Hydrodynamic interactions between many spheres

Maria L. Ekiel-Jeżewska
PMMH ESPCI, Paris, France
(on leave from: IPPT PAN, Warszawa, Poland)
mekie@ippt.gov.pl

November 20, 1998

Abstract

This paper is an introductory guide to many-particle hydrodynamic interactions. Basic concepts of the fluid mechanics are assumed to be known. Experience in the Stokes equations is useful but not necessary. The study is estimated to fit five sessions about three hours each.

Auguste Rodin: “Nothing else that I have done satisfies me as much, because nothing else cost me so much effort...”

Contents

1 Introduction 2

I Formulation of the problem 3

II Developing basic concepts: work sheets 3

2 Principles of work 4

3 Simplifying: analogy between Stokesian hydrodynamics and electrostatics 4

3.1 Reasoning by analogy 4

3.2 Units 4

3.3 Basic equation 4

3.4 Uniqueness theorem 5

3.4.1 Electrostatics 5

3.4.2 Stokesian hydrodynamics 5

3.5 Boundary conditions 5

3.6 Friction problem 6

3.7 Green function 6

3.7.1 Definition 7

3.7.2 Derivatives of the Green functions $\mathcal{G}$ and $\mathcal{P}$ 7

3.8 Boundary integral equations 7

3.8.1 Electrostatics 8

3.8.2 Stokesian hydrodynamics 8
Auguste Rodin: “...human thought is limited by comparison with what nature transmits directly to us and imposes on us. All that is necessary is to follow the model; character results from its unity.”

The text consists of three parts: a brief formulation of the problem, work sheets for own studies aiming to develop the basic concepts and a short concluding overview to indicate how these concepts are useful in construction of the theory modelling quantitatively many-particle systems [Felderhof 1976a] [Cichocki 1995].

The structure of the curriculum emerged from the method of effective learning by inquiry [McDermott], redesigned and extended according to our previous experience, to specific needs of the subject and to feedback received from participants of a course guided by a preliminary version of the worksheets presented here. [McDermott], a non-standard textbook for teachers, called “a set of laboratory-based modules” by the authors, provides a method to develop habits of effective learning, based on active inquiry, application of scientific reasoning and cooperative work in small groups. Connection with reality is essential in this learning pattern, since it provides the motivation leading to a personal engagement (students start from their own observations) and it serves as the natural objective evaluation of own understanding (students make experiments, which verify predictions of physical models which they have just constructed). A discrepancy between own predictions and reality becomes a driving force to learn. An extension of this approach to study a theoretical curriculum has been neither straightforward nor simplistic. Actually, it required a general analysis of creative learning principles.

The course has been guided by a continuous struggle to take care of integrity of the learning process, namely to make its mission and its vision clear, its specific goals apparent, its structure simple and evolving to fit students’ needs and capacities. Therefore we have started with a very specific formulation of the goal. We have tried not only to identify and to keep the right sequence of steps building on each other, but also to make this sequence apparent in advance. We have shown how to make various side connections, giving a chance to see possible generalizations and applications, to establish a relation
with own experience, and to recognize an own direction of further studies. The necessary attitude of the instructor reaching out for integrity was to be first of all a student, challenged to develop a new deeper insight into learning/teaching techniques as well as into physical and mathematical aspects of the hydrodynamic interactions, to make the own learning evident, and to allow other learners for influencing what and how they learn. This curriculum would be never developed without its practical application.

These principles have led to the structure of the learning process presented here. The problem and the goal were formulated specifically in the announcement about the course (see Appendix) sent to scientists and students working on problems related to hydrodynamic interactions. This information was important to decide for participation in a non-standard activity. Originally the curriculum had been designed to be based only on active group work rather than passive listening to lectures. However, the participating scientists demanded to be conscious not only of a direction and goals, but also of a perspective of their studies, important in choosing what to investigate further. Therefore the structure has been modified. Work sheets (Sec. 2.4) served as a guide in own studies carried out in small groups of 2-4 people during 4 sessions about 2.5 hours each. An overview concluding lecture (Sec. 5.3) was added at the end of the course as a closure and as an application of the participants’ own inquires. A similar pattern of education had been earlier developed and tested in *Glazek, Masłowski, Wiecekowski*.

**Part I**

**Formulation of the problem**

How to determine the behavior of N spheres in low Reynolds number incompressible fluid flow (N up to several hundred)?

We will concentrate on the following ‘friction problem’:
If translational and rotational velocities of the spheres are given, as well as an ambient fluid flow in which they have been immersed, then what are the forces and torques they exert on the fluid?

This approach can be afterwards adjusted to solve also the twin ‘mobility problem’:
If an ambient fluid flow and external forces and torques acting on the spheres are given, then what are their translational and rotational velocities? *Felderhof 1988*

Our goal is to inquire the basic structure and tools of the technique developed in *Felderhof 1976a*–*Cichocki 1995*.

**Part II**

**Developing basic concepts: work sheets**

Auguste Rodin: “I forced myself to express in each swelling of the torso or of the limbs the efflorescence of a muscle or of a bone which lay deep beneath the skin. And so the truth of my figures, instead of being merely superficial, seems to blossom from within to outside, like life itself.”

Following *McDermott*, in this part we used different type styles to distinguish between a text guiding independent work (written like this sentence), general informations (slanted) and additional remarks (small letters).
2 Principles of work

The idea is to make the whole problem a subject of your own active inquiry, carried out and discussed in small groups of 2-4 persons, on the basis of work sheets written specially for you.

Questions raised during our sessions will help to identify separate steps to be made, building subsequently on each other. Each step consists of the problems (formulated as a separate subsection of the work sheets) to be solved by you. You may find it useful to keep a written record of your work.

The end of each subsection is a point to conclude – first to share your reasoning with each other, and next to discuss your results with me, giving me your comments and questions. To keep track of time we will indicate in a ‘calendar of progress’ when your group has finished each subsection.

If you have a problem blocking your progress, and any of your group cannot solve it, please ask me for help.

Our interests and background vary. Therefore it is reasonable to divide into groups of a similar attitude. The content of work sheets is the same for all members of a group, but it can be different for different groups, according to your specific needs. If you find it useful, don’t hesitate to change a group and/or to demand for a curriculum related to your own questions on the problem.

There is a collection of references quoted in the instruction. They are to be read only to such an extent which you find relevant and useful to solve the problems posed in the work sheets and to answer your own questions.

