BESOV SPACES INDUCED BY DOUBLING WEIGHTS

ATTE REIJONEN

ABSTRACT. Let $1 \leq p < \infty$, $0 < q < \infty$ and $\nu$ be a two-sided doubling weight satisfying
\[
\sup_{0<r<1} \left( \frac{(1-r)^q}{\nu(t) dt} \int_0^r \frac{\nu(s)}{(1-s)^q} ds \right)^{1/p} < \infty.
\]
The weighted Besov space $B_{\nu}^{p,q}$ consists of those $f \in H^p$ such that
\[
\int_0^1 \left( \int_0^{2\pi} |f'(re^{i\theta})|^p d\theta \right)^{q/p} \nu(s) ds < \infty.
\]

Our main result gives a characterization for $f \in B_{\nu}^{p,q}$ depending only on $|f|$, $p$, $q$ and $\nu$.

1. INTRODUCTION AND CHARACTERIZATIONS

Let $\mathbb{D}$ be the open unit disc of the complex plane $\mathbb{C}$ and $\mathbb{T}$ the boundary of $\mathbb{D}$. The set of all analytic functions in $\mathbb{D}$ is denoted by $\mathcal{H}(\mathbb{D})$. For $0 < p < \infty$, the Hardy space $H^p$ consists of those $f \in \mathcal{H}(\mathbb{D})$ such that
\[
\|f\|_{H^p} = \sup_{0<r<1} M_p(r,f) < \infty,
\]
where
\[
M_p(r,f) = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p}.
\]
The Hardy space $H^\infty$ is the set of all bounded functions in $\mathcal{H}(\mathbb{D})$. Moreover, we recall that a measurable function $f$ on $\mathbb{T}$ belongs to $L^p(\mathbb{T})$ for some $p \in (0,\infty)$ if
\[
\|f\|_{L^p} = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^p d\theta < \infty.
\]
Alternatively, the Hardy space $H^p$ for $0 < p < \infty$ can be characterized as follows: $f \in H^p$ if and only if $f \in \mathcal{H}(\mathbb{D})$, non-tangential limit $f(e^{i\theta})$ exists almost everywhere on $\mathbb{T}$ and $f(e^{i\theta}) \in L^p(\mathbb{T})$.

In particular, $\|f\|_{H^p} = \|f\|_{L^p}$ for $0 < p < \infty$ and $f \in H^p$. This is due to Hardy’s convexity and the mean convergence theorems. These results and much more can be found in classic book [8] by P. Duren.

A function $\nu : \mathbb{D} \to [0,\infty)$ is called a (radial) weight if it is integrable over $\mathbb{D}$ and $\nu(z) = \nu(|z|)$ for all $z \in \mathbb{D}$. For $0 < p, q < \infty$ and a weight $\nu$, the weighted mixed norm space $A_{\nu}^{p,q}$ consists of those $f \in \mathcal{H}(\mathbb{D})$ such that
\[
\|f\|_{A_{\nu}^{p,q}} = \left( \int_0^1 M_{\nu}^q(r,f) \nu(r) dr \right)^{1/q} < \infty.
\]

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If \( \nu(z) = (1 - |z|)^\alpha \) for \(-1 < \alpha < \infty\), then the notation \( A_\alpha^{p,q} \) is used for \( A_\nu^{p,q} \). In this note, we study class \( \mathcal{D} \) of so-called two-sided doubling weights, which originates from the work of J. A. Peláez and J. Rättyä [19, 20]. For the definition of \( \mathcal{D} \) we have to define two wider classes. For a weight \( \nu \), set
\[
\tilde{\nu}(z) = \nu(|z|) = \int_{|z|}^1 \nu(s) \, ds, \quad z \in \mathbb{D}.
\]
If a weight \( \nu \) satisfies the condition \( \tilde{\nu}(r) \leq C\tilde{\nu}\left(\frac{1-r}{2}\right) \) for all \( 0 \leq r < 1 \) and some \( C = C(\nu) > 0 \), then we write \( \nu \in \tilde{\mathcal{D}} \). Correspondingly, \( \nu \in \mathcal{D} \) if there exist \( K = K(\nu) > 1 \) and \( C = C(\nu) > 1 \) such that
\[
\tilde{\nu}(r) \geq C\tilde{\nu}\left(1 - \frac{1-r}{K}\right), \quad 0 \leq r < 1.
\]
Class \( \mathcal{D} \) is the intersection of \( \tilde{\mathcal{D}} \) and \( \mathcal{D} \). In addition, we define the following subclass of \( \tilde{\mathcal{D}} \): \( \nu \in \tilde{\mathcal{D}}_p \) for some \( p \in (0, \infty) \) if the condition
\[
\tilde{\mathcal{D}}_p(\nu) = \sup_{0 \leq r < 1} \left(1 - r\right)^p \tilde{\nu}(r) \int_0^r \frac{\nu(s)}{(1-s)^p} \, ds < \infty \tag{1.1}
\]
is satisfied. As a concrete example, we mention that \( \nu_1(z) = (1 - |z|)^\alpha \) and \( \nu_2(z) = (1 - |z|)^\alpha \left(\log \frac{e}{|z|}\right)^\beta \) for any \( \beta \in \mathbb{R} \) belong to \( \mathcal{D} \cap \tilde{\mathcal{D}}_p \) if and only if \(-1 < \alpha < p - 1 \). Additional information about weights can be found in [18, 19, 20]. Some basic properties are recalled also in Section 2.

Define the weighted Besov space \( B_\nu^{p,q} \) by \( B_\nu^{p,q} = \{ f : f' \in A_\nu^{p,q} \} \cap H^p \). For \(-1 < \alpha < \infty \) and \( \nu(z) = (1 - |z|)^\alpha \), the notation \( B_\nu^{p,q} \) is used for \( B_\nu^{p,q} \). The space \( B_\nu^{p,q} \) is the main research objective of this note. Hence it is worth pointing out that the definition is rational, which means that \( H^p \) is not a subset of \( \{ f : f' \in A_\nu^{p,q} \} \) in general, or conversely. The family of Blaschke products offers examples for the case where \( f \in H^\infty \) and \( f' \notin A_\nu^{p,q} \); see for instance [24]. Moreover, it would be natural that certain lacunary series \( g \) lie out of \( H^p \), while \( g' \in A_\nu^{p,q} \).

Arguments for this kind of examples can be found in M. Pavlović’s book [15], which contains numerous important observations on the topic of this note. The existence of both examples, of course, depends on \( p, q \) and \( \nu \). In other words, under certain hypotheses for \( p, q \) and \( \nu \), an inclusion relation between \( \{ f : f' \in A_\nu^{p,q} \} \) and \( H^p \) might be valid. However, this is not the case in general.

For \( 0 < p < \infty \) and \( f \in L^p(\mathbb{T}) \), the \( L^p \) modulus of continuity \( \omega_p(t, f) \) is defined by
\[
\omega_p(t, f) = \sup_{0 < h < t} \left( \int_0^{2\pi} |f(e^{i(\theta + h)}) - f(e^{i\theta})|^p \, d\theta \right)^{1/p}, \quad 0 < t \leq 2\pi.
\]
We interpret \( \omega_p(t, f) = \omega_p(2\pi, f) \) for \( t > 2\pi \). It is a well-known fact that, for \( 0 < p, q < \infty \), \(-1 < \alpha < q - 1 \) and \( f \in H^p \), the derivative of \( f \) belongs to \( A_\alpha^{p,q} \) if and only if
\[
\int_0^\infty \frac{\omega_p(t, f)^q}{t^{\alpha - q}} \, dt < \infty.
\]

This result originates from E. M. Stein’s book [26, Chapter V, Section 5], and the complete version is a consequence of [17, Theorems 2.1 and 5.1] or [12, Theorem 1.2] by M. Pavlović and M. Jevtić. Our first theorem is a partial generalization of the result. Its proof uses some ideas from [8, 21, 22].

**Theorem 1.** Let \( 1 \leq p < \infty \), \( 0 < q < \infty \) and \( \nu \in \mathcal{D} \). Then \( \nu \in \tilde{\mathcal{D}}_p \) if and only if there exists a constant \( C = C(p, q, \nu) > 0 \) such that
\[
\int_{1/2}^1 \omega_p(1 - r, f)^q \frac{\nu(r)}{(1 - r)^q} \, dr \leq C \| f' \|_{A_\nu^{p,q}} \tag{1.2}
\]
for all \( f \in H^p \).
Note that (1.2) is valid also if $0 < p, q < \infty$, $\nu$ is a weight and there exists $\beta = \beta(q, \nu) < q - 1$ such that $\nu(r)/(1 - r)^{\beta}$ is increasing for $0 \leq r < 1$. This is due to [17, Theorem 5.1] and its proof. Even though the result is valid also for $0 < p < 1$, Theorem 1 is more useful for our purposes. In particular, it is worth underlining that the hypothesis 1 takes the form. Even

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Note that (1.2) is valid also if $0 < p, q < \infty$ and $\nu \in \mathcal{D} \cap \mathcal{D}_q$. Then there exist positive constants $C_1$ and $C_2$ depending only on $p, q$ and $\nu$ such that

\[
\|f\|_{A^{p,q}_0} \leq C_1 \left( \int_0^1 \left( \int_0^{2\pi} |f(e^{i\theta})| \, d\mu_{r\nu}(\theta) \right)^p \, d\nu \right)^{1/p} \|f\|_{H^p},
\]

for all $f \in H^p$.

