Duality and confinement in D=3 models driven by condensation of topological defects

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We study the interplay of duality and confinement in Maxwell-like three-dimensional models induced by the condensation of topological defects driven by quantum fluctuations. To this end we check for the confinement phenomenon, in both sides of the duality, using the static quantum potential as a testing ground. Our calculations are done within the framework of the gauge-invariant but path-dependent variables formalism which are alternative to the Wilson loop approach. Our results show that the interaction energy contains a linear term leading to the confinement of static probe charges.

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I. INTRODUCTION

This work is aimed at studying the duality symmetry for certain 3D models coupled to sources of different dimensions that eventually condense due to quantum fluctuations, using the Quevedo- Trugenberger phenomenology [1] to the Julia-Toulouse mechanism [2]. In general studies in Field Theory, the presence of the duality between two models is verified through their equations of motion and the algebra of the observables. However, since quantum fluctuations may eventually drive the condensation of topological defects destroying the duality, we should be able to look for a distinct property to be used as a point of proof in the condensed phase. In this paper we propose to use the effective potential between two static charges as such a testing ground for the existence of duality, after quantum fluctuations drive the condensation of topological defects. In principle one could check for this proposal studying the interplay between confinement and duality in an $U(1)$ gauge theory in $D$ spacetime dimensions for Maxwell-like theories of totally anti-symmetric tensors of arbitrary rank. However, due to technical difficulties in the computation of the effective static potential in arbitrary $D$ dimensions we shall restrict ourselves to the case $D = 3$.

Based on the common knowledge coming from the continuum abelian gauge theory, the assertion that the such a theory has a confining phase may sound strange. In fact the existence of a phase structure for the continuum abelian $U(1)$ gauge theory was obtained by including the effects due to compactness of $U(1)$ group that dramatically change the infrared properties of the model. These results, first found by Polyakov [3], have been confirmed by many distinct techniques basically due to the contribution of the vortices into the partition function of the theory. The condensation of these topological defects then lead to a structural change of the conventional vacuum of the theory into a dual superconductor vacuum. An interesting approach to this problem has recently been proposed by Kondo who derived the effective potential from a partition function that includes the contribution of all topologically non-trivial sectors of the theory [4].

In a previous paper [5] we have approached the problem in a phenomenological way using the Julia-Toulouse mechanism [2], as proposed by Quevedo and Trugenberger [1], that considers the condensation of topological defects. This study was undertaken for theories of compact anti-symmetric gauge tensors of arbitrary ranks in $D$ spacetime dimensions that appear as low-energy effective field theories of strings. More specifically, using the Quevedo-Trugenberger phenomenology [1] we studied the low-energy field theory of a pair of compact massless anti-symmetric tensor fields, say $A_p$ and $B_q$ with $p + q + 2 = D$, coupled magnetically and electrically, respectively, to a large set of $(q-1)$-branes, characterized by charge $e$ and a Chern-Kernel $\Lambda_{p+1}$ [6], that eventually condense. It has been argued that the effective theory that results displays the confinement property. The results of [5] show that the phenomenological action proposed in [1] incorporates automatically the contribution of the condensate of topological defects to the vacuum of the model or, alternatively, the non-trivial topological sectors as in [4].
II. DUALITY AND CONFINEMENT

A. Duality in the Dilute Phase

Consider for a moment a dual pair of massless antisymmetric tensors, \( A_p \) and \( \tilde{A}_q \), that represent the potentials for a pair of Maxwell-like theories, coupled to closed charged branes of dimension \((p - 1)\) and \((q - 1)\). In this dilute phase, duality is manifest in the following sense. From the point of view of the \( A_p \) tensor, the \((p - 1)\)-brane is electric while the \((q - 1)\)-brane is magnetic. This situation, is illustrated by the following diagram

\[
\begin{array}{cc}
(q - 1)\text{-brane} & (p - 1)\text{-brane} \\
MC & A_p & EC
\end{array}
\]

and is formally described by the following action

\[
S_A = \int \frac{1}{2} \frac{(-1)^p}{(p + 1)!} [F_{p+1}(A_p) - gA_{p+1}]^2 + e A_p J^p(\Omega) .
\]

