ON DIMENSION GROWTH OF MODULAR IRREDUCIBLE REPRESENTATIONS OF SEMISIMPLE LIE ALGEBRAS

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Abstract. In this paper we investigate the growth with respect to $p$ of dimensions of irreducible representations of a semisimple Lie algebra $g$ over $\mathbb{F}_p$. More precisely, it is known that for $p \gg 0$, the irreducibles with a regular rational central character $\lambda$ and $p$-character $\chi$ are indexed by a certain canonical basis in the $K_0$ of the Springer fiber of $\chi$. This basis is independent of $p$. For a basis element, the dimension of the corresponding module is a polynomial in $p$. We show that the canonical basis is compatible with the two-sided cell filtration for a parabolic subgroup in the affine Weyl group defined by $\lambda$. We also explain how to read the degree of the dimension polynomial from a filtration component of the basis element. We use these results to establish conjectures of the second author and Ostrik on a classification of the finite dimensional irreducible representations of $W$-algebras, as well as a strengthening of a result by the first author with Anno and Mirkovic on real variations of stabilities for the derived category of the Springer resolution.

To the memory of Bertram Kostant.

1. INTRODUCTION

In this paper we study the representation theory of semisimple Lie algebras over algebraically closed fields of big positive characteristic. More precisely, let $G$ be a semisimple algebraic group (of adjoint type) over $\mathbb{C}$ and $g$ be its Lie algebra. Then $g$ is defined over $\mathbb{Z}$ so for an algebraically closed field $\mathbb{F}$ of characteristic $p$ we can define the form $g_\mathbb{F}$ over $\mathbb{F}$. The universal enveloping algebra $U(g_\mathbb{F})$ is finite over its center, namely, we have a central algebra embedding $S(g_\mathbb{F}(1)) \hookrightarrow U(g_\mathbb{F}), x \mapsto x^p - x^p$, where the superscript $(1)$ indicates the Frobenius twist and the superscript $[p]$ stands for the restricted $p$-th power map $g_\mathbb{F}(1) \to g_\mathbb{F}$. The image is known as the $p$-center. In particular, all irreducible representations of $g_\mathbb{F}$ are finite dimensional. Below we will assume that $p \gg 0$ (although some statements hold under weaker assumptions).

Let $\mathfrak{h}$ denote a Cartan subalgebra of $g$. We have an identification $U(g_\mathbb{F})^{G_\mathbb{F}} \cong \mathbb{F}[\mathfrak{h}^*]^W$ (the Harish-Chandra isomorphism), the central subalgebra $U(g_\mathbb{F})^{G_\mathbb{F}} \subset U(g_\mathbb{F})$ is known as the Harish-Chandra center. Fix $\lambda \in \mathfrak{h}^*$ and consider the corresponding central reduction $U_{\lambda, \mathbb{F}}$ of the algebra $U(g_\mathbb{F})$. Further, for $\chi \in g_\mathbb{F}(1)^*$ we can consider the further central reduction $U_{\lambda, \mathbb{F}}^{\chi}$, this is a finite dimensional algebra. Obviously, every irreducible representation of $U(g_\mathbb{F})$ factors through exactly one irreducible quotient $U_{\lambda, \mathbb{F}}^{\chi}$ (some of these quotients are zero).

The study of the representation theory of the algebras $U_{\lambda, \mathbb{F}}^{\chi}$ can be easily reduced to the case when the element $\chi$ is nilpotent, see [KW, Theorem 2]. Here the algebra $U_{\lambda, \mathbb{F}}^{\chi}$ is nonzero if and only if $\lambda \in \mathfrak{h}_p^*$. Let us recall some results of the first author and collaborators on the representation theory of $U_{\lambda, \mathbb{F}}^{\chi}$.

Consider the flag variety $\mathcal{B}$ for $g$ (over $\mathbb{C}$). Let $e$ be a nilpotent element in $g$ in the orbit corresponding to that of $\chi$ (since $p \gg 0$, there is a natural bijection between the nilpotent

MSC 2010: 17B20, 17B35, 17B50.
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1. For any regular $\lambda \in \frak{h}_R^*$, the basis $\mathfrak{B}$ of $K_0(\text{Coh}^{\infty}(\mathcal{B}_e))$ is compatible with the filtration $K_0(\text{Coh}^{\infty}(\mathcal{B}_e))_{\leq c}$.

2. Let $b \in \mathfrak{B}$ lie in $K_0(\text{Coh}^{\infty}(\mathcal{B}_e))_{\leq c}$ but not in smaller filtration pieces. Then the degree of the polynomial $\dim V_{\lambda, p}(b)$ in $p$ equals $\dim \mathcal{O}_c/2$. 

(1) $K_0(\mathcal{U}_{\mathcal{A}, p}^{\infty} \text{-mod}) \xrightarrow{\sim} K_0(\text{Coh}(\mathcal{B}_e)) \xrightarrow{\sim} H_*(\mathcal{B}_e, \mathbb{C})$

(in the present paper all $K_0$-groups will be over $\mathbb{C}$ but, in fact, the first isomorphism holds over $\mathbb{Z}$).

There is a way to identify classes of simples under this isomorphism conjectured by Lusztig and proved in [BM]. The space $K_0(\text{Coh}(\mathcal{B}_e))$ admits a $q$-deformation, the equivariant $K$-theory group $K_0(\text{Coh}^{\infty}(\mathcal{B}_e))$ for a contracting action of $\mathbb{C}^\times$ on $\mathcal{B}_e$, [Lu2, Section 6]. Then, according to [BM], there is a canonical basis $\mathfrak{B}$ in $K_0(\text{Coh}^{\infty}(\mathcal{B}_e))$ such that the classes of simples in $K_0(\text{Coh}(\mathcal{B}_e))$ are the specializations of the elements of $\mathfrak{B}$ to $q = 1$. The only thing that we need to know about $\mathfrak{B}$ is that it is independent of $p$ (and depends not on $\lambda$ itself but on its $p$-alcove, we will not need this).

A big problem with this canonical basis is that it is very implicit. For example, it is unclear how to compute the dimensions of the irreducible modules. The goal of this paper is to get a more explicit information about the canonical bases elements and about dimensions of the corresponding simple modules. More precisely, we want to understand the dependence of the dimensions on $p$.

First, let us recall that $K_0(\text{Coh}^{\infty}(\mathcal{B}_e))$ is a module over the affine Hecke algebra $\mathcal{H}_q(W^\alpha)$. Here and below we write $W^\alpha$ for the affine Weyl group of $\frak{g}$, i.e., $W^\alpha = W \ltimes Q$, where $Q$ is a root lattice.

Now pick a finite localization $R$ of $\mathbb{Z}$ and a dominant regular element $\lambda \in \frak{h}_R^\ast$. Then for $p \gg 0$, we can reduce $\lambda$ to an element in $\frak{h}_R^\ast$. Further, pick $b \in \mathfrak{B}$, and let $V_{\lambda, p}(b)$ denote the corresponding simple in $\mathcal{U}_{\mathcal{A}, p}^{\infty} \text{-mod}$. Then (for $\lambda$ and $b$ fixed) $\dim V_{\lambda, p}(b)$ is known to be a polynomial in $p$ assuming $p$ satisfies some congruence conditions depending on $\lambda$. Our first goal is to determine the degree of this polynomial.

Note that $\lambda$ determines a proper standard parabolic subgroup $W_{[\lambda]} \subset W^\alpha$. Namely, we consider the action of $W^\alpha$ on $\frak{h}_R^\ast$. Let $\lambda^\circ$ be the intersection of $W^\alpha \lambda$ with the fundamental alcove. For $W_{[\lambda]}$ we take the standard parabolic subgroup generated by the simple reflections corresponding to the walls containing $\lambda^\circ$. For example, when $\lambda \in Q$, we have $W_{[\lambda]} = W$ (as a standard parabolic subgroup of $W^\alpha$).

Consider the partition of $W_{[\lambda]}$ into two-sided cells. This partition also determines a partition of the irreducible $W_{[\lambda]}$-modules (or $\mathcal{H}_q(W_{[\lambda]})$-modules for generic $q$) into families. We filter the module $K_0(\text{Coh}^{\infty}(\mathcal{B}_e))$ according to two-sided cells for $W_{[\lambda]}$. Namely, given a two-sided $c$ for $W_{[\lambda]}$, let $K_0(\text{Coh}^{\infty}(\mathcal{B}_e))_{\leq c}$ denote the intersection of $K_0(\text{Coh}^{\infty}(\mathcal{B}_e))$ with the sum of all irreducible $\mathcal{H}_q(W_{[\lambda]})$-submodules in the localized $K_0$ that belong to families indexed by two-sided cells $c' \leq c$.

The following is the main result of the paper. Let us recall that from a two-sided cell $c$ in $W_{[\lambda]}$ we can recover a nilpotent orbit $\mathcal{O}_c$ in $\frak{g}$, see Section 2.2 for more details.

**Theorem 1.1.** The following are true:

1. For any regular $\lambda \in \frak{h}_R^*$, the basis $\mathfrak{B}$ of $K_0(\text{Coh}^{\infty}(\mathcal{B}_e))$ is compatible with the filtration $K_0(\text{Coh}^{\infty}(\mathcal{B}_e))_{\leq c}$.

2. Let $b \in \mathfrak{B}$ lie in $K_0(\text{Coh}^{\infty}(\mathcal{B}_e))_{\leq c}$ but not in smaller filtration pieces. Then the degree of the polynomial $\dim V_{\lambda, p}(b)$ in $p$ equals $\dim \mathcal{O}_c/2$. 

orbits in $\frak{g}$ and in $\frak{g}_R^{(1)}$. Consider the corresponding Springer fiber $\mathcal{B}_e$. In [BMR2], for a regular $\lambda$, the authors have constructed identifications
Remark 1.2. There is a classical analog of (2) for categories in characteristic 0 such as category $\mathcal{O}$. There the result is that the Gelfand-Kirillov dimension of the module corresponding to a canonical basis element equals $\dim \mathcal{O}_c/2$. So part (2) means that the degree of the dimension polynomial is the modular analog of the Gelfand-Kirillov dimension. Heuristically this can justified as follows: a module of Gelfand-Kirillov dimension $d$ has “the same size” as the space of sections of a coherent sheaf on $\mathfrak{g}^*$ with support of dimension $d$, while a module in characteristic $p$ whose dimension $D$ is expressed by a polynomial in $p$ of degree $d$ has the same size as the space of sections of such a coherent sheaf restricted to the Frobenius neighborhood of a point, see also Remark 4.6.

We can also reformulate (2) as follows. We will see below that there is a unique primitive ideal $\mathcal{J} \in \mathcal{U}$ such that the simple corresponding to $b$ is annihilated by the reduction of $\mathcal{J}$ mod $p$. We will see that $\mathcal{O}_c$ is the associated variety of $\mathcal{J}$ so that the degree of the dimension polynomial is $\frac{1}{2}\text{GK-dim}(\mathcal{U}/\mathcal{J})$. We expect that an analog of this result holds in a much greater generality, for example, for quantizations of symplectic singularities.

Let us discuss some applications of Theorem 1.1. First, it allows us to prove conjectures of the second author and Ostrik on the classification of finite dimensional irreducible modules over the finite $W$-algebra $\mathcal{W}$ for $(\mathfrak{g}, e)$, see [LO, Section 7.6]. This is Theorem 5.2 in the paper. In particular, this theorem implies that the $K_0$ of the finite dimensional representations of $\mathcal{W}$ with central character $\lambda$ coincides with $\bigoplus_e K_0(\text{Coh}(\mathcal{B}_e)_e)$, where the sum is taken over all two-sided cells in $W_{[\lambda]}$ such that $\mathcal{O}_e = Ge$. In fact, for such $e$ we have $K_0(\text{Coh}(\mathcal{B}_e))_e = 0$. The first author and Kazhdan plan to use part (1) and the result mentioned in the previous sentence to study restrictions of characters for unipotent irreducible representations of $p$-adic groups.

Another application that motivated the main result is a strengthened version of the result of [ABM]. The central point of loc. cit. is the definition of real variation of stabilities, a concept partly inspired by the notion of a Bridgeland stability condition on a triangulated category, and a theorem asserting that the categories of $\mathcal{U}_{\lambda,\mathcal{F}}$-modules give rise to such a structure. Let us describe it in more detail. The above identification $K_0(\mathcal{U}_{\lambda,\mathcal{F}}\text{-mod}) \xrightarrow{\sim} K_0(\text{Coh}(\mathcal{B}_{e,\mathcal{F}}))$ comes from an equivalence of triangulated categories $\mathcal{L}_{\lambda} : D^b(\mathcal{U}_{\lambda,\mathcal{F}}\text{-mod}^\chi) \rightarrow D^b(\text{Coh}_{\mathcal{B}_e}(\mathcal{T}^*\mathcal{B}_e))$, (the definition of $\mathcal{L}_{\lambda}$ is recalled below after Lemma 2.2). Here $\lambda$ is an element of the root lattice such that $\lambda = \bar{\lambda} \mod p$; $\text{Coh}_{\mathcal{B}_e}(\mathcal{T}^*\mathcal{B}_e)$ denotes the category of coherent sheaves on $\mathcal{T}^*\mathcal{B}_e$ set-theoretically supported on the closed subvariety $\mathcal{B}_e$, while $\mathcal{U}_{\lambda,\mathcal{F}}\text{-mod}^\chi$ is the category of modules over $\mathcal{U}_{\lambda,\mathcal{F}}$ where the kernel of $\chi$ acts nilpotently. The image of the abelian category $\mathcal{U}_{\lambda,\mathcal{F}}\text{-mod}^\chi$ under the equivalence $\mathcal{L}_{\lambda}$, i.e. the corresponding $t$-structure on $D^b(\text{Coh}_{\mathcal{B}_e}(\mathcal{T}^*\mathcal{B}_e))$ depends only on the $p$-algebra of $\bar{\lambda}$, not on $\lambda$ itself. Thus we get a collection of $t$-structures on the derived category of coherent sheaves indexed by alcoves; although the above construction applies to varieties of large finite characteristic only, the $t$-structures admits a canonical lift to $D^b(\text{Coh}_{\mathcal{B}_c}(\mathcal{T}^*\mathcal{B}_c))$.

It turns out to be a part of a real variation of stability conditions; the content of this statement is as follows: for two neighboring alcoves sharing a codimension one face the derived equivalence between the corresponding abelian categories is a perverse equivalence governed by a certain polynomial map $Z : \mathfrak{t}_e^* \rightarrow K_0(\text{Coh}(\mathcal{B}_e))^*$ called the central charge map.

A conjecture stated in loc. cit. [ABM, Remark 6] asserts that a similar property should hold for two alcoves symmetric relative to a higher codimension face of the affine coroot stratification of $\mathfrak{t}_e^*$. In Section 6 we deduce (a statement essentially equivalent to) that
conjecture from Theorem 1.1. Again, we expect a similar statement to hold for all (or at least for a wide class of) symplectic singularities.

Acknowledgements. We would like to thank Pavel Etingof for a kind permission to use his results in this paper, see Section 4.3, and Victor Ostrik for numerous discussions related to the project. The work of R.B. was partially supported by the NSF under the grant DMS-1601953. The work of I.L. has been funded by the Russian Academic Excellence Project ‘5-100’ and was partially supported by the NSF under the grant DMS-1501558.

