Metric factorizability and equivalence of brane world models with Brans-Dicke theory

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Abstract

In the standard brane world models, the bulk metric ansatz is usually assumed to be factorizable in brane and bulk coordinates. However it is not self evident that it is always possible to factorize the bulk metric. Using gradient expansion scheme, which involves, expansion of bulk quantities in terms of the brane to bulk curvature ratio, as perturbative parameter, we have explicitly shown that upto second order in perturbative expansion, metric factorizability is a valid assumption. We have also argued from our result that the same should be true for all orders in the perturbation scheme. We further establish that the non-local terms present in the bulk gravitational field equation can be replaced by radion field and the effective action on the brane obtained thereof resembles Brans-Dicke theory of gravity.

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1 Introduction

Conjecture of the existence of more than four spacetime dimension has serious implications in high energy physics. Such higher dimensional spacetime appear quiet naturally in the context of string theory. Recently there has been progress in this regime, specially for theories with extra spatial dimensions. The common perception for all these theories correspond to the fact that gravity can access the whole of spacetime including the extra dimensions (compositely known as bulk), while the standard model fields are localized on four dimensional sub-manifold (known as brane). One of the main motivation behind these models, has been to explain the large hierarchy between the Planck scale \( M_P \sim 10^{18}\text{GeV} \) and the Electro-weak scale \( m_W \sim 10^2\text{GeV} \).

First such model was proposed by Arkani-Hamed et.al. \cite{1,2}. In this model the extra dimensions were assumed to be large, such that the five-dimensional Planck scale differs from the four-dimensional Planck scale by a factor of the volume of these extra dimensions. Thus by assuming more than one extra dimension and a large volume (still within experimental bound), the five-dimensional Planck scale can be brought down to four-dimensional Electro-weak scale. However in this case the extra dimensions are assumed to be flat.

From the gravitational viewpoint it is more tempting to take the bulk geometry as warped, while the brane(s) as flat. This was first realized in a set-up proposed by Randall and Sundrum (RS) \cite{3}, where two branes were held fixed at orbifold fixed points with \( S^1/Z_2 \) symmetry. Due to exponential warping the Planck scale in one brane (the Planck brane) was brought down to Electro-weak scale in the other brane, known as visible brane. Such a warped model was also extended to one brane with an infinitely extended bulk \cite{4}. In this work, we however, focus into two branes warped geometry model.

The separation between the branes in RS model may not be constant and needs to be stabilized. Such a stabilization mechanism, was proposed in \cite{5,6}, while the stabilization for time dependent scenario was discussed in \cite{7}. Particle phenomenology of various matter fields in this scenario was discussed in \cite{8–13} with interesting consequences. Recently, these ideas are also being put forward in the context of various alternative gravity theories \cite{14–18}.

All these results depend on a crucial fact, factorizability of metric ansatz. However there exists objections against this assumption of factorizability and it is also not self evident, why the metric ansatz should be factorizable \cite{19}. In this work, we have tried to address this issue using low energy effective action, obtained by solving the bulk equations. The bulk equations in general are not exactly solvable and a convenient way to handle the situation at low energy is to expand the bulk variables in terms of the ratio of four-dimensional curvature to bulk curvature. This method, known as Gradient expansion method was developed by Kanno and Soda \cite{20–23}. In \cite{24} the gradient expansion method has been used to show that the factorizable metric ansatz is valid upto linear in this perturbative expansion. In this work we obtain the second order correction to the metric in this gradient expansion scheme, which leads to the effective action upto second order. This also exhibits the factorizable nature, which in turn enables us to generalize our result to include higher order corrections. We conclude that at any order, the metric is factorizable and thus factorizability of the metric is a valid assumption.

Along with the issue of factorizability of the metric ansatz we also address the equivalence of this bulk-brane system with scalar-tensor or Brans-Dicke theory of gravity. The solution to bulk equations intrinsically inherit non-local terms which, as we have argued, can be traded off through the radion field. This equivalence was shown earlier in \cite{22} for first order perturbative corrections through the gradient expansion method. We have reformulated the previous method and show explicitly that upto second order of perturbative expansion, when the non-local terms are eliminated, the field equation on the brane becomes local and equivalent to that of Brans-Dicke theory of gravity. We also argue that this
result can be generalized to arbitrary higher orders in the perturbative expansion. The same assertion also follows from the effective action, i.e. the effective action can be written explicitly in the Brans-Dicke form.

The paper is organized as follows: In Sec. 2 we review the gradient expansion method and evaluate the second order correction to the bulk metric. Then in Sec. 3 we use the bulk metric in order to determine the effective action and the equation of motion it corresponds to. Along with these we also present the criteria for obtaining second order field equation from this effective action. Finally in Sec. 4 we establish the equivalence of this bulk-brane system with Brans-Dicke theory of gravity. We then conclude with a short discussion on our results.

