SMALLNESS AND COMPARISON PROPERTIES FOR MINIMAL DYNAMICAL SYSTEMS

JULIAN BUCK

Abstract. We introduce the dynamic comparison property for minimal dynamical systems which has applications to the study of crossed product $C^*$-algebras. We demonstrate that this property holds for a large class of systems which includes all examples where the underlying space is finite-dimensional, as well as for an explicit infinite-dimensional example, showing that the it is strictly weaker than finite-dimensionality in general.

1. Introduction

The role of topological dynamics in the classification program for nuclear $C^*$-algebras enjoys a rich and long history. Transformation group $C^*$-algebras associated to Cantor minimal systems were studied and classified in [16] and [5]. The case for general finite-dimensional compact metric spaces was analyzed in [12], [8], and [18], where it is shown that the transformation group $C^*$-algebra $C^*(Z, X, h)$ is classifiable so long as projections separate traces. In [18], Toms and Winter additionally prove that $C^*(Z, X, h)$ is stable under tensoring with the Jiang-Su algebra $Z$, assuming only the finite-dimensionality of $X$.

Very little is known in the case where $X$ is an infinite-dimensional space. Giol and Kerr [3] have constructed examples of infinite-dimensional minimal dynamical systems such that the associated transformation group $C^*$-algebras have perforation in their K-theory and Cuntz semigroups (such examples lie outside the $Z$-stable class). The existence of such minimal systems demonstrates the intractability of classifying $C^*$-algebras associated to general infinite-dimensional dynamical systems and indicates the need for regularity properties which rule out the type of behavior exhibited by their examples. In this paper we survey some existing regularity properties, and introduce a new one which holds for all finite-dimensional minimal systems as well as for at least some infinite-dimensional examples. Applications to crossed product $C^*$-algebras are referenced which will appear in subsequent papers.

We would like to thank N. Christopher Phillips for his numerous suggestions and insights, as much of it was completed at the University of Oregon under his supervision as part of the author’s Ph.D. thesis. We would also like to thank Taylor Hines, David Kerr, and Ian Putnam for several helpful conversations and ideas related to aspects of this paper.

Date: 27 June 2013.

2010 Mathematics Subject Classification. Primary 37B05, Secondary 46L35.
2. Preliminaries

Notation 2.1. Throughout, we let $X$ be an infinite compact metrizable space, and let $h: X \to X$ be a minimal homeomorphism. The corresponding minimal dynamical system $(X, h)$ will sometimes be denoted simply by $X$, with the homeomorphism $h$ understood. We denote by $M_h(X)$ the space of $h$-invariant Borel probability measures on $X$. Whenever necessary, it will be assumed that $X$ is a metric space with metric $d$. In this case, for $x \in X$ and $\varepsilon > 0$, we will denote the $\varepsilon$-ball centered at $x$ by

$$B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}.$$ 

We denote the boundary of a set $A \subset X$ by $\partial A$. In particular, if $U \subset X$ is open then $\partial U = \overline{U} \setminus U$, and if $C \subset X$ is closed then $\partial C = C \cap \text{int}(C)$.

Definition 2.2. A Borel set $E \subset X$ is called universally null if $\mu(E) = 0$ for all $\mu \in M_h(X)$.

Lemma 2.3. Let $(X, h)$ be as in Notation 2.1. If $U \subset X$ is open and non-empty, then $\mu(U) > 0$ for all $\mu \in M_h(X)$. Moreover, if $C \subset X$ is closed and $\mu(C) < \mu(U)$ for all $\mu \in M_h(X)$, then $\inf_{\mu \in M_h(X)} [\mu(U) - \mu(C)] > 0$. In particular, $\inf_{\mu \in M_h(X)}[\mu(U)] > 0$ for any non-empty open set $U \subset X$.

Proof. It is well-known that if $U \subset X$ is open and non-empty, then the minimality of $h$ implies that $X = \bigcup_{n=-\infty}^{\infty} h^n(U)$. Suppose that $\mu(U) = 0$ for some $\mu \in M_h(X)$. Then the $h$-invariance of $\mu$ implies that

$$1 = \mu(X) = \mu \left( \bigcup_{n=-\infty}^{\infty} h^n(U) \right) \leq \sum_{n=-\infty}^{\infty} \mu(h^n(U)) = \sum_{n=-\infty}^{\infty} \mu(U) = 0,$$

a contradiction. Now, the map $\gamma_U: M_h(X) \to [0, 1]$ given by $\gamma_U(\mu) = \mu(U)$ is a lower-semicontinuous function by Proposition 2.7 of [13], and strictly positive by the above argument. By an entirely analogous argument as in the proof of Proposition 2.7 of [13], if $C \subset X$ is closed the map $\gamma_C: M_h(X) \to [0, 1]$ given by $\gamma_C(\mu) = \mu(C)$ is upper-semicontinuous. Define $\gamma_{U,C}: M_h(X) \to [0, 1]$ by $\gamma_{U,C}(\mu) = \gamma_U(\mu) - \gamma_C(\mu) = \mu(U) - \mu(C)$. Since the sum of upper-semicontinuous functions is upper-semicontinuous, and the negative of a positive upper-semicontinuous function is lower-semicontinuous, it follows that $\gamma_{U,C} = -[\gamma_U - \gamma_C]$ is lower-semicontinuous. By assumption, $\gamma_{U,C}$ is strictly positive. The compactness of $M_h(X)$ implies that $\gamma_{U,C}$ achieves a lower bound $\delta > 0$ on $M_h(X)$. But then $\inf_{\mu \in M_h(X)} [\mu(U) - \mu(C)] \geq \delta > 0$, as claimed. The final observation follows immediately from the previous one by taking $C = \emptyset$. \hfill $\Box$

Lemma 2.4. Let $(X, h)$ be as in Notation 2.1. For any $\varepsilon > 0$ and any closed set $F \subset X$ with $F$ universally null, there is a non-empty open set $E \subset X$ such that $F \subset E$ and $\mu(E) < \varepsilon$ for all $\mu \in M_h(X)$.

Proof. Define a sequence $(E_n)_{n=0}^{\infty}$ of open sets by $E_n = \{x \in X : \text{dist}(x, F) < 1/n\}$. Then $\overline{E_{n+1}} \subset E_n$ for all $n \in \mathbb{N}$, and $\bigcap_{n=0}^{\infty} E_n = F$. Choose continuous functions $f_n: X \to [0, 1]$ with $f_n = 1$ on $\overline{E_{n+1}}$ and $\text{supp}(f_n) \subset E_n$. Then $f_n \geq f_{n+1}$ for all $n \in \mathbb{N}$. Now each $f_n$ defines an affine function $\hat{f}_n$ on $M_h(X)$ by

$$\hat{f}_n(\mu) = \int_X f_n \, d\mu.$$
Applying the Dominated Convergence Theorem, we conclude that
\[
\lim_{n \to \infty} \hat{f}_n(\mu) = \lim_{n \to \infty} \int_X f_n \, d\mu = \int_X \lim_{n \to \infty} f_n \, d\mu = \mu(F) = 0
\]
for all $\mu \in M_h(X)$. It follows that the monotone decreasing sequence $(\hat{f}_n)_{n=1}^\infty$ of continuous functions converges pointwise to the continuous affine function $\hat{f} = 0$ on the compact set $M_h(X)$, and so Dini’s Theorem implies that the convergence is uniform. Therefore, there is an $N \in \mathbb{N}$ such that $\hat{f}_N(\mu) < \varepsilon$ for all $\mu \in M_h(X)$. Finally, set $E = E_{N+1}$. Then $F \subseteq E$, and $f_N|_F = 1$ implies that
\[
\mu(E) \leq \int_X f_N \, d\mu = \hat{f}_N(\mu) < \varepsilon
\]
for all $\mu \in M_h(X)$.

**Corollary 2.5.** Let $(X, h)$ be as in Notation 2.1

1. For any $\varepsilon > 0$ and any non-empty open set $U \subseteq X$, there is a non-empty open set $E \subseteq U$ such that $\mu(E) < \varepsilon$ for all $\mu \in M_h(X)$.
2. For any $\varepsilon > 0$ and any non-empty open set $U \subseteq X$ with $\partial U$ universally null, there is a closed set $K \subseteq U$ with $\text{int}(K) \neq \emptyset$ such that $\mu(U \setminus K) < \varepsilon$ for all $\mu \in M_h(X)$.
3. For any $\varepsilon > 0$ and any non-empty open set $U \subseteq X$ with $\partial U$ universally null, there is an open set $E \subseteq U$ with $\overline{E} \subseteq U$ such that $\mu(U \setminus \overline{E}) < \varepsilon$ for all $\mu \in M_h(X)$.
4. For any $\varepsilon > 0$ and any closed set $K \subseteq X$ with $\partial K$ universally null, there is an open set $E \subseteq X$ such that $K \subseteq E$ and $\mu(E \setminus K) < \varepsilon$ for all $\mu \in M_h(X)$.

**Proof.**

1. Fix any point $x_0 \in U$, and let $F = \{x_0\}$, which is easily seen by minimality to satisfy $\mu(F) = 0$ for all $\mu \in M_h(X)$. Let $\delta > 0$ be such that $B(x_0, \delta) \subseteq U$, and replace the sets $E_n$ in the proof of Lemma 2.4 by $E_n = B(x_0, \delta/n)$. This ensures that the set $E = E_{N+1}$ satisfies $E \subseteq U$.
2. Let $F = \partial U$, let $E = E_{N+1}$ be as in the proof of Lemma 2.5 but with $N$ chosen so large that $X \setminus \overline{E} \neq \emptyset$, and set $K = U \cap (X \setminus \overline{E})$. Note that $K \subseteq U$ since $\text{dist}(X \setminus \overline{E}, X \setminus U) > 1/(N+1)$, and that $\text{int}(K) = U \cap (X \setminus \overline{E}) \neq \emptyset$. Finally, $U \setminus K \subseteq U \setminus (U \cap (X \setminus \overline{E})) \subseteq E$ implies $\mu(U \setminus K) \leq \mu(E) < \varepsilon$ for all $\mu \in M_h(X)$.
3. This is a convenient restate mant of (2), with $\text{int}(K) = E$ and $K = \overline{E}$.
4. Let $F = \partial K$, let $E$ be as in the proof of Lemma 2.6, and set $U = K \cup E$. Then $U$ is open since $U = \text{int}(K) \cup \partial(K) \cup E = \text{int}(K) \cup E$ (as $\partial(K) \subseteq E$), and $\mu(U \setminus K) = \mu(E) < \varepsilon$ for all $\mu \in M_h(X)$.

□

3. Topologically Small and Thin Sets

The following theorem is the well-known Rokhlin tower construction, where the space $X$ is decomposed in terms of a closed set $Y \subseteq X$ and the “first return times to $Y$” for the points of $X$. We show that a Rokhlin tower can be made compatible with some given partition of $X$ by sets with non-empty interior, in the sense that the interior of each level in the tower is contained in exactly one set of the partition.
Theorem 3.1. Let \((X, h)\) be as in Notation 2.1. Let \(Y \subset X\) be a closed set with \(\text{int}(Y) \neq \emptyset\). For \(y \in Y\), define \(r(y) = \min \{m \geq 1: h^m(y) \in Y\}\). Then \(\sup_{y \in Y} r(y) < \infty\), so there are finitely many distinct values \(n(0) < n(1) < \cdots < n(l)\) in the range of \(r\). For \(0 \leq k \leq l\), set
\[
Y_k = \{y \in Y: r(y) = n(k)\} \quad \text{and} \quad Y_k^c = \text{int}(\{y \in Y: r(y) = n(k)\}).
\]

Then:
1. the sets \(h^j(Y_k^c)\) are pairwise disjoint for \(0 \leq k \leq l\) and \(0 \leq j \leq n(k) - 1\);
2. \(\bigcup_{k=0}^l Y_k = Y\);
3. \(\bigcup_{k=0}^l \bigcup_{j=0}^{n(k)-1} h^j(Y_k) = X\).

