THE PROPERTIES OF BERTRAND CURVES IN DUAL SPACE

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Abstract. In this study, we investigate Bertrand curves in three dimensional dual space $D^3$ and we obtain the characterizations of these curves in dual space $D^3$. Also we show that involutes of a curve constitute Bertrand pair curves.

1. Introduction

In the study of differential geometry, the characterizations of the curves and the corresponding relations between the curves are significant problem. It is well known that many important results in the theory of the curves in $E^3$ were given by G. Monge and then G. Darboux detected the idea of moving frame. After this Frenet defined moving frame and special equations which are used in mechanics, kinematics and physics.

A set of orthogonal unit vectors can be built, if a curve is differentiable in an open interval, at each point. These unit vectors are called Frenet frame. The Frenet vectors along the curve, define curvature and torsion of the curve. The frame vectors, curvature and torsion of a curve constitute Frenet apparatus of the curve.

It is certainly well known that a curve can be explained by its curvature and torsion except as to its position in space. The curvature ($\kappa$) and torsion ($\tau$) of a regular curve help us to specify the shape and size of the curve. Such as; If $\kappa = \tau = 0$, then the curve is a geodesic. If $\kappa \neq 0$ (constant) and $\tau = 0$, then the curve is a circle with radius $\frac{1}{\kappa}$. If $\kappa \neq 0$ (constant) and $\tau \neq 0$ (constant), then the curve is a helix.

Bertrand curves can be given as another example of that relation. Bertrand curves are discovered in 1850, by J. Bertrand who is known for his applications of differential equations to physics, especially thermodynamics. A Bertrand curve in $E^3$ is a curve such that its principal normal vectors are the principal normal vectors of an other curve. It is proved in most studies on the subject that the characteristic property of a Bertrand curve is the existence of a linear relation between its curvature and torsion as:

$$\lambda \kappa + \mu \tau = 1$$

with constants $\lambda$, $\mu$ where $\lambda \neq 0$ (see [6]).

Dual numbers were defined by W.K.Clifford (1849-1879). After him E. Sudy used dual numbers and dual vectors to clarify a mapping from dual unit sphere to three dimensional Euclidean space $E^3$. This mapping is called Study mapping. Study mapping corresponds the dual points of a dual unit sphere to the oriented lines in

2000 Mathematics Subject Classification. 53A04 , 53A25 , 53A40.

Key words and phrases. Bertrand curves, dual space.
For the dual number \( \hat{a} \) as a type of dual unit. The elements of the set \( D \) are called the dual part of \( \hat{a} \) and \( a^* \in \mathbb{R} \) is called the dual part of \( a \).

Two inner operations and equality on \( D \) are defined for \( \hat{a} = a + \varepsilon a^* \) and \( \hat{b} = b + \varepsilon b^* \), as:

1) \( + : D \times D \rightarrow D \)
\[ \hat{a} + \hat{b} = (a + \varepsilon a^*) + (b + \varepsilon b^*) = (a + b) + \varepsilon(a^* + b^*) \]

is called the addition in \( D \).

2) \( \cdot : D \times D \rightarrow D \)
\[ \hat{a} \cdot \hat{b} = (a + \varepsilon a^*) \cdot (b + \varepsilon b^*) = a.b + \varepsilon(ab^* + ba^*) \]

is called the multiplication in \( D \).

3) \( \hat{a} = \hat{b} \) if and only if \( a = b \) and \( a^* = b^* \).

Also the set \( D = \{ \hat{a} = a + \varepsilon a^* | a, a^* \in \mathbb{R} \} \) forms a commutative ring with the following operations

1) \( (a + \varepsilon a^*) + (b + \varepsilon b^*) = (a + b) + \varepsilon(a^* + b^*) \)
2) \( (a + \varepsilon a^*) \cdot (b + \varepsilon b^*) = a.b + \varepsilon(ab^* + ba^*) \).

