Gales and supergales are equivalent for defining constructive Hausdorff dimension

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August 16, 2002

Abstract

We show that for a wide range of probability measures, constructive gales are interchangeable with constructive supergales for defining constructive Hausdorff dimension, thus generalizing a previous independent result of Hitchcock [2] and partially answering an open question of Lutz [5].

1 Introduction

Various constructive, computable, and subcomputable versions of classical Lebesgue measure have been proposed and studied [1] in order to quantify the sizes of complexity classes (in the broadest sense of the word), and also to clarify the idea of “random sequence.” In particular, the notion of constructive measure and Martin-Löf randomness has a rich history [7, 11].

An useful idea related to Lebesgue measure is that of classical Hausdorff dimension, which provides a fine-grained gauge of the sizes of null sets. For example, Hausdorff dimension is a primary tool for classifying geometric fractals [6]. As with Lebesgue measure, various resourcebounded versions of Hausdorff dimension have been studied [1, 3]. In [3], Lutz concentrates on the constructive Hausdorff dimension of individual sequences and (even) individual finite strings. Constructive dimension turns out to be closely related to Kolmogorov complexity [8], so several results about the latter idea [3, 12, 1] inform the former.

Due to a result of Schnorr, constructive measure may be defined either in terms of constructive martingales or in terms of constructive supermartingales (see below for definitions). Lutz defined constructive dimension in terms of supergales for general computable probability measures on the Cantor space [3], but left open the question of whether this definition is equivalent to the analogous one using gales (see below for definitions). Hitchcock has recently shown that for the uniform probability measure, gales and supergales are indeed equivalent [2]. In the current paper, we prove the equivalence for a wide range of probability measures—those satisfying a certain reasonable balance condition defined below.

Unfortunately, our techniques do not work for all computable probability measures. It is still an open question whether this balance condition can be weakened or eliminated.

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2 Preliminaries

We borrow most of our definitions and notation from [1], which should be consulted for more details. Let \( \mathbb{R}, \mathbb{Q}, \) and \( \mathbb{N} \) be the set of real numbers, rational numbers, and nonnegative integers, respectively. Let \( \{0,1\}^* \) be the set of finite binary strings, and let \( \mathcal{C} \) be the set of infinite binary sequences, i.e., the Cantor space. We let \( \lambda \) denote the empty string, and we let \( |w| \) denote the length of \( w \in \{0,1\}^* \). For any \( x, y \in \{0,1\}^* \cup \mathcal{C} \) we write \( x \sqsubseteq y \) to mean \( x \) is a prefix of \( y \), and we write \( x \sqsubset y \) to mean \( x \) is a proper prefix of \( y \). A set \( U \subseteq \{0,1\}^* \) is a prefix set if no element of \( U \) is a proper prefix of any other element in \( U \).

A function \( d : \{0,1\}^* \to \mathbb{R} \) is computable if there is a computable function \( d' : \{0,1\}^* \times \mathbb{N} \to \mathbb{Q} \) such that for all \( x \in \{0,1\}^* \) and \( r \in \mathbb{N} \),

\[
|d'(x,r)-d(x)| \leq 2^{-r}.
\]

A real number \( s \in \mathbb{R} \) is computable if the constant function \( d(w) = s \) is computable. A function \( d : \{0,1\}^* \to \mathbb{R} \) is lower semicomputable if \( d \) is lower semicomputable combined with \( d' : \{0,1\}^* \to \mathbb{R} \) is weakly computable if there is a computable function \( d' : \{0,1\}^* \times \mathbb{N} \to \mathbb{Q} \) such that for all \( x \in \{0,1\}^* \) and \( r \in \mathbb{N} \),

1. \( d'(x,r) \leq d'(x,r+1) < d(x) \) and
2. \( d(x) = \lim_{r \to \infty} d'(x,r) \).

Equivalently, \( d \) is lower semicomputable iff the set \( \{(x,q) \in \{0,1\}^* \times \mathbb{Q} \mid q < d(x)\} \) is computably enumerable (c.e.). We can define upper semicomputability similarly, whence computability is equivalent to upper and lower semicomputability combined.

A probability measure on \( \mathcal{C} \) is a function \( \nu : \{0,1\}^* \to [0,\infty) \) such that \( \nu(\lambda) = 1 \) and

\[
\nu(w) = \nu(w0) + \nu(w1)
\]

for all \( w \in \{0,1\}^* \). The uniform probability measure \( \mu \) is defined as \( \mu(w) = 2^{-|w|} \).

