WHEN THE NUMBER OF DIVISORS IS A QUADRATIC RESIDUE

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Abstract. Let \( q > 2 \) be a prime number and define \( \lambda_q := \left( \frac{n}{q} \right) \) where \( \tau(n) \) is the number of divisors of \( n \) and \( \left( \frac{\cdot}{q} \right) \) is the Legendre symbol. When \( \tau(n) \) is a quadratic residue modulo \( q \), then \((\lambda_q \ast 1)(n)\) could be close to the number of divisors of \( n \). This is the aim of this work to compare the mean value of the function \( \lambda_q \ast 1 \) to the well known average order of \( \tau \). The proof reveals that the results depend heavily on the value of \( \left( \frac{2}{q} \right) \). A bound for short sums in the case \( q = 5 \) is also given, using profound results from the theory of integer points close to certain smooth curves.

1. Introduction and main result

If \( \lambda = (-1)^{\Omega} \) is the Liouville function, then

\[
L(s, \lambda) = \frac{\zeta(2s)}{\zeta(s)} \quad (\sigma > 1).
\]

This implies the convolution identity

\[
\sum_{n \leq x} (\lambda \ast 1)(n) = \left\lfloor x^{1/2} \right\rfloor.
\]

Define \( \lambda_3 := \left( \frac{n}{3} \right) \) where \( \tau(n) \) is the number of divisors of \( n \) and \( \left( \frac{\cdot}{3} \right) \) is the Legendre symbol modulo 3. Then from Proposition \[3\] below

\[
L(s, \lambda_3) = \frac{\zeta(3s)}{\zeta(s)} \quad (\sigma > 1)
\]

implying the convolution identity

\[
\sum_{n \leq x} (\lambda_3 \ast 1)(n) = \left\lfloor x^{1/3} \right\rfloor.
\]

Now let \( q > 2 \) be a prime number and define \( \lambda_q := \left( \frac{n}{q} \right) \) where \( \left( \frac{\cdot}{q} \right) \) is the Legendre symbol modulo \( q \). Our main aim is to investigate the sum

\[
\sum_{n \leq x} (\lambda_q \ast 1)(n).
\]

When \( \tau(n) \) is a quadratic residue modulo \( q \), one may wonder if \((\lambda_q \ast 1)(n)\) has a high probability to be equal to the number of divisors of \( n \). It then could be interesting to study its average order and to compare it to that of \( \tau \), i.e.

\[
\sum_{n \leq x} \tau(n) = x(\log x + 2\gamma - 1) + O(x^{\theta + \varepsilon})
\]

where \( \frac{1}{4} \leq \theta \leq \frac{131}{316} \), the left-hand side being established by Hardy \[5\], the right-hand side being the best estimate to date due to Huxley \[6\]. The main result of this paper can be stated as follows.

Theorem 1. Let \( q > 3 \) be a prime number.

\( \therefore \) If \( q \equiv \pm 1 \pmod{8} \)

\[
\sum_{n \leq x} (\lambda_q \ast 1)(n) = x\zeta(q)P_q(1)\left\{ \log x + 2\gamma - 1 + q\zeta'(q) + \frac{P'_q}{P_q}(1) \right\} + O_{q, x}\left(x^{\max(1/c_q, \theta) + \varepsilon}\right)
\]

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where $\theta$ is defined in (1), $c_q$ is given in (2) and

$$P_q(1) = \prod_p \left(1 + \sum_{m=c_q}^{q-1} \left\{ (\frac{m+1}{q}) - (\frac{m}{q}) \right\} \frac{1}{p^m} \right)$$

$$\frac{P_q'(1)}{P_q(1)} = -\sum_p \log p \left( \frac{\sum_{m=c_q}^{q-1} \left\{ (\frac{m+1}{q}) - (\frac{m}{q}) \right\} m}{1 + \sum_{m=c_q}^{q-1} \left\{ (\frac{m+1}{q}) - (\frac{m}{q}) \right\} m} \right)$$

\[\triangleright\] If $q \equiv \pm 1 \pmod{24}$

$$\sum_{n \leq x} (\lambda_q \ast 1)(n) = x^{1/2} \zeta\left(\frac{1}{2}\right) R_q\left(\frac{1}{2}\right) + O_{q,x}\left(x^{1/3+\varepsilon}\right)$$

where

$$R_q\left(\frac{1}{2}\right) := \prod_p \left(1 + \sum_{m=3}^{q-1} \left\{ (\frac{m+1}{q}) + (\frac{m}{q}) \right\} \frac{1}{p^{m/2}} \right).$$

\[\triangleright\] If $q \equiv \pm 5 \pmod{24}$, there exists $c > 0$ such that

$$\sum_{n \leq x} (\lambda_q \ast 1)(n) \ll_q x^{1/2} e^{-c(\log x)^{3/5} (\log \log x)^{1/5}}.$$

