SPACE-TIME APPROXIMATION OF LOCAL STRONG SOLUTIONS TO THE 3D STOCHASTIC NAVIER–STOKES EQUATIONS

DOMINIC BREIT AND ALAN DODGSON

ABSTRACT. We consider the 3D stochastic Navier–Stokes equation on the torus. Our main result concerns the temporal and spatio-temporal discretisation of a local strong pathwise solution. We prove optimal convergence rates for the energy error with respect to convergence in probability, that is convergence of order 1 in space and of order (up to) 1/2 in time. The result holds up to the possible blow-up of the (time-discrete) solution. Our approach is based on discrete stopping times for the (time-discrete) solution.

1. INTRODUCTION

We are concerned with the numerical approximation of the 3D stochastic Navier–Stokes equations which read as

\[
\begin{align*}
\frac{du}{dt} &= \mu \Delta u dt - (\nabla u) u dt - \nabla p dt + \Phi(u) dW \\
\text{div } u &= 0 \\
u(0) &= u_0
\end{align*}
\tag{1.1}
\]

\(P\)-a.s. in \(Q_T := (0, T) \times \Omega\), where \(T > 0\), \(\mu > 0\) is the viscosity and \(u_0\) is a given initial datum. The momentum equation is driven by a cylindrical Wiener process \(W\) and the diffusion coefficient \(\Phi(u)\) takes values in the space of Hilbert-Schmidt operators; see Section 2.1 for details.

The analysis of (1.1) has a long history starting with the paper [2], where a semi-deterministic approach is applied. A further milestone is the existence of martingale solutions to (1.1) shown in [16]. These solutions are weak in the analytical sense (derivatives exist only in the sense of distributions and singularities may occur) and weak in the probabilistic sense (the probability space is not a priori given but is an integral part of the solution). By now most results from the deterministic case have found their stochastic counterpart, an overview is given in [15] and [25]. Since the well-posedness of (1.1) (and its deterministic version) is a big open problem, the existence of weak solutions is the best one can hope unless one is satisfied with a local-in-time result. There exists various results concerning local strong pathwise solutions to (1.1), cf. [1, 12, 19, 23, 24]. These solutions are defined on a given stochastic basis and are regular with respect to the spatial variable but only exist up to a stopping time (a precise formulation is given in Definition 2.1). The only information about the latter we have is that it is \(P\)-a.s. strictly positive. It is yet unclear if the presence of noise changes the well-posedness for (1.1). On the one hand, there are results based on the method of convex integration showing that stochastic perturbations do not render the ill-posedness of problems in fluid mechanics, cf. [8, 21, 22]. On the other hand, it was recently proved in [17] that a carefully chosen transport noise in (1.1) can delay the blow-up of the vorticity.
There is less known about the numerical approximation of (1.1), though there was recently some progress on the 2D case. In particular, it is shown in [7] and [13] that for any \( \xi > 0 \),
\[
(1.2) \quad \mathbb{P} \left[ \max_{1 \leq m \leq M} \| u(t_m) - u_{h\_m} \|_{L^2}^2 + \sum_{m=1}^M \tau \| \nabla u(t_m) - \nabla u_{h\_m} \|_{L^2}^2 > \xi \left( h^{2\beta} + \tau^{2\alpha} \right) \right] \to 0
\]
as \( h, \tau \to 0 \) (where \( \alpha < \frac{1}{4} \) and \( \beta < 1 \) are arbitrary); see also [3, 4] for related results. Here \( u \) is the solution to (1.1) and \( u_{h\_m} \) the approximation of \( u(t_m) \) with discretisation parameters \( \tau = T/M \) (in time) and \( h \) (in space). The relation (1.2) tells us that the convergence with respect to convergence in probability is of order (almost) \( 1/2 \) in time and \( 1 \) in space. A similar convergence result for the pathwise error is not expect due to non-Lipschitz nonlinearity in (1.1). A result such as (1.2) heavily relies on the spatial regularity of the solution and can consequently not be expected for the 3D problem we are interested here. The only reachable outcome is the approximation of a martingale solution, which converges in law to the solution up to taking a subsequence. A corresponding result has been proved in [11].

In this paper we take a different perspective and study the approximation of local strong pathwise solutions. We prove a counterpart of (1.2) which holds locally in time, that is, up to a discrete stopping time which replaces \( M \) in maximum and sum. We obtain a result for the temporal discretisation in Theorem 4.10 as well as for the error between time- and space-time discretisation; see Theorem 5.3. Both combined give the convergence rate for this spatio-temporal discretisation; see Theorem 5.1. The analysis of the temporal error in Section 4.2 is reminiscent of the estimates for the space-time error for the 2D Dirichlet problem from [9]. They rely on a discrete version of the stopping time for the continuous solution and estimates for the latter. The analysis of the error between time- and space-time discretisation in Section 5 is more delicate and has not been performed in [9]. Building up on ideas from [10] we introduce a discrete stopping time for the time-discrete solution which announces the blow-up, similar to the stopping time for the continuous solution. In Lemma 4.4, we prove that for \( \tau \to 0 \) we can perform a given number of time steps with a high probability before the blow-up. These replaces the strict positivity of the stopping time in the continuous set-up and justifies the subsequent analysis.

We believe that our approach will be applicable to a wide range of stochastic PDEs which are well-posed locally in time, in particular stochastic Euler equations and stochastic compressible Navier–Stokes equations. Surprisingly, results for the numerical approximation of local solutions to stochastic PDEs do not seem to exist so far.

## 2. Mathematical framework

### 2.1. Probability setup.

Let \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \) be a stochastic basis with a complete, right-continuous filtration. The process \( W \) is a cylindrical \( \mathcal{U} \)-valued Wiener process, that is, \( W(t) = \sum_{j \geq 1} \beta_j(t) e_j \) with \( (\beta_j)_{j \geq 1} \) being mutually independent real-valued standard Wiener processes relative to \( (\mathcal{F}_t)_{t \geq 0} \), and \( (e_j)_{j \geq 1} \) a complete orthonormal system in a separable Hilbert space \( \mathcal{U} \). Let us now give the precise definition of the diffusion coefficient \( \Phi(u) \) taking values in the set of Hilbert-Schmidt operators \( L_2(\mathcal{U}; \mathbb{H}) \), where \( \mathbb{H} \) can take the role of various Hilbert spaces. We assume that \( \Phi(u) \in L_2(\mathcal{U}; L^2(\Omega)) \) for \( u \in L^2(\Omega) \), and \( \Phi(u) \in L_2(\mathcal{U}; L^2(\Omega)) \) for \( u \in W^{1,2}(\Omega) \), together with
\[
\begin{align*}
(2.1) \quad & \| \Phi(u) - \Phi(v) \|_{L_2(\mathcal{U}; L^2)} \leq c \| u - v \|_{L^2} \quad \forall u, v \in L^2(\Omega), \\
(2.2) \quad & \| \Phi(u) \|_{L_2(\mathcal{U}; W^{1,2})} \leq c (1 + \| u \|_{W^{1,2}}) \quad \forall u \in W^{1,2}(\Omega), \\
(2.3) \quad & \| D\Phi(u) \|_{L_2(\mathcal{U}; L^2(\Omega))] \) \leq c \quad \forall u \in L^2(\Omega).
\end{align*}
\]
If we are interested in higher regularity some further assumptions are in place and we require additionally that $\Phi(u) \in L^2(\Omega; W^{2,2}(\mathbb{T}^3))$ for $u \in W^{2,2}(\mathbb{T}^3)$, together with
\begin{equation}
\| \Phi(u) \|_{L^2(\Omega; W^{2,2})} \leq c(1 + \| u \|^2_{W^{1,4}} + \| u \|^2_{W^{2,2}}) \quad \forall u \in W^{2,2}(\mathbb{T}^3),
\end{equation}
\begin{equation}
\| D^2 \Phi(u) \|_{L^2(\Omega; L(L_2(\mathbb{T})) \times L^2(\mathbb{T})^3))} \leq c \quad \forall u \in L^2(\mathbb{T}^3).
\end{equation}

Assumption (2.1) allows us to define stochastic integrals. Given an $(\mathcal{F}_t)$-adapted process $u \in L^2(\Omega; C([0,T]; L^2(\mathbb{T}^3)))$, the stochastic integral
\[ t \mapsto \int_0^t \Phi(u) \, dW \]
is a well-defined process taking values in $L^2(\mathbb{T}^3)$ (see [14] for a detailed construction). Moreover, we can multiply by test functions to obtain
\[ \int_0^t \Phi(u) \, dW, \varphi \right\}_{L^2} = \sum_{j=1}^n \int_0^t \langle \Phi(u) e_j, \varphi \rangle_{L^2} \, d\beta_j \quad \forall \varphi \in L^2(\mathbb{T}^3). \]

Similarly, we can define stochastic integrals with values in $W^{1,2}(\mathbb{T}^3)$ and $W^{2,2}(\mathbb{T}^3)$ respectively if $u$ belongs to the corresponding class.

### 2.2. The concept of solutions.

We give the definition of a strong pathwise solution to (1.1) which exists up to a stopping time $t$. The velocity field belongs $\mathbb{P}$-a.s. to $C([0,t]; W^{2,2}(\mathbb{T}^3))$.

**Definition 2.1.** Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a given stochastic basis with a complete right-continuous filtration and an $(\mathcal{F}_t)$-cylindrical Wiener process $W$. Let $u_0$ be an $\mathcal{F}_0$-measurable random variable with values in $W^{2,2}_{\text{div}}(\mathbb{T}^3)$. The tuple $(u, t)$ is called a local strong pathwise solution to (1.1) with the initial condition $u_0$ provided
(a) $t$ is a $\mathbb{P}$-a.s. strictly positive $(\mathcal{F}_t)$-stopping time;
(b) the velocity field $u$ is $(\mathcal{F}_t)$-adapted and
\[ u(\cdot \wedge t) \in C([0,T]; W^{2,2}_{\text{div}}(\mathbb{T}^3)) \cap L^2(0,T; W^{3,2}_{\text{div}}(\mathbb{T}^3)) \quad \mathbb{P}\text{-a.s.,} \]
(c) the momentum equation
\begin{align}
&\int_{\mathbb{T}^3} u(t \wedge t) \cdot \varphi \, dx - \int_{\mathbb{T}^3} u_0 \cdot \varphi \, dx \\
&= -\int_0^{t \wedge t} \int_{\mathbb{T}^3} \nabla u \cdot \varphi \, dx \, ds + \mu \int_0^{t \wedge t} \int_{\mathbb{T}^3} \Delta u \cdot \varphi \, dx \, ds + \int_0^{t \wedge t} \int_{\mathbb{T}^3} \Phi(u) \, dW \cdot \varphi \, dx \\
&\quad \text{holds $\mathbb{P}$-a.s. for all $\varphi \in C_0^\infty(\mathbb{T}^3)$ and all $t \geq 0$.}
\end{align}

We finally define a maximal strong pathwise solution.

**Definition 2.2 (Maximal strong pathwise solution).** Fix a stochastic basis with a cylindrical Wiener process and an initial condition as in Definition 2.1. A triplet
\[(u, (t_R)_{R \in \mathbb{N}}, t)\]
is a maximal strong pathwise solution to system (1.1) provided
(a) $t$ is a $\mathbb{P}$-a.s. strictly positive $(\mathcal{F}_t)$-stopping time;
(b) $(t_R)_{R \in \mathbb{N}}$ is an increasing sequence of $(\mathcal{F}_t)$-stopping times such that $t_R < t$ on the set $[t < \infty]$, $\lim_{R \to \infty} t_R = t$ $\mathbb{P}$-a.s., and
\begin{equation}
t_R := \inf \{ t \in [0, \infty) : \| u(t) \|_{W^{2,2}} \geq R \} \quad \text{on} \quad [t < \infty],
\end{equation}
with the convention that $t_\infty = \infty$ if the set above is empty;
(c) each triplet $(u, t_R)$, $R \in \mathbb{N}$, is a local strong pathwise solution in the sense of Definition 2.1.
The following result can be proved along the lines of [23], where the stochastic Navier–Stokes equations on the whole space \( \mathbb{R}^3 \) are considered with fractional differentiability \( \sigma \in (3/2, 2) \). As mentioned on [23, page 2] the case of differentiability \( \sigma = 2 \) we are interested in is even easier.

**Theorem 2.3.** Suppose that (2.1)–(2.5) hold, and that \( \mathbf{u}_0 \in L^2(\Omega, W^{1,2}_\text{div}(\mathbb{T}^3)) \). Then there is a unique maximal global strong pathwise solution to (1.1) in the sense of Definition 2.2.

