Nonholonomic Algebroids, Finsler Geometry, and Lagrange–Hamilton Spaces

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Abstract

We elaborate an unified geometric approach to classical mechanics, Riemann–Finsler spaces and gravity theories on Lie algebroids provided with nonlinear connection (N–connection) structure. There are investigated the conditions when the fundamental geometric objects like the anchor, metric and linear connection, almost symplectic and related almost complex structures may be canonically defined by a N–connection induced from a regular Lagrangian (or Hamiltonian), in mechanical models, or by generic off–diagonal metric terms and nonholonomic frames, in gravity theories. Such geometric constructions are modelled on nonholonomic manifolds provided with nonintegrable distributions and related chains of exact sequences of submanifolds defining N–connections. We investigate the main properties of the Lagrange, Hamilton, Finsler–Riemann and Einstein–Cartan algebroids and construct and analyze exact solutions describing such objects.

Keywords: Lie algebroids, Lagrange, Hamilton and Riemann–Finsler spaces, nonlinear connection, nonholonomic manifold, geometric mechanics and gravity theories

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1 Introduction

The theory of Lie algebroids (see a mathematical background, discussion, first applications and bibliography in [4, 22, 9, 32]) has received recently considerable attention in geometric mechanics [64, 21, 23, 14], control theory [8], geometry of gauge fields, string and gravity [38, 39, 49, 50, 51, 52]. In the present paper, we study the canonical realization of the Lagrange–Hamilton, Riemann–Finsler and Einstein–Cartan geometry (see details, for instance, in Refs. [28, 3, 29, 30, 2]; for extensions to superspaces, spinors and noncommutative spaces, see [43, 48] on Lie algebroids enabled with nonlinear connection structure and analyze some important examples of such nonholonomic configurations and exact solutions of the Einstein equations modelling algebroid structures.

There are two general approaches to geometrization of mechanics on the tangent/cotangent bundle:

Roughly speaking, the first approach follows the idea to describe the mechanics in terms of sympletic geometry by developing certain procedures of geometrization of the Euler–Lagrange equations (we cite a recent a review [19] of results in mechanics and classical field theory based on the (multi)sympletic formalism, differential forms, jets ....).

In a quite alternative form, i.e. in the second approach, the Lagrange and Hamilton mechanics was geometrized [28, 29, 30, 43, 48] by using the methods of Finsler geometry, see also [27, 3, 2] for alternative researches and applications, on tangent and vector bundles and generalizations to Clifford bundles, superbundles and projective modules in noncommutative geometry.... Such spaces are enabled with nonholonomic structures, i.e. nonintegrable distributions, defined by nonlinear connections (in brief, N–connection). Following this approach, the N–connection and the bulk of fundamental geometric structures (metric, canonical linear connection, almost sympletic and almost complex structures...) are derived in general form starting from a regular (for simplicity) Lagrangian and/or Hamiltonian. In such a case, the geometric constructions are not related to the particular properties of corresponding systems of partial differential equations, symmetries and constraints of mechanical and field models, i. e. to the Euler–Lagrange equations which are equivalently transformed into a (semi)spray configuration of "nonlinear" geodesics, but canonically defined by certain classes of Sasaki type metrics, distinguished linear connections

\[\text{in general, with spinor, supersymmetric, quantum group and another type of noncommutative variables}\]
(adapted to the N–connection) and corresponding torsions and curvatures.

The first mentioned approach was recently developed into some descriptions of mechanics on Lie algebroids \[64, 21, 23, 14\] and the second one has natural extensions to the noncommutative geometry of mechanics, Clifford–Lagrange spaces and nonholonomic Dirac operators \[48\]. Of course, both approaches are inter–related: For instance, certain (semi) sprays and N–connection configurations were considered in the mentioned first direction of researches and the almost sympletic/ complex/ tangent structures were derived for the second ones. There are also works applying both types of geometrizations of mechanics with explicit purposes to elaborate a geometric quantization formalism for nonholonomic mechanics \[11, 60\].

There are a number of features motivating rigorous studies of new types of nonholonomic algebroid structures and their applications. Here, we enumerate twelve already distinguished directions:

The first one come from the geometry of nonholonomic manifolds and bundles (in our case provided with N–connection structure). Such spaces with nonholonomic distributions are characterized by generalized Lie type nonholonomy relations for frames and admit quotients by the structure Lie group which requests definition of a new class of nonholonomic Lie algebroids (in brief, Lie N–algebroids). This is not a trivial academic procedure of modelling physical theories and geometries on spaces provided with algebroid structure because for general nonholonomic manifolds there is a not completely solved problem of definition the curvature tensor\[^{2}\]. In this work, we shall prove that it is possible to construct curvature tensors for very general classes on nonholonomic manifolds defined by nonlinear and linear connections on Lie N–algebroids.

The second direction arises from the modern gravity theories, in general, with nontrivial torsion and nonmetricity. There are classes of such space–times (considered also in this work) when the generic off–diagonal metric, nonholonomic frames and nonlinear/linear connections mimic certain Lie N–anholonomic algebroid structures constructed as exact solutions of the generalized gravitational field equations (in particular, of the Einstein equations). This can be related to new directions in constructing exact solutions with Lie N–algebroid symmetries and investigation of their symmetries, singular and nonholonomic configurations for which the application of the methods of algebroid theory are crucial. We give in this paper some explicit examples of such Einstein–Cartan algebroids.

There is a third motivation coming from the Riemann–Finsler geome-

\[^{2}\text{see detailed discussions and references, for instance, in }11, 48\]
try, almost Kahler models and their generalizations. Such geometries are naturally defined on tangent bundles and higher order extensions provided with nonlinear connection structure, or on manifolds of even dimensions (or containing embedding of such even dimension submanifolds) provided with exact chains of submanifolds prescribing N–connection and associated nonholonomic frame structures. It is not possible to define such Finsler–Lagrange structures on general vector bundles with different dimensions of the fibers and basic manifold. In another turn, the Lie algebroid constructions related to the tangent bundles of the associated vector bundles, allows to define new types of Finsler geometries (as well, their Lagrange and/or Hamilton geometry extensions) because the even dimensions arise naturally for the geometric objects transferred on 'tangents' to the fibers of a vector bundle. In this work, we define explicitly and investigate such new type of Finsler/Lagrange/Hamilton algebroids. As a matter of principle, such algebroid constructions are defined by subclasses of structure functions of the Einstein–Cartan (and more general metric–affine) algebroids.

The forth set of arguments for the theory of Lie of N–algebroids results from the mentioned geometrization of mechanics with the Lagrange and Hamilton functions defined on the Lie structure group quotients of the tangent, respectively, cotangent bundles. This direction will be extended by considering nonholonomic configurations (canonically defined by the fundamental Lagrange or Hamilton functions and their respective homogeneous variants for the Finsler and Cartan geometry). We study an application of such Lie N–algebroid methods in order to elaborate a rigorous geometric formalism for the optic–mechanical modelling of gravitational processes like in analogous gravity.

There is the fifth approach related to investigation of the control systems on the Lie algebroids [24, 8]. We shall not work in this direction in this paper but we note here that the optimal control theory having explicit relations to the Lagrange mechanics will obtain a number of new features and possibilities by applying the formalism of Lie N–algebroids provided with metric and distinguished connection (by N–connection structures).

The sixth direction appears as the jet formalism elaborated on Lie algebroids and related to models of classical field theory, time–dependent and higher order mechanics, see the first results in [9, 25, 26]. This concerns a further elaboration of algebroid multisympletic models for noholonomic mechanics and classical field theories, sigma models, gravitational and string actions and various type of topological theories.

We plan to elaborate a seventh direction devoted to N–connections and
field dynamics on Lie algebroid jets, revising the presymplectic formalism on
the spaces of Cauchy data, classification of infinitesimal symmetries, conserv-
ation lows in the geometric context of multisympletic geometry and Ehres-
mann connections [17]. The set of the invariants and conservation lows on
such algebroid spaces will be completed by the corresponding fundamental
system of nearly autoparallel maps and conservation lows in the past inves-
tigated for the (pseudo) Riemannian and generalized Finsler spaces [41, 10],
see also Chapters 3 and 8 in [45].

The eight direction may be related to the already stated approaches to
the "Lie algebroid" gauge theories and gravity models (of string gravity and
Einstein type) which in our opinion has certain perspectives in gauge gravity
modelling of the general relativity, brane physics and string gravity. Such
constructions may be derived from the gauge locally anisotropic gravity and
noncommutative gauge gravity [55, 46, 53], see also Chapters 2 and 7 in [45].
The Lie algebroid variants with singular maps, anchors and nonholonomic
structure seem to solve a number of problems concerning nonsemisimple
realizations of gravitational gauge theories and broken symmetries in such
models.

The ninth new direction of the geometric and physical applications of
the algebroid mathematics is related to the Clifford algebroid structures and
various type of spinor–Finsler/Lagrange/Hamilton geometries and redefin-
tion of mechanics on spinor bundles provided with N–connection structures
[42, 44, 58, 59] and Chapter 6 in [45]. This is not a trivial rewriting of the
Lagrange or Hamilton formalism in spinor terms. An explicit geometriza-
tion of mechanics both in terms of sympletic and N–connection structures on
algebroids allows to work directly with singular, nonholonomic and quotient
symmetries which can be related to the quantum "world" via spinor vari-
ables which in such cases are modelled by nonholonomic Clifford structures,
nonlinear connections, connections and curvatures on such spaces.

It became already explicit the tenth direction which leads from the
Clifford–Lagrange and Clifford–Riemann–Finsler geometry to the noncom-
mutative geometry. Following the geometry of nonholonomic frames, the
Lagrange–Finsler algebroids can be treated as Riemann–Cartan manifolds
provided with corresponding prescribed types of N–connection and Lie alge-
broid structures. The Riemann geometry can be 'extracted' from the non-
commutative geometry via the Dirac operator formalism [7, 12, 35]. A gen-
eralized Dirac operator approach was already elaborated for the generalized
Riemann–Finsler and Lagrange-Hamilton spaces [48] which emphasizes the
possibility to define noncommutative extensions of the Clifford–Lagrange–
Hamilton algebroids.
The eleventh approach may be considered in connection to mechanical integrators, numerical methods and applications in economics [18]. In this case, it would be necessary the elaboration of a discrete geometry and a corresponding calculus for nonholonomic algebroid structures. The specific point would be that numeric and analytic methods of the theory of differential equations will have to be elaborated in a form preserving the prescribed algebroid configuration.

Finally, in the twelfth, less distinguished direction with possible new subdirections, the mentioned classical commutative and noncommutative geometry and algebroid methods seem to have a perspective to the geometric quantization, spin networks and path quantum gravity, Fedosov spaces, Hopf algebras and quantum group geometry, see references and algebroid related discussions in [4]. It also follows from the constructions with Lie algebroids and nonholonomic geometries presented, for instance in [28], and has a direct relation to the cohomology of Jacobi manifolds and algebroids [15, 16], but this can be considered as a long term program of our further researches.

The purpose of this work is to elaborate the theory of Lie N–anholonomic algebroids and to present a set of strong arguments and motivations for such constructions, derived from the geometric mechanics and gravity theory. We shall follow in the bulk the first mentioned four directions but also formulate a Lie algebroid nonholonomic geometric background for a future work related to the rest of eight directions.

This paper is organized into six sections. In Section 2, we recall some necessary results on Lie algebroids and their prolongations. In Section 3, we clarify the relevance of the geometry of nonlinear connections to geometric models of mechanics on tangent bundles and Lie algebroids. We elaborate the almost Hermitian model of Lagrange mechanics on Lie algebroids and define the canonical nonlinear connection, metric and distinguished connection, almost complex and sympletic structures all induced by regular Lagrangians. The theory of Lie algebroids provided with nonlinear connection structure is formulated in Section 4. We investigate the main classes of nonlinear and linear connections and prove the main theorems on torsions and curvatures of such nonholonomic manifolds provided with, in general, nonintegrable distributions. In Section 5, we define the Finsler and Hamilton algebroids and their generalizations and prove that there are canonical Lie algebroid structures determined by the canonical sympletic, metric and nonlinear connections induced by corresponding Lagrangians, Hamiltonians and/or Finsler–Cartan fundamental functions. Section 6 is devoted to a proof that a certain class of Lie algebroids can be associated to exact solutions in gravity theories, parametrized by generic off–diagonal metrics and
nonholonomic frames. We formulate some criteria when the gravitational processes may be modelled by optical and continuous mechanics media and can geometrized in the Lie algebroid approach. Finally, some examples of exact solutions defining such nonholonomic algebroid configurations are constructed and analyzed.

2 Preliminaries: Lie Algebroids and Prolongations

The section is an overview of the results and conventions on Lie algebroids and vector/tangent bundles to be applied and developed in this work.

2.1 Definition of Lie algebroids

Let $\mathcal{E} = (E, \pi, M)$ be a vector bundle defined by surjective projection $\pi : E \longrightarrow M$ when the dimensions of the base and total manifolds are respectively $\dim M = n$ and $\dim E = n + m$. A Lie algebroid $A \cong (E, [\cdot, \cdot], \rho)$ is defined as the vector bundle $E$ provided with algebroid structure $(\cdot, \cdot, \rho)$, where $[\cdot, \cdot]$ is a Lie bracket on the $C^\infty(M)$–module of sections of $E$, denoted $\text{Sec}(E)$, and the 'anchor' $\rho$ is defined as a bundle map $\rho : E \rightarrow TM$ ($TM$ is the tangent bundle to $M$) such that

$$[X, fY] = f[X, Y] + \rho(X)(f)Y$$

for $X, Y \in \text{Sec}(E)$ and $f \in C^\infty(M)$. The anchor also induces an homomorphism of $C^\infty(M)$-modules $\rho : \text{Sec}(A) \rightarrow \mathfrak{X}^1(M)$ where $\mathfrak{X}^r(M)$ and $\mathfrak{X}^\ast(M)$ denote, respectively, the spaces of differential $r$–forms and $r$–multivector fields on $M$.

Let us state the typical notations for abstract (coordinate) indices given with respect to an arbitrary or coordinate basis. For a local basis on $\mathcal{E}$ we write $e_\alpha = (e_i, v_\alpha)$. The small Greek indices $\alpha, \beta, \gamma, \ldots$ are to be considered general ones when the values $1, 2, \ldots, n + m$ and $i, j, k, \ldots$ and $a, b, c, \ldots$ label respectively the geometrical objects on the base and typical fiber. The local coordinates of a point $u \in \mathcal{E}$ are written $u = (x, u)$, or $u^\alpha = (x^i, u^a)$, where $u^a(u)$ is the $a$-th coordinate with respect to the basis $v_\alpha$ and $(x^i)$ are local coordinates with respect to the basis $e_i$ on $M$. For our purposes, it is convenient to write the local coordinates on a tangent bundle, when $\mathcal{E} = TM$, in the form $(x^i, y^k)$, where $i, j, k, \ldots = 1, \ldots, n$. We shall write also

Sometimes, we shall write for this vector bundle only the symbol $E$, if there is not confusion.
briefly $E$ or $\mathcal{E}$ instead of the set of sections of the total space $\text{Sec}(E)$, if such a notation will not result in ambiguities.

In local form, the Lie algebroid structure is defined by its structure functions $\rho_a^i(x)$ and $C_{ab}^i(x)$ on $M$, determined by the relations

$$\rho(v_a) = \rho_a^i(x) e_i,$$  

(1)

$$[v_a, v_b] = C_{ab}^c(x) v_c$$  

(2)

and subjected to the structure equations

$$\rho_a^j \frac{\partial \rho_b^i}{\partial x^j} - \rho_b^j \frac{\partial \rho_a^i}{\partial x^j} = \rho_c^i C_{ab}^c \quad \text{and} \quad \sum_{\text{cyclic}(a,b,c)} (\rho_a^j \frac{\partial C_{bc}^d}{\partial x^j} + C_{af}^d C_{bc}^f) = 0.$$  

(3)

Roughly speaking, the concept of Lie algebroid $\mathcal{A}$ substitutes that of the tangent bundle $TM$, when an element $\sigma \in E$ is considered as a generalized “velocity” and the actual velocity $V$ is obtained via the anchor map, $V = \rho(\sigma)$. Subjected to the conditions (3), the image $\rho(E)$ defines an integrable generalized distribution; therefore, $M$ is foliated by the leaves of $\rho(E)$. We will say that an algebroid is transitive if it has only one leaf which is obviously equal to $M$. This property holds if and only if the map $\rho$ is surjective (we shall construct mechanical models on such manifolds). For certain gravity models, it is possible that $\mathcal{A}$ is not transitive but the restriction of a Lie algebroid to a leaf $q \subset M, E_q \rightarrow q$ is transitive (we can also consider a trivial embedding of $E_q$ into a higher dimension which allows a surjective map). One says that a Lie algebroid is locally transitive at a point $x \in M$ if $\rho_x : E_x \rightarrow T_x M$ is surjective. In this case, the point $x$ is contained in a leaf of maximal dimension.

If $\mathcal{A}$ is a Lie algebroid and $E^*$ is the dual of $E$, we can introduce the differential of $E$, $d^E : \text{Sec}(\wedge^k E^*) \rightarrow \text{Sec}(\wedge^{k+1} E^*)$ as follows

$$d^E \lambda(X_0, X_1, ..., X_k) = \sum_{r=1}^k (-1)^r \rho(X_r) \left( \lambda(X_0, ..., \widehat{X}_r, ..., X_k) \right)$$

$$+ \sum_{r < r'} (-1)^{r + r'} \rho(X_r) \left( \lambda([X_r, X_{r'}], X_0, ..., \widehat{X}_r, ..., \widehat{X}_{r'}, ..., X_k) \right)$$

for $\lambda$ being an element from the set of sections of the $E$–valued $k$–forms, $\text{Sec}(\wedge^k E^*)$, and $X_0, X_1, ..., X_k \in \text{Sec}(E)$, where $\widehat{X}_r$ means that this term is omitted under summation. It is obvious that $(d^E)^2 = 0$. The trivial
examples of such differentials are those for a function \( f \in C^\infty(M) \) and \( \theta = \theta_a v^a \in \text{Sec}(E^*) \) when, respectively,

\[
d^E f = \frac{\partial f}{\partial x^i} \rho^i_a v^a \quad \text{and} \quad d^E \theta = \left( \rho^i_a \frac{\partial \theta_b}{\partial x^i} - \frac{1}{2} \theta_{bc} C_{ab}^c \right) v^a \wedge v^b,
\]

where \((d^E f)(X) = \rho(X)(f)\). Therefore,

\[
d^E x^i = \rho^i_a v^a \quad \text{and} \quad d^E v^a = -\frac{1}{2} C_{bc}^a v^b \wedge v^c.
\]

We also define the Lie derivative with respect to \( X \) as the operator \( L^E_X : \text{Sec}(\Lambda^k E^*) \rightarrow \text{Sec}(\Lambda^k E^*) \) given by \( L^E_X = i_X \circ d^E + d^E \circ i_X \).

On the other hand, for a function \( f \in C^\infty(M) \), one introduces the 'complete' and 'vertical' lifts to \( E \) of \( f \) defined respectively by \( {}^c f(u) = \rho(u)(f) \) and \( {}^v f(u) = f(\pi(u)) \) for all \( u \in E \). Let us consider a section \( X \) of \( E \). The vertical lift of \( X \) is a vector field on \( E \) given by \( {}^v X(u) = {}^v X(\pi(u))_u \), for all \( u \in E \), where a canonical isomorphism is defined by the map \( v_u : E_{\pi(u)} \rightarrow T_u(E_{\pi(u)}) \).

Let us consider the notion of complete lift of a section. The complete lift \( {}^c X \) of a section \( X \) of \( E \) is the vector field on \( E \) which satisfies the following properties:

1. \( {}^c X \) is \( \pi \)-projectable on \( \rho(X) \)
2. \( {}^c X(\hat{\mu}) = \hat{L}^E_X \mu \), for all \( \mu \in \text{Sec}(E^*) \).

Here \( \hat{\mu} \) denotes the linear function on \( E \) defined by \( \hat{\mu}(u) = \mu(\pi(u))(u) \), for all \( u \in E \).

Now, we briefly consider the notion of prolongation of Lie algebroids [for details, see [14]], in order to generate a prolongation over a smooth map; our notations are different, being adapted to those from Lagrange and Finsler geometry [28, 29, 48]. The underlying motivation for prolongations is that of formulating the “second order dynamical models” on \( E \) and relating such constructions to similar models on \( TM \).

### 2.2 The prolongation of a Lie algebroid

We consider a local basis \( \{ v_a \} \) of \( \text{Sec}(E) \). If \( u \in E, \pi(u) = x \in M, \) and \( x^i \) are local coordinates around \( x \), we have \( u = u^a v_a \) and the bundle coordinates on \( E \) are \( (x^i, u^a) \). In this case, the map \( \rho \) acts in the form \( \rho(x^i, u^a) = (x^i, \rho^i_a(x)) \) and \( \rho(v_a) = \rho^a_i(x) \frac{\partial}{\partial x^i} \).
Let $E \stackrel{\pi}{\rightarrow} M$ be a Lie algebroid with Lie bracket $[,]$ and anchor map $\rho : E \rightarrow TM$. Consider the prolongation $\mathcal{L}^n E$ of $E$ as a subset $\mathcal{L}^n E \subset E \times TE$ defined by

$$\mathcal{L}^n E \doteq \{(u, z) \in E \times TE/\rho(u) = (T\pi)(z)\}$$

where $T\pi : TE \rightarrow TM$ is the tangent map to $\pi$. In particular, we have the prolongation of $E$ over the vector bundle projection $\pi : E \rightarrow M$. In the case when $E$ is the standard algebroid on $TM$, then $\mathcal{L}^n E = T(TM)$.

The space $\mathcal{L}^n E$ is fibred over $E$ by the projection $\pi : \mathcal{L}^n E \rightarrow E$, given by $\pi(u, z) = \tau_E(z)$ where $\tau_E : TE \rightarrow E$ is the tangent projection. It is also interesting to define the projection into the second factor: $\rho : \mathcal{L}^n E \rightarrow TE$, given by $\rho(u, z) = z$.

We denote respectively the section $s \in \text{Sec}(E)$ and the sections of the modules of vector fields $v_1 \in \mathcal{X}(E)$, $c_1 \in \mathcal{X}(E)$, and $v_2 \in \mathcal{X}(L^n E)$, $c_2 \in \text{Sec}(L^n E)$, and define respectively the vertical and complete lifts of sections of $E$ into sections of $L^n E$. In particular

$$c_2(u) = (s(\pi(u)), v_2(u)) \text{ and } v_2(u) = (0, v_2(u)). \quad (5)$$

There is an unique Lie algebroid structure $(\pi[, ], \pi \rho)$ on $\mathcal{L}^n E$ which can be defined by

$$\pi[v_2, v_1] = 0, \quad \pi[c_2, v_1] = v_2[s, s], \quad \pi[c_2, c_2] = c_2[s, s].$$

For the vertical and complete lifts of functions we have:

$$\pi \rho(c_2)(c_1 f) = c_2(\rho(s)(f)), \quad \pi \rho(c_2)(v_2 f) = v_2(\rho(s)(f)), \quad \pi \rho(v_2)(c_1 f) = v_2(\rho(s)(f)), \quad \pi \rho(v_2)(v_2 f) = 0.$$

Other two interesting geometric objects on $\mathcal{L}^n E$ are the Liouville section $\pi \Delta$ and the vertical endomorphism $\pi S$. The object $\pi \Delta$ is just the section of $\pi$ defined by $\pi \Delta(u) = (0, v_2 u)$, for all $u \in E$ and the object $\pi S$ is the section of the vector bundle $\mathcal{L}^n E \oplus (\mathcal{L}^n E)^* \rightarrow E$ defined by

$$\pi S(v_2) = 0, \quad \pi S(c_2) = v_2 s,$$

for all sections $s$ of $E$. For $E = TM$, one has $\pi S = s$ and $\pi \Delta = \Delta$. We will say that a section $\text{Sec}$ of $\pi \pi$ is a second order differential equation (SODE) or a semispray on $E$ if $\pi S(\text{Sec}) = \pi \Delta$.