Furthermore, if the Riemann hypothesis is true, then for $x$ sufficiently large

$$\sum_{n \leq x} (\lambda_q \ast 1)(n) \ll_{q,c} x^{1/4} e^{(\log \sqrt{q})^{1/2} (\log \log \sqrt{q})^{3/2+\varepsilon}}.$$

Example 2.

$$\sum_{n \leq x} (\lambda_2 \ast 1)(n) \approx 0.454 x (\log x + 2\gamma + 0.784) + O_{\varepsilon}\left(x^{1/2+\varepsilon}\right).$$

$$\sum_{n \leq x} (\lambda_{23} \ast 1)(n) \approx 0.899 x (\log x + 2\gamma - 0.678) + O_{\varepsilon}\left(x^{131/416+\varepsilon}\right).$$

$$\sum_{n \leq x} (\lambda_{13} \ast 1)(n) \approx 1.969 x^{1/2} + O_{\varepsilon}\left(x^{1/3+\varepsilon}\right).$$

$$\sum_{n \leq x} (\lambda_5 \ast 1)(n) \ll x^{1/2} e^{-c(\log x)^{3/5} (\log \log x)^{1/5}}.$$

2. Notation

In what follows, $x \geq e^4$ is a large real number, $\varepsilon \in (0, \frac{1}{4})$ is a small real number which does not need to be the same at each occurrence, $s := \sigma + it \in \mathbb{C}$, $q$ always denotes an odd prime number, $\left( \frac{\cdot}{q} \right)$ is the Legendre symbol modulo $q$ and define

$$\lambda_q := \left( \frac{x}{q} \right)$$

where $\tau(n) := \sum_{d|n} 1.$ Also, 1 is the constant arithmetic function equal to 1.

For any arithmetic functions $F$ and $G$, $L(s, F)$ is the Dirichlet series of $F$, the Dirichlet convolution product $F \ast G$ is defined by

$$(F \ast G)(n) := \sum_{d|n} F(d)G(n/d)$$

and $F^{-1}$ is the Dirichlet convolution inverse of $F$. If $r \in \mathbb{Z}_{\geq 2}$, then

$$a_r(n) := \begin{cases} 1, & \text{if } n = m^r; \\ 0, & \text{otherwise}. \end{cases}$$

For some $c > 0$, set

$$\delta_c(x) := e^{-c(\log x)^{3/5} (\log \log x)^{-1/5}}$$

and

$$\omega(x) := e^{(\log x)^{1/2} (\log \log x)^{5/2+c}}.$$
Proposition 3. Let \( q \geq 3 \) be a prime number. For any \( s \in \mathbb{C} \) such that \( \sigma > 1 \)

\( \triangleright \) If \( q \equiv \pm 1 \pmod{8} \)

\[
L(s, \lambda_q) = \zeta(qs) \zeta(s) \prod_p \left( 1 + \sum_{m=c_q}^{q-1} \left\{ \frac{m+1}{q} - \frac{m}{q} \right\} \frac{1}{p^{ms}} \right)
\]

where

\[
c_q := \begin{cases} 
2, & \text{if } q \equiv \pm 7 \pmod{24}; \\
\geq 4, & \text{if } q \equiv \pm 1 \pmod{24}.
\end{cases}
\]

\( \triangleright \) If \( q \equiv \pm 3 \pmod{8} \)

\[
L(s, \lambda_q) = \frac{\zeta(qs) \zeta(2s)}{\zeta(s)} \prod_p \left( 1 + \sum_{m=d_q}^{q-1} \left\{ \frac{m+1}{q} + \frac{m}{q} \right\} \frac{1}{p^{ms}} \right)
\]

where

\[
d_q := \begin{cases} 
2, & \text{if } q \equiv \pm 5 \pmod{24} \text{ or } q = 3; \\
3, & \text{if } q \equiv \pm 11 \pmod{24}.
\end{cases}
\]

Proof. Set \( \chi_q := \left( \frac{q}{p} \right) \) for convenience. From [8, Lemma 2.1], we have

\[
L(s, \lambda_q) = \prod_p \left( 1 + \sum_{\alpha=1}^{\infty} \frac{\chi_q(\alpha+1)}{p^{\alpha s}} \right) = \prod_p \left( 1 + p^s \sum_{\alpha=2}^{\infty} \frac{\chi_q(\alpha)}{p^{\alpha s}} \right)
\]

\[
= \prod_p \left\{ 1 + p^s \left( \left( 1 - \frac{1}{p^s} \right)^{-1} \sum_{m=1}^{q-1} \left( \frac{m}{q} \right) \frac{1}{p^{ms}} - p^{-s} \right) \right\}
\]

\[
= \prod_p \left( 1 - \frac{1}{p^s} \right)^{-1} \sum_{m=1}^{q-1} \left( \frac{m}{q} \right) \frac{1}{p^{(m-1)s}}
\]

\[
= \zeta(qs) \prod_p \left( 1 + \sum_{m=2}^{q-1} \left( \frac{m}{q} \right) \frac{1}{p^{(m-1)s}} \right).
\]

