Multi-bump solutions for logarithmic Schrödinger equations

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Abstract
We study spatially periodic logarithmic Schrödinger equations:
\[
\begin{cases}
-\Delta u + V(x)u = Q(x)u \log u^2, & u > 0 \text{ in } \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N),
\end{cases}
\] (LS)
where \(N \geq 1\) and \(V(x), Q(x)\) are spatially 1-periodic functions of class \(C^1\). We take an approach using spatially 2L-periodic problems \((L \gg 1)\) and we show the existence of infinitely many multi-bump solutions of \((LS)\) which are distinct under \(\mathbb{Z}^N\)-action.

1 Introduction
We study the existence of solutions of the following spatially periodic logarithmic Schrödinger equation:
\[
\begin{cases}
-\Delta u + V(x)u = Q(x)u \log u^2, & u > 0 \text{ in } \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N),
\end{cases}
\] (LS1)
where \(N \in \mathbb{N}\) and \(V, Q \in C^1(\mathbb{R}^N, \mathbb{R})\) satisfy
\begin{equation}
(A1) \quad V(x), Q(x) > 0 \text{ for all } x \in \mathbb{R}^N;
\end{equation}

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(A2) \( V(x) \), \( Q(x) \) are 1-periodic in each \( x_i \) \((i = 1, 2, \ldots, N)\), that is,

\[
V(x_1, \cdots, x_i+1, \cdots, x_N) = V(x_1, \cdots, x_i, \cdots, x_N),
\]

\[
Q(x_1, \cdots, x_i+1, \cdots, x_N) = Q(x_1, \cdots, x_i, \cdots, x_N)
\]

for all \( x = (x_1, x_2, \cdots, x_N) \in \mathbb{R}^N \) and \( i = 1, \ldots, N \).

**Remark 1.1.** Substituting \( u \) with \( \lambda v \) \((\lambda > 0)\) in \([LS1]\), we get

\[
-\Delta v + (V(x) - Q(x) \log \lambda^2) v = Q(x) v \log v^2.
\]

Under the conditions (A2) and \( Q(x) > 0 \) in \( \mathbb{R}^N \), the positivity of \( V(x) \) is not essential. In fact, replacing \( V(x) \) with \( \tilde{V}(x) \equiv V(x) - Q(x) \log \lambda^2 \), we get \( \tilde{V}(x) > 0 \) for a suitable \( \lambda > 0 \) and thus we can recover positivity for any \( V(x) \).

\([LS1]\) has applications to physics (e.g. quantum mechanics, quantum optics etc. See Zloshchastiev [24] and references therein).

Formally solutions of \([LS1]\) are characterized as critical points of

\[
I_{\infty}(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 - \frac{1}{2} \int_{\mathbb{R}^N} Q(x)(u^2 \log u^2 - u^2).
\]

However \( \int_{\mathbb{R}^N} u^2 \log u^2 \) is not well-defined on \( H^1(\mathbb{R}^N) \) due to the behavior of \( s^2 \log s^2 \) as \( s \to 0 \) and thus we cannot apply standard critical point theories to \( I_{\infty}(u) \). To overcome such difficulties, d’Avenia-Montefusco-Squassina [9], Squassina-Szulkin [22] and Ji-Szulkin [12] applied non-smooth critical point theory for lower semi-continuous functionals. In [9], d’Avenia, Montefusco and Squassina dealt with the case, where \( V \) and \( Q \) are constant, and they showed the existence of infinitely many radially symmetric possibly sign-changing solutions. They also showed the unique (up to translations) positive solution is so-called \( \text{Gaussson } \lambda e^{-|x|^2/2} \) for suitable constants \( \lambda, \mu > 0 \). See Biadynicki-Birula and Mycielski [23] for the \( \text{Gaussson } \) and related topics. See also [10] for logarithmic Schödinger equations with fractional Laplacian. In [22], Squassina and Szulkin considered spatially periodic \( V(x) \) and \( Q(x) \) and they showed the existence of a ground state and infinitely many possibly sign-changing solutions, which are geometrically distinct under \( Z^N \)-action. Here we say that \( u(x), v(x) \in H^1(\mathbb{R}^N) \) are geometrically distinct if and only if \( u(x+n) \neq v(x) \) for all \( n \in Z^N \). In [12], Ji and Szulkin also applied non-smooth variational framework to obtain a ground state and infinitely many solutions for logarithmic Schrödinger equation under the setting: \( Q(x) \equiv 1 \) and \( \lim_{|x| \to \infty} V(x) = \sup_{x \in \mathbb{R}^N} V(x) \in (-\infty, \infty) \) or \( V(x) \to \infty \) as \( |x| \to \infty \). To get infinitely many solutions, symmetry of functional \( I_{\infty}(u) \) (i.e. evenness) and pseudo-index theories are important in [9][12][22]. We also refer to Cazenave [4] and Guerrero-López-Nieto [11] for approaches using a Banach space with a Luxemburg type norm or penalization.

In this paper, we take another approach, which is inspired by Coti Zelati-Rabinowitz [5] and Chen [5], and we try to construct solutions of \([LS1]\) through spatially 2\( L \)-periodic solutions:

\[
\begin{aligned}
-\Delta u + V(x)u &= Q(x)u \log u^2 \quad \text{in } \mathbb{R}^N, \\
u(x + 2Ln) &= u(x) \quad \text{for all } x \in \mathbb{R}^N \text{ and } n \in Z^N.
\end{aligned}
\]
That is, first we find a solution $u_L(x)$ of (L1) and second, after a suitable shift, we take a limit as $L \to \infty$ to obtain a solution of (LS1). See also Rabinowitz [18] and Tanaka [23] for earlier works.

The main purpose of this paper is to find multi-bump positive solutions of (LS1) with this approach. In particular, we will show the existence of infinitely many geometrically distinct positive solutions under $\mathbb{Z}^N$-action. We note that in our argument symmetry of the functional is not important. Actually we find critical points of the following modified functional $J_\infty(u)$, which is not even and whose critical points are non-negative solutions of (LS1) (see Section 2 below).

$$J_\infty(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 - \int_{\mathbb{R}^N} Q(x)G(u).$$

Here $G(u) = \int_0^u g(s) \, ds$ and $g(u)$ is defined in (2.1).

To state our main result, we need some preliminaries. We set

$$\mathcal{D} = \{ u \in H^1(\mathbb{R}^N) ; \int_{\mathbb{R}^N} u^2 |\log u|^2 < \infty \},$$

$$\mathcal{N}_\infty = \{ u \in \mathcal{D} \setminus \{ 0 \} ; \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 - Q(x)g(u)u = 0 \}.$$

Any non-trivial non-negative solution of (LS1) lies in $\mathcal{N}_\infty$ and we consider

$$b_\infty = \inf_{u \in \mathcal{N}_\infty} J_\infty(u). \quad (1.2)$$

We denote

$$\mathcal{K}_\infty = \{ u \in \mathcal{D} ; u \text{ is a positive solution of (LS1)} \},$$

$$[J_\infty = c]_\infty = \{ u \in \mathcal{D} ; J_\infty(u) = c \},$$

$$[J_\infty \leq c]_\infty = \{ u \in \mathcal{D} ; J_\infty(u) \leq c \} \quad \text{for } c \in \mathbb{R}.$$

By the periodicity of $V$ and $Q$, we note that $\mathcal{K}_\infty$ is invariant under $\mathbb{Z}^N$-action. Following [6, 7, 19], to discuss multiplicity of positive solutions, we assume that for $\alpha > 0$ small

$$(\mathcal{K}_\infty \cap [J_\infty \leq b_\infty + \alpha])/{\mathbb{Z}^N} \text{ is finite.} \quad (\ast)$$

We note that if (\ast) does not hold, (LS1) has infinitely many geometrically distinct solutions. We choose a finite set

$$\mathcal{F}_\infty = \{ u^i \in \mathcal{K}_\infty \setminus \{ 0 \} ; 1 \leq i \leq m \}$$

such that $u^i \neq w^j(\cdot + n)$ for $1 \leq i, j \leq m$, $i \neq j$, $n \in \mathbb{Z}^N$, and $\mathcal{K}_\infty \cap [J_\infty = b_\infty]_\infty = \{ w^i(\cdot + n) ; 1 \leq i \leq m; n \in \mathbb{Z}^N \}$.

Now we can state our main result, which deals with 2-bump solutions.

**Theorem 1.** Assume (A1), (A2) and (\ast). Then for any $r > 0$, there exists $R_r > 0$ such that for any $P \in \mathbb{Z}^N$ with $|P| > 5R_r$ and $\omega$, $\omega' \in \mathcal{F}_\infty$,

$$\mathcal{K}_\infty \cap B_{2r}^{\infty}(\omega + \omega'(\cdot - P)) \neq \emptyset.$$

Here we use notation:

$$B_r^{\infty}(u) = \{ v \in H^1(\mathbb{R}^N) ; \| v - u \|_{H^1(\mathbb{R}^N)} < r \} \quad \text{for } u \in H^1(\mathbb{R}^N) \text{ and } r > 0.$$
We remark that such multi-bump type solutions were constructed via variational methods firstly by Séré [19] and Coti Zelati-Rabinowitz [6] for Hamiltonian systems and by Coti Zelati-Rabinowitz [7] for nonlinear elliptic equations. See also Alama-Li [1], Liu-Wang [14, 15], Montecchiari [16] and Séré [20].

**Remark 1.2.** In Theorem 1, we state the existence of 2-bump solutions. In a similar way, we can find $k$-bump solutions for any $k \in \mathbb{N}$.

To show Theorem 1, as we stated earlier, we will find a solution of (LS1) through 2$L$-periodic problems (PL) with large $L \in \mathbb{N}$:

\[
\begin{cases}
- \Delta u + V(x)u = Q(x)g(u) & \text{in } \mathbb{R}^N, \\
u(x + 2Ln) = u(x) & \text{for all } x \in \mathbb{R}^N \text{ and } n \in \mathbb{Z}^N.
\end{cases}
\]  

(PL)

That is, first we find a solution of (PL), which has 2-bumps in $[-L,L]^N$ — we call such a solution 2$L$-periodic 2-bump — and second we take a limit as $L \to \infty$. Such approaches were taken by Coti Zelati-Rabinowitz [8] and Chen [5] for Hamiltonian systems and strongly indefinite nonlinear Schrödinger equations. They succeeded to construct multi-bump type periodic solutions through periodic multi-bump solutions.

Solutions of (PL) is characterized as critical points of

\[
J_L(u) = \frac{1}{2} \int_{D_L} |\nabla u|^2 + V(x)u^2 - \frac{1}{2} \int_{D_L} Q(x)G(u) : E_L \to \mathbb{R},
\]

where

\[
D_L = [-L,L]^N, \\
E_L = \{ u \in H^1_{\text{loc}}(\mathbb{R}^N); u(x) \text{ is } 2L\text{-periodic in } x_i \text{ for all } i = 1, 2, \cdots, N \}.
\]

In our problem, the approach using (PL) gives us a merit that the functional $\int_{D_L} u^2 \log u^2$ and $J_L(u)$ are well-defined and of class $C^1$ since we work essentially in a bounded domain $D_L$ for (1.1). Thus for (1.1) we are in a better situation than the original problem in $\mathbb{R}^N$ and we can apply the standard critical point theory to $J_L(u)$. Our result for (PL) is the following theorem, in which we prove the existence of 2$L$-periodic 2-bump solutions.

**Theorem 2.** Assume (A1), (A2) and (3). Then for any $r > 0$, there exists $R_{r0} > 0$ such that for any $L \in \mathbb{N}$, $P \in \mathbb{Z}^N$ with $|P| > 5R_{r0}$, $L > 2|P|$, and $\omega, \omega' \in \mathcal{F}_\infty$,

\[
\mathcal{K}_L \cap B^L_r(\Phi_L(\psi_{R,r,0}\omega) + \Phi_L(\psi_{R,r,0}\omega')(\cdot - P)) \neq \emptyset.
\]

In Theorem 2 we use notation:

\[
\mathcal{K}_L = \{ u \in E_L; u \text{ is a positive solution of (PL)} \}, \\
B^L_r(u) = \{ v \in E_L; \| v - u \|_{H^1(D_L)} < r \} \text{ for } u \in H^1(\mathbb{R}^N) \text{ and } r > 0.
\]

and $\psi_{R_r}(x)$ is a suitable cut-off function around 0 and $\Phi_L(\psi_{R,r}\omega)(x)$ is a 2$L$-periodic extension of $\psi_{R_r}(x)$. See Section 2 below for a precise definition.

This paper is organized as follows: In Section 2 first we introduce truncation of nonlinearity $u \log u^2$ and a modified functional together with its fundamental
where we study (LS1), we introduce the following modified problem:

2.1 A modified problem

We give a proof of our concentration-compactness type result (Proposition 3.1). It enables us to take a limit as \( L \to \infty \) in (LS1) to obtain a solution of our original problem (LS1). In Proposition 3.1, uniform estimates of \( \int_{D_L} H(u_j) \) as well as \( \| u_j \|_{H^1(D_L)} \) for Palais-Smale type sequences \( (u_j)_{j=1}^{\infty} \) \( (u_j)_{j=1}^{\infty} \) \( u_j \in E_{L_j}, \ L_j \to \infty \) is important. In Section 4, we define mountain pass value \( b_L \) for \( J_L(u) \) and we study its behavior as \( L \to \infty \). It enables us to show that \( b_\infty \) defined in (2.1) is achieved and the existence of the ground state (Theorem 3 and Corollary 3.1) is proved. Moreover, it is also important to show the existence of 2-bump solutions in Section 5. In Section 5, we give proofs to Theorems 1 and 2. A gradient estimate in annular type neighborhood (Proposition 5.2) and deformation lemma (Lemma 5.3) are important. Our deformation flow is constructed so that it keeps invariant for suitable constant \( R \gg 1 \) and \( \rho > 0 \). Together with the elliptic decay estimate in \( D_L \setminus (B_R(0) \cup B_R(P)) \), this property enables us to get a critical point \( u_L \) of \( J_L(u) \) with an estimate on \( \int_{D_L} u^2 \log u^2 \) independent of \( L \). In Appendix, we give a proof of our concentration-compactness type result (Proposition 3.1).