Fix a probability measure \( \nu \) and a real number \( s \in [0,\infty) \). A \( \nu\)-\( s \)-supergale is a function \( d : \{0,1\}^* \to [0,\infty) \) such that for all \( w \in \{0,1\}^* \),

\[
\nu(w)^sd(w) \geq \nu(w0)^sd(w0) + \nu(w1)^sd(w1). \tag{1}
\]

A \( \nu\)-\( s \)-gale is a \( \nu\)-\( s \)-supergale that satisfies (1) with equality.

An \( s \)-supergale (respectively \( s \)-gale) is a \( \mu \)-\( s \)-supergale (respectively \( \mu \)-\( s \)-gale).
A \( \nu \)-supermartingale (respectively \( \nu \)-martingale) is a \( \nu \)-\( 1 \)-supergale (respectively \( \nu \)-\( 1 \)-gale).
A supermartingale (respectively martingale) is a \( \mu \)-supermartingale (respectively \( \mu \)-martingale).

For example, an \( s \)-supergale \( d \) satisfies

\[
d(w) \geq 2^{-s}[d(w0) + d(w1)].
\]

or any \( \nu\)-\( s \)-supergale \( d \), we define its success set \( S^\infty[d] \subseteq \mathcal{C} \) by

\[
z \in S^\infty[d] \iff \limsup_{w \sqsubseteq z} d(w) = \infty.
\]

For \( z \in S^\infty[d] \) we say that \( d \) succeeds on \( z \). It is well-known that a set \( X \subseteq \mathcal{C} \) has Lebesgue measure zero iff there is a martingale \( d \) with \( X \subseteq S^\infty[d] \).

A \( \nu\)-\( s \)-supergale is constructive if it is lower semicomputable.
Definition 2.1 Let $\nu$ be a probability measure on $\mathbb{C}$. A set $X \subseteq \mathbb{C}$ has constructive $\nu$-measure zero if there is a constructive $\nu$-martingale $d$ with $X \subseteq S^\infty[d]$. We say that $X$ has constructive $\nu$-measure one if $\mathbb{C} - X$ has constructive $\nu$-measure zero. A sequence $R \in \mathbb{C}$ is $\nu$-random if $\{R\}$ does not have constructive $\nu$-measure zero. We let $\text{RAND}_\nu$ be the set of all $\nu$-random sequences.

It is well-known that $\text{RAND}_\nu$ has $\nu$-measure one for all computable $\nu$ [5]. The most important case of Definition 2.1 is when $\nu = \mu$. The following characterization of $\mu$-randomness was proved in [11]. For a definition of Martin-Löf randomness, see [7] or [11].

Theorem 2.2 (Schnorr) A sequence $R \in \mathbb{C}$ is $\mu$-random if and only if $R$ is random in the sense of Martin-Löf.

Schnorr also essentially proved that for computable $\nu$, Definition 2.1 does not change if we replace “$\nu$-martingale” with “$\nu$-supermartingale.” That is,

Theorem 2.3 (Schnorr [11, 10, 5]) Let $\nu$ be a computable probability measure on $\mathbb{C}$. A set $X \subseteq \mathbb{C}$ has constructive $\nu$-measure zero if and only if there is a constructive $\nu$-supermartingale $d$ with $X \subseteq S^\infty[d]$.

Lutz [4, 5] develops constructive Hausdorff dimension as an analog both to classical Hausdorff dimension and to constructive measure.

Definition 2.4 Let $\nu$ be a probability measure on $\mathbb{C}$. Let $X \subseteq \mathbb{C}$ be a set of sequences, and let

$$\mathcal{G}(X) = \{s \in [0, \infty) \mid \text{there is a constructive } \nu-s\text{-superale } d \text{ with } X \subseteq S^\infty[d]\}.$$ 

Then $\inf \mathcal{G}(X)$ is the constructive $\nu$-dimension of $X$ and is written $\dim_\nu(X)$.

It is easy to show that $\dim_\nu(X)$ is always at most one, and if $\dim_\nu(X) < 1$, then $X$ has constructive $\nu$-measure zero.