If \( q \equiv \pm 1 \pmod{8} \), then \( \left( \frac{2}{q} \right) = 1 \) and

\[
L(s, \lambda_q) = \zeta(qs) \zeta(s) \prod_p \left( 1 - \frac{1}{p^s} + \left( 1 - \frac{1}{p^s} \right) \sum_{m=2}^{q-1} \left( \frac{m}{q} \right) \frac{1}{p^{(m-1)s}} \right)
\]

where

\[
\left( 1 - \frac{1}{p^s} \right) \sum_{m=2}^{q-1} \left( \frac{m}{q} \right) \frac{1}{p^{(m-1)s}} = \sum_{m=2}^{q-1} \left( \frac{m}{q} \right) \left( \frac{1}{p^{(m-1)s}} - \frac{1}{p^{ms}} \right)
\]

\[
= \sum_{m=2}^{q-2} \left( \frac{m+1}{q} \right) \frac{1}{p^{ms}} - \sum_{m=2}^{q-1} \left( \frac{m}{q} \right) \frac{1}{p^{ms}}
\]

\[
= \left( \frac{2}{q} \right) \frac{1}{p^s} + \sum_{m=2}^{q-1} \left\{ \left( \frac{m+1}{q} \right) - \left( \frac{m}{q} \right) \right\} \frac{1}{p^{ms}} - \frac{q}{q} \frac{1}{p^{(q-1)s}}
\]

\[
= \sum_{m=2}^{q-1} \left\{ \left( \frac{m+1}{q} \right) - \left( \frac{m}{q} \right) \right\} \frac{1}{p^{ms}} + \frac{1}{p^s}.
\]

Similarly, if \( q \equiv \pm 3 \pmod{8} \), then \( \left( \frac{2}{q} \right) = -1 \) and

\[
L(s, \lambda_q) = \frac{\zeta(qs) \zeta(2s)}{\zeta(s)} \prod_p \left( 1 + \frac{1}{p^s} + \left( 1 + \frac{1}{p^s} \right) \sum_{m=2}^{q-1} \left( \frac{m}{q} \right) \frac{1}{p^{(m-1)s}} \right)
\]
where
\[
\left(1 + \frac{1}{p^s}\right) \sum_{m=2}^{q-1} \left(\frac{m}{q}\right) \frac{1}{p^{(m-1)s}} = \sum_{m=2}^{q-1} \left(\frac{m}{q}\right) \frac{1}{p^{(m-1)s}} + \frac{1}{p^ms} = \sum_{m=1}^{q-1} \left(\frac{m+1}{q}\right) \frac{1}{p^{m}} + \frac{q}{q} \frac{1}{p^{m}} - \frac{1}{p^{m}a}
\]
\[
= \left(\frac{2}{q}\right) \frac{1}{p^s} + \frac{q}{q} \frac{1}{p^{m}} - \frac{1}{p^{m}a}
\]
\[
= \sum_{m=2}^{q-1} \left(\frac{m+1}{q}\right) \frac{1}{p^{m}} - \frac{1}{p^{m}a}
\]

We achieve the proof noting that, if \(q \equiv \pm 1 \pmod{24}\), then \(\left(\frac{2}{q}\right) = \left(\frac{q}{2}\right) = 0\) and, similarly, if \(q \equiv \pm 11 \pmod{24}\), then \(\left(\frac{2}{q}\right) = 0\) whereas \(\left(\frac{q}{2}\right) = 2\).

4. Proof of Theorem

4.1. The case \(q \equiv \pm 1 \pmod{8}\). For \(s > 1\), we set
\[
G_q(s) = \zeta(qs) \prod_p \left(1 + \sum_{m=q}^{q-1} \left(\frac{m+1}{q}\right) \frac{1}{p^{m}}\right) = \zeta(qs)P_q(s) := \sum_{n=1}^{\infty} \frac{g_q(n)}{n^s}.
\]

First observe that \(c_q < q\) in the case \(q \equiv \pm 1 \pmod{24}\). Indeed, among the \(q - 4\) integers \(m \in \{4, \ldots, q-1\}\), it is known from [3] p.76 that there are \(\frac{1}{4}(q-3)-3\) of them such that \(\left(\frac{m}{q}\right) = \left(\frac{m+1}{q}\right)\). Consequently there are \(\frac{1}{4}(q+1)\) integers \(m \in \{4, \ldots, q-1\}\) verifying \(\left(\frac{m}{q}\right) \neq \left(\frac{m+1}{q}\right)\), and the inequality follows.

