Manipulation of matter waves using Bloch and Bloch–Zener oscillations

B M Breid, D Witthaut and H J Korsch

FB Physik, Technische Universität Kaiserslautern, D-67653 Kaiserslautern, Germany
E-mail: korsch@physik.uni-kl.de

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Abstract. We present theoretical and numerical results on the dynamics of ultracold atoms in an accelerated single- and double-periodic optical lattice. In the single-periodic potential Bloch oscillations can be used to generate fast directed transport with very little dispersion. The dynamics in the double-periodic system is dominated by Bloch–Zener oscillations, i.e. the interplay of Bloch oscillations and Zener tunnelling between the subbands. Apart from directed transport, the latter system permits various interesting applications, such as widely tunable matter wave beam splitters and Mach–Zehnder interferometry. As an application, a method for efficient probing of small nonlinear mean-field interactions is suggested. Furthermore, the influence of the nonlinearity onto the Bloch bands, the breakdown of adiabaticity and the stability of the dynamics is discussed and analysed by numerical results.
1. Introduction

The experimental progress in storing and controlling ultracold atoms in optical lattices (see, e.g. [1, 2]) has led to a variety of spectacular results in the last decade, for instance the superfluid to Mott insulator phase transition [3]. Also the field of linear and nonlinear atom optics has benefited a lot from cooling and storing atoms in optical lattices. Early results include the observation of Bloch oscillations in accelerated lattices [4, 5] and coherent pulsed output from a BEC in a vertical lattice under the influence of gravity [6]. Today it is a matter of routine to prepare a wave packet in a state of well defined quasi-momentum by accelerating the optical lattice.

Combining Bloch oscillations and Zener tunnelling between Bloch bands offers new possibilities to control the dynamics of cold atoms. However, in a usual cosine-shaped optical potential, the band gaps decrease rapidly with increasing energy. A matter wave packet tunnelling from the ground band to the first excited band will therefore also tunnel to even higher bands and finally escape to infinity. Indeed this happens, e.g. in the Kasevich experiment [6, 7].

A discussion of Zener tunnelling in optical lattices and a method to measure it can be found, e.g. in [8, 9].

However, systems can be constructed that avoid decay and still allow Zener tunnelling between certain (mini)bands. In fact, this can be achieved by introducing a second, double-periodic potential as it has been generated recently [10] by combining optical lattices based on virtual two-photon and four-photon processes. This leads to a splitting of the ground band into two minibands that are still energetically well separated from all excited bands. A matter wave packet under the influence of an external field will Bloch oscillate, whereas Zener tunnelling between the minibands will lead to a splitting of the wave packet and to interference. Zener
tunnelling between minibands has also been observed in different systems as, e.g. in optical superlattices for light waves [11].

In this paper, we investigate the dynamics of cold atoms in a one-dimensional (1D) double-periodic potential, which is governed by the Schrödinger equation

\[ i\hbar \frac{\partial}{\partial t} \Psi(x, t) = \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + U \cos \frac{2\pi x}{d} + \varepsilon U \cos \frac{\pi x}{d} + F(t)x \right] \Psi(x, t). \] (1)

Here, \( d \) denotes the fundamental period, \( F \) is the strength of the external field and \( U \) and \( \varepsilon U \) are the amplitudes of the two optical lattices, where the double-periodic potential is weak, \( \varepsilon \ll 1 \). For convenience we use scaled units such that the fundamental period is \( d_s = 2\pi \) and the amplitude of the deeper lattice is \( U_s = 1 \),

\[ x_s = \frac{2\pi}{d} x, \quad \hbar_s = \frac{2\pi}{d\sqrt{Um}} \hbar, \quad F_s(t) = \frac{d}{2\pi U} F(t), \quad t_s = \frac{2\pi}{d} \sqrt{\frac{U}{m} t}, \] (2)

which leads to the dimensionless Schrödinger equation

\[ i\hbar_s \frac{\partial}{\partial t_s} \Psi(x, t) = \left[ -\frac{\hbar_s^2}{2} \frac{\partial^2}{\partial x_s^2} + \cos (x_s) + \varepsilon \cos \left( \frac{x_s}{2} \right) + F_s(t)x_s \right] \Psi(x, t). \] (3)

Unless otherwise stated, the parameter values are chosen as \( \hbar_s = 2.828 \) and \( F_s = 0.0011 \), which corresponds to the experimental set-up of the Arimondo group [12] for \( \varepsilon = 0 \). These are typical experimental dimensions. In the following we will omit the index \( s \) to simplify notation.

This paper is organized as follows: the non-interacting regime is discussed in sections 2 and 3, while the role of mean-field (MF) interactions is investigated in section 4. We start by reviewing some important results for the single-periodic potential, i.e. \( \varepsilon = 0 \), in section 2. Furthermore we discuss in section 2.2 a shuttling mechanism for transporting wave packets in optical lattices by flipping the external field. It is shown that the transport velocity is independent of the field strength and that dispersion is negligible. The case of a double-periodic potential is then discussed in section 3, starting with a brief description of the dynamics of Bloch–Zener oscillations. Combining this effect with the shuttling transport mechanism offers the possibility to construct a highly controllable matter wave beam splitter as described in section 3.3. Based on the previous results, we discuss the possibility of matter wave Mach–Zehnder interferometry in section 3.4. In section 4.1, the influence of MF interactions in Bose–Einstein condensates (BECs) on to the Bloch bands and the breakdown of adiabaticity is analysed. The stability of the dynamics in dependence of the second shallow lattice is discussed. As a possible application of Mach–Zehnder interferometry we finally show in section 4.2 how one could probe small MF interactions in BECs.

2. Single-periodic potentials

2.1. Bloch oscillations

Bloch oscillations of quantum particles in periodic potentials under the influence of a static external field \( F(t) = F \) have been predicted already in 1928 [13]. The recent experimental
progress with cold atoms in optical lattices has triggered a renewed theoretical interest in this topic (see [14]–[16] for recent reviews).

