Symbolic-computation study of integrable properties for the (2+1)-dimensional Gardner equation with the two-singular-manifold method

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Abstract

The singular manifold method from the Painlevé analysis can be used to investigate many important integrable properties for the nonlinear partial differential equations. In this paper, the two-singular-manifold method is applied to the (2+1)-dimensional Gardner equation with two Painlevé expansion branches to determine the Hirota bilinear form, Bäcklund transformation, Lax pairs and Darboux transformation. Based on the obtained Lax pairs, the binary Darboux transformation is constructed and the $N \times N$ Grammian solution is also derived by performing the iterative algorithm $N$ times with symbolic computation.

PACS numbers: 02.30.Jr; 02.30.Ik; 05.45.Yv; 02.70.Wz

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1. Introduction

Arising from the Painlevé analysis proposed by Weiss, Tabor and Carnevale [1], the singular manifold method (SMM) has been successfully used to investigate the typical integrable properties for many integrable nonlinear partial differential equations (NPDEs), such as the Lax pair [2, 3], auto-Bäcklund transformation [2, 3, 4], nonclassical Lie symmetry [5] and Hirota bilinear formulation [6]. In Refs. [4, 7], it is shown the SMM has turned out to be capable of obtaining some special classes of solutions for non-integrable NPDEs. However, due to the existence of several Painlevé expansion branches for some given NPDEs like the modified Korteweg-de Vries (mKdV) equation [8], Sine-Gordon (SG) equation [8] and modified Kadomtsev-Petviashvili (KP) equation [9], in this situation the SMM is not feasible to exploit the integrable properties of these equations. Therefore, Refs. [8, 10, 11] have generalized the SMM and developed the two-singular-manifold method to uncover information about integrable character.

Different from the usual expansion, the two-singular-manifold method involves two truncated Painlevé expansions at the constant level term, which contains two different singular manifolds at a time. This approach has been applied to the mKdV equation [8, 12], SG equation [8], classical Boussinesq system [10, 11], Mikhailov-Shabat system [10], generalized dispersive long wave equation [9], modified KP equation [9], and so on. With this method, not only the auto-Bäcklund transformation and Lax pair can be obtained, but also the Darboux transformation can be constructed in terms of the truncated Painlevé expansions in both the NPDE and its Lax pair [9, 13, 14]. In addition, the relationship relating the singular manifolds and Hirota τ-function can be precisely established [9, 10, 11, 12].

Permeation of symbolic computation among various fields of science and engineering remarkably helps the investigations on the nonlinear partial differential equations (NPDEs) [15, 16, 17, 18]. Symbolic computation has increased the ability of a computer to deal with a large amount of complicated and tedious algebraic calculations.

In this paper, by virtue of the symbolic computation, we will investigate the integrable properties for the (2+1)-dimensional Gardner equation [19, 20]

\[ u_t - u_{xxx} - 6 \beta u u_x + \frac{3}{2} \alpha^2 u^2 u_x - 3 \int u_{yy} \, dx + 3 \alpha u_x \int u_y \, dx = 0, \tag{1.1} \]

where \( \alpha \) and \( \beta \) are two arbitrary constants. When \( u_y = 0 \), Eqn. (1.1) reduces to the well-known (1+1)-dimensional Gardner equation. For \( \alpha = 0 \), Eqn. (1.1) is the KP equation, while it is the modified KP equation with \( \beta = 0 \). Therefore, the (2+1)-dimensional Gardner equation could be regarded as a combined KP and modified KP equation. Eqn. (1.1) is completely integrable in the sense that it has been solved by the inverse spectral transform method [20]. Refs. [20, 21, 22, 23, 24] have presented its wide classes of analytical solutions including the rational solution, quasi-periodic solution, soliton solution and non-decaying real solutions.
In the following sections, with the help of symbolic computation, we will apply the two-singular-manifold method to the (2+1)-dimensional Gardner equation to determine the Hirota bilinear form, Bäcklund transformation and Lax pairs. Based on the obtained Lax pairs, we will construct the binary Darboux transformation and perform symbolic computation on the iterative algorithm to generate the Grammian solutions.

2. Hirota bilinear form

To begin with, we rewrite Eqn. (1.1) as the following system

\[ \begin{align*}
    u_t - u_{xxx} - 6 \beta u u_x + \frac{3}{2} \alpha^2 u^2 u_x - 3 v_y + 3 \alpha u_x v &= 0, \\
    v_x &= u_y.
\end{align*} \] (2.1)

Then, we expand the solutions of System (2.1) in a generalized Laurent series

\[ \begin{align*}
    u &= \sum_{j=0}^{\infty} u_j \chi^{-a+j}, \\
    v &= \sum_{j=0}^{\infty} v_j \chi^{-b+j},
\end{align*} \] (2.2)

where \( \chi = \chi(x,y,t) \), and \( u_j = u_j(x,y,t) \), \( v_j = v_j(x,y,t) \) are analytical functions in the neighborhood of a non-characteristic movable singularity manifold \( \chi(x,y,t) = 0 \), while \( a \) and \( b \) are two integers to be determined. By the analysis of the leading terms, we obtain

\[ a = 1, \quad b = 1, \quad u_0 = 2 \epsilon \frac{\chi_x}{\alpha}, \quad v_0 = 2 \epsilon \frac{\chi_y}{\alpha}, \] (2.3)