Let us consider local coordinates $x^i$ on $M$ and $(x^i, u^a)$ on $\mathcal{E}$, a local basis $v_0$ of sections of $E$ and the Lie algebroid structure functions $\rho^{\alpha}_\beta(x)$ and
$C^a_{bc}(x)$. We can define a local basis for the considered vertical and complete lifts,

$$
{c^i}_a = \rho^i_a \frac{\partial}{\partial x^i} - C^{b}_{ac} u^c \frac{\partial}{\partial u^b} \quad \text{and} \quad v^i_a = \frac{\partial}{\partial u^a} \tag{6}
$$

which transform any section $s = s^a v_a$ of $E$, respectively, into the vector fields $^v s$ and $^c s$, when

$$
{c^i} s = s^a \rho^i_a \frac{\partial}{\partial x^i} + \left( \rho^i_a \frac{\partial s^b}{\partial x^i} - s^d C^{b}_{da} \right) u^a \frac{\partial}{\partial u^b} \quad \text{and} \quad v^i s = s^a \frac{\partial}{\partial u^a}.
$$

These are local expressions, for a complete definition see Ref. [14].

Following the rule (5) and putting $z^a = ^c e^a$ and $v^a = ^v e^a$, we may transform the local frame (6) into a local basis $(z^a, v^a)$ of $L^\pi E$ when for $s = s^a v_a \in \text{Sec}(E)$ we have

$$
{c^i} s = (\rho^i_a \frac{\partial s^a}{\partial x^i} u^b) v_a + s^a z_a \quad \text{and} \quad v^i s = s^a \frac{\partial}{\partial u^a}.
$$

Hereafter we shall use primed indices $a', b', ...$ running the same values as $a, b, ...$ if one would be necessary to distinguish the objects decomposed with respect to the bases of type $z_{a'}$ from those decomposed with respect to the bases of type $v_a$. It is convenient to introduce a new local basis $\tilde{c}_A = (\tilde{z}_a, \tilde{v}_a)$ on sections of $L^\pi E$

$$
\tilde{c}_A \equiv (\tilde{z}_a = C^b_{ac} u^c v_b + z_a, \tilde{v}_a = v_a) \tag{7}
$$

with the components satisfying the typical Lie algebroid structure relations (1) and (2), respectively,

$$
\pi \rho(\tilde{v}_a) = \frac{\partial}{\partial u^a}, \quad \pi \rho(\tilde{z}_a) = \rho^i_a \frac{\partial}{\partial x^i}
$$

and

$$
\pi [\tilde{v}_a, \tilde{v}_a] = 0, \quad \pi [\tilde{z}_a, \tilde{v}_a] = 0, \quad \pi [\tilde{z}_a, \tilde{z}_a] = C^e_{ab} \tilde{z}_a.
$$

With respect to the $\tilde{c}_A = (\tilde{z}_a, \tilde{v}_a)$, for an element $\omega = \gamma^a \tilde{z}_a + \zeta^a \tilde{v}_a \in L^\pi E$, we can define the natural local coordinates $(x^i, u^a, \gamma^a, \zeta^a)$ on $L^\pi E$, when the point $\omega \in \pi \pi(\pi^{-1}(x))$ [for a vector bundle projection $\pi : L^\pi E \to E$ and $x \in M$, and $(x^i, u^a)$ considered also as the coordinates of the point $\pi (\omega) \in \pi^{-1}(x)$] may be expressed in coordinate form

$$
\omega = \gamma^a \tilde{z}_a (\pi (\omega)) + \zeta^a \tilde{v}_a (\pi (\omega)).
$$
We note that \( \pi(\omega) = \pi(u, z) = \tau_E(z) \) which for
\[
u = \gamma^a e_a \quad \text{and} \quad z = \gamma^a \rho^i \frac{\partial}{\partial x^i} + \zeta^a \frac{\partial}{\partial u^a}
\]
results in the coordinate expression \( \pi(u, z, \gamma^a, \zeta^a) = \tau_E(z) = (x^i, u^a) \). In coordinate form, the anchor map is defined
\[
\pi(x^i, u^a, \gamma^a, \zeta^a) = \tau_E(z) = (x^i, u^a).
\]

In coordinate form, the anchor map is defined
\[
\pi(x^i, u^a, \gamma^a, \zeta^a) = (x^i, u^a, \rho^i \gamma^a, \zeta^a).
\]

We can elaborate a differential form calculus by stating an abstract differential operator \( d^\mathcal{L} \equiv d^{\mathcal{L}E} \) acting in the form
\[
d^\mathcal{L} f = \rho^i \frac{\partial f}{\partial x^i} \bar{z}^a + \frac{\partial f}{\partial u^a} \bar{v}^a, \quad \text{(8)}
\]
\[
d^\mathcal{L} \bar{z}^a = -\frac{1}{2} C^a_{be} \bar{z}^b \wedge \bar{z}^e, \quad d^\mathcal{L} \bar{v}^a = 0,
\]
where the local basis \( \bar{z}^A = (\bar{z}^a, \bar{v}^a) \) is the dual to \( \bar{c}_A = (\bar{z}_a, \bar{v}_a) \).

3 N–anholonomic Lie Algebroids

In this section, we formulate an approach to the theory of Lie algebroids provided with a general N–connection structure. We define and investigate the main properties of the metric and nonlinear and connection structures and compute their torsions and curvatures and related almost Hermitian models of N–anholonomic manifolds.

3.1 Lie Algebroids with N–connection structure

For convenience, we consider the main concepts and formulas for the N–connection geometry both on vector bundles and related Lie algebroids.

3.1.1 Lie N–anholonomic algebroids

We start with the definition of nonlinear connection for vector bundles:

**Definition 3.1** A nonlinear connection (in brief, N–connection) \( \mathcal{N} \) on a vector bundle \( \mathcal{E} \) is defined by using the exact sequence
\[
0 \rightarrow v\mathcal{E} \xrightarrow{i} T\mathcal{E} \rightarrow T\mathcal{E}/v\mathcal{E} \rightarrow 0,
\]
and giving a morphism \( \mathcal{N} : T\mathcal{E} \rightarrow v\mathcal{E} \) such that \( \mathcal{N} \circ i \) is the identity in the vertical subbundle \( v\mathcal{E} \) (the kernel \( \ker \pi^\top \notdiv v\mathcal{E} \), for \( \pi^\top : T\mathcal{E} \rightarrow TM \)) where \( i : v\mathcal{E} \rightarrow T\mathcal{E} \) is the inclusion mapping.
We remit the reader to Refs. \cite{28,29,13,48} for historical remarks and discussions on E. Cartan and A. Kawaguchi first definitions of N–connections, in Finsler geometry, and further generalizations (by Ehresmann, Barthel, Miron, Grifone and others) to various type of spaces.\cite{4}

We can say equivalently that a N–connection is defined by a Whitney sum

\[ T\mathcal{E} = h\mathcal{E} \oplus v\mathcal{E} \]

globally splitting \( T\mathcal{E} \) into conventional horizontal (h–) subspace, \( h\mathcal{E} \), and vertical (v–) subspace, \( v\mathcal{E} \), (subbundles). In general, a decomposition of type (9) defines a nonintegrable (or nonholonomic; in literature, one uses also the equivalent term ’anholonomic’) distribution. Such spaces are called ’N–anholonomic’ because their nonholonomy is related to the N–connection structure: we follow the conventions from \cite{48,11}.

**Definition 3.2**  
A Lie algebroid \( A = (\mathcal{E}, [\cdot, \cdot], \rho) \) is N–anholonomic (in brief, it is a Lie N–algebroid) if the vector bundle \( \mathcal{E} \) is provided with N–connection structure.\cite{5}

A section \( \mathbf{X} \) of \( \mathcal{E} \) has the h– and v–decompositions

\[ \mathbf{X} = \mathbf{X}^a e_a = (X \equiv -X = X^i e_i, \ *X = X^b v_b) \]

and 1-section of \( \mathcal{E}^* \), \( \Phi \in \text{Sec}(\mathcal{E}^*) \) has the h– and v–decomposition

\[ \Phi = (\Phi \equiv -\Phi = \Phi^i e^i, \ *\Phi = \Phi^b v^b), \]

where the rule of summing on repeating indices is used. Following the convention of \cite{28,29,13}, we call respectively such objects to be distinguished vectors and forms (in brief, d–vectors and d–forms) being adapted to the global decomposition induced by the N–connection. In a similar manner, we can introduce d–tensor, d–connection, d–spinor ... objects if such objects are adapted to the N–connection structure. We emphasize that the v–components, those labelled with indices \( a, b, c, ... \) are just those which would be subjected to the algebroid structure conditions given by a couple \( ([\cdot, \cdot], \rho) \). The constructions for N–anholonomic algebroids have to be adapted to the N–connection structure.

\footnote{The term ”non-linear” connection has already used in algebroid theory (for instance, in \cite{9}) for certain connections related to sections of algebroids and associated bundles: such approaches do not follow the general theory of N–connections in bundle spaces and/or on nonholonomic manifolds.}

\footnote{We shall use ’boldfaced’ symbols for algebroids, bundles and manifolds and, in general, for the geometrical objects defined on such spaces if it would be necessary to emphasize that they are provided with (or adapted to) a N–connection structure.}
3.1.2 Geometric structures induced by N–connections

A N–connection may be described by its coefficients,

$$N = N^a_i(u) dx^i \otimes \frac{\partial}{\partial y^a} = N^b_i(u)e^i \otimes v_b,$$

where we underlined the indices defining the coefficients with respect to a local coordinate basis. The well known class of linear connections consists on a particular subclasses with the coefficients being linear on $u\alpha$, i.e. $N^a_i(x, u) = \Gamma^a_{b \alpha}(x)u^b$.

On any Lie algebroid and its associated vector bundle we can consider 'vielbein' transforms, stated by nondegenerated matrices $A_{\alpha \beta}(u)$ and their inverse ones $A^{\alpha \beta}(u)$, from local coordinate frame $e_{\alpha} = \partial_{\alpha} = (e_i = \partial_i, v_b = \partial_a)$ and, respectively, coordinate co–frames, $e^{\alpha} = du^{\alpha} = (dx^i, du^a)$ to any general ones $e_\alpha = (e_i, v_a)$ and, respectively, $e^\alpha = (e^i, v^a)$,

$$e_\alpha = A_{\alpha \beta}(u)e_\beta \text{ and } e^\alpha = A^{\alpha \beta}(u)e^\beta,$$  \hspace{0.5cm} (10)

where the point $u = (x, u) \in E$ has the coordinates $u^\alpha = (x^i, u^a)$. In general, such frame transforms are not adapted to the Lie algebroid and/or N–connection structure. Nevertheless, by straightforward computations we can prove:

**Proposition 3.1** The Lie algebroid and N–connection structures prescribe a subclass of local frames related via a subclass of matrix transforms $A_{\alpha \beta}$ and $A^{\alpha \beta}$ from (10) linearly depending on $N^a_i(x, u)$ and parametrized in the form generating N–adapted frames

$$e_\alpha = (e_i = \partial_i - N^b_i v_b, v_b)$$  \hspace{0.5cm} (11)

and dual coframes

$$e^\alpha = (e^i, v^b = v^b + N^b_i dx^i),$$  \hspace{0.5cm} (12)

for any $v_b = A^b_i \partial^i$ satisfying the condition $v_c | v^b = \delta^b_c$.

The the Lie algebroid structure can be adapted to the N–connection and resulting frame structures (11) and (12). This can be done following

---

\textsuperscript{6}the symbols $\partial$ and $d$ are used correspondingly for usual partial derivatives and differentials
the procedure: Let us re-define the coefficients of the anchor and structure functions with respect to the $e^\alpha$ and $e_\alpha$, when

$$\tilde{\rho}_b(x, u) \rightarrow \tilde{\rho}_b^i(x, u) = A^i_b(x, u) \tilde{\rho}_b(x),$$

$$C^f_{db}(x, u) \rightarrow C^f_{db}(x, u) = A^f_d(x, u) A^b_k(x, u) C^f_{kb}(x),$$

where the transform $A$-matrices are linear on coefficients $N^a_i$ as can be obtained from the formulas in the above presented Proposition. In terms of N–adapted anchor $\tilde{\rho}_b^i(x, u)$ and structure functions $C^f_{db}(x, u)$ (which depend also on variables $u^a$), the structure equations of the Lie algebroids (1), (2) and (3) transform respectively into

$$\tilde{\rho}_i^e = \tilde{\rho}_i^b(x, u) e_i, \quad (13)$$

$$[v_d, v_b] = C^f_{db}(x, u) v_f \quad (14)$$

and

$$\sum_{cyclic(a,b,e)} \left( \tilde{\rho}_b^i e_j(\tilde{\rho}_i^b) - \tilde{\rho}_a^j e_j(\tilde{\rho}_b^a) \right) = \tilde{\rho}_e^j C^e_{ab}, \quad (15)$$

for $Q^{f'b'e'}_{f'bej} = e^{f'}_b e^{e'}_{e} e^{f'}_j (e^b_d e^e_i e^f_l)$ computed for the values $e^{b'}_d$ and $e^{f'}_l$ taken from

$$e^a_\alpha(x, u) = \begin{bmatrix} e^i_\alpha(x, u) & N^b_i(x, u) e^a_b(x, u) \\ 0 & e^a_a(x, u) \end{bmatrix}, \quad (16)$$

$$e^\beta_\beta(x, u) = \begin{bmatrix} e^i_\beta(x, u) & -N^k_i(x, u) e^k_\beta(x, u) \\ 0 & e^a_a(x, u) \end{bmatrix}, \quad (17)$$

where $e^a_\alpha = e^a_\alpha \partial_a$ and $e^\beta_\beta = e^\beta_\beta du^\beta$ depending linearly on $N^b_i(x, u)$. We note that the operators $e_\alpha$ and $e^\alpha$ (respectively defined by (11) and (12)) are "N–elongated" partial derivatives and differentials defining a N–adapted differential calculus on N–anholonomic manifolds. In the limit $N^a_i \rightarrow 0$ obtain the usual Lie algebroid constructions for holonomic manifolds and bundles.

Following J. Grifone [13], we introduce the curvature of a N–connection $\Omega$ as the Nijenhuis tensor

$$N_v(X, Y) = \frac{1}{2} \left[ * X, \ * Y \right] + \left[ * X, Y \right] - \left[ * Y, X \right] - \left[ X, \ * Y \right]$$

15
for any $X, Y \in \mathcal{X}(E)$ associated to the vertical projection "$\star$" defined by this N–connection:

$$\Omega \doteq -N_v$$

written in the Lie algebroid and N–adapted form

$$\Omega = \frac{1}{2} \Omega^b_{ij} e^i \wedge e^j \otimes v_b$$

with coefficients

$$\Omega^a_{ij} = e[jN^a_i] = e_jN^a_i - e_iN^a_j + N^b_i v_b N^a_j - N^b_j v_b N^a_i.$$  

The vielbeins (11) satisfy the nonholonomy (equivalently, anholonomy) relations

$$[e_\alpha, e_\beta] = W^\gamma_{\alpha \beta} e_\gamma$$

with nontrivial anholonomy coefficients $W^a_{jk} = \Omega^a_{jk}(x, u), W^b_{ic} = v_e N^b_i(x, u)$ and $W^b_{ac} = C^b_{ac}(x)$ reflecting the fact that the Lie algebroid is N–anholonomic.

J. Vilms [63] showed that any N–connection in a vector bundle $E$ provides a linear connection in the vertical subbundle $vE$. Such linear connections are called Berwald connections after the name of the geometer who introduced them originally in the Finsler geometry, see details and historical remarks in [28, 29, 2, 48]. The construction also holds for the Lie algebroids:

**Definition 3.3** The Berwald connection with local coefficients

$$\overline{N}^a_{bi} \doteq v_b N^a_i(x, y) \text{ and } \overline{N}^a_{be} \doteq 0$$

is associated to a N–connection $N = \{N^a_i\}$ and defines a covariant derivative $\overline{\nabla}$ on sections in the vertical vector subbundle $vE$.

By using local expressions

$$\overline{\nabla}_{e_i} v_b = v_b(N^a_i) v_c \text{ and } \overline{\nabla}_{e_c} v_b = 0,$$

a d–vector $X = X^i e_i + X^b v_b \in E$ and section $^*B = B^b v_b \in vE$, we prove by direct component computations:

**Proposition 3.2** The Berwald covariant derivative $\overline{\nabla}$ has the local expression

$$\overline{\nabla}_X (^*B) \doteq \overline{\nabla}X \cdot \overline{\nabla} = \left[ X^i \left( \partial_i B^a - (v_b N^a_i) B^b \right) + X^b v_b B^a \right] v_a.$$
By definition, the Berwald connection is different from the notion of $E$–connection in a vector bundle (see, for instance, [9, 8, 14, 5]) which was introduced for sections in holonomic Lie algebroids and vector bundle maps. Roughly speaking, the standard Lie algebroid constructions consider vielbeins, connections, metrics, maps... on sections of vector bundles related to the structure functions $\rho^i_a(x)$ and $C^{ij}_{ab}(x)$ depending on base coordinates $x^i$.

### 3.2 N–connections on prolongations of Lie algebroids

The aim of this subsection is to elaborate a general N–connection formalism on any $\mathcal{L}^\pi E$ provided with a nonintegrable distribution of type (71). Let us consider the projection $\text{pr} : \mathcal{L}^\pi E \rightarrow E$, when $(u, z) \mapsto \text{pr}(u, z) = u$, and introduce the vertical subbundle $v \mathcal{L}^\pi E = \{(u, z) \in \mathcal{L}^\pi E; \pi(u, z) = 0\}$.

In this case, one has the projection $\text{pr} |_{v \mathcal{L}^\pi E} : v \mathcal{L}^\pi E \rightarrow E$ in a vector subbundle of $\text{pr} : \mathcal{L}^\pi E \rightarrow E$ allowing to define an exact sequence of vector bundles:

$$
0 \rightarrow v \mathcal{L}^\pi E \xrightarrow{\circ i} \mathcal{L}^\pi E \xrightarrow{p} \mathcal{L}^\pi E / v \mathcal{L}^\pi E \rightarrow 0 \quad (18)
$$

We denote by $h : \mathcal{L}^\pi E \rightarrow z \mathcal{L}^\pi E$ the complementary projection corresponding to $h : \mathcal{L}^\pi E \rightarrow v \mathcal{L}^\pi E$ when there are satisfied the conditions $h^2 = h, v^2 = v, vh = hv = 0$ and $h + v = \text{Id}$.

**Definition 3.4** A nonlinear connection (N–connection) $\circ N$ on a Lie algebroid $\mathcal{L}^\pi E$ is defined by an exact sequence (18), i.e. by a morphism of subspaces $\circ N : \mathcal{L}^\pi E \rightarrow v \mathcal{L}^\pi E$ such that interior product $\circ \pi$ results in identity in the vertical subbundle $v \mathcal{L}^\pi E$ (the kernel $\ker \circ \pi \cap v \mathcal{L}^\pi E$, for $\circ \pi : \mathcal{L}^\pi E \rightarrow TE$) where $\circ i : v \mathcal{L}^\pi E \rightarrow \mathcal{L}^\pi E$ is an inclusion mapping.

We can say, equivalently, that a N–connection is defined by a Whitney sum of type (71) defining a general nonholonomic structure $\mathcal{L}^\pi E = h \mathcal{L}^\pi E \oplus v \mathcal{L}^\pi E$.

---

7 The label "\(\circ\)" points that the geometric objects are defined just for algebroid configurations and not for some nonholonomic vector bundles or nonholonomic manifolds.

8 It should be emphasized that we can similarly define a N–connection by using the splitting $T \mathcal{L}^\pi E = h T \mathcal{L}^\pi E \oplus v T \mathcal{L}^\pi E$, but this would be a higher order structure if $E = TM$, when $\mathcal{L}^\pi TM = TT M$. 

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**Definition 3.5** A (prolongated) Lie algebroid \( L^\pi E \cong (E, \pi [\cdot, \cdot], \pi \rho) \) is \( N \)-anholonomic (in brief, is a prolongation Lie \( N \)-algebroid) if it is provided with a \( N \)-connection structure \( \circ N \).

A \( N \)-connection \( \circ N \) may be described by its coefficients,

\[
\circ N = N^b_a z^a \otimes \tilde{v}_b,
\]

where \( \tilde{c}_A = (\tilde{z}_b, \tilde{v}_a) \) are defined by (7) playing the role of local coordinate base (nevertheless, being in general nonholonomic because of nontrivial structure functions \( C^c_{bc}(x) \)) on \( L^\pi E \). There are also nonholonomic operators (local bases)

\[
c_A = (z_a = \tilde{z}_a - N^b_a v_b, \ v_a = \tilde{v}_a) \quad (19)
\]

and

\[
c^A = (z^a = \tilde{z}^a, \ v^a = v^a + N^a_{b'} z^{b'}) \quad (20)
\]

which for a general \( \circ N \) define respectively "\( N \)-elongated" partial derivatives and differentials stating a \( N \)-adapted differential calculus on prolongated Lie \( N \)-algebroids.

**Theorem 3.1** Any \( N \)-connection \( \circ N \) on a prolongated Lie algebroid \( L^\pi E \) induces a \( N \)-connection \( N \) on the associated vector bundle \( E \). The inverse statement also holds true.

**Proof.** Let us consider a Lie algebroid \( N \)-connection

\[
\circ N = N^b_a z^a \otimes \tilde{v}_b = \rho^i_a \partial X^i \otimes \partial u^b,
\]

where we have used the formulas (74), (6) and (7). Identifying \( N^i_{b'} = N^a_{b'} \rho^i_a \), we obtain a \( N \)-connection on \( E \). In order to proof the inverse statement, we have consider a \( N \)-connection on the vector bundle and to ‘lift’ it to \( L^\pi E \) by using similar inverse formulas. \( \square \)

The curvature \( \circ \Omega \) of a \( N \)-connection \( \circ N \) is just the Nijenhuis tensor on \( L^\pi E \),

\[
\circ \Omega = -\circ N_v = \frac{1}{2} \Omega^b_{ae'} \tilde{z}^a \wedge \tilde{z}^{e'} \otimes \tilde{v}_b,
\]

with the coefficients

\[
\Omega^a_{be'} = \rho^j_{b'} \frac{\partial N^a_{e'}}{\partial x^j} - \rho^j_a \frac{\partial N^b_{e'}}{\partial x^j} + N^e_{b'} \frac{\partial N^a_{e'}}{\partial u^e} - N^e_{a'} \frac{\partial N^b_{e'}}{\partial u^e}. \quad (21)
\]

computed as those from (78) but for a general (not derived from a Lagrangian) \( N \)-connection. We omitted the label "\( \circ \)" in formulas (79) because
the algebroid character of geometric objects is identified already by "primed" indices of type $a', b'...$ (we shall use this rule for our further considerations in order to omit dubbing of labels).

