Application of Dictionary Learning in Alleviating Computational Burden of EEG Source Localization

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Abstract—Two techniques are proposed to alleviate the computational burden of MULTiple Signal Classification (MUSIC) algorithm applied to Electroencephalogram (EEG) source localization. A significant reduction was achieved by parsing the cortex surface into smaller regions and nominating only a few regions for the exhaustive search inherent in the MUSIC algorithm. The nomination procedure involves a dictionary learning phase in which each region is assigned an atom matrix. Moreover, a dimensionality reduction step provided by excluding some of the electrodes is designed such that the Cramer-Rao bound of localization is maintained. It is shown by simulation that computational complexity of the MUSIC-based localization can be reduced by up to 80%.

Index Terms—Complexity Reduction, Dictionary Learning, Subspace Correlation, Cramer-Rao bound, MULTiple Signal Classification (MUSIC).

I. INTRODUCTION

Electroencephalograph (EEG) source localization is a non-invasive brain imaging technique suitable for real-time monitoring of cortical activity due to its high temporal resolution. Localization of cortical sources is used in analysing the brain signals for clinical applications such as estimation of seizure foci in epileptic patients [1, 2]. In fact, the equivalent current dipoles [3] represent the cortical activity within a small brain region and the number of active dipoles is assumed to be much smaller than the number of electrodes [8]. Thus the equivalent current dipole techniques posses an inherent sparse nature. These methods involve an exhaustive search over a 3 dimensional grid of hypothetical dipole locations and are suffering from extensive iterations and computational burden. To reduce the computational load, the FOcal Underdetermined System Solver (FOCUSS) [9] and Low Resolution Electromagnetic Tomography (LORETA) [10] are the most well-known sparsity imposing techniques. However, they lead to smeared regions of activity and lack the required spatial resolution to distinguish the adjacent active regions [6].

Most state-of-art techniques suffer from enormous iterations as well as high dimensionality due to large number of electrodes. To reduce the number of iterations, a wiser approach is to identify the cortical areas with lower probabilities of having active dipoles and exclude those areas from exhaustive search. In this regard, [17] tries to adaptively apply compressed sensing to the inverse problem. However it requires an assumption for the dipole orientation and its performance will degrade drastically with an incorrect assumption the dipole orientations.

This paper addresses the problem of high computational burden, which imposes a need for bulky and power-hungry processing blocks. The objective is to reduce the computational complexity aimed to show feasibility of a real-time and mobile VLSI implementation. Our proposed method parses the cortical surface into smaller regions and a dictionary learning phase is proposed to learn the atom vectors from each region. A few regions are nominated for exhaustive search based on the subspace correlation of their atom vectors with the measured signal. In addition, although it was shown that increasing the number of electrodes will improve the localization accuracy [11, 12], the computational complexity will grow quadratically with the number of electrodes. Thus, it is desirable to keep the number of electrodes minimum, while maintaining the localization accuracy. A mathematical model to represent trade-off between the localization accuracy and computational complexity due to the number of electrodes is still missing. In this work, we will introduce a mathematical framework to choose the best set of electrodes to compensate between the accuracy and the complexity.

This paper is outlined as follow. In section II the basics of the inverse problem and the challenges associated with its real-time implementation are discussed. The proposed candidate point reduction technique is brought in Section III. Section IV formulates the problem of electrode selection. The simulation results and the complexity analysis are argued in section V.

II. PRELIMINARIES

In the conventional equivalent current dipole model, the brain activity originates from a number of electric current dipoles [2], where the location, orientation, and the magnitude of each dipole plays a key role in determination of the recoded signal from the EEG electrodes. In fact, the position of the dipoles contains valuable information about brain functionality and has been exploited in several clinical applications and cognitive neuroscience studies. The ill-posed problem of estimating the locations and the moments of the current dipoles...
from the recorded EEG signal is called the inverse problem. The mathematical formulation of the inverse-problem and the associated presumptions are brought in the following sections.

### A. Data model

Considering the EEG signal recorded using an array of $M$ electrodes, the samples are denoted by an $M \times 1$ vector of $y(t) = [y_1(t), y_2(t), \ldots, y_M(t)]^T$, where $y_i(t)$ is the sample from the $i^{th}$ electrode at time $t$. Let such signal be originating from $N$ dipoles ($N \ll M$) at the surface of the cortex. Assume the $i^{th}$ dipole is associated with a fixed location vector $\mathbf{r}_i = [r^i_x, r^i_y, r^i_z]^T$ for $i = 1, \ldots, N$, where $r^i_x$, $r^i_y$, and $r^i_z$ are the Cartesian coordinates of the location. In addition, the orientation and the magnitude of the $i^{th}$ dipole are jointly quantified by a moment vector $\mathbf{s}_i(t) = [s_i^x(t), s_i^y(t), s_i^z(t)]^T$, where $s_i^x(t)$, $s_i^y(t)$, and $s_i^z(t)$ are the moments along the $x$, $y$, and $z$ axes, respectively. Assume three unit dipoles at location $\mathbf{r}_i$ orientated along the $x$, $y$, and $z$ axes. The response of such hypothetical unit dipoles on the $M$ electrodes are called the lead field vectors (LFV) and are denoted by $M \times 1$ vectors $g^x_i(\mathbf{r}_i)$, $g^y_i(\mathbf{r}_i)$, and $g^z_i(\mathbf{r}_i)$. Stacking up these three LFVs into an $M \times 3$ matrix, the lead field matrix (LFM) of the $i^{th}$ dipole is formed as

$$G_i(\mathbf{r}_i) = [g^x_i(\mathbf{r}_i), g^y_i(\mathbf{r}_i), g^z_i(\mathbf{r}_i)].$$  

