A Characterization of GRW Spacetimes

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Abstract: We show presence a special torse-forming vector field (a particular form of torse-forming of a vector field) on generalized Robertson–Walker (GRW) spacetime, which is an eigenvector of the de Rham–Laplace operator. This paves the way to showing that the presence of a time-like special torse-forming vector field $\xi$ with potential function $\rho$ on a Lorentzian manifold $(M,g)$, dim$M > 5$, which is an eigenvector of the de Rham Laplace operator, gives a characterization of a GRW-spacetime. We show that if, in addition, the function $\xi(\rho)$ is nowhere zero, then the fibers of the GRW-spacetime are compact. Finally, we show that on a simply connected Lorentzian manifold $(M,g)$ that admits a time-like special torse-forming vector field $\xi$, there is a function $f$ called the associated function of $\xi$. It is shown that if a connected Lorentzian manifold $(M,g)$, dim$M > 4$, admits a time-like special torse-forming vector field $\xi$ with associated function $f$ nowhere zero and satisfies the Fischer–Marsden equation, then $(M,g)$ is a quasi-Einstein manifold.

Keywords: generalized Robertson–Walker spacetime; special torse-forming vector fields; de Rham–Laplace operator; quasi-Einstein manifold

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1. Introduction

It is well known that through cosmological considerations the space being homogeneous and isotropic in the large scale, picks the Robertson–Walker metrics. It amounts to the fact that an $n$-dimensional spacetime, $n > 3$, acquires the form $I \times N$, with metric $g = -dt^2 + \varphi^2g$, where $I$ is an open interval, $\varphi$ is a smooth positive function defined on $I$, and $(N,g)$ is an $(n-1)$-dimensional Riemannian manifold of constant curvature. An $n$-dimensional generalized Robertson–Walker spacetime (GRW-spacetime) is $I \times \varphi N$, with metric $g = -dt^2 + \varphi^2 \gamma$, where $(N,\gamma)$ is an $(n-1)$-dimensional Riemannian manifold (cf. [1,2]). An interesting characterization of GRW-spacetime was obtained by Chen (cf. [3]), by proving that a Lorentzian manifold $(M,g)$ admits a non-trivial time-like concircular vector field, if, and only if, it is a GRW-spacetime. Additionally, for interesting characterizations of GRW-spacetimes using torse-forming vector fields and Weyl tensors, we refer to (cf. [4,5]).

A concircular vector field $\xi$ on a semi-Riemannian manifold $(M,g)$ satisfies:

$$\nabla_U \xi = \rho U, \quad U \in \mathcal{X}(M),$$

where $\rho$ is a scalar, $\nabla$ is a Levi–Civita connection, and $\mathcal{X}(M)$ is the Lie algebra of smooth vector fields on $M$ (cf. [5–7]). For other characterizations of GRW-spacetimes, we refer to (cf. [2,3,8,9]).
Yano generalized concircular vector fields by introducing a torse-forming vector field on semi-Riemannian manifold \((M,g)\) (cf. [10]), defined by:

\[
\nabla_U \xi = \rho U + \alpha(U) \xi, \quad U \in \mathfrak{X}(M),
\]

where \(\alpha\) is a 1-form called the torse 1-form. Naturally, if \(\alpha = 0\), then a torse-forming vector field is a concircular vector field. These vector fields are also used in characterizing a GRW-spacetime (cf. [2,4]). In [11], Chen considered an interesting special class of torse-forming vector field, requiring \(\xi\) to be nowhere zero and satisfying \(\alpha(\xi) = 0\), that is the torse-forming vector field is perpendicular to the dual-vector field to torse form \(\alpha\), called torqued vector fields.

In the present paper, we introduce on a Lorentzian manifold a special type of torse-forming vector field. A unit time-like torse-forming vector field \(\xi\) on a Lorentzian manifold \((M,g)\) is said to be a special torse-forming vector field if it satisfies:

\[
\nabla_U \xi = \rho(U + \eta(U) \xi), \quad U \in \mathfrak{X}(M),
\]

where \(\rho\) is a non-zero function and \(\eta\) is 1-form dual to \(\xi\). We call \(\rho\) the potential function of the special torse-forming vector field \(\xi\). Note that for a special torse-forming vector field, using Equation (1), we have \(\alpha(U) = -\rho \eta(U)\), that is \(\xi\) is a torse-forming vector field, which is parallel to the vector field dual \(\alpha\) as opposed to the torqued vector field where \(\xi\) is orthogonal to the vector field dual \(\alpha\). Moreover, from the definition of special torse-forming vector field \(\xi\) on a Lorentzian manifold, it follows that under no situation, it reduces to a concircular vector field.

