On Biproducts and Extensions

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Abstract

We describe in which ways the Radford biproducts of certain eight-dimen-
sional Yetter-Drinfel’d Hopf algebras over the elementary abelian group
of order 4 can be written as extensions of Hopf algebras.

Introduction

In their article [10], the authors described two semisimple Yetter-Drinfel’d Hopf
algebras of dimension 8 over the group ring of the elementary abelian group of
order 4. The purpose of the construction of these Yetter-Drinfel’d Hopf algebras
was to show that the core of a group-like element is not always completely
trivial. It was not even mentioned in this article that, via the Radford biproduct
construction (cf. [23]), these Yetter-Drinfel’d Hopf algebras give rise to ordinary
Hopf algebras of dimension 32. The purpose of the present article is to discuss
how these Hopf algebras fit into the general theory of semisimple Hopf algebras
of prime power dimension, as found for example in [6], [7], [8], [13], and [14].
As we will see, these ordinary Hopf algebras also behave differently from other
known examples.

Every semisimple Hopf algebra of prime power dimension can in principle be
constructed as an iterated crossed product (cf. [20], Theorem 3.5). However,
for the dimensions where the description is more concrete, or even so concrete
that all semisimple Hopf algebras of that given dimension can be classified,
like in the articles just cited, the Hopf algebras often contain a commutative
Hopf subalgebra of prime index, a situation sometimes also encountered for
semisimple Hopf algebras that are not of prime power dimension, like in [15]
and [21]. The examples constructed here do not contain such a Hopf subalgebra,
as we show in Paragraph 4.4. More generally, we show that they cannot be
constructed as central or cocentral abelian extensions.

Let us state the main results of the article more precisely, while we simultaneously
explain its structure. Section 1 contains preliminaries on Yetter-Drinfel’d
Hopf algebras, Radford biproducts, and extensions of Hopf algebras. In Section 2
we consider the eight-dimensional Yetter-Drinfel’d Hopf algebra $A$ over
the elementary abelian group of order 4 that appears in [10], Section 2. We
describe the arising Radford biproduct $B$, which has dimension 32, give a pre-
sentation of $B$ in terms of generators and relations, and compute its center. In
Section 3 we show that the groups of group-like elements of both $B$ and $B^*$
are elementary abelian of order 8, which means in particular that they have
eight one-dimensional representations. In Theorem 3.2 we show that, up to
isomorphism, both $B$ and $B^*$ have in addition two irreducible representations
of dimension 2 and one irreducible representation of dimension 4.

In Section 4 we show that $B$ contains a unique sixteen-dimensional Hopf sub-
algebra $N$. It is normal and isomorphic to the Hopf algebra $H_{d_{1,1}}$ from the
classification appearing in [6], Table 1. Our Hopf algebra $B$ therefore fits into
exactly one extension of the type

$$N \hookrightarrow B \twoheadrightarrow Z$$

with $\dim N = 16$ and $\dim Z = 2$. This extension is not abelian.

In Section 5 we show that the Hopf subalgebra $N$ also arises as the unique
sixteen-dimensional quotient of $B$. By construction, $N$ is also a Radford bipro-
duct, and the quotient map $\pi_N : B \to N$ is induced from a map between the
corresponding Yetter-Drinfel’d Hopf algebras. Although $\pi_N$ does not restrict to
the identity on $N$, it is conormal, and therefore $B$ fits into exactly one extension
of the type

$$U \hookrightarrow B \twoheadrightarrow N$$

with $\dim U = 2$ and $\dim N = 16$. This extension is not abelian. We use these
facts to describe the structure of the Grothendieck ring of $B$ in Paragraph 5.3.

In Section 6 we begin by determining the eight-dimensional Hopf subalgebras
of $B$. There are three, denoted by $M_1$, $M_2$, and $M_3$, all of which are contained
in $N$. The algebra $M_1$ is just the span of the group-like elements, while the other
two are dual to the group ring of the dihedral group. We show in Paragraph 6.2
that, of the three Hopf subalgebras, only $M_2$ is normal, so that $B$ fits into
exactly one extension of the type

$$M \hookrightarrow B \twoheadrightarrow Q$$

with $\dim M = 8$ and $\dim Q = 4$. In this case, $M = M_2 \cong K[D_8]$, while
$Q \cong K[\mathbb{Z}_2 \times \mathbb{Z}_2]$. This extension is abelian, but neither central nor cocentral.

In Paragraph 6.3 we determine the four-dimensional Hopf subalgebras of $B$.
There are seven, but only one of them is normal. Consequently, the Hopf
algebra $B$ fits into exactly one extension of the type

$$P \hookrightarrow B \twoheadrightarrow F$$

with $\dim P = 4$ and $\dim F = 8$. In this case, $P \cong K[\mathbb{Z}_2 \times \mathbb{Z}_2]$ and $F \cong K[D_8]$.
This extension is abelian, but neither central nor cocentral.
Besides the Yetter-Drinfel’d Hopf algebra $A$, the authors considered in [10], Section 3 a second eight-dimensional Yetter-Drinfel’d Hopf algebra over $K[Z_2 \times Z_2]$, denoted here by $A'$. These two algebras are certainly not isomorphic, because $A$ is commutative, while $A'$ is not. However, the corresponding Radford biproducts $B$ and $B'$ are isomorphic, as we show in Section 7, so that it is not necessary to carry out the same analysis for $B'$.

Let us now state the conventions that are used throughout this article. Our base field $K$ will be algebraically closed of characteristic zero. The multiplicative group of $K$ will be denoted by $K^\times = K \setminus \{0\}$.

All vector spaces will be defined over $K$. The dual space of a vector space $V$ will be denoted by $V^* = \text{Hom}(V, K)$, and the dual map of a linear map $f$, also called its transpose, will be denoted by $f^*$.

The group algebra of a group $G$, which we will also call its group ring, will be denoted by $K[G]$. The corresponding dual space $K[G]^*$, the dual group ring, will be denoted by $K^G$. The character group of a group $G$ will be denoted by $\hat{G} = \text{Hom}(G, K^\times)$. Its elements will be interchangeably called multiplicative characters, one-dimensional characters, or one-dimensional representations, a terminology that we will use not only for groups, but also for algebras. Clearly, the multiplicative characters of the group algebra correspond bijectively to the multiplicative characters of the group via restriction.

All algebras are assumed to have a unit element, and algebra homomorphisms are assumed to preserve these unit elements. The center of an algebra $A$ will be denoted by $Z(A)$, which should not be confused with a certain quotient Hopf algebra $Z$ already mentioned above. The set of group-like elements in a coalgebra $A$ will be denoted by $G(A)$. The subalgebra generated by elements $a_1, \ldots, a_n$ of an algebra $A$ will be denoted by $K\langle a_1, \ldots, a_n \rangle$, whereas the subgroup of a group $G$ generated by elements $g_1, \ldots, g_n$ will be denoted by $\langle g_1, \ldots, g_n \rangle$. The augmentation ideal of a Hopf algebra $H$ will be denoted by $H^+ := \ker(\varepsilon_H)$. The image of an element $a$ in a quotient space will be denoted by $\bar{a}$. Finally, the symbol $\subset$ denotes non-strict inclusion.

As already discussed, the article is divided into sections, which are divided further into relatively small paragraphs. A reference to Proposition 2.2 refers to the unique proposition in Paragraph 2.2, and definitions, theorems, lemmas, and corollaries are referenced in the same way.

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1 Preliminaries

1.1 In this section, we consider a finite-dimensional Hopf algebra \( H \) with coproduct \( \Delta_H \), counit \( \varepsilon_H \), and antipode \( S_H \). Recall that a Yetter-Drinfel’d module over \( H \) is a vector space \( V \) that is both a left \( H \)-module and a left \( H \)-comodule in such a way that the compatibility condition

\[
h(1)v(1) \otimes h(2)v(2) = (h(1),v)^{(1)} h(2) \otimes (h(1),v)^{(2)}
\]

is satisfied for all \( h \in H \) and \( v \in V \), where we have used the notation

\[
\delta(v) = v^{(1)} \otimes v^{(2)} \in H \otimes V
\]

for the coaction \( \delta \). The category \( H_{YD} \) of all Yetter-Drinfel’d modules, whose morphisms are the \( H \)-linear and colinear maps, is a braided monoidal category with the braiding \( \sigma_{V,W} : V \otimes W \to W \otimes V \) given by

\[
\sigma_{V,W}(v \otimes w) = (v^{(1)},w) \otimes v^{(2)}
\]

We collect a number of facts about Yetter-Drinfel’d modules that will be needed in the sequel:

(1) The dual space \( V^* \) of a finite-dimensional Yetter-Drinfel’d module \( V \) can be considered as a Yetter-Drinfel’d module over the dual Hopf algebra \( H^* \) by using the module structure

\[
\gamma \cdot \varphi = (\gamma \otimes \varphi) \circ \delta
\]

for \( \gamma \in H^* \) and \( \varphi \in V^* \), and the comodule structure \( \delta(\varphi) = \varphi^{(1)} \otimes \varphi^{(2)} \in H^* \otimes V^* \) with the defining property

\[
\varphi^{(1)}(h) \varphi^{(2)}(v) = \varphi(h \cdot v)
\]

where \( \varphi \in V^* \), \( h \in H \), and \( v \in V \) (cf. [27], Proposition 1.2 and Lemma 1.3). With respect to these structures, the transpose of an \( H \)-linear and \( H \)-colinear map is \( H^* \)-linear and \( H^* \)-colinear.

(2) If \( A \) and \( A' \) are algebras in the Yetter-Drinfel’d category, so that their multiplication maps \( \mu_A \) and \( \mu_{A'} \) as well as their unit maps \( \eta_A \) and \( \eta_{A'} \) are morphisms in the category, then their tensor product \( A \otimes A' \) is also an algebra in this category when endowed with the multiplication map

\[
\mu_{A \otimes A'} = (\mu_A \otimes \mu_{A'}) \circ (\text{id}_A \otimes \sigma_{A',A} \otimes \text{id}_{A'})
\]

and the unit map \( \eta_{A \otimes A'} = \eta_A \otimes \eta_{A'} \).

(3) A Hopf algebra in the Yetter-Drinfel’d category \( H_{YD} \), or a Yetter-Drinfel’d Hopf algebra for short, is an algebra \( A \) that is simultaneously a coalgebra, both in this category, so that comultiplication and counit are multiplicative.
Here, the algebra structure on $A \otimes A$ is the one introduced above, so that the multiplicativity of the coproduct means that

$$\Delta_A \circ \mu_A = (\mu_A \otimes \mu_A) \circ (\sigma_{A,A} \otimes \sigma_{A,A}) \circ (\Delta_A \otimes \Delta_A)$$

or, if we use the notation $\Delta_A(a) = a(1) \otimes a(2)$ for the coproduct,

$$\Delta_A(aa') = a(1) \left( a(2)(1), a'(1) \right) \otimes a(2)(2)a'(2)$$

We also require that $A$ has an antipode $S_A$, i.e., a convolution inverse of the identity. As shown in [10], Lemma 1.2, such an antipode is automatically a morphism in the category.

1.2 Yetter-Drinfel’d Hopf algebras give rise to ordinary Hopf algebras via the Radford biproduct construction, which was originally defined in [23], Theorem 1:

**Definition.** Let $A$ be a Yetter-Drinfel’d Hopf algebra over $H$. The Radford biproduct $B := A \star H$ is the tensor product of $A$ and $H$ as a vector space, endowed with the smash product multiplication

$$(a \star h)(a' \star h') := a \left( h(1), a' \right) \star h(2)h'$$

with corresponding unit element $1_B := 1_A \star 1_H$, comultiplication

$$\Delta_B(a \star h) := \left( a(1) \star a(2)(1)h(1) \right) \otimes \left( a(2)(2) \star h(2) \right)$$

with corresponding counit $\varepsilon_B(a \star h) := \varepsilon_A(a)\varepsilon_H(h)$, and antipode

$$S_B(a \star h) := \left( 1_A \star S_H(a^{(1)}h) \right) \left( S_A(a^{(2)}) \star 1_H \right)$$

With respect to these structures, it is an ordinary Hopf algebra. The notation $a \star h$ instead of $a \otimes h$ is used to signify that the algebra and coalgebra structures are not the canonical structures on the tensor product.

Two results about Radford biproducts will be needed in the sequel:

1. If $A$ is a finite-dimensional Yetter-Drinfel’d Hopf algebra over $H$, then $A^*$ is a Yetter-Drinfel’d Hopf algebra over $H^*$ with respect to the module and comodule structures described in Paragraph 1.1 and the standard dualizations of the algebra and coalgebra structures (cf. [27], loc. cit.), so that we can form the biproduct $A^* \star H^*$. It is isomorphic to $B^*$ via the canonical isomorphism of the underlying vector spaces.

2. The biproduct $B$ is semisimple if $A$ and $H$ are semisimple. Similarly, it is cosemisimple if $A$ and $H$ are cosemisimple (cf. [23], Proposition 3 and Proposition 4; [25], Paragraph 2.14 and the references cited there).

Further details on Yetter-Drinfel’d Hopf algebras and biproducts can be found in [19], § 10.6, [23], and [27].
1.3 A generalization of the smash products just encountered in Paragraph 1.2 are crossed products (cf. [19], Definition 7.1.1): Here we have, for a Hopf algebra $H$ and an algebra $A$, not only a weak action, or measuring, $\mapsto: H \otimes A \to A$, but also a cocycle $\tau: H \times H \to A$, and the multiplication on the underlying vector space $A \otimes H$, which is then commonly denoted by $A \#\tau H$, is given by the formula
\[
(a \# h)(a' \# h') := a(h(1) \to a')\tau(h(2), h'(1))\#h(3)h'(2)
\]
Such a crossed product has the following universal property:

**Proposition.** Suppose that $B$ is an algebra and that $f_A: A \to B$ as well as $f_H: H \to B$ are $K$-linear maps. Then the following assertions are equivalent:

1. There exists an algebra homomorphism $f: A \#\tau H \to B$ such that $f(a \# 1_H) = f_A(a)$ and $f(1_A \# h) = f_H(h)$

2. $f_A$ is an algebra homomorphism, $f_H(1_H) = 1_B$, and we have $f_A(h(1) \to a)f_H(h(2)) = f_H(h)f_A(a)$ as well as $f_A(\tau(h(1), h'(1)))f_H(h(2)h'(2)) = f_H(h)f_H(h')$

**Proof.** To show that the first assertion implies the second, we define the inclusion maps $i_A: A \to A \#\tau H$ and $i_H: H \to A \#\tau H$ via $i_A(a) := a \# 1_H$ and $i_H(h) := 1_A \# h$, and also define $f_A := f \circ i_A$ and $f_H := f \circ i_H$. Then we have $f_H(1_H) = f(1_A \# 1_H) = 1_B$ and
\[
f_H(h)f_A(a) = f(1_A \# h)f(a \# 1_H) = f \left((1_A \# h)(a \# 1_H)\right)
\]
\[
= f \left((h(1) \to a) \# h(2))\right) = f_A(h(1) \to a)f_H(h(2))
\]
as well as
\[
f_H(h)f_H(h') = f(1_A \# h)f(1_A \# h') = f \left((1_A \# h)(1_A \# h')\right)
\]
\[
= f \left(\tau(h(1), h'(1)) \# h(2)h'(2)\right) = f_A(\tau(h(1), h'(1)))f_H(h(2)h'(2))
\]
To show that the second assertion implies the first, we define $f(a \# h) := f_A(a)f_H(h)$
Then $f$ preserves the unit element, and we have
\[
f((a \# h)(a' \# h')) = f(a(h(1) \to a')\tau(h(2), h'(1))\#h(3)h'(2)) = f_A(a)f_A(h(1) \to a')f_A(\tau(h(2), h'(1)))f_H(h(3)h'(2)) = f_A(a)f_A(h(1) \to a')f_H(h(2))f_H(h') = f_A(a)f_H(h)f_A(a')f_H(h') = f(a \# h)f(a' \# h')
\]
as asserted.  

\[\square\]
There are two observations about this universal property that are worth recording. First, if the equation
\[
 f_A(h(1) ↦ a) f_H(h(2)) = f_H(h) f_A(a)
\]
that appears in the universal property is satisfied for all \( h \in H \), but only for a specific \( a \in A \), and in the same way for another specific element \( a' \in A \), then it holds for their product \( aa' \), as we have by the measuring property that
\[
 f_A(h(1) ↦ (aa')) f_H(h(2)) = f_A((h(1) ↦ a)(h(2) ↦ a')) f_H(h(3))
\]
\[
 = f_A(h(1) ↦ a) f_A(h(2) ↦ a') f_H(h(3))
\]
\[
 = f_A(h(1) ↦ a) f_H(h(2)) f_A(a') = f_H(h) f_A(a) f_A(a') = f_H(h) f_A(aa')
\]
Arguing inductively, this shows that it is sufficient to verify this equation for all \( a \) in a generating set of \( A \), as long as it holds for all \( h \in H \) for each of these generators.

