DEFORMATIONS OF THE KDV HIERARCHY AND RELATED
SOLITON EQUATIONS

EDWARD FRENKEL

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Abstract. We define hierarchies of differential–$q$-difference equations, which are $q$–deformations of the equations of the generalized KdV hierarchies. We show that these hierarchies are bihamiltonian, one of the hamiltonian structures being that of the $q$–deformed classical $W$–algebra of $\mathfrak{sl}_N$, defined by Reshetikhin and the author. We also find $q$–deformations of the mKdV hierarchies and the affine Toda equations.

1. Introduction.

The $N$th Korteweg-de Vries (KdV) hierarchy is a bihamiltonian integrable system defined on the space $M'_N$ of $N$th order differential operators on the circle of the form

$$\partial^N + u_1(z)\partial^{N-2} + \ldots + u_{N-2}(z)\partial + u_{N-1}(z),$$

where $\partial \equiv \partial/\partial z$ [1, 2]. The space of local functionals on $M'_N$, considered as a Poisson algebra with respect to one of the hamiltonian structures of this system, is the classical $W$–algebra associated to $\mathfrak{sl}_N$, or the Adler–Gelfand–Dickey algebra.

Recently, N. Reshetikhin and the author [3] defined a Poisson algebra $W_q(\mathfrak{sl}_N)$ which is a $q$–deformation of the classical $W$–algebra of $\mathfrak{sl}_N$. This Poisson algebra consists of functionals on the space $M'_{q,N}$ of $N$th order $q$–difference operators of the form

$$D^N - t_1(z)D^{N-1} + \ldots + (-1)^{N-1}t_{N-1}(z)D + (-1)^N,$$

where $[D,f](z) = f(zq)$. It is natural to ask whether there is an integrable hierarchy on $M'_{q,N}$, which is hamiltonian with respect to this Poisson structure. In this work we construct such a hierarchy and show that it also has another compatible hamiltonian structure, i.e. is bihamiltonian. Moreover, the hamiltonians of this $q$–deformed KdV hierarchy have the following heredity property: the hamiltonian of the $n$th equation of the hierarchy with respect to one of the Poisson structures coincides with the hamiltonian of the $(n + N)$th equation with respect to the other structure. This property is well-known for the hamiltonians of the ordinary KdV hierarchies [1, 2].

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The KdV hierarchies are closely related to the modified KdV (mKdV) hierarchies and the affine Toda equations, see [4, 5, 6, 7]. Very briefly, the Nth mKdV hierarchy is the pull-back of the Nth KdV hierarchy by the so-called Miura transformation. The hamiltonian structure of the mKdV hierarchy is that of a classical Heisenberg algebra. The Nth affine Toda equation is a non-local equation on the phase space of the Nth mKdV hierarchy, whose local conservation laws are the hamiltonians of the mKdV hierarchy.

Using a q–deformation of the Miura transformation defined in [3], we construct a q–deformations of the Nth mKdV hierarchy and the affine Toda equation. We show that our q–deformed mKdV hierarchy is hamiltonian, and its hamiltonians are conserved with respect to the evolution of the q–deformed affine Toda equation.

This paper was greatly inspired by my joint work with B. Feigin and A. Odesskii [9] on commutative subalgebras of the q–deformed classical Virasoro algebra (i.e. \( W_q(\mathfrak{sl}_2) \)) and its elliptic deformations. The heredity property in the case of \( W_q(\mathfrak{sl}_2) \) is a result of [9], which is generalized here to the case of \( W_q(\mathfrak{sl}_N) \).

In this paper we use the general technique of discrete Lax equations, developed by B. Kuperschmidt [10]. Although he considers in [10] another class of Lax operators – difference rather than q–difference, many of his ideas and results can be applied in our context. But there are also many differences with [10], most importantly, in hamiltonian formalism. We would also like to mention a recent paper of D. Gieseker [11], where another differential-difference deformation of the KdV hierarchy is considered.

It would be interesting to find solutions of the q–deformed hierarchies, in particular, analogues of the soliton solutions of the ordinary hierarchies. It would also be interesting to associate such q–deformed hierarchies to other Lie algebras.

The paper is arranged as follows. In Sect. 2 we recall basic facts about the KdV, mKdV and the affine Toda equations in the form suitable for generalization. In Sect. 3 we construct the q–deformed KdV hierarchies, and in Sects. 4 and 5 we construct the q–deformed mKdV hierarchies and affine Toda equations, respectively. The proofs of the results presented here will appear in [12].

2. THE CLASSICAL EQUATIONS.

2.1. The KdV hierarchy. Let us we recall the construction of the Nth KdV hierarchy. For more details, the reader may consult the original papers [1, 2] and the book [13]. Consider the Nth order differential operator \( L \) of the form

\[
\partial^N + u_0(z)\partial^{n-1} + u_1(z)\partial^{n-2} + \ldots + u_{N-2}(z)\partial + u_{N-1}(z).
\]

(2.1)

Let \( \mathcal{R}_N = \mathbb{C}[\partial^nu_i]_{i=0,\ldots,N-1,n\geq 0} \) be the ring of differential polynomials in \( u_i(z) \)'s, and \( \mathcal{P}_N \) be the ring of pseudo-differential operators with coefficients in \( \mathcal{R}_N \). For any rational number \( \alpha \), there is a unique ath power \( L^\alpha \) of \( L \) in \( \mathcal{P}_N \). The nth equation of
the $N$th KdV hierarchy can be written in the Lax form as
\begin{equation}
\partial_{\tau_n} L = [L, (L^{n/N})_+],
\end{equation}
where $P_+$ is the differential part of the pseudo-differential operator $P$. The non-trivial equations correspond to $n$, which are not divisible by $N$.

2.2. Hamiltonian structures of the KdV hierarchy. The equations (2.2) can be written in Hamiltonian form with respect to two different Poisson structures [1, 2, 13]. These two Poisson structures are compatible in the sense that any linear combination of them is again a Poisson structure. Moreover, the Hamiltonian of the $n$th equation of the hierarchy with respect to the first structure coincides with the Hamiltonian of the $(n+N)$th equation with respect to the second structure. In other words, for each $n$ not divisible by $N$ there exists a local functional of $u_i(z)$’s, $H_n$, such that the $n$th equations can be written as
\[ \partial_{\tau_n} L = \{L, H_{n+N}\}_1 \]
and as
\[ \partial_{\tau_n} L = \{L, H_n\}_2, \]
where the subscript distinguishes the first and the second Poisson structures. This property was discovered by Magri [14] for $N = 2$, and by Gelfand and Dickey [2] for general $N$. We call it the heredity property.

