GENERATION OF BIANCHI TYPE V BULK VISCOSOUS COSMOLOGICAL MODELS WITH TIME DEPENDENT $\Lambda$-TERM

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Abstract

Bianchi type V bulk viscous fluid cosmological models are investigated with dynamic cosmological term $\Lambda(t)$. Using a generation technique (Camci et al., 2001), it is shown that the Einstein’s field equations are solvable for any arbitrary cosmic scale function. Solutions for particular forms of cosmic scale functions are also obtained. The cosmological constant is found to be decreasing function of time, which is supported by results from recent type Ia supernovae observations. Some physical and geometrical aspects of the models are also discussed.

Keywords: cosmology, Bianchi type V universe, cosmological constant

1 Introduction

The study of Bianchi type V cosmological models create more interest as these models contain isotropic special cases and permit arbitrary small anisotropy levels at some instant of cosmic time. This property makes them suitable as model of our universe. The homogeneous and isotropic Friedman-Robertson-Walker (FRW) cosmological models, which are used to describe standard cosmological models, are particular case of Bianchi type I, V and IX universes, according to whether the constant curvature of the physical three-space, $t = \text{constant}$, is zero, negative or positive. These models will be interesting to construct cosmological models of the types which are of class one. Present cosmology is
based on the FRW model which is completely homogeneous and isotropic. This is in agreement with observational data about the large scale structure of the universe. However, although homogeneous but anisotropic models are more restricted than the inhomogeneous models, they explain a number of observed phenomena quite satisfactorily. This stimulates the research for obtaining exact anisotropic solution for Einstein’s field equations (EFEs) as a cosmologically accepted physical models for the universe (at least in the early stages). Roy and Prasad [1] have investigated Bianchi type V universes which are locally rotationally symmetric and are of embedding class one filled with perfect fluid with heat conduction and radiation. Bianchi type V cosmological models have been studied by other researchers (Farnsworth [2], Maartens and Nel [3], Wainwright et al. [4], Collins [5], Meena and Bali [6], Pradhan et al. [7, 8]) in different context.

Models with a dynamic cosmological term $\Lambda(t)$ are becoming popular as they solve the cosmological-constant problem in a natural way. There is a significant observational evidence for the detection of Einstein’s cosmological constant, $\Lambda$ or a component of material content of the universe, that varies slowly with time and space and so acts like $\Lambda$. Recent cosmological observations by High-z Supernova Team and Supernova Cosmological Project (Garnavich et al. [9], Perlmutter et al. [10], Riess et al. [11], Schmidt et al. [12]) strongly favour a significant and positive $\Lambda$ with the magnitude $\Lambda(\bar{G}h/c^3) \approx 10^{-123}$. These observations on magnitudes and red-shift of type Ia supernova suggest that our universe may be accelerating one with a large function of the cosmological density in the form of cosmological $\Lambda$-term. Earlier researches on this topic, are contained in Lodovico [13], Weinberg [14], Dolgov [15] – [17], Bertolami [18], Ratra and Peebles [19], Carrol, Press and Turner [20]. Some of the recent discussions on the cosmological-constant “problem” and consequence on cosmology with a time-varying cosmological-constant have been discussed by Tsagas and Maartens [21], Sahni and Starobinsky [22], Peeble [23], Padmanabhan [24], Vishwakarma [25], and Pradhan et al. [26]. This motivates us to study the cosmological models, where $\Lambda$ varies with time.

The distribution of matter can be satisfactorily described by a perfect fluid due to the large scale distribution of galaxies in our universe. However, observed physical phenomena such as the large entropy per baryon and the remarkable degree of isotropy of the cosmic microwave background radiation, suggest analysis of dissipative effects in cosmology. Furthermore, there are several processes which are expected to give rise to viscous effects. These are the decoupling of neutrinos during the radiation era and the decoupling of radiation and matter during the recombination era. Bulk viscosity is associated with the GUT phase transition and string creation. Misner [27] has studied the effect of viscosity on the evolution of cosmological models. The role of viscosity in cosmology has been investigated by Weinberg [28]. Nightingale [29], Heller and Klimek [30] have obtained a viscous universes without initial singularity. The model studied by Murphy [31] possessed an interesting feature in which big bang type of singularity of infinite space-time curvature does not occur to be a finite past. However, the relationship assumed by Murphy between the viscosity coefficient
and the matter density is not acceptable at large density. Thus, we should consider the presence of material distribution other than a perfect fluid to obtain a realistic cosmological models (see Grøn [32] for a review on cosmological models with bulk viscosity). The effect of bulk viscosity on the cosmological evolution has been investigated by a number of authors in the framework of general theory of relativity. This motivates to study cosmological bulk viscous fluid model.

In recent years, several authors (Hajj-Boutros [33], Hajj-Boutros and Sfeila [34], Ram [35], Mazumder [36] and Pradhan and Kumar [37]) have investigated the solutions of EFEs for homogeneous but anisotropic models by using some different generation techniques. Bianchi spaces I − IX are useful tools in constructing models of spatially homogeneous cosmologies (Ellis and MacCallum [38], Ryan and Shepley [39]). From these models, homogeneous Bianchi type V universes are the natural generalization of the open FRW model which eventually isotropize. Recently Camci et al. [40] derived a new technique for generating exact solutions of EFEs with perfect fluid for Bianchi type V space-time. Very recently Pradhan et al. [41] have obtained Bianchi type V perfect fluid cosmological models with time dependent Λ-term.

In this paper, in what follows, we will discuss Bianchi type V cosmological models obtained by augmenting the energy-momentum tensor of a bulk viscous fluid by a term that represents a cosmological constant varying with time, and later generalize the solutions of Refs. [35, 40, 41]. This paper is organized as follows: The field equations and the generation technique are presented in Section 2. We relate three of the metric variables by solving the off-diagonal component of EFEs, and find a second integral which is used to relate the remaining two metric variables. In Section 3, for the particular form of each metric variables, some solutions are presented separately and solutions of Camci et al. [40], Ram [35] and Pradhan et al. [41] are shown to be particular cases of our solutions. Kinematical and dynamical properties of all solutions are also studied in this section. In Section 4, we give the concluding remarks.