By studying the classical weight $\nu(z) = (1 - |z|)^{\alpha}$, where $-1 < \alpha < q - 1$, we obtain K. M. Dyakonov’s [9, Proposition 2.2(a)] as a direct consequence of Theorem 2. Hence it does not come as a surprise that the proofs of [9, Theorem 2.1] and Theorem 2 have some similarities. Nonetheless, it is worth mentioning that the presence of general weights complicates the argument; and consequently, our proof is quite technical. Note also that Theorem 2 plays an essential role in the proof.

Our main result below gives a characterization for functions $f$ in $B^{p,q}_0$ depending only $|f|$, $p$, $q$ and $\nu$. This result improves B. Boe’s [3, Theorem 1.1], which concentrates only on the case where $1 \leq p, q < \infty$, $-1 < \alpha < q - 1$ and $\nu(z) = (1 - |z|)^{\alpha}$. It also generalizes the essential contents of [2, Proposition 2.4] and [9, Proposition 2.2(b)] made by A. Aleman and K. M. Dyakonov, respectively.

**Theorem 3.** Let $1 \leq p < \infty$, $0 < q < \infty$ and $\nu \in \mathcal{D} \cap \mathcal{D}_q$. Then there exist positive constants $C_1$ and $C_2$ depending only on $p, q$ and $\nu$ such that

\[
\|f\|_{A^{p,q}_0} \leq C_1 \left( F_1(f) + F_2(f) \right) \leq C_2 \left( \|f\|_{A^{p,q}_0} + \|f\|_{H^p} \right), \quad f \in H^p,
\]

where

\[
F_1(f) = \int_0^1 \left( \int_0^{2\pi} \left| f(e^{i\theta}) \right| d\mu_{r\nu}(\theta) \right)^p \, d\nu \frac{\nu(r)}{(1 - r)^q} \, dr
\]

and

\[
F_2(f) = \int_0^1 \left( \int_0^{2\pi} \left| f(e^{i\theta}) \right| - \left| \int_0^{2\pi} f(e^{i\theta}) \, d\mu_{r\nu}(\theta) \right| \right)^p \, d\nu \frac{\nu(r)}{(1 - r)^q} \, dr.
\]

Before we talk about the argument of Theorem 3, recall the inner-outer factorization. An inner function is a member of $H^\infty$ having unimodular radial limits almost everywhere on $\mathbb{T}$. For $0 < p \leq \infty$, an outer function for $H^p$ takes the form

\[
O_\phi(z) = \exp \left( \frac{1}{2\pi} \int_0^{2\pi} e^{i\theta} + z}{e^{i\theta} - z} \log \phi(e^{i\theta}) \, d\theta \right), \quad z \in \mathbb{D},
\]
where $\phi$ is a non-negative function in $L^p(\mathbb{T})$ and $\log \phi \in L^1(\mathbb{T})$. The inner-outer factorization asserts that $f \in H^p$ can be represented as the product of an inner and outer function; see for instance [8, Theorem 2.8]. It is worth noting that the factorization is unique, and

$$|f(\xi)| = |O_\phi(\xi)| = \phi(\xi)$$  \hspace{1cm} (1.5)

for almost every $\xi \in \mathbb{T}$ if $O_\phi$ is the outer function from the factorization of $f$. Equation (1.5) is due to the definition of inner functions, Poisson integral formula, harmonicity of $\log |O_\phi(z)|$ and fact that

$$|O_\phi(z)| = \exp \left( \int_0^{2\pi} \log \phi(e^{i\theta}) \, d\mu_\phi(\theta) \right), \quad z \in \mathbb{D}.$$ 

The last inequality in (1.4) can be proved by applying Theorem 2. In the argument of the first inequality, the inner-outer factorization, Schwarz-Pick lemma and an upper estimate for $|O_\phi|$ from [6] are the main tools. It is worth underlining that this Bôe’s idea to make an upper estimate for $|f'|$ by using the factorization seems to be quite effective. Another way to prove results like Theorem 3 is to use a modification of Theorem 2 together with the well-known equation

$$\int_0^{2\pi} |f(e^{i\theta}) - f(z)|^2 \, d\mu_\phi(\theta) = \int_0^{2\pi} |f(e^{i\theta})|^2 \, d\mu_\phi(\theta) - |f(z)|^2, \quad z \in \mathbb{D};$$

but this Dyakonov’s method has the obvious defect that it works only when $f \in H^2$. The advantage of this method in the case where $2 \leq p < \infty$, $0 < q < \infty$, $q/2 - 1 < \alpha < q - 1$ and $\nu(z) = (1 - |z|)^\alpha$ is that $F_1(f) + F_2(f)$ in Theorem 3 can be replaced by

$$\int_0^1 \left( \int_0^{2\pi} \left( \int_0^{2\pi} |f(e^{i\theta})|^2 \, d\mu_\nu(\theta) - |f(re^{i\theta})|^2 \right)^{q/2} \, dt \right) \frac{\nu(r)}{(1-r)^q} \, dr,$$

see [9, Proposition 2.2(b)]. It is an open problem to prove a corresponding estimate for general weights.

Next we give an example which shows that the hypothesis $\nu \in \mathcal{D} \cap \mathcal{D}_q$ in Theorem 3 for $p \geq 2$ is sharp in a certain sense. Note that the example is a modification of [22, Example 8]. Before the statement we fix some notation. Write $f \lesssim g$ if there exists a constant $C > 0$ such that $f \leq C g$, while $f \asymp g$ is understood analogously. If $f \leq g$ and $f \geq g$, then we write $f \asymp g$.

**Example 4.** Let $2 \leq p < \infty$, $q = p$, $\nu(z) = (1 - |z|)^{p-1}$ and $A(z)$ be the two-dimensional Lebesgue measure $dx dy$. Let $f$ be an inner function such that

$$\int_{\{|z|: |f(z)| < \varepsilon\}} \frac{dA(z)}{1 - |z|^2} = \infty$$

for some $\varepsilon \in (0, 1)$. The existence of such $f$ is guaranteed by [7, Theorem 5]. Then

$$F_1(f) + F_2(f) = F_1(f) \asymp \int_{\mathbb{D}} (1 - |f(z)|)^p (1 - |z|)^{-1} \, dA(z) = \infty,$$

while

$$\|f'\|_{A^{p,p}}^q + \|f'\|_{H^p}^q = \|f'\|_{A^{p,p-1}}^p + 1 < \infty$$

by the well-known inclusion

$$H^p \subset \{ g : g' \in A^{p,p} \}, \quad 2 \leq p < \infty,$$

which originates from [13].

We close the section by explaining how the remainder of this note is organized. Auxiliary results on weights are recalled in the next section. The utility of Theorem 3 is demonstrated in Sections 3 and 4. More precisely, in Section 3 we prove the factorization which states that, for any $f \in B^{p,q}_0$, there exist $f_1, f_2 \in B^{p,q}_0 \cap H^\infty$ such that $f = f_1/f_2$. Section 4 begins with a result giving a sufficient and necessary condition guaranteeing that the product of $f \in H^p$
and an inner function belongs to $\mathcal{B}_p^q$. As a consequence of this theorem, we obtain some results on zero sets of $\mathcal{B}_p^q$. Sections 3 and 4 consist of the proofs of Theorems 12 and 13 respectively.

2. Auxiliary results on weights

In this section, we recall some basic properties of weights in $\mathcal{D}$ and $\tilde{\mathcal{D}}$. These properties are needed in next sections. Another reason for these results is to help the reader to understand the nature of weights in $\mathcal{D}$. We begin with a result which is essentially [19, Lemma 3]; see also [18, Lemma 2.1].

**Lemma A.** Let $\nu$ be a weight. Then the following statements are equivalent:

(i) $\nu \in \tilde{\mathcal{D}}$.

(ii) There exist $C = C(\nu) > 0$ and $\beta = \beta(\nu) > 0$ such that

$$\hat{\nu}(r) \leq C \left( \frac{1 - r}{1 - s} \right)^{\beta} \hat{\nu}(s), \quad 0 \leq r \leq s < 1.$$

(iii) There exist $C = C(\nu) > 0$ and $\gamma = \gamma(\nu) > 0$ such that

$$\int_0^r \left( \frac{1 - r}{1 - s} \right)^{\gamma} \nu(s) \, ds \leq C \hat{\nu}(r), \quad 0 \leq r < 1.$$

(iv) The estimate

$$\int_0^1 s^\alpha \nu(s) \, ds \leq \hat{\nu} \left( 1 - \frac{1}{x} \right), \quad 1 \leq x < \infty,$$

is satisfied.