The dynamics of Bloch oscillations is illustrated in figure 1 in real space and in momentum space. A handwaving explanation of these oscillations can be given easily, assuming that the external field tilts the energy of the Bloch bands in real space. Quite often, all higher bands are energetically far from the ground band and the Bloch oscillation can already be understood within a single-band approximation. A wave packet in the ground band is accelerated in real space by the external field and reflected at the edges of the band, which gives rise to an oscillating motion. Within this picture, the spatial extension of these oscillations can be estimated as

$$L = \frac{\Delta}{F},$$  \quad (4)

where $\Delta$ denotes the energy width of the Bloch band. The oscillation period is given by the characteristic Bloch time

$$T_B = \frac{2\pi h}{dF}.$$  \quad (5)

However, a rigorous calculation shows that the Bloch bands do not exist any longer for $F \neq 0$. The spectrum of the rescaled Wannier–Stark Hamiltonian

$$\hat{H}_{WS} = -\frac{\hbar^2}{2} \frac{\partial^2}{\partial x^2} + V(x) + F x, \quad V(x + d) = V(x),$$  \quad (6)

is continuous with embedded resonances, the so-called Wannier–Stark ladder of resonances [18, 19]. The corresponding eigenvalues are arranged in ladders $E_{\alpha,n} = E_{\alpha,0} + ndF$, where $\alpha$ denotes the ladder index and $n$ denotes the site index. The eigenstates within one ladder are related by a spatial translation $\psi_{\alpha,n}(x) = \psi_{\alpha,0}(x - nd)$, respectively $\psi_{\alpha,n}(k) = e^{-ink} \psi_{\alpha,0}(k)$ in momentum space.
The dynamics of Bloch oscillations is now readily understood in the Wannier–Stark eigenbasis [14]. For weak fields, the dynamics takes place almost exclusively in the lowest ladder $\alpha = 0$. The dynamics of an initial wave packet $\Psi(k, 0) = \sum_n c_{\alpha=0,n} \psi_{\alpha=0,n}(k)$ is then given by

$$\Psi(k, t) = \sum_n c_{0,n} e^{-iE_{0,n}t/\hbar} \psi_{0,n}(k)$$

$$= \sum_n c_{0,n} e^{-i(E_{0,0} + nF)t/\hbar} e^{-i\vec{k}\cdot\vec{r}} \psi_{0,0}(k)$$

$$= e^{-iE_{0,0}t/\hbar} \psi_{0,0}(k) C(k + Ft/\hbar). \tag{7}$$

The function $C(k + Ft/\hbar)$ is the discrete Fourier transformation of the coefficients $c_{0,n}$ evaluated at the point $k + Ft/\hbar$. For a broad initial wave packet it is a $2\pi/d$-periodic series of narrow peaks. The dynamics in momentum space shown in figure 1 is now easily understood: the function $C(k)$ moves under an envelope given by the Wannier–Stark function $\psi_{0,0}(k)$. In real space, this periodic motion yields the familiar Bloch oscillations.

In view of the introduction of efficient matter wave beam splitters in double-periodic potentials in section 3.3, we also discuss a very simple mechanism of beam splitting using Bloch oscillations. At the time $t = T_B/2$, when a wave packet with average initial quasi-momentum $\kappa = 0$ is just crossing the edge of the Brillouin zone, this wave packet consists of two fractions with opposite momentum (see figure 1). In the case of completely switching off the periodic potential and also the Stark field at $t = T_B/2$, the two fractions move in opposite directions according to their momentum (see figure 2). The main disadvantage of this method is the strong dispersion of the free wave packet. Furthermore, the split wave packet is no longer located in a periodic potential. But periodic potentials are often desired for further experiments. Switching on the potential again would cause even stronger dispersion. In contrast, the splitting of a wave packet within a periodic potential can be done easily and with only little loss by a Bloch–Zener oscillation as will be shown in section 3.3.

2.2. Shutting transport

It is well known that a time-dependent external field $F(t)$ may lead to transport or dynamical localization [14, 20]. An effective way of transporting a wave packet with low loss in an optical

Figure 2. Splitting of a gaussian wave packet in position space as described in the text. Shown is $|\Psi(x, t)|$ as a colormap plot for $|\epsilon| = 0$.\n
\[\text{Figure 2.} \]
The basic idea is simple. Within half of a Bloch period $T_B/2$, the wave packet will be displaced by $L = \Delta / F_0$ and will return to its initial position within the next half period as shown in figure 1. However, if the direction of the external field is flipped, also the direction of motion flips. This shuttling transport is illustrated in figure 3, where the modulus of the wavefunction $|\Psi(x, t)|$ is plotted for such an alternating external field (cf [14]). The simulation has been done with Hamiltonian (3) which takes all Bloch bands into account. Note that anyway the dynamics considered here is based on Bloch oscillations which take place within the ground band. In that sense, the transport mechanism presented here is a single-band effect. Since all other bands are energetically separated, their influence on the dynamics can be neglected. A stronger influence of higher bands due to a stronger external force would increase the dispersion.

This transport mechanism has some remarkable features that can be analysed in the single-band tight-binding approximation. First of all, the transport velocity $v_{\text{trans}}$ is independent of the field strength $F_0$, which can be concluded easily. Within half of the Bloch period, the wave packet is displaced by $L = \Delta / F_0$ (cf equation (4)), which leads to the estimate

$$
 v_{\text{trans}} = \frac{L}{T_B/2} = \frac{\Delta / F_0}{\pi \hbar / dF_0} = \frac{d\Delta}{\pi \hbar}.
$$

Secondly, as observed in figure 3, the width of the wave packet is nearly conserved. No dispersion can be detected in this figure. In fact one can prove within a tight-binding model, that the width

$$
\Delta x^2(t) = \langle x^2 \rangle_t - \langle x \rangle_t^2
$$

Figure 3. Shuttling transport of a gaussian wave packet in a flipped field. Shown is $|\Psi(x, t)|$ as a colourmap plot (lower panel). Parameters are the same as in section 1, except that the external field $F(t)$ is periodically flipped as illustrated in the upper panel.
of an initially broad gaussian wave packet is in leading order given by

$$\Delta_\chi^2(nT_B) - \Delta_\chi^2(0) \approx \frac{\Delta^2 d^4 n^2}{8 F_0^2 \sigma_x^4}$$  \hspace{1cm} (11)$$

(see appendix). The dispersion vanishes rapidly with increasing spatial width $\sigma_x$ of the initial wave packet. For comparison, a free gaussian wave packet of a particle with mass $m$ spreads as

$$\Delta_\chi^2(t) - \Delta_\chi^2(0) = \frac{\hbar^2}{4m^2} \frac{t^2}{\sigma_x^4}.$$  \hspace{1cm} (12)$$

In conclusion, dispersion is negligible for all relevant applications. Finally we want to point out, that the presented transport mechanism differs from the transport in a quantum ratchet (see, e.g. [21, 22]) since the underlying double-periodic lattice is spatially symmetric and the direction of the transport depends on the initial sign of $F$ and the initial state.

