Deformations of calibrations,
Calabi-Yau, HyperKähler,
$G_2$ and Spin(7) structures

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Abstract. We shall develop a new deformation theory of geometric structures in terms of closed differential forms. This theory is a generalization of Kodaira-Spencer theory and further we obtain a criterion of unobstructed deformations. We apply this theory to certain geometric structures: Calabi-Yau, HyperKähler, $G_2$ and Spin(7) structures and show that these deformation spaces are smooth in a systematic way.

§0. Introduction

Let $X$ be a compact Riemannian manifold with vanishing Ricci curvature. Then the list of holonomy group of $X$ includes four interesting classes of the holonomy groups: $SU(n)$, $Sp(m)$, $G_2$ and $Spin(7)$ [1]. The Lie group $SU(n)$ arises as the holonomy group of Calabi-Yau manifolds and $Sp(m)$ is the holonomy group of HyperKähler manifolds. The Lie groups $G_2$ and $Spin(7)$ occur as the holonomy groups of 7 and 8 dimensional manifolds respectively. There are many intriguing common properties between these four geometries. One of the most remarkable
property is smoothness of the deformation spaces of these geometric structures. In the case of Calabi-Yau manifolds, Tian and Todorov show that smoothness of the deformation space (Kuranishi space) by using Kodaira-Spencer theory [19],[20]. Joyce obtains smooth moduli spaces of $G_2$ and Spin(7) structures respectively [10,11,12]. His method of construction of moduli spaces are different from Tian-Todorov’s one since $G_2$ and Spin(7) manifolds are real manifolds and we can not apply the deformation theory of complex manifolds. Hitchin shows a significant and suggestive construction of deformation spaces of a Calabi-Yau structures on real 6 manifolds and $G_2$ structures on 7 manifolds [9]. It must be noted that these four geometries are defined by certain closed differential forms on real manifolds. In this paper we shall obtain a direct and unified construction of deformation spaces of these geometric structures on real manifolds in terms of these closed differential forms. We shall show that the deformation spaces of these geometric structures are smooth in a systematic way. In the case of Calabi-Yau manifolds, we consider a real compact $2n$ manifold with a pair of a closed complex $n$ form $\Omega$ and a symplectic form $\omega$. We show that a certain pair $(\Omega, \omega)$ defines a Calabi-Yau metric (Ricci-flat Kähler metric) on $X$. Hence the deformation space of Calabi-Yau metrics on $X$ arises as the deformation space of certain pairs of closed forms $(\Omega, \omega)$ (see section 4-2 for precise definition of Calabi-Yau structures ). In section 1, we discuss a general deformation theory of geometric structures defined by closed differential forms. Let $V$ be a real $n$ dimensional vector space. Then we consider the linear action $\rho$ of $G = \text{GL}(V)$ on the direct sum of skew-symmetric tensors,

$$\rho: \text{GL}(V) \to \bigoplus_{i=1}^{l} \text{End}(\wedge^{p_i} V^*).$$

Let $\Phi_0^* = (\phi_0^1, \phi_0^2, \cdots, \phi_0^l)$ be an element of $\bigoplus_{i=1}^{l} \wedge^{p_i} V^*$. Then we have the orbit $O = O_{\Phi_0^*}(V)$ under the action of $G$ as

$$O_{\Phi_0^*}(V) = \{ \rho_g \Phi_0^* = (\rho_g \phi_0^1, \cdots, \rho_g \phi_0^l) \in \bigoplus_{i=1}^{l} \wedge^{p_i} V^* \mid g \in G \}.$$

Then the orbit $O = O_{\Phi_0^*}$ is regarded as a homogeneous space $G/H$, where $H$ is the isotropy group. Let $X$ be a real $n$ dimensional compact
manifold. Then we define a homogeneous space bundle $A_\mathcal{O}(X) \to X$ by

$$A_\mathcal{O}(X) = \bigcup_{x \in X} O_{\Phi_0}(T_xX).$$

Then we define $E_\mathcal{O}(X)$ to be the set of global sections $\Gamma(X, A_\mathcal{O}(X))$. We denote by $\tilde{M}_\mathcal{O}(X)$ the set of closed forms in $E_\mathcal{O}(X)$. Let $\Phi^0$ be an element of $\tilde{M}_\mathcal{O}(X)$. As we shall show that $E_\mathcal{O}(X)$ is regarded as a infinite dimensional homogeneous space (a Hilbert manifold). Hence we have the tangent space $T_{\Phi^0}E_\mathcal{O}(X)$ of $E_\mathcal{O}(X)$. We denote by $\mathcal{H}$ the Hilbert space consisting of closed forms in $\bigoplus_i \wedge^p_i$. Then the space $\tilde{M}_\mathcal{O}(X)$ is the intersection between the Hilbert space $\mathcal{H}$ and the Hilbert manifold $E_\mathcal{O}(X)$. We define an infinitesimal tangent space of $\tilde{M}_\mathcal{O}(X)$ by the intersection $\mathcal{H} \cap T_{\Phi^0}E_\mathcal{O}(X)$. Then we shall discuss if the infinitesimal tangent space is regarded as the tangent space of actual deformations.

**Definition 1-4.** A closed element $\Phi^0 \in E^1(X)$ is unobstructed if there exists an integral curve $\Phi_t(\alpha)$ in $\tilde{M}_\mathcal{O}(X)$ for each infinitesimal tangent vector $\alpha \in \mathcal{H} \cap T_{\Phi^0}E_\mathcal{O}(X)$ such that

$$\frac{d}{dt} \Phi_t(\alpha)|_{t=0} = \alpha$$

An orbit $\mathcal{O}$ is unobstructed if any $\Phi^0 \in \tilde{M}_\mathcal{O}(X)$ is unobstructed for all compact $n$ dimensional manifold $X$.

We shall prove the following theorem in section 3.

**Theorem 1-5 (Criterion of unobstructedness).** We assume that an orbit $\mathcal{O}$ is elliptic (see definition 1-1 in section one). If the map $p^2 : H^2(\#_{\Phi^0}) \to \bigoplus_i H^{p_i+1}_{DR}(X)$ is injective, then $\Phi^0$ is unobstructed (see section 1 for $p^2$).

In section 2 we obtain preliminary results. In section 3 we try to construct a deformation of calibrations as a formal power series in $t$. Then we encounter obstructions to deformation of calibrations. A primary
obstruction is discussed in section 3-1. If the primary obstruction 
vanishes, then we have the second obstruction. Successively we have higher 
obstructions to deformations. Explicit description of higher obstructions 
are given in section 3-2. In section 3-3, we prove our criterion of unob-
structedness (Theorem 1-5). If the criterion holds, then all obstruction 
vanish simultaneously. Hence we have a deformation of calibrations as 
a formal power series in \( t \). Further we prove the power series uniformly 
converges. In section 4, 5 and 6 we shall show that our criterion holds for 
Calabi-Yau, HyperKähler, \( G_2 \) and Spin(7) structures. In section 4-1 we 
define a \( SL_n(\mathbb{C}) \) structure as a certain complex form \( \Omega \), which defines the 
almost complex structure \( I_{\Omega} \) with trivial canonical line bundle. Then the 
integrability of the almost complex structure \( I_{\Omega} \) is given by a closeness 
of the complex differential form \( \Omega \). We show that the orbit of \( SL_n(\mathbb{C}) \) 
structures is elliptic and satisfies the criterion. In section 4-2, we define a 
Calabi-Yau structure as a certain pair consisting \( SL_n(\mathbb{C}) \) structure \( \Omega \) and 
a real symplectic form \( \omega \). Then we prove that the orbit corresponding to 
a Calabi-Yau structure is elliptic and satisfies the criterion. Hence the 
orbit of Calabi-Yau structures is unobstructed. Background material of 
Calabi-Yau manifolds is found in [1]. Our primary obstruction of \( SL_n(\mathbb{C}) \) 
structures corresponds to the one of Kodaira-Spencer theory. Then our 
result is regarded as another proof of unobstructedness by using cali-
brations. Our direct proof reveals a geometric meaning of unobstructed 
deformations. (we do not use Calabi-Yau’s theorem to obtain a smooth 
deformation space of Calabi-Yau structures). It must be noted that 
Kawamata and Ran give algebraic proof of unobstructed deformations. 
In section 5, we show the orbit corresponding to a HyperKähler structure 
is also elliptic and satisfies the criterion. In section 6 and 7 we discuss 
unobstructedness of \( G_2 \) and Spin (7) structures respectively.

§1. Deformation spaces of calibrations

Let \( V \) be a real vector space of dimension \( n \). We denote by \( \wedge^p V^* \) the 
vector space of \( p \) forms on \( V \). Let \( \rho_p \) be the linear action of \( G = GL(V) \) 
on \( \wedge^p V^* \). Then we have the action \( \rho \) of \( G \) on the direct sum \( \bigoplus_i \wedge^{p_i} V^* \)
by
\[ \rho: GL(V) \longrightarrow \bigoplus_{i=1}^{l} \text{End}(\wedge^{p_i} V^*), \]
\[ \rho = (\rho_{p_1}, \cdots, \rho_{p_l}). \]
We fix an element \( \Phi^0_V = (\phi^0_1, \phi^0_2, \cdots, \phi^0_l) \in \bigoplus_i \wedge^{p_i} V^* \) and consider the \( G \)-orbit \( O = O_{\Phi^0_V} \) through \( \Phi^0_V \):
\[ O_{\Phi^0_V} = \{ \Phi_V = \rho_g \Phi^0_V \in \bigoplus_i \wedge^{p_i} V^* \mid g \in G \} \]
The orbit \( O_{\Phi^0_V} \) can be regarded as a homogeneous space, \( O_{\Phi^0_V} = G/H \),
where \( H \) is the isotropy group
\[ H = \{ g \in G \mid \rho_g \Phi^0_V = \Phi^0_V \}. \]
We denote by \( A_O(V) = A(V) \) the orbit \( O_{\Phi^0_V} = G/H \). The tangent space \( E^1(V) = T_{\Phi^0_V} A(V) \) is given by
\[ E^1(V) = T_{\Phi^0_V} A(V) = \{ \dot{\rho}_\xi \Phi^0_V \in \bigoplus_i \wedge^{p_i} V^* \mid \xi \in \mathfrak{g} \}, \]
where \( \dot{\rho} \) denotes the differential representation of \( \mathfrak{g} \). The vector space \( E^1(V) \) is the quotient space \( \mathfrak{g}/\mathfrak{h} \). We also define a vector space \( E^0(V) \) by the interior product,
\[ E^0(V) = \{ i_v \Phi^0_V = (i_v \phi^0_1, \cdots, i_v \phi^0_l) \in \bigoplus_i \wedge^{p_i-1} V^* \mid v \in V \}. \]
\( E^2(V) \) is defined as a vector space spanned by the following set,
\[ E^2(V) = \text{Span}\{ \alpha \wedge i_v \Phi^0_V \in \bigoplus_i \wedge^{p_i+1} V^* \mid \alpha \in \wedge^2 V^*, i_v \Phi^0_V \in E^0(V) \}. \]
We also define \( E^k(V) \) for \( k \geq 0 \) by
\[ E^k(V) = \text{Span}\{ \beta \wedge i_v \Phi^0_V \in \bigoplus_i \wedge^{p_i+k-1} V^* \mid \beta \in \wedge^k V^*, i_v \Phi^0_V \in E^0(V) \}. \]
Let \(\{e_1, \ldots, e_n\}\) be a basis of \(V\) and \(\\{\theta^1, \ldots, \theta^n\}\) the dual basis of \(V^*\). Then we see that \(\hat{\rho}_\xi \Phi^0_V\) is written as

\[
\hat{\rho}_\xi \Phi^0_V = \sum_{ij} \xi^j_i \theta^j \wedge i_{e_i} \Phi^0_V,
\]

where \(\xi = \sum_{ij} \xi^j_i \theta^j \otimes e_i\) and \(i_{e_i}\) denotes the interior product. Hence we have the graded vector space \(E(V) = \bigoplus_k E^k(V)\) generated by \(E^0(V)\) over \(\wedge^* V^*\). Then we have the complex by the exterior product of a nonzero \(u \in V^*\),

\[
E^0(V) \xrightarrow{\wedge u} E^1(V) \xrightarrow{\wedge u} E^2(V) \xrightarrow{\wedge u} \cdots.
\]

**Definition 1-1 (elliptic orbits).** An orbit \(O_{\Phi^0_V}\) is an elliptic orbit if the complex

\[
E^0(V) \xrightarrow{\wedge u} E^1(V) \xrightarrow{\wedge u} E^2(V) \xrightarrow{\wedge u} \cdots
\]

is exact for any nonzero \(u \in V^*\). In other words, if \(\alpha \wedge u = 0\) for \(\alpha \in E^k(V)\), then there exists \(\beta \in E^{k-1}(V)\) such that \(\alpha = \beta \wedge u\) for \(k \geq 1\).

**Remark.** If \(\alpha \wedge u = 0\), then we have \(\alpha = \beta \wedge u\) for some \(\beta \in \bigoplus_i \wedge^{p_i-1}\) since the de Rham complex is elliptic. However \(\beta\) is not an element of \(E^0(V)\) in general, (Note that \(E^0(V)\) is a subspace of \(\bigoplus_i \wedge^{p_i-1}\)). For instance, we take \(\Phi^0_V\) as a real symplectic form \(\omega\) on a real 2\(n\) dimensional vector space \(V\). Then \(E^0 = \wedge^1\) and \(E^1 = \wedge^2\). Hence \(O_{\Phi^0_V}\) is elliptic. However if \(\Phi^0_V\) is a degenerate 2 form on \(V\), i.e., \(\omega^n = 0\), then \(O_{\Phi^0_V}\) is not elliptic.

Let \(X\) be a compact real manifold of dimension \(n\). We define \(A_{O}(T_x X)\) by using an identification \(h: T_x X \cong V\). The subspace \(A_{O}(T_x X) \subset \bigoplus_i \wedge^{p_i} T_x^* X\) is independent of a choice of the identification \(h\). Hence we define the \(G/H\)-bundle \(A(X) = A_{O}(X)\) by

\[
A_{O}(X) = \bigcup_{x \in X} A(T_x X) \rightarrow X.
\]
We denote by $\mathcal{E}^1 = \mathcal{E}^1(X)$ the set of $C^\infty$ global sections of $\mathcal{A}(X)$,

$$\mathcal{E}^1(X) = \Gamma(X, \mathcal{A}(X)).$$

Let $\Phi^0$ be a closed element of $\mathcal{E}^1$. Then we have the vector spaces $E^k(T_xX)$ for each $x \in X$ and $k \geq 0$. We define the vector bundle $E^k_x = E^k$ over $X$ as

$$E^k_x = E^k := \bigcup_{x \in X} E^k(T_xX) \longrightarrow X.$$ 

for each $k \geq 0$. (Note that the fibre of $E^1$ is $\mathfrak{g}/\mathfrak{h}$.) Then we define the graded module $\Gamma(E)$ over $\Gamma(\wedge^*)$ as $\bigoplus_k \Gamma(E_k^k)$, where $\Gamma$ denotes the set of global $C^\infty$ sections and $\wedge^p$ is the sheaf of germs of smooth $p$ forms on $X$.

**Theorem 1-2.** $\Gamma(E)$ is the differential graded module in $\bigoplus_k \Gamma(\bigoplus_i \wedge^{p_i+k-1})$ with respect to the exterior derivative $d$.

