On three dimensional coupled bosons

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Abstract

In this work we study two complex scalar fields coupled through a quadratic interaction in 2 + 1 dimensions using the method of bilinears as suggested by Rajeev[8]. The resulting theory can be formulated as a classical theory. We study the linear approximation, and show that there is a possible bound state in a range of coupling constants.

1 Introduction

Quantum field theory has been an essential tool for the modeling of various physical phenomena. One of the major problems in field theory is the understanding of relativistic bound states. The standart way to look at field theories is via perturbation theory around the free theory, and bound state problems are difficult to formulate in this approach. The most common way is to use a Bethe-Salpeter approach for the four-point function (for two particle bound states) and find a self-consistency condition for the bound state solution. Typically this requires various approximations which may break down in the highly relativistic cases.

One of the most successful applications of this approach is within the large-$N_c$ approximation: in his classic paper, ’t Hooft obtained a bound state equation for mesons in two dimensions in the large-$N_c$ where $N_c$ refers to the color for the nonabelian $SU(N_c)$ gauge theory [1]. This leads to a singular integral equation for the possible masses of the mesonic excitations. This equation is expressed in terms of the wave function of the meson given as a function of the fractional light-cone momentum. The analysis of this integral equation in [2] shows that there are only bound states corresponding to positive eigenvalues with finite
multiplicity, and these eigenvalues tend to infinity. The scalar version of this model is worked out in [3] using the original approach of ’t Hooft and in [4] via a Hamiltonian approach to the large-$N_c$ limit. These relativistic equations behave in a very similar way to the standard ’t Hooft equation. In two dimensions we can generalize the Yang-Mills Lagrangian, since the gauge fields are not dynamical, by means of a nondynamical scalar field. The large-$N_c$ limit meson bound state equation of these models have some other interesting features [5]. In [6], Aoki has generalized these bound state equations for bosons and fermions coupled via $SU(N_c)$ gauge theory. A good presentation of many two dimensional models using the bilocal fields in the path integral formalism within the large-$N_c$ limit is given in [7]. In this article several interesting bound state equations are derived, and further references are given.

In [8] Rajeev has formulated the large-$N_c$ model as a classical field theory using color invariant bilinears, and he has shown that the phase space of the theory is the restricted Grassmannian. The knowledge of the phase space allows one to make a variational ansatz for the baryons in this theory, which correspond to the large fluctuations of the field(see [9]). Further details of this approach are given in [10]. Toprak and the author have extended this work to $SO(N_c)$ gauge theory of bosons and fermions and obtained variants of the ’t Hooft equation for these cases [11, 12]. The adjoint matter fields in the large-$N_c$ limit yields again singular integral equations for possible mesonic strings, they exhibit a very similar bound state structure as the original model, but these equations are more complicated due to the fact that mesons are now color invariant strings of operators [13, 14, 15, 16].

The two dimensional Yukawa model is analyzed within the light-cone method in [17, 18, 19]. These models are more complicated due to nonlocal renormalization effects, it is possible to get an integral equation for bound states using some further approximations. A four dimensional extension of these ideas are given in [20]. The common feature of all these bound state equations is that they are singular integral equations. In the gauge theory cases these singular integral equations are rather restrictive in that they only allow for a discrete spectrum. In the other cases this is not necessarily true. There is usually a finite number (typically one) bound state. There are investigations in three dimensional QCD for the bound state equations of mesons (see the recent article [21]). Four dimensional realistic theories are very complicated since one has to deal with renormalization. The author is not knowledgable enough about these realistic bound state equations, some information can be found in the review article [22] (see also [23] for a review of renormalization in the light-front point of view and some non-perturbative applications in this formalism).

In this article we will apply a certain kind of mean field theory, which is a large-$N_f$ limit to two coupled complex bosons (we call it flavor symmetry to emphasize that it is not gauged). This theory is simple since it does not require coupling constant and wave function renormalization in the perturbation theory. Defining the scalar field around the free field theory may not be so interesting from a physical point of view, but we regard this as an interesting toy model. The linear approximation yields a bound state for the composite of two bosons. We will apply the methods of Rajeev [10] and formulate it as a classical field theory of bilinears. In this case (unlike the gauge theory case) this is only an approximation since the theory does not have to be restricted to this flavor invariant sector. To avoid repetitions we sometimes refer to our work on complex bosons in [24]. There is an
interesting Bethe-Salpeter treatment\footnote{I am grateful to E. Langmann for pointing this out to me} of bound states in the broken phase of $\phi^4$ theory in \cite{25}, in some sense this is similar to the model we work with.