2 Field equations and generation technique

In this section, we review the solutions obtained by Pradhan et al [41]. The usual energy-momentum tensor is modified by addition of a term

\[ T^{(\text{vac})}_{ij} = -\Lambda(t)g_{ij}, \]

where \( \Lambda(t) \) is the cosmological term and \( g_{ij} \) is the metric tensor. Thus the new stress energy-momentum tensor in presence of bulk stress is given by

\[ T_{ij} = (\bar{p} + \rho)u_iu_j - \bar{p}g_{ij} - \Lambda(t)g_{ij}, \]

where

\[ \bar{p} = p + \xi u^i u_i \]

Here, \( \rho, p, \bar{p}, \xi \) and \( u \) are, respectively, the energy density, isotropic pressure, effective pressure, bulk viscous coefficient and the fluid four-velocity vector of distribution such that \( u^iu_i = 1 \). In general, \( \xi \) is a function of time.
We consider the space-time metric of the spatially homogeneous Bianchi type V of the form
\[ ds^2 = dt^2 - A^2(t)dx^2 - e^{2\alpha x} [B^2(t)dy^2 + C^2(t)dz^2], \]
(4)
where \( \alpha \) is a constant. For the energy momentum tensor (2) and Bianchi type V space-time (4), Einstein’s field equations
\[ R_{ij} - \frac{1}{2} R g_{ij} = -8\pi T_{ij} \]
(5)
yield the following five independent equations
\[
\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4 B_4}{AB} - \frac{\alpha^2}{A^2} = -8\pi(\bar{p} + \Lambda), \quad \text{(6)}
\]
\[
\frac{A_{44}}{A} + \frac{C_{44}}{C} + \frac{A_4 C_4}{AC} - \frac{\alpha^2}{A^2} = -8\pi(\bar{p} + \Lambda), \quad \text{(7)}
\]
\[
\frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{B_4 C_4}{BC} - \frac{\alpha^2}{A^2} = -8\pi(\bar{p} + \Lambda), \quad \text{(8)}
\]
\[
\frac{A_4 B_4}{AB} + \frac{A_4 C_4}{AC} + \frac{B_4 C_4}{BC} - \frac{3\alpha^2}{A^2} = 8\pi(\rho - \Lambda), \quad \text{(9)}
\]
\[
2A - \frac{B_4}{B} - \frac{C_4}{C} = 0. \quad \text{(10)}
\]
Here and in what follows the suffix 4 by the symbols \( A, B, C \) and \( \rho \) denote differentiation with respect to \( t \). The Bianchi identity \( T_{ij}^{\; j} = 0 \) takes the form
\[ \rho + (\rho + p)\theta = 0. \quad \text{(11)} \]
It is worth noting here that our approach suffers from a lack of Lagrangian approach. There is no known way to present a consistent Lagrangian model satisfying the necessary conditions discussed in this paper.

The physical quantities expansion scalar \( \theta \) and shear scalar \( \sigma^2 \) have the following expressions:
\[ \theta = u^i_{;i} = \frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C} \quad \text{(12)} \]
\[ \sigma^2 = \frac{1}{2} \sigma_{ij} \sigma^{ij} = \frac{1}{3} \left[ \theta^2 - \frac{A_4 B_4}{AB} - \frac{A_4 C_4}{AC} - \frac{B_4 C_4}{BC} \right]. \quad \text{(13)} \]
Integrating Eq. (10) and absorbing the integration constant into \( B \) or \( C \), we obtain
\[ A^2 = BC \quad \text{(14)} \]
without any loss of generality. Thus, elimination of \( \bar{p} \) from Eqs. (6) - (8) gives the condition of isotropy of pressures
\[ 2B + \left( \frac{B_4}{B} \right)^2 = 2 \frac{B_{44}}{B} + \left( \frac{C_4}{C} \right)^2, \quad \text{(15)} \]
which on integration yields

\[
\frac{B_4}{B} - \frac{C_4}{C} = \frac{k}{(BC)^{3/2}},
\]

(16)

where \( k \) is a constant of integration. Hence for the metric function \( B \) or \( C \) from the above first order differential Eq. (16), some scale transformations permit us to obtain new metric function \( B \) or \( C \).

Firstly, under the scale transformation \( dt = B^{1/2}d\tau \), Eq. (16) takes the form

\[
CB_{\tau} - BC_{\tau} = kC^{-1/2},
\]

(17)

where subscript represents derivative with respect to \( \tau \). Considering Eq. (17) as a linear differential equation for \( B \), where \( C \) is an arbitrary function, we obtain

\[
(i)B = k_1C + kC \int \frac{d\tau}{C^{3/2}},
\]

(18)

where \( k_1 \) is an integrating constant. Similarly, using the transformations \( dt = B^{3/2}d\tilde{\tau} \), \( dt = C^{1/2}dT \), and \( dt = C^{3/2}d\tilde{T} \) in Eq. (16), after some algebra we obtain respectively

\[
(ii)B(\tilde{\tau}; k_2, k) = k_2C \exp \left(k \int \frac{d\tilde{\tau}}{C^{3/2}} \right),
\]

(19)

\[
(iii)C(T; k_3, k) = k_3B - kB \int \frac{dT}{B^{3/2}},
\]

(20)

and

\[
(iv)C(\tilde{T}; k_4, k) = k_4B \exp \left(k \int \frac{d\tilde{T}}{B^{3/2}} \right),
\]

(21)

where \( k_2, k_3 \) and \( k_4 \) are constants of integration. Thus choosing any given function \( B \) or \( C \) in cases (i), (ii), (iii) and (iv), one can obtain \( B \) or \( C \) and hence \( A \) from (14).