where \( \epsilon = \pm 1 \). It is easy to see that \( u_0 \) and \( v_0 \) can take two values so that System (2.1) has two different Painlevé expansion branches. By using two different singular manifolds \( \phi \) and \( \varphi \) [9, 13, 14], we take the truncated Painlevé expansion at the constant level term

\[ \begin{align*}
    u' &= u + 2 \alpha \frac{\phi_x}{\phi} - \frac{\varphi_x}{\varphi}, \\
    v' &= v + 2 \alpha \frac{\phi_y}{\phi} - \frac{\varphi_y}{\varphi},
\end{align*} \] (2.4)

where the singular manifold \( \phi \) corresponds to \( \epsilon = 1 \) and \( \varphi \) to \( \epsilon = -1 \), which can also be regarded as an auto-Bäcklund transformation between two different solutions \( (u', v') \) and \( (u, v) \) for System (2.1), when singular manifolds \( \phi \) and \( \varphi \) satisfy the truncation conditions.

Motivated by Expressions (2.4), we introduce the dependent variable transformations

\[ \begin{align*}
    u &= \frac{2}{\alpha} \frac{g_x}{g} - \frac{f_x}{f} = \frac{2}{\alpha} \left( \log \frac{g}{f} \right)_x, \\
    v &= \frac{2}{\alpha} \frac{g_y}{g} - \frac{f_y}{f} = \frac{2}{\alpha} \left( \log \frac{g}{f} \right)_y,
\end{align*} \] (2.5)

to transform System (2.1) into the Hirota bilinear form. Substituting Expressions (2.5) back
into System (2.1), we obtain

\[
\left( \frac{D_x g \cdot f}{g f} - \frac{D_x^2 g \cdot f}{g f} \right)_x + 3 \left( \frac{D_x^2 g \cdot f}{g f} \right)_x \left( \frac{D_x g \cdot f}{g f} \right) + 3 \frac{D_x^2 g \cdot f}{g f} \left( \frac{D_x g \cdot f}{g f} \right)_x - 2 \left( \frac{D_x g \cdot f}{g f} \right)_x^3 - \frac{6 \beta}{\alpha} \left( \frac{D_x g \cdot f}{g f} \right)_x^3 + 2 \left( \frac{D_x g \cdot f}{g f} \right)_x^3 - 3 \left( \frac{D_x g \cdot f}{g f} \right)_y + 6 \frac{D_x g \cdot f}{g f} \left( \frac{D_x g \cdot f}{g f} \right)_x = 0, \quad (2.6)
\]

where \( D \) is the well-known Hirota bilinear operator. \( D^m D^l D^l g \cdot f = (\partial_x - \partial_{x'})^m (\partial_y - \partial_{y'})^n (\partial_t - \partial_{t'})^l g(x, y, t)f(x', y', t') \mid_{x'=x, y'=y, t'=t}. \) (2.7)

With symbolic computation, Eqn. (2.6) can be split into

\[
(D_y + D_x^2 - 2 \beta / \alpha D_y) g \cdot f = 0, \quad (2.8)
\]

\[
(D_t - D_x^3 + 3 D_x D_y - 6 \beta / \alpha D_y) g \cdot f = 0, \quad (2.9)
\]

which are the Hirota bilinear form of Eqn. (1.1). By the perturbation technique, one can assume the functions \( f \) and \( g \) in powers of a small parameter \( \varepsilon \) to obtain the multi-soliton solutions of Eqn. (1.1) from Eqns. (2.8) and (2.9).

It is noted that the key step for the Hirota method is to seek for the suitable dependent variable transformation for a given NPDE to be transformed into the Hirota bilinear form. If we do not know how to do this, then there is little prospect of being able to use the Hirota method. However, the truncated expansion in Painlevé analysis can provide us with a useful clue in finding such desired transformations. In fact, the Hirota bilinear forms for a large class of NPDEs can be obtained in terms of the Painlevé truncated expansion \( [32, 33, 34] \).

3. Bilinear Bäcklund transformation

For an integrable NPDE, the existence of a Bäcklund transformation seems to be widely accepted \( [27, 28] \). In this section, from Eqns. (2.8) and (2.9), we will derive a bilinear Bäcklund transformation between two different solutions \( u = \frac{2}{\alpha} (\log g / f)_x \) and \( u' = \frac{2}{\alpha} (\log g' / f')_x \) for Eqn. (1.1), by considering the following two equations,

\[
P1 = \left[ (D_y + D_x^2 - 2 \beta / \alpha D_y) g \cdot f' \right] g' f' - g f \left[ (D_y + D_x^2 - 2 \beta / \alpha D_y) g' \cdot f' \right], \quad (3.1)
\]

\[
P2 = \left[ (D_t - D_x^3 + 3 D_x D_y - 6 \beta / \alpha D_y) g \cdot f \right] g' f' - g f \left[ (D_t - D_x^3 + 3 D_x D_y - 6 \beta / \alpha D_y) g' \cdot f \right]. \quad (3.2)
\]