Moreover, given any finite partition \(\mathcal{P}\) of \(X\) (consisting of sets with non-empty interior), there exist closed sets \(Z_0, \ldots, Z_m \subset Y\) and non-negative integers \(t(0) \leq t(1) \leq \cdots \leq t(m)\) such that with \(Z_k^{(0)} = Z_k \setminus \partial Z_k\) (which may be empty) for \(0 \leq k \leq m\), we have:
1. the sets \(h^j(Z_k^{(0)})\) are pairwise disjoint for \(0 \leq k \leq m\) and \(0 \leq j \leq t(k) - 1\);
2. \(\bigcup_{k=0}^m Z_k = Y\);
3. \(\bigcup_{k=0}^m \bigcup_{j=0}^{t(k)-1} h^j(Z_k) = X\);
4. for \(0 \leq k \leq m\) and \(0 \leq j \leq t(k) - 1\), the set \(h^j(Z_k^{(0)})\) is contained in exactly one \(P \in \mathcal{P}\).

Proof. The finiteness of \(r(y)\) and all statements concerning the sets \(Y_k\) are shown in [9, 10], and [12] (as well as other places). Now suppose we have a finite partition \(\mathcal{P}\) of \(X\) consisting of sets with non-empty interior. For each \(0 \leq k \leq l\), the set
\[
B_k = \left\{ h^{-j} (h^j(Y_k) \cap P) : 0 \leq j \leq n(k) - 1, P \in \mathcal{P} \right\}
\]
is a cover of \(Y_k\) by a finite collection of sets with non-empty interior. Write \(B_k = \{B_1, \ldots, B_N\}\) for an appropriate choice of \(N \in \mathbb{N}\). Let \(\mathcal{C}_k\) be the collection of all sets of the form \(D = \bigcap_{i=1}^m C_i\), where each for each \(i\), there is a \(j \in \{1, \ldots, N\}\) such that either \(C_i = B_j\) or \(C_i = Y_k \setminus B_j\). Set \(\mathcal{C}^0 = \bigcup_{k=0}^m \mathcal{C}_k\) and \(\mathcal{C} = \{\mathcal{D}: D \in \mathcal{C}^0\}\), both of which are finite collections of sets. Write \(\mathcal{C} = \{Z_0', \ldots, Z_m'\}\), and for \(0 \leq i \leq m\), set \(t(i) = n(k)\) where \(Z_i = \mathcal{D}\) and \(D \in \mathcal{C}_k\). Without loss of generality, arrange the order of the sets \(Z_0', \ldots, Z_m'\) so that \(t(0) \leq t(1) \leq \cdots \leq t(m)\). Finally, define \(Z_k\) and \(Z_k^{(0)}\) for \(0 \leq k \leq m\) by
\[
Z_0 = Z_0', \quad Z_k = Z_k' \setminus \bigcup_{j=0}^{k-1} Z_j, \quad Z_k^{(0)} = Z_k \setminus \partial Z_k.
\]
Then \(Z_0, \ldots, Z_m\) is a cover of \(Y\) by closed sets with the desired properties. \(\Box\)

In applications of the Rokhlin tower construction to \(C^*\)-algebras, it is often technically important to have some control over the boundary \(\partial Y\) of the closed set \(Y \subset X\) used as the base. In [8] the sets employed are taken to have universally null boundaries, but this restriction will be too weak for our purposes. Instead, we need to insist the boundaries of the sets used be small in a more topological sense. In [12] this is accomplished by restricting to the situation where \(X\) is a compact smooth manifold and \(h\) is a minimal diffeomorphism, then requiring that \(\partial Y\) satisfy a certain transversality condition. Definition 3.2 that follows, which first appeared in [17], is an attempt to formulate an analogous property for the case of a more general compact metric space. For our purposes, we will often find it convenient to
use another form of a smallness property for closed sets, which is given in Definition 3.4. The connection between these two definitions is considered in Proposition 3.10.

**Definition 3.2.** Let \((X, h)\) be as in Notation 2.1. A closed subset \(F \subset X\) is said to be **topologically \(h\)-small** if there is some \(m \in \mathbb{Z}_+\) such that whenever \(d(0), d(1), \ldots, d(m)\) are \(m + 1\) distinct elements of \(\mathbb{Z}\), then \(h^{d(0)}(F) \cap h^{d(1)}(F) \cap \cdots \cap h^{d(m)}(F) = \emptyset\). The smallest such constant \(m\) is called the **topological smallness constant**.

We assemble some basic results about topologically \(h\)-small sets.

**Lemma 3.3.** Let \((X, h)\) be as in Notation 2.1.

1. Let \(F \subset X\) be topologically \(h\)-small with topological smallness constant \(m\), let \(K \subset F\) be closed, and let \(d \in \mathbb{Z}\). Then \(K\) is topologically \(h\)-small with smallness constant at most \(m\), and \(h^{d}(F)\) is topologically \(h\)-small with smallness constant \(m\).
2. The intersection of arbitrarily many topologically \(h\)-small sets is topologically small.
3. Let \(F_1, \ldots, F_n \subset X\) be topologically \(h\)-small, where \(F_j\) has topological smallness constant \(m_j\) for \(1 \leq j \leq n\). Then \(F = \bigcup_{j=1}^{n} F_j\) is topologically \(h\)-small with smallness constant \(m = \sum_{j=1}^{n} m_j\).

**Proof.**

1. This is immediate from the definition.
2. This follows immediately from (1), since the intersection of arbitrarily many closed sets is a closed subset of any one of them.
3. With \(m = \sum_{j=1}^{n} m_j\), let \(d(0), \ldots, d(m) \in \mathbb{Z}\) be \(m + 1\) distinct integers. Let \(S\) be the collection of all functions \(s: \{0, \ldots, m\} \to \{1, \ldots, n\}\). By the pigeonhole principle, for any \(s \in S\) there is a \(j \in \{1, \ldots, n\}\) such that \(\text{card}(s^{-1}(j)) > m_j\). Since \(F_j\) has topological smallness constant \(m_j\), it follows that \(\bigcap_{i \in s^{-1}(j)} h_i(F_j) = \emptyset\). This implies that
   \[
   \bigcap_{j=0}^{m} h^{d(j)}(F_{s(j)}) \subset \bigcap_{i \in s^{-1}(j)} h_i(F_j) = \emptyset,
   \]
   which gives
   \[
   \bigcap_{j=0}^{m} h^{d(j)}(F) = \bigcup_{s \in S} \bigcap_{j=0}^{m} h^{d(j)}(F_{s(j)}) = \emptyset,
   \]
   as required.

**Definition 3.4.** Let \((X, h)\) be as in Notation 2.1. Let \(F \subset X\) be closed and let \(U \subset X\) be open. We write \(F \preceq U\) if there exist \(M \in \mathbb{N}, U_0, \ldots, U_M \subset X\) open, and \(d(0), d(M) \in \mathbb{Z}\) such that:

1. \(F \subset \bigcup_{j=0}^{M} U_j\);
2. \(h^{d(j)}(U_j) \subset U\) for \(0 \leq j \leq M\);
3. the sets \(h^{d(j)}(U_j)\) are pairwise disjoint for \(0 \leq j \leq M\).

We say the closed set \(F\) is **thin** if \(F \preceq U\) for every non-empty open set \(U \subset X\).

It is clear that any closed subset of a thin set is thin, and hence the intersection of arbitrarily many thin sets is thin. It is also clear that if \(F\) is thin, then so is \(h^n(F)\) for any \(n \in \mathbb{Z}\).
Lemma 3.5. Let \((X, h)\) be as in Notation 2.1. Suppose that \(F \subset X\) is closed and \(U \subset X\) is open with \(F \prec U\). Then there is an open set \(V \subset X\) such that \(F \subset V\) and \(\nabla \prec U\).

Proof. Since \(F \prec U\), there exist \(M \in \mathbb{N}, U_0, \ldots, U_M \subset X\) open, and \(d(0), \ldots, d(M) \in \mathbb{Z}\) such that \(F \subset \bigcup_{j=0}^{M} U_j\) and such that the sets \(h^{-d(j)}(U_j)\) are pairwise disjoint subsets of \(U\). Let \(E = \bigcup_{j=0}^{M} U_j\), and use \(X\) locally compact Hausdorff to choose an open set \(V\) with \(\nabla \subset V \subset \nabla \subset E\). Then \(\nabla \prec U\) using the same open sets \(U_j\) and integers \(d(j)\) as for \(F\).

\[\Box\]

Lemma 3.6. Let \((X, h)\) be as in Notation 2.1. If \(F \subset X\) is thin, then \(F\) is universally null.

Proof. Let \(\varepsilon > 0\) be given, let \(\mu \in \mathcal{M}_h(X)\), and choose \(N \in \mathbb{N}\) such that \(1/N < \varepsilon\). Since the action of \(h\) on \(X\) is free, there is a point \(x \in X\) such that \(x, h(x), \ldots, h^N(x)\) are distinct. Choose disjoint open neighborhoods \(U_0, \ldots, U_N\) of these points, and let \(U = \bigcap_{j=0}^{N} h^{-j}(U_j)\), which is an open neighborhood of \(x\) such that \(U, h(U), \ldots, h^N(U)\) are pairwise disjoint. Then using the \(h\)-invariance of \(\mu\), it follows that

\[(N + 1)\mu(U) = \sum_{j=0}^{N} \mu(h^j(U)) = \mu\left(\bigcup_{j=0}^{N} h^j(U)\right) \leq \mu(X) = 1,\]

which gives \(\mu(U) < 1/N < \varepsilon\). Since \(F\) is thin, we have \(F \prec U\), and so there exist \(M \in \mathbb{N}, U_0, \ldots, U_M \subset X\) open, and \(d(0), \ldots, d(M) \in \mathbb{Z}\) such that \(F \subset \bigcup_{j=0}^{M} U_j\) and such that the sets \(h^{d(j)}(U_j)\) are pairwise disjoint subsets of \(U\) for \(0 \leq j \leq M\). Then again using the \(h\)-invariance of \(\mu\), we have

\[\mu(F) \leq \mu\left(\bigcup_{j=0}^{M} U_j\right) \leq \sum_{j=0}^{M} \mu(U_j) \leq \sum_{j=0}^{M} \mu(h^{d(j)}(U_j))
= \mu\left(\bigcup_{j=0}^{M} h^{d(j)}(U_j)\right) \leq \mu(U) < \varepsilon.\]

Since \(\varepsilon > 0\) was arbitrary, it follows that \(\mu(F) = 0\).

\[\Box\]

Lemma 3.7. Let \((X, h)\) be as in Notation 2.1.