**Proof.** Since $\Gamma(E)$ is the graded module generated by $\Gamma(E^0)$, it is suffices to prove that $di_v^* \Phi^0$ is an element of $\Gamma(E^1)$ for $v \in \Gamma(TX)$. We denote by $\text{Diff}(X)$ the group of diffeomorphisms of $X$. Then there is the action of $\text{Diff}(X)$ on differential forms on $X$ and we see that $\mathcal{E}^1(X)$ is invariant under the action of $\text{Diff}(X)$. An element of $\Gamma(E^0)$ is given as $i_v^* \Phi^0 = (i_v \phi_1, \ldots, i_v \phi_l)$, where $v \in \Gamma(TX)$. Since $\Phi^0$ is closed, we have

$$di_v^* \Phi^0 = L_v \Phi^0.$$ 

The vector field $v$ generates the one parameter group of transformation $f_t$. Then $L_v \Phi^0 = \frac{d}{dt} f_t^* \Phi^0|_{t=0}$. Since $\mathcal{E}^1(X)$ is invariant under the action of $\text{Diff}(X)$, $f_t^* (\Phi^0) \in \mathcal{E}^1(X)$. Since the tangent space of $\mathcal{E}^1$ at $\Phi^0$ is $\Gamma(E^1)$, $L_v \Phi^0 \in \Gamma(E^1)$. Hence $di_v^* \Phi^0 \in \Gamma(E^1)$. From definition of $E^k(V)$, we see that $da \in \Gamma(E^k)$ for all $a \in \Gamma(E^{k-1})$ for all $k$. □

Then from theorem 1-2, we have a complex $\#_{\Phi^0}$

$$(\#_{\Phi^0}) \quad \Gamma(E^0) \xrightarrow{d_0} \Gamma(E^1) \xrightarrow{d_1} \Gamma(E^2) \xrightarrow{d_2} \cdots,$$
where $\Gamma(E^i)$ is the set of $C^\infty$ global sections for each vector bundle and $d_i = d|_{E^i}$ for each $i = 0, 1, 2$. The complex $\#_{\Phi^0}$ is a subcomplex of the direct sum of the de Rham complex (For simplicity, we call this complex the de Rham complex):

$$
\begin{array}{cccccccc}
\Gamma(E^0) & \xrightarrow{d_0} & \Gamma(E^1) & \xrightarrow{d_1} & \Gamma(E^2) & \xrightarrow{d_2} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\Gamma(\oplus_i \wedge^{p_i-1}) & \xrightarrow{d} & \Gamma(\oplus_i \wedge^{p_i}) & \xrightarrow{d} & \Gamma(\oplus_i \wedge^{p_i+1}) & \xrightarrow{d} & \cdots
\end{array}
$$

If $\mathcal{O}$ is an elliptic orbit, the complex $\#_{\Phi^0}$ is an elliptic complex for all closed $\Phi^0 \in \mathcal{E}_1$ on any $n$ dimensional compact manifold $X$ (Note that the complex in definition 1-1 is the symbol complex of $\#_{\Phi^0}$). Then we have a finite dimensional cohomology group $H^k(\#_{\Phi^0})$ of the elliptic complex $\#_{\Phi^0}$. Since $\#_{\Phi^0}$ is a subcomplex of deRham complex, there is the map $p^k$ from the cohomology group of the complex $\#_{\Phi^0}$ to de Rham cohomology group:

$$p^k: H^k(\#_{\Phi^0}) \longrightarrow \bigoplus_i H^{p_i-k-1}(X, \mathbb{R}).$$

where

$$H^k(\#_{\Phi^0}) = \{ \alpha \in \Gamma(E^k) \mid d_k \alpha = 0 \}/\{ d \beta \mid \beta \in \Gamma(E^{k-1}) \}.$$

Let $\mathcal{O}$ be an orbit in $\oplus_i \wedge^{p_i} V^*$. Then we define the moduli space $\mathcal{M}_\mathcal{O}(X)$ by

$$\mathcal{M}_\mathcal{O}(X) = \{ \Phi \in \mathcal{E}^1 \mid d\Phi = 0 \}/\text{Diff}_0(X),$$

where $\text{Diff}_0(X)$ is the identity component of the group of diffeomorphisms for $X$. We denote by $\widetilde{\mathcal{M}}_\mathcal{O}(X)$ the set of closed elements in $\mathcal{E}^1$. We have the natural projection $\pi: \widetilde{\mathcal{M}}_\mathcal{O}(X) \to \mathcal{M}_\mathcal{O}(X)$. Let $\Phi^0$ be an element of $\widetilde{\mathcal{M}}_\mathcal{O}(X)$. As we shall show that $\mathcal{E}(X)$ is regarded as an infinite dimensional homogeneous space (a Hilbert manifold). Hence we have the tangent space $T_{\Phi^0}\mathcal{E}(X)$. We denote by $\mathcal{H}$ the Hilbert space
consisting of closed forms. Then the space $\tilde{M}_O(X)$ is the intersection between the Hilbert space $\mathcal{H}$ and the Hilbert manifold $\mathcal{E}(X)$. We define an infinitesimal tangent space of $\tilde{M}_O$ by the intersection $\mathcal{H} \cap T_{\Phi^0} \mathcal{E}$. Since $T_{\Phi^0} \mathcal{E}(X) = E^1$, the infinitesimal tangent space is written as

$$\mathcal{H} \cap T_{\Phi^0} \mathcal{E}(X) = \mathcal{H} \cap E^1.$$ 

Then we shall discuss if the infinitesimal tangent space is regarded as the tangent space of actual deformations.

**Definition 1-4.** A closed element $\Phi^0 \in \mathcal{E}^1(X)$ is unobstructed if there exists an integral curve $\Phi_t(a)$ in $\tilde{M}(X)$ for each $a \in \mathcal{H} \cap E^1$ such that

$$\left. \frac{d}{dt} \Phi_t(a) \right|_{t=0} = a$$

An orbit $O$ is unobstructed if any $\Phi^0 \in \tilde{M}_O(X)$ is unobstructed for all compact $n$ dimensional manifold $X$. (see section 3 for the precise statement with respect to Sobolev norms.)

We shall prove the following theorems in section 2.

**Theorem 1-5 (Criterion of unobstructedness).** We assume that an orbit $O$ is elliptic. If the map $p^2: H^2(\#_{\Phi^0}) \rightarrow \oplus_i H^{p_i+1}_{DR}(X)$ is injective, then $\Phi^0$ is unobstructed.

§2. Preliminary results

Let $X$ be a manifold and we denote by $\wedge^*$ the differential forms on $X$. Let $P$ be a linear operator acting on $\wedge^*$. Then the operator $P: \wedge^* \rightarrow \wedge^*$ is a derivative if $P$ satisfies the followings:

$$P(s + t) = P(s) + P(t),$$

$$P(s \wedge t) = P(s) \wedge t + s \wedge P(t).$$
An anti-derivative \( Q \) is also a linear operator defined by the following:

\[
Q(s + t) = Q(s) + Q(t), \\
Q(s \wedge t) = Q(s) \wedge t + (-1)^{|s|} s \wedge Q(t),
\]

where \(|s|\) denotes the degree of a differential form \( s \). Then the exterior derivative is the anti-derivative and the differential representation \( \hat{\rho}_a \) is a derivative for each \( a \in \text{End}(TX) \).

**Lemma 2-1.** The commutator \([\hat{\rho}_a, d]\) = \( \hat{\rho}_a \circ d - d \circ \hat{\rho}_a \) is the anti-derivative. We denote by \( L_a \) the commutator \([\hat{\rho}_a, d]\).

**Proof.** In general the commutator of a derivative \( P \) and an anti-derivative \( Q \) is an anti-derivative if \( Q \) preserves degrees of differential forms.

The operator \( L_a \) is regarded as a generalizations of the Lie derivative. Indeed we have

**Lemma 2-2.** The commutator \( L_a \) is expressed as

\[
L_a : \wedge^n \rightarrow \wedge^{n+1},
\]

\[
L_a \eta(u_0, u_1, \cdots, u_n) = \sum_{i=0}^{n} (-1)^i L_{a u_i} \eta(u_0, \cdot \cdot \cdot \hat{i} \cdot \cdot \cdot, u_n), \\
- \sum_{i<j} (-1)^{i+j} \eta(a[u_i, u_j], u_0, \cdot \cdot \cdot \hat{i} \hat{j} \cdot \cdot \cdot, u_n)
\]

where \( \eta \) is an \( n \) form and \( a \in \text{End}(TX) \) maps a vector \( u_i \) to \( au_i \in TX \) and we denote by \( L_{au_i} \) the ordinary Lie derivative.

**Proof.** It is sufficient to show the lemma with respect to vectors \( \{u_i\} \) satisfying \([u_i, u_j] = 0\). Then we have

\[
(\hat{\rho}_a d\eta)(u_0, \cdots, u_n) = \sum_i (-1)\hat{i}(i_{au_i} d\eta)(u_0, \cdot \cdot \cdot \hat{i} \cdot \cdot \cdot, u_n)
\]

\[
(d\hat{\rho}_a \eta)(u_0, \cdots, u_n) = - \sum_i (-1)^i (d i_{au_i} \eta)(u_0, \cdot \cdot \cdot \hat{i} \cdot \cdot \cdot, u_n).
\]
Hence from $L_{a_{u_i}} = di_{a_{u_i}} + i_{a_{u_i}}d$, we have the result.

we also have a description of the commutator between $L_a$ and $\hat{\rho}_a$.

**Lemma 2-3.**

$$[L_a, \hat{\rho}_b] = i_{N(a,b)} - L_{ab},$$

where $a, b \in \text{End}(TX) \cong \wedge^1 \otimes T$ and a tensor $N(a, b) \in \wedge^2 \otimes T$ is given by the following

$$N(a, b)(u, v) = ab[u, v] + ba[u, v] + [au, bv] - [av, bu] - a[bu, v] + a[bv, u] - b[au, v] + b[av, u],$$

for $u, v \in TX$, and $i_{N(a, a)}$ is the composition of the interior product and the wedge product of the tensor $N(a, a) \in TX \otimes \wedge^2$.

**Remark.** The tensor $N(a, b)$ is a generalization of the Nijenhuis tensor.

**Proof of lemma 2-3.** For $a, b \in \text{End}(TX)$, we have the tensor $N(a, b) \in \wedge^2 \otimes TX$. Then $i_{N(a,b)}$ is the linear operator from $\wedge^* \rightarrow \wedge^{*+1}$. We see that $i_{N(a,b)}$ is an anti-derivative. By lemma 2-1, $L_{ab}$ is an anti-derivative, where $ab$ denotes the composition of endmorphisms. As in proof of lemma 2-1, the commutator $[L_a, \hat{\rho}_a]$ is also an anti-derivative. Hence it sufficient to show that the identity in lemma 2-3 for functions and 1 forms. For a function $f$, we have $[L_a, \hat{\rho}_a]f = -\hat{\rho}_aL_af = -L_{a^2}f$. Since $i_{N(a,b)}f = 0$, we have the identity. For a one form $\theta$ by lemma 2-2,
we have

\[ L_a \hat{\rho}_b \theta (u, v) = (L_{au} \hat{\rho}_b \theta ) (v) - (L_{av} \hat{\rho}_b \theta ) (u) + \hat{\rho}_b \theta (\hat{\rho}_a [u, v]) \]
\[ = au(\hat{\rho}_b \theta (v)) - av(\hat{\rho}_a \theta (u)) + \theta (ba[u, v]) \]
\[ - \hat{\rho}_b \theta ([au, v]) + \hat{\rho}_b \theta ([av, u]). \]

\[ \hat{\rho}_b L_a \theta (u, v) = (L_a \theta ) (\hat{\rho}_b u, v) + (L_a \theta ) (u, \hat{\rho}_b v) \]
\[ = (L_{abu} \theta ) (v) - (L_{av} \theta ) (\hat{\rho}_b u) + \theta (a[bu, v]) \]
\[ + (L_{au} \theta ) (\hat{\rho}_b v) - (L_{av} \theta ) (u) + \theta (a[u, bv]) \]
\[ = (abu)(\theta v) - (av)(\theta (bu)) + \theta (a[bu, v]) \]
\[ - \theta ([abu, v]) + \theta ([av, bu]) + \theta (a[u, bv]) \]
\[ + (au)\theta (bv) - (abv)\theta (u) \]
\[ - \theta ([au, bv]) + \theta ([abv, u]). \]

Hence the commutator is given by

\[ [L_a, \hat{\rho}_b] \theta (u, v) = - (abu)(\theta v) + \theta ([abu, v]) + (abv)\theta (u) - \theta ([abv, u]) \]
\[ + \theta (ba[u, v]) + \theta ([au, bv]) - \theta ([av, bu]) \]
\[ - \theta (a[bu, v]) + \theta (a[bv, u]) - \theta (b[au, v]) + \theta (b[av, u]) \]
\[ = - L_{ab} \theta (u, v) + i_{N(a, b)} \theta. \]

□

Lemma 2-4. We assume that \( \Phi \) and \( \hat{\rho}_a \Phi \) are closed forms respectively. Then \( d\hat{\rho}_a \hat{\rho}_a \Phi \) is an element of \( \Gamma(E^2) \).

Proof.

\[ d\hat{\rho}_a \hat{\rho}_a \Phi = \hat{\rho}_a d\hat{\rho}_a \Phi - L_a \hat{\rho}_a \Phi = -L_a \hat{\rho}_a \Phi \]
\[ = - \hat{\rho}_a L_a \Phi - i_{N(a, a)} \Phi + L_{a^2} \phi. \]
Since \( L_a \Phi = \hat{\rho}_a d\Phi - d\hat{\rho}_a \Phi = 0 \), we have

\[
d\hat{\rho}_a \hat{\rho}_a \Phi = -i_{N(a,a)} \Phi + L_{a^2} \Phi.
\]

Since \( G(a, a) \in \Lambda^2 \otimes T \cong \Lambda^1 \otimes \text{End}(TX) \), then it follows from our definition of \( E^2 \) that

\[
i_{N(a,a)} \Phi \in \Gamma(E^2).
\]

Since \( L_{a^2} \Phi = -d\hat{\rho}_{a^2} \Phi \in d\Gamma(E^1) \subset \Gamma(E^2) \). Hence we have the result. \( \square \)

We denote by \( G = G(a, a) \) the operator \( i_{N(a,a)} - L_{a^2} \). Then we consider the commutator \( [\hat{\rho}_a, G(a, a)] \). For simplicity we write this by \( \text{Ad}_{\hat{\rho}_a} G(a, a) (= \text{Ad}_{\hat{\rho}_a} G), \)

\[
\text{Ad}_{\hat{\rho}_a} G(a, a) = [\hat{\rho}_a, G(a, a)].
\]

The \( k \)th composition of commutator is denoted by

\[
\text{Ad}^k_{\hat{\rho}_a} G = [\hat{\rho}_a, [\hat{\rho}_a, \cdots [\hat{\rho}_a, G(a, a)], \cdots]],
\]

where \( \text{Ad}_{\hat{\rho}_a} G(a, a) \) acts on differential forms.