3. Double-periodic potentials

3.1. Bloch–Zener oscillations

In this section, we study the coherent superposition of Bloch oscillations as described in section 2 and Zener tunnelling, denoted as Bloch–Zener oscillations. The dynamics is given by the scaled Schrödinger equation (3) with a double-periodic potential, i.e. $\varepsilon \neq 0$.

In the field-free case, the additional double-periodic potential leads to a splitting of the ground Bloch band into two minibands. This is shown in figure 4, where the dispersion relation for $\varepsilon = 0.121$ is compared to the one of the single-periodic system $\varepsilon = 0$. Because of the large energy gap between the two minibands and the next higher band, the dynamics of the system is almost exclusively affected by these minibands. Therefore, the dynamics is expected to be similar to the two-band tight-binding model (see [23]).

In the presence of a constant external field $F$, the spectrum of (3) consists of Wannier–Stark ladders instead of Bloch bands [19]. Due to the double-periodic potential, even these ladders split up into two ‘miniladders’ just like the Bloch bands of the field free system. This splitting

**Figure 4.** Dispersion relation of the system (3) with the parameters $\hbar = 2.828$, $F = 0$. Left panel: $\varepsilon = 0$ (bands ‘folded’ into the reduced Brillouin zone). Right panel: $\varepsilon = 0.121$. 

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was rigorously proven for a corresponding two-band tight-binding system recently [23], where it was also shown that the parameters of the tight-binding system can be chosen in a way to reach periodic reconstruction of an arbitrary initial wave packet. We expect to find a similar behaviour for the more realistic potential considered here.

In order to discuss the general features of the dynamics of Bloch–Zener oscillations, we expand a given wave packet in the Wannier–Stark eigenstates of the system (1)

\[ \hat{H}|\psi_{\alpha,n}\rangle = E_{\alpha,n}|\psi_{\alpha,n}\rangle, \]  

(13)

where \(\alpha = 0, 1\) denotes the miniladder index and \(n\) denotes the site index. For weak external fields \(F\), decay can be neglected and only the two miniladders corresponding to the lowest minibands (the two Wannier–Stark resonances with the least decay) have to be taken into account. The decay in a double-periodic lattice under the influence of a strong external field has been studied in [24].

The two energy ladders can be written as

\[ E_{0,n} = E_0 + 2nF \quad \text{and} \quad E_{1,n} = E_1 + 2nF, \]  

(14)

where \(E_0\) and \(E_1\) are the energy offsets of the two Wannier–Stark miniladders. The eigenstates with different site indices are related by a spatial translation

\[ |\psi_{\alpha,n}(x)\rangle = |\psi_{\alpha,0}(x - 2nd)\rangle. \]  

(15)

Now, an arbitrary initial wave packet \(|\Psi(t = 0)\rangle\) can be expanded in the Wannier–Stark basis

\[ |\Psi(t = 0)\rangle = \sum_n c_{\alpha=0,n}|\psi_{\alpha=0,n}(x)\rangle + \sum_n c_{\alpha=1,n}|\psi_{\alpha=1,n}(x)\rangle. \]  

(16)

Writing the Wannier–Stark states in the Bloch basis and using the phase change

\[ |\chi_{0,\kappa}(x + 2d)\rangle = e^{i2d\kappa}|\chi_{0,\kappa}(x)\rangle \quad \text{and} \quad |\chi_{1,\kappa}(x + 2d)\rangle = e^{i2d\kappa}|\chi_{1,\kappa}(x)\rangle \]  

(17)

of the Bloch waves under spatial translations, the time evolution of the wave packet \(|\Psi(t)\rangle\) is given by

\[ \langle \chi_{0,\kappa}|\Psi(t)\rangle = e^{-i(E_0 - Ft)/\hbar} \left[ a_{0,0}(\kappa)C_0\left(\kappa + \frac{Ft}{\hbar}\right) + a_{1,0}(\kappa)e^{-i(E_1 - E_0)/\hbar}C_1\left(\kappa + \frac{Ft}{\hbar}\right) \right], \]  

(18)

\[ \langle \chi_{1,\kappa}|\Psi(t)\rangle = e^{-i(E_0 - Ft)/\hbar} \left[ b_{0,0}(\kappa)C_0\left(\kappa + \frac{Ft}{\hbar}\right) + b_{1,0}(\kappa)e^{-i(E_1 - E_0)/\hbar}C_1\left(\kappa + \frac{Ft}{\hbar}\right) \right], \]  

(19)

where \(|\chi_{0,\kappa}\rangle\) and \(|\chi_{1,\kappa}\rangle\) are the Bloch waves \((F = 0)\) of the ground and the first excited miniband. The functions \(C_0(\kappa)\) and \(C_1(\kappa)\) are the discrete Fourier transforms of the expansion coefficients \(c_{0,n}\) and \(c_{1,n}\), respectively, and the functions \(a_{0,0}(\kappa), a_{1,0}(\kappa), b_{0,0}(\kappa)\) and \(b_{1,0}(\kappa)\) are the coefficients of the Wannier–Stark functions in the \(\kappa\)-basis. Note that all six functions are \(\pi/d\)-periodic in \(\kappa\).

This result, which is an extension of the corresponding tight-binding equations [23], leads to some interesting effects. The dynamics of the two band system is characterized by two timescales.
The functions $C_0(\kappa)$ and $C_1(\kappa)$ are reconstructed at multiples of

$$T_1 = \frac{\pi \hbar}{dF},$$  \hspace{1cm} (20)

whereas the exponential function $e^{-i(\hbar/\bar{h})(E_1 - E_0)t}$ has a period of

$$T_2 = \frac{2\pi \hbar}{E_1 - E_0}.$$  \hspace{1cm} (21)

The period $T_1$ is just half of the Bloch time $T_B = \frac{2\pi \hbar}{dF}$ of the single-periodic system, $\varepsilon = 0$, which we take as a reference time in the following. If a wavefunction consists only of states of a single energy ladder, one of the functions $C_0$, $C_1$ is zero for all times $t$. In this case the initial state is reconstructed up to a global phase after a period $T_1$ which is just an ordinary Bloch oscillation.

