Construction of Solitary Two-Wave Solutions for a New Two-Mode Version of the Zakharov-Kuznetsov Equation

Imad Jaradat * and Marwan Alquran †

Department of Mathematics and Statistics, Jordan University of Science and Technology, P.O. Box 3030, Irbid 22110, Jordan; marwan04@just.edu.jo
* Correspondence: iajaradat@just.edu.jo
† These authors contributed equally to this work.

Received: 5 June 2020; Accepted: 8 July 2020; Published: 10 July 2020

Abstract: A new two-mode version of the generalized Zakharov-Kuznetsov equation is derived using Korsunsky’s method. This dynamical model describes the propagation of two-wave solitons moving simultaneously in the same direction with mutual interaction that depends on an embedded phase-velocity parameter. Three different methods are used to obtain exact bell-shaped soliton solutions and singular soliton solutions to the proposed model. Two-dimensional and three-dimensional plots are also provided to illustrate the interaction dynamics of the obtained two-wave exact solutions upon increasing the phase-velocity parameter.

Keywords: two-mode Zakharov-Kuznetsov equation; bell-shaped soliton solutions; Kudryashov expansion method; simplified bilinear method

MSC: 35C08; 74J35

1. Introduction

The work on nonlinear partial differential equations has been developed to get an insight through qualitative and quantitative features of many models arise in diverse fields, such as electro-magnetic waves, optics, nerve pulses, nonlinear dynamics, condensed matter physics and others. One of the essential properties of most of nonlinear equations is to capture the perfect balance between dispersion and nonlinearity effects which results in soliton pulse. The study of soliton solutions for nonlinear equations has been integrated by suggesting and developing ansatze methods that produce different types of solitons. Such compatible methods include Bernoulli sub-equation function method, \((G'/G)\)-expansion method, sine-cosine method, simplified-bilinear method, Kudryashov method, Unified methods, and many others listed in [1–8]. Motivated by exploring new physical insights for new models arise in physical sciences, we aim to propose new mathematical modification in the construction of one of the well-known physical models and recognize their dynamical soliton solutions. The suggested model to be addressed in this work is the Zakharov-Kuznetsov (ZK) equation.

The Zakharov-Kuznetsov (ZK) equation was first established to model the propagation of weakly nonlinear ion-acoustic waves in plasma, which includes cold ions and hot-isothermal electrons in a medium with a uniform magnetic field [9,10]. Moreover, it also describes the \((2 + 1)\)-dimensional modulations of a KdV soliton equation in fluid mechanics [11]. The standard ZK equation reads

\[
 u_t + ku_x + (u_{xx} + u_{yy})_x = 0. \quad (1)
\]
It has been shown that Equation (1) is not integrable by means of the inverse scattering transform test [12]. Thus, it is a difficult task to study it if compared with other integrable equations. In order to study the dynamics of ion-acoustic waves in cold-ion plasma when the behavior of electrons is not isothermal, Schamel [13] has derived a new (1 + 1)-dimensional ZK equation with a fractional power nonlinear term as follows:

$$u_t + u^{1/2}_x + au_{xxx} = 0. \quad (2)$$

By means of the sine-cosine ansatz method [14], some special forms of exact solutions to the fractional ZK equation have been reported. Other related studies on the ZK equation can be found in [15,16]. A more general form of the (2 + 1)-dimensional ZK equation takes the form

$$u_t + (u_{xx} + u_{yy} + ku^p)_x = 0. \quad (3)$$

In this paper, we aim to derive a two-mode version of the ZK equation given in (3). Two-mode equations are nonlinear partial differential equations of second-order in the time coordinate, and they describe the dynamics of the two-wave solitons propagating in the same direction, which overlap with one another without changing their shapes.

