A SHORT PROOF OF KOTZIG’S THEOREM

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Abstract. A new shortest proof of Kotzig’s Theorem about graphs with unique perfect matching is presented in this paper. It is well known that Kotzig’s theorem is a consequence of Yeo’s Theorem about edge-colored graph without alternating cycle. We present a proof of Yeo’s Theorem based on the same ideas as our proof of Kotzig’s theorem.

1. Introduction

The well-known theorem of A. Kotzig was proved for the first time in [K].

Theorem 1. (A. Kotzig, 1959) Let $G$ be a connected graph with unique perfect matching. Then $G$ has a bridge that belongs to this matching.

However, the proof in [K] was tedious. The shortest proof of Kotzig’s theorem that is known now is to derive it from the following theorem of A. Yeo [Y].

We denote by $G-e$ graph $G$ without the edge $e$ and by $G-v$ graph $G$ without the vertex $v$ and all edges incident to it.

Theorem 2. (A. Yeo, 1997) All edges of a graph $G$ are colored such that there is no alternating cycles (i.e. each cycle has two adjacent edges of the same colors). Then $G$ contains a vertex $v$ such that every connected component of $G-v$ is joined to $v$ with edges of one color.

This theorem have rather short and elegant proof using the method of alternating chains. Let us also mention, that a particular case of Yeo’s Theorem for coloring with two colors was proved by Grossman and Haggkvist in 1982 [GH].

Our short proof of the Theorem 1 is based on analyzing of a minimal counterexample. Also we show that our method works in Theorem 2.

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2. Proofs

Proof of Theorem 1. Suppose the statement of theorem is false. Consider a counterexample $G$ with a minimum number of edges. Let $F$ the set of all edges of the unique perfect matching and $\overline{F} = E(G) \setminus F$. Consider two cases.

1° There exists a bridge $a \in \overline{F}$ of the graph $G$. Consider the graph $G-a$. Clearly, it has exactly two connected components. If any of these components has a bridge that belongs to $F$, then the graph $G$ also has such a bridge. We obtain a contradiction. Hence, each connected component has a second perfect matching. Clearly, then $G$ also has second matching. This is a contradiction.

2° Set $\overline{F}$ contains no bridges of the graph $G$. Each vertex is incident to an edge of $\overline{F}$, otherwise $G$ has a bridge that belongs to $F$. Furthermore, at least one vertex is incident to at least two edges of $\overline{F}$, since otherwise $G$ is an even cycle and hence, it has two perfect matchings.

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Thus, $|F| < |\overline{F}|$. Since $G$ is a minimal counterexample after deleting from $G$ any edge of $\overline{F}$ a bridge that belongs to $F$ appears in the resulting graph. Hence, there are two edges $a_1, a_2 \in \overline{F}$ such that both graphs $G - a_1$ and $G - a_2$ have the same bridge $b \in F$. Hence, the graph $G - b$ has two bridges $a_1$ and $a_2$, and the graph $G - \{a_1, a_2\}$ has three connected components.

Returning the bridge $b$ we obtain two connected components $X, Y$ in $G - \{a_1, a_2\}$. Since $a_1$ and $a_2$ are not bridges in $G$, we can denote them $a_1 = x_1y_1$ and $a_2 = x_2y_2$, where $x_1, x_2 \in X, y_1, y_2 \in Y$ (see Fig. 1 left).

Denote by $F_x$ the edges of matchings $F$ lying in the component $X$. Let us contract $x_1y_1$, $Y$ and $y_2x_2$ in edge $x_1x_2$ (possibly multiple edges or a loop can appear, see Fig. 1) and obtain a graph $G_x$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Solid lines - edges from $F$, dotted - from $\overline{F}$.}
\end{figure}

Obviously, if $G_x$ has a bridge of $F_x$, then $G$ has a bridge of $F$. Hence, there is $F_x$ - a matching in $G_x$ that is different from $F_x$. If $x_1x_2 \notin F_x$, then there is a second matching $F' = (F \setminus F_x) \cup F_2$ in the graph $G$, but it’s impossible. Then $F_x'$ contains an edge $x_1x_2$. One can similarly define the matching $F_y$ and the graph $G_y$, and prove that there exists another matching $F_y'$ in $G_y$ that contains $y_1y_2$. Then the graph $G$ has a matching $F'' = (F_y' \setminus \{x_1x_2\}) \cup (F_y' \setminus \{y_1y_2\}) \cup \{x_1y_1, x_2y_2\}$, different from $F$, contradiction.

**Proof of Theorem 2.** Assume the contrary and consider a minimal counterexample $G$ (at first we minimize the number of vertices, after that the number of edges).

