A Dynamical 2-dimensional Fuzzy Space

M. Burić\textsuperscript{1}, J. Madore\textsuperscript{2}
\textsuperscript{1}Faculty of Physics, P.O. Box 368
11001 Belgrade
\textsuperscript{2}Laboratoire de Physique Théorique
Université de Paris-Sud, Bâtiment 211, F-91405 Orsay

Abstract

The noncommutative extension of a dynamical 2-dimensional space-time is given and some of its properties discussed. Wick rotation to euclidean signature yields a surface which has as commutative limit the doughnut but in a singular limit in which the radius of the hole tends to zero.

PACS: 02.40.Gh, 04.60.Kz

1 Introduction and notation

There is a very simple argument due to Pauli that the quantum effects of a gravitational field will in general lead to an uncertainty in the measurement of space coordinates. It is based on the observation that two ‘points’ on a quantized curved manifold can never be considered as having a purely space-like separation. If indeed they had so in the limit for infinite values of the Planck mass, then at finite values they would acquire for ‘short time intervals’ a time-like separation because of the fluctuations of the light cone. Since the ‘points’ are in fact a set of four coordinates, that is scalar fields, they would not then commute as operators. This effect could be considered important at least at distances of the order of Planck length, and perhaps greater. This is one motivation to study noncommutative geometry. A second motivation, which is the one we consider ours, is the fact that it is possible to study noncommutative differential geometry, and there is no reason to assume that even classically coordinates commute at all length scales. One can consider for example coordinates as order parameters as in solid-state physics and suppose that singularities in the gravitational field become analogs of core regions; one must go beyond the classical approximation to describe them. A straightforward and conservative way is to represent coordinates by operators. The space-time manifold is thus replaced with an algebra generated by a set of noncommutative ‘coordinates’. The essential element which allows us to interpret a noncommutative algebra as a space-time is the possibility \cite{1,2} to introduce a differential structure on the former.

We define noncommutative ‘space’ as an associative \*-algebra $\mathbb{A}$ generated by a set of hermitean ‘coordinates’ $x^i$ which in some limit tend to the (real) coordinates $\tilde{x}^i$ of a
manifold; the latter we identify as the classical limit of the geometry. We suppose that the center of the algebra $A$ is trivial. The coordinates satisfy a set of commutation relations

$$[x^i, x^j] = i\kappa J^{ij}(x^k). \quad (1.1)$$

The parameter $\kappa$ is introduced to describe the fundamental area scale on which noncommutativity becomes important. It is presumably of order of the Planck area $G\hbar$; the commutative limit is defined by $\kappa \to 0$. We stress that we do not imply that all noncommutative geometries are suitable as noncommutative models of space-time, any more so than for example an exotic $\mathbb{R}^4$ would be suitable as commutative models.

In order to define the differential structure on $A$ we use a noncommutative version of Cartan’s frame formalism [3]. In ordinary geometry a vector field can be defined as a derivation of the algebra of smooth functions; this definition can be used also when the algebra is noncommutative. A derivation, we recall, is a linear map which satisfies the Leibniz rule; sometimes this is modified to a ‘twisted’ Leibniz rule, [4, 5]. The set of all derivations we denote by $\text{Der}(A)$. The classical notion of the moving frame is generalized in the following way. The frame on $A$ is a set of $n$ inner derivations $e_a$, generated by ‘momenta’ $p_a$:

$$e_a f = [p_a, f]. \quad (1.2)$$

We assume that the momenta also generate the whole algebra $A$. For inner derivations, the Leibniz rule is the Jacobi identity. An alternative way to define the frame is to use the 1-forms $\theta^a$ dual to $e_a$ such that the relation

$$\theta^a(e_b) = \delta^a_b \quad (1.3)$$

holds. The module of 1-forms we denote by $\Omega^1(A)$. To define the left hand side of the equation (1.3), that is the basic forms $\theta^a$, we first define the differential, exactly as in the classical case, by the condition

$$df(e_a) = e_a f, \quad (1.4)$$

and the multiplication of 1-forms by elements of the algebra $A$ by

$$fdg = f e_a g \theta^a, \quad dgf = e_a g f \theta^a. \quad (1.5)$$

Since every 1-form can be written as sum of such terms, the definition of differential is complete. In particular, since

$$f \theta^a(e_b) = f \delta^a_b = (\theta^a f)(e_b), \quad (1.6)$$

we conclude that the frame necessarily commutes with all the elements of the algebra $A$. The 1-form $\theta$ defined as $\theta = -p_a \theta^a$ can be considered as an analog of the Dirac operator in ordinary geometry. It implements the action of the exterior derivative on elements of the algebra, that is $df = -[\theta, f]$.

