Abstract. We consider collections of $N$ chordal random curves obtained from a critical lattice model on a planar graph, in the limit when a fine-mesh graph approximates a simply-connected domain. We define and study candidates for such limits in terms of conformally invariant collections of random curves, generated via iterated Loewner equations. These curves are a natural “domain Markov extension” of the earlier introduced local multiple SLE initial segments to global multiple SLE curves. For realizing them as scaling limits, we provide two a priori results to guarantee the precompactness of the discrete random curves and to allow promoting a discrete domain Markov property to the scaling limit. These results essentially only take as input certain crossing conditions, very similar to those introduced by Kemppainen and Smirnov, and they allow the identification of scaling limits via the martingale strategy of classical SLE convergence proofs. The use of these results is exemplified with convergence proofs in various lattice models.

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1. Introduction

1.1. Background. The scaling limits of critical random models on lattices, as the lattice mesh size tends to zero, are studied in physics via Conformal field theory \cite{Pol70, BPZ84a, BPZ84b, Car88}. One mathematical approach to proving conformal invariance in such limits is to characterize the scaling limits of some discrete interfaces in terms of conformally invariant random curves. A breakthrough in this approach was the observation by Schramm \cite{Sch00} that if such conformally invariant scaling limits exist and inherit the domain Markov property — a domain reduction property prominent in many simple lattice models — they belong to a one-parameter family of random curve models, called Schramm-Loewner evolutions (SLEs). This has led to the identification of scaling limits in various lattice models in terms of SLE type curves; see \cite{Smi01, LSW04, SS05, CN07, Zha08, SS09, CDCH14} on chordal SLEs and \cite{LSW04, Zha08, HK13, Izy15, KS15, LV16, Wu18, Izy17, KS18, BPW18, GW18} on other SLE type curves. This paper is concerned with SLE type models and convergence results for multiple simultaneous chordal curves.

All SLE convergence proofs consist of two parts: precompactness and identification. Precompactness means that any sequence of discrete curves on lattices of decreasing mesh sizes has a weakly convergent subsequence. The identification of any subsequential weak limit then proves weak convergence along the entire sequence. These two parts are in a typical proof very different in spirit: the precompactness relies on verifying certain \textit{a priori} crossing estimates that are \textit{non-specific}, in the sense that they hold in a wide range of lattice models. A machinery of precompactness results then applies for the curves \cite{AB99, KS17, Kar18}. The identification, following the nowadays established strategy of \cite{Smi06}, relies on finding an observable in the lattice model that is a martingale under growing an interface, then promoting this martingale to the subsequential limit by a strong enough convergence of the observable, and finally showing that the obtained continuous martingale characterizes the scaling limit. In contrast to the precompactness part, the identification step relies on the exact, highly \textit{model-specific} relations in the martingale and its convergence, and it seems to be the bottleneck in finding SLE convergence proofs. This paper derives non-specific \textit{a priori} results that allow the use of such martingale identification of multiple SLE type scaling limits, and provides several examples of convergence proofs.
Two SLE variants have then been proposed to describe the scaling limit of multiple simultaneous chordal interfaces: the local [BBK05, Dub07, Gra07, KP16] and global [KL07, Law09, PW19, BPW18] multiple SLEs, and both models have their advantages and disadvantages when working with convergence results.

Let us briefly discuss local multiple SLEs first — see Section 2 for a more formal introduction and Figure 1.1(left) for an illustration. Consider a simply-connected domain $\Lambda$ with $2N$ distinct marked boundary points $p_1, \ldots, p_{2N}$. The local multiple SLE on some disjoint neighbourhoods $U_1, \ldots, U_{2N}$ of the marked boundary points in $\Lambda$ yields $2N$ curve initial segments (“localizations”) starting from a point $p_i$, $1 \leq i \leq 2N$, up to exiting the corresponding neighbourhood $U_i$. These initial segments are described explicitly via Loewner’s equation, as suitably weighted chordal SLE measures of initial segments in the localization neighbourhoods. One important motivation for studying multiple SLEs is that these weights are given by the most central objects of Conformal field theory, the correlation functions, see e.g. [BBK05, Gra07, KKP17b]. Back to scaling limits, an advantage of the local multiple SLEs is its similarity to the chordal SLE, while its disadvantages are that it only describes initial segments and that a new martingale observable needs to be introduced for a convergence proof.

The global multiple SLE on $(\Lambda; p_1, \ldots, p_{2N})$, in turn, describes collections of $N$ mutually non-crossing random curves $\gamma_1, \ldots, \gamma_N$, pairing the marked boundary points in some predetermined manner, see Figure 1.1(right) for an illustration. Given the SLE parameter $\kappa \in (0, 8)$ it is defined, following [PW19, BPW18] (see also [MS16, MSW16]), as the stationary distribution of the discrete time Markov chain on collections of $N$ curves, where at each time step one curve is resampled as a chordal $SLE(\kappa)$ in the domain left for it by the remaining curves. Such a stationary distribution is proven to be unique [BPW18] and exist [PW19] for $\kappa \in (0, 4)$; also the case $\kappa \in (4, 8)$ is conjectured in [BPW18]. This definition is rather implicit, but it can be shown to yield a local multiple SLE($\kappa$) if $\kappa \in (0, 4)$ [PW19]. We will not rely on global multiple SLEs in this paper, but we will show that the obtained scaling limits satisfy the above Markov chain stationarity.

From the lattice model point of view, a great benefit of the global multiple SLE is that it often yields miraculously short convergence proofs, provided that the convergence of the corresponding one-curve lattice model to chordal $SLE(\kappa)$ has been established. Namely, lattice models with domain Markov property satisfy a discrete version of this curve resampling stationarity, and with some a priori estimates, it can be promoted to a subsequential scaling limit; see [BPW18] for examples. In particular, no new martingale observable is needed after to the one-curve convergence. Nevertheless, convergence proofs of this type only hold for conditioned lattice models, where the pairing of the boundary points

**Figure 1.1.** Left: A schematic illustration of the local multiple SLE in a domain $\Lambda$ with eight marked boundary points and their localization neighbourhoods. Right: A schematic illustration of the global multiple SLE with the pairing $\{1, 6\}, \{2, 3\}, \{4, 5\}, \{7, 8\}$ of the boundary points. Conditional on the red curves, the blue one is a chordal SLE in the subdomain of $\Lambda$ left for it, shaded in the figure.
by the interfaces is predetermined. Such a conditioning may appear slightly unnatural, for instance for magnetization cluster interfaces in the Ising magnetism model. To find an unconditional scaling limit, one would thus need to solve the probabilities of the different pairings of boundary points as in \[PW18\] for the Ising model and \[Dub06\], \[KW11a\], \[KW11b\], \[KKP17a\], \[PW19\] for some other models. This seems not to be easy. Indeed, in lattice models with the discrete domain Markov property, such pairing probabilities yield, under growing an interface, conditional pairing probabilities, and are hence martingales. Proving their convergence should thus be roughly equivalent to an SLE identification step with the usual martingale strategy; see \[Smi01\], \[KS18\] for examples.

Finally, we remark that the connection of these two multiple SLE type models is not completely clear. For \(\kappa \in (0, 4]\) the initial segments of a global multiple SLEs are local multiple SLEs \[PW19\] (see also \[Wu18\] on \(N = 2\) curves), but for \(\kappa \in (4, 8]\) such a connection remains conjectural. Furthermore, in this paper we will provide a warning example (with \(\kappa = 6\) and \(N \geq 3\)) showing that curves whose initial segments in any localization neighbourhoods are local multiple SLEs are not necessarily global multiple SLEs.

1.2. Contributions of this paper. In this paper, we show how the convergence of multiple simultaneous chordal interfaces can be proven following the classical strategy of \[Smi06\]. We characterize such limits in terms of explicit Loewner growth processes similar to local multiple SLEs, and show that such scaling limits are convex combinations of global multiple SLEs with different pairings. Roughly speaking, this takes three ingredients.

First, we propose a natural “domain Markov extension” of local multiple SLEs to full curves, which we call local-to-global multiple SLEs. The well-definedness of the obtained curves follows by realizing them as scaling limits. (For \(\kappa \in (0, 4]\) it could also be done based on global multiple SLEs being local, but we avoid taking this or other SLE theory as logical inputs, consistently relying only on arguments based on the underlying lattice models.)

Second, we provide two important non-specific results related to the convergence of lattice models: a straightforward generalization of the precompactness conditions \[KS17\], \[Kar18\] for multiple curves, and a result showing that any subsequential scaling limit inherits a domain Markov property from the discrete model. By the latter property, identifying one initial segment of one curve as a local multiple SLE suffices to identify the full collection of full curves as its domain Markov extension. The a priori results needed for these non-specific results to hold are the discrete domain Markov property and a crossing condition, very similar to that in \[KS17\] to guarantee precompactness. In particular, these conditions are known to be satisfied in most well-studied lattice models. As a by-product of the domain Markov type properties, we also obtain the connection to global multiple SLEs.

Third, a convergence proof requires an identification step, in this case identifying one initial segment of one curve as a local multiple SLE. We review three priorly known convergence proofs in Ising, FK-Ising and percolation models, and two new proofs, in detail for the multiple harmonic explorer curves and a sketch for the uniform spanning tree branches. Also FK cluster model is discussed.

Except for referring to the precompactness results of \[KS17\], \[Kar18\], the paper is self-contained. We have tried minimize the amount of logical inputs taken, as well as the a priori estimates required from the lattice models.

1.3. Related work. Apart from the related work mentioned so far, let us mention some references that address similar underlying principles.

One motivation and a Conformal field theory approach to the study of multiple SLEs is their description as chordal SLEs weighted by correlation functions. Some central notions of Conformal field theory, such as fusion and conformal blocks, do not arise when studying single SLEs. This is not the perspective of this paper, but should be kept in mind, see e.g. \[BBK05\], \[Gra07\], \[Dub07\], \[Dub15a\], \[Dub15b\], \[Pel16\], \[KKP17b\] for more.

The idea of working with non-specific results based on crossing estimates dates back to \[AB99\], \[KS17\]. We also have to prove precompactness in different topologies and the agreement of the different weak
limits, similarly to [KS17, Kar18]. As regards the non-specific result on the domain Markov property, the
non-triviality of promoting the discrete domain Markov property to a scaling limit has been addressed
recently in, e.g., [GW18, BPW18]. It should be noticed the latter non-specific results in this paper
take very little inputs and follow (essentially) once precompactness is verified with the standard crossing
estimates, cf. [KS17].

The idea of proposing scaling limit random models that are well-defined due to being scaling limits is
present in SLE literature at least in [LSW04, Zha08, Lzy17, BPW18].

This work was initiated in attempt to answer Conjecture 4.3 in the author’s earlier paper [KKP17a],
whose proof is now sketched in Section 6.3. Multiple SLE type models have since then attracted quite
some attention, see [Lzy17, Wu18, PW19, KS18, BPW18, PW18].

1.4. **Organization.** This paper is organized as follows. In Section 2, we introduce the local multiple SLE
and propose its domain Markov extension, the local-to-global multiple SLE. Section 3 contains some
preliminaries used throughout the paper. Section 4 addresses non-specific results on precompactness
and contains our first main result, Theorem 4.1. Section 5 addresses non-specific results on domain
Markov property, with the main theorem 5.8 guaranteeing that identification of one initial segment
actually identifies the full collection of full curves. We also give a variant of that theorem, suited for the
local multiple SLE collection of initial segments, as well as some consequences. For the ease of reading,
Sections 4 and 5 are arranged so that the statements of the main results are given first, and the technical
proofs are postponed to the end of the section. Finally, in Section 6 we give various applications of these
results, addressing several lattice models.

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2. **Multiple SLE type models**

The purpose of this section is to give a sufficient overview of $N -$ SLE type random curve models. We
emphasize that the results of this paper, describing scaling limits as SLE type curves, do not take SLE
type as logical inputs, but only rely properties of the converging lattice model\(^1\).

We begin with a brief exposition of local multiple SLEs in Section 2.1. No new results are introduced
there. Then, in Section 2.2 we define local-to-global multiple SLEs. This definition is new, ans it will
describe the scaling limits in our main results. We assume that the reader is familiar with the basic
properties of the most well-known SLE variant, the chordal SLE; see, e.g., the text books [Law05, BN14,
Kem17].

2.1. **Local multiple SLE.** The local multiple SLE is a generalization of the chordal SLE to handle
collections of $N$ simultaneous chordal SLE type random curves connecting $2N$ boundary points, first
proposed in [Dub07]. Similarly to the chordal SLE, it is defined via conformal invariance and the hulls of
a random Loewner growth process in $\mathbb{H}$. However, the definition of the local multiple SLE does not give
full chordal curves, but in stead only initial segments. A familiar example of an analogous restriction
the chordal SLE($\kappa$) between two real points $x_1$ and $x_2$ [Dub07, Lemma 3].

\(^1\)To be very precise, there is one small exception to this rule, namely the a posteriori argument in the proof of
Proposition 5.20, using the fact that the chordal SLE($\kappa$) has no boundary visits if and only if $\kappa \leq 4$ [RS05]. This result
is used only after convergence of a lattice model has already been proven, to yield a more convenient description of the
scaling limit.
Fundamentally, local multiple SLEs arise as multiple random random curve initial segments that satisfy
local multiple SLE is then a measure on collections of curve initial segments from
above. The initial segment is described by a Loewner equation up to the hitting time
as the covariance rule and degeneracy PDEs \([BPZ84a]\) for primary boundary fields of conformal weight
obtained as a part of the identification of a scaling limit; see Section 6.4.5 for a conctrete example.

As a second remark, the conditions \([\text{PDE}}) and \([\text{COV}) arise in the derivation of \([\text{Dub07}\) by purely

2.1.1. \textbf{Partition functions}. The definition of the local multiple SLE(\(\kappa\)) with \(2N\) boundary points relies
on a partition function \(Z\). A function \(Z\) defined on a chamber \(X_{2N} = \{(x_1, \ldots, x_{2N}) \in \mathbb{R}^{2N}: x_1 < \ldots < x_{2N}\}\) is called an \(N\)-SLE(\(\kappa\)) partition function if it is positive, \(Z(x_1, \ldots, x_{2N}) > 0\) for all \((x_1, \ldots, x_{2N}) \in X_{2N}\), and satisfies the linear partial differential equations (PDEs)

\[
\text{(PDE)} \quad \left[ \frac{\kappa}{2} \frac{\partial^2}{\partial x_j^2} + \sum_{i \neq j}^{N} \left( \frac{2}{x_i - x_j} \frac{\partial}{\partial x_i} - \frac{2h}{(x_i - x_j)^2} \right) \right] Z(x_1, \ldots, x_{2N}) = 0 \quad \text{for all } j = 1, \ldots, 2N,
\]

where \(h = h(\kappa) = \frac{6 - \kappa}{2\kappa}\) and the Möbius covariance

\[
\text{(COV)} \quad Z(x_1, \ldots, x_{2N}) = \prod_{i=1}^{2N} \mu'(x_i)^h \times Z(\mu(x_1), \ldots, \mu(x_{2N}))
\]

for all \(\mu(z) = \frac{az + b}{cz + d}\), with \(a, b, c, d \in \mathbb{R}\), \(ad - bc > 0\), such that \(\mu(x_1) < \cdots < \mu(x_{2N})\).

Remarks. Characterizing the positive solutions to \([\text{PDE}}) and \([\text{COV})\, and hence all local multiple SLEs,
is a long-standing task, recently completed for \(\kappa \in (0, 4]\), and still partly open for \(\kappa \in (4, 8]\) \([FK15a, FK15b, FK15c, FK15d, KP16, KKP17a, Wu18, PW19, BPW18]\). We stress that \emph{the results in this paper do not rely on the analysis of these PDE solutions}. In stead, we assume that partition functions are obtained as a part of the identification of a scaling limit; see Section 6.4.5 for a conctrete example.

As a second remark, the conditions \([\text{PDE}}) and \([\text{COV}) arise in the derivation of \([\text{Dub07}\) by purely

2.1.2. \textbf{One-curve marginals in }\mathbb{H}. Let us now describe the marginal law of the initial segment from
the \(j\)th marked boundary point in a local multiple SLE(\(\kappa\)) in \(\mathbb{H}\), given the partition function \(Z\) as above.
The initial segment is described by a Loewner equation up to the hitting time \(T_j\) of \((\mathbb{H} \setminus U_j)\),
where \(U_j \subset \mathbb{H}\) is the localization neighbourhood. Let us denote the real boundary points by \(p_i = x_i, 1 \leq i \leq 2N\), and assume that \(-\infty < x_1 < \ldots < x_{2N} < +\infty\). We will also need to assume that the

More formally, the local multiple SLE is defined in simply-connected domains \(\Lambda\) with \(2N\) distinct boundary points (or in more general, prime ends) \(p_i, 1 \leq i \leq 2N\), numbered counterclockwise, and their localization neighbourhoods \(U_i\) which are closed neighbourhoods of \(p_i\) in \(\Lambda\), and pairwise disjoint, \(U_i \cap U_j = \emptyset\) if \(i \neq j\), and such that \(\Lambda \setminus U_i\) is simply-connected for all \(i\). See Figure 1.1(left) for an illustration. The local multiple SLE is then a measure on collections of curve initial segments from \(p_i\) up to the exit time of \(U_i\), i.e., the first hitting time of \((\Lambda \setminus U_i)\).

In this paper, we will restrict our attention to SLEs with parameter \(\kappa \in (0, 8]\).
localization neighbourhood $U_j$ is bounded (in other words, it is a compact $\mathbb{H}$-hull). Then, the marginal law of the $j$:th initial segment is described by the Loewner differential equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_{j,t}},$$

where the driving function $W_{j,t}$, for $t \in [0,T_j]$ is determined by the system stochastic differential equations

$$\begin{cases}
    dW_{j,t} = \sqrt{\kappa} dB_t + \kappa \partial_j (\log Z(W_{1:t}, \ldots, W_{2N,t})) dt \\
    dW_{i,t} = \frac{2\mu}{W_{i,t} - W_{j,t}}, \quad i \neq j.
\end{cases}$$

Here $Z$ is the partition function of the local multiple SLE, and two partition functions that are not constant multiples of each others will yield different multiple SLE measures. From basic SDE theory, the driving function stopped at $T_j$ is a measurable random variable in the topology of Section 3.3.2.

**Remarks.** First, by absolute continuity with respect to the chordal SLE (see [Dub07] or [KP16 Section A.3]), the one-curve marginals up to the exit time of $U_j$ enjoy many good properties of the chordal SLE. For instance, for the model in $\mathbb{H}$, the local multiple SLE hulls are indeed curves [RS05] and share the same fractal dimension depending on $\kappa$ [DeF08]. Likewise, for $\kappa < 8$, their conformal images in a bounded domain ($A; p_1, \ldots, p_{2N}$), with marked prime ends where radial limits exist (see Section 3.2.2), are curves and measurable random variables the topology of Section 3.3.1 see [Kar18 Proposition 5.2].

Second, if there are $N = 1$ curves, condition (COV) for scalings $\mu$ alone determines a solution to (PDE), unique up to scaling, namely $Z(x_1, x_2) \propto (x_2 - x_1)^{-2\mu}$. Then, the growth process (2.2) coincides with the chordal SLE$(\kappa)$ from $x_1$ to $x_2$, appearing in, e.g., [Dub07 Lemma 3]. For general $N$, the variables $W^{(i)}_t, i \neq j$ are the conformal images of the boundary points, $W_1 = g_t(x_i)$, and $W^{(j)}_t$ is the conformal image of the tip of the growing curve at time $t$.

### 2.1.3. Curve collections in $\mathbb{D}$. Let us finally address the local multiple SLE$(\kappa)$ as a collection of curves. Due to using the topology of compact curves in this paper (see Section 3.3.1) we will now use the unit disc $\mathbb{D}$ as our reference domain in stead of $\mathbb{H}$. Thus, we consider the domain $(\mathbb{D}; \tilde{p}_1, \ldots, \tilde{p}_{2N})$ with localization neighbourhoods $U_1, \ldots, U_{2N}$.

We fix a point $\tilde{p}_\infty \in \partial \mathbb{D}$ on the counterclockwise arc of $\partial \mathbb{D}$ from $\tilde{p}_{2N}$ to $\tilde{p}_1$, and a conformal map $\psi$ taking $(\mathbb{D}; \tilde{p}_\infty)$ to $(\mathbb{H}, \infty)$. (Hence $-\infty < \psi(\tilde{p}_1) < \ldots < \psi(\tilde{p}_{2N}) < +\infty$.)

The local multiple SLE in this setup is a collection of curve initial segments, defined via the regular conditional laws of the driving function of the $j$:th initial segment, conditional on the initial segments $1, 2, \ldots, (j - 1)$, for each $j$. (For basics of regular conditional laws, see Appendix A.) Denote by $\nu_j$ the $j$:th initial segment, up to the hitting time $T_j$ of $\mathbb{D} \setminus U_j$. Given a partition function $Z$, the local multiple SLE is now defined as follows.

**Definition 2.1.** Given the previous initial segments $\nu_1, \ldots, \nu_{j-1}$ and a conformal map $\psi_j$ (where $\psi_0 = \psi$) from the connected component of $\mathbb{D} \setminus (\nu_1 \cup \ldots \cup \nu_{j-1})$ adjacent to the marked boundary points $\lambda_j(T_1), \ldots, \lambda_{j-1}(T_{j-1}), p_1, \ldots, p_{2N}$, the regular conditional law of the $j$:th initial segment $\nu_j$, is the conformal image under $\psi_j^{-1}$ of the curve given by multiple SLE one-curve marginal (2.2) in the localization neighbourhood $\psi_j^{-1}(U_j)$ of $\mathbb{H}$. The map $\nu_j$ is $g_{T_j} \circ \psi_{j-1}$, where $g_{T_j}$ is given by (2.1) and $T_j$ is the exit time of $\psi_{j-1}(U_j)$ by the growth process in $\mathbb{H}$.

**Important remarks.** The definition above does not depend on the choice of the reference point $\tilde{p}_\infty$ and the conformal map $\psi$ due to the conformal invariance of the local multiple SLE initial segments in $\mathbb{H}$.

Even if in the above definition, the initial segments are sampled in the order from 1 to 2N, any order of sampling will produces the same law of the curves, see [KP16 Sampling procedure A.3].
An alternative and perhaps more fundamental way to state the definition above would be in terms of the regular conditional laws of the driving functions of $\psi_{j-1}(\lambda_j)$ being given by (2.2). To see the equivalence, first by [Kar18, Proposition 5.2] (stated for chordal SLEs, holding for multiple SLEs by absolute continuity) and Corollary [C.5] the driving functions of $\lambda_j$ and the curves $\lambda_j$ are measurable functions of each other. Then, as discussed in Section 4.3 of this paper, specifically commutative diagram (4.12), the collections of driving functions of $\lambda_j$ and $\psi_{j-1}(\lambda_j)$ are measurable functions of each other. By these two-way measurabilities, one can see the equivalence of conditional-law descriptions via curves or driving functions. We have nevertheless chosen to postpone the further treatment in terms of driving functions to later sections to keep the notation minimal in this introductory section.

2.1.4. **General domains.** For a general bounded simply-connected domain $\Lambda$ with marked prime ends $p_1, \ldots, p_{2N}$ where radial limits exist (see Section 3.2.2) and their localization neighbourhoods $U_1, \ldots, U_{2N}$, the local multiple SLE is the conformal image of a local multiple SLE in $\mathbb{D}$, with the boundary points and localization neighbourhoods chosen according to the conformal images.

2.1.5. **Continuous stopping times.** The (capacity at the) hitting time $T_j$, introduced in the previous paragraph, is not continuous in many of the topologies that we will impose on curves. This is illustrated in Figure B.1 in Appendix B. Because the main focus of this article is on weak convergence results, we will often have to use the continuous modifications $\tau_j$ of the hitting times $T_j$. These stopping times are introduced in more detail in Appendix B — for a busy reader it suffices to us to know that they are conformally invariant and satisfy $\tau_j > T_j$. It then follows from the “local commutation property” of [Dmb07] that if a collection of initial segments $\lambda_1([0, \tau_1]), \ldots, \lambda_{j-1}([0, \tau_{j-1}])$ satisfies the regular conditional law property of the previous subsection, then also the shorter initial segments $\lambda_1([0, T_1]), \ldots, \lambda_{j-1}([0, T_{j-1}])$ satisfy the same property. Thus, treating continuous stopping times should be regarded merely as a technicality arising from weak convergence.

2.2. **Local-to-global multiple SLE.** We now define the local-to-global multiple SLE, which is a natural domain Markov extension of the local multiple SLEs in the preceding subsection, and the main object of interest in this paper.

2.2.1. **Unconditional and conditional random curve models.** A link pattern of $N$ links is a partition of $\{1, 2, \ldots, 2N\}$ into $N$ disjoint pairs $\{\{a_1, b_1\}, \ldots, \{a_N, b_N\}\}$, called links, such that the real-line points $a_i$ and $b_i$, for all $1 \leq i \leq N$, can be connected by pairwise disjoint curves in the upper half-plane. The set of all link patterns of $N$ links is denoted by $\text{LP}_N$. We use link patterns to encode in which way some chordal curves pair $2N$ marked boundary points of a simply-connected domain. Note also that due to parity reasons, every link of a link pattern must contain one odd and one even boundary point.

We will define separately local-to-global multiple SLEs and conditional local-to-global multiple SLEs. The unconditional versions arise as scaling limits of interface models when no condition is imposed on the link pattern formed by the interfaces in the corresponding lattice model. Similarly, the conditional version will be scaling limits of $N$ interfaces conditional on each particular link pattern $\alpha \in \text{LP}_N$.

2.2.2. **The definitions.** Let us begin with the unconditional version of the local-to-global multiple SLE. Similarly to local multiple SLE, the definition relies on conformal invariance, Loewner growth processes for suitable initial segments, and regular conditional laws given the initial segments. A new ingredient is induction the number $N$ of curves.

Let $(\Lambda; p_1, \ldots, p_{2N})$ be a bounded simply-connected planar domain with $2N$ marked prime ends with radial limits (indexed counterclockwise). Suppose that we are given a family of local multiple SLE($\kappa$) partition functions $Z_N$, for $N$ up to some value (possibly all $N \in \mathbb{N}$). We define the local-to-global multiple SLE($\kappa$) as the following random curves.
1) (Induction.) If \( N = 1 \), we define the symmetric multiple \( \text{SLE}(\kappa) \) to be the usual chordal \( \text{SLE}(\kappa) \) on \((\Lambda; p_1, p_2)\). Assume now that the unconditional multiple \( \text{SLE}(\kappa) \) with partition functions \( Z_N \), is defined for \( k \) curves (in any bounded domain with degenerate prime ends), for all \( 1 \leq k \leq N-1 \), and define it for \( N \) curves as follows.

2) (Conformal invariance.) Let \( \phi : \Lambda \to \mathbb{D} \) be a conformal map taking our domain of interest \((\Lambda; p_1, \ldots, p_{2N})\) to \((\mathbb{D}; \tilde{p}_1, \ldots, \tilde{p}_{2N})\). We will define the symmetric multiple \( \text{SLE}s \) below in \((\mathbb{D}; \tilde{p}_1, \ldots, \tilde{p}_{2N})\) as random curves \((\gamma_{\mathbb{D};1}, \ldots, \gamma_{\mathbb{D};N})\) and then in \((\Lambda; p_1, \ldots, p_{2N})\) as the conformal image curves \((\phi^{-1}(\gamma_{\mathbb{D};1}), \ldots, \phi^{-1}(\gamma_{\mathbb{D};N}))\).

3) (Initial segments; Figure 2.1.) Denote by \( \lambda_0 \) the initial segments of the random curve \( \gamma_{\mathbb{D};1} \) in \( \mathbb{D} \) starting from the boundary point \( \tilde{p}_1 \), until the continuous modification of the hitting time of the \( \delta \)-neighbourhood of the boundary arc \((\tilde{p}_2 \tilde{p}_{2N})\). For all \( \delta > 0 \), \( \lambda_0 \) is described by the local multiple \( \text{SLE} \) growth process (2.2) (with the partition function \( Z_N \)). As \( \delta \downarrow 0 \), \( \lambda_0 \) almost surely tend to a closed curve \( \lambda_0(0) \) from \( \tilde{p}_1 \) to \((\tilde{p}_2 \tilde{p}_{2N})\).

4a) (Conditional laws for \( \kappa \in (0, 4] \); Figure 2.2) The initial segment \( \lambda_0(0) \) will almost surely hit the arc \((\tilde{p}_2 \tilde{p}_{2N})\) at some even-index marked boundary point, and forms one full random curve, \( \lambda_0(0) = \gamma_{\mathbb{D};1} \) (Figure 2.2(left)). The regular conditional distribution of the remaining curves \( \gamma_{\mathbb{D};2}, \ldots, \gamma_{\mathbb{D};N} \) are two independent local-to-global multiple \( \text{SLE}(\kappa) \)s in the relevant connected components of \( \mathbb{D} \setminus \gamma_{\mathbb{D};1} \) and with the relevant marked boundary points (the brown and green domains, curves, and boundary points in Figure 2.2(right)).

4b) (Conditional laws for \( \kappa \in (4, 8) \); Figure 2.3) The initial segment \( \lambda_0(0) \) will almost surely not hit the arc \((\tilde{p}_2 \tilde{p}_{2N})\) at any of the marked boundary points \( \tilde{p}_2, \ldots, \tilde{p}_{2N} \), and thus \( \mathbb{D} \setminus \lambda_0(0) \) has two connected components adjacent to the remaining boundary points \( \tilde{p}_2, \ldots, \tilde{p}_{2N} \): one with an even and one with an odd number of them; see Figure 2.3(left). Declare the tip of the initial segment \( \lambda_0(0) \) as a new marked boundary point in the "odd" component (in brown in Figure 2.3(right)), so that both components now have an even number of boundary points. The regular conditional distribution of the remainder of the curves \( \gamma_{\mathbb{D};1}, \ldots, \gamma_{\mathbb{D};N} \) are two independent local-to-global multiple \( \text{SLE}(\kappa) \)s in the relevant connected components of \( \mathbb{D} \setminus \gamma_{\mathbb{D};1} \) and with the relevant marked boundary points (the brown and green domains, curves, and boundary points in Figure 2.2(right)).

For \( 0 < \kappa \leq 4 \), we will also consider conditional lattice models. The conditional local-to-global multiple \( \text{SLE} \) is defined almost identically, except that the collection of partition functions \( Z_\alpha \) is now indexed by \( N \) and link patterns \( \alpha \in \text{LP}_N \). Only step (4) is slightly modified:

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\[ \text{Figure 2.1. A schematic illustration of step (3) in the definition of the local-to-global multiple SLE.} \]
Figure 2.2. A schematic illustration of step (4a) in the definition of the local-to-global multiple SLE for $\kappa \in (0, 4]$.

Figure 2.3. A schematic illustration of step (4b) in the definition of the local-to-global multiple SLE for $\kappa \in (4, 8]$.

4a') The initial segment $\lambda_{(0)}$ will almost surely hit the arc $(\tilde{p}_2 \tilde{p}_2 N)$ at the even-index marked boundary point linked to 1 in the link pattern $\alpha$, and forms one full random curve, $\lambda_{(0)} = \gamma_{D;1}$. The regular conditional distribution of the remaining curves $\gamma_{D;2}, \ldots, \gamma_{D;N}$ are two independent conditional SLE($\kappa$):s in the relevant connected components of $D \setminus \gamma_{D;1}$, with the relevant marked boundary points, and conditional on the relevant link patterns.

Important remarks. The existence of curves given by the definition above is not immediate, and we will rely on realizing them as scaling limits. Some non-trivial obstacles that we will take care of are:

- The operation of $\phi^{-1}$ maps curves to curves and the obtained collection of curves ($\gamma_{D;1}, \ldots, \gamma_{D;2N}$) is a measurable random variable (cf. [Kar18]).
- The initial segment $\lambda_{(0)}$ exists as a closed curve up to and including its end point (cf. [PWIS]).
- Being regular conditional laws requires some regularity properties from the local-to-global multiple SLEs.

The above regular conditional laws can be interpreted as a sampling procedure, and above we always sampled the initial segment of $\gamma_{D;1}$ first. This is only for definiteness; one can start by sampling the initial or final segment adjacent to a boundary point of choice. This will be proven basing on the underlying discrete models.

In step (4) we did not specify how the boundary points in the two connected components are re-labelled (although parities must be preserved). This need not be done due to the properties that we will require of the underlying lattice models.
Only very specific multiple SLE partition functions should yield the initial segments of the above type. These are called symmetric and pure partition functions in \[\text{KPI16, PW19, Wu18}\], corresponding to the unconditional and conditional cases, respectively, and finding such functions is not always easy. Nevertheless, in the strategy of this paper, the partition functions will be found as a step in the limit identification.

### 3. Preliminaries

This section introduces some notations, definitions, and concepts that are used throughout the paper.

#### 3.1. Lattice models.

**3.1.1. Discrete random curve models.** We start with the setup and notation that we refer to as discrete random curve models. Various examples will be given in Section 6.