For the second observation, note that \( i_H \) is convolution-invertible (cf. [19], Proposition 7.2.7). If \( \bar{i}_H \) denotes its convolution-inverse, then \( \bar{f}_H := f \circ \bar{i}_H \) is the convolution-inverse of \( f_H = f \circ i_H \). Thus the two main conditions of the second assertion can also be stated in the form
\[
 f_A(h ↦ a) = f_H(h(1)) f_A(a) \bar{f}_H(h(2))
\]
and
\[
 f_A(\tau(h, h')) = f_H(h(1)) f_H(h'(1)) \bar{f}_H(h(2) h'(2))
\]

In the case that the cocycle is trivial in the sense that \( \tau(h, h') = \varepsilon_H(h) \varepsilon_H(h') 1_A \), the crossed product reduces to the smash product. Then the twisted module condition (cf. [19], Lemma 7.1.2) reduces to the usual module condition, so that the weak action is an ordinary action, which we will denote by a dot again. The preceding proposition now yields the following universal property of the smash product:

**Corollary.** Suppose that \( B \) is an algebra and that \( f_A : A \to B \) and \( f_H : H \to B \) are algebra homomorphisms such that
\[
 f_A(h(1), a) f_H(h(2)) = f_H(h) f_A(a)
\]
Then the map \( f : A \# H \to B \) defined via
\[
 f(a \# h) := f_A(a) f_H(h)
\]
is an algebra homomorphism.

For the smash product, there is a second point to be made, which is similar, but not identical, to the one made above in the case of a general crossed product: If the hypothesis in the preceding corollary holds for all \( a \in A \) and two
elements \( h, h' \in H \), it holds for their product, because the weak action is then an ordinary action and we therefore have

\[
\begin{align*}
 f_A(h_1 h'_1, a) f_H(h_2 h'_2) &= f_A(h_1 h'_1, a) f_H(h_2) f_H(h'_2) \\
 &= f_H(h) f_A(h'_1, a) f_H(h'_2) \\
 &= f_H(h) f_H(h') f_A(a) = f_H(h h') f_A(a)
\end{align*}
\]

Arguing inductively, this shows that it is sufficient to verify this hypothesis for all \( h \) in a generating set of \( H \), as long as it holds for all \( a \in A \) for each of these generators.

1.4 As we will explain in Paragraph 1.5 below, crossed products appear naturally when studying Hopf algebra extensions, which are defined as follows (cf. [17], Definition 5.6):

**Definition.** Consider a sequence

\[
N \xhookrightarrow{\iota_N} B \xrightarrow{\pi_F} F
\]

of finite-dimensional Hopf algebras and Hopf algebra maps \( \iota_N \) and \( \pi_F \), where \( \iota_N \) is injective and \( \pi_F \) is surjective. This sequence is called an extension of \( F \) by \( N \) if

\[
\iota_N(N) = \{ b \in B \mid b(1) \otimes \pi_F(b(2)) = b \otimes 1_F \}
\]

In this situation, \( \iota_N(N) \) is a normal Hopf subalgebra of \( B \) (cf. [2], Section 1.2).

Hopf algebra extensions can have additional properties:

1. The extension is called abelian if \( N \) is commutative and \( F \) is cocommutative. Since \( K \) is algebraically closed of characteristic zero, it follows from [19], Theorem 2.3.1 that \( N \cong K^\Gamma \) and \( F \cong K[L] \) for two finite groups \( \Gamma \) and \( L \).

2. The extension is called central if \( \iota_N(N) \subset Z(B) \), the center of \( B \). This happens in particular if \( N \) has dimension 2, since \( N \) is then necessarily semisimple and is spanned by its unit element and its integral, both of which are central in view of [12], Lemma 2.16.

3. The extension is called cocentral if \( \pi_F^*(F^*) \subset Z(B^*) \), the center of \( B^* \).

4. We say that two extensions \( N \hookrightarrow B \twoheadrightarrow F \) and \( N \hookrightarrow B' \twoheadrightarrow F \) are equivalent if there is a Hopf algebra isomorphism \( f : B \to B' \) that induces the identity maps on \( N \) and \( F \) (cf. [18], Definition 1.4).

Further details on Hopf algebra extensions can be found in [1], [2], [4], [5], [17], and [18].
1.5 The connection between crossed products and Hopf algebra extensions arises via bicrossed products, which are defined as follows:

**Definition.** Let $N$ and $F$ be Hopf algebras equipped with a weak action, or measuring, $\rightarrow: F \otimes N \to N$, a cocycle $\tau: F \times F \to N$, and dually a weak coaction $\rho: F \to F \otimes N$ and a dual cocycle $\kappa: F \to N \otimes N$. The bicrossed product $N \#^\tau_\kappa F$ is a Hopf algebra with $N \otimes F$ as underlying vector space, crossed product multiplication

$$(x \#^\tau t)(y \#_s^s) := x(t_{(1)} \rightarrow y)\tau(t_{(2)}, s_{(1)}) \#^\tau t_{(3)} \kappa(t_{(2)})$$

as in Paragraph 1.3, and dually cocrossed product comultiplication

$$\Delta(x \#^\tau t) := x_{(1)} \kappa_1(t_{(1)}) \#^{\tau(1)} x_{(2)} \kappa_2(t_{(2)}) t_{(2)} \#^\tau t_{(3)}$$

where we have, in analogy to Paragraph 1.1, used the notation

$$\rho(t) = t^{(1)} \otimes t^{(2)} \in F \otimes N$$

even though this is a right, not a left, weak coaction. For the dual cocycle, we have used the notation

$$\kappa(t) = \kappa_1(t) \otimes \kappa_2(t) \in N \otimes N$$

The necessary compatibility conditions on the four structure elements $\rightarrow$, $\tau$, $\rho$, and $\kappa$ can be found in [1], Theorem 2.20; [2], Section 3.1; or [3], Kapitel 5.

Every bicrossed product yields the extension

$$N \xrightarrow{\iota_N} N \#^\tau_\kappa F \xrightarrow{\pi_F} F$$

where $\iota_N(x) := x \#_1^F$ and $\pi_F(x \#^\tau t) := \varepsilon_N(x)t$. In the finite-dimensional case considered here, in fact every extension is equivalent to such a bicrossed product extension (cf. [1], Theorem 3.2.14; [2], Proposition 3.1.12 and Theorem 3.1.17; [17], Page 130).

It is not difficult to see from the formula for the crossed product multiplication given above that, in a central bicrossed product extension, the weak action of $F$ on $N$ must be trivial. Dually, for a cocentral extension $B = N \#^\tau\kappa F$, the weak coaction $\rho$ is trivial, because in $B^*$ the weak action of $N^*$ on $F^*$ is trivial.

2 The biproduct in the first case

2.1 If we use the notation $\mathbb{Z}_2 = \{0, 1\}$ for the group with two elements, the group $G := \mathbb{Z}_2 \times \mathbb{Z}_2$ contains the four elements

$$g_1 := (0, 0) \quad g_2 := (1, 0) \quad g_3 := (0, 1) \quad g_4 := (1, 1)$$
with \( g_1 = 1_G \) as its unit element when we write \( G \) multiplicatively, as we will in the sequel. In [10], Section 2, the authors defined a commutative Yetter-Drinfel’d Hopf algebra \( A \) over the group algebra \( H := K[G] \) as the algebra generated by two commuting variables \( x \) and \( y \) subject to the defining relations

\[
\begin{align*}
x^4 &= 1 \\
y^2 &= \frac{1}{2}(1 + \zeta x + x^2 - \zeta x^3)
\end{align*}
\]

where \( \zeta \) is a not necessarily primitive fourth root of unity. With the help of a primitive fourth root of unity \( \iota \), it is possible to introduce the elements \( \omega_1 := 1_A \),

\[
\begin{align*}
\omega_2 &:= \frac{1}{2}(1 + \iota \zeta^2)x + \frac{1}{2}(1 - \iota \zeta^2)x^3 \\
\omega_3 &:= \frac{1}{2}(1 - \iota \zeta^2)x + \frac{1}{2}(1 + \iota \zeta^2)x^3 \\
\omega_4 &:= x^2
\end{align*}
\]

and

\[
\begin{align*}
\eta_1 &:= y \\
\eta_2 &:= x^3y \\
\eta_3 &:= x^2y \\
\eta_4 &:= xy
\end{align*}
\]

According to [10], Proposition 2.4, these eight elements form a basis of \( A \), and the coalgebra structure is determined by the fact that they are group-like. As also shown there, the set \( \{ \omega_1, \omega_2, \omega_3, \omega_4 \} \) is closed under multiplication and in fact forms a group isomorphic to \( \mathbb{Z}_2 \times \mathbb{Z}_2 \). The products of all basis elements are computed there; here we only record that \( \omega_4 \eta_1 = \omega_3 \) and \( \omega_4 \eta_2 = \omega_4 \eta_4 \). In [10], Paragraph 2.2, the \( H \)-action on \( A \) is introduced on the generators as

\[
\begin{align*}
g_2.x &= x^3 \\
g_2.y &= x^3y \\
g_3.x &= x \\
g_3.y &= x^2y
\end{align*}
\]

which according to [10], Lemma 2.4 means for the basis elements that

\[
\begin{align*}
g_3.\omega_1 &= \omega_1 \\
g_2.\omega_4 &= \omega_4 \\
g_2.\omega_2 &= \omega_3 \\
g_3.\eta_{g_j} &= \eta_{g_j}
\end{align*}
\]

where we have used the notation \( \eta_{g_j} := \eta_j \). In particular, we have \( g_3.\eta_1 = \omega_4 \eta_1 \) for all \( i = 1, 2, 3, 4 \). The coaction of \( H \) on \( A \) is derived from the action by defining

\[
\delta(a) := \frac{1}{4} \sum_{g,g' \in G} \theta(g,g') g \otimes g'.a
\]

where \( \theta \) is the nondegenerate symmetric bicharacter described in [10], Paragraph 2.2 and determined by the condition

\[
\begin{pmatrix}
\theta(g_2, g_2) & \theta(g_2, g_3) \\
\theta(g_3, g_2) & \theta(g_3, g_3)
\end{pmatrix} = \begin{pmatrix}
\zeta^2 & -1 \\
-1 & 1
\end{pmatrix}
\]

on its fundamental matrix.

We will need the following two explicit forms, the first of which is recorded in [10], Paragraph 4.3:

\[
\begin{align*}
\delta(\omega_2) &= \frac{1}{2}(g_1 + g_3) \otimes \omega_2 + \frac{1}{2}(g_1 - g_3) \otimes \omega_3 \\
\delta(\eta_1) &= \frac{1}{4}(g_1 + g_2 + g_3 + g_4) \otimes \eta_1 + \frac{1}{4}(g_1 + \zeta^2 g_2 - g_3 - \zeta^2 g_4) \otimes \eta_2 \\
&\quad+ \frac{1}{4}(g_1 - g_2 + g_3 - g_4) \otimes \eta_3 + \frac{1}{4}(g_1 - \zeta^2 g_2 - g_3 + \zeta^2 g_4) \otimes \eta_4
\end{align*}
\]
Because $\eta_1 = y$, the last formula also yields the value of the coaction on one of the generators. According to [10], Proposition 2.2, the value on the other generator is

$$
\delta(x) = \frac{1}{2}(g_1 + g_3) \otimes x + \frac{1}{2}(g_1 - g_3) \otimes x^3
$$

$$
\delta(x^3) = \frac{1}{2}(g_1 - g_3) \otimes x + \frac{1}{2}(g_1 + g_3) \otimes x^3
$$

where it is also stated that $x^2 = \omega_4$ is coinvariant.

It is possible to expand the powers of $x$ explicitly in terms of the group-like elements: If we multiply the definition of $\omega_2$ by $1 + \zeta^2$ and the definition of $\omega_3$ by $1 - \zeta^2$, we obtain

$$(1 + \zeta^2)\omega_2 = x^3 + \zeta^2 x$$

$$(1 - \zeta^2)\omega_3 = x^3 - \zeta^2 x$$

Adding these two formulas, we get

$$x^3 = \frac{1}{2}(1 + \zeta^2)\omega_2 + \frac{1}{2}(1 - \zeta^2)\omega_3$$

Subtracting these formulas, we obtain $2\zeta^2 x = (1 + \zeta^2)\omega_2 - (1 - \zeta^2)\omega_3$ or

$$x = \frac{1}{2}(1 - \zeta^2)\omega_2 + \frac{1}{2}(1 + \zeta^2)\omega_3$$

It follows from these considerations that the space

$$C := \text{Span}(1, x, x^2, x^3) = \text{Span}(\omega_1, \omega_2, \omega_3, \omega_4)$$

is a Yetter-Drinfel’d Hopf subalgebra of $A$. As shown in [10], Paragraph 4.3, it is the so-called core of $\eta_1$; it is trivial, but not completely trivial in the sense explained there.

2.2 For this specific Yetter-Drinfel’d Hopf algebra $A$ over the group algebra $H$, we now form the biproduct $B := A \ast H$. It is generated by the four elements

$$u := x \ast 1_H \quad v := y \ast 1_H \quad r := 1_A \ast g_2 \quad s := 1_A \ast g_3$$

These generators can be used to give a presentation of $B$:

**Proposition.** The four generators satisfy the relations

1. $u^4 = 1$, $uv = vu$, $v^2 = \frac{1}{2}(1 + \zeta u + u^2 - \zeta u^3)$
2. $r^2 = 1$, $rs = sr$, $s^2 = 1$
3. $ru = u^3 r$, $rv = u^3 vr$, $su = us$, $sv = u^2 vs$

These relations are defining.
Proof. The first set of relations is a direct consequence of the defining relations of $A$, and the second set is a consequence of the relations inside the group $G$. The third set follows from the definition of the action of $G$ on $A$ and the multiplication in a smash product from Equation (1.3). We have

$$ru = (1_A \star g_2)(x \star 1_H) = (g_2 \cdot x) \star g_2 = x^3 \star g_2 = u^3r$$

and similarly $rv = (g_2 \cdot y) \star g_2 = (x^3 \cdot y) \star g_2$ as well as $su = (g_3 \cdot x) \star g_3 = us$ and $sv = (g_3 \cdot y) \star g_3 = (x^2 \cdot y) \star g_3$. To see that the relations are defining, we consider the abstract algebra given by this presentation. The computations just carried out yield a surjective algebra homomorphism from this abstract algebra to $B$, which has dimension 32. On the other hand, because every pair of the abstract generators satisfies a commutation relation, the abstract algebra is spanned by the 32 elements $w^i u^j v^k s^l$, where the exponents take the values $i = 0, 1, 2, 3$ and $j, k, l = 0, 1$. Therefore the abstract algebra has at most dimension 32. As it maps surjectively onto a space of dimension 32, it must have exactly this dimension, and the map must be an isomorphism.