(2)

where \( A_{p+1} \) and \( \Omega_{q+1} \) are the Chern-Kernel of the \((q - 1)\)-brane and the \((p - 1)\)-brane, respectively. For the \( D = 3 \) case where \( p = 1 \) and \( q = 0 \), Eq.(2) represents the action of a vector field coupled minimally to an electric charge and non-minimally to a magnetic instanton. This system displays electric and magnetic symmetries

\[
\begin{align*}
\delta_E A_\mu &= \partial_\mu \xi ; \quad \delta_E \Omega_\mu = \partial_\mu \xi ; \quad \text{electric} \\
\delta_M A_\mu &= g \chi_\mu ; \quad \delta_M \Lambda_{\mu\nu} = \partial_{[\mu} \chi_{\nu]} ; \quad \text{magnetic}
\end{align*}
\]

(3)

with \( \xi \) and \( \chi_\mu \) being the corresponding gauge functions. The new magnetic symmetry [7, 8] appears due to the presence of magnetic sources and is manifest as the invariance of the “generalized” field tensor \( H_{p+1} = [F_{p+1}(A_p) - gA_{p+1}] \). The conventional (electric) gauge symmetry, manifest by \( F_{\mu\nu}(A) \) keeps the minimal coupling with the electric pole invariant thanks to the current conservation condition coming from the fact that we are considering closed branes. This
second term is also invariant under the magnetic symmetry if the charges of the branes satisfy the Dirac quantization condition,

\[ eg = 2\pi n ; \quad n \in \mathbb{Z}. \tag{4} \]

From the point of view of the dual field \(\tilde{A}_q\), the electric and magnetic couplings with the branes have reversed character, as depicted by the diagram,

\[ \begin{array}{c}
\text{EC} \\
(q - 1) - \text{brane} \\
\text{MC}
\end{array} \quad \begin{array}{c}
\tilde{A}_q \\
(p - 1) - \text{brane}
\end{array} \]

whose action reads,

\[ S_{\tilde{A}} = \int \frac{1}{2} \left[ (-1)^q \left[ \tilde{F}_{q+1}(\tilde{A}_q) - e \Omega_{q+1} \right] \right]^2 + g \tilde{A}_q J^q(\Lambda). \tag{6} \]

This dual action is invariant under a corresponding pair of electric and magnetic gauge transformation. It is known that upon duality transformation from one picture to the other, a brane-brane term like,

\[ S_{\text{brane-brane}} = eg \int \Lambda_{p+1} e^{p+1,q+1} \Omega_{q+1} \tag{7} \]

will also be induced. Such a term however disappear from the partition function thanks to the charge quantization condition (4) and the fact that the Chern-kernels are integer functions, rendering the above integral as an integer that represents the intersection number of the two branes. It is worth noticing at this juncture that such a duality equivalence \((\ast_0)\), illustrated as,

\[ \begin{array}{c}
\text{EC} \\
(q - 1) - \text{brane} \\
\ast_0 \\
\text{MC}
\end{array} \quad \begin{array}{c}
\tilde{A}_q \\
(p - 1) - \text{brane}
\end{array} \]

is not the conventional textbook duality since the potential tensors involved are not, in general, of the same rank and, correspondingly, the branes are not of the same dimensions. For the particular \(D = 3\) case one tensor is of rank-1 (vector) and the other is of rank-0 (scalar) while the sources are a monopole and an instanton. The classification of these topological defects as electric or magnetic will depend on which point of view is taken, as discussed above. When both objects are of the same dimension (and the tensors are of the same rank) we have self-duality. This is the usual electric-magnetic duality discussed in the Maxwell theory in \(D=4\), for instance, being straightforward then that the condensation of either the electric and the magnetic sources leads to identical physical results.

However, when the topological defects are of different dimensions duality has to be understood in the more general sense discussed above. Therefore without self-duality it seems doubtful that the eventual condensation of either branes lead to the very same physical picture. In fact, as we will show clearly in section III by the computation of the effective static potential, the phenomenon of confinement occurs irrespective of which brane condense. In particular, we will show that for \(D = 3\) two electric instantons are confined when immersed in the condensate of magnetic instantons while the electric charges experience the same result when immersed in the condensate of magnetic monopoles.

