Extension of the de Broglie-Bohm theory to the Ginzburg-Landau equation

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The de Broglie-Bohm approach permits to assign well defined trajectories to particles that obey the Schroedinger equation. We extend this approach to electron pairs in a superconductor. In the stationary regime this extension is completely natural; in the general case additional postulates are required. This approach gives enlightening views for the absence of Hall effect in the stationary regime and for the formation of permanent currents.

Keywords: de Broglie-Bohm, Ginzburg-Landau, quantum force, quantum potential, Hall effect in superconductors, beables

I. INTRODUCTION

If the evolution of a particle obeys the Schroedinger equation, then the de Broglie-Bohm quantum theory (dBB) provides a deterministic description of its motion by assuming that, besides the classical forces, an additional “quantum force” \( \nabla (\hbar^2 \nabla^2 |\psi|/2m|\psi|) \) acts on the particle. The dBB theory remains applicable in the presence of magnetic fields. Variations and extensions of dBB have recently been considered.

In this article we want to extend dBB to the motion of Cooper pairs in a superconductor, which obey the Ginzburg-Landau equation:

\[
\frac{1}{2m} \left(-i\hbar \nabla - \frac{q}{c} A\right)^2 \psi + \alpha \psi + \beta |\psi|^2 \psi = -\gamma \frac{\partial \psi}{\partial t} .
\] (1)

Here we follow the notation of Ref. 8, that makes Eq. (1) look similar to the Schroedinger equation: \( \psi \) is the “order parameter”, \( t \) is the time, \( q \) and \( m \) are the charge and the mass of a pair, \( \hbar \) and \( c \) are Planck’s constant and the speed of light, \( \alpha, \beta \) and \( \gamma \) are material constants, and \( A \) is the vector potential of the electromagnetic field. We have chosen a gauge such that the scalar potential vanishes.

In this letter we will interpret the order parameter \( \psi = |\psi|e^{i\varphi} \) as the wavefunction for Cooper pairs. As usual, \( |\psi|^2 \) will have the meaning of density of particles, \( \hbar \nabla \varphi \) will be the canonic momentum, and \( \mathbf{v} = (\hbar \nabla \varphi - \frac{q}{c} \mathbf{A})/m \), the velocity of the pairs. However, the equation of motion will not be based on the usual Hamilton–Jacobi formalism, but just on Newton’s second law. The price we will have to pay for this approach is some basic knowledge of fluid mechanics (e.g. Ref. 9).

II. STATIONARY SITUATIONS

In this case the right hand side of Eq. (1) vanishes. The remaining equation differs qualitatively from the Schroedinger equation, since it is nonlinear. We shall see that, nevertheless, the dBB theory can be smoothly extended.

A. Equation of Motion

For a stationary situation, conservation of particles gives \( \nabla \cdot (|\psi|^2 \mathbf{v}) = 0 \), which leads to

\[
2(\hbar \nabla \varphi - \frac{q}{c} \mathbf{A}) \cdot \nabla |\psi| + |\psi| (\hbar \nabla^2 \varphi - \frac{q}{c} \nabla \cdot \mathbf{A}) = 0 .
\] (2)

Expanding Eq. (1), using Eq. (2), and defining

\[
-Q_{\text{stat}} = \frac{\hbar^2}{2m} \frac{\nabla^2 |\psi|}{|\psi|} - \beta |\psi|^2 ,
\] (3)

we arrive at

\[
-Q_{\text{stat}} = \frac{1}{2m} (\hbar \nabla \varphi - \frac{q}{c} \mathbf{A})^2 + \alpha .
\] (4)
Taking the gradient at both sides of Eq. (4) and noting that in the stationary regime $a = (v \cdot \nabla)v$ is the convective time derivative of the velocity, and hence the acceleration of a pair, we obtain

$$-\nabla Q_{\text{stat}} + qv \times B = ma, \tag{5}$$

with $B = \nabla \times A$. This means that, in addition to the magnetic force, the quantum force $-\nabla Q_{\text{stat}}$ acts on every pair. As a matter of principle, $-\nabla Q_{\text{stat}}$ is not only the quantum force, but it also includes the force exerted by the lattice. Note that the only physical input required for Eq. (5) is Eq. (1) (with vanishing r.h.s.) and the interpretations made in the Introduction; the material parameters, the applied field or the geometry of the superconductor have no influence.

B. The Hall Effect

We shall see that Eq. (5) provides an intuitive way to explain the absence of Hall effect in a superconductor in a stationary state. The Hall effect was discovered in 1879, well before quantum effects were suspected to exist. Historically, it turned out that an apparently trivial classical explanation could be given to the effect (except for the existence of positive carriers). Only when the Hall voltage turned out not to be a linear function of the magnetic field, the effect was called “quantum”.

In the case of a superconductor carrying a stationary current perpendicular to a magnetic field that penetrates the sample, there is clearly a Lorentz force that acts on the flowing electrons; therefore, it seems obvious to anticipate the presence of an electric force that balances it, implying a Hall voltage. However, early experiments found that the Hall voltage drops to zero when a sample becomes superconducting. Moreover, if there were an electric field in the superconductor, it would produce normal currents and dissipation. At least in the case of permanent currents, this is not what is observed.

