A new bound for Vizing’s conjecture

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Abstract. For any graph $G$, we define the power $\pi(G)$ as the minimum of the largest number of neighbors in a $\gamma$-set of $G$, of any vertex, taken over all $\gamma$-sets of $G$. We show that $\gamma(G \square H) \geq \frac{\pi(G)}{2\pi(G) - 1} \gamma(G) \gamma(H)$. This implies that for any graphs $G$ and $H$, $\gamma(G \square H) \geq \frac{\gamma(G)}{2\gamma(G) - 1} \gamma(G) \gamma(H)$, and if $G$ is claw-free or $P_4$-free, $\gamma(G \square H) \geq \frac{2}{3} \gamma(G) \gamma(H)$, where $\gamma(G)$ is the domination number of $G$.

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1. Introduction

The famous conjecture of Vadim G. Vizing (1963) \cite{vizing} is the simple statement for any two graphs $G$ and $H$,

\begin{equation}
\gamma(G \square H) \geq \gamma(G) \gamma(H).
\end{equation}

The survey \cite{survey} discusses many past results and contemporary approaches to the problem. For more recent partial results see \cite{suen2010}, \cite{tarr}, \cite{bresar}, \cite{brause}, and \cite{brause2}. The best current bound for the conjectured inequality was shown in 2010 by Suen and Tarr \cite{suen2010},

\begin{equation}
\gamma(G \square H) \geq \frac{1}{2} \gamma(G) \gamma(H) + \frac{1}{2} \min\{\gamma(G), \gamma(H)\}
\end{equation}

In this paper we define the power of a graph $\pi(G)$ and apply the Contractor-Krop overcount technique \cite{brause} to the method of Brešar \cite{bresar} to show that for any graphs $G$ and $H$, $\gamma(G \square H) \geq \frac{\pi(G)}{2\pi(G) - 1} \gamma(G) \gamma(H)$, which immediately implies the bound $\gamma(G \square H) \geq \frac{\gamma(G)}{2\gamma(G) - 1} \gamma(G) \gamma(H)$. By results of \cite{brause} and \cite{brause2}, it follows that if $G$ is claw-free graph or a cograph, for any graph $H$, $\gamma(G \square H) \geq \frac{2}{3} \gamma(G) \gamma(H)$.

Our argument relies on bounding the horizontal domination of vertically undominated cells and is a generalization of the argument in \cite{brause2}. We hope that others will find our approach valuable as our method is quite different from the “double projection” argument of \cite{suen2010}.
1.1. Notation. All graphs $G(V,E)$ are finite, simple, connected, undirected graphs with vertex set $V$ and edge set $E$. We may refer to the vertex set and edge set of $G$ as $V(G)$ and $E(G)$, respectively. For more on basic graph theoretic notation and definitions we refer to Diestel [6].

For any graph $G = (V,E)$, a subset $S \subseteq V$ dominates $G$ if $N[S] = G$. The minimum cardinality of $S \subseteq V$, so that $S$ dominates $G$ is called the domination number of $G$ and is denoted $\gamma(G)$. We call a dominating set that realizes the domination number a $\gamma$-set.

The Cartesian product of two graphs $G_1(V_1,E_1)$ and $G_2(V_2,E_2)$, denoted by $G_1 \square G_2$, is a graph with vertex set $V_1 \times V_2$ and edge set $E(G_1 \square G_2) = \{((u_1,v_1),(u_2,v_2)) : v_1 = v_2 \text{ and } (u_1,u_2) \in E_1, \text{ or } u_1 = u_2 \text{ and } (v_1,v_2) \in E_2\}$.

A graph $G$ is claw-free if $G$ contains no induced $K_{1,3}$ subgraph, and a cograph or $P_4$-free if it contains no induced $P_4$ subgraph.

If $D = \{v_1, \ldots, v_k\}$ is a minimum dominating set of $G$, then for any $i \in [k]$, define the set of private neighbors for $v_i$, $P_i = \{v \in V(G) - D : N(v) \cap D = \{v_i\}\}$. For $S \subseteq [k]$, $|S| \geq 2$, we define the shared neighbors of $\{v_i : i \in S\}$, $P_S = \{v \in V(G) - D : N(v) \cap D = \{v_i : i \in S\}\}$.