1. If \(F_1, F_2 \subset X\) are closed and \(V_1, V_2 \subset X\) are open such that \(F_1 \prec V_1, F_2 \prec V_2\), and \(V_1 \cap V_2 = \emptyset\), then \(F_1 \cup F_2 \prec V_1 \cup V_2\).
2. The union of finitely many thin sets in \(X\) is thin.

Proof. To prove (1), simply observe that since \(V_1 \cap V_2 = \emptyset\), the union of a pairwise disjoint collection of subsets of \(V_1\) and a pairwise disjoint collection of subsets of \(V_2\) is still pairwise disjoint.

For (2), it is sufficient to prove that the union of two thin sets is thin. Let \(F_1, F_2 \subset X\) be thin closed sets, and let \(U \subset X\) be a non-empty open set. Since \(h\) is minimal there must be distinct points \(x_1, x_2 \subset U\). Let \(V_1 \subset U\) and \(V_2 \subset U\) be disjoint open neighborhoods of \(x_1\) and \(x_2\) respectively. Then \(F_1 \prec V_1\) and \(F_2 \prec V_2\), and now part 1 implies that \(F_1 \cup F_2 \prec V_1 \cup V_2 \subset U\), which proves that \(F_1 \cup F_2\) is thin.

\[\Box\]

Lemma 3.8. Let \((X, h)\) be as in Notation 2.1. Let \(F \subset X\) be a thin closed set, and let \(U \subset X\) be open. Then there exist \(M \in \mathbb{N}, F_0, \ldots, F_M \subset X\) closed, and \(d(0), \ldots, d(M) \in \mathbb{Z}\) such that:

\[\Box\]
(1) \( F \subseteq \bigcup_{j=0}^{M} F_j \);
(2) \( h^{d(j)}(F_j) \subseteq U \) for \( 0 \leq j \leq M \);
(3) the sets \( h^{d(j)}(F_j) \) are pairwise disjoint for \( 0 \leq j \leq M \).

**Proof.** Since \( F \) is thin, we have \( F \prec U \), and so there exist \( M \in \mathbb{N}, U_0, \ldots, U_M \subset X \) open, and \( d(0), \ldots, d(M) \in \mathbb{Z} \) such that \( F \subseteq \bigcup_{j=0}^{M} U_j \) and the sets \( h^{d(j)}(U_j) \) are pairwise disjoint subsets of \( U \) for \( 0 \leq j \leq M \). Now temporarily fix \( j \in \{0, \ldots, M\} \).

For each \( x \in U_j \), let \( V^j_x \) be a neighborhood of \( x \) such that \( V^j_x \subset U_j \). Then \( \{V^j_x : x \in U_j, 0 \leq j \leq M\} \) is an open cover for \( F \), hence it contains a finite subcover. For \( 0 \leq j \leq M \) let \( S_j \) be the (possibly empty) collection of all sets \( V^j_x \) that appear in the finite subcover for \( F \), and set \( F_j = \bigcup_{V \in S_j} V \). Note that \( F_j = \emptyset \) if the collection \( S_j \) is empty. Then each \( F_j \) is closed (being the union of finitely many closed sets) and satisfies \( F_j \subset U_j \). It follows that the sets \( h^{d(j)}(F_j) \) are pairwise disjoint subsets of \( U \) for \( 0 \leq j \leq M \). \( \Box \)

**Lemma 3.9.** Suppose that \( d_0, \ldots, d_m \) are \( m+1 \) distinct integers, and that \( n_1, n_2 \) are distinct integers (but not necessarily distinct from the \( d_i \)). Then the set

\[
\{d_i + n_j : 0 \leq i \leq m, j = 1, 2\}
\]

contains at least \( m + 2 \) distinct integers.

**Proof.** Without loss of generality, suppose that \( d_0 < d_1 < \cdots < d_m \) and \( n_1 < n_2 \). Then we have

\[
d_0 + n_1 < d_1 + n_1 < \cdots < d_m + n_1 < d_m + n_2,
\]

which provides \( m + 2 \) distinct integers in the set \( \{d_i + n_j : 0 \leq i \leq m, j = 1, 2\} \). \( \Box \)

**Proposition 3.10.** Let \((X, h)\) be as in Notation 2.1. If \( F \subset X \) is topologically \( h \)-small, then \( F \) is thin.

**Proof.** The proof is by induction on the smallness constant \( m \). First consider the case where the smallness constant is \( m = 1 \). Then given \( j, k \in \mathbb{Z} \) with \( j \neq k \), we have \( h^j(F) \cap h^k(F) = \emptyset \). Let \( U \subset X \) be open and non-empty, and let \( V_0 \subset U \) be open and non-empty with \( \nabla_0 \subset X \). By Lemma 2.6 \( \{h^n(V_0) : n \in \mathbb{Z}\} \) is an open cover for \( F \), so there exists a finite subcover \( \{h^{d(0)}(V_0), \ldots, h^{d(M)}(V_0)\} \). Set \( F_j = F \cap h^{d(j)}(V_0) \). Then the sets \( h^{d(j)}(F_j) \) are closed, disjoint (since \( h^{d(j)}(F_j) \subset h^{d(j)}(F) \) and these sets are disjoint) and satisfy \( h^{d(j)}(F_j) \subset \nabla_0 \subset U \). Since \( X \) is normal, there exist disjoint open sets \( W_0, \ldots, W_M \subset X \) such that \( h^{d(j)}(F_j) \subset W_j \). Finally, for \( 0 \leq j \leq M \) set \( U_j = h^{d(j)}(W_j \cap U) \). Then \( F \subset \bigcup_{j=0}^{M} U_j \), and the sets \( h^{d(j)}(U_j) \) are pairwise disjoint (being subsets of the \( W_j \)) and contained in \( U \).

Now let \( m \geq 1 \), and suppose that closed sets which are topologically \( h \)-small with smallness constant \( m \) are thin. Let \( F \subset X \) be topologically \( h \)-small with smallness constant \( m + 1 \). For \( j, k \in \mathbb{Z} \) with \( j \neq k \), define \( F_{j,k} = h^j(F) \cap h^k(F) \). We claim that the sets \( F_{j,k} \) are topologically \( h \)-small with smallness constant \( m \). To see this, let \( d_0, \ldots, d_m \) be \( m + 1 \) distinct integers, and let \( j, k \in \mathbb{Z} \) with \( j \neq k \). By Lemma 3.9, the set \( \{d_i + l : 0 \leq i \leq m, l = j, k\} \) contains at least \( m + 2 \) distinct integers. It follows that

\[
h^{d_0}(F_{j,k}) \cap \cdots \cap h^{d_m}(F_{j,k}) = \bigcap_{i=0}^{m} (h^{d_i+j}(F) \cap h^{d_i+k}(F)) = \emptyset,
\]
which proves the claim. Now choose disjoint, non-empty open sets $V_1, V_2 \subset U$, and choose disjoint, non-empty open sets $Z_1, Z_2$ with $\overline{Z_1} \subset V_1$ and $\overline{Z_2} \subset V_2$. By Lemma 2.8, the collection $\{h^n(Z_1) : n \in \mathbb{Z}\}$ is an open cover for $F$, so it contains a finite subcover $\{h^{-n_0}(Z_1), \ldots, h^{-n_K}(Z_1)\}$. Set $T = \{(j, k) : 0 \leq j < k \leq K\}$ and for each $(j, k) \in T$ define $D_{j,k} = h^{-n_j}(F) \cap h^{-n_k}(F) \cap \overline{Z}_1$, which is a closed subset of $F_{n_j,n_k}$. By the earlier claim, $D_{j,k}$ is topologically $h$-small with smallness constant $m$, and so it is thin by the induction hypothesis. Choose pairwise disjoint open sets $S_{j,k} \subset Z_2$ for $(j, k) \in T$. Since each $D_{j,k}$ is thin, there exist $M(j, k) \in \mathbb{N}$, $U^{(0)}_{j,k,0}, \ldots, U^{(0)}_{j,k,M(j,k)} \subset X$ open, and $d_{j,k}(0), \ldots, d_{j,k}(M(j,k)) \in \mathbb{Z}$ such that:

1. $D_{j,k} \subset \bigcup_{i=0}^{M(j,k)} U^{(i)}_{j,k,i}$;
2. $h^{d_{j,k}(i)}(U^{(i)}_{j,k,i}) \subset S_{j,k}$;
3. the sets $h^{d_{j,k}(i)}(U^{(i)}_{j,k,i})$ are pairwise disjoint for $0 \leq i \leq M(j,k)$.

Now set

$$D = \bigcup_{(j,k) \in T} h^{-n_j}(D_{j,k}) \quad \text{and} \quad W_0 = \bigcup_{(j,k) \in T} h^{-n_j} \left( \bigcup_{i=0}^{M(j,k)} U^{(i)}_{j,k,i} \right).$$

Then $D$ is closed, $W_0$ is open, and $D \subset W_0$. Choose $W \subset X$ open such that $D \subset W \subset \overline{W} \subset W_0$. For $0 \leq j \leq K$, set $F_j = h^{-n_j}(Z_1) \cap (X \setminus W) \cap F$, which is closed. Let $x \in F$ and suppose $x \notin W$. For some $j \in \{0, \ldots, K\}$, we have $x \in h^{-n_j}(Z_1)$. Then $x \in F$, $x \in h^{-n_j}(Z_1)$, and $x \in X \setminus W$, so $x \notin F_j$. It follows that $\{F_0, \ldots, F_K, W\}$ covers $F$. Next suppose that $x \in h^{n_j}(F_j) \cap h^{n_k}(F_k)$ for some $(j, k) \in T$. Then there are $x_j \in F_j$ and $x_k \in F_k$ such that $h^{n_j}(x_j) = x = h^{n_k}(x_k)$. Since $F_j, F_k \subset F$ we certainly have $x \in h^{n_j}(F) \cap h^{n_k}(F)$. Moreover, $x_j = h^{-n_j}(x) \in h^{-n_j}(Z_1)$, which gives $x \in Z_1$. It follows that $x \in D_{j,k}$, and so also $x_j = h^{-n_j}(x) \in h^{-n_j}(D_{j,k}) \subset W$. This implies $x_j \notin F_j$, a contradiction. Therefore, the sets $h^{n_j}(F_j)$ are pairwise disjoint. Since $h^{n_j}(F_j) \subset Z_1$, they are all subsets of $V_1$. Using the normality of $X$, choose non-empty pairwise disjoint open sets $U_0^{(0)}, \ldots, U_K^{(0)} \subset X$ such that $h^{n_j}(F_j) \subset U_j^{(0)} \subset V_1$. For an appropriate $M \in \mathbb{N}$, re-index the sets

$$\left\{h^{-n_0}(U^{(0)}_0), \ldots, h^{-n_K}(U^{(0)}_K)\right\} \cup \left\{h^{-n_j}(U^{(0)}_{j,k,i}) : (j,k) \in T, 0 \leq i \leq M(j,k)\right\}$$

and

$$\{n_0, \ldots, n_K\} \cup \{n_j + d_{j,k}(i) : (j,k) \in T, 0 \leq i \leq M(j,k)\}$$

as $\{U_0, \ldots, U_M\}$ and $\{d(0), \ldots, d(M)\}$ respectively. Then $F \subset \bigcup_{i=0}^{M} U_j$ and the sets $h^{d(j)}(U_j)$ are pairwise disjoint subsets of $U$ for $0 \leq j \leq M$. It follows that $F$ is thin, completing the induction.

