ON RESOLVABILITY OF LINDELÖF GENERATED SPACES

MARIA A. FILATOVA AND ALEXANDER V. OSIPOV

Abstract. In this paper we study the properties of $P$ generated spaces (by analogy with compactly generated). We prove that a regular Lindelöf generated space with uncountable dispersion character is resolvable. It is proved that Hausdorff hereditarily $L$-spaces are $L$-tight spaces which were defined by István Juhász, Jan van Mill in (Variations on countable tightness, arXiv:1702.03714v1). We also prove $\omega$-resolvability of regular $L$-tight space with uncountable dispersion character.

Keywords: resolvable space, $k$-space, tightness, $\omega$-resolvable space, Lindelöf generated space, $P$ generated space, $P$-tightness.

1. Introduction

A Hausdorff space is said to be a $k$-space (also called compactly generated) if it has the final topology with respect to all inclusions $K \to X$ of compact subspaces $K$ of $X$, so that a set $A$ in $X$ is closed in $X$ if and only if $A \cap K$ is closed in $K$ for all compact subspaces $K$ of $X$. A space is Fréchet-Uryson if, whenever a point $x$ is in the closure of a subset $A$, there is a sequence from $A$ converging to $x$; it is proved in [1] that a space is hereditarily $k$, i.e., every subspace is a $k$-space, if and only if it is Fréchet-Uryson. For examples, any locally compact, or the first countably Hausdorff spaces are $k$-spaces.

An interesting common generalizations of $k$-spaces and the notion of tightness was given by A.V. Arhangel’skii and D.N. Stavrova in [2]. They defined the $k$, $k_1$, $k_2$, and $P$-tightness.
$k^*$-tightness as follows: the $k$-tightness of $X$ does not exceed $\tau$ ($t_k(X) \leq \tau$) if and only if for every $A \subseteq X$ that is not closed there exists a $\tau$–compact $B \subseteq X$ for which $A \cap B$ is not closed in $X$ (a set $B$ is called $\tau$–compact if $B = \bigcup \{B_\alpha : \alpha \in \tau\}$, where $B_\alpha$ is a compact subset of $X$ for all $\alpha \in \tau$); the $k_1$–tightness of $X$ does not exceed $\tau$ ($t_{k_1}(X) \leq \tau$) if and only if for every $A \subseteq X$ and every $x \in \overline{A}$ there exists a $\tau$–compact $B \subseteq X$ such that $x \in \overline{A \cap B}$; the $k^*$–tightness of $X$ does not exceed $\tau$ ($t_k^*(X) \leq \tau$) if and only if for every $A \subseteq X$ and every $x \in \overline{A}$ there exists a $\tau$–compact $B \subseteq A$ such that $x \in \overline{B}$.

In [4] István Juhász and Jan Van Mill defined and studied nine attractive natural tightness conditions for topological spaces. They called a space $\mathcal{P}$–tight, if for all $x \in X$ and $A \subseteq X$ such that $x \in A$, there exists $B \subseteq A$ such that $x \in \overline{B}$ and $B$ has the property $\mathcal{P}$.

István Juhász and Jan Van Mill considered in [4] the following properties $\mathcal{P}$ that a subspace of a topological space might have:

- $\omega D$ Countable discrete;
- $\omega N$ Countable and nowhere dense;
- $C_2$ Second-countable;
- $\omega$ Countable;
- $hL$ Hereditarily Lindelöf;
- $\sigma$-cmp $\sigma$–compact;
- $ccc$ The countable chain condition;
- $L$ Lindelöf;
- $wL$ Weakly Lindelöf.

Inspired by the researches above, in the first part of this paper we introduce and study $\mathcal{P}$ generated spaces (by analogy with $k$-spaces). Since the space with property $\mathcal{P}$ is not necessarily closed, there are two ways to determine the $\mathcal{P}$ generated spaces.

**Definition 1.** A topological space $X$ is $\mathcal{P}$–space ($\mathcal{P}$ generated space) if a subspace $A$ is closed in $X$ if and only if $A \cap P$ is closed in $P$ for any subspace $P \subseteq X$ which has the property $\mathcal{P}$.

**Definition 2.** A topological space $X$ is $\mathcal{P}c$–space if a set $A$ is closed in $X$ if and only if $A \cap P$ is closed in $P$ (or, is the same one, in $X$) for any closed subspace $P \subseteq X$ which has the property $\mathcal{P}$.

In this paper we will consider the property $\mathcal{P} \in \{\omega N, C_2, \omega, hL, \sigma$-cmp, $ccc, L, wL\}$ because each $\omega D$-space is discrete. Note that every space with countable tightness (i.e. $\omega$-tight) is $\omega$–space [11].