The differential is real if $(df)^* = df^*$. This is assured if the derivations $e_a$ are real: $e_a f^* = (e_a f)^*$, which is the case if the momenta $p_a$ are antihermitean.

Let us mention further properties of the module structure defined by (1.4-1.6). The exterior product is a map from the tensor product of two copies of the module of 1-forms into the module of 2-forms; we shall identify the latter as a subset of the former and write the product as

$$\theta^a \theta^b = P^{ab}_{cd} \theta^c \otimes \theta^d. \quad (1.7)$$
The $P_{cd}^{ab}$ are complex numbers which satisfy the projector condition and the hermiticity [6]:

$$P_{cd}^{ab}P_{ef}^{cd} = P_{ef}^{ab}, \quad \bar{P}_{cd}^{ab}P_{ef}^{dc} = P_{ef}^{ba}.$$  \hfill (1.8)

The basis 1-forms anticommute for $P_{abcd}^{ab} = \frac{1}{2} (\delta_a^c \delta_d^b - \delta_b^c \delta_d^a)$. The exterior derivative of $\theta^a$ is a 2-form,

$$d\theta^a = -\frac{1}{2} C_{bc}^{a} \theta^b \theta^c.$$  \hfill (1.9)

The $C_{bc}^{a}$ are called the structure elements.

The relation $d^2 = 0$ and the consistency of the relation (1.6) with the differential, $d(f \theta^a - \theta^a f) = 0$, have nontrivial consequences. The structure elements are linear in the momenta

$$C_{bc}^{a} = F_{bc}^{a} - 2p_d P^{(ad)}_{bc}.$$  \hfill (1.10)

Furthermore, the momenta obey a quadratic relation

$$2p_d p_c P_{cd}^{ab} - p_c F_{ab}^{c} - K_{ab} = 0.$$  \hfill (1.11)

The $F_{bc}^{a}$ and $K_{ab}$ are complex numbers. From (1.10) it follows immediately that

$$e_a C_{bc}^{a} = 0.$$  \hfill (1.12)

This relation must be also satisfied in the commutative limit and constitutes a constraint on the frame. A frame has four degrees of freedom in two dimensions; the constraint subtracts one therefrom.

## 2 Fuzzy doughnut

Having outlined the main features of the frame formalism, let us discuss it in a simple 2-dim case. Clearly, every choice of a frame $\theta^a$ implements a different differential structure. On the other hand, the conditions (1.10, 1.11) constrain the possible choices quite rigidly. This can easily be seen in low dimensions: one can readily ‘solve’ a family of 2-dim metrics with one Killing vector. We shall exhibit all possible frames which yield differential calculi based on inner derivations. As a frame we choose

$$\theta^0 = f(x) dt, \quad f > 0, \quad \theta^1 = dx.$$  \hfill (2.1)

The frame relations can be written as

$$dx x = x dx, \quad dx t = t dx, \quad dt x = x dt, \quad dt t = (t + i k F) dt,$$  \hfill (2.2)

and imply $dJ^{01} = 0$. We have set

$$F = J^{01} \frac{d}{dx} \log f.$$  \hfill (2.3)

The differential structure of the algebra can be written as

$$(dx)^2 = 0, \quad dx dt = -dt dx, \quad (dt)^2 = -\frac{1}{2} i k F' dx dt$$  \hfill (2.4)

or as the relations

$$(\theta^1)^2 = 0, \quad \theta^0 \theta^1 = -\theta^1 \theta^0,$$  \hfill (2.5)

$$(\theta^0)^2 = \frac{1}{2} i k f F' \theta^0 \theta^1 = 2 \epsilon \theta^0 \theta^1.$$  \hfill (2.6)
We have introduced here a parameter \( \epsilon \) define by
\[
\epsilon = k \mu^2, \quad \mu^2 = \frac{1}{4} f F'. \tag{2.7}
\]
It follows from the frame properties that the mass scale \( \mu \) is a constant.