With the aid of the Hirota bilinear operator identities (see Appendix (A.1 - A.5)), symbolic computation on Eqns. (3.1) and (3.2) yields

\[
P1 = \left[ (D_y + D_x^2 - 2 \beta / \alpha D_y) g \cdot g' \right] f f' - g g' \left[ (D_y + D_x^2 - 2 \beta / \alpha D_y) f \cdot f' \right] - 2 D_x (g f') \cdot (D_x f \cdot g'), \quad (3.3)
\]

\[
P2 = 3 D_x (D_x g \cdot f') \cdot (D_x f \cdot g') - 3 D_x (g f') \cdot (D_y f \cdot g') - 3 D_y (g f') \cdot (D_x f \cdot g') + \left[ (D_t - D_x^3 + 3 D_x D_y - 6 \beta / \alpha D_y) g \cdot g' \right] f f' - \left[ (D_t - D_x^3 + 3 D_x D_y - 6 \beta / \alpha D_y) f \cdot f' \right] g g'. \quad (3.4)
\]
Thus, Eqns. (3.3) and (3.4) can be further decoupled into the following equations

\begin{align}
D_x f \cdot g' &= \eta(t) g f', \quad (3.5a) \\
\eta(t) D_x g \cdot f' + D_y f \cdot g' + \gamma(t) g f' &= 0, \quad (3.5b) \\
[D_y + D_x^2 - 2 \beta / \alpha D_y + \xi(t)] g \cdot g' &= 0, \quad (3.5c) \\
[D_y + D_x^2 - 2 \beta / \alpha D_y + \xi(t)] f \cdot f' &= 0, \quad (3.5d) \\
[D_t - D_x^3 + 3 D_x D_y - 6 \beta / \alpha D_y + \zeta(t)] g \cdot g' &= 0, \quad (3.5e) \\
[D_t - D_x^3 + 3 D_x D_y - 6 \beta / \alpha D_y + \zeta(t)] f \cdot f' &= 0, \quad (3.5f)
\end{align}

where \( \eta(t), \gamma(t), \xi(t) \) and \( \zeta(t) \) are all arbitrary differentiable functions of \( t \). Eqns. (3.5) constitute the bilinear Bäcklund transformation for Eqn. (1.1), from which more complicated solutions can be progressively constructed beginning with a seed solution. Additionally, it can also be of use for the investigation on other integrable properties \([25, 26, 27, 28, 29, 30, 31]\), like the nonlinear superposition formula, Lax pair, conservation laws, etc.

### 4. Lax pairs with symbolic computation

In this section, by the two-singular-manifold method, the Lax pairs of the (2+1)-dimensional Gardner equation will be derived. With symbolic computation, we insert Expressions (2.4) into System (2.1), and get

\begin{align}
\frac{\phi_x}{\phi} \frac{\varphi_x}{\varphi} &= A \frac{\phi_x}{\phi} + B \frac{\varphi_x}{\varphi}, \quad (4.1) \\
12 \beta u - 3 \alpha^2 u^2 + 2 v_1^2 - 6 \alpha v - 2 w_1 + 6 \tau_1^2 + 6 \alpha u_x + 8 v_{1x} &= 0, \quad (4.2) \\
12 \beta u - 3 \alpha^2 u^2 + 2 v_2^2 - 6 \alpha v - 2 w_2 + 6 \tau_2^2 - 6 \alpha u_x + 8 v_{2x} &= 0, \quad (4.3) \\
3 \tau_1 y - v_1 v_{1x} - w_{1x} + 3 \tau_1 \tau_{1x} + v_{1xx} &= 0, \quad (4.4) \\
3 \tau_2 y - v_2 v_{2x} - w_{2x} + 3 \tau_2 \tau_{2x} + v_{2xx} &= 0, \quad (4.5)
\end{align}

with

\begin{align}
A &= \frac{1}{2 \alpha} \left( \alpha v_1 - 2 \beta + \alpha^2 u + \alpha \tau_1 \right), \quad (4.6) \\
B &= \frac{1}{2 \alpha} \left( \alpha v_2 + 2 \beta - \alpha^2 u - \alpha \tau_2 \right), \quad (4.7)
\end{align}

where \( v_i, w_i \) and \( \tau_i \ (i = 1, 2) \) are defined as

\begin{align}
v_1 &= \frac{\phi_{xx}}{\phi_x}, \quad v_2 = \frac{\varphi_{xx}}{\varphi_x}, \quad (4.8) \\
w_1 &= \frac{\phi_t}{\phi_x}, \quad w_2 = \frac{\varphi_t}{\varphi_x}, \quad (4.9) \\
\tau_1 &= \frac{\phi_y}{\phi_x}, \quad \tau_2 = \frac{\varphi_y}{\varphi_x}, \quad (4.10)
\end{align}
We use the derivatives of Expression (4.1) with respect to $x$, $y$ and $t$, so as to obtain

\begin{align*}
A_x &= A(v_2 - A - B), \\
A_y &= (A \tau_2)_x + AB(\tau_2 - \tau_1), \\
A_t &= (A w_2)_x + AB(w_2 - w_1), \\
B_x &= B(v_1 - A - B), \\
B_y &= (B \tau_1)_x - AB(\tau_2 - \tau_1), \\
B_t &= (B w_1)_x - AB(w_2 - w_1). \\
\end{align*}

(4.11) \quad (4.12) \quad (4.13) \quad (4.14) \quad (4.15) \quad (4.16)

By virtue of Eqns. (4.6) − (4.16), it is easy to check that the following two relationships are satisfied

\begin{align*}
(AB)_x &= AB(\tau_2 - \tau_1), \\
[AB(2A - 2B + v_1 - v_2 - 3\tau_1 - 3\tau_2)]_x &= AB(w_1 - w_2).
\end{align*}

