Optimal Capital Injections with the Risk of Ruin: A Stochastic Differential Game of Impulse Control and Stopping Approach

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Abstract

We consider an investment problem in which an investor performs capital injections to increase the liquidity of a firm for it to maximise profit from market operations. Each time the investor performs an injection, the investor incurs a fixed transaction cost. In addition to maximising their terminal reward, the investor seeks to minimise risk of loss of their investment (from a possible firm ruin) by exiting the market at some point in time. We show that the problem can be reformulated in terms of a new stochastic differential game of control and stopping in which one of the players modifies a (jump-)diffusion process using impulse controls and an adversary chooses a stopping time to end the game. We show that the value of this game can be computed by solving a double obstacle problem described by a quasi-variational inequality. We then characterise the value of the game via a set of HJBI equations, considering both games with zero-sum and non-zero-sum payoff structures. Our last result demonstrates that the solution to the investment problem is recoverable from the Nash equilibrium strategies of the game.

Keywords: Impulse control, stochastic differential games, optimal stopping, jump diffusion, Hamilton-Jacobi-Bellman equation, optimal liquidity control, lifetime ruin, transaction costs.

1 Introduction

There are numerous environments in which financial agents incur fixed or minimal costs when adjusting their investment positions; trading environments with transaction costs, real options pricing and real estate and large-scale infrastructure investing are a few important examples. The study of optimal investments by an economic agent who seeks to minimise the probability that they go bankrupt within their lifetime is known as the probability of lifetime ruin problem. The problem was introduced by [MR00] and studied in depth by [You04].

Despite the breadth of the literature concerning the lifetime ruin problem and the widespread occurrence and influence of transaction costs on investment behaviour, current models within the literature have yet to include those in which the investor faces financial transaction costs. The absence of transaction costs within the theoretical analysis limits the scope of application of the lifetime ruin model to a wide number of instances within financial systems. The objective of this paper is therefore to generalise the lifetime ruin problem to problems the investor now faces transaction costs when modifying their position.

In order to tackle this problem, we introduce a new stochastic differential game of control and stopping in which the controller uses impulse controls to modify the dynamics of

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One exception is [Zer10] in which a single-controller problem is analysed - however, in [Zer10] the controller’s action space is limited to two actions wherein the model in [Zer10] can thus be viewed as regime switching model with switching costs.
Capital Injections with the Risk of Ruin

We specifically concern ourselves with an optimal firm liquidity control problem with lifetime ruin in which an investor performs capital injections to increase the liquidity of a firm. In this instance, the investor seeks to maximise his capital injections to buoy the firm’s liquidity process whilst seeking to minimise the probability of loss of investment by exiting the market (selling all firm holdings).

Our analysis shows that by representing the investment problem as stochastic differential game of control and stopping with impulse controls enables the optimal solution of the problem to be computed. A significant component of this paper is therefore concerned with studying the stochastic game which leads to a full characterisation the value of the game using PDEs (HJBI equations). We then generalise the results to cover the game with a non-zero-sum payoff structure and generalise the results to provide a characterisation of the Nash equilibrium of the non-zero-sum stochastic differential game enabling the optimal solution for the investment problem to be computed using solutions to a joint set of PDEs.

Theoretical Background

Impulse control problems are stochastic control models in which the cost of control is bounded below by some fixed positive constant which prohibits continuous control, thus augmenting the problem to one of finding both an optimal sequence of times to apply the control policy, in addition to determining optimal control magnitudes. We refer the reader to [EBY11a] as a general reference to impulse control theory and to [VLVP07; PS10] for articles on applications. Additionally, matters relating to the application of impulse control models have been surveyed extensively in [Kor99]. Impulse control frameworks therefore underpin the description of financial environments with transaction costs and liquidity risks and more generally, applications of optimal control theory in which the system dynamics are modified by a sequence of discrete actions.

Stochastic differential games with impulse control (in which two players modify the system dynamics) have recently appeared in the stochastic impulse control literature. Deterministic versions of a game in were first studied by [Yon94; TY93] - in the model presented in [Yon94], impulse controls are restricted to use by one player and the other uses continuous control. Similarly, in [Zha11] stochastic differential games in which one player uses impulse control and the other uses continuous controls were studied. Using a verification argument, the conditions under which the value of game is a solution to a HJBI equation is also shown in [Zha11]. In [Cos12], a stochastic differential game in which both players use impulse control is analysed using viscosity theory.

Problems that combine both discretionary stopping and stochastic optimal control have attracted much attention over recent years; in particular there is a notable amount of literature on models of this kind in which a single controller uses absolutely continuous controls to modify the system dynamics. Discretionary stopping and stochastic optimal control problems in which the controller exercises modifications through the drift component of the state process (using absolutely continuous controls) have been studied by [KO02; Ben92; KS99a; KW00]. Another version of these problems which has attracted significant interest is problems in which the controller acts to modify the system dynamics by finite variations of the state process - such problems have been studied by [DZ94; IKW00].

A related family of models has recently emerged in which the task of controlling the system dynamics and exit time is divided between two players who act according to separated interests [OKB13a; NZ14]. Controller-Stopper games were introduced by Maitra & Sudderth in [MS96]; however, the game remains to be studied extensively notwithstanding notable papers such as [KS99a; KS99b] who study a game in which the underlying system dynamics are given by a one-dimensional diffusion within a given interval in $\mathbb{R}$. Other papers on the topic include [KSW01] and [BH13]; in the latter, a multidimensional model is studied wherein the state process is controlled on a diffusion in a multidimensional Euclidean space. A game-theoretic approach to stochastic optimal control problems with discretionary stopping has been used to analyse the lifetime ruin problem in [BY11a; OKB13a; NZ14] amongst others.

Within these models, the task of controlling the investment process and selecting the market exit time is assigned to two individual players who each seek to maximise some form of the same objective payoff functional. Game-theoretic formulations of the optimal
stochastic control with discretionary stopping model can be viewed as generalised versions of
the single controller models wherein the investor is now allowed to seek multiple objectives
which are each defined over multiple payoff functions.

Within the body of literature concerning stochastic differential games of control and
stopping however, the set of controller is restricted to an absolutely continuous class of
controls (e.g. \[EBY11b; EBY11a; MS96; KS99b; KZ08\]). This renders the aforementioned
models unsuitable for prescribing solutions for investment problems with fixed minimal costs
as continuous adjustments would result in immediate ruin.

**Organisation**

The paper is organised as follows: in section 2, we give a complete description of the
optimal liquidity control problem and construct the main investment model of the paper. In
section 3, we give a technical description of the game and introduce some of the underlying
concepts required in the script. In section 4 we prove some preliminary results that underpin
the main analysis which is performed in sections 5, 6 and 7 though we postpone some of the
technical proofs to the appendix. In section 5, we study the controller-stopper
game with impulse controls with a jump-diffusion process and prove a verification result. In
section 6, we generalise the results of section 5 to non-zero-sum games. In section 7 we apply
the results of section 5 to derive the optimal investment strategies the model in section 2.

We initiate the paper with the optimal liquidity control with lifetime ruin problem that
the model studied in this paper addresses; however, the general results found in the paper are
broadly applicable. The problem is one of minimising the probability of lifetime ruin whilst
maximising some utility criterion. For a complete treatment of the background and origins
of this problem, we refer the reader to \[OkB13a; NZ14; BY11b\] and references therein.

The following presentation of the problem is loosely based on the problems presented in
\[BY11b; Mk07; JMZ09\].

**2 Investment Problem**

We concern ourselves with the lifetime ruin problem within an investment context. The
objective of this section is to develop a framework through which the optimal policies of
a probability of lifetime ruin model in which (fixed) transaction costs are present can be
characterised.

The problem of how an investor should inject capital to raise a firm’s liquidity process in
order to maximise the investor’s terminal reward is known as the optimal capital injections
problem. In this environment, the investor injects capital into the firm to increase available
liquidity in order that the firm be able to pursue its market objectives.

The central task of the optimal liquidity control problem is to characterise the optimal
sequence of timing and magnitude of the capital injections to be performed by the investor.
The problem of when capital injections should be performed (and when dividends should be
paid) by the firm is an area of active research within theoretical actuarial science to which
a great deal of attention has been focused. Current models within the literature, the opti-

cmal capital injections and dividends model is represented as a single-player impulse control

problem in which the controller seeks the optimal sequence of capital injections. In \[Kor99\]
a model in which the firm can seek to raise capital (by issuing new equity) to be injected
so as to allow the firm to remain solvent is considered. We refer the reader to \[Kor99\] and
\[Zer10\] and references therein for exhaustive discussions.

The problem we address in this section is one in which a firm investor seeks to both max-
imise the availability of liquidity to the firm whilst minimising the risk of loss of investment
due to firm ruin. Following the notion of ruin in classical ruin theory, we define ruin as the
first time at which some surplus process (or liquidity process) goes negative.

In the problem we study, the investor faces transaction costs so that and each capital
injection incurs some fixed minimal cost. The investor seeks to maximise their terminal re-


turns by performing the maximal sequence of capital injections at selected times that their
wealth process can tolerate. However, the investor also seeks an optimal time to exit the
market by selling all firm holdings before firm bankruptcy.

In the following construction, we formulate the optimal liquidity control and the lifetime
ruin problem as a stochastic differential game in which two players each seek to fulfill one of
the investor’s objectives. Some of the ideas for the following description of the problem are
loosely adapted from the (continuous control) descriptions of problem presented in \[BY11a;
Mk07\].
Description of The Problem

We now provide a description of the lifetime ruin and optimal liquidity control problems. First we outline the key features of the general problem. In particular we will introduce three separate processes, namely the firm’s liquidity process and the investor’s wealth process after which we will be in a position to construct a complete description of the problem. Finally, we will describe the problem with fixed or minimally bounded costs which defines the problem we wish to solve.

We start firstly by describing the firm’s liquidity process $X_s = X(s, \omega) \in \mathbb{R} \times \Omega$ at time $s \in [t, T]$ - a stochastic process defined over some time horizon $T \in [0, +\infty[.$

When there are no capital injections, the firm’s liquidity process evolves according to the following expression:

$$dX_s = e(r-1)ds + \sigma_f(s, X^t,x_0)dB_s(s) + S_f(s), \quad X^t,x_0 := x \quad ; \mathbb{P} - a.s. \quad (1)$$

where $x \in \mathbb{R}$ is the firm’s initial surplus, $e \in \mathbb{R}^+$ is a constant that describes the firm’s rate of expenditure and $r \in (0, 1)$ is the firm’s rate of return on capital. The term $S_f(s)$ captures the stochasticity of the firm’s liquidity process and is given by $S_f(s) := \int \gamma_f(X(s-), z) N_f(ds, dz)$ where $N_f(ds, dz) \equiv N_f(ds, dz) - \nu(ds)dz$ is a compensated $\mathcal{F}$-Poisson random measure where $\nu(\cdot) := \mathbb{E}[(0, 1], V])$ for $V \subset \mathbb{R}\setminus 0,$ and $B_s(s) \in \mathbb{R}$ is a 1–dimensional standard Brownian motion; coupled with the functions $\gamma_f : [t, T] \times \mathbb{R} \to \mathbb{R}$ and $\gamma_f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}.$

Each time the investor performs a capital injection, the investor incurs a cost - the cost function $c$ associated to the injections is given by $c(\tau, z) := \exp^{-\tau}(\kappa_f + (1 + \lambda)z),$ where $\kappa_f$ is a fixed transaction cost and the parameter $\lambda > 0$ determines the proportional cost for an injection of size $z$. Since performing continuous actions would result in immediate bankruptcy, the investor’s capital injections must be performed over a discrete sequence of investments.

The investor therefore performs a sequence of capital injections $\{z_k\}_{k \in \mathbb{Z}}$ over the horizon of the problem which are performed over a sequence of intervention times $\{\tau_k(\omega)\}_{k \in \mathbb{N}}.$ We denote the investor’s control by the double sequence $(\tau, Z) \equiv \sum_{k \in \mathbb{N}} z_k \cdot 1_{(\tau_k \leq T)} \in \Phi$ where the set $\mathcal{Z}$ is a feasible set of investor capital injections and $T \subseteq [t, T]$ is the set of intervention times and $\Phi \subseteq T \times \mathcal{Z}.$

Denote by $e \in \mathbb{R}^+$ the proportion of capital flows expended by the firm and by $T_s^{\tau,Z} := \sum_{m \geq 1} z_m \cdot 1_{(\tau_m < \tau \land s)}$ - the investor’s capital injections process. With capital injections, the firm liquidity at time $s \in [t, T]$ is then given by the following expression:

$$X_s = x + \int_t^s (r - e) X^{t,x}(\tau, z) dr + \int_t^s \sigma_f(t, X^{t,x}(\tau, z)) dB_s(t) + T^{t,z}_s \mathcal{F}$$

$$+ \int_t^s \gamma_f(X^{t,x}(\tau, z), z) N_f(dr, dz), \quad \mathbb{P} - a.s. \quad (2)$$

In order to complete the description of the investor’s problem we construct the notion of ruin facing the investor. As in [OkB13b] and in the sense given by [PAH98; FS02a], let $\theta$ be a convex risk measure, then we can write the risk measure associated to the problem is given by:

$$\theta(X) = \sup_{Q \in \mathcal{M}_a} \mathbb{E}_Q[-X] - \chi(Q), \quad (3)$$

where $\mathcal{M}_a$ is some family of measures s.th. $\mathcal{Q} \ll \mathbb{P}$ and where $\mathbb{E}_Q$ denotes the expectation w.r.t. $Q \in \mathcal{M}_a$ and $\chi : \mathcal{M}_a \to \mathbb{R}$ is some convex (penalty) function.

Since the investor seeks to minimise risk of null returns, the investor seeks to exit the market by selling all holdings at a point $\rho(\Omega) \in \mathcal{T}$ that minimises the risk $\theta(X)$ of the investor’s returns falling below $m$ (after firm ruin) before $T,$ where $\rho \in \mathcal{T}$ where $\mathcal{T} \subseteq [t, T]$ is a set of $\mathcal{F}$–measurable stopping times. From now on we will consider only the case $m = 0.$

We now observe that since the investor seeks to exit the market in advance of firm ruin, we can describe the investor’s optimal stopping problem by the following representation

$$\inf_{\rho \in \mathcal{T}} \left[ \sup_{Q \in \mathcal{M}_a} \mathbb{E}_Q[-X] - \chi(Q) \right], \quad (4)$$

We assume that $\gamma_f$ and $\gamma_f$ are deterministic, uniformly continuous, measurable functions.
where the function $\chi : \mathcal{M}_a \to \mathbb{R}$ is a given function.

The firm’s liquidity process is therefore raised by capital injections performed by the investor, however, in performing capital injections, the investor’s wealth is reduced since liquidity is transferred from investor to firm. The investor however receives a return on capital through some running stream and some terminal reward after liquidating all holdings in the firm.

The investor’s wealth at time $s \leq T$ is $Y_s = Y(s, \omega)$ is a stochastic process; denote by $\pi \in [0, 1]$ the portion of the investor’s wealth invested in risky assets and by $\hat{J}_s^{(\tau, Z)} := \sum_{m \geq 1} \exp^{-\delta t_m} \left[ (1 + \lambda) z_m + \kappa t \right] \cdot 1_{\{r_m < T \wedge \rho \}}$ which is the total deductions from the investor’s wealth process due to the injections, then $Y_s$ is given by the following:

$$Y_s = y + \int_t^{s \wedge \rho} \Gamma Y_{t+}(\tau, Z) dr - \hat{J}_s^{(\tau, Z)} + \int_t^{s \wedge \rho} \pi \sigma I(r, Y^{t+}(\tau, Z)) dB_1(r) \quad \text{for} \quad t \leq s \leq T$$

$$= \int_t^{s \wedge \rho} \pi \gamma l(Y_{t+}(\tau, Z), z) \bar{N}_I(dr, dz), \quad \mathbb{P} - \text{a.s.}$$

(5)

where $\delta, r_0, \mu_R \in \mathbb{R} \in \mathbb{R}$ are constants describing that are the investor’s discount rate, the interest rate and the expected rate of return on the risky assets. The constant $\Gamma$ is given by $\Gamma := (1 - \pi) r_0 + \pi \mu_R$. The term $\bar{N}_I(ds, dz) \equiv N_I(ds, dz) - \nu(dz)ds$ is a compensated $\mathcal{F}$–Poisson random measure and $B_I(s) \in \mathbb{R}$ is a 1–dimensional standard Brownian motion.

If we now interpret optimality of the stopping time $\varrho(\Omega)$ in a sense of risk-minimal w.r.t. the risk measure $\theta$ we can reformulate the problem in (4) and the investor’s maximisation problem in terms of a decoupled pair of objective functions. Focusing firstly on the investor’s capital injection problem, we can write the problem as:

Find an admissible strategy $(\hat{\tau}, \hat{Z}) \in \Phi$ s.t.:

$$(\hat{\tau}, \hat{Z}) \in \arg \sup_{(\tau, Z) \in \Phi} J^{(1)}_I(s, x, y, (\tau, Z), \rho)$$

(6)

where

$$J^{(1)}_I(s, x, y, (\tau, Z), \rho) = \mathbb{E} \left[ \sum_{m \geq 1} e^{-\delta t_m} z_m \cdot 1_{\{r_m < T \wedge \rho \}} \right]$$

(7)

$\forall (s, x, y) \in [t, T] \times \mathbb{R} \times \mathbb{R}, \rho \in \mathcal{T}$.

The following expression represents the investor’s optimal stopping problem which seeks an optimal time to exit the market:

Find an admissible strategy $\hat{\rho} \in \mathcal{T}$ s.t.:

$$\hat{\rho} \in \inf_{\rho \in \mathcal{T}} \sup_{Q \in \mathcal{M}_a} J^{(2)}_I(s, x, y, (\tau, Z), \hat{\rho})$$

(8)

where

$$J^{(2)}_I(s, x, y, (\tau, Z), \hat{\rho}) = - e^{-\delta (r_0 + \rho)} \left( X^{t+}_{\tau_0 \wedge \rho} + \lambda_T \right),$$

(9)

where $\lambda_T := \bar{\lambda}_T \delta_{r_0 \wedge \rho}$ and $\tau_0 := \inf\{ s \in [t, T] : X_s, Y_s < 0 \} \wedge T$.