(1)

The output of the $i^{th}$ dipole on $M$ EEG electrodes is the product of the LFM, $G_i(\mathbf{r}_i)$, and the dipole’s moment $\mathbf{s}_i(t)$. Considering all the $N$ active dipoles, the overall output is the superposition of the signals from all active dipoles as

$$y(t) = \sum_{i=1}^{N} G_i(\mathbf{r}_i)\mathbf{s}_i(t) + \mathbf{n}(t) = \mathbf{G}(\mathbf{r})\mathbf{s}(t) + \mathbf{n}(t),$$

(2)

where the $M \times 3N$ matrix $\mathbf{G}(\mathbf{r})$ is the total LFM defined as $\mathbf{G}(\mathbf{r}) = [G_1(\mathbf{r}_1), \ldots, G_N(\mathbf{r}_N)]$ with $\mathbf{r} = [\mathbf{r}_1, \ldots, \mathbf{r}_N]$. The $3N \times 1$ total moment vector is $\mathbf{s}(t) = [s_1^x(t), s_1^y(t), \ldots, s_N^x(t)]^T$, and the $M \times 1$ measurement noise vector $\mathbf{n}(t)$ is independently identically distributed random noise captured on the $M$ electrodes with a Gaussian distribution $N(0, \sigma_n^2 I_M)$.

The objective of the inverse-problem is to estimate the locations and the moments of the active dipoles using the output signal $y(t)$. The various solutions proposed in the literature for the inverse-problem can be divided into three categories of extended source scanning, tensor-based, and Bayesian approaches[2], among which the extended source scanning approach including the Linearly Constrained Minimum Variance (LCMV) and MUSIC have received considerable attention due to their accuracy in identifying closely located dipoles. In general, the solutions of the inverse-problem seek to minimize the regularized least square error as the following cost function

$$\mathbf{r}, \mathbf{S} = \arg \min_{\mathbf{r}, \mathbf{S}} ||\mathbf{Y} - \mathbf{G}(\mathbf{r})\mathbf{S}||_2 + \lambda f(\mathbf{S}),$$

(3)

where $\mathbf{Y}$ and $\mathbf{S}$ are $M \times T$ and $N \times T$ matrices generated by stacking up the $T$ time samples as $\mathbf{Y} = [y(1), y(2), \ldots, y(T)]$, and $\mathbf{S} = [s(1), s(2), \ldots, s(T)]$. $f(\mathbf{S})$ and $\lambda$ are the regularization term and its coefficient incorporating some constraint such as sparsity.

### B. Contributions

In the extended source scanning approach, the entire cortex surface is divided into a fine grid of candidate points as the hypothetical candidate locations of the active dipoles. Let $\mathcal{R} = \{\mathbf{r}_i\}_{i=1}^{N_c}$ denote the set of all candidate locations, where $N_c$ is the total number of candidate grid points. In the extended source scanning approach, Eqn. (3) is iteratively assessed for all $N_c$ grid points. Among all the grid points, the $N$ points minimizing the right hand side of Eqn. (3) are picked as the solution. According to [15], Eqn. (3) can be substituted by a generalized eigenvalue problem leading to a computational complexity of order $O(N_c M^2 N)$ for such an exhaustive search. In this paper, we will devise two techniques to alleviate the burden of the extended source scanning technique by reducing the number of candidate grid points $N_c$ and the number of electrodes $M$. Fig. [I] illustrates the steps involved in our proposed reduction techniques and the concept is briefly described as follows:

- The inverse problem is a sparse problem in the sense that the number of active dipoles is significantly less than the number of grid points across the cortex surface ($N \ll N_c$). Thus, a thorough scanning over all the $N_c$ points is not computationally efficient. A wiser approach is to parse the cortex surface into several areas and to nominate the areas with higher probability of including active dipoles for the exhaustive search. Such a strategy will significantly reduce the complexity of scanning by excluding a portion of the candidate points. In Section III the proposed candidate point reduction technique is clarified in details.

- Although it is shown that the accuracy of localization is improved by increasing the number of electrodes, the computational complexity will grow cubically with $M$. Thus, a new on-demand method is devised to pick the best set of $M_{sel}$ electrodes maintaining the localization accuracy, where $M_{sel}$ is the desired number of electrodes. Section IV expresses the electrode selection technique.

