GLOBAL DISPERSIVE ESTIMATES FOR DEFOCUSING NONLINEAR SCHröDINGER EQUATIONS

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Abstract. We consider the defocusing nonlinear Schrödinger equation with a
gauge invariant power-like nonlinearity. We prove global dispersive estimates
which make it possible to infer scattering in the conformal energy space for
all short range nonlinearities. In addition, in a semi-classical scaling, after
rescaling the solution thanks to a suitable distortion of the natural dispersion
rate, we establish a uniform bound on the modulus of the solution in some
space providing compactness properties. We discuss the consequences of these
estimates in the light of both scattering theory and semi-classical analysis.

1. Introduction

We consider the defocusing nonlinear Schrödinger equation on \( \mathbb{R}^d \),
\[
i\varepsilon \partial_t u^\varepsilon + \frac{\varepsilon^2}{2} \Delta u^\varepsilon = |u^\varepsilon|^{2\sigma} u^\varepsilon, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d,
\]
in the energy-subcritical case, \( 0 < \sigma < 2/(d-2)_+ \) (the case of sums of such
nonlinearities is discussed in Section 5). Two régimes are considered here: either
\( \varepsilon = 1 \) is fixed, or the semi-classical limit \( \varepsilon \to 0 \) is addressed. In the second case, if
we assume that the initial data \( u^\varepsilon_0 = u^\varepsilon|_{t=0} \) are such that \( u^\varepsilon_0, \varepsilon \nabla u^\varepsilon_0, x u^\varepsilon_0 \) are uniformly
bounded in \( L^2(\mathbb{R}^d) \), and \( u^\varepsilon_0 \) is uniformly bounded in \( L^{2\sigma+2}(\mathbb{R}^d) \), we show that global
dispersive estimates are available, uniformly in \( \varepsilon \in [0, 1] \). The precise statement is
given in Theorem 1.5 below. Before getting to the details, we present the context
of such a result. In order to simplify the presentation, we restrict our attention to
the case of positive times, the case of negative times being rather similar.

1.1. Large time behavior for the nonlinear Schrödinger equation. In the
case \( \varepsilon = 1 \), it is well-known that for \( \lambda \in \mathbb{R} \) and \( \sigma > 0 \), the large time behavior of the solution of the Cauchy problem
\[
i\partial_t u + \frac{1}{2} \Delta u = \lambda|u|^{2\sigma} u, \quad u|_{t=0} = u_0,
\]
depends on the sign of \( \lambda \) and the size of \( \sigma \). We summarize here some results which
can be found in e.g. [7, 12, 37, 46]. For \( \lambda < 0 \), finite time blow-up may occur if
\( \sigma \geq 2/d \), and states of the form \( u(t, x) = e^{i\omega t} \psi(x) \) exist for all \( 0 < \sigma < 2/(d-2)_+ \).
On the other hand, in the defocusing case \( \lambda > 0 \), the Cauchy problem (1.2) is
globally well-posed in \( H^1(\mathbb{R}^d) \): if \( u_0 \in H^1(\mathbb{R}^d) \), then there exists a unique solution
\( u \in L^\infty(\mathbb{R}; H^1(\mathbb{R}^d)) \cap L^{\frac{2\sigma+4}{2\sigma}}_{\text{loc}}(\mathbb{R}; L^{2\sigma+2}(\mathbb{R}^d)) \). By homogeneity, we may now assume
\( \lambda = 1 \), and consider
\[
i\partial_t u + \frac{1}{2} \Delta u = |u|^{2\sigma} u, \quad u|_{t=0} = u_0.
\]
One passes from local well-posedness to global well-posedness thanks to the following
conservations:

- Mass: \( \frac{d}{dt} \|u(t)\|_{L^2(\mathbb{R}^d)}^2 = 0 \).
In the mass-supercritical case \( \sigma > 2/d \), Morawetz estimates make it possible to show that the nonlinear solution behaves asymptotically like a linear solution in the large time limit,

\[
\exists u_+ \in H^1(\mathbb{R}^d), \quad \|u(t) - e^{i\frac{t}{2}\Delta}u_+\|_{H^1(\mathbb{R}^d)} \underset{t \to \infty}{\to} 0.
\]

See [17, 18, 19] for the case of equality). For \( 1/d < \sigma < \sigma^{*} \), the pseudo-conformal conservation law has only been exploited for values of \( \sigma \) which are not too weak. Above the Strauss exponent, \( \sigma \geq \sigma_0(d) := \frac{2 - d + \sqrt{d^2 + 12d + 4}}{4d} \), asymptotic completeness in \( H^1 \cap L^2_t \) (called conformal energy space) is known (see [14] for the case of equality). For \( 1/d < \sigma < \sigma_0(d) \), a weaker convergence is known [48],

\[
\|u(t) - e^{i\frac{t}{2}\Delta}u_+\|_{L^2(\mathbb{R}^d)} \underset{t \to \infty}{\to} 0, \quad e^{-i\frac{t}{2}\Delta}u(t) \rightharpoonup u_+ \quad \text{weakly in } H^1(\mathbb{R}^d).
\]

In the case \( \sigma > \max(1/d, 2/(d + 2)) \), and for small data, it is proven in [44] that asymptotic completeness holds in \( H^1 \cap L^2_t \). The authors in [35] show that if a suitable bound is available on the evolution of the (fractional) momentum of the nonlinear solution, then scattering holds, thus replacing a smallness assumption by an a priori information. In the case \( \sigma \leq 1/d \), such a comparison of the dynamics is impossible, due to long range effects: the above limit is possible only in the trivial case \( u \equiv 0 \equiv u_+ \) [4]. In the (scattering) critical case \( \sigma = 1/d \), it is known that the
simplified dynamics which describes the large time behavior of \( u \) involves not only \( e^{it\Delta} \), but also a nonlinear phase modification; see \[15, 20, 29, 13\]. These results require in addition stronger regularity and localization properties than above, as well as smallness assumptions.

In this paper, we show among other results that if \( u_0 \in H^1 \cap L^2_1 \), then \( u(t, \cdot) \) has the same dispersive rate as linear solutions (in \( L^{2\sigma+2} \)), regardless of the size of \( u_0 \), for any \( 0 < \sigma < 2/(d-2) + \). As a consequence, (short range) scattering in \( H^1 \cap L^2_1 \) is established on the whole range \( 1/d < \sigma < 2/(d-2) + \). The fact that this whole range can be covered, without loss of regularity, seems to be new.

**Theorem 1.1.** Let \( d \geq 1 \), and \( 1/d < \sigma < 2/(d-2) + \). For any \( u_0 \in H^1 \cap L^2_1(\mathbb{R}^d) \), there exist a unique solution \( u \in L^\infty_{loc}(\mathbb{R}; H^1 \cap L^2_1(\mathbb{R}^d)) \cap L^{4\sigma+4}_{loc}(\mathbb{R}; L^{2\sigma+2}(\mathbb{R}^d)) \) to (1.3), and a unique \( u_+ \in H^1 \cap L^2_1(\mathbb{R}^d) \) such that

\[
\| e^{-i\frac{4}{d} \Delta} u(t) - u_+ \|_{H^1 \cap L^2_1} \rightarrow 0.
\]

This result follows from classical Strichartz estimates and from the following global dispersive properties:

**Theorem 1.2.** Let \( u_0 \in H^1 \cap L^2_1(\mathbb{R}^d) \), and let

\[
\begin{align*}
    u &\in C(\mathbb{R}; H^1 \cap L^2_1(\mathbb{R}^d)) \cap L^{4\sigma+4}_{loc}(\mathbb{R}; L^{2\sigma+2}(\mathbb{R}^d)) \\
    v &\text{ be the solution to (1.3). For } t \geq 1, \text{ define } v \text{ by}
\end{align*}
\]

\[
u(t, x) = \frac{1}{|t|^{d/2}} \left( \frac{x}{|x|} \right) e^{-i \frac{|x|^2}{4|t|}}.
\]

There exists \( C \) depending on \( \| u_0 \|_{H^1 \cap L^2_1(\mathbb{R}^d)} \) such that for all \( t \geq 1,

\[
\frac{1}{|t|^{d/2}} \| \nabla v(t) \|_{L^2(\mathbb{R}^d)}^2 + \frac{1}{|t|^{d\sigma}} \| v(t) \|_{L^{2\sigma+2}(\mathbb{R}^d)}^{2\sigma+2} \leq C \frac{1}{|t|^{\min(2, d\sigma)}}.
\]

**Remark 1.3.** We will see in Remark 2.1 that the above estimates can also be inferred rather directly from the pseudo-conformal conservation law. We present a somehow more direct approach to derive these estimates.

**Remark 1.4.** The change of unknown function (1.7) is exactly the same as the one used in e.g. [20], relying on the decomposition

\[
e^{it\Delta} = M(t)D(t)FM(t), \quad M(t) = e^{it\Delta/4}, \quad D(t)\varphi(x) = \frac{1}{|t|^{d/2}} \varphi \left( \frac{x}{|x|} \right),
\]

so (1.7) also reads \( u = MDv \). However, it seems that the Hamiltonian identity invoked to prove Theorem 1.2 had not been written before. In the next subsection, we will also use a slightly modified dispersion to infer extra compactness properties.

### 1.2. Semi-classical limit

1.2.1. Linear Schrödinger equation. The semi-classical régime \( \varepsilon \rightarrow 0 \) for the linear Schrödinger equation

\[
i\varepsilon \partial_t u^\varepsilon + \frac{\varepsilon^2}{2} \Delta u^\varepsilon = 0, \quad u^\varepsilon|_{t=0} = u_0^\varepsilon,
\]

has attracted much attention, with especially three important possible scalings regarding the initial data (see e.g. [16, 22, 10]):

- WKB states: \( u_0^\varepsilon(x) = a_0(x)e^{i\phi_0(x)/\varepsilon} \), with e.g. \( a_0, \phi_0 \in \mathcal{S}(\mathbb{R}^d) \), and \( \phi_0 \)

real-valued.

