Logged Rewriting for Monoids

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Abstract

A rewriting system is a set of equations over a given set of terms called rules that characterize a system of computation and is a powerful general method for providing decision procedures of equational theories, based upon the principle of replacing subterms of an expression with other terms. In particular, a string rewriting system is usually associated with a monoid presentation. At the first level the problem is to decide which combinations of the generators are equivalent under the given rules; Knuth-Bendix completion of the string rewriting system is one of the most successful mechanisms for solving this problem. At the second level, the problem involves determining which combinations of rules are equivalent. Logged rewriting is a technique which not only transforms strings but records the transformation in terms of the original system of rules. The relations between combinatorial, homotopical and homological finiteness conditions for monoids prompt us to consider using computer-friendly rewriting systems to calculate homotopical and homological structure from monoid presentations.

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1 Introduction

The idea of making a note of which rules are used as they are applied is quite a simple one and it would be easy to regard it as too trivial to spend any time on. However, when we look into the algebraic structure of the records themselves, things become a lot less trivial. The background to this work includes the papers by Squier, Lafont, Prouté, Otto, Cremanns, Anick, Kobayashi, Pride and others on finiteness conditions for monoids. The ‘combinatorial’ finiteness condition is that a monoid has a finite complete presentation. This implies the homological finiteness condition $FP_\infty$ (Anick, 1986; Squier, 1987; Kobayashi, 1990) and also the ‘homotopical’ finiteness condition FDT (Squier, 1994). It is also known that the weaker homological condition $FP_3$ is not sufficient for either FDT or the existence of a complete rewriting system in the case of monoids (Squier, 1994).

Our aim is to use enhanced rewriting procedures to explicitly provide:

i) A finite complete rewriting system (combinatorial specification).
ii) A finite set of homotopy generators for $P^{(2)}\Gamma$ (homotopical specification).
iii) A (small) finitely generated resolution (homological specification).

Logged rewriting for group presentations (Heyworth and Wensley, 1999) gives procedures for representing consequences of the relations of the presentation as elements of a pre-crossed module and algorithms for computing generators of the modules of identities among relations. In the monoid case the relations are given by pairs of terms and the structure of a crossed module is not appropriate to represent consequences of relations. It is well known (Stell, 1994; Street, 1992) that sesquicategories or 2-categories can be used to model rewriting systems. It has been proved (Pride, 1999; Gilbert, 1996) that when the rewriting system comes from a group presentation, the 2-category can be identified with the crossed module of the presentation.

In the special case of groups, various results are known. In particular there are methods for calculating a set of generators for the kernel $\Pi_2$ of the crossed module of ‘consequences’, which is useful for constructing resolutions and calculating (co)homology. For the case where the rewriting system does not present a group we detail the algebraic structure of the analogue of $\Pi_2$; presenting an algorithm for computing a set of generators for it; and provide justification that the constructions we make give combinatorial, homotopical and (co)homological detail in the same spirit as $\Pi_2$.

In the case of monoids, logged rewriting techniques have further applications. Specifically, we have so far examined applications to the analysis of coset and double coset systems and used logged rewriting to provide an alternative to the Reidemeister-Schreier algorithm for finding presentations of subgroups (Brown et al, 2004; Ghani and Heyworth, 2003). Additionally, we show
in Section 8 that logged rewriting techniques are easily generalised to Kan extensions where they provide a proof technique for a wide range of decidability problems solvable by string rewriting (Brown and Heyworth, 2000).

2 Logged Rewriting Systems

A **monoid presentation** is given as a pair \( \mathcal{P} = \text{mon}(X|R) \) where \( X \) is a set of generators and \( R \) is a set of pairs \((l, r)\) of elements of the free monoid \( X^* \). The monoid presented, \( M \) is the quotient obtained by factoring \( X^* \) by \( =_R \), the congruence generated by \( R \). The quotient monoid morphism will be denoted \( \theta : X^* \rightarrow M \).

We will assume that the reader is familiar with standard string rewriting as in (Book and Otto, 1993). The notation we use follows the usual conventions. The set \( R \) is a **rewriting system** for the monoid \( M \) and its elements are referred to as **rules**. The reduction relation generated by \( R \) on the free monoid \( X^* \) is denoted by \( \rightarrow_R \), the reflexive, transitive closure is denoted \( \rightarrow^*_R \) and the reflexive, symmetric, transitive closure \( \leftrightarrow^*_R \) coincides with the congruence \( =_R \). For convenience we assume that \( R \) is compatible with an admissible well-ordering \( > \); i.e. for all pairs \((l, r) \in R\), we have \( l > r \). This ensures that the relation \( \rightarrow_R \) is Noetherian.

The main aim of this section is to formally define a **logged rewrite system** for \( \mathcal{P} \). Such a system must not only reduce any word in \( X^* \) to an irreducible word (unique if the rewriting system is complete) but must also express the actual reduction as a consequence of the original monoid relations. The reader who does not wish to get into the details at this stage may wish to think of a consequence of the monoid relations as a sequence of rewrites recorded as: [prefix, rule, direction of rule, suffix] which must give a valid rewrite.

It is important to identify the algebraic framework for these ‘consequences’ in order to understand what we may do with them. Formally, one represents consequences of group relations by elements of a crossed module. Consequences of monoid relations cannot be represented in that framework; essentially this is because the free monoid does not have inverses. However, it is well known that general string rewriting systems may be modelled by sesquicategories or 2-categories (Benson, 1975; Stell, 1994; Street, 1992). Therefore to every monoid presentation we shall associate a sesquicategory. Its 2-cells correspond to possible sequences of rewrites and inverse rewrites between strings in the free monoid, with respect to the given rewriting system. Formally:

**Definition 2.1 (Sesquicategory of Rewrites)**

The sesquicategory \( SQ(\mathcal{P}) \) of a monoid presentation \( \mathcal{P} \) consists of the following:
• a single 0-cell which is denoted \(*\),
• a free monoid of 1-cells which are the elements of \(X^*\),
• a collection of 2-cells which are sequences

\[ \alpha = u_1\alpha_1^e v_1 \cdots u_n\alpha_n^e v_n \]

where \(u_1, \ldots, u_n, v_1, \ldots, v_n \in X^*\), \(\alpha_1, \ldots, \alpha_n \in R \cup \{1\}\) and \(\epsilon_1, \ldots, \epsilon_n = \pm 1\) such that \(u_i \text{tgt}(\alpha_i^\epsilon) v_i = u_{i+1} \text{src}(\alpha_{i+1}^\epsilon) v_{i+1}\) for \(i = 1, \ldots, n-1\).