The expressions (7) and (8) fully express the investor’s set of objectives. We can combine the expressions (7) and (8) to construct a single objective function $\Pi$ given by the following expression $\forall (t, x, y) \in [t, T] \times \mathbb{R} \times \mathbb{R}$:

Find an admissible strategy $(\hat{\rho}, (\hat{\tau}, \hat{Z})) \in \mathcal{T} \times \Phi$ s.t.:

$$\hat{\rho} \in \arg \inf_{\rho \in \mathcal{T}} \Pi(s, x, y, (\hat{\tau}, \hat{Z}), \rho)$$

(10)

$$(\hat{\tau}, \hat{Z}) \in \arg \sup_{(\tau, Z) \in \Phi} \Pi(s, x, y, (\tau, Z), \rho)$$

(11)

3We observe that the problem in (9) can be viewed as a zero-sum game between two players; namely a player that controls the measure $\mathbb{Q}$ which may be viewed as an adverse market and the investor who selects the stopping time $\rho \in T$. Games of this type are explored in [NZ14] and [BM10].

4We shall hereon specialise to the case $\chi \equiv 0$ in which case the risk measure $\theta$ is called coherent.
where

\[
\Pi(s, x, y, (\tau, Z), \rho) = \mathbb{E} \left[ \sup_{Q \in M_a} \mathbb{E}_Q \left[ -e^{-\delta (\tau_S \wedge \rho)} \left( X^{t,x,(\tau,Z)}_{\tau_S \wedge \rho} + \lambda_T \right) \right] + \sum_{m \geq 1} e^{-\delta m} \varepsilon_m \cdot 1_{\{\tau_m < \tau_S \wedge \rho\}} \right],
\]

(12)

It can now be seen that the problem is now to find the interdependent set \((\hat{\rho}, (\hat{\tau}, \hat{Z})) \in \mathcal{T} \times \Phi\). If we now think of the two objectives \(7\) and \(9\) as being assigned to two individual players, we recognise the pair of problems \(7\) and \(9\) as jointly representing a stochastic differential game of control and stopping in which the controller modifies the system dynamics using impulse controls.

The problem involves a risk-minimising investor seeks to maximise their liquidity input into the firm through capital injections whilst seeking an optimal exit time with concern for a suboptimal early ruin. The underlying structure of the model is a stochastic differential game of control and stopping in which the investor has dual objectives. Each of the investor’s objectives is delegated to an individual player who plays in such a way as to maximise their own objective whilst seeking an optimal response to the other player.

In section 5, we provide a PDE characterisation of a general formulation of stochastic differential games of control and stopping involving impulse controls. We then apply the results to show that the optimal investment strategy for the problem can be recovered from the (saddle point) equilibrium strategies of a stochastic differential game of control and stopping with impulse controls.

3 Current Literature

Since its introduction to the literature, a considerable amount of work has been dedicated to the study of the lifetime ruin problem in addition to a number of variants of the problem. Variations of the original problem include models with stochastic consumption [BY11a], stochastic volatility [EBY11b], ambiguity aversion [EBY11a] amongst many other works. Clearly, the probability of lifetime ruin model can be extended to address an analogous problem within the context of an investor who holds some portfolio of risky assets who seeks to both maximise their return whilst finding the optimal time to exit the market.

A common approach to study the lifetime ruin problem is to model the problem as an optimal stochastic control problem in which the controller seeks both an optimal investment strategy (modelled using absolutely continuous controls) and an optimal time to sell all market holdings. Thus, in general the lifetime ruin problem in which the investor also seeks to maximise their returns can be formulated as an optimal stochastic control problem with discretionary stopping.

In general, lifetime ruin problems in which the investor also seeks to maximise some performance criterion can be reformulated as stochastic differential games. The intuition behind this is that given a sufficient player aversion to lifetime ruin, nature can be viewed as a second player with the first player responding to nature’s actions in such a way that seeks to avoid the occurrence of lifetime ruin.

In [BY11a], it is shown that the single investor portfolio problem in a Black-Scholes market in which an investor seeks to both maximise a running reward and minimise the probability of lifetime bankruptcy exhibits duality with controller-stopper games. Indeed, in [BY11a] it is shown that the value function of the investment problem is the convex dual of the value of a controller-stopper game. Similarly, in [OkB13a] an investor portfolio problem with discretionary stopping is analysed by studying an optimal stopping-stochastic control differential game and proving an equivalence.

In [OkB13a], the value for a game in which the stopper seeks to minimise a convex risk measure defined over a common (zero-sum) payoff objective is characterised in terms of a Hamilton-Jacobi-Bellman Variational Inequality (HJBVI) to which it is proven that the value is a viscosity solution. The inclusion of a convex risk measure, as outlined in [78], [79], serves as a means by which risk attitudes of the investor are encapsulated into the model, Furthermore, the zero-sum payoff structure of the model implies that the strategies are appropriate for the extraction of optimal strategies in worst-case scenario analyses.
Contributions

This paper introduces a controller-stopper game in which the controller uses impulse controls; the results cover a general setting in which the underlying state process is a jump diffusion process. We extend existing game-theoretic impulse control results to now cover games in which i) the underlying state process is a jump-diffusion process and in contrast to ii) the payoff is no longer restricted to a zero-sum structure.

To our knowledge, this paper is the first to deal with a jump-diffusion process within a stochastic differential game in which the players use impulse controls to modify the state process. Lastly, also to the best of our knowledge, this paper is the first to provide results pertaining to a non-zero-sum payoff structure within stochastic differential games for a controller-stopper game in which impulse controls used.

A related paper to the current is \[\text{[OkB13a]}\] in which conditions for a HJBI equation are proved for controller-stopper games in which the controller uses continuous controls.

In the following section we describe the details of a general version of the controller-stopper game thereafter, we prove two key results: we firstly prove a set of verification theorems that characterise the conditions for a HJBI equation in non-zero-sum and zero-sum games. As in the Dynkin game case and controller-controller case, the HJBI equation is an obstacle problem in particular, the HJBI equation is an obstacle quasi-variational inequality.

We begin by giving a canonical description of the game dynamics, starting with the zero sum game.

The Dynamics: Canonical Description

Let \(\mathcal{C}(U; G)\) be the set of continuous functions from some set \(U \subseteq \mathbb{R}\) to a field \(G\). The index \(s \in [t, \tau_S]\) is time which runs continuously over some random and possibly finite time horizon \(\tau_S\). We denote the coordinate mapping on \(\mathcal{C}([t, \tau_S]; \mathbb{R}^p)\) by \(B_s(\omega_B) = \omega_B(s)\) and denote also by \(\mathcal{F} = \{\mathcal{F}_s\}_{s \in [t, \tau_S]}\) the completed natural filtration and define \(\{\mathcal{F}_s\}_{s \in [t, \tau_S]}\) to be \(\{\mathcal{F}_s\}_{s \in [t, \tau_S]}\) restricted to the interval \([t, s]\) (uncompleted natural filtration). Correspondingly, we also denote by \(\mathcal{W}_{t,v}\) a \(\sigma\)-algebra generated by the paths in \(\mathcal{C}([t, \tau_S]; \mathbb{R}^p)\) up to time \(t'\) and let \(B(s) \in \mathbb{R}^p\) be a \(p\)-dimensional standard Brownian motion with state space \(S\). \(\tilde{N}(ds, dz) = N(ds, dz) - \nu(dz)dt\) is a \(\mathcal{F}\)-Poisson random measure with \(\nu(\cdot) := \mathbb{E}[\mathcal{N}(1, \cdot)]\) is a Lévy measure; both \(\tilde{N}(ds, dz)\) and \(B(s)\) are supported by the filtered probability space \(\mathcal{F}\) is the filtration of the probability space\((\Omega, \mathcal{F}, \mathcal{P}, \mathcal{F} = \{\mathcal{F}_s\}_{s \in [t, \tau_S]}\). We assume that \(N\) and \(B\) are independent. As in \([\text{TY93a}]\), we note that the above specification of the filtration ensures stochastic integration and hence, the controlled jump diffusion is well defined (this is proven in \([\text{Zha11}]\)).

We suppose then that the uncontrolled passive state \(X \in S \subset \mathbb{R}^p(p \in \mathbb{N})\), evolves according to a (jump-)diffusion on \(\mathcal{C}([t, \tau_S]; \mathbb{R}^p), (\mathcal{F}_{t,s})_{s \in [t, \tau_S]}, \mathcal{P}_0\) that is to say for \(s \in [t, \tau_S]\), \(X \in S \subseteq \mathbb{R}^p\) the state process obeys the following SDE:

\[
dX^{t,x_0}_s = \mu(s, X^{t,x_0}_s)ds + \sigma(s, X^{t,x_0}_s)dB(s) + \int \gamma(X_{s-}, z)\tilde{N}(ds, dz), \quad X^{t,x_0}_t := x_0. \tag{13}
\]

\(\forall s \in [t, \tau_S], (t, x_0) \in [t, \tau_S] \times \mathbb{R}^p; \mathcal{P}\)-a.s.

The generator of \(X\) (the uncontrolled process) is:

\[
\mathcal{L}\phi(x) = \sum_{i=1}^p \mu_i(x) \frac{\partial \phi}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^p (\sigma \sigma^T)_{ij}(x) \frac{\partial^2 \phi}{\partial x_i \partial x_j} + I\phi(x) \tag{14}
\]

\(\forall x \in [t, \tau_S] \times \mathbb{R}^p\), where \(I\) is the integro-differential operator defined by:

\[
I\phi(x) := \sum_{j=1}^T \int_{\mathbb{R}^p} \{\phi(x + \gamma^j(x, z_j)) - \phi(x) - \nabla \phi(x) \gamma^j(x, z_j)\} \nu_j(dz_j), \tag{15}
\]

\(\forall x \in [t, \tau_S] \times \mathbb{R}^p\).

The state process is influenced by impulse controls \(u \in U\) exercised by player I where \(u(s) = \sum_{j \geq 1} \xi_j \cdot 1_{\{\tau_j \leq \tau_S\}}(s)\) for all \(s \in [t, \tau_S]\). The impulses \(\{\xi_j\} \in \mathcal{Z} \subset \mathcal{S}\) are exercised by player I who intervenes at \(\mathcal{F}\)-measurable stopping times \(\{\tau_i\}\) where \(t < \tau_1 < \tau_2 < \ldots < \) and where \(\mathcal{S} \subset \mathbb{R}^p\) is a given set. We assume \(U \subseteq \mathbb{R}^p\) is a convex cone which is the set of admissible control actions for player I and \(\mathcal{Z}\) is the set of admissible impulse values. Indeed, if we suppose that an impulse \(\zeta \in \mathcal{Z}\) determined by some admissible policy \(w\) is applied at some \(\mathcal{F}\)-measurable stopping time \(\tau\) when the state is \(x' = X^{t,x_0}_\tau(\tau-),\) then the state
immediately jumps from \( x' = X^{t,x_0}(\tau-) \) to \( X^{t,x_0,u}(\tau) = \Gamma(x', \zeta) \) where \( \Gamma : \mathbb{R}^p \times \mathcal{Z} \to \mathbb{R}^p \) is called the impulse response function and \( (t, x_0) \in [t, \tau_S] \times \mathbb{R}^p \). We assume that the impulses \( \xi_j \in \mathcal{Z} \) are \( U^- \) valued and are \( \mathcal{F} \)-measurable for all \( j \in \mathbb{N} \).

For notational convenience, as in \( \text{OkB13a} \), we will use \( u = [\tau_j, \xi_j]_{j \geq 1} \) to denote the control policy \( u = \sum_{j \geq 1} \xi_j \cdot 1_{\{\tau_j \leq \tau_s\}}(s) \in \mathcal{U} \) which consists of \( \mathcal{F} \)-measurable stopping times \( \{\tau_j\}_{j \in \mathbb{N}} \) and \( \mathcal{F} \)-measurable impulse interventions \( \{\xi_j\}_{j \in \mathbb{N}} \).

The evolution of the state process with interventions is described by the equation:

\[
X_r = x + \int_t^r \mu(s, X_s^{t,x_0,u})ds + \int_t^r \sigma(s, X_s^{t,x_0,u})dB_s + \sum_{j \geq 1} \xi_j \cdot 1_{\{\tau_j \leq \tau_s\}}(r) + \int_t^r \gamma(X(s-), z)N(ds, dz),
\]

for all \( r \in [t, \tau_S]; \mathbb{P} \)-a.s.

The game is s.t.h. player II can choose some \( \mathcal{F} \)-measurable stopping time \( \rho \in [t, \tau_S] \) at which point the process is stopped and both players receive a terminal cost (reward) \( G(X) \). Player I has a cost function which is also the player II gain (or profit) function. The corresponding payoff functions are given by the following expression which player I (resp., player II) minimises (resp., maximises):

\[
J^{\rho,u}(x) = J(t, x_0; u, \rho) = \mathbb{E}\left[ \int_t^{\tau_{\rho}\wedge \tau_S} f(s, X_s^{t,x_0,u})ds + \sum_{m \geq 1} c(\tau_m, \xi_m) \cdot 1_{\{\tau_m \leq \tau_S \wedge \rho\}} + G(\rho, X_{\tau_S \wedge \rho}) \right], 
\]

where \( x := (t, x_0) \) and where the functions \( f : [t, \tau_S] \times \mathbb{R}^p \to \mathbb{R} \), \( G : [t, \tau_S] \times \mathbb{R}^p \to \mathbb{R} \) are deterministic functions which we shall refer to as the running cost function and the bequest function respectively.

The results contained in this paper are built exclusively under the following set of assumptions unless otherwise stated:

**Standing Assumptions**

**A.1.1. Lipschitz Continuity**

We assume there exist real-valued constants \( c_\mu, c_\sigma > 0 \) and \( c_\gamma(\cdot) \in L^1 \cap L^2(\mathbb{R}^l, \nu) \) s.t. \( \forall s \in [t, \tau_S], \forall x, y \in \mathbb{R}^p \) and \( \forall z \in \mathbb{R}^l \) we have:

\[
|\mu(s, x) - \mu(s, y)| \leq c_\mu|x - y| \\
|\sigma(s, x) - \sigma(s, y)| \leq c_\sigma|x - y| \\
\int_{|z| \geq 1} |\gamma(x, z) - \gamma(y, z)| \leq c_\gamma(z)|x - y|.
\]

**A.1.2. Lipschitz Continuity**

We also assume the Lipschitzianity of the running functions \( h, g, \psi \) and \( \phi \) that is, we assume the existence of real-valued constants \( c_h, c_g, c_\psi, c_\phi > 0 \) s.t. \( \forall s \in [t, \tau_S], \forall (x, y) \in \mathbb{R}^p \) we have for \( R \in \{h, g, k, l, \psi, \phi\} \):

\[
|R(s, x) + R(s, y)| \leq c_R|x - y|.
\]

**A.2. Growth Conditions**

We assume the existence of a real-valued constants \( d_\mu, d_\sigma > 0 \) and \( d_\gamma(\cdot) \in L^1 \cap L^2(\mathbb{R}^l, \nu), \rho \in [0, 1] \) s.t. \( \forall s, x \in [t, \tau_S] \times \mathbb{R}^p \) and \( \forall z \in \mathbb{R}^l \) we have:

\[
|\mu(s, x)| \leq d_\mu(1 + |x|^\rho) \\
|\sigma(s, x)| \leq d_\sigma(1 + |x|^\rho) \\
\int_{|z| \geq 1} |\gamma(x, z)| \leq d_\gamma(1 + |x|^\rho).
\]

We also make the following assumptions on the cost function \( c : [t, \tau_S] \times \mathbb{R}^p \to \mathbb{R} \):

**A.3.**

Let \( \tau, \tau' \in [t, \tau_S] \) be \( \mathcal{F} \)-measurable stopping times s.t.h. \( t \leq \tau < \tau' \leq \tau_S \) and let \( \xi, \xi' \in \mathcal{Z} \) be measurable impulse interventions. Then we assume that the following statements hold:

\[
c(\tau, \xi) \leq c(\tau, \xi) + c(\tau, \xi'), \quad (18) \\
c(\tau, \xi) \geq c(\tau', \xi). \quad (19)
\]
A.4.

We also assume that the there exists a constant \( \lambda_c > 0 \) s.th. \( \inf_{\xi \in \mathcal{Z}} c(s, \xi) \geq \lambda_c \forall s \in [t, T] \) where \( \xi \in \mathcal{Z} \) is a measurable impulse intervention.

Assumptions A.1.1 and A.2 ensure the existence and uniqueness of a solution to (13) (c.f. [BY11]). Assumption A.3 (i) (subadditivity) is required in the proof of the uniqueness of the value function. Assumption A.3 (ii) (the player cost function is a decreasing function in time) and may be interpreted as a discounting effect on the cost of interventions. Assumption A.1.2 is required to prove the regularity of the value function (see for example [Yon94] and for the single-player case, see for example [Mk07]). Assumption A.3 (ii) was introduced (for the two-player case) in [JMZ09] though is common in the treatment of single-player case problems (e.g. [Mk07, OkB13a]). Assumption A.4 is integral to the definition of the impulse control problem.

Throughout the script we adopt the following standard notation (e.g. [TY93, OkB13a, NZ14]):

**Notation**

Let \( \tilde{\Omega} \) be a bounded open set on \( \mathbb{R}^{p+1} \). Then we denote by: \( \overline{\Omega} \) - The closure of the set \( \Omega \).

\( \partial \Omega \) - the parabolic boundary \( \Omega \) i.e. the set of points \( (s, x) \in \mathbb{S} \) s.th. \( R > 0, Q(s, x; R) \not\subset \tilde{\Omega} \).

\( C^{1,2}([t, T]; \mathbb{R}) \), \( \Omega \) - \( \{ h \in C^{1,2}(\Omega) : \partial_h \delta_h, \partial_{x_r}, h \in C(\Omega) \} \), where \( \partial_h \) and \( \partial_{x_r} \) denote the temporal differential operator and second spatial differential operator respectively.

\( \nabla \phi = (\partial_x \phi, \ldots, \partial_x \phi) \) - The gradient operator acting on some function \( \phi \in C^1([t, T] \times \mathbb{R}^p) \).

\( C_{gd}([a, b]; U) \) - The set of c\'{a}dl\'{a}g functions that map \([a, b] \mapsto U\) for some set \( U \subset \mathbb{R}^p \).

