On Cooperation in Multi-Terminal Computation and Rate Distortion

Milad Sefidgaran and Aslan Tchamkerten

Abstract

A receiver wants to compute a function of two correlated sources separately observed by two transmitters. In the system model of interest, one of the transmitters may send some data to the other transmitter in a cooperation phase before both transmitters convey data to the receiver. What is the minimum number of noiseless bits that need to be communicated by each transmitter to the receiver for a given number of cooperation bits?

This paper investigates both the function computation and the rate distortion versions of this problem; in the first case, the receiver wants to compute the function exactly and in the second case the receiver wants to compute the function within some distortion.

For the function computation version, a general inner bound to the rate region is exhibited and shown to be tight in a number of cases: the function is partially invertible, full cooperation, one-round point-to-point communication, two-round point-to-point communication, and cascade. As a corollary, it is shown that one bit of cooperation may arbitrarily reduce the amount of information both transmitters need to convey to the receiver.

For the rate distortion version, an inner bound to the rate region is exhibited which always includes, and sometimes strictly, the convex hull of Kaspi-Berger’s related inner bounds.

I. INTRODUCTION

Distributed function computation has been a long studied source coding problem in information theory. The point-to-point case was investigated in the context of interactive communication by Orlitsky and Roche [16] who derived the rate region for one-round and two-round communication using the concept of conditional characteristic graph defined and developed by Körner [12] and Witsenhausen [23], respectively. The generalization to \( m \geq 1 \) round communication was considered by Ma and Ishwar [15].

The first setting with more than one source can be attributed to Slepian and Wolf who investigated the multiple access configuration where a receiver wants to recover the sources perfectly [20], [2]. Later, Körner and Marton considered the specific setting where a receiver wants to compute the sum modulo two of two binary sources and derived the rate region in the specific case where the sources have a symmetric distribution [13]. This result has
been generalized for sum modulo $p$ for an arbitrary prime number $p$ in [7] and [26]. More recently, we derived the rate regions for the setting where the function is partially invertible (and arbitrary distributions) and the case of independent sources (and arbitrary functions) [17].

Except for these cases, the rate region for two sources remains an open problem in general. For instance, for the sum modulo two and arbitrary distributions, the best known rate region inner bound is the one obtained Ahlswede and Han [1]. Building on this work, Huang and Skoglund derived an achievable rate region for a certain class of polynomial functions which is larger than the Slepian-Wolf rate region [10], [8], [9]. Finally, a variation of the problem where the receiver wants to compute some subspace generated by the sources has been investigated by Lalitha et al. [14].

The cascade network configuration has been considered in the cases where there is no side information at the receiver by Cuff et al. [4] and where the sources form a Markov chain by Viswanathan [22]. The general case was recently investigated in [19].

The aforementioned network configurations—point-to-point, multiple access, and cascade— are special cases of the network configuration depicted in Fig. 1. Two sources, $X$ and $Y$, are separately observed by two transmitters, and a receiver wants to compute a function $f(X,Y)$ of the sources. Transmitter-$X$ first sends some information to transmitter-$Y$ at rate $R_0$ (cooperation phase), then transmitter-$X$ and transmitter-$Y$ send information to the receiver at rate $R_X$ and $R_Y$, respectively.\footnote{There exists a similar problem with the difference that the sent message from transmitter-$X$ to transmitter-$Y$ can be heard by the receiver too. This problem has been considered in the context of rate distortion problem by Kaspi and Berger [11] and in the context of function computation by Ericsson and Korner [6].} This paper investigates this setting in the context of both function computation and rate distortion.

The first part of the paper is devoted to function computation. The main result is a general rate region inner bound that is tight in a number of special cases:

- unlimited cooperation, \textit{i.e.}, when transmitter-$Y$ knows $X$;
- the function is partially invertible—\textit{i.e.}, when $X$ is a function of $f(X,Y)$;
- one and two-round point-to-point communication for which we recover the results in [16];
- cascade network for which we recover the results of [4], [21];
• no cooperation: invertible function and arbitrary function, or, arbitrary function and independent sources for which we recover the results of [18].

An interesting illustration of the second case shows that the sum rate per cooperation rate, \( R_X + R_Y / R_0 \), can be arbitrarily large.

In the second part of the paper we consider the problem where the receiver wants to recover some functions \( f_1(X, Y) \) and \( f_2(X, Y) \) within some distortions. For the special case where \( f_1(X, Y) = X \) and \( f_2(X, Y) = Y \) Kaspi and Berger [11] proposed two inner bounds. The first one is a general inner bound while the second one is valid and tight in the full cooperation case only. These bounds easily generalize to arbitrary functions by using similar arguments as those used by Yamamoto in [25, Proof of Theorem 1] to extend Wyner and Ziv’s result [24, Theorem 1] from identity functions to arbitrary functions.

Building on ideas used to establish the inner bound for the function computation problem, we derive a new inner bound for the rate distortion problem which always includes, and in certain cases strictly, the convex hull of Kaspi-Berger’s inner bounds [11, Theorems 5.1 and 5.4] generalized to arbitrary functions.

The paper is organized as follows. In Section II we formally state the problem and provide some background material and definitions. In Section III, we present our results in two subsections: function computation and rate distortion problems, and in Section IV we provide a proof sketch of our main result.

II. PROBLEM STATEMENT AND PRELIMINARIES

We use calligraphic fonts to denote the range of the corresponding random variable. For instance, \( \mathcal{X} \) denotes the range of \( X \) and \( \mathcal{T} \) denotes the range of \( T \).

Let \( \mathcal{X}, \mathcal{Y}, \mathcal{F}, \mathcal{F}_1, \) and \( \mathcal{F}_2 \) be finite sets. Further, define

\[
\begin{align*}
f : \mathcal{X} \times \mathcal{Y} &\to \mathcal{F} \\
f_i : \mathcal{X} \times \mathcal{Y} &\to \mathcal{F}_i \quad i \in \{1, 2\} \\
d_i : \mathcal{F}_i \times \mathcal{F}_i &\to \mathbb{R}^+ \quad i \in \{1, 2\}.
\end{align*}
\]

Let \( \{(x_i, y_i)\}_{i=1}^{\infty} \) be independent instances of random variables \( (X, Y) \) taking values over \( \mathcal{X} \times \mathcal{Y} \) and distributed according to \( p(x, y) \). Define \( X \overset{\text{def}}{=} X_1, \ldots, X_n \) and

\[
\begin{align*}
f(X, Y) &\overset{\text{def}}{=} f(X_1, Y_1), \ldots, f(X_n, Y_n).
\end{align*}
\]

Similarly define \( f_1(X, Y) \) and \( f_2(X, Y) \).

Next, we recall the notions of achievable rate tuples for function computation and rate distortion. For function computation it is custom to consider asymptotic zero block error probability whereas for rate distortion it is custom to consider bit average distortion.

**Definition 1 (Code).** An \((n, R_0, R_X, R_Y)\) code for the function computation problem consists of three encoding
functions

\[ \varphi_0 : \mathcal{X}^n \to \{1, 2, ..., 2^{nR_0}\} \]
\[ \varphi_X : \mathcal{X}^n \to \{1, 2, ..., 2^{nR_X}\} \]
\[ \varphi_Y : \mathcal{Y}^n \times \{1, 2, ..., 2^{nR_0}\} \to \{1, 2, ..., 2^{nR_Y}\} \]

and a decoding function

\[ \psi : \{1, 2, ..., 2^{nR_X}\} \times \{1, 2, ..., 2^{nR_Y}\} \to \mathcal{F}_n. \]

The corresponding error probability is defined as

\[ P(\psi(\varphi_X(X), \varphi_Y(\varphi_0(X), Y)) \neq f(X, Y)). \]

An \((n, R_0, R_X, R_Y)\) code for the rate distortion problem consists of three encoding functions defined as for the function computation problem, and two decoding functions

\[ \psi_i : \{1, 2, ..., 2^{nR_X}\} \times \{1, 2, ..., 2^{nR_Y}\} \to \mathcal{F}_n, \quad i \in \{1, 2\}. \]

The corresponding average distortions are defined as\(^2\)

\[ Ed_i(f_i(X, Y), \psi_i(\varphi_X(X), \varphi_Y(\varphi_0(X), Y))) = \frac{1}{n} \sum_{j=1}^{n} Ed_i(f_i(X_j, Y_j), \psi_i(\varphi_X(X), \varphi_Y(\varphi_0(X), Y))) \]

for \(i \in \{1, 2\}\). In the above expression \(\psi_i(\varphi_X(X), \varphi_Y(\varphi_0(X), Y)))\) refers to the \(j\)th component of the length \(n\) vector \(\psi_i(\varphi_X(X), \varphi_Y(\varphi_0(X), Y)))\).

**Definition 2** (Function Computation Rate Region). A rate tuple \((R_0, R_X, R_Y)\) is achievable if, for any \(\varepsilon > 0\) and all \(n\) large enough, there exists an \((n, R_0, R_X, R_Y)\) code whose error probability is no larger than \(\varepsilon\). The rate region is the closure of the set of achievable rate tuples \((R_0, R_X, R_Y)\).

**Definition 3** (Rate Distortion Region). Let \(D_1, D_2\) be two non-negative constants. A rate tuple \((R_0, R_X, R_Y)\) is achievable with distortions \(D_1\) and \(D_2\) if, for any \(\varepsilon > 0\) and all \(n\) large enough, there exists an \((n, R_0, R_X, R_Y)\) code whose average distortions are no larger than \(D_1\) and \(D_2\), respectively. The rate distortion region with respect to \(D_1\) and \(D_2\) is the closure of the set of achievable rate tuples \((R_0, R_X, R_Y)\) with distortions \(D_1\) and \(D_2\).

The problems we consider are the characterizations of

i. the function computation rate region for given function \(f(x, y)\) and distribution \(p(x, y)\);

ii. the rate distortion region for given functions \(f_1(x, y), f_2(x, y)\), distribution \(p(x, y)\), and distortion constraints \(D_1, D_2\).