• left and right actions of the 1-cells upon the 2-cells (whiskering) i.e. for any rewrite \(\alpha\) and any elements \(u\) and \(v\) of the free monoid we say that \(u\alpha v\) is a rewrite and \(\text{src}(u\alpha v) = \text{usrc}(\alpha)v\) and \(\text{tgt}(u\alpha v) = \text{utgt}(\alpha)v\).

• identity rewrites for each string \(w \in X^*\), denoted \(1_w\) where \(\text{src}(1_w) = \text{tgt}(1_w) = w\) with the property that

\[ u \cdot 1_w \cdot v = 1_{uwv} \]

for all \(u, v\) in \(X^*\).

• a partial (‘vertical’) composition of rewrites, defined so that \(\alpha \cdot \beta\) is a rewrite with \(\text{src}(\alpha \cdot \beta) = \text{src}(\alpha)\) and \(\text{tgt}(\alpha \cdot \beta) = \text{tgt}(\beta)\) whenever \(\text{tgt}(\alpha) = \text{src}(\beta)\).

For the above definition it can be verified that the sesquicategory axioms hold with respect to vertical composition and the whiskering action.

Further, we shall allow rewrites to be cancelled by the reverse application of the rewriting sequence. The formal inverse of any rewrite \(\alpha\) is denoted \(\alpha^{-1}\) where \(\text{src}(\alpha^{-1}) = \text{tgt}(\alpha)\) and \(\text{tgt}(\alpha^{-1}) = \text{src}(\alpha)\) and we allow that

\[ \alpha \cdot \alpha^{-1} = 1_{\text{src}(\alpha)} \text{ for all rewrites } \alpha. \]

This gives \(\cdot\) a groupoid structure, so we may refer to the sesquigroupoid \(SQ(P)\).

In the case where we can apply the rule \(\alpha\) to one substring of a string and the rule \(\beta\) to another substring which is completely disjoint from the first, it is natural to regard the order in which the rules are actually applied as immaterial. This interchangability of non-overlapping rewrites is captured by the interchange law on the sesquicategory, giving us a 2-category. We shall denote the set of 2-cells in \(SQ(P)\) by \(C_2\).

**Definition 2.2 (2-category of Rewrites)**

The 2-category of rewrites \(C_2(P)\) is obtained by factoring the 2-cells \(C_2\) of \(SQ(P)\) by the interchange law:

\[ I = \{(\text{src}\beta \cdot \text{tgt}\alpha) \beta, \text{src}\alpha \beta \cdot \alpha \text{tgt}\beta) : \alpha, \beta \in C_2\} \]

Specifically, the set of pairs of \(I\) generates a relation on \(C_2\)

\[ \{(\gamma \cdot u\alpha_1 v \cdot \delta, \gamma \cdot u\alpha_2 v \cdot \delta : (\alpha_1, \alpha_2) \in I, u, v \in X^*, \gamma, \delta \in C_2\} \]
and the reflexive, symmetric, transitive closure of this is $=_{I}$, which preserves both vertical composition and whiskering. Congruence classes are formally denoted with square brackets so $[\alpha]_{I}$ denotes the class of $C_2$ under $=_{I}$ that contains $\alpha$. Whiskering and vertical composition are preserved and so may be applied to the congruence classes: $u[\alpha]_{I}v = [u\alpha v]_{I}$ for all $u, v \in X^{*}$ and $[\alpha]_{I} \cdot [\beta]_{I} = [\alpha \cdot \beta]_{I}$. A horizontal composition of the congruence classes may also be defined: $[\alpha]_{I} \circ [\beta]_{I} = [\alpha \text{src}(\beta) \cdot \text{tgt}(\alpha) \beta]_{I}$.

In the case of term rewriting one does not always wish to factor out by the interchange law as it destroys the notion of length (number of steps) of a rewrite. In the case of string rewriting we do not have to worry about notions of length of derivation, thus we use the 2-category. However, it should be noted that whilst rewrites may be represented uniquely in the sesquicategory, the word problem for the 2-category is generally unsolvable (generalisation of a crossed module). Like many, for convenience, we abuse notation a little, representing rewrites that should strictly be written as classes $[\alpha]_{I}$ by non-unique representatives in the sesquicategory $\alpha$. So a pair of rewrites $\alpha, \beta \in C_2$, are equivalent if and only if $[\alpha]_{I} = [\beta]_{I}$.

In the context of groups, the sesquicategory associated to a monoid presentation is well known. Pride proved that if the monoid presentation involved is obtained from a group presentation then the associated 2-category is isomorphic (as a crossed module) to the free crossed module associated to the group presentation \cite{Gilbert1996, Pride1999}. Logged rewriting for groups was established by using the crossed module structure for the logs. We now formally define logged rewriting using the 2-category associated with a monoid presentation.

**Definition 2.3 (Logged Rewriting System)**

A logged rewriting system for a presentation $P$ of a monoid $M$ is a collection of 2-cells (rewrites) \[ \mathcal{L} = \{\alpha_1, \ldots, \alpha_n\} \]

of the associated 2-category $C_2(P)$ so that the underlying rewriting system

\[ R_{\mathcal{L}} = \{(\text{src}(\alpha_1), \text{tgt}(\alpha_1)), \ldots, (\text{src}(\alpha_n), \text{tgt}(\alpha_n))\} \]

is a rewriting system for $M$.

A rewriting system $R$ on a monoid $M$ generates a reduction relation

\[ \rightarrow_{R} = \{(ulv, urv) : (l, r) \in R, u, v \in M\}. \]

The reflexive, transitive closure of this relation is denoted $\rightarrow^{*}_{R}$, and the reflexive, symmetric, transitive closure is denoted $\leftrightarrow^{*}_{R}$ and coincides with the congruence generated by $R$ on $M$, denoted $=_{R}$. The logged reduction of a string by a rule $\alpha$ is written as: $u\alpha v : u\text{src}(\alpha)v \rightarrow ut\text{tgt}(\alpha)v$ and the rewrite recorded is $u\alpha v$. 5
If the elements \((l, r)\) of a rewriting system on a free monoid \(X^*\) are ordered such that \(l > r\) with respect to some well-ordering on \(X^*\), then the resulting reduction system is **Noetherian**; i.e. an irreducible element is reached after finitely many reductions. A reduction system is **confluent** if for any string \(w\) there exists a unique irreducible string \(\bar{w}\) such that \(w \rightarrow_R \bar{w}\). A rewrite system is said to be **complete** if the corresponding reduction relation is both Noetherian and confluent. This is a desirable property, since any pair of strings \(w_1, w_2\) can be reduced in a finite number of steps to their irreducible forms \(\bar{w}_1\) and \(\bar{w}_2\) which will be equal if and only if \(w_1 =_R w_2\); i.e. the word problem is decidable.