The division of two dual numbers \( \hat{a} = a + \varepsilon a^* \) and \( \hat{b} = b + \varepsilon b^* \) provided \( b \neq 0 \) can be defined as
\[ \frac{\hat{a}}{\hat{b}} = \frac{a + \varepsilon a^*}{b + \varepsilon b^*} = \frac{a}{b} + \varepsilon\frac{a^*b - ab^*}{b^2} \]

The set
\[ D^3 = D \times D \times D = \left\{ \begin{array}{c} \vec{a} + \varepsilon \vec{a}^* = (a_1 + \varepsilon a_1^*, a_2 + \varepsilon a_2^*, a_3 + \varepsilon a_3^*) \\ \vec{a} + \varepsilon \vec{a}^* = (a_1, a_2, a_3) + \varepsilon (a_1^*, a_2^*, a_3^*) \\ \varepsilon \vec{a}^* = (a_1^*, a_2^*, a_3^*) \\ \vec{a} \in \mathbb{R}^3, \ a^* \in \mathbb{R}^3 \end{array} \right\} \]

is a module on the ring \( D \) which is called \( D \)-Module and the elements are dual vectors consisting of two real vectors.

The inner product and vector product of \( \vec{a} = \vec{a} + \varepsilon \vec{a}^* \in D^3 \) and \( \vec{b} = \vec{b} + \varepsilon \vec{b}^* \in D^3 \) are given by
\[ \langle \vec{a}, \vec{b} \rangle = \langle \vec{a}, \vec{b} \rangle + \varepsilon \left( \langle \vec{a}, \vec{b} \rangle + \langle \vec{a}, \vec{b} \rangle \right) \]
\[ \vec{a} \times \vec{b} = (\hat{a}_2\hat{b}_3 - \hat{a}_3\hat{b}_2, \hat{a}_3\hat{b}_1 - \hat{a}_1\hat{b}_3, \hat{a}_1\hat{b}_2 - \hat{a}_2\hat{b}_1) \]
where $\hat{a}_i = a_i + \varepsilon a^*_i$, $\hat{b}_i = b_i + \varepsilon b^*_i \in \mathbb{D}$, $1 \leq i \leq 3$.

The norm $\|\overrightarrow{a}\|$ of $\overrightarrow{a}$ is defined by

$$\|\overrightarrow{a}\| = \sqrt{\langle \overrightarrow{a}, \overrightarrow{a} \rangle} = \|\overrightarrow{a}\| + \varepsilon \langle \overrightarrow{a}, \overrightarrow{a} \rangle$$

where $a \neq 0$.

If the norm of $\overrightarrow{a}$ is 1, then $\overrightarrow{a}$ is called dual unit vector.

Let

$$\hat{\alpha} : I \subset \mathbb{D} \rightarrow \mathbb{D}^3$$

be a dual space curve with differentiable vectors $\overrightarrow{\alpha}(\lambda)$ and $\overrightarrow{\alpha}^*(\lambda)$. The dual arc-length parameter of $\overrightarrow{\alpha}(\lambda)$ is defined as

$$s = \int_{t_1}^{t} \|\overrightarrow{\alpha}(\lambda)\| \, d\lambda.$$

Now we will give dual Frenet vectors of the dual curve

$$\hat{\alpha} : I \subset \mathbb{D} \rightarrow \mathbb{D}^3$$

with the dual arc-length parameter $s$. Then

$$\frac{d\overrightarrow{\alpha}}{ds} = \frac{d\overrightarrow{\alpha}}{d\lambda} \frac{d\lambda}{ds} = \overrightarrow{T}$$

is called the unit tangent vector of $\overrightarrow{\alpha}(s)$. The norm of the vector $\frac{d\overrightarrow{T}}{ds}$ which is given by