**Corollary 3.11.** Let $(X, h)$ be as in Notation 2.1. Let $F \subset X$ be closed and topologically $h$-small. Then $F$ is universally null.

**Proof.** This follows immediately from Proposition 3.10 and Lemma 3.6. □

The next lemma will play a crucial role in the proof of our main result. It provides a strong decomposition property for thin sets.

**Lemma 3.12.** Let $(X, h)$ be as in Notation 2.1. Let $\varepsilon > 0$ be given, and let $F \subset X$ be thin. Then for any non-empty open set $U \subset X$ there exist $M \in \mathbb{N}$, closed sets $F_j \subset X$ for $0 \leq j \leq M$, open sets $T_j, V_j, W_j \subset X$ for $0 \leq j \leq M$, continuous functions $f_0, \ldots, f_M : X \to [0, 1]$, and $d(0), \ldots, d(M) \in \mathbb{Z}$ such that:
(1) $F \subset \bigcup_{j=0}^{M} F_j$;
(2) $h^{-d(j)}(F_j) \subset T_j \subset \overline{T}_j \subset V_j \subset \overline{V}_j \subset W_j \subset U$ for $0 \leq j \leq M$;
(3) $\sum_{j=0}^{M} f_j = 1$ on $\bigcup_{j=0}^{M} h^{d(j)}(T_j)$;
(4) $\text{supp}(f_j \circ h^{-d(j)}) \subset W_j$ for $0 \leq j \leq M$;
(5) the sets $W_j$ are pairwise disjoint and $\sum_{j=0}^{M} \mu(W_j) < \varepsilon$ for all $\mu \in M_h(X)$.

**Proof.** Since $U$ is open and non-empty, Corollary 2.5 implies there is a non-empty open set $E \subset U$ with $\mu(E) < \varepsilon$ for all $\mu \in M_h(X)$. Since $F$ is thin, we can apply Lemma 3.8 to $F$ and $E$, which implies there exist $M \in \mathbb{N}$, $F_0, \ldots, F_M \subset X$ closed, and $k(0), \ldots, k(M) \in \mathbb{Z}$ such that $F \subset \bigcup_{j=0}^{M} F_j$ and such that the sets $h^{k(j)}(F_j)$ are pairwise disjoint subsets of $E$. For $0 \leq j \leq M$, we set $d(j) = -k(j)$. Since $X$ is normal, we may choose for $0 \leq j \leq M$ open sets $W_j$ with $F_j \subset W_j \subset E$ such that the $W_j$ are pairwise disjoint. Now we can use the compactness of $X$ to obtain open sets $T_j, V_j \subset X$ such that

$$h^{-d(j)}(F_j) \subset T_j \subset \overline{T}_j \subset V_j \subset \overline{V}_j \subset W_j.$$ 

For $0 \leq j \leq M$ choose continuous functions $g_j: X \to [0, 1]$ such that $g_j = 1$ on $h^{d(j)}(\overline{V}_j)$ and $\text{supp}(g_j) \subset h^{d(j)}(W_j)$. Then $\sum_{j=0}^{M} g_j(x) \geq 1$ for all $x \in \bigcup_{j=0}^{M} h^{d(j)}(\overline{V}_j)$. By the continuity of the $g_j$, there is an open set $Q \subset X$ such that $\bigcup_{j=0}^{M} h^{d(j)}(\overline{V}_j) \subset Q$ and $\sum_{j=0}^{M} g_j(x) \geq \frac{1}{2}$ for all $x \in Q$. Choose a continuous function $f: X \to [0, 1]$ such that $f = 1$ on $\bigcup_{j=0}^{M} h^{d(j)}(\overline{V}_j)$ and $\text{supp}(f) \subset Q$. Now, for $0 \leq j \leq M$, define continuous functions $f_j: X \to [0, 1]$ by

$$f_j(x) = \begin{cases} f(x)g_j(x) \left( \sum_{i=0}^{M} g_i(x) \right)^{-1} & \text{if } x \in Q \\ 0 & \text{if } x \notin Q \end{cases}$$

Then for any $x \in \bigcup_{j=0}^{M} h^{d(j)}(\overline{V}_j)$, we have

$$\sum_{j=0}^{M} f_j(x) = \sum_{j=0}^{M} f(x)g_j(x) \left( \sum_{i=0}^{M} g_i(x) \right)^{-1} = \left( \sum_{i=0}^{M} g_i(x) \right)^{-1} \sum_{j=0}^{M} g_j(x) = 1.$$ 

In particular, $\sum_{j=0}^{M} f_j = 1$ on $\bigcup_{j=0}^{M} h^{d(j)}(T_j)$. Moreover, $\text{supp}(f_j) = \text{supp}(g_j) \subset h^{d(j)}(W_j)$, which implies that $\text{supp}(f_j \circ h^{-d(j)}) = \text{supp}(g_j \circ h^{-d(j)}) \subset W_j$. Finally, as the $W_j$ are pairwise disjoint subsets of $E$ for $0 \leq j \leq M$, it follows that for any $\mu \in M_h(X)$, we have

$$\sum_{j=0}^{M} \mu(W_j) = \mu \left( \bigcup_{j=0}^{M} W_j \right) \leq \mu(E) < \varepsilon,$$

which completes the proof. \hfill $\square$

### 4. Small Boundary Properties

In [3], Giol and Kerr established that classification of transformation group $C^*$-algebras by their Elliott invariants is intractable for minimal dynamical systems having positive mean dimension (in the sense of [14]). For this reason, we only wish to consider systems with strong enough regularity properties to imply the system has mean dimension zero. For minimal dynamical systems, the following definition
Lemma 4.3. Let \((X, h)\) be as in Notation \([2.1]\). We say \((X, h)\) has the small boundary property if for every \(x \in X\) and every open neighborhood \(U\) of \(x\), there is an open neighborhood \(V\) of \(x\) such that \(V \subset U\) and \(\partial V\) is universally null.

As noted earlier, being universally null might be seen as a weak form of smallness. We thus introduce a new definition where this is replaced with one of our stronger topological conditions.

Definition 4.4. We say \((X, h)\) has the topological small boundary property if whenever \(F, K \subset X\) are disjoint compact sets, then there exist open sets \(U, V \subset X\) such that \(F \subset U\), \(K \subset V\), \(\overline{U} \cap \overline{V} = \emptyset\), and \(\partial U\) is topologically small.

Definition 4.2 first appeared in \([17]\) as a purely topological analogue of a transversality property for manifolds.

Lemma 4.3. Let \((X, h)\) be as in Notation \([2.1]\).

(1) If \((X, h)\) has the topological small boundary property, then whenever \(F \subset X\) is a compact set, there is an open set \(U \subset X\) such that \(F \subset U\) and \(\partial U\) is topologically small.

(2) \((X, h)\) has the topological small boundary property if and only if for any \(x \in X\) and any open neighborhood \(U\) of \(x\), there is an open neighborhood \(V\) of \(x\) such that \(V \subset U\) and \(\partial V\) is topologically small.

Proof. (1) This follows immediately by applying the condition of the topological small boundary property to the compact sets \(F\) and \(K = \emptyset\), then disregarding the open set \(U\) obtained.

(2) If \((X, h)\) has the topological small boundary property, \(x \in X\), and \(U\) is an open neighborhood of \(x\), then by the topological small boundary property, there exist open sets \(V, W \subset X\) such that \(\{x\} \subset V, X \setminus U \subset W, \overline{V} \cap \overline{W} = \emptyset\), and \(\partial V\) is topologically small. Since \(X \setminus U \subset \overline{W}\), it follows that \(\overline{V} \subset U\), as required. For the converse, let \(F, K \subset X\) be compact. Since \(X\) is normal, there exist open sets \(E, W \subset X\) such that \(F \subset E, K \subset W,\) and \(E \cap W = \emptyset\). By assumption, for each \(x \in F\) there is an open set \(U_x \subset X\) such that \(x \in U_x \subset \overline{U}_x \subset E\) and \(\partial U_x\) is topologically small. Since \(F\) is compact the open cover \(\{U_x : x \in X\}\) contains a finite subcover \(\{U_{x_1}, \ldots, U_{x_n}\}\) for \(F\). Set \(U = \bigcup_{j=1}^n U_{x_j}\). Then \(F \subset U \subset \overline{U} \subset W\), and \(\partial U\) is topologically small, since \(\bigcup_{j=1}^n \partial U_{x_j}\) is topologically small by Lemma \([3.3][4]\) and \(\partial U \subset \bigcup_{j=1}^n \partial U_{x_j}\). Finally, the compactness and local compactness of \(K\) imply there is an open set \(V\) with \(K \subset V \subset \overline{V} \subset W\). Then \(\overline{U} \cap \overline{V} = \bigcup_{j=1}^n \overline{U}_{x_j} \cap \overline{V} \subset E \cap W = \emptyset\). It follows that \((X, h)\) has the topological small boundary property.

Corollary 4.4. Let \((X, h)\) be as in Notation \([2.1]\). If \((X, h)\) has the topological small boundary property, then \((X, h)\) has the small boundary property. Consequently, \((X, h)\) has mean dimension zero.

Proof. Since Corollary \([3.1][1]\) implies that topologically \(h\)-small sets are universally null, this follows immediately from Proposition \([13][2]\).
Proposition 4.5. Let \((X, h)\) be as in Notation 2.1. Assume in addition that \(\dim(X) = d < \infty\). Then \((X, h)\) has the topological small boundary property.

Proof. Let \(x \in X\) and let \(U \subset X\) be an open neighborhood of \(X\). Let \(\varepsilon = \text{dist}(x, X \setminus U) > 0\). We apply Lemma 3.7 of [7] with \(i > 1/\varepsilon\) there to obtain closed sets \(F_1, \ldots, F_n \subset X\) which cover \(X\) such that for each \(1 \leq j \leq n\), \(\text{int}(F_j) \neq \emptyset\), \(\text{diam}(F_j) < 1/i\), \(\text{int}(F_j) = F_j\), and such that, whenever \(k(0), \ldots, k(d) \in \mathbb{Z}\) are distinct, then

\[ h^{k(0)}(\partial F_j) \cap h^{k(1)}(\partial F_j) \cap \cdots \cap h^{k(d)}(\partial F_j) = \emptyset. \]

It follows that each \(\partial F_j\) is topologically \(h\)-small. Now, there is some \(M \in \{0, \ldots, n\}\) such that \(x \in F_M\). Then \(\text{diam}(F_M) < 1/i < \varepsilon\) implies that, for any \(y \in F_M\), we have \(d(x, y) < \text{dist}(x, X \setminus U)\). It follows that \(y \in U\), and thus \(F_M \subset U\). Then \(V = \text{int}(F_j)\) is an open neighborhood of \(x\) such that \(V \subset \nabla \subset U\) and \(\partial V\) is topologically \(h\)-small. Now Lemma 4.3[2] implies that \((X, h)\) has the topological small boundary property. \(\square\)

Proposition 4.6. Let \((X_n, h_n)_{n=1}^{\infty}\) be a sequence of dynamical systems consisting of infinite compact metrizable spaces \(X_n\) and homeomorphisms \(h_n : X_n \to X_n\). Let \(X = \prod_{n \in \mathbb{N}} X_n\) and \(h = \prod_{n \in \mathbb{N}} h_n\). Assume that each system \((X_n, h_n)\) has the topological small boundary property. Then the dynamical system \((X, h)\) has the topological small boundary property. If each system \((X_n, h_n)\) is minimal, then so is \((X, h)\).

