A FAST ALGORITHM FOR THE PRODUCT STRUCTURE OF PLANAR GRAPHS

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Abstract. Dujmović et al. (FOCS2019) recently proved that every planar graph $G$ is a subgraph of $H \boxtimes P$, where $\boxtimes$ denotes the strong graph product, $H$ is a graph of treewidth 8 and $P$ is a path. This result has found numerous applications to linear graph layouts, graph colouring, and graph labelling. The proof given by Dujmović et al. is based on a similar decomposition of Pilipczuk and Siebertz (SODA2019) which is constructive and leads to an $O(n^2)$ time algorithm for finding $H$ and the mapping from $V(G)$ onto $V(H \boxtimes P)$. In this note, we show that this algorithm can be made to run in $O(n \log n)$ time.

1 Introduction

The strong product $G_1 \boxtimes G_2$ of two graphs $G_1$ and $G_2$ is the graph whose vertex set is the Cartesian product $V(G_1) \times V(G_2)$ in which the vertices $(v,x)$ and $(w,y)$ are adjacent if and only if

- $v = w$ and $xy \in E(G_2)$;
- $vw \in E(G_1)$ and $x = y$; or
- $vw \in E(G_1)$ and $xy \in E(G_2)$.

Dujmović et al. [5] recently proved the following product structure theorem for planar graphs:

**Theorem 1** (Dujmović et al. 2019). For any $n$-vertex planar graph $G$, there exists a graph $H$ of treewidth at most 8 and a path $P$ such that $G$ is a subgraph of $G^+ := H \boxtimes P$.

Though still very new, Theorem 1 has been used to solve a number of longstanding open problems on planar graphs:

- Theorem 1 has been used to show that the queue-number of every planar graph is upper bounded by a constant. This solves an open problem of Heath, Leighton, and Rosenberg posed in 1992 [10].
- Theorem 1 has been used to show that the nonrepetitive chromatic number of every planar graph is upper bounded by a constant. This solves an open problem of Alon et al. [1] posed in 2002.
- Theorem 1 has been used to produce (asymptotically) optimal labelling schemes for planar graphs [6]. This (asymptotically) resolves a problem of Kannan, Naor, and Rudich posed in 1988 [11, 12].
• Theorem 1 has been used to make significant improvements on the best-known bounds for $p$-centered colourings of planar graphs [2]. This gives the strongest result thus far on a question motivated by the work of Nešetřil and Ossona de Mendez from 2006 [16, 17] and posed explicity by Dvořák in 2016 [14].

The proof of Theorem 1 given by Dujmović et al. is based on a similar decomposition of Pilipczuk and Siebertz [18] which is constructive and leads to an $O(n^2)$ time algorithm for finding $H$ and the mapping from $V(G)$ onto $V(H \boxdot P)$ [5, Section 10]. Given the number of applications of Theorem 1 (and that more are likely to be found), it is natural to ask if this running-time can be improved. In this paper, we provide a faster algorithmic version of Theorem 1:

**Theorem 2.** For any $n$-vertex planar graph $G$, there exists a graph $H$ of treewidth at most 8 and a path $P$ such that $G$ is a subgraph of $G^+ := H \boxdot P$.

Furthermore, there exists an algorithm that, given $G$ as input, runs in $O(n \log n)$ time and produces the graph $H$, the path $P$, and an injective function $\varphi : V(G) \to V(G^+)$ such that, for each edge $vw \in E(G)$, $\varphi(v)\varphi(w) \in E(G^+)$. 

The remainder of this paper is organized as follows. Section 2 reviews the proof of Theorem 1 and the resulting $O(n^2)$ time algorithm. Section 3 describes the $O(n \log n)$ time algorithm. Section 4 discusses some of the implications and generalizations of this work.

