Integer Quantum Hall Effect with Realistic Boundary Condition: Exact Quantization and Breakdown.

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Abstract

A theory of integer quantum Hall effect (QHE) in realistic systems based on von Neumann lattice is presented. We show that the momentum representation is quite useful and that the quantum Hall regime (QHR), which is defined by the propagator in the momentum representation, is realized. In QHR, the Hall conductance is given by a topological invariant of the momentum space and is quantized exactly. The edge states do not modify the value and topological property of $\sigma_{xy}$ in QHR. We next compute distribution of current based on effective action and find a finite amount of current in the bulk and the edge, generally. Due to the Hall electric field in the bulk, breakdown of the QHE occurs. The critical electric field of the breakdown is proportional to $B^{3/2}$ and the proportional constant has no dependence on Landau levels in our theory, in agreement with the recent experiments.
§1. Introduction

We study the quantum Hall effect in realistic systems in the present paper. Especially finite size effects and finite current effects are investigated in details.

Two-dimensional electrons in a strong perpendicular magnetic field have discrete energies with a magnetic field dependent finite degeneracy per area at each energy. When degenerate electrons are distributed uniformly, distance between two electrons has a minimum value. Conjugate momentum, hence, is defined on a torus. Thus the momentum space is compact. The energy space seems not to be compact but physical requirement to the propagator at $p_0 = \pm \infty$ or $\pm i\infty$ leads the energy space defined by the propagator compact. A representation of having these properties in a manifest manner was constructed based on von Neumann lattice of guiding center coordinates, and exact low energy theorems were given based on this representation $^1,^2$. Namely, it was shown that Hall conductance $\sigma_{xy}$ is a topological invariant of a mapping from the momentum space to a space defined by the propagator, and the quantization of the $\sigma_{xy}$ as $(e^2/h)N$ at the plateau, thus, is exact in systems of interactions and disorders. One-particle properties in systems of short range impurities, boundary potential, and periodic array of potentials have been obtained $^3$ also based on the same representation. Localization is shown easily in our method.

From our previous investigations, the $\sigma_{xy}$ becomes a topological invariant of the momentum space and is quantized as $(e^2/h)N$ exactly at the plateau if all the one-particle states around Fermi energy are either localized or have finite energy gap. Hereafter we call this energy region as quantum Hall regime(QHR). In finite system with boundary, there are edge states$^4$ which are extended along the edge and have continuous energy across the Fermi energy, generally. They may give a finite correction to the quantized value. We will show in the present work that a resummation formula of the $\sigma_{xy}$ with the momentum representation is valid in finite systems as well and QHR is defined based on the propagator in the momentum representation. The outside regions of the continuous band of propagator in the
resummation formula agrees with QHR. Perturbative series converges well in QHR. Consequently, a finite size correction disappears in QHR. Niu and Thouless\textsuperscript{5} claimed before that the finite size correction is of exponential type. Our results show that even an exponential correction does not exist in QHR.

The quantized Hall conductance\textsuperscript{6} is used as a standard of resistance and to determine fine structure constant, $\alpha$. Its precise value is needed in testing quantum electrodynamics. The theoretical foundation of the quantum Hall effect (QHE) is, however, still insufficient. Especially there is a controversy in current distribution in a finite system and in finite size corrections of the QHE. Breakdown of the QHE is also observed recently if the current becomes large.

Experiments are done with semi-conductors of Hall bar geometry. There are edges in such systems. Concerning edge states, a controversy is the following. In one method for a proof of QHE, Böttiker-Landauer\textsuperscript{7} formula is used. It is assumed that the one-dimensional edge states are the only current carrying states around the Fermi energy and are connected with the leads. The quantization of the Hall conductance due to the one-dimensional edge states was derived. A correction may be caused by backscattering of edge states. In other approaches, the bulk states carry the currents and their roles are important. The edge states might give a correction to the quantized value, because they are extended in one direction and have continuous energies. The role of edge states and bulk states are thus reversed. Two approaches are thus very different. It is a purpose of the present work to resolve these controversies and to give a foundation of the quantum Hall effect in finite systems.

We study a finite size effect and a finite current effect based on von Neumann lattice representation in the present paper. We show that the momentum representation and resummations of the diagrams based on the momentum representations are valid in finite systems and the QHR is defined as the outside region of the continuous energy band using the resummation formula of the $\sigma_{xy}$. It is shown that the quantization is exact in QHR of
finite systems despite the fact that the edge states have continuous energies and cross Fermi energy. There are current flow both in the bulk and near the edges generally and due to the Hall electric field at the bulk, the QHR becomes narrower as the current becomes larger and eventually disappears at a critical value. QHE is broken, then. We estimate a critical Hall electric field and find an agreement with the experiments.

The paper is organized in the following manner. We review our representation of two-dimensional electrons in the magnetic field based on magnetic von Neumann lattice in the rest of Section 1. In Section 2, one-particle properties are studied. Exact low energy theorem concerning the slope of current correlation function in infinite system are given in Section 3. Topological invariant expression of Hall conductance is given and the corrections due to interactions and disorders are shown to disappear in QHR. Finite system with cylinder geometry is discussed in Section 4 and finite systems with Hall bar geometry, which is the geometry of realistic experiments, are discussed in Section 5. In Section 6, the current distribution is computed with a use of effective potential. Breakdown of QHE is also discussed. A critical value of Hall electric field from our theory is proportional to $B^{3/2}$ and the proportional constant is independent of Landau levels. These properties are in agreement with the recent experiments by Kawaji et al. Summary is given in Section 7.

We review our representation\textsuperscript{1,2,3} of two-dimensional electrons in a strong perpendicular magnetic field here. It is based on von Neumann magnetic lattice\textsuperscript{9} and has excellent properties for a purpose of studying one-particle properties and of giving the exact low energy theorem of quantum field theory in quantum Hall system. Localized states are studied with the coordinate representation and extended states are studied with the momentum representation.

A one-body Hamiltonian of a planar charged particle with a strong perpendicular magnetic field,
\[ H = \frac{(\vec{p} + e\vec{A})^2}{2m}, \quad \partial_1 A_2 - \partial_2 A_1 = B, \] (1.1)

is expressed as

\[ H = \frac{e^2 B^2}{2m}(\xi^2 + \eta^2), \] (1.2)

with two-dimensional relative coordinates in the magnetic field \(^{10}\). They are defined by

\[
\begin{align*}
\xi &= \frac{1}{eB}(p_y + eA_y), \\
\eta &= -\frac{1}{eB}(p_x + eA_x), \\
[\xi, \eta] &= -\frac{i\hbar}{eB},
\end{align*}
\] (1.3)

and commute with another set of coordinates, center coordinates, \((X, Y)\) defined by

\[
\begin{align*}
X &= x - \xi, \\
Y &= y - \eta, \\
[X, Y] &= \frac{i\hbar}{eB}, \\
[\xi, X] &= [\xi, Y] = [\eta, X] = [\eta, Y] = 0.
\end{align*}
\] (1.4)

It is convenient to use variables \((X, Y)\) and \((\xi, \eta)\), instead of \((x, y)\) for studying the electrons in the magnetic field. A set of well localized functions,

\[
\begin{align*}
f_l(\xi, \eta) \otimes |R_{m,n}\rangle,
\end{align*}
\] (1.5)
defined by

\[ H_0 f_l(\xi, \eta) = E_l f_l(\xi, \eta), \quad E_l = \frac{\hbar e B}{m} (l + \frac{1}{2}), \]

\[ |R_{m,n}\rangle = (-1)^{m+n+1} e^{A^\dagger \sqrt{\pi} (m+i) - A \sqrt{\pi} (m-i)} |0\rangle, \]

\[ \langle R_{m_1,n_1}|R_{m_2,n_2}\rangle = e^{i\pi [(m_1-m_2+1)(n_1-n_2+1)-1]} e^{-\pi/2} [(m_1-m_2)^2+(n_1-n_2)^2], \]

\[ \sum_{m_1,n_1} \langle R_{m_1,n_1}|R_{m_2,n_2}\rangle = \sum_{m_2,n_2} \langle R_{m_1,n_1}|R_{m_2,n_2}\rangle = 0, \quad (1.6) \]

\[ R_{m,n} = a(m,n), \quad m, n : \text{integer}, \quad a = \sqrt{\frac{2\pi \hbar}{eB}}, \]

\[ A|0\rangle = 0, \quad A = \sqrt{\frac{\hbar}{2e}} (X + iY), \]

\[ A|R_{m,n}\rangle = \sqrt{\pi} (m+i)|R_{m,n}\rangle, \]

is a complete set and is used as base functions. Dual basis \( \tilde{R}_{m,n} \) is defined, with a Green’s function \( G(m_1, n_1; m_2, n_2) \), by

\[ \langle \tilde{R}_{m_1,n_1}| = \sum_{m_2,n_2} G(m_1, n_1; m_2, n_2) \langle R_{m_2,n_2}|, \]

\[ \sum_{m',n'} G(m_1, n_1; m', n') \langle R_{m',n'}|R_{m_2,n_2}\rangle = \delta_{m_1,m_2} \delta_{n_1,n_2} - 1/N, \quad (1.7) \]

\[ \sum_{m,n} = N. \]

\( G(m_1, n_1; m_2, n_2) \) is well localized in small \( |m_1 - m_2| \) and \( |n_1 - n_2| \) region as is shown in Ref.(3). We expand the electron field as,

\[ \Psi = \sum_{l,m,n} a_l(m,n) f_l(\xi, \eta) \otimes |R_{m,n}\rangle, \]

\[ \Psi^\dagger = \sum_{l,m,n} b_l(m,n) f_l(\xi, \eta) \otimes (\tilde{R}_{m,n}), \quad (1.8) \]

and regard coefficients \( a_l(m,n) \) and \( b_l(m,n) \) as quantized operators. The free kinetic term,
potential term, and the electromagnetic current are expressed with these new variables as
\[
\int d\vec{x} \Psi^\dagger(x) \frac{(\vec{p} + e\vec{A})^2}{2m} \Psi(x) = \sum_{l,m,n} E_l b(m,n) a(m,n),
\]
\[
\int d\vec{x} \Psi^\dagger(x) \Psi(x) V(x) = \sum_{l,m,n} b_l(m_1,n_1) a_l(m_2,n_2) V_{l_1,l_2}(m_1,n_1;m_2,n_2),
\]
\[
J_\mu(x) = \sum_{l,m,n} b_l(m_1,n_1) a_l(m_2,n_2) \Gamma_{\mu}^{l_1,l_2}(m_1,n_1;m_2,n_2;x),
\]
\[
V_{l_1,l_2}(m_1,n_1;m_2,n_2) = \int \frac{d^2k}{(2\pi)^2} \langle \tilde{R}_{m_1,n_1} e^{i\vec{k}\vec{X}} | R_{m_2,n_2} \rangle (f_{l_1} e^{i\vec{k}\vec{X}} f_{l_2}) V(\vec{k}),
\]
Advantages of the present representation are summarized in the following:

(1) The kinetic term is diagonal in the Landau level index and two-dimensional lattice coordinates and only the potential term has off diagonal terms. Hence the localization by short range random impurities in energy regions \(|E - E_l| > \delta\) with a small impurity dependent parameter \(\delta\) are shown easily in the present method.