- \(\Gamma = (V(\Gamma), E(\Gamma))\) is a (possibly infinite) connected planar graph with fixed planar embedding, such as \(\mathbb{Z}^2\), or an isoradial graph as in \[\text{CSS11}\], or a more general graph as in \[\text{Che16}\]. We call \(\Gamma\) a lattice.
- \(\Lambda_\Gamma\) is a simply-connected planar domain, whose boundary consists of edges and vertices in \(\Gamma\).
- \(\mathcal{G} = (V, E)\) is the following graph: its vertices \(V\) consist of interior vertices \(V^\circ = V(\Gamma) \cap \Lambda\) and boundary vertices \(V^\partial = V(\Gamma) \cap \partial \Lambda\). Its edges \(E\) consist of \(E = E(\Gamma) \cap \Lambda_{\mathcal{G}}\). We call boundary edges \(\partial E \subset E\) the edges that connect \(V^\circ\) to \(V^\partial\), and edges running between interior vertices are called interior edges \(E^\circ\). We call \(\mathcal{G}\) the simply-connected subgraph of \(\Gamma\) corresponding to \(\Lambda_\Gamma\).
- Let \(\mathcal{G}\) be as above \(e_1, \ldots, e_{2N} \in \partial E\) be \(2N\) distinct boundary edges, indexed counterclockwise along \(\partial \Lambda_\Gamma\). A measure with random curves on \((\mathcal{G}; e_1, \ldots, e_{2N})\) is a pair \((P^{(\mathcal{G}; e_1, \ldots, e_{2N})}, (\gamma_{e_1}, \ldots, \gamma_{e_{2N}}))\) of a probability measure and a measurable random variable \((\gamma_{e_1}, \ldots, \gamma_{e_{2N}})\), supported on \(N\)-tuples of (vertex-)disjoint simple paths on the graph \(\mathcal{G}\), pairing the boundary edges \(e_1, \ldots, e_{2N}\), and otherwise running in the interior edges \(E^\circ\). We choose here the convention that every path travels from odd to even boundary boundary edge, \(\gamma_{e_2}\) starting from \(e_1, \gamma_{e_{2N}}\) from \(e_3\), etc.
- A discrete (random) curve model on \(\Gamma\) is a collection of measures with random curves \((P^{(\mathcal{G}; e_1, \ldots, e_{2N})}, (\gamma_{e_1}, \ldots, \gamma_{e_{2N}}))\). This collection is indexed by some positive integers \(N\) and some simply-connected subgraphs \((\mathcal{G}; e_1, \ldots, e_{2N})\) of the lattice \(\Gamma\) with \(2N\) distinct marked boundary edges.
- Given a discrete random curve model, we define a discrete conditional (random) curve model as a collection measures

\[
P_\alpha^{(\mathcal{G}; e_1, \ldots, e_{2N})}[\cdot] = P^{(\mathcal{G}; e_1, \ldots, e_{2N})}[\cdot | \text{curves form the link pattern } \alpha] = P^{(\mathcal{G}; e_1, \ldots, e_{2N})}[\cdot | \alpha]
\]

with the associated random curves. This collection is indexed by \(N\), graphs \((\mathcal{G}; e_1, \ldots, e_{2N})\), and link patterns \(\alpha \in \text{LP}_N\) such that the measures \(P^{(\mathcal{G}; e_1, \ldots, e_{2N})}\) and the conditioning on \(\alpha\) make sense.

#### 3.1.2. Discrete domain Markov property.** A key property of the discrete random curve models will be the discrete domain Markov property (DDMP). To be able to define the DDMP with a reasonably light notation, we will make one more assumption on the discrete random curve models. Namely, we will assume that if \((P^{(\mathcal{G}; e_1, \ldots, e_{2N})}, (\gamma_1, \ldots, \gamma_N))\) is one measure with random curves in our discrete random curve model, and if we denote \((\hat{e}_1, \ldots, \hat{e}_{2N}) = (e_3, e_4, \ldots, e_{2N}, e_1, e_2)\), then also \((P^{(\mathcal{G}; \hat{e}_1, \ldots, \hat{e}_{2N})}, (\hat{\gamma}_1, \ldots, \hat{\gamma}_N))\) is a pair in that model, and have the equality in distribution

\[
(\hat{\gamma}_1, \ldots, \hat{\gamma}_N) \overset{(d)}{=} (\gamma_2, \ldots, \gamma_N, \gamma_1).
\]

If this is satisfied, we say that a symmetric random curve model has alternating boundary conditions. In other words, we may re-label the edges from \(e_1, \ldots, e_{2N}\) to \(\hat{e}_1, \ldots, \hat{e}_{2N}\), using any cyclic permutation...
that labels the edges counterclockwise and preserves the parities, and our random curve model yields the same random curves on \((G; e_1, \ldots, e_{2N})\) and \((\tilde{G}; \tilde{e}_1, \ldots, \tilde{e}_{2N})\); there are hence two kinds of boundary segments between the marked edges.

Informally speaking, a discrete random curve model satisfies the DDMP if conditioning the model on \(G\) on an initial segment or a full curve is equivalent to reducing the graph \(G\) by that initial segment or curve. Formally, let the collection of pairs \((\mathbb{P}^{(G; e_1, \ldots, e_{2N})}, (\gamma_1, \ldots, \gamma_N))\) be a discrete random curve model on \(\Gamma\) with alternating boundary conditions. We say that it satisfies the DDMP the following hold:

i) Consider the curves given by the random curve model on \((G; e_1, \ldots, e_{2N})\), and for some \(1 \leq j \leq 2N\), condition them on a sequence of vertices \(\lambda(0), \lambda(1), \ldots, \lambda(t)\), appearing in this order as the first vertices along the random curve adjacent to \(e_j = (\lambda(0), \lambda(1))\), started from that edge. Denote by \(G_t\) the graph corresponding to the simply-connected domain \(\Lambda_t\), whose boundary consists of \(\partial \Lambda\) and the graph path \(\lambda(0), \lambda(1), \ldots, \lambda(t-1)\). Now, on this condition, the remaining parts of the curves given by the random curve model on \((G; e_1, e_2, \ldots, e_{2N})\) are in distribution equal to the random curve model on \((G_t; e_1, \ldots, e_{j-1}, (\lambda(t-1), \lambda(t)), e_{j+1}, \ldots, e_{2N})\).

ii) Condition the curves given by the random curve model on \((G; e_1, e_2, \ldots, e_{2N})\), where \(N \geq 2\), on the full random curve \(\gamma_{G_t}\), that reaches the boundary via \(e_j\). The simple curve \(\gamma_{G_t}\), the conditional law the remaining random curves is the following: the curves in \(G_t\) are independent of those in \(G_L\), the marginal law of the curves on each of these, say for definiteness \(G_L\), is (up to relabelling the curves) given by the random curve model on \((G_L; \tilde{e}_1, \ldots, \tilde{e}_{2M})\), where \(\tilde{e}_1, \ldots, \tilde{e}_{2M}\) are those of the marked boundary edges \(e_1, e_2, \ldots, e_{2N}\) left in \(G_L\), relabelled counterclockwise in such a way that the parities are preserved.

Remarks. As a consequence, analogues of properties (i) and (ii) hold for the conditional measures \(\mathbb{P}^{(G; e_1, e_2, \ldots, e_{2N})}_G\). The analogue of (i) is obvious. In the analogue of (ii), the conditional laws of the curves on \(G_L\) and \(G_R\) are independent of each other and given by the conditional random curve model on \(G_L\) and \(G_R\), where the link pattern formed by the curves between the boundary edges in \(G_L\) and \(G_R\) are those inherited from \(\alpha\).

Property (i) can be equivalently stated in terms of stopping times. Then, in stead of some fixed first vertices \(\lambda(0), \lambda(1), \ldots, \lambda(t)\), one conditions on the first vertices \(\lambda(0), \lambda(1), \ldots, \lambda(t)\) up to a stopping time \(\tau\) in the filtration \(\mathcal{F}_1, \mathcal{F}_2, \ldots\), where \(\mathcal{F}_t\) is the sigma-algebra generated by the first \(t\) vertices \(\lambda(0), \ldots, \lambda(t)\).

From measures with \(N = 1\) curves, only property (i) is required.

Property (ii) can be used inductively to deduce the distribution of the remaining curves, given any collection of full random curves. In particular, if we condition on all but one curves, so that the remaining one has to stay on the simply-connected subgraph \(\tilde{G}\) with marked boundary edges \(e_{k_1}, e_{k_2}\), with \(k_1\) odd, then the remaining one is described by the random curve model on \((\tilde{G}; \tilde{e}_{k_1}, e_{k_2})\).

3.2. Approximations of planar domains. In this subsection, we introduce the concepts related to approximations of planar domains. All notations and conventions are identical to the previous paper of the author [Kar18].

3.2.1. Prime ends. We are dealing with simply-connected domains \(\Lambda\) with possibly a very rough boundary. The notion of boundary points must thus be replaced with that of prime ends. Define first the following.

- A cross cut \(S\) is an open Jordan arc in \(\Lambda\) such that \(\overline{S} = S \cup \{a, b\}\), with \(a, b \in \partial \Lambda\).
- A sequence \((S_n)_{n \in \mathbb{N}}\) of cross cuts is a null chain if \(S_n \cap S_{n+1} = \emptyset\) for all \(n\), \(S_n\) separates \(S_{n+1}\) from \(S_0\) in \(\Lambda\) for all \(n \geq 1\), and \(\text{diam}(S_n) \to 0\) as \(n \to \infty\).
Radial limits of conformal maps

Points. A conformal map $\phi : \Lambda \to \mathbb{D}$ induces a bijection $\hat{\phi}$ between the prime ends of $\Lambda$ and $\partial \mathbb{D}$, such that if a null chain $S_n$ determines a prime end $p$ of $\Lambda$, then $\hat{\phi}(S_n)$ is a null chain in $\mathbb{D}$ and determines the prime end $\hat{\phi}(p) \in \partial \mathbb{D}$ [Pom92, Theorem 2.15]. In this sense, prime ends are “the conformal notion of boundary points”.

3.2.2. Radial limits of conformal maps. We will extend the conformal map $\phi^{-1} : \mathbb{D} \to \Lambda$ to $\partial \mathbb{D}$ by radial limits whenever they exist: denote by $P_\varepsilon : \overline{\mathbb{D}} \to \mathbb{D}$ the radial projection on $\mathbb{D}$,

$$P_\varepsilon(z) = \frac{z}{|z|} \min\{1 - \varepsilon, |z|\},$$

where $0 < \varepsilon < 1$, and for $z \in \partial \mathbb{D}$ denote by $\phi^{-1}(z)$ the limit

$$\lim_{\varepsilon \to 0} \phi^{-1} \circ P_\varepsilon(z)$$

whenever it exists (by Fatou’s theorem, it exists for Lebesgue-almost every $z \in \partial \mathbb{D}$ if $\Lambda$ is bounded).

It holds true that the existence and value of such a radial limit of a conformal map $\phi^{-1}$ at some $z = \hat{\phi}(p) \in \partial \mathbb{D}$ only depends on the corresponding prime end $p$ of $\Lambda$, but not the choice of the conformal map $\phi : \Lambda \to \mathbb{D}$ [Pom92, Corollary 2.17]. We thus say that radial limits exist at $p$ or do not exist at $p$. In particular, radial limits exist at degenerate prime ends $p$. We will often restrict our consideration to prime ends $p$ with radial limits, and we will then with a slight abuse of notation also denote the radial limit point in $\mathbb{C}$ by $p$.

3.2.3. Carathéodory convergence of domains. The notion of domain approximations that we use will be Carathéodory convergence. Let $(\Lambda_n)_{n \in \mathbb{N}}$ and $\Lambda$ be simply-connected open sets $\Lambda, \Lambda_n \subseteq \mathbb{C}$, all containing a common point $u$. We say that $\Lambda_n \to \Lambda$ in the sense of kernel convergence with respect to $u$ if

i) every $z \in \Lambda$ has some neighbourhood $V_z$ such that $V_z \subset \Lambda_n$ for all large enough $n$; and

ii) for every point $p \in \partial \Lambda$, there exists a sequence $p_n \in \partial \Lambda_n$ such that $p_n \to p$.

Let $\phi_n$ be the Riemann uniformization maps from $\Lambda_n$ to $\mathbb{D}$ normalized at $u$, i.e., $\phi_n(u) = 0$ and $\phi_n'(u) > 0$. Let $\phi$ be the Riemann uniformization map from $\Lambda$ to $\mathbb{D}$. The kernel convergence $\Lambda_n \to \Lambda$ with respect to $u$ holds if and only if the inverses $\phi_n^{-1}$ converge uniformly on compact subsets of $\mathbb{D}$ to $\phi^{-1}$ [Pom92, Theorem 1.8]. Then, also $\phi_n \to \phi$ uniformly on compact subsets of $\Lambda$.

It is easy to see that if $\Lambda_n \to \Lambda$ in the sense of kernel convergence with respect to $u$, then the same convergence holds with respect to any $\hat{u} \in \Lambda$, taking the tail of the sequence $\Lambda_n$ if needed. We then say that $\Lambda_n \to \Lambda$ in the Carathéodory sense as $n \to \infty$, or that $\Lambda_n$ are Carathéodory approximations of $\Lambda$. Working from the point of view of uniformization maps, this relates to the following elementary lemma, whose proof we leave to the reader.

Lemma 3.1. $\Lambda_n \to \Lambda$ in the Carathéodory sense if and only if there exist some conformal maps $\phi_n : \Lambda_n \to \mathbb{D}$ and $\phi : \Lambda \to \mathbb{D}$ such that $\phi_n^{-1}$ converge uniformly on compact subsets of $\mathbb{D}$ to $\phi^{-1}$.

For domains with marked prime ends, we say $(\Lambda_n; p_1^{(n)}, \ldots, p_m^{(n)}) \to (\Lambda; p_1, \ldots, p_m)$ in the Carathéodory sense as $n \to \infty$, if $\Lambda_n \to \Lambda$ in the Carathéodory sense and $(\hat{\phi}_n(p_1^{(n)}), \ldots, \hat{\phi}_n(p_m^{(n)})) \to (\hat{\phi}(p_1), \ldots, \hat{\phi}(p_m))$ as $n \to \infty$, where $\hat{\phi}_n$ and $\hat{\phi}$ are the induced maps of prime ends.
3.2.4. Close approximations of prime ends with radial limits. A Carathéodory approximation of domains \((\Lambda_n; p_1^{(n)}, \ldots, p_m^{(n)}) \to (\Lambda; p_1, \ldots, p_m)\) allows wild behaviour of the boundaries at the marked prime ends. We wish to consider compact curves ending at these prime ends. For such compact curves to exist at all, the prime ends \(p_1^{(n)}, \ldots, p_m^{(n)}\) and \(p_1, \ldots, p_m\) must possess radial limits. Furthermore, to avoid bad boundary approximations, we need to restrict to close Carathéodory approximations. Informally, being a close approximation means that a chordal curve in \(\Lambda_n\) starting from \(p_1^{(n)}\) is not forced to wiggle a macroscopic distance to enter into \(\Lambda_n \cap \Lambda\). This concept was introduced by the author in [Kar18], and we repeat the definition below.

Assume that \((\Lambda_n; p_1^{(n)}, \ldots, p_m^{(n)}) \to (\Lambda; p_1, \ldots, p_m)\) in the Carathéodory sense, and that the radial limits exist at the prime ends \(p_1^{(n)}, \ldots, p_m^{(n)}\) and \(p_1, \ldots, p_m\) of the respective domains. We say that \(p_1^{(n)}\) are close approximations of a prime end \(p_1\), if \(p_1^{(n)} \to p_1\) as \(n \to \infty\) (as points in \(\mathbb{C}\)), and in addition the following holds: for any \(r > 0\), \(r < d(p_1, u)\) (where \(u\) denotes the reference point of the approximation \(\Lambda_n \to \Lambda\)), denote by \(S_r\) the connected component of \(\partial B(p_1, r)\) disconnecting \(p_1\) from \(u\) in \(\Lambda\) that lies innermost, i.e., closest to \(p_1\) in \(\Lambda\). Such a component exists by the existence of radial limits at the prime end \(p_1\). Let \(w_r \in S_r\) be any fixed reference point; the precise choice makes no difference. Now, \(p_1^{(n)}\) are close approximations of \(p_1\) if for any fixed \(0 < r < d(p_1, u)\), taking a large enough \(n\), \(p_1^{(n)}\) is connected to \(w_r\) inside \(\Lambda_n \cap B(p_1, r)\).

3.3. The different metric spaces. In this subsection, we introduce the different metric spaces. All notations and conventions are identical to the previous paper of the author [Kar18].

3.3.1. Space of plane curves modulo reparametrization. A planar curve is a continuous function \(\gamma : [0, 1] \to \mathbb{C}\). Define an equivalence relation \(\sim\) on curves: \(\gamma \sim \hat{\gamma}\) if

\[
\inf_{\psi} \left\{ \sup_{t \in [0,1]} |\gamma(t) - \hat{\gamma} \circ \psi(t)| \right\} = 0,
\]

where the infimum is taken over all reparametrizations (continuous increasing bijections) \(\psi : [0, 1] \to [0, 1]\). The space of curves modulo this equivalence relation is denoted by \(X(\mathbb{C})\).

We equip \(X(\mathbb{C})\) with the following metric. For two curves \(\gamma, \hat{\gamma}\) the distance between their equivalence classes \([\gamma]\) and \([\hat{\gamma}]\) in this metric is

\[
(3.1) \quad d([\gamma], [\hat{\gamma}]) = \inf_{\psi} \left\{ \sup_{t \in [0,1]} |\gamma(t) - \hat{\gamma} \circ \psi(t)| \right\},
\]

where the infimum is taken over all reparametrizations \(\psi\). The closed subset of \(X(\mathbb{C})\) consisting of curves that stay in \(\overline{\mathbb{D}}\) is denoted by \(X(\overline{\mathbb{D}})\). The spaces \(X(\mathbb{C})\) and \(X(\overline{\mathbb{D}})\) are both complete and separable. We will in this paper only study curves \(\gamma\) via the space \(X(\mathbb{C})\). As there is thus no danger of confusion, we will denote the equivalence class \([\gamma]\) by \(\gamma\) for short.

The space \(X(\mathbb{C})^N\) of collections of \(N\) curves modulo reparametrization is equipped with the metric

\[
(3.2) \quad d((\gamma_1, \ldots, \gamma_N), (\hat{\gamma}_1, \ldots, \hat{\gamma}_N)) = \max_{1 \leq i \leq N} d(\gamma_i, \hat{\gamma}_i),
\]

where the distance on the right-hand side is given by \((3.1)\). This space is complete and separable, too.

3.3.2. Space of continuous functions. We equip the space \(C\) of continuous functions \(W : \mathbb{R}_{\geq 0} \to \mathbb{R}\) with the metric of uniform convergence over compact subsets

\[
(3.2) \quad d(W, \hat{W}) = \sum_{n \in \mathbb{N}} 2^{-n} \min\{1, \sup_{t \in [0,n]} |\hat{W}_t - W_t|\}.
\]
The space $C$ is then complete and separable. The space $C^N$ of collections of continuous functions will be equipped with the metric
\[ d((W_1, \ldots, W_N), (\tilde{W}_1, \ldots, \tilde{W}_N)) = \max_{1 \leq i \leq N} d(W_i, \tilde{W}_i), \]
where the distance on the right-hand side is given by (3.2). This space is complete and separable, too.

4. Precompactness theorems

4.1. The main precompactness theorem.

4.1.1. Setup and notation. The setup and notation for the main theorem 4.1 of this subsection is the following.

Let $(\Gamma_n)_{n \in \mathbb{N}}$ be a sequence of lattices, and $(G_n; e_1^{(n)}, \ldots, e_{2N}^{(n)})$ for each $n$, simply-connected subgraphs of $\Gamma_n$, with $N$ giving the number of boundary edges fixed. Assume that for each $n$, we have a discrete random curve model on $\Gamma_n$, defined on the subgraph $(G_n; e_1^{(n)}, \ldots, e_{2N}^{(n)})$. Denote the measures with random curves on $(G_n; e_1^{(n)}, \ldots, e_{2N}^{(n)})$ by $(\mathbb{P}(G_n; e_1^{(n)}, \ldots, e_{2N}^{(n)}; \gamma_{G_n1}, \ldots, \gamma_{G_n2N})) = (\mathbb{P}^{(n)}; (\gamma_1^{(n)}, \ldots, \gamma_{2N}^{(n)}))$.

Let $\Lambda_n = \Lambda_{G_n}$ be the simply-connected domains corresponding to $G_n$, and let $p_1^{(n)}, \ldots, p_{2N}^{(n)}$ be the prime ends of $\Lambda_n$ where the edges $e_1^{(n)}, \ldots, e_{2N}^{(n)}$, respectively, land. Assume that $(\Lambda_n; p_1^{(n)}, \ldots, p_{2N}^{(n)})$ are close Carathéodory approximations of a domain $(\Lambda; p_1, \ldots, p_{2N})$ with marked prime ends where radial limits exist. (The limiting prime ends need not be distinct for the statement and proof of Theorem 4.1, but they will be in all the applications in this paper.) Let $\phi_n : \Lambda_n \to \mathbb{D}$ and $\phi : \Lambda \to \mathbb{D}$ be any conformal maps such that $\phi_n^{-1} \to \phi^{-1}$ uniformly over compact subsets of $\mathbb{D}$. Denote $(\gamma_1^{(n)}; \ldots, \gamma_{2N}^{(n)}) = (\phi_n(\gamma_1^{(n)}), \ldots, \phi_n(\gamma_{2N}^{(n)})) \in X(\mathbb{D})^N$. We also assume that $\Lambda_n$ are uniformly bounded.

Fix a point $\tilde{p}_\infty \in \partial \mathbb{D}$ on the counterclockwise arc of $\partial \mathbb{D}$ from $\tilde{p}_{2N} = \phi(p_{2N})$ to $\tilde{p}_1 = \phi(p_1)$, and a conformal map $\psi$ taking $(\mathbb{D}; \tilde{p}_\infty)$ to $(\mathbb{H}, \infty)$. Let $U_j$ be a localizations neighbourhoods of $\tilde{p}_j$ in $\mathbb{D}$, for each $j$, i.e., $U_j$ only contain the marked boundary point $\tilde{p}_j$ and $\psi(U_j)$ are compact $\mathbb{H}$-hulls. (The neighbours need not be disjoint.) For each $1 \leq j \leq 2N$, denote by $\lambda_j^{(n)}$ the initial segment of the one of the random curves $\gamma_{\mathbb{D};1}, \ldots, \gamma_{\mathbb{D};2N}$ adjacent to $p_j^{(n)}$, started from that boundary point, up to the continuous modification $\tau_j^{(n)}$ of the hitting time $T_j^{(n)}$ of $\overline{(\mathbb{D} \setminus U_j)}$ by the curve. Let $W_j^{(n)}$ be the driving function of the curve $\psi(\lambda_j^{(n)})$ in $\mathbb{H}$, stopped at the half-plane capacity corresponding to $\tau_j^{(n)}$.

Conditional discrete random curve models are studied in an identical notation, with the only difference that also the link pattern $\alpha \in \mathbb{P}N$ is fixed, and $\mathbb{P}^{(n)}$ then denotes the corresponding conditional measures, $\mathbb{P}^{(n)}[\cdot] = \mathbb{P}(G_n; e_1^{(n)}, \ldots, e_{2N}^{(n)}; \cdot | \alpha]$.

4.1.2. Statement of the theorem. We now state the main theorem of this section, giving the precompactness results needed for convergence proofs to local-to-global multiple SLEs. Analogues of this result for models with only one curve have been given in [KS17, Kar18], see also [AB99, Wu18].

**Theorem 4.1.** Consider the setup and notation of Section 4.1.1 for discrete random curve models (resp. conditional discrete random curve models). Assume that the discrete curve models on $\Gamma_n$ (resp. on which we impose the conditioning) have alternating boundary conditions and satisfy the DDMP. Assume in addition that the collection of one-curve measures $\mathbb{P}^{(G, e_1, e_2)}$ in these random curve models, indexed by $n$ and simply-connected subgraphs $G$ of $\Gamma_n$, satisfy the equivalent conditions (C) and (G), as defined below. Then, the following hold:

(A) The measures $\mathbb{P}^{(n)}$ are precompact in the following senses:

i) as laws of the collections of curves $(\gamma_1^{(n)}, \ldots, \gamma_N^{(n)})$ on the space $X(C)^N$;

ii) as laws of the collections of curves $(\gamma_{\mathbb{D};1}^{(n)}, \ldots, \gamma_{\mathbb{D};N}^{(n)})$ on the space $X(\mathbb{D})^N \subset X(C)^N$;
iii) as laws of the curves $\gamma_j^{(n)}$ on the space $X(\overline{\mathbb{D}}) \subset X(\mathbb{C})$, for any $j$; and
iv) as laws of the driving functions $W_j^{(n)}$ on the space $\mathcal{C}$, for any $j$.

In other words, there exist subsequences $(n_k)_{k \in \mathbb{N}}$ such that the random objects above converge weakly.

B) For a subsequence $(n_k)_{k \in \mathbb{N}}$, a weak convergence takes place in topology (i), $(\gamma_1^{(n_k)}, \ldots, \gamma_N^{(n_k)}) \to (\gamma_1, \ldots, \gamma_N)$, if and only if it takes place in topology (ii), $(\gamma_1^{(n_k)}, \ldots, \gamma_N^{(n_k)}) \to (\gamma_1, \ldots, \gamma_N)$. Furthermore, we then have the equality
\[(\gamma_1, \ldots, \gamma_N) \overset{(d)}{=} (\phi^{-1}(\gamma_1), \ldots, \phi^{-1}(\gamma_N))\]
in distribution\footnote{More precisely, the random variable $(\phi^{-1}(\gamma_1), \ldots, \phi^{-1}(\gamma_N))$ in $X(\mathbb{C})^N$ denotes the following: the map $\phi^{-1}$, as extended by radial limits to $\partial\mathbb{D}$ whenever possible, is almost surely defined on all points of the curves $\gamma_i$, $1 \leq i \leq N$. Picking a parametrization of the curves $\gamma_i : [0, 1] \to \mathbb{C}$, the functions $t \mapsto \phi^{-1}(\gamma_i(t))$ are almost surely curves. The collection of curves $(\phi^{-1}(\gamma_1), \ldots, \phi^{-1}(\gamma_N))$, as an element of $X(\mathbb{C})^N$, is almost surely equal to an $X(\mathbb{C})^N$-valued random variable measurable with respect to the sigma algebra of $X(\mathbb{C})^N \subset X(\mathbb{C})$. This $X(\mathbb{C})^N$-valued random variable is denoted, slightly abusively, by $(\phi^{-1}(\gamma_1), \ldots, \phi^{-1}(\gamma_N))$ in the statement.}
and $(\gamma_1, \ldots, \gamma_N)$ has the unique distribution on $X(\overline{\mathbb{D}})^N$ satisfying this equality.

C) For a subsequence $(n_k)_{k \in \mathbb{N}}$, a weak convergence takes place in topology (iii), $\lambda_j^{(n_k)} \to \lambda_j$, if and only if it takes place in topology (iv), $W_j^{(n_k)} \to W_j$. Furthermore, $\lambda_j$ and $W_j$ are then Loewner transforms of each other\footnote{More precisely, the curve $\lambda_j \in X(\overline{\mathbb{D}})$ almost surely has a Loewner transform, and the Loewner driving function obtained from this transform is almost surely equal to a $\mathcal{C}$-valued random variable measurable with respect to the sigma algebra of $\lambda_j \in X(\mathbb{C})$. This random variable is in distribution equal to $W_j$. Conversely, the driving function $W_j$ almost surely has a Loewner transform curve, and the curve in $X(\overline{\mathbb{D}})$ obtained from this transform is almost surely equal to an $X(\overline{\mathbb{D}})$-valued random variable measurable with respect to the sigma algebra of $W_j \in \mathcal{C}$. This random variable is in distribution equal to $\lambda_j$.}

D) If the weak convergences of part (B) above takes place, then do the weak convergences of part (C).

Informally speaking, Theorem 4.1 above proves two commutative diagrams:
\[
\begin{array}{ccc}
(\gamma_1^{(n)}, \ldots, \gamma_N^{(n)}) & \overset{\text{conformal}}{\longrightarrow} & (\gamma_1^{(n)}, \ldots, \gamma_N^{(n)}) \\
\downarrow n \to \infty & & \downarrow n \to \infty \\
(\gamma_1, \ldots, \gamma_N) & \overset{\text{conformal}}{\longrightarrow} & (\gamma_1, \ldots, \gamma_N)
\end{array}
\]
and
\[
\begin{array}{ccc}
\lambda_j^{(n)} & \overset{\text{Loewner}}{\longrightarrow} & W_j^{(n)} \\
\downarrow n \to \infty & & \downarrow n \to \infty \\
\lambda_j & \overset{\text{Loewner}}{\longrightarrow} & W_j
\end{array}
\]

Remark 4.2. The assumptions that the limiting prime ends $p_1, \ldots, p_{2N}$ possess radial limits and that $p_1^{(n)}, \ldots, p_{2N}^{(n)}$ are their close approximations are only needed in order to study the curves $(\gamma_1^{(n)}, \ldots, \gamma_N^{(n)})$ in the natural planar topology (i). Removing these assumptions, statements (A)(ii)–(iv) and (C) still hold. Also (D) holds, with the modification that “weak convergences of part (B)” should be replaced with “weak convergence in topology (ii)”.

4.1.3. Hypotheses of the theorem. The hypotheses of Theorem 4.1 i.e., the equivalent conditions (C) and (G), are the well-established crossing conditions of Kemppainen and Smirnov [KS17]. The same hypotheses will later be used in Theorem 4.4 where we recall some prior results from [KS17] and [Kar18]. The latter, and hence conditions (C) and (G) below, are given in a more general setup with measures.
$P(n)$ with random curves $\gamma^{(n)}$ on some simply-connected planar graphs $(G_n; e_1^{(n)}, e_2^{(n)})$. These measures need not originate in a random curve model on a lattice $\Gamma_n$ (and in particular not a DDMP model).

Both conditions (C) and (G) require the following filtrations: consider $\gamma^{(n)}$ as a path on the graph $G^{(n)}$. Let $F^{(n)}_m$ be the sigma algebras generated by the $m$ first vertices of $\gamma^{(n)}$. We call $(F^{(n)}_1, F^{(n)}_2, \ldots)$ the filtration of the path $\gamma^{(n)}$. We denote by $T_2^{(n)}$ the ending time of the path $\gamma^{(n)}$, i.e., the time when $\gamma^{(n)}$ uses the edge $e_2^{(n)}$.

Let us start with condition (G). Let $0 < r < R$. Denote open annuli by,

$$A(z, r, R) = B(z, R) \setminus B(z, r).$$

Let $\Lambda_G$ be a simply-connected domain. (In our case, there is always an underlying planar graph $G$, hence the notation.) We say that an annulus $A(z, r, R)$ is on the boundary of a simply-connected domain $\Lambda_G$ if $B(z, r) \cap \partial \Lambda_G \neq \emptyset$. Let $p_1^{(G)}$ and $p_2^{(G)}$ be prime ends of $\Lambda_G$. A chordal curve $\gamma_G$ from $p_1^{(G)}$ to $p_2^{(G)}$ in $\Lambda_G$ makes an unforced crossing of $A(z, r, R)$ if for some connected component $C$ of $A(z, r, R) \cap \Lambda_G$ which does not disconnect $p_1^{(G)}$ from $p_2^{(G)}$ in $\Lambda_G$, there exists a subinterval $[t_0, t_1] \subset [0, 1]$ such that $\gamma_G([t_0, t_1])$ intersects both connected components of $C \setminus A(z, r, R)$, but for $t \in (t_0, t_1)$ we have $\gamma(t) \in C$.

Condition (G): We say that the measures $P(n)$ with random curves $\gamma^{(n)}$ satisfy condition (G) if for all $\varepsilon > 0$ there exists $M > 0$, independent of $n$, such that the following holds for all stopping times $u^{(n)} < T_2^{(n)}$ with respect to the filtrations of the paths $\gamma^{(n)}$: for any annulus $A(z, r, R)$ with $R/r \geq M$ on the boundary of $\Lambda_n \setminus \gamma^{(n)}([0, u^{(n)})])$, we have

$$P(n)[\gamma^{(n)}([u^{(n)}, T_2^{(n)}]) \text{ makes a crossing of } A(z, r, R) \text{ unforced in } \Lambda_n \setminus \gamma^{(n)}([0, u^{(n)})]) \mid F^{(n)}_u] \leq \varepsilon.$$

Let us now work towards condition (C). A topological quadrilateral $(Q; S_0, S_1, S_2, S_3)$ consists of a planar domain $Q$ homeomorphic to a square, and arcs $S_0, S_1, S_2, S_3$ of its boundary, indexed counterclockwise, that correspond to the closed edges of the square under the homeomorphism. There is a one-parameter family of classes of conformally equivalent topological quadrilaterals with labelled sides, and the equivalence class of $(Q; S_0, S_1, S_2, S_3)$ is captured by the modulus $m(Q)$. It is the unique $L > 0$ such that there exists a biholomorphism between $Q$ and the rectangle $(0, L) \times (0, 1)$, so that the sides $S_0, S_1, S_2, S_3$ of $Q$ correspond to the edges of the rectangle, and $S_0$ to $\{0\} \times [0, 1]$. (There is an alternative terminology and notation: $m(Q)$ is the extremal distance $d_Q(S_0, S_2)$ of $S_0$ and $S_2$ in $Q$, see, e.g., [Ahlfors 1973 Chapter 4].)

Let $\Lambda_G$ be a simply-connected planar domain. We say that a topological quadrilateral $(Q; S_0, S_1, S_2, S_3)$ is on the boundary of $\Lambda_G$, if $Q \subset \Lambda_G$ and $S_1, S_3 \subset \partial \Lambda_G$, while $S_0$ and $S_2$ lie inside $\Lambda_G$, except for their end points. Let $p_1^{(G)}$ and $p_2^{(G)}$ be prime ends of $\Lambda_G$. A chordal curve $\gamma_G$ from $p_1^{(G)}$ to $p_2^{(G)}$ in $\Lambda_G$ is said to make a crossing of $Q$ if there is a subinterval $[t_0, t_1] \subset [0, 1]$ such that $\gamma_G([t_0, t_1])$ intersects both $S_0$ and $S_2$, but for $t \in (t_0, t_1)$ we have $\gamma_G(t) \in Q$. The crossing is unforced if $Q$ does not disconnect $p_1^{(G)}$ from $p_2^{(G)}$ in $\Lambda_G$.