### 3 Logged Completion

The Knuth-Bendix algorithm attempts to convert an arbitrary rewriting system into a complete one by adding rules compatible with the ordering to the system to try to force confluence. The key concept here is that of **critical pairs** which are pairs of reductions which can be applied to the same string to obtain two different results. The important critical pairs are associated with the **overlaps** of the rules in the rewriting system. When considering normal critical pairs we only care about the sources and targets of the rewrites and the relevant information identifying the overlap. When we are dealing with a logged rewriting system it is necessary to think of the sequences of rules giving the instructions permitting both of the rewrites and to include these logs as part of the critical pair information.

**Definition 3.1 (Logged Critical Pairs)**

An overlap occurs between the logged rewrites \(\alpha_1 : l_1 \rightarrow r_1\) and \(\alpha_2 : l_2 \rightarrow r_2\) of \(\mathcal{L}\) whenever one of the following is true:

\[
i) \quad u_1 l_1 v_1 = l_2, \quad ii) \quad u_1 l_1 = l_2 v_2, \quad iii) \quad l_1 v_1 = u_2 l_2, \quad iv) \quad l_1 = u_2 l_2 v_2.
\]

for some \(u_1, u_2, v_1, v_2 \in X^*\). The logged critical pair resulting from the overlap is a whiskered pair \((u_1 \alpha_1 v_1, u_2 \alpha_2 v_2)\) for the appropriate \(u_1, u_2, v_1, v_2 \in X^*\).

Given a monoid presentation \(P = \text{mon}(X|R)\) we can associate to it an **initial logged rewriting system** \(\mathcal{L}_{\text{init}}\) which consists of one 2-cell \(\alpha\) for each rule \((l, r)\) of \(R\) with \(\text{src}(\alpha) = l\) and \(\text{tgt}(\alpha) = r\). These 2-cells are the generators of sesquigroupoid associated to the presentation.

If the initial logged rewriting system is not complete then we can attempt to transform it into a complete logged rewriting system, by adding 2-cells which will make the underlying rewriting system complete, in a version of the Knuth-Bendix algorithm which records information that is usually discarded. Clearly, this recorded completion terminates exactly when the usual completion procedure would terminate.
Algorithm 3.2 (Logged Knuth-Bendix Procedure)

LKB1: (Input) Let $P$ be a presentation of a monoid with generators $X$ and relations $(l_1, r_1), \ldots, (l_n, r_n)$ where $l_1 > r_1, \ldots, l_n > r_n$ for some well-ordering on the free monoid $X^*$. Define $\mathcal{L}_{\text{init}}$ to be the set of 2-cells or logged rules $\{\alpha_1, \ldots, \alpha_n\}$, where $\text{src}(\alpha_i) = l_i$ and $\text{tgt}(\alpha_i) = r_i$ for $i = 1, \ldots, n$.

LKB2: (Initialise) Set $\mathcal{L}_{\text{all}} = \mathcal{L}_{\text{init}}$; $\mathcal{L}_{\text{new}} = \mathcal{L}_{\text{init}}$; and let $C$ be the empty list.

LKB3: (Search for Overlaps and Record Critical Pairs) Whenever an overlap occurs between the rewrites $\alpha_a \in \mathcal{L}_{\text{all}}$ and $\alpha_n \in \mathcal{L}_{\text{new}}$, record the associated critical pair by adding the element $(u_a, \alpha_a, v_a, u_n, \alpha_n, v_n)$ to the list $C$ (where $u_a, v_a, u_n$ or $v_n$ may be the identity element).

LKB4: (Attempt to Resolve Critical Pairs) Set $\mathcal{L}_{\text{new}} = \emptyset$. For every element of $C$ consider the pair $(u_ar_a v_a, u_nr_n v_n)$, reducing each string by $\mathcal{L}_{\text{all}}$ to the irreducible strings $z_a$ and $z_n$ respectively. If $z_a = z_n$ then the critical pair is said to resolve and it can be removed from $C$. Otherwise we must add a new logged rule to the system. If $\beta_a$ and $\beta_n$ are the logs of the reductions to $z_a$ and $z_n$ then the new logged rule is $\gamma = \beta_a^{-1} \cdot u_a \alpha_a^{-1} v_a \cdot u_n \alpha_n v_n \cdot \beta_n$ if $z_a > z_n$ and $\gamma = \beta_n^{-1} \cdot u_n \alpha_n^{-1} v_n \cdot u_a \alpha_a v_a \cdot \beta_a$ if $z_n > z_a$. Add $\gamma$ to $\mathcal{L}_{\text{new}}$ and to $\mathcal{L}_{\text{all}}$.

LKB5: (Loop) If $\mathcal{L}_{\text{new}}$ is non-empty then loop to LKB3. Otherwise the procedure terminates: all critical pairs of $\mathcal{L}_{\text{all}}$ have been tested, and resolve.

LKB6: (Output) Output $\mathcal{L}_{\text{all}}$, a complete logged rewriting system for $P$.

The immediate application for logged rewriting systems is in the provision of witnesses for computation. An ordinary complete rewriting system can determine whether or not two strings $s_1$ and $s_2$ represent the same element of the monoid; a logged rewriting system produces a proof in terms of a sequence of specific applications of the original monoid relations which will transform $s_1$ into $s_2$. This is a fairly shallow application, although variations on it are useful in more complex algorithms such as [Heyworth and Wensley, 1999].

4 Endorewrites

Deeper information about the presentation can be gained by studying the interaction of the relations with each other, known in group theory as the identities among relations. The identities themselves represent rewrite sequences which start at a word, send it through various transformations and return it to its original form. For the monoid case, we decided to refer to such rewrites as endorewrites. In the case of monoids, the structure is necessarily less simple.
than the kernel of a crossed module map. Note that we will continue to identify rewrites which should strictly be written as classes \([\alpha]_I \) by (non-unique) representatives in the sesquicategory \(\alpha\). So \(\alpha, \beta \in EQ \subseteq C_2\), are equal as rewrites if and only if \([\alpha]_I = [\beta]_I\).

**Definition 4.1 (Endorewrites)**

A 2-cell \(\alpha \in C_2\) is an endorewrite on a string \(w\) if \(\text{src}(\alpha) = \text{tgt}(\alpha) = w\).

The set of all endorewrites is actually the equaliser object of the two maps \(\text{src}, \text{tgt} : C_2 \to X^*\) in the category of sets. We denote it \(EQ\).