There are essentially two formulas for the Hamiltonians of the KdV hierarchies. One of them is [1, 2]
\[ H_n = \frac{n}{N} \int \text{Res} L^{n/N} dz, \]
where $\text{Res} P$ stands for the $\partial^{-1}$–coefficient of $P$. The other formula is $H_n = N \int f_n dz$, where $f_n$ is uniquely determined by the formula
\[ \partial = L^{1/N} + \sum_{i \geq 0} f_i L^{-i/N}, \]
see [15]. From these formulas we see that $H_n$ is a local functional, i.e. it has the form $\int R dz$, where $R \in \mathcal{R}_N$.

V. Drinfeld and V. Sokolov [4] have shown that the phase space $\mathcal{M}_N$ of the $N$th KdV hierarchy, which consists of the differential operators of the form (2.1), can be obtained by Hamiltonian reduction from a hyperplane in the dual space to the affine algebra $\mathfrak{gl}_N$. The Poisson structure induced on $\mathcal{M}_N$ by this reduction coincides with the second Poisson structure of the corresponding KdV hierarchy. The space of all local functionals on $\mathcal{M}_N$ is the classical $\mathcal{W}$–algebra $\mathcal{W}_N$ associated to $\mathfrak{gl}_N$. The Hamiltonians $H_n$ of the KdV hierarchy commute with each other with respect to both Poisson structures. Thus, they span an infinite-dimensional commutative subalgebra of the $\mathcal{W}$–algebra.
2.3. Reduction to the submanifold $u_0(z) = 0$. Equations (2.2) imply: $\partial_{\tau_n} u_0(z) = 0$. Therefore we can set $u_0(z)$ to be equal to any function. The standard choice is $u_0(z) = 0$. Then formula (2.2) defines a hierarchy of equations on the submanifold $M'_N \subset M_N$ of operators of the form (1.1). The two Poisson structures on $M_N$ discussed above can be restricted to $M'_N$, and the equations are hamiltonian with respect to these restrictions. The Poisson algebra of local functionals on $M'_N$ is the classical $W$-algebra associated to $\mathfrak{sl}_N$.

2.4. The mKdV hierarchy. Now consider the space $F_N$ of $N$–tuples of first order differential operators

\begin{equation}
(\partial + v_1(z), \ldots, \partial + v_N(z)).
\end{equation}

Consider the map $\mu_i : F_N \to M_N$, which sends an $N$–tuple (2.3) to

$$L_i = (\partial + v_i(z))(\partial + v_{i+1}(z)) \ldots (\partial + v_{i+N-1}(z)).$$

Here and below we identify all indices modulo $N$. This map is called the $i$th Miura transformation. We can pull back the KdV hierarchy from $M_N$ to $F_N$ using one of these maps. The corresponding hierarchy on $F_N$ is called the $N$th mKdV hierarchy.

The equations of this hierarchy can be written in the Lax form as follows. Consider the operator

\begin{equation}
L = \begin{pmatrix}
0 & \partial + v_1 & 0 & \ldots & 0 \\
0 & 0 & \partial + v_2 & \ldots & 0 \\
& \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \partial + v_N & \ldots & \partial + v_{N-1} \\
\partial + v_N & 0 & 0 & \ldots & 0
\end{pmatrix}.
\end{equation}

There exists a unique matrix pseudo-differential operator of the form

$$P = \begin{pmatrix}
P_1 & 0 & \ldots & 0 \\
0 & P_2 & \ldots & 0 \\
& \ldots & \ldots & \ldots \\
0 & 0 & \ldots & P_N
\end{pmatrix},$$

where

$$P_i = \partial + \sum_{m \geq 0} P_i^{[m]} \partial^{-m},$$

and $P_i^{[m]}$ are differential polynomials in $v_i$, $i = 1, \ldots, N$, such that $[L, P] = 0$. The $n$th equation of the $N$th mKdV hierarchy reads

\begin{equation}
\partial_{\tau_n} L = [L, (P^n)_+].
\end{equation}

In writing the mKdV equations in this form we followed the idea of Kuperschmidt [10], Ch. IV. We could not find this particular representation of mKdV hierarchy in
2.5. Hamiltonian structure of the mKdV hierarchy. The space $F_N$ is isomorphic to the product of a hyperplane in the dual space to the homogeneous Heisenberg subalgebra of $\hat{sl}_N$ and the space of functions on the circle. Therefore it has a Poisson structure, which is the product of the Kirillov-Kostant Poisson structure on the hyperplane and the trivial structure on the space of functions. We call the Poisson algebra of local functionals on $F_N$, the classical Heisenberg algebra and denote it by $H_N$. Explicitly, we have:

$$\{v_i(z), v_i(w)\} = -\frac{N-1}{N} \delta'(\frac{w}{z}),$$

$$\{v_i(z), v_j(w)\} = \frac{1}{N} \delta'(\frac{w}{z}), \quad i \neq j,$$

where $\delta'(x) = \sum_{m \in \mathbb{Z}} m x^m$. In particular, $v_1(z) + \ldots + v_N(z)$ lies in the kernel of this Poisson structure.

Equation (2.5) is Hamiltonian with respect to this Poisson structure [4, 5, 6], i.e. it can be written as

$$\partial_t v_i(z) = \{v_i(z), H_n\}, \quad i = 1, \ldots, N,$$

where

$$H_n = \frac{1}{n} \int \text{Res} \text{tr} P^n dz.$$

Moreover, the map $\mu_1$ is Hamiltonian, if we consider $\mathcal{M}_N$ as a Poisson manifold with respect to the second Poisson structure [6]. Thus, $\mu_1$ defines a homomorphism of Poisson algebras $\mathcal{W}_N \to H_N$. The Hamiltonian $H_n$ is the image of the $n$th Hamiltonian $H_n$ of the corresponding KdV hierarchy under this homomorphism.

Remark 1. Explicitly, $P_i$ is the pull-back of $(L_i)^{1/N}$ under the $i$th Miura transformation.

Equations (2.5) imply that $\partial_t(v_1(z) + \ldots + v_N(z)) = 0$. Therefore we can set $\sum_{i=1}^N v_i(z) = 0$. This gives us a submanifold $\mathcal{F}_N' \subset \mathcal{M}_N$. The restriction of $\mu_1$ to $\mathcal{F}_N'$ is a Hamiltonian map $\mathcal{F}_N' \to \mathcal{M}_N'$.

2.6. The affine Toda equations. Closely related to the mKdV hierarchy is the affine Toda equation

$$\partial_t \phi_i(z) = e^{\phi_{i+1}(z) - \phi_i(z)} - e^{\phi_i(z) - \phi_{i-1}(z)}, \quad i = 1, \ldots, N,$$

where $\partial_t \phi_i(z) = v_i(z)$. The affine Toda equation can be represented in the Lax form as follows:

$$\partial_t L = [L, A L^{-1}],$$
where $A$ is a matrix of the form

$$
A = \begin{pmatrix}
0 & A_1 & 0 & \ldots & 0 \\
0 & 0 & A_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & A_{N-1} \\
A_N & 0 & 0 & \ldots & 0
\end{pmatrix}.
$$

Again, we could not find this representation of the affine Toda equation in the literature, but similar representations have been given in [4, 5].

