Linear and nonlinear optical signals in probability and phase-space representations

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Abstract. Review of different representations of signals including the phase-space representations and tomographic representations is presented. The signals under consideration are either linear or nonlinear ones. The linear signals satisfy linear quantumlike Schrödinger and von Neumann equations. Nonlinear signals satisfy nonlinear Schrödinger equations as well as Gross–Pitaevskii equation describing solitons in Bose–Einstein condensate. The Ville–Wigner distributions for solitons are considered in comparison with tomographic-probability densities describing solitons completely. Different kinds of tomographies — symplectic tomography, optical tomography and Fresnel tomography are reviewed. New kind of map of the signals onto probability distributions of discrete photon number-like variable is discussed. Mutual relations between different transformations of signal functions are established in explicit form. Such characteristics of the signal-probability distribution as entropy is discussed.

1. Introduction
Complex functions of the position or time are used to describe different processes in physics. In information and signal processing, the functions of time are analyzed. In quantum theory, the wave functions, which are functions of position, are studied.

In signal analysis, the main notion is analytic signal \( f(t) \), with the parameter \( t \) being time. The complex function \( f(t) \) describes a signal of any nature like electromagnetic signal, seismic, acoustic signal, etc. There are several approaches to analyze signals. The most important and traditional is the Fourier analysis of a signal. It provides information on what frequencies are present in the decomposition of the signal into series of harmonics. If the signal is periodic in time, one uses the Fourier transform, which gives the discrete sum of harmonics. For nonperiodic signals, one uses the continuous sum (Fourier integral) [1, 2] of harmonics, which represents the Fourier transform of the generic signal. For the function \( f(x) \), which depends on the spatial variable \( x \), the Fourier analysis (which uses the integral Fourier transform) provides information on what wavelengths are present in the Fourier decomposition of the function under study. Fourier analysis is employed in all areas where the result of measurements is expressed in terms of a function. For example, the Fourier transform was successfully applied in studying the electromagnetic signals in semiconductor-laser physics [3, 4].

In modern research, Fourier analysis was generalized in several directions. Some of generalizations are reviewed, for example, in [5–7]. One of the generalizations is fractional Fourier transform. The kernel of the fractional Fourier transform is a Gaussian, which for some values of parameters of the transform provides the kernel of the standard Fourier transform.

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The Fourier transform is related to the translation group representation. The fractional Fourier transform is related to the inhomogeneous symplectic group representation, which contains the translation group as its subgroup.

Signal analysis is based on studying the properties of a complex time-dependent function $f(t)$ called the analytic signal. Signal analysis is an essential ingredient in information processing. Fourier analysis provides the function $F(\omega)$ describing the frequency structure of a signal. Fourier analysis is equivalent to applying an invertible map $f(t) \leftrightarrow F(\omega)$ of analytic signal on Fourier component of the signal.

There exist other methods to study signals where invertible (up to a phase factor) maps of the analytic-signal function onto a function of two variables (time–frequency quasidistributions) \([5, 8]\), for example, the Ville–Wigner quasidistributions \([9, 10]\) $f(t) \leftrightarrow W(t, \omega)$ are used, which have been introduced to describe a joint time–frequency distribution of the signal. This approach is important for signals with fluctuations. For the case of a signal $f(x)$ dependent on a spatial variable $x$, the Ville–Wigner function $W(x, k)$ describes the quasiprobability distribution of the position–wave vector.

Formally the analytic signal $f(t)$ or $f(x)$ is equivalent to the complex wave function $\psi(x)$ describing a system's state in quantum mechanics. In view of this, the results of quantum theory can be applied to signal analysis and vice versa. This is important since in analyzing the wave function of a quantum state, new results were obtained within the framework of the tomographic approach to measuring quantum states \([11–16]\).

The analogy of the complex wave function of the quantum state in quantum mechanics and the complex analytic signal in information processing and signal analysis was studied in \([17]\) where the formal identity of the Green function of the quantum harmonic oscillator and the kernel of fractional Fourier transform was considered (see also \([6]\)). The properties of some integral transforms used in information processing like the Ville–Wigner transform \([9, 10]\), the Radon transform \([18]\), the Gabor transform \([19]\), and the wavelet transform in connection with quasidistributions like the Wigner function \([9]\), the Husimi–Kano Q-function \([20]\), and the Glauber–Sudarshan P-distribution \([21]\) are important in different applications. The tomographic probability and its properties as well as noncommutative tomography of analytic signals of different nature are reviewed in \([17]\).