B. Duality in the Condensed Phase

In this subsection we use the Quevedo and Trugenberger phenomenology to study the interplay between duality and confinement. To this end we consider the low-energy field theory of a pair of independent massless anti-symmetric tensor fields, say \(A_p\) and \(B_q\) with \(p + q + 2 = D\), coupled electrically (magnetically) and magnetically (electrically) to a large set of \((q - 1)\)-branes, characterized by charge \(e\) and a Chern-Kernel \(\Lambda_{p+1}\) (resp., \((p - 1)\)-branes with charge
g and Chern-kernel $\Omega_{q+1}$ [6], that eventually condense. It will be explicitly shown, for the D=3 example of vector and scalar fields coupled to instantons and monopoles, that the induced effective theories display the confinement property by providing a clear-cut derivation of the effective potential for a pair of static, very massive point probes immersed in the condensate.

Firstly consider that the fields $B_q$ and $A_p$ are electrically (EC) and magnetically (MC) coupled to a $e$-charged, $(q-1)$-brane, respectively, according to the following diagram

$$S_e = \int \frac{1}{2} \frac{(-1)^q}{(q+1)!} [H_{q+1}(B_q)]^2 + e B_q J^q(\Lambda) + \frac{1}{2} \frac{(-1)^p}{(p+1)!} [F_{p+1}(A_p) - e\Lambda_{p+1}]^2.$$  

Upon the condensation phenomenon, the Chern-Kernel $\Lambda_{p+1}$ will become the new massive mode of the effective theory.

Next, we consider the dual picture where the $B_q$ and $A_p$ fields are magnetically and electrically coupled to a $g$-charged, $(p-1)$-brane, respectively, according to the following diagram

$$S_g = \int \frac{1}{2} \frac{(-1)^p}{(p+1)!} [F_{p+1}(A_p)]^2 + g A_p J^p(\Omega) + \frac{1}{2} \frac{(-1)^q}{(q+1)!} [H_{q+1}(B_q) - g\Omega_{q+1}]^2$$  

in which case the Chern-Kernel $\Omega_{q+1}$ will become the new massive mode of the effective theory after the condensation.

Our compact notation here goes as follows. The field strengths are defined as $F_{p+1}(A_p) = F_{µ_1,µ_2,\ldots,µ_{p+1}} = \partial[µ_1 A_{µ_2,\ldots,µ_{p+1}}]$ and $H_{q+1}(B_q) = H_{µ_1,µ_2,\ldots,µ_{q+1}} = \partial[µ_1 B_{µ_2,\ldots,µ_{q+1}}]$, while the Chern-kernels $\Lambda_{p+1} = \Lambda_{µ_1,\ldots,µ_{p+1}}$ and $\Omega_{q+1} = \Omega_{µ_1,\ldots,µ_{q+1}}$ are totally anti-symmetric objects of rank $(p+1)$ and $(q+1)$, respectively. The conserved currents $J^q(\Lambda)$ and $J^p(\Omega)$ are given by delta-functions over the world-volume of the $(q-1)$-brane and $(p-1)$-brane [9]. These conserved currents can be rewritten in terms of the kernel $\Lambda_{p+1}$ and $\Omega_{q+1}$ as

$$J^q(\Lambda) = \frac{1}{(p+1)!} \epsilon^{q,α,p+1}_β \partial_α \Lambda_{p+1},$$

$$J^p(\Omega) = \frac{1}{(q+1)!} \epsilon^{p,α,q+1}_β \partial_α \Omega_{q+1},$$

and $\epsilon^{q,α,p+1}_β = \epsilon^{µ_1,\ldots,µ_q,α,ν_1,\ldots,ν_{p+1}}$.

We review next how to constructed the effective interacting action, in the condensed phase, between the anti-symmetric tensor field $B_q$ ($A_p$) and the degrees of freedom of the condensate $\Lambda_{p+1}$ ($\Omega_{q+1}$). To this end, after the condensate is integrated out, we compute the effective quantum potential for a pair of static probe living inside the condensate, within the framework of the gauge-invariant but path-dependent variables formalism. This will disclose the dependence of the confinement properties with the condensation parameters coming from the Julia–Toulouse mechanism.
We are now ready to discuss the consequences of the Julia-Toulouse mechanism over the action (10). Since the manipulations are quite general, the result for the dual action (12) follows analogously. The initial theory, before condensation, displays two independent fields coupled to a \((q-1)\)-brane. The nature of the two couplings are however different. The \(A_p\) tensor, that is magnetically coupled to the brane, will then be absorbed by the condensate after phase transition. On the other hand, the electric coupling, displayed by the \(B_q\) tensor, becomes a \(B \wedge F(\Lambda)\) topological term after condensation. Indeed, the distinctive feature is that after condensation, the Chern-Kernel \(\Lambda_{p+1}\) is elevated to the condition of propagating field. The new degree of freedom absorbs the degrees of freedom of the tensor \(A_p\) this way completing its longitudinal sector. The new mode is therefore explicitly massive. Since \(A_p \rightarrow \Lambda_{p+1}\) there is a change of rank with dramatic consequences. The last term in (10), displaying the magnetic coupling between the field-tensor \(F_{p+1}(A_p)\) and the \((q-1)\)-brane, becomes the mass term for the new effective theory in terms of the tensor field \(\Lambda_{p+1}\) and a new dynamical term is induced by the condensation. It is consequential that the minimal coupling of the \(B_q\) tensor becomes responsible for another contribution for the mass, this time of topological nature. Indeed the second term (10) becomes an interacting \("BF\)-term" between the remaining propagating modes, inducing the appearance of topological mass, in addition to the induced condensate mass. The final result reads