2.7. Hamiltonian structure of the affine Toda equation. The affine Toda equation can be written in Hamiltonian form [5]:

$$
\partial_t v_i(z) = \{v_i(z), H\}, \quad i = 1, \ldots, N,
$$

where

$$
H = \sum_{i=1}^N \int e^{\phi_{i+1}(z) - \phi_i(z)} dz.
$$

The Hamiltonian $H$ is not a local functional, and hence does not lie in the classical Heisenberg algebra $\mathcal{H}_N$. Because of that, the Poisson bracket of $H$ with any element of $\mathcal{H}_N$ can only be defined as a linear operator acting from $\mathcal{H}(\mathfrak{sl}_N)$ to another vector space, see [5] and also [7, 8]. However, we can still define local integrals of motion of the affine Toda equation as elements in the kernel of the linear operator $\{\cdot, H\}$ [5, 7, 8]. These quantities are conserved with respect to the evolution defined by the equation (2.6).

Since the affine Toda equation has Lax representation with the same operator $L$ as the equations of the corresponding mKdV hierarchy, the Hamiltonians of the mKdV hierarchy are integrals of motion of the affine Toda equation [4, 5], i.e. $\{H_n, H\} = 0$ for all $n > 0$. The converse is also true: the local functionals which commute with $H$ are precisely the linear combinations of $H_n$ [7, 8]. Therefore this property can be taken as the definition of the Hamiltonians $H_n$, as it was done in [7, 8].

3. Deformation of the KdV hierarchy.

3.1. The phase space and related rings. Consider the space $\mathcal{M}_{q,N}$ of $q$–difference operators of the form

$$
L = D^N - t_1(z)D^{N-1} + \ldots + (-1)^{N-1}t_{N-1}(z)D + (-1)^N t_N(z),
$$

where

$$
t_i(z) = \sum_{m \in \mathbb{Z}} t_i[m] z^{-m}
$$

is a Laurent series for each $i = 1, \ldots, N$, and $[D \cdot f](z) = f(zq)$. Let us define the rings $\mathcal{R}_{q,N}$ and $\mathcal{P}_{q,N}$ associated to these $q$–difference operators, which are analogous to the rings $\mathcal{R}_N$ and $\mathcal{P}_N$. 
Consider the ring \( \mathcal{R}^{(0)}_{q,N} = \mathbb{C}[t_i(zq^j)]_{i=1, \ldots, N; j \in \mathbb{Z}} \). The operator \( D \) acts naturally on \( \mathcal{R}^{(0)}_{q,N} \). The ring \( \mathcal{R}_{q,N} \) is the smallest ring containing \( \mathcal{R}^{(0)}_{q,N} \) as a subring, on which the operator \( 1 + D + \ldots + D^{N-1} \) is invertible. To construct it, we adjoin to \( \mathcal{R}_{q,N} \) the Fourier coefficients of elements of solutions of the equations \( (1 + D + \ldots + D^{N-1})f(z) = R(z) \) for all \( R(z) \in \mathcal{R}^{(0)}_{q,N} \). This gives us a ring \( \mathcal{R}^{(1)}_{q,N} \). Then we adjoin to \( \mathcal{R}^{(1)}_{q,N} \) the solutions of these equations for all elements \( R \) of \( \mathcal{R}^{(1)}_{q,N} \), which are not in \( \mathcal{R}^{(0)}_{q,N} \), etc. The inductive limit of the rings \( \mathcal{R}^{(i)}_{q,N} \), which are obtained this way, is the ring \( \mathcal{R}_{q,N} \).

If we write \( t_i(z) = \sum_m t_i[m]z^{-m} \), then the Fourier coefficient \( R[m] \) of an element \( R(z) = \sum_m R[m]z^{-m} \) of \( \mathcal{R}_{q,N} \) is a linear combination of expressions of the form

\[
\sum_{m_1+\ldots+m_k=M} c(m_1, \ldots, m_k)t_{i_1}[m_1] \ldots t_{i_k}[m_k],
\]

where \( c(m_1, \ldots, m_k) \) is a rational function in \( q^{m_1}, \ldots, q^{m_k} \), which has no poles for integral values of \( m_i \)'s if \( q \) is generic. We denote the Fourier coefficient \( R[0] \) of \( R(z) \) by \( \int R(z) \).

Given an operator of the form (3.1), we can substitute the coefficients \( t_i[m], i = 1, \ldots, N - 1; m \in \mathbb{Z} \), into an expression like (3.2) and get a number. Therefore Fourier coefficients of elements of \( \mathcal{R}_{q,N} \) define functionals on the space \( \mathcal{M}_{q,N} \). We call all expressions of the form (3.2), \( q \)-local functionals. The leading term in the \( (1 - q) \)-expansion of a \( q \)-local functional is a local functional. Note that there are \( q \)-local functionals, which can not be represented as Fourier coefficients of elements of \( \mathcal{R}_{q,N} \). We denote the space of all \( q \)-local functionals by \( \mathcal{L}_{q,N} \), and its subspace, spanned by the 0th Fourier coefficients of elements of \( \mathcal{R}_{q,N} \) by \( \mathcal{L}_{q,N} \). We have a map \( \int : \mathcal{R}_{q,N} \to \mathcal{L}_{q,N} \), which sends \( R(z) \) to \( R[0] \).

Now let \( \mathcal{P}_{q,N} \) be the ring consisting of “pseudo-difference” operators of the form

\[
\sum_{n \leq M} R_n(z)D^n,
\]

where each \( R_n(z) \in \mathcal{R}_{q,N} \), and it may be non-zero for any negative \( n \).

**Lemma 1.** There exists a unique element \( P \) of \( \mathcal{P}_{q,N} \), such that \( P^N \) is equal to \( L \) given by formula (3.1).

We will denote \( P^n \) by \( L^{n/N} \). Also, for each \( R = \sum_n R_n D^n \) from \( \mathcal{R}_{q,N} \), we set:

\[
R_+ = \sum_{n \geq 0} R_n D^n, \quad R_- = \sum_{n < 0} R_n D^n, \quad \text{and} \quad \text{Res} R = R_0.
\]

**3.2. Lax form of the equations.** For each \( n = 1, 2, \ldots \), the \( n \)th equation of the \( q \)-deformed KdV is by definition the Lax equation

\[
\partial_r L = [L, (L^{n/N})_+] = -[L, (L^{n/N})_-].
\]
From the first equality it follows that $\partial_{\tau_n} L$ has the form $\sum_{n=1}^{\infty} w_n D^n$, while from the second equality it follows that it has the form $\sum_{n=-\infty}^{N-1} w_n D^n$. Hence $\partial_{\tau_n}$ defined this way is a difference operator of order $N - 1$, and formula (3.3) makes sense.