Thus this Dirichlet series is absolutely convergent in the half-plane \(s > \frac{1}{c_q}\) where \(c_q\) is given in [2], so that
\[
\sum_{n \leq x} g_q(n) \ll_{q,\varepsilon} x^{1/c_q+\varepsilon}.
\]

By partial summation, we infer
\[
\sum_{n \leq x} \frac{g_q(n)}{n} = \zeta(q)P_q(1) + O \left(x^{-1+1/c_q+\varepsilon}\right)
\]
\[
\sum_{n \leq x} \frac{g_q(n)}{n} \log \frac{x}{n} = \zeta(q)P_q(1) \log x + qP_q(1)\zeta'\left(q\right) + P_q'(1)\zeta\left(q\right) + O \left(x^{-1+1/c_q+\varepsilon}\right).
\]

From Proposition [3] \(\lambda_q * 1 = g_q * \tau\). Consequently
\[
\sum_{n \leq x} \left(\lambda_q * 1\right) (n) = \sum_{d \leq x} g_q(d) \sum_{k \leq x/d} \tau(k)
\]
\[
= \sum_{d \leq x} g_q(d) \left\{\frac{x}{d} \log \frac{x}{d} + \frac{\gamma - 1}{d} \frac{x}{d} + O \left(\left(\frac{x}{d}\right)^{\theta+\varepsilon}\right)\right\}
\]
\[
= x \left\{\zeta(q)P_q(1) \log x + qP_q(1)\zeta'\left(q\right) + P_q'(1)\zeta\left(q\right) + (2\gamma - 1)\zeta\left(q\right)P_q(1)\right\}
\]
\[
+ O \left(x^{\max(1/c_q,\theta)+\varepsilon}\right)
\]

where \(\theta\) is defined in [1] and where we used
\[
x^{-\varepsilon} \sum_{d \leq x} \frac{|g_q(d)|}{d^\theta} \ll \begin{cases} x^{1/c_q-\theta}, & \text{if } c_q^{-1} \geq \theta; \\ 1, & \text{otherwise}. \end{cases}
\]
4.2. The case $q \equiv \pm 11 \pmod{24}$. For $\sigma > 1$, we set

$$H_q(s) = \zeta(qs) \prod_p \left( 1 + \sum_{m=3}^{q-1} \left\{ \frac{m+1}{q} + \frac{m}{q^s} \right\} \right)^{1/p^{ms}} := \zeta(qs)R_q(s) := \sum_{n=1}^{\infty} \frac{h_q(n)}{n^s}.$$\)

Since $q > 5$, this Dirichlet series is absolutely convergent in the half-plane $\sigma > \frac{1}{3}$, so that

$$\sum_{n \leq x} |h_q(n)| \ll q^{-\epsilon} x^{1/3+\epsilon}.$$\)

From Proposition 3, $\lambda_q \ast 1 = h_q \ast a_2$, hence

$$\sum_{n \leq x} (\lambda_q \ast 1)(n) = \sum_{d \leq x} h_q(d) \left\lfloor \frac{x}{d} \right\rfloor = x^{1/2} \sum_{d \leq x} \frac{h_q(d)}{\sqrt{d}} + O \left( x^{1/3+\epsilon} \right) = x^{1/2} H_q \left( \frac{1}{2} \right) + O \left( x^{1/3+\epsilon} \right).$$\)

4.3. The case $q \equiv \pm 5 \pmod{24}$. In this case, it is necessary to rewrite $L(s, \lambda_q)$ in the following shape.

Lemma 4. Assume $q \equiv \pm 5 \pmod{24}$. For any $\sigma > 1$, $L(s, \lambda_q) = \frac{K_q(s)}{\zeta(s)\zeta(2s)}$ with

$$K_q(s) := \begin{cases} 
\zeta(5s), & \text{if } q = 5 \\
\zeta(4s)L_q(s), & \text{if } q \equiv \pm 19, \pm 29 \pmod{120} \\
\zeta(s)\zeta(4s), & \text{if } q \equiv \pm 43, \pm 53 \pmod{120}
\end{cases}$$\)

where

$$L_q(s) := \zeta(qs) \prod_p \left( 1 + \frac{2(p^{2s} + p^s + 1)}{p^{3s} - p^{3s}} + \frac{p^{2s} + 1}{p^{3s} - 1} \sum_{m=6}^{q-1} \left\{ \frac{m+1}{q} + \frac{m}{q^s} \right\} \frac{1}{p^{ms}} \right)$$\)

and

$$\mathcal{L}_q(s) := \zeta(qs) \prod_p \left( 1 - \frac{2p^{2s} - 1}{(p^{2s} - 1)^3 (p^{2s} + 1)} + \frac{p^{2s}}{(p^{2s} - 1)^3 (p^{2s} + 1)} \sum_{m=6}^{q-1} \left\{ \frac{m+1}{q} + \frac{m}{q^s} \right\} \frac{1}{p^{ms}} \right).$$\)