### III. REDUCING THE NUMBER OF CANDIDATE POINTS

The proposed candidate point reduction (CPR) technique is based on excluding a portion of the grid points across the cortex surface for the optimization search in Eqn. (3). The CPR technique comprises three steps. First, the entire cortex surface is broken into $M$ equiareal subregions, where $M$ is number of electrodes. The subsequent steps will nominate $L$ regions out of the the total of $M$ for the exhaustive search. The second step involves a novel dictionary learning algorithm called Orthogonal Component Extraction (OCE) which is devised to assign atom vectors to each region. Finally, the space correlations of the recorded signal subspace and the space spanned by the each region’s atom vectors are repeatedly assessed for all regions. The regions with the $L_r (L_r \geq N)$ highest subspace correlations are nominated for the exhaustive search of optimizing Eqn. (3). The choice of the number of nominated regions ($L_r$) affects the localization error and the details on how to adjust it will be discussed in section V.

As it is depicted in fig. VIthe CPR technique involves a sequence of three steps including clustering grid points into
equiareal regions, learning the dictionary from each regions, and nominating regions for optimization search. The details of these three steps is brought in the following.

A. Step1: Clustering Grid Point

The LFM for the hypothetical unit dipoles in the candidate locations can be derived using several approaches ranging from sophisticated methods such as Finite Element Method (FEM) \[18\] and Boundary Element Method (BEM) \[19\] to simple multi-layer spherical model \[20\]. Apart from the choice of volume conduction model, the following assumptions can be adopted \[21\]:

1-Any set of \( P < M \) LFVs are linearly independent.
2-LFVs of grid points spatially adjacent to each other are highly correlated.

Although in this study we will resort to the simple concentric 4-shell spheres as the volume conduction model \[20\], the above characteristics are common among all the existing approaches in the literature. According to the second property, if an LFM has a low subspace correlation with the output signal, one can conclude the LFVs of the adjacent points will also have a low correlation and thus can be removed from the exhaustive search. To promote the idea of eliminating the neighbouring points from exhaustive search, we shall parse the inner shell representing the cortex surface into \( M \) isosurface regions. The set of spatially adjacent grid point indices confined to the boundaries of the \( k^{th} \) region is represented by \( I_k \) and \( R_k \) is the set of their corresponding locations, where

\[
I_k = \{ 1 \leq i \leq N, \vec{r}_i \in R_k \}, \quad k = 1, \ldots, M. \quad (4)
\]

Stacking up the LFVs of all hypothetical unit dipoles belonging to the \( k^{th} \) region and oriented along the \( x \)-axis, a new \( M \times |I_k| \) lead field matrix \( G_k^x \) is defined, where \( |I_k| \) is the cardinality of the set \( I_k \). Columns of \( G_k^x \) are the lead field vectors corresponding to points located inside the \( k^{th} \) region, \( \vec{r}_i, i \in I_k \), and represented as \( \{ G_k^x (\vec{r}_i) \}_{i \in I_k} \). Similarly, \( G_k^y \) and \( G_k^z \) contain the LFVs of the hypothetical unit dipoles along the \( y \) and \( z \) axes in the \( k^{th} \) region. In Step 2, the newly defined LFMs will be used for learning the dictionary elements of each region.

B. Step 2: Dictionary Learning

The objective of this step is to learn a set of dictionary elements, where any LFV can be sparsely represented as a linear combination of these elements. We shall refer to the dictionary elements as atom vectors. The proposed learning technique, called the Orthogonal Component Extractor (OCE), will learn three atom vectors from each region. Let us denote the three atoms associated with \( k^{th} \) region by \( d_k^x, d_k^y, \) and \( d_k^z \). The \( M \times M \) dictionary matrices \( D^x, D^y, \) and \( D^z \) contain the atom vectors from all the regions and are defined as follows:

\[
D^x = [d_1^x, d_2^x, \ldots, d_M^x], \quad D^y = [d_1^y, d_2^y, \ldots, d_M^y], \quad D^z = [d_1^z, d_2^z, \ldots, d_M^z]. \quad (5)
\]

To simplify the notations, the equations that are held for all the three matrices are briefly indicated once by replacing those matrices with \( D^\ell, \ell \in \{x, y, z\} \). In the following the OCE method which is an iterative learning algorithm leading to orthonormal atom vectors will be described. In fact, the atom vectors obtained as the final result of the OCE has the following two characteristics:

- The atom vectors from different regions are orthonormal.
  \[
  D^\ell D^{\ell H} = I_M, \quad \ell \in \{x, y, z\}. \quad (6)
  \]

- The LFVs in each region can be written as a linear combination of the atom vectors. In other words, OCE seeks to minimize the second norm of the error for each region as follows:
  \[
  e_k^\ell = \min_{\Phi_k^\ell} ||G_k^\ell - D^\ell \Phi_k^\ell||^2, \quad \ell \in \{x, y, z\}, \quad k = 1, \ldots, M, \quad (7)
  \]
  where \( \Phi_k^\ell \) is the \( M \times |I_k| \) matrix of coefficients.

- The matrices of coefficients \( \Phi_k^\ell, \ell \in \{x, y, z\} \) are sparse. In other word, The columns of \( \Phi_k^\ell \) which are obtained by projecting the LFVs onto the atom vectors have only a few nonzero elements.

Algorithm \[1\] depicts the proposed OCE algorithm to derive the dictionary matrices \( D^\ell, \ell \in \{x, y, z\} \). First, the atom vectors are initialized using the principle components of the LFVs inside each region. Next, an iterative procedure is performed including two steps of estimating the residuals and deriving the principle component of the residuals, in which the dictionary atoms are updated. These two steps are brought in the following.