**Remark 2.** The equation (3.3) can be viewed as the integrability condition for the system

$$L \Psi = \lambda \Psi, \quad \partial_{\tau_n} \Psi = (L^{n/N})_+ \Psi.$$  

Note that the first formula looks similar to formulas for the spectra of transfermatrices appearing in Bethe ansatz, see [3] and references therein. □

We call the hierarchy of Lax equations (3.3) the $N$th $q$–deformed KdV hierarchy. These equations define flows on the space of operators of the form (3.1) with $t_i(z)$'s belonging to a sufficiently large class of functions of real or complex variable. In the present work, we do not discuss analytic aspects of these flows, but focus on the algebraic structures, which underlie them. Such approach proved very effective in the case of the ordinary KdV hierarchies.

From the algebraic point of view, formula (3.3) defines a derivation $\partial_{\tau_n}$ on the ring $R_{q,N}$ and a linear operator on the space $L_{q,N}$. The latter can clearly be extended to a linear operator $\hat{L}_{q,N} \rightarrow \hat{L}_{q,N}$. The following proposition is analogous to Proposition 1.7 from [10], Ch. III.

**Proposition 1.** The derivations $\partial_{\tau_n}$ are non-trivial for all $n$ not divisible by $N$ and they commute with each other. The $q$–local functionals $\int \text{Res} L^{n/N}$ are conserved with respect to all $\partial_{\tau_m}$, i.e. $\partial_{\tau_n} \cdot \int \text{Res} L^{n/N} = 0, \forall n, m$. Moreover, $\int \text{Res} L^{n/N} \neq 0$, if $n$ is not divisible by $N$.

**Remark 3.** Denote $D = 1 - D$. The proof of the fact that the functionals $\int \text{Res} L^{n/N}$ are conserved with respect to all $\partial_{\tau_n}$ relies on the property of the map $\int : R_{q,N} \rightarrow L_{q,N}$ that it vanishes on the image of $D$, i.e. $\int (D R) = 0, \forall R \in R_{q,N}$. In general, we may want to allow the coefficients $t_i(z)$ of the operator (3.1) to belong to a class of functions other than Laurent power series. Accordingly, we may consider another space of functionals $L'_{q,N}$. In that case, conservation laws of the equations (3.3) are $\phi(\text{Res} L^{n/N})$, where $\phi$ is an arbitrary map $R_{q,N} \rightarrow L'_{q,N}$, which vanishes on the image of $D$. For example, in some cases we can use the Jackson integral as $\phi$. □

**Remark 4.** Consider the equations

$$\partial_{\tau_n} P = [P, (P^n)_+] = 0, \quad n = 1, 2, \ldots,$$

where $P$ is an operator of the form $D + \sum_{n<0} p_n D^n$. These equations define a commuting hierarchy of flows on the space of such operators. If we impose the condition $P^N = L$, where $L$ is of the form (3.1), then this hierarchy is equivalent to the one defined by formula (3.3). If we do not impose this condition, we obtain a hierarchy, which is $q$–deformation of the KP hierarchy. □
We define two compatible Poisson structures on the space $M_{q,N}$, i.e. we define two Poisson brackets on the space $\tilde{L}_{q,N}$ of functionals on $M_{q,N}$. It is sufficient to define the Poisson brackets between $t_i[m]$ and $t_j[n]$, or between the power series $t_i(z)$ and $t_j(w)$.

We set

\begin{equation}
\{t_i(z), t_j(w)\}_1 = \begin{cases} \delta \left( \frac{wq^{N-j}}{z} \right) t_N(z)t_{i+j-N}(w) - \delta \left( \frac{w}{zq^{N-i}} \right) t_{i+j-N}(z)t_N(w), & i, j \neq N, i+j \geq N \\ 0, & \text{otherwise} \end{cases}
\end{equation}

and

\begin{equation}
\{t_i(z), t_j(w)\}_2 = \sum_{m \in \mathbb{Z}} \left( \frac{w}{z} \right)^m \frac{(1-q^m)(1-q^{m(N-j)})}{1-q^{mN}} t_i(z)t_j(w)
+ \sum_{r=1}^{\min(i,N-j)} \delta \left( \frac{wq^r}{z} \right) t_{i-r}(w)t_{j+r}(z)
- \sum_{r=1}^{\min(i,N-j)} \delta \left( \frac{w}{zq^{j-i+r}} \right) t_{i-r}(z)t_{j+r}(w), & i \leq j.
\end{equation}

In these formulas $\delta(x) = \sum_{m \in \mathbb{Z}} x^m$, and we use the convention that $t_0(z) \equiv 1$.

Remark 5. The parameter $q$ in formula (3.6) corresponds to $q^{-2}$ in notation of [3].

These Poisson structures are $q$–deformations of the first and second Poisson structures of the Nth KdV hierarchy, respectively. The Poisson structure $\{\cdot, \cdot\}_2$ given by (3.6) was defined in [3] (modulo the relation $t_N(z) = 1$). The space $\tilde{L}_{q,N}$, considered as a Poisson algebra with respect to this structure, is a $q$–deformation of the classical $W$–algebra of $gl_N$.

Note that these two brackets preserve the subspace $L_{q,N}$ of $\tilde{L}_{q,N}$ and hence define two Poisson brackets on it as well.

**Proposition 2.** The Poisson structures $\{\cdot, \cdot\}_1$ and $\{\cdot, \cdot\}_2$ are compatible in the sense that any linear combination of them is also a Poisson structure.

The next theorem shows that the Nth $q$–deformed KdV hierarchy defined by (3.3) is bihamiltonian with respect to these two Poisson structures. Moreover, the hamiltonians of the hierarchy have the heredity property characteristic for the hamiltonians of the KdV hierarchies.

**Theorem 1.** Let $H_n = \frac{N}{n} \int \text{Res} L^{n/N}$. The equation $\partial_{\tau_n} L = [L, (L^{n/N})_+]$ can be represented in hamiltonian form as

$\partial_{\tau_n} L = \{L, H_{n+N}\}_1 \quad \text{and} \quad \partial_{\tau_n} L = \{L, H_n\}_2$. 
The Hamiltonians $H_n$ commute with each other with respect to both Poisson structures:
\[ \{H_n, H_m\}_1 = \{H_n, H_m\}_2 = 0. \]

The proof of this theorem will appear in [9] for the case $N = 2$ and in [12] for the general case.

Remark 6. The first $N - 1$ Hamiltonians $H_n, n = 1, \ldots, N - 1$, commute with all elements of $\tilde{\mathcal{L}}_{q,N}$ with respect to the Poisson structure $\{\cdot, \cdot\}_1$.