3 Simplifying: analogy between Stokesian hydrodynamics and electrostatics

Auguste Rodin: “The most remote antiquity is my habitat. I want to link the past to the present; to return to memory, judge it, and contrive to complete it. Symbols are the guidelines of humanity. They are no lies.”

3.1 Reasoning by analogy

Read an introduction from [McDermott], p. 90.

Electrostatics is simpler than Stokesian hydrodynamics. Therefore developing the analogy and showing ‘how its corresponding parts are alike’ helps to understand the basic concepts and processes of the complex technique we are going to study. To put the emphasis on foundations in this section we assume that there is no external ambient flow, which will be added to the system in Sec. 4.

3.2 Units

To allow for an easy comparison with [Kim, Karilla] and [Jackson], our reference textbooks, we will use SI units in hydrodynamics and CGS units in electrostatics, i.e. we assume that $k=1$.

How the unit of charge (so-called statcoulomb) is defined in CGS system?
How does this unit relate to centimeter, second and gram?
Calculate how many coulombs it is.
Reference: [Jackson], Appendix 4

3.3 Basic equation

The basic equation of both electrostatics and hydrodynamics can be written in a general form as:

$$L_0\Psi = s$$  \hspace{1cm} (1)
where Ψ is a physical field to be found, $L_0$ is a differential operator, and $s$ is a known source distribution. In electrostatics Ψ is the scalar potential field $\Phi$ and $s$ is the charge density $\rho$.

Specify the operator $L_0$ in electrostatics. Use CGS units.

Specify the meaning of an unknown field Ψ, a differential operator $L_0$ and a given source $s$ in Stokesian hydrodynamics. What are the similarities and the differences in comparison to electrostatics?

Compare your analogy with analogies developed by other groups.

The eq. (1) needs to be supplemented by boundary conditions.

Specify what do you understand as the fluid boundaries in the friction problem.

References: [Jackson], Sec. 1.7, [Kim, Karilla], Sec. 1.2.3.

### 3.4 Uniqueness theorem

**Guiding question**

In electrostatics the solution to the Poisson equation is determined uniquely by specifying on the boundary:
- the normal component of the electrostatic field $E$ (so-called Neumann condition) or
- the potential $\Phi$ (so-called Dirichlet condition).

Predict what is a hydrodynamic analogue of this theorem.

#### 3.4.1 Electrostatics

Use the Green’s identity ([Jackson], Sec. 1.8):

$$\int_V (\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi) d^3r = \oint_S \phi \nabla \psi \cdot n dA$$

(2)

(Where S is a surface surrounding V, $n$ is the unit vector normal to S) to show the uniqueness of solutions to the Poisson equation (1) for Dirichlet and Neumann boundary conditions ([Jackson], Sec. 1.9). Specify what is the meaning of uniqueness in both cases.

#### 3.4.2 Stokesian hydrodynamics

Develop a similar proof for the Stokesian hydrodynamics.

Use the Gauss theorem for a tensor $K$:

$$\int_V \partial_i K_{i...l} d^3r = \oint_S K_{i...l} n_i dA$$

(3)

to formulate a useful generalization of the Green’s identity (2) for vector functions $\psi$, $\phi$.

State the uniqueness theorem. Specify what do you mean by uniqueness.

Hint: Find out a vector analog of the following scalar theorem used in electrostatics:

If $\nabla \Phi = 0$, then $\Phi = \text{const}(r)$.

References: [Kim, Karilla], Exercise 2.1 and Sec. 2.2.1, [Pozrikidis], Sec. 1.5.

Compare your findings with results obtained by other groups.

Answer the guiding question.

### 3.5 Boundary conditions

In Stokesian hydrodynamics from now on we will restrict to the stick boundary conditions, i.e. to the fluid velocity $\mathbf{v}(\mathbf{r})$ at the boundary equal to the rigid motion velocity of the boundary itself:

$$\mathbf{v}(\mathbf{r}) = \mathbf{U} + \mathbf{\Omega} \times \mathbf{r}$$

(4)
However, according to [Felderhof 1976b], [Felderhof 1988], the formalism is valid for a more general class of the so-called slip boundary conditions, namely

\[
\begin{align*}
\mathbf{t} \cdot \mathbf{v}(r) &= \mathbf{t} \cdot (\mathbf{U} + \mathbf{\Omega} \times r) + l \mathbf{t} \cdot \mathbf{\sigma}(r) \cdot \mathbf{n} \\
\mathbf{n} \cdot \mathbf{v}(r) &= \mathbf{n} \cdot \mathbf{U}
\end{align*}
\]

at the boundary \( S \)

where \( \mathbf{t} \) is a unit vector tangential to the boundary surface, \( l \) vary from 0 (for the stick boundary conditions) to \( \infty \) (the so-called perfect slip boundary conditions), \( \mathbf{\sigma} \) is the fluid stress tensor:

\[
\sigma_{ij} = \mu \left( \partial_i v_j + \partial_j v_i \right) - p \delta_{ij}.
\]

**A supplementary problem: slip boundary conditions.** In Stokesian hydrodynamics the slip boundary conditions on the sphere surface are defined as [Felderhof 1976b]:

\[
\begin{align*}
\mathbf{t} \cdot \mathbf{v} - \mathbf{t} \cdot (\mathbf{U} + \mathbf{\Omega} \times r) &= l \mathbf{t} \cdot \mathbf{\sigma} \cdot \mathbf{n} \\
\mathbf{n} \cdot \mathbf{v} &= \mathbf{n} \cdot \mathbf{U}
\end{align*}
\]

where \( \mathbf{t} \) is a unit vector tangential to the boundary surface, \( l \) vary from 0 (the so-called no slip or stick boundary conditions) to \( \infty \) (the so-called perfect slip boundary conditions).

Do slip conditions determine a unique solution to the Stokes equations? Prove your statement.

In Sec. 3 we assume that the fluid is motionless at infinity: \( \mathbf{v}|_\infty = 0 \). In the following sections we will consider the general case of any conditions at infinity.

Construct simple electrostatic analogues of:

A) \((\mathbf{v}_i, \mathbf{\Omega}_i)\) - a pair consisting of translational and rotational velocities of a body \( i \),

B) a body at rest in a viscous fluid,

C) a rigidly moving body in a viscous fluid,

Specify analogous equations for the boundary conditions in all cases.