The resolution of the paradox is provided by Eq. (5): the Lorentz force is balanced by the quantum force $-\nabla Q_{\text{stat}}$ in order to complete the total force required to keep the pairs on their trajectories, and no electric force is required.

Later experiments did find a Hall voltage in superconductors (e.g. Ref. 14). However, in these cases the voltage is due to the motion of vortices and cannot be described as a stationary situation.

C. An Example

As an illustration, let us examine a situation with constant field and cylindric symmetry. Let the field be $B = B_0 \hat{z}$ and the vector potential, $A = \frac{1}{2}B_0 r \hat{\theta}$, where $r$, $\theta$ and $z$ are cylindric coordinates. Cylindric symmetry implies that $\varphi$ is a linear function of $\theta$. Together with single-valuedness, this enables us to write $\varphi = -n\theta$, where $n$ is an integer. It follows that $v = (1/m)(|q|B_0 r/2c - n\hbar/r)\hat{\theta}$ and therefore the sum of the Lorentz and the centrifugal force is

$$F_{\text{Lorentz}} + F_{\text{centrifugal}} = \frac{1}{mr} \left( \frac{n^2 \hbar^2}{r^2} - \frac{q^2 B_0^2 r^2}{4c^2} \right) \hat{r}.$$

This force points away from or towards the $z$-axis, depending on whether $r$ is smaller or greater than $r_B = (2n\hbar/|q|B_0)^{1/2}$. On the other hand, using Eq. (4) we find

$$Q_{\text{stat}} = \frac{1}{2m} \left( \frac{n\hbar}{r} - \frac{|q|B_0 r}{2c} \right)^2 + \alpha,$$

(The quantum potential is maximal at $r = r_B$.) We can now immediately verify that $-\nabla Q_{\text{stat}} = -F_{\text{Lorentz}} - F_{\text{centrifugal}}$.

III. GENERAL CASE

Now Eq. (1) is not only nonlinear; it is also nonunitary. Accordingly, the number of Cooper pairs is not conserved and any model that is limited to a fixed number of particles cannot describe the physical situation. We can still envision a deterministic evolution in the spirit of Refs. 13 and 16 (“beables”). The objects of the model are the order parameter $\psi$, which acts as a field, and the density of particles. We still postulate that particles move with velocity $(\hbar \nabla \varphi - \frac{2}{\hbar A})/m$. The order parameter, the density and the velocity depend on position and on time. The evolution
of the system will consist of two processes: motion of pairs and creation (or destruction) of pairs. In order to have a consistent model, we require that if the density of pairs is initially \(|\psi|^2\), it will remain given by \(|\psi|^2\) as the system evolves.

A. Pair Trajectories

We expand Eq. (10) and factor out \(e^{i\varphi}\). Equating the imaginary parts we obtain

\[
\nabla \cdot (|\psi|^2 \mathbf{v}) = 2\gamma|\psi|^2 \frac{\partial \varphi}{\partial t}
\]

(6)

and, from the real parts

\[
-Q_{\text{stat}} = \alpha + \frac{m}{2} \nu^2 + \frac{\gamma \hbar}{2|\psi|^2} \frac{\partial|\psi|^2}{\partial t}.
\]

(7)

We define now

\[
-Q_{\text{dyn}} \equiv -\frac{\hbar}{2|\psi|^2} \left( \frac{\partial|\psi|^2}{\partial t} - \frac{1}{\gamma} \nabla \cdot (|\psi|^2 \mathbf{v}) \right).
\]

(8)

Taking the gradient at both sides of Eq. (7), using Eq. (6), noting that the acceleration is now \(a = (\mathbf{v} \cdot \nabla)\mathbf{v} + \partial \mathbf{v} / \partial t\) and the electric field is \(\mathbf{E} = -(1/c)\partial \mathbf{A} / \partial t\) gives

\[
-\nabla (Q_{\text{stat}} + Q_{\text{dyn}}) + q \left( \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) = ma,
\]

(9)

so that the motion of the pairs can still be interpreted as having Newtonian trajectories, provided that the quantum force \(-\nabla (Q_{\text{stat}} + Q_{\text{dyn}})\) is added to the electromagnetic force. Note that now the quantum force is not a function of \(|\psi|\) only, but is still a functional of \(\psi\).

In regions where \(|\psi|\) is nearly constant and uniform, \(-\nabla Q_{\text{dyn}} \approx \frac{\hbar}{2|\psi|^2} \nabla^2 \mathbf{v}\). In this case \(-\nabla Q_{\text{dyn}}\) behaves as a viscous force, with viscosity coefficient \(\frac{\hbar}{2\gamma^2}|\psi|^2\).