- Coherent states: \( u_0^\varepsilon(x) = \frac{1}{\varepsilon^{d/2}} a_0 \left( \frac{x - q_0}{\varepsilon} \right) e^{i\phi_0(x - q_0)/\varepsilon} \), for \( (q_0, p_0) \in \mathbb{R}^d \) an initial center in phase space, and e.g. \( a_0 \in \mathcal{S}(\mathbb{R}^d) \).

- Concentrated states: \( u_0^\varepsilon(x) = \frac{1}{\varepsilon^{d/2}} a_0 \left( \frac{x}{\varepsilon} \right) \), with e.g. \( a_0 \in \mathcal{S}(\mathbb{R}^d) \).
Our main results will include WKB initial states, but not the other two above examples, so we focus the discussion in this subsection on the case of WKB analysis. In the simple case of $(1.3)$ where there is no external potential, we have an explicit representation formula of the solution, valid for all time $t \neq 0$,

$$u^\varepsilon(t, x) = \frac{1}{(2i\pi \varepsilon)^{d/2}} \int_{\mathbb{R}^d} e^{i \frac{|x-y|^2}{2\varepsilon}} u^\varepsilon_0(y) dy,$$

which yields, in the case of an initial WKB state,

$$u^\varepsilon(t, x) = \frac{1}{(2i\pi \varepsilon)^{d/2}} \int_{\mathbb{R}^d} e^{i \frac{|x-y|^2}{2\varepsilon} + i \frac{\phi(y)}{\varepsilon}} a_0(y) dy.$$

In the presence of such an oscillatory integral, a natural reflex consists in trying to apply the stationary phase formula. For a smooth, at most quadratic, phase function $\phi_0$ and typically $a_0 \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, this is always possible on an time interval $[0, T]$ for some $T > 0$ independent of $\varepsilon$, and the outcome is the same as what is obtained by WKB methods, namely

$$u^\varepsilon(t, x) \sim_{\varepsilon \to 0} \left(a(t, x) + \varepsilon a^{(1)}(t, x) + \ldots\right) e^{i \phi(t, x)/\varepsilon},$$

in the sense of asymptotic expansion (see e.g. [3] Chapter 1]). The phase $\phi$ solves a Hamilton–Jacobi equation (eikonal equation),

$$\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 = 0, \quad \phi_{t=0} = \phi_0.$$

For “many” initial phases $\phi_0$, the solution to the eikonal equation develops a singularity in finite time ($\nabla \phi$ solves the Burger’s equation). The appearance of a singularity corresponds to the presence of a caustic, which is a set in $\mathbb{R} \times \mathbb{R}_d^d$ where one can no longer apply the stationary phase formula in $(1.3)$, due to degenerate stationary points. There, WKB expansion breaks down because each function involved (not only $\phi$, but also $a, a^{(1)}, \ldots$) becomes singular. See e.g. [20] for a very precise description of this phenomenon. There are at least two ways to understand the notion of caustic. Geometrically, this corresponds to the set where the rays of geometric optics (or classical trajectories, in the context of Schrödinger equation) form an envelop (think of the characteristics in Burger’s equation for the present case). Analytically, this corresponds to an amplification of the wave function on this set (the term caustic comes from the Greek kaustikos, after the verb katein=to burn). Typically, if we consider

$$\phi_0(x) = -\frac{|x|^2}{2},$$

then for $t < 1$,

$$\phi_0(t, x) = \frac{|x|^2}{2(t-1)}, \quad a(t, x) = \frac{1}{(1-t)^{d/2}} a_0 \left( \frac{x}{1-t} \right),$$

and

$$u^\varepsilon(1, x) = \frac{1}{(2i\varepsilon)^{d/2}} e^{i \frac{|x|^2}{2\varepsilon}} \int_{\mathbb{R}^d} e^{-i \frac{\dot{\eta}}{\varepsilon} x} a_0(y) dy = \frac{1}{\varepsilon^{d/2}} \hat{a}_0 \left( \frac{x}{\varepsilon} \right) e^{i \frac{|x|^2}{\varepsilon}}.$$

We readily compute, for $\sigma > 0$,

$$\|u^\varepsilon(1)\|_{L^{2\sigma+2}(\mathbb{R}^d)} = e^{-\frac{d\varepsilon}{2\sigma}} \|\hat{a}_0\|_{L^{2\sigma+2}(\mathbb{R}^d)},$$

whereas of course $\|u^\varepsilon(0)\|_{L^{2\sigma+2}(\mathbb{R}^d)} = \|a_0\|_{L^{2\sigma+2}(\mathbb{R}^d)} = O(1)$. The refined $L^p$ estimates for oscillatory integrals performed in [34] show that this amplification of Lebesgue norms is generic, and that the geometrical considerations from [20] may not suffice to grasp the right order of magnitude of these norms.
1.2.2. Semi-classical limit in the nonlinear Schrödinger equation. In the case of (1.1) with WKB initial data like above, the phenomenon of caustic is different. On the one hand, WKB expansions can be justified on some time interval \([0,T]\) with \(T > 0\) independent of \(\varepsilon > 0\) (see [27] [1] [15]), on which the solution to the limiting system remains smooth. Unlike in the linear case, phase and amplitude cannot be decoupled, and the limiting system involves the compressible Euler equation

\[
\begin{aligned}
\partial_t \rho + \text{div}(\rho v) &= 0, \\
\partial_t u + u \cdot \nabla u + \nabla \rho^\sigma &= 0.
\end{aligned}
\]

Like in the case of Burger’s equation, one must expect the appearance of singularities in the solution: for instance, if the data \(a_0, \phi_0 \in C_c^\infty\) (compactly supported), then regardless of their size, singularities appear in finite time; see [12]. However, we readily observe that the amplification of Lebesgue norms of the solution to (1.1) cannot be similar to the caustic phenomenon described above. Indeed, the scaling \((t,x) \mapsto (\varepsilon t, \varepsilon x)\) shows that the conservation laws evoked in Subsection 1.1 become:

\[
\frac{d}{dt} \Vert u^\varepsilon(t) \Vert^2_{L^2(\mathbb{R}^d)} = 0,
\]

\[
\frac{d}{dt} \left( \frac{\varepsilon^2}{2} \Vert \nabla u^\varepsilon(t) \Vert^2_{L^2(\mathbb{R}^d)} + \frac{1}{\sigma + 1} \Vert u^\varepsilon(t) \Vert^2_{L^{2\sigma + 2}(\mathbb{R}^d)} \right) = 0,
\]

\[
\frac{d}{dt} \left( \frac{1}{2} \Vert (x + i\varepsilon \nabla) u^\varepsilon \Vert^2_{L^2} + \frac{\varepsilon^2}{2} \sigma + 1 \Vert u^\varepsilon \Vert^2_{L^{2\sigma + 2}} \right) = \frac{t}{\sigma + 1} (2 - d\sigma) \Vert u^\varepsilon \Vert^2_{L^{2\sigma + 2}}.
\]

In the case of WKB initial data, the conservations of the mass and energy show that \(u^\varepsilon(t, \cdot)\) is bounded in \(L^2 \cap L^{2\sigma + 2}(\mathbb{R}^d)\) uniformly in \(\varepsilon > 0\) and \(t \in \mathbb{R}\); this rules out the amplification phenomenon described above. In addition, if \(\sigma \geq 2/d\), the semi-classical pseudo-conformal conservation law yields

\[
\Vert u^\varepsilon(t) \Vert_{L^{2\sigma + 2}}^{2\sigma + 2} \lesssim \frac{1}{t^2}, \quad t \geq 1.
\]

On the other hand, recall that linear solutions satisfy, in the case \(\varepsilon = 1\),

\[
\Vert u(t) \Vert_{L^{2\sigma + 2}}^{2\sigma + 2} \lesssim \frac{1}{t^{d\sigma}}, \quad t \geq 1.
\]

So unless the nonlinearity is mass-critical, \(\sigma = 2/d\), the power of \(t\) is not as good as in the linear case, but it may already be surprising that we obtain an estimate which is uniform in \(\varepsilon\). This is due to the fact that we consider a semi-classical régime where nonlinear effects are very strong (see [8] for more precise discussions).

In this paper, we prove in particular that if the initial datum is a WKB state, or a finite sum of WKB states (a case where even a local description of the semi-classical limit is still missing), then we have, among other estimates,

\[
\Vert u^\varepsilon(t) \Vert_{L^{2\sigma + 2}}^{2\sigma + 2} \leq \frac{C}{t^{\min(2,d\sigma)}}, \quad t \geq 1,
\]

with \(C > 0\) independent of \(\varepsilon\).

To shorten the discussions, we choose not to describe the results which are available in the one-dimensional cubic case thanks to inverse scattering methods, since this particular equation is completely integrable; see e.g. [32].

1.2.3. Main result. We now state our main result regarding the semi-classical régime.

**Theorem 1.5.** Let \(d \geq 1\), \(0 < \sigma < 2/(d-2)_+\), and \(u_0^\varepsilon \in H^1 \cap L^2(\mathbb{R}^d)\). Consider the solution \(u^\varepsilon\) to (1.1) such that \(u^\varepsilon_{|t=0} = u_0^\varepsilon\).