### 3.6 Friction problem

The friction problem for \( N \) bodies is the following:

If given: \( \mathbf{\Omega}_\alpha, \mathbf{U}_\alpha, \mathbf{v}_0(r) \), then what are \( \mathbf{F}_\beta, \mathbf{T}_\beta \) (\( \alpha, \beta = 1, ..., N \))?\

In this section we assume that the ambient fluid flow vanishes: \( \mathbf{v}_0(r) = 0 \). In such a case motion of \( N \) bodies with velocities \( \mathbf{U}_\beta + \mathbf{\Omega}_\beta \times (\mathbf{r} - \mathbf{r}_\beta) \) result in forces \( \mathbf{F}_\alpha \) and torques \( \mathbf{T}_\alpha \) exerted on the fluid by the body \( \alpha \), determined by the \( N \)-particle friction matrix \( \mathbf{\zeta} \):

\[
\begin{pmatrix}
\mathbf{F}_\alpha \\
\mathbf{T}_\alpha
\end{pmatrix} = \mathbf{\zeta}_{\alpha\beta} 
\begin{pmatrix}
\mathbf{U}_\beta \\
\mathbf{\Omega}_\beta
\end{pmatrix}
\]

(7)

Explain how does (7) follow from Stokes equations. Does \( \mathbf{\zeta} \) depend on:

- position in the fluid, \( \mathbf{r} \),
- position of a body \( \beta, \mathbf{r}_\beta \),
- translational velocity of a body \( \beta, \mathbf{U}_\beta \) and
- rotational velocity of a body \( \beta, \mathbf{\Omega}_\beta \)?

If yes, explain how. If no, why not? What is the dimension of \( \mathbf{\zeta} \)?

Are the \( \alpha\beta \) components of the 2-particle friction matrix equal to \( \mathbf{\zeta}_{\alpha\beta} \), the corresponding components of the \( N \)-particle friction matrix from eq. (7)? Support your answer by a reasoning.

Develop an electrostatic analogue of the friction problem (make it as simple as possible). What are the electrostatic analogues of the quantities appearing in eq. (7)? Explain.

In Sec. 3.4 we consider \( N \) rigid bodies of an arbitrary shape. Later we will concentrate on \( N \) spheres only.

### 3.7 Green function

Green function \( G \) will help us to solve our friction problem, in a similar way it helps to solve its electrostatic analogue.
Guiding question

Predict if the following statement is always true, true only under special supplementary conditions (if yes, specify them) or false; explain your reasoning:

\[
\Psi(r) = \int d^3 r' G(r, r') s(r')
\] (8)

3.7.1 Definition

Green function \( G \) is a solution to the equation:

\[
L_0(r) G(r, r') = \delta(r - r')
\] (9)

Is \( G \) a scalar, a vector or a tensor in: A) electrostatics; B) hydrodynamics?

Write down (9) explicitly for electrostatics and for hydrodynamics, indicating arguments and all components.

Specify \( G \) in electrostatics ([Jackson], Sec. 1.10) and in hydrodynamics ([Kim, Karilla], Sec. 2.4.1) if the Dirichlet boundary conditions vanish at infinity. Make your definition consistent with your choice of \( L_0 \).

From now on we will assume that \( G \) is the Green function for an infinite system.

However, the formalism presented in Sec. 3.7 has been developed for any Green function \( G \) ([Hasimoto], [Felderhof, 1988], corresponding to a container or to periodic boundary conditions [Hasimoto], [Felderhof, 1988], [Cichocki, Felderhof, 1989]).

3.7.2 Derivatives of the Green functions \( G \) and \( P \)

The Green functions \( G \) and \( P \) satisfy the following identities:

\[
\mu \nabla^2 G_{ij}(\mathbf{R}) - \partial_i P_j(\mathbf{R}) = -\delta_{ij} \delta^3(\mathbf{R})
\] (10)

\[
\partial_i G_{ij}(\mathbf{R}) = 0
\] (11)

\[
\nabla^2 P_j(\mathbf{R}) = \partial_j \delta^3(\mathbf{R})
\] (12)

\[
\partial_j P_j(\mathbf{R}) = \delta^3(\mathbf{R}).
\] (13)

where \( \mathbf{R} = r' - r \) and all derivatives are taken with respect to \( r' \): \( \partial_i \equiv \partial/\partial r_i' \).

Although \( G \) and \( P \) are functions, but their derivatives are distributions. We need to have a clear prescription how to evaluate such derivatives. Each of them can be understood as a limit of a sequence of functions.

Show that the Green functions \( G \) and \( P \) for the infinite system (our choice from Sec. 3.7):

\[
G_{ij}(\mathbf{R}) = \frac{1}{8\pi\mu} \left( \frac{\delta_{ij}}{R} + \frac{R_i R_j}{R^3} \right)
\] (14)

\[
P_j(\mathbf{R}) = \frac{1}{4\pi} R_j + P_{0j},
\] (15)

can be obtained as the following limits: \( G = \lim_{a \to 0} G^a \) and \( P = \lim_{a \to 0} P^a \), where

\[
G^a_{ij}(\mathbf{R}) = \frac{1}{8\pi\mu} (-\partial_i \partial_j + \delta_{ij} \nabla^2) (R^2 + a^2)^{1/2}
\] (16)

\[
P^a_j(\mathbf{R}) = \frac{1}{8\pi} \partial_j \nabla^2 (R^2 + a^2)^{1/2}
\] (17)

Explain the procedure how to evaluate \( D G \) and \( D P \) for a differential operator \( D \).

Apply this prescription to verify eq. (11).

3.8 Boundary integral equations

Boundary integral equations are useful if there is a closed surface \( S \) surrounding a volume \( V \) and the field \( \Psi \) is defined both inside and outside.
3.8.1 Electrostatics

Use the Green’s theorem:
\[
\int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) \, d^3 r = \oint_S \left[ \phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right] \, dA
\]
(18)
to derive the following expressions for the electrostatic field potential \( \Phi \) ([Jackson], Sec. 1.10):
\[
\int_V \rho(r') \, G(r, r') \, d^3 r' + \frac{1}{4\pi} \oint_S \left[ G(r, r') \frac{\partial \Phi(r')}{\partial n'} - \Phi(r) \frac{\partial G(r, r')}{\partial n'} \right] \, dA' = \begin{cases} \Phi(r) & r \in V \\ 0 & r \notin V \end{cases}
\]
(19)

Reference: [Jackson], Sec. 1.6, 1.8.