\section{The model in the light-cone and large-$N_f$ limit}

We write down a $U(N_f)$ invariant action for two complex scalars $\phi_a$ and $\phi_b$,

\begin{equation}
S = \int d^3x \left( \partial^\mu \phi_a^\dagger \partial_\mu \phi_a + \partial^\mu \phi_b^\dagger \partial_\mu \phi_b - m_a^2 \phi_a^\dagger \phi_a - m_b^2 \phi_b^\dagger \phi_b - \frac{\lambda}{4} \left[ (\phi_a^\dagger \phi_a)^2 + (\phi_b^\dagger \phi_b)^2 - 2(\phi_a^\dagger \phi_b)(\phi_b^\dagger \phi_a) \right] \right). \tag{1}
\end{equation}

We introduced a common coupling $\lambda$, for the two complex fields $\phi_{aa}$ and $\phi_{ba}$. We assume that the internal index $\alpha$ takes the values $1, ..., N_f$. If we do not have the last term which couples the two fields the action would have a $U(N_f) \times U(N_f)$ symmetry, and the interaction term explicitly breaks this. The interaction may look discomforting, but it is easy to see that it is always positive. The classical ground state of the massive theory is when both fields are set to zero. This means that we can quantize the theory around its classical minimum by introducing creation-annihilation operators for the Fourier modes, as we will see below.

Perturbatively, there is only one type of divergence after normal ordering as we will comment later on \cite{26}. To apply the methods developed by Rajeev, we will use the light-cone coordinates, introduce $x^+ = \frac{1}{\sqrt{2}}(x^0 + x^3)$ and $x^- = \frac{1}{\sqrt{2}}(x^0 - x^3)$, and $x^3$ remains as the transverse coordinate. We choose $x^+$ as time (that is our evolution variable). A good review of light-front methods is given in \cite{27}, a good discussion of the scalar field in the light-front is also given in \cite{28} and in \cite{29}. The three dimensional scalar field theory has been investigated from different points of view in the articles \cite{30, 31, 26, 32}.

We use basically the same conventions in our previous work \cite{24, 33}.

\begin{equation}
S = \int dx^+ dx^- dx^1 \left( \frac{1}{2} \phi_a^\dagger (-2\partial_-) \partial_+ \phi_a + \frac{1}{2} \phi_b^\dagger (-2\partial_-) \partial_+ \phi_b - \phi_a^\dagger (m_a^2 - \partial^2_1) \phi_a - \phi_b^\dagger (m_b^2 - \partial^2_1) \phi_b - \frac{\lambda}{4} \left[ (\phi_a^\dagger \phi_a)^2 + (\phi_b^\dagger \phi_b)^2 - 2(\phi_a^\dagger \phi_b)(\phi_b^\dagger \phi_a) \right] \right). \tag{2}
\end{equation}

The action is first order in time $x^+$, this means that we are already in the Hamiltonian formalism. The Hamiltonian can be read off directly,

\begin{equation}
H = \int dx^- dx^1 \left( \phi_a^\dagger (-\partial_1^2 + m_a^2) \phi_a + \phi_b^\dagger (-\partial_1^2 + m_b^2) \phi_b + \frac{\lambda}{4} \left[ (\phi_a^\dagger \phi_a)^2 + (\phi_b^\dagger \phi_b)^2 - 2(\phi_a^\dagger \phi_b)(\phi_b^\dagger \phi_a) \right] \right). \tag{2}
\end{equation}