\( |\cdot| \) - The Euclidean norm to which \((x, y)\) is the associated scalar product acting between two vectors belonging to some finite dimensional space.

As in [TY93], we will use the notation \( u \equiv [\tau_j, \xi_j]_{j \geq 1} \) to denote the control policy \( u = \sum_{j \geq 1} \xi_j \cdot 1_{\{\tau_j \leq T \}}(s) \in U \) which consists of \( \mathcal{F} \) measurable stopping times \( \{\tau_j\}_{j \in \mathbb{N}} \) and \( \mathcal{F} \) measurable impulse interventions \( \{\xi_j\}_{j \in \mathbb{N}} \).

# Statement of Main Results

In this paper, we prove two key results for the game that characterise the value HJBI in both zero-sum and non-zero-sum impulse controller-stopper stochastic differential games.

We prove a verification theorem (Theorem 5.1) for stochastic differential games with a jump-diffusion process and in which one of the players uses impulse controls and the other player chooses when to end the game. In doing so, we show that the value of the game must satisfy a double obstacle quasi-variational inequality:

\[
\begin{align*}
\max \{ \min \left[ -\frac{\partial V}{\partial s} - LV - f, V - G \right], V - \mathcal{M} V \} &= 0 \\
V(X^{t, x, u}(\tau_s \wedge \rho)) &= G(X^{t, x, u}(\tau_s \wedge \rho)).
\end{align*}
\]

where \( L \) is the local stochastic generator operator associated to the process \( \tilde{\xi} \) and \( \mathcal{M} \) is the non-local intervention operator - we will use \( L \) to denote the local stochastic generator for the controlled process, where it will not cause confusion we will also employ the shorthand \( r(s, X_S) \equiv r(X_S) \) where \( r \in \mathcal{F} \), \( G \) - the constituent functions of the payoff function \( J \).

In the non-zero-sum case we have the following result:

**Theorem 6.2.** Denote by \( \phi_i \) the objective function for the non-zero-sum game for player \( i \in \{1, 2\} \), then the functions \( \phi_i \) satisfy the following quasi-variational inequalities \( \forall y \in \mathbb{R}^p, x \in [t, T_s] \times \mathbb{R}^p \):

\[
\begin{align*}
\max \{ \partial_x \phi_1(x) + L \phi_1(x) + f_1(x), \phi_1(x) - M_1 \phi_1(x) \} &= 0 \\
\max \{ \partial_x \phi_2(x) + L \phi_2(x) + f_2(x), \phi_2(x) - G_2 \phi_2(x) \} &= 0 \\
\phi_1(\tau_S, y) &= G(\tau_S, y).
\end{align*}
\]

Having proven these results, we then implement the analysis to prove the following set of results relating to the optimal liquidity control and lifetime ruin investment problem stated in section 2:

**Theorem 7.1.** Suppose that the firm’s liquidity process \( x \) evolves according to (2) and suppose that the investor’s wealth process \( \rho \) evolves according to (3), then the sequence of optimal capital injections \( \{\hat{\tau}, \hat{Z}\} \equiv [\hat{\tau}_j, \hat{z}_j]_{j \in \mathbb{N}} \equiv \sum_{j \geq 1} \hat{z}_j \cdot 1_{\{\hat{\tau}_j \leq \hat{\rho} \wedge T \}}(s) \) is characterised
by the investment times \( \{ \hat{\tau}_j \} \in \mathbb{N} \) and magnitudes \( \{ \hat{z}_j \} \in \mathbb{N} \) where \( [\hat{\tau}_j, \hat{z}_j] \in \mathbb{N} \) are constructed recursively via the following expressions:

1. \( \hat{\tau}_0 \equiv t_0 \) and \( \hat{\tau}_{j+1} = \inf \{ s > \tau_j ; Y^{(\hat{\tau}, \hat{z})}_{(t, s)}(s) \geq \hat{y} | \hat{y} \in \mathbb{R}, Y \in S \} \wedge \hat{\rho} \),
2. \( \hat{z}_j = \hat{y} - y(\hat{\tau}_j) \).

The fixed duplet \((\hat{y}, \hat{y})\) is determined by the following equations:

\[
\phi_2(\hat{y}) = \alpha_1, \quad \phi_2(\hat{y}) = \phi_2,0(\hat{y}) - (\alpha_1 + \alpha_1(\hat{y} - y)), \quad \phi_2^2(\hat{y}) = \alpha_2. \tag{21}
\]

where the function \( \phi_2 \) is given by (21) and the function \( \phi_{2, 0} \) is given by:

\[
\phi_{2, 0}(x) = c(y^{d_1} - y^{d_2}), \tag{24}
\]

where the constants \( d_1, d_2, c \in \mathbb{R} \) are given in (158) and (159) - (161).

The investor’s non-investment region is given by:

\[
D_2 = \{ Y \in S, s \in [t, T] ; Y(s) < \hat{y} \}, \tag{25}
\]

The investor exits the market at \( \hat{\rho} \in \mathcal{T} \) where the exit time is defined by:

\[
\hat{\rho} = \inf \{ s > t ; X^{(\hat{\tau}, \hat{z})}_{(t, s)}(s) \cdot Q(s) \notin D_1 | X \in S, Q \in \mathbb{R} \} \wedge \tau_S, \tag{26}
\]

where the process \( Q(s) \) is determined by:

\[
Q(r) = Q(t) \exp \left\{ -\frac{1}{2} \sigma^2 r + \sigma f B_f(r) + \int_t^r \int_{\mathbb{R}} \left( \ln(1 + \hat{\theta}_1(s, z)) - \hat{\theta}_1(s, z) \right) \hat{N}_f(ds, dz) \right\}. \tag{27}
\]

and the set \( D_1 \) (non-stopping region) is defined by:

\[
D_1 = \left\{ \left( \frac{\partial}{\partial s} + \mathcal{L}^\theta \right) \psi(s, \cdot) > 0 \right| s > t \}. \tag{28}
\]

where the operator \( \mathcal{L}^\theta \) corresponds to the stochastic generator of the controlled process.

Theorem 7.1 says that the investor performs discrete capital injections over a sequence of intervention times \( \{ \hat{\tau}_k \} \in \mathbb{N} \) over the time horizon of the problem. The decision to invest is determined by the investor’s wealth process - in particular, at the points at which the investor’s wealth process reaches \( \hat{y} \), then the investor performs capital injections of magnitudes \( \{ \hat{z}_k \} \in \mathbb{N} \) to increase the firm’s liquidity levels in order to provide the firm with maximal liquidity to perform market operations. This in turn maximises the liquidity that the investor makes available to the firm whilst the investor remains in the market after which the investor liquidates all investment holdings. However, if the firm’s liquidity process exits the region \( D_1 \), in order to avoid the prospect of loss on investment, the investor immediately exits the market by liquidating all market holdings in the firm.

The fixed duplet \((\hat{y}, \hat{y})\) is determined by (21) - (23). The non-stopping region \( D_1 \) is defined by (28). The function \( \psi \) is the investor’s value function and the operator \( \mathcal{L}^\theta \) is the (controlled) stochastic generator. In section 7 we provide a full characterisation of the investor’s value function.

From Theorem 7.1. we also arrive at the following result that enables us to state the exact points at which the investor performs an injection, when the investor exits the market and when the investor does nothing:

**Corollary 7.2.** For the optimal liquidity control and lifetime ruin problem, the investor’s wealth process \( x \) lies within a space that splits into three regions: a region in which the investor performs a capital injection - \( I_1 \), a region in which no action is taken - \( I_2 \) and lastly a region in which the investor exits the market by selling all firm holdings - \( I_3 \). Moreover, the three regions are characterised by the following expressions:

\[
I_1 = \{ y \geq \hat{y}, y, \hat{y} \in S \},
I_2 = \{ q x > \omega^*, y < \hat{y}, x, y, \hat{y} \in S; \omega^*, q \in \mathbb{R} \},
I_3 = \{ q x \leq \omega^* | x \in S; q, \omega^* \in \mathbb{R} \},
\]

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where \( q \) is the value of the process \( Q \) and the fixed duplet \((\tilde{y}, \tilde{y})\) and the value \( \omega^* \) are determined by (21) - (23) and (171) respectively.

To our knowledge, this paper is the first to deal with a jump-diffusion process within a stochastic differential game in which the players use impulse controls to modify the state process. Additionally, to our knowledge, this is the first game that involves impulse controls in which the role of one of the players is to stop the game at a desirable point.

We now give some definitions which we shall need to describe the system dynamics modified by impulse controls:

**Definition 4.1.** Denote by \( T_{(t, \tau)} \) the set of all \( \mathcal{F} \)-measurable stopping times in the interval \([t, \tau']\), where \( \tau' \) is some stopping time s.t. \( \tau' \leq \tau_S \), if \( \tau' = \tau_S \) then we will denote by \( T \equiv T_{(t, \tau_S)} \). Let \( u = \{\tau_j, \xi_j\}_j \in \mathbb{N} \) be a control policy where \( \{\tau_j\}_j \in \mathbb{N} \) and \( \{\xi_j\}_j \in \mathbb{N} \) are \( \mathcal{F}_\tau \)-measurable stopping times and interventions respectively, then we denote by \( \mu_{t, \tau}(u) \) the number of impulses the agent executes within the interval \([t, \tau]\) under the control policy \( u \) for some \( \tau \in T \).

**Definition 4.2.** Let \( u \) be an impulse control policy. We say that an impulse control is admissible on \([t, \tau_S]\) if the number of impulse interventions is finite \( \mathbb{P}-a.s. \), that is to say we have that \( E[\mu_{t, \tau_S}(u)] < \infty \).

We shall hereon use the symbol \( \mathcal{U} \) (resp., \( T \)- which belongs to the set of all \( \mathcal{F} \)-measurable stopping times in \([t, \tau_S]\)) to denote the set of admissible controls for player I (resp., player II). Given two player I controls \( u \in \mathcal{U} \) and \( u' \in \mathcal{U} \); we interpret the notion \( u \equiv u' \) on \([t, \tau_S]\) iff \( P(u = u') \) a.e. on \([t, \tau_S]\) = 1.

Similarly, given two player II stopping times \( \rho \in \mathcal{T} \) and \( \rho' \in \mathcal{T} \), we interpret the notion \( \rho \equiv \rho' \) on \([t, \tau_S]\) analogously.

**Definition 4.3.** Let \( u(s) = \sum_{j \geq 1} \xi_j \cdot 1_{(\tau_j, r_{\tau, \tau_S})}(s) \in \mathcal{U} \) be an impulse control defined over \([t, \tau_S]\), further suppose that \( \tau \) and \( \tau' \) are two \( \mathcal{F} \)-measurable stopping times contained within the interval \([t, \tau_S]\), then we define the restriction \( u_{(t, \rho)} \) of the impulse control \( u(s) \) to be \( u_{(t, \rho)}(s) = \sum_{j \geq 1} \xi_j \cdot 1_{(\tau_j, \rho_{\tau, \tau_S})}(s) \) where \( \tau \leq s < \tau' \).

We denote by \( \mathcal{U}_{(t, \tau_S)} \equiv \mathcal{U} \) the (restricted) set of admissible controls over the interval \([\tau, \tau']\).

**Strategies**

A player strategy is a map from the other player’s set of controls to the player’s own set of controls. An important feature of the players’ strategies is that they are non-anticipative - neither player may guess in advance, the future behavior of other players given his current information.

We formalise this condition by constructing non-anticipative strategies which were used in the viscosity solution approach to differential games in [FR02]. Non-anticipative strategies were introduced by [FS02b; KO02; Ben92; KS99a]. Hence, in this game, one of the players chooses his control and the other player responds by selecting a control according to some strategy.

**Definition 4.4.** A non-anticipative strategy on \([t, \tau_S]\) for Player I is a measurable mapping which we shall denote by \( \alpha : [t, \tau_S] \times \Omega \times T \to \mathcal{U} \) and for any stopping time \( \tau : \Omega \to T \) and any \( \mathcal{F} \)-measurable player II stopping times \( \rho_1, \rho_2 \in \mathcal{T} \) with \( \rho_1 \equiv \rho_2 \) on \([t, \tau]\) we have that \( \alpha(\rho_1) \equiv \alpha(\rho_2) \) on \([t, \tau]\).

We define the Player II non-anticipative strategy \( \beta : [t, \tau_S] \times \Omega \times \mathcal{U} \to \mathcal{T} \) analogously. Hence, \( \alpha \) and \( \beta \) are Elliott-Kalton strategies.

We denote the set of all non-anticipative strategies for Player I (resp., Player II) by \( \mathcal{A}_{(t, \tau_S)} \) (resp., \( \mathcal{B}_{(t, \tau_S)} \)).

**Remark 4.5.** The intuition behind definition 4.4 is as follows: suppose player I uses the control \( u_1 \in \mathcal{U} \) and the system follows a path \( \omega \) and that player II employs the strategy \( \beta \in \mathcal{B}_{(t, \tau_S)} \) against the control \( u_1 \). If in fact player II cannot distinguish between the control \( u_1 \) and some other player I control \( u_2 \in \mathcal{U} \) then controls \( u_1 \) and \( u_2 \) induce the same response from the player II strategy that is to say \( \beta(u_1) \equiv \beta(u_2) \).

Note that when \( \mathcal{U} \) is a singleton the game is degenerate and collapses into a classical optimal stopping problem for player II with a value function and solution as that in ch.3 in [KW00]. Similarly, when \( T \) is a singleton the game collapses into a classical impulse control problem for player I with a value function and solution as that in ch.7 in [KW00].

**Definition 4.6.** Suppose we denote the space of measurable functions by \( \mathcal{H} \), suppose also that the function \( \phi : [t, \tau_S] \times \mathbb{R}^p \to \mathbb{R}^p \) s.t. \( \phi \in \mathcal{H} \). Let \( \tau \in [t, \tau_S] \) and \( \rho \in \mathcal{T} \) be \( \mathcal{F} \)-measurable stopping times; we define the [non-local] Player I intervention operator
\[ M : \mathcal{H} \to \mathcal{H} \text{ acting at } \tau \text{ by the following expression:} \]
\[ M[\phi] := \inf_{z \in \mathbb{Z}} [\phi(\Gamma(\tau-), z)] + c(\tau, z) \cdot 1_{\{\tau \leq \tau_S\}} \]
where \( \Gamma : \mathbb{R}^p \times \mathbb{Z} \to \mathbb{R}^p \) is the impulse response function defined earlier.

**Remark 4.7.** Suppose that the value of the game exists and that we denote the value by \( V \). If \( V \in \mathcal{H} \), then the term \( MV(s, x) \) that is, the non-local intervention operator \( M \) acting the value function associated to the game, represents the value of the player I strategy that consists of performing the best possible intervention at some given time \( s \in [t, \tau_S] \) when the state is at \( x \in \mathbb{R}^p \), then performing optimally thereafter.

Suppose \( \tau \in [t, \tau_S] \) is some intervention time then the equality \( MV(\tau, x) = V(\tau, x) \) holds at the points of intervention \( \forall x \in \mathbb{R}^p \), we note however, that an immediate intervention may not be optimal; that is we have the following lemma:

**Lemma 4.8.** Suppose that the value of the game \( V \) exists and that \( \tau \in [t, \tau_S] \), then the non-local intervention operator \( M \) satisfies the following inequality pointwise \( \forall (s, x) \in [t, \tau_S] \times \mathbb{R}^p \):
\[ MV(s, x) \geq V(s, x). \] 

We give a statement of the following result without proof:

**Lemma 4.9.** (Lemma 3.10 in [Mk07]) The non-local intervention operator \( M \) is continuous wherein we can deduce the existence of a constants \( c_1, c_2 > 0 \) s.th. \( \forall x, y \in \mathbb{R}^p \) and \( s < s' \in [t, \tau_S] \):
\[ |MV^+(s, x) - MV^+(s, y)| \leq c_1|x - y|. \]

\[ |MV^+(t, x) - MV^+(s, x)| \leq c_2|t - s|^{\beta}. \]

A proof of the result is reported in [Mk07].

## 5 Stochastic Differential Games of Impulse Control and Stopping

We now study the zero-sum case of the game. The following theorem provides the conditions under which, if a sufficiently smooth function can be found then we have the value function of the game. Thus the following verification theorem characterises the conditions in which the value of the game satisfies a HJBI equation.

We will later use the conditions of Theorem 5.1 in a practical sense to derive the optimal investment strategy for the optimal liquidity control and lifetime ruin model presented in section 7.

**Theorem 5.1.** [Verification Theorem for Zero-Sum Controller-Stopper Games with Impulse Control]

Suppose the problem is to find \( \phi(x) \) and \((\hat{u}, \hat{\rho}) \in \mathcal{U} \times \mathcal{T}\) s.th. for all \( x \in [t, \tau_S] \times \mathbb{R}^p \):
\[ \phi(x) = \sup_{\rho \in \mathcal{T}} \left( \inf_{u \in \mathcal{U}} J^{(u, \rho)}(x) \right) = \inf_{u \in \mathcal{U}} \left( \sup_{\rho \in \mathcal{T}} J^{(u, \rho)}(x) \right) = J^{(\hat{u}, \hat{\rho})}(x), \]
where \( (\hat{u}, \hat{\rho}) \in \mathcal{U} \times \mathcal{T} \) exists, it is an optimal pair consisting of the optimal control for player I and the optimal stopping time for player II (resp.).

Let \( \tau \) be some \( \mathcal{F} \)-measurable stopping time and denote by \( \hat{X}(\tau) = X(\tau^-) + \Delta_N X(\tau) \), where \( \Delta_N X(\tau) \) denotes a jump at some \( \mathcal{F}_\tau \)-measurable time \( \tau \) due to \( N \). Suppose that the value of the game exists. Denote by \( X(s) \equiv X(s, \cdot), \forall s \in \mathbb{R} \) and \( \phi \equiv \phi(\cdot, X) \equiv \phi(X), f(\cdot, X) \equiv f(X) \), \( \forall X \in \mathcal{S} \).