$$\frac{d\overrightarrow{T}}{ds} = \frac{d\overrightarrow{\alpha}}{ds} \frac{d\lambda}{ds} = \frac{d^2\overrightarrow{\alpha}}{d\lambda^2} = \hat{\kappa} \overrightarrow{N}$$

is called curvature function of $\overrightarrow{\alpha}(s)$. Here $\hat{\kappa} : I \rightarrow \mathbb{D}$ is nowhere pure-dual. Then the unit principal normal vector of $\overrightarrow{\alpha}(s)$ is defined as

$$\frac{\overrightarrow{N}}{\hat{\kappa}} = \frac{d\overrightarrow{T}}{d\lambda} \frac{d\lambda}{ds}$$

The vector $\overrightarrow{B} = \overrightarrow{T} \times \overrightarrow{N}$ is called the binormal vector of $\overrightarrow{\alpha}(s)$. Also we call the vectors $\overrightarrow{T}, \overrightarrow{N}, \overrightarrow{B}$ Frenet trihedron of $\overrightarrow{\alpha}(s)$ at the point $\hat{\alpha}(s)$. The derivatives of dual Frenet vectors $\overrightarrow{T}, \hat{\kappa}, \overrightarrow{B}$ can be written in matrix form as

$$\begin{bmatrix} \overrightarrow{T}' \\ \overrightarrow{N}' \\ \overrightarrow{B}' \end{bmatrix} = \begin{bmatrix} 0 & \hat{\kappa} & 0 \\ -\hat{\kappa} & 0 & \hat{\tau} \\ 0 & -\hat{\tau} & 0 \end{bmatrix} \begin{bmatrix} \overrightarrow{T} \\ \overrightarrow{N} \\ \overrightarrow{B} \end{bmatrix}$$

which are called Frenet formulas. The function $\hat{\tau} : I \rightarrow \mathbb{D}$ such that $\frac{d\overrightarrow{B}}{d\lambda} = -\hat{\tau}\overrightarrow{N}$ is called the torsion of $\overrightarrow{\alpha}(s)$ which is nowhere pure-dual.
For a general parameter $t$ of a dual space curve $\vec{\alpha}$, the curvature and torsion of $\vec{\alpha}$ can be calculated as:

$$\hat{\kappa} = \frac{\left\| \vec{\alpha}' \times \vec{\alpha}'' \right\|}{\left\| \vec{\alpha}' \right\|^3}, \quad \hat{\tau} = \frac{\det(\vec{\alpha}', \vec{\alpha}'', \vec{\alpha}''')}{\left\| \vec{\alpha}' \times \vec{\alpha}'' \right\|}.$$

3. Bertrand curves in $\mathbb{D}^3$

In this section, we define Bertrand curves in dual space $\mathbb{D}^3$ and give characterizations and theorems for these curves.

**Definition 1.** Let $\mathbb{D}^3$ be the dual space with standard inner product $\langle \cdot, \cdot \rangle$ and $\vec{\alpha}$ and $\vec{\beta}$ be the dual space curves. If there exists a corresponding relationship between the dual space curves $\vec{\alpha}$ and $\vec{\beta}$ so that the principal normal vectors of $\vec{\alpha}$ and $\vec{\beta}$ are linear dependent to each other at the corresponding points of the dual curves, then $\vec{\alpha}$ and $\vec{\beta}$ are called Bertrand curves in $\mathbb{D}^3$.

Let the curves $\vec{\alpha}$ and $\vec{\beta}$ be Bertrand curves in $\mathbb{D}^3$, parameterized by their arc-length $s$ and $\hat{s}$, respectively. Let $\left\{ \vec{T}(s) , \vec{N}(s) , \vec{B}(s) \right\}$ indicate the unit Frenet frame along $\vec{\alpha}$ and $\left\{ \vec{\tau}(s) , \vec{\nu}(s) , \vec{\rho}(s) \right\}$ the unit Frenet frame along $\vec{\beta}$. Also $\hat{\kappa}(s) = \kappa(s) + \varepsilon \kappa^*(s)$ and $\hat{\tau}(s) = \tau(s) + \varepsilon \tau^*(s)$ are the curvature and torsion of $\vec{\alpha}$, respectively. Similarly, $\hat{\kappa}^*(s) = \kappa^*(s) + \varepsilon \kappa^{**}(s)$ and $\hat{\tau}^*(s) = \tau^*(s) + \varepsilon \tau^{**}(s)$ are the curvature and torsion of $\vec{\beta}$, respectively.