**Lemma 2-5.** \( \text{Ad}^k_{\hat{\rho}_a} G(a, a) \Phi^0 \) is an element of \( \Gamma(E^2) \).

**Proof.** At first we consider \( \text{Ad}_{\hat{\rho}_a} G(a, a) \Phi^0 \). By lemma 2-3, we have

\[
\text{Ad}_{\hat{\rho}_a} G(a, a) \Phi^0 = [\hat{\rho}_a, G(a, a)] \Phi^0
\]

\[
= [\hat{\rho}_a, i_{N(a,a)}] \Phi^0 - [\hat{\rho}_a, L_{a^2}] \Phi^0
\]

\[
= [\hat{\rho}_a, i_{N(a,a)}] \Phi^0 + G(a^2, a) \Phi^0.
\]

Since \( N(a^2, a) \in \Lambda^1 \otimes \text{End}(TX) \), as in lemma 2-4 \( G(a^2, a) \Phi^0 \) is an element of \( \Gamma(E^2) \). We see that \([\hat{\rho}_a, i_{N(a,a)}] \) is given by the interior product of the tensor \( \hat{\rho}_a(N(a,a)) \in \Lambda^1 \otimes \text{End}(TX) \), where \( \hat{\rho}_a \) acts on the tensor \( N(a,a) \). Hence \([\hat{\rho}_a, i_{N(a,a)}] \Phi^0 \) is an element of \( \Gamma(E^2) \). Therefore \( \text{Ad}_{\hat{\rho}_a} G(a, a) \Phi^0 \in \Gamma(E^2) \). By induction, we see that \( \text{Ad}^k_{\hat{\rho}_a} G(a, a) \Phi^0 \) is an element of \( \Gamma(E^2) \). \( \square \)
§3-1 Primary obstruction

In this section we use the same notations as in section one and two. The background material are found in [4],[6],[15],and [17]. Our treatment of the construction are similar as one in gauge theory [3],[13]. Let \( X \) be a real \( n \) dimensional compact manifold. We fix a Riemannian metric \( g \) on \( X \). (Note that this metric does not depend on calibration \( \Phi \).) We denote by \( C^\infty(X,\wedge^p) \) the set of smooth \( p \) forms on \( X \). Let \( L^2_s(X,\wedge^p) \) be the Sobolev space and suppose that \( s > k + \frac{n}{2} \), i.e., the completion of \( C^\infty(X,\wedge^p) \) with respect to the Sobolev norm \( \| \cdot \|_s \), where \( k \) is sufficiently large (see [6] for instance). Then we have the inclusion \( L^2_s(X,\wedge^p) \rightarrow C^k(X,\wedge^n) \). We define \( \mathcal{E}_s^1 \) by

\[
(3-1-1) \quad \mathcal{E}_s^1 = C^k(X,\mathcal{A}_\mathcal{O}(X)) \cap L^2_s(X,\bigoplus_{i=1}^l \wedge^{p_i}).
\]

Then we have

**Lemma 3-1-1.** \( \mathcal{E}_s^1 \) is a Hilbert manifold (see [15] for Hilbert manifolds). The tangent space \( T_{\Phi^0}\mathcal{E}_s^1 \) at \( \Phi^0 \) is given by

\[
T_{\Phi^0}\mathcal{E}_s^1 = L^2_s(X,E^1).
\]

**Proof.** We denote by \( \exp \) the exponential map of Lie group \( G = \text{GL}(n,\mathbb{R}) \). Then we have the map \( k_x \)

\[
(3-1-2) \quad k_x : E^1(T_x X) \rightarrow \mathcal{A}(T_x X),
\]

by

\[
(3-1-3) \quad k_x(\hat{\rho}_\xi \Phi^0(x)) = \rho_{\exp \xi \Phi^0}(x).
\]

for each tangent space \( T_x X \). From 3-1-2, we have the map \( k \)

\[
(3-1-4) \quad k : L^2_s(E^1) \rightarrow \mathcal{E}_s^1,
\]
by

\[ k|_{E^1(T_xX)} = k_x. \]

The map \( k \) defines local coordinates of \( E^1_s \). \( \Box \)

Let \( \text{GL}(TX) \) be the group of gauge transformations, i.e.,

\[
\begin{align*}
  TX & \xrightarrow{g} TX \\
  \downarrow & \quad \downarrow \\
  X & \xrightarrow{id} X
\end{align*}
\]

\( g \in \text{GL}(TX) \) acts on \( E_\mathcal{O}(X) \) by

\[ \Phi \mapsto \rho_g(\Phi) \]

The tangent space \( T_{\Phi_0}E(X) \) is \( E^1(X) \),

\[ E^1(X) = \{ \hat{\rho}_a\Phi^0 | a \in \text{End}(TX) \} \]

where \( \hat{\rho} \) is the differential representation of \( \rho \). We denote by \( H(TX) \)
be the gauge transformations with structure group \( H \), i.e., the isotropy group. Then by lemma 3-1, \( E \) is regarded as the infinite dimensional homogeneous space \( \text{GL}(TX)/H(TX) \). Let \( \mathcal{H} \) be the closed subspace of \( L^2_s(X, \oplus_{i=1}^{l} \wedge^p_i) \) consisting of closed forms. Then \( \widetilde{\mathcal{M}}_s \) is the intersection between \( E \) and \( \mathcal{H} \). The image \( dE^0_\#(X) \) is given by

\[ div_\Phi^0 = L_v\Phi^0, \]

where \( L_v \) is the Lie derivative with respect to \( v \in TX \). Hence the cohomology \( H^1(\#) \) of the complex \( \#_{\Phi_0} \) is considered as the infinitesimal tangent space of the moduli space \( \mathcal{M}(X) = \widetilde{\mathcal{M}}(X)/\text{Diff}_0(X) \). However, the moduli space may not be a manifold in general. This is because the infinitesimal tangent space may not exponentiate to the actual deformations. Then there exists an obstruction. This is a general problem of
deformation. In our situation, we must show that the intersection \( E \cap H \) is a manifold. In order to obtain a deformation space, we shall construct a deformation of \( \Phi^0 \) in terms of a power series in \( t \). We consider a formal power series in \( t \):

\[
a(t) = a_1 t + \frac{1}{2!} a_2 t^2 + \frac{1}{3!} a_3 t^3 + \cdots \in \text{End}(TX)[t],
\]

where \( a_k \in \text{End}(TX) \). We define a formal power series \( g(t) \) by,

\[
g(t) = \exp a(t) \in GL(TX)[t]
\]

For simplicity, we put \( a = a(t) \). The gauge group \( GL(TX) \) acts on differential forms by \( \rho \). This action \( \rho \) is written in terms of the differential representation \( \hat{\rho} \),

\[
\rho_{g(t)} \Phi^0 = \Phi^0 + \hat{\rho}_a \Phi^0 + \frac{1}{2!} \hat{\rho}_a \hat{\rho}_a \Phi^0 + \frac{1}{3!} \hat{\rho}_a \hat{\rho}_a \hat{\rho}_a \Phi^0 + \cdots
\]

\[
= \Phi^0 + \hat{\rho}_{a_1} \Phi^0 t + \frac{1}{2}(\hat{\rho}_{a_2} \Phi^0 + \hat{\rho}_{a_1} \hat{\rho}_{a_1} \Phi^0) t^2 + \cdots,
\]

where \( \hat{\rho} \) is just written as

\[
\hat{\rho}_{a(t)} \Phi^0 = \sum_{k=1}^{\infty} \frac{1}{k!} \hat{\rho}_{a_k} \Phi^0 t^k.
\]

The equation what we want to solve is,

\[
(d_{\rho_{g(t)}} \Phi^0 = 0.
\]

We must find a power series \( a = a(t) \) satisfying the condition \((eq_*)\). At first we take \( a_1 \) such that \( d\hat{\rho}_{a_1} \Phi^0 = 0 \). Then it remains to determine \( a_2, a_3, \cdots \) satisfying \((eq_*)\). \( d\rho_{g(t)} \Phi^0 \) is written as a power series,

\[
d\rho_{g(t)} \Phi^0 = \sum_{k=1}^{\infty} \frac{1}{k!} dR_k t^k,
\]
where \( R_k \) denotes the homogeneous part of degree \( k \). Hence the equality 
\[ d\rho_{g(t)}\Phi^0 = 0 \] 
is reduced to the system of infinitely many equations 
\[ (eq^*) \quad dR_k = 0, \quad k = 1, 2, \ldots \]
By our assumption 
\[ d\hat{\rho}_{a_1}\Phi^0 = 0, \] 
we already have \( dR_1 = 0 \). Thus in order to obtain \( a(t) \), it suffices to determine \( a_k \) satisfying \( (eq^*) \) by induction on \( k \). By \( (3-1-6) \), the term of the second order \( dR_2 \) is given as 
\[ (3-1-8) \quad dR_2 = \frac{1}{2!} (d\hat{\rho}_{a_2}\Phi^0 + d\hat{\rho}_{a_1}\hat{\rho}_{a_1}\Phi^0) \]
We denote by \( Ob_2(a_1) \) the quadratic term, 
\[ (3-1-9) \quad Ob_2(a_1) = \frac{1}{2!}(d\hat{\rho}_{a_1}\hat{\rho}_{a_1}\Phi^0) \]
Then by lemma 2-4 in section 2, \( Ob_2 \) is an element of \( \Gamma(E^2) \), which is explicitly written as 
\[ (3-1-10) \quad Ob_2(a_1) = -\frac{1}{2!}(-i N(a_1,a_1) + L a_1^2)\Phi^0, \]
Since \( Ob_2(a_1) \) is a d-closed form, this defines a representative of the cohomology group \( H^2(\#) \). In order to determine \( a_2 \) satisfying \( dR_2 = 0 \), we must solve the equation, 
\[ (eq_2) \quad \frac{1}{2!} d\hat{\rho}_{a_2}\Phi^0 = -Ob_2(a_1). \]
The L.H.S of \( (eq_2) \) cohomologically vanishes in \( H^2(\#) \). Hence if the class 
\[ [Ob_2(a_1)] \in H^2(\#) \] 
does not vanishes, there exists no solution \( a_2 \) of \( eq_2 \) and no deformation with \( a_1 \). In this sense we call the class \( [Ob_2(a_1)] \) the obstruction to deformation of \( \Phi^0 \) (the primary obstruction). If 
\[ [Ob_2(a_1)] \] 
vanishes, then we have a solution \( a_2 \) by 
\[ (3-1-11) \quad \frac{1}{2!} \hat{\rho}_{a_2}\Phi^0 = -d_1^* G_\#(Ob_2(a_1)), \]
where \( G_\# \) denotes the Green operator of the complex \( \# \). It is quite remarkable that the representative \( Ob_2(a_1) \) is d-exact form. Hence \( Ob_2(a) \) is in kernel of the map \( p^2: H^2(\#) \rightarrow \bigoplus_i H^{p_i+1}(X) \). Hence we obtain a nice criterion of unobstructedness.
**Theorem 3-1-2.** If the map \( p^2: H^2(\#) \to \bigoplus_i H^{p_i+1}(X) \) is injective, the obstruction class \([Ob_2(a_1)]\) vanishes.

§3-2 Higher obstructions

Similarly we obtain infinitely many obstructions to deformation of \( \Phi^0 \). We define an operator \( G(a, a) \) on \( \wedge^* \) by

\[
(3-2-1) \quad G(a, a) = i_{N(a, a)} - L_{a^2},
\]

where \( a = a(t) \in \text{End}(TX)[t] \). We denote its \( k \)th homogeneous part by \( G(a, a)_k \). Then by lemma 2-4, we have

\[
(3-2-2) \quad Ob_2(a_1) = -\frac{1}{2!} G(a, a)_2.
\]

We assume that \( a_1, a_2, \cdots a_{k-1} \) are determined satisfying \( dR_1 = 0, dR_2 = 0, \cdots, dR_{k-1} = 0 \). Then \( dR_k \) is written as

\[
(3-2-3) \quad dR_k = d\hat{\rho}_a \Phi^0 + \sum_{l=2}^{k} \frac{1}{l!} d\hat{\rho}_a^l \Phi^0.
\]

We define \( Ob_k(a_{<k}) \) as \( \sum_{l=2}^{k} \frac{1}{l!} (d\hat{\rho}_a^l)_k \Phi^0 \), where \( a_{<k} = a_1 t + \frac{1}{2!} a_2 t^2 + \cdots + \frac{1}{(k-1)!} a_{k-1} t^{k-1} \). Then we have

**Proposition 3-2-1.**

\[
 dR_k = \frac{1}{k!} d\hat{\rho}_{a_k} \Phi^0 + Ob_k(a_{<k}),
\]

where \( Ob_k \) is written as

\[
 Ob_k(a) = \left( -\frac{1}{2!} G(a, a) \Phi^0 + \frac{1}{l!} [\hat{\rho}_a, G(a, a)] \Phi^0 - \cdots - (-1)^{k-1} \frac{1}{(k-1)!} [\hat{\rho}_a, \cdots, [\hat{\rho}_a, G(a, a)] \cdots] \right)_k \Phi^0 = (f(Ad_{\hat{\rho}_a}) G(a, a))_k \Phi^0,
\]
where \( f(x) \) is a convergent sequence,

\[
(3-2-3) \quad f(x) = \frac{1}{2!} + \frac{1}{3!} x - \frac{1}{4!} x^2 - \cdots = -\frac{e^{-x} - 1 + x}{x^2}
\]

and \( \text{Ad}_{\hat{\rho}_a} \) is the adjoint operator \([\hat{\rho}_a, \cdot]\). Substituting \( \text{Ad}_{\hat{\rho}_a} \) into \( f(x) \), we have an operator \( f(\text{Ad}_{\hat{\rho}_a}) \). This operator consists of commutators. Hence \( f(\text{Ad}_{\hat{\rho}_a})\Phi^0 \) is essentially the interior product of \( \Phi^0 \) with respect to a tensor of type \( T \otimes \Lambda^2 \). Hence we see that \( \text{Ob}_k(a_{<k}) \in E^2 \).