The Dirichlet series $L_q$ is absolutely convergent in the half-plane $\sigma > \frac{1}{5}$, and the Dirichlet series $\mathcal{L}_q$ is absolutely convergent in the half-plane $\sigma > \frac{1}{6}$.

Proof. From Proposition 3, we immediately get

\[ (3) \]

$$L(s, \lambda_q) = \frac{\zeta(5s)}{\zeta(s)\zeta(2s)}.$$\)

Now suppose $q > 5$ and $q \equiv \pm 5 \pmod{24}$. In this case, $\left(\frac{3}{q}\right) + \left(\frac{2}{q}\right) = -2$ and $\left(\frac{4}{q}\right) + \left(\frac{4}{q}\right) = 0$ so that we may write by Proposition 3

$$L(s, \lambda_q) = \frac{\zeta(qs)\zeta(2s)}{\zeta(s)\zeta(2s)} \prod_p \left( 1 - \frac{2}{p^{2s} + \sum_{m=4}^{q-1} \left\{ \frac{m+1}{q} + \frac{m}{q^s} \right\} \frac{1}{p^{ms}} \right)$$\)

where

$$K_q(s) := \zeta(qs) \prod_p \left( 1 - \frac{1}{(p^{2s} - 1)^2} + \frac{p^{4s}}{(p^{2s} - 1)^2} \sum_{m=4}^{q-1} \left\{ \frac{m+1}{q} + \frac{m}{q^s} \right\} \frac{1}{p^{ms}} \right).$$\)
Assume \( q \equiv \pm 19, \pm 29 \pmod{120} \). Then
\[
\left( \frac{5}{q} \right) + \left( \frac{4}{q} \right) = \left( \frac{6}{q} \right) + \left( \frac{5}{q} \right) = 2.
\]

\( K_q(s) \) can therefore be written as
\[
K_q(s) = \zeta(qs) \prod_p \left( 1 + \frac{p^s + 2}{p^s (p^{2s} - 1)^2} + \frac{q^{4s}}{(p^{2s} - 1)^2} \sum_{m=6}^{q-1} \left( \frac{m+1}{q} \right) \frac{1}{p^m} \right)
\]
\[
= \zeta(qs) \zeta(4s) \prod_p \left( 1 + \frac{2(p^s - 1) + 1}{p^s - p^{2s}} + \frac{q^{2s} + 1}{p^{2s} - 1} \sum_{m=6}^{q-1} \left( \frac{m+1}{q} \right) \frac{1}{p^m} \right)
\]
\[
= \zeta(4s) L_q(s).
\]

Similarly, if \( q \equiv \pm 43, \pm 53 \pmod{120} \), then
\[
\left( \frac{5}{q} \right) + \left( \frac{4}{q} \right) = \left( \frac{6}{q} \right) + \left( \frac{5}{q} \right) = 0.
\]

Hence
\[
K_q(s) := \zeta(qs) \prod_p \left( 1 - \frac{1}{(p^{2s} - 1)^2} + \frac{p^{4s}}{(p^{2s} - 1)^2} \sum_{m=6}^{q-1} \left( \frac{m+1}{q} \right) \frac{1}{p^m} \right)
\]
\[
= L_q(s) \zeta(4s).
\]

The proof is complete.

We now are in a position to prove Theorem 1 in the case \( q \equiv \pm 5 \pmod{24} \).

Assume first that \( q \equiv \pm 19, \pm 29 \pmod{120} \) and let \( \ell_q(n) \) be the \( n \)-th coefficient of the Dirichlet series \( L_q(s) \). From Lemma 4 \( \lambda_q * 1 = \ell_q * a_1 * a_2^{-1} \) and therefore
\[
\sum_{n \leq x} (\lambda_q * 1) (n) = \sum_{d \leq x} \ell_q(d) \sum_{m \leq (x/d)^{1/4}} M \left( \frac{1}{m^2} \sqrt{\frac{x}{d}} \right) = \sum_{d \leq x} \ell_q(d) L \left( \sqrt{\frac{x}{d}} \right).
\]