Suppose now that the dual momenta exist. The duality relations are
\[
[p_0, t] = f^{-1}, \quad [p_0, x] = 0, \\
[p_1, t] = 0, \quad [p_1, x] = 1. \tag{2.8}
\]
If we denote \([p_0, p_1] = L_{01}\), the Jacobi identities imply the relations
\[
[p_0, J^{01}] = 0, \quad [p_1, J^{01}] = 0, \\
[t, L_{01}] = -f f^{-2}, \quad [x, L_{01}] = 0. \tag{2.9}
\]
One can conclude again that \( J^{01} \) is constant and also that \( L_{01} \) is a function of \( x \) alone.
We set \( J^{01} = 1 \). It follows that, neglecting the integration constants, the ‘Fourier transformation’ between the position and momentum generators is given by
\[
p_0 = -\frac{1}{ik} \int f^{-1}, \quad p_1 = -\frac{1}{ik} t. \tag{2.10}
\]
Each of the pairs \((t, x)\) and \((p_0, p_1)\) generates the algebra.

The array \( P_{\alpha \beta \gamma \delta} \) we write as
\[
P_{\alpha \beta \gamma \delta} = \frac{1}{2} \delta^{[\alpha}_{\beta} \delta^{\gamma]}_{\delta] + i \epsilon Q_{\alpha \beta \gamma \delta}. \tag{2.11}
\]
In dimension two, if we assume that metric depends on \( x \), that is on \( p_0 \) only, we find that
\[
P_{\alpha \beta} p_\alpha p_\beta = \frac{1}{2} [p_\alpha, p_\beta] + i \epsilon Q^{00} p^2 \tag{2.12}
\]
and therefore \( L_{01} \) is given by
\[
L_{01} = K_{01} + p_0 F_{01}^0 - 2i \epsilon p_0^2 Q_{00}^{00}. \tag{2.13}
\]
The structure elements are
\[
C_{01}^0 = F_{01}^0 - 4i \epsilon p_0 Q_{00}^{00}. \tag{2.14}
\]
Symmetry and reality of the product (1.8) imply that \( Q_{\alpha \beta \gamma \delta} \) has the following non-vanishing elements:
\[
Q^{10}_{00} = -Q^{01}_{00} = 1, \quad Q^{00}_{01} = -Q^{00}_{10} = 1. \tag{2.15}
\]
We set also
\[
K_{01} = \frac{1}{ik}, \quad F_{01}^0 = -ib \mu, \tag{2.16}
\]
while \( C_{01}^{01} \) is determined by the constraint
\[
C_{01}^{01} F_{01}^0 = C_{01}^{01} + C_{01}^{00} = -2i \epsilon C_{00}^0. \tag{2.17}
\]
We have then finally the expressions
\[
L_{01} = (ik)^{-1}(1 - b \mu^{-1}(i \epsilon p_0) - 2 \mu^{-2}(i \epsilon p_0)^2), \tag{2.18}
\]
\[
C_{01}^0 = -ib \mu - 4i \epsilon p_0. \tag{2.19}
\]
and a differential equation for $p_0$:

$$-i\epsilon \frac{dp_0}{dx} = \mu^2 - i\epsilon b \mu p_0 - 2(i\epsilon p_0)^2. \quad (2.20)$$

There are three cases to be considered. The simplest is the case with $\mu^2 \to \infty$. The equation (2.20) reduces to

$$-ik \frac{dp_0}{dx} = 1. \quad (2.21)$$

One finds the relations

$$ikp_0 = -x, \quad f(x) = 1. \quad (2.22)$$

This is noncommutative Minkowski space.

An equally degenerate case is the case $\mu^2 \to \infty$ and $\epsilon b = c \mu$. Equation (2.20) can be written in the form

$$-i\bar{k} \frac{dp_0}{dx} = 1 - icp_0. \quad (2.23)$$

One finds the solution

$$ip_0 = c^{-1}(e^{-k^{-1}cx} - 1), \quad f(x) = e^{k^{-1}cx}. \quad (2.24)$$

This is noncommutative de Sitter space; it can be brought to the usual form by the change of variables

$$t' = 2t, \quad \mu x' = 2c^{-1}(e^{-cx} - 1). \quad (2.25)$$

The case which interests us the most is that with $\mu$ finite. With $b = 0$ (that is, with $F^{a}_{bc} = 0$) the equation (2.20) becomes

$$-i\epsilon \frac{dp_0}{dx} = \mu^2 - 2(i\epsilon p_0)^2. \quad (2.26)$$

If we fix $\beta^2 = 2\mu^2 > 0$, the equation for $p_0$ becomes

$$\frac{1}{\beta} \frac{d}{dx} (-2i\epsilon\beta^{-1}p_0) = 1 - (-2i\epsilon\beta^{-1}p_0)^2. \quad (2.27)$$

