position \( z_0 = (x_0 + iy_0) \) and the strength \( \kappa \) of the free vortex and the circulation \( \Gamma \) around the airfoil, under the constraint that the vortex becomes stationary and Kutta condition is fulfilled at the trailing edge of the airfoil. Note that the \( \mu \) means here is equivalent to the sum \( \Gamma + \Delta \Gamma \) in Figure (5.13) in the steady state when the wake vortex is carried to infinity.

### 5.3.2.1 Conditions For The Vortex To Be Stationary

The complex velocity of the free vortex is

\[
(u - iv)_{z_0} = \left( \frac{dw}{dz} - \frac{i\kappa}{2\pi} \frac{1}{z - z_0} \right) \frac{\zeta^2}{\zeta'_0 - 1} \left[ e^{-i\alpha} - \left( \frac{a}{\zeta'_0} \right)^2 e^{i\alpha} - \frac{i\kappa}{\pi} \frac{1}{\zeta'_0 - \frac{a^2}{\xi'_0}} + \frac{i}{2\pi} (\Gamma + \kappa) \right] \tag{5.50}
\]

Taylor’s series expansion about the point \( \zeta'_0 \) results in
\[
\frac{d\zeta'}{dz} = \left( \frac{d\zeta''}{dz} \right)_0 - \left( \frac{d^2z}{d\zeta'^2} \right)_0 \left( \frac{d\zeta'}{dz} \right)_0 (\zeta' - \zeta'_0) + \mathcal{O}(\zeta' - \zeta'_0)^2 \tag{5.51}
\]

\[
\frac{1}{z - z_0} = \left( \frac{d\zeta'}{dz} \right)_0 \frac{1}{\zeta' - \zeta'_0} - \frac{1}{2} \left( \frac{d^2z}{d\zeta'^2} \right)_0 \left( \frac{d\zeta'}{dz} \right)_0 + \mathcal{O}(\zeta' - \zeta'_0) \tag{5.52}
\]

which give the limiting value of \(-\frac{2\zeta_0}{(\zeta'_0 - 1)^2}\) for the expression contained within the second square brackets in Equation (5.50). To require that the vortex be stationary we get, by making the right-hand side of Equation (5.50) equal to zero,

\[
\zeta_0 e^{-i\alpha} - \zeta_0 \left( \frac{a}{\zeta_0} \right)^2 e^{i\alpha} + \frac{i\Gamma}{\pi} \frac{\zeta_0}{\zeta'_0} + \frac{i\kappa}{\pi} \left[ \frac{\zeta_0}{\zeta'_0} \left( 1 - \frac{1}{1 - \frac{a^2}{|\zeta'_0|^2}} \right) - \frac{1}{S_0 - 1} \right] = 0 \tag{5.53}
\]

After substituting \(\zeta_0\) and \(\zeta'_0\) from Equations (5.44) and (5.45) and separating it into real and imaginary parts, Equations (5.53) becomes

\[
R_1 + \frac{\Gamma}{\pi} R_2 + \frac{\kappa}{\pi} R_3 = 0 \tag{5.54}
\]

\[
I_1 + \frac{\Gamma}{\pi} I_2 + \frac{\kappa}{\pi} I_3 = 0 \tag{5.55}
\]

in which

\[
R_1 = \left( \rho_0 - \frac{a^2}{\rho_0} \right) \cos(\theta_0 - \alpha) + \delta \cos(\mu - \alpha) - \frac{\delta a^2}{\rho_0^2} \cos(\alpha + \mu - 2\theta_0) \tag{5.56}
\]

\[
R_2 = -\frac{\delta}{\rho_0} \sin(\mu - \theta_0) \tag{5.57}
\]

\[
R_3 = \frac{\delta a^2}{\rho_0(\rho_0^2 - a^2)} \sin(\mu - \theta_0) - \frac{1}{2} \left( \frac{1}{S_1} - \frac{1}{S_2} \right) \tag{5.58}
\]

\[
I_1 = \left( \rho_0 + \frac{a^2}{\rho_0} \right) \sin(\theta_0 - \alpha) + \delta \sin(\mu - \alpha) - \frac{\delta a^2}{\rho_0^2} \sin(\alpha + \mu - 2\theta_0) \tag{5.59}
\]

\[
I_2 = 1 + \frac{\delta}{\rho_0} \cos(\mu - \theta_0) \tag{5.60}
\]
\[ I_3 = -\frac{a^2}{\rho_0^2 - a^2} \left[ 1 + \frac{\delta}{\rho_0} \cos(\mu - \theta_0) \right] - \frac{1}{2} \left( \frac{1}{S_1} - \frac{1}{S_2} \right) \times (\rho_0 \cos \theta_0 + \delta \sin \mu) + \frac{1}{2} \left( \frac{1}{S_1} - \frac{1}{S_2} \right) \]  