**Definition 3.6** The Lie algebroid Berwald connection with local coefficients

$$\overline{N}_{be'} = \tilde{v}_b(N^a_{e'}) \text{ and } \overline{N}_{be'} = 0$$

is associated to a N–connection $^oN = \{N^a_{b'}\}$ and defines a covariant derivative $\overline{\nabla}$ on sections in the vertical vector subbundle $vL^pE$.

One holds the

**Proposition 3.3** The Berwald covariant derivative $\overline{\nabla}$ on $L^pE$ has the local expression

$$\overline{\nabla}_X (^*B) \equiv ^oX \cdot \overline{\nabla} = \left[ X^{b'} (\tilde{v}_b B^a - (\tilde{v}_c N^c_{b'}) B^a) + ^*X^c \tilde{v}_{e'} B^a \right] \tilde{v}_a.$$  

**Proof.** It is evident if we component computations with

$$\overline{\nabla}_{\tilde{z}_a'} \tilde{v}_b = \tilde{v}_b (N^c_{a'}) \tilde{v}_c \text{ and } \overline{\nabla}_{\tilde{v}_c} \tilde{v}_b = 0,$$

for a d–vector $^oX = X^A \tilde{c}_A = X^{a'} \tilde{z}_{a'} + X^{b'} \tilde{v}_b \in L^pE$ and section $^*B = B^b \tilde{v}_b \in vE$ mapped into $B^b \tilde{v}_b \in vL^pE$. □

We emphasize that the N–connection formalism is a natural one for investigating physical systems with mixed sets of holonomic–anholonomic variables. The imposed anholonomic constraints (anisotropies) are characterized by the coefficients of N–connection defining a global splitting of the components of geometrical objects with respect to some 'horizontal' (holonomic) and 'vertical' (anisotropic) directions. In brief, we shall use respectively the terms h- and/or v–components, h- and/or v–indices, and h-hand/or v–subspaces which on Lie algebroids are correspondingly substituted into z– and v–components.

A N–connection structure on $L^pE$ defines the algebra of tensorial distinguished (by the N–connection structure) fields $dT (L^pE)$ (d–fields, d–tensors, d–objects, if to follow the terminology from [28, 29, 43]) on $L^pE$ introduced as the tensor algebra $T = \{T^pr_{qs}\}$ of the distinguished tangent bundle $V(d)$, $p_d : hL^pE \oplus vL^pE \to L^pE$. An element $t \in T^pr_{qs}$, a d–tensor field of type $\left( p \atop q \atop r \atop s \right)$, can be written in local form as

$$t = t_{e_1'...e_{p'a'...a'}}^{c_1'...c_{q'b'...b'}} (u) \text{ } Z_{c_1'} \otimes ... \otimes Z_{c_{p'}} \otimes v_{a_1} \otimes ... \otimes v_{a_r} \otimes v^{e_1} \otimes ... \otimes v^{e_q} \otimes Z_{b_1'} \otimes ... \otimes Z_{b_{r'}}.$$  

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3.3 Algebroid d–connection and d–metric structures

The Lie algebroid d–objects are defined in a coordinate free form as geometric objects adapted to the N–connection structure on a prolongated Lie algebroid $\mathcal{L}^\pi \mathcal{E}$. In coordinate form, we can characterize such objects (linear connections, metrics or any tensor field) by certain group and coordinate transforms adapted to the global space splitting (71) into $z$- and $v$–subspaces ($z$–projections on $\mathcal{L}^\pi \mathcal{E}$ play the role of h–projections on $\mathcal{E}$).

3.3.1 d–connections

We analyze the general properties of the class of linear connections which are adapted to the N–connection structure on $\mathcal{L}^\pi \mathcal{E}$.

**Definition 3.7** A d–connection $\mathcal{D}$ on $\mathcal{L}^\pi \mathcal{E}$ is defined as a linear connection conserving under a parallelism the global decomposition (71).

A N–connection induces decompositions of d–tensor indices into sums of horizontal and vertical parts, for example, for every d–vector $\circ X \in \mathcal{V}(d)$ and its dual, i. e. 1-form, $\circ X$ we have respectively

$$\circ X = \circ X + \star X \quad \text{and} \quad \circ X = \circ X + \star X.$$  

For simplicity, we shall not use boldface symbols for d–vectors and d–forms as well we shall omit the Lie algebroid label ”$\circ$” if this will not result in ambiguities. We can associate to every d–covariant derivation $\mathcal{D}_X = \circ X \mathcal{D}$ two new operators of $z$– and $v$–covariant derivations, $\mathcal{D}_X = \circ D_X + \star D_X$, defined respectively

$$\circ D_X \circ Y = \mathcal{D}_X \circ Y \quad \text{and} \quad \star D_X \circ Y = \mathcal{D}_X \star Y,$$

for which the following conditions hold:

$$\mathcal{D}_X \circ Y = \circ D_X \circ Y + \star D_X \circ Y,$$

(22)

$$\circ D_X f = (\circ X)f \quad \text{and} \quad \star D_X f = (\star X)f,$$

for any $\circ X$, $\circ Y \in \mathcal{L}^\pi \mathcal{E}$ and any function $f$ on $\mathcal{E}$.

The N–adapted components $\Gamma^A_{BC}$ of a d–connection $\mathcal{D}_A = \circ c_A \mathcal{D}$ are defined by the equations

$$\mathcal{D}_A \circ c_B = \Gamma^E_{AB} \circ c_E,$$
from which one immediately follows

\[ \Gamma^E_{AB} = (D_A c_B) | c^E. \] (23)

The operations of c- and v-covariant derivations, \( \circ D_c = (L^a_{bc}, L^a_{a'c'}) \) and \( \ast D_c = (K^a_{bc}, K^a_{a'c'}) \) (see (22)) are introduced as the corresponding c- and v-parametrizations of (23),

\[
\begin{align*}
L^a_{b'c'} &= (D_{e'} z_{b'}) | z^{a'}, \\
L^a_{a'c'} &= (D_c z_{b'}) | v^a.
\end{align*}
\] (24)

\[
\begin{align*}
k^a_{b'c'} &= (D_{e'} z_{b'}) | z^{a'}, \\
k^a_{a'c'} &= (D_c z_{b'}) | v^a.
\end{align*}
\] (25)

A set of h-components (24) and v-components (25), distinguished in the form \( \Gamma^E_{AB} = (L^a_{b'c'}, L^a_{a'c'}, K^a_{b'c'}, K^a_{a'c'}) \), completely defines the local action of a d-connection \( \mathcal{D} \) in \( \mathcal{E}^E \). For instance, having taken a Lie algebroid d-tensor field of type \( \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \), \( s = s^{b'c} e_f z_{b'} \otimes v_a \otimes z^{e'} \otimes v^f \), and a d-vector \( \circ X = X^a z_{a'} + X^b v_b \) we can write

\[ \mathcal{D}_X s = \circ D_X s + \ast D_X s = \left( X^d s^{b'c} e_f | d + X^e s^{b'c} e_f | c \right) z_{b'} \otimes v_a \otimes z^{e'} \otimes v^f, \]

where the z-covariant derivative is

\[ s^{b'c} e_f | d = z_d s^{b'c} e_f + L^a_{h'd} s^{b'c} e_f b + L^{b'c} e_f | a = L^a_{e'd} s^{b'c} e_f - L^e_{a'd} s^{b'c} e_f - L^c_{f} s^{b'c} e_f \]

and the v-covariant derivative is

\[ s^{b'c} e_f | c = v_c s^{b'c} e_f + K^a_{h'c} s^{b'c} e_f + K^a_{d'e} s^{b'c} e_f - K^a_{e'd} s^{b'c} e_f - K^a_{f} s^{b'c} e_f. \]

For a scalar function \( f \) we have

\[
\begin{align*}
\circ D_a f &= z_a f - N^b_{a'} v_b f = z_a f + C^b_{a'e} u^e v_b f - N^b_{a'} v_b f, \\
&= \rho^i_a \frac{\partial f}{\partial x^i} - N^b_{a'} \frac{\partial f}{\partial u^b}, \\
\ast D_a f &= v_c f = v_a f = \frac{\partial f}{\partial u^a},
\end{align*}
\]

where the action of the N-elongated operators \( c_A = (z_a, v_a) \) is stated consequently by the formulas (19), (74), (6) and (7). We note that such formulas are written in abstract index form and specify for d-connections the covariant derivation rule.
### 3.3.2 Metric structures and d–metrics

We consider arbitrary metric structures on a Lie algebroid $\mathcal{L}^\pi\mathbf{E}$ and state the possibility to adapt them to N–connection structures.

**Definition 3.8** A metric $\circ g$ on a Lie algebroid $\mathcal{L}^\pi\mathbf{E}$ is defined as a symmetric covariant tensor field of type $(0,2)$, $g_{AB}$, being nondegenerate and of constant signature.

We write a N–connection $\circ N = \{ N^b_a \}$ and a metric structure $\circ g = g_{AB} \tilde{c}^A \otimes \tilde{c}^B$ (26) on $\mathcal{L}^\pi\mathbf{E}$, where we underline the indices considering that the basis $\tilde{c}^A$ being dual to (7). The introduced geometric objects are mutually compatible if there are satisfied the conditions

$$\circ g(\tilde{z}_a, \tilde{v}_b) = 0, \text{ or equivalently, } \circ g_{ab} - N^c_{ba} h_{ac} = 0, \quad (27)$$

where $h_{ab} \equiv \circ g(\tilde{v}_a, \tilde{v}_b)$ and $g_{ba} \equiv \circ g(\tilde{z}_b, \tilde{v}_a)$ resulting in

$$N^b_c = h_{ab} \circ g_{ca} \quad (28)$$

(the matrix $h_{ab}$ is inverse to $h_{ab}$; for simplicity, we have not underlined the indices in the last formula). We obtain a $z$–$v$–decomposition of metric (in brief, d–metric)

$$\circ g(\circ X, \circ Y) = z \circ g(\circ X, \circ Y) + v \circ g(\circ X, \circ Y), \quad (29)$$

where the d-tensor $z \circ g(\circ X, \circ Y) = g(X, *Y)$ is of type $\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ and the d-tensor $v \circ g(\circ X, \circ Y) = h( *X, *Y)$ is of type $\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$. With respect to a N–coframe (75), the d–metric (29) is written

$$\circ g = g_{AB} \tilde{c}^A \otimes \tilde{c}^B = g_{a'b'} \tilde{z}_a' \otimes \tilde{z}_b' + h_{ab} \tilde{v}^a \otimes \tilde{v}^b, \quad (30)$$

---

9Metric structures are considered also in the case of transitive Courant algebroids when the Lie type bracket is changed into a more general product with deformations induced by the metric structure [62]. In our case, we shall also obtain certain deformations of the Lie structure by the N–connections which can be treated as some off–diagonal metric terms.
where \( g_{a'b'} \doteq \circ \mathbf{g}(z_{a'}', z_{b'}') \). The d–metric \( \mathbf{g} \) can be equivalently written in "off–diagonal" form \( \mathbf{g}^{(30)} \) if the basis of dual vectors consists from the coordinate differentials,

\[
\circ \mathbf{g}_{AB} = \begin{bmatrix} g_{a'b'} + N^a_{a'} N^b_{b'} h_{ab} & N^c_{a'} h_{ac} \\ N^c_{a'} h_{be} & h_{ab} \end{bmatrix}. \tag{31}
\]

It is easy to check that one holds the relations

\[
\circ \mathbf{g}_{AB} = e^A_A e^B_B \circ \mathbf{g}_{AB}
\]

or, inversely,

\[
\circ \mathbf{g}_{AB} = e^A_A e^B_B \circ \mathbf{g}_{AB}
\]

for respective vielbein transforms which prove that a N–connection structure can be associated to a prescribed ansatz of vielbein transforms

\[
A^A_A = e^A_A \quad \text{and} \quad A^B_B = e^B_B
\]

in a particular case \( e^a_a = \delta^a_a \) with \( \delta^a_a \) being the Kronecker symbol, defining a global splitting of \( \mathbf{L}^{\pi} \mathbf{E} \) into z– and v–subspaces with the N–vielbein structure

\[
c_A = e^A_A c_A \quad \text{and} \quad e^B = e^B_B c^B.
\]

A metric, for instance, parametrized in the form \( \mathbf{g}^{(31)} \) is generic off–diagonal if it can not be diagonalized by any coordinate transforms. If the anholonomy coefficients \( \mathbf{g}^{(78)} \) vanish for a such parametrization, we can define certain coordinate transforms to diagonalize both the off–diagonal form \( \mathbf{g}^{(31)} \) and the equivalent d–metric \( \mathbf{g}^{(30)} \).

**Definition 3.9** The nonmetricity d–field

\[
Q = Q_{AB} c^A \otimes c^B
\]

on a Lie algebroid \( \mathbf{E}^{\pi} \mathbf{E} \) provided with a N–connection structure is defined by a d–tensor field with the coefficients

\[
Q_{AB} \doteq -\mathcal{D} \circ \mathbf{g}_{AB}
\]

where the covariant derivative \( \mathcal{D} \) is for a d–connection \( \Gamma^E_A = \Gamma^E_{AB} c^B \), see \( \mathbf{g}^{(23)} \) with the respective splitting \( \Gamma^E_{AB} = (L^e_{b'e'}; K^a_{b'e'}; K^a_{b'e}) \), in order to be adapted to the N–connection structure.
A linear connection $D_X$ is compatible with a d–metric $\circ g$ if
\[ D_X \circ g = 0, \quad (35) \]
i. e. if $Q_{AB} \equiv 0$. In a space provided with N–connection structure, the metricity condition \((35)\) may split into a set of compatibility conditions on v– and v– subspaces. We should consider separately which of the conditions
\[ \circ D(\circ g) = 0, \quad \star D(\circ g) = 0, \quad \circ D(\star g) = 0, \quad \star D(\star g) = 0 \quad (36) \]
are satisfied, or not, for a given d–connection $D = (\Gamma^E_{AB})$.

**Definition 3.10** *A prolonged N–anholonomic algebroid $L^nE$ is metric–affine if it is provided with a nontrivial nonmetricity structure $Q = Q_{AB} c^A \otimes c^B$.***

By acting on forms with the covariant derivative $D$, on a metric–affine N–anholonomic algebroid, we can also define another very important geometric objects (the 'gravitational field potentials' on Lie algebroids):
\[ T^A \equiv Dc^A = dc^A + \Gamma^A_B \wedge c^B, \quad (37) \]
and
\[ R^A_B \equiv D\Gamma^A_B = d\Gamma^A_B - \Gamma^E_B \wedge \Gamma^A_E. \quad (38) \]

The Bianchi identities are
\[ DQ_{AB} \equiv R_{AB} + R_{BA}, \quad DT^A \equiv R^A_E \wedge c^E \quad \text{and} \quad DR^A_E \equiv 0, \quad (39) \]
where we stress the fact that $Q_{AB}, T^A$ and $R^A_B$ are called also the strength fields of a metric–affine theory on a N–anholonomic algebroids (we use the terms considered in \[47\]).

### 3.4 Torsions and Curvatures on Lie N–Algebroids

We define and calculate the components of torsion and curvature of a general d–connection $D$ on a Lie N–algebroid $L^nE$.

#### 3.4.1 d–torsions and N–connections

We give a definition being equivalent to \([37]\) but in d–operator form:
The torsion $T$ of a $d$–connection $D = (\circ D, \ast D)$ on $\mathcal{L}^sE$ is defined as an operator ($d$–tensor field) adapted to the $N$–connection structure
\[
T (\circ X, \circ Y) = D \circ X \circ Y - D \circ Y \circ X - [\circ X, \circ Y].
\]
(40)

One holds the following $c$– and $v$–decompositions
\[
T (\circ X, \circ Y) = T (\circ X, \circ Y) + T (\ast X, \circ Y) + T (\circ X, \ast Y) + T (\ast X, \ast Y).
\]
(41)

We consider the projections:
\[
cT (\circ X, \circ Y) \div \circ T (\circ X, \circ Y),
\]
\[
vT (\circ X, \circ Y) \div \ast T (\circ X, \circ Y),
\]
\[
\ast T (\circ X, \circ Y),
\]
and say that, for instance, $\circ T (\circ X, \circ Y)$ is the $z(zz)$–torsion of $D$,
\[
\ast T (\circ X, \circ Y)
\]
is the $v(zz)$–torsion of $D$ and so on.

The torsion $\mathcal{(41)}$ is locally determined by five $d$–tensor fields, $d$–torsions ($N$–adapted $z$–$v$–decompositions with respect to $c_A = (z_a^i, v_a)$ and $c^A = (z^a_i, v^a)$, when it is convenient to use primed abstract indices for the $c$–components of local bases) defined
\[
T_{a'd'}^b = \circ T (z_{b'}, z_{a'}) | z^a_i, \quad T^a_{bc} = \ast (z_{b'}, z_{a'}) | v^a,
\]
\[
T_{a'b}^c = \circ T (v_{b'}, z_{a'}) | z^a_i, \quad T_{a'b}^c = \ast T (v_{b'}, z_{a'}) | v^a,
\]
\[
T_{a'b}^c = \ast T (v_c, v_{b'}) | v^a.
\]

Using the formulas (19), (20), and (21), we can calculate the $c$– and $v$–components of torsion (41) for a $d$–connection, i.e. we prove

**Theorem 3.2** The torsion $T_{ABC} (D) = (T_{a'b'}^a, T_{a'b}^a, T_{a'b'}^a, T_{a'b}^a, T_{a'b'}^a)$ of a given $d$–connection $T_{ABC} (D) = (L_{a'b'}^a, L_{a'b}^a, K_{a'b'}^a, K_{a'b}^a)$ (23) is defined by the corresponding $z$– and $v$–components ($d$–torsions)
\[
T_{a'b'}^a = -T_{a'b'}^a = L_{a'b'}^a - L_{a'b'}^a, \quad T_{a'b}^a = -T_{a'b}^a = K_{a'b'}^a,
\]
\[
T_{a'b'}^a = -T_{a'b'}^a = \Omega_{a'b'}^a, \quad T_{a'b}^a = -T_{a'b}^a = K_{a'b}^a - K_{a'b}^a,
\]
\[
T_{a'b}^a = -T_{a'b}^a = \frac{\partial N_{a'b}}{\partial u^a} - L_{a'b}^a.
\]
(42)

We note that for (pseudo) Riemannian structures on Lie $N$–algebroids the $d$–torsions can be induced by the $N$–connection coefficients and reflect the nonholonomic character of the the corresponding manifold provided with a nonintegrable distribution. Such objects vanishes when we transfer our considerations with respect to holonomic bases for a trivial $N$–connection and zero "vertical" dimension.
3.4.2 d–curvatures and N–connections

In operator form, the curvature \([35]\) is stated from the

**Definition 3.12** The curvature \( \mathcal{R} \) of a \( d \)-connection \( \mathcal{D} = (\circ D, \ast D) \) on \( \mathcal{L}^n \mathcal{E} \) is defined as an operator (\( d \)-tensor field) adapted to the \( N \)-connection structure

\[
\mathcal{R}(\circ X, \circ Y) \circ Z = (D \circ X \circ Y - D \circ Y \circ X - D| \circ X, \circ Y) \circ Z. \tag{43}
\]

One holds certain properties for the \( z \)- and \( v \)-decompositions of curvature, \( \mathcal{R}(\circ) = (\circ R(\ast R)) \), when

\[
\begin{align*}
\ast R(\circ X, \circ Y) \circ Z &= 0, \quad \ast R(\circ X, \circ Y) \circ Z = 0, \\
\mathcal{R}(\circ X, \circ Y) \circ Z &= \circ R(\circ X, \circ Y) \circ Z + \ast R(\circ X, \circ Y) \ast Z,
\end{align*}
\]

where, for instance, \( \circ Z = (\circ Z, \ast Z) \). From \([13]\) and the equation

\[
\mathcal{R}(\circ X, \circ Y) = -\mathcal{R}(\circ Y, \circ X),
\]

we get that the curvature of a \( d \)-connection \( \mathcal{D} \) in \( \mathcal{L}^n \mathcal{E} \) is completely determined by the following six \( d \)-tensor fields (\( d \)-curvatures):

\[
\begin{align*}
R^a_{e'b'c'} &= z^{a'}|\mathcal{R}(z_{e'}, z_{b'}) \circ z_{c'} \circ v_b, \quad R^a_{bb'e'} &= v^a|\mathcal{R}(z_{e'}, z_{b'}) \circ v_b, \quad P^a_{b'e'c'} = v^a|\mathcal{R}(v_{c'}, z_{c'}) \circ v_b, \quad S^a_{b'ec'} = v^a|\mathcal{R}(v_{d'}, v_{c'}) \circ v_b.
\end{align*}
\]

By a direct computation, using \([19], [20], [24], [25] \) and \([44] \), we prove

**Theorem 3.3** The curvature

\[
\mathcal{R}^A_{\text{BEM}} = (P^a_{e'b'c'}, P^a_{b'e'c'}, P^a_{b'ec'}, S^a_{b'ec'}, S^a_{bcd})
\]

of a \( d \)-connection \( \Gamma^A_{BC} \) \([23]\) is defined by the corresponding \( z \)- \( v \)-components

\( d \)-curvatures)

\[
\begin{align*}
R^a_{e'b'c'} &= z_c(L^a_{e'c'}) - z_b(L^a_{e'b'} + L^d_{e'c'} - L^d_{e'b'} - L^d_{e'c'} - K^a_{b'e'} \circ \Omega^a_{b'c'})(45) \\
R^a_{bb'e'} &= z_b(L^a_{bb'} - z_{b'}(L^a_{b'e'}) + L^c_{bb'} K^d_{c'a} - L^d_{b'e'} K^a_{d'a} - L^c_{ab} K^d_{c'e'} + K^c_{b'd} T^d_{e'a} + K^c_{b'k} T^d_{d'a} \\
P^a_{e'b'a} &= v_a(L^a_{e'b'}) - (z_a(K^c_{a'b} + L^a_{d'a} K^d_{ba} - L^a_{bda} K^c_{da} - L^d_{aab} K^c_{bd}) \\
S^a_{b'ec} &= v_e(K^a_{b'e} - v_e(K^a_{b'c}) - v_b(K^a_{b'c}) + v_e(K^a_{b'c}) - K^a_{b'c} K^a_{b'c} + v_c(K^a_{b'c}) - K^a_{b'c} K^a_{b'c} \\
S^a_{bcd} &= v_d(K^a_{bc}) - v_e(K^a_{bd}) + K^a_{bc} K^a_{ed} - K^a_{bd} K^a_{ec}.
\end{align*}
\]

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The components of the Ricci d-tensor

\[ R_{AB} \overset{\dagger}{= \mathcal{R}^{E}_{ABE}} \]

with respect to a locally adapted frame \( \text{(19)} \) has four h–v–components, 
\[ R_{AB} = (R^{'a}b', R^{'a}a, R_{aa'}, S_{ab}), \]
where 
\[ R^{'a}b' = R^{\prime c}a'b', \quad R^{'a}a = -2P^{'b}a = -P^{'b'}a', \]
\[ R_{aa'} = 1P_{aa'} = P_{bb'}, \quad S_{ab} = S_{abc}. \] (46)

We point out that because, in general, \( 1P_{aa'} \neq 2P^{'b}a \) the Ricci d–tensor is non symmetric.\(^{10}\)

Having defined a d–metric of type \( \text{(30)} \) in \( L^{\pi}E \), we can introduce the scalar curvature \( \overline{R} \) of a d–connection \( D \),
\[ \overline{R} = g^{AB}R_{AB} = \circ R + {\ast}S, \] (47)
where \( \circ R = g^{ab}R_{ab} \) and \( {\ast}S = h^{ab}S_{ab} \) and define the distinguished form of the Einstein tensor (the Einstein d–tensor),
\[ G_{AB} \overset{\dagger}{= R_{AB} - \frac{1}{2} g_{AB} \overline{R}}. \] (48)

The Ricci and Bianchi identities \( \text{(39)} \) of d–connections are formulated in terms of z- and v–forms on vector bundle \( \text{[28, 29]} \). In a similar form, by using operators on \( L^{\pi}E \), we can formulate them in component form on Lie N–algebroids (we omit in this work such cumbersome formulas).

