Explain Influence Maximization with Sobol Indices

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Abstract
Due to its vast application on online social networks, Influence Maximization (IM) has garnered considerable attention over the last couple of decades. Current IM research lacks human-comprehensible explanations of how the seed set results in the influence effect, hence reducing the trustworthiness of existing solutions despite their applicability. Due to the intricacy of IM, the majority of current research concentrate on estimating first-order spreading power and often disregard the interplay between flows dispersed from different seeds. This study uses Sobol indices, the cornerstone of variance-based sensitivity analysis, to decompose the influence effect to individual seeds and their interactions. The Sobol indices are tailored for IM contexts by modeling the seed selection as binary variables. This explanation method is universally applicable to all network types, IM techniques, and diffusion models. Based on the explanation method, a general framework dubbed SobolIM is proposed to improve the performance of current IM studies by over-selecting nodes followed by an elimination strategy. Experiments on synthetic and real-world graphs demonstrate that the explanation of the impact effect can dependably identify the key high-order interaction between seeds across a variety of networks and IM methods. SobolIM is empirically proved to be superior on effectiveness and competitive on efficiency.

1 Introduction
As online social networks draw and maintain hundreds of millions of users during the past few decades, influence maximization (IM), which was originally proposed by Domingos and Richardson [11] in 2001, attracts a great attention from researchers and practitioners. IM is a series of problems in which a seed set consisting of \( k \) nodes is selected to maximize the expected number of influenced nodes, usually within a few time steps [25]. In the 2003 milestone work, Kempe et al. proved that this series of problems was NP-hard and simply evaluating the influence of a seed set was \( \#P \)-hard [22]. They also presented a few classic diffusion models including the Independent Cascade (IC) model and the Linear Threshold (LT) model in the same work. Despite its high complexity, IM is widely applied to viral marketing [6], rumor control [3, 18], social recommendation [48], and infectious disease containment [29]. The quality of the IM methods could easily result in loss of million dollars or even human lives, thus its trustworthiness are highly demanded.

However, existing IM methods’ dependability suffers from their lack of human-comprehensible explanations on the source of the influence. The absence of an interpretable IM algorithm mostly comes from: (1) Transparency and scalability issue of simulations approaches. As the first
approximation attempt, a simulation-based greedy algorithm was proposed in [22]. It guaranteed a \((1 - \frac{1}{e})\) approximation ratio since the influence function was found to be monotone and submodular. However, due to the high complexity of the evaluation of a seed set, it was impossible to be applied to the enormous-scaled online social networks. Similarly, a thread of simulation-based methods are developed to increase the performance or reduce the complexity [25, 14, 8]. Although great effort has been made to accelerate the process of the simulation-based methods, the complexity is still unacceptably high for the enormous online social networks [1, 40]. Most importantly, the opaqueness of the simulation makes this thread of methods impossible to be explained or improved by evaluating the diffusion process [26].

2 Lack of understanding impact overlaps among seeds.

To alleviate the heavy computational burden of simulations, some researchers turned to proxy-based methods in which the spreading power of the nodes is estimated by certain proxies. They started from simple heuristic measures such as degree, PageRank [30] and eigen-centrality [51] and later turned to several influence-aware or diffusion model-aware proxies [5, 23, 6, 7, 42, 15, 46, 50] to better estimate the influence effect brought by the seed. However, due to the intricate interactions among the nodes within the seed set [22, 5, 26], the actual influence effect of seeds selected by those proxies are not guaranteed to match the proxies’ initial intention (which we will show in the experiments). When computing the proxies of the nodes, the heuristics completely disregard impact overlaps and linearly add the effects of all seeds. Other approaches simply update the proxies by altering the spreading power of the seeds’ first-order neighbors. Due to the overlaps, proxy-based approaches tend to overstate the entire impact of the seed set’s influence, rendering them erroneous.

current researches on IM problems solely focus on finding a good seed set within an acceptable time, ignoring the combinatorial effects among seeds. The explanation of the seed set such as which nodes or even which combinations of nodes are more important than others has been understudies.