Suppose also that there exists a function \( \phi \) that satisfies technical conditions (T1) - (T4) and the following conditions:

(i) \( \phi \in C^{1,2}([t, \tau_S], S) \cap C([t, \tau_S], \tilde{S}). \)

(ii) \( \phi \leq \mathcal{M}\phi \) on \( S \) and \( \phi \geq G(X) \) on \( S \) and the regions \( D_1 \) and \( D_2 \) are defined by:
\[ D_1 = \{ X \in S; \phi(X) < \mathcal{M}\phi(X) \} \text{ and } D_2 = \{ X \in S; \phi(X) > G(X) \} \]
where we refer to \( D_1 \) (resp., \( D_2 \)) as the player I (resp., player II) continuation region.

(iii) \( \frac{\partial \phi}{\partial s} + \mathcal{L}\phi(X^{-u}(s)) + f(X^{-u}(s)) \geq 0 \forall u \in \mathcal{U}, X \in S \setminus \partial D_1, s \in [t, \tau_S] \).

(iv) \( \frac{\partial \phi}{\partial s} + \mathcal{L}\phi(X^{-\hat{u}}(s)) + f(X^{-\hat{u}}(s)) = 0 \) in \( D_1 \cap D_2, s \in [t, \tau_S] \).
(v) For \( u \in \mathcal{U} \), define \( \rho_D = \rho_D^u = \inf\{s > t, X^{\cdot,u}(s) \notin D_2 \} \) and specifically, \( \tilde{\rho}_D = \tilde{\rho} = \inf\{s > t, X^{\cdot,u}(s) \notin D_2 \} \).

(vi) \( X^{\cdot,u}(\tau_S) \in \partial S \) \( \mathbb{P} \) \(-\text{a.s.} \) on \( \tau_S < \infty \) and \( \phi(X^{\cdot,u}(s)) \rightarrow G(X^{\cdot,u}(\tau_S \land \rho)) \cdot 1_{\{\tau_S < \infty\}} \) as \( s \rightarrow \tau_S \land \rho \) \( \mathbb{P} \) \(-\text{a.s.} \) for all \( X \in S, u \in \mathcal{U} \).

Put \( \tau_0 = t \) and define \( \hat{u} := [\hat{\tau}_j, \hat{\xi}_j] \in \mathbb{N} \) inductively by:
\[
\hat{\tau}_{j+1} = \inf\{s > \hat{\tau}_j; X^{\cdot,\hat{u},\hat{\xi}_j}(s) \notin D_1 \} \land \tau_S \land \rho,
\]
then \((\hat{u}, \hat{\rho}) \in \mathcal{U} \times \mathcal{T}\) are an optimal pair for the game, that is to say that we have:
\[
\phi(x) = \inf_{u \in \mathcal{U}} \left( \sup_{\rho \in \mathcal{T}} J^{(u,\rho)}(x) \right) = \sup_{\rho \in \mathcal{T}} \left( \inf_{u \in \mathcal{U}} J^{(u,\rho)}(x) \right) \tag{31}
\]
for all \( x \in [t, \tau_S] \times \mathbb{R}^p \).

Theorem 5.1 provides a characterisation of the value of the game in terms of a dynamic programming equation (which is simply the non-linear PDE in (iv)). In particular, Theorem 5.1 says that given some solution to the non-linear PDE in (iv), then this solution coincides with the value of the game from which we can calculate the optimal controls for each player.

Before stating the proof of Theorem 5.1, we make the following set of remarks which also applies to Theorem 5.3.:

**Remark 5.2.** For the jump-diffusion process considered here, by Lemma 3.7 in [CG14] we can automatically conclude that \( \xi_k \in \text{argmin}_{x \in \mathbb{Z}} \phi(\Gamma(X^{(\cdot)}(\tau_k -), z)) + c(\tau_k, z) \forall k \in \mathbb{N}, X \in S, \tau_k \in [t, \tau_S] \) where \( \tau_k \) is an \( \mathcal{F} \) \(-\text{measurable} \) stopping time exists for the game considered here.

To prove Theorem 5.1, we firstly require the following result:

**Theorem (Approximation Theorem) (Theorem 3.1 in [KW00])**

Let \( \hat{D} \subset S \) be an open set and let us assume that \( X(\tau_S) \in \partial S \) and suppose that \( \partial \hat{D} \) is a Lipschitz surface. Let \( \psi: S \rightarrow \mathbb{R} \) be a function s.t. \( \psi \in C^1(S) \cap C(\bar{S}) \) and \( \psi \in C^2(S \setminus \partial \hat{D}) \) and suppose the second order derivatives of \( \psi \) are locally bounded near \( \partial \hat{D} \); then there exists a sequence of functions \( \{\psi_m\}_{m=1}^{\infty} \subset C^2(S) \cap C(\bar{S}) \) s.t.:

\[
\begin{align*}
\lim_{m \rightarrow \infty} \psi_m & \rightarrow \psi \text{ pointwise dominatedly in } S, \\
\lim_{m \rightarrow \infty} \frac{\partial \psi_m}{\partial x_i} & \rightarrow \frac{\partial \psi}{\partial x_i} \text{ pointwise-dominatedly in } S, \\
\lim_{m \rightarrow \infty} \frac{\partial^2 \psi_m}{\partial x_i \partial x_j} & \rightarrow \frac{\partial^2 \psi}{\partial x_i \partial x_j} \text{ and } \lim_{m \rightarrow \infty} \mathcal{L} \psi_m \rightarrow \mathcal{L} \psi \text{ pointwise dominatedly in } S \setminus \partial \hat{D}.
\end{align*}
\]

We are now in a position to prove the theorem; some ideas for the proof come from [EBY11] and [W83]:

**Proof of Theorem 5.1.**

Let us fix the player I control \( \hat{u} \in \mathcal{U} \) and let us define \( \rho_m = \rho \land m; m = 1, 2, \ldots \). By Dynkin’s formula for jump-diffusion processes (see for example Theorem 1.24 in [KW00]) we have:

\[
\mathbb{E}[\phi(X^{t,x,0,\hat{u}}(\hat{\tau}_j))] = \mathbb{E}[\phi(X^{t,x,0,\hat{u}}(\hat{\tau}_{j+1}))] = -\mathbb{E} \left[ \int_{\tau_j}^{\tau_{j+1}} \frac{\partial \phi}{\partial s} + \mathcal{L} \phi(X^{t,x,0,\hat{u}}(s)) ds \right]. \tag{32}
\]

Summing (32) from \( j = 0 \) to \( j = k \) for some \( 0 < k < \mu(m, \rho_m)(\hat{u}) - 1 \) (recall the definition of \( \mu(m, \rho_m)(\hat{u}) \) from definition 4.1) and observe that using (iv) we have that \(- (\partial_s + \mathcal{L}) \phi = f\), hence we have that:

\[
\phi(x) - \sum_{j=1}^{k} \mathbb{E} \phi(X^{t,x,0,\hat{u}}(\hat{\tau}_j)) = \mathbb{E} \phi(X^{t,x,0,\hat{u}}(\hat{\tau}_{k+1})) = -\mathbb{E} \left[ \int_{t}^{\tau_{k+1}} \left( \frac{\partial \phi}{\partial s} + \mathcal{L} \phi(X^{t,x,0,\hat{u}}(s)) \right) ds \right] = \mathbb{E} \left[ \int_{t}^{\tau_{k+1}} f(X^{t,x,0,\hat{u}}(s)) ds \right]. \tag{33}
\]

Now by definition of the non-local intervention operator \( \mathcal{M} \) and by choice of \( \hat{\xi}_j \in \mathbb{Z} \), we have that:
\[
\phi(X^{t,x,0,\hat{u}}(\hat{\tau}_j)) = \phi(\Gamma(X^{t,x,0,\hat{u}}(\hat{\tau}_j), \hat{\xi}_j)) = \mathcal{M} \left[ \phi(X^{t,x,0,\hat{u}}(\hat{\tau}_j^-)) \right] + c(\hat{\tau}_j, \hat{\xi}_j), \tag{34}
\]
hence after deducting $\phi(\hat{X}^{t,x_0,\hat{u}}(\hat{\tau}^-))$ from both sides we find:

$$M[\phi(\hat{X}^{t,x_0,\hat{u}}(\hat{\tau}^-))] - \phi(\hat{X}^{t,x_0,\hat{u}}(\hat{\tau}^-)) + c(\hat{\tau}^-, \hat{\xi}^-) = \phi(X^{t,x_0,\hat{u}}(\tau_j)) - \phi(\hat{X}^{t,x_0,\hat{u}}(\hat{\tau}^-)), \quad (35)$$

and by (vi) we readily observe that: $\phi(X^{t,x_0,\hat{u}}(\tau_s)) - \phi(\hat{X}^{t,x_0,\hat{u}}(\tau_s)) = 0$, hence after plugging (35) into (33) we obtain the following:

$$\phi(x) - \sum_{j=1}^{k} E[M[\phi(X^{t,x_0,\hat{u}}(\tau^-))] - \phi(X^{t,x_0,\hat{u}}(\tau^-))] - E[\phi(\hat{X}^{t,x_0,\hat{u}}(\hat{\tau}_{k+1}))]
= E \left[ \int_t^{\hat{\tau}_{k+1}} f(X^{t,x_0,\hat{u}}(\tau^-))(s) ds + \sum_{j=1}^{k} c(\hat{\tau}^-, \hat{\xi}^-) \cdot 1_{\{\hat{\tau}^- \leq \tau_s\}} \right].$$

(36)

Note that our choice of $\hat{\xi} \in \mathcal{Z}$ induces equality in (36).

Since the number of interventions in (36) is bounded above by $\mu_{(t,\rho_m \wedge \tau_S)}(\hat{u})$ for some $m < \infty$ and (36) holds for any $k \in \mathbb{N}$, taking the limit as $k \to \infty$ in (36) gives:

$$\phi(x) - \sum_{j=1}^{\mu_{(t,\rho_m \wedge \tau_S)}(\hat{u})} E[M[\phi(X^{t,x_0,\hat{u}}(\tau^-))] - \phi(X^{t,x_0,\hat{u}}(\tau^-))]
= E \left[ \int_t^{\rho_m \wedge \tau_S} f(X^{t,x_0,\hat{u}}(\rho_m \wedge \tau_S))(s) ds + \sum_{j=1}^{\mu_{(t,\rho_m \wedge \tau_S)}(\hat{u})} c(\hat{\tau}^-, \hat{\xi}^-) \cdot 1_{\{\hat{\tau}^- \leq \rho_m \wedge \tau_S\}} \right].$$

(37)

Now $\lim_{m \to \infty} \sum_{j=1}^{\mu_{(t,\rho_m \wedge \tau_S)}(\hat{u})} E[M[\phi(X^{t,x_0,\hat{u}}(\tau^-))] - \phi(X^{t,x_0,\hat{u}}(\tau^-))] = 0$ since also by (vi) we have that $\phi(X^{t,x_0,\hat{u}}(\tau^-)) = 0 \text{ P-a.s.}$ when $\hat{\tau}^- = \tau_s$, we can then deduce the statement by Lemma 4.9 i.e. using the H"{o}lder continuity of the non-local operator $M$. Similarly, we have by (vi) that $\phi(X^{\cdot,\hat{u}}(s)) \to G(X^{\cdot,\hat{u}}(\tau_S)) \cdot 1_{\{\tau_S < \infty\}}$ as $t \to \tau_S^+ \text{ P-a.s.}$.

Now since $\rho_m \wedge \tau_S \to \rho \wedge \tau_S$ as $m \to \infty$, we can exploit the quasi-left continuity of $X$ (for further details see [MDG10] Proposition 1.2.26 and Proposition 1.3.27) and the continuity properties of $f$, we find that there exists some $c > 0$ s.t.:

$$\left| \lim_{m \to \infty} \phi(X^{t,x_0,\hat{u}}(\rho_m \wedge \tau_S)) + \lim_{m \to \infty} \int_t^{\rho_m \wedge \tau_S} f(X^{t,x_0,\hat{u}}(\rho_m \wedge \tau_S))(s) ds \right|
\leq c \left| \lim_{m \to \infty} \left( 1 + |\hat{X}|^{t,x_0,\hat{u}}(\rho_m \wedge \tau_S) \right) + \int_t^{\rho_m \wedge \tau_S} |X^{t,x_0,\hat{u}}(\rho_m \wedge \tau_S)| ds \right|
\leq c(1 + \tau_S)(1 + \sup_{s \in [t,\tau_S]} |X^{t,x_0,\hat{u}}(s)|) \in L^1.$$

Hence, taking the limit as $m \to \infty$ and using the Fatou lemma and (37), we find that:

$$\phi(x) = \sum_{j=1}^{\mu_{(t,\rho_m \wedge \tau_S)}(\hat{u})} E \left[ M[\phi(X^{t,x_0,\hat{u}}(\tau^=_j))] - \phi(X^{t,x_0,\hat{u}}(\tau^=_j))] \right]
+ E[\phi(X^{t,x_0,\hat{u}}(\rho_m \wedge \tau_S))]
+ \int_t^{\rho_m \wedge \tau_S} f(X^{t,x_0,\hat{u}}(s)) ds + \sum_{j=1}^{\mu_{(t,\rho_m \wedge \tau_S)}(\hat{u})} c(\hat{\tau}^-, \hat{\xi}^-) \cdot 1_{\{\hat{\tau}^- \leq \rho_m \wedge \tau_S\}}
= \lim_{m \to \infty} \inf \left[ \sum_{j=1}^{\mu_{(t,\rho_m \wedge \tau_S)}(\hat{u})} E [M[\phi(X^{t,x_0,\hat{u}}(\tau^=_j))] - \phi(X^{t,x_0,\hat{u}}(\tau^=_j))] 
+ \phi(X^{t,x_0,\hat{u}}(\rho_m \wedge \tau_S)) + \int_t^{\rho_m \wedge \tau_S} f(X^{t,x_0,\hat{u}}(s)) ds + \sum_{j=1}^{\mu_{(t,\rho_m \wedge \tau_S)}(\hat{u})} c(\hat{\tau}^-, \hat{\xi}^-) \cdot 1_{\{\hat{\tau}^- \leq \rho_m \wedge \tau_S\}} \right]
\geq \lim_{m \to \infty} \inf \left[ \sum_{j=1}^{\mu_{(t,\rho_m \wedge \tau_S)}(\hat{u})} c(\hat{\tau}^-, \hat{\xi}^-) \cdot 1_{\{\hat{\tau}^- \leq \rho_m \wedge \tau_S\}} \right],$$

where we have used that $\sum_{j=1}^{\mu_{(t,\rho_m \wedge \tau_S)}(\hat{u})} c(\hat{\tau}^-, \hat{\xi}^-) = \sum_{j \geq 1} c(\hat{\tau}^-, \hat{\xi}^-) \cdot 1_{\{\hat{\tau}^- \leq \rho_m \wedge \tau_S\}}$.

Since this holds for all $\rho \in T$ we observe that:

$$\phi(x) \geq \sup_{\rho \in T} \left[ \sum_{j=1}^{\rho \wedge \tau_S} c(\hat{\tau}^-, \hat{\xi}^-) \cdot 1_{\{\hat{\tau}^- \leq \rho \wedge \tau_S\}} \right].$$

(38)
After which we easily deduce that:

$$\phi(x) \geq \inf_{u \in U} \sup_{\rho \in T} \left[ G(X_{t,x_0}, (\rho \wedge \tau_0)) + \int_t^{\rho \wedge \tau_0} f(X_{t,x_0}, u(s)) \, ds + \sum_{j \geq 1} c(\tau_j, \xi_j) \cdot 1_{\{\tau_j \leq \rho \wedge \tau_0\}} \right].$$

(39)

For the second part of the proof, let us fix \( \rho' \in T(t, \tau_0) \) as in (v) and define:

$$\rho_D = \rho_D^0 = \inf_{s \in [t, \tau_0]} \{ s > t; X_{t,s}^\pi (s) \notin D_2 \}. \quad (40)$$

Now we choose a sequence \( \{D_{2,m}\}_{m=1}^\infty \) of open sets s.t. the set \( D_{2,m} \) is compact with \( D_{2,m} \subset D_{2,m+1} \) and \( D_2 = \bigcup_{m=1}^\infty D_{2,m} \) and choose \( \rho_D(m) = m \wedge \inf_{s \geq t} X_{t,s}^\pi (s) \notin D_{2,m}. \) By Dynkin’s formula for jump-diffusion processes and (iii) we have:

$$\phi(x) + \sum_{j=1}^k E[\phi(X_{t,x_0}^\pi(\tau_j)) - \phi(X_{t,x_0}^\pi(\tau_j^-))] = -E \left[ \int_t^{\tau_{k+1}} \frac{\partial \phi}{\partial s} + \mathcal{L}[\phi(X_{t,x_0}^\pi(s))] \, ds \right] - E \left[ \int_t^{\tau_{k+1}} f(X_{t,x_0}^\pi(s), u(s)) \, ds \right]. \quad (41)$$

Hence,

$$\phi(x) + \sum_{j=1}^k E[\phi(X_{t,x_0}^\pi(\tau_j)) - \phi(X_{t,x_0}^\pi(\tau_j^-))] \leq E \left[ \int_t^{\tau_{k+1}} f(X_{t,x_0}^\pi(s), u(s)) \, ds + \phi(X_{t,x_0}^\pi(\tau_{k+1})) \right]. \quad (42)$$

Now by definition of \( \mathcal{M} \) we find that:

$$\phi(X_{t,x_0}^\pi(\tau_j^-)) = \phi(\Gamma(X_{t,x_0}^\pi(\tau_j^-), \xi_j)) \geq \mathcal{M}\phi(X_{t,x_0}^\pi(\tau_j^-)) - c(\tau_j, \xi_j). \quad (44)$$

Subtracting \( \phi(X_{t,x_0}^\pi(\tau_j^-)) \) from both sides of (44) and summing and negating, we find that:

$$\sum_{j=1}^k E[\phi(X_{t,x_0}^\pi(\tau_j)) - \phi(X_{t,x_0}^\pi(\tau_j^-))] \geq k \mathcal{M}\phi(X_{t,x_0}^\pi(\tau_j^-)) - \phi(X_{t,x_0}^\pi(\tau_j^-)) - \sum_{j=1}^k E[c(\tau_j, \xi_j)]. \quad (45)$$