(4.17) \quad (4.18)

Therefore, Eqns. (4.11) − (4.16) can be simplified as

\begin{align*}
A_x &= A(v_2 - A - B), \\
A_y &= [A(\tau_2 + B)]_x, \\
A_t &= [A w_2 - AB(2A - 2B + v_1 - v_2 - 3\tau_1 - 3\tau_2)]_x, \\
B_x &= B(v_1 - A - B), \\
B_y &= [B(\tau_1 - A)]_x, \\
B_t &= [B w_1 + AB(2A - 2B + v_1 - v_2 - 3\tau_1 - 3\tau_2)]_x.
\end{align*}

(4.19) \quad (4.20) \quad (4.21) \quad (4.22) \quad (4.23) \quad (4.24)

Through introducing the changes as

\begin{align*}
A &= \frac{\psi_x^+}{\psi^+}, \quad B &= \frac{\psi_x^-}{\psi^-},
\end{align*}

(4.25)

Eqns. (4.19) − (4.24) can be linearized into

\begin{align*}
\psi_{xx}^- &= \psi_x^-(v_1 - A), \\
\psi_y^- &= \psi_x^-(\tau_1 - A), \\
\psi_t^- &= \psi_x^-[w_1 + A(2A - 2B + v_1 - v_2 - 3\tau_1 - 3\tau_2)], \\
\psi_{xx}^+ &= \psi_x^+(v_2 - B), \\
\psi_y^+ &= \psi_x^+(\tau_2 + B), \\
\psi_t^+ &= \psi_x^+[w_2 - B(2A - 2B + v_1 - v_2 - 3\tau_1 - 3\tau_2)].
\end{align*}

(4.26) \quad (4.27) \quad (4.28) \quad (4.29) \quad (4.30) \quad (4.31)

Symbolic computation on Eqns. (4.26) − (4.31) with the substitution of Eqns. (4.2) and (4.3)

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gives rise to
\[ \psi_y^- = -\psi_{xx}^- - \left( \alpha u - \frac{2 \beta}{\alpha} \right) \psi_x^-, \quad (4.32a) \]
\[ \psi_t^- = 4 \psi_{xxx}^- + \left( 6 \alpha u - \frac{12 \beta}{\alpha} \right) \psi_{xx}^- + \left( \frac{12 \beta^2}{\alpha^2} - 6 \beta u + \frac{3}{2} \alpha^2 u^2 - 3 \alpha v + 3 \alpha u_x \right) \psi_x^-, \quad (4.32b) \]

and
\[ \psi_y^+ = \psi_{xx}^+ - \left( \alpha u - \frac{2 \beta}{\alpha} \right) \psi_x^+, \quad (4.33a) \]
\[ \psi_t^+ = 4 \psi_{xxx}^+ - \left( 6 \alpha u - \frac{12 \beta}{\alpha} \right) \psi_{xx}^+ + \left( \frac{12 \beta^2}{\alpha^2} - 6 \beta u + \frac{3}{2} \alpha^2 u^2 - 3 \alpha v - 3 \alpha u_x \right) \psi_x^+. \quad (4.33b) \]

By direct calculation, it is found that Eqn. (1.1) can be derived from the compatibility conditions \( \psi_y^- = \psi_t^- \). Thus, Eqs. (4.32) and (4.33) are two different types of Lax pairs of the \((2+1)\)-dimensional Gardner equation. It is noted that through the following gauge transformations
\[ \psi^- = \exp \left\{ \frac{\beta}{\alpha} x + \frac{\beta^2}{\alpha^2} y + \frac{\beta^3}{\alpha^3} t \right\} \Gamma^- , \]
\[ \psi^+ = \exp \left\{ -\frac{\beta}{\alpha} x - \frac{\beta^2}{\alpha^2} y - \frac{\beta^3}{\alpha^3} t \right\} \Gamma^+ , \]
Lax pairs (4.32) and (4.33) can be respectively transformed into
\[ \Gamma_y^- = -\Gamma_{xx}^- - \alpha u \Gamma_x^- - \beta u \Gamma^- , \quad (4.34a) \]
\[ \Gamma_t^- = 4 \Gamma_{xxx}^- + 6 \alpha u \Gamma_{xx}^- + \left( 6 \beta u + \frac{3}{2} \alpha^2 u^2 - 3 \alpha v + 3 \alpha u_x \right) \Gamma_x^- + \left( 3 \beta u_x + \frac{3}{2} \alpha \beta u^2 - 3 \beta v \right) \Gamma^- , \quad (4.34b) \]

and
\[ \Gamma_y^+ = \Gamma_{xx}^+ - \alpha u \Gamma_x^+ + \beta u \Gamma^+ , \quad (4.35a) \]
\[ \Gamma_t^+ = 4 \Gamma_{xxx}^+ - 6 \alpha u \Gamma_{xx}^+ + \left( 6 \beta u + \frac{3}{2} \alpha^2 u^2 - 3 \alpha v - 3 \alpha u_x \right) \Gamma_x^+ + \left( 3 \beta u_x - \frac{3}{2} \alpha \beta u^2 + 3 \beta v \right) \Gamma^+ . \quad (4.35b) \]

The compatibility conditions \( \Gamma_y^+ = \Gamma_t^- \) can also give rise to Eqn. (1.1). Note that this form of Lax pair (4.34) has been presented in Ref. [19], and other three Lax pairs are given here for the first time.
To this stage, with the two-singular-manifold method, we have obtained the Lax pairs of the (2+1)-dimensional Gardner equation, i.e., Systems (4.32)−(4.35). In the next section, we will construct the relationship between the singular manifolds φ, ϕ and eigenfunctions ψ−, ψ+.