To address these issues, this paper introduces one of the most significant variance-based sensitivity analysis methods, namely Sobol indices [39], into the explanation of the justification of the seed set selected by a certain IM method. Initially designed to evaluate the relative importance of the input variables over the variance of the output variable, Sobol indices are widely adopted in physics and economics [20, 43]. In IM problems, the separate influence effect of each node within the seed set can be estimated by its Sobol Total index, while the overlaps among the nodes’ influence are evaluated by the high-order Sobol indices. Figure 1 demonstrates the connection between the IM problem and the Sobol indices. The left of the figure shows the influence effect of a seed set of four nodes. Among them, three nodes (node 1, 2, 3) are close to each other while the last one (node 4) is far away. During the \(t\) steps of the propagation, the influence flows coming out of the seeds may meet each other, creating an influence overlap that diminishes the total influence effect. While conducting the sensitive analysis with the Sobol indices, the contribution of each node and the influence overlaps among the nodes can be quantified as shown on the right side of the figure. After obtaining the indices, the seeds can be ranked accordingly. This ranking method can be utilized to select the seed to be removed when the budget constraint shrinks . According to this method, an improvement framework \text{SobolIM} is proposed to increase the performance of the proxy-based IM algorithms. This paper’s primary contributions include:

• Design a universal method to explain the influence effect of a given seed set: Given any seed set, the contribution to the influence effect of each seed can be evaluated with the proposed method. Thus, it can serve as a universal post-hoc explanation of a solution regardless the diffusion model and the IM algorithm.
Quantify the influence overlaps: To the best of our knowledge, this paper serves as the first discussion on the quantification of the influence overlaps among the seeds despite that overestimation brought by this overlap is widely recognized.

Propose a novel framework to improve IM performance: SobollIM manages to reduce, if not totally eliminate, the performance loss from the influence overlaps by selecting seed candidates over the budget constraint and removing the least important candidate iteratively until the constraint is met.

Conduct extensive experiments on synthetic and real-world networks: The effectiveness of our explanation method is evaluated on both synthetic dataset and real-world data. Results show that the evaluations provided by this framework are robust across different networks. SobollIM is compared with multiple baselines and achieves a superior performance.

2 Related work

Explainable AI While AI performs a more and more important role in our socioeconomic life, its opaqueness raises a concern about the difficulties in understanding its decisions [16]. This concern is not only related to ethics but also about safety [10] and industrial liability [24]. Thus, explainable AI merges as a subfield of AI and tries to provide human-understandable explanations. In their 2021 survey of explainable AI, Bodria et al. [2] reviewed the literature [16, 41, 13, 36, 4] and provided a multi-perspective taxonomy for explanation methods. One of the distinctions is that explanation methods are divided into explanation by model design and post-hoc explanation to the predictions. Although most IM methods are not based on machine learning algorithms, the seed selection procedure still lacks transparency. Due to the high stake of the quality of the solution, explanations towards either the selection methods or the selected seed set are necessary. One of the greatest challenges met by IM algorithms is how to gain trust from the domain experts so that they can faithfully follow the recommendation from the algorithms rather than discard it. For example, Yadav et al. [45] stated that during the implementation of their IM algorithm HEALER, it was challenging to prove that the generated solution was superior to the solutions preferred by the homeless shelter officials.

Our explanation method falls into the category of post-hoc explanation to the seed set generated by any IM algorithms. A model-level explanation would be too costly right now since calculating the influence overlaps for the whole network is a high complexity problem. When the size of the network increase, the number of possible combinations increases exponentially. On one hand, in a post-hoc explanation task for IM where there are $k$ seeds, the number of 2-node combinations among seeds will be $\binom{k}{2} = \frac{k(k-1)}{2}$, and it is $\binom{k}{3} = \frac{k(k-1)(k-2)}{3!}$ for 3-node combinations, and so on. On the other hand, in a model-level task where all nodes on an $N$-node graph have to be considered, the size of all combinations is significantly larger. That is to say $\binom{N}{2} = \frac{N(N-1)}{2}$ due to $N \gg K$.

Variance-Based Sensitivity Analysis Sensitive analysis studies the proportion based on which the uncertainty lying within the output of a model is distributed to the multiple sources of uncertainty in the input factors [34]. Variance-based methods take a major part in sensitive analysis and have been dated back to 1970s [9]. Since then, these methods have been well studied and widely adopted by researchers and practitioners [37, 35]. Among them, Sobol indices [39] is considered as a significant milestone. However, variance-based methods cannot be directly utilized on IM problems since the input factors are not clearly defined and the influence effect has not been represented as a nonlinear function. In the following section, we will adapt Sobol indices and a few of its important inheritors to the IM problems and utilize them to provide a human-understandable explanation for a given seed set.