• Suppose \(0 < \sigma \leq 2/d\). There exists a function \(\tau \in C^\infty(\mathbb{R}; \mathbb{R})\) (independent of \(\varepsilon\)) such that

\[
\hat{\tau}(t) \to 1 \quad \text{as } t \to \infty,
\]

\[
\tau(t) \sim t, \quad t \to \infty.
\]
and, if we define $v^\varepsilon$ by

$$
(1.11) \quad u^\varepsilon(t,x) = \frac{1}{\tau(t)^{d/2}} v^\varepsilon \left( t, \frac{x}{\tau(t)} \right) e^{i \frac{\tau(t)}{d} \frac{\varepsilon^2}{2} \frac{\tau(t)}{\sigma+1}},
$$

then the following estimates hold. If we set

$$
E_0^\varepsilon = \frac{\varepsilon^2}{2} \| \nabla u_0^\varepsilon \|_{L^2}^2 + \frac{d \sigma}{4} \int_{\mathbb{R}^d} |y|^2 |u_0^\varepsilon(y)|^2 dy + \frac{1}{\sigma+1} \| u_0^\varepsilon \|_{L^{2\sigma+2}}^{2\sigma+2},
$$

then for all $t \geq 0$,

$$
\frac{\varepsilon^2}{2} \| \nabla v^\varepsilon(t) \|_{L^2}^2 + \frac{d \sigma}{4} \int_{\mathbb{R}^d} |y|^2 |v^\varepsilon(t,y)|^2 dy + \frac{1}{\sigma+1} \| v^\varepsilon(t) \|_{L^{2\sigma+2}}^{2\sigma+2} \leq E_0^\varepsilon.
$$

- Suppose $2/d \leq \sigma < 2/(d-2)$. If we set

$$
E_0^\varepsilon = \frac{\varepsilon^2}{2} \| \nabla u_0^\varepsilon \|_{L^2}^2 + \frac{1}{2} \int_{\mathbb{R}^d} |y|^2 |u_0^\varepsilon(y)|^2 dy + \frac{1}{\sigma+1} \| u_0^\varepsilon \|_{L^{2\sigma+2}}^{2\sigma+2},
$$

then $v^\varepsilon$, defined by (1.11) with $\tau(t) = \sqrt{1+t^2}$, satisfies, for all $t \geq 0$,

$$
\frac{\varepsilon^2}{2} \| \nabla v^\varepsilon(t) \|_{L^2}^2 + \frac{1}{2} \int_{\mathbb{R}^d} |y|^2 |v^\varepsilon(t,y)|^2 dy + \frac{1}{(\sigma+1)\sqrt{1+t^2}} \| v^\varepsilon(t) \|_{L^{2\sigma+2}}^{2\sigma+2} \leq E_0^\varepsilon.
$$

Assume that $0 < \sigma \leq 2/d$ and that $u_0^\varepsilon$ is such that $u_0^\varepsilon, \varepsilon \nabla u_0^\varepsilon, xu_0^\varepsilon$ are bounded in $L^2(\mathbb{R}^d)$, and $u_0^\varepsilon$ is bounded in $L^{2\sigma+2}(\mathbb{R}^d)$, uniformly in $\varepsilon \in [0,1]$. Then the family $(|v^\varepsilon(t,\cdot)|)_{t,\varepsilon}$ is bounded in $(L_2^2 \cap L^{2\sigma+2})(\mathbb{R}^d)$, uniformly in $\varepsilon \in [0,1]$ and $t \in \mathbb{R}$. Typically, with the classical concentration-compactness results of Pierre-Louis Lions [39, 38] in mind, if the $L^2$-norm of $u_0^\varepsilon$ is independent of $\varepsilon$, then the boundedness of the $L^{2\sigma+2}$-norm rules out concentration, while the boundedness of the first moment of $|v^\varepsilon|$ in $L^2$ rules out dispersion, as well as escape to infinity. This may also be seen from the inequality

$$
norm{f}_{L^p(\mathbb{R}^d)} \lesssim \norm{f^{1-d/(p-1/2)} \norm{xf}_{L^2(\mathbb{R}^d)}^{d/(p-1/2)}}_{L^2(\mathbb{R}^d)}, \quad \mathrm{max} \left(1, \frac{2d}{d+2} \right) \leq p \leq 2.
$$

Note that in this case, the assumption $u_0^\varepsilon \in H^1(\mathbb{R}^d)$ implies $u_0^\varepsilon \in L^{2\sigma+2}(\mathbb{R}^d)$ thanks to Sobolev embedding. However the uniform boundedness of $u_0^\varepsilon$ and $\varepsilon \nabla u_0^\varepsilon$ in $L^2(\mathbb{R}^d)$ implies

$$
\| u_0^\varepsilon \|_{L^{2\sigma+2}(\mathbb{R}^d)} \lesssim \varepsilon^{-\frac{d}{2\sigma+2}},
$$

an estimate which is sharp in the case of concentrating initial data,

$$
u_0^\varepsilon(x) = \frac{1}{\varepsilon^{d/2}} u_0 \left( \frac{x}{\varepsilon} \right).
$$

Therefore, the uniform boundedness of $u_0^\varepsilon$ in $L^{2\sigma+2}(\mathbb{R}^d)$ is crucial for the conclusion of Theorem 1.5 to hold.

The arguments in the proof of Theorem 1.5 are inspired by the approach followed in [10] for the logarithmic Schrödinger equation, and then generalized to the semi-classical régime with in addition more precise estimates in [24].

The rest of this paper is organized as follows. In Section 2 we prove Theorem 1.5. In Section 3 we investigate some consequences of Theorem 1.5 in terms of Strichartz estimates and scattering theory, and establish Theorem 1.4. Section 4 addresses more specifically the semi-classical limit. Finally, some possible extensions are discussed in Section 5.

2. Proof of dispersive estimates

In this section, we prove Theorems 1.2 and 1.6.
2.1. A general computation. We begin by resuming the change of unknown function \((1.11)\), since it generalizes \((1.3)\). Set
\[ u^\varepsilon(t, x) = \frac{1}{\tau(t)^{d/2}} v^\varepsilon(t, \frac{x}{\tau(t)}) e^{i\tau(t) \frac{\xi_0^2}{2}}, \]
where for simplicity we require \(\tau(0) = 1\) and \(\dot{\tau}(0) = 0\), so that \(v^\varepsilon|_{t=0} = u_0^\varepsilon\). For a given function \(\tau\) (still to be defined), the above formula obviously defines \(v^\varepsilon\).

We will need the function \(v^\varepsilon\) to have an \(H^1\)-regularity: in view of the quadratic oscillation, the \(H^1\) regularity of \(v^\varepsilon\) is ensured by the property \(u^\varepsilon \in L_2^\infty(\mathbb{R}; H^1 \cap L_2^2)\), which in turn follows from classical results provided that \(u_0^\varepsilon \in H^1 \cap L_2^2\). In this case, we also have
\[ v^\varepsilon \in L_2^\infty(\mathbb{R}; H^1 \cap L_2^2(\mathbb{R}^d)) \cap L_2^{4+\varepsilon}(\mathbb{R}; L_2^{2\varepsilon+2}(\mathbb{R}^d)). \]

Direct computations show that \(\varepsilon = v^\varepsilon(t, y)\) solves the non-autonomous nonlinear Schrödinger equation
\[ i\varepsilon \partial_t v^\varepsilon + \frac{\varepsilon^2}{2\tau(t)^2} \Delta v^\varepsilon = \tau(t) \frac{\dot{\tau}(t)}{2} v^\varepsilon + \frac{1}{\tau(t)^{d\sigma}} |v^\varepsilon|^{2\sigma} v^\varepsilon, \quad \varepsilon|_{t=0} = u_0^\varepsilon. \]

This is where different choices for the function \(\tau\) appear to be interesting.

2.2. Standard dispersion: proof of Theorem 1.2. For the standard dispersion, \(\tau\) is affine in \(t\). We may either set \(\tau(t) = t + 1\) to meet the above requirements concerning the initial data for \(\tau\), or set \(\tau(t) = t\) like in Theorem 1.2 up to a time translation consisting in considering \(t = 1\) as the new initial time \((u^\varepsilon|_{t=1}\) enjoys the same regularity as \(u_0^\varepsilon\)). We follow the second option in order to remain consistent with Theorem 1.2 and set \(\varepsilon = 1\). The function \(v\) solves
\[ i\partial_t v + \frac{1}{2\tau^2} \Delta v = \frac{1}{\tau(1)} |v|^{2\sigma} v, \quad v|_{t=1} \in H^1 \cap L_2^2(\mathbb{R}^d). \]

Even though the equation has become non-autonomous, it still enjoys a Hamiltonian structure, with a time dependent Hamiltonian,
\[ H(t, v) = \mathcal{E}(t) := \frac{1}{2} \frac{\|\nabla v(t)\|_{L_2}^2}{\tau_0(t)} + \frac{1}{(\sigma + 1) t^{d\sigma}} \|v(t)\|_{L_2^{2\sigma+2}}^{2\sigma+2}. \]

We readily compute
\[ \dot{\mathcal{E}}(t) = -\frac{2}{t} \mathcal{E}_k(t) - \frac{d\sigma}{t} \mathcal{E}_p(t), \]
an evolution law which is straightforward on a formal level, and may be justified by standard procedures (see e.g. [12]). Since both \(\mathcal{E}_k\) and \(\mathcal{E}_p\) are nonnegative, we infer
\[ \dot{\mathcal{E}}(t) \leq -\frac{\min(2, d\sigma)}{t} \mathcal{E}(t), \]
and Gronwall lemma yields, for \(t \geq 1\),
\[ \mathcal{E}(t) \leq \frac{\mathcal{E}(1)}{t^{\min(2, d\sigma)}}, \]
which proves Theorem 1.2

Remark 2.1 (Link with the pseudo-conformal conservation law). In view of \((1.6)\),
\[ \|\nabla v(t)\|_{L_2(\mathbb{R}^d)} = \|(x + it\nabla) u(t)\|_{L_2(\mathbb{R}^d)}, \]
and \((2.2)\) appears as a rewriting of the pseudo-conformal conservation law \((1.5)\), as
\[ \frac{d}{dt} \left( \frac{1}{2t^2} \|(x + it\nabla) u\|_{L_2}^2 + \frac{1}{\sigma + 1} \|u\|_{L_2^{2\sigma+2}}^{2\sigma+2} \right) = -\frac{1}{t^3} \|(x + it\nabla) u\|_{L_2}^2 - \frac{d\sigma}{t(\sigma + 1)} \|u\|_{L_2^{2\sigma+2}}^{2\sigma+2}. \]
We believe that our approach is a little bit more flexible: in the next subsection, we use distorted dispersions, providing new compactness properties, and in Section 5 we show how to extend Theorem 1.2 to the case of finitely many power-like nonlinearities.