We study the role of a time-like special torse-forming vector field \(\xi\) on a Lorentzian manifold \((M,g)\) in characterizing GRW-spacetimes. It is achieved by using the de Rham–Laplace operator \(\Box\) (cf. [12]) and a time-like special torse-forming vector field \(\xi\) with potential function \(\rho\) on a connected Lorentzian manifold \((M,g)\), \(\text{dim}M > 5\), through showing that \(\Box \xi = \sigma \xi\) holds for a smooth function \(\sigma\), if, and only if, \((M,g)\) is a GRW-spacetime (see Theorem 1). We also show that if the function \(\xi(\rho)\) is nowhere zero on \(M\), then the fibers of GRW-spacetime \(I \times_g N\) are compact (see Theorem 2).

If \(\xi\) is a special torse-forming vector field on a simply connected Lorentzian manifold \((M,g)\), then the dual-1-form \(\eta\) is closed (see Equation (15)), and, therefore, there is a function \(f\) such that \(\eta = df\). Thus, the special torse-forming vector field \(\xi\) on a simply connected Lorentzian manifold \((M,g)\) satisfies \(\xi = \nabla f\), call this function \(f\) the associated function of \(\xi\). Recall that a Lorentzian manifold \((M,g)\) is said to be a quasi-Einstein manifold (cf. [13]) if its Ricci tensor has the following expression:

\[
\text{Ric} = f_1 g + f_2 \beta \otimes \beta,
\]

where \(f_1, f_2\) are scalars and \(\beta\) is a 1-form on \(M\). Exact solutions of the Einstein field equations can provide very important information about quasi-Einstein manifolds. For example, the Robertson–Walker spacetimes are quasi-Einstein manifolds. For this reason, the study of quasi-Einstein manifolds is important. It is shown that if the associated function \(f\) of the special torse-forming vector field \(\xi\) on a simply connected Lorentzian manifold \((M,g)\), \(\text{dim}M > 4\), satisfies (i) \(f\) is nowhere zero and (ii) \(f\) is a solution of the Fischer–Marsden equation, then \((M,g)\) is a quasi-Einstein manifold (see Theorem 3). Additionally, it is shown that if the scalar curvature \(\tau\) of a simply connected Lorentzian manifold \((M,g)\), \(\text{dim}M \geq 4\), is a constant and possesses a special torse-forming vector field \(\xi\) with potential function \(\rho\) and associated function \(f\) satisfying the above two conditions, then the potential function \(\rho\) is an eigenfunction of the Laplace operator \(\Delta\) (see Corollary 1).
2. Preliminaries

Let $\varphi$ be a smooth function on an $n$-dimensional connected Lorentzian $(M, g)$. The Hessian operator $H_\varphi$ is defined by:

$$H_\varphi(V) = \nabla_V \nabla \varphi, \quad V \in \mathfrak{X}(M),$$

where $\nabla \varphi$ is the gradient of $\varphi$ and Hessian $\text{Hess}(\varphi)$ is defined by (cf. [14]):

$$\text{Hess}(\varphi)(U_1, U_2) = g(H_\varphi(U_1), U_2), \quad U_1, U_2 \in \mathfrak{X}(M).$$

The Laplacian $\Delta \varphi$ of the function $\varphi$ is given by $\Delta \varphi = \text{div}(\nabla \varphi)$, and it satisfies:

$$\Delta \varphi = \text{tr}H_\varphi.$$ (6)

Let $\xi$ be a time-like special torse-forming vector field on a Lorentzian $(M, g)$. Then, using the expression for the curvature tensor field

$$R(F_1, F_2)F_3 = \nabla_{F_1} \nabla_{F_2} F_3 - \nabla_{F_2} \nabla_{F_1} F_3 - \nabla_{[F_1, F_2]} F_3, \quad F_1, F_2, F_3 \in \mathfrak{X}(M)$$

and Equation (2), we compute:

$$R(F_1, F_2)\xi = F_1(\rho)F_2 - F_2(\rho)F_1 + (F_1(\rho)\eta(F_2) - F_2(\rho)\eta(F_1))\xi + \rho^2(\eta(F_2)F_1 - \eta(F_1)F_2).$$