Inserting (45) into (43) gives:

$$\phi(x) + \sum_{j=1}^k E[\mathcal{M}\phi(X_{t,x_0}^\pi(\tau_j^-)) - \phi(X_{t,x_0}^\pi(\tau_j^-))] \leq E \left[ \int_t^{\tau_{k+1}} f(X_{t,x_0}^\pi(s), u(s)) \, ds + \sum_{j=1}^k c(\tau_j, \xi_j) \cdot 1_{\{\tau_j \leq \rho_D(m) \wedge \tau_0\}} \right]. \quad (46)$$

Then letting \( k \to \infty \) in (46) gives:

$$\phi(x) \leq -\sum_{j=1}^{\mu(t, \rho_D(m) \wedge \tau_0)} E[\mathcal{M}\phi(X_{t,x_0}^\pi(\tau_j^-)) - \phi(X_{t,x_0}^\pi(\tau_j^-))] + E[\phi(X_{t,x_0}^\pi(\rho_D(m) \wedge \tau_0))]$$

$$+ \int_t^{\rho_D(m) \wedge \tau_0} f(X_{t,x_0}^\pi(s)) \, ds + \sum_{j \geq 1} c(\tau_j, \xi_j) \cdot 1_{\{\tau_j \leq \rho_D(m) \wedge \tau_0\}}. \quad (47)$$

Again, using the quasi-left continuity of \( X \) we find that:

$$\lim_{m \to \infty} [\mu(t, \rho_D(m) \wedge \tau_0)](u) = \mu(t, \rho_D(u) \wedge \mu(t, \tau_0) \vee \mu(t, \tau_0)) \leq \mu(t, \rho_D(u) \wedge \mu(t, \tau_0), \mu(t, \tau_0)) \leq \mu(t, \rho_D(u) \wedge \mu(t, \tau_0)), \text{ hence}$$

we have that: \( \lim_{m \to \infty} \sum_{j=1}^{\mu(t, \rho_D(m) \wedge \tau_0)} E[\mathcal{M}\phi(X_{t,x_0}^\pi(\tau_j^-)) - \phi(X_{t,x_0}^\pi(\tau_j^-))] = 0. \)
Moreover, as in part (i), using the fact that \(\rho_D(m) \land \tau_S \to \rho_D \land \tau_s\) as \(m \to \infty\), we can deduce the existence of a constant \(c > 0\) s.t.:

\[
\lim_{m \to \infty} \phi(\hat{X}^{t,x_0,u}(\rho_D(m) \land \tau_s)) + \lim_{m \to \infty} \int_t^{\rho_D(m) \land \tau_s} f(X^{t,x_0,u}(s))\,ds \leq c \lim_{m \to \infty} (1 + |\hat{X}^{t,x_0,u}(\rho_D(m) \land \tau_s)|) + \int_t^{\rho_D(m) \land \tau_s} |X^{t,x_0,u}(s)|\,ds
\]

\[
\leq c(1 + \tau_S)(1 + \sup_{s \in [t,\tau_S]} |X^{t,x_0,u}(s)|) \in \mathbb{L}^1.
\]

Moreover, using (vi), we observe that: \(\lim_{m \to \infty} \phi(\hat{X}^{t,x_0,u}(\rho_D(m))) = \phi(X^{t,x_0,u}(\rho_D))\). Hence, by the dominated convergence theorem after taking the limit \(m \to \infty\) in (47) we find that:

\[
\phi(x) \leq \mathbb{E}
\left[
\int_t^{\rho_D \land \tau_s} f(X^{t,x_0,u}(s))\,ds + \sum_{j \geq 1} c(\tau_j, \xi_j) \cdot 1_{\{\tau_j \leq \rho_D \land \tau_s\}} + G(\hat{X}^{t,x_0,u}(\rho_D \land \tau_S))\right].
\]

(48)

Since this holds for all \(u \in U\) we have that:

\[
\phi(x) \leq \inf_{u \in U} \mathbb{E}
\left[
\int_t^{\rho_D \land \tau_s} f(X^{t,x_0,u}(s))\,ds + \sum_{j \geq 1} c(\tau_j, \xi_j) \cdot 1_{\{\tau_j \leq \rho_D \land \tau_s\}} + G(\hat{X}^{t,x_0,u}(\rho_D \land \tau_S))\right],
\]

(49)

from which clearly we have that:

\[
\phi(x) \leq \sup_{\rho \in T} \inf_{u \in U} \mathbb{E}
\left[
\int_t^{\rho \land \tau_s} f(X^{t,x_0,u}(s))\,ds + \sum_{j \geq 1} c(\tau_j, \xi_j) \cdot 1_{\{\tau_j \leq \rho \land \tau_s\}} + G(\hat{X}^{t,x_0,u}(\rho \land \tau_s))\right].
\]

(50)

where we observe that by (50) and (51) we can conclude that:

\[
\inf_{u \in U} \sup_{\rho \in T} J^{(u,\rho)}(x) \leq \phi(x) \leq \sup_{\rho \in T} \inf_{u \in U} J^{(u,\rho)}(x).
\]

(51)

However, since for all \(u \in U, \rho \in T\) and \(x \in \mathbb{R}_p\) we have:

\[
\inf_{u \in U} (\sup_{\rho \in T} J^{(u,\rho)}(x)) \geq \sup_{\rho \in T} (\inf_{u \in U} J^{(u,\rho)}(x)).
\]

Moreover, choosing \(u = \hat{u}\) in (51), by (iv) we find equality, hence:

\[
\phi(x) = \mathbb{E}
\left[
\int_t^{\hat{\rho} \land \tau_s} f(X^{t,x_0,u}(s))\,ds + \sum_{j \geq 1} c(\hat{\tau}_j, \hat{\xi}_j) \cdot 1_{\{\hat{\tau}_j \leq \hat{\rho} \land \tau_s\}} + G(\hat{X}^{t,x_0,u}(\hat{\rho} \land \tau_s))\right].
\]

(52)

from which we find that:

\[
\phi(x) = \inf_{u \in U} \sup_{\rho \in T} J^{(u,\rho)}(x) = \sup_{\rho \in T} \inf_{u \in U} J^{(u,\rho)}(x),
\]

(53)

and hence we deduce the thesis.

**Corollary 5.3.** The sample space splits into three regions that represent a region in which player I applies impulse interventions \(I_1\), a region for player II stops the game \(I_2\), and a region \(I_3\) in which no action is taken by neither player; moreover the three regions are characterised by the following expressions:

\[
I_1 = \{x \in [t, \tau_S] \times \mathbb{R}_p : V(x) = MV(x), LV(x) + f(x) \geq 0\},
\]

\[
I_2 = \{x \in [t, \tau_S] \times \mathbb{R}_p : V(x) = G(x), LV(x) + f(x) \leq 0\},
\]

\[
I_3 = \{x \in [t, \tau_S] \times \mathbb{R}_p : V(x) < MV(x), V(x) > G(x); LV(x) + f(x) = 0\}.
\]

## 6 Stochastic Differential Games of Impulse Control and Stopping with Non-Zero-Sum Payoff

In this section, we study the game as studied in section 5 however we now extend the results to a non-zero-sum stochastic differential game. The results of this section are loosely
based on [KZ08] where we make the necessary adjustments to accommodate both impulse controls and the action of the stopper. We start by proving a non-zero-sum verification theorem for the game in which both players use impulse controls to modify the state process and lastly adapt the impulse controller-stopper game in section 2 to the non-zero-sum setting.

Suppose firstly that the uncontrolled passive state \( X \in S \subset \mathbb{R}^p (p \in \mathbb{N}) \), evolves according to a (jump-)diffusion on \( (C([t, \tau_S]; \mathbb{R}^p), (\mathcal{F}_{t,s})_{s \in [t, \tau_S]}, F, \mathbb{P}_0) \) as in sections 2 and 3.

We decouple the objective performance functionals so that we now consider the following payoff functionals:

\[
J_1^{(u, \rho)}(x) = E \left[ \int_t^{\rho \wedge \tau_S} f_1(X^{t, x_0, u}(s)) ds - \sum_{j \geq 1} c_1(\tau_j, \xi_j) \cdot 1_{\{\tau_j \leq \rho \wedge \tau_S\}} + G_1(X^{t, x_0, u}(\rho \wedge \tau_S)) \right] \tag{54}
\]

\[
J_2^{(u, \rho)}(x) = E \left[ \int_t^{\rho \wedge \tau_S} f_2(X^{t, x_0, u}(s)) ds - \sum_{j \geq 1} c_2(\tau_j, \xi_j) \cdot 1_{\{\tau_j \leq \rho \wedge \tau_S\}} + G_2(X^{t, x_0, u}(\rho \wedge \tau_S)) \right],
\tag{55}
\]

where \((t, x) \in [t, \tau_S] \times \mathbb{R}^p, \tau_j \in [t, \tau_S]; \xi_j \in \mathcal{Z}\) are \( \mathcal{F} \)-measurable intervention values \( \forall j \in \mathbb{N} \) and \( u \in \mathcal{U} \) is an admissible controls for player I. The cost functions \( c_1 \) and \( c_2 \) share the same properties as \( c \) in section 2 we assume also that the functions \( G_1 \) and \( G_2 \) are Lipschitz continuous and bounded.

We can observe the functional \( J_1^{(u, \rho)}(x) \) (resp., \( J_2^{(u, \rho)}(x) \)) defines the payoff received by the player I (resp., player II) during the game with initial point \( x \in [t, \tau_S] \times \mathbb{R}^p \) when player I uses the control \( u \in \mathcal{U} \) and player II decides to stop the game at time \( \rho \in \mathcal{T} \).

Since we are now handling a game with a non-zero-sum payoff structure, we must adapt the definition of the non-local intervention operator \( \mathcal{M}_1 : \mathcal{H} \to \mathcal{H} \) (c.f. definition 4.6) to

\[
\mathcal{M}_1[\phi] := \sup_{x \in \mathcal{Z}} \{ \phi(\Gamma(X(\tau-), z)) - c(\tau, z) \cdot 1_{\{\tau_1 \leq \tau_S\}} \},
\tag{56}
\]

where \( \phi : [t, \tau_S] \times \mathbb{R}^p \to \mathbb{R} \) is s.t. \( \phi \in C([t, \tau_S]; \mathbb{R}^p) \), \( \tau \) is some \( \mathcal{F} \)-measurable stopping time and as before, \( \gamma : \mathbb{R}^p \times \mathcal{Z} \to \mathbb{R} \) is the impulse response function.

**Definition 6.1.** [Nash Equilibrium] We say that a pair \((\hat{u}, \hat{\rho}) \in \mathcal{U} \times \mathcal{T}\) is a Nash equilibrium of the stochastic differential game with impulse controls \( u(s) = \sum_{j \geq 1} \xi_j \cdot 1_{\{\tau_j < \tau_S\}}(s) \in \mathcal{U} \) for all \( s \in [t, \tau_S] \) if the following statements hold:

\[
J_1^{(\hat{u}, \hat{\rho})}(x) \geq J_1^{(u, \rho)}(x) \tag{57}
\]

\forall u \in \mathcal{U} \text{ and } \forall x \in [t, \tau_S] \times \mathbb{R}^p,

\[
J_2^{(\hat{u}, \hat{\rho})}(x) \geq J_2^{(u, \rho)}(x) \tag{58}
\]

\forall \rho \in \mathcal{T} \text{ and } \forall x \in [t, \tau_S] \times \mathbb{R}^p.

Condition (i) states that given some fixed player II stopping time \( \hat{\rho} \in \mathcal{T} \), player I cannot profitably deviate from playing the control policy \( \hat{u} \in \mathcal{U} \). Analogously, condition (ii) is the equivalent statement given the player I’s control policy is fixed as \( \hat{u} \), player II cannot profitably deviate from \( \hat{\rho} \in \mathcal{T} \). We therefore see that \((\hat{u}, \hat{\rho}) \in \mathcal{U} \times \mathcal{T}\) is an equilibrium in the sense of a Nash equilibrium since neither player has an incentive to deviate given their opponent plays the equilibrium policy.

As in [Mk07], we generalise our zero-sum Theorem 5.1 to cover non-zero-sum payoff structure with the use of a Nash Equilibrium solution concept.

As in the zero-sum case, we can give a heuristic motivation of the key features of the verification theorem for the game when the payoff structure is non-zero-sum by studying the complete repertoire of tactics that each player can employ throughout the horizon of the game (see supplementary material).

**Theorem 6.2.** [Verification Theorem for Non-Zero-Sum Controller-Stopper Games with Impulse Control]

Let \( \tau_j, \rho \in \mathcal{T} \) be \( \mathcal{F} \)-measurable stopping times. Denote by \( X^{\cdot, u} \equiv X \) for any \( u \in \mathcal{U} \) and suppose that there exist functions \( \phi, i \in \{1, 2\} \) s.t. conditions (T1) - (T4) hold (see appendix) and additionally:

(i’) \( \phi \in C^{1,2}([t, \tau_S], S) \cap C([t, \tau_S], \hat{S}) \).

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(ii') $\phi_1 \geq M_1 \phi_1$ on $S$ and $\phi_2 \geq G_2(X)$ on $S$
and the regions $D_1$ and $D_2$ are defined by: $D_1 = \{ X \in S; \phi_1(X) > M_1 \phi_1(X) \}$ and
$D_2 = \{ X \in S; \phi_2(X) > G_2(X) \}$ where we refer to $D_1$ (resp., $D_2$) as the player I (resp.,
player II) continuation region.

(iii') $\frac{\partial \phi_1}{\partial s} + \mathcal{L} \phi_1(X^{t,u(s)}) + f_1(X^{t,u(s)}) \leq \frac{\partial \phi_1}{\partial s} + \mathcal{L} \phi_1(X^{t,u(s)}) + f_1(X^{t,u(s)}) \leq 0$
for $X \in S \setminus \partial D_1$

(iv') $\frac{\partial \phi_1}{\partial s} + \mathcal{L} \phi_1(X^{t,u(s)}) + f_1(X^{t,u(s)}) = 0$ in $D_1, i \in \{1, 2\}$.

(v') For $u \in U$ define $\rho_D = p_D^u = \inf\{s > t, X^{t,u(s)} \notin D_2 \}$ and specifically, $\rho_D = \hat{\rho} = \inf\{s > t, X^{t,u(s)} \notin D_2 \}$

(vi') $X^{t,u}(\tau_S) \in \mathcal{D}, \mathbb{P}$-a.s. on $\tau_S < \infty$ and $\phi_i(X^{t,u}(s)) \rightarrow G_i(X^{t,u}(\tau_S \wedge \rho)) \cdot 1_{\{\tau_S < \infty\}}$ as $s \rightarrow \tau_S \wedge \rho$, $\mathbb{P}$-a.s., $i \in \{1, 2\}$ $\forall X \in S, u \in U$.

Put $\hat{\tau}_0 \equiv t$ and define $\hat{u} \in [\hat{\tau}_j, \hat{\xi}_j]_{j \in \mathbb{N}}$ inductively by $\hat{\tau}_{j+1} = \inf\{s > \tau_j; X^{t,u}(s) \notin D_1 \} \wedge (\tau_S \wedge \rho)$, then $(\hat{u}, \hat{\rho}) \in U \times \mathcal{T}$ is a Nash equilibrium for the game; that is to say that we have $\forall x \in [t, \tau_S] \times \mathbb{R}^p$:

$$\phi_1(x) = \sup_{u \in U} J_1^{(u, \hat{\rho})}(x) = J_1^{(\hat{u}, \hat{\rho})}(x)$$

and

$$\phi_2(x) = \sup_{\rho \in \mathcal{T}} J_2^{(\hat{u}, \rho)}(x) = J_2^{(\hat{u}, \hat{\rho})}(x).$$

Proof

As in the proof of Theorem 5.1, let us fix the player II control $\hat{\rho} \in \mathcal{T}$; we firstly appeal
to the Dynkin formula for jump-diffusions, hence for $X = X^{t,u}$ we have the following:

$$\mathbb{E}[\phi_1(X^{t,x_0,u}(\tau_{j+1}))] - \mathbb{E}[\phi_1(X^{t,x_0,u}(\tau_j))] = \mathbb{E}\left[\int_{\tau_j}^{\tau_{j+1}} \frac{\partial \phi_1}{\partial s} + \mathcal{L} \phi_1(X^{t,x_0,u}(s))ds\right].$$

(61)

Summing (61) from $j = 0$ to $j = k$ for some $k: t < \tau_{k+1} < \hat{\rho}$ implies that:

$$- \phi_1(x) - \sum_{j=1}^{k} \mathbb{E}[\phi_1(X^{t,x_0,u}(\tau_j)) - \phi_1(X^{t,x_0,u}(\tau_j^-))] + \mathbb{E}[\phi_1(X^{t,x_0,u}(\tau_{k+1}^-))]$$

$$E\left[\int_{t}^{\tau_{k+1}} \frac{\partial \phi_1}{\partial s} + \mathcal{L} \phi_1(X^{t,x_0,u[t,t]})ds\right].$$

(62)

Now by (iii') we have that:

$$\frac{\partial \phi_1}{\partial s} + \mathcal{L} \phi_1(X^{t,x_0,u[t,t]}) \leq \frac{\partial \phi_1}{\partial s} + \mathcal{L} \phi_1(X^{t,x_0,u[t,t]}) + (f_1(X^{t,x_0,u(t)}(s)) - f_1(X^{t,x_0,u(t)}(s))) \leq -f_1(X^{t,x_0,u[t,t]})$$

Hence inserting (63) into (62) yields

$$- \phi_1(x) - \sum_{j=1}^{k} \mathbb{E}[\phi_1(X^{t,x_0,u}(\tau_j)) - \phi_1(X^{t,x_0,u}(\tau_j^-))] + \mathbb{E}[\phi_1(X^{t,x_0,u}(\tau_{k+1}^-))]$$

$$E\left[\int_{t}^{\tau_{k+1}} \frac{\partial \phi_1}{\partial s} + \mathcal{L} \phi_1(X^{t,x_0,u[t,t]})ds\right] \leq - E\left[\int_{t}^{\tau_{k+1}} f_1(X^{t,x_0,u[t,t]}(s))ds\right].$$

(64)

