Local boundary controllability in classes of differentiable functions for the wave equation

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Abstract

The well-known fact following from the Holmgren-John-Tataru uniqueness theorem is a local approximate boundary $L_2$-controllability of the dynamical system governed by the wave equation. Generalizing this result, we establish the controllability in certain classes of differentiable functions in the domains filled up with waves.

1 Introduction

The paper deals with a local approximate boundary controllability of dynamical systems governed by the wave equation. This property means that the states of the system (waves) initiated by the boundary sources (controls), constitute $L_2$-complete sets in the domains, which the waves fill up. Such a result is derived from the fundamental Holmgren-John-Tataru uniqueness theorem [10] by the scheme proposed by D.L. Russel [9]. The $L_2$-controllability is a cornerstone of the boundary control method (BC-method), which is an approach to inverse problems based upon their relations to control and system theory [1, 2].

In this paper, we show that completeness of waves also holds in the certain classes of differentiable functions.

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tutorship of the author in 1997 at the physical faculty of the St-Petersburg State University. However, it was never published in official issues. The given variant is a revised and extended version of [3]. Recently, prof. G.Nakamura informed the author about certain interest to this kind of results. It is the reason, which has stimulated to return to this subject.

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2 Dynamical systems

All the function classes and spaces are real.

Initial boundary value problem

Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with the \( C^\infty \)-smooth boundary \( \Gamma \); \( T > 0 \), \( Q^T := \Omega \times (0, T) \), \( \Sigma^T := \Gamma \times [0, T] \). Consider the problem

\[
\begin{align*}
  u_{tt} + Au &= 0 & \text{in } Q^T \quad (2.1) \\
  u|_{t=0} &= u_t|_{t=0} & \text{in } \Omega \quad (2.2) \\
  u|_{\Sigma^T} &= f , & (2.3)
\end{align*}
\]

where \( f \) is a boundary control, \( A \) is a differential expression of the form

\[
A = -\sum_{i,j=1}^n \partial_x^i a^{ij}(x) \partial_x^j
\]

with the coefficients \( a^{ij} \in C^\infty(\overline{\Omega}) \) provided

\[
a^{ij}(x) = a^{ji}(x); \sum_{i,j=1}^n a^{ij}(x) \xi_i \xi_j \geq \mu |\xi|^2, \quad x \in \overline{\Omega}, \quad \xi = \{\xi_1, \ldots, \xi_n\} \in \mathbb{R}^n
\]

with a constant \( \mu > 0 \). Let \( u = u^f(x,t) \) be a solution (wave); the following is the list of its known properties.

(i) Let

\[
\mathcal{M}^T := \{ f \in C^\infty(\Sigma^T) \mid \text{supp } f \subset \Gamma \times (0, T) \}
\]

be the class of smooth controls vanishing near \( t = 0 \). This class is dense in \( L_2(\Sigma^T) \). For \( f \in \mathcal{M}^T \), problem \((2.1)-(2.3)\) has a unique classical solution
$u^f \in C^\infty(\overline{Q^T})$. Since the operator $A$, which governs the evolution of waves, doesn’t depend on $t$ and $\partial_t \mathcal{M}^T = \mathcal{M}^T$, this solution satisfies

$$u^f_t = u^f_t; \quad u^f_{tt} = u^f_{tt} - Au^f. \quad (2.5)$$

(ii) The map $f \mapsto u^f$ defined on $\mathcal{M}^T$ is continuous from $L^2(\Sigma^T)$ to $C([0, T]; L^2(\Omega))$ \cite{3}. As such, it can be extended onto $L^2(\Sigma^T)$. From this point on, for $f \in L^2(\Sigma^T)$, we define a (generalized) solution $u^f$ of the class $C([0, T]; L^2(\Omega))$ as the image of $f$ via the extended map.

(iii) By $H^s(\ldots)$ we denote the Sobolev classes. For $f \in H^{2p}(\Sigma^T)$ provided $(\partial_t)^j f|_{t=0}, \ j = 0, 1, \ldots, 2p - 1$, one has $u^f \in C([0, T]; H^{2p}(\Omega))$ \cite{5-7}.