1) Derivation of the Residuals: The first step in learning the atom vectors is to define some \( M \times |I_k| \) matrices called the residuals and denoted as \( R_k^\ell, k = 1, \ldots, M \). The residuals are derived for each region specifically. Initially, the residuals of each region contain the LFVs of the associated region as its columns. Next, for each region a recursively applied projection will remove the subspaces spanned by the atoms of the other regions except for atoms of its own region. Mathematically formulating such a procedure, assume the dictionary matrices and the residual of the \( k^{th} \) region learned at \( j^{th} \) iteration be denoted by \( D^\ell(j) \) and \( R_k^\ell(j) \), respectively. Using the dictionary
matrices learned in the current iteration, \( D^t(j) \), the projection matrices \( P^t_k \), \( k = 1, \ldots, M \) \( \ell \in \{x,y,z\} \) are derived for each region as follows:

\[
P^t_k = I_M - [D^t(j)]_k [D^t(j)]_k^H [D^t(j)]_k^{-1} [D^t(j)]_k^H, \quad (8)
\]

where \([D^t(j)]_k\) is an \( M \times (M - 1) \) matrix obtained by removing the \( k^{th} \) column of \( D^t(j) \). Such projection will map the new residual, \( R^t_k(j + 1) \), onto a subspace orthogonal to atom vectors from all the other regions. Algorithm 2 describes this procedure in mathematical form.

2) Updating the Dictionary: The principle component of the residuals is derived for all the regions and is assigned as the updated atoms. Let the covariance matrix of the residuals, \( R \), for each region as follows:

\[
\text{Algorithm 1 Complete Dictionary Learning Algorithm}
\]

1: \( \bullet \) First iteration: Initialize the atom vectors
2: \( j = 1 \)
3: \( \text{for all regions } k = 1, \ldots, M \) do
   4: \( \text{Compute the principle component of LFVs:} \)
   5: \( Q^t_k(j) = 1/|k| R^t_k(j), \quad \ell \in \{x,y,z\} \)
   6: \( \text{Derive the covariance matrix of the LFVs} \)
   7: \( Q^t_k(j) = U^t_k \Lambda^t_k U^t_k^H, \quad \ell \in \{x,y,z\} \)
   8: \( \text{Factorize the covariance matrix using eigenvalue decomposition} \)
   9: \( d^t_k(j) = [U^t_k]_1 \)
10: \( \text{Assign the first principle} \)
11: \( j = j + 1 \)
12: \( \text{end for} \)
13: \( \bullet \) Repeat Until Convergence
14: \( j = j + 1 \)
15: \( \text{for all regions } k = 1, \ldots, M \) do
16: \( \text{Derive Residuals } R^t_k(j) \text{ using Algorithm 2} \)
17: \( \text{Update the Dictionary using the principle component of residuals} \)
18: \( Q^t_k(j) = 1/|k| R^t_k(j) R^t_k(j)^H \)
19: \( Q^t_k(j) = U^t_k \Lambda^t_k U^t_k^H, \quad \ell \in \{x,y,z\} \)
20: \( d^t_k(j) = [U^t_k]_1 \)
21: \( \text{end for} \)

\[
Q^t_k(j) = 1/|k| R^t_k(j) R^t_k(j)^H, \quad \ell \in \{x,y,z\}. \quad (9)
\]

Factoring \( Q^t_k(j) \) using eigenvalue decomposition leads to

\[
Q^t_k(j) = U^t_k \Lambda^t_k U^t_k^H, \quad (10)
\]

where \( U^t_k \) is the matrix of the eigenvectors and \( \Lambda^t_k \) is a diagonal matrix of eigenvalues \( \delta_1 > \delta_2 > \ldots > \delta_M \). The principal component of \( R^t_k(j) \) is the eigenvector corresponding to the largest eigenvalue \( \delta_1 \) denoted as \([U^t_k]_1\). This principle component will be assigned as the updated atom vector \( d^t_k(j + 1) \). Algorithm 2 illustrates updating the dictionary steps. Once the atom vectors are learned, the nomination of regions are carried out based on the subspace correlation of the signal with the atom vectors from each region. \( L \) regions with highest subspace correlation will be picked for exhaustive search. The successive iteration of the steps in Algorithm 1 will lead to orthogonality of the atom vectors. In fact, the orthogonality property will help in a situation where multiple active dipoles with different moment vectors are present. For instance, consider a case of two dipoles where one strong active dipole and another weak dipole are located randomly around the cortex surface. Since the LFVs from the adjacent regions are highly correlated, the orthogonality property will help prevent nominating the regions adjacent to the strong dipole. In fact, being orthogonal, the atom vectors of each region is offering a basis component that does not exists in the atoms from other regions. If an atom vector has a high correlation with the signal, such a component is present in the signal and that is a sign to nominate the corresponding region for exhaustive search. The details on how to find the subspace correlations are embodied in Step 3.

\[
\text{Algorithm 2 Deriving Residuals for Each Region}
\]