Or equivalently:

$$\phi_1(x) + \sum_{j=1}^{k} \mathbb{E}[\phi_1(X^{t,x_0,u}(\tau_j)) - \phi_1(X^{t,x_0,u}(\tau_j^-))] - \mathbb{E}[\phi_1(X^{t,x_0,u}(\tau_{k+1}^-))]$$

$$E\left[\int_{t}^{\tau_{k+1}} f_1(X^{t,x_0,u[t,t]}(s))ds\right].$$

(65)
We now use analogous arguments to (64) - (65). Indeed, by definition of $\mathcal{M}_1$ we find that:
\begin{equation}
\phi(\hat{X}^{t,x_0,u}(\tau_j)) = \phi(\Gamma(\hat{X}^{t,x_0,u}(\tau_j^-)), \xi_j) \leq \mathcal{M}_1 \phi(\hat{X}^{t,x_0,u}(\tau_j^-)) + c(\tau_j, \xi_j).
\end{equation}
(66)
After subtracting $\phi(\hat{X}^{t,x_0,u}(\tau_j^-))$ from both sides of (66), summing then negating, we find that:
\begin{equation}
\sum_{j=1}^{k} \mathbb{E}[\phi(\hat{X}^{t,x_0,u}(\tau_j)) - \phi(\hat{X}^{t,x_0,u}(\tau_j^-))]
\leq \sum_{j=1}^{k} \mathbb{E}[\mathcal{M}_1 \phi(\hat{X}^{t,x_0,u}(\tau_j^-)) - \phi(\hat{X}^{t,x_0,u}(\tau_j^-)) + \sum_{j=1}^{k} \mathbb{E}[c(\tau_j, \xi_j)].
\end{equation}
(67)
After inserting (67) into (65) we find that:
\begin{align*}
\phi_1(x) &\geq \mathbb{E} \left[ \phi_1(\hat{X}^{t,x_0,u}(\tau_{k+1}^-)) - \sum_{j=1}^{k} [\mathcal{M}_1 \phi_1(X^{t,x_0,u}(\tau_j^-)) - \phi_1(\hat{X}^{t,x_0,u}(\tau_j^-))] + \int_{t}^{T_{k+1}} f_1(X^{t,x_0,u(\tau_j^-)}(s))ds \right] \\
&\geq \mathbb{E} \left[ \phi_1(\hat{X}^{t,x_0,u}(\tau_{k+1}^-)) - \sum_{j=1}^{k} [\mathcal{M}_1 \phi_1(X^{t,x_0,u}(\tau_j^-)) - \phi_1(\hat{X}^{t,x_0,u}(\tau_j^-))] \\
&\quad - \sum_{j=1}^{k} c_1(\tau_j, \xi_j) + \int_{t}^{T_{k+1}} f_1(X^{t,x_0,u(\tau_j^-)}(s))ds \right] \\
&+ \int_{t}^{T_{k+1}} f_1(X^{t,x_0,u(\tau_j^-)}(s))ds - \sum_{j=1}^{k} c_1(\tau_j, \xi_j) \cdot 1_{\{\tau_j \leq \rho \land \tau_k\}}.
\end{align*}
(68)
Define $\hat{\rho}_m \equiv \hat{\beta}_m(u) = \hat{\rho} \land \tau m; m = 1, 2, \ldots$. As in the zero-sum case, since the number of interventions in (65) is bounded above by $\mu(\hat{\rho}_m \land \tau m)(u) \land m$ for some $m < \infty$ and (68) holds for any $k \in \mathbb{N}$, taking the limit as $k \to \infty$ in (65) gives:
\begin{align*}
\phi_1(x) &\geq \mathbb{E} \left[ \phi_1(\hat{X}^{t,x_0,u}(\tau_{k+1}^-)) - \sum_{j=1}^{k} [\mathcal{M}_1 \phi_1(X^{t,x_0,u}(\tau_j^-)) - \phi_1(\hat{X}^{t,x_0,u}(\tau_j^-))] \\
&\quad + \int_{t}^{T_{k+1}} f_1(X^{t,x_0,u(\tau_j^-)}(s))ds - \sum_{j=1}^{k} c_1(\tau_j, \xi_j) \cdot 1_{\{\tau_j \leq \rho \land \tau_k\}} \right].
\end{align*}
(69)
Now, $\lim_{m \to \infty} \sum_{j=1}^{\mu(\hat{\rho}_m \land \tau m)(u) \land m} \mathbb{E}[\mathcal{M}_1 \phi_1(X^{t,x_0,u}(\tau_j^-)) - \phi_1(\hat{X}^{t,x_0,u}(\tau_j^-))] \not= 0$
and
\begin{align*}
\lim_{m \to \infty} \mathbb{E}[\phi_1(X^{t,x_0,u}(\tau_{\mu(\hat{\rho}_m \land \tau m)(u) \land m}))] = \phi_1(X^{t,x_0,u}(\hat{\rho} \land \tau m)) = G_1(\hat{X}^{t,x_0,u}(\hat{\rho} \land \tau m)).
\end{align*}
Indeed, by (v') we have that $\lim_{m \to \infty} \mathbb{E}[\phi_1(X^{t,x_0,u}(\tau_{\mu(\hat{\rho}_m \land \tau m)(u) \land m}))] = \phi_1(X^{t,x_0,u}(\hat{\rho} \land \tau m)).$
Thus, after taking the limit $m \to \infty$ in (69) and noting that by definition, $\lim_{m \to \infty} \hat{\rho}_m = \hat{\rho}$, we have that:
\begin{align*}
\phi_1(x) &\geq \mathbb{E} \left[ \int_{t}^{\hat{\rho} \land \tau m} f_1(X^{t,x_0,u(\tau_j^-)}(s))ds - \sum_{j=1}^{\hat{\rho} \land \tau m} c_1(\tau_j, \xi_j) \cdot 1_{\{\tau_j \leq \hat{\rho} \land \tau m\}} + G_1(X^{t,x_0,u}(\hat{\rho} \land \tau m)) \right].
\end{align*}
(70)
Since this holds for all $u \in \mathcal{U}$ we find:
\begin{align*}
\phi_1(x) &\geq \sup_{u \in \mathcal{U}} \mathbb{E} \left[ G_1(\hat{X}^{t,x_0,u(\hat{\rho} \land \tau m)}(s))ds - \sum_{j=1}^{\hat{\rho} \land \tau m} c_1(\tau_j, \xi_j) \cdot 1_{\{\tau_j \leq \hat{\rho} \land \tau m\}} \right].
\end{align*}
(71)
Hence, we find that $\forall x \in [t, \tau_S] \times \mathbb{R}^p$:
\begin{align*}
\phi_1(x) &\geq \sup_{u \in \mathcal{U}} j_1^{(u, \hat{\rho})}(x).
\end{align*}
(72)
Now, applying the above arguments with the controls $(\hat{\nu}, \hat{\rho})$ yields the following equality
$\forall x \in [t, \tau_S] \times \mathbb{R}^p$:
\begin{align*}
\phi_1(x) = \sup_{\rho \in T} j_1^{(\hat{u}, \rho)}(x) = J_1^{(\hat{u}, \hat{\rho})}(x).
\end{align*}
(73)
To prove (50) - (60), we firstly fix $\hat{u} \in \mathcal{U}$ as in (iv’), we again define $\rho_m = \rho \land m; m = 1, 2, \ldots$. Now, by the Dynkin formula for jump diffusions and by (iv’) and (61) - (62), we have that:

\[
\begin{aligned}
E[\phi_2(\hat{X}^t,x_0,\hat{u}(\rho_m))] - \phi_2(x) - \sum_{j=1}^{\mu_{t,\rho_m}(\hat{u})} E[\phi_2(\hat{X}^t,x_0,\hat{u}(\hat{\tau}_j))] & = E\left[\int_t^{\tau_{\mu_{t,\rho_m}(\hat{u})}} \left(\frac{\partial \phi_2}{\partial s} + \mathcal{L}\phi_2(\hat{X}^t,x_0,\hat{u}(\hat{s}))\right) ds\right] = -E\left[\int_t^{\tau_{\mu_{t,\rho_m}(\hat{u})}} f_2(\hat{X}^t,x_0,\hat{u}(\hat{s})) ds\right],
\end{aligned}
\]

which (as before, similar to (55)) and by our choice of $\hat{\xi}_j \in \mathcal{Z}$, implies

\[
\phi_2(x) + \sum_{j=1}^{\mu_{t,\rho_m}(\hat{u})} E[\mathcal{M}_1 \phi_2(\hat{X}^t,x_0,\hat{u}(\hat{\tau}_j))] - \phi_2(\hat{X}^t,x_0,\hat{u}(\hat{\tau}_j)) = E\left[\phi_2(\hat{X}^t,x_0,\hat{u}(\hat{\tau}_{\mu_{t,\rho_m}(\hat{u})})) + \int_t^{\tau_{\mu_{t,\rho_m}(\hat{u})}} f_2(\hat{X}^t,x_0,\hat{u}(s)) ds - \sum_{j=1}^{\mu_{t,\rho_m}(\hat{u})} c_2(\hat{\tau}_j, \hat{\xi}_j) \cdot 1\{\hat{\tau}_j \leq \rho_m\} \right],
\]

which we may rewrite as

\[
\begin{aligned}
\phi_2(x) & = E\left[\phi_2(\hat{X}^t,x_0,\hat{u}(\hat{\tau}_{\mu_{t,\rho_m}(\hat{u})})) + \int_t^{\tau_{\mu_{t,\rho_m}(\hat{u})}} f_2(\hat{X}^t,x_0,\hat{u}(s)) ds \right.
\end{aligned}
\]

\[
\begin{aligned}
& - \sum_{j=1}^{\mu_{t,\rho_m}(\hat{u})} c_2(\hat{\tau}_j, \hat{\xi}_j) \cdot 1\{\hat{\tau}_j \leq \tau_{\mu_{t,\rho_m}(\hat{u})}\} - \sum_{j=1}^{\mu_{t,\rho_m}(\hat{u})} E[\mathcal{M}_1 \phi_2(\hat{X}^t,x_0,\hat{u}(\hat{\tau}_j))] - \phi_2(\hat{X}^t,x_0,\hat{u}(\hat{\tau}_j))] + \sum_{j=1}^{\mu_{t,\rho_m}(\hat{u})} E[\mathcal{M}_1 \phi_2(\hat{X}^t,x_0,\hat{u}(\hat{\tau}_j))] - \phi_2(\hat{X}^t,x_0,\hat{u}(\hat{\tau}_j))] \right].
\end{aligned}
\]

(74)

Now, since $\mu_{t,\rho_m}(\hat{u}) \to \mu_{t,\rho \land \tau} \hat{u}$ as $m \to \infty$ and $\lim_{a \to \tau} \phi_i(\hat{X}^t,x_0,\hat{u}(s)) = G_i(\hat{X}^t,x_0,\hat{u}(\tau_S))$, $i \in \{1, 2\}$ using (v) and $\hat{\tau}_{\rho \land \tau} \equiv \rho \land \tau_S$ then using (74) and by the Fatou lemma we find that:

\[
\begin{aligned}
\phi_2(x) & \geq \lim_{m \to \infty} \inf_{\tau \in \mathcal{T}} E\left[\phi_2(\hat{X}^t,x_0,\hat{u}(\hat{\tau}_{\mu_{t,\rho_m}(\hat{u})})) \right.
\end{aligned}
\]

\[
\begin{aligned}
& + \int_t^{\tau_{\mu_{t,\rho_m}(\hat{u})}} f_2(\hat{X}^t,x_0,\hat{u}(s)) ds \right.
\end{aligned}
\]

\[
\begin{aligned}
& - \sum_{j=1}^{\mu_{t,\rho_m}(\hat{u})} c_2(\hat{\tau}_j, \hat{\xi}_j) \cdot 1\{\hat{\tau}_j \leq \tau_{\mu_{t,\rho_m}(\hat{u})}\} - \sum_{j=1}^{\mu_{t,\rho_m}(\hat{u})} E[\mathcal{M}_1 \phi_2(\hat{X}^t,x_0,\hat{u}(\hat{\tau}_j))] - \phi_2(\hat{X}^t,x_0,\hat{u}(\hat{\tau}_j))] + \sum_{j=1}^{\mu_{t,\rho_m}(\hat{u})} E[\mathcal{M}_1 \phi_2(\hat{X}^t,x_0,\hat{u}(\hat{\tau}_j))] - \phi_2(\hat{X}^t,x_0,\hat{u}(\hat{\tau}_j))] \right].
\end{aligned}
\]

(75)

Since this holds for all $\rho \in \mathcal{T}$ we find that:

\[
\phi_2(x) \geq \sup_{\rho \in \mathcal{T}} E\left[G_2(\hat{X}^t,x_0,\hat{u}(\rho \land \tau_S)) + \int_t^{\rho \land \tau_S} f_2(\hat{X}^t,x_0,\hat{u}(s)) ds - \sum_{j=1}^{\rho \land \tau_S} c_2(\hat{\tau}_j, \hat{\xi}_j) \cdot 1\{\hat{\tau}_j \leq \rho \land \tau_S\} \right].
\]

(76)

Hence, we find that $\forall \ x \in [t, \tau_S] \times \mathcal{R}^p$

\[
\phi_2(x) \geq \sup_{\rho \in \mathcal{T}} G(\hat{u}, \rho) (x).
\]

(77)

Now, applying the above arguments with the controls $(\hat{u}, \hat{\rho})$ yields the following equality $\forall \ x \in [t, \tau_S] \times \mathcal{R}^p$:

\[
\phi_2(x) = \sup_{u \in \mathcal{U}} J(\hat{u}, \hat{\rho})(x) \quad (78)
\]

We therefore observe using (63) in conjunction with (63) and that $(\hat{u}, \hat{\rho})$ is a Nash equilibrium and hence the thesis is proven.

In full analogy to Corollary 5.3, we can readily arrive at the following corollary to Theorem 5.3:

**Corollary 6.4.** The sample space splits into three regions that represent a region in which the controller performs impulse interventions $\mathcal{I}_1$, a region in which the stopper stops
the process $I_2$ and a region in which no action is taken by either player $I_3$; moreover the three regions are characterised by the following expressions:

\[
I_1 = \left\{ x \in [t, \tau_S] \times \mathbb{R}^p : V_1(x) = M_1 V_1(x), \mathcal{L}V_1(x) + f_1(x) \geq 0 \right\},
\]

\[
I_2 = \left\{ x \in [t, \tau_S] \times \mathbb{R}^p : V_2(x) = G_2(x), \mathcal{L}V_2(x) + f_2(x) \geq 0 \right\},
\]

\[
I_3 = \left\{ x \in [t, \tau_S] \times \mathbb{R}^p : V_1(x) < M_1 V_1(x), V_2(x) < G_1(x); \mathcal{L}V_2(x) + f_2(x) = 0, j \in \{1, 2\} \right\}.
\]

## 7 The Optimal Liquidity Control and Lifetime Ruin Problem

We now revisit the optimal liquidity control and lifetime ruin problem in section 4 and solve the model presented in section 2. In the following analysis, we use the results of the stochastic differential game of impulse control and stopping to solve our model. Before stating results, using (14) and (2), we firstly make the following observation on the stochastic generator $\mathcal{L}^\theta$ which is given by the following expression $(s, x, y, q) \in [t, T] \times S^2 \times \mathbb{R}$:

\[
\mathcal{L}^\theta \psi(s, \cdot) = (r - e) x \frac{\partial \psi}{\partial x} + \left[ \Gamma y \frac{\partial \psi}{\partial y} + \frac{1}{2} \sigma^2 \frac{\partial^2 \psi}{\partial x^2} + \frac{1}{2} \sigma^2 \frac{\partial^2 \psi}{\partial y^2} + 2 \sigma \gamma x \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y} - \frac{1}{2} q^2 \frac{\partial^2 \psi}{\partial y^2} \right] x - \psi(s, x, y, q) - x \gamma f(s, y, q) + \theta h(s, y) \nu(dz).
\]

We now restate Theorem 7.1:

**Theorem 7.1.** Suppose that the firm’s liquidity process $x$ evolves according to (2) and suppose that the investor’s wealth process $y$ evolves according to (5) then the sequence of optimal capital injections $(\tilde{t}, \tilde{Z}) \equiv [\tilde{t}_j, \tilde{z}_j]_{j \in \mathbb{N}} \equiv \sum_{j \geq 1} \tilde{z}_j \cdot 1_{\{\tilde{t}_j \leq \rho \wedge T\}}(s)$ is characterised by the investment times $\{\tilde{t}_j\}_{j \in \mathbb{N}}$ and magnitudes $\{\tilde{z}_j\}_{j \in \mathbb{N}}$ where $[\tilde{t}_j, \tilde{z}_j]_{j \in \mathbb{N}}$ are constructed recursively via the following expressions:

(i) $\tilde{t}_0 \equiv t_0$ and $\tilde{t}_{j+1} = \inf\{s > \tau_S : Y^{(t, \tilde{Z})_{[\cdot, \cdot]}(s)}(x) \geq \tilde{y}\} \in \mathbb{R}, Y \in S \} \wedge \hat{\rho},$

(ii) $\tilde{z}_j \equiv \tilde{y} - y(\tilde{t}_j)$.

The fixed duplet $(\tilde{y}, \tilde{y})$ is determined by the following equations:

\[
\phi_2(\tilde{y}) = \alpha_1,
\]

\[
\phi_2(\tilde{y}) = \phi_2(0)(\tilde{y}) - (\kappa_1 + \alpha_1(\tilde{y} - y)),
\]

\[
\phi_2(\tilde{y}) = \alpha_1.
\]

where the function $\phi_2$ is given by (80) and the function $\phi_{2,0}$ is given by:

\[
\phi_{2,0}(x) = c(y^{d_1} - y^{d_2}),
\]

where the constants $d_1, d_2, c \in \mathbb{R}$ are such as in (158) and (159) - (161).

The investor’s non-investment region is given by:

\[
D_2 = \left\{ Y \in S, s \in [t, T]; Y(s) < \tilde{y} \right\},
\]

(84)

The investor exits the market at $\hat{\rho} \in \mathcal{T}$ where the exit time is defined by:

\[
\hat{\rho} = \inf\{s > t; X^{(t, \tilde{Z})_{[\cdot, \cdot]}(s)}(s) \cdot Q(s) \notin D_1 | X \in S, Q \in \mathbb{R} \} \wedge \tau_S,
\]

(85)

where the process Q is determined by the optimal choice of $\tilde{\theta} = \tilde{\theta}_0, \tilde{\theta}_1$ and the set $D_1$ (non-stopping region) is defined by:

\[
D_1 = \left\{ \left( \frac{\partial \psi}{\partial s} + \mathcal{L}^\theta \right) \psi(s, \cdot) > 0 \right\} \left\{ s > t \right\}.
\]

(86)
where the operator $\mathcal{L}^\hat{\theta}$ corresponds to the stochastic generator of the controlled process.