(2) Functions used as basis and dual basis are well localized around center coordinate \(R_{m,n}\). Hence the multi-pole expansion of current operator itself and of commutation relations between the current and field operators are applicable. Due to these properties it is a straightforward matter to derive Ward-Takahashi identity and related identities as well as the exact low energy theorem. Coordinate dependence of a potential \(V(\vec{x})\) is preserved in \(V_{l_1,l_2}(m_1,m_2;n_1,n_2)\) with a smeared form. For instance, a short range potential,
\[
V(\vec{x}) = g\delta(x),
\]
is transformed into
\[
V_{0,0}(m_1,m_2;n_1,n_2) = \frac{g}{a^2} e^{-\frac{\pi}{4}(m_1^2+n_1^2+m_2^2+n_2^2)} [1 + O(e^{-\frac{\pi}{4}})],
\]
which is a short range potential with an extension of few magnetic distance. Other components \(V_{l_1,l_2}(m_1,m_2;n_1,n_2)\) have the same properties.
(3) Apart from disorder potentials, translational invariance is manifest and Fourier transformation of the operators can be defined, because magnetic translations commute each other and constitute abelian group in our lattice system. Extended states can be studied with the momentum representation.

(4) The coordinates are defined on lattice sites. Since the momentum defined by Fourier transformation from lattice sites is defined on a torus, which is compact, the $\sigma_{xy}$ becomes a topological invariant in momentum space manifestly. Neither artificial boundary condition in configuration space nor periodic potential are needed.

§2. One-particle states in finite systems with disorders and interactions

We study one-particle properties of systems with one short range impurity, dilute short range impurities, boundary potential, and interactions between electrons in this section. Localized states are studied with the coordinate representation and the extended states are studied with the momentum representation. It is shown that the QHR which is defined by the propagator in the momentum representation exists in finite system if the magnetic field is of suitable strength.

(2-a) Disorder potentials

A Hamiltonian,

$$\int d\vec{x}\Psi^\dagger(x)[\frac{\vec{p}^2 + eA^2}{2m} + V(x)]\Psi(x) = \sum_{l_1,l_2,R_1,R_2} b_{l_1}(R_1)[E_{l_1}\delta_{l_1,l_2}\delta_{R_1,R_2} + V_{l_1,l_2}(R_1,R_2)]a_{l_2}(R_2)$$

(2.1)

discribes the systems with a disorder potential $V(x)$. The kinetic term becomes diagonal and the potential term becomes non-diagonal. In fact this term becomes a lattice kinetic term if a suitable periodic potential is used for $V(x)$, as was shown in Ref(3). Eigenstates are extended, then. For a short range potential, on the other hand, eigenstates of energy $E$ in region, $|E - E_l| > \delta$ with a small parameter $\delta$, becomes localized. This is easily seen from
the eigenvalue equation,

\[ \sum_{l_2, \vec{R}_2} V_{l_1,l_2}(\vec{R}_1, \vec{R}_2) u_{l_2}^{(\alpha)}(\vec{R}_2) = (E^{(\alpha)} - E_{l_1}) u_{l_1}^{(\alpha)}(\vec{R}_1). \]  

(2.2)

Eigenvector \( u_{l_1}^{(\alpha)}(m_1, n_1) \) has the same coordinates dependence as the transformed potential term \( V_{l_1,l_2}(\vec{R}_1, \vec{R}_2) \) provided \( E - E_l \neq 0 \). Hence an eigenvector is localized around a short range impurity if its energy \( E \) is away from Landau level energy \( E_l \). We confirmed this property of wave functions by solving the equation numerically. Wave functions of the energy \( E - E_l \neq 0 \) have spatial extensions of few magnetic lengths.

Many short range impurity problem is studied as easily as one impurity problem if impurities are dilute and random. Localized wave functions around one short range impurity decrease very fast and almost vanishes at nearby impurities if distances between impurities are much larger than the magnetic distance. Corrections of localized wave functions due to nearby impurities are thus very small and is treated perturbatively. If \( E - E_l \) is also very small, the situation is different and the energy denominator becomes very small and compatible to the small numerator in perturbative series. The correction can become order one, and perturbative series may be divergent, then, and wave function becomes extended. Consequently, one-particle property depends on its energy in the system with dilute random impurities. Wave function is localized if its energy \( E \) is in \( |E - E_l| > \delta \) with impurity dependent small parameter \( \delta \). The propagator in the momentum representation has no singularity in this energy region.

(2-b) Boundary potentials

A potential \( V_0 \theta(x) \) shows a potential barrier in the positive \( x \) region with a boundary at \( x = 0 \). Due to a translational invariance in \( y \)-direction, it is convenient to project the
potential and wave functions to those of definite momenta, as

\[ V_{l_1,l_2}(m_1, n_1; m_2, n_2) = \frac{a}{2\pi} \int_{-\pi/a}^{\pi/a} dp_y e^{ip_y(n_1-n_2)a}V_{l_1,l_2}(m_1; m_2; p_y), \]

\[ u_l(m, n) = \sqrt{\frac{a}{2\pi}} \int_{-\pi/a}^{\pi/a} dp_y e^{ip_y n_a}u_l(m, p_y). \]

The eigenvalue equation becomes,

\[ \sum_{l_2,m_2} \{E_{l_1}\delta_{l_1,l_2}\delta_{m_1,m_2} + V_{l_1,l_2}(m_1; m_2; p_y)\}u_l^{(\alpha)}(m_2, p_y) = E^{(\alpha)}u_l^{(\alpha)}(m_1, p_y). \] (2.4)

We solved the equation numerically. Edge states have continuous energy and wave functions are extended in y-direction and are confined in x-direction. As the boundary potential gives steep electric field in x-direction, edge states have current with definite direction and are regarded as chiral modes. They have been computed in the present representation and were given in Figs. 7 and 8 of Ref. (3). They have continuous energies in the wide range and bridge from one Landau level to next Landau level. Hence there are always one-dimensionally extended states around an arbitrary value of Fermi energy.

In systems with finite width in the x-direction, there are two edges. The wave functions change their properties depending upon their widths. To see them, we solve systems of potential wells in the x-direction numerically. As are shown in Fig.1 and Fig.2, if the width is large, two edge states are separated, but if the width is small, they are combined. In wide systems, one-dimensionally extended states have small overlap with the momentum eigenstates that are used in our method of computing the Hall conductance. We will see that they give no correction to the quantized Hall conductance in certain situation.

Now we find out when the edge terms and impurities give no correction. In the section 5 of the present paper, we apply the momentum representation of the \( \sigma_{xy} \) as in Ref. (1). The edge potential term is treated as perturbative term in the Hamiltonian and in the propagator. In this expansion, infra-red divergence is not involved unless the Fermi energy agrees to the Landau level energy. Moreover, the series converges well if the energy difference, \( E - E_l \),
is much larger than the perturbative energy of the states projected into the momentum eigenstates. These momentum eigenstates, although they are fictitious, play the important roles in our calculations and a band width of these fictitious extended states is important and is estimated in the following. If the Fermi energy is in the outside of this fictitious band, the convergence of the perturbative expansion is very good, and the low-energy theorem for the $\sigma_{xy}$ is derived easily. It should be noted that this can occur even if the real one-dimensional edge states bridge from one Landau level to next Landau level and have zero energy around the Fermi energy.

For estimation of the band width in the momentum representation we study eigenvalue equation of the Hamiltonian in one method, and we study the propagator in another method.

The band width in the first method is estimated directly from Eq.(2.2). The boundary potential in magnetic lattice representation $V_{l_1,l_2}(\vec{R}_1, \vec{R}_2)$ decreases rapidly with the distance between the coordinates $\vec{R}_1$ or $\vec{R}_2$ and the boundary and behave as

$$V(\vec{R}_1, \vec{R}_2) \sim V_0 \text{Max}(e^{-\frac{\pi}{a}(\frac{L_1}{2})^2}, e^{-\frac{\pi}{a}(\frac{L_2}{2})^2}),$$

(2.5)

where $V_0$ is the potential height and $L_1(L_2)$ is the distance. Hence the equation(2.2) becomes at a point $\vec{R}$ in the middle of the system,

$$|E^{(\alpha)} - E_{l_1}| = \frac{|\sum_{l_2} V_{l_1,l_2}(\vec{R}, \vec{R}') u^{(\alpha)}_{l_2}(\vec{R}')|}{|u^{(\alpha)}_{l_1}(\vec{R})|} \leq \frac{V_0 e^{-\frac{\pi}{a}(\frac{L}{2})^2}}{|u^{(\alpha)}_{l_1}(\vec{R})|}.$$  

(2.6)

An extended eigenstate which does not vanishes in whole space satisfies,

$$|u^{(\alpha)}_l(\vec{R})| \geq \frac{c}{\sqrt{L}}.$$  

(2.7)

By combining Eq.(2.6) and Eq.(2.7), we have

$$|E^{(\alpha)} - E_{l}| \leq cV_0 \sqrt{L} e^{-\frac{\pi}{a}(\frac{L}{2})^2}.$$  

(2.8)

The band width, $\Delta_{\text{edge}}$, thus satisfies,

$$\Delta_{\text{edge}} < cV_0 \sqrt{L} e^{-\frac{\pi}{a}(\frac{L}{2})^2}.$$  

(2.9)

The extended states in the momentum representation are generated from the extended states
which have energies within Eq.(2.8).

In the second method, we study the propagator. As is expressed in Ref.(1), the energy of the momentum eigenstates in the system of disorders can be defined and is expressed from diagrams of Fig.3. We calculate the lowest non-trivial order contribution here. The characteristic properties of the full order are known from them. The self-energy correction of the momentum \( p \) up to \( O(V^2) \) is given by

\[
\langle p | V | p \rangle + \frac{1}{E - E_l} \sum_{p_i \neq p} |\langle p | V | p_i \rangle|^2,
\]

where the state has definite momentum and the boundary potential of Eq.(5.5) and impurity potential are used. Since the boundary potential is not vanishing only in the finite region around the boundary of few magnetic length, the above quantity has the magnitude,

\[
\frac{aV_0}{L} + \frac{cV^2_0}{E - E_l} \left( \frac{a}{L} \right)^2,
\]

with magnetic field independent constant \( c \) and a width \( L \). If the magnetic field is strong enough, or the width \( L \) is large enough, then the self-energy correction due to the boundary potential or due to the boundary potential and the impurity potential becomes arbitrary small. Hence the width of the fictitious band which appears in our computation can be (much) smaller than the Landau level spacing if the \( L \) is large enough or the magnetic field is strong enough. If the Fermi energy is in the outside of this fictitious band, the system is in the QHR. The QHR can be realized in the finite systems.

(2-c) Electron interactions

Interactions also remove the degeneracy of Landau levels. We estimate band width due to Coulomb interactions based on perturbative calculation for momentum eigenstates of Fig.4. Within the \( l \)-th Landau level space, the propagator is written as,

\[
\frac{1}{p_0 - E_l - \Sigma(\vec{p}, p_0)},
\]

\[
\Sigma(\vec{p}) = \nu \int \frac{d\vec{k}}{(2\pi)^2} V(\vec{k}) |\langle f_l | e^{i\vec{k}\cdot\vec{x}} | f_l \rangle|^2,
\]

where \( \nu \) is the filling factor in the \( l \)-th Landau level. The eigenvalue of the energy is obtained
from the pole of the above propagator. The energy shift from the degenerate value $E_l$ is given by $\Sigma(\vec{p})$, which vanishes at $\nu = 0$, and agrees with

$$\Delta E = \int \frac{d\vec{k}}{(2\pi)^2} V(\vec{k}) |\langle f_l | e^{i k \xi} | f_l \rangle|^2 = e^2 \sqrt{2eB}. \quad (2.13)$$

at $\nu = 1$ in the lowest Landau level space. The difference of the energy correction between $\nu = 1$ and $\nu = 0$ gives the band width due to interactions. We have thus

$$\Delta_{\text{int}} = e^2 \sqrt{2eB}, \quad (2.14)$$

which has a weaker magnetic field dependence than Landau levels spacing.

