Iterative method for solution of radiation emission/transmission matrix equations

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An iterative method is derived for image reconstruction. Among other attributes, this method allows constraints unrelated to the radiation measurements to be incorporated into the reconstructed image. A comparison is made with the widely used Maximum-Likelihood Expectation-Maximization (MLEM) algorithm.

Imaging by radiation emission or transmission effectively produces a set of linear equations to be solved. For example, in the case of coded aperture imaging, the solution is a “reconstructed” set of radiation sources, while in the case of x-ray interrogation, the solution is a set of attenuation coefficients for the voxels comprising the volume through which the x-ray beam passes.

The linear equations have the form

\[ d_i = \sum_{j=1}^{J} M_{ij} \mu_j \]  

where the set \( \{d_i\} \) corresponds to the radiation intensity distribution recorded at a detector (a detector pixel is labeled by the index \( i \)), the set \( \{\mu_j\} \) is the solution, and the matrix element \( M_{ij} \) connects the known \( d_i \) to the unknown \( \mu_j \). Typically the matrix \( M \) is non-square so that \( \{\mu_j\} \) cannot be obtained by standard matrix methods. (And note that, when the set of equations is large, it can be difficult to ascertain \textit{a priori} whether the equation set is over- or under-determined.)

In any case the matrix equation \( d = M \mu \) may be solved by the iterative method that is derived as follows. Clearly this method will feature a relation between \( \mu_j^{(n)} \) and \( \mu_j^{(n-1)} \), where \( n \) is the iteration number. Consider the two equations for \( \mu_j^{(n)} \) and \( \mu_j^{(n-1)} \),

\[ d_i^{(n)} = \sum_j M_{ij} \mu_j^{(n)} \]  

\[ d_i^{(n-1)} = \sum_j M_{ij} \mu_j^{(n-1)} \]  

and rewrite the latter as

\[ d_i = \frac{d_i^{(n-1)}}{\sum_j M_{ij} \mu_j^{(n-1)}} \sum_j M_{ij} \mu_j^{(n-1)}. \]  

Then the relationship between \( \mu_j^{(n)} \) and \( \mu_j^{(n-1)} \) is obtained by setting \( \sum_i d_i^{(n)} = \sum_i d_i \):

\[ \sum_i \left( \sum_j M_{ij} \mu_j^{(n)} \right) = \sum_i \left( \frac{d_i^{(n-1)}}{\sum_j M_{ij} \mu_j^{(n-1)}} \sum_j M_{ij} \mu_j^{(n-1)} \right) \]

\[ \sum_j \left\{ \mu_j^{(n)} \sum_i M_{ij} \right\} = \sum_j \left\{ \mu_j^{(n-1)} \sum_i \left( \frac{d_i}{d_i^{(n-1)}} M_{ij} \right) \right\} \]

\[ \mu_j^{(n)} = \mu_j^{(n-1)} \frac{1}{\sum_i M_{ij}} \sum_i \left( \frac{d_i}{d_i^{(n-1)}} M_{ij} \right). \]  

Note that this last equation can be written

\[ \mu_j^{(n)} = \mu_j^{(n-1)} \frac{\sum d_i}{\sum M_{ij}} \]  

where the last factor is essentially a weighted average of all \( d_i / d_i^{(n-1)} \). Thus the set \( \{\mu_j^{(n)}\} \) approaches a solution \( \{\mu_j\} \) by requiring \( \sum d_i^{(n)} = \sum d_i \) at each iteration; in effect, by requiring all \( d_i^{(n)} \to d_i \).

The iteration procedure alternates between use of Eq. (3) and Eq. (6) until all \( d_i^{(n)} \) are as close to \( d_i \) as desired. For the first \( (n = 1) \) iteration, an initial set \( \{\mu_j^{(0)}\} \) is chosen, which produces the set \( \{d_i^{(0)}\} \) according to Eq. (3). These values are used in Eq. (6), so producing the set \( \{\mu_j^{(1)}\} \). And so on... That a final set \( \{\mu_j^{(n)}\} \) is a solution \( \{\mu_j\} \) to the matrix equation \( d = M \mu \) is verified by checking that all \( d_i^{(n)} = d_i \) to within a desired tolerance.

Some cautions and opportunities follow from this simple derivation of Eq. (6). A caution is that, in the event the equation set is under-determined, different initial sets \( \{\mu_j^{(0)}\} \) will lead to different final sets \( \{\mu_j\} \) that satisfy the matrix equation. The corresponding opportunity is that this problem may be mitigated to some extent by the addition, to the original set of equations, of linear equations that further constrain the \( \mu_j \) (perhaps derived from, for example, independent knowledge of some of the contents of a container under interrogation). In general the \( d_i \) appearing in a constraint equation will have nothing to do with radiation intensity.

