Recovering a Gaussian distribution from its minimum.

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Abstract

Let \( X = (X_1, X_2, X_3) \) be a Gaussian random vector such that \( X \sim N(0, \Sigma) \). We consider the problem of determining the matrix \( \Sigma \), up to permutation, based on the knowledge of the distribution of \( X_{\text{min}} := \min(X_1, X_2, X_3) \). Particularly, we establish a connection between this identification problem and a geometric identification problem in the context of the theory of the circular radon transform.

Keywords: Identifiability, circular radon transform, competing risk.

1 Introduction

A Gaussian random vector is a vector valued random variable \( X = (X_1, \ldots, X_n) \) whose components \( X_1, \ldots, X_n \) are jointly Gaussian. The mean vector \( \mu \), of a Gaussian random vector \( X = (X_1, \ldots, X_n) \) is defined by \( \mu = (\mu_1, \ldots, \mu_n)^T \), where \( \mu_i = E(X_i) \) for \( i = 1, \ldots, n \). Moreover, the correlation matrix \( \Sigma \) of \( X \) is an \( n \times n \) matrix defined by \( \Sigma := (\Sigma_{ij}) \), where \( \Sigma_{ij} = \text{Cov}(X_i, X_j) \) for \( 1 \leq i, j \leq n \). The statement “\( X \) is a Gaussian random vector with mean vector \( \mu \) and correlation matrix \( \Sigma \)” will be compactly written as \( X \sim N(\mu, \Sigma) \).

Suppose that \( X = (X_1, \ldots, X_n) \sim N(0, \Sigma) \) and let \( X_{\text{min}} \) be the random variable defined by \( X_{\text{min}} = \min(X_1, \ldots, X_n) \). It is clear that the knowledge of \( \Sigma \) allow us to determine the distribution of \( X_{\text{min}} \). On the other hand, a natural question arises: Does the distribution of \( X_{\text{min}} \) determine the matrix \( \Sigma \)? In other words, is it possible to recover the matrix \( \Sigma \) from the distribution of \( X_{\text{min}} \)? The purpose of this paper is to solve this problem affirmatively, under the assumption that \( \mathbf{1}^T \Sigma^{-1} > 0 \), which in particular covers the case where the correlations are negative [9,10].

This problem seems to have originated from an econometrics supply-demand problem posed to Anderson and Ghurye [3]. Additionally, a similar set of problems were previously known in the context of competing and complimentary risks [4,5,11]. These kind of problems of identification have been the subject of interest to many authors, including [3,6,8,10,13]. Particularly, in the same setting established above, the authors in [4] studied this recovery problem under the hypothesis \( \rho_{ij} \sigma_i < \sigma_j \). In [8,11] it was studied the case of common correlations. And, in [9,10] it was studied the case of non-negative correlations. The novelty of our approach consists in tackling the problem via a geometric approach, reducing it to a recovering problem in the context of circular radon transform. This allows us to significantly extend previous results. Moreover, from this approach much intuition

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is gained regarding the ‘backstage’ geometric difficulty implicit in this and similar recovery problems, particularly improving previous results established in [9] and [10].

This paper is organized as follows: Section 2 is devoted to develop some preliminary geometric notions. Section 3 contains the proof of the main result of the article, while Section 4 presents in detail the proof of Lemma 4 which is a key component in the proof of the main theorem.

2 Preliminaries

2.1 Generalized square roots

Definition 1 We say that $N$ is a generalized square root of a matrix $M$ (or, for the purpose of this article, just a square root of $M$) if $NN^t = M$.

Now, if $M$ is an $n \times n$ positive definite symmetric matrix, it has an eigendecomposition $PDP^t$, where $P$ is a unitary complex matrix and $D$ is a real diagonal matrix whose main diagonal contains the corresponding positive eigenvalues. Thus, the matrix $N = PD^{1/2}$ is a generalized square root of $M$, (where $D^{1/2}$ is a square root of $D$ in the ‘usual’ sense).

Note that, given an orthogonal matrix $O$ and any square root $N$ of $M$, the matrix $NO$ is also a square root of $M$. Moreover, each square root for $M$ can be obtained by right multiplication of $N$ with an orthogonal matrix. In other words, the orthogonal group acts transitively by right multiplication on the set of square roots of $M$.