Condition (C): We say that the measures $P(n)$ with random curves $\gamma^{(n)}$ satisfy condition (C) if for all $\varepsilon > 0$ there exists $M > 0$, independent of $n$, such that the following holds for all stopping times $u^{(n)} < T_2^{(n)}$ with respect to the filtrations of the paths $\gamma^{(n)}$: for any topological quadrilateral $Q$ with $m(Q) \geq M$ on the boundary of $\Lambda_n \setminus \gamma^{(n)}([0, u^{(n)})])$, we have

$$P(n)[\gamma^{(n)}([u^{(n)}, T_2^{(n)}]) \text{ makes a crossing of } Q \text{ unforced in } \Lambda_n \setminus \gamma^{(n)}([0, u^{(n)})]) \mid F^{(n)}_u] \leq \varepsilon.$$

Remark 4.3. If the measures $P(n)$ with random curves $\gamma^{(n)}$, for each $n$, originate in random curve models on $\Gamma_n$ with the DDMP, we know that conditioning on an initial segment $\gamma^{(n)}([0, u^{(n)})])$ is equivalent to reducing the graph $G^{(n)}$ by that segment. Thus, we may assume that $u^{(n)} = 0$ in the conditions above, with the cost that in stead of merely the graphs $(G_n; e_1^{(n)}, e_2^{(n)})$, we will have to consider all simply-connected subgraphs $(G; e_1, e_2)$ of the lattices $\Gamma_n$ that may appear as such reduced graphs.

4.2. Proof of Theorem 4.1
4.2.1. **An analogous theorem for \( N = 1 \) curve.** The proof of Theorem 4.1 relies heavily on the analogue of that theorem for \( N = 1 \) curves, given in [KS17] and [Kar18]. To state this analogue, consider the setup described in Section 4.1.1 with \( N = 1 \), and omitting the assumption that the measures with random curve \((\mathbb{P}^{(n)}, \gamma)\) on \((\mathbb{G}_n; e_1^n, e_2^n)\) originate in some random curve model. Note that the choice of conformal maps \( \phi_n \) is free in Section 4.1.1 as long as they converge. In the special case of the conformal maps \( \phi_n \) chosen so that in addition \((\Lambda_n; p_1^{(n)}, p_2^{(n)})\) maps to \((-1, 1)\), denote by \( \gamma^{(n)} = \phi_n(\gamma^{(n)}) \) the curves from \(-1, 1\) in \( \mathbb{D} \). Denote by \( V^{(n)} \) the Loewner driving functions of the curves \( \gamma^{(n)} \) (where a conformal map \((\mathbb{D}; -1, 1) \rightarrow (\mathbb{H}; 0, \infty)\) is fixed independent of \( n \)).

**Theorem 4.4.** ([KS17] Theorems 1.5 and 1.7 and [Kar18] Theorem 4.4 and Proposition 4.7) In the setup and notation given above, suppose that the measures with random curves \((\mathbb{P}^{(n)}, \gamma^{(n)})\) satisfy the equivalent conditions (C) and (G). Then the following hold:

**A)** The measures \( \mathbb{P}^{(n)} \) are precompact in the following senses:

i) as laws of the curves \( \gamma^{(n)} \) on the space \( X(C) \);

ii) as laws of the curves \( \gamma^{(n)}_D \) (or \( \gamma^{(n)}_D \), obtained with any converging conformal maps) on the space \( X(\mathbb{D}) \);

iii) as laws of the driving functions \( V^{(n)} \) on the space \( C \).

**B)** If for some subsequence \((n_k)_{k \in \mathbb{N}}\) weak convergence takes place in one of the topologies above, it also takes place in the two other ones. Furthermore, denoting the respective weak limits by \( \gamma, \gamma_D, \gamma_D \), and \( V \), it holds that \( \gamma_D \) and \( V \) are Loewner transforms of each other, while \( \gamma \) and \( \gamma_D \) satisfy

\[
\gamma^{(d)} = \phi^{-1}(\gamma_D),
\]

and \( \gamma_D \) has the unique distribution on \( X(\mathbb{D}) \) satisfying this.

The statements that \( \gamma_D \) and \( V \) are Loewner transforms of each other and \( \gamma^{(d)} = \phi^{-1}(\gamma_D) \) are formally interpreted as in Theorem 4.1. Taking the conformal maps so that \( \gamma_D = \gamma_D \), this theorem can be summarized in the commutative diagram

\[
\begin{array}{ccc}
\gamma^{(n)} & \xrightarrow{\text{conformal}} & \gamma^{(n)}_D \\
\downarrow & & \downarrow \\
\gamma & \xrightarrow{\text{conformal}} & \gamma_D
\end{array}
\]

\[
\begin{array}{ccc}
\gamma^{(n)} & \xrightarrow{\text{Loewner}} & V^{(n)} \\
\downarrow & & \downarrow \\
\gamma & \xrightarrow{\text{Loewner}} & V.
\end{array}
\]

4.2.2. **One-curve marginals for general \( N \).** To prove Theorem 4.1 our strategy is based on establishing Theorem 4.4 for the one-curve marginal laws. We thus start with an analogue of condition (G) for multiple curves. Note that the assumptions of the below lemma hold in the setup of Theorem 4.1.

**Lemma 4.5.** Let \( \Gamma_n \) be a sequence of lattices, and assume that we have, for each \( n \in \mathbb{N} \), a discrete random curve model \((\mathbb{P}^{[\Gamma_n]}; \gamma_{1:n}, \ldots, \gamma_{N:n})\) on some simply-connected subgraphs \( \mathcal{G} \) of \( \Gamma_n \). Assume that these models have alternating boundary conditions and satisfy the DDMP. Fix \( N \in \mathbb{N} \) and \( \alpha \in \mathbb{L}_N \), and from these symmetric curve models, extract the collection of conditioned measures \( \mathbb{P}^{[\Gamma_n]} \{ \alpha \} = \mathbb{P}^{[\Gamma_n]} \{ \gamma_{1:n}, \ldots, \gamma_{2N} \}, \ldots \} with random curves \( \{ \gamma_{1:n}, \ldots, \gamma_{2N} \} \), indexed by \( n \) and the simply-connected subgraphs \( \mathcal{G} \) of \( \Gamma_n \) on which these measures make sense. This collection of measures with random curves satisfies the condition (multi-G) below.

**Condition (multi-G):** We say that a collection of measures with random curves \( \{ \mathbb{P}^{[\Gamma_n]} \{ \alpha \} \}_{\alpha} \), where the link pattern formed by the curves \( \gamma_{1:n}, \ldots, \gamma_{2N:n} \) is always \( \alpha \), satisfies condition (multi-G) if for all \( \varepsilon > 0 \) there exists \( M > 0 \) such that the following holds: for any annulus \( A(z, r, R) \) with \( R/r \geq M \) on the boundary of the simply-connected domain corresponding to \( \mathcal{G} \),

\[
\mathbb{P}^{[\Gamma_n]} \{ \gamma_{1:j} \text{ makes an unforced crossing of } A(z, r, R) \} \leq \varepsilon.
\]
Condition (multi-G) is a direct analogue of condition (G) in the case $\tau = 0$ for multiple curves. Note that we need to fix the link pattern $\alpha$ in order to be able to talk about forced and unforced crossings of $\gamma^r_j$.

**Proof of Lemma 4.5.** We prove the proposition by induction on $N$. In the base case $N = 1$ condition (multi-G) becomes simply condition (G) with $\tau = 0$, and holds by assumption. Assume now that the claim holds for each number of curves $\ell = 1, \ldots, N$ and any link patterns with that number of curves, with some $M = M(\ell, \varepsilon)$. (We may assume that $M(\ell, \varepsilon)$ does not depend on the link pattern since, for any $\ell$, there are finitely many link patterns.) Let us study the model with $N + 1$ curves, and fix an index $j \in \{1, \ldots, N + 1\}$ of the considered curve. To satisfy (4.4) in Condition (multi-G) it clearly suffices to show that for any $j$

\begin{equation}
(4.5)
\quad p =: [\gamma^r_j \text{ makes an unforced crossing of } A(z, r, R)] \leq \varepsilon/(N + 1).
\end{equation}

We claim that (4.5) holds when we choose $M(N + 1, \varepsilon) = M(N, \varepsilon/(4N + 4))M(1, \varepsilon/(4N + 4))$. For the rest of this proof, let us fix the index $n$ of our random curve model and the subgraph $(G; e_1, \ldots, e_{2N+2})$ of $\Gamma_n$, and show that $p \leq \varepsilon/(N + 1)$ irrespective of these choices. We will also drop all subscripts $G$ for short.

Notice first that the curve $\gamma_j$ lies in the connected component $\Lambda'$ of $\Lambda \setminus \{\gamma_1, \ldots, \gamma_{2j-2}, \gamma_{2j+1}, \ldots, \gamma_{2N}\}$ containing the $j$:th odd-index edge $e_{2j-1}$. Let us denote the corresponding simply-connected subgraph of $G$ by $G'$, and the two marked boundary edges left in that graph by $e'_1, e'_2$, where $e'_1 = e_{2j-1}$ is the odd one. By the DDMP, when we condition on the remaining curves $(\gamma_1, \ldots, \gamma_{2j-2}, \gamma_{2j+1}, \ldots, \gamma_{2N})$, $\gamma_j$ is in distribution equal to $\gamma^r_j$ from the pair $(G', e'_1, e'_2; \gamma^r_j)$ in our random curve model on $\Gamma_n$.

Consider now the event in (4.5) that $\gamma_j$ makes an unforced crossing of $A(z, r, R)$. Let us first study the case that $\gamma_j$ makes this unforced crossing from the outside of the annulus $A(z, r, R)$. (Formally, $\partial B(z, R)$ separates the crossed component $C$ of $A(z, r, R)$ in $\Lambda$ from the end points of $\gamma_j$.) Denote the probability of such an unforced crossing from outside by $p'$. Divide $A(z, r, R)$ into boundary annuli $A(z, r, r')$ and $A(z, r', R)$, where $r'/r = M(1, \varepsilon/(4N + 4))$ and $R/r' = M(N, \varepsilon/(4N + 4))$. A crossing of $C$ includes a crossing of the inner subannulus $A(z, r, r')$. At least one of the following two thus has to occur:

i) there is a component $C'$ of $A(z, r, r')$ in $\Lambda'$, with $C' \subset C$ for some non-disconnecting component of $A(z, r, R)$ in $\Lambda$, such that $C'$ is disconnecting in $\Lambda'$.

Case (i) occurs with probability $\leq \varepsilon/(4N + 4)$, by the conditional law of $\gamma_j$ deduced above. In case (ii), recall that $C$ is separated from the end points of $\gamma_j$ in $\Lambda$, by $\partial B(z, R)$. It follows that $C'$ is separated from the end points of $\gamma_j$ in $\Lambda$, and hence also in $\Lambda'$, by $\partial B(z, r')$. On the other hand, $C'$ is disconnecting in $\Lambda'$ if and only if both the clockwise and counterclockwise boundary arcs of $\partial \Lambda'$ from $e'_1$ to $e'_2$ touch $C'$. Likewise, since $C$ does not disconnect $e'_1$ from $e'_2$ in $\Lambda$, we know that one of the arcs of $\partial \Lambda$ from $e'_1$ to $e'_2$, say for definiteness the clockwise one, does not touch $C$ and is thus separated from $C$ by $\partial B(z, R)$. In other words, for $C'$ to be disconnecting in $\Lambda'$, one of the remaining curves $\gamma_i$, $i \neq j$, starting and ending on the clockwise arc of $\partial \Lambda$, has to cross $\partial B(z, R)$, then enter $C$ and enter it deep enough to touch $\partial B(z, r')$, and finally touch $C'$. In particular, this curve $\gamma_i$ crosses the annulus $A(z, r', R)$ inside $C$. Now, study the component $\Lambda''$ of $\Lambda \setminus \gamma_j$ containing $\gamma_i$, and let $C'' \subset C$ be the component of $A(z, r', R)$ in $\Lambda''$ containing a crossing of $\gamma_i$ (if there are several, pick one). We claim that the crossing of $C''$ by $\gamma_i$ is unforced in $\Lambda''$: indeed, the clockwise boundary of $\Lambda''$ between the end points of $\gamma_i$ is contained in that of $\Lambda$ between the end points of $\gamma_j$, and we already know that the latter does not touch $C$. Thus, the small clockwise boundary arc of $\Lambda''$ between the end points of $\gamma_i$ does not touch the smaller set $C''$, so the crossing of $C''$ is unforced in $\Lambda''$. This holds for any $C$, and $C''$ is always a connected component of the same annulus. Thus, by the DDMP, such an unforced crossing by $\gamma_i$, for some $i$, occurs with probability $\leq M(N, \varepsilon/(4N + 4))$. 


Finally, summing up the contributions of cases (i) and (ii) above, we notice that \( p' \leq 2\varepsilon/(4N + 4) \). Crossings of \( A(z, r, R) \) from the inside are treated similarly. We thus obtain
\[
p \leq 2\varepsilon/(4N + 4) + 2\varepsilon/(4N + 4) = \varepsilon/(N + 1),
\]
as required. \( \square \)

The lemma above allows us to apply Theorem 4.4 for the one-curve marginals of the random curve collections in Theorem 4.1.

**Corollary 4.6.** Consider the setup of Theorem 4.1 in the version where the measures \( \mathbb{P}^{(n)} \) are those conditional on a fixed link pattern \( \alpha \in \text{LP}_N \), and let the assumptions in that theorem hold. Then, the measures with random curves \( (\mathbb{P}^{(n)}, \gamma_j^{(n)}) \), for any fixed \( j \), satisfy condition (G), and hence all the consequences of Theorem 4.4.

**Proof.** Condition (G) for the curve \( \gamma_j^{(n)} \) follows immediately by combining Remark 4.3 and condition (multi-G) obtained in Lemma 4.5. \( \square \)

### 4.2.3. Proofs of the statements about curves.

We can now rather straightforwardly prove the statements of Theorem 4.1 that only employ random variables in the spaces of curves \( X(\mathbb{C}) \) and \( X(\mathbb{R}) \).

**Proof of Theorem 4.1(A)(i)–(iii).** Let us first prove part (i). Consider first Theorem 4.4 with the measures \( \mathbb{P}^{(n)} \) being those conditional on a fixed link pattern \( \alpha \in \text{LP}_N \). Recall that by Prohorov’s theorem, tightness and precompactness are equivalent for measures on Polish spaces (i.e., complete separable metric spaces). Thus, by Corollary 4.6 on the measures with random curves \( (\mathbb{P}^{(n)}, \gamma_j^{(n)}) \) and Theorem 4.4(A)(i), we know that \( \mathbb{P}^{(n)} \) are tight as laws of the random curves \( \gamma_j^{(n)} \). In other words, for all \( \varepsilon > 0 \), there exists a (sequentially) compact set \( K^{(j)}_\varepsilon \subset X(\mathbb{C}) \) such that \( \mathbb{P}^{(n)}(\gamma_j^{(n)} \in K^{(j)}_\varepsilon) \geq 1 - \varepsilon/N \) for all \( n \). This holds for all \( 1 \leq j \leq N \).

Next, take for all \( 1 \leq j \leq N \) the sets \( K^{(j)}_\varepsilon \subset X(\mathbb{C}) \) as above. Their product set \( K^{(1)}_\varepsilon \times \ldots \times K^{(N)}_\varepsilon \subset X(\mathbb{C})^N \) is a (sequentially) compact set in the space \( X(\mathbb{C})^N \). Furthermore, it clearly holds that
\[
\mathbb{P}^{(n)}([\gamma_1^{(n)}, \ldots, \gamma_N^{(n)}]) \in K^{(1)}_\varepsilon \times \ldots \times K^{(N)}_\varepsilon \geq 1 - \varepsilon.
\]
Thus, the measures \( \mathbb{P}^{(n)} \) are tight in topology (i). Using Prohorov’s theorem to the converse direction, we deduce that they are precompact. This proves Theorem 4.1(A)(i) for measures \( \mathbb{P}^{(n)} \) conditional on a fixed link pattern \( \alpha \in \text{LP}_N \).

For the symmetric measures \( \mathbb{P}^{(n)}(\cdot) \), notice that
\[
\mathbb{P}^{(n)}(\cdot) = \sum_{\alpha \in \text{LP}_N} \mathbb{P}^{(n)}(\cdot|\alpha)\mathbb{P}^{(n)}(\cdot|\alpha).
\]

Since there are finitely many link patterns \( \alpha \in \text{LP}_N \), we can extract a subsequence so that the numbers \( \mathbb{P}^{(n)}(\cdot|\alpha) \) converge for all \( \alpha \in \text{LP}_N \). Then, we can use the precompactness of the conditional measures \( \mathbb{P}^{(n)}(\cdot|\alpha) \), deduced in the previous paragraph, to extract a further subsequence where the conditional measures \( \mathbb{P}^{(n)}(\cdot|\alpha) \) converge weakly as laws of \( (\gamma_1^{(n)}, \ldots, \gamma_N^{(n)}) \in X(\mathbb{C})^N \), for all \( \alpha \in \text{LP}_N \). Then, also the symmetric measures \( \mathbb{P}^{(n)} \) converge weakly as laws of \( (\gamma_1^{(n)}, \ldots, \gamma_N^{(n)}) \) along this subsequence.

The proof of part (ii) is identical to the proof of part (i) above.

For part (iii), the initial segments \( \lambda_j^{(n)} \) obtained from the continuous stopping time is a continuous function of the full curves \( (\gamma_1^{(n)}, \ldots, \gamma_N^{(n)}) \), see Appendix B. Thus, the weak convergence of the former follows from that of the latter. \( \square \)
Proof of Theorem 4.1(B). The proof for \( N = 1 \) curve is given by the author in [Kar18, Theorem 4.4 and Proposition 4.7]. For \( N \geq 2 \), the proof essentially identical, and we thus only outline the proof here. Let us first give the proof in the case when the measures \( P(n) \) are those conditional on a fixed link pattern \( \alpha \in \mathcal{L}_{N} \). By Corollary 4.6, the curves \( \gamma_{j}^{(n)} \), for all \( j \), satisfy condition (G), which is the hypothesis in [Kar18, Theorem 4.4 and Proposition 4.7]. The proofs of [Kar18] can now be straightforwardly repeated for the collections of curves \( (\gamma_{1}^{(n)}, \ldots, \gamma_{2N}^{(n)}) \).

To handle the case when the measures \( P(n) \) are not conditional on some link pattern, we have to be careful with Condition (G), which does not make sense any more now that a curve \( \gamma_{j}^{(n)} \) does not have a single target point. Condition (G) only appears in the proof of [Kar18, Theorem 4.4 and Proposition 4.7]. Unfortunately, the driving functions \( \gamma_{j}^{(n)} \) are precompact, as detailed in Theorem 4.4. Since there are finitely many link patterns \( \alpha \in \mathcal{L}_{N} \), we can make the conclusion of that lemma hold for all of them simultaneously. Since \( P(n) \) is a convex combination of such conditional measures, we then obtain the conclusion of that lemma also for curves \( \gamma_{j}^{(n)} \) under \( P(n) \), irrespective of the convex weights \( P(n)[\alpha] \). The rest of the arguments in [Kar18, Theorem 4.4 and Proposition 4.7] can be repeated straightforwardly.

4.2.4. Proofs of the statements about driving functions. Let us now prove the statements about driving functions in Theorem 4.4. The strategy will be once again to first prove the theorem in the case when the random curves are conditional on some particular link pattern \( \alpha \in \mathcal{L}_{N} \). Let us introduce some notation in that case. Note first that under the assumptions of Theorem 4.1, Corollary 4.6 and the conclusions of Theorem 4.4 hold for the curves \( \gamma_{j}^{(n)} \) starting from the odd boundary points. The same deduction also applies for their reversals, starting from the even boundary points. In particular, if we choose the conformal maps \( \phi_{n} \) so that the \( i \)-th boundary point, for a fixed \( 1 \leq i \leq 2N \), maps to \(-1\) and the other endpoint of that curve to \(+1\), the driving functions \( V_{i}^{(n)} \) of the corresponding curves are precompact, as detailed in Theorem 4.4. Unfortunately, the driving functions \( W_{i}^{(n)} \) in Theorem 4.4 are driving functions of some conformal images of the curves described by \( V_{i}^{(n)} \). Let us relate \( W_{i}^{(n)} \) and \( V_{i}^{(n)} \), and let us for a moment fix \( i \) and omit the subscripts \( i \).

Now, more precisely, \( V^{(n)} \), parametrized by time \( t \), are the Loewner driving functions of some random growing hulls \( K_{i}^{(n)} \) from zero to infinity in \( \mathbb{H} \). Likewise \( W^{(n)} \), parametrized by time \( s \), are the driving functions of \( \varpi_{n}(K_{i}^{(n)}) \); here \( \varpi_{n} \) are suitable conformal (Möbius) maps \( \mathbb{H} \to \mathbb{H} \) that converge uniformly over compacts, \( \varpi_{n} \to \varpi \), and the time parametrization \( s \) is different than \( t \), \( s = s^{(n)}(t) \). Recalling that \( W^{(n)} \) is stopped when the corresponding hulls \( \varpi_{n}(K_{i}^{(n)}) \) reach the continuous exit time of the localization neighbourhood \( \psi(U_{i}) \) of the \( i \)-th boundary point in \( \mathbb{H} \); we denote that that value of \( s \) by \( \sigma^{(n)} \).

Lemma 4.7. Under the notation above and assumptions of Theorem 4.4, the continuously stopped driving functions \( W_{s \wedge \sigma^{(n)}}^{(n)} \) are precompact in \( \mathcal{C} \). Furthermore, if the driving functions \( V_{i}^{(n)} \) converge weakly in \( \mathcal{C} \) to \( V_{i} \), describing some random growing hulls \( K_{i} \), then the driving functions \( W_{s \wedge \sigma^{(n)}}^{(n)} \) converge weakly in \( \mathcal{C} \) to \( W_{s \wedge \sigma} \), describing the hulls \( \varpi(K_{i}) \) up to their continuous exit time \( \sigma \) of \( \psi(U_{i}) \).

This lemma can be summarized in the commutative diagram

\[
\begin{array}{ccc}
V^{(n)} & \xrightarrow{\text{formal}} & W_{s \wedge \sigma^{(n)}}^{(n)} \\
\downarrow_{n \to \infty} & & \downarrow_{n \to \infty} \\
V & \xrightarrow{\text{formal}} & W_{s \wedge \sigma} 
\end{array}
\]

(4.8)

Proof of Lemma 4.7. The proof becomes more transparent if we operate with the conformal maps \( \varpi_{n} \) and \( \varpi \) on driving functions in stead of their hulls, with the understanding that after this operation, the
driving functions are stopped at their continuous exit times of $\psi(U_i)$. For instance, we replace $W_{s\wedge\sigma(n)}^{(n)}$ by $\varpi_n(V_i^{(n)})$ and $W_{s\wedge\sigma}$ by $\varpi(V_i)$. By Corollary C.4, conformal maps operate continuously on driving functions (when interpreted this way), so $\varpi_n(V_i^{(n)})$ and $\varpi(V_i)$ are measurable random variables.

Now, by Theorem 4.4 the functions $V_i^{(n)}$ are precompact. It thus suffices to show that if $V_i^{(n)} \to V_i$ weakly in $C$, then also $\varpi_n(V_i^{(n)}) \to \varpi(V_i)$ weakly in $C$. Take thus $f: C \to \mathbb{R}$ a bounded, Lipschitz continuous test function, and compute

$$|E^n[f(\varpi_n(V_i^{(n)}))] - E[f(\varpi(V_i))]| \leq |E^n[f(\varpi_n(V_i^{(n)})) - f(\varpi(V_i^{(n)}))]| + |E^n[f(\varpi(V_i^{(n)}))] - E[f(\varpi(V_i))]|.$$

Consider first the latter term on the right-hand side of (4.9). By Corollary C.4, the mapping $\varpi$ operates continuously on driving functions, and hence we deduce that $\varpi(V_i^{(n)}) \to \varpi(V_i)$ weakly in $C$. Thus, the latter term tends to zero as $n \to \infty$.

Consider now the former term on the right-hand side of (4.9). Since the random variables $V_i^{(n)} \in C$ are tight and $f$ is bounded, we may with an arbitrarily small error restrict our consideration to the case where $V_i^{(n)}$ belongs to a suitable compact set of the space $C$. Now, by Corollary C.6, we have

$$d_C(\varpi_n(\cdot), \varpi(\cdot)) \to 0 \quad \text{as } n \to \infty,$$

uniformly over any compact set of $C$. Since $f$ is Lipschitz, it follows that also

$$|f(\varpi_n(\cdot)) - f(\varpi(\cdot))| \to 0 \quad \text{as } n \to \infty,$$

uniformly over any compact set of $C$. This shows that the former term on the right-hand side of (4.9) tends to zero as $n \to \infty$.

Having analyzed both terms of (4.9), we now deduce that

$$|E^n[f(\varpi_n(V_i^{(n)}))] - E[f(\varpi(V_i))]| \to 0 \quad \text{as } n \to \infty.$$

This shows that $\varpi_n(V_i^{(n)}) \to \varpi(V_i)$ weakly in $C$, and hence completes the proof. \hfill \Box

Remark 4.8. Lemma 4.7 and its proof also apply for the slightly different definition of the stopping times $\sigma(n)$, as in Remarks C.3 and C.7.

With the above lemma at hand, we can finish the proof of Theorem 4.1.

Proof of Theorem 4.1.(iv). In the case where the measures $\mathbb{P}^{(n)}$ are those conditional on a link pattern $\alpha \in LP_N$, the precompactness of the stopped driving function $W_i^{(n)}$ was stated and proven in Lemma 4.7 above. In the general case, it follows from Equation (4.7).

Proof of Theorem 4.1.(C). We prove the equivalence as two implications.

Implication 1: Let us first assume that the initial segments $\lambda_i^{(n)}$ converge weakly, $\lambda_i^{(n)} \to \lambda_i$. Start by showing that the corresponding stopped driving functions $W_i^{(n)}$ then converge weakly to the stopped driving functions $W_{i,s \wedge \sigma_i^{(n)}}$ of $\lambda_i$. (We suppress here and in continuation the subsequence notation $n_k$.) By the precompactness of $W_{i,s \wedge}$, it suffices to prove that the claimed convergence holds for some further subsequence. Employing this strategy, we can pick a subsequence of $n$ so that the probabilities $\mathbb{P}^{(n)}[\alpha]$ of all link patterns $\alpha \in LP_N$ converge. It thus suffices to prove the claim subsequentially for the conditional measures $\mathbb{P}^{(n)}[\alpha]$. Now, for the conditional measure $\mathbb{P}^{(n)}_\alpha$, apply Corollary 4.6 and thus Theorem 4.4 to deduce the commutative diagram (4.3), where the curves $\gamma_i^{(n)}$ and $\gamma_i^{(n)}$ are those starting from the boundary point $i$ in the link pattern $\alpha$, and $V^{(n)}$ their driving functions. We can pick a subsequence so that all the weak convergences in that diagram hold. By Kar18 Proposition 4.3, also the curves $\gamma_i^{(n)}$ obtained by a
Let us still consider the setup of Section 4.1.1. The prime ends\( p \) presented here.

We yet have to prove the random variable measurability claimed in Theorem 4.1(C). Here \( W_{i,s}^{(n)} \) is a measurable function of \( V_{i,s}^{(n)} \) by Corollary C.4 and \( V_{i,s}^{(n)} \) is a measurable function of \( \lambda_i \) by the commutative diagram (4.3) and [Kar18, Proposition 4.3] (see also Appendix B on the measurability of \( \lambda_i \)).

Theorem 4.1 can for the discussion of this subsection be relaxed as in Remark 4.2.

Proof of Theorem 4.1(D). We finish the proof of Theorem 4.1 with a triviality.

We yet have to prove the random variable measurability claimed in Theorem 4.1(C). Here \( \frac{\partial}{\partial t} \frac{\partial}{\partial \sigma} \) is a measurable function of \( \frac{\partial}{\partial s} \) and \( \frac{\partial}{\partial \tau} \).

Implication 2: Let us now assume that the stopped driving functions \( W_{i,s}^{(n)} \) converge weakly, \( W_{i,s}^{(n)} \rightarrow W_{i,s}^{(n)} \). Show first that the corresponding initial segments \( \lambda_i^{(n)} \) then also converge weakly, \( \lambda_i^{(n)} \rightarrow \lambda_i \), where \( \lambda_i \) is the curve described by the driving function \( W_{i,s}^{(n)} \) up to time \( \sigma_i \). As in the previous implication, it suffices to prove this claim subsequentially for the conditional measures \( P^i \).

The argument to prove the subsequential convergence is now identical to the previous implication. To argue the measurabilities of random variables, \( \lambda_i \) is a measurable function of \( V_{i,s}^{(n)} \) by the arguments above, and \( V_{i,s}^{(n)} \) with respect to \( W_{i,s}^{(n)} \) by Corollary C.5. This finishes the proof of Theorem 4.1(C).

We finish the proof of Theorem 4.1 with a triviality.

**Proof of Theorem 4.1(D).** An initial segment \( \lambda_i \) up to a continuous exit time are a continuous function of the full curves \((\gamma_1;\bar{\mathbb{D}},\ldots,\gamma_N;\bar{\mathbb{D}})\), see Appendix B.

### 4.3. A precompactness result for local multiple SLEs

We have now in complete detail proven Theorem 4.1 sufficient for the main results of this paper. In this subsection, we will discuss some slight improvements to that theorem, holding under the same assumptions. These improvements are required in order to prove convergence of the collection of initial segments to a local multiple SLEs, in stead of convergence of full curves to a local-to-global multiple SLEs. We will keep the discussion in this subsection on an informal level, trusting that the interested reader can formalize the arguments presented here.

Let us still consider the setup of Section 4.1.1. The prime ends \( p_1,\ldots,p_{2N} \) of \( \Lambda \) now need to be distinct, and their localization neighbourhoods \( U_1,\ldots,U_{2N} \) disjoint, but otherwise the assumptions of Theorem 4.1 can for the discussion of this subsection be relaxed as in Remark 4.2.

The local multiple SLE describes the collection of initial segments \( \lambda_1^{(n)},\ldots,\lambda_{2N}^{(n)} \in X(\bar{\mathbb{D}}) \) in terms of their iterated driving functions \( \bar{W}_1^{(n)},\ldots,\bar{W}_{2N}^{(n)} \in C \). These are described as follows: the first function \( \bar{W}_1^{(n)} \) is the driving function of \( \psi(\lambda_1^{(n)}) \) stopped at the continuous exit time \( \sigma_1^{(n)} \) of \( \psi(U_1) \), i.e., \( \bar{W}_1^{(n)} = W_1^{(n)} \).

Given the first one, denote by \( g_{\lambda_2^{(n)}} \) the mapping-out function of the initial segment \( \psi(\lambda_1^{(n)}) \), obtained from the Loewner equation with the first driving function \( \bar{W}_1^{(n)} \). Then, \( \bar{W}_2^{(n)} \) is the driving function of the mapped-out second initial segment \( g_{\lambda_2^{(n)}}(\psi(\lambda_2^{(n)})) \). Similarly, the further iterated driving functions are defined as the driving functions of the conformally mapped initial segment, when one maps out the previous initial segments. Note that the local multiple SLE is defined via its iterated driving functions, see Section 2.1.3.

Now, our precompactness result for the local multiple SLEs is the precompactness of the collections of driving functions \( \{\bar{W}_1^{(n)},\ldots,\bar{W}_{2N}^{(n)}\} \in C^{2N} \) and the commutative diagram

\[
\begin{array}{ccc}
(\lambda_1^{(n)},\ldots,\lambda_{2N}^{(n)}) & \xrightarrow{\text{iter. Loewner}} & (\bar{W}_1^{(n)},\ldots,\bar{W}_{2N}^{(n)}) \\
\downarrow n \to \infty & & \downarrow n \to \infty \\
(\lambda_1,\ldots,\lambda_{2N}) & \xleftarrow{\text{iter. Loewner}} & (\bar{W}_1,\ldots,\bar{W}_{2N}),
\end{array}
\]

formally interpreted similarly to all previous commutative diagrams.
Let us sketch the proof of this result. First, one utilizes the earlier commutation relations and to establish the diagram

\[(\lambda_1^{(n)}, \ldots, \lambda_{2N}^{(n)}) \xleftarrow{\text{Loewner}} (W_1^{(n)}, \ldots, W_2^{(n)}) \]

\[(\lambda_1, \ldots, \lambda_{2N}) \xrightarrow{\text{Loewner}} (W_1, \ldots, W_{2N}),\]

(4.11)

with the usual (not iterated) driving functions. This diagram is a direct multicurve analogue of diagram (4.2), i.e., Theorem 4.1(C). The tightness of the collections of curves (resp. driving functions) follows from that of the individual curves (resp. driving functions) similarly to the proof of Theorem 4.1(A)(i) (see especially Equation (4.6)). The commutation in diagram (4.11) follows by straightforwardly repeating for multiple curves the arguments that yielded the analogous commutation relation for a single initial segment, needed to prove Theorem 4.1(C), given in [KS17] and the previous subsection.

Having proven (4.11), one next establishes the commutative diagram

\[(W_1^{(n)}, \ldots, W_{2N}^{(n)}) \xleftarrow{\text{iterated}} (\tilde{W}_1^{(n)}, \ldots, \tilde{W}_{2N}^{(n)}) \]

\[(W_1, \ldots, W_{2N}) \xrightarrow{\text{iterated}} (\tilde{W}_1, \ldots, \tilde{W}_{2N}),\]

(4.12)

This diagram follows from the observation that the mapping \(f\) from the original driving functions \((W_1^{(n)}, \ldots, W_{2N}^{(n)})\) to the iterated ones \((\tilde{W}_1^{(n)}, \ldots, \tilde{W}_{2N}^{(n)})\) is a continuous bijection, whose inverse is also continuous. Let us now illustrate how this is proven for \(N = 1\) with two initial segments — larger \(N\) are treated identically.