Recent studies \([22–26]\) demonstrate the emerging interest in the problem under consideration. There exist different kinds of maps of a complex function of one variable $\psi(x)$ onto functions of two or more variables. Among these maps, there is the Ville–Wigner map \([9, 10]\), which connects the complex function $\psi(x)$ with real function of two variables $W(q, p)$. Recently the map of the complex function $\psi(x)$ onto the positive probability distribution function of two variables $w(X, \theta)$ \([11]\) and of three variables $w(X, \mu, \nu)$ (called symplectic tomogram) \([14, 27]\) was introduced (see also \([6, 17]\)). The complex function can be considered as a wave function $\psi(x)$ of the quantum system in quantum mechanics or as analytic signal $f(t)$ in signal analysis. The function $\psi(x)$ can be also considered as a soliton solution to a nonlinear equation like nonlinear Schrödinger equation. The tomographic approach to soliton solutions of nonlinear Schrödinger equation (nonlinear signals) was suggested in \([28]\) and it was applied to study solitons of Gross–Pitaevskii equation \([29]\) describing the Bose–Einstein condensate (BEC) in \([30]\).

The symplectic tomographic map is based on linear symplectic transform in the phase space. Partial cases of the tomographic map yield optical tomography \([11]\) and Fresnel tomography \([31, 32]\). The advantage of tomograms is the fact that they are standard positive probability distributions. Due to this, one can construct the corresponding entropy, in view of standard Shannon scheme. The symplectic entropy of signal was introduced in \([33]\) and for solitons in \([34]\).

Our aim is to review the tomographic map for analytic signals, quantumlike systems and soliton solutions of nonlinear equations. In particular, we consider the symplectic entropy...
associated to complex wave function of quantum system as well as to analytic signal. Also we discuss the Ville–Wigner function of analytic signal and its relation to tomographic map of the signal, including nonlinear signals.

2. Symplectic, optical and Fresnel tomograms

The density matrix \( \rho_\psi(x, x') = \psi(x)\psi^*(x') \) of pure state \( |\psi\rangle \) can be mapped onto the real Wigner function on the phase space [9]

\[
W_\psi(q, p) = \int \rho_\psi(q + \frac{u}{2}, q - \frac{u}{2}) e^{-ipu} \, du. \tag{1}
\]

Ville used this map in the analytic-signal theory.

The inverse of the Fourier transform (1) defining the Wigner function in terms of the density matrix reads

\[
\psi(x)\psi^*(x') = \frac{1}{2\pi} \int W_\psi\left(\frac{x + x'}{2}, p\right) e^{ip(x-x')} \, dp. \tag{2}
\]

Let us now construct the tomographic map.

To do this, we use the integral Radon transform of the Wigner function

\[
w(X, \mu, \nu) = \int W_\psi(q, p) \delta(X - \mu q - \nu p) \frac{dq \, dp}{2\pi}. \tag{3}
\]

It is easy to get the inverse transform [14]

\[
W(q, p) = \int w(X, \mu, \nu) \exp\left[i\left(X - \mu q - \nu p\right)\right] \frac{dX \, d\mu \, d\nu}{2\pi}. \tag{4}
\]

The expression for symplectic tomogram in terms of the wave function reads [27]

\[
w(X, \mu, \nu) = \frac{1}{2\pi|\nu|} \left| \int \psi(y) \exp\left(\frac{i\mu}{2\nu} y^2 - \frac{iX}{\nu} y\right) \, dy \right|^2, \tag{5}
\]

with normalization condition

\[
\int w(X, \mu, \nu) \, dX = 1.
\]

From relation (5), the homogeneity property follows [35]

\[
w(\lambda X, \lambda \mu, \lambda \nu) = (|\lambda|)^{-1} w(X, \mu, \nu), \tag{6}
\]

which, in reality, means that the tomogram is a function of two real variables.

The inverse transform which relates the tomogram to the complex function (density matrix) reads

\[
\psi(X)\psi^*(X') = \frac{1}{2\pi} \int w(Y, \mu, X - X') \exp\left[i\left(Y - \mu \frac{X + X'}{2}\right)\right] \, d\mu \, dY. \tag{7}
\]

For \( \mu = \cos \theta \) and \( \nu = \sin \theta \), one has the optical tomogram \( w(X, \theta) \)

\[
w(X, \theta) = \frac{1}{2\pi|\sin \theta|} \left| \int \psi(y) \exp\left(\frac{i \cot \theta}{2} y^2 - \frac{iX}{\sin \theta} y\right) \, dy \right|^2, \tag{8}
\]

where the integrand is similar to the Green function of the quantum harmonic oscillator and the tomogram coincides with the modulus squared of the fractional Fourier transform of \( \psi(y) \).