The quantization at equal time following Dirac gives

\begin{equation}
[\hat{\phi}_a^\dagger(x^-, x^1), \hat{\phi}_{a\beta}(y^-, y^1)] = -i \frac{\delta^3_{\alpha\beta}}{4} \text{sgn}(x^- - y^-) \delta(x^1 - y^1), \tag{3}
\end{equation}
and the same rule applies for \( \phi_b \). We recall that the field can be expanded in terms of creation-annihilation operators at initial light-front time (since the classical minimum is the zero configuration for the fields),

\[
\hat{\phi}_{aa}(x^-, x^1) = \int \frac{dp_- dp_1}{\sqrt{2|p_-|}} a^\alpha(p_-, p_1) e^{-ip_- x^- - p_1 x^1} \\
\hat{\phi}_{ba}(x^-, x^1) = \int \frac{dp_- dp_1}{\sqrt{2|p_-|}} b_\alpha(p_-, p_1) e^{-ip_- x^- - p_1 x^1},
\]

where we use \([dp] = \frac{dp}{2\pi}\). (To properly define everything we should assume that this expansion is given for \((-\infty, -\epsilon_0] \cup [\epsilon_0, \infty)\) at the end we take \(\epsilon_0 \to 0\) limit).

The creation and annihilation operators now satisfy,

\[
[a_\alpha(p_-, p_1), a^{\alpha\dagger}(q_-, q_1)] = \text{sgn}(p_-) \delta^\alpha_\beta \delta[p_- - q_-] \delta[p_1 - q_1], \\
[b_\alpha(p_-, p_1), b^{\alpha\dagger}(q_-, q_1)] = \text{sgn}(p_-) \delta^\alpha_\beta \delta[p_- - q_-] \delta[p_1 - q_1].
\]

Since the fields are complex valued annihilation and creation operators are not related. We introduce a vacuum state \(|0\rangle >\) for the Fock space construction,

\[
a_\alpha(p_-, p_1)|0\rangle = 0 \quad \text{for} \quad p_- > 0 \quad \text{and} \quad a^{\alpha\dagger}(p_-, p_1)|0\rangle = 0 \quad \text{for} \quad p_- < 0 \\
b_\alpha(p_-, p_1)|0\rangle = 0 \quad \text{for} \quad p_- > 0 \quad \text{and} \quad b^{\alpha\dagger}(p_-, p_1)|0\rangle = 0 \quad \text{for} \quad p_- < 0.
\]

It is important to keep in mind that the operator \(a_\alpha(p_-, p_1)\) for \(p_- < 0\), creates an antiparticle of momentum \((-p_-, -p_1)\) and similarly for \(b_\alpha\) (which one can see by rewriting the above expansions in a more conventional way, by separating particle and antiparticle operators). We define normal ordering rules with respect to this vacuum as usual and denote it by a colon : \(\phi_a \ldots \phi_a :\), we will be using for the computations the following relation,

\[
a^{\alpha\dagger}(p_-, p_1) a_\beta(q_-, q_1) := a^{\alpha\dagger}(p_-, p_1) a_\beta(q_-, q_1) - \frac{\delta^\alpha_\beta}{2} (1 - \text{sgn}(p_-)) \delta[p_- - q_-] \delta[p_1 - q_1], \tag{4}
\]

and exactly the same for \(b\) quanta.

We have the following Hamiltonian in the quantized theory,

\[
\hat{H} = \int dx^1 \frac{dx^-}{4} \left( \hat{\phi}_a^\dagger (m_a^2 - \partial^2) \hat{\phi}_a : + : \hat{\phi}_b^\dagger (m_b^2 - \partial^2) \hat{\phi}_b : \\
+ \frac{\lambda}{4} [\hat{\phi}_a^\dagger \hat{\phi}_a]^2 : + : (\hat{\phi}_b^\dagger \hat{\phi}_b)^2 : - 2 : (\hat{\phi}_a^\dagger \hat{\phi}_b) (\hat{\phi}_b^\dagger \hat{\phi}_a) : \right)
\]

As it stands the Hamiltonian would not be a well-defined operator for finite \(N_f\) theory, we need to introduce mass renormalization terms which correspond in the diagramatic language the setting-sun diagrams\([30, 31, 28]\). When we take the large-\(N_f\) limit these counter terms become of smaller order, therefore the Hamiltonian as written will have a well-defined limit.