### 3.5 Classes of d–connections

In this section, we analyze a set of linear connections and associated co-variant derivations being important for modelling of mechanics and gravitational field interactions on Lie N–algebroids provided with anholonomic frame structure and generic off–diagonal metrics.

#### 3.5.1 The Levi–Civita connection and N–connections

By definition, the Levi–Civita connection \( \circ \nabla = \{ \nabla^{A}_{BE} \} \) on \( L^{\pi}E \), with coefficients
\[ \nabla_{ABE} = \circ g(c_{A}, \nabla_{E}c_{B}) = g_{AF} \nabla_{E}^{F}, \] (49)

\(^{10}\)We note that we consider such h– and v–splitting which are adapted to the N–connection decomposition into subspaces as the Whitney sum: a h–component can not be transformed into a v–term and inversely.
is torsionless,
\[ \nabla T^A = \circ \nabla c^A = dc^A + \nabla \Gamma^A_{BE} \wedge e^B = 0, \]
and metric compatible, \( \circ \nabla (\circ g) = 0 \). The formula (49) states that the operator \( \circ \nabla \) can be considered on spaces provided with N–connection structure but this linear connection is not adapted to the N–connection splitting (71), i. e. it is not a d–connection, see Definition 3.7 (so, we do not use a 'bold-faced' symbol for the Levi–Civita connection). One holds

**Theorem 3.4** If a Lie N–algebroid \( \mathcal{L}^{\circ} \mathbf{E} \) is provided with both N–connection \( \circ N \) and d–metric \( \circ g = \{ g_{AB} \} \) structures, there is a unique linear symmetric and torsionless connection \( \circ \nabla = \{ \nabla_E \} \), being metric compatible such that \( \nabla_E g_{AB} = 0 \) for \( g_{AB} = (g_{a'b'}, h_{ab}) \), see (30), with the coefficients

\[ \nabla \Gamma_{ABE} = \circ g (c_A, \nabla_E c_B) = g_{AD} \nabla \Gamma^D_{BE}, \]

where

\[ \nabla \Gamma_{ABE} = \frac{1}{2} \left[ c_B g_{AE} + c_E g_{BA} - c_A g_{EB} \right. \]
\[ + g_{AK} W^K_{EB} + g_{BK} W^K_{AE} - g_{EK} W^K_{BA} \left. \right] \]

with respect to N–adapted frames \( c_A \) and N–coframes \( c^A \), defined similarly to (24) and (25) but using the operator \( \nabla \),

\[ \nabla \Gamma^E_{AB} = \left( \nabla L^a_{b'} c', \nabla L^a_{bc'}, \nabla K^a_{bc'}, \nabla K^a_{b'c'} \right), \]

defined similarly to (24) and (25) but using the operator \( \nabla \),

\[ \nabla L^a_{b'} = (\nabla c \bar{z}_b) | z^a, \quad \nabla L^a_{bc} = (\nabla d \bar{v}_b) | v^a, \]
\[ \nabla K^a_{b'} = (\nabla c \bar{z}_b) | z^a, \quad \nabla K^a_{bc} = (\nabla c \bar{v}_b) | v^a. \]
In explicit form, the components \( L^a_{b,c}, L^a_{b,d}, K^a_{bc}, \) and \( K^a_{bc} \) are defined by the formula \( \text{(51)} \), the N–adapted frame \( c_A \) and coframe \( c^B \) and a d–metric \( ^g = (g_{a'b'}, h_{ab}) \).

### 3.5.2 The canonical d–connection and the Levi–Civita connection

We search a d–connection which is similar to the Levi–Civita connection satisfying the metricity conditions adapted to the N–connection and possessing some flexibility on existing of nontrivial d–torsion components.

**Proposition 3.4** There are metric d–connections \( D = (\circ D, * D) \) in a Lie N–algebroid \( \mathcal{L}^*E \), see \( \text{(22)} \), satisfying the metricity conditions if and only if

\[
\circ D_a'g_{b'c'} = 0, \quad * D_a'g_{b'c'} = 0, \quad \circ D_a'h_{ab} = 0, \quad * D_a'h_{ab} = 0. \tag{52}
\]

A proof consists from an explicit example:

**Definition 3.13** The canonical d–connection \( \hat{D} = (\circ \hat{D}, * \hat{D}) \), equivalently \( \hat{\Gamma}^E_{AB} = \hat{\Gamma}^E_{AB}c^B \), is defined by the h– v–components

\[
\hat{\Gamma}^E_{AB} = \left( \hat{L}^a_{b,c}, \hat{L}^a_{b,c}, \hat{K}^a_{b,c}, \hat{K}^a_{b,c} \right),
\]

where

\[
\begin{align*}
\hat{L}^a_{b,c} &= \frac{1}{2} g^{a'e'} \left( z_{a'c} g_{b'c'} + z_{b'c} g_{a'c'} - z_{a'c} g_{b'c'} \right), \\
\hat{L}^a_{b,c'} &= v_b \left( N^a_{c'} \right) + \frac{1}{2} h^{ac} \left( z_{c} h_{bc} - v_c \left( N^d_{c} \right) h_{db} \right) - v_b \left( N^d_{c} \right) h_{db}, \\
\hat{K}^a_{b,c} &= \frac{1}{2} g^{a'e'} \left( z_{e'c} g_{b'd'} - z_{e'c} g_{b'd'} \right), \\
\hat{K}^a_{b,c} &= \frac{1}{2} h^{ad} \left( v_c h_{bd} + v_b h_{cd} - v_d h_{bc} \right).
\end{align*}
\]

satisfying the torsionless conditions for the c–subspace and v–subspace, respectively, \( \hat{T}^a_{b,c'} = 0 \) and \( \hat{T}^a_{b,c} = 0 \).

By straightforward calculations with \( \text{(53)} \) we can verify that the conditions \( \text{(52)} \) are satisfied and that the d–torsions are really subjected to the conditions \( \hat{T}^a_{b,c'} = 0 \) and \( \hat{T}^a_{b,c} = 0 \) (see section \( \text{3.4} \)). We emphasize that the canonical d–torsion possess nonvanishing torsion components,

\[
\begin{align*}
\hat{T}^a_{b,c'} &= -\hat{T}^a_{c'b'} + \Omega^a_{b,c'}, \quad \hat{T}^a_{b,c} = -\hat{T}^a_{a'b'} = \hat{K}^a_{b,c}, \\
\hat{T}^a_{b,b'} &= -\hat{T}_{b,b'} = v_b \left( N^a_{b'} \right) - \hat{L}^a_{b,b'}
\end{align*}
\]

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which reflects a nontrivial anholonmic frame structure on Lie N–algebroids.

**Proposition 3.5** The components of the Levi–Civita connection $\Gamma^{E}_{\nabla BA}$ and the components of the canonical d–connection $\hat{\Gamma}^{E}_{BA}$ are related by formulas

$$\Gamma^{E}_{\nabla BA} = \left( \hat{L}_{\hat{b}' c'}, \hat{L}_{\hat{b}' d'}, -v_{b} \left( N^{a}_{b} \right), \hat{K}_{\hat{b}' c' a}, \frac{1}{2} g^{a_{c}' e} \Omega_{b d e}, \hat{K}_{\hat{b} c} \right),$$

(54)

where $\Omega^{a}_{a'b'}$ is the N–connection curvature (69).

The proof follows from an explicit calculus with respect to the N–adapted frame (19) and N–adapted coframe (20) in (50) (equivalently, in (51)) and re–groupation of the components as to distinguish the z- and v–components (53) for $g_{AB} = (g_{a'a'}, h_{ab})$.

**3.5.3 The set of metric d–connections**

Let us define the set of all possible metric d–connections, satisfying the conditions (52) and being constructed only form $g_{a'a'}, h_{ab}$ and $N_{a'}$, and their partial derivatives. Such d–connections satisfy the conditions for d– torsions that $T_{a'b'c'} = 0$ and $T_{a'bc} = 0$ and can be generated by two procedures of deformation of the connection

$$\hat{\Gamma}^{E}_{BA} \rightarrow [K] \Gamma^{E}_{BA} = \Gamma^{E}_{BA} + Z^{E}_{AB},$$

or

$$\hat{\Gamma}^{\gamma}_{\alpha \beta} = \hat{\Gamma}^{\gamma}_{\alpha \beta} + [M] Z^{\gamma}_{\alpha \beta}.$$

**Theorem 3.5** Every deformation d–tensor (equivalently, distortion, or deflection)

$$Z^{E}_{AB} = \left\{ Z^{a}_{b'd'}, \frac{1}{2} g^{a'd'} \circ D_{b'} g_{d'c'}, Z^{a}_{bb'} = \frac{1}{2} h^{ac} \circ D_{b} h_{cb}, \right.$$

$$Z^{a}_{a'd'} = \frac{1}{2} g^{a'd'} \ast D_{a} g_{d'c'}, Z^{a}_{bc} = \frac{1}{2} h^{ad} \ast D_{c} h_{db} \right\}$$

transforms a d–connection $\Gamma^{E}_{AB} = \left( L^{a}_{b' c'}, L^{a'}_{bc}, K_{b' c'}^{a}, K_{bc}^{a} \right)$ (23) into a metric d–connection

$$[K] \Gamma^{E}_{AB} = \left( L^{a}_{b' c'} + Z^{a}_{b'd'}, L^{a'}_{bc} + Z^{a}_{b'c'}, K_{b' c'}^{a} + Z^{a}_{b'd'}, K_{bc}^{a} + Z^{a}_{bc} \right).$$

**Proof.** The proof consists from a straightforward verification that the conditions (52) are satisfied for $[K] \Gamma_{AB} = \left\{ [K] \Gamma^{E}_{AB} \right\}$ and $g_{AB} = (g_{a'a'}, h_{ab})$. We note that this metrization procedure contains additional covariant derivatives of the d–metric coefficients, defined by arbitrary d–connection, not only N–adapted derivatives of the d–metric and N–connection coefficients as in the case of the canonical d–connection.\[\Box\]
Theorem 3.6 For a fixed d–metric structure \( [M] \), \( g_{AB} = (g_{a'b'}, h_{ab}) \), on a Lie N–algebroid \( L^\pi E \), the set of metric d–connections

\[
[M] \Gamma^E_{AB} = \hat{\Gamma}^E_{AB} + [M] Z^E_{AB}
\]

is defined by the deformation d–tensor

\[
[M] Z^E_{AB} = \{ [M] Z^a_{b'c'} = [-] O^l_{c'm} Y^{m'}_{l'y'}, [M] Z^a_{bc} = [-] O^{ea}_{bd} Y^{d}_{ec'}, [M] Z^{a'}_{b'a} = [+] O^{m'c'}_{a'b'} Y^{l'}_{b'c'}, [M] Z^{a}_{bc} = [+] O^{ea}_{bd} Y^{d}_{ec'} \}
\]

where the so–called Obata operators are

\[
[\pm] O^l_{c'm} = \frac{1}{2} \left( \delta^l_{c'} \delta^m_{l'} \pm g_{c'm} g^{d'a'} \right) \quad \text{and} \quad [\pm] O^{ea}_{bd} = \frac{1}{2} (\delta^e_b \delta^d_a \pm h_{ba} h^{ea})
\]

and \( Y^{m'}_{l'y'}, Y^{m'}_{ec'}, Y^{k'}_{m'c'}, Y^{d}_{ec'} \) are arbitrary d–tensor fields.

**Proof.** The proof consists from a direct verification of the fact that the conditions (52) are satisfied on \( L^\pi E \) for \( [M] \mathcal{D} = \{ [M] \Gamma^E_{AB} \} \). We note that the relation (54) between the Levi–Civita and the canonical d–connection is a particular case of \([M] Z^E_{AB}\), when \( Y^{m'}_{l'y'}, Y^{m'}_{ec'} \) and \( Y^{d}_{ec'} \) are zero, but \( Y^{k'}_{m'c'} \) is taken to have \( [+] O^{m'c'}_{a'b'} Y^{l'}_{b'c'} = \frac{1}{2} g^{c'b'} \Omega^{a}_{a'b'} h_{ca} \).

4 N–Connections, Geometric Mechanics and Lie Algebroids

The general idea on geometrization of mechanics on the tangent /cotangent bundle and/or on Lie algebroids is that a regular (for simplicity) Lagrangian, or Hamiltonian, define the fundamental geometric objects of the model. There were elaborated two general approaches: In the first one, the basic geometric constructions are derived from the so–called Poincaré–Cartan 1-form, the Poincaré–Cartan 2-form and the energy function (see [1, 20]) permitting us to geometrize the Euler–Lagrange equations in terms of the (pre-) sympletic geometry. In the second approach, there are emphasized the (semi) spray configuration and associated N–connection, canonical metric and linear connection, almost complex/sympletic ... structures, all adapted to a N–connection (see [30]). In this Section, we state the main results and outline the proofs for Lie algebroid constructions in the second approach to geometrization of mechanics and classical field theory. We refer to [64, 8, 14, 19] and [28, 29, 48] for respective details and proofs on algebroid geometrization of the Euler–Lagrange equations and N–connections on the tangent bundle, i.e. to details on the first approach.
4.1 Lie algebroids, vector bundles, and the Lagrange formalism

Let $\tilde{T}M \cong TM \setminus \{0\}$, $\dim M = n$, where $\{0\}$ means the null section of the tangent bundle $\tau_M : TM \to M$, and $L : TM \to \mathbb{R}$ be a Lagrangian function. Fixing the local coordinates $(x^i, y^j)$, the elements of the Hessian are defined

$$L_{g_{ij}} \doteq \frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j}, \quad (55)$$

when, for simplicity, we consider that the regularity condition is satisfied, i.e. $\text{rank}(L_{g_{ij}}) = n$. We shall also use the matrix $(L_{g^{ij}})$ inverse to $(L_{g_{ij}})$.

Definition 4.1 A Lagrange space is a pair $L^n = (M, L)$ defined by a regular Lagrangian $L(x^i, y^k)$ for which $L_{g_{ik}}$ is of constant signature on $\tilde{T}M$.

We can elaborate a similar construction on a Lie algebroid $A$ with $\pi : E \to M$ and $\pi_* : \mathcal{L}^\pi E \to E$ for a Lagrangian $l : E \to \mathbb{R}$ which defines a Lagrange fundamental function $l(x, u)$ on $\tilde{E} \doteq E/\{0\}_E$, where $\{0\}_E$ is the null section of the vector bundle $\pi : E \to M$, with

$$l_{g_{ab}} \doteq \frac{1}{2} \frac{\partial^2 l}{\partial u^a \partial u^b}, \quad (56)$$

being nondegenerate, i.e. $\text{rank}(l_{g_{ab}}) = m$, where $m$ is the dimension of the typical fiber of $E$, and of constant signature.

Definition 4.2 A Lagrange algebroid is a pair $L^A = (E, l)$ defined by a regular Lagrangian $l(x^i, u^a)$ for which $l_{g_{ab}}$ is of constant signature on $\tilde{E}$.

Let us define the basic geometric objects necessary for a geometrization of the Euler–Lagrange equations in the usual context (see, for instance, [19] [20]):

1. The Poincaré–Cartan 1–form

$$\theta_L \doteq S^*(dL) = \hat{p}_i dx^i,$$

where $S$ is the vertical endomorphism on $TM$, and the generalized momenta is

$$\hat{p}_i = \frac{\partial L}{\partial y^i}.$$
2. The Poincaré–Cartan 2-form $\omega_L \doteq -d\theta_L$,

$$
\omega_L = 2 g_{ij} dy^i \wedge dx^j + \frac{\partial^2 L}{\partial y^i \partial x^j} dx^j \wedge dx^i = dx^i \wedge d\hat{p}_i.
$$

3. The energy function

$$
E_L \doteq C_M L - L = y^i \hat{p}_i - L,
$$

where $\Delta = y^i \partial / \partial y^i$ is the Liouville vector field on $TM$.

A vector field $\xi$ on $TM$ is said to be a second order differential equation (SODE, or a semispray) if $S\xi = \Delta$. This allows us to express

$$
\xi = y^i \frac{\partial}{\partial x^i} + \xi^i(x, y) \frac{\partial}{\partial y^i}.
$$

A curve $\gamma : \mathbb{R} \to M$, parametrized $\gamma(t) = \{x^i(t)\}$, with the canonical extension to $TM$, $\dot{\gamma}(t) = dx/dt = \{x^i(t), y^i(t)\}$, is a solution of the SODE $\xi$ if and only if it is satisfied the equation

$$
\frac{d^2 x^i}{dt^2} = \xi^i(t, x^i, \frac{dx^i}{dt})
$$

for $y^i = dx^i/dt$, that is, if $\dot{\gamma}$ is an integral curve of $\xi$.

The introduced geometric objects can be redefined for a Lie algebroid $\mathcal{L}^\pi E$ and its dual $\mathcal{L}^\pi E^*$ (see [23, 14] for an intrinsic definition).

1. The Poincaré–Cartan 1–section

$$
\theta_l \doteq \pi S^*(dE^l) = l^{\hat{p}_a} e^a \in Sec((\mathcal{L}^\pi E)^*),
$$

where the general momenta is

$$
l^{\hat{p}_a} \doteq \frac{\partial l}{\partial u^a}.
$$

2. The Poincaré–Cartan 2-section $\omega_l \doteq -d\theta_l$,

$$
\omega_l = 2 g_{ab} \pi^a \wedge \pi^b + \left( \frac{1}{2} l^{\hat{p}_e} \pi^e \wedge \pi^j \right) z^a \wedge z^b.
$$

3. The energy function

$$
E_l \doteq \pi \Delta(l) - l = l^{\hat{p}_a} u^a - l.
$$
We shall use additional “algebroid” labels like “◦”, “l”, ... (on the left and right, upper or lower ones, for convenience) for certain algebroid constructions if it would be necessary to distinguish them from some geometric objects on the vector/tangent bundle spaces.

In an intrinsic way, the variational Euler–Lagrange equations \([23]\) can be geometrized in terms of the introduced three geometrical objects, respectively, on \(T M\) or on \(L^\pi E\). One holds:

**Theorem 4.1**  
a) For any regular Lagrangian \(L\), there is a unique SODE, which is called the Euler–Lagrange vector field:

\[
\xi_L = y^i \frac{\partial}{\partial x^i} + \frac{1}{2} L g^{ik}(\frac{\partial L}{\partial x^k} - y^j \frac{\partial^2 L}{\partial x^j \partial y^k}) \frac{\partial}{\partial y^i}
\]
on \(T M\) such that

\[
i_{\xi_L} \omega_L = dE_L
\]
and its solutions are solutions of the Euler–Lagrange equations

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial y^i} \right) - \frac{\partial L}{\partial x^i} = 0
\] (60)
and \(y^i = \dot{x}^i\);

b) For any regular Lagrangian \(l\), there is a unique SODE, which is called the Euler–Lagrange section:

\[
\xi_l = u^a c_a + \frac{1}{2} l g^{ab}(\rho^b \frac{\partial l}{\partial x^i} - \rho^i u^c \frac{\partial^2 l}{\partial x^i \partial u^b} + u^c C^e_{cb} \frac{\partial l}{\partial u^e})v_a
\]
on \(L^\pi E\) such that

\[
i_{\xi_l} \omega_l = d^E E_L
\]
and its solutions are solutions of the Euler–Lagrange equations

\[
\frac{d}{dt} \left( \frac{\partial l}{\partial u^a} \right) - \rho^a \frac{\partial l}{\partial x^i} + C_{ab}^e u^a \frac{\partial l}{\partial u^e} = 0
\] (61)
for

\[
\dot{x}^i = \rho^i u^b.
\] (62)

**Proof.** The proof of the results equivalent to the part b) of the Theorem is considered in \([14, 23]\). It transforms into a usual one for the geometric mechanics if the trivial Lie algebroid structures are considered on \(T M\).
4.2 Geometric structures defined by Lagrangians

A Lagrange space also defines another important geometric objects and structures (see [30] and references therein, for more details).

4.2.1 The Euler–Lagrange equations as ’nonlinear’ geodesic equations

For certain purposes of geometric mechanics, it is enough to consider that the solutions of the Euler–Lagrange equations are defined by a set of nonlinear geodesic equations.

**Theorem 4.2** The Euler–Lagrange equations a) (60) and b) (61) are equivalent to the corresponding ’nonlinear’ geodesic equations

a) on Lagrange spaces,

\[ \frac{dy^i}{dt} + 2 L^i(x^k, y^j) = 0 \]  

where

\[ 2 L^i(x^k, y^j) = \frac{1}{2} g^{ij} \left( \frac{\partial^2 L}{\partial y^i \partial x^k} y^k - \frac{\partial L}{\partial x^i} \right), \]  

and

b) on Lagrange algebroids,

\[ \frac{du^a}{dt} + 2 l^a(x^k, u^b) = 0 \]  

where

\[ 2 l^a(x^k, u^b) = \frac{1}{2} g^{ab} \left( \frac{\partial^2 l}{\partial u^b \partial x^i} \rho^i_c u^c - C^e_{bc} u^c \frac{\partial l}{\partial u^e} - \rho^i_b \frac{\partial l}{\partial x^i} \right) \]  

**Proof.** The proof of part b) of this theorem follows from a straightforward computation

\[ \frac{d}{dt} \left( \frac{\partial l}{\partial u^a} \right) = \frac{\partial^2 l}{\partial u^a \partial x^i} \frac{dx^i}{dt} + \frac{\partial^2 l}{\partial u^a \partial u^b} \frac{du^b}{dt}, \]

\[ = \frac{\partial^2 l}{\partial u^a \partial x^i} \rho^i_b u^b + 2 l^a \frac{du^b}{dt}, \]  

where we have taken into account the formulas (56) and (62). Introducing the (67) into the Euler–Lagrange equations (61) and re-grouping the terms in order to emphasize the value (66), we obtain the formula (65). The proof of the part a) may be considered as a trivial limit for the Lie algebroid structures on TM, which is outlined in explicit form in Refs. [28, 30]. □
4.2.2 Canonical semispray and N–connection

The semisprays of the previous nonlinear geodesic equations are related to a fundamental geometric object, the N–connection defined canonically by the Lagrangian.