In [17], the overlapping of phase-locked waves and the over-taking waves have been observed, and their corresponding phase-speeds are found close to each other. These phenomena have been seen in the model of second-order in time of the Korteweg–de Vries equation, which reads as [18]

$$U_{\tau\tau} + (c_1 + c_2)U_{\chi\tau} + c_1 c_2 U_{\chi\chi} + \left( (\alpha_1 + \alpha_2) \frac{\partial}{\partial \tau} + (\alpha_1 c_2 + \alpha_2 c_1) \frac{\partial}{\partial \chi} \right) U U_{\chi} + \left( (\beta_1 + \beta_2) \frac{\partial}{\partial \tau} + (\beta_1 c_2 + \beta_2 c_1) \frac{\partial}{\partial \chi} \right) U_{\chi\chi\chi} = 0, \quad (4)$$

where

- $\chi$ and $\tau$ are the scaled space and time coordinates.
- $U(\chi, \tau)$ is the height of the water’s free surface above the flat bottom.
- $c_1$ and $c_2$ are the phase velocities.
- $\alpha_1$ and $\alpha_2$ are the linearity parameters.
- $\beta_1$ and $\beta_2$ are the dispersion parameters.

The model given in (4) describes the propagation of two-mode waves with the same dispersion relation and different phase-velocities, nonlinearity, and dispersion parameters [19]. In [18], Korsunsky reformulated (4) by using the following new variables defined as

$$u = \frac{(\alpha_1 + \alpha_2) U}{\sqrt{\beta_1 + \beta_2}}, \quad x = \frac{\chi - \frac{\alpha_1 + \alpha_2}{2 \beta_1 + \beta_2} \tau}{\sqrt{\beta_1 + \beta_2}}, \quad t = \frac{\tau}{\sqrt{\beta_1 + \beta_2}}, \quad (5)$$

and proposing the constraints: $|\frac{\alpha_1 - \alpha_2}{\alpha_1 + \alpha_2}| \leq 1$, $|\frac{\beta_1 - \beta_2}{\beta_1 + \beta_2}| \leq 1$, and $c_2 \leq c_1$. Accordingly, (4) is converted into

$$u_{tt} - s^2 u_{xx} + \left( \frac{\partial}{\partial t} - \alpha s \frac{\partial}{\partial x} \right) u u_x + \left( \frac{\partial}{\partial t} - \beta s \frac{\partial}{\partial x} \right) u_{xxx} = 0. \quad (6)$$

Equation (6) is regarded as a two-mode KdV equation with $u = u(x, t)$ being the field function, $\alpha$, $\beta$, respectively, are the nonlinearity and dispersion parameters that are less than 1, and $s$ is the interaction phase velocity. Note here, that $\alpha$, $\beta$ and $s$ appear in (6) can be obtained if we let $c_1 = -c_2 = s$ and simplify algebraically (4). It is clear that, in case of $s = 0$, no interaction occurs, and integrating
with respect to the time \( t \), (6) is reduced to the standard KdV equation that describes the propagation of a single-moving wave.

Motivated by the Korsunsky’s technique, Wazwaz [20–23] has established the two-mode versions of Sharma–Tasso–Olver equation, fourth-order Burgers’ equation, fifth-order KdV equation, higher-order modified KdV and the KP equations, and has obtained multiple-kink solutions by adopting the simplified Hirota’s method. Furthermore, other two-mode models have been established by using Korsunsky’s scheme and their solutions have been obtained by means of simplified bilinear method, tanh-coth method, and the \((G'/G)\)-expansion method. Such types of two-mode equations have been derived for coupled Burgers’ equation, coupled KdV equation, coupled modified KdV equation, KdV–Burgers’ equation, third-order Fisher equation, Kuramoto-Sivashinsky equation, and higher-order Boussinesq-Burgers system [24–30]. Furthermore, in [31,32], the two-mode KdV equation and the two-mode Sharma-Tasso-Olver equation have been revisited and more new solitary wave solutions have been obtained. Moreover, the two-mode concept has been applied to the Schrödinger equations [33,34]. The dynamics of the two-mode phenomena have also been investigated in [35–37]. We should note here that the aforementioned works are devoted in presenting new techniques will be further developed for the study of two-mode models.