3.4. Reduction to the submanifold $t_N(z) = 1$. Equations (3.3) imply that $\partial_{t_n} t_N(z) = 0$. Therefore we can set $t_N(z) = 1$. This gives us a submanifold $\mathcal{M}'_{q,N} \subset \mathcal{M}_{N,q}$, which consists of the $q$–difference operators of the form
\begin{equation}
L = D^N - t_1(z)D^{N-1} + \ldots + (-1)^{N-1}t_{N-1}(z)D + (-1)^N.
\end{equation}

Formula (3.3) defines a hierarchy of differential–$q$-difference equations on the space $\mathcal{M}'_{q,N}$. Formulas (3.5) and (3.6), in which we set $t_N(z) = 1$, define two compatible Poisson structures on $\mathcal{M}'_{q,N}$ with respect to which the equations (3.3) are Hamiltonian. The Poisson algebra corresponding to the second structure is a $q$–deformation of the classical $\mathcal{W}$–algebra of $\mathfrak{sl}_N$, defined in [3].

There is a nice formula for the first Poisson structure (3.5) on $\mathcal{M}'_{q,N}$, which is analogous to the formula defining the first Poisson structure of the ordinary KdV hierarchies. As we mentioned earlier, elements of the space $\tilde{\mathcal{L}}_{q,N}$ can be considered as functionals on the space $\mathcal{M}_{q,N}$ of $q$–difference operators of the form (3.1). On the other hand, each element $X$ from $\mathcal{P}_{q,N}$ of the form $\sum_{i=1}^{N-1} x_i D^{-i}$ defines a linear functional $\ell_X$ on $\mathcal{M}_{q,N}$ by the formula
\[ \ell_X(L) = \int \text{Res} \, LX. \]

The Poisson structure (3.5) is uniquely determined by the following formula for the Poisson bracket of the linear functionals:
\[ \{\ell_X, \ell_Y\}_1(L) = \int \text{Res}(L[X, Y]). \]

It would be interesting to find a similar formula for the second Poisson structure.

3.5. The second construction of Hamiltonians. Following Kuperschmidt [10], Ch. IX, we can give another construction of the Hamiltonians $H_n$. The idea of this construction goes back to [15] (see Sect. 2 above).

Let again $P$ be the $N$th root of $L$ given by (3.1). We can uniquely represent $D$ as
\[ D = P + \sum_{i \geq 0} f_i P^{-i}, \]
where \( f_i \in \mathcal{R}_{q,N} \). Now introduce elements \( h_n \in \mathcal{R}_{q,N} \) via the generating series

\[
\sum_{n>0} h_n t^n = -\log \left(1 + \sum_{i \geq 0} f_n t^{-n-1}\right),
\]

where

\[
-\log(1 - x) = \sum_{m>0} \frac{x^m}{m}.
\]

**Proposition 3.** \( h_n \) equals \( \frac{1}{n} \) Res \( L^{n/N} \) up to a total difference, i.e.

\[
f_n - \frac{1}{n} \text{Res} L^{n/N} = (1 - D) g_n
\]

for some \( g_n \in \mathcal{R}_{q,N} \). Therefore \( H_n = N \int f_n \).

The proof of this proposition is analogous to the proof of Theorem 2.24 of [10], Ch. IX. Following [10] we can define the \( \tau \)-function of the \( N \)th \( q \)-deformed KdV hierarchy in the same way as for the ordinary hierarchies.

**3.6. Examples and \( q \to 1 \) limit.** We have:

\[
P = L^{1/N} = D + b(z) + \ldots,
\]

where \( b(z) \) is defined by the equation

\[
(1 + D + \ldots + D^{N-1})b(z) = -t_1(z).
\]

The first equation of the \( N \)th \( q \)-deformed KdV hierarchy is therefore:

\[
\partial_\tau L = [L, D + b(z)],
\]

which gives

\[
(3.8) \quad \partial_\tau t_i(z) = t_i(z)(b(zq^{N-1}) - b(z)) + t_{i+1}(zq) - t_{i+1}(z),
\]

where \( t_{N+1}(z) = 0 \). The hamiltonian with respect to the second structure is

\[
H_1 = N \int \text{Res } P = Nb[0] = -t_1[0].
\]

From formula (3.6) we find that equation (3.8) is equivalent to the equation

\[
\partial_\tau t_i(z) = \{t_i(z), H_1\}_2.
\]

Let us now consider the case \( N = 2 \) more closely. We can reduce the \( q \)-deformed KdV equations to the manifold \( \mathcal{M}^2_{2,q} \) of operators of the form \( L = D^2 - t(z)D + 1 \). To simplify notation let us write \( t_n \) for \( t[n] \). The first two non-trivial hamiltonians are: \( H_1 = -t_0 \), and

\[
H_3 = -\frac{1}{2} t_0 + \frac{1}{3} \sum_{i+j+k=0} \frac{1}{(1 + q^i)(1 + q^j)(1 + q^k)} t_it_jt_k.
\]

\[\text{DEFORMATIONS OF SOLITON EQUATIONS} \quad 11\]
The Poisson brackets are:

\[ \{t(z), t(w)\}_1 = \delta \left( \frac{wq}{z} \right) - \delta \left( \frac{w}{zq} \right), \]

\[ \{t(z), t(w)\}_2 = \sum_{m \in \mathbb{Z}} \left( \frac{w}{z} \right)^m \frac{1 - q^m}{1 + q^m} t(z)t(w) + \delta \left( \frac{wq}{z} \right) - \delta \left( \frac{w}{zq} \right) \]

(see [3]).

The equations of the hierarchy are \( \partial_{r_n} t(z) = \{t(z), H_n\}_2 \). For \( n = 1 \) have from (3.8):

\[ \partial_{r_1} t(z) = t(z)(b(zq) - b(z)) \]

where \( b(z) \) satisfies \( b(z) + b(zq) = -t(z) \). In terms of Fourier coefficients, we have:

\[
\partial_{r_1} t(z) = -\sum_{i,j} \frac{1 - q^{i+j}}{(1 + q^i)(1 + q^j)} t_{ij} t_{ij} z^{-i-j},
\]

\[
\partial_{r_2} t(z) = -\frac{3}{2} \sum_{i,j} \frac{1 - q^{i+j}}{(1 + q^i)(1 + q^j)} t_{ij} t_{ij} z^{-i-j} + \sum_{i,j,k,l} \frac{1 - q^{i+j+k}}{(1 + q^i)(1 + q^j)(1 + q^{-i-j})(1 + q^{i+j+k})} t_{ij} t_{kl} t_{kl} z^{-i-j-k-l}.
\]

The equations of the \( N \)th KdV hierarchy can be recovered from the equations of the \( N \)th \( q \)-deformed KdV hierarchy in the limit \( q \to 1 \). Let us explain this in the case \( N = 2 \). Set \( t(z) = 2 - h^2 u(z) + o(h^2) \), where \( q = e^h \). Then

\[ D^2 - t(z)D + 1 \leftrightarrow h^2(\partial^2 + u(z)) + o(h^2). \]

Hence the manifold \( \mathcal{M}_{2,q}' \) becomes \( \mathcal{M}_2' \). In [3] it was shown that if we multiply the right hand side by \( h \), then the leading term of the Poisson structure (3.9) gives us in the limit \( q \to 1 \) the Poisson structure of the classical Virasoro algebra, which is the second Poisson structure of the KdV hierarchy. It is also easy to see that the first Poisson structure of the \( q \)-deformed KdV hierarchy becomes in the limit \( q \to 1 \) the first Poisson structure of the KdV hierarchy.