3 Main Result

Can we alternatively define constructive $\nu$-dimension as in Definition 2.4 replacing “$\nu$-superale” with $\nu-s$-gale? The proof of Theorem 2.3 does not generalize to $s < 1$. We show, however, that for certain $\nu$ (including $\mu$), one can in fact make the replacement above in Definition 2.4 without changing it. To do this we prove a weaker analog of Theorem 2.3 for $s < 1$ (Theorem 3.2, below). The case for $\nu = \mu$ was shown by Hitchcock [2]. Our more general proof has some elements similar to his, even though it was arrived at independently. Our result does not hold for all computable $\nu$; we need the following definition.

Definition 3.1 Let $\nu$ be a computable probability measure on $\mathbb{C}$. We say that $\nu$ is well-balanced if there are constants $0 < \alpha < 1$ and $C > 0$ such that for all $w \in \{0, 1\}^*$,

$$0 < \nu(w) \leq C\alpha^{|w|}.$$ 

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Note that the uniform measure $\mu$ is well-balanced ($\alpha = \frac{1}{2}$ and $C = 1$). More generally, if
\[
\liminf_{w \in \{0,1\}^*, b \in \{0,1\}} \frac{\nu(wb)}{\nu(w)} > 0,
\]
then $\nu$ is well-balanced (but not conversely).

The following theorem, which should be compared with Theorem 2.3, immediately implies that $\nu$-s-gales are equivalent to $\nu$-s-supergales for defining constructive Hausdorff $\nu$-dimension, for all well-balanced $\nu$.

**Theorem 3.2** Let $\nu$ be a well-balanced computable probability measure on $C$, and let $s \in [0, \infty)$. For every constructive $\nu$-s-supergale $d$ and every computable $s' > s$, there is a $\nu$-$s'$-gale $d'$ such that $S^\infty[d] \subseteq S^\infty[d']$.

**Proof.** We generalize an argument made in [11] about martingales. Let $\nu$ be as in the statement of the theorem. For arbitrary $U \subseteq \{0,1\}^*$ and $t > 0$ define
\[
d_{U}^t(w) = \frac{1}{\nu(w)^t} \left( \sum_{u \in U} \nu(wu)^t + \sum_{n < |w| : w[0..(n-1)] \in U} \frac{\nu(w[0..(n-1)])^t}{2^{|w|-n}} \right). \tag{2}
\]
This definition makes sense provided the first sum on the right-hand side converges. Clearly, this will be true for all $w$ if it is true for $w = \lambda$, i.e., if
\[
d_{U}^t(\lambda) = \sum_{u \in U} \nu(u)^t < \infty.
\]
Assume that $d_{U}^t(\lambda)$ is indeed bounded. It then follows that $d_{U}^t$ is a $\nu$-$t$-gale. This can be seen as follows: if $U$ is a prefix set, then we may argue as in [11]—at most one term on the right-hand side of (2) is nonzero, and so we have two cases: some prefix of $w$ is in $U$, or otherwise. The equation for a $\nu$-$t$-gale is easy to check in either case. Now an arbitrary $U$ (not necessarily a prefix set) can be partitioned into the union $U = V_0 \cup V_1 \cup V_2 \cup \cdots$ of pairwise disjoint prefix sets
\[V_i = \{w \in U \mid \text{exactly } i \text{ many proper prefixes of } w \text{ are in } U\},\]
and it is then clear from (2) that
\[d_{U}^t = d_{V_0}^t + d_{V_1}^t + d_{V_2}^t + \cdots.
\]
Thus $d_{U}^t$ is the sum of $\nu$-$t$-gales and so is a $\nu$-$t$-gale. Note that if $t$ is computable and $U$ is c.e., then $d_{U}^t$ is constructive.

Let $0 \leq s < s'$ and $d$ be as in the statement of the theorem, and let $c, \epsilon > 0$ be such that $\nu(x) \leq 2^{c-\epsilon|x|}$ for every $x \in \{0,1\}^*$. Such $c$ and $\epsilon$ exist because $\nu$ is well-balanced. We may assume without loss of generality that $d(\lambda) \leq 1$. For any $k \in \mathbb{N}$ we then have
\[
\sum_{w \in \{0,1\}^k} d(w)\nu(w)^s \leq d(\lambda) \leq 1
\]
by Lemma 3.3 in [3]. For each $i \in \mathbb{N}$ let
\[U_i = \{w \in \{0,1\}^* \mid d(w) > 2^i\}.
\]
Then for all $i, k \in \mathbb{N}$,