The solution to this equation is given by

$$ikp_0 = -\beta^{-1} \tanh(\beta x), \quad (2.28)$$

with

$$f(x) = \cosh^2(\beta x) \quad (2.29)$$

and

$$F = -2i\beta^2 k p_0 = 2\beta \tanh(\beta x). \quad (2.30)$$

The frame corresponding to this solution is given by

$$\theta^0 = \cosh^2(\beta x)dt = \frac{1}{2}(1 + \cosh(2\beta x))dt, \quad \theta^1 = dx. \quad (2.31)$$

Frames of similar type have appeared [7, 8, 9] in 2-dimensional dilaton gravity. The connection and the curvature of the analogous commutative moving frame are

$$\tilde{\omega}^0_1 = \tilde{\omega}^1_0 = F \tilde{\theta}^0, \quad (2.32)$$

$$\tilde{\Omega}^0_1 = \tilde{\Omega}^1_0 = -(F' + F^2)\theta^0\theta^1 = -f^{-1} f'' \tilde{\theta}^0 \tilde{\theta}^1. \quad (2.33)$$
The solution is a completely regular manifold of Minkowski signature. In the limit \( \beta \to 0 \)
\[ ikp_0 = -x, \quad f = 1, \tag{2.34} \]
one finds Minkowski space. In ‘tortoise’ coordinate \( x^* \), \( x^* = \int \frac{dx}{f(x)} \), the frame is given by
\[ \theta^0 = \frac{1}{1 - x^{*2}} dt, \quad \theta^1 = \frac{1}{1 - x^{*2}} dx^*. \tag{2.35} \]
From (2.10) we see that \( x^* = -ikp_0 \).

Under a Wick rotation \( u = 2i\beta x, \quad v = t \), \( (2.36) \)
the frame (2.1) becomes
\[ \theta^0 = \frac{1}{2}(1 + \cos u)dv, \quad \theta^1 = \frac{1}{2i\beta} du, \tag{2.37} \]
and the corresponding commutative line element has the form
\[ d\tilde{s}^2 = \frac{1}{4}(1 + \cos \tilde{u})^2 d\tilde{v}^2 + \frac{1}{4}\beta^{-2} d\tilde{u}^2. \tag{2.38} \]
This is the surface of the torus embedded in \( \mathbb{R}^3 \):
\[ \tilde{x} = \frac{1}{2}(1 + \cos \tilde{u}) \cos \tilde{v}, \quad \tilde{y} = \frac{1}{2}(1 + \cos \tilde{u}) \sin \tilde{v}, \quad \tilde{z} = \frac{1}{2}\beta^{-1} \sin \tilde{u}, \tag{2.39} \]
and for this reason we call this metric the ‘fuzzy doughnut’. It is a singular axially-symmetric surface of Gaussian curvature
\[ \tilde{K} = 2\beta^2(1 - \tan^2 \frac{1}{2} \tilde{u}). \tag{2.40} \]
The doughnut is defined by the coordinate range \( 0 \leq \tilde{u} \leq 2\pi, \quad 0 \leq \tilde{v} \leq 2\pi \), with a singularity at the point \( \tilde{u} = \pi \). In spite of the singularity, the Euler characteristic is given by
\[ e[A] = \frac{1}{4\pi} \epsilon_{ab} \int \tilde{\Omega}^{ab} = -\frac{1}{2\pi} \int \tilde{\Omega}^{01} = -\frac{1}{2\pi} \int d\tilde{\omega}^{01} = 0 \tag{2.41} \]
as it should be. If we suppose the same domain in the Wick rotated real-\( t \) region, then
\[ 0 \leq x \leq \beta^{-1} \pi, \quad 0 \leq t \leq 2\pi. \tag{2.42} \]
As \( \beta \to \infty \) the doughnut becomes more and more squashed, and this domain becomes an elementary domain in the limiting Minkowski space.

3 Noncommutative differential geometry

In order to investigate the differential-geometric structure of the fuzzy doughnut we mention first the definition the linear connection and the metric, specified in the frame formalism; for details we refer to [3, 10]. Note that when the momenta exist the metric is given; otherwise there is a certain ambiguity which must be determined by field equations. Next, we will apply these definitions in the weak-field approximation \( \epsilon \to 0 \), and find the first noncommutative corrections to the classical doughnut geometry.
To define a linear connection one needs a ‘flip’ $\sigma$, $\sigma(\theta^a \otimes \theta^b) = S^{ab}_{\ cd} \theta^c \otimes \theta^d$, which in the present notation is equivalent to a 4-index set of complex numbers $S^{ab}_{\ cd}$ which we can write as