2 Gradient Expansion and Higher Order Terms

The metric ansatz for the five dimensional spacetime is taken in Gaussian normal coordinates, where we denote the brane coordinates by \( x^\mu \) and the bulk coordinate by \( y \), such that

\[
d s^2 = h_{\mu\nu}(y, x^\mu)dx^\mu dx^\nu + dy^2
\]  

Thus the metric in general is not taken as factorizable. The branes are assumed to be moving in the coordinate chart where they are placed at

\[
y = \phi_+(x^\mu); \quad y = \phi_-(x^\mu)
\]

and in the literature they are often quoted as moduli fields. In order to determine the brane geometry we need to solve the bulk equations. The form of the metric ansatz suggests that the extrinsic curvature on \( y = \text{constant} \) hypersurface can be found through its decomposition into traceless and trace part as,

\[
K_{\mu\nu} = -\frac{1}{2} \frac{\partial h_{\mu\nu}}{\partial y}; \quad K_{\mu\nu} = \Sigma_{\mu\nu} + \frac{1}{4} h_{\mu\nu} K; \quad K = -\frac{\partial \ln \sqrt{-h}}{\partial y}
\]

Using these properties of extrinsic curvature into the bulk equations lead to the equations,

\[
\partial_y \Sigma_{\mu\nu} - K \Sigma_{\mu\nu} = - \left[ R_{\mu\nu}(h) - \frac{1}{4} h_{\mu\nu} R(h) \right]
\]

\[
\frac{3}{4} K^2 - \Sigma^{\alpha\beta} \Sigma_{\alpha\beta} = R(h) + \frac{12}{\ell^2}
\]

\[
\nabla_\nu \Sigma^{\nu}_{\mu} - \frac{3}{4} \nabla_\mu K = 0
\]

where the covariant derivatives are with respect to the metric \( h_{\mu\nu} \), and all the curvature components i.e. Ricci tensor and Ricci scalar are to be determined using \( h_{\mu\nu} \). In general we should first solve Eq. (4) and integrate over \( y \) to get \( \Sigma_{\mu\nu} \) and then we may solve for \( K \) from Eq. (5), to get \( K_{\mu\nu} \), which finally can be integrated to obtain \( h_{\mu\nu} \). However as Curvature components depends on \( h_{\mu\nu} \), this procedure cannot be implemented in general. This poses a serious problem, which can be bypassed by observing that we are seeking a low energy effective theory, where the brane matter energy density can be assumed to be much smaller compared to bulk cosmological constant. This implies that four dimensional curvature is much smaller compared to the five dimensional one and the gradient expansion scheme can be applicable.

At zeroth order, the curvature terms can be neglected in comparison to the extrinsic curvature terms. Being isotropic at this order, the anisotropic term \( \Sigma_{\mu\nu} \) vanishes. Then the metric at zeroth order is,
$h_{\mu\nu} = a^2(y)g_{\mu\nu}(x)$, with the standard warp factor, $a(y) = e^{-y/\ell}$. This iteration scheme helps to write the metric $h_{\mu\nu}$ as a sum of tensors constructed from $g_{\mu\nu}$. Thus the metric have the form of a perturbative series expansion,

$$h_{\mu\nu} = a^2(y) [g_{\mu\nu}(x) + f_{\mu\nu}(y, x) + q_{\mu\nu}(y, x) + \cdots]; \quad a(y) = e^{-y/\ell}$$

where $f_{\mu\nu}(y, x)$ corresponds to leading order correction, and $q_{\mu\nu}$ represents the second order correction. After calculating second order corrections a pattern will emerge from which the effective action can be determined at all order. This we will elaborate at a later stage.

In a similar manner we can expand both extrinsic curvature and trace free part as,

$$K_{\mu\nu} = 1/\ell \delta_{\mu\nu} + K_{\mu\nu}^{(1)} + K_{\mu\nu}^{(2)} + \cdots$$

$$\Sigma_{\mu\nu} = 0 + \Sigma_{\mu\nu}^{(1)} + \Sigma_{\mu\nu}^{(2)}$$

In the above expansion, objects with superscript $(1)$ denote first order corrections, while those with superscript $(2)$ denotes second order corrections and so on. We briefly discuss the first order formulation, leading to a possible solution for $f_{\mu\nu}$, then we shall elaborate the second order calculation in order to get the tensor $q_{\mu\nu}$. These will be used later to get the effective action.

### 2.1 First Order

The first order equations are obtained by considering terms containing $K_{\mu\nu}^{(1)}$ and $\Sigma_{\mu\nu}^{(1)}$ appearing once in the expressions. For example, $K^2 = (16/\ell^2) + (8/\ell)K^{(1)}$, where we have used the result that at zeroth order $K^{(0)} = (4/\ell)$. Similar considerations apply to $\Sigma_{\mu\nu}$ as well with the fact that at zeroth order it vanishes. Thus the bulk equations at first order take the forms,

$$\partial_y \Sigma_{\mu\nu}^{(1)} - (4/\ell)\Sigma_{\mu\nu}^{(1)} = - \left[ R_{\mu\nu}^{(1)}(h) - \frac{1}{4} \delta_{\mu\nu} R^{(1)}(h) \right]$$