\(^2\)We use \(E\) to denote expectation.
Conditional characteristic graphs play a key role in function computation problem [23], [12], [17]. Below we introduce a general definition of conditional characteristic graph.

**Remark 1.** Given two random variables $X$ and $V$, where $X$ ranges over $\mathcal{X}$ and $V$ over subsets of $\mathcal{X}$, we write $X \in V$ whenever $P(X \in V) = 1$.

Recall that an independent set of a graph $G$ is a subset of vertices no two of which are connected. The set of independent sets of $G$ is denoted by $\Gamma(G)$.

**Definition 4** (Generalized Conditional Characteristic Graph). Let $L, K$, and $S$ be arbitrary discrete random variables with $(L, K, S) \sim p(l, k, s)$. Let $f : S \rightarrow \mathbb{R}$ be a function such that $H(f(S)|L, K) = 0$. The conditional characteristic graph $G_{L|K}(f)$ of $L$ given $K$ with respect to the function $f$ is the graph whose vertex set is $L$ and such that $l_1 \in L$ and $l_2 \in L$ are connected if for some $s_1, s_2 \in S$, and $k \in K$

i. $p(l_1, k, s_1) \cdot p(l_2, k, s_2) > 0$,

ii. $f(s_1) \neq f(s_2)$.

When there is no ambiguity for the function $f$, the above conditional characteristic graph is denoted by $G_{L|K}$.

**Definition 5** (Conditional Graph Entropy [16]). Given $(L, K, S) \sim p(l, k, s)$ and $f : S \rightarrow \mathbb{R}$ such that $H(f(S)|L, K) = 0$, the conditional graph entropy $H(G_{L|K}(f))$ is defined as

$$H(G_{L|K}(f)) \overset{\text{def}}{=} \min_{V-L-K} \min_{L \in V \in \Gamma(G_{L|K})} I(V; L|K)$$

where $V - L - K$ refers to the standard Markov chain notation.

III. RESULTS

In the first part of this section we consider the function computation problem formulation and in the second part of the section we consider the corresponding rate distortion formulation.

A. Computation

Given a finite set $S$, we use $M(S)$ to denote the collection of all multisets of $S$. A multiset of a set $S$ is a collection of elements from $S$ possibly with repetitions, e.g., if $S = \{0, 1\}$, then $\{0, 1, 1\}$ is a multiset.

---

3*I.e.*, a sample of $V$ is a subset of $\mathcal{X}$.

4A multiset of a set $S$ is a collection of elements from $S$ possibly with repetitions, e.g., if $S = \{0, 1\}$, then $\{0, 1, 1\}$ is a multiset.
Theorem 1 (Inner Bound – Computation). $(R_0, R_X, R_Y)$ is achievable whenever

\begin{align*}
R_0 &> I(X; U|Y) \\
R_X &> I(V; X|T, W) \\
R_Y &> I(U, Y; W|V, T) \\
R_X + R_Y &> I(X, Y; V, T, W) + I(U; W|V, X, T, Y),
\end{align*}

for some $T$, $U$, $V$, and $W$ with alphabets $T$, $U$, $V$, and $W$, respectively, that satisfy

\begin{align*}
T - U - X - Y \\
V - (X, T) - (U, Y) - W,
\end{align*}

and

\begin{align*}
X &\in V \in M(\Gamma(G_{T,X|T,U,Y})) \\
(U, Y) &\in W \in M(\Gamma(G_{T,U,Y|T,V})).
\end{align*}

Moreover, the following cardinality bounds hold

\begin{align*}
|T| \leq |X| + 4 \\
|V| \leq (|X| + 4) \cdot |X| + 1 \\
|W| \leq |U| \cdot |Y| + 1.
\end{align*}

The last part of the theorem says that the achievable rate region (1) is maximal for random variables $T$, $V$, and $W$ defined over sets whose cardinalities are bounded as in (4). Note that in the graphs $G_{T,X|T,U,Y}$ and $G_{T,U,Y|T,V}$ the random variable $T$ can be interpreted as a time sharing random variable over a set of conditional characteristic graphs.

The rate region characterized in Theorem 1 turns out to be tight in a number of interesting cases which we now list. The first case holds when the function is partially invertible with respect to $X$, i.e., when $X$ is a function of $f(X, Y)$.

Theorem 2 (Partially Invertible Function). The inner bound is tight when $f(X, Y)$ is partially invertible with
Fig. 2. Minimum sum-rate $R_X + R_Y$ as a function of the cooperation rate $R_0$ for the partially invertible function of Example 1 with $a = 3$ and $b = 10.$

respect to $X.$ In this case, the rate region reduces to

$$R_0 \geq I(X; U|Y)$$

$$R_X \geq H(X|U, W)$$

$$R_Y \geq I(Y; W|X, U)$$

$$R_X + R_Y \geq H(X) + I(Y; W|U),$$

for some $U$ and $W$ with alphabets $\mathcal{U}$ and $\mathcal{W}$, respectively, that satisfy

$$U - X - Y$$

$$X - (U,Y) - W,$$

and

$$Y \in W \in M(\Gamma(G_{U,Y|X,U})).$$

Moreover, the following cardinality bounds hold

$$|\mathcal{U}| \leq |\mathcal{X}| + 4$$

$$|\mathcal{W}| \leq (|\mathcal{X}| + 4) \cdot |\mathcal{Y}| + 1.$$  \hspace{1cm} (7)

In the following example we apply Theorem 2 to show that one bit of cooperation may arbitrarily reduce the minimum sum rate $R_X + R_Y$.

**Example 1.** Let $a \geq 2$ and $b \geq 1$ be two natural numbers. Let $X$ be uniform over $\{1, 2, \ldots, a\}$, and let $Y = (Y_1, Y_2, \ldots, Y_a)$ where the $Y_i$'s, $i \in \{1, 2, \ldots, a\}$, are independent random variables, each of them uniformly distributed over $\{1, \ldots, 2^b\}$ and independent of $X.$ The receiver wants to recover $X$ and $Y_X$, i.e., $f(X, Y) =$
From Theorem 2 and the fact that $X$ and $Y$ are independent the rate is given by

$$R_0 \geq I(X;U)$$
$$R_X \geq H(X|U)$$
$$R_Y \geq I(Y;W|U)$$
$$R_X + R_Y \geq H(X) + I(Y;W|U)$$

for some $U$ and $W$ that satisfy (5), (6), and (7).

We evaluate the sum rate constraint. Since $X$ is uniformly distributed, $H(X) = \log_2(a)$. Now, due to the independence of $X$ and $Y$ and the Markov chain $U - X - Y$ we have

$$H(Y|U) = H(Y) = a \cdot b. \quad (8)$$

Further, by Definition 4, for each $u \in U$, $(u,y) = (u,(y_1,y_2,\cdots,y_a))$ and $(u,y') = (u,(y'_1,y'_2,\cdots,y'_a))$ are connected in $G_{U,Y|X,U}$ if and only if $y_x \neq y'_x$ for some $x \in A_u \overset{\text{def}}{=} \{x : p(x,u) > 0\}$.

Hence, because $W$ satisfies (6), conditioned on $U = u$ the maximum number of elements in an independent set $w \in W$ that contains vertices $(u,y)$, $y \in Y$, is $2^{b(a - |A_u|)}$. Therefore,

$$H(Y|W,U = u) = b \cdot (a - |A_u|), \quad (9)$$

by letting $W$ take as values maximal independent sets.

Equations (8) and (9) give

$$\min_W I(Y;W|U) = b \cdot \sum_{u \in \mathcal{U}} |A_u| \cdot p(u),$$

and therefore

$$R_0 = \log_2(a) + \sum_{x,u} p(x,u) \cdot \log_2 p(x|u)$$
$$R_X + R_Y = \log_2(a) + b \cdot \sum_{u \in \mathcal{U}} |A_u| \cdot p(u) \quad (10)$$

for any valid choice of $U$. By considering all random variables $U$ over alphabets of no more than $a + 4$ elements and that satisfy the Markov chain $U - X - Y$, one can numerically evaluate the minimum achievable sum rate for all values of $R_0$ using the above equations. Fig. 2 shows the minimum achievable sum rate $R_X + R_Y$ as a function of $R_0$ for $a = 4$ and $b = 10$.

$^5$We use $|A_u|$ to denote the cardinality $|A_u|$. 
Choosing $U \in \{0, 1\}$ in (10) such that
\[
p(U = 0 | X = 1) = p(U = 0 | X = 2) = p(U = 1 | X = 3) = p(U = 1 | X = 4) = 1
\]
\[
p(U = 0 | X = 3) = p(U = 0 | X = 4) = p(U = 1 | X = 1) = p(U = 1 | X = 2) = 0
\]
shows that
\[
R_0 = 1
\]
\[
R_X + R_Y = 2 + 2 \cdot b
\] (11)
is achievable.

When $R_0 = 0$ the minimum sum rate is given by
\[
\min(R_X + R_Y) = 2 + 4 \cdot b
\] (12)
from [17, Theorem 3] and using the fact that the function is partially invertible and that the sources are independent.\(^6\)

From (11) and (12) we deduce that one bit of cooperation decreases the sum rate by at least $2 \cdot b$, which can be arbitrarily large since $b$ is an arbitrary natural number.

The next three theorems provide three other cases where Theorem 1 is tight. In each of them, one of the links is rate unlimited.

When there is full cooperation between transmitters, i.e., when transmitter-$Y$ has full access to source $X$, the setting is captured by the condition $R_0 > H(X|Y)$ and is depicted in Fig. 3(a).

**Theorem 3 (Full Cooperation).** The inner bound is tight when
\[
R_0 > H(X|Y).
\]
In this case, the rate region reduces to

\[ R_0 \geq H(X|Y) \]
\[ R_Y \geq H(f(X,Y)|T) \]
\[ R_X + R_Y \geq H(f(X,Y)) + I(X; T|f(X,Y)) \]

for some \( T \) with alphabet \( T \) that satisfies

\[ T - X - Y, \]

with cardinality bound

\[ |T| \leq |\mathcal{X}| + 1. \]

In the following example, we derive the rate region for a partially invertible function when there is no cooperation and when there is full cooperation.