The homogeneity property (6) implies that the particular values of the symplectic tomogram,
e.g., $w(1, \mu, \nu)$, $w(X, 1, \nu)$, and $w(X, \mu, 1)$, determine the whole tomogram and, consequently, the complex function $\psi(x)$.

Now we show that both tomograms can be obtained in two different and realizable processes. The optical tomogram can be rewritten in terms of the fractional Fourier transform

$$w(X, \theta) = \sqrt{\frac{1}{2\pi}} \int \psi(y) \exp \left[ i \cot \frac{\theta}{2} \left( \frac{y^2 + X^2}{\sin \theta} \right) - iXy \right] dy.$$  \hspace{1cm} (9)

The phase factor $\exp \left[ (iX^2/2) \cot \theta \right]$ does not change the value of the tomogram.

A tomogram presented in such form coincides with the value of modulus squared of the wave function of an “oscillator” at the point $x$ at the time moment $t$ if the initial value of the wave function at the time moment $t = 0$ is equal to $\psi(y)$.

This means that to reconstruct the initial value of the wave function $\psi(x)$, including both the amplitude $|\psi(x)|$ and phase $\phi(x)$,

$$\psi(x) = |\psi(x)| \exp i\phi(x),$$

one can measure the tomogram, i.e., the amplitude squared of the wave function which evolves in the quadratic potential well.

This situation can be perfectly realized in optical fibers with a parabolic profile of the refractive index, the so-called “selfoc” (linear propagation). In fact, the light beams in optical fibers obey the Schrödinger-like equation which follows from the Helmholtz equation in the Fock–Leontovich approximation [36]. But the time $t$ in the Schrödinger equation is replaced by the longitudinal coordinate $z$ and the Planck’s constant, by the wavelength.

Thus, to measure the input field amplitude and phase, it is sufficient to measure the tomogram which is the field intensity in each cross-section of the fiber given by the longitudinal coordinate $0 < z \leq 2\pi$.

Another possibility is related to the formula

$$w(X, 1, \nu) = \sqrt{\frac{1}{2\pi i \nu}} \int \exp \frac{i(X - y)^2}{2\nu} \psi(y) dy.$$  \hspace{1cm} (10)

This formula is equivalent to formula (5), in which we put $\mu = 1$ and added the nonessential phase factor $\exp \left( iX^2/2\nu \right)$.

One can see that the tomogram is expressed in terms of the Fresnel integral. The tomography based on the Fresnel integral was suggested in [37, 38] and described in detail in [32]; this tomography was called “Fresnel tomography” and it is different from the scheme of reconstructing the Wigner function proposed in [39] where the Fresnel transform is also used.

Thus the tomogram for “time moment” $\nu$ is equivalent to the intensity of a freely propagating signal. In fact, the kernel in (10) is the Green function of a free particle. Since due to homogeneity the Fresnel tomogram $w(X, 1, \nu)$ is equivalent to the tomogram $w(X, \mu, \nu)$, while measuring the intensity of a freely propagating signal, one measures both the phase and amplitude of the input signal $\psi(y)$.

If one measures the field in an optical fiber, the structure of the output field can be evaluated by measuring free propagation of the light beam.

There is a peculiarity in using formula (10). For complete reconstructing of the amplitude, one needs to know the intensity for arbitrary large values of time (or longitudinal coordinate $z$). In practice, the length or duration can be chosen to fit the appropriate accuracy of the measurement (window).
3. The chirped Gaussian case

In order to verify the validity of the method, we have numerically simulated the reconstruction of a 1D complex Gaussian chirped field given by the following normalized form:

\[ \psi(x) = \left( \frac{2}{\pi \sigma^2} \right)^{1/4} \exp \left[ -\frac{x^2}{\sigma^2} + i\alpha x^2 \right] \]  

(11)

having a chirp \( \alpha \) and a width determined by the parameter \( \sigma \). The complex field given by Eq. (11) can be retrieved from its tomographic representation \( w(X, \mu, x) \) by the following inversion integral

\[ \psi(x)\psi^*(0) = \frac{1}{2\pi} \int \int w(X, \mu, x) \exp \left[ i \left( X - \frac{\mu x}{2} \right) \right] dX d\mu. \]