The form of any added constraints, and the initial choice \( \{\mu_j^{(0)}\} \), must allow all \( \mu_j^{(n)} \to \mu_j \) and \( d_i^{(n)} \to d_i \) monotonically. In particular, care should be taken when a constraint has one or more coefficients \( M_{ij} < 0 \), as that affects the denominator \( \sum_i M_{ij} \) in Eq. (3) (a straightforward fix may be to reduce the magnitudes of all \( M_{ij} \) coefficients and \( d_i \) in that constraint equation by a multiplicative factor). In any event, the acceptability of a
constraint equation is easily ascertained by monitoring the behavior $d_i^{(n)} \rightarrow d_i$ for that constraint.

Note that all solutions $\{\mu_j\}$ to a set of equations that includes additional constraints with $d_i > 0$ and all $M_{ij} \geq 0$ are accessible from sets $\{\mu_j^{(0)}\}$ of initial values, and further that any set $\{\mu_j^{(0)}\}$ will produce a solution $\{\mu_j\}$. This suggests that, for this implementation of constraints, a superposition of many solutions may give a good “probabilistic” reconstruction. To achieve this, consider that the innumerable solutions to the set of equations may be regarded as points in a $J$-dimensional space ($J$ is the number of elements in a solution $\{\mu_j\}$). These points must more-or-less cluster, producing a cluster centroid that is itself a solution. While the centroid solution $\{\mu_j^{(c)}\}$ has no intrinsic special status (as all cluster points represent equally likely reconstructions), it may be taken to represent the particular set of equations. The cluster size, which indicates the degree to which solutions are similar to one another, should decrease as constraints are added. A logical measure of the cluster size is

$$\sigma_{\text{cluster}} = \left\langle (x_c - x_k) \cdot (x_c - x_k) \right\rangle_k^{1/2} \quad (7)$$

where $x_c$ is the centroid vector and $x_k$ is the vector corresponding to the $k$th solution. Thus the quantity $\sigma_{\text{cluster}}/\sqrt{J}$, which represents the standard deviation of the innumerable values of an arbitrary element $\mu_j$, is a useful measure of the variation among solutions $\{\mu_j\}$. In general it is desirable that the variation among solutions be much less than the variation within the centroid solution, which is

$$\sigma_{\mu}^{(c)} = \left\langle \left( \mu_j^{(c)} - \bar{\mu}^{(c)} \right)^2 \right\rangle_j^{1/2} \quad (8)$$

where $\bar{\mu}^{(c)} = \left\langle \mu_j^{(c)} \right\rangle_j$. In that case ($\sigma_{\text{cluster}} J^{-1/2} \ll \sigma_{\mu}^{(c)}$) the centroid solution is little changed by additional constraints, so suggesting that the centroid solution $\{\mu_j^{(c)}\}$ may be regarded as the sought-after reconstruction.

Another caution follows from the fact that the denominator $\sum_j M_{ij}$ in Eq. (5) is the sum of all elements in column $j$ of matrix $M$. This iterative method can of course be used without explicitly converting a set of linear equations into a matrix equation (or several sets of equations into a single matrix equation), but in that event very careful attention must be paid to get the factors $\sum_j M_{ij}$ right.

It may be noticed that Eq. (5) is similar to the so-called Maximum-Likelihood Expectation-Maximization (MLEM) algorithm (see refs. [1] and [2] for derivations of the latter, and see numerous papers in the recent imaging literature for applications of it). The MLEM, which is derived from physical considerations having to do with radiation emission and detection, purports to find the set $\{\mu_j\}$ that maximizes the probability $P\{\{d_i\} | \{\mu_j\}\}$, which is the probability of realizing the observed set $\{d_i\}$ given a set $\{\mu_j\}$. This is in contrast to the derivation above, which leads to Eq. (5) as simply a method to find a solution to a matrix equation (a set of linear equations).

The derivation presented here makes clear how the iterative procedure should be implemented for an application, and allows constraints to be added to the original set of equations (those produced by the imaging exercise) thereby enabling a more-accurate reconstruction when the original equation set is under-determined.

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[1] L. A. Shepp and Y. Vardi, “Maximum likelihood reconstruction for emission tomography,” IEEE Trans. Med. Imag. MI-1, 113 (1982).

[2] K. Lange and R. Carson, “EM reconstruction algorithms for emission and transmission tomography,” J. Comput. Assist. Tomography 8, 306 (1984).

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