$$S^{ab}_{\ cd} = \delta^b_c \delta^a_d + i \epsilon^{abcd}. \tag{3.1}$$

The covariant derivative of a 1-form $\xi$ is given by $D\xi = \sigma(\xi \otimes \theta) - \theta \otimes \xi$. In particular

$$D\theta^a = -\omega^{a}_{\ bc} \otimes \theta^c = -(S^{ab}_{\ cd} - \delta^b_c \delta^a_d) p_b \theta^c \otimes \theta^d = -i \epsilon T^{ab}_{\ cd} p_b \theta^c \otimes \theta^d, \tag{3.2}$$

so the connection-form coefficients are linear in the momenta

$$\omega^a_{\ bc} = \omega^a_{\ be} \theta^b = i \epsilon p_d T^{ad}_{\ be} \theta^b. \tag{3.3}$$

On the left-hand side of the last equation is a quantity $\omega^a_{\ bc}$ which measures the variation of the metric; on the right-hand side is the array $T^{ad}_{\ bc}$ which is directly related to the anti-commutation rules for the 1-forms, and more important the momenta $p_d$ which define the frame. As $\kappa \to 0$ the right-hand side remains finite and $\omega^a_{\ bc} \to \tilde{\omega}^a_{\ bc}$. The identification is only valid in the weak-field approximation. The connection is torsion-free if the components satisfy the constraint

$$\omega^a_{\ ef} P^{ef}_{\ bc} = \frac{1}{2} C^a_{\ bc}. \tag{3.4}$$

The metric is a bilinear map $g : \Omega^1(\mathcal{A}) \otimes \Omega^1(\mathcal{A}) \to \mathcal{A}$. It can be defined using the frame as

$$g(\theta^a \otimes \theta^b) = g^{ab}. \tag{3.5}$$

The bilinearity of the metric implies that the $g^{ab}$ must belong to the center of the algebra; they must be complex numbers. This is a familiar property of the moving frame formalism. Here we take the frame to be orthonormal in the commutative limit and thus to first order the metric is

$$g^{ab} = \eta^{ab} + i \epsilon h^{ab}. \tag{3.6}$$

In general it is unclear whether the noncommutative extension can always be defined so that it possesses all usual properties. In the present formalism the metric is ‘real’ if it satisfies the condition $\tilde{g}^{ab} = S^{ab}_{\ cd} g^{cd}$. The definition of ‘symmetry’ of the metric is ambiguous: it can be defined either using the projection, $P^{ab}_{\ cd} g^{cd} = 0$, or the flip $S^{ab}_{\ cd} g^{cd} = c g^{ab}$. We shall see that the present example prefers the former definition.

In general a connection is metric-compatible if the condition

$$\omega^i_{\ kl} g^{ij} + \omega^i_{\ ln} S^{il}_{\ km} g^{mn} = 0 \tag{3.7}$$

is satisfied. This can be written in a more familiar form $D_\mathbf{i} g^{jk} = 0$ if one introduce an appropriately twisted ‘covariant derivative’. Linearized, the condition reads

$$T^{(ac}_{\ d} b) = 0. \tag{3.8}$$

One should note that not all of the usual conditions on a metric are necessarily satisfied. It is therefore of interest to see, in specific examples, which of them can be imposed. Although the relations are algebraic, they form a set of equations difficult to solve in full generality even in dimension two. They simplify somewhat if one use a perturbation expansion.
In our 2-dim model the frame is of the form
\[ \theta^0 = f(x)dt, \quad \theta^1 = dx. \] (3.9)

The torsion-free metric-compatible connection and the curvature are classically given by the expressions (2.33). From these expressions we see that the geometry is flat only if \( f(x) \) is linear in \( x \). We recall that \( \epsilon = k\mu^2 \). To first order the fuzzy calculus differs from the commutative limit in the two relations
\[ \theta^{(0}\theta^1) = -2i\epsilon q\theta^0)^2, \quad (\theta^0)^2 = i\epsilon q[0\theta^1], \] (3.10)
which to first order reduce to
\[ \theta^{(0}\theta^1) = 0, \quad (\theta^0)^2 = 2i\epsilon q\theta^0\theta^1. \] (3.11)

The quantity \( q \) which we have introduced in (3.11) is a constant: \( q = 0 \) in the cases of flat and of de Sitter noncommutative space, \( q = 1 \) in the fuzzy doughnut case. We will restrict our considerations to the latter.