### 2 The Original Proof/Algorithm

Throughout this paper we use standard graph theory terminology as used in the textbook by Diestel [3]. Every graph $G$ that we consider is finite, simple, and undirected, and has vertex set denoted by $V(G)$ and edge set denoted by $E(G)$.

Let $T$ be a tree rooted at some node $r$ and, for each node $v$ of $T$, let $P_T(v)$ denote the path in $T$ from $v$ to $r$. The $T$-depth of a node $v$ in $T$ is the length of $P_T(v)$. A path $P$ in $T$ is a **vertical path** if no two nodes of $P$ have the same $T$-depth. Every node $w$ in $P_T(v)$ is a **$T$-ancestor** of $v$ and $v$ is a **$T$-descendant** of every node $w$ in $P_T(v)$. Note that $v$ is both a $T$-ancestor and $T$-descendant of itself. A $T$-ancestor or $T$-descendant $x$ of $v$ is **strict** if $x \neq v$.

For a graph $G$ and a partition $\mathcal{P}$ of $V(G)$, the **quotient graph** $G/\mathcal{P}$ is the graph whose vertices $V(G/\mathcal{P})$ are the sets in $\mathcal{P}$ and in which an edge $XY \in E(G/\mathcal{P})$ if and only if there exists $x \in X$ and $y \in Y$ with $xy \in E(G)$. Dujmović et al. [5] prove Theorem 1 by first adding edges to a planar graph $G_0$ to complete it to a triangulation $G$, computing a breadth-first spanning tree $T$ of $G$ and then applying the following result to $G$ and $T$:

**Theorem 3.** For any $n$-vertex triangulation $T$ and any spanning tree $T$ of $G$, there exists a partition $\mathcal{P}$ of $V(G)$ such that each $P \in \mathcal{P}$ induces a vertical path in $T$ and the quotient graph $H := G/\mathcal{P}$ has treewidth at most 8.
Deriving Theorem 1 from Theorem 3 is just a matter of checking definitions. The graph $H$ in Theorem 1 is the same graph $H$ in Theorem 3. The path $P$ in Theorem 1 is simply the path $0, 1, 2, \ldots, h$ where $h$ is the maximum depth of any node in $T$. Each vertex $v \in V(G)$ maps to the node $q(v) := (X, y)$ where $X$ is the set in $P$ that contains $v$ and $y$ is the depth of $v$ in $T$. It is straightforward to check (using the definition of $\boxdot$ and the fact that $T$ is a breadth-first search tree) that for any edge $vw \in E(G)$, $q(v)q(w) \in E(H \boxdot P)$.

Therefore, we will focus on giving a fast algorithm for Theorem 3, from which we immediately obtain Theorem 2. We begin by describing the proof of Dujmović et al. [5], which is inductive, and leads naturally to a recursive algorithm. Refer to Figure 1. The algorithm is initialized with a breadth-first-search tree $T$ of the triangulation $G$. Each recursive invocation of the algorithm is given as input:

1. A cycle $F$ in $G$.
   The subgraph of $G$ induced by the vertices of $F$ and the vertices of $G$ in the interior of $F$ is a near-triangulation, $N$. The following are preconditions on the cycle $F$:
   (P1) The root $r$ of $T$ is not in the interior of $F$, i.e., $r \notin V(N) \setminus V(F)$.
   (P2) For every vertex $v \in V(N) \setminus V(F)$, and every $T$-descendant $w$ of $v, w \in V(N) \setminus V(F)$.
   (P3) Prior to this recursive invocation, every vertex of $F$ is already included in some part of the partition $P$ and no vertex in $V(N) \setminus V(F)$ is included in any part of $P$.

2. Three edges $e_1, e_2$, and $e_3$ of $F$ that we will call portals.
   Removing $e_1, e_2$, and $e_3$ from $F$ splits $F$ into three non-empty paths $P_1, P_2$, and $P_3$ where, for each $i \in \{1, 2, 3\}$, neither endpoint of $e_i$ is included in $P_i$. The portals satisfy the following precondition:
   (P4) For each $i \in \{1, \ldots, 3\}$, $V(P_i)$ is contained in the union of at most two elements of $P$.