In the following theorems, we obtain the characterizations of a dual Bertrand curve.

**Theorem 1.** Let $\vec{\alpha}$ and $\vec{\beta}$ be two curves in $\mathbb{D}^3$. If $\vec{\alpha}$ and $\vec{\beta}$ are Bertrand curves, then

$$d\left(\vec{\alpha}(s), \vec{\beta}(s)\right) = \hat{c}$$

where $s \in I \subset \mathbb{D}$ and $\hat{c} \in \mathbb{D}(\text{constant})$. 
Proof. Let $\vec{\alpha}$ and $\vec{\beta}$ be Bertrand curves.

If $\vec{\alpha}$ and $\vec{\beta}$ are Bertrand curves, then we write from Figure 3.1

(3.1) $\vec{\beta}(s) = \vec{\alpha}(s) + \hat{\lambda}(s)\hat{N}(s)$

for the vectors of $\vec{\alpha}$ and $\vec{\beta}$.

If we differentiate the equation (3.1) with respect to $s$ and use the Frenet equations, we get

(3.2) $\frac{d\vec{\beta}}{ds}(s) = \left(1 - \hat{\lambda}(s)\hat{\kappa}(s)\right)\vec{T}(s) + \hat{\lambda}'(s)\hat{N}(s) + \hat{\lambda}(s)\hat{\tau}(s)\vec{B}(s)$.

If we take the inner product of the equation (3.2) with $\vec{N}(s)$ both sides,

$0 = \hat{\lambda}'(s)$

is found. Then

$\hat{\lambda}(s) = \hat{c}$

where $\hat{c} \in \mathbb{D}(\text{constant})$. If we use

$d\left(\vec{\alpha}(s), \vec{\beta}(s)\right) = \left\|\vec{\beta}(s) - \vec{\alpha}(s)\right\|$

and the equation (3.1), we obtain

$d\left(\vec{\alpha}(s), \vec{\beta}(s)\right) = \hat{c}$

where $s \in I \subset \mathbb{D}$ and $\hat{c} \in \mathbb{D}(\text{constant})$.

Namely we mean that the distance between the corresponding points of the dual Bertrand curves is constant. □
Theorem 2. Let $\vec{\alpha}$ and $\vec{\beta}$ be two curves in $\mathbb{D}^3$. If $\vec{\alpha}$ and $\vec{\beta}$ are Bertrand curves, then the dual angle between the tangent vectors at the corresponding points of the dual Bertrand curves is constant.

Proof. Let $\vec{\alpha}$ and $\vec{\beta}$ be two Bertrand curves in $\mathbb{D}^3$.

If the dual angle between the tangent vectors $\vec{T}(s)$ and $\vec{T}'(s)$ at the corresponding points of $\vec{\alpha}$ and $\vec{\beta}$ is

$$\phi = \varphi + \varepsilon\varphi^* \in \mathbb{D},$$

then we write

$$\vec{T}'(s) = \cos \phi \vec{T}(s) + \sin \phi \vec{B}(s).$$

If we differentiate the equation (3.3), we get

$$\begin{align*}
\frac{d\cos \phi}{ds} &= 0.
\end{align*}$$

So

$$\cos \phi = \text{constant}$$

is found where $\phi = \varphi + \varepsilon\varphi^* \in \mathbb{D}$. This completes the proof.