3 Sobol-Based Influence Decomposition

In this work, we make a quantitative evaluation of each node’s actual influence effect and the reason causing this evaluation based on variance-based sensitivity analysis tools closely related to Sobol index. Regarding each factor’s contribution, popular explainable AI approaches include the marginal contribution, Shapley [33], and other local interpretation methods such as LIME [31] and Grad-cam [38]. Among them, the marginal contribution does not separate the interaction effects from the main effect. The Shapley value, to guarantee its efficiency property $\sum_{j=1}^{d} \phi_j = val([d])$, distribute
the interaction among the seeds and tend to underestimate the relative importance of the nodes with larger influence overlaps with others. The local interpretation methods focus only on certain neighborhoods, while the IM problems consider the global effect of the seeds. Also, they do not consider the interaction between the focal factor and the others. After careful consideration, we select Sobol indices as the most appropriate method to explain the influence effect of a seed set. In this section, the problem description is elaborated in 3.1. After briefly introducing these tools in section 3.2, we will discuss how to adapt them to the settings of the IM problems in 3.3, such as how the indices are estimated based on the influence effect of the seed set and what they indicate in the explanation of the influence effect.

### 3.1 Problem Description

Given the graph $G$, a budget constraint $k \in \mathbb{R}^+$, and an influence maximization method, a $k$-sized seed set $S$ can be generated to approximately maximize the eventual influence. Our goal is to estimate each seed’s relative importance within the seed set and quantify the influence overestimation brought by the overlaps. Based on the explanation method, we propose an improvement framework named SobolIM. Given any existing algorithm $M$ solving an IM problem $P$, we have:

$$\text{SobolIM}(M(P)) \geq M(P)$$  \hspace{1cm} (1)

In our research, the IC model [22] is adopted to exhibit the function of the proposed framework since its influence overlap effect is more straightforward and significant. It is a classic propagation model which is often used in studying influence in networks [23, 49, 46, 27]. A network is denoted by a graph $G = (V, E, A)$ where $V$ and $E$ represent vertices and edges respectively. Each edge is weighted by the activation probability $a_{ij} \in A \in \mathbb{R}^{N \times N}$ where $A$ stands for the weighted adjacency matrix. For the sake of simplicity, we define the graph $G$ as undirected, but this work can be easily generated to directed graphs. The propagation spreads in multiple discrete time steps $t_j$. At $t_0$, some initial nodes are active. In each time step, each active node tries to activate its neighbors with the corresponding probability $a_{ij}$. Once a node is activated, it cannot turn back to inactive status. Each node can only attempt to activate its neighbors once.

Although the interactions among the seeds during the propagation are more complicated in the LT model, our explanation method can still estimate the relative importance of each seed and the SobolIM framework can improve corresponding IM algorithms with no modification. The only difference is on how to explain the influence overlaps or the possible confluence effect with the high order Sobol indices. This gap requires future investigation.

### 3.2 Sobol Indices

The original Sobol index evaluates the first-order effect of a subset $\Psi$ of all input factors based on the variance in the output it is accounted for. Only the contribution made by $\Psi$ independently counts for the first-order effect, that is to say, any variance of the output brought by the interaction between any factor in $\Psi$ and other factors out of $\Psi$ is excluded from the calculation of this index. This effect can be written as:

$$V_\Psi = V_{X_\Psi}(E_{X_{\sim\Psi}}(Y|X_\Psi))$$  \hspace{1cm} (2)

When $\Psi$ is a singleton of $i$ we have the first-order effect of the single factor $i$:

$$V_i = V_{X_i}(E_{X_{\sim i}}(Y|X_i))$$  \hspace{1cm} (3)

where $X_{\sim i}$ represents all the input factors except for $X_i$ and the inner conditional expectation is the weighted mean of $Y$ taken over all possible combinations of $X_{\sim i}$ while fixing $X_i$ at one of its possible value. After all conditional expectations are calculated, the outer variance is calculated based on $V_{X_i} = E[(E_{X_{\sim i}}(Y|X_i) - \mu)^2]$ where $\mu$ stands for the mean of all the conditional expectations.