The width in our momentum representation due to edge states, Eq.(2.9) or Eq.(2.11), becomes (much) smaller than the Landau level spacing with a strong magnetic field or with a large $L_x$. The width due to interactions, Eq.(2.14), also becomes much smaller than the level’s spacing. Hence with a strong magnetic field, there are wide energy region which have no singularity due to two-dimensional extended states. The localized states due to short range impurities can have energies in these regions. We call these regions as quantum Hall regime(QHR). QHR exists in systems of boundary, disorder and interactions if the magnetic field is strong enough. Whole spectrum is shown in Fig.5.

§3. Low energy theorem of $\sigma_{xy}$

Low energy theorem of $\sigma_{xy}$ is satisfied in QHR with impurities and interactions. We study many-body problems with (quantized) field operators $\{a_l(m, n; t), b_l(m, n; t)\}$ which are defined on lattice sites,

$$\vec{R}_{mn} = a(m, n), \quad a = \sqrt{\frac{2\pi \hbar}{eB}} \quad (3.1)$$

and satisfy an equal time commutation relation,

$$\{a_{l_1}(m_1, n_1; t_1), b_{l_2}(m_2, n_2; t_2)\} \delta(t_1 - t_2) = \delta_{l_1, l_2} \delta_{m_1, m_2} \delta_{n_1, n_2} \delta(t_1 - t_2). \quad (3.2)$$

Operator of zero momentum was not included originally in operator set, but is added for
computational convenience\(^1\). This operator decouples from physical space due to the constraints of von Neumann lattice coherent states.

The propagator, vertex function, and current correlation functions in translationally invariant systems are defined by,

\[
\int dt_1 dt_2 \sum_{X_{1,2}} e^{i(p_1 x_1 - p_2 x_2)} \langle T \{ a_{t_1}(\vec{X}_1, t_1) b_{t_2}(\vec{X}_2, t_2) \} \rangle = \frac{1}{a^2} (2\pi)^3 \delta(p_1 - p_2) S_{t_1, t_2}(p_1),
\]

\[
\int dt_1 dt_2 dx \sum_{X_{1,2}} e^{i(qx + p_1 x_1 - p_2 x_2)} \langle T \{ j_\mu(x) a_{t_1}(\vec{X}_1, t_1) b_{t_2}(\vec{X}_2, t_2) \} \rangle = i \frac{(2\pi)^3}{a^2} \delta(p_1 - p_2 + q) S_{t_1, t_2}(p_1) \Gamma^\mu_{\nu}(p_1, p_2) S_{\nu, t_2}(p_2),
\]

\[
\int dx_1 dx_2 e^{i(q_1 x_1 - q_2 x_2)} \langle T \{ j_{\mu_1}(x_1) j_{\mu_2}(x_2) \} \rangle = (2\pi)^3 \delta(q_1 - q_2) \pi_{\mu_1, \mu_2}(q_1).
\]

\(T(\cdot)\) denotes time-ordered product and momenta conjugate to lattice coordinates are defined on a torus. Hall conductance in QHR is computed from the time ordered current correlation functions and agree to that of retarded product used in Kubo formula\(^1\). Owing to the current conservation,

\[
\partial_\mu j^\mu = 0,
\]

and the commutation relations Eq.(3.2) and their representation Eq.(1.9), the above Green’s functions satisfy\(^1,2\),

\[
q^\mu \pi_{\mu\nu}(q) = \pi_{\mu\nu}(q) q^\nu = 0,
\]

\[
q^\mu \Gamma_\mu(p_1, p_2) = S^{-1}(p_1) R(p_2) - L(p_1) S^{-1}(p_2),
\]

\[
R_{t_1, t_2}(p) = \delta_{t_1, t_2} + i q_x [d_x(p) \delta_{t_1, t_2} + \vec{d}_{x, t_1, t_2}] + i q_y [d_y(p) \delta_{t_1, t_2} + \vec{d}_{y, t_1, t_2}],
\]

\[
L_{t_1, t_2}(p) = \delta_{t_1, t_2} + i q_x [d_x(p) \delta_{t_1, t_2} + \vec{d}_{x, t_1, t_2}] + i q_y [d_y(p) \delta_{t_1, t_2} + \vec{d}_{y, t_1, t_2}],
\]

\[
d_i(p) = d'_i(p) = \sum_{X_1 - X_2} e^{i p(X_1 - X_2)} \langle \tilde{X}_1 | (\vec{X} - \vec{X}_1)_i | \vec{X}_2 \rangle,
\]

\[
\tilde{d}_{i, t_1, t_2}(p) = d_{i, t_1, t_2}(p) = \langle f_{t_1} | (\tilde{X})_i | f_{t_2} \rangle.
\]

The explicit forms of \(d_i(p)\) and \(\tilde{d}_i(p)\) are given in Appendix A. The Hall conductance is the
slope of $\pi_{\mu\nu}(q)$ at the origin and is written as
\[
\sigma_{xy} = \left. \frac{1}{3!} \epsilon^{\mu\nu\rho} \frac{\partial}{\partial q^\rho} \pi_{\mu\nu}(q) \right|_{q=0} = \frac{e^2}{2\pi} N_w, 
\]
\[N_w = \frac{1}{24\pi^2} \int d^3 p \epsilon^{\mu\nu\rho} \text{Tr}[\frac{\partial \tilde{S}^{-1}(p)}{\partial p_\rho} \tilde{S}(p) \frac{\partial \tilde{S}^{-1}(p)}{\partial p_\mu} \tilde{S}(p) \frac{\partial \tilde{S}^{-1}(p)}{\partial p_\nu} \tilde{S}(p)].\]

with the transformed propagator
\[
\tilde{S}(p) = V(p)S(p)U(p),
\]
\[U(p) \frac{\partial}{\partial p_i} U^{-1}(p) + l_i(p) = 0,
\]
\[- \frac{\partial V^{-1}(p)}{\partial p_i} V(p) + r_i(p) = 0,
\]
\[l_i(p) = r_i(p) = d_i(p)\delta_{l_1,l_2} + \bar{d}_i(p)\delta_{l_1,l_2}.
\]

We have used a fact that the vertex function which is transformed by the above matrices as
\[
\tilde{\Gamma}_\mu(p_1,p_2) = U^{-1}(\vec{p}_1)\Gamma_\mu(p_1,p_2)V^{-1}(\vec{p}_2)
\]
satisfies the standard Ward-Takahashi identity,
\[
\tilde{\Gamma}_\mu(p,p) = \frac{\partial \tilde{S}^{-1}(p)}{\partial p_\mu}
\]
\[\text{(3.10)}\]

It should be noted that the relations Eqs.(3.4)~(3.10) are satisfied in interacting systems as well.

The meaning of $N_w$: The $N_w$ of Eq.(3.7) is a winding number of mapping from the momentum space to a matrix space defined by the propagator $\tilde{S}_{l_1,l_2}(p)$. The momentum in spatial direction is defined on a torus. The integration region of the momentum in temporal direction is also regarded as a compact space, when the propagator is non-singular at $p_0 = \pm\infty$. Physically, there is no singularity at $p_0 = \pm\infty$ in theories we study here and magnetic field effect should disappear at $p_0 = \pm\infty$. Furthermore, the energy integration region becomes a closed path in complex plane, which is compact, if the difference $N_w(E_{F_1}) - N_w(E_{F_2})$ is computed. Hence the momentum in temporal direction is also regarded as compact.
Consequently, the momentum space is regarded as a compact space. The matrix space is of infinite dimensions and seems to have a difficulty in defining a topological invariant. However, since the energy of Landau level \( E_l \) diverges if \( l \to \infty \), matrix space which corresponds to finite energy is of finite dimensional. It includes \( SU(2) \) space as a subspace. Hence \( N_w \) generally agrees to an integer as an elements of \( \pi_3(G), \ G \supset SU(2) \).

Computation of \( N_w \): 

(i) Free system

We compute \( N_w \) in a free system without disorders and interactions first. Let substitute Eq.(3.6), (3.8), and free propagator \( S_0(p) \) expressed by \( (p_0 - E_l \mp i\epsilon)^{-1} \delta_{l_1,l_2} \) into Eq.(3.7), then we find that the integrand of Eq.(3.7) is independent of \( \vec{p} \) and is given by

\[
-\frac{i}{4\pi^2 eB} \sum_l \frac{1}{p_0 - E_l \pm i\epsilon}.
\]

We have, then,

\[
N_w = l, \quad E_l < E_F < E_{l+1}.
\]

\( N_w \) depends on the Fermi energy and agrees to an integer at \( E \neq E_l \) as is shown in Fig.6. The value is ambiguous at \( E = E_l \) due to degeneracy of Landau levels. By adding momentum dependent small energy, the degeneracy is removed and we have a unique value of \( N_w \). The value thus obtained is proportional to electron filling factor \( \nu \), as

\[
N_w = \nu,
\]

which leads the following classical value of the Hall conductance:

\[
\sigma_{xy} = \frac{e^2}{2\pi N}.
\]

(ii) Systems with impurities
Degeneracy of Landau levels are partially removed by impurities and there appear one-particle energy levels of having energy $E$ at $E \neq E_l$. It was shown that the energy eigenstates of $E$ at $|E - E_l| > \delta$, are localized if the impurities are dilute, random and short range based on perturbative expansion method in von Neumann magnetic lattice representation, where $\delta$ is a system dependent small constant. Numerical calculations in random system also show that the wave functions are localized at $E \neq E_l$ with the localization length being inversely proportional to some power of $E - E_l$, $(E - E_l)^{-\lambda}$, $\lambda > 0$. Since the wave functions of localized states have finite extensions and their energies are discrete, it is possible to compute the contributions of the localized states to $\sigma_{xy}$ perturbatively, if the Fermi energy is in the range $|E_F - E_l| > \delta$.