(iv) The inverse matrix $\{a_{ij}\} := \{a^{ij}\}^{-1}$ determines a Riemannian metric $d\tau^2 = \sum_{ij} a_{ij} dx^i dx^j$ in $\Omega$ and the corresponding distance $\text{dist}_A$. Denote $\tau(x) := \text{dist}_A(x, \Gamma)$ and

$$\Omega^r := \{x \in \overline{\Omega} \mid \tau(x) < r\}, \ r > 0.$$

The well-known finiteness of the domain of influence principle for the hyperbolic problem (2.1)–(2.3) holds and implies the following equivalent relations

$$\text{supp} \ u^f(\cdot, t) \subset \overline{\Omega}, \ t > 0; \quad \text{supp} \ u^f \subset \{(x, t) \in \overline{Q^T} \mid t \geq \tau(x)\} \quad (2.6)$$

(see, e.g., \cite{5}). So, $\Omega^T$ is the subdomain filled with waves at the final moment $t = T$. Under our assumptions on $\Omega$ and $a^{ij}$, the value

$$T_{\text{fill}} := \inf \{T > 0 \mid \Omega^T = \Omega\}$$

is finite; we call it a filling time.

**Dual problem**

The problem

$$v_{tt} + Av = 0 \quad \text{in} \ Q^T \quad (2.7)$$

$$v|_{t=T} = 0, \ v_t|_{t=T} = y \quad \text{in} \ \overline{\Omega} \quad (2.8)$$

$$v|_{\Sigma^T} = 0 \quad (2.9)$$

is called dual to problem (2.1)–(2.3); let $v = v^u(x, t)$ be its solution. The following is the list of its known properties.
(i*) For $y \in C^\infty_0(\Omega)$, problem \ref{eq:2.7}--\ref{eq:2.9} has a unique classical solution $v^y \in C^\infty(Q_T)$. The map $y \mapsto v^y$ acts continuously from $L_2(\Omega)$ to $C([0,T];H^1_0(\Omega))$ \cite{6,7}.

(ii*) Let $\partial_{\nu,A} := \sum_{i,j=1}^n a^{ij} \cos(\nu, x^j) \partial x^i$ be the conormal derivative at the boundary $\Gamma$ (here $\nu$ is the Euclidean normal). The map $y \mapsto \partial_{\nu,A} v^y|_{\Sigma_T}$ is continuous from $L_2(\Omega)$ to $L_2(\Sigma_T)$ \cite{6,7}.

(iii*) By the finiteness of the domain of influence principle for the hyperbolic problem \ref{eq:2.7}--\ref{eq:2.9}, the trace $\partial_{\nu,A} v^y|_{\Sigma_T}$ is determined by the values of $y|_{\Omega_T}$ (does not depend on $y|_{\Omega\setminus\Omega_T}$). In particular, if $y|_{\Omega_T} = 0$ then $\partial_{\nu,A} v^y|_{\Sigma_T} = 0$ holds.

Spaces and operators

Here we consider the above introduced problems as dynamical systems and endow them with the standard attributes of control and system theory.

- The Hilbert space of controls $\mathcal{F}^T := L_2(\Sigma_T)$ is an outer space of the system \ref{eq:2.1}--\ref{eq:2.3}.

  The Hilbert space $\mathcal{H} := L_2(\Omega)$ is called an inner space. It contains the subspace $\mathcal{H}^T := \{ y \in \mathcal{H} \mid \text{supp } y \subset \overline{\Omega_T} \}$ of functions supported in the subdomain filled up with waves at the final moment $t = T$.

- The map $W^T : \mathcal{F}^T \to \mathcal{H}$, $W^T f := u^f(\cdot,T)$ is a control operator. By the property (ii), it is continuous.

  The map $O^T : \mathcal{H} \to \mathcal{F}^T$, $O^T y := \partial_{\nu,A} v^y|_{\Sigma_T}$ associated with the system \ref{eq:2.7}--\ref{eq:2.9} is an observation operator. The well-known fact is the duality relation

  $$O^T = (W^T)^*,$$

  which is derived by integration by parts (see, e.g., \cite{1,2}).