The basis elements of $A$ can be used in a similar way to define

$$c_i := \omega_i \star 1_H \quad d_i := \eta_i \star 1_H \quad h_i := 1_A \star g_i$$

for $i = 1, 2, 3, 4$. Then the elements $c_i h_j$ together with the elements $d_i h_j$, for $i, j = 1, 2, 3, 4$, form a basis of $B$. Note that $c_1 = h_1 = 1_B$. Except for $u$, the above generators are among these elements, as $v = d_1$, $r = h_2$, and $s = h_3$. Because the canonical map from $A$ to $B$ is an algebra homomorphism, the relations that we have derived in Paragraph 2.1 yield corresponding relations in $B$. In particular, the elements $c_i$ commute with elements $d_j$, and we have

$$c_3 = c_4 c_2 \quad d_3 = c_4 d_1 \quad d_4 = c_3 d_2$$

As in the proof of the preceding proposition, Equation (1.3), the formula for the multiplication in a smash product, yields the commutation relations

$$h_3 c_i = c_i h_3 \quad h_j c_4 = c_4 h_j \quad h_2 c_2 = c_2 h_2 \quad h_i d_{g_j} = d_{g_j} h_i$$

where we have used the analogous notation $d_{g_i} := d_i$. We also note that

$$(c_2 h_2)^2 = c_2 h_2 c_2 h_2 = c_2 c_3 h_2 h_2 = c_2 c_3 = c_4$$

so that $(c_2 h_2)^3 = c_3 h_2$ and $(c_2 h_2)^4 = 1$.

2.3 The form of the coproduct can be derived from Equation (1.4). According to [23], Fact 2.11, the group-like elements of $B$ are of the form $a \star g_j$, where $a \in A$ is group-like and coinvariant. This implies that we have $\Delta_B(c_4) = c_4 \otimes c_4$ and $\Delta_B(h_j) = h_j \otimes h_j$. The formula for the coaction of $\omega_2$ yields

$$\Delta_B(c_2) = \frac{1}{2} c_2 (h_1 + h_3) \otimes c_2 + \frac{1}{2} c_2 (h_1 - h_3) \otimes c_3$$

$$= \frac{1}{2} ((h_1 + h_3) \otimes c_1 + (h_1 - h_3) \otimes c_4) (c_2 \otimes c_2)$$
while the formula for the coaction of \( \eta_1 \) yields
\[
\Delta_B(d_1) = \frac{1}{4} d_1(h_1 + h_2 + h_3 + h_4) \otimes d_1 + \frac{1}{4} d_1(h_1 + \zeta^2 h_2 - h_3 - \zeta^2 h_4) \otimes d_2 \\
+ \frac{1}{4} d_1(h_1 - h_2 + h_3 - h_4) \otimes d_3 + \frac{1}{4} d_1(h_1 - \zeta^2 h_2 - h_3 + \zeta^2 h_4) \otimes d_4 \\
= \frac{1}{4} (d_1 \otimes d_1) ((h_1 + h_3) \otimes (c_1 + c_4) + (h_2 + h_4) \otimes (c_1 - c_4)) \\
+ \frac{1}{4} (d_1 \otimes d_2) ((h_1 - h_3) \otimes (c_1 + c_4) + \zeta^2(h_2 - h_4) \otimes (c_1 - c_4))
\]

It should be noted that the coproduct of \( B \) is completely determined by the formulas above, since they also allow to compute the coproduct of \( c_3 = c_4 c_2 \) and \( d_j = h_j d_1 h_j \) by multiplicativity. The counit is given on these elements by \( \varepsilon_B(c_i) = \varepsilon_B(d_i) = \varepsilon_B(h_i) = 1 \) for all \( i = 1, 2, 3, 4 \).

An alternative way of determining the coproduct is to record its values on the generators:

**Proposition.** We have \( \Delta_B(r) = r \otimes r \) and \( \Delta_B(s) = s \otimes s \). Furthermore, we have
\[
\Delta_B(u) = \frac{1}{2} (u \otimes u + u \otimes u^3 + u^3 \otimes u - u^3 \otimes u^3)
\]
and
\[
\Delta_B(v) = \frac{1}{4} v(1 + r + s + rs) \otimes v + \frac{1}{4} v(1 - \zeta^2 r - s + \zeta^2 rs) \otimes uv \\
+ \frac{1}{4} v(1 - r + s - rs) \otimes u^2 v + \frac{1}{4} v(1 + \zeta^2 r - s - \zeta^2 rs) \otimes u^3 v
\]

The counit is given on generators by \( \varepsilon_B(u) = \varepsilon_B(v) = \varepsilon_B(r) = \varepsilon_B(s) = 1 \).

**Proof.** We have already said that \( r = h_2 \) and \( s = h_3 \) are group-like. Since \( v = d_1 \), the formula for \( \Delta_B(v) \) is just a restatement of the corresponding one for \( \Delta_B(d_1) \). According to [10], Proposition 2.3, we have
\[
\Delta_A(x) = \frac{1}{2} (x \otimes x + x \otimes x^3 + x^3 \otimes x - x^3 \otimes x^3)
\]
With the help of the formulas for \( \delta(x) \) and \( \delta(x^3) \) recalled in Paragraph 2.1, Equation (1.4) then yields
\[
\Delta_B(u) = \frac{1}{4} (x \ast (g_1 + g_3)) \otimes (x \ast 1_H) + \frac{1}{4} (x \ast (g_1 - g_3)) \otimes (x^3 \ast 1_H) \\
+ \frac{1}{4} (x \ast (g_1 + g_3)) \otimes (x \ast 1_H) + \frac{1}{4} (x \ast (g_1 + g_3)) \otimes (x^3 \ast 1_H) \\
+ \frac{1}{4} (x^3 \ast (g_1 + g_3)) \otimes (x \ast 1_H) + \frac{1}{4} (x^3 \ast (g_1 - g_3)) \otimes (x^3 \ast 1_H) \\
- \frac{1}{4} (x^3 \ast (g_1 - g_3)) \otimes (x \ast 1_H) - \frac{1}{4} (x^3 \ast (g_1 + g_3)) \otimes (x^3 \ast 1_H)
\]
By adding the first and the second as well as the third and the fourth line, this expression becomes

$$\Delta_B(u) = \frac{1}{2}(x \ast g_1) \otimes (x \ast 1_H) + \frac{1}{2}(x \ast 1_H) \otimes (x^3 \ast 1_H)$$

$$+ \frac{1}{2}(x^3 \ast g_3) \otimes (x \ast 1_H) - \frac{1}{2}(x^3 \ast g_3) \otimes (x^3 \ast 1_H)$$

Using that $g_1 = 1_H$, we get the formula for $\Delta_B(u)$. The form of the counit follows directly from the defining equations $\varepsilon_A(x) = \varepsilon_A(y) = 1$ in [10], Paragraph 2.3. \hfill \Box

2.4 To complete the description of $B$, we compute the values of the antipode on the generators:

**Proposition.** We have $S_B(u) = \frac{1}{2}(u + su + u^3 - su^3)$ and

$$S_B(v) = \frac{1}{4}(1 + u + u^2 + u^3) v + \frac{1}{4} \left( \zeta 1 + \zeta^3 u - \zeta u^2 - \zeta^3 u^3 \right) rv$$

$$+ \frac{1}{4} \left( 1 - u + u^2 - u^3 \right) sv + \frac{1}{4} \left( -\zeta 1 + \zeta^3 u + \zeta u^2 - \zeta^3 u^3 \right) rsv$$

as well as $S_B(r) = r$ and $S_B(s) = s$.

**Proof.** The forms of $S_B(r)$, $S_B(s)$, and $S_B(u)$ follow from Equation (1.5), where the third also requires the equation for $\delta(x)$ given in Paragraph 2.1 as well as the facts that $S_A(x) = x$ and $S_A(x^3) = x^3$, which follow from [10], Paragraph 2.5. The form of $S_B(v)$ can be derived in the same way, but the computation is more tedious: Using the formula for the coaction of $y = \eta_1$ given in Paragraph 2.1 and the equations for $S_A(\eta_i)$ from [10], loc. cit. yields

$$S_B(v) = \frac{1}{8}(h_1 + h_2 + h_3 + h_4) (d_1 + \zeta^3 d_2 + d_3 - \zeta^3 d_4)$$

$$+ \frac{1}{8}(h_1 + \zeta^2 h_2 - h_3 - \zeta^2 h_4) (\zeta^3 d_1 + d_2 - \zeta^3 d_3 + d_4)$$

$$+ \frac{1}{8}(h_1 - h_2 + h_3 - h_4) (d_1 - \zeta^3 d_2 + d_3 + \zeta^3 d_4)$$

$$+ \frac{1}{8}(h_1 - \zeta^2 h_2 - h_3 + \zeta^2 h_4) (-\zeta^3 d_1 + d_2 + \zeta^3 d_3 + d_4)$$

The assertion now follows by collecting like terms, writing the result in terms of the generators, and using their commutation relations. \hfill \Box

2.5 In order to determine the Wedderburn decomposition of $B$, it will be helpful to know the center of $B$. It is described in the following theorem:
**Theorem.** The center $Z(B)$ of $B$ has dimension 11. The four sets of vectors

1. $c_1, c_4, c_2 + c_3, d_1 + d_2 + d_3 + d_4$
2. $(c_1 + c_2 + c_3 + c_4)h_2, (d_1 + d_2 + d_3 + d_4)h_2$
3. $(c_1 + c_4)h_3, (c_2 + c_3)h_3, (d_1 + d_2 + d_3 + d_4)h_3$
4. $(c_1 + c_2 + c_3 + c_4)h_4, (d_1 + d_2 + d_3 + d_4)h_4$

Together form a basis of $Z(B)$.

**Proof.** (1) Suppose that $b = \sum_{j=1}^{4} a_j \ast g_j$ lies in the center of $B$. That $b$ commutes with elements of the form $h = 1_A \ast g$ for some $g \in G$ means that

$$\sum_{j=1}^{4} a_j \ast g_j g = bh = hb = (1_A \ast g) \sum_{j=1}^{4} a_j \ast g_j = \sum_{j=1}^{4} (g \cdot a_j) \ast g_j g$$

Thus, for each $j = 1, 2, 3, 4$, we must have $g \cdot a_j = a_j$, and therefore $a_j$ lies in the space of invariants, which we denote by $A^H$.

(2) On the other hand, $b$ commutes with elements of the form $b' = a' \ast 1_H$ provided that

$$\sum_{j=1}^{4} a' a_j \ast g_j = b'b = bb' = \left( \sum_{j=1}^{4} a_j \ast g_j \right) (a' \ast 1_H) = \sum_{j=1}^{4} (g_j \cdot a') a_j \ast g_j$$

This condition is equivalent to the condition

$$a' a_j = (g_j \cdot a') a_j$$

for all $j = 1, 2, 3, 4$.

(3) In fact, it is sufficient to test this condition in the two cases $a' = x$ and $a' = y$, because $b$ will be contained in the center if and only if it commutes with the four generators $u, v, r, s$ introduced at the beginning of Paragraph 2.2. The generators $r$ and $s$ are of the form treated in Step (1), and the generators $u$ and $v$ correspond to the two cases $a' = x$ and $a' = y$.

(4) Discussing each $j$ separately, we see that the condition $a' a_j = (g_j \cdot a') a_j$ is vacuous for $j = 1$. For $j = 2$, the condition states that

$$xa_2 = (g_2 \cdot x) a_2 = x^3 a_2 \quad \text{and} \quad ya_2 = (g_2 \cdot y) a_2 = x^3 ya_2$$

in other words, $a_2 = x^2 a_2$ and $a_2 = x^3 a_2$, which means that $xa_2 = a_2$.

For $j = 3$, the condition states that

$$xa_3 = (g_3 \cdot x) a_3 = xa_3 \quad \text{and} \quad ya_3 = (g_3 \cdot y) a_3 = x^2 ya_3$$

The first part is vacuous and the second is equivalent to the condition $x^2 a_3 = a_3$.  

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For \( j = 4 \), the condition states that
\[
x a_4 = (g_4.x)a_4 = x^3 a_4 \quad \text{and} \quad y a_4 = (g_4.y)a_4 = xy a_4
\]
in other words, \( a_4 = x^2 a_4 \) and \( a_4 = x a_4 \). Obviously, the second part of the condition implies the first.

(5) The group-like elements of \( A \) are divided into four orbits under the action of \( G \), and \( A^H \) is spanned by the sum of the elements in each of these orbits, i.e., is spanned by the four elements
\[
\omega_1, \omega_4, \omega_2 + \omega_3, \eta_1 + \eta_2 + \eta_3 + \eta_4
\]
It is immediate from the definition of these elements that \( \omega_1 = 1_A, \omega_4 = x^2, \omega_2 + \omega_3 = x + x^3, \) and \( \eta_1 + \eta_2 + \eta_3 + \eta_4 = (1_A + x + x^2 + x^3)y \). We can therefore expand \( a_j \) in the form
\[
a_j = \lambda_{1,j}1_A + \lambda_{2,j}x^2 + \lambda_{3,j}(x + x^3) + \lambda_{4,j}(1_A + x + x^2 + x^3)y
\]
For \( j = 1 \), we saw above that there are no extra conditions on \( a_1 \), so the coefficients \( \lambda_{k,1} \) can be arbitrary. For \( j = 2 \), the condition \( xa_2 = a_2 \) implies that \( \lambda_{1,2} = \lambda_{2,2} = \lambda_{3,2} \), and so \( a_2 \) is a linear combination of the vectors
\[
1_A + x + x^2 + x^3 \quad \text{and} \quad (1_A + x + x^2 + x^3)y
\]
For \( j = 3 \), the condition \( x^2a_3 = a_3 \) implies only that \( \lambda_{1,3} = \lambda_{2,3} \), and so \( a_3 \) is a linear combination of the three vectors
\[
1_A + x^2 \quad x + x^3 \quad (1_A + x + x^2 + x^3)y
\]
The condition on \( a_4 \) is the same as the condition on \( a_2 \), and so \( a_4 \) is a linear combination of the same vectors as \( a_2 \), which can, as we have already seen, also be written as
\[
\omega_1 + \omega_2 + \omega_3 + \omega_4 \quad \text{and} \quad \eta_1 + \eta_2 + \eta_3 + \eta_4
\]
Rewriting these results in terms of \( c_i \) and \( d_i \) instead of \( \omega_i \) and \( \eta_i \) yields the assertion, where the four items in the list correspond to the index values \( j = 1, j = 2, j = 3, \) and \( j = 4 \) in the preceding discussion.

3 Representations

3.1 As we pointed out in Paragraph 1.2, the dual space \( A^* \) is a Yetter-Drinfel’d Hopf algebra over \( H^* \), and the corresponding bi-product \( A^* \ast H^* \) is isomorphic to \( B^* \). The Hopf algebra \( H^* = K[G]^* \) is itself a group algebra, namely the group algebra of the character group \( G \), whose elements are considered as elements of \( H^* \) by linear extension. In our situation, the nondegenerate symmetric
bicharacter \( \theta \) described in Paragraph 2.1 yields a specific bijection between \( G \) and \( \hat{G} \): For \( i = 1, 2, 3, 4 \), we define the character \( \gamma_i \) by

\[
\gamma_i(g_j) = \theta(g_i, g_j)
\]

for all \( j = 1, 2, 3, 4 \). We then have \( \hat{G} = \{ \gamma_1, \gamma_2, \gamma_3, \gamma_4 \} \), where \( \gamma_1 = \varepsilon_H \).

The action of \( \gamma_i \) on \( A^* \) is closely related to the action of \( g_i \) on \( A \): For \( \varphi \in A^* \) and \( a \in A \), we have by Equation (1.1) and [10], Lemma 4.3 that

\[
(\gamma_i, \varphi)(a) = (\gamma_i \otimes \varphi)(a) = \gamma_i(a^{(1)}) \varphi(a^{(2)}) = \varphi(g_i^{-1}a)
\]

where of course \( g_i = g_i^{-1} \) in our situation.