Let us first argue that \(f\) is continuous. The first functions are identical, \(\tilde{W}_1^{(n)} = W_1^{(n)}\). For the second functions, \(W_2^{(n)}\) is the driving function of the second initial segment \(\psi(\lambda_2^{(n)})\), and \(\tilde{W}_2^{(n)}\) that of the mapped-out second initial segment \(\tilde{\psi}(\lambda_2^{(n)})\). Now, the mapping-out function \(g_{\lambda_1^{(n)}}\) depends continuously on the first driving function \(W_1^{(n)}\) (equipping analytic functions with the topology of uniform convergence over compacts), see [Kem17] Lemma 5.1, and the driving function of the conformal image \(g_{\tilde{\psi}(\lambda_2^{(n)})}\) thus depends continuously on the conformal map \(g_{\lambda_1^{(n)}}\) and the original driving function \(W_2^{(n)}\), see Corollary C.2 and Remark C.3. This shows that \(f\) is continuous.

To argue that \(f\) is bijective and its inverse is continuous, we can find \(f^{-1}\) and prove its continuity identically to the previous paragraph. The only difference is that here we have to use the inverse mapping-out function \(g_{\lambda_1^{(n)}}^{-1}\), whose continuity follows from [Kem17] Lemma 5.8. This finishes our sketch of proof of the commutative diagram (4.12).

Finally, combining the commutative diagrams (4.11) and (4.12) yields (4.10).

5. LOCAL-TO-GLOBAL PROPERTIES

In this section, we will assume that the study of a discrete random curve model is in a phase where the precompactness conditions of Theorem 4.1 have been verified, and the scaling limit \(W_\gamma\) of the driving function of an initial segment has been identified as that of a local multiple SLE. The objective is to show that the discrete domain Markov property is (under some reasonable assumptions on the discrete curve model) inherited to a domain Markov type property of the scaling limit, so that this identification of the scaling limit of one initial segment actually identifies the scaling limit of the full collection of full curves.
Our results of this type are stated in two theorems, Theorem 5.2 addressing the convergence of collections of initial segments to local multiple SLEs, and Theorem 5.8 addressing convergence of collections of full curves to local-to-global multiple SLEs. The latter has two alternative assumptions and proofs, a simpler one with a strong \textit{a priori} estimate only possible for SLE scaling limits with $\kappa \leq 4$, and a longer one with an assumption applicable for general $\kappa < 8$. Finally, in Theorems 5.10 and 5.11 we explicate the relation of the obtained scaling limits to the global multiple SLEs of [PW19, BPW18], introduced in Section 4.

5.1. **Statements of the main theorems.** This subsection contains the statements of the main theorems of this section. The rest of this section will constitute their proofs.

5.1.1. **Notations and domain discretizations.** We will continue in the notation and setup introduced in Section 4.1.1. The only difference is that in some statements, we should only assume that the domains $(\Lambda_n; p_1^{(n)}, ..., p_{2N}^{(n)})$ with marked boundary points are uniformly bounded and converge in the Carathéodory sense to $(\Lambda; p_1, ..., p_{2N})$, without additional requirements on existence of radial limits at $p_1, ..., p_{2N}$ or closeness of approximations (cf. Remark 4.2). If the assumptions in Section 4.1.1 are relaxed in this way, we say that we have \textit{relaxed regularity at marked boundary points}.

5.1.2. **Assumptions on the discrete curve models.** Throughout this section, we consider discrete random curve models under the following setup and assumptions.

We have a sequence of lattices $\Gamma_n$, and a discrete random curve model on each lattice. These lattices describe a scaling limit, in the sense that the maximal length of a lattice edge in any bounded domain tends to zero as $n \to \infty$. (In many applications $\Gamma_n$ are just scalings of $\Gamma_1$, and we may thus think of having a single random curve model on $\Gamma_1$.) The assumptions imposed on the random curve models in Theorem 4.1 are satisfied, i.e., they have alternating boundary conditions, satisfy the DDMP, and the collection of one-curve measures $P^{(G, e_1, e_2)}_{\Gamma_n}$ in these random curve models, indexed by $n$ and simply-connected subgraphs $G$ of $\Gamma_n$, satisfy the equivalent conditions (C) and (G).

The above assumptions are well established properties for many discrete curve models. To state our nontrivial standing assumption about convergence of driving functions, consider the setup of Section 4.1.1 with relaxed regularity at marked boundary points. By Theorem 4.1 and Remark 4.2 the stopped driving functions $W_{j}^{(n)}$ of the curve initial segment starting from the $j$:th boundary point, for any $j$ and any localization neighbourhood $U_j$ in $\mathbb{D}$, are precompact. Throughout this section, we will assume that the weak limit of any such initial segment has been in that case identified as a local multiple SLE with partition function $Z_N$:

**Assumption 5.1.** For any $(G_n; e_1^{(n)}, ..., e_{2N}^{(n)})$ converging in the Carathéodory sense, and any localization neighbourhood $U_j$, the stopped driving functions $W_{j}^{(n)}$ of the random curves converge weakly in $C$,

$$W_{j}^{(n)} \to W_j \quad \text{as } n \to \infty,$$

where $W_j$ is the driving function of the local multiple SLE, stopped at the continuous exit time of $U_j$, and with partition function $Z_N$.

The partition functions $Z_N$ above, indexed by $N$, should be partition functions with the same parameter $\kappa \in (0, 8)$. We will mostly suppress the notation $\kappa$ in our discussions. Let us make two remarks about this assumption.

First, it is largely for simplicity that we restrict our consideration to unconditional measures, or partition function $Z_N$ here. The proofs and statements of Theorem 5.2 and Theorem 5.8 under Assumption 5.5 (i.e., the \textit{a priori} estimate only possible for $\kappa \leq 4$) can be straightforwardly generalized to convergence of conditional discrete curve models. Assumption 5.1 above should then be modified so that the convergence holds for the conditional discrete curve models with any link pattern $\alpha \in \text{LP}_N$, to local multiple SLEs.
with the corresponding partition functions $Z_n$. However, the alternative Assumption 5.6 in Theorem 5.8 is suitable for general $0 < \kappa < 8$, intrinsically requires considering non-conditional measures.

Second, note that whether Assumption 5.1 holds or not is independent of the choice of conformal maps from $\Lambda_n$ to the upper half-plane $\mathbb{H}$, as long as the conformal maps converge to that of $\Lambda$. Namely, the local multiple SLE is conformally invariant, and on the other hand, so is the weak limit of $W_j^{(n)}$, by applying the commutative diagram (4.2) with different conformal maps.

5.1.3. **Convergence to local multiple SLEs.** Our first theorem states, roughly, that condition (C), discrete the domain Markov property, and the identification of one initial segment in Assumption 5.1 together identify the collection of initial segments as a local multiple SLE.

**Theorem 5.2.** Consider the setup of Section 4.1.1 with relaxed regularity at marked boundary points, and let the discrete curve models satisfy the assumptions of Section 5.1.4. Then, the iterated driving functions $(W_1^{(n)}, \ldots, W_{2N}^{(n)}) \in C^{2N}$ (resp. the initial segments $(\lambda_1^{(n)}, \ldots, \lambda_{2N}^{(n)}) \in X(\mathbb{D})^{2N}$) converge weakly to the iterated driving functions (resp. to the initial segments) of the local multiple SLE in $(\mathbb{D}; p_1, \ldots, p_{2N})$, stopped at the continuous exit times of $U_1, \ldots, U_{2N}$, and with partition function $Z_N$.

**Corollary 5.3.** If the regularity at marked boundary points is not relaxed above, then the weak convergence to local multiple SLE also takes place in the sense that initial segments of any subsequential weak limit in topology (i) of Theorem 4.1 are in distribution equal to the conformal images $(\phi^{-1}(\lambda_1), \ldots, \phi^{-1}(\lambda_{2N}))$ of local multiple SLE curves $(\lambda_1, \ldots, \lambda_{2N})$ in $\mathbb{D}$.

5.1.4. **Additional assumptions on the discrete curve models.** For the other main result of this section, we need two more assumptions on our discrete curve models, in addition to those in Section 5.1.2.

First, informally, the discrete curve models must allow increasingly dense-mesh discretizations of any desired limiting domain. This assumption is necessary in the proofs since contrary to the case of local multiple SLEs, we cannot directly rely the Carathéodory stability of the scaling limit, but it has to be deduced via the discrete curve model.

**Assumption 5.4.** For any bounded simply-connected domain $(\Lambda; p_1, \ldots, p_{2N})$ with marked prime ends that possess radial limits, there exist close lattice approximations $(\mathcal{G}^{(n)}; e_1^{(n)}, \ldots, e_{2N}^{(n)})$ on the graphs $\Gamma^{(n)}$, such that the measures $\mathbb{P}(\mathcal{G}^{(n)}; e_1^{(n)}, \ldots, e_{2N}^{(n)})$ are defined and $\mathcal{G}^{(n)}$ lies inside of $\Lambda$ for all large enough $n$.

Second, in order to handle full curves in stead of initial segments, an initial segment in a very large neighbourhood must yield some information about the target of the curve we are following. There are two alternative assumptions ensuring this, the first one possible only for discrete models corresponding to (multiple) SLEs with parameter $\kappa \leq 4$, and the second one applicable for all $0 < \kappa < 8$.

**Assumption 5.5.** For any fixed sequence of Carathéodory converging graphs $(\mathcal{G}_n; e_1^{(n)}, \ldots, e_{2N}^{(n)})$, we have the following. For any $\delta' > 0$ and any $\varepsilon > 0$, taking $\delta$ small enough, the following holds. Denote by $(\gamma_{D;1}^{(n)}, \ldots, \gamma_{D;N}^{(n)}) \in E(\delta, \delta')$ if some of the curves $(\gamma_{D;1}^{(n)}, \ldots, \gamma_{D;N}^{(n)})$ visits the $\delta$-neighbourhood of $\partial D$ outside of the $\delta'$-neighbourhoods of its end points (see Figure 5.1). Then,

$$
\mathbb{P}(\mathcal{G}^{(n)}; e_1^{(n)}, \ldots, e_{2N}^{(n)})(\gamma_{D;1}^{(n)}, \ldots, \gamma_{D;N}^{(n)}) \in E(\delta, \delta') < \varepsilon
$$

for all large enough $n$.

The assumption alternative to Assumption 5.5 is a slightly improved version of condition (C), which we call condition (C').

---

5 As regards the Carathéodory stability of local multiple SLE driving functions, we rather consider it as an input from basic SDE theory than SLE theory.
Assumption 5.6. The collection of measures with random curves \( (\mathbb{P}(G;e_1,\ldots,e_{2N}),(\gamma_{G;1},\ldots,\gamma_{G;N})) \), indexed by \( n \) and all simply-connected subgraphs \((G;e_1,\ldots,e_{2N})\) of \( \Gamma_n \) on which the random curve model of \( \Gamma_n \) is defined, satisfies condition (C') below.

Let us formulate condition (C'). For this purpose, we need some new terminology. Consider a measure \( \Gamma \) indexed by some of the curves \( ~p_2,\ldots,\gamma_{G;2},\ldots,\gamma_{G;3} \) unforced for the collection of random curves, if it occurs in a connected component \( N,\varepsilon \). We say that a crossing of a boundary annulus \( A(z,r,\varepsilon) \) is unforced for the collection of random curves if for all \( N,\varepsilon \) such that for an annulus \( A(z,r,\varepsilon) \) which is unforced for the collection of random curves of the corresponding planar graph \( G \), as defined in Section 4.1.3. We say that a crossing of \( Q \) by some of the curves \( \gamma_{G;1},\ldots,\gamma_{G;N} \) is unforced for the collection of random curves if the sides \( S_1, S_3 \) of \( Q \) that lie on \( \partial \Lambda_G \) are entirely inside marked boundary arcs of the same parity. Equivalently, the mark boundary edges \( e_1,\ldots,e_{2N} \) all land outside of \( Q \) on \( \partial \Lambda_G \), and both connected components of \( \Lambda_G \setminus Q \) contain an even number of them.

Condition (C') is now stated as follows: a collection of measures with random curves \( (\mathbb{P}(G;e_1,\ldots,e_{2N}),(\gamma_{G;1},\ldots,\gamma_{G;N})) \) satisfies condition (C') if for all \( \varepsilon > 0 \) there exists \( M = M(N,\varepsilon) > 0 \), such that for any topological quadrilateral \( Q \) with \( m(Q) \geq M \) on the boundary of \( \Lambda_G \), we have

\[
\mathbb{P}(G;e_1,\ldots,e_{2N})[\text{crossing of } Q \text{ unforced for the collection of random curves } (\gamma_{G;1},\ldots,\gamma_{G;N})] \leq \varepsilon.
\]

for all the measures with random curves.

Let us make some remarks about condition (C'). First, taking \( N = 1 \), it becomes condition (C) with the stopping time 0, i.e., for DDMP models. Second, condition (C') is not compatible with conditional discrete curve models; it is easy to come up with examples of graphs \( (G;e_1,\ldots,e_{2N}) \) and \( Q \), where conditioning on a link pattern \( \alpha \) forces a crossing of \( Q \) which is unforced for the collection of random curves.

As a final and most important remark, we give a sufficient alternative condition in terms of annuli. We say that a crossing of a boundary annulus \( A(z,r,\varepsilon) \) by some of the curves \( \gamma_{G;1},\ldots,\gamma_{G;N} \) is unforced for that collection of random curves, if it occurs in a connected component \( C \) of \( A(z,r,\varepsilon) \cap \Lambda_G \) such that \( \partial C \cap \partial \Lambda_G \) lies entirely inside marked boundary arcs of the same parity.

Condition (G'): a collection of measures with random curves \( (\mathbb{P}(G;e_1,\ldots,e_{2N}),(\gamma_{G;1},\ldots,\gamma_{G;N})) \) satisfies condition (G') if for all \( \varepsilon > 0 \) there exists \( M = M(N,\varepsilon) > 0 \), such that for an annulus \( A(z,r,\varepsilon) \) with
Theorem 5.8. Consider the setup of Section 4.1.1 with relaxed regularity at marked boundary points, and let discrete curve models satisfy the assumptions of Section 5.1.2, as well as Assumptions 5.4, and 5.1.5. Convergence to local-to-global multiple SLEs, see [KS17, Proof of Proposition 2.6].

Proof. The proof is identical to showing that condition (G) implies condition (C) for the one-curve models, see [KST17, Proof of Proposition 2.6]. □

5.1.5. Convergence to local-to-global multiple SLEs. We now state the main theorem of this section.

Theorem 5.8. Consider the setup of Section 4.1.1 with relaxed regularity at marked boundary points, and let discrete curve models satisfy the assumptions of Section 5.1.2, as well as Assumptions 5.4, and either 5.5 or 5.6. Then, the curves \((\gamma_{1}^{(n)}, \ldots, \gamma_{N}^{(n)})\) converge weakly in \(C^{N}\) to the local-to-global multiple SLE with partition function \(Z_{N}\) on the domain \((\Omega; \tilde{p}_{1}, \ldots, \tilde{p}_{2N})\). If the regularity at marked boundary points is not relaxed from Section 4.1.1, then also the curves \((\gamma_{1}^{(n)}, \ldots, \gamma_{N}^{(n)})\) converge weakly in \(C^{N}\) to the local-to-global multiple SLE with partition function \(Z_{N}\) on the domain \((\Lambda; p_{1}, \ldots, p_{2N})\).

5.1.6. Relation to global multiple SLE. The final theorems of this section connect the limits of Theorem 5.8 to the global multiple SLEs. We however emphasize once again that the SLE convergence proof in Theorem 5.8 by no means relies on global multiple SLEs.

First, since Theorem 5.8 addresses scaling limits unconditional curve models, while the global multiple SLEs address the conditional ones, the two models cannot be the same. Proposition 5.9 below however guarantees that the scaling limits from Theorem 5.8 are convex combinations of measures satisfying the Markov stationarity that defines global multiple SLEs for \(\kappa \in (0, 4]\) and conjecturally also for \(\kappa \in (4, 8)\).

Proposition 5.9. The scaling limits \((\gamma_{1}, \ldots, \gamma_{N})\) from Theorem 5.8 satisfy the following property: for any \(1 \leq j \leq N\), the regular conditional law of \(\gamma_{j}\) given all the other curves is the chordal SLE in between the remaining marked boundary points in the remaining domain; the boundary points almost surely lie adjacent to the same simply-connected component of the complement of the remaining curves in \(\Omega\), so this chordal SLE makes sense.

Next, an interesting question is if all the link patterns \(\alpha \in \mathbb{L}_{P_{N}}\) appear with positive probability in the scaling limits of Theorem 5.8. The answer is positive at least if Assumption 5.6 holds. In the sense formalized below, this means that Theorem 5.8 also guarantees convergence of the conditional discrete models to global multiple SLEs.

Theorem 5.10. Suppose that the assumptions of Theorem 5.8 (with relaxed regularity at marked boundary points), including Assumption 5.6, are satisfied. Then, all link patterns \(\alpha \in \mathbb{L}_{P_{N}}\) appear with positive probability in the scaling limit \((\gamma_{1}, \ldots, \gamma_{N})\). In particular, the conditional discrete models converge weakly to a measure satisfying the local multiple SLE, which defines the local multiple SLE if \(\kappa \leq 4\) and conjecturally also for \(\kappa \in (4, 8)\).

Conversely, suppose now that the conditional discrete models are known to converge to global multiple SLEs with \(\kappa \in (0, 4]\). Theorem 5.11 below guarantees that the conditional models then also converge in the sense of Theorem 5.8 to conditional local-to-global multiple SLEs.

To be precise, we say that the conditional discrete curves converge to global multiple SLE(\(\kappa\)), for short, if the following holds. The discrete random curves \((\gamma_{1}^{(n)}, \ldots, \gamma_{N}^{(n)})\) obtained from corresponding conditional discrete models \(P_{\alpha}^{(n)}[\cdot] = P^{(n)}[\cdot|\alpha]\) converge weakly to the global multiple SLE(\(\kappa\)) on \((\Omega; \tilde{p}_{1}, \ldots, \tilde{p}_{2N})\) with link pattern \(\alpha\), and this holds for any \(\alpha \in \mathbb{L}_{P_{N}}\) and any Carathéodory converging domain approximations \((\mathcal{G}^{(n)}, e_{1}^{(n)}, \ldots, e_{2N}^{(n)})\).
Theorem 5.11. Suppose that the assumptions imposed on the discrete curve models in the precompactness theorem 4.7 and in (the usually trivial) Assumption 5.4 are satisfied. Suppose also that the conditional discrete curves of the discrete curve models converge to global multiple SLE(κ), for some κ ≤ 4. Then, also the remaining Assumptions 5.7 and 5.8 of Theorem 5.8 are satisfied by the conditional measures P_{α}^{(n)}, the former in its conditional form and with the partition functions Z_{α} as given in [PW19 Equation (3.7)]. Thus, Theorem 5.8 holds in the form giving convergence to conditional local-to-global multiple SLEs.

5.2. Proof of Theorem 5.2

Lemma 5.12. In the setup of Theorem 5.2 let (\tilde{W}_{1},...\tilde{W}_{2N}) ∈ C^{2N} be any subsequential scaling limit of the iterated driving functions (\tilde{W}_{1}^{(n)},...\tilde{W}_{2N}^{(n)}). Let f : C^{m} → R, where 1 ≤ m ≤ 2N − 1, and g : C → R be bounded continuous test functions. Then, we have

\[ E[f(\tilde{W}_{1},...\tilde{W}_{m})g(\tilde{W}_{m+1})] = E[f(\tilde{W}_{1},...\tilde{W}_{m})E_{\tilde{W}_{1},...\tilde{W}_{m}}^{N,SLE}(g(\tilde{W}_{m+1}))], \]

where by the random variable W_{m+1} ∈ C under the measure P_{\tilde{W}_{1},...\tilde{W}_{m}}^{N,SLE}, we mean the driving function of the local multiple SLE with partition function Z_{N}, when growing the initial segment starting from the (m + 1):st boundary point, and with the initial configuration of the marked boundary points being where the m first iterated growth processes \tilde{W}_{1},...\tilde{W}_{m} end, up to the stopping time corresponding to the continuous exit time of U_{j}.

Proof. Let us assume that a converging subsequence has been extracted, so that (\tilde{W}_{1}^{(n)},...\tilde{W}_{2N}^{(n)}) → (\tilde{W}_{1},...\tilde{W}_{2N}) weakly in C^{2N}. Start with the triangle inequality,

\[ |E[f(\tilde{W}_{1},...\tilde{W}_{m})g(\tilde{W}_{m+1})] − E[f(\tilde{W}_{1},...\tilde{W}_{m})E_{\tilde{W}_{1},...\tilde{W}_{m}}^{N,SLE}(g(\tilde{W}_{m+1}))]| \leq |E[f(\tilde{W}_{1},...\tilde{W}_{m})g(\tilde{W}_{m+1})] − E[f(\tilde{W}_{1}^{(n)},...\tilde{W}_{m}^{(n)})g(\tilde{W}_{m+1}^{(n)})]| \]

\[ + |E[f(\tilde{W}_{1}^{(n)},...\tilde{W}_{m}^{(n)})g(\tilde{W}_{m+1}^{(n)})] − E[f(\tilde{W}_{1}^{(n)},...\tilde{W}_{m}^{(n)})E_{\tilde{W}_{1}^{(n)},...\tilde{W}_{m}^{(n)}}^{N,SLE}(g(\tilde{W}_{m+1}^{(n)}))]| \]

\[ + |E[f(\tilde{W}_{1}^{(n)},...\tilde{W}_{m}^{(n)})E_{\tilde{W}_{1}^{(n)},...\tilde{W}_{m}^{(n)}}^{N,SLE}(g(\tilde{W}_{m+1}^{(n)}))] − E[f(\tilde{W}_{1},...\tilde{W}_{m})E_{\tilde{W}_{1},...\tilde{W}_{m}}^{N,SLE}(g(\tilde{W}_{m+1}))]|. \]

We claim that, taking a large enough n, the right-hand side of [5.1] can be made arbitrarily small. The first term becomes arbitrarily small by the weak convergence (\tilde{W}_{1}^{(n)},...\tilde{W}_{2N}^{(n)}) → (\tilde{W}_{1},...\tilde{W}_{2N}), likewise the third one. (This uses the fact that the stopped local multiple SLE driving function W_{m+1} is continuous with respect to the initial configuration of the marked boundary points.)

Let us examine the second term of [5.1]. First, by the DDMP,

\[ E^{(n)}[f(\tilde{W}_{1}^{(n)},...\tilde{W}_{m}^{(n)})g(\tilde{W}_{m+1}^{(n)})] = E^{(n)}[f(\tilde{W}_{1}^{(n)},...\tilde{W}_{m}^{(n)})E_{\tilde{W}_{1}^{(n)},...\tilde{W}_{m}^{(n)}}^{N,SLE}(g(\tilde{W}_{m+1}^{(n)}))], \]

where we denoted by E_{\tilde{W}_{1}^{(n)},...\tilde{W}_{m}^{(n)}}^{(n)} the measure from the discrete random curve model on Γ_{n}, on the graph obtained by reducing the original graph G^{(n)} by the initial segments described by the driving functions (\tilde{W}_{1}^{(n)},...\tilde{W}_{m}^{(n)}). Under this measure, \tilde{W}_{m+1}^{(n)} is the driving function of conformal image of the (m + 1):st curve initial segment λ_{m+1}^{(n)}, after mapping-out of the previous initial segments λ_{1}^{(n)},...λ_{m}^{(n)}.

Let us state the next step of the proof as a separate lemma.

Lemma 5.13. For any fixed compact set K ⊂ C^{m}, we have the convergence

\[ E_{\tilde{W}_{1}^{(n)},...\tilde{W}_{m}^{(n)}}^{(n)}[g(\tilde{W}_{m+1}^{(n)})] \rightarrow E_{\tilde{W}_{1},...\tilde{W}_{m}}^{N,SLE}[g(\tilde{W}_{m+1})] \text{ as } n \rightarrow \infty, \]

uniformly over (\tilde{W}_{1}^{(n)},...\tilde{W}_{m}^{(n)}) describing possible initial segments and belonging to K.
Proof. Assume for a contradiction than such uniform convergence does not occur, i.e., for infinitely many \( n \), there exist deterministic iterated driving functions \((\tilde{V}_1^{(n)}, \ldots, \tilde{V}_m^{(n)})\) in \( K \) that can each appear as iterated driving functions \((\tilde{W}_1^{(n)}, \ldots, \tilde{W}_m^{(n)})\) of the initial segments in our lattice models (i.e., they describe lattice curves), and

\[
\mathbb{E}^{(n)}_{(\tilde{V}_1^{(n)}, \ldots, \tilde{V}_m^{(n)})} [g(\tilde{W}_{m+1}^{(n)})] - \mathbb{E}^{N,SLE}_{(\tilde{V}_1^{(n)}, \ldots, \tilde{V}_m^{(n)})} [g(W_{m+1})] > \delta \tag{5.3}
\]

for some \( \delta > 0 \).

Now, by compactness, we may extract a convergent subsequence (which we suppress in notation), \((\tilde{V}_1^{(n)}, \ldots, \tilde{V}_m^{(n)}) \to (\tilde{V}_1, \ldots, \tilde{V}_m)\). In Assumption 5.1, we assumed that the convergence of a single driving function to local multiple SLE is verified, so this implies\(^6\)

\[
\mathbb{E}^{(n)}_{(\tilde{V}_1^{(n)}, \ldots, \tilde{V}_m^{(n)})} [g(\tilde{W}_{m+1}^{(n)})] \overset{n \to \infty}{\longrightarrow} \mathbb{E}^{N,SLE}_{(\tilde{V}_1, \ldots, \tilde{V}_m)} [g(W_{m+1})].
\]

On the other hand, the continuity of the local multiple SLE driving function with respect to the initial configuration implies

\[
\mathbb{E}^{N,SLE}_{(\tilde{V}_1^{(n)}, \ldots, \tilde{V}_m^{(n)})} [g(W_{m+1})] \overset{n \to \infty}{\longrightarrow} \mathbb{E}^{N,SLE}_{(\tilde{V}_1, \ldots, \tilde{V}_m)} [g(W_{m+1})].
\]

These two convergences contradict (5.3), proving the lemma. \( \square \)

Let us now finish the proof of Lemma 5.12 by bounding the second term of (5.1). First, by Prohorov’s theorem, weak convergence implies tightness. Since the functions \( f \) and \( g \) are bounded, we can thus with arbitrarily small error assume that \((\tilde{W}_1^{(n)}, \ldots, \tilde{W}_m^{(n)}) \in K \) for a suitable compact set \( K \subset \mathbb{C}^m \). Applying then (5.2) and Lemma 5.13 we observe that the second term of (5.1) tends to zero as \( n \to \infty \). This concludes the proof. \( \square \)

Proof of Theorem 5.2. By the commutative diagram (4.10), it suffices to prove the weak convergence of the iterated driving function. Consider a subsequential weak limit \((\tilde{W}_1, \ldots, \tilde{W}_{2N})\). Lemma 5.12 with fixed \( g : \mathbb{C} \to \mathbb{R} \), holds for all continuous and bounded \( f : \mathbb{C}^m \to \mathbb{R} \). Thus, the weak limit \((\tilde{W}_1, \ldots, \tilde{W}_{2N})\) satisfies

\[
\mathbb{E}[g(\tilde{W}_{m+1}) \mid \sigma(\tilde{W}_1, \ldots, \tilde{W}_m)] = \mathbb{E}^{N,SLE}_{(\tilde{W}_1, \ldots, \tilde{W}_m)} [g(W_{m+1})].\tag{5.4}
\]

The right-hand side is a continuous function of \((\tilde{W}_1, \ldots, \tilde{W}_m)\) by the stability of local multiple SLE with respect to the initial configuration.

By Proposition A.1 from the appendices, the fact that (5.4) holds for all continuous functions \( g : \mathbb{C} \to \mathbb{R} \) means that the regular conditional law of the \((m+1):st\) iterated driving function \( \tilde{W}_{m+1} \) given the previous ones \((\tilde{W}_1, \ldots, \tilde{W}_m)\) is the local multiple SLE growth driving function, launched from the boundary point configuration where the previous ones \((W_1, \ldots, W_m)\) end. Inductively on \( m \), this shows that \((\tilde{W}_1, \ldots, \tilde{W}_m)\) are local multiple SLE iterated driving functions. \( \square \)

Proof of Corollary 5.3. By the commutative diagram (4.1), depicting Theorem 4.1(B), the initial segments of any subsequential weak limit \((\gamma_1, \ldots, \gamma_N)\) are the conformal images of the initial segments \((\lambda_1, \ldots, \lambda_{2N})\) on \( \mathbb{D} \). \( \square \)

5.3. Proof of Theorem 5.8 under Assumption 5.5. In this subsection, we present the proof of Theorem 5.8 under Assumption 5.5. This proof is easier and notationally lighter than the one under the alternative assumption 5.6. By “assumptions of Theorem 5.8” we refer to the set of assumptions with Assumption 5.5.

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\(^6\) The stopping times used here are slightly different than in Assumption 5.1. However, this does not change the weak convergence by Remark 4.8.
5.3.1. **Identifying the scaling limit of one-curve marginals.** Note that by of Theorem 4.1 and Remark 4.2, the curves \( (\gamma_{D;1}) \) are precompact. Fix a subsequential scaling limit \((\gamma_{D;1}, \ldots, \gamma_{D;N})\), and consider the marginal law of one curve. For notational simplicity, we choose this special curve to be \( \gamma_{D;1} \) in this and following computations, but the straightforward analogues hold for all curves \( \gamma_{D;j} \), as well as their reversals. Let us denote by \( \lambda(\delta) \) the initial segment of \( \gamma_{D;1} \), up to the continuous exit time of a very large localization neighbourhood \( U(\delta) \) of the first boundary point, consisting of all of \( \bar{D} \) except a \( \delta \)-neighbourhood of the arc \((\hat{p}_2\hat{p}_{2N})\) of the other boundary points. Note that by Assumption 5.1, \( \lambda(\delta) \) is described by the local multiple SLE growth process, for any subsequential scaling limit.

**Lemma 5.14.** Under the setup and assumptions of Theorem 5.8, the curve \( \gamma_{D;1} \) almost surely visits the boundary \( \partial D \) only at its end points, and only at times 0 and 1, and \( \lambda(\delta) \to \gamma_{D;1} \) almost surely as \( \delta \downarrow 0 \).

**Proof.** The property that the curve \( \gamma_{D;1} \) almost surely visits the boundary \( \partial D \) only at its end points follows straightforwardly from Assumption 5.5 and Portmanteau’s theorem on weak convergence. The fact that these boundary visits occur at times 0 and 1, i.e., the end points are almost surely not double points of the curve \( \gamma_{D;1} \), follows from the annulus crossing estimates [KS17, Theorem 1.5].

The convergence \( \lambda(\delta) \to \gamma_{D;1} \) almost surely as \( \delta \downarrow 0 \) is proven by the following argument: the curve \( \lambda(\delta) \) is the initial segment of \( \gamma_{D;1} \), as grown up to the continuous exit time \( \tau(\delta) \) of \( U(\delta) \). For a fixed realization of \( \gamma_{D;1} \), it is easy to show that as \( \delta \downarrow 0 \), the stopping times \( \tau(\delta) \) tend to the hitting time of the arc \((\phi(p_2)\phi(p_{2N}))\). Since the boundary visits of \( \gamma_{D;1} \) a.s. only occur at times 0 and 1, we deduce that, a.s., \( \tau(\delta) \to 1 \) as \( \delta \downarrow 0 \), and thus also \( \lambda(\delta) \to \gamma_{D;1} \). \( \square \)

**Corollary 5.15.** In the setup of Theorem 5.8, the marginal law of the curve \( \gamma_{D;1} \) is the same for all subsequential scaling limits \((\gamma_{D;1}, \ldots, \gamma_{D;N})\). It is the local multiple SLE which, when defined in the increasing neighbourhoods \( U(\delta) \) as \( \delta \downarrow 0 \), almost surely yields a continuous closed curve between two marked boundary points.

Note that the above properties of the local multiple SLE initial segment from \( \hat{p}_1 \) to \( (\hat{p}_2\hat{p}_{2N}) \) would be difficult to prove directly by SLE theory, but are now easy by the underlying full curve \( \gamma_{D;1} \) obtained from the lattice model.

**Proof of Corollary 5.15.** Let us first identify the marginal scaling limit \( \gamma_{D;1} \). It suffices to show that for any bounded continuous function \( g : X(\bar{D}) \to \mathbb{R} \), the expectation \( \mathbb{E}[g(\gamma_{D;1})] \) is the same for all subsequential scaling limits \((\gamma_{D;1}, \ldots, \gamma_{D;N})\). Almost sure convergence implies weak convergence, so by Lemma 5.14 we have

\[
\mathbb{E}[g(\gamma_{D;1})] = \lim_{\delta \downarrow 0} \mathbb{E}[g(\lambda(\delta))],
\]

for any subsequential limit \( \gamma_{D;1} \). By Assumption 5.1, \( \lambda(\delta) \) is the local multiple SLE initial segment, and in particular, the right-hand side above the same for any subsequential scaling limit. The fact that the local multiple SLE determines a full curve is an immediate consequence of Lemma 5.14. \( \square \)

5.3.2. **Identifying the full scaling limit.** We now prove the following statements inductively on the number of curves \( N \). Statement (ii) in the proposition below is Theorem 5.8.