Theorem 3. Let $\vec{\alpha}$ and $\vec{\beta}$ be two curves in $\mathbb{D}^3$. If $\vec{\alpha}$ and $\vec{\beta}$ are Bertrand curves, $\hat{\kappa}(s)$ and $\hat{\tau}(s)$ are the curvature and torsion of $\vec{\alpha}$, $\hat{\kappa}'(s)$ and $\hat{\tau}'(s)$ are the curvature and torsion of $\vec{\beta}$, respectively, then

$$\hat{\lambda}\hat{\kappa}(s) + \hat{\mu}\hat{\tau}(s) = 1$$

where $\hat{\lambda}, \hat{\mu} \in \mathbb{D}$ are constant.

Proof. Let $\vec{\alpha}$ and $\vec{\beta}$ be Bertrand curves. If the dual angle between the tangent vectors $\vec{T}(s)$ and $\vec{T}'(s)$ at the corresponding points of $\vec{\alpha}$ and $\vec{\beta}$ is

$$\phi = \varphi + \varepsilon\varphi^* \in \mathbb{D},$$

then from the previous proof we have

$$\vec{T}'(s) = \cos \phi \vec{T}(s) + \sin \phi \vec{B}(s).$$

From the equation (3.2) we write

$$\begin{align*}
\frac{d\hat{s}}{ds} \vec{T}'(s) &= \left(1 - \hat{\lambda}\hat{k}(s)\right) \vec{T}(s) + \hat{\lambda}\hat{\tau}(s) \vec{B}(s).
\end{align*}$$

In above equations, if we take into account

$$\frac{d\hat{s}}{ds} = \hat{a} \text{ (constant)}$$
then we get

\[(3.8) \quad 1 - \hat{\lambda}\hat{\kappa}(s) = \cot \phi \hat{\tau}(s).\]

We can write

\[(3.9) \quad \hat{\mu} = \hat{\lambda} \cot \phi \text{ (constant)}.\]

Finally, from the equations (3.8) and (3.9), we find

\[\hat{\lambda}\hat{\kappa}(s) + \hat{\mu}\hat{\tau}(s) = 1.\]

\[\square\]

**Theorem 4.** Let \(\vec{\alpha}\) be a plane curve in \(D^3\). If \(\vec{\beta}\) and \(\vec{\gamma}\) are the involutes of \(\vec{\alpha}\), then \(\vec{\beta}\) and \(\vec{\gamma}\) are Bertrand curves in \(D^3\).

**Proof.** Let \(\vec{\beta}\) and \(\vec{\gamma}\) be the involutes of \(\vec{\alpha}\).

It can be written from [4]

\[(3.10) \quad \hat{\tau}^\circ(s) = \frac{\hat{\kappa}(s).\hat{\tau}'(s) - \hat{\tau}'(s).\hat{\tau}(s)}{\hat{\kappa}(s).\hat{c}(s) - \hat{\tau}'(s).\left(\hat{\kappa}^2(s) + \hat{\tau}^2(s)\right)}\]

where \(\hat{\kappa}(s), \hat{\tau}(s)\) and \(\hat{\kappa}'(s), \hat{\tau}'(s)\) are the curvature and torsion of \(\vec{\alpha}\) and \(\vec{\beta}\), respectively, and \(\hat{c} \in D\text{(constant)}\).

If \(\vec{\alpha}\) is a plane curve,

\[(3.11) \quad \hat{\tau}(s) = 0.\]

From the equation (3.10) and (3.11), we have

\[\hat{\tau}^\circ(s) = 0.\]

So \(\vec{\beta}\) is a plane curve. Similarly, \(\vec{\gamma}\) is a plane curve. Consequently, the principal normal vectors of \(\vec{\beta}\) and \(\vec{\gamma}\) are linear dependent to each other at the corresponding
points of the dual curves. In that case $\vec{\beta}$ and $\vec{\gamma}$ curves are Bertrand curves in $\mathbb{D}^3$. □

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