\[
S_{\text{cond}} = \int \frac{(-1)^q}{2(q+1)!} [H_{q+1}(B_q)]^2 + e B_q \epsilon^{\alpha\beta\gamma} F_{\alpha\beta}(B_q) \partial_\gamma \Lambda_{p+1} + \frac{(-1)^{p+1}}{2(p+2)!} [F_{p+2}(\Lambda_{p+1})]^2 - \frac{(-1)^{p+1}(p+1)!}{2} m^2 \Lambda_{p+1}^2 \tag{14}
\]

where \(m = \Theta/e\) where \(\Theta\) represents the condensate density.

There has been a drastic change in the physical scenario. To see this we need first to obtain an effective action for the \(B_q\) tensor that includes the effects of the condensate. To this end on one integrates out the condensate field \(\Lambda_{p+1}\) to obtain, after some algebra [5]

\[
S_{\text{eff}}^{(e)} = \int \frac{(-1)^q}{2(q+1)!} H_{q+1}(B_q) \left(1 + \frac{\epsilon^2}{\Box + m^2}\right) H_{q+1}(B_q) \tag{15}
\]

Analogously, for the condensation of the \(g\)-charged brane, we obtain from (12) the following effective action,

\[
S_{\text{eff}}^{(g)} = \int \frac{(-1)^{p+1}}{2(p+1)!} F_{p+1}(A_p) \left(1 + \frac{g^2}{\Box + \mu^2}\right) F_{p+1}(A_p), \tag{16}
\]

where \(\mu = \Theta/g\) is the mass parameter for this condensate.

In the next section we shall examine the duality versus confinement issue. To this end we shall consider a specific example involving a Maxwell field coupled electrically and magnetically to a monopole and an instanton while the scalar fields couples electrically and magnetically to the instanton and the monopole. After the condensation of the monopole we end up with two Maxwell fields (the massive one being the condensate) coupled topologically to each other. On the other hand after the condensation of the instanton we end up with a massless scalar field coupled to a massive Kalb-Ramond potential carrying the degrees of freedom for the condensate.

### III. INTERACTION ENERGY

Our aim in this Section is to calculate the interaction energy for the effective theories computed above, Eqs.(15) and (16), between appropriate external probe for each specific model. For \(D = 3\) the effective actions that naturally incorporate the contents of the dual superconductor effects via Julia-Toulouse mechanism, for \(p = 1\) and \(q = 0\), are

\[
S_{\text{eff}}^{(pole)} = \int -\frac{1}{4} F_{\mu\nu} \left(1 + \frac{\epsilon^2}{\Delta^2 + m^2}\right) F^{\mu\nu} - A_\mu J^\mu \tag{17}
\]

and

\[
S_{\text{eff}}^{(inst)} = \int -\frac{1}{2} \partial_\mu \phi \left(1 + \frac{g^2}{\Delta^2 + \mu^2}\right) \partial^\mu \phi - \phi J \tag{18}
\]

where the external current will be chosen so as to represent the presence of the two point probes.