For $i \in [k]$, let $Q_i = \{v_i\} \cup P_i$. We call $Q = \{Q_1, \ldots, Q_k\}$ the cells of $G$. For any $I \subseteq [k]$, we write $Q_I = \bigcup_{i \in I} Q_i$ and call $\mathcal{C}(\bigcup_{i \in I} Q_i) = \bigcup_{i \in I} Q_i \cup \bigcup_{S \subseteq I} P_S$ the chamber of $Q_I$. We may write this as $\mathcal{C}_I$.

For a vertex $h \in V(H)$, the $G$-fiber, $G^h$, is the subgraph of $G \square H$ induced by $\{(g,h) : g \in V(G)\}$.

For a minimum dominating set $D$ of $G \square H$, we define $D^h = D \cap G^h$. Likewise, for any set $S \subseteq [k]$, $P_S^h = P_S \times \{h\}$, and for $i \in [k]$, $Q_i^h = Q_i \times \{h\}$. By $v_i^h$ we mean the vertex $(v_i,h)$. For any $I \subseteq [k]$, we write $\mathcal{C}_I^h$ to mean the chamber of $Q_I^h$.

We may write $\{v_i : i \in I\}$ for $\{v_i^h : i \in I^h\}$ when it is clear from context that we are talking about vertices of $G \square H$ and not vertices of $G$.

For $i \in [k]$ and $h \in V(H)$, we say that the cell $Q_i^h$ is vertically dominated if $Q_i^h \cap D \neq \emptyset$. A cell which is not vertically dominated is vertically undominated. Any vertex $v \in G \times H$ is vertically dominated if $\{v\} \times N_H[h] \cap D \neq \emptyset$ and vertically undominated, otherwise.

Recently, Chellali et al. [4] considered uniformly restricted types of dominating sets. For any graph $G$ and subset of vertices $S$, they defined $S$ to be a $[j,k]$-set if for every vertex $v \in V - S$, $j \leq |N(v) \cap S| \leq k$. For $k \geq 1$, the $[1,k]$-domination number of $G$, written $\gamma_{[1,k]}(G)$, is the minimum cardinality of a $[1,k]$-set in $G$. A $[1,k]$-set with cardinality $\gamma_{[1,k]}(G)$ is called a $\gamma_{[1,k]}(G)$-set.

Among other results, they showed that if $G$ is a claw-free, then $\gamma(G) = \gamma_{[1,2]}(G)$, and that the same result holds if $G$ is $P_4$-free.

**Definition 1.1.** For a fixed $\gamma$-set $D$ of $G$, the allegiance of $D$ with respect to $G$, $a_G(D) = \max_{v \in V(G)} \{|D \cap N[v]|\}$.

**Definition 1.2.** The power of a graph $G$, $\pi(G) = \min_D \{a_G(D)\}$ taken over all $\gamma$-sets $D$ of $G$.

Notice that the power of a graph $G$ is the minimum $k$ so that $\gamma_{[1,k]}(G) = \gamma(G)$. 
1.2. A Useful Inequality. Although the following inequality is elementary, we provide the proof for completeness.

**Proposition 1.3.** If

$$f(t_1, \ldots, t_n) = \sum_{i=1}^{n} i \times t_i$$

subject to

$$\sum_{i=1}^{n} t_i = 1 \text{ and } t_1 \geq \sum_{i=2}^{n} (i - 1)t_i,$$

for real valued $t_i, 1 \leq i \leq n$, then

$$f(t_1, \ldots, t_n) \leq \frac{2n - 1}{n}$$

and equality is attained when $t_i = 0$ for $1 < i < n$.

**Proof.** We induct on $n$. If $n = 2$, notice that $t_1 = 1 - t_2$, which means that $f(t_1, t_2) = 1 + t_2$. Since $t_2 \leq 1 - t_2$ we see that $t_2 \leq \frac{1}{2}$ which promptly implies $f(t_1, t_2) \leq \frac{3}{2}$.