### 2.3. Finite elements

We work with a standard finite element set-up for incompressible fluid mechanics, see e.g. [18]. We denote by \( \mathcal{T}_h \) a quasi-uniform subdivision of \( \mathbb{T}^3 \) into simplices of maximal diameter \( h > 0 \). For \( K \subset \mathbb{R}^3 \) and \( \ell \in \mathbb{N}_0 \) we denote by \( \mathcal{P}_\ell(K) \) the polynomials on \( K \) of degree less than or equal to \( \ell \). Let us characterize the finite element spaces \( V^h(\mathbb{T}^3) \) and \( P^h(\mathbb{T}^3) \) as

\[
V^h(\mathbb{T}^3) := \{ \mathbf{v}_h \in W^{1,2}(\mathbb{T}^3) : \mathbf{v}_h|_K \in \mathcal{P}_i(K) \ \forall K \in \mathcal{T}_h \},
\]

\[
P^h(\mathbb{T}^3) := \{ \pi_h \in L^2(\mathbb{T}^3) : \pi_h|_K \in \mathcal{P}_j(K) \ \forall K \in \mathcal{T}_h \}.
\]

We will assume that \( i, j \in \mathbb{N} \) to get (2.9) below. In order to guarantee stability of our approximations we relate \( V^h(\mathbb{T}^3) \) and \( P^h(\mathbb{T}^3) \) by the discrete inf-sup condition, that is we assume that

\[
\sup_{\mathbf{v}_h \in V^h(\mathbb{T}^3)} \frac{\int_{\mathbb{T}^3} \text{div} \mathbf{v}_h \pi_h \, dx}{\| \text{div} \mathbf{v}_h \|_{L^2}} \geq C \| \pi_h \|_{L^2} \quad \forall \pi_h \in P^h(\mathbb{T}^3),
\]

where \( C > 0 \) does not depend on \( h \). This gives a relation between \( i \) and \( j \) (for instance the choice \((i,j) = (1,0)\) is excluded whereas \((i,j) = (2,0)\) is allowed). Finally, we define the space of discretely solenoidal finite element functions by

\[
V_{\text{div}}^h(\mathbb{T}^3) := \left\{ \mathbf{v}_h \in V^h(\mathbb{T}^3) : \int_{\mathbb{T}^3} \text{div} \mathbf{v}_h \, dx = 0 \ \forall \pi_h \in P^h(\mathbb{T}^3) \right\}.
\]

Let \( \Pi_h : L^2(\mathbb{T}^3) \to V_{\text{div}}^h(\mathbb{T}^3) \) be the \( L^2(\mathbb{T}^3) \)-orthogonal projection onto \( V_{\text{div}}^h(\mathbb{T}^3) \). The following results concerning the approximability of \( \Pi_h \) are well-known (see, for instance [20]). There is \( c > 0 \) independent of \( h \) such that we have

\[
\int_{\mathbb{T}^3} \left| \mathbf{v} - \frac{\Pi_h \mathbf{v}}{h} \right|^2 \, dx + \int_{\mathbb{T}^3} |\nabla \mathbf{v} - \nabla \Pi_h \mathbf{v}|^2 \, dx \leq c \int_{\mathbb{T}^3} |\nabla \mathbf{v}|^2 \, dx
\]

for all \( \mathbf{v} \in W^{1,2}_{\text{div}}(\mathbb{T}^3) \), and

\[
\int_{\mathbb{T}^3} \left| \frac{\mathbf{v} - \Pi_h \mathbf{v}}{h} \right|^2 \, dx + \int_{\mathbb{T}^3} |\nabla \mathbf{v} - \nabla \Pi_h \mathbf{v}|^2 \, dx \leq ch^2 \int_{\mathbb{T}^3} |\nabla^2 \mathbf{v}|^2 \, dx
\]

for all \( \mathbf{v} \in W^{2,2}_{\text{div}}(\mathbb{T}^3) \). Similarly, if \( \Pi^j_h : L^2(\mathbb{T}^3) \to P^h(\mathbb{T}^3) \) denotes the \( L^2(\mathbb{T}^3) \)-orthogonal projection onto \( P^h(\mathbb{T}^3) \), we have

\[
\int_{\mathbb{T}^3} \left| \frac{p - \Pi^j_h p}{h} \right|^2 \, dx \leq c \int_{\mathbb{T}^3} |\nabla p|^2 \, dx
\]

for all \( p \in W^{1,2}(\mathbb{T}^3) \), and

\[
\int_{\mathbb{T}^3} \left| \frac{p - \Pi^j_h p}{h} \right|^2 \, dx \leq ch^2 \int_{\mathbb{T}^3} |\nabla^2 p|^2 \, dx
\]

for all \( p \in W^{2,2}(\mathbb{T}^3) \). Note that (2.11) requires the assumption \( j \geq 1 \) in the definition of \( P^h(\mathbb{T}^3) \), whereas (2.10) also holds for \( j = 0 \).
3. Regularity of solutions

3.1. Estimates for the continuous solution. In this section we derive the crucial estimates for the continuous solution, which hold up to the stopping time \( t_R \) for some \( R \gg 1 \).

**Lemma 3.1.** Let \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \) be a given stochastic basis with a complete right-continuous filtration and an \((\mathcal{F}_t)\)-cylindrical Wiener process \( W \). Suppose that \( u_0 \) is an \( \mathcal{F}_0 \)-measurable random variable with values in \( W_{0, \text{div}}^2(\mathbb{T}^3) \). Let \((u, (t_R))_{R \in \mathbb{N}, t}\) be the maximal strong pathwise solution to (1.1), cf. Definition 2.2.

(a) Assume that \( u_0 \in L^r(\Omega, L^2_{\text{div}}(\mathbb{T}^3)) \) for some \( r \geq 2 \) and that \( \Phi \) satisfies (2.1). Then we have

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \int_{\mathbb{T}^3} |u(t \wedge t_R)|^2 \, dx + \int_0^{T \wedge t_R} \int_{\mathbb{T}^3} |\nabla u|^2 \, dx \, dt \right] \leq c \mathbb{E} \left[ 1 + \|u_0\|_{L^2}^2 \right].
\]

(b) Assume that \( u_0 \in L^r(\Omega, W^{1,2}_{0, \text{div}}(\mathbb{T}^3)) \) for some \( r \geq 2 \) and that \( \Phi \) satisfies (2.1)–(2.3). Then we have

\[
\mathbb{E} \left[ \left( \sup_{0 \leq t \leq T} \int_{\mathbb{T}^3} |\nabla \varphi(t \wedge t_R)|^2 \, dx + \int_0^{T \wedge t_R} \int_{\mathbb{T}^3} |\nabla^2 u|^2 \, dx \, dt \right)^{\frac{r}{2}} \right] \leq c R^3 \mathbb{E} \left[ 1 + \|u_0\|_{W^{1,2}}^2 \right].
\]

(c) Assume that \( u_0 \in L^r(\Omega, W^{2,2}(\mathbb{T}^3)) \) for some \( r \geq 2 \) and that assumptions (2.1)–(2.5) hold. Then we have

\[
\mathbb{E} \left[ \left( \sup_{0 \leq t \leq T} \int_{\mathbb{T}^3} |\nabla^2 u(t \wedge t_R)|^2 \, dx + \int_0^{T \wedge t_R} \int_{\mathbb{T}^3} |\nabla^3 u|^2 \, dx \, dt \right)^{\frac{r}{2}} \right] \leq c R^3 \mathbb{E} \left[ 1 + \|u_0\|_{W^{2,2}}^2 \right].
\]

Here \( c = c(r, T) \) is independent of \( R \).

**Proof.** Let us suppose that \( u_0 \in L^\infty(\Omega, W^{2,2}(\mathbb{T}^3)) \). This assumption can be removed eventually by truncating \( u_0 \). Similar to [24] we consider the solution to a truncated problem. For \( R > 1 \) and \( \zeta \in C_0^\infty([0, 2]) \) with \( 0 \leq \zeta \leq 1 \) and \( \zeta = 1 \) in \([0, 1] \) we set \( \zeta_R := \zeta(R^{-1} \cdot) \). Let \( u^R \) be an \((\mathcal{F}_t)\)-adapted process with

\[
u^R \in C([0, T]; W_{0, \text{div}}^{2,2}(\mathbb{T}^3)) \cap L^2(0, T; W^{3,2}_{\text{div}}(\mathbb{T}^3)) \quad \mathbb{P}\text{-a.s.}
\]

such that

\[
\int_{\mathbb{T}^3} u^R(t) \cdot \varphi \, dx = \int_{\mathbb{T}^3} u_0 \cdot \varphi \, dx + \int_0^t \int_{\mathbb{T}^3} \zeta_R(\|u^R\|_{W^{2,2}}) u^R \otimes u^R : \nabla \varphi \, dx \, ds
\]

\[
- \mu \int_0^t \int_{\mathbb{T}^3} \nabla u^R : \nabla \varphi \, dx \, ds + \int_0^t \zeta_R(\|u^R\|_{W^{2,2}}) \int_{\mathbb{T}^3} \Phi(u^R) \cdot \varphi \, dx \, dW,
\]

holds \( \mathbb{P}\)-a.s. for all \( \varphi \in W^{1,2}_{0, \text{div}}(\mathbb{T}^3) \) and all \( t \in [0, T] \). It can be shown by means of a Glarking approximation (and higher order energy estimates which hold thanks to the periodic boundary conditions) that a unique strong pathwise solution to (3.4) exists. We finally note that \( u^R(\cdot \wedge t_R) = u(\cdot \wedge t_R) \) such that is sufficient to prove the claimed estimates for \( u^R \) instead of \( u \).

Estimate (3.1) is the standard a priori estimate which can be proved by applying Itô’s formula to the functional \( t \mapsto \|u^R\|_{L^2}^2 \) and using the cancellation of the convective term.

As far as (3.2) is concerned we can apply Itô’s formula to \( t \mapsto \|\nabla u^R\|_{L^2}^2 \) and use (3.4), which yields

\[
\int_{\mathbb{T}^3} |\nabla u^R|^2 \, dx = \int_{\mathbb{T}^3} |\nabla u_0|^2 \, dx + 2 \int_0^t \int_{\mathbb{T}^3} \zeta_R(\|u^R\|_{W^{2,2}})(u^R \cdot \nabla) u^R \cdot \Delta u^R \, dx \, ds
\]
and apply Itô’s formula to the mapping
\[\int T_3 |\Delta u^R|^2 \, dx + 2 \int_0^t \int T_3 \zeta_R(\|u^R\|_{W^{2,2}}) \Phi(u^R) \cdot \Delta u^R \, dx \, dW \]
\[+ \sum_{k=1}^\infty \int_0^t \left( \zeta_R(\|u^R\|_{W^{2,2}}) \int T_3 \nabla \{\Phi(u^R) e_k\} \, dx \right)^2 \, ds \]
\[=: I(t) + \cdots + V(t).\]

We only show how to estimate the convective term II, which significantly differs from the 2D case. We refer to [9, Lemma 3.1], where it is shown how to control the remaining integrals independently of \(R\). We have by definition of \(\zeta_R\) and the embedding \(W^{2,2}(\mathbb{T}^3) \hookrightarrow L^\infty(\mathbb{T}^3)\)

\[\Pi(t) \leq 2 \int_0^t \zeta_R(\|u^R\|_{W^{2,2}}) \|u^R\|_{L^\infty} \|\Delta u^R\|_{L^2} \, ds \leq cR^3.\]

For (c) we argue similarly to (b) and apply Itô’s formula to the mapping \(t \mapsto \int T_3 |\Delta u^R(t)|^2 \, dx\) which shows

\[\int T_3 |\Delta u|^2 \, dx = \int T_3 |\Delta u_0|^2 \, dx + 2 \int_0^t \int T_3 \zeta_R(\|u^R\|_{W^{2,2}}) (u^R \cdot \nabla) u^R \cdot \Delta^2 u^R \, dx \, ds \]
\[- 2\mu \int_0^t \int T_3 |\nabla \Delta u|^2 \, dx \, ds + 2 \int_0^t \zeta_R(\|u^R\|_{W^{2,2}}) \int T_3 \Phi(u^R) \cdot \Delta^2 u^R \, dx \, dW \]
\[+ \sum_{k=1}^\infty \int_0^t \left( \zeta_R(\|u^R\|_{W^{2,2}}) \int T_3 \nabla^2 \{\Phi(u^R) e_k\} \, dx \right)^2 \, ds \]
\[=: \Pi(t) + \cdots + X(t).\]

It holds using the embedding \(W^{2,2}(\mathbb{T}^3) \hookrightarrow W^{1,4}(\mathbb{T}^3)\) that

\[\Pi(t) \leq 2 \int_0^t \zeta_R(\|\nabla u^R\|_{L^2}) \|u^R\|_{L^\infty} \|\nabla^2 u^R\|_{L^2} \|\nabla \Delta u^R\|_{L^2} \, ds \]
\[+ 2 \int_0^t \zeta_R(\|\nabla u^R\|_{L^2}) \|\nabla u^R\|_{L^2}^2 \|\nabla \Delta u^R\|_{L^2} \, ds \]
\[\leq cR \int_0^t \|\nabla^2 u^R\|_{L^2} \|\nabla \Delta u^R\|_{L^2} \, ds \]
\[\leq c(\delta) R^2 \int_0^t \|\nabla^2 u^R\|_{L^2}^2 \, ds + \delta \int_0^t \|\nabla \Delta u^R\|_{L^2}^2 \, ds \]

for any \(\delta > 0\). The expectation of the first term can be controlled by the estimates from (b). As for the stochastic terms, we obtain

\[X(t) \leq \sum_{k=1}^\infty \int_0^t \int T_3 \left| \zeta_R(\|u^R\|_{W^{2,2}}) \Delta \Phi(u^R) e_k \right|^2 \, dx \, ds \]
\[\leq c \int_0^t \zeta_R(\|u^R\|_{W^{2,2}}) (1 + \|u^R\|_{W^{1,4}}^4 + \|u^R\|_{W^{2,2}}^2) \, ds \]
\[\leq cR^2 \int_0^t \left( 1 + \|\nabla^2 u^R\|_{L^2}^2 \right) \, ds \]

as well as

\[\mathbb{E} \left[ \sup_{0 \leq t \leq T} |X(t)|^2 \right] \leq c \mathbb{E} \left[ \left( \int_0^T \zeta_R(\|u^R\|_{W^{2,2}}) (1 + \|u^R\|_{W^{1,4}}^4 + \|u^R\|_{W^{2,2}}^2) \|\nabla \Delta u^R\|_{L^2}^2 \, dt \right)^2 \right] \]
\[\leq cR^2.\]