What is the hydrodynamic analogue of the electrostatic \( \frac{\partial \Phi(r)}{\partial n} \)?

Does (19) answer the guiding question? Explain.

3.8.2 Stokesian hydrodynamics

If there is no ambient flow, the hydrodynamic analogue of identities (19) is given as:
\[
\int_V \rho f_j \, j \, G_{jk} \, d^3 r' + \oint_S \left\{ G_{jk} \sigma_{lj} - v_j [\mu (\partial_l G_{jk} + \partial_j G_{lk}) - \delta_{jl} P_{lk}] \right\} n'_l \, dA' = \begin{cases} v_k(r) & r \in V \\ 0 & r \notin V \end{cases}
\]
(20)
\[
-\int_V \rho f_j \, P_j \, d^3 r' + \oint_S \left\{ -P_{jk} \sigma_{lj} + \mu v_j [(\partial_j P_{lj} + \partial_l P_{lj})] \right\} n'_l \, dA' = \begin{cases} p(r) & r \in V \\ 0 & r \notin V \end{cases}
\]
(21)

Are \( \sigma \) and \( v \) taken at \( r \) or \( r' \)? Put the missing order of arguments of the Green functions \( G \) and \( P \) – \( (r, r') \) or \( (r', r) \) – into eqs (20)-(21).

What are the symmetry properties of the Green functions \( G \) and \( P \)? Which of them are general, and which are due to the specific symmetries (no fluid motion at infinity) of the Oseen functions (14)-(15)?

How does the unit vector \( n' \) point: out or into the fluid?

How do the equations simplify if there is no external forces acting on the fluid other than gravity?

Reference: [Pozrikidis], Sec. 2.3, [Kim, Karilla], Sec. 2.4.2, [Happel, Brenner], Sec. 3.4.

Eqs (20)-(21) are valid if there is a closed boundary of any shape inside a fluid. How would you modify them to describe a rigid body in a fluid? Explain.

The integral representation (20)-(21) still does not allow to address the guiding question from Sec. 3.7 – just the opposite, it seems to contain a term differing in form from eq. (8). We will come back to this problem in Sec. 4.2, but first we will learn how to take into account the existence of an ambient flow around a particle.

4 Particle in ambient flow

Auguste Rodin: “First, I usually create my stone children without cloths. Then all I have to do is to throw some drapery over them...”

4.1 Ambient flow

How to use the results of Sec. 3 to construct solutions of the Stokes equation in the presence of an ambient flow?
4.1.1 Definitions

1. The ambient flow is a solution of the Stokes equation with given boundary conditions.

2. The external ambient flow \(v_\theta(r)\) is a solution of the Stokes equation with a given boundary condition at infinity:

\[
v_\theta(r)|_\infty = V(\theta, \phi)
\]

Specify what is the ambient flow for the boundary condition:

\[
v_\theta(r)|_\infty = V_0.
\]

Explain your reasoning.

Give examples of other external ambient flows.

Construct example of an ambient flow which has the same boundary conditions at infinity as a certain external ambient flow but which is different. Predict how this construction can be in agreement with the uniqueness theorem from Sec. 3.4. (We will come back to this example in Sec. 4.2.1.)

4.1.2 Equivalence of solutions

Assume that \(v(r)\) is the solution of the Stokes equation with the boundary conditions at infinity and on closed surfaces \(S_\alpha, \alpha = 1, ..., N:\)

\[
v(r)|_{S_\alpha} = U + \Omega \times (r|_{S_\alpha} - r_\alpha)
\]

\[
v(r)|_{\infty} = V(\theta, \phi)
\]

It is often said that the bodies \(S_\alpha\) are immersed in the ambient flow \(v_\theta(r)\) given by (22).

Does \(v(r) - v_\theta(r)\) satisfy the Stokes equation? If yes, then specify the boundary conditions and explain your reasoning. If no, then why not. Explain how two solutions – in the presence and in the absence of an ambient flow – are equivalent.

4.2 Formalism of induced forces

Generalize the eqs (20)-(21) from Sec. 3.8.2 to describe a closed boundary in an ambient external flow \(v_\theta(r)\). Explain your reasoning.

The starting point of [Felderhof 1988] (eq. 2.7) is the conjecture that the formal solution to the Stokes equation can be written as:

\[
v(r) - v_\theta(r) = \int G(r, r') f(r') d^3 r'
\]

where \(f(r)\) is the total force density exerted on the fluid.

To understand and to justify this statement we will derive a formula in the form of (26) from the boundary integral equations given in Sec. 3.8.2. To this goal we will first develop the concept of induced surface force density.

4.2.1 The concept of induced forces

Construct a hydrodynamic analogue of the electrostatic relation between the induced surface charge density and the electrostatic field. How do you interpret the meaning of the adjective induced describing a surface density on the boundary in both cases?

The goal is to express the boundary integrals given in Sec. 3.8.4 in terms of a surface induced force density (depending on the fluid stress tensor at the surface) rather than in terms of the fluid velocity \(v\) at the boundary.

The question is how to achieve it - the eqs (20)-(21) depend on both \(\sigma\) and \(v\) at the boundary. The idea is to first introduce an artificial fluid flow inside the rigid solid particles. Namely, to construct the inside solution to the Stokes equation with the same stick boundary conditions (24) at the particle surface as those which determine the real outside solution. Next, to determine what are these induced forces, using the eqs (20)-(21), which you have just generalized for a non-vanishing ambient flow.
Reference: [Cox, Brenner], Sec. 2 and [Mazur, Bedeaux], Sec. 3-4.

Solve the Stokes equation inside a volume $V$ with the stick boundary conditions (24) and the ambient flow given by (25). Hint: Make use of the uniqueness theorem.

**A supplementary problem: uniqueness theorem revised.** The combination of the inside and the outside solutions is different than the ambient flow, although both satisfy the same boundary conditions at infinity. Explain how this construction can be in agreement with the uniqueness theorem from Sec. 3.4. Compare with your reasoning from Sec. 4.1.1, where you also constructed example of an ambient flow which had the same boundary conditions at infinity as a certain different external ambient flow.

### 4.2.2 Justification of the formalism

Write down two sets of the generalized eqs (20)-(21):
- for $V$ being the interior of all the particles and
- for $V$ being the real fluid.

Combine both sets and make use of the stick boundary conditions to eliminate the surface integrals including values of the fluid velocity at the boundary.

Derive an expression for the induced force density in terms of the fluid stress tensor at the boundary.