**Proposition 5.16.** Under the setup and assumptions of Theorem 5.8, the following hold.

i) The local-to-global multiple SLE with partition function \( Z_N \), on any domain \( (\Lambda; p_1, \ldots, p_{2N}) \) with \( 2N \) distinct marked prime ends with radial limits, exists as a random variable in \( X(\mathbb{C})^N \). Furthermore, interpreting its conditional-law definition as a sampling procedure of curve initial segments from the local multiple SLE in a given order, sampling the initial or final segments as local multiple SLEs in any order yields the same distribution of full curves.


ii) The weak limits \((\gamma_{D;1}, \ldots, \gamma_{D;N})\) and \((\gamma_1, \ldots, \gamma_N)\) (the latter only when considering non-relaxed regularity at marked boundary points) of the curves \((\gamma_{D;1}^{(n)}, \ldots, \gamma_{D;N}^{(n)})\) and \((\gamma_1^{(n)}, \ldots, \gamma_N^{(n)})\), are local-to-global multiple SLE with partition function \(Z_N\), on domains \((\mathbb{D}; p_1, \ldots, p_{2N})\) and \((\Lambda; p_1, \ldots, p_{2N})\), respectively.

iii) The multiple SLE of part (i) above is Carathéodory stable in the following precise sense: if \((\Lambda_m; p_1^{(m)}, \ldots, p_{2N}^{(m)})\) and \((\Lambda; p_1, \ldots, p_{2N})\) are uniformly bounded simply-connected planar domains with \(2N\) distinct marked prime ends with radial limits, the former being close Carathéodory approximations of the latter as \(m \to \infty\), then the local-to-global multiple SLEs on \((\Lambda_m; p_1^{(m)}, \ldots, p_{2N}^{(m)})\) converge weakly in \(X(\mathbb{C})^N\) to local-to-global multiple SLE on \((\Lambda; p_1, \ldots, p_{2N})\).

**Proof of the case base** \(N = 1\). For the base case \(N = 1\), we will prove the claim using the weak convergence \(\gamma_{D;1}^{(n)} \to \gamma_{D;1}\). For the existence in part (i), in the preceding subsection, we defined the marginal law of one curve in the local-to-global multiple SLE as the weak limit of the curves \(\gamma_{D;1}^{(n)}\). Thus, if there is only one curve, the local-to-global multiple SLE on the unit disc exists as this weak limit. The existence in general domains follows by Assumption 5.4 and Theorem 4.11(B). For the order of sampling in part (i), there are two possible starting points from which we can grow the curve \(\gamma_{D;1}\) when sampling with the Loewner growth processes. Lemma 5.14 and the discussion on one-curve marginals holds for both starting points, so sampling the local multiple SLE growth from either starting point, we get the Loewner description of the limiting curve \(\gamma_{D;1}\). This finishes part (i) in the base case \(N = 1\).

As \(N = 1\), part (ii) follows directly from the identification of one-curve marginals in Corollary 5.15.

Part (iii) can be proven by the following argument, relying on Assumption 5.4. For each \(m\), let \(\gamma_m^{(n)}\) be the discrete curves on the lattice approximations \((\Lambda_{n;m}; p_1^{(n,m)}, p_2^{(n,m)})\) of \((\Lambda_m; p_1^{(m)}, p_2^{(m)})\) given by Assumption 5.4 and let \(\gamma_m\) denote the local-to-global multiple SLE on \((\Lambda_m; p_1^{(m)}, p_2^{(m)})\), i.e., the weak limit of \(\gamma_m^{(n)}\) as \(n \to \infty\). Recall that weak convergence is metrizable. It is easy to see that one can define inductively an increasing sequence \(n(m)\) such that \((\Lambda_{n(m);m}; p_1^{(n(m),m)}, p_2^{(n(m),m)})\) are close Carathéodory approximations of \((\Lambda; p_1, p_2)\), and the distance of \(\gamma_m^{(n(m))}\) and \(\gamma_m\) in the metric of weak convergence tends to zero as \(m \to \infty\). By part (ii), \(\gamma_m^{(n(m))}\) tends weakly to \(\gamma\) as \(m \to \infty\), so also the distance of \(\gamma\) and \(\gamma_m\) in the metric of weak convergence tends to zero as \(m \to \infty\).

**Proof of the induction step.** Let us now assume that the three properties in the statement of Proposition 5.10 hold for any number of curves \(1, 2, \ldots, (N-1)\), and show that they then also hold for \(N\) curves. We first prove property (ii). Let us start with an analogy of Lemma 5.12.

**Lemma 5.17.** Any subsequential limit \((\gamma_{D;1}, \ldots, \gamma_{D;N})\) of the curves \((\gamma_{D;1}^{(n)}, \ldots, \gamma_{D;N}^{(n)})\) satisfies, for any bounded continuous functions \(g : X(\mathbb{D})^{N-1} \to \mathbb{R}\), and \(f : (\mathbb{D}) \to \mathbb{R}\),

\[
\mathbb{E}[f(\gamma_{D;1})g(\gamma_{D;2}, \ldots, \gamma_{D;N})] = \mathbb{E}[f(\gamma_{D;1})\mathbb{E}_{D \setminus \gamma_{D;1}}^{(N-1)-SLE}\{g(\eta_1, \ldots, \eta_{N-1})\}],
\]

where \(\mathbb{E}_{D \setminus \gamma_{D;1}}^{(N-1)-SLE}\) on the right-hand side denotes the following: \((\eta_1, \ldots, \eta_{N-1})\) are the local-to-global multiple SLE of \((N-1)\) curves on \(D \setminus \gamma_{D;1}\) with the remaining marked boundary points. If \(D \setminus \gamma_{D;1}\) is not simply connected, it should be interpreted as two independent local-to-global multiple SLEs on the two connected components of \(D \setminus \gamma_{D;1}\) that are adjacent to the remaining marked boundary points.

Note that the one or two connected components of \(D \setminus \gamma_{D;1}\) adjacent to the remaining marked boundary points are (almost surely) simply-connected by Lemma 5.14 and the local-to-global multiple SLEs on them exist by the inductive assumption (i).

---

7 The statement in the special case \(N = 1\) can be seen as more or less standard properties of chordal SLEs. Instead, we use here arguments that are less standard for chordal SLEs, but generalize to \(N \geq 2\).
Proof of Lemma 5.10: Assume for notational simplicity that a weakly converging subsequence has been extracted, so that $(\gamma_{D;1}^{(n)}, \ldots, \gamma_{D;N}^{(n)}) \to (\gamma_{D;1}, \ldots, \gamma_{D;N})$. Start with the triangle inequality:

$$|E[f(\gamma_{D;1}^{(n)})g(\gamma_{D;2}^{(n)}, \ldots, \gamma_{D;N}^{(n)})] - E[f(\gamma_{D;1})][E^{(N-1)-\text{SLE}}_{D \setminus \gamma_{D;1}^{(n)}}[g(\eta_1, \ldots, \eta_{N-1})]]|$$

(5.5)

$$\leq |E[f(\gamma_{D;1}^{(n)})g(\gamma_{D;2}^{(n)}, \ldots, \gamma_{D;N}^{(n)})] - E[f(\gamma_{D;1}^{(n)})][E^{(N-1)-\text{SLE}}_{D \setminus \gamma_{D;1}^{(n)}}[g(\eta_1, \ldots, \eta_{N-1})]]|$$

$$+ |E[f(\gamma_{D;1}^{(n)})][E^{(N-1)-\text{SLE}}_{D \setminus \gamma_{D;1}^{(n)}}[g(\eta_1, \ldots, \eta_{N-1})]] - E[f(\gamma_{D;1})][E^{(N-1)-\text{SLE}}_{D \setminus \gamma_{D;1}^{(n)}}[g(\eta_1, \ldots, \eta_{N-1})]]|$$

where the SLE curves on both $D \setminus \gamma_{D;1}$ and $D \setminus \gamma_{D;1}^{(n)}$ run between the limiting marked points $\tilde{p}_1, \ldots, \tilde{p}_{2N} \in \partial D$.

We claim that all terms in (5.5) can be made arbitrarily small by choosing $n$ large enough. For the first term, this holds by the weak convergence. Likewise, the third term follows by weak convergence: namely,

$$\gamma_{D;1} \mapsto E^{(N-1)-\text{SLE}}_{D \setminus \gamma_{D;1}^{(n)}}[g(\eta_1, \ldots, \eta_{N-1})]$$

is a continuous function of the curve $\gamma_{D;1}$ not visiting $\partial D$ except at its end points, by the inductive assumption (iii).

For the second term, notice that by the DDMP,

$$E^{(n)}[f(\gamma_{D;1}^{(n)})g(\gamma_{D;2}^{(n)}, \ldots, \gamma_{D;N}^{(n)})] = E^{(n)}[f(\gamma_{D;1}^{(n)})][E^{(n)}_{D \setminus \gamma_{D;1}^{(n)}}[g(\gamma_{D;2}^{(n)}, \ldots, \gamma_{D;N}^{(n)})]]$$

The rest is similar to the proof of Lemma 5.12 by tightness of the sequence $\gamma_1^{(n)}$, we take a compact set $K_\varepsilon \subset X(D)$, containing $1 - \varepsilon$ of probability mass of $\gamma_{D;1}^{(n)}$ for all $n$. It then suffices to show that the convergence

(5.6)

$$E^{(n)}_{D \setminus \gamma_{D;1}^{(n)}}[g(\gamma_{D;2}^{(n)}, \ldots, \gamma_{D;N}^{(n)})] \to E^{(N-1)-\text{SLE}}_{D \setminus \gamma_{D;1}}[g(\eta_1, \ldots, \eta_{N-1})]$$

is uniform over $\gamma_{D;1}^{(n)} \in K_\varepsilon$. The intersection of a compact set with a closed set is compact. Thus, by Assumption 5.5 we may assume that $K_\varepsilon$ is such that $\gamma_{D;1}^{(n)}$ never visits at distance $< \delta$-neighbourhood of $\partial D$, except at distance $\leq \delta'$ from its end points.

Assume for a contradiction that the convergence (5.6) is not uniform over $K_\varepsilon$. I.e., there exist deterministic curves $\nu^{(n)} \in K_\varepsilon$, each possible to be observed as the curves $\gamma_{D;1}^{(n)}$, such that for some $\ell > 0$ and infinitely many $n$ we have

(5.7)

$$|E^{(n)}_{D \setminus \nu^{(n)}}[g(\gamma_{D;2}^{(n)}, \ldots, \gamma_{D;N}^{(n)})] - E^{(N-1)-\text{SLE}}_{D \setminus \nu^{(n)}}[g(\eta_1, \ldots, \eta_{N-1})]| > \ell.$$

By compactness, extract a convergent subsequence (which we suppress in notation), $\nu^{(n)} \to \nu$ for which (5.7) holds. Now, by the inductive assumption (iii), we have

(5.8)

$$E^{(N-1)-\text{SLE}}_{D \setminus \nu}[g(\eta_1, \ldots, \eta_{N-1})] \to E^{(N-1)-\text{SLE}}_{D \setminus \nu}[g(\eta_1, \ldots, \eta_{N-1})].$$

where the multiple SLE in $D \setminus \nu$ makes sense as we restricted the boundary visits of $\nu$ by our choice of $K_\varepsilon$.

Now, recall that the curves $\gamma_2^{(n)}, \ldots, \gamma_N^{(n)}$ originate in a DDMP lattice model where the one-curve model satisfies the conformally invariant condition (C). DDMP and condition (C) clearly also hold for the conformal images $\gamma_2^{(n)}, \ldots, \gamma_N^{(n)}$. Also the assumptions imposed on the discrete domains in Theorem 4.1 hold for the curves $\gamma_2^{(n)}, \ldots, \gamma_N^{(n)}$, and thus we can deduce precompactness by that theorem. Now, by inductive assumption (ii), the conformal images of the curves $\gamma_2^{(n)}, \ldots, \gamma_N^{(n)}$ when the two connected components of $\phi_n^{-1}(D \setminus \nu^{(n)})$ are both mapped to $D$, tend weakly to a local-to-global multiple SLE.
But these are also the conformal images of $\gamma_{D;2}^{(n)}, \ldots, \gamma_{D;N}^{(n)}$ on $\mathbb{D}$, so by Theorem 4.1(B), also the curves $(\gamma_{D;2}^{(n)}, \ldots, \gamma_{D;N}^{(n)})$ converge weakly to the local-to-global multiple SLE in the two connected components of $\mathbb{D} \setminus \nu$. Thus,

$$E^{n}_{D;\nu} [g(\gamma_{D;2}^{(n)}, \ldots, \gamma_{D;N}^{(n)})] \to E^{(N-1)-\text{SLE}}_{D;\nu} [g(\eta_1, \ldots, \eta_{N-1})].$$

The two convergences (5.8) and (5.9) together contradict (5.7), finishing the proof. □

Let us now finish the induction step in the proof of Proposition 5.16. Notice that Lemma 5.17 implies that for any bounded continuous function $g : X^N \to \mathbb{R}$, we have

$$E[g(\gamma_{D;2}, \ldots, \gamma_{D;N}) \mid \sigma(\gamma_{D;1})] = E^{(N-1)-\text{SLE}}_{D;\gamma_{D;1}} [g(\eta_1, \ldots, \eta_{N-1})].$$

(By inductive assumption (iii), the right-hand side above is a continuous function of $\gamma_{D;1}$.) By Proposition A.1, this shows that the regular conditional law of $(\gamma_{D;2}, \ldots, \gamma_{D;N})$ given $\gamma_{D;1}$ is the local-to-global multiple SLE of $(N-1)$ curves. Since the marginal law of $\gamma_{D;1}$ is by Corollary 5.15 the one-curve marginal of the local-to-global multiple SLE of $N$ curves, this identifies the law of the curves $(\gamma_{D;1}, \ldots, \gamma_{D;N})$ as the local-to-global multiple SLE. For domains with non-relaxed regularity at marked boundary points, the weak convergence of $(\gamma_1^{(n)}, \ldots, \gamma_N^{(n)})$ is guaranteed by Theorem 4.1(B). This proves the induction step for property (ii).

For property (i), the local-to-global multiple SLE exists as the scaling limit in the proof of property (ii). The independence on sampling order follows from that in the discrete case. The proof of property (iii) is identical to the base case $N = 1$. □

5.4. **Proof of Theorem 5.8 under Assumption 5.6** We now prove Theorem 5.8 under Assumption 5.6.

5.4.1. Identifying the scaling limit of up-to-swallowing initial segments. Let us start an analogue of Section 5.3.1. Continue in the notation introduced there. Denote by $\lambda_{(0)}$ the initial segment of $\gamma_{D;1}$ up to hitting the closed boundary arc $(\hat{p}_2\hat{p}_2N) \subset \partial \mathbb{D}$. We call $\lambda_{(0)}$ the up-to-swallowing initial segment of $\gamma_{D;1}$. Denote by $\vartheta(\delta)$ and $\vartheta_{(0)}$ the remainder of the curve $\gamma_{D;1}$ after the initial segments $\lambda(\delta)$ and $\lambda_{(0)}$, respectively. Repeating the arguments of Lemma 5.14 and Corollary 5.15 one readily obtains:

**Lemma 5.18.** For any subsequential scaling limit $(\gamma_{D;1}, \ldots, \gamma_{D;N})$, we have $\lambda(\delta) \to \lambda_{(0)}$ and $\vartheta(\delta) \to \vartheta_{(0)}$ almost surely as $\delta \downarrow 0$. In particular, the marginal law of $\lambda_{(0)}$ is the local multiple SLE for all subsequential limits.

By the almost sure convergence above, the local multiple SLE initial segment from $\hat{p}_1$ to $(\hat{p}_2\hat{p}_2N)$ exists as a closed curve and yields $\lambda_{(0)}$, analogously to Corollary 5.15.

One would expect that $\lambda_{(0)}$ terminates at an even-index marked boundary point if and only if the SLE has parameter $\kappa \in (0, 4]$. This indeed holds true, by an a posteriori proof, relying on the properties of the chordal SLE($\kappa$). In order not to mix a posteriori and a priori properties of the scaling limits, we will first finish the proof of Theorem 5.8 without knowledge of where $\lambda_{(0)}$ terminates, and return to this discussion in Proposition 5.20 in Section 5.4.4.

5.4.2. Distance from the initial segment to marked boundary arcs. With the marginal law of $\lambda_{(0)}$ determined, note that by definition, $\lambda_{(0)}$ only visits the boundary arc $(\hat{p}_2\hat{p}_2N) \subset \partial \mathbb{D}$ at its end point. We now explicate two simple but important consequences of this trivial observation.
First, let $B_1$ and $B_2$ be closed marked boundary arcs of $\mathbb{D}$, between some neighbouring marked boundary points $\tilde{p}_1, \ldots, \tilde{p}_{2N}$. Assume that $B_1$ and $B_2$ are not adjacent to the boundary point $\tilde{p}_1$ and not adjacent to each other. As $\delta' \downarrow 0$, we have the following approximation of events:

\begin{equation}
(5.10) \quad \{\lambda_0 \text{ visits } \delta'\text{-close to } B_1\} \cap \{\lambda_0 \text{ visits } \delta'\text{-close to } B_2\} \downarrow \{\lambda_0 \text{ visits } B_1\} \cap \{\lambda_0 \text{ visits } B_2\} = \emptyset.
\end{equation}

In particular, taking $\delta'$ small enough, we can make the probability of the event (5.10) arbitrarily small.

Second, let $B_1$ and $B_2$ now be closed marked boundary arcs of $\partial \mathbb{D}$, not adjacent to the boundary point $\tilde{p}_1$ but possibly adjacent to each other. Fix a (small) $\delta' > 0$. This time, as $\delta'' \downarrow 0$, we have

\begin{equation}
(5.11) \quad \{\lambda_0 \text{ visits } \delta''\text{-close to the } \delta'\text{-interior of } B_1\} \cap \{\lambda_0 \text{ visits } \delta''\text{-close to } B_2\} \downarrow \{\lambda_0 \text{ visits the } \delta'\text{-interior of } B_1\} \cap \{\lambda_0 \text{ visits } B_2\} = \emptyset.
\end{equation}

Useful interpretations of these computations play a role analogous to Assumption 5.5 in the proof of Theorem 5.8, see illustration in Figure 5.2.

5.4.3. Identifying the full scaling limit. To complete the proof of Theorem 5.8 under Assumption 5.6, one proves Proposition 5.16 with that assumption in stead of Assumption 5.5. The proof of the Proposition remains identical, except for Lemma 5.17, which now has to be replaced by the following.

**Lemma 5.19.** Under Assumption 5.6 in the setup of Theorem 5.8, any subsequential limit $(\gamma_{\mathbb{D};1}, \ldots, \gamma_{\mathbb{D};N})$ of the curves $(\gamma_{\mathbb{D};1}^{(n)}, \ldots, \gamma_{\mathbb{D};N}^{(n)})$ satisfies the following: the tip $\lambda_0(1)$ is almost surely not an odd-index marked boundary point, and for any bounded, non-negative, Lipschitz continuous test functions $g$ :
\[ X(\mathbb{D})^N \to \mathbb{R}, \text{ and } f : X(\mathbb{D}) \to \mathbb{R}, \]
\[ \mathbb{E}[f(\lambda_0(0))g(\vartheta_0; \gamma_D; 2; \ldots; \gamma_D; N)] \]
\[ = \mathbb{E}[\{ \lambda_0(1) \text{ is an even-index marked boundary point} \} \mathbb{E}^{(N-1)\text{-SLE}}_{\mathbb{D}\setminus\lambda_0} [g(w, \eta_1, \ldots, \eta_{N-1})] \]
\[ + \mathbb{E}[\{ \lambda_0(1) \text{ is not a marked boundary point} \} f(\lambda_0)\mathbb{E}^{\text{SLE}}_{\mathbb{D}\setminus\lambda_0} [\mathbb{E}^{\text{SLE}}_{\mathbb{D}\setminus\lambda_0} g(o, e)], \]
where the notation \( \mathbb{E}^{(N-1)\text{-SLE}}_{\mathbb{D}\setminus\lambda_0} \) in the first expectation, i.e., when \( \lambda_0(0) \) traverses between two marked boundary points, is interpreted as the similar notation in Lemma 5.17; the notations \( \mathbb{E}^{\text{SLE}}_{\mathbb{D}\setminus\lambda_0} \) and \( \mathbb{E}^{\text{SLE}}_{\mathbb{D}\setminus\lambda_0} \) in turn are multiple SLE expectations in the components of \( \mathbb{D}\setminus\lambda_0 \) with an even and odd number of marked boundary points, respectively, between the remaining marked boundary points and, in the odd component, the tip of the initial segment \( \lambda_0(0) \) (the notation \( g(o, e) \) is a slightly abusive shorthand since the original argument \( \vartheta_0, \gamma_D; 2; \ldots; \gamma_D; N \) of \( g \) should actually be replaced by the obvious re-labelling of the curves \( o, \gamma_D; 2; \ldots; \gamma_D; N \)).

**Proof.** First, the annulus crossing condition (G) guarantees that \( \gamma_D; 1 \) will almost surely not hit an odd boundary point. Next, note that it then suffices to prove the claim for nonnegative functions \( f \) such that for some \( \delta > 0 \), \( f(\lambda_0(0)) \) takes the value 0 if \( \lambda_0(0) \) visits the \( \delta \)-neighbourhood of some odd-index boundary point other than the first one. Indeed, the case of a general \( f \) then follows by taking increasing approximations of \( f \) and using Monotone convergence. We will assume this property of \( f \), keeping \( \delta > 0 \) fixed throughout the proof.

Next, take a small auxiliary radius \( \delta' < \tilde{\delta} \) and denote by \( \lambda_0(0)(1) \) the tip of the curve \( \lambda_0(0) \). By our assumption on \( f \), either \( B(\lambda_0(1), \delta') \) contains an even-index marked boundary point, or it contains no marked boundary points, or \( f(\lambda_0(0)) = 0 \). We will treat the two first nontrivial cases separately, and in the end combine the results again in the limit \( \delta' \downarrow 0 \). To formalize this, write \( f = 1 \cdot f \) and decompose 1 into a sum continuous cutoff functions of \( \lambda_0(0) \); to be explicit, for instance
\[ c_1^{(\delta')}(\lambda_0(0)) = \min \left\{ \frac{d(\lambda_0(0)(1), B(\tilde{\vartheta}, \delta') \cup \ldots \cup B(\tilde{\vartheta}, \delta'))}{\delta'} ; 1 \right\}, \]
\[ c_2^{(\delta')}(\lambda_0(0)) = 1 - c_1^{(\delta')}(\lambda_0(0)). \]
We observe that both \( c_1^{(\delta')}f \) and \( c_2^{(\delta')}f \) are bounded, Lipschitz continuous, nonnegative functions, for each \( \delta' > 0 \). The function \( c_1^{(\delta')}f \) takes nonzero values only when the tip \( \lambda_0(1)(1) \) is at a distance \( \geq \delta' \) from all the marked boundary points \( \tilde{\vartheta}_2, \ldots, \tilde{\vartheta}_{2N} \), whereas the function \( c_2^{(\delta')}f \) takes nonzero values only when the tip \( \lambda_0(1)(1) \) is at a distance \( \leq 2\delta' \) from some even-index boundary point.

**The term \( c_2^{(\delta')}f \):** Consider first the term \( c_2^{(\delta')}f \) and start by computing
\[ \mathbb{E}[c_2^{(\delta')}f(\lambda_0(0))g(\vartheta_0; \gamma_D; 2; \ldots; \gamma_D; N)] \]
\[ = \mathbb{E}[c_2^{(\delta')}f(\lambda_0(0))g(\vartheta_0; \gamma_D; 2; \ldots; \gamma_D; N)] + o_0^{(\delta')}(1) \]
\[ = \mathbb{E}[c_2^{(\delta')}f(\lambda_0^{(n)}(0))g(\vartheta_0^{(n)}; \gamma_D; 2; \ldots; \gamma_D; N)] + o_0^{(\delta')}(1) + o_0^{(\delta')}(1) \]
where the equality \[ 5.13 \] holds by the almost sure convergences in Lemma 5.18 and \( o_0^{(\delta')}(1) \) stands for \( o(1) \) as \( \delta \downarrow 0 \) for any fixed \( \delta' \), while \[ 5.14 \] holds by the weak convergence of the discrete models and \( o_0^{(\delta')}(1) \) stands for \( o(1) \) as \( n \to \infty \) for any fixed \( \delta \) and \( \delta' \).

Recall that the function in the expectation \[ 5.14 \] takes values \( \neq 0 \) only when the tip \( \lambda_0^{(n)}(1) \) is at a distance \( \leq 2\delta' \) from some even-index boundary point \( \tilde{\vartheta}_2, \ldots, \tilde{\vartheta}_{2N} \); call it \( w \). Assume that \( \delta < \delta' \) so that this is possible and condition on such a \( \lambda_0^{(n)}(0) \). Now, when having grown the initial segment \( \lambda_0^{(n)}(0) \) only up to the first hitting of \( B(w, 2\delta') \), the circle arcs \( \partial B(w, 2\delta') \) and \( \partial B(w, 2\sqrt{\delta'}) \) allow one to define two topological quadrilaterals of modulus \( 1/o_0^{(\delta')}(1) \) that separate the tip of that smaller segment and the boundary point \( w \) from all other marked boundary points; see Figure 5.3. By Condition (G'), it thus occurs with probability \( \geq 1 - o_0^{(\delta')}(1) \) that \( \gamma_D; 1 \) actually connects to the boundary point \( w \), where the term
and $\delta$ in (5.13) within the same error: Lipschitz continuous, we can replace $\delta$ with a factor $c$.

Let $\vartheta_{(\delta)}^{(n)}$ now be a continuous cutoff function taking value 1 if $\vartheta_{(\delta)}^{(n)}$ never exits the ball $B(w, 2\sqrt{\delta'})$ and 0 if it exits the ball $A(w, 3\sqrt{\delta'})$. By the above paragraph, making an error $o_{\delta'}(1)$ uniform over $n$ and $\delta$ with $\delta < \delta'$, we can add a factor $c(\delta')(\vartheta_{(\delta)}^{(n)})$ to (5.14). Likewise, since the function $g$ is bounded and Lipschitz continuous, we can replace $\vartheta_{(\delta)}^{(n)}$ in its argument by the curve consisting of the single point $w$ in (5.13) within the same error:

$$E^{(n)}[c(\delta')(\vartheta_{(\delta)}^{(n)})c_2(\lambda_{(\delta)}^{(n)})g(w, \gamma_{(\delta)}^{(n)}, \ldots, \gamma_{(\delta)}^{(n)}); \delta, \delta')] = o_{\delta'}(1) + o_{n, \delta, \delta'}(1) + o_{(\delta')(1)}.$$

The limit of the above expectation as $n \to \infty$ can now be treated as in the proof of Lemma 5.17 by using the inductively assumed Proposition 5.16(ii) for the curves $\gamma_{(\delta)}^{(n)}, \ldots, \gamma_{(\delta)}^{(n)}$ (the estimate in Figure 5.2(left) replaces Assumption 5.9). This yields

$$E^{(n)}[c(\delta')(\vartheta_{(\delta)}^{(n)})c_2(\lambda_{(\delta)}^{(n)})g(w, \gamma_{(\delta)}^{(n)}, \ldots, \gamma_{(\delta)}^{(n)}); \delta, \delta')] = E[c(\delta')(\vartheta_{(\delta)}^{(n)})c_2(\lambda_{(\delta)}^{(n)})g(\omega, \gamma_{(\delta)}^{(n)}, \ldots, \gamma_{(\delta)}^{(n)}); \delta, \delta')] + o_{(\delta')(1)}.$$

where the last step used the weak convergence $\gamma_{(\delta)}^{(n)} \to \gamma_{(\delta)}^{(n)}$. (This is possible since when $c(\delta')(\vartheta_{(\delta)}^{(n)})c_2(\lambda_{(\delta)}^{(n)}) \neq 0$, the expectation $E^{(N-1)-SLE}_{\omega, \gamma_{(\delta)}^{(n)}, \ldots, \gamma_{(\delta)}^{(n)}}$ makes sense by the estimate in Figure 5.2(left) and is a continuous function.
of $\gamma_{\Delta;1}$ by inductive assumption (iii).) Finally, substituting this back, we have

$$
\mathbb{E}[c_2^{(\delta')}(\vartheta(\delta))c_2^{(\delta')}(\lambda(\delta))f(\lambda(\delta))E_{D;\gamma_{\Delta;1}}^{(N-1)-\text{SLE}_6}[g(w,\eta_1,\ldots,\eta_{N-1})] + o_\delta(1) + o_\delta^{(\delta,\delta')}(1) + o_\delta^{(\delta')}(1).
$$

Note that in the above equation, $n$ only appears in the Landau-o-terms. We would like to achieve this for $\delta$ and $\delta'$, too. Thus, study the expectation above first in the limit $\delta \downarrow 0$ using the almost sure convergence of Lemma 5.18, and then in the limit $\delta' \downarrow 0$. Using Bounded convergence theorem in these limits yields

$$
\mathbb{E}[c_2^{(\delta')}(\vartheta(\delta))c_2^{(\delta')}(\lambda(\delta))f(\lambda(\delta))E_{D;\gamma_{\Delta;1}}^{(N-1)-\text{SLE}_6}[g(w,\eta_1,\ldots,\eta_{N-1})] + o_\delta(1) + o_\delta^{(\delta,\delta')}(1) + o_\delta^{(\delta')}(1)
$$

Now, the three Landau-o-terms above can be all made simultaneously arbitrarily small by choosing first $\delta'$ small enough, then $\delta$ small enough, and then $n$ large enough. In addition, Condition (G) for the curves $\gamma_{\Delta;1}^{(n)}$ guarantees that $\gamma_{\Delta;1}^{(n)}$ almost surely only visits an even-index marked boundary point once, namely, its target point in the end of the curve. Thus, we obtain

$$
\lim_{\delta' \downarrow 0} \mathbb{E}[\{\vartheta(\delta) \text{ is a point}\} \{\lambda(\delta) \text{ ends at even-index boundary point } w\}]f(\lambda(0))E_{D;\gamma_{\Delta;1}}^{(N-1)-\text{SLE}_6}[g(w,\eta_1,\ldots,\eta_{N-1})] + o_\delta(1) + o_\delta^{(\delta,\delta')}(1) + o_\delta^{(\delta')}(1),
$$

This finishes our treatment of the term $c_2^{(\delta')f}$.

**The term $c_1^{(\delta')f}$:** Consider next the term $c_1^{(\delta')f}$ and start by computing

$$
\mathbb{E}[c_1^{(\delta')}(\lambda(\delta))f(\lambda(\delta))g(\eta_0,\gamma_{\Delta;2},\ldots,\gamma_{\Delta;N})]
$$

Recall that $c_1^{(\delta')}(\lambda(\delta))f(\lambda(\delta))$ takes nonzero values only when the tip $\lambda(\delta)(1)$ is at a distance $\geq \delta'$ from all the marked boundary points. Condition on such a $\lambda(\delta)$, and assume that $\sqrt{\delta} < \delta'$ so that the tip of $\lambda(\delta)$ is very close to $\partial\mathbb{D}$. In that case, removing the curve $\lambda(\delta)$ and the ball $B(\lambda(\delta)(1),\sqrt{\delta})$ at its tip, the remaining boundary $\partial\mathbb{D} \setminus \lambda(\delta) \setminus B(\lambda(\delta)(1),\sqrt{\delta})$ will consist of two connected components, containing altogether $N-1$ marked boundary points. Both of these boundary arcs contain a nonzero amount of boundary points, and it is natural to call even the arc with an even number of them, and the other one odd. We call the connected components of $\mathbb{D} \setminus \lambda(\delta) \setminus B(\lambda(\delta)(1),\sqrt{\delta})$ adjacent to these arcs even and odd, respectively. Note also that given $\lambda(\delta)$, we know the indices of the curves $\gamma_{\Delta;1}^{(n)}, \gamma_{\Delta;2}^{(n)}, \ldots$ that start from the even and odd arc, as each curve starts from an odd-index boundary point. We denote by $e^{(n)}$ the collection of curves starting from the even arc and by $o^{(n)}$ the rest, i.e., the remainder curve $\vartheta(\delta)$ and the curves starting in the odd components. We will also denote, with a slight abuse of notation in the curve indexing

$$
g(\vartheta^{(n)}(\delta),\gamma_{\Delta;2}^{(n)},\ldots,\gamma_{\Delta;N}^{(n)}) = g(o^{(n)},e^{(n)}).
$$

First, identically to the case of the term $c_2^{(\delta')f}$, observe that given $\lambda(\delta)$ there is a conditional probability $1 - o_\delta(1)$ that the even curves $e^{(n)}$ only intersect one connected component of $\mathbb{D} \setminus \lambda(\delta) \setminus B(\lambda(\delta)(1),\sqrt{\delta})$. 
Denote this event by $P(\delta)$. Note that on this event, the curves $\mathbf{e}^{(n)}$ are only adjacent to the even component boundary points. Note also that $P(\delta)$ only depends on $\lambda^{(n)}(\delta)$ and the even curves $\mathbf{e}^{(n)}$. With a slight abuse of notation, we will denote the corresponding Borel set on the space of collections of curves $X(\overline{\mathbb{D}})^N$ also by $P(\delta)$.

Next, take $\delta'' > 0$ with $\delta < \delta'' < \delta'$. We denote $\lambda(\delta) \in E(\delta', \delta'')$ if the tip $\lambda^{(n)}(\delta)(1)$ is at a distance $\geq \delta'$ from all the marked boundary points (i.e., $c_1^{(\delta')}(\lambda^{(n)}(\delta)) f(\lambda^{(n)}(\delta)) \neq 0$), and the curve $\lambda(\delta)$ visits at distance $\leq \delta''$ from some marked boundary arc not closest to the tip of $\lambda^{(n)}(\delta)$ or adjacent to $\tilde{p}_1$. It follows from the deduction in Figure 5.2 that for the measure $P$ of the weak limit $\lambda(\delta)$

$$\mathbb{P}[\lambda(\delta) \in E(\delta', \delta'')] = o(\delta'') \big(1\big),$$

where the $o(\delta'')(1)$-term is independent of $\delta$ apart from the requirement $\delta < \delta'$. Note that $E(\delta', \delta'')$ is a closed set in $X(\overline{\mathbb{D}})$. By Portmanteau’s theorem on weak convergence, and the weak convergence $\lambda^{(n)}(\delta) \to \lambda(\delta)$, it follows that

$$\mathbb{P}^{(n)}[\lambda^{(n)}(\delta) \in E(\delta', \delta'')] = o(\delta'') \big(1\big),$$

where $o(\delta'')(1)$ is small uniformly over all $n$ large enough, $n > n_0(\delta, \delta', \delta'')$.