**Example 2.** Let \( f(x, y) = (-1)^y \cdot x \), with \( \mathcal{X} = \mathcal{Y} = \{0, 1, 2\} \), and

\[ p(x, y) = \begin{bmatrix} .21 & .03 & .12 \\ .06 & .15 & .16 \\ .03 & .12 & .12 \end{bmatrix}. \]

The rate region when \( R_0 = 0 \) was derived in [17, Example 4] and is depicted by the gray area in Fig. 4. With full cooperation, \( i.e., R_0 = H(X|Y) = 1.38 \), using Theorem 3 the rate region is the union of the gray and the black areas in Fig. 4. Note that the black area, which represents the difference between the two regions is non-symmetric with respect to \( X \) and \( Y \), as can be expected.

When \( R_Y \) is unlimited, \( i.e., \) when the receiver looks over the shoulder of transmitter-\( Y \), the setting is captured by condition \( R_Y > R_0 + H(Y) \) and reduces to point-to-point communication as depicted in Fig. 3(c) with the transmitter
observing $X$ and the receiver observing $Y$. The rate region for this case was established in [16, Theorem 1].

**Theorem 4** (One-Round Point-to-Point Communication). *The inner bound is tight when*

$$R_Y > R_0 + H(Y).$$

*In this case, the rate region reduces to*

$$R_0 + R_X \geq H(G_{X|Y}).$$

When condition $R_X > H(X)$ holds, the situation reduces to the two-round communication setting depicted in Fig. 3(b). The receiver, having access to $X$, first conveys information to transmitter-$Y$, which then replies.

**Theorem 5** (Two-Round Point-to-Point Communication). *The inner bound is tight when*

$$R_X > H(X).$$

*In this case, the rate region reduces to*

$$R_0 \geq I(X;U|Y)$$

$$R_X \geq H(X)$$

$$R_Y \geq I(Y;W|X,U)$$

(13)

for some $U$ and $W$ with alphabets $\mathcal{U}$ and $\mathcal{W}$, respectively, that satisfy

$$U - X - Y$$

$$X - (U,Y) - W,$$

and

$$Y \in W \in M(\Gamma(G_{U,Y|X,U})), $$

with cardinality bounds

$$|\mathcal{U}| \leq |\mathcal{X}| + 2$$

$$|\mathcal{W}| \leq (|\mathcal{X}| + 2) \cdot |\mathcal{Y}| + 1.$$
Finally, when $R_X = 0$ there is no direct link between transmitter-$X$ and the receiver and the situation reduces to the cascade setting depicted in Fig. 3(d). The rate region for this case was established in [4, Theorem 3.1] (see also [21, Theorem 2]).

**Theorem 6 (Cascade).** The inner bound is tight when

$$R_X = 0.$$  

In this case, the rate region reduces to

$$R_0 \geq H(G_X|Y)$$

$$R_Y \geq H(f(X,Y)).$$

**B. Rate Distortion**

Theorem 1 gives an inner bound to the rate distortion problem (see Definition 3) with zero distortions when both distortion functions are the same. It turns out that this inner bound is in general larger than the rate region obtained by Kaspi and Berger in [11, Theorem 5.1] for zero distortions. The reason for this lies in Kaspi and Berger’s achievable scheme which their inner bound relies upon. For any distortions their scheme implicitly allows the receiver to perfectly decode whatever is transmitted from transmitter-$X$ to transmitter-$Y$. By contrast, we do not impose this constraint in the achievability scheme that yields Theorem 1. More generally, by relaxing this constraint it is possible to achieve an achievable rate region that contains, and in certain cases strictly, the rate region given by [11, Theorem 5.1]. This is given by Theorem 7 below. For the specific full cooperation case, Theorem 7 reduces to [11, Theorems 5.4]. As a result, Theorem 7 always includes the convex hull of the two regions [11, Theorems 5.1 and 5.4] generalized to arbitrary functions, and this inclusion is strict in certain cases.

**Theorem 7 (Inner Bound – Rate Distortion).** $(R_0, R_X, R_Y)$ is achievable with distortions $D_1$ and $D_2$ whenever

$$R_0 > I(X;U|Y)$$

$$R_X > I(V;X|T,W)$$

$$R_Y > I(U,Y;W|V,T)$$

$$R_X + R_Y > I(X,Y;V,T,W) + I(U;W|V,X,T,Y)$$

for some $T$, $U$, $V$, and $W$ with alphabets $T$, $U$, $V$, and $W$, respectively, that satisfy

$$T - U - X - Y$$

$$V - (X,T) - (U,Y) - W,$$
and if there exist functions $g_1(V,T,W)$ and $g_2(V,T,W)$ such that
\[ E_d(i(f_i(X,Y), g_i(V,T,W)) \leq D_i, i \in \{1,2\}. \]

with cardinality bounds
\[
|T| \leq |X| + 4 \\
|V| \leq (|X| + 4) \cdot |X| + 1 \\
|W| \leq |U| \cdot |Y| + 1.
\]

To obtain the general inner bound [11, Theorem 5.1] it suffices to let $T = U$ in Theorem 7. To obtain the specific full cooperation inner bound [11, Theorem 5.4], it suffices to let $U = X$ and let $V$ be a constant in Theorem 7. Hence, Theorem 7 always includes the convex hull of the two schemes [11, Theorems 5.1 and 5.4]. The following two examples show that this inclusion is strict in certain cases.

In the first example one of the distortion functions is defined on both sources $X$ and $Y$, while in the second example the distortion functions are defined on each sources separately, as considered by Kaspi and Berger (see [11, Section II]).

**Example 3.** Let $(X = (X_1, X_2), Y)$ where $X_1$ and $Y$ are uniformly distributed over $\{1,2,3\}$ and $X_2$ is a Bern($p$) random variable with $p \leq 1/2$. Random variables $X_1$, $X_2$, and $Y$ are supposed to be independent. Define the binary function $f(X_1,Y)$ to be equal to 1 whenever $X_1 = Y$ and equal to 0 otherwise. The goal is to reconstruct $f(X_1,Y)$ with average Hamming distance equal to zero (i.e., $D_1 = 0$) and $X_2$ with average Hamming distance $D_2 \leq p$.

For any value of $R_0$, the achievable scheme [11, Theorem 5.1] gives
\[
R_X + R_Y > H(X_1) + H_b(p) - H_b(d).
\]
(14)

To see this note that the achievable scheme that yields [11, Theorem 5.1] is so that whatever transmitter-$X$ sends to transmitter-$Y$ will be retransmitted to the receiver. Therefore, the sum rate is at least as large as the point-to-point rate distortion problem where the transmitter has access to $X$ and the receiver, who has access to $Y$, wants to recover $f(X_1,Y)$ and $X_2$ with distortions 0 and $D_2$, respectively. For the point-to-point case, due to the independence of $(X_1,Y)$ and $X_2$, the infimum of sum rate is at least
\[
R_0(f(X_1,Y)) + R_2.
\]

Here $R_0(f(X_1,Y))$ is the infimum of number of bits for recovering $f(X_1,Y)$ with zero distortion, which is equal to $H(X_1)$ due to [16, Theorem 2], and $R_2$ is the infimum of number of bits for recovering $X_2$ with distortion $D_2 \leq p$ and is equal to $H_b(p) - H_b(d)$ by [3, Theorem 10.3.1]. Inequality (14) then follows.
Now, for the scheme [11, Theorem 5.4] the infimum of sum rate for $R_0 > H(X|Y)$ is

$$R_X + R_Y = H(f(X_1, Y)) + H_b(p) - H_b(d).$$  \hfill (15)

Therefore, from (14) and (15) the time sharing of [11, Theorems 5.1] and [11, Theorems 5.4] gives

$$R_X + R_Y > q \cdot H(f(X_1, Y)) + (1 - q) \cdot H(X_1) + H_b(p) - H_b(d)$$  \hfill (16)

for $q \in [0, 1]$. To have an average cooperation at most equal to $R_0$, the time-sharing constant $q$ should be less than $\frac{R_0}{H(X|Y)}$. This is because the scheme [11, Theorem 5.4] needs, on average, more than $H(X|Y)$ cooperation bits.

Now, since $H(f(X_1, Y)) < H(X_1)$, the bigger $q$ is, the smaller the right-hand side of (16), and therefore

$$R_X + R_Y \geq \frac{R_0}{H(X|Y)} \cdot H(f(X_1, Y)) + \left(1 - \frac{R_0}{H(X|Y)}\right) \cdot H(X_1) + H_b(p) - H_b(d).$$  \hfill (17)

We now turn to Theorem 7. By letting $U = X_1$, $T$ be a constant, $W = f(X_1, Y)$, and $V = \text{Bern}(\frac{p-d}{1-2d})$ with\footnote{We use $\oplus$ to denote the sum modulo 2.}

$$p_{V|X,Y}(V|X,Y) = p_{V|X_2}(V|X_2) = \frac{p_Z(X_2 \oplus V) \cdot p_V(V)}{p_{X_2}(X_2)},$$

where $Z = \text{Bern}(d)$, Theorem 7 gives for $R_0 > H(X_1|Y)$ the sum rate

$$R_X + R_Y = H(f(X_1, Y)) + H_b(p) - H_b(d),$$  \hfill (18)

which can be checked to be strictly below the right-hand side of (17) for $H(X_1|Y) < R_0 < H(X|Y)$.