How will you calculate the total force acting on a body in terms of the fluid stress tensor, and how in terms of the induced force density?

How does the induced force density relate to $f$ in eq. (26)? What is the range of integration in eq. (26)?

Use the generalized eqs (20)-(21) to derive an equation for $p(\mathbf{r})$ similar to eq. (26).

Specify the properties of the Green function used in this section. How could they be justified?

### 4.3 Method of reflections

#### 4.3.1 The difficulty

Assume that $\mathbf{v}_\alpha$ is the solution of the one-particle friction problem, i.e. the solution to the Stokes equation with the boundary conditions (25) and the one-particle version of (24).

Does $\mathbf{v} = \sum_{\alpha=1}^{N} \mathbf{v}_\alpha$ satisfy Stokes equations? Explain your reasoning.

Does $\mathbf{v}$ satisfy the boundary conditions (25) at infinity? If yes, then why? If not, then how could you construct $\mathbf{v}'$, another combination of $\mathbf{v}_\alpha$, satisfying (25)?

Calculate $\mathbf{v}$ (and $\mathbf{v}'$) at the surface $S_\alpha$ of the body $\alpha$. How do they compare to the boundary conditions (24)?

Are $\mathbf{v}$ (and $\mathbf{v}'$) solutions to the N-particle friction problem? Explain your reasoning.

#### 4.3.2 Construction

*Method of reflections* is an iteration procedure to construct an approximate solution to the N-particle friction problem, building it from $N$ single-particle solutions. At each step corrections are added to decrease discrepancy between the boundary conditions and the actual value of the approximate solution on the body surfaces. It means that at each step we modify the single particle solutions by a better adjustment of their boundary conditions. The N-particle solution $\mathbf{v}$ is formally written as:

$$\mathbf{v} = \sum_{\alpha=1}^{N} \mathbf{v}_\alpha \quad (27)$$

Each single particle solution $\mathbf{v}_\alpha$ for particle $\alpha$ is given by the following formal expansion:

$$\mathbf{v}_\alpha = \mathbf{v}_0 + \mathbf{w}_\alpha + \sum_{\beta \neq \alpha} \mathbf{w}_{\beta \alpha} + \sum_{\gamma \neq \beta} \sum_{\beta \neq \alpha} \mathbf{w}_{\gamma \beta \alpha} + \ldots \quad (28)$$
Each $w_{\gamma\alpha}$ satisfies the Stokes equations.
Reference: [Kim, Karilla], Sec. 8.1.

Specify what are the boundary conditions at infinity and at the surface $S_\alpha$ for $v_0$, for $w_\alpha$, for $w_{\beta\alpha}$, and for $w_{\gamma\beta\alpha}$.

What is the approximate value of $v$ at the surface $S_\alpha$ after:
- the first
- the second
- the n-th iteration step?

What is the effective ambient flow in which the particle $\alpha$ is immersed before:
- the first,
- the second and
- the n-th iteration step?

4.3.3 Interpretation

Eq. (28) can be interpreted as a summation over all incident and outgoing ”waves” in multiple scattering (in a sequence of reflections).

Specify what are the incident and the outgoing ”waves” scattered by a particle $\beta$ at the first, the second and the n-th iteration step.

Predict if the iteration procedure is convergent. Give arguments.

Part III

Exploring the structure: a lecture.

Auguste Rodin: “For the first time I saw separate pieces, arms, heads or feet; then I attempted the figure as a whole. Suddenly, I grasped what unity was...”

5 Application of the basic concepts

Reference: [Felderhof 1988], Sec. 2.

5.1 Reformulation of the friction problem

Forces and torques exerted by the fluid on the surface $S_\alpha$ of the particle $\alpha$ are given in terms of the fluid velocity $v$ and pressure $p$ as:

$$ F_\alpha = -\int_{S_\alpha} \sigma \cdot n_\alpha \, dA $$
$$ T_\alpha = -\int_{S_\alpha} (r - r_\alpha) \times (\sigma \cdot n_\alpha) \, dA $$

where $v, p$ satisfy the Stokes equations:

$$ \mu \nabla^2 v - \nabla p = 0 $$
$$ \nabla \cdot v = 0 $$

with given boundary conditions:

$$ v(r) \rightarrow V(\theta, \phi) \quad r \rightarrow \infty $$
\[ v(r) = w_\alpha(r) \equiv U_\alpha + \Omega_\alpha \times (r - r_\alpha) \quad r \in S_\alpha \] (34)

The surface normal \( n \) in (29)-(30) points into the particle.

To solve (29)-(34) we will apply the tools developed so far, in the following way:

1. We evaluate the ambient flow \((v_0, p_0)\) as the solution to (31)-(32) with (33).
2. We use the formalism of induced forces (Sec. 4.2) and the concept of equivalent solution (Sec. 4.1) to replace (29)-(34) by:

\[
F_\alpha = \int f \, d^3r \quad (35)
\]

\[
T_\alpha = \int (r - r_\alpha) \times f \, d^3r \quad (36)
\]

\[
w_\alpha(r) - v_0(r) = \int G(r, r') f(r') \, d^3r' \quad \text{for} |r - r_\alpha| \leq a \quad (37)
\]

with \( G \) given by (14), and with \( f \) – non-vanishing on the particle surfaces only – related to the fluid stress tensor \( \sigma \) as in [Felderhof 1976a]:

\[
f(r) = \sum_\alpha f_\alpha(r) \quad (38)
\]

\[
f_\alpha(r) = \sigma \cdot n_\alpha \delta(|r - r_\alpha| - a) \quad (39)
\]

Eq. (37) has been obtained by substitution of (34) into (26).

### 5.2 Induced forces in terms of the boundary conditions for the fluid velocities

The reformulated friction problem means solving (37) for \( f \). First we note that due to linearity of the Stokes equations \( f \) depends linearly on the boundary conditions of the equivalent solution, i.e. the rigid motion of the spheres minus the ambient flow: \( w(r) - v_0(r) \), where

\[
w(r) = \sum_\alpha w_\alpha(r) \quad (40)
\]

and \( w_\alpha \) are given by (34). We write it as:

\[
f(r) = \int Z(r, r') [w_\alpha(r') - v_0(r')] \, d^3r' \quad (41)
\]

and \( Z \) vanishes if \( r \) or \( r' \) is located outside a particle.