3 Generation of new solutions

We consider the following four cases:

3.1 Case (I): Let \( C = \tau^n \), \( n \) is a real number satisfying \( n \neq \frac{2}{5} \).

In this case, Eq. (18) gives

\[
B = k_1\tau^n + \frac{2k}{2 - 5n} \tau^{1 - 3n/2}
\]

(22)
and then from (14), we obtain

\[ A^2 = k_1 \tau^{2n} + \frac{2k}{2 - 5n} \tau^{-n/2}. \]  

(23)

Hence the metric (4) reduces to the new form

\[ ds^2 = (k_1 \tau^n + 2\ell \tau^{\ell_1}) \left[ d\tau^2 - \tau^n dx^2 \right] - e^{2\alpha x} \left[ (k_1 \tau^n + 2\ell \tau^{\ell_1})^2 dy^2 + \tau^{2n} dz^2 \right] \]

(24)

where

\[ \ell = \frac{k}{2 - 5n} \text{ and } \ell_1 = 1 - \frac{3n}{2}. \]

For this derived model (24), the effective pressure, energy density and cosmological constant are given by

\[ 8\pi(\overline{p} + \Lambda) = \left( k_1 \tau^n + 2\ell \tau^{\ell_1} \right)^{-3} \left[ -2k_1^2 n(n - 1) \tau^{2n-2} - k_1 \ell n(10 - 13n) \tau^{-\ell_1 + 2n} \right] \]

\[ - \frac{\ell^2(4 + 4n - 11n^2)}{2} \tau^{-3n} + \alpha^2 \tau^{-n} \left( k_1 \tau^n + 2\ell \tau^{\ell_1} \right)^{-1}, \]  

(25)

\[ 8\pi(\rho - \Lambda) = \left( k_1 \tau^n + 2\ell \tau^{\ell_1} \right)^{-3} \left[ 3k_1^2 n^2 \tau^{2n-2} + 3k_1 \ell n(2 - n) \tau^{-\ell_1 + 2n} \right] \]

\[ + \frac{\ell^2(4 + 4n - 11n^2)}{2} \tau^{-3n} - 3\alpha^2 \tau^{-n} \left( k_1 \tau^n + 2\ell \tau^{\ell_1} \right)^{-1}. \]  

(26)

We assume for the specification of \( \xi \), that the fluid obeys an equation of state of the form

\[ p = \gamma \rho, \]  

(27)

where \( 0 \leq \gamma \leq 1 \) is a constant.

Thus, given \( \xi(t) \), we can solve the system for the physical quantities. In most of investigations involving bulk viscosity is assumed to be a simple power function of the energy density (Pavon et al. [42]; Maartens [43]; Zimdahl [44]; Santos et al. [45])

\[ \xi(t) = \xi_0 \rho^w, \]  

(28)

where \( \xi_0 \) and \( w \) are real constants. For small density, \( w \) may even be equal to unity as used in Murphy’s work [46] for simplicity. If \( w = 1 \), Eq. (28) may correspond to a radiative fluid (Weinberge [14]). Near the big bang, \( 0 \leq w \leq \frac{1}{2} \) is a more appropriate assumption (Belinskii and Khalatnikov [47]) to obtain realistic models.

For simplicity and realistic models of physical importance, we consider the following two cases (\( w = 0, 1 \)):
3.1.1 Model I: Solution for $w = 0$

When $w = 0$, Eq. (25) reduces to $\xi = \xi_0 = \text{constant}$. Hence in this case Eq. (25), with the use of (26), (27) and (28), leads to

$$8\pi(1 + \gamma)\rho = D_1^{-3}\left[n(n + 2)k_1^2r^{2(n-1)} + 2(5n - 2)k_1\ell n\tau^{-(\ell_1 + 2n)}\right] + D_1^{-1}\left[-2\alpha_2\tau^{-n} + 24\pi\xi_0\left(k_1n\tau^{n-1} + \frac{1}{2}\ell(2 - n)\tau^{-\frac{3n}{2}}\right)D_1^{-\frac{1}{2}}\right],$$

(29)

where

$$D_1 = k_1\tau^n + 2\ell\tau^{\ell_1}.$$  

Eliminating $\rho(t)$ from Eqs. (26) and (29), we obtain

$$8\pi(1 + \gamma)\Lambda = D_1^{-3}\left[(1 - 2n - 3n\gamma)nk_1^2r^{2(n-1)} - (10 - 13n + 3(2 - n)\gamma)k_1\ell n\tau^{-(\ell_1 + 2n)}

- \frac{1}{2}(4 + 4n - 11n^2)(1 + \gamma)\ell^2\tau^{-3n}\right] +

(1 + 3\gamma)\alpha_2^2\tau^{-n}D_1^{-1} + 24\pi\xi_0\left[nk_1\tau^{n-1} + \frac{1}{2}\ell(2 - n)\tau^{-\frac{3n}{2}}\right]D_1^{-\frac{1}{2}},$$

(30)

3.1.2 Model II: Solution for $w = 1$

When $w = 1$, Eq. (28) reduces to $\xi = \xi_0\rho$. Hence in this case Eq. (25), with the use of (26), (27) and (28), leads to

$$8\pi\rho = \frac{n(n + 2)k_1^2r^{2(n-1)} + 2n(5n - 2)k_1\ell r^{-(\ell_1 + 2n)} - 2\alpha_2^2\tau^{-n}D_1^{-2}}{D_1^3\left[(1 + \gamma) - 3\xi_0\left(nk_1\tau^{n-1} + \frac{1}{2}\ell(2 - n)\tau^{-\frac{3n}{2}}\right)D_1^{-\frac{1}{2}}\right]},$$

(31)