2 Preliminaries

2.1 A modified problem

To study (LS1), we introduce the following modified problem:

\[
\begin{cases}
-\Delta u + V(x)u = Q(x)g(u) & \text{in } \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N),
\end{cases}
\]

(2.1)

where \( g(u) = -h(u) + f(u) \) and

\[
h(u) = \begin{cases} 
-u \log u^2, & \text{if } |u| \leq e^{-1}, \\
2e^{-1}, & \text{if } |u| > e^{-1}, \\
-2e^{-1}, & \text{if } |u| < -e^{-1},
\end{cases}
\]

\[
f(u) = \begin{cases} 
0, & \text{if } u \leq e^{-1}, \\
2e^{-1} + u \log u^2, & \text{if } u > e^{-1}.
\end{cases}
\]

We note that

\[
g(u) = -h(u) + f(u) = \begin{cases} 
\log u^2, & \text{if } u \geq -e^{-1}, \\
2e^{-1}, & \text{if } u < -e^{-1}.
\end{cases}
\]

As we will see in Lemma 5.3, non-zero solutions of (LS2) are positive solutions of (LS1). We set

\[
\begin{align*}
H(u) &= \int_0^u h(s) \, ds = \begin{cases} 
-\frac{1}{2} u^2 \log u^2 + \frac{1}{2} u^2, & \text{if } |u| \leq e^{-1}, \\
\frac{1}{2} |u| - \frac{1}{2e^2}, & \text{if } |u| > e^{-1},
\end{cases} \\
F(u) &= \int_0^u F(s) \, ds, \quad G(u) = -H(u) + F(u) = \int_0^u g(s) \, ds.
\end{align*}
\]
We note that $H(u)$ is a convex part of $-\frac{1}{4}u^2 \log u^2 + \frac{1}{2}u^2$, which is sub-quadratic near 0, and $F(u)$ is a super-quadratic part of $\frac{3}{2}u^2 \log u^2 - \frac{1}{2}u^2$, which is modified for $u < 0$ so that solutions of (LS2) correspond to non-negative solutions of (LS1).

Here we give some properties of $g(u), h(u)$ and $f(u)$.

**Lemma 2.1.** (i) $g \in C^1(\mathbb{R} \setminus \{0\}) \cap C(\mathbb{R})$ and $\frac{g(s)}{s}$ is strictly increasing in $(0, \infty)$ and strictly decreasing in $(-\infty, 0)$;

(ii) $f \in C^1(\mathbb{R})$ is positive and increasing in $[0, \infty)$. Moreover

- (a) $f(s)s \geq 2F(s) \geq 0$ for all $s \in \mathbb{R}$;
- (b) for any $p > 2$ there is a constant $C_p > 0$ depending on $p$, such that
  
  \[ |f(s)| \leq C_p |s|^{p-1}, \quad F(s) \leq C_p |s|^p \quad \text{for all } s \in \mathbb{R}; \]

(iii) $h \in C(\mathbb{R})$ is positive, increasing and concave in $[0, \infty)$ and $H(s)$ is convex on $\mathbb{R}$. Moreover

- (a) For all $s > 0$ and $\theta \in [0, 1]$
  
  \[ \theta h(s) \leq h(\theta s), \quad \theta^2 H(s) \leq H(\theta s), \quad \frac{1}{2}h(s)s \leq H(s) \leq h(s)s; \]

- (b) For all $s, t \in \mathbb{R}$ and $\theta \in (0, 1]$
  
  \[ |h(s)t| \leq \theta H(s) + \frac{1}{\theta} H(t). \]

We will give a proof of Lemma 2.1 at the end of this section.

We note that for $u \in H^1(\mathbb{R}^N)$

\[ \int_{\mathbb{R}^N} u^2 |\log u|^2 < \infty \quad \text{if and only if} \quad \int_{\mathbb{R}^N} H(u) < \infty. \]

and we can write

\[ \mathcal{D} = \{ u \in H^1(\mathbb{R}^N); \int_{\mathbb{R}^N} H(u) < \infty \}. \]

By Lemma 2.1 we have

**Lemma 2.2.** (i) $u \mapsto \int_{\mathbb{R}^N} F(u); H^1(\mathbb{R}^N) \to \mathbb{R}$ is of class $C^1$;

(ii) For any $u, v \in \mathcal{D}$, $\int_{\mathbb{R}^N} h(u)v$ is well-defined;

(iii) $C_0^\infty(\mathbb{R}^N)$ is dense in $\mathcal{D}$ in the following sense: for any $u \in \mathcal{D}$, there exists a sequence $\{ \varphi_n \}_{n=1}^\infty \subset C_0^\infty(\mathbb{R}^N)$ such that

\[ \| \varphi_n - u \|_{H^1} \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} H(\varphi_n) \to \int_{\mathbb{R}^N} H(u). \]
We say that \( u \in H^1(\mathbb{R}^N) \) is a weak solution of (\text{LS1}) (resp. (\text{LS2})) if \( u \) satisfies \( u \in \mathcal{D} \) and
\[
\int_{\mathbb{R}^N} \nabla u \nabla \varphi + V(x)u\varphi - Q(x)u\log u^2 = 0
\]
(resp. \( \int_{\mathbb{R}^N} \nabla u \nabla \varphi + V(x)u\varphi - Q(x)g(u)\varphi = 0 \)) for all \( \varphi \in C_c^\infty(\mathbb{R}^N) \).

Next lemma ensures that any solution of (\text{LS2}) is a nonnegative solution of (\text{LS1}). In what follows, for \( u \in \mathbb{R} \), we denote \( u^+ = \max\{u, 0\} \), \( u^- = \max\{-u, 0\} \).

**Lemma 2.3.** For \( u \in \mathcal{D} \), the following statements are equivalent:

(i) \( u(x) \) is a non-negative weak solution of (\text{LS1});

(ii) \( u(x) \) is a weak solution of (\text{LS2}).

Moreover, any non-trivial solution of (\text{LS2}) are positive on \( \mathbb{R}^N \).

**Proof.** We show just (ii) implies (i). Suppose \( u \in \mathcal{D} \) is a solution to (\text{LS2}). We take \( \varphi = u^- \) in (2.6). Since \( f(u^-)u^- = 0 \) and \( h(u^-)u^- = -h(u^-)u^- \),
\[
\int_{\mathbb{R}^N} (|\nabla u^-|^2 + V(x)(u^-)^2 + h(u^-)u^-) = 0.
\]
So \( u^- = 0 \) and \( u \geq 0 \). Moreover, if \( u \neq 0 \), then \( u > 0 \) by the strong maximum principle. \( \square \)

To study the solution to (\text{LS2}), we consider the energy functional \( J_\infty : E_\infty \to \mathbb{R} \cup \{+\infty\} \) associated to (\text{LS2}):
\[
J_\infty(u) = \frac{1}{2}\|u\|_{E_\infty}^2 - \int_{\mathbb{R}^N} Q(x)G(u),
\]
where we write \( E_\infty = H^1(\mathbb{R}^N) \) and we introduce the following inner product and norm to \( E_\infty \):
\[
\langle u, v \rangle_{E_\infty} = \int_{\mathbb{R}^N} \nabla u \nabla v + V(x)uv,
\]
\[
\|u\|_{E_\infty} = \sqrt{\langle u, u \rangle_{E_\infty}} \quad \text{for } u, v \in E_\infty.
\]

To study (\text{LS2}), we introduce the notion of derivatives and critical points of \( J_\infty \).

**Definition 2.1.**

(i) For \( u, v \in \mathcal{D} \), we define
\[
J_\infty'(u)v \equiv \langle u, v \rangle_{E_\infty} - \int_{\mathbb{R}^N} Q(x)g(u)v.
\]
We note that \( \int_{\mathbb{R}^N} Q(x)h(u)v \) is well-defined by Lemma 2.2 (ii).

(ii) We say that \( u \in E_\infty \) is a critical point of \( J_\infty \) if \( u \in \mathcal{D} \) and \( J_\infty'(u)v = 0 \) for all \( v \in \mathcal{D} \). We also say that \( c \in \mathbb{R} \) is a critical value for \( J_\infty \) if there exists a critical point \( u \in E_\infty \) such that \( J_\infty(u) = c \). We also use notation \( \mathcal{K}_\infty = \{ u \in \mathcal{D}; J_\infty'(u)v = 0 \text{ for all } v \in \mathcal{D} \} \).
Remark 2.1. By Lemma 2.2 (iii), $u \in \mathcal{D}$ is a critical point of $J_\infty$ if and only if
\[ J'_\infty(u)\varphi = 0 \quad \text{for all } \varphi \in C^\infty_0(\mathbb{R}^N). \]

Remark 2.2. There exists $\rho_\infty > 0$ such that
\[ \|w\|_{E_\infty} \geq \rho_\infty \quad \text{for all } w \in \mathcal{K}_\infty \setminus \{0\}. \]

In fact, we have for small $\rho_\infty > 0$
\[ J'_\infty(u)u \geq \frac{1}{4}\|u\|_{E_\infty}^2 \quad \text{for all } u \in \mathcal{D} \text{ with } \|u\|_{E_\infty} \leq \rho_\infty. \]

At the end of this section, we give a proof of Lemma 2.1

Proof of Lemma 2.1. We can easily show (i) and (ii). Here we give a proof to (iii). (iii-a) By the concavity of $s \mapsto h(s); [0, \infty) \to \mathbb{R}$, we have (2.2). Integrating (2.2), we get (2.3). We can also verify (2.4) easily.

(iii-b) It suffices to show (2.5) for $s, t > 0$ and $\theta \in (0, 1]$. Setting $H^*(y) = \sup_{z \in \mathbb{R}}(yz - H(z))$, i.e., the convex conjugate of $H(x)$, we have the following Fenchel’s inequality:
\[ xy \leq H(x) + H^*(y) \quad \text{for all } x, y \in \mathbb{R}. \]

Setting $x = t$, $y = \theta h(s)$, we have
\[ \theta h(s)t \leq H^*(\theta h(s)) + H(t). \]

To show (2.5), it suffices to show
\[ H^*(\theta h(s)) \leq \theta^2 H(s). \tag{2.7} \]

We set $\phi(z) = \theta h(s)z - H(z)$. Then we have $H^*(\theta h(s)) = \sup_{z \in \mathbb{R}} \phi(z)$. Noting by (2.4)
\[ \phi'(0) = \theta h(s) \geq 0, \quad \phi'(\theta s) = \theta h(s) - h(\theta s) \leq 0, \]
\[ \phi(z) \text{ takes its maximum in } [0, \theta s] \text{ and there exists } a \in [0, 1] \text{ such that} \]
\[ H^*(\theta h(s)) = \phi(a\theta s) = a\theta^2 h(s) - H(a\theta s) \leq 2a\theta^2 H(s) - (a\theta)^2 H(s) = (2a - a^2)\theta^2 H(s) \leq \theta^2 H(s). \]

Here we used (2.3) and (2.4). Thus we obtain (2.7). \hfill \Box

2.2 2L-periodic problems

As stated in Introduction, we construct a solution of (LS2) through 2L-periodic problems.

For $L \in \mathbb{N}$, we set $D_L = [-L, L]^N$ and we consider a Hilbert space
\[ E_L = \{ u(x) \in H^1_{loc}(\mathbb{R}^N); u(x + 2Ln) = u(x) \text{ for all } n \in \mathbb{Z}^N \} \]

with an inner product and a norm:
\[ \langle u, v \rangle_{E_L} = \int_{D_L} \nabla u \nabla v + V(x)uv, \]
\[ \|u\|_{E_L} = \sqrt{\langle u, u \rangle_{E_L}} \quad \text{for } u, v \in E_L. \]
We denote the dual space of $E_L$ and its norm by $E'_L$ and $\|\cdot\|_{E'_L}$.

For a subset $U \subset D_L$, we also denote

$$\|u\|_{E_L(U)} = \left( \int_U |\nabla u|^2 + V(x)u^2 \right)^{\frac{1}{2}}$$

for $u \in E_L$.

For $u \in E_L$ we define $J_L(u)$ by

$$J_L(u) = \frac{1}{2} \|u\|_{E_L}^2 - \int_{D_L} Q(x)G(u).$$

The equation corresponding to $J_L(u)$ is (PL). We note that we are essentially working in a bounded domain $D_L$ and $\int_{D_L} Q(x)H(u)$ is well-defined on $E_L$. We have

**Lemma 2.4.**

(i) $E_L \to \mathbb{R}$; $u \mapsto \int_{D_L} Q(x)H(u)$ is of class $C^1$;

(ii) $J_L(u) \in C^1(E_L, \mathbb{R})$ and any critical point of $J_L(u)$ is a weak solution of (PL), that is,

$$\int_{D_L} \nabla u \nabla \varphi + V(x)u\varphi - Q(x)g(u)\varphi = 0 \quad \text{for all } \varphi \in E_L,$$

We denote the critical point set of $J_L$ by

$$K_L = \{u \in E_L; J'_L(u) = 0\}.$$

As in Lemma 2.3 we have

**Lemma 2.5.** Any solution to (PL), i.e. any critical point of $J_L(u)$, is nonnegative.

As a fundamental property of $J_L(u)$, we see $J_L(u)$ possesses a mountain pass geometry uniformly in $L \in \mathbb{N}$.

**Lemma 2.6.**

(i) There are constants $C_1$, $r_1 > 0$ independent of $L$ such that

$$J_L(u) \geq C_1 \quad \text{for } \|u\|_{E_L} = r_1;$$

(ii) there exists $u_0 \in H^1_0(D_L)$ such that

$$\|\Phi_L(u_0)\|_{E_L} > r_1 \quad \text{and} \quad J_L(\Phi_L(u_0)) < 0.$$

where $r_1$ is the number in (i). Here we regard $u_0(x)$ as an element in $H^1_0(D_L)$ and $\Phi_L(u_0) \in E_L$ is a $2L$-periodic extension of $u_0(x)$. In particular, there exists a constant $C_2 > 0$ independent of $L \in \mathbb{N}$ such that

$$\max_{t \in [0,1]} J_L(t\Phi_L(u_0)) \leq C_2.$$

**Proof.** (i) Since $H(u) \geq 0$,

$$J_L(u) \geq \frac{1}{2} \|u\|_{E_L}^2 - C_p \|Q\|_{L^\infty} \int_{D_L} |u|^p \geq \frac{1}{2} \|u\|_{E_L}^2 - C_p \|u\|_{E_L}^p.$$
Thus the conclusion (i) holds for small \( r_1 > 0 \) and for a suitable constant \( C_1 > 0 \).

(ii) We choose \( u_1 \in H^1_0(D_1) \setminus \{0\} \) with \( u_1(x) \geq 0 \) in \( D_1 \) and consider \( J_L(t\Phi_L(u_1)) \) for \( t > 0 \). Here \( \Phi_L(u_1) \) is a \( 2L \)-periodic extension of \( u_1(x) \).

\[
J_L(t\Phi_L(u_1)) = \frac{1}{2} t^2 \|\Phi_L(u_1)\|^2_{E_L} + \int_{D_1} Q(x)G(t\Phi_L(u_1))
= \frac{1}{2} t^2 \|\Phi_L(u_1)\|^2_{E_L} + \frac{1}{2} \int_{D_1} Q(x)((tu_1)^2 \log((tu_1)^2) - (tu_1)^2)
= \frac{1}{2} t^2 \{ \|u_1\|^2_{H^1(D_1)} + \int_{D_1} Q(x)u_1^2 - \int_{D_1} Q(x)u_1^2 \log u_1^2 \}
- \frac{1}{2} t^2 \log t \int_{D_1} Q(x)u_1^2.
\]

Since \( J_L(t\Phi_L(u_1)) \rightarrow -\infty \) as \( t \rightarrow \infty \), we can choose a \( t_1 > 0 \) such that \( J_L(t_1\Phi_L(u_1)) < 0 \). Thus for \( u_0 = t_1u_1 \), the conclusion (ii) holds.