\( Z \) has a matrix form, with \( Z_{\alpha\beta} \) relating \( f_\alpha \), the forces acting on the particle \( \alpha \), to the boundary conditions \( w_\beta - v_0 \), on the surfaces of all the particles \( \beta \). Using a simplified notation we write it as:

\[
f_\alpha = Z_{\alpha\beta} [w_\beta - v_0] \quad (42)
\]

Therefore our goal is to find the N-body friction kernels \( Z_{\alpha\beta}(r, r') \). We will do it in two steps:

1. Simplification: one particle \( \alpha \) in the fluid flow. We will have a look how to evaluate the one-particle friction kernel \( Z_0(\alpha) \).
2. Multiplication: many particles. We will see how to express \( Z \) in terms of one-particle friction kernels \( Z_0(\alpha) \).

### 5.3 Multipole expansion

Assume that we know the N-particle friction kernel \( Z_{\alpha\beta} \). The question is how to find the forces and the torques. Substituting (41) into (33) and (34) we get:

\[
F_\alpha = \int d^3r \int d^3r' Z_{\alpha\beta}(r, r') [w_\beta(r') - v_0(r')] \quad (43)
\]

\[
T_\alpha = \int d^3r \int d^3r' (r - r_\alpha) \times Z_{\alpha\beta}(r, r') [w_\beta(r') - v_0(r')] \quad (44)
\]
If there is no ambient flow, then:

\[ w_\alpha(r) - v_0(r) = U_\alpha + \Omega_\alpha \times (r - r_\alpha) \]  

and we have:

\[
\begin{pmatrix}
F_\alpha \\
T_\alpha
\end{pmatrix} = \begin{pmatrix}
\zeta^{\alpha tt} & \zeta^{\alpha tr} \\
\zeta^{\alpha rt} & \zeta^{\alpha rr}
\end{pmatrix}_{\alpha\beta} \begin{pmatrix}
U_\beta \\
\Omega_\beta
\end{pmatrix}
\]  

The friction matrix elements can be written as:

\[
< b_{i\alpha} | Z_{\alpha\beta} | b_{j\beta} >
\]  

with

\[
< b_{0\alpha} | = | b_{0\alpha} > = 1 \theta_\alpha(r)
\]

\[
< b_{1\alpha} | = | b_{1\alpha} > = -\epsilon_{ijk}(r - r_\alpha)_k \theta_\alpha(r)
\]

and the scalar product defined as:

\[
< a | b > = \int a^*(r) \cdot b(r) d^3r
\]

If there is no ambient flow, then:

\[ w_\beta(r) - v_0(r) = \sum \frac{(r - r_\beta)^p}{p!} \nabla^p[w_\beta(r_\beta) - v_0(r_\beta)] \]  

We construct a complete set of functions \( b_\beta^p = \) - combinations of \( (r - r_\beta)^p \) and a complete set of "velocity multipoles" \( C_\beta^p = \) - combinations of \( \nabla^p[w_\beta(r_\beta) - v_0(r_\beta)] \). Instead of eq. (46) we now have:

\[
F_\alpha = < b_{0\alpha} | Z_{\alpha\beta} | b_{p\beta} > C_\beta^p
\]

\[
T_\alpha = < b_{1\alpha} | Z_{\alpha\beta} | b_{p\beta} > C_\beta^p
\]

or equivalently:

\[
\begin{pmatrix}
F_\alpha \\
T_\alpha
\end{pmatrix} = \begin{pmatrix}
\zeta^{\alpha tt} & \zeta^{\alpha tr} \\
\zeta^{\alpha rt} & \zeta^{\alpha rr}
\end{pmatrix}_{\alpha\beta} C_\beta^p
\]

In particular:

\[
C_\beta^0 = U_\beta
\]

\[
C_\beta^1 = \Omega_\beta + \frac{1}{2} \nabla \times v_0
\]

The questions remain how to choose the basis \( b_\beta^p \), how to expand the N-particle friction kernel \( Z \) and \( Z_0 \), and how to truncate. First we need to get acquainted with the N-particle and the one-particle friction kernels \( Z \) and \( Z_0 \).

### 5.4 Single particle solution

The single particle solution in an ambient flow \( w_0 \) is given as [26], [11]:

\[
v(r) - w_0(r) = \int \left[ \int G(r, r'') Z_0(r'', r') d^3r'' \right] [w(r') - w_0(r')] d^3r'
\]

Note that \( r', r'' \) are inside the particle, while \( r \) has no such restriction. To shorten notation we write [11] and [57] as:

\[
f = Z_0[w - w_0]
\]

\[
v - w_0 = G Z_0[w - w_0]
\]
5.5 Multiple scattering

We can describe our many-particle system as the system consisting of a sphere \( \alpha \) in an ambient flow created by the other bodies. But this ambient flow is also unknown, and it depends on the position and the velocity field will be approximated by the sum of all one-particle solutions \( \mathbf{v}_\alpha \) in a given ambient flow, evaluated from (59). Since this is not a many-particle solution, then \( \mathbf{v}_\beta \) will change the ambient flow in which particle \( \alpha \) is immersed. It will be taken into account through modification of the ambient flow entering the next step of the iteration.

The interpretation of this iteration in terms of a “multiple scattering” (or “reflection”) on each particle \( \beta \) has been made in Sec. 4.3.3.

To carry out the multiple scattering we need to specify what is the total ambient flow \( \mathbf{w}_n \), in which the particle \( \alpha \) is immersed after each step \( n \) of the iteration procedure. \( \mathbf{w}_n \) is the “wave” outgoing from the step \( n \) and incident to step \( n + 1 \). It consists of the incident ambient flow \( \mathbf{w}_{n-1} \) and corrections coming from the other particles \( \beta \neq \alpha \), evaluated from eq. (59) in step \( n - 1 \). Therefore:

\[
\mathbf{w}_n = \sum_{\beta \neq \alpha} [\mathbf{v}_{n\beta} - \mathbf{w}_{n-1}] + \mathbf{w}_{n-1}
\]  

(60)

\( \mathbf{v}_{n\alpha} - \mathbf{w}_{n-1}, n = 1, 2, \ldots \) correspond to the subsequent terms in the scattering expansion given in eq. (28). In Sec. 4.3.3 we have already analyzed the boundary conditions for them. Therefore, with the help of (59)-(60) the multiple scattering process made in step \( n \) can be described as:

\[
\mathbf{v}_{1\alpha} - \mathbf{v}_0 = \mathcal{G} \mathcal{Z}_{0\alpha} [\mathbf{w}_\alpha - \mathbf{v}_0] \quad \text{for } n \geq 2
\]