$$\frac{6}{\ell} K^{(1)} = R^{(1)}(h)$$

$$\nabla_{\nu} \Sigma_{(1)\mu} - \frac{3}{4} \nabla_{\mu} K^{(1)} = 0$$

Here, the covariant derivatives are with respect to the metric $g_{\mu\nu}$ and $R^{(1)}(h)$ imply the Ricci scalar calculated using $a^2(y)g_{\mu\nu}$. Similar conclusion can be reached for the Ricci tensor as well. For this reason we will henceforth provide the curvature components with respect to the metric $g_{\mu\nu}$ only, with $a^2(y)$ taken out. This reduces the first order equation (11) to the form,

$$K^{(1)} = \frac{\ell}{6a^2} R(g)$$

Similarly by integrating over $y$ in Eq. (10) leads to the first order trace less part of the extrinsic curvature as,

$$\Sigma_{\mu\nu}^{(1)} = \frac{\ell}{2a^2} \left( R_{\mu\nu}^{(1)}(g) - \frac{1}{4} \delta_{\mu\nu} R(g) \right) + \frac{1}{a^2} \chi_{\mu\nu}^{(1)}(x)$$

$$\chi_{\mu}^{\mu} = 0; \quad \nabla_{\mu} \chi_{\nu}^{\mu} = 0$$
Thus in this particular situation along with the boundary condition, we get the first order correction as

\[ f_{\mu\nu}(y, x) = \frac{\ell^2}{2a^2} \left( R_{\mu\nu} - \frac{1}{6} g_{\mu\nu} R \right) - \frac{\ell}{2a} \Sigma_{\mu\nu}(x) + C_{\mu\nu}(x) \]

Using this, the first order corrected metric \( h_{\mu\nu} \) turns out to have the following expression

\[ h_{\mu\nu} = a^2(y) \left[ g_{\mu\nu} - \frac{\ell^2}{2a^2} \left( R_{\mu\nu} - \frac{1}{6} g_{\mu\nu} R \right) - \frac{\ell}{2a} \Sigma_{\mu\nu}(x) + C_{\mu\nu}(x) \right] \]

Having obtained the metric with the first order correction term included, we now proceed to calculate the second order correction in greater detail.

### 2.2 Second Order

At second order the bulk equations contain single power of 2nd order objects, double power of first order objects, and so on. For example, at second order our expression would include only \( \Sigma^{(1)}_{\mu} \) but we can have terms like, \( K^{(1)}_{\mu} \Sigma_{\nu}^{(1)} \). However we also ignore the terms like \( \chi^{\mu}_{\nu} \chi^{\nu}_{\mu} \) as they only correspond to some constants with respect to \( y \) (since these are integration constants of the first order equation). Thus at second order the bulk equations reduce to the following form

\[ \partial_y \Sigma^{(2)}_{\mu} - \frac{4}{\ell} \Sigma^{(2)}_{\mu} = - \left( R^{(2)}_{\mu\nu} - \frac{1}{4} R^{(2)} \delta^{\mu}_{\nu} \right) + K^{(1)} \Sigma^{(1)}_{\mu} \]

\[ K^{(2)} = \frac{\ell}{6} \left[ -3 \left( K^{(1)} \right)^2 + \Sigma^{(1)}_{\alpha} \Sigma^{(1)}_{\mu} + R^{(2)} \right] \]

\[ \partial_y K^{(2)} - \frac{2}{\ell} K^{(2)} = \frac{1}{4} \left( K^{(1)} \right)^2 + \Sigma^{(1)}_{\mu} \Sigma^{(1)}_{\mu} \]

In order to obtain the Ricci tensor and scalar at second order we should use the metric corrected up to the first order, i.e. the result provided in Eq. (19). Thus in second order we have the following expression:

\[ R^{(2)}_{\alpha\beta} \]

\[ = \frac{\ell^2}{2a^4} \left[ R^{(2)}_{\mu\nu} R^{\mu}_{\beta} - \frac{1}{6} R R^{\mu}_{\beta} - \frac{1}{4} \delta^{\beta}_{\mu} \left( R^{(2)}_{\nu\rho} R^{\nu}_{\rho} - \frac{1}{6} R^2 \right) \right. \\

\[ - \frac{1}{2} \left( \nabla_{\mu} \nabla_{\beta} R^{\mu\alpha} + \nabla_{\mu} \nabla^{\alpha} R^{\mu}_{\beta} \right) + \frac{1}{3} \nabla^{\alpha} \nabla_{\beta} R + \frac{1}{2} \square R^{\alpha}_{\beta} - \frac{1}{12} \delta^{\beta}_{\alpha} \square R \]

\[(23)\]
From which we arrive at the expression for Ricci scalar at second order as,

\[ R^{(2)} = \frac{\ell^2}{2a^4} \left( R^\mu_\nu R^\nu_\mu - \frac{1}{6} R^2 \right) \]  

Note that while deriving the above results we have used the fact that at second order the curvature tensor can be obtained in the local inertial frame and can be written in terms of derivatives of the metric. From the bulk equations, with the help of Ricci scalar at second order, the trace of the extrinsic curvature at second order turns out to be,

\[ K^{(2)} = \frac{\ell}{6a^4} \left[ -\frac{3}{4} \left( K^{(1)} \right)^2 + \Sigma^{(1)\alpha} \Sigma^{(1)\beta} + R^{(2)} \right] \]

\[ = \frac{\ell^3}{8a^4} \left( R^\beta_\delta R^\delta_\beta - \frac{2}{9} R^2 \right) \]  