Whenever the commensurability condition

$$\frac{T_1}{T_2} = \frac{E_1 - E_0}{2dF} = \frac{s}{r} \quad \text{with } r, s \in \mathbb{N}$$  \hspace{1cm} (22)

is fulfilled, the functions (18) and (19) reconstruct at multiples of the Bloch–Zener time

$$T_{BZ} = sT_2 = rT_1$$  \hspace{1cm} (23)

up to a global phase shift. Furthermore the dynamics of the occupation probability of the two minibands at multiples of the time $T_1$ can be expressed in the form (cf [23])

$$\frac{\pi}{2d} \int_{-\pi/2d}^{\pi/2d} |\langle \chi_{0,\kappa} | \Psi(nT_1) \rangle|^2 \, d\kappa = X + Y \cos \left( \frac{E_1 - E_0}{dF} \pi n + \varphi \right),$$  \hspace{1cm} (24)

$$\frac{\pi}{2d} \int_{-\pi/2d}^{\pi/2d} |\langle \chi_{1,\kappa} | \Psi(nT_1) \rangle|^2 \, d\kappa = 1 - \left[ X + Y \cos \left( \frac{E_1 - E_0}{dF} \pi n + \varphi \right) \right],$$  \hspace{1cm} (25)

where $X$ and $Y$ are real positive numbers. This follows from the equations (18) and (19) by a straightforward calculation. In the case of $T_2$ and $T_1$ being commensurate, these equations read

$$\frac{\pi}{2d} \int_{-\pi/2d}^{\pi/2d} |\langle \chi_{0,\kappa} | \Psi(nT_1) \rangle|^2 \, d\kappa = X + Y \cos \left( \frac{2\pi}{r} \frac{s}{r} n + \varphi \right),$$  \hspace{1cm} (26)

$$\frac{\pi}{2d} \int_{-\pi/2d}^{\pi/2d} |\langle \chi_{1,\kappa} | \Psi(nT_1) \rangle|^2 \, d\kappa = 1 - \left[ X + Y \cos \left( \frac{2\pi}{r} \frac{s}{r} n + \varphi \right) \right],$$  \hspace{1cm} (27)

where one recognizes the complete reconstruction at multiples of $T_{BZ}$. In the case of $T_2$ and $T_1$ being incommensurate, $\varphi = 0$ holds whenever only one of the bands is initially occupied.
We now present some numerical results for the dynamics given by the Schrödinger equation (3) with a period-doubled potential. The time evolution has been calculated using a split-operator method \[25\], where the initial state is a real \((\kappa_0 = 0)\) normalized gaussian

\[\Psi(t = 0, x) \sim e^{-(x/60)^2}\]  \hspace{1cm} (28)

throughout this section. Figure 5 shows the results for two different values of \(\epsilon\). The time \(t\) is given in units of the Bloch time \(T_B\) of the single-band system \(\epsilon = 0\).

In order to understand the dynamics of the wave packet, it is instructive to consider the time evolution in quasi-momentum space. Remember that the motion of a wave packet under the influence of a constant force \(F\) in quasi-momentum space (cf figure 4) follows the acceleration theorem

\[\kappa(t) = \kappa_0 - Ft/\hbar.\]  \hspace{1cm} (29)

The main part of the wave packet shows a superposition of Bloch oscillations and Zener tunnelling between the two minibands, the Bloch–Zener oscillations. Zener tunnelling takes place almost exclusively when the wave packet reaches the edge of the reduced Brillouin zone, where the tunnelling rate strongly depends on \(\epsilon\). In figure 5 one clearly sees the splitting of the wave packet due to Zener tunnelling around \(t = T_B/4\). In general, the fractions from the two bands interfere, which gives rise to splitting and reconstruction. The parameter \(\epsilon\) in figure 5 is chosen in a way that reconstruction appears after one or two Bloch times, respectively.

The group velocity of a wave packet is proportional to the slope of the dispersion relation. At the edge of the Brillouin zone the slope of the wave packet does not change for the fraction of the wave packet which tunnels into the other miniband but does so for the fraction remaining in its miniband. The results are two interfering oscillations of different amplitudes as shown in figure 5.

The fractions of the wave packet in higher bands escape to \(-\infty\) quite quickly. Depending on the slope of the particular band (see figure 4) this happens initially in the direction of the positive or negative \(x\)-axis.

### 3.2. Transport

In section 2.2, we presented a transport mechanism based on a periodic field flip. The whole process can be understood within a single-band approximation. Even for a two-band system
Table 1. Sequence of sign parameters for figure 6.

| $t/T_B$ | 0–0.5 | 0.5–1 | 1–1.5 | 1.5–2 |
|-------|-------|-------|-------|-------|
| $\epsilon$ | + | − | − | + |
| $F$ | + | + | − | − |

Figure 6. Transport of a gaussian wave packet. Shown is $|\Psi(x, t)|$ as a colourmap plot for $|\epsilon| = 0.0825$. The parameter sequence is shown in table 1.

one can achieve transport of a gaussian wave packet by switching the signs of some system parameters. In order to reach transport not only the field strength $F$ but also the amplitude of the double-periodic part of the potential in equation (3), the parameter $\epsilon$ has to be flipped periodically. We can achieve transport by the sequence of parameters given in table 1.

Apart from loss, this transport can be continued arbitrarily far by repetition of the parameter sequence. In figure 6 the modulus of the wavefunction is shown. One can easily see the fractions of the wave packet which move away from the main part at the beginning. They are caused by the occupation of higher bands as described in the previous section. Whenever the split wave packet interferes at the edge of the Brillouin zone, the loss increases.

3.3. Beam splitter

The motion of a wave packet under the influence of a constant force $F$ in quasi-momentum space (cf figure 4) follows the acceleration theorem (29). Whenever a wave packet reaches the edge of the Brillouin zone, it can partially tunnel into the other miniband, leading to a splitting of the single-particle wave function in position space (cf figure 5). A permanent splitting of the wave packet can be achieved by transporting the two fractions into opposite directions. Thus one can realize a beam splitter in the period-doubled system (3) by applying the parameter sequence shown in table 2 (cf figure 7). The two branches of a Bloch–Zener oscillation at $t = 0.5 T_B$ are transported in opposite directions by switching the sign of $F$ twice, once at $t = 0.5 T_B$ and once at $t = T_B$. Since the value of $\epsilon$ is set to zero after $t = 0.5 T_B$, the transport process which separates the two branches is the same as the transport process described in section 2.2. After $t = T_B$ the field strength is constant and $\epsilon = 0$. Therefore the two wave packets continue performing ordinary Bloch oscillations.