- The set of waves

  $$\mathcal{U}^T := \{ u^f(\cdot,T) \mid f \in \mathcal{F}^T \} = W^T \mathcal{F}^T = \text{Ran } W^T \subset \mathcal{H}$$

  is called reachable (at the moment $t = T$). By the first relation in \ref{eq:2.6}, the embedding

  $$\mathcal{U}^T \subset \mathcal{H}^T,$$  \hspace{1cm} $T > 0$

  holds. The general operator equality implies

  $$\text{Ker } O^T = \mathcal{H} \ominus \text{Ran } (O^T)^* \overset{\text{2.10}}{=} \mathcal{H} \ominus \text{Ran } W^T = \mathcal{H} \ominus \overline{\mathcal{U}^T}$$
(see, e.g., [4]), whereas (2.12) leads to
\[
\text{Ker } O^T \supset \mathcal{H} \ominus \mathcal{H}^T, \quad T > 0
\] (2.13)
that corresponds to the property (iii*).

## 3 Controlability

### $L_2$-controllability

One of the central results of the boundary control theory, which plays the crucial role for the BC-method, is that the embedding (2.12) is dense:
\[
\mathcal{U}^T = \mathcal{H}^T, \quad T > 0
\] (3.1)
(see [1, 2]). In particular, for $T > T_{\text{fill}}$ one has $\mathcal{U}^T = \mathcal{H}$. As was mentioned in Introduction, (3.1) is derived from the Holmgren-John-Tataru Theorem on uniqueness of continuation of the solutions to the wave equation across a non-characteristic surfaces [10]. This result means that any function supported in the domain $\Omega^T$ filled with waves can be approximated (in the $L_2$-metric) by a wave $u^f(\cdot, T)$ with the properly chosen control $f \in \mathcal{F}^T$. In control theory such a property is referred to as a local approximate boundary controllability of system (2.1)–(2.3).

Since $\text{Ker } O^T = \mathcal{H} \ominus \mathcal{U}^T$, property (3.1) leads to the equality
\[
\text{Ker } O^T = \mathcal{H} \ominus \mathcal{H}^T, \quad T > 0
\] (3.2)
which refines (2.13) and is interpreted as an observability of the dual system (2.7)–(2.9). It means that the wave $v^y$ isn’t observed at the boundary during the interval $0 \leq t \leq T$ if and only if the velocity perturbation $y$, which initiates the wave process, is separated from the boundary: $\text{dist}_{\mathcal{A}}(\text{supp } y, \Gamma) \geq T$. In particular, for $T > T_{\text{fill}}$ one has $\text{Ker } O^T = \{0\}$. The duality ‘controllability–observability’ is a very general fact of the system theory.

Later on we’ll use the following quite evident consequence of the observability (3.2).

### Proposition 1

Let $0 < \delta < T$. The relation $O^T y|_{T \times [\delta, T]} = 0$ implies $y \in \mathcal{H} \ominus \mathcal{H}^{T-\delta}$ that is equivalent to $\text{supp } y \subset \Omega \setminus \Omega^{T-\delta}$. 

5
Spaces $\mathcal{D}_s$

As is well known, the operator

$$A_0 : \mathcal{H} \to \mathcal{H}, \quad \text{Dom} \, A_0 = H^2(\Omega) \cap H^1_0(\Omega), \quad A_0 y := Ay$$

is positive definite in $\mathcal{H}$ and has a purely discrete spectrum $\{\lambda_k\}_{k \geq 1} : 0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots; \lambda_k \to \infty$. Let $\{e_k\}_{k \geq 1} : A_0 e_k = \lambda_k e_k$ be the basis of its eigenfunctions normalized by $(e_k, e_l)_{\mathcal{H}} = \delta_{kl}$.

For $s > 0$, define the Hilbert space of functions

$$\mathcal{D}_s := \text{Dom} \, A_0^{s/2}, \quad (y, w)_{\mathcal{D}_s} := (A_0^{s/2} y, A_0^{s/2} w)_{\mathcal{H}}$$

and note the relations $\mathcal{D}_s \subset H^s(\Omega)$ and $\mathcal{D}_s \supset \mathcal{D}_{s'}$ as $s < s'$ (see [8]). This space contains the subspace

$$\mathcal{D}_s^T := \left\{ y \in \mathcal{D}_s \mid \text{supp} \, y \subset \Omega^T \cup \Gamma \right\}$$

of functions supported in the filled domain. The definition easily implies

$$\mathcal{D}_s^T = \bigcup_{0 < \delta < T} \mathcal{D}_{s-\delta}^T.$$  \hfill (3.4)

Introduce the (sub)class of smooth controls

$$\mathcal{M}_0^T := \{ f \in \mathcal{M} \mid \partial_t^{2p} f|_{t=T} = 0, \ p = 0, 1, 2, \ldots \}$$

and note that $\partial_t^{2p} \mathcal{M}_0^T \subset \mathcal{M}_0^T$ holds for all $p \geq 0$. Let

$$\mathcal{W}_0^T := \{ u^f(\cdot, T) \mid f \in \mathcal{M}_0^T \} = W^T \mathcal{M}_0^T$$

be the corresponding reachable set.