**Example 4.** Let $X$ and $Y$ be random variables taking values in $\{-1, 0, +1\}$ with probabilities

$$p(x, y) = \begin{cases} 0 & \text{if } (x, y) = (-1, +1) \text{ or } (x, y) = (+1, -1), \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Define the distortion function

$$d_1(x, \hat{x}) = \begin{cases} 1 & \text{if } x \cdot \text{sign}((\hat{x}) = -1, \\ 0 & \text{otherwise.} \end{cases}$$

where

$$\text{sign}(\hat{x}) = \begin{cases} +1 & \text{if } \hat{x} > 0, \\ -1 & \text{if } \hat{x} < 0, \\ 0 & \text{if } \hat{x} = 0, \end{cases}$$

Define $d_2(y, \hat{y})$ as $d_1$. We consider the rate region for distortion pairs $(D_1, D_2) = (0, 0)$. 


We claim that
1. for any value of $R_0$ in [11, Theorem 5.1]
   \[ R_X + R_Y > 1.03; \]  
   (19)
2. the infimum of sum rate in [11, Theorem 5.4] under full cooperation $R_0 > H(X|Y) = 1.25$ is
   \[ R_X + R_Y = 0.85; \]  
   (20)
3. from Theorem 7 it is possible to achieve for any $R_0 > 0.38$, the sum rate
   \[ R_X + R_Y = 0.85. \]  
   (21)

From 1. and 2. it can be concluded that any time sharing of the schemes [11, Theorem 5.1] and [11, Theorem 5.4] that achieves $R_0 = 0.39$, yields a sum rate bigger than 0.89, which is larger than the sum rate achieved by Theorem 7.

The proofs of Claims 1.-3. are deferred to the Appendix.

IV. ANALYSIS

Proof of Theorem 1: Pick $T, U, V,$ and $W$ as in the theorem. These random variables together with $(X,Y)$ are distributed according to some distribution $p(v,x,t,u,y,w)$.

The coding procedure consists of two phases. In the first phase transmitter-$X$ sends $(T(X),U(T(X)))$ to transmitter-$Y$. In the second phase, both transmitters send $T(X)$ to the receiver. In addition to this message, transmitter-$X$ and transmitter-$Y$ send $V(X,T(X))$ and $W(Y,T(X), U(T(X)))$, respectively, to the receiver. As can be seen, only part of the message sent from transmitter-$X$ to transmitter-$Y$, $T(X)$, is retransmitted from both transmitters to the receiver while for the other part, $U(T(X))$, a function of it $W(Y,T(X), U(T(X)))$ is sent by transmitter-$Y$ to the receiver. Details follow.

For $t \in T$, $v \in \Gamma(G_{T,X|T,U,Y})$, and $w \in \Gamma(G_{T,U,Y|T,V})$, define $\tilde{f}(v,t,w)$ to be equal to $f(x,y)$ for all $(t,x) \in v$ and $(t,u,y) \in w$ such that $p(x,t,u,y) > 0$. Further, for $t = (t_1, \ldots, t_n)$, $v = (v_1, \ldots, v_n)$, and $w = (w_1, \ldots, w_n)$ let
\[
\tilde{f}(v,t,w) \overset{\text{def}}{=} \tilde{f}(v_1,t_1,w_1), \ldots, \tilde{f}(v_n,t_n,w_n).
\]

Generate $2^{nI(X:T)}$ sequences
\[ t^{(i)} = (t_1^{(i)}, t_2^{(i)}, \ldots, t_n^{(i)}), \]
i $\in \{1, 2, \ldots, 2^{nI(X:T)}\}$, i.i.d. according to the marginal distribution $p(t)$.

For each codeword $t^{(i)}$, generate $2^{nI(X:U|T)}$ sequences
\[ u^{(j)}(t^{(i)}) = (u_1^{(j)}(t_1^{(i)}), u_2^{(j)}(t_2^{(i)}), \ldots, u_n^{(j)}(t_n^{(i)})), \]
$j \in \{1, 2, \ldots, 2^{nI(X:U|T)}\}$, i.i.d. according to the marginal distribution $p(u|t)$, and randomly bin each sequence $(t^{(i)}, u^{(j)}(t^{(i)}))$ uniformly into $2^{nR_0}$ bins. Similarly, generate $2^{nI(V;X|T)}$ and $2^{nI(U,Y:W|T)}$ sequences

$$v^{(k)}(t^{(i)}) = (v_1^{(k)}(t_1^{(i)}), v_2^{(k)}(t_2^{(i)}), \ldots, v_n^{(k)}(t_n^{(i)})),$$

and

$$w^{(l)}(t^{(i)}) = (w_1^{(l)}(t_1^{(i)}), w_2^{(l)}(t_2^{(i)}), \ldots, w_n^{(l)}(t_n^{(i)})),$$

respectively, i.i.d. according to $p(v|t)$ and $p(w|t)$, respectively, and randomly and uniformly bin each sequence $(t^{(i)}, v^{(k)}(t^{(i)}))$ and $(t^{(i)}, w^{(l)}(t^{(i)}))$ into $2^{nR_X}$ and $2^{nR_Y}$ bins, respectively. Reveal the bin assignment $\phi_0$ to both the encoders and the bin assignments $\phi_X$ and $\phi_Y$ to the encoders and the decoder.

**Encoding**

First phase: Transmitter-$X$ tries to find a sequence $(t, u(t))$ that is jointly typical with $x$, i.e., $^8(t, u(t), x) \in \mathcal{A}_{\epsilon'}^{(n)}(T, U, X)$ and sends the index of the bin that contains this sequence, i.e., $\phi_0(t, u(t)) \overset{\text{def}}{=} q_0$, to transmitter-$Y$.

Second phase: Transmitter-$X$ tries to find a unique $v(t)$ that is jointly typical with $(x, t)$, i.e., $(v(t), x, t) \in \mathcal{A}_{\epsilon'}^{(n)}(V, X, T)$ and sends the index of the bin that contains $(t, v(t))$, i.e., $\phi_X(t, v(t)) \overset{\text{def}}{=} q_X$, to the receiver.

Transmitter-$Y$ upon receiving the index $q_0$, first tries to find a unique $(\hat{t}, \hat{u}(\hat{t}))$ such that $(\hat{t}, \hat{u}(\hat{t}), y) \in \mathcal{A}_{\epsilon'}^{(n)}(T, U, Y)$ and such that $\phi_0(\hat{t}, \hat{u}(\hat{t})) = q_0$. Then, it tries to find a unique $w(\hat{t})$ that is jointly typical with $(\hat{u}(\hat{t}), y)$, i.e., $(w(\hat{t}), \hat{u}(\hat{t}), y) \in \mathcal{A}_{\epsilon'}^{(n)}(W, U, Y)$ and sends the index of the bin that contains $(\hat{t}, w(\hat{t}))$, i.e., $q_Y = \phi_Y(\hat{t}, w(\hat{t}))$, to the receiver.

If a transmitter cannot find an index as above, it declares an error, and if there is more than one index, the transmitter selects one of them randomly and uniformly.

**Decoding:** Given the index pair $(q_X, q_Y)$, declare $\hat{f}(\hat{v}(\hat{t}), \hat{t}, w(\hat{t}))$ if there exists a unique jointly typical $(\hat{v}, \hat{t}, \hat{w}) \in \mathcal{A}_{\epsilon'}^{(n)}(V, T, W)$ such that $\phi_X(\hat{t}, \hat{v}(\hat{t})) = q_X$ and $\phi_Y(\hat{t}, \hat{w}(\hat{t})) = q_Y$, and such that $\hat{f}(\hat{v}(\hat{t}), \hat{t}, \hat{w}(\hat{t}))$ is defined. Otherwise declare an error.

**Probability of Error:** In each of the two phases there are two types of error.

In the first phase, the first type of error occurs when no $(t, u(t))$ is jointly typical with $x$. The probability of this error is negligible for $n$ large enough, due to the covering lemma (Lemma 5 in the second Appendix).

The second type of error occurs if $(\hat{t}, \hat{u}(\hat{t})) \neq (t, u(t))$. By symmetry of the scheme, this error probability, is the same as the average error probability conditioned on the transmitter-$X$ selecting $T^{(1)}$ and $U^{(1)}(T^{(1)})$. So, we consider the error event

$$\mathcal{E}' = \{(\hat{T}, \hat{U}(\hat{T})) \neq (T^{(1)}, U^{(1)}(T^{(1)}))\}.$$  \hspace{1cm} (22)

$^8\mathcal{A}_{\epsilon'}^{(n)}(X, Y)$ is the set of jointly $\epsilon$-typical $n$-sequences. See the second Appendix for more details.
Define the following events

\[ \mathcal{E}_{i,j}^\prime \overset{\text{def}}{=} \{(T^{(i)}, U^{(j)}(T^{(i)})) \in A_{q_0}^n(T, U, Y), \phi_0(T^{(i)}, U^{(j)}(T^{(i)})) = q_0\}. \]

Hence we have

\[
P(\mathcal{E}') \leq P(\bigcup_{j \neq 1} \mathcal{E}_{1,j}^\prime) + \sum_{i \neq 1} P(\mathcal{E}_{i,1}^\prime) + \sum_{i \neq 1, j \neq 1} P(\mathcal{E}_{i,j}^\prime). \tag{23}
\]

According to the properties of jointly typical sequences (Lemmas 1, 2, 3, and 4), for any \( \varepsilon' > \varepsilon'' > 0 \) we have

- \( P(\mathcal{E}_{1,1}^\prime) \leq \delta'(\varepsilon', \varepsilon'') \) due to the encoding process and the Markov chain \( T - U - X - Y \);
- for \( j \neq 1 \),
  \[ P(\mathcal{E}_{1,j}^\prime) \leq 2^{-n(I(U;X|T)-\delta'_1(\varepsilon'))} \cdot 2^{-nR_0}; \]
- for \( i \neq 1 \),
  \[ P(\mathcal{E}_{i,1}^\prime) \leq 2^{-n(I(T,U;Y)-\delta'_2(\varepsilon'))} \cdot 2^{-nR_0}; \]
- for \( i \neq 1 \) and \( j \neq 1 \),
  \[ P(\mathcal{E}_{i,j}^\prime) \leq 2^{-n(I(T,U;Y)-\delta'_3(\varepsilon'))} \cdot 2^{-nR_0}; \]

where \( \delta'(\varepsilon', \varepsilon''), \delta'_1(\varepsilon'), \delta'_2(\varepsilon'), \) and \( \delta'_3(\varepsilon') \) tend to zero as \( \varepsilon' \) tends to zero.