We assume, hereafter, that the system is in quantum Hall regime where all the one-particle states around Fermi energy are localized and the momentum conserving propagator in the momentum representation has no singularity. We show that only the momentum conserving amplitude contributes to the linear coefficient of the current correlation function first. We decompose the current correlation function into the momentum conserving part $\pi_{\mu\nu}^{(1)}(q)$ and the momentum non-conserving part $\pi_{\mu\nu}^{(2)}(q_1, q_2)$.

$$\pi_{\mu\nu}(q_1, q_2) = (2\pi)^3 \delta(q_1 - q_2) \pi_{\mu\nu}^{(1)}(q_1) + \pi_{\mu\nu}^{(2)}(q_1, q_2).$$ (3.15)

In the latter, $\vec{q}_1$ and $\vec{q}_2$ are momenta carried by currents and are independent each other. This term is not zero generally due to a lack of translational invariance. Current is conserved, hence the identities,

$$q^\mu \pi_{\mu\nu}^{(1)}(q) = \pi_{\mu\nu}^{(1)}(q)q^\nu = 0,$$

$$q_1^\mu \pi_{\mu\nu}^{(2)}(q_1, q_2) = \pi_{\mu\nu}^{(2)}(q_1, q_2)q_2^\nu = 0,$$ (3.16)

are satisfied. In quantum Hall regime, $\pi_{\mu\nu}^{(1)}(q)$ and $\pi_{\mu\nu}^{(2)}(q_1, q_2)$ have no singularity at the origin $q_\mu = 0$ or $q_1 \mu = q_2 \mu = 0$, because one particle states have discrete energies and their wave functions have finite spatial extensions. Singularity is not generated. Thus they are
expanded as,

\[
\pi^{(1)}_{\mu \nu}(q) = \pi^{(1)}_{\mu \nu}(0) + \pi^{(1)}_{\mu \nu, \rho}(0) q^\rho + O(q^2),
\]

\[
\pi^{(2)}_{\mu \nu}(q_1, q_2) = \pi^{(2)}_{\mu \nu}(0, 0) + \pi^{(2)}_{\mu \nu, \rho}(0, 0) q_1^\rho + \pi^{(2)}_{\rho, \mu \nu}(0, 0) q_2^\rho + O(q^2).
\]

The coefficients are finite. We plug these forms to Eq.(3.16), and we have,

\[
\pi^{(1)}_{\mu \nu}(0) = 0,
\]

\[
\pi^{(1)}_{\mu \nu, \rho}(0) + \pi^{(1)}_{\mu \rho, \nu}(0) = 0, \tag{3.18}
\]

\[
\pi^{(1)}_{\mu \nu, \rho}(0) + \pi^{(1)}_{\rho, \mu \nu}(0) = 0.
\]

The slope of \(\pi^{(1)}_{\mu \nu}(q)\) at the origin is, hence, totally anti-symmetric, and is written with one constant \(c\), as

\[
\pi^{(1)}_{\mu \nu, \rho}(0) = c \epsilon_{\mu \nu \rho}. \tag{3.19}
\]

We have, also,

\[
\pi^{(2)}_{\mu \nu}(0, 0) = 0,
\]

\[
\pi^{(2)}_{\mu \nu, \rho}(0, 0) = 0, \tag{3.20}
\]

\[
\pi^{(2)}_{\rho, \mu \nu}(0, 0) = 0,
\]

by substituting the second form, Eq.(3.17), into the second equation of Eq.(3.16), because \(\vec{q}_1\) and \(\vec{q}_2\) are independent. Thus the linear term vanishes if \(\vec{q}_1\) and \(\vec{q}_2\) are different and independent. In fact, even in systems with translational invariance in one direction where one component of \(\vec{q}_i\) are the same, the linear term vanishes. This is understandable easily from the fact that there is only one free parameter left in Eq.(3.16), and one additional condition makes non-trivial solution disappear. The slope of the current-current correlation function at the origin is due to the momentum conserving part, \(\pi^{(1)}_{\mu \nu}(q)\), and the momentum non-conserving part, \(\pi^{(2)}_{\mu \nu}(q_1, q_2)\), does not contribute. This will be used also in the next part when interaction effects are studied.

(iii) Resummation formula of the \(\sigma_{xy}\) in the momentum representation.
We use a representation of the momentum conserving part, \( \pi^{(1)}_{\mu\nu}(q) \), in terms of a momentum conserving part of propagator and a momentum conserving part of vertex part. They are defined by summing the momentum conserving parts in perturbative expansions. The details of the resummation formula have been given in Ref.(1). Here, we give only the momentum conserving part of propagator in a system of impurity potentials or boundary potentials as an example. QHR is defined as the outside region of the energy band of the momentum conserving part of the propagator. In order to write the propagator with a momentum representation in the system of disorder potential \( V \), we start from,

\[
\langle p_1 | \frac{1}{E - (H_0 + V)} | p_2 \rangle = \frac{1}{E - E_0(p_1)} [\delta_{p_1, p_2} + \langle p_1 | V | p_2 \rangle \frac{E - E_0(p_1)}{E - E_0(p_1)} + \langle p_1 | V | p' \rangle \langle p' | V | p_2 \rangle + \cdots],
\]

(3.21)

\[H_0 | p \rangle = E_0(p) | p \rangle,\]

where \( H_0 \) is assumed to be invariant under translations. Eigenvalue \( E_0(p) \) could have a dependence on the momentum. In quantum Hall system, \( H_0 \) is given by

\[H_0 = \sum E_l b_l^\dagger(R) a_l(R),\]

(3.22)

and the energy eigenvalue, \( E_0(p) \), thus is constant, \( E_l \). Due to the non-invariant term, \( V \), the momentum is not conserved and \( p_2 \) is different from \( p_1 \) generally. The momentum conserving propagator within the \( l \)-th Landau level space is defined by the perturbative series as,

\[S(p) = \langle p | \frac{1}{E - (H_0 + V)} | p \rangle = \frac{1}{E - E_l} [1 + \langle p | V | p \rangle \frac{E - E_l}{E - E_l} + \langle p | V | p' \rangle \langle p' | V | p \rangle E - E_l + \cdots].\]

(3.23)

This can be written in the following form:

\[S(p) = \frac{1}{E - E_l} \left[ 1 + \frac{\langle p | V | p \rangle}{E - E_l} + \sum_{p' = p} \frac{\langle p | V | p' \rangle \langle p' | V | p \rangle}{E - E_l} + \cdots \right.

\[+ \sum_{p' \neq p} \frac{\langle p | V | p' \rangle \langle p' | V | p \rangle}{E - E_l} + \cdots \right]\]

(3.24)

\[= \frac{1}{E - E_l} \left[ 1 + \frac{\langle p | V | p \rangle}{E - E_l} + \left( \frac{\langle p | V | p \rangle}{E - E_l} \right)^2 + \left( \frac{\langle p | V | p \rangle}{E - E_l} \right)^3 + \cdots \right.

\[+ \sum_{p' \neq p} \frac{\langle p | V | p' \rangle \langle p' | V | p \rangle}{E - E_l} + \cdots \right] = \frac{1}{E - E_l - \Sigma(E, p)},\]
\[
\Sigma(E, p) = \langle p | V | p \rangle + \sum_{p' \neq p} \frac{\langle p | V | p' \rangle \langle p' | V | p \rangle}{E - E_l} + \cdots.
\] (3.25)

The above formula is equivalent to write the full Green’s function in interacting systems with a self-energy part and a free part. The self-energy part is defined from the one-particle irreducible part in standard manner. In the present system, the one-particle irreducible part is defined from the momentum, and the self-energy part \(\Sigma(E, p)\) is defined based on it. It is easy to see it in diagrams. They are given in Fig.3.

In the above perturbative expansion, if the energy denominator \(E - E_l\) does not vanish, there is no infra-red divergence. Moreover the series converge fast if each term in \(\Sigma(E, p)\) is small. In fact, the boundary potential \(V\) has a non-zero value only in narrow regions near the boundaries. Hence the matrix element from the boundary potential, \(\langle p | V | p' \rangle\) is of order \(V_0 a / L\). There are the following \(\Delta_1\) and \(\Delta_2\):

For such \(E\) that satisfies

\[
|E - E_l| > \Delta_1,
\] (3.26)

\(\Sigma(E, p)\) satisfies,

\[
\Sigma(E, p) < \Delta_2,
\] (3.27)

\(\Delta_2 < \Delta_1\),

where \(\Delta_1\) is of order \(V_0 a / L\). The convergence of Eq.(3.25) is good then, and \(S(p)\) has no singularity. Hence the energy region of Eq.(3.26) is regarded as QHR. They have the width of order \(a / L\) and becomes very small if \(L\) is large or the magnetic field is strong. It should be noticed that \(\Delta_1\) is finite and QHR exists if \(\Delta_1\) is smaller than Landau level energy spacing \(E_{l+1} - E_l\). This is possible even in a finite \(L\) and a finite magnetic field case.

\(S_0(p)\) in free systems are replaced with \(S(p)\) in systems with disorders. In QHR, \(S(p)\) has no singularities and the perturbative expansions due to disorders which modifies \(S_0(p)\) into \(S(p)\), converge well. Their effects in the \(\sigma_{xy}\) are also studied by perturbative expansions and
they do not give any corrections to the quantized Hall conductance from the Coleman-Hill argument. Consequently the quantization of the $\sigma_{xy}$ is exact in QHR that is defined by $S(p)$ of Eq.(3.23). The fact that the QHR is defined by the momentum representation, $S(p)$, becomes important when we discuss the edge states generated by the boundary potential.

(iv) Systems with interactions in quantum Hall regime, $|E_F - E_l| > \delta$.

Since all the one-particle states are localized in the energy region, $|E - E_l| > \delta$, current correlation function has no singularity if the Fermi energy is in this energy region. Since $\delta$ and $\Delta_1$ should be the same order, we identify them. We investigate this energy region here and show that the interactions do not give any corrections to $\sigma_{xy}$ in QHR.

As was shown in the previous part, the momentum non-conserving part of the current correlation function has no first derivative at the origin and does not contribute to $\sigma_{xy}$. A momentum conserving part in a system with interactions also has no first derivative at the origin if that agrees to a (regular) limit of a momentum non-conserving part.

In QHR, $\pi_{\mu\nu}(q)$ has no singularity in $q_\mu$. Perturbative higher order diagrams due to interactions and disorders also have no singularity. They are computed by integrating a product of propagators and vertices. By cutting one of bosonic propagators and regarding them as two external lines with different momenta, following Coleman and Hill\textsuperscript{12}, we are able to regard the original amplitude as a limit of momentum non-conserving amplitudes, $\tilde{\pi}_{\mu\nu}^{(2)}(q_1, q_2)$, which satisfy the same identity as $\eta_{\mu\nu}^{(2)}(q_1, q_2)$ of Eq.(3.14). Hence, $\tilde{\pi}_{\mu\nu}(q_1, q_2)$ has no linear term in $q_\mu^{(i)}$ and $\tilde{\pi}_{\mu\nu}(q_1, q_2)|_{q_2 \rightarrow q_1}$ does not contribute to $\sigma_{xy}$ in QHR, as well.

Only the momentum conserving part that is isolated from the momentum non-conserving parts and is not connected with any non-conserving parts contributes to $\sigma_{xy}$. There are two kind of momentum conserving parts that are isolated from non-conserving parts. The first type is the lowest order diagram, i.e., one loop diagram. This is obviously isolated from any other diagrams. The second type is similar to the first one, but is a loop diagram with a dressed momentum dependent propagator. The momentum dependent propagator expresses
extended states and has a singularity within a finite band width, which is estimated later. If the Fermi energy is in the outside of the continuous energy band region, all the particle states around the Fermi energy are localized and the system is regarded as QHR.

Thus the higher order terms do not contribute to $\sigma_{xy}$ in QHR. If there are ultra-violet divergences, physical observables are computed with renormalized quantities. The bare charge is replaced with the renormalized charge. Consequently, $\sigma_{xy}$ is given by exactly quantized value $(e^2/h)N$ in QHR with the renormalized charge $e$.

§4. Cylinder geometry

We study electron system of a cylinder geometry with a length $L$. For simplicity we assume $L$ is an integer multiple of lattice spacing $a$. Wave functions in center coordinates $\vec{R}_{mn}$ satisfy,

$$\Psi(\vec{R}_{m,n} + L\vec{e}_x) = \Psi(\vec{R}_{m,n}), \quad (4.1)$$

and the corresponding discrete momentum $p_x$ satisfies,

$$Lp_x = 2\pi n, \quad n = \text{integer}. \quad (4.2)$$

If the $L$ is much larger than magnetic lattice spacing $a$, localized one particle states are essentially the same as those of the previous infinite system, because they have finite spatial extensions and are insensitive to the boundary conditions. Hence one-particle states in the energy region $|E - E_l| > \tilde{\delta}$ with a slightly modified $\tilde{\delta}$ are localized. This energy region corresponds to QHR. We study a finite size effect of $\sigma_{xy}$ in QHR.