**Proof.** In the case \( k = 1 \) we have the proposition. We shall prove the proposition by induction on \( k \). We assume that proposition holds for all \( l < k \). Then we have

\[
(3-2-4) \quad dR_l = -(L_a)_l \Phi^0 + (f(\text{Ad}_{\hat{\rho}_a})G(a,a))_l \Phi^0.
\]

We put \((L_a)_{<k}\) as

\[
(L_a)_{<k} = \sum_{l=2}^{k-1} (L_a)_l.
\]

If \( dR_l = 0 \) \((l < k)\), from our assumption we have

\[
(L_a)_{<k} \Phi^0 = -(f(\text{Ad}_{\hat{\rho}_a})G(a,a))_{<k} \Phi^0
\]

\[
= \left( -\frac{1}{2!} G(a,a) + \frac{1}{3!} [\hat{\rho}_a, G(a,a)] - \frac{1}{4!} [\hat{\rho}_a, [\hat{\rho}_a, G(a,a)]] + \cdots \right)_{<k} \Phi^0.
\]

\[
(3-2-5)
\]

\[
= \sum_{l=2}^{k} (-1)^{l-1} \frac{1}{l!} (\text{Ad}_{\hat{\rho}_a}^{l-2}G(a,a))_{<k} \Phi^0
\]
Then by using lemma 2-3, we have

\[
d(\rho e^a)_k \Phi^0 = \sum_{l=1}^{k} \frac{1}{l!} (d\hat{\rho}^l_a)_k \Phi^0
\]
\[
= - (L_a)_k \Phi^0 - \frac{1}{2!} (G(a, a) + 2\hat{\rho}_a L_a)_k \Phi^0
\]
\[
- \frac{1}{3!} (G(a, a)\hat{\rho}_a + 2\hat{\rho}_a G(a, a) + 3\rho_a \hat{\rho}_a L_a)_k \Phi^0
\]
\[
- \frac{1}{4!} (G(a, a)\hat{\rho}_a \rho_a + 2\hat{\rho}_a G(a, a)\rho_a + 3\rho_a \hat{\rho}_a G(a, a) + 4\hat{\rho}_a \rho_a \hat{\rho}_a L_a)_k \Phi^0 - \cdots .
\]

(3-2-6)

Since the degree of \( a = a(t) \) is greater than or equal to one, we have

\[
(3-2-7) \quad (\hat{\rho}_a^m L_a)_k = (\hat{\rho}_a^m (L_a)_{<k})_k.
\]

Hence from (3-2-7), we substitute (3-2-5) into (3-2-6) and we have

\[
d(\rho e^a)_k \Phi^0 = -(L_a)_k \Phi^0 - \frac{1}{2!} G(a, a)_k \Phi^0
\]
\[
- \frac{1}{2!} 2(\hat{\rho}_a (-\frac{1}{2!} G(a, a) + \frac{1}{3!} \text{Ad} \hat{\rho}_a G(a, a) + \cdots ))_k \Phi^0
\]
\[
- \frac{1}{3!} (G(a, a)\hat{\rho}_a + 2\hat{\rho}_a G(a, a))_k \Phi^0 - \frac{1}{3!} 3(\hat{\rho}_a \rho_a (-\frac{1}{2!} G(a, a) + \cdots ))_k \Phi^0 - \cdots .
\]
Then we calculate each homogeneous part with respect to $a$ and we have

\[ d(\rho_{e^a})_k \Phi^0 = -(L_a)_k \Phi^0 - \frac{1}{2!} G(a, a)_k \Phi^0 \]

\[ + \left( \frac{2}{2!2!} \hat{\rho}_a G(a, a) - \frac{2}{3!} \hat{\rho}_a G(a, a) - \frac{1}{3!} G(a, a) \hat{\rho}_a \right)_k \Phi^0 \]

\[ + \left( -\frac{2}{2!3!} \hat{\rho}_a[\hat{\rho}_a, G(a, a)] + \frac{3}{3!2!} \hat{\rho}_a \hat{\rho}_a G(a, a) \right)_k \Phi^0 \]

\[ + \frac{1}{4!} (-G(a, a) \hat{\rho}_a \hat{\rho}_a - 2 \hat{\rho}_a G G(a, a) \hat{\rho}_a - 3 \hat{\rho}_a \hat{\rho}_a G(a, a))_k \Phi^0 + \cdots \]

\[ = -(L_a)_k \Phi^0 - \frac{1}{2!} G(a, a)_k \Phi^0 + \frac{1}{3!} [\hat{\rho}_a, G(a, a)]_k \Phi^0 \]

\[ + \left( -\frac{1}{2!} G(a, a) \hat{\rho}_a \hat{\rho}_a + \left( \frac{2}{4!} + \frac{2}{3!2!} \right) \hat{\rho}_a G \hat{\rho}_a + \left( \frac{3}{4!} + \frac{3}{3!2!} \right) \hat{\rho}_a \hat{\rho}_a G \right)_k \Phi^0 + \cdots \]

\[ = -(L_a)_k \Phi^0 - \frac{1}{2!} G(a, a)_k \Phi^0 + \frac{1}{3!} [\hat{\rho}_a, G(a, a)]_k \Phi^0 - \frac{1}{4!} [\hat{\rho}_a, [\hat{\rho}_a, G(a, a)]]_k \Phi^0 + \cdots \]

\[ = -(L_a)_k \Phi^0 + \sum_{l=2}^{k} (-1)^{l-1} \frac{1}{l!} \text{Ad}_{\hat{\rho}_a}^{l-2} G(a, a)_k \Phi^0 \]

\[ \square \]

We determine $a_k$ such that

\[ (eq_k) \quad \frac{1}{k!} d\hat{\rho}_{a_k} \Phi^0 = -\text{Ob}_a(a_{<k}) \]

In order that there exists a solution of $eq_k$, it is necessary that $[\text{Ob}_k] = 0 \in H^2(\#)$. If $[\text{Ob}_k] = 0$, we define $a_k$ by

\[ (3-2-8) \quad \frac{1}{k!} \hat{\rho}_{a_k} \Phi^0 = -d^* G_\#(\text{Ob}_k(a_{<k})). \]

Since $\text{Ob}_k(a_{<k})$ is $d$-exact, then we also have a criterion,

**Theorem 3-2-2.** If $p^2$ is injective, then $\text{Ob}_k(a_{<k})$ vanishes for all $k$.

Thus we construct a power series $a(t)$ satisfying $d\rho_{a(t)} \Phi^0 = 0$. Next we must prove that this power series $a(t)$ converges for sufficiently small $t$. 
§3-3 CRITERION OF UNOBSSTRUCTEDNESS

We rewrite definition 1-4 by using the Sobolev norm.

**Definition 1-4.** A closed element $\Phi^0 \in \mathcal{E}_s^1(X)$ is unobstructed if there exists an integral curve $\Phi_t(a)$ in $\widetilde{M}_s(X)$ for each $a \in E^1_s \cap \mathcal{H}$ such that

$$\frac{d}{dt} \Phi_t(a)|_{t=0} = a$$

An orbit $\mathcal{O}$ is unobstructed if any $\Phi^0 \in \widetilde{M}_s(X)$ is unobstructed for all compact $n$ dimensional manifold $X$.

The rest of this subsection is devoted to the proof theorem 1-5 (criterion of unobstructedness). Our method is similar to the one of Kodaira-Spencer theory. See the extremely helpful book by Kodaira [14] for technical details.

**Proof of theorem 1-5.** We already have a formal power series $a(t)$ such that

$$d\rho_{g(t)} \Phi^0 = 0.$$  

Hence it is sufficient to prove that $a(t)$ uniformly converges with respect to the Sobolev norm $\|\|_s$. Since $(L_a)k \Phi^0 = L_{a_k} \Phi^0 = -d\hat{\rho}_{a_k} \Phi^0$ and $dR_k = 0$, $a_k$ satisfies

$$\frac{1}{k!}d\hat{\rho}_{a_k} \Phi^0 = Ob_k.$$  

As in section 3-2, $Ob_k$ is an element of $\Gamma(E^2)$. $Ob_k$ is also written as

$$Ob_k = \frac{1}{2!}d\hat{\rho}^2_{a_{<k}} \Phi^0 + \cdots + \frac{1}{(k-1)!}d\hat{\rho}^{k-1}_{a_{<k}} \Phi^0$$

By (3-3-2), we see that $Ob_k$ is an exact form. Hence if the map $p^2: H^2(\#) \to \oplus_i H^{p_i+1}(X)$ is injective, then the class $[Ob_k] \in H^2(\#)$ vanishes. Hence we obtain a solution of the equation (3-3-1) by

$$\frac{1}{k!}\hat{\rho}_{a_k} \Phi^0 = -d^*_1 G_\#(Ob_k) \in E^1.$$
We assume that $a_k$ belongs to the orthogonal complement of the Lie algebra $H$, where $H$ is the isotropy group of $\Phi^0$. Hence $a_k$ is defined uniquely by $\hat{\rho}_a \Phi^0$ and we have the estimate

$$\|a_k\|_s = C_1 \|\hat{\rho}_a \Phi^0\|_s$$

Hence by (3-3-3), we define a formal power series,

$$a = \sum_{k=1}^{\infty} \frac{1}{k!} a_k t^k.$$

Given two power series $P(t) = \sum_k p_k t^k$ and $Q(t) = \sum_k q_k t^k$, if $p_k < q_k$ for all $k$, we denote it by

$$P(t) \ll Q(t).$$

We denote by $(P)_k$ the homogeneous part of degree $k$ of $P(t)$. Let $A(t)$ be a convergent series given by

$$A(t) = \frac{b}{16c} \sum_{k=1}^{\infty} c^k t^k,$$

with $b > 0, c > 0$. $b$ and $c$ will be determined later. As regards $A(t)$ we have the following inequality (see section 5-3 in [14]),

$$A(t)^l \ll \left(\frac{c}{b}\right)^{l-1} A(t).$$

Fix a natural number $s$. We shall show by induction on $k$ if we choose appropriate large $b$ and $c$,

$$(*_k) \quad \|a_{\leq k}\|_s \ll A(t),$$

where $\|a_{\leq k}\|_s = \sum_{l=1}^{k} \frac{1}{l!} \|a_l\|_s t^l$. We assume $*_k-1$ holds and make an estimate $\|a_k\|_s$. By (3) we have the inequalities for constants $C_2, C_3$,

$$\frac{1}{k!} \|a_k\|_s = C_1 \frac{1}{k!} \|\hat{\rho}_a \Phi^0\|_s = C_1 \|d_1^* G\#(Ob_k)\|_s$$

$$< C_2 \|G\#(Ob_k)\|_{s+1} < C_3 \|Ob_k\|_{s-1}.$$
By theorem 3-2-1, we have an estimate,

$$\|O_{b_k}\|_{s-1} < C_4 f(2\|a\|) \left( \frac{1}{2!} \|G(a,a)\|_{s-1} + \frac{1}{3!} 2\|a\| \|G(a,a)\|_{s-1} + \cdots + \frac{1}{k!} 2^{k-2} \|a\| \|G(a,a)\|_{s-1} \right)$$

where $f(x) = \frac{1}{x^2}(e^x - 1 - x)$. We have an estimate of $G(a,a)$,

$$\|G(a,a)\|_{s-1} < C_5 \|a\| \|a\|_s$$

Hence by (3-3-5),

$$\|O_{b_k}\|_{s-1} < C_6 \left( \frac{1}{2!} + \frac{1}{3!} 2\|a\|_s + \cdots + \frac{1}{k!} 2^{k-1} \|a\|_{s-2} \|a\|_s^2 \right),$$

where $C_6$ is a constant. By the hypothesis of the induction,

$$\|O_{b_k}\|_{s-1} < C_6 \left( \frac{1}{2!} + \frac{1}{3!} 2A(t) + \cdots + \frac{1}{k!} 2^{k-1} A(t)^{k-2} A(t)^2 \right)_k$$

where $p = \frac{b}{c}$. We define $p$ by $C_6 \frac{1}{2p}(e^{2p} - 1 - 2p) = 1$. Then we obtain

$$\|O_{b_k}\|_{s-1} < A_k(t)$$

Therefore we have

$$\frac{1}{k!} \|a_k\|_s < C_3 A_k(t).$$

Since $A(t)$ is a convergent series for sufficiently small $t$, we see that $a(t)$ uniformly convergents. □

Further we assume that

$$d\hat{\rho}_{a_1} \Phi^0 = 0,$$

$$d^*\hat{\rho}_{a_1} \Phi^0 = 0,$$
where $d_1^*$ is the adjoint operator and

$$
\begin{array}{cccc}
0 & \rightarrow & E^0 & \rightarrow \ E^1 & \rightarrow \ \cdots \ .
\end{array}
$$

We also apply elliptic regularity to $\rho_{g(t)}\Phi^0$. As in our construction, we have

$$
\begin{align*}
\dd \rho_{g(t)}\Phi^0 &= \hat{\rho}_a\Phi^0 + \sum_{l=2}^{\infty} \frac{1}{l!} d\hat{\rho}_a^l\Phi^0 = 0 \\
\quad d_0^*\hat{\rho}_a\Phi^0 &= 0
\end{align*}
$$

Hence $\hat{\rho}_a\Phi^0$ is a weak solution of an elliptic differential equation,

$$
\begin{align*}
\triangle\#\hat{\rho}_a\Phi^0 + d_1^*(\sum_{l=2}^{\infty} \frac{1}{l!} d\hat{\rho}_a^l\Phi^0) &= 0
\end{align*}
$$

Hence we obtain

**Theorem 3-3-1.** If $p^2$ is injective, then there exists a solution of the equation * for all tangent $[\hat{\rho}_a\Phi^0] \in H^1(\#\Phi^0)$. i.e., There exists a smooth form $\rho_{\exp a(t)}\Phi^0 \in \tilde{\mathcal{M}}(X)$ such that

$$
(\rho_{\exp a(t)}\Phi^0)^t = \hat{\rho}_a\Phi^0
$$

\section*{§4. Calabi-Yau structures}

**§4-1.** Let $V$ be a real $2n$ dimensional vector space. We consider the complex vector space $V \otimes \mathbb{C}$ and a complex form $\Omega \in \wedge^n V^* \otimes \mathbb{C}$. The vector space $\text{ker} \, \Omega$ is defined as

$$
\text{Ker} \, \Omega = \{ v \in V \otimes \mathbb{C} \mid i_v \Omega = 0 \},
$$

where $i_v$ denotes the interior product.
Definition 4-1-1 (SL\(_n(\mathbb{C})\) structures). A complex \(n\) form \(\Omega\) is an SL\(_n(\mathbb{C})\) structure on \(V\) if \(\dim_{\mathbb{C}} \ker \Omega = n\) and \(\ker \Omega \cap \overline{\ker \Omega} = \{0\}\), where \(\overline{\ker \Omega}\) is the conjugate vector space.

We denote by \(A_{SL}(V)\) the set of SL\(_n(\mathbb{C})\) structures on \(V\). We define the almost complex structure \(I_{\Omega}\) on \(V\) by

\[
I_{\Omega}(v) = \begin{cases} 
-\sqrt{-1}v & \text{if } v \in \ker \Omega, \\
\sqrt{-1}v & \text{if } v \in \overline{\ker \Omega}.
\end{cases}
\]

So that is, \(\ker \Omega = T^{0,1}V\) and \(\overline{\ker \Omega} = T^{1,0}V\) and \(\Omega\) is a non-zero \((n,0)\) form on \(V\) with respect to \(I_{\Omega}\). Let \(J\) be the set of almost complex structures on \(V\). Then \(A_{SL}(V)\) is the \(\mathbb{C}^*\)–bundle over \(J\). We denote by \(\rho\) the action of the real general linear group \(G = GL(V) \cong GL(2n, \mathbb{R})\) on the complex \(n\) forms,

\[
\rho: GL(V) \longrightarrow \text{End}(\wedge^n(V \otimes \mathbb{C}))^*.
\]

For simplicity we denote by \(\wedge^n_{\mathbb{C}}\) complex \(n\) forms. Since \(G\) is a real group, \(A_{SL}(V)\) is invariant under the action of \(G\). Then we see that the action of \(G\) on \(A_{SL}(V)\) is transitive. The isotropy group \(H\) is defined as

\[
H = \{ g \in G \mid \rho_g \Omega = \Omega \}.
\]

Then we see \(H = \text{SL}(n, \mathbb{C})\). Hence the set of SL\(_n(\mathbb{C})\) structures \(A_{SL}(V)\) is the homogeneous space,

\[
A_{SL}(V) = G/H = GL(2n, \mathbb{R})/\text{SL}(n, \mathbb{C}).
\]

(Note that the set of almost complex structures \(J = \text{GL}(2n, \mathbb{R})/\text{GL}(n, \mathbb{C})\).)