5. Relationship between the singular manifolds and eigenfunctions

Using Eqns. (4.8)−(4.10) and (4.25), we rewrite Eqns. (4.26)−(4.31) as

\[
\begin{align*}
\psi_{xx}^− &= \frac{\phi_{xx}^−}{\phi_x} - \frac{\psi_x^−}{\psi^−}, \\
\psi_{xx}^+ &= \frac{\phi_{xx}^+}{\phi_x} - \frac{\psi_x^+}{\psi^+}, \\
\psi_x^− &= \frac{\phi_y^−}{\phi_x} - \frac{\psi_x^−}{\psi^−}, \\
\psi_x^+ &= \frac{\phi_y^+}{\phi_x} - \frac{\psi_x^+}{\psi^+}, \\
\psi_{t}^− &= \frac{\phi_t^−}{\phi_x} + \frac{\psi_t^−}{\psi^−} \left( 2 \frac{\psi_x^−}{\psi^−} - \frac{2 \phi_{xx}^−}{\phi_x} - \frac{\phi_{xx}^−}{\phi_x} - 3 \frac{\phi_y^−}{\phi_x} - 3 \frac{\phi_y^−}{\phi_x} \right), \\
\psi_{t}^+ &= \frac{\phi_t^+}{\phi_x} - \frac{\psi_t^+}{\psi^+} \left( 2 \frac{\psi_x^+}{\psi^+} - \frac{2 \phi_{xx}^+}{\phi_x} - \frac{\phi_{xx}^+}{\phi_x} - 3 \frac{\phi_y^+}{\phi_x} - 3 \frac{\phi_y^+}{\phi_x} \right).
\end{align*}
\]

Integrating Eqns. (5.1) and (5.2) with respect to x yields

\[
\begin{align*}
\phi_x &= \psi_x^− \psi^+, \\
\phi_x &= \psi_x^− \psi^−.
\end{align*}
\]

Then, by substituting Eqns. (5.7) and (5.8) into Eqns. (5.3)−(5.6), we obtain

\[
\begin{align*}
\phi_y &= \psi_y^+ \psi^− + \psi_x^− \psi_x^+, \\
\phi_y &= \psi_y^+ \psi^− - \psi_x^− \psi_x^+, \\
\phi_t &= \psi_t^− \psi_x^− - 2 \psi_x^− \psi_x^+ + 2 \psi_x^+ \psi_y^− + 4 \psi_x^− \psi_y^+, \\
\phi_t &= \psi_t^− \psi_x^− + 2 \psi_x^− \psi_x^+ - 2 \psi_x^+ \psi_y^− - 4 \psi_x^− \psi_y^+.
\end{align*}
\]

From Eqns. (5.7)−(5.12), it is shown that the singular manifolds φ and ϕ are determined by the eigenfunctions ψ− and ψ+. So, by defining the singular manifolds φ and ϕ in the abbreviated form as

\[
\begin{align*}
\phi &= \Delta (\psi^−, \psi^+), \\
\phi &= \Omega (\psi^−, \psi^+),
\end{align*}
\]

Eqns. (5.7)−(5.12) can be written as
Here, we can see that Eqns. (5.15)–(5.20) establish the relationship between two singular manifolds \( \phi, \varphi \) and eigenfunctions \( \psi^-, \psi^+ \), and possess the following relation
\[
\Delta(\psi^-, \psi^+) + \Omega(\psi^-, \psi^+) = \psi^- \psi^+.
\] (5.21)

6. Binary Darboux transformation

The Darboux transformation method is a powerful tool to get the analytical solutions for the integrable NPDEs \([35, 36]\). The most obvious advantage of this method lies in its iterative algorithm, which is purely algebraic and can be easily achieved on the symbolic computation system. By virtue of the SMM and Darboux transformation, starting from the seed solution and solving the corresponding linear equation or system, one can obtain wide classes of exact analytical solutions for a NPDE, such as the soliton solutions, periodic solutions and rational solutions \([35, 36, 37, 38, 39, 40]\).

Based on the Lax pairs (4.32) and (4.33) obtained in Section 3, we can construct the binary Darboux transformation of Eqn. (1.1). Although Eqns. (2.4) and (2.5) are considered as an auto-Bäcklund transformation, it is actually not convenient to generate more and more complicated solutions in a recursive manner, because it only involves the transformation for potentials. In comparison, the Darboux transformation not only has the potential transformation, but also establishes the relationship between the new and old eigenfunctions. Next, we turn our attention to the transformation of eigenfunctions.