**Proposition 2** The embedding $\mathcal{W}_0^T \subset \mathcal{D}_s^T$ holds for all $s > 0$ and $T > 0$.

Indeed, if $f \in \mathcal{M}_0^T$ then $u^f(\cdot, T) \in C^\infty(\Omega)$, $u^f(\cdot, T)|_{\Gamma} \overset{2.3}{=} f|_{t=T} = 0$ and, hence, $u^f(\cdot, T) \in \text{Dom} \, A_0$. Therefore,

$$A_0 u^f(\cdot, T) = A u^f(\cdot, T) \overset{2.5}{=} -u^{f_{tt}}(\cdot, T) \in \text{Dom} \, A_0$$

since $f_{tt} \in \mathcal{M}_0^T$. Thus, we have $u^f(\cdot, T) \in \text{Dom} \, A_0^2$. Going on in the evident way, we get $u^f(\cdot, T) \in \text{Dom} \, A_0^p$ with any integer $p \geq 1$. Hence, $u^f(\cdot, T) \in \mathcal{D}_s$ for all $s > 0$. In the mean time, supp $u^f(\cdot, T) \subset \Omega^T \cup \Gamma$ and, hence, $u^f(\cdot, T) \in \mathcal{D}_s^T$. The Proposition is proven.
**D**-controllability

The following result is referred to as an approximate boundary **D**-controllability of system (2.1)–(2.3).

**Theorem 1** The relation

\[ U_0^T = D_s^T, \quad s > 0, \quad T > 0 \]  

(3.5)

(the closure in **D**) is valid. In particular, for \( T > T_{\text{fill}} \) one has \( U_0^T = D_s \).

**Proof.**

1. **Spectral representation.** Recall that \( \{ \lambda_k \}_{k \geq 1} \) and \( \{ e_k \}_{k \geq 1} \) are the spectrum and basis (in \( \mathcal{H} \)) of eigenfunctions of the operator \( A_0 \). As is easy to check, the system

\[ \{ e^s_k \}_{k \geq 1} : \quad e^s_k := \lambda_k^{-s/2} e_k \]

constitutes an orthogonal normalized basis in \( D_s \).

The system

\[
\begin{align*}
v_{tt} + A v &= 0 \quad \text{in } \Omega \times \mathbb{R} \\
v|_{t=T} &= 0, \quad v|_{t=T} = y \quad \text{in } \Omega \\
v|_{\Gamma \times \mathbb{R}} &= 0
\end{align*}
\]

(3.6)

(3.7)

(3.8)

is an extension of the dual system to all times. For a \( y \in C_0^\infty(\Omega) \) it has a unique classical solution \( v^y \in C^\infty(\Omega \times \mathbb{R}) \). Applying the Fourier method to problem (2.7)–(2.9), one easily derives

\[
v^y(\cdot, t) = \sum_{k=1}^{\infty} \alpha_k \frac{\sin \sqrt{\lambda_k}(t - T)}{\sqrt{\lambda_k}} e_k; \quad \alpha_k = (y, e_k)_{\mathcal{H}}.
\]

(3.9)

Note that \( v^y(\cdot, t) \) is odd w.r.t. \( t = T \).

2. **Regularization.** For an arbitrary \( y \in \mathcal{H} \), the (generalized) solution \( v^y(\cdot, t) \) is also represented by the right hand side of (3.9) but may not belong to the classes **D**. Here we provide a procedure, which improves smoothness of solutions to the dual system.

