A REMARK ON NORM INFLATION FOR NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. We consider semilinear Schrödinger equations with nonlinearity that is a polynomial in the unknown function and its complex conjugate, on $\mathbb{R}^d$ or on the torus. Norm inflation (ill-posedness) of the associated initial value problem is proved in Sobolev spaces of negative indices. To this end, we apply the argument of Iwabuchi and Ogawa (2012), who treated quadratic nonlinearities. This method can be applied whether the spatial domain is non-periodic or periodic and whether the nonlinearity is gauge/scale-invariant or not.

1. Introduction. We consider the initial value problem for semilinear Schrödinger equations:

$$
\begin{cases}
  i \partial_t u + \Delta u = F(u, \bar{u}), & (t, x) \in [0, T] \times Z, \\
  u(0, x) = \phi(x),
\end{cases}
$$

where the spatial domain $Z$ is of the form $Z = \mathbb{R}^{d_1} \times \mathbb{T}^{d_2}$, $d_1 + d_2 = d$, and $F(u, \bar{u})$ is a polynomial in $u, \bar{u}$ without constant and linear terms, explicitly given by

$$
F(u, \bar{u}) = \sum_{j=1}^{n} \nu_j u^{p_j} \bar{u}^{q_j}
$$

with mutually different indices $(p_1, q_1), \ldots, (p_n, q_n)$ satisfying $p_j \geq 2$, $0 \leq q_j \leq p_j$ and non-zero complex constants $\nu_1, \ldots, \nu_n$.

The aim of this article is to prove norm inflation for the initial value problem (1) in some negative Sobolev spaces. We say norm inflation in $H^s(Z)$ ("NI$_s$" for short) occurs if for any $\delta > 0$ there exist $\phi \in H^\infty$ and $T > 0$ satisfying

$$
\|\phi\|_{H^s} < \delta, \quad 0 < T < \delta
$$

such that the corresponding smooth solution $u$ to (1) exists on $[0, T]$ and

$$
\|u(T)\|_{H^s} > \delta^{-1}.
$$

Clearly, NI$_s$ implies the discontinuity of the solution map $\phi \mapsto u$ (which is uniquely defined for smooth $\phi$ locally in time) at the origin in the $H^s$ topology, and hence the ill-posedness of (1) in $H^s$. However, NI$_s$ is a stronger instability property of the flow than the discontinuity, which only requires $0 < T \lesssim 1$ and $\|u(T)\|_{H^s} \gtrsim 1$.

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Let us begin with the case of single-term nonlinearity:
\[
\begin{aligned}
  i\partial_t u + \Delta u = \nu u|u|^{p-1}u, \\
  u(0, x) = \phi(x),
\end{aligned}
\]
where \( p \geq 2 \) and \( 0 \leq q \leq p \) are integers, \( \nu \in \mathbb{C} \setminus \{0\} \) is a constant. The equation is invariant under the scaling transformation \( u(t, x) \mapsto \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x) \) (\( \lambda > 0 \)), and the critical Sobolev index \( s \) for which \( \|\lambda^{\frac{2}{p-1}} \phi(\lambda)\|_{H^s} = \|\phi\|_{H^s} \) is given by
\[
s = s_c(d, p) := \frac{d}{2} - \frac{2}{p-1}.
\]
The scaling heuristics suggests that the flow becomes unstable in \( H^s \) for \( s < s_c(d, p) \). In addition, we will demonstrate norm inflation phenomena by tracking the transfer of energy from high to low frequencies (that is called “high-to-low frequency cascade”), which naturally restrict us to negative Sobolev spaces. In fact, we will show \( N_\Lambda \) with any \( s < \min\{s_c(d, p), 0\} \) for any \( Z \) and \( (p, q) \), as well as with some negative but scale-subcritical regularities for specific nonlinearities. Precisely, our result reads as follows:

**Theorem 1.1.** Let \( Z \) be a spatial domain of the form \( \mathbb{R}^{d_1} \times \mathbb{T}^{d_2} \) with \( d_1 + d_2 = d \geq 1 \), and let \( p \geq 2 \), \( 0 \leq q \leq p \) be integers. Then, the initial value problem (2) exhibits \( N_\Lambda \) in the following cases:

(i) \( Z \) and \( (p, q) \) are arbitrary, \( s < \min\{s_c(d, p), 0\} \).

(ii) \( d, p, s \) satisfy \( s = s_c(d, p) = -\frac{d}{2} \); that is, \( (d, p, s) = (1, 3, -\frac{1}{2}) \) and \( (2, 2, -1) \).

(iii) \( d = 1 \), \( (p, q) = (2, 0) \), \( (2, 2) \) and \( s < -1 \).

(iv) \( Z = \mathbb{R}^d \) with \( 1 \leq d \leq 3 \), \( (p, q) = (2, 1) \) and \( s < -\frac{1}{4} \).

(v) \( Z = \mathbb{R}^{d_1} \times \mathbb{T}^{d_2} \) with \( d_1 + d_2 \leq 3 \), \( d_2 \geq 1 \), \( (p, q) = (2, 1) \) and \( s < 0 \).

(vi) \( Z = \mathbb{T}^d \), \( (p, q) = (4, 1), (4, 2), (4, 3) \) and \( s < 0 \).

There is an extensive literature on the ill-posedness of nonlinear Schrödinger equations, and a part of the above theorem has been proved in previous works.

Concerning ill-posedness in the sense of norm inflation, Christ, Colliander, and Tao [10] treated the case of gauge-invariant power-type nonlinearities \( \pm|u|^{p-1}u \) on \( \mathbb{R}^d \) and proved \( N_{\Lambda} \) when \( 0 < s < s_c(d, p) \) or \( s \leq -\frac{d}{2} \) (with some additional restriction on \( s \) if \( p \) is not an odd integer). For the remaining range of regularities \( -\frac{d}{2} < s < 0 \) (when \( s_c \geq 0 \)) they proved the failure of uniform continuity of the solution map. Note that this milder form of ill-posedness is not necessarily incompatible with well-posedness in the sense of Hadamard, for which continuity of the solution map is required. Moreover, since their argument is based on scaling consideration and some ODE analysis, it does not apply in any obvious way to the cases of periodic domains, non gauge-invariant nonlinearities, and complex coefficients. Later, Carles, Dumas, and Sparber [6] and Carles and Kappeler [7] studied norm inflation in Sobolev spaces of negative indices for the problem with smooth nonlinearities (i.e., \( \pm|u|^{p-1}u \) with an odd integer \( p \geq 3 \)) in \( \mathbb{R}^d \) and in \( \mathbb{T}^d \), respectively. They used a geometric optics approach to obtain \( N_{\Lambda} \) for \( d \geq 2 \) and \( s < -\frac{1}{p} \) in the \( \mathbb{R}^d \) case and for \( s < 0 \) in the \( \mathbb{T}^d \) case with the exception of \( (d, p) = (1, 3) \) for which \( s < -\frac{2}{3} \) was was
assumed. (See [5, 1] for related ill-posedness results.) In fact, they showed stronger instability property than NI for these cases; that is, norm inflation with infinite loss of regularity (see Proposition 1 below for the definition). Our argument, which evaluates each term in the power series expansion of the solution directly, is different from the aforementioned works. Note that, for smooth nonlinearities, Theorem 1.1 covers all the remaining cases in the range \( s < \min\{ s_c(d,p), 0 \} \) and extends the result to the (partially) periodic setting as well as to the case of general nonlinearities with complex coefficients. Moreover, our argument also gives another proof of the results in [6, 7] on NI with infinite loss of regularity; see Proposition 1 for the precise statement.

The one-dimensional cubic equation with nonlinearity \( \pm |u|^2 u \) has been attracting particular attention due to its various physical backgrounds and complete integrability. Note also that this is the only \( L^2 \)-subcritical case among smooth and gauge-invariant nonlinearities. In spite of the \( L^2 \) subcriticality, the equation becomes unstable below \( L^2 \) due to the Galilean invariance, both in \( \mathbb{R} \) and in \( \mathbb{T} \). In fact, the initial value problem was shown to be globally well-posed in \( L^2 \) [39, 3], whereas it was shown in [23, 9] for \( \mathbb{R} \) and in [4, 9] for \( \mathbb{T} \) that the solution map fails to be uniformly continuous below \( L^2 \). Ill-posedness below \( L^2(\mathbb{T}) \) was established in the periodic case by the lack of continuity of the solution map [11, 32] and by the non-existence of solutions [17]. Nevertheless, one can show a priori bound in some Sobolev spaces below \( L^2 \) [27, 12, 28, 17], which prevents norm inflation. Recent results in [29, 24] finally gave a priori bound on \( H^s \) for \( s > -\frac{1}{2} \), both in \( \mathbb{R} \) and in \( \mathbb{T} \). We remark that NI at \( s = -\frac{1}{2} \) shown in Theorem 1.1 ensures the optimality of these results.\(^3\) In [24, Theorem 4.7], Killip, Vişan and Zhang also derived a priori bound of the solutions in the norm which is logarithmically stronger than the critical \( H^{-\frac{1}{2}} \). Motivated by this result, in addition to Theorem 1.1 (ii) we also show norm inflation for the one-dimensional cubic equation in some “logarithmically subcritical” spaces; see Proposition 3 below.