(12)

The tomogram of the field can be calculated from Eqs. (5) and (11). The Gaussian distribution can be integrated and the tomogram can be written in the following form using, according to the notation of the previous sections, the symbol \( \nu \) instead of \( x \):

\[ w(X, \mu, \nu) = \left( \frac{2\sigma^2}{\pi} \right) \exp \left\{ -\frac{2\sigma^2 X^2}{4\nu^2 + \sigma^4 (\mu + 2\alpha \nu)^2} \right\}. \]

(13)

Equation (13) shows that the tomogram of the field is still characterized by a Gaussian distribution law but its width \( \omega \) is, in general, different from the width of the original field and it can be expressed in terms of the phase-space variables \( \mu \) and \( \nu \), chirp parameter \( \alpha \), and width \( \sigma \), namely,

\[ \omega^2 = \frac{(\mu + 2\alpha \nu)^2 \sigma^4 + 4\nu^2}{2\sigma^2}. \]

(14)

Recalling the general homogeneity property of the tomogram we can obtain the full dependence of the tomogram from its three space variables, in view of homogeneity property of tomogram:

\[ w(X, \mu, \nu) = \frac{1}{|\mu|} w \left( \frac{X}{\mu}, 1, \frac{\nu}{\mu} \right). \]

(15)

According to the Fresnel-based interpretation of the optical tomogram, \( w(X/\mu, 1, \nu/\mu) \) represents the intensity of the field at distance \( \nu \) and Eq. (15) tells us that the general dependence of the tomogram from its three real variables \( X, \mu, \) and \( \nu \) can be recovered from a set of measurements of the intensity distributions of the propagated field performed at different distances with a varying scale factor \( 1/|\mu| \). Equation (15) can be verified immediately from the defining expression of the field tomogram given by Eq. (13).

**Squeeze tomograms (discrete random variable \( n \))**

We describe another type of tomogram which we call squeeze tomogram [40] \( W_{sq}(n, \mu, \nu) \), where \( n = 0, 1, 2, \ldots \) has the physical meaning of the number of photons in the quantum state of light under consideration. We define the tomogram of the state with density operator \( \hat{\rho} \) by the relation

\[ W_{sq}(n, \mu, \nu) = \langle n| \hat{S}(\mu, \nu) \hat{\rho} \hat{S}^\dagger(\mu, \nu)|n \rangle \]

\[ = \langle n| \hat{S}(\lambda) \hat{R}(\theta) \hat{\rho} \hat{R}^\dagger(\theta) \hat{S}^\dagger(\lambda)|n \rangle. \]

(16)

Here \( \hat{S}(\mu, \nu) = \hat{S}(\lambda) \hat{R}(\theta) \), where \( \hat{S}(\lambda) \) and \( \hat{R}(\theta) \) are the squeezing and rotation operators, respectively. They have the form

\[ \hat{S}(\lambda) = \exp \left[ \frac{i\lambda}{2} (\hat{q}\hat{p} + \hat{p}\hat{q}) \right], \quad \hat{R}(\theta) = \exp \left[ \frac{i\theta}{2} (\hat{q}^2 + \hat{p}^2) \right]. \]

(17)
The scaling parameter $\lambda$ and rotation angle $\theta$ are connected with symplectic transform parameters $\mu$ and $\nu$ by $\mu = e^\lambda \cos \theta$, $\nu = e^{-\lambda} \sin \theta$.

The squeeze tomogram can be interpreted as the diagonal matrix element in a Fock basis of the scaled and rotated density operator

$$\hat{\varrho}^{\mu \nu} = \hat{S}(\mu, \nu) \hat{\varrho} \hat{S}^\dagger(\mu, \nu).$$

(18)

Since the squeezing and rotation are unitary operators, the Hermitian nonnegative density operator $\hat{\varrho}^{\mu \nu}$ has positive diagonal matrix elements in the Fock basis. These matrix elements (tomogram) have the physical meaning of photon distribution function in the state described by the density operator $\hat{\varrho}^{\mu \nu}$. To measure the tomogram one has to take the initial photon state with density operator $\hat{\varrho}$. Then one needs to rotate the quadratures as it is done in the homodyne detection scheme. The rotated state has to be squeezed by applying the squeezing operator $\hat{S}^\dagger(\lambda)$. Measuring the photon statistics in the obtained state with density operator $\hat{\varrho}^{\mu \nu}$ one gets the squeeze tomogram $W_{sq}(n, \mu, \nu)$. This tomogram is the normalized probability distribution of the discrete random variable $n$. The tomogram is normalized, satisfying the equality

$$\sum_{n=0}^{\infty} W_{sq}(n, \mu, \nu) = 1.$$  

(19)

The tomogram depends on the number of photons $n$ and two real parameters $\mu$ and $\nu$. The number of the parameters is sufficient to characterize the quantum state completely, since it is determined by the Wigner function depending on two real variables $q$ and $p$.