This equation will be used below to determine which of the group-like elements in \( B^* \) are central. But first, we record the group-like elements of \( B \):

**Lemma.** We have

\[
G(B) = \langle c_4 \rangle \times \langle h_2 \rangle \times \langle h_3 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2
\]

Moreover, \( c_4 \) is the only nontrivial central group-like element of \( B \).

**Proof.** As we already mentioned in Paragraph 2.3, the group-like elements of \( B \) are of the form \( a \star g_j \), where \( a \in A \) is group-like and coinvariant. Of the eight group-like elements \( \omega_1, \omega_2, \omega_3, \omega_4, \eta_1, \eta_2, \eta_3, \eta_4 \) of \( A \), only \( \omega_1 = 1_A \) and \( \omega_4 \) are coinvariant. Because these are also invariant, all the group-like elements of \( B \) commute, so that they form the elementary abelian group of order 8 stated above. As we have seen explicitly in the proof of Theorem 2.5 that \( h_2, h_3, \) and \( h_4 \) are not central, \( c_4 \) is the only nontrivial central group-like element of \( B \).

Alternatively, this lemma can be stated in terms of the generators introduced at the beginning of Paragraph 2.2 as we have \( c_4 = u^2, h_2 = r, \) and \( h_3 = s \).

The group of group-like elements of \( B^* \) has a very similar form:

**Proposition.** There are three elements \( \chi_1, \chi_2, \chi_3 \in G(B^*) \) that satisfy

1. \( \chi_1(d_i) = -1 \) and \( \chi_1(c_i) = \chi_1(h_i) = 1 \)
2. \( \chi_2(h_2) = \chi_2(h_4) = -1 \) and \( \chi_2(c_i) = \chi_2(d_i) = \chi_2(h_3) = 1 \)
3. \( \chi_3(h_3) = \chi_3(h_4) = -1 \) and \( \chi_3(c_i) = \chi_3(d_i) = \chi_3(h_2) = 1 \)

for all \( i = 1, 2, 3, 4 \). With these elements, we have

\[
G(B^*) = \langle \chi_1 \rangle \times \langle \chi_2 \rangle \times \langle \chi_3 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2
\]

Moreover, \( \chi_1 \) is the only nontrivial central group-like element of \( B^* \).
Proof. (1) Since $B^* \cong A^* \star H^*$, it follows as in the preceding lemma that the group-like elements of $B^*$ are of the form $\rho \star \gamma$, where $\gamma \in G$ and $\rho \in G(A^*)$ is $H^*$-coinvariant. In view of Equation (1.2), a linear form is $H^*$-coinvariant if and only if it is an $H$-linear map to the base field, and similarly Equation (1.1) implies that it is $H^*$-invariant if and only if it is an $H$-coinvariant map to the base field. On the other hand, Equation (3.1) above shows that a linear form is $H^*$-invariant if and only if it is $H$-coinvariant, so that all these four properties are equivalent.

(2) According to [10], Paragraph 2.1, there are eight one-dimensional characters of $A$, which were denoted there by $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3$, $\varepsilon_4$, $\rho_1$, $\rho_2$, $\rho_3$, and $\rho_4$. Their definition depended on the exact order of $\zeta$, but the only $H$-linear ones were $\varepsilon_1$ and $\varepsilon_4$, where $\varepsilon_1 = \varepsilon_A$ was the counit of $A$, and $\varepsilon_4$ was in each of the cases determined by the condition that $\varepsilon_4(x) = 1$ and $\varepsilon_4(y) = -1$. Thus $B^*$ contains eight group-like elements, namely

$\varepsilon_1 \gamma_1 \quad \varepsilon_1 \gamma_2 \quad \varepsilon_1 \gamma_3 \quad \varepsilon_1 \gamma_4 \quad \varepsilon_4 \gamma_1 \quad \varepsilon_4 \gamma_2 \quad \varepsilon_4 \gamma_3 \quad \varepsilon_4 \gamma_4$

The $H^*$-invariance of $\varepsilon_1$ and $\varepsilon_4$ and the multiplication formula in a smash product stated in Equation (1.3) together imply that these elements commute; since $\varepsilon_4^2 = \varepsilon_A$, they all have at most order 2.

(3) We set $\chi_1 := \varepsilon_4 \star \gamma_1$ and then have $\chi_1(d_i) = -1$ and $\chi_1(c_i) = \chi_1(h_i) = 1$ as required. By letting $\chi_2 := \varepsilon_1 \star \gamma_2$, the requirements $\chi_2(h_2) = \chi_2(h_4) = -1$ and $\chi_2(c_i) = \chi_2(d_i) = \chi_2(h_3) = 1$ are also satisfied. But for $\gamma_2$, we have according to the definition that $\gamma_2(g_2) = \zeta^2$ and $\gamma_2(g_3) = -1$, while $\gamma_4(g_2) = -\zeta^2$ and $\gamma_4(g_3) = -1$. So if $\zeta^2 = 1$, we set $\chi_3 := \varepsilon_1 \star \gamma_2$, while we set $\chi_3 := \varepsilon_1 \star \gamma_4$ if $\zeta^2 = -1$. It is obvious that these three characters generate $G(B^*)$.

(4) Because $\varepsilon_4$ is $H^*$-invariant, we have for $\varphi \in A^*$ and $\gamma \in H^*$ that

$$(\varphi \star \gamma)\chi_1 = \varphi(\gamma_1, \varepsilon_4) \star \gamma_2 = \varphi \varepsilon_4 \star \gamma = \chi_1(\varphi \star \gamma)$$

so that $\chi_1$ is central. On the other hand, we have for $i = 2, 3, 4$ that

$$(\varepsilon_A \star \gamma_i)(\varphi \star \varepsilon_H) = (\gamma_i, \varphi) \star \gamma_i \quad \text{but} \quad (\varphi \star \varepsilon_H)(\varepsilon_A \star \gamma_i) = \varphi \star \gamma_i$$

Since in general $\gamma_i \cdot \varphi \neq \varphi$ by Equation (3.1) above, $\varepsilon_A \star \gamma_i$ is not central. This shows that $\chi_1$ is the only nontrivial central group-like element in $B^*$. $\square$

At least part of the preceding proposition can also be derived with the help of the presentation of $B$ given in Proposition 2.2. The values of our characters on the generators $u$, $v$, $r$, and $s$ introduced there are given in the following table:

|     | u  | v  | r  | s  |
|-----|----|----|----|----|
| $\chi_1$ | 1  | -1 | 1  | 1  |
| $\chi_2$ | 1  | 1  | -1 | 1  |
| $\chi_3$ | 1  | 1  | 1  | -1 |
We also note that the existence of nontrivial central group-like elements in $B$ and $B^*$ is consistent with [10], Theorem 1, which states under our assumptions on the base field that a semisimple Hopf algebra of prime power dimension always contains such elements.

3.2 We have already mentioned in Paragraph 1.2 that biproducts are semisimple and cosemisimple if their two factors are. Here, this implies that the biproduct $B$ under consideration is semisimple and cosemisimple, because the Yetter-Drinfel’d Hopf algebra $A$ is semisimple by [10], Proposition 2.1 and cosemisimple by [10], Proposition 2.4. But we can determine the Wedderburn decomposition of $B$ and $B^*$ precisely:

**Theorem.** Both $B$ and $B^*$ have eight 1-dimensional, two 2-dimensional, and one 4-dimensional irreducible representations.

**Proof.** Since $\text{dim } Z(B) = 11$ by Theorem 2.5 and $|G(B^*)| = 8$ by Proposition 3.1, there are exactly three irreducible representations of $B$ of dimension larger than 1. The sum of the squares of the dimensions of these three representations is $32 - 8 = 24$, which implies that $B$ has, up to isomorphism, two 2-dimensional and one 4-dimensional irreducible representations.

To determine the Wedderburn decomposition of $B^*$, recall from Paragraph 1.2 that $B^*$ is the smash product of $A^*$ and $H^* = K[\hat{G}]$ as an algebra. Therefore, the correspondence between the irreducible representations of $B^*$ and the irreducible representations of $A^*$ is described by Clifford theory, which in our situation yields the following form of the irreducible representations: Because the coalgebra $A$ has a basis consisting of group-like elements, $A^*$ is a commutative semisimple algebra whose simple modules are one-dimensional and given by evaluation at one of these group-like elements. Its primitive idempotents are therefore the dual basis elements

$$p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4$$

of the basis $\omega_1, \omega_2, \omega_3, \omega_4, \eta_1, \eta_2, \eta_3, \eta_4$. The action of $\hat{G}$ on $A^*$ permutes these primitive idempotents. According to the theory, we have to choose a representative $\rho$ in each orbit, consider the stabilizer $T_\rho \subset \hat{G}$, often called the inertia group, of this representative, and find the simple $T_\rho$-modules. For each such simple $T_\rho$-module $V$, the induced module $\text{Ind}_{\hat{T}}^G V$ can be endowed with a $B^*$-module structure that makes it simple (cf. [3], Proposition (11.16); [9], Corollary 3.5). This yields a bijective correspondence between the isomorphism classes of simple $T_\rho$-modules in the various orbits and the isomorphism classes of simple $B^*$-modules, the so-called Clifford correspondence.

It follows from Equation (3.1) that $\hat{G}$ permutes the primitive idempotents in $A^*$ in the same way as $G$ permutes the group-like elements in $A$ (cf. [9], Equation (2.3)). From Paragraph 2.1 we know that the orbits of the $G$-action on the set of group-like elements of $A$ are $\{\omega_1\}$, $\{\omega_4\}$, $\{\omega_2, \omega_3\}$, and $\{\eta_1, \eta_2, \eta_3, \eta_4\}$.
Therefore, the orbits of the \( \hat{G} \)-action on the set of primitive idempotents in \( A^* \) are \( \{p_1\}, \{p_4\}, \{p_2, p_3\}, \) and \( \{q_1, q_2, q_3, q_4\} \).

Since \( T_{p_1} = T_{p_4} = \hat{G} \), the Clifford correspondence yields four 1-dimensional irreducible representations for each of the orbits \( \{p_1\} \) and \( \{p_4\} \), so eight 1-dimensional irreducible representations in total. Since \( T_{q_1} = \{1_G\} \), the Clifford correspondence yields one 4-dimensional irreducible representation for the orbit \( \{q_1, q_2, q_3, q_4\} \). Finally, since \( g_3 \) fixes \( \omega_2 \), it follows from Equation (3.1) that \( \gamma_3 \) fixes \( p_2 \), so that \( T_{p_2} = \{1_G, \gamma_3\} \). The two 1-dimensional representations of \( T_{p_2} \) therefore give rise to two 2-dimensional irreducible representations of \( B^* \).

In Paragraph 5.3 we will describe in addition the product in the Grothendieck ring of \( B \).

4 Hopf subalgebras of dimension 16

4.1 The algebra \( C \) introduced in Paragraph 2.1 is a Yetter-Drinfel’d Hopf subalgebra of \( A \), and therefore the biproduct \( B = A \ast H \) contains the biproduct \( N := C \ast H \)

as a Hopf subalgebra. Clearly, \( N \) has dimension 16, and the elements \( c_i h_j \) for \( i, j = 1, 2, 3, 4 \) form a basis of \( N \). This implies that \( N \) is generated as an algebra by \( c_2, c_4, h_2, \) and \( h_3 \). This generating set has the advantage of being close to the group-like elements of \( A \). Alternatively, \( N \) is generated by the elements \( u, r, \) and \( s \) introduced in Paragraph 2.2 which satisfy the relations

\[
\begin{align*}
    u^4 &= 1, \\
    ru &= u^3 r, \\
    su &= us, \\
    r^2 &= 1, \\
    rs &= sr, \\
    s^2 &= 1
\end{align*}
\]

from Proposition 2.2. An argument that is very similar to the one used there shows that these relations are defining. Comparing this presentation to the usual presentation of the dihedral group (cf. [24], Section 5.3), we see that \( N \) is isomorphic to the group ring \( K[D_8 \times \mathbb{Z}_2] \) as an algebra, but not as a coalgebra, as it is not cocommutative, as for example the formula for the coproduct of \( c_2 \) in Paragraph 2.3 shows. This issue will be revisited at the end of Paragraph 1.2.

The groups of group-like elements of \( N \) and \( N^* \) have the following form:

Lemma. We have

\[
\text{G}(N) = \langle c_4 \rangle \times \langle h_2 \rangle \times \langle h_3 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2
\]

and also \( \text{G}(N^*) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \).
Proof. Since all the group-like elements of $B$ are already contained in $N$, we have $G(N) = G(B)$, so that the first assertion follows from Lemma 3.1.

As in the proof of Proposition 3.1, we have $N^* \cong C^* \ast H^*$, and under this isomorphism a group-like element $\chi \in G(N^*)$ decomposes in the form $\rho \ast \gamma$, where $\gamma \in \hat{G}$ and $\rho \in G(C^*)$ is $H^*$-coinvariant. We also discussed there that $H^*$-coinvariance is equivalent to $H^*$-invariance as well as to $H$-linearity and $H$-colinearity. If $\chi' \in G(N^*)$ is another group-like element that decomposes in the form $\rho' \ast \gamma'$, the $H^*$-invariance implies that $\chi \chi'$ corresponds to $\rho \rho' \ast \gamma \gamma'$.

There are clearly four algebra homomorphisms from $C$ to $K$, determined by the image of $x$, which must be a fourth root of unity. Out of these, two are $H^*$-linear, namely $\varepsilon'_1 := \varepsilon_C$, the counit, which satisfies $\varepsilon'_1(x) = 1$, and the character $\varepsilon'_2$ determined by $\varepsilon'_2(x) = -1$. It also satisfies $\varepsilon'_2(\omega_2) = \varepsilon'_2(\omega_3) = -1$, so that $\varepsilon'_2 = \varepsilon_C$. This shows that $|G(N^*)| = 8$ and $G(N^*) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

In analogy with Paragraph 3.1, we choose generators $\chi'_1, \chi'_2, \text{ and } \chi'_3$ of $G(N^*)$ whose values on the generators of $N$ are given by the following table:

|     | $u$ | $r$ | $s$ |
|-----|-----|-----|-----|
| $\chi'_1$ | $-1$ | $1$ | $1$ |
| $\chi'_2$ | $1$  | $-1$| $1$ |
| $\chi'_3$ | $1$  | $1$ | $-1$|

Explicitly, $\chi'_1$ corresponds to $\varepsilon'_2 \ast \gamma_1$, while $\chi'_2$ corresponds to $\varepsilon'_1 \ast \gamma_3$, and $\chi'_3$ corresponds to $\varepsilon'_1 \ast \gamma_2$ if $\zeta^2 = 1$ and to $\varepsilon'_1 \ast \gamma_4$ if $\zeta^2 = -1$.

The preceding lemma is consistent with a general fact about semisimple Hopf algebras of dimension 16 that are neither commutative nor cocommutative: Such a Hopf algebra has an abelian group of group-likes of order 8 if and only if the dual Hopf algebra does (cf. [6], Proposition 3.1).

The isomorphism $G(N^*) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ can also be established in a more direct way that does not depend on the biproduct perspective:

Corollary. If $\mathbb{Z}_2 = \{1, -1\}$ is written multiplicatively, the map

$$G(N^*) \to \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \; \chi \mapsto (\chi(u), \chi(r), \chi(s))$$

is an isomorphism of groups.

Proof. For $\chi \in G(N^*)$, the relation $\chi(ru) = \chi(u^3r)$ implies that $\chi(u)^2 = 1$. This condition and the relations $r^2 = s^2 = 1$ imply that $\chi$ must map all three generators $u, r,$ and $s$ to $\pm 1$. So the above map is well-defined, and also injective, because $\chi$ is determined by its values on the generators. On the other hand, it is
not complicated to check that the above defining relations of $N$ are satisfied by an arbitrary choice of these signs, so the map is also surjective. Our assertion will therefore follow if we can show that it is a group homomorphism.