The differentials of the frame are given by
\[ d\theta^0 = -C_{01}^{0}\theta^0\theta^1, \quad d\theta^1 = 0, \] (3.12)
with \( C_{01}^{0} = -4i\epsilon p_0Q_{01}^{00} = -4i\epsilon p_0 \). The only non-vanishing components of the connection are
\[ \omega^0 = \omega^1 = -4i\epsilon p_0\theta^0 = F\theta^0, \] (3.13)
so from (3.3) we find
\[ T^{00}_{01} = T^{10}_{00} = -4. \] (3.14)

Metric-compatibility (3.8) is satisfied to first order. Also, to the same order the condition that the torsion vanish (3.4) is satisfied by the values we obtain. The curvature 2-form has components
\[ \Omega^0 = -(F' + F^2)\theta^0\theta^1, \quad \Omega^0 = \Omega^1 = 2i\epsilon F^2\theta^0\theta^1. \] (3.15)

We must define a ‘real’, ‘symmetric’ metric. There are in principle four possible ways to define it depending on which of two possible ways one chooses to define symmetry, and whether or not one includes a twist in the extension of the metric to the tensor product. In the present example one sees that the only consistent choice is the following
\[ h_{ab} = -2Q_{abcd}\eta_{cd} = -2Q_{-}^{ab} \] (3.16)
where we denoted \( Q_{-}^{ab}cd = \frac{1}{2}Q_{-}^{ab}\eta_{cd} \). Thus for the symmetric and real metric we obtain
\[ g^{ab} = \eta^{ab} + i\epsilon h^{ab} = \begin{pmatrix} -1 & 2i\epsilon \\ 0 & 1 \end{pmatrix}. \] (3.17)
4 Higher-order effects

To find the second order corrections to our system, we write the 4-index tensors as matrices ordering the indices \((01, 10, 11, 00)\). Let \(P_0\) and \(S_0\) be respectively the canonical projector and the flip

\[
P_0 = \frac{1}{2} \begin{pmatrix}
1 & -1 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad S_0 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]  

(4.1)

and denote the second order projection and flip by \(P\) and \(S\)

\[
P = P_0 + i\epsilon Q + (i\epsilon)^2 Q_2, \\
S = S_0 + i\epsilon T + (i\epsilon)^2 T_2.
\]  

(4.2)

(4.3)

The projector constraints are, in matrix notation,

\[
P^2 = P, \quad \bar{P}P = \hat{P},
\]  

(4.4)

where \(\hat{A}^{ab\cd} = A^{ba\cd} = (S_0A)^{ab\cd}\). To first order the projector conditions become

\[
\bar{Q} = Q, \quad \hat{Q} = Q_+.
\]  

(4.5)

The twist constraints are

\[
\hat{S}\bar{S} = 1, \quad \bar{S}P + \bar{P}\hat{P} = 0, \quad SP + P = 0.
\]  

(4.6)

The last two identities are equivalent if \(\bar{P}\hat{P} = \hat{P}\), the condition already imposed in (4.4).

One can easily check the first order solution of the previous section. In the matrix notation it is given by

\[
Q_- = \begin{pmatrix}
0 & 0 \\
\tau & 0
\end{pmatrix}, \quad Q_+ = \begin{pmatrix}
0 & -\tau^* \\
0 & 0
\end{pmatrix}, \quad Q = Q_- + Q_+ = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0
\end{pmatrix},
\]  

(4.7)

and

\[
T = -2 \begin{pmatrix}
0 & \tau\tau^* \\
\tau\tau^*\sigma_1 & 0
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -4 \\
0 & 0 & 0 & 0 \\
-4 & 0 & 0 & 0
\end{pmatrix},
\]  

(4.8)

where we introduced the matrix \(\tau\) and its transpose \(\tau^*\), \(\tau = \begin{pmatrix}
0 & 0 \\
1 & -1
\end{pmatrix}\).

The constraints (4.4-4.6) can be solved to second order using inner automorphisms of the matrix algebra. Write \(P = W^{-1}P_0W\), with \(W\) arbitrary nonsingular \(4 \times 4\) matrix. We see immediately that \(P^2 = P\). To satisfy the second condition of (4.4) on \(P\) it is sufficient to require that

\[
\hat{W}S_0 = S_0W,
\]  

(4.9)
and to recall $\dot{S} = S_0 S$. Let $W = \exp(i\epsilon B)$. To second order

$$P = P_0 + i\epsilon [P_0, B] + \frac{1}{2}(i\epsilon)^2[[P_0, B], B],$$

(4.10)

so the two expansions coincide if

$$Q = [P_0, B], \quad Q_2 = \frac{1}{2}[[P_0, B], B] = \frac{1}{2}[Q, B].$$

(4.11)