The class of $L$-spaces generalizes the class of spaces with countable tightness and the class of $k$-spaces. Below (Theorem 1) we get the example of the space with countable tightness, but is not $Lc$-space.

Note that the relationships between the spaces of $\mathcal{P}$–tight where the property $\mathcal{P} \in \{\omega N, \omega D, C_2, \omega, hL, \sigma$-cmp, $ccc, L, wL\}$ were considered in [4].

We summarize the relationships between $\mathcal{P}$–spaces ($\mathcal{P}$–s) and $\mathcal{P}$–tight ($\mathcal{P}$–t) where the property $\mathcal{P} \in \{\omega N, \omega, hL, \sigma$-cmp, $ccc, L, wL\}$ in next diagrams.
In 1943 E. Hewitt \cite{5} called a topological space $\tau$-resolvable, if it can be represented as a union of $\tau$ dense disjoint subsets. A $2$-resolvable space is called resolvable, and irresolvable space is one which is not resolvable. He also defined the dispersion character $\Delta(X)$ of a space $X$ as the smallest size of a non-empty open subset of $X$. A topological space $X$ is called maximally resolvable if it is $\Delta(X)$-resolvable.

The $\omega$-resolvability of Lindel"of spaces whose dispersion character is uncountable was proved by I. Juhasz, L. Soukup, Z. Szentmiklossy in \cite{6}. E. Hewitt in \cite{5} constructed an example of countable irresolvable normal space, so condition $\Delta(X) > \omega$ is natural. V.I. Malyskin constructed an example of irresolvable Hausdorff Lindel"of space with uncountable dispersion character in \cite{7}, therefore, resolvability of regular Lindel"of spaces was studied.

The resolvability of locally compact spaces was proved by Hewitt in 1943 \cite{5}. The resolvability (maximal resolvability) of $k$-spaces was proved by N.V. Velichko in 1976 \cite{8} (E.G. Pytkeev in 1983 \cite{9}). In connection with these results, the question of the resolvability of $L$-spaces is natural.

Second part of this paper is devoted to resolvability of regular $L$-spaces of uncountable dispersion character. We also prove $\omega$-resolvability of regular hereditarily $L$-spaces of uncountable dispersion character.

Throughout this paper the symbol $\omega$ denotes the smallest infinite cardinal, $\omega_1$ stands for the smallest uncountable cardinal. For a subset $A$ of a topological space $X$, the closure and interior set of $A$ are respectively denotes by $\overline{A}$ (or $[A]$) and $Int(A)$. We assume that all spaces are Hausdorff. Notation and terminology are taken from \cite{10}.

2. $\mathcal{P}$ GENERATED SPACE

By Definitions \cite{1} and \cite{2}, a $\mathcal{P}c$-space is a $\mathcal{P}$-space. The following example shows that the converse is not true.
Theorem 1. There exists a space $X$ such that $X$ is a $\omega$-space and $t_k(X) = t_k^*(X) = t(X) = \omega$.

Proof. Let $X = \beta N$, $M = \beta N \setminus N$. In a $x \in X$ we put the base of neighborhoods $\mathcal{B}(x) = \{x\}$, if $x \in N$ and $\mathcal{B}(x) = \{(U(x) \setminus M) \cup \{x\}\}$ where $U(x)$ is open in $\beta N$, if $x \in M$.

Now we show that $X$ is a $\omega$-space. Consider the set $A \subset X$ for which $A \cap P$ is closed in $P$ for any countable subspace $P \subseteq X$. Let $x \in \overline{A}$. The set $P_1 = (A \cap N) \cup \{x\}$ is a countable. Then $A \cap P_1$ is closed in $P_1$ therefore $x \in A$.

Let us note that if $P$ is a countable tightness is $\omega$-space. Consider the set $\mathcal{P}$ of $\omega$-spaces. The converse statements are not true. Theorem 1 shows that even $\omega$-spaces do not generalize a notion of countable tightness. Thus, a natural generalization of spaces with countable tightness and $k$-spaces is $\mathcal{P}$-spaces.

Theorem 2. There exists a space $X$ such that $X$ is a $\omega$-space, but is not $\omega c$-space.

Proof. For example, let $D$ be the infinite discrete space of cardinality continuum $\mathfrak{c}$, and $X = \beta D$ be the Čech-Stone compactification of $D$. Being the compact space, $X$ is a $\omega$-space. Consider $p \in X \setminus \{\bigcup \overline{A} : A \in [D]^{\leq \omega}\}$. Since the space $D$ has not an uncountable Lindelöf subspace, there is no Lindelöf subspace $B \subset D$ such that $p \in \overline{B}$.