Eliminating $\rho(t)$ between (26) and (31), we obtain

$$8\pi\Lambda = \left[n(n + 2)k_1^2r^{2(n-1)} + 2n(5n - 2)k_1\ell r^{-(\ell_1 + 2n)} - 2\alpha_2^2\tau^{-n}D_1^{-2}\right]

D_1^{-3}\left[(1 + \gamma) - 3\xi_0\left(nk_1\tau^{n-1} + \frac{1}{2}\ell(2 - n)\tau^{-\frac{3n}{2}}\right)D_1^{-\frac{1}{2}}\right]

- \frac{1}{D_1^3}\left[3n^2k_1^2r^{2(n-1)} + 3n(2 - n)k_1\ell r^{-(\ell_1 + 2n)} + \frac{1}{2}(4 + 4n - 11n^2)\ell^2\tau^{-3n} - 3\alpha_2^2\tau^{-n}D_1^{-2}\right].$$

(32)

From Eqs. (29) and (32), we observe that at the time of early universe the energy density $\rho(t)$ and cosmological constant $(\Lambda(t))$ decrease with time increases (see Figure 1). We also observe that the value of $\Lambda$ is small and positive at late times which is supported by recent type Ia supernovae observations [9] – [12]. In Model II, from Eqs. (31) and (32), we find that for large range of parameters the energy density decreases with time very sharply and becomes negative (even if it is positive initially) and then remains negative throughout.
the evolution. The cosmological constant shows singular behaviour closer to origin. Then it shows eretic behaviour and later stage it remain negative constant value. It seems that Model II may not be physical model of the universe.

The metric (24) is a four-parameter family of solutions to EFEs with a bulk viscous fluid. Using the scale transformation $dt = B^2 d\tau$ in Eqs. (12) and (13) for this case, the scalar expansion $\theta$ and the shear $\sigma$ have the expressions:

$$\theta = 3 \left[ k_1 n^{-1} + \frac{\ell (2 - n)}{2} \right] \left( k_1^{-1} + 2 \ell \right)^{-3/2}$$

$$\sigma = \frac{1}{2} k^{-3/2} \left( k_1^{1/2} + 2 \ell \right)^{-3/2}$$

Eqs. (33) and (34) lead to

$$\frac{\sigma}{\theta} = \frac{k}{6} \left[ k_1 n^{-1} + \frac{\ell (2 - n)}{2} \right]^{-1}$$

Now, we consider four subcases for the parameters $\Lambda$, $n$, $k$, $k_1$ whether zero or not.

In subcase $\Lambda = 0$ and $\xi_0 = 0$, metric (24) with expressions $p$, $\rho$, $\theta$ and $\sigma$ for this model are same as that of solution (18) of Camci et al. [40]. If we set $\xi_0 = 0$, the metric (24) gives the solution obtained by Pradhan et al. [41].

In subcase $\Lambda = 0$, $n = 0$, after a suitable inverse time transformation, we find that

$$ds^2 = dt^2 - K_1 (t + t_0)^2/3 dx^2 - e^{2\alpha x} \left[ K_1 (t + t_0)^{4/3} dy^2 + dz^2 \right],$$

where $t_0$ is a constant of integration and $K_1 = (3k/2)^{2/3}$. The expressions $p$, $\rho$, $\theta$ and $\sigma$ for this model are not given here, since it is observed that the physical properties of this one are same as that of the solution (24) of Ram [35].

In subcase $\Lambda = 0$, $k = 0$, after inverse time transformation and rescaling, the metric (24) reduces to

$$ds^2 = dt^2 - K_2 (t + t_1)^{4m/2} \left[ dx^2 + e^{2\alpha x} (dy^2 + dz^2) \right],$$

where $t_1$ is a constant of integration and $K_2 = (24)^{2m/2}$. For this solution, when $n = 1$ and $\alpha = 0$, we obtain Einstein and de Sitter [48] dust filled universe. For $K_2 = 1$, $t_1 = 0$ and $n = 2m/(2m)$, where $m$ is a parameter in Ram’s paper [35], the solution (37) reduces to the metric (13) of Ram [35]. In later case, if also $\alpha = 0$, then we get the Minkowski space-time.
Now, in subcase $\Lambda = 0, k_1 = 0$, after some algebra the metric (24) takes the form
\[
 ds^2 = dt^2 - 2\ell K_3(t + t_2)^{2/3} \left[ dx^2 + e^{2\alpha x} (at^{m_1} dy^2 + a^{-1}t^{-m_1} dz^2) \right], \tag{38}
\]
where $t_2$ is a constant, $at^{m_1} = 2\ell K_3 (t + t_2)^{2(2-n)/3}$ and $K_3 = \left[ \frac{3(2-n)}{4\sqrt{2\ell}} \right]^{2/3}$.

For $t_2 = 0$, $k = \frac{2}{3}$ and $n = 0$ from (38), we obtain that the solution (24) of Ram [35].

Some Physical aspects of model:

The model (24) has barrel singularity at $\tau = \tau_0$ given by
\[
 \tau_0 = \left[ \frac{k_1 (5n - 2)}{2k} \right]^{2/(2-5n)}.
\]
which corresponds to $t = 0$. For $n \neq 2/5$ from (24), it is observed that at the singularity state $\tau = \tau_0$, $p$, $\rho$, $\Lambda$, $\theta$ and $\sigma$ are infinitely large. At $t \to \infty$, which corresponds to $\tau \to \infty$ for $n < 2/5$ and $k > 0$, or $\tau \to 0$ for $n > 2/5$ and $k < 0$, $p$, $\rho$, $\Lambda$, $\theta$ and $\sigma$ vanish. Therefore, for $n \neq 2/5$, the solution (24) represents an anisotropic universe exploding from $\tau = \tau_0$, i.e. $t = 0$, which expands for $0 < t < \infty$. We also find that the ratio $\sigma/\theta$ tends to a finite limit as $t \to \infty$, which means that the shear scalar does not tend to zero faster than the expansion. Hence the model does not approach isotropy for large values of $t$.