2. Preliminaries

2.1. Harish-Chandra bimodules and primitive ideals. Let us write $\mathcal{U}$ for $U(\mathfrak{g})$ and $\mathcal{U}_\lambda$ for the central reduction of $\mathcal{U}$ at $\lambda \in \mathfrak{h}^*$.

Recall that by a Harish-Chandra (shortly, HC) $\mathcal{U}$-bimodule one means a finitely generated $\mathcal{U}$-bimodule with locally finite adjoint action of $\mathfrak{g}$. In this paper we will only consider the bimodules where the adjoint $\mathfrak{g}$-action integrates to an action of $G := \text{Ad}(\mathfrak{g})$. Every HC bimodule admits a so called good filtration, i.e., a $G$-stable filtration such that the associated graded is finitely generated as a module over $S(\mathfrak{g})$ (since the filtration is $G$-stable the left and the right actions of $\mathfrak{g}$ on the associated graded coincide).

We will write $HC(\mathcal{U})$ for the category of HC $\mathcal{U}$-bimodules and $D^b_{HC}(\mathcal{U}$ -bimod) for the full subcategory of $D^b(\mathcal{U}$ -bimod) of all objects with HC homology. We note that $D^b_{HC}(\mathcal{U}$ -bimod) is closed under taking the derived tensor products.

Inside $HC(\mathcal{U})$ we will consider three kinds of subcategories defined by the central character conditions. Fix $\lambda, \lambda' \in \mathfrak{h}^*$. We consider the subcategory $1^\lambda HC_\lambda^{\lambda'}$ of all HC bimodules with genuine central characters $\lambda$ on the left and $\lambda'$ on the right. When $\lambda = \lambda'$, we write $HC(\mathcal{U}_\lambda)$ for $1^\lambda HC_\lambda^{\lambda}$. Note that $HC(\mathcal{U}_\lambda)$ is a monoidal category. We can also consider the larger subcategory $\infty^\lambda HC_{\lambda'}^{\lambda'}$, where the central characters on the left and on the right are generalized. Finally, there is an intermediate category $\infty^\lambda HC_\lambda^{\lambda'}$. Note that $1^\lambda HC_\lambda^{\lambda'}$ contains all simple objects in $\infty^\lambda HC_{\lambda'}^{\lambda'}$ and all objects in $\infty^\lambda HC_{\lambda'}^{\lambda'}$ have finite length. Because of this, the $K_\mu$’s of these categories are the same.

Now suppose that $\lambda$ is regular. Let $\mu \in W\lambda$ be anti-dominant meaning that $\langle \alpha^\vee, \mu \rangle \not\in \mathbb{Z}_{\geq 0}$ for any positive coroot $\alpha^\vee$. Fix this $\mu$ (it is not unique unless $\lambda$ is integral). Consider the block $\mathcal{O}(\mu)$ of the BGG category $\mathcal{O}$ spanned by the simples $L(\mu u)$ (with highest weight $u\mu - \rho$), where $u$ is in the integral Weyl group $W_{\mu,\text{int}}$ of $\mu$. Recall that this group is generated by all reflections $s_\alpha$ such that $\langle \alpha^\vee, \mu \rangle \in \mathbb{Z}$.

Then there is the Bernstein-Gelfand equivalence $\infty^\lambda HC_{\lambda'}^{\lambda} \overset{\sim}{\rightarrow} \mathcal{O}(\mu)$ given by $\mathcal{M} \mapsto \mathcal{M} \otimes_{\mathcal{U}_\lambda} \Delta(w_0\mu)$, where $w_0$ is the longest element in $W_{\mu,\text{int}}$ (so that $\Delta(w_0\mu)$ is projective in $\mathcal{O}(\mu)$). In particular, the simples in $HC(\mathcal{U}_\lambda)$ are labelled by $u \in W_{\mu,\text{int}}$. Note that there is a natural isomorphism $W_{|\mu|} \overset{\sim}{\rightarrow} W_{\mu,\text{int}}$: namely, let $w_1 \in W^a$ be the minimal length element such that (in the notation of the introduction) $\mu = w_1\mu^a$. Then an isomorphism $W_{|\mu|} \overset{\sim}{\rightarrow} W_{\mu,\text{int}}$ is given by $w \mapsto \text{pr}(w_1^{-1}w w_1)$, where we write pr for the projection $W^a \rightarrow W$. This defines a bijection $\text{Irr}(HC(\mathcal{U}_\lambda)) \overset{\sim}{\rightarrow} W_{|\lambda|}$. Let us write $\mathcal{M}_w$ for the simple HC $\mathcal{U}_\lambda$-bimodule corresponding to $w \in W_{|\lambda|}$.

Let $\mathcal{M}$ be a HC $\mathcal{U}$-bimodule. By the associated variety, $V(\mathcal{M})$, we mean the support of $\text{gr}\mathcal{M}$ in $\mathfrak{g}$, where the associated graded is taken with respect to any good filtration. We note that $V(\text{Tor}_{|\lambda|}(\mathcal{M}_1, \mathcal{M}_2)) \subset V(\mathcal{M}_1) \cap V(\mathcal{M}_2)$.

Let us fix a nilpotent orbit $\mathcal{O} \subset \mathfrak{g}$. We can consider the subcategories $HC_{\partial \mathcal{O}}(\mathcal{U}) \subset HC_{\overline{\mathcal{O}}}(\mathcal{U})$ of all $\mathcal{M} \in HC(\mathcal{U})$ with $V(\mathcal{M}) \subset \partial \mathcal{O}$ (resp., $V(\mathcal{M}) \subset \overline{\mathcal{O}}$). These are tensor ideals in $HC(\mathcal{U})$. 


So we can form the quotient category $HC_{\mathcal{O}}(\mathcal{U})$ that also carries the tensor product. Let $HC^{ss}_{\mathcal{O}}(\mathcal{U})$ denote the full subcategory of semisimple objects in $HC_{\mathcal{O}}(\mathcal{U})$. One can show, using, for example, [L2, Corollary 1.3.2], that the subcategory $HC^{ss}_{\mathcal{O}}(\mathcal{U})$ is closed under taking the tensor products. Moreover, it is a rigid monoidal category. This has the following corollary.

**Corollary 2.1.** For simple HC bimodules $\mathcal{M}, \mathcal{M}' \in HC_{\mathcal{O}}(\mathcal{U})$, there are $M_1 \subset M_2 \subset \mathcal{M} \otimes_{\mathcal{U}} \mathcal{M}'$ with $M_1, \mathcal{M} \otimes_{\mathcal{U}} \mathcal{M}'/M_2 \subset HC_{\mathcal{O}}(\mathcal{U})$, while $M_2/M_1$ is the sum of simple HC bimodules with associated variety $\overline{\mathcal{O}}$.

Let us proceed to primitive ideals (=annihilators of irreducible representations). We write $\text{Prim}(\mathcal{U}_\lambda)$ for the set of primitive ideals in $\mathcal{U}_\lambda$. By the Duflo theorem, every primitive ideal in $\mathcal{U}_\lambda$ is the annihilator $\mathcal{J}(\lambda')$ of some irreducible module $L(\lambda')$ with $\lambda' \in W\lambda$. Inside $\text{Prim}(\mathcal{U}_\lambda)$ we can consider the subset $\text{Prim}_0(\mathcal{U}_\lambda)$ of all $\mathcal{J}$ such that $V(\mathcal{U}_\lambda/\mathcal{J}) = \mathcal{O}$.

Suppose that $\lambda$ is regular. We have a surjection $W_{[\lambda]} \twoheadrightarrow \text{Prim}(\mathcal{U}_\lambda)$ that sends $w \in W_{[\lambda]}$ to the left annihilator of $\mathcal{M}_w$, let us denote it by $\mathcal{J}_w$. We have $\mathcal{J}_w = \mathcal{J}(w\mu)$ in our previous notation. The right annihilator of $\mathcal{M}_w$ is $\mathcal{J}_w^{-1}$.

Now suppose $\lambda_0$ is singular (and dominant). Pick a strictly dominant element $\mu$ in the root lattice and let $\lambda = \lambda_0 + \mu$ so that, in particular, $\lambda$ is regular dominant. We have $\mathcal{J}(w'\lambda_0) = \mathcal{J}(w\lambda_0)$ provided $\mathcal{J}(w'\lambda) = \mathcal{J}(w\lambda)$, see [Ja, Section 5.4-5.8]. This gives the embedding $\text{Prim}(\mathcal{U}_\lambda_0) \hookrightarrow \text{Prim}(\mathcal{U}_\lambda)$ whose image consists of the primitive ideals $\mathcal{J}(w\lambda)$, where $w$ is longest in $wW_{\lambda_0}$.

The embedding sends $\text{Prim}(\mathcal{U}_\lambda_0)$ to $\text{Prim}(\mathcal{U}_\lambda)$.

### 2.2. Hecke algebras, cells, and HC bimodules

For a Weyl group $W$ we can consider its Hecke algebra $H_q(W)$ which comes with the distinguished basis $c_w, w \in W$, known as the Kazhdan-Lusztig basis (we use the convention, where the elements $c_w$ are sign-positive with respect to the standard basis $T_w$). This basis allows us to define the so called *two-sided pre-order* on the basis elements. Namely, consider the two-sided based (=spanned by basis elements as a $\mathbb{C}[q^{\pm 1}]$-module) ideal $I_w$. Set $w \leq w'$ if $I_w \subset I_{w'}$. The equivalence classes for this pre-order are known as the *two-sided cells*. The induced order on the set of two-sided cells will also be denoted by $\leq$. Similarly, we can consider left based ideals, and have the pre-order $\leq^L$ and the equivalence relation $\sim^L$ on $W$. The equivalences classes are known as the *left cells*.

The two-sided cells and left cells naturally define subquotients of $H_q(W)$ that are bimodules and left modules, respectively (called two-sided and left cell modules). The two-sided cell modules allow to partition irreducible representations of $H_q(W)$ (and of $CW$ when $W$ is of finite type) into subsets called families.

Now let us discuss a connection between the Hecke algebras and HC bimodules. Let $W$ be the Weyl group of $\mathfrak{g}$. The category $D^b_{HC}(\mathcal{U}_\lambda$-bimod) is monoidal with respect to $\bullet \otimes^{L}_{\mathcal{U}_\lambda} \bullet$. This monoidal structure equips $K_0(\text{HC}(\mathcal{U}_\lambda))$ with an algebra structure. The resulting algebra is $CW_{[\lambda]}$. The class $\mathcal{M}_w$ corresponds to the specialization of $c_{w^{-1}}$ to $q = 1$. The simple reflections in $CW_{[\lambda]}$ correspond to the so called wall-crossing bimodules in $K_0(\text{HC}(\mathcal{U}_\lambda))$.

Let us recall the definition of these bimodules. For $\lambda \in \mathfrak{h}^*$, let us write $D^b_{\mathcal{O}}$ for the sheaf of $(\lambda - \rho)$-twisted differential operators. Pick $w \in W_{[\lambda]}$ and view it as an element of $W_{\mu, \text{int}}$ as before. Set $\psi = w_0\mu - w^{-1}w_0\mu$, this is an element of the root lattice. Let $\mathcal{W}C_w$ denote the $D^b_{\mathcal{O}^{w_0\lambda}} = D^b_{\mathcal{O}^{w_0\lambda}}$-bimodule quantizing the line bundle $\mathcal{O}(\psi)$ on $T^*B$. Then the wall-crossing bimodule $\mathcal{W}C_w$ is the global sections of $\mathcal{W}C_w$.

Moreover, we get a homomorphism $\text{Br}_{W_{[\lambda]}} \to D^b_{HC}(\mathcal{U}_\lambda$-bimod) sending the natural generators of the braid groups to the wall-crossing bimodules, see [M, Section L.3] or [BMR1, Section 2] (that treats the positive characteristic case).
Let us now discuss the representation theoretic meaning of cells. We have \( J_w \subset J_{w'} \) if and only if \( w \leq^L w' \) and hence \( J_w = J_{w'} \) if and only if \( w \sim^L w' \), this follows from combining [Lu4, Lemma 7.4] and [Jo1, Theorem 3.10]. So if \( w, w' \) are in the same two-sided cell, then \( V(M_w) = V(M_{w'}) \) (and the converse is true for integral \( \lambda \)). So to a two-sided cell we can assign a nilpotent orbit in \( g \), let us denote it by \( O_c \). It is easy to see that \( c < c' \) implies \( O_c \subset O_{c'} \).

Recall that every left cell contains a so called distinguished (a.k.a. Duflo) involution, say \( d \). The corresponding simple HC bimodule \( M_d \) is the socle of \( U_\lambda / J_d \). Moreover, the quotient \( V((U_\lambda / J_d)/M_d) \subset \partial O_c \).

Now let us discuss asymptotic Hecke algebras. To any Weyl group \( W \) Lusztig assigned the so called asymptotic Hecke algebra \( J = J(W) \) that is a unital associative algebra (say, over \( \mathbb{C} \)) together with a distinguished basis \( t_w, w \in W \). The unit in \( J \) is the element \( \sum d t_d \), where the sum is taken over all distinguished involutions in \( W \). There is a homomorphism \( CW \to J \) that is known to be an isomorphism when \( W \) is of finite type.

Note that we have \( t_w t_{w'} = 0 \) when \( w, w' \) lie in two different two-sided cells. So we get a decomposition \( J = \bigoplus_c J_c \), where \( J_c \) is the ideal in \( J \) with basis \( t_w, w \in c \). Note that (for \( W \) of finite type) the irreducible \( W \)-modules that belong to a two-sided cell \( c \) are precisely the modules obtained by pullback \( J_c \). Moreover, if \( \sigma \) is a left cell in \( W \) with distinguished involution \( d \), then the left cell module \( [\sigma] \) is \( J(W)_d \).

Now let us give a categorical interpretation of the algebra \( J(W)[\lambda] \). Consider the rigid monoidal category \( \bigoplus_c HC^{ss}(U_\lambda) \), where the sum is taken over all nilpotent orbits in \( g \) (some summands may be zero). Note that \( HC^{ss}(U_\lambda) \) splits as \( \bigoplus_c HC^{ss}(U_\lambda) \), where the summand \( HC^{ss}(U_\lambda) \) is spanned by the \( M_w 's \) with \( w \in c \). So our category can be written as \( \bigoplus_c HC^{ss}(U_\lambda) \), where the sum is taken over all two-sided cells in \( W[\lambda] \). Then, by the work of Joseph, e.g., [Jo2], see also [BFO], it is known that \( K_0(HC^{ss}(U_\lambda)) = J_c(W[\lambda]) \).

### 2.3. Localization in characteristic \( p \)

Let us explain results of [BMR2, BM] related to the localization in characteristic \( p \gg 0 \).

Pick a regular dominant element \( \lambda \in \mathfrak{h}_Q^* \). Let \( x \) be the least common denominator of the coefficients of the simple roots in \( \lambda \). In what follows we assume that \( p + 1 \) is divisible by \( x \) so that \( (p + 1) \lambda \) lies in the root lattice.