For $\chi, \chi' \in G(N^*)$, we clearly have $(\chi\chi')(r) = \chi(r)\chi'(r)$, since $r$ is group-like, and the same equation holds for $s$. For $u$, we have by Proposition 2.3 that

$$(\chi\chi')(u) = \frac{1}{2} (\chi(u)\chi'(u) + \chi(u)\chi'(u))^3 + \chi(u)^3\chi(s)\chi'(u) - \chi(u)^3\chi(s)\chi'(u)^3) = \chi(u)\chi'(u)$$

so that the requirement is also fulfilled for this generator.\[\square\]

We note that the fact that $\chi(u) = \pm 1$ recorded in the preceding proof implies that $\chi(u) = \chi(c_2) = \chi(c_3)$ for all $\chi \in G(N^*)$.

Of the eight group-like elements of $N^*$, four are central:

**Proposition.** The group $G(N^*) \cap Z(N^*)$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and is generated by $\chi'_1$ and $\chi'_2$.

**Proof.** If $\chi \in G(N^*)$ corresponds to $\rho \ast \gamma \in C^* \ast H^*$ as in the proof of the preceding lemma, we have for $\varphi \in N^*$ in view of the $H$-colinearity of $\rho$ that

$$(\varphi\chi)(a \ast h) = \varphi(a_1) \ast a_2^{(1)} h_1^{(1)}) \varphi(a_2^{(2)}) \gamma(h_2)) = \varphi(a_1) \ast h_1^{(1)}) \varphi(a_2^{(2)}) \gamma(h_2))$$

on the one hand and

$$(\chi\varphi)(a \ast h) = \rho(a_1) \gamma(a_2^{(1)} h_1^{(1)}) \varphi(a_2^{(2)}) \ast h_2))$$

on the other hand, so that $\chi$ is central if and only if

$$a_1 \ast h_1^{(1)} \rho(a_2^{(2)}) \gamma(h_2)) = \rho(a_1) \gamma(a_2^{(1)} h_1^{(1)}) a_2^{(2)} \ast h_2)$$

for all $a \in C$ and all $h \in H$.

If $\chi = \chi'_1$, we have $\rho = \varepsilon'_2$ and $\gamma = \gamma_1 = \varepsilon_H$, so that this condition becomes

$$a_1 \ast h_1^{(1)} \rho(a_2^{(2)}) = \rho(a_1) a_2^{(2)} \ast h$$

which is satisfied since $C$ is cocommutative. So $\chi'_1$ is central.

If $\chi = \chi'_2$, we have $\rho = \varepsilon'_3 = \varepsilon_C$ and $\gamma = \gamma_3$, so that this condition becomes

$$a \ast h_1^{(1)} \gamma_3(h_2) = \gamma_3(a^{(1)} h_1^{(1)}) a^{(2)} \ast h$$

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which will hold if and only if $\gamma_3(a^{(1)})a^{(2)} = a$ for all $a \in C$. But because $\gamma_3(g_3) = \theta(g_3,g_3) = 1$, the formulas for $\delta(x)$ and $\delta(x^3)$ given in Paragraph 2.1 show that this is indeed the case. So $\chi'_3$ is also central.

If $\chi = \chi'_3$, we have $\rho = \varepsilon'_1 = \varepsilon_C$ and $\gamma = \gamma_2$ or $\gamma = \gamma_4$, depending on the order of $\zeta$. But if $a = \omega_2$, the formula for $\delta(\omega_2)$ given in Paragraph 2.1 yields that $\gamma_2(a^{(1)})a^{(2)} = \gamma_4(a^{(1)})a^{(2)} = \omega_3$. So the condition is not met in both cases and $\chi'_3$ is not central.

4.2 A characteristic property of $N$ is the following:

**Proposition.** The Hopf algebra $N$ has exactly three quotient Hopf algebras of dimension 8. All of these quotients are cocommutative and isomorphic to either $K[Z_2 \times Z_2 \times Z_2]$ or $K[D_8]$.

**Proof.** (1) If $N$ has a quotient Hopf algebra $F$ of dimension 8, then $N^*$ has a Hopf subalgebra $F^*$ that has index 2 and is therefore necessarily normal by [11], Proposition 2. Thus $N^*$ fits into an extension of the form

$$ F^* \hookrightarrow N^* \twoheadrightarrow M $$

and so $N$ fits into an extension of the form

$$ M^* \hookrightarrow N \twoheadrightarrow F $$

where $M^*$ is normal and has dimension 2. As pointed out in Paragraph 1.3, $M^*$ is then central, and therefore $M^* = K(g)$ for a central group-like element $g$ of order 2, so that $F \cong N/((M^*)^+ N)$.

(2) It follows from Lemma 4.1 that $N$ contains exactly three central group-like elements of order 2, namely $c_4$, $h_3$, and $c_4 h_3$. Therefore it has exactly three quotient Hopf algebras of dimension 8, namely

$$ F_1 := N/ (K(c_4)^+ N) \quad F_2 := N/ (K(h_3)^+ N) \quad F_3 := N/ (K(c_4 h_3)^+ N) $$

In all three of these quotient spaces, the images of the group-like elements $r = h_2$ and $s = h_3$ are group-like.

(3) In $F_1$, we have $\bar{c}_4 = \bar{u}^2 = 1_{F_1}$, so that $F_1$ is generated by $\bar{c}_2, \bar{h}_2$, and $\bar{h}_3$, or alternatively by $\bar{u}, \bar{r},$ and $\bar{s}$, since $\bar{u} = \bar{c}_2$ in $F_1$. We have seen in Proposition 2.3 that

$$ \Delta_B(u) = \frac{1}{2} (u \otimes u + u \otimes u^3 + u^3 \otimes u - u^3 \otimes u^3) $$

In $F_1$, this equation reduces to $\Delta_{F_1} (\bar{u}) = \bar{u} \otimes \bar{u}$. In fact, the relations recorded at the beginning of Paragraph 1.4 imply that $\bar{u}^2 = \bar{r}^2 = \bar{s}^2 = 1$ and that these three generators commute in $F_1$, which shows that $F_1 \cong K[Z_2 \times Z_2 \times Z_2]$. 

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(4) In $F_2$, we have $\bar{h}_3 = \bar{s} = 1_{F_2}$, so that $F_2$ is generated by $\bar{c}_2$, $\bar{c}_4$, and $\bar{h}_2$, or alternatively by $\bar{u}$ and $\bar{r}$. We have seen in Paragraph 2.8 that

$$\Delta_B(c_2) = \frac{1}{2} c_2 (1_H + h_3) \otimes c_2 + \frac{1}{2} c_2 (1_H - h_3) \otimes c_3$$

In $F_2$, this reduces to $\Delta_{F_2}(c_2) = \bar{c}_2 \otimes \bar{c}_2$. Now we know from Equation (2.2) that $(c_2 h_2)^2 = c_4$, so that $(c_2 h_2)^3 = c_3 h_2$ and $(c_2 h_2)^4 = 1$. This shows that $F_2$ is already generated by $\bar{c}_2$ and $\bar{h}_2$ alone. Also, we have

$$(c_2 h_2)^3 h_2 = c_3 = h_2 (c_2 h_2)$$

In view of the presentation of the dihedral group $D_8$ already mentioned in Paragraph 1.1, this implies that $F_2 \cong K[D_8]$.

(5) In $F_3$, we have $\bar{u}^2 = \bar{s}$, so that $F_3$ is generated by $\bar{u}$ and $\bar{r}$. The formula for $\Delta_B(u)$ already recalled above gives $\Delta_{F_3}(\bar{u}) = \bar{u} \otimes \bar{u}$ also in this case. The relations recorded at the beginning of Paragraph 1.1 yield directly that $\bar{u}^4 = 1$ and $\bar{r} \bar{u} = \bar{u}^3 \bar{r}$, so that also $F_3 \cong K[D_8]$. As in the previous step, $F_3$ is also already generated by $\bar{c}_2$ and $\bar{h}_2$ alone.

From the classification results in [6], Table 1, we know that there are exactly two Hopf algebras of dimension 16 for which both the group of group-likes and the group of group-likes of the dual are elementary abelian of order 8, namely those denoted there by $H_{d,1,1}$ and $H_{d,-1,1}$. The Hopf algebra $H_{d,-1,1}$ is isomorphic to $H_8 \otimes K\mathbb{Z}_2$, where $H_8$ is the Kac-Paljutkin Hopf algebra of dimension 8. Since $H_8 \otimes K\mathbb{Z}_2$ has a noncocommutative quotient isomorphic to $H_8$, the proposition above implies that $N$ must be isomorphic to $H_{d,1,1}$. By [6], Remark 1.4, we then know that $N$ is isomorphic to $K[D_8 \times \mathbb{Z}_2]_J$, which is triangular, and that $N$ is self-dual. The twisting cocycle $J$ is described in [6], Section 7, Example 2. In particular, we see again that $N$ is isomorphic to $K[D_8 \times \mathbb{Z}_2]$ as an algebra.

4.3 In fact, we can construct an explicit isomorphism between $N$ and $H_{d,1,1}$. As described in [6], Paragraph 3.2, Page 629ff, the Hopf algebra $H_{d,1,1}$ is a bicrossed product $K[\Gamma] \#^\theta K[L]$ for $\Gamma = \langle x \rangle \times \langle y \rangle \times \langle z \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $L = \langle t \rangle \cong \mathbb{Z}_2$ in which the cocycle and the coaction are trivial. Here, and in the present paragraph only, we use the notation from [6], loc. cit., so that $\theta$ does not denote the nondegenerate bicharacter from Paragraph 2.1 but rather a dual cocycle like the one that was denoted by $\kappa$ in Paragraph 1.5. Furthermore, $x$ and $y$ do not denote the generators of $A$ introduced in Paragraph 2.1 but rather two of the three generators of $\Gamma$ already appearing above. These generators give rise to the primitive idempotents $e_{i,j,k}$ in $K[\Gamma]$ via the formula

$$e_{i,j,k} := \frac{1}{8} (1 + (-1)^i x) (1 + (-1)^j y) (1 + (-1)^k z)$$

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where the indices $i$, $j$, and $k$ each take the value $0$ or $1$. Conversely, the generators can be expressed in terms of the primitive idempotents via the formulas

$$x = \sum_{i,j,k=0}^1 (-1)^i e_{i,j,k}, \quad y = \sum_{i,j,k=0}^1 (-1)^j e_{i,j,k}, \quad z = \sum_{i,j,k=0}^1 (-1)^k e_{i,j,k}$$

The action $\rightarrow$ of the bicrossed product is determined by the facts that the unit element acts as the identity, while $t$ acts as the algebra automorphism that takes the values

$$t \rightarrow x = y \quad t \rightarrow y = x \quad t \rightarrow z = z$$
on the generators. Using the primitive idempotents, the dual cocycle $\theta$ is defined as the map that sends $t$ to

$$\theta(t) := \sum_{i,j,k,l,m,n=0}^1 (-1)^{k(l+m)} e_{i,j,k} \otimes e_{l,m,n}$$

and $1$ to $1 \otimes 1$. Alternatively, we can express $\theta(t)$ in terms of the generators. To do this, note that

$$(-1)^{k(l+m)} = \frac{1}{2} (1 + (-1)^k + (-1)^{l+m} - (-1)^k(-1)^{l+m})$$

since both sides are equal to $1$ when $k = 0$ and to $(-1)^{l+m}$ when $k = 1$. Therefore we have

$$\theta(t) = \frac{1}{2} \sum_{i,j,k,l,m,n} \left(1 + (-1)^k + (-1)^{l+m} - (-1)^k(-1)^{l+m}\right) e_{i,j,k} \otimes e_{l,m,n}$$

$$= \frac{1}{2} \left( \sum_{i,j,k} e_{i,j,k} + \sum_{i,j,k} (-1)^k e_{i,j,k} \right) \otimes \sum_{l,m,n} e_{l,m,n}$$

$$+ \frac{1}{2} \left( \sum_{i,j,k} e_{i,j,k} - \sum_{i,j,k} (-1)^k e_{i,j,k} \right) \otimes \sum_{l,m,n} (-1)^{l+m} e_{l,m,n}$$

$$= \frac{1}{2} ((1 + z) \otimes 1 + (1 - z) \otimes xy)$$

The bicrossed product $H_{d,1,1} = K[\Gamma][#^\theta K[L]$ just described is isomorphic to our Hopf subalgebra $N$.

**Theorem.** There exists a Hopf algebra isomorphism $f : H_{d,1,1} \rightarrow N$ satisfying

$$f(x) = h_2 \quad f(y) = c_4 h_2 \quad f(z) = h_3 \quad f(\bar{t}) = c_2$$

where $a \in K[\Gamma]$ is identified with $a#1 \in H_{d,1,1}$ and $\bar{t} := 1#t \in H_{d,1,1}$. 

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Proof. Because its cocycle is trivial by assumption, our crossed product is indeed a smash product, so that we can use the universal property of a smash product stated in Corollary 1.3. Since both $\Gamma$ and $G(N)$ are elementary abelian of order 8, there is an algebra homomorphism $f_{K[\Gamma]}: K[\Gamma] \to N$ defined by

$$f_{K[\Gamma]}(x) = h_2 \quad f_{K[\Gamma]}(y) = c_4 h_2 \quad f_{K[\Gamma]}(z) = h_3$$

Similarly, as $c_2$ has order 2, there is an algebra homomorphism $f_{K[L]}: K[L] \to N$ with $f_{K[L]}(t) = c_2$. As we pointed out in Paragraph 1.3, it is sufficient to verify the hypothesis of Corollary 1.3 on the generating set $x, y, z$, where it holds since

$$f_{K[\Gamma]}(t \to x)f_{K[L]}(t) = f_{K[\Gamma]}(y)f_{K[L]}(t) = c_4 h_2 c_2 = c_2 h_2 = f_{K[L]}(t)f_{K[\Gamma]}(x)$$
$$f_{K[\Gamma]}(t \to y)f_{K[L]}(t) = f_{K[\Gamma]}(x)f_{K[L]}(t) = h_2 c_2 = c_2 h_2 = f_{K[L]}(t)f_{K[\Gamma]}(y)$$
$$f_{K[\Gamma]}(t \to z)f_{K[L]}(t) = f_{K[\Gamma]}(z)f_{K[L]}(t) = h_3 c_2 = c_2 h_3 = f_{K[L]}(t)f_{K[\Gamma]}(z)$$

Thus Corollary 1.3 yields an algebra homomorphism $f: H_{d,1,1} \to N$ that satisfies $f(a \# 1) = f_{K[\Gamma]}(a)$ and $f(1 \# c) = f_{K[L]}(c)$, and hence the asserted equations.

To see that $f$ is a coalgebra homomorphism, it suffices to check the comultiplicativity of $f$ on the generators of $H_{d,1,1}$, namely $x, y, z$, and $t$. Since $x, y,$ and $z$ as well as their images are group-like, it is enough to verify this condition on $t$:

$$(f \otimes f)(\Delta(t)) = (f \otimes f)(\frac{1}{2}((1 + z) \otimes 1 + (1 - z) \otimes xy)(t \otimes t))$$

$$= \frac{1}{2}((1_B + h_3) \otimes 1_B + (1_B - h_3) \otimes c_4)(c_2 \otimes c_2)$$

$$= \Delta_B(c_2) = \Delta_B(f(t))$$

It is easily checked on the generators that $f$ preserves the counit.

Since $N$ is generated by $h_2, c_4 h_2, h_3$, and $c_2$, the map $f$ is surjective. Therefore, since $\dim H_{d,1,1} = \dim N = 16$, it is even bijective.