By the time the recursive invocation terminates, each vertex of $N - V(F)$ is included in some part of the partition $P$. Let $f$ denote the number of inner triangular faces of $N$. The base case occurs when $f = 1$ so $N$ consists of a single triangle $(F)$. In this case (P3) implies that each vertex of $N$ is already included in $P$ and there is nothing to do so the algorithm returns immediately.

If $f > 1$, the paths $P_1, P_2$, and $P_3$, along with the breadth-first search tree $T$ are used to partition the vertices of $N$ into three colour classes, as follows. Each vertex $v \in P_i$ has colour $c(v) = i$. For each vertex $v \in V(N) \setminus V(F)$, (P1) implies that $P_1(v)$ contains some first vertex $v_F$ of $F$. The vertex $v$ is assigned the colour $c(v) = c(v_F)$.

By Sperner’s Lemma, $N$ contains a triangular face $\tau = x_1x_2x_3$ that is trichromatic, i.e., $c(x_i) = i$ for each $i \in \{1, 2, 3\}$. (Note that 0, 1, 2, or 3 vertices of $\tau$ may be in $V(F)$.) The edges of $F, \tau$, and the paths in $T$ from each $x_i$ to the first vertex of $P_i$ define a graph $M$ with at most 4 interior faces, one of which is $\tau$. Each of the other (at most three) interior faces does not contain $x_i$ for some $i \in \{1, 2, 3\}$. For each $i \in \{1, 2, 3\}$, we let $Q_i$ denote the face that does not contain $x_i$. Observe that, for each $i \in \{1, 2, 3\}$, $Q_i$ contains no vertex of $P_i$.

For each $i \in \{1, 2, 3\}$, let $Z_i$ be the path, in $T$, from $x_i$ up to, but not including the first vertex in $P_i$. Note that $Z_i$ may be empty, which occurs when $x_i$ is a vertex of $P_i$. Let
Figure 1: A single recursive step from Dujmović et al. [5].
$Y := V(Z_1) \cup V(Z_2) \cup V(Z_3)$. The algorithm adds $V(Z_1)$, $V(Z_2)$ and $V(Z_3)$ to the partition $P$ and then recurses on each of $Q_1$, $Q_2$, and $Q_3$.

We now argue that $Q_1$ satisfies preconditions (P1)–(P3). The face $Q_1$ is a cycle in $G$ that is contained in the cycle $F$, so $Q_1$ satisfies precondition (P1). The vertices of $Q_1$ are contained in $V(P_2) \cup V(P_3) \cup Y$. Therefore every vertex of $Q_1$ is contained in some part of $P$, so $Q_1$ satisfies precondition (P3). For each vertex $w$ in the interior of $Q_1$, every $T$-ancestor of $w$ is either in $Y$, $F$, or the exterior of $F$. Therefore $Q_1$ satisfies precondition (P2).

Next we describe the three portals used when recursing on $Q_1$. The cycle $Q_1$ contains at least one vertex each from $V(P_2)$ and $V(P_3)$ and therefore also contains the portal $e_1$, which is also used as one of the three portals in the recursive invocation. If $V(Z_2) \cup V(Z_3)$ is non-empty, then $Q_1$ contains two edges $e_2'$ and $e_3'$ where $e_2'$ has an endpoint in $V(P_3)$ and an endpoint in $V(Z_2) \cup V(Z_3)$ and where $e_3'$ has an endpoint in $V(P_2)$ and an endpoint in $V(Z_2) \cup V(Z_3)$. In this case, the edges $e_1$, $e_2'$, and $e_3'$ are used as the three portal in the recursive invocation on $Q_1$. Note that $e_1$, $e_2'$ and $e_3'$ satisfy precondition (P4) since the vertices of $P_1'$—the path from $e_1'$ to $e_3'$ on $Q_1$ that does not contain $e_1$—are contained in the union of $V(Z_2)$ and $V(Z_3)$, which are included in $P$.