B. Creation and Annihilation of Particles

We build a model for formation and destruction of pairs as follows. We define the depairing potential as

\[
\psi_{\text{dep}} \equiv 2 \left( \frac{\gamma}{1 + \frac{1}{\gamma}} \right) \left( Q_{\text{stat}} + \alpha + \frac{m}{2} \nu^2 \right) + 2\gamma Q_{\text{dyn}}.
\]

(10)

We postulate that at intervals of time \(\tau\) the system checks itself. (For typical non-stationary processes in superconductors, characteristic times are of the order of \(10^{-13}\) s.) At the end of each interval we divide the sample into “depairing cells” such that the volume integral in each cell obeys \(|\int Q_{\text{dep}}|\psi|^2 dV| = h/\tau\). Since the integral of \(Q_{\text{dep}}|\psi|^2\) over the entire sample is not necessarily integer, some marginal region of the sample will in general be left out of the cells. In order to have a well defined division we need some additional criterion; for instance, we may require that the average boundary area of the cells be as small as possible. This criterion would discourage elongated cells or merging of cells with positive and negative \(Q_{\text{dep}}\); also, the regions left out of the cells will be those where \(|Q_{\text{dep}}|\psi|^2\) is small. Finally, we postulate that if \(\int Q_{\text{dep}}|\psi|^2 dV\) is negative, then a new Cooper pair forms at the center of the cell and, if it is positive, the pair closest to the center is destroyed.

We have to prove that the evolution of the pair density predicted by this model is the same as that predicted by Eq. (11). The pair density changes due to two processes: flow of pairs and net pair creation. The increase of density per unit time do to flow is \(-\nabla \cdot (|\psi|^2 \mathbf{v})\). The increase due to creation equals the net number of negative cells per unit volume, \(-Q_{\text{dep}}|\psi|^2 \tau / h\), multiplied by the frequency \(1/\tau\). (We have assumed that the cells and \(\tau\) are sufficiently small to be treated as a continuum, with \(Q_{\text{dep}}|\psi|^2\) uniform in the analyzed region.) Substituting Eqs. (6) and (8) into Eq. (10) we obtain that the total rate of density increase is \(\partial|\psi|^2 / \partial t\), as required.

For fast processes we expect that typical depairing cells will have microscopic sizes, but if the particle density is small or a process is almost stationary, these cells can be large. We could build an alternative model by postulating the existence of depairing cells with fixed positions and sizes, which destroy or create a pair every time that the integral \(\int \int Q_{\text{dep}}|\psi|^2 dV / dt\) in the cell changes by \(\pm h\).
C. Example

We consider ring of radius \( R \), sufficiently thin to be treated as one-dimensional, threaded by a magnetic flux \( \Phi \). For negative times the ring is in the normal state, but at \( t = 0 \) it is instantaneously cooled and becomes superconducting.

In Eq. (11) cooling below the critical temperature is implemented by changing the value of \( \alpha \) from positive to negative. It is well known that if \( \Phi \) is not an integer multiple of the quantum of flux \( \Phi_0 = \frac{ch}{|q|} \), then a permanent current flows around the ring. A classical-minded question naturally arises: since there are no electromagnetic fields in the ring, how do the charges know to start moving when the temperature is lowered?

In order to answer this question we first have to calculate \( \psi \). Let us restrict our attention to the period during which \( |\psi| \) is so small that the nonlinear term in Eq. (11) can be neglected. For a thin ring Eq. (11) takes the one-dimensional form

\[
\frac{\hbar^2}{2mR^2} \left( \frac{i}{\partial \theta} - \frac{\Phi}{\Phi_0} \right)^2 \psi + \alpha \psi = -\hbar \gamma \frac{\partial \psi}{\partial t}.
\]

We write \( \psi = \sum_{n=-\infty}^{\infty} T_n(t)e^{-ni\theta} \) and linearity permits to treat each harmonic separately. We obtain

\[
T_n(t) = T_n(0)e^{\lambda_n t}, \quad \lambda_n = -\frac{\alpha}{\hbar \gamma} - \frac{\hbar(n - \Phi/\Phi_0)^2}{2m \gamma R^2}.
\]

Since the ring was initially in the normal state, the initial values \( T_n(0) \) are very small. (They don’t vanish due to thermal fluctuations.)

For simplicity, let us consider the case in which \( \lambda_n > 0 \) for just one value of \( n \) (the closest to \( \Phi/\Phi_0 \)), which will be denoted by \( \tilde{n} \). This means that \( T_{\tilde{n}} \) will increase with time, whereas \( T_n \) will remain negligible for \( n \neq \tilde{n} \). As a consequence, \( |\psi| \) will be independent of position.

After \( \psi \) is known, we can evaluate the potentials. We obtain \( Q_{\text{stat}} = 0 \), \( Q_{\text{dyn}} = \hbar \gamma \lambda_{\tilde{n}} \) and \( Q_{\text{dep}} = -2\hbar \lambda_{\tilde{n}} < 0 \). Since these potentials are independent of position, it follows that the pairs are not accelerated. Current arises due to their formation, with velocity \( \left( \hbar/mR \right)(\Phi/\Phi_0 - \tilde{n})\dot{\theta} \).

IV. CONCLUSION

We have extended the de Broglie-Bohm theory, which was built for particles that obey the Schroedinger equation, to the case of electron pairs in a superconductor, which obey a more complex (nonlinear and nonunitary) equation. For the stationary regime this extension is completely natural; in the general case, in which the number of pairs is not conserved, additional postulates are required.

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