Following the thoughts of Sobol index, the high-order interaction effects $V_{ij}$ among input factors can also be calculated. For instance, the $2_{nd}$-order interaction effect between $X_i$ and $X_j$ can be calculated by:

$$V_{Hij} = V_{X_iX_j}(E_{X_{\sim ij}}(Y|X_i, X_j)) - V_i - V_j$$  \hspace{1cm} (4)

Furthermore, Homma and Saltelli introduced the total effect index [19] based on the idea proposed in [21]. It measures the total effect, including the first and all higher order effects, of an input factor.
on the variance of output. Different from in the Shapley value, the whole high order effect is added to each of the involved factor’s total effect. Therefore, the sum of the total effects of a group of complementary factor sets is greater than the total variance of the output, given that there is any interaction among the sets. Since this index is closely related to the original Sobol index, this paper will refer to it as Sobol Total index from now on. It can be written as:

$$\text{total effect} = E_{X_{\sim i}}(V_{X_{i}}(Y|X_{-i}))$$  \hspace{1cm} (5)

Based on the known identity that $V(Y) = V_{X_{i}}(E_{X_{\sim i}}(Y|X_{i})) + E_{X_{\sim i}}(V_{X_{i}}(Y|X_{-i}))$ \hspace{1cm} (28), it is easy to observe that both Equation \hspace{1cm} (3) and Equation \hspace{1cm} (5) can be normalized by $V(Y)$ since they both range between zero and $V(Y)$. After the normalization, the two indices can be formally identified as:

$$S_i = \frac{V_{X_{i}}(E_{X_{\sim i}}(Y|X_{i}))}{V(Y)}, \hspace{1cm} S_{Ti} = \frac{E_{X_{\sim i}}(V_{X_{i}}(Y|X_{-i}))}{V(Y)} = 1 - \frac{V_{X_{\sim i}}(E_{X_{i}}(Y|X_{-i}))}{V(Y)} \hspace{1cm} (6)$$

At the same time, Equation \hspace{1cm} (4) can be rewritten as follow after being normalized by $V(Y)$:

$$S_{Hij} = S_{ij} - S_i - S_j \hspace{1cm} (7)$$

Similarly, we can calculate the $3^{rd}$-order interaction among factors $X_i, X_j, \text{and} X_k$ as $S_{Hijk} = S_{ijk} - S_{Hij} - S_{Hjk} - S_{Hik} - S_i - S_j - S_k$ and so on. Given there are totally $n$ factors in the model, we have

$$\sum_i S_i + \sum_i \sum_{j>i} S_{Hij} + \cdots + S_{H12\ldots n} = 1 \hspace{1cm} (8)$$

### 3.3 Indices Estimation in IM Problems

The influence effect of a certain seed set has not been previously defined as a nonlinear function of a set of input factors. To apply the Sobol indices into the IM context, we need to first identify the output, the input factors, and the function connecting the two. Given a seed set $\Omega$ and the graph $G$ representing the network where the propagation happens, we can estimate the influence effect, which is the output we care about, within $T$ time steps or until the propagation fully stops. Given a fixed seed set, the corresponding influence effect has an uncertainty that comes from solely the probabilities the nodes activating each other. This uncertainty is not what we are interested in thus we remove this uncertainty by taking the average of results from multiple rounds of MC simulations as the estimated total influence effect. Although evaluating the influence effect of the seed set bears a $\#P$-hardness complexity, the MC simulations carried out for the estimation of the Sobol indices are relatively feasible given $k \ll n$. Assuming the influence effect of each combination of the seeds is estimated by $r$ rounds of MC simulations, the total number of MC simulations is $2^r r$. As a comparison, the number of MC simulations needed for the naive greedy IM algorithm \hspace{1cm} (22) proposed by Kempe et al. is $\frac{n!}{(n-k)!}$ which is $O(n!)$ when $n \gg k$.