The \( h \)-expansion of the \( n \)th hamiltonian of the \( q \)-deformed KdV hierarchy is \( H_n = \text{const} + h^{n+1} H_n^{(0)} + o(h^{n+1}) \), where \( H_n^{(0)} \) is the \( n \)th hamiltonians of the KdV hierarchy. Therefore if we rescale the derivation \( \partial_{r_n} \) as \( \partial_{r_n}' = h^{n+1} \partial_{r_n} \), then the equation \( \partial_{r_n} t(z) = \{t(z), H_n\}_2 \) gives us in the limit \( q \to 1 \) the \( n \)th equation of the KdV hierarchy. In particular, the equation corresponding to \( H_2 \) (see above) gives us the KdV equation itself.

Note that the hamiltonian \( H_n \) is non-zero if and only if \( n \) is odd. The hamiltonians \( H_1, H_3, \ldots \), of the \( q \)-deformed KdV hierarchy have odd degrees as series in \( t_m \). It will be shown in [9] that there exists another infinite set \( J_2, J_4, \ldots \), of series in \( t_m \) of
even degrees, which commute with each other and with all $H_n$ with respect to both Poisson brackets. The first of them is

$$J_2 = \sum_{i>0} \frac{i q^i}{1 - q^{2i}} t_i t_{-i}.$$  

Technically, the series $J_n$ do not lie in the Poisson algebra $\mathcal{L}_{2,q}$. But we can enlarge $\mathcal{L}_{2,q}$ by adjoining series of the form (3.2), where $c(m_1, \ldots, m_k)$ polynomially depend on $m_1, \ldots, m_k$. The series $H_n$’s and $J_n$’s will then constitute a commutative subalgebra in this Poisson algebra. We expect that the elements $J_n$ also satisfy the heredity property, which the $H_n$’s satisfy. It would be interesting to construct explicitly the corresponding hamiltonian equations.

The hamiltonians of the $N$th $q$–deformed KdV hierarchy have degrees which are not divisible by $N$. We conjecture that there exists an additional set of series of degrees divisible by $N$, which commute with these hamiltonians with respect to both Poisson structures.

**Remark 7.** In [10] the equations of the Toda lattice hierarchy (which should not be confused with the affine Toda equations) were represented in the Lax form as $\partial_q L = [L, (L^n)_+]$, where $L = \zeta + p + v\zeta^{-1}$ and $\zeta$ is a difference operator. Although this $L$–operator resembles the $L$–operator of the $q$–deformed KdV hierarchy $D^2 - t_1(z)D + t_2(z)$, the equations of the two hierarchies are different. □

4. **Deformations of the mKdV hierarchies.**

4.1. **The phase space.** Consider the space $\mathcal{F}_{q,N}$ of $N$–tuples of $q$–difference operators

\begin{equation}
(D - \Lambda_1(z), \ldots, D - \Lambda_N(z)).
\end{equation}

We define the ring $\mathcal{J}_{q,N}$ in the same way as the ring $\mathcal{R}_{q,N}$. Let $\mathcal{J}_{q,N}^{(0)}$ be the ring generated by $\Lambda_i(zq^j), i = 1, \ldots, N; j \in \mathbb{Z}$. We take as $\mathcal{J}_{q,N}$, the smallest ring containing $\mathcal{J}_{q,N}^{(0)}$ as a subring, on which the operator $(1 + D + \ldots + D^{N-1})$ is invertible. Let $\mathcal{G}_{q,N}$ be the span the 0th Fourier coefficients of elements of $\mathcal{J}_{q,N}$. The space $\mathcal{G}_{q,N}$ lies in a larger space $\tilde{\mathcal{G}}_{q,N}$, which is defined in the same way as $\tilde{\mathcal{L}}_{q,N}$. We have a map $f : \mathcal{J}_{q,N} \to \mathcal{G}_{q,N}$.

Introduce the ring $\mathcal{I}_{q,N}$ of “pseudo-difference” operators of the form $\sum_{n \leq M} J_n D^n$, where each $J_n \in \mathcal{J}_{q,N}$.

In [3] the $q$–deformed Miura transformation was defined. The $i$th $q$–deformed Miura transformation $\mu_{i,q}$ is the map $\mathcal{F}_{q,N} \to \mathcal{M}_{q,N}$, which sends the $N$–tuple (4.1) to

\begin{equation}
L_i = (D - \Lambda_i(z)) (D - \Lambda_{i+1}(z)) \ldots (D - \Lambda_{N+i-1}(z)),
\end{equation}

where we identify $\Lambda_{N+j}(z) \equiv \Lambda_j(z)$.
4.2. Lax form of the equations. We want to write down the equations on $F_{q,N}$, which are the pull-backs of the equations of the $N$th $q$–deformed KdV hierarchy. The idea of the following construction of these equations is due to Kuperschmidt, see [10], Ch. IV.

Consider the matrix

$$L = \begin{pmatrix} 0 & D - \Lambda_1 & 0 & \ldots & 0 \\ 0 & 0 & D - \Lambda_2 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ D - \Lambda_N & 0 & 0 & \ldots & 0 \end{pmatrix}.$$  

(4.3)

**Lemma 2.** There exists a unique matrix

$$P = \begin{pmatrix} P_1 & 0 & \ldots & 0 \\ 0 & P_2 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & P_N \end{pmatrix},$$

where each $P_i$ is an element of $I_{q,N}$ of the form

$$P_i = D + \sum_{m \geq 0} P_i^{[m]} D^{-m},$$

which commutes with $L$.

For each $n = 1, 2, \ldots$, consider the Lax equation

$$\partial_t^n L = [L, (P^n)_+].$$

(4.4)

We call the hierarchy of equations (4.4) the $N$th $q$–deformed mKdV hierarchy.

Since

$$L^N = \begin{pmatrix} L_1 & 0 & \ldots & 0 \\ 0 & L_2 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & L_N \end{pmatrix},$$

where $L_i$ is given by formula (4.2), we see that $P_i$ is the pull-back of $P = (L_i)^{1/N}$ via the $i$th Miura transformation, and so equation (4.4) on $(\Lambda_1(z), \ldots, \Lambda_N(z))$ implies equation (3.3) on each $L_i, i = 1, \ldots, N$. Hence the $q$–deformed mKdV hierarchy is really a pull-back of the $q$–deformed KdV hierarchy.