(5.61)

where

\[ S_1 = 1 + \rho_0^2 + \delta^2 + 2\rho_0 \delta \cos(\mu - \theta_0) - 2\rho_0 \cos \theta_0 - 2\delta \cos \mu \]  

(5.62)

\[ S_1 = 1 + \rho_0^2 + \delta^2 + 2\rho_0 \delta \cos(\mu - \theta_0) + 2\rho_0 \cos \theta_0 + 2\delta \cos \mu \]  

(5.63)

### 5.3.3 Kutta Condition at the Trailing back Edge

The Joukowski airfoil design has a cusp at the trailing edge where the upper cover and lower cover of the airfoil are tangent. Kutta condition for this type of edge is that the velocity there is finite. It demands in Equation (5.47) that \( \frac{dw}{d\zeta'} = 0 \) at \( \zeta = 1 \), that is, at \( \zeta' = ae^{i\beta} \) as indicated in Figure (5.14). Starting from Equation (5.48) and after some manipulation, this requirement becomes

\[ \sin(\alpha + \beta) = \frac{\Gamma + \kappa}{2\pi a} - \frac{\kappa}{\pi} F \]  

(5.64)

where

\[ F = -\frac{1}{2} e^{-i\beta} \left( \frac{1}{\zeta' - \zeta_0} - \frac{1}{\zeta' - \frac{a^2}{\zeta_0}} \right) \]  

(5.65)

\[ = -\frac{\rho_0^2 - a^2}{2a[\rho_0^2 + a^2 - 2a\rho_0 \cos(\theta_0 + \beta)]} \]

Let us go a little bit further to calculate the velocity at the trailing edge where the right-hand side of Equation (5.47) has the indeterminate form. The complex velocity at the trailing edge the will be evaluated accordingly.
\[(u - iv)_{\xi=1} = \left( \frac{d^2 w}{d\xi'^2} \right) \left( \frac{d^2 z}{d\xi'^2} \right) \zeta' = ae^{-i\beta} = \begin{bmatrix} \frac{a^2}{\zeta'^3} e^{i\alpha} - i \frac{\Gamma + \kappa}{2\pi \zeta'^2} - i \frac{\kappa}{2\pi} \left( \frac{1}{\zeta' - \zeta'_0} - \frac{1}{\zeta' - \frac{a^2}{\zeta'_0}} \right) \times \left( \frac{1}{\zeta' - \zeta'_0} - \frac{1}{\zeta' - \frac{a^2}{\zeta'_0}} \right) \end{bmatrix} \zeta' = ae^{-i\beta} \]

The expression is gotten by differentiating Equations (5.48) and (5.49) and by using the fact that \( \frac{d^2 z}{d\xi'^2} = 1 \) at \( \zeta' = 1 \). After simplifying by using of Equations (5.64) and (5.65), the result shows that the velocity at the trailing edge creates an angle \(-2\beta\) with the \( x \) axis and has a magnitude

\[ V_t = \frac{\cos(\alpha + \beta)}{a} - \frac{2\rho_0 \sin(\theta_0 + m\beta)}{\rho_0^2 + a^2 - 2a\rho_0 \cos(\theta_0 + \beta)} \times \left[ \frac{\Gamma + \kappa}{2\pi a} - \sin(\alpha + \beta) \right] \]  

(5.66)

A negative value of \( V_t \) shows the direction of the flow at the trailing edge is opposite where it flows from the trailing to the leading edge, which is in contradiction to the situation in the present flow issue. For that, the solution corresponding to \( V_t < 0 \) is thrown away even if it satisfies the Kutta condition represented by Equation (5.64).