Suppose the statement true for $n \leq k - 1$. We show it true for $n = k$. Let $t_1 + \cdots + t_{k-1} = s$. Then by the induction hypothesis, $\frac{1}{s} \sum_{i=1}^{k-1} (i \times t_i) \leq \frac{2k-3}{k-1}$ and equality is achieved when $t_2 = \cdots = t_{k-2} = 0$. Hence, $\sum_{i=1}^{k-1} (i \times t_i)$ is maximized when $t_2 = \cdots = t_{k-2} = 0$.

We consider the resulting expression, $g(t_1, t_{k-1}, t_k) = t_1 + (k-1)t_{k-1} + kt_k$ subject to $t_1 + t_{k-1} + t_k = 1$ and $t_1 \geq (k-2)t_{k-1} + (k-1)t_k$.

Notice that $t_1 = 1 - t_{k-1} + t_k$ which we can substitute into the constraining inequality to obtain $t_{k-1} \leq \frac{1-kt_k}{k-1}$. Furthermore, since $t_1 + t_k \leq 1$ and $t_1 \geq (k-1)t_k$, we see that $t_k \leq \frac{1}{k}$.

Note that

$$g(t_1, t_{k-1}, t_k) = 1 - t_{k-1} - t_k + (k-1)t_{k-1} + kt_k$$

$$= 1 + (k-2)t_{k-1} + (k-1)t_k$$

$$\leq 1 + (k-2)\left(\frac{1-kt_k}{k-1} + (k-1)t_k\right)$$

$$= \frac{2k-3}{k-1} + \frac{1}{k-1}t_k$$

$$\leq \frac{2k-3}{k-1} + \frac{1}{k-1}$$

$$= \frac{2k-1}{k}.$$  

\[ \square \]

2. A New Bound

**Theorem 2.1.** For any graphs $G$ and $H$,

$$\gamma(G \Box H) \geq \frac{\pi(G)}{2\pi(G) - 1} \gamma(G)\gamma(H).$$
Proof. For any graphs $G$ and $H$, let $\Gamma = \{v_1, \ldots, v_k\}$ be a minimum dominating set of $G$ and $D$ be a minimum dominating set of $G \sqcup H$.

Our proof is composed of increasingly refining labelings of the vertices of $D$. In all instances, for any $S \subseteq [k]$, if $v \in P_S$, then $v$ may only be labeled by a subset of $S$. For example, if $v \in P_{i,j}$, then $v$ may be labeled by $i$, $j$, or $\{i,j\}$. We call labelings that follow this property faithful. For any fixed label $i$, we project vertices that contain $i$ in their label onto $H$ and produce a dominating set of $H$. We show a bound on the label overcount to produce the desired inequality.

For any $h \in V(H)$, suppose the fiber $G^h$ contains $\ell_h(= \ell)$ vertically undominated cells $\{Q_{i_1}^h, \ldots, Q_{i_\ell}^h\}$ for $0 \leq \ell \leq k$. We set $I^h = \{i_1, \ldots, i_\ell\}$.

We apply the procedure Labeling 1 to the vertices of $D$. For $v \in D^h \cap Q_i^h$ for any $h \in V(H)$ and $1 \leq i \leq k$, we label $v$ by $\{j : v \in N[v_j], j \in [k]\}$. If $v \in D^h$ is a shared neighbor of some subset $S$ of $\{v_i : i \in I^h\}$, then it is a member of $P_S^h$ for some $S \subseteq I^h$, and we label $v$ by $S$. If $v \in D^h$ is a member of $P_S^h$ where $S = R \cup T$ for nonempty $R \subseteq I^h$ and $T \subseteq ([k] - I^h)$, then we label $v$ by $R$. This completes Labeling 1.

We relabel the vertices of $D$, doing so in $D^h$ for fixed $h \in H$, stepwise, until we exhaust every $h \in V(H)$. This procedure, which we call Labeling 2, is described next.

For every $h \in V(H)$, we list the labels of vertices of $D^h$, and write them in row $h$. This produces a two-dimesional array of $|H|$ rows of labels, some of which may be empty. For an arbitrary $h \in V(H)$, we perform two alterations to the labels in row $h$ which we call the internal and external alterations. In each of these procedures we make the exception which we denote the dominion rule: if $v_i^h \in D^h$ with label $S$, then any alteration of $S$ must retain the label $i$.