One can find the relation of squeeze tomogram to the Wigner function. The connection of squeeze tomogram with the Wigner function can be presented in the integral form [40]

$$W_{sq}(n, \mu, \nu) = \int W(q, p) K_W(q, p, n, \mu, \nu) dq dp.$$  

(20)

The kernel of the integral transform has the form

$$K_W(q, p, n, \mu, \nu) = \frac{(-1)^n}{\pi} \exp \left( -|z|^2 / 2 \right) L_n \left( |z|^2 \right),$$

(21)

with

$$|z|^2 = \frac{2q^2}{\mu^2 + \nu^2} + 2 \left( \mu^2 + \nu^2 \right) \times \left[ p - \left( \frac{\sqrt{2}}{1 - \sqrt{1 - 4\mu^2 \nu^2}} - \frac{\mu}{\nu(\mu^2 + \nu^2)} \right) q \right]^2.$$  

(22)

The squeeze tomogram is a new characteristic of the quantum state.

4. Nonlinear Gross–Pitaevskii equation in tomographic representation

The states of Bose–Einstein condensates (BEC) are described by solutions of nonlinear Gross–Pitaevskii equation [29]. In comparison with nonlinear Schrödinger equation, this equation contains an additional linear potential-energy term (e.g., a harmonic-oscillator potential energy of a trap); it can be presented in the form (see, for example, [41])

$$i \frac{\partial \psi}{\partial s} = -\frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + U(x)\psi + q_0 |\psi|^2 \psi.$$  

(23)
For $q_0 = 0$, one has linear Schrödinger equation.
For $U(x) = 0$, one has nonlinear (cubic) Schrödinger equation.

Using the change of variables given by the tomographic map, one can obtain the Gross–Pitaevskii equation in the tomographic form

$$\frac{\partial w(X, \mu, \nu, s)}{\partial s} - \mu \frac{\partial w(X, \mu, \nu, s)}{\partial \nu} - 2q_0 \text{Im} \left\{ \int \frac{dy \, dy'}{2\pi} \right\} w(y, \mu', 0, s) = 0.$$  

(24)

One can see that the equation obtained contains two contributions. One contribution is the nonlinear potential of BEC, and the other is related to an external potential. The potential energy can be chosen as the harmonic oscillator potential $U = \omega^2 x^2 / 2$.

For $q_0 = 0$, one has von Neumann equation in tomographic form.
For $U(x) = 0$, one has nonlinear Schrödinger equation in tomographic form.

5. Quasidistributions, wavelets and tomograms
Below we present a unified general construction of three types of maps.

The first class consists of wavelet-type transforms, the second one of quasidistributions, and the third class of tomographic transforms. Quasidistributions are transforms like the Wigner–Ville one.

The general setting for our construction is as follows [42].

Signals $f(t)$ (or wave functions) are considered to be vectors $| f \rangle$. With $\alpha$ being a set of parameters, $\{U(\alpha)\}$ is a family of operators. In this setting, three types of transforms are defined.

Consider a reference vector $| h \rangle$ chosen in such a way that out of the set $\{U(\alpha) | h \rangle = | h, \alpha \rangle\}$ a complete set of vectors can be chosen to serve as a basis. Two of the transforms considered are given by scalar products

$$W_f^{(h)}(\alpha) = \langle h, \alpha | f \rangle, \quad Q_f(\alpha) = \langle f, \alpha | f \rangle.$$  

(25)

We will denote transforms of the $W_f^{(h)}$-type as wavelet-type transforms and those of the $Q_f$-type as quasidistribution transforms.

In general, if $U(\alpha)$ are unitary operators, there are self-adjoint operators $B(\alpha)$ such that

$$W_f^{(h)}(\alpha) = \langle h | e^{iB(\alpha)} | f \rangle, \quad Q_f^{(B)}(\alpha) = \langle f | e^{iB(\alpha)} | f \rangle.$$  

(26)

In this case, another transform can be defined by means of the Dirac delta function

$$M_f^{(B)}(X) = \langle f | \delta\left(B(\alpha) - X\right) | f \rangle.$$  

(27)

Equation (27) defines what we call the tomographic transform of an analytic signal or tomogram. In contrast to the quasiprobabilities, the transform $M_f^{(B)}(X)$ is positive and it can be correctly interpreted as a probability distribution.