If $(V(Z_2) \cup V(Z_3))$ is empty—because $x_2 \in V(P_2)$ and $x_3 \in V(P_3)$—then we artificially create two portals $e_2'$ and $e_3'$ for the recursive invocation by taking any two edges of $Q_1$ other than $e_1$. Clearly, this choice of $e_2'$ and $e_3'$ also satisfies precondition (P4).

The recursive invocations on $Q_2$ and $Q_3$ are done similarly, but rotating the values 1, 2, 3. After these three recursive invocations, every vertex in $N - V(F)$ is included in some part of $P$, so the recursive invocation is complete. Dujmović et al. then show that the contraction $H := G/P$ has treewidth at most 8. There is no need to repeat their argument here. Instead we discuss the running time of this algorithm.

Recall that $f$ denotes the number of inner faces in the near-triangulation $N$. By having each vertex of $G$ store a pointer to its parent in $T$ and storing $G$ using a representation that simultaneously represents $G$ and its dual graph $G^*$, the colouring of the vertices of $N$ can be done in $O(f)$ time and then the inner triangular faces of $N$ can be traversed in $O(f)$ time to find the trichromatic triangle $\tau$. The rest of the work (adding $Z_1$, $Z_2$, and $Z_3$ to $P$ and preparing the recursive invocations on $Q_1$, $Q_2$, and $Q_3$) is also easily implemented in $O(f)$ time, so the running time of the algorithm is given by the recurrence

$$T(f) \leq \begin{cases} a & \text{for } f \leq 1 \\ a \cdot f + T(f_1) + T(f_2) + T(f_3) & \text{for } f \geq 2 \end{cases}$$

where $a$ is a sufficiently large constant and, for each $i \in \{1, \ldots, 3\}$, $f_i$ is the number of faces of $G$ contained in the interior of $Q_i$. Note that $f_1 + f_2 + f_3 = f - 1$ (since $\tau$ is not contained in $Q_1$, $Q_2$, or $Q_3$). An easy inductive proof shows that $T(f) \leq a \cdot f \cdot (f + 1)/2 = O(f^2)$.

The recursive procedure described above is used to prove Theorem 3 as follows. Given an $n$-vertex triangulation $G$ and a spanning tree $T$ of $G$:

1. Define one of the faces incident to the root $r$ of $T$ to be the outer face of $G$ and let $r$, $x$, and $y$ denote the three vertices on the outer face of $G$.  


2. Place \([r, x, \text{and } y]\) in the partition \(P\) and run the recursive procedure described above on the cycle \(F := rxy\) with the portals \(e_1 = rx, e_2 = xy\) and \(e_3 = yr\).

The first step of this procedure runs in constant time. The second step requires \(\Theta(f^2) = \Theta(n^2)\) time in the worst case.

3 A Faster Algorithm

To obtain a faster algorithm we will create an algorithm (part of) whose running time satisfies the recurrence:

\[
T(f) \leq \begin{cases} 
  a & \text{for } f \leq 1 \\
  a \cdot (1 + \min\{f_1, f_2, f_3\}) + T(f_1) + T(f_2) + T(f_3) & \text{for } f \geq 2 
\end{cases}
\]

It is straightforward to show, by induction, that \(T(f) \leq (a/3)f \log_3(f) = O(a \cdot f \log f)\). The value of \(a\) here depends on the running time of an operation on a certain data structure described below.

Our algorithm makes use of a data structure that preprocesses a \((n+1)\)-vertex tree \(T\) with root \(r\) whose nodes are all initially uncoloured and that supports the following operations:

1. \(\text{Colour}(v, c)\): Set the colour \(c(v)\) of the node \(v\) of \(T\) to some integer value \(c\).