Let us assume that
\[
\psi'^- = \psi^- + f_1 \phi + f_2 \varphi, \quad \psi'^+ = \psi^+ + g_1 \phi + g_2 \varphi,
\] (6.1)

where \( \psi^- \) and \( \psi^+ \) respectively correspond to the solutions of Lax pairs (4.32) and (4.33), while \( f_i \) and \( g_i \) (\( i = 1, 2 \)) are two differentiable functions to be determined, the new eigenfunctions \( \psi'^- \) and \( \psi'^+ \) also satisfy Lax pairs (4.32) and (4.33) except that \( (u, v) \) is replaced by \( (u', v') \),
namely,

\[
\psi'_y = -\psi'_x - \left( \alpha u' - \frac{2\beta}{\alpha} \right) \psi'_x,
\]

(6.3a)

\[
\psi'_t = \left( \frac{12\beta^2}{\alpha^2} - 6\beta u' + \frac{3}{2} \alpha^2 u'^2 - 3\alpha v' + 3\alpha u'_x \right) \psi'_x
+ 4 \psi'_{x\!x\!x} + \left( 6\alpha u' - \frac{12\beta}{\alpha} \right) \psi'_{x\!x},
\]

(6.3b)

and

\[
\psi'^+_y = \psi'^+_x - \left( \alpha u' - \frac{2\beta}{\alpha} \right) \psi'^+_x,
\]

(6.4a)

\[
\psi'^+_t = \left( \frac{12\beta^2}{\alpha^2} - 6\beta u' + \frac{3}{2} \alpha^2 u'^2 - 3\alpha v' - 3\alpha u'_x \right) \psi'^+_x
+ 4 \psi'^{+\!x\!x\!x} - \left( 6\alpha u' - \frac{12\beta}{\alpha} \right) \psi'^{+\!x\!x},
\]

(6.4b)

Substituting Eqns. (2.4) – (2.5) and Ansatzs (6.1) – (6.2) into Eqns. (6.3) – (6.4) and equating to zero the coefficients of like powers \(\phi\) and \(\phi\), with symbolic computation, yield the following set of equations:

\[
f_{1x} = B f_1 - \frac{\phi_x \psi'^-_{x\!x}}{B},
\]

(6.5)

\[
f_{1y} = \frac{\psi'^-_{x\!x}}{\psi'} f_1 + \frac{\phi_x}{B} \left( -\psi'^-_{1y} - A \psi'_{1x} \right),
\]

(6.6)

\[
f_{1t} = \frac{\psi'^-_{x\!x}}{\psi'} f_1 + \frac{\phi_x \psi'^-_{1x}}{B} \left( -\psi'^+_{1t} - 4 A \frac{\psi'^-_{1t}}{\psi'_{1x}} + 4 A \frac{\beta}{\alpha} - 2 \alpha A u - 4 A \tau_2 - 4 A B \right),
\]

(6.7)

\[
f_2 = 0,
\]

(6.8)

and

\[
g_{2x} = A g_2 - \frac{\phi_x \psi'^+_{1x}}{A},
\]

(6.9)

\[
g_{2y} = \frac{\psi'^+_{x\!x}}{\psi'^+} g_2 + \frac{\phi_x}{A} \left( -\psi'^+_{1y} + B \psi'^+_{1x} \right),
\]

(6.10)

\[
g_{2t} = \frac{\psi'^+_{x\!x}}{\psi'^+} g_2 + \frac{\phi_x \psi'^+_{1x}}{A} \left( -\psi'^-_{1t} - 4 B \frac{\psi'^-_{1t}}{\psi'_{1x}} + 4 B \frac{\beta}{\alpha} + 2 \alpha B u + 4 B \tau_1 - 4 B A \right),
\]

(6.11)

\[
g_1 = 0.
\]

(6.12)

From Eqns. (6.5) – (6.7) and (6.9) – (6.11), \(f_1\) and \(g_2\) can be determined as

\[
f_1 = -\psi^- \Delta (\psi'^-_{1}, \psi'^+),
\]

(6.13)

\[
g_2 = -\psi^+ \Omega (\psi'^-_{1}, \psi'^+).
\]

(6.14)
Now, the relation between the new and old eigenfunctions has been constructed. Therefore, we arrive at the Darboux transformation for System (2.1) in the form

\[ u' = u + \frac{2}{\alpha} \left\{ \log \left[ \frac{\Delta(\psi^-, \psi^+)}{\Omega(\psi^-, \psi^+)} \right] \right\}_x, \quad (6.15) \]

\[ v' = v + \frac{2}{\alpha} \left\{ \log \left[ \frac{\Delta(\psi^-, \psi^+)}{\Omega(\psi^-, \psi^+)} \right] \right\}_y, \quad (6.16) \]

\[ \psi^- = \psi_1^- - \psi^- \frac{\Delta(\psi_1^-, \psi^+)}{\Delta(\psi^-, \psi^+)}, \quad (6.17) \]

\[ \psi^+ = \psi_1^+ - \psi^+ \frac{\Omega(\psi_1^-, \psi^+)}{\Omega(\psi^-, \psi^+)}. \quad (6.18) \]