**Lemma 4.2 (Endorewrite Structure)**

The set of all endorewrites \(EQ\) is the disjoint union of the sets \(EQ_w\) for \(w \in X^*\) where

\[
EQ_w = \{ \alpha : \text{src}(\alpha) = \text{tgt}(\alpha) = w \}.
\]

Each \(EQ_w\) is closed under vertical composition; and their union \(EQ\) is additionally closed under horizontal composition and whiskering.

Vertical composition is defined only within subsets \(EQ_w\). Horizontal composition is defined across subsets: if \(\alpha \in EQ_w\) and \(\alpha' \in EQ_{w'}\), then \(\alpha \circ \alpha' \in EQ_{ww'}\).

Whiskering means that for any substring \(s\) of a string \(w\) there is an injective mapping \(EQ_s \to EQ_w\) defined by \(\alpha \mapsto u\alpha v\) where \(uv = w\).

**Lemma 4.3 (Conjugate Endorewrites)**

If \(\theta(w_1) = \theta(w_2)\) then for every \(\beta \in C_2\) such that \(\text{src}(\beta) = w_1\) and \(\text{tgt}(\beta) = w_2\) there exists a bijection \(\Phi_\beta : EQ_{w_1} \to EQ_{w_2}\) defined by \(\alpha \mapsto \beta^{-1} \cdot \alpha \cdot \beta\).

Thus the elements of \(EQ_{w'}\) are all conjugates of elements of \(EQ_w\) so it becomes logical that we should only seek generators for endorewrites \(EQ_w\) of one representative string \(w\) for each monoid element \(\theta(w)\). The next lemma helps to make this concrete.

**Lemma 4.4 (Partial Action of Rewrites on Endorewrites)**

There is a partial function \(EQ \times (C_2, \cdot) \to EQ\), defined by \(\alpha^\beta = \beta^{-1} \cdot \alpha \cdot \beta\) for \(\alpha \in EQ\) and \(\beta \in C_2\) such that \(\text{src}(\alpha) = \text{tgt}(\alpha) = \text{src}(\beta)\). This satisfies the following properties:

i) \(\alpha^{1_{\text{src}}} = \alpha\) for all \(\alpha \in EQ\).

ii) \(\alpha^{(\beta_1 \cdot \beta_2)} = (\alpha^{\beta_1})^{\beta_2}\) for all \(\alpha, \beta_1, \beta_2 \in C_2\) such that \(\text{src}(\alpha) = \text{tgt}(\alpha) = \text{src}(\beta_1)\) and \(\text{tgt}(\beta_1) = \text{src}(\beta_2)\).

iii) \(u\alpha^\beta v = (uav)^{uv^\beta}\) for all \(u, v \in X^*\) whenever \(\alpha^\beta\) is defined.

iv) \((\alpha_1 \cdot \alpha_2)^\beta = \alpha_1^\beta \cdot \alpha_2^\beta\) for all \(\alpha_1, \alpha_2 \in EQ\) and \(\beta \in C_2\) such that \(\text{tgt}(\alpha_1 \circ \alpha_2) = \text{src}(\beta)\).

The first two properties are the categorical equivalent of the properties required for a partial monoid action, the second two show that the partial action preserves the whiskering and vertical composition operations in \(EQ\). All the properties follow from the definitions of \((C_2, \cdot)\), identity 2-cells \(1_{\text{src}(\alpha)}\) and the definition of \(\alpha^\beta\).
Intuitively, $\alpha$ is like a circular walk: conjugating by $\beta$ just means that we first walk down an additional path to the start of $\alpha$, retracing our steps back along that path once the circular walk $\alpha$ is completed. Clearly the circular walk is not much more interesting for having this initial path added to it and a guidebook that suggested all conjugates of $\alpha$ were distinct jaunts would be absurd. Thus we factor $EQ$ by this partial action and consider $\alpha$ to be equivalent to all its possible conjugates. Formally:

**Lemma 4.5 (Classes of Endorewrites)**
Let $C_2(P)$ be the 2-category of rewrites for a monoid presentation $P$ and let $EQ$ be the set of all endorewrites. Then define

$$J = \{ (\alpha, \beta^{-1} \cdot \alpha \cdot \beta) : \alpha \in EQ, \beta \in C_2 \text{ and } \text{src} \alpha = \text{src} \beta \}.$$ 

Let $=_{I+J}$ be the smallest congruence on $E$ with respect to $\cdot$ and whiskering which contains both $J$ and the interchange law $I$. Then the quotient $EQ^J = EQ / =_{I+J}$ is well-defined, preserving both vertical composition and whiskering.

To conclude this section we observe the following lemma.

**Lemma 4.6 (Structure of $EQ_w$)**
Let $C_2(P)$ be the 2-category of rewrites for a monoid presentation $P$ of a monoid $M$. Then $EQ^I_w$, the set of classes of endorewrites on any string $w \in X^*$, is a $\mathbb{Z}M$-bimodule with respect to vertical composition and whiskering.

**Proof** Vertical composition of conjugacy classes of $EQ_w$ gives an abelian group structure: it is associative, with identity is $[1_w]_{I+J}$; the inverse of $[\alpha]_{I+J}$ is $[\alpha^{-1}]_{I+J}$; and if $\alpha_1, \alpha_2 \in EQ_w$ then

$$[\alpha_1]_{I+J} \cdot [\alpha_2]_{I+J} = [\alpha_1 \cdot \alpha_2]_{I+J} = [\alpha_2 \cdot \alpha_1]_{I+J} = [\alpha_2 \cdot \alpha_1 \cdot \alpha_2^{-1}]_{I+J} = [\alpha_1 \cdot \alpha_1]_{I+J}.$$

The left and right whiskering actions of $X^*$ on $EQ^I_w$ restrict to well-defined left and right actions of $M$ since $u_1 \alpha v_1 =_{I+J} u_2 \alpha v_2$, when $\theta(u_1) = \theta(u_2)$ and $\theta(v_1) = \theta(v_2)$ since:

$$u_1 \alpha v_1 = u_1 \cdot \alpha \cdot 1 v_1 =_{I+J} u_2 \cdot \alpha \cdot 1 v_2 = u_2 \alpha v_2.$$

\square

**Remark 4.7** Note that horizontal composition is not abelian: if $\alpha : w \to w$ and $\beta : z \to z$ then $\alpha \circ \beta : wz \to wz$ whilst $\beta \circ \alpha : zw \to zw$ and generally we cannot expect that $\theta(wz) = \theta(zw)$. 