The isomorphism $f$ can be used to relate the one-dimensional representations of the two Hopf algebras. In particular, the one-dimensional characters $\chi_1', \chi_2'$, and $\chi_3'$ of $N$ defined in Paragraph 4.1 become the one-dimensional characters $\chi_1 \circ f, \chi_2' \circ f$, and $\chi_3' \circ f$ of $H_{d,1,1}$, which were denoted in [9], Page 633 by $\chi, \varphi$, and $\psi$, respectively.

4.4 The sixteen-dimensional Hopf subalgebra $N$ of $B$ introduced in Paragraph 4.1, which is, as just discussed, isomorphic to $H_{d,1,1}$ and therefore a twisting of the group algebra of $D_8 \times \mathbb{Z}_2$, is in fact the only such Hopf subalgebra:
Theorem. The Hopf subalgebra $N$ is the unique Hopf subalgebra of $B$ of dimension 16. It is normal, but neither commutative nor cocommutative. The Hopf algebra $B$ therefore fits into exactly one extension of the type

$$N \hookrightarrow B \twoheadrightarrow Z$$

with $\dim N = 16$ and $\dim Z = 2$. This extension is not abelian.

Proof. Suppose that $N'$ is a Hopf subalgebra of $B$ of dimension 16. As in the proof of Proposition 4.2, it follows from [11], Proposition 2 that $N'$ is normal. Therefore, $B$ fits into the extension

$$N' \hookrightarrow B \twoheadrightarrow Z$$

where $\iota_{N'}$ is the inclusion map and $\pi_Z$ is the canonical projection to the quotient $Z := B/BN'$ of dimension 2. By dualization, we obtain the extension

$$Z^* \xrightarrow{\pi_Z^*} B^* \xrightarrow{\iota_{N'}^*} N'^*$$

As pointed out in Paragraph 1.4, the image $\pi_Z^*(Z^*)$ is then central in $B^*$, and is of course a group algebra by [19], Theorem 2.3.1. But by Proposition 4.1 the only nontrivial central group-like element of $B^*$ is $\chi_1$, so that $\pi_Z^*(Z^*)$ is spanned by $\varepsilon_B$ and $\chi_1$. In particular, $\pi_Z^*(Z^*)$ together with its embedding into $B^*$ is uniquely determined, so that dually $B$ has a unique two-dimensional quotient Hopf algebra, namely $Z$. But the quotient determines the Hopf subalgebra as the space of coinvariants (cf. [19], Lemma 3.4.2 and Proposition 3.4.3). Therefore, $N'$ is uniquely determined, in other words, we have $N' = N$. We have already seen in Paragraph 4.1 that $N$ is neither commutative nor cocommutative, so the corresponding extension is not abelian.

As we mentioned in the introduction, many of the known semisimple Hopf algebras of prime power dimension contain a large commutative Hopf subalgebra of prime index. The result above shows that $B$ is not of this kind.

4.5 We have seen in Proposition 4.2 that $N$ has exactly three quotient Hopf algebras of dimension 8. It has also exactly three Hopf subalgebras of dimension 8:

**Proposition.** The Hopf algebra $N$ has exactly three Hopf subalgebras of dimension 8, namely

$$M_1 := K\langle c_4, h_2, h_3 \rangle \quad M_2 := K\langle c_2, c_4, h_3 \rangle \quad M_3 := K\langle c_2 h_2, h_3 \rangle$$

All these three subalgebras are commutative and normal in $N$.

Proof. (1) Suppose that $M$ is an eight-dimensional Hopf subalgebra of $N$. As already mentioned twice, we then know from [11], Proposition 2 that $M$ is
normal in \( N \). The quotient \( N/NM^+ \) has dimension 2, and the transpose of the quotient map yields an embedding

\[(N/NM^+)^* \to N^*\]

As in the proof of Theorem 4.4, the two-dimensional image of this map must have the form \( \text{Span}(\varepsilon, \chi) \) for a nontrivial central group-like element \( \chi \) of order 2. This means that the original quotient map is essentially the map

\[N \to K \times K, \ b \mapsto (\varepsilon(b), \chi(b))\]

By [19], Proposition 3.4.3, the original Hopf subalgebra can be recovered as the space of coinvariants, i.e., as the space

\[M = \{ b \in N \mid b = b(1)\chi(b(2)) \}\]

By tracing this argument backwards, we also see that, for a given nontrivial central group-like element \( \chi \) of order 2, the last equation yields an eight-dimensional normal Hopf subalgebra \( M \) of \( N \).

(2) From Proposition 4.1, we know that \( N^* \) contains exactly three nontrivial central group-like elements, namely \( \chi'_1, \chi'_2, \) and \( \chi'_1 \chi'_2 \). The computations in the proof of that proposition show that, if \( b = a \ast h \in N \), we have

\[b(1)\chi'_1(b(2)) = a(1)\varepsilon'_2(a(2)) \ast h\]

Now the map \( C \to C, \ a \mapsto a(1)\varepsilon'_2(a(2)) \) maps \( \omega_1 \) and \( \omega_4 \) to themselves and \( \omega_2 \) and \( \omega_3 \) to their negatives. This implies that

\[\{ b \in N \mid b = b(1)\chi'_1(b(2)) \} = K\langle c_4, h_2, h_3 \rangle = M_1\]

(3) For \( b = a \ast h \in N \), the computations in the proof of Proposition 4.1 also show that

\[b(1)\chi'_2(b(2)) = a \ast h(1)\gamma_3(h(2))\]

Since \( \gamma_3(g_3) = \theta(g_3, g_3) = 1 \), but \( \gamma_3(g_2) = \theta(g_3, g_2) = -1 \), we see that

\[\{ b \in N \mid b = b(1)\chi'_2(b(2)) \} = K\langle c_2, c_4, h_3 \rangle = M_2\]

(4) Finally, in the case of the third element \( \chi'_1 \chi'_2 \), the computations in the proof of Proposition 4.1 yield for \( b = a \ast h \in N \) that

\[b(1)(\chi'_1 \chi'_2)(b(2)) = a(1)\varepsilon'_2(a(2)) \ast h(1)\gamma_3(h(2))\]

By combining the considerations for the first two elements, we see that

\[\{ b \in N \mid b = b(1)(\chi'_1 \chi'_2)(b(2)) \} = \text{Span}(1, c_4, h_3, c_4 h_3, c_2 h_2, c_3 h_2, c_2 h_4, c_3 h_4)\]

Now we know from Equation 2.22 that \( (c_2 h_2)^2 = c_4 \), so that \( (c_2 h_2)^3 = c_3 h_2 \) and \( (c_2 h_2)^4 = 1 \). This shows that this space is indeed equal to \( M_3 = K\langle c_2 h_2, h_3 \rangle \).
The first two of these eight-dimensional Hopf subalgebras have a very natural interpretation: According to Lemma 4.1, the subalgebra \( M_1 \) is just the span of the group-like elements \( G(N) \). The Hopf algebra \( M_2 \) is the biproduct

\[
M_2 = C \star K[\langle g_3 \rangle]
\]

which can be formed because the formulas for \( \delta(x) \) and \( \delta(x^3) \) in Paragraph 2.1 show that \( C \) is indeed a Yetter-Drinfel’d Hopf algebra over \( K[\langle g_3 \rangle] \). Note that, because the action of \( g_3 \) on \( C \) is trivial, \( M_2 \) is in fact the tensor product of \( C \) and \( K[\langle g_3 \rangle] \) as an algebra. The Hopf subalgebra \( M_3 \) is less obvious. However, we will see in Paragraph 4.6 that \( M_2 \) and \( M_3 \) are isomorphic.

The first two Hopf subalgebras can also be easily expressed in terms of the generators \( u, r, \) and \( s \) introduced in Paragraph 2.2, namely as

\[
M_1 = K\langle u^2, r, s \rangle \quad \quad M_2 = K\langle u, s \rangle
\]

The third one looks more complicated when expressed in this way: \( M_3 \) is generated by \( \frac{1}{2}(1 + i\zeta^2)ur + \frac{1}{2}(1 - i\zeta^2)u^3r \) together with \( s \).

4.6 Semisimple Hopf algebras of dimension 8 have been classified in [13]. So the question arises which of these Hopf algebras the algebras \( M_1, M_2, \) and \( M_3 \) actually are. For \( M_1 \), this is not difficult: As we already said above, it is the group algebra of the group \( G(N) \approx Z_2 \times Z_2 \times Z_2 \). The remaining two, being commutative, must be dual group algebras, as we mentioned in Paragraph 1.4.

It turns out that they are both isomorphic to the dual group algebra of the dihedral group \( D_8 \):

**Proposition.** We have \( M_2 \cong M_3 \cong K^{D_8} \).

**Proof.** (1) Because \( M_2 = K\langle c_2, c_4, h_3 \rangle \) is commutative and \( K \) is algebraically closed, we must have \(|G(M_2^*)| = 8\). Since \( c_2^2 = c_4^2 = h_3^2 = 1 \), any multiplicative character \( \chi \in G(M_2^*) \) satisfies

\[
\chi(c_2) = \pm 1 \quad \chi(c_4) = \pm 1 \quad \chi(h_3) = \pm 1
\]

and therefore, since \(|G(M_2^*)| = 8\), all possible combinations of \( \pm 1 \) must yield multiplicative characters.

For two multiplicative characters \( \chi, \chi' \in G(M_2^*) \), we have on the last two generators

\[
(\chi\chi')(c_4) = \chi(c_4)\chi'(c_4) \quad \text{and} \quad (\chi\chi')(h_3) = \chi(h_3)\chi'(h_3)
\]

because these generators are group-like. For the third generator \( c_2 \), the formula

\[
\Delta_B(c_2) = \frac{1}{2}c_2(1_B + h_3) \otimes c_2 + \frac{1}{2}c_2(1_B - h_3) \otimes c_2c_4
\]

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for its coproduct, which was given in Paragraph 2.3 and in particular shows that $M_2$ is not cocommutative, implies that

$$\chi'(c_2) = \frac{1}{2} (1 + \chi(h_3) + \chi'(c_4) - \chi(h_3)\chi'(c_4)) \chi(c_2)\chi'(c_2)$$

If we apply this formula in the case $\chi = \chi'$, it shows that $G(M_2^*)$ has exactly two elements of order 4, defined by $\chi(c_2) = \pm 1$ and $\chi(c_4) = \chi(h_3) = -1$, and five elements of order 2, so that again $G(M_2^*) \cong D_8$. 

(2) In $M_3 = K(c_2 h_2, h_3)$, we have by Equation 2.2 that $(c_2 h_2)^2 = c_4$, so that $(c_2 h_2)^3 = c_3 h_2$ and $(c_2 h_2)^4 = 1$, as we have already recalled in the proof of Proposition 4.5. Each of the eight multiplicative characters therefore maps the first generator $c_2 h_2$ to a fourth root of unity and the second generator $h_3$ to $\pm 1$, and any fourth root of unity and any sign can arise in this way. The formula for the coproduct of $c_2$ given in Paragraph 2.3 now yields

$$\Delta_B(c_2 h_2) = \frac{1}{2} ((c_2 h_2 + c_2 h_2 h_3) \otimes c_2 h_2 + (c_2 h_2 - c_2 h_2 h_3) \otimes c_3 h_2)$$

This shows on the one hand that $M_3$ is a subbialgebra, and therefore a Hopf subalgebra, of $B$, and on the other hand that $M_3$ is not cocommutative.

For two multiplicative characters $\chi, \chi' \in G(M_3^*)$, we have for the second generator that $\chi(h_3)\chi'(h_3)$ and for the first generator that

$$\chi'(c_2 h_2) = \frac{1}{2} (1 + \chi(h_3) + \chi'(c_2 h_2)^2 - \chi(h_3)\chi'(c_2 h_2)^2) \chi(c_2 h_2)\chi'(c_2 h_2)$$

If we apply this formula in the case $\chi = \chi'$, it shows that $G(M_3^*)$ has also exactly two elements of order 4, defined by $\chi(c_2 h_2) = \pm 1$ and $\chi(h_3) = 1$, and five elements of order 2, so that again $G(M_3^*) \cong D_8$. 

We note that there is a second way to see that $N$ has exactly three eight-dimensional Hopf subalgebras: It follows from the classification of semisimple Hopf algebras of dimension 16 that $N \cong H_{d,1,1}$ is self-dual (cf. 3, Remark 1.4). Since $N$ has, by Proposition 4.5 up to isomorphism exactly three quotient Hopf algebras of dimension 8, self-duality implies that $N \cong N^*$ has exactly three Hopf subalgebras of dimension 8. Our discussion above, however, gives their explicit form.
5 Hopf subalgebras of dimension 2

5.1 The four-dimensional Yetter-Drinfel’d Hopf subalgebra $C$ of $A$ introduced in Paragraph 2.1 also arises as a quotient of $A$:

**Proposition.** There is a unique homomorphism $\pi_C : A \to C$ of Yetter-Drinfel’d Hopf algebras defined on generators as

$\pi_C(x) := \omega_4 = x^2 \quad \pi_C(y) := \omega_2$

Its values on the group-like elements are given by the following table:

| $a$ | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\omega_4$ | $\eta_1$ | $\eta_2$ | $\eta_3$ | $\eta_4$ |
|-----|------------|------------|------------|------------|-----------|-----------|-----------|-----------|
| $\pi_C(a)$ | $\omega_1$ | $\omega_4$ | $\omega_4$ | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\omega_2$ | $\omega_3$ |

**Proof.** It is immediate from the defining relations of $A$ that there is such an algebra homomorphism, and it is also immediate from the definitions that it takes the specified values on the group-like elements. Because it takes group-like elements to group-like elements, it is a coalgebra homomorphism that satisfies $S_C \circ \pi_C = \pi_C \circ S_A$. It commutes with the action of $g_2$ since

$\pi_C(g_2.x) = \pi_C(x^3) = x^2 = g_2.\pi_C(x)$

and

$\pi_C(g_2.y) = \pi_C(x^3y) = \omega_4\omega_2 = \omega_3 = g_2.\pi_C(y)$

Similarly, it commutes with the action of $g_3$ since

$\pi_C(g_3.x) = \pi_C(x) = x^2 = g_3.\pi_C(x)$

and

$\pi_C(g_3.y) = \pi_C(x^2y) = \omega_2 = g_3.\pi_C(y)$

It is therefore $H$-linear and, because the coaction is derived from the action via Equation (2.1), also colinear.

On the Radford biproduct, $\pi_C$ induces the surjective Hopf algebra homomorphism

$\pi_N : B \to N$, $a \star h \mapsto \pi_C(a) \star h$

which on the generators introduced in Paragraph 2.2 takes the values

$\pi_N(u) = u^2 \quad \pi_N(v) = \frac{1}{2}(1 + i\zeta^2)u + \frac{1}{2}(1 - i\zeta^2)v$

as well as $\pi_N(r) = r$ and $\pi_N(s) = s$. Note that the second equation can also be stated as $\pi_N(d_1) = c_2$. By comparing the table in Paragraph 3.1 to the table in Paragraph 4.1 we see that $\pi_N$ relates the group-like elements of $B^*$ and the group-like elements of $N^*$ via

$\chi_1 = \chi_1' \circ \pi_N \quad \chi_2 = \chi_2' \circ \pi_N \quad \chi_3 = \chi_3' \circ \pi_N$

We will denote the kernel of $\pi_N$ by $I$, so that $B/I \cong N$. 

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5.2 The ideal $I$ just defined is a Hopf ideal of dimension 16. We will now show that it is the only such ideal. In preparation, we prove a simple lemma:

Lemma. The space $U := \text{Span}(c_1, c_4)$ is the only normal Hopf subalgebra of $B$ of dimension 2. It is central.

Proof. We have already stated in Paragraph 1.4 that a normal Hopf subalgebra $U'$ of $B$ of dimension 2 is central. Since it is commutative, cocommutative, and semisimple, $U'$ is spanned by the unit element and a nontrivial central group-like element. But according to Lemma 3.1, the only nontrivial central group-like element is $c_4$, and so the assertion holds.