7. Iteration of binary Darboux transformation and Grammian solutions

In this section, we will perform the binary Darboux transformation \( N \) times and progressively generate the analytical Grammian solutions. After the second iteration of the binary Darboux transformation, the new potential functions and eigenfunctions are expressed as

\[ u[2] = u' + \frac{2}{\alpha} \left( \frac{\phi'_x}{\phi'} - \frac{\varphi'_x}{\varphi'} \right) = u + \frac{2}{\alpha} \left\{ \log \left( \frac{\Phi[2]}{\Sigma[2]} \right) \right\}_x, \quad (7.1) \]

\[ v[2] = v' + \frac{2}{\alpha} \left( \frac{\phi'_y}{\phi'} - \frac{\varphi'_y}{\varphi'} \right) = v + \frac{2}{\alpha} \left\{ \log \left( \frac{\Phi[2]}{\Sigma[2]} \right) \right\}_y, \quad (7.2) \]

with

\[ \Phi[2] = \phi \phi' = \Delta(\psi^-, \psi^+) \Delta(\psi'^-, \psi'^+), \quad (7.3) \]

\[ \Sigma[2] = \varphi \varphi' = \Omega(\psi^-, \psi^+) \Omega(\psi'^-, \psi'^+), \quad (7.4) \]

where \( \phi' \) and \( \varphi' \) are the new singular manifolds for \( (u', v') \), while \( (\phi, \varphi) \) and \( (\Phi[2], \Sigma[2]) \) are the first and second iterative \( \tau \)-functions in the Hirota method [10, 11, 12]. From Eqns. (7.3) and (7.4), one can immediately calculate that

\[ \Phi[2] = \left| \begin{array}{cc} \Delta(\psi_1^-, \psi_1^+) & \Delta(\psi_2^-, \psi_1^+) \\ \Delta(\psi_1^-, \psi_2^+) & \Delta(\psi_2^-, \psi_2^+) \end{array} \right|, \quad (7.5) \]

\[ \Sigma[2] = \left| \begin{array}{cc} \Omega(\psi_1^-, \psi_1^+) & \Omega(\psi_2^-, \psi_1^+) \\ \Omega(\psi_1^-, \psi_2^+) & \Omega(\psi_2^-, \psi_2^+) \end{array} \right|. \quad (7.6) \]

Taking \( \psi_i^- \) and \( \psi_i^+ \) \((i = 1, 2, 3)\) as the solutions of the Lax pairs (4.32) and (4.33) for \( (u, v) \) and iterating the Darboux transformation, we can get the following results:

\[ u[3] = u + \frac{2}{\alpha} \left\{ \log \left( \frac{\Phi[3]}{\Sigma[3]} \right) \right\}_x, \quad (7.7) \]

\[ v[3] = v + \frac{2}{\alpha} \left\{ \log \left( \frac{\Phi[3]}{\Sigma[3]} \right) \right\}_y. \quad (7.8) \]
Following the same procedure above, we iterate the Darboux transformation $N$ times with symbolic computation and obtain

$$u[N] = u + \frac{2}{\alpha} \left\{ \log \left( \frac{\Phi[N]}{\Sigma[N]} \right) \right\}_x,$$

$$v[N] = v + \frac{2}{\alpha} \left\{ \log \left( \frac{\Phi[N]}{\Sigma[N]} \right) \right\}_y,$$

with

$$\Phi[N] = \begin{vmatrix} \Delta(\psi_1^-, \psi_1^+) & \Delta(\psi_1^-, \psi_2^+) & \cdots & \Delta(\psi_1^-, \psi_N^+) \\ \Delta(\psi_2^-, \psi_1^+) & \Delta(\psi_2^-, \psi_2^+) & \cdots & \Delta(\psi_2^-, \psi_N^+) \\ \vdots & \vdots & \ddots & \vdots \\ \Delta(\psi_N^-, \psi_1^+) & \Delta(\psi_N^-, \psi_2^+) & \cdots & \Delta(\psi_N^-, \psi_N^+) \end{vmatrix},$$

$$\Sigma[N] = \begin{vmatrix} \Omega(\psi_1^-, \psi_1^+) & \Omega(\psi_1^-, \psi_2^+) & \cdots & \Omega(\psi_1^-, \psi_N^+) \\ \Omega(\psi_2^-, \psi_1^+) & \Omega(\psi_2^-, \psi_2^+) & \cdots & \Omega(\psi_2^-, \psi_N^+) \\ \vdots & \vdots & \ddots & \vdots \\ \Omega(\psi_N^-, \psi_1^+) & \Omega(\psi_N^-, \psi_2^+) & \cdots & \Omega(\psi_N^-, \psi_N^+) \end{vmatrix},$$

where $\psi_i^-$ and $\psi_i^+$ ($i = 1, 2, \cdots, N$) satisfy Lax pairs (4.32) and (4.33).

In illustration, we take $u = v = 0$ as the seed solutions for Lax pairs (4.32) and (4.33), yielding

$$\psi_1^- = \exp \{k_1 x + l_1 y + w_1 t\},$$

$$\psi_1^+ = \exp \{p_1 x + m_1 y + n_1 t\},$$

$$\psi_2^- = \exp \{k_2 x + l_2 y + w_2 t\},$$

$$\psi_2^+ = \exp \{p_2 x + m_2 y + n_2 t\},$$

where $l_i = \frac{2 \gamma}{\alpha} k_i - k_i^2$, $w_i = 4k_i^3 - \frac{12 \beta}{\alpha} k_i^2 + \frac{12 \gamma}{\alpha^2} k_i$, $m_i = \frac{2 \gamma}{\alpha} p_i + p_i^2$, $n_i = 4p_i^3 + \frac{12 \beta}{\alpha} p_i^2 + \frac{12 \gamma}{\alpha^2} p_i$ with $k_i$ and $p_i$ ($i = 1, 2$) as arbitrary constants.