1: \( \text{for all regions } k = 1, \ldots, M \) do
2: \( \bullet \) The Initialization Phase:
3: \( R^t_k(j) = G^t_k, \quad \ell \in \{x,y,z\} \)
4: \( \bullet \) The Projection Phase:
5: \( P^t_k = I_M - [D^t(j)]_k [D^t(j)]_k^H [D^t(j)]_k^{-1} [D^t(j)]_k^H, \)
6: \( R^t_k(j) = P^t_k R^t_k(j), \)
7: \( \text{where } P^t_k \text{ is the projection matrix derived from atom vectors associated with } k^{th} \text{ region.} \)
4: \( \text{end for} \)

C. Step 3: Nominating The Regions

The final step in nominating the regions for exhaustive search includes finding the correlation of the signal subspace and the space spanned by the atom vectors. The definition of the subspace correlation is brought in the following. Assume two matrices \( A \) and \( B \) and let the subspaces spanned by their columns be denoted by \( \text{subspc}(A) \) and \( \text{subspc}(B) \), respectively. We are seeking two vectors \( a \in \text{subspc}(A) \) and \( b \in \text{subspc}(B) \) with the highest inner product as the correlation between these two subspaces as follows:

\[
c = \max_{a \in \text{subspc}(A)} \max_{b \in \text{subspc}(B)} \frac{|a^H b|}{||a||_2 ||b||_2} = 1 \quad (11)
\]

In order to find such vectors, one shall first factorize the matrices \( A \) and \( B \) using singular value decomposition as \( A = U_A \Sigma_A V_A^H \) and \( B = U_B \Sigma_B V_B^H \). It is shown that the highest singular value of \( C = U_A^H U_B = U_C \Sigma_C V_C^H \) is equal to the correlation of the two subspaces spanned by \( A \) and \( B \) \([22]\). The solution can also be derived using the generalized eigenvalue problem as \([13]\)

\[
\rho = \arg \max_{\phi} \frac{\phi^H U_A^H U_B U_B^H U_A \phi}{\phi^H \phi}, \quad (12)
\]

where \( U_A \) and \( U_B \) are formed by retaining the eigenvectors of \( U_A \) and \( U_B \) corresponding to nonzero eigenvalues. In the third step of nominating regions to reduce the candidate points, we
shall repeatedly compute the correlation between the signal subspace and the space spanned by atom vectors of each region. The covariance matrix of the signal is estimated as

$$R_y = \frac{1}{T} \sum_{t=0}^{T-1} y(t)y(t)^H,$$

where T is the number of time samples. Factorizing the covariance matrix, $R_y = U_y \Sigma_y V_y^H$. Let the singular values be ordered as $\delta_1 > \delta_2 > \ldots > \delta_M$. Define the new $M \times 3P$ matrix $\tilde{U}_y$ containing the eigenvectors of $R_y$ associated with the highest 3P eigenvalues. The subspace spanned by columns of $\tilde{U}_y$ is called the signal subspace. Let us define the $M \times 3$ matrix $D_k = [d_{k,1}, d_{k,2}, d_{k,3}]$ containing the atom vectors learned from the $k^{th}$ region. In order to nominate L regions, the following generalized eigenvalue problem assessed for all regions will give the subspace correlations as:

$$\rho_k = \max_{\phi_k} \frac{\phi_k^H D_k^H \tilde{U}_y \tilde{U}_y^H D_k \phi_k}{\phi_k^H \phi_k} \quad k = 1, \ldots, M,$$

where $\phi_k$ is a 3x1 vector of coefficients. Once all the subspace correlations are assessed for all the regions, the L regions with the highest $\rho_k$s will be nominated for exhaustive search.

IV. Reducing the Number of Electrodes

The objective of this section is to select the best set of $M_{sel}$ EEG electrodes out of the total of M, such that a certain localization performance is maintained. The localization performance is quantified using the error covariance matrix as the input and produces a scalar output. The choices for $f$ are the sum of eigenvalues, maximum eigenvalue, or the determinant of the error covariance matrix [23]. In this study, we resort to the determinant Cramer-Rao bound to quantify the localization accuracy. Thus, the problem of choosing the electrodes can be formulated as follows:

$$\bar{w} = \arg\min_{w} \lambda |M_{sel} - \|w\|_2| - \log \det \mathbb{E}\left(\left((\text{diag}(s(t))) \otimes I_3 \right) \mathbf{J}^H \mathbf{J} (\text{diag}(s(t))) \otimes I_3 \right)^H \right),$$

(23)

The first term is the Lagrangian function imposing the number of selected electrodes to be $M_{sel}$, while the second term compensates for the accuracy of localization. A gradient descent algorithm is used to solve the optimization problem in Eqn. (23). However, since elements of $\bar{w}$ are discrete binary values of 0s and 1s, we need to substitute $\bar{w}$ with a surrogate matrix $\bar{w}$ with continuous diagonal values using a sigmoid function as

$$\bar{w}_i(x_i) = \frac{1}{1 + e^{-\gamma x_i}}, \quad x_i \in \mathbb{R}, \quad i = 1, \ldots, M,$$