Another method to split the wave packet is given by the parameter sequence in table 3 which leads to the dynamics shown in figure 8. Here we again achieve separation of the two branches
Table 2. Sequence of signs of parameters for figure 7 (0 indicates that the parameter is set to zero).

| $t/T_B$ | 0–0.5 | 0.5–1 | 1–1.5 | 1.5–3 |
|---------|--------|--------|--------|--------|
| $\varepsilon$ | +      | 0      | 0      | 0      |
| $F$     | +      | –      | +      | +      |

Figure 7. Splitting of a gaussian wave packet in position space. Shown is $|\Psi_1(x, t)|$ as a colour map plot for $|\varepsilon| = 0.0825$. The parameter sequence is shown in table 2.

by switching the sign of $F$. The main difference to the case above is that $\varepsilon$ is held constant during the whole process. Thus the two fractions of the split wave packet show Bloch–Zener oscillations instead of ordinary Bloch oscillations.

It is remarkable that the whole splitting process takes place with very little loss. Furthermore the process shown in figure 7 can be used to separate the two fractions of the wave packet at nearly arbitrary distance. To clarify this, figure 9 shows the dynamics of a gaussian wave packet for the parameter sequence in table 4. Because the transport of the wave packet takes place for $\varepsilon = 0$, the loss decreases strongly with an increasing width of the wave packet (see section 2.2). This is proven in the tight-binding approximation in the appendix.

In addition, the control of the occupation in both branches of the split wave packet is quite easy by Bloch–Zener oscillations. For the splitting process analogous to figure 7, only the occupation in the upper and lower branch at $t = T_B/2$ is relevant. This occupation can be adjusted by the variation of $\varepsilon$ (see figure 10).

In the above considerations, $\varepsilon$ was chosen in a way that the wave packet would reconstruct after a single Bloch time as long as the parameters are not switched. This choice is not mandatory. Even if $\varepsilon$ is chosen in a way that there is no reconstruction, we obtain two clearly distinguishable wave packets at $t = T_B/2$. In the picture of Bloch bands, these are just the fractions in the two different minibands. They are separated in position space by the different group velocities respectively the different slopes of the dispersion relations of the two lowest minibands. Those fractions of the wave packet, which tunnel at the edge of the Brillouin zone, will have a different group velocity to the remaining part of the wave packet. The tunnelling fraction of the wave packet can be controlled by the choice of $\varepsilon$ which gives an approximate measure for the band gap. Therefore we obtain a nearly pure Bloch oscillation for small $\varepsilon$ which transports the wave
Table 3. Sequence of signs of parameters for figure 8.

| $t/T_B$ | 0–0.5 | 0.5–1 | 1–1.5 | 1.5–3 |
|--------|-------|-------|-------|-------|
| $\varepsilon$ | + | + | + | + |
| $F$ | + | − | − | − |

Figure 8. Splitting of a gaussian wave packet in position space. Shown is $|\Psi(x, t)|$ as a colourmap plot for $|\varepsilon| = 0.0825$. The parameter sequence is shown in table 3.

Table 4. Sequence of signs of parameters for figure 9 (0 indicates that the parameter is set to zero).

| $t/T_B$ | 0–0.5 | 0.5–1 | 1–1.5 | 1.5–2 | 2–6 | 6–9 |
|--------|-------|-------|-------|-------|-----|-----|
| $\varepsilon$ | + | 0 | 0 | 0 | 0 | 0 |
| $F$ | + | − | + | − | ± Alternating with $T_B/2$ | + |

Figure 9. Splitting of a gaussian wave packet in position space. Shown is $|\Psi(x, t)|$ versus space and time with $|\varepsilon| = 0.0825$. The parameter sequence is shown in table 4.
packet to the lower position range. By the choice of large $\varepsilon$, the bands become separated and thus the Zener transitions are suppressed. Hence the wave packet stays in the lower miniband respectively in the upper position range.

The intermediate range $-0.2 \leq \varepsilon \leq 0.2$, where the occupation probability varies strongly with $\varepsilon$ is of special interest. In this area, the occupation of both branches of the beam splitter can be adjusted very exactly (compare figure 10). The distribution follows approximately the Landau–Zener formula [26, 27] according to which the tunnelling probability dependence of $\varepsilon$ is a gaussian distribution.

In conclusion, the splitting of a wave packet within a periodic potential can be done easily and with very little loss by a Bloch–Zener oscillation as shown in figures 7 and 9.

3.4. Mach–Zehnder interferometry

A very useful application of Bloch–Zener oscillations is matter wave interferometry. To this end we consider a Bloch–Zener oscillation that reconstructs after one Bloch time (see section 3.1). Because the wave packet splits up into two spatially separated parts in the meantime, we can insert an additional potential into one branch as illustrated in figure 11. Here, we apply a constant potential of strength $V_0$ in the range of $-195 \leq x \leq 195$ and within the time $0.45 T_B \leq t \leq 0.55 T_B$. After one Bloch time $T_B$, when both parts of the wave packet interfere again, we consider the probability density $|\Psi(x, T_B)|^2$. Figure 12 shows the squared modulus of the wavefunction $|\Psi(x, T_B)|^2$ in a range of $-800 \leq x \leq 200$ at the time $t = T_B$ versus the strength of the potential $V_0$.

One clearly sees that the probability distribution of the wave packet varies between the two output branches with the strength of the potential $V_0$. Depending on the phase shift that a fraction of the wave packet receives within its branch, we obtain constructive interference in the upper or the lower branch. The wave packet in the upper range is interpreted as the occupation of the lower band, the one in the lower range can be seen as the occupation of the upper band. In order to describe the interference effect more quantitatively, we integrate $|\Psi(x, T_B)|^2$ over the relevant regions. Figure 13 shows the $V_0$-dependence of the probability to find the wave packet at $t = T_B$ in the upper and the lower region of figure 12. Obviously, the probability oscillates with the
Figure 11. One branch of the split gaussian wave packet receives a phase shift by the additional square well potential $V_0$. Shown is $|\Psi(x, t)|^2$ as a colourmap plot for $\epsilon = 0.0825$ and $V_0 = 0$. The position of the square well potential (green) and the probability density considered in figure 12 at $t = T_B$ (red) are sketched.