The following property of \( J_L(u) \) is based on a special feature of our nonlinearity:

\[
G(s) - \frac{1}{2} g(s)s = \frac{1}{2} s^2 \quad \text{for } s \geq 0. \tag{2.8}
\]

It is useful to check Palais-Smale condition and concentration-compactness type result for \( J_L(u) \).

**Lemma 2.7.** For \( \delta > 0 \) there exists a constant \( C_3 > 0 \) independent of \( L \in \mathbb{N} \) such that for any \( L \in \mathbb{N} \) and \( u \in E_L \) with

\[
\|J_L(u)\|_{E_L^*} \leq \delta, \tag{2.9}
\]

we have

(i) \( \|u^-\|_{E_L} \leq \delta, \int_{D_L} H(u^-), \int_{D_L} h(u^-)u^- \leq C_3\delta^2 \);

(ii) Moreover, assume that for \( M > 0 \)

\[
J_L(u) \leq M. \tag{2.10}
\]

Then there exists a constant \( C_4(M, \delta) > 0 \) independent of \( L \in \mathbb{N} \) such that

\[
\|u\|_{E_L}, \int_{D_L} H(u), \int_{D_L} h(u)u \leq C_4(M, \delta).
\]

(iii) For a critical point \( u \in E_L \) of \( J_L(u) \), we have

\[
J_L(u) = \frac{1}{2} \int_{D_L} Q(x)u^2. \tag{2.11}
\]

**Proof.** Suppose that \( L \in \mathbb{N} \) and \( u \in E_L \) satisfies (2.9). We note that \( g(u^-) = h(u^-)u^- \). It follows from \( |J_L'(u)u^-| \leq \delta\|u^-\|_{E_L} \) that

\[
\|u^-\|^2_{E_L} + \int_{D_L} Q(x)h(u^-)u^- \leq \delta\|u^-\|_{E_L}.
\]

We can easily get (i) from Lemma 2.4 (iii-a).
To show (ii), we note by (2.4) and (2.8) that
\[
J_L(u) - \frac{1}{2} J'_L(u) u = - \int_{D_L} Q(x) (G(u) - \frac{1}{2} g(u) u) \\
= \frac{1}{2} \int_{D_L} Q(x) (u^+)^2 + \int_{D_L} Q(x) (H(u^-) - \frac{1}{2} h(u^-) u^-) \\
\geq \frac{1}{2} \int_{D_L} Q(x) (u^+)^2.
\]

It follows from (2.9)–(2.10) that
\[
\frac{1}{2} \int_{D_L} Q(x) (u^+)^2 \leq M + \frac{\delta}{2} \|u\|_{E_L}.
\]
Thus, by (i),
\[
\|u\|_{L^2(D_L)} \leq C_5(M, \delta) (1 + \|u\|_{E_L})^{\frac{1}{2}},
\]
where \(C_5(M, \delta) > 0\) is independent of \(L \in \mathbb{N}\). By Lemma 2.1 and Gagliardi-Nirenberg inequality, we have
\[
\int_{D_L} Q(x) F(u) \leq C_p \|Q\|_{L^\infty} \|u\|_{L^p(D_L)}^p \leq C'_p \|u\|_{E_L}^{\theta p} \|u\|_{L^p(D_L)}^{(1-\theta)p} \\
\leq C'_p C_5(M, \delta)^{(1-\theta)p} \|u\|_{E_L}^{\theta p} (1 + \|u\|_{E_L})^{(1-\theta)p/2},
\]
where \(\theta \in (0, 1)\) satisfies
\[
\frac{1}{p} = \left(\frac{1}{2} - \frac{1}{N}\right) \theta + \frac{1}{2} (1 - \theta), \quad \text{that is, } \theta p = N\left(p - \frac{2}{2}\right).
\]
Thus it follows from \(J_L(u) \leq M\) that
\[
\frac{1}{2} \|u\|_{E_L}^2 + \int_{D_L} Q(x) H(u) - C'_p C_5(M, \delta)^{(1-\theta)p} \|u\|_{E_L}^{\theta p} (1 + \|u\|_{E_L})^{(1-\theta)p/2} \leq M.
\]
Choosing \(p\) close to 2 so that \(\theta p + (1-\theta)p/2 < 2\), we can see that (ii) holds for a suitable constant \(C_4(M, \delta) > 0\) independent of \(L \in \mathbb{N}\).

(2.11) follows from (2.8).

2.3 Some notation

At the end of this section, we give some notation which will be used repeatedly in the following sections.

We denote
\[
2^* = \begin{cases} \frac{2N}{N-2}, & \text{if } N \geq 3, \\ \infty, & \text{if } N = 1, 2. \end{cases}
\]

We will use the following subsets of \(\mathbb{R}^N\) frequently.

\[
D_1(n) = \{x \in \mathbb{R}^N; x - n \in D_1\} \quad \text{for } n \in \mathbb{Z}^N, \\
B_r(y) = \{x \in \mathbb{R}^N; |x - y| < r\} \quad \text{for } r > 0 \text{ and } y \in \mathbb{R}^N.
\]
We also denote
\[ B^L_r(u) = \{ v \in E_L; \|u - v\|_{E_L} < r \} \] for \( u \in E_L, r > 0 \),
\[ B^{\infty}_r(u) = \{ v \in E_\infty; \|u - v\|_{E_\infty} < r \} \] for \( u \in E_\infty, r > 0 \).

For \( c \in \mathbb{R} \) and \( L \in \mathbb{N} \) we denote
\[ [J_L \leq c]_L = \{ u \in E_L; J_L(u) \leq c \}, \]
\[ [J_\infty \leq c]_\infty = \{ u \in D; J_\infty(u) \leq c \}. \]

In a similar way, we use \([J_L \geq c]_L, [J_\infty \geq c]_\infty \) etc.

In what follows, for \( u \in H^1_0(D_L) \), we denote its \( 2L \)-periodic extension by \( \Phi_L(u) \in E_L \). We choose and fix a function \( \psi(x) \in C^\infty_c(\mathbb{R}^N) \) such that
\[ \psi(x) = \begin{cases} 1 & \text{for } |x| \leq \frac{1}{4}, \\ 0 & \text{for } |x| \geq \frac{1}{2}, \end{cases} \]
\[ |\nabla \psi(x)| \leq 8 \text{ for all } x \in \mathbb{R}^N. \]

For \( s > 0 \), we set \( \psi_s(x) = \psi(\frac{x}{s}) \). We note that for \( 0 < R \leq L, u \in H^1_0(\mathbb{R}^N) \),
\[ (\psi_R u)(x) \equiv \psi_R(x)u(x) \in H^1_0(D_L) \]
Thus \( \Phi_L(\psi_R u)(x) \), i.e., \( 2L \)-periodic extension of \( \psi_R u \), is an element of \( E_L \).

3 Concentration-compactness type result

In this section we give a concentration-compactness type result. It will play an important role when we take a limit as \( L \to \infty \).

To state our concentration-compactness type result, we need notation:
\[ \text{dist}_L(y, y') = \min_{n \in \mathbb{Z}^N} |y - y' - 2Ln|. \]
\( \text{dist}_L(y, y') \) is a distance in \( \mathbb{R}^N / \sim_L \), where an equivalence relation \( \sim_L \) is given by
\[ y \sim_L y' \text{ if and only if } y - y' = 2Ln \text{ for some } n \in \mathbb{Z}^N. \]

Our concentration-compactness type result is the following

**Proposition 3.1.** Assume that \( (L_j)_{j=1}^\infty \subset \mathbb{N} \) and \( u_j \in E_{L_j} \) \( (j = 1, 2, \cdots) \) satisfy for some \( c > 0 \)
\[ L_j \to \infty, \quad J_{L_j}(u_j) \to c > 0, \quad \|J'_{L_j}(u_j)(E_{L_j})\| \to 0 \text{ as } j \to \infty. \] (3.1)

Then
(i) \( \|u_j\|_{E_{L_j}}, \int_{D_{L_j}} h(u_j)u_j, \int_{D_{L_j}} H(u_j) \) are bounded;
(ii) There exists \( m \in \mathbb{N} \), \( (w^\ell_{\ell=1}^m) \subset K_\infty \setminus \{0\} \) and subsequence \( (j_k)_{k=1}^\infty \) and sequences \( (y^\ell_{k})_{k=1}^\infty \subset \mathbb{R}^N \) with \( y^\ell_{jk} \in D_{L_{jk}} \) \( (\ell = 1, 2, \cdots, m) \) such that
(a) For \( \ell \neq \ell' \),
\[ \text{dist}_{L_{jk}}(y^\ell_{jk}, y^\ell'_{jk}) \to \infty \text{ as } k \to \infty. \] (3.2)
(b) For any \( R_{jk} > 0 \) with \( R_{jk} \leq L_{jk} \) and \( R_{jk} \to \infty \) as \( k \to \infty \)

\[
\| u_{jk} - \sum_{\ell=1}^{m} \Phi_L(\psi_{R_{jk}} w^\ell)(x - y^\ell_{jk}) \|_{E_{L_{jk}}} \to 0 \quad \text{as} \quad k \to \infty. \quad (3.3)
\]

(c)

\[
c = \sum_{\ell=1}^{m} J_\infty(w^\ell) \quad (3.4)
\]

(d)

\[
\int_{D_{L_{jk}}} H(u_{jk}) \to \sum_{\ell=1}^{m} \int_{\mathbb{R}^N} H(w^\ell) \quad (3.5)
\]

To show Proposition 3.1, for a sequence \( u_{j} \in E_{L_{j}} (j = 1, 2, \cdots) \) satisfying (3.1) we need to give estimates of \( \int_{\mathbb{R}^N} H(u_{j}) \) as well as \( \| u_{j} \|_{E_{L_{j}}} \).

First we show

Lemma 3.2. For any \( q \in (2, 2^*) \) there exists a constant \( C_q > 0 \) independent of \( L \in \mathbb{N} \) such that for \( L \in \mathbb{N} \), \( u \in E_{L} \), it holds that

\[
\| u \|_{L^q(D_L)} \leq C \left( \sup_{n \in \mathbb{Z}^N} \| u \|_{L^q(D_1(n))} \right)^{q-2} \| u \|_{E_L}^2. \quad (3.6)
\]

Proof. For \( u \in E_L \), \( n \in \mathbb{Z}^N \), \( q \in (2, 2^*) \), by Sobolev inequality we have

\[
\| u \|_{L^q(D_1(n))}^q = \| u \|_{L^q(D_1(n))}^{q-2} \| u \|_{L^q(D_1(n))}^2 \leq C \left( \sup_{n \in \mathbb{Z}^N} \| u \|_{L^q(D_1(n))} \right)^{q-2} \| u \|_{E_L}^2(D_1(n)).
\]

Summing up for \( n \in \{-L, -L+1, \cdots, L-1\}^N \), we get (3.6).

Next we show

Lemma 3.3. Let \( q \in (2, 2^*) \) and suppose that \( L_j, v_j \ (j = 1, 2, \cdots) \) satisfy \( L_j \to \infty \), \( v_j \in E_{L_j} \) and

\[
\| J'_{L_j}(v_j) \|_{(E_{L_j})^*} \to 0 \quad \text{and} \quad \sup_{j \in \mathbb{N}} \| v_j \|_{E_{L_j}} < \infty.
\]

Moreover suppose that

\[
\sup_{n \in \mathbb{N}} \| v_j \|_{L^q(D_1(n))} \to 0.
\]

Then

\[
\| v_j \|_{E_{L_j}} \to 0, \quad \int_{D_{L_j}} H(v_j) \to 0, \quad J_{L_j}(v_j) \to 0.
\]

Proof. By Lemma 3.2 we have

\[
\| v_j \|_{L^q(D_{L_j})} \leq C \left( \sup_{n \in \mathbb{Z}^N} \| v_j \|_{L^q(D_1(n))} \right)^{q-2} \| v_j \|_{E_{L_j}}^2 \to 0,
\]

13
which implies $\int_{D_{L_j}} Q(x)f(v_j)v_j \to 0$. Thus
\[
\|v_j\|_{E_{L_j}}^2 + \int_{D_{L_j}} Q(x)H(v_j) \leq \|v_j\|_{E_{L_j}}^2 + \int_{D_{L_j}} Q(x)h(v_j) v_j
\]
\[= J'_{E_{L_j}}(v_j)v_j + \int_{D_{L_j}} Q(x) f(v_j)v_j \to 0.
\]
Therefore we have the conclusion of Lemma 3.3.

Proof of Proposition 3.1 is rather lengthy and we will give a proof in Appendix.

4 One bump solutions

As we observed in Lemma 2.6, $J_L(u)$ has a mountain pass geometry uniformly in $L \in \mathbb{N}$. We define mountain pass values for $J_L(u)$ by
\[
b_L = \inf_{\gamma \in \Gamma_L} \max_{\tau \in [0,1]} J_L(\gamma(\tau)),
\]
where
\[
\Gamma_L = \{ \gamma \in C([0,1], E_L); \gamma(0) = 0, J_L(\gamma(1)) < 0 \}.
\]

Our main result in this section is the following theorem.

**Theorem 3.** Assume (A1) and (A2). Then we have

(i) For any $L \in \mathbb{N}$, $b_L$ is attained by a positive solution $u_L(x) \in E_L$ of (PL).

(ii) $b_L$ is characterized as
\[
b_L = \inf_{u \in N_L} J_L(u),
\]
where
\[
N_L = \{ u \in E_L \setminus \{0\}; J'_L(u)u = 0 \}.
\]

(iii) $b_L \to b_{\infty}$ as $L \to \infty$, where $b_{\infty}$ is defined in (1.2).

As a corollary to Theorem 3, we have

**Corollary 4.1** (c.f. Theorem 1.2 of [22]). Assume (A1) and (A2). Let $b_{\infty}$ be a number defined in (1.2). Then $b_{\infty} \in (0, \infty)$ and $b_{\infty}$ is achieved by a positive solution of (LS1).

In this section, we will first prove that $b_L$ is a critical value of $J_L$, and then use the concentration compactness (Proposition 3.1) to get a nonzero critical point for $J_{\infty}$. Then we will prove $\lim_{L \to \infty} b_L = b_{\infty}$. By Lemma 2.6 we know for every $L \in \mathbb{N}$, $J_L$ has mountain-pass geometry. To prove $b_L$ is a critical value of $J_L$, we only need to prove that $J_L$ satisfies the (PS) condition. The first statement (i) in Theorem 3 follows from mountain pass theorem using the following lemma.

**Lemma 4.2.** For any $L \in \mathbb{N}$, $J_L(u)$ satisfies the Palais-Smale condition. More precisely, if a sequence $(u_k)_{k=1}^\infty \subset E_L$ satisfies
\[
(JL(u_k))_{k=1}^\infty \text{ is bounded and } \|J'_L(u_k)\|_{E_L^*} \to 0, \quad (4.1)
\]
then $(u_k)_{k=1}^\infty$ has a convergent subsequence in $E_L$. 