(24)

where $\gamma$ is a large positive number and $x_i$ is a continuous variable. As can be seen, for positively large values of $x_i$, $\bar{w}_i$ goes to 1, while for negatively large values of $x_i$, $\bar{w}_i$ goes to 0. Thus, $\bar{w}$ mimics the selecting vector $w$, while its elements are continuous and differentiable. From Eqn. (23), it is clear that we need to have a knowledge of the dipole moments and derivative of lead field vectors $\mathbf{g}_i(r)$, $i = 1, \ldots, M$ to solve such optimization problem and pick the best set of $M_{sel}$ electrodes. However, before doing the localization, the moment vectors, the lead field vectors, and their derivatives with respect to dipole locations are unknown. To tackle this problem, we shall resort to an approximation of the $s(t)$ and $\mathbf{J}$ based on the nominated regions obtained in Section III. In fact, $N$ regions with the highest subspace correlations $\rho_k$, $i = 1, \ldots, N$ in Eqn. (14) are picked. The average of the derivative of the lead field vectors inside those regions will be substituted to form an approximation for $J$ as following. Assume the case where there are $N$ active dipoles and atom vectors of regions.
Fig. 2: The spatial spectra derived from reciprocal of right hand side of Eqn. (3) using a generated EEG signal with SNR = 5 dB. (a) Regular MUSIC is applied to an EEG signal with one active dipole. (b) The reduced candidate technique is applied to the same signal in previous part. The number of nominated regions is \( L = 5 \). (c) Regular MUSIC for the case with two active dipoles. (d) The candidate reduction technique is applied to the signal in part (c), with the number of nominated regions being 5. (e) The electrode selection technique with the dipole moments and the dipole locations being known beforehand. The regular MUSIC is applied using the selected electrodes. (e) The proposed blind electrode selection technique.

Fig. 3: The sensor locations and labels used to assess the proposed candidate reduction technique. The selected electrodes are demonstrated in red, while the excluded ones are in white. (a) 59 electrodes. (b) 79 electrodes. (c) 107 electrodes.

\[ v_i, i = 1, \ldots, N \] have the \( N \) highest subspace correlation with the signal. The approximation of the Jacobian, \( \mathbf{J} \), is denoted by \( \mathbf{J} \) and is defined as

\[
\mathbf{J} = [\nabla_{v_1} \mathbf{G}_{v_1}, \ldots, \nabla_{v_N} \mathbf{G}_{v_N}]_{M \times 9N},
\]

where

\[
\frac{\partial \mathbf{g}^s_i}{\partial v_i} = \frac{1}{|I_i|} \sum_{j \in I_i} \nabla_i v_j \mathbf{g}^s_j,
\]

Likewise the \( s(t) \) is approximated by \( \bar{s}(t) \) by

\[
\bar{s}(t) = (\mathbf{G}^H \mathbf{G})^{-1} \mathbf{G}^H y(t),
\]

where \( \mathbf{G} = [\mathbf{G}_{v_1}, \ldots, \mathbf{G}_{v_N}] \) and

\[
\bar{G}_{v_i} = \frac{1}{|I_i|} \sum_{j \in I_i} \mathbf{G}_j.
\]

is the average of the all lead field matrices in region \( v_i \).

V. SIMULATION RESULTS

A. Performance Analysis

To assess the performance of the proposed techniques, the EEG was generated using the four-shell spherical model as the volume conduction in [20]. The radius of the brain and the skull are assumed to be 90mm and 94 mm respectively. The location of active dipoles were randomly chosen over the cortex surface. The dipole moments \( s_i(t) = [s_i^x(t), s_i^y(t), s_i^z(t)] \), \( i = 1, \ldots, N \) were assigned independently and identically distributed (i.i.d.) following the normal distribution as \( s_i^\ell \sim (0, 1), i = 1, \ldots, N, \ell \in \{x, y, z\} \). Thus, each active dipole is randomly oriented and its orientation and magnitude are changing over time. Next, the dipole moments are multiplied by the LFVs corresponding to their dipole locations and summed up to form the EEG signal. An i.i.d. white Gaussian noise with a variance specified by the signal to noise ratio (SNR) is added to the signal using \( \text{SNR} = \frac{\mathbb{E}[(s(t))^2]}{\mathbb{E}[(n(t))^2]} \). In Fig. 2(a), the spatial spectra of the regular MUSIC algorithm for an EEG signal with one active dipole is depicted. The spatial spectra is derived by finding the reciprocal of the right hand side of Eqn. (4) for all the grid points across the cortex. The EEG signal is emulated using 123 electrodes with 5dB of SNR. The actual location of the active dipole is marked by a white star, while the estimated location as the output of the regular MUSIC is denoted by a white square. The locations of the EEG electrodes are represented by black dots. Fig. 2(b) depicts the spatial spectra for the same signal in part (a). The proposed candidate reduction technique explained in Section III is applied, where the cortex surface is divided into 123 equiareal regions and \( L = 5 \) regions are nominated for the exhaustive search. The exhaustive search involves assessing Eqn. (3) for all the grid points inside the nominated regions. Fig. 2(c) and (d) are the spatial spectra of the regular MUSIC and the proposed candidate grid point reduction technique, respectively for an emulated EEG signal with two
picking the best knowledge of the active dipole moments and locations. After selection algorithm in Fig. 2(f), where we have no previous same EEG signal is used to assess the proposed blind electrode trodes to derive the spatial spectra as depicted in Fig. 2(e). The dots. The regular MUSIC is applied using the 30 selected elec-

active dipoles. Likewise, the EEG signal is generated using 123 electrodes with SNR = 5 dB. In part (d), \( L = 5 \) regions are picked for deriving the spatial spectra. As it is observed, the regions containing the active dipoles are well among the nominated regions.