To evaluate the importance of a single seed within the set, we need to quantify the difference between the influence effects before and after including the seed into the seed set. Intuitively, we can write the relationship between the influence effect and the seed set as a model:

$$\text{Influence} = f(\Omega) \hspace{1cm} (9)$$

where $f$ is a non-linear function of a seed set $\Omega$ whose size is $k$. In IM problems, a node can be either selected or not, while cannot be partially selected. To evaluate the difference a node brings, we can force a uniform distribution to the mask such that it can be 0 or 1 with a 50%/50% probability. Similar to Sobol notation, $\Omega_i$ denotes the selection of the $i$-th seed (1 for selected, 0 for not selected). $\Omega_{\sim i}$ represent a subset of seed set that excludes the $i$-th seed.

After defining the function, three Sobol indices are applied into the IM scenario (i.e., Equation \hspace{1cm} (6) and \hspace{1cm} (7)). We will analyze the first-order index first followed by the Sobol Total index, and lastly the high order Sobol indices. The first-order Sobol index of the node $i$ evaluates the influence it brings with no overlaps with other nodes. This is the part of the network that is activated by the node $i$ as the seed while no other seeds could successfully activate it. Representing the selection of the node $i$
When taking with a binary variable, the first-order Sobol index can be calculated inspired by Equation 6:

\[ S_i = \frac{V_{\Omega_i}(E_{\Omega_i}(Y|\Omega_i))}{V(Y)} \]  

(definition of 1st-order Sobol index)

\[ = \frac{[E_{\Omega_i}(Y|\Omega_i = 1) - \Gamma]^2 + [E_{\Omega_i}(Y|\Omega_i = 0) - \Gamma]^2}{2 \cdot V(Y)} \]

(\( \Gamma = E_{\Omega_i}(Y|\Omega_i = 1) + E_{\Omega_i}(Y|\Omega_i = 0) \))

\[ = 2 \cdot \frac{(E_{\Omega_i}(Y|\Omega_i = 0) - E_{\Omega_i}(Y|\Omega_i = 1))^2}{2 \cdot V(Y)} \]

\[ = \frac{(\sum_{i=1}^{\Omega_{\Omega_i}} Y_{i} - Y_{\Omega_i=0})^2}{4^k \cdot V(Y)} \]

(10)

The Sobol Total index measures the amount of influence lost if node \( i \) is not included in the seed set. It includes not only the part of the network activated by node \( i \), but also the part that could be activated by this node and some other nodes at the same time. Since \( k \) is the size of seed set \( \Omega \), the size of \( \Omega_{\Omega_i} \) is \( k - 1 \), indicating that there are \( 2^{k-1} \) combinations of \( \Omega_{\Omega_i} \). According to Equation 6, the total Sobol index can be written as:

\[ S_{T_i} = \frac{\sum_{\Omega_{\Omega_i}} V_{\Omega_i}(Y|\Omega_{\Omega_i})}{2^{k-1} \cdot V(Y)} = \frac{\sum_{\Omega_{\Omega_i}} (Y_{i=1} - Y_{i=0})^2}{4 \cdot 2^{k-1} \cdot V(Y)} = \frac{\sum_{\Omega_{\Omega_i}} (Y_{i=1} - Y_{i=0})^2}{2^{k+1} \cdot V(Y)} \]

(11)

Define \( \Delta_{\Omega_{\Omega_i}} = |Y_{i=1} - Y_{i=0}| \) as the difference of influence effects of \( i \) given seed sets \( \Omega_{\Omega_i} \), we can rewrite Equations [10] and [11] as:

\[ S_i = \frac{(\sum_{\Omega_{\Omega_i}} \Delta_{\Omega_{\Omega_i}})^2}{4^k \cdot V(Y)}, \quad S_{T_i} = \frac{\sum_{\Omega_{\Omega_i}} (\Delta_{\Omega_{\Omega_i}})^2}{2^{k+1} \cdot V(Y)} \]

(12)