The Korsunsky’s scheme to construct two-mode equations has the following scaled form

\[
\frac{\partial u}{\partial t} - s^2 u_{xx} + \left( \frac{\partial}{\partial t} - \alpha s \frac{\partial}{\partial x} \right) N(u, u_x, ...) + \left( \frac{\partial}{\partial t} - \beta s \frac{\partial}{\partial x} \right) L(u_{xx}, u_{xxx}, ...) = 0. \tag{7}
\]

Here \( N(u, u_x, ...) \) and \( L(u_{xx}, u_{xxx}, ...) \) are the nonlinear and the linear terms of the model, respectively. In this work we extend (7) to construct \((2 + 1)\)-dimensional two-mode equations that will take the form

\[
\frac{\partial u}{\partial t} - s^2 u_{xx} - s^2 u_{yy} + \left( \frac{\partial}{\partial t} - \alpha_1 s \frac{\partial}{\partial x} - \alpha_2 s \frac{\partial}{\partial y} \right) N(u, u_x, u_y, ...) \\
+ \left( \frac{\partial}{\partial t} - \beta_1 s \frac{\partial}{\partial x} - \beta_2 s \frac{\partial}{\partial y} \right) L(u_{xx}, u_{yy}, u_{xy}, u_{xxy}, ...) = 0, \tag{8}
\]

where \( u = u(x, y, t) \). Applying (8) on (3), we introduce the following \((2 + 1)\)-dimensional two-mode Zakharov-Kuznetsov (TMZK) equation

\[
\frac{\partial u}{\partial t} - s^2 u_{xx} - s^2 u_{yy} + \left( \frac{\partial}{\partial t} - \alpha_1 s \frac{\partial}{\partial x} - \alpha_2 s \frac{\partial}{\partial y} \right) \left\{ (ku^p)_x \right\} \\
+ \left( \frac{\partial}{\partial t} - \beta_1 s \frac{\partial}{\partial x} - \beta_2 s \frac{\partial}{\partial y} \right) \left\{ u_{xx} + u_{yy} \right\} = 0. \tag{9}
\]

We aim to study the solutions of the TMZK equation by implementing three different integration techniques: the sech-csch method, the Kudryashov’s expansion method, and the simplified bilinear method.

2. Bell-Shaped Soliton Solutions

To obtain a bell-shaped soliton solution for the TMZK equation, we consider the new variable \( z = ax + by - ct \) to convert (9) into the following reduced-order differential equation

\[
(c^2 - (a^2 + b^2)s^2) u - a(a^2 + b^2)(c + \beta_1 as + \beta_2 bs) u'' - ak(c + \alpha_1 as + \alpha_2 bs) u^p = 0, \tag{10}
\]
where now $u = u(z)$. Then, we assume that the solution of (10) takes the form \[u(z) = A \sech^q(z).\] (11)

We substitute (11) in (10), and we collect the coefficients of the same powers of $\sech(z)$ to get the following outputs

\[
\begin{align*}
\sech^q(z) & : A(c^2 - (a^2 + b^2)s^2) - aAq^2(a^2 + b^2)(c + \beta_1as + \beta_2bs), \\
\sech^{pq}(z) & : -akAp(c + \alpha_1as + \alpha_2bs), \\
\sech^{q+2}(z) & : -aqA(1 + q)(a^2 + b^2)(c + \beta_1as + \beta_2bs). 
\end{align*}
\] (12)

Equating the power indices of $\sech^{pq}$ against $\sech^{q+2}$, and setting the coefficient of the same power of $\sech$ to zero, leads to the following system of equations

\[
\begin{align*}
0 &= pq - (q + 2), \\
0 &= c^2 - (a^2 + b^2)s^2 - aq^2(a^2 + b^2)(c + \beta_1as + \beta_2bs), \\
0 &= akAp(c + \alpha_1as + \alpha_2bs) + aqA(1 + q)(a^2 + b^2)(c + \beta_1as + \beta_2bs). 
\end{align*}
\] (13)

Since the above system of equations involves many parameters, we require some reasonable restrictions. We may set

\[
\begin{align*}
\alpha_1 &= \alpha_2 = \gamma_1, \\
\beta_1 &= \beta_2 = \gamma_2.
\end{align*}
\] (14)