- The role of the smoothing kernel is played by a function

\[
\phi_\varepsilon(t) := \varepsilon^{-1} \phi(\varepsilon^{-1} t) \xrightarrow{\varepsilon \to 0} \delta(t),
\]
where \( \phi \in C^\infty(\mathbb{R}) \) satisfies

\[
\phi \geq 0; \quad \phi(-t) = \phi(t); \quad \text{supp} \, \phi \subset [-1, 1]; \quad \int_{\mathbb{R}} \phi(t) \, dt = 1.
\]

Let \( y \in C^\infty_0(\Omega) \). The function

\[
v_\varepsilon(\cdot, t) := [\phi \ast v^y](\cdot, t) = \int_{\mathbb{R}} \phi_\varepsilon(\eta) v^y(\cdot, t - \eta) \, d\eta, \quad t \in \mathbb{R}
\]

(the convolution w.r.t. time) is also \( C^\infty \)-smooth in \( \overline{\Omega} \times \mathbb{R} \) and odd w.r.t. \( t = T \). Integrating in (3.9), one easily gets its spectral representation:

\[
v_\varepsilon(\cdot, t) = \sum_{k=1}^{\infty} \beta_\varepsilon^k \alpha_k \sin \sqrt{\lambda_k} t \, e_k, \quad \alpha_k = (y, e_k)_{\mathcal{H}}, \quad \beta_\varepsilon^k = \int_{-\varepsilon}^{\varepsilon} \phi_\varepsilon(\eta) \cos \sqrt{\lambda_k} \eta \, d\eta = \int_{-1}^{1} \phi(t) \cos \varepsilon \sqrt{\lambda_k} t \, dt
\]

with \( |\beta_\varepsilon^k| \leq 1 \). Taking into account the properties of \( \phi \), one can easily derive

\[
\beta_\varepsilon^k \rightarrow 1, \quad k \geq 1; \quad \lambda_k^{s/2} \beta_\varepsilon^k \rightarrow 0, \quad s \geq 0, \quad \varepsilon > 0.
\]

Comparing (3.9) with (3.11), we see that \( v_\varepsilon^y \) is a solution to problem (3.6)–(3.8) satisfying

\[
v_\varepsilon^y \big|_{t=T} = 0, \quad (v_\varepsilon^y)_t \big|_{t=T} = \sum_{k=1}^{\infty} \beta_\varepsilon^k \alpha_k e_k =: y_\varepsilon.
\]

So, we have \( v_\varepsilon^y = v_\varepsilon^y \). Also, note that \( y_\varepsilon \in \mathcal{D}^s \) for all \( s > 0 \) by virtue of the second relation in (3.12).

- By the aforesaid, the operator (regularizer) \( R_\varepsilon : \mathcal{H} \rightarrow \mathcal{H}, \, R_\varepsilon y := y_\varepsilon \) is well defined on \( C^\infty_0(\Omega) \). Estimating

\[
\|y_\varepsilon\|^2_{\mathcal{H}_s} = \sum_{k=1}^{\infty} \lambda_k^s (\beta_\varepsilon^k)^2 \alpha_k^2 \leq \text{const} \sum_{k=1}^{\infty} \alpha_k^2 = \text{const} \|y\|^2_{\mathcal{H}} \quad (\varepsilon > 0),
\]

we see that \( R_\varepsilon \) acts continuously from \( \mathcal{H} \) to \( \mathcal{D}^s \).

Representation (3.13) implies

\[
R_\varepsilon e_k = \beta_\varepsilon^k e_k, \quad (3.14)
\]
i.e., the regularizer is diagonal in the eigenbasis of $A_0$. Hence, we have $R_\varepsilon e_k^\varepsilon = \beta_k^\varepsilon e_k^\varepsilon$ in $\D_s$. Since $\beta_k^\varepsilon$ are uniformly bounded, $R_\varepsilon$ is continuous as an operator in $\D_s$ and its norm is bounded uniformly w.r.t. $\varepsilon$. In the mean time, by the first relation in (3.12), the regularizer converges to the identical operator $I$ on the dense set span $\{e_k\}_{k \geq 1}$ as $\varepsilon \to 0$. As a result, the convergence $R_\varepsilon \to I$ in the strong operator topology in $\D_s$ does occur.