9
5 Critical Pairs

In this chapter we shall prove the intuitively reasonable idea that all distinct circular routes come from examining the reconnection of non-trivial diverging paths and thus provide a method for identifying all the interesting endorewrites of any completable rewriting system. Our main result requires that we first identify exactly what we mean by ‘a generating set of endorewrites’.

A generating set for $EQ$ must be a set of endorewrites $E$ such that any other endorewrite of $EQ$ is equivalent under the interchange law together with the conjugacy congruence $=_{I+J}$, to a product of whiskered elements and inverse elements of $E$. Formally:

**Definition 5.1 (Generating Set for $EQ$)**

A generating set for the endorewrites $EQ$ associated with a monoid presentation is a set $E \subseteq EQ$ such that for any $\gamma \in EQ$ there exist $\alpha_1, \ldots, \alpha_n \in E$ such that

$$\gamma = _{I+J} u_1\alpha_1v_1 \cdots u_n\alpha_n v_n$$

for some $u_1, \ldots, u_n, v_1, \ldots, v_n \in X^*$ and $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$.

Our main theorem claims that a set of generating endorewrites $E$, can be produced from the critical pairs which result from overlaps of the completed rewriting system. In order to prove the theorem we use digraph arguments, a digraph being associated with each endorewrite coming from a critical pair in the following way:

**Lemma 5.2 (Digraphs associated with Endorewrites)**

Given two strings $w$ and $z$, any pair of logged reductions $\alpha_1, \alpha_2 : w \rightarrow z$ is represented by a labelled digraph which is associated uniquely with an endorewrite.

**Proof**

Let $\alpha_1, \alpha_2 : w \rightarrow z$ in $C_2$. Then we have a digraph $D(\alpha_1, \alpha_2)$, as shown, and the associated endorewrite $\delta(\alpha_1, \alpha_2) = \alpha_1\alpha_2^{-1} \in EQ_w$ can be obtained by reading the labels anticlockwise from the edges, beginning at the vertex which is greatest with respect to $>$ on $X^*$.

**Remark 5.3 (Resolved Critical Pairs Yield Endorewrites)**

If $C$ is the set of all logged critical pairs and $EQ$ is the set of all endorewrites of a complete logged rewriting system, then there is a map $\delta : C \rightarrow EQ$ associating an endorewrite with each critical pair. In detail, if $c = (\alpha_1, \alpha_2)$ is a logged critical pair then there is a string $w$ which may be rewritten in two ways $- \alpha_1 : w \rightarrow w_1$ and $\alpha_2 : w \rightarrow w_2$, where $\alpha_1, \alpha_2 \in C_2$. Since the pair can be resolved, there exists a string $z$ so that $\beta_1 : w_1 \rightarrow z$ and $\beta_2 : w_2 \rightarrow z$, for some rewrite sequences $\beta_1, \beta_2$ in $C_2$. It is immediate that $\delta(c) = \alpha_1 \beta_1 \beta_2^{-1} \alpha_2^{-1}$ is an endorewrite on $w$. 
We now observe that endorewrites resulting from critical pairs are trivial when the critical pair involves disjoint rules or a conjugate of the endorewrite resulting from the reduction of the minimal string on which the same overlap occurs.

**Lemma 5.4 (Overlaps and Endorewrites)**

If \( \alpha_1 : l_1 \rightarrow r_1 \) and \( \alpha_2 : l_2 \rightarrow r_2 \) are rules of a complete logged rewriting system \( \mathcal{L} \), such that they may both be applied to a string \( w \) then:

i) if the rules overlap on \( w \) then the endorewrite of the critical pair is equivalent to a whiskering of the endorewrite given by a resolution of the same pair of rules applied to the minimal string on which the same overlap occurs.

ii) if the rules do not overlap on \( w \) then resolution of the critical pair yields the trivial identity.

**Proof**

In case (i) the rules overlap on \( w \) so there exist \( u_1, v_1, v_2, x, y, z \in X^* \) such that \( w = xyz \) and either \( y = u_1l_1v_1 = l_2 \) or \( y = u_1l_1 = l_2v_2 \). In either case we can write \( y = u_1l_1v_1 = l_2v_2 \) and the logged reductions of \( y \) are \( u_1\alpha_1v_1 : y \rightarrow u_1r_1v_1 \) and \( \alpha_2v_2 : y \rightarrow r_2v_2 \). By completeness there are logged reductions \( \beta_1 : u_1r_1v_1 \rightarrow t \) and \( \beta_2 : r_2v_2 \rightarrow t \) such that \( \gamma = u_1\alpha_1v_1 \cdot \beta_1 \cdot \beta_2^{-1} \cdot \alpha_2^{-1}u_2 \) is an endorewrite. The critical pair of reductions on \( w \) are \( xu_1r_1v_1z \) and \( xo_2u_2z : w \rightarrow xr_2v_2z \). This pair can be resolved by \( x\beta_1z : xu_1r_1v_1z \rightarrow xt \) and \( x\beta_2z : xr_2v_2z \rightarrow xtz \). The endorewrite associated to it is \( x\gamma z \).

In case (ii) the rules do not overlap on \( w \) so there exist \( x, y, z \in X^* \) such that \( w = xl_1yl_2z \) and the logged reductions shown in the digraph on the right apply. This yields the endorewrite \( xo_1y_2z \cdot xr_1y_2z \cdot xl_1y_2z \cdot xo_1y_2z \), which is equivalent under the interchange law to \( lw \).

\[ \square \]

**Lemma 5.5 (Digraph of Reduction Sequences)**

For any critical pair of logged reduction sequences, there exists a finite digraph which is the union of digraphs resulting from resolving critical pairs as in Lemma 5.4.
**Proof** Given two logged reduction sequences $\alpha : w \rightarrow w_1 \rightarrow \cdots \rightarrow w_m \rightarrow z$ and $\alpha' : w \rightarrow w_{m+1} \rightarrow \cdots \rightarrow w_n \rightarrow z$, we define a digraph $D$. The vertices $V(D)$ are the distinct words occurring in these sequences, and there is an edge labelled $\alpha_i$ from $w_i$ to $w_j$ if $w_i \rightarrow w_j$ is a reduction step labelled by $\alpha_i$ in one of the two given reduction sequences. The pair of reduction sequences $(\alpha, \alpha')$ yield the endorewrite $\gamma = \delta(\alpha, \alpha')$ in the way described in Lemma 5.4. We now add to the graph (if the graph is drawn, this looks like subdivision into small confluence diagrams, the proof was originally phrased in ‘diamonds’). Note that the vertices are ordered with respect to $>$ in $X^*$.