We now define as an approximation a large-\(N_f\) limit and restrict the theory to the flavor invariant sector. This is to be taken as an approximation to the full quantum theory. We introduce a set of flavor invariant operators, which are directly written in the momentum
representation, and to simplify notation we write \( p \) to denote \( p_-, p_1 \) collectively, and \( \delta[p-q] = \delta[p_- - q_-]\delta[p_1 - q_1] \),

\[
\begin{align*}
\hat{N}_a(p, q) &= \frac{2}{N_f} : a^\alpha(p_-, p_1)a_\alpha(q_-, q_1) : \\
\hat{N}_b(p, q) &= \frac{2}{N_f} : b^\alpha(p_-, p_1)b_\alpha(q_-, q_1) : \\
\hat{C}(p, q) &= \frac{2}{N_f} a^\alpha(p_-, p_1)b_\alpha(q_-, q_1) \\
\hat{\bar{C}}(p, q) &= \frac{2}{N_f} b^\alpha(p_-, p_1)a_\alpha(q_-, q_1).
\end{align*}
\]

Note that \( \hat{C} \) and \( \hat{\bar{C}} \) are just hermitian conjugates of each other.

The idea behind the papers \([8, 10]\) is that when we take the large-\( N_f \) limit the flavor invariant operators have smaller and smaller fluctuations, and if we compute their commutator, for example for \( N_a \) with itself, we get,

\[
[\hat{N}_a(p, q), \hat{N}_a(s, t)] = \frac{2}{N_f} \left( \hat{N}_a(p, s)\text{sgn}(p_-)\delta[q - r] - \hat{N}_a(r, q)\text{sgn}(p_-)\delta[p - s] \\
- (\text{sgn}(p_-) - \text{sgn}(q_-))\delta[p - s]\delta[q - r] \right).
\]

We assume that when we let \( N_f \to \infty \) there are proper large-\( N_f \) limits for these bilinears restricted to the flavor invariant states. As a result the theory becomes classical, the expectation values of flavor invariant operators factorize as \( N_f \to \infty \) \([10, 34]\). Thus we may postulate a set of Poisson brackets for these classical variables:

\[
\begin{align*}
\{N_a(p, q), N_a(s, t)\} &= -2i\left( N_a(p, s)\text{sgn}(p_-)\delta[q - r] - N_a(r, q)\text{sgn}(p_-)\delta[p - s] \\
&\quad - (\text{sgn}(p_-) - \text{sgn}(q_-))\delta[p - s]\delta[q - r] \right) \\
\{N_b(p, q), N_b(s, t)\} &= -2i\left( N_b(p, s)\text{sgn}(p_-)\delta[q - r] - N_b(r, q)\text{sgn}(p_-)\delta[p - s] \\
&\quad - (\text{sgn}(p_-) - \text{sgn}(q_-))\delta[p - s]\delta[q - r] \right) \\
\{C(p, q), \bar{C}(s, t)\} &= -2i\left( N_a(p, t)\text{sgn}(q_-)\delta[q - s] - N_b(s, q)\text{sgn}(t_-)\delta[p - t] \\
&\quad - (\text{sgn}(p_-) - \text{sgn}(q_-))\delta[p - t]\delta[q - s] \right) \\
\{N_a(p, q), C(s, t)\} &= -2i\text{sgn}(q_-)\delta[q - s]C(p, t) \\
\{N_b(p, q), C(s, t)\} &= 2i\text{sgn}(q_-)\delta[p - t]C(s, q) \\
\{N_a(p, q), \bar{C}(s, t)\} &= 2i\text{sgn}(p_-)\delta[p - t]\bar{C}(s, q) \\
\{N_b(p, q), \bar{C}(s, t)\} &= -2i\text{sgn}(q_-)\delta[q - s]\bar{C}(p, t),
\end{align*}
\]

and all the other Poisson brackets being zero.

There are constraints coming from the \( U(N_f) \) invariance, if we restrict the theory to the flavor invariant sector. They are very similar to the ones found in \([24]\),

\[
\begin{align*}
(\epsilon N_a + \epsilon)^2 + \epsilon C\epsilon C^\dagger &= 1, \\
(\epsilon N_b + \epsilon)^2 + \epsilon C^\dagger\epsilon C &= 1, \\
N_a\epsilon C + C\epsilon N_b + \epsilon C + C\epsilon &= 0,
\end{align*}
\]

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and the hermitian conjugate of the last equation. These conditions are derived using similar techniques to [33, 24]. Here we are using \( \epsilon(p, q) = \text{sgn}(p_-)\delta[p_--q_-]\delta[p_1-q_1], \) and a shorthand for the operator products, for example, \( N_a C \) means,

\[
(N_a C)(p, s) = \int [dq_-dq_1]N_a(p_-, p_1; q_-, q_1)C(q_-, q_1; s_-, s_1),
\]

and similarly for the others.