In subcase $\Lambda = 0, k_1 = 0$, the ratio (35) tends to zero, then the model approaches isotropy i.e. shear scalar $\sigma$ goes to zero. For the model (37), $p$ and $\rho$ tends to zero as $t \to \infty$; the model would give an essentially empty universe at large time. The dominant energy condition given by Hawking and Ellis [49] requires that
\[
 \rho + p \geq 0, \quad \rho + 3p \geq 0 \tag{39}
\]
Thus, we find for the model (37) that $n(2-n) \geq 0$. Hence for the values $0 \leq n \leq 2$, the universe (37) satisfies the strong energy condition i.e. $\rho + 3p \geq 0$. Also this model is shear-free and expanding.

In subcase $\Lambda = 0, k_1 = 0$, for $n \neq 2/5, 2$, it is observed from relations (24) - (34) that $p$, $\rho$, $\theta$ and $\sigma$ are infinitely large at the singularity state $t = -t_2$. When $t \to \infty$, these quantities vanish. We also find that the ratio $\sigma/\theta$ is a constant. This shows that the cosmological model (38) does not approach isotropy for large value of $t$. In this model the dominant energy conditions (39) are then verified for $6 - 5n - 25n^2 \geq 0$. Since $n \neq 2/5$, the model (38) satisfies the strong energy condition for $-3/5 \leq n \leq 2/5$.

In each of subcases, all the obtained solutions (36), (37) and (38) satisfy the Bianchi identity given in Eq. (10).
3.2 Case (II): Let $C = \tilde{\tau}^n$, ($n$ is a real number satisfying $n \neq 2/3$).

In this case Eq. (19) gives

$$B = k_2 \tilde{\tau}^n \exp \left( M \tilde{\tau}^{\ell_1} \right)$$

(40)

and from (14), we obtain

$$A^2 = k_2 \tilde{\tau}^{2n} \exp \left( M \tilde{\tau}^{\ell_1} \right)$$

(41)

where $M = \frac{k}{\ell_1}$. Hence the metric (11) reduces to the form

$$ds^2 = \tilde{\tau}^{4(1-\ell_1)/3} \left[ \tilde{\tau}^{2(1-\ell_1)/3} e^{3M \tilde{\tau}^{\ell_1}} d\tau^2 - e^{M \tilde{\tau}^{\ell_1}} dx^2 
- e^{2ax} \left( e^{2M \tilde{\tau}^{\ell_1}} dy^2 + dz^2 \right) \right],$$

(42)

where the constant $k_2$ is taken, without any loss of generality, equal to 1. This metric is a three-parameter family of solutions to EFEs with a perfect fluid.

For the above model, the distribution of matter and nonzero kinematical parameters are obtained as

$$8\pi(\bar{p} + \Lambda) = 2n \tilde{\tau}^{2(\ell_1-2)} + 3nk \tilde{\tau}^{3\ell_1-4}$$

$$+ \frac{k^2}{2} \tilde{\tau}^{4(\ell_1-1)} + \alpha^2 \tilde{\tau}^{4(\ell_1-1)/3} e^{-3M \tilde{\tau}^{\ell_1}},$$

$$8\pi(\rho - \Lambda) = 3n^2 \tilde{\tau}^{2(\ell_1-2)} + 3nk \tilde{\tau}^{3\ell_1-4}$$

$$+ \frac{k^2}{2} \tilde{\tau}^{4(\ell_1-1)} - 3\alpha^2 \tilde{\tau}^{4(\ell_1-1)/3} e^{-3M \tilde{\tau}^{\ell_1}}.$$  

(43)

(44)

For simplicity and realistic models of physical importance, we consider the following two cases ($w = 0, 1$):

### 3.2.1 Model I: Solution for $w = 0$

When $w = 0$, Eq. (28) reduces to $\xi = \xi_0 = \text{constant}$. Hence in this case Eq. (13), with the use of (14), (27) and (28), leads to

$$8\pi(1+\gamma)\rho = n(2+3n)\tilde{\tau}^{2(\ell_1-2)} + 6nk \tilde{\tau}^{(3\ell_1-4)} + k^2 \tilde{\tau}^{4(\ell_1-1)} - 2D_3 + 24\pi \xi_0 D_2,$$

(45)

where

$$D_2 = n\tilde{\tau}^{(\ell_1-2)} + \frac{1}{2}k^2 \tilde{\tau}^{2(\ell_1-1)},$$

$$D_3 = \alpha^2 \tilde{\tau}^{4(\ell_1-1)} e^{-3M \tilde{\tau}^{\ell_1}}.$$

Eliminating $\rho(t)$ from Eqs. (14) and (15), we obtain

$$8\pi(1+\gamma)\Lambda = n(2-3n\gamma)\tilde{\tau}^{2(\ell_1-2)} + 3nk(1-\gamma)\tilde{\tau}^{(3\ell_1-4)}$$

$$+ \frac{1}{2}k^2(1-\gamma)\tilde{\tau}^{4(\ell_1-1)} + (1-3\gamma)D_3 + 24\pi \xi_0 D_2.$$  

(46)
3.2.2 Model II: Solution for $w = 1$

When $w = 1$, Eq. (28) reduces to $\xi = \xi_0 \rho$. Hence in this case Eq. (43), with the use of (44), (27) and (28), leads to

$$8\pi \rho = \frac{n(2 + 3n)\hat{\tau}^{2(\ell_1 - 2)} + 6nk\hat{\tau}(3\ell_1 - 4) + k^2\hat{\tau}^4(\ell_1 - 1) - 2D_3}{(1 + \gamma) - 3\xi_0 D_2}. \quad (47)$$

Eliminating $\rho(t)$ from Eqs. (44) and (47), we obtain

$$8\pi \Lambda = \frac{n(2 + 3n)\hat{\tau}^{2(\ell_1 - 2)} + 6nk\hat{\tau}(3\ell_1 - 4) + k^2\hat{\tau}^4(\ell_1 - 1) - 2D_3}{(1 + \gamma) - 3\xi_0 D_2} - 3n(2)\hat{\tau}^{2(\ell_1 - 2)} - 3nk\hat{\tau}(3\ell_1 - 4) - \frac{1}{2}k^2\hat{\tau}^4(\ell_1 - 1) + 3D_3. \quad (48)$$

In Mode I, from Eqs. (45) and (46), we observe that the energy density $\rho(t)$ and cosmological constant $(\Lambda(t))$ decrease with time increases (see Figure 2). Here we find energy density always positive. We also observe that the value of $\Lambda$ is small and positive at late times which is supported by recent type Ia supernovae observations $[9] - [12]$. From Eqs. (47) and (48), we observe that Model II has the similar behaviour as Model I, so it is not reproduced here.