Solving (13) based on (14), we reach at the following findings

\[
\begin{align*}
q &= \frac{2}{p - 1} : \quad p \neq 1, \\
c &= \pm \sqrt{a^2 + b^2}s,
\end{align*}
\] (15)

under the condition $\gamma_1 = \pm \sqrt{a^2 + b^2}$. Therefore, the two-mode bell-shaped soliton solution for (9) is

\[u(x, y, t) = A \sech^{\frac{2\gamma}{p} - 1}(ax + by \pm \sqrt{a^2 + b^2}st).\] (16)

Figure 1, presents the dynamics of overlapping the obtained two-mode bell-shaped soliton solutions given by Equation (16) upon increasing the phase velocity $s$. Some remarks regarding the $(2 + 1)$-dimensional TMZK Equation (9) are as follows:

- If $u(z) = A \csch^q(z)$ is considered instead of (11), a two-mode singular soliton solution for (9) will be obtained as $u(x, y, t) = A \csch^{\frac{2\gamma}{p} - 1}(ax + by \pm \sqrt{a^2 + b^2}st)$, provided that $\gamma_1 = \pm \sqrt{a^2 + b^2}$.
- The obtained solution given in (16) preserve its bell-type shape when $p$ varies.
- The overlapping of the obtained two-wave solutions does not change the shapes of these waves.
Figure 1. Interaction of the two-wave solutions given by (16) upon increasing the phase velocity, for \( s = 0.5, 1, 3 \), respectively. Here \( p = 2 \).

3. Kudryashov Expansion Method

In this section we solve the TMZK equation for the particular case \( p = 2 \) by means of the Kudryashov expansion method. In particular, we study the following equation

\[
\frac{\partial^2 u}{\partial t^2} - s^2 \frac{\partial^2 u}{\partial x^2} - s^2 \frac{\partial^2 u}{\partial y^2} + \left( \frac{\partial}{\partial t} - \alpha_1 s \frac{\partial}{\partial x} - \alpha_2 s \frac{\partial}{\partial y} \right) \left( ku \frac{\partial u}{\partial x} \right) + \left( \frac{\partial}{\partial t} - \beta_1 s \frac{\partial}{\partial x} - \beta_2 s \frac{\partial}{\partial y} \right) \left( (u_{xx} + u_{yy})_x \right) = 0. \tag{17}
\]

By using the new variable \( \zeta = ax + by - ct \), (17) is converted into

\[
(c^2 - (a^2 + b^2)s^2) u - a(a^2 + b^2)(c + \beta_1 as + \beta_2 bs)u'' - a k u^2(c + \alpha_1 as + \alpha_2 bs)u^2 = 0, \tag{18}
\]

with \( u = u(\zeta) \). The Kudryashov’s method [39,40] assumes the solution of (18) as a finite series in the variable \( Z \):

\[
u(\zeta) = \sum_{i=0}^{n} b_i Z^i. \tag{19}\]

The variable \( Z \) is the solution of the nonlinear differential equation

\[
Z' = \mu Z(Z - 1). \tag{20}\]

Applying the separable method on (20) gives

\[
Z(\zeta) = \frac{1}{1 + e^{\mu \zeta}}. \tag{21}\]
Performing the balance procedure on the terms $u''$ and $u^2$, gives $n = 2$ and accordingly we write (19) as

$$u(\zeta) = b_0 + b_1 Z + b_2 Z^2.$$  \hspace{1cm} (22)

Differentiating both (20) and (22), leads to

$$Z'' = \mu^2 Z(Z - 1)(2Z - 1),$$  \hspace{1cm} (23)

and

$$u'(z) = b_1 Z' + 2b_2 ZZ',$$
$$u''(z) = b_1 Z'' + 2b_2(ZZ'' + (Z')^2).$$ \hspace{1cm} (24)

Now, we substitute (20) through (24) in (18) to get a finite series in $Z$ whose coefficients are identical to zero. To be able to solve the resulting system, we require the following two reasonable constraints:

$$\alpha_1 = \alpha_2 = \gamma_1,$$
$$\beta_1 = \beta_2 = \gamma_2.$$ \hspace{1cm} (25)

Now, by solving the resulting system along with (25), we reach at the following outputs.