Clearly, $G$ is connected and has no cut-vertex. Vertex is called monochrome if all edges incident to it are of the same color. And vertex $v$ of $G$ is called cut-color if no connected component of $G - v$ is joined to $v$ with edges of more than one color. Consider several cases.

1° There is an edge $b_1b_2$, such that the graph $G - b_1b_2$ has no monochrome vertex.

Since $G$ is a minimal counterexample there is a cut-color vertex $v$ in $G - b_1b_2$. Graph $G - b_j$ is connected, hence, $b_j \neq v$. Let $X_1$ be a subgraph of the graph $G - b_1b_2$ induced on the vertex $v$ together with the connected component of $G - b_1b_2 - v$ that contains $b_1$. Now we add to $X_1$ the vertex $c$ and edge $b_1c$ of the same color as $b_1b_2$ and edge $v$ of any unique color. Denote the obtained graph by $G_1$ (see Fig. 2). Graph $G_1$ is less than $G$, otherwise the part $X_2$ has only one vertex and hence, the vertex $b_2$ in $G - b_1b_2$ is monochrome.

The graph $G_1$ has no monochrome vertices and no cut-color vertices ($G_1$ has no cut-vertex). Hence, there is the alternating cycle in $G_1$, it must contain $C$ (otherwise, there is such a cycle in $G$). Consequently, there is an alternating path from $v$ to $b_1$ in $X_1$ with color of last edge different from the color of edge $b_1b_2$. Similarly, there is such a path from $v$ to $b_2$. Two edges incident to $v$ in these paths have different colors. Otherwise, if their colors coincide all edges incident to $v$ in both parts $X_1$ and $X_2$ have the same color, and therefore, $v$ is monochrome, which is impossible. Taking these two paths and adding to them the edge $b_1b_2$ we obtain an alternating cycle in $G$. This contradicts our assumption.
There is a vertex $c$ of degree 2 such that the graph $G - c$ has no monochrome vertex.

Since $G$ is a minimal counterexample, there appears a cut-color vertex $v$ in $G - c$. Vertex $c$ is incident to two edges $cb_1$ and $cb_2$ of distinct color. Let $X_1$ be a subgraph of the graph $G - c$ induced on the vertex $v$ together with the connected component of $G - c - v$ that contains $b_1$. Then we construct (similarly $1^°$ case) an alternating path from $v$ to $b_1$ with last edge with distinct color from $b_1c$ in the part $X_1$ and an alternating path from $v$ to $b_2$ with last edge with distinct color from $b_2c$ in the part $X_2$. Glue these paths together with the edges $b_1c$ and $cb_2$, we obtain an alternating cycle in $G$. This contradicts our assumption.

There are two adjacent vertices $c_1, c_2$ of degree 2.

Let $c_1b_1$ and $c_2b_2$ be the other edges incident to $c_1$ and $c_2$, respectively. If these two edges have different colors, then after deleting the edge $c_1c_2$ and gluing the vertices $c_1$ and $c_2$ we obtain a smaller counterexample. If $c_1b_1$ and $c_2b_2$ have the same color, we delete the vertices $c_1$ and $c_2$ and add a new edge $b_1b_2$ with the same color as $b_1c_1$. Clearly, we obtain a smaller counterexample.

Consider the remaining cases.

Let us construct a digraph on the vertices of $G$ using his edges. We draw an arc $\vec{ab}$ if $ab$ is an edge of $G$ and the $b$ is a monochrome vertex of the graph $G - ab$. (Maybe the arc $\vec{ba}$ is drawn too).

Let $x$ be the number of vertices of degree 2 in $G$. Then the graph $G$ has at least $\frac{2x + 3(v - x)}{2} = 1.5v - 0.5x$ edges. Since there is no situation of case $2^°$, at least one arc starts at each vertex of degree 2. Obviously, at least two arcs end at each vertex of degree 2. Since vertices of degree 2 are not adjacent, the number of arcs is at least $x + e \geq 1.5v + 0.5x$.

Let two arcs (corresponding to the edges $e_1, e_2$ of graph $G$) end at the vertex $d$. Let $E_d$ be the set of all edges incident to $d$. Then edges of $E_d \setminus \{e_1\}$ have the same color and edges of $E_d \setminus \{e_2\}$ have the same color. But vertex $d$ is not monochrome in $G$, clearly, $d$ has degree 2. Then the sum of incoming degrees of vertices does not exceed $2x + (v - x) = v + x$. But then we have $1.5v + 0.5x \leq v + x$, hence, $v = x$. But we have two adjacent vertices of degree 2. We obtain a contradiction.

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