Since the work of Kenig, Ponce, and Vega [22], non gauge-invariant nonlinearities have also been intensively studied. In [2], Bejenaru and Tao proposed an abstract framework for proving ill-posedness in the sense of discontinuity of the solution map. They considered the quadratic NLS (2) on \( \mathbb{R} \) with nonlinearity \( u^2 \) and obtained a complete dichotomy of Sobolev index \( s \) into locally well-posed (\( s \geq -1 \)) and ill-posed (\( s < -1 \)) in the sense mentioned above. Their argument is based on the power series expansion of the solution, and they proved ill-posedness by observing that high-to-low frequency cascades break the continuity of the first nonlinear term in the series. A similar dichotomy was shown for other quadratic nonlinearities \( \ddot{u}^2 \) in [25, 26] by employing the idea of [2].

Later, Iwabuchi and Ogawa [20] considered the nonlinearity \( \ddot{u}^2 \) in \( \mathbb{R}, \mathbb{R}^2 \) and refined the idea of [2] to prove ill-posedness in the sense of NI for \( s < -1 \) in \( \mathbb{R} \) and \( s \leq -1 \) in \( \mathbb{R}^2 \). In particular, in the two-dimensional case they could complement the local well-posedness result in \( H^s(\mathbb{R}^2) \), \( s > -1 \), which had been obtained in [25]. Note that the original argument of [2] is not likely to yield norm inflation phenomena nor discontinuity of the solution map at the threshold regularity such as \( s = -1 \) in the above \( \mathbb{R}^2 \) case. We will have more discussion on this issue in the next section. Another quadratic nonlinearity \( uu \) was investigated by the same

\(^3\)The one-dimensional cubic problem was not treated in the first version of this article. We would like to thank T. Oh for drawing our attention to this case.
method in [21], where for \( \mathbb{R}^d \) with \( d = 1, 2, 3 \) they proved norm inflation in Besov spaces \( B_{2,\sigma}^{-1/4} \) of regularity \(-\frac{1}{4}\) with \( 4 < \sigma \leq \infty \).

It turns out that the method of Iwabuchi and Ogawa [20] proving norm inflation has a wide applicability. The purpose of the present article is to apply this method to NLS with general nonlinearities. In the last few years the method has been used to a wide range of equations; see for instance [30, 31, 19, 8, 37]. In [33, 37], norm inflation based at general initial data was proved for NLS and some other equations.

We make some additional remarks on Theorem 1.1.

**Remark 1.** (i) Concerning one-dimensional periodic cubic NLS below \( L^2 \), the renormalized (or Wick ordered) equation

\[
i\partial_t u + \partial_x^2 u = \pm (|u|^2 - 2\int_T |u|^2) u
\]

is known to behave better than the original one (2) with nonlinearity \( \pm |u|^2u \); see [34] for a detailed discussion. We note that our proof can be also applied to the renormalized cubic NLS. In fact, the solutions constructed in Theorem 1.1 is smooth [34] for a detailed discussion. We note that our proof can be also applied to the renormalized cube tori, whether rational or irrational; that is, \( \mathbb{Z} \) considered. Hence, it can be easily adapted to the problem on general anisotropic equation that exhibit norm inflation.

(ii) In the periodic setting, our proof does not rely on any number theoretic consideration. Hence, it can be easily adapted to the problem on general anisotropic tori, whether rational or irrational; that is, \( \mathbb{Z} \), except for the \( \mathbb{Z} \) of regularity \(-\frac{1}{4}\) with \( 4 < \sigma \leq \infty \).

(iii) When \( Z = \mathbb{R} \) and \( (p, q) = (4, 2) \), the example in [15, Example 5.3] suggests that a high-to-low frequency cascade leads to instability of the solution map when \( s < -\frac{1}{8} \). However, our argument does not imply NI for \(-\frac{1}{8} \leq s < -\frac{1}{2}\) so far.

There are far less results on ill-posedness for multi-term nonlinearities than for (2). However, such nonlinear terms naturally appear in application. For instance, the nonlinearly \( 6u^5 - 4u^3 \) appears in a model related to shape-memory alloys [13], and \( (u + 2\bar{u} + u\bar{u})u \) is relevant in the study of asymptotic behavior for the Gross-Pitaevskii equation (see e.g. [18]). Note that norm inflation for a multi-term nonlinearity does not immediately follow from that for each nonlinear term. Our next result concerns the equation (1) of full generality:

**Theorem 1.2.** The initial value problem (1) exhibits NI whenever \( s \) satisfies the condition in Theorem 1.1 for at least one term \( u^p \bar{u}^q u \) in \( F(u, \bar{u}) \), except for the case where \( Z = \mathbb{T} \) and \( F(u, \bar{u}) \) contains \( u\bar{u} \).

When \( Z = \mathbb{T} \) and \( F(u, \bar{u}) \) contains \( u\bar{u} \), NI occurs in the following cases:

4Essentially, they also proved NI for \( s < -1/4 \), i.e., the case (iv) of our Theorem 1.1.

5In the first version of this article, we only considered gauge-invariant smooth nonlinearities \( \nu |u|^{2k}u \), \( k \in \mathbb{Z}_{\geq 0} \) and linear combinations of them. Note, however, that the method of Iwabuchi and Ogawa [20] had been applied before only to quadratic nonlinearities and it was the first result dealing with nonlinearities of general degrees in a unified manner. The authors of [8, 33] informed us that their proofs of norm inflation results followed the argument in the first version of this article. We also remark that an estimate proved in the first version (Lemma 3.5 below) was employed later in [31, 19, 37].

6In [37] non gauge-invariant nonlinearities were first treated in a general setting. In fact Theorem 1.1 follows as a corollary of [37, Proposition 2.5 and Corollary 2.10]. However, we decide to include the non gauge-invariant cases in the present version in order to state Theorem 1.2 (for multi-term nonlinearities) with more generality.
(i) $s < 0$ if $F(u, \bar{u})$ has a quintic or higher term, or one of $u^3\bar{u}, u^2\bar{u}^2, u\bar{u}^3$.
(ii) $s < -\frac{1}{6}$ if $F(u, \bar{u})$ has $u^3$ or $\bar{u}^4$ but no other quartic or higher terms.
(iii) $s \leq -\frac{1}{2}$ if $F(u, \bar{u})$ has a cubic term but no quartic or higher terms.
(iv) $s < 0$ if $F(u, \bar{u})$ has no cubic or higher terms.

In the above theorem, the range of regularities is restricted when $Z = \mathbb{T}$ and $F(u, \bar{u})$ has $uu$; note that the nonlinear term $uu$ by itself leads to NI for $s < 0$ as shown in Theorem 1.1. This restriction seems unnatural and an artifact of our argument.

The rest of this article is organized as follows. In the next section, we recall the idea of [2, 20] and discuss some common features and differences between them. Section 3 is devoted to the proof of Theorem 1.1 for the single-term nonlinearities. Then, in Section 4 we see how to treat the multi-term nonlinearities, proving Theorem 1.2. In Appendices, we consider norm inflation with infinite loss of regularity in Section A and inflation of various norms with the critical regularity for the one-dimensional cubic problem in Section B.

2. **Strategy for proof.** We will use the power series expansion of the solutions to prove norm inflation. To see the idea, let us consider the simplest case of quadratic nonlinearity $u^2$ in (2). This amounts to considering the integral equation

\[ u(t) = e^{it\Delta} \phi - i \int_0^t e^{i(t-\tau)\Delta} (u(\tau) \cdot u(\tau)) \, d\tau =: \mathcal{L}[\phi](t) + \mathcal{N}[u, u](t), \quad t \in [0, T]. \]  

(3)

We first recall the argument of Bejenaru and Tao [2]. By Picard’s iteration, the power series $\sum_{k=1}^{\infty} U_k[\phi]$ with

\[ U_1[\phi] := \mathcal{L}[\phi], \quad U_2[\phi] := \mathcal{N}[\mathcal{L}[\phi], \mathcal{L}[\phi]], \]
\[ U_3[\phi] := \mathcal{N}[\mathcal{L}[\phi], \mathcal{N}[\mathcal{L}[\phi], \mathcal{L}[\phi]]] + \mathcal{N}[\mathcal{N}[\mathcal{L}[\phi], \mathcal{L}[\phi]], \mathcal{L}[\phi]], \]
\[ \vdots \]
\[ U_k[\phi] := \sum_{k_1, k_2 \geq 1, k_1 + k_2 = k} \mathcal{N}[U_{k_1}[\phi], U_{k_2}[\phi]] \quad (k \geq 2) \]

formally gives a solution to (3). To justify this, we basically need the linear and bilinear estimates

\[ \| \mathcal{L}[\phi] \|_S \leq C \| \phi \|_D, \quad \| \mathcal{N}[u_1, u_2] \|_S \leq C \| u_1 \|_S \| u_2 \|_S \]  

(4)

for the space of initial data $D$ and some space $S \subset C([0, T]; D)$ in which we construct a solution. In fact, they showed (roughly speaking) the following:

Assume that (4) holds with the Banach space $D$ of initial data and some Banach space $S$. Then, (i) for any $k \geq 1$ the operators $U_k : D \rightarrow S$ are well-defined and satisfies $\| U_k[\phi] \|_S \leq (C \| \phi \|_D)^k$, and (ii) there exists $\varepsilon_0 > 0$ (depending on the constants in (4)) such that the solution map $\phi \mapsto u[\phi] := \sum_{k=1}^{\infty} U_k[\phi]$ is well-defined on $B_D(\varepsilon_0) := \{ \phi \in D \mid \| \phi \|_D \leq \varepsilon_0 \}$ and gives a solution to (3).