• Fix $\delta \in (0, T)$ and a positive $\varepsilon < \delta$. For the controls $f \in \F_T$ provided $\text{supp } f \subset \Gamma \times [\delta, T]$, the operation $f \mapsto f_\varepsilon$:

$$f_\varepsilon(\cdot, t) := \int_0^T [\phi_\varepsilon(t - \eta) - \phi_\varepsilon(2T - t - \eta)] f(\cdot, \eta) \, d\eta, \quad 0 \leq t \leq T \quad (3.15)$$

is well defined. With regard to the definition of the class $\mathcal{M}_0^T$ and properties of the kernel $\phi_\varepsilon$, one can easily check that $f_\varepsilon \in \mathcal{M}_0^T$. Note that the latter implies

$$W^T f_\varepsilon = u^{f_\varepsilon}(\cdot, T) \in \U_0^T \subset \D^T_s.$$ 

Proposition 3 For any admissible $f \in \F_T$ and $y \in \mathcal{H}$, the relation

$$(f_\varepsilon, O^T y)_{\F_T} = (f, O^T R_\varepsilon y)_{\F_T} \quad (3.16)$$

holds.

Indeed, let $y \in C^\infty_0(\Omega)$, so that $v^y$ is classical and smooth in $\overline{\Omega} \times \mathbb{R}$. Applying $\partial_{\nu_A}$ in (3.10), we get

$$\partial_{\nu_A} v_\varepsilon^y = \phi_\varepsilon \ast \partial_{\nu_A} v^y \quad \text{on } \Gamma \times \mathbb{R}. $$

By the evenness/oddness of $\phi_\varepsilon$ and $v^y$, for the times $t < T$ the right hand side can be written in the form

$$[\phi_\varepsilon \ast \partial_{\nu_A} v^y](\cdot, t) = \int_{-\infty}^T [\phi_\varepsilon(t - \eta) - \phi_\varepsilon(2T - t - \eta)] \partial_{\nu_A} v^y(\cdot, \eta) \, d\eta. \quad (3.17)$$

Taking into account (3.15), (3.17) and changing the order of integration, one
derives
\[
(f, O^T y)_{\mathcal{F}} = \int_{\Sigma_T} f_\varepsilon(\gamma, t) \, \partial_{\nu_A} v^y(\gamma, t) \, d\Gamma \, dt =
\]
\[
= \int_{\Sigma_T} f(\gamma, t) \left[ \int_{-\varepsilon}^\varepsilon \phi_\varepsilon(\eta) \, \partial_{\nu_A} v^y(\gamma, t - \eta) \, d\eta \right] \, d\Gamma \, dt =
\]
\[
= \int_{\Sigma_T} f(\gamma, t) \, \partial_{\nu_A} v^y(\gamma, t - \eta) \, d\Gamma \, dt =
\]
\[
= \int_{\Sigma_T} f(\gamma, t) \, \partial_{\nu_A} v^\varepsilon(\gamma, t - \eta) \, d\Gamma \, dt = (f, O^T R_\varepsilon y)_{\mathcal{F}}.
\]

Thus, we get (3.16) for the given \( y \). Since such \( y \)'s constitute a dense set in \( \mathcal{H} \), whereas the operator \( O^T R_\varepsilon \) is continuous, we extend (3.16) to all \( y \in \mathcal{H} \).

The Proposition is proven.

3. Completing the proof of Theorem. Let \( z \in D^T_s \ominus \mathcal{F}^T_0 \) (the orthogonality in \( D^T_s \)); we are going to show that \( z = 0 \). Recall that the scalar product in \( D^T_s \) is
\[
(y, w)_{D^T_s} = (A^s_{1/2} y, A^s_{1/2} w)_{\mathcal{H}} = \sum_{k=1}^{\infty} \lambda^s_k (y, e_k)_\mathcal{H} (w, e_k)_\mathcal{H}.
\]

Let \( \alpha_k = (z, e_k)_\mathcal{H} \). Fix a \( \delta \in (0, T) \) and positive \( \varepsilon < \delta \). By the choice of \( z \), for \( f \in \mathcal{F}^T \) provided \( \text{supp} \, f \subset \Gamma \times [\delta, T] \) one derives
\[
0 = (W^T f_\varepsilon, z)_{D^T_s} = \sum_{k=1}^{\infty} \lambda^s_k (W^T f_\varepsilon, e_k)_\mathcal{H} (z, e_k)_\mathcal{H} \overset{2.10}{=} \sum_{k=1}^{\infty} \lambda^s_k \alpha_k (f_\varepsilon, O^T e_k)_{\mathcal{F}} =
\]
\[
= \sum_{k=1}^{\infty} \lambda^s_k \alpha_k (f, O^T R_\varepsilon e_k)_{\mathcal{F}} \overset{3.14}{=} \sum_{k=1}^{\infty} \lambda^s_k \alpha_k \beta^s_k (f, O^T e_k)_{\mathcal{F}} =
\]
\[
= \left( f, O^T \sum_{k=1}^{\infty} \lambda^s_k \alpha_k \beta^s_k e_k \right)_{\mathcal{F}} \overset{3.13}{=} (f, O^T A^s_0 z_\varepsilon)_{\mathcal{F}}.
\]

the continuity of \( O^T : \mathcal{H} \to \mathcal{F}^T \) being in the use. Since \( f \) is arbitrary on \( \Gamma \times [\delta, T] \), we see that
\[
(O^T A^s_0 z_\varepsilon)_{\Gamma \times [\delta, T]} = 0.
\]