The Hall conductance in QHR of a cylinder geometry is given by,

$$\sigma_{xy} = \frac{e^2}{2\pi} \sum_{p_x} \int dp^0 dp_y \epsilon_{\mu\nu\rho} \frac{1}{24\pi^2} \text{Tr} \left[ \frac{\partial S^{-1}}{\partial p_\mu} \tilde{S} \frac{\partial S^{-1}}{\partial p_\nu} \tilde{S} \frac{\partial S^{-1}}{\partial p_\rho} \tilde{S} \right], \quad (4.3)$$

where the $p_x$ integration in Eq.(3.7) was replaced with the discrete momentum summation. By the replacement, the integral is generally changed depending upon the form of the
integrand. In the present case, however, the integrand becomes,
\[
\epsilon_{\mu \nu \rho} \frac{1}{24\pi^2} \text{Tr} \left[ \frac{\partial S^{-1}}{\partial p_\mu} S \frac{\partial S^{-1}}{\partial p_\nu} S \frac{\partial S^{-1}}{\partial p_\rho} S \right] = -i \frac{1}{4\pi^2 eB} \sum_l \frac{1}{p_0 - E_l \pm i\epsilon},
\]
where \(-i\epsilon(+i\epsilon)\) is taken if the energy \(E_l\) is less than(larger than) Fermi energy, and does not depend on \(p_x\). Hence \(\sigma_{xy}\) is unchanged. \(\sigma_{xy}\) in the cylinder geometry is identical to that of the infinite system. There is no finite size correction in quantized Hall conductance in QHR.

The effects of impurities and interactions in QHR are studied in the same way as infinite systems. They do not give any corrections. We will describe this in more detail in the next section.

§5. Hall bar geometry

We study realistic systems which have Hall bar geometry in this section.

(5-1) finite system

Realistic experiments are done with electronic systems of Hall bar geometry. In one direction a Hall bar system has potential barriers which confine electrons in inside. There are gates in perpendicular direction through which electrons move in and move out. So it would be sufficient to study a system that is finite in one direction due to potential barrier and infinite in another direction.

The potential barrier in the positive \(x\) region is expressed with the following potential term:
\[
H_I = \int d\bar{x}V(x)\Psi^\dagger(x)\Psi(x),
\]
\[
V(x) = V_0(\theta(-x) + \theta(x - L)).
\]
The potential is invariant under translation in the \(y\)-direction, hence the eigenstates are the plane wave in this direction. The eigenvalue equations were solved before in the present representation. The edge states which have the \(p_y\) dependent continuous energy in the
region $E_l < E < E_l + V_0$ and are localized in the $x$-direction are found. They cross the Fermi energy. Since edge states have the continuous energy around the Fermi energy, they could contribute to conductance, generally. However, they are confined along the edges and move in one direction without reflected. We see that they do not contribute to the conductance, then.

The one-dimensional edge states near the Fermi energy are chiral and electrons move in one direction. Their properties are unchanged by adding disorder potentials, because in the wave equation for a chiral mode with a potential term,

$$[i \frac{\partial}{\partial x} + V(x)]\Psi = E\Psi,$$

the potential can be removed completely by a gauge transformation $^{13}$,

$$\Psi = e^{i\phi(x)}\tilde{\Psi},$$

$$i \frac{\partial}{\partial x}\tilde{\Psi} = E\tilde{\Psi}. \tag{5.3}$$

Hence the phase factor of chiral modes are modified by the potential but the waves are neither reflected nor localized. They carry electric current but do not contribute to conductance in the above situation.

We use the resummation formula in the momentum representation and study the finite size effect. If one edge state at one side interacts with one edge state at another side, extended states can be formed, then the propagator in the momentum representation has singularities. The effect of the edge states and the boundary potentials are found by writing the Hamiltonian as,

$$H = \sum b_{l_1}(m_1, n_1)a_{l_2}(m_2, n_2)\{E_{l_1}\delta_{l_1,l_2}\delta_{m_1,m_2}\delta_{n_1,n_2} + V_{l_1,l_2}(m_1, n_1;m_2, n_2)\}. \tag{5.4}$$

The transformed potential $V_{l_1,l_2}(m_1, n_1;m_2, n_2)$ vanishes in inside region and becomes diagonal form $V_0\delta_{l_1,l_2}\delta_{m_1,m_2}\delta_{n_1,n_2}$ in outside region. In narrow boundary region of few magnetic lengths, it has off diagonal term. The potential and eigenfunctions are transformed in the
$y$-direction as Eq.(2.3), and satisfy Eq.(2.4). We solved the equations numerically. Energy eigenvalues and corresponding eigenfunctions are given in Fig.2. Obviously, edge states are localized in narrow region in $x$-direction and have energy in a range $E_l < E < E_l + V_0$. This is understandable easily from the properties of $V_{l_1, l_2}(m_1, n_1; m_2, n_2)$.

Edge state at one boundary could couple with edge state at another boundary if their distance is small and there are impurities and interactions. They could give singularities in the momentum representation, $S(p)$, then, and contribute to the conductance. To study their effects, it is convenient to write the previous Hamiltonian, Eq.(5.4), in the following manner:

$$H = H_0 + H_1 + H_2,$$

$$H_0 = \sum_{\vec{X} \text{ inside}} E_l b_l(\vec{X}_1)a_l(\vec{X}_1),$$

$$H_1 = \sum_{\vec{X} \text{ outside}} (E_l + V_0)b_l(\vec{X})a_l(\vec{X}),$$

$$H_2 = \sum_{\text{boundary}} (\delta V)_{l_1, l_2}(\vec{X}_1, \vec{X}_2)b_{l_1}(\vec{X}_1)a_{l_2}(\vec{X}_2).$$

(5.5)

Single particle energy from $H_0$ and $H_1$ are $E_l$ or $E_l + V_0$ respectively. $(\delta V)_{l_1, l_2}(\vec{X}_1, \vec{X}_2)$ is not vanishing only when $\vec{X}_1$ and $\vec{X}_2$ are in the narrow boundary regions of few magnetic distances, hence a perturbative treatment of $H_2$ is possible. When the Fermi energy is slightly larger than $E_l$ but is much smaller than $E_l + V_0$, the one particle eigenstates of $H_1$ decouple and contribute to conductance only through virtual effects. Hence we take into account $H_0$ and $H_2$ meanwhile. $H_2$ is taken perturbatively and virtual effects are studied later. We study particles confined in the finite inside region.

We express the field operators of the Landau level in momentum representation as,

$$a_l(\vec{X}) = \sum_{p_x} \int_{-\pi/a}^{\pi/a} dp_y \frac{1}{a} e^{ip\vec{X}} a_l(\vec{p}),$$

$$b_l(\vec{X}) = \sum_{p_x} \int_{-\pi/a}^{\pi/a} dp_y \frac{1}{a} e^{ip\vec{X}} b_l(\vec{p}),$$

(5.6)

$$p_x = \frac{2\pi}{L_x} n_x + \alpha.$$
In Eq.(5.6), \( n_x \) is an integer and a parameter \( \alpha \) depends on boundary condition. The current operator, \( \tilde{J}_\mu \), which is expressed with these operators, is conserved within the Hilbert space of confined particles. It satisfies,

\[
\partial^\mu \tilde{J}_\mu = C,
\]

\[
C = \sum_{\vec{X} \text{ or } \vec{Y} \text{ in outside}} b(X)\Gamma a(Y),
\]

\[
\langle \text{Confined particle}\vert C \vert \text{Confined particle} \rangle = 0.
\]

The current operators, \( \tilde{J}_\mu \), satisfies the same commutation relation as that of infinite system, Eq.(3.4). Hence the identities of the infinite system, Eq.(3.5), are satisfied, but slight modifications are necessary. Current correlation function is expanded with the momentum carried by current, which becomes discrete, and its slope at the origin, which is simple derivative, is proportional to the conductance. One loop diagram contributes in QHR, from Coleman-Hill theorem. The momentum has a discrete component, from Eq.(5.8), hence the Hall conductance agrees to that of the torus geometry, Eq.(4.3), which has no finite size correction and is independent of \( L_x \).

Coleman-Hill theorem in finite systems.

In the present geometry the \( x \)-coordinate is defined in a finite region and the corresponding momentum, \( p_x \), becomes discrete. Hence the derivations of Ward-Takahashi identity and related low energy theorems should be re-examined.

We define the current correlation function, vertex function, and the propagator as Eq.(3.3). For a convenience, we use a notation \( j_\mu(x) \) instead of \( \tilde{j}_\mu(x) \) of the previous part. We concentrate to the cases where Green’s functions in configuration space vanish in the outside region due to the finite Fermi energy that is smaller than \( V_0 \). Integrations of the \( x \) coordinate, then, are defined in a finite region, \( 0 \leq x \leq L \), and the corresponding momenta thus become discrete from completeness.

The current expectation value is connected with external vector potential through the
current correlation function as
\[ J_\mu(x) = \int dy \pi_{\mu\nu}(x, y) A_\nu(y), \] (5.8)
\[ \pi_{\mu\nu}(p_1, p_2) = \int dx_1 dx_2 e^{ip_1 x_1 + ip_2 x_2} \pi_{\mu\nu}(x_1, x_2). \]

The current correlation function has a translational invariant term and a non-invariant term.

They are expressed by
\[ \pi_{\mu\nu}(p_1, p_2) = \pi^{(1)}_{\mu\nu}(p_1, p_2) + \pi^{(2)}_{\mu\nu}(p_1, p_2), \]
\[ \pi^{(1)}_{\mu\nu}(p_1, p_2) = (2\pi)^2 \delta(2)(p_1 + p_2) L \delta^{(1)}_{p_1^x + p_2^x, 0} \pi^{(1)}_{\mu\nu}(p_1), \]
\[ \pi^{(2)}_{\mu\nu}(p_1, p_2) = (2\pi) \delta^{(1)}(p_1^0 + p_2^0) \pi^{(2)}_{\mu\nu}(p_1, p_2; p_1^0), \] (5.9)
or \[ \pi^{(2)}_{\mu\nu}(p_1, p_2) = (2\pi)^2 \delta^{(1)}(p_1^0 + p_2^0) \delta^{(1)}(p_1^y + p_2^y) \pi^{(2)}_{\mu\nu}(p_1^0, p_2^0), \]
where \( \delta(x) \) is Dirac delta function and \( \delta_{m,0} \) is Kronecker delta. If the Fermi energy is in the energy gap region or in the localized state region where all the energy eigenstates around the Fermi energy are localized, \( \pi^{(1)}_{\mu\nu}(p) \) and \( \pi^{(2)}_{\mu\nu}(p_1, p_2) \) have no singularities. They satisfy,
\[ p^\mu \pi^{(1)}_{\mu\nu}(p) = p^\nu \pi^{(1)}_{\mu\nu}(p) = 0, \] (5.10)
\[ p_1^0 \pi^{(2)}_{0\nu} + p_1^i \pi^{(2)}_{i\nu} = p_2^0 \pi^{(2)}_{\mu\nu}(p_1, p_2) + p_2^i \pi^{(2)}_{\mu i} = 0, \]
and are expanded with momenta as
\[ \pi^{(1)}_{\mu\nu}(p_1) = C \epsilon_{\mu\nu \rho} p_1^\rho + \text{higher power}, \]
\[ \pi^{(2)}_{\mu\nu}(p_1, p_2) = O(p_1^2), \] (5.11)
from the arguments of Section 3. Note that the coefficient \( C \) and higher power terms are defined uniquely by the slope or the higher order curvaures at the origin and are defined by the simple derivatives, even though the momentum is not infinitesimal quantity in finite systems. We see that \( C \) contributes to the Hall conductance, because the total current is given by,
\[ \int dx_0 dx'_2 \int_0^L dx_1 J_y(x_1) = \int \frac{dp_0^0 dp_2^0}{(2\pi)^2} \frac{1}{L} \sum_{p_2^2} \{ \pi^{(1)}_{y,\nu}(0, p_2) + \pi^{(2)}_{y,\nu}(0, p_2) \} A_\nu(p_2). \] (5.12)