For the point view of our main results Lemma A(iii) is interesting because it states that $\nu \in \tilde{\mathcal{D}}$ if and only if $\nu \in \tilde{\mathcal{D}}_p$ for some $p > 0$. This means that $\tilde{\mathcal{D}} = \bigcup_{p > 0} \tilde{\mathcal{D}}_p$. Nevertheless, Lemma A(ii) gives maybe the most interesting description for $\tilde{\mathcal{D}}$. Together with its $\mathcal{D}$ counterpart below it offers a very practical characterization for weights in $\mathcal{D}$. Essentially this characterization says that $\hat{\nu}$ is normal in the sense of A. L. Shields and D. L. Williams [25].

**Lemma B.** Let $\nu$ be a weight. Then $\nu \in \mathcal{D}$ if and only if there exist $C = C(\nu) > 0$ and $\alpha = \alpha(\nu) > 0$ such that

$$\hat{\nu}(s) \leq C \left( \frac{1 - s}{1 - r} \right)^{\alpha} \hat{\nu}(r), \quad 0 \leq r \leq s < 1.$$

Lemma B originates from [21], and it can be proved in a corresponding manner as Lemma A(ii). See in particular the proof of [18, Lemma 2.1].

By the definition of class $\tilde{\mathcal{D}}_p$, it is clear that $\tilde{\mathcal{D}}_p \subset \tilde{\mathcal{D}}_{p+\varepsilon}$ for any $\varepsilon > 0$. Next we state [21, Lemma 3], which shows that also the converse inclusion is true for sufficiently small $\varepsilon = \varepsilon(\nu, p) > 0$. The proof of this result is based on integration by parts. Note that $\tilde{\mathcal{D}}_p(\nu)$ in the statement is defined by (1.1).

**Lemma C.** If $0 < p < \infty$ and $\nu \in \tilde{\mathcal{D}}_p$, then $\nu \in \tilde{\mathcal{D}}_{p-\varepsilon}$ for any $\varepsilon \in \left( 0, \frac{p}{\mathcal{D}_p(\nu) + 1} \right)$.

The last result of this section is [22, Lemma 5], which shows that $\nu \in \mathcal{D}$ in the norm $\|f\|_{\mathcal{A}^p_\nu}$ can be replaced by $\hat{\nu}(z)/(1 - |z|)$ without losing any essential information.

**Lemma D.** Let $0 < p, q < \infty$ and $\nu$ be a weight.

(i) If $\nu \in \mathcal{D}$, then there exists $C = C(\nu) > 0$ such that

$$\|f\|_{\mathcal{A}^q_\nu} \geq C \int_0^1 M_\nu^q(r, f) \frac{\hat{\nu}(r)}{1 - r} \, dr, \quad f \in \mathcal{H}(\mathbb{D}).$$
We prove the following lemma.

\textbf{Lemma D.} Let \( f \in H^p \) for some \( p > 0 \). Then there exist \( f_1, f_2 \in H^\infty \) such that \( f = f_1/f_2 \). This is an important consequence of classical factorization \([3, \text{Theorem 2.1}]\) by F. and R. Nevanlinna.

The main purpose of this section is to give the following \( B^p \) counterpart for the above-mentioned result.

\textbf{Theorem 5.} Let \( 1 \leq p < \infty \), \( 0 < q < \infty \) and \( \nu \in \mathcal{D} \cap \hat{\mathcal{D}}_q \). If \( f \in B^p \nu \), then there exist \( f_1, f_2 \in B^p \nu \cap H^\infty \) such that \( f = f_1/f_2 \) and \( f_2 \) is an outer function.

It is worth mentioning that \([10, \text{Theorem 9.19}]\) is a similar type of result as \textbf{Theorem 5} with a different hypothesis for \( \nu \). Moreover, we note that \textbf{Theorem 5} generalizes \([2, \text{Corollary 2.7}]\), \([6, \text{Theorem 3.4}]\) and \([9, \text{Corollary 3.4}]\). For its argument we need an extension of \([6, \text{Theorem 3.3}]\). Note that a part of the next pages is really inspired by \([6]\).

\textbf{Proposition 6.} Let \( 1 \leq p < \infty \), \( 0 < q < \infty \), \( \nu \in \mathcal{D} \cap \hat{\mathcal{D}}_q \) and \( f \in H^p \) be the product of an inner function \( I \) and an outer function \( O_\phi \). Then there exists a constant \( C = C(p, q, \nu) > 0 \) such that

\[
\|O'_{\max\{\phi,1\}}\|_{\mathcal{A}^q_{p,q}} + \|(IO_{\min\{\nu,1\}})'\|_{\mathcal{A}^q_{p,q}} + \|O_{\max\{\phi,1\}}\|_{H^p} \leq C \left( \|f\|_{\mathcal{A}^q_{p,q}} + \|f\|_{H^p} + 1 \right).
\]

Before the proof of \textbf{Proposition 6} we note that the quantities \( F_1(f) \) and \( F_2(f) \) in \textbf{Theorem 3} are used repeatedly in the future.

\textbf{Proof.} Let us begin by noting that \( |O_\phi(e^{i\theta})| = \phi(e^{i\theta}) \), \( |O_{\max\{\phi,1\}}(e^{i\theta})| = \max\{\phi(e^{i\theta}),1\} \) and

\[
\max\{\phi(e^{i\theta}),1\} - \phi(e^{i\theta}) = \frac{\max\{\phi(e^{i\theta}),1\} - \phi(e^{i\theta})}{\max\{\phi(e^{i\theta}),1\}} \leq |O_{\max\{\phi,1\}}(z)| \left( 1 - \frac{\phi(e^{i\theta})}{\max\{\phi(e^{i\theta}),1\}} \right)
\]

for all \( z \in \mathbb{D} \) and almost every \( \theta \in [0, 2\pi) \). Using these facts together with Jensen’s inequality \([10, \text{Chapter I, Lemma 6.1}]\) and the definition of outer functions, we obtain

\[
\int_0^{2\pi} |O_{\max\{\phi,1\}}(e^{i\theta})| \, d\mu_z(\theta) - \int_0^{2\pi} |O_\phi(e^{i\theta})| \, d\mu_z(\theta) \\
\leq |O_{\max\{\phi,1\}}(z)| \left( 1 - \int_0^{2\pi} \frac{\phi(e^{i\theta})}{\max\{\phi(e^{i\theta}),1\}} \, d\mu_z(\theta) \right) \\
= |O_{\max\{\phi,1\}}(z)| \left( 1 - \int_0^{2\pi} \exp \left( \log \phi(e^{i\theta}) - \log \max\{\phi(e^{i\theta}),1\} \right) \, d\mu_z(\theta) \right) \\
\leq |O_{\max\{\phi,1\}}(z)| \left( 1 - \frac{|O_\phi(z)|}{|O_{\max\{\phi,1\}}(z)|} \right), \quad z \in \mathbb{D}.
\]
Consequently, the obvious inequality $|f(z)| \leq |O_p(z)|$ yields

$$\int_0^{2\pi} |O_{\max(\phi,1)}(e^{i\theta})| d\mu_z(\theta) - |O_{\max(\phi,1)}(z)| \leq \int_0^{2\pi} |f(e^{i\theta})| d\mu_z(\theta) - |f(z)|, \quad z \in \mathbb{D}. \quad (3.1)$$

Write $z = re^{it}$. Raising both sides of (3.1) to power $p$, integrating from 0 to $2\pi$ with respect to $dt$, then raising both sides to power $q/p$ and finally integrating from 0 to 1 with respect to $\nu(r)dr/(1 - r)^q$, we obtain $F_1(\max(\phi,1)) \leq F_1(f)$.

Next we show $F_2(\max(\phi,1)) \leq F_2(f)$. Set

$$\Gamma_1 = \Gamma_1(z, \phi) = \left\{ \theta \in [0, 2\pi) : \int_0^{2\pi} \max\{\phi(e^{is}), 1\} d\mu_z(s) \leq \phi(e^{i\theta}) \right\}$$

and

$$\Gamma_2 = \Gamma_2(z, \phi) = \left\{ \theta \in [0, 2\pi) : \int_0^{2\pi} \phi(e^{is}) d\mu_z(s) \leq \phi(e^{i\theta}) \right\}, \quad z \in \mathbb{D}.$$

Then elementary calculations yield

$$\int_0^{2\pi} \max\{\phi(e^{i\theta}), 1\} - \int_0^{2\pi} \max\{\phi(e^{is}), 1\} d\mu_z(s) \leq \phi(e^{i\theta}) \quad (3.2)$$

Consequently, we obtain $F_2(\max(\phi,1)) \leq F_2(\phi) = F_2(f)$ by doing a similar integral procedure as above. Now Theorem 3 together with the inequalities for $F_1(f)$ and $F_2(f)$ gives

$$\|O'_{\max(\phi,1)}\|_{L^q(\mathbb{C})}^q + \|O_{\max(\phi,1)}\|_{L^p}^{q} \leq \|f\|_{L^q(\mathbb{C})}^q + \|f\|_{L^p}^{q} + 1. \quad (3.3)$$