Using the above bounds, the probability of error in (23) can be bounded as

\[
P(\mathcal{E}') \leq \delta'(\varepsilon', \varepsilon'') + 2^n I(U;X|T) \cdot 2^{-n(I(U;Y|T)-\delta'_1(\varepsilon'))} \cdot 2^{-nR_0}
+ 2^n I(T;X) \cdot 2^{-n(I(T,U;Y)-\delta'_2(\varepsilon'))} \cdot 2^{-nR_0}
+ 2^n I(U,T;X) \cdot 2^{-n(I(T,U;Y)-\delta'_3(\varepsilon'))} \cdot 2^{-nR_0}
\]

Hence the error probability can be made to vanish whenever \( n \) tends to infinity as long as

\[
R_0 > I(X; U, T) - I(U, T; Y) = I(X; U, T|Y) = I(X; U|Y), \tag{24}
\]

where the equalities are due to the Markov chain \( T - U - X - Y \).

In the second phase, the first type of error occurs when no \( v(t) \), respectively no \( w(t) \), is jointly typical with \( (x, t) \), respectively with \( (u(t), y) \). The probability of each of these two errors is negligible for \( n \) large enough, due to the covering lemma (Lemma 5 in the second Appendix). Hence, the probability of the first type of error is negligible.

The second type of error refers to the Slepian-Wolf coding procedure. By symmetry of the scheme, the average error probability of the Slepian-Wolf coding procedure, is the same as the average error probability conditioned on
the transmitters selecting $T^{(1)}$, $U^{(1)}(T^{(1)})$, $V^{(1)}(T^{(1)})$ and $W^{(1)}(T^{(1)})$. Note that if the transmitted messages are decoded correctly at the decoder, then there is no error due to the Claim 3.b. of Lemma 1 and the definitions of $T$, $V$, $W$, and $\hat{f}(V,T,W)$.

We now consider the error event

$$E = \{(T, \hat{V}(T), \hat{W}(T)) \neq (T^{(1)}, V^{(1)}(T^{(1)}), W^{(1)}(T^{(1)}))\}. \tag{25}$$

and assume that the transmitters selected $T^{(1)}$, $U^{(1)}(T^{(1)})$, $V^{(1)}(T^{(1)})$ and $W^{(1)}(T^{(1)})$.

Define the following events,

$$E_{i,k,l} \overset{\text{def}}{=} \{(T^{(i)}, V^{(k)}(T^{(i)}), W^{(l)}(T^{(i)})) \in A_{\epsilon}(n)(T, V, W),$$

$$\phi_{X}(T^{(i)}, V^{(k)}(T^{(i)})), \phi_{Y}(T^{(i)}, W^{(l)}(T^{(i)})) = (q_{X}, q_{Y})\}.$$ We have

$$P(E) = P(E_{1,1,1}) \cup \left( \bigcup_{k \neq 1} E_{1,k,1} \right) \cup \left( \bigcup_{l \neq 1} E_{1,1,l} \right) \cup \left( \bigcup_{k \neq 1, l \neq 1} E_{1,k,l} \right) \cup \left( \bigcup_{i \neq 1, k, l} E_{i,k,l} \right)$$

$$\leq P(E_{1,1,1}) + \sum_{k \neq 1} P(E_{1,k,1}) + \sum_{l \neq 1} P(E_{1,1,l}) + \sum_{k \neq 1, l \neq 1} P(E_{1,k,l}) + \sum_{i \neq 1, k, l} P(E_{i,k,l}). \tag{26}$$

According to the properties of jointly typical sequences (Lemmas 1, 2, 3, and 4), for any $\epsilon > \epsilon' > 0$ we have

- $P(E_{1,1,1}) \leq \delta(\epsilon, \epsilon')$ due to the encoding process and the Markov chain $V - (T, X) - (U, Y) - W$;
- for $k \neq 1$,
  $$P(E_{1,k,1}) \leq 2^{-n(I(V;W|T) - \delta_{1}(\epsilon))} \cdot 2^{-nR_{X}};$$
- for $l \neq 1$,
  $$P(E_{1,1,l}) \leq 2^{-n(I(V;W|T) - \delta_{2}(\epsilon))} \cdot 2^{-nR_{Y}};$$
- for $k \neq 1, l \neq 1$,
  $$P(E_{1,k,l}) \leq 2^{-n(I(V;W|T) - \delta_{3}(\epsilon))} \cdot 2^{-nR_{X}} \cdot 2^{-nR_{Y}};$$
- for $i \neq 1$,
  $$P(E_{i,k,l}) \leq 2^{-n(I(V;W|T) - \delta_{4}(\epsilon))} \cdot 2^{-nR_{X}} \cdot 2^{-nR_{Y}};$$

where $\delta(\epsilon, \epsilon')$, $\delta_{1}(\epsilon)$, $\delta_{2}(\epsilon)$, $\delta_{3}(\epsilon)$, and $\delta_{4}(\epsilon)$ tend to zero as $\epsilon$ tends to zero.
Using the above bounds, the probability of error in (26) can be bounded as

\[ P(\mathcal{E}) \leq \delta(\varepsilon, \varepsilon') + 2^{n I(V;X|T)} \cdot 2^{-n(I(V;W|T) - \delta_1(\varepsilon))} \cdot 2^{-n R_X} \\
+ 2^{n I(U;Y;W|T)} \cdot 2^{-n(I(V;W|T) - \delta_2(\varepsilon))} \cdot 2^{-n R_Y} \\
+ 2^{n I(V;X|T)} \cdot 2^{n I(U;Y;W|T)} \cdot 2^{-n(I(V;W|T) - \delta_3(\varepsilon))} \cdot 2^{-n R_X} \cdot 2^{-n R_Y} \\
+ 2^{n I(X;U)} \cdot 2^{n I(V;X|T)} \cdot 2^{n I(U;Y;W|T)} \cdot 2^{-n(I(V;W|T) - \delta_3(\varepsilon))} \cdot 2^{-n R_X} \cdot 2^{-n R_Y}. \]

Hence the error probability goes to zero whenever \( n \) goes to infinity and inequalities (1) are satisfied.

We now prove that for calculating the rate region it is sufficient to consider random variables with cardinality bounds (4). Suppose \((V, X, T, U, Y, W) \sim p(v, x, t, u, y, w)\) satisfies (2) and (3).

**Cardinality of \( T \):** We want to bound the cardinality of \( T \) by \(|\mathcal{X}| + 4\). Suppose that \(|\mathcal{X}| \geq |\mathcal{X}| + 5\).

We keep \( p(v, x, u, y, w|t) \) unchanged. The Markov chains (2) concludes that \( p(y,x|u) \), and \( p(w|y, u, x) \) remain also unchanged. This guarantees that the Markov chains (2) hold for any new probability distribution \( p'(t), t \in T \).

Then we assign a new probability distribution \( p'(t), t \in T \), such that \( p'(t) = 0 \) for at least \(|T| - (|\mathcal{X}| + 4)\) elements and remove these elements from \( T \). The cardinality is now at most \(|\mathcal{X}| + 4\). We choose this new probability distribution in a way that \( p(x, y) \) and the right-hand sides of (1) remain unchanged. This guarantees that the achievable rate region remains unchanged. Note that since the support sets of the new random variables are subsets of the previous support sets, the conditions (3) are satisfied. We now show how to choose a new probability distribution with the desired characteristics.

The new probability distribution \( p'(t) \) should satisfy

\[ \sum_{t \in T} p'(t) = 1. \quad (27) \]

To keep \( p(x, y) \) unchanged we keep \( p(x) \) and \( p(y|x) \). For this, \( p'(t) \) should satisfy

\[ \sum_{t \in T} p(x = i|t) \cdot p'(t) = p(x = i)|_{p(t)} \quad 1 \leq i \leq |\mathcal{X}| - 1, \quad (28) \]

where \( p(x = i)|_{p(t)} \) is the original distribution of \( X \).

Consider the right-hand side of the first term in (1):

\[ I(X;U|Y)|_{p(t)} = H(X|Y)|_{p(t)} - H(X|U,Y)|_{p(t)}. \]

If \( p(x, y) \) remains unchanged, so remains \( H(X|Y) \). Hence, for keeping the value of \( I(X;U|Y)|_{p(t)} \), we should keep the value of \( H(X|U,Y)|_{p(t)} \), i.e., we should have

\[ \sum_{t \in T} a_t \cdot p'(t) = b, \quad (29) \]

where \( a_t = \sum_{u,y} H(X|U = u, Y = y)p(y|u)p(u|t) \) and \( b = H(X|U,Y)|_{p(t)} \). Note that the \( a_t \)'s do not depend on \( p'(t) \).
Similarly, for keeping the right-hand sides of the other terms in (1), \( p'(t) \) should satisfy the set of linear equations

\[
\sum_{t \in \mathcal{T}} I(V; X|W; T = t) \cdot p'(t) = I(V; X|T, W)|_{p(t)},
\]

(30)

\[
\sum_{t \in \mathcal{T}} I(U, Y; W|V; T = t) \cdot p'(t) = I(U, Y; W|V, T)|_{p(t)},
\]

(31)

\[
\sum_{t \in \mathcal{T}} (H(X, Y|V, T = t, W) + I(U; W|V, X, T = t, W)) \cdot p'(t) = (H(X, Y|V, T, W) + I(U; W|V, X, T, W))|_{p(t)}.
\]

(32)

Combining, we deduce that the distribution \( p'(t) \) should satisfy the set of \( m = |\mathcal{X}| + 4 \) linear equations (27)-(32).