The traceless part of the extrinsic curvature can be obtained by integrating Eq. (20) over the extra coordinate which leads to the following expression

\[ \Sigma^{(2)\mu} = -\frac{\ell^2 y}{2a^4} S^{\mu}_\nu + \frac{\ell^3}{a^4} t^{\mu}_\nu(x) \]  

where for convenience we have defined a second rank tensor \( S^{\mu}_\nu \) as,

\[ S^{\alpha\beta} = R^{\alpha}_\mu R^\mu_\beta - \frac{1}{3} R R^{\alpha}_\beta - \frac{1}{4} g^{\alpha\beta} \left( R^{\mu}_\nu R^{\nu}_\mu - \frac{1}{3} R^2 \right) - \frac{1}{2} \left( \nabla^\mu \nabla^\alpha R^\alpha_\mu + \nabla^\alpha \nabla^\mu R^\mu_\alpha \right) + \frac{1}{3} \nabla^\alpha \nabla^\beta R + \frac{1}{2} \square R^{\alpha\beta} - \frac{1}{12} g^{\alpha\beta} \square R \]  

Note that the tensor \( S^{\mu}_\nu \) is transverse and traceless. Also in the expression for \( \Sigma^{(2)\mu} \), \( t^{\mu}_\nu \) is an arbitrary integration constant just like \( \chi^{\mu}_\nu \) in the first order. This tensor satisfies the following properties,

\[ t^{\alpha}_\alpha = 0; \quad \nabla^\alpha t^{\alpha}_\mu = 0; \]  

The traceless nature of \( t^{\mu}_\nu \) follows from the fact that, \( \Sigma^{(2)\mu}_\nu \) is also traceless. Then we can obtain the extrinsic curvature at second order as,

\[ K^{(2)\beta}_\alpha = \Sigma^{(2)\alpha}_\beta + (1/4) \delta^{\alpha}_\beta K^{(2)} \]  

Thus the second order correction to \( h^{\mu}_\nu \) can be obtained from the differential equation,

\[ -\frac{1}{2} \partial_y h^{(2)}_{\alpha\beta} = -\frac{\ell^2 y}{2a^4} S^{\alpha\beta} + \frac{\ell^3}{a^4} t^{\alpha}_\beta + \frac{\ell^4}{2a^4} S^{\mu}_\nu \left( R^{\mu}_\nu R^{\nu}_\mu - \frac{2}{9} R^2 \right) \]

This expression can be integrated to obtain the second order correction to the metric. We can now add these zeroth order, first order and second order corrections, in order to obtain the expression for \( h^{\mu}_\nu \) up to second order as,

\[ h^{\mu}_\nu = a^2(y) \left[ g^{\mu}_\nu - \frac{\ell^2}{2a^2} \left( R^{\mu}_\nu - \frac{1}{6} g^{\mu}_\nu R \right) - \frac{\ell}{2a^2} \chi^{\mu}_\nu(x) + C^{\mu}_\nu(x) \right. \]

\[ + \left. \left( \frac{\ell^3 y}{4a^4} - \frac{\ell^4}{16a^4} \right) S^{\mu}_\nu - \frac{\ell^4}{2a^4} t^{\mu}_\nu(x) - \frac{\ell^4}{64a^4} g^{\mu}_\nu \left( R^{\alpha\beta}_\alpha R^{\alpha\beta} - \frac{1}{3} R^2 \right) + B^{\mu}_\nu(x) \right] \]  

Where, \( B^{\mu}_\nu \) is again a constant of integration. Thus having obtained the metric \( h^{\mu}_\nu \) which includes corrections till second order, we now calculate the effective action, constructed out of it.
3 Effective Action

In order to determine the effective action we need to evaluate the determinant of $h_{\mu\nu}$. For this purpose we will use Eq. (7) and the following expression for the determinant,

$$ h_{\mu\nu} = \frac{1}{24} \epsilon_{\mu\nu\rho\sigma} h_{\alpha\gamma} h_{\beta\delta} h_{\mu\rho} h_{\nu\sigma} \epsilon_{\alpha\beta\gamma\delta} $$

where

$$ \epsilon_{\alpha\beta\mu\nu} = - \epsilon_{\alpha\beta\mu\nu} \quad (32) $$

$$ \epsilon_{\alpha\beta\mu\nu} \epsilon_{\gamma\beta\mu\nu} = -6 \delta_{\gamma} \quad (33) $$

$$ \epsilon_{\alpha\beta\mu\nu} \epsilon_{\gamma\rho\mu\nu} = -2 \left( \delta_{\gamma} \rho - \delta_{\rho} \gamma \right) \quad (34) $$