By Proposition II the latter implies
\[
A^s_0 z_\varepsilon = 0 \quad \text{in} \quad \Omega^{T-\delta}.
\]

(3.18)
Next, for a $y \in \mathcal{D}^{-\delta}_s$ we have
\[
(y, z_\varepsilon)_{\mathcal{D}_s} = (A_0^{\varepsilon/2} y, A_0^{\varepsilon/2} z_\varepsilon)_{\mathcal{H}} = (y, A_0^s z_\varepsilon)_{\mathcal{H}} \equiv 0,
\]
i.e., $z_\varepsilon \in \mathcal{D}_s^T \cap \mathcal{D}_s^{T-\delta}$. Tending $\varepsilon \to 0$, we get $z_\varepsilon = R_\varepsilon z \to z$ in $\mathcal{D}_s$ and conclude that $(y, z)_{\mathcal{D}_s} = 0$ for any $y \in \mathcal{D}_s^{T-\delta}$ and $\delta \in (0, T)$. Referring to (3.4), we arrive at $z = 0$ and, thus, prove Theorem 1.

$H^1_0$-controllability

In the rest of the paper we consider certain applications of Theorem 1.

In terms of the Riemannian geometry in $\Omega$ determined by the metric $d\tau^2 = a_{ij} dx^i dx^j$, the subdomain $\Omega^T$ is a near-boundary layer of the thickness $T$. It increases as $T$ grows. Recall that $\tau(x) = \text{dist}_A(x, \Gamma)$. For $T < T_{\text{fill}}$, the boundary of the layer consists of two parts: $\partial \Omega^T = \Gamma \cup \Gamma^T$, where
\[
\Gamma^T := \{x \in \Omega \mid \tau(x) = T\}
\]
is a surface equidistant to $\Gamma$.

The smoothness of $\Gamma$ provides $\mathcal{D}_1 = H^1_0(\Omega)$, i.e., these spaces consist of the same reserve of functions, whereas the norms $\| \cdot \|_{\mathcal{D}_1}$ and $\| \cdot \|_{H^1_0(\Omega)}$ are equivalent. Hence, one has
\[
\mathcal{D}^T_1 \equiv \{y \in H^1_0(\Omega) \mid \text{supp } y \subset \Omega^T \cup \Gamma\} = H^1_0(\Omega^T)
\]
(the closure in $H^1$-metric), the latter equality being valid since the compactly supported functions are dense in $H^1_0(\Omega^T)$. As a result, we arrive at
\[
\mathcal{U}^T_0 = H^1_0(\Omega^T), \quad T > 0. \quad (3.19)
\]

$H^1$-controllability

Using the wider class of controls $\mathcal{M}^T$ instead of $\mathcal{M}^T_0$, one extends the corresponding reachable set from $\mathcal{U}^T_0$ to
\[
\mathcal{U}^T_* := \{u^f(\cdot, T) \mid f \in \mathcal{M}^T\} = W^T \mathcal{M}^T,
\]
so that $\mathcal{U}^T_0 \subset \mathcal{U}^T_* \subset \mathcal{U}^T$ holds.
For $T > 0$, define the class
\[
H^1_+(\Omega^T) := \{ y \in H^1(\Omega) \mid \text{supp } y \subset \Omega^T \cup \Gamma \}
\]
(the closure in $H^1(\Omega)$). Its elements differ from the ones of $H^1_0(\Omega)$ by that $y|_\Gamma = 0$ is cancelled: $H^1_0(\Omega) = \{ y \in H^1(\Omega^T) \mid y|_\Gamma = 0 \}$.