14
Proof. Suppose that \((u_k)_{k=1}^\infty \subset E_L\) satisfies (4.1). By Lemma 2.7, we can deduce that \((u_k)_{k=1}^\infty \subset E_L\) is a bounded sequence in \(E_L\). Thus, extracting a subsequence if necessary, we may assume that for some \(u_0 \in E_L\)

\[ u_k \rightharpoonup u_0 \text{ weakly in } E_L. \]

Since we are working in a bounded domain \(D_L\) essentially, we can prove \(u_k \to u_0\) strongly in \(E_L\) in a standard way.

To show the second statement (ii) in Theorem 3, we need the following Lemma 4.3.

(i) For \(L \in \mathbb{N}\) and for any \(u \in E_L\) with \(u^+ \neq 0\), we have

(a) \(\{tu; t \geq 0\} \cap \mathcal{N}_L \neq \emptyset\) and there exists a unique \(t_u > 0\) such that \(t_uu \in \mathcal{N}_L\).

(b) \(J(tu) < J(t_uu)\) for all \(t \in (0, \infty) \setminus \{t_u\}\).

(ii) For any \(u \in \mathcal{N}_L\), we have

\[ \sup_{t \in [0, \infty)} J_L(tu) \leq J_L(u). \]

In particular, there exists \(\gamma_u(t) \in \Gamma_L\) such that \(\max_{t \in [0,1]} J_L(\gamma_u(t)) \leq J_L(u)\).

Proof. For \(u \in E_L\) with \(u^+ \neq 0\), we set \(\phi(t) = J_L(tu)\) for \(t \in [0, \infty)\). Then

\[ \phi'(t) = t\|u\|_{E_L}^2 - \int_{D_L} Q(x)g(tu)u \]

\[ = t \left( \|u\|_{E_L}^2 - \int_{D_L} Q(x) \frac{g(tu)}{t}u \right). \]

Thus by Lemma 2.7 \(\phi'(t)\) is a strictly decreasing function of \(t \in (0, \infty)\). Since \(u^+ \neq 0\), we can also easily see that \(\phi'(t) < 0\) for large \(t > 0\) and \(\phi'(t) > 0\) for small \(t > 0\). Thus there exists a unique \(t_u \in (0, \infty)\) such that \(\phi'(t_u) = 0\) and

\[ \phi'(t) > 0 \quad \text{for } t \in (0, t_u), \]

\[ \phi'(t) < 0 \quad \text{for } t \in (t_u, \infty). \]

Noting \(\{tu; t > 0\} \cap \mathcal{N}_L = \{tu; \phi'(t) = 0\}\), we have (i).

Noting also \(u^+ \neq 0\) for all \(u \in \mathcal{N}_L\) and setting \(\gamma_u(t) = tMu\) for a large constant \(M \gg 1\), (ii) follows from (i).

In a similar way to Lemma 4.3, we have

Lemma 4.4. For any \(u \in \mathcal{N}_\infty\),

\[ \sup_{t \in [0, \infty)} J_\infty(tu) \leq J_\infty(u). \]

Proof of (ii) of Theorem 3. By lemma 4.3 for any \(u \in \mathcal{N}_L\) there exists a path \(\gamma_u \in \Gamma_L\) such that \(\max_{t \in [0,1]} J_L(\gamma_u(t)) \leq J_L(u)\). Thus we have

\[ b_L \leq \max_{t \in [0,1]} J_L(\gamma_u(t)) \leq J_L(u). \]
Since $u \in N_L$ is arbitrary, we have
\[
b_L \leq \inf_{v \in N_L} J_L(u). \tag{4.2}
\]
On the other hand, since $J_L(u^+) - \frac{1}{2} J'_L(u^+)u^+ = \frac{1}{2} \int_{D_L} Q(x)(u^+)^2 \geq 0$ by (4.3), we have for any $\gamma \in \Gamma_L$\[
\frac{1}{2} J'_L(\gamma(1)^+ \gamma(1)^+ \leq J_L(\gamma(1)^+) \leq J_L(\gamma(1)) < 0.
\]
Thus there exists $t_0 \in (0, 1)$ such that $J'_L(\gamma(t_0)^+)\gamma(t_0)^+ = 0$, that is, $\gamma(t_0)^+ \in N_L$. Thus,
\[
\inf_{v \in N_L} J_L(v) \leq J_L(\gamma(t_0)^+) \leq \max_{t \in [0, 1]} J_L(\gamma(t)^+) \leq \max_{t \in [0, 1]} J_L(\gamma(t)).
\]
Since $\gamma \in \Gamma_L$ is arbitrary, we have $b_L \geq \inf_{v \in N_L} J_L(v)$. Together with (4.2), we have (ii) of Theorem 3.

Finally we show (iii) of Theorem 3 and Corollary 4.1.

**Proof of (iii) of Theorem 3 and Corollary 4.1.** First we remark that there are constants $C_1, C_2 > 0$ such that
\[
b_L \in [C_1, C_2] \quad \text{for all } L \in \mathbb{N},
\]
which follows from Lemma 2.10.

For any $\varepsilon > 0$, we can find $u_0 \in N_{\infty}$ such that $J_{\infty}(u_0) < b_\infty + \varepsilon$. By Lemma 4.4 we have
\[
\max_{t \in [0, \infty)} J_{\infty}(tu_0) = J_{\infty}(u_0) < b_\infty + \varepsilon.
\]
Considering $\psi_L u_0 \in H^1_0(D_L)$ for $L \in \mathbb{N}$, we have
\[
\max_{t \in [0, \infty)} \int_{D_L} \frac{1}{2} \|\nabla(\psi_L u_0)\|^2 - Q(x)G(t\psi_L u_0) \rightarrow \max_{t \in [0, \infty]} J_{\infty}(tu_0) \quad \text{as } L \rightarrow \infty.
\]
That is,
\[
\max_{t \in [0, \infty)} J_L(t\Phi_L(\psi_L u_0)) \rightarrow \max_{t \in [0, \infty)} J_{\infty}(tu_0) < b_\infty + \varepsilon \quad \text{as } L \rightarrow \infty,
\]
from which we have
\[
\limsup_{L \to \infty} b_L \leq \limsup_{L \to \infty} \max_{t \in [0, \infty)} J_L(t\Phi_L(\psi_L u_0)) \leq b_\infty + \varepsilon.
\]
Thus we have $\limsup_{L \to \infty} b_L \leq b_\infty$.

On the other hand, we assume that $b_{L_k} \rightarrow c = \liminf_{L \to \infty} b_L \in [C_1, b_\infty]$ for some subsequence $L_k \to \infty$. By Proposition 3.1 we have
\[
c = \sum_{\ell=1}^{m} J_{\infty}(w^\ell)
\]
for some $m \geq 1$ and $w^\ell \in K_\infty \setminus \{0\} \subset N_\infty$. Thus we have $c \geq b_\infty = \inf_{u \in N_\infty} J_\infty(u)$. Therefore we have $b_L \to b_\infty$ as $L \to \infty$ and there exists a $w \in K_\infty \setminus \{0\} \subset N_\infty$ such that
\[
b_\infty = J_{\infty}(w).
\]
This completes the proof of (iii) of Theorem 3 and Corollary 4.1.
5 Multi-bump solutions

In this section, we assume that there exists an \( \alpha \in (0, \frac{1}{10}b_\infty) \) such that
\[
(\mathcal{K}_\infty \cap [J_\infty \leq b_\infty + \alpha])_\infty / \mathbb{Z}^N \text{ is finite.} \tag{*}
\]

Under the assumption \((\ast)\), choosing \( \alpha > 0 \) smaller if necessary, we may also assume that
\[
\mathcal{K}_\infty \cap [J_\infty = b_\infty]_\infty = \mathcal{K}_\infty \cap [0 < J_\infty \leq b_\infty + \alpha]_\infty.
\]

We choose and fix \( \omega, \omega' \in F_\infty \), arbitrary and for large \( R \gg 1 \) we choose \( P \in \mathbb{Z}^N \) and \( L \in \mathbb{N} \) such that
\[
5R \leq |P| \quad \text{and} \quad 2|P| \leq L. \tag{\#}
\]

We try to find a critical point in a neighborhood of \( \Omega_{R,P,L} = \Phi_L(\psi_R\omega)(\cdot) + \Phi_L(\psi_R\omega')(\cdot - P) \).

In what follows, we always assume that \( R, P, L \) satisfy \((\#)\).

Later for a fixed \( P \in \mathbb{Z}^N \) with \(|P| > 5R\), we take a limit as \( L \to \infty \) to obtain our main Theorem 2.

5.1 A gradient estimate and deformation argument

To show the existence of a critical point in a neighborhood of \( \Omega_{R,P,L} \), estimates of \( J'_L(u) \) in annular neighborhoods of \( \Omega_{R,P,L} \) are important. We need the following notation to state our estimates.

We note that under \((\#)\)
\[
B_{2R}(0), B_{2R}(P) \subset D_L, \quad B_{2R}(0) \cap B_{2R}(P) = \emptyset.
\]

For \( R \gg 1, P \in \mathbb{Z}^N, L \in \mathbb{N} \) satisfying \((\#)\), we set for \( t \in [\frac{R}{2}, 2R] \),
\[
A_{t,P,L} = D_L \setminus (B_t(0) \cup B_t(P)).
\]

We also use notation for \( t \in [\frac{R}{2}, 2R] \) and \( u \in E_L \)
\[
\|u\|_{E_L(A_{t,P,L})}^2 = \int_{A_{t,P,L}} |\nabla u|^2 + V(x)u^2
\]
and we define \( J_{A_{t,P,L}}(u) : E_L \to \mathbb{R} \) by
\[
J_{A_{t,P,L}}(u) = \frac{1}{2}\|u\|_{E_L(A_{t,P,L})}^2 - \int_{A_{t,P,L}} Q(x)G(u).
\]

We also denote for \( c \in \mathbb{R} \)
\[
[J_{A_{t,P,L}} \leq c]_L = \{u \in E_L; J_{A_{t,P,L}}(u) \leq c\} \text{ etc.}
\]

First we need the following lemma.
Lemma 5.1. There exists $r_0 > 0$ and $R_0 \geq 1$ with the following properties:

(I) For any $L \in \mathbb{N}$

$$J_L(u), J'_L(u) \geq \frac{1}{4} \| u \|^2_{E_L}$$

for all $\| u \|_{E_L} \leq 2r_0$.

In particular,

$$\| u \|_{E_L} \geq 2r_0$$

for all $u \in K L \setminus \{0\}$.

(II) $\| u \|_{E_\infty} \geq 2r_0$ for all $u \in K \setminus \{0\}$;

(III) For any $R, P, L$ with $R \geq 1$ and $L$, $J_{AR,P,L}(u), u \in E_L; \| u \|_{E_L(A_R,P,L)} \leq r_0 \to \mathbb{R}$ is strictly convex and it holds that for $t \in \left[\frac{L}{2}, 2R\right]

$$J_{AR,P,L}(u) \geq \frac{1}{4} \| u \|^2_{E_L(A_R,P,L)} + \int_{A_t,P,L} Q(x) f(u) \leq \frac{1}{8} \| u \|^2_{E_L(A_R,P,L)}$$

for all $u \in E_L$ with $\| u \|_{E_L(A_R,P,L)} \leq 2r_0$.

(IV) For $R, P, L$ with $R \geq R_0$ and $L$, we have

$$\| \psi_R v - (\psi_R v')(\cdot - n) \|_{E_\infty}, \| \psi_R v - (\psi_R v')(\cdot - n') \|_{E_\infty} \geq 4r_0$$

for all $v, v' \in F_\infty$ with $v \neq v'$ and $n, n' \in \mathbb{Z}^N \setminus \{0\}$.

(V) For $R, P, L$ with $R \geq R_0$ and $L$ and for $u \in \mathcal{B}^{(L)}_{R_0}(\Omega_{R,P,L})$

$$\frac{1}{2} \int_{D_L} Q(x) u^2 \in (2b_\infty - \alpha, 2b_\infty + \alpha).$$

(VI) For any $x_0 \in \mathbb{R}^N$, if $u(x) \in H^1(B_2(x_0))$ satisfies

$$-\Delta u + V(x) u = Q(x) g(u)$$

in $B_2(x_0)$,

$$\| u \|_{H^1(B_2(x_0))} \leq r_0.$$

Then $u$ satisfies

$$\| u \|_{L^\infty(B_1(x_0))} \leq \frac{1}{e}.$$

Proof. (I)–(III) can be checked easily for $r_0 > 0$ small independent of $R, P, L$. For (IV), we set

$$\mu_1 = \inf \{ \| v - v' \|_{E_\infty}; v, v' \in K \setminus [J_\infty = b_\infty], v \neq v' \} = \inf \{ \| v - v'(\cdot - n) \|_{E_\infty}, \| v - v'(\cdot - n') \|; v, v' \in F_\infty, v \neq v', n, n' \in \mathbb{Z}^N, n' \neq 0 \} > 0.$$

Choosing $r_0 \in (0, \frac{1}{4} \mu_1]$, we can see (5.2) holds for large $R$. 

18
For (V), we note that \( \frac{1}{2} \int_{\mathbb{R}^N} Q(x)\omega^2 = \frac{1}{2} \int_{\mathbb{R}^N} Q(x)\omega^2 = b_\infty \). Since
\[
\frac{1}{2} \int_{D_L} Q(x)\Omega_{R,P,L}^2 \to \frac{1}{2} \int_{\mathbb{R}^N} Q(x)\omega^2 + \frac{1}{2} \int_{\mathbb{R}^N} Q(x)\omega^2 = 2b_\infty
\]
as \( R \to \infty \) (\( L \geq 10R \)), there exists \( R_0 \geq 1 \) such that
\[
\frac{1}{2} \int_{D_L} Q(x)\Omega_{R,P,L}^2 \in (2b_\infty - \frac{1}{2}a, 2b_\infty + \frac{1}{2}a)
\]
for \( R, P, L \) with \( R \geq R_0 \) and \( \left[ \right] \). Thus, choosing \( r_0 > 0 \) small, we have (V).
Proof of (IV) is rather lengthy. We give it in the appendix.

In what follows, we assume \( r \in (0, r_0) \) without loss of generality and we try to find a critical point of \( J_L \) in \( B_{r_0}^{(L)}(\Omega_{R,P,L}) \). We will use Lemma 5.1 repeatedly.