Using the boundedness of the involved functions, we can add the indicator functions $\mathbb{I}_{P(\delta)}(\lambda^{(n)}(\delta), e^{(n)})$ and $\mathbb{I}_{E(\delta', \delta'')} \mathcal{C}(\lambda^{(n)}(\delta))$ to (5.16) within errors of $o(1)$ and $o(\delta'') \big(1\big)$, respectively:

$$\mathbb{E}^{(n)} \big[ \mathbb{I}_{E(\delta', \delta'')} \mathcal{C}(\lambda^{(n)}(\delta)) f(\lambda^{(n)}(\delta)) E^{(n)}(\mathbb{D} \setminus \lambda^{(n)}(\delta)) \mathbb{I}_{P(\delta)}(\lambda^{(n)}(\delta), e^{(n)}) g(o^{(n)}, e^{(n)}) \big] + o(\delta'')(1) + o_n(\delta, \delta')(1) + o(\delta')(1)
$$

$$= \mathbb{E}^{(n)} \big[ \mathbb{I}_{E(\delta', \delta'')} \mathcal{C}(\lambda^{(n)}(\delta)) f(\lambda^{(n)}(\delta)) E^{(n)}(\mathbb{D} \setminus \lambda^{(n)}(\delta)) \mathbb{I}_{P(\delta)}(\lambda^{(n)}(\delta), e^{(n)}) \mathbb{E}^{(n)}(\mathbb{D} \setminus \lambda^{(n)}(\delta) \setminus e^{(n)}) g(o^{(n)}, e^{(n)}) \big]$$

$$+ o(\delta'')(1) + o_n(\delta, \delta')(1) + o(\delta')(1),$$

where the latter step used the DDMP.

Next, we replace the odd curves by SLEs: By tightness we may assume that $e^{(n)}$ and $\lambda^{(n)}(\delta)$ belong to a compact set $K_\varepsilon$ carrying a large probability mass. The intersection of $K_\varepsilon$ with the closed sets $E(\delta, \delta', \delta'') \mathcal{C}$ and $P(\delta)$ is compact.

**Claim**: We can choose the former compact sets $K_\varepsilon$ suitably, so that uniformly over all $e^{(n)}$ and $\lambda^{(n)}(\delta)$ in the latter compact sets, we have the convergence as $n \to \infty$

$$\mathbb{E}^{(n)}(\mathbb{D} \setminus \lambda^{(n)}(\delta) \setminus e^{(n)}) \big[ g(o^{(n)}, \cdot) \big] - \mathbb{E}^{(n)}(\mathbb{D} \setminus \lambda^{(n)}(\delta) \setminus e^{(n)}) \big[ g(o^{(n)}, \cdot) \big] = o_n(\delta, \delta', \delta'')(1),$$

also uniformly over any arguments $\delta$ of $g$; here the $m$-SLE curves $o$ now run in $\mathbb{D} \setminus \lambda^{(n)}(\delta) \setminus e^{(n)}$ between the tip of $\lambda^{(n)}(\delta)$ and the limiting boundary points $\tilde{p}_1, \ldots, \tilde{p}_{2N}$ on $\partial \mathbb{D}$. The proof of this claim is based on the inductively assumed SLE convergence in Proposition 5.16(ii). We have chosen to leave this proof to the reader, since we present a very similar but perhaps more difficult proof in the next paragraph.

Substituting (5.17) into (5.15) yields

$$\mathbb{E}^{(n)} \big[ \mathbb{I}_{E(\delta', \delta'')} \mathcal{C}(\lambda^{(n)}(\delta)) f(\lambda^{(n)}(\delta)) E^{(n)}(\mathbb{D} \setminus \lambda^{(n)}(\delta)) \mathbb{I}_{P(\delta)}(\lambda^{(n)}(\delta), e^{(n)}) \mathbb{E}^{(n)}(\mathbb{D} \setminus \lambda^{(n)}(\delta) \setminus e^{(n)}) g(o^{(n)}, e^{(n)}) \big]$$

$$+ o(\delta'')(1) + o_n(\delta, \delta', \delta'')(1) + o(\delta')(1),$$

Now, we remove the even curves from the domain of the SLEs: Equip both $\mathbb{D} \setminus \lambda^{(n)}(\delta) \setminus e^{(n)}$ and $\mathbb{D} \setminus \lambda^{(n)}(\delta)$ with the odd amount of marked boundary points $\tilde{p}_1, \ldots, \tilde{p}_{2N}$ on the odd side and one at the tip of.

---

8We will later wish to repeat almost identical computations for the even curves, and for that purpose, it is more transparent to leave this argument unspecified.
the curve $\lambda^{(n)}_\delta$, we keep these marked points implicit in the notation. Now, by tightness, $\gamma^{(n)}_{E;1}$ lie on a compact set $K_\varepsilon$ with probability $1 - \varepsilon$.

**Claim**: the sets $K_\varepsilon$ may be chosen so that uniformly over $\gamma^{(n)}_{E;1}$ with $\lambda^{(n)}_\delta \in E(\delta'; \delta'')$ and $e^{(n)}$ such that $(\lambda^{(n)}_\delta, e^{(n)}) \in \overline{P}(\delta)$, we have the convergence

$$\|E^{\text{SLE}}_{D\setminus \lambda^{(n)}_\delta \setminus e^{(n)}}[g(o, \cdot)] - E^{\text{SLE}}_{D\setminus \lambda^{(n)}_\delta}[g(o, \cdot)]\| = o_\delta^{(\delta', \delta'')} (1)$$

for any arguments $\cdot$ of $g$ that are same in both expectations; here the $o_\delta^{(\delta', \delta'')} (1)$ term is independent of $n$.

**Proof of claim.** Fix $\delta'$ and $\delta''$, and take a sequence of $\delta$s converging to zero. We suppress the sequence notation, as well as all subsequence notations to come. Assume for a contradiction that the claim does not hold, i.e., we can choose a (sub)sequence of $n = n(\delta)$s (not necessarily growing to infinity!) and deterministic curves $\nu^{(\delta)} \in K_\varepsilon$, with initial segments $\nu^{(\delta)}$ and arguments $a_\delta$ of $g$, satisfying

$$\|E^{\text{SLE}}_{D\setminus \nu^{(\delta)} \setminus e^{(n)}}[g(o, a_\delta)] - E^{\text{SLE}}_{D\setminus \nu^{(\delta)}}[g(o, a_\delta)]\| > \ell$$

for some $\ell > 0$.

We would now like to use the Carathéodory stability of multiple SLEs, i.e., inductive assumption (iii). First, recall that domains that are bounded from inside and outside are sequentially compact with respect to Carathéodory convergence (with respect to a reference point in the domain bounding from inside). Here, the domains $D \setminus \nu^{(\delta)} \setminus e^{(n)}$ and $D \setminus \nu^{(\delta)}$ are bounded from inside due to the event of $E(\delta', \delta'')$, and we may thus assume that they converge in the Carathéodory sense (with a suitable reference point at a distance $< \delta''$ from the odd boundary arc of $\partial D$). It is easily deduced that both sequences of domains converge to the same limit. By Schwarz reflection of conformal maps over $\partial D$, this Carathéodory convergence can be extended to domains with the marked boundary points on $\partial D$. Also closeness of these boundary approximations is trivial. To apply inductive assumption (iii), we have to reach these conclusions for the marked boundary points at the tip of $\nu^{(\delta)}_\delta$.

First, by the compactness of $K_\varepsilon$, we may assume that the curves $\nu^{(\delta)}_\delta$ converge, $\nu^{(\delta)}_\delta \to \nu$ in $X(\overline{D})$. Assume that the curves $\nu^{(\delta)}$ and $\nu$ come as parametrized representatives such that this uniform convergence takes place as functions, too. Next, recall that the curves $\gamma^{(n)}_{E;1}$ satisfy condition (G) by Corollary 4.6. In the proof of its consequence [KST7] Theorems 1.5 (stated as Theorem 4.4 in this paper), the compact sets $K_\varepsilon$ of $X(\overline{D})$ are chosen so that the curves in them can be described by a Loewner equation. Choosing our $K_\varepsilon$ in this manner, we thus know that $\nu$ has a Loewner description (when mapped to $\mathbb{H}$ so that its end point is at infinity).

Now, by compactness of the interval $[0, 1]$, we may assume that the times $T_0$ at which $\nu^{(\delta)}_\delta$ is stopped to obtain $\nu^{(\delta)}_\delta$ converge, $T_0 \to T$. It follows that $\nu^{(\delta)}_\delta = \nu^{(\delta)}([0, T_0]) \to \nu([0, T])$ in $X(\overline{D})$. It also follows that $\nu([0, T])$ hits $\partial \mathbb{D}$ at time $T$. Also, since $\nu$ has a Loewner description and connects to the odd component, the tip $\nu(T)$ of its initial segment $\nu([0, T])$ is on the boundary of the odd component of $D \setminus \nu([0, T])$. Now, the Carathéodory convergence $(D \setminus \nu^{(\delta)}_\delta; \nu^{(\delta)}_\delta(T_0)) \to (D \setminus \nu([0, T]); \nu(T))$ can be deduced, e.g., by showing that the harmonic measures in $D \setminus \nu^{(\delta)}_\delta$ of the boundary segment from $\tilde{p}_2$ clockwise to the tips of the curves $\nu^{(\delta)}_\delta$ converge to those with curves $\nu$. (The same holds for the Carathéodory convergence $(D \setminus \nu^{(\delta)}_\delta \setminus e^{(n)}; \nu^{(\delta)}_\delta(T_0)) \to (D \setminus \nu([0, T]); \nu(T))$ with any $e^{(n)}$ such that $\overline{P}(\delta)$ occurs.)

We yet need to show that the existence of radial limits and closeness in these Carathéodory approximations. For the first property, it is easy to see that the boundary of $D \setminus \nu([0, T])$ is locally connected, using the continuity of $\nu$. This implies (among stronger consequences) that radial limits exist at the prime end $\nu(T)$ of $D \setminus \nu([0, T])$, see [Rom92 Theorem 2.1]. Closeness of the approximations follows by comparing how $\nu^{(\delta)}([T_0, 1])$ and $\nu([T, 1])$ exit small neighbourhoods of $\nu(T)$. 


Finishing the proof is now easy. Due to the inductive assumption (iii) and the multiple SLEs \( o \) in the domains \( \mathbb{D} \setminus \nu^{(\delta)}_0 \) and \( \mathbb{D} \setminus \nu^{(\delta)}_2 \) converging weakly to the same limit \( O \), the multiple SLEs on the odd component of \( \mathbb{D} \setminus \nu(0, T) \). In particular, both are hence tight. On the other hand, by the Arzelà–Ascoli theorem and \( g \) being bounded and Lipschitz, we may pick a further subsequence so that \( g(o, \nu_\delta) \) converges as functions of \( o \), uniformly over compacts, to some function \( h(o) \). Combining this with the tightness, it follows that both expectations in (5.19) tend to \( \mathbb{E}[h(O)] \), a contradiction.

Let us continue the proof we were working on. Substituting (5.18) into (5.15), we get

\[
\begin{align*}
(5.15) &= \mathbb{E}^{(n)}\left[ \mathbb{P}_{E(O, \delta, \epsilon)}(\lambda^{(n)}_\delta)(\lambda^{(n)}_\delta: f(\lambda^{(n)}_\delta)) \mathbb{E}_{D^{\lambda^{(n)}_\delta}}\left[ \mathbb{E}_{D^{\lambda^{(n)}_\delta}}\left[ g(o, e^{(n)}) \right] \right] \right] \\
&\quad + o_n^{(\delta')}(1) + o_n^{(\delta', \delta'')}(1) + o_n^{(\delta', \delta')}(1).
\end{align*}
\]

Removing the indicator functions by identical arguments as they were introduced with, and using Fubini’s theorem, we get

\[
(5.15) = \mathbb{E}^{(n)}\left[ e^{(\delta')}(\lambda^{(n)}_\delta: f(\lambda^{(n)}_\delta)) \mathbb{E}_{D^{\lambda^{(n)}_\delta}}\left[ \mathbb{E}_{D^{\lambda^{(n)}_\delta}}\left[ g(o, e^{(n)}) \right] \right] \right] + o_n^{(\delta')}(1) + o_n^{(\delta', \delta'')}(1) + o_n^{(\delta', \delta')}(1).
\]

Now, the next steps are to introduce similar indicator functions as before, but switching the roles of odd and even curves in the definition of \( P(\delta) \). Then, repeating the uniform convergence arguments (5.19) and (5.15) for the even curves, one obtains

\[
(5.15) = \mathbb{E}^{(n)}\left[ e^{(\delta')}(\lambda^{(n)}_\delta: f(\lambda^{(n)}_\delta)) \mathbb{E}_{D^{\lambda^{(n)}_\delta}}\left[ \mathbb{E}_{D^{\lambda^{(n)}_\delta}}\left[ g(o, e^{(n)}) \right] \right] \right] + o_n^{(\delta')}(1) + o_n^{(\delta', \delta'')}(1) + o_n^{(\delta', \delta')}(1).
\]

Note that the only discrete curve left above is \( \lambda^{(n)}_\delta \). Next, we would like to use the weak convergence \( \lambda^{(n)}_\delta \to \lambda_\delta \). For this purpose, we will again have to restrict our consideration on the compact sets \( K_\delta \) that guarantee the Loewner regularity of \( \gamma^{(n)}_{\mathbb{D}:1} \). As in the proof of the claim above, we can choose the compact sets \( K_\delta \) so that for any converging sequence of curves \( \nu^{(n)} \to \nu \) in \( K_\delta \), on the additional events of \( P(\delta) \) and \( E(\delta', \delta'')^C \), it holds that the tip of the initial segment \( \nu_\delta \) is a prime end with radial limits in \( \mathbb{D} \setminus \nu_\delta \), and \( (\mathbb{D} \setminus \nu^{(n)}_\delta; \nu^{(n)}_\delta(1)) \) are close Carathéodory approximations of \( (\mathbb{D} \setminus \nu_\delta; \nu(1)) \). Thus, by the inductive assumption (iii), the expectation

\[
(5.20) \quad \mathbb{E}_{D^{\lambda^{(n)}_\delta}}\left[ \mathbb{E}_{D^{\lambda^{(n)}_\delta}}\left[ g(o, e^{(n)}) \right] \right]
\]

with respect to two independent multiple SLEs in \( \mathbb{D} \setminus \lambda^{(n)}_\delta \), is a continuous function of \( \gamma^{(n)}_{\mathbb{D}:1} \) on the intersection of \( \gamma^{(n)}_{\mathbb{D}:1} \in K_\delta \) with \( P(\delta) \) and \( E(\delta', \delta'')^C \). Now, by tightness, take a compact set in \( X(\overline{\mathbb{D})}^N \) carrying a large probability mass and intersect it with \( \gamma^{(n)}_{\mathbb{D}:1} \in K_\delta \) and \( P(\delta) \) and \( E(\delta', \delta'')^C \). The latter set is also compact with a large probability mass. Now, by Tietze’s extension theorem, the continuous function (5.20) on the latter compact set can be continued to yield a bounded continuous function the whole space \( X(\overline{\mathbb{D}})^N \). Using now the weak convergence \( (\gamma^{(n)}_{\mathbb{D}:1}, \ldots, \gamma^{(n)}_{\mathbb{D}:N}) \to (\gamma_{\mathbb{D}:1}, \ldots, \gamma_{\mathbb{D}:N}) \), we obtain

\[
(5.15) = \mathbb{E}^{(n)}\left[ e^{(\delta')}(\lambda^{(n)}_\delta: f(\lambda^{(n)}_\delta)) \mathbb{E}_{D^{\lambda^{(n)}_\delta}}\left[ \mathbb{E}_{D^{\lambda^{(n)}_\delta}}\left[ g(o, e^{(n)}) \right] \right] \right] + o_n^{(\delta')}(1) + o_n^{(\delta', \delta'')}(1) + o_n^{(\delta', \delta')}(1).
\]

Finally, fix a realization of \( \gamma_{\mathbb{D}:1} \) that can be described by the Loewner equation; it is then easy to argue that \( \mathbb{D} \setminus \lambda^{(n)}_\delta \), with the marked boundary points on \( \partial \mathbb{D} \) and at the tip of \( \lambda^{(n)}_\delta \), are close Carathéodory approximations of \( \mathbb{D} \setminus \lambda_\delta \). Thus, the almost sure convergence \( \lambda^{(n)}_\delta \to \lambda_\delta \) and the Bounded convergence theorem, we have

\[
(5.15) = \mathbb{E}^{(n)}\left[ e^{(\delta')}(\lambda^{(n)}_\delta: f(\lambda^{(n)}_\delta)) \mathbb{E}_{D^{\lambda^{(n)}_\delta}}\left[ \mathbb{E}_{D^{\lambda^{(n)}_\delta}}\left[ g(o, e^{(n)}) \right] \right] \right] + o_n^{(\delta')}(1) + o_n^{(\delta', \delta'')}(1) + o_n^{(\delta', \delta')}(1).
\]
We would like to find the limit of (5.16) as \( \delta \downarrow 0 \). The expectation above can be treated using the Bounded convergence theorem, yielding
\[
\mathbb{E}[c^{(F)}_1(\lambda(0))f(\lambda(0))\mathbb{E}_{\mathbb{D}\backslash\lambda(0)}^{\text{SLE}}[\mathbb{E}_{\mathbb{D}\backslash\lambda(0)}^{\text{SLE}}[g(\mathbf{o}, \mathbf{e})]]] = \mathbb{E}[\mathbb{I}\{\lambda(0)(1) \text{ is not a marked boundary point}\}f(\lambda(0))\mathbb{E}_{\mathbb{D}\backslash\lambda(0)}^{\text{SLE}}[\mathbb{E}_{\mathbb{D}\backslash\lambda(0)}^{\text{SLE}}[g(\mathbf{o}, \mathbf{e})]] + o(\delta).
\]
Finally, by taking first \( \delta' \) small enough, and then \( \delta'' \), and then \( \delta \), and then \( n \) large enough, all the Landau \( o \)-terms above can all be made arbitrarily small. Thus, as \( \delta' \downarrow 0 \), we have
\[
\lim_{\delta' \downarrow 0} = \mathbb{E}[\mathbb{I}\{\lambda(0)(1) \text{ is not a marked boundary point}\}f(\lambda(0))\mathbb{E}_{\mathbb{D}\backslash\lambda(0)}^{\text{SLE}}[\mathbb{E}_{\mathbb{D}\backslash\lambda(0)}^{\text{SLE}}[g(\mathbf{o}, \mathbf{e})]]].
\]
This finishes our discussion on the second term.

**Conclusion:** Finally, combining the analyses of the two terms above, we observe that
\[
\mathbb{E}[f(\lambda(0))g(\partial(0), \gamma_{\mathbb{D}:2}, \ldots, \gamma_{\mathbb{D}:N})] = \mathbb{E}[c^{(F)}_1(\lambda(0))f(\lambda(0))g(\partial(0), \gamma_{\mathbb{D}:2}, \ldots, \gamma_{\mathbb{D}:N})] + \mathbb{E}[c^{(D)}_1(\lambda(0))f(\lambda(0))g(\partial(0), \gamma_{\mathbb{D}:2}, \ldots, \gamma_{\mathbb{D}:N})]
\]
any \( \delta' \in (0, \delta) \) = \( \mathbb{E}[c^{(F)}_2(\lambda(0))f(\lambda(0))g(\partial(0), \gamma_{\mathbb{D}:2}, \ldots, \gamma_{\mathbb{D}:N})] + \mathbb{E}[c^{(D)}_2(\lambda(0))f(\lambda(0))g(\partial(0), \gamma_{\mathbb{D}:2}, \ldots, \gamma_{\mathbb{D}:N})]
\]
(limit \( \delta' \downarrow 0 \) = \( \mathbb{E}[\mathbb{I}\{\lambda(0)(1) \text{ is an even-index marked boundary point}\}f(\lambda(0))\mathbb{E}_{\mathbb{D}\backslash\lambda(0)}^{(N-1)\text{SLE}}[g(w, \eta_1, \ldots, \eta_{N-1})] + \mathbb{E}[\mathbb{I}\{\lambda(0)(1) \text{ is not a marked boundary point}\}f(\lambda(0))\mathbb{E}_{\mathbb{D}\backslash\lambda(0)}^{\text{SLE}}[\mathbb{E}_{\mathbb{D}\backslash\lambda(0)}^{\text{SLE}}[g(\mathbf{o}, \mathbf{e})]],
\]
and thus the claim holds.

The proof of Proposition 5.16 with Assumption 5.6 can now be finished identically to the case with Assumption 5.5.

5.4.4. **Termination points of initial segments.** Let us return to the question where the initial segments \( \lambda_{(0)} \) terminate, left open in Section 5.4.1.

**Proposition 5.20.** For scaling limits with SLE parameter \( \kappa \in (0, 4) \), the initial segment \( \lambda_{(0)} \) almost surely terminates at an even-index marked boundary point. For scaling limits with \( \kappa \in (4, 8) \), \( \lambda_{(0)} \) almost surely does not terminate at an even-index marked boundary point.

**Proof of Proposition 5.20** for \( 4 < \kappa < 8 \). Consider first the case \( \kappa \in (4, 8) \). Take any subsequential limit \( \gamma_{\mathbb{D}:1} \). By condition (G) for \( \gamma_{\mathbb{D}:1}^{(n)} \), the end point of \( \gamma_{\mathbb{D}:1} \) is almost surely not a double point of that curve, so we can study the final segment of \( \gamma_{\mathbb{D}:1} \) (the initial segment of the reversed curve \( \gamma_{\mathbb{D}:1} \)) to answer whether \( \gamma_{\mathbb{D}:1} \) hits \( \partial \mathbb{D} \) somewhere else before hitting an even-index marked boundary point. By Theorem 5.2, the initial segments converge to a local multiple SLE initial segment. Now, a chordal SLE(\( \kappa \)) initial segment with \( \kappa \in (4, 8) \) almost surely hits the boundary outside of its starting point in any small neighborhood of the end points. By absolute continuity, so does the local multiple SLE initial segment. Thus, irrespective of which local multiple SLE initial segment turns out to be the final segment of \( \gamma_{\mathbb{D}:1} \) we can conclude that \( \lambda(0) \) almost surely does not terminate at an even-index marked boundary point.

In order to prove Proposition 5.20 for \( \kappa \in (0, 4] \), we will first need to prove Proposition 5.9.

**Proof of Proposition 5.9.** By Proposition 5.16(i), we can freely choose the order in which we inductively sample the different up-to-swallowing initial segments to obtain the collection of curves \( \gamma_{\mathbb{D}:1}, \ldots, \gamma_{\mathbb{D}:N} \). Sampling in an order that leaves \( \gamma_{\mathbb{D}:1} \) last, it follows that \( \gamma_{\mathbb{D}:1} \) is a chordal SLE in the domain left for it.

**Proof of Proposition 5.20** for \( 0 < \kappa \leq 4 \). By Proposition 5.9, \( \gamma_{\mathbb{D}:1} \) is a chordal SLE(\( \kappa \)) in the domain left for it. It follows that \( \gamma_{\mathbb{D}:1} \) almost surely only visits \( \partial \mathbb{D} \) at its end points.
5.5. Proofs of Theorems 5.10 and 5.11

Proof of Theorem 5.10. We will show by induction over $N$ that all link patterns $\alpha \in \text{LP}_N$ have a probability $\geq p$ to occur in the scaling limit $(\gamma_{G;1}, \ldots, \gamma_{G;N})$, given that the distances between the marked boundary points $\tilde{p}_1, \ldots, \tilde{p}_{2N}$ are bounded from below by some number ($p$ of course depends on this number). The base case $N = 1$ is obvious, since there is only one link pattern.

Let us sketch the induction step with $N \geq 1$. Fix $\alpha \in \text{LP}_N$, and a small tubular neighbourhood of the straight line segment connecting $\tilde{p}_1$ in $\mathbb{D}$ to its pair boundary point given by $\alpha$. There is a positive probability that the usual chordal SLE($\kappa$) from $\tilde{p}_1$ to $\tilde{p}_\infty$ has an initial segment in $U(\delta)$ (fixed but small $\delta$) that stays inside this tubular neighbourhood. (This follows from an analogous property of the Brownian motion: there is a positive probability that the driving function of the chordal SLE stays close to that of the straight line.) By absolute continuity (see, e.g., [KP16] for the explicit Radon-Nikodym derivatives) the initial segment $\lambda_{(\delta)}$ of the local multiple SLE also has a positive probability to stay in this tube.

By weak convergence, this also holds for the discrete initial segments $\lambda^{(n)}_{(\delta)}$. Now, Assumption 5.6 (see especially Figure 5.3) guarantees that the curve $\gamma^{(n)}_{(\delta)}$ is then likely to pair the boundary point $\tilde{p}^{(n)}_1$ to its pair given by $\alpha$, and so that its remainder $\rho^{(n)}_{(\delta)}$ after $\lambda^{(n)}_{(\delta)}$ stays close to the tip of $\lambda^{(n)}_{(\delta)}$. The same conclusion holds for the weak limit $\gamma_{G;1}$. Now, we have obtained a positive probability that $\gamma_{G;1}$ connects $\tilde{p}_1$ to its pair in $\alpha$ and stays close to the corresponding straight line. By the conditional law definition of the local-to-global multiple SLE and the inductive assumption, the remaining curves also have a positive probability to pair the marked boundary as given by $\alpha$.

□

Proof of Theorem 5.11. The driving function of a global multiple SLE($\kappa$) one-curve marginal has been identified in [PW19]. Together with the precompactness theorem 4.1, this guarantees that Assumption 5.1 holds in its conditional form. Assumption 5.5 holds in the $\kappa \leq 4$ case a posteriori, relying on chordal SLE($\kappa$):s having no boundary visits, and the weak convergence to global multiple SLEs.

□

6. Application examples

In this section, we show how our main results can be applied to deduce the convergence of multiple simultaneous random curves in various random models. Also relation to prior literature is discussed.

We will in this section deduce multiple SLE convergence for various discrete curve models using Theorem 5.11. For simplicity, we have chosen to state the convergence results in the topology of curves, $X(\mathbb{C})$. Analogous convergences to local or local-to-global multiple SLEs naturally also hold in the other topologies of Theorems 5.2 and 5.8 and Corollary 5.3 and in these topologies also under the relaxed boundary regularity assumptions. Also the connection to global multiple SLEs, given in Theorem 5.10, holds.

6.1. Three priorly known examples: Ising, FK-Ising and Percolation. We start by discussing three models for which convergence results for multiple curves have appeared in prior literature. The purpose of this discussion is to demonstrate the applicability and practical application of our results.

6.1.1. The Ising model. Consider first the Ising model on the faces of the square lattice $\mathbb{Z}^2$ at critical temperature. We consider this model in simply-connected subgraphs $(G; e_1, \ldots, e_{2N})$ of $\mathbb{Z}^2$. We set boundary conditions that fix the spins on faces (edge-)adjacent to the boundary $\partial \Lambda_G$ of the corresponding planar domain, and the spin signs of these boundary conditions alter precisely at the edges $e_1, \ldots, e_{2N}$. (This of course puts some limitations on the subgraph $(G; e_1, \ldots, e_{2N})$.) The random curves $(\gamma_{G;1}, \ldots, \gamma_{G;N})$ in $(G; e_1, \ldots, e_{2N})$ are the magnetization cluster interfaces that surround the clusters adjacent to the boundary, with the convention of turning left when there are multiple ways to choose the
interface. See, e.g., [BPW18] for a precise definition of the model, boundary conditions and the random curves.

Consider now the lattices $\delta_n \mathbb{Z}^2 = \Gamma_n$, where $\delta_n \downarrow 0$ as $n \to \infty$, and their simply-connected subgraphs $(\mathcal{G}^{(n)}; e_1^{(n)}, \ldots, e_{2N}^{(n)})$ converging to some domain $(\Lambda; p_1, \ldots, p_{2N})$ in the Carathéodory sense. Study these discretizations under the assumptions and notation of Section 4.1.1.

**Proposition 6.1.** In the setup described above, the Ising interfaces $(\gamma_1^{(n)}, \ldots, \gamma_N^{(n)})$ converge weakly in $X(\mathbb{C})$ to the local-to-global multiple SLE(3) in $(\Lambda; p_1, \ldots, p_{2N})$ with the partition functions

$$Z_N(x_1, \ldots, x_{2N}) = \text{Pf}\left(\frac{1}{x_i - x_j}\right)_{i,j=1}^{2N},$$

where $\text{Pf}(\cdot)$ denotes the Pfaffian of a matrix.

Note that it is not immediate, but verified in [KP16, Proposition 4.6], that (6.1) actually is a local multiple SLE partition function, as defined in Section 2.1.1.

**Proof of Proposition 6.1.** We wish to apply Theorem 4.1 to deduce precompactness and Theorem 5.8 to identify the scaling limit. In order to apply these results, we have to check that the discrete curve model satisfies their assumptions.

Assumptions of Theorem 4.1:

- Alternating boundary conditions and DDMP are trivially satisfied.
- Condition (C) for the one-curve model is non-trivial, but has been verified in [CDCH13, Corollary 1.7].

In addition to the above, applying Theorem 5.8 requires the following assumptions to be satisfied:

- Assumption 5.1, i.e., convergence of driving functions to local multiple SLE with the partition function (6.1), holds by [Izy17, Theorem 1.1]. In the case of $N = 1$ curve, Assumption 5.1, i.e., convergence to usual chordal SLE, was verified in [CDCH+14, Theorem 1].
- Assumption 5.4 holds trivially, and 5.6, i.e., Condition (C'), is also a direct consequence of [CDCH13, Corollary 1.7].

We have now verified all the assumptions of Theorems 4.1 and Theorem 5.8 and the conclusions of the latter thus hold. □

**Prior results on the Ising model.** The convergence of multiple Ising interfaces is by now understood rather completely, and the above proposition hence only provides a new proof for a known result, and a slightly different characterization of the weak limit. Convergence of one initial segment to that of a local multiple SLE was established in [Izy17, Theorem 1.1] via martingale observables. The weak convergence of full curves under the conditional measures $\mathbb{P}^{(n)}[\cdot | \alpha]$ to global multiple SLEs, for any link pattern $\alpha$, was established in [BPW18, Proposition 1.3]. Later on, [PW18, Theorem 1.1] established the convergence of the connection probabilities $\mathbb{P}^{(n)}[\alpha]$. Combining this with the convergence of the conditional measures $\mathbb{P}^{(n)}[\alpha]$, the weak convergence of the full curves under the unconditional measures $\mathbb{P}^{(n)}[\cdot | \alpha]$, follows. Interestingly, the results of [PW18] rely on the local convergence of [Izy17]. This manifests the principle from Section 4 proving the convergence of $\mathbb{P}^{(n)}[\alpha]$ is roughly equivalent to finding a converging martingale observable. The two-interface case is discussed in [Wu18].
6.1.2. **Percolation.** Consider now the critical percolation on the faces of the honeycomb lattice $H$, i.e., colouring each face independently either black or white, both with probability $1/2$. We consider this model in simply-connected subgraphs $(\mathcal{G}; e_1, \ldots, e_{2N})$ of $H$, fixing the colours of the faces adjacent to a boundary vertex, so that these boundary conditions alter colour precisely at the edges $e_1, \ldots, e_{2N}$. The random curves $(\gamma_{e_1}, \ldots, \gamma_{e_{2N}})$ in $(\mathcal{G}; e_1, \ldots, e_{2N})$ are the outer boundaries of the black or white clusters adjacent to the boundary, see Figure 6.1.

Consider now the lattices $\delta_nH = \Gamma_n$, where $\delta_n \downarrow 0$ as $n \to \infty$, and their simply-connected subgraphs $(\mathcal{G}^{(n)}; e_1^{(n)}, \ldots, e_{2N}^{(n)})$ converging to some domain $(\Lambda; p_1, \ldots, p_{2N})$ in the Carathéodory sense. Study these discretizations under the assumptions and notation of Section 4.1.1.

**Proposition 6.2.** In the setup described above, the percolation interfaces $(\gamma_{e_1}^{(n)}, \ldots, \gamma_{e_{2N}}^{(n)})$ converge weakly in $X(\mathbb{C})$ to the local-to-global multiple SLE(6) in $(\Lambda; p_1, \ldots, p_{2N})$ with the partition functions

$$Z_N(x_1, \ldots, x_{2N}) = 1.$$  

(6.2)

**Proof.** It is trivial to check that (6.2) are local multiple SLE partition functions with $\kappa = 6$ (this was observed, e.g., in [KP16, Proposition 4.9]). Observe also that the local multiple SLE initial segment from $p_1$ is then equal in distribution to the initial segment of a chordal SLE(6) from $p_1$ targeting at, say, $p_2$ (the precise choice of target is irrelevant due to the locality of the chordal SLE(6)).

We now check the assumptions of Theorem 4.1:

- Alternating boundary conditions and DDMP are trivially satisfied.
- Condition (G) for the one-curve model follows from the Russo–Seymour–Welsh estimates.

The additional assumptions for Theorem 5.8:

- Assumption 5.1 holds since the initial segment both in the percolation model and in the local multiple SLE (6.2) are independent of the number and locations of the other marked boundary

---

**Figure 6.1.** Left: A simply-connected subgraph $\mathcal{G}$ of $H$, with boundary faces altering colour between black and white over the marked boundary edges $e_1, \ldots, e_6$. The remaining faces are coloured gray. Right: Percolation colouring of the remaining faces, and the obtained random curves on $H$ bounding the black and white clusters adjacent to the boundary of $\mathcal{G}$.
points. Thus, the proof of convergence to chordal SLE for $N = 1$ interface suffices. The latter has been addressed by various authors, see, e.g., Smi01, CN07, Be07.