Then we obtain,

$$ \sqrt{-G} = \sqrt{-h} = a^4 \sqrt{-g} \left[ 1 - \frac{\ell^2}{6a^2 R} - \frac{\ell^4}{16a^4} \left( 3R_{\alpha\beta} R^{\alpha\beta} - \frac{8}{9} R^2 \right) \right]^{1/2} $$

$$ = a^4 \sqrt{-g} \left[ 1 - \frac{\ell^2}{12a^2 R} - \frac{\ell^4}{32a^4} \left( 3R_{\alpha\beta} R^{\alpha\beta} - R^2 \right) \right] \quad (35) $$

Having obtained the bulk metric, it is now trivial to calculate the bulk action, with second order correction terms included. For that purpose we substitute the determinant $\sqrt{-h}$ which includes second order correction, to the bulk action. With this factors included in the bulk action we arrive at,

$$ S_{\text{bulk}} = \frac{1}{2\kappa^2} \int d^5 x \sqrt{-G} \left[ \mathcal{R} + \frac{12}{\ell^2} \right] $$

$$ = -\frac{8}{\kappa^2 \ell^2} \int d^4 x \sqrt{-g} \left[ \frac{\ell}{4} (a_+^4 - a_+^4) - \frac{\ell^4 R}{24} (a_+^2 - a_-^2) \right. $$

$$ \left. - \frac{3\ell^4}{32} (\phi_+ - \phi_-) \left( R_{\beta\gamma} R^{\beta\gamma} - \frac{1}{3} R^2 \right) \right] \quad (36) $$

where $\phi_+$ and $\phi_-$ are the respective brane positions defined through Eq. (2). Also we have defined the warp factor $a^2$ at the position of the branes $\phi_+$ and $\phi_-$ as $a_+^2$ and $a_-^2$ respectively. The next thing to
calculate is the action corresponding to brane tension. For this we require the induced metric on each brane, with the following expression

\[
g_{\alpha\beta}^\pm(y = \phi_\pm, x) = a^2(\phi_\pm)[g_{\alpha\beta}(x) + f_{\alpha\beta}(\phi_\pm, x) + q_{\alpha\beta}(\phi_\pm, x)] + \partial_\alpha \phi_\pm \partial_\beta \phi_\pm
\]  

(37)

Then the determinant of the induced metric turns out to have the following expression

\[
\sqrt{-g_\pm} = a^4(\phi_\pm) \sqrt{-g} \left[1 + \frac{1}{a^2(\phi_\pm)} \partial_\mu \phi_\pm \partial^\mu \phi_\pm - \frac{\ell^2}{6a^2} R - \frac{\ell^4}{16a^4} \left(3R_{\alpha\beta}R^{\alpha\beta} - \frac{8}{9} R^2\right)\right]^{1/2}
\]

\[
= a^4(\phi_\pm) \sqrt{-g} \left[1 + \frac{1}{2a^2(\phi_\pm)} \partial_\mu \phi_\pm \partial^\mu \phi_\pm - \frac{\ell^2}{12a^2} R - \frac{3\ell^4}{32a^4} \left(R_{\alpha\beta}R^{\alpha\beta} - \frac{1}{3} R^2\right)\right]
\]  

(38)

With the help of the above equation, the action on the two branes can be written as

\[
S_\pm = \mp \frac{6}{\kappa^2 \ell} \int d^4x \sqrt{-g_\pm} \left[\frac{4}{\ell} + \frac{1}{a^2(\phi_\pm)} \Box \phi_\pm + \frac{1}{6a^2(\phi_\pm)} \partial_\mu \phi_\pm \partial^\mu \phi_\pm + \frac{\ell^2}{6a^2(\phi_\pm)} R + \frac{\ell^4}{4a^4(\phi_\pm)} \left(R_{\mu\nu}R^{\mu\nu} - \frac{2}{9} R^2\right)\right]
\]  

(39)

Then simple addition of the two actions \(S_+\) and \(S_-\) leads to,

\[
S_+ + S_- = -\frac{6}{\kappa^2 \ell} \int d^4x \sqrt{-g} \left[\left(\frac{a_+^4}{a_-^4} - \frac{a^4}{a_-^4}\right) - \frac{\ell^2}{12} \left(a_+^2 - a_-^2\right)
\]

\[+ \frac{1}{2} \left(a_+^2 \partial_\mu \phi_+ \partial^\mu \phi_+ - a_-^2 \partial_\mu \phi_- \partial^\mu \phi_-\right)\right]
\]  

(40)

Finally we need to calculate the counter term provided by Gibbons and Hawking. For that we need to calculate the extrinsic curvature, or more importantly its trace, which has the following expression

\[
K_\pm = \left[\frac{4}{\ell} + \frac{1}{a^2(\phi_\pm)} \Box \phi_\pm + \frac{1}{6a^2(\phi_\pm)} \partial_\mu \phi_\pm \partial^\mu \phi_\pm + \frac{\ell^2}{6a^2(\phi_\pm)} R + \frac{\ell^4}{4a^4(\phi_\pm)} \left(R_{\mu\nu}R^{\mu\nu} - \frac{2}{9} R^2\right)\right]
\]  

(41)