Next, consider some coarser topologies on $D$ and $S$ induced by the norms $\| \|_D$ and $\| \|_S$ weaker than $\| \|_D$ and $\| \|_S$, respectively. They claimed the following:
Assume further that the solution map $\phi \mapsto u[\phi]$ given above is continuous from $(B_D(\varepsilon_0), \|\cdot\|_{D'})$ (i.e., $B_D(\varepsilon_0)$ equipped with the $D'$ topology) to $(S, \|\cdot\|_S)$. Then, for each $k$ the operator $U_k$ is continuous from $(B_D(\varepsilon_0), \|\cdot\|_{D'})$ to $(S, \|\cdot\|_S)$.

To show the continuity of $U_k$ in coarser topologies, by its homogeneity one can restrict to sufficiently small initial data. Then, by the estimates (4), contribution of higher order terms $\sum_{k'>k} U_{k'}[\phi]$ can be made arbitrarily small compared to $U_k[\phi]$. Combining this fact with the hypothesis that $\sum_{k\geq 1} U_k[\phi]$ is continuous, one can show the claim by an induction argument on $k$.

Now, this claim gives a way to prove ill-posedness in coarse topologies. Namely, one can show the discontinuity of the solution map $\phi \mapsto \sum_{k=1}^\infty U_k[\phi]$ in coarse topologies by simply establishing the discontinuity of the (more explicit) map $\phi \mapsto U_k[\phi]$ for at least one $k$.

We next recall Iwabuchi and Ogawa’s result [20], which settled the aforementioned two-dimensional case. Indeed, the argument in [20] is similar to that of [2] in that it exploits the power series expansion and shows that one term in the series exhibits instability and dominates all the other terms. Now, we notice that the existence time $T > 0$ is allowed to shrink for the purpose of establishing norm

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7It is worth noticing that the continuity of $U_k$ from $(B_D(\varepsilon_0), \|\cdot\|_{D'})$ to $(S, \|\cdot\|_S)$ does not imply its continuity from $(D, \|\cdot\|_{D'}$) to $(S, \|\cdot\|_S)$ in general, even though $U_k$ can be defined for all functions in $D$. By the $k$-linearity of $U_k$, the latter continuity is equivalent to the boundedness: $\|U_k[\phi]\|_S \leq C\|\phi\|_{D'}$. Hence, only disproving the boundedness of $U_k$ in coarse topologies (which may imply that the solution map is not $k$ times differentiable) is not sufficient to conclude the discontinuity of the solution map.
inflation, while in [2] it is fixed and uniform with respect to the initial data. The main difference of the argument in [20] from that of [2] is that they worked with the estimates like
\[
\|\mathcal{L}[\phi]\|_{S_T} \leq C\|\phi\|_{D}, \quad \|\mathcal{N}[u_1, u_2]\|_{S_T} \leq C T^\delta\|u_1\|_{S_T}\|u_2\|_{S_T}
\] (5)
for the data space \(D, S_T \subset C([0, T]; D)\), and \(\delta > 0\), and consider the expansion up to different times \(T\) according to the initial data. In fact, this enables us to take a sequence of initial data which is unbounded in \(D\) (but converges to 0 in a weaker norm), and such a set of initial data actually yields unbounded sequence of solutions. Another feature of the argument in [20] is that higher-order terms were estimated directly in \(D\) by using properties of specific initial data they chose; in [2] these terms were simply estimated in \(D\) by (4) that hold for general functions.

At a technical level, another novelty in [20] is the use of modulation space \(M_{2,1}^2\) as \(D\) instead of Sobolev spaces. The bilinear estimate in (5) is then straightforward thanks to the algebra property of \(M_{2,1}^2\).

Finally, we remark that the strategies of [2, 20] work well in the case that the operator \(U_k\) involves a significant high-to-low frequency cascade, as mentioned in [2]. However, the situation is different in the case of system of equations, as there are more than one regularity indices and one cannot simply order two pairs of regularity indices; see e.g. [30], where the argument of [20] was employed to derive norm inflation from nonlinear interactions of “high\times low→high” type.

3. Proof of Theorem 1.1. Let us first consider the case of single-term nonlinearity and prove Theorem 1.1. The argument in this section basically follows that in [20]. Since the coefficient \(\nu \neq 0\) plays no role in our proof, we assume \(\nu = 1\) for simplicity. We write
\[
\mu_{p,q}(z_1, \ldots, z_p) := \prod_{l=1}^{q} z_l \prod_{m=q+1}^{p} \bar{z}_m, \quad \mu_{p,q}(z) := \mu_{p,q}(z, \ldots, z),
\]
so that \(u^q \bar{u}^{p-q} = \mu_{p,q}(u)\).

Definition 3.1. For \(\phi \in L^2(Z)\), we (formally) define
\[
U_1[\phi](t) := e^{it\Delta} \phi, \\
U_k[\phi](t) := -i \sum_{k_1, \ldots, k_p \geq 1} \int_0^t e^{i(t-\tau)\Delta} \mu_{p,q}(U_{k_{1}}, \ldots, U_{k_{p}}[\phi])(\tau) \, d\tau, \quad k \geq 2.
\]
Note that \(U_k[\phi] = 0\) unless \(k \equiv 1 \mod p - 1\).

The expansion \(u = \sum_{k=1}^{\infty} U_k[\phi]\) of a (unique) solution \(u\) to (2) will play a crucial role in the proof. To make sense of this representation, we use modulation spaces. The notion of modulation spaces was introduced by Feichtinger in the 1980s [14] and nowadays it has become one of the common tools in the study of nonlinear evolution PDEs; see e.g. the survey [38] and references therein.
Definition 3.2. Let $A > 0$ be a dyadic number. Define the space $M_A$ as the completion of $C_0^\infty(Z)$ with respect to the norm

$$\|f\|_{M_A} := \sum_{\xi \in AZ^d} \|\hat{f}\|_{L^2(\xi + Q_A)},$$

where $Q_A := [-A, A]^d$.

Remark 2. We consider the space $M_A$ with $A < 1$ only when $Z = \mathbb{R}^d$. For $Z = \mathbb{R}^{d_1} \times \mathbb{T}^{d_2}$, the $L^2(\xi + Q_A)$ norm in the above definition means the $L^2$ norm restricted onto $(\xi + Q_A) \cap \hat{Z}$, where $\hat{Z} := \mathbb{R}^{d_1} \times \mathbb{Z}^{d_2}$. If $Z = \mathbb{T}^d$, the space $M_1$ coincides with the Wiener algebra $\mathcal{F}L^1(\mathbb{T}^d)$.

We will only use the following properties of the space $M_A$. The proof is elementary, and thus it is omitted.

Lemma 3.3. (i) $M_A \cong_A M_1$, $H^{\frac{d}{2} + \varepsilon} \hookrightarrow M_1 \hookrightarrow L^2 (\varepsilon > 0)$.
(ii) There exists $C = C(d) > 0$ such that for any $f, g \in M_A$, we have

$$\|fg\|_{M_A} \leq CA^\frac{d}{2}\|f\|_{M_A}\|g\|_{M_A}.$$

Since the space $M_A$ is a Banach algebra and the linear propagator $e^{it\Delta}$ is unitary in $M_A$, we can easily show the following multilinear estimates.

Lemma 3.4. Let $A \geq 1$ be a dyadic number and $\phi \in M_A$ with $\|\phi\|_{M_A} \leq M$. Then, there exists $C > 0$ independent of $A$ and $M$ such that

$$\|U_k[\phi](t)\|_{M_A} \leq t^{\frac{k-1}{p-1}}(CA^\frac{d}{2}M)^{k-1}M$$

for any $t \geq 0$ and $k \geq 1$.

Proof. Let $\{a_k\}_{k=1}^\infty$ be the sequence defined by

$$a_1 = 1, \quad a_k = \frac{p-1}{k-1} \sum_{k_1, \ldots, k_p \geq 1 \atop k_1 + \cdots + k_p = k} a_{k_1} \cdots a_{k_p}, \quad (k \geq 2).$$

As observed in [2, Eq. (16)], one can show inductively that $a_k \leq C^k$ for some $C > 0$. To be more precise, we state it as the following lemma. The $p = 2$ case can be found in [31, Lemma 4.2] with a detailed proof.

Lemma 3.5. Let $\{b_k\}_{k=1}^\infty$ be a sequence of nonnegative real numbers such that

$$b_k \leq C \sum_{k_1, \ldots, k_p \geq 1 \atop k_1 + \cdots + k_p = k} b_{k_1} \cdots b_{k_p}, \quad k \geq 2$$

for some $p \geq 2$ and $C > 0$. Then, we have

$$b_k \leq b_1 C_0^{k-1}, \quad k \geq 1; \quad C_0 := \frac{\pi^2}{6}(Cp^2)^{\frac{1}{p+1}}b_1.$$

By Lemma 3.5, it holds $a_k \leq C_0^{k-1}$ for some $C_0 > 0$. Thus, it suffices to show

$$\|U_k[\phi](t)\|_{M_A} \leq a_k t^{\frac{k-1}{p-1}}(C_1 A^\frac{d}{2}M)^{k-1}M, \quad t \geq 0, \quad k \geq 1.$$
for some $C_1 > 0$. This is trivial if $k = 1$. Let $k \geq 2$, and assume the above estimate for $U_1, U_2, \ldots, U_{k-1}$. Using Lemma 3.3, we have
\[
\|U_k[\phi](t)\|_{M_A} \leq CA^{\frac{d}{2}(p-1)} \sum_{k_1, \ldots, k_p \geq 1, k_1 + \cdots + k_p = k} \int_0^t \prod_{j=1}^p \|U_{k_j}[\phi](\tau)\|_{M_A} d\tau
\]
\[
\leq CA^{\frac{d}{2}(p-1)}(CA^{\frac{d}{2}}M)^{k-p} M^p \sum_{k_1, \ldots, k_p \geq 1, k_1 + \cdots + k_p = k} a_{k_1} \cdots a_{k_p} \int_0^t \tau^{\frac{k-p}{2}} dt
\]
\[
= C a_k C_1^{k-p}(A^{\frac{d}{2}} M)^{k-1} M t^{\frac{k-p}{2}}.
\]
The estimate for $U_k$ follows by setting $C_1$ to be $C^{\frac{d}{2}}$ with the constant $C$ in the last line, which is independent of $k$. \hfill \square

A standard argument (cf. [2, Theorem 3]) with Lemma 3.3 (ii) and Lemma 3.4 shows the following local well-posedness of (2) in $M_A$.