We perform the internal alteration,

(1) For every pair of labels $S$ and $T$ in row $h$, if $|S \cap T| > 1$, then remove one common element from $S$ and another from $T$, arbitrarily, subject to the dominion rule. Repeat this step.
(2) If $|S| = 1$, $|T| > 1$, and $S \cap T \neq \emptyset$, then remove the label of $S$ from $T$, following the dominion rule.
(3) If $|S| = |T| = 1$, then make no changes to $S$ or $T$.
(4) Otherwise, if $|S| > 1$, $|T| > 1$, $|S \cap T| = 1$, then remove the common element from one of $S$ or $T$ arbitrarily, subject to the dominion rule.

We repeat this internal alteration for every row $h \in V(H)$ until every pair of labels in a row is a pair of singletons or mutually disjoint.

We perform the external alteration to the array obtained from the internal alteration. Choose any $h \in V(H)$ and suppose $N(h) = \{h_1, \ldots, h_n\}$. For every label $S$ in row $h$, we consider labels $T$ of row $h_i$ for $i = 1, \ldots, n$, and repeat the relabeling from the internal alteration,

(1) Set $i = 1$. 
(2) For every label \( S \) in row \( h \) and \( T \) in row \( h_i \), if \( |S \cap T| > 1 \), then remove one common element from \( S \) and another from \( T \), arbitrarily, subject to the dominion rule. Repeat this step.

(3) If \( |S| = 1 \), \( |T| > 1 \), and \( S \cap T \neq \emptyset \), then remove the label of \( S \) from \( T \), following the dominion rule.

(4) If \( |S| > 1 \), \( |T| = 1 \), and \( S \cap T \neq \emptyset \), then remove the label of \( T \) from \( S \), following the dominion rule.

(5) If \( |S| = |T| = 1 \), then make no changes to \( S \) or \( T \).

(6) Otherwise, if \( |S| > 1 \), \( |T| > 1 \), and \( |S \cap T| = 1 \), then remove the common element from one of \( S \) or \( T \) arbitrarily, subject to the dominion rule.

(7) Let \( i = i + 1 \) and repeat this relabeling until \( i = n + 1 \).

After all alterations are performed for every row \( h \in V(H) \), we confer the labels in the rows to the corresponding vertices of \( D \). This completes Labeling 2.

Define the index set \( I^h_I = [k] - I^h = \{i_{i+1}, \ldots, i_k\} \) for vertically dominated cells of \( G^h \). We relabel those vertices of \( D^h \cap \mathcal{C}^h_{I^h} \) which are shared neighbors of \( \{v^h_i : i \in I^h_I\} \) so that all labels on these vertices are singletons and the labeling remains faithful. We call this procedure Labeling 3. For any \( h \in V(H) \), if \( v \in D^h \cap \mathcal{C}^h_{I^h} \) is a shared neighbor of some vertices of \( \{v^h_i : i \in I^h_I\} \), with label \( S \), then choose any element of \( S \) and label \( v \) by that element. Repeat this procedure for every \( h \in V(H) \). This completes Labeling 3.

For \( h \in V(H) \), let \( S^h_i \) be the vertices of \( D^h \) which have labels with more than one element. That is, vertices \( v \) labeled by some \( X \) so that \( |X| > 1 \). Say \( |S^h_i| = s \) and \( S^h_i = \{x_1, \ldots, x_s\} \). Let \( m_i \) be the cardinality of the label on \( x_i \in S^h_i \). For each vertex in \( S^h_i \), we place each element from the label on that vertex in the set \( J^h_i \). For example, if \( S^h_i \) contains vertices with labels \( \{i_1, i_2\} \) and \( \{i_3, i_4\} \), then \( J^h_i = \{i_1, i_2, i_3, i_4\} \).