**Theorem 4.3** The coefficients a) \( L^G_i(x^k, y^k) \) and b) \( l^G_a(x^k, u^b) \) define respectively:

a) the solutions of both type equations (60) and (63) as paths of the canonical semispray

\[
\xi_L = y^i \frac{\partial}{\partial x^i} - 2 L^G_i \frac{\partial}{\partial y^i}
\]

and the canonical N–connection structure on Lagrange space,

\[
L^N_{ij} \triangleq \frac{\partial L^G_i(x, y)}{\partial y^j},
\]

and

b) the solutions of both type equations (61) and (65) as paths of the canonical semispray

\[
\xi_l = u^a \rho^i_a \frac{\partial}{\partial x^i} - 2 l^G_a \frac{\partial}{\partial u^a}
\]

and the canonical N–connection structure on Lagrange algebroid,

\[
l^N_{ab} \triangleq \frac{\partial l^G_a(x, u)}{\partial u^b}.
\]

**Proof.** The idea to proof the part a) of this theorem is to show that the coefficients \( L^N_j \) define a local distribution of h- and v–subspaces, \( hT_xM \) and \( vT_xM \), for any point \( x \in M \). Unifying the construction on all points, \( \bigcup_x \), we get a global splitting on \( TT_M \), i.e. a Whitney sum,

\[
TT_M = hTM \oplus vTM,
\]

as a nonintegrable distribution (nonholonomic structure) into horizontal (h) and vertical (v) subspaces. This is equivalent to the definition of N–connection, see, for instance, [28, 29] and the discussion in next Section, related to the formula (9) when \( E = TM \). The proof of the part b) is related to a similar proof (via local distributions and their globalization) of existence of a Whitney sum decomposition

\[
\mathcal{L}^E = h \mathcal{L}^h E \oplus v \mathcal{L}^v E
\]
defined just by $\tilde{l}N^a_b(x, u)$ and $l(x, u)$.

For various geometric applications it is enough to show that such $N$–coefficients prescribe a canonical nonholonomic frame structure (on the Lagrange spaces or on the Lagrange algebroid) as we shall do in the next subsection. □

We note that in the presented Proof we consider a nonholonomic (non-integrable) distribution just for $\mathcal{L}^\pi E$ because for $E = TM$ we get $\mathcal{L}^\pi E = TTM$ and (71) transforms into (70). As a matter of principle, for instance, by considering such splitting on sets of sections with the attempt to define the Ehresmann connection like in the usual approach [19, 20], it is more useful to consider splitting of $T\mathcal{L}^\pi E$. In this work, we shall not consider such type of higher order $N$–connections which for $E = TM$ are defined for $TTTM$.

4.2.3 Canonical nonholonomic frames

Any regular Lagrangians $a)$ $L(x, y)$ and $b)$ $l(x, u)$ prescribe respectively classes of local (co)frames defined by the canonical $N$–connection.

**Proposition 4.1** There are preferred local nonholonomic (co) bases (equivalently, vielbeins$^{11}$) induced linearly by the coefficients of the $N$–connection structure:

- **on Lagrange spaces,**

  $$e_\alpha = (e_i, v_i) = (e_i = \frac{\partial}{\partial x^i} - L^j_i \frac{\partial}{\partial y^i}, v_i = \frac{\partial}{\partial y^i}) \quad (72)$$

  and

  $$e^\beta = (e^i, v^i) = (e^i = dx^i, v^i = dy^i + L^i_j dx^j), \quad (73)$$

  and

- **on Lagrange algebroids,**

  $$c_A = (z_{a'} = \tilde{z}_{a'} - \tilde{l}N^b_a \tilde{v}_b, \quad v_a = \tilde{v}_a) \quad (74)$$

  and

  $$c^A = (z^a = \tilde{z}^a, \quad v^a = \tilde{v}^a + \tilde{l}N^a_b \tilde{z}_b) \quad (75)$$

where $1 \leq a, b \leq m$.

$^{11}$this term is largely used in modern physical literature
Proof. The proof follows from the presented formulas for \( e_\alpha, e^\beta \) and \( c_A, c^A \) depending linearly on \( N^j_i \) and, respectively, \( N^a_b \). For instance, the vielbeins (72) satisfy the nonholonomy relations

\[
\left[ e_\alpha, e_\beta \right] = e_\alpha e_\beta - e_\beta e_\alpha = W^\gamma_{\alpha\beta} e_\gamma
\]

with (antisymmetric) nontrivial anholonomy coefficients

\[
1 W^k_{ij} = \frac{\partial N^k_i}{\partial y^j} \quad \text{and} \quad 2 W^k_{ji} = \Omega^k_{ij}
\]

where

\[
\Omega^k_{ij} = \frac{\partial N^k_i}{\partial x^j} - \frac{\partial N^k_j}{\partial x^i} + N^p_i \frac{\partial N^k_j}{\partial y^p} - N^p_j \frac{\partial N^k_i}{\partial y^p}.
\]

Similar nonholonomy relations hold for the algebroid vielbein \( c_A = (z_a, v_a) \),

\[
\left[ c_A, c_B \right] = c_A c_B - c_B c_A = W^D_{AB} c_D
\]

but there are additional nontrivial anholonomy coefficients determined by the nontrivial Lie algebroid structure, \( \omega W^D_{AB} = (1 W^b_{ca}, 2 W^b_{ca}, 3 W^b_{ca}) \), with

\[
1 W^b_{c'a} = \frac{\partial N^b_{c'a}}{\partial u^a}, \quad 2 W^b_{ca} = C^a_{bc}, \quad \text{and} \quad 3 W^b_{ca} = \Omega^b_{ca}
\]

where

\[
\Omega^b_{c'a'} = \rho_{c'}^j \frac{\partial N^b_{a'}}{\partial x^j} - \rho_{a'}^j \frac{\partial N^b_{c'}}{\partial x^j} + N^e_{c'} \frac{\partial N^b_{a'}}{\partial u^e} - N^e_{a'} \frac{\partial N^b_{c'}}{\partial u^e}.
\]

So, we conclude that the nonholonomy coefficients are defined by the N–connection structure. In the trivial case, we obtain holonomic bases. □

We note that we omitted, for simplicity, in this Proposition, the labels "\( L \)", "\( l \)" and "\( \circ \)" for the canonical Lagrange N–connection coefficients and another objects because the abstract indices label already the space for which the geometric objects are considered. Such formulas hold true for arbitrary N–connections (see next Section).

In order to preserve a relation with our previous denotations [43, 48], we note that \( e_\alpha = (e_i, v_i) \) and \( e^\alpha = (e^i, v^i) \) are, respectively, the former \( \delta_\alpha = \delta/\partial u^\alpha = (\delta_i, \partial_i) \) and \( \delta^\alpha = \delta u^\alpha = (dx^i, \delta y^i) \) which emphasize that the operators (72) and (73) define, correspondingly, certain ‘N–elongated’ partial derivatives and differentials which are more convenient for calculations on nonholonomic manifolds. In a similar manner we can argue that the N–connection elongates certain Lie algebroid local frames to generate vielbeins both adapted to the N–connection and algebroid structure.
4.3 Almost Hermitian Mechanics and Lie Algebroids

Any Lagrange space can be “lifted” to an almost Kahlerian structure over $\tilde{TM}$, see Refs. [28, 29]. Similar lifts can be defined for Lagrange algebroids:

4.3.1 Canonical almost complex structures

By explicit constructions, one proves

**Proposition 4.2** The canonical $N$-connections a) $L^{N}_i \, (68)$ and b) $L^{N}_a \, (69)$ naturally induce, respectively,

a) an almost complex structure $F : \mathcal{X}(\tilde{TM}) \to \mathcal{X}(\tilde{TM})$, defined for a tangent bundle $TM$,

$$F(e_i) = v_i \text{ and } F(v_i) = -e_i,$$

and

$$F = -v_i \otimes e^i + e_i \otimes v^i \quad (80)$$

satisfies the condition $F \bullet F = -I$, i.e. $F^\alpha_\beta F^\beta_\gamma = -\delta^\alpha_\gamma$, where $\delta^\alpha_\gamma$ is the Kronecker symbol and $\mathcal{X}(\tilde{TM})$ denotes the module of vector fields on $\tilde{TM}$;

b) an almost complex (algebroid) structure $\circ F : \mathcal{X}(\mathcal{L}^\pi E) \to \mathcal{X}(\mathcal{L}^\pi E)$,

$$\circ F(z_a) = v_a \text{ and } \circ F(v_a) = -z_a,$$

such that

$$\circ F = -v_a \otimes z^a + z_a \otimes v^a \quad (81)$$

satisfies the condition $\circ F \bullet \circ F = -I$, i.e. $F^A_B F^B_D = -\delta^A_D$.

4.3.2 Canonical metric structures

A regular Lagrangian defines also the canonical metric on Lagrange spaces (algebroids) constructed by using Sasaki type lifts from $M$ to $TM$ (to $\mathcal{L}^\pi E$) where the metric tensor is $Lg_{ab} \, (55)$ (or $l_{ab} \, (56)$).

**Theorem 4.4** There are canonical metric structures

a) on $\tilde{TM}$, i.e.

$$Lg = Lg_{\alpha\beta} e^\alpha \otimes e^\beta = Lg_{ij} e^i \otimes e^j + Lg_{ij} v^i \otimes v^j \quad (82)$$

and
b) on $L^zE$,

$$
\mathbf{g} = \mathbf{g}_{AB} \mathbf{e}^A \otimes \mathbf{e}^B = \mathbf{g}_{a'b'} \mathbf{z}^{a'} \otimes \mathbf{z}^{b'} + \mathbf{g}_{ab} \mathbf{v}^a \otimes \mathbf{v}^b \quad (83)
$$

called distinguished metrics (d–metrics) defined by the corresponding Lagrangians.

It is possible to prove that $L\mathbf{g}$ and $l\mathbf{g}$ does not depend on local transformations of coordinates but, respectively, on $L$ and $l$.

In a standard manner, by using the metric (82) (or (83)) it is possible to construct the Levi–Civita connection $L\nabla$ (or $l\nabla$) on a Lagrange space (algebroid) which, by definition, satisfies both the metricity, $L\nabla(L\mathbf{g}) = 0$, (or $l\nabla(l\mathbf{g}) = 0$) and the torsionless conditions. From a formal point of view, this geometrizes the Lagrange mechanics in terms of a (pseudo) Riemannian model on the Lagrange space (or algebroid). But a such approach would consider a linear connection structure which is not adapted to the N–connection (i. e. to the nonholonomic distribution) which is defined canonically by the Lagrangian, see section 4.3.4.

The nonholonomic frames (72) (respectively, (74)) induce naturally a torsion structure via the anholonomy coefficients (76) (respectively, (78)). We can elaborate a covariant N–adapted differential calculus by considering a class of linear connections preserving the global splitting (70) (respectively, (71)), called distinguished (by the N–connection) connections, in brief, d–connections. Such connections may be chosen to satisfy the metricity condition, but contain a nontrivial torsion component which is defined by the Lagrangian and, in general, another geometric/physical terms.

### 4.3.3 Canonical almost sympletic structures

A regular Lagrangian defines a canonical almost sympletic structure via the canonical N–connection, almost complex structure and metric constructed on tangent bundles and/or on Lie algebroids.

**Theorem 4.5** There are almost Kahlerian models of the a) Lagrange spaces and b) Lagrange algebroids defined respectively by

a) triads $K^{2n} = (\tilde{T}M, L\mathbf{g}, \mathcal{F})$ with the induced almost sympletic 2–form

$$
L\omega = L\omega_{\alpha\beta} \mathbf{e}^\alpha \wedge \mathbf{e}^\beta = Lg_{ij} v^i \wedge e^j \quad (84)
$$

and

---

for details, see Refs [28, 29, 43] and the subsection 3.3.1 in this work.
b) triads $^\circ K^{2m} = (\mathcal{L}^2, l^g, ^\circ F)$ with the induced almost sympletic 2–section

$$^l_\omega = l^\omega_{AB} e^A \wedge e^B = l^g_{ab} v^a \wedge z^b.$$ (85)

**Proof.** It is evident if we define, correspondingly,

$$a) L^\omega(e_\alpha, e_\beta) \overset{\text{def}}{=} L^g(F e_\alpha, e_\beta)$$ and

$$b) {^l_\omega(c_A, c_B) \overset{\text{def}}{=} l^g( ^\circ F c_A, c_B)$$

and consider the components of formulas (82), (80) and (83), (81). □

### 4.3.4 Canonical d–connection structures

It should noted that the almost Kahler manifolds $K^{2n}$ and $^\circ K^{2m}$ transform into Kahlerian spaces if the N–connection structure is integrable for the corresponding Lagrange space and Lagrange algebroid.

**Definition 4.3** A linear connection $\tilde{D}$ on $T M$ ($\tilde{\mathcal{D}}$ on $\mathcal{L}^2$) is said to be a distinguished connection (d–connection) if it preserves by parallelism (i. e. by parallel transports defined by the corresponding covariant derivative) the vertical and horizontal distributions (70) on $T M$ (71) on $\mathcal{L}^2$.

We consider a particular class of d–connections:

**Definition 4.4** A normal (or natural) d–connection $D$, or $\mathcal{D}$, is adapted to the almost sympletic structure $F$ for Lagrange spaces, or $^\circ F$ for Lagrange algebroids, when (respectively)

$$D_X F = 0, \quad \mathcal{D} \circ_X ^\circ F = 0,$$

for any vector field $X$ on $T M$, or $^\circ X$ on $TE$.

A normal d–connection $D$ is characterized by its coefficients,

$$\Gamma^\alpha_\beta_\gamma = \left( L^i_{jk}(x, y), \; K^i_{jk}(x, y) \right)$$

on $T M$, where

$$D^e_k e_j \overset{\text{def}}{=} L^i_{jk} e_i, \quad D^e_k v_j \overset{\text{def}}{=} L^i_{jk} v_i, \quad D_v^e k e_j \overset{\text{def}}{=} K^i_{jk} e_i, \quad D_v^e v_j \overset{\text{def}}{=} K^i_{jk} v_i,$$

and (for Lie algebroids)

$$\mathcal{D} = \left( L^e_{ab}(x, u), \; K^e_{ab}(x, u) \right)$$

on $TE$, denoted

$$\mathcal{D}^e_c a c_b \overset{\text{def}}{=} L^e_{ab} c_e, \quad \mathcal{D}^e_c a v_b \overset{\text{def}}{=} L^e_{ab} v_e, \quad \mathcal{D}^e v_a c_b \overset{\text{def}}{=} K^e_{ab} c_e, \quad \mathcal{D}^e v_a v_b \overset{\text{def}}{=} K^e_{ab} v_e.$$
**Definition 4.5** A d–connection a) \( \hat{\mathbf{D}} \), or b) \( \tilde{\mathbf{D}} \), is a) h– and/or v–metric, and b) c– and/or v–metric, respectively, if there are satisfied the conditions:

a) for Lagrange spaces,

\[
\hat{D}_k \ g_{ij} = 0 \quad \text{and/or} \quad *\hat{D}_k \ g_{ij} = 0,
\]

and

b) for Lagrange algebroids,

\[
\tilde{D}_a \ g_{bc} = 0 \quad \text{and/or} \quad *\tilde{D}_a \ g_{bc} = 0;
\]

a such connection is metric (compatible) if it satisfies both h- and v–metricity conditions.

The torsion of a d–connection, for instance of \( \tilde{\mathbf{D}} \), can be defined in component free form

\[
\tilde{T}(\circ X, \circ Y) \equiv \tilde{\mathbf{D}}_{\circ X} \circ Y - \tilde{\mathbf{D}}_{\circ Y} \circ X - [\circ X, \circ Y].
\]

Any d–vector decompose in its z– and v–components, \( \circ Y = zY + vY \), for \( zY = Y^a z^a \) and \( vY = Y^a v^a \). Considering such projections, we can decompose the torsion \( \tilde{T}(c_A, c_B) \) into N–adapted components

\[
z\tilde{T}(z_a, z_b), z\tilde{T}(z_a, v_b), z\tilde{T}(v_a, v_b), v\tilde{T}(z_a, z_b), v\tilde{T}(z_a, v_b), v\tilde{T}(v_a, v_b).
\]

**Theorem 4.6** There are almost Kahlerian models of the a) Lagrange spaces and b) Lagrange algebroids defined by respective unique (canonical) almost Kahlerian d–connections a) \( \hat{\mathbf{D}} \) on \( TM \) and b) \( \tilde{\mathbf{D}} \) on \( \mathcal{L}^n E \) which preserve by parallelism the vertical distributions and satisfy the conditions:

a) one holds the compatibility with the almost Kahlerian structure

\[
\hat{D}_X \ g = \hat{D}_X \omega = 0 \quad \text{and} \quad D_X F = 0,
\]

for any vector field \( X \) on \( TM \), and the property of vanishing of the complete “horizontal” and “vertical” torsions, i. e.

\[
h\tilde{T}(e_i, e_j) = 0 \quad \text{and} \quad v\tilde{T}(v_i, v_j) = 0;
\]
b) one holds the compatibility with the almost Kahlerian structure
\[ \hat{D} \circ_X l_g = \hat{D} \circ_X l_{\omega^o} = 0 \quad \text{and} \quad \hat{D} \circ_X l_F = 0, \]

for any section \( \circ X \) on \( \mathcal{L}^n E \), and the property of vanishing of the complete "horizontal" and "vertical" torsions, i.e.
\[ z\hat{T}(z_a, z_b) = 0 \quad \text{and} \quad v\hat{T}(v_a, v_b) = 0; \]

Proof. Let us state that the almost Kahlerian d–connections
\[ a) \hat{\mathcal{L}}_{jk} = \frac{1}{2} L_{g^h} \left( e_k L_{ghj} + e_j L_{ghk} - e_h L_{gjk} \right), \]
\[ b) \hat{\mathcal{K}}_{jk} = \frac{1}{2} L_{g^h} \left( v_k L_{ghj} + v_j L_{ghk} - v_h L_{gjk} \right), \]

and
\[ a) \hat{\mathcal{L}}_{ab}^e = \frac{1}{2} L_{g^c} \left( z_b L_{gca} + z_a L_{gcb} - z_c L_{gab} \right), \]
\[ b) \hat{\mathcal{K}}_{bc}^a = \frac{1}{2} L_{g^c} \left( v_b L_{gca} + v_a L_{gcb} - v_c L_{gab} \right). \]

By straightforward calculations with covariant derivatives defined by the the coefficients \( \hat{\mathcal{L}} \) and \( \hat{\mathcal{K}} \) we can verify that one holds true all conditions of the theorem. □

The existence of canonical almost complex and almost symplectic structures defined by Lagrangian and/or N–connection is very important for elaborating an approach to geometric quantization of mechanical systems modelled on nonholonomic manifolds [11] as well for a rigorous definition of nonholonomic (anisotropic) Clifford structures and spinors in commutative and noncommutative spaces [43, 48].

5 Finsler and Hamilton Algebroids

The theory of N–connections and adapted metric and linear connection structures on prolonged Lie algebroids, elaborated in the previous Section, gives rise to a number of possibilities to construct geometric and physical models on Lie N–anholonomic algebroids and their duals. The aim of this Section is to consider two such geometries (the Finsler and generalized Lagrange algebroids) realized on Lie algebroids and two dual algebroid geometries (the Hamilton and Cartan algebroids).
5.1 Finsler and generalized Lagrange algebroids

We analyze in brief two possibilities for the algebroid constructions presented in Section 3: 1) to constrain them to a Finsler like configuration and 2) to extend them for generalized Lagrange configurations.

5.1.1 Finsler algebroids

The Finsler geometry was modelled on tangent bundles (the canonical ways with metric compatible, or not, d–connections, see [27, 3, 28, 29, 2]) and, more recently, as embedding of N–anholonomic spaces of even dimensions into Riemann–Cartan and metric–affine manifolds, superbundles and Clifford bundles and on finite projective modules for noncommutative spaces, see references and discussion from [43, 47, 48]. There is a possibility to define a new class of "singular" Finsler geometries on Lie algebroids with associated vector bundles (which are not tangent bundles, or their mimics on manifolds of odd dimensions). Such algebroid Finsler configurations can be modelled on prolongated Lie algebroids.

Definition 5.1

A Finsler algebroid $F_A = (E, f^2)$ is a Lie algebroid $\mathcal{L}^\pi E$ provided with a fundamental Finsler function $f : E \to \mathbb{R}$ satisfying the conditions:

1. $f$ is a scalar differentiable function on the manifold $\tilde{E} = E/\{0\}$ and continuous on the null section of $\pi : E \to M$;

2. $f$ is a positive function, homogeneous on the fibers of $E$, i. e. $f(x, \lambda u) = \lambda f(x, u)\lambda \in \mathbb{R}$;

3. the Hessian of $f^2$ with elements

$$f_{ab} = \frac{1}{2} \frac{\partial^2 f^2}{\partial u^a \partial u^b}$$

is positively defined on $\tilde{E}$.

There are two general ways to model Finsler algebroids on $\mathcal{L}^\pi E$:

The first one is to say that they consist a particular case of Lagrange algebroids (see Definition 4.2) when $l = f^2$ and the Hessian (56) transforms into (88). In this case, we can define an almost Hermitian model of Finsler
algebroids and restrict all results of the Sections 3 and 4 for metric compatible canonical $d$–connections derived from $l = f^2$ and respective canonical $N$–connection $f^{N}_{ab} \rightarrow f^{N}_{ab}$, see (69), when

$$f^{N}_{ab} = \frac{\partial f G^{a}}{\partial u^{b}} = \frac{1}{4} \frac{\partial}{\partial u^{b}} \left\{ f g^{ac} \left( u^{e} \rho^{i}_{e} \rho^{j}_{c} \partial^{2} f^{2}_{i} \partial u^{e} \partial x^{j} - \rho^{i}_{c} \partial f^{2}_{i} \partial x^{j} \right) \right\},$$

the Sasaki type of Finsler $d$–metric, on $L_{\pi E}$, $l_{g} = \left[ \circ g, \ast g \right]$, see (83), i.e.

$$f_{g} = f_{g}^{AB} c^{A} \otimes c^{A} = f_{gab} z^{a} \otimes z^{b} + f_{gab} v^{a} \otimes v^{b},$$

(89)

and canonical $d$–connection computed similarly to (87)

$$\tilde{L}_{ab}^{e} = \frac{1}{2} f g^{ac} \left( z_{b} f_{gca} + z_{a} f_{gc} - z_{c} f_{gab} \right),$$

(90)

$$\tilde{K}_{bc}^{a} = \frac{1}{2} f g^{ac} \left( v_{b} f_{gca} + v_{a} f_{gc} - v_{c} f_{gab} \right),$$

(91)

where the $N$–elongation of the operators $z_{b}$ is defined by $f^{N}_{ab}$.