2. \(\text{FirstColour}(w)\): Return the colour of the first coloured node on the path from node \(w\) to the root of \(T\).

Gabow and Tarjan [8] prove the following result:

**Theorem 4** (Gabow and Tarjan (1985)). Any rooted \(n\)-vertex tree \(T\) can be processed in \(O(n)\) time so that any sequence consisting of a total of \(m \geq n\) \(\text{Colour}(v, c)\) and \(\text{FirstColour}(w)\) operations can be performed in \(O(m)\) time.

In the remainder of this section, we will show how Theorem 4 can be used to achieve the desired running time. In particular, we will show that the number of calls to \(\text{FirstColour}(w)\) made in our algorithm is \(O(n \log n)\) and the number of calls to \(\text{Colour}(v, c)\) is \(O(n)\). Since we are only interested in the overall running-time of our algorithm, we will treat each such call as taking \(O(1)\) time.

It is worth noting that the algorithm we now describe produces exactly the same partition \(P\) produced by the algorithm of Dujmović et al. and therefore \(P\) has all the properties described by Dujmović et al. In particular, the quotient graph \(H := G/P\) has treewidth at most 8.

As before, each recursive step takes as input the cycle \(F\) and the three portals \(e_1, e_2,\) and \(e_3\). Additionally, the algorithm requires that the vertices of \(P_1, P_2\) and \(P_3\) are coloured.

\[2\]In the terminology of Gabow and Tarjan [8], \(\text{Colour}(v, c)\) corresponds to marking a node \(v\) and \(\text{FirstColour}(w)\) corresponds to find the nearest marked ancestor of \(w\).
with three different colours. More precisely, there are three distinct integers \( c_1, c_2 \) and \( c_3 \) such that \( c(v) = c_i \) for each \( v \in V(P_i) \) and each \( i \in \{1, 2, 3\} \).

The algorithm searches for the trichromatic triangle \( \tau \) beginning from the portals. Refer to Figure 2. Step 0 of the search begins with \( e_{i,0} = e_i \) and \( t_{i,0} \) as the unique triangular inner face of \( N \) with \( e_i \) on its boundary, for each \( i \in \{1, 2, 3\} \). In Step \( j \) of the search, the algorithm has three triangles \( t_{i,j} \) and three edges \( e_{i,j} \) where \( e_{i,j} \) is an edge of \( t_{i,j} \) for each \( i \in \{1, 2, 3\} \). Using the data structure for \( T \), the algorithm checks, for each \( i \in \{1, 2, 3\} \), the colours of \( t_{i,j} / \)’s three vertices by calling \( \text{FirstColour} \). \(^3\) If \( t_{i,j} \) is trichromatic for at least one \( i \in \{1, 2, 3\} \), then the algorithm has found the necessary trichromatic triangle \( \tau \) and this step is complete. Otherwise, for each \( i \in \{1, 2, 3\} \), the triangle \( t_{i,j} \) contains another bichromatic edge \( e_{i,j+1} \neq e_{i,j} \) and this edge bounds another triangular face \( t_{i,j+1} \neq t_{i,j} \) of \( N \). The algorithm then continues to Step \((j + 1)\) of the search using the triangles \( t_{i,j+1} \) and edges \( e_{i,j+1} \) for each \( i \in \{1, 2, 3\} \). The fact that this algorithm terminates (and would even terminate if the search were limited to any one of the portals) follows from a classic proof of Sperner’s Lemma in 2-dimensions.

Suppose the search for \( \tau \) succeeds when \( \tau = t_{i,k} \) in Step \( k \). Thus, for each \( i \in \{1, 2, 3\} \), the algorithm has searched the sequence of triangles \( t_{i,0}, \ldots, t_{i,k} \). Each of the shorter subsequences \( S_i := t_{i,0}, \ldots, t_{i,k-1} \) consists entirely of bichromatic triangles. Each sequence \( S_i \) contains \( k \) bichromatic triangles whose vertices are coloured with \( \{c_1, c_2, c_3\} \setminus c_i \).