In Fig. 4(e), the sensor selection optimization problem in Eqn. (23) is solved with the presumptive knowledge of the active dipole locations and moments. \( M_{sel} = 30 \) electrodes are picked among the 123 electrodes which are represented by pentagons while excluded electrodes are shown using black dots. The regular MUSIC is applied using the 30 selected electrodes to derive the spatial spectra as depicted in Fig. 2(e). The same EEG signal is used to assess the proposed blind electrode selection algorithm in Fig. 2(f), where we have no previous knowledge of the active dipole moments and locations. After picking the best \( L = 5 \) regions, the approximate matrices \( \mathbf{J} \) and \( \mathbf{s}(t) \) are derived using Eqn. (25) and Eqn. (28). 30 electrodes are picked out of the total of 123 electrodes. Similar to part (e), the included electrodes are represented by pentagons while the excluded ones are depicted by black dots in Fig. 2(f). The MUSIC algorithm is used to derive the spatial spectra inside the nominated regions using the selected electrodes. It is observed that the selected electrodes are mostly around the area close to the active dipole locations.

To assess the performance of the candidate reduction technique with proposed EEG, a Monte Carlo simulation is done where the EEG signal is generated using sets of 59, 79, and 107 electrodes. The location and label of employed electrodes are displayed in red in Fig. 3. The active dipoles are placed randomly across the cortex surface with random moment vectors following the normal distribution. Fig. 4, 5, and 6 show the performance in terms of mean squared error (MSE) with respect to the number of nominated regions for three cases of one \((N = 1)\), two \((N = 2)\), and three \((N = 3)\) active dipoles respectively. Three different SNR values of \(-5\), 0, and 5 dB are being investigated for the three sets of electrodes in Fig. 5. It is observed that the performance is improved by increasing the number of nominated regions \( L \), while this growth will also lead to an increase in computational load. In addition, a higher SNR will improve the MSE for the same set of employed electrodes and a fixed value of nominated regions \( L \). Moreover, increasing the number of electrodes and consequently the number of regions, will decrease the MSE. For example in fig. 5 for the signal with SNR=−5 dB, the performance will degrade as the number of electrodes changes from 107 to 79, and to 59. Fig 7 compares the MSE of the candidate reduction technique with the regular MUSIC. The performances of regular MUSIC for 59 EEG electrode set for three values of SNR = −5 dB, = 0 dB, and = 5 dB are displayed with colored solid lines as well as the performance for the proposed candidate reduction technique in black. The number of active dipoles is two \((N = 2)\). It is observed that by increasing \( L \), the performance will eventually approach those of the regular MUSIC. Thus, the \( L \) should be adjusted to have a feasible compensation between the accuracy of
The advantage is that some steps are carried out off-
sensor selection techniques involves some extra steps at the
implementing the reduction techniques including CPR and
of selected electrodes, while the computational load grows
Likewise, the performance improves by increasing the number
that comes
Fig. 8 compares the performance of the random sensor
localization and computational complexity. Fig. 8 compares
mean squared error with respect to the number of selected
electrodes for EEG signal with two active dipoles and two
different values of SNRs. The performance of the randomly
selected electrodes is marked by squares, while our proposed
blind electrode selection technique is marked by triangles.

**B. Complexity Reduction Analysis**

According to performance analysis, the MSE is improved
and tends to the values of regular MUSIC algorithm as the
number of candidate regions grows. However, that comes
at the cost of linearly increasing computational complexity.
Likewise, the performance improves by increasing the number
of selected electrodes, while the computational load grows
quadratically with the number of electrodes. On the other hand,
implementing the reduction techniques including CPR and
dictionary selection techniques involves some extra steps at the
beginning. The advantage is that some steps are carried out off-
line and only once. The results will be stored to be used later.
For example, among the three steps of the CPR, the clustering
grid point step and the learning dictionary step will be done
only once. The same learned atom vectors are used to nominate
regions every time a new data is going to be analyzed. Thus, the
learned tom vectors are stored to be used in real time.

In addition in the sensor selection technique, \( \mathbf{G} \) and \( \mathbf{J} \) are
estimated once from Eqn. (29) and Eqn. (25) and are stored to
be used later in the sensor selection algorithm. However every
time a new data is fed in, the following steps are performed
in real-time to implement the proposed reduction techniques.
First, \( L \) regions are nominated. The computational complexity
of the nomination step which involves the eigenvalue problem
of Eqn. (14) is of order \( O(M^3) \), where \( M \) is the initial number
electrodes. Next, \( S(t) \) is estimated using Eqn. (28) with a
computational complexity of \( O(L^2 M) \). The sensor selection
algorithm picks \( M_{\text{sel}} \) electrodes by iteratively optimizing
Eqn. (23) with a complexity of order \( O(M^2 N_c^2) \). Once \( L \)
regions are nominated and \( M_{\text{sel}} \) electrodes are picked, the
regular MUSIC is applied to the nominated regions which is
shown to have a computational complexity of \( O(M_{\text{sel}} N_c L) \).
Since \( M_{\text{sel}} L < M \ll N_c \), the final step has the highest
contribution to the complexity. In fact considering the total
complexity, the load of other steps can be ignored. In addition
we have \( O(M_{\text{sel}}^2 N_c^2 N_e^2) < O(M^2 N_c^2) \) which means our
proposed algorithm can reduce the complexity significantly
depending on the choice of \( L \) and \( M_{\text{sel}} \). Table I shows the
computational complexity of each step and compares them to
regular MUSIC algorithm. The number of nominated regions
and the number of selected electrodes should be chosen to
have a balance between the performance and computational
burden. Here, the Alkai Information Criterion (AIC) is used
to quantify the tradeoff as