It is easy to find the appropriate solution

$$B = \begin{pmatrix} 0 & -\tau^* \\ -\tau & 0 \end{pmatrix}.$$  

(4.12)

The solution for $T_2$ is

$$T_2 = \frac{1}{2}TS_0T.$$  

(4.13)

To summarize,

$$P = \begin{pmatrix} 1/2 - \epsilon^2 & -1/2 + \epsilon^2 & 0 & -i\epsilon \\ -1/2 + \epsilon^2 & 1/2 - \epsilon^2 & 0 & i\epsilon \\ 0 & 0 & 0 & 0 \\ i\epsilon & -i\epsilon & 0 & 2\epsilon^2 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 - 8\epsilon^2 & 0 & 0 & -4i\epsilon \\ 0 & 0 & 1 & 0 \\ -4i\epsilon & 0 & 0 & 1 - 8\epsilon^2 \end{pmatrix}.$$  

(4.14)

The second-order metric is given by

$$g = g_0 - \frac{1}{2}i\epsilon S_0Tg_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + 2i\epsilon \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$  

(4.15)

It is real and symmetric. We saw in Section 3 that this metric is compatible with the connection

$$\omega^a_b = S^{ac}_{\ db}p_c + \theta\delta^a_b, \quad \theta = -p_0\theta^a$$  

(4.16)

to first order in the expansion parameter. We have not succeeded in finding a connection which is metric-compatible to second order. The non-metricity was found in noncommutative spaces defined by some string-theory models before [13].

5 Further speculations

Several models have been found which illustrate a close relation between noncommutative geometry in its ‘frame-formalism’ version and classical gravity. Heuristically, but incorrectly, one can formulate the relation by stating that gravity is the field which appears when one quantizes the coordinates much as the Schrödinger wave function encodes the uncertainty resulting from the quantization of phase space.

The first and simplest of these is the fuzzy sphere [14] which is a noncommutative geometry which can be identified with the 2-dimensional (euclidean) ‘gravity’ of the 2-sphere. The algebra in this case is an $n \times n$ matrix algebra; if the sphere has radius $r$ then the parameter $r/n$ can be interpreted as a lattice length. With the identification this model illustrates how gravity can act as an ultraviolet cutoff, a regularization which is very similar to the ‘point splitting’ technique which has been used when quantizing a field in classical curved backgrounds. It can also be compared with the screening of electrons in plasma physics, which gives rise to a Debye length proportional to the
inverse of the electron-number density \( n \). The analogous ‘screening’ of an electron by virtual electron-positron pairs is responsible for the reduction of the electron self-energy from a linear to logarithmic dependence on the classical electron radius. We refer to a noncommutative geometry as ‘fuzzy’ if the algebra and the representation are such so that there is a true ‘screening of points’. A notable counterexample would be the irrational noncommutative torus. Other models have been found which illustrate the identification including an infinite series in all dimensions.

In the present paper yet another model is given, one which although representing a classical manifold of dimension 2 is of interest because the classical ‘gravity’ which arises has a varying Gaussian curvature. The authors will leave to a subsequent article the delicate task of explaining exactly which property of the metric makes it ‘quantizable’. This geometry could furnish a convenient model to study noncommutative effects, for example in the colliding-\( D \)-brane description of the Big-Bang proposed by Turok & Steinhardt [15]. The 2-space describing the time evolution of the separation of the branes has been shown to be conveniently described using Rindler coordinates. One can blur this geometry by using the metric and connection described here. The flat geometry would have to be replaced by the one given in this section; in the limit \( q \to 0 \) it would become flat.

The doughnut example is of importance in that it is the first explicit construction of an algebra and differential calculus which is singularity-free in the Minkowski-signature domain and which has a non-constant curvature. There are two aspects of this problem. To construct a classical manifold from a differential calculus is relatively simple once one has constructed the frame. One takes formally the limit and uses the so constructed moving frame to define the metric. This is contained in the upper right of the following little diagram

\[
\begin{array}{c}
\text{Fuzzy} \\
\text{Frame} \\
\downarrow \\
\text{Fuzzy} \\
\text{Geometry}
\end{array}
\quad \rightarrow 
\begin{array}{c}
\text{Classical} \\
\text{Frame} \\
\downarrow \\
\text{Classical} \\
\text{Geometry}
\end{array}
\tag{5.1}
\]

More difficult is the construction of a ‘fuzzy geometry’ which would fill in the lower left of the diagram and would be such that the classical geometry is a limit thereof. But this step is very important since it gives an extension of the right-hand side into what could eventually be a domain of quantum geometry. It is the box in the to-be-constructed lower left corner where possibly one can find an interesting extension of the metric containing correction terms which describe the noncommutative structure.