Refer to the second part of Figure 2. Consider again the graph \( M \) with faces \( Q_1, Q_2, Q_3 \) and \( \tau \). For each \( i \in \{1, 2, 3\} \), each face in \( S_i \) is contained in \( Q_i \). Since \( f_i \) counts the number of triangular faces of \( N \) contained in \( Q_i \), this implies that \( f_i \geq k \) for each \( i \in \{1, 2, 3\} \). Therefore, \( \min\{f_1, f_2, f_3\} \geq k \). On the other hand, the search for for \( \tau \) took \( 1 + k \) steps, each of which performs three \( \text{FirstColour}(w) \) queries and therefore the entire search runs in time \( O(1 + k) \subseteq O(1 + \min\{f_1, f_2, f_3\}) \).

Next, the algorithm prepares the three subproblems defined by \( Q_1, Q_2, \) and \( Q_3 \) on which to recurse. To do this it follows the path, in \( T \), from each vertex \( x_i \) of \( \tau \) to the first vertex of \( P_i \), calling \( \text{Colour}(v, c_4) \) for each vertex \( v \) it encounters with any integer \( c_4 \notin \{c_1, c_2, c_3\} \).

Finally, in preparing each subproblem \( Q_i \) for the recursive invocation, it may be necessary to recolour an already coloured vertex \( v \) of \( F \) with the colour \( c_4 \) before making the recursive call and then recolouring \( v \) with its original colour once the recursion is complete. This corresponds to introducing an artificial portal adjacent to an edge of \( \tau \) contained in \( F \).

### 3.1 Running-Time Analysis

We analyze the running time of the preceding algorithm by analyzing two parts separately.

During each recursive invocation, the algorithm does work to find the trichromatic triangle \( \tau \). The time associated with this is \( O(1 + k) \) where \( k \geq \min\{f_1, f_2, f_3\} \). As already described above, this leads to a recurrence of the form \( T(f) \leq O(\min\{f_1, f_2, f_3\}) + T(f_1) + \)

\(^3\)Note that each vertex \( w \) of \( t_{i,j} \) satisfy the precondition for the argument \( w \) of \( \text{FirstColour}(w) \) since every vertex of \( F \) is already coloured and, by (P1), the path \( P_T(w) \) contains at least one vertex of \( F \).
Figure 2: Searching for the trichromatic triangle $\tau$ beginning at the portals $e_1$, $e_2$, and $e_3$. In this example, $\tau = t_{1,6}$ is found after $k + 1 = 7$ steps.
\( T(f_2) + T(f_3) \) which resolves to \( O(f \log f) \). In the initial call, \( f = 2n - 3 \) is the number of inner faces of \( G \), so the total running time attributable to this part of the algorithm is \( O(n \log n) \).

In addition to this, the algorithm does other work in preparing inputs for recursive calls. Once \( \tau \) is identified, the previously uncoloured vertices of \( Y \) are coloured with a call to \( \text{Colour}(v, c) \). Since each of the \( n \) vertices of \( G \) are coloured at most once for the first time, this takes a total of \( O(n) \) time. In addition to this, preparing the subproblem \( Q_1, Q_2, \) and \( Q_3 \) may require an additional 6 calls to \( \text{Colour}(v, c) \). However, the number of recursive invocations of the algorithm is exactly \( 2n - 3 \), since the triangle \( \tau \) identified during the invocation is not included in any of the recursive calls. Therefore, the total running time attributable to these parts of the algorithm is \( O(n) \). This completes the proof of our main theorem:

**Theorem 5.** There exists an algorithm that, given any \( n \)-vertex triangulation \( T \) and any breadth-first-spanning tree \( T \) of \( G \), runs in \( O(n \log n) \) time and finds a partition \( \mathcal{P} \) of \( V(G) \) such that each \( P \in \mathcal{P} \) induces a vertical path in \( T \) and the quotient graph \( H := G/\mathcal{P} \) has treewidth at most \( 8 \).