\(^1\)In [9, Lemma 3.1] (c) a completely different approach is used for the corresponding estimate due to problems related to Dirichlet boundary conditions.
The proof is complete. □

3.2. Stochastic pressure decomposition. Since we will be working with discretely divergence-free function spaces in the finite-element analysis, it is inevitable to introduce the pressure function. For $\varphi \in C^\infty(\mathbb{T}^3)$ we can insert

$$\mathcal{P}\varphi := \varphi - \nabla \Delta^{-1} \text{div} \varphi$$

in (2.6). Here $\Delta^{-1}$ is the solution operator to the Laplace equation with respect to periodic boundary conditions on $\mathbb{T}^3$. Note that $\Delta^{-1}$ satisfies

$$\begin{align*}
(3.5) & \quad \Delta^{-1} : W^{-2,p}(\mathbb{T}^3) \to L^p(\mathbb{T}^3), \\
(3.6) & \quad \Delta^{-1} : W^{-1,p}(\mathbb{T}^3) \to W^{1,p}(\mathbb{T}^3), \\
(3.7) & \quad \Delta^{-1} : W^{r,p}(\mathbb{T}^3) \to W^{r+2,p}(\mathbb{T}^3),
\end{align*}$$

for all $p \in (1, \infty)$ and all $r \in \mathbb{N}$, where $W^{-k,p}(\mathbb{T}^3) = (W^{k,p}(\mathbb{T}^3))'$ for $k \in \mathbb{N}$. We obtain

$$\begin{align*}
\int_{\mathbb{T}^3} u(t \wedge t_R) \cdot \varphi \, dx - \int_0^{T \wedge t_R} \int_{\mathbb{T}^3} \mu \Delta u \cdot \varphi \, dx \, dt + \int_0^{T \wedge t_R} \int_{\mathbb{T}^3} (\nabla u) u \cdot \varphi \, dx \, dt &= \int_{\mathbb{T}^3} u(0) \cdot \varphi \, dx + \int_0^{t \wedge t_R} \int_{\mathbb{T}^3} \pi_{\text{det}} \text{div} \varphi \, dx \, dt \\
&\quad + \int_{\mathbb{T}^3} \int_0^{t \wedge t_R} \Phi(u) \, dW \cdot \varphi \, dx + \int_{\mathbb{T}^3} \int_0^{t \wedge t_R} \Phi^\pi \, dW \cdot \varphi \, dx,
\end{align*}$$

where

$$\begin{align*}
\pi_{\text{det}} &= -\Delta^{-1} \text{div} (\nabla u), \\
\Phi^\pi &= -\nabla \Delta^{-1} \text{div} \Phi(u).
\end{align*}$$

This corresponds to the stochastic pressure decomposition introduced in [5] (see also [6, Chap. 3] for a slightly different presentation). In the following we will analyse how the regularity of $u$ transfers to $\pi_{\text{det}}$ and $\Phi^\pi$. Arguing as in [7, Corollary 2.5] and using (3.5)–(3.7) we obtain

$$E\left(\left(\sup_{t \in [0,T \wedge t_R]} \|\pi_{\text{det}}\|^2_{L^2_{W^{r,2}}} \right)^{\frac{p}{2}}\right) \leq c E\left(\left(\sup_{0 \leq t \leq T} \|u(t \wedge t_R)\|^2_{W^{r,2}} + \int_0^{T \wedge t_R} \|u\|^2_{L^2_{W^{r,2}}(t \wedge t_R)} \, dt\right)^{\frac{p}{2}}\right)$$

as well as

$$E\left(\left(\sup_{t \in [0,T \wedge t_R]} \|\Phi^\pi\|^2_{L^2_{W^{r,2}}(U_{W^{r,2}})}\right)^{\frac{p}{2}}\right) \leq c E\left[1 + \sup_{0 \leq t \leq T} \|u(t \wedge t_R)\|^2_{W^{r,2}}\right]$$

for $\ell \in \{0, 1, 2\}$ (note that $W^{0,p}(\mathbb{T}^3) = L^p(\mathbb{T}^3)$ for $p \in [1, \infty]$). Consequently, Lemma 3.1 implies the following.

Lemma 3.2. (a) Under the assumptions of Lemma 3.1 (a) we have

$$E\left(\left(\int_0^{T \wedge t_R} \|\pi_{\text{det}}\|^2_{L^2_{W^{r,2}}} \, dt + \sup_{0 \leq t \leq T} \|\Phi^\pi(t \wedge t_R)\|^2_{L^2_{W^{r,2}}(U_{W^{r,2}})}\right)^{\frac{p}{2}}\right) \leq c E\left[1 + \|u_0\|_{L^2_{W^{r,2}}}\right].$$

(b) Under the assumptions of Lemma 3.1 (b) we have

$$E\left(\left(\int_0^{T \wedge t_R} \|\pi_{\text{det}}\|^2_{W^{r,2}} \, dt + \sup_{0 \leq t \leq T} \|\Phi^\pi(t \wedge t_R)\|^2_{L^2_{W^{r,2}}(U_{W^{r,2}})}\right)^{\frac{p}{2}}\right) \leq c R^p E\left[1 + \|u_0\|_{W^{r,2}}\right].$$

(c) Under the assumptions of Lemma 3.1 (c) we have

$$E\left(\left(\int_0^{T \wedge t_R} \|\pi_{\text{det}}\|^2_{W^{r,2}} \, dt + \sup_{0 \leq t \leq T} \|\Phi^\pi(t \wedge t_R)\|^2_{L^2_{W^{r,2}}(U_{W^{r,2}})}\right)^{\frac{p}{2}}\right) \leq c R^p E\left[1 + \|u_0\|_{W^{r,2}}\right].$$

Here $c = c(r, T)$ is independent of $R$. 

Note: The proof of Lemma 3.2 involves advanced mathematical concepts and techniques that are typically beyond the scope of introductory courses in fluid dynamics or stochastic processes. It is important to understand the definitions and properties of the spaces and operators used, as well as the implications of the stochastic pressure decomposition.
Corollary 3.3. \( (a) \) Let the assumptions of Lemma 3.1 \( (b) \) be satisfied for some \( r > 2 \). Then we have

\[
E \left[ \left( \|u(t_R \wedge \cdot)\|_{C^0([0,T];L^2_t)} \right)^2 \right] \leq c R^{2r} E \left[ 1 + \|u_0\|_{W^{2,2}} \right]
\]

for all \( \alpha < \frac{1}{r} \).

\( (b) \) Let the assumptions of Lemma 3.1 \( (c) \) be satisfied for some \( r > 2 \). Then we have

\[
E \left[ \left( \|u(t_R \wedge \cdot)\|_{C^0([0,T];W^{1,2}_t)} \right)^2 \right] \leq c R^{2r} E \left[ 1 + \|u_0\|_{W^{2,2}} \right]
\]

for all \( \alpha < \frac{1}{r} \).

Here \( c = c(r,T,\alpha) \) is independent of \( R \).

4. Time discretisation

We now consider a temporal approximation of \((1.1)\) on an equidistant partition of \([0,T]\) with mesh size \( \tau = T/M \), and set \( t_m = m\tau \). Let \( u_0 \) be an \( \mathcal{F}_0 \)-measurable random variable with values in \( L^2_{\text{div}}(\mathbb{T}^3) \). For \( 1 \leq m \leq M \), we aim at constructing iteratively a sequence of \( \mathcal{F}_m \)-measurable random variables \( u_m \) with values in \( W^{1,2}_{\text{div}}(\mathbb{T}^3) \) such that for every \( \varphi \in W^{1,2}_{\text{div}}(\mathbb{T}^3) \) it holds true \( \mathbb{P}\)-a.s.

\[
\int_{\mathbb{T}^3} u_m : \varphi \, dx - \tau \int_{\mathbb{T}^3} (u_m \otimes u_{m-1}) : \nabla \varphi \, dx
= -\mu \tau \int_{\mathbb{T}^3} \nabla u_m : \nabla \varphi \, dx + \int_{\mathbb{T}^3} u_{m-1} : \varphi \, dx + \int_{\mathbb{T}^3} \Phi(u_{m-1}) \Delta_m W : \varphi \, dx,
\]

where \( \Delta_m W = W(t_m) - W(t_{m-1}) \). For given \( u_{m-1} \) and \( \Delta_m W \), verifying the existence of a unique \( u_m \) solving \((4.1)\) is straightforward since the problem is linear in \( u_m \).

Lemma 4.1. Assume that \( u_0 \in L^{2q}(\Omega, L^2_{\text{div}}(\mathbb{T}^3;\mathbb{R}^2)) \) for some \( q \in \mathbb{N} \) and that \((2.1)\) holds. Then the iterates \((u_m)_{m=1}^M\) given by \((4.1)\) satisfy the following estimate uniformly in \( M \):

\[
E \left[ \max_{1 \leq m \leq M} \|u_m\|_{L^2_t}^{2q} + \tau \sum_{m=1}^M \|u_m\|_{L^2_t}^{2q-2} \|\nabla u_m\|_{L^2_t}^2 \right] \leq c,
\]

where \( c = c(q,T,\Phi,u_0) > 0 \).

Proof. The proof of \((4.2)\) is identical to \([11, \text{Lemma 3.1}]\). Note that, different from \([11]\), we consider a semi-implicit algorithm, which does not impact the proof since the convective term still cancels when testing with \( u_m \). \( \square \)

4.1. Estimates for the time-discrete solution. In order to obtain “local-in-time” estimates we consider a truncated variant of \((4.1)\), similarly to \((3.4)\). Let \( u^R \in W^{1,2}_{\text{div}}(\mathbb{T}^3) \) be such that for every \( \varphi \in W^{1,2}_{\text{div}}(\mathbb{T}^3) \), it holds true \( \mathbb{P}\)-a.s. that

\[
\int_{\mathbb{T}^3} u^R \cdot \varphi \, dx - \tau \int_{\mathbb{T}^3} \zeta_R(\|u^R_{m-1}\|_{W^{2,2}}) u^R \otimes u^R_{m-1} : \nabla \varphi \, dx
= -\mu \tau \int_{\mathbb{T}^3} \nabla u^R_m : \nabla \varphi \, dx + \int_{\mathbb{T}^3} u^R_{m-1} : \varphi \, dx
+ \int_{\mathbb{T}^3} \zeta_R(\|u^R_{m-1}\|_{W^{2,2}}) \Phi(u^R_{m-1}) \Delta_m W : \varphi \, dx
\]

where \( \zeta_R \) is a cut-off function.

As in \([7, \text{Corollary 2.6}]\), we can combine Lemmas 3.1 and 3.2 to conclude the following result concerning the time regularity of \( u \).

Corollary 3.3. \( (a) \) Let the assumptions of Lemma 3.1 \( (b) \) be satisfied for some \( r > 2 \). Then we have

\[
\int_{\mathbb{T}^3} u^R \cdot \varphi \, dx - \tau \int_{\mathbb{T}^3} \zeta_R(\|u^R_{m-1}\|_{W^{2,2}}) u^R \otimes u^R_{m-1} : \nabla \varphi \, dx
= -\mu \tau \int_{\mathbb{T}^3} \nabla u^R_m : \nabla \varphi \, dx + \int_{\mathbb{T}^3} u^R_{m-1} : \varphi \, dx
+ \int_{\mathbb{T}^3} \zeta_R(\|u^R_{m-1}\|_{W^{2,2}}) \Phi(u^R_{m-1}) \Delta_m W : \varphi \, dx
\]

where \( \zeta_R \) is independent of \( R \).

Here \( c = c(r,T,\alpha) \) is independent of \( R \).
with \( u_0^R = u_0 \). Arguing as for Lemma 4.1 and noticing that the convective term in (4.3) still cancels when testing with \( u_m^R \) once can show for \( q \in \mathbb{N} \)

\[
\mathbb{E}\left[ \max_{1 \leq m \leq M} \| u_m^R \|_{L_x^2}^{2q} + \tau \sum_{m=1}^{M} \| u_m^R \|_{L_x^2}^{2q-2} \| \nabla u_m^R \|_{L_x^2}^2 \right] \leq c
\]

where \( c = c(q,T,\Phi,u_0) > 0 \) is independent of \( R \). We are now going to prove \( R \)-dependent estimates for \( (u_m^R)_{m=1}^{M} \) which we transfer eventually to \( (u_m)_{m=1}^{M} \) by introducing a suitable discrete stopping time.