An almost complex structure \(I\) defines a complex subspace \(T^{1,0}\) of dimension \(n\). Hence we have the map \(J \longrightarrow \text{Gr}(n, \mathbb{C}^{2n})\). We also have the map from \(A_{SL}(V)\) to the tautological line bundle \(L\) over the Grassmannian \(\text{Gr}(n, \mathbb{C}^{2n})\) removed \(0\)–section. Then we have the diagram:

\[
\begin{array}{ccc}
A_{SL}(V) & \longrightarrow & L \setminus 0 \\
\downarrow \mathbb{C}^* & & \downarrow \\
J & \longrightarrow & \text{Gr}(n, \mathbb{C}^{2n})
\end{array}
\]
$\mathcal{A}_{SL}(V)$ is embedded as a smooth submanifold in $n$–forms $\wedge^n$. This is Plücker embedding described as follows,

$$
\begin{array}{c}
\mathcal{A}_{SL}(V) \\ \downarrow \mathbb{C}^* \\
\mathcal{J} \\
\end{array} \longrightarrow 
\begin{array}{c}
L \setminus 0 \\ \rightarrow \\
Gr(n, \mathbb{C}^{2n}) \\
\rightarrow \\
\mathbb{C}P^n.
\end{array}
$$

Hence the orbit $O_{SL} = \mathcal{A}_{SL}(V)$ is a submanifold in $\wedge^n$ defined by Plücker relations. Let $X$ be a real $2n$ dimensional compact manifold. Then we have the $G/H$ bundle $\mathcal{A}_{SL}(X)$ over $X$ as in section 1. We denote by $\mathcal{E} = \mathcal{E}^1_{SL}$ the set of smooth global sections of $\mathcal{A}_{SL}(X)$. Then we have the almost complex structure $I_\Omega$ corresponding to $\Omega \in \mathcal{E}^1$. Then we have

**Lemma 4-1-2.** If $\Omega \in \mathcal{E}^1$ is closed, then the almost complex structure $I_\Omega$ is integrable.

**Proof.** Let $\{\theta_i\}_{i=1}^n$ be a local basis of $\Gamma(\wedge^{1,0})$ with respect to $\Omega$. From Newlander-Nirenberg’s theorem it is sufficient to show that $d\theta_i \in \Gamma(\wedge^{2,0} \oplus \wedge^{1,1})$ for each $\theta_i$. Since $\Omega$ is of type $\wedge^{n,0}$,

$$
\theta_i \wedge \Omega = 0.
$$

Since $d\Omega = 0$, we have

$$
d\theta_i \wedge \Omega = 0.
$$

Hence $d\theta_i \in \Gamma(\wedge^{2,0} \oplus \wedge^{1,1})$. \(\square\)

Then we define the moduli space of $SL_n(\mathbb{C})$ structures on $X$ by

$$
\mathcal{M}_{SL}(X) = \{ \Omega \in \mathcal{E}^1_{SL} \mid d\Omega = 0 \} / \text{Diff}_0(X).
$$

From lemma 4-1-2 we see that $\mathcal{M}_{SL}(X)$ is the $\mathbb{C}^*$–bundle over the moduli space of integrable complex structures on $X$ with trivial canonical line bundles.
Proposition 4-1-3. The orbit $O_{SL}$ is elliptic.

Proof. Let $\wedge^{p,q}$ be $(p,q)$--forms on $V$ with respect to $I_{\Omega^0} \in A_{SL}(V)$. In this case we see that

$$E^0 = \wedge^{n-1,0}$$
$$E^1 = \wedge^{n,0} \oplus \wedge^{n-1,1}$$
$$E^2 = \wedge^{n,1} \oplus \wedge^{n-1,2}.$$

Hence we have the complex:

$$\wedge^{n-1,0} \overset{\wedge u}{\longrightarrow} \wedge^{n,0} \oplus \wedge^{n-1,1} \overset{\wedge u}{\longrightarrow} \wedge^{n,1} \oplus \wedge^{n-1,2} \overset{\wedge u}{\longrightarrow} \cdots,$$

for $u \in V$. Since the Dolbeault complex is elliptic, we see that the complex $0 \rightarrow E^1 \rightarrow E^2 \rightarrow \cdots$ is exact. $\square$

Proposition 4-1-4. Let $I_{\Omega}$ be the complex structure corresponding to $\Omega \in \mathcal{E}^1$. If $\partial \bar{\partial}$ lemma holds for the complex manifold $(X, I_{\Omega})$, then $H^2(\#) \cong H^{n,1}(X) \oplus H^{n-1,2}(X)$ and $p^2 : H^2(\#) \rightarrow H^{n+1}(X, \mathbb{C})$ is injective. In particular, if $(X, I_{\Omega})$ is Kählerian, $p^2$ is injective.

Proof. As in proof of proposition 4-1-3 the complex $\#_\Omega$ is given as

$$\Gamma(\wedge^{n-1,0}) \overset{d}{\longrightarrow} \Gamma(\wedge^{n,0} \oplus \wedge^{n-1,1}) \overset{d}{\longrightarrow} \Gamma(\wedge^{n,1} \oplus \wedge^{n-1,2}) \overset{d}{\longrightarrow} \cdots.$$

Then we have the following double complex:

$$\begin{array}{c}
\Gamma(\wedge^{n,0}) \overset{\bar{\partial}}{\longrightarrow} \Gamma(\wedge^{n,1}) \overset{\bar{\partial}}{\longrightarrow} \Gamma(\wedge^{n,2}) \overset{\bar{\partial}}{\longrightarrow} \cdots \\
\partial \uparrow & \partial \uparrow & \partial \uparrow \\
\Gamma(\wedge^{n-1,0}) \overset{\bar{\partial}}{\longrightarrow} \Gamma(\wedge^{n-1,1}) \overset{\bar{\partial}}{\longrightarrow} \Gamma(\wedge^{n-1,2}) \overset{\bar{\partial}}{\longrightarrow} \cdots \\
\partial \uparrow & \partial \uparrow & \partial \uparrow \\
\Gamma(\wedge^{n-2,0}) \overset{\bar{\partial}}{\longrightarrow} \Gamma(\wedge^{n-2,1}) \overset{\bar{\partial}}{\longrightarrow} \Gamma(\wedge^{n-2,2}) \overset{\bar{\partial}}{\longrightarrow} \cdots
\end{array}$$
Let $a = x + y$ be a closed element of $\Gamma(\Lambda^{n,1}) \oplus \Gamma(\Lambda^{n-1,2})$. Then we have the following equations,

(1) \hspace{1cm} \bar{\partial} y = 0,

(2) \hspace{1cm} \bar{\partial} x + \partial y = 0.

Using the Hodge decomposition, we have

(3) \hspace{1cm} y = \text{Har}(y) + \bar{\partial}(\bar{\partial}^* G_{\bar{\partial}} y),

where $G_{\bar{\partial}}$ is the Green operator with respect to the $\bar{\partial}$–Laplacian and $\text{Har}(y)$ denotes the harmonic component of $y$. We also have

(4) \hspace{1cm} x = \text{Har}(x) + \partial(\partial^* G_{\partial} x),

where $G_{\partial}$ is the Green operator with respect to the $\partial$–Laplacian and $\text{Har}(x)$ denotes the harmonic component of $x$. We put $s = \partial^* G_{\partial} x$ and $t = \bar{\partial}^* G_{\bar{\partial}} y$ respectively. Then we have from (2)

(5) \hspace{1cm} \bar{\partial}\partial s + \partial\bar{\partial} t = \bar{\partial}\partial(s - t) = 0.

Applying $\partial\bar{\partial}$-lemma, we see from (5) that there exists a $\gamma \in \Lambda^{n-1,0}$ such that

(6) \hspace{1cm} \partial(s - t) = \partial\bar{\partial} \gamma.

Hence we have from (4),

\[
\begin{align*}
x &= \text{Har}(x) + \partial s = \text{Har}(x) + \partial t + \bar{\partial}(-\partial \gamma) \\
y &= \text{Har}(y) + \bar{\partial} t.
\end{align*}
\]

Thus if $\text{Har}(x) = 0$ and $\text{Har}(y) = 0$, then $a$ is written as $a = x + y = d(t - \bar{\partial} \gamma)$ where $t - \bar{\partial} \gamma \in E^1 \cong \Lambda^{n,0} \oplus \Lambda^{n-1,1}$. It implies that the map $p^2: H^2(\#) \to H^{n+1}(X, \mathbb{C})$ is injective and $H^2(\#) \cong H^{n,1}(X) \oplus H^{n-1,2}(X)$.

Remark. Similarly we see that

\[
H^0(\#) \cong H^{n-1,0}(X), \hspace{1cm} H^1(\#) \cong H^{n,0}(X) \oplus H^{n-1,1}(X).
\]
§4-2. Calabi-Yau structures. Let $V$ be a real vector space of $2n$ dimensional. We consider a pair $\Phi = (\Omega, \omega)$ of a $\text{SL}_n(\mathbb{C})$ structure $\Omega$ and a real symplectic structure $\omega$ on $V$,

$$\Omega \in \mathcal{A}_{SL}(V),$$

$$\omega \in \wedge^2 V^*, \quad \omega \wedge \cdots \wedge \omega \neq 0.$$ 

We define $g_{\Omega, \omega}$ by

$$g_{\Omega, \omega}(u, v) = \omega(I_{\Omega}u, v),$$

for $u, v \in V$.

**Definition 4-2-1 (Calabi-Yau structures ).** A *Calabi-Yau structure* on $V$ is a pair $\Phi = (\Omega, \omega)$ such that

1. $\Omega \wedge \omega = 0$, $\overline{\Omega} \wedge \omega = 0$ 
2. $\Omega \wedge \overline{\Omega} = c_n \omega \wedge \cdots \wedge \omega$ 
3. $g_{\Omega, \omega}$ is positive definite.

where $c_n$ is a constant depending only on $n$, i.e,

$$c_n = (-1)^{\frac{n(n-1)}{2}} \frac{2^n}{i^n n!}.$$ 

From the equation (1) we see that $\omega$ is of type $\wedge^{1,1}$ with respect to the almost complex structure $I_{\Omega}$. The equation (2) is called Monge-Ampère equation.

**Lemma 4-2-2.** Let $\mathcal{A}_{CY}(V)$ be the set of Calabi-Yau structures on $V$. Then There is the transitive action of $G = \text{GL}(2n, \mathbb{R})$ on $\mathcal{A}_{CY}(V)$ and $\mathcal{A}_{CY}(V)$ is the homogeneous space

$$\mathcal{A}_{CY}(V) = \text{GL}(2n, \mathbb{R})/\text{SU}(n).$$
**Proof.** Let \( g_{\Omega, \omega} \) be the Kähler metric. Then we have a unitary basis of \( TX \). Then the result follows from (1) and (2). \( \square \)

Hence the set of Calabi-Yau structures on \( V \) is the orbit \( \mathcal{O}_{CY} \),

\[
\mathcal{O}_{CY} \subset \wedge^n (V \otimes \mathbb{C})^* \oplus \wedge^2 V^*.
\]

Let \( V \) be a real \( 2n \)-dimensional vector space with a Calabi-Yau structure \( \Phi^0 = (\Omega^0, \omega^0) \). We define the complex Hodge star operator \( *_{\mathbb{C}} \) by

\[
\alpha \wedge *_{\mathbb{C}} \beta = \langle \alpha, \beta \rangle \Omega^0,
\]

where \( \alpha, \beta \in \wedge^{*,0} \). The complex Hodge star operator \( *_{\mathbb{C}} \) is a natural generalization of the ordinary Hodge star \( * \),

\[
*_{\mathbb{C}} : \wedge^{i,0} \to \wedge^{n-i,0}.
\]

The vector space \( E^0 \) is, by definition,

\[
E^0_{CY} (V) = \{ (i_v \Omega^0, i_v \omega^0) \in \wedge^{n-1,0} \oplus \wedge^1 | v \in V \}
\]

The map \( TX \to \wedge^{n-1,0} \) is given by \( v \mapsto i_v \Omega^0 \). Then we see that this map is an isomorphism. Hence the projection to the first component defines an isomorphism:

\[
E^0_{CY} \longrightarrow \wedge^{n-1,0},
\]

\[
(i_v \Omega^0, i_v \omega^0) \mapsto i_v \Omega^0
\]

The \( E^1_{CY} \) is the tangent space of Calabi-Yau structures \( \mathcal{A}_{CY}(X) \). Hence by (1) and (2) of definition 4-2-1, the vector space \( E^1 (V) = E^1_{CY} (V) \) is the set of \( (\alpha, \beta) \in \wedge^n_{\mathbb{C}} \oplus \wedge^2 \) satisfying equations

\[
\alpha \wedge \omega^0 + \Omega^0 \wedge \beta = 0,
\]

\[
\alpha \wedge \overline{\Omega^0} + \Omega^0 \wedge \overline{\alpha} = nc_{n} \beta \wedge (\omega^0)^{n-1}
\]

(4)
Let $P^{p,q}$ be the primitive cohomology group with respect to $\omega^0$. Then we have the Lefschetz decomposition,

$$
\alpha = \alpha^{n,0} + \alpha^{n-1,1} + \alpha^{n-2,0} \wedge \omega^0 \in P^{n,0} \oplus P^{n-1,1} \oplus P^{n-2,0} \wedge \omega^0,
$$

(5)

$$
\beta = \beta^{2,0} + \beta^{1,1} + \beta^{0,0} \wedge \omega^0 + \beta^{0,2} \in P^{2,0} \oplus P^{1,1}_R \oplus P^{0,0} \wedge \omega^0 \oplus P^{0,2},
$$

where $\beta^{2,0} = \overline{\beta^{0,2}}$ and $P^{1,1}_R$ denotes the real primitive forms of type $(1, 1)$. Then equation (4) is written as

$$
\alpha^{n-2,0} \wedge \omega \wedge \omega + \Omega \wedge \beta^{0,2} = 0,
$$

(6)

$$
\alpha^{n,0} \wedge \overline{\Omega} = n c_n \beta^{0,0} \omega^n
$$

(7)

Then we see that (6) gives a relation between $\alpha^{n-2,0}$ and $\beta^{2,0}$ and (7) also describes a relation between $\alpha^{n,0}$ and $\beta^{0,0}$. Since there is no relation between the primitive parts $P^{n-1,1}$ and $P^{1,1}_R$, the kernel of the projection $E^1_{CY} \to \wedge^{n,0} \oplus \wedge^{n-1,1}$ is given by the primitive forms $P^{1,1}_R$. Hence we have an exact sequence:

$$
0 \to P^{1,1}_R \to E^1_{CY} \to \wedge^{n,0} \oplus \wedge^{n-1,1} \to 0.
$$

(8)

The vector space $E^2_{CY}$ is the subspace of $\wedge^{n,1} \oplus \wedge^{n-1,2} \oplus \wedge^3_R$. We also consider the projection to the first component and we have an exact sequence:

$$
0 \to (\wedge^{2,1} \oplus \wedge^{1,2})_R \to E^2_{CY} \to \wedge^{n,1} \oplus \wedge^{n-1,2} \to 0,
$$

(9)

where $(\wedge^{2,1} \oplus \wedge^{1,2})_R$ denotes the real part of $\wedge^{2,1} \oplus \wedge^{1,2}$. Let $X$ be a $2n$ dimensional compact Kähler manifold. We denote by $\wedge^{i,j}$ (global) differential forms on $X$ of type $(i, j)$. The real primitive forms of type $(i, j)$ is denoted by $P^{i,j}_R$. Then we have a complex of forms on $X$ by using the exterior derivative $d$:

$$
0 \to P^{1,1}_R \xrightarrow{d} (\wedge^{2,1} \oplus \wedge^{1,2})_R \xrightarrow{d} \cdots.
$$

(10)
proposition 4-2-3. The cohomology groups of the complex (10) are respectively given by

\[ \mathbb{P}^{1,1}_\mathbb{R}, \quad (H^{2,1}(X) \oplus H^{1,2}(X))_\mathbb{R}, \]

where \( \mathbb{P}^{1,1}_\mathbb{R} \) denotes the harmonic and primitive forms.