Integration of Eqns. (5.15)–(5.20) with respect to $x$, $y$ and $t$ results in

$$\Delta(\psi_1^-, \psi_1^+) = \frac{k_1}{k_1 + p_1} \psi_1^- \psi_1^+ + \delta_1,$$
\[ \Delta (\psi_2^-, \psi_2^+) = \frac{k_2}{k_2 + p_2} \psi_2^- \psi_2^+ + \delta_2, \]  
(7.20)
\[ \Delta (\psi_1^-, \psi_2^+) = \frac{k_1}{k_1 + p_2} \psi_1^- \psi_2^+ + \delta_3, \]  
(7.21)
\[ \Delta (\psi_2^-, \psi_1^+) = \frac{k_2}{k_2 + p_1} \psi_2^- \psi_1^+ + \delta_4, \]  
(7.22)
\[ \Delta (\psi_1^-, \psi_1^+) = \frac{k_1}{k_1 + p_1} \psi_1^- \psi_1^+ + \delta_1, \]  
(7.23)
\[ \Omega (\psi_1^-, \psi_1^+) = \frac{p_1}{k_1 + p_1} \psi_1^- \psi_1^+ - \delta_1, \]  
(7.24)
\[ \Omega (\psi_2^-, \psi_2^+) = \frac{p_2}{k_2 + p_2} \psi_2^- \psi_2^+ - \delta_2, \]  
(7.25)
\[ \Omega (\psi_1^-, \psi_2^+) = \frac{p_1}{k_1 + p_2} \psi_1^- \psi_2^+ - \delta_3, \]  
(7.26)
\[ \Omega (\psi_2^-, \psi_1^+) = \frac{p_2}{k_2 + p_1} \psi_2^- \psi_1^+ - \delta_4, \]  
(7.27)

where \( \delta_i \ (i = 1, 2, 3, 4) \) are all arbitrary constants. Particularly, setting \( \delta_1 = \delta_2 = 1 \) and \( \delta_3 = \delta_4 = 0 \), the \( \tau \)-functions can be expressed as follows:

\[ \phi = 1 + k_1 F_1, \]  
(7.28)
\[ \varphi = p_1 F_1 - 1, \]  
(7.29)
\[ \Phi[2] = 1 + k_1 F_1 + k_2 F_2 + k_1 k_2 R F_1 F_2, \]  
(7.30)
\[ \Sigma[2] = 1 - p_1 F_1 - p_2 F_2 + p_1 p_2 R F_1 F_2, \]  
(7.31)

where

\[ F_1 = \exp \left\{ (k_1 + p_1) x + (l_1 + m_1) y + (w_1 + n_1) t + \theta_1 \right\}, \]  
(7.32)
\[ F_2 = \exp \left\{ (k_2 + p_2) x + (l_2 + m_2) y + (w_2 + n_2) t + \theta_2 \right\}, \]  
(7.33)
\[ R = \frac{(k_1 - k_2)(p_1 - p_2)}{(k_1 + p_1)(k_2 + p_2)}, \]  
(7.34)
\[ e^{-(k_1 + p_1) \theta_1} = k_1 + p_1, \]  
(7.35)
\[ e^{-(k_2 + p_2) \theta_2} = k_2 + p_2. \]  
(7.36)

It is of interest to note that Expressions (7.28)−(7.31) are very similar to the forms of soliton solutions obtained with the Hirota method [41]. Substituting Expressions (7.28)−(7.31) into Eqns. (6.15), (6.16), (7.1) and (7.2), one can get the one and two soliton solutions in the sense of Refs. [9, 41, 42, 43].

8. Conclusions

The singular manifold method from the Painlevé analysis plays a vital role in investigating many important integrable properties for the NPDEs. In this paper, we have successfully applied the two-singular-manifold method to the (2+1)-dimensional Gardner equation and
derived its Hirota bilinear form, bilinear Bäcklund transformation, Lax pairs, as well as binary Darboux transformation. With the help of symbolic computation, we have performed the $N$-time iterative algorithm of binary Darboux transformation to generate the $N \times N$ Grammian solution.

Acknowledgments

We express our thanks to Prof. Y. T. Gao and Ms. X. H. Meng for their valuable comments. This work has been supported by the Key Project of Chinese Ministry of Education (No. 106033), by the Specialized Research Fund for the Doctoral Program of Higher Education (No. 20060006024), Chinese Ministry of Education, and by the National Natural Science Foundation of China under Grant No. 60372095.

Appendix: Identity properties of the bilinear operator

The following identities are used in the derivation of the bilinear Bäcklund transformation.

\[
D_x c_0 a(x) \cdot a(x) = D_x a(x) \cdot c_0 a(x) = 0, \\
(D_x a \cdot b) c d - a b (D_x c \cdot d) = (D_x a \cdot c) b d - a c (D_x b \cdot d) = D_x (a d) \cdot (b c), \\
(D_x^2 a \cdot b) c d - a b (D_x^2 c \cdot d) = (D_x^2 a \cdot c) b d - a c (D_x^2 b \cdot d) - 2 D_x a d \cdot (D_x b \cdot c), \\
(D_x^3 a \cdot b) c d - a b (D_x^3 c \cdot d) = (D_x^3 a \cdot c) b d - a c (D_x^3 b \cdot d) - 3 D_x (D_x a \cdot d) \cdot (D_x b \cdot c), \\
(D_x D_y a \cdot b) c d - a b (D_x D_y c \cdot d) = (D_x D_y a \cdot c) b d - a c (D_x D_y b \cdot d) - D_x (a d) \cdot (D_y b \cdot c) - D_y (a d) \cdot (D_x b \cdot c).
\]

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