Proof. Let \(x \in X\) and let \(U \subset X\) be a neighborhood of \(x\). For each \(n \in \mathbb{N}\), let \(\pi_n : X \to X_n\) denote the projection map. We may write \(U = \prod_{n \in \mathbb{N}} U_n\), where each \(U_n \subset X_n\) is open and \(U_n = X_n\) for all but finitely many choices of \(n\). Without loss of generality we may assume that there is an \(N \in \mathbb{N}\) such that \(U_n \neq X_n\) for \(1 \leq n \leq N\) and \(U_n = X_n\) for \(n \geq N\). For each \(1 \leq n \leq N\), \(U_n = \pi_n(U)\) is an open neighborhood of \(x\) in \(X_n\), and the topological small boundary property for \((X_n, h_n)\) implies that there exists an open neighborhood \(V_n \subset X_n\) of \(\pi_n(x)\) such that \(V_n \subset \nabla \subset U_n\) with \(\partial V_n\) topologically \(h\)-small. Let \(m_n\) denote the topological smallness constant for \(\partial V_n\). Define an open neighborhood \(V \subset X\) of \(x\) by

\[ V = \left( \prod_{n \leq N} V_n \right) \times \left( \prod_{n > N} X_n \right). \]

Then \(x \in V \subset \nabla \subset U\) and \(\partial V\) is topologically \(h\)-small, with topological smallness constant \(m = \max\{m_n : 1 \leq n \leq N\}\). \(\square\)

Taylor Hines [6] has constructed an example of a minimal infinite-dimensional dynamical system which is of the form considered in Proposition 4.6. Since that work has not yet been published, we describe the system here.

Corollary 4.7. There is a minimal dynamical system \((X, h)\) with \(\dim(X) = \infty\) that has the topological small boundary property.

Proof. Let \(\{\theta_n : n \in \mathbb{N}\}\) be a collection of irrational numbers \(\theta_n\) that is rationally independent. For \(n \in \mathbb{N}\) let \(h_{\theta_n} : S^1 \to S^1\) be given by \(h_{\theta_n}(\zeta) = e^{2\pi i \theta_n} \zeta\). For \(n \in \mathbb{N}\), let \(X_n = \prod_{j=1}^{n} S^1\) and \(h_n = \prod_{j=1}^{n} h_{\theta_j}\). Then each system \((X_n, h_n)\) is minimal by the rational independence of the irrational numbers \(\theta_j\). Define \((X, h) = \lim(X_n, h_n)\), where the connecting maps \(X_{n+1} \to X_n\) are the obvious projection maps. Then \(X = \prod_{n \in \mathbb{N}} S^1\) and \(h = \prod_{n \in \mathbb{N}} h_{\theta_n}\). Each \(X_n\) is finite-dimensional, and so \((X_n, h_n)\)
has the topological small boundary property by Proposition 4.8. Now the result follows by Proposition 4.6.

**Lemma 4.8.** Suppose that \((X, h)\) has the topological small boundary property, and let \(\varepsilon > 0\) be given. Then for any closed set \(F \subset X\) and any open set \(U \subset X\) with \(F \subset U\), there exist \(K \subset X\) closed and \(V \subset X\) open such that \(F \subset K \subset V \subset \overline{V} \subset U\) and such that \(\mu(V \setminus K) < \varepsilon\) for all \(\mu \in M_h(X)\).

**Proof.** Let \(\varepsilon, F,\) and \(U\) be as given in the statement of the Lemma. The topological small boundary property implies there are open sets \(W, T \subset X\) such that \(F \subset W\), \(X \setminus U \subset T\), \(\overline{W} \cap T = \emptyset\), and \(\partial W\) is topologically \(h\)-small. The compactness of \(F\) and \(\overline{W}\) imply that

\[
\delta = \min \left\{ \operatorname{dist}(F, X \setminus W), \operatorname{dist}(\overline{W}, X \setminus U) \right\} > 0.
\]

For \(n \geq 1\) define \(E_n = \{x \in X : \operatorname{dist}(x, \partial W) < \delta/2^n\}\). An argument entirely analogous to that given in the proof of Lemma 4.4 implies that there is an \(N \in \mathbb{N}\) such that \(\mu(E_N) < \varepsilon\) for all \(\mu \in M_h(X)\). Now set \(K = \overline{W} \setminus E_N\) and \(V = W \cup E_N\). If \(x \in F\) then \(\operatorname{dist}(x, \partial W) = \operatorname{dist}(x, X \setminus W) \geq \delta > \delta/2^N\), and so \(x \notin E_N\), which implies that \(F \subset K\). Since \(\partial W \subset E_N\) we have \(V = W \cup E_N = \overline{W} \cup E_N\), and so \(K \subset \overline{W} \subset V\). Let \(x \in \overline{E}_N\). Then

\[
\operatorname{dist}(x, X \setminus U) = \operatorname{dist}(x, \partial U) \geq \operatorname{dist}(\overline{W}, \partial U) - \operatorname{dist}(x, \partial W) > \delta - \delta/2^N > 0,
\]

which implies that \(x \in U\). This shows that \(\overline{E}_N \subset U\), and as \(\overline{W} \subset U\), it follows that \(\overline{V} = \overline{W} \cup \overline{E}_N \subset U\). Finally, \(V \setminus K = E_N\) by construction, and so we have \(\mu(V \setminus K) < \varepsilon\) for all \(\mu \in M_h(X)\).

**Proposition 4.9.** Suppose that \((X, h)\) has the topological small boundary property, and let \(\varepsilon > 0\) be given. Then for any closed set \(F \subset X\) and any open set \(U \subset X\) with \(F \subset U\), there is an closed set \(K \subset X\) and an open set \(V \subset X\) such that

1. \(F \subset \operatorname{int}(K) \subset K \subset V \subset \overline{V} \subset U\);
2. \(\partial K, \partial U\) are topologically \(h\)-small;
3. \(\mu(V \setminus K) < \varepsilon\) for all \(\mu \in M_h(X)\).

**Proof.** By Lemma 4.8 there is a closed set \(K_0 \subset X\) and an open set \(V_0 \subset X\) such that \(F \subset K_0 \subset V_0 \subset \overline{V}_0 \subset U\) and such that \(\mu(V_0 \setminus K_0) < \varepsilon\) for all \(\mu \in M_h(X)\). The topological small boundary property implies there exist open sets \(E_0, E_1 \subset X\) such that \(K_0 \subset E_0, X \setminus V_0 \subset E_1\), \(\overline{E}_0 \cap \overline{E}_1 = \emptyset\), and \(\partial E_0\) and \(\partial E_0\) is topologically \(h\)-small. We also immediately obtain that \(\overline{E}_0 \subset V_0\). Using the topological small boundary property again, we obtain open sets \(W_0, W_1 \subset X\) such that \(\overline{W}_0 \subset W_0, \overline{E}_1 \subset W_1\), \(\overline{W}_0 \cap \overline{W}_1 = \emptyset\), and \(\partial W_1\) is topologically \(h\)-small. Set \(K = \overline{E}_0\) and \(V = X \setminus \overline{W}_1\).

To see that (1) is satisfied, observe first that \(F \subset K_0 \subset E_0 = \operatorname{int}(K) \subset K\). Next, \(\overline{W}_0 \cap \overline{W}_1 = \emptyset\) implies that \(\overline{W}_0 \subset V\), and so \(K = \overline{E}_0 \subset W_0 \subset \overline{W}_0 \subset V\). Finally, if \(x \in V\) then \(x \notin \overline{W}_1\), and so \(x \notin X \setminus V_0\), which implies that \(x \in V_0\). This gives \(V \subset V_0\), and so \(\overline{V} \subset V_0 \subset U\). Together, these three observations give (1). For (2), we observe that by construction, \(\partial K = \partial \overline{E}_0 = \partial E_0\) and \(\partial V = \partial (X \setminus \overline{W}_1) = \partial W_1\). Therefore, both \(\partial K\) and \(\partial V\) are topologically \(h\)-small. Finally, the containments \(K_0 \subset K \subset V \subset V_0\) imply that \(\mu(V \setminus K) \leq \mu(V_0 \setminus K_0) < \varepsilon\) for all \(\mu \in M_h(X)\). This gives (3) and completes the proof.

**Corollary 4.10.** Suppose that \((X, h)\) has the topological small boundary property.
(1) For any $\varepsilon > 0$ and any open set $U \subset X$ with $\partial U$ universally null, there is an open set $V \subset X$ such that $\overline{V} \subset U$, $\partial V$ is topologically $h$-small, and $\mu(U \setminus \overline{V}) < \varepsilon$ for all $\mu \in M_h(X)$.
(2) For any $\varepsilon > 0$, any open set $U \subset X$, and any closed set $F \subset U$ with $\partial F$ universally null, there is an open set $V \subset X$ such that $F \subset V \subset \overline{V} \subset U$, $\partial V$ is topologically $h$-small, and $\mu(V \setminus F) < \varepsilon$ for all $\mu \in M_h(X)$.
(3) For any $\varepsilon > 0$ and any closed set $F \subset X$ with $\partial F$ universally null, there is an open set $V \subset X$ such that $F \subset V$, $\partial V$ is topologically $h$-small, and $\mu(V \setminus F) < \varepsilon$ for all $\mu \in M_h(X)$.

**Proof.**

(1) By Corollary 2.5(2), there is a closed set $K \subset U$ such that $\mu(U \setminus K) < \varepsilon$ for all $\mu \in M_h(X)$. Applying Proposition 4.9 to $K$ and $U$, we obtain an open set $V \subset X$ such that $K \subset V \subset \overline{V} \subset U$ and $\partial V$ is topologically $h$-small. Then $K \subset \overline{V}$ implies that $\mu(U \setminus \overline{V}) \leq \mu(U \setminus K) < \varepsilon$ for all $\mu \in M_h(X)$.

(2) By Corollary 2.5(4), there is an open set $E \subset X$ such that $F \subset E$ and $\mu(E \setminus F) < \varepsilon$ for all $\mu \in M_h(X)$. Applying Proposition 4.9 to $F$ and $U \cap E$ (which is an open set containing $F$), we obtain an open set $V \subset X$ such that $F \subset V \subset \overline{V} \subset U \cap E \subset U$ and $\partial V$ is topologically $h$-small. Then $V \subset U \cap E \subset E$ implies that $\mu(V \setminus F) \leq \mu(E \setminus F) < \varepsilon$ for all $\mu \in M_h(X)$.

(3) This follows immediately from (2) by taking $U = X$.

5. THE DYNAMIC COMPARISON PROPERTY

In Lemma 2.5 of [4], it is shown that if $(X, h)$ is a Cantor minimal system and $A, B \subset X$ are compact-open sets with $\mu(A) < \mu(B)$ for all $\mu \in M_h(X)$, then in fact there is a decomposition $\bigcup A_i$ of $A$ such that the sets $A_i$ can be translated disjointly into $B$. The following definition is an attempt to formulate a similar condition for more general minimal dynamical systems. In general, we cannot expect non-trivial compact-open sets to exist, and so we must instead use open and closed sets with some assumed good behavior of their boundaries.