**Corollary 1.** Let $A \geq 1$ be dyadic, and $M > 0$. If $0 < T \ll (A^{d/2} M)^{-p-1}$, then for any $\phi \in M_A$ with $\|\phi\|_{M_A} \leq M$ the following holds.

(i) A unique solution $u$ to the integral equation associated with (2),
\[
u(t) = e^{i t \Delta} \phi - i \int_0^t e^{i(t-\tau)\Delta} \mu_{p, q}(u(\tau)) d\tau, \quad t \in [0, T]
\]
exists in $C([0, T]; M_A)$.

(ii) The solution $u$ given in (i) has the expression
\[
u = \sum_{k=1}^{\infty} U_k[\phi] = \sum_{i=0}^{\infty} U_{(p-1)i+1}[\phi],
\]
which converges absolutely in $C([0, T]; M_A)$.

**Proof.** (i) Let
\[
\Psi_\phi[u](t) := e^{i t \Delta} \phi - i \int_0^t e^{i(t-\tau)\Delta} \mu_{p, q}(u(\tau)) d\tau,
\]
then from Lemma 3.3 (ii) we have
\[
\|\Psi_\phi[u]\|_{L^\infty(0, T; M_A)} \leq \|\phi\|_{M_A} + CTA^{\frac{d}{2}(p-1)} ||u||_{L^\infty(0, T; M_A)}^p
\]
and that $\Psi$ is a contraction on a ball in $C([0, T]; M_A)$ if $TA^{\frac{d}{2}(p-1)} \|\phi\|_{M_A}^{p-1} \ll 1$.

(ii) The series $u = \sum_{k=1}^{\infty} U_k[\phi]$ converges in $C([0, T]; M_A)$ by virtue of Lemma 3.4. By uniqueness, it suffices to show that $u$ solves the equation (6). Let $u_K := \sum_{k=1}^K U_k[\phi]$, so that $u = \lim_{K \to \infty} u_K$ in $C([0, T]; M_A)$. We see that $\Psi_\phi[u_K] - u_K$ consists of $k$-linear terms in $\phi$ with $K + 1 \leq k \leq pK$, and we can show
\[
\|\Psi_\phi[u_K] - u_K\|_{L^\infty(0, T; M_A)} \leq C(TA^{\frac{d}{2}} M)^K M
\]
by an argument similar to Lemma 3.4. By letting $K \to \infty$, we obtain $\Psi_\phi[u] = u$. \hfill \square

**Remark 3.** (i) In $M_A$ we have unconditional local well-posedness. In particular, the embedding (Lemma 3.3 (i)) shows that the unique solution with initial data in some high-regularity Sobolev space exists on a time interval $[0, T]$ and coincides with the solution constructed in Corollary 1.
(ii) In the following proof of Theorem 1.1 we will take initial data that are localized in frequency on several cubes of side length $O(A)$ located in $\{\xi \gg \max(1, A)\}$. For such initial data the $L^2$ norm is comparable with the $M_A$ norm, but much smaller than the Sobolev norms of positive indices. In the $L^2$-supercritical cases (i.e., $s_c(d, p) > 0$), no reasonable well-posedness is expected in $L^2$, while the use of higher Sobolev space would verify the power series expansion only on a smaller time interval. In this regard, the space $M_A$ is suitable for our purpose.

Let $N, A$ be dyadic numbers to be specified so that $N \gg 1$ and $0 < A \ll N$ ($1 \leq A \ll N$ when $Z$ has a periodic direction). In the proof of norm inflation, we will use initial data $\phi$ of the following form:

$$\hat{\phi} = rA^{-\frac{d}{2}}N^{-s} \chi_\Omega$$

with a positive constant $r$ and a set $\Omega$ satisfying

$$\Omega = \bigcup_{\eta \in \Sigma} (\eta + Q_A)$$

for some $\Sigma \subset \{\xi \in \mathbb{R}^d : |\xi| \sim N\}$ s.t. $#\Sigma \leq 3$. \(\text{(7)}\)

Note that $\|\phi\|_{M_A} \sim rN^{-s}$, $\|\phi\|_{H^s} \sim r$.

We derive Sobolev bounds of $U_k[\phi](t)$ with $\phi$ satisfying the above condition.

**Lemma 3.6.** There exists $C > 0$ such that for any $\phi$ satisfying (7) and $k \geq 1$, we have

$$\left| \text{supp} \ U_k[\phi](t) \right| \leq C^k A^d, \quad t \geq 0.$$

**Proof.** Since the $\xi$-support of $U_k[\phi]$ is determined by a spatial convolution of $k$ copies of $\hat{\phi}$ or $\hat{\phi} = \hat{\phi}(-\cdot)$, it is easily seen that

$$\text{supp} \ U_k[\phi](t) \subset \bigcup_{\eta \in S_k} (\eta + QA)$$

for all $t \geq 0$, where $S_1 := \Sigma$ and

$$S_k := \{\eta \in \mathbb{R}^d : \eta = \sum_{l=1}^{k} \eta_l, \eta_l \in \Sigma \cup (-\Sigma) \ (1 \leq l \leq k)\}, \quad k \geq 2.$$

Since $#S_k \leq 6^k$, we have

$$\left| \text{supp} \ U_k[\phi](t) \right| \leq |Q_{kA}| #S_k \leq (kA)^d 6^k \leq C^k A^d.$$  \(\square\)

**Lemma 3.7.** Let $\phi$ satisfy (7). Assume that $s < 0$. Then, there exists $C > 0$ depending only on $d, p, s$ such that the following holds.

(i) $\|U_1[\phi](T)\|_{H^s} \leq Cr$ for any $T \geq 0$.

(ii) $\|U_k[\phi](T)\|_{H^s} \leq Cr(Cp)^{k-1}A^{-\frac{d}{2}}N^{-s}f_s(A)$ for any $T \geq 0$ and $k \geq 2$, where

$$\rho := rA^{-\frac{d}{2}}N^{-s}T^{\frac{1}{2s}}, \quad f_s(A) := \|\langle \xi \rangle^s\|_{L^2(|\xi| \leq A)}.$$

**Proof.** (i) is easily verified. For (ii), we see that

$$\|U_k[\phi](t)\|_{H^s} \leq \|\langle \xi \rangle^s\|_{L^2(\text{supp} \ U_k[\phi](t))} \sup_{t \in \mathbb{R}^d} \left| \text{supp} \ U_k[\phi](t, \xi) \right|$$

$$\leq \|\langle \xi \rangle^s\|_{L^2(\text{supp} \ U_k[\phi](t))} \sum_{k_1, \ldots, k_p \geq 1 \atop k_1 + \cdots + k_p = k} \int_0^t \left| v_k(\tau) \right| \cdots \left| v_{kp}(\tau) \right| d\tau,$$
where \( v_{k_1} \) is either \( \overline{U_{k_1}[\phi]} \) or \( \overline{\hat{U}_{k_1}[\phi]} \). By Young’s inequality, the above is bounded by
\[
\left\| \langle \xi \rangle^s \right\|_{L^2(\operatorname{supp} \overline{U_{k}[\phi]}(t))} \sum_{k_1, \ldots, k_p \geq 1} \int_0^t \left\| v_{k_1}(\tau) \right\|_{L^2} \left\| v_{k_2}(\tau) \right\|_{L^2} \prod_{l=3}^p \left\| v_{k_l}(\tau) \right\|_{L^2} d\tau
\leq \left\| \langle \xi \rangle^s \right\|_{L^2(\operatorname{supp} \overline{U_k[\phi]}(t))} \sum_{k_1, \ldots, k_p \geq 1} \int_0^t \prod_{l=3}^p \left\| \left. \sup_{\lambda \geq 1} \right|_{\lambda \geq 1} \right\|_{L^2} d\tau.
\]

Since \( s < 0 \), for any bounded set \( D \subset \mathbb{R}^d \) it holds that
\[
\left\| \langle \xi \rangle^s \right\|_{L^2(D)} \leq \left\| \langle \xi \rangle^s \right\|_{L^2(B_D)} \quad (\lambda > 0),
\]
where \( B_D \subset \mathbb{R}^d \) is the ball centered at the origin with \( |D| = |B_D| \). This implies that \( \left\| \langle \xi \rangle^s \right\|_{L^2(D)} \leq \left\langle \langle \xi \rangle^s \right\rangle_{L^2(B_D)} \). Moreover, it follows from Lemma 3.4 with \( M = CrN^{-s} \) that
\[
\left\| U_k[\phi](t) \right\|_{L^2} \leq \left\| U_k[\phi](t) \right\|_{M_A} \leq Ct^\frac{k-1}{2}(CrA^\frac{k}{2}N^{-s})^{k-1}rN^{-s}, \quad k \geq 1.
\]
Hence, we apply Lemma 3.6 to bound the above by
\[
\left\| \langle \xi \rangle^s \right\|_{L^2(A)} \times C^\frac{k}{2} A^\frac{(d-2)}{2} \sum_{k_1, \ldots, k_p \geq 1} \int_0^t \prod_{l=1}^p \left[ C_t^\frac{k-1}{2}(CrA^\frac{k}{2}N^{-s})^{k-1}rN^{-s} \right] d\tau
\leq C^k \left\| \langle \xi \rangle^s \right\|_{L^2(A)} A^\frac{(d-2)}{2} \left( k-p \right) (rN^{-s})^k \int_0^t \tau^\frac{k-1}{2} d\tau
\leq f(A)A^{\frac{d}{2} \left( k-2 \right)} (CrN^{-s})^{k-1} t^\frac{k-1}{2},
\]
which is the desired one. \( \square \)

We observe the following lower bounds on the \( H^s \) norm of the first nonlinear term in the expansion of the solution.