The action corresponding to the Gibbon-Hawking counter term has the expression,

\[
S_{GH} = \frac{2}{\kappa^2} \int d^4x \sqrt{-g_+} K_+ - \frac{2}{\kappa^2} \int d^4x \sqrt{-g_-} K_-
\]

\[
= \frac{2}{\kappa^2} \int d^4x \sqrt{-g} \left[\frac{4}{\ell} \left(a_+^4 - a_-^4\right) - \frac{\ell}{6} \left(a_+^2 - a_-^2\right)
\]

\[+ \frac{3}{\ell} \left(a_+^2 \partial_\mu \phi_+ \partial^\mu \phi_+ - a_-^2 \partial_\mu \phi_- \partial^\mu \phi_-\right)\right]
\]  

(42)

Thus substitution of the bulk action, brane tension and Gibbon-Hawking counterterm leads to the complete four dimensional effective action which has the expression,

\[
S_{tot} = S_{bulk} + S_+ + S_- + S_{GH}
\]

\[
= \frac{\ell}{2\kappa^2} \int d^4x \sqrt{-g} \left[\left(a_+^4 - a_-^4\right) R + \frac{6}{\ell^2} \left(a_+^2 \partial_\mu \phi_+ \partial^\mu \phi_+ - a_-^2 \partial_\mu \phi_- \partial^\mu \phi_-\right)
\]

\[- \frac{3\ell^4}{32} \left(\phi_+ - \phi_-\right) \left(R_{\alpha\beta}R^{\alpha\beta} - \frac{1}{3} R^2\right)\right]
\]  

(43)
From the above analysis it comes out that the third order terms would be associated with \(a^{-2}\), the fourth order terms will be connected to \(a^{-4}\) and so on. Thus the nth order term would be associated with \(a^{-2(n-2)}\) term. Also these terms would be independent of the part depending on the brane coordinates. Thus the effective action would contain terms like \(R_{\alpha\beta} R^{\alpha\beta} \times (a_+^{-2} - a_-^{-2})\). All these terms will appear with the extra dimensional part separated where only the difference enters the picture. Thus the effective action would be factorizable at all orders and only the difference between these moduli fields appear in the effective action.

In order to understand the effective action in greater detail, we vary the action with respect to \(g_{\mu\nu}\) with the assumption of fixed brane, i.e. \(\phi_+\) and \(\phi_-\) are assumed to be independent of \(x^\mu\). Then equation of motion in absence of any matter field obtained from arbitrary variation turns out to be,

\[
\frac{\ell^2}{2\kappa^2} \left( a_+^2 - a_-^2 \right) \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = \frac{\ell^3}{64\kappa^2} \left( \phi_+ - \phi_- \right) \left( 6 R_{\mu\alpha} R^\alpha_{\nu} - 2 R R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \left( 3 R_{\alpha\beta} R^{\alpha\beta} - R^2 \right) \right) + 2 \nabla_\mu \nabla_\nu R - 6 \nabla_\alpha \nabla_\mu R^\alpha_{\nu} + 3 \nabla_\alpha \nabla^\alpha R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \left( 3 \nabla_\alpha \nabla_\beta R^{\alpha\beta} - 2 \Box R \right) = 0
\] (44)

However this equation contains higher order derivatives of the metric. In order to avoid appearances of any ghost field, these higher derivative terms must vanish. This conditions yields the following equation,

\[
2 \nabla_\mu \nabla_\nu R - 6 \nabla_\alpha \nabla_\mu R^\alpha_{\nu} + 3 \nabla_\alpha \nabla^\alpha R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \left( 3 \nabla_\alpha \nabla_\beta R^{\alpha\beta} - 2 \Box R \right) = 0
\] (45)

The interesting aspect of this equation is that the trace part leads to, \(\nabla_\mu \nabla_\nu G^{\mu\nu} = 0\), which is automatically satisfied by Bianchi identity.

### 4 Equivalence with Scalar-Tensor Gravity

In first order we have two arbitrary constants, \(C_{\mu\nu}(x)\) and \(\chi_{\mu\nu}(x)\), both of them being independent of the extra coordinate, while depending on the brane coordinates. Let us exploit these two tensors and obtain some simplified results. Firstly, we can use \(C_{\mu\nu}\) such that \(f_{\mu\nu}(y = \phi_+, x) = 0\). This can be seen explicitly from Eq. (16) which under the above condition, reduces to the following form:

\[
f_{\mu\nu}(y = \phi_+, x) = \frac{\ell^2}{2a_+^2} \left( R_{\mu\nu} - \frac{1}{6} g_{\mu\nu} R \right) - \frac{\ell}{2a_+} \chi_{\mu\nu}(x) + C_{\mu\nu}(x) = 0
\]

Then we can use \(\chi_{\mu\nu}\) to set \(f_{\mu\nu}(y = \phi_-, x) = 0\). This can be achieved by using the arbitrary constant \(\chi_{\mu\nu}\) such that we readily obtain the following equation to be satisfied by \(\chi_{\mu\nu}\), so that this leads to

\[
f_{\mu\nu}(y = \phi_-, x) = \frac{\ell^2}{2} \left( \frac{1}{a_+^2} - \frac{1}{a_-^2} \right) \left( R_{\mu\nu} - \frac{1}{6} g_{\mu\nu} R \right) + \frac{\ell}{2} \left( \frac{1}{a_+^4} - \frac{1}{a_-^4} \right) \chi_{\mu\nu}(x) = 0
\]