The technique for computing the effective potential, which is distinguished by particular attention to gauge invariance, has been developed in [10]. However due to the absence of gauge symmetry in one end of the duality, an extension of that technique will be proposed to deal with this issue here. Other then that our notation is defined in the Appendix-A where the reader will also find an alternative derivation of such results, see Eq.(A25). We start with
the analysis of the theory (17) that display the coupling of a vector field to a pole. To this end consider the potential [10],

\[ V \equiv q (A_0(0) - A_0(y)), \tag{19} \]

where the physical scalar potential is given by

\[ A_0(x, y) = \int_0^1 d\lambda x^i E_i(\lambda x), \tag{20} \]

and \( i = 1, 2 \). This follows from the vector gauge-invariant field expression [11]:

\[ A_\mu(x) \equiv A_\mu(x) + \partial_\mu \left( - \int^{x} d\xi A_\mu(\xi) \right), \tag{21} \]

where the line integral is along a space-like path from the point \( \xi \) to \( x \), on a fixed time slice, see Eq.(A6). The gauge-invariant variables (21) commute with the sole first constraint (Gauss' law), confirming that these fields are physical variables [12]. Note that Gauss' law for the present theory reads

\[ \partial_i \Pi^i_L = J^0, \tag{22} \]

where \( \Pi^i_L \) refers to the longitudinal part of \( \Pi^i \equiv \left( 1 - \frac{m^2}{\sqrt{1 - t^2}} \right) E^i \), and \( E^i \) is the electric field. For \( J^0(t, x) = q\delta^{(2)}(x) \) the electric field is given by

\[ E^i = q \left( 1 - \frac{m^2}{\sqrt{2}} \right) \partial^i G(x), \tag{23} \]

where

\[ G(x) = \frac{1}{2\pi} K_0(M|x|); \quad M^2 \equiv m^2 + e^2, \tag{24} \]

is the Green function for the Proca operator in \( D = 3 \). As a consequence, Eq.(20) becomes

\[ A_0(t, x) = q \left( 1 - \frac{m^2}{\sqrt{2}} \right) G(x), \tag{25} \]

after subtraction of self-energy terms. Our next task is the computation of the second term on the right-hand side of Eq. (25). We will make use of the Green function, defined in Eq.(A17). Using this in (25) we then obtain

\[ \frac{G}{\nabla^2} = - \frac{1}{8\pi M^2} (I_1 + I_2), \tag{26} \]

where the \( I_1 \) and \( I_2 \) terms are given by

\[ I_1 = \int_{-\infty}^{\infty} dt \frac{1}{\sqrt{1 + t^2}} \frac{1}{t^2} \frac{1}{1 - \gamma Mrt}, \tag{27} \]

and

\[ I_1 = \int_{-\infty}^{\infty} dt \frac{1}{\sqrt{1 + t^2}} \frac{1}{t^2} \frac{1}{1 + \gamma Mrt}, \tag{28} \]

here \( |x| \equiv r \). The integrals (27) and (28) have been explicitly computed in Appendix-A. As a consequence, Eq. (26) reduces to

\[ \frac{G}{\nabla^2} = \frac{r}{4M}. \tag{29} \]
Finally, making use of (29) in (25), the potential for a pair of point-like opposite charges \( q \) located at \( 0 \) and \( L \), becomes

\[
V \equiv q (A_0 (0) - A_0 (L)) = -\frac{q^2}{2\pi} K_0 (ML) + \frac{q^2 m^2}{4M} L, \tag{30}
\]

where \( |L| \equiv L \). This potential displays the conventional screening part, encoded in the Bessel function, and the linear confining potential. As expected, the confinement disappears in the dilute phase \( (m \to 0) \).

Next we perform the analysis of the theory (18) that display the coupling of a scalar field to an instanton. From (18) we obtain the following equation of motion

\[
\left( \Box + g^2 + \mu^2 \right) \Box \phi = J. \tag{31}
\]

Next, we restrict ourselves to static scalar fields which allowed us to replace \( \Box \phi = -\nabla^2 \phi \). It implies that (31) becomes,

\[
\frac{\nabla^2 - M^2}{\nabla^2 - \mu^2} \phi = -\frac{J}{\nabla^2} \tag{32}
\]

where now \( M^2 = \mu^2 + g^2 \). From this we see that

\[
\phi = \left( 1 - \frac{\mu^2}{\nabla^2} \right) \left( -\frac{J}{\nabla^2 - M^2} \right). \tag{33}
\]

As before we note that for \( J (t, x) = q \delta^{(2)} (x) \), the scalar field is given by

\[
\phi = q \left( 1 - \frac{\mu^2}{\nabla^2} \right) G (x), \tag{34}
\]

where \( G (x) \) is the massive Green function defined before, Eq.(24), with \( M \to M \). It must now be observed that Eq.(34) is identical to Eq.(25), leading to the same effective potential also for the scalar field case, confirming that the claimed duality between the two models persists after the condensation of the topological defects.