### 4 Discussion

Another variant of Theorem 3 described by Dujmović et al. gives a partition \( \mathcal{P} \) of \( V(G) \) such that \( G/\mathcal{P} \) has treewidth at most 3 and each part \( Y \in \mathcal{P} \) is the union of at most 3 vertical paths in \( T \). The algorithm described here also gives an \( O(n \log n) \) time algorithm for this variant.

#### 4.1 Other Graph Classes

Theorem 1 has been generalized to a number of graph classes including bounded-genus graphs [5], apex-minor free graphs [5], graphs of bounded-degree from proper-minor closed families [4], and \( k \)-planar graphs [7]. In all cases, these generalizations ultimately involve decomposing the input graph into a number of planar subgraphs and applying Theorem 1 to each of these planar graphs.

In at least two cases, the extra work done in these generalizations can be done in \( O(n \log n) \) time. Combined with Theorem 2, this gives \( O(n \log n) \) time algorithms for the corresponding generalizations of Theorem 1.

- For graphs \( G \) of fixed Euler genus \( g \), the result of Dujmović et al. [5] only requires finding a genus-\( g \) embedding of \( G \), computing a breadth-first spanning tree \( T \) of \( G \), and computing any spanning-tree \( D \) of the dual graph that does not cross edges of \( T \). The two spanning trees \( T \) and \( D \) can be computed in \( O(n) \) time using standard algorithms. The genus-\( g \) embedding of \( G \) can be computed in \( O(n) \) time using an algorithm of Mohar [15].

- Given a \( k \)-plane embedding of a \( k \)-planar graph \( G \), the result of Dujmović, Morin, and Wood [7] applies Theorem 1 directly to the planar graph obtained by adding a dummy vertex at every point where a pair of edges crosses.

While the problem of testing \( k \)-planarity of a graph is NP-complete, even for \( k = 1 \) [9, 13, 19], there are a number of graph classes that are \( k \)-planar and in which an
embedding can be found easily. These include (appropriate representations of) map graphs, bounded-degree string graphs, powers of bounded-degree planar graphs, and \( k \)-nearest-neighbour graphs of points in \( \mathbb{R}^2 \) [7, Section 8].

4.2 Applications

The algorithm presented here applies immediately to the four applications of Theorem 1 discussed in the introduction.

- There exists an algorithm that, given an \( n \)-vertex planar graph \( G \), runs in \( O(n \log n) \) time and computes a 49 queue layout of \( G \) [5].

- There exists an algorithm that, given an \( n \)-vertex planar graph \( G \), runs in \( O(n \log n) \) time and computes a nonrepetitive colouring of \( G \) using at most 768 colours [4].

- There exists an algorithm that, given an \( n \)-vertex planar graph \( G \), runs in \( O(n \log n) \) time and computes a \((1 + o(1)) \log n\)-bit adjacency labelling of \( G \) [6].

- There exists an algorithm that, given an \( n \)-vertex planar graph \( G \) and an integer \( p \), runs in \( O(p^3 n \log n) \) time and computes \( p \)-centered colouring of \( G \) using at most \( 3(p + 1)(p^3) \) colours [2].

Prior to this work, the bottleneck in all these algorithms was the \( \Theta(n^2) \) worst-case running time of the algorithm for computing the decomposition of Theorem 1.

4.3 Future Work

The obvious open problem left by our work is that of finding a faster algorithm. Can the running-time in Theorem 2 be improved to \( O(n) \)?

Acknowledgement

Part of this research was conducted during the Eighth Workshop on Geometry and Graphs, held at the Bellairs Research Institute, January 31–February 7, 2020. The author is grateful to the other organizers and participants for providing a stimulating research environment.

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