Let \( \mathbb{F} \) be an algebraically closed field of large enough characteristic \( p \). Recall that \( B_\mathbb{F} \) stands for the flag variety for \( G \) over \( \mathbb{F} \). Then we have the sheaf \( \mathcal{D}^\lambda_{B_\mathbb{F}} \) that is an Azumaya algebra on \( T^* B_{\mathbb{F}}^{(1)} \). Note that the categories \( \text{Coh}(\mathcal{D}^\lambda_{B_\mathbb{F}}), \text{Coh}(\mathcal{D}^\lambda_{B_\mathbb{F}}) \) are abelian equivalent, say via twist with a line bundle \( O(\mu) \), where \( \mu \) is a weight congruent to \( \lambda' - \lambda \mod p \).

We have \( R \Gamma(\mathcal{D}^\lambda_{B_\mathbb{F}}) = U_{\lambda, \mathbb{F}} \). It was shown in [BMR2, Section 3.2] that the derived global section functor \( R \Gamma : D^b(\text{Coh}(\mathcal{D}^\lambda_{B_\mathbb{F}})) \to D^b(U_{\lambda, \mathbb{F}} \text{-mod}) \) is an equivalence. Further, it was checked in [BMR2, Section 5.4] that \( \mathcal{D}^\lambda_{B_\mathbb{F}} \) splits in the formal neighborhood \( B_{\mathbb{F}}^{(1)} \) of the Springer fiber in \( T^* B_{\mathbb{F}}^{(1)} \).

Pick a splitting bundle \( \mathcal{V}_{\chi, \mathbb{F}} \). This gives rise to the abelian equivalence

\[
\mathcal{V}_{\chi, \mathbb{F}} \otimes \bullet : \text{Coh}_\chi(T^* B_{\mathbb{F}}^{(1)}) \xrightarrow{\sim} \text{Coh}_\chi(\mathcal{D}^\lambda_{B_\mathbb{F}})),
\]

where the subscript \( \chi \) refers to the subcategory of sheaves set-theoretically supported at the Springer fiber. So we arrive at the derived equivalence \( D^b(\text{Coh}(T^* B_{\mathbb{F}}^{(1)})) \xrightarrow{\sim} D^b(U_{\lambda, \mathbb{F}} \text{-mod}) \) given by \( M \mapsto R \Gamma(\mathcal{V}_{\chi, \mathbb{F}} \otimes M) \). The following was shown in [BMR2, Lemma 6.2.5]:
**Lemma 2.2.** Fix $\lambda'$ in the root lattice such that $\lambda' = \lambda \mod p$. Then there exists a canonical choice of the splitting bundle $\mathcal{V}_{\lambda',\mathbb{F}}$ (recall it is defined up to a twist with a line bundle) such that the class $[\mathcal{V}_{\lambda',\mathbb{F}}] \in K_0(B^{(1)}_{\lambda',\mathbb{F}})$ is the pull-back of $[(\text{Fr}_{\mathcal{B}_{\mathbb{F}}})_*\mathcal{O}((p-1)\rho + \lambda')]$.

The resulting equivalence is denoted by $\mathcal{L}_{\lambda'}$.

Below we always choose $\mathcal{V}_{\lambda',\mathbb{F}}$ as in the lemma.

**Proposition 2.3.** The following is true:

1. The images of the classes of simple $\mathcal{U}_{\lambda',\mathbb{F}}\mod\chi$-modules in $K_0(\text{Coh}(\mathcal{B}_{\chi}))$ are independent of $p$ (as long as $p \gg 0$).
2. The dimensions of the simple $\mathcal{U}_{\lambda',\mathbb{F}}\mod\chi$-modules are polynomials in $p$ provided $p+1$ is divisible by $x$.

**Proof.** (1) for general $\lambda$ follows from the case $\lambda = 0$ which is (a) in [BM, Corollary 5.1.8]. (2) follows easily from (1) and Lemma 2.2, compare to [BMR2, Section 6.2].

We now discuss actions of algebras of interest on the above Grothendieck groups. Recall that $K_0(\text{Coh}(\mathbb{F}^*(\mathcal{B}_c)))$ is a module over the affine Hecke algebra $\mathcal{H}_q(W^\alpha)$. In particular, $K_0(\text{Coh}(\mathcal{B}_c)) \cong \mathcal{H}_q(\mathcal{B}_c, \mathbb{C})$ acquires an action of $W^\alpha$.

As was shown in [BMR1, Section 2] the latter action is categorified by an action of the affine braid group $B_{\text{aff}}$ on $\mathcal{D}^b(\mathcal{U}_{\lambda',\mathbb{F}}\mod\chi) \cong \mathcal{D}^b(\text{Coh}_{\mathcal{B}_{\mathbb{F}}}(T^*\mathcal{B}_{\mathbb{F}}))$, while the former one is categorified by a compatible action on the derived category of a graded version of $\mathcal{U}_{\lambda',\mathbb{F}}\mod\chi$, which is derived equivalent to $\text{Coh}_{\mathcal{B}_{\mathbb{F}}}(T^*\mathcal{B}_{\mathbb{F}})$ (see [BM, 5.3.1, 5.3.2]).

For future reference we mention a standard property of this action. For a simple reflection $\alpha$ we let $\tilde{s}_\alpha$ denote the corresponding generator of the affine braid group.

**Lemma 2.4.** For a simple reflection $\alpha$ and an irreducible module $L \in \mathcal{U}_{\lambda',\mathbb{F}}\mod\chi$ the object $\tilde{s}_\alpha(L)$ either lies in the abelian category $\mathcal{U}_{\lambda',\mathbb{F}}\mod\chi$ or is isomorphic to $L[1]$.

**Proof.** Consider the full embedding $\mathcal{U}_{\lambda',\mathbb{F}}\mod\chi \to \mathcal{U}_{\mathbb{F}}\mod\chi$, where the target category consists of all $\mathfrak{g}_{\mathbb{F}}$-modules where the kernel of the central ideal corresponding to $(\lambda, \chi)$ acts nilpotently. By [BR, Theorem 1.3.1], we have a compatible $B_{\text{aff}}$ action on $\mathcal{D}^b(\mathcal{U}_{\mathbb{F}}\mod\chi)$, and it suffices to check the same statement in $\mathcal{D}^b(\mathcal{U}_{\mathbb{F}}\mod\chi)$. We have the exact reflection functor $\Xi_\alpha$ acting on $\mathcal{U}_{\mathbb{F}}\mod\chi$ and an exact triangle

$$L \mapsto \Xi_\alpha(L) \to \tilde{s}_\alpha(L).$$

Recall that $\Xi_\alpha = T_{\mu \to \lambda} \circ T_{\lambda \to \mu}$ is a composition of two biadjoint translation functors for a weight $\mu$ on the $\alpha$-wall. If $T_{\lambda \to \mu}(L) = 0$ then $\Xi_\alpha(L) = 0$ and $\tilde{s}_\alpha(L) \cong L[1]$. If $T_{\lambda \to \mu}(L) \neq 0$, then the adjunction arrow $L \to T_{\mu \to \lambda} \circ T_{\lambda \to \mu}(L)$ is nonzero, hence it is injective provided that $L$ is irreducible. Thus in this case $\tilde{s}_\alpha(L) \cong \Xi_\alpha(L)/L$ is concentrated in homological degree zero.

**Remark 2.5.** It is natural to expect that the aforementioned action of the affine braid group on the derived categories of coherent sheaves factors through the standard categorification of the affine Hecke algebra; the latter can be defined either using constructible sheaves on the affine flag variety, or using the theory of Soergel bimodules. For a base field of characteristic zero this follows from the main result of [B], see also [BY] for the relation to Soergel bimodules. For a base field of positive characteristic (which is the setting related to $\mathfrak{g}$-modules in positive characteristic as explained above) this question is still open, to the authors’ knowledge.
3. Lengths

This section contains a number of results that will be used to prove Theorem 1.1.

3.1. Reduction of HC bimodules to characteristic $p$. The proof of Proposition 3.3 will be based on considering reductions of HC bimodules to characteristic $p$.

Let us start by discussing $R$-forms of Harish-Chandra bimodules. The category of HC bimodules is defined over $\mathbb{Q}$, the Bernstein-Gelfand equivalence shows that $\lambda HC^\lambda$ is split over $\mathbb{Q}$ because the category $O$ is split over the rationals. Recall that an abelian category equivalent to a category of modules over a finite dimensional algebra over a field is called split if the endomorphism algebras of all simples coincide with the field.

Clearly, there is a finite localization $R$ of $\mathbb{Z}$ such that the tensor category $D^b_{HC}(U_\lambda)$ is defined over $R$. All simples are defined over $R$ as well, let us fix some $R$-lattices $\mathcal{M}_{w,R}, w \in W_\lambda$. Note that we can still talk about HC $U_\lambda$-bimodules: these are bimodules $\mathcal{M}$ that admit a bounded from below good filtration (such that the left and the right actions of $R[N]$ on $\text{gr} \ \mathcal{M}$ coincide and the $R[N]$-module $\text{gr} \ \mathcal{M}$ is finitely generated – here $N$ stands for the nilpotent cone of $g$). In particular, every HC $U_\lambda$- (or $U_R$-) bimodule becomes flat over $R$ after a finite localization. Note also that any Tor of any two HC $U_\lambda$-bimodules is again HC.

For a primitive ideal $\mathcal{J} \subset U_\lambda$ we set $\mathcal{J}_R := U_\lambda \cap \mathcal{J}$.

Let $V = V(\mu)$ denote the irreducible $G$-module with highest weight $\mu$. For $m \in \mathbb{Z}_{>0}$, we write

$$V^m := \bigoplus_{\mu(\rho',\mu) \leq m} V(\mu)^{\dim V(\mu)}.$$

We will also impose the following conditions that we can achieve by a finite localization of $R$ (in (c3),(c4) we fix $m$ and then further localize $R$). Here (c2) follows from Corollary 2.1, that is an analogous statement over $\mathbb{C}$.

(c1) For every distinguished involution $d$, we have an inclusion $\mathcal{M}_{d,R} \hookrightarrow U_{\lambda,R}/\mathcal{J}_{d,R}$ and the quotient is filtered by bimodules $\mathcal{M}_{w,R}$ with $w <^L d$.

(c2) For every $w_1, w_2 \in c$, there are $U_{\lambda,R}$-subbimodules $M_1 \subset M_2 \subset M_{w_1,R} \otimes_{U_{\lambda,R}} M_{w_2,R}$ such that

- both $M_1$ and $M_{w_1,R} \otimes_{U_{\lambda,R}} M_{w_2,R}/M_2$ are filtered by $\mathcal{M}_{w,R}$’s, where $w$ lie in cells strictly less then $c$
- and $M_2/M_1$ is isomorphic to the direct sum of $\mathcal{M}_{w,R}$’s for $w \in c$.

(c3) Both pr$_\lambda(V^m \otimes M_w)$ and its complement in $V^m \otimes M_w$ are defined over $R$ for all $w \in W_\lambda$.

(c4) pr$_\lambda(V^m \otimes M_w)_R$ is filtered by $\mathcal{M}_{w,R}$’s. Moreover, there is a quotient of pr$_\lambda(V^m \otimes M_w)_R$ isomorphic to the direct sum of $\mathcal{M}_{w,R}$’s that gives $\text{head}(\text{pr}_\lambda(V^m \otimes M_w))$ after base change to $\mathbb{C}$ (recall that by the head we mean the maximal semisimple quotient).

(c5) The wall-crossing bimodules are defined over $R$ and define a homomorphism $Br_{W_\lambda} \to D^b_{HC}(U_{\lambda,R})$.

(c6) All Tor$_{U_{\lambda,R}}^i(M_{w_1,R}, M_{w_2,R})$ are filtered by $\mathcal{M}_{w,R}$’s (note that after a finite localization of $R$ only finitely many of these Tor’s are nonzero because $U_\lambda$ has finite homological dimension). The analogous result is true for Tor$_{U_{\lambda,R}}^i(M_{w_1,R}, U_{\lambda,R}/\mathcal{J}_{w_2,R})$.
Lemma 3.1. Under the identification $K_0(\text{Coh}(B_\lambda)) \cong K_0(U_{\lambda, F} \text{-mod}^\times)$, the two actions of $W_{[\lambda]}$ (i.e., the one defined above and the one restricted from the $W^a$-action in Section 2.3) coincide.

Proof. The $W^a$-action on $K_0(\text{Coh}(B_\lambda))$ corresponds to the action on $K_0(U_{\lambda, F} \text{-mod}^\times)$ by the wall-crossing functors, [R, Section 5.4]. By [BMR1, Theorem 2.1.4], the wall-crossing functors through the walls defined by the simple roots for $W_{[\lambda]}$ are given by taking the derived tensor products with the wall-crossing bimodules. □

Below we are also going to use the following lemma.

Lemma 3.2. Let $L$ be a simple $U_{\lambda, F}$-module and let $d$ be a distinguished involution such that $J_d$ is a maximal primitive ideal with $J_{d,R}L = 0$. Then the following is true:

1. For any $w \in W$, all simple constituents of $\text{Tor}^*_{U_{\lambda, R}}(M_{w,R}, L)$ are annihilated by $J_{w,R}$.
2. Let $L'$ be a simple constituent of $\text{Tor}^*_{U_{\lambda, R}}(M_{w,R}, L)$ and let $J_{d'}$ be a maximal primitive ideal such that $J_{d,R}L' = 0$. Then the two-sided cell of $d'$ is less than or equal to that of $d$.
3. Assume, in addition, that $d \not\prec_L w^{-1}$. Let $L'$ be a simple constituent of $\text{Tor}^*_{U_{\lambda, R}}(M_{w,R}, L)$ and $d' \in W$ be as in (2). Then $d' \prec_L w$.

Proof. Let us take a resolution of $L$ by free $U_{\lambda,R}$-modules:

$$\cdots \to U_{\lambda,R}^{\oplus n_2} \to U_{\lambda,R}^{\oplus n_1} \to L \to 0.$$ 

Then $\text{Tor}^*_{U_{\lambda, R}}(M_{w,R}, L)$ is the homology of the complex

$$\cdots \to M_{w,R}^{\oplus n_2} \to M_{w,R}^{\oplus n_1} \to 0.$$ 

The individual terms of this complex are annihilated by $J_{w,R}$ hence so is the homology. This proves (1).

Let us prove (2). Since $J_{d,R}$ annihilates $L$, we have

$$M_{w,R} \otimes_{U_{\lambda,R}} J_{d,R} L = \left( M_{w,R} \otimes_{U_{\lambda,R}} (U_{\lambda,R}/J_{d,R}) \right) \otimes_{U_{\lambda,R}/J_{d,R}} J_{d,R} L.$$ 

Note that all simples $M_{w'}$ appearing in $\text{Tor}^*_{U_{\lambda}}(M_{w},U_{\lambda}/J_{d})$ satisfy $w' \leq_L w$, $w'^{-1} \leq_L d$. This, together with (c6), implies (2). In (3), since $d \not\prec_L w^{-1}$, we have $w' \prec_L w$. (3) follows. □

3.2. Results on growth of lengths. Recall that $\lambda \in b_Q^*$ is regular and $p$ is large enough, in particular, $\lambda$ is well-defined and regular mod $p$. Let $L$ be a simple in $U_{\lambda, F} \text{-mod}^\times$.

Given $m$, we always choose $p$ large enough for $V(\mu)$ to be irreducible mod $p$ provided $\langle \rho^\vee, \mu \rangle \leq m$ and $V_F^m \otimes V_F^m$ to be semisimple.