In what follows we shall concentrate on the study of some properties of $\mathcal{P}$-spaces. Most of all we are interested in the resolvability of $L$-space. The following statements will be useful for these purpose.

Theorem 3. Let the property $\mathcal{P} \in \{\omega N, \omega, hL, \sigma$-cmp, $ccc, L, wL\}$. Then the following properties of a space $X$ are equivalent.

(i) $X$ is a $\mathcal{P}$-space;

(ii) a set $A \subset X$ is non-closed in $X$ if and only if there exists $P \subseteq X$ which have a property $\mathcal{P}$ such that $P \cap A$ is non-closed in $X$.

Proof. (i) $\Rightarrow$ (ii). Let $X$ is a $\mathcal{P}$-space, $A \subset X$ is non-closed in $X$. Then there exists $P \subseteq X$ which have a property $\mathcal{P}$ such that $P \cap A$ is non-closed in $P$, therefore $P \cap A$ is non-closed in $X$. Let for a set $A$ there exists $P \subseteq X$ which have a property $\mathcal{P}$ such that $P \cap A$ is non-closed in $X$. Consider $x \in \overline{P} \cap A \setminus (P \cap A)$. If $x \in A$ then $x \in \overline{A} \setminus P$, i.e. $P \cap A$ is non-closed in $P$, consequently a set $A$ is non-closed in $X$. If $x \in P$ then $x \in \overline{A} \setminus A$, i.e. a set $A$ is non-closed in $X$.
(ii) → (i). Consider $A \subset X$ such that $A \cap P$ is closed in $P$ for any subspace $P \subset X$ which has the property $\mathcal{P}$. Suppose that $A \neq \emptyset$. Then exists $P \subset X$ which have a property $\mathcal{P}$ such that $P \cap A$ is non-closed in $X$. Consider $x \in \overline{P \cap A \setminus (P \cap A)}$. Let $P_1 = P \cup \{x\}$. Then $P_1$ has the property $\mathcal{P}$ and $P_1 \cap A$ is not closed in $P_1$, contradiction. □

**Corollary 2.** A space $X$ is $\mathcal{P}$-space if and only if for any $A$ non-closed in $X$ there are $x \in \overline{A \setminus A}$ and $P \subset X$ with a property $\mathcal{P}$ such that $x \in \overline{P \cap A}$.

**Corollary 3.** For any non-isolated point $x$ in $\mathcal{P}$-space $X$ there is a subspace $P \subset X$ with a property $\mathcal{P}$ such that $x \in \overline{P \setminus \{x\}}$.

We give the following definition.

**Definition 3.** The property $\mathcal{P}$ is ICS (independent of the containing subspace) in $X$ if $P \subset X$ has a property $\mathcal{P}$ in $X$ if and only if $P$ has a property $\mathcal{P}$ in $Y$ for any subset $Y$ such that $P \subset Y$.

The next theorem shows that hereditarily $\mathcal{P}$-space is $\mathcal{P}$-tight if property $\mathcal{P}$ is ICS in $X$.

**Theorem 4.** Let property $\mathcal{P}$ be ICS in $X$. A space $X$ is hereditarily $\mathcal{P}$-space if and only if for any $A \subset X$ and $x \in \overline{A \setminus A}$ there exists $P \subset A$ which have a property $\mathcal{P}$ such that $x \in \overline{P}$.

**Proof.** Let $X$ be a hereditarily $\mathcal{P}$-space, $A \subset X$ and $x \in \overline{A \setminus A}$. The subspace $B = A \cup \{x\}$ is $\mathcal{P}$-space, $x$ is non-isolated point in $B$. Then there is $P \subset A$ with a property $\mathcal{P}$ such that $x \in \overline{P}$.

Converse, let $B \subset X$, $A \subset B$ and $A \cap P$ is closed in $B$ for any $P \subset B$ which have a property $\mathcal{P}$. If $x \in \overline{A \setminus A}$ then there exists $P \subset A$ which have a property $\mathcal{P}$ such that $x \in \overline{P}$, i.e. $A \cap P$ is non-closed in $B$. □

**Corollary 4.** Let property $\mathcal{P}$ be ICS in $X$. A space $X$ is $\mathcal{P}$-tight if and only if each subspace $Y$ of $X$ is $\mathcal{P}$-tight.

Note that $\omega$N-tight is not hereditarily property. The real numbers $\mathbb{R}$ is $\omega$N-tight, but its subspace $Y = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ is not $\omega$N-tight.

### 3. On resolvability of $L$-spaces

The maximal resolvability of $\omega$-tight spaces whose dispersion character is uncountable was proved by E.G. Pytkeev in [9]. Therefore $\omega$-spaces with uncountable dispersion character are maximally resolvable. A. Bella and V.I. Malykhin was proved that $\omega$N-tight spaces are maximally resolvable [12].