We will have further convergence conditions coming from the super-renormalizability. These should be regarded as sufficiently restrictive conditions to keep the system’s evolution in phase space. The Hamiltonian puts more stringent conditions on the admissible class of observables, we believe its domain should be dense inside the phase space. Correct normalization should be then found using the Hamiltonian as a quadratic form on the space of these variables and demanding this form to be finite for all physical states. To state our conditions we recall that the one-particle Hilbert spaces of bosons are divided into positive and negative energy subspaces will be Hilbert-Schmidt, whereas the operators acting between positive and negative energy subspaces will be Hilbert-Schmidt, whereas the operators acting between the same subspaces will be trace class [35]. We write explicitly the one for \( C: C(u_-, u_1; v_-, v_1) \) is trace class for \( u_-v_+ > 0 \), and Hilbert-Schmidt for \( u_-v_- < 0 \), and the same for the other variables. Note that these are consistent with the constraints on the system, which defines the geometry of the phase space. The constraints and the convergence conditions can be cast into a coherent geometric picture: it defines a homogeneous manifold of \( U_1((\mathcal{H}_a \oplus \mathcal{H}_b)_+, (\mathcal{H}_a \oplus \mathcal{H}_b)_-) \), but we will not use it in this work.

We can rewrite the large-\( N_f \) Hamiltonian in terms of these variables,

\[
H = \frac{1}{4} \int [dp][m_a^2 + p_1^2]N_a(p, p) + \frac{1}{4} \int [dp][m_b^2 + p_1^2]N_b(p, p) + \frac{\lambda}{64} \int [dpdqdsdt] \frac{\delta[p-q+s-t]}{\sqrt{|p-q-s-t|}} \left[ N_a(p, q)N_a(s, t) + N_b(p, q)N_b(s, t) - 2C(p, q)\bar{C}(s, t) \right]
\]

This defines our large-\( N_f \) approximation, and in principle we can calculate the equations of motion of the basic observables by,

\[
\frac{\partial O(u, v)}{\partial x^+} = \{O(u, v), H\}, \tag{7}
\]

where \( O \) refers to any one of \( N_a, N_b, C, \bar{C} \). The resulting equations are nonlinear integral equations and we also have the constraint to satisfy. It would be interesting to study this system using a variational ansatz. We will leave the analysis of the full system to a future work and look at a linearized version.

### 3 Linearization and a possible bound state

To get a better feeling about the system we can start with a linear approximation. This means we should linearize the constraint as well as the equations of motion. The linearization of the constraint gives us, for our basic variables,

\[
(1 + \text{sgn}(u_-)\text{sgn}(v_-))N_a(u_-, u_1; v_-, v_1) = 0 \quad (1 + \text{sgn}(u_-)\text{sgn}(v_-))N_b(u_-, u_1; v_-, v_1) = 0
\]
\[(\text{sgn}(u_-) + \text{sgn}(v_-))C(u_-, u_1, v_-, v_1) = 0.\]  

We will be using the last one in our computations, it says that the light-cone momenta should be opposite to each other: \(C(u_-, u_1; v_-, v_1) = 0\) if \(u_- v_- > 0\), and nonzero if \(u_- v_- < 0\). The same conditions hold for \(N_a\) and \(N_b\) as well. In the linear approximation we will search for a possible bound state of \(a\) and \(b\) particles. In principle we can compute the linearized equations for all the other variables, but they will not lead to a solution for the bound state nor a solution for a resonance: they only have scattering states. Therefore we work only with the composite \(C(u, v)\), let us choose \(u_+ > 0, v_- < 0\). The equations of motion of \(C(u, v)\) for \(u_+ > 0, v_- < 0\), in the linear approximation become,

\[
\frac{\partial C(u, v)}{\partial x^+} = \{C(u, v), H\} = \frac{i}{2} \left[ \frac{m_a^2 + u_1^2}{u_-} - \frac{m_b^2 + v_1^2}{v_-} \right] C(u, v) - i \frac{\lambda}{8\pi} \int \frac{[dpdq]}{\sqrt{|p_- q_+ u_- v_-|}} \delta[p_- q_+ + v_- u_-] \delta[p_1 q_1 + v_1 u_1] C(u, v)
\]