**Lemma 4.2.** Assume that \( u_0 \in L^{2q}(\Omega, W_{\text{div}}^{1,2}(\mathbb{T}^3)) \) for some \( q \in \mathbb{N} \) and suppose that (2.1)–(2.3) hold. Then the iterates \( (u_m^R)_{m=1}^{M} \) given by (4.3) satisfy the following estimates uniformly in \( M \):

\[
\mathbb{E}\left[ \max_{1 \leq m \leq M} \| u_m^R \|_{W_x^{2q}}^{2q} + \tau \sum_{m=1}^{M} \| u_m^R \|_{W_x^{2q-2}}^{2q-2} \| \nabla^2 u_m^R \|_{L_x^2}^2 \right] + \mathbb{E}\left[ \sum_{m=1}^{M} \| u_m^R \|_{W_x^{2q-2}}^{2q-2} \| \nabla(u_m^R - u_{m-1}^R) \|_{L_x^2}^2 \right] \leq ce^{cR^2},
\]

where \( c = c(q,T,\Phi,u_0) > 0 \) is independent of \( R \).

**Proof.** The proof is reminiscent of [10, Lemma 2.8] but differs in the estimates for the convective term and the nonlinear diffusion coefficient. We proceed formally; a rigorous proof can be obtained using a Galerkin approximation. We test (4.3) by \( \Delta u_m \), obtaining

\[
\int_{\mathbb{T}^3} \nabla(u_m^R - u_{m-1}^R) \cdot \nabla u_m^R \, dx + \tau \int_{\mathbb{T}^3} \| \Delta u_m^R \|_{L_x^2}^2 \, dx
\]

\[
= -\tau \int_{\mathbb{T}^3} \zeta_R(\| u_m^R - u_{m-1}^R \|_{W_x^{2q-2}})(\nabla u_m^R)(\nabla u_{m-1}^R) \cdot \Delta u_m^R \, dx
\]

\[
+ \zeta_R(\| u_m^R - u_{m-1}^R \|_{W_x^{2q-2}}) \int_{\mathbb{T}^3} \| \nabla u_{m-1}^R \|_{W_x^{2q-2}} \| \nabla u_m^R \|_{L_x^2} \| \Delta u_m^R \|_{L_x^2} \, dx
\]

For \( \delta > 0 \) we have by the definition of \( \zeta_R \) and the embedding \( W^{2,2}(\mathbb{T}^3) \hookrightarrow L^{\infty}(\mathbb{T}^3) \)

\[
\int_{\mathbb{T}^3} \zeta_R(\| u_m^R - u_{m-1}^R \|_{W_x^{2q-2}})(\nabla u_m^R)(\nabla u_{m-1}^R) \cdot \Delta u_m^R \, dx
\]

\[
\leq \int_{\mathbb{T}^3} \zeta_R(\| u_m^R - u_{m-1}^R \|_{W_x^{2q-2}}) \| u_m^R - u_{m-1}^R \|_{L_x^{\infty}} \| \nabla u_m^R \|_{L_x^2} \| \Delta u_m^R \|_{L_x^2}
\]

\[
\leq cR^2 \| u_m^R - u_{m-1}^R \|_{L_x^2}^2
\]

Summing up and choosing \( \delta \) sufficiently small shows that

\[
\frac{1}{2} \int_{\mathbb{T}^3} \| \nabla u_m^R \|_{L_x^2}^2 \, dx + \frac{1}{2} \sum_{m=1}^{M} \int_{\mathbb{T}^3} \| \nabla(u_m^R - u_{m-1}^R) \|_{L_x^2}^2 \, dx + \frac{\tau}{2} \sum_{m=1}^{M} \int_{\mathbb{T}^3} \| \Delta u_m^R \|_{L_x^2}^2 \, dx
\]

\[
\leq \frac{1}{2} \int_{\mathbb{T}^3} \| \nabla u_{m} \|_{L_x^2}^2 \, dx + cR^2 \sum_{m=1}^{M} \tau \| \nabla u_m^R \|_{L_x^2}^2 + 2\mathcal{M}_m^1 + 2\mathcal{M}_m^2,
\]

where

\[
\mathcal{M}_m^1 = \sum_{n=1}^{m} \zeta_R(\| u_{n-1}^R - u_n^R \|_{W_x^{2q-2}}) \int_{\mathbb{T}^3} \int_{t_{n-1}}^{t_n} \Phi(u_{m-1}^R) \, dW \cdot \Delta u_{n-1}^R \, dx,
\]

\[
\mathcal{M}_m^2 = \sum_{n=1}^{m} \zeta_R(\| u_{n-1}^R - u_n^R \|_{W_x^{2q-2}}) \int_{\mathbb{T}^3} \int_{t_{n-1}}^{t_n} \Phi(u_{m-1}^R) \, dW \cdot \Delta(u_{n-1}^R - u_{n-1}^R) \, dx.
\]
By the discrete Gronwall lemma we have \( \mathbb{P} \)-a.s.
\[
\frac{1}{2} \max_{1 \leq m \leq M} \int_{\mathbb{T}^3} |\nabla u_m^R|^2 \, dx + \frac{1}{2} \sum_{n=1}^M \int_{\mathbb{T}^3} |\nabla (u_n^R - u_{n-1}^R)|^2 \, dx + \frac{1}{2} \sum_{n=1}^M \tau \int_{\mathbb{T}^3} |\Delta u_n^R|^2 \, dx \\
\leq c e^{cR^2} \left( \int_{\mathbb{T}^3} |\nabla u_0|^2 \, dx + \max_{1 \leq m \leq M} |\mathcal{U}_m| + \max_{1 \leq m \leq M} |\mathcal{M}_m| \right).
\]

Since \( u_{m-1} \) is \( \mathcal{F}_{t_{m-1}} \)-measurable we know that \( \mathcal{M}_m^1 \) is an \( (\mathcal{F}_t) \)-martingale. Consequently, by the Burkholder-Davis-Gundy inequality, (2.1) and Young’s inequality, and for \( k > 0 \)
\[
\mathbb{E} \left[ \max_{1 \leq m \leq M} |\mathcal{M}_m^1| \right] \leq c \mathbb{E} \left[ \left( \sum_{n=1}^M \tau \xi_k(||u_{n-1}^R||_{W^{2,2}})^2 \|\Phi(u_{n-1}^R)\|_{L^2(u;L^2)} ||\Delta u_{n-1}^R||_{L^2}^2 \right)^{\frac{2}{2-k}} \right] \\
\leq c \mathbb{E} \left[ \left( \sum_{n=1}^M \tau \xi_k(||u_{n-1}^R||_{W^{2,2}}) (1 + ||u_{n-1}^R||_{L^2}^2) ||\Delta u_{n-1}^R||_{L^2}^2 \right)^{\frac{2}{2-k}} \right] \\
\leq c \mathbb{E} \left[ \sum_{n=1}^M \tau R^2 ||\Delta u_{n-1}^R||_{L^2}^2 \right] \\
\leq c(\delta) R^2 + \delta \mathbb{E} \left[ \sum_{n=1}^{M-1} \tau ||\Delta u_n^R||_{L^2}^2 \right].
\]

Furthermore, we have
\[
\mathbb{E} \left[ \max_{1 \leq m \leq M} |\mathcal{M}_m^R| \right] \leq \delta \mathbb{E} \left[ \sum_{n=1}^M \|\nabla (u_n^R - u_{n-1}^R)\|_{L^2}^2 \right] + c(\delta) \mathbb{E} \left[ \sum_{n=1}^M \|\Phi(u_{n-1}^R)\|_{L^2(u;W^{1,2})} \right] \\
\leq \delta \mathbb{E} \left[ \sum_{n=1}^M \|\nabla (u_n^R - u_{n-1}^R)\|_{L^2}^2 \right] + c(\delta) \mathbb{E} \left[ \tau \sum_{n=1}^M \|\Phi(u_{n-1}^R)\|_{L^2(u;W^{1,2})} \right] \\
\leq \delta \mathbb{E} \left[ \sum_{n=1}^M \|\nabla (u_n^R - u_{n-1}^R)\|_{L^2}^2 \right] + c(\delta) \mathbb{E} \left[ \tau \sum_{n=1}^M (1 + ||u_{n-1}^R||_{W^{2,2}}) \right] \\
\leq \delta \mathbb{E} \left[ \sum_{n=1}^M \|\nabla (u_n^R - u_{n-1}^R)\|_{L^2}^2 \right] + c(\delta)
\]

due to Young’s inequality, Itô-isometry, (2.2) and (4.4). Absorbing the \( \delta \)-terms we conclude for \( q = 1 \). The case \( q \geq 2 \) follows similarly by multiplying with \( ||u_n^R||_{W^{q-2,2}} \) and iterating (see [11, Lemma 3.1] for details).

**Lemma 4.3.** Assume that \( u_0 \in L^2(\Omega, W^{2,2}_{\text{div}}(\mathbb{T}^3)) \) for some \( q \in \mathbb{N} \) and suppose that (2.1)–(2.5) hold. Then the iterates \( (u_m^R)_{m=1}^M \) given by (4.3) satisfy the following estimates uniformly in \( M \):

\[
\mathbb{E} \left[ \max_{1 \leq m \leq M} ||u_m^R||_{W^{2,2}}^2 \right] + \sum_{m=1}^M \tau ||u_m^R||_{W^{2,2}}^2 ||\nabla u_m^R||_{L^2}^2 \right] \\
\leq c cR^2, \tag{4.7}
\]

where \( c = c(q,T,\Phi, u_0) > 0 \) is independent of \( R \).

**Proof.** We argue similarly to the proof of Lemma 4.2 testing this time (4.3) by \( \Delta^2 u_m \) obtaining
\[
\int_{\mathbb{T}^3} \Delta (u_m^R - u_{m-1}^R) : \Delta u_m^R \, dx + \mu \int_{\mathbb{T}^3} |\nabla u_m^R|^2 \, dx
\]
\[-\int_{T^3} \zeta_R(\|u_{m-1}\|_{L^2}) (\nabla u_m)(\nabla u_m) \cdot \Delta^2 u_m \, dx\]

\[\leq cR\|\nabla^2 u_m\|_{L^2}^2 \Delta \nabla u_m\|_{L^2}^2 \leq c(\delta) R^2 \|\nabla^2 u_m\|_{L^2}^2 + \delta \|\Delta \nabla u_m\|_{L^2}^2.
\]

Summing up, choosing \(\delta\) sufficiently small and using continuity of \(\nabla^2 \Delta^{-1}\) on \(L^2(T^3)\) shows

\[
\frac{1}{2} \int_{T^3} |\nabla^2 u_m|^2 \, dx + \frac{1}{2} \sum_{n=1}^m \int_{T^3} |\nabla^2 (u_n - u_{n-1})|^2 \, dx + \frac{1}{2} \sum_{m=1}^M \tau \int_{T^3} |\nabla^3 u_m|^2 \, dx
\]

\[
\leq \frac{1}{2} \int_{T^3} |\nabla^2 u_0|^2 \, dx + cR^2 \sum_{n=1}^m \tau \|\nabla^2 u_n\|_{L^2}^2 + \mathcal{M}_1 + \mathcal{M}_2,
\]

where

\[
\mathcal{M}_1 = -\sum_{n=1}^m \zeta_R(\|u_{n-1}\|_{L^2}) \int_{T^3} \int_{t_n-\tau}^{t_n} \nabla \{\Phi(u_{m-1})\} \, dW \cdot \Delta \nabla u_m \, dx,
\]

\[
\mathcal{M}_2 = \sum_{n=1}^m \zeta_R(\|u_{n-1}\|_{L^2}) \int_{T^3} \int_{t_n-\tau}^{t_n} \Delta \{\Phi(u_{m-1})\} \, dW \cdot \Delta (u_n - u_{n-1}) \, dx.
\]

By the discrete Gronwall lemma we have \(P\)-a.s.

\[
\frac{1}{2} \max_{1 \leq m \leq M} \int_{T^3} |\nabla^2 u_m|^2 \, dx + \frac{1}{2} \sum_{m=1}^M \int_{T^3} |\nabla^2 (u_n - u_{n-1})|^2 \, dx + \frac{1}{2} \sum_{m=1}^M \tau \int_{T^3} |\nabla^3 u_m|^2 \, dx
\]

\[
\leq cR^2 \left( \int_{T^3} |\nabla^2 u_0|^2 \, dx + \max_{1 \leq m \leq M} |\mathcal{M}_1| + \max_{1 \leq m \leq M} |\mathcal{M}_2| \right).
\]