The Hopf ideal $I$ and the Hopf subalgebra $U$ are related as follows:

Theorem. The ideal $I$ is the only Hopf ideal of $B$ of dimension 16. It is normal. We have $I = BU^+$ and $U = B^{\text{co} N}$. The Hopf algebra $B$ therefore fits into exactly one extension of the type

$$U \hookrightarrow B \xrightarrow{\pi} N$$

with $\text{dim} U = 2$ and $\text{dim} N = 16$. Since $N$ is neither commutative nor cocommutative, this extension is not abelian.

Proof. If $I'$ is a Hopf ideal of $B$ of dimension 16, we can set $N' := B/I'$ and denote the quotient map by $\pi_{N'}: B \to N'$. Then $\pi_{N'}^*(N'^*)$ is a Hopf subalgebra of $B^*$ of index 2. As in the proof of Theorem 4.4, it follows from [11], Proposition 2 that $\pi_{N'}^*(N'^*)$ is normal in $B^*$. This implies that $I'$ is normal and that $\pi_{N'}$ is a conormal surjection (cf. [19], Section 3.4, Page 36), which in turn implies that $U' := B^{\text{co} N'}$ is a normal Hopf subalgebra of dimension 2. By the preceding lemma, we have $U' = U$, and consequently $I' = BU^+$. This shows that $I'$ is uniquely determined, and therefore $I' = I$.

5.3 As promised in Paragraph 3.2, the preceding results enable us to describe the product in the Grothendieck ring $K_0(B)$. From Proposition 3.1, we know that

$$G(B^*) = \langle \chi_1 \rangle \times \langle \chi_2 \rangle \times \langle \chi_3 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

which describes the one-dimensional representations. By Theorem 5.2, the Hopf algebra $B$ has two irreducible representations of dimension 2, which we will denote by $\pi_1$ and $\pi_2$, and one irreducible representation of dimension 4, which we will denote by $\rho$.

In Theorem 5.2, we have proved that $B$ has a unique quotient Hopf algebra $N$ of dimension 16, which is isomorphic to the Hopf algebra $H_{d1,1}$ from [6], Table 1. Clearly, the irreducible representations of $N$ can be pulled back along the quotient map $\pi_N$ to irreducible representations of $B$, so that we obtain a ring homomorphism from $K_0(N)$ to $K_0(B)$. The one-dimensional and the two-dimensional
irreducible representations of $B$ arise in this way; for the one-dimensional representations, we have already seen that at the end of Paragraph 5.1 and we have also indicated at the end of Paragraph 4.3 how the one-dimensional representations of $N$ correspond to the one-dimensional representations of $H_{d:1;1}$. Therefore, the description of the Grothendieck ring of $H_{d:1;1}$ given in [6], Paragraph 5.1 implies on the one hand that the two-dimensional irreducible representations, and also the one-dimensional representations, are self-dual, and on the other hand that the products of the one-dimensional with the two-dimensional irreducible representations are given by the table

|   | $\chi_1$ | $\chi_2$ | $\chi_3$ |
|---|---|---|---|
| $\pi_1$ | $\pi_1$ | $\pi_1$ | $\pi_2$ |
| $\pi_2$ | $\pi_2$ | $\pi_2$ | $\pi_1$ |

while the products of the two-dimensional irreducible representations are given by the table

|   | $\pi_1$ | $\pi_2$ |
|---|---|---|
| $\pi_1$ | $1 + \chi_1 + \chi_2 + \chi_1\chi_2$ | $\chi_3 + \chi_1\chi_3 + \chi_2\chi_3 + \chi_1\chi_2\chi_3$ |
| $\pi_2$ | $\chi_3 + \chi_1\chi_3 + \chi_2\chi_3 + \chi_1\chi_2\chi_3$ | $1 + \chi_1 + \chi_2 + \chi_1\chi_2$ |

Note that the self-duality of the irreducible representations implies that the Grothendieck ring of $N$ is commutative, so that, in the above tables, it does not matter which factor in the product comes first.

It remains to treat the products that involve $\rho$. Because it is the only four-dimensional irreducible representation, it is also self-dual, and we have $\chi_i\rho = \rho$ for $i = 1, 2, 3$. Furthermore, as in the case of $N$, self-duality implies that the Grothendieck ring of $B$ is commutative.

Since $\rho$ does not occur in the decomposition of $\chi_i\pi_j$ and $\pi_i\pi_j$, it follows from [22], Theorem 9 that no one-dimensional or two-dimensional irreducible representation occurs in the decomposition of $\pi_j\rho$, so that

$$\pi_1\rho = \pi_2\rho = 2\rho$$

Similarly, since $\rho$ occurs in the decomposition of $\chi_i\rho$ with multiplicity 1 and in the decomposition of $\pi_i\rho$ with multiplicity 2, we get in the same way from [22], Theorem 9 that

$$\rho^2 = 1 + \chi_1 + \chi_2 + \chi_3 + \chi_1\chi_2 + \chi_1\chi_3 + \chi_2\chi_3 + \chi_1\chi_2\chi_3 + 2\pi_1 + 2\pi_2$$

Note that the first eight summands are precisely the eight elements of $G(B^*)$. 33
6 Hopf subalgebras of dimensions 4 and 8

6.1 From Theorem 4.4 and Theorem 5.2 we know that neither $B$ nor $B^*$ fits into an abelian extension of a Hopf algebra of dimension 2 by a commutative Hopf algebra of dimension $2^n$. Thus neither $B$ nor $B^*$ can be constructed as an extension of the type used in [6], [7], [8], and [13] to classify certain semisimple Hopf algebras of prime power dimension. We will now show that nevertheless $B$ fits into an abelian extension of a Hopf algebra of dimension 4 by a commutative Hopf algebra of dimension 8, but this extension, which is essentially unique, is neither central nor cocentral.

**Theorem.** The Hopf algebra $B$ has exactly three Hopf subalgebras of dimension 8, namely

$$M_1 := K\langle c_4, h_2, h_3 \rangle \quad M_2 := K\langle c_2, c_4, h_3 \rangle \quad M_3 := K\langle c_2 h_2, h_3 \rangle$$

**Proof.** Suppose that $M$ is an eight-dimensional Hopf subalgebra of $B$. From Theorem 3.2 we know that $B^*$ has eight 1-dimensional, two 2-dimensional, and one 4-dimensional irreducible representations. Furthermore, it follows from Theorem 4.4 that the unique quotient Hopf algebra of $B^*$ of dimension 16 is the dual $N^*$ of the Hopf subalgebra $N$ of $B$ defined in Paragraph 4.1. Since no Hopf algebra of dimension 16 or 8 can have a four-dimensional irreducible representation, the canonical restriction mappings from $B^*$ to $N^*$ and $M^*$, which are dual to the inclusion mappings, must both contain the sixteen-dimensional ideal corresponding to the four-dimensional irreducible representation in their respective kernels. For the canonical map from $B^*$ to $N^*$, this kernel must in fact be equal to this ideal. This shows that the restriction mapping from $B^*$ to $M^*$ factors over the restriction mapping from $B^*$ to $N^*$, which implies that $M \subset N$. But then Proposition 4.5 implies that $M = M_1, M = M_2,$ or $M = M_3$. 

We have seen in Proposition 4.5 that $M_1, M_2,$ and $M_3$ are normal in $N$. They are not all normal in $B$.

**Proposition.** $M_1$ and $M_3$ are not normal in $B$.

**Proof.** (1) If $M_1$ were normal in $B$, the ideal $BM_1^+ = M_1^+ B$ would be two-sided, and in the corresponding quotient algebra $B/(BM_1^+)$, which would be four-dimensional by [19], Corollary 8.4.7, we would have

$$\bar{c}_4 = \bar{h}_2 = \bar{h}_3 = 1$$

Therefore we would also have $\bar{h}_4 = \bar{h}_2 \bar{h}_3 = 1$ and $\bar{c}_3 = \bar{c}_4 \bar{c}_2 = \bar{c}_2$ as well as $\bar{d}_3 = \bar{c}_4 \bar{d}_1 = \bar{d}_1$ and $\bar{d}_4 = \bar{c}_4 \bar{d}_2 = \bar{d}_2$, so that

$$B/(BM_1^+) = \text{Span}(1, \bar{c}_2, \bar{d}_1, \bar{d}_2)$$

But as we would then also have

$$\bar{d}_1 = \bar{h}_2 \bar{d}_1 = \bar{d}_2 \bar{h}_2 = \bar{d}_2$$

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it would follow that $B/(BM^+_1) = \text{Span}(1, \bar{c}_2, \bar{d}_1)$, which would be a contradiction.

(2) The argument showing that $M_3$ is not a normal Hopf subalgebra of $B$ is very similar. If it were, $BM^+_3 = M^+_3B$ would be a two-sided ideal of $B$, and the corresponding quotient algebra $B/(BM^+_3)$ would again have dimension 4. By considering the basis of $M_3$ given in the proof of Proposition 4.5, we see that we would have

$$\bar{c}_4 = \bar{h}_3 = \bar{c}_2\bar{h}_2 = \bar{c}_3\bar{h}_2 = \bar{c}_2\bar{h}_4 = \bar{c}_3\bar{h}_4 = 1$$

in the quotient algebra. In this quotient, we would also have

$$\bar{c}_2 = \bar{c}_4, \bar{d}_3 = \bar{d}_1$$

as well as $\bar{d}_3 = \bar{d}_1, \bar{d}_4 = \bar{c}_4\bar{d}_2 = \bar{d}_2$, so that

$$B/(BM^+_3) = \text{Span}(1, \bar{c}_2, \bar{d}_1, \bar{d}_2)$$

In this case, we would have

$$\bar{h}_2\bar{d}_1 = \bar{d}_2\bar{h}_2 = \bar{d}_2\bar{c}_2 = \bar{c}_2\bar{d}_2 = \bar{h}_2\bar{d}_2$$

and therefore $\bar{d}_1 = \bar{d}_2$, so that we would arrive again at the contradiction that $B/(BM^+_3) = \text{Span}(1, \bar{c}_2, \bar{d}_1)$.

6.2 On the other hand, there is a normal eight-dimensional Hopf subalgebra:

**Lemma.** $M_2$ is normal in $B$.

**Proof.** Recall that $M_2$ is generated by $c_2, c_4$, and $h_3$. From the description of the generators $\chi_1, \chi_2, \chi_3$ of $G(B^*)$ in Proposition 3.1, we know that

$$\chi_1(c_2) = \chi_1(c_4) = \chi_1(h_3) = 1$$

as well as $\chi_2(c_2) = \chi_2(c_4) = \chi_2(h_3) = 1$, so that $\chi_1$ and $\chi_2$ take the generators of $M_2$ to 1. By Corollary 3.1 and the remark following it, this also holds for $\chi_1\chi_2$.

As in the proof of Proposition 4.5, the algebra homomorphism

$$\pi: B \to K^4, \ b \mapsto (\varepsilon_B(b), \chi_1(b), \chi_2(b), (\chi_1\chi_2)(b))$$

is essentially the transpose of the inclusion map from $\text{Span}(\varepsilon_B, \chi_1, \chi_2, \chi_1\chi_2)$ to $B^*$, and is therefore surjective. Because $c_4$ and $h_3$ are group-like, our computations above show that these two generators are contained in the spaces $B^{\text{co}\pi}$ and $\text{co}^{\pi}B$ of right and left coinvariants, which are eight-dimensional by [19], Corollary 8.4.7. The formula for the coproduct of $c_2$ given in Paragraph 2.3 shows that this also holds for $c_2$, so that $M_2$ is completely contained in these spaces. But as $M_2$ is itself eight-dimensional, we have

$$B^{\text{co}\pi} = \text{co}^{\pi}B = M_2$$

By [19], Lemma 3.4.2, this implies that $M_2$ is normal in $B$. 

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The preceding proof also yields the form of the quotient:

**Theorem.** The Hopf subalgebra $M := M_2$ is the unique normal Hopf subalgebra of $B$ of dimension 8. Thus $B$ fits into exactly one extension of the type

$$M \hookrightarrow B \twoheadrightarrow Q$$

with $\dim M = 8$ and $\dim Q = 4$. In this case, $M \cong K^{D_8}$ and $Q \cong K[\mathbb{Z}_2 \times \mathbb{Z}_2]$. This extension is abelian, but neither central nor cocentral.

**Proof.** (1) That $M_2$ is the unique normal Hopf subalgebra of $B$ of dimension 8 follows from Theorem 6.1, Proposition 6.1, and the lemma above. Therefore, we have a unique extension with the stated properties. From Proposition 4.6, we know that $M_2 \cong K^{D_8}$.

(2) The proof of the lemma above also yields that the quotient $Q := B/(BM_2^*)$ is isomorphic to the dual of $\text{Span}(\varepsilon_B, \chi_1, \chi_2, \chi_1 \chi_2) \cong K[\mathbb{Z}_2 \times \mathbb{Z}_2]$. Since this Hopf algebra is self-dual, we have $Q \cong K[\mathbb{Z}_2 \times \mathbb{Z}_2]$ as well. Without referring to the proof of the lemma, this can be seen as follows: Since $\dim Q = 4$, the quotient must be a group algebra $Q = K[L]$ for a group $L$ of order 4, so that $L \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $L \cong \mathbb{Z}_4$. Then $Q^*$ is isomorphic to the group algebra $K[L]$ of the character group, and the transpose $\pi^*_Q$ of the quotient map $\pi_Q$ yields an injective group homomorphism from $\hat{L}$ to $G(B^*)$, which is elementary abelian of order 8 by Proposition 3.1. Therefore $L \cong \hat{L}$ must be elementary abelian of order 4.

(3) The extension is not central, for example because $M_2$ contains the noncentral element $c_2$. From Proposition 3.1, we know that $B^*$ has only one nontrivial central group-like element. So $\pi^*_Q(Q^*)$ cannot be central in $B^*$, which means that this extension is not cocentral. \qed

**6.3** We have just seen that $B$ fits into exactly one abelian extension of a Hopf algebra of dimension 4 by a Hopf algebra of dimension 8. We will now show that $B$ also fits into exactly one abelian extension of a Hopf algebra of dimension 8 by a Hopf algebra of dimension 4, which is also neither central nor cocentral.

**Theorem.** The Hopf algebra $B$ has seven Hopf subalgebras of dimension 4. Precisely one of them, namely $K(c_4, h_3)$, is normal.

**Proof.** (1) Since every semisimple Hopf algebra of dimension 4 over an algebraically closed field of characteristic zero is isomorphic to a group algebra, every Hopf subalgebra of $B$ of dimension 4 is a group algebra of a subgroup of order 4 of $G(B)$. From Lemma 5.1, we know that $G(B)$ is an elementary abelian group of order 8, or equivalently a three-dimensional vector space over the field with two elements. A subgroup of order 4 is then the same as a two-dimensional subspace. This subspace has a one-dimensional orthogonal complement in the dual vector space, and this orthogonal complement determines the subspace. A one-dimensional subspace over the field with two elements is determined by
its unique nonzero vector. In the dual space, there are seven nonzero vectors, and correspondingly there are seven subgroups of order 4 of $G$. Explicitly, these are the groups $G_1 := \langle c_4, h_3 \rangle = \{1_B, c_4, h_3, c_4h_3\}$ and

$$G_2 := \{1_B, c_4, h_2, c_4h_2\} \quad G_3 := \{1_B, c_4, h_4, c_4h_4\}$$

$$G_4 := \{1_B, h_2, h_3, h_4\} \quad G_5 = \{1_B, h_2, c_4h_3, c_4h_4\}$$

$$G_6 := \{1_B, h_3, c_4h_2, c_4h_4\} \quad G_7 := \{1_B, h_4, c_4h_2, c_4h_3\}$$

(2) Suppose that, for one of these groups $G_i$, the group algebra $P := K[G_i]$ is normal in $B$. Because $G(B) = G(N)$, we then have $P \subset N$. It follows directly from [19], Definition 3.4.1 that $P$ is also normal in $N$. By [19], Corollary 8.4.7, the quotient $N/(NP^*)$ has dimension 4. It is therefore a group algebra of an abelian group of order 4. The four distinct multiplicative characters in $G((N/(NP^*))^*)$ yield by pullback along the quotient map a subgroup of order 4 in $G(N^*)$ whose elements vanish on $NP^*$, and therefore take the value 1 on $G_i$.