Let us make an ansatz, in the light-front direction we make a 't Hooft like choice with respect to the relative momentum variable \(\zeta = u_+/(u_- - v_-)\). This variable now satisfies \(0 < \zeta < 1\), and we set \(C(u_-, u_1; v_-, v_1) = \tilde{f}(\zeta; u_1, v_1) e^{iP_+ x^+}\). Notice that \(P_- = u_- - v_- > 0\), and we introduce a relativistically invariant mass variable \(\mu^2 = 2P_+ |u_- - v_-| - (u_1 + (-v_1))^2\), which will be the mass of the bound state (recall that the momentum \(v_1\) denotes an antiparticle with momentum \(-v_1\), thus \(u_1 + (-v_1)\) is the total transversal momentum of the two particle state). After some manipulations, similar to the ones in [8, 11], this gives us an eigenvalue equation for the invariant mass:

\[
\mu^2 \tilde{f}(\zeta; u_1, v_1) = \left[ \frac{m_a^2 + (u_1 - \zeta(u_1 + (-v_1)))^2}{\zeta} + \frac{m_b^2 + (v_1 - (1 - \zeta)(u_1 + (-v_1)))^2}{1 - \zeta} \right] \tilde{f}(\zeta; u_1, v_1) - \frac{\lambda}{8\pi} \int_0^1 d\eta \int_{-\infty}^\infty \frac{[dpdq]}{\sqrt{\eta(1 - \eta)\zeta(1 - \zeta)}} \delta[p_1 q_1 - (u_1 + (-v_1))] \tilde{f}(\eta; p_1, q_1, u_1, v_1)
\]

(Note that this form reduces to \(\mu = m_a + m_b\) if we set \(\lambda = 0\), and choose the function \(f\) properly). We may equivalently use a new set of variables, \(R = u_1 - \zeta(u_1 + (-v_1))\) and \(u_1 + (-v_1)\), relative transversal light-front momentum and transversal total momentum, respectively, instead of the above variables. If we write everything in terms of these new set of variables we get,

\[
\mu^2 \tilde{f}(\zeta; R, u_1 + (-v_1)) = \left[ \frac{m_a^2 + R^2}{\zeta} + \frac{m_b^2 + (\zeta R)^2}{1 - \zeta} \right] \tilde{f}(\zeta; R, u_1 + (-v_1)) - \frac{\lambda}{8\pi} \int_0^1 d\eta \int_{-\infty}^\infty \frac{[dQ][dpq]}{\sqrt{\eta(1 - \eta)\zeta(1 - \zeta)}} \delta[p_1 q_1 - (u_1 + (-v_1))] \tilde{f}(\eta; Q, p_1 + (-q_1)u_1 + (-v_1))
\]

(The Jacobian of this transformation in the integral is one). The total momentum integral can be done due to the delta function and we end up with,

\[
\mu^2 \tilde{f}(\zeta; R, u_1 + (-v_1)) = \left[ \frac{m_a^2 + R^2}{\zeta} + \frac{m_b^2 + (\zeta R)^2}{1 - \zeta} \right] \tilde{f}(\zeta; R, u_1 + (-v_1))
\]
We see that the total momentum $u_1 + (-v_1)$ is conserved, thus we can factor out the transversal center of mass motion by assuming $\tilde{f}(\zeta; u_1, v_1) = f(\zeta, R)g(u_1 + (-v_1))$:

$$\mu^2 f(\zeta, R) = \left[\frac{m_a^2 + R^2}{\zeta} + \frac{m_b^2 + R^2}{1 - \zeta}\right] f(\zeta, R) - \frac{\lambda}{16\pi^2} \int_{-\infty}^{\infty} dQ \int_{0}^{1} d\eta \frac{f(\eta, Q)}{\sqrt{\eta(1 - \eta)\zeta(1 - \zeta)}}.$$  \hspace{1cm} (9)