By (3.3) it suffices to show

$$\|\left( IO_{\min(\phi,1)} \right)'\|_{L^q(\mathbb{C})}^q \leq \|f\|_{L^q(\mathbb{C})}^q + \|f\|_{L^p}^{q}. \quad (3.4)$$

Since

$$\phi(e^{i\theta}) - \min\{\phi(e^{is}), 1\} \geq \min\{\phi(e^{i\theta}), 1\} \left( \frac{\phi(e^{i\theta})}{\min\{\phi(e^{is}), 1\}} - 1 \right),$$

we obtain

$$\int_0^{2\pi} |\phi(e^{i\theta})| d\mu_z(\theta) - |\phi(z)| \geq \int_0^{2\pi} |\min\{\phi(e^{is}), 1\}| d\mu_z(\theta) - |\min\{\phi, 1\}(z)|, \quad z \in \mathbb{D},$$

by arguing as above using Jensen’s inequality. It follows that

$$\int_0^{2\pi} |f(e^{i\theta})| d\mu_z(\theta) - |f(z)| = \left( \int_0^{2\pi} |\phi(e^{i\theta})| d\mu_z(\theta) - |\phi(z)| \right) + |\phi(z)|(1 - |I(z)|)$$

$$\geq \left( \int_0^{2\pi} |\min\{\phi(e^{is}), 1\}| d\mu_z(\theta) - |\min\{\phi, 1\}(z)| \right) + |\min\{\phi, 1\}(z)|(1 - |I(z)|)$$

$$= \int_0^{2\pi} |\min\{\phi(e^{is}), 1\}| d\mu_z(\theta) - |\min\{\phi, 1\}(z)|, \quad z \in \mathbb{D}.$$
Hence it is easy to deduce $F_1(IO_{\min(\phi,1)}) \leq F_1(f)$. Since
\[ F_2(IO_{\min(\phi,1)}) = F_2(O_{\min(\phi,1)}) \leq F_2(O_\phi) = F_2(f) \]
can be shown by using a modification of (3.2), the desired estimate (3.4) follows from Theorem 3. This completes the proof.

Now we can easily prove Theorem 5 by using Proposition 6.

Proof of Theorem 5. By the inner-outer factorization, there exist an inner function $I$ and an outer function $O_\phi$ such that $f = IO_\phi$. Since $O_\phi = O_{\min(\phi,1)}O_{\max(\phi,1)}$, we have $f = f_1/f_2$, where $f_1 = IO_{\min(\phi,1)}$ and $f_2 = 1/O_{\max(\phi,1)}$. Applying Proposition 6 together with the inequalities
\[ |O_{\min(\phi,1)}(z)| \leq 1 \leq |O_{\max(\phi,1)}(z)| \]
and
\[ |f_2'(z)| \leq |O_{\max(\phi,1)}(z)|^2f_2'(z) = |O_{\max(\phi,1)}(z)|, \quad z \in \mathbb{D}, \]
we can check that $f_1$ and $f_2$ belong to $\mathcal{B}^{p,q}_{0} \cap H^\infty$. Moreover, it is obvious that $f_2$ is an outer function. Hence the proof is complete.

4. Product of $f \in H^p$ and an inner function in $\mathcal{B}^{p,q}_{0}$

Theorem 7 below gives a sufficient and necessary condition guaranteeing that the product of $f \in H^p$ and an inner function belongs to $\mathcal{B}^{p,q}_{0}$. This result generalizes [3, Corollary 3.2], the essential contents of [6, Corollary 3.1] and [9, Theorem 3.2].

Theorem 7. Let $1 \leq p < \infty$, $0 < q < \infty$, $\nu \in \mathcal{D} \cap \hat{\mathcal{D}}_q$, $f \in H^p$ and $I$ be an inner function. Then $fI \in \mathcal{B}^{p,q}_{0}$ if and only if $f \in \mathcal{B}^{p,q}_{0}$ and
\[ \int_0^1 \left( \int_0^{2\pi} \left( \frac{|f(re^{it})|(1 - |I(re^{it})|)}{1 - r} \right)^p dt \right)^{q/p} \nu(r) dr < \infty. \]

Proof. We have
\[ \int_0^{2\pi} |fI(e^{it})|d\mu_z(\theta) - |fI(z)| = \left( \int_0^{2\pi} |f(e^{it})|d\mu_z(\theta) - |f(z)| \right) + |f(z)|(1 - |I(z)|) \]
for all $z \in \mathbb{D}$. Write $z = re^{it}$. Raising both sides of (4.1) to power $p$, integrating from 0 to $2\pi$ with respect to $dt$, then raising both sides to power $q/p$, integrating from 0 to 1 with respect to $\nu(r)dr/(1-r)^q$ and finally splitting the right-hand side into two parts by using well-known inequalities, we obtain
\[ F_1(fI) = F_1(f) + \int_0^1 \left( \int_0^{2\pi} \left( \frac{|f(re^{it})|(1 - |I(re^{it})|)}{1 - r} \right)^p dt \right)^{q/p} \nu(r) dr. \]
Since
\[ F_2(fI) + \|fI\|_H^q = F_2(f) + \|f\|_H^q, \]
the assertion follows from Theorem 5.

Recall that a subspace $X$ of $H^p$ satisfies the $F$-property if the hypothesis $fI \in X$, where $f \in H^p$ and $I$ is an inner function, implies $f \in X$. The $F$-property for $\mathcal{B}^{p,q}_{0}$ is a direct consequence of Theorem 7. However, it is worth mentioning that if one just aims to prove the $F$-property for $\mathcal{B}^{p,q}_{0}$, our argument is not maybe the simplest one, taking into account the length of proofs of Theorem 5 and its auxiliary results. Ideas for an alternative proof can be found, for instance, in [16, Section 5.8.3].

A sequence $\{z_n\} \subset \mathbb{D}$ is said to be a zero set of $\mathcal{B}^{p,q}_{0}$ if there exists $f \in \mathcal{B}^{p,q}_{0}$ such that $\{z : f(z) = 0\} = \{z_n\}$. Here each zero $z_n$ is repeated according to its multiplicity and function $f$ is not identically zero. Applying Theorem 5, we make some observations on zero sets of
More precisely, we concentrate on the case where \( \{z_n\} \) is separated, which means that there exists \( \delta = \delta(\{z_n\}) > 0 \) such that \( d(z_n, z_k) > \delta \) for all \( n \neq k \), where

\[
d(z, w) = \left| \frac{z - w}{1 - \overline{z}w} \right|, \quad z, w \in \mathbb{D},
\]

is the pseudo-hyperbolic distance between points \( z \) and \( w \). Before these results some basic properties of Hardy spaces are recalled.

For \( \{z_n\} \subset \mathbb{D} \) satisfying the Blaschke condition \( \sum_n (1 - |z_n|) < \infty \) and a point \( \theta \in [0, 2\pi) \), the Blaschke product with zeros \( \{z_n\} \) is defined by

\[
B(z) = e^{i\theta} \prod_n \left| \frac{z_n - z}{z_n \overline{z}_n} \right|, \quad z \in \mathbb{D}.
\]

For \( z_n = 0 \), the interpretation \( |z_n|/z_n = -1 \) is used. By factorization [8, Theorem 2.5] made by F. Riesz, we know that any \( f \in H^p \) for some fixed \( p \in (0, \infty) \) can be represented in the form \( f = Bg \), where \( B \) is a Blaschke product and \( g \in H^p \) does not vanish in \( \mathbb{D} \). More precisely, Beurling factorization [8, Theorem 2.8] asserts that \( g \) is the product of an outer function and a singular inner function

\[
S(z) = \exp \left( \int_{\mathbb{T}} \frac{z + \xi}{z - \xi} d\sigma(\xi) + i\theta \right), \quad z \in \mathbb{D},
\]

where \( \theta \in [0, 2\pi) \) is a constant and \( \sigma \) a positive measure on \( \mathbb{T} \), singular with respect to the Lebesgue measure. Consequently, every zero set of \( B^{p,q}_v \) satisfies the Blaschke condition. With these preparations we are ready to state and prove the following result.