Let us write $pr_\lambda$ for the projection $U_F \text{-mod}^\times \to U_F \text{-mod}^\times_\lambda$. For a module $M \in U_F \text{-mod}^\times$, let us write $\ell(M)$ for its length. We want to understand the behavior of the length $\ell(pr_\lambda(V_F^m \otimes L))$ as a function of $m$.

Proposition 3.3. Let $L$ be an irreducible $U_{\lambda,F}$-module. Let $J \subset U_{\lambda}$ be a maximal primitive ideal with the following property: $L$ is annihilated by $J_R$. Let $\mathcal{O}$ be the associated variety of $U_{\lambda}/J$. Then there are real numbers $c, C$ with $0 < c < C$ such that for all $m \in \mathbb{Z}_{>0}$ and $p \gg m$ we have

$$c \leq \frac{\ell(pr_\lambda(V_F^m \otimes L))}{m^{\dim \mathcal{O}}} < C.$$
We will deduce this from an analogous result for Harish-Chandra bimodules. Namely, let $\mathcal{M} \in \underset{\chi}{\bigotimes} \text{HC}^1_{\lambda}$. Then we can consider the HC bimodule $\text{pr}_{\lambda}(V^m \otimes \mathcal{M}) \in \underset{\chi}{\bigotimes} \text{HC}^1_{\lambda}$ and its length $\ell(\text{pr}_{\lambda}(V^m \otimes \mathcal{M}))$.

**Proposition 3.4.** Let $\mathcal{M} \in \underset{\chi}{\bigotimes} \text{HC}^1_{\lambda}$ with $V(\mathcal{M}) = \overline{O}$. Let $m \in \mathbb{Z}_{>0}$. Then there are $0 < c < C$ such that for all $m \in \mathbb{Z}_{>0}$ we have

$$c < \frac{\ell(\text{pr}_{\lambda}(V^m \otimes \mathcal{M}))}{m \dim O} < C.$$  

**3.3. Lengths for HC bimodules.** In this section we will prove Proposition 3.4 and work over $\mathbb{C}$. We are going to bound $\ell(\text{pr}_{\lambda}(V^m \otimes \mathcal{M}))$ by two degree $\dim O$ polynomials in $m$, where $V(\mathcal{M}) = \overline{O}$.

Under the Bernstein-Gelfand equivalence $\underset{\chi}{\bigotimes} \text{HC}^1_{\lambda} \sim \mathcal{O}(\mu)$ (in the notation of Section 2.1) $U_{\lambda}$ maps to the indecomposable projective $\Delta(w_{\lambda}\mu)$. Every indecomposable projective in $\underset{\chi}{\bigotimes} \text{HC}^1_{\lambda}$ appears as a summand in an object of the form $\text{pr}_{\lambda}(V_0 \otimes U_{\lambda})$ for a suitable finite dimensional $G$-module $V_0$ that we fix from now on.

**Lemma 3.5.** Let $\mathcal{M} \in \underset{\chi}{\bigotimes} \text{HC}(U^1_{\lambda})$ have associated variety $\overline{O}$. Then

$$\dim \text{Hom}_{\text{bimod}}(U_{\lambda}, V^m \otimes \mathcal{M})$$

is a degree $\dim O$ polynomial in $m$.

**Proof.** Note that $\text{Hom}_{\text{bimod}}(U_{\lambda}, V^m \otimes \mathcal{M}) = \text{Hom}_{\text{bimod}}(U(\mathfrak{g}), V^m \otimes \mathcal{M})$ because $V^m \otimes \mathcal{M}$ has genuine central character $\lambda$ on the right. Also $\text{Hom}_{\text{bimod}}(U(\mathfrak{g}), V^m \otimes \mathcal{M}) = (V^m \otimes \mathcal{M})^G = \text{Hom}_G(V^{m*}, \mathcal{M})$. So

$$\dim \text{Hom}_{\text{bimod}}(U_{\lambda}, V^m \otimes \mathcal{M}) = \dim \text{Hom}_G(V^{m*}, \mathcal{M}).$$

Set $\mathcal{M}' := \text{gr} \mathcal{M}$ with respect to some good filtration, this is a finitely generated $G$-equivariant $\mathbb{C}[\mathfrak{g}]$-module. Clearly, $\dim \text{Hom}_G(V^{m*}, \mathcal{M}) = \dim \text{Hom}_G(V^{m*}, \mathcal{M}')$. Consider the filtration $\mathcal{M}'_{\leq i}$ on $\mathcal{M}'$ given by $\mathcal{M}'_{\leq i}$ being the sum of the isotypic components of $\mathcal{M}'$ with $\langle \rho^\vee, \mu \rangle \leq i$. This filtration is compatible with the similarly defined filtration on $\mathbb{C}[\mathfrak{g}]$.

It is well known that for any finitely generated commutative $G$-algebra $A$, the algebra $\text{gr} A$ (for the filtration $A = \bigcup_i A_{\leq i}$) is finitely generated and for any finitely generated $G$-equivariant $A$-module $M$, the $\text{gr} A$-module $\text{gr} M$ is finitely generated. This is because $\text{gr} A = (\mathbb{C}[G/U \times G/U]^T \otimes A)^G$ and a similar equality holds for $\text{gr} M$, here $U$ is a maximal unipotent subgroup of $G$.

It follows that the GK dimensions of $M, \text{gr} M$ are the same. Since $\dim \text{Hom}_G(V^{m*}, \mathcal{M}') = \dim \mathcal{M}'_{\leq m} = \dim \text{gr} \mathcal{M}'_{\leq m}$, the left hand side of this equality is the Hilbert polynomial of the graded module $\text{gr} \mathcal{M}'$. But the GK-dimension of $\mathcal{M}'$ is $\dim O$ and our claim follows. □

**Proof of Proposition 3.4.** Let us prove that $\ell(\text{pr}_{\lambda}(V^m \otimes \mathcal{M})) \geq Q(m)$, where $Q$ is a degree $\dim O$ polynomial. Note that $\dim \text{Hom}_{\text{bimod}}(U_{\lambda}, V^m \otimes \mathcal{M})$ is the multiplicity of the simple bimodule covered by $U_{\lambda}$ in $V^m \otimes \mathcal{M}$. By Lemma 3.5, this multiplicity is a degree $\dim O$ polynomial in $m$, say $Q(m)$. On the other hand, it is clear that $\ell(\text{pr}_{\lambda}(V^m \otimes \mathcal{M})) \geq \dim \text{Hom}_{\text{bimod}}(U_{\lambda}, V^m \otimes \mathcal{M}) = Q(m)$.

Let us prove that $\ell(\text{pr}_{\lambda}(V^m \otimes \mathcal{M})) \leq \tilde{Q}(m)$, where $\tilde{Q}$ is also a degree $\dim O$ polynomial. Let $V_0$ be as in the second paragraph of Section 3.3. Then

$$\ell(\text{pr}_{\lambda}(V^m \otimes \mathcal{M})) \leq \dim \text{Hom}_{\text{bimod}}(V_0 \otimes U_{\lambda}, V^m \otimes \mathcal{M})$$

$$= \dim \text{Hom}_{\text{bimod}}(U_{\lambda}, V^m \otimes (V_0^* \otimes \mathcal{M})) =: \tilde{Q}(m).$$
Applying Lemma 3.5 to $V_0^* \otimes \mathcal{M}$, we see that $\tilde{Q}(m)$ is a degree dim $\mathcal{O}$ polynomial in $m$. □

In the proof of Proposition 3.3 we will also need the following lemma.

**Lemma 3.6.** There is a constant $0 < c_0 < 1$ such that, for any object $\mathcal{M} \in \overline{\lambda}^1 \text{HC}(\mathcal{U})_\lambda$ we have $\ell(\text{head}(\mathcal{M})) \geq c_0 \ell(\mathcal{M})$.

*Proof.* Note that $\overline{\lambda}^1 \text{HC}(\mathcal{U})_\lambda$ is equivalent to the category of modules over a finite dimensional algebra. We claim that in any such category

$$\ell(\text{head}(\mathcal{M})) > \frac{1}{L} \ell(\mathcal{M})$$

for every module $\mathcal{M}$, where $L$ is the maximum of lengths of the indecomposable projectives. Suppose that we know (2) for all $\mathcal{M}'$ with $\ell(\mathcal{M}') < \ell(\mathcal{M})$. Pick a simple constituent $L$ in $\text{head}(\mathcal{M})$. Let $P_L$ be the projective cover of $L$. We have a homomorphism $\varphi : P_L \to M$ whose composition with $M \to \text{head}(\mathcal{M})$ coincides with $P_L \to L \hookrightarrow \text{head}(\mathcal{M})$. Clearly, $\text{head}(\text{coker}\varphi) = \text{head}(\mathcal{M})/L$. Applying the induction hypothesis to $\text{im}\varphi$ and $\text{coker}\varphi$ we finish the proof of this lemma. □

### 3.4. Lengths in characteristic $p$

To prove Proposition 3.3 we will need the following technical lemma.

**Lemma 3.7.** Let $L$ be an irreducible $\mathcal{U}_F^\lambda$-module such that $\mathcal{J} = \mathcal{J}_{w^{-1}}$ is a maximal primitive ideal with $\mathcal{J}_RL = 0$. Then the following is true:

1. $\mathcal{M}_{w,R} \otimes_{\mathcal{U}_R} L \neq 0$.
2. there is $c_1 > 1$ independent of $p$ such that $\ell(\mathcal{M}_{w,R} \otimes_{\mathcal{U}_R} L) < c_1$.

We will first deduce Proposition 3.3 from this lemma and then prove it.

*Proof of Proposition 3.3.* Let $\mathcal{J}$ be a maximal primitive ideal in $\mathcal{U}_\lambda$ such that $L$ is annihilated by $\mathcal{J}_R$. By (c3), $\text{pr}_\lambda(V_R^m \otimes_R L) = \text{pr}_\lambda(V^m \otimes \mathcal{J}_R \otimes_{\mathcal{U}_R} L)$. Thanks to Proposition 3.4, what we need to prove is that there are constants $0 < c < C$ such that $c \leq \ell(\mathcal{M}_R \otimes_{\mathcal{U}_F} L)/\ell(\mathcal{M}) \leq C$ for any $\mathcal{M} \in \overline{\lambda}^1 \text{HC}(\mathcal{U}_R)_\lambda$ whose right annihilator is $\mathcal{J}_R$. By (1) of Lemma 3.7 combined with Lemma 3.6 and (c4), we can set $c := c_0$ from Lemma 3.6. By Lemma 3.7, we can set $C := c_1$. □

*Proof of Lemma 3.7.* Let us prove (1). Pick $w'$ such that $\mathcal{M}_{w'} \otimes_{\mathcal{U}_\lambda} \mathcal{M}_w$ has $\mathcal{M}_d$ as a direct summand in $\text{HC}(\mathcal{U}_\lambda)$ ($w'$ exists because $\text{HC}(\mathcal{U}_\lambda)$ is a rigid monoidal category). Set $w_1 = w', w_2 = w$ in (c2) and let $M_1 < M_2$ be as in (c2). The equality $\mathcal{M}_{w,R} \otimes_{\mathcal{U}_R} L \neq 0$ will follow once we show that $((\mathcal{M}_{w',R} \otimes_{\mathcal{U}_R} \mathcal{M}_{w,R})/M_1) \otimes_{\mathcal{U}_R} L \neq 0$.

First, let us check that $\mathcal{M}_{d,R} \otimes_{\mathcal{U}_R} L \neq 0$. By the choice of $d$, we have $(\mathcal{U}_\lambda/R, \mathcal{J}_{d,R}) \otimes_{\mathcal{U}_R} L \neq 0$. So $\mathcal{M}_{d,R} \otimes_{\mathcal{U}_R} L \neq 0$ as long as $\text{Tor}_i^{\mathcal{U}_R}((\mathcal{U}_\lambda/R)/(\mathcal{J}_{d,R}), L)$ does not have $L$ in its Jordan-Hoelder series for $i = 0,1$. This is a consequence of (3) Lemma 3.2. We conclude that $\mathcal{M}_{d,R} \otimes_{\mathcal{U}_R} L \neq 0$.

From here we deduce that $(M_2/M_1) \otimes_{\mathcal{U}_R} L \neq 0$. Similarly to the previous paragraph this implies $((\mathcal{M}_{w',R} \otimes_{\mathcal{U}_R} \mathcal{M}_{w,R})/M_1) \otimes_{\mathcal{U}_R} L \neq 0$. This finishes the proof of (1).

Let us prove (2). It is enough to prove this statement with $\mathcal{M}_{w,R}$ replaced with a bimodule that covers it, e.g., $\text{pr}_\lambda(V_0 \otimes \mathcal{U}_\lambda, R)$, where $V_0$ is as in the beginning of Section 3.3. Note that bimodule is projective as a right module. On the level of $K_0$ the operator $\text{pr}_\lambda(V_0 \otimes \mathcal{U}_\lambda, R) \otimes_{\mathcal{U}_R} \bullet$ is the multiplication by some element, say $y$, of $\text{CW}[\lambda]$ independent of $p$. For $b \in \mathfrak{B}$, we can expand $yb = \sum_{b' \in \mathfrak{B}} m_{bb'} b'$. Then $c_1 = \max_{b \in \mathfrak{B}} \sum_{b'} m_{bb'}$ satisfies the conditions of (2). □
4. Proof of Theorem 1.1

In this section we will prove Theorem 1.1. Part (1) is proved in Section 4.1, while the proof of part (2) occupies the remainder of the section. We will describe the main steps of the proof in Section 4.2.

4.1. Proof of part (1) of Theorem 1.1. For a two-sided cell $c$ in $W_{[\lambda]}$ consider the full subcategory $\mathcal{U}_{\lambda,F} - \text{mod}^\lambda_{<c}$ that is the Serre span of all simples annihilated by $\mathcal{J}_{w,R}$ with $w \in c$. Note that $D^b(\mathcal{U}_{\lambda,F} - \text{mod}^\lambda)_{<c}$ is a submodule category for the action of $D^b_{HC}(\mathcal{U}_{\lambda,R} - \text{mod})$, this follows from (2) of Lemma 3.2.

We have the following result.

**Proposition 4.1.** All irreducible representations of $W_{[\lambda]}$ occurring in

$$K_0(\mathcal{U}_{\lambda,F} - \text{mod}^\lambda_{<c})/K_0(\mathcal{U}_{\lambda,F} - \text{mod}^\lambda_{<c})$$

belong to the two-sided cell $c$.

**Proof.** We need to prove two statements:

1. $C\mathcal{W}_{[\lambda],<c}K_0(\mathcal{U}_{\lambda,F} - \text{mod}^\lambda_{<c}) \subset K_0(\mathcal{U}_{\lambda,F} - \text{mod}^\lambda_{<c})$.
2. $C\mathcal{W}_{[\lambda],<c}K_0(\mathcal{U}_{\lambda,F} - \text{mod}^\lambda_{<c}) = K_0(\mathcal{U}_{\lambda,F} - \text{mod}^\lambda_{<c})$.