We write these equations in the matrix form

\[
A_{n \times m} \times Z_{m \times 1} = B_{n \times 1},
\]

(33)

where \( n = |\mathcal{T}| \), where \( Z \) denotes the vector of \( p'(t), t \in \mathcal{T} \), where \( A \) denotes the matrix of coefficients (constants on the left-hand side of (27)-(32)), and \( B \) the vector of constants on the right-hand side of equations (27)-(32).

We want to find a positive solution of \( Z \) in the above equation where \( Z_i = 0 \) for at least \( n - m \) indices \( 1 \leq i \leq n \). We find such a solution recursively, i.e., we show that if \( n > m \), then we can find a solution \( S \) which has at least one zero entry, say \( S_i \). Then, we set \( n := n - 1 \), remove the corresponding column of \( A \) and corresponding row of \( Z \) and repeat the procedure.

We now show that if \( n > m \), then there exists a non-negative solution for (33) with at least one zero entry. We know that (33) has at least one non-negative solution, which is the vector \( p(t) \). Therefore, if we find another solution with at least one negative entry then, since the space solution of (33) is convex, there exists a solution with at least one zero entry.

Since \( n > m \), there exists a column which is a linear combination of the other columns. Without loss of generality, suppose \( A_m = \sum_{i=1}^{m-1} a_i A_i \), where \( A_i \) is the \( i \)-th column of \( A \). Now, if \( Z = [Z_1, \cdots, Z_m]^T \) is a non-negative solution, then

\[
Z' = [Z_1 + c \cdot a_1, Z_2 + c \cdot a_2, \cdots, Z_{m-1} + c \cdot a_{m-1}, Z_k - c]^T,
\]

is also a solution for any value of \( c \). By a suitable choice of \( c \), \( Z_k - c \) is negative which completes the proof.

**Cardinalities of \( \mathcal{V} \) and \( \mathcal{W} \):** To bound the cardinalities of \( \mathcal{V} \) and \( \mathcal{W} \) by \( |\mathcal{T}| \times |\mathcal{X}| + 1 \) and \( |\mathcal{U}| \times |\mathcal{Y}| + 1 \), respectively, one proceeds as in [17, Proof of Theorem 1] by means of Carathéodory’s theorem.

**Proof of Theorem 2:** For achievability it suffices to let \( T = U \) and \( V = X \) in Theorem 1.

Now for the converse. Let \( C_0 = \varphi_0(X) \) be the message received by transmitter-Y and let \( C_X = \varphi_X(X) \) and \( C_Y = \varphi_Y(C_0, Y) \) be the received messages at the receiver from transmitter-X and transmitter-Y respectively. Suppose that

\[
P(\psi(C_X, C_Y) \neq f(X, Y)) \leq \epsilon'_n,
\]
where \( \varepsilon_n' \to 0 \) when \( n \to \infty \). Using Fano’s inequality we have

\[
H(f(X, Y)|C_X, C_Y) \leq \varepsilon_n
\]

where \( \varepsilon_n \to 0 \) when \( n \to \infty \).

We start by showing that the Markov chain

\[
f(X_i, Y_i) - (C_0, X^n_i, Y^n_i - 1, C_Y) - C_X
\]

holds. We have

\[
p(f(x_i, y_i)|c_0, x^n_i, y^n_i - 1, cy, cx) = \sum_{x^n_i - 1} p(f(x_i, y_i)|c_0, x^n_i, y^n_i - 1, cy, cx) \cdot p(x^n_i - 1|c_0, x^n_i, y^n_i - 1, cy, cx)
\]

\[
= \sum_{x^n_i - 1} p(x^n_i - 1|c_0, x^n_i, y^n_i - 1, cy, cx) \cdot \sum_{y_i} p(f(x_i, y_i)|x_i, y_i) \cdot p(y_i|c_0, x^n_i, y^n_i - 1, cy)
\]

\[
= \sum_{x^n_i - 1} p(x^n_i - 1|c_0, x^n_i, y^n_i - 1, cy, cx) \cdot \sum_{y_i} p(f(x_i, y_i)|x_i, y_i) \cdot p(y_i|c_0, x^n_i, y^n_i - 1, cy)
\]

\[
= \sum_{x^n_i - 1} p(f(x_i, y_i)|c_0, x^n_i, y^n_i - 1, cy) \cdot p(x^n_i - 1|c_0, x^n_i, y^n_i - 1, cy, cx)
\]

where \((a)\) is due to the fact that \( C_X \) is a function of \( X^n_i \). This gives the desired Markov chain.

Now, by taking \( U_i = \{C_0, X^n_{i+1}, Y_{i-1}^n\} \), \( W_i = \{C_Y, Y^n_i - 1\} \), the Markov chains \( U_i - X_i - Y_i \) and \( X_i - (U_i, Y_i) - W_i \) hold and

\[
H(f(X_i, Y_i)|X_i, U_i, W_i) \overset{(a)}{=} H(f(X_i, Y_i)|X_i, U_i, W_i, C_X)
\]

\[
\leq H(f(X_i, Y_i)|C_Y, C_X)
\]

\[
\leq H(f(X, Y)|C_Y, C_X)
\]

\[
\leq \varepsilon_n,
\]

where \((a)\) is true due to the Markov chain \((34)\).
Then, we have

\[
\begin{align*}
nR_0 & \geq \log_2 |C_0| \\
& \geq H(C_0) \\
& \geq I(C_0; X^n_1 | Y^n) \\
& \geq \sum_{i=1}^{n} H(X_i | Y_i) - H(X_i | Y_i^{i-1}, C_0, X_{i+1}^n, Y_i) \\
& = \sum_{i=1}^{n} I(X_i; U_i | Y_i) \\
\end{align*}
\]

(35)

and

\[
\begin{align*}
nR_X & \geq \log_2 |C_X| \\
& \geq H(C_X) \\
& \geq I(C_X; X^n_1 | C_0, C_Y) \\
& \overset{(a)}{=} H(X^n_1 | C_0, C_Y) - \varepsilon \\
& \geq \sum_{i=1}^{n} H(X_i | X_{i+1}^n, C_0, C_Y, Y_i^{i-1}) - \varepsilon \\
& = \sum_{i=1}^{n} H(X_i | U_i, W_i) - \varepsilon \\
\end{align*}
\]

(36)

where (a) comes from the fact that \( X \) can be recovered knowing \( (C_X, C_Y) \), since the function \( f(X, Y) \) is partially invertible with respect to \( X \). Further,

\[
\begin{align*}
nR_Y & \geq \log_2 |C_Y| \\
& \geq H(C_Y) \\
& \geq I(C_Y; Y^n_1 | C_0, X^n_1) \\
& = \sum_{i=1}^{n} [H(Y_i | X_{i+1}^n, C_0, Y_i^{i-1}) - H(Y_i | X_{i+1}^n, C_0, Y_i^{i-1}, C_Y, X_i)] \\
& = \sum_{i=1}^{n} I(Y_i; W_i | X_i, U_i). \\
\end{align*}
\]

(37)
and

\[ n(R_X + R_Y) \geq \log_2 |(C_X, C_Y)| \geq H(C_X, C_Y) \]
\[ \geq I(C_X, C_Y; X^n_1, Y^n_1) \]
\[ \overset{(a)}{\geq} H(X^n_1) + H(Y^n_1|X^n_1) - H(Y^n_1|X^n_1, C_X, C_Y) - \varepsilon \]
\[ = H(X^n_1) + H(Y^n_1|X^n_1, C_0) - H(Y^n_1|X^n_1, C_0, C_X, C_Y) - \varepsilon \]
\[ \geq \sum_{i=1}^{n} [H(X_i) + H(Y_i^n|X_i^{i+1}, C_0, Y_i^{i-1}, X_i) - H(Y_i^n|Y_i^{i-1}, X_i^{i+1}, C_0, C_Y, X_i)] - \varepsilon \]
\[ = \sum_{i=1}^{n} [H(X_i) + I(Y_i, W_i|X_i, U_i)] - \varepsilon \] \tag{38}

where (a) comes from the fact that \( X \) can be recovered knowing \( (C_X, C_Y) \).

Let \( Q \) be a uniform random variable over \( \{1, 2, \ldots, n\} \). Let

\[ X = X_Q \]
\[ Y = Y_Q \]
\[ U = (U_Q, Q) \]
\[ W = (W_Q, Q). \]

Note that knowing \( U \) or \( W \), one knows \( Q \).

In the remaining part of the proof, we first show that \( U \) and \( W \) satisfy the inequalities of the theorem, the Markov chains (5), and the equality

\[ H(f(X, Y)|X, U, W) = 0. \] \tag{39}

Then, based on \( W \), we introduce a new random variable \( W' \) such that \( U \) and \( W' \) satisfy the inequalities of the theorem as well as the Markov chains (5) and the relation (6), which completes the proof.

We start by showing that \( U \) and \( W \) satisfy the Markov chains

\[ U - X - Y \]
\[ X - (U, Y) - W. \] \tag{40}

For the first Markov chain, we have

\[ H(Y|X, U) = \sum_{q=1}^{n} \frac{1}{n} H(Y_q|X_q, U_q, q) \overset{(a)}{=} \sum_{q=1}^{n} \frac{1}{n} H(Y_q|X_q) = H(Y|X), \]

where (a) is due to the Markov chain \( U_q - X_q - Y_q. \)
For the second Markov chain, we have

\[ I(X; W|U, Y) = n \sum_{q=1}^{n} I(X_q; W_q|U_q, Y_q) = 0, \]

where \((a)\) is due to the Markov chain \(X_q - (U_q, Y_q) - W_q\).

Equation (39) holds due to

\[ H(f(X, Y)|X, U, W) = n \sum_{q=1}^{n} H(f(X_q, Y_q)|X_q, U_q, W_q) \leq \varepsilon_n \]

and the fact that \(\varepsilon_n\) can be chosen arbitrarily small.