$$b_0 = 0,$$
$$b_1 = \frac{12(a^2 + b^2)\mu^2}{k},$$
$$b_2 = -b_1,$$
$$c = \frac{1}{2}(a(a^2 + b^2)\mu^2 \mp \sqrt{(a^2 + b^2)h}),$$
$$h = 4s^2 + 4as(a\gamma_1 + b\gamma_2)\mu^2 + a^2(a^2 + b^2)\mu^4.$$ \hspace{1cm} (26)

Therefore, the two-wave solution of the TMZK model (17) is given by

$$u(x, y, t) = \frac{12\mu^2}{k} (a^2 + b^2) \exp \left( \frac{t}{\sqrt{(a^2 + b^2)h + a(a^2 + b^2)}} \right) \exp \left( \frac{2(ax + by) + t}{\sqrt{(a^2 + b^2)h + a(a^2 + b^2)}} \right) \left( \frac{d + e^2}{(d + e^2)^2} \right)^2.$$ \hspace{1cm} (27)

Figure 2 is an overview of the dynamics of the profiles of solutions for the interaction of the two-wave solitons given by (27) upon increasing the phase velocity $s$. 
4. Simplified Bilinear Method

In this section, we use the simplified bilinear method \cite{41,42} to find the one-soliton solution for (17). This method requires the following functions

\begin{align*}
h(x, y, t) &= ax + by - ct, \quad (28) \\
v(x, y, t) &= e^{b(x,y,t)}, \quad (29) \\
g(x, y, t) &= 1 + B v(x, y, t) : B = \pm 1, \quad (30) \\
u(x, y, t) &= A(\ln g(x, y, t))_{xx}. \quad (31)
\end{align*}

Substituting (29) in the linear terms of (17) and solving for \( c \), we get

\[ c = \frac{1}{2} \left( a^3 + ab^2 + \sqrt{a^6 + 2a^2b^2 + a^2b^4 + 4a^2s^2 + 4b^2s^2 + 4ab^2(a + b)s\gamma_2} \right). \quad (32) \]

Considering the result obtained in (32), we substitute (31) in (17) and we solve it for \( A \) under the constraints \( \alpha_1 = \alpha_2 = \gamma_1 \) and \( \beta_1 = \beta_2 = \gamma_2 \), to get

\[ A = \frac{12a^5 + 24a^3b^2 + 12a \left( b^4 + 2b^2s\gamma_2 \right) - 12a^2\Delta - 12b^2(-2bs\gamma_1 + \Delta)}{a^3k + a^3b^2k + 2a^3ks\gamma_1 - a^2k(-2bs\gamma_1 + \Delta)}, \quad (33) \]

where \( \Delta = \sqrt{(a^2 + b^2)(a^4 + a^2b^2 + 4s^2) + 4ab^2(a + b)s\gamma_2} \). Therefore, the one-soliton solution of Equation (17) is

\begin{align*}
u(x, y, t) &= -a^2B^2e^{2ax+2by-\frac{t}{2} \left( a^3 + ab^2 - \sqrt{a^6 + 2a^2b^2 + a^2b^4 + 4a^2s^2 + 4b^2s^2 + 4ab^2(a + b)s\gamma_2} \right)} \\
&\quad \times \left( 1 + Be^{ax+by-\frac{t}{2} \left( a^3 + ab^2 - \sqrt{a^6 + 2a^2b^2 + a^2b^4 + 4a^2s^2 + 4b^2s^2 + 4ab^2(a + b)s\gamma_2} \right)} \right)^{-2} \\
&\quad + \frac{a^3B^2e^{ax+by-\frac{t}{2} \left( a^3 + ab^2 - \sqrt{a^6 + 2a^2b^2 + a^2b^4 + 4a^2s^2 + 4b^2s^2 + 4ab^2(a + b)s\gamma_2} \right)}}{1 + Be^{ax+by-\frac{t}{2} \left( a^3 + ab^2 - \sqrt{a^6 + 2a^2b^2 + a^2b^4 + 4a^2s^2 + 4b^2s^2 + 4ab^2(a + b)s\gamma_2} \right)}}. \quad (34)
\end{align*}
It is worth to mention that for $B = 1$, (34) gives the bell-shaped soliton solution, while for $B = -1$, it gives the singular soliton-solution.