However instead of being so restrictive, we can impose another but single boundary condition, \(f_{\mu\nu}(y = \phi_+, x) = f_{\mu\nu}(y = \phi_-, x)\). Then we obtain the equation satisfied by \(\chi_{\mu\nu}\) to be identical to Eq. (47), where \(C_{\mu\nu}\) remains arbitrary.
Then in a similar manner we can use \( t_{\mu\nu} \) to set \( q_{\mu\nu}(y = \phi_+, x) = q_{\mu\nu}(y = \phi_-, x) \). The same argument holds at all order. Thus we have

\[
\begin{align*}
  h_{\mu\nu}(y = \phi_+, x) &= a^2 (\phi_+) \left[ g_{\mu\nu} + f_{\mu\nu} (\phi_+, x) + q_{\mu\nu} (\phi_+, x) + \ldots \right] \\
  h_{\mu\nu}(y = \phi_-, x) &= a^2 (\phi_-) \left[ g_{\mu\nu} + f_{\mu\nu} (\phi_-, x) + q_{\mu\nu} (\phi_-, x) + \ldots \right]
\end{align*}
\] (48)

where, we readily obtain, \( f_{\mu\nu} (\phi_+, x) = f_{\mu\nu} (\phi_-, x) \) as well as, \( q_{\mu\nu} (\phi_+, x) = q_{\mu\nu} (\phi_-, x) \). Thus we obtain,

\[
\begin{align*}
  h_{\mu\nu} (\phi_-, x) &= \Omega^2 h_{\mu\nu} (\phi_+, x) ; \\
  \Omega^2 &= a^2 (\phi_-) / a^2 (\phi_+) = \exp \left[ \frac{2 (\phi_+ - \phi_-)}{\ell} \right]
\end{align*}
\] (50)

Note that since the branes are not fixed the factor \( \Omega \) depends on brane coordinates. Also \( \Omega \) depends only on the separation, \( \phi_- - \phi_+ \), i.e. on the radion field. Thus we observe that in general for any order in the gradient expansion scheme, we can have the relation (50), where metric on the brane located at \( y = \phi_- \) is connected to the metric on the brane located at \( y = \phi_+ \) by a conformal factor. Thus the Ricci, tensor, Ricci scalar and Einstein tensor in the two branes are related through the following relation:

\[
R_{\mu\nu} = R_{\mu\nu}^+ + \frac{1}{\Omega^2} \nabla_\mu \nabla_\nu \Omega^2 + \frac{3}{2\Omega^4} \nabla_\mu \nabla_\nu \nabla_\alpha \nabla^\alpha \Omega^2 + a^2 (\phi_+) h_{\mu\nu}^+ \nabla_\alpha \nabla^\alpha \Omega^2
\] (51a)

\[
R^- = \frac{1}{\Omega^2} \left[ R^+ + \frac{3}{2\Omega^4} \nabla_\mu \nabla_\nu \Omega^2 - \frac{3}{2\Omega^4} \nabla_\mu \nabla_\nu \nabla_\alpha \nabla^\alpha \Omega^2 \right]
\] (51b)

\[
G_{\nu}^{(-)\mu} = G_{\nu}^{(+)\mu} + M_{\nu}^\mu
\]

\[
G_{\nu}^{(+)\mu} + \frac{1}{\Omega^2} \nabla_\nu \nabla_\mu \Omega^2 - \frac{3}{2\Omega^4} \nabla_\nu \nabla_\mu \nabla_\alpha \nabla^\alpha \Omega^2 - \delta_{\nu}^{\mu} \frac{1}{\Omega^2} \nabla_\alpha \nabla^\alpha \Omega^2 + \Omega^2 \frac{3}{4\Omega^4} \nabla_\alpha \nabla^\alpha \nabla_\alpha \nabla^\alpha \Omega^2
\] (51c)

where the object \( M_{\nu}^\mu \) is defined through Eq. (51c). We therefore have the following Einstein’s equation on the two branes,

\[
\frac{\ell}{2} G_{\nu}^{(+)\mu} = \frac{\kappa^2}{2} T_{\nu}^{(+)\mu}
\] (52a)

\[
\frac{\ell}{2} G_{\nu}^{(-)\mu} = \frac{\ell}{2} (G_{\nu}^{(+)\mu} + M_{\nu}^\mu) = \frac{\kappa^2}{2} T_{\nu}^{(-)\mu}
\] (52b)