We can reduce this again to a functional equation for the unknown eigenvalue, by using the standard techniques,

$$\frac{\lambda}{16\pi^2} \int_{-\infty}^{\infty} \int_{0}^{1} d\eta dQ \frac{R^2 + m_a^2 + \mu^2 \eta^2 + (m_b^2 - m_a^2 - \mu^2)\eta}{\sqrt{\mu^2 \eta^2 + (m_b^2 - m_a^2 - \mu^2)\eta + m_a^2}} = 1.$$ \hspace{1cm} (10)

The integrand will have no poles if the quadratic expression involving $\eta$ has no real roots, or if it has a double root. This is the case if $|m_a - m_b| \leq \mu \leq m_a + m_b$. We assume this case, the last one is clear it says that the bound state cannot be of bigger mass, the other one says that the fundamental quanta should be stable against decay, if for example $m_a > m_b$ then it would be favorable to have $m_a \rightarrow m_b + \mu$. Then we can evaluate the integral in any way we want, first we take the $Q$ integral, this gives us,

$$\int_{0}^{1} \frac{d\eta}{\sqrt{\mu^2 \eta^2 + (m_b^2 - m_a^2 - \mu^2)\eta + m_a^2}} = \frac{16\pi}{\lambda}.$$ \hspace{1cm} (11)

The next integral can be done and simplified into

$$\frac{1}{\mu} \ln \left[\frac{m_a + m_b + \mu}{m_a + m_b - \mu}\right] = \frac{16\pi}{\lambda},$$ \hspace{1cm} (12)

which is valid when $|m_a - m_b| \leq \mu \leq m_a + m_b$. We may study the small coupling limit of this expression. In this case we expect that the bound state mass becomes very close to the two mass threshold, then we can write

$$\frac{m_a + m_b - \mu}{m_a + m_b + \mu} \approx e^{-16\pi(m_a + m_b)/\lambda}, \quad \mu \approx (m_a + m_b)(1 - 2e^{-16\pi(m_a + m_b)/\lambda}),$$ \hspace{1cm} (13)

which is consistent if we take $\lambda/(m_a + m_b) << 1$. the other extreme is interesting as well, $\mu \approx m_a - m_b$ (assuming $a$ is the heavier particle), this implies a critical coupling $\lambda_c$ beyond which our methods break down, due to the appearance of a tachyon,

$$\frac{16\pi}{\lambda_c} = \frac{1}{m_a - m_b} \ln \left[\frac{m_a}{m_b}\right] \quad \text{or} \quad \lambda_c = \frac{16\pi}{\ln\left[\frac{m_a}{m_b}\right]}(m_a - m_b).$$ \hspace{1cm} (14)

This critical coupling is pushed to higher and higher values if $b$ becomes lighter and lighter with respect to the $a$ particle. For any given value of the coupling constant in the interval $(0, \lambda_c]$, there is a solution for the bound state energy. Hence we see that there is a composite
bound state for these values of the coupling constants. It is not clear what happens beyond this value, it is possible that the linear approximation breaks down, it is also possible that the large-$N$ limit is not a good approximation beyond a certain value. The other possibility is that the naive vacuum is not a true vacuum of the quantum theory and we should redefine the vacuum of the system. We are not able to analyze these possibilities at the moment.

Let us compare this with the results we would have found if we looked at a 1+1-dimensional version of the same model. Then the bound state equation could be written in terms of the fractional light-cone momentum $\zeta$ only. There is no transversal component and we have only one integral to compute. As a result we find the equation that should be satisfied by the eigenvalue $\mu$, (with the condition $|m_a - m_b| \leq \mu \leq m_a + m_b$),

$$\frac{1}{\sqrt{4m_a^2m_b^2 - z^2}} \left[ \arctan \left( \frac{2m_a^2 + z}{\sqrt{4m_a^2m_b^2 - z^2}} \right) + \arctan \left( \frac{2m_b^2 + z}{\sqrt{4m_a^2m_b^2 - z^2}} \right) \right] = \frac{8\pi}{\lambda^2}; \quad (15)$$