\[
AIC = \log(MSE) + \lambda_L L + \lambda_{M_{\text{sel}}} M_{\text{sel}}. \tag{30}
\]

where \( \lambda_L \) and \( \lambda_{M_{\text{sel}}} \) are some constants. Fig. 9
depicts the AIC derived for \( \text{SNR} = -5, 0, 5 \) dB (denoted by different marks on the diagram) and three EEG electrodes sets in fig 4
with \( M_{\text{sel}} = 59, 79, 107 \). The trade off between the complexity
and performance can be seen in all the EEG signals. The nadir
of the AIC can be picked as the best choice for the value of
\( L \). Based on the average AIC depicted by a black solid line
in fig 9, \( L \) is chosen to be 16.

In order to quantify the reduction analysis, the reduction rate
(RR) is defined as follows:

\[
RR = \frac{\text{The number of investigated grid points}}{\text{The total number of grid points across the cortex}} \times \frac{M_{\text{sel}}}{M}. \tag{31}
\]

Fig. 10 shows the RR for the three EEG electrode sets in Fig. 4
\( (M = 59, 79, 107) \), with \( L = 16 \) and \( M_{\text{sel}} = 30 \). It is observed
that the computational complexity is almost one-fifth of the
regular MUSIC.

**C. Real EEG Experiment**

We used the auditory mismatch negativity provided by
center for neuroimaging in University College London [24].
The EEG data is a part of an experiment auditory oddball...
TABLE I: COMPARISON OF THE COMPUTATIONAL COMPLEXITY, REDUCED MUSIC VERSUS REGULAR MUSIC

| Algorithm       | Covariance matrix calculation | principle component extraction | Nominating L regions | Selection of Electrodes | Evaluating the cost function |
|-----------------|-------------------------------|--------------------------------|---------------------|-------------------------|-----------------------------|
| Proposed CPR and Sensor selection | $O(TM^2)$ | $O(M^3)$ | $O(N^2M)$ | $O(L^2M)$ | (with $L>N$) $O(L^2M)$ |
| Regular MUSIC   | $O(TM^2)$ | $O(M^3)$ | -        | -          | $O(M^2N_{sel}N_c)$        |

Fig. 9: The AIC with respect to number of nominated regions for EEG signals with three different SNR values, SNR = $-5$, 0, and 5 dB distinguished using different marks and three sets of electrodes in Fig. 3 depicted in different colors, $N = 2$. The average of all the AICs is illustrated by a black solid line.

Fig. 10: The reduction rate for the choice of $L = 16$ and $M_{sel} = 30$

EEG signal using 128 scalp electrodes and 128 Hz of sampling frequency. According to [25], the experiment involves a subject listening to two different tones of standard and deviant with 500Hz and 550Hz of frequencies respectively. The tones occur in a random sequence with 20% of deviant tone events (120 events) and 80% of standard tone events (480 events). Each event lasts 1.5 seconds. In order to reduce the noise, the rare EEG signal is averaged over all rare events to obtain a single 1.5 second of auditory oddball EEG. The 4 shell concentric spheres model proposed in [20] with brain, cerebrospinal fluid, skull, and scalp is used as the volume conduction model. The cortex radius is assumed to be 90mm and the electrode positions provided in [24] are mapped onto the surface of scalp modeled by a sphere with radius of 94mm. Next, the candidate reduction technique proposed in section III was applied. Based on the results from complexity analysis, $L = 15$ regions were nominated to keep a balance between computational complexity and performance. Fig. 11(a) depicts the top view of the spatial spectra obtained by applying the regular MUSIC, while its side view from left is depicted in fig. 11(b). The spatial spectra of the reduced MUSIC for the top and side views are depicted in fig. 11(c) and (d). According to [26, 27], the auditory oddball signal is originating from bilateral superior temporal lobe, inferior parietal lobe, hippocampus, and thalamus.

VI. CONCLUSION

The two techniques proposed in this paper reduce the complexity burden of the MUSIC-based EEG source localization by up to 80%. Such reduction will make the real-time, low-power, compact hardware implementation of the EEG source localization feasible to be used in long-term monitoring of the brain activity.

Fig. 11: (a) A top view of the spatial spectra of the auditory oddball derived using regular MUSIC source localization. (b) View of the back side (c) Proposed reduced MUSIC algorithm, top view (d) reduced MUSIC, view from back side

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