Above equation gives expression for the Ricci tensor $\text{Ric}$ of the Lorentzian manifold $(M, g)$:

$$\text{Ric}(V, \xi) = -(n - 4)V(\rho) + \left(\xi(\rho) + (n - 3)\rho^2\right)\eta(V), \quad V \in \mathfrak{X}(M).$$ (7)

Note that the Ricci operator $Q$ of the Lorentzian manifold $(M, g)$ is given by $\text{Ric}(U, V) = g(U, V), U \in \mathfrak{X}(M)$, and, therefore, Equation (7) implies:

$$Q\xi = -(n - 4)\nabla \rho + \left(\xi(\rho) + (n - 3)\rho^2\right)\xi$$ (8)

and:

$$\text{Ric}(\xi, \xi) = -(n - 3)\left(\xi(\rho) + \rho^2\right).$$ (9)

The Laplace operator $\Delta$ acting on vector fields on the Lorentzian manifold $(M, g)$ is defined by:

$$\Delta U = \sum_{i=1}^{n} \left(\nabla_{v_i} \nabla_{v_i} U - \nabla_{\nabla_{v_i} v_i} U\right), \quad U \in \mathfrak{X}(M),$$ (10)

where $\{v_1, \ldots, v_n\}$ is a local orthonormal frame on $M$. The de Rham–Laplace operator $\Box$ on the Lorentzian manifold $(M, g)$ is $\Box : \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by (cf. [12]):

$$\Box U = \Delta U + QU, \quad U \in \mathfrak{X}(M).$$ (11)

**Lemma 1.** Let $\xi$ be a time-like special torse-forming vector on an $n$-dimensional Lorentzian manifold $(M, g)$ with potential function $\rho$. Then:

$$\Box \xi = -(n - 5)\nabla \rho + 2(n - 2)\rho^2 \xi.$$ (12)

**Proof.** Using Equation (2), for $U \in \mathfrak{X}(M)$, we have:

$$\nabla_U \nabla_U \xi - \nabla_{\nabla_U U} \xi = U(\rho)U + U(\rho)\eta(U)\xi + \rho^2\|U\|^2 \xi + 2\rho^2 \eta(U)^2 \xi + \rho^2 \eta(U)U.$$
Since $\xi$ is a time-like unit vector field, choosing a local frame $\{v_1, \ldots, v_{n-1}, \xi\}$ on $M$, where $v_i$, $i = 1, \ldots, n - 1$ are spacelike unit vector fields in the above equation, to conclude:

$$\Delta \xi = \nabla \rho - \left(\xi(\rho) - (n - 1)\rho^2\right) \xi.$$ 

Thus, using Equations (8) and (11) with the above equation, we conclude:

$$\Box \xi = -(n - 5)\nabla \rho + 2(n - 2)\rho^2 \xi.$$

3. Characterizing GRW Spacetimes

Consider an n-dimensional GRW-spacetime $M = I \times N$ with metric $g = -dt^2 + \varphi^2 g$. Then, $\xi = \frac{\partial}{\partial t}$ is a time-like unit vector field on $(M, g)$. Let $\nabla$ be the Levi–Civita connection on $(M, g)$. Then, for a $U \in \mathcal{X}(M)$, we have $U = h \xi + \varphi, \varphi \in \mathcal{X}(N)$. If we denote by $\eta = dt$, then $\eta(U) = g(U, \xi) = -h$, where $\eta(\xi) = g(\xi, \xi) = -1$. Using fundamental equations for the warped product (cf. [8]), we have:

$$\nabla_U \xi = \nabla_h \xi + \varphi = \nabla_{\xi} \xi = \frac{\xi(\varphi)}{\varphi} E = \frac{\xi(\varphi)}{\varphi} (U + h \xi) = \frac{\xi(\varphi)}{\varphi} (U + \eta(U) \xi).$$