2.3. Distorted dispersion: proof of Theorem 1.5. With the semi-classical limit in mind, it is important to obtain (uniform) compactness properties for \( v^\varepsilon \).

The presence of a (time dependent) harmonic potential will be crucial to show that \( v^\varepsilon \) is not dispersive. To do so, we modulate the function \( \tau \).

More precisely, we first impose that the harmonic potential has the same time dependent coefficient as the nonlinearity (up to some constant that will turn out to be irrelevant),

\[
\tau \ddot{\tau} = \frac{c_0}{\sigma} \quad \tau(0) = 1, \quad \dot{\tau}(0) = 0.
\]

The most natural operation consists in regrouping the powers of \( \tau \) together, multiplying the outcome by \( \dot{\tau} \), and integrating between 0 and \( t \). This yields, in view of our initial data,

\[
\dot{\tau}(t)^2 = 2c_0 \frac{d\sigma}{d\tau} \left( 1 - \frac{1}{\tau(t)\sigma} \right).
\]

We now choose

\[
c_0 = \frac{d\sigma}{2},
\]

so the forthcoming formulas are lighter. We have more generally:

**Lemma 2.2.** Let \( \alpha > 0 \). The ordinary differential equation

\[
\ddot{\tau} = \frac{\alpha}{2\tau^{1+\alpha}} \quad \tau(0) = 1, \quad \dot{\tau}(0) = 0,
\]

has a unique, global, smooth solution \( \tau \in C^\infty(\mathbb{R}; \mathbb{R}_+) \). In addition, its large time behavior is given by

\[
\dot{\tau}(t) \to 1 \quad \text{as} \quad t \to \infty, \quad \text{hence} \quad \tau(t) \sim t.
\]

**Proof.** The local existence stems directly from the Cauchy-Lipschitz theorem. The only possible obstruction to the global propagation of regularity is the cancellation of \( \tau \), which is impossible in view of the above relation, now becoming

\[
\dot{\tau}(t)^2 = 1 - \frac{1}{\tau(t)^\alpha},
\]

which implies \( \tau(t) \geq 1 \). Therefore, \( \tau \in C^\infty(\mathbb{R}; \mathbb{R}_+) \). The equation shows that \( \tau \) is strictly (but not uniformly) convex. If it was bounded, \( \tau(t) \leq M \), then we would have

\[
\dot{\tau}(t)^2 \geq 1 - \frac{\alpha}{2M^{1+\alpha}} > 0,
\]

hence a contradiction after two integrations. Therefore, \( \tau(t_n) \to \infty \) for some sequence \( t_n \to \infty \), and since \( \tau \) is convex, \( \tau(t) \to \infty \) as \( t \to \infty \). Hence \( \dot{\tau}(t)^2 \to 1 \), and since \( \tau \) is necessarily increasing for \( t \geq 0 \), \( \dot{\tau}(t) \to 1 \), and the comparison of diverging integrals yields \( \tau(t) \sim t \).

**Remark 2.3.** The case \( \alpha = 0 \) is considered in [10], and leads to an extra logarithmic growth for \( \dot{\tau} \) and \( \tau \), showing that solutions to the logarithmic Schrödinger equation disperse faster than solutions to nonlinear Schrödinger equations with power-like, gauge invariant, nonlinearity.
With our latest choice for \( \tau \), the equation satisfied by \( v^\varepsilon \) reads

\[
\begin{align*}
    i\varepsilon \partial_t v^\varepsilon + \frac{\varepsilon^2}{2\tau(t)^2} \Delta v^\varepsilon &= \frac{d\sigma}{2\tau(t)^d\sigma} |y|^2 v^\varepsilon + \frac{1}{\tau(t)^d\sigma} |v^\varepsilon|^{2\sigma} v^\varepsilon, \quad v^\varepsilon_{|t=0} = u_0^\varepsilon.
\end{align*}
\]

Obviously, the \( L^2 \)-norm of \( v^\varepsilon \) is independent of time (and it is the same as that of \( u^\varepsilon \)). Like before, this equation is non-autonomous, and has a dissipative structure:

**Lemma 2.4.** Consider \( \tau \) given by

\[
\tau = \frac{d\sigma}{2\tau_1+d\sigma}, \quad \tau(0) = 1, \quad \dot{\tau}(0) = 0.
\]

Define the kinetic pseudo-energy

\[
E^\varepsilon_K(t) = \frac{\varepsilon^2}{2\tau(t)^2} \| \nabla v^\varepsilon(t) \|_{L^2(\mathbb{R}^d)}^2,
\]

and the potential pseudo-energy

\[
E^\varepsilon_p(t) = \frac{d\sigma}{4\tau(t)^d\sigma} \int_{\mathbb{R}^d} |y|^2 |v^\varepsilon(t, y)|^2 dy + \frac{1}{(\sigma+1)\tau(t)^d\sigma} \int_{\mathbb{R}^d} |v^\varepsilon(t, y)|^{2\sigma+2} dy.
\]

Then the total pseudo-energy \( E^\varepsilon = E^\varepsilon_K + E^\varepsilon_p \) satisfies

\[
\dot{E}^\varepsilon(t) = -\frac{\dot{\tau}(t)}{\tau(t)} \left( 2E^\varepsilon_K(t) + d\sigma E^\varepsilon_p(t) \right).
\]

Like in the previous subsection, this lemma is readily established by direct calculations and standard justification. For the computations, note that we may distinguish terms involving the derivative of the time-dependent factors in front of the integrals, leading to the terms

\[-\frac{\dot{\tau}(t)}{\tau(t)} \left( 2E^\varepsilon_K(t) + d\sigma E^\varepsilon_p(t) \right),\]

and terms involving differentiation under the integral, whose total contribution is zero again due to the Hamiltonian structure of the equation.

Gronwall lemma readily implies

\[
E^\varepsilon(t) \leq \frac{E^\varepsilon(0)}{\tau(t)^{\min(2,d\sigma)}},
\]

Therefore, if \( d\sigma \leq 2 \),

\[
E^\varepsilon(t) \leq \frac{E^\varepsilon(0)}{\tau(t)^{d\sigma}},
\]

which is the estimate stated in the first case of Theorem 1.5.

For the second case of Theorem 1.5, \( \sigma \geq 2/d \), we modify the definition of \( \tau \), and impose \( \alpha = 2 \) inLemma 2.2. We find that \( \tau \) solution to

\[
\dot{\tau} = \frac{1}{\tau}, \quad \tau(0) = 1, \quad \dot{\tau}(0) = 0,
\]

is given explicitly by \( \tau(t) = \sqrt{1+t^2} \). This dispersion is rather familiar, since it appears when computing the evolution of \( e^{-ix^2/2} \) under the linear Schrödinger flow. Lemma 2.4 becomes:

**Lemma 2.5.** Consider \( \tau(t) = \sqrt{1+t^2} \). Define the kinetic pseudo-energy

\[
E^\varepsilon_K(t) = \frac{\varepsilon^2}{2\tau(t)^2} \| \nabla v^\varepsilon(t) \|_{L^2(\mathbb{R}^d)}^2,
\]

and the potential pseudo-energies

\[
E^\varepsilon_{p1}(t) = \frac{1}{2\tau(t)^2} \int_{\mathbb{R}^d} |y|^2 |v^\varepsilon(t, y)|^2 dy, \quad E^\varepsilon_{p2}(t) = \frac{1}{(\sigma+1)\tau(t)^d\sigma} \int_{\mathbb{R}^d} |v^\varepsilon(t, y)|^{2\sigma+2} dy.
\]
Then the total pseudo-energy $\mathcal{E}^\varepsilon = \mathcal{E}_k^\varepsilon + \mathcal{E}_{p1}^\varepsilon + \mathcal{E}_{p2}^\varepsilon$ satisfies
\[
\dot{\mathcal{E}}^\varepsilon(t) = -\frac{\dot{\tau}(t)}{\tau(t)} \left( 2\mathcal{E}_k^\varepsilon(t) + 2\mathcal{E}_{p1}^\varepsilon(t) + d\sigma \mathcal{E}_{p2}^\varepsilon(t) \right).
\]

Since now $\min(2,d\sigma) = 2$, Gronwall lemma yields the second part of Theorem 1.3.

3. Strichartz estimates and scattering

3.1. Strichartz estimates. We recall the standard Strichartz estimates related to the Schrödinger equation in a semi-classical scaling (see e.g. [6]). Duhamel’s formula associated to the equation
\[
\varepsilon \partial_t u^\varepsilon + \frac{\varepsilon^2}{2} \Delta u^\varepsilon = F, \quad u^\varepsilon_{|t=0} = u_0^\varepsilon,
\]
reads
\[
u^\varepsilon(t) = e^{i\varepsilon \frac{\varepsilon^2}{2} \Delta} u_0^\varepsilon - \varepsilon \int_0^t e^{i\varepsilon \frac{\varepsilon^2}{2} \Delta} F(s) ds.
\]
To simply the discussion, we leave out endpoint Strichartz estimates.