**Corollary 8.** Let \( 1 \leq p < \infty \), \( \nu \in \mathbb{D} \cap \mathbb{D}_p \), and assume that \( \{z_n\} \) is a finite union of separated sequences and zero set of \( B^{p,q}_v \). Then there exists an outer function \( O_\phi \in B^{p,q}_v \) such that

\[
\sum_n |O_\phi(z_n)|^p \frac{\nu(z_n)}{(1 - |z_n|)^{p-1}} < \infty.
\]

**Proof.** Let \( \{z_n\} = \bigcup_{j=1}^M \{z_n^j\} \), where \( M \in \mathbb{N} \) and each \( \{z_n^j\} \) is separated. Let \( B \) be the Blaschke product with zeros \( \{z_n\} \), \( S \) a singular inner function and \( O_\phi \) an outer function such that \( BSO_\phi \in B^{p,q}_v \). By Theorem [7] we know that \( O_\phi \) and \( BO_\phi \) belong to \( B^{p,q}_v \). For \( w \in \mathbb{D} \) and \( 0 < r < 1 \), set

\[
\Delta(w, r) = \{ z : d(z, w) < r \} \quad \text{and} \quad \Lambda(w, r) = \{ z : |w - z| < r(1 - |w|) \}.
\]

Since each \( \{z_n^j\} \) is separated, we find \( R_j, \delta_j \in (0, 1) \) such that, for a fixed \( j \), discs \( \Lambda(z_n^j, R_j) \) are pairwise disjoint and the inclusion \( \Delta(z_n^j, \delta_j) \subset \Lambda(z_n^j, R_j) \) is valid for every \( n \). Hence \( \hat{\nu} \) is essentially constant in each disc \( \Delta(z_n^j, \delta_j) \) by Lemma [A][ii]). Moreover,

\[
|B(z)| \leq \left| \frac{z_n^j - z}{1 - \overline{z}_n^j z} \right| \leq \delta_j, \quad z \in \Delta(z_n^j, \delta_j).
\]
Using these facts together with the subharmonicity of $|O_\phi|^p$, we obtain

$$
\sum_n |O_\phi(z_n)|^p \frac{\hat{\nu}(z_n)}{(1 - |z_n|)^{p-1}} = \sum_{j=1}^M \sum_n |O_\phi(z_n^j)|^p \frac{\hat{\nu}(z_n^j)}{(1 - |z_n^j|)^{p-1}} \\
\leq \sum_{j=1}^M \sum_n \int_{\Delta(z_n^j, \delta_j)} |O_\phi(z)|^p \frac{\hat{\nu}(z)}{(1 - |z|)^{p+1}} dA(z) \\
= \sum_{j=1}^M \sum_n \int_{\Delta(z_n, \delta_j)} |O_\phi(z)|^p \frac{\hat{\nu}(z)}{(1 - |z|)^{p+1}} dA(z) \\
\leq \sum_{j=1}^M (1 - \delta_j)^{-p} \sum_n \int_{\Delta(z_n, \delta_j)} \left( \frac{|O_\phi(z)| (1 - |B(z)|)}{1 - |z|} \right)^p \frac{\hat{\nu}(z)}{(1 - |z|)} dA(z) \\
\leq \int_{\mathbb{D}} \left( \frac{|O_\phi(z)| (1 - |B(z)|)}{1 - |z|} \right)^p \frac{\hat{\nu}(z)}{(1 - |z|)} dA(z),
$$

where $dA(z)$ is the two-dimensional Lebesgue measure. Now it suffices to show that the last integral in (4.2) is finite.

Set $\psi(z) = \hat{\nu}(z)/(1 - |z|)$ for $z \in \mathbb{D}$. Note that $\hat{\nu}(r) = \psi(r)$ for $0 \leq r < 1$ by Lemmas [A] ii) and [B]. Moreover, integrating by parts, one can show that $\nu \in \mathcal{D}_p$ if and only if

$$
\frac{(1 - r)^p}{\psi(r)} \int_0^r \frac{\hat{\nu}(s)}{(1 - s)^{p+1}} ds \approx 1, \quad r \to 1^-.
$$

In particular, $\psi \in \mathcal{D} \cap \mathcal{D}_p$ by the hypotheses of $\nu$. Since Lemma [D] implies $BO_\phi$ in $\mathcal{B}^{p,p}_\psi$, Theorem [I] gives

$$
\int_{\mathbb{D}} \left( \frac{|O_\phi(z)| (1 - |B(z)|)}{1 - |z|} \right)^p \frac{\hat{\nu}(z)}{(1 - |z|)} dA(z) < \infty.
$$

This completes the proof. \qed

Recall that a sequence $\{z_n\} \subset \mathbb{D}$ is said to be uniformly separated if

$$
\inf_{n \in \mathbb{N}} \prod_{k \neq n} \frac{|z_k - z_n|}{1 - \overline{z}_k z_n} > 0;
$$

and a finite union of uniformly separated sequences is called a Carleson-Newman sequence. It is worth mentioning that any Carleson-Newman sequence is a finite union of separated sequences satisfying the Blaschke condition, but the converse statement is not true. For $1 < p < \infty$, $p - 2 < \alpha < p - 1$ and a Carleson-Newman sequence $\{z_n\}$, we can give a sufficient and necessary condition for $\{z_n\}$ to be a zero set of $\mathcal{B}_\alpha^{p,p}$. This is a straightforward consequence of Theorem [7] Corollary [8] and the reasoning made in paper [4] by N. Arcozzi, D. Blasi and J. Pau.

**Corollary 9.** Let $1 < p < \infty$, $p - 2 < \alpha < p - 1$ and $\{z_n\}$ be a Carleson-Newman sequence. Then $\{z_n\}$ is a zero set of $\mathcal{B}_\alpha^{p,p}$ if and only if there exists an outer function $O_\phi \in \mathcal{B}_\alpha^{p,p}$ such that

$$
\sum_n |O_\phi(z_n)|^p (1 - |z_n|)^{\alpha + 2 - p} < \infty. \quad (4.4)
$$
Proof. Let $B$ be the Blaschke product with zeros $\{z_n\}$ and $O_\phi \in B^{p,p}_\alpha$ an outer function satisfying (4.3). Then [14] Theorem 3.5 together with some elementary calculations gives
\[
\int_{\mathbb{D}} \left( \frac{|O_\phi(z)|(1-|B(z)|)}{1-|z|} \right)^p (1-|z|)^\alpha \, dA(z) 
\leq 2 \int_{\mathbb{D}} |O_\phi(z)|^p \sum_n \frac{1-|z_n|^2}{|1-z_n|} \left( 1-|z| \right)^{\alpha+1-p} \, dA(z) 
\leq \sum_n \int_{\mathbb{D}} |O_\phi(z_n)|^p \frac{1-|z_n|^2}{|1-z_n|} \left( 1-|z| \right)^{\alpha+1-p} \, dA(z) 
\leq \sum_n \int_{\mathbb{D}} |O_\phi(z_n)-O_\phi(z)|^p \frac{1-|z_n|^2}{|1-z_n|} \left( 1-|z| \right)^{\alpha+1-p} \, dA(z) 
=: I_1 + I_2.
\]
Following the reasoning in the proof of [4, Proposition 3.2], it is easy to check that $I_1$ and $I_2$ are finite. More precisely, estimating in a natural manner, one can show
\[
I_1 \leq \sum_n |O_\phi(z_n)|^p (1-|z_n|)^{\alpha+2-p} < \infty.
\]
In the argument of $I_2 \leq \|O_\phi\|_{A^{p,p}}^p < \infty$, [5] Lemma 2.1 and the hypothesis that $\{z_n\}$ is a Carleson-Newman sequence play key roles.

Since $O_\phi \in B^{p,p}_\alpha$ and the first integral in (4.5) is finite, $BO_\phi$ belongs to $B^{p,p}_\alpha$ by Theorem 7. Consequently, the implication $\Leftarrow$ is valid. The converse implication is a direct consequence of Corollary 8. Hence the proof is complete. \qed

It is an open problem to prove a $B^{p,p}_\alpha$ counterpart of Corollary 9. One could try prove such result, for instance, assuming $\nu \in D \cap D^p$ and
\[
\sup_{0 \leq r < 1} \frac{(1-r)^{p-1}}{\nu(r)} \int_r^1 \frac{\nu(s)}{(1-s)^{p-1}} \, ds < \infty.
\]
In this case, the implication $\Leftarrow$ is the problematic part. An idea to approach this problem is to follow the argument of [4, Proposition 3.2] and aim to apply therein [3, Theorem 3.1] instead of [5, Lemma 2.1]. The down side of this method is that it leads to laborious computations of Bekollé-Bonami weights.

Corollaries 8 and 9 are related to some main results in [15] by J. Pau and J. A. Peláez. In particular, the equivalence (i) $\Rightarrow$ (ii) in [15] Theorem 1 follows from Corollary 9 by setting $p = 2$. Moreover, Corollary 8 shows that the implication (i) $\Rightarrow$ (ii) in [15] Theorem 1 is valid also if $\{z_n\}$ in the statement is a finite union of separated sequences. Applying the last observation, we can also replace a Carleson-Newman sequence in [15] Corollary 1 by a finite union of separated sequences: If $0 < \alpha < 1$, $\{z_n\}$ is a finite union separated sequences and zero set of $B^{2,2}_\alpha$, then
\[
\int_0^{2\pi} \log \left( \sum_n \frac{(1-|z_n|)^{\alpha+1}}{|e^{i\theta}-z_n|^2} \right) d\theta < \infty.
\]
This result offers a practical way to construct Blaschke sequences which are not zero sets of $B^{2,2}_\alpha$; see [15] Theorem 2 and its proof.

Note that (4.2) and (4.5) together with the estimates for $I_1$ and $I_2$ are valid also if outer function $O_\phi$ is replaced by an arbitrary $f \in H^p$. Using this observation and Theorem 7 we can rewrite Corollary 9 in the following form.