**Algorithm 5.5 (Digraph Filling/Construction)**

\begin{itemize}
  \item **D1:** (Initialise) Given $D$ as defined above, set $V$ to be the set of vertices in $D$ and set $i = 1$.
  \item **D2:** (Select a Vertex) If $V$ is empty, go to step D7. Otherwise, set $v_i$ to be the maximum vertex in $V$ and remove $v_i$ from $V$.
  \item **D3:** (Test and Resolve) If the vertex is not the source of two distinct arrows in $D$ then discard it and go back to step D2. Otherwise, consider the corresponding two reductions $\beta_{i,1} : v_i \rightarrow v_{i,1}$ and $\beta_{i,2} : v_i \rightarrow v_{i,2}$. The critical pair $(\beta_{i,1}, \beta_{i,2})$ can be resolved since $L$ is a complete rewrite system so we have $\gamma_{i,1} : v_{i,1} \rightarrow z_i$ and $\gamma_{i,2} : v_{i,2} \rightarrow z_i$.
  \item **D4:** (Create New Digraph) Define $D_i$ to be the digraph

$$
\begin{array}{c}
\gamma_{i,1} & \rightarrow & v_{i,1} & \leftarrow & \beta_{i,1} \\
& & & & \\
\gamma_{i,2} & \rightarrow & v_{i,2} & \leftarrow & \beta_{i,2} \\
\end{array}
$$

  \item **D5:** (Add to Digraph) Add $D_i$ to $D$, identifying the vertices which have the same labels.
  \item **D6:** (Loop) Increment $i$ by 1 and go to step D2.
  \item **D7:** (Terminate) Output $D$.
\end{itemize}

We note firstly that $L$ is finite, so there are only finitely many rules which can be applied; secondly, any finite word can only be reduced in a finite number of ways; finally, the system is noetherian, so there are no infinite reduction sequences. This means that the procedure will terminate, giving a finite digraph $D$ which is the union of the digraphs $D_i$, which are all of the type considered in Lemma 5.4.

**Lemma 5.6 (Digraph Compositions)**
The product (at the base point) of the endorewrites associated (in the sense of Lemma 5.2) with the sub-digraphs is equivalent under the interchange law to the endorewrite associated with the original digraph.

**Proof**

Consider the composition of digraphs of the type described, remembering that each edge is associated uniquely to a particular log of the reduction. The endorewrites associated to the two digraphs are $\alpha_1 \cdot \gamma_1^{-1} \cdot \beta_1^{-1}$ and $\gamma_1 \cdot \alpha_2 \cdot \beta_2^{-1}$. Composing them from the base point $w$ gives us $\alpha_1 \cdot \gamma_1^{-1} \cdot \beta_1^{-1} \cdot \beta_1 \cdot (\gamma_1 \cdot \alpha_2 \cdot \beta_2^{-1}) \cdot \beta_1^{-1}$ which is equivalent in the sesquigroupoid to $\alpha_1 \cdot \alpha_2 \cdot \beta_2^{-1} \cdot \beta_1^{-1}$, the endorewrite given by taking the boundary of the composite. The fact that the order of the digraph endorewrites is not important corresponds with the fact that $EQ$ is abelian.

Combining Lemma 5.5 and Lemma 5.6 with Lemma 5.4 we can deduce that any digraph can be identified with a product of whiskered endorewrites and inverse endorewrites of $E$. This allows us to prove the main theorem:

**Theorem 5.7 (Critical Pairs give a Set of Generators for $EQ$)**

Let $L_{init}$ be the initial logged rewriting system for a monoid presentation, and let $L_{comp}$ be a completion. Let $C$ be the set of all logged critical pairs resulting from overlaps of the system $L_{comp} \cup L_{init}$. Then

$$E = \{ \delta(c) : c \in C \}$$

is a generating set of endorewrites.

**Proof** Let $\gamma$ be an endorewrite on some string $w$. Then consider the critical pair $(\gamma, 1_w)$. Using Algorithm 5.5 we can construct a digraph $D$ whose associated endorewrite is $\gamma$ and whose sub-digraphs yield a product of whiskered elements of $E$ and their inverses which is equivalent to $\gamma$ by Lemma 5.6.

**6 Example**

This small example illustrates our methods for computing a complete set of generators for the endorewrites of a monoid presentation from the overlaps of a complete logged rewriting system.

Consider the monoid presentation

$$\text{mon}\langle e, s \mid e^2 = e, s^3 = s, s^2 e = e, es^2 = e, sese = ese, eses = ese \rangle.$$
Using the short-lex ordering with $s > e$, labelling the relations $\alpha_1, \ldots, \alpha_6$ we have the complete logged rewriting system consisting of the following six rules:

$$
\alpha_1 : e^2 \rightarrow e, \quad \alpha_2 : s^3 \rightarrow s, \quad \alpha_3 : s^2 e \rightarrow e, \\
\alpha_4 : e^2 s \rightarrow e, \quad \alpha_5 : s e s e \rightarrow e s e, \quad \alpha_6 : e s e s \rightarrow e s e.
$$

Consider the overlap of $\alpha_2$ and $\alpha_3$ on the string $w = s^3 e$. Reducing it by $\alpha_2 e$ we get $se$ which is irreducible. Alternately, we can reduce $w$ by $sa_3$ and similarly get $se$. Thus we have an endorewrite of $se$, i.e. $\alpha_2 e \cdot sa_3^{-1}$. Continuing in this way, considering all the overlaps of the logged system the following twenty six endorewrites can be computed:

Endorewrites of $e$: $\alpha_2 s e \cdot s^2 a_3^{-1}$, $\alpha_1 s^2 \cdot \alpha_4 \cdot \alpha_1^{-1} \cdot e a_4^{-1}$, $\alpha_1 e \cdot e a_1^{-1}$, $\alpha_3 e \cdot \alpha_1 \cdot \alpha_3^{-1} \cdot s^2 a_1^{-1}$, $\alpha_3 s^2 \cdot \alpha_4 \cdot \alpha_3^{-1} \cdot s^2 a_4^{-1}$, $\alpha_4 s^2 \cdot e s a_2^{-1}$ and $\alpha_4 e \cdot e a_3^{-1}$.

Endorewrites of $s$: $\alpha_2 s^2 \cdot s^2 a_2^{-1}$.

Endorewrites of $s^2$: $\alpha_2 s \cdot s a_2^{-1}$.

Endorewrites of $es$: $\alpha_4 s \cdot e a_2^{-1}$.

Endorewrites of $se$: $\alpha_2 e \cdot s a_3^{-1}$.