Our main result in this subsection is

**Proposition 5.2.** For any \( r_1, r_2 \in (0, r_0] \) with \( r_1 < r_2 \) and for any \( \rho > 0 \), there exists \( R_1 = R_1(r_1, r_2, \rho) \geq R_0, v_1 = v_1(r_1, r_2, \rho) > 0 \) such that for all \( R, P, L \) with \( R \geq R_1 \) and \( \left[ \right] \), if
\[
u \in \left( B_{r_2}^{(L)}(\Omega_{R,P,L}) \setminus B_{r_1}^{(L)}(\Omega_{R,P,L}) \right) \cup \left( (B_{r_2}^{(L)}(\Omega_{R,P,L}) \cap \{J_{L_{\rho,L}} \geq \rho\}) L \right),
\]
then there exists \( \varphi_u \in E_L \) with \( \|\varphi_u\|_{E_L} \leq 1 \) such that
(a) \( J_L'(u)\varphi_u \geq \nu_1 \);
(b) moreover if \( J_{L_{\rho,L}}(u) \geq \rho \), then
\[
J_{L_{\rho,L}}'(u)\varphi_u \geq 0.
\]

**Proof.** Proof is divided into two parts.

**Step 1.** For any \( 0 < r_1 < r_2 \leq r_0 \) there exists \( \tilde{R}_1 = \tilde{R}_1(r_1, r_2) > 0 \) and \( \tilde{\nu}_1 = \tilde{\nu}_1(r_1, r_2) > 0 \) such that for any \( R, P, L \) with \( R \geq \tilde{R}_1 \) and \( \left[ \right] \)
\[
\|J_L'(u)\|_{E_L} \geq \tilde{\nu}_1 \text{ for all } u \in B_{r_2}^{(L)}(\Omega_{R,P,L}) \setminus B_{r_1}^{(L)}(\Omega_{R,P,L}).
\]

In fact, if the conclusion does not hold, there exist sequences \( R_j, P_j, L_j, u_j \) satisfying \( \left[ \right] \) and \( u_j \in E_{L_j} \) such that
\[
R_j \to \infty,
\]
\[
u_j \in B_{r_2}^{(L_j)}(\Omega_{R_j,P_j,L_j}) \setminus B_{r_1}^{(L_j)}(\Omega_{R_j,P_j,L_j}),
\]
\[
\|J_L'(u_j)\|_{E_{L_j}} \to 0.
\]

Clearly \( \|u_j\|_{E_{L_j}} \) is bounded. By Lemma 2.7 we have
\[
\|u_j\|_{E_{L_j}}, \int_{D_{L_j}} H(u_j^-), \int_{D_{L_j}} h(u_j^+)u_j^- \to 0 \text{ as } j \to \infty.
\]
Thus,
\[
J_{L_j}(u_j) = \frac{1}{2} J'_{L_j}(u_j) u_j - \int_{D_{L_j}} Q(x)(G(u_j) - \frac{1}{2}g(u_j)u_j)
\]
\[
= \frac{1}{2} \int_{D_{L_j}} Q(x)u_j^2 + o(1) \quad \text{as } j \to \infty.
\]

Thus by Lemma 5.1 (V), we have
\[
J_{L_j}(u_j) \in (2b_\infty - \alpha, 2b_\infty + \alpha) \quad \text{for large } j.
\]

Applying Proposition 3.1, we have for some \(m \in \mathbb{N}\) and \((w^\ell)_{\ell=1}^m \subset K_\infty \setminus \{0\}\)
\[
J_{L_j}(u_j) \to \sum_{\ell=1}^m J_\infty(w^\ell) \geq mb_\infty
\]
after extracting a subsequence. By (5.4), we have \(m \in \{1, 2\}\). On the other hand, by (5.3), we have for some constant \(C > 0\)
\[
\|u_j\|_{H^1(B_{R_j}(0))}, \|u_j\|_{H^1(B_{R_j}(P_j))} \geq C \quad \text{for all } j,
\]
from which we can see \(m = 2\).

Thus we can find for some \(w_1, w_2 \in F_\infty\) and \((y_1^j, y_2^j) \in \mathbb{Z}^N\)
\[
\|u_j - \Phi_{L_j}(\psi_{R_j}w^\ell)(\cdot - y_\ell^j)\|_{E_{L_j}} \to 0.
\]

By Lemma 5.1 (IV), it follows from (5.3) that \(w^1 = \omega, w^2 = \omega', y_1^j = 0, y_2^j = P_j\).
Therefore
\[
\|u_j - \Phi_{L_j}(\psi_{R_j}w^\ell) - \Phi_{L_j}(\psi_{R_j}w')(\cdot - P_j)\|_{E_{L_j}} \to 0.
\]

In particular, we have
\[
u_j \in B_{r_1}^{(L_j)}(\Phi_{L_j}(\psi_{R_j}w) + \Phi_{L_j}(\psi_{R_j}w')(\cdot - P_j)) \quad \text{for large } j,
\]
which contracts with (5.3). Thus we have the conclusion of Step 1.

Step 2: For any \(\rho > 0\) there exists \(\tilde{R}_2 = \tilde{R}_2(\rho) > 0\) and \(\tilde{v}_2 = \tilde{v}_2(\rho) > 0\) such that for any \(R, P, L\) with \(R \geq \tilde{R}_2\) and \([\tilde{\nu}]\) and for any
\[
u_j \in B_{r_1}^{(L)}(\Omega_{R,P,L}) \cap [J_{A_{R,P,L}} \geq \rho L],
\]
there exists \(\varphi_u \in E_L\) such that
\[
\|\varphi_u\|_{E_L} \leq 1,
\]
\[
J'_{\varphi_u}(u) \geq \tilde{v}_2,
\]
\[
J'_{A_{R,P,L}}(u) \varphi_u \geq 0.
\]

First we remark that (5.5) implies
\[
\|u\|_{E_L(A_{p,R,L})} \leq \|u - \Omega_{R,P,L}\|_{E_L} + \|\Omega_{R,P,L}\|_{E_L(A_{p,R,L})}
\]
\[
\leq r_0 + \|\Omega_{R,P,L}\|_{E_L(A_{p,R,L})}.
\]

20
Since \( \| \Omega_{R,R,L} \|_{E_L(A_{R_2},P,L)} \to 0 \) as \( R \to \infty \) uniformly in \( P, L \) satisfying (2), we may assume that for large \( \tilde{R}_2 \geq 0 \), (5.5) implies
\[
\| u \|_{E_L(A_{\tilde{R}_2},P,L)} \leq 2r_0. \tag{5.9}
\]

Denoting the maximal integer less than \( \frac{R}{2} \) by \( \lceil \frac{R}{2} \rceil \), we have
\[
\sum_{j=1}^{\lceil \frac{R}{2} \rceil} \| u \|_{E_L(A_{\tilde{R}_2+j-1,P,L} \setminus A_{\tilde{R}_2+j,P,L})}^2 \leq \| u \|_{E_L(A_{\tilde{R}_2},P,L)}^2 \leq 4r_0^2.
\]

There exists \( j_u \in \{ 1, 2, \ldots, \lceil \frac{R}{2} \rceil \} \) such that
\[
\| u \|_{E_L(A_{\tilde{R}_2+j_u-1,P,L} \setminus A_{\tilde{R}_2+j_u,P,L})} \leq 2r_0 \sqrt{\frac{1}{2}}. \tag{5.10}
\]

We choose a \( 2L \)-periodic function \( \chi(x) \in C^1(\mathbb{R}^N) \) such that
\[
\chi(x) \in [0,1], \quad |\nabla \chi(x)| \leq 2 \quad \text{for all } x \in \mathbb{R}^N,
\]
\[
\chi(x) = \begin{cases} 
0 & \text{for } x \in D_L \setminus A_{\tilde{R}_2+j_u-1,P,L}, \\
1 & \text{for } x \in A_{\tilde{R}_2+j_u,P,L}.
\end{cases}
\]

We take \( \tilde{R}_2 \) larger so that for \( R \geq \tilde{R}_2 \), (5.10) implies
\[
\left| \int_{A_{\tilde{R}_2+j_u-1,P,L} \setminus A_{\tilde{R}_2+j_u,P,L}} \nabla u \nabla \chi u + V(x) \chi u^2 - Q(x) f(u) \chi u \right| \leq \frac{\rho}{2}. \tag{5.11}
\]

Under the assumption \( J_{A_{R,P,L}}(u) \geq \rho \), we have by (5.11)
\[
J'_{A_{\tilde{R}_2+j_u,P,L}}(u)(\chi u) = J'_{A_{\tilde{R}_2+j_u,P,L}}(u) u = \| u \|_{E_L(A_{\tilde{R}_2+j_u,P,L})}^2 + \int_{A_{\tilde{R}_2+j_u,P,L}} Q(x) h(u) u - \int_{A_{\tilde{R}_2+j_u,P,L}} Q(x) f(u) u \\
\geq \frac{1}{2} \| u \|_{E_L(A_{\tilde{R}_2+j_u,P,L})}^2 + \int_{A_{\tilde{R}_2+j_u,P,L}} Q(x) h(u) u \\
\geq \frac{1}{2} \| u \|_{E_L(A_{\tilde{R}_2+j_u,P,L})}^2 + \int_{A_{\tilde{R}_2+j_u,P,L}} Q(x) H(u) \\
\geq \frac{1}{2} \| u \|_{E_L(A_{R,P,L})}^2 + \int_{A_{R,P,L}} Q(x) H(u) \\
\geq J_{A_{R,P,L}}(u) \geq \rho. \tag{5.12}
\]
It follows from (5.11) and (5.12) that
\[ J_1(u)(
\begin{align*}
J_1(u)(\chi u) & = J_{A^{\mu,j_2+r,P,L}}(u)u \\
& + \int_{A^{\mu,j_2-1,r,P,L} \setminus A^{\mu,j_2+r,P,L}} \nabla u \nabla (\chi u) + V(x)\chi u^2 + Q(x)h(u)\chi u - Q(x)f(u)\chi u \\
& \geq \rho + \int_{A^{\mu,j_2-1,r,P,L} \setminus A^{\mu,j_2+r,P,L}} \nabla u \nabla (\chi u) + V(x)\chi u^2 - Q(x)f(u)\chi u \\
& \geq \frac{1}{2}\rho.
\end{align*}
\]

By (5.10), clearly we have
\[ \|\chi u\|_{E_L}^2 = \int_{D_L} |\nabla (\chi u)|^2 + V(x)\chi u^2 \]
\[ \leq \int_{D_L} 2|\nabla \chi|^2|u|^2 + 2\chi^2|\nabla u|^2 + V(x)\chi u^2 \]
\[ \leq C\|u\|^2_{E_L(A^{\mu,j_2},P,L)} \leq 4Cr_0^2.\]

Thus, setting \( \varphi_u = \frac{\chi u}{\|\chi u\|_{E_L}} \), we have (5.6)–(5.8) for a suitable constant \( \hat{\nu}_2 > 0 \) independent of \( R \geq \hat{R}_2 \).

**Step 3: Conclusion**

Setting \( R_1 = \max\{\hat{R}_1, \hat{R}_2\}, \nu_1 = \min\{\hat{\nu}_1, \hat{\nu}_2\} \), we have the conclusion of Proposition 5.2. \( \square \)

We have the following deformation result from the previous lemma.

**Lemma 5.3.** Let \( 0 < r_1 < r_2 \leq r_0 \) and \( \rho > 0 \) and suppose that \( R, P, L \) satisfies (9) and \( R \geq R_1 \), where \( R_1 \) is given in Proposition 5.2. Moreover assume that
\[ K_1 \cap B_{r_2}(\Omega_{R,P,L}) = \emptyset. \] (5.13)

Then, for any \( 0 < \varepsilon < \frac{\nu_1}{2}(r_2 - r_1) \) and \( \nu > \varepsilon \) there exists \( \tilde{\eta} \in C(E_L, E_L) \) such that

(i) \( \tilde{\eta}(u) = u \) for all \( u \in (E_L \setminus B_{r_2}(\Omega_{R,P,L})) \) \( \cup [J_L \leq 2b_\infty - \eta L]; \)

(ii) \( J_L(\tilde{\eta}(u)) \leq J_L(u) \) for all \( u \in E_L; \)

(iii) if \( u \in B_{r_1}(\Omega_{R,P,L}) \cap [J_L \leq 2b_\infty + \varepsilon L], \) then \( \tilde{\eta}(u) \in B_{r_2}(\Omega_{R,P,L}) \cap [J_L \leq 2b_\infty - \varepsilon L]; \)

(iv) if \( u \in B_{r_1}(\Omega_{R,P,L}), \) then \( \tilde{\eta}(u) \in B_{r_2}(\Omega_{R,P,L}); \)

(v) for \( u \in B_{r_1}(\Omega_{R,P,L}), \) if \( J_{A_{r_1,P,L}}(u) \leq \rho, \) then \( J_{A_{r_1,P,L}}(\tilde{\eta}(u)) \leq \rho. \)

**Proof.** It follows from (5.13) that for some constant \( \nu_L > 0 \)
\[ \|J_L(u)\|_{E_L} \geq \nu_L \quad \text{for all} \ u \in B_{r_2}(\Omega_{R,P,L}). \] (5.14)
In fact, if a sequence \((u_j)_{j=1}^\infty \subset B^{(L)}_{r_2}(\Omega_{R,P,L})\) satisfies \(\|J'_L(u_j)\|_{E_L^*} \to 0\), we can see that, after extracting a subsequence, \(u_j \to u_0 \in B^{(L)}_{r_2}(\Omega_{R,P,L})\) in \(E_L\) and \(J'_L(u_0) = 0\). This is a contradiction to (5.13). We may assume \(\nu \leq \nu_1\) without loss of generality.

By Proposition 5.2 and (5.14), there exists a locally Lipschitz continuous vector field \(V : B^{(L)}_{r_2}(\Omega_{R,P,L}) \to E_L\) such that

1. \(\|V(u)\|_{E_L} \leq 1\) for all \(u \in E_L\);
2. \(J'_L(u)V(u) \geq \frac{1}{2}\nu_1\) for all \(u \in B^{(L)}_{r_2}(\Omega_{R,P,L})\);
3. \(J'_L(u)V(u) \geq \frac{1}{2}\nu_2\) for all \(u \in B^{(L)}_{r_2}(\Omega_{R,P,L}) \setminus B^{(L)}_{r_1}(\Omega_{R,P,L})\);
4. \(J'_{A_{R,P,L}}(u)V(u) \geq 0\) for all \(u \in B^{(L)}_{r_2}(\Omega_{R,P,L})\) with \(J'_{A_{R,P,L}}(u) \geq \rho\).

Such a vector field can be constructed in a standard way using a partition of unity (c.f. Appendix of Rabinowitz [17]).

We define locally Lipschitz functions \(\chi_1(u), \chi_2(u) : E_L \to [0,1]\) by

\[
\begin{align*}
\chi_1(u) &= \begin{cases} 0 & \text{for } u \notin B^{(L)}_{r_2}(\Omega_{R,P,L}), \\ 1 & \text{for } u \in B^{(L)}_{r_2}(\Omega_{R,P,L}), \end{cases} \\
\chi_2(u) &= \begin{cases} 1 & \text{for } u \in [J_L \geq 2b_\infty - \varepsilon]_L, \\ 0 & \text{for } u \in [J_L \leq 2b_\infty - \varepsilon]_L. \end{cases}
\end{align*}
\]

We consider an initial value problem in \(E_L\)

\[
\begin{cases}
\frac{d\eta}{dt} = -\chi_1(\eta)\chi_2(\eta)V(\eta), \\
\eta(0,u) = u.
\end{cases}
\]  

(5.15) has a unique solution \(\eta(t,u) \in C([0,\infty) \times E_L, E_L)\) which satisfies for all \(t \in [0,\infty)\) and \(u \in E_L\)

1. \(\eta(t,u) = u\) for all \(u \in (E_L \setminus B^{(L)}_{r_2}(\Omega_{R,P,L})) \cup [J_L \leq 2b_\infty - \varepsilon]_L\);
2. \(\frac{d}{dt}J_L(\eta(t,u)) \leq 0\) for all \(u \in E_L\);
3. \(\frac{d}{dt}J_L(\eta(t,u)) \leq -\frac{\rho}{\nu_1}\) if \(\eta(t,u) \in (B^{(L)}_{\frac{r_1}{2}}(\Omega_{R,P,L}) \setminus B^{(L)}_{r_1}(\Omega_{R,P,L})) \cap [J_L \geq 2b_\infty - \varepsilon]_L\);
4. \(\frac{d}{dt}J_L(\eta(t,u)) \leq -\frac{\rho}{\nu_1}\) if \(\eta(t,u) \in B^{(L)}_{r_1}(\Omega_{R,P,L}) \cap [J_L \geq 2b_\infty - \varepsilon]_L\);
5. \(\frac{d}{dt}J_{A_{R,P,L}}(\eta(t,u)) \leq 0\) if \(\eta(t,u) \in B^{(L)}_{r_2}(\Omega_{R,P,L}) \cap [J_{A_{R,P,L}} \geq \rho]_L\).