We obtain further by Burkholder-Davis-Gundy inequality, (2.2) and Young's inequality, and for \(\delta > 0\)

\[
\mathbb{E} \left[ \max_{1 \leq m \leq M} |\mathcal{M}_1| \right] \leq c \mathbb{E} \left[ \left( \sum_{n=1}^M \tau \zeta_R(\|u_{n-1}\|_{L^2})^2 \|\Phi(u_{n-1})\|^2_{L^2} \|\Delta \nabla u_{n-1}\|_{L^2}^2 \, dx \right)^\frac{1}{2} \right]
\]

\[
\leq c \mathbb{E} \left[ \left( \sum_{n=1}^M \tau \zeta_R(\|u_{n-1}\|_{L^2}) (1 + \|u_{n-1}\|_{L^2}^2) \|\Delta \nabla u_{n-1}\|_{L^2}^2 \right)^\frac{1}{2} \right]
\]

\[
\leq c \mathbb{E} \left[ \left( \sum_{n=1}^M \tau R^2 \|\Delta \nabla u_{n-1}\|_{L^2}^2 \right)^\frac{1}{2} \right].
\]
we have

\[ \leq c(\delta) R^2 + \delta \mathbb{E} \left[ \sum_{n=1}^{M-1} \tau \| \Delta \nabla u_n^R \|_{L^2}^2 \right]. \]

Furthermore, we have

\[
\mathbb{E} \left[ \max_{1 \leq m \leq M} |\tilde{M}_m^2| \right] \leq \delta \mathbb{E} \left[ \sum_{n=1}^{M} \| \nabla^2 (u_n^R - u_{n-1}^R) \|_{L^2}^2 \right] \\
+ c(\delta) \mathbb{E} \left[ \sum_{n=1}^{M} \zeta_R (\| u_{m-1}^R \|_{W^{2,2}}^2)^2 \right] \mathbb{E} \left[ \int_{t_{n-1}}^{t_n} \Delta \Phi(u_{n-1}^R) \, dW \right]^2 \\
\leq \delta \mathbb{E} \left[ \sum_{n=1}^{M} \| \nabla^2 (u_n^R - u_{n-1}^R) \|_{L^2}^2 \right] \\
+ c(\delta) \mathbb{E} \left[ \tau \sum_{n=1}^{M} \zeta_R (\| u_{m-1}^R \|_{W^{2,2}}^2) \| \Phi(u_{n-1}^R) \|_{L^2(\mathbb{R})}^2 \right] \\
\leq \delta \mathbb{E} \left[ \sum_{n=1}^{M} \| \nabla^2 (u_n^R - u_{n-1}^R) \|_{L^2}^2 \right] \\
+ c(\delta) \mathbb{E} \left[ \tau \max_{1 \leq m \leq M} \| u_m^R \|_{W^{2,2}}^2 \right] + c(\delta) R^2 \mathbb{E} \left[ \sum_{n=1}^{M} \| \nabla^2 u_n^R - u_{n-1}^R \|_{L^2}^2 \right]
\]
due to Young’s inequality, Itô-isometry and (2.4) (together with the embedding $W^{2,2}(\mathbb{T}^3) \hookrightarrow W^{1,4}(\mathbb{T}^3)$). Absorbing the $\delta$-terms and applying Lemma 4.2 yields the claim for $q = 1$, whereas the general case follows again by iteration. \qed

For $R > 0$, we define the (discrete) $(\mathbb{G}_{\ell_m})$-stopping time

\[
\mathbb{G}_R^\ell := \min_{0 \leq m \leq M} \left\{ \tau_m : \max_{0 \leq n \leq m} \| u_n \|_{W^{2,2}} \geq R \right\},
\]

where we set $\mathbb{G}_R^\ell = \ell M$ if the set above is empty. Note that $\mathbb{G}_R^\ell \in \{ \ell M \}_{m=0}^M$, with random index $\ell R \in \mathbb{N}_0 \cap [0, M]$, such that $\mathbb{G}_R^\ell = \ell t_m$. The crucial point is now to show a counterpart of the strict positivity of the stopping time from the continuous solution, cf. Definition 2.1. This is the content of the next lemma, which states that $\mathbb{G}_R^\ell \geq \tau$ with high probability.

**Lemma 4.4.** Suppose that the assumptions from Lemma 4.3 (with $q = 1$) hold and let $R = R(\tau)$ be chosen such that $\tau e^{c R^2} \to 0$ as $\tau \to 0$. Then we have for any $\ell \in \mathbb{N}$

\[
\lim_{\tau \to 0} \mathbb{P} \left( \{ \mathbb{G}_R^\ell \leq \ell t \} \right) = 0.
\]

**Proof.** Arguing as in the proof of Lemma 4.3 we have

\[
\frac{1}{2} \max_{1 \leq m \leq \ell} \int_{\mathbb{T}^3} |\nabla^2 \mathbb{M}_m^R|^2 \, dx + \frac{1}{2} \sum_{n=1}^{\ell} \int_{\mathbb{T}^3} |\nabla^2 (u_n^R - u_{n-1}^R)|^2 \, dx + \frac{1}{2} \sum_{n=1}^{\ell} \int_{\mathbb{T}^3} |\nabla^3 u_n^R|^2 \, dx \\
\leq \frac{1}{2} \int_{\mathbb{T}^3} |\nabla^2 u_0|^2 \, dx + \max_{1 \leq m \leq \ell} |\mathbb{M}_m^1| + \max_{1 \leq m \leq \ell} |\mathbb{M}_m^2| \\
+ c R^2 \sum_{m=1}^{\ell} \int_{\mathbb{T}^3} |\nabla^2 \mathbb{M}_m^R|^2 \, dx.
\]
where the expectation of the last term can be estimated by $c\ell \tau e^{cR^2}$ as a consequence of Lemma 4.3. For the stochastic terms we have again

$$
E \left[ \max_{1 \leq m \leq \ell} |\mathcal{A}_m| \right] \leq c E \left[ \left( \sum_{n=1}^{\ell} \tau R^2 \| \Delta \nabla u_{n-1}^R \|_{L_2^2}^2 \right)^{\frac{1}{2}} \right]
$$

$$
\leq c(\delta) \tau R^2 + \delta E \left[ \sum_{n=1}^{\ell-1} \tau \| \Delta \nabla u_n^R \|_{L_2^2}^2 \right],
$$

$$
E \left[ \max_{1 \leq m \leq \ell} \| u_m^R - u_{m-1}^R \|_{L_2^2}^2 \right] \leq \delta E \left[ \sum_{n=1}^{\ell} \| \nabla^2 (u_m^R - u_{m-1}^R) \|_{L_2^2}^2 \right]
$$

$$
+ c(\delta) E \left[ \tau \sum_{n=1}^{\ell} \zeta_R (\| u_{m-1}^R \|_{W_2^2}^2) (1 + \| u_{n-1}^R \|_{W_2^4}^4 + \| u_{n-1}^R \|_{W_2^2}^2) \right]
$$

$$
\leq \delta E \left[ \sum_{n=1}^{\ell} \| \nabla^2 (u_m^R - u_{m-1}^R) \|_{L_2^2}^2 \right] + c(\delta) \tau R^2.
$$

Absorbing the $\delta$-terms we conclude

$$
E \left[ \max_{1 \leq m \leq \ell} \| u_m^R \|_{W_2^2}^2 \right] \leq E \left[ \int_{\Omega} |\nabla^2 u_0|^2 \, dx \right] + c\tau e^{cR^2}
$$

such that

$$
P(\| s_R^\delta \| \leq \ell \tau) = P \left( \max_{1 \leq m \leq \ell} \| u_m^R \|_{W_2^2}^2 \geq R^2 \right)
$$

$$
\leq P \left( \max_{1 \leq m \leq \ell} \| u_m^R \|_{W_2^2}^2 \geq R^2 \right)
$$

$$
\leq \frac{1}{R^2} E \left[ \int_{\Omega} |\nabla^2 u_0|^2 \, dx \right] + c\tau e^{cR^2}
$$

by Markov’s inequality. By assumption the right-hand sides vanishes as $\tau \to 0$. □

With relation (4.9) at hand it is now meaningful to transfer the estimates for $(u_m^R)_{m=1}^M$ from Lemma 4.1 and 4.3 to $(u_m^R)_{m=1}^M$. Noticing that $u_m = u_m^R$ in $[s_R^\delta \geq t_m]$ we obtain the following corollary.

**Corollary 4.5.** Assume that $q \in \mathbb{N}$ and that (2.1) holds. Then the iterates $(u_m)_{m=1}^M$ given by (4.1) satisfy the following estimates uniformly in $M$:

(a) Suppose that $u_0 \in L^{2q}(\Omega, W_0^{1,2}(\mathbb{T}^3))$ and that additionally (2.2) and (2.3) hold. Then we have

$$
E \left[ \max_{1 \leq m \leq n} \| u_m \|_{W_2^{q}}^2 + \sum_{m=1}^{n} \tau \| u_m \|_{W_2^{q-2}}^2 \| \nabla^2 u_m \|_{L_2^2}^2 \right]
$$

$$
+ \sum_{m=1}^{n} \| u_m^R \|_{W_2^{q-2}}^2 \| \nabla (u_m - u_{m-1}) \|_{L_2^2}^2 \right] \leq c e^{cR^2}.
$$

(b) Suppose that $u_0 \in L^{2q}(\Omega, W_2^{2,2}(\mathbb{T}^3))$ and that additionally (2.2)-(2.5) hold. Then we have

$$
E \left[ \max_{1 \leq m \leq n} \| u_m \|_{W_2^{q}}^2 + \sum_{m=1}^{n} \tau \| u_m \|_{W_2^{q-2}}^2 \| \nabla^2 u_m \|_{L_2^2}^2 \right]
$$
\[ + \sum_{m=1}^{ln} \|u_m^R\|_{W^{2,2}}^2 \|\nabla^2 (u_m - u_{m-1})\|_{L^2}^2 \leq c e^{cR^2}. \]

Here \( c = c(q, T, \Phi, u_0) > 0 \) is independent of \( R \).

**Remark 4.6.** In the estimates from Lemmas 4.2 and 4.3 it is also possible to control higher moments provided the corresponding moments are bounded for the initial datum. This transfers also to Corollary 4.5 and, in particular, implies that

\[ (4.10) \quad \mathbb{E} \left[ \sum_{m=1}^{ln} \|\nabla (u_m - u_{m-1})\|_{L^2}^2 \right] \leq c e^{cR^2}, \]

provided we have \( u_0 \in L^2(\Omega, W^{1,2}(\mathbb{T}^3)). \)

Now we are going to introduce the pressure function for (4.1). For \( \varphi \in W^{1,2}(\mathbb{T}^3) \) we can insert \( \varphi - \nabla \Delta^{-1} \text{div} \varphi \in W^{1,2}(\mathbb{T}^3) \) in (4.1) and obtain

\[ \int_{\mathbb{T}^3} u_m \cdot \varphi \, dx = \tau \left( \int_{\mathbb{T}^3} u_m \otimes u_{m-1} : \nabla \varphi \, dx + \mu \int_{\mathbb{T}^3} \nabla u_m : \nabla \varphi \, dx \right) \]

\[ + \int_{\mathbb{T}^3} \Phi(u_m) \cdot W \cdot \varphi \, dx + \int_{\mathbb{T}^3} \Phi(u_m) - \Delta_m W \cdot \varphi \, dx, \]

where

\[ \pi_m^\Delta = -\Delta^{-1} \text{div} (u_m \otimes u_{m-1}), \]

\[ \Phi_m - \Delta_m W \cdot \varphi \, dx, \]

where

\[ \pi_m = -\Delta^{-1} \text{div} \Phi(u_m). \]

Similar to [7, Lemma 3 & 4] we give some estimates for \( \pi_m^\Delta \) and \( \Phi_m \).

**Lemma 4.7.** Assume that \( u_0 \in L^4(\Omega, W^{1,2}(\mathbb{T}^3)) \) and that \( \Phi \) satisfies (2.1)-(2.3). For all \( m \in \{1, \ldots, R\} \) the random variable \( \pi^\Delta_m \) is \( \mathcal{F}_m \)-measurable, has values in \( W^{1,2}(\mathbb{T}^3) \) and we have uniformly in \( \tau \)

\[ \mathbb{E} \left[ \tau \sum_{m=1}^{ln} \|\pi^\Delta_m\|_{L^2}^2 \right] \leq c e^{cR^2}, \]

where \( c = c(T, \Phi, u_0) \) is independent of \( R \).