**Corollary 10.** Let $1 < p < \infty$, $p-2 < \alpha < p-1$, $f \in H^p$ and $B$ be a Blaschke product associated with a Carleson-Newman sequence $\{z_n\}$. Then $fB \in B^{p,p}_\alpha$ if and only if $f \in B^{p,p}_\alpha$ and
\[
\sum_n |f(z_n)|^p (1-|z_n|)^{\alpha+2-p} < \infty.
\]
Corollary 10 is a partial improvement of the main result in M. Jevtić’s paper [11]. More precisely, this paper contains an extended counterpart of Corollary 10 (in the sense of $p$ and $q$) with the defect $f \equiv 1$. It is also worth mentioning that Corollary 10 is not valid if the Carleson-Newman sequence $\{z_n\}$ is replaced by an arbitrary Blaschke sequence. This can be shown by studying the case where $f \equiv 1$ and $B$ is a Blaschke product with zeros on the positive real axis. More precisely, the counter example follows from [23, Theorem 1], which asserts that all such Blaschke products belong to $B^p_q$ for $1/2 < p < \infty$ and $p - 3/2 < \alpha < \infty$.

Theorem 7 for $f \equiv 1$ (or Theorem 3 for inner functions) has also extended counterpart [22, Theorem 1].

**Theorem E.** Let $0 < p, q < \infty$ and $\nu \in \mathcal{D}_q$. Then $\nu \in \mathcal{D}_q$ if and only if

$$\|I'\|_{A^p_q} \geq \int_0^1 \left( \int_0^{2\pi} \left( \frac{1 - |r e^{i\theta}|}{1 - r} \right)^p d\theta \right)^{q/p} \nu(r) \, dr$$

for all inner functions $I$. Here the comparison constants may depend only on $p$, $q$ and $\nu$.

Theorem [22] confirms that the hypothesis $\nu \in \mathcal{D}_q$ in Theorems 3 and 4 is sharp in a certain sense. Studying the argument of this result in [22], we can also deduce that the proof of Theorem 3 is more straightforward when $f$ is an inner function, and the statement is valid for all $0 < p < \infty$. It is also worth mentioning that results like Theorem E have turned out to be useful in the theory of inner functions. Several by-products of Theorem E can be found in [22, 24].

5. PROOF OF THEOREM 1

Before the proof of Theorem 1 we recall [22, Lemma 6], which is a modification of [11, Lemma 5].

**Lemma F.** If $0 < p \leq 1$ and $g : [0, 1) \to [0, \infty)$ is measurable, then

$$\left( \int_0^1 g(s) \, ds \right)^p \leq 2 \int_0^1 \sup_{0 \leq x \leq s} g(x)^p (1 - s)^{p-1} \, ds$$

for $0 \leq r < 1$.

**Proof of Theorem 1.** Let $\frac{4}{5} \leq s < 1$ and choose $n = n(s) \in \mathbb{N} \setminus \{1, 2, 3, 4\}$ such that $1 - \frac{1}{n} \leq s < 1 - \frac{1}{n+1}$. Set $f_n(z) = z^n$ for $z \in \mathbb{D}$. Since

$$|e^{in\theta} - e^{inh}|^2 = |1 - e^{in\theta}|^2 = 2(1 - \cos(nh)) = 2n^2h^2 \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(nh)^2(k-1)}{(2k)!} \geq 2n^2h^2 \left( \frac{1}{2} - \frac{n^2h^2}{24} \right) \geq \frac{2}{3}n^2h^2, \quad 0 < h < \frac{2}{n},$$

we have

$$\int_{1/2}^1 \omega_p(1 - r, f_n)^q \frac{\nu(r)}{(1 - r)^q} \, dr = \left( \int_{1/2}^{1-2/n} + \int_{1-2/n}^1 \right) \sup_{0 < h < 1 - r} |1 - e^{inh}|^q \frac{\nu(r)}{(1 - r)^q} \, dr \geq \int_{1/2}^{1-2/n} |1 - e^{i2\pi q} r| \frac{\nu(r)}{(1 - r)^q} \, dr + \int_{1-2/n}^s |1 - e^{i\pi (1-r)}| q \frac{\nu(r)}{(1 - r)^q} \, dr \geq \int_{1/2}^{1-2/n} \frac{\nu(r)}{(1 - r)^q} \, dr + \int_{1-2/n}^s (n + 1)^q \nu(r) \, dr \geq \int_{1/2}^{1-2/n} \frac{\nu(r)}{(1 - r)^q} \, dr + (1 - s)^{-q} \int_{1-2/n}^s \nu(r) \, dr \geq \int_{1/2}^s \frac{\nu(r)}{(1 - r)^q} \, dr \approx \int_0^s \frac{\nu(r)}{(1 - r)^q} \, dr.$$
Using the hypothesis $\nu \in \widehat{\mathcal{D}}$ together with Lemma 4(iv)(ii) in a similar manner as in the proof of [21, Theorem 1], we obtain

$$
\|f_n\|_{A_p^q}^q = n^q \int_0^1 r^{q(n-1)+1} \nu(r) \, dr = n^q \int_{1-\frac{1}{n+1}}^1 \nu(r) \, dr
$$

Finally combining the estimates above and using the inequality

$$
\int_0^t \frac{\nu(r)}{(1-r)^q} \, dr \lesssim 1 = \frac{\tilde{\nu}(t)}{(1-t)^q}, \quad 0 < t < \frac{4}{5},
$$

we deduce that if $\nu \in \widehat{\mathcal{D}}$ and (1.2) is satisfied for all $f \in H_p$, then $\nu \in \widehat{\mathcal{D}}_p$. Hence it suffices to prove the converse statement.

Let $f \in H_p$, $0 \leq \theta < 2\pi$, $\frac{1}{2} < r < 1$ and $0 < h < \frac{1}{P}$. Set $\rho = r - h$ and $\Gamma$ be the contour which goes first rapidly from $re^{i\theta}$ to $\rho e^{i\theta}$, then along the circle $\{ z : |z| = \rho \}$ to $\rho e^{i(\theta+h)}$ and finally rapidly to $re^{i(\theta+h)}$. Since

$$
f(re^{i(\theta+h)}) - f(re^{i\theta}) = \int_{\Gamma} f'(z) \, dz,
$$

we have

$$
|f(re^{i(\theta+h)}) - f(re^{i\theta})| \leq \int_{\rho}^r |f'(se^{i\theta})| \, ds + \int_{\theta}^{\theta+r} |f'(\rho e^{i\theta})| \, dt + \int_{\rho}^r |f'(se^{i(\theta+h)})| \, ds.
$$

Consequently, the discrete and continuous forms of Minkowski’s inequality, a change of variable and Hardy’s convexity theorem yield

$$
\left( \int_0^{2\pi} |f(re^{i(\theta+h)}) - f(re^{i\theta})|^p \, d\theta \right)^{1/p} \leq \left( \int_0^{2\pi} \left( \int_{\rho}^r |f'(se^{i\theta})| \, ds \right)^p \, d\theta \right)^{1/p}
$$

$$
+ \left( \int_0^{2\pi} \left( \int_{\rho}^r |f'(\rho e^{i(x+\theta)})| \, dx \right)^p \, d\theta \right)^{1/p} + \left( \int_0^{2\pi} \left( \int_{\rho}^r |f'(se^{i(\theta+h)})| \, ds \right)^p \, d\theta \right)^{1/p}
$$

$$
\leq 2 \int_{\rho}^r M_p(s, f') \, ds + M_p(\rho, f') \leq 3 \int_{\rho-h}^r M_p(s, f') \, ds.
$$

Note that the deduction above can be found, for instance, in the proof of [8, Theorem 5.4].

By raising both sides of (5.1) to power $q$, adding $\sup_{0<h<1-t}$ and then integrating from 1/2 to $r$ with respect to $\nu(t) \, dt/(1-t)^q$, we obtain

$$
\frac{\nu(t)}{(1-t)^q} dt \leq \frac{\nu(t)}{(1-t)^q} dt.
$$

Letting $r \to 1^-$, using the monotone and mean convergence theorems together with the hypothesis $f \in H_p$, we deduce

$$
\int_{1/2}^1 \omega_p(1-t, f)^q \frac{\nu(t)}{(1-t)^q} \, dt \leq \int_{1/2}^1 \left( \int_{1/2}^1 M_p(s, f') \, ds \right)^q \frac{\nu(t)}{(1-t)^q} \, dt.
$$