The scalar of expansion $\theta$ and the shear $\sigma$ are obtained as

$$\theta = 3 \left[ n\hat{\tau}^{\ell_1 - 2} \frac{k}{2} \hat{\tau}^{2(\ell_1 - 1)} \right], \quad (49)$$

$$\sigma = \frac{k}{2} \hat{\tau}^{2(\ell_1 - 1)} e^{-3M\hat{\tau}}, \quad (50)$$

From Eqs. (49) and (50), we have

$$\frac{\sigma}{\theta} = \frac{k}{6(n\hat{\tau} - \ell_1 + \frac{k}{2})}. \quad (51)$$

In subcase $\Lambda = 0$ and $\xi_0 = 0$, metric (42) with expressions $p$, $\rho$, $\theta$ and $\sigma$ for this model are same as that of solution (27) of Camci et al. [40].

In sub-case $\Lambda = 0$, $\xi_0 = 0$, $\ell_1 = 1$ (i.e. $n = 0$), we find a similar solution to (36), and hence this subclass is omitted. For $k = 0$, the ratio (51) is zero and hence there is no anisotropy.

After a suitable coordinate transformation, the metric (42) can be written as

$$ds^2 = dt^2 - K_4(t + t_3)^{2\ell_1} \left[ dx^2 + e^{2\alpha x}(dy^2 + dz^2) \right], \quad (52)$$

where $t_3$ is a constant and $K_4 = \left[ \frac{2}{(2 - 3M_1)} \right]^{2M_1}$, $M_1 = \frac{2n}{2 + 3n} \neq \frac{2}{3}$, where $M_1$ is a new parameter. When $M_1 = 0$ and $\ell_1 = 0$, from (52), we get the solution (12).
Some physical aspects of model:

The models have singularity at $\tilde{\tau} \to -\infty$ for $\ell_1 > 0$ or $\tilde{\tau} \to 0$ for $\ell_1 < 0$, which corresponds to $t \to 0$. It is a point type singularity for $\ell_1 > 0$ whereas it is a cigar or a barrel singularity according as $\ell_1 < 0$. At $t \to \infty$, which correspond to $\tilde{\tau} \to \infty$ for $\ell_1 > 0$ or $\tilde{\tau} \to 0$ for $\ell_1 < 0$, from Eqs. (45) - (50), we obtain that for $\ell_1 > 0$, $p, \rho \to 0$, and $\sigma, \theta \to 0$ ($k > 0$), $-\infty$ ($k < 0$); for $\ell_1 < 0$, similar the above ones. Then, clearly, for a realistic universe, it must be fulfill $\tilde{\tau} \to -\infty$, $n$, and $k$ are positive and $\ell_1$ is an odd positive number; as $\tilde{\tau} \to 0$, $k$ is positive, and $\ell_1$ an even negative number. Also, since $\lim_{\tilde{\tau} \to \infty} \sigma, \theta \neq 0$, therefore these models do not approach isotropy for large values of $\tilde{\tau}$.

In subcase $k = 0$ for the metric (52), the effective pressure, density and cosmological constant are given by

$$8\pi(\bar{p} + \Lambda) = \frac{\ell_1(2 - 3\ell_1)}{(t + t_3)^2} + \frac{\alpha^2}{K_4(t + t_3)^{2\ell_1}},$$

$$8\pi(\rho - \Lambda) = \frac{3\ell_1}{(t + t_3)^2} - \frac{3\alpha^2}{K_4(t + t_3)^{2\ell_1}}.$$  (54)

When $\Lambda = 0$ and $\xi_0 = 0$, the pressure and energy density are same as that of given in Eq. (44) of paper Camci et al. [40]. In this case, the weak and strong energy conditions (39) for this solution are identically satisfied when $\ell_1(1 - \ell_1) \geq 0$ i.e. $0 \leq \ell_1 \leq 1$. This model is shear-free and expanding with $\theta = \frac{\tau}{t + t_3}$.

3.3 Case (III) : Let $B = T^n$ ($n$ is a real number).

In this case Eq. (20) gives

$$C = k_3T^n - 2\ell T^{\ell_1}$$  (55)

and then from (14), we obtain

$$A^2 = k_3T^{2n} - 2\ell T^{\ell_1 + n}$$  (56)

Hence the metric (14) takes the new form

$$ds^2 = \left(k_3T^n - 2\ell T^{\ell_1}\right)[dt^2 - T^n dx^2] -$$

$$e^{2\alpha x}\left[T^{2n}dy^2 + \left(k_3T^n - 2\ell T^{\ell_1}\right)^2 dz^2\right]$$  (57)