Notice that a square root $N$ of a positive definite matrix $M$ is a natural change of variables between the Hilbert space induced by the internal product $\langle u,v \rangle_M := u^tMv$ and the Euclidean Hilbert space endowed with the usual internal product $\langle u,v \rangle := u^tv$, in the sense that $\langle u,v \rangle_M = \langle Nu,Nv \rangle$. A particular use of this fact appears when we consider a Gaussian random vector $X \sim N(0,\Sigma)$. In this case we will have that $X \overset{D}{=} NU$, where $N$ is any square root of $\Sigma$ and $U \sim N(0,1)$.

We will impose an additional condition on the square roots of $\Sigma$: If a square root $N$ of $\Sigma$ satisfies that $Ne_1$ is a positive multiple of the vector $1 = (1,\ldots,1)^t$, then $N$ will be called a standard square root of $\Sigma$. Notice that, if this is the case, $Ne_1 = \kappa^{-1}1$, with $\kappa := \sqrt{\Sigma^{-1}1}$. A different representation for $\Sigma$ is given by

\[ \Sigma = \left\{ \sum_{i=1}^{n} c_i v_i : c_i \geq 0 \text{ for } i = 1,\ldots,n \right\} , \]

where, for $i = 1,\ldots,n$, the vector $v_i$ is a positive multiple of $A^{-1}e_i$. The vectors $v_1,\ldots,v_n$ are collectively called the directions of $C_A$.

In the following let us assume that $\Sigma^{-1}1 > 0$ and let $N$ be a standard square root of $\Sigma$. Notice that $CN$ (except for the origin) is completely contained in the half space $\{w \in \mathbb{R}^n : e_1^t w > 0\}$. Indeed, if $0 \neq u \in CN$, then

\[ e_1^t u = \lambda 1^t \left( N^{-1} \right)^t N^{-1} Nu = \lambda 1^t \Sigma^{-1} Nu = \lambda (\Sigma^{-1}1)^t Nu > 0. \]

\[ \overset{1^t}{\leq} \text{ denotes the componentwise order, that is for a given vectors } v = (v_1,\ldots,v_n)^t, w = (w_1,\ldots,w_n)^t \in \mathbb{R}^n, v \geq w \text{ sii } v_i \geq w_i, \text{ for each } i = 1,\ldots,n. \]
3 How to recover $\Sigma$ from the distribution of $X_{\min}$?

3.1 Reducing the problem

Given a Gaussian random vector $X = (X_1, X_2, X_3) \sim N(0, \Sigma)$, we define $m_\Sigma$ as the tail distribution of $X_{\min} = \min(X_1, X_2, X_3)$:

$$m_\Sigma(t) := \Prob(X_{\min} \geq t),$$

In the following, we will assume that $\Sigma^{-1} \mathbf{1} > 0$.

**Lemma 2** Let $N$ be a standard square root of $\Sigma$. Then $\Sigma$ is uniquely determined (up to a permutation), from the vertices of the section $T_N$ and the parameter $\kappa := \sqrt{\Tr\Sigma^{-1}}$.

**Proof.** Let $w_i = (1, \alpha_i, \beta_i), i = 1, 2, 3$ be the vertices of the triangle $T_N$ and let $W$ be the matrix with columns $w_1, w_2, w_3$. Define $\mu_1, \mu_2, \mu_3$ by the relation $[\mu_1, \mu_2, \mu_3]^T := \kappa W^{-1}e_1$ and let $\Lambda := \text{diag}(\mu_1, \mu_2, \mu_3)$. We assert that there exists a permutation matrix $P$ such that

$$N = P^t \Lambda^{-1} W^{-1}.$$

From this statement the lemma would clearly follow since we would have that

$$\Sigma = P^t \Lambda^{-1} W^{-1} (W^{-1})^t (\Lambda^{-1})^t P.$$

In order to prove the assertion notice that, up to ordering and scaling, the vectors $w_1, w_2, w_3$ are equal to the directions $N^{-1}e_1, N^{-1}e_2, N^{-1}e_3$, thus

$$(N^{-1}e_1, N^{-1}e_2, N^{-1}e_3) = (\lambda_{\sigma(1)} w_{\sigma(1)}, \lambda_{\sigma(2)} w_{\sigma(2)}, \lambda_{\sigma(3)} w_{\sigma(3)})$$

for some permutation $\sigma$ and some $\lambda_1, \lambda_2, \lambda_3 > 0$. This is equivalent to saying that $W \Lambda' P = N^{-1}$, where $P$ is the matrix permutation associated to $\sigma$ and $\Lambda' := \text{diag}(\lambda_1, \lambda_2, \lambda_3)$.