E. Hewitt in [5] constructed an example of countable irresolvable normal space. Such space is obviously $\omega$-tight ($hL$-, $\sigma$-cmp-, ccc-, $L$-, $wL$-tight) space with countable dispersion character. Therefore, in studying the resolvability of these spaces, it is necessarily to consider the spaces with uncountable dispersion character.

In this section we prove $\omega$-resolvability of regular $L$-tight spaces which dispersion character is uncountable. Consequently, $hL$-, $\sigma$-cmp-, ccc-, $wL$-tight spaces are also $\omega$-resolvable (if $X$ is regular space then properties ccc-tight, $L$-tight, $wL$-tight are equivalent).
The $\omega$-resolvability of $\omega D$-spaces was proved by P.L. Sharma and S. Sharma in [13]. V.I. Malychin constructed an example of irresolvable Hausdorff Lindelöf space with uncountable dispersion character [7].

It is clear that any Lindelöf space is $L$-space. Therefore, the resolvability of $L$-spaces must be investigated in the class of regular spaces.

We will use a following

**Theorem 5.** (Hewitt’s criterion of resolvability [12]) A topological space $X$ is resolvable ($\tau$-resolvable) if and only if for all open subset $U$ of $X$ there exist nonempty resolvable ($\tau$-resolvable) subspace without isolated points.

A.G. El’kin proved the following elegant result

**Theorem 6.** (Theorem 1 in [3]) Let $X$ be a collectionwise Hausdorff $\sigma$-discrete normal space that satisfies the following condition:

(*) For each point $x \in X$ there exists a discrete set $D \subset X$ such that $x \in \overline{D} \setminus D$.

Then the space $X$ is $\omega$-resolvable.

Recall that a set $D \subset X$ is strongly discrete if for every $x \in D$ there is an open neighborhood $U_x$ such that $U_x \cap U_y = \emptyset$ for $x \neq y$.

A point $x$ of a space $X$ is called lsd-point (a limit point of a strongly discrete subspace) if there exists a strongly discrete subspace $D \subset X$ such that $x \in \overline{D} \setminus D$.

By using idea of the proof of El’kin’s Theorem 6, P.L. Sharma and S. Sharma proved the following result [13].

**Theorem 7.** (P.L. Sharma, S. Sharma) Let $X$ be a $T_1$-space. If each $x \in X$ is a limit point of a strongly discrete subspace of $X$ then $X$ is a $\omega$-resolvable.

**Definition 4.** A discrete set $D$ of cardinality $\omega_1$ is a correct discrete, if any $Y \subseteq D$ such that $|Y| = \omega_1$ has a $\omega_1$-accumulation point.

The following lemma is a generalized version of the lemma 2.1 in [15].

**Lemma 1.** Let $X$ be a regular space such that for each $x \in X$ there exists a correct discrete set $D_x$ of cardinality $\omega_1$ such that $x$ is a $\omega_1$-accumulation point of $D_x$. Then $X$ is resolvable.

**Proof.** By Theorem 5 it suffices to prove that each non-empty open set $V$ of $X$ contains dense-in-itself $Y$ such that $\overline{Y}$ is resolvable.

Denote by $P(D)$ the set of $\omega_1$-accumulation points of a correct discrete set $D$. Let us note that $P(D) = P(D)$ and for any open set $U$ such that $P(D) \subset U$, the set $D \setminus U$ is at most countable.

Let $y \in V$, $D_y \subset V$ is a correct discrete set of cardinality $\omega_1$, such that $y \in P(D_y)$. Let $Y_1 = D_y$, $Y_n = \bigcup\{D_y : y \in Y_{n-1}\}$, where $D_y$ is correct discrete set, $y$ is $\omega_1$-accumulation of $D_y$, we put $Y = \bigcup_{n=1}^{\infty} Y_n$. It is clear that $|Y| = \omega_1$ and any $y \in Y$ is a $\omega_1$-accumulation point of correct (in $\overline{Y}$) discrete subset $D_y$ of $Y$.

Now we prove that $\overline{Y}$ is resolvable.

First, we renumber the space $Y = \{y_\alpha : \alpha < \omega_1\}$.
For $\alpha < \omega_1$ we construct sets $A^i_\alpha \subset \overline{Y}$, $A_i = \bigcup_{\alpha < \omega_1} A^i_\alpha$, where $i = 1, 2$, such that $A_1 \cap A_2 = \emptyset$ and $\overline{A_1} = \overline{A_2} \supset Y$.