Hence it suffices to show $\mathcal{I} \leq \|f'\|_{A_p^q}^q$. Note that the argument of this estimate uses ideas from [22].
If $q \leq 1$, then Lemma 1 with the choice $g(s) = M_p(s, f')$, Hardy’s convexity theorem, Fubini’s theorem, the hypothesis $\nu \in \mathring{D}_q$ and Lemma 2 give

\[
\mathcal{I} \leq \int_0^1 \frac{\nu(t)}{(1-t)^q} \left( \int_0^t \sup_{0 \leq x \leq s} M_p^q(x, f')(1-s)^{q-1} \, ds \, dt \right)
\]

\[
= \int_0^1 \frac{\nu(t)}{(1-t)^q} \left( \int_t^1 M_p^q(s, f')(1-s)^{q-1} \, ds \, dt \right)
\]

\[
= \int_0^1 M_p^q(s, f')(1-s)^{q-1} \int_0^{t(s)} \frac{\nu(t)}{(1-t)^q} \, dt \, ds
\]

\[
\leq \int_0^1 M_p^q(s, f')(1-s)^{q-1} \frac{\nu(s)}{(1-s)^q} \, ds \approx \|f\|^q_{A_p^q}
\]

for all $f \in \mathcal{H}$. Hence the assertion for $q \leq 1$ is proved. If $q > 1$, $0 < \varepsilon < q/(\mathring{D}_q(\nu) + 1)$ and $h(s) = (1-s)^{\frac{q-1-s}{q}}$, then Hölder’s inequality and Fubini’s theorem yield

\[
\mathcal{I} \leq \int_0^1 \int_t^1 M_p^q(s, f') h(s)^q \, ds \left( \int_t^1 h(r)^{-\frac{s}{1-t}} \, dr \right)^{q-1} \frac{\nu(t)}{(1-t)^q} \, dt
\]

\[
= \int_0^1 \frac{\nu(t)}{(1-t)^{q-\varepsilon}} \int_t^1 M_p^q(s, f')(1-s)^{q-1-\varepsilon} \, ds \, dt
\]

\[
= \int_0^1 M_p^q(s, f')(1-s)^{q-1-\varepsilon} \int_0^{t(s)} \frac{\nu(t)}{(1-t)^{q-\varepsilon}} \, dt \, ds
\]

for all $f \in \mathcal{H}$. Since $\nu \in \mathring{D}_{q-\varepsilon}$ by Lemma 3, the assertion for $q > 1$ follows from Lemma 4. This completes the proof.

Since

\[
\int_0^{1/2} \omega_p(1-t, f)^q \frac{\nu(t)}{(1-t)^q} \, dt \leq 2^q \|f\|^q_{H^p} \int_0^{1/2} \nu(t) \, dt
\]

by Minkowski’s inequality, Theorem 5 has the following consequence.

Corollary 11. Let $1 \leq p < \infty$, $0 < q < \infty$ and $\nu \in \mathcal{D} \cap \mathring{D}_q$. Then there exists a constant $C = C(p, q, \nu) > 0$ such that

\[
\int_0^1 \omega_p(1-r, f)^q \frac{\nu(r)}{(1-r)^q} \, dr \leq C \left( \|f\|^q_{A_p^q} + \|f\|^q_{H^p} \right)
\]

for all $f \in H^p$.

Note that Corollary 11 is a part of Theorem 2. We state it here as an independent result because it is needed for the proof of Theorem 2.

6. PROOF OF THEOREM 2

We go directly to the proof of Theorem 2.

Proof of Theorem 2. Let $0 \leq r < 1$ and $0 \leq t < 2\pi$. Since

\[
\int_0^{2\pi} \frac{e^{i\theta} \, d\theta}{(e^{i\theta} - re^{i\theta})^2} = 0,
\]

Cauchy’s integral formula gives

\[
|f'(re^{i\theta})| = \left| \frac{1}{2\pi(1-r^2)} \int_0^{2\pi} \frac{f(e^{i\theta}) - f(re^{i\theta})}{(e^{i\theta} - re^{i\theta})^2} \, d\theta \right|
\]

\[
\leq \frac{1}{1-r} \int_0^{2\pi} |f(e^{i\theta}) - f(re^{i\theta})| \, d\mu_{re^{i\theta}}(\theta), \quad f \in H^1.
\]
Raising both sides to power $p$, integrating from $0$ to $2\pi$ with respect to $dt$, then raising both sides to power $q/p$ and finally integrating from $0$ to $1$ with respect to $\nu(r)\,dr$, we obtain
\[
\|f\|_{A^q_r}^q \leq \int_0^1 \left( \int_0^{2\pi} \left( \int_0^{2\pi} |f(e^{i\theta}) - f(\rho e^{i\tau})| \, d\mu_{\rho e^{i\tau}}(\theta) \right)^p \, dt \right)^{q/p} \frac{\nu(r)}{(1-r)^q} \, dr \tag{6.1}
\]
for all $f \in H^p$.

Let $f \in H^p$, $0 < q \leq 1$, and set
\[
\mathcal{I}(r) = \left( \int_0^{2\pi} \left( \int_0^{2\pi} |f(e^{i\theta}) - f(\rho e^{i\tau})| \, d\mu_{\rho e^{i\tau}}(\theta) \right)^p \, dt \right)^{q/p}.
\]
By the proof of [9, Theorem 2.1], we know that
\[
\mathcal{I}(r) \leq \left( \sum_{k=0}^{\infty} 2^{-k} \omega_p(2^k(1-r), f) \right)^q.
\] (6.2)
Hence the sub-additivity of $\mathcal{I}(r)$ and Fubini’s theorem give
\[
\int_0^1 \frac{\mathcal{I}(r)\nu(r)}{(1-r)^q} \, dr \leq \sum_{k=0}^{\infty} 2^{-qk} \int_0^1 \omega_p(2^k(1-r), f)^q \frac{\nu(r)}{(1-r)^q} \, dr.
\] (6.3)
Next we show that the weight $\nu(r)$ in the right-hand side can be replaced by $\tilde{\nu}(r)$, without losing any essential information.

Set $\psi(z) = \tilde{\nu}(z)/(1-|z|)$ for $z \in \mathbb{D}$, and remind that $\tilde{\nu}(r) = \hat{\psi}(r)$ for $0 \leq r < 1$ by Lemmas [A](ii) and [B]. In particular, $\psi$ belongs to class $\mathcal{D}$; and thus, there exist $K = K(\psi) > 1$ and $C = C(\psi) > 1$ such that
\[
\hat{\psi}(r) \geq C\hat{\psi} \left( 1 - \frac{1-r}{K} \right), \quad 0 \leq r < 1.
\] (6.4)
Let $k \in \mathbb{N} \cup \{0\}$ and $r_n = 1 - K^{-n}$ for $n \in \mathbb{N} \cup \{0\}$. Using (6.4) together with Lemma [A](ii), we obtain
\[
(C-1)\hat{\psi}(r_{n+1}) = C\hat{\psi} \left( 1 - \frac{1-r_n}{K} \right) - \hat{\psi}(r_{n+1}) \leq \hat{\psi}(r_n) - \hat{\psi}(r_{n+1})
\]
\[
= \int_{r_n}^{r_{n+1}} \psi(r) \, dr \leq \hat{\psi}(r_n) \approx \hat{\psi}(r_{n+1}).
\] (6.5)
Now Minkowski’s inequality, the monotonicity of $\omega_p(s, f)$ with $s$ and (6.4) yield
\[
\int_0^1 \omega_p(2^k(1-r), f)^q \frac{\nu(r)}{(1-r)^q} \, dr \leq \sum_{n=1}^{\infty} \int_{r_n}^{r_{n+1}} \omega_p(2^k(1-r), f)^q \frac{\nu(r)}{(1-r)^q} \, dr + \|f\|_{H^p}^q
\]
\[
\leq \sum_{n=1}^{\infty} \omega_p(2^k(1-r_n), f)^q \frac{\tilde{\nu}(r_n)}{(1-r_{n+1})^q} + \|f\|_{H^p}^q
\]
\[
= \sum_{n=1}^{\infty} \omega_p(2^k(1-r_n), f)^q \frac{\hat{\psi}(r_n)}{(1-r_{n+1})^q} + \|f\|_{H^p}^q
\]
\[
= \sum_{n=0}^{\infty} \omega_p(2^k(1-r_{n+1}), f)^q \frac{\hat{\psi}(r_{n+1})}{(1-r_n)^q} + \|f\|_{H^p}^q
\]
\[
= \sum_{n=0}^{\infty} \int_{r_n}^{r_{n+1}} \omega_p(2^k(1-r), f)^q \frac{\hat{\psi}(r)}{(1-r)^q} \, dr + \|f\|_{H^p}^q
\]
\[
= \int_0^1 \omega_p(2^k(1-r), f)^q \frac{\hat{\nu}(r)}{(1-r)^{q+1}} \, dr + \|f\|_{H^p}^q.
\] (6.6)
It is worth noting that a similar deduction works also in the opposite direction.
Using (6.3) and (6.6), we obtain
\[
\int_0^1 \frac{I(r)\nu(r)}{(1-r)^q} \, dr \leq \left[ \sum_{k=0}^{\infty} 2^{-kq} \int_0^1 \omega_p(2^k(1-r), f)^q \frac{\nu(r)}{(1-r)^{q+1}} \, dr + \|f\|_{H^p}^q \right] \\
\quad + \sum_{k=0}^{\infty} 2^{-kq} \int_{1-2^{-k}}^1 \omega_p(2^k(1-r), f)^q \frac{\nu(r)}{(1-r)^{q+1}} \, dr \\
=: I_1 + I_2.
\]
Minkowski’s inequality, (4.3) with \(p\) being replaced by \(q\) and Lemma 13 yield
\[
I_1 \leq \|f\|_{H^p}^q \sum_{k=0}^{\infty} 2^{-kq} \int_0^1 \frac{\nu(r)}{(1-r)^{q+1}} \, dr \\
\leq \|f\|_{H^p}^q \nu \sum_{n=0}^{\infty} 2^{-nq} = \|f\|_{H^p}^q
\]
for some \(\alpha = \alpha(\nu) > 0\). The continuity of \(\nu\), changes of variables, Fubini’s theorem and the hypothesis \(\nu \in D\) give
\[
I_2 = \int_0^1 \frac{\omega_p(1-s, f)^q}{(1-s)^{q+1}} \int_0^1 \nu \left(1 - 2^{-k}(1-s)\right) \, dk \, ds \\
= \frac{1}{\log 2} \int_0^1 \frac{\omega_p(1-s, f)^q}{(1-s)^{q+1}} \int_s^1 \frac{\nu(x)}{1-x} \, dx \, ds \\
= \int_0^1 \frac{\omega_p(1-s, f)^q}{(1-s)^{q+1}} \nu(s) \, ds.
\]
Summarizing, we have shown
\[
\int_0^1 \frac{I(r)\nu(r)}{(1-r)^q} \, dr \leq \int_0^1 \frac{\omega_p(1-s, f)^q}{(1-s)^{q+1}} \nu(s) \, ds + \|f\|_{H^p}^q. \quad (6.7)
\]
Applying a similar argument as in (6.6), we can replace \(\nu(s)\) in the right-hand side of (6.7) by \(\nu(s)(1-s)\). Consequently, (6.11) and Corollary 11 imply (1.3) for all \(f \in H^p\). Hence the assertion for \(q \leq 1\) is proved.