Consequently,

$$W [\lambda_1, \lambda_2, \lambda_3]^t = W \Lambda' \mathbf{1} = W \Lambda' P \mathbf{1} = N^{-1} \mathbf{1} = \kappa e_1.$$

Thus $[\lambda_1, \lambda_2, \lambda_3]^t := \kappa W^{-1} e_1$, implying that $\Lambda = \Lambda'$ from where the assertion follows. ■

Let $N$ be a standard root of $\Sigma$. Notice that for any $u \in \mathbb{R}^3$ the condition $Nu \geq t \mathbf{1}$ is equivalent to $N(u - t \kappa e_1) \geq 0$, and therefore $\{ u : Nu \geq t \mathbf{1} \} = t \kappa e_1 + C_N$. Then, we have that for $t > 0$,

$$m_\Sigma(t) = \Prob(X_{\min} \geq t) = \mu(t \kappa e_1 + C_N),$$

where $\mu$ is the standard Gaussian measure $N(0, I)$ in $\mathbb{R}^3$. Moreover, if $\mu_\sigma$ stands for the Gaussian measure $N(0, \sigma^2 I)$, it is the case that

$$m_\Sigma(t) = \mu_{1/(t \kappa)}(e_1 + C_N).$$
Lemma 3  As \( t \to \infty \), \( \ln m_{\Sigma}(t) \sim -\frac{t^2 \kappa^2}{2} \). In particular,

\[
\kappa^2 = -\frac{1}{2} \lim_{t \to +\infty} \frac{\ln m_{\Sigma}(t)}{t^2}.
\]

Proof. The rate function of the sequence of measures \( \mu_{\sigma} \), where \( \sigma \to 0 \), is given by \( I(u) = \|u\|^2 / 2 \), (see [15]). Therefore,

\[
\ln \mu_{\sigma}(e_1 + C_N) \sim -\sigma^{-2} \inf_{u \in (e_1 + C_N)} I(u);
\]

On the other hand, it is clear that

\[
\inf_{u \in (e_1 + C_N)} I(u) = I(e_1) = \frac{1}{2}.
\]

Therefore, as \( t \to \infty \), we have that

\[
\ln (m_{\Sigma}(t)) = \ln \mu_{1/t\kappa}(e_1 + C_N) \sim -\frac{t^2 \kappa^2}{2}.
\]

We define the (two-dimensional) circular transform of a function \( f : \mathbb{R}^2 \to \mathbb{R} \) by

\[
R_f(\rho) := \int_0^{2\pi} f(\rho \cos \theta, \rho \sin \theta) d\theta.
\]

For a measurable set \( S \subseteq \mathbb{R}^2 \), we define the circular transform by

\[
R_S(\rho) := R_{I(-e_1)}(\rho) = \int_0^{2\pi} I_S(\rho \cos \theta, \rho \sin \theta) d\theta, \quad \rho > 0,
\]

where \( I_S \) denotes the characteristic function of \( S \). Notice that, since the angular measure is invariant under orthogonal transformations, the circular transform is also invariant under orthogonal transformations. In other words if \( O \) is an orthogonal transformation, it is the case that \( R_{f \circ O} = R_f \). On the other hand, if \( R_f = R_g \) for given functions \( f, g \), it is not necessarily the case that \( f \circ O = g \) for some orthogonal transformation (see example 5). In spite of this, we have a positive result in this direction when \( f \) and \( g \) correspond to characteristic functions of triangles enclosing the origin (that is, such that \( 0 \) belongs to its interior).

Lemma 4  If \( T \) is a triangle enclosing the origin, then \( T \) can be recovered (up to an orthogonal transformation), from \( R_T \).

Due to its length, we relegate the proof of Lemma 4 to Section [3]. Although moderately technical, it relies on elementary geometry and basic linear algebra. The condition that the triangles enclose the origin is necessary, as the following example shows:
Example 5  Consider the triangles depicted in the figure below

It is clear that the triangles have the same circular transform. Therefore, for the triangles

the circular transform is also the same. However, it is clear that they are not orthogonally equivalent.

3.2 The main Theorem

Now we are able to state and prove the main result of this article:

Theorem 6 Suppose that $X = (X_1, X_2, X_3) \sim N(0, \Sigma)$. Let $X_{\min}$ be the random variable defined by $X_{\min} = \min(X_1, X_2, X_3)$ and assume that $1^T \Sigma^{-1} > 0$. Then, the distribution of $X_{\min}$ uniquely determines $\Sigma$ up to permutation equivalence. More exactly, if $Y = (Y_1, Y_2, Y_3) \sim N(0, \Sigma_0)$ and $X_{\min} \overset{d}{=} Y_{\min}$, there exists a permutation matrix $P$ such that $\Sigma_0 = P \Sigma P^t$.