We consider $\partial_t^n$ as a derivation of the ring $\mathcal{I}_{q,N}$ and as a linear operator on the space $\mathcal{G}_{q,N}$. We have the following analogue of Proposition 1.

**Proposition 4.** The derivations $\partial_t^n$ are non-trivial for all $n$ not divisible by $N$ and they commute with each other. The functionals $\int \text{Tr Res} P^n$ are conserved with respect to all $\partial_t^n$. Moreover, these functionals are non-zero, if $n$ is not divisible by $N$. 
4.3. Hamiltonian form of the equations. We define a Poisson structure on $\mathcal{F}_{q,N}$ by the formulas [3]

$$\{\Lambda_i(z), \Lambda_i(w)\} = \sum_{m \in \mathbb{Z}} \left( w \frac{1}{z} \right)^m \frac{(1 - q^m)(1 - q^{m(N-1)})}{1 - q^{mN}} \Lambda_i(z)\Lambda_i(w),$$  \hspace{1cm} (4.5)

$$\{\Lambda_i(z), \Lambda_j(w)\} = -\sum_{m \in \mathbb{Z}} \left( wq^{N+i-j-1} \right)^m \frac{(1 - q^m)^2}{1 - q^{mN}} \Lambda_i(z)\Lambda_j(w), \hspace{1cm} i < j. \hspace{1cm} (4.6)$$

**Theorem 2.** Let $H_n = \frac{1}{n} \int \text{Tr} \text{Res} P^n$. The equation $\partial_{t_n} L = [L, (P^n)_+]$ can be represented in hamiltonian form as

$$\partial_{t_n} \Lambda_i(z) = \{\Lambda_i(z), H_n\}, \hspace{1cm} i = 1, \ldots, N.$$ 

The hamiltonians $H_n$ commute with each other: $\{H_n, H_m\} = 0$.

**Proposition 5.** If we consider the space $\mathcal{M}_{q,N}$ as a Poisson manifold with respect to the Poisson structure $\{\cdot, \cdot\}_2$, then the first Miura transformation $\mu_{1,q}$ is a hamiltonian map. This map sends $H_n$ to $H_n$. Equations (4.4) imply that

$$\partial_{t_n}(\Lambda_1(z)\Lambda_2(z) \ldots \Lambda_N(z)) = 0$$

for all $n$. Therefore the $q$–deformed mKdV hierarchy can be restricted to the submanifold $\mathcal{F}_{q,N}'$ of $\mathcal{F}_{q,N}$ defined by the equation

$$\Lambda_1(z)\Lambda_2(z) \ldots \Lambda_N(z) = 1.$$ 

The Poisson structure on $\mathcal{F}_{q,N}$ can also be restricted to $\mathcal{F}_{q,N}'$. The Miura transformation $\mu_{1,q}$ provides a hamiltonian map $\mathcal{F}_{q,N}' \rightarrow \mathcal{M}_{q,N}'$.

Finally, we can recover the ordinary mKdV equations from equations (4.4) in the limit $q \rightarrow 1$, if we set $\Lambda_i(z) = 1 - hv_i(z) + o(h)$, where $q = e^h$.

5. Deformation of the affine Toda equations.

5.1. Lax form of the equation. Consider the Lax equation

$$\partial_t L = [L, AL^{-1}],$$  \hspace{1cm} (5.1)

where $A$ is a matrix of the form

$$A = \begin{pmatrix}
0 & A_1(z) & 0 & \ldots & 0 \\
0 & 0 & A_2(z) & \ldots & 0 \\
0 & 0 & 0 & \ldots & A_{N-1}(z) \\
A_N(z) & 0 & 0 & \ldots & 0
\end{pmatrix}.$$
From this equation we find:

\[ \partial_t \Lambda_i(z) = A_i(z) - (D - \Lambda_i(z))A_{i+1}(z)(D - \Lambda_{i+1}(z))^{-1}, \]

for all \( i = 1, \ldots, N \). Since

\[ (D - \Lambda_i(z))A_{i+1}(z)(D - \Lambda_{i+1}(z))^{-1} = \]

\[ A_{i+1}(zq) + (A_{i+1}(zq)\Lambda_{i+1}(z) - A_{i+1}(z)\Lambda_i(z))(D - \Lambda_{i+1})^{-1}, \]

we obtain from (5.2) that

\[ A_i(zq) = \Lambda_{i-1}(z)\Lambda_i(z)^{-1}A_i(z), \]

and

\[ \partial_t \Lambda_i(z) = A_i(z) - A_{i+1}(zq). \]

The function \( A_i(z) \) appearing on the right hand side of this equation is determined by the \( q \)–difference equations (5.4). In the limit \( q \to 1 \) this equation gives rise to the differential equation satisfied by the function \( e^{\phi_i(z) - \phi_{i-1}(z)} \) appearing on the right hand side of the affine Toda equation (2.6). Indeed, if we set \( \Lambda_i(z) = 1 - hv_i(z) + o(h) \), and \( A_i(z) = a_i(z) + O(h) \), we obtain from (5.4):

\[ \partial_t a_i(z) = (v_i(z) - v_{i-1}(z))a_i(z), \]

which shows that \( a_i(z) = e^{\phi_i(z) - \phi_{i-1}(z)} \). Therefore equation (5.5) becomes in the limit \( q \to 1 \) the \( N \)th affine Toda equations. We call equation (5.5) the \( q \)–deformed \( N \)th affine Toda equation. Note that \( \partial_t(\Lambda_1(z) \ldots \Lambda_N(z)) = 0. \) Therefore we can set \( \Lambda_1(z) \ldots \Lambda_N(z) = 1. \)

5.2. Hamiltonian form of the equation. We now want to represent the \( q \)–deformed affine Toda equation in hamiltonian form. The construction of the corresponding Poisson structure is analogous to the construction of the Poisson structure of the ordinary affine Toda equation [5], see also [7, 8].