At a point of measurement there is no electric field, because that is a gate, and derivative of \( A_\nu \) vanishes. Hence, \( \pi^{(2)}_{\mu\nu}(p_1, p_2) \) and higher terms of \( \pi^{(1)}_{\mu\nu}(p) \) thus do not contribute to the Hall conductance.
Higher order correction due to interactions in quantum Hall regime does not exist in $C$, as in the infinite system. Higher order corrections of $\pi^{(1)}_{\mu\nu}(p)$ come from diagrams in which all the internal momenta satisfy boundary condition. If an internal momentum does not satisfy boundary condition of Eq.(5.6), the momentum in that direction is not conserved. Consequently, this kind of higher order diagram contributes to only $\pi^{(2)}_{\mu\nu}(p_1,p_2)$, and does not contribute to $C$. Examples are given in Fig.7.

As was done by Coleman and Hill, in order to study $\pi^{(1)}_{\mu\nu}(p)$ we study new diagrams in which internal lines of $\pi^{(1)}_{\mu\nu}(p)$ are cut and two different momenta, $l_1$ and $l_2$, are given to these lines. Whenever $l_1 + l_2 \neq 0$, two current have different momenta. These diagrams then correspond to $\pi^{(2)}_{\mu\nu}(p_1,p_2)$, and are written as $\pi^{(2)}_{\mu\nu}(p_1,p_2) = \delta^{(3)}(p_1 + p_2 + l_1 + l_2)\tilde{\pi}_{\mu\nu}(p_1,p_2)$, and do not contribute to $C$ from the above argument. Amplitude $\tilde{\pi}_{\mu\nu}(p_1,p_2)$ have no singularity in QHR and linear terms in $p_1$ or $p_2$ are not allowed, because $p_1$ and $p_2$ are independent. Then, $\tilde{\pi}_{\mu\nu}(p,p)$ has no linear term also. Thus, none of higher order diagrams contribute to $C$ and only the one loop diagram contributes to $C$.

The linear coefficient is written as a topological invariant. Since the current conservation and the commutation relations are the same as the infinite system, the transformed vertex part as Eq.(3.8) and (3.9), hence satisfies,

$$q^\mu \tilde{\Gamma}_\mu(p_1,p_2) = \tilde{S}^{-1}(p_1) - \tilde{S}^{-1}(p_1 + q).$$  \hspace{1cm} (5.13)

Consequently, the derivative $\frac{\partial \tilde{S}^{-1}}{\partial p_\mu}$ is obtained by comparing a linear coefficient of both sides as

$$\frac{\partial \tilde{S}^{-1}}{\partial p_\mu} = \tilde{\Gamma}_\mu(p,p).$$  \hspace{1cm} (5.14)

The above equation is satisfied with the simple derivative in the left hand side, despite of the fact that one component of the momentum is discrete. Using this formula, the linear
coefficient of the current correlation function is expressed as,

\[ \sigma_{xy} = \frac{e^2}{3!} \sum \epsilon^{\mu\nu\rho} \frac{\partial}{\partial p_{\rho}} \pi_{\mu\nu}(p) \bigg|_{p=0} = \frac{e^2}{2\pi^2} \frac{1}{24\pi^2} \sum q_x \int d^2 q \epsilon_{\mu\nu\rho} \text{Tr} \left[ \frac{\partial S^{-1}}{\partial p_{\mu}} \tilde{S} \frac{\partial S^{-1}}{\partial p_{\nu}} S \frac{\partial S^{-1}}{\partial p_{\rho}} \tilde{S} \right]. \]  

(5.15)

This formula gives topologically invariant expression of the Hall conductance in finite systems.

(5-2) Boundary effect

We study off-diagonal term \( H_2 \) of Eq.(5.5). This term does not conserve momentum in the \( x \)-direction. Hence perturbative expansion of this term gives higher order diagrams in which the total momentum is not conserved in the \( x \)-direction in addition to the momentum conserving diagrams. These higher order diagrams do not contribute to the \( \sigma_{xy} \) as far as they are treated perturbatively. Perturbative treatment is good if the Fermi energy is in the outside of the continuous band of the momentum conserving propagator, but may not be good if the Fermi energy is in the inside of fictitious band of section 2. Real extended states could be formed by boundary potentials in a finite energy region around the center of Landau level. The width of the fictitious band in our momentum representation can be made small in certain conditions discussed before. The QHR is in the outside region of the energy bands of real extended states and of fictitious extended states. The width of the bands was estimated in Section 2 and was given by Eqs.(2.9) and (2.11). Suitably large \( L_x \) or strong magnetic field make the widths much smaller than the Landau level energy spacing, \( E_l - E_{l-1} = eB/m \). There are enough spacings for the localized states, then, and QHR do exist in this situation.

In QHR, the momentum non-conserving term gives no contribution to the slope of current correlation function from the arguments of Eqs.(5.11) and (5.12). \( H_2 \) in Eq.(5.5) does not conserve momentum in the \( x \)-direction but conserves in the \( y \)-direction. Consequently, impurities and interactions do not modify the value of the quantized Hall conductance there.
In the inside of extended states energy band of $S(p)$, the expression Eq.(5.15) is valid but there are corrections to the quantized $\sigma_{xy}$ and to $\sigma_{xx}$. Their contributions were found before, and are given by

$$\sigma_{xy} = \frac{e^2}{2\pi} N + \epsilon,$$

$$\sigma_{xx} = \epsilon', \quad (5.16)$$

where $\epsilon$ and $\epsilon'$ are small parameters that are proportional to the number of the extended states.

(5-3) Virtual effects (renormalization effect)

Higher energy states, such as higher Landau level states, and the states in the outside region expressed in $H_1$, give only virtual effect in higher order corrections. In the infinite system, by using the Ward-Takahashi identity derived from the current conservation, the charge renormalization factor cancels exactly with the vector potential renormalization factor and the final formula of the Hall conductance,

$$\sigma_{xy} = \frac{e^2}{h} N_w,$$

$$N_w = \frac{1}{24\pi^2} \int dq \epsilon_{\mu\nu\rho} \text{Tr}[\frac{\partial S^{-1}(p)}{\partial p_\mu} \tilde{S}(p) \frac{\partial S(p)}{\partial p_\nu} \tilde{S}(p) \frac{\partial S^{-1}(p)}{\partial p_\rho} \tilde{S}(p)] \quad (5.17)$$

was given. Now in a finite system, the current conservation and the Ward-Takahashi identity are satisfied. Hence we have the formula,

$$\sigma_{xy} = \frac{e^2}{h} N'_w,$$

$$N'_w = \frac{1}{24\pi^2} \int d^2q \sum_i \epsilon_{\mu\nu\rho} \text{Tr}[\frac{\partial S^{-1}(p)}{\partial p_\mu} \tilde{S} \frac{\partial S(p)}{\partial p_\nu} \tilde{S} \frac{\partial S^{-1}(p)}{\partial p_\rho} \tilde{S}] \quad (5.18)$$

By combining Eq.(4.4) and Eq.(5.18), we have the Hall conductance in a QHR of finite
system as,
\[ \sigma_{xy} = \frac{e^2}{h} N, \quad N = \text{integer} \] (5.19)

The \(\sigma_{xy}\) has no finite size corrections in QHR.

(5-4) Finite current effects.

The linear response formula, Eq.(5.8), is valid actually if the current and the vector potential are infinitesimal,
\[ \delta J_{\mu}(x) = \int dy \pi_{\mu \nu}(x, y) \delta A_{\nu}(y). \] (5.20)

In order to know the relation for a finite current, we integrate the above relation. That is possible once the current correlation function is computed in a system of a finite vector potential. Our formula and theorem can be applied to such correlation function. Then the infinitesimal current, infinitesimal potential, and current correlation function depend implicitly on the vector potential. We parametrize them with a parameter \(s\), under boundary conditions
\[ J_\mu(x, 0) = 0, \quad J_\mu(x, 1) = J_\mu(x), \]
\[ A_\mu(x, 0) = 0, \quad A_\mu(x, 1) = A_\mu(x), \] (5.21)
\[ \delta J_\mu(x, s) = \int dy \pi_{\mu \nu}(x, y; A_\mu(s)) \delta A_\nu(y, s). \]

A total current satisfies a similar equation,
\[ \delta I_x(s) = \sigma_{xy}[A_\mu(s)] \delta V_y(s), \]
\[ V_y(0) = 0, \quad V_y(1) = V_y, \] (5.22)
\[ I_x(0) = 0, \quad I_x(1) = I_x. \]

Now, the \(\sigma_{xy}\) could have the same value during the change from \(s = 0\) to \(s = 1\), if the electronic system is in the same QHR. Then the relation for a finite current is obtained by integrating Eq.(5.22) with \(s\), and becomes
\[ I_x = \sigma_{xy} V_y, \quad \sigma_{xy} = \frac{e^2}{h} N. \] (5.23)

Note that this is possible only in the plateau region\(^14\). On the other hand, if the system
moves from one QHR to extended states region or to another QHR, then we are not sure if linear relation between a finite current and a finite voltage is satisfied. The relation may become complicated, then. We will give more arguments of finite current effects, especially on a breakdown of the QHE, in the next section.

The relation between the finite total current and the finite total voltage becomes linear in the plateau regions, and the conductance $\sigma_{xy}$ is quantized. It behaves as Fig.8.

§6. Current distribution, Büttiker-Landauer formula, and Breakdown of QHE.

In our proof of the quantum Hall effect given in the previous parts, current distribution is irrelevant. The Hall conductance is the ratio between the total current in one direction and total voltage in another direction and so is quantized exactly in QHR. The current density is not uniform and varies with spatial region, generally. We study its implication in this section.

There is a completely different approach of the quantum Hall effect from ours. In its proof, it is assumed that edge states are the only current carrying states around the Fermi energy and is used Büttiker-Landauer formula for one-dimensional systems. The formula may be valid only under the assumption. The current distribution is thus important. Two approaches give totally different result when the current becomes larger. We study the current distribution and finite current effect in this section. We will see that in general situation, the bulk states as well as the edge states carry the current. Hall electric field is thus generated at the bulk and it leads Landau level broadening. QHR becomes narrow consequently and vanishes eventually. Breakdown of QHE occurs.

(6-1) Current distribution
In order to compute the current distribution, we compute an effective action of an external vector potential. The vector potential is regarded as either the external potential added to the system or as a Lagrange multiplier which is expressing a condition of a finite external current. Variational principle to the effective action gives the current distribution.