From Theorem 7.1 we immediately arrive at the following result:

**Corollary 7.2.** For the optimal liquidity control and lifetime ruin problem, the investor’s wealth process $x$ lies within a space that splits into three regions: a region in which the investor performs a capital injection - $I_1$, a region in which no action is taken - $I_2$ and lastly a region in which the investor exits the market by selling all firm holdings - $I_3$. Moreover, the three regions are characterised by the following expressions:

$$
I_1 = \{ y \geq \hat{y}|y, \hat{y} \in S \},
$$
$$
I_2 = \{ qx > \omega^*, y < \hat{y}|x, y, \hat{y} \in S; \omega^*, q \in \mathbb{R} \},
$$
$$
I_3 = \{ qx \leq \omega^*|x \in S; q, \omega^* \in \mathbb{R} \},
$$

where $q$ is the value of the process $Q$ and the fixed duplet $(\hat{y}, \hat{y})$ and the value $\omega^*$ are determined by (21) - (23) and (171) respectively.

**Theorem 7.3.** The investor’s problem reduces to the following double obstacle variational inequality:

$$
\inf \{ \sup [\psi(s, \cdot) - (\kappa_1 + \alpha_1(\hat{y} - y)), - \left( \frac{\partial}{\partial s} + \mathcal{L}^\hat{\theta} \right) \psi(s, \cdot)], \psi(s, \cdot) - G(s, \cdot) \} = 0, \quad (87)
$$

where $G(s, \cdot) = e^{-\delta s}(g_1xq + \lambda T + g_2y)$.

The investor’s optimal stopping time $\hat{\rho} \in [t, T]$ is given by:

$$
\hat{\rho} = \inf \{ s > t; X(s)Q(s) \notin D_1 | s \in [t, T] \} \wedge \tau_S, \quad (88)
$$

where the set $D_1$ is the investor’s non-stopping region defined by (25).

Theorem 7.1 and Theorem 7.3 are underpinned by the following results:

**Lemma 7.4.** The optimal choice of $\hat{\theta} = (\hat{\theta}_0, \hat{\theta}_1)$ corresponds to the measure $Q$ which is defined by:

$$
dQ = \text{d}Qd\hat{\theta} \quad (89)
$$

and the process $Q$ is determined by the expression $\forall s \in [t, T]$:

$$
Q(s) = Q(t) \exp \left\{ \frac{1}{2} \sigma^2_s - \sigma_f B_f(s) + \int_t^s \int_\mathbb{R} \left( \ln(1 + \hat{\theta}_1(r, z)) - \hat{\theta}_1(r, z) \right) N_1(\text{d}r, \text{d}z) \right\}, \quad (90)
$$

where $\hat{\theta}_1$ is a solution to the equation $H(\psi) = 0$ where $H$ is given by:

$$
H(\psi) = \int_\mathbb{R} \left( \| \Xi(\psi) \| - 1 \right) \nu(dz), \quad (91)
$$

where $\Xi(\psi) := (1 - \hat{\theta}_1(z))(1 + \gamma_f(z))$ and $k$ is a solution to (91).

The following result provides a complete characterisation of the investor’s value function:

**Proposition 7.5** The value function $\psi$ for the investor’s (joint) problem (10) - (13) is given by:

$$
\psi(s, x, y, q) = \begin{cases} 
\exp^{-\delta s}(\phi_2(y) - q^{-1}(\kappa_1 + \alpha_1(\hat{y} - y))) + \phi_\omega(xq), & S \setminus D_2 \\
\exp^{-\delta s}(\phi_2(y) + \phi_\omega(xq)), & S \setminus D_1 \\
\exp^{-\delta s}(\phi_2(y) + \phi_\omega(xq)), & D_1 \cap D_2
\end{cases} \quad (92)
$$

where the functions $\phi_2$ and $\phi_\omega$ are given by (93) - (94).

$$
\phi_2(y) = \begin{cases} 
\left( c(y^d_1 - y^d_2) - (\kappa_1 + \alpha_1(\hat{y} - y)) \right), & y \geq \hat{y} \\
\left( c(y^d_1 - y^d_2) \right), & y < \hat{y}
\end{cases} \quad (93)
$$

$$
\phi_\omega(xq) = \begin{cases} 
\exp^{-\delta s}(\phi_2(y) + \phi_\omega(xq)), & qx \leq \omega^* \\
\exp^{-\delta s}(\phi_2(y) + \phi_\omega(xq)), & qx > \omega^*
\end{cases} \quad (94)
$$

where $a$, $d_1$, $d_2$ and $\omega^*$ are constants given by (156) - (158).

Proposition 7.5 provides a complete characterisation of the value function for the investor’s problem.
Appendix

Technical Conditions for (T1) - (T4).

(T1) Assume that $\mathbb{E} [ \int_1^T 1_{\partial D}(X,u(s))ds ] = 0$ for all $X \in S, u \in U$ where $D = D_1 \cup D_2$.

(T2) $\partial D$ is a Lipschitz surface - that is to say that $\partial D$ is locally the graph of a Lipschitz continuous function: $\phi \in C^2(S,\partial D)$ with locally bounded derivatives.

(T3) The sets $\{ \phi^{-1}(X, u)(\tau_m) \}; \tau_m \in [t, \tau_S], \forall m \in \mathbb{N}$ and $\{ \phi^{-1}(X, u)(\rho); \rho \in T \}$ are uniformly integrable $\forall x \in S, u \in U$.

(T4) $\mathbb{E} [\phi(X, u)(\tau_m)] + [\phi(X, u)(\rho)] + \int_1^T |L\phi(X, u(s))|ds < \infty, \forall$ intervention times $\tau_m \in [t, \tau_S], \rho \in T, u \in U$.

Proof of Theorem 7.3.

We prove the theorem by applying the Theorem 5.1 to the model. We wish to fully characterise the optimal investment strategies for the investor. To put problem (7) - (10) in terms of the framework of Theorem 5.1 we firstly note that we now seek the triplet $(\hat{\theta}, (\tau, Z), \hat{\rho}) \in \mathbb{R}^2 \times [t, T] \times \Phi \times T$ with $\Phi \subseteq U$ and $T \subseteq [t, T]$ s.t.: 

$$J^{\hat{\theta}, \hat{\rho}, \hat{\tau}}(t, y_1, y_2, y_3) = \sup_{\rho \in \mathbb{T}} \left( \inf_{\tau \in [t, T]} \left( \inf_{\theta \in \mathbb{R} \times \mathbb{R}} J^{\theta, \rho, \tau}(t, y_1, y_2, y_3) \right) \right),$$

(95)

where

$$J^{\theta, \rho, \tau}(t, y_1, y_2, y_3) = \mathbb{E} \left[ -\sum_{m \geq 1} e^{-\delta \tau_m} [\kappa_1 + \alpha_1(\tau_m)z_m] \cdot 1_{\{\tau_m < \tau_S\}} + e^{-\delta (\tau_S + \rho)} \left( g_1 Y_1(t, y_1, \tau_S, \rho) Y_3 + \lambda T \right) + g_2 Y_2(t, y_2, \tau, Z) \right],$$

(96)

and $\theta = (\theta_0, \theta_1) \in \mathbb{R} \times \mathbb{R}$ and the dynamics of the state processes $Y := (Y_1, Y_2, Y_3)$ are expressed via the following:

$$dY_0(s) = dt,$$

(97)

$$dY_1(s) = dX(s); X(t) = y_1,$$

(98)

$$dY_2(s) = dY(s); Y(t) = y_2,$$

(99)

$$dY_3(s) = -Y_3(s)[\theta_0(s)dB_1(s) + \int_{\mathbb{R}} \theta_1(s, z)\tilde{N}_f(ds, dz)],$$

(100)

so that $Y_1, Y_2$ are processes which represent the firm liquidity processes and the investor’s wealth process respectively. The processes $Y_0$ and $Y_3$ represent time and market adjustments to the investor’s wealth process respectively and lastly, we have the following relations for the state process coefficients:

$$\mu(\cdot, y_2) = \Gamma y_2, \quad \mu(\cdot, y_1) = (r - \epsilon)y_1,$$

(101)

We will restrict ourselves to the case when $\forall y_1, y_2 \in S:

$$\sigma(\cdot, y_2) = \sigma_f y_2, \quad \gamma(\cdot, y_2) = 0, \quad \gamma_f(\cdot, y_1) = \gamma_f y_1, \quad \kappa(\cdot, y_1) = \kappa \rho y_1, \quad Y_2(t) = 0$$

(102)

For the case that includes jumps (i.e $\gamma_f \neq 0, \theta_1 \neq 0$), we will also impose a set of conditions on the firm’s discounted rate of return (in particular that it is greater than 1) and the discount rate, that is we assume the following conditions hold:

$$\frac{r - \epsilon}{\sigma_f} > 1 \quad \text{and} \quad \delta < \sigma_f(1 - \sigma_f).$$

(103)

The continuation regions $D_2$ and $D_1$ for the controller and the stopper respectively now take the form:

$$D_2 = \{ y_2 \in S, s \in [t, \tau_S]; \psi(s, y_1, y_2, y_3) < M_1\psi(s, y_1, y_2, y_3) \},$$

(104)

$$D_1 = \{ y_1 > 0; \psi(s, y_1, y_2, y_3) - G(s, y) > 0 \},$$

(105)
where given some \( \phi \in \mathcal{H} \) the intervention operator \( \mathcal{M}_1 \phi \) is given by:

\[
\mathcal{M}_1 \phi(s, y_1, y_2, y_3) = \inf_{\zeta \in \mathcal{I}} \{ \phi(s, y_1, y_2 - \zeta, y_3) - (\kappa_1 + \alpha_f \zeta), \zeta > 0 \}
\]  

(106)

for all \((s, y) \in [t, T] \times S^2 \times \mathbb{R}\) and the stopping time \( \hat{\rho} \) is defined by:

\[
\hat{\rho} = \inf\{s > t; \psi(s, \cdot) \notin D_1|s \in [t, T]\} \wedge \tau_S
\]  

(107)

Our first task is to characterise the value of the game. Now by the conditions of Theorem 5.1, we observe that the following expressions must hold \( \forall s \in [t, T] \):

\[
\psi(s, y_1, y_2, y_3) = e^{-\delta s}(g_1 y_1 y_3 + \lambda_T), \quad \forall y_1 \in S, \forall y_3 \in \mathbb{R} \quad \text{(condition (ii))}
\]  

(108)

\[
\psi(s, y_1, y_2, y_3) \geq e^{-\delta s}(g_1 y_1 y_3 + \lambda_T), \quad \forall y_1 \in S, \forall y_3 \in \mathbb{R} \quad \text{(condition (v))}
\]  

(109)

\[
\frac{\partial \psi}{\partial s} + \mathcal{L}^a \psi(s, y) \geq 0, \quad \forall (y_1, y_2, y_3) \in S^2 \times \mathbb{R} \quad \text{(condition (vi))}
\]  

(110)

\[
\inf_{\theta \in \mathbb{R} \times \mathbb{R}} \{ \frac{\partial \psi}{\partial s} + \mathcal{L}^a \psi(s, y) \} = 0 \quad \forall (y_1, y_2) \in D_1 \cup D_2 \quad \text{(condition (vi))}
\]  

(111)

Now using (97) to (100) we find that the generator is given by the following expression:

\[
\mathcal{L}^a \psi(s, y) = (r - e) y_1 \frac{\partial \psi}{\partial y_1} + \Gamma y_2 \frac{\partial \psi}{\partial y_2} + \frac{1}{2} \sigma^2 y_1^2 \frac{\partial^2 \psi}{\partial y_1^2} + \frac{1}{2} \sigma^2 y_2^2 \frac{\partial^2 \psi}{\partial y_2^2}
\]

\[
+ \frac{1}{2} \sigma^2_0 y_3^2 \frac{\partial^2 \psi}{\partial y_3^2} - \theta_0 y_1 y_3 \sigma f \frac{\partial \psi}{\partial y_1} + \int_{\mathbb{R}} \{ \psi(s, y_1 + y_1 f(z), y_2, y_3 - y_3 \theta_1(z))
\]

\[
- \psi(s, y_1, y_2, y_3) - y_1 \gamma f(z) \frac{\partial \psi}{\partial y_1} + y_3 \theta_1(z) \frac{\partial \psi}{\partial y_3} \} \nu(dz),
\]

(112)

\[
\sup_{\rho \in \mathbb{R}} \left\{ \inf_{(t, z) \in \mathbb{R}} \left( \inf_{\theta \in \mathbb{R}} f_\rho((t, z), \theta)(y) \right) \right\} = 0
\]

(113)

By (111) and (112) we readily deduce that the first order condition on \( \hat{\theta}_0 \) is given by the following expression:

\[
\hat{\theta}_0 y_3^2 \frac{\partial^2 \psi}{\partial y_3^2} - y_1 y_3 \sigma f \frac{\partial \psi}{\partial y_1} \frac{\partial^2 \psi}{\partial y_3} = 0,
\]

(114)

which after some simple manipulation we find that:

\[
\hat{\theta}_0 = y_1 y_3 - y_1 \sigma f \psi(\frac{\partial^2 \psi}{\partial y_3})^{-1},
\]

(115)

Now by (vi) of Theorem 5.1 we have that on \( D_1 \):

\[
\frac{\partial \psi}{\partial s} + \mathcal{L}^a \psi = 0,
\]

(116)

(here \( f = 0 \)) which implies that:

\[
0 = \frac{\partial \psi}{\partial s} + (r - e) y_1 \frac{\partial \psi}{\partial y_1} + \Gamma y_2 \frac{\partial \psi}{\partial y_2} + \frac{1}{2} \sigma^2 y_1^2 \frac{\partial^2 \psi}{\partial y_1^2} + \frac{1}{2} \sigma^2 y_2^2 \frac{\partial^2 \psi}{\partial y_2^2}
\]

\[
- \frac{1}{2} \sigma^2_0 y_3^2 \left( \frac{\partial^2 \psi}{\partial y_3^2} \right)^2 \left( \frac{\partial^2 \psi}{\partial y_3^2} \right)^{-1} + \int_{\mathbb{R}} \{ \psi(s, y_1 + y_1 f(z), y_2, y_3 - y_3 \theta_1(z)) - \psi(s, y_1, y_2, y_3) - y_1 \gamma f(z) \frac{\partial \psi}{\partial y_1} + y_3 \theta_1(z) \frac{\partial \psi}{\partial y_3} \} \nu(dz),
\]

(117)

Let us try as our candidate function \( \psi(y) = e^{-\delta s} y_3 [\phi_2(y_2, \phi(\omega))] \), where \( \omega := y_1 y_3 \). Then after plugging our expression for \( \psi \) into (113) we find that:

\[
0 = -\delta [\phi_2(y_2) + \phi(\omega)] + (r - e) \omega \phi'(\omega) + \Gamma \phi_2(y_2) + \frac{1}{2} \sigma^2 \phi_2''(y_2) + \frac{1}{2} \sigma^2 y_3 \phi''(\omega) - \theta_0 \sigma f \phi(2\phi_2'(\omega) + \omega \phi''(\omega)) - \theta_0 \sigma f \phi(2\phi_2'(\omega) + \omega \phi''(\omega))
\]

\[
+ \int_{\mathbb{R}} \{ (1 - \theta_1(z)) [\phi_2(y_2) + \phi(\omega)] + \phi(\omega)(1 + \gamma f(z))(1 - \theta_1(z)) \} \nu(dz),
\]

(118)

\[
-\delta [\phi_2(y_2) + \phi(\omega)] + \omega \phi'(\omega) + \phi(\omega)(1 + \gamma f(z))(1 - \theta_1(z)) \}
\]

\[
0 = \frac{1}{2} \sigma^2 y_3 \left( \frac{\partial^2 \psi}{\partial y_3^2} \right) \left( \frac{\partial^2 \psi}{\partial y_3^2} \right)^{-1} + \int_{\mathbb{R}} \{ (1 - \theta_1(z)) [\phi_2(y_2) + \phi(\omega)] + \phi(\omega)(1 + \gamma f(z))(1 - \theta_1(z)) \} \nu(dz),
\]

(119)
and (115) now becomes:

\[
\hat{\theta}_0 = \sigma_f \frac{y_1(2y_3\phi'_\omega(\omega) + y_3\omega\phi''_\omega(\omega))}{y_3(2y_1\phi'_\omega(\omega) + y_1\omega\phi''_\omega(\omega))} = \sigma_f. \tag{120}
\]

Hence, substituting (120) into (119) we find that:

\[
0 = -\delta [\phi_\omega(\omega) + \phi_2(\omega)] + (r - \sigma_f^2)\omega\phi'_\omega(\omega) + \Gamma y_2\phi'_2(\omega) + \frac{1}{2}\pi^2\sigma_f^2\phi''_\omega(\omega)
\]

\[
+ \int_{\mathbb{R}} \left\{ (1 - \theta_1(z)) \left[ \phi_2(y_2) + \phi_\omega(\omega(1 + \gamma_f(z))(1 - \theta_1(z))) \right] - (1 - \theta_1(z)) \phi_\omega(\omega) + \omega\phi'_\omega(\omega)(\theta_1(z) - \gamma_f(z)) \right\} \nu(dz), \tag{121}
\]

Additionally, our first order condition on \( \hat{\theta}_1 \) becomes:

\[
\int_{\mathbb{R}} \left\{ \phi_\omega(\omega \Xi) + \omega \Xi \phi'_\omega(\omega \Xi) - \phi_\omega - \omega \phi'_\omega \right\} \nu(dz) = 0. \tag{122}
\]

where \( \Xi(\hat{\theta}_1) := (1 - \hat{\theta}_1(z))(1 + \gamma_f(z)) \)

Note that by combining (99) with (120) we immediately arrive at Lemma 7.4.