### 5.3.3.1 Method of Solution

We now possess three algebraic equations, Equations (5.54), (5.55), and (5.64), for three of the four unknowns \( \kappa, \Gamma, \rho_0 \) and \( \theta_0 \). The system of these equations can be rewrite in a more suitable form. Equations (5.54) and (5.55) are first merged to obtain

\[ \frac{\Gamma}{\pi} = \frac{R_1 I_3 - R_3 I_1}{R_2 I_3 - R_3 I_2} \]  

(5.67)

\[ \frac{\kappa}{\pi} = \frac{R_1}{R_3} - \frac{\Gamma R_2}{\pi R_3} \]  

(5.68)

Upon substitution into Equation (5.64) a single equation results

\[ \sin(\alpha + \beta) + \frac{I}{2a} \frac{R_1 I_3 - R_3 I_1}{R_2 I_3 - R_3 I_2} + \left( \frac{1}{2a} - F \right) \times \left( \frac{R_1}{R_3} - \frac{R_2 R_1 I_3 - R_3 I_1}{R_3 R_2 I_3 - R_3 I_2} \right) = 0, \]  

(5.69)
which has only two variables $\rho_0$ and $\theta_0$. The issue is solved according to the coming procedure. For the assumed $\alpha$, $\delta$ and $\mu$, the values of $a$ and $\beta$ is computed from Equation (5.42) and (5.43), and the airfoil shape is drawn by mapping the circle that has the radius $a$ at the origin of the $\zeta$ plane through the transformations Equation. (5.40) and (5.41). We look for all possible places at where a free vortex be stationary by seeking along the radial direction for every of the many values of $\theta_0$ chosen between 0 and $2\pi$. For a given value of $\theta_0$, if no value of $\rho_0$ can be found that will satisfy Equation (5.69), it means that no vortex can be trapped in that particular direction. Otherwise, the coordinates $(\rho_0, \theta_0)$ that found are used to find the $\Gamma$ and $\kappa$ from Equation. (5.67) and (5.68) and to determine the position $(x_0, y_0)$ of the vortex in the physical plane-$z$ using the Equation (5.40) and (5.41). Loci of the free vortex for both constants $\alpha$ and $\kappa$ are sketched in the vicinity of the airfoil after throwing away the solution in which $V_t < 0$. To test how the distribution of the desired pressure on the airfoil upper and lower covers is get changed by the external vortex, First, we find out the velocity $V$ the airfoil surface from the transformation of the velocity on the circle in the $\zeta$ plane using Equation (5.47). The pressure coefficient, which is defined as the pressure difference between the local and a far-away station divided by the dynamic pressure at infinity, is then computed from the equation

$$C_p = 1 - \left(\frac{V}{U}\right)^2,$$

(5.70)

The stream mapping is the imaginary part of the complex potential shown in Equation (5.46). Applying the Blasius theorem to the present problem having a vortex out of the airfoil, we get a lift of $\rho U (\Gamma + \kappa)$ per unit span of the wing. We define $L$ as this lift divided by $\rho U$, so that

$$L = \Gamma + \kappa$$

(5.71)
Bibliography

[1] Saeid Abbasbandy. Iterated he’s homotopy perturbation method for quadratic riccati differential equation. *Applied Mathematics and Computation*, 175(1):581–589, 2006.

[2] Naum Il’ich Akhiezer and Izrail Markovich Glazman. *Theory of linear operators in Hilbert space*. Courier Corporation, 2013.

[3] Weizhu Bao and Qinglin Tang. Numerical study of quantized vortex interaction in the ginzburg-landau equation on bounded domains. *Communications in Computational Physics*, 14(3):819–850, 2013.

[4] Belal Batiha. A new efficient method for solving quadratic riccati differential equation. *International Journal of Applied Mathematics Research*, 4(1):24, 2015.

[5] Heinz H Bauschke and Patrick L Combettes. Hilbert spaces. In *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, pages 27–47. Springer, 2017.

[6] John Baylis. Advanced calculus of a single variable. *The Mathematical Gazette*, 101(551):367–368, 2017.

[7] Carl M Bender and Steven A Orszag. *Advanced mathematical methods for scientists and engineers I: Asymptotic methods and perturbation theory*. Springer Science & Business Media, 2013.

[8] Sterling K Berberian. *Introduction to Hilbert space*, volume 287. American Mathematical Soc., 1999.