For four-parameters family of solution (57), the physical and kinematical quantities are given by

$$8\pi(\bar{p} + \Lambda) = \left[-\frac{\ell^2}{2}(11n^2 - 4n - 4)T^{-3n} + \ell k_3n(13n - 10)T^{\ell_1 + n} -$$
Eliminating \( \rho \) leads to
\[
2k_3^2 n(n-1)T^{2n-2} \left( k_3T^n - 2\ell T^\ell t \right)^{-3} + \alpha^2 T^{-n} \left( k_3T^n - 2\ell T^\ell t \right)^{-1},
\] (58)
\[
8\pi(\rho - \Lambda) = \left[ -\frac{\ell^2(11n^2 - 4n - 14)}{2}T^{-3n} - 3k_3n(2-n)T^\ell t+n+\gamma \right]
\]
where
\[
\Lambda = 1 \left( 1 + \gamma \right) \rho - \frac{3k_3^2n^2T^{2n-2}}{2} \left( k_3T^n - 2\ell T^\ell t \right)^{-3} - 3\alpha^2 T^{-n} \left( k_3T^n - 2\ell T^\ell t \right)^{-1}.
\] (59)

### 3.3.1 Model I: Solution for \( w = 0 \)

When \( w = 0 \), Eq. (23) reduces to \( \xi = \xi_0 = \) constant. Hence in this case Eq. (58), with the use of (59), (27) and (28), leads to
\[
8\pi(1+\gamma)\rho = \left[ n(n+2)k_3^2T^{2(n-1)} + 16n(n-1)k_3\ell T^\ell t+n+-n(2n^2 - 4n - 9)\ell^2 T^{-3n} \right] D_4^{-3}
\]
\[
- 2\alpha^2 T^{-n} D_4^{-1} + 24\pi\xi_0 D_5 D_4^{-\frac{5}{2}},
\] (60)
where
\[
D_4 = k_3T^n - 2\ell T^\ell t,
\]
\[
D_5 = k_3nT^{n-1} + \frac{1}{2}(n-2)\ell T^{\frac{3}{2}}.
\]

Eliminating \( \rho(t) \) from Eqs. (59) and (60), we obtain
\[
8\pi(1+\gamma)\Lambda = \left[ \left( 1 - \frac{1}{2}\ell^2(1-\gamma)(11n - 4) - \ell^2(7\gamma - 2) \right) T^{-3n}
\right.
\]
\[
+ n\ell k_3 \left\{ 13n - 10 + 3\gamma(2-n) \right\} T^\ell t+n-nk_3^2(2n - 2 + 3n\gamma)T^{2(n-1)} \right] D_4^{-3}
\]
\[
\left. + (1 + 3\gamma)\alpha^2 T^{-n} D_4^{-n} + 24\pi\xi_0 D_5 D_4^{-\frac{5}{2}}. \right]
\] (61)

### 3.3.2 Model II: Solution for \( w = 1 \)

When \( w = 1 \), Eq. (28) reduces to \( \xi = \xi_0 \rho \). Hence in this case Eq. (58), with the use of (59), (27) and (28), leads to
\[
8\pi\rho = \frac{1}{\left( 1 + \gamma \right) - 3\xi_0 D_5 D_4^{-\frac{5}{2}}} \left[ -\ell^2(11n^2 - 4n - 9)T^{-3n} + 16n(n-1)k_3\ell T^\ell t+n
\right.
\]
\[
\left. + n(n+2)k_3^2 T^{2(n-1)} \right] D_4^{-3} - \alpha^2 T^{-n} D_3^{-1}. \] (62)

Eliminating \( \rho(t) \) from Eqs. (59) and (62), we obtain
\[
8\pi\Lambda = \frac{1}{\left( 1 + \gamma \right) - 3\xi_0 D_5 D_4^{-\frac{5}{2}}} \left[ \left( -\ell^2(11n^2 - 4n - 9)T^{-3n} + 16n(n-1)k_3\ell T^\ell t+n
\right.
\right.
\]
\[
\left. + n(n+2)k_3^2 T^{2(n-1)} \right] D_4^{-3} - \alpha^2 T^{-n} D_3^{-1} -
\]
In Mode I, from Eqs. (60) and (61), we observe that the energy density \( \rho(t) \) and cosmological constant \( (\Lambda(t)) \) are decreasing functions of time (see Figure 2). The energy density is always positive. We also observe that the value of \( \Lambda \) is small and positive at late times which is supported by recent type Ia supernovae observations [9] – [12]. From Eqs. (62) and (63), we observe that Model II has the similar behaviour as Model I, so it is not reproduced here.

The scale of expansion and the shear are obtained as

\[
\theta = 3 \left[ \frac{\ell(n-2)}{2} T^{-3n/2} + k_3 n T^{n-1} \right] \left( k_3 T^n - 2\ell T^{\ell_1} \right)^{-3/2},
\]

(64)

\[
\sigma = k T^{-3n/2} \left( k_3 T^n - 2\ell T^{\ell_1} \right)^{-3/2}.
\]

(65)

From (64) and (65), we get

\[
\frac{\sigma}{\theta} = k \left[ k_3 n T^{-\ell_1+n} + \frac{\ell(n-2)}{2} \right]^{-1}.
\]

(66)

In subcase \( \Lambda = 0 \) and \( \xi = 0 \), metric (57) with expressions \( p, \rho, \theta \) and \( \sigma \) for this model are same as that of solution (34) of Camci et al. [40]. If we set \( \xi_0 = 0 \), this metric (57) represents the solution obtained by Pradhan et al. [41].

In subcase \( \Lambda = 0, \xi = 0, n = 0 \), after an inverse transformation, metric (57) reduces to the form

\[
ds^2 = dt^2 - K_5(t + t_4)^{2/3} dx^2 - e^{2\alpha x} \left[ dy^2 + K_6^2(t + t_4)^{4/3} dz^2 \right],
\]

(67)

where \( t_4 \) is an integrating constant. This model is different from the model (36) by a change of scale.

In subcase \( \Lambda = 0, k = 0 \), same model as (37) is obtained.