Let us first prove the following lemma:

Lemma 7 Let $N$ be a standard square root of $\Sigma$ (see definition 1) where $1^T \Sigma^{-1} > 0$. Then $R_{TN}$ is identifiable from $m_\Sigma(t) := \Pr(X_{\min} \geq t)$. That is, $m_\Sigma(t)$ uniquely determines $R_{TN}$.

Proof. From eq. (1), we have that

$$m_\Sigma(t) = \mu(\kappa_1 + C_N) = \frac{1}{(2\pi)^{3/2}} \iiint e^{-\frac{1}{2}||u||^2} du,$$

where $\kappa_1 = 1^T \Sigma^{-1}$. Equivalently,

$$m_\Sigma(t) = \frac{1}{(2\pi)^{3/2}} \int_0^\infty \int_{(u_1 - \kappa t)T_N} e^{-\frac{1}{2}(u_2^2 + u_3^2)} du_2 du_3 du_1,$$
Now, a change of variables leads to
\[
m_{\Sigma}(t) = \frac{1}{(2\pi)^{3/2}} \int_{0}^{\infty} e^{-\frac{1}{2}(x^2 + \kappa t)} \int_{xT_N} e^{-\frac{1}{2}(y^2 + z^2)} \, dy \, dz \, dx
\]
\[
= e^{-\frac{1}{2}\kappa t^2} \int_{0}^{\infty} e^{-\kappa tx} e^{-\frac{1}{2}x^2} \int_{xT_N} e^{-\frac{1}{2}(y^2 + z^2)} \, dy \, dz \, dx.
\]
Therefore, if \( \mathcal{L} \) denotes the Laplace transform and \( h \) is the function defined by
\[
h(x) := e^{-\frac{1}{2}x^2} \int_{xT_N} e^{-\frac{1}{2}(y^2 + z^2)} \, dy \, dz,
\]
we have that
\[
(2\pi)^{3/2} e^{\frac{1}{2}t^2} m_{\Sigma}(t/\kappa) = \int_{0}^{\infty} e^{-tx} \left[ e^{-\frac{1}{2}x^2} \int_{xT_N} e^{-\frac{1}{2}(y^2 + z^2)} \, dy \, dz \right] \, dx = \mathcal{L}(h)(t).
\]
Since the Laplace transform is injective over the functions of polynomial growth on \((0, \infty)\), in particular, we have
\[
\mathcal{L}^{-1} \left( (2\pi)^{3/2} e^{\frac{1}{2}t^2} m_{\Sigma}(t/\kappa) \right)(\sqrt{2x}) = h(\sqrt{2x}) = e^{-x} \int_{\sqrt{2x}T_N} e^{-\frac{1}{2}(y^2 + z^2)} \, dy \, dz,
\]
or equivalently,
\[
\frac{e^x}{x} \mathcal{L}^{-1} \left( (2\pi)^{3/2} e^{\frac{1}{2}t^2} m_{\Sigma}(t/\kappa) \right)(\sqrt{2x}) = \frac{1}{x} \int_{\sqrt{2x}T_N} e^{-\frac{1}{2}(y^2 + z^2)} \, dy \, dz.
\]  
(4)
On the other hand, it is clear that
\[
\frac{1}{x} \int_{\sqrt{2x}T_N} e^{-\frac{1}{2}(y^2 + z^2)} \, dy \, dz = 2 \int_{T_N} e^{-x(y^2 + z^2)} \, dy \, dz
\]
\[
= 2 \int_{(0, \infty) \times (0, 2\pi)} e^{-x\rho^2} I_{TN}(\rho \cos \theta, \rho \sin \theta) \rho \, d\theta \, d\rho
\]
\[
= \int_{0}^{+\infty} \left[ \int_{0}^{2\pi} I_{TN}(\sqrt{\rho} \cos \theta, \sqrt{\rho} \sin \theta) \, d\theta \right] \rho \, d\rho
\]
\[
= \mathcal{L}(g)(x),
\]
where \( g \) denotes the function \( g(\rho) := \int_{0}^{2\pi} I_{TN}(\sqrt{\rho} \cos \theta, \sqrt{\rho} \sin \theta) \, d\theta \).
Therefore, it follows that
\[
\mathcal{L}^{-1} \left( \frac{1}{x} \int_{\sqrt{2x}T_N} e^{-\frac{1}{2}(y^2 + z^2)} \, dy \, dz \right)(\rho^2) = \int_{0}^{2\pi} I_{TN}(\rho \cos \theta, \rho \sin \theta) \, d\theta.
\]  
(5)
Finally, by combining Eqs. 4 and 5, we obtain
\[ R_{T_N}(\rho) = \mathcal{L}^{-1} \left( \frac{e^x}{\sqrt{2\pi}} \mathcal{L}^{-1} \left( (2\pi)^{3/2} e^{\frac{1}{4}t^2} m_{\Sigma}(t/\kappa) \right) \left( \sqrt{2\pi} \right) \right) (\rho^2). \]