Thus,

$$\nabla_U \xi = \rho(U + \eta(U) \xi), \quad U \in \mathcal{X}(M), \quad \rho = \frac{\xi(\varphi)}{\varphi}, (12)$$

this proves, $\xi$ is a special torse-forming vector field on the GRW-spacetime $(M, g)$. Now, using the expression for the Ricci tensor for the warped product $I \times N$ (cf. [8]), we have:

$$\text{Ric}(\xi, E) = 0, \quad E \in \mathcal{X}(N),$$

which implies $Q(\xi) = \lambda \xi$ for a smooth function $\lambda$ on $I$. Furthermore, choosing a local frame $\{v_1, \ldots, v_{n-1}\}$ on $N$, we have a local orthonormal frame $\{\xi, v_1, \ldots, v_{n-1}\}$ on $M$. Then, using Equation (12), we have $\nabla_\xi \xi = 0$, $\nabla_v \xi = \rho v, v(\rho) = 0$, and:

$$\nabla_v \nabla_v \xi = \rho \nabla_v v, \nabla_\xi \nabla_\xi = 0.$$

Furthermore, using Equation (12), we have:

$$\nabla \nabla_\xi v = \rho (\nabla_\xi v + g(\nabla_\xi v, \xi) \xi) = \rho \nabla_\xi v - \rho g(v, \nabla_\xi v) \xi = \rho \nabla_\xi v - \rho^2 \xi.$$

Thus, the rough Laplace operator $\Delta$ acting on $\xi$ is given by:

$$\Delta \xi = \left(\nabla_\xi \nabla_\xi - \nabla_\xi \nabla_\xi \xi\right) + \sum_{i=1}^{n-1} \left(\nabla_{v_i} \nabla_{v_i} \xi - \nabla_{v_i} v_i \xi\right) = (n - 1)\rho^2 \xi.$$

Now, we see that the de Rham–Laplace operator $\Box$ acting on $\xi$ is given by:

$$\Box \xi = \left((n - 1)\rho^2 + \lambda\right) \xi.$$

Hence, GRW-spacetime $(M, g)$ admits a special torse-forming vector field $\xi$, which is an eigenvector of the de Rham–Laplace operator $\Box$.

**Theorem 1.** An $n$-dimensional connected Lorentzian manifold $(M, g)$, $n > 5$, is a GRW-spacetime $I \times N$, if, and only if, it admits a time-like special torse-forming vector field $\xi$, which is an eigenvector of the de Rham–Laplace operator on $(M, g)$.
Proof. Let \((M, g)\) be a connected Lorentzian manifold, \(n > 5\), \(\xi\) be a time-like special torse-forming vector field on \((M, g)\) with \(\Box_{\xi} = \lambda \xi\), \(\lambda\) being a scalar. We denote by \(\nabla\) the Levi–Civita connection on \((M, g)\); using Equation (2), we have:

\[
\nabla_{\xi} \xi = 0. \tag{13}
\]

Define a smooth distribution \(D\) on \(M\) by:

\[
D = \{U \in \mathfrak{x}(M) : \eta(U) = 0\}. \tag{14}
\]

Note that Equation (2) gives:

\[
d\eta(U, V) = g(\nabla_U \xi, V) - g(\nabla_V \xi, U) = 0, \quad U, V \in \mathfrak{x}(M), \tag{15}
\]

that is the dual-1-form \(\eta\) to \(\xi\) is closed. Thus, for \(E, F \in D\), we have \(\eta([E, F]) = -d\eta(E, F) = 0\), that is \([E, F] \in D\), proving that the distribution \(D\) is integrable. Let \(N\) be a leaf of \(D\). Then, \(N\) is a hypersurface of \(M\) with unit normal \(\xi\). Using Equation (2), we observe that for \(E \in \mathfrak{x}(N)\),

\[
\nabla_{E} \xi = \rho E, \tag{16}
\]

that is the shape operator \(S\) of \(N\) is given by:

\[
S(E) = -\rho E, \quad E \in \mathfrak{x}(N). \tag{17}
\]

Now, as \(\Box_{\xi} = \lambda \xi\), where \(\lambda\) is a scalar on \(M\), using Lemma 1, we get:

\[
-(n - 5)\nabla \rho + 2(n - 2)\rho^2 \xi = \lambda \xi. \tag{18}
\]

On taking the inner product in above equation with \(\xi\) yields

\[
\lambda = (n - 5)\xi(\rho) + 2(n - 2)\rho^2
\]

and substituting this value of \(\lambda\) in Equation (18), we have:

\[
-(n - 5)\nabla \rho = (n - 5)\xi(\rho)\xi. \tag{19}
\]

Above equation on taking the inner product with \(E \in \mathfrak{x}(N)\), gives \((n - 5)E(\rho) = 0\), and the assumption \(n > 5\) implies \(E(\rho) = 0\), that is \(\rho\) is a constant on the hypersurface \(N\). Therefore, Equation (17) implies that \(N\) is a totally umbilical hypersurface of \(M\). Moreover, the orthogonal complementary distribution \(D^\perp\) to \(D\) is one-dimensional spanned by \(\xi\), and by Equation (13), the integral curves of the distribution \(D^\perp\) are geodesics on \(M\). Thus, \((M, g)\) is the warped product \(I \times_{\phi} N\) (cf. [15]), that is \((M, g)\) is a GRW-spacetime.