$$\sum_{w \in U_i \cap \{0,1\}^k} \nu(w)^s \leq 2^{-i} \sum_{w \in \{0,1\}^k} d(w) \nu(w)^s \leq 2^{-i},$$

and hence,

$$\sum_{w \in U_i \cap \{0,1\}^k} \nu(w)^s' = \sum_{w \in U_i \cap \{0,1\}^k} \nu(w)^s \nu(w)^{s' - s} \leq 2(s' - s)(c - \epsilon k) \sum_{w \in U_i \cap \{0,1\}^k} \nu(w)^s \leq 2(s' - s)(c - \epsilon k) - i.$$

This in turn yields, for all $i \in \mathbb{N}$,

$$d_{U_i}^{s'}(\lambda) = \sum_{w \in U_i} \nu(w)^s' = \sum_{k \in \mathbb{N}} \sum_{w \in U_i \cap \{0,1\}^k} \nu(w)^s' \leq \sum_{k \in \mathbb{N}} 2(s' - s)(c - \epsilon k) - i = 2^{-i} \frac{2(s' - s)c}{1 - 2^{-2(s' - s)c}} < \infty,$$

which means that $d_{U_i}^{s'}$ is well-defined.

Finally we define, as in [1],

$$d' = \sum_{i \in \mathbb{N}} i \cdot d_{U_i}^{s'}, \quad (3)$$

We have $d'(\lambda) \leq C \cdot \sum_{i \in \mathbb{N}} i 2^{-i} < \infty$ (C is a positive constant). Being the sum of $\nu$-$s'$-gales, $d'$ itself is a $\nu$-$s'$-gale, and since the set $U = \{(i, w) \mid w \in U_i\}$ is clearly c.e., $d'$ is constructive. For any $z \in C$, suppose $z \in S_\infty[d]$. Then for each $i \in \mathbb{N}$ there is a prefix $w_i$ of $z$ such that $d(w_i) \geq 2^i$, i.e., $w_i \in U_i$. But then by (3) we have $d_{U_i}^{s'}(w_i) \geq 1$, whence $d'(w_i) \geq i$. Thus $d'$ succeeds on $z$. \[\square\]

**Remark.** In the special case where $\nu = \mu$, (3) reduces to

$$d'(w) = \sum_{i \in \mathbb{N}} i \left( \sum_{u : w \in U_i} 2^{-s'|u|} + \sum_{n < |w| : w \in [0..(n-1)] \in U_i} 2(s' - 1)(|w| - n) \right).$$

**Corollary 3.1** Let $\nu$ be a well-balanced computable probability measure on $C$. Let $X \subseteq C$ be a set of sequences, and let

$$\mathcal{G}(X) = \{ s \in [0, \infty) \mid \text{there is a constructive } \nu-s\text{-gale } d \text{ with } X \subseteq S_\infty[d] \}.$$

Then $\inf \mathcal{G}(X) = \dim_\nu(X)$.

**Corollary 3.2** (Hitchcock [2]) Let $X \subseteq C$ be a set of sequences, and let

$$\mathcal{G}(X) = \{ s \in [0, \infty) \mid \text{there is a constructive } s\text{-gale } d \text{ with } X \subseteq S_\infty[d] \}.$$

Then $\inf \mathcal{G}(X) = \dim_\mu(X)$. 

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4 Further Research

If $\nu$ is not well-balanced, then the $d'$ defined in the proof of Theorem 3.2 may not exist. There are clearly some (ill-balanced) $\nu$ and constructive $\nu$-s-supergales $d$ such that the first sum in (2) with $U = U_i$ and $t = s'$ is unbounded. Perhaps one can restrict the sets $U_i$ in some way to bound the sum.

If not, perhaps there is a condition on $\nu$ that is strictly weaker than being well-balanced but still suffices to prove the theorem. One possible candidate is the following: there is an $\epsilon > 0$ such that $S^\infty[f] = \emptyset$, where $f$ is the $\epsilon$-gale defined by $f(w) = 2^{|w|}\nu(w)$. If this is the case (and if $\nu$ is computable and $> 0$), then we’ll say that $\nu$ is weakly balanced. Clearly, well-balanced implies weakly balanced, but the converse does not hold. Does Theorem 3.2 hold for all weakly balanced $\nu$?

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