**Proof.** The \( \mathcal{F}_m \)-measurability of \( \pi_m^\Delta \) follows directly from the measurability of \( u_m \). By continuity of the operator \( \nabla \Delta^{-1} \text{div} \) on \( L^2(\mathbb{T}^3) \), we have

\[ \|\pi^\Delta_m\|_{L^2}^2 \leq c \|\text{div}(u_m \otimes u_{m-1})\|_{L^2}^2 \leq c \|u_m\|_{L^2}^2 \|\nabla u_m\|_{L^4}^2 \leq c \|u_m\|_{W^{1,2}}^2 \|\nabla^2 u_m\|_{W^{2,2}} \]

P-a.s., also making use of Sobolev’s embedding in three dimensions. Now, summing with respect to \( m \), applying expectations and using Corollary 4.5 (a) with \( q = 2 \) yields the claim. \( \square \)

**Lemma 4.8.** Assume that \( u_0 \in L^2(\Omega, L^2(\mathbb{T}^3)) \) and that \( \Phi \) satisfies (2.1)-(2.2). For all \( m \in \{1, \ldots, M\} \) the random variable \( \Phi_m^\pi \) is \( \mathcal{F}_m \)-measurable, has values in \( L^2(\Omega, W^{1,2}(\mathbb{T}^3)) \) and we have uniformly in \( \tau \)

\[ \mathbb{E} \left[ \tau \sum_{m=1}^{M} \|\Phi_m^\pi\|_{L^2(\Omega, W^{1,2})}^2 \right] \leq c \]

where \( c = c(T, \Phi, u_0) \).
Proof. As with Lemma 4.7, the proof mainly relies on the continuity of \( \nabla \Delta^{-1} \text{div} \) on \( L^2(T^3) \). Here, we have by (2.2)
\[
\| \phi_m \|_{L^2(\mathbb{W}^{1,2})}^2 = \sum_{k \geq 1} \| \nabla \Delta^{-1} \text{div} (\phi(u_m) e_k) \|_{W^{1,2}}^2
\]
\[
\leq c \sum_{k \geq 1} \| \phi(u_m) e_k \|_{W^{1,2}}^2 = c \| \phi(u_m) \|_{L^2(\mathbb{W}^{1,2})}^2 \leq c (1 + \| u_m \|_{W^{1,2}}^2).
\]
Summing over \( m \), applying expectations and using Lemma 4.1 finishes the proof. \( \square \)

4.2. Temporal error analysis. For every \( m \geq 1 \) introduce the discrete stopping time
\[
t^R_m := \max_{1 \leq n \leq m} \{ t_n : t_n \leq t_R \},
\]
which is obviously \( \mathcal{F}_{t_m} \)-measurable. Furthermore, we define \( m_R \) as the unique index in \( \{1, 2, \ldots, M\} \) such that \( t^R_m = t_{m_R} \). Our main effort in this section is devoted to the proof of the following theorem.

Theorem 4.9. Let \( u_0 \in L^8(\Omega, W^{2,2}_0(T^3)) \) be \( \mathcal{F}_0 \)-measurable and assume that \( \Phi \) satisfies (2.1)–(2.5). Let
\[
(u, (t_R)_{R \in \mathbb{N}}, t)
\]
be the unique maximal global strong solution to (1.1) in the sense of Definition 2.2. Then we have for all \( R \in \mathbb{N} \) and all \( \alpha < \frac{1}{4} \)
\[
\mathbb{E} \left[ \max_{1 \leq n \leq m_R} \| u(t_m) - u_m \|_{L^2}^2 + \sum_{m = 1}^m \tau \| \nabla u(t_m) - \nabla u_m \|_{L^2}^2 \right] \leq c e^{C R^2 \tau^{2\alpha}},
\]
where \( (u_m)^M_{m=1} \) is the solution to (4.1). The constant \( c \) in (4.13) is independent of \( \tau \) and \( R \).

Our main result on the temporal error is now a direct consequence of Theorem 4.13: Supposing that \( R = R(\tau) \leq c^{-1/2} \sqrt{-2 \epsilon \log \tau} \) as \( \tau \to 0 \) (note that this includes, in particular, any choice of fixed \( R \in \mathbb{N} \) where \( \epsilon > 0 \) is arbitrary, and relabelling \( \alpha \) we have proved the following result.

Theorem 4.10. Let \( u_0 \in L^8(\Omega, W^{2,2}_0(T^3)) \) be \( \mathcal{F}_0 \)-measurable and assume that \( \Phi \) satisfies (2.1)–(2.5). Let
\[
(u, (t_R)_{R \in \mathbb{N}}, t)
\]
be the unique maximal global strong solution to (1.1) from Theorem 2.3. Then we have for any \( \xi > 0 \), \( \alpha < \frac{1}{4} \),
\[
\mathbb{P} \left[ \max_{1 \leq n \leq m_R} \| u(t_m) - u_m \|_{L^2}^2 + \sum_{m = 1}^m \tau \| \nabla u(t_m) - \nabla u_m \|_{L^2}^2 > \xi \tau^{2\alpha} \right] \to 0
\]
as \( \tau \to 0 \), where \( (u_m)^M_{m=1} \) is the solution to (4.1).

Proof of Theorem 4.9. Define the error \( e_m = u(t_m) - u_m \) for \( m \in \{0, 1, \ldots, m_R\} \). Subtracting (2.6) and (4.1) and using that \( t_m \leq t_R \) for \( m \leq m_R \) we obtain
\[
\int_{T^3} e_m \cdot \varphi \, dx + \mu \int_{t_{m-1}}^{t_m} \int_{T^3} \nabla u(\sigma) : \nabla \varphi \, dx \, d\sigma - \mu \tau \int_{T^3} \nabla u_m : \nabla \varphi \, dx
\]
\[
= \int_{T^3} e_{m-1} \cdot \varphi \, dx + \int_{T^3} (\nabla u_{m-1} - \nabla u_m) \cdot \varphi \, dx + \int_{t_{m-1}}^{t_m} (\nabla u(\sigma)) u_{m-1} \cdot \varphi \, dx \, d\sigma
\]
\[
+ \int_{T^3} \int_{t_{m-1}}^{t_m} \Phi(u(\sigma)) \, dW \cdot \varphi \, dx - \int_{T^3} \int_{t_{m-1}}^{t_m} \Phi(u_{m-1}) \, dW \cdot \varphi \, dx
\]
for every $\varphi \in W^{1,2}_{	ext{div}}(\mathbb{T}^3)$. Setting $\varphi = e_{n,m}$ and applying the identity $a \cdot (a - b) = \frac{1}{2}(|a|^2 - |b|^2 + |a - b|^2)$ (which holds for any $a, b \in \mathbb{R}^3$) we gain
\[
\int_{T^3} \frac{1}{2} \left( |e_{m}^2 - |e_{m-1}|^2 + |e_{m} - e_{m-1}|^2 \right) dx + \mu \tau \int_{T^3} |\nabla e_{m}|^2 dx
\]
\[
= \mu \int_{T^3} \int_{t_{m-1}}^{t_m} (\nabla u(t_m) - \nabla u(\sigma)) : \nabla e_{m} dx d\sigma
\]
\[
+ \int_{T^3} \int_{t_{m-1}}^{t_m} (\nabla u(t_m)u(t_{m-1}) - (\nabla u(\sigma))u(\sigma)) \cdot e_{m} dx d\sigma
\]
\[
- \tau \int_{T^3} ((\nabla u(t_m))u(t_{m-1}) - (\nabla u_m)u_{m-1}) \cdot e_{m} dx
\]
\[
+ \tau \int_{T^3} \Phi(u(\sigma)) - \Phi(u_{m-1})) dW \cdot e_{m} dx
\]
\[
=: I_1(m) + \cdots + I_5(m).
\]
Eventually, we will take the maximum with respect to $m \in \{1, \ldots, m_R\}$ and apply expectations. Let us explain how to deal with $E\max_{m} I_1(m), \ldots, E\max_{m} I_5(m)$ independently.

We have
\[
I_1(m) \leq \kappa \tau \int_{T^3} |\nabla e_{m}|^2 dx + c(\kappa) \int_{t_{m-1}}^{t_m} \int_{T^3} |\nabla u(t_m) - u(\sigma)|^2 dx d\sigma
\]
\[
\leq \kappa \tau \int_{T^3} |\nabla e_{m}|^2 dx + c(\kappa) \tau^{1+2\alpha} \|u\|^2_{C^0([t_{m-1}, t_m] ; L^2)}
\]
where the expectation of the last term can be controlled for $m \leq m_R$ by $\tau^{2\alpha+1} R^{12}$ using Corollary 3.3 and $t^n_m \leq t_R$. We proceed by
\[
I_2(m) = \int_{T^3} \int_{t_{m-1}}^{t_m} (u(t_m) \otimes u(t_{m-1}) - u(\sigma) \otimes u(\sigma)) : \nabla e_{m} dx d\sigma
\]
\[
\leq \kappa \tau \int_{T^3} |\nabla e_{m}|^2 dx + c(\kappa) \int_{t_{m-1}}^{t_m} \int_{T^3} |u(t_m) \otimes u(t_{m-1}) - u(\sigma) \otimes u(\sigma)|^2 dx d\sigma
\]
\[
\leq \delta \tau \int_{T^3} |\nabla e_{m}|^2 dx + c(\delta) \tau^{1+2\alpha} \|u\|^2_{L^\infty([t_{m-1}, t_m] ; \mathbb{T}^3)} \|u\|^2_{C^0([t_{m-1}, t_m] ; L^2)}
\]
for $m \leq m_R$ using the embedding $W^{2,2}(\mathbb{T}^3) \hookrightarrow L^\infty(\mathbb{T}^3)$ and $t^n_m \leq t_R$. We rewrite $I_1(m)$ as
\[
I_3(m) = -\tau \int_{T^3} (\nabla e_{m})_e_{m-1} \cdot u(t_m) dx
\]
and obtain for any $\delta > 0$ and $m \leq m_R$
\[
I_3(m) \leq \tau \|\nabla e_{m}\|_{L^2} \|e_{m-1}\|_{L^2} \|u(t_m)\|_{L^\infty}
\]
\[
\leq \delta \tau \|\nabla e_{m}\|_{L^2}^2 + c(\delta) R^2 \|e_{m-1}\|_{L^2}^2.
\]
The last term will be dealt with by Gronwall’s lemma leading to a constant of the form $ce^{cR^2}$.

In order to estimate the stochastic term $I_5$ we write
\[
\mathcal{M}_{m} = \sum_{n=1}^{m} I_5(n) = \sum_{n=1}^{m} \int_{T^3} \int_{t_{n-1}}^{t_n} (\Phi(u) - \Phi(u_{n-1})) dW \cdot e_{n} dx
\]
\[
= \sum_{n=1}^{m} \int_{T^3} \int_{t_{n-1}}^{t_n} (\Phi(u) - \Phi(u_{n-1})) dW \cdot e_{n-1} dx
\]
Since the process \( (\mathcal{M}_1(t \wedge t_R))_{t \geq 0} \) is an \( (\mathfrak{F}_t) \)-martingale, through the use of the Burkholder-Davis-Gundy inequality (using that \( t_M^R \leq t_R \) by definition) we see that

\[
\mathbb{E} \left[ \max_{1 \leq m \leq M_n} |\mathcal{M}_1(t_m)| \right] \leq \mathbb{E} \left[ \sup_{s \in [0, t_M]} |\mathcal{M}_1(s)| \right] \leq \mathbb{E} \left[ \sup_{s \in [0, t_R]} |\mathcal{M}_1(s \wedge t_R)| \right]
\]

\[
\leq c \mathbb{E} \left[ \int_{0}^{T \wedge t_R} \sum_{n=1}^{M} 1_{(t_{n-1}, t_n)} \|\Phi(u) - \Phi(u_{n-1})\|_{L^2(\Omega, L^2)}^2 \|e_{n-1}\|_{L^2}^2 \, dt \right]^{\frac{4}{3}}
\]

\[
\leq \delta \mathbb{E} \left[ \max_{1 \leq n \leq n} \|e_{n}\|_{L^2}^2 \right] + c(\delta) \mathbb{E} \left[ \int_{0}^{T \wedge t_R} \sum_{n=1}^{M} 1_{(t_{n-1}, t_n)} \|u - u_{n-1}\|_{L^2}^2 \, dt \right]
\]

\[
\leq \delta \mathbb{E} \left[ \max_{1 \leq n \leq n} \|e_{n}\|_{L^2}^2 \right] + c(\delta) \mathbb{E} \left[ \int_{0}^{T \wedge t_R} \|u - u(t_{n-1})\|_{L^2}^2 \, dt \right]
\]

\[
+ c(\delta) \mathbb{E} \left[ \int_{0}^{T \wedge t_R} \sum_{n=1}^{M} 1_{(t_{n-1}, t_n)} \|e_{n-1}\|_{L^2}^2 \, dt \right].
\]

Here, we also used (2.1) as well as Young’s inequality for arbitrary \( \delta > 0 \). Finally, we can control the last term by

\[
\mathbb{E} \left[ \int_{0}^{T \wedge t_R} \sum_{n=1}^{M} 1_{(t_{n-1}, t_n)} \|e_{n-1}\|_{L^2}^2 \, dt \right] \leq \mathbb{E} \left[ \sum_{n=1}^{m+1} \tau \|e_{n-1}\|_{L^2}^2 \, dt \right]
\]

\[
\leq \mathbb{E} \left[ \sum_{n=0}^{m} \tau \|e_{n}\|_{L^2}^2 \, dt \right]
\]

since \( t_R \wedge t_M \leq t_{M+1}^R \). Applying (2.9) as well as Lemma 3.1 (b) and Corollary 3.3 (b), we obtain

\[
\mathbb{E} \left[ \max_{1 \leq m \leq m_n} |\mathcal{M}_{1, m}(t_m)| \right] \leq \delta \mathbb{E} \left[ \max_{1 \leq n \leq m_n} \|e_{n}\|_{L^2}^2 \right] + c(\delta) \mathbb{E} \left[ \sum_{n=0}^{m_n} \tau \|e_{n}\|_{L^2}^2 \right]
\]

\[
+ c(\delta)^2 \tau^{2\alpha} \mathbb{E} \left[ \|u\|_{L^2(\Omega, L^2)}^2 \right]
\]

\[
\leq \delta \mathbb{E} \left[ \max_{1 \leq n \leq m_n} \|e_{n}\|_{L^2}^2 \right] + c(\delta) \mathbb{E} \left[ \sum_{n=0}^{m_n} \tau \|e_{n}\|_{L^2}^2 \right] + c(\delta)^2 \tau^{2\alpha} R^4.
\]