Introduce new power series \( Q_i(z) \) as solutions of the \( q \)–difference equations

\[ Q_i(zq) = \Lambda_i(z)Q_i(z). \]

Let \( \mathcal{J}_{q,N}^+ \) be the tensor product \( \mathcal{J}_{q,N} \otimes \mathbb{C}[Q_1(z), Q_1(z)^{-1}, \ldots, Q_N(z), Q_N(z)^{-1}] \) where each \( Q_i(z) \) satisfies equation (5.6). Note that according to formula (5.4) we can express \( A_i(z) \) in terms of \( Q_j(z) \)'s:

\[ A_i(z) = Q_{i-1}(z)Q_i(z)^{-1}. \]

Let \( \mathcal{G}_{q,N}^+ \) be the span of the 0th Fourier coefficients of elements of \( \mathcal{J}_{q,N}^+ \), and \( \mathcal{I} \) be the corresponding map \( \mathcal{J}_{q,N}^+ \to \mathcal{G}_{q,N}^+ \).
We can extend the Poisson bracket on $G_{q,N}$ to a partial Poisson bracket $\{\cdot,\cdot\} : G_{q,N} \times G_{q,N} \to G_{q,N}$ by postulating

$$\{\Lambda_i(z), Q_i(w)\} = -\sum_{m \in \mathbb{Z}} \left(\frac{w}{z}\right)^m \frac{1 - q^{m(N-1)}}{1 - q^{mN}} \Lambda_i(z)Q_i(w),$$

$$\{\Lambda_i(z), Q_j(w)\} = \sum_{m \in \mathbb{Z}} \left(\frac{wq^{N+i-j-1}}{z}\right)^m \frac{1 - q^m}{1 - q^{mN}} \Lambda_i(z)\Lambda_j(w), \quad i < j,$n

$$\{\Lambda_i(z), Q_j(w)\} = \sum_{m \in \mathbb{Z}} \left(\frac{wq^{i-j-1}}{z}\right)^m \frac{1 - q^m}{1 - q^{mN}} \Lambda_i(z)\Lambda_j(w), \quad i > j.$$n

Note that these formulas are consistent with formulas (4.5) and (4.6) if we take into account equation (5.6).

Denote

$$S_i(z) = Q_i(w)Q_{i+1}(wq)^{-1}, \quad i = 1, \ldots, N.$$n

From the Poisson brackets above we derive:

$$\{\Lambda_i(z), S_i(w)\} = -\delta \left(\frac{w}{z}\right) \Lambda_i(z)S_i(w) = -\delta \left(\frac{w}{z}\right) \Lambda_{i+1}(zq),$$

$$\{\Lambda_i(z), S_{i-1}(w)\} = \delta \left(\frac{w}{z}\right) \Lambda_i(z)S_{i-1}(w) = \delta \left(\frac{w}{z}\right) \Lambda_i(z),$$

$$\{\Lambda_i(z), S_j(w)\} = 0, \quad j \neq i, i - 1.$$n

These formulas lead us to the following result.

**Proposition 6.** The $q$–deformed affine Toda equation can be presented in hamiltonian form as

$$\partial_t \Lambda_i(z) = \{\Lambda_i(z), H\}, \quad i = 1, \ldots, N,$$n

where

$$H = \sum_{i=1}^{N} \int Q_i(w)Q_{i+1}(wq)^{-1}.$$n

Consider the case $N = 2$. We can set $\Lambda_2(z) = \Lambda_1(z)^{-1}$. Denote $\Lambda(z) = \Lambda_1(z)$. Let $Q(z)$ be a solution of the equation $Q(zq) = \Lambda(z)Q(z)$. The equation (5.5) reads in this case

$$\partial_t \Lambda(z) = Q(z)^{-2} - Q(zq)^2.$$n

This is a $q$–deformation of the sine-Gordon equation. The hamiltonian is

$$H = Q(z)Q(zq) + Q(z)^{-1}Q(zq)^{-1}.$$n

In general, the operator $\partial_t$ defines operators $J_{q,N} \to J_{q,N}^+$ and $G_{q,N} \to G_{q,N}^+$. Since the $q$–deformed affine Toda equation can be represented in the Lax form with the same Lax operator $L$ as the equations of the $q$–deformed mKdV hierarchy, we conclude that
the conservations laws $H_n$ of this hierarchy are also conserved with respect to the $q$–deformed affine Toda equation.

**Theorem 3.** The hamiltonians $H_n$ of the $q$–deformed mKdV hierarchy commute with the hamiltonian of the $q$–deformed affine Toda equation: $\{H_n, H\} = 0$.

**Remark 8.** Each summand $\{\cdot, S_i(z)\}$ of the linear operator $\{\cdot, H\} : J_{q,N} \to J_{q,N}^+$ gives us a map $J_{q,N} \to J_{q,N}^+$, where $J_{q,N}^+ = J_{q,N} \otimes S_i(z)$. Hence $\{J, H\} = 0$ if and only if $\{J, \int S_i(z)\} = 0$ for all $i = 1, \ldots, N$.

5.3. Classical $\mathcal{W}$–algebra as algebra of integrals of motion. The hamiltonian equation corresponding to the hamiltonian $\sum_{i=1}^{N-1} \int e^{\phi_{i+1} - \phi_i} dz$ is the Toda field equation associated to the Lie algebra $\mathfrak{sl}_N$ (finite Toda equation for shorthand). We can construct a $q$–analogue of this equation:

$$\partial_t \Lambda_i(z) = \{\Lambda_i(z), \sum_{i=1}^{N-1} \int S_i(z)\}.$$ 

The next proposition shows that the algebra of local integrals of motion of this equation contains the $q$–deformation of the classical $\mathcal{W}$–algebra of $\mathfrak{sl}_N$.

**Proposition 7.** For all $J \in J_{q,N}$, which lie in the image of $R_{q,N}$ under the map $\mu_{1,q}$, $\{J, \int S_i(z)\} = 0, i = 1, \ldots, N - 1$.

For all $G \in G_{q,N}$, which lie in the image of $L_{q,N}$ under the map $\mu_{1,q}$, $\{G, \int S_i(z)\} = 0, i = 1, \ldots, N - 1$.

We conjecture that the images of $R_{q,N}$ and $L_{q,N}$ exhaust all elements in the intersection of kernels of the operators $\{\cdot, \int S_i(z)\} : J_{q,N} \to J_{q,N}^+$ and $G_{q,N} \to G_{q,N}^+$.

The operators $\int S_i(z), i = 1, \ldots, N - 1$, are the classical limits of the screening operators constructed in [16, 17] in the case $N = 2$ and in [18, 19] in general case. In [17, 18, 19] it was proved that the quantum $\mathcal{W}$–algebra of $\mathfrak{sl}_N$ commutes with the screening operators. Proposition 7 is the classical version of this fact.

We can define spaces of quantum integrals of motion of the $q$–deformed finite and affine Toda equations as the intersections of kernels of the screening operators corresponding to $\int S_i(z)$, where $i = 1, \ldots, N - 1$ in the finite case, and $i = 1, \ldots, N$ in the affine case. The latter is clearly a subalgebra of the former. We expect that the integrals of motion of the quantum affine Toda equation constitute a commutative subalgebra in the quantum $\mathcal{W}$–algebra. We also expect that the spaces of classical and quantum integrals of motion are isomorphic as vector spaces. This is true for the ordinary (i.e. not $q$–deformed) affine Toda equations [20, 7]. In that case the quantum integrals of motion can be interpreted as conservation laws of certain perturbations of conformal field theories with $\mathcal{W}$–symmetry [21].
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DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, CAMBRIDGE, MA 02138, USA