**IV. FINAL REMARKS**

We have used the confinement as a criterium to study duality for a pair of antisymmetric tensors coupled to topological defects that eventually condense. To this end we have computed the effective static potential for a general effective theory in the condensed phase using the Quevedo-Trugenberger formalism. This result was successfully used as a testing ground for duality using the interaction energy between two point-like probes but, for technical reasons, we had to settle for the specific case of \( D = 3 \) only. According to the Quevedo-Trugenberger phenomenology, the condensation mechanism for a couple of massless antisymmetric tensors is responsible for the appearance of mass and the jump of rank in the magnetic sector while the electric sector becomes a BF coupling with the condensate. The condensate absorbs and replaces one of the tensors and becomes the new massive propagating mode but does not couple directly to the probe charges. The effects of the condensation are however felt through the BF coupling with the condensate. It is therefore not surprising that they become manifest in the interaction energy for the effective theory. Our results show that in both sides of the duality the interaction energy in fact contains a linear confining term. This is an important result showing that the effective potential is a key tool to corroborate the existence of duality, which, otherwise are only suggested by other, very formal approaches. Extension of this approach to check for duality in higher dimensions using the confinement criterium is presently under investigation by the authors.

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APPENDIX A: COMPUTATION OF THE CONFINING POTENTIAL

Our aim in this Appendix is to recover the confining potential for the effective theory (17) computed between external probe sources. To do this, we will compute the expectation value of the energy operator $H$ in the physical state $|\Phi\rangle$ describing the sources, which we will denote by $\langle H \rangle_{\Phi}$. Our starting point is the effective Lagrangian Eq. (17):

$$\mathcal{L}_{eff} = -\frac{1}{4} F_{\mu\nu} \left( 1 + \frac{m^2}{\Box + e^2} \right) F^{\mu\nu} - A_0 J^0,$$  \hspace{1cm} (A1)

where $J^0$ is an external current.

Once this is done, the canonical quantization of this theory from the Hamiltonian point of view follows straightforwardly. The canonical momenta read $\Pi^\mu = - \left( 1 + \frac{m^2}{\Box + e^2} \right) F^{0\mu}$ with the only non-vanishing canonical Poisson brackets being

$$\{ A_\mu (t, x) , \Pi^\nu (t, y) \} = \delta^\nu_\mu \delta (x - y).$$  \hspace{1cm} (A2)

Since $\Pi_0$ vanishes we have the usual primary constraint $\Pi_0 = 0$, and $\Pi^i = \left( 1 + \frac{m^2}{\Box + e^2} \right) F^{i0}$. The canonical Hamiltonian is thus

$$H_C = \int d^3 x \left\{ -\frac{1}{2} \Pi^0 \left( 1 + \frac{m^2}{\Box + e^2} \right)^{-1} \Pi^0 + \Pi^i \partial_i A_0 + \frac{1}{4} F_{ij} \left( 1 + \frac{m^2}{\Box + e^2} \right) F^{ij} + A_0 J^0 \right\}.$$  \hspace{1cm} (A3)

Time conservation of the primary constraint $\Pi_0$ leads to the secondary Gauss-law constraint

$$\Gamma_1(x) \equiv \partial_i \Pi^i - J^0 = 0.$$  \hspace{1cm} (A4)

The preservation of $\Gamma_1$ for all times does not give rise to any further constraints. The theory is thus seen to possess only two constraints, which are first class, therefore the theory described by (A1) is a gauge-invariant one. The extended Hamiltonian that generates translations in time then reads

$$H = H_C + \int d^2 x \left( c_0 (x) \Pi_0 (x) + c_1 (x) \Gamma_1 (x) \right),$$

where $c_0 (x)$ and $c_1 (x)$ are the Lagrange multiplier fields. Moreover, it is straightforward to see that $\dot{A}_0 (x) = [A_0 (x), H] = c_0 (x)$, which is an arbitrary function. Since $\Pi_0 = 0$ always, neither $A^0$ nor $\Pi^0$ are of interest in describing the system and may be discarded from the theory. Then, the Hamiltonian takes the form

$$H = \int d^2 x \left\{ -\frac{1}{2} \Pi^i \left( 1 + \frac{m^2}{\Box + e^2} \right)^{-1} \Pi^i + \frac{1}{4} F_{ij} \left( 1 + \frac{m^2}{\Box + e^2} \right) F^{ij} + c (x) \left( \partial_i \Pi^i - J^0 \right) \right\},$$  \hspace{1cm} (A5)

where $c(x) = c_1 (x) - A_0 (x)$.