**Lemma 3.8.** The following estimates hold for any \( s \in \mathbb{R} \).

(i) Let \((p, q)\) and \(Z = \mathbb{R}^d_1 \times T^d_2\) be arbitrary. For \( 1 \leq A \ll N \), we define the initial data \( \phi \) by (7) with \( \Sigma = \{ Ne_d, -Ne_d, 2Ne_d \} \), where \( e_d := (0, \ldots, 0, 1) \in \mathbb{R}^d \).

If \( 0 < T \ll N^{-2} \), then we have
\[
\left\| U_p[\phi](T) \right\|_{H^s} \gtrsim r \rho^{p-1} A^{-\frac{d}{2}} N^{-s} f_s(A).
\]

(ii) Let \((p, q) = (2, 1)\) and \(Z = \mathbb{R}^d_1 \), \( 1 \leq d \leq 3 \). For \( N \gg 1 \), define \( \phi \) by
\[
\hat{\phi} := rN^{\frac{d}{2}-s} \chi_{N\epsilon_d + \tilde{Q}_{N^{-1}}} \quad \text{with} \quad r > 0, \quad \tilde{Q}_{N^{-1}} := \left[ -\frac{1}{2}, \frac{1}{2} \right]^d - \left[ -\frac{1}{2N}, \frac{1}{2N} \right].
\]

Then, for any \( 0 < T \ll 1 \) we have
\[
\left\| U_2[\phi](T) \right\|_{H^s} \gtrsim r^2 N^{-2s-\frac{1}{2}} T.
\]

(iii) Let \((p, q) = (2, 1)\) and \(Z = \mathbb{R}^d_1 \times T^d_2\) with \( d_1 + d_2 \leq 3 \), \( d_2 \geq 1 \). Define \( \phi \) by (7) with \( A = 1 \), \( \Sigma = \{ Ne_d \} \). Then, for any \( 0 < T \ll 1 \) we have
\[
\left\| U_2[\phi](T) \right\|_{H^s} \gtrsim r^2 N^{-2s} T.
\]
(iv) Let \((p, q) = (4, 1)\) or \((4, 2)\) or \((4, 3)\) and \(Z = \mathbb{T}\). Define \(\phi\) by (7) with \(A = 1, \Sigma = \{-N, 2N, 3N\}\). Then, for any \(T > 0\) we have

\[
\|U_1[\phi](T)\|_{H^s} \gtrsim N^{-4s} T.
\]

**Proof.** Note that

\[
\hat{U}_1[\phi](T, \xi) \equiv c e^{-iT|\xi|^2} \int_t^T \prod_{l=1}^q \bar{\phi}(\xi_l) \prod_{m=q+1}^p \bar{\phi}(\xi_m) \int_0^T e^{i\Phi} dt,
\]

where

\[
\Gamma := \{ (\xi_1, \ldots, \xi_p) \mid \sum_{l=1}^q \xi_l - \sum_{m=q+1}^p \xi_m = \xi \}, \quad \Phi := |\xi|^2 - \sum_{l=1}^q |\xi_l|^2 + \sum_{m=q+1}^p |\xi_m|^2.
\]

(i) If we restrict \(\xi\) to \(Q_A\), we have

\[
\hat{U}_1[\phi](T, \xi) = c (r A^{-\frac{d}{2}} N^{-s}) e^{-iT|\xi|^2} \sum_{(\eta_1, \ldots, \eta_p)} \int_{\Gamma} \prod_{l=1}^q \chi_{\eta_l + Q_A}(\xi_l) \int_0^T e^{i\Phi} dt,
\]

where the sum is taken over the set

\[
\{(\eta_1, \ldots, \eta_p) \in \{\pm Ne_d, 2Ne_d\}^p \mid \sum_{l=1}^q \eta_l - \sum_{m=q+1}^p \eta_m = 0\},
\]

which is non-empty for any \((p, q)\). Since \(|\Phi| \lesssim N^2\) in the integral, for \(0 < T \ll N^{-2}\) we have

\[
|\hat{U}_1[\phi](T, \xi)| \gtrsim (r A^{-\frac{d}{2}} N^{-s})^p (A^d)^{p-1} T \chi_{p^{-1}Q_A}(\xi),
\]

and thus

\[
\|U_1[\phi](T)\|_{H^s} \gtrsim (r A^{-\frac{d}{2}} N^{-s})^p (A^d)^{p-1} T \|\xi|^s\|_{L^2(\mathbb{C}^p)} \sim r^p A^{-\frac{d}{2}} N^{-s} f_s(A).
\]

(ii) In this case we have

\[
\hat{U}_2[\phi](T, \xi) = c (r N^{-\frac{d}{2}-s})^2 e^{-iT|\xi|^2} \int_{\xi_1 - \xi_2 = \xi} \chi_{Q_{N-1}}(\xi_1 - Ne_d) \chi_{Q_{N-1}}(\xi_2 - Ne_d) \int_0^T e^{i\Phi} dt,
\]

and in the integral, for \(\xi = \xi_1 - \xi_2 \in Q_{N-1}\),

\[
\Phi = |\xi|^2 - |\xi_1|^2 + |\xi_2|^2
\]

\[
= |\xi|^2 - \|\xi_1 - Ne_d\|^2 + \|\xi_2 - Ne_d\|^2 - 2(\xi_1 - \xi_2) \cdot Ne_d = O(1).
\]

Hence, if \(0 < T \ll 1\), we have

\[
|\hat{U}_2[\phi](T)| \gtrsim (r N^{-\frac{d}{2}-s})^2 N^{-1} T \chi_{2^{-1}Q_{N-1}}, \quad \|U_2[\phi](T)\|_{H^s} \gtrsim (r N^{-\frac{d}{2}-s})^2 N^{-\frac{d}{2}} T
\]

for any \(s \in \mathbb{R}\).

(iii) Similarly to (ii), we see that

\[
\hat{U}_2[\phi](T, (\xi', 0)) = c (r N^{-s})^2 e^{-iT|\xi|^2} \int_{\xi_1 - \xi_2 = \xi'} \chi_{\{1/2, 1/2\}^d}(\xi_1') \chi_{\{1/2, 1/2\}^d}(\xi_2') \int_0^T e^{i\Phi} dt,
\]

9If \(p\) is even, we can choose \(\eta_l\) to be \(Ne_d\) or \(-Ne_d\) so that \(\sum_{l=1}^q \eta_l - \sum_{m=q+1}^p \eta_m = 0\). If \(p\) is odd, we choose \(\eta_1 = 2Ne_d\) and \(\eta_2\) to be \(Ne_d\) or \(-Ne_d\) so that the output from these two frequencies is either \(Ne_d\) or \(-Ne_d\). Then, the other \(\eta_j\) can be chosen as for \(p\) even.
where the integral in $\xi' = (\xi_1, \ldots, \xi_{d-1})$ vanishes if $Z = T$. In the integral, 
\[ \Phi = \|((\xi', 0)^2 - |(\xi'_1, N)|^2 + |(\xi'_2, N)|^2 = O(1). \]

Hence, if $0 < T \ll 1$, we have 
\[ \|U_2[\phi](T)\|_{H^s} \geq \|\langle \xi \rangle^s \hat{U}_2[\phi](T)\|_{L^2(Q_{1/2})} \gg (r N^{-s})^2 T \]
for any $s \in \mathbb{R}$.

(iv) We first consider $(p, q) = (4, 1)$; the case of $(4, 3)$ is treated in the same way.

Observe that 
\begin{align*}
\{ (\eta_1, \ldots, \eta_4) \in \{-N, 2N, 3N\}^4 \mid \eta_1 - \eta_2 - \eta_3 - \eta_4 = 0 \} &= \{ (3N, -N, 2N, 2N), (3N, 2N, -N, 2N), (3N, 2N, 2N, -N) \}.
\end{align*}

Therefore, we have 
\[ \hat{U}_4[\phi](T, 0) = c(r N^{-s})^4 \sum_{\xi_1, \ldots, \xi_4 \in \mathbb{Z}} 4 \chi_{(-N, 2N, 3N)}(\xi) \int_0^T e^{i t \Phi} dt \]
\[ = 3c(r N^{-s})^4 \int_0^T e^{i t (0^2 - (3N)^2 + (-N)^2 + (2N)^2 + (2N)^2)} dt = 3c(r N^{-s})^4 T, \]
which implies 
\[ \|U_4[\phi](T)\|_{H^s} \gg (r N^{-s})^4 T \]
for any $s \in \mathbb{R}$ and $T > 0$.