(61)

\[
\mathbf{v}_{(n+1)\alpha} - \mathbf{w}_n = -\mathcal{G} \mathcal{Z}_{0\alpha} (\mathbf{w}_n - \mathbf{w}_{n-1}) \quad \text{for } n \geq 2
\]

(62)

The multiple scattering (60)-(62) is equivalent to the construction of the fluid velocity \( \mathbf{v} \) by the method of reflections [Kim, Karilla], [Happel, Brenner]. Eqs (60), (61) and (62) specify the ambient flow:

\[
\mathbf{w}_1 - \mathbf{v}_0 = \sum_{\beta \neq \alpha} \mathcal{G} \mathcal{Z}_{0\beta} [\mathbf{w}_\beta - \mathbf{v}_0] \quad \text{for } n \geq 2
\]

(63)

\[
\mathbf{w}_{n+1} - \mathbf{w}_n = -\sum_{\beta \neq \alpha} \mathcal{G} \mathcal{Z}_{0\beta} (\mathbf{w}_n - \mathbf{w}_{n-1}) \quad \text{for } n \geq 2
\]

(64)

and eq. (68) gives the induced forces:

\[
\mathbf{f}_{1\alpha} = \mathcal{Z}_{0\alpha} [\mathbf{w}_\alpha - \mathbf{v}_0] \quad \text{for } n \geq 2
\]

(65)

\[
\mathbf{f}_{(n+1)\alpha} = \mathcal{Z}_{0\alpha} (\mathbf{w}_\alpha - \mathbf{w}_n) = \mathbf{f}_{n\alpha} + \mathcal{Z}_{0\alpha} (\mathbf{w}_{n-1} - \mathbf{w}_n)
\]

(66)

Therefore the multiple scattering expansion of \( \mathbf{f}_\alpha \) reads to the following form of the N-particle friction kernel \( \mathcal{Z}_{\alpha\beta} \) in eq. (12):

\[
\mathcal{Z}_{\alpha\beta} = \mathcal{Z}_{0\alpha} \delta_{\alpha\beta} - (1 - \delta_{\alpha\beta}) \mathcal{Z}_{0\alpha} \mathcal{G} \mathcal{Z}_{0\beta} + \mathcal{Z}_{0\alpha} \sum_{\gamma \neq \alpha, \beta} \mathcal{G} \mathcal{Z}_{0\gamma} \mathcal{G} \mathcal{Z}_{0\beta} - ...
\]

(67)

Auguste Rodin: “Yes, form I have looked at and understood, it can be learnt: but the genius of form has yet to be studied.”

Acknowledgements

I thank François Feuillebois for inviting me to guide the course on hydrodynamic interactions between many spheres – and the same for providing me the motivation to develop the work presented here. I benefited from the educational structure of Physique Thermique, Laboratoire de Physique et Mecanique des Milieux Heterogenes, École Supérieure de Physique et de Chimie Industrielles de la Ville de Paris, where the sessions took place in the framework of a non-standard educational activity. Discussions with the participants of the course, Paul Chaikin, François Feuillebois, Pierre-Emmanuel Jabin, Nicolas Lecoq, Daniel Lhuillier, Michel Martin were essential for developing this paper, although I remain responsible for all the possible mistakes. I thank Stanisław Glazek, Tomasz Masłowski and Marek Więckowski for sharing with me their teaching experience. My stay at ESPCII has been supported by the French Ministry of Education and Research.
Appendix. Announcement about the course

Between June 15 and July 10, 1998 I will guide a course on

HYDRODYNAMIC INTERACTIONS BETWEEN MANY SPHERES

4 sessions, 2.5 hours each.

GOAL:
To inquire the basic structure and tools of the modern theory, which has been developed by Felderhof, Jones, Cichocki, Schmitz and their coworkers for 20 years, and resulted in numerical packages allowing for accurate calculations of hydrodynamic interactions.

SUBJECT:
How to determine the behavior of N spheres in low Reynolds number incompressible flow (N between several and several hundred). Namely:
If an ambient fluid flow and external forces & torques acting on them are given, then what are their translational & rotational velocities (mobility problem)?
If their translational & rotational velocities and an ambient fluid flow are given, then what are the forces & torques they exert on the fluid (friction problem)?

KEY WORDS:
Stokes equations, stick boundary conditions, ambient flow, Green function, induced forces, friction kernel, generalized resistance matrix, generalized mobility matrix, multiple scattering expansion, force multipole moments, vector harmonics, rotational invariance.

IDEA:
To simplify, but knowing how to reach for complexity.
Since the course is meant to be a first step needed to be done before making a more sophisticated analysis, then we will concentrate on basic concepts applied to a simple system. In particular, the following problems treated by this theory will be mentioned, but will not be discussed: slip boundary conditions, Green function other than Oseen tensor, lubrication phenomena, averaging procedure leading to evaluation of transport coefficients, mobility and friction problem for non-spherical shapes of particles (built from spheres). Analogy with electrostatics and quantum mechanics will be outlined, since you may later find it helpful in carrying out calculations.

ATTITUDE:
To make it useful.
Therefore first of all you are welcome to participate in making a plan of the course, by e-mailing me your suggestions what you would like to gain, what do you need it for and which concepts from those listed above are of your interest and which are not. Secondly, to help you in applying the technique, the sessions will be based on your active inquiry in small groups and your own solving of some basic problems rather than on passive listening to a lecture. Finally, your comments on time allocated to this activity are appreciated.

CONTACT:
If you want to participate, reply by e-mail before Thursday, June 11. Please let me know what are your time limitations - it will help me to fix the day of the week and the hour of our sessions.