Endorewrites of $ese$: $\alpha_1 s e s a_6 \cdot \alpha_4^{-1} \cdot s e \cdot e a_6^{-1}$, $s a_5 \cdot \alpha_5 \cdot a_3^{-1} s e$, $\alpha_3 s e s \cdot s^2 a_6^{-1}$, $\alpha_4 s e \cdot e s a_3^{-1}$, $\alpha_5 e \cdot s e s a_1^{-1}$, $\alpha_5 s^2 \cdot e s a_4 \cdot \alpha_5^{-1} \cdot s e s a_4^{-1}$, $\alpha_5 s \cdot \alpha_6 \cdot a_5^{-1} \cdot s a_6^{-1}$, $\alpha_5 s e s \cdot \alpha_6 s e \cdot e s a_1^{-1} \cdot s a_6^{-1} \cdot s e a_5^{-1} \cdot s e s a_6^{-1}$, $\alpha_6 s e \cdot \alpha_6 e \cdot e s a_3^{-1}$, $\alpha_6 s^2 \cdot e s a_4 \cdot \alpha_6^{-1} \cdot s e s a_1^{-1}$, $\alpha_6 s \cdot \alpha_6 e \cdot e s a_4^{-1}$, $\alpha_6 e \cdot e s a_1 \cdot a_5^{-1} \cdot s e a_5^{-1}$, $\alpha_6 e s \cdot e s a_1 \cdot a_6^{-1} \cdot e \cdot e s a_6^{-1}$ and $\alpha_5 s e \cdot e a_5 \cdot a_5^{-1} \cdot s a_1^{-1} \cdot s e a_5^{-1}$.

These endorewrites generate all possible endorewrites of the system, but we note that generating sets obtained in this way are unlikely to be minimal generating sets. For example, in this case there is a relation between the three endorewrites $\alpha_2 s \cdot s a_2^{-1}$, $\alpha_2 e \cdot s a_3^{-1}$, and $\alpha_2 s e \cdot s^2 a_3^{-1}$, in that the third can be obtained from the first two in the following way:

$$(\alpha_2 s \cdot s a_2^{-1}) e \cdot s(\alpha_2 e \cdot s a_3^{-1}) = \alpha_2 s e \cdot s^2 a_3^{-1}.$$ 

Unfortunately, the fact that this problem generalises the word problem for crossed modules means that reducing the generating set can be rather ad-hoc since there are no normal forms for the 2-cells.

### 7 Homotopical and Homological Interpretations

We promised, in the introduction, that our results would enable homotopical and homological specifications of the monoid. It is well known that the existence of a finite complete rewriting system for a monoid presentation implies the homological finiteness conditions $FP_3$ ([Squier, 1987]) and the stronger
condition $\text{FP}_\infty$ [Anick 1986; Kobayashi 1990] as well as the homotopical condition of having finite derivation type (FDT) [Cremanns 1995; Squier 1994].

The addition made by this paper, in considering logged rewriting systems, is that our algorithms enable the specification of the structures which the properties are based upon.

In the homotopical case, it is immediate to observe that the set $E$ of generating endorewrites suffices as a set of homotopy generators in the sense of [Cremanns 1995]. In detail: if $\alpha$ is any cycle of the graph whose objects are all strings and whose invertible edges are all rewrites, then $\alpha$ corresponds to the digraph of an endorewrite and it turns out that the product of the subdigraphs is homotopically equivalent to $\alpha$ for the same reasons as the associated endorewrite is equivalent to the composite of the endorewrites of the subdigraphs.

In terms of homology, the specification of $E$, similar to the analogous case of $\Pi_2$ for groups, enables us to construct a resolution. Specifically, we have an exact sequence of free, finitely generated $\mathbb{Z}M$-modules:

$$C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\delta_0} \mathbb{Z} \xrightarrow{0}.$$ 

Given our specification of a finite set of homotopy generators, further details of the resolution can be found in [Cremanns 1995] in the proof of the fact that FDT implies $\text{FP}_3$.

For lower dimensional topology and cohomological dimensions for monoids, Pride [Pride 1993; 1995; 1999] has developed geometric methods; using a calculus of pictures, with spherical pictures representing the relations between the relations, which may be identified with our endorewrites. His method for determining a generating set differs significantly from ours; involving picking an ‘obvious’ set of pictures and then using picture operations to prove that they generate all spherical pictures for the presentation. The key word here is ‘obvious’ – whether an obvious set of pictures can be identified depends upon the shape of the presentation and its relation to presentations for which generating sets of pictures are known. In the case of groups substantial research means that many shapes of presentation can be recognised, but in the case of monoids, presentations are less recognisable.

Our generating set of endorewrites is determined algorithmically, dependent on the successful completion of the presentation. The rewriting method has the clear advantage of being able to be applied like brute force in cases where the pictures are not obvious, or potentially in complex examples where the pictures may be too complex to be identified by eye. More interesting than comparing the two methods, however is to consider using them in combination – rewriting can provide an initial set of pictures for unrecognisable monoid presentations and picture calculus can then operate on the result to refine...
and reduce the set and present is as something more ascetically pleasing and expressive than the strings of letters representing whiskered 2-cells.

An alternative to looking at standard resolutions of a group by \( \mathbb{Z}G \)-modules as in \cite{Pride:1999} is to consider crossed resolutions. One reason for interest in these is because their stronger invariance with respect to the presentation makes them potentially more useful in the classification of topological structures such as knots via crossed resolutions of their intertwining monoids.

Recall the group case: a crossed complex (over groupoids) is a sequence

\[
\cdots \xrightarrow{\delta_{n+1}} C_n \xrightarrow{\delta_n} C_{n-1} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_2} C_2 \xrightarrow{\delta_1} C_1 \xrightarrow{\delta_0} C_0
\]

such that

i) \( C_1 \) is a groupoid with \( C_0 \) as its set of vertices and \( \delta^1, \delta^0 \) as its source and target maps.

ii) For \( n \geq 2 \), \( C_n \) is a totally disconnected groupoid over \( C_0 \) and for \( n \geq 3 \), the groups at the vertices of \( C_n \) are abelian.

iii) The groupoid \( C_1 \) operates on the right of each \( C_n \) for \( n \geq 2 \) by an action denoted \((x, a) \mapsto x^a\).

iv) For \( n \geq 2 \), \( \delta_n : C_n \to C_{n-1} \) is a morphism of groupoids over \( C_0 \) and \( C_1 \) acts on itself by conjugation.

v) \( \delta_n \delta_{n-1} = 0 : C_n \to C_{n-2} \) for \( n \geq 3 \) and \( \delta_2 \delta^0 = \delta_2 \delta^1 : C_2 \to C_0 \).

vi) If \( c \in C_2 \) then \( \delta_2(c) \) operates trivially on \( C_n \) for \( n \geq 3 \) and operates on \( C_2 \) by conjugation by \( c \).