(1) follows from (1) of Lemma 3.2. Let us prove (2). Let $L \in \mathcal{U}_{\lambda,F} - \text{mod}^\lambda_{<c}$ be a simple object annihilated by $\mathcal{J}_{d,R}$, where $d \in c$. Tautologically ($\mathcal{U}_{\lambda,R}/\mathcal{J}_{d,R}) \otimes_{\mathcal{U}_{\lambda,R}/\mathcal{J}_{d,R}} \mathcal{L} = L$. Let us write $V$ for $\text{Span}_{w \in L_d}(\mathcal{M}_{w^-1})$. Note that $C\mathcal{W}_{[\lambda],<c}V = V$. Indeed, all $W$-irreducibles appearing in $V$ belong to the families indexed by two-sided cells $\leq c$.

So we have $[\mathcal{U}_{\lambda}/\mathcal{J}_{d}] = \sum_{w_1,w_2}[\mathcal{M}_{w_1}]\mathcal{M}_{w_2}$ (an equality in $V$), where $w_1$ runs over $c$ and $w_2$ over the elements such that $w_2^{-1} \leq L d$. It follows that $[L]$ belongs to the linear span of the classes of the form

$$[(\mathcal{M}_{w_1,R} \otimes_{\mathcal{U}_{\lambda,R}} \mathcal{M}_{w_2,R})_{\mathcal{U}_{\lambda,R}/\mathcal{J}_{d,R}} L] = [\mathcal{M}_{w_1}]\mathcal{M}_{w_2,R} \otimes_{\mathcal{U}_{\lambda,R}} L].$$

Note that $\mathcal{M}_{w_2,R} \otimes_{\mathcal{U}_{\lambda,R}} L \in D^b(\mathcal{U}_{\lambda,F} - \text{mod}^\lambda)_{<c}$ by (1) of Lemma 3.2. We deduce that $[L] \in C\mathcal{W}_{[\lambda],<c}K_0(\mathcal{U}_{\lambda,F} - \text{mod}^\lambda_{<c})$. This implies (2). □

**Proof of (1) of Theorem 1.1.** Proposition 4.1 and an easy induction on $c$ show that

$$K_0(\mathcal{U}_{\lambda,F} - \text{mod}^\lambda_{<c}) = K_0(\mathcal{U}_{\lambda,F} - \text{mod}^\lambda_{<c})$$

This establishes the claim of part (1) at $q = 1$.

To prove the full claim one uses the graded lifts mentioned in Section 2.3. Namely, let us write $\mathcal{C}^{gr}$ for the graded lift of $\mathcal{C} := \mathcal{U}_{\lambda,F} - \text{mod}^\lambda$. We still have the two-sided cell filtration $\mathcal{C}^{gr}_{<c}$ on $\mathcal{C}^{gr}$ that is closed under the grading shifts and lifts the filtration $\mathcal{U}_{\lambda,F} - \text{mod}^\lambda_{<c}$.

We claim that $D^b(\mathcal{C}^{gr}_{<c})$ is invariant under the braid group $B_{[\lambda]}$, hence its Grothendieck group is invariant under the corresponding Hecke algebra. It is enough to check that, for a simple module $L \in \mathcal{C}^{gr}_{<c}$ and a generator $\tilde{s}_\alpha \in B_{[\lambda]}$, we have $\tilde{s}_\alpha(L) \in D^b(\mathcal{C}^{gr}_{<c})$. In view of Lemma 2.4 this follows from $s_\alpha([L]) \in K^0(\mathcal{C}^{gr}_{<c})$, which has already been proven.

So $K_0(\mathcal{C}^{gr}_{<c})/K_0(\mathcal{C}^{gr}_{<c})$ is an $\mathcal{H}_q(W_{[\lambda]})$-module flat over $\mathbb{C}[q, q^{-1}]$ that specializes to

$$K_0(\mathcal{U}_{\lambda,F} - \text{mod}^\lambda_{<c})/K_0(\mathcal{U}_{\lambda,F} - \text{mod}^\lambda_{<c})$$

at $q = 1$. The latter factors through the quotient corresponding to $c$ hence so is the former. □
4.2. Outline of the proof of (2) of Theorem 1.1. Below we will prove part (2) of Theorem 1.1. We will start with the $\chi = 0$ case.

**Proposition 4.2.** (2) of Theorem 1.1 holds when $\chi = 0$.

Here we will use an easy adaptation of an argument due to Etingof that relates the degrees of dimension polynomials with the GK dimensions of simples in the category $\mathcal{O}$ (in characteristic 0). We will reduce the case of general $\chi$ to $\chi = 0$ by using the degeneration map $K_0(U_{\lambda, R}^\chi$-mod) $\to K_0(U_{\lambda, R}^0$-mod). This map can be shown to be independent of $p$. Hence it preserves the dimension polynomials. The most nontrivial step is to show that the degeneration of a simple module that lies in $K_0(U_{\lambda, R}^\chi$-mod) but not in the lower filtration terms lies in $K_0(U_{\lambda, R}^0$-mod)$_{\leq c}$ (this is straightforward) but not in the lower filtration terms (this is harder, we need Proposition 3.3 to handle this part).

Before we proceed to proving part (2), let us reformulate it. For this we need the following lemma.

**Lemma 4.3.** Let a simple $L \in U_{\lambda, R}^\chi$-mod$_{\leq c}$ but not in lower filtration terms. Then there is a unique maximal primitive ideal $J = J_d \in \text{Prim}(U_{\lambda})$ with $J_d L = 0$. We have $d \in c$.

**Proof.** Let $J_1, J_2$ be two maximal primitive ideals with the required property. Assume, in addition, that the two-sided cell corresponding to $J_1$ is maximal possible. By (c6), $U_{\lambda, R}/(J_1, R + J_2, R)$ is filtered by $M_{w, R}$ for $w$ lying in two-sided cells smaller than $c$. If $M_{w, R} \otimes_{U_{\lambda, R}} L \neq 0$, then $J_{w^{-1}, R} L = 0$. The two-sided cell of $J_{w^{-1}}$ is strictly smaller than those of $J_1, J_2$. A contradiction. The inclusion $d \in c$ follows from the definition of $U_{\lambda, R}^\chi$-mod$_{\leq c}$. □

**Remark 4.4.** So (2) of Theorem 1.1 says that the dimension polynomial of $L$ has degree equal to $\frac{1}{2}$ GK- dim$(U_{\chi}/J)$, where $J$ is the maximal primitive ideal in $U_{\chi}$ such that $J_d L = 0$. This statement makes sense for other classes of quantizations, e.g. for those of symplectic singularities and we expect it to hold in this setting.

### 4.3. Etingof’s construction

We will prove Proposition 4.2 by adapting an argument due to Etingof from categories $\mathcal{O}$ for type A rational Cherednik algebras to BGG categories $\mathcal{O}$.

Let us explain this argument. Let $R$ be a finite localization of $\mathbb{Z}$ such that $\lambda \in \mathfrak{h}_R^*$. Then we can consider the Verma module $\Delta_R(\lambda)$ with highest weight $\lambda + \rho$. Note that $\Delta_R(\lambda)$ is naturally graded with highest vector in degree 0 and the operators $L_n$ are graded quotients. Let $F$ to $Etingof$.

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Let us explain this argument. Let $R$ be a finite localization of $\mathbb{Z}$ such that $\lambda \in \mathfrak{h}_R^*$. Then we can consider the Verma module $\Delta_R(\lambda)$ with highest weight $\lambda + \rho$. Note that $\Delta_R(\lambda)$ is naturally graded with highest vector in degree 0 and the operators $L_n$ have degree 1 for a simple root $\alpha$. Let $p$ be a prime number invertible in $R$. Set $\Delta_{R, \lambda} := \mathbb{F}_p \otimes_R \Delta_R(\lambda), \Delta_{\lambda}(\lambda) := \mathbb{Q} \otimes_R \Delta_R(\lambda)$. Let $L_{\lambda}(\lambda)$ be the unique irreducible quotient of $\Delta_{\lambda}(\lambda)$. The module $\Delta_{R, \lambda}$ has a unique graded simple quotient, let us denote it by $L_{R, \lambda}(\lambda)$. The modules $L_{R, \lambda}(\lambda')$ for $\lambda' \in W\lambda$ are absolutely irreducible and pairwise non-isomorphic and so their base changes to $\mathbb{F}$ form a complete collection of the irreducibles in $U_{\lambda, R}^0$-mod. Note that $L_{\lambda}(\lambda)$ and $L_{R, \lambda}(\lambda)$ are graded quotients. Let $L^i_{\lambda}(\lambda), L^i_{R, \lambda}(\lambda)$ denote the $i$th graded component.

The following lemma is due to Etingof (in the Cherednik case).

**Lemma 4.5.** Suppose that $p$ is sufficiently large. Then there is a positive integer $N_\lambda$ independent of $p$ such that $\dim L^i_{R, \lambda}(\lambda) = \dim L^i_{\lambda}(\lambda)$ for $i < p/N_\lambda$.

**Proof.** Recall that, for any $\lambda' \in \mathfrak{h}_R^*$, the module $\Delta_R(\lambda')$ has a unique (up to scaling by elements of $R$) contravariant form $B_{\lambda'}$, and different graded components are orthogonal with respect to $B_{\lambda'}$. We assume that $B$ is nondegenerate on the highest weight component.

We can also consider a one-parameter deformation $\Delta_R[\ell](\lambda + t \rho)$, where $t$ is an independent variable. The module $\Delta_R[\ell](\lambda + t \rho)$ comes with a contravariant form $B_{\lambda + t \rho}$. Let us write $B^i_{\lambda}$ for the restriction of $B_{\lambda}$ to $\Delta^i_{R}(\lambda)$. The notation $B^i_{\lambda + t \rho}$ has the similar meaning.
We note that $L_Q(\lambda)$ is the quotient of $\Delta_Q(\lambda)$ by the radical of $B_{\lambda,Q}$, the specialization of $B_\lambda$ to $\mathbb{Q}$ and, similarly, $L_{F_p}(\lambda)$ is the quotient of $\Delta_{F_p}(\lambda)$ by the radical of $B_{\lambda,F_p}$. So we need to check that for $i < p/N_\lambda$, the radicals of $B_{\lambda,F_p}^i$ and of $B_{\lambda,Q}^i$ have the same dimension.

Consider the finitely generated $\mathbb{K}[t]$-module $\Delta_{\mathbb{K}[t]}^i(\lambda + t \rho)$, where $\mathbb{K} = \mathbb{Q}$ or $\mathbb{F}_p$. The form $B_{\lambda+t\rho,\mathbb{K}}^i$ defines a descending filtration $F^j\Delta_{\mathbb{K}[t]}^i(\lambda)$ (the Jantzen filtration); by definition, $F^i\Delta_{\mathbb{K}}^i(\lambda) = \Delta_{\mathbb{K}}^i(\lambda)$ and $F^j\Delta_{\mathbb{K}}^i(\lambda)$ is the radical of the well-defined form $t^{-j}B_{\lambda+t\rho,\mathbb{K}}^i|_{t=0}$ on $F^j\Delta_{\mathbb{K}}^i(\lambda)$. It is clear that dim $F^j\Delta_{\mathbb{K}}^i(\lambda) \geq$ dim $F^j\Delta_{\mathbb{K}}^i(\lambda)$ for all $p$. The equality dim $F^j\Delta_{\mathbb{K}}^i(\lambda) = \dim F^j\Delta_{\mathbb{K}}^i(\lambda)$ for all $j$ (that will immediately imply what we need, which is the $j = 1$ case) for $i < p/N_\lambda$ will follow if we check that the order of vanishing of $f^i_p(t) := \det B_{\lambda+t\rho,\mathbb{F}_p}^i$ at $t = 0$ coincides with the order of vanishing of $f^i(t) := \det B_{\lambda+t\rho,\mathbb{Q}}^i$ at $t = 0$ – for example, the former equals $\sum_j \dim F^j\Delta_{\mathbb{F}_p}^i(\lambda)$. Clearly, $f^i_p(t)$ is obtained from $f(t)$ by reduction mod $p$.

The polynomial $f^i(t)$ can be decomposed as $C^i t^{n_0} \prod_{j=1}^{k} (t - z_j^i)^{n_j}$, where $C^i \in \mathbb{R}$ and $z_j^i$ are nonzero elements that lie, a priori, in the algebraic closure of $\mathbb{Q}(t)$. In fact, $f^i(z) = 0$ means that there is a singular vector in $\Delta_{\lambda + z \rho}(\lambda)$, where $0 < i' \leq i$. In particular, there is $w \in W, w \neq 1$, such that $\langle \lambda + z \rho - w(\lambda + z \rho), \rho' \rangle = i'$. This is equivalent to

\[ z = \langle \rho - w \rho, \rho' \rangle^{-1}(i' - \langle \lambda - w \lambda, \rho \rangle) \]

In particular, $f^i(z) = 0$ implies $z \in R$.

Therefore what we need to prove is that, for $i < p/N_\lambda$, we have $C^i, z_j^i \neq 0$ modulo $p$.

Let us show that $C^i$ is nonzero mod $p$. Consider the baby Verma module $\Delta_{\mathbb{F}_p[t]}^i(\lambda + t \rho) = \Delta_{\mathbb{F}_p[t]}(\lambda + t \rho)/(\mathbb{F}_p[t])^{i}(\lambda + t \rho)$. Note that $\Delta_{\mathbb{F}_p[t]}^i(\lambda + t \rho) = \Delta_{\mathbb{F}_p[t]}^i(\lambda + t \rho)$ as long as $i < p$.

It is well-known that for a generic $z \in \mathbb{F}$ the module $\Delta_{\mathbb{F}}^i(\lambda + z \rho)$ is irreducible. In particular, $f^i_p(t) \neq 0$ as a polynomial so $C^i \neq 0$ mod $p$.

Now let us show that $z_j^i \neq 0$ mod $p$ using (3). Recall that $x$ stands for the least common multiple of the denominators of the coordinates of $\lambda$ in the basis of simple roots. Then $xi' - x(\lambda - w \lambda, \rho)$ is a nonzero multiple of $p$. Clearly as long as $p$ is large enough, there is $N_\lambda$ such that for $i < p/N_\lambda$, we have $0 < xi' - x(\lambda - w \lambda, \rho) < p$.

\[ \square \]

4.4. **Proof of Proposition 4.2.** Let us deduce Proposition 4.2 from Lemma 4.5.

**Proof of Proposition 4.2.** The GK dimension of $L(\lambda)$ equals to $\frac{1}{2} \text{GK-dim}(U_\lambda \otimes \text{Ann } L(\lambda))$ and the latter coincides with $\dim \mathcal{O}_c/2$. Therefore $\sum_{j=0}^{i} \dim L^j(\lambda)$ is a polynomial in $i$ of degree $\dim \mathcal{O}_c/2$. So dim $L_{\mathbb{F}}(\lambda) \geq \sum_{j=0}^{i} \dim L^j_{\mathbb{F}}(\lambda) = \sum_{j=0}^{i} \dim L^j(\lambda)$, where the equality follows from Lemma 4.5, as long as $i < p/N_\lambda$. This shows that dim $L_{\mathbb{F}}(\lambda)$ is bounded below by a degree $\dim \mathcal{O}_c/2$ polynomial in $p$.