**Definition 5.1.** Let $(X, h)$ be as in Notation 2.1. We say $(X, h)$ has the dynamic comparison property if whenever $U \subset X$ is open and $C \subset X$ is closed with $\partial C, \partial U$ universally null and $\mu(C) < \mu(U)$ for every $\mu \in M_h(X)$, then there are $M \in \mathbb{N}$, continuous functions $f_j : X \to [0, 1]$ for $0 \leq j \leq M$, and $d(0), \ldots, d(M) \in \mathbb{Z}$ such that $\sum_{j=0}^M f_j = 1$ on $C$, and such that the sets $\text{supp}(f_j \circ h^{-d(j)})$ are pairwise disjoint subsets of $U$ for $0 \leq j \leq M$.

The next lemma gives a condition that implies the dynamic comparison property holds for systems with the topological small boundary property, and is easier to verify because of the assumed additional structure for the closed and open sets involved.

**Lemma 5.2.** Let $(X, h)$ be as in Notation 2.1 and assume that $(X, h)$ has the topological small boundary property. Suppose that $X$ has the property that if whenever $F \subset X$ is closed with $\text{int}(F) \neq \emptyset$ and $\partial F$ topologically $h$-small, $E \subset X$ is open, and there exists an open set $E_0 \subset E$ with $\overline{E_0} \subset E$, $E_0 \cap F = \emptyset$, $\partial E_0$ topologically $h$-small, and $\mu(F) < \mu(E_0)$ for every $\mu \in M_h(X)$, then there exist $M \in \mathbb{N}$, continuous functions $f_j : X \to [0, 1]$ for $0 \leq j \leq M$, and $d(0), \ldots, d(M) \in \mathbb{Z}$ such that $\sum_{j=0}^M f_j = 1$ on $F$, and such that the sets $\text{supp}(f_j \circ h^{-d(j)})$ are pairwise
disjoint subsets of $E$ for $0 \leq j \leq M$. Then $(X, h)$ has the dynamic comparison property.

**Proof.** Let $U \subset X$ be open and let $C \subset X$ be closed with $\mu(C) < \mu(U)$ for every $\mu \in M_h(X)$. Set $\delta = \inf_{\mu \in M_h(X)} [\mu(U) - \mu(C)] > 0$. By Corollary [1.10][1], there is an open set $U_0 \subset U$ with $U \cap U_0 \subset C \subset X$ and $\mu(U \setminus U_0) < \delta/3$ for all $\mu \in M_h(X)$. For any $\mu \in M_h(X)$, we have

$$\mu(U_0) - \mu(C) = [\mu(U) - \mu(C)] - [\mu(U) - \mu(U_0)] < \delta - \delta/3 = 2\delta/3 > 0.$$

First suppose that $C \subset \overline{U_0}$. Set $M = 0$ and $d(0) = 0$, and choose a continuous function $f_0 : X \to [0, 1]$ such that $f_0 = 1$ on $U_0$ and $\text{supp}(f_0) \subset U$. Then $\sum_{j=0}^M f_j = f_0 = 1$ on $C$, and $\text{supp}(f_0 \circ h^{-d(0)}) = \text{supp}(f_0) \subset U$ as required.

So we may assume that $C \cap (X \setminus U_0) \neq \emptyset$. By Corollary [1.10][1], there is an open set $V \subset U_0$ such that $V \subset U_0$, $\partial V$ is topologically $h$-small, and $\mu(U_0 \setminus V) < \delta/3$ for all $\mu \in M_h(X)$. For any $\mu \in M_h(X)$, we have

$$\mu(V) - \mu(C) = [\mu(U_0) - \mu(C)] - [\mu(U_0) - \mu(V)] > 2\delta/3 - \delta/3 = \delta/3 > 0.$$

Now, set $\epsilon = \inf_{\mu \in M_h(X)} [\mu(V) - \mu(C)] > 0$. Applying Corollary [1.10][2] three times, we obtain open sets $G_0, G_1, G_2 \subset X$ such that

$$C \cap \overline{V} \subset G_0 \subset \overline{G_0} \subset G_1 \subset \overline{G_1} \subset G_2 \subset \overline{G_2} \subset U_0,$$

with $\partial G_i$ topologically $h$-small for $i = 0, 1, 2$ (and so also $\partial G_i$ universally null for $i = 0, 1, 2$) and $\mu(G_0 \setminus C \cap \overline{V}) < \epsilon/4$, $\mu(G_1 \setminus \overline{G_0}) < \epsilon/4$, and $\mu(G_2 \setminus G_1) < \epsilon$ for all $\mu \in M_h(X)$. Set $F_0 = C \setminus G_0$, $E_0 = U \setminus \overline{G_1}$, and $E_1 = V \setminus \overline{G_2}$. Then:

1. $F_0$ is closed and non-empty, since $C \cap (X \setminus U_0) \neq \emptyset$ implies $C \cap (X \setminus G_0) \neq \emptyset$;
2. $E_1$ and $E_0$ are both open and non-empty, and by construction we have $E_1 \subset \overline{E_1} \subset E_0$;
3. $E_1 \cap F_0 = \emptyset$;
4. $\partial F_0$, $\partial E_0$, and $\partial E_1$ are universally null, being subsets of the universally null sets $\partial C \subset \partial G_0$, $\partial U \subset \partial G_1$, and $\partial V \subset \partial G_2$ respectively;
5. Observing $C \cap \overline{V} \subset C \cap G_0$, for every $\mu \in M_h(X)$ we have

$$\mu(E_1) - \mu(F_0) - \mu(C \setminus G_0) = \mu(V) - \mu(V \setminus G_2) - [\mu(C) - \mu(C \cap G_0)]$$

$$\geq [\mu(V) - \mu(C)] + \mu(C \cap G_0)$$

$$- |\mu(C \cap V) + \mu(G_2 \setminus \overline{G_1}) + \mu(G_1 \setminus \overline{G_0}) + \mu(G_0 \setminus (C \cap \overline{V})|]$$

$$\geq \epsilon - [\mu(G_2 \setminus G_1) + \mu(G_1 \setminus G_0) + \mu(G_0 \setminus (C \cap \overline{V})|]$$

$$> \epsilon - 3\epsilon/4$$

$$= \epsilon/4 > 0.$$

Now Corollary [1.10][1] gives an open set $E \subset E_1$ such that $\overline{E} \subset E_1$, $\partial E$ is topologically $h$-small, and $\mu(E \setminus \overline{E}) < \epsilon/16$ for all $\mu \in M_h(X)$. Since $F_0$ is disjoint from $\overline{E}$, there is an open set $W_0 \subset X$ such that $F_0 \subset W_0$ and $W_0 \cap \overline{E} = \emptyset$. Corollary [1.10][2] then implies that there is an open set $W \subset X$ such that $F_0 \subset W \subset \overline{W} \subset W_0$, $\partial W$ is topologically $h$-small, and $\mu(W \setminus F_0) < \epsilon/16$ for all $\mu \in M_h(X)$. Now set $F = \overline{W}$,
Choose a continuous function \( \mu(F) = \mu(W) \), and \( E \cap F = \emptyset \). For any \( \mu \in M_b(X) \), we have
\[
\begin{align*}
\mu(E) - \mu(F) &= \mu(E) - \mu(W) \\
&= [\mu(E_1) - \varepsilon/16] - [\mu(F_0) + \varepsilon/16] \\
&= [\mu(E_1) - \mu(F_0)] - \varepsilon/8 \\
&> \varepsilon/8 \\
&> 0.
\end{align*}
\]

It follows that the sets \( F \) and \( E \) satisfy the conditions for the property given in the statement of the Lemma. Therefore, there exist \( M \in \mathbb{N} \), continuous functions \( f_0, \ldots, f_M : X \to [0, 1] \), and \( d(0), \ldots, d(M) \in \mathbb{Z} \) such that \( \sum_{j=0}^{M} f_j = 1 \) on \( F \), and such that the sets \( \text{supp}(f_j \circ h^{-d(j)}) \) are pairwise disjoint subsets of \( E \) for \( 0 \leq j \leq M \).

Choose a continuous function \( f_{M+1} : X \to [0, 1] \) such that \( f_{M+1} = 1 \) on \( G_1 \) and \( \text{supp}(f_{M+1}) \subset G_2 \), and set \( d(M + 1) = 0 \). Now for any \( x \in C \), either \( x \in F_0 \) or \( x \in G_0 \cap C \). If \( x \in F_0 \) then in particular \( x \in F \), and so \( \sum_{j=0}^{M+1} f_j(x) = \sum_{j=0}^{M} f_j(x) = 1 \).

If \( x \in G_0 \cap C \) then in particular \( x \in C \), and so \( \sum_{j=0}^{M+1} f_j(x) \geq 1 \) for all \( x \in C \). From the continuity of the \( f_j \), there is an open set \( S \subset X \) such that \( C \subset S \) and \( \sum_{j=0}^{M+1} f_j(x) \geq \frac{1}{2} \) for all \( x \in S \). Choose a continuous function \( f : X \to [0, 1] \) such that \( f = 1 \) on \( C \) and \( \text{supp}(f) \subset S \). For \( 0 \leq j \leq M+1 \), define a continuous function \( g_j : X \to [0, 1] \) by
\[
g_j(x) = \begin{cases} 
 f(x) f_j(x) \left( \sum_{i=0}^{M+1} f_i(x) \right)^{-1} & \text{if } x \in S \\
 0 & \text{if } x \notin S.
\end{cases}
\]

Then for any \( x \in C \), we have
\[
\sum_{j=0}^{M+1} g_j(x) = \left( \sum_{i=0}^{M+1} f_i(x) \right)^{-1} \sum_{j=0}^{M+1} f(x) f_j(x) = \left( \sum_{i=0}^{M+1} f_i(x) \right)^{-1} \sum_{j=0}^{M+1} f_j(x) = 1.
\]

Moreover, \( g_j(x) = 0 \) for any \( x \in X \) where \( f_j(x) = 0 \), which implies that \( \text{supp}(g_j) \subset \text{supp}(f_j) \). It follows that \( \text{supp}(g_j \circ h^{-d(j)}) \subset \text{supp}(f_j \circ h^{-d(j)}) \) for \( 0 \leq j \leq M+1 \). This immediately gives pairwise disjointness of the sets \( \text{supp}(g_j \circ h^{-d(j)}) \) for \( 0 \leq j \leq M \), since the sets \( \text{supp}(f_j \circ h^{-d(j)}) \) are pairwise disjoint for \( 0 \leq j \leq M \). Further, all of these sets are contained in \( U \) as \( E \subset U \). Finally, \( \text{supp}(g_{M+1} \circ g^{-d(M+1)}) = \text{supp}(g_{M+1}) \subset \text{supp}(f_{M+1}) = \text{supp}(f_{M+1} \circ h^{-d(M+1)}) \subset G_2 \subset U \), and \( E \cap G_2 = \emptyset \).

Thus, the sets \( \text{supp}(g_j \circ h^{-d(j)}) \) are pairwise disjoint subsets of \( U \) for \( 0 \leq j \leq M+1 \). It follows that \( (X, h) \) has the dynamic comparison property. \( \Box \)

**Lemma 5.3.** Let \( (X, h) \) be as in Notation 2.1. Suppose that \( F \subset X \) is closed and \( E \subset X \) is open such that

1. \( F \cap \overline{E} = \emptyset \);
2. \( \mu(\partial F) = 0 \) and \( \mu(\partial E) = 0 \) for all \( \mu \in M_h(X) \);
3. \( \mu(F) < \mu(E) \) for all \( \mu \in M_h(X) \).