For a normalized vector $| f \rangle$, $\langle f | f \rangle = 1$, the tomogram is a normalized function

$$\int M_f^{(B)}(X) \, dX = 1.$$
and, therefore, it may be interpreted as a probability distribution for the random variable $X$
corresponding to the observable defined by the operator $B(\alpha)$.

The three classes of transforms are mutually related

\[ M_f^{(B)}(X) = \frac{1}{2\pi} \int Q_f^{(kB)}(\alpha) e^{-ikX} \, dk, \]
\[ Q_f^{(B)}(\alpha) = \int M_f^{(B/p)}(X) e^{ipX} \, dX, \]
\[ Q_f^{(B)}(\alpha) = W_f^{(f)}(\alpha), \]

where

\[ |f_1\rangle = |h\rangle + |f\rangle, \quad |f_3\rangle = |h\rangle - |f\rangle, \quad |f_2\rangle = |h\rangle + i |f\rangle, \quad |f_4\rangle = |h\rangle - i |f\rangle. \]

Another important case concerns the operators $U(\alpha)$, which can be represented as

\[ U(\alpha) = e^{ib(\alpha)} P_h e^{-ib(\alpha)}, \]

with $P_h$ being a projector on a reference vector $|h\rangle$.

This creates a quasidistribution of the Husimi–Kano type

\[ H_f^{(h)}(\alpha) = \langle f | U(\alpha) | f \rangle. \]

The tomograms and Wigner functions of solitons can be described in view of the constructions
presented above. Also wavelet transforms can be applied to the soliton solutions of nonlinear
equations.

The construction introduced unifies practically all known integral transform and provides a
possibility to elucidate the mutual relations of the integral transforms.

6. Entropy associated with generic tomographic transformations

Since the map (27)

\[ M_f^{(B)}(X) = \langle f | \delta \left( B(\alpha) - X \right) | f \rangle \]

yields the fair probability distribution, one can introduce entropy associated with a vector in a
Hilbert space represented by a complex function (wave function or analytic signal).

In fact, one uses known Shannon construction [43].

For a given probability $P(x)$, it is defined as

\[ S = -\langle \ln P \rangle = - \sum_x P(x) \ln P(x), \]

where $x$ is considered as a discrete random variable from some domain.

The entropy describes a measure of uncertainty of our knowledge of the possibility to get a
value of variable $x$.

We introduce entropy, in view of the probability distribution determined by tomogram (27),
using the definition

\[ S_f(\alpha) = - \int dX \left\{ \operatorname{Tr} \delta \left( B(\alpha) - X \right) | f \rangle \langle f | \right\} \ln \left\{ \operatorname{Tr} \delta \left( B(\alpha) - X \right) | f \rangle \langle f | \right\}. \]  

(28)

The entropy introduced depends on the state (signal) vector and parameters determining the
operator $B(\alpha)$. Formula (28) in terms of generic tomographic transform $M_f^{(B)}$ reads

\[ S_f(\alpha) = - \int M_f^{(B)}(X) \ln M_f^{(B)}(X) \, dX. \]  

(29)
The notion of Shannon entropy was used to define the entropy for states of spin quantum systems [44], for signals [33], as well as for solitons [34].

For symplectic tomography, entropy reads

\[ S(\mu, \nu) = - \int w(X, \mu, \nu) \ln w(X, \mu, \nu) \, dX. \]

Analogous entropies can be introduced for optical and Fresnel tomograms since they are related directly to symplectic tomogram. For example, entropy related to optical tomogram and to Fresnel tomogram of a quantum state \( |\psi\rangle \) is given as the function of angle \( \theta \) and parameter \( \nu \), respectively, by the relations

\[
S_\psi(\theta) = - \int dX \left\{ \frac{1}{2\pi|\sin \theta|} \left| \int \psi(y) \exp \left( \frac{i\cot \theta}{2} y^2 - \frac{iX}{\sin \theta} y \right) dy \right|^2 \right\} 
\times \ln \left\{ \frac{1}{2\pi|\sin \theta|} \left| \int \psi(y) \exp \left( \frac{i\cot \theta}{2} y^2 - \frac{iX}{\sin \theta} y \right) dy \right|^2 \right\}
\]

and

\[
S_\psi(\nu) = - \int dX \left\{ \frac{1}{\sqrt{2\pi i\nu}} \left| \int \exp \frac{i(X-y)^2}{2\nu} \psi(y) dy \right|^2 \right\} 
\times \ln \left\{ \frac{1}{\sqrt{2\pi i\nu}} \left| \int \exp \frac{i(X-y)^2}{2\nu} \psi(y) dy \right|^2 \right\}.
\]