In the second way of elaborating Finsler geometries on Lie algebroids, one may consider that a Lagrange structure is a singular Finsler structure on higher dimension and to follow the idea that such geometries possess nontrivial nonmetricity $d$–tensors (of Berwald or Chern type$^{13}$). Here, we present some details on Berwald type nonmetricity for Finsler algebroids:

A $d$–connection of Berwald type (see, for instance, Ref. [28, 29, 2] on such configurations in Finsler and Lagrange geometry) and denoted $\left[ B \right] D = \left( \circ [B] D, \ast [B] D \right)$

$$\left[ B \right] \Gamma_{A}^{E} = \left[ B \right] \Gamma_{AB}^{E} c^{B},$$

with $c$– and $v$–components

$$\left[ B \right] \Gamma_{AB}^{E} = \left( \tilde{L}_{bc'}^{a}, v_{b} \left( f^{N}_{a'}, 0, \tilde{K}_{bc}^{a} \right) \right),$$

(91)

with $\tilde{L}_{jk}$ and $\tilde{K}_{bc}^{a}$ taken as in (90), satisfying only partial metric compatibility conditions for a $d$–metric (89), $f_{g} = \left( f_{gab}, f_{gab'} \right)$ on $L_{\pi E}$

$$\circ [B] D_{c'} f_{gab'} = 0 \text{ and } \ast [B] D_{c} f_{gab} = 0.$$

$^{13}$see discussions in [2] and [38]; the models satisfying the metricity conditions admit a more simple geometric and physical interpretation of interactions with spinor and gravity fields, but in another turn the "nonmetricity" physics also presents certain interest.
This is an example of d–connections which may possess nontrivial nonmetricity components, \( [\mathcal{B}] Q_{ABC} = ([\mathcal{B}] Q_{ca'b'}, [\mathcal{B}] Q_{a'ab}) \) with
\[
[\mathcal{B}] Q_{ca'b'} = \hat{\Gamma}^c_{a'b'} D_c f g_{a'b'} \quad \text{and} \quad [\mathcal{B}] Q_{a'ab} = \hat{\Gamma}^a_{a'ab} D_a f g_{ab}.
\] (92)

So, the Berwald d–connection defines a metric–affine algebroid \( \mathcal{L}^\pi \mathcal{E} \) provided with \( N \)–connection structure of Finsler type.

If \( \hat{L}^{jk} \) and \( \hat{K}^{\alpha}_{bc} \) vanish, we obtain a Berwald type connection
\[
\Gamma_{\alpha\beta}^\gamma = \left( 0, v_b \left( f N^{a}_{a'} \right), 0, 0 \right)
\] induced only by the canonical Finsler \( N \)–connection structure. It defines a vertical covariant derivation \( \star \Gamma^c_{a'b'} \) acting in the \( \mathcal{L}^\pi \mathcal{E} \), with the coefficients being partial derivatives on \( v \)–coordinates \( u^a \) of the \( N \)–connection coefficients \( f N^{a}_{b} \).

We can generalize the Berwald connection (91) to contain any prescribed values of d–torsions \( T^{a'}_{b'c'} \) and \( T^{a}_{bc} \) from the c- \( v \)–decomposition (42), but redefined with respect to the canonical Finsler d–connection (90). We can check by a straightforward calculations that the d–connection
\[
[\mathcal{B} \tau] \Gamma^E_{AB} = \left( \hat{L}^{a'}_{b'c'} + \tau^{a'}_{b'c'}, v_b \left( f N^{a}_{a'} \right), 0, \hat{K}^{\alpha}_{bc} + \tau^{\alpha}_{bc} \right)
\] (93)
with
\[
\tau^{a'}_{b'c'} = \frac{1}{2} f g^{a'd'} \left( f g_{bf} T^f_{dc} + f g_{ef} T^f_{db} - f g_{df} T^f_{bc} \right) \quad \text{and} \quad \tau^{\alpha}_{bc} = \frac{1}{2} f g^{ad} \left( f g_{bf} T^f_{dc} + f g_{ef} T^f_{db} - f g_{df} T^f_{bc} \right)
\] (94)
results in \( [\mathcal{B} \tau] T^{a'}_{b'c'} = T^{a'}_{b'c'} \) and \( [\mathcal{B} \tau] T^{a}_{bc} = T^{a}_{bc} \). The d–connection (93) has nonvanishing nonmetricity components, \( [\mathcal{B}] Q^{\alpha}_{\alpha\beta\gamma} = \left( [\mathcal{B}] Q_{ca'b'}, [\mathcal{B}] Q_{a'ab} \right) \).

In general, by using the metrization procedure (see Theorem 3.5) we can also construct metric d–connections with prescribed values of d–torsions \( T^{a'}_{b'c'} \) and \( T^{a}_{bc} \), or to express, for instance, the Levi–Civita connection via the coefficients of an arbitrary metric d–connection.

We can express a general affine Berwald d–connection
\[
[\mathcal{B}] \Gamma^{E}_{A} = [\mathcal{B}] \Gamma^{E}_{AB} c^{B}
\] via its deformations from the Levi–Civita connection \( \Gamma^A \nabla_B \),
\[
[\mathcal{B}] \Gamma^{E}_{A} = \nabla \Gamma^{E}_{A} + [\mathcal{B}] Z^{E}_{A},
\] (95)
\n∇Γ^A_E\ being\ expressed\ as\ (51)\ (equivalently,\ defined\ by\ (50))\ and
\n\ [Br]Z_{AB} = c_B\ [Br]T_A - c_A\ [Br]T_B + \frac{1}{2}\ (c_A) c_B\ [Br]T_E\ c^E\ (96)
\n\ + (c_A) [Br]Q_{BE}\ c^E - (c_B) [Br]Q_{AE}\ c^E + \frac{1}{2}\ [Br]Q_{AB}
\n\ defined\ with\ prescribed\ d–torsions\ [Br]T_{a'b'c'} = T_{a'b'c'}\ and\ [Br]T^a_{bc} = T^a_{bc},
\nwhere,\ for\ simplicity,\ we\ have\ omitted\ the\ label\ "Γ".\ Such\ formulas\ hold\ true\ for\ any\ d–connection\ expressed\ via\ deformations\ of\ a\ metric\ compatible\ d–connection.\ The\ Berwald\ d–connection\ defines\ a\ particular\ subclass\ of\ metric–affine\ connections\ being\ adapted\ to\ the\ N–connection\ structure\ and\ with\ prescribed\ values\ of\ d–torsions.

If\ the\ deformations\ of\ d–metrics\ in\ (95)\ are\ considered\ with\ respect\ to\ the\ canonical\ d–connection\ \[\Gamma^A_{BC}\] with\ z–v–coefficients\ (53),\ we\ can\ construct\ a\ set\ of\ canonical\ metric–affine\ d–connections.\ Such\ d–connections\ Γ^E_A = Γ^E_{AB}c^B\ are\ defined\ via\ deformations

\nΓ^A_B = \hat{Γ}^A_B + \hat{Z}^A_B,\quad (97)
\n\hat{Γ}^A_B\ being\ the\ canonical\ d–connection\ (23)\ and
\n\hat{Z}^A_{AB} = c_B\ T_A - c_A\ T_B + \frac{1}{2}\ (c_A) c_B\ T_A\ c^E\ (98)
\n\ + (c_A) [Br]Q_{BE}\ c^E - (c_B) Q_{AE}\ c^E + \frac{1}{2}\ [Br]Q_{AB}
\nwhere\ T_A\ and\ Q_{AB}\ are\ arbitrary\ torsion\ and\ nonmetricity\ structures.

A\ metric–affine\ d–connection\ Γ^E_A\ can\ also\ be\ considered\ as\ a\ deformation\ from\ the\ Berwald\ connection\ \[Br]Γ^E_{AB}\n\nΓ^A_B = [Br]Γ^A_{AB} + [Br] \hat{Z}^A_B,\quad (99)
\n[Br]Γ^E_{AB}\ being\ the\ Berwald\ d–connection\ (93)\ and
\n\[Br] \hat{Z}^A_{AB} = c_B\ T_A - c_A\ T_B + \frac{1}{2}\ (c_A) c_B\ T_E\ c^E\ (100)
\n\ + (c_A) [Br]Q_{BE}\ c^E - (c_B) Q_{AE}\ c^E + \frac{1}{2}\ [Br]Q_{AB}.

The\ z– and\ v–splitting\ of\ formulas\ can\ be\ computed\ by\ introducing\ the\ adapted\ N–frames\ (74)\ and\ (75)\ and\ d–metric\ \[g = (g_{a'b'}, h_{ab})\] into\ (51),\ (95)\ and\ (96)\ for\ the\ general\ Berwald\ d–connections.\ In\ a\ similar\ form,\ we
can compute splittings of connections by introducing the N–frames and d–metric into (23), (97) and (98) for the metric affine canonic d–connections and, respectively, into (93), (99) and (100) for the metric–affine Berwald d–connections.

Finally, we note that for the respective classes of d–connections, the components of the torsion and curvature tensors may be defined by introducing the corresponding connections (51), (53), (91), (93), (95), (97) and (99) into the general formulas for torsion (37) and curvature (38) on spaces provided with N–connection structure.

5.1.2 Generalized Lagrange algebroids

The d–metric (83) was introduced for the canonical geometric modelling of Lagrange mechanics on Lie N–algebroids. There are physical arguments to consider more general configurations than those for the Lagrange algebroids. For instance, J. L. Synge [40] considered a metric of type

\[ g_{ij}(x, V(x)) = g_{ij}(x) + (1 - u^2(x, V(x)))V_i \]

in order to study the propagation of electromagnetic waves in a medium with the index of refraction \( n^i(x, V(x)) = 1/u^i(x, V(x)), V^2 = V_i V_j g_{ij} \) where \( g_{ij}(x) \) is the background (pseudo) Riemannian metric of the medium and \( V_i(x) \) is the velocity of the medium. There were also considered metrics of type

\[ g_{ij} = e^{\sigma(x, y)} g_{ij}(x, y), \text{ or } g_{ij} = e^{\sigma(x, y)} g_{ij}(x) \]

related to physical processes in dispersive media or in general relativity: a detailed study of the relativistic optic and mechanical and electromagnetic models resulting in generalizations of the Finsler, Lagrange and (pseudo) Riemannian geometries is contained in Chapters XI and XII of the monograph [29] (see [43] on such generalizations suggested from higher energy physics). In order to model such processes on Lie algebroids, we have to introduce into consideration classes of d–metrics with more general coefficients and N–connections than \( {^t g_{ab}} (54) \) and \( {^t N_{b}^a} (68) \) on \( \mathcal{L} \gamma E \), i. e. d–metrics of type

\[ g_{l}^l = g_{AB} c^A \otimes c^A = g_{a'b}(x, u) z^{a'} \otimes z^{b'} + g_{ab}(x, u) v^a \otimes v^b \] (101)

with arbitrary \( g_{ab}(x, u) \) and \( v^a \) elongated by arbitrary \( N_{b}^a(x, u) \), see (75).

**Definition 5.2** A generalized Lagrange algebroid is a pair \( GLA = (E, g_{ab}) \) defined by a nongenerated and constant signature \( g_{ab} \) on \( \tilde{E} \).
On GLA, it is also possible to define a canonical N–connection defined
only by \( g_{ab} \) and to associate a semispray configuration. To do this, we
introduce the absolute energy

\[
\varepsilon(x, u) \doteq g_{ab}(x, u)u^a u^b
\]

and consider the action integral

\[
I = \int_0^1 \varepsilon(x^k(t), \dot{x}^i(t) = \rho^i_b u^b(t)) dt.
\]

For a regular system, when the auxiliary d–tensor

\[
\varepsilon g_{ab} \doteq \frac{1}{2} \frac{\partial^2 \varepsilon}{\partial u^a \partial u^b}
\]

is nondegenerated, we can compute the Euler–Lagrange equations

\[
d\varepsilon \frac{\partial^2 \varepsilon}{\partial u^a \partial x^i} \rho^i_c u^c + C_{bc}^e u^c \frac{\partial \varepsilon}{\partial u^e} - \rho^i_b \frac{\partial \varepsilon}{\partial x^i}
\]

This follows just from the Theorem 4.2 b) if \( l \to \varepsilon \). The canonical N–
connection is

\[
\varepsilon N^a_b \doteq \frac{\partial \varepsilon g^a}{\partial u^b}.
\]

We may prove all results of the Section 3 (to define the canonical d–
connection, the almost complex and almost symplectic structure, ....) for
the generalized Lagrange algebroids, satisfying the regularity condition, by
substituting the absolute energy instead of the Lagrange function. This
results in

**Corollary 5.1** Any generalized Lagrange algebroid defined by a metric ten-
sor \( g_{ab} \) can be modelled equivalently as a Lagrange algebroid provided with
a corresponding absolute energy function \( \varepsilon(x, u) \) if the regularity conditions
are satisfied.
For some explicit purposes, it could be more convenient to work directly with \( g_{ab} \), instead of \( \varepsilon^{ab} g_{ab} \), and with the d–metric (101) which is a particular case of (30) (when \( h_{ab} = g_{ab} \)). In this case, we can introduce the canonical d–connection (53) and compute the respective d–torsions (42) and d–curvatures (45).

5.2 Hamilton–Cartan algebroids

The Hamilton mechanics can be geometrized on cotangent bundles \((T^*M, \pi, M)\) where \( T^*M \) is dual to \( TM \) (see, for instance, a summary of approaches in Refs. [30, 19]). There were elaborated Lie algebroid geometrizations of the Hamilton equations in terms of symplectic and Poisson structures on algebroids (see details and references in [4, 14]). The aim of this section is to outline the main features of the Hamilton mechanics realized in terms of prolongations to Lie algebroids (with associated covector, dual, bundles) provided with N–connection structure, defined by a Lagrangian and/or Hamiltonian. It should be noted here that the Hamilton algebroids have been considered also in Ref. [34] following the formalism from [8].

We shall use the concept a covector bundle \( E^* = (E^*, \pi, M) \) where \( E^* \) is dual to \( E \). When \( E = TM \), we obtain the particular case of (co) tangent bundle. The local coordinates of a point \( \tilde{u} = (x, p) \in E^* \) are denoted \( \tilde{u}^\alpha = (x^i, p_a) \) where the coordinates \( p_a \) are dual to \( u^a \). The local bases and cobases on \( E^* \) are denoted, for instance, \( \tilde{e}_\alpha = (e_i, \tilde{e}_a) \) and \( \tilde{e}^\alpha = (e^i, \tilde{e}_a) \). A dual Lie algebroid (prolongated to a Lie algebroid) is defined as a usual one but associated to a covector bundle \( E^* \). It should be emphasized here that we use the term "coalgebroid" induced from covector/ cotangent bundle but not from "co–algebra".

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14 We preserve all conventions and denotations introduced in the previous sections with respect to Lie algebroids, vector bundles and manifolds, in general, provided with N–connection structure. If would be necessary, we shall use the labels "..." and/or "*" in order to emphasize that some geometric objects refer just to certain dual spaces or, equivalently, co–spaces, and call such objects like co–vectors, but we shall omit the term co–algebroid in order to avoid confusions with the concept of coalageras. It should be noted that under dualization the upper/lower indices will be transformed into the corresponding inverse ones, lower/uper, indices. Following such formal rules, we can re–derive all formulas for the dual spaces directly from the similar formulas for 'non–dual' objects. Nevertheless, in some cases, the relation between the objects of Lagrange and Hamilton geometry in not a complete formal duality: we have to take into account the Legendre transforms which introduce more sophisticate constructions for the Hamilton spaces and geometries.
5.2.1 Prolongations to Lie algebroids on dual vector bundles

Let $\mathcal{L}^*E \cong (E, \pi[,\cdot], \pi\rho)$ be a prolongation Lie N–algebroid derived for a Lie N–algebroid $(E, [\cdot,\cdot], \rho)$ with associated vector bundle $\pi: E \rightarrow M$ when the algebroids and related vector bundles, in general, are provided with mutually compatible N–connection structures. We also consider $^*\pi: E^* \rightarrow M$ to be the vector bundle projection of the dual bundle $E^*$ to $E$. For trivial N–connection structures, we can write respectively $E^*$ and $E$ (we shall also use ”not–boldfaced” symbols if the constructions will not depend on existence of the N–connection structure).

The prolongation $\mathcal{L}^{*\pi} E$ of $E$ over $^*\pi$ is defined by the set of elements satisfying the conditions

$$\mathcal{L}^{*\pi} E = \{ (u, \tilde{z}) \in E \times T^* E / \rho(b) = (T^* \pi)(\tilde{z}) \}.$$ 

We call $\mathcal{L}^{*\pi} E = (E^*, \pi^*[\cdot,\cdot], \pi^*\rho)$ a prolongation N–algebroid over $E^*$, of rank 2m, with fibers isomorphic to $(E, E^*)$ with the Lie algebroid structure $(\pi^*[\cdot,\cdot], \pi^*\rho)$ defined for $^*\pi$ and $E^*$ instead of $\pi$ and $E$.

We note that if $E = TM$ the Lie algebroid $\mathcal{L}^{*\pi} E$ transforms into the standard Lie algebroid $(T(T^*M), [,\cdot], \pi^*\rho = Id)$.

The space $\mathcal{L}^{*\pi} E$ is fibred over $E^*$ by the projection $^*\pi: \mathcal{L}^{*\pi} E \rightarrow E^*$, given by $^*\pi(u, \tilde{z}) = ^*\pi_E(\tilde{z})$ where $^*\pi_E: TE^* \rightarrow E^*$ is the standard Lie algebroid projection. It is also interesting to define the projection into the second factor: $^*\pi^*\rho: \mathcal{L}^{*\pi} E \rightarrow TE$, given by $^*\pi^*\rho(u, \tilde{z}) = \tilde{z}$. Stating a local basis $\{\tilde{v}^a\}$ of $\text{Sec}(E^*)$, for $p \in E^*, \pi(p) = x \in M$, and $x^i$ are local coordinates around $x$, we have $p = p_a \tilde{v}^a$ and the bundle coordinates on $E^*$ are $(x^i, p_a)$.

We denote respectively the section $\tilde{s} \in \text{Sec}(E^*)$ and the sections of the modules of vector fields $^v\tilde{s} \in \mathfrak{X}(E^*)$, $^c\tilde{s} \in \mathfrak{X}(E^*)$, and $^v\tilde{s} \in \mathfrak{X}(\mathcal{L}^{*\pi} E)$, $^c\tilde{s} \in \text{Sec}(\mathcal{L}^{*\pi} E)$, and define the corresponding vertical and complete lifts of sections of $E$ into sections of $\mathcal{L}^{*\pi} E$. One holds the expressions

$$^c\tilde{s}(u) = (\tilde{s}(\pi(p)), ^c\tilde{s}(p)) \quad \text{and} \quad ^v\tilde{s}(p) = (0, ^v\tilde{s}(p)).$$  \hspace{1cm} (103)

There is an unique Lie algebroid projection structure $(^*\pi[,\cdot], ^*\pi\rho)$ on $\mathcal{L}^{*\pi} E$ which defined by

$$^*\pi^*[^v\tilde{s}, ^v\tilde{s}] = 0, \quad ^*\pi^*[^c\tilde{s}, ^v\tilde{s}] = ^v[\tilde{s}, \tilde{s}], \quad ^*\pi^*[^c\tilde{s}, ^c\tilde{s}] = ^c[\tilde{s}, \tilde{s}].$$

For the lifts of functions we write

$$^\pi\rho(\ ^c\tilde{s})(^c f) = ^c(\rho(\tilde{s})(f)), \quad ^\pi\rho(\ ^v\tilde{s})(^v f) = ^v(\rho(\tilde{s})(f)),$$

$$^\pi\rho(\ ^v\tilde{s})(^c f) = ^v(\rho(\tilde{s})(f)), \quad ^\pi\rho(\ ^v\tilde{s})(^v f) = 0.$$
We denote the local coordinates on $M$ and $E^*$ respectively by $x^i$ and $(x^i, p_a)$ and consider the Lie algebroid structure functions $\rho^i_a(x)$ and $C^a_{bc}(x)$. The local bases for the considered vertical and complete lifts are written

\[c_{\hat{e}a} = \rho^i_a \frac{\partial}{\partial x^i} - C^b_{ae} p_b \frac{\partial}{\partial p_e}\]  

and

\[v_{\hat{e}a} = \frac{\partial}{\partial p_a}\]  

transforming any section $s = s_a \hat{v}^a$ of $E^*$, respectively, into the vector fields $v_s$ and $c_s$, when

\[c_s = s^a \rho^i_a \frac{\partial}{\partial x^i} - \left( \rho^i_a \frac{\partial s^b}{\partial x^i} - s^d C^b_{da} \right) p_b \frac{\partial}{\partial p_a}\]  

and

\[v_s = s_a \frac{\partial}{\partial p_a}\]  

These are local expressions, for a complete definition see Ref. [14].

The relations (103) for $z_a' = c_{\hat{e}a}'$ and $\hat{v}^a = v_{\hat{e}a}$, in terms of the basis $\{z_a', \hat{v}^a\}$ of sections of $\mathcal{L}^*\pi : E^* \rightarrow E^*$, we may transform the local frame (104) into

\[c_s = s'^a \rho^i_a \frac{\partial}{\partial x^i} - \left( \rho^i_a \frac{\partial s'^b}{\partial x^i} - s'^d C^b_{da} \right) p_b \frac{\partial}{\partial p_a}\]  

and

\[v_s = s_a \frac{\partial}{\partial p_a}\]  

It is convenient to introduce a new local basis on sections of $\mathcal{L}^*\pi E$ over $E^*$,

\[\dot{c}_A = (z_a' = z_a + C^b_{ae} p_b \hat{v}^e, \hat{v}^a = \hat{v}^a)\]  

with the components satisfying the typical Lie algebroid structure relations (1) and (2). Defining

\[\pi (\dot{z}_a) = \rho^i_a \frac{\partial}{\partial x^i}; \quad \pi (\hat{v}^a) = \frac{\partial}{\partial p_a}\]  

one obtains

\[*^{\pi} [\dot{z}_a', \dot{z}_b'] = C^c_{ab} \dot{z}^c e', \quad *^{\pi} \left[ \dot{z}_a', \hat{v}^b \right] = 0, \quad *^{\pi} \left[ \hat{v}^a, \hat{v}^a \right] = 0.\]  

With respect to the (105) for an element $\omega = \gamma^a \dot{z}_a' + \zeta_a \hat{v}^a \in \mathcal{L}^{*\pi} E$, we can define the natural local coordinates $(x^i, p_a, \gamma^a, \zeta_a)$ on $\mathcal{L}^{*\pi} E$, when the point $\omega \in *^{\pi\pi}(\pi^{*-1}(x))$ for a vector bundle projection $*^{\pi\pi} : \mathcal{L}^{*\pi} E \rightarrow E^*$ and $x \in M$, and $(x^i, p_a)$ considered also as the coordinates of the point $*^{\pi\pi}(\omega) \in *^{\pi}(x)$] may be expressed in coordinate form

\[\omega = \gamma^a \dot{z}_a' ( *^{\pi\pi}(\omega)) + \zeta_a \hat{v}^a ( *^{\pi\pi}(\omega)).\]  

In coordinate form, the anchor map is defined

\[\pi (x^i, p_a, \gamma^a, \zeta_a) = (x^i, p_a, \rho^i_a \gamma^a, \zeta_a).\]
We can elaborate a differential form calculus by stating an abstract differential operator $d^*L \equiv d L^*E$ acting in the form
\[
d^*L f = \rho^a_i \frac{\partial f}{\partial x^i} \check{z}^a + \frac{\partial f}{\partial p^a} \check{v}^a, \\
d^*L \check{z}^a = -\frac{1}{2} C^a_{be} \check{z}^b \wedge \check{z}^e, \\
d^*L \check{v}^a = 0,
\]
where the local basis $\check{c}^A = (\check{z}^a, \check{v}^a)$ is the dual to $\check{c}^A = (\check{z}_a, \check{v}_a)$. Such formulas generalize on $L^*E$ the similar ones \(4\) defined by $f \in C^\infty(M)$ and $\theta = \theta_b \check{v}^a \in \text{Sec}(E^*)$, compare also with the formulas \(8\) for $L^\pi E$.