Then there exist continuous functions \( g_0, g_1 : X \to [0, 1] \) such that \( g_0 = 1 \) on \( F \), \( \text{supp}(g_0) \subset X \setminus \overline{E} \), \( \text{supp}(g_1) \subset E \), and
\[
\inf_{\mu \in M_h(X)} \int_X g_1 - g_0 \, d\mu > 0.
\]
Moreover, with \( g = g_1 - g_0 \), there exist \( N_0 \in \mathbb{N} \) and \( \sigma > 0 \) such that for all \( N \geq N_0 \) and \( x \in X \), we have

\[
\frac{1}{N} \sum_{j=0}^{N-1} g(h^j(x)) \geq \sigma.
\]

**Proof.** Since \( F \cap \overline{E} = \emptyset \), the normality of \( X \) gives open sets \( V_0, V_1 \subset X \) such that \( F \subset V_0, \overline{V_1} \subset V_1 \), and \( V_0 \cap V_1 = \emptyset \). Let \( \varepsilon = \frac{1}{3} \inf_{\mu \in M_h(X)} [\mu(E) - \mu(F)] > 0 \). Corollary 2.5 implies there exist and open set \( F \subset W \subset X \) and a compact set \( K \subset X \) such that \( F \subset W, K \subset E, \mu(U \setminus F) < \varepsilon \) for all \( \mu \in M_h(X) \), and \( \mu(E \setminus K) < \varepsilon \) for all \( \mu \in M_h(X) \). Set \( W_0 = V_0 \cap W \), which satisfies \( F \subset W_0 \), \( W_0 \cap \overline{E} \subset W_0 \cap V_1 = \emptyset \), and \( 0 < \mu(W_0 \setminus F) \leq \mu(W \setminus F) < \varepsilon \) for all \( \mu \in M_h(X) \). Now choose continuous functions \( g_0 \) and \( g_1 \) such that \( g_0 = 1 \) on \( F \), \( \text{supp}(g_0) \subset W_0 \), \( g_1 = 1 \) on \( K \), and \( \text{supp}(g_1) \subset E \). Observing that

\[
\mu(K) - \mu(W_0) = (\mu(E) - \mu(F)) - (\mu(E) - \mu(K)) - (\mu(W_0) - \mu(F))
\]

\[
= (\mu(E) - \mu(F)) - (\mu(E \setminus K)) - (\mu(W_0 \setminus F))
\]

\[
> 3\varepsilon - \varepsilon - \varepsilon
\]

\[
= \varepsilon
\]

for all \( \mu \in M_h(X) \), it follows that \( \inf_{\mu \in M_h(X)} [\mu(K) - \mu(W)] \geq \varepsilon > 0 \). Since \( g_0 \leq \chi_{W_0} \) and \( \chi_K \leq g_1 \), we obtain

\[
\inf_{\mu \in M_h(X)} \int_X g_1 - g_0 \, d\mu \geq \inf_{\mu \in M_h(X)} \int_X \chi_K - \chi_{W_0} \, d\mu = \inf_{\mu \in M_h(X)} [\mu(W_0) - \mu(K)] \geq \varepsilon > 0.
\]

Noting that, by the previous calculation, the function \( g = g_1 - g_0 \) satisfies

\[
\inf_{\mu \in M_h(X)} \int_X g \, d\mu > 0,
\]

we define \( \sigma > 0 \) by

\[
\sigma = \frac{1}{2} \inf_{\mu \in M_h(X)} \int_X g \, d\mu.
\]

Suppose that no \( N_0 \in \mathbb{N} \) as in the statement of the Lemma exists. Then there exist sequences \( (N_k)_{k=1}^{\infty} \subset \mathbb{N} \) and \( (x_k)_{k=1}^{\infty} \subset X \) such that for all \( k \in \mathbb{N} \) we have

\[
\frac{1}{N_k} \sum_{j=0}^{N_k-1} g(h^j(x_k)) \leq \sigma.
\]

Passing to subsequences \( (N_{k(l)})_{l=1}^{\infty} \) and \( (x_{k(l)})_{l=1}^{\infty} \) (if necessary) and applying the pointwise ergodic theorem (see the remark after Theorem 1.14 of [19]) yields

\[
\int_X g \, d\mu = \lim_{l \to \infty} \frac{1}{N_{k(l)}} \sum_{j=0}^{N_{k(l)}-1} g(h^j(x_{k(l)})) \leq \sigma
\]

for every \( \mu \in M_h(X) \). We conclude that

\[
\sup_{\mu \in M_h(X)} \int_X g \, d\mu \leq \sigma = \frac{1}{2} \inf_{\mu \in M_h(X)} \int_X g \, d\mu,
\]

a contradiction. \( \square \)
Lemma 5.4. Suppose that $(X, h)$ has the topological small boundary property. Then for any $N \in \mathbb{N}$, there exists a closed set $Y \subset X$ such that $\text{int}(Y) \neq \emptyset$, $\text{int}(Y) = Y$, $\partial Y$ is topologically $h$-small, and the sets $Y, h(Y), \ldots, h^N(Y)$ are pairwise disjoint.

Proof. Since the action of $h$ on $X$ is free, for $y \in X$ the iterates $y, h(y), \ldots, h^N(y)$ are all distinct elements of $X$. Choose pairwise disjoint open neighborhoods $W_0, \ldots, W_N$ of these points, and set $W = \bigcap_{j=0}^N h^{-j}(W_j)$. Then the iterates $W, h(W), \ldots, h^N(W)$ are pairwise disjoint. Let $F = \{y\}$ and $K = X \setminus W$. Apply the topological small boundary property to obtain open sets $U, V \subset X$ such that $F \subset U$, $K \subset V$, $\overline{U} \cap \overline{V} = \emptyset$, and $\partial U$ is topologically $h$-small. Setting $Y = \overline{U}$, it follows that $\text{int}(Y) = U \neq \emptyset$, $\text{int}(Y) = Y$, and $\partial Y$ is topologically $h$-small. Finally, as $X \setminus W \subset V$ and $Y \cap \overline{V} = \emptyset$, it follows that $Y \subset W$, and hence the sets $Y, h(Y), \ldots, h^N(Y)$ are pairwise disjoint.

Lemma 5.5. Let $(X, h)$ be as in Notation 2.1. Let $Y \subset X$ be closed with $\text{int}(Y) \neq \emptyset$ and $\partial Y$ topologically $h$-small. Adopt the notation of Theorem 3.1. Then $\partial(h^j(Y))$ is thin for $0 \leq k \leq l$ and $0 \leq j \leq n(k) - 1$.

Proof. By Proposition 3.10, $\partial Y$ is thin. For $0 \leq j \leq n(k) - 1$, we have $\partial h^j(Y_k) = h^j(\partial Y_k)$, and since translates of thin sets are thin, it suffices to prove that each of the sets $\partial Y_k$ is thin. But $\partial Y_k \subset \bigcup_{j=0}^{l-1} h^j(\partial Y)$, and this set is thin by Lemma 3.7 since it is a finite union of translates of thin sets.

Theorem 5.6. Let $(X, h)$ be as in Notation 2.1 and suppose that $(X, h)$ has the topological small boundary property. Then $(X, h)$ has the dynamic comparison property.

Proof. Let $C \subset X$ be closed and $U \subset X$ be open such that $\mu(C) < \mu(U)$ for all $\mu \in M_h(X)$. By Lemma 5.2, we may assume that $\text{int}(C) \neq \emptyset$, $\partial C$ is topologically $h$-small, and that there is an open set $U_0 \subset U$ such that $\overline{U_0} \subset U$, $\partial U_0$ is topologically $h$-small, $\overline{U_0} \cap C = \emptyset$, and $\mu(C) < \mu(U_0)$ for all $\mu \in M_h(X)$. Applying Lemma 5.3 to $C$ and $U_0$, there exist continuous functions $g_0, g_1 : X \to [0, 1]$ such that $g_0 = 1$ on $C$, $\text{supp}(g_0) \subset X \setminus \overline{U_0}$, $\text{supp}(g_1) \subset U_0$, and

$$\inf_{\mu \in M_h(X)} \int_X g_1 - g_0 \, d\mu > 0.$$  

Moreover, with $g = g_1 - g_0$, there exists $N_0 \in \mathbb{N}$ and $\sigma > 0$ such that for all $N \geq N_0$ and $x \in X$, we have

$$\frac{1}{N} \sum_{j=0}^{N-1} g(h^j(x)) \geq \sigma.$$  

By Lemma 5.4, there exists a closed set $Y \subset X$ with $\text{int}(Y) \neq \emptyset$ such that $\partial Y$ is topologically $h$-small, and that the sets $Y, h(Y), \ldots, h^N(Y)$ are pairwise disjoint. Following the notation of Theorem 5.1, we construct the Rokhlin tower over $Y$ by first return times to $Y$, then apply the second statement of Theorem 3.1 with the partition $\mathcal{P} = \{U_0, C, X \setminus (U_0 \cup C)\}$ of $X$ by sets with non-empty interior (discarding the third set if it is empty). For convenience, we will use $Y_0, \ldots, Y_l$ and $n(0) \leq n(1) \leq \cdots \leq n(l)$ for the base spaces and first return times in the tower.
compatible with $\mathcal{P}$, and set $Y_k^{(0)} = Y_k \setminus \partial Y_k$. (Note that since these $Y_k$ are the sets $Z_k$ in Theorem 3.1, it may be the case that $Y_k^{(0)} = \emptyset$.) We set
\[
F = X \setminus \left( \bigcup_{k=0}^{l} \bigcup_{j=0}^{n(k)-1} h^j(Y_k^{(0)}) \right).
\]
For each $k \in \{0, \ldots, l\}$, the column $\{h^j(Y_k) : 0 \leq j \leq n(k) - 1\}$ has height at least $N_0$. Thus, for any $x \in Y_k$ we have
\[
\frac{1}{n(k)} \sum_{j=0}^{n(k)-1} g(h^j(x)) \geq \sigma > 0.
\]
For $S \subset X$ and $k \in \{0, \ldots, l\}$ define
\[
N(S, k) = \{ n \in \{0, 1, \ldots, n(k) - 1\} : h^n(Y_k) \subset S \}.
\]
Letting $\chi = \chi_{U_0} - \chi_C$, we observe that $g_0 = 1$ on $C$ implies that $\chi_C \leq g_0$ and $\text{supp}(g_1) \subset U_0$ implies that $g_1 \leq \chi_{U_0}$. Combining these inequalities gives $g \leq \chi$, and so
\[
0 < \sigma \leq \frac{1}{n(k)} \sum_{j=0}^{n(k)} g(h^j(x)) \leq \frac{1}{n(k)} \sum_{j=0}^{n(k)} \chi(h^j(x)) = \frac{\text{card}(N(U_0, k)) - \text{card}(N(C, k))}{n(k)}.
\]
It follows that for $0 \leq k \leq l$, we have $\text{card}(N(U_0, k)) > \text{card}(N(C, k))$ (that is, more levels in the column $\{h^j(Y_k) : 0 \leq j \leq n(k) - 1\}$ are contained in $U_0$ than are contained in $C$) and so there is an injective map $\varphi_k : N(C, k) \to N(U_0, k)$. If we order $N(C, k)$ as $\{s_k(0), \ldots, s_k(L_k)\}$ and order $N(U_0, k)$ as $\{t_k(0), \ldots, t_k(L_k)\}$, then one way to represent the injection $\varphi_k$ is by $\varphi_k = (d_k(0), \ldots, d_k(L_k)) \in \mathbb{Z}^{L_k}$ where, for $0 \leq m \leq L_k$, the integer $d_k(m)$ satisfies
\[
h^{d_k(m)}(h^{s_k(m)}(Y_k)) \subset h^{t_k(m)}(Y_k).
\]
Next, we claim that the closed set $F$ is thin. Since the finite union of thin sets is thin by Lemma 3.7 it clearly suffices to prove that $\partial h^j(Y_k)$ is thin for each $0 \leq k \leq l$, $0 \leq j \leq n(k) - 1$. Now, $\partial C$ and $\partial U_0$ are both topologically $h$-small, hence thin. Since $\partial(X \setminus (U_0 \cup C)) = \partial(U_0 \cup C) \subset \partial U_0 \cup \partial C$, it follows that the boundaries of all sets in the partition $\mathcal{P}$ are thin. As the only processes used in the construction of the Rokhlin tower compatible with this partition are translation by powers of $h$, finite unions, and finite intersections, it follows that it is sufficient to prove that the boundaries $\partial h^j(Y_k)$ in a standard Rokhlin tower (without any condition about compatibility with respect to a partition) are thin. This is true by Lemma 3.5 and consequently $F$ is thin.