Next, we consider $(p, q) = (4, 2)$, which is very similar to the above. Since 
\begin{align*}
\{ (\eta_1, \ldots, \eta_4) \in \{-N, 2N, 3N\}^4 \mid \eta_1 + \eta_2 - \eta_3 - \eta_4 = 0 \} &= \{ (\eta_1, \ldots, \eta_4) \in \{-N, 2N, 3N\}^4 \mid \{\eta_1, \eta_2\} = \{\eta_3, \eta_4\} \},
\end{align*}
we have 
\[ \hat{U}_4[\phi](T, 0) = c(r N^{-s})^4 \sum_{\xi_1, \ldots, \xi_4 \in \mathbb{Z}} 4 \chi_{(-N, 2N, 3N)}(\xi) \int_0^T e^{i t \Phi} dt = 15c(r N^{-s})^4 T, \]
and the same estimate holds.

Now, we are in a position to prove norm inflation.

**Proof of Theorem 1.1.** We first recall that $U_k[\phi] = 0$ unless $k \equiv 1 \mod p - 1$. If the initial data $\phi$ satisfies (7), Corollary 1 guarantees existence of the solution to (2) and the power series expansion in $M_A$ up to time $T$ whenever $\rho = r A^{\frac{2}{3}} N^{-s} T \frac{1}{p} \ll 1$.

Case 1: General $Z$ and $(p, q)$, $s < \min\{s_c(d, p), 0\}$.

Take $\phi$ as in Lemma 3.8 (i). From Lemmas 3.7 and 3.8, under the conditions 
\[ T \ll N^{-2}, \quad \rho \ll 1, \quad r \rho^{p-1} A^{-\frac{2}{3}} N^{-s} f_s(A) \gg r, \]
we have 
\[ \|u(T)\|_{H^s} \sim \|U_p[\phi](T)\|_{H^s} \sim r \rho^{p-1} A^{-\frac{2}{3}} N^{-s} f_s(A). \]

Now, we set 
\begin{align*}
& r = (\log N)^{-1}, \quad A \sim (\log N)^{-\frac{q+1}{3q+1}} N, \quad T = (A^{-\frac{4}{3}} N^s)^{p-1},
\end{align*}

so that \( \rho = (\log N)^{-1} \ll 1 \). The super-critical assumption \( s < s_c(d, p) = \frac{d}{2} - \frac{2}{p-1} \) ensures that

\[
T \sim (\log N)^{\frac{d(p+1)}{2(d+p)} (p-1) N^{(s - \frac{d}{2})(p-1)}} \ll N^{-2}.
\]

Moreover, since \( f_s(A) \gtrsim A^{\frac{d}{2} + s} \) for any \( s < 0 \) and \( A \geq 1 \), we see that

\[
r \rho^{-1} A^{-\frac{d}{2}} N^{-s} f_s(A) \gtrsim r \rho^{-1} A^s N^{-s} \sim \log N \gg (\log N)^{-1} = r.
\]

Therefore, (8) is fulfilled and we have \( \|u(T)\|_{H^s} \gtrsim \log N \). Noticing \( \|\phi\|_{H^s} \sim r = (\log N)^{-1} \) and \( T \ll N^{-2} \), we show norm inflation by letting \( N \to \infty \).

Case 2: \( Z = \mathbb{R} \) or \( T \), \( (p, q) = (2, 0) \) or \( (2, 2) \), \( -\frac{3}{2} \leq s < -1 \).

We take the same initial data \( \phi \) as in Case 1, but with

\[
r = (\log N)^{-1}, \quad A = 1, \quad T = (\log N)^{-1} N^{-2}.
\]

Then, \( T \ll N^{-2} \), \( \rho = (\log N)^{-2} N^{-2-s} \ll 1 \) by \( s \geq -\frac{3}{2} \) and

\[
r \rho^{-1} A^{-\frac{d}{2}} N^{-s} f_s(A) \sim r \rho N^{-s} = (\log N)^{-3} N^{-2-2s} \gg 1 \gg r
\]

by \( s < -1 \). Hence, (8) holds and we have \( \|u(T)\|_{H^s} \sim (\log N)^{-3} N^{-2-2s} \gg 1 \), which together with \( \|\phi\|_{H^s} \sim r \ll 1 \) and \( T \ll 1 \) shows norm inflation by taking \( N \) large.

Case 3: \( Z = \mathbb{R} \) or \( T \), \( p = 3 \), \( s = -\frac{1}{2} \).

Take the same \( \phi \) as in Case 1, but with

\[
r = (\log N)^{-\frac{3}{2}}, \quad A \sim (\log N)^{-\frac{1}{2}} N, \quad T = (\log N)^{-\frac{3}{2}} N^{-2}.
\]

Then, \( T \ll N^{-2} \), \( \rho \sim (\log N)^{-\frac{1}{2}} \ll 1 \) and

\[
r \rho^{-1} A^{-\frac{d}{2}} N^{-s} f_s(A) \sim r \rho^2 A^{-\frac{d}{2}} N^{-\frac{3}{2}} (\log A)^{\frac{1}{2}} \sim (\log N)^{\frac{1}{2}} \gg 1 \gg r.
\]

Hence, (8) holds and we have \( \|u(T)\|_{H^s} \sim (\log N)^{\frac{1}{2}} \gg 1 \), which implies norm inflation as well.

Case 4: \( Z = \mathbb{R}^2 \) or \( \mathbb{R} \times T \) or \( T^2 \), \( (p, q) = (2, 0) \) or \( (2, 2) \), \( s = -1 \).

We follow the argument in Case 1 again, but with

\[
r = (\log N)^{-\frac{3}{2}}, \quad A \sim (\log N)^{-\frac{1}{2}} N, \quad T = (\log N)^{-\frac{3}{2}} N^{-2}.
\]

Then, \( T \ll N^{-2} \), \( \rho \sim (\log N)^{-\frac{1}{2}} \ll 1 \) and

\[
r \rho^{-1} A^{-\frac{d}{2}} N^{-s} f_s(A) \sim r \rho A^{-1} N (\log A)^{\frac{1}{2}} \sim (\log N)^{\frac{1}{2}} \gg 1 \gg r.
\]

Hence, (8) holds and we have \( \|u(T)\|_{H^s} \sim (\log N)^{\frac{1}{2}} \gg 1 \), which shows \( N \text{L}_1 \).

Case 5: \( Z = \mathbb{R}^{d_1} \times T^{d_2} \) with \( d_1 + d_2 \leq 3 \), \( d_2 \geq 1 \), \( (p, q) = (2, 1) \), and \( \frac{d}{2} - 2 \leq s < 0 \).

Take \( \phi \) as in Lemma 3.8 (iii) and choose \( r, T \) as \( r = (\log N)^{-1} \) and \( T = N^s \), which implies

\[
T \ll 1, \quad \rho \sim r N^{-s} T = (\log N)^{-1} \ll 1, \quad r \rho N^{-s} \sim (\log N)^{-2} N^{-s} \gg 1 \gg r.
\]

From Lemmas 3.7 and 3.8, we have \( \|u(T)\|_{H^s} \sim \|\mathbb{U}_2[\phi](T)\|_{H^s} \sim (\log N)^{-2} N^{-s} \gg 1 \), and norm inflation occurs.

Case 6: \( Z = T \), \( (p, q) = (4, 1) \) or \( (4, 2) \) or \( (4, 3) \), and \( -\frac{1}{6} \leq s < 0 \).

Take \( \phi \) as in Lemma 3.8 (iv), and then take \( r = (\log N)^{-1} \) and \( T = N^{3s} \), which implies

\[
T \ll 1, \quad \rho \sim r N^{-s} T^{\frac{1}{2}} = (\log N)^{-1} \ll 1, \quad r \rho N^{-s} \sim (\log N)^{-2} N^{-s} \gg 1 \gg r.
\]
Again, we have \( \|u(T)\|_{H^s} \sim \|U_2[\phi](T)\|_{H^s} \sim (\log N)^{-4}N^{-s} \gg 1 \).

Case 7: \( Z = \mathbb{R}^d \) with \( 1 \leq d \leq 3 \), \( (p, q) = (2, 1) \), and \( \frac{d}{2} - 2 \leq s < -\frac{1}{4} \).

In this case the data \( \phi \) is taken as in Lemma 3.8 (ii) and does not satisfy (7), so we need to modify the previous argument.

We use anisotropic modulation space \( \tilde{M} \) defined by the norm
\[
\|f\|_{\tilde{M}} := \sum_{\xi \in \mathbb{Z}^{d-1} \times N^{-1} \mathbb{Z}} \|\hat{f}\|_{L^2(\xi + \tilde{\gamma}_N^{-1})}.
\]
We have the product estimate
\[
\|fg\|_{\tilde{M}} \lesssim N^{-\frac{1}{2}}\|f\|_{\tilde{M}}\|g\|_{\tilde{M}}
\]
in this space. Thus, we follow the proof of Lemma 3.4 to obtain
\[
\|U_k[\phi](t)\|_{\tilde{M}} \leq C\rho N^{-\frac{1}{4}-s}T^{k-1}N^{-s}
\]
for any \( k \geq 1 \), which is used to justify the expansion of the solution in \( \tilde{M} \) up to time \( T \) such that \( T := rN^{-\frac{1}{4}-s}T \ll 1 \). Then, by the same argument as in the proofs of Lemmas 3.6 and 3.7, we see that
\[
|\text{supp } \tilde{U}_k[\phi](t)| \leq C^kN^{-1}, \quad \|U_k[\phi](T)\|_{H^s} \leq C^k(\rho N)^{k-1}N^{-s}.
\]
In particular, \( \|U_2[\phi](T)\|_{H^s} \sim r\tilde{\rho}N^{-s} \) for \( 0 < T \ll 1 \) by Lemma 3.8 (iii).