Therefore, equation 6 gives us an expression for \( R_{T_N} \) in terms of \( m_{\Sigma}(t) \) and \( \kappa \). However, from lemma 8, we know that \( \kappa \) is recoverable from \( m_{\Sigma}(t) \), thus the right-hand side of 6 is determined from \( m_{\Sigma}(t) \), and the result follows. ■

**Proof of Theorem 6.** From lemma 8, \( \kappa \) is recoverable from \( m_{\Sigma}(t) \) using eq. 6. Moreover, from lemma 4, we can recover \( R_{T_N} \) from \( m_{\Sigma}(t) \) using the formula stated in eq. 6, where \( N \) is a standard square root of \( \Sigma \). Therefore, using Lemma 4, we can recover \( T_N \) up to an orthogonal transformation, that is, we can recover \( OT_N \) where \( O \) is an (unknown) orthogonal transformation in \( \mathbb{R}^2 \). Now, we define \( \tilde{O} = \left[ \begin{array}{cc} 1 & 0 \\ 0 & O \end{array} \right] \). Notice that \( OT_N = T_N\tilde{O} \), so that \( OT_N \) is the section associated with the standard root \( \tilde{N} := NO \). Therefore, since we can recover \( T_N \) and \( \kappa \), from lemma 3, we can recover \( \Sigma \) up to permutation equivalence. ■

### 4 The circular transform of a triangle

We can think of the circular transform as a systematic ‘scan’ that recognizes the mass at distance \( \rho \) from the origin. When the scan encounters an abrupt change of medium, its smoothness is momentarily lost. For instance, in order to recover an acute triangle \( T \) containing the origin, by detecting changes in the smoothness of \( R_T \) we can determine the distance from the origin to the sides and vertices of the triangle and then rely on a geometric construction to recover \( T \). However, we take a detour from this approach and we use a more concise tool. Namely, that the circular transform of some ‘basic triangles’ form a linearly independent set (lemma 3). Then, we use such basic triangles as building blocks to recover more complex geometric figures, in particular, any triangle containing the origin.

#### 4.1 Parametric form of a triangle

Let \( T \) be a triangle with vertices \( x, y, z \in \mathbb{R}^2 \) and such that 0 belongs to the interior of \( T \). Also, let \( n_{xy}, n_{yz}, n_{zx} \) be the closest points to the origin from the lines \( xy, yz \) and \( zx \) respectively (we will call them the *heights* of \( T \)). We define the *parametric form* of the triangle \( T \) as the ordered sequence of distances \( (\| x \|, \| n_{xy} \|, \| y \|, \| n_{yz} \|, \| z \|, \| n_{zx} \|) \). It is an easy geometric fact that this list determines the triangle \( T \) up to rotation.

As an intermediate step, we require the following lemma, which states that, if the terms of the parametric form can be recovered by pairs, then the parametric form of the triangle can be recovered.

**Lemma 8** Consider a triangle \( T \) with parametric form \((\| x \|, \| n_{xy} \|, \| y \|, \| n_{yz} \|, \| z \|, \| n_{zx} \|) \). Let \( \alpha, \beta, \gamma, \eta_1, \eta_2, \eta_3 \) satisfy
\[
\left\{ (\eta_1, \alpha), (\eta_1, \beta), (\eta_2, \beta), (\eta_3, \alpha) \right\} = \left\{ (\| n_{xy} \|, \| x \|), (\| n_{xy} \|, \| y \|), (\| n_{yz} \|, \| y \|), (\| n_{yz} \|, \| z \|), (\| n_{zx} \|, \| z \|), (\| n_{zx} \|, \| x \|) \right\}
\]