Conversely, we have already seen that a GRW-spacetime \(I \times_{\phi} N\) admits a special torse-forming vector field \(\xi\), which is an eigenvector of \(\Box\). \(\Box\)

In the above result we have seen that the presence of a time-like special torse-forming vector field \(\xi\) on a Lorentzian manifold \((M, g)\) satisfying \(\Box_{\xi} = \lambda \xi\) for scalar \(\lambda\) is a GRW-spacetime \(I \times_{\phi} N\). It is interesting to observe if in addition \(\xi(\rho)\) is nowhere zero, then this condition has effect on the topology of \(N\).

**Theorem 2.** Let \(\xi\) be a time-like special torse-forming vector field with potential function \(\rho\) on an \(n\)-dimensional complete and connected Lorentzian manifold \((M, g)\), \(n > 5\). If \(\xi\) is an eigenvector of the de Rham–Laplace operator on \((M, g)\) and the function \(\xi(\rho)\) is nowhere zero, then \((M, g)\) is GRW-spacetime \(I \times_{\phi} N\), with \(N\) compact.
Proof. Let \( \xi \) be a time-like special torse-forming vector field on a Lorentzian manifold \((M, g)\), \(n > 5\), with \( \xi \) being an eigenvector of the de Rham Laplace operator on \((M, g)\) and the function \( \xi(\rho) \neq 0 \) everywhere on \( M \). Since \( n > 5 \), Equation (19) implies:

\[
\nabla \rho = -\xi(\rho)\xi.
\]

(20)

As \( \xi \) is a time-like unit vector field and \( \xi(\rho) \) is nowhere zero, the above equation implies that \( \nabla \rho \) is nowhere zero on \( M \). Therefore, the potential function \( \rho : M \to E \) is a submersion, and each fiber \( F_x = \rho^{-1}(\{\rho(x)\}) \), \( x \in M \), is an \((n - 1)\)-dimensional smooth manifold; as \( \{\rho(x)\} \) is compact in \( E \), we obtain that \( F_x \) is compact. Consider a smooth vector field:

\[
\mathbf{u} = -\frac{\xi}{\xi(\rho)}
\]

that has no zeros on \( M \). Then, it follows that \( \mathbf{u}(\rho) = -1 \) and \( \mathbf{u} \) has a local flow \( \{\phi_s\} \) that satisfies:

\[
\rho(\phi_s(x)) = \sigma(x) - s.
\]

(21)

Recall the escape Lemma (cf. [16]), which states that if \( \gamma \) is a integral curve of \( \mathbf{u} \) whose maximal domain is not all of \( E \), then the image of \( \gamma \) cannot lie in any compact subset of \( M \).

Using the escape lemma and Equation (21) on a complete and connected \( M \), we obtain that \( \mathbf{u} \) is complete and has global flow \( \{\phi_s\} \). Now, define \( f : E \times F_x \to M \) by:

\[
f(s, u) = \phi_s(u), \quad u \in F_x.
\]

Then, \( f \) is smooth, and for each \( u \in M \), we find \( s \in E \) such that \( \phi_s(u) = y \in F_x \), satisfying \( u = \phi_{-s}(y) \). Thus, \( f(-s, y) = u \), that is \( f \) is an onto map. We observe that, on taking \((s_1, u_1), (s_2, u_2) \) in \( E \times F_x \) satisfying \( f(s_1, u_1) = f(s_2, u_2) \), we have \( \phi_{s_1}(u_1) = \phi_{s_2}(u_2) \), and using Equation (21), we obtain \( \rho(u_1) - s_1 = \rho(u_2) - s_2 \). As \( u_1, u_2 \in F_x \), \( \rho(u_1) = \rho(u_2) \), and we obtain \( s_1 = s_2 \). Thus, using \( \phi_{s_1}(u_1) = \phi_{s_2}(u_2) \), we arrive at \( u_1 = u_2 \), that is \( f \) is one-to-one. Furthermore, we have:

\[
f^{-1}(u) = (-s, y) = (-s, \phi_s(u)),
\]

which is smooth. Hence, \( f : E \times F_x \to M \) is a diffeomorphism, where \( F_x \) is a compact subset of \( M \). Using Theorem 3.1, we see that \( I \times N \) is diffeomorphic to \( E \times F_x \), and as the open interval \( I \) is diffeomorphic to \( E \), the fiber \( N \) must be diffeomorphic to \( F_x \). As \( F_x \) is compact, we obtain that \( N \) is compact. \( \Box \)