The quantization of the theory requires the removal of nonphysical variables, which is done by imposing a gauge condition such that the full set of constraints becomes second class. A convenient choice is found to be [10]

$$\Gamma_2 (x) \equiv \int_{C_{\xi}} dz^\nu A_\nu (z) \equiv \int_{0}^{1} d\lambda x^i A_i (\lambda x) = 0,$$  \hspace{1cm} (A6)

where $\lambda (0 \leq \lambda \leq 1)$ is the parameter describing the spacelike straight path $x^i = \xi^i + \lambda (x - \xi)^i$, and $\xi$ is a fixed point (reference point). There is no essential loss of generality if we restrict our considerations to $\xi^i = 0$. In this case, the only non-vanishing equal-time Dirac bracket is

$$\{ A_i (x) , \Pi^j (y) \}^* = \delta_i^j \delta (x - y) - \partial_i^x \int_{0}^{1} d\lambda x^j \delta (\lambda x - y).$$  \hspace{1cm} (A7)

In passing we recall that the transition to quantum theory is made by the replacement of the Dirac brackets by the operator commutation relations according to

$$\{ A, B \}^* \rightarrow (-i) [A, B].$$  \hspace{1cm} (A8)
We now turn to the problem of obtaining the interaction energy between point-like sources in the model under consideration. The state \( \Phi \rangle \) representing the sources is obtained by operating over the vacuum with creation/annihilation operators. We want to stress that, by construction, such states are gauge invariant. In the case at hand we consider the gauge-invariant stringy \( \{ \nabla (y) \Psi (y') \rangle \), where a fermion is localized at \( y' \) and an anti-fermion at \( y \) as follows [12],

\[
| \Phi \rangle \equiv \{ \nabla (y) \Psi (y') \rangle = \bar{\psi} (y) \exp \left( i q \int_{y'} d z_i A_i (z) \right) \psi (y') | 0 \rangle , \tag{A9}
\]

where \( | 0 \rangle \) is the physical vacuum state and the line integral appearing in the above expression is along a space-like path starting at \( y' \) and ending \( y \), on a fixed time slice. It is worth noting here that the strings between fermions have been introduced in order to have a gauge-invariant function \( \langle \Phi | \). In other terms, each of these states represents a fermion-antifermion pair surrounded by a cloud of gauge fields sufficient to maintain gauge invariance. As we have already indicated, the fermions are taken to be infinitely massive (static).

From our above discussion, we see that \( \langle H \rangle_\Phi \) reads

\[
\langle H \rangle_\Phi = \langle \Phi | \int d^2 x \left\{ - \frac{1}{2} \Pi_i \left( 1 + \frac{m^2}{\Box + e^2} \right)^{-1} \Pi^i + \frac{1}{4} F_{ij} \left( 1 + \frac{m^2}{\Box + e^2} \right) F^{ij} \right\} | \Phi \rangle . \tag{A10}
\]

Consequently, we can write Eq.(A10) as

\[
\langle H \rangle_\Phi = \langle \Phi | \int d^2 x \left\{ - \frac{1}{2} \Pi_i \left( 1 - \frac{m^2}{\nabla^2 - e^2} \right)^{-1} \Pi^i \right\} | \Phi \rangle , \tag{A11}
\]

where, in this static case, \( \Box = - \nabla^2 \). Observe that when \( m = 0 \) we obtain the pure Maxwell theory, as already mentioned. From now on we will suppose \( m \neq 0 \).