Consider $y_1$. Let $D_1$ be a correct (in $X$) discrete space of cardinality $\omega_1$, such that $y_1 \in P(D_1)$. Let $A^1_1 = D_1$, $A^2_1 = P(D_1)$, $P_1 = P(D_1)$. Obviously, $A^1_1 \cap A^2_1 = \emptyset$ and $\overline{A^1_1} \cap \overline{A^2_1} \supset P_1$.

Suppose that for each $\alpha < \beta < \omega_1$ we construct sets $P_\alpha \subset \overline{Y}$, $D_\alpha \subset Y$, $A^1_\alpha$, $A^2_\alpha$ with the following properties.

1. If $y_\alpha \in \left[ \bigcup_{\eta < \alpha} P_\eta \right]$, then $P_\alpha = D_\alpha = \emptyset$, $A^i_\alpha = \bigcup_{\eta < \alpha} A^i_\eta$, $i = 1, 2$.

2. If $y_\alpha \notin \left[ \bigcup_{\eta < \alpha} P_\eta \right]$ and $y_\alpha \in \left[ \bigcup_{\eta < \alpha} A^1_\eta \right] \cap \left[ \bigcup_{\eta < \alpha} A^2_\eta \right]$, then $P_\alpha = \{ y_\alpha \}$, $D_\alpha = \emptyset$, $A^1_\alpha = \bigcup_{\eta < \alpha} A^1_\eta$, $A^2_\alpha = \bigcup_{\eta < \alpha} A^2_\eta$.

3. If $y_\alpha \notin \left[ \bigcup_{\eta < \alpha} P_\eta \right]$ and $y_\alpha \notin \left[ \bigcup_{\eta < \alpha} A^1_\eta \right] \cap \left[ \bigcup_{\eta < \alpha} A^2_\eta \right]$, then $D_\alpha$ is a correct discrete set of cardinality $\omega_1$, such that $y_\alpha \in P(D_\alpha)$, $\left( \bigcup_{\eta < \alpha} D_\eta \right) \cap D_\alpha = \emptyset$, $P_\alpha = P(D_\alpha)$. In this case $A^1_\alpha = \bigcup_{\eta < \alpha} A^1_\eta \cup P_\alpha$, $A^2_\alpha = \bigcup_{\eta < \alpha} A^2_\eta \cup D_\alpha$.

4. $A^1_\alpha \cap A^2_\alpha = \emptyset$; $A^i_\alpha$ form a monotonically nondecreasing sequence (by $\alpha$), $\left[ \bigcup_{\eta < \alpha} P_\eta \right] \subset \left[ A^i_\alpha \right]$, and $A^i_\alpha \subset \bigcup_{\eta < \alpha} (P_\eta \cup D_\eta)$, $i = 1, 2$.

Consider $y_\beta$. If $y_\beta \in \left[ \bigcup_{\alpha < \beta} P_\alpha \right]$, we suppose $P_\beta = D_\beta = \emptyset$, $A^i_\beta = \bigcup_{\alpha < \beta} A^i_\alpha$, $i = 1, 2$. It is easy to see $\left[ \bigcup_{\alpha < \beta} P_\alpha \right] \subset \left[ A^i_\beta \right]$, $i = 1, 2$ and $A^1_\beta \cap A^2_\beta = \emptyset$.

Indeed, by construction, $\left[ \bigcup_{\alpha < \beta} P_\alpha \right] \subset \left[ A^i_\beta \right]$, $i = 1, 2$, then $\left[ \bigcup_{\alpha < \beta} P_\alpha \right] \subset \left[ \bigcup_{\alpha < \beta} P_\alpha \right] \subset \left[ A^i_\beta \right]$.

Let $y_\beta \notin \left[ \bigcup_{\alpha < \beta} P_\alpha \right]$, but $y_\beta \in \left[ \bigcup_{\alpha < \beta} A^1_\alpha \right] \cap \left[ \bigcup_{\alpha < \beta} A^2_\alpha \right]$, then $P_\beta = \{ y_\beta \}$, $D_\beta = \emptyset$, $A^i_\beta = \bigcup_{\alpha < \beta} A^i_\alpha$, $i = 1, 2$.

It is easy to see that $A^i_\beta$ satisfy item 4.

Finally the last case $y_\beta \notin \left[ \bigcup_{\alpha < \beta} P_\alpha \right]$, and $y_\beta \notin \left[ \bigcup_{\alpha < \beta} A^1_\alpha \right] \cap \left[ \bigcup_{\alpha < \beta} A^2_\alpha \right]$.

Suppose, for definiteness, $y_\beta \notin \left[ \bigcup_{\alpha < \beta} A^1_\alpha \right]$. Consider a neighborhood $U_\beta$ of $y_\beta$, such that the closure of $U_\beta$ is disjoint with $\left[ \bigcup_{\alpha < \beta} A^1_\alpha \right]$ and $\left[ \bigcup_{\alpha < \beta} P_\alpha \right]$.
In this neighborhood there is at most a countable of points of the set \( \bigcup_{\alpha<\beta} A^2 \) because it intersect with \( U_\beta \) can only with \( D_\alpha, \alpha<\beta \) (by 4).