Now we set \(\tilde{\eta}(u) = \eta(\chi_2(\eta), u)\). Then \(\tilde{\eta} \in C(E_L, E_L)\) has the desired properties (i)–(v).

Here we show just (iii). We argue indirectly and suppose that \(u \in B^{(L)}_{r_1}(\Omega_{R,P,L}) \cap [J_L \leq 2b_\infty + \varepsilon]_L\) satisfies

\[
J_L(\eta(t,u)) > 2b_\infty - \varepsilon \quad \text{for all } t \in [0, \frac{4\varepsilon}{\nu_1}].
\]  

(5.16) and consider two cases:
Case 1: \( \eta(t, u) \in B_{\frac{4 \epsilon}{\nu_L}}^{(L)}(\Omega_{R,P,L}) \) for all \( t \in [0, \frac{4 \epsilon}{\nu_L}] \),

Case 2: \( \eta(t, u) \notin B_{\frac{4 \epsilon}{\nu_L}}^{(L)}(\Omega_{R,P,L}) \) for some \( t \in [0, \frac{4 \epsilon}{\nu_L}] \).

If Case 1 occurs, we have
\[
\frac{d}{dt} J_L(\eta(t, u)) = -J_L'(\eta(t, u)) \nabla(\eta(t, u)) \leq \frac{\nu_L}{2} \text{ for all } t \in \left[0, \frac{4 \epsilon}{\nu_L}\right].
\]

Thus
\[
J_L(\eta(u)) = J_L(\eta(\frac{4 \epsilon}{\nu_L}, u)) \leq J_L(u) - 2\epsilon \leq 2b_\infty - \epsilon.
\]

which is in contradiction to (5.16).

If Case 2 occurs, we can find an interval \([t_0, t_1] \subset [0, \frac{4 \epsilon}{\nu_L}]\) such that
\[
\eta(t_0, u) \in \partial B_{\frac{4 \epsilon}{\nu_L}}^{(L)}(\Omega_{R,P,L}), \quad \eta(t_1, u) \in \partial B_{\frac{4 \epsilon}{\nu_L}}^{(L)}(\Omega_{R,P,L}),
\]
\[
\eta(t, u) \in B_{\frac{4 \epsilon}{\nu_L}}^{(L)}(\Omega_{R,P,L}) \setminus B_{\frac{4 \epsilon}{\nu_L}}^{(L)}(\Omega_{R,P,L}) \text{ for all } t \in (t_0, t_1).
\]

Since \( \frac{d}{dt} \|\eta(t, u)\|_{E_L} \leq \|\nabla(\eta(t, u))\|_{E_L} \leq 1 \) for all \( t \), we have \( t_1 - t_0 \geq \frac{24}{\nu_L} \). Thus we have
\[
J_L(\eta(u)) = J_L(\eta(\frac{4 \epsilon}{\nu_L}, u)) \leq J_L(\eta(t_1, u))
\]
\[
\leq J_L(\eta(t_0, u)) - \frac{\nu_L}{2} (t_1 - t_0)
\]
\[
\leq J_L(\eta(t_0, u)) - \frac{\nu_L r_2 - r_1}{2}
\]
\[
\leq J_L(u) - \frac{\nu_L r_2 - r_1}{2},
\]

which is also in contradiction to (5.16).

\[\square\]

### 5.2 Minimizing problem in \( A_{R,P,L} \)

To find a critical point of \( J_L(u) \), we need to solve the minimizing problem in \( A_{R,P,L} \). A decay property of a unique minimizer will play an important role later.

For \( R, P, L \) with \( R \geq 1 \), \( \bar{\rho} > 0 \), we set
\[
O(A_{R,P,L}, \rho) = \{ u \in E_L; \|u\|_{E_L(A_{R,P,L})} \leq r_0, J_{A_{R,P,L}}(u) \leq \rho \}
\]
and for \( u \in O(A_{R,P,L}, \rho) \) we also set
\[
K_{A_{R,P,L}}(u) = \{ v \in E_L; v = u \text{ in } D_L \setminus A_{R,P,L}, \|v\|_{E_L(A_{R,P,L})} \leq r_0 \}.
\]

We have the following existence result.

**Proposition 5.4.** Suppose \( \rho \in (0, \frac{1}{4} r_0^2) \). Then for any \( u \in O(A_{R,P,L}, \rho) \), the following minimizing problem has a unique minimizer \( v = v(A_{R,P,L}; u) \in K_{A_{R,P,L}}(u) \):
\[
\inf_{v \in K_{A_{R,P,L}}(u)} J_{A_{R,P,L}}(v).
\]

Moreover
Proof. By Lemma 5.1 (I), we have

(i) $O(AR_P,L,\rho) \rightarrow EL_\rho; u \mapsto v(AR_P,L;u)$ is continuous;

(ii) $\|v(AR_P,L;u)\|_{EL(AR_P,L)} < r_0$ for all $u \in O(AR_P,L,\rho)$;

(iii) $v(AR_P,L;u)(x) = 0$ in $AR_P,L$ if $u = 0$ in $AR_P,L$;

(iv) $J_L(v(AR_P,L;u)) \leq J_L(u)$, $J_{AR_P,L}(v(AR_P,L;u)) \leq J_{AR_P,L}(u)$ for all $u \in O(AR_P,L,\rho)$;

Proof. By Lemma 5.1 (I), we have

(1) $J_{AR_P,L}(v) \geq \frac{1}{4} \|v\|_{EL(AR_P,L)}^2$ for all $\|v\|_{EL(AR_P,L)} \leq r_0$;

(2) $J_{AR_P,L}(v)$ is strictly convex on $K_{AR_P,L}(u)$.

Thus, under the assumption $\rho \in (0, \frac{1}{4} r_0^2)$, we have

$$\inf_{v \in K_{AR_P,L}(u)} J_{AR_P,L}(v) \leq J_{AR_P,L}(u) \leq \rho < \frac{1}{4} r_0^2 \leq \inf_{v \in \partial K_{AR_P,L}(u)} J_{AR_P,L}(v)$$

and the infimum (5.17) is achieved in $int K_{AR_P,L}(u)$. Moreover the minimizer $v(AR_P,L;u) \in int K_{AR_P,L}(u)$ is unique. It is easy to see that (i)–(iv) hold.

We have the following decay estimate for the unique minimizer $v(AR_P,L;u)$ obtained in Proposition 5.4.

**Lemma 5.5.** There exist constants $R_2 > 0$, $A_1$, $A_2 > 0$ independent of $P$, $L$ such that if $R \geq R_2$, then

$$|v(AR_P,L;u)(x)|, \ |\nabla v(AR_P,L;u)(x)| \leq A_1 e^{-A_2 R} \text{ for all } x \in A_{2R,P,L}.$$

**Proof.** We know $v(x) = v(AR_P,L;u)(x)$ is the unique solution in $K_{AR_P,L}(u)$ for

$$\begin{cases} -\Delta v + V(x)v + Q(x)h(v) - Q(x)f(v) = 0 \text{ in } AR_P,L, \\ v = u \text{ in } DL \setminus AR_P,L. \end{cases}$$

By (VI) of Lemma 5.1 we see that $v(x)$ satisfies

$$|v(x)| \leq \frac{1}{e} \text{ in } A_{R+1,P,L}.$$

In particular, $v(x)$ solves

$$-\Delta v + V(x)v + Q(x)h(v) = 0 \text{ in } A_{R+1,P,L}.$$

By the maximal principle, we can get the exponential decay of $v$ in $A_{2R+2,P,L}$, that is,

$$|v(x)| \leq A_1 e^{-A_2 R} \text{ in } A_{2R+2,P,L}.$$

By the regularity argument, we also have the exponential decay of $\nabla v$ in $A_{2R,P,L}$. 

$\Box$
5.3 Proof of Theorem 2

For any given \( r \in (0, \frac{1}{3}r_0] \) we try to find a critical point \( J_L(u) \) in \( B^L_{\infty}(\Omega_{R,P,L}) \) for large \( R, P, L \) with \( \Box \) to prove Theorem 2.

Proof of Theorem 2. Proof of Theorem 2 is divided into several steps.

Step 1: Choice of parameters \( r_1, r_2, \rho, \varepsilon, \tau \)

First we choose parameters which will be used in the proof of Theorem 2.

For any given \( r \in [0, \frac{1}{3}r_0] \), we set \( r_1 = \frac{1}{2}r \), \( r_2 = r \), \( \rho = \frac{1}{2}r^2 \). We apply Proposition [5.2] to obtain \( R_1 = R_1(r_1, r_2, \rho), \nu_1 = \nu_1(\rho) > 0 \).

Next we choose \( T > 1 \) such that

\[
J_{\infty}(T\omega), \quad J_{\infty}(T\omega') < 0.
\]

We also choose \( \delta_0 > 0 \) with a property: for \( \theta > 0 \)

\[
J_{\infty}(\theta \omega) \geq b_0 - \delta_0 \quad (\text{resp.} \quad J_{\infty}(\theta \omega') \geq b_0 - \delta_0) \implies \| \theta \omega - \omega \|_{E_{\infty}} \leq \frac{r}{4} \quad (\text{resp.} \quad \| \theta \omega' - \omega' \|_{E_{\infty}} < \frac{r}{4}).
\]

Next we set \( \tau = b_{\infty}/3 \) and we choose

\[
\varepsilon \in \left(0, \min\{\tau, \frac{\delta_0}{2}, \frac{\nu_1}{4}(r_2 - r_1)\}\right).
\]

To show Theorem 2, we argue indirectly and assume

\[
\mathcal{K}_L \cap B^L_{\infty}(\Omega_{R,P,L}) = \emptyset.
\]

The by Lemma 5.3 there exists \( \tilde{\eta} \in C(E_L, E_L) \) satisfying properties (i)--(v) of Lemma 5.3.

For \( (\theta_1, \theta_2) \in [0,T]^2 \), we set

\[
G(\theta_1, \theta_2)(x) = \theta_1 \Phi_L(\psi_{R\omega})(x) + \theta_2 \Phi_L(\psi_{R\omega}')(x - P).
\]

We choose

\[
R_3 \geq \max\{R_1, R_2\}
\]

such that for \( R, P, L \) with \( R \geq R_3 \) and \( \Box \)

\[
J_L(G(\theta_1, \theta_2)) \leq b_{\infty} + \tau \quad \text{for all} \quad (\theta_1, \theta_2) \in \partial([0,T]^2);
\]

\[
J_L(G(\theta_1, \theta_2)) \leq 2b_{\infty} + \varepsilon \quad \text{for all} \quad (\theta_1, \theta_2) \in [0,T]^2;
\]

\[
J_L(G(\theta_1, \theta_2)) \geq 2b_{\infty} - \varepsilon \quad \text{implies} \quad G(\theta_1, \theta_2) \in B^L_{\infty}(\Omega_{R,P,L});
\]

\[
G(\theta_1, \theta_2)(x) = 0 \quad \text{in} \quad A_{R,P,L} \quad \text{for all} \quad (\theta_1, \theta_2) \in [0,T]^2;
\]

\[
b_L \geq b_{\infty} - \frac{\varepsilon}{8} \quad \text{for all} \quad L \geq 10R;
\]

\[
A_1 e^{-2A_2R}(3 + \| V \|_{L^\infty} + 2\| Q \|_{L^\infty} A_2 R) \omega_{N-1} (2R + 1)^N \leq \frac{\varepsilon}{2};
\]

where \( A_1, A_2 \) are constant appeared in Lemma 5.3 (c.f. (5.20)).

To find \( R_3 \), we note that

\[
J_L(\theta_1 \Phi_L(\psi_{R\omega})) \rightarrow J_{\infty}(\theta_1 \omega) \leq b_{\infty};
\]

\[
J_L(\theta_2 \Phi_L(\psi_{R\omega}')) \rightarrow J_{\infty}(\theta_2 \omega') \leq b_{\infty};
\]

\[
J_L(G(\theta_1, \theta_2)) = J_L(\theta_1 \Phi_L(\psi_{R\omega})) + J_L(\theta_2 \Phi_L(\psi_{R\omega}')) \rightarrow J_{\infty}(\theta_1 \omega) + J_{\infty}(\theta_2 \omega');
\]

\[
\|(1 - \psi_R) \omega\|_{E_{\infty}}, \quad \|(1 - \psi_R) \omega\|''_{E_{\infty}} \rightarrow 0;
\]

\[
((1 - \psi_R) \omega)''_{E_{\infty}} \rightarrow 0;
\]

\[
(5.26)
\]
as $R \to \infty$ uniformly in $(\theta_1, \theta_2) \in [0, T]^2$.

We can easily find $R_3 \geq \max\{R_1, R_2\}$ such that (5.20)–(5.21) hold for $R \geq R_3$. For (5.22), we note that $J_L(G(\theta_1, \theta_2)) \geq 2b_{\infty} - \varepsilon$ implies $J_L(\theta_1, \Phi_L(\psi_R \omega))$, $J_L(\theta_2, \Phi_L(\psi_R \omega')) \geq b_{\infty} - \frac{2\varepsilon}{L}$ for large $R$ and thus $J_{\infty}(\theta_1 \omega), J_{\infty}(\theta_2 \omega) \geq b_{\infty} - 2\varepsilon \geq b_{\infty} - \delta_0$ for large $R$. Thus from (5.18) and (5.20), we have (5.22). Properties (5.20) and (5.21) follow from the definition of $G(\theta_1, \theta_2)$ and Theorem 5. Property (5.24) is easily checked.

Step 2: Definition of $\hat{G}(\theta_1, \theta_2)$ and its properties

We set

$$\hat{G}(\theta_1, \theta_2) = \tilde{h}(G(\theta_1, \theta_2)) \in C([0, T]^2, E_L),$$

where $\tilde{h} \in C(E_L, E_L)$ is defined in Lemma 5.3. $\hat{G}(\theta_1, \theta_2)$ has the following properties:

1. $\hat{G}(\theta_1, \theta_2) = G(\theta_1, \theta_2)$ for all $(\theta_1, \theta_2) \in \partial([0, T]^2)$;
2. $J_L(\hat{G}(\theta_1, \theta_2)) \leq 2b_{\infty} - \varepsilon$ for all $(\theta_1, \theta_2) \in [0, T]^2$;
3. $\|\hat{G}(\theta_1, \theta_2)\|_{E_L(A_{R.P.L})} \leq r_0, J_{A_{R.P.L}}(\hat{G}(\theta_1, \theta_2)) \leq \frac{1}{2} \varepsilon^2$ for all $(\theta_1, \theta_2) \in [0, T]^2$.