- Assumption 5.4 holds trivially, and 5.6, i.e., Condition (C'), is verified via condition (G'), which in turn is a direct consequence of the Russo–Seymour–Welsh estimates.

We can now apply Theorem 5.8 to complete the proof. □

**Prior results on percolation.** Percolation interfaces are very well understood. (Indeed, the main reason for our discussion on it is the warning example of Section 6.2.) Convergence results to multiple SLE type curves have been addressed in [KS18, Section 3] and [BPW18, Remark 1.5]. Also the scaling limit of the full collection of percolation interfaces has been identified [CN06].

### 6.1.3. The FK cluster and FK-Ising models.

**Definition of the models.** Let us discuss the FK cluster model on the square lattice — the FK-Ising model is later addressed as an important special case. We follow the conventions of the literature, and refer the reader to, e.g., [DCS07] for a good introduction.

First, colour the squares of $\mathbb{Z}^2$ black and white in a chessboard manner. The black (resp. white) squares form a scaled and rotated $\mathbb{Z}^2$ lattice, which we call the black (resp. white) lattice. These lattices are mutual planar duals. In the original $\mathbb{Z}^2$ lattice, take a simply-connected subgraph $G$ whose boundary consists of $N$ black and $N$ white squares; by a black (resp. white) segment mean here that the $\mathbb{Z}^2$ squares inside $G$ edge-adjacent to that boundary segment are all black (resp. white). The $2N$ marked boundary edges $e_1, \ldots, e_{2N}$ of $G$ separate black and white boundary-neighbouring squares (top left in Figure 6.2).

Next, on subgraph of the black lattice inside $G$, we impose wired boundary conditions, i.e., the black squares adjacent to the each of black boundary segment are identified, producing $N$ black boundary segment vertices. Call this graph $B_G$. On the white squares inside $G$, we impose slightly different boundary conditions: the white squares adjacent to the white boundary segments are all identified, producing a single one white boundary segment vertex. Denote this graph by $W_G$. The graphs $B_G$ and $W_G$ are mutual planar duals.

Then, we run the FK cluster model on $B_G$ with parameters $p$ and $q$: choose a random subgraph $\omega$ of $B_G$, whose vertices are all vertices in $B_G$ but whose edges are a subset of the edges of $B_G$, so that the probability of each different such subgraph $\omega$ is proportional to

$$p^{\#\{\text{edges of } B_G \text{ present in } \omega\}} (1-p)^{\#\{\text{edges of } B_G \text{ not present in } \omega\}} q^{\#\{\text{connected components of } \omega\}}.$$  

We will only consider the self-dual parameters satisfying $p = \sqrt{q}/(1 + \sqrt{q})$; this means that the dual subgraph $\omega^*$ of $W_G$, consisting of all the vertices of $W_G$ and the edges of $W_G$ not crossed by $\omega$, is in distribution equal to the FK clustered model $W_G$ with the same parameters $q$ and $p = \sqrt{q}/(1 + \sqrt{q})$. We will identify $\omega$ (resp. $\omega^*$) with the subgraph of the black (resp. white) lattice obtained from the edges of $\omega$ (resp. $\omega^*$) and the edges connecting black (resp. white) vertices of same black (resp. white) boundary segment (bottom left in Figure 6.2).

Finally, the related random curve model is obtained from the loop representation of the FK clusters, which we describe next. First, we modify $G$ slightly: every corner of the $\mathbb{Z}^2$ lattice is rounded by putting there a small square, making the lattice into a square-octagon lattice, which we denote by $L$. Round the corners of the graph $G$ to obtain a simply-connected subgraph of $L$, i.e., include the small squares at concave corners of $G$ and exclude the ones at the convex or 180° corners (top right in Figure 6.2). Slightly abusively, let us in continuation refer by $G$ to this subgraph of $L$. Now, with our convention of regarding $\omega$ (resp. $\omega^*$) as a subgraph of the black (resp. white) lattice, two opposite sides of each small square of $G$ are crossed by exactly one edge from either $\omega$ or $\omega^*$; this is visible in Figure 6.2 (bottom left). In particular, $\omega$ can thus be bijectively encoded into the pairs of opposite non-crossed small-square edges of $G$. Let us add to this collection of edges of $G$ all the black-boundary edges of $L \cap \partial \Lambda_G$ and all the edges of $G$ originating from $\mathbb{Z}^2$ and not on $\partial \Lambda_G$. The bijection with $\omega$ of course pertains. However,
in the new collection of edges of $G$, each vertex of $G$ has either 0 or 2 edges adjacent to it: the edges form a collection of disjoint simple loops on $G$. This is the loop representation of the FK cluster model. Each loop is adjacent to black (resp. white) squares of $\mathbb{Z}^2$ from exactly one connected component of $\omega$ (resp. $\omega^*$). We can sample $\omega$ via sampling its loop representation, in which case the probability of a loop configuration in proportional to

$$\sqrt{q}^\#\{\text{loops}\}. \tag{6.3}$$

Consider now those loops that contain the black boundary segments of $\partial \Lambda G$. In addition to the boundary segments, this collection of loops contains $N$ chordal paths inside $G$, pairing the marked boundary edges $e_1, \ldots, e_{2N}$. The measures with random curves $(P_{(G; e_1, \ldots, e_{2N})}, (\gamma_{G; 1}, \ldots, \gamma_{G; N}))$ are the FK cluster loop representations and these chordal paths on $G$ (bottom right in Figure 6.2).

The FK-Ising model is the FK cluster model with parameters $q = 2$ and $p = \sqrt{q}/(1 + \sqrt{q}) = \sqrt{2}/(1 + \sqrt{2})$. 
Precompactness of the FK cluster models. It has been conjectured that the $N$ random curves in the self-dual FK cluster model introduced above, with parameter $q \in [0, 4)$, converges to SLE type scaling limits, with the SLE parameter $\kappa$ depending on the cluster model parameter via

$$\kappa = \frac{4\pi}{\arccos(-\sqrt{2}/2)}$$

For such predictions, see, e.g., [Sch07, Smi09] for $N = 1$ curve and chordal SLEs, [BPW18] for general $N$ and global multiple SLEs. Regarding such convergence proofs, the precompactness part has been established [DCST17, DCST17] but the limit identification step is missing, except in the FK-Ising case $q = 2$. We now check that also when following the convergence proof strategy and of this paper, only the limit identification step, i.e., Assumption 5.1 is missing.

Proposition 6.3. The discrete curve models obtained from the loop representation of the FK cluster model with $q \geq 1$ satisfy the assumptions of Theorem 5.1. Also the assumptions of Theorem 5.8 except for possibly Assumption 5.4 hold.

Proof. For the assumptions of Theorem 4.1, the discrete models clearly have alternating boundary conditions and satisfy the DDMP. Condition (G) for the one-curve model has been verified in [DCST17]. As regards the assumptions of Theorem 5.8, Assumption 5.4 holds trivially. Assumption 5.6, i.e., Condition (C'), is verified via condition (G'), which in turn is proven identically to condition (G) in [DCST17].

Convergence of two FK-Ising interfaces. Let us now discuss the weak convergence in the FK-Ising model, i.e., the FK cluster model with $q = 2$ with two curves. We keep the discussion here largely informal, referring to the more complete account in [KS15, KS18] for those parts. Multiple interfaces in FK cluster and FK Ising models have been studied priorly in [KS15, KS18, BPW18], and the scaling limits in the setups considered below could be identified (with slightly different characterizations) by combining results from those papers. Following [KS18], we consider a slightly modified FK model, so that in the loop representation probabilities 5.3 boundary-touching loops are not counted.

Note first that Proposition 6.3 applies for the FK-Ising model. (Conditions (C) and (C') can also be verified directly then [CDCH13].) Thus, in order to apply the main theorem 5.8 of this paper, it remains to verify Assumption 5.1. To that end, first, the scaling limit of $N = 1$ curve has been identified in [CDCH14, Theorem 2] as a chordal SLE(16/3). For $N = 2$ curves, the driving process of the initial segment of one curve has been identified in [KS15, Equation (94)]. Recalling that the proof of Theorem 5.5 is based on an induction over $N$, we can thus apply it for the FK-Ising model with $N = 2$ curves. We conclude the following.

Proposition 6.4. The curves $(\gamma_{D;1}^{(n)}, \gamma_{D;2}^{(n)})$ under the FK-Ising model with $N = 2$ curves converges weakly to the following limit: the up-to-swallowing initial segment $\lambda(0)$ is described by the Loewner growth in [KS15, Equation (94)]. Given $\lambda(0)$, the regular conditional laws of the remainder of the curves are two independent chordal SLE(16/3) curves in the respective domains of $\mathbb{D}\setminus\lambda(0)$, with the three remaining marked boundary points and one at the tip of $\lambda(0)$.

The curves $(\gamma_{D;1}^{(n)}, \gamma_{D;2}^{(n)})$ under the FK-Ising model conditional on a link pattern were studied in [KS18, Theorem 1.1]. The initial segment $\lambda(0)$ is then described by the hypergeometric SLE(16/3). The following convergence of a pair of curves was stated there without explicit proof.

Proposition 6.5. Proposition 6.4 holds for the curves $(\gamma_{D;1}^{(n)}, \gamma_{D;2}^{(n)})$ under the FK-Ising model conditional on a link pattern, with $\lambda(0)$ changed to the hypergeometric SLE of [KS18, Equation (2)].

Proof. Consider first the unconditional scaling limit of Proposition 6.4. The boundary point $\hat{p}_1$ connects to $\hat{p}_2$ (resp. $\hat{p}_3$) if and only if the tip of $\lambda(0)$ is on the arc $(\hat{p}_2, \hat{p}_3)$ (resp. $(\hat{p}_3, \hat{p}_1)$) of $\partial \mathbb{D}$, by Proposition 5.20. Both of these occur with positive probability, given explicitly in [KS18, Equation (4)].
unconditional scaling limit of Proposition 6.4 on the tip of \( \lambda(0) \) lying one of these arcs, say \((\tilde{p}_2, \tilde{p}_3)\). On the one hand, Proposition 6.4 gives the regular conditional distributions of the remaining curves as chordal SLEs. On the other hand, this conditioning reveals the link pattern and thus [KSIS, Theorem 1.1] tells that the law of \( \lambda(0) \) under this condition is the hypergeometric SLE. □

6.2. A warning example: not all local multiple SLEs are global. The identification of the scaling limit of percolation in the previous subsection was particularly interesting due to the following consequence.

**Proposition 6.6.** Let \( N \geq 3 \) and \( \tilde{p}_1, \ldots, \tilde{p}_{2N} \in \partial D \) be any \( 2N \) distinct points. There is a collection of \( N \) chordal random curves in \( X(D) \), pairing the boundary points \( \tilde{p}_1, \ldots, \tilde{p}_{2N} \), such that the initial and final segments of these curves in any localization neighbourhoods are those of the local multiple SLE(6) in Proposition 6.2, but the full curves are not the local-to-global multiple SLE(6) in Proposition 6.2.

**Proof.** Start from the local-to-global multiple SLE(6) in Proposition 6.2. By Proposition 5.16, it has the following property: if the up-to-swallowing initial segment from \( \tilde{p}_1 \) (in blue in Figure 6.3(left)) hits the arc \((\tilde{p}_2, \tilde{p}_{2N})\) in \((\tilde{p}_4, \tilde{p}_5)\) and is disjoint from the up-to-swallowing initial segment from \( \tilde{p}_4 \) (in red in Figure 6.3(left)), which hits the arc \((\tilde{p}_5, \tilde{p}_3)\) in \((\tilde{p}_1, \tilde{p}_2)\), then \( \tilde{p}_1 \) and \( \tilde{p}_4 \) are connected by a random curve. Denote this event by \( E \). By the Russo–Seymour–Welsh estimates, \( E \) has a positive probability. On the event \( E \), the random curve from \( \tilde{p}_1 \) to \( \tilde{p}_4 \) is a concatenation of three curves (in order 1–2–3, in blue, green, and red in Figure 6.3(left), respectively): 1) the up-to-swallowing initial segment from \( \tilde{p}_1 \) to \( (\tilde{p}_4, \tilde{p}_5) \); 2) the reversal of the up-to-swallowing initial segment from \( \tilde{p}_4 \) to \( (\tilde{p}_5, \tilde{p}_3) \); and 2) a chordal SLE(6) between the tips of these two up-to-swallowing initial segments, in the domain restricted by them.

Now, on the event \( E \), let us replace the curve (2) above by a hyperbolic geodesic, i.e., the chordal SLE(0), in the same domain; see Figure 6.3(right). It is elementary to verify that after this replacement, we obtain a different family of random curves, whose all localizations are nevertheless the same as before this replacement operation. This proves the claim. □

**Remark 6.7.** The counterexample in the proof above is conformally invariant, and may be defined via conformal maps in any domain \((\Lambda; p_1, \ldots, p_{2N})\) with marked prime end that possess radial limits.

6.3. Outline of a new example: UST branches. Let us return to direct applications of our main theorems 4.1 and 5.8. The next discrete model that we will study is the uniform spanning tree (UST).

Verifying the assumptions of these theorems in that model would take up some space, and thus we only
Figure 6.4. A uniform spanning tree, with the paths to the boundary from the interior vertices of the odd edges $e_1, e_3,$ and $e_5$ reaching the boundary each via a different even edge $e_2, e_4,$ or $e_6.$

Outline the proofs in this subsection. Also, we will for simplicity restrict our consideration in this paper to the lattice $\mathbb{Z}^2,$ even if all the results could be derived on any isoradial lattice, as defined in [CS11].

Let $(G; e_1, \ldots, e_{2N})$ be a simply-connected subgraph of $\mathbb{Z}^2$ with marked boundary edges. Consider the uniform spanning tree on the graph $G/\partial$ obtained by identifying all the boundary vertices of $G.$ Each interior vertex $v \in V$ thus connects to the boundary vertices $\partial V$ by a unique path on such a tree. Condition the UST on $G/\partial$ on the event that that such boundary paths from the interior vertices of the odd edges $e_1, e_3, \ldots, e_{2N-1}$ reach $\partial V$ via the even edges $e_2, e_4, \ldots, e_{2N},$ each using a different even edge; see Figure 6.4 for illustration. (This conditioning making sense puts some very mild limitations on the subgraph $(G; e_1, \ldots, e_{2N}).$) The probability measures $P^{(G; e_1, \ldots, e_{2N})}$ that we are interested in are these conditional USTs, and the random chordal curves $\gamma_{G;1}, \ldots, \gamma_{G;N}$ are the chordal graph paths consisting of the odd edges $e_1, e_3, \ldots, e_{2N-1}$ and the boundary paths from their interior vertices. We call $\gamma_{G;1}, \ldots, \gamma_{G;N}$ UST boundary branches. This model is also sometimes called multiple loop-erased random walks (LERWs), due to the connection of the discrete models [Wil96].

Consider now the lattices $\delta_n \mathbb{Z}^2 = \Gamma_n,$ where $\delta_n \downarrow 0$ as $n \to \infty,$ and their simply-connected subgraphs $(G^{(n)}; e_1^{(n)}, \ldots, e_{2N}^{(n)})$ as above, converging to some domain $(\Lambda; p_1, \ldots, p_{2N})$ in the Carathéodory sense. Study these discretizations under the assumptions and notation of Section 4.1.1.

Theorem 6.8. In the setup described above, the UST boundary branches $(\gamma_{1}^{(n)}, \ldots, \gamma_{N}^{(n)})$, both unconditional and conditional on a link pattern $\alpha \in \text{LP}_N,$ converge weakly in $X(\mathbb{C})$ to the local-to-global multiple SLE(2) in $(\Lambda; p_1, \ldots, p_{2N}).$ The scaling limit in the conditional case is described by local multiple SLEs with the partition functions

$$Z_{\alpha}(x_1, \ldots, x_{2N})$$

given in [KKP17a, Equation (3.14)], and in the unconditional case by

$$Z_{N}(x_1, \ldots, x_{2N}) = \sum_{\alpha \in \text{LP}_N} Z_{\alpha}(x_1, \ldots, x_{2N}).$$

(6.5)

An alternative expression for $Z_{N}$ is given in [PW19, Lemma 4.12].

Proof precompactness. Let us first verify the assumptions of Theorem 4.1 on the UST discrete curve model. First, the model clearly has alternating boundary conditions. Actually, by the bijection argument in [KKP17a, Lemma 3.1], we can re-label the edges $e_1, \ldots, e_{2N}$ to $\hat{e}_1, \ldots, \hat{e}_{2N}$ counterclockwise starting
from any edge, and the UST models on \((\mathcal{G};e_1,\ldots,e_{2N})\) and \((\mathcal{G};\hat{e}_1,\ldots,\hat{e}_{2N})\) yield the same distribution of random curves.

The DDMP follows from the fact that for the UST on any graph \(\mathcal{G}\), the UST conditional on a subtree is in distribution a UST on the graph obtained by identifying the vertices of that subtree. Now, property (ii) in the definition of the DDMP follows from this property of the UST, likewise property (i) when conditioning on a branch initial segment from an even boundary edge \(e_2,e_4,\ldots,e_{2N}\). For property (i) with a branch initial segment from an odd-index boundary edge, use the re-labelling argument of the previous paragraph.

Finally, Condition (C) for the one-curve model has been verified in [KS17, Theorem 4.18]. The assumptions of Theorem [1.1] for discrete curve model have thus been verified.  

**Outline of identification.** The verification of the additional assumptions imposed in Theorem 6.8 is postponed to a follow-up paper. The outline is the following:

1) Assumption 5.1, i.e., convergence of driving functions to local multiple SLE with the partition function (6.4) or (6.5): the precompactness part guarantees the existence of subsequential limits of the driving functions \(W_j\). Any subsequential limit is identified as a local multiple SLE(2) initial segment via a martingale observable, as in classical SLE convergence proofs. There are several alternative martingale observables, one (in the \(\alpha\)-conditional model) being the ratio of partition functions \(Z_\beta/Z_\alpha\), with any \(\beta \in \text{LP}_N\), in the notation of [KKP17a, Theorem 3.12]. With the expression given there for this observable and some discrete harmonic analysis, one can prove the convergence of the observable.

2) Assumption 5.4 holds trivially.

3) We verify Assumption 5.5 for the conditional models — if it holds for the conditional models with any link pattern \(\alpha\), it clearly holds for the unconditional model. In the conditional case, construct the uniform spanning tree by Wilson’s algorithm [Wil96]; the paths \(\gamma_{\mathcal{G};1},\ldots,\gamma_{\mathcal{G};N}\) are then loop-erasures of suitable random walk excursions (see, e.g., [KKP17a Corollary 3.5(c)]), conditional on the loop-erasure paths not crossing. Assumption 5.5 is now first verified for the (traces of) the underlying random walk excursions; it is thus also satisfied by their loop-erasures. Finally, one shows that these loop-erasures are vertex-disjoint with a uniformly positive probability, and hence Assumption 5.5 also holds for the loop-erasures conditional on this vertex-disjointness.

This finishes the outline of the identification step in Theorem 6.8. We conclude by remarking that step (3), which took a lengthy outline above, relies on tools required for the martingale argument in step (1); in other words, here as in the case of all other models, it is the verification of Assumption 5.1 that is the core of the convergence proof.

Convergence results for a single UST branch have been given at least in [LSW04, Zha08, YY11, LV16, CW19]. Convergence results for multiple branches have been predicted in various sources, e.g., [KW11a, KKP17a, Wu18].

6.4. **A complete new example: the harmonic explorer.** In this subsection we give a new and complete example of multiple SLE convergence given by our main theorems. The model we consider is the natural multiple-curve generalization of the harmonic explorer on the honeycomb lattice. The harmonic explorer was introduced as a toy model for the study of the level lines of the discrete Gaussian free field. One very useful simplification in moving to the harmonic explorer was the arrival of the DDMP. This is also the reason why we consider the harmonic explorer but not the discrete Gaussian free field.

In the one-curve case, the convergence of the harmonic explorer to the chordal SLE(4) was proven in [SS05], and the proof here employs similar ideas. The convergence of discrete Gaussian free field level lines to chordal SLE(4) was later proven in [SS09]. Multiple SLE(4)s have been studied via the (continuous) Gaussian free field in [PW19], but we are not aware of a prior lattice model convergence result addressing multiple SLE(4).
Figure 6.5. A simply-connected subgraph $G$ of $H$, and its boundary faces, coloured black and white. In order to grow an edge at index $4$, one launches a random walk on the faces of $H$ from the face $F$ right in front of the edge $e_4$. If the random walk hits the boundary faces of $G$ on a white face (green trajectory), one colours $F$ white. If the random walk first hits a black face (purple trajectory), one colours $F$ black.

6.4.1. A first definition. The multiple harmonic explorer has to our knowledge not appeared anywhere previously, but it is a straightforward generalization of the harmonic explorer. We give a first definition here, and will soon find a useful equivalent definition.

Consider the honeycomb lattice $H$, and its simply-connected subgraph $(G; e_1, \ldots, e_{2N})$ with distinct marked boundary edges. Colour of the faces adjacent to a boundary vertex black or white, so that colour of the boundary faces changes precisely at the edges $e_1, \ldots, e_{2N}$, say for definiteness so that boundary arcs counterclockwise from odd to even are black and even to odd white.

Let $G_t$ be a graph with marked boundary edges as above (we suppress the edges in the notation; note also that the marked boundary edges determine boundary colouring and vice versa). Given such $G_t$, we now define a procedure that yields $G_{t+1}$, also of the above type. We call this procedure growing an edge at $i$, where $1 \leq i \leq 2N$. Call the interior vertex of $e_i$ its tip vertex. There are three cases.

1) If the tip vertex also adjacent to some marked boundary edge of $G_t$ other than $e_i$, set $G_{t+1} = G_t$.

Otherwise, observe that the tip vertex of $e_i$ is adjacent to three faces of $G_t$. Two of these faces are the boundary faces of $G_t$ on either side of $e_i$. Call the third one $F$. We then determine a colour to $F$ as follows.

2) If $F$ is a boundary face of $G_t$ it is already coloured in $G_t$.

3) If $F$ is not a boundary face of $G_t$, launch a simple random walk on the faces of $G_t$ from $F$. If it first hits the boundary faces of $G_t$ at a black face, colour $F$ black, and otherwise colour $F$ white.

Note that cases (2) and (3) can be summarized as

\[ P[F \text{ is black}] = H_{G_t}\{F; \text{black boundary of } G_t\}, \]

where $H_{G_t}\{F; \text{black}\}$ denotes the harmonic measure on the faces of $G_t$ of the black boundary, as seen from $F$.

In cases (2) and (3) above, $G_{t+1}$ is obtained from $G_t$ by declaring the tip vertex of $e_i$ a boundary vertex. The $i$:th marked boundary edge of $G_{t+1}$ then starts from this vertex, and goes either clockwise
or counterclockwise along the boundary of $F$, the direction chosen so that the boundary colourings of $F$ in $G_{t+1}$ is as determined above. All other marked boundary edges of $G_{t+1}$ are the same as in $G_t$.

Finally, we define the discrete random curves $(\gamma_{G;1}, \ldots, \gamma_{G;N})$ given by the multiple harmonic explorer on $(G; e_1, \ldots, e_{2N})$. These are the curves obtained by growing edges in the following order: start from $G_1 = (G; e_1, \ldots, e_{2N})$ as above. Then, inductively, $G_{t+1}$ is obtained from $G_t$ by growing the edge at $i$, where $i \equiv t$ modulo $2N$. I.e., we grow edges at $1, 2, \ldots, 2N, 1, 2, \ldots, 2N, 1, 2, \ldots$. Each growing of an edge is independent from the previous ones. We continue this until the graphs $G_t$ stabilize, i.e., growing any edge leads to case (1) above.

### 6.4.2. An equivalent definition.

Let $G_1 = (G; e_1, \ldots, e_{2N}), G_2, \ldots$ be as above. Let $F_1$ be the face at the tip of the boundary edge $e_1$ of $G$, and $F_2$ the at the tip of $e_2$. Suppose that we grow an edge of $G_1$ at $2$, not at $1$ as in the first definition of the harmonic explorer, or equivalently colour $F_2$. We have

$$
\mathbb{P}[F_2 \text{ coloured first is black}] = H_{G_1^*}(F_2; \text{black boundary of } G_1).
$$

Suppose now that we first grow an edge at $1$ and then at $2$. Then, we observe that

$$
(6.6)
$$

$$
\mathbb{P}[F_2 \text{ coloured second is black}] = H_{G_2^*}(F_2; \text{black boundary of } G_1) + H_{G_2^*}(F_2; F_1) \mathbb{P}[F_1 \text{ is black}]
$$

$$
= H_{G_2^*}(F_2; \text{black boundary of } G_1) + H_{G_2^*}(F_2; F_1) H_{G_1^*}(F_1; \text{black boundary of } G_1)
$$

$$
= H_{G_1^*}(F_2; \text{black boundary of } G_1),
$$

where the last step follows by the strong Markov property of the simple random walk. From the two computations above, we observe that the probability of colouring $F_2$ black does not depend on whether it was coloured first or second. Switching the roles of $F_1$ and $F_2$ in the computation above, we observe that growing and edge (i) first at $1$ and then at $2$, or (ii) first at $2$ and then at $1$, the pairs of edges grown in cases (i) and (ii) are equal in distribution.

Using the the argument above one can deduce that the edges of the multiple harmonic explorer can be grown in an order chosen freely.

**Proposition 6.9.** (Equivalent definition of the multiple harmonic explorer.) Let $G_1 = (G; e_1, \ldots, e_{2N})$ be as above. Inductively, let $G_{t+1}$ be obtained from $G_t$ by growing the edge at $i = i(t)$, where the index $i(t)$ only depends on the graphs $G_1, \ldots, G_t$ up to time $t$. Assume that the indices $i(t)$ are chosen so that $G_{t+1} \neq G_t$ if it is possible to choose such an $i(t)$. The curves obtained by growing edges in this manner are in distribution equal to the multiple harmonic explorer on $G_1$. Furthermore, conditional on any sequence of first the graphs $G_1, \ldots, G_t$, the remainder of the curves is distributed as a multiple harmonic explorer on $G_t$.

A rule to determine $i(t)$ given $G_1, \ldots, G_t$ such that $G_{t+1} \neq G_t$ if possible is called a valid growth rule. The obtained process is called harmonic explorer under a valid growth rule. Note also that by the above proposition, the one-curve harmonic explorer studied, e.g., in [SS05] is the special case $N = 1$, namely under the growth rule of always growing an edge at $1$, which is valid if $N = 1$.

**Proof of Proposition 6.9.** Start from the first definition of the multiple harmonic explorer, i.e., growing edges at indices $1, 2, 3, \ldots, 2N, 1, 2, 3, \ldots$. By the argument above, we can swap the growth order of edges at $1$ and $2$, growing edges at indices $2, 1, 3, \ldots, 2N, 2, 1, 3, \ldots$, and obtain same distribution of curves. Similarly, we can swap the indices of any two subsequent growth steps. Any permutation $\sigma$ of $2N$ is a composition of such swaps, so we can grow the edges in order $\sigma(1), \sigma(2), \ldots, \sigma(2N), \sigma(1), \sigma(2), \ldots$ and still obtain same distribution of curves. Thus, the first index $i(1) = \sigma(1)$ of the edge to be grown can be chosen freely, and conditional on the obtained graph $G_2$, the remainder of the curves (obtained by growing edges at indices $\sigma(2), \ldots, \sigma(2N), \sigma(1), \sigma(2), \ldots$) is a harmonic explorer in $G_2$. Both claims now follow inductively. \(\square\)
6.4.3. **The convergence theorem.** Consider now the scaled honeycomb lattices \( \delta_n H = \Gamma_n \), where \( \delta_n \downarrow 0 \) as \( n \to \infty \), and their simply-connected subgraphs \( (G^{(n)}; e_1^{(n)}, \ldots, e_{2N}^{(n)}) \) as above, converging to some domain \((\Lambda; p_1, \ldots, p_{2N})\) in the Carathéodory sense. Study these discretizations under the assumptions and notation of Section 4.1.1.

**Theorem 6.10.** In the setup described above, the multiple harmonic explorer interfaces \( (\gamma_1^{(n)}, \ldots, \gamma_N^{(n)}) \) converge weakly in \( X(\mathbb{C}) \) to the local-to-global multiple SLE(4) in \((\Lambda; p_1, \ldots, p_{2N})\) with the partition functions

\[
Z_N(x_1, \ldots, x_{2N}) = \prod_{1 \leq k < \ell \leq 2N} (x_\ell - x_k)^{1/2(1-\ell/k)}.
\]

The fact that \((6.7)\) actually is a local multiple SLE partition function, as defined in Section 2.1.1, is verified in [KP16, Proposition 4.8]. Note that combining this theorem with Theorems 4.1 we also obtain the convergence of the conditional discrete curve models to the global multiple SLEs of [PW19, BPW18]. Furthermore, combining this with Theorem 5.11 we obtain their convergence of the conditional local-to-global multiple SLEs.

**Proof of Precompactness in Theorem 6.10.** Let us first prove the precompactness part of Theorem 6.10 by verifying the assumptions of Theorem 4.1 for the discrete curve model. Using Proposition 6.9, one can easily find valid growth rules that show that the harmonic explorer has alternating boundary conditions and satisfies the DDMP. Condition (G) for the one-curve model is verified in [SS05, Proposition 6.3].

The identification step will be done in Section 6.4.6.

6.4.4. **A discrete martingale.** Let us return to the discrete model: consider the harmonic explorer with a valid growth rule on a simply-connected subgraph \( G_1 = (G; e_1, \ldots, e_{2N}) \) of \( H \). Let \( z \) be a face of \( G_1 \). Note that it is also a face of \( G_t \), for any \( t \). Define

\[
M_t(z) = H_{G_t^*}(z; \text{black boundary of } G_t).
\]

**Proposition 6.11.** \( M_t(z) \) is for any face \( z \) an \( \mathcal{F}_t \)-martingale, where \( \mathcal{F}_t \) is the sigma algebra of the graphs \( G_1, \ldots, G_t \).

**Proof.** \( M_t(z) \) is clearly bounded and \( \mathcal{F}_t \)-adapted, so it remains to show the conditional expectation property of discrete martingales. Let \( F_1 \) be the face coloured to obtain \( G_{t+1} \) from \( G_t \). Note that given \( G_t \), we know the face \( F_t \), since the growth rule is valid. The computation is now identical to (6.6):

\[
\mathbb{E}[M_{t+1}(z) | \mathcal{F}_t] = H_{G_{t+1}^*}(z; \text{black boundary of } G_t) + H_{G_{t+1}^*}(z; F_t)\mathbb{P}[F_t \text{ is black } | \mathcal{F}_t]
\]

\[
= H_{G_{t+1}^*}(z; \text{black boundary of } G_t) + H_{G_{t+1}^*}(F_t)\mathbb{H}_{G_t^*}(F_t; \text{black boundary of } G_t)
\]

\[
= H_{G_t^*}(z; \text{black boundary of } G_t) = M_t(z).
\]

This concludes the proof. \( \square \)

6.4.5. **Verifying Assumption 5.1.** Let us now verify Assumption 5.1 for the multiple harmonic explorer. This is the core of the limit identification step in the proof of Theorem 6.10. Note that in Assumption 5.1 we worked with Carathéodory converging graphs \( (G^{(n)}; e_1^{(n)}, \ldots, e_{2N}^{(n)}) \), but with relaxed regularity at marked boundary points, see Section 5.1.1. Nevertheless, by Remark 4.2 since the assumptions of Theorem 4.1 for the discrete curve model were verified in the precompactness part of Theorem 6.10 we know that the stopped driving functions \( W_j^{(n)} \) are precompact also under this relaxed boundary regularity. Let us fix \( j \), assume that a convergent subsequence has been extracted, and suppress it in notation, so that

\[
W_j^{(n)} \overset{n \to \infty}{\longrightarrow} W_j \quad \text{weakly in } C.
\]

We now claim that Assumption 5.1 is now satisfied:
**Proposition 6.12.** Any weak limit \( W_j \) as above is the stopped driving function of the local multiple SLE(4) with partition function \( \{6.7\} \).

For \( i \neq j \), let \( W_{i; t} \) describe the locations of the other marked boundary points under the Loewner equation driven by \( W_{j; t} \):

\[
dW_{i; t} = \frac{2dt}{W_{i; t} - W_{j; t}}.
\]

Let \( g_t \) be the mapping-out functions of this Loewner equation, so \( W_{i; t} = g_t(W_{i; 0}) \). For any point \( \zeta \in \mathbb{H} \) outside of the localization neighbourhood \( U_j \) of the \( j \):th boundary point, define for times \( t \leq \tau_j \)

\[
\mathcal{M}_t(\zeta) = \frac{1}{\pi} \Re \left( \log(g_t(\zeta) - W_{2N;j; t}) - \log(g_t(\zeta) - W_{2N-1;j; t}) + \ldots - \log(g_t(\zeta) - W_{1;j; t}) \right)
\]

\[
= \frac{1}{\pi} \Re \left( \sum_{i=1}^{2N} (-1)^i \log(g_t(\zeta) - W_{i;t}) \right),
\]

and for \( t \geq \tau_j \) set \( \mathcal{M}_t(\zeta) = \mathcal{M}_{\tau_j}(\zeta) \). (Here \( \log \) denotes the natural complex logarithm; we choose the branch cut on the negative imaginary axis.) Note that \( \mathcal{M}_t(\zeta) \) is the continuum harmonic measure in \( \mathbb{H} \) of the counterclockwise odd-to-even marked boundary arcs between the marked boundary points \( W_{1; t}, \ldots, W_{2N;j; t} \) as seen from \( g_t(\zeta) \). In other words, \( \mathcal{M}_t(\zeta) \) is the direct continuum analogue of the discrete martingale \( M_t(\zeta) \) in the previous subsection.