Since symplectic tomogram, e.g., of soliton in BEC obeys to generalized nonlinear Fokker–Planck equation (kinetic equation), the corresponding entropy introduced is a characteristic of the solutions of this kinetic equation. The meaning of this characteristic needs further investigation but we think that potentially the entropy (or Shannon information) could be useful in analysis of signals from the viewpoint of their informativity, if the signals are some linear or nonlinear perturbations (e.g., solitons) obeying the nonlinear dynamical equations.

The entropy can be also introduced using squeeze tomogram, i.e.,

\[ S_{sq}(\mu, \nu) = - \sum_{n=0}^{\infty} W_{sq}(n, \mu, \nu) \ln W_{sq}(n, \mu, \nu). \]

7. Conclusions

We have shown that linear and nonlinear signals can be associated to different kinds of tomograms which are fair probability-distribution functions related to the Ville–Wigner function of the signals.

New notion of tomographic entropy of the signals was introduced.

Examples of symplectic tomograms, squeeze tomograms and Fresnel tomograms demonstrate the possibility of extending the set of different characteristics of analytic signal (wave function, density matrix).

Nonlinear equations (nonlinear Schrödinger equation, Gross–Pitaevskii equation) were presented in tomographic form.

In a future publication, we hope to extend the suggested approach to other kinds of dynamic systems, both classical and quantum ones. The general scheme of this extension is the same as for using the Fourier integral transform. For any given dynamical equation describing a signal or some quantum process, one can apply the Radon transform or its discrete analog to
transform the equation into new tomographic form. The procedure can provide an equation for the Radon component of the signal (an analog of the Fourier component of the signal). In some cases, the equation for the Radon component can be of the form more suitable for the explicit (or approximate) solving than the initial equation. An advantage of the tomographic representation is that the Radon component of the signal under study is the standard probability distribution.