For each \( \alpha \) such that \( D_\alpha \neq \emptyset \) and \( D_\alpha \) is a discrete, \( P(D_\alpha) \subset \overline{Y} \setminus [U_\beta] \), hence, \( D_\alpha \cap U_\beta \) is at most a countable. It follows that \( \left( \bigcup_{\alpha<\beta} D_\alpha \right) \cap U_\beta \) has a cardinality < \( \omega_1 \) and, by 4, \( \left( \bigcup_{\alpha<\beta} A^2 \right) \cap U_\beta \) is countable. Let \( D_\beta \) be a correct discrete set such that \( D_\beta \subset U_\beta, y_\beta \in P(D_\beta) \) and \( D_\beta \cap \left( \bigcup_{\alpha<\beta} A^2 \right) = \emptyset. \)

Let \( P_\beta = P(D_\beta), A^1_\beta = D_\beta \cup \left( \bigcup_{\alpha<\beta} A^1_\alpha \right), A^2_\beta = D_\beta \cup \left( \bigcup_{\alpha<\beta} A^2_\alpha \right). \)

By construction, \( A^1_\beta, A^2_\beta \) are disjoint sets and has the property 4.

We prove that \( Y \subset \left[ \bigcup_{\alpha<\omega_1} P_\alpha \right. \). Suppose that \( Y \setminus \left[ \bigcup_{\alpha<\omega_1} P_\alpha \right. \neq \emptyset, \) consider \( y_\gamma \in Y \setminus \left[ \bigcup_{\alpha<\omega_1} P_\alpha \right. \). Then \( P_\gamma = \emptyset, \) because if \( P_\gamma \neq \emptyset, \) then \( y_\gamma \in P_\gamma. \) By 1, 2, 3, we have \( y_\gamma \in \left[ \bigcup_{\alpha<\gamma} P_\alpha \right. \subset \left[ \bigcup_{\alpha<\omega_1} P_\alpha \right. , \) a contradiction.

Let \( A_i = \bigcup_{\alpha<\omega_1} A^i_\alpha, i = 1, 2. \) By construction, \( A_1 \cap A_2 = \emptyset \) and \( |A_1| = |A_2| \supset Y. \) So we have that \( Y \) is resolvable.

\[ \square \]

**Corollary 5.** Let \( X \) be a regular space, \( Y = \{x \in X : \exists \text{ countable discrete set } D_x \text{ such that } x \in \overline{D_x} \setminus D_x \) or \( \exists \) correct discrete \( D_x \) of cardinality \( \omega_1 \) such that \( x \in \overline{D_x} \setminus D_x \) and \( \overline{Y} = X. \) Then \( X \) is resolvable.

**Proof.** Let \( A = \{x \in X : \exists \text{ countable discrete set } D_x \text{ such that } x \in \overline{D_x} \setminus D_x \} \) and \( B = Y \setminus A. \) By Theorem 4, the set \( \text{Int}(A) \) is resolvable. Hence, there are disjoint sets \( A_1, A_2 \subset \text{Int}(A) \) such that \( A_i \supset \text{Int}(A) \) for \( i = 1, 2. \) If \( \text{Int}(A) = \emptyset \) then we put \( A_i = \emptyset \) for \( i = 1, 2. \)

Now we prove that if \( \text{Int}(B) \neq \emptyset \) then \( \text{Int}(B) \) is resolvable. Let \( W \) be an non-empty open set such that \( \overline{W} \subset \text{Int}(B). \) Note that there is a set \( Y \subset W \) such that \( |Y| = \omega_1 \) and any \( y \in Y \) is a \( \omega_1 \)-accumulation point of correct (in \( Y \) and also in \( W \)) discrete subset \( D_y \) of \( Y. \) Then, by Lemma 4, \( \text{Int}(B) \) is resolvable.