Finally to show that \(U\) and \(W\) satisfy the inequalities of the theorem, consider the following equalities

\[
\begin{align*}
I(X; U|Y) &= \frac{1}{n} \sum_{q=1}^{n} I(X_q; U_q|Y_q) \\
H(X|U, W) &= \frac{1}{n} \sum_{q=1}^{n} H(X_q|U_q, W_q) \\
I(Y; W|X, U) &= \frac{1}{n} \sum_{q=1}^{n} I(Y_q; W_q|X_q, U_q) \\
[H(X) + I(Y, W|X, U)] &= \frac{1}{n} \sum_{q=1}^{n} H(X_q) + I(Y_q, W_q|X_q, U_q). \\
\end{align*}
\]

This, together with (35), (36), (37), and (38) shows that \(U\) and \(W\) satisfy the inequalities of the theorem.

Until here we have shown that \(U\) and \(W\) satisfy the inequalities of the theorem, the Markov chains (5), and the equation (39).

The last step consists in defining a new random variable \(W'\) such that \(U\) and \(W'\) satisfy the inequalities of the theorem, the Markov chains (5), and equality (6), which completes the proof. To do this we need the following definition.

**Definition 6** (Support set of a random variable), [17, Definition 6] Let \((V, X) \sim p(v, x)\) where \(V\) is a random variable taking values in some countable set \(V = \{v_1, v_2, \ldots\}\). The support set of \(X\) with respect to \(V\) is the random variable \(S_X(V)\) defined as

\[ S_X(v_j) = (j, s_j = \{x : p(v_j, x) > 0\}) \quad v_j \in V. \]

Moreover, random variable \(S\) is defined as

\[ S = s_j \iff V = v_j \quad j = 1, 2, \ldots \]

Note that \(V\) and \(S_X(V)\) are in one-to-one correspondence by definition. In the sequel, with a slight abuse of notation we write \(Z \in S_X(V)\) whenever \(Z \in S\) and write \(S_X(V) \in \mathcal{A}\) whenever \(S \in \mathcal{A}\).
Let $W' = S_{(U,Y)}(W)$. According to Definition 6 and relations (39) and (40), $U$ and $W'$ satisfy

\[ X - (U,Y) - W' \]

\[ H(f(X,Y)|X,U,W') = 0 \tag{41} \]

and the inequalities of theorem. To conclude the proof it remains to show that

\[ (U,Y) \in W' \in M(\Gamma(G_{U,Y}|X,U)). \]

That $(U,Y) \in W'$ follows directly from the fact that $W' = S_{(U,Y)}(W)$. We show that $W' \in M(\Gamma(G_{U,Y}|X,U))$ by contradiction. Suppose that $w' \in W'$ is not an independent set in $G_{U,Y}|X,U$. Notice that for any $u_i, u_j \in U$, with $u_i \neq u_j$, and $y_i, y_j \in Y$, $(u_i, y_i)$ and $(y_j, y_j)$ are not connected in $G_{U,Y}|X,U$. Hence, there exists some $u \in U$ and $y_i, y_j \in Y$ such that $(u, y_i), (u, y_j) \in w'$, i.e.,

\[ p(u, y_i, w') \cdot p(u, y_j, w') > 0. \tag{42} \]

Now, $(u, y_i)$ and $(u, y_j)$ are connected in $G_{U,Y}|X,U$. This means that there exists some $x \in X$ such that

\[ p(x, u, y_i) \cdot p(x, u, y_j) > 0 \]

\[ f(x, y_i) \neq f(x, y_j). \tag{43} \]

The relations (42), (43) and the Markov chain $X - (U,Y) - W'$, imply that

\[ p(x, u, w') > 0 \]

\[ p(y_i|x, u, w') \cdot p(y_j|x, u, w') > 0 \]

\[ f(x, y_i) \neq f(x, y_j). \]

From these relations one concludes that

\[ H(f(X,Y)|X,U,W') \geq H(f(X,Y)|X = x, U = u, W' = w') \cdot p(x, u, w') > 0, \]

which contradicts (41).

Proof of Theorem 3:

Achievability: Letting $U = X$, $V = \text{Constant}$, $W = f(X,Y)$, and using the Markov chain $T - X - f(X,Y)$, gives the desired result. Note that in this case the cardinality bound can be tightened using Caratheodory’s theorem as in [17, Proof of Theorem 1].

Converse: Let $C_0 = \varphi_0(X)$ denote the message sent by transmitter-$X$ to transmitter-$Y$. Let $C_X = \varphi_X(X)$ and $C_Y = \varphi_Y(C_0, Y)$ be the messages sent by the transmitters to the receiver. Further, suppose that

\[ P(\psi(C_X, C_Y) \neq f(X, Y)) \leq \epsilon_n', \]
where $\epsilon_n' \to 0$ when $n \to \infty$. Using Fano’s inequality we have

$$H(f(X, Y)|C_X, C_Y) \leq \epsilon_n,$$

where $\epsilon_n \to 0$ when $n \to \infty$.

Letting $T_i \overset{\text{def}}{=} (f(X_{i+1}^n, Y_{i+1}^n), C_X)$ the Markov chain $T_i - X_i - Y_i$ holds. Moreover, we have

$$nR_Y \geq \log_2 |C_Y| \geq H(C_Y)$$

$$\geq I(C_Y, f(X_1^n, Y_1^n)|C_X)$$

$$\geq \sum_{i=1}^{n} H(f(X_i, Y_i)|f(X_{i+1}^n, Y_{i+1}^n), C_X) - \epsilon$$

$$= \sum_{i=1}^{n} H(f(X_i, Y_i)|T_i) - \epsilon,$$ (44)

and

$$n(R_X + R_Y) \geq \log_2 |C_X, C_Y| \geq H(C_X, C_Y)$$

$$= I(C_X, C_Y; X_1^n, f(X_1^n, Y_1^n))$$

$$= I(C_X, C_Y; f(X_1^n, Y_1^n)) + I(C_X, C_Y; X_1^n|f(X_1^n, Y_1^n))$$

$$\geq H(f(X_1^n, Y_1^n)) + I(C_X, C_Y; X_1^n|f(X_1^n, Y_1^n)) - \epsilon$$

$$= \sum_{i=1}^{n} H(f(X_i, Y_i)) + H(X_i|f(X_i, Y_i)) - H(X_i|X_{i-1}^i, f(X_1^n, Y_1^n), C_X, C_Y) - \epsilon$$

$$\geq \sum_{i=1}^{n} H(f(X_i, Y_i)) + H(X_i|f(X_i, Y_i)) - H(X_i|C_X, f(X_{i+1}^n, Y_{i+1}^n), f(X_i, Y_i)) - \epsilon$$

$$= \sum_{i=1}^{n} H(f(X_i, Y_i)) + I(X_i; f(X_{i+1}^n, Y_{i+1}^n), C_X|f(X_i, Y_i)) - \epsilon$$

$$= \sum_{i=1}^{n} H(f(X_i, Y_i)) + I(X_i; T_i|f(X_i, Y_i)) - \epsilon.$$ (45)

Let $Q$ be a uniform random variable over $\{1, 2, \cdots, n\}$. Let

$$X = X_Q$$

$$Y = Y_Q$$

$$T = (T_Q, Q)$$

Since the knowledge of $T$ gives $Q$ we have

$$H(Y|X, T) = \sum_{q=1}^{n} \frac{1}{n} H(Y_q|X_q, T_q, q) \overset{(a)}{=} \sum_{q=1}^{n} \frac{1}{n} H(Y_q|X_q) = H(Y|X)$$
where \((a)\) is due to the Markov chain \(T_q - X_q - Y_q\). Hence, the Markov chain

\[ T - X - Y \quad (46) \]

holds.

Moreover, we have the following equalities

\[
H(f(X,Y)|T) = \frac{1}{n} \sum_{q=1}^{n} H(f(X_q,Y_q)|T_q)
\]

\[
H(f(X,Y)) + I(X;T|f(X,Y)) = \frac{1}{n} \sum_{q=1}^{n} H(f(X_q,Y_q)) + I(X_q;T_q|f(X_q,Y_q)).
\]

This, together with (44), (45), and (46) completes the proof.

**Proof of Theorem 5:** From the converse of [16, Theorem 3] we deduce that if a rate pair \((R_0, R_Y)\) is achievable, then there exist random variables \(U\) and \(W\) that satisfy (13) and

\[
U - X - Y
\]

\[
X - (U,Y) - W
\]

\[
H(f(X,Y)|X,U,W) = 0.
\]

Finally, the same argument as the final argument of the converse proof of Theorem 2 shows that \(W' = S_{U,Y}(W)\) satisfies the above relations, the inequalities of the theorem, and

\[
Y \in W' \in \Gamma(G_{U,Y}|X,U).
\]

The cardinality bounds for \(U\) and \(W\) can be derived using the same methods as used in the proof of Theorem 1 for bounding the cardinalities of \(T\) and \(W\), respectively.

**References**

[1] R. Ahlswede and T. Han. On source coding with side information via a multiple-access channel and related problems in multi-user information theory. *Information Theory, IEEE Transactions on*, 29(3):396 – 412, May 1983.

[2] T. Cover. A proof of the data compression theorem of slepian and wolf for ergodic sources (corresp.). *Information Theory, IEEE Transactions on*, 21(2):226 – 228, March 1975.

[3] T. Cover and J. A. Thomas. *Elements of information theory (2. ed.).* Wiley, 2006.

[4] P. Cuff, H.I. Su, and A. El Gamal. Cascade multiterminal source coding. In *Information Theory Proceedings (ISIT), 2009 IEEE International Symposium on*, pages 1199 –1203, June 2009.

[5] A. El Gamal and Y. H. Kim. *Network Information Theory*. Cambridge University Press, 2012.

[6] T. Ericson and J. Körner. Successive encoding of correlated sources. *Information Theory, IEEE Transactions on*, 29(3):390 – 395, May 1983.

[7] T. S. Han and K. Kobayashi. A dichotomy of functions \(f(x,y)\) of correlated sources \((x,y)\) from the viewpoint of the achievable rate region. *Information Theory, IEEE Transactions on*, 33:69–76, January 1987.