Now we see from Corollary 4.11 that the subgroup of $G(N^*)$ of characters that take the value 1 on the group generated by $r = h_2$ and $s = h_3$ has only order 2 and is generated by $\chi_1'$. On the other hand, every element of $G(N^*)$ takes the value 1 on $u^2 = c_4$. Thus the fact that there are four distinct elements of $G(N^*)$ that map $G_i$ to 1 rules out that $i = 4, 5, 6, \text{ or } 7$; in other words, $G_i$ must contain $c_4$, and therefore $P$ must contain $U := \text{Span}(c_1, c_4)$.

(3) Consequently, we have a surjective map $B/(B \cdot U^+) \to B/(BP^+)$. By composing it with the inverse of the isomorphism $\pi_B : B/(BU^+) \to N$ from Theorem 5.2, we obtain an eight-dimensional quotient of $N$. From Proposition 4.2 we know that $N$ has exactly three quotients of dimension 8, denoted there by $F_1$, $F_2$, and $F_3$.

The multiplicative character $\chi_2' \in G(N^*)$ satisfies $\chi_2'(c_4) = \chi_2'(u^2) = 1$ and $\chi_2'(h_3) = \chi_2'(s) = 1$ and therefore factors to a multiplicative character on all three quotients $F_1$, $F_2$, and $F_3$. As discussed in Paragraph 5.1, we have $\chi_2 = \chi_2 \circ \pi_N$, so $\chi_2$ factors to a multiplicative character of $B/(BP^+)$. In particular, $\chi_2$ maps all elements of $G_i$ to 1. This implies that $i = 1$. We have therefore shown that, if $P$ is a four-dimensional normal Hopf subalgebra of $B$, then $P = K[G_1]$.

(4) To see that $K[G_1]$ is indeed normal in $B$, we argue as in the proof of Lemma 6.2. Recall that $F_2 = N/(K(h_3)^+N)$ and consider the Hopf algebra homomorphism that arises as the composition

$$B \xrightarrow{\pi_B} N \to F_2$$

This composition maps the group-like elements $c_4$ and $h_3$ to $1_{F_2}$, so that they are both contained in the spaces $B^{co F_2}$ and $co F_2 B$ of right and left coinvariants. As these spaces are four-dimensional by [19], Corollary 8.4.7, we have

$$B^{co F_2} = co F_2 B = K[G_1]$$

By [19], Lemma 3.4.2, this implies that $K[G_1]$ is normal in $B$. \hfill \Box
There is a somewhat more direct way to see that $K[G_i]$ is not normal in $B$ for $i \neq 1$, which uses an argument that is similar to the proof of Proposition 6.1 and proceeds on a case by case basis. We illustrate this argument by giving it in the case $i = 3$. If we assume that $P = K[G_3]$ is normal in $B$, we know from [19], Corollary 3.4.4 that $BP^+ = P^+B$. In the eight-dimensional quotient algebra $B/(BP^+)$, we have $$\bar{c}_4 = \bar{h}_4 = 1$$ and therefore also $\bar{h}_2 = \bar{h}_3$, $\bar{c}_2 = \bar{c}_3$, $\bar{d}_1 = \bar{d}_3$, and $\bar{d}_2 = \bar{d}_4$. This implies that $$B/(BP^+) = \text{Span}(1, \bar{c}_2, \bar{d}_1, \bar{d}_2, \bar{h}_2, \bar{h}_1 \bar{h}_2, \bar{d}_3 \bar{h}_2)$$ But since $$\bar{d}_1 = \bar{h}_4 \bar{d}_1 = \bar{d}_4 \bar{h}_4 = \bar{d}_4 = \bar{d}_2$$ the algebra $B/(BP^+)$ is spanned by only seven elements, which is a contradiction.

The preceding theorem has the following consequence:

Corollary. The Hopf algebra $B$ fits into exactly one extension of the type $$P \hookrightarrow B \twoheadrightarrow F$$ with $\dim P = 4$ and $\dim F = 8$. In this case, $P = K\langle c_4, h_3 \rangle \cong K[\mathbb{Z}_2 \times \mathbb{Z}_2]$ and $F \cong K[D_8]$. This extension is abelian, but neither central nor cocentral.

Proof. We have seen in the preceding theorem that $P = K\langle c_4, h_3 \rangle$ is the unique normal Hopf subalgebra of $B$ that has dimension 4, but in the proof we have also seen that the arising quotient is isomorphic to $F_2$. From Paragraph 4.2, we know that $F_2 \cong K[D_8]$. The corresponding extension is therefore abelian. It is not a central extension, because $h_3$ is not central.

The commutator factor group of $D_8$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, and therefore $D_8$ has four one-dimensional representations. This implies that $G(F_2^*) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. But from Proposition 3.1, we know that $B^*$ has only one nontrivial central group-like element, so $F_2^*$ cannot be contained in the center of $B^*$. In other words, this extension is not cocentral.

We note that, in terms of the generators introduced at the beginning of Paragraph 2.2, we have $P = K\langle u^2, s \rangle$.

7 The biproduct in the second case

7.1 So far, we have only treated the Yetter-Drinfel’d Hopf algebra $A$ and its associated Radford biproduct $B = A \star H$. In [10], Section 3, the authors also described a second example of a Yetter-Drinfel’d Hopf algebra over $H$ of
dimension 8, denoted here by $A'$. As the algebra $A$ introduced in Paragraph 2.1, it is a Yetter-Drinfel’d Hopf algebra over the group algebra $H = K[G]$ of the elementary abelian group $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, and it is also generated by two elements, denoted here by $x'$ and $y'$. But in contrast to the previous case, these generators do not commute and satisfy instead the defining relations

$$x'^4 = 1, \quad x'y' = y'x'^3, \quad y'^2 = \frac{1}{2}(\zeta(1 + x' - \zeta x'^2 + x'^3))$$

where $\zeta$ is again a not necessarily primitive fourth root of unity.

The action of $G$ on $A'$ is defined by the same formulas as the action of $G$ on $A$, and the coaction is again derived from the action via Equation (2.1), using the same bicharacter $\theta$. As in Paragraph 2.1, we introduce the elements $\omega'_1 := 1_{A'}$, $\omega'_2 := \frac{1}{2}(1 + \imath \zeta^2)x' + \frac{1}{2}(1 - \imath \zeta^2)x'^3$, $\omega'_3 := \frac{1}{2}(1 - \imath \zeta^2)x' + \frac{1}{2}(1 + \imath \zeta^2)x'^3$, $\omega'_4 := x'^2$, and

$$\eta'_1 := y', \quad \eta'_2 := x'^5y', \quad \eta'_3 := x'^2y', \quad \eta'_4 := x'y'$$

where $\imath$ is a fixed primitive fourth root of unity unrelated to $\zeta$. According to [10], Proposition 3.4, these eight elements form a basis of $A'$, and the coalgebra structure is determined by the fact that they are group-like. By [10], Paragraph 3.3, this means for $x'$ that

$$\Delta_{A'}(x') = \frac{1}{2}(x' \otimes x' + x' \otimes x'^3 + x'^3 \otimes x' - x'^3 \otimes x'^3)$$

As in Paragraph 2.2, we can form the biproduct $B' := A' \star H$. It is generated by the four elements

$$u' := x' \star 1_H, \quad v' := y' \star 1_H, \quad r' := 1_{A'} \star g_2, \quad s' := 1_{A'} \star g_3$$

In the same way as before, it is possible to derive a presentation of $B'$:

**Proposition.** The four generators satisfy the relations

1. $u'^4 = 1$, $u'v' = v'u'^3$, $v'^2 = \frac{1}{2}(\zeta(1 + u' - \zeta u'^2 + u'^3))$
2. $r'^2 = 1$, $r's' = s'r'$, $s'^2 = 1$
3. $r'u' = u'^3r'$, $r'v' = u'^3v'r'$, $s'u' = u's'$, $s'v' = u'^2v's'$

These relations are defining.

We can also, as in Paragraph 2.2, introduce the elements that correspond to the basis elements of $A'$ by defining

$$c'_i := \omega'_i \star 1_H, \quad d'_i := \eta'_i \star 1_H, \quad h'_i := 1_{A'} \star g_i$$
for $i = 1, 2, 3, 4$. Then the elements $c'_ih'_j$ together with the elements $d'_ih'_j$, for $i, j = 1, 2, 3, 4$, form a basis of $B'$. Note that $c'_1 = h'_1 = 1_B$. Except for $u'$, the above generators are among these elements, as $v' = d'_1$, $r' = h'_2$, and $s' = h'_3$.

Since the coaction of $G$ on $A'$ is defined in the same way as the coaction of $G$ on $A$, the elements $c'_4$ and $h'_j$ are group-like. Exactly as in Paragraph 2.3, we

\[
\Delta_{B'}(c'_2) = \frac{1}{2} c'_2(h'_1 + h'_3) \otimes c'_2 + \frac{1}{2} c'_2(h'_1 - h'_3) \otimes c'_3
\]

for the coproduct of $c'_2$ and

\[
\Delta_{B'}(d'_1) = \frac{1}{4} d'_1(h'_1 + h'_2 + h'_3 + h'_4) \otimes d'_1 + \frac{1}{4} d'_1(h'_1 + \zeta^2 h'_2 - h'_3 - \zeta^2 h'_4) \otimes d'_2
\]

\[
+ \frac{1}{4} d'_1(h'_1 - h'_2 - h'_3 + h'_4) \otimes d'_3 + \frac{1}{4} d'_1(h'_1 - \zeta^2 h'_2 - h'_3 - \zeta^2 h'_4) \otimes d'_4
\]

for the coproduct of $d'_1$. As in Paragraph 2.3, the coproduct is also determined by its values on the generators:

**Lemma.** We have $\Delta_{B'}(r') = r' \otimes r'$ and $\Delta_{B'}(s') = s' \otimes s'$. Furthermore, we have

\[
\Delta_{B'}(u') = \frac{1}{2}(u' \otimes u' + u' \otimes u^3 + u^3s' \otimes u' - u'^3 \otimes u'^3)
\]

and

\[
\Delta_{B'}(v') = \frac{1}{4} v'(1 + r' + s' + r's') \otimes v' + \frac{1}{4} v'(1 - \zeta^2 r' - s' + \zeta^2 r's') \otimes u'v'
\]

\[
+ \frac{1}{4} v'(1 - r' + s' - r's') \otimes u'^2v' + \frac{1}{4} v'(1 + \zeta^2 r' - s' - \zeta^2 r's') \otimes u'^3v'
\]

The counit is given on generators by $\varepsilon_{B'}(u') = \varepsilon_{B'}(v') = \varepsilon_{B'}(r') = \varepsilon_{B'}(s') = 1$.

**7.2** The Hopf algebra $B'$ is a Radford biproduct by construction, and as in the case of $B$, it is reasonable to ask whether, and in which ways, it can also be written as an extension. Fortunately, it is not necessary to repeat this analysis for $B'$, because $B'$ is isomorphic to $B$, as we show now. We begin with the following lemma:

**Lemma.** There is an algebra homomorphism $f_{A'} : A' \rightarrow B$ with the property that $f_{A'}(x') = u$ and $f_{A'}(y') = vr$

**Proof.** We have to check the defining relations of $A'$, using Proposition 2.2. It is obvious that $f_{A'}(x')^4 = 1$. For the second relation, we have

\[
f_{A'}(y')f_{A'}(x')^3 = vru^3 = vur = uvr = f_{A'}(x')f_{A'}(y')
\]
For the third relation, we have

\[ f_{A'}(y')^2 = vrvr = v(u^3vr)r = v^2u^3 = \frac{1}{2}(1_B + \zeta u + u^2 - \zeta u^3)u^3 \]

\[ = \frac{1}{2}(u^3 + \zeta 1_B + u - \zeta u^2) = \frac{1}{2}(1_B + f_{A'}(x') - \zeta f_{A'}(x')^2 + f_{A'}(x')^3) \]

as required. \(\square\)

The algebra homomorphism \(f_{A'}\) can be extended to the entire biproduct:

**Proposition.** The map \(f : B' = A' \ast H \to B = A \ast H\) given by

\[ f(a' \ast h) = f_{A'}(a')(1_A \ast h) \]

is an algebra isomorphism.

**Proof.** We apply the universal property of a smash product stated in Corollary \([3]\). For the map \(f_H\) required there, we use the canonical embedding

\[ f_H : H \to A \ast H, \ h \mapsto 1_A \ast h \]

of \(A\) into the biproduct \(B\). We have to check that

\[ f_{A'}(h_{(1)}a')f_H(h_{(2)}) = f_H(h)f_{A'}(a') \]

As pointed out after that corollary, we only need to check this on the algebra generators \(g_2\) and \(g_3\) of \(H\). In both cases, the condition can be written in the form

\[ f_{A'}(g_i.a') = f_H(g_i)f_{A'}(a')f_H(g_i)^{-1} \]

In this form, both sides depend multiplicatively on \(a'\), so that it is sufficient to verify this equation for \(a' = x'\) and \(a' = y'\). Hence we have to consider four cases:

1. The case \(h = g_2, \ a' = x'\): We then have
   \[ f_H(g_2)f_{A'}(x')f_H(g_2)^{-1} = rur^{-1} = u^3 = f_{A'}(x'^3) = f_{A'}(g_2.x') \]
2. The case \(h = g_2, \ a' = y'\): We then have
   \[ f_H(g_2)f_{A'}(y')f_H(g_2)^{-1} = r(vr)r^{-1} = rv = u^3vr = f_{A'}(x'^3 y') = f_{A'}(g_2.y') \]
3. The case \(h = g_3, \ a' = x'\): We then have
   \[ f_H(g_3)f_{A'}(x')f_H(g_3)^{-1} = sus^{-1} = u = f_{A'}(x') = f_{A'}(g_3.x') \]
4. The case \(h = g_3, \ a' = y'\): We then have
   \[ f_H(g_3)f_{A'}(y')f_H(g_3)^{-1} = svrs^{-1} = u^2vsvr = f_{A'}(x'^2 y') = f_{A'}(g_3.y') \]
Because the elements $u$, $vr$, $r$, and $s$ are contained in its image, $f$ is surjective, and therefore bijective by dimension considerations.

We note that, on the generators of $B'$, the map $f$ takes the values

$$f(u') = u, \quad f(v') = vr, \quad f(r') = r, \quad f(s') = s$$

These equations can be used to give an alternative proof of the result above, which is based on Proposition 7.1. They are also used in the proof of the following corollary:

**Corollary.** The map $f$ is a Hopf algebra isomorphism between $B$ and $B'$.

**Proof.** We still have to check that $f$ is a coalgebra homomorphism, which amounts to the conditions $\Delta_B \circ f = (f \otimes f) \circ \Delta_{B'}$ and $\varepsilon_B \circ f = \varepsilon_{B'}$. It is sufficient to check both conditions on the generators, where the second condition is obvious. The first condition is also obvious on the generators $u'$, $r'$, and $s'$, because their coproducts and the coproducts of the respective images are given essentially by the same formulas. For $v'$, we see by comparing Proposition 2.3 and Lemma 7.1 that

$$(f \otimes f)(\Delta_{B'}(v')) = \Delta_B(v)(r \otimes r) = \Delta_B(v)\Delta_B(r) = \Delta_B(vr) = \Delta_B(f(v'))$$

which completes our proof, because a bialgebra homomorphism commutes automatically with the antipode.

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