In the IM scenario, the high order Sobol indices can quantify the influence overlaps among the nodes and help with identifying the critical overlaps that make the IM algorithms misjudge the value of selecting a certain node as one of the seeds. The larger a node’s total high order Sobol indices are, which can be calculated either by summing the high order Sobol indices from all orders together or by subtracting the first-order Sobol index from the Sobol Total index, the more the actual influence it results in would shrink from its value identified by the IM algorithms. The calculation of the high order Sobol indices can also be adapted to the IM settings and simplified. Based on [2], the first-order Sobol index of a size-\( s \) subset \( \Psi \subseteq \Omega \) can be calculated by:

\[ S_{\Psi} = \frac{V_{\Psi}(E_{\Omega_{\Psi}}(Y|\Psi))}{V(Y)} \]

\[ = \sum_{\Psi} \left( \frac{\sum_{\Omega_{\Psi}} Y_{|\Omega_{\Psi}=\Psi=1} - E(\Psi))^2}{2^{k-s} \cdot V(Y)} \right) \cdot \frac{1}{2^s \cdot V(Y)} \]

(13)

Because:

\[ E_{\Psi}(E_{\Omega_{\Psi}}(Y|\Psi)) = \frac{1}{2^s} \sum_{\Psi} \frac{\sum_{\Omega_{\Psi}} Y_{|\Omega_{\Psi}=\Psi=1}}{2^{k-s}} = \frac{\sum_{\Psi} Y_{\Omega_{\Psi}=\Psi}}{2^k} = E(\Psi) \]

(14)

When taking \( s = 1 \) we can see that the equation at the left of [12] is a special case of Equation [13]. The high order Sobol indices can be calculated in an iterative manner starting from the second-order to the \( k \)-th order.

4 SobolIM: A Framework to Improve IM Algorithms

Now that the relative importance of a seed within the seed set can be evaluated by the Sobol Total index, we can conclude that removing the node with the lowest Sobol Total index will produce the smallest influence loss when the budget constraint drops from \( k \) to \( k - 1 \). Based on this finding, we propose SobolIM, a general framework that can improve the performance of any proxy-based IM method in two steps, namely over-selection of seed candidates and elimination of the nodes with less impact on the influence effect.
Since SobollIM is a framework that includes no strategy for node selection, it must be combined with an existing IM method. The underlying logic behind this framework to sacrifice an acceptable amount of efficiency in exchange for a better influence result. Thus, it works best with proxy-based IM methods whose efficiency is high while the performance suffers due to the ignorance of the influence overlaps. Comparatively, simulation-based IM methods have better performance, but their time complexity is usually too high to be applied to online social networks. Thus, applying SobollIM to simulation-based methods to further decrease the time efficiency makes no sense.

4.1 Over-Selection

In the first step of SobollIM, \( ak \) \((a \in N^+)\) candidates are selected with the target IM algorithm. Proxy-based IM algorithms would select nodes with high spreading power assuming that there is no influence overlap exists. Thus, the first \( ak \) nodes selected in this step are expected to have relatively high influences. For IM algorithms with no updates to the proxies during the selection procedure such as degree and eigen-centrality, the time cost of selecting \( ak \) nodes is same as selecting \( k \) nodes since the proxy is only calculated and ranked once. For those who update their proxies after each seed is selected, the time cost will increase \( a \) times. The time for each iteration of the node selection and proxy update remains the same while the number of iterations raises from \( k \) to \( ak \). For example, the time complexity of the Pi IM algorithm \([50]\) is \( O(k \cdot n^3) \). The corresponding time complexity for SobollIM’s first step is \( O(ak \cdot n^3) \). However, given that \( k \ll n \), the time complexities would be on the same scale when \( a \) is a small positive integer.

4.2 Elimination

To satisfy the hard budget constraint, \( k \) seeds need to be selected from the \( ak \) candidates. This selection process is accomplished by the Sobol Total indices. Note that the Sobol Total index is to measure the variance lost when holding the factor stable with all other factors still in the model, thus selecting the nodes with the highest indices from the candidates or removing the \((a-1)k\) lowest-ranking nodes at one time does not guarantee the best result. Conversely, removing the node ranked lowest always results in the least loss of influence. The elimination of the extra nodes must be done one at a time iteratively until there are exactly \( k \) nodes left in the set.