[8] S. Huang and M. Skoglund. Computing polynomial functions of correlated sources: Inner bounds. In *Information Theory and its Applications (ISITA), 2012 International Symposium on*, pages 160 –164, October 2012.
In this Appendix we prove the claims stated in Example 4.

1. Suppose $T'$, $V'$ and $W$ satisfy the conditions of [11, Theorem 5.1], i.e.,

$$T' - X - Y$$

$$V' - (X, T') - (T', Y) - W$$

(47)
and that there exist functions \( g_1(T', V', W) \) and \( g_2(T', V', W) \) such that
\[
\begin{align*}
\mathbb{E} d_1(X, g_1(V', T', W)) &= 0 \\
\mathbb{E} d_2(Y, g_2(V', T', W)) &= 0.
\end{align*}
\] (48)

With this choice of auxiliary random variables \( T', V', \) and \( W \) the sum-rate constraint in [11, Theorem 5.1] becomes
\[
R_X + R_Y > I(X,Y; V', T', W).
\] (49)

The minimum of the right-hand side of (49) over \((T', V', W')\) can be restricted to the case where \( V \) is a constant. To see this, replace \( T' \) by \( T \triangleq (T', V') \) and let \( V \) be a constant. The random variables \( T', V, \) and \( W \) satisfy (47) and (48) and give the same-rate constraint as in (49).

We now want to find the minimum of
\[
I(X, Y; T, W)
\] (50)

for some \( T \) and \( W \) with
\[
|T| \leq 7 \\
|W| \leq 5,
\]
that satisfy
\[
T - X - Y \\
X - (T,Y) - W
\] (51)

and such that there exist functions \( g_1(T,W) \) and \( g_2(T,W) \) such that
\[
\begin{align*}
\mathbb{E} d_1(X, g_1(T, W)) &= 0 \\
\mathbb{E} d_2(Y, g_2(T, W)) &= 0.
\end{align*}
\] (52)

Since [11, Theorem 5.1] is a special case of Theorem 7 with \( U = \text{Constant} \), we can apply the cardinality bounds established in Theorem 7 and deduce that
\[
|T| \leq 7 \\
|W| \leq 5.
\]

The minimum of (50) with the above cardinality bounds can in principle be numerically evaluated to obtain
\[
\min I(X,Y; T, W) = 1.03.
\]
However, the number of degrees of freedom in the minimization still makes the
problem intractable on a regular desktop computer. As it turns out, for the problem at hand the cardinality bound \(|W| \leq 5\) can be tightened to \(|W| \leq 2\), which then allows to obtain the desired minimum in a matter of seconds on a regular computer.

2. The sum-rate constraint in [11, Theorem 5.4] is

\[ R_X + R_Y > I(X,Y;T,W) \]

for some \(T\) and \(W\) that satisfy

\[ T - X - Y, \]

and such that there exist functions \(g_1(W)\) and \(g_2(W)\) such that

\[
\mathbb{E}[d_1(X,g_1(W))] = 0 \\
\mathbb{E}[d_2(Y,g_2(W))] = 0. \tag{53}
\]

Since \(I(X,Y;T,W) \geq I(X,Y;W)\), by letting \(T\) be a constant decreases the sum-rate constraint. We now want to find the infimum \(I(X,Y;W)\) over \(W\)’s that satisfy (53) for some \(g_1(W)\) and \(g_2(W)\).

The distortion criteria (53) imply that for any \(w \in W\) with \(p(w) > 0\), we should have

\[
P(W = w|X = -1) \cdot P(W = w|X = +1) = 0 \\
P(W = w|Y = -1) \cdot P(W = w|Y = +1) = 0.
\]

Because of the symmetry of \(X\) and \(Y\), \(I(X,Y;W)\) is minimized for the random variable \(W \in \{w_-, w_+\}\) with probability distribution

\[
p(w_-|x,y) = \begin{cases} 
1 & \text{if } x = -1 \text{ or } y = -1 \\
\frac{1}{2} & \text{if } (x,y) = (0,0) \\
0 & \text{otherwise}
\end{cases}
\]

\[
p(w_+|x,y) = \begin{cases} 
1 & \text{if } x = +1 \text{ or } y = +1 \\
\frac{1}{2} & \text{if } (x,y) = (0,0) \\
0 & \text{otherwise}.
\end{cases}
\]

This \(W\) satisfies the Markov chain and distortion criteria constraints of [11, Theorem 5.4] and gives

\[
\inf(R_X + R_Y) = I(X,Y;W) = 1 - \frac{1}{7} = 0.85
\]
3. Let $T, V$ be constants and $U \in \{u_-, u_+\}$ have the probability distribution

$$p(u_{-}\mid x) = \begin{cases} 1 & \text{if } x = -1 \\ \frac{1}{2} & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$p(u_{+}\mid x) = \begin{cases} 1 & \text{if } x = +1 \\ \frac{1}{2} & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

Let $W \in \{w_-, w_+\}$ have the probability distribution

$$p(w_{-}\mid u, y) = \begin{cases} 1 & \text{if } (u, y) \in \{(u_-, -1), (u_-, 0), (u_+, -1)\} \\ 0 & \text{otherwise} \end{cases}$$

$$p(w_{+}\mid u, y) = \begin{cases} 1 & \text{if } (u, y) \in \{(u_+, +1), (u_+, 0), (u_-, +1)\}, \\ 0 & \text{otherwise}. \end{cases}$$

Random variables $T$, $U$, $V$, and $W$ satisfy the Markov chains and distortion criteria of Theorem 7. These random variables give the sum rate

$$R_{X} + R_{Y} > I(X, Y; W) = 1 - \frac{1}{7} = 0.85$$

with

$$R_{0} > I(X; U \mid Y) = 0.38.$$
Jointly typical sequences satisfy the following properties:

**Lemma 1.** [16, Corollary 2]. [5, Page 27]

1. Let \((X^n, Y^n) \sim \prod_{i=1}^{n} p_{X,Y}(x_i, y_i)\). Then
   \[
P((X^n, Y^n) \in \mathcal{A}_\varepsilon^{(n)}(X, Y)) \geq 1 - \delta(\varepsilon).
   \]

2. \((1 - \delta(\varepsilon))2^{nH(X,Y)(1-\varepsilon)} \leq |\mathcal{A}_\varepsilon^{(n)}(X, Y)| \leq 2^{nH(X,Y)(1+\varepsilon)}.

3. Let \(p(x^n, y^n) = \prod_{i=1}^{n} p_{X,Y}(x_i, y_i)\). Then, for each \((x^n, y^n) \in \mathcal{A}_\varepsilon^{(n)}(X, Y),\)
   a. \(x^n \in \mathcal{A}_\varepsilon^{(n)}(X)\) and \(y^n \in \mathcal{A}_\varepsilon^{(n)}(Y);\)
   b. \(p_{X,Y}(x_i, y_i) > 0\) for all \(1 \leq i \leq n;\)
   c. \(2^{-nH(X,Y)(1+\varepsilon)} \leq p(x^n, y^n) \leq 2^{-nH(X,Y)(1-\varepsilon)};\)
   d. \(2^{-nH(X|Y)(1+\varepsilon)} \leq p(x^n|y^n) \leq 2^{-nH(X|Y)(1-\varepsilon)}.

**Lemma 2** (Conditional Typicality Lemma). Let \((X, Y) \sim p(x, y)\). Suppose that \(x^n \in \mathcal{A}_\varepsilon^{(n)}(X)\) and \(Y^n \sim p(y^n|x^n) = \prod_{i=1}^{n} p_{Y|X}(y_i|x_i)\). Then, for \(\varepsilon > \varepsilon'\)
   \[
P((x^n, Y^n) \in \mathcal{A}_\varepsilon^{(n)}(X, Y)) \geq 1 - \delta(\varepsilon, \varepsilon').
   \]

**Lemma 3** (Markov Lemma). [16, Lemma 23] Let \(X - Y - Z\) form a Markov chain. Suppose that \((x^n, y^n) \in \mathcal{A}_\varepsilon^{(n)}(X, Y)\) and \(Z^n \sim p(z^n|y^n) = \prod_{i=1}^{n} p_{Z|Y}(z_i|y_i)\). Then, for \(\varepsilon > \varepsilon'\)
   \[
P((x^n, y^n, Z^n) \in \mathcal{A}_\varepsilon^{(n)}(X, Y, Z)) \geq 1 - \delta(\varepsilon, \varepsilon').
   \]

**Lemma 4.** [16, Corollary 4] Let \((X, Y) \sim p_{X,Y}(x, y)\) with marginal probability distributions \(p_X(x)\) and \(p_Y(y)\).
Let \((X', Y') \sim \prod_{i=1}^{n} p_{X,Y}(x'_i, y'_i)\). Then,
\[
(1 - \delta(\varepsilon)) \cdot 2^{-n(I(X;Y) + 2\varepsilon H(Y))} \leq P((X', Y') \in \mathcal{A}_\varepsilon^{(n)}(X, Y)) \leq 2^{-n(I(X;Y) - 2\varepsilon H(Y))}.
\]

**Lemma 5** (Covering Lemma). [5, Lemma 3.3] Let \((X, \hat{X}) \sim p_{X,\hat{X}}(x, \hat{x})\). Let \(X^n \sim \prod_{i=1}^{n} p_{X}(x_i)\) and
\[
\{\hat{X}^n(m), m \in \mathcal{B}\} \text{ with } |\mathcal{B}| \geq 2^{nR}.
\]

be a set of random sequences independent of each other and of \(X^n\), each distributed according to \(\prod_{i=1}^{n} p_{\hat{X}}(\hat{x}_i(m))\).
Then, there exists \(\delta(\varepsilon)\) that tends to zero as \(\varepsilon \to 0\) such that
\[
\lim_{n \to \infty} P((X^n, \hat{X}^n(m)) \notin \mathcal{A}_\varepsilon^{(n)}(X, \hat{X}^n) \text{ for all } m \in \mathcal{B}) = 0,
\]
if
\[
R > I(X; \hat{X}) + \delta(\varepsilon).
\]