4. Lorentzian Manifolds as Quasi-Einstein Manifolds

Fischer–Marsden considered the following differential equation on a semi-Riemannian manifold \((M, g)\) (cf. [17]):

\[
(\Delta f)g + fRic = Hess(f),
\]

(22)

where \( f \) is a smooth function on \( M \). We call the above differential equation the Fischer–Marsden equation. This differential equation is closely associated with Einstein spaces. A generalization of Einstein manifolds was considered in [13], where the authors defined quasi-Einstein manifolds. A semi-Riemannian manifold \((M, g)\) is said to be a quasi-Einstein manifold if its Ricci tensor satisfies Equation (3). In this section, we use a unit time-like special torse-forming vector field \( \xi \) on a Lorentzian manifold \((M, g)\) to find conditions under which \((M, g)\) is a quasi-Einstein manifold.

Let \( \xi \) be a time-like special torse-forming vector field on a simply connected Lorentzian manifold \((M, g)\). On using Equations (2) and (15), we have \( d\eta = 0 \), that is \( \eta \) is a closed 1-form and \( M \) is simply connected \( \eta = df \) (exact) for a smooth function \( f \) on \( M \). Thus, for a time-like special torse-forming \( \xi \) on a simply connected Lorentzian manifold \((M, g)\), we have:

\[
\xi = \nabla f
\]

(23)
and we call the smooth function $f$ in Equation (23) the associated function of $\xi$.

**Theorem 3.** Let $\xi$ be a time-like special torse-forming vector field on an $n$-dimensional simply connected Lorentzian manifold $(M, g)$, $n > 4$, with potential function $\rho$ and associated function $f$. If $f$ is a nowhere zero solution of the Fischer–Marsden equation, then $(M, g)$ is a quasi-Einstein manifold.

**Proof.** Using Equations (2) and (23), we have:

$$H_f(U) = \rho(U + \eta(U)\xi),$$

which implies:

$$Hess(f) = \rho g + \rho \eta \otimes \eta, \quad \Delta f = (n - 3)\rho. \quad (24)$$

Since $f$ satisfies Fischer–Marsden equation, using Equations (22) and (24), we have:

$$f Ric = -(n - 4)\rho g + \rho \eta \otimes \eta. \quad (25)$$

As $f$ is nowhere zero, we have:

$$Ric = -(n - 4)\left(\rho f^{-1}\right)g + \left(\rho f^{-1}\right)\eta \otimes \eta.$$ 

Hence, $(M, g)$ is a quasi-Einstein manifold. \(\square\)

If simply connected Lorentzian manifold $(M, g)$ has scalar curvature $\tau = trQ$, using above result we have the following result that gives a relation between $\rho$ and $f$ of the time-like special torse-forming vector field $\xi$ on $(M, g)$.

**Corollary 1.** Let $\xi$ be a time-like special torse-forming vector field on an $n$-dimensional simply connected Lorentzian manifold $(M, g)$, $n \geq 4$, with potential function $\rho$ and associated function $f$. If $f$ is a solution of the Fischer–Marsden equation, then:

$$\rho = -\frac{\tau}{(n - 3)^2}f.$$ 

In particular, if the scalar curvature $\tau$ of $(M, g)$ is a constant, then the potential function $\rho$ is an eigenfunction of the Laplace operator $\Delta$.

**Proof.** Let $\xi$ be a time-like special torse-forming vector field on a simply connected Lorentzian manifold $(M, g)$, $n \geq 4$, with potential function $\rho$ and associated function $f$. Suppose $f$ satisfies Equation (22). Then, Equation (25), gives

$$f \tau = -(n - 4)(n - 2)\rho - \rho = -(n - 3)^2\rho.$$

Hence,

$$\rho = -\frac{\tau}{(n - 3)^2}f.$$ 

Now, if $\tau$ is a constant, then the above equation in view of Equation (24) implies:

$$\Delta \rho = -\frac{\tau}{(n - 3)}\rho,$$

that is the potential function $\rho$ is an eigenfunction of $\Delta$. \(\square\)

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