Proof. By using Kähler identity, we see that a closed primitive form of type \((1, 1)\) is harmonic. Hence the first cohomology group of the complex (10) is \( \mathbb{P}^{1,1}_\mathbb{R} \). Let \( q \) be a real \( d \)-exact form of type \( \wedge^{(2,1)} \oplus \wedge^{(1,2)} \). The applying \( \partial \overline{\partial} \)-lemma, we show that \( q \) is written as

\[ q = da, \]

where \( a = d^* \eta \in \wedge^{1,1}_\mathbb{R} \) and \( \eta \in (\wedge^{2,1} \oplus \wedge^{1,2})_\mathbb{R} \). We shall show that there exists \( k \in \wedge^1 \) such that \( d^* \eta + dk \in P^{1,1}_\mathbb{R} \). By the Lefschetz decomposition, the three form \( \eta \) is written as

\[ \eta = s + \theta \wedge \omega^0, \]

where \( s \in (P^{2,1} \oplus P^{1,2})_\mathbb{R} \), and \( \theta \in \wedge^1_\mathbb{R} \). Let \( \wedge_\omega^0 \) be the contraction with respect to the Kähler form \( \omega^0 \). Since \( \wedge_\omega^0 \) and \( d^* \) commutes,

\[ \wedge_\omega^0 d^* \eta = d^* \wedge_\omega^0 \eta = d^* (s + \theta \wedge \omega^0) = d^* \theta. \]

On the other hand, applying Kähler identity again, we have

\[ \wedge_\omega^0 dk = d \wedge_\omega^0 k + \sqrt{-1}d^*_c k = \sqrt{-1}d^*_c k, \]

where \( d^*_c = \partial^* - \overline{\partial}^* \). Since \( k \in \wedge^1 \),

\[ d^*_c k = (\partial^* - \overline{\partial}^*) k = \partial^* k^{1,0} - \overline{\partial}^* k^{0,1} = (\partial^* + \overline{\partial}^*) (k^{1,0} - k^{0,1}). \]
Hence if we define $k$ by

$$k = \sqrt{-1}(\theta^{1,0} - \theta^{0,1}),$$

then

$$\wedge_0^\omega (d^* \eta + dk) = d^* \theta + \sqrt{-1}d^* (k^{1,0} - k^{0,1}) = d^* \theta + (-d^* \theta^{1,0} - d^* \theta^{0,1}) = 0.$$

Hence each exact form $q$ of type $(\wedge^{2,1} \oplus \wedge^{1,2})_\mathbb{R}$ is given by

$$q = d(d^* \eta + dk),$$

where $d^* \eta + dk \in P^{1,1}_\mathbb{R}$. Thus the second cohomology group of the complex $(10)$ is $(H^{2,1}(X) \oplus H^{1,2}(X))_\mathbb{R}$. □

**Theorem 4-2-4.** The cohomology groups of the complex $\#_{CY}$:

$$0 \longrightarrow E^0_{CY} \xrightarrow{d} E^1_{CY} \xrightarrow{d} E^2_{CY} \xrightarrow{d} \cdots,$$

is respectively given by

$$H^0(\#_{CY}) = H^{n-1,0}(X),$$

$$H^1(\#_{CY}) = H^{n,0}(X) \oplus H^{n-1,1}(X) \oplus P^{1,1}_\mathbb{R},$$

$$H^2(\#_{CY}) = H^{n,1}(X) \oplus H^{n-1,2}(X) \oplus (H^{2,1}(X) \oplus H^{1,2}(X))_\mathbb{R},$$

In particular, $p^k$ is injective for $k = 0, 1, 2$.
Proof. By (8) and (9), we have the following diagram:

At first we shall consider $H^2(\#_{CY})$. We assume that $(s, t) \in E^2_{CY}$ is written as an exact form, i.e., $(s, t) = (da, db)$. Let $a$ be an element of $\wedge^{n,0} \oplus \wedge^{n-1,1}$. There is a splitting map $\lambda: \wedge^{n,0} \oplus \wedge^{n-1,1} \rightarrow \wedge^2$ such that $(a, \lambda(a))$ is an element of $E^1_{CY}$. Hence

$$(da, d\lambda(a)) \in E^2_{CY}.$$ 

By (10), we see that

$$db - d\lambda(a) \in (\wedge^{2,1} \oplus \wedge^{1,2})_R.$$ 

Then by proposition 4-2-3, there exists $p \in P^{1,1}_R$ such that

$$db - d\lambda(a) = dp.$$ 

Hence $(s, t)$ is written as

$$(s, t) = (da, db) = (da, d(\lambda(a) + p)),$$
where $(a, \lambda(a) + p) \in E_{\mathcal{CY}}^1$. Hence we see that

$$H^1(\#_{\mathcal{CY}}) = H^{n,1}(X) \oplus H^{n-1,2}(X) \oplus (H^{2,1}(X) \oplus H^{1,2}(X))_\mathbb{R}.$$ 

Next we shall consider $H^1(\#_{\mathcal{CY}})$. Let $(a, b)$ be an element of $E_{\mathcal{CY}}^1$ and we assume that $(a, b) = (d\eta, d\gamma)$. Then $s$ is written as $s = i_v \Omega^0$ for some $v \in TX$. By our definition $E_{\mathcal{CY}}^0$, $(i_v \Omega^0, i_v \omega^0)$ is an element of $E_{\mathcal{CY}}^0$. Hence $d\gamma - di_v \omega^0 \in P^{1,1}_\mathbb{R}$. By proposition 4-2-3, a $d$-exact, primitive form vanishes. Thus $dt - di_v \omega^0 = 0$. Hence $(a, b) = (d\eta, d\gamma) = (di_v \Omega^0, di_v \omega^0)$, where $(i_v \Omega^0, i_v \omega^0) \in E_{\mathcal{CY}}^0$. Hence we see that

$$H^1(\#_{\mathcal{CY}}) = H^{n,0}(X) \oplus H^{n-1,1}(X) \oplus P^{1,1}_\mathbb{R}(X).$$

Similarly we see that $E_{\mathcal{CY}}^0(X) = H^{n-1,0}(X)$.

§5 HyperKähler structures

Let $V$ be a $4n$ dimensional real vector space. A hyperKähler structure on $V$ consists of a metric $g$ and three complex structures $I, J$ and $K$ which satisfy the followings:

(1) \quad $g(u, v) = g(Iu, Iv) = g(Ju, Jv) = g(Ku, Kv)$, \quad for $u, v \in V$,

(2) \quad $I^2 = J^2 = K^2 = IJK = -1$.

Then we have the fundamental two forms $\omega_I, \omega_J, \omega_K$ by

$$\omega_I(u, v) = g(Iu, v), \quad \omega_J(u, v) = g(Ju, v),$$

$$\omega_K(u, v) = g(Ku, v).$$

We denote by $\omega_C$ the complex form $\omega_J + \sqrt{-1} \omega_K$. Let $\mathcal{A}_{HK}(V)$ be the set of pairs $(\omega_I, \omega_C)$ corresponding to hyperKähler structures on $V$. As in section one $\mathcal{A}_{HK}(V)$ is the subset of $\wedge^2 \oplus \wedge^2_C$ and the group $\text{GL}(4n, \mathbb{R})$ acts on $\mathcal{A}_{HK}(V)$. Then we see that $\mathcal{A}_{HK}(V)$ is $\text{GL}(4n, \mathbb{R})$–orbit with the isotropy group $\text{Sp}(n)$,

(4) \quad $\mathcal{A}_{HK}(V) = \text{GL}(4n, \mathbb{R})/\text{Sp}(n)$.

We denote by $\mathcal{O}_{HK}$ the orbit $\mathcal{A}_{HK}(V)$. 

Theorem 5-1. The orbit $O_{HK}$ is elliptic and unobstructed.

We shall prove theorem 5-1. Let $\Phi^0 = (\omega_I^0, \omega_J^0, \omega_K^0)$ be a hyperKähler structure on a $4n$ dimensional vector space $V$. We denote by $\omega_C^0$ the complex symplectic form $\omega_J^0 + \sqrt{-1}\omega_K^0$. Then we consider the pair $(\omega_I^0, \omega_C^0)$.

The vector space $E^k_{HK}$ are respectively given by

$E^0_{HK} = \{ (iv\omega_I^0, iv\omega_C^0) | v \in TX \}$

$E^1_{HK} = \{ (\hat{\rho}_a\omega_I^0, \hat{\rho}_a\omega_C^0) | a \in \text{End}(TX) \}$.

Then we consider the projection to the second component and we have the diagram:

$$
\begin{array}{cccccc}
0 & \rightarrow & E^0_{HK} & \rightarrow & E^1_{HK} & \rightarrow \ E^2_{HK} & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \wedge^{1,0} & \rightarrow & \wedge^{2,0} \oplus \wedge^{1,1} & \rightarrow & \wedge^{3,0} \oplus \wedge^{2,1} \oplus \wedge^{1,2} & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0
\end{array}
$$

Let $I, J, K$ be the three almost complex structures on $V$. Then we denote by $\wedge^{1,1}_I$ forms of type $(1, 1)$ with respect to $I$. Similarly $\wedge^{1,1}_J$ (resp. $\wedge^{1,1}_K$) denotes forms of type $(1, 1)$ w.r.t $J$ (resp. $K$). We define $\wedge^{2}_{HK}$ by the intersection between them,

$$\wedge^{2}_{HK} = \wedge^{1,1}_I \cap \wedge^{1,1}_J \cap \wedge^{1,1}_K.$$

Note that $a \in \wedge^{2}_{HK}$ is the primitive form with respect to $I, J, \text{and } K$.

When we identify two forms with $so(4m)$, we have the decomposition:

$$\wedge^2 = sp(4m) \oplus so(4m)/sp(4m).$$

Then $\wedge^{2}_{HK}$ corresponds to $sp(4m)$. Hence the dimension of $\wedge^{2}_{HK}$ is $2m^2 + m$. We also see that

$$
\begin{align*}
\dim_{\mathbb{R}} E^1_{HK} &= \dim_{\mathbb{R}} gl(4m, \mathbb{R})/sp(4m) = 14m^2 - m, \\
\dim_{\mathbb{R}} \wedge^{2,0} \oplus \wedge^{1,1} &= 12m^2 - 2m
\end{align*}
$$
In fact we see that the kernel of the map $E_{HK}^1 \to \wedge^{2,0} \oplus \wedge^{1,1}$ is given by $\wedge_{HK}^2$. We also define $\wedge_{HK}^3$ by real forms of type $(\wedge^{2,1} \oplus \wedge^{2,1})_\mathbb{R}$ for each $I, J, K$. Then we also see that the kernel of the map $E_{HK}^2 \to \wedge^{3,0} \oplus \wedge^{2,1} \oplus \wedge^{1,2}$ is $\wedge_{HK}^3$. We consider the following complex:

$$(HK) \quad 0 \longrightarrow \wedge_{HK}^2 \longrightarrow \wedge_{HK}^3 \longrightarrow \cdots$$

As in proof of Calabi-Yau structures, we see that the cohomology groups of the complex $(HK)$ are respectively given by

$$\mathbb{H}_{HK}^2 = \{ \text{real harmonic forms of type}(1, 1) \text{r.t} I, J, K \}$$
$$\mathbb{H}_{HK}^3 = \{ \text{real harmonic forms of type } \wedge^{2,1} \oplus \wedge^{1,2} \text{ r.t } I, J, K \}.$$

Hence we have the following:

$$
\begin{array}{ccccccc}
& & & 0 & & & 0 \\
& & & \downarrow & & & \downarrow \\
& & & \wedge_{HK}^2 & \longrightarrow & \wedge_{HK}^3 & \longrightarrow & \cdots \\
& & & \downarrow & & & \downarrow \\
& & & 0 & \longrightarrow & \wedge_{HK}^0 & \longrightarrow & \wedge_{HK}^1 & \longrightarrow & E_{HK}^0 & \longrightarrow & E_{HK}^1 & \longrightarrow & \cdots \\
& & & \downarrow & & & \downarrow & & \downarrow & & \downarrow \\
& & & 0 & \longrightarrow & \wedge^{1,0} & \longrightarrow & \wedge^{2,0} \oplus \wedge^{1,1} & \longrightarrow & \wedge^{3,0} \oplus \wedge^{2,1} \oplus \wedge^{1,2} & \longrightarrow & \cdots \\
& & & \downarrow & & & \downarrow & & \downarrow & & \downarrow \\
& & & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \\
\end{array}
$$

**Theorem 5-2.** The cohomology groups of the complex $\#_{HK}$ are given by

$$
\begin{align*}
H^0(\#_{HK}) &= H^{1,0}(X) \\
H^1(\#_{HK}) &= H^{2,0}(X) \oplus H^{1,1}(X) \oplus \mathbb{H}_{HK}^2, \\
H^3(\#_{HK}) &= H^{3,0}(X) \oplus H^{2,1}(X) \oplus H^{1,2}(X) \oplus \mathbb{H}_{HK}^3.
\end{align*}
$$
In particular, the map $p^k$ is injective for $k = 0, 1, 2$.

Proof. The proof is essentially same as in the case of Calabi-Yau structures. Let $\lambda$ be the splitting map $\wedge^{2,0} \oplus \wedge^{1,1} \to E_{HK}^1$. Let $(s, t)$ be an element of $E_{HK}^2$. We assume that $(s, t) = (da, db)$ for $b \in \wedge^{2,0} \oplus \wedge^{1,1}$ and $a \in \wedge^2$. By using the splitting map $\lambda$, we have $(\lambda(b), b) \in E_{HK}^1$. Then $(d\lambda(b), db) \in E_{HK}^2$. Hence $da - d\lambda(b) \in \wedge^3_{HK}$. Then there is an element $\gamma \in \wedge^2_{HK}$ such that $da - d\lambda(b) = d\gamma$.

Hence $(s, t) = (da, db) = (d(\lambda(b) + \gamma), db)$, where $((\lambda(b) + \gamma, b) \in E_{HK}^1$. Thus we have

$$H^2(#_{HK}) = H^{3,0}(X) \oplus H^{2,1}(X) \oplus H^{1,2}(X) \oplus \mathbb{H}^3_{HK}.$$

Similarly we see that

$$H^1(#_{HK}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus \mathbb{H}^2_{HK}$$

$$H^0(#_{HK}) = H^{1,0}(X)$$

□

Proof of theorem 5-1. This follows from theorem 5-2 and theorem 1-5. □

§6. $G_2$ structures

Let $V$ be a real 7 dimensional vector space with a positive definite metric. We denote by $S$ the spinors on $V$. Let $\sigma^0$ be an element of $S$ with $\|\sigma^0\| = 1$. By using the natural inclusion $S \otimes S \subset \wedge^*V^*$, we have a calibration by a square of spinors,

$$\sigma^0 \otimes \sigma^0 = 1 + \phi^0 + \psi^0 + \text{vol},$$

where vol denotes the volume form on $V$ and $\phi^0$ (resp. $\psi^0$) is called the associative 3 form (resp. coassociative 4 form). Our construction
of these forms in terms of spinors is written in chapter IV §10 of [16] and in section 14 of [8]. Background materials of $G_2$ geometry are found in [7], [10,12] and [18]. We also have an another description of $\phi^0$ and $\psi^0$.