5.2.2 Hamilton equations and Poisson brackets on Lie algebroids

We introduce the Liouville section (1–form) of $L^*E$,
\[
h \check{\theta} \equiv p_a \check{v}^a
\]
and, following the rules \(106\), we can derive that the 2–form
\[
h \check{\omega} \equiv -d^*L h \check{\theta} = \check{z}^a \wedge \check{v}^a + \frac{1}{2} C^a_{be} \check{z}^b \wedge \check{z}^e
\]
defines a canonical sympletic structure which is nondegenerate and satisfies the condition $d^*L h \check{\omega} = 0$. For the standard Lie algebroid with $E = TM$ the $\check{\theta}_{TE}$ and $\check{\omega}_{TM}$ are respectively the usual Liouville 1-form and the canonical sympletic 2–form on $T^*M$.

Let $\check{h} : E^* \to \mathbb{R}$ be a map defining a Hamiltonian function $\check{h}(x^i, p_a)$ which, for simplicity, satisfies the regularity condition when the Hessian
\[
h_{\check{g}}^{ab} \equiv \frac{1}{2} \frac{\partial^2 \check{h}}{\partial p^a \partial p^b}
\]
is nondegenerated and of constant signature. There is a unique section
\[
\check{\xi}_h = \frac{\partial \check{h}}{\partial p_a} \check{z}_a - \left(C^c_{ab} p^e \frac{\partial \check{h}}{\partial p^b} + \rho^i_a \frac{\partial \check{h}}{\partial x^i} \right) \check{v}^a \in \text{Sec}(L^*E),
\]
inducing a vector filed $\rho^* \check{\xi}_h$ on $E^*$ \[10\]
\[
\rho^* \check{\xi}_h = \rho^a_i \frac{\partial \check{h}}{\partial p_a} \frac{\partial}{\partial x^i} - \left(C^c_{ab} p^e \frac{\partial \check{h}}{\partial p^b} + \rho^i_a \frac{\partial \check{h}}{\partial x^i} \right) \frac{\partial}{\partial p_a},
\]
\[\text{for a standard Lie algebroid, } \check{\xi} \text{ transforms into a usual Hamilton vector field.}\]
satisfying the equation
\[
\tilde{\xi}_h \tilde{\omega}_E = d^*\tilde{h}
\] (111)
for a given \( d\tilde{h} \in \text{Sec}(L^\pi E)^* \). Such formulas sketch the proof of a theorem (the "dual" of Theorem 4.1 for Hamilton structures on Lie algebroids):

**Theorem 5.1** The integral curves of the section \( \tilde{\xi}_h \) (110) (with induced vector field \( \rho^\pi(\tilde{\xi}_h) \)) defining the solution of (111) satisfy the Hamilton equations for \( \tilde{h}(x^i, p_a) \),

\[
\frac{dx^i}{dt} = \rho_a^i \frac{\partial \tilde{h}}{\partial p_a},
\]
\[
\frac{dp_a}{dt} = - \left( C_{ab}^e p_e \frac{\partial \tilde{h}}{\partial p_b} + \rho_a^i \frac{\partial \tilde{h}}{\partial x^i} \right).
\] (112)

The dual bundle \( E^* \) admits a linear Poisson structure, a 2–vector field, \( \Lambda_{E^*} \) such that

\[
[\Lambda_{E^*}, \Lambda_{E^*}] = 0
\]
and \( \Lambda_{E^*}(df, df') \) is a linear function for any linear functions \( f, f' \) on \( E^* \). The local coordinate expression is

\[
\Lambda_{E^*} = \rho_a^i \frac{\partial}{\partial p_a} \wedge \frac{\partial}{\partial x^i} + \frac{1}{2} p_e C_{ab}^e \frac{\partial}{\partial p_a} \wedge \frac{\partial}{\partial p_b}.
\] (113)

This structure induces a linear Poisson bracket

\[
\{f, w\}_{E^*} \doteq \Lambda_{E^*}(d^{TE^*}f, d^{TE^*}w)
\]
where the operator \( d^{TE^*} \) is defined by the rules (106) but on \( TE^* \). In a particular case of local coordinates, we have

\[
\{x^i, x^j\}_{E^*} = 0, \quad \{p_a, x^j\}_{E^*} = \rho_a^i \quad \text{and} \quad \{p_a, p_b\}_{E^*} = p_e C_{ab}^e,
\]
see details in [23, 14].

For Hamiltonian sections, we can naturally use \( h \tilde{\omega} \) (108) satisfying (111), instead of \( \Lambda_{E^*} \), in order to define

\[
\{f, w\}_{E^*} \doteq - h \tilde{\omega}(\tilde{\xi}_f, \tilde{\xi}_w).
\]

The formula for energy \( E_l \) on a Lagrange algebroid [39] can be rewritten in the form

\[
\tilde{h} = p_a u^a - \frac{l}{2}
\]
which relates the integral curves of the Euler–Lagrange equations to the integral curves of the Hamilton equations if the regularity conditions are satisfied. This relation stated in a more rigorous geometric form by the Legendre transform associated to $l$ which is a smooth map

$$\text{Leg}_l: E \to E^*; \quad \text{Leg}_l(a)(b) = l(\theta(a))(z)$$

defined with respect to the Poincare–Cartan 1–form $^1\theta \in \text{Sec}((L^\pi E)^*)$ [57], for any $a, b \in E_x$, $x \in M$ and $z \in L^\pi E_{|a}$ such that $pr_1(z) = b$ when $pr_1: L^\pi E \to E$ is the restriction to $L^\pi E$ of the first canonical projection $pr_1: E \times TE \to E$. The transform $\text{Leg}_l$ induces a map for the prolongated algebroids,

$$\mathcal{L}\text{Leg}_l: \mathcal{L}^\pi E \to \mathcal{L}^{*\pi} E; \quad (\mathcal{L}\text{Leg}_l)(b, X_a) = (b, (T_a \text{Leg}_l)(X_a)),$$

where $a, b \in E$ and $(b, X_a) \in (\mathcal{L}^\pi E)_a \subseteq E_{\pi(a)} \times T_a E$ and $T\text{Leg}_l: TE \to TE^*$ being the tangent map of $\text{Leg}_l$. The map $\mathcal{L}\text{Leg}_l$ is well defined because $\pi^* \circ \text{Leg}_l = \pi$.

In local coordinates, the introduced Legendre maps are parametrized respectively

$$\text{Leg}_l(x^i, u^a) \mapsto (x^i, p_a = \frac{\partial l}{\partial u^a})$$

and

$$\mathcal{L}\text{Leg}_l(x^i, u^a; z^a, v^a) \mapsto (x^i, p_a; z^a, \rho_b \zeta^b \frac{\partial p_a}{\partial x^i} + v^b l_{ab} \) .$$

Using such coordinate maps, we can prove by straightforward computations that under Legendre transforms the Poincare forms $^1\theta$ [57] and $^1\omega$ [58] transform respectively into $h^\theta$ [107] and $h^\omega$ [108]. This deduces that the pair $(\text{Leg}_l, \mathcal{L}\text{Leg}_l)$ defines a morphism between the prolongated Lie algebroids $\mathcal{L}^\pi E$ and $\mathcal{L}^{*\pi} E$ with compatible Lie algebroid structure functions.

5.2.3 Hamilton algebroids and Cartan algebroids

The Hamilton algebroids can be introduced as geometric mechanics structures on $\mathcal{L}^{*\pi} E$ and $\mathcal{L}^\pi E$ related by Legendre transforms:

**Definition 5.3** A Hamilton algebroid is a pair $\mathcal{H}\mathcal{A} = (E^*, \ h)$ defined by a regular Hamiltonian $h(x^i, p_a)$ being differentiable on $E^* \equiv E^*/\{0^*\}$ and continuous on the null sections of $0^*: E^* \to M$, with a nondegenerated and constant signature Hessian $h^{\gamma\delta}_{\gamma\delta}$ [109] on $E^*$.
There are possibilities to restrict and/or to generalize this definition:

Definition 5.4  A Cartan algebroid $C.A = (E^*, \, \, g)$ is a Hamilton algebroid with positive $g = \tilde{g}(x^i, p_a)$ on $E^*$ and 1–homogeneous with respect to the momenta $p_a$, i. e. $\tilde{g}(x^i, \lambda p_a) = \lambda \tilde{g}(x^i, p_a)$.

Roughly speaking, the Cartan algebroids are Finsler algebroids but defined on $L^\pi E$ and $E^*$ (with some additional geometric structures related to the Legendre transforms).

In a more general case, we can consider prolongated Lie algebroid structures defined by an arbitrary nondegeneratd d–tensor field $\tilde{g}^{ab}$, co–metric on $\tilde{E}^*$, not obligatory defined as the second derivative on the momenta from a Hamiltonian. Nevertheless, the geometry of such generalized Hamilton algebroids $GHA = (E^*, \, \, \tilde{g}^{ab})$ can be modelled similarly to that of the usual Hamilton algebroids by introducing an additional dual global ”energy” function

$$\tilde{\varepsilon}(x, u) \doteq \tilde{g}^{ab}(x, p)p_ap_b$$

like for the generalized Lagrange algebroids.

For simplicity, in this work we shall consider the main geometric constructions only for the Hamilton algebroids. We shall sketch the idea of such proofs.

Canonical N–connections on Hamilton algebroids

The results of Theorems 4.2 and 4.3 reformulated for the Hamilton algebroids are stated by

Theorem 5.2  The set of coefficients

$$h_{N_{ab}} = \frac{1}{4} \left\{ \, h_{g_{ab}}, \tilde{h} \right\}_{E^*} - \frac{1}{4} \left( g_{ac}\rho_b^l \frac{\partial^2 \tilde{h}}{\partial p_c \partial x^i} + g_{bc}\rho_a^l \frac{\partial^2 \tilde{h}}{\partial p_c \partial x^i} \right)$$

(114)

defines a canonical N–connection structure $h\tilde{N} = (h_{N_{ab}})$ on $L^\pi E$ constructed only from $h$ and $h\tilde{g}^{ab}$ and its dual, $h_{g_{ab}}$, in a form related to the canonical N–connection $l\tilde{N}$ (62) defined by the corresponding Lagrangian $l(x^i, u^a)$ on $L^\pi E$.

Proof. Let us introduce the locally adapted base for such Hamilton algebroids,

$$\tilde{\xi}_A = (z_a' = \tilde{z}_a' - N_a'\alpha', \, \, \tilde{\rho}^a = \tilde{\alpha}^a = \tilde{\alpha}^a)$$

(115)
and the duals
\[ \tilde{v}^A = (z^{a'}, \tilde{z}^{a'}, \tilde{v}_a = \tilde{v}_a + N_{b'a} z^{b'} ) \] (116)

where the algebroid indices \( A = (a', a), B = (b', b), \ldots \) running the values \( a', b', a, \ldots b = 1, 2, \ldots, m \). The construction of such N–elongated operators is similar to that from the Proposition 4.1, but in our case we use geometrical objects defined on prolongation coalgebroids (for instance, labelled in the form \( \tilde{v} \) in order to emphasize the difference from the similar ones on usual algebroids; for simplicity, we shall omit a such label, and the left label ”\( h \)”, for \( N_{b'a} \) when denotations will not give rise to ambiguities). By local computations we can verify that \( h N_{a'b} \) define a local distribution which can be globalized to the Whitney sum
\[ \mathcal{L}^{*\pi}E = h \mathcal{L}^{*\pi}E \oplus v \mathcal{L}^{*\pi}E \]

which is an equivalent definition of the N–connection on a Lie algebroid, see (119). We may conclude that the coefficients (114) define a N–connection on \( \mathcal{L}^{*\pi}E \). It is a more cumbersome task to prove that \( h N_{a'b} = \rho_i \partial G_b / \partial x_i \) defines a nonlinear geodesic semispray configuration \( G_b \) which is equivalent both to the Euler–Lagrange equations (61) and the equivalent (for regular Legendre transforms) Hamilton equations (112). Such computations are equivalent to those outlined for the proof of Theorems 4.2 and 4.3 but re–derived in algebroid terms on \( \mathcal{L}^{*\pi}E \).

**Definition 5.5** A N–connection \( \tilde{N}_{a'b} \) on \( \mathcal{L}^{*\pi}E \) is symmetric if its torsion d–tensor
\[ \tau_{ab} = \frac{1}{2} ( \tilde{N}_{ab} - \tilde{N}_{ba} ) = 0. \] (117)

From (114), one follows that the canonical N–connection is symmetric, i. e.
\[ h \tau_{ab} = 0. \]

**Definition 5.6** The curvature d–tensor \( \tilde{\Omega} \) of a N–connection \( \tilde{N}_{ab} \) on \( \mathcal{L}^{*\pi}E \) is defined by the components
\[ \tilde{\Omega}_{ab'e'} = z_{e'} N_{b'a} - z_{b'} N_{e'a}. \]

A N–connection distribution on \( \mathcal{L}^{*\pi}E \) is integrable if and only if \( \tilde{\Omega} = 0 \). There is also a d–connection (Berwald type) defined by the N–connection coefficients:
**Definition 5.7** The Lie algebroid Berwald $d$–connection with local coefficients

\[
\overline{N}_{a' e}^b \equiv \check{v}^b (N_{a' e}) \quad \text{and} \quad \overline{N}_{be}^a \equiv 0
\]

is associated to a $N$–connection $N_{ab}$ and defines a covariant derivative $\overline{D}$ on sections in the vertical vector subbundle $v \mathcal{L}^{*\pi}E$.

One holds (the proof is similar to that for the prolongation Lie algebroids, see formula (21) and Proposition 3.3) the

**Proposition 5.1** The Berwald covariant derivative $\overline{D}$ on $\mathcal{L}^{*\pi}E$ has the local expression

\[
\overline{D}_X (\ast B) = \check{X} \cdot \overline{D} = \left[ \check{X}^b (z_b B^a - \check{v}^c (\check{N}_{cb}^a)) B^a \right] + \ast \check{X}^e \check{v}_e B^a.
\]

**Canonical almost complex structures on coalgebroids**

The $N$–connection splitting on prolongation coalgebroids defines a corresponding class of almost complex structures, which for the canonical configurations are defined by the Hamiltonians.

**Proposition 5.2** A canonical $N$–connection $^hN_{a'b} (114)$ induces, naturally an almost complex (coalgebroid) structure $\check{F} : \mathcal{X}(\mathcal{L}^{*\pi}E) \to \mathcal{X}(\mathcal{L}^{*\pi}E)$,

\[
\check{F}(z_a) = \check{v}_a \quad \text{and} \quad \check{F}(\check{v}_a) = -z_a,
\]

when

\[
\check{F} = ^h g_{ab} \check{v}^a \otimes z^b - ^h g^{ab} z_a \otimes \check{v}_a
\]

satisfies the condition $\check{F} | \check{F} = -I$, i.e. $\check{F}^A_B \check{F}^B_K = -\delta^A_K$, where $\delta^A_K$ is the Kronecker symbol and $\mathcal{X}$ denotes the module of vector fields on $\check{E}^*$. \[\square\]

**Canonical metric and sympletic structures**

A regular Hamiltonian induces a canonical metrics on the corresponding Hamilton algebroid:

**Theorem 5.3** There is a canonical metric structures on $\mathcal{L}^{*\pi}E$,

\[
^h g = ^h g_{AB} \check{c}^A \otimes \check{c}^B = \eta_{ab} \check{z}^a \otimes \check{z}^b + ^h g_{ab} \check{v}_b \otimes \check{v}_b
\]

called distinguished metrics (d–metrics) defined by the corresponding Lagrangians and, induced by such Lagrangians, canonical $N$–connections.
Proof. The existence follows from a Sasaki type lift of $h\hat{g}^{ab}$ to $L^*\pi E$ by emphasizing the $(E, E^*)$ structure of fibers. □

In modern gravity, one considers models with cofiber metrics resulting in more general d–metrics then (119),

$$\hat{g} = \hat{g}_{AB} z^A \otimes z^B + \hat{h}^{ab} \hat{v}_a \otimes \hat{v}_b$$

where $\hat{g}_{ab}$ and $\hat{h}^{ab}$ are respectively some independent z- and v– components.

Definition 5.8 A Lie algebroid (coalgebroid) with associated vector bundle $\mathcal{E} = (E, \pi, M)$ (covector bundle $\mathcal{E} = (E^*, \pi^*, M)$) is said to be sympletic if it admits a sympletic structure $\omega$ on the sections of the bundle $\wedge^2 E^* \to M$ such that 1) the map $\omega(x) : E_x \times E_x \to \mathbb{R}$ is nondegenerate and $d^E \omega = 0$.  

A regular Hamiltonian defines a canonical almost sympletic structure derived from the canonical N–connection (114), almost complex structure (118) and the canonical Poisson structure (113) which in terms of the N–elongated partial derivatives (115) defines the N–adapted Poisson bracket

$$\{f_1, f_2\}_{E^*} = \hat{v}^a (f_1) \hat{z}_a (f_2) - \hat{v}^a (f_2) \hat{z}_a (f_1)$$

for any $f_1$ and $f_2$ on $L^*\pi E$.

Theorem 5.4 A Hamilton algebroid is a sympletic algebroid with the canonical sympletic structure defined by the 2–form

$$h\omega = h\omega_{AB} \hat{c}^A \wedge \hat{c}^B \div \hat{v}_a \otimes \hat{z}^b$$

and canonical Poisson structure $\{f_1, f_2\}_{E^*}$ on $L^*\pi E$.

Proof. We compute, using formulas (115) and (114),

$$h\omega = \hat{v}_a \otimes \hat{z}^b = \hat{v}_a \otimes \hat{z}^b + h\tau_{ab} \hat{z}^a \wedge \hat{z}^b$$

and

$$\{f_1, f_2\}_{E^*} = \hat{v}^a (f_1) \hat{z}_a (f_2) - \hat{v}^a (f_2) \hat{z}_a (f_1) - 2 h\tau_{ab} \hat{v}^a (f_1) \hat{z}_a (f_2)$$

where $h\tau_{ab} = 0$.  

In a more general context, the Hamilton algebroids may be described in terms of the Lichnerowicz–Poisson and/or H–Chevalley–Eilenberg cohomologies [15, 16] where the Lie algebroids of Jacobi manifolds were considered. For simplicity, in this work we do not concern topological properties of the Lie N–algebroids.
The next step is to show that $d \, h_\omega = 0$ which implies $d \tilde{v}_a \wedge \tilde{z}^a = 0$ for the N–connection $h_{ab}$. We have that the exterior differential
\[
d \tilde{v}_a = -d \left( h_{ab} \right) \wedge \tilde{z}^b,
\]
where $d \left( h_{ab} \right)$ is symmetric on indices $a$ and $b$. So, the antisymmetric product with $\tilde{z}^a$ vanishes, which proofs $d \tilde{v}_a \wedge \tilde{z}^a = 0$ and $d \, h_\omega = 0$. We can conclude that $h_\omega$ is a closed 2–form defining a sympletic structure. □

### 5.2.4 Canonical anchors for the Lagrange–Hamilton algebroids

It is possible to model the Hamilton algebroids as almost Kahlerian structures:

**Theorem 5.5** There is canonical almost Kahlerian model of the Hamilton algebroids defined by the respective unique almost Kahlerian d–connection $^*\hat{D}$ on $\mathcal{L}^{*\pi}E$ which preserves by parallelism the vertical distribution and satisfies the conditions:

1. there is compatibility with the almost Kahlerian structure
   \[ ^*\hat{D} \big|_{\mathcal{H}_X} h_\omega = 0, \quad ^*\hat{D} \big|_{\mathcal{H}_X} h_\omega = 0 \text{ and } \hat{D} \big|_{\mathcal{H}_X} \hat{F} = 0; \]

2. the complete "horizontal", i.e. z-component, and "vertical" torsions vanish, i.e.
   \[ z \big|_{\mathcal{V}_T} \hat{T}(z_a, z_b) = 0 \quad \text{and} \quad v \big|_{\mathcal{V}_T} \hat{T}(\hat{v}^a, \hat{v}^a) = 0; \]
   for any vector field $\hat{X}$ on $T E^*$.

**Proof.** The proof is similar to that of Theorem 4.6 but re–written for $^*\hat{D}$ on $\mathcal{L}^{*\pi}E$ in terms of dual objects indices. The almost Kahlerian d–connection $h^\hat{T}_{\beta\gamma} = \left( h^\hat{L}_{ab}, h^\hat{K}_{a}^{bc} \right)$, equivalent to $^*\hat{D}$, is defined by the coefficients

\[
h^\hat{L}_{ab} = \frac{1}{2} \, h^{ac} \left\{ z_b \left( h_{ca} \right) + z_a \left( h_{cb} \right) - z_c \left( h_{ab} \right) \right\}, \quad (122)\]
\[
h^\hat{K}_{a}^{bc} = \frac{1}{2} \, h^{ac} \left\{ v^b \left( h_{ca} \right) + v^a \left( h_{cb} \right) - v^c \left( h_{ab} \right) \right\}.
\]

By straightforward calculations with covariant derivatives defined by the the coefficients (122) we can verify that one holds true all conditions of the theorem. □
Any symplectic structure on a symplectic manifold \((M, \omega)\) induces a corresponding isomorphism related to the bracket operator \(\{.,.\}\),

\[
T^*M \xrightarrow{\{.,.\}} TM
\]

where \(\tilde{\omega}(v) \equiv \omega(v,.\) and \(\tilde{\omega}^{-1}\) denotes the pulling back of the standard bracket on \(\mathcal{X}(M)\) to define the bracket operation for the differential 1-forms \(\Omega^1(M) \equiv \text{Sec}(T^*M)\). This transforms \(T^*M\) into a Lie algebroid with anchor \(\rho = -\tilde{\omega}^{-1}\) defined by the symplectic structure (as it was observed in \[65\] and investigated in details for the case of Lie algebroids of the Poisson manifolds in \[61\] and \[4\]).

The algebroids considered in this work (Lagrange–Finsler and Hamilton–Cartan ones) posses canonical symplectic structures defined by the corresponding canonical N–connections induced by the fundamental Lagrange or Hamilton functions. The symplectic structure induces canonical anchor maps on such N–anholonomic manifolds.