[9] Richard Bertram, Joel Tabak, Wondimu Teka, Theodore Vo, Martin Wechselberger, Vivien Kirk, and James Sneyd. *Mathematical Analysis of Complex Cellular Activity*. Springer, 2015.
[10] Changbum Chun and Rathinasamy Sakthivel. Homotopy perturbation technique for solving two-point boundary value problems—comparison with other methods. *Computer Physics Communications*, 181(6):1021–1024, 2010.

[11] Manuel Colera and Miguel Pérez-Saborid. An efficient finite differences method for the computation of compressible, subsonic, unsteady flows past airfoils and panels. *Journal of Computational Physics*, 2017.

[12] Lokenath Debnath and Piotr Mikusiński. *Hilbert spaces with applications*. Academic press, 2005.

[13] Guido Fano and SM Blinder. Mathematical methods in quantum mechanics. In *Twenty-First Century Quantum Mechanics: Hilbert Space to Quantum Computers*, pages 43–84. Springer, 2017.

[14] Guido Fano and SM Blinder. *Twenty-First Century Quantum Mechanics: Hilbert Space to Quantum Computers: Mathematical Methods and Conceptual Foundations*. Springer, 2017.

[15] Galina Filipuk, Yoshishige Haraoka, and Sławomir Michalik. *Analytic, Algebraic and Geometric Aspects of Differential Equations*. Springer, 2017.

[16] Valerii I Gromak, Ilpo Laine, and Shun Shimomura. *Painleve differential equations in the complex plane*, volume 28. Walter de Gruyter, 2002.

[17] Baokui Guo, Silong Peng, Xiyuan Hu, and Pengcheng Xu. Complex-valued differential operator-based method for multi-component signal separation. *Signal Processing*, 132:66–76, 2017.

[18] Kansari Haldar. *Decomposition analysis method in linear and nonlinear differential equations*. CRC Press, 2015.

[19] Ji-Huan He. Homotopy perturbation technique. *Computer methods in applied mechanics and engineering*, 178(3):257–262, 1999.

[20] Zhao Huan, Gao Yuan, Wang Chao, et al. Effective robust design of high lift nlf airfoil under multi-parameter uncertainty. *Aerospace Science and Technology*, 2017.

[21] Zhi-Gang Huang and Jun Wang. The radial oscillation of entire solutions of complex differential equations. *Journal of Mathematical Analysis and Applications*, 431(2):988–999, 2015.
[22] Zhoushen Huang, W Zhu, Daniel P Arovas, Jian-Xin Zhu, and Alexander V Balatsky. Invariance of topological indices under hilbert space truncation. *arXiv preprint arXiv:1705.05949*, 2017.

[23] Sergey Isaev, Paul Baranov, Igor Popov, Alexander Sudakov, and Alexander Usachov. Improvement of aerodynamic characteristics of a thick airfoil with a vortex cell in sub-and transonic flow. *Acta Astronautica*, 132:204–220, 2017.

[24] Victor Isakov. *Inverse problems for partial differential equations*, volume 127. Springer, 2006.

[25] M Jalaal, DD Ganji, and F Mohammadi. He's homotopy perturbation method for two-dimensional heat conduction equation: Comparison with finite element method. *Heat Transfer Asian Research*, 39(4):232–245, 2010.

[26] Ali Joohy, Mohammed S Mechee, and Ghassan A Al-Juaifri. A new method of solving third order non-linear ordinary complex differential equation by generalizing prelle-singer method. *arXiv preprint arXiv:1805.01785*, 2018.

[27] Soon-Mo Jung and Jaiok Roh. The linear differential equations with complex constant coefficients and schrodinger equations. *Applied Mathematics Letters*, 66:23–29, 2017.

[28] Zhuang Kang, Wenchi Ni, and Liping Sun. A numerical investigation on capturing the maximum transverse amplitude in vortex induced vibration for low mass ratio. *Marine Structures*, 52:94–107, 2017.

[29] Yue Kuen Kwok. Applied complex variables. for scientists and engineers, 2002.

[30] Yue Kuen Kwok. *Applied complex variables for scientists and engineers*. Cambridge University Press, 2010.

[31] Serge Lang. *Complex analysis*, volume 103. Springer Science and Business Media, 2013.