Definition 3.1. Let $d \geq 1$. A pair $(q,r)$ is admissible if $2 \leq r < \frac{2}{d-2} + 2$ (2 ≤ $r \leq \infty$ if $d = 1$), and
\[
\frac{2}{q} = \delta(r) := d \left( \frac{1}{2} - \frac{1}{r} \right).
\]

Lemma 3.2. Denote $S_0^\varepsilon(t) = e^{i\varepsilon \frac{\varepsilon^2}{2} \Delta}.$

- Homogeneous Strichartz estimate. For any admissible pair $(q,r)$, there exists $C_q$ independent of $\varepsilon$ such that
\[
\varepsilon^{1/q} \| S_0^\varepsilon \varphi \|_{L^q(\mathbb{R};L^r)} \leq C_q \| \varphi \|_{L^2}, \quad \forall \varphi \in L^2(\mathbb{R}^d).
\]

- Inhomogeneous Strichartz estimate. For a time interval $I$, denote
\[
D_I^\varepsilon(F)(t,x) = \frac{1}{\varepsilon} \int_{I \cap (\tau \leq t)} S_0^\varepsilon(t - \tau) F(\tau,x) d\tau.
\]
For all admissible pairs $(q_1, r_1)$ and $(q_2, r_2)$, and any interval $I$, there exists $C = C_{q_1,q_2}$ independent of $\varepsilon$ and $I$ such that
\[
\varepsilon^{1+1/q_1+1/q_2} \| D_I^\varepsilon(F) \|_{L^{q_1}(I;L^{r_1})} \leq C \| F \|_{L^{q_2}_t(L^r(I;L^{r_2}))},
\]
for all $F \in L^{q_2}(I;L^{r_2}(\mathbb{R}^d)).$

Strichartz estimates will be used in the proof of Theorem 1.1 in the next subsection. Before that, they will serve as a comparison with the mixed space-time estimates stemming from Theorem 1.3. To simplify the presentation, we present $L_1^t L_2^x$ estimates for $2 \leq r \leq 2\sigma + 2$ only, by interpolation between the estimates of the $L_2^x$ and the $L_{2\sigma+2}^{2\sigma+2}$ norms for $u^\varepsilon$. Interpolation between the $L_x^{2\sigma+2}$ and the $H^1$ norms of $u^\varepsilon$ would yield estimates for $2\sigma + 2 \leq r < 2/(d-2) +$, but they are less concise to write in the case $\sigma < 2/d$, and we will not need them here.
3.1.1. Case $0 < \sigma \leq 2/d$. The first case of Theorem 1.5 implies
\[ \|u^\varepsilon(t)\|_{L^{2\sigma+2}} \leq (\sigma + 1) \frac{\mathcal{E}_0}{\tau(t)^{(2\sigma+2)/d}}, \]
where the notation $\delta(r)$ was introduced in Definition 2.1. Let $2 \leq r \leq 2\sigma + 2$: by interpolation, and without optimizing the dependence in $\varepsilon$ concerning the initial data,
\[ \|u^\varepsilon(t)\|_{L^r} \leq \frac{C(\|u_0^\varepsilon\|_{L^2}, \mathcal{E}_0)}{\tau(t)^{(r-2)/2}}. \]
In view of the properties of $\tau$, $t \mapsto \frac{1}{\tau(t)^{1/d(r)}} \in L^p(0, \infty)$ if an only if $p > 1/\delta(r)$.

Typically, if $(q, r)$ is admissible, $q = 2/\delta(r)$, and so
\[ \|u^\varepsilon\|_{L^qL^r_x} \leq C(\|u_0^\varepsilon\|_{L^2}, \mathcal{E}_0), \]

but obviously, the same estimate holds with $q$ replaced by $p$ for all $p > 1/\delta(r)$. This information is crucial to prove Theorem 1.1 all the way down to $\sigma > 1/d$. There are at least two surprising aspects in this estimate:

- A better integrability in time than what can be expected by using only Strichartz estimates. Such decay in time can classically be obtained by the Gagliardo–Nirenberg estimate adapted to the vector-fields $x + i\varepsilon t \nabla$,
\[ \|f\|_{L^r} \leq \frac{C}{(\varepsilon t)^{\delta(r)}} \|f\|_{L^2}^{1-\delta(r)} \|(x + i\varepsilon t \nabla)f\|_{L^2}^{\delta(r)}, \]
and the whole point is to show that $(x + i\varepsilon t \nabla)u^\varepsilon$ remains bounded in $L^2$, an information difficult to obtain as $\sigma$ decreases to zero.

- A remarkable dependence upon $\varepsilon$: in the case of WKB initial data for instance, $u_0 = a_0 e^{i\phi_0/\varepsilon}$, with $a_0 \in C^\infty_c(\mathbb{R}^d)$ and $\phi_0 \in C^\infty(\mathbb{R}^d; \mathbb{R})$, $\|u_0^\varepsilon\|_{L^2}$ and $\mathcal{E}_0$ are bounded uniformly in $\varepsilon$, so the above $L^pL^r$ estimate involve constants uniform in $\varepsilon$, an aspect which does not follow from Strichartz estimates as stated in Lemma 4.2 nor from the use of the vector-field $x + i\varepsilon t \nabla$ evoked above. This is due to very strong nonlinear effects (see also the discussion about the notion of caustic).

3.1.2. Case $2/d \leq \sigma < 2/(d-2)_+$. In this case, Theorem 1.5 implies
\[ \|u^\varepsilon(t)\|_{L^{2\sigma+2}} \leq (\sigma + 1) \frac{\mathcal{E}_0}{\tau(t)^{(2\sigma+2)/d}}. \]
By interpolation, for $2 \leq r \leq 2\sigma + 2$,
\[ \|u^\varepsilon(t)\|_{L^r} \leq \frac{C(\|u_0^\varepsilon\|_{L^2}, \mathcal{E}_0)}{\tau(t)^{(r-2)/2}}. \]
We infer $u^\varepsilon \in L^p_t L^r_x$ provided that
\[ 2p > \delta(r)/d, \]
and note that admissible pairs are allowed, $p = q = 2/\delta(r)$, only if $d \leq 3$.

We note that we can have a better dispersive estimate in terms of time decay, and for a larger range of Lebesgue exponents, but this involves a singular factor in terms of $\varepsilon$. Indeed, Theorem 1.5 implies
\[ \|\varepsilon \nabla u^\varepsilon(t)\|_{L^2} \leq \mathcal{E}_0, \]
hence by Gagliardo-Nirenberg inequality,
\[ \|u^\varepsilon(t)\|_{L^r} \lesssim \varepsilon^{-\delta(r)} \|u_0^\varepsilon\|_{L^2}^{1-\delta(r)} \left(\frac{\mathcal{E}_0}{\varepsilon}\right)^{\delta(r)/2}, \quad 2 \leq r < \frac{2d}{(d-2)_+}. \]
Back to $u^\varepsilon$ in view of (1.11), this yields
\[
\|u^\varepsilon(t)\|_{L^r} \lesssim \frac{1}{(\varepsilon t)^{\delta(r)}} \|u_0^\varepsilon\|_{L^2}^{1-\delta(r)} (\varepsilon t)^{\delta(r)/2},
\]
$2 \leq r < \frac{2d}{d-2}.$

For $\varepsilon = 1$, this is exactly the same estimate as in the case $\sigma \leq 2/d$, and corresponds to the dispersive estimate provided by the pseudo-conformal conservation law, as discussed in Subsection 1.1 in agreement with Remark 2.1. This will be used in the next subsection.

3.2. Scattering in the conformal energy space: proof of Theorem 1.1

The idea is to resume the same estimates as those used to prove local existence in (1.3) via Strichartz estimates, but now using the time decay provided by Theorem 1.2.

To lighten the presentation, we write the proof without $\varepsilon$ like in Theorem 1.1 but the result is valid also for (1.1) for any fixed $\varepsilon > 0$, and the underlying dependence upon $\varepsilon$ will be analyzed in Section 4.

For $t_0 \geq 0$, we write Duhamel’s formula related to (1.3) between times $t_0$ and $t \geq t_0$,
\[
\begin{aligned}
   u(t) &= e^{i \frac{\varepsilon t - t_0}{\varepsilon} \Delta} u(t_0) - i \int_{t_0}^{t} e^{i \frac{\varepsilon s - t_0}{\varepsilon} \Delta} \left( |u|^{2\sigma} u \right)(s) ds.
\end{aligned}
\]

Let
\[
A \in \{\text{Id, } \nabla, x + it\nabla\}.
\]

Such an $A$ commutes with the Schrödinger operator, and acts on gauge invariant nonlinearities “like a derivative”, in the sense that the pointwise estimate
\[
|A(|u|^{2\sigma} u)| \leq C |u|^{2\sigma} |Au|
\]
holds for some constant $C$ independent of $t$ and $u$. For $A \in \{\text{Id, } \nabla\}$, this is obvious.