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References
[1] Fourier J B J 1888 Théorie analytique de la chaleur Oeuvres de Fourier Tome premier, edited by G Darboux (Paris, Gauthiers-Villars)
[2] Howel K B 1996 Fourier transforms The transforms and applications handbook p. 95, edited by A D Poularikas (Boca Raton, Florida, CRC Press & IEEE Press)
[3] Bachert H, Eliseev P G, Man’ko M A, Raab S, Strakhov V P and Tran Mihn Thai 1975 IEEE Quantum Electron. QE-11/1 Pt. 2, 510; 1975 Sov. J. Quantum Electron. 45 1102
[4] Man’ko M A 1979 Rozprawy Elektrotechniczne 425 731
[5] Cohen L 1996 IEEE Trans. Signal Process. 44 1080
[6] Man’ko M A 1999 J. Russ. Laser Res. 20 226
[7] Man’ko M A 2000 Fractional Fourier analysis and quantum propagators Quantum theory and symmetries p. 226, edited by H-D Doebner et al. (Singapore, World Scientific)
[8] Cohen L 1989 Proc. IEEE 77 941
[9] Wigner E P 1932 Phys. Rev. 40 749
[10] Ville J 1948 Cables et Transmission 2 61
[11] Bertrand J and Bertrand P 1987 Found. Phys. 17 397
[12] Vogel K and Risken H 1989 Phys. Rev. A 40 2847
[13] Smithey D T, Beck M, Raymer M G and Faridani A 1993 Phys. Rev. Lett. 70
[14] Mancini S, Man’ko V I and Tombesi P 1995 Quantum Semiclass. Opt. 7 615
[15] D’Ariano G M, Mancini S, Man’ko V I and Tombesi P 1996 Quantum Semiclass. Opt. 8 1017
[16] Mancini S, Man’ko V I and Tombesi P 1996 Phys. Lett. A 213 1; 1997 Found. Phys. 27 801
[17] Man’ko M A 2000 J. Russ. Laser Res. 21 411
[18] Radon J 1917 Über die bestimmung von funktionen durch ihre integralwerte längs gewisse mannigfaltigkeiten Breichte Sachsische Akademie der Wissenschaften, Leipzig, Mathematische-Physikalische Klasse 69 S. 262
[19] Gabor D 1946 Theory of communication IEE J. 93 429
[20] Husimi K 1940 Proc. Phys. Math. Soc. Jpn 23 264
[21] Sudarshan E C G 1963 Phys. Rev. Lett. 10 277
[22] Kano Y 1965 J. Math. Phys. 6 1913
[23] Mehta C L and Sudarshan E C G 1965 Phys. Rev. B 138 274
[24] Glauber R J 1963 Phys. Rev. 131 2766
[25] Chountasis S, Vourdas A and Bendjaballah C 1999 Phys. Rev. A 60 3467
[26] Winscha A 2000 J. Mod. Opt. 47 33
[27] De Nicola S, Fedele R, Man’ko M A and Man’ko V I 2002 Wigner picture and tomographic representation of envelope solitons Nonlinear physics. Theory and Experiment. II p. 372, edited by M J Ablowitz et al. (Singapore, World Scientific)
[28] Gross E P 1961 Nuovo Cim. 20 454
[29] Arkhipov A S and Man’ko V I 2003 J. Opt. B: Quantum Semiclass. Opt. 5 227
[30] Castaños O, López-Peña R, Man’ko M A and Man’ko V I 2003 J. Opt. B: Quantum Semiclass. Opt. 5 227
[31] Arkhipov A S and Lozovik Yu E 2003 Zh. Éksp. Teor. Fiz. 125 1; 2003 Phys. Lett. A 319 217
[32] Arkhipov A S, Lozovik Yu E and Sharapov V A 2004 Phys. Rev. A 69 022116
[33] Arkhipov A S and Man’ko V I 2004 J. Russ. Laser Res. 25 468
[34] De Nicola S, Fedele R, Man’ko M A and Man’ko V I 2003 J. Opt. B: Quantum Semiclass. Opt. 5 95
[35] Gross E P 1961 Sov. Phys. JETP 13 451
[36] De Nicola S, Fedele R, Man’ko M A and Man’ko V I 2004 J. Russ. Laser Res. 25 1
Fedele R, Shukla P K, De Nicola S, Man’ko M A, Man’ko V I and Cataliotti F S 2004 JETP Lett. 80 535; 2005 Phys. Scr. T116 10

[31] De Nicola S, Fedele R, Man’ko M A and Man’ko V I 2004 Acta Physica Hungarica B 20 261
[32] De Nicola S, Fedele R, Man’ko M A and Man’ko V I 2005 Theor. Math. Phys. 144 1206
[33] Man’ko M A 2001 J. Russ. Laser Res. 22 168
[34] De Nicola S, Fedele R, Man’ko M A and Man’ko V I 2003 Eur. Phys. J. B 36 385
[35] Man’ko V I, Rosa L and Vitale P 1998 Phys. Rev. A 58 3291
[36] Leontovich M A and Fock V A 1946 Zh. Éksp. Teor. Fiz. 16 557
[37] M. A. Man’ko 2002 Wigner function and tomograms in signal processing and quantum computing. Talk at Wigner Centennial Conference (Pecs, Hungary, July 2002)
[38] De Nicola S, Fedele R, Man’ko M A and Man’ko V I 2002 Tomographic analysis of envelope solitons: Concepts and applications Abstracts of the International Workshop ‘Optics in Computing’; International Optical Congress ‘Optics XXI Centenary’ (St. Petersburg, Russia, November 2002)
[39] Lougovski P, Solano E, Zhang Z M, Walter H, Mack H and Schleich W P 2003 Phys. Rev. Lett. 91 010401
[40] Castaños O, Lópe-Peña R, Man’ko M A and Man’ko V I 2004 J Phys. A: Math. Gen. 37 8529
[41] Karkuszewski Z P, Sacha K and Zakrzewski J 2001 Phys. Rev. A 63 061601R
[42] Man’ko M A, Man’ko V I and Mendes R V 2001 J. Phys. A: Math. Gen. 34 8321
Man’ko M A 2001 J. Russ. Laser Res. 22 505; 2002 Unified view of tomographic and other transforms in signal analysis Group 24: Physical and mathematical aspects of symmetries. Conference series 173 887, edited by J-P Gazeau et al. (Bristol, Institute of Physics)
[43] Shannon C E 1948 Bell Tech. J. 27 379 (1948)
[44] Man’ko O V and Man’ko V I 2004 J. Russ. Laser Res. 25 115