Maria Ekiel-Jeżewska

References

[Felderhof 1976a] B.U. Felderhof, Force density induced on a sphere in linear hydrodynamics I. Fixed sphere, stick boundary conditions, Physica 84A (1976) 557

[Felderhof 1976b] B.U. Felderhof, Force density induced on a sphere in linear hydrodynamics II. Moving sphere, mixed boundary conditions, Physica 84A (1976) 569
[Felderhof 1977] B.U. Felderhof, *Hydrodynamic interactions between two spheres*, Physica 89A (1977) 373

[Jones 1978] R.B. Jones, *Hydrodynamic interaction of two permeable spheres I: the method of reflections*, Physica 92A (1978) 545

[Felderhof, Jones 1978] B.U. Felderhof, R.B. Jones, *Fächer theorems for spherically symmetric polymers in solution*, Physica 93A (1978) 457

[Reuland et al. 1978] P. Reuland, B.U. Felderhof, R.B. Jones, *Hydrodynamic interaction of two spherically symmetric polymers*, Physica 93A (1978) 465

[Schmitz, Felderhof 1978] R. Schmitz, B.U. Felderhof, *Creeping flow about a sphere*, Physica 92A (1978) 423

[X] SchMITZ 1980] R. Schmitz, *Force multipole moments for a spherically symmetric particle in solution*, Physica 102A (1980) 161

[X] SchMITZ, Felderhof 1982a] R. Schmitz, B.U. Felderhof, *Creeping flow about a spherical particle*, Physica 113A (1982) 90

[Schmitz, Felderhof 1982b] R. Schmitz, B.U. Felderhof, *Friction matrix for two spherical particles with hydrodynamic interaction*, Physica 113A (1982) 103

[Schmitz, Felderhof 1982c] R. Schmitz, B.U. Felderhof, *Mobility matrix for two spherical particles with hydrodynamic interactions*, Physica 116A (1982) 163

[Felderhof, Jones 1983] B.U. Felderhof, R.B. Jones, *Cluster expansion of the diffusion kernel of a suspension of interacting Brownian particles*, Physica 121A (1983) 329

[Felderhof, Jones 1987a] B.U. Felderhof, R.B. Jones, *Addition theorems for spherical wave solutions of the vector Helmholtz equation*, J. Math. Phys. 28 (1987) 836

[Felderhof, Jones 1987b] B.U. Felderhof, R.B. Jones, *Convective motion and transfer of force by many-body hydrodynamic interaction*, Physica 146A (1987) 404

[Jones, Schmitz 1988] R.B. Jones, R. Schmitz, *Mobility matrix for arbitrary spherical particles in solution*, Physica 149A (1988) 373

[Felderhof 1988] B.U. Felderhof, *Many-body hydrodynamic interactions in suspensions*, Physica 151A (1988) 1

[Cichocki et al. 1988] B. Cichoki, B.U. Felderhof, R. Schmitz, *Hydrodynamic interactions between two spherical particles*, PCH PhysicoChemicalHydrodynamics, 10, 383 (1988)

[Cichocki, Felderhof 1988a] B. Cichocki, B.U. Felderhof, *Short-time diffusion coefficients and high-frequency viscosity of dilute suspensions of spherical Brownian particles*, J. Chem. Phys. 89 (1988) 1049

[Cichocki, Felderhof 1988b] B. Cichocki, B.U. Felderhof, *Long-time self-diffusion coefficient and zero-frequency viscosity of dilute suspensions of spherical Brownian particles*, J. Chem. Phys. 89 (1988) 3705

[Cichocki, Felderhof 1988c] B. Cichocki, B.U. Felderhof, *Renormalized cluster expansion for multiple scattering in disordered systems*, J. Stat. Phys. 51 (1988) 57

[Cichocki, Felderhof 1989a] B. Cichocki, B.U. Felderhof, *Sedimentation and self-diffusion in suspensions of spherical particles*, Physica 154A (1989) 213

[Felderhof, Jones 1989] B.U. Felderhof, R.B. Jones, *Displacement theorems for spherical solutions of the linear Navier-Stokes equations*, J. Math. Phys. 30, 339 (1989)

[Felderhof 1989] B.U. Felderhof, *Hydrodynamic interactions in suspensions with periodic boundary conditions*, Physica 159A (1989) 1

[Cichocki, Felderhof 1989b] B. Cichocki, B.U. Felderhof, *Periodic fundamental solution of the linear Navier-Stokes equations*, Physica 159A (1989) 19
[Cichocki,Felderhof 1993] B. Cichocki, B.U. Felderhof, Influence of hydrodynamic interactions on self-diffusion and stress relaxation in a semidilute suspension of hard spheres, Physica 198A (1993) 423

[Cichocki et al. 1994] B. Cichocki, B.U. Felderhof, K. Hinsen, E. Wajnryb, J. Bławzdziewicz, Friction and mobility of many spheres in Stokes flow, J. Chem. Phys. 100 (1994) 3780

[Cichocki,Hinsen 1995] B. Cichocki, K. Hinsen, Stokes drag on conglomerates of spheres, Phys. Fluids 7 (1995) 285

[Cichocki 1995] B. Cichocki, Hydrodynamic interactions, in: Continuum Models and Discrete Systems, ed. K.Z. Markov, World Scientific 1996, p. 15.

[McDermott] L.C. McDermott and the Physics Education Group at the University of Washington, Physics by Inquiry, Wiley 1996

[Głazek,Masłowski,Wieckowski] St.D. Głazek, T. Masłowski and M. Wieckowski, Foundations of Renormalization in Quantum Mechanics, unpublished, 1997

[Jackson] J.D. Jackson, Classical Electrodynamics, Wiley, New York 1975

[Kim, Karilla] S. Kim, S.J. Karilla, Microhydrodynamics: Principles and Selected Applications, Butterworth - Heinemann, 1991

[Pozrikidis] C. Pozrikidis, Boundary integral and singularity methods for linearized viscous flow, Cambridge University Press 1992

[Hasimoto] H. Hasimoto, On the periodic fundamental solutions of the Stokes equations and their application to viscous flow past a cubic array of spheres, J. Fluid Mech. 5 (1959) 317

[Cox,Brenner] R. G. Cox, H. Brenner, Effect of finite boundaries on the Stokes resistance of an arbitrary particle: Part 3. Translation and rotation, J. Fluid Mech. 28 (1967) 391

[Mazur,Bedeaux] P. Mazur, D. Bedeaux, A generalization of Fazén's theorem to nonsteady motion of a sphere through an incompressible fluid in arbitrary flow, Physica 76 (1974) 235

[Happel,Brenner] J. Happel, H. Brenner, Low Reynolds Number Hydrodynamics, McGraw-Hill, 1963

[Lamb] H. Lamb, Hydrodynamics, Cambridge University Press, 1975

[Edmonds] A.R. Edmonds, Angular momentum in quantum mechanics, Princeton University Press 1974

[*] H. Pinet, Rodin. The hands of genius, trans. C. Palmer, Thames and Huston 1997

[**] G. Néret, Auguste Rodin. Sculptures and Drawings, transl. C. Miller, Benedikt Taschen 1994