In the case $A = x + it\nabla$, this follows from the factorization (1.6). Applying the operator $A$ to the equation, and using Strichartz estimates with $r_j = 2\sigma + 2$ yields
\[
\|Au\|_{L^r_t L^2} + \|Au\|_{L^{r_j}_t L^{2\sigma+2}} \lesssim \|Au(t_0)\|_{L^2} + \|u|^{2\sigma}Au\|_{L^{r_j}_t L^r},
\]
where we have denoted
\[
\|f\|_{L^r_t L^r} = \|f\|_{L^{r_q(t_0,t;L^r(\mathbb{R}^d))}},
\]
and where $q$ is such that $(q, 2\sigma + 2) = (q, r)$. Note that the nonlinearity is $(2\sigma + 1)$-homogeneous and that
\[
\begin{aligned}
   \frac{1}{r'} = \frac{2\sigma + 1}{r}, & \quad \text{for} \ r = 2\sigma + 2, \\
   \frac{1}{q'} = \frac{2\sigma + 1}{q}, & \quad \text{where} \ k = \frac{4\sigma(\sigma + 1)}{2 - (d - 2)\sigma}.
\end{aligned}
\]

Then Hölder inequality implies
\[
(3.1) \quad \|Au\|_{L^r_t L^2} + \|Au\|_{L^{r_j}_t L^{2\sigma+2}} \lesssim \|Au(t_0)\|_{L^2} + \|u|^{2\sigma}Au\|_{L^{r_j}_t L^r}.
\]

The main point to notice is stated in the following lemma:

Lemma 3.3. Let $d \geq 1$, $1/d < \sigma < 2/(d - 2)_+$, and $u_0 \in H^1 \cap L^2_t(\mathbb{R}^d)$. Then the solution $u$ to (1.3) satisfies
\[
u \in L^k(\mathbb{R}; L^{2\sigma+2}(\mathbb{R}^d)), \quad \text{where} \ k = \frac{4\sigma(\sigma + 1)}{2 - (d - 2)\sigma}.
\]
Proof of Lemma 3.3. Since \( u \in L^\infty(\mathbb{R}; H^1(\mathbb{R}^d)) \subset L^\infty(\mathbb{R}; L^{2\sigma+2}(\mathbb{R}^d)) \), we consider large time only, and invoke Theorem 1.2. We have seen in the previous subsection that for \( t \geq 1 \) (recall that \( r = 2\sigma + 2 \)),
\[
\|u(t)\|_{L^\infty} \lesssim \frac{1}{t^{d/2}}.
\]
The conclusion of the lemma holds provided that \( k\delta(r) > 1 \), a condition which reads, since \( \sigma < 2/(d-2)_+ \),
\[
\frac{4\sigma(\sigma + 1)}{2 - (d-2)\sigma} \times \frac{d\sigma}{2\sigma + 2} > 1 \iff 2d\sigma^2 + (d-2)\sigma - 2 > 0.
\]
The discriminant of this equation is \((d-2)^2 + 8d = (d+2)^2\), so the above condition boils down to \( \sigma > 1/d \).

In view of this lemma, we can split \( \{ t \geq 0 \} \) into finitely many intervals on which \( \|u\|_{L^{\infty}L^r} \) is sufficiently small so that (3.2) yields
\[
\|Au\|_{L^\infty L^2} + \|Au\|_{L^t L^r} \lesssim \|Au(t_0)\|_{L^2},
\]
up to doubling the implicit constant at each step. Therefore, \( Au \in L^\infty(\mathbb{R}_+; L^2) \cap L^q(\mathbb{R}_+; L^q) \). Using Duhamel’s formula and Strichartz estimate again, we have, for \( t_2 > t_1 > 0 \),
\[
\|Au(t_2) - Au(t_1)\|_{L^2} \lesssim \|u\|_{L^q(t_1,t_2; L^{2\sigma+2})} \|Au\|_{L^q(t_1,t_2; L^q)} \xrightarrow{t_1,t_2 \to \infty} 0,
\]
and Theorem 1.1 follows, since \( e^{it\Delta} \) is unitary on \( H^s(\mathbb{R}^d) \), and in view of the other classical factorization
\[
x + it \nabla = e^{it\frac{1}{2}\Delta} x e^{-it\frac{1}{2}\Delta}.
\]

4. Semi-classical limit

4.1. Compactness. In this discussion, the framework that we have in mind is typically the case where the initial are a finite sum of WKB states,
\[
u_0^\varepsilon(x) = \sum_{j=1}^N a_j(x) e^{i\theta_j(x)/\varepsilon}.
\]
In the case of a single initial WKB states, a local in time description of the semi-classical limit for \( u^\varepsilon \) is available (when the nonlinearity is smooth, that is \( \sigma \) is an integer); see [5] and references therein. On the other hand, as soon as \( N \geq 2 \), even a formal description of the short time semi-classical limit of \( u^\varepsilon \) is missing. This is due to the fact that nonlinear effects are very important (this regime is supercritical as far as geometric optics is concerned, see [5]), causing complex interactions of rapid oscillations.

Proposition 4.1. Let \( d \geq 1 \), \( 0 < \sigma \leq 2/d \), and \( u_0^\varepsilon \) be such that \( u_0^\varepsilon, \varepsilon \nabla u_0^\varepsilon, xu_0^\varepsilon \) are bounded in \( L^2(\mathbb{R}^d) \), and \( u_0^\varepsilon \) is bounded in \( L^{2\sigma+2}(\mathbb{R}^d) \), uniformly in \( \varepsilon \in ]0,1[ \). Let \( u^\varepsilon \) as in Theorem 1.3.

1. For fixed \( \varepsilon > 0 \), there exists \( t_k \to \infty \) and \( v_{k+}^\varepsilon \in L^2 \cap L^{2\sigma+2}(\mathbb{R}^d) \) such that
\[
\left\| u^\varepsilon(t_k, \cdot) - v_{k+}^\varepsilon \right\|_{L^2(\mathbb{R}^d)} \xrightarrow{k \to \infty} 0.
\]

2. For fixed \( \varepsilon > 0 \), the family \( \{ |u^\varepsilon(t, \cdot)| \}_{t \geq 0} \) has only one accumulation point in \( L^2(\mathbb{R}^d) \) as \( t \to \infty \). In particular, there exists \( 0 \leq v_+^\varepsilon \in L^2 \cap L^{2\sigma+2}(\mathbb{R}^d) \) such that \( |v_{k+}^\varepsilon| = v_+^\varepsilon \) for any subsequence \( t_k \) like above, and
\[
\left\| u^\varepsilon(t, \cdot) - v_+^\varepsilon \right\|_{L^2(\mathbb{R}^d)} \xrightarrow{t \to \infty} 0.
\]
(3) The family \((v^\varepsilon_k)_{\varepsilon \in [0,1]}\) is uniformly bounded in \(L^2_1 \cap L^{2\sigma+2}(\mathbb{R}^d)\), and \(v^\varepsilon_+\) possesses at least one accumulation point in \(L^2(\mathbb{R}^d)\).

(4) For fixed \(t \geq 0\), there exists \(\varepsilon_k \to 0\) and \(v_t \in L^2_1 \cap L^{2\sigma+2}(\mathbb{R}^d)\) such that
\[
\|v^\varepsilon_k(t,\cdot) - v_t\|_{L^2(\mathbb{R}^d)} \to 0.
\]

(5) The family \((v_t)_{t \geq 0}\) is uniformly bounded in \(L^2_1 \cap L^{2\sigma+2}(\mathbb{R}^d)\), and \(|v_t|\) has only one accumulation point in \(L^2(\mathbb{R}^d)\) as \(t \to \infty\).

Proof. (1) Since \(|v^\varepsilon(t)|_{L^2} = \|u_0\|_{L^2}\) for all \(t\), there exists \(t_k \to \infty\) such that
\[
v^\varepsilon(t_k,\cdot) \to v^\varepsilon_+\quad \text{weakly in } L^2(\mathbb{R}^d)
\]
as \(k \to \infty\).

Consider the family of probability densities
\[
f^\varepsilon_t(y) = \frac{|v^\varepsilon(t,y)|^2}{\|u_0\|_{L^2(\mathbb{R}^d)}^2},
\]
with \(\varepsilon > 0\) fixed for the moment. Since \(v^\varepsilon(t,\cdot)\) is bounded in \(L^2_1 \cap L^{2\sigma+2}(\mathbb{R}^d)\), \(f^\varepsilon_t\) is bounded in \(L^1_1 \cap L^{\sigma+1}(\mathbb{R}^d)\), therefore is uniformly integrable from Dunford-Pettis Theorem, and does not lose mass in the limit: \(f^\varepsilon_{t_k} := |v^\varepsilon_+|^2/\|u_0\|_{L^2}^2\) is a probability density, and so the convergence in \(L^2\) is strong,
\[
v^\varepsilon(t_k,\cdot) \to v^\varepsilon_+\quad \text{strongly in } L^2(\mathbb{R}^d)
\]
as \(k \to \infty\).

(2) Introduce the notations
\[
\rho^\varepsilon = |v^\varepsilon|^2, \quad J^\varepsilon = \varepsilon \operatorname{Im}(\overline{v^\varepsilon} \nabla v^\varepsilon).
\]
Then \((2.3)\) implies
\[
\partial_t \rho^\varepsilon + \frac{1}{\tau(t)^2} \operatorname{div} J^\varepsilon = 0.
\]

Cauchy-Schwarz inequality yields
\[
\int_{\mathbb{R}^d} |J^\varepsilon| \leq \|v^\varepsilon\|_{L^2}\|\varepsilon \nabla v^\varepsilon\|_{L^2} = \|u_0\|_{L^2} \times \tau(t) \sqrt{\mathcal{E}^\varepsilon_k(t)},
\]
hence
\[
\|\partial_t \rho^\varepsilon\|_{W^{-1,1}(\mathbb{R}^d)} \lesssim \frac{1}{\tau(t)} \sqrt{\mathcal{E}^\varepsilon_k(t)} \leq \frac{C}{\tau(t)^{1+\frac{d}{2}}},
\]
for \(C > 0\) independent of \(t \geq 0\) and \(\varepsilon \in [0,1]\). The right hand side is then obviously integrable, so \(\partial_t \rho^\varepsilon \in L^1(\mathbb{R}^d; W^{-1,1}(\mathbb{R}^d))\), and \(|(v^\varepsilon(t,\cdot))_{t \geq 0}\) has only one accumulation point in \(L^2(\mathbb{R}^d)\).