Assuming that it takes \( r \) rounds of simulations to estimate a seed set’s influence effect, the total number of simulation rounds required for calculating the indices throughout the elimination process is:

\[
Total \text{ Rounds} = 2^{ak} \cdot r + 2^{ak-1} \cdot r + \ldots + 2^{k+1} \cdot r = (2^{ak+k} - 2^{k+1}) \cdot r
\]

5 Experiment

The proposed explanation method and SobollIM are tested on a collection of synthetic and real-world data sets. All experiments are run on a x64 PC with a 16-core Intel Core i9-9900K, NVIDIA RTX 2080 SUPER, and 32GB RAM. The simulations to estimate the influence effects and the Sobol indices are carried out utilizing NDLib \([32]\), an open-source package for studying diffusion processes and dynamics.

5.1 Dataset

The framework is tested mainly on (1) real-world data sets, including CiteSeer, Cora, and PubMed \([47]\) to mimic the sophisticated online social network structure. Since the IM problems traditionally focus on connected networks, the largest connected component of each network is utilized as the network from which the seed set is selected. The edges are randomly weighted between 0.40 to 0.80 uniformly; (2) The synthetic graphs representing pseudo social networks are generated using NetworkX \([17]\). The graphs include connected Watts-Strogatz small-world graphs \([44]\) and Erdős–Rényi random graphs \([12]\). Each graph has 5000 nodes, and the average degree is approximately 10. The edges are weighted in a similar manner with the real world data.
5.2 IM Methods

To evaluate the legitimacy of our evaluation of the selected seed set and the corresponding IM method, we select a few popular proxy-based IM algorithms and two heuristics as our candidates. SobolIM is compared also with these baselines to evaluate its performance: (1) NetShield (NS) [42]: Tong et al. measure the vulnerability of a graph by the first eigenvalue. A set of \( k \) nodes are selected such that they can each greatly decrease the vulnerability and are not connected to each other. NetShield is more of an influence blocking algorithm, thus the seed set selected is expected to have low performance on spreading influence. (2) Sigma [46]: The spreading powers of the nodes are estimated by \( \sum I \cdot A^t \) where \( I \) is a unit column vector, \( A \) represents the weighted adjacency matrix and \( t \) stands for the number of time steps of the propagation. After selecting each node, the selected node and the edges connecting to it are removed from the graph, then the proxy for the rest nodes are updated accordingly. (3) \( \Pi \) [50]: Considering the existence of cycles in networks, a node can be reached multiple times by a seed in different step lengths. Thus, the nodes spreading powers can be better estimated by \( I \cdot (J - \prod_{r=1}^R (1 - A^r)) \) where \( J \) is an all-one matrix and \( \prod \) is the element-wise product of matrices. (4) Degree Centrality (DEG): The first \( k \) nodes with the highest degree centrality from the target graph are selected. (5) Eigenvector Centrality (EIG): Similarly, the first \( k \) nodes are selected based on their eigenvector centrality.

5.3 Result

The empirical study generally consists of three major parts. The first part is to evaluate the legitimacy of the node importance ranking; the second part is to verify the relationship between the influence overlaps and the high order Sobol indices; and the last part is to compare the SobolIM with the baseline IM methods to evaluate its performance and time efficiency.

On each of the real-world graphs, five seed sets are generated by the IM algorithms respectively. Each seed set contains five seeds. The five seeds are ranked based on their Sobol Total indices. The ranking is evaluated by the marginal contribution which is the difference of estimated influence effects before and after including the node into the seed set. The result of the seed set generated by Method \( \Pi \) on the largest connected component of Cora is presented in Table 1. We can clearly observe a strong positive correlation between a node’s Sobol Total index and its marginal contribution. Experiments with seed sets generated by all five algorithms on each graph show similar results, proving that the measure of the node’s relative importance is robust across graphs and IM methods.

| Selected Seeds Index | 1358 | 306  | 1701 | 1986 | 1810 |
|----------------------|------|------|------|------|------|
| Variance Accounted   | 32084.95 | 15792.13 | 8944.68 | 15150.50 | 7183.58 |
| Sobol Total Index    | 0.4640 | 0.2284 | 0.1293 | 0.2191 | 0.1039 |
| Marginal Contribution| 202.8  | 95.8  | 67.4  | 85.2  | 48.7  |