Now let us show that dim $L_{\mathbb{F}}(\lambda)$ is bounded from above by a degree $\dim \mathcal{O}_c/2$ polynomial in $p$. Note that $L_{\mathbb{F}}(\lambda)$ is a quotient of the baby Verma module $\Delta_{\mathbb{F}}^i(\lambda)$. We have $\Delta_{\mathbb{F}}^j(\lambda) = 0$ for $j \geq 2(\rho, \rho')p$ hence $L_{\mathbb{F}}^j(\lambda) = 0$ for such $j$. We claim that, for any $m_1 > 1 > m_2$, there is a constant $M$ such that

\[ \sum_{j=0}^{m_1 p} \dim L_{\mathbb{F}}^j(\lambda) \leq M \sum_{j=0}^{m_2 p} \dim L_{\mathbb{F}}^j(\lambda), \]

this will establish the upper bound thanks to Lemma 4.5.
Let us write $U_{\leq k}(n^-)$ for $k$th filtration term with respect to the PBW filtration on $U(n^-)$ and $U^{\leq i}(n^-)$ for $\bigoplus_{j=0}^{i} U^j(n^-)$. For any $i$, we have

$$\bigoplus_{j=0}^{i} L^j_F(\lambda) = U^{\leq i}(n^-)L^0_F(\lambda).$$

Note that the filtrations $U^{\leq i}(n^-)$ and $U_{\leq k}(n^-)$ are compatible in the sense that there are constants $c_1 < 1 < c_2$ such that $U_{\leq c_1 i}(n^-) \subset U^{\leq i}(n^-) \subset U_{\leq c_2 i}(n^-)$. So, thanks to (5), (4) will follow if we show that, for any $m \in \mathbb{Z}_{>1}$ there is $M$ such that for any $i$ we have

$$\dim U_{\leq mi}(n^-) \leq M \dim U^{\leq i}(n^-).$$

Let $x_1, \ldots, x_n$ be a basis of $n^-$. (6) will follow if we show that every element of $U_{\leq mi}(n^-)$ can be written as a sum of elements of the form $PQ$, where $P$ is an ordered monomial in $x_1^{i+1}, \ldots, x_n^{i+1}$ of degree $\leq m$ and $Q$ is an element of $U^{\leq i}(n^-)$. The latter claim follows from the analogous one on the associated graded level, which is straightforward.

**Remark 4.6.** Let us mention an alternative way to prove the upper bound for $\dim L^i_F(\lambda)$ established above. Let $M$ be an object in category $\mathcal{O}$ over $\mathbb{C}$ of Gelfand-Kirillov dimension $d$. We can find a $\mathfrak{g}$-module $M_R$ is defined over $R$ with $M \cong M_R \otimes_R \mathbb{C}$, and consider its based change $M_{\mathbb{F}}$ to a field of almost any prime characteristic. Let $\overline{M_{\mathbb{F}}}$ be the reduction of $M_{\mathbb{F}}$ by the zero $p$-central character. Then one can check that:

1. $\dim \overline{M_{\mathbb{F}}} = O(p^d)$.
2. The space $K_0(\mathcal{U}^{\geq}_{\lambda, \mathbb{F}} - \text{mod}_c')$ is spanned by classes $\dim \overline{L_{\mathbb{F}}}$, where $L$ runs over the set of irreducible module in category $\mathcal{O}$ belonging to cells $c' \leq c$.

These two properties clearly imply the upper bound.

4.5. **Degeneration map.** We have a one parameter subgroup $\gamma : \mathbb{F}^x \to G_{\mathbb{F}}$ with $\gamma(t)\chi = t^2 \chi$. Via $\gamma$, the group $\mathbb{F}^x$ acts on the sheaf of algebras $\mathcal{U}_{\mathbb{F}}|_{\mathbb{F}x}$, where the action on the base $\mathbb{F}x$ is by dilations. This gives rise to the degeneration map $\delta : K_0(\mathcal{U}^{\lambda}_{\mathbb{F}} - \text{mod}) \to K_0(\mathcal{U}^0_{\mathbb{F}} - \text{mod})$.

Since $\mathbb{F}^x$ acts trivially on the Harish-Chandra center, we see that the map restricts to

$$\delta : K_0(\mathcal{U}^{\lambda}_{\mathbb{F}} - \text{mod}) \to K_0(\mathcal{U}^0_{\mathbb{F}} - \text{mod}).$$

The following standard lemma summarizes basic properties of the degeneration map (7).

**Lemma 4.7.** The following are true.

1. Under the identifications $K_0(\mathcal{U}^{\lambda}_{\mathbb{F}} - \text{mod}) \cong H_*(\mathcal{B}_c, \mathbb{C}), K_0(\mathcal{U}^0_{\mathbb{F}} - \text{mod}) \cong H_*(\mathcal{B}, \mathbb{C})$ the map $\delta$ coincides with the push-forward map $H_*(\mathcal{B}_c, \mathbb{C}) \to H_*(\mathcal{B}, \mathbb{C})$. In particular, it is independent of $p$ and $W^a$-equivariant.
2. The map $\delta$ intertwines the endomorphisms $\text{pr}_\lambda(V(\mu)_{\mathbb{F}} \otimes \bullet)$ for $p \gg \langle \rho', \mu \rangle$.
3. The map $\delta$ preserves the dimension polynomials.

**Proof.** (2) and (3) are straightforward, let us prove (1). We can consider the categories $\text{Coh}(\mathcal{D}^t_{\mathbb{F}})^{\mathbb{F}x}$ (here $t = 0, 1$) of all coherent sheaves of $\mathcal{D}^t_{\mathbb{F}}$-modules supported at the preimage of $t\chi$ and the corresponding derived category $D^b(\text{Coh}(\mathcal{D}^{t}_{\mathbb{F}}))^{\mathbb{F}x}$. The derived equivalence $R\Gamma$ induces identifications $K_0(\text{Coh}(\mathcal{D}^t_{\mathbb{F}}))^{\mathbb{F}x} \cong K_0(\mathcal{U}_{\mathbb{F}}^{\lambda} - \text{mod})^{\mathbb{F}x}$ that are compatible with the degeneration maps. Also note that the Harish character isomorphisms intertwine the degeneration maps. So it remains to show that the identifications $K_0(\text{Coh}(\mathcal{B}_{1x})) \sim K_0(\text{Coh}(\mathcal{D}^t_{\mathbb{F}}))^{\mathbb{F}x}$ intertwine the degeneration maps. This is a consequence of Lemma 2.2 and the projection formula. \qed
4.6. Proof of (2) of Theorem 1.1. We start with the following proposition.

**Proposition 4.8.** Let \( L \) be a simple in \( \mathcal{U}_{\lambda,F}^\chi \)-mod but not in the smaller filtration terms. Then the projection of \( \delta([L]) \) to the \( c \)-isotypic component of \( K_0(\mathcal{U}_{\lambda,F}^\chi \text{-mod})_{\leq c} \) is nonzero.

**Proof.** Assume the contrary. Let \( \delta[L] = \sum_{i=1}^k [L_i] \), where \( L_i \) are simples in \( \mathcal{U}_{\lambda,F}^\chi \)-mod. Note that \( k \) is independent of \( p \) by (1) of Lemma 4.7 and the fact that the basis of simples is independent of \( p \). By our assumption, \( L_i \in \mathcal{U}_{\lambda,F}^0 \text{-mod}_{\leq c} \).

Pick a dominant weight \( \mu \) and suppose that \( p \) is very large. Let \( M_\mu \) denote the functor \( pr_\lambda(V(\mu)_F \otimes \bullet) \). By (2) of Lemma 4.7, we have

\[
\delta([M_\mu L]) = \sum_{i=1}^k [M_\mu L_i].
\]

Note that for \( N \in \mathcal{U}_F^\chi \)-mod, we have

\[
\ell([N]) \leq \ell(\delta[N]).
\]

By Proposition 3.3 combined with Lemma 4.3,

\[
\sum_{\mu \vdash (p^\vee,\mu) \leq m} \dim V(\mu) \ell(M_\mu L)
\]
grows (with respect to \( m \)) faster than

\[
\sum_{i=1}^k \sum_{\mu \vdash (p^\vee,\mu) \leq m} \dim V(\mu) \ell(M_\mu L_i)
\]

But combining (8) with (9), we see that (11) \( \geq \) (10). A contradiction. \( \square \)

**Proof of (2) of Theorem 1.1.** By (3) of Lemma 4.7, the map \( \delta : K_0(\mathcal{U}_{\lambda,F}^\chi \text{-mod}) \to K_0(\mathcal{U}_{\lambda,F}^0 \text{-mod}) \) preserves the dimension polynomials. For an irreducible \( L \in \mathcal{U}_{\lambda,F}^\chi \text{-mod}_{\leq c} \) (but not in smaller filtration components), we have \( \delta([L]) = \sum_{i=1}^k [L_i] \), where \( k \) is independent of \( p \), the simple \( L_i \) belongs to a two-sided cell \( c_i \leq c \) and there is \( i \) such that \( c_i = c \). The dimension polynomial for \( L \) is the sum of the dimension polynomials for the \( L_i \)'s. Note that \( \mathcal{O}_{c_i} \subset \mathcal{O}_c \). By Proposition 4.2, the dimension polynomial of \( L \) has degree \( \dim \mathcal{O}_c/2 \). \( \square \)

5. Application to W-algebras

In this section we will use Theorem 1.1 to prove conjectures from [LO, Section 7.6] on the classification of finite dimensional irreducible representations of W-algebras.

5.1. Background on W-algebras. Finite W-algebras (below we omit the adjective “finite”) were introduced by Premet in [P1] (with alternative constructions later given by the second author). These are associative algebras constructed from pairs \( (\mathfrak{g}, e) \), where \( \mathfrak{g} \) is a semisimple Lie algebra over \( \mathbb{C} \) and \( e \in \mathfrak{g} \) is a nilpotent element. Such a W-algebra is a quantization of the transverse Slodowy slice to the adjoint orbit \( \mathcal{O} \) of \( e \). The reader is referred to the survey article [L1] for details.

Let us recall Premet’s definition. Include \( e \) into an \( \mathfrak{sl}_2 \)-triple \( (e, h, f) \). The element \( h \) induces the grading on \( \mathfrak{g} \) by eigenvalues of \( \text{ad}(h) : \mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i) \). Let, as before, \( \chi = (e, \cdot) \). The form \( \omega(x,y) = \langle \chi, [x,y] \rangle \) is symplectic on \( \mathfrak{g}(-1) \). Let us pick a lagrangian subspace \( \ell \subset \mathfrak{g}(-1) \). Form a subalgebra \( \mathfrak{m} \subset \mathfrak{g} \) by \( \mathfrak{m} = \bigoplus_{i \leq -2} \mathfrak{g}(i) \oplus \ell \). Note that \( \chi \) is the character
of $\mathfrak{m}$ and that $\dim \mathfrak{m} = \frac{1}{2} \dim \mathcal{O}$, where we write $\mathcal{O}$ for the orbit of $e$. Then, by definition, the $W$-algebra $\mathcal{W}$ is the quantum Hamiltonian reduction $[U(\mathfrak{g})/U(\mathfrak{g})\{x - \langle \chi, x \rangle | x \in \mathfrak{m}\}]^{[\mathfrak{g}]}$.

Let us list some important properties of the $W$-algebra.

1) The algebra $\mathcal{W}$ is naturally independent of the choice of $\ell$ as was demonstrated in [GG]. Moreover, it comes with a Hamiltonian action of the group $Q = Z_G(e, h, f)$ by automorphisms.

2) Next, $\mathcal{W}$ comes with a filtration induced from the filtration on $U(\mathfrak{g})$, where $\deg \mathfrak{g}(i) = i+2$. The associated graded for this filtration is $\mathbb{C}[S]$, the algebra of functions on the Slodowy slice $S = e + h_0(f)$.

3) Also note that the definition of $\mathcal{W}$ via the Hamiltonian reduction yields a homomorphism $U(\mathfrak{g})^G \to \mathcal{W}$. As was checked by Ginzburg, see the footnote for [P2, Question 5.1], this homomorphism is an isomorphism onto the center of $\mathcal{W}$. So for $\lambda \in \mathfrak{h}^*$ we can talk about the central reduction $\mathcal{W}_\lambda$.

Now let us discuss a reduction mod $p$ for $W$-algebras. Note that $\mathcal{W}$ is defined over some finite localization $R$ of $\mathbb{Z}$; we can take the Hamiltonian reduction $\mathcal{W}_R$ of $\mathcal{U}_R$ and the properties 1), 2), 3) still hold. So we can reduce mod $p$ and get the algebra $\mathcal{W}_p := \mathbb{F} \otimes_R \mathcal{W}_R$.

As Premet proved, see, for example, [P4, Theorem 2.1], one has a central inclusion $\mathbb{F}[S^{(1)}] \hookrightarrow \mathcal{W}_p$. In [P3, Proposition 4.1] Premet checked that one has an isomorphism $\mathcal{U}_p^\lambda \cong \text{End}_{\mathcal{F}(\mathfrak{m}_p)} \mathcal{W}_p^\lambda$.

**Remark 5.1.** Consider $\mathcal{U}_p|_{S^{(1)}} = \mathbb{F}[S^{(1)}] \otimes_{\mathbb{F}[\mathfrak{g}]} \mathcal{U}_p$. One can strengthen Premet’s result and show that $\mathcal{U}_p/\mathcal{U}_p\{x - \langle \chi, x \rangle, x \in \mathfrak{m}_p\}|_{S^{(1)}}$ is a Morita equivalence bimodule between $\mathcal{U}_p|_{S^{(1)}}$ and $\mathcal{W}_p$. This follows from [T]. From here we see that $\mathcal{U}_p/\mathcal{U}_p|_{S^{(1)}}$ is a Morita equivalence bimodule between $\mathcal{U}_p|_{S^{(1)}}$ and $\mathcal{W}_p$.

**5.2. Restriction functor for HC bimodules.** In this section we will recall results from [L2] on the restriction functor between the category of HC $\mathcal{U}$-bimodules and the category of HC $\mathcal{W}$-bimodules.

Namely in [L2] the second author has constructed a functor $\bullet : \text{HC}(\mathcal{U}) \to \text{HC}^Q(\mathcal{W})$ to the category of $Q$-equivariant HC $\mathcal{W}$-bimodules (introduced in that paper) with the following properties:

1) The functor $\bullet$ is exact, tensor, $\mathbb{C}[\mathfrak{h}^*]^\mathcal{W}$-bilinear and sends $\mathcal{U}$ to $\mathcal{W}$,

2) it maps $\text{HC}_{\mathcal{O}}(\mathcal{U})$ to the category $\text{Bimod}_{\mathcal{O}}^Q(\mathcal{W})$ of finite dimensional $Q$-equivariant $\mathcal{W}$-bimodules,

3) and kills $\text{HC}_{\partial \mathcal{O}}(\mathcal{U})$.

4) There is a functor $\bullet : \text{Bimod}_{\mathcal{O}}^Q(\mathcal{W}) \to \text{HC}_{\mathcal{O}}(\mathcal{U})$ that is right adjoint to $\bullet$.

5) For $\mathcal{M} \in \text{HC}_{\mathcal{O}}(\mathcal{U})$, the kernel and the cokernel of the adjunction unit $\mathcal{M} \to (\mathcal{M})^\dagger$ are supported on $\partial \mathcal{O}$.

6) Let $\mathcal{M} \in \text{HC}(\mathcal{U})$ and $\mathcal{N} \subset \mathcal{M}$ be a $Q$-stable subbimodule of finite codimension. Then there is a unique maximal subbimodule $\mathcal{M}' \subset \mathcal{M}$ with $\mathcal{M}' = \mathcal{N}$ and $\text{V}(\mathcal{M}/\mathcal{M}') = \mathcal{O}$.