Further in subcase \( \Lambda = 0, \xi = 0 \), we see that the metric (57) takes the form

\[
ds^2 = dt^2 - 2\ell K_6(t + t_5)^{2/3} \left[ dx^2 + e^{2\alpha x} \left( b t^{m_2} dy^2 + b^{-1} t^{-m_2} dz^2 \right) \right],
\]

(68)

where \( t_5 \) is a constant, \( b t^{m_2} = 2\ell K_6 \left( t + t_5 \right)^{2(2-n)/6^{(2-n)}} \) and \( K_6 = \left[ \frac{3(2-n)}{4\sqrt{2}\ell} \right]^{2/3} \). This metric is only different from (37) by a change of sign. Also, in each of subcase the physical and kinematical properties of obtained metric are same as that of Case(I). Therefore, we do not consider here them.
3.4 Case (IV) : Let \( B = \tilde{\tau}^n \), where \( n \) is any real number.

In this case Eq. (21) gives

\[
C = k_4 \tilde{\tau}^n \exp \left( \frac{k}{\ell_1} \tilde{\tau} \right) \tag{69}
\]

and then from (14), we obtain

\[
A^2 = k_4 \tilde{\tau}^{2n} \exp \left( \frac{k}{\ell_1} \tilde{\tau} \right) \tag{70}
\]

Hence the metric (4) reduces to

\[
ds^2 = \tilde{\tau}^{2n} \exp \left( \frac{k}{\ell_1} \tilde{\tau} \right) \left[ \tilde{\tau}^n \exp \left( \frac{2k}{\ell_1} \tilde{\tau} \right) - dx^2 \right] \\
- e^{2\alpha x} \left[ dy^2 + \exp \left( \frac{2k}{\ell_1} \tilde{\tau} \right) - dz^2 \right], \tag{71}
\]

where, without any loss of generality, the constant \( k_4 \) is taken equal to 1. Expressions for physical and kinematical parameters for the model (71) are not given here, but it is observed that the properties of the metric (71) are same as that of the solution (42), i.e. the Case (II).

4 Concluding remarks

In this paper we have described a new exact solutions of EFES for Bianchi type V spacetime with a bulk viscous fluid as the source of matter and cosmological term \( \Lambda \) varying with time. Using a generation technique followed by Camci et al. [40], it is shown that the Einstein’s field equations are solvable for any arbitrary cosmic scale function. Starting from particular cosmic functions, new classes of spatially homogeneous and anisotropic cosmological models have been investigated for which the fluids are acceleration and rotation free but they do have expansion and shear. For \( \alpha = 0 \) in the metric (4), we obtained metrics as LRS Bianchi type I model (Hajj-Boutros [33], Hajj-Boutros and Sfeila [34], Ram [35], Mazumder [36], Pradhan and Kumar [37]) and Pradhan et al. [41]. It is also seen that the solutions obtained by Camci et al. [19], Ram [35], Pradhan and Kumar [37] and Bianchi type V models studied by Pradhan et al. [41] are particular cases (except one) of our solutions.

The cosmological constants in all models given in Section 3 are decreasing functions of time (except one, Model II of Case (I)) and they all approach a small positive value at late times which are supported by the results from recent supernova observations recently obtained by the High-z Supernova Team and Supernova Cosmological Project (Garnavich et al. [9], Perlmutter et al. [10]).
The features of these new solutions are that from our wide range of choice of parameters for Model I of Cases (I), (II), and (III), it is apparent from figures that energy density $\rho(t)$ and dynamic cosmological term $\Lambda(t)$ are decreasing function of time, remain positive and small at later stage during evolution. These two quantities remain finite and do not become zero at later stage. Hence this seem to be physically viable to explore physical mechanism for further detail study depending on relevance of the physical problem. These studies show that the bulk viscous effect is apparent on $\Lambda(t)$. In many astrophysical situations the viscosity calculated using statistical consideration do not give satisfactory viscous number. Hence the specific thing about bulk viscosity will depend on the details of the viscous nature of the matter. We have explored Model II of Cases (I), (II) and (III). So we feel that it is not necessary to display behaviour of $\rho(t)$ and $\Lambda(t)$. We have studied these models and no anomaly has been observed.

Model II of Case (I) is explored for variety of parameters. We find that many cases energy density decreases sharply and becomes negative very fast and later stage also remains negative but there is a slight increase at later stage and remains constant (negative constant), which we feel undesirable feature. So it is not discussed in the paper. Also in many cases it initially oscillatory. In this case $\Lambda(t)$ shows very peculiar behaviour. Only in few cases $\Lambda$ is initially negative then becomes positive but in all cases during initial evolution $\Lambda$ has singularities near $t$ close to zero and a few at later finite time. The cause of this behaviour is not very apparent from the model. Otherwise in wide range of parameters $\Lambda$ shows erratic behaviour. Most of the cases singular and erratic behaviour of $\Lambda(t)$ is persistent and cause is not known. So we do not discuss further. This may require detailed study separately and this paper is not appropriate place.

Acknowledgements

The authors would like to thank the Inter-University Centre for Astronomy and Astrophysics, Pune, India for providing facility where this work was carried out.

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Figure 1: (Top) Plot of $\rho \rightarrow \tau$ and (bottom) $\Lambda \rightarrow \tau$ for parameters $n = 1/4$, $k_1 = 1$, $\gamma = 0.5$, $\alpha = 1$, $\xi_0 = 1$ and rest of the constants are set to 1.
Figure 2: (Top) Plot of $\rho \to \tilde{\tau}$ and (bottom) $\Lambda \to \tilde{\tau}$ for parameters $n = 0.45$, $K = 2.0$, $k_3 = 2$, $\gamma = 0.5$, $\alpha = 1$, $\xi_0 = 1$, $M = 1$ and rest of the constants are set to 1.
Figure 3: (Top) Plot of $\rho \rightarrow T$ and (bottom) $\Lambda \rightarrow T$ for parameters $n = 0.45$, $K = 2.0$, $k_3 = 1$, $\gamma = 0.5$, $\alpha = 1$, $\xi_0 = 1$ and rest of the constants are set to 1.