We have not succeeded however to completely extend this geometry to all orders in the noncommutativity parameter \( ie \). This will be considered in a subsequent article. There is evidence that the extension will involve a non-vanishing value of the torsion 2-form. The metric is extended into the noncommutative domain so as to maintain such formal properties as reality and symmetry. The interpretation however as a length requires more attention when the ‘coordinates’ do not commute.

Last, but not least, our example illustrates even better than the fuzzy sphere the way in which quantum mechanics is modified by geometry and the important role which noncommutative geometry plays in understanding the relation between the two. The ‘momenta’ which we introduce are the natural curved-space generalization of the canonical momentum operators of ordinary quantum mechanics. In the present formalism they generate the algebra as well as do the coordinates. Once the algebra
is given the noncommutative structure of space-time is manifest in the commutation
relations \([x^i, x^j]\) and the appropriate curved-space version of quantum mechanics is
defined by the relations (1.2). The two structures are intimately enmeshed by the Fourier
transform as well as Jacobi identities. If the right-hand side of (1.2) reduces to the
Kronecker symbol when \(f = x^i\) then the space is flat; because of the Jacobi identities
only in this case can quantum mechanics be consistent with a commutative space-time
structure.

Acknowledgment

Part of this work was done while the authors were visiting ESI in Vienna. They would
like to thank H. Grosse, as well as M. Axenides, M. Floratos, T. Grammatikopoulos,
J. Mourad, T. Schücker and G. Zoupanos for enlightening conversations. The research
was supported in part by the Grant No. 1468 of the Serbian Ministry of Science.

References

[1] S. L. Woronowicz, “Differential calculus on compact matrix pseudogroups,”
Commun. Math. Phys. 122 (1989) 125.

[2] A. Connes, Noncommutative Geometry. Academic Press, 1994.

[3] J. Madore, An Introduction to Noncommutative Differential Geometry and its
Physical Applications. No. 257 in London Mathematical Society Lecture Note
Series. Cambridge University Press, second ed., 2000. 2nd revised printing.

[4] M. Dimitrijevic, L. Jonke, L. Moller, E. Tsouchnika, J. Wess, and
M. Wohlgenannt, “Deformed field theory on \(\kappa\)-spacetime,” Euro. Phys. Jour. C
C31 (2003) 129–138, hep-th/0307149

[5] A. Dimakis and F. Mueller-Hoissen, “Automorphisms of associative algebras and
noncommutative geometry,” J. Phys. A: Math. Gen. 37 (2004) 2307–2330,
math-ph/0306058

[6] B. L. Cerchiai, G. Fiore, and J. J. Madore, “Geometrical tools for quantum
euclidean spaces,” Commun. Math. Phys. 217 (2001), no. 3, 521–554,
math.QA/0002007

[7] J. P. S. Lemos and P. M. Sa, “The black holes of a general two-dimensional
dilaton gravity theory,” Phys. Rev. D49 (1994) 2897–2908, gr-qc/9311008

[8] J. Gegenberg and G. Kunstatter, “Solitons and black holes,” Phys. Lett. B413
(1997) 274–280, hep-th/9707181

[9] D. Grumiller, W. Kummer, and D. V. Vassilevich, “Dilaton gravity in two
dimensions,” Phys. Rep. 369 (2002) 327–430, hep-th/0204253.

[10] M. Burić, M. Maceda, and J. Madore, “On the resolution of space-time
singularities III,” in Geometric Methods In Physics, A. Odzijewicz,
A. Strasburger, S. T. Ali, J.-P. Antoine, T. Friedrich, J.-P. Gazeau, Z. Hasiewicz,
and M. Schlichenmaier, eds., pp. –. 2004. Bialowieza, Poland, July 2004.

[11] J. Mourad, “Linear connections in non-commutative geometry,” Class. and
Quant. Grav. 12 (1995) 965.
[12] M. Dubois-Violette, J. Madore, T. Masson, and J. Mourad, “On curvature in noncommutative geometry,” *J. Math. Phys.* 37 (1996), no. 8, 4089–4102, q-alg/9512004.

[13] B. Sazdović, “Torsion and nonmetricity in the stringy geometry,” hep-th/0304086.

[14] J. Madore, “The fuzzy sphere,” *Class. Quant. Grav.* 9 (1992) 69–88.

[15] N. Turok and P. J. Steinhardt, “Beyond inflation: A cyclic universe scenario,” *Physica Scripta* (2004) hep-th/0403020.