(3) We distinguish two cases:

Either \(\liminf_{\varepsilon \to 0} \|u_0\|_{L^2(\mathbb{R}^d)} = 0\), or \(\liminf_{\varepsilon \to 0} \|u_0\|_{L^2(\mathbb{R}^d)} = \eta > 0\).

In the first case, the zero function is obviously an accumulation point. In the second case, we can resume the argument of \((1)\), with now \(t\) fixed, in the limit \(\varepsilon \to 0\), since, at least along a sequence \(\varepsilon_k \to 0\),
\[
\int_{\mathbb{R}^d} \left(\frac{|v^\varepsilon_k(y)|^4}{\|u_0\|_{L^2}^2} + |y|^2 |v^\varepsilon_k(y)|^2 \right) dy \leq \frac{1}{\eta} \sup_{t \geq 0} \int_{\mathbb{R}^d} |v^\varepsilon_k(t,y)|^4 dy
\]
\[
+ \frac{1}{\eta^2} \sup_{t \geq 0} \int_{\mathbb{R}^d} |y|^2 |v^\varepsilon_k(t,y)|^2 dy,
\]
so the family of probability densities \(|v^\varepsilon_+|^2/\|u_0\|_{L^2}^2\) is bounded in \(L^1_2 \cap L^2(\mathbb{R}^d)\), and we conclude like in \((1)\).
(4) We can resume the same arguments as above:

Either \( \liminf_{\varepsilon \to 0} \| u_0^\varepsilon \|_{L^2(\mathbb{R}^d)} = 0 \), or \( \liminf_{\varepsilon \to 0} \| u_0^\varepsilon \|_{L^2(\mathbb{R}^d)} = \eta > 0 \).

In the first case, we may choose \( v_t = 0 \). In the second case, we proceed as in (1) and (3).

(5) is proved like (2). \qed

This result calls for several comments.

In (2) and (5), no extraction of subsequence is needed only if the modulus of \( v \) is considered. At least in the case \( \sigma \leq 1/d \), which involves long range scattering theory, this is certainly sharp. Indeed, in the case \( d = 1 = \sigma \) for instance, we have, with \( \varepsilon = 1 \) since \( \varepsilon \) plays no role at this stage,

\[
u(t, x) \sim \frac{1}{\sqrt{t}} \hat{u}_+ \left( \frac{x}{t} \right) e^{i \varepsilon J \frac{1}{2} t (\hat{u}_+ (x))^2 \log t},
\]

(see e.g. [30]), so in view of (1.7),

\[
v(t, y) \sim \hat{u}_+ (y) e^{i \varepsilon J (\hat{u}_+ (y))^2 \log t},
\]

and the logarithmic oscillation in time explains the role of subsequences in the case of \( v \), and the fact that they are not needed in the case of \( |v| \). Note in passing that in the situation \( 0 < \sigma < 1/d \), such an information seems to be new. This will be discussed more precisely in Section 5. In the short range case \( \sigma > 1/d \), (1.3) suggests that \( v_+^\varepsilon = \hat{a}^\varepsilon_+ \), where \( \hat{a}^\varepsilon_+ \) is the asymptotic state for fixed \( \varepsilon > 0 \) with a semi-classical scaling.

Considering now the semi-classical limit \( \varepsilon \to 0 \), one has to consider in general that \( v^\varepsilon (t, \cdot) \) is rapidly oscillatory, for fixed \( t \geq 0 \). This is obvious in the case where the initial data is a finite sum of WKB states,

\[
u_0^\varepsilon (x) = v^\varepsilon (0, x) = \sum_{j=1}^{N} a_j (x) e^{i \phi_j(x)/\varepsilon},
\]

with \( N \geq 2 \). But even in the case of a single initial WKB state, \( N = 1 \), this corresponds to the observations made in the numerical simulations presented in [6]; as long as the solution to some compressible Euler equation remains smooth, we have

\[
u^\varepsilon (t, x) = A^\varepsilon (t, x) e^{i \phi^\varepsilon (t, x)/\varepsilon},
\]

with \( A^\varepsilon \) and \( \phi^\varepsilon \) bounded in some Sobolev spaces, uniformly in \( \varepsilon \), with

\[
A^\varepsilon (t, \cdot) \underset{\varepsilon \to 0}{\longrightarrow} A(t, \cdot), \quad \phi^\varepsilon (t, \cdot) \underset{\varepsilon \to 0}{\longrightarrow} \phi(t, \cdot), \text{ in } H^s(\mathbb{R}^d).
\]

On the other hand, after the appearance of singularities in the above mentioned Euler equation (in finite time), the function \( A^\varepsilon \) has become rapidly oscillatory (in terms of \( \varepsilon \)), which means that the situation is probably at least as involved as in the case \( N \geq 2 \) as far as \( v^\varepsilon \) is concerned. Still, in the case \( N = 1 \), \( \| u_0^\varepsilon \|_{L^2} = \| a_1 \|_{L^2} \) is independent of \( \varepsilon \), and we know that all the accumulation points of \( (v^\varepsilon (t, \cdot))_{t \in [0,1]} \) in \( L^2 \) have the same \( L^2 \)-norm.

Proposition 4.1 does not cover the mass-supercritical case \( \sigma > 2/d \). Theorem 1.5 does not make it possible to claim that \( \| v^\varepsilon (t, \cdot) \|_{L^{2\sigma+2}} \) is uniformly bounded in time and \( \varepsilon \). On the other hand, we know that \( v^\varepsilon (t, \cdot) \) is uniformly bounded in \( H^1_t \cap L^2_x \), with the definition \( \| \psi^\varepsilon \|_{H^1} = \| \psi^\varepsilon \|_{L^2} + \| \varepsilon \nabla \psi^\varepsilon \|_{L^2} \). To mimic the arguments of the
proof of Proposition [4.1], it would be enough to prove that \( |v^\varepsilon(t, \cdot)|^2 \) is uniformly bounded in \( L \log L \). The logarithmic Sobolev inequality yields (see e.g. [36])
\[
\int_{\mathbb{R}^d} |v^\varepsilon(t, y)|^2 \log |v^\varepsilon(t, y)|^2 dy \leq \frac{1}{\pi} \| \varepsilon \nabla v^\varepsilon(t) \|^2_{L^2(\mathbb{R}^d)} + \left( \log \|v^\varepsilon(t)\|^2_{L^2(\mathbb{R}^d)} - (1 + \log \varepsilon) \right) \|v^\varepsilon(t)\|^2_{L^2(\mathbb{R}^d)},
\]
and the uniform boundedness in \( L \log L \) is missed by a \( \log 1/\varepsilon \) factor (unless of course \( \|u_0^\varepsilon\|^2_{L^2} = O \left( (\log 1/\varepsilon)^{-1} \right) \)). We conclude this discussion by the remark that like in the proof of point (2), \( \rho^\varepsilon = |v^\varepsilon|^2 \) satisfies
\[
\|\partial_t \rho^\varepsilon\|_{W^{-1,1}(\mathbb{R}^d)} \lesssim \frac{1}{\tau(t)} \sqrt{E_\xi(t)} \leq \frac{C}{\tau(t)^2},
\]
where the final estimate is adapted in view of Theorem [15]. Therefore, for fixed \( \varepsilon > 0 \), there is at most one accumulation point in \( L^2(\mathbb{R}^d) \), but since \( \sigma > 2/d \), this is already known from scattering theory: the only possibly interesting new information seems to be the uniform boundedness in \( \varepsilon \).

4.2. On the notion of caustic. In this subsection, we specify the form of the initial data as
\[
u_0^\varepsilon(x) = \varepsilon^\beta a_0(x)e^{i\phi_0(x)/\varepsilon},
\]
with \( \beta \geq 0, \phi_0 \in C^\infty(\mathbb{R}^d; \mathbb{R}) \) and, say, \( a_0 \in C^\infty_c(\mathbb{R}^d) \).

The case \( \beta = 0 \) corresponds to a supercritical régime as far as WKB analysis is concerned (see e.g. [8]). Theorem [15] implies that for all \( 0 < \sigma < 2/(d - 2)_+ \),
\[
\|u^\varepsilon(t)\|_{L^2(\mathbb{R}^d)} + \frac{1}{t} \times \|xu^\varepsilon(t)\|_{L^2(\mathbb{R}^d)} + t^{(2\sigma + 2)} \|u^\varepsilon(t)\|_{L^{2\sigma+2}(\mathbb{R}^d)} \leq C,
\]
for some \( C > 0 \) independent of \( \varepsilon > 0 \) and \( t \geq 0 \). This estimate is incompatible with the usual notion of caustic, since in the linear case, the \( L^{2\sigma+2} \)-norm of the solution becomes unbounded on the caustic set as \( \varepsilon \to 0 \). The same caustic phenomenon remains in the case with an external potential,
\[
i\varepsilon \partial_t u_{\infty}^\varepsilon + \frac{\varepsilon^2}{2} \Delta u_{\infty}^\varepsilon = V(x)u_{\infty}^\varepsilon.
\]
In the linear case, the rapid oscillation are described, before the caustic, by a Hamilton-Jacobi equation, whose solution may become singular in finite time, precisely on the caustic set. In the linear case, this geometrical phenomenon coincides with the amplification of the order of magnitude of \( u^\varepsilon \) in the limit \( \varepsilon \to 0 \). In the case of \( \varepsilon = 0 \), the Hamilton-Jacobi equation is replaced (essentially) by a compressible Euler equation, whose solution may develop singularities in finite time, as recalled in the introduction. However, there is no amplification of \( u^\varepsilon \), at least in \( L^2 \cap L^{2\sigma+2} \) (and most likely in \( L^\infty \) as well, as suggested by the numerical simulations from [3] [6]). This suggests that the notion of caustic must be adapted in this case, for the geometrical phenomenon and the analytical phenomenon which coincide in the linear case no longer do.