**Claim 2.2.** \( \mathcal{C}_j^h \) is dominated by vertices of \( D^h \).

**Proof.** Suppose \( v \in \mathcal{C}_j^h \) is dominated by \( u \in D^{h'} \) for some \( h' \in N_H(h) \).

Let \( S \) be the label on \( u \). Since Labeling 2 has been performed, for any vertex \( w \in S^h_i \) with labels \( T \), \( S \cap T = \emptyset \) unless \( |S| = |T| = 1 \). If \( |T| = 1 \), then \( w \notin S^h_i \), contradicting our assumption. Otherwise, elements of \( S \) are not in \( J^h_i \) leading to \( v \notin \mathcal{C}_j^h \) which is a contradiction. \( \square \)

Set \( D^h_I = D^h - (D^h \cap \mathcal{C}_j^h) \). By Claim 2.2 we can let \( E^h_{J^h_I} \) be a minimum subset of vertices of \( D^h_I \) so that \( (D^h \cap \mathcal{C}_j^h) \cup E^h_{J^h_I} \) dominates \( \mathcal{C}_j^h \). That is, \( E^h_{J^h_I} \) is a set of vertices with neighbors in \( \mathcal{C}_j^h \), which along with vertices in \( D^h \cap \mathcal{C}_j^h \), dominate \( \mathcal{C}_j^h \).

**Claim 2.3.** For every \( h \in H \), \( |E^h_{J^h_I}| \geq \sum_{i=1}^s (m_i - 1) \).

**Proof.** Set \( j = |E^h_{J^h_I}| \) and notice that \( E^h_{J^h_I} \cup S^h_i \) dominates \( \mathcal{C}_j^h \). If we let \( I' = [k] - J^h_I \), then \( E^h_{J^h_I} \cup S^h_i \cup \left( \bigcup_{i \in I'} v^h_i \right) \) dominates \( G^h \). We note here that
some elements of \( E^h_i \) may also be elements of \( \bigcup_{i \in I} v_i^h \). However, such a set contains at most \( j + s + k - \sum_{i=1}^{s} m_i \) vertices, which must be at least \( k \). Thus, \( \sum_{i=1}^{s} m_i \leq j + s \) and we obtain the desired inequality. \( \square \)

Notice that for a fixed \( i, 1 \leq i \leq k \), projecting all vertices with labels containing \( i \) to \( H \) produces a dominating set of \( H \). Call the set of such vertices of \( D \) labeled \( i \), \( D_i \). Summing over all \( i \) we count vertices which have one label once and vertices with labels of cardinality \( m_i, m_i \) times, for every \( h \in V(H) \).

For any \( i \in [k] \), let \( F_i \) be the set of those vertices of \( D \) with labels of cardinality \( i \). We see that \( \gamma(G) \gamma(H) \leq \sum_{i=1}^{k} |D_i| = \sum_{i=1}^{k} i \times |F_i| = \sum_{i=1}^{\pi(G)} i \times |F_i| \)

Define \( t_i = \frac{|F_i|}{|D|} \), for \( 1 \leq i \leq \pi(G) \). Note that \( \sum_{i=1}^{\pi(G)} t_i = 1 \). We apply Proposition 1.3 to find that \( (2.1) \leq \frac{2\pi(G)-1}{\pi(G)} |D| \), and thus,

\[
|D| \geq \frac{\pi(G)}{2\pi(G) - 1} \gamma(G) \gamma(H).
\]

\( \square \)

2.1. Some Consequences. By definition, \( \pi(G) \leq \gamma(G) \) and \( \pi(G) \leq \Delta(G) \), which immediately implies the following.

**Corollary 2.4.** For any graphs \( G \) and \( H \),

\[
\gamma(G) \gamma(H) \geq \frac{\gamma(G)}{2\gamma(G) - 1} \gamma(G) \gamma(H)
\]

We note that this bound is an improvement to formula (1.2) when either \( \gamma(G) < \frac{\gamma(H)+1}{2} \) or \( \gamma(H) < \frac{\gamma(G)+1}{2} \).

**Corollary 2.5.** For any graphs \( G \) and \( H \),

\[
\gamma(G) \gamma(H) \geq \frac{\Delta(G)}{2\Delta(G) - 1} \gamma(G) \gamma(H)
\]

Furthermore, since \( \gamma(G) = \gamma_{[1,2]}(G) \) holds when \( G \) is claw-free or a cograph \([4]\), we have the next result.

**Corollary 2.6.** If \( G \) is claw-free or \( P_4 \)-free, and \( H \) is any graph, \( \gamma(G) \gamma(H) \geq \frac{2}{3} \gamma(G) \gamma(H) \).

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