**Lemma 6.13.** \( \mathcal{M}_t(\zeta) \) is for any \( \zeta \) as above a continuous bounded martingale with respect to the filtration \( \mathcal{F}_t \) of \( W_{j; t} \).

**Proof.** \( \mathcal{M}_t(\zeta) \) is clearly continuous, bounded, and \( \mathcal{F}_t \)-adapted. It remains to verify the conditional expectation property of martingales. For this, we will show that

\[
\mathcal{M}_t(\zeta) = \mathbb{E}[\mathcal{M}_{\tau_j}(\zeta) | \mathcal{F}_t],
\]

since any conditional expectation is a martingale. The above holds if and only if for all continuous bounded functions \( f_t : \mathbb{C} \to \mathbb{R} \) of \( W_j \), measurable with respect to \( \mathcal{F}_t \), i.e., only depending on \( W_j \) up to time \( t \), we have

\[
\mathbb{E}[\mathcal{M}_t(\zeta)f_t(W_j)] = \mathbb{E}[\mathcal{M}_{\tau_j}(\zeta)f_t(W_j)].
\]

Let us verify \( \{6.9\} \). Consider the analogue of \( \mathcal{M}_t(\zeta) \) with the discrete driving function \( W_{j; t}^{(n)} \), i.e., define for times \( t \leq \tau_j^{(n)} \)

\[
\mathcal{M}_t^{(n)}(\zeta) = \frac{1}{\pi} \Re \left( \log(g_t^{(n)}(\zeta) - W_{2N;j; t}^{(n)}) - \log(g_t^{(n)}(\zeta) - W_{2N-1;j; t}^{(n)}) + \ldots - \log(g_t^{(n)}(\zeta) - W_{1;j; t}^{(n)}) \right)
\]

and for \( t \geq \tau_j^{(n)} \) set \( \mathcal{M}_t^{(n)}(\zeta) = \mathcal{M}_{\tau_j^{(n)}}^{(n)}(\zeta) \). Note again that \( W_{i; t}^{(n)} \) describe the locations of the other discrete marked boundary points under the Loewner equation driven by \( W_{j; t}^{(n)} \). Now, if \( W_{i; t}^{(n)} \) were launched from the same locations as \( W_{i; t} \), \( \mathcal{M}_t^{(n)}(\zeta) \) and \( \mathcal{M}_t(\zeta) \) would just be continuous functions of \( W_{j; t}^{(n)} \) and \( W_{i; t} \), respectively. It takes a standard harmonic measure argument to show that a small change in the launching location does not play a role, and hence by the weak convergence \( W_{j; t}^{(n)} \to W_j \) (using also the continuity of the stopping times), we get

\[
\mathbb{E}[\mathcal{M}_{\tau_j}(\zeta)f_t(W_j)] = \lim_n \mathbb{E}^{(n)}[\mathcal{M}_{\tau_j^{(n)}}^{(n)}(\zeta)f_t(W_{j; t}^{(n)})].
\]

Similarly, for \( \mathcal{M}_t = \mathcal{M}_{t \land \tau_j} \) and \( \mathcal{M}_t^{(n)} \), we get

\[
\mathbb{E}[\mathcal{M}_t(\zeta)f_t(W_j)] = \lim_n \mathbb{E}^{(n)}[\mathcal{M}_t^{(n)}(\zeta)f_t(W_{j; t}^{(n)})].
\]
Let us next relate \( M^{(n)} \) to the discrete martingales \( M^{(n)} \) under the harmonic explorer in \( G^{(n)} \), given by Proposition 6.11. Consider first \( M^{(n)} \). Due to the convergence of discrete harmonic measures to the continuous ones, which is uniform over the family of discrete domains bounded from inside and outside [CS11], we have

\[
\mathbb{E}^{(n)}[\mathcal{M}^{(n)}(\tau_j(n), \zeta) f_t(W_j^{(n)})] = \mathbb{E}^{(n)}[M^{(n)}(\tau_j(n), \zeta) f_t(W_j^{(n)}) + o_n(1)];
\]

(6.12)

where \( z^{(n)} \) is the face of \( G^{(n)} \) whose conformal image in \( \mathbb{H} \) contains \( \zeta \); \( [\tau_j^{(n)}] \) is the first time after \( \tau_j^{(n)} \) when the lattice initial segment reaches a vertex; and \( o_n(1) \) denotes \( o(1) \) as \( n \to \infty \), and is uniform over \( t \) and the possible initial segments up to time \( [\tau_j^{(n)}] \), or, equivalently, over the driving functions \( W_j^{(n)} \).

Arguing identically, we also have

\[
\mathbb{E}[\mathcal{M}_t^{(n)}(\zeta) f_t(W_j^{(n)})] = \mathbb{E}[M^{(n)}(\tau_j^{(n)}, \zeta) f_t(W_j^{(n)}) + o_n(1)],
\]

(6.13)

where the notations are defined analogously to the above.

Now, using the discrete martingale property of \( M^{(n)} \) and the uniformity of the \( o(1) \) terms in (6.12) and (6.13), we deduce

\[
\mathbb{E}[\mathcal{M}_t^{(n)}(\zeta) f_t(W_j^{(n)})] = \mathbb{E}[\mathcal{M}_t^{(n)}(\zeta) f_t(W_j^{(n)})] + o_n(1).
\]

Substituting this into (6.11) and (6.10), we observe that

\[
\mathbb{E}[\mathcal{M}_t(\zeta) f_t(W_j)] - \mathbb{E}[\mathcal{M}_t(\zeta) f_t(W_j)] = \lim_{n \to \infty} (0 + o_n(1)) = 0.
\]

This shows that (6.9) holds and finishes the proof of the lemma.

**Proof of Proposition 6.12** Observe first that the derivative of the expression (6.8) defining \( \mathcal{M}_t(\zeta) \) with respect to \( W_{j,t} \) is never zero. Thus, by the Implicit function theorem, \( W_{j,t} \) can be expressed as a smooth function of \( \mathcal{M}_t(\zeta) \) and \( W_{i,t}, i \neq j \). In particular, since \( \mathcal{M}_t(\zeta) \) is a continuous bounded martingale and \( W_{i,t} \) is continuously differentiable in time, it follows that \( W_{j,t} \) is a semimartingale.

Let us now apply Itô calculus to the (complex) process

\[
A_t = \sum_{i=1}^{2N} (-1)^i \log(g_i(\zeta) - W_{i,t}),
\]

whose real part is a martingale by Lemma 6.13. We obtain

\[
dA_t = \sum_{i=1}^{2N} (-1)^i \frac{1}{g_i(\zeta) - W_{i,t}} \left( \frac{2dt}{g_i(\zeta) - W_{j,t}} - \frac{2dt}{W_{i,t} - W_{j,t}} \right)
\]

\[+ (-1)^j \frac{1}{g_j(\zeta) - W_{j,t}} \left( \frac{2dt}{g_j(\zeta) - W_{j,t}} - dW_{j,t} \right) + 1/2(-1)^j \frac{1}{(g_j(\zeta) - W_{j,t})^2} d\langle W_j, W_j \rangle_t.
\]

Now, \( A_t \) is a semimartingale, and consist thus of a local martingale part and a finite variation (f.v.) part. For the real part of \( A_t \) to be a martingale, the real part of the f.v. part of \( A_t \) must vanish. After some simplifications, we express the f.v. part as

\[
d[f.v. \text{ part of } A_t] = \sum_{i=1}^{2N} (-1)^i \frac{2dt}{g_i(\zeta) - W_{j,t}} \left( - \frac{1}{W_{i,t} - W_{j,t}} \right) - (-1)^j \frac{1}{g_j(\zeta) - W_{j,t}} d[f.v. \text{ part of } W_{j,t}]
\]

\[+ (-1)^j \frac{1}{(g_j(\zeta) - W_{j,t})^2} (2dt - d\langle W_j, W_j \rangle_t/2).
\]

Furthermore, the real part of the above must vanish for a continuum of \( \zeta \)'s. On the other hand, the above is a second degree polynomial of the complex variable \( 1/(g_j(\zeta) - W_{j,t}) \) with real coefficients.
Now, the real part of such a polynomial vanishes on an open set of \( \zeta \)'s if and only if the coefficients of \( 1/(g_t(\zeta) - W_{j,t}) \) and \( 1/(g_t(\zeta) - W_{j,t})^2 \) both vanish. The latter gives

\[
2dt - d\langle W_j, W_j \rangle_t/2 = 0 \quad \Rightarrow \quad \langle W_j, W_j \rangle_t = 4t,
\]

and the former gives

\[
d[f.v. \text{ part of } W_{j,t}] = \sum_{i=1}^{2N} (-1)^{i-j} \left( -\frac{2dt}{W_{i,t} - W_{j,t}} \right) = 4(\partial_j \log Z_N)(W_{1,t}, \ldots, W_{2N,t})dt,
\]

where \( Z_N \) is given by (6.7). Equations (6.14) and (6.15) and the initial value \( W_{j,0} \) together identify the local semimartingale \( W_{j,t} \), giving the stochastic integral representation

\[
dW_{j,t} = \sqrt{4} dB_t + 4(\partial_j \log Z_N)(W_{1,t}, \ldots, W_{2N,t})dt,
\]

where \( B_t \) is a standard Brownian motion. By definition, this means that \( W_{j,t} \) is a local multiple SLE(4) driving function with partition function (6.7). This concludes the proof. \( \square \)

**Remark 6.14.** A sophisticated guess for the partition function \( Z_N \) from [KP16, Proposition 4.8] streamlined the proof above. However, this is not an inevitable logical input: \( Z_N \) is determined (up to a multiplicative constant) by requiring that (6.15) holds for all \( 1 \leq j \leq 2N \).

6.4.6. **Finishing the proof of Theorem 6.10**

**Identification part of Theorem 6.10** Let us now finish the proof of Theorem 6.10 by verifying that the assumptions needed for applying Theorem 5.8 are satisfied. Assumption 5.1 was just verified in Proposition 6.12. Assumption 5.4 holds trivially, and 5.6 is verified via condition (G'): condition (G) is verified in [SS05, Lemma 6.3] based on the discrete martingale of Proposition 6.11 for \( N = 1 \) curves. The identical computation with general \( N \) proves condition (G'). We can now apply Theorem 5.8 to conclude the proof of Theorem 6.10. \( \square \)

**APPENDIX A. ON REGULAR CONDITIONAL LAWS**

In this appendix, we present for completeness some basic facts about regular conditional laws and their relation to conditional expectations. The notations in this appendix are independent of the notations in the rest of the article.

A.1. **Regular conditional law given a sigma algebras.** Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space and \( Y : (\Omega, \mathcal{F}) \to (G, \mathcal{G}) \) a measurable random variable. The regular conditional law of \( Y \) given a sub-sigma algebra \( \mathcal{H} \) of \( \mathcal{F} \) is a map \( \mu : \Omega \times \mathcal{G} \to \mathbb{R} \) such that

i) \( \mu_\omega \) is a probability measure on \( (G, \mathcal{G}) \) for all \( \omega \in \Omega \);  
ii) \( \omega \mapsto \mu_\omega[B] \) is, for all \( B \in \mathcal{G} \), measurable \( (\Omega, \mathcal{H}) \to (\mathbb{R}, \mathcal{B}) \), where \( \mathcal{B} \) denotes the standard Borel sigma algebra of \( \mathbb{R} \); and  
iii) \( \mathbb{P}[\omega \in A, Y \in B] = \mathbb{E}[\mathbb{1}_A(\omega) \mu_\omega[B]] \) for all \( A \in \mathcal{H} \) and \( B \in \mathcal{G} \).
A.2. Regular conditional law given a random variable. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. Let \(X : (\Omega, \mathcal{F}) \to (E, \mathcal{E})\) and \(Y : (\Omega, \mathcal{F}) \to (G, \mathcal{G})\) be random variables taking values in complete separable metric spaces \(E\) and \(G\), respectively, with the Borel sigma algebras \(\mathcal{E}\) and \(\mathcal{G}\). We say that a map \(\lambda\) from \(E \times \mathcal{G}\) to \(\mathbb{R}\), denoted \((x, B) \mapsto \lambda_x[B]\), is the (regular) conditional law of \(Y\) given \(X\) if

i) \(\lambda_x\) is a probability measure on \((G, \mathcal{G})\) for all \(x \in E\);

ii) \(x \mapsto \lambda_x[B]\) is, for all \(B \in \mathcal{G}\), a measurable map from \((E, \mathcal{E}) \to (\mathbb{R}, \mathcal{B})\); and

iii) for all \(A \in \mathcal{E}\) and \(B \in \mathcal{G}\),

\[
\mathbb{P}[X \in A, Y \in B] = \mathbb{E}[\mathbb{I}_A(X)\lambda_X[B]] = \int_{x \in E} \mathbb{I}_A(x)\lambda_x[B]dP_X(x),
\]

where \(P_X\) denotes the law of \(X\) on \((E, \mathcal{E})\).

We observe that the conditional law \(\lambda\) of \(Y\) given \(X\), and the conditional law \(\mu\) of \(Y\) given the sigma algebra \(\sigma(X) \subset \mathcal{F}\) generated by \(X\), are related by \(\mu_x[\cdot] = \lambda_X(x)[\cdot]\) in the following precise sense. First, given a conditional law \(\lambda\) of \(Y\) given \(X\), and taking \(\mu_x[B] = \lambda_X(x)[B]\) one readily observes that \(\mu\) is the conditional law of \(Y\) given \(\sigma(X)\). Conversely, assume that we are given the conditional law \(\mu\) of \(Y\) given \(\sigma(X)\). By the Doob–Dynkin Lemma for random variables in complete separable metric spaces [Tar18, Lemma 5], a measurable random variable \((\Omega, \sigma(X)) \to (\mathbb{R}, \mathcal{B})\) can be expressed as a measurable function of \(X\), so \(\mu[B] : (\Omega, \sigma(X)) \to (\mathbb{R}, \mathcal{B})\) generates a function \(\lambda[B] : (E, \mathcal{E}) \to (\mathbb{R}, \mathcal{B})\) such that \(\mu_x[B] = \lambda_X(x)[B]\). One then readily observes that such a \(\lambda\) is the conditional law of \(Y\) given \(X\).

A.3. Existence and almost sure uniqueness. The regular conditional law of \(Y\) given \(X\), as defined above, exists and is unique in an almost sure sense: for the existence, a regular conditional law of \(Y\) taking values in a complete separable metric space, given any sub-sigma algebra of \(\mathcal{F}\), exists [Dur10]. As observed in the previous paragraph, the law of \(Y\) given \(X\) (or \(\sigma(X)\)) thus also exists. For the uniqueness, if \(\lambda\) and \(\hat{\lambda}\) are two regular conditional laws of \(Y\) given \(X\), then for any fixed \(B \in \mathcal{G}\), we have that \(\lambda_X[B] = \hat{\lambda}_X[B]\) almost surely. Since the sigma algebra \(\mathcal{G}\) can be generated by a countable collection of sets \(B\), it follows that also \(\lambda_X = \hat{\lambda}_X\) as measures on \((G, \mathcal{G})\), almost surely.

A.4. Conditional expectations determine the conditional law. It is well known that the expectations \(\mathbb{E}[f(X)]\) of all continuous bounded functions \(f(X)\) determine the law of a random variable \(X\) on a metric space. For instance, this result implies that the weak limit of a weakly converging sequence of random variables is unique. We will next prove an analogous result for conditional laws, stating that all conditional expectations of continuous bounded functions determine the conditional law of a random variable. This characterization, labelled (a) below, will be of key importance in this paper when identifying regular conditional laws in weak limits.

Proposition A.1. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and \(X : (\Omega, \mathcal{F}) \to (E, \mathcal{E})\) and \(Y : (\Omega, \mathcal{F}) \to (G, \mathcal{G})\) be random variables taking values in complete separable metric spaces \(E\) and \(G\), respectively, with the Borel sigma algebras \(\mathcal{E}\) and \(\mathcal{G}\). Let \(\lambda_x\) be probability measures on \((G, \mathcal{G})\) for all \(x \in E\). Then, the following are equivalent:

a) for all bounded, non-negative, Lipschitz continuous functions \(f : (G, \mathcal{G}) \to (\mathbb{R}, \mathcal{B})\), the function

\[
F : x \mapsto \int_{y \in G} f(y)d\lambda_x(y)
\]

is measurable \((E, \mathcal{E}) \to (\mathbb{R}, \mathcal{B})\) and we have

\[
\mathbb{E}[f(Y) \mid \sigma(X)] = F(X)
\]

b) \(\lambda\) is the conditional law of \(Y\) given \(X\).
c) for all \( \mathbb{P} \)-integrable measurable functions \( h : (E \times G, \mathcal{E} \otimes \mathcal{G}) \rightarrow (\mathbb{R}, \mathcal{B}) \), the function
\[
H : x \mapsto \int_{y \in G} h(x, y) d\lambda_x(y)
\]
is measurable \( (E, \mathcal{E}) \rightarrow (\mathbb{R}, \mathcal{B}) \) and we have
\[
\mathbb{E}[h(X, Y) \mid \sigma(X)] = H(X).
\]

Proof. We prove the implications (c) \( \Rightarrow \) (b) \( \Rightarrow \) (a) \( \Rightarrow \) (c).

To see the implication (c) \( \Rightarrow \) (b), by assumption, property (i) of regular conditional distributions holds. Properties (ii) and (iii) follow by taking \( h(x, y) = \mathbb{1}_B(y) \) in (c), where \( B \in \mathcal{G} \), in which case \( H(x) = \lambda_x[B] \).

For the implication (b) \( \Rightarrow \) (a), study the class of bounded measurable functions \( f : (G, \mathcal{G}) \rightarrow (\mathbb{R}, \mathcal{B}) \) for which the corresponding function \( F : (E, \mathcal{E}) \rightarrow (\mathbb{R}, \mathcal{B}) \)
\[
F : x \mapsto \int_{y \in G} f(y) d\lambda_x(y)
\]
is measurable and
\[
\mathbb{E}[\mathbb{1}_A(X) f(Y)] = \mathbb{E}[\mathbb{1}_A(X) F(X)]
\]
for all \( A \in \mathcal{E} \). By assumption, for all \( B \in \mathcal{G} \), we have \( \mathbb{1}_B(y) \) belongs to this class of functions. By the “standard machine” of integration theory, one then readily finds that all bounded, non-negative, continuous functions of \( y \) belong to this class.

For the implication (a) \( \Rightarrow \) (c), take first an open set \( B \subset G \) and a sequence of functions \( f_n : (G, \mathcal{G}) \rightarrow (\mathbb{R}, \mathcal{B}) \) given by \( f_n(y) = 1 - \max\{1 - n \cdot d(y, G \setminus B), 0\} \), so that \( f_n \) are non-negative, bounded, and Lipschitz continuous each, and they increase to \( \mathbb{1}_B \) pointwise, i.e., \( f_n(y) \uparrow \mathbb{1}_B(y) \) for all \( y \in G \). Denote
\[
F_n(x) = \int_{y \in G} f_n(y) d\lambda_x(y).
\]
By monotone convergence, \( F_n(x) \uparrow \lambda_x[B] \) for all \( x \in E \). As an increasing limit of measurable functions, \( x \mapsto \lambda_x[B] \) is thus measurable. Finally, one deduces that \( \mathbb{E}[\mathbb{1}_B(Y) \mid \sigma(X)] = \lambda_X[B] \) by starting from
\[
\mathbb{E}[f_n(Y) \mathbb{1}_A(X)] = \mathbb{E}[F_n(X) \mathbb{1}_A(X)]
\]
which holds by assumption for all \( A \in \mathcal{E} \), and the using monotone convergence on both sides.

Next, study the class \( \mathcal{H} \) of bounded measurable functions \( h : (E \times G, \mathcal{E} \otimes \mathcal{G}) \rightarrow (\mathbb{R}, \mathcal{B}) \) for which
\[
H : x \mapsto \int_{y \in G} h(x, y) d\lambda_x(y)
\]
is measurable \( (E, \mathcal{E}) \rightarrow (\mathbb{R}, \mathcal{B}) \) and
\[
\mathbb{E}[h(X, Y) \mid \sigma(X)] = H(X).
\]
By the previous paragraph, for all open sets \( B \subset G \), we have \( \mathbb{1}_B(y) \in \mathcal{H} \). The same then also holds for all closed \( B \subset G \). Thus, taking \( A \subset E \) closed and \( B \subset G \) closed, we also have \( \mathbb{1}_A(x) \mathbb{1}_B(y) \in \mathcal{H} \). Next, it is a standard exercise in integration theory to show that \( \mathcal{H} \) is a monotone class of functions, and that products of closed sets \( A \times B \subset E \times G \) are a \( \pi \)-system that generate the sigma algebra \( \mathcal{E} \otimes \mathcal{G} \). Thus, by the monotone class theorem, \( \mathcal{H} \) contains all bounded measurable functions \( f : (E \times G, \mathcal{E} \otimes \mathcal{G}) \rightarrow (\mathbb{R}, \mathcal{B}) \). This can be extended to all integrable functions by the “standard machine”. This shows that (a) \( \Rightarrow \) (c). \( \square \)
Lemma B.1. In the setup described above, the stopping times and stopped driving functions converge, \( \tau^{(n)} \to \tau \) in \( \mathbb{R} \) and \( \hat{V}^{(n)}_{t^{(n)}\tau} \to \hat{V}_{t\tau} \) in \( \mathcal{C} \).

**Proof.** Denote by \( K_t^{(n)} \) and \( K_t \) the growing hulls corresponding to the driving functions \( V_t^{(n)} \) and \( V_t \), respectively. Combining \([\text{Kem17}, \text{Lemma 5.1}]\) and \([\text{Lemma 3.1}]\) we see that for any fixed time \( s \in [0, \mathbb{S}] \), \( \mathbb{H} \setminus K_t^{(n)} \) converge to \( \mathbb{H} \setminus K_t \) in the Carathéodory sense. This convergence also holds, if we equip the

**Appendix B. Continuous modifications of exit times**

The exit time of a localization neighbourhood is not a continuous function of the curves in the topologies that we consider in this paper, see Figure [B.1]. We define here the continuous modifications of the exit times of localization neighbourhoods. These stopping times will also be conformally invariant as precised shortly. Let us thus consider them in \( \mathbb{H} \). Let \( U_1, \ldots, U_{2N} \) be bounded localization neighbourhoods of the local multiple SLE in \( \mathbb{H} \). Let \( \hat{U}_1, \ldots, \hat{U}_{2N} \) be “strictly larger” localization neighbourhoods in the sense that \( d(U_i, \mathbb{H} \setminus \hat{U}_i) > 0 \) for all \( i \), but so that the latter are also valid localization neighbourhoods for the local multiple SLE. Such \( \hat{U}_1, \ldots, \hat{U}_{2N} \) always exist, and the conformal invariance mentioned above holds assuming that localization neighbourhoods actually come as such pairs \( U_1, \ldots, U_{2N}, \hat{U}_1, \ldots, \hat{U}_{2N} \).

The continuous modification \( \tau_i \) of the exit time of \( U_i \) will then be between the exit times \( T_i \) and \( S_i \) of \( U_i \) and \( \hat{U}_i \), respectively, i.e., \( T_i < \tau_i < S_i \).

Now, the topological quadrilateral \( \hat{U} \setminus U_i \) (real-line segments being two opposite sides) can be conformally mapped to a rectangle \( (0, 1) \times (0, L) \), with a unique \( L > 0 \). Fix the reference point \( w_i \in \hat{U} \setminus U_i \) corresponding to the center point of this rectangle.

Denote by \( h_t^{(i)}(z) \) the value at \( w_i \) of the following harmonic function on \( \mathbb{H} \setminus K_t^{(i)} \) (where \( K_t^{(i)} \) is the hull growing from \( U_i \)): it takes boundary values \( 0 \) on \( \mathbb{R} \setminus K_t^{(i)} \) and inside \( U_i \); for boundary points \( z \in K_t^{(i)} \) with \( z \in \hat{U}_i \setminus U_i \), it takes the boundary values given by the corresponding \( x \)-coordinate in the rectangle \( (0, 1) \times (0, L) \); and for boundary points \( z \in U_i \setminus \hat{U}_i \) it takes the boundary value one. Clearly, \( h_t^{(i)} = 0 \) if \( K_t^{(i)} \subseteq U_i \), and \( h_t^{(i)} \) is increasing in \( t \). On the other hand, an easy Brownian motion argument gives a lower bound \( h_{t_{\text{low}}}^{(i)}(z) \) to the value \( h_{S_i}^{(i)}(z) \) at the exit time \( S_i \) of \( \hat{U}_i \). Let \( H_t^{(i)} \) be the harmonic measure of \( K_t^{(i)} \) in \( \mathbb{H} \), as seen from \( w_i \), so \( H_t^{(i)} \) is strictly increasing in \( t \). We define the continuous modification \( \tau_i \) of the exit time \( T_i \) to be the first time when the product \( h_t^{(i)} H_t^{(i)} \) reaches the level \( h_{t_{\text{low}}}^{(i)} H_{0_{(i)}}^{(i)} \).

The following result guarantees the continuity of the modified exit times. Let \( V \) be the driving function of a Loewner chain starting inside \( U_i \). Let \( V^{(n)} \) be a sequence of driving functions, such that \( \hat{V}^{(n)}_{s} \to \hat{V}_{s} \) uniformly over \( s \in [0, S] \). Denote the continuous exit times of \( U_i \) by the respective hulls by \( \tau^{(n)} \) and \( \tau \).

**Lemma B.1.** In the setup described above, the stopping times and stopped driving functions converge, \( \tau^{(n)} \to \tau \) in \( \mathbb{R} \) and \( \hat{V}^{(n)}_{t^{(n)}\tau_{\tau}} \to \hat{V}_{t\tau} \) in \( \mathcal{C} \).
Appendix C. Deterministic Loewner equation and conformal maps

In this appendix, we provide some analysis of the deterministic Loewner equation under conformal maps. Consider the following setup. Let $V^{(n)}$ be a sequence of continuous (deterministic) driving functions, and assume that they converge in the space $\mathcal{C}$ of continuous functions, $V^{(n)} \to V$. These functions generate by Loewner’s equation some growing hulls $K_i^{(n)}$ (resp. $K_i$), starting their growth from $V^{(n)}_{t=0}$ (resp. $V_{t=0}$). Let $N$ be a localization neighbourhood (a hull) of these starting points, and let $\hat{N}$ be a larger one, so that $d(N, \mathbb{H} \setminus \hat{N}) > 0$. Let $\varpi_n$ be conformal maps from $\hat{N}$ to some subset of $\mathbb{H}$, such that the real line segment of $\partial \hat{N}$ maps to real line under $\varpi_n$. (For instance, $\varpi_n$ could be conformal (Möbius) maps $\mathbb{H} \to \mathbb{H}$ or mapping-out functions of some hull disjoint from $\hat{N}$.) Assume that $\varpi_n$ converge to another conformal map, $\varpi_n \to \varpi$, uniformly over the compact set $\hat{N}$. The main task of this appendix is to show that the Loewner driving functions of the hulls $\varpi_n(K_i^{(n)})$ converge to that of $\varpi(K_i)$.

Recall first that some growing family of hulls can be described by a Loewner equation driven by a continuous function if and only if satisfies the local growth property (see [Kem17] for details). This characterization readily implies that the conformal images $\varpi_n(K_i^{(n)})$ (resp. $\varpi(K_i)$) of interest here can be described by a Loewner driving function $\hat{V}_s^{(n)}$ (resp. $\hat{V}_s$), at least up to the time $\hat{S}_n$ of exiting $\varpi_n(\hat{N})$ (resp. $\hat{S}$ of exiting $\varpi(\hat{N})$). (We denote by $s$ the capacity parametrization after the conformal maps.) Note that these times depend on $n$, so it is way more convenient to observe that at the exit time $S$ of the smaller localization neighbourhood $\varpi(N)$ by $\varpi(K_i)$, also the hulls $\varpi_n(K_i^{(n)})$ stay inside their larger neighbourhoods $\varpi_n(\hat{N})$, for all $n$ large enough. (This follows since $\mathbb{H} \setminus K_i^{(n)}$ converge to $\mathbb{H} \setminus K_i$ in the Carathéodory sense, for all fixed $t$.) Thus, we will study the driving processes up to the time $S$.

The main result of this appendix is the following.

**Proposition C.1.** In the setup and notation above, if $V^{(n)} \to V$ in $\mathcal{C}$, i.e., uniformly over compacts, then $\hat{V}_s^{(n)} \to \hat{V}_s$ uniformly over $s \in [0, S]$.

The statement above is certainly not surprising. However, note that differences compared to typical references addressing Loewner equation and conformal maps, e.g., [Law05] Section 4.6, are that we cannot apply Itô calculus, and that the conformal maps $\varpi_n$ depend on $n$. With some effort, similar ideas can be used to prove this proposition. We have chosen not to include the proof of this proposition in this version of the paper.
C.1. Some consequences. Let us list some consequences of Proposition [C.1]. Continue in the notation introduced before the statement of that lemma.

Take localization neighbourhoods $N \subset N_2 \subset \hat{N}$, such that $d(N, \mathbb{H} \setminus N_2), d(N_2, \mathbb{H} \setminus \hat{N}) > 0$. Denote the exit times of $\varpi(N)$ and $\varpi(N_2)$ by the growing hulls $K_\sigma$, with $K_{\sigma(t)} = \varpi(K_t)$, by $S$ and $S_2$ respectively, and assume that the continuous modification of the exit time $\sigma$ of $\varpi(N)$ is chosen using the pair of neighbourhoods $\varpi(N)$ and $\varpi(N_2)$, so that $S < \sigma < S_2$. Denote the similar continuous exit times by $\hat{K}_{\sigma(n)}$ by $\sigma(n)$. Let $V^{(n)} \to V$ and $\varpi_n \to \varpi$ as before.

**Corollary C.2.** In the notation above, $\sigma^{(n)} \to \sigma$ in $\mathbb{R}$ and $\hat{V}^{(n)}_{s,\sigma^{(n)}} \to \hat{V}_{s,\sigma}$ in $\mathcal{C}$.

**Proof.** Apply Proposition [C.1] and with the localization neighbourhoods $N_2 \subset \hat{N}$. This implies that $\hat{V}^{(n)}_s \to \hat{V}_s$ uniformly over $s \in [0, S_2]$. Combining with Lemma B.1 proves the claim. □

**Remark C.3.** Above $\sigma$ and $\sigma^{(n)}$ are the continuous exit times of $\varpi(N)$, defined via the same neighbourhoods $\varpi(N)$ and $\varpi(N_2)$. The statement of Corollary C.2 also holds if we instead define $\sigma^{(n)}$ to be the continuous exit times of $\varpi_n(N_2)$, with the neighbourhoods $\varpi_2(N)$ and $\varpi_n(N_2)$.

Taking $\varpi_n = \varpi$ for all $n$, we obtain an important special case of Corollary C.2.

**Corollary C.4.** In the notation above, $\sigma \in \mathbb{R}$ and $\hat{V}_{s,\sigma} \in \mathcal{C}$ are continuous functions of $V \in \mathcal{C}$.

Denote by $\tau$ the continuous modification of the exit time of $N$, with the pair $N_2$. Recall that the continuous exit times are conformally invariant, so $s(\tau) = \sigma$. Corollary C.4 now implies a stronger tool.

**Corollary C.5.** Equip the space of Loewner driving functions, in $t$ and $s$, stopped at exit times $\tau$ and $\sigma$, with the topology of $\mathcal{C}$. The mapping from $V_{t,\tau}$ to $\hat{V}_{s,\sigma}$ is a continuous bijection and its inverse is continuous.

**Proof.** By Corollary C.4, $V_{t,\tau}$ maps continuously to $\hat{V}_{s,\sigma}$, the driving function of its conformal images, and $V_{s,\sigma}$ to $\hat{V}_{t,\tau}$, the driving function of the conformal preimages. The bijectivity follows. □

Consider now another special case of Corollary C.2: take $V^{(n)} = V$ for all $n$ so that comparing $\hat{V}^{(n)}$ and $\hat{V}$ means comparing the effect of the different conformal maps. The below corollary shows that this effect is small uniformly over the choice of $V$:

**Corollary C.6.** Take $V^{(n)} = V$ for all $n$, and fix the sequence of conformal maps $\varpi_n \to \varpi$. Then, we have $d_c(\hat{V}^{(n)}_{s,\sigma^{(n)}}, \hat{V}_{s,\sigma}) \to 0$ as $n \to \infty$, uniformly over $V \in \mathcal{C}$, for any compact set $\mathcal{C}$ in the space of continuous functions $\mathcal{C}$.

**Proof.** For notational reasons, let us equip growing hulls, stopped at the continuous modification of the exit time of $N$, with the topology of their driving functions. For instance, we denote $d_c(\hat{V}^{(n)}_{s,\sigma^{(n)}}, \hat{V}_{s,\sigma}) = d(\varpi_n(K_1), \varpi(K_1))$.

Assume now for a contradiction that for some $\delta$ and infinitely many values of $n$, there exist $V^{(n)} \in \mathcal{C}$ such that

$$d(\varpi_n(K^{(n)}_1), \varpi(K^{(n)}_1)) > \delta,$$

where $K^{(n)}_1$ are the growing hulls generated by $V^{(n)}$. By compactness, we may extract a subsequence (which we suppress in notation) so that $V^{(n)}$ converge, $V^{(n)} \to V$ in $\mathcal{C}$. Let $K_t$ be the growing hulls corresponding to $V$. Now, compute

$$d(\varpi_n(K^{(n)}_1), \varpi(K^{(n)}_1)) \leq d(\varpi_n(K^{(n)}_1), \varpi(K_1)) + d(\varpi(K_1), \varpi(K^{(n)}_1)).$$

By Corollary C.2, both terms on the right-hand side above converge to 0 as $n \to \infty$. This is a contradiction, proving the claim. □

**Remark C.7.** Also Corollary C.6 holds with the alternative choice of stopping times $\sigma^{(n)}$ in Remark C.3.
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