Since \( L(z) \ll z \delta_c(z) \) for some \( c > 0 \)
\[
\sum_{n \leq x} (\lambda_q * 1) (n) \ll x^{1/2} \sum_{d \leq x} \frac{\ell_q(d)}{\sqrt{d}} \delta \left( \sqrt{\frac{x}{d}} \right)
\]
\[
\ll x^{1/2} \left( \sum_{d \leq x} + \sum_{\sqrt{x} < d \leq x} \right) \frac{|\ell_q(d)|}{\sqrt{d}} \delta \left( \sqrt{\frac{x}{d}} \right)
\]
\[
\ll x^{1/2} \delta_c \left( x^{1/4} \right) + x^{1/2} \sum_{d > \sqrt{x}} \frac{|\ell_q(d)|}{\sqrt{d}}.
\]

The Dirichlet series \( L_q(s) := \sum_{n=1}^{\infty} \ell_q(n) n^{-s} \) is absolutely convergent in the half-plane \( \sigma > \frac{1}{4} \), consequently
\[
\sum_{d \leq x} |\ell_q(d)| \ll q, \epsilon \ z^{1/5+\epsilon}
\]
and by partial summation
\[
\sum_{d > x} \frac{|\ell_q(d)|}{\sqrt{d}} \ll q, \epsilon \ z^{-3/10+\epsilon}.
\]

We infer that
\[
\sum_{n \leq x} (\lambda_q * 1) (n) \ll x^{1/2} \delta_c \left( x^{1/4} \right) + x^{7/20+\epsilon} \ll x^{1/2} \delta_c \left( x^{1/4} \right).
\]

Now suppose that the Riemann hypothesis is true. By [11], which is a refinement of [9], we know that \( M(z) \ll q, z^{1/2} \omega(z) \). The method of [9] [11] may be adapted to the function \( L \) yielding
\[
L(z) \ll q, z^{1/2} \omega(z) \log z.
\]

Observe that, for any \( a \geq 2, \epsilon > 0 \) and \( z \geq e^{\epsilon} \)
\[
\log z \exp \left( \sqrt{\log z (\log \log z)^a} \right) \ll \exp \left( \sqrt{\log z (\log \log z)^{a+\epsilon}} \right)
\]
so that \( L(z) \ll e^{z} \log z \) and hence

\[
\sum_{n \leq x} (\lambda_q \star 1)(n) \ll x^{1/4} \sum_{d \leq x} \frac{|\lambda_q(d)|}{d^{1/4}} \omega \left( \sqrt{\frac{x}{d}} \right) \ll x^{1/4} \omega \left( \sqrt{x} \right)
\]

achieving the proof in that case. The case \( q = 5 \) is similar but simpler since \( \lambda_5 \star 1 = a_5 \star a_2^{-1} \) by \( \text{[3]} \).

Finally, when \( q \equiv \pm 43, \pm 53 \pmod{120} \), we proceed as above. Let \( \nu_q(n) \) be the \( n \)-th coefficient of the Dirichlet series \( \mathcal{L}_q(s) \). Then \( \lambda_q \star 1 = \nu_q \star a_2^{-1} \star a_2^{-1} \) from Lemma \( \text{[4]} \) so that

\[
\sum_{n \leq x} (\lambda_q \star 1)(n) = \sum_{d \leq x} \nu_q(d) \sum_{m \leq (x/d)^{1/4}} \mu(m)M \left( \frac{1}{m^2} \sqrt{\frac{x}{d}} \right)
\]

and estimating trivially yields

\[
\sum_{n \leq x} (\lambda_q \star 1)(n) \ll x^{1/2} \sum_{d \leq x} \frac{|\nu_q(d)|}{\sqrt{d}} \sum_{m \leq (x/d)^{1/4}} \frac{1}{m^2} \delta_c \left( \frac{1}{m^2} \sqrt{\frac{x}{d}} \right)
\]

and we complete the proof as in the previous case. \( \square \)

**Remark 5.** Let us stress that a bound of the shape

\[
\sum_{n \leq x} (\lambda_q \star 1)(n) \ll x^{1/4 + \varepsilon}
\]

for all \( x \) sufficiently large and small \( \varepsilon > 0 \), is a necessary and sufficient condition for the Riemann hypothesis. Indeed, if this estimate holds, then by partial summation the series \( \sum_{n=1}^{\infty} (\lambda_q \star 1)(n)n^{-s} \) is absolutely convergent in the half-plane \( \sigma > \frac{5}{4} \). Consequently, the function \( K_q(s) \zeta(2s)^{-1} \) is analytic in this half-plane. In particular, \( \zeta(2s) \) does not vanish in this half-plane, implying the Riemann hypothesis, proving the necessary condition, the sufficiency being established above.