Furthermore, since the wave speed $c$ has two different values as given by Equation (32), the soliton solution given by (34) describes the propagation of two-wave solitons moving simultaneously in the same direction with mutual interaction that depends on an embedded phase-velocity parameter and with no change in their shapes.

5. Conclusions

In this work, we have introduced a new nonlinear partial differential equation called the two-mode Zakharov-Kuznetsov (TMZK) equation. This model represents the overlapping of moving two-wave solitons that are propagating simultaneously, in the same direction. Three different methods are used to obtain exact soliton solutions of the TMZK equation. Both two-dimensional and three-dimensional plots are provided to show the profiles of the obtained soliton solutions upon increasing the phase-velocity parameter.

We should point here that the suggested model is proposed for the first time in this work. This study is adhered only with the mathematical modification of ZK equation and obtaining different forms of solutions but with the same soliton type which is found to be of bell-shaped type. In a future work, we aim to find a connection between these new equations with possible applications arise in physical sciences. Furthermore, one future goal, is to find possible soliton solutions of other two-mode equations that are relevant from the physics and engineering point of view by using either the Korsunsky’s method or similar integration techniques.

Author Contributions: Conceptualization, I.J. and M.A.; methodology, I.J. and M.A.; software, I.J. and M.A.; validation, I.J. and M.A.; formal analysis, I.J. and M.A.; investigation, I.J. and M.A.; resources, I.J. and M.A.; data curation, I.J. and M.A.; writing—original draft preparation, I.J.; writing—review and editing, M.A.; visualization, I.J.; project administration, M.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Acknowledgments: We thank the anonymous reviewers for their thorough review and highly appreciate the comments and suggestions, which significantly contributed to improving the quality of the publication.

Conflicts of Interest: The authors declare no conflict of interest.

References
1. Baskonus, H.M.; Koç, D.A.; Gülsu, M.; Bulut, H. New wave simulations to the $(3 + 1)$-dimensional modified KdV-Zakharov-Kuznetsov equation. *AIP Conf. Proc.* 2017, 1863, 560085.
2. Yavuz, M.; Yokus, A. Analytical and numerical approaches to nerve impulse model of fractional-order. *Numer. Methods Partial. Differ. Eq.* 2020. [CrossRef]
3. Baskonus, H.M.; Bulut, H. On the complex structures of Kundu-Eckhaus equation via improved Bernoulli sub-equation function method. *Waves Random Complex Media* 2015, 25, 720–728. [CrossRef]
4. Alquran, M.; Qawasmeh, A. Classifications of solutions to some generalized nonlinear evolution equations and systems by the sine-cosine method. *Nonlinear Stud.* 2013, 20, 263–272.
5. Ali, M.; Alquran, M.; Jaradat, I.; Baleanu, D. Stationary wave solutions for new developed two-waves’ fifth-order Korteweg–de Vries equation. *Adv. Diff. Eq.* 2019, 2019, 263. [CrossRef]
6. Ali, M.; Alquran, M.; Jaradat, I.; Afouna, N.A.; Baleanu, D. Dynamics of integer-fractional time-derivative for the new two-mode Kuramoto-Sivashinsky model. *Rom. Rep. Phys.* 2020, 72, 103.
7. Alquran, M.; Jaradat, I.; Ali, M.; Baleanu, D. The dynamics of new dual-mode Kawahara equation: Interaction of dual-waves solutions and graphical analysis. *Phys. Scr.* 2020, 95, 045216. [CrossRef]
8. Alquran, M.; Jaradat, I.; Ali, M.; Al-Ali, N.; Momani, S. Development of spreading symmetric two-waves motion for a family of two-mode nonlinear equations. *Heliyon* 2020, 6, e04057. [CrossRef] [PubMed]
9. Zakharov, V.E.; Kuznetsov, E.A. On three-dimensional solitons. *Sov. Phys. Sov. Phys.* 1974, 39, 285–288.
10. Monro, S.; Parkes, E.J. The derivation of a modified Zakharov-Kuznetsov equation and the stability of its solutions. *J. Plasma Phys.* 1999, 62, 305–317. [CrossRef]
11. Monro, S.; Parkes, E.J. Stability of solitary-wave solutions to a modified Zakharov-Kuznetsov equation. *J. Plasma Phys.* 2000, 64, 411–426. [CrossRef]