**Lemma 1** For any $T > 0$, the relation
\[
\overline{\mathcal{W}^T_+} = H^1_+ (\Omega^T)
\]
(the closure in $H^1(\Omega)$) is valid. In particular, for $T > T_{\text{fill}}$ one has $\overline{\mathcal{W}^T_+} = H^1(\Omega)$.

**Proof.**

- The well-known geometric fact, which is popularly referred to as a variant of the ‘collar theorem’, is that there exists a domain $\hat{\Omega} \supset \Omega$ with the properties listed below. The objects related with it are marked with dots.

  1. The boundary $\partial \hat{\Omega} =: \hat{\Gamma}$ is $C^\infty$-smooth. The coefficients $\hat{a}^{ij} \in C^\infty(\hat{\Omega})$ obey the ellipticity conditions (2.4) with a constant $\hat{\mu} > 0$ and satisfy $\hat{a}^{ij}|_{\Omega} = a^{ij}$.

  2. The distance $\text{dist}_{\hat{\Omega}}$ in $\hat{\Omega}$ is such that $\hat{\Gamma}^{\eta} = \Gamma$ for some $\eta > 0$, and, respectively, $\hat{\Gamma}^{T+\eta} = \Gamma^T$ ($T > 0$).

- Let $y \in H^1_+ (\Omega^T)$. To prove the Lemma, it suffices to construct a sequence $\{f_j\}_{j \geq 1} \subset \mathcal{M}^T_+$ such that $u^{ij}(\cdot, T) \rightarrow y$ in $H^1(\Omega)$. We do it as follows.

  Extend $y$ to a function $\hat{y} \in H^1_+(\hat{\Omega}^{T+\eta}) : \hat{y}|_{\Omega} = y$. Such an extension does exist owing to smoothness of $\hat{\Gamma}$: see, e.g., [8].

  Consider problem (2.1)–(2.3) in $\hat{\Omega} \times (0, T+\eta)$. By (3.19), one can choose the controls $\{f_j\} \subset \mathcal{M}^T_0$ so that $u^{ij}(\cdot, T + \eta) \rightarrow \hat{y}$ in $H^1(\hat{\Omega})$. Correspondingly, the convergence $u^{ij}(\cdot, T + \eta)|_{\Omega} \rightarrow \hat{y}|_{\Omega} = y$ holds in $H^1(\Omega)$.

- Return to problem (2.1)–(2.3) in $\Omega \times (0, T)$ and put $\{f_j\} \subset \mathcal{F}^T : f_j(\cdot, t) := u^{ij}(\cdot, t + \eta)|_{\Gamma}, \ 0 \leq t \leq T$. Recalling the properties (2.5) and (2.6), one can easily verify that $f_j \in \mathcal{M}^T_+$ holds and provides $u^{ij}(\cdot, t) = u^{ij}(\cdot, t + \eta)|_{\Omega} \rightarrow y$.

The Lemma is proven.
**$H^p$- and $C^m$-controllability**

The result of Lemma 1 can be easily generalized as follows.

- In the spaces $H^p(\Omega)$ for $p = 0, 1, 2, \ldots$ define the subspaces
  
  $$H^p_s(\Omega^T) := \{ y \in H^p(\Omega) \mid \text{supp}\, y \subset \Omega^T \cup \Gamma \}, \quad T > 0.$$ 

  The relation
  
  $$\overline{H^p_s(\Omega^T)} = H^p_s(\Omega^T), \quad T > 0$$

  (the closure in $H^1(\Omega)$) is valid for all $T > 0$.

- Let $m = 0, 1, 2, \ldots$. In the spaces $C^m(\Omega)$, define the subspaces
  
  $$C^m_s(\Omega^T) := \{ y \in C^m(\Omega) \mid \text{supp}\, y \subset \Omega^T \cup \Gamma \}, \quad T > 0.$$ 

  By the Sobolev embedding theorems, for $s \geq m+1+\left\lceil \frac{n}{2} \right\rceil$ the relation $H^s(\Omega) \subset C^m(\Omega)$ holds [3]. As a simple consequence, one has
  
  $$\overline{C^m_s(\Omega^T)} = C^m_s(\Omega^T), \quad T > 0$$

  (the closure in $C^m(\Omega)$).

- Note in conclusion that all the above obtained results are valid in the case $A = - \sum_{i,j=1}^{n} \partial_{x_i} a^{ij}(x) \partial_{x_j} + q$ with $q \in C^\infty(\Omega)$.

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