5. **A short interval result for the case \( q = 5 \)**

5.1. **Introduction.** This section deals with sums of the shape

\[
\sum_{x < n \leq x + y} (\lambda_5 \star 1)(n)
\]

where \( x^\varepsilon \leq y \leq x \). From Theorem \( \text{[11]} \)

\[
\sum_{x < n \leq x + y} (\lambda_5 \star 1)(n) \ll x^{1/2} e^{-c \left( \log x^{1/4} \right)^{3/5} \left( \log \log x^{1/4} \right)^{-1/5}}
\]

and if the Riemann hypothesis is true, then

\[
\sum_{x < n \leq x + y} (\lambda_5 \star 1)(n) \ll e^{c \left( \log x \right)^{1/2} \left( \log \log x \right)^{-3/2 + \varepsilon}}
\]

The purpose is to improve significantly upon these estimates when \( y = o(x) \), by using fine results belonging to the theory of integer points near a suitably chosen smooth curve. To this end, we need the following additional specific notation. Let \( \delta \in (0, \frac{1}{4}) \), \( N \in \mathbb{Z}_{\geq 1} \) large, \( f : [N, 2N] \rightarrow \mathbb{R} \) be any map, and define \( R(f, N, \delta) \) to be the number of elements of the set of integers \( n \in [N, 2N] \) such that \( \|f(n)\| < \delta \), where \( \|x\| \) is the distance from \( x \) to its nearest integer. Note that the trivial bound is given by

\[
\sum_{x < n \leq x + y} (\lambda_5 \star 1)(n) \ll \sum_{x < n \leq x + y} \tau(n) \ll y \log x.
\]

5.2. **Tools from the theory.** In what follows, \( N \in \mathbb{Z}_{\geq 1} \) is large and \( \delta \in (0, \frac{1}{4}) \). The first result is \( \text{[7]} \) Theorem 5] with \( k = 5 \). See also \( \text{[2]} \) Theorem 5.23 (iv)].

**Lemma 6** (5th derivative test). Let \( f \in C^5[N, 2N] \) such that there exist \( \lambda_4 > 0 \) and \( \lambda_5 > 0 \) satisfying \( \lambda_4 = N \lambda_5 \) and, for any \( x \in [N, 2N] \)

\[
|f^{(4)}(x)| \asymp \lambda_4 \quad \text{and} \quad |f^{(5)}(x)| \asymp \lambda_5.
\]

Then

\[
R(f, N, \delta) \ll N \lambda_5^{1/15} + N \delta^{1/6} + (\delta \lambda_4^{-1})^{1/4} + 1.
\]
Lemma 9. Let \( f \in C^1 \left( [N, 2N] \right) \) such that there exist \( \lambda_1 > 0 \) such that \( |f'(x)| \geq \lambda_1 \). Then
\[
\mathcal{R}(f, N, \delta) \ll N \lambda_1 + N \delta + \delta \lambda_1^{-1} + 1.
\]
This result is essentially a consequence of the mean value theorem.

The second tool is \( [4] \) Theorem 7] with \( k = 3 \).

Lemma 8. Let \( s \in \mathbb{Q}^* \setminus \{ \pm 2, \pm 1 \} \) and \( X > 0 \) such that \( N \leq X^{1/s} \). Then there exists a constant \( c_3 := c_3(s) \in (0, \frac{1}{2}) \) depending only on \( s \) such that, if
\[
N^2 \delta \leq c_3
\]
then
\[
\mathcal{R} \left( \frac{X}{n^s}, N, \delta \right) \ll (XN^{3-s})^{1/7} + \delta (XN^{59-s})^{1/21}.
\]

Our last result relies the short sum of \( \lambda_5 \ast 1 \) to a problem of counting integer points near a smooth curve.

Lemma 9. Let \( 1 \leq y \leq x \). Then
\[
\sum_{x < n \leq x + y} (\lambda_5 \ast 1)(n) \ll \max_{(16y^2 x^{-1})^{1/5} < N \leq (2x)^{1/5}} \mathcal{R} \left( \frac{x}{n^5}, N, \frac{y}{\sqrt{N^{3/5}}} \right) \log x + y x^{-1/2} + x^{-1/3} y^{2/5}.
\]