Hence, there are disjoint sets \( B_1, B_2 \subset \text{Int}(B) \) such that \( \overline{B_i} \supset \text{Int}(B) \) for \( i = 1, 2. \) If \( \text{Int}(B) = \emptyset \) then we put \( B_i = \emptyset \) for \( i = 1, 2. \)

Let \( Z = X \setminus (\text{Int}(A) \cup \text{Int}(B)), A_z = A \cap Z, B_z = B \cap Z. \) Consider \( C = \text{Int}(A_z \cup B_z). \) Let \( C_1 = A \cap C, C_2 = B \cap C \) for \( C \neq \emptyset. \) Note that \( A \cap B = \emptyset, \) hence, \( C_1 \cap C_2 = \emptyset, \) \( \text{Int}(C_1) = \text{Int}(C_2) = \emptyset \) and \( C = C_1 \cup C_2. \) It follows that \( C \subset C_i \) for \( i = 1, 2. \) If \( C = \emptyset \) then we assume \( C_1 = C_2 = \emptyset. \)

Let \( D = X \setminus (\text{Int}(A) \cup \text{Int}(B) \cup \overline{C}). \) If \( D = \emptyset \) we assume \( D_1 = (A \cup B) \cap D \) and \( D_2 = (X \setminus (A \cup B)) \cap D. \) Note that \( D_1 \) and \( D_2 \) are disjoint sets and are dense in \( D. \) If \( D = \emptyset \) then we assume \( D_1 = D_2 = \emptyset. \) Note that \( X = \overline{\text{Int}(A) \cup \text{Int}(B) \cup \overline{C} \cup \overline{D}}. \)
Finally, let \( X_i = A_i \cup B_i \cup C_i \cup D_i \) for \( i = 1, 2 \). It is clear \( X = \overline{X_i} \) for \( i = 1, 2 \) and \( X_1 \cap X_2 = \emptyset \).

The following theorem is the main result of this work.

**Theorem 8.** The regular \( L \)-space with uncountable dispersion character is resolvable.

**Proof.** Let \( U \) be an open subset of \( X \) such that each Lindelöf subspace of \( U \) at most countable. Note that if \( U \neq \emptyset \) then \( U \) is maximally resolvable \([9]\) because it is a \( \omega \)-tight and has uncountable dispersion character.

Consider a set \( Z = \bigcup \{ U, \text{where } U \text{ is an open subset of } X \text{ such that each Lindelöf subspace of } U \text{ at most countable } \} \). Then the set \( Z \) is resolvable (as the union of resolvable subspaces \([14]\)) or \( Z = \emptyset \).

Let \( Y = X \setminus \overline{Z} \). By definition of \( Y \) and Corollary \([8]\) any open subset of \( Y \) contains uncountable Lindelöf subspace \( L \). If \( L \) is a heriditaraly Lindelöf then it contains subspace \( M \) of uncountable dispersion character. Then \( M \) is resolvable (see \([6]\)). Let \( H = \bigcup \{ M, \text{where } M \text{ is a heriditaraly Lindelöf of uncountable dispersion character } \} \). The set \( H \) is resolvable (as the union of resolvable subspaces \([13]\)) or \( H = \emptyset \).

Let \( K = Y \setminus \overline{H} \). It remains to prove that \( K \) is resolvable. Let us note that if a subspace \( L \subset K \) is not heridataraly Lindelöf, then it contains a closed set \( F \) which is not \( G_\delta \)-set. Let \( K \neq \emptyset \).

We claim that \( K \) has the conditions of Corollary \([6]\). Let \( V \) be an open set such that \( V \subset K \). Consider \( L \subset \overline{V} \) such that \( L \) is Lindelöf, but is not heridataraly Lindelöf and \( F \subset L \) such that \( F \) is a closed set of \( L \), but is not \( G_\delta \)-set in \( L \).

By induction on \( \alpha \), we construct the required discrete \( D \subset L \).

Let \( x_1 \in L \setminus F \). By the regularity of \( L \), there are open sets \( U_1 \) and \( V_1 \) such that \( U_1 \cap V_1 = \emptyset \), \( x_1 \in U_1 \) and \( F \subset V_1 \).

We constructed \( x_\alpha, V_\alpha, U_\alpha \) for \( \alpha < \beta \) such that \( \overline{V_\alpha} \cap \overline{U_\alpha} = \emptyset \), \( x_\alpha \in U_\alpha \) and \( F \subset V_\alpha \) and

1. \( x_\alpha \in \bigcap_{\gamma < \alpha} \overline{V_\gamma} \);
2. \( x_\alpha \notin U_\alpha \) for \( \alpha \neq \gamma \);
3. \( x_\alpha \notin F \);
4. \( \bigcup_{\gamma < \alpha} \{ x_\gamma \} \) are closed subsets of \( L \) for \( \alpha < \beta \).

If \( \beta = \omega_1 \) or \( \bigcup_{\gamma < \alpha} \{ x_\gamma \} \) is not closed set then inductive process is completed.

If \( \bigcup_{\gamma < \alpha} \{ x_\gamma \} \) is closed set then there is \( x_\beta \in \bigcap_{\gamma < \beta} \overline{V_\gamma} \setminus F \). There are open sets \( V_\beta \) and \( U_\beta \) such that \( F \subset V_\beta, x_\beta \in U_\beta, \overline{U_\beta} \cap \overline{V_\beta} = \emptyset, \overline{V_\beta} \cap \bigcup_{\gamma < \beta} \{ x_\gamma \} = \emptyset \) and \( \overline{U_\beta} \cap \bigcup_{\gamma < \beta} \{ x_\gamma \} = \emptyset \).