where $\mu^2 = m_a^2 + m_b^2 + z$ and we wrote $\lambda^2$ for the coupling constant since it has dimensions mass-squared. To analyze the behaviour it is more natural to define the dimensionless variables, $\tilde{\zeta} = z / 2m_am_b$, and $\sigma = m_a/m_b$, and rescale the coupling $\tilde{\lambda}^2 = \lambda^2 / m_am_b$,

$$\frac{1}{\sqrt{1 - \tilde{\zeta}^2}} \left[ \arctan \left( \frac{\sigma + \tilde{\zeta}}{\sqrt{1 - \tilde{\zeta}^2}} \right) + \arctan \left( \frac{1/\sigma + \tilde{\zeta}}{\sqrt{1 - \tilde{\zeta}^2}} \right) \right] = \frac{16\pi}{\tilde{\lambda}^2}. \quad (16)$$

Note that now $\tilde{\zeta}$ satisfies $-1 < \tilde{\zeta} < 1$. If we take the limit $\tilde{\zeta} \to -1^+$, this corresponds to $\mu \to |m_a - m_b|^+$ and the other limit $\tilde{\zeta} \to 1^-$ corresponds to $\mu \to (m_a + m_b)^-$. If we assume $\tilde{\zeta} \approx 1^-$ we see that the bound states satisfy a relation

$$\tilde{\zeta} \approx 1 - \frac{\tilde{\lambda}^4}{128}, \quad \text{or} \quad \mu \approx (m_a + m_b) \left[ 1 - \frac{\lambda^4}{128m_am_b(m_a + m_b)^2} \right]. \quad (17)$$

and if we take $\tilde{\lambda}$ sufficiently small this is consistent. Notice that in 2+1 dimensions we have an exponential behaviour in the inverse coupling, which is nonanalytic in the coupling constant (around zero), as opposed to this power law change. If we look at the other extreme we see that there is a finite limit for $\tilde{\zeta} \to -1^+$, in fact it is equal to 1. This implies a critical coupling again, beyond which our methods predict a tachyonic state,

$$\tilde{\lambda}^2_c = 16\pi, \quad \text{or} \quad \lambda^2_c = 16\pi m_am_b.. \quad (18)$$

This is to be compared with the result in equation (14), which is sensitive to the mass difference.

Let us comment on the convergence conditions in this context. Since we are looking for a normalizable solution it looks natural to demand

$$\int_0^1 d\zeta \int_{-\infty}^{\infty} [dR]|f(\zeta, R)|^2 < \infty. \quad (19)$$

In fact this is right, and we could see this from our Hilbert-Schmidt condition,

$$\int_{u_-v_- < 0} [du_-dv_-][du_1dv_1]|C(u_-, u_1; v_-, v_1)|^2 < \infty. \quad (20)$$
If we now make the above change of variables by calling $u_+ - v_+ = P_+$ we have

$$\frac{1}{\pi} \int_0^1 \! d\zeta \int_0^{\infty} \! P_- [dP_-] \int [d(u_1 + (-v_1))dR] |C(P_-, \zeta; R, (u_1 + (-v_1)))|^2 < \infty,$$

(21)

In our case we are restricting $P_-$ to the surface $2P_-P_+ = \mu^2 + (u_1 + (-v_1))^2$ (for fixed $\mu$, $P_+$), this means we should reinterpret the above normalization as

$$\int_0^1 \! d\zeta \int [dR] |f(\zeta, R)|^2 \frac{\mu^2 + (u_1 + (-v_1))^2}{2P_+} |g(u_1 + (-v_1))|^2 [d(u_1 + (-v_1))] < \infty,$$

(22)

(notice that $P_+$ is not allowed to be zero), which means two separate conditions,

$$\int_0^1 \! d\zeta \int [dR] |f(\zeta, R)|^2 < \infty \quad \int \frac{\mu^2 + (u_1 + (-v_1))^2}{2P_+} |g(u_1 + (-v_1))|^2 [d(u_1 + (-v_1))] < \infty.$$

(23)

The second one simply is a Sobolev type condition, which states that the energy of the transversal center of momentum component should also be finite. We see that our solution actually satisfies a stronger condition for equations of motion to make sense.

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