We decompose $V$ into a real 6 dimensional vector space $W$ and the one dimensional vector space $\mathbb{R}$. Let $(\Omega^0, \omega^0)$ be an element of Calabi-Yau structure on $W$ and $t$ a nonzero 1 form on $\mathbb{R}$. Then the 3 form $\phi^0$ and the 4 form $\psi^0$ are respectively written as

$$\phi^0 = \omega^0 \wedge t + \text{Im} \Omega^0, \quad \psi^0 = \frac{1}{2} \omega^0 \wedge \omega^0 - \text{Re} \Omega^0 \wedge t.$$  

Then as in section 1, we define $G_2$ orbit $\mathcal{O} = \mathcal{O}_{G_2}$ as

$$\mathcal{O}_{G_2} = \{ (\phi, \psi) = (\rho g \phi^0, \rho g \psi^0) \mid g \in \text{GL}(V) \}.$$  

Note that the isotropy group is the exceptional Lie group $G_2$. We denote by $\mathcal{A}_{G_2}(V)$ the orbit $\mathcal{O}_{G_2}$. Let $X$ be a real 7 dimensional compact manifold. Then we define a $GL(7, \mathbb{R})/G_2$ bundle $\mathcal{A}_{G_2}(X)$ by

$$\mathcal{A}_{G_2}(X) = \bigcup_{x \in X} \mathcal{A}_{G_2}(T_x X).$$

Let $\mathcal{E}^1_{G_2}$ be the set of smooth global sections of $\mathcal{A}_{G_2}(X)$,

$$\mathcal{E}^1_{G_2}(X) = \Gamma(X, \mathcal{A}_{G_2}(X)).$$

Then the moduli space of $G_2$ structures over $X$ is given as

$$\mathcal{M}_{G_2}(X) = \{ (\phi, \psi) \in \mathcal{E}^1_{G_2} \mid d\phi = 0, d\psi = 0 \} / \text{Diff}_0(X).$$

We shall prove unobstructedness of $G_2$ structures.

**Theorem 6-1.** The orbit $\mathcal{O}_{G_2}$ is elliptic and satisfies the criterion.

The rest of this section is devoted to prove theorem 6-1. In the case of $G_2$, each $E^i$ is written as

$$E^0 = E^0_{G_2} = \{ (i_v \phi^0, i_v \psi^0) \in \wedge^2 \oplus \wedge^3 \mid v \in V \}$$

$$E^1 = E^1_{G_2} = \{ (\rho_\xi \phi^0, \rho_\xi \psi^0) \in \wedge^3 \oplus \wedge^4 \mid \xi \in \mathfrak{g}l(V) \}$$

$$E^2 = E^2_{G_2} = \{ (\theta \wedge \phi, \theta \wedge \psi) \in \wedge^4 \oplus \wedge^5 \mid \theta \in \wedge^1, (\phi, \psi) \in E^1_{G_2} \}. $$
The Lie group $G_2$ is a subgroup of $SO(7)$ and we see that $G_2 = \{ g \in GL(V) \mid \rho_g \phi^0 = \phi^0 \}$. Hence we have the metric $g_\phi$ corresponding to each 3 form $\phi$. Let $*_{\phi}$ be the Hodge star operator with respect to the metric $g_\phi$. Then a non linear operator $\Theta(\phi)$ is defined as

\begin{equation}
\Theta(\phi) = *_{\phi} \phi.
\end{equation}

According to [10], the differential of $\Theta$ at $\phi$ is described as

\begin{equation}
J(\phi) = d\Theta(a)_\phi = \frac{4}{3} *_{\pi_1(a)} + *_{\pi_7(a)} - *_{\pi_{27}(a)},
\end{equation}

for each $a \in \wedge^3$, where we use the irreducible decomposition of 3 forms on $V$ under the action of $G_2$,

\begin{equation}
\wedge^3 = \wedge^3_1 + \wedge^3_7 + \wedge^3_{27},
\end{equation}

and each $\pi_i$ is the projection to each component for $i = 1, 7, 27$, (see also [9] for the operator $J$). From (1) the orbit $O_{G_2}$ is written as

\begin{equation}
O_{G_2} = \{ (\phi, \Theta(\phi)) \mid \phi \in \wedge^3 \}.
\end{equation}

Since $E^1_{G_2}(V)$ is the tangent space of the orbit $O_{G_2}$ at $(\phi^0, \psi^0)$, from (2) the vector space $E^1_{G_2}(V)$ is also written as

\begin{equation}
E^1_{G_2}(V) = \{ (a, Ja) \in \wedge^3 \oplus \wedge^4 \mid a \in \wedge^3 \}.
\end{equation}

Let $X$ be a real 7 dimensional compact manifold and $(\phi^0, \psi^0)$ a closed element of $E^1_{G_2}(X)$. Then we have a vector bundle $E^i_{G_2}(X) \rightarrow X$ by

\begin{equation}
E^i_{G_2}(X) = \bigcup_{x \in X} E^i_{G_2}(T_x X),
\end{equation}

for each $i = 0, 1, 2$. Then we have the complex $\#_{G_2}$,

\[ 0 \rightarrow \Gamma(E^0_{G_2}) \xrightarrow{d_0} \Gamma(E^1_{G_2}) \xrightarrow{d_1} \Gamma(E^2_{G_2}) \rightarrow \cdots. \]
The complex $\#_{G_2}$ is a subcomplex of the de Rham complex,

\[
\begin{array}{cccccccc}
0 & \longrightarrow & \Gamma(E^0_{G_2}) & \overset{d_0}{\longrightarrow} & \Gamma(E^1_{G_2}) & \overset{d_1}{\longrightarrow} & \Gamma(E^2_{G_2}) & \overset{d_2}{\longrightarrow} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \Gamma(\wedge^2 \oplus \wedge^3) & \overset{d}{\longrightarrow} & \Gamma(\wedge^3 \oplus \wedge^4) & \overset{d}{\longrightarrow} & \Gamma(\wedge^4 \oplus \wedge^5) & \overset{d}{\longrightarrow} & \cdots .
\end{array}
\]

Then we have the map $p^1: H^1(\#_{G_2}) \to H^3(X) \oplus H^4(X)$ and $p^2: H^2(\#_{G_2}) \to H^4(X) \oplus H^5(X)$. The following lemma is shown in [9].

**Lemma 6-2.** Let $a^3 = db^2$ be an exact 3 form, where $b^2 \in \Gamma(\wedge^2)$. If $dJdb^2 = 0$, then there exists $\gamma^2 \in \Gamma(\wedge^2_7)$ such that $db^2 = d\gamma^2$.

We shall show that $p^1$ is injective by using lemma 6-2.

**Proposition 6-3.** Let $\alpha = (a^3, a^4)$ be an element of $\Gamma(E^1_{G_2})$. We assume that there exists $(b^2, b^3) \in \Gamma(\wedge^2 \oplus \wedge^3)$ such that

\[
(a^3, a^4) = (db^2, db^3).
\]

Then there exists $\gamma = (\gamma^2, \gamma^3) \in \Gamma(E^0_{G_2})$ satisfying

\[
(db^2, db^3) = (d\gamma^2, d\gamma^3).
\]

**Proof.** From (5) an element of $\Gamma(E^1_{G_2})$ is written as

\[
(a^3, a^4) = (a^3, Ja^3).
\]

From (7) we have

\[
(8) \quad dJdb^2 = da^4 = ddb^3 = 0
\]

From lemma 6-2 we have $\gamma^2 \in \Gamma(\wedge^2_7)$ such that

\[
(9) \quad db^2 = d\gamma^2.
\]
Since $\gamma \in \Gamma(\wedge^2_7)$, $\gamma$ is written as

$$\gamma = i_v \phi^0,$$

where $v$ is a vector field. Since $\phi^0$ is closed, $d\gamma$ is given by the Lie derivative,

$$d\gamma = di_v \phi^0 = L_v \phi^0.$$

Then since $\text{Diff}_0$ acts on $\mathcal{E}^1_{G_2}$, $(L_v \phi^0, L_v \psi^0) = (di_v \phi^0, di_v \psi^0)$ is an element of $\Gamma(E^1_{G_2})$. Hence from (5), we see

$$di_v \psi^0 = Jdi_v \phi^0 = Jd\gamma^2.$$

From (12) we have

$$db^2, db^3 = (db^2, Jdb^2) = (di_v \phi, di_v \psi),$$

where $(i_v \phi^0, i_v \psi^0) \in \Gamma(E^0_{G_2})$. □

Next we shall show that $p^2$ is injective.

**Lemma 6-4.** Let $V$ be a real 7 dimensional vector space with a $G_2$ structure $\Phi^0_V$. Let $u$ be a non-zero one form on $V$. Then for any two form $\eta$ there exists $\gamma \in \wedge^2_{14}$ such that

$$u \wedge J(u \wedge \eta) = u \wedge J(u \wedge \gamma) = -2 * \|u\| \gamma,$$

$$i_v \gamma = 0,$$

where $v$ is the vector which is metrical dual of the one form $u$ and $*$ is the Hodge star operator.

**Proof.** The two forms $\wedge^2$ is decomposed into the irreducible representations of $G_2$,

$$\wedge^2 = \wedge^2_7 \oplus \wedge^2_{14}.$$
We denote by $\eta_7$ the $\wedge^2_7$-component of $\eta \in \wedge^2$. The subspace $u \wedge \wedge^2$ is defined by $\{u \wedge \eta \in \wedge^3|\eta \in \wedge^2\}$. We also denote by $u \wedge \wedge^2_7$ the subspace $\{u \wedge \eta_7 \in \wedge^3|\eta \in \wedge^2\}$. Then we have the orthogonal decomposition,

\[(6-4-1)\quad u \wedge \wedge^2 = u \wedge \wedge^2_7 \oplus (u \wedge \wedge^2_7)^\perp,\]

where $(u \wedge \wedge^2_7)^\perp$ is the orthogonal complement. By the decomposition $6-4-1$, $u \wedge \eta$ is written as

\[(6-4-2)\quad u \wedge \eta = u \wedge \eta_7 + u \wedge \hat{\eta}.\]

for $\hat{\eta} \in \wedge^2$. Then we see that

\[(6-4-3)\quad i_v(u \wedge \hat{\eta}) \in \wedge^2_{14}.\]

Since $\eta_7$ is expressed as $i_w \phi^0$ for $w \in V$, we have

\[
u \wedge J(u \wedge \eta_7) = u \wedge J(u \wedge i_w \phi^0) = u \wedge J\hat{\rho}_a \phi^0,\]

where $a = w \otimes u \in V \otimes V^* \cong \text{End}(V)$. Since $J\hat{\rho}_a \phi^0 = \hat{\rho}_a \psi^0$,

\[
u \wedge J\hat{\rho}_a \phi^0 = u \wedge \hat{\rho}_a \psi^0 = u \wedge (u \wedge i_w \psi^0) = 0.\]

Hence

\[(6-4-4)\quad u \wedge J(u \wedge \eta_7) = 0.\]

Then by $6-4-2$ we have

\[(6-4-5)\quad u \wedge J(u \wedge \eta) = u \wedge J(u \wedge \hat{\eta}).\]

$\hat{\eta}$ is written as

\[(6-4-6)\quad \hat{\eta} = \frac{1}{2||u||^2}(i_v(u \wedge \hat{\eta}) + u \wedge i_v \hat{\eta}).\]
We define $\gamma$ by

$$\gamma = \frac{1}{2\|u\|^2} i_v (u \wedge \hat{\eta}).$$

By 6-4-3, $\gamma \in \wedge^2_{14}$. By 6-4-5,6 we have

$$u \wedge J (u \wedge \eta) = u \wedge J (u \wedge \gamma).$$

Since $\gamma \in \wedge^2_{14}$, $\gamma \wedge \psi^0 = 0$. Then it follows that

$$\psi^0 \wedge u \wedge \gamma = 0.$$

We also have $*\gamma = -\gamma \wedge \phi^0$ from $\gamma \in \wedge^2_{14}$. Since $i_v \gamma = 0$, we have $u \wedge (*\gamma) = 0$. Thus

$$\phi^0 \wedge u \wedge \gamma = 0.$$

By 6-4-8 and 6-4-9, we have

$$u \wedge \gamma \in \wedge^3_{27}.$$

Then by 6-4-7,

$$u \wedge J (u \wedge \eta) = u \wedge J (u \wedge \gamma)$$

$$= -u \wedge *(u \wedge \gamma) = -i_v u \wedge \gamma$$

$$= -2\|u\|^2 (*\gamma)$$

\[\square\]

**Proposition 6-5.** Let $E^2_{G_2}(V)$ be the vector space as in before. Then we have an exact sequence,

$$0 \longrightarrow \wedge^5_{14} \longrightarrow E^2_{G_2}(V) \longrightarrow \wedge^4 \longrightarrow 0$$

**Proof.** The map $E^2_{G_2} \rightarrow \wedge^4$ is the projection to the first component. We denote by Ker the Kernel of the map $E^2_{G_2} \rightarrow \wedge^4$. We shall show that Ker
\[ \cong \wedge^5_{14}. \] Let \( \{v_1, v_2, \cdots, v_7\} \) be an orthonormal basis of \( V \). We denote by \( \{u^1, u^2, \cdots, u^7\} \) the dual basis of \( V^* \). Let \( (s, t) \) be an element of \( E_{G_2}^2(V) \), where \( s \in \wedge^4 \) and \( t \in \wedge^5 \). Then we have the following description:

\begin{align*}
(6-5-1) \quad s &= u^1 \wedge a_1 + u^2 \wedge a_2 + \cdots + u^7 \wedge a_7, \\
(6-5-2) \quad t &= u^1 \wedge Ja_1 + u^2 \wedge Ja_2 + \cdots + u^7 \wedge Ja_7.
\end{align*}

where \( a_1, a_2, \cdots, a_7 \in \wedge^3 \) satisfying
\[ i_{v_l} a_m = 0, \forall l < m. \]

We assume that \( (s, t) \in \text{Ker} \). Then \( s = 0 \). By 6-5-1, we see that \( u^l \wedge a_l = 0 \), for all \( l \). Hence each \( a_l \) is written as

\begin{align*}
(6-5-3) \quad a_l &= u^l \wedge \eta_l
\end{align*}

where \( \eta_l \in \wedge^2 \). By (6-5-2) we have

\begin{align*}
(4-4) \quad t &= \sum_{l=1}^7 u^l \wedge J(u^l \wedge \eta_l).
\end{align*}

Then it follows from lemma 6-4 there exists \( \gamma_l \) such that
\[ t = \sum_{l=1}^7 u^l \wedge J(u^l \wedge \gamma_l) = -2 \sum_{l=1}^7 \| u^l \|^2 (\ast \gamma_l), \]
where \( \gamma_l \in \wedge^2_{14} \). Hence \( t \in \wedge^5_{14} \). Therefore we see that \( \text{Ker} = \wedge^5_{14} \). \( \Box \)

**Lemma 6-6.** Let \( X \) be a compact 7 dimensional manifold with \( G_2 \) structure \( \Phi^0 \), (i.e., \( d\Phi^0 = 0 \)). Then for any two form \( \eta \) there exists \( \gamma \in \wedge^2_{14} \) such that

\[ dJd\eta = dJd\gamma = -\ast \Delta \gamma, \]
\[ d^*\gamma = 0. \]
Proof. We denote by $d\wedge^2$ the closed subspace $\{d\eta|\eta \in \wedge^2\}$. Since $d\wedge^2 = \{d\eta_7|\eta_7 \in \wedge_7^2\}$ is the closed subspace of $d\wedge^2$, we have the decomposition,

$$d\wedge^2 = d \wedge^2_7 \oplus (d\wedge^2_7)^\perp$$

where $(d\wedge^2_7)^\perp$ denotes the orthogonal subspace of $d\wedge^2_7$. By 6-6-1, $d\eta$ is written as

$$d\eta = d\eta_7 + d\hat{\eta},$$

where $d\hat{\eta} \in (d\wedge^2_7)^\perp$. Hence we have

(6-6-2) \hspace{1cm} d^*d\hat{\eta} \in \wedge^2_{14}

As in the proof of lemma 6-4, $\eta_7$ is written as $i_w \phi^0$ for some $w \in TX$. Hence

(6-6-3) \hspace{1cm} dJd\eta_7 = dJd_iw\phi^0 = dJL_w\phi^0 = dL_w\psi^0 = di_w\psi^0 = 0.