By construction, if \( |D| = \omega \) then \( \overline{D} \setminus D \neq \emptyset \).

If \( |D| = \omega_1 \), but \( D \) contains a countable is not closed (in \( K \)) subset \( D_1 \), then \( D = D_1 \).

If \( |D| = \omega_1 \) and each countable subset of \( D \) is closed set in \( K \) then \( D \) is a correct discrete set in Lindelöf space \( L \) and, hence, \( D \) is a correct discrete set in \( V \).
So $K$ is resolvable. Then $X$ is resolvable as the union of resolvable subspaces. □

Note that a regular $wL$-space is a $L$-space.

**Corollary 6.** Let $\mathcal{P} \in \{\omega N, \omega, hL, \sigma\text{-}cmp, ccc, L, wL\}$. The regular $\mathcal{P}$-space with uncountable dispersion character is resolvable.

**Lemma 2.** Let $X$ be a regular resolvable space and $\Delta(X) > \omega$. Then there are disjoint dense in $X$ subsets $Y_1, Y_2$ of $X$ such that $\Delta(Y_1) > \omega$.

**Proof.** Let $X = X_1 \cup X_2$ and $X_i = X$ for $i = 1, 2$. Suppose that $\Delta(X_1) = \Delta(X_2) = \omega$. Consider an open set $U_1$ of $X$ such that $\Delta(U_1 \cap X_1) = \omega$. Then $\Delta(U_1 \cap X_2) > \omega$. Let $Z^1_i = U_1 \cap X_2$, $Z^2_i = U_1 \cap X_1$. If $\Delta(X_1 \setminus \overline{U_1}) > \omega$ then inductive process is completed. Suppose that $\Delta(X_1 \setminus \overline{U_1}) = \omega$. Let $U_2$ be an open set of $X$ such that $\Delta(X_1 \setminus \overline{U_2}) = \omega$. $Z^1_2 = U_2 \cap X_2$, $Z^2_2 = U_2 \cap X_1$ and so on. By inductive process, we construct a disjoint family $\{U_\alpha, \alpha < \beta\}$ of open subsets of $X$ such that $\Delta(X \setminus \bigcup_{\alpha < \beta} U_\alpha)$ and disjoint sets $Z^1_\alpha \subset U_\alpha$, $Z^2_\alpha \subset U_\alpha$. Let $Y_1 = \bigcup_{\alpha < \beta} Z^1_\alpha$ and $Y_2 = \bigcup_{\alpha < \beta} Z^2_\alpha$. It is clear that $X = \overline{Y_i}$ for $i = 1, 2$ and $\Delta(Y_1) > \omega$. □

**Theorem 9.** The regular $L$-tight space $X$ of uncountable dispersion character is $\omega$-resolvable.

**Proof.** Let $X = Y_1 \cup X_1$, where $X_1, Y_1$ are disjoint and dense in $X$. Let $\Delta(Y_1) > \omega$. Let $Y_1 = Y_2 \cup X_2$, where $X_2, Y_2$ are disjoint dense in $Y_1$ and $\Delta(Y_2) > \omega$. A space $Y_1$ is dense in $X$, consequently $X_2, Y_2$ are dense in $X$ too. And so on. By inductive process, we construct a countable disjoint family $\{X_\alpha, \alpha \in \omega\}$ of dense in $X$ sets.

Note that a regular $wL$-tight is a $L$-tight.

**Corollary 7.** Let $\mathcal{P} \in \{\omega N, \omega, hL, \sigma\text{-}cmp, ccc, L, wL\}$. The regular $\mathcal{P}$-tight space with uncountable dispersion character is $\omega$-resolvable.

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MARIA ALEXANDROVNA FILATOVA
Ural Federal University,
19 Mira str.,
620002, Yekaterinburg, Russia
Krasovskii Institute of Mathematics and Mechanics,
16 S.Kovalevskaya str.
620990, Yekaterinburg, Russia
E-mail address: MA.Filatova@urfu.ru

ALEXANDER VLADIMIROVICH OSIPOV
Krasovskii Institute of Mathematics and Mechanics,
16 S.Kovalevskoy str.,
620990, Yekaterinburg, Russia;
Ural Federal University,
19 Mira str.,
620002, Yekaterinburg, Russia;
Ural State University of Economics,
62, 8th of March str.,
620219, Yekaterinburg, Russia.
E-mail address: oab@list.ru