A crossed complex \( C \) is free if \( C_1 \) is a free groupoid (on some graph \( \Gamma_1 \)) and \( C_2 \) is a free crossed \( C_1 \)-module (for some \( \lambda : \Gamma_2 \to C_1 \)) and for \( n \geq 3 \), \( C_n \) is a free \( \pi_1 C \)-module on some \( \Gamma_n \) where \( \pi_1 C \) is the fundamental groupoid of the crossed complex; i.e. the quotient of the groupoid \( C_1 \) by the normal, totally disconnected subgroupoid \( \delta_2(C_2) \).

A crossed complex \( C \) is exact if for \( n \geq 2 \)

\[
\text{Ker}(\delta_n : C_n \to C_{n-1}) = \text{Im}(\delta_{n+1} : C_{n+1} \to C_n).
\]

If \( C \) is an free exact crossed complex and \( G \) is a groupoid then \( C \) together with an isomorphism \( \pi_1 C \to G \) (or, equivalently, \( C \) with a quotient morphism \( C_1 \to G \) whose kernel is \( \delta_2(C_2) \)) is called a crossed resolution of \( G \). It is a free crossed resolution if \( C \) is also free.

In the case of monoids, we propose a similar structure. Let \( \mathcal{P} = \text{mon}(X, R) \) be a monoid presentation. If we can find a complete rewriting system for \( R \) then we can construct the following sequence:

\[
\cdots \xrightarrow{\delta_{n+1}} C_n \xrightarrow{\delta_n} C_{n-1} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} \text{tgt} \xrightarrow{\text{src}} C_1 \xrightarrow{\delta_1} C_0
\]
Define $C_0$ to be the monoid $M$ which is presented by $\mathcal{P}$. Define $C_1$ to be the free monoid $X^*$ and let $\delta_1 : C_1 \to C_0$ be the quotient morphism. Then let $\text{src}, \text{tgt} : C_2 \to C_1$ be the 2-category of rewrites, but instead of a right action of $C_1$ we have a two-sided action; instead of a crossed module $\delta_2 : C_2 \to C_1$ we have a 2-category $\text{src}, \text{tgt} : C_2 \to C_1$ and instead of $C_1$ being a groupoid, it is a category. Then let $C_3$ be a family of free $\mathbb{Z}M$-bimodules: its objects are the elements of $M$ and its arrows are of the form $\epsilon_1(m_1e_1n_1) + \epsilon_2(m_2e_2n_2) + \cdots \epsilon_k(m_ke_kn_k) : m \to m$ when $m_1e_1n_1 \cdot m_2e_2n_2 \cdot \cdots \cdot m_ke_kn_k$ is an endorewrite in $EQ_w$ for some $\theta(w) = m$. For higher levels $n > 3$ we can define $C_n$ to be the free $\mathbb{Z}M$-bimodule on a set of generators for $\text{Ker}(\delta_{n-1})$.

We find that $C$ is a crossed complex and we have maps $b_{i,j} : C_i \times C_j \to C_{i+j}$ - whiskering in the case of $C_0$ operating on the left and right of $C_i$ for $i > 0$. Then $C_1$ has 2 multiplications under the operations of $C_0$ which coincide only if $C_1$ is a monoid in the category of groupoids (interchange law). There are no inverses in dimension 0, but inverses at all higher levels. From the definitions we deduce exactness: $\text{Ker}(\delta_n) = \text{Im}(\delta_{n+1})$.

This appears to be identifiable with the structure of a crossed differential algebra, that is a crossed complex $C$ with a morphism $C \otimes C \to C$ which gives a monoid structure on $C$ (these are defined in detail in [Tonks, 1993]). We are still investigating how useful this enhanced style of resolution may be in the monoid case, so we won’t pursue the details of the construction further in this paper.

8 Generalised Logged String Rewriting

In [Brown and Heyworth, 2000] it was shown that the familiar string rewriting methods can be applied to problems of computing left Kan extensions over the category of sets. Structures such as monoid and category presentations, induced actions of groups and monoids, equivalence and conjugacy classes, equalisers and pushouts all turn out to be special cases of left Kan extensions over Sets and thus string rewriting methods can be applied to all these variations on the word problem.

Since string rewriting for Kan extensions can be achieved by embedding in a monoid, it is unnecessary to go through the detail of the sesquigroupoid whose 2-cells possess the structure for the logged rules. However, since we don’t need to embed in a monoid in order for the string rewriting methods to work, we briefly outline the alternative sesquigroupoid.

Let $(E, \epsilon)$ be the left Kan extension of the category action $X : A \to \text{Sets}$ along the functor $F : A \to B$. We assume that the data for the Kan extension is given as a finite presentation $\mathcal{P}$, consisting of generating graphs for $A$ and $B$, a set of relations for $B$ and the action of functors $F$ and $X$ being defined for every
object and arrow of the generating graph of $A$. The 2-category $C_2$ associated with the presentation of the Kan extension has 0-cells $(\bigsqcup_{A\in \text{Ob}A} X A) \sqcup \text{Ob}B$ and 1-cells $\{(s_x : x \to FA) \mid x \in X A, A \in \text{Ob}A\} \sqcup \text{Arr}B$. The 2-cells are the rewrites and inverse rewrites, with vertical composition as before, but clearly, whiskering and horizontal compositions are partial operations dependent on whether paths can be composed.

In conjunction with (Brown and Heyworth, 2000), this observation enables logged rewriting techniques to be applied to a wide range of problems, including category presentations, equivalence relations, induced actions, pushouts and coset systems. In each case, interpretations and potential applications of the endorewrites requires further investigation.

9 Implementations and Further Applications

Techniques of logged rewriting have been implemented by the first author as GAP functions which will eventually be submitted as a package. Applications of logged rewriting were explored in (Heyworth and Wensley, 1999) where the group version was implemented, providing a new algorithmic method for the construction of crossed resolutions of groups; in (Ghani and Heyworth, 2003) where the logged completion methods give an alternative to the Reidemeister-Schreier method of computing a subgroup presentation; and in (Brown et al, 2004) we show how endorewrites for double coset rewriting systems reveal information about the subgroups.

Further work could pursue other potential applications, including in Petri nets, concurrency and the analysis of knot quandles; as well as generalising the techniques to Gröbner bases where the endorewrites can be identified with syzygies.

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