The case \( \beta > 0 \) is also of some interest. Since the nonlinearity is homogeneous, it is equivalent to consider
\[
i\varepsilon \partial_t w^\varepsilon + \frac{\varepsilon^2}{2} \Delta w^\varepsilon = \varepsilon^\alpha |w^\varepsilon|^{2\sigma} w^\varepsilon, \quad w^\varepsilon(0, x) = a_0(x)e^{i\phi_0(x)/\varepsilon},
\]
with
\[w^\varepsilon = \varepsilon^{-\beta} u^\varepsilon, \quad \alpha = 2\sigma \beta.\]
If \( \alpha \geq 1 \), then the rapid oscillations of \( w^\varepsilon \) (and hence of \( u^\varepsilon \)) are described by the same eikonal equation as in the linear case (see e.g. [3] Chapter 1). In particular,
the solution may become singular in finite time due to the formation of a “geometric” caustic, in the standard sense. However, the amplification of the solution on the caustic may be altered, due to nonlinear effects. This can be seen directly from the conservation of the energy for $w^\varepsilon$, without invoking Theorem 1.5, whose main interest in that case concerns large times. The conservation of the Hamiltonian for $w^\varepsilon$ yields

$$\frac{d}{dt} \left( \frac{\varepsilon^2}{2} \| \nabla w^\varepsilon(t) \|^2_{L^2} + \frac{\varepsilon^\alpha}{\sigma + 1} \| w^\varepsilon(t) \|^2_{L^{2\sigma+2}} \right) = 0,$$

hence

$$\| w^\varepsilon(t) \|^2_{L^{2\sigma+2}} = \mathcal{O} \left( \varepsilon^{-\alpha} \right).$$

Therefore, whatever the geometric nature of the caustic, the amplification of the $L^{2\sigma+2}$-norm of $w^\varepsilon$ is limited by the above estimate. Suppose for instance, like in the introduction, that

$$\phi_0(x) = -\frac{|x|^2}{2},$$

hence

$$\phi(t, x) = \frac{|x|^2}{2(t-1)}.$$

We have seen that in the linear case, the $L^{2\sigma+2}$-norm of the solution is of order exactly $\varepsilon^{-\sigma}$ at the focal point. Thus, if $\alpha < d\sigma$, the amplification is decreased by nonlinear effects. The above discussion is an illustration of the heuristics presented in [31], according to which nonlinear effects may be of a different nature away from or near a caustic.

5. Some extensions

In this final section, we discuss some extensions for which we sketch the main arguments, as well as some lines for possible improvements of known results. We mostly address aspects related to scattering theory.

5.1. Several powers. Assume $\varepsilon = 1$, and suppose that in (1.3), finitely many powers of the same type are involved,

$$i\partial_t u + \frac{1}{2} \Delta u = \sum_{j=1}^N \lambda_j |u|^{2\sigma_j} u, \quad u|_{t=0} = u_0,$$

with $\lambda_j > 0$ for all $j$, and $1/d < \sigma_1 < \sigma_2 < \cdots < \sigma_N < 2/(d-2)$. For $u_0 \in H^1 \cap L^2_1(\mathbb{R}^d)$, the Cauchy problem is classically solved, essentially like in the case of a single power (see e.g. [12]). Then the conclusions of Theorem 1.1 remain valid, with $\sigma$ replaced by $\sigma_1$. To see this, resume the proof of Theorem 1.2, and set, for $t \geq 1$,

$$\mathcal{E}(t) = \frac{1}{2t^2} \| \nabla v(t) \|^2_{L^2} + \sum_{j=1}^N \frac{\lambda_j}{(\sigma_j + 1)} \| v(t) \|_{L^{2\sigma_j+2}}^{2\sigma_j+2},$$

The same argument as in Section 2 yields, for $t \geq 1$,

$$\mathcal{E}(t) \leq \frac{\mathcal{E}(1)}{\min(2, d\sigma_1)},$$

from which we infer like in Subsection 3.1,

$$\| u(t) \|_{L^{2\sigma_j+2}(\mathbb{R}^d)} \lesssim \frac{1}{t^{\sigma_1}}, \quad t \geq 1.$$

In the case $\sigma_1 \geq 2/d$, we can even replace $\sigma_1$ by $\sigma_j$ in the above estimate, following the discussion at the end of Subsection 3.1.
Resuming the details of the proof of Theorem 1.1, we see that it suffices to prove
\[ k_j \times \frac{d_2}{2\sigma_j + 2} > 1, \quad \forall j \in \{1, \ldots, N\}, \quad \text{where } k_j := \frac{4\sigma_j(\sigma_j + 1)}{2 - (d - 2)\sigma_j}. \]

For \( j = 1 \), this is exactly what we have proved in Section 3. For \( j \geq 2 \), we distinguish two cases:

If \( d \geq 2 \), the expected property stems from
\[ 2 - (d - 2)\sigma_j \leq 2 - (d - 2)\sigma_1 < 2d\sigma_1^2 < 2d\sigma_1\sigma_j, \]
where the second inequality is precisely the one that we proved in Lemma 3.3.

If \( d = 1 \), we write
\[ 2 + \sigma_j < 2\sigma_1\sigma_j \iff 2 < (2\sigma_1 - 1)\sigma_1 < (2\sigma_1 - 1)\sigma_2, \]
where \( 1 < \sigma_1 < \sigma_2 \).

Remark 5.1 (Energy-critical case). Formally, we may allow \( \sigma_N = \frac{2}{d - 2} \) when \( d \geq 3 \), provided that we know that the Cauchy problem is globally well-posed in the conformal energy space. See [47] for such results in the case \( N = 2 \).

5.2. Anisotropic scattering. In [59, 50], the Cauchy problem and the large time behavior of the solution of
\[ i\partial_t u + \frac{1}{2}\Delta_{\mathbb{R}^d} u - \frac{1}{2}H_y u = |u|^{2\sigma} u, \quad u|_{t=0} = u_0, \]
was studied in the case \( H_y = \Delta_{M^d} \), where \( M^d \) is a \( d \)-dimensional compact manifold (with more specifically \( M^d = T \) in [59]). The goal is to measure the role of dispersive effects on \( \mathbb{R}^d \) compared to the confining role of \( M^d \), as far as the large time behavior is concerned. Similarly, in [2], the short range nonlinear scattering theory is studied, with \( H_y = -\Delta_{\mathbb{R}^d} + |y|^2 \) a (confining) harmonic oscillator. In [28], the authors consider \( n = 1 \), and \( H_y = -\Delta_{\mathbb{T}^d} \), in the cubic case \( \sigma = 1 \), corresponding to long range scattering in the only dispersive direction. Mimicking the approach of Theorem 1.2 and defining \( v \) as
\[ u(t, x, y) = \frac{1}{t^{n/2}} v \left( t, \frac{x}{t}, y \right) e^{i|x|^2/2t}, \quad t \geq 1, \]
on other dispersive estimates are available, provided that the Cauchy problem is solved globally in time, with a suitable regularity for the solution. Note that the above change of unknown function corresponds exactly to the approach followed in [11] in a case similar to [28], and more precisely, [11] uses dispersive estimates obtained in the same spirit as in this paper ([11, Proposition 4]).

5.3. Wave operators. Proving the existence of wave operators consists in solving (1.3), but with a prescribed datum at infinite time, typically
\[ e^{-i\frac{1}{2}t u(t)|_{t=\infty} = u_+. \]

In general, it is easier to prove the existence of wave operators rather than asymptotic completeness. In our framework, we would like to solve
\[ i\partial_t v + \frac{1}{2t^2} \Delta v = \frac{1}{t^{n/2}} |v|^{2\sigma} v, \quad v|_{t=\infty} = \hat{u}_+, \]
on some time interval \([T, \infty[\) with \( T \geq 1 \) possibly very large (classically, the problem on arbitrary finite time intervals is solved by other techniques). Such an approach requires more precise estimates than the mere energy inequality used in this paper.
5.4. **Long range scattering.** In the study of long range scattering for (1.3), with \( \sigma = 1/d \), a crucial step consists in describing the modulus of \( u \) in the large time limit, since after that it becomes possible to reconstruct the phase modification needed to take long range effects into account. The information provided by Theorem 1.2 is of this nature, maybe not in a sufficiently convenient function space (controlling \( |u| \) in \( L^\infty \) is very helpful). Also, it seems that so far, the long range scattering has not been studied for (1.3) in the case \( 0 < \sigma < 1/d \). The lack of regularity of the nonlinearity is an important technical difficulty, which was bypassed in the analogous case of (generalized) Hartree nonlinearities, see [24, 25, 26] and references therein. Again, maybe the dispersive properties provided by Theorem 1.2 or the more complete estimate given by Theorem 1.5 (yielding moreover \( v \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^d)) \)) can help progressing on the analysis of long range effects.

5.5. **Numerical analysis.** The numerical simulation of (1.1) in the semi-classical limit \( \varepsilon \to 0 \) is a difficult challenge in general, due to rapid oscillations, see e.g. [3, 33]. A fortiori, proving error estimates in the case of (1.1) in which the dependence upon \( \varepsilon \) is not (too much) more singular than in numerical observations is a very challenging issue. In the case of a single initial WKB state, and locally in time, such error estimates were established in [11], for the time discretization known as splitting method (as a matter of fact, the proof does not even cover the whole time interval corresponding to the lifespan of smooth solutions to the Euler equations). To extend such a result, Theorem 1.5 may provide important information, inasmuch as the “nonlinear potential” \( |u|^{2\sigma} \) is controlled a little bit more precisely than what is available thanks to the standard conservation laws.

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