We will need relative versions of (2)-(5), compare to [L5, Section 3.3.2]. Namely, let us pick an affine subspace $\mathfrak{h}^1 \subset \mathfrak{h}^*$ and write $\mathcal{U}_{\mathfrak{h}^1} := \mathbb{C}[\mathfrak{h}^1] \otimes_{\mathbb{C}[\mathfrak{h}^*]^\mathcal{W}} \mathcal{U}$, $\mathcal{W}_{\mathfrak{h}^1} := \mathbb{C}[\mathfrak{h}^1] \otimes_{\mathbb{C}[\mathfrak{h}^*]^\mathcal{W}} \mathcal{W}$. Then we get a $\mathbb{C}[\mathfrak{h}^1]$-bilinear exact tensor functor $\bullet : \text{HC}(\mathcal{U}_{\mathfrak{h}^1}) \to \text{HC}^Q(\mathcal{W}_{\mathfrak{h}^1})$.

Let us write $\text{HC}_{\mathcal{O}}(\mathcal{U}_{\mathfrak{h}^1})$ for the full subcategory of $\text{HC}(\mathcal{U}_{\mathfrak{h}^1})$ consisting of HC bimodules $\mathcal{M}$ with $\text{V}(\mathcal{M}) \cap \mathcal{N} \subset \mathcal{O}$. The notation $\text{HC}_{\partial \mathcal{O}}(\mathcal{U}_{\mathfrak{h}^1})$ has the similar meaning. We also write
HC^Q_{\lambda}(W_{\lambda}) for the category of all bimodules finitely generated (as left, or equivalently, right) modules over $\mathbb{C}[h^1]$. Analogs of (2)-(5) are as follows.

(2’) $\bullet$, maps $HC^Q_{\lambda}(U_{\lambda})$ to $HC^Q_{\lambda}(W_{\lambda})$,

(3’) $\bullet$, annihilates $HC_{\partial Q}(U_{\lambda})$.

(4’) There is a functor $\bullet : HC^Q_{\lambda}(W_{\lambda}) \to HC_{\partial Q}(U_{\lambda})$ right adjoint to $\bullet$.

(5’) For $M \in HC_{\partial Q}(U_{\lambda})$, the kernel and the cokernel of the adjunction unit $M \to (M_{\lambda})^\dagger$ are supported on $HC_{\partial Q}(U_{\lambda})$.

5.3. Results on finite dimensional irreducible $W$-modules. Let us state our results on the classification of finite dimensional irreducible $W$-modules. For this, we will need to recall one of the main results of [L2]. Since $Q$ acts on $W$ by automorphisms, it also acts on the set $Irr_{\text{fin}}(W)$ of the isomorphism classes of finite dimensional irreducible $W$-modules. Since the action of $Q$ on $W$ is Hamiltonian, the action on $Irr_{\text{fin}}(W)$ descends to an action of component group $A(= A_0) := Q/Q^\circ$.

One of the main results of [L2], see Section 1.2 there, was a natural identification $Irr_{\text{fin}}(W)/A \cong \text{Prim}_0(U)$: it sends $J \in \text{Prim}_0(U)$ to the $A$-orbit of the irreducible representations of $W/J$, this is well-defined and gives a bijection by (6) of Section 5.2. So to finish the classification of the finite dimensional irreducible $W$-modules we need, for every primitive ideal $J \in \text{Prim}_0(U)$, to compute the stabilizer $H_J$ (defined up to conjugacy) in the $A$-orbit over $J$.

Fix a central character $\lambda$ and assume for time being that it is regular. Let us write $W_{\lambda,c}$ for the semisimple finite dimensional quotient of $W_{\lambda}$ whose simple representations are precisely the irreducibles lying over the primitive ideals corresponding to the two-sided cell $c$. Recall the Springer representation $Spr_0 := H_{\text{top}}(B_\lambda, \mathbb{C})$ of $W \times A$. Also recall that $W_{[\lambda]}$ can be regarded as a subgroup of $W$ via $W_{[\lambda]} \hookrightarrow W^a \rightarrow W$. Let us write $Spr_{0,c}$ for the sum of all irreducibles $W_{[\lambda]}$-submodules in the Springer representation that belong to the family of irreducible $W$-modules indexed by the cell $c$.

**Theorem 5.2.** Let $c$ be a two-sided cell in $W_{[\lambda]}$ and let $\mathcal{O} = \mathcal{O}_c$. Then the following is true.

1. Let $J \in \text{Prim}_0(U_{\lambda})$ correspond to a left cell $\sigma \subset W_{[\lambda]}$ and let $H_J$ denote a stabilizer in the $A$-orbit in $Irr_{\text{fin}}(W_{\lambda})$ lying over $J$. Then the $A$-module $\text{Hom}_{W_{[\lambda]}([\sigma], \text{Spr}_0)}$ coincides with the $A$-module induced from the trivial $H_J$-module.

2. We have an isomorphism $K_0(W_{\lambda,c}-\text{mod}) \cong Spr_{0,c}$ of $W_{[\lambda]} \times A$-modules.

When $\lambda$ is integral, this theorem is the main result of [LO], see Theorem 1.1 and (iii) of Theorem 7.4 there. Note that (1) is sufficient to determine $H_J$ (at least in all cases but 2). Indeed, the group $A$ is abelian for all nilpotent orbits but twelve in the exceptional Lie algebras, see, e.g., [CM, Section 8.4]. If $A$ is abelian, then $H_J$ is just the kernel of the $A$-action on $\text{Hom}_{W_{[\lambda]}([\sigma], \text{Spr}_0)}$. Out of these twelve cases, in ten cases we have $A = S_3$, where, clearly, the induced module determines a subgroup uniquely. In the two remaining cases we have $A = S_4$ (in $F_4$) and $A = S_5$ (in $E_8$), we haven’t checked for general $\lambda$ if (1) determines $H_J$ uniquely (though for an integral $\lambda$ this is indeed the case).

To finish this section we let explain what happens for singular central characters. The situation is very similar to the integral case considered in [LO]. Let $\lambda_0$ be a singular dominant element in $h^\ast$. Pick a dominant element $\mu$ in the root lattice so that $\lambda := \lambda_0 + \mu$ is strictly dominant. As was explained in Section 2.1, $\text{Prim}_0(U_{\lambda_0}) \hookrightarrow \text{Prim}_0(U_{\lambda})$. This gives rise to the partitions $\text{Prim}_0(U_{\lambda_0}) = \bigsqcup_c \text{Prim}_c(U_{\lambda_0})$, $Irr_{\text{fin}}(W_{\lambda_0}) = \bigsqcup_c \text{Irr}(W_{\lambda_0,c})$. 


**Corollary 5.3.** Let $\mathcal{J}_0 \subset \text{Prim}_c(\mathcal{U}_\lambda_0)$, let $\mathcal{J}$ be the corresponding ideal in $\text{Prim}_c(\mathcal{U}_\lambda)$. Then the $A$-orbit over $\mathcal{J}_0$ coincides with $A/H_{\mathcal{J}}$ and $K_0(\mathcal{W}_{\lambda_0,c}\text{-mod}) = \text{Spr}_{\mathcal{G},c}^{\mathcal{W}_{\lambda_0}}$.

**5.4. Reduction of representations mod $p$.** Now fix a dominant rational $\lambda \in \mathfrak{h}^*$. Recall, [L4, Theorem 1.3], that $\mathcal{W}_\lambda$ has a minimal ideal of finite codimension, say $\mathcal{I}$. By definition, this ideal is defined over $\mathbb{Q}$. For a finite localization $R$ of $\mathbb{Z}$, set $\mathcal{I}_R := \mathcal{W}_\lambda R \cap \mathcal{I}$. We assume that $\text{gr} \mathcal{W}_R = R[S]$ and $\text{gr} \mathcal{W}_{\lambda,R} = R[S \cap \mathcal{N}]$, this can be achieved after a finite localization of $R$.

**Lemma 5.4.** After a finite localization of $R$, we get $\mathcal{I}_R^2 = \mathcal{I}_R$.

**Proof.** Note that $\text{gr} \mathcal{W}_{\lambda,R}/\mathcal{I}_R$ is a finitely generated commutative $R$-algebra. So after a finite localization of $R$ we can achieve that $\mathcal{W}_{\lambda,R}/\mathcal{I}_R$ is a free finite rank $R$-module. Note that $\mathcal{W}_{\lambda,R}$ is Noetherian because of $\text{gr} \mathcal{W}_{\lambda,R} = R[S \cap \mathcal{N}]$. In particular, $\mathcal{I}_R$ is a finitely generated left $\mathcal{W}_{\lambda,R}$-module. It follows that $\mathcal{I}_R/\mathcal{I}_R^2$ is a finitely generated module over $\mathcal{W}_{\lambda,R}/\mathcal{I}_R$ and hence a finite rank $R$-module. Note that $\mathcal{I}_Q$ is still the minimal ideal of finite codimension in $\mathcal{W}_Q$. So $\mathcal{I}_Q = \mathcal{I}_Q^2$. It follows that $\mathcal{I}_R/\mathcal{I}_R^2$ is a finitely generated torsion $R$-module hence it is killed by a finite localization of $R$.

This lemma shows that $(\mathbb{F} \otimes_R (\mathcal{W}_{\lambda,R}/\mathcal{I}_R))\text{-mod}$ is a Serre subcategory in $\mathcal{W}_{\lambda,F}\text{-mod}$.

After replacing $R$ with a finitely generated algebraic extension, we can assume that $\mathcal{W}_{\lambda,F}/\mathcal{I}_{\mathcal{Frac}(R)}$ is split. So there is a natural bijection $\text{Irr}_{\text{fin}}(\mathcal{W}_\lambda) \cong \text{Irr}(\mathcal{W}_{\lambda,F}/\mathcal{I}_{\mathcal{Frac}(R)})$. So, for $L \in \text{Irr}(\mathcal{W}_\lambda)$, we can talk about its reduction $L_F \mod p$. For standard reasons, $L_F$ is irreducible. As was checked in [BL, Section 6.5], $L_F$ has central character $\chi$. So we get an inclusion $\text{Irr}_{\text{fin}}(\mathcal{W}_\lambda) \hookrightarrow \text{Irr}(\mathcal{W}_{\lambda,F})$. Recall, Section 5.1, that the target is naturally identified with $\text{Irr}(\mathcal{U}_{\lambda,F})$.

**Proposition 5.5.** For $p \gg 0$, the image of $\text{Irr}_{\text{fin}}(\mathcal{W}_\lambda)$ in $\text{Irr}(\mathcal{U}_{\lambda,F})$ consists of the simples with degree of dimension polynomial equal to $\dim \mathcal{O}/2$.

**Proof.** The simples in $\text{Irr}(\mathcal{U}_{\lambda,F})$ with degree of dimension polynomial equal to $\dim \mathcal{O}/2$ correspond to the simples in $\text{Irr}(\mathcal{W}_{\lambda,F})$ whose dimension is independent of $p$. Let $d$ be the maximal dimension of these representations. Let $\mathcal{I}_{R}$ be the ideal in $\mathcal{W}_R$ generated by the elements $\sum_{i \in S_{2d}} \text{sgn}(\sigma) a_{\sigma(1)} \ldots a_{\sigma(2d)}$ for $a_i \in W_R, i = 1, \ldots, 2d$. By the Amitsur-Levitsky theorem, the ideal $\mathcal{I}_R$ vanishes on all representations of $\mathcal{W}_F$ of dimension $\leq d$. Arguing as in [L3, Lemma 5.1], we see that $\mathcal{I}_R$ is of finite codimension. So $\mathcal{I}_R \supset \mathcal{I}$. It follows that after a finite localization of $R$, we have $\mathcal{I}_R \supset \mathcal{I}_R$. So any irreducible representation of $W_{\lambda,F}$ of dimension $\leq d$ factors through $\mathcal{W}_{\lambda,F}/\mathcal{I}_F$ for $p$ large enough. This finishes the proof.

**5.5. Proof of Theorem 5.2.** Let us prove Theorem 5.2 in the case when $\lambda$ is rational.

**Proof of Theorem 5.2 for rational $\lambda$.** Let us start by proving (2). From Proposition 5.5 combined with (2) of Theorem 1.1 we know that $K_0(\mathcal{W}_{\lambda,c}\text{-mod}) = H_*(\mathcal{B}_c, \mathbb{C})_c$, the sum of all irreducible $W_{[\lambda]}$-submodules in $H_*(\mathcal{B}_c, \mathbb{C})_c$ that belong to $c$. This is with respect to the standard embedding $W_{[\lambda]} \hookrightarrow W^a$. What remains to show is that

$$H_*(\mathcal{B}_c, \mathbb{C})_c \overset{\sim}{\rightarrow} H_{\text{top}}(\mathcal{B}_c, \mathbb{C})_c$$

where now the action on the right hand side is via $W_{[\lambda]} \hookrightarrow W^a \rightarrow W$ and the map is induced by the natural projection $H_*(\mathcal{B}_c, \mathbb{C}) \rightarrow H_{\text{top}}(\mathcal{B}_c, \mathbb{C})$. According to Dodd, [D, Section 7], $K_0(\mathcal{W}_{\lambda,c}\text{-mod}) \subset H_*(\mathcal{B}_c, \mathbb{C})$ projects injectively to $H_{\text{top}}(\mathcal{B}_c, \mathbb{C})$. The projection $H_*(\mathcal{B}_c, \mathbb{C}) \rightarrow$
$H_{\text{top}}(\mathcal{B}_c, C)$ is $W^a$-equivariant, where on the target space $W^a$ acts via the projection $W^a \to W$, and so intertwines the actions of $W_{[\lambda]}$. This implies (12) and finishes the proof of (2).

Let us now deduce (1) from (2). The restriction functor $\bullet : \text{HC}_C(U_\lambda) \to \text{Bimod}_f^q(W_\lambda)$ recalled in Section 5.3 equips $K_0(\mathcal{W}_{\lambda,c,-\text{mod}})$ with an action of $K_0(\text{HC}_c^*(U_\lambda)) = J_c(W_{[\lambda]})$. By results of Dodd, [D, Section 8], this action is the same as the $J_c(W_{[\lambda]})$-action on $K_0(U_{\lambda,\mathbb{C}}^{-\text{mod}})_c$.

By the description of the $J_c$-action in the previous paragraph, the span of classes of the irreducible modules lying over $\mathcal{J}$ equals the span of classes of the irreducible modules lying over $\mathcal{J}$ equals $t_0K_0(\mathcal{W}_{\lambda,c,-\text{mod}})$.

So $t_0K_0(\mathcal{W}_{\lambda,c,-\text{mod}})$ is nothing else but $\text{Hom}_{\mathcal{W}_{[\lambda]}([\sigma]_s\text{Spr}_C)}$. On the other hand, the span is $A$-stable and is the $A$-representation induced from the trivial representation of $H_\mathcal{J}$. This finishes the proof of (1). \qed

Now let us reduce the proof of Theorem 1.1 to the case when $\lambda$ is rational. To state our main technical result we need some notation.