We study the system in which the vector potential couples with electron field in a gauge invariant manner as,

$$
\int \! dx \left[ \Psi^\dagger \left(i\hbar \frac{\partial}{\partial t} + cA_0\right) \Psi - \Psi^\dagger \left(\vec{\nabla} + e\vec{A}_0 + e\vec{A}\right)^2 \frac{1}{2m} \Psi - \Psi^\dagger \Psi V_b(x)\right],
$$

$$|\vec{\nabla} \times \vec{A}_0| = B,$$

$$V_b : \text{boundary potential.}$$

We integrate the electron fields $\Psi(x)$ and $\Psi^\dagger(x)$ and obtain the effective action of $A_0$ and $\vec{A}$. Long distance phenomena are represented by low dimensional terms of the effective action,

$$\frac{1}{2} \int \! dx \left[ c(x) F_{0x}^2 + e(x) F_{0y}^2 - d(x) F_{ij}^2 \right] + \frac{\sigma_{xy}}{2} \int \! dx \epsilon_{\mu\nu\rho} A^\mu \partial^\nu A^\rho + \cdots,$$

$$F_{0x} = \partial_0 A_x - \partial_x A_0, \quad F_{0y} = \partial_0 A_y - \partial_y A_0, \quad F_{xy} = \partial_x A_y - \partial_y A_x.$$

The $x$-integration is defined in a finite region from 0 to $L$. The coefficients $c(x)$, $e(x)$, and $d(x)$ of the above action are the constants in the bulk and change their values near the edge. They are computed as,

$$c(x) = e(x) = \frac{e^2 m}{4\pi eB},$$

$$d(x) = \frac{3}{4\pi} \frac{e^2}{m},$$

in the bulk. Near the edges they change the values. Because we need only $c(x)$ here, we give $c(x)$ near the edges for a boundary potential with a constant slope, $E_b$.

$$c(x) = \frac{e^2 m}{8\pi eB} - \frac{B}{4E_b} \sqrt{\frac{2}{eB\pi}} e^{-2eB\Delta x^2},$$

where $\Delta x$ is the distance between the boundary and the coordinate $x$. The first term comes from inter Landau levels and the second term comes from intra Landau levels. They have opposite sign. For high potential barrier, the first term is dominant and $c(x)$ is positive definite in all the regions. We study this case here and discuss general cases in Appendix C.
The total current in the $x$-direction is given by,

$$I_y = \int_0^L dx J_y = \sigma_{xy} \{ A^0(L) - A^0(0) \}. \quad (6.5)$$

We assume the translational invariance in the $y$-direction and obtain a stationary solution of the effective action under the constraint of total current $I_y$ with time independent ansatz:

$$\partial_0 A_i = 0, \; \partial_y A_0 = 0. \quad (6.6)$$

Then we minimize the following action,

$$\frac{1}{2} \int d\vec{x} [c(x)(\partial_x A_0)^2] - \mu \sigma_{xy} \int_0^L dx \partial_x A_0(y_2). \quad (6.7)$$

Euler-Lagrange equation is given by,

$$\partial_x (c(x)\partial_x A_0) = 0, \quad (6.8)$$

and is solved as

$$c(x)\partial_x A_0 = \text{const} = C_0,$$

$$\partial_x A_0 = \frac{C_0}{c(x)}, \; c(x) \neq 0. \quad (6.9)$$

The constant $C_0$ is given from the condition of total current,

$$C_0 \int_{x_1}^{x_2} dy \frac{1}{c(x)} = \frac{I_y}{\sigma_{xy}} = \text{const.} \quad (6.10)$$

Obviously, if the coefficient $c(x)$ were constant, the local electric field, as well as the local current density, would be uniform. On the other hand, if $c(x)$ varies, electric field varies also. In fact from Eq.(6.4), $c(x)$ decreases toward the edge and the electric field and current density increases toward the edge. The edge current is only a portion of the total current. In an exceptional case, where $c(x)$ vanishes or becomes negative at the edge and stays at the constant value in the bulk, there is an energy minimizing solution which has a current only at the edge region. In this situation, it is obvious that Büttiker-Landauer formula could be
applied. Our formula is applicable to general situations. A general discussion concerning a connection between the sign of $c(x)$ and the current distribution is given in Appendix C. It will be shown that edge current states may make a transition to bulk current states.

(6-2) Finite current effects and breakdown of QHE

Potential thus obtained modifies one-particle properties. Their effects become important if the magnitude of the total current is not infinitesimal but is finite. We study the system of the uniform electric field. Landau levels with the uniform electric field are not degenerate in energy and have a finite width, as is given in Appendix D. If the width in the momentum representation exceeds the Landau level energy spacing, QHR disappears. Then, breakdown of QHE occurs.

In the von Neumann lattice representation it is easy to find the width in the momentum representation. From Eq.(D.2), the gauge invariant width is given by,

$$eEa.$$  \hspace{1cm} (6.11)

QHR disappears, when the band width exceeds the Landau level’s energy spacing, $\hbar \omega_c$, for all Landau levels without spin effect or for energy splitting from Zeeman energy due to magnetic moment $\mu$, $\mu B$. The critical electric field satisfies,

$$eE_c a = \hbar \omega_c,$$  \hspace{1cm} (6.12)

or

$$eE'_c a = \mu B,$$  \hspace{1cm} (6.13)

$$\omega_c = \frac{eB}{m^*}, \ a = \sqrt{\frac{2\pi\hbar}{eB}} = \sqrt{2\pi l_B}. $$ \hspace{1cm} (6.14)

where $l_B$ is the magnetic length used usually. Thus, the critical electric field, $E_c$ for Landau
level splitting and $E'_c$ for spin splitting are given by,

$$E_c = \frac{\hbar \omega_c}{ea} = N_1 B^{3/2},$$

$$E'_c = \frac{\mu B}{ea} = N_2 B^{3/2},$$

$$N_1 = 25.4 \times 10^3 \text{[Vm}^{-1}\text{T}^{-3/2}],$$

$$N_2 = 0.84 g \times 10^3 \text{[Vm}^{-1}\text{T}^{-3/2}],$$

(6.15)

In both cases the critical electric fields are proportional to $B^{3/2}$ and the numerical constants $N_1$ and $N_2$ are constant and independent of Landau levels. $g$-factor extracted from the experiment is 7.3 and $N_2$ becomes comparable to $N_1$. The critical electric fields for even plateaus and odd plateaus are computed from the previous values as,

$$E^{\text{odd}}_c = N^{\text{odd}} B^{3/2},$$

$$E^{\text{even}}_c = N^{\text{even}} B^{3/2},$$

$$N^{\text{odd}} = N_2 = 6.48 \times 10^3 \text{[Vm}^{-1}\text{T}^{-3/2}],$$

$$N^{\text{even}} = N_1 - N_2 = 19.0 \times 10^3 \text{[Vm}^{-1}\text{T}^{-3/2}].$$

(6.16)

These critical fields are compared with the recent experiments of Kawaji et al. Both of them have $B^{3/2}$ behavior and are independent from Landau levels. Our values of $N^{\text{odd}}$ and $N^{\text{even}}$, however, are much larger than the experimental observations.

The critical electric field has been estimated before from naive overlappings of wave functions by Eaves and Sheard and from semi-classical method by Trugman, and Nicopoulos and Trugman. In the former method, the critical electric field behaves as $B^{3/2}$, and its magnitude thus obtained was similar to the current values. In the latter method, the critical electric field behaves as $B$ instead of $B^{3/2}$. We estimate the band width of the extended states due to Hall electric field first and compute the critical electric field from a condition that the bands overlapp. Our method is thus quite natural and leads to reasonable qualitative agreements, in the $B$-dependence and the level dependence of the critical electric fields with the experiments.
Edge current systems make transition to bulk current systems, from Appendix C. Consequently, the systems of only the edge currents should show the breakdown of the QHE in two steps. In the first step, an edge current system becomes to a bulk current system, and in the second step, the QHE is broken.

§7. Summary

In the present work, we have shown that the quantum Hall regime (QHR) is realized in finite two-dimensional electron systems if the magnetic field is strong enough and that the quantized Hall conductance has no finite size correction in QHR. They are shown by the use of magnetic von Neumann lattice representation, which has been used by us before and is quite useful for studying one-particle properties and for showing the connection of Hall conductance with the topological invariant and the absence of corrections in quite general systems with disorders and interactions. The momentum representation is used, and QHR is defined based on the momentum representation of the propagator.

In magnetic von Neumann lattice representation, base functions and dual base functions are local functions and have values around rectangular lattice coordinates. It is easy to apply the momentum representation in this method. Then Ward-Takahashi identity is expressed with simple and transparent form by use of multi-pole expansion technique. The derivation of the exact low energy theorem about the $\sigma_{xy}$ is given based on them. From the theorem, the $\sigma_{xy}$ at the QHR is quantized exactly as $\left(\frac{e^2}{h}\right)N$ and has no correction from finite size effect, disorders and interactions. The edge states are extended along the boundary and have continuous energies across Fermi energy. Nevertheless they have small overlap with the momentum eigenstates, and the QHR is realized at the outside region of the singularities of the propagator in the momentum representation, $S(p)$. Edge states carry the electromagnetic current together with the bulk extended states. Both states contribute to Hall conductance and Hall conductance is quantized in QHR. Hall conductivity, on the other hand, may depend on spatial region and is not quantized generally.
We studied the current distribution in Section 6 and found that the bulk states as well as the one-dimensional edge states carry the current generally. In QHR, the bulk extended states have energy gap but the edge states have no energy gap. Around Fermi energy, there are only one-dimensional edge states. Büttiker-Landauer formula may be applied to them, then.

The current in QHR does not cause energy dissipation. One-dimensional chiral modes near Fermi energy carry a current without energy dissipation and two-dimensional extended states at the bulk with finite energy gap also carry a current without energy dissipation. In our approach, we have studied combined total current and total voltage, and we have shown that the $\sigma_{xy}$ are quantized exactly.

In Büttiker-Landauer approach, only the states near Fermi energy are studied and it was shown that their contributions to $\sigma_{xy}$ gives the quantized $\sigma_{xy}$. We discussed when the current flows only in the edges. These systems, however, are changed to bulk current systems if the current exceeds the critical value.

The current in the bulk produces the Hall electric field in the bulk. Due to the electric field, the bulk one particle states are modified to become extended energetically, and QHR becomes narrow and eventually disappears at a critical current. A theoretical estimation is made. The results are consistent with the recent experiment by Kawaji et al and others\textsuperscript{8}. The critical electric fields are proportional to $B^{3/2}$ and are independent of Landau levels in consistent with the experiment by Kawaji et al and others\textsuperscript{8}, but their magnitudes are substantially smaller.