Finally, we note that the torsions and curvatures on Hamilton algebroids may be globally defined to be compatible to the N–connection structure and computed in z– and v–component form following the geometric formalism presented in section 3.3.1 (on d–connections, in the general canonical case related to d–metrics of type ) and in section 3.4 (on d–torsions and d–curvatures, adapted to the N–elongated bases on algebroids, see \[120\]).

### 6 Einstein–Cartan Algebroid Structures

In the previous sections, we demonstrated that the theory of Lie algebroids provided with nontrivial N–connections has a natural background from geometric mechanics and Finsler geometry. The aim of this section is to demonstrate that certain nontrivial Lie algebroid and N–connection structures can be defined by generic off–diagonal metrics and nonholonomic frames in gravity theories and that such gravity configurations may be also modelled by analogous optic–mechanical geometries. Further developments on exact solutions in gravity possessing Lie algebroid symmetry can be found in Refs.\[49, 50, 51, 52\], see also some recent results on gerbe extensions \[56\].

#### 6.1 N–connections and algebroid structures in gravity

For the geometric models of gravity and string theories with nonholonomic frame (vielbein) structure, one does not work on the tangent bundle \(TM\) but on a general manifold \(V\), \(\dim V = n + m\), which is a (pseudo) Riemannian...
space or a certain generalization with possible torsion and nonmetricity fields [37, 48].

6.1.1 \textbf{N–anholonomic manifolds}

Let us consider a metric tensor $g$ on the manifold $V$ with the coefficients defined with respect to a local coordinate basis $du^a = \left(dx^i, du^a\right)$,\footnote{the indices run correspondingly the values $i,j,k,... = 1,2,...,n$ and $a,b,c,... = 1,2,...,m$.}

$$g = g_{\alpha\beta}(u)du^\alpha \otimes du^\beta$$

where

$$g_{\alpha\beta} = \begin{bmatrix} g_{ij} + N^a_i N^b_j h_{ab} & N^e_i h_{ae} \\ N^f_j h_{be} & h_{ab} \end{bmatrix}.$$ (123)

\textbf{Definition 6.1} A manifold $V$ is N–anholonomic if it is provided with a $N$–connection structure $N = \{N^a_j\}$ defining a global splitting

$$TV = hV \oplus vV,$$ (124)

which, in general, is a nonholonomic distribution on $TV$.

Such nonintegrable distributions may be derived for any generic off–diagonal metrics parametrized in the form (123) which can not be diagonalized by coordinate transforms.

\textbf{Theorem 6.1} A metric $g$ with the coefficients (123) models a N–anholonomic manifold if and only if a nonholonomic vielbein structure

$$e_{\nu} = (e_i, v_b) = (e_i = \frac{\partial}{\partial x^i} - N^a_i \frac{\partial}{\partial u^a}, v_b = \frac{\partial}{\partial u^b})$$ (125)

and

$$e^\mu = (e^i, v^b) = (e^i = dx^i, v^b = du^b + N^b_i dx^i),$$ (126)

is prescribed on $TV$.

\textbf{Proof.} Performing a frame transform

$$e_{\alpha} = e_{\alpha}^\mu \partial_{\nu}^\mu \quad \text{and} \quad e_{\beta} = e_{\beta}^\alpha du^\alpha$$
with the coefficients

\[ e^\alpha_\alpha(u) = \begin{bmatrix} e_{i}^{i}(u) & N_{i}^{b}(u)e_{b}^{\alpha}(u) \\ 0 & e_{a}^{a}(u) \end{bmatrix} \]  

(127)

and

\[ e^{\beta}_{\beta}(u) = \begin{bmatrix} e_{i}^{i}(u) & -N_{k}^{k}(u)e_{i}^{k}(u) \\ 0 & e_{a}^{a}(u) \end{bmatrix}, \]  

(128)

we write equivalently the metric \( g \) in the form

\[ g = g_{\alpha\beta}(u) e^\alpha \otimes e^\beta = g_{ij}(u) e^i \otimes e^j + h_{cb}(u) v^c \otimes v^b, \]  

(129)

where \( g_{ij} \equiv g(e_i, e_j) \) and \( h_{cb} \equiv g(v_c, v_b) \) and \( e_\nu = (e_i, v_b) \) and \( e^\mu = (e^i, v^b) \) are, respectively, just the values (125) and (126), i.e. vielbeins of type (72) and (73), but in our case considered for arbitrary \( N_{b}^{i}(u) \) of dimension \( n \times m \) stating a splitting of the manifold into submanifolds of dimensions \( n \) and \( m \). This defines a special class of nonholonomic manifolds provided with a global splitting into conventional "horizontal" and "vertical" subspaces (124) (similar to (9) but not for the vector bundles) induced by the "off–diagonal" terms \( N_{b}^{i}(u) \) and the prescribed type of nonholonomic frame structure. The d–metric (129) is a generalization of the (82) to the case of arbitrary (non–Lagrange) metric and N–connection coefficients.

If the manifold \( V \) is (pseudo) Riemannian, there is a unique linear connection (the Levi–Civita connection) \( \nabla \), which is metric, \( \nabla g = 0 \), and torsionless, \( \nabla T = 0 \) but this connection is not adapted to the nonintegrable distribution induced by \( N_{b}^{i}(u) \). In order to construct exact solutions parametrized by generic off–diagonal metrics, or for investigating nonholonomic frame structures in gravity models with nontrivial torsion, it is more convenient to work with more general classes of linear connections which are \( N \)–adapted but contain nontrivial torsion coefficients because of nontrivial nonholonomy coefficients \( W^{\alpha}_{\beta\gamma} \). Such geometric constructions can be adapted to the prescribed N–connection structure on Riemann–Cartan spaces provided with corresponding classes of nonholonomic (N–adapted) frames.

For splitting into subspaces of dimensions \( n \) and \( m \) of a Riemann–Cartan space of dimension \((n + m)\), (the (pseudo) Riemannian configurations can be treated as particular cases), the Lagrange and Finsler type geometries were modelled by \( N \)–anholonomic structures as exact solutions of gravitational field equations [57]. It was concluded that the geometry of any Riemann...
space of dimension $n + m$ (where $n, m \geq 2$; we can consider $n, m = 1$ as special degenerated cases), provided with off–diagonal metric structure of type (123) can be equivalently modelled, by vielbein transforms of type (127) and (128) as a geometry of nonholonomic manifolds enabled with $N$–connection structure $N^b_i(u)$ and d–metric (129), see details in [47]. For certain special conditions when $n = m$, $N^b_i = L N^b_i$ (68) and the metric (129) is of type (82), a such Riemann–Cartan space of even dimension is 'nonholonomically' equivalent to a Lagrange space (for the corresponding homogeneity conditions one obtains the equivalence to a Finsler space).

Roughly speaking, by prescribing a corresponding vielbein structure, we can model a Lagrange, or Finsler, geometry on a Riemannian manifold and, inversely, a Riemannian geometry is 'not only a Riemannian one' but could be also a generalized Finsler space.

It is known the fact that the first example of Finsler geometry was considered in 1854 in the famous B. Riemann’s habilitation thesis (see historical details and discussion in Refs. [2, 29, 47]) who, for simplicity, restricted his considerations only to the curvatures defined by quadratic forms on hypersurfaces. Sure, for B. Riemann, it was unknown the fact that if we consider general (nonholonomic) frames with associated nonlinear connections (the E. Cartan’s moving frame geometry, see Refs. in [6]) and off–diagonal metrics, the Lagrange and Finsler geometry may be derived naturally even from quadratic metric forms adapted to the $N$–connection structure.

6.1.2 Lie N–algebroids modelled on N–anholonomic manifolds

For some additional parametrizations, a subclass of d–metrics of type (129) (equivalently, a subclass of metrics of type (123)) models a prolongation Lie algebroid provided with $N$–connection structure and Sasaki type d–metric:

**Theorem 6.2** A d–metric (129) on a $N$–anholonomic manifold $V$, for\[\text{dim } V = n + m,\]
defines a d–metric $\circ g$ of type (50) on $(vV,vV)$ of dimension $2m$, modelling a Lie N–algebroid with structure functions $\rho^i_j(x)$ and $C^{ab}_d(x)$, if and only if one holds the parametrizations:

\[
\begin{align*}
      g_{ij}(x,u) &= g_{d'v'}(x,u)\rho^a_i(x,u)\rho^b_j(x,u), \\
      h_{ab}(x,u) &= *h_{d'v'}(x,u) e^a_{\ a}(x) e^b_b(x), \\
      N^a_i(x,u) &= \rho^a_i(x,u)N^a_a(x,u)
\end{align*}
\]

for any values $\rho^a_i(x,u)$ and $e^a_{\ a}(x)$ for which the inverse $e^a_{d'}(x)$ $\partial/\partial u^a$ defining the conditions

\[
e_{d'} e^b_b - e_{d'} e^{d'}_{\ a} = C_{d'v'}^{a'}(x) e_{d'}
\]

for any values $\rho^a_i(x,u)$ and $e^a_{\ a}(x)$ for which the inverse $e^a_{d'}(x) = e^{d'}_{\ a}(x) \partial/\partial u^a$ satisfying the conditions

\[
\text{dim } V = n + m.
\]
and
\[ \rho_j^b(x) = g^{ji}(x, u) \ast h_{a'b'}(x, u) \rho_i^{a'}(x, u). \] (132)

**Proof.** Let us introduce the values (130) into (129) and define
\[ c^a = dx_i \rho_i^a(x, u) \]
and
\[ v^a = e^a_{\alpha''}(x) du^\alpha'' \]
which together with theirs duals prescribe a class of N–adapted (to \( N^a_{\alpha'} \)) frames of type \( c_A = (z_{a'}, v_a) \) (74) and \( v_A = (z^a_{\alpha'}, v^a) \) (75) derived by using vielbeins of type \((z_{a'}, v_a) \) (6) and \((\tilde{z}_{a'}, \tilde{v}_a) \) (7). Now we can consider the d–metric
\[ \overset{0}{g} = g_{a'b'}z_{a'} \otimes z_{b'} + h_{ab}v^a \otimes v^b, \] (133)
\[ z^a_{\alpha'} = z^a_{\alpha'} \] and \( v^a = v^a + N^a_{\alpha'} z^a_{\alpha'} \),
of type (30) on \((vV, vV)\). The values \( \rho_j^b = g^{ji} \ast h_{a'b'} \rho_i^{a'} \), see formula (132), the metric coefficients and \( \rho_i^{a'} \) depending on \( x- \) and \( u- \)variables must be related in a form to generate \( \rho_j^b(x) \) depending only on the \( x- \)variables, and \( C_{ab}^d(x) \) from (131): this defines the structure constants of a Lie N–algebroid.
We conclude that the data
\[ \left( g_{ij}, h_{ab}, N^a_i, \rho_{i}^{a'}, C_{ab}^d \right) \]
for a N–anholonomic manifold \( V \) can be transformed into the data
\[ \left( g_{ij}, *h_{ab}, N^a_{\beta'}, \rho_{\beta'}^i = g^{ji} \ast h_{a'b'} \rho_i^{a'}, C_{ab}^d \right) \]
modelling a Lie N–algebroid on \((vV, vV)\) (the inverse statement is also true) if the conditions of the Theorem are satisfied. \( \square \)

The Theorem 6.2 motivates the concept:

**Definition 6.2** A Lie N–algebroid is called Riemann–Cartan if it is provided with a d–metric and d–connection structure induced from a Riemann–Cartan manifold.

The next two sections are devoted to general properties and explicit examples of Riemann–Cartan and related Einstein–Cartan algebroids.
6.2 Mechanical and optical modelling of gravitational processes

During the last decade, an intensive research is devoted to analogous models of gravity when some important gravitational effects like black hole radiation, gravitational–electromagnetic interactions, nearly horizon effects... are modelled by optic and continuous media models, see outlines of results and references in [33, 36]. The unified geometric approach both to mechanics and gravity theories elaborated as some examples of N–anholonomic manifolds and/or algebroids give rise to new principles in ideas on such analogous modelling of field interactions.

Any gravitational theory defined by a generic off–diagonal metric on a N–manifold \( V \), \( \dim V = n + m \), induces a canonical Lagrange N–algebroid provided with a \( d \)–metric \( *g \) of type (101) on \( (vV,vV) \) of dimension \( 2m \). By a corresponding class of nonholonomic transforms of the \( v \)–subspace, we can ‘map’ a such gravity into a Lagrange algebroid:

**Theorem 6.3** Any \( d \)–metric (129) with the coefficients satisfying the conditions (130)–(132), transforms into a \( d \)–metric of type (101) for a generalized Lagrange algebroid,

\[
g^l g = g_{AB} \tilde{c}^A \otimes \tilde{c}^A = g_{a'b'}(x,u) z^a' \otimes z'^b + g_{ab}(x,u) \tilde{v}^a \otimes \tilde{v}^b,
\]

where \( g_{ab}(x,u) = A_{a'}^a A_{b'}^b \star h_{a'b'} \), see the Lie N–algebroid \( d \)–metric (133), and the \( v \)–vielbeins \( \tilde{v}^a = A_{a'}^a v^{a'} \), \( A_{a'}^a \) being inverse to \( A_a^{a'} \), are subjected to a nonholonomic relation of type

\[
\tilde{v}_a \tilde{v}_b - \tilde{v}_a \tilde{v}_b = \star W^c_{ab} \tilde{v}_c
\]

for \( \tilde{v}_a = A_{a'}^a v_{a'} \).

**Proof.** The Theorem 6.2 states that (129) transforms into (133) which is not a generalized Lagrange algebroid \( d \)–metric because, in general, \( g_{ab} \neq \star h_{ab} \). A formal equality of the \( c \)– and \( v \)–components of the \( d \)–metric on the Lie N–algebroid can be obtained by a certain nonholonomic frame transform in the \( v \)–subspaces when \( g_{ab}(x,u) = A_{a'}^a A_{b'}^b \star h_{a'b'} \) for some \( \tilde{v}^a = A_{a'}^a v^{a'} \), which is just the \( d \)–metric (134). The nonholonomy coefficients \( \star W^c_{ab} \) depend both on \( A_{a'}^a, C^a_{be} \) and \( N^a_{a'} \) : they can be computed in explicit form by commutating the vielbeins \( \tilde{v}_a \) when \( v^a = v^a + N^a_{a'} \varepsilon'^a \), see (133). □

For the \( d \)–metric (134), we can introduce the "absolute energy" (102), in this case of gravitational origin, and define the nonlinear geodesic congruences, on Lagrange algebroids (derived as Riemann–Cartan algebroids), as
Euler–Lagrange equations on Lie algebroids \[61\]. We can investigate some particular cases like in section 5.1.2 when \( g_{ab} \) is derived from
\[ g_{ij}(x, V(x)) = 0 g_{ij}(x) + (1 - u^2(x, V(x))) V_i \] or \( g_{ij} = e^{\sigma(x,y)} 0 g_{ij}(x) \), which can be related to various type of optic – continuous media mechanics processes.

6.3 Exact solutions with algebroid structure

We conclude this work by constructing some explicit examples of classes of \( d \)-metrics which describe exact solutions of the Einstein equations in string gravity and, for more particular cases, in Einstein–Cartan gravity and general relativity. Such solutions define Lie \( N \)-alegebroid configurations which are called Einstein–Cartan algebroids.

6.3.1 A class four dimensional \( N \)-anholonomic manifolds

The gravity field equations in string gravity can be written in effective form in terms of differential forms on a four dimensional \( N \)-anholonomic manifold (see, for instance, Refs. \[47\])
\[ \eta_{\alpha \beta \gamma} \wedge \tilde{R}^\beta_\gamma = \tilde{Y}_\alpha, \] (135)
where \( \tilde{R}^\beta_\gamma \) is the curvature 2–form for the canonical \( d \)-connection, \( Y_\alpha \) denote all possible sources defined by using the canonical \( d \)-connection and \( \eta \div \ast 1 \) is the volume form with the Hodge operator "\( \ast \)". \( \eta_\alpha \div e_\alpha \eta, \eta_{\alpha \beta} \div e_\beta \eta_\alpha, \eta_{\alpha \beta \gamma} \div e_\gamma \eta_{\alpha \beta}, .... \)

Let us consider a four dimensional metric ansatz for the \( d \)-metric \[129\] and frame \[73\] when \( u^\alpha = (x^1, x^2, y^3 = v, y^4); i = 1, 2 \) and \( a = 3, 4 \) and the coefficients
\[ g_{ij} = diag[g_1(x^1, x^2), g_2(x^1, x^2)] \], \( h_{ab} = diag[h_3(x^k, v), h_5(x^k, v)] \),
\[ N^3_i = w_i(x^k, v), N^4_i = n_i(x^k, v) \] (136)
are some functions of necessary smooth class. The partial derivatives are denoted \( a^* = \partial a/\partial x^1, a' = \partial a/\partial x^2, a^* = \partial a/\partial v \).
Theorem 6.4 The nontrivial components of the Ricci d–tensors for the canonical d–connection are

\[ R_1^1 = R_2^2 = -\frac{1}{2g_1g_2} [g_2^{**} - \frac{g_1^{*}g_2^{*}}{2g_2} - \frac{(g_2^{*})^2}{2g_2} + g_1^{**} - \frac{g_1^{*}g_2^{*}}{2g_2} - \frac{(g_1^{*})^2}{2g_1}], \]

\[ R_3^3 = R_4^4 = -\frac{1}{2h_3h_4} [h_3^{**} - h_4^{*}(\ln |h_3h_4|)^*], \quad (137) \]

\[ R_{3i} = -w_i^j \frac{\beta}{2h_4} - \frac{\alpha_i}{2h_4}, \quad R_{4i} = -\frac{h_4}{2h_3} [n_i^{**} + \gamma n_i^*], \]

\[ \alpha_i = \partial_i h_4^{*} - h_4^{*} \partial_i \ln |h_3h_4|, \quad \beta = h_4^{**} - h_4^{*} [\ln |h_3h_4|]^*, \]

\[ \gamma = 3h_4^{*}/2h_4 - h_3^{*}/h_4 \]

for \( h_3^* \neq 0 \) and \( h_4^* \neq 0 \).

Proof. Details for such computation are provided in Ref. [54].

Corollary 6.1 The Einstein equations (135) for the ansatz (136) are compatible for vanishing sources and if and only if the nontrivial components of the source, with respect to the frames (72) and (73), are any functions of type

\[ \hat{Y}_1 = \hat{Y}_2 = \Upsilon_1(x^1, x^2, v), \quad \hat{Y}_3 = \hat{Y}_4 = \Upsilon_3(x^1, x^2). \]

Proof. The proof, see details in [54], follows from the Theorem 6.4 with the nontrivial components of the Einstein d-tensor, \( \hat{G}^{\alpha\beta} = \hat{R}^{\alpha\beta} - \frac{1}{2} \delta^{\alpha\beta} \hat{R} \), computed to satisfy the conditions

\[ G_1^1 = G_2^2 = -R_3^3(x^1, x^2, v), \quad G_3^3 = G_4^4 = -R_1^1(x^1, x^2). \]

Having the values (137), we can prove [54] the

Theorem 6.5 The system of gravitational field equations (135) defined for the ansatz (136) can be solved in general form if there are given certain values of functions \( g_1(x^1, x^2) \) (or, inversely, \( g_2(x^1, x^2) \)), \( h_3(x^i, v) \) (or, inversely, \( h_4(x^i, v) \)) and of sources \( \Upsilon_1(x^1, x^2, v) \) and \( \Upsilon_3(x^1, x^2) \).

So, we defined a class of four dimension N–anholonomic manifolds as exact solutions of the Einstein equations for the canonical d–connection.
6.3.2 Generation of Einstein–Cartan algebroid structures

On \((vV,vV)\), which being associated to the data satisfying the conditions of the Theorem 6.5 is 2+2–dimensional, we introduce additional parametrizations modelling a Lie N–anholonomic structure stated by the Theorem 6.2.

Let us label the local bases by corresponding abstract indices, \(z_{a'} = (z_1', z_2')\) corresponding to \(e_i = (e_1, e_2)\) and \(v_{a''} = (v_1'', v_2'')\) corresponding to \(v_b = (v_3, v_4)\). For simplicity, we shall work with d–metrics which are diagonalized with respect to the corresponding N–anholonomic frames, i. e.

\[
g_{ij} = (g_1(x^k), g_2(x^k)), \quad h_{ab} = (h_3(x^k, v), h_4(x^k, v))
\]

and

\[
g_{a'\nu} = (g_1(x^k, v), g_2(x^k, v)), \quad * h_{a''\nu} = (h_1''(x^k, v), h_2''(x^k, v)),
\]

and consider nontrivial components of projections

\[
\rho^{a'}_i = \left(\rho^{a'}_1(x^k, v), \rho^{a'}_2(x^k, v)\right)
\]

and vielbein components \(e_1^3(x) = e_{11}(x)\) and \(e_2^4(x) = e_{22}(x)\), which induces nontrivial \(C^a_{\nu\nu\epsilon\nu}(x)\).

The Lie algebroid structure functions are chosen \(C^a_{\nu''\epsilon\nu}(x)\) and

\[
\rho^i_{a''} = \left(\rho^1_{a''}(x^k), \rho^2_{a''}(x^k)\right)
\]

when in order to satisfy the conditions \(130\)–\(132\) we have to satisfy the relations

\[
g_1(x) = g_1'(x^k, v) \left[\rho^1_1(x^k, v)\right]^2, \quad g_2(x) = g_2'(x^k, v) \left[\rho^2_2(x^k, v)\right]^2 \tag{138}
\]

\[
h_3(x, u) = h_{1\nu}(x, u) \left[e_{13}''(x)\right]^2, \quad h_4(x, u) = h_{2\nu}(x, u) \left[e_{24}''(x)\right]^2
\]

and

\[
\rho^1_1(x^k) = \frac{1}{g_1(x^k)} h_{1\nu}(x^k, v) \rho^1_1(x^k, v), \tag{139}
\]

\[
\rho^2_2(x^k) = \frac{1}{g_2(x^k)} h_{2\nu}(x^k, v) \rho^2_2(x^k, v).
\]

Having chosen the data \(\rho^{a'}_i = \left(\rho^{a'}_1, \rho^{a'}_2\right)\), we can compute \(N^a_i = \rho^{a'}_i N^{a'}_{a'}\).
The data \( \{g_1, g_2, h_3, h_b\} \) are given from an ansatz (136) solving the Einstein equations (135). We may consider in infinite set of Lie algebroid anchors \( \{\rho_1', \rho_2'\} \) and algebroid d–metrics \( \{g_1', g_2', h_{1'}', h_{2'}\} \) related in a compatible way, via (138) and (139), to some functions \( \{\rho_1', \rho_2', e_{[1]}, e_{[2]}\} \) with \( e_{[1]} \) and \( e_{[2]} \) related to the structure coefficients \( C_{\nu'\alpha'}(x) \). This points to the fact that an exact solution of the Einstein equations (vacuum or nonvacuum type) may parametrize an infinite set of Lie algebroid structures which was emphasized in Refs. [39, 38]. This is not surprising because the Lie algebroids are specific spaces defined by singular maps. We can induce a more explicit Lie algebroid configuration by fixing compatible frames of reference, boundary conditions and some classes of symmetries describing two (interrelated) theoretical models on the N–anholonomic manifold and on a corresponding Lie N–algebroids.

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