Now, we take \( r = (\log N)^{-1} \ll 1 \), \( T = (\log N)^3N^{2s+\frac{1}{4}} \ll 1 \), so that \( \tilde{\rho} = (\log N)^2N^s \ll 1 \), \( r\tilde{\rho}N^{-s} = \log N \gg r \). From the estimates above, we have \( \|u(T)\|_{H^s} \sim \log N \gg 1 \), which shows norm inflation.

4. Proof of Theorem 1.2. Here, we see how to use the estimates for single-term nonlinearities for the proof in the multi-term cases. We write \( p := \max_{1 \leq j \leq n} p_j \).

For the initial value problem (1), the \( k \)-th order term \( U_k[\phi] \) in the expansion of the solution is given by
\[
U_k[\phi] := -i\sum_{j=1}^n \sum_{k_1, \ldots, k_{p_j} \geq 1}_{k_1 + \cdots + k_{p_j} = k} \int_0^t e^{i(t-\tau)\Delta} \mu_{p_j} \left(U_{k_1}[\phi](\tau), \ldots, U_{k_{p_j}}[\phi](\tau)\right) d\tau
\]
for \( k \geq 2 \) inductively.

The following lemmas are verified in the same manner as Lemmas 3.5, 3.4, and Corollary 1.

Lemma 4.1. Let \( \{b_k\}_{k=1}^\infty \) be a sequence of nonnegative real numbers such that
\[
b_k \leq \sum_{j=1}^n C_j \sum_{k_1, \ldots, k_{p_j} \geq 1}_{k_1 + \cdots + k_{p_j} = k} b_{k_1} \cdots b_{k_{p_j}}, \quad k \geq 2
\]
for some \( p_1, \ldots, p_n \geq 2 \) and \( C_1, \ldots, C_n > 0 \). Then, we have
\[
b_k \leq b_1 C_0^{k-1}, \quad k \geq 1, \quad C_0 = \max_{1 \leq j \leq n} \frac{\prod_{i=1}^{p_j} (nC_i p_j^2)\frac{1}{p_j}}{6} b_1.
\]

Lemma 4.2. There exists \( C > 0 \) such that for any \( \phi \in M_A \) with \( \|\phi\|_{M_A} \leq M \) we have
\[
\|U_k[\phi](t)\|_{M_A} \leq \frac{\|\phi\|_{M_A}}{M} M^{-\frac{k-1}{2}}(CA^2M)^{k-1}M
\]
for any \( 0 \leq t \leq 1 \) and \( k \geq 1 \).
Lemma 4.3. Let $\phi \in M_A$ with $\|\phi\|_{M_A} \leq M$. If $T > 0$ satisfies $A^{\frac{3}{4}}MT^{\frac{1}{4}} \ll 1$, then a unique solution $u \in C([0,T]; M_A)$ to (1) exists and has the expansion $u = \sum_{k=1}^{\infty} U_k[\phi]$.

The next lemma can be verified similarly to Lemma 3.7.

Lemma 4.4. Let $\phi$ satisfy (7) and $s < 0$. Then, the following holds.

(i) $\|U_1[\phi](T)\|_{H^s} \leq C r$ for any $T \geq 0$.

(ii) $\|U_k[\phi](T)\|_{H^s} \leq Cr(Cp)^{k-1}A^{-\frac{3}{4}} T^{-s} N^{-s} f_s(A)$ for any $0 \leq T \leq 1$ and $k \geq 2$, where

$$\rho = rA^{\frac{3}{4}} T^{-s} \sum_{k=1}^{\infty} \left( \sum_{q=0}^{p} \nu_{p,q} \mu_{p,q}(u) \right)$$

We now begin to prove Theorem 1.2.

Proof of Theorem 1.2. We divide the proof into two cases: (I) One of the terms of order $p$ (highest order) is responsible for norm inflation, or (II) a lower order term determines the range of regularities for norm inflation. Note that (II) occurs only when $Z = \mathbb{R}, p = 3, F(u, \bar{u})$ has the term $u \bar{u}$ and $s \in (-\frac{1}{2}, -\frac{3}{4})$.

(I): Rewrite the nonlinear terms as

$$F(u, \bar{u}) = \sum_{q=0}^{p} \nu_{p,q} \mu_{p,q}(u) + \text{(terms of order less than } p).$$

Note that $\nu_{p,q}$ may be zero but $(\nu_{0,0}, \ldots, \nu_{p,p}) \neq (0, \ldots, 0)$.

We divide the series into four parts:

$$\sum_{k=1}^{\infty} U_k[\phi] = U_1[\phi] + \left\{ \sum_{k=2}^{\infty} U_k[\phi] - \left( -i \sum_{q=0}^{p} \nu_{p,q} \int_0^t e^{i(t-\tau)} \mu_{p,q}(U_1[\phi](\tau)) d\tau \right) \right\}$$

$$+ \left( -i \sum_{q=0}^{p} \nu_{p,q} \int_0^t e^{i(t-\tau)} \mu_{p,q}(U_1[\phi](\tau)) d\tau \right) + \sum_{k=p+1}^{\infty} U_k[\phi] =: U_1[\phi] + U_{\text{low}}[\phi] + U_{\text{main}}[\phi] + U_{\text{high}}[\phi].$$

Note that $U_{\text{low}}[\phi] = 0$ if $p = 2$.

The following lemma indicates how $U_{\text{low}}$ is dominated by $U_{\text{main}}$, and how the contributions of the $(p+1)$ terms in $U_{\text{main}}$ can be separated.

Lemma 4.5. We have the following:

(i) Let $\phi$ satisfy (7) and $s < 0$. Let $0 < T \leq 1$, and assume that

$$\rho = rA^{\frac{3}{4}} T^{-s} \ll 1.$$

Then, if $p \geq 3$,

$$\|U_{\text{low}}[\phi](T)\|_{H^s} \lesssim r^2 N^{-2s} f_s(A) T^{\frac{s}{s-1}}.$$

(ii) Let $q_s \in \{0, 1, \ldots, p\}$ be such that $\nu_{p,q_s} \neq 0$. Then, for any $T \geq 0$ there exists $j \in \{0, 1, \ldots, p\}$ such that

$$\|U_{\text{main}}[e^{i\frac{2\pi}{T} t} \phi](T)\|_{H^s} \gtrsim \|G_{q_s}[\phi](T)\|_{H^s},$$

where

$$G_{q_s}[\phi](t) := -i \int_0^t e^{i(t-\tau)} \mu_{p,q_s}(U_1[\phi](\tau)) d\tau; \quad U_{\text{main}}[\phi] = \sum_{q=0}^{p} \nu_{p,q} G_q[\phi](t).$$
This implies the claim. 

Hence, if $\nu \geq 0$ since $0 < T \leq 1$ implies $rA^{\frac{1}{2}}N^{-s}T^{-\frac{1}{2} - 1} \leq \rho \ll 1$, we have

$$\|U_{low}[\phi](T)\|_{H^s} \leq \sum_{k=2}^{p} Cr(CrA^{\frac{1}{2}}N^{-s}T^{-\frac{1}{2} - 1})^{k-1}A^{-\frac{1}{2}}N^{-s}f_s(A).$$

Since $0 < T \leq 1$ implies $rA^{\frac{1}{2}}N^{-s}T^{-\frac{1}{2} - 1} \leq \rho \ll 1$, we have

$$\|U_{low}[\phi](T)\|_{H^s} \leq r \cdot rA^{\frac{1}{2}}N^{-s}T^{-\frac{1}{2}} \cdot A^{-\frac{1}{2}}N^{-s}f_s(A).$$

(ii) We observe that $\zeta_p := e^{i\pi T}$ satisfies $\sum_{j=0}^{p} \zeta_{2qj} = 0$ if $q \neq 0 \mod p + 1$. Since $G_q[\zeta_p\phi] = \sum_{j=0}^{p} \zeta_{2qj}G_q[\phi]$, for any $0 \leq q_* \leq p$ it holds that

$$\sum_{j=0}^{p} \zeta_{(2q_*-p)j}U_{main}[\zeta_p\phi] = \sum_{j=0}^{p} \zeta_{2(q_*-p)j}U_{main}[\zeta_p\phi] = \sum_{j=0}^{p} \zeta_{2(q_*-p)j}U_{main}[\zeta_p\phi] = \sum_{j=0}^{p} \nu_{p,q_*}G_q[\phi] = (p + 1)\nu_{p,q_*}G_q[\phi].$$

Hence, if $\nu_{p,q_*} \neq 0$, by the triangle inequality we see that

$$\sum_{j=0}^{p} \|U_{main}[\zeta_p\phi](T)\|_{H^s} \geq (p + 1)\nu_{p,q_*} \|G_q[\phi](T)\|_{H^s}.$$ 

This implies the claim. 

By Lemma 4.5, the proof is almost reduced to the case of single-term nonlinearities, as we see below.