In particular, $G(\theta_1, \theta_2) \in O(A_{R.P.L}, \frac{1}{2} \varepsilon^2)$ for all $(\theta_1, \theta_2) \in [0, T]^2$.

(1) and (2) follow from Lemma 5.3 and (5.20)–(5.22). For (3), we note that $\tilde{h}(u) = u$ or $\tilde{h}(u) \in B_{L_2}^R(\Omega_{R.P.L})$ holds for all $u \in E_L$. By (5.23), we have $\|\hat{G}(\theta_1, \theta_2)\|_{E_L(A_{R.P.L})} \leq r_0$ for all $(\theta_1, \theta_2) \in [0, T]^2$. We also deduce $J_{A_{R.P.L}}(\hat{G}(\theta_1, \theta_2)) \leq \frac{1}{2} \varepsilon^2$ from Lemma 5.3.

Step 3: Definition of $\tilde{G}(\theta_1, \theta_2)$ and its properties

By the property (3) of $\hat{G}(\theta_1, \theta_2)$, we can define

$$\tilde{G}(\theta_1, \theta_2) = v(A_{R.P.L}; \hat{G}(\theta_1, \theta_2)) \in C([0, T]^2, E_L).$$

$\tilde{G}(\theta_1, \theta_2)$ has the following properties:

1. $\tilde{G}(\theta_1, \theta_2) = G(\theta_1, \theta_2)$ for all $(\theta_1, \theta_2) \in \partial([0, T]^2)$;
2. $J_L(\tilde{G}(\theta_1, \theta_2)) \leq 2b_{\infty} - \varepsilon$ for all $(\theta_1, \theta_2) \in [0, T]^2$;
3. $\|\tilde{G}(\theta_1, \theta_2)\|_{E_L(A_{R.P.L})} \leq r_0$ for all $(\theta_1, \theta_2) \in [0, T]^2$;
4. $|\nabla \tilde{G}(\theta_1, \theta_2)| \leq A_1 e^{-A_2 R}$ for all $x \in A_{R.P.L}$ and $(\theta_1, \theta_2) \in [0, T]^2$.

These properties follow from Proposition 5.4 and Lemma 5.5.

Step 4: Definition of $G(\theta_1, \theta_2)$ and its properties

27
We define \( g \) by our choice of \( R \). By Lemma 5.1 (III), we have

\[
\hat{J}_A \leq 1 \leq 2 \varepsilon.
\]

To see (2), we write

\[
\hat{J}_2 \leq \hat{J}_1 \leq \hat{J}_3.
\]

\( \hat{J}_1, \hat{J}_2, \hat{J}_3 \) have the following properties:

1. \( \hat{J}_1, \hat{J}_2, \hat{J}_3 \) and \( \hat{J}_4 \) are \( 2L \)-periodic functions in \( 0 \leq 1 \leq 0 \).

2. \( J_{\text{Le}}(\hat{J}_1, \hat{J}_2) \leq 2b_{\infty} - \frac{1}{2} \varepsilon \) for all \( (\theta_1, \theta_2) \in [0, T]^2 \).

To see (2), we write \( \hat{u} = \hat{G}(\theta_1, \theta_2) \), \( \hat{u} = \hat{G}(\theta_1, \theta_2) \) and we compute

\[
J_{\text{Le}}(\hat{u}) - J_{\text{Le}}(\hat{u}) = J_{A_{2R.P.L}}(\hat{u}) - J_{A_{2R.P.L}}(\hat{u}) \leq J_{A_{2R.P.L}}(\hat{u}).
\]

Here we used Lemma 5.1 (III). Since \( |\hat{u}| \leq 2|\hat{u}| + |\hat{u}| \leq 2A_1 e^{-A_2 R} \), we have

\[
J_{A_{2R.P.L}}(\hat{u}) = \frac{1}{2} \int_{A_{2R.P.L} \setminus A_{2R+1.P.L}} |\nabla \hat{u}|^2 + V(x)\hat{u}^2 - Q(x)| \hat{u}^2 | \log \hat{u}^2 
\]

\[
\leq \frac{1}{2} A_1 e^{-A_2 R} \left( 3 + |\hat{u}| + |Q||L^\infty| \omega_{N-1}(2R + 1)^N \right) \leq \frac{1}{2} \varepsilon. 
\]

By our choice of \( R_3 \), we have \( J_{A_{2R+1,P.L}}(\hat{u}) < \frac{1}{2} \varepsilon \). Thus we have

\[
J_{Le}(\hat{u}) = J_{Le}(\hat{u}) + (J_{Le}(\hat{u}) - J_{Le}(\hat{u})) \leq J_{A_{2R,P.L}}(\hat{u}) + J_{A_{2R,P.L}}(\hat{u}) 
\]

\[
\leq 2b_{\infty} - \varepsilon + \frac{1}{2} \varepsilon = 2b_{\infty} - \frac{1}{2} \varepsilon.
\]
Thus we get (2).

**Step 5: An intersection result and the end of proof**

By the property (5.28), for any curve \( \gamma(s) : [0, 1] \to [0, T]^2 \) with \( \gamma(0) \in \{0\} \times [0, T], \gamma(1) \in \{T\} \times [0, T] \) (resp. \( \gamma(0) \in [0, T] \times \{0\}, \gamma(1) \in [0, T] \times \{T\} \)), a path \( \mathcal{P}_1(\gamma(s)) \) (resp. \( \mathcal{P}_2(\gamma(s)) \)) is a path joining 0 and \( T\Phi_L(\psi_R \omega') \) (resp. \( T\Phi_L(\psi_R \omega')(-c - P) \)). Noting \( J_L(T\Phi_L(\psi_R \omega')) \), \( J_L(T\Phi_L(\psi_R \omega')) < 0 \), they can be regarded as a sample path corresponding to mountain pass theorem.

As in Proposition 3.4 of Coti Zelati-Rabinowitz [8], there exists a \((\overline{\vartheta}_1, \overline{\vartheta}_2) \in [0, T]^2 \) such that

\[
J_L((\overline{\vartheta}_1, \overline{\vartheta}_2)) \geq b_L, \quad J_L((\overline{\vartheta}_1, \overline{\vartheta}_2)) \geq b_L.
\]

Thus we have

\[
J_L((\overline{\vartheta}_1, \overline{\vartheta}_2)) = J_L((\overline{\vartheta}_1, \overline{\vartheta}_2)) + J_L((\overline{\vartheta}_1, \overline{\vartheta}_2)) \geq 2b_L.
\]

Therefore we have

\[
2b_\infty - \frac{1}{2} \epsilon \geq 2b_L.
\]

This contradicts with (5.24) and thus \( K_{\omega} \neq \emptyset \). Thus, choosing \( R_3 > R_3 \), we complete the proof of Theorem 2. \( \square \)

### 5.4 Proof of Theorem 1

Finally we give a proof of our Theorem 1.

**Proof of Theorem 1** Let \( R_3 > 0 \) be a number given in (5.19). We take \( R_3 \geq R_3 \) such that

\[
\|\omega - \psi_{R_3} \omega\|_{E_\infty}, \quad \|\omega' - \psi_{R_3} \omega'\|_{E_\infty} \leq \frac{r}{2}.
\]

(5.30)

We fix \( P \in \mathbb{Z}^N \) with \( |P| \geq 5R_3 \). By Theorem 2 for any \( L \geq 2|P| \) there exists a critical point \( u_L \in K_L \cap B_r^{(L)}(\Omega_{R_3}, P, L) \). By Lemma 2.7 (ii) and Lemma 5.1 we have

\[
J_L(u_L) = \frac{1}{2} \int_{D_L} Q(x)u_L^2 \in [2b_\infty - \alpha, 2b_\infty + \alpha].
\]

We apply our concentration-compactness result (Proposition 5.1) to \( u_L \) \( (L = 2|P| + 1, 2|P| + 2, \ldots) \). After extracting a subsequence \( L_j \to \infty \), we have for some \( \omega_0 \in K_\infty \)

\[
\|u_{L_j} - \Phi_j(\psi_{L_j/2} \omega_0)\|_{E_{L_j}} \to 0, \quad (5.31)
\]

\[
J_{L_j}(u_{L_j}) \to J_\infty(\omega_0) \quad \text{as} \quad j \to \infty. \quad (5.32)
\]

In fact, if not, we have \( m \geq 2 \) in the statement of Proposition 5.1, and for some sequence \( (y_j)_{j=1}^\infty \subset \mathbb{R}^N \) with \( y_j \in D_{L_j} \) and \( |y_j| \to \infty \)

\[
\liminf_{j \to \infty} \|u_{L_j}\|_{H^1(B_{L_j/3}(y_j))} > 0,
\]
which is in a contradiction to $u_* \in B^1_L(\Omega_{R_1, P, L})$. It easily follows from \((5.31) - (5.32)\) that
\[
\left\| w_0 - \Omega_{R_1, P, L} \right\|_{E_*} = \lim_{j \to \infty} \left\| \Phi_{L_j}(\psi_{L_j}/2w_0) - \Omega_{R_1, P, L} \right\|_{E_{L_j}} \\
\leq \lim_{j \to \infty} \left\| \Phi_{L_j}(\psi_{L_j}/2w_0) - u_{L_j} \right\|_{E_{L_j}} + \lim_{j \to \infty} \left\| u_{L_j} - \Omega_{R_1, P, L} \right\|_{E_{L_j}} \\
\leq \rho,
\]
which implies by \((5.30)\)
\[
\left\| w_0 - (\omega + \omega'(\cdot - P)) \right\|_{E_*} \leq 2\rho.
\]

\[\Box\]

A Proofs of Proposition \(3.1\) and (VI) of Lemma \(5.1\)

A.1 Proof of Proposition \(3.1\)

In the following proof, we use an idea from Jeanjean-Tanaka [13].

**Proof of Proposition \(3.1\)**. We assume $L_j, u_j (j = 1, 2, \cdots)$ satisfy the assumption \((3.1)\) of Proposition \(3.1\). By Lemma \(2.7\) we have for some $A > 0$ independent of $j$
\[
\left\| u_j \right\|_{E_{L_j}}, \int_{D_{L_j}} H(u_j), \int_{D_{L_j}} h(u_j)u_j \leq A \quad \text{for all } j.
\]

**Step 1**: After extracting a subsequence, there exists a sequence $y_j^1 \in \mathbb{Z}^N$ and $w^1 \in K_\infty \setminus \{0\}$ such that
\[
y_j^1 \in D_{L_j}, \\
u_j(x + y_j^1) \rightharpoonup w^1(x) \quad \text{weakly in } H^1_{loc}(\mathbb{R}^N).
\]

For $q \in (2, 2^*)$, we set
\[
d_j = \sup_{n \in \mathbb{Z}^N} \left\| u_j \right\|_{L^q(D_1(n))} \quad \text{for } j = 1, 2, \cdots.
\]
If $d_j \to 0$ as $j \to \infty$, by Lemma \((3.3)\) we have $J_{L_j}(u_j) \to 0$, which contradicts with \((3.1)\). Thus, after extracting a subsequence if necessary, we may assume $d_j \to d_0 > 0$ and there exists $y_j^1 \in \mathbb{Z}^N$ such that
\[
\left\| u_j(\cdot + y_j^1) \right\|_{L^1(D_1(0))} \to d_0 > 0.
\]
We may also assume that there exists $w^1 \in H^1_{loc}(\mathbb{R}^N) \setminus \{0\}$ such that
\[
u_j(\cdot + y_j^1) \rightharpoonup w^1 \quad \text{weakly in } H^1_{loc}(\mathbb{R}^N).
\]
We claim \( w^1 \in K_\infty \setminus \{0\} \). In fact, for any \( L \in \mathbb{N} \)
\[
\|w^1\|_{E_\infty(D_L)}^2 \leq \limsup_{j \to \infty} \|u_j(\cdot + y_j^1)\|_{E_{L_j}(D_L)}^2 \\
\leq \limsup_{j \to \infty} \|u_j\|_{E_{L_j}}^2 \\
\leq A,
\]
\[
\int_{D_L} H(w^1) \leq \limsup_{j \to \infty} \int_{D_L} H(u_j(\cdot + y_j^1)) \\
\leq \limsup_{j \to \infty} \int_{D_{L_j}} H(u_j(\cdot + y_j^1)) \\
\leq A.
\]
Since \( A \) is independent of \( L \), we have \( w^1 \in E_\infty \) and \( \int_{\mathbb{R}^N} H(w^1) < \infty \). Thus \( w^1 \in D \). Next we see \( w^1 \in K_\infty \).

Next we assume that there exists \( m_0 \in \mathbb{N} \), \( w^\ell \in K_\infty \setminus \{0\} \), \((y_j^\ell)_{j=1}^{\infty} \subset \mathbb{Z}^N \) with \( y_j^\ell \in D_{L_j} \) \((\ell = 1, 2, \cdots, m_0)\) such that
\[
\text{dist}_{L_j}(y_j^\ell, y_j^{\ell'}) \to \infty \quad \text{for} \quad \ell \neq \ell', \quad (A.1)
\]
\[
u_j(\cdot + y_j^\ell) \rightharpoonup w^\ell \text{ weakly in } H^1_{\text{loc}}(\mathbb{R}^N) \quad \text{for all } \ell = 1, 2, \cdots, m_0. \quad (A.2)
\]
For \( R_j \in \mathbb{N} \) with \( R_j \to \infty \) and \( R_j \leq L_j \), we set
\[
\tilde{w}_j = \sum_{\ell=1}^{m_0} \Phi_{L_j}(\psi_{R_j} w^\ell)(\cdot - y_j^\ell) \in E_{L_j}.
\]
Taking a subsequence if necessary, we may assume
\[
\lim_{j \to \infty} \|u_j - \tilde{w}_j\|_{E_{L_j}}, \lim_{j \to \infty} \|u_j\|_{E_{L_j}} \text{ exist.}
\]
We show
\[
\text{Step 2: } \lim_{j \to \infty} \|u_j - \tilde{w}_j\|_{E_{L_j}}^2 = \lim_{j \to \infty} \|u_j\|_{E_{L_j}}^2 - \sum_{\ell=1}^{m_0} \|w^\ell\|_{E_{L_j}}^2.
\]
It follows from \((A.2)\) and
\[
\psi_{R_j} w^\ell \rightharpoonup w^\ell \text{ in } E_\infty \text{ as } j \to \infty \text{ for all } \ell = 1, 2, \cdots, m_0
\]
that
\[
\langle u_j, \Phi_{L_j}(\psi_{R_j} w^f)(\cdot - y^f_j) \rangle_{E_{L_j}} = \langle u_j(\cdot + y^f_j), \Phi_{L_j}(\psi_{R_j} w^f) \rangle_{E_{L_j}}
\]
\[
\rightarrow \|w^f\|^2_{E_{\infty}} \quad \text{for all } \ell = 1, 2, \ldots , m_0,
\]
(A.3)

\[
\langle \Phi_{L_j}(\psi_{R_j} w^f)(\cdot - y^f_j), \Phi_{L_j}(\psi_{R_j} w^f)(\cdot - y^f_j) \rangle_{E_{L_j}}
\]
\[
\rightarrow \begin{cases} 
\|w^f\|^2_{E_{\infty}} & \text{if } \ell = \ell', \\
0 & \text{if } \ell \neq \ell'
\end{cases}
\]
(A.4)
as \( j \rightarrow \infty \).