We can decouple (121) after which we find that when \( y \in D_1 \cap D_2 \) we have that:

\[
i) \quad -\delta \phi_\omega(\omega) + (r - \sigma_f^2)\omega\phi'_\omega(\omega) + \int_{\mathbb{R}} \left\{ (1 - \theta_1(z)) \left[ \phi_2(y_2) + \phi_\omega(\omega(1 + \gamma_f(z))(1 - \theta_1(z))) \right] - (1 - \theta_1(z)) \phi_\omega(\omega) + \omega\phi'_\omega(\omega)(\theta_1(z) - \gamma_f(z)) \right\} \nu(dz) = 0, \tag{123}
\]

\[
ii) \quad -\delta \phi_2(y_2) + \Gamma y_2\phi'_2(y_2) + \frac{1}{2}\pi^2\sigma_f^2\phi''_\omega(y_2) = 0, \tag{124}
\]

We can solve the Cauchy-Euler equation (124) - after performing some straightforward calculations we find that:

\[
\phi_2(y_2) = c_1 y_2^{d_1} + c_2 y_2^{d_2} \tag{125}
\]

for some (as yet undetermined) constants \( c_1 \) and \( c_2 \) and the constants \( d_1 \) and \( d_2 \) are given by:

\[
d_1 = \frac{1}{2} - \frac{1}{\pi^2\sigma_f^2} \left( \sqrt{(\Gamma - \frac{1}{2}\pi^2\sigma_f^2)^2 + 2\pi\sigma_f^2\delta + \Gamma} \right) \tag{126}
\]

\[
d_2 = \frac{1}{2} + \frac{1}{\pi^2\sigma_f^2} \left( \sqrt{(\Gamma - \frac{1}{2}\pi^2\sigma_f^2)^2 + 2\pi\sigma_f^2\delta - \Gamma} \right) \tag{127}
\]

Since \( \psi(t) = Y_2(t) = 0 \), we easily deduce that \( c_2 = -c_1 \), after which we deduce that \( \phi_2 \) is given by the following expression:

\[
\phi_2(y_2) = c(y_2^{d_1} - y_2^{d_2}) \tag{128}
\]

where \( c := c_1 = -c_2 \) is some as of yet undetermined constant.

To obtain an expression for the function \( \phi_\omega \), in light of (128) we conjecture that \( \phi_\omega \) takes the form:

\[
\phi_\omega = a\omega^k \tag{129}
\]

where \( a \) and \( k \) are some constants. Using (129) and (128), we find the following:

\[
\mathcal{L}^0 \phi_\omega(\omega) = a\omega^k p(k) \tag{130}
\]

where the operator \( \mathcal{L}^0 \) is defined by the following expression for some function \( \phi \in C^{1,2}([t, \tau_S], \mathbb{R}) \):

\[
\mathcal{L}^0[\phi(\omega)] := -\delta \phi_\omega(\omega) + (r - \sigma_f^2)\omega\phi'_\omega(\omega)
\]

\[
+ \int_{\mathbb{R}} \left\{ (1 - \hat{\theta}_1(z)) \left[ \phi(\omega(1 + \gamma_f(z))(1 - \hat{\theta}_1(z))) - \phi(\omega) \right] + \omega\phi'_\omega(\omega)(\hat{\theta}_1(z) - \gamma_f(z)) \right\} \nu(dz), \tag{131}
\]
and $p(k)$ is defined by:

$$p(k) := -\delta + (r - e - \sigma_f^2)k + \int_{\mathbb{R}} \left\{ (1 - \hat{\theta}_1(z))\Xi(\hat{\theta}_1)k - 1 \right\} \nu(dz), \quad (132)$$

where $\Xi(\hat{\theta}_1) := (1 - \hat{\theta}_1(z))(1 + \gamma_f(z))$.

Hence using (133), (132) becomes:

$$\int_{\mathbb{R}} (\Xi - 1)\nu(dz) = 0. \quad (133)$$

We now make the following observations:

$$\begin{align*}
    p(0) &= -\delta < 0, \\
    p(1) &= r - e - \sigma_f^2 - \int_{\mathbb{R}} (\hat{\theta}_1(z) - \gamma_f(z)) \nu(dz) > 0, \mathbb{P} - \text{a.s.} \quad (135)
\end{align*}$$

We now split the analysis into two parts in which we study the investor’s capital injections (impulse control) problem and the investor’s optimal stopping problem separately. We then recombine the two problems to construct our solution to the problem.

**The Investor’s Capital Injections Problem**

We firstly tackle the investor’s capital injections problem, in particular we wish to ascertain the form of the function $\phi_2$ and describe the intervention region and the optimal size of the investor’s capital injections.

Our ansatz for the continuation region $D_2$ is that it takes the form:

$$D_2 = \{y_2 > \tilde{y}_2, |y_2, \tilde{y}_2 \in S\}. \quad (138)$$

Therefore by (ii) of Theorem 5.1 for $y_2 \notin D_2$ we have that

$$\psi(s, y) = M\psi(s, y) = \inf \{\psi(s, y_1, y_2 - \zeta, y_3) + (\kappa_I + \alpha_I \zeta), \zeta > 0\}$$

$$\iff \phi_2(y_2) = \inf \{\phi_2(y_2 - \zeta) + (\kappa_I + \alpha_I \zeta), \zeta > 0\}. \quad (139)$$

Let us define the function $h$ by the following expression:

$$h(\zeta) = \phi_2(y_2 - \zeta) - (\kappa_I + \alpha_I \zeta). \quad (140)$$

Hence we see that the first order condition for the minima $\hat{\zeta}(y_2) \in \mathbb{Z}$ of the function $h$ is

$$h'(\zeta) = \phi_2'(y_2 - \zeta) = \alpha_I \quad (141)$$

Let us now consider a unique point $\tilde{y}_2 \in (0, \tilde{y}_2)$ s.th.:

$$\phi_2'(\tilde{y}_2) = \alpha_I. \quad (142)$$

and

$$\tilde{y}_2 = y_2 - \hat{\zeta}(y_2) \text{ or } \zeta(\tilde{y}_2) = \tilde{y}_2 - y_2. \quad (143)$$

Then, after imposing a continuity condition at $y_2 = \tilde{y}_2$, by (139) we have that

$$\phi_2(\tilde{y}_2) = \phi_2(y_2) - (\kappa_I + \alpha_I (\tilde{y}_2 - y_2)) \quad (144)$$

where $\phi_{2,0}(y_2) = \phi_2(y_2)$ on $D_2$ where $\phi_2$ is given by (128). Additionally, be construction of $\tilde{y}_2$ we have that:

$$\phi_2'(\tilde{y}_2) = \alpha_I. \quad (145)$$
Hence we deduce that the function $\phi_2$ is given by the following expression:

$$
\phi_2(y_2) = \begin{cases} 
(\epsilon(y_2^d - y_2^{d_2}) + (\kappa_I + \alpha_I(y_2 - y_2^{d_2})), & y_2 > \tilde{y}_2 \\
(\epsilon(y_2^{d_1} - y_2^{d_2})), & y_2 \leq \tilde{y}_2
\end{cases}
$$

where $d_1$ and $d_2$ are given by (126) - (127).

We can use the system of equations (132), (141) and (145) to compute the constants $a$, $\tilde{y}_2$ and $\bar{y}_2$.

**The Investor’s Optimal Stopping Problem**

Our ansatz for the continuation region $D_1$ is that it takes the form:

$$
D_1 = \{ \omega = y_1y_3 < y_1y_3^* = \omega^*|y_1, y_3, y_3^* \in \mathbb{R} \} \tag{147}
$$

If we assume that the high contact principle holds, in particular if we have differentiability at $\omega^*$ then, using (129) we obtain the following equations:

(i) $a \omega^* = g_1 \omega^* + \lambda_T$,

(ii) $a k \omega^* = g_1$,

by continuity and differentiability at $\omega^*$. Since the system of equations (i) - (ii) completely determine the constants $a$ and $\omega^*$, we can compute the values of $\omega^*$ and $a$ in (129), after which we find:

$$
\omega^* = \frac{\lambda_T k}{g_1 (1 - k)}, \quad a = \left( \frac{g_1 k}{\lambda_T k} \right)^{(1 - k)} \tag{148}
$$

**The Investor’s Value Function and Joint Problem**

Using (100) and (120) we now see that the process $Y_3$ is determined by the expression:

$$
dY_3(s) = -\left[ \sigma_f Y_3 dB_f(s) + Y_3(s) \int_{\mathbb{R}} \hat{\theta}_1(s, z) \tilde{N}_f(ds, dz) \right]. \tag{149}
$$

$\mathbb{P}$-a.s., where $\hat{\theta}_1$ is determined by the equation (c.f. (153)):

$$
\int_{\mathbb{R}} (\Xi^k - 1) \nu(dz) = 0, \tag{150}
$$

where $\Xi(\hat{\theta}_1) := (1 - \hat{\theta}_1(z))(1 + \gamma_f(z))$.

Using Itô’s formula for Itô-Lévy processes, we can solve (150), moreover since

$$
\mathbb{E}_Q \left[ X + \lambda_T \right] = \mathbb{E}_P \left[ \left( X + \lambda_T \right) Y_3 \right], \quad \text{(c.f. (9))},
$$

the process $Y_3$ represents the Radon-Nikodym derivative of the measure $Q$ with respect to the measure $\mathbb{P}$ (i.e. $Y_3(s) = dP^\mathbb{Q}/dP$). Combining these two statements and denoting $Y_3$ by $Q$ immediately gives the result stated in Lemma 7.4.

Our last task is to combine the results together and fully characterise the investor’s value function. We firstly note that putting the above results together yields the following double obstacle variational inequality:

$$
\sup \{ \inf \left[ \psi(s, y) - (\kappa_I + \alpha_I(y_2 - y_2^{d_2})), -\left( \frac{\partial}{\partial y_3} + L^\theta \right) \psi(s, y) \right], \psi(s, y) - G(s, y) \} = 0, \tag{151}
$$

where $y = (y_1, y_2, y_3)$ and $G(s, y) = e^{-\delta s}(g_1 y_1 y_3 + g_2 y_2)$ and the investor’s stopping time is given by:

$$
\hat{\rho} = \inf \{ t > 0; Y_1(t)Y_2(t) \notin D_1 | s \in [t, T] \}, \tag{152}
$$

where the stochastic generator $L^\theta$ is defined via the following expression:

$$
L^\theta \psi(s, y) = (r - \rho) \frac{\partial \psi}{\partial y_1} + \gamma y_2 \frac{\partial \psi}{\partial y_2} + \frac{1}{2} \sigma_f \frac{\partial^2 \psi}{\partial y_1^2} + \frac{1}{2} \sigma_f \frac{\partial^2 \psi}{\partial y_2^2} + \frac{1}{2} \sigma_f \frac{\partial^2 \psi}{\partial y_3^2} - \theta_0 y_1 y_3 \frac{\partial^2 \psi}{\partial y_1 \partial y_3} + \int_{\mathbb{R}} \left\{ \psi(s, y_1 + y_1 \gamma_f(z), y_2, y_3 - y_2 \theta_1(z)) - \psi(s, y_1, y_2, y_3) - y_1 \gamma_f(z) \frac{\partial \psi}{\partial y_1} + y_2 \theta_1(z) \frac{\partial \psi}{\partial y_1} \right\} \nu(dz),
$$

27
and where the function $\hat{\theta}_1$ satisfies the first order condition (150).

The double obstacle problem in (87) characterises the value for the game, this proves Theorem 7.3.

Combining (129) and (140) and using (108) shows that the value function $\psi(s, y)$ is given by:

$$\psi(s, y) = \begin{cases} e^{-\delta s} y_3 (\phi_2(y_2) + y_3^{-1} (\kappa_1 + \alpha_1(\hat{y}_2 - y_2))) & \text{if } y_3 \leq \omega^* \\ e^{-\delta (T\land \rho)} [y_1 y_3 + \lambda \tau + g_2 y_2] & \text{if } y_1 y_3 > \omega^* \end{cases}$$

(153)

where the functions $\phi_2$ and $\phi_3$ are given by the following:

$$\phi_3(y_1, y_3) = \begin{cases} e^{-\delta s} [g_1 y_1 y_3 + \lambda \tau] & \text{if } y_1 y_3 \leq \omega^* \\ a y_1^k y_3^k, & \text{if } y_1 y_3 > \omega^* \end{cases}$$

(154)

and $\forall \ y_2 \in S$:

$$\phi_2(y_2) = \begin{cases} c(y_2^d_1 - y_2^d_2) - (\kappa_1 + \alpha_1(\hat{y}_2 - y_2)), & \text{if } y_2 > \hat{y}_2 \\ c(y_2^d_1 - y_2^d_2), & \text{if } y_2 \leq \hat{y}_2 \end{cases}$$

(155)

The constants $a, \omega^*$ are given by:

$$\omega^* = \frac{\lambda \tau k}{g_1 (1 - k)}, \quad a = \left(\frac{g_1}{k}\right)^k \left(\frac{\lambda \tau k}{1 - k}\right)^{1-k}$$

(156)

and the constants $d_1$ and $d_2$ are given by:

$$d_1 = \frac{1}{2} - \frac{1}{\pi^2 \sigma_1^2} \left(\sqrt{(1 - \frac{1}{2} \pi^2 \sigma_1^2)^2 + 2 \pi \sigma_1^2 \delta + \Gamma}\right)$$

(157)

$$d_2 = \frac{1}{2} + \frac{1}{\pi^2 \sigma_1^2} \left(\sqrt{(1 - \frac{1}{2} \pi^2 \sigma_1^2)^2 + 2 \pi \sigma_1^2 \delta - \Gamma}\right)$$

(158)

The constants $c, \hat{y}_2, \tilde{y}_2$ are determined by the set of equations:

$$\hat{y}_2^d_1 - \hat{y}_2^d_1 + \hat{y}_2^d_2 - \hat{y}_2^d_2 = c^{-1}(\alpha_1(\hat{y}_2 - \tilde{y}_2) - \kappa_1)$$

(159)

$$d_1 \hat{y}_2^{d_1-1} - d_2 \hat{y}_2^{d_2-1} = \alpha_1 c^{-1}$$

(160)

$$d_1 \hat{y}_2^{d_1-1} - d_2 \hat{y}_2^{d_2-1} = \alpha_1 c^{-1}$$

(161)

and the constant $k$ is a solution to the equation $p(k) = 0$ where the function $p$ is given by:

$$p(k) := -\delta + (r - e - 1)k + k \int_{\mathbb{R}} (\hat{\theta}_1(z) - \gamma_f(z)) \nu(dz)$$

(162)

where $\hat{\theta}_1$ is a solution to (91). This proves Proposition 7.5.

Though obtaining a closed analytic solution to $p(k) = 0$ represents a difficult task, the solution may be approximated using numerical methods. As we show in the following section, the analytic intractability of the equation $p(k) = 0$ is alleviated when the jumps in the diffusion processes are removed.

**The Case $\gamma_1 \equiv 0, \theta_1 \equiv 0$**

If the investor’s liquidity process contains no jumps (i.e. $\gamma_f \equiv 0$ and $\theta_1 \equiv 0$ in (2) and (100) (resp.)) then we can obtain closed analytic solutions for the parameters of the function $\phi_\omega$. Indeed, when $\gamma_1 \equiv 0$ and $\theta_1 \equiv 0$ using (102) we see that the expression for $p(k)$ reduces to:

$$p(k) := -\delta + (r - e - \sigma_f^2)k$$

(163)

We can therefore solve for $k$ after which we find that the function $\phi_\omega$ is given by:

$$\phi_\omega(y_1, y_3) = \begin{cases} e^{-\delta s} [g_1 y_1 y_3 + \lambda \tau], \quad y_1 y_3 \leq \omega^* \\ a y_1^k y_3^k, \quad y_1 y_3 > \omega^* \end{cases}$$

(164)

For the case that includes jumps in the firm liquidity process, we assume that the firm’s discounted rate of return is greater than 1 and the discount rate is relatively small compared to the volatility parameter $\sigma_f$ as given in condition (103).
where
\[ k = \frac{\delta}{r - e - \sigma_f^2}, \]  
(165)
and where the constants \(a, \omega^*, d_1, d_2\) are determined by (156)–(158) and the constants \(c, \gamma, \tilde{y}\) are determined by the set of equations:
\[
\begin{align*}
\tilde{y}'^d_1 - \tilde{y}'^d_2 + \tilde{y}'^d_2 - \tilde{y}'^d_2 &= e^{-1}(\alpha_1(\tilde{y} - \tilde{y}) - \kappa_1) \\
d_1 \tilde{y}'^d_1 - d_2 \tilde{y}'^d_2 &= \alpha_1 c^{-1} \\
d_1 \tilde{y}'^d_1 - d_2 \tilde{y}'^d_2 &= \alpha_1 c^{-1},
\end{align*}
\]  
(166)
(167)
(168)

We therefore immediately arrive at the following result: The following lemma provides a complete characterisation of the value function for the investor’s problem when the liquidity process contains no jumps:

**Lemma 7.6.** For the case in which the investor’s liquidity process contains no jumps (i.e. \(\gamma_f \equiv 0\) in (2)) we can obtain the following (closed analytic expression) for the function \(\psi\):
\[
\psi(s, x, y, q) = \begin{cases}
 e^{-\delta s} q \{ \phi_2(y) - q^{-1}(\kappa_1 + \alpha_1(\tilde{y} - y)) + \phi_\omega(xq) \}, & S \setminus \partial D_2 \\
 e^{-\delta s} (y_1 xq + \lambda_1 + g_2 y), & S \setminus \partial D_1 \\
 q e^{-\delta s} (\phi_2(y) + \phi_\omega(xq)), & D_1 \cap D_2
\end{cases}
\]  
(169)

where the function \(\phi_2\) is given by (159) and the function \(\phi_\omega\) is given by the following:
\[
\phi_\omega(xq) = \begin{cases}
 e^{-\delta(T \wedge \beta)}(y_1 xq + \lambda_1), & xq \leq \omega^* \\
 k a^k q^k, & xq > \omega^*,
\end{cases}
\]  
(170)

where \(k, a\) and \(\omega^*\) are given by:
\[
k = \frac{\delta}{r - e - \sigma_f^2}, \quad \omega^* = \frac{\lambda_1 k}{g_1 (1 - k)}, \quad a = \left( g_1 k \right)^k \left( \frac{\lambda_1 k}{1 - k} \right)^{1-k}.
\]  
(171)

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