Pick a regular central character $\lambda \in \mathfrak{h}^*$. Let $W_0$ denote a minimal parabolic subgroup of $W$ containing $W_{[\lambda]}$, we can conjugate $\lambda$ and assume that $W_0$ is standard, while $\lambda$ is still dominant. We can write $\lambda = \lambda_1 + \lambda_2$, where $\lambda_1$ lies in $(\mathfrak{h}^*)_W^0$ and $\lambda_2$ lies in the orthogonal complement to $(\mathfrak{h}^*)_W^0$. Note that $\lambda_2$ is rational.

**Proposition 5.6.** For all dominant regular rational $\lambda' \in \lambda_2 + (\mathfrak{h}^*)_W^0$ satisfying $W_{[\lambda']} = W_{[\lambda]}$ the following is true. Let $\sigma$ be a left cell in $W_{[\lambda]}$ and let $\mathcal{J}, \mathcal{J}'$ be primitive ideals in $\mathcal{U}$ with central characters $\lambda, \lambda'$ corresponding to the left cell $\sigma$. Then $H_\mathcal{J} = H_\mathcal{J'}$.

**Proof.** The proof is in several steps.

**Step 1.** Since $\lambda$ is not rational, $W_0 \neq W$. By [LO, Proposition 5.7] for any integral dominant $\mu \in (\mathfrak{h}^*)_W^0$ for the ideal $\mathcal{J}'$ with central character $\lambda + \mu$ (corresponding to $\mathcal{J}$ under the isomorphism Prim$(U_\lambda) \cong \text{Prim}(U_{\lambda+\mu})$) we have $H_{\mathcal{J}'_\mu} = H_{\mathcal{J}_\mu}$. So in the proof we can assume that $\lambda_1$ is Zariski generic in $(\mathfrak{h}^*)_W^0$. To simplify the notation we will write $\mathfrak{h}^1$ for $\lambda_2 + (\mathfrak{h}^*)_W^0$.

**Step 2.** A standard argument, see, for example, the proof of [L3, Lemma 5.1], shows that there is an ideal $\mathcal{I} \subset \mathcal{W}_{\mathfrak{h}^1}$ such that $\mathcal{W}_{\mathfrak{h}^1}/\mathcal{I}$ is finitely generated over $\mathbb{C}[\mathfrak{h}^1]$ and for a Weil generic $\hat{\lambda} \in \mathfrak{h}^1$ the specialization $\mathcal{I}_{\hat{\lambda}}$ is the minimal ideal of finite codimension in $\mathcal{W}_{\hat{\lambda}}$. Let us write $\mathcal{J}$ for the kernel of $U_{\mathfrak{h}^1}_* \to (\mathcal{W}_{\mathfrak{h}^1}/\mathcal{I})^\mathfrak{h}^1$. Note that, thanks to (4') of Section 5.2, $V(U_{\mathfrak{h}^1}/\mathcal{J}) \cap \mathcal{N} = \mathcal{O}$. So $\mathcal{W}_{\mathfrak{h}^1}/\mathcal{J}$ is finitely generated over $\mathbb{C}[\mathfrak{h}^1]$. Since $\mathcal{J}_{\hat{\lambda}} \subset \mathcal{I}$, we can replace $\mathcal{I}$ with $\mathcal{J}_{\hat{\lambda}}$ and assume that $\mathcal{I} = \mathcal{J}_{\hat{\lambda}}$.

**Step 3.** For a dominant regular $\hat{\lambda}$ with $W_{[\hat{\lambda}]} = W_{[\lambda]}$, let us write $\mathcal{J}_{\hat{\lambda}}$ for the primitive ideal with central character $\hat{\lambda}$ corresponding to the left cell $\sigma$. Let us prove that for a Zariski generic $\hat{\lambda} \in \mathfrak{h}^1$ with $W_{[\hat{\lambda}]} = W_{[\lambda]}$ we have $\mathcal{J}_{\hat{\lambda}} \subset \mathcal{J}_\lambda$. Note that this inclusion is automatic provided $\mathcal{I}_\lambda$ is the minimal ideal of finite codimension in $\mathcal{W}_{\hat{\lambda}}$.

Pick $w \in \sigma$. For $\hat{\lambda} \in \mathfrak{h}^1$ consider the Verma module $\Delta(w\hat{\lambda})$. These Verma modules form a flat family over $\mathfrak{h}^1$, let us denote the corresponding $\mathcal{U}_{\mathfrak{h}^1}$-module by $\Delta_{\mathfrak{h}^1}$. Inside we have a $\mathcal{U}_{\mathfrak{h}^1}$-submodule $\overline{\mathcal{J}}\Delta_{\mathfrak{h}^1}$. Consider the quotient $\Delta_{\mathfrak{h}^1}/\overline{\mathcal{J}}\Delta_{\mathfrak{h}^1}$. Its specialization to $\hat{\lambda}$ is $\Delta(w\hat{\lambda})/\overline{\mathcal{J}}_{\hat{\lambda}}\Delta(w\lambda)$. It is nonzero for a Weil generic $\hat{\lambda}$. It follows that it is nonzero for a Zariski generic $\hat{\lambda}$ as well. The claim in the first paragraph of this step follows.

**Step 4.** The algebra $\mathcal{W}_{\mathfrak{h}^1}/\mathcal{I}$ comes with a $Q$-action. Since the algebra $\mathcal{W}_{\mathfrak{h}^1}/\mathcal{I}$ is a finitely generated module over $\mathbb{C}[\mathfrak{h}^1]$, we see that the number of irreducible representations is the same for two Zariski generic specializations (this is a version of the Tits deformation argument). Moreover, for two nearby generic parameters, there is a natural bijection between
the irreducibles. Being natural, this bijection preserves the stabilizers in $Q$. And when the parameters are not nearby, the monodromy may appear but it does not change the stabilizers in $A$.

Step 5. Recall, Step 1, that we can assume that $\lambda$ is Zariski generic in $\mathfrak{h}^1$. Similarly, we can assume that $\lambda'$ is Zariski generic. Now the claim of the proposition follows from Step 4.

\[ \square \]

**Proof of Theorem 5.2 for general $\lambda$.** (1) for $\lambda$ immediately follows from Proposition 5.6 and (1) for $\lambda'$ proved above. To prove (2) we can argue as follows. Take a Weil generic $\lambda$ with $W_\lambda = W_\lambda'$. Then we have the degeneration maps (compare to [BL, Section 11.1])

\[ K_0(W_\lambda \text{-mod}_{fin}) \to K_0(W_{\lambda'} \text{-mod}_{fin}), K_0(W_\lambda \text{-mod}_{fin}) \to K_0(W_\lambda' \text{-mod}_{fin}). \]

By the proof of Proposition 5.6, we see that both these maps are isomorphisms. They are also $W_\lambda$-invariant. This implies (2).

\[ \square \]

5.6. **Proof of Corollary 5.3.** Let $\lambda_0, \lambda$ be as before the statement of Corollary 5.3. Let $V$ be a finite dimensional $G$-module. We can define the endo-functor $V \otimes \bullet$ of $W$-mod as the tensor product with the bimodule $(V \otimes U)$. Using this we can define translation functors $\mathcal{T}_{\lambda_0 \rightarrow \lambda} : W$-mod$_{\lambda_0} \to W$-mod$_{\lambda_0}$, $\mathcal{T}_{\lambda \rightarrow \lambda_0} : W$-mod$_{\lambda} \to W$-mod$_{\lambda_0}$ in a standard way. They enjoy properties similar to those of the usual translation functors (because $\bullet \circ$ is a tensor functor):

1. On $K_0(W_\lambda$-mod$_{fin})$ the composition $\mathcal{T}_{\lambda_0 \rightarrow \lambda} \circ \mathcal{T}_{\lambda_0 \rightarrow \lambda}$ is the multiplication by $|W_{\lambda_0}|$.
2. On $K_0(W_\lambda$-mod$_{fin})$ the composition $\mathcal{T}_{\lambda_0 \rightarrow \lambda} \circ \mathcal{T}_{\lambda \rightarrow \lambda_0}$ acts as $\sum_{w \in W_{\lambda_0}} w$.

This implies the equality $K_0(W_\lambda$-mod$_{fin}) = K_0(W_\lambda$-mod$_{fin}) W_{\lambda_0}$. Also, for $w$ longest in its right $W_{\lambda_0}$-coset, the maps $[\mathcal{T}_{\lambda_0 \rightarrow \lambda}], [\mathcal{T}_{\lambda \rightarrow \lambda_0}]$ map between $K_0(W/\mathcal{J}(w_{\lambda_0}) \text{-mod})$ and $K_0(W/\mathcal{J}(w_{\lambda}) \text{-mod})$, which together with (1) implies that $H_{\mathcal{J}(w_{\lambda_0})} = H_{\mathcal{J}(w_{\lambda})}$.

6. **APPLICATION TO REAL VARIATION OF STABILITY CONDITIONS**

In this section we use Theorem 1.1 to essentially realize the idea sketched in [ABM, Remark 6].

We now describe a simplified version of a construction of [ABM]. Let $A$ be an abelian category and let $\zeta : \mathbb{C} \to K_0(A)^*$ be a polynomial map. We assume that for some $r > 0$ we have

\[ \langle [M], \zeta(x) \rangle \in \mathbb{R}_{>0} \quad \forall \ x \in (0, r), \ M \in A, \ M \neq 0. \]

In this situation we get a filtration on $A$ by Serre subcategories where $A_{>d}$ consists of objects $M$ such that the polynomial $x \mapsto \langle [M], \zeta(x) \rangle$ has a zero of order at least $d$ at zero. We say that a derived equivalence $\phi : D^b(A) \to D^b(A')$ is a perverse equivalence governed by $\zeta$ if there exists a filtration on $A'$ by Serre subcategories such that $D^b(A_{>d}) = \phi D^b(A_{>d})$, while the functor $\text{gr}_d(\phi) : D^b(A_{>d}/A_{\geq d}) \to D^b(A'_{>d}/A'_{\geq d})$ sends $A_{>d}/A_{\geq d}$ to $(A_{>d}/A_{\geq d})[d]$.

We now set $A = U_{A, x}$-mod$^X$. Let $\xi : \mathbb{R} \to t^X_{\mathbb{R}}$ be an affine linear functional sending zero to a face $F$ in the closure of the fundamental alcove $A_0$; we assume that $\xi(\mathbb{R}_{>0})$ intersects $A_0$.

The central charge map $Z : t \to K_0(\text{Coh}(B_t))^*$ was defined in [ABM]. We use identification (1) for a choice of $\lambda$ in the fundamental $p$-alcove to get a map $t \to K_0(A)^*$ which we also denote by $Z$. We set $\zeta = Z \circ \xi$. Then [ABM, Proposition 1(a)] implies that the positivity condition (13) holds for some $r > 0$. 

\[ \square \]
The face $F$ determines a proper subset in the set of vertices of the affine Dynkin graph. Let $W_F$ be the corresponding finite Weyl group and $w_F$ be the longest element in $W_F$; let $\bar{w}_F$ be the canonical (minimal length) lift of $w_F$ to the affine braid group $B_{aff}$. Note that a path in the complement to affine coroot hyperplanes whose end-points are contained in $t^*_R$ defines an element in $B_{aff}$; the element $\bar{w}_F$ corresponds to the path $[0,1] \rightarrow t^*_\xi$, $x \mapsto \xi(R \exp(2\pi i x))$ for a small $R > 0$.

Recall the action of $B_{aff}$ on $D^b(A)$.

The main result of this section is as follows.

**Theorem 6.1.** The functor $\bar{w}_F : D^b(A) \rightarrow D^b(A)$ is a perverse equivalence governed by $\zeta$.

**Proof.** We let $A_{\geq d}$ be the filtration introduced earlier in this section and we let $A'_{\geq d}$ denote the Serre subcategory generated by $U_{\lambda,R}$-mod$_c$ for all cells $c$ in $W_F$ such that $a(c) \geq d$.

The result clearly follows from the following two statements.

a) We have $A_{\geq d} = A'_{\geq d}$.

b) The functor $\bar{w}_F[-d]$ induces a $t$-exact functor on $D^b(A'_{\geq d}/A'_{\geq d})$.

We claim that (a) follows from Theorem 1.1. To see this, choose a regular rational weight $\lambda$ with $W_{[\lambda]} = W_F$. Furthermore, we can and will assume that $\lambda$ satisfies the following assumptions: it can be written as $\lambda = \mu + \nu$, where $\nu$ is an integral weight and $\mu$ lies in the closure of the fundamental alcove $A_0$, while $\mu + t \nu$ lies in $A_0$ for small (equivalently, for some) $t > 0$ (equivalently, $\mu' + t \nu$ lies in $A_0$ for all $\mu' \in F$ and small $t > 0$, where the bound on $t$ depends on $\mu'$). Choose a large prime $p$ such that $(p+1)\lambda$ is an integral weight. Then $\tilde{\lambda} := (p+1)\mu + \nu$ is an integral weight satisfying: $\tilde{\lambda} = \lambda \mod p$ and $\tilde{\lambda} \in A_0$.

For $M \in A$ consider the polynomial $D_M$, such that for an integral weight $\eta$ such that $\tilde{\lambda} + \frac{\eta + \rho}{p} \in A_0$ we have $\dim(T_{\lambda \rightarrow \eta}(M)) = D_M(\lambda)$, where $T$ denotes the translation functor (the existence of $D_M$ follows from [BMR2, Theorem 6.2.1]). By Theorem 1.1 for a $M \in A'_{\geq d}$, $M \notin A'_{\geq d}$ we have $\deg_p(D_M) = \dim(A) - d$.

On the other hand the central charge $Z$ of $[ABM]$ satisfies (see the proof of [ABM, Proposition 1]):

$$\langle Z\left(\frac{\eta + \rho}{p}\right), [M]\rangle = p^{-\dim(A)}D_M(\eta).$$

It follows that the order of zero of the polynomial $\zeta_M(t) = \langle Z(\mu + t \nu), [M]\rangle$ at $t = 0$ equals $\dim(A) - \deg_p(D_M)$, this proves (a).

We now sketch the proof of (b). We use the fact that the braid group $B_F$ of the Coxeter group $W_F$ acts the category $D^b(U_{\lambda,R}\text{-mod})$ compatibly with its action on $D^b(U_{\lambda,F}\text{-mod})$.

Note that the action of $\bar{w}_F$ is given by the derived tensor product with the wall-crossing $U_{\lambda}$-bimodule $W_{C_{w_F}}$ corresponding to the element $w_F$. By [L6, Theorem 3.1], the functor $W_{C_{w_F}}$ is a perverse equivalence with

$$U_{\lambda}\text{-mod}_{\geq d} = \{ M \in U_{\lambda}\text{-mod} \mid \dim V(U_{\lambda}/\text{Ann}(M)) \leq \dim N - 2d \}.$$

As was shown in the proof of that theorem the statement reduces to vanishing of Tor’s involving $W_{C_{w_F}}$ and the quotients of $U_{\lambda}$ by the minimal ideals with given dimensions of associated varieties. This vanishing was checked in the proof. Now this vanishing over $\mathbb{C}$ implies the analogous vanishing over $R$ (after a finite localization) and hence the claim that the endo-functor $W_{C_{w_F}} = W_{C_{w_F,R}} \otimes_{U_{\lambda,R}} \bullet$ of $D^b(U_{\lambda,F}\text{-mod}^\lambda)$ is perverse with respect to the filtration $A'_{\geq d}$.

$\square$
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