Proof. Using (3), we get
\[
\sum_{n \leq x} (\lambda_5 \ast 1)(n) = \sum_{d \leq \sqrt{x}} \mu(d) \left( \frac{x}{d^5} \right)^{1/5}
\]
so that
\[
\sum_{x < n \leq x + y} (\lambda_5 \ast 1)(n) \ll \sum_{d \leq \sqrt{x}} \mu(d) \left( \left[ \frac{x + y}{d^5} \right]^{1/5} - \left[ \frac{x}{d^5} \right]^{1/5} \right) + \sum_{\sqrt{x} < d \leq x + y} \mu(d)
\ll \sum_{d \leq \sqrt{x}} \left( \left[ \frac{x + y}{d^5} \right]^{1/5} - \left[ \frac{x}{d^5} \right]^{1/5} \right) + y x^{-1/2}
\ll \sum_{d \leq \sqrt{x}} \sum_{x < d^2 n \leq x + y} 1 + y x^{-1/2}
\ll \sum_{n \leq (2x)^{1/5}} \sum_{x < d^2 n \leq x + y} \left[ \frac{x + y}{n^5} \right]^{1/5} - \left[ \frac{x}{n^5} \right]^{1/5} + x^{-1/3} y^{2/5} + y x^{-1/2}
\ll \max_{(16y^2 x^{-1})^{1/5} < N \leq (2x)^{1/5}} \mathcal{R} \left( \frac{x}{n^5}, N, \frac{y}{\sqrt{N^{3/5}}} \right) \log x + y x^{-1/2} + x^{-1/3} y^{2/5}
\]
and for any integers \( N \in (16y^2 x^{-1})^{1/5}, (2x)^{1/5} \) and \( n \in [N, 2N] \)
\[
\sqrt{\frac{x + y}{n^5}} - \sqrt{\frac{x}{n^5}} < \frac{y}{\sqrt{N^{3/5}}} < \frac{1}{4}
\]
so that the sum does not exceed
\[
\ll \max_{(16y^2 x^{-1})^{1/5} < N \leq (2x)^{1/5}} \mathcal{R} \left( \frac{x}{n^5}, N, \frac{y}{\sqrt{N^{3/5}}} \right) \log x + x^{-1/3} y^{2/5} + y x^{-1/2}
\]
as asserted. \( \square \)

5.3. The main result.

Theorem 10. Assume \( y \leq c_3 x^{1/20} \) where \( c_3 := c_3 \left( \frac{5}{4} \right) \) is given in (3). Then
\[
\sum_{x < n \leq x + y} (\lambda_5 \ast 1)(n) \ll \left( x^{1/12} + y x^{-4/9} \right) \log x.
\]
Furthermore, if \( y \leq c_3 x^{10/36} \)
\[
\sum_{x < n \leq x + y} (\lambda_5 \ast 1)(n) \ll x^{1/12} \log x.
\]
**Proof.** We split the first term in Lemma 4 into three parts, according to the ranges
\[(16y^2 x^{-1})^{1/5} < N \leq 2x^{1/10}, \quad 2x^{1/10} < N \leq 2x^{1/6} \quad \text{and} \quad 2x^{1/6} < N \leq (2x)^{1/5}.
\]
In the first case, we use Lemma 8 with 
\[
\lambda_4 = (xN^{-13})^{1/2} \quad \text{and} \quad \lambda_5 = (xN^{-15})^{1/2}
\]
which yields
\[
\max_{(16y^2 x^{-1})^{1/5} < N \leq 2x^{1/10}} \mathcal{R} \left( \frac{x}{n^5}, N, \frac{y}{\sqrt{N^3 x}} \right) \ll x^{1/12} + x^{-1/40} y^{1/6} + x^{-3/20} y^{1/4}.
\]

For the second range, we use Lemma 8 with 
\[X = x^{1/2}, \ s = \frac{5}{2}, \text{ and } \delta = y \left( N^3 x \right)^{-1/2}.
\]
Notice that the conditions 
\[N > 2x^{1/10} \quad \text{and} \quad y \leq c_3 x^{11/20}
\]
ensure that 
\[\delta < \frac{1}{2} \quad \text{and} \quad N^2 \delta \leq c_3.
\]
We get
\[
\max_{2x^{1/10} < N \leq 2x^{1/6}} \mathcal{R} \left( \frac{x}{n^5}, N, \frac{y}{\sqrt{N^3 x}} \right) \ll x^{1/12} + yx^{-4/9}.
\]

The last range is easily treated with (4), giving
\[
\max_{2x^{1/6} < N \leq (2x)^{1/5}} \mathcal{R} \left( \frac{x}{n^5}, N, \frac{y}{\sqrt{N^3 x}} \right) \ll x^{1/12} + yx^{-3/4}.
\]

Using Lemma 9 we finally get
\[
\sum_{x < n \leq x+y} \lambda_5 \ast 1 (n) \ll \left( x^{1/12} + x^{-1/40} y^{1/6} + x^{-3/20} y^{1/4} + yx^{-4/9} \right) \log x + x^{-1/5} y^{2/5}
\]
and note that 
\[x^{-1/40} y^{1/6} + x^{-3/20} y^{1/4} + x^{-1/5} y^{2/5} \ll x^{1/12} \quad \text{as soon as} \quad y \ll x^{13/20}.
\]
This completes the proof of the first estimate, the second one being obvious.

\[\square\]

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