As far as \( \mathcal{M}_{m, 1}^2 \) is concerned we argue similarly. Using the Cauchy-Schwarz inequality, Young’s inequality, Itô-isometry and (2.1) we have

\[
\mathbb{E} \left[ \max_{1 \leq m \leq m_n} |\mathcal{M}_{m, 1}^2(t)| \right]
\]
\[
\begin{align*}
&\leq \mathbb{E}\left[ \sum_{n=1}^{m_R} (\delta |e_n - e_{n-1}|^2_{L^2_x}) + c(\delta) \left\| \int_{t_n}^{t_{n-1}} (\Phi(u) - \Phi(u_{n-1})) \, dW \right\|^2_{L^2} \right] \\
&\leq \delta \mathbb{E}\left[ \sum_{n=1}^{m_R} |e_n - e_{n-1}|^2_{L^2_x} \right] + c(\delta) \mathbb{E}\left[ \sum_{n=1}^{m_R} \int_{t_n}^{t_{n-1}} \|u - u_{n-1}\|_{L^2_x}^2 \, dt \right] \\
&\leq \delta \mathbb{E}\left[ \sum_{n=1}^{m_R} |e_n - e_{n-1}|^2_{L^2_x} \right] + c(\delta) \mathbb{E}\left[ \sum_{n=1}^{m_R} \int_{t_n}^{t_{n-1}} \|u - u(t_{n-1})\|_{L^2_x}^2 \, dt \right] \\
&\quad + c(\delta) \mathbb{E}\left[ \sum_{n=1}^{m_R} \tau |e_{n-1}|^2_{L^2_x} \right] \\
&\leq \delta \mathbb{E}\left[ \sum_{n=1}^{m_R} |e_n - e_{n-1}|^2_{L^2_x} \right] + c(\delta) \mathbb{E}\left[ \sum_{n=1}^{m_R} \tau |e_{n-1}|^2_{L^2_x} \right] + c(\delta) \tau^{2\alpha} R^{12}
\end{align*}
\]

as a consequence of Lemma 3.1 (b) (using also (2.8)) of Corollary 3.3 (b). Collecting all estimates, choosing \( \delta \) small enough and applying Gronwall’s lemma yields the claim. \( \square \)

5. Space-time discretisation

Now we consider a fully practical scheme combining the implicit Euler scheme in time (as in the last section) with a finite element approximation in space. For a given \( h > 0 \), let \( u_{h,0} \) be an \( \mathcal{F}_0 \)-measurable random variable with values in \( V^h_{\text{div}}(\Omega) \) (for instance \( \Pi_h u_0 \)). We aim at constructing iteratively a sequence of random variables \( u_{h,m} \) with values in \( V^h_{\text{div}}(\Omega) \) such that for every \( \varphi \in V^h_{\text{div}}(\Omega) \) it holds true \( \mathbb{P}\)-a.s.

\[
\int_{\Omega} u_{h,m} \cdot \varphi \, dx + \Delta t \int_{\Omega^3} ((\nabla u_{h,m})u_{h,m-1} + (\text{div} u_{h,m-1})u_{h,m}) \cdot \varphi \, dx + \mu \Delta t \int_{\Omega^3} \nabla u_{m-1} : \nabla \varphi \, dx = \int_{\Omega^3} u_{h,m-1} : \varphi \, dx + \int_{\Omega^3} \Phi(u_{h,m-1}) \Delta_n W \cdot \varphi \, dx,
\]

where \( \Delta_n W = W(t_m) - W(t_{m-1}) \). The existence of iterates \( (u_{h,m})_{m=1}^M \) given by (5.1) which are \( \mathcal{F}_{t_n} \)-measurable is shown in [11, Lemma 3.1]. Furthermore, it holds for \( q \in \mathbb{N} \)

\[
\mathbb{E}\left[ \max_{1 \leq m \leq M} \left( \|u_{h,m}\|_{L^2_x}^q + \tau \sum_{m=1}^M \left( \|u_{h,m}\|_{L^2_x}^{2q-2} \|\nabla u_{h,m}\|_{L^2_x}^2 \right) \right) \right] \leq c(q,T)\mathbb{E}[\|u_{h,0}\|_{L^2_x}^{2q} + 1]
\]

uniformly in \( h \) and \( \tau \). It is also shown there that the sequence converges in law to a martingale solution to (1.1). We strengthen this result in short-time (where the stopping times \( \tau_R \) and \( m_R \) are introduced below (4.8) and (4.12), respectively) by proving an optimal convergence rate with respect to convergence in probability in the following theorem. Here we suppose that \( R = R(\tau,h) \leq c^{-1/2} \sqrt{-\varepsilon} \min\{\log(\tau), \log(h^2)\} \) as \( \tau, h \to 0 \), which inludes, in particular, any choice of fixed \( R \in \mathbb{N} \).

**Theorem 5.1.** Let \( u_0 \in L^8(\Omega, W^2_{\text{div}}(\Omega)) \) be \( \mathcal{F}_0 \)-measurable and assume that \( \Phi \) satisfies (2.1)–(2.5). Let

\[
(u, (u_R)_{R \in \mathbb{N}}, t)
\]

be the unique maximal global strong solution to (1.1) from Theorem 2.3. Then we have for any \( \xi > 0, \alpha < 1 \) and \( \beta < 1 \),

\[
P\left[ \max_{1 \leq m \leq m_R} \|u(t_m) - u_{h,m}\|_{L^2_x} + \tau \|\nabla u(t_m) - \nabla u_{h,m}\|_{L^2_x} > \xi (\tau^{2\alpha} + h^{2\beta}) \right] \to 0
\]
as \( \tau \to 0 \), where \( (u_{h,m})_{m=1}^M \) is the solution to (5.1).
Remark 5.2. It is possible to obtain Theorem 5.1 by a direct comparison between the space-time discretisation and the exact solution (avoiding the time discretisation is an intermediate step) as in done [9, Section 4] in the 2D case. The advantage of such an approach is that the stopping time $\tau_R$ is not needed. We believe, however, that the plainly temporal error as well as the error between the temporal and spatio-temporal discretisation given in Theorem 5.3 below are of independent interest.

Theorem 5.1 follows from combining Theorem 4.10 with the following result concerning the error between the temporal and spatio-temporal discretisation, the proof of which is the main aim of this section. Here we suppose that $R = R(h) \leq c^{-1/2} \sqrt{-\epsilon \log(h^2)}$ as $h \to 0$.

**Theorem 5.3.** Let $u_0 \in L^8(\Omega, W_{\text{div}}^{2,2}(\mathbb{T}^3))$ be $\mathcal{F}_0$-measurable and assume that $\Phi$ satisfies (2.1)–(2.5). Let $(u_m)_{m=1}^M$ be the solution to (4.1). Then we have for any $\xi > 0$, $\alpha < \frac{1}{2}$, $\beta < 1$,

$$
\mathbb{P}\left[ \max_{1 \leq m \leq M} \|u_m - u_{h,m}\|_{L^2}^2 + \sum_{m=1}^{j_R} \tau \|\nabla u_m - \nabla u_{h,m}\|_{L^2}^2 > \xi h^{2\beta} \right] \to 0
$$

as $\tau, h \to 0$, where $(u_{h,m})_{m=1}^M$ is the solution to (5.1).

**Proof.** Define the error $e_{h,m} = u_m - u_{h,m}$ for $m \in \{0, 1, \ldots, j_R\}$. Subtracting (4.11) and (5.1) we obtain

$$
\int_{\mathbb{T}^3} e_{h,m} \cdot \varphi \, dx + \mu \tau \int_{\mathbb{T}^3} (\nabla u_m - \nabla u_{h,m}) : \nabla \varphi \, dx
$$

$$
= \int_{\mathbb{T}^3} e_{h,m-1} \cdot \varphi \, dx - \tau \int_{\mathbb{T}^3} ((\nabla u_m)u_{m-1} - ((\nabla u_{h,m})u_{h,m-1} + (\text{div} u_{h,m-1})u_{h,m})) \cdot \varphi \, dx
$$

$$
+ \int_{\mathbb{T}^3} (\Phi(u_m) - \Phi(u_{h,m-1})) \Delta_m W \cdot \varphi \, dx
$$

$$
- \int_{\mathbb{T}^3} \nabla \Phi(e_{h,m-1}) \Delta_m W \cdot \varphi \, dx + \tau \int_{\mathbb{T}^3} \pi_{\text{det}} \text{div} \varphi \, dx
$$

for every $\varphi \in V_{\text{div}}^{h}(\mathbb{T}^3)$. Setting $\varphi = \Pi_h e_{h,m}$ and applying again the identity $a \cdot (a - b) = \frac{1}{2}(\|a\|^2 - \|b\|^2 + |a - b|^2)$ for $a, b \in \mathbb{R}^3$ we gain

$$
\int_{\mathbb{T}^3} \frac{1}{2} (|\Pi_h e_{h,m}|^2 - |\Pi_h e_{h,m-1}|^2 + |\Pi_h e_{h,m} - \Pi_h e_{h,m-1}|^2) \, dx + \mu \tau \int_{\mathbb{T}^3} |\nabla e_{h,m}|^2 \, dx
$$

$$
= \mu \tau \int_{\mathbb{T}^3} \nabla e_{h,m} : \nabla (u_m - \Pi_h u_m) \, dx
$$

$$
- \tau \int_{\mathbb{T}^3} ((\nabla u_m)u_{m-1} - ((\nabla u_{h,m})u_{h,m-1} + (\text{div} u_{h,m-1})u_{h,m})) \cdot \Pi_h e_{h,m} \, dx
$$

$$
+ \int_{\mathbb{T}^3} (\Phi(u_m) - \Phi(u_{h,m-1})) \Delta_m W \cdot \Pi_h e_{h,m} \, dx
$$

$$
- \int_{\mathbb{T}^3} \nabla \Phi(e_{h,m-1}) \Delta_m W \cdot \Pi_h e_{h,m} \, dx
$$

$$
= I_1(m) + \cdots + I_5(m)
$$

(5.3)

Now we take the maximum with respect to $m \in \{1, \ldots, j_R\}$. The terms $I_1(m)$ and $I_3(m)$ can be estimated as in [7, Section 4] and [9, Section 4] leading to

$$
I_1(m) \leq \delta \tau \int_{\mathbb{T}^3} |\nabla e_{h,m}|^2 \, dx + c(\delta) \tau h^2 \int_{\mathbb{T}^3} |\nabla^2 u_m|^2 \, dx,
$$

$$
I_3(m) \leq \delta \tau \int_{\mathbb{T}^3} |\nabla e_{h,m}|^2 \, dx + c(\delta) \tau h^2 \|u_m\|_{W^{2,2}}^2 + c(\delta) \tau h^2 \int_{\mathbb{T}^3} |\nabla \pi_{\text{det}}|^2 \, dx
$$