Now, set $Q = \left\{ k : 0 \leq k \leq l, Y_k^{(0)} \neq \emptyset \right\}$, $Q' = \{0, \ldots, l\} \setminus Q$, and define
\[
\varepsilon = \frac{1}{2} \min_{k \in Q} \inf_{\mu \in M_h(X)} \mu(Y_k^{(0)}) > 0.
\]
Apply Lemma 3.12 with $F$, $U \setminus U_0$, and $\varepsilon$. We obtain $M \in \mathbb{N}$, and for $0 \leq i \leq M$ open sets $T_i, V_i, W_i \subset X$, closed sets $F_i \subset X$, continuous functions $b_i : X \to [0, 1]$, and integers $r(i)$ such that:
(1) $h^{-r(i)}(F_i) \subset T_i \subset \overline{T_i} \subset V_i \subset \overline{V_i} \subset W_i \subset U \setminus U_0$ for $0 \leq i \leq M$;
(2) $\sum_{i=0}^{M} b_i = 1$ on $\bigcup_{i=0}^{M} h^{r(i)}(\overline{V_i})$;
(3) $\text{supp}(b_i \circ h^{-r(i)}) \subset W_i$ for $0 \leq i \leq M$;
By the choice of \( x \), it follows that for \( k \in Q \) and \( 0 \leq j \leq n(k) - 1 \), we have
\[
\mu \left( h^j(Y^{(0)}_k) \setminus \bigcup_{i=0}^{M} h^{r(i)}(W_i) \right) \geq \mu(h^j(Y^{(0)}_k)) - \mu \left( \bigcup_{i=0}^{M} h^{r(i)}(W_i) \right)
\]
\[
\geq 2\varepsilon - \sum_{i=0}^{M} \mu(h^{r(i)}(W_i))
\]
\[
= 2\varepsilon - \sum_{i=0}^{M} \mu(W_i)
\]
\[
> \varepsilon
\]
for every \( \mu \in M_h(X) \), and therefore
\[
\inf_{\mu \in M_h(X)} \mu \left( h^j(Y^{(0)}_k) \setminus \bigcup_{i=0}^{M} h^{r(i)}(W_i) \right) \geq \varepsilon > 0.
\]
This implies the sets \( h^j(Y^{(0)}_k) \setminus \bigcup_{i=0}^{M} h^{r(i)}(W_i) \) are non-empty whenever \( k \in Q \).
It follows that for \( k \in Q \), each set \( h^j(Y_k) \setminus \bigcup_{i=0}^{M} h^{r(i)}(V_i) \) is a non-empty closed subset of \( h^j(Y_k) \). Now for \( k \in Q \) and \( 0 \leq m \leq L_k \) choose a continuous function \( f_{m,k} : X \to [0,1] \) such that \( f_{m,k} = 1 \) on \( h^{s_k(m)}(Y_k) \setminus \bigcup_{i=0}^{M} h^{r(i)}(V_i) \) and \( \text{supp}(f_{m,k}) \subseteq h^{s_k(m)}(Y_k) \setminus \bigcup_{i=0}^{M} h^{r(i)}(\overline{V_i}) \). Now we have collections of continuous functions
\[
\{ b_i : 0 \leq i \leq M \} \cup \{ f_{m,k} : k \in Q, 0 \leq m \leq L_k \}
\]
and associated integers
\[
\{ r(i) : 0 \leq i \leq M \} \cup \{ d_k(m) : k \in Q, 0 \leq m \leq L_k \}. \]
For any \( x \in C \), if \( x \in \bigcup_{k \in Q} \bigcup_{m=0}^{L_k} \left( h^{s_k(m)}(Y_k) \setminus \bigcup_{i=0}^{M} h^{r(i)}(V_i) \right) \), then \( f_{m,k}(x) \neq 0 \) for some \( k \in Q \) and some \( m \in \{ 0, \ldots, L_i \} \). Otherwise, \( x \in \bigcup_{i=0}^{M} h^{r(i)}(V_i) \), and \( b_i(x) \neq 0 \) for some \( 0 \leq i \leq M \). (Notice that if \( x \in \bigcup_{k \in Q} \bigcup_{m=0}^{L_k} h^{s_k(m)}(Y_k) \), then in fact \( x \in F \), and so also \( x \in \bigcup_{i=0}^{M} h^{r(i)}(V_i) \).) Now re-order the two collections above as \( \{ f_j^{(0)} : 0 \leq j \leq K \} \) and \( \{ d(j) : 0 \leq j \leq K \} \) for an appropriate \( K \in \mathbb{N} \). Then
\[
\sum_{j=0}^{K} f_j^{(0)}(x) > 0 \text{ for all } x \in C. \]
Since \( C \) is compact and the \( f_j^{(0)} \) are continuous, there must be a \( \omega > 0 \) such that \( \sum_{j=0}^{K} f_j^{(0)}(x) \geq \omega \) for all \( x \in C \). Again using continuity, we can choose an open set \( S \subset X \) such that \( C \subset S \) and \( \sum_{j=0}^{K} f_j^{(0)}(x) \geq \frac{1}{2}\omega \) for all \( x \in S \). Choose a continuous function \( f : X \to [0,1] \) such that \( f(x) = 1 \) for all \( x \in C \), and \( \text{supp}(f) \subset S \). For \( 0 \leq j \leq K \) define continuous functions \( f_j : X \to [0,1] \) by
\[
f_j(x) = \begin{cases} 
  f(x)f_j^{(0)}(x) \left( \sum_{i=0}^{K} f_i^{(0)}(x) \right)^{-1} & \text{if } x \in S \\
  0 & \text{if } x \notin S.
\end{cases}
\]
Then for any \( x \in C \),
\[
\sum_{j=0}^{K} f_j(x) = \left( \sum_{i=0}^{K} f_i^{(0)}(x) \right)^{-1} \sum_{j=0}^{K} f(x)f_j^{(0)}(x) = \left( \sum_{i=0}^{K} f_i^{(0)}(x) \right)^{-1} \sum_{j=0}^{K} f_j^{(0)}(x) = 1.
\]
Moreover, supp$(f_j) \subset \text{supp}(f_j^{(0)})$ for $0 \leq j \leq K$. If $f_j^{(0)} = b_i$ for some $0 \leq i \leq M$, then
\[
\text{supp}(f_j^{(0)} \circ h^{-d(j)}) = \text{supp}(b_i \circ h^{-r(i)}) \subset W_i \subset U \setminus U_0
\]
and the sets $W_i$ are pairwise disjoint. Therefore the sets supp$(f_j^{(0)} \circ h^{-d(j)})$ are pairwise disjoint for all choices of $j$ where $f_j^{(0)} \in \{b_i: 0 \leq i \leq M\}$. Next, if $f_j^{(0)} = f_{m,k}$ for some $k \in Q$ and some $0 \leq m \leq L_k$, then
\[
\text{supp}(f_j^{(0)} \circ h^{-d(j)}) = \text{supp}(f_{m,k} \circ h^{-d_k(m)}) = h^{t_k(m)}(Y_k) \subset U_0.
\]
Moreover, the definition of the functions $f_{m,k}$ implies that
\[
\text{supp}(f_{m,k} \circ h^{-d_k(m)}) \subset h^{d(m)} \left( h^{s_k(m)}(Y_k) \setminus \bigcup_{i=0}^{M} h^{r(i)}(V_i) \right),
\]
so that in particular, for $k \in Q$ the set supp$(f_{m,k} \circ h^{-d_k(m)})$ is a subset of $h^{t_k(m)}(Y_k^{(0)})$ (which is non-empty by the choice of $k$). Since the sets $h^{t_k(m)}(Y_k^{(0)})$ are pairwise disjoint, the sets supp$(f_j^{(0)} \circ h^{-d(j)})$ are pairwise disjoint for all choices of $j$ where $f_j^{(0)} \in \{f_{m,k}: k \in Q, 0 \leq m \leq L_k\}$. Moreover, the sets $W_i$ are pairwise disjoint from the sets $h^{t_k(m)}(Y_k^{(0)})$, as $U \setminus \overline{U}_0$ is certainly disjoint from $U_0$. Therefore, the sets supp$(f_j^{(0)} \circ h^{-d(j)})$ are pairwise disjoint subsets of $U$ for all $0 \leq j \leq K$. It follows that the sets supp$(f_j \circ h^{-d(j)})$ are pairwise disjoint subsets of $U$ for all $0 \leq j \leq K$. This completes the proof. \hfill \Box

**Corollary 5.7.** Let $(X, h)$ be a minimal dynamical system, where dim$(X) < \infty$. Then $(X, h)$ has the dynamic comparison property.

**Proof.** By Proposition 4.5 $(X, h)$ has the topological small boundary property. Theorem 5.6 then implies $(X, h)$ has the dynamic comparison property. \hfill \Box

In [1] and [2], the dynamic comparison property will be used to study properties of crossed product $C^*$-algebras $C^*(\mathbb{Z}, C(X, A), \beta)$, where $\beta$ is an automorphism whose restriction to $C(X)$ is the action induced by a minimal homeomorphism. In particular, it plays a key role in showing that given a nonzero positive element of $C(X, A)$, there is a non-zero positive element of $C(X)$ which is Cuntz subequivalent to it. Taylor Hines has suggested that the dynamic comparison property may be a topological analogue for strict comparison of positive elements. For this to be reasonable, something like the following should hold:

**Conjecture 5.8.** Let $(X, h)$ be as in Notation 2.1 and assume that $(X, h)$ has the dynamic comparison property. Then the transformation group $C^*$-algebra $C^*(\mathbb{Z}, X, h)$ has strict comparison of positive elements.

This result would be independent of the proof in [18] that $C^*(\mathbb{Z}, X, h)$ is $\mathcal{Z}$-stable for finite-dimensional $X$, since the dynamic comparison property holds for infinite-dimensional systems such as that in Corollary 4.7. It would also verify a conjecture of Toms and Phillips (namely, that the radius of comparison for $C(X)$ is approximately half the mean dimension of $X$) for all systems with the dynamic comparison property (including all those with the topological small boundary property).
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