Figure 12. Shown is a colourmap plot of the squared modulus of the wavefunction $|\Psi(x, t = T_B)|^2$ at $t = T_B$ versus the strength of the square well potential $V_0$ for $\epsilon = 0.0825$.

Phase shift in the upper branch (compare figure 11) caused by $V_0$. The period of one oscillation is $\Delta V_0 \approx 0.069$. This can be estimated by the phase shift in the upper branch

$$\frac{\Delta V_0 \ t_{\text{int}}}{\hbar} = 2\pi \quad \text{with} \quad t_{\text{int}} = \frac{1}{10} T_B = \frac{1}{10} \frac{\hbar}{F}.$$  

Thus we get

$$\Delta V_0 \approx 10 \cdot 2\pi \cdot F = 10 \cdot 2\pi \cdot 0.0011 \approx 0.069.$$  

It is remarkable that the probability in the upper branch never vanishes, whereas this is the case for the probability in the lower branch. The reason for this is that the occupation of the interfering branches is not exactly equal. If desired, this can be achieved, however, by an adequate choice of $\epsilon$ taken from figure 10. In the above examples we chose $\epsilon$ such that the wave packet reconstructs...
Figure 13. Integral of $|\Psi(x, t = T_B)|^2$ in figure 12 once over a range of $-800 \leq x \leq -300$ (blue) and once over a range of $-300 \leq x \leq 200$ (red, dashed), giving the density in the two output branches of the interferometer.

Figure 14. Splitting of a gaussian wave packet in position space. Shown is $|\Psi(x, t)|^2$ as a colormap plot for $\varepsilon = 0.0825$. The parameter sequence can be found in table 5.

for $V_0 = 0$, i.e. without an additional potential. The achieved contrast

$$\max \left[ \int_{-300}^{200} |\Psi|^2 \, dx \right] - \min \left[ \int_{-300}^{200} |\Psi|^2 \, dx \right] \approx 0.977$$

is very good and can even be improved by the choice of equally occupied bands. Therefore, the above method is suitable for probing weak potentials in the path of the wave packet.

Finally we want to point out that the splitting of a wave packet can be easily extended by repeated splitting (see figure 14). Thus even more complex interferometers could be realized. The parameter sequence for the variation of $F$ in figure 14 is given in table 5.

Mach–Zehnder interferometry by repeated Landau–Zener tunnelling in the energy domain was previously discussed for different systems (see, e.g. [28]–[30]). In contrast, Bloch–Zener oscillations also lead to a spatial separation of the two branches, which resembles much closer the original Mach–Zehnder set-up. Other interferometer set-ups using Bragg interactions to experimentally control the atomic motion are described, e.g. in [31].
4. Effects of atom–atom interaction

During recent years, the number of experiments investigating the dynamics of BECs in optical lattices increased rapidly. In this paper, we restrict ourselves to the ‘one-dimensional mean-field (1D-MF) regime’ as defined in [32] where the system is well described by the Gross–Pitaevskii or nonlinear Schrödinger equation (GPE, see, e.g. [32]–[34]).

\[
i\hbar \frac{\partial}{\partial t} \Psi(x, t) = \left[ -\frac{\hbar^2}{2} \frac{\partial^2}{\partial x^2} + \cos x + \varepsilon \cos \frac{x}{2} + Fx + g|\Psi(x, t)|^2 \right] \Psi(x, t).
\] (33)

The conditions to be in the 1D-MF regime are \(a_{3D}^2/a_0^2 \ll a_{3D} N_{1D}, a_{3D} n_{1D} \ll 1\) and \(a_{3D} N_{1D} \ll a_0^2/a_{3D}^3\), where the latter condition is automatically fulfilled by the former [33, 35]. Here \(n_{1D}\) is the 1D density, \(a_{3D}\) is the 3D s-wave scattering length and \(a_0^2 \propto 1/\omega_r\) is the oscillator length which is given by the radial harmonic trapping frequency \(\omega_r\). For a detailed discussion of the 1D-MF regime see [32, 33]. The above conditions have been derived for the case of a purely harmonic potential, but similar reasoning for our lattice scenario implies that equation (33) should hold as long as in addition to the above conditions, \(a_{3D} \ll a_1\) is fulfilled, where \(a_1\) is the extension of a Wannier state.

The wave function is normalized to \(\|\Psi\| = 1\) leading to a factor of \(N\) in the effective 1D interaction strength \(g\) which is given by \(g = g_{3D}/(2\pi a_0^2)\) [32]. Here, \(g_{3D} = 4\pi\hbar^2 a_{3D} N/M\) is the 3D interaction strength, \(N\) is the particle number and \(M\) is the mass of one particle. In terms of the 1D s-wave scattering length \(a_{1D}\) this reads \(g = 2\hbar^2 N/(Ma_{1D})\) where \(a_{1D} = a_0^2/a_{3D}\) [32]. For a detailed description of the 1D scattering length see [35]. Thus, from the mathematical point of view, the difference to the single-particle Hamiltonian is the nonlinear MF potential \(g|\Psi(x, t)|^2\).

The effective interaction constant can be tuned by changing the number of particles in the condensate, the size of the radial confinement \(a_0\), or directly by using a Feshbach resonance. It was shown that a weak nonlinear interaction leads to damping and revival phenomena of Bloch oscillations [36, 37] while a stronger interaction destroys these coherence effects immediately [38, 39]. In the following we will discuss the effects of the MF interaction on Bloch–Zener oscillations and we propose an interferometric method to probe very weak nonlinearities in section 4.2.

4.1. Nonlinear Bloch bands and instability

In the nonlinear case one can also define Bloch states as stationary solutions of the GPE (33) of the form \(\Psi(x) = e^{ix}\tilde{u}_\kappa(x)\), where \(\kappa\) is the quasi-momentum and \(u_\kappa(x) = \tilde{u}_\kappa(x + d)\). The Bloch functions fulfill the stationary GPE

\[
-\frac{\hbar^2}{2} \left[ \frac{\partial}{\partial x} + ik \right]^2 u_\kappa(x) + \left[ \cos x + \varepsilon \cos \frac{x}{2} + g|u_\kappa(x)|^2 \right] u_\kappa(x) = \mu u_\kappa(x).
\] (34)
In the following examples we fix the normalization of the Bloch states as
\[
\frac{1}{4\pi} \int_0^{4\pi} |u_\kappa(x)|^2 \, dx = 1, \tag{35}
\]
i.e. the mean density is unity, \( \bar{n} = 1 \). Due to the nonlinearity of the GPE, novel nonlinear eigenstates may be found if the nonlinearity \( g \) exceeds a critical value. These states are not eigenstates in the sense of linear algebra, rather they should be interpreted as stationary states of a nonlinear dynamical system. For a periodic potential as in equation (34), the novel nonlinear eigenvalues appear as looped Bloch bands. For a repulsive nonlinearity, \( g > 0 \), the loops appear at the edges of the lowest band if \( g \bar{n} \) exceeds a critical value \( g_c \) \[40\].