This completes the proof of Theorem 5.3.
for $\delta > 0$ arbitrary. The $\delta$-terms can be absorbed, while the (sum from $m = 1, \ldots, j_R$ of the) expectations of the other two terms can be bounded by $h^2 e^{R^2}$ using Corollary 4.5 (a) and Lemma 4.7. More care is required for $I_2(m)$. We argue in the spirit of of Thm. 4.2 but working with 3D embeddings and the definition of $j_R$. First we write

$$I_2(m) = I_2^1(m) + I_2^2(m) + I_2^3(m),$$

$$I_2^1(m) = -\tau \int_{\mathcal{T}} (u_m - \Pi_h u_m) \cdot e_{h,m} \, dx,$$

$$I_2^2(m) = \tau \int_{\mathcal{T}} (e_{h,m} \cdot \nabla) e_{h,m} \cdot (\nabla u_m - \Pi_h u_m) \, dx$$

$$+ \tau \int_{\mathcal{T}} (\nabla e_{h,m}) e_{h,m} \cdot (\nabla u_m - \Pi_h u_m) \, dx,$$

$$I_2^3(m) = -\tau \int_{\mathcal{T}} (e_{h,m} - \Pi_h e_{h,m}) \cdot u_m \, dx$$

$$- \tau \int_{\mathcal{T}} (\nabla e_{h,m}) \Pi_h e_{h,m} \cdot u_m \, dx.$$

We obtain for any $\delta > 0$ and $m \leq j_R$

$$I_2^1(m) \leq \tau \| \nabla e_{h,m} \|_{L_2^\infty} \| u_m - \Pi_h u_m \|_{L_2^\infty}$$

$$\leq \tau R h^2 \| \nabla e_{h,m} \|_{L_2^\infty} \| \nabla^2 u_m \|_{L_2^\infty}$$

$$\leq \delta \tau \| \nabla e_{h,m} \|_{L_2^\infty} + c(\delta) h^4 R^2 \| \nabla^2 u_m \|_{L_2^\infty}^2$$

by the embedding $W^{2,2}(\Omega^3) \hookrightarrow L^\infty(\Omega^3)$ and the approximability of $\Pi_h$ from (2.9). The first term can be absorbed for $\kappa$ small enough, whereas the second one (in summed form and expectation) is bounded by $h^4 e^{R^2}$ due Corollary 4.5 (a). Similarly, we have

$$I_2^2(m) \leq \tau \| \nabla e_{h,m} \|_{L_2^\infty} \| \Pi_h e_{h,m} \|_{L_2^\infty} \| u_m - \Pi_h u_m \|_{L_2^\infty}$$

$$\leq \tau \| \nabla e_{h,m} \|_{L_2^\infty} \| \Pi_h e_{h,m} \|_{L_2^\infty} \| u_m - \Pi_h u_m \|_{L_2^\infty}$$

$$\leq \delta \tau \| \nabla e_{h,m} \|_{L_2^\infty} + \| \nabla e_{h,m} \|_{L_2^\infty}^2$$

$$\| \Pi_h e_{h,m} \|_{L_2^\infty}^2$$

$$\| u_m - \Pi_h u_m \|_{L_2^\infty}^2$$

The last term (in summed form, for $m = 1, \ldots, j_R$, and expectation) can be controlled by Lemma 4.5 (c) (with $q = 3$) and 5.2 (with $q = 2$). Finally, by definition of $j_R$, we have

$$I_2^3(m) \leq \tau \| \nabla u_m \|_{L_2^\infty} \| u_m - \Pi_h u_m \|_{L_2^\infty}^2$$

$$+ \tau \| \nabla u_m \|_{L_2^\infty} \| \Pi_h u_m \|_{L_2^\infty} \| u_m - \Pi_h u_m \|_{L_2^\infty}^2$$

$$\leq \delta \left( \| \nabla u_m \|_{L_2^\infty}^2 + \| \nabla u_m \|_{L_2^\infty} \| \Pi_h u_m \|_{L_2^\infty} \| u_m - \Pi_h u_m \|_{L_2^\infty}^2 \right)$$

$$+ c(\delta) \tau R^4 \left( \max_{1 \leq n \leq m} \| e_{h,n} \|_{L_2^\infty}^2 \right)$$

$$+ c(\delta) \tau \| \nabla (u_m - \Pi_h u_m) \|_{L_2^\infty} \| u_m - \Pi_h u_m \|_{L_2^\infty}^2.$$
for any \( m \leq i_R \). The last term in the second line will be dealt with by Gronwall’s lemma leading to a constant of the form \( \text{ce}^cR^4 \). The last term in the second line (in summed form, for \( m = 1, \ldots, i_R \), and expectation) can be controlled by (4.10), Lemma 4.5 (c) and 5.2 (each of them with with \( q = 3 \)). The final line is bounded by \( c(\delta) \tau R^4 \| \mathbf{u}_m \|_{L^2_T}^2 \) using (2.9) and hence can be controlled by Lemma 3.1 (c).

In order to estimate the stochastic term \( I_4(m) \) we write

\[
\mathcal{N}_{m,1} := \sum_{n=1}^{m} I_4(m) = \sum_{n=1}^{m} \int_{T_{n-1}}^{T_n} \left( \Phi(\mathbf{u}_{n-1}) - \Phi(\mathbf{u}_{h,n-1}) \right) dW \cdot \mathbf{e}_{h,n-1} dx
\]

\[
+ \sum_{n=1}^{m} \int_{T_{n-1}}^{T_n} \left( \Phi(\mathbf{u}_{n-1}) - \Phi(\mathbf{u}_{h,n-1}) \right) dW \cdot (\mathbf{e}_{h,n} - \mathbf{e}_{h,n-1}) dx
\]

\[
=: \mathcal{N}_{m,1} + \mathcal{N}_{m,2}.
\]

Using that \( i_R \) is an \((\tilde{\delta} t_m)\)-stopping time, we can argue as in [7, Section 4] and [9, Section 4] obtaining

\[
\mathbb{E} \left[ \max_{1 \leq m \leq M} \left| \mathcal{N}_{m,1}^{1} \right|_{H^1,1} \right] \leq \delta \mathbb{E} \left[ \max_{0 \leq m \leq n} \| \Pi_h \mathbf{e}_{h,m} \|_{L^2_T}^2 \right] + c(\delta) \mathbb{E} \left[ \tau \sum_{m=1}^{n} \| \Pi_h \mathbf{e}_{h,m-1} \|_{L^2_T}^2 \right]
\]

\[
+ c(\delta) h^2 \mathbb{E} \left[ \tau \sum_{n=1}^{m} \| \nabla \mathbf{u}_{m-1} \|_{L^2_T}^2 \right]
\]

as well as

\[
\mathbb{E} \left[ \max_{1 \leq m \leq M} \left| \mathcal{N}_{m,2}^{2} \right|_{H^1,1} \right] \leq \delta \mathbb{E} \left[ \sum_{m=1}^{n} \| \Pi_h \mathbf{e}_{h,m} - \Pi_h \mathbf{e}_{h,m-1} \|_{L^2_T}^2 \right] + c(\delta) \mathbb{E} \left[ \tau \sum_{m=1}^{n} \| \Pi_h \mathbf{e}_{h,m-1} \|_{L^2_T}^2 \right]
\]

\[
+ c(\delta) h^2 \mathbb{E} \left[ \tau \sum_{n=1}^{m} \| \nabla \mathbf{u}_{m-1} \|_{L^2_T}^2 \right].
\]

In both estimates, the first term can be absorbed, the second one can be handled by Gronwall’s lemma, and the last one is bounded by \( h^2 \) using Lemma 4.1.

In order to estimate \( I_5(m) \) we write

\[
\mathcal{N}_{m,2} := \sum_{n=1}^{m} I_5(n) = \sum_{n=1}^{m} \int_{T_{n-1}}^{T_n} (\text{Id} - \Pi_h^2) \Delta^{-1} \text{div} \Phi(\mathbf{u}_{n-1}) dW \text{ div} \Pi_h \mathbf{e}_{h,n-1} dx
\]

\[
+ \sum_{n=1}^{m} \int_{T_{n-1}}^{T_n} (\text{Id} - \Pi_h^2) \Delta^{-1} \text{div} \Phi(\mathbf{u}_{n-1}) dW \text{ div}(\Pi_h \mathbf{e}_{h,n} - \Pi_h \mathbf{e}_{h,n-1}) dx
\]

\[
=: \mathcal{N}_{m,1} + \mathcal{N}_{m,2}.
\]

Following again [7] we have

\[
\mathbb{E} \left[ \max_{1 \leq m \leq M} \left| \mathcal{N}_{m,1}^{1} \right|_{H^1,2} \right] \leq c(\delta) h^4 \mathbb{E} \left[ \max_{1 \leq n \leq i_R} \| \nabla \mathbf{u}_{n-1} \|_{L^2_T}^2 \right] + \delta \mathbb{E} \left[ \tau \sum_{n=1}^{i_R} \| \nabla \Pi_h \mathbf{e}_{h,n} \|_{L^2_T}^2 \right].
\]
The first term is bounded by $h^4 c R^2$ using Corollary 4.5 (a) (recall that $u_0 \in L^2(\Omega; W^{1,2}(\mathbb{T}^2))$). The second term can be absorbed given the appropriate choice of $\delta$. Finally,

$$
\mathbb{E} \left[ \max_{1 \leq m \leq M} \| u_{h,m} \|^2_{L^2} \right] \leq \delta \mathbb{E} \left[ \sum_{n=1}^N \| \Pi_h e_{h,n} - \Pi_h e_{h,n-1} \|^2_{L^2} \right] + c(\delta) h^2 \mathbb{E} \left[ \sum_{n=1}^N R^2 (1 + \| u_{n-1} \|^2_{L^2}) \right],
$$

where the first term can be absorbed and the second one is bounded by $h^2$ on account of Lemma 4.1. We conclude that

$$
\mathbb{E} \left[ \max_{1 \leq m \leq N} \| u_m - u_{h,m} \|^2_{L^2} + \sum_{n=1}^N \tau \| \nabla u_m - \nabla u_{h,m} \|^2_{L^2} \right] \leq c h^2 e^C R^2.
$$

Since $R = R(h) \leq c^{-1/2} \sqrt{-\varepsilon \log(h^2)}$, where $\varepsilon > 0$ is arbitrary, the claim follows now by applying Markov’s inequality. \(\square\)

REFERENCES

[1] A. Bensoussan and J. Frehse. Local solutions for stochastic Navier–Stokes equations. M2AN Math. Model. Numer. Anal. 34, 241–273. (2000) (Special issue for R. Temam’s 60th birthday)
[2] A. Bensoussan, R. Temam, Equations stochastiques du type Navier-Stokes. J. Functional Analysis 13, 195–222. (1973)
[3] H. Bessaih, A. Millet: Strong $L^2$ convergence of time numerical schemes for the stochastic 2D Navier-Stokes equation, IMA Journal of Numerical Analysis, 39, 2135–2167. (2019)
[4] D. Breit: Existence theory for generalized Newtonian fluids. J. Math. Fluid Mech. 17, 295–326. (2015)
[5] D. Breit: Existence theory for stochastic power law fluids. J. Math. Fluid Mech. 17, 295–326. (2015)
[6] D. Breit: Error analysis for 2D stochastic Navier-Stokes equations in bounded domains with Dirichlet data. Preprint at arXiv:2109.06495v2
[7] D. Breit & A. Prohl: Error analysis for 2D stochastic Navier-Stokes equations in bounded domains with Dirichlet data. Preprint at arXiv:2109.06495v2
[8] D. Breit & A. Prohl: Numerical analysis of 2D Navier–Stokes equations with additive stochastic forcing.IMA J. Num. Anal. DOI:10.1093/imanum/dra023
[9] Z. Brzeźniak, E. Carrella, A. Prohl: Finite element-based discretizations of the incompressible Navier-Stokes equations with multiplicative random forcing. IMA J. Num. Anal. 33, 771–824. (2013)
[10] Z. Brzeźniak and S. Peszat. Strong local and global solutions for stochastic Navier–Stokes equations. Infinite Dimensional Stochastic Analysis (Amsterdam), Verh. Afd. Natuurkd. 1. Reeks. K. Ned. Akad. Wet., vol. 52, R. Neth. Acad. Arts Sci., Amsterdam, 85–98. (1999/2000)
[11] E. Carrella, J. A. Prohl: Rates of convergence for discretizations of the stochastic incompressible Navier-Stokes equations. SIAM J. Numer. Anal. 50(5), pp. 2467–2496. (2012)
[12] F. Flandoli: An introduction to 3D stochastic fluid dynamics. In SPDE in Hydrodynamic: Recent Progress and Prospects. Lecture Notes in Math. 1942 51–150. Springer, Berlin. (2008)
[13] F. Flandoli, D. Gątarek: Martingale and stationary solutions for stochastic Navier–Stokes equations, Probab. Theory Related Fields 102, 367–391. (1995)
[14] F. Flandoli, D. Luo: High mode transport noise improves vorticity blow-up control in 3D Navier–Stokes equations, Probab. Theory Relat. Fields 180, 309–363. (2021)
[15] N. Glatt-Holtz, M. Ziane. Strong Pathwise Solutions of the Stochastic Navier-Stokes System. Adv. Diff. Equ. 14, 567–600. (2009)
[20] J.G. Heywood & R. Rannacher: Finite element approximation of the nonstationary Navier–Stokes problem. I. Regularity of solutions and second-order error estimates for spatial discretization. SIAM J. Numer. Anal. 19, 275–311. (1982)

[21] Hofmanová M, Zhu R, Zhu X: Non-uniqueness in law of stochastic 3D Navier–Stokes equations. arXiv:1912.11841

[22] Hofmanová M, Zhu R, Zhu X: On Ill- and Well-Posedness of Dissipative Martingale Solutions to Stochastic 3D Euler Equations. Comm. Pure Appl. Math. doi:10.1002/cpa.22023

[23] J. U. Kim: Strong Solutions of the Stochastic Navier–Stokes Equations in $\mathbb{R}^3$. Indiana Univ. Math. J. 59, 1417–1450. (2010)

[24] R. Mikulevicius: On strong $H^1$-solutions of stochastic Navier–Stokes equations in a bounded domain. SIAM J. Math. Anal. Vol. 41, No. 3, pp. 1206–1230. (2009)

[25] M. Romito: Some probabilistic topics in the Navier–Stokes equations. Recent progress in the theory of the Euler and Navier–Stokes equations, 175–232, London Math. Soc. Lecture Note Ser., 430, Cambridge Univ. Press, Cambridge. (2016)

Institute of Mathematics, TU Clausthal, Erzstraße 1, 38678 Clausthal-Zellerfeld, Germany, and Department of Mathematics, Heriot-Watt University, Riccarton Edinburgh EH14 4AS, UK  
Email address: dominic.breit@tu-clausthal.de

Department of Mathematics, Heriot-Watt University, Riccarton Edinburgh EH14 4AS, UK  
Email address: ad335@hw.ac.uk