Then, a parametric form of \( T \) is given by
\[
(\alpha, \eta_1, \beta, \eta_2, \gamma, \eta_3).
\]
Proof. Without loss of generality, we can assume that \( \alpha = \|x\|, \beta = \|y\| \) and \( \gamma = \|z\| \). If it is the case that \( \alpha, \beta \) and \( \gamma \) are different numbers, then it is clear that \( \|n_{xy}\| = \eta_1, \|n_{yz}\| = \eta_2 \) and \( \|n_{xz}\| = \eta_3 \). On the other hand, if two of these numbers are equal, say \( \alpha = \beta \neq \gamma \), then necessarily \( \|n_{xy}\| = \eta_1 \) and \( \{\|n_{yz}\|, \|n_{xz}\|\} = \{\eta_2, \eta_3\} \) where the two possible choices lead to isomorphic triangles, therefore, by appropriately interchanging \( y \) and \( z \), we have that \( \|n_{yz}\| = \eta_2 \) and \( \|n_{xz}\| = \eta_3 \). Finally, if it is the case that \( \alpha = \beta = \gamma \), then we have that \( \{\|n_{xy}\|, \|n_{yz}\|, \|n_{xz}\|\} = \{\eta_1, \eta_2, \eta_3\} \) and in fact any assignment leads to isomorphic triangles. Therefore, appropriately interchanging \( x, y \) and \( z \), we have that \( \|n_{xy}\| = \eta_1, \|n_{yz}\| = \eta_2, \|n_{xz}\| = \eta_3 \). \( \blacksquare \)

4.2 Some properties of the circular transform

A family of functions \( \{f_\alpha (x)\}_{\alpha \in \Lambda} \) is said to be linearly independent in an interval \( I \) if for any finite set \( F \subset \Lambda \), \( \sum_{\alpha \in F} c_\alpha f_\alpha (x) = 0 \) for all \( x \in I \) implies that \( c_\alpha = 0 \) for all \( \alpha \in F \).

For any \( a, b > 0 \), we define the triangle

\[
T(a, b) := \text{conv} \left( (0, 0), (a, 0), (a, \sqrt{b^2 - a^2}) \right)
\]

where \( \text{conv} \) stands for the convex closure in \( \mathbb{R}^2 \). We also define the function \( \Phi_{a,b} : (0, +\infty) \to \mathbb{R} \) by

\[
\Phi_{a,b}(\rho) := R_{T(a,b)}(\rho).
\]

Lemma 9 The set of functions \( \{\Phi_{a,b}\}_{a,b>0} \) is linearly independent in \((0, +\infty)\).

We first prove the following lemma.

Lemma 10 Consider an increasing sequence of real numbers \( 0 < a_0 < \cdots < a_m \). Then, for every \( x \in (0, 1/a_m^2) \) and every \( i,j = 1, \ldots, m, i \neq j \), it is the case that

\[
a_i^2 (1 - a_i^2 x) \neq a_j^2 (1 - a_j^2 x)
\]

Proof. Let \( x \in (0, 1/a_m^2) \) be fixed. Since \( 0 < x < 1/a_m^2 \leq 2/a_i^2 \), for all \( i = 1, \ldots, m \), we have that \( a_i \in \left( 0, \sqrt{2/x} \right) \) for \( i = 0, \ldots, m \). The result follows from the fact that the function \( f(t) := t^2 (1 - t^2 x) \) is increasing in \((0, \sqrt{2/x})\). \( \blacksquare \)

Proof of Lemma 9 Given \( 0 < a_0 < \cdots < a_m \), consider the functions \( \{\psi_{a_i}(x)\}_{i=0}^m \), where

\[
\psi_{a_i}(x) := (1 - a_i^2 x)^{-1/2}.
\]

The Wronskian of these functions in \((0, 1/a_m^2)\) is given by

\[
W(\psi_{a_0}, \ldots, \psi_{a_m}) = C \begin{vmatrix} 1 & \cdots & 1 \\ a_0^2 (1 - a_0^2 x)^{-1} & \cdots & a_m^2 (1 - a_m^2 x)^{-1} \\ \vdots & \ddots & \vdots \\ a_0^{2m} (1 - a_0^2 x)^{-m} & \cdots & a_m^{2m} (1 - a_m^2 x)^{-m} \end{vmatrix},
\]

where

\[
C = \prod_{i=0}^{m} \frac{(2i)!}{i!} \prod_{i=0}^{m} (1 - a_i^2 x)^{-1/2}.
\]