Thus $dJd\eta = dJd\hat{\eta}$. By the Hodge decomposition, we have

$$\hat{\eta} = Harm(\hat{\eta}) + dd^*G\hat{\eta} + d^*dG\hat{\eta},$$

where $Harm(\hat{\eta})$ is the harmonic part of $\hat{\eta}$ and $G$ denotes the Green operator. We define $\gamma$ by

$$\gamma = d^*dG\hat{\eta}.$$ 

Then by Chern’s theorem ($\pi_7G = G\pi_7$) and 6-6-2, we see that $\gamma \in \wedge^2_{14}$. Then $d\hat{\eta} = d\gamma$ and $d^*\gamma = 0$. Since $\gamma \in \wedge^2_{14}$, we have $\gamma \wedge \psi^0 = 0$ and $\ast \gamma = -\gamma \wedge \phi^0$. Hence we have

(6-6-4) \hspace{1cm} d\gamma \wedge \phi^0 = 0,

(6-6-5) \hspace{1cm} d\gamma \wedge \psi^0 = 0.

Hence it follows from 6-6-4,5 that

(6-6-6) \hspace{1cm} d\gamma \in \wedge^3_{27}.

Then by 6-6-6,

$$dJd\gamma = -d^*d\gamma = -\ast \triangle \gamma.$$ 

By 6-6-3,

$$dJd\eta = -\ast \triangle \gamma.$$ 

□
Proposition 6-7.

\[ H^2(\#G_2) = H^4(X) \oplus H^5_{14}(X). \]

In particular,

\[ p^2: H^2(\#G_2) \to H^4(X) \oplus H^5(X) \]

is injective.

Proof. Let \((s, t)\) be an element of \(E^2_{G_2}(X)\). We assume that \(s, t\) are exact forms respectively, i.e.,

\[(6-7-1) \quad s = da, \quad t = db,\]

for some \(a \in \wedge^3\) and \(b \in \wedge^4\). Then we shall show that there exists \(\tilde{a} \in \wedge^3\) such that \(da = d\tilde{a}\) and \(db = dJa\). Since \((da, dJa)\) is an element of \(E^2_{G_2}\), it follows from proposition 6-5 that

\[(6-7-2) \quad db - dJa \in \wedge^5_{14}.\]

We shall show that there exists \(\eta \in \wedge^2\) satisfying,

\[(6-7-3) \quad db = dJ(a + d\eta)\]

In order to solve the equation \((6-7-3)\), we apply lemma 6-6. Then there exists \(\gamma \in \wedge^2_{14}\) such that

\[(6-7-4) \quad dJd\eta = -\ast \triangle \gamma \quad d^*\gamma = 0.\]

Substituting \((6-7-4)\) to the equation \((6-7-3)\), we have

\[(6-7-5) \quad -\ast \triangle \gamma = db - dJa\]

Then by \((6-7-2)\), there exists a solution \(\gamma\) of the equation \((6-7-5)\),

\[\gamma = -G \ast (db - dJa) \in \wedge^2_{14}.\]
Hence if we set $\tilde{a} = a + d\gamma$, $(s, t)$ is written as
\[ s = d\tilde{a} = d(a + d\gamma), \]
\[ t = dJ\tilde{a} = dJ(a + d\gamma). \]

Therefore $p^2 : H^2(\#G_2) \to H^4(X) \oplus H^5(X)$ is injective. Furthermore we consider harmonic forms $\mathbb{H}^4(X)$ and $\mathbb{H}^5_{14}(X)$. By Chern’s theorem $H^4(X) \oplus H^5_{14}(X) \cong \mathbb{H}^4(X) \oplus \mathbb{H}^5_{14}(X)$. Since the complex $\#G_2$ is elliptic, $H^2(\#G_2)$ is represented by harmonic forms of the complex $\#G_2$, i.e.,
\[ H^2(\#G_2) \cong \mathbb{H}^2(\#G_2). \]
Then we see that there is the injective map
\[ \mathbb{H}^4(X) \oplus \mathbb{H}^5_{14}(X) \to \mathbb{H}^2(\#G_2). \]
Since $p^2$ is injective, we have
\[ H^2(\#G_2) \cong H^4(X) \oplus H^5_{14}(X). \]

$\square$

§7. Spin(7) structures

Let $V$ be a real 8 dimensional vector space with a positive definite metric. We denote by $S$ the spinors of $V$. Then $S$ is decomposed into the positive spinor $S^+$ and the negative spinor $S^-$. Let $\sigma_0^+$ be a positive spinor with $\|\sigma_0^+\| = 1$. Then under the identification $S \otimes S \cong \wedge^* V$, we have a calibration by the square of the spinor,
\[ \sigma_0^+ \otimes \sigma_0^+ = 1 + \Phi^0 + \text{vol}, \]
where vol denotes the volume form on $V$ and $\Phi^0$ is called the Cayley 4 form on $V$ (see [8], [16] for our construction in terms of spinors). Background materials of Spin(7) geometry are found in [11,12] and [18].
we decompose $V$ into a real 7 dimensional vector space $W$ and the one dimensional vector space $\mathbb{R}$,

$$V = W \oplus \mathbb{R}.$$ 

Then a Cayley 4 form $\Phi^0$ is defined as

$$\phi^0 \wedge \theta + \psi^0 \in \wedge^4 V^*,$$

where $(\phi^0, \psi^0) \in \mathcal{O}_{G_2}(W)$ and $\theta$ is non zero one form on $\mathbb{R}$. We define an orbit $\mathcal{O}_{Spin(7)} = \mathcal{A}_{Spin(7)}(V)$ by

$$\mathcal{O}_{Spin(7)} = \{ \rho_g \Phi^0 \mid g \in GL(V) \}.$$

Since the isotropy is $Spin(7)$, the orbit $\mathcal{O}_{Spin(7)}$ is written as

$$\mathcal{O}_{Spin(7)} = GL(V)/Spin(7).$$

Let $X$ be a real 8 dimensional compact manifold. Then we define $\mathcal{A}_{Spin(7)}(X)$ by

$$\mathcal{A}_{Spin(7)}(X) = \bigcup_{x \in X} \mathcal{A}_{Spin(7)}(T_x X) \rightarrow X.$$

We denote by $\mathcal{E}_{Spin(7)}^1(X)$ the set of global section of $\mathcal{A}_{Spin(7)}(X)$,

$$\mathcal{E}_{Spin(7)}^1(X) = \Gamma(X, \mathcal{A}_{Spin(7)}(X)).$$

Then we define the moduli space of $Spin(7)$ structures over $X$ as

$$\mathcal{M}_{Spin(7)}(X) = \{ \Phi \in \mathcal{E}_{Spin(7)}^1 \mid d\Phi = 0 \}/Diff_0(X).$$

The following theorem is shown in [11,12]
**Theorem 7-1.** [11,12] The moduli space $\mathcal{M}_{\text{Spin}(7)}(X)$ is a smooth manifold with
\[
\dim \mathcal{M}_{\text{Spin}(7)}(X) = b^4_1 + b^4_7 + b^4_{35},
\]
where Harmonic 4 forms on $X$ is decomposed into irreducible representations of Spin(7),
\[
\mathbb{H}^4(X) = \mathbb{H}^4_1 \oplus \mathbb{H}^4_7 \oplus \mathbb{H}^4_{27} \oplus \mathbb{H}^4_{35},
\]
each $b^4_i$ denoted $\dim \mathbb{H}^4_i$, for $i = 1, 7, 27$ and 35.

Note that $\mathbb{H}^4(X)$ is decomposed into self dual forms and anti-self dual forms,
\[
\mathbb{H}^4(X) = \mathbb{H}^+ \oplus \mathbb{H}^-,
\]
where
\[
\mathbb{H}^+(X) = \mathbb{H}^4_1 \oplus \mathbb{H}^4_7 \oplus \mathbb{H}^4_{27}, \quad \mathbb{H}^- = \mathbb{H}^4_{35}.
\]
We shall show the unobstructedness of $G_2$ structures by using our method in section one.

**Theorem 7-2.** The orbit $\mathcal{O}_{\text{Spin}(7)}$ is elliptic and satisfies the criterion.

Since Spin(7) is a subgroup of SO(8), we have the metric $g_{\Phi^0}$ for each $\Phi^0 \in \mathcal{O}_{\text{Spin}(7)}$. For each $\Phi^0 \in \mathcal{O}_{\text{Spin}(7)}(V)$, $\wedge^3$ and $\wedge^4$ are orthogonally decomposed into the irreducible representations of Spin(7),
\[
\wedge^3 = \wedge^3_8 \oplus \wedge^3_{48},
\]
\[
\wedge^4 = \wedge^+ \oplus \wedge^- = (\wedge^4_1 \oplus \wedge^4_7 \oplus \wedge^4_{27}) \oplus \wedge^4_{35},
\]
where $\wedge^p_i$ denotes the irreducible representation of Spin(7) of $i$ dimensional. We denote by $\pi_i$ the orthogonal projection to each component. Let $X$ be a real 8 dimensional compact manifold with a closed form $\Phi^0 \in \mathcal{E}^1_{\text{Spin}(7)}(X)$. Let $g_{\Phi^0}$ be the metric corresponding to $\Phi^0$. Then there is a unique parallel positive spinor $\sigma^+_0 \in \Gamma(S^+)$ with
\[
\sigma^+_0 \otimes \sigma^+_0 = 1 + \Phi^0 + \text{vol},
\]
where $S^+ \otimes S^+$ is identified with the subset of Clifford algebra $\text{Cliff} \cong \wedge^*$ (see [16]). By using the parallel spinor $\sigma_0^+$, the positive and negative spinors are respectively identified with following representations,

$$\Gamma(S^+) \cong \Gamma(\wedge^4_1 \oplus \wedge^4_7),$$

$$\sigma^+ \mapsto \sigma^+ \otimes \sigma_0^+,$$

(1)

$$\Gamma(S^-) \cong \Gamma(\wedge^3_8),$$

$$\sigma^- \mapsto \sigma^- \otimes \sigma_0^+,,$$

(2)

where $\sigma^\pm \in \Gamma(S^\pm)$. Under the identification (1) and (2), The Dirac operator $D^+: \Gamma(S^+) \to \Gamma(S^-)$ is written as

$$\pi_8 \circ d^*: \Gamma(\wedge^4_1 \oplus \wedge^4_7) \to \Gamma(\wedge^3_8).$$

In particular $\text{Ker} \pi_8 \circ d^*$ are Harmonic forms in $\Gamma(\wedge^4_1 \oplus \wedge^4_7)$. Hence we have

**Lemma 7-3.**

$$\text{Ker} \pi_8 \circ d^* = H^1_1(X) \oplus H^4_{1\#}(X).$$

In the case of $\text{Spin}(7)$, each $E^i = E^i_{\text{Spin}(7)}$ is given by

$$E^0_{\text{Spin}(7)} = \wedge^3_8, \quad E^1_{\text{Spin}(7)} = \wedge^4_1 \oplus \wedge^4_7 \oplus \wedge^4_-.$$

Let $\alpha$ be an element of $\Gamma(E^1_{\text{Spin}(7)}(X))$. We assume that

$$d\alpha = 0, \quad \pi_8 d^* \alpha = 0,$$

(3)

So that is, $\alpha$ is an element of $H^1_1(\#)$, where $\#$ is the complex

$$0 \quad \longrightarrow \quad \Gamma(E^0_{\text{Spin}(7)}) \quad \overset{d_0}{\longrightarrow} \quad \Gamma(E^1_{\text{Spin}(7)}) \quad \overset{d_1}{\longrightarrow} \quad \Gamma(E^2_{\text{Spin}(7)}) \quad \longrightarrow \quad \cdots$$

$$\quad \overset{\vdots}{\longrightarrow} \quad \Gamma(\wedge^3_8) \quad \overset{d}{\longrightarrow} \quad \Gamma(\wedge^4_1 \oplus \wedge^4_7 \oplus \wedge^-) \quad \overset{d}{\longrightarrow} \quad \Gamma(\wedge^5) \quad \longrightarrow \quad \cdots$$
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(Note that $d_0^* = \pi_S d^*$.) We decompose $\alpha$ into the self-dual form and the anti-self-dual form,

$$\alpha = \alpha^+ + \alpha^- \in \Gamma(\wedge^+) \oplus \Gamma(\wedge^-).$$

From (3) we have

$$d\alpha^+ + da^- = 0$$
$$\pi_S \ast d\alpha^+ - \pi_S \ast d\alpha^- = 0.$$

Hence we have $\pi_S d^* \alpha^+ = 0$. From lemma 7-3, we see that $d\alpha^+ = 0$. Hence we also have $d\alpha^- = 0$ and it implies that $\alpha$ is a harmonic form with respect to the metric $g_{\Phi_0}$. Hence the map $p: H^1(#) \cong \mathbb{H}^1(#) \to H^4(X) \cong \mathbb{H}^4(X)$ is injective.

**Theorem 7-4.** The cohomology groups of the complex $\#_{Spin(7)}$ are respectively given by

$$H^0(\#_{Spin(7)}) \cong H^3_8(X),$$
$$H^1(\#_{Spin(7)}) \cong H^4_1(X) \oplus H^4_7(X) \oplus H^4_-(X),$$
$$H^2(\#_{Spin(7)}) = H^5(X),$$

In particular $p^1$ and $p^2$ are respectively injective.

**Proof.** It is sufficient to show that $H^2(\#_{Spin(7)}) = H^5(X)$. Since anti-self dual forms $\wedge^4_-$ is the subset of $E^1_{Spin(7)}$, we see that our result.

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