Next, from our above Hamiltonian analysis, we note that

\[
\Pi_i (x) \{ \nabla (y) \Psi (y') \} = \bar{\Psi} (y) \Psi (y') \Pi_i (x) | 0 \rangle + q \int_{y'} d z_i \delta^{(2)} (z - x) | \Phi \rangle . \tag{A12}
\]

As a consequence, Eq.(A11) becomes

\[
\langle H \rangle_\Phi = \langle H \rangle_0 + V^{(1)} + V^{(2)} , \tag{A13}
\]

where \( \langle H \rangle_0 = \langle 0 | H | 0 \rangle \). The \( V^{(1)} \) and \( V^{(2)} \) terms are given by:

\[
V^{(1)} = - \frac{q^2}{2} \int d^2 x \int_y^{y'} d z_i \delta^{(2)} (x - z') \frac{1}{\nabla^2 - M^2} d^2 x \int_y^{y'} d z_i \delta^{(2)} (x - z) , \tag{A14}
\]

and

\[
V^{(2)} = \frac{q^2 m^2}{2} \int d^2 x \int_y^{y'} d z_i \delta^{(2)} (x - z') \frac{1}{\nabla^2 - M^2} d^2 x \int_y^{y'} d z_i \delta^{(2)} (x - z) , \tag{A15}
\]

where \( M^2 \equiv m^2 + e^2 \) and the integrals over \( z_i \) and \( z_i' \) are zero except on the contour of integration.

The \( V^{(1)} \) term may look peculiar, but it is just the familiar Bessel interaction plus self-energy terms. In effect, expression (A14) can also be written as

\[
V^{(1)} = \frac{q^2}{2} \int_y^{y'} d z_i \partial_i \int_y^{y'} d z_i \partial_i G (z', z) , \tag{A16}
\]

where \( G \) is the Green function

\[
G (z', z) = \frac{1}{2 \pi} K_0 (M | z' - z |) . \tag{A17}
\]
Employing Eq.(A17) and remembering that the integrals over \( z_i \) and \( z'_i \) are zero except on the contour of integration, expression (A16) reduces to the familiar Bessel interaction after subtracting the self-energy terms, that is,

\[
V^{(1)} = -\frac{q^2}{2\pi} K_0 (M|y - y'|). \tag{A18}
\]

We now turn our attention to the calculation of the \( V^{(2)} \) term, which is given by

\[
V^{(2)} = \frac{q^2 m^2}{2} \int_y^{y'} dz^i \int_y^{y'} dz^i G(z', z). \tag{A19}
\]

It is appropriate to observe here that the above term is similar to the one found for the system consisting of a gauge field interacting with an external background[13]. Notwithstanding, in order to put our discussion into context it is useful to summarize the relevant aspects of the calculation described previously [13]. In effect, as was explained in Ref. [13], by using the integral representation of the Bessel function

\[
K_0 (x) = \int_0^\infty \cos(x \sinh t)dt = \int_0^\infty \frac{\cos(xt)}{\sqrt{t^2 + 1}} dt, \tag{A20}
\]

where \( x > 0 \), expression (A19) can also be written as

\[
V^{(2)} = \frac{q^2 m^2}{2\pi M^2} \int_0^\infty \frac{dt}{t^2 + \frac{\varepsilon^2}{\sqrt{t^2 + \varepsilon^2}}} (1 - \cos(MLt)) , \tag{A21}
\]

where \( L \equiv |y - y'| \). Now let us calculate integral (A21). For this purpose we introduce a new auxiliary parameter \( \varepsilon \) by making in the denominator of integral (A21) the substitution \( t^2 \to t^2 + \varepsilon^2 \). Thus it follows that

\[
V^{(2)} \equiv \lim_{\varepsilon \to 0} V^{(2)} = \lim_{\varepsilon \to 0} \frac{q^2 m^2}{2\pi M^2} \int_0^\infty \frac{dt}{t^2 + \frac{\varepsilon^2}{\sqrt{t^2 + \varepsilon^2}}} (1 - \cos(MLt)) . \tag{A22}
\]

A direct computation on the \( t \)-complex plane yields

\[
\tilde{V}^{(2)} = \frac{q^2 m^2}{4M^2} \left( \frac{1 - e^{-ML\varepsilon}}{\varepsilon} \right) \frac{1}{\sqrt{1 - \varepsilon^2}}. \tag{A23}
\]

Taking the limit \( \varepsilon \to 0 \), expression (A23) then becomes

\[
V^{(2)} = \frac{q^2 m^2}{4M}|y - y'|. \tag{A24}
\]

From Eqs.(A18) and (A24), the corresponding static potential for two opposite charges located at \( y \) and \( y' \) may be written as

\[
V(L) = -\frac{q^2}{2\pi} K_0 (ML) + \frac{q^2 m^2}{4M} L, \tag{A25}
\]

where \( L \equiv |y - y'| \).

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