Thus we observe

\[
\frac{\kappa^2}{\ell} \left[ \frac{1}{\Omega^2} T_{\nu}^{(+)\mu} - \frac{1}{\Psi} \nabla_\nu \nabla^\mu \right] = \frac{1}{\Psi} G_{\nu}^{(+)\mu} - \frac{1}{\Psi} \left( G_{\nu}^{(+)\mu} + M_{\nu}^\mu \right) = G_{\nu}^{(+)\mu} - \frac{1}{\Psi} M_{\nu}^\mu
\] (53)

where we have defined \( \Psi = 1 - \Omega^2 \). The field equation now leads to the following form

\[
G_{\nu}^{(+)\mu} = \frac{\kappa^2}{\ell} \left[ \frac{1}{\Omega^2} T_{\nu}^{(+)\mu} - \frac{1}{\Psi} \nabla_\nu \nabla^\mu \right] + \frac{1}{\Psi} M_{\nu}^\mu
\]

\[
= \frac{\kappa^2}{\ell} \left[ \frac{1}{\Omega^2} T_{\nu}^{(+)\mu} - \frac{1}{\Psi} \nabla_\nu \nabla^\mu \right] - \frac{1}{\Psi} \left( (\nabla_\nu \nabla^\mu - \delta_{\nu}^{\mu} \nabla_\alpha \nabla^\alpha) \Psi + \omega (\Psi) \left( \nabla_\nu \nabla^\mu - \frac{1}{2} \delta_{\nu}^{\mu} \nabla_\alpha \nabla^\alpha \Psi \right) \right)
\] (54)
where we have introduced a new function, $\omega(\Psi) = 3\Psi / 2(1 - \Psi)$. Again eliminating $G^{(+)}_{\mu}$ from Eqs. (52a) and (52b) with contraction of the indices we arrive at,

$$\Box \Psi + \frac{2\omega + 3}{2} \nabla_\alpha \Psi \nabla^\alpha \Psi = \frac{\kappa^2}{\ell} \frac{1}{2\omega + 3} \left( T^{(-)}_{\mu} - T^{(+)}_{\mu} \right)$$

(55)

Note that the field equation for gravity given in Eq. (54) and the field equation for $\Psi$ provided by Eq. (55) holds for any order in the gradient expansion scheme. Thus the field equation for $\Psi$ or equivalently for the radion field is determined by the trace of the stress energy tensor at both the branes. The remarkable thing about these field equations are that they hold for all orders in the perturbation scheme and is equivalent to Brans-Dicke field equations for gravity.

In order to make the circle complete let us write down the effective equation entirely in terms of $\Omega$. For that we note the following identities,

$$\partial_\alpha \Omega^2 = \Omega^2 \frac{2}{\ell} (\partial_\alpha \phi_+ - \partial_\alpha \phi_-)$$

(56)

$$\partial_\alpha \Omega^2 \partial^\alpha \Omega^2 = \frac{4\Omega^4}{\ell^2} \left[ (\partial_\alpha \phi_+)^2 + (\partial_\alpha \phi_-)^2 - 2\partial_\alpha \phi_- \partial^\alpha \phi_+ \right]$$

(57)

$$\phi_+ - \phi_- = \frac{\ell}{2} \ln \Omega^2$$

(58)

However in order to get a proper idea we put $\phi_+ = \text{constant}$, such that $a_+^2 = 1$. Thus upto second order the effective equation turns out to have the following form from Eq. (43) as,

$$S_{tot} = \frac{\ell}{2\kappa^2} \int d^4 x \sqrt{-g} \left[ (1 - \Omega^2) R - \frac{3}{2\Omega^2} \partial_\mu \Omega^2 \partial^\mu \Omega^2 - \frac{3\ell^5}{64} \ln \Omega^2 \left( R_\beta R^\beta - \frac{1}{3} R^2 \right) \right]$$

$$= \frac{\ell}{2\kappa^2} \int d^4 x \sqrt{-g} \left[ \Psi R - \frac{\omega(\Psi)}{\Psi} \partial_\mu \Psi \partial^\mu \Psi - \frac{3\ell^5}{64} \ln \Omega^2 \left( R_\beta R^\beta - \frac{1}{3} R^2 \right) \right]$$

(59)

which resembles the action for Brans-Dicke theory of gravity. Thus even at the level of the effective action the bulk-brane system is equivalent to Brans-Dicke or scalar-tensor theories of gravity.

5 Discussion

In this work, our main aim was to address two important aspects related to brane world models. First, the issue of factorizability of the metric ansatz and secondly equivalence of Brans-Dicke theory with this brane world model. The first result was derived earlier in connection to gradient expansion scheme upto first order in the brane to bulk curvature ratio. In this work we have generalized the previous result to second order in the gradient expansion scheme, by deriving the second order correction to the bulk metric. This ultimately leads to validity of this result to arbitrary order of the perturbative expansion in brane to bulk curvature ratio. Through this exercise we have achieved the bulk metric upto second order and then we have applied this bulk metric in order to calculate the effective action correct upto second order. It turns out that the effective action gets factorized into the extra dimension or radion part and brane part even when the second order corrections are included (this generalizes previous results derived only upto first order). Also by generalizing our result we can argue that factorizability is a valid assumption upto all order in this perturbative expansion scheme.
Secondly, we have used the non-local factors present from the bulk field equations, in order to express the gravitational field equation on the brane in terms of the radion field and bulk metric. The field equation on the brane obtained in this way turns out to be equivalent to scalar-tensor theory of gravity. The same result has also been obtained from the effective action obtained earlier in this work. Thus we can conclude that the two brane system is equivalent to Brans-Dicke theory as far as the effective description is considered and this is true for all orders in the perturbative gradient expansion valid at low energies.

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