The emergence of looped Bloch bands is illustrated in figure 15. A loop is observed in the ground miniband at the edge of the Brillouin zone for \( g \bar{n} = 0.4 > g_c \), while the first excited miniband is flattened. No loop has yet emerged for \( g \bar{n} = 0.1 < g_c \), however, the ground miniband is sharpened.

For small external fields \( F \) the system may follow these nonlinear Bloch bands adiabatically up to the point where the nonlinear Bloch state vanishes in a bifurcation, i.e. up to the edge of the loop. Here, adiabaticity breaks down and the Zener tunnelling probability to the first excited band does not tend to zero even in the adiabatic limit \( F \to 0 \). But even below \( g_c \) the Zener tunnelling probability from the ground to the first excited band is enhanced while the tunnelling probability from the first excited to the ground band is reduced \[38\], \[40\]–[42].

Experiments showing the dependence of Zener tunnelling on MF interactions have been carried out in the last years \[39, 43, 44\]. However, systems with looped bands, i.e. \( g \bar{n} > g_c \), are hard to realize in a conventional optical lattices. The double-periodic potential discussed in the present paper now offers the unique possibility to tune the band gap \( \delta \) between the lowest and the first excited miniband and thus \( g_c \) in a wide range by adjusting the relative strength \( \epsilon \) of the second lattice. This is shown on the right-hand side of figure 15, where the dependence of the band gap \( \delta \) and \( g_c \) on \( \epsilon \) is plotted. The critical nonlinearity \( g_c \) is approximately proportional to \( \delta \). Both the band gap \( \delta \) and \( g_c \) increase nearly linearly with \( \epsilon \). However this is not correct in the limit.
\( \varepsilon = 0 \), where one recovers a common single-periodic optical lattice. Then the Brillouin zone is again \([-1/2, +1/2]\) and no loops emerge at \( \kappa = \pm 1/4 \).

The dynamical effects of the nonlinear MF interaction are studied by numerically integrating the Gross–Pitaevskii-equation (3) using a split-step method [25, 45]. As an initial state, we use a gaussian wave packet

\[ \Psi(x, t = 0) \sim e^{-x^2/4\sigma^2} \]  

with \( \sigma = 40\pi \), which is projected onto the ground band and then renormalized to unity. Figure 16 shows the dynamics of the squared modulus of the wave function for \( F = 0.005 \), \( g = 10 \) and \( \varepsilon = 0 \) (left panel), \( \varepsilon = 0.1 \) (middle panel) and \( \varepsilon = 0.4 \) (right panel). For \( \varepsilon = 0 \) one observes the familiar Bloch oscillations. The effects of the MF interaction are weak and can be explained in terms of nonlinear dephasing [36].

For \( \varepsilon \neq 0 \), the fundamental period of the potential is doubled. As discussed in the preceding sections, the period and the spatial width of the Bloch oscillations will be halved as long as Zener tunnelling can be neglected. This is observed in figure 16 for \( \varepsilon = 0.4 \). For \( \varepsilon = 0.1 \) however, the dynamics becomes unstable at the edge of the Brillouin zone. This is understood in terms of the breakdown of adiabaticity due to looped Bloch bands discussed above. For the given density, a loop has emerged in the ground band for \( \varepsilon = 0.1 \) but not for \( \varepsilon = 0.4 \). Thus we face the unfamiliar effect that a weaker perturbation (a smaller value of \( \varepsilon \)) has a much stronger effect on the dynamics.

In order to analyse this effect quantitatively, we have calculated the time evolution of the wave packet (36) for different values of \( F \) and \( g \). Figure 17 shows the dependence of the autocorrelation after half of the Bloch time on the nonlinearity \( g \) and the field strength \( F \). Here, one should keep in mind that we use the Bloch time of the single periodic system as reference timescale. In the linear case, \( g = 0 \), perfect Bloch oscillations are found for \( F \rightarrow 0 \) and the autocorrelation tends to one. Parts of the wave function will tunnel to the excited miniband for \( F > 0 \), such that the autocorrelation decreases with \( F \). The nonlinear MF interaction disturbs this coherent dynamics and thus the autocorrelation decreases with \( g \). This decrease is most significant in the adiabatic limit, i.e. for small values of \( F \). We conclude that double-periodic optical potentials are particularly suitable to study the dynamical instability due to looped Bloch bands.

**Figure 16.** Nonlinear Bloch–Zener oscillations. Time evolution of the probability density for \( \hbar = 2.828 \), \( F = 0.005 \), \( g = 10 \) and \( \varepsilon = 0 \) (left panel), \( \varepsilon = 0.1 \) (middle panel) and \( \varepsilon = 0.4 \) (right panel).
4.2. Probing small nonlinearities

Here, we will demonstrate that Bloch–Zener oscillations can be used to probe very weak nonlinearities, the effect of which would be negligible otherwise. This can be achieved by the Mach–Zehnder interferometer set-up introduced in the preceding section. Here the phase shift in both branches of the interferometer is caused by the nonlinear MF potential. A difference of the condensate density in both branches will lead to a phase shift of both wave packets depending on the interaction constant $g$. Thus, the condensate density in the two output branches of the interferometer will also vary with $g$. To analyse this effect the time evolution of a normalized gaussian wave packet (36) with $\sigma = 10$ over one Bloch period was calculated numerically for $\varepsilon = 0.104$ and different values of the interaction strength $g$. The resulting density in both output branches of the interferometer calculated by integrating $|\Psi(x, T_B)|^2$ over the respective spatial intervals (cf figure 10) are plotted in figure 18. One observes that the output depends strongly on the interaction strength even for quite small values of $g$. For the considered parameter values nonlinear damping or dephasing effects are still negligible [36]. The only noticeable effect of the nonlinearity is the influence on the interferometer output. Therefore it should be possible to probe very weak interactions with an interferometric set-up as described here.