8
The matrix in the expression of $W (\psi_{a_0}, \ldots, \psi_{a_m})$ is a Vandermonde matrix, with generators $a_i^2 (1 - a_i^2 x)^{-1/2}$, for $i = 0, \ldots, m$. It follows from Lemma 3 that these generators are all different. Therefore the Wronskian $W (\psi_{a_0}, \ldots, \psi_{a_m})$ is non zero in $(0, 1/a_m^2)$. Consequently, for any fixed interval $I \subseteq (0, 1/a_m^2)$, the functions $\{\psi_{a_i}(x)\}_{i=0}^m$ are linearly independent in $I$. Therefore, the functions $\{a_i x \psi_{a_i}(x)\}_{i=0}^m$ are linearly independent in $I \subseteq (0, 1/a_m^2)$. In consequence, the change of variables $x = 1/p^2$, produces linearly independent functions in $I \subseteq (a_m, +\infty)$, given by

\[
\frac{a_i}{\rho \sqrt{\rho^2 - a_i^2}}, \quad i = 0, \ldots, m.
\]

Moreover, their antiderivatives $\{\arccos (a_i/\rho) + \gamma_i\}_{i=0}^m$, where $\gamma_0, \ldots, \gamma_m$ are arbitrary constants, will be also linearly independent in $I \subseteq (a_m, +\infty)$.

Now, in order to prove that the functions $\{\Phi_{a,b}\}_{a,b>0}$ are linearly independent in $(0, +\infty)$, we assume that

\[
\sum_{i=0}^l \sum_{j=0}^{m_i} c_{i,j} \Phi_{a_i,j,b_i}(\rho) = 0, \quad \forall \rho > 0;
\]

where $0 < a_{i,0} < \cdots < a_{i,m_i}$ for $i = 0, \ldots, l$, $0 < b_1 < \cdots < b_l$ and all the coefficients $c_{i,j}$ are nonzero (that is, we assume the existence of a nontrivial minimal dependent set).

Now, eq. (4.2) implies that $\Phi_{a_i,j,b_i}(\rho) = 0$, whenever $i < l$ and $\rho \in (b_{i-1}, b_i)$, therefore

\[
\sum_{j=0}^{m_i} c_{i,j} \Phi_{a_i,j,b_i}(\rho) = 0, \quad \forall \rho \in (b_{i-1}, b_i).
\]

Now, since

\[
\Phi_{a,b}(\rho) = \arccos (a/b) I (\rho \leq a) + [\arccos (a/b) - \arccos (a/\rho)] I (a \leq \rho < b),
\]

the eq. (4) restricted to the interval $(a_{i,m_i}, b_i)$ takes the form

\[
\sum_{j=0}^{m_i} c_{i,j} [\arccos (a_{i,j}/\rho) - \arccos (a_{i,j}/b_i)] = 0, \quad \forall \rho \in (a_{i,m_i}, b_i),
\]

which contradicts the fact that the set $\{\arccos (a_{i,j}/\rho) + \gamma_j\}_{j=0}^m$, with $\gamma_j = -\arccos (a_{i,j}/b_i), \ j = 0, \ldots, m$ is linearly independent in the interval $(a_{i,m_i}, b_i)$. ■

### 4.3 Decomposing the triangle

In order to prove Lemma 4 we will distinguish the following three cases:

- **Case I:** All the heights of $T$ lie in the interior of the sides. That is $n_{xy} = s_1 x + (1 - s_1) y,$ $n_{yz} = s_2 y + (1 - s_2) z$ and $n_{xz} = s_3 x + (1 - s_3) z$, for some $s_1, s_2, s_3 \in (0, 1)$.

- **Case II:** There is one height that does not lie in the interior of the corresponding side. Say, without loss of generality, $n_{xy} = sx + (1 - s) y$, where $s > 1$.

- **Case III:** There is one height that lies over a vertex. Say, without loss of generality, $n_{xy} = x$.

We define the basic subtriangles of the triangle $T$ as the following six right triangles:

\[
T_1 = \text{conv} (0, x, n_{xy}), \ T_2 = \text{conv} (0, y, n_{xy}), \ T_3 = \text{conv} (0, y, n_{yz}),
\]

\[
T_4 = \text{conv} (0, z, n_{yz}), \ T_5 = \text{conv} (0, z, n_{xz}), \ T_6 = \text{conv} (0, x, n_{xz}).
\]

Notice that these triangles are non degenerate (that is, of positive area), except in case III, in which $T_1$ is a degenerate triangle. We should also point out the following elementary facts:

Let $I_A : \mathbb{R}^2 \rightarrow \{0, 1\}$ denotes the characteristic function on the set $A \subseteq \mathbb{R}^2$. Now,
• In case I,

\[ I_T = \sum_{i=1}^{6} I_{T_i}. \]  

(8)

• In case II,

\[ I_T = -I_{T_1} + \sum_{i=2}^{6} I_{T_i}, \]  

(9)

and the subtriangle \( T_1 \) is not congruent to any other of the subtriangles.