Case 1: General $Z$ and $p, s < \min\{s_c(d, p), 0\}$.

Let us take the initial data $\phi$ as in Lemma 3.8 (i), and assume

$$\rho \equiv rA^{-\frac{1}{2}}N^{-s}T^{-\frac{1}{2} - 1} \ll 1, \quad 0 < T \ll N^{-2}.$$ 

Lemma 4.4 (ii) yields that

$$\|U_{high}[\zeta_p\phi](T)\|_{H^s} \ll r\rho A^{-\frac{1}{2}}N^{-s}f_s(A),$$

while Lemma 4.5 (ii) and Lemma 3.8 (i) imply that

$$\|U_{main}[\zeta_p\phi](T)\|_{H^s} \sim r\rho^{p-1}A^{-\frac{1}{2}}N^{-s}f_s(A) \gg \|U_{high}[\zeta_p\phi](T)\|_{H^s}$$

for an appropriate $j$. Hence, from Lemma 4.4 (i) and Lemma 4.5 (i),

$$\|u(T)\|_{H^s} \geq \frac{1}{2}\|U_{main}[\zeta_p\phi](T)\|_{H^s} - \|U_{low}[\zeta_p\phi](T)\|_{H^s} - \|U_{1}[\zeta_p\phi](T)\|_{H^s} \geq C^{-1}r\rho^{p-1}A^{-\frac{1}{2}}N^{-s}f_s(A) - C(r^2N^{-2s}f_s(A)T^{\frac{1}{2}} + r).$$

If we take the same choice for $r, A, T$ as in Case 1 of the proof of Theorem 1.1:

$$r \equiv (\log N)^{-1}, \quad A \sim (\log N)^{-\frac{1}{2} + \frac{1}{p+1}}N, \quad T = (A^{-\frac{1}{2}}N^s)^{p-1},$$

all the required conditions for norm inflation are satisfied when $p = 2$. Even for $p \geq 3$, it suffices to check that

$$r\rho^{p-1}A^{-\frac{1}{2}}N^{-s}f_s(A) \gg r^2N^{-2s}f_s(A)T^{\frac{1}{2}}.$$ 

This is equivalent to $\rho^{p-2} \gg T^{\frac{1}{2} - \frac{1}{p+1}}$, which we can easily show.

Case 2-4-5-7: $p = 2$. We need to deal with the following situations:

- $d = 1, \nu_{2,1} = 0, -\frac{3}{2} \leq s < -1$;
- $d = 2, \nu_{2,1} = 0, s = -1$;
- $Z = R^{d_1} \times T^{d_2}$ with $d_1 + d_2 \leq 3, d_2 \geq 1, \nu_{2,1} \neq 0, \text{ and } \frac{d}{2} - 2 \leq s < 0$;
• $Z = \mathbb{R}^d$, $1 \leq d \leq 3$, $\nu_2, 1 \neq 0$, $\frac{d}{2} - 2 \leq s < -\frac{1}{4}$,
which correspond to Cases 2, 4, 5, and 7 in the proof of Theorem 1.1, respectively.
As seen in the preceding case, we do not have to care about $U_{low}$ and the proof is the same as the single-term cases, except that we need to pick up the appropriate one among $u^2$, $u\bar{u}$, $\bar{u}^2$ by using Lemma 4.5 (ii).

Case 3: $d = 1$, $p = 3$, $s = -\frac{1}{2}$,
we take the initial data $e^{i\frac{\pi}{2}}\hat{\phi}$ with $\phi$ as in (7) and parameters $r, A, T$ as in Case 3 for Theorem 1.1.
Following the argument in Case 1, it suffices to check the condition for $\|U_{main}\|_{L^\infty} \gg \|U_{low}\|_{L^\infty}$:
$$rp^2 A^{-\frac{1}{2}} N^\frac{1}{2} f_{-\frac{1}{2}}(A) \gg r^2 N f_{-\frac{1}{2}}(A)T.$$ 
Actually, we see that L.H.S. $\sim (\log N)^{\frac{1}{2}} \gg (\log N)^{\frac{1}{2}} N^{-1} \sim$ R.H.S.

Case 6: $Z = \mathbb{R}$, $p = 4$, $(\nu_{4,1}, \nu_{4,2}, \nu_{4,3}) \neq (0, 0, 0)$, $s \in [-\frac{1}{5}, 0]$.
Similarly, we take $e^{i\frac{\pi}{2}}\hat{\phi}$ with parameters $r, A, T$ as in Case 6 for Theorem 1.1.
It suffices to verify the condition
$$rp^3 N^{-s} \gg r^2 N^{-2s}T^{\frac{1}{2}},$$
and in fact it holds that L.H.S. $\sim (\log N)^{-4} N^{-s} \gg (\log N)^{-2} N^{-\frac{1}{2}} \sim$ R.H.S.

(II): Recall that we claim NI $\sim (\log N)^{-4} N^{-s}$ as in (7) with parameters $r, A, T$ as in Case 7 for Theorem 1.1.
By Lemmas 4.3 and 4.4, we can expand the solution whenever $r = N^{-\frac{1}{2}-s} T^{\frac{1}{2}} \ll 1$ and we have
$$\sum_{k \geq 4} \|U_k[\phi](T)\|_{L^\infty} \lesssim rp^3 N^{\frac{1}{2}-s} f_s(N^{-1}) \sim r^4 N^{-\frac{1}{2}-2s} T^{\frac{1}{2}}$$
for $0 < T \leq 1$. For $U_3$, observing that the Fourier support is in the region $|\xi| \sim N$, we modify the estimate in Lemma 4.4 to obtain
$$\|U_3[\phi](T)\|_{L^\infty} \lesssim rp^3 N^{\frac{1}{2}-s} \cdot \|\xi|^s\|_{L^2(supp \hat{U_3}[\phi])} \sim r^3 N^{-1-2s} T.$$ 
For $U_2$ the contribution from $u^2$ and $\bar{u}^2$ has the Fourier support in high frequency, thus being dominated by the contribution from $u\bar{u}$. By Lemma 3.8 (ii), we have
$$\|U_2[\phi](T)\|_{L^\infty} \gtrsim r^2 N^{-\frac{1}{2}-2s} T$$
if $0 < T \ll 1$. We set $r = (\log N)^{-1}$ and $T = (\log N)^3 N^{2s+\frac{1}{2}}$ as before (Case 7 in the single-term case), then it holds that $T \ll 1$, $\rho = (\log N)^{\frac{1}{2}} N^{-\frac{1}{2}} \ll 1$ and
$$\|u(T)\|_{L^\infty} \gtrsim C^{-1} r^2 N^{-\frac{1}{2}-2s} T - C(r^3 N^{-1-2s} T + r^4 N^{-\frac{1}{2}-4s} T^{\frac{1}{2}}) \gtrsim \log N \gg 1$$
for $s \in [-\frac{4}{5}, -\frac{1}{5})$, which gives the claimed norm inflation.
This concludes the proof of Theorem 1.2.

Appendix A. Norm inflation with infinite loss of regularity. In this section, we derive norm inflation with infinite loss of regularity for the problem with smooth gauge-invariant nonlinearities:
$$\begin{cases}
i\partial_t u + \Delta u = \pm |u|^{2\nu} u, & t \in [0, T], \quad x \in Z = \mathbb{R}^{d-d_2} \times T^{d_2}, \\
u(0, x) = \phi(x),
\end{cases}$$

(9)
where $\nu$ is a positive integer. The initial value problem (9) on $\mathbb{R}^d$ is invariant under the scaling $u(t, x) \mapsto \lambda^2 u(\lambda^2 t, \lambda x)$, and the critical Sobolev index is $s_c(d, 2\nu + 1) = \frac{d}{2} - \frac{1}{\nu}$, which is non-negative except for the case $d = \nu = 1$.

**Proposition 1.** We assume the following condition on $s$:

- If $d = \nu = 1$, then $s < -\frac{2}{3}$;
- if $d \geq 2$, $\nu = 1$ and $d_2 = 0, 1$ (i.e., $Z = \mathbb{R}^d$ or $\mathbb{R}^{d-1} \times T$), then $s < -\frac{1}{3}$;
- if $d \geq 1$, $\nu \geq 2$ and $d_2 = 0$ (i.e., $Z = \mathbb{R}^d$), then $s < -\frac{1}{2\nu+1}$;
- otherwise, $s < 0$.

Then, $N_k$ with infinite loss of regularity occurs for the initial value problem (9): For any $\delta > 0$ there exist $\phi \in H^\infty$ and $T > 0$ satisfying $\|\phi\|_{H^\sigma} < \delta$, $0 < T < \delta$ such that the corresponding smooth solution $u$ to (2) exists on $[0, T]$ and $\|u(t)\|_{H^\sigma} > \delta^{-1}$ for all $\sigma \in \mathbb{R}$.

**Remark 4.** (i) The proofs of Theorems 1.1 and 1.2 are easily adapted to yield $N_k$ with finite loss of regularity in most cases. However, we only consider here infinite loss of regularity.

(ii) The coefficient of the nonlinearity is not important in the proof, and the same result holds for any non-zero complex constant.

(iii) To show infinite loss of regularity, we need to use the nonlinear interactions of very high frequencies which create a significant output in low frequency $\{|\xi| \leq 1\}$. Except for the case $d = \nu = 1$, there are such interactions that are also resonant; i.e., there exist non-zero vectors $k_1, \ldots, k_{2\nu+1} \in \mathbb{Z}^2$ satisfying

$$
\sum