12. Liu, Y.; Wang, X. The construction of solutions to Zakharov-Kuznetsov equation with fractional power nonlinear terms. *Adv. Diff. Eq.* 2019, 2019, 134. [CrossRef]

13. Schamel, H. A modified Korteweg-de-Vries equation for ion acoustic waves due to resonant electrons. *J. Plasma Phys.* 1973, 9, 377–387. [CrossRef]

14. Wazwaz, A.M. Exact solutions with solitons and periodic structures for the Zakharov-Kuznetsov (ZK) equation and its modified form. *Commun. Nonlinear Sci. Numer. Simul.* 2005, 10, 597–606. [CrossRef]

15. Schamel, H. A modified Korteweg–de-Vries equation for ion acoustic waves due to resonant electrons. *J. Plasma Phys.* 1973, 9, 377–387. [CrossRef]

16. Wazwaz, A.M. Exact solutions with solitons and periodic structures for the Zakharov-Kuznetsov (ZK) equation and its modified form. *Commun. Nonlinear Sci. Numer. Simul.* 2005, 10, 597–606. [CrossRef]

17. Wazwaz, A.M. Two-mode Sharma-Tasso-Olver equation and two-mode fourth-order Burgers equation: Multiple kink solutions. *Alexandria Eng. J.* 2018, 57, 2151–2155. [CrossRef]

18. Wazwaz, A.M. Two-mode coupled Burgers equation: Multiple-kink solutions and other exact solutions. *Alexandria Eng. J.* 2018, 57, 2151–2155. [CrossRef]

19. Yassin, O.; Alquran, M. Constructing new solutions for some types of two-mode nonlinear equations. *Appl. Math. Inform. Sci.* 2018, 12, 361–367. [CrossRef]

20. Alquran, M.; Jarrah, A. Jacobi elliptic function solutions for a two-mode KdV equation. *Optik* 2018, 172, 822–825. [CrossRef]
36. Jaradat, I.; Alquran, M.; Ali, M. A numerical study on weak-dissipative two-mode perturbed Burgers’ and Ostrovsky models: Right-left moving waves. *Eur. Phys. J. Plus.* 2018, 133, 164. [CrossRef]

37. Alquran, M.; Jaradat, I.; Baleanu, D. Shapes and dynamics of dual-mode Hirota-Satsuma coupled KdV equations: Exact traveling wave solutions and analysis. *Chin. J. Phys.* 2019, 58, 49–56. [CrossRef]

38. Alquran, M.; Ali, M.; Al-Khaled, K. Solitary wave solutions to shallow water waves arising in fluid dynamics. *Nonlinear Stud.* 2012, 19, 555–562.

39. Kudryashov, N.A. One method for finding exact solutions of nonlinear differential equations. *Commun. Nonlinear Sci. Numer. Simul.* 2012, 17, 2248–2253. [CrossRef]

40. Wang, L.; Shen, W.; Meng, Y.; Chen, X. Construction of new exact solutions to time-fractional two-component evolutionary system of order 2 via different methods. *Opt. Quant. Electron.* 2018, 50, 297. [CrossRef]

41. Alquran, M.; Jaradat, H.M.; Al-Shara, S.; Awawdeh, F. A New Simplified Bilinear Method for the N-Soliton Solutions for a Generalized FmKdV Equation with Time-Dependent Variable Coefficients. *Int. J. Nonlin. Sci. Num.* 2015, 16, 259–269. [CrossRef]

42. Jaradat, H.M.; Awawdeh, F.; Al-Shara, S.; Alquran, M.; Momani, S. Controllable dynamical behaviors and the analysis of fractal Burgers hierarchy with the full effects of inhomogeneities of media. *Rom. J. Phy.* 2015, 60, 324–343.

© 2020 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).