Let \(1 < q < \infty\). Then (6.2), the continuous form of Minkowski’s inequality, (6.6) and well-known inequalities give
\[
\int_0^1 \frac{I(r)\nu(r)}{(1-r)^q} \, dr \leq \left[ \sum_{k=0}^{\infty} 2^{-kq} \int_0^1 \omega_p(2^k(1-r), f)^q \frac{\nu(r)}{(1-r)^{q+1}} \, dr \right]^{1/q} \nu(r)^{1/q} \\
\leq \left[ \sum_{k=0}^{\infty} 2^{-kq} \left( \int_0^1 \omega_p(2^k(1-r), f)^q \frac{\nu(r)}{(1-r)^{q+1}} \, dr \right)^{1/q} \nu(r)^{1/q} \right]^{q} \\
\leq \left[ \sum_{k=0}^{\infty} 2^{-kq} \left( \int_0^1 \omega_p(2^k(1-r), f)^q \frac{\nu(r)}{(1-r)^{q+1}} \, dr \right)^{1/q} \nu(r)^{1/q} \right]^{q} + \|f\|_{H^p}^q \\
\leq \left[ \sum_{k=0}^{\infty} 2^{-kq} \left( \int_0^1 \omega_p(2^k(1-r), f)^q \frac{\nu(r)}{(1-r)^{q+1}} \, dr \right)^{1/q} \nu(r)^{1/q} \right]^{q} + \|f\|_{H^p}^q \\
\quad + \left( \sum_{k=0}^{\infty} 2^{-kq} \left( \int_{1-2^{-k}}^1 \omega_p(2^k(1-r), f)^q \frac{\nu(r)}{(1-r)^{q+1}} \, dr \right)^{1/q} \nu(r)^{1/q} \right) \\
=: I_3 + I_4.
\]
Minkowski's inequality, \( (4.3) \) with \( p \) being replaced by \( q \) and Lemma 1 yield

\[
I_3 \lesssim \|f\|_{H^q}^q \left( \sum_{k=0}^\infty 2^{-k} \left( \int_0^{1-2^{-k}} \frac{\hat{\nu}(r)}{(1-r)^{q+1}} \, dr \right)^{1/q} \right) \]

\[
\lesssim \|f\|_{H^q} \left( \sum_{k=0}^\infty \hat{\nu}(1-2^{-k})^{1/q} \right)^q = \|f\|_{H^p}^q.
\]

By Lemma 1 there exists a constant \( \alpha = \alpha(\nu) > 0 \) such that

\[
\hat{\nu} \left( 1 - 2^{-k}(1 - s) \right) \lesssim 2^{-\alpha k} \hat{\nu}(s), \quad 0 \leq s \leq 1 - 2^{-k}(1 - s) < 1.
\]

Using this together with a change of variable and modification of \( (6.6) \), we get

\[
I_4 = \left( \sum_{k=0}^\infty \left( \int_0^1 \omega_p(2^k(1 - s), f)^q \frac{\hat{\nu}(1-2^{-k}(1 - s))}{(1-s)^{q+1}} \, ds \right)^{1/q} \right)^q \]

\[
\lesssim \left( \sum_{k=0}^\infty 2^{-\alpha k/q} \left( \int_0^1 \omega_p(1-s, f)^q \frac{\hat{\nu}(s)}{(1-s)^{q+1}} \, ds \right)^{1/q} \right)^q \]

\[
\lesssim \int_0^1 \omega_p(1-s, f)^q \frac{\nu(s)}{(1-s)^q} \, ds + \|f\|_{H^p}^q.
\]

Finally \( (6.1), (6.8) \) and Corollary 11 imply \( (1.3) \) for all \( f \in H^p \). This completes the proof. \( \square \)

### 7. PROOF OF THEOREM 3

Before the proof of Theorem 3 we recall the following result, which is a part of the argument of [6, Theorem 1.1].

**Lemma G.** If \( O \phi \) is an outer function, then

\[
|O \phi(z)| \leq \frac{4}{1 - |z|} \left\{ \int_0^{2\pi} |\phi(e^{i\theta}) - \int_0^{2\pi} \phi(e^{i\theta}) \, d\mu(z)(s) \right| \, d\mu(z)(\theta) + \int_0^{2\pi} \phi(e^{i\theta}) \, d\mu(z)(h) - |O \phi(z)| \right\}
\]

for all \( z \in \mathbb{D} \).

**Proof of Theorem 3.** Let \( f \in H^p \). Then there exist an inner function \( I \) and an outer function \( O \phi \) such that \( f = IO \phi \). Hence the Schwarz-Pick lemma, Lemma 13, and the fact that \( \phi(\xi) = |f(\xi)| \) for almost every \( \xi \in \mathbb{T} \) yield

\[
|f'(z)|(1 - |z|) \leq (|I(z)O \phi(z)| + |I'(z)O \phi(z)\| (1 - |z|) \]

\[
\leq |O \phi(z)|(1 - |z|) + 2|O \phi(z)|(1 - |I(z)|) \]

\[
\leq 4 \int_0^{2\pi} |f(e^{i\theta})| - \int_0^{2\pi} |f(e^{i\theta})| \, d\mu(z)(s) \right| \, d\mu(z)(\theta) + 4 \left( \int_0^{2\pi} |f(e^{i\theta})| \, d\mu(z)(h) - |f(z)| \right)
\]

for all \( z \in \mathbb{D} \). Write \( z = re^{i\theta} \). Raising both sides of \( (7.1) \) to power \( p \), integrating from 0 to \( 2\pi \) with respect to \( dt \), then raising both sides to power \( q/p \), integrating from 0 to 1 with respect to \( \nu(r)dr/(1-r)^q \) and finally splitting the right-hand side into two parts, we obtain

\[
\|f\|_{A^p, q}^q \lesssim F_1(f) + F_2(f),
\]

which is the first inequality in \( (1.4) \).

Set

\[
\Gamma = \Gamma(z, f) = \left\{ \theta \in [0, 2\pi) : \int_0^{2\pi} |f(e^{i\theta})| \, d\mu(z)(s) \leq |f(e^{i\theta})| \right\}, \quad z \in \mathbb{D}.
\]
Then elementary calculations together with the subharmonicity of $|f|$ yield
\[ \int_0^{2\pi} |f(e^{i\theta})| - \int_0^{2\pi} |f(e^{i\xi})|d\mu_z(s) \, d\mu_z(\theta) = 2 \int_\Gamma \left( |f(e^{i\theta})| - \int_0^{2\pi} |f(e^{i\xi})|d\mu_z(s) \right) \, d\mu_z(\theta) \leq 2 \int_\Gamma \left( |f(e^{i\theta})| - |f(z)| \right) \, d\mu_z(\theta), \quad z \in \mathbb{D}. \]

It follows that
\[ \int_0^{2\pi} |f(e^{i\theta})| - \int_0^{2\pi} |f(e^{i\xi})|d\mu_z(s) \, d\mu_z(\theta) + \left( \int_0^{2\pi} |f(e^{i\theta})|d\mu_z(h) - |f(z)| \right) \leq 4 \int_0^{2\pi} |f(e^{i\theta}) - f(z)| \, d\mu_z(\theta), \quad z \in \mathbb{D}. \]

Doing a corresponding integration procedure for this estimate as above and applying Theorem 2 we obtain
\[ F_1(f) + F_2(f) \leq \|f\|_{H^p}^p + \|f\|_{H^p}^p, \]
which is the last inequality in (1.4). This completes the proof. \hfill \Box

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University of Eastern Finland, P.O.Box 111, 80101 Joensuu, Finland
E-mail address: atte.reijonen@uef.fi