Thus we have
\[
\|u_j - \tilde{w}_j\|^2_{E_{L_j}} = \|u_j\|^2_{E_{L_j}} - 2\langle u_j, \tilde{w}_j \rangle_{E_{L_j}} + \|\tilde{w}_j\|^2_{E_{L_j}}
\]
\[
= \|u_j\|^2_{E_{L_j}} - 2 \sum_{\ell=1}^{m_0} \langle u_j, \Phi_{L_j}(\psi_{R_j} w^f)(\cdot - y^f_j) \rangle_{E_{L_j}}
\]
\[
+ \sum_{\ell=1}^{m_0} \sum_{\ell' = 1}^{m_0} \langle \Phi_{L_j}(\psi_{R_j} w^f)(\cdot - y^f_j), \Phi_{L_j}(\psi_{R_j} w^f)(\cdot - y^f_{j'}) \rangle_{E_{L_j}}
\]
\[
\rightarrow \|u_j\|^2_{E_{L_j}} - 2 \sum_{\ell=1}^{m_0} \|w^f\|^2_{E_{\infty}} \quad \text{as } j \rightarrow \infty.
\]

Next we set
\[
\tilde{d}_j = \sup_{n \in \mathbb{Z}^N} \|u_j - \tilde{w}_j\|_{L^4(D_{L_j})}.
\]

After extracting a subsequence, we may assume \( \lim_{j \rightarrow \infty} \tilde{d}_j \) exists. We consider 2 cases:

Case 1: \( \tilde{d}_j \rightarrow 0 \) as \( j \rightarrow \infty \),

Case 2: \( \tilde{d}_j \not\rightarrow 0 \) as \( j \rightarrow \infty \).

**Step 3:** If Case 1 occurs, we have \( \|u_j - \tilde{w}_j\|_{E_{L_j}} \rightarrow 0 \) as \( j \rightarrow \infty \).

In fact, if \( \tilde{d}_j \rightarrow 0 \), we have by Lemma 3.2
\[
\|u_j - \tilde{w}_j\|_{L^4(D_{L_j})} \rightarrow 0.
\]
(A.5)

We have
\[
\|u_j - \tilde{w}_j\|^2_{E_{L_j}} = \langle u_j, u_j - \tilde{w}_j \rangle_{E_{L_j}} - \langle \tilde{w}_j, u_j - \tilde{w}_j \rangle_{E_{L_j}}
\]
\[
= \int_{D_{L_j}} (u_j(h(u_j) - u_j) - \langle \tilde{w}_j, u_j - \tilde{w}_j \rangle_{E_{L_j}}
\]
\[
+ \int_{D_{L_j}} Q(x)f(u_j)(u_j - \tilde{w}_j) - \langle \tilde{w}_j, u_j - \tilde{w}_j \rangle_{E_{L_j}}
\]
\[
= \alpha(1) - (I) + (II) - (III) \quad \text{as } j \rightarrow \infty.
\]
(A.6)

By (A.5), we can see that
\[
(II) \rightarrow 0 \quad \text{as } j \rightarrow \infty.
\]
(A.7)
For (III), we have

\[(III) = \left( \sum_{\ell=1}^{m_0} \Phi_{L_j}(\psi_{R_j} w^\ell)(\cdot - y_j^\ell), u_j - \sum_{\ell=1}^{m_0} \Phi_{L_j}(\psi_{R_j} w^\ell)(\cdot - y_j^\ell) \right)_{E_{L_j}} \]

\[= \sum_{\ell=1}^{m_0} \left( \Phi_{L_j}(\psi_{R_j} w^\ell), u_j + y_j^\ell - \Phi_{L_j}(\psi_{R_j} w^\ell) \right)_{E_{L_j}} \]

\[= \sum_{\ell \neq \ell'}^{m_0} \left( \Phi_{L_j}(\psi_{R_j} w^\ell), \Phi_{L_j}(\psi_{R_j} w^{\ell'}) \right)_{E_{L_j}} \]

\[\to 0 \quad \text{as } j \to \infty, \quad (A.8)\]

which follows from \((A.3)-(A.4)\). For (I), we fix \(\theta \in (0, 1]\) small and \(\rho \geq 1\) large. We use notation:

\[B(\rho, \ell, j) = \{ x \in D_{L_j}; \quad \text{dist}_{L_j}(x, y_j^\ell) < \rho \}, \]

that is,

\[D_{L_j} \setminus B(\rho, \ell, j) = D_{L_j} \setminus \bigcup_{n \in \mathbb{Z}^N} B_{\rho}(y_j^\ell + 2L_j n). \]

We note that

\[B(\rho, \ell, j) \cap B(\rho, \ell', j) = \emptyset \quad \text{for } \ell \neq \ell', \]

provided

\[\rho < \frac{1}{2} \min_{\ell \neq \ell'} \text{dist}_{L_j}(y_j^\ell, y_j^{\ell'}). \quad (A.9)\]

By \((A.1)\) we remark that for any \(\rho \geq 1\) \((A.9)\) holds for large \(j\).

We compute

\[(I) = \int_{D_{L_j}} Q(x) h(u_j)(u_j - \tilde{\omega}_j) \]

\[= \int_{D_{L_j} \setminus \bigcup_{n=1}^{m_0} B_{(\rho, \ell, j)}} Q(x) h(u_j)(u_j - \tilde{\omega}_j) + \sum_{\ell=1}^{m_0} \int_{B(\rho, \ell, j)} Q(x) h(u_j)(u_j - \tilde{\omega}_j) \]

\[= (I1) + \sum_{\ell=1}^{m_0} (I2)_\ell. \]

By Lemma 2.1 (iii-b),

\[(I1) \geq - \int_{D_{L_j} \setminus \bigcup_{n=1}^{m_0} B_{(\rho, \ell, j)}} Q(x) h(u_j) \tilde{\omega}_j \]

\[= - \sum_{\ell=1}^{m_0} \int_{D_{L_j} \setminus \bigcup_{n=1}^{m_0} B_{(\rho, \ell, j)}} Q(x) h(u_j) \Phi_{L_j}(\psi_{R_j} w^\ell)(\cdot - y_j^\ell) \]

\[\geq - \|Q\|_{L^\infty} \sum_{\ell=1}^{m_0} \left( \theta \int_{D_{L_j} \setminus \bigcup_{n=1}^{m_0} B_{(\rho, \ell, j)}} H(u_j) \right. \]

\[\left. + \frac{1}{\theta} \int_{D_{L_j} \setminus \bigcup_{n=1}^{m_0} B_{(\rho, \ell, j)}} H(\Phi_{L_j}(\psi_{R_j} w^\ell)(\cdot - y_j^\ell)) \right) \]

\[\geq - \|Q\|_{L^\infty} \sum_{\ell=1}^{m_0} \left( \theta A + \frac{1}{\theta} \int_{\mathbb{R}^{N} \setminus B_{\rho}(0)} H(w^\ell) \right). \]
We also have
\[
(I2)_\ell = \int_{B(\rho, J, j)} Q(x) h(u_j)(u_j - \sum_{\ell'=1}^{m_0} \Phi_{L_j}(\psi_{R_j} w^{\ell'})(\cdot - y^{\ell'}_j))
\]
\[
= \int_{B_\rho(0)} Q(x) h(u_j(\cdot + y^{\ell}_{j})) (u_j(\cdot + y^{\ell}_{j}) - w^{\ell'}) + o(1)
\]
\[
\rightarrow 0 \quad \text{as } j \to \infty.
\]
Thus we have
\[
\liminf_{j \to \infty} (I) \geq -\|Q\|_{L^\infty} \sum_{\ell=1}^{m_0} \left( \theta A + \frac{1}{\theta} \int_{\mathbb{R}^N \setminus B_\rho(0)} H(w^{\ell'}) \right).
\]
Since \(\theta \in (0, 1]\) and \(\rho \geq 1\) are arbitrary, we have
\[
\liminf_{j \to \infty} (I) \geq 0.
\]
(A.10)

Thus by (A.6)–(A.10), we have \(\|u_j - \tilde{w}_j\| \to 0\).

Next we deal with Case 2.

Step 4: If Case 2 occurs, there exists \(w^{m_0+1} \in K_{\infty} \setminus \{0\}\) and a sequence \((y^{m_0+1}_j)_{j=1}^{\infty} \subset \mathbb{Z}^N\) such that
\[
\text{dist}_{L_j}(y^{\ell}_{j}, y^{m_0+1}_j) \to \infty \quad \text{for all } \ell = 1, 2, \ldots, m_0,
\]
\[
u_j(\cdot + y^{m_0+1}_j) \rightharpoonup w^{m_0+1} \quad \text{weakly in } H^1_{\text{loc}}(\mathbb{R}^N)
\]
as \(j \to \infty\).

In fact, we choose \(y^{m_0+1}_j \in \mathbb{Z}^N\) such that
\[
\|u_j - \tilde{w}_j\|_{L^q(D_1(y^{m_0+1}_j))} = \tilde{d}_j.
\]
(A.11) follows from (A.1) and (A.2). By (A.11), we can see
\[
\|\Phi_{L_j}(\psi_{R_j} w^{\ell'})\|_{L^q(D_1(y^{m_0+1}_j))} \to 0 \quad \text{as } j \to \infty
\]
for \(\ell = 1, 2, \ldots, m_0\) and thus \(u_j(\cdot + y^{m_0+1}_j)\) has a non-zero weak limit \(w^{m_0+1}\). As in Step 1, we can see \(w^{m_0+1} \in K_{\infty} \setminus \{0\}\).

Step 5: Conclusion

We follow a recursive procedure. We start with \(m = 1\) and use Step 1 to find \(w^1\) and \((y^1_j)_{j=1}^{\infty}\). If it satisfies \(\sup_{n \in \mathbb{Z}^N} \|u_j - \Phi_{L_j}(\psi_{R_j} w^1)(\cdot - y^1_j)\|_{L^q(D_1(n))} \to 0\), we are done by Step 3. Otherwise, we use Step 4 to get \(w^2\) and \((y^2_j)_{j=1}^{\infty}\) and continue this procedure. Next we prove this procedure stops in finite steps.

By Step 2, we have
\[
\sum_{\ell=1}^{m} \|w^{\ell'}\|_{E_{\infty}}^2 \leq \lim_{j \to \infty} \|u_j\|_{E_{L_j}}^2 \leq A.
\]

On the other hand, by Remark 2.2
\[
m\rho^2 \leq A.
\]
Thus, this procedure must end in finite steps. Therefore there exists \((w^\ell)^m_{\ell=1} \subset K_\infty \setminus \{0\}\) and \((y^\ell)^\infty_{j=1}\) such that (3.2)–(3.3) hold. We show here (3.4) and (3.5).

By (2.8),

\[
J_{L,j}(u_j) = J_{L,j}(u_j) - \frac{1}{2} J'_{L,j}(u_j) + o(1)
\]

Thus (3.3) implies

\[
J_{L,j}(u_j) \to \frac{1}{2} \sum_{\ell=1}^m \int_{D_{L,j}} Q(x)(w^\ell)^2 = \sum_{\ell=1}^m J_\infty(w^\ell). \tag{A.12}
\]

This is nothing but (3.4).

By (A.12) and (3.3), we have

\[
\int_{D_{L,j}} Q(x)H(u_j) = J_{L,j}(u_j) - \frac{1}{2} \|u_j\|^2_{E_{L,j}} + \int_{D_{L,j}} Q(x)F(u_j)
\]

\[
\to \sum_{\ell=1}^m \left( J_\infty(w^\ell) + \frac{1}{2} \|w^\ell\|^2_{E_\infty} - \int_{R^N} Q(x)F(w^\ell) \right)
\]

\[
= \sum_{\ell=1}^m \int_{R^N} Q(x)H(w^\ell).
\]

Clearly we have for all \(R \geq 1\)

\[
\int_{\bigcup_{\ell=1}^m B(R,\ell,j)} Q(x)H(u_j) \to \sum_{\ell=1}^m \int_{B_R(0)} Q(x)H(w^\ell).
\]

Thus for any \(\varepsilon > 0\) we can find a large \(R_\varepsilon > 1\) such that

\[
\int_{D_{L,j} \setminus \bigcup_{\ell=1}^m B(R_\varepsilon,\ell,j)} Q(x)H(u_j) < \varepsilon,
\]

from which we can show (3.5). \(\Box\)

A.2 Proof of (VI) of Lemma 5.1

We need the following lemma to prove (VI) of Lemma 5.1

Lemma A.1 (Subsolution estimate, Theorem C.1.2 of [21]): Suppose \(v \in H^1(B_2(x_0))\) solves

\[-\Delta v + \tilde{V}(x)v = 0 \quad \text{in} \; B_2(x_0).\]

Then

\[|v(x_0)| \leq C \int_{B_1(x_0)} |v|,\]
where $C > 0$ is a constant depending only on the following quantities:

$$\sup_{x \in B_1(x_0)} \int_{|y-x| \leq 1} \tilde{V}(y) \, dy \quad \text{if } N = 1;$$

$$\sup_{x \in B_1(x_0)} \int_{|y-x| \leq \frac{1}{2}} \log(|x-y|^{-1}) \tilde{V}(y) \, dy \quad \text{if } N = 2;$$

$$\sup_{x \in B_1(x_0)} \int_{|y-x| \leq 1} |x-y|^{2-N} \tilde{V}(y) \, dy \quad \text{if } N \geq 3.$$

**Proof of (VI) of Lemma 5.1.** We may assume that $r_0 \in (0, 1]$ and $\|u\|_{H^1(B_2(x_0))} \leq r_0 \leq 1$.

We apply Lemma A.1 to obtain

$$|u(x)| \leq C \int_{B_1(x)} |u(y)| \, dy \quad \text{for } x \in B_1(x_0),$$

where $C > 0$ is a constant depending only on

$$\left( V(x) + Q(x) \frac{h(u)}{u} - Q(x) \frac{f(u)}{u} \right)^- \leq Q(x) \frac{f(u)}{u} \quad \text{in } B_2(x_0).$$

Thus $C > 0$ does not depend on $u$ with $\|u\|_{H^1(B_2(x_0))} \leq 1$. Therefore we have

$$|u(x)| \leq C \|u\|_{L^1(B_1(x))} \leq C' r_0.$$

Choosing $r_0 > 0$ small, we get the conclusion.

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