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Appendix A

The explicit form of matrices of Eqs. (3.5) and (3.6) are following:

\[
\begin{align*}
&d_x(p) = d'_x(p) = -i\frac{a}{2}\left(\frac{\partial}{\partial p_x} - i\frac{\partial}{\partial p_y}\right) \log \alpha(p_x, p_y), \\
&d_y(p) = d'_y(p) = -i\frac{a}{2}\left(i\frac{\partial}{\partial p_x} + \frac{\partial}{\partial p_y}\right) \log \alpha(p_x, p_y), \\
&\alpha(p_x, p_y) = \sum e^{ip_x(m_1-m_1')} + e^{ip_y(n_1-n_1')} \langle \vec{R}_1 | \vec{R}'_1 \rangle,
\end{align*}
\]

(A.1)

\[
\begin{align*}
&\bar{d}_x(p)_{l_1,l_2} = \bar{d}'_x(p)_{l_1,l_2} = \frac{-i}{\sqrt{2eB}}(\sqrt{l_2}\delta_{l_1,l_2-1} - \sqrt{l_2+1}\delta_{l_1,l_2+1}), \\
&\bar{d}_y(p)_{l_1,l_2} = \bar{d}'_y(p)_{l_1,l_2} = \frac{1}{\sqrt{2eB}}(\sqrt{l_2}\delta_{l_1,l_2-1} + \sqrt{l_2+1}\delta_{l_1,l_2+1}),
\end{align*}
\]

(A.2)

\[
\begin{align*}
&\bar{d}_i \text{ satisfy the commutation relation:} \\
&[\bar{d}_x(p), \bar{d}_y(p)] = [\bar{d}'_x(p), \bar{d}'_y(p)] = -i \frac{1}{eB}.
\end{align*}
\]

(A.3)

\[
\begin{align*}
&d_x(p) \text{ is given in Fig.9.}
\end{align*}
\]

Appendix B

We study finite size correction of a topological invariant defined by a propagator, \(S_{l_1,l_2}(p)\),

\[
N_w = \frac{1}{24\pi^2} \int d^3p \epsilon_{\mu\nu\rho} \text{Tr}[\partial S^{-1} \frac{\partial S^{-1}}{\partial p_\mu} S \frac{\partial S^{-1}}{\partial p_\nu} S \frac{\partial S^{-1}}{\partial p_\rho} S].
\]

(B.1)

\(N_w\) agrees with the integer if the momentum space is compact and the matrix space of \(S_{l_1,l_2}(p)\) includes SU(2) group as a subspace. We see in the main text that the finite size correction does not appear in QHE. Finite temperature effect is similar to the finite size effect.
integration in Eq.(B-1) is replaced with a discrete summation at finite temperature, and $N_w$ of QHE is given by,

$$N_w = \begin{cases} N, & \text{finite size),} \\ \sum_{l=1}^{N} (\coth \frac{\beta}{2}|\mu - E_l| + 1)/2, & \text{finite temperature),} \end{cases}$$

$$\beta = \frac{1}{kT}, \quad \mu : \text{chemical potential.} \quad (B.2)$$

Finite size effect and finite temperature effect of $N_w$ depend on the form of the propagator. We compute these effects of ground state$^{16}$ of Dirac field, which has

$$S^{-1}(p) = \gamma_\mu p^\mu + m,$$

$$\gamma_0 = i\tau_3,$$

$$\gamma_1 = i\tau_1,$$

$$\gamma_2 = i\tau_2. \quad (B.3)$$

$N_w$ is given by,

$$N_w = \frac{1}{2} \frac{|m|}{m} \left\{ \coth \frac{|m| L}{2} \right\}, \quad \text{(finite size),}$$

$$\coth \frac{|m| L}{2}, \quad \text{(finite temperature),}$$

$$N_w = \frac{1}{2} \frac{|m|}{m} \left\{ \coth \frac{|m| L}{2} \right\}, \quad \text{(finite size),}$$

$$\coth \frac{|m| L}{2}, \quad \text{(finite temperature),}$$

where $L$ is the width in $x$-direction. Thus the topological invariant $N_w$ of the Dirac theory has the both of finite size correction and finite temperature correction of exponential type$^{17}$.

### Appendix C

We study a system described by one-dimensional Landau-Ginzburg type static energy for $A_0$,

$$U = \int_0^L dx \left\{ \frac{1}{2} c(x)(\partial_x A_0)^2 + \frac{1}{4}\lambda(x)(\partial_x A_0)^4 \right\}, \quad (C.1)$$

with a constraint,

$$\sigma_{xy} I_y = \int_0^L dx \partial_x A_0 = \text{constant.} \quad (C.2)$$

In writing (C.1), We have ignored higher derivative terms such as $(\partial_x^2 A_0)^2$. Hence a self-consistency of Ref.18 is not considered here. Owing to translational invariance in $y$-direction,
we use one-dimensional form. We regard the coefficient $c(x)$ and $\lambda(x)$ are known and study solutions of Euler-Lagrange equation under the above constraint. This gives the electric field and the current density.

Depending on sign of the function $c(x)$, solutions have totally different properties.

1. $c(x) > 0$, $\lambda(x) > 0$

Euler-Lagrange equation from (C-1) is given by,

$$\partial_x[(\partial_x A_0)(c(x) + \lambda(x)\partial_x A_0^2)] = 0. \quad (C.3)$$

Integrating (C-3), we have

$$\partial_x A_0\{c(x) + \lambda(x)\partial_x A_0^2\} = C_0 = \text{constant}. \quad (C.4)$$

The electric field $\partial_x A_0$ is given by solving (C-4). For small $\partial_x A_0$, the solution is approximately given by,

$$\partial_x A_0 = \frac{C_0}{C(x)}. \quad (C.5)$$

This solution has electric field in whole region and the current flows in the bulk and at the edges. The non-zero constant $C_0$ is determined from the constraint (C-2).

2. $c(x) > 0$, $\lambda(x) > 0$ in the bulk and $c(x) < 0$, $\lambda(x) > 0$ at the edges.

Euler-Lagrange equation is the same as before and is given in (C-3). Because $c(x)$ is negative near the edge regions, a new type of solution, which corresponds to $C_0 = 0$, exists. If $C_0 = 0$, we have

$$c(x) + \lambda(x)(\partial_x A_0)^2 = 0, \quad (C.6)$$

or $\partial_x A_0 = 0$.

In a region where $c(x) > 0$, the electric field vanishes. In another region where $c(x) < 0$, either the electric field vanishes or the electric field satisfies the first equation of (C-6). They
are determined from the constraint of total current (C-2). This type of solutions have electric field only at the edge regions. The bulk has neither electric field nor electric current. Hence this corresponds to edge states.

The edge current region where the first solution of (C-6) is satisfied is determined from the constraint (C-2), hence it depends on the total current. A relation between the width of the edge current region and the total current is given in Fig. 9. This type of solution disappears if the current exceeds a critical value which satisfies,

\[ \int_{0, c(x) < 0}^{L} dx \sqrt{-\frac{c(x)}{\lambda(x)}} = \sigma_{xy} J_c. \]  

(C.7)

In obtaining Fig. 9, we have used,

\[ \lambda(x) = \lambda_0 = \text{constant}, \]

\[ c(x) = \begin{cases} c, & \text{at the bulk,} \\ -c + dx^2, & \text{near the edge, } x \leq \sqrt{\frac{2c}{d}}. \end{cases} \]  

(C.8)

The solution becomes that of \( C_0 \neq 0 \) if the current exceeds the critical value, and is obtained by solving (C-4). It has current in the whole region. The current distributions of the solutions are given in Fig. 10.

**Appendix D**

In the von Neumann lattice representation, it is easy to compute energy spectrum in the momentum representation. Hamiltonian of a system with a uniform electric field, \( E \), in the \( x \)-direction is given by,

\[ H = H_0 + H_1, \]

\[ H_0 = \sum_{l, \tilde{R}_1} E_l b_l(\tilde{R}_1) a_l(\tilde{R}_1), \]  

\[ H_1 = eE \sum_{l, \tilde{R}_1} b_l(\tilde{R}_1) \left[ d_{xl1, l2} \delta_{\tilde{R}_1, \tilde{R}_2} + \delta_{l_1, l_2} \alpha_m \delta_{\tilde{R}_1, \tilde{R}_2} + \delta_{l_1, l_2} d_x(\tilde{R}_1 - \tilde{R}_2) \right] a_{l_2}(\tilde{R}_2), \]  

(D.1)

where the zero-momentum state is not included in the sets \( \{ b_l(\tilde{R}) \} \) and \( \{ a_l(\tilde{R}) \} \) from the constraint of the minimum coherent states. \( d_{xl1, l2} \) is given in Eq. (A.3) and the Fourier
The transform of $d_l(\vec{R})$ is given in Eqs.(3.6) and (A.1). The first term of $H_1$ gives an inter Landau level mixings and the second term gives the static energy due to the electric field. These two terms do not contribute to the width of the Landau levels in the momentum representation. The last term, on the other hand, gives the intrinsic momentum dependent energy, $d_x(\vec{p})$. In the momentum representation, $d_x(\vec{p})$ and $d_y(\vec{p})$ are expressed as,

$$
\begin{align*}
  d_x(\vec{p}) &= eEA^{-2}p_y + \frac{\partial}{\partial p_x}\theta(\vec{p}), \\
  d_y(\vec{p}) &= \frac{\partial}{\partial p_y}\theta(\vec{p}),
\end{align*}
$$  

(D.2)

with a suitable gauge function $\theta(\vec{p})$ in the momentum space. A gauge invariant quantity, $\oint d_i(\vec{p})d\vec{p}$, is used for defining the gauge invariant energy width. The invariant width is given by, $eEA^{-2}(2\pi/a) = eEA$. Hence the Landau levels in systems of the constant electric field in the $x$-direction have the width, $eEA$. Hence the width of the extended Landau levels is inversly proportional to $\sqrt{B}$. QHR disappears if the above width agrees to the Landau level’s energy spacing, which is proportional to $B$. The critical electric field of the breakdown of QHE is proportional to $B^{3/2}$ and the proportional constant is independent of Landau levels, in agreement with the experiments.
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FIGURE CAPTIONS

1) Energy eigenvalues (a), eigenfunctions (c) at \( p_y = \pi/2a \), and their currents (b) are shown for a narrow potential well of a width, \( a \). Left moving modes and right moving modes are not separated. In (c), potential barrier is denoted by shadow region.

2) Energy eigenvalues (a), eigenfunctions (c) at \( p_y = \pi/2a \), and their currents (b) are shown for a wide potential well of a width, \( 5a \). Left moving modes and right moving modes are separated. Edge states are confined in narrow edge regions. In (c), potential barrier is denoted by shadow region.

3) Lowest order self-energy diagram which removes the degeneracy of Landau levels due to potentials is shown.

4) Lowest order self-energy diagram which removes the degeneracy of Landau levels due to interactions is shown.

5) One-particle spectrum in the presence of the edge region is shown. From Eqs.(2.9), (2.11), and (2.14), in narrow regions around the center of Landau levels, there are extended states. At outside of this regions there are localized states and one-dimensional edge states. These are regarded as QHR.

6) \( E_F \) dependence of the topological invariant, Eq.(3.7), in free system is shown.

7) An example of higher order diagrams is shown. If the internal line does not satisfy the boundary condition, these diagrams do not contribute to \( \pi^{(1)}_{\mu\nu}(p) \) but contribute to \( \pi^{(2)}_{\mu\nu}(p_1, p_2) \).

8) Hall conductance \( \sigma_{xy} \) in realistic system is given. In quantum Hall regimes where there are only localized states and one-dimensionaly extended states, \( \sigma_{xy} \) agrees with \( (e^2/h)N \).

9) The width of edge current solution is obtained from the constraint Eq.(C-2), for the second case where \( c(x) \) becomes negative toward the edges. Edge current solutions
disappear if the current exceeds the critical value which satisfies (C.7).

10) A function $c(x)$ of (C-8), and the current distribution for a small current case and for a large current case are shown. The current is restricted in narrow regions near the edges if $J_y < J_c$ but the current spreads into the whole region if $J_y > J_c$. 