• In case III,

\[ I_T = \sum_{i=2}^{6} I_{T_i}. \]  

(10)

Given a basis \( \mathcal{B} = \{v_i\}_{i=1}^{d} \) of a vector space and an expression

\[ w = \sum_j \beta_j v_j \]  

(11)

where \( \{t_j\}_j \) is an arbitrary sequence in \( \{1, \ldots, d\} \), we say that eq. (11) is irreducible if \( \sum_{j=1}^{d} |\alpha_j| = \sum_j |\beta_j| \) where \( \alpha_1, \ldots, \alpha_d \) are the coefficients of \( w \) in the basis \( \mathcal{B} \).

**Lemma 11** \( R_T \) belongs to the linear span of \( \{\Phi_{a,b}\}_{a,b>0} \). More exactly, if \( \eta_1 := n_{xy}, \eta_2 := n_{yz}, \eta_3 := n_{xz}, \)

1. In case I,

\[ R_T = \Phi_{||\eta_1||,||x||} + \Phi_{||\eta_1||,||y||} + \Phi_{||\eta_2||,||y||} + \Phi_{||\eta_2||,||z||} + \Phi_{||\eta_3||,||x||} + \Phi_{||\eta_3||,||z||}; \]

2. In case II,

\[ R_T = -\Phi_{||\eta_1||,||x||} + \Phi_{||\eta_1||,||y||} + \Phi_{||\eta_2||,||y||} + \Phi_{||\eta_2||,||z||} + \Phi_{||\eta_3||,||x||} + \Phi_{||\eta_3||,||z||}; \]

3. In case III,

\[ R_T = \Phi_{||\eta_1||,||y||} + \Phi_{||\eta_2||,||y||} + \Phi_{||\eta_2||,||z||} + \Phi_{||\eta_3||,||x||} + \Phi_{||\eta_3||,||z||}. \]

Moreover, all of the previous linear representations are irreducible.

**Proof.** By using the linearity of the circular transform, all three formulas follow directly from Eqs. (8), (9) and (10) respectively. The irreducibility is clear in cases I and III due to the positivity of the coefficients. On the other hand, to show the irreducibility in case II we must show that the function \( \Phi_{||\eta_1||,||x||} \) is not in the set

\[ \{\Phi_{||\eta_1||,||y||}, \Phi_{||\eta_2||,||y||}, \Phi_{||\eta_2||,||z||}, \Phi_{||\eta_3||,||x||}, \Phi_{||\eta_3||,||z||}\}, \]

Indeed this is true since, in this case, the triangle \( T_1 \) is not congruent to any other subtriangle. \( \blacksquare \)
4.4 Recovering the triangle

Proof of Lemma 4. From Lemma 11, $R_T$ can be expressed in basis $\{\Phi_{a,b}\}_{a,b>0}$, say

$$R_T = \sum c_{a,b} \Phi_{a,b}. \quad (12)$$

Moreover, depending if $\sum c_{a,b}$ is equal to 6, 4 or 5, we can distinguish if we are in case I, II or III respectively. Notice also that the representation in eq. (12) can be expressed as:

$$R_T = c\Phi_{a_1,b_1} + \Phi_{a_1,b_2} + \Phi_{a_2,b_2} + \Phi_{a_3,b_3} + \Phi_{a_3,b_1},$$

for some $a_i, b_i, i = 1, \ldots, 3$ and where $c$ is equal to 1, $-1$ or 0, depending if we are in case I, II or III. We claim that $(b_1, a_1, b_2, a_2, b_3, a_3)$ is a parametric representation of $T$. To see this, notice that in the cases I and II, from the uniqueness of the linear representation we have that

$$\left\{ (a_1, b_1), (a_1, b_2), (a_2, b_2), \right\} = \left\{ (\|n_{xy}\|, \|x\|), (\|n_{xy}\|, \|y\|), (\|n_{yz}\|, \|y\|), \right\}.$$

Then, from Lemma 8 it follows that $(b_1, a_1, b_2, a_2, b_3, a_3)$ is a parametric representation of $T$. In case III the claim follows the same argument, by adding the missing pair $(a_1, a_1)$.

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