[32] Alex Laratro, Maziar Arjomandi, Benjamin Cazzolato, and Richard Kelso. Self-noise and directivity of simple airfoils during stall: An experimental comparison. *Applied Acoustics*, 127:133–146, 2017.

[33] Jinyi Li, Yang Cao, Guoqing Wu, Zifan Miao, and Jiawei Qi. Aerodynamic stability of airfoils in lift-type vertical axis wind turbine in steady solver. *Renewable Energy*, 111:676–687, 2017.
[34] Xiaoyan Ma, Liping Wei, and Zhongjin Guo. He’s homotopy perturbation method to periodic solutions of nonlinear jerk equations. *Journal of Sound and Vibration*, 314(1):217–227, 2008.

[35] Mohammed Mechee, F Ismail, Zahir Moosajee Hussain, and Zailan Siri. Direct numerical methods for solving a class of third-order partial differential equations. *Applied Mathematics and Computation*, 247:663–674, 2014.

[36] Mohammed S Mechee, Ghassan A Al-Juaifri, and Ali K Joohy. Modified homotopy perturbation method for solving generalized linear complex differential equations. *Applied Mathematical Sciences*, 11(51):2527–2540, 2017.

[37] A Neamaty and R Darzi. Comparison between the variational iteration method and the homotopy perturbation method for the sturm-liouville differential equation. *Boundary Value Problems*, 2010(1):317369, 2010.

[38] MM Oueslati, AW Dahmouni, and S Ben Nasrallah. Effects of sudden change in pitch angle on oscillating wind turbine airfoil performances. *Engineering Analysis with Boundary Elements*, 81:21–34, 2017.

[39] Belkacem Said-Houari. Differential equations: Methods and applications, 2016.

[40] Anton Savin and Boris Sternin. *Introduction to Complex Theory of Differential Equations*. Birkh"auser, 2017.

[41] Mehmet Sezer and Mustafa G"ulsu. Approximate solution of complex differential equations for a rectangular domain with taylor collocation method. *Applied mathematics and computation*, 177(2):844–851, 2006.

[42] Mehmet Sezer, Mustafa G"ulsu, and Bekir Tanay. A taylor collocation method for the numerical solution of complex differential equations with mixed conditions in elliptic domains. *Applied mathematics and computation*, 182(1):498–508, 2006.

[43] Y Sibuya. Ordinary differential equations in complex domain, 1976.

[44] Yasutaka Sibuya. *Linear differential equations in the complex domain: problems of analytic continuation*, volume 82. American Mathematical Soc., 2008.

[45] Abhishek Subramanian, S Arun Yogesh, Hrishikesh Sivanandan, Abhijit Giri, V Madhavan, M Vivek, and V Ratna Kishore. Effect of airfoil and solidity on
performance of small scale vertical axis wind turbine using three dimensional cfd model. *Energy*, 2017.

[46] Charles Swartz. *An introduction to functional analysis*, volume 157. CRC Press, 1992.

[47] Harkrishan Lal Vasudeva. Elements of hilbert spaces and operator theory, 2017.

[48] Ahmet Yıldırım. Application of the homotopy perturbation method for the fokker–planck equation. *International Journal for Numerical Methods in Biomedical Engineering*, 26(9):1144–1154, 2010.

[49] Ahmet Yıldırım. He's homotopy perturbation method for nonlinear differential-difference equations. *International Journal of Computer Mathematics*, 87(5):992–996, 2010.

[50] cSuayip Yüzbaşcı and Mehmet Sezer. A collocation method to find solutions of linear complex differential equations in circular domains. *Applied Mathematics and Computation*, 219(22):10610–10626, 2013.

[51] Li Zhang and Fangqi Chen. Bifurcations and stability analysis for nonlinear oscillations of an airfoil. *Chaos, Solitons and Fractals*, 103:220–231, 2017.

[52] Wanglong Zhang, Zhiyong Zhang, Zhihua Chen, and Qizhong Tang. Main characteristics of suction control of flow separation of an airfoil at low reynolds numbers. *European Journal of Mechanics-B/Fluids*, 65:88–97, 2017.

[53] Dennis G Zill and Patrick Shanahan. *A first course in complex analysis with applications*. Jones and Bartlett Learning, 2009.