Table 1: The Legitimacy of Each Seed’s Relative Importance

Besides the high order Sobol indices, there is no other measure to quantify the influence overlaps among the seeds. Thus, we intuitively predict the amount of a overlap by the distance among the involved node. When all the nodes involved are closer to each, the influence overlaps tends to be larger since the influence flows coming out of the seeds meet each other in the early stage of the propagation. The overlap will be small if at least one pair of the nodes are far away from each other. The result shown in Table 2 is the second-order Sobol indices of all pairs of the seeds selected by DEG on the largest connected component of Cora graph. The other high order Sobol indices also show such a relationship with the largest distance among the involved nodes.

| Pairs | 1422 | 1422 | 1422 | 582  | 582  | 582  | 1214 | 1214 | 2782  |
|-------|------|------|------|------|------|------|------|------|-------|
|       | 582  | 1214 | 2782 | 1943 | 1214 | 2782 | 1943 | 2782 | 1943  |
| High Order Sobol Index | 12.03 | 764.78 | 763.14 | 546.10 | 0.00 | 0.80 | 3.52 | 523.55 | 295.41 |
| Distance | 6 | 2 | 2 | 2 | 7 | 7 | 7 | 1 | 2 |

Table 2: High order Sobol indices for quantifying the influence overlaps

Regarding the effectiveness and the efficiency of SobolIM, we compare the influence performance of the five baseline IM methods before and after they are combined with the SobolIM framework. The experiments are carried out on all five graphs. Before applying the framework, each IM algorithm
generates a seed set consists of 5 nodes. The framework selects 10 candidate nodes with the corresponding algorithm then eliminates 5 of them. The final influence effect is measured by the mean of the results from 100 simulations. The detailed result is presented in Table 3.

| Algorithm | Cora (n=2485) | CiteSeer (n=2120) | PubMed (n=19717) | WS (n=5000) | ER (n=5000) |
|-----------|---------------|------------------|------------------|-------------|-------------|
| **P**     |               |                  |                  |             |             |
| NS B      | 534.47        | 358.53           | N/A*             | 376.6       | 578.25      |
| A         | 665.16        | 459.59           | N/A*             | 924.35      | 4580.85     |
| F R       | 1367.47       | 703.51           | 11267.25         | 1378.83     | 4601.39     |
| Sigma B   | 1356.26       | 618.85           | 10483.07         | 1062.16     | 4548.36     |
| A         | 1356.26       | 701.62           | 11371.41         | 1194.26     | 4604.14     |
| DEG B     | 1293.26       | 615.86           | 10943.06         | 1333.31     | 4548.36     |
| A         | 1346.05       | 489.24           | 11355.70         | 1445.23     | 4601.49     |
| EIG B     | 1260.31       | 418.05           | 5011.86          | 1155.55     | 4565.64     |
| A         | 1346.05       | 81.69            | 7691.62          | 1260.54     | 4588.57     |

Table 3: Influence maximization performance (P) and time cost (T in s) before (B) and after (A) applying SobolIM. Italic bold for the best performance, bold for the second best.

* NetShield cannot be applied to the PubMed graph due to its high memory requirement.

We observe a significant increase on influence effect after **SobolIM** is applied in most scenarios, with a ratio up to 60%. Only one algorithm on Cora returns the same seed set after **SobolIM** is applied. The performance enhancement effect is greater when the original IM algorithm fails to generate an effective seed set. In most cases, the framework can achieve a boost of around 10% on the influence effect. As predicted, the time cost on generating the seed set increases too. The increase ratio is smaller on a larger graph. This is the time cost on eliminating extra nodes is not related to the graph size, while the selection time scales with the graph size. The time efficiency of **SobolIM** can be further enhanced by data analyzing tools that are faster than Pandas. Another significant finding is that after **combining the degree heuristic with SobolIM**, it always achieves the best or the second best performance among the algorithms. With a relatively low time cost and memory requirement, it can serve as a great candidate on any graphs.

6 Conclusion

This article provides a universal explanation method for the influence effect of a seed set. The seeds’ relative importance is evaluated by the Sobol Total index. The influence overlaps among seeds are quantified for the first time using the high order Sobol indices. The calculation of the Sobol indices are applied to the IM context and simplified. Experiments show that the explanation we provided is robust across graphs and IM algorithms. A novel improvement framework **SobolIM** is proposed to increase the performance of any proxy-based IM algorithm. Empirical experiments on synthetic and real-world data sets proved that **SobolIM** achieves superior performance while controlling the time cost at an acceptable level.
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