5. Conclusion and outlook

In conclusion, we have investigated the possibility of engineering the dynamics of matter waves in periodic potentials. A wave packet in a periodic potential under the influence of a static field performs an oscillatory motion, the Bloch oscillation. Directed quasi dispersion-free transport can be realized by a periodic field flip.

Introducing a weak additional, double-periodic potential offers even richer opportunities. The Bloch bands split into two minibands, so that the interplay between Zener tunnelling and Bloch oscillations, denoted as Bloch–Zener oscillations, becomes important. Tunnelling between the minibands leads to splitting of a wave packet and interference phenomena. Combined with the shuttling transport mechanism, one can implement highly controllable matter wave beam splitters. Furthermore, Bloch–Zener oscillations provide a natural Mach–Zehnder interferometer.
Figure 18. Occupation probability in the output branches of a nonlinear Mach–Zehnder interferometer. The integral of $|\Psi(x, T_B)|^2$ once over a range of $-800 \leq x \leq -300$ (lower output branch, solid blue line) and once over a range of $-300 \leq x \leq 200$ (upper output branch, dashed red line) are plotted versus the interaction strength $g$ for $\varepsilon = 0.104$.

for matter waves. This interferometer can be used, e.g. to probe weak nonlinear MF interactions in BECs. Furthermore, the influence of the MF interaction on the dynamics by changing the shape of the Bloch bands and the possibility of breakdown of adiabaticity have been analysed and illustrated with numerical examples.

The techniques described in this paper should be experimentally realizable without major problems, for instance in setups as described in [10]. A possible application of the Mach–Zehnder interferometer could be the detection of the MF interaction in an atom laser beam.

Up to now we considered only weak external fields $F$, for which decay is negligible. Since Wannier–Stark eigenstates are resonance states [19], the eigenenergies (the Wannier–Stark ladders) are generally complex, i.e. we obtain decay for strong fields. Decay rates for the nonlinear Wannier–Stark problem were first calculated only recently [46, 47]. The role of decay is a bit more involved in double-periodic potentials since the dynamics usually takes place in two miniladders instead of a single one. The splitting, altering and shifting of resonant tunnelling peaks of the decay rate for the two miniladders are discussed in more detail in [24].

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Appendix. Proof that the dispersion of a gaussian wave packet is proportional to $t^2/\sigma^4$

This result can be proven rigorously in a single-band tight-binding approximation. Here, the Hamiltonian reads

$$\hat{H}(t) = -\frac{\Delta}{4} \sum_n \langle n+1 | \langle n | + | n-1 \rangle \langle n |) + dF(t) \sum_n n |n\rangle \langle n |,$$

(A.1)
where $|n\rangle$ is the Wannier state located at the $n$th lattice position. The initial state in the Wannier basis

$$|\Psi(0)\rangle = \sum_{n} c_n |n\rangle$$  \hspace{1cm} (A.2)

is assumed to be gaussian, i.e.

$$c_n \sim e^{-n^2/4\sigma_n^2}. \hspace{1cm} (A.3)$$

The time evolution of the position expectation value of the wave packet and its square can be calculated conveniently using the Lie-algebraic approach introduced in [20]. If the initial state is symmetric with respect to the origin and the coefficients $c_n$ are real, as assumed throughout this section, the results from [20] simplify and one finds

$$\langle \hat{N} \rangle_t = 2K |\chi_t| \sin(\phi_t) \quad \text{and} \quad \langle \hat{N}^2 \rangle_t = \langle \hat{N}^2 \rangle_0 + 2|\chi_t|^2 (1 - L \cos (2\phi_t)) \hspace{1cm} (A.4)$$

with the position operator

$$\hat{N} = \sum_{n} n|n\rangle \langle n| \hspace{1cm} (A.5)$$

and with the coefficients

$$\eta_t = \int_{0}^{t} \frac{dF(\tau)}{\hbar} d\tau \quad \text{and} \quad \chi_t = -\frac{\Delta}{4\hbar} \int_{0}^{t} e^{-i\eta_{\tau}} d\tau =: |\chi_t| e^{-i\phi_t}. \hspace{1cm} (A.6)$$

The time evolution depends on the initial state through the coherence parameters

$$K = \sum_{n} c_{n-1} c_n \quad \text{and} \quad L = \sum_{n} c_{n-2} c_n. \hspace{1cm} (A.7)$$

The dispersion of a wave packet subjected to the Bloch shuttle $F(t) = \pm F_0$ is given by the increase of the width

$$\Delta_N^2(t) = \langle \hat{N}^2 \rangle_t - \langle \hat{N} \rangle_t^2 \hspace{1cm} (A.8)$$

of the wave packet

$$\Delta_N^2(nT_B) - \Delta_N^2(0) = 2n^2 |\chi_t|^2 (1 + L - 2K^2). \hspace{1cm} (A.9)$$

For a broad initial state one can replace the sums in (A.7) by integrals which yields

$$K = \frac{\int e^{-(n-1)^2/4\sigma_n^2} e^{-n^2/4\sigma_n^2} dn}{\int e^{-n^2/2\sigma_n^2} dn} = e^{-1/8\sigma_n^2},$$

$$L = \frac{\int e^{-(n-2)^2/4\sigma_n^2} e^{-n^2/4\sigma_n^2} dn}{\int e^{-n^2/2\sigma_n^2} dn} = e^{-1/2\sigma_n^2}. \hspace{1cm} (A.10)$$
Thus the dispersion is given by
\[ \Delta_n^2(nT_B) - \Delta_n^2(0) = \frac{2\Delta^2}{F_0^2d^2} \left( 1 + e^{-1/2\sigma_n^2} - 2e^{-1/4\sigma_n^2} \right) n^2 = \frac{\Delta^2}{8F_0^2d^2\sigma_n^4} n^2 + O(\sigma_n^{-6})n^2, \] (A.11)
where \( \chi_{TB} = i\frac{\Delta}{\Delta_1} \) has been inserted. Since we still have to change from the Wannier basis to the position space we set
\[ \sigma_x \approx d\sigma_n \quad \text{and} \quad \Delta_x(t) \approx d/D_{\Delta 1}N(t). \] (A.12)

Thus the dispersion is in leading order given by
\[ \Delta_n^2(nT_B) - \Delta_n^2(0) \approx \frac{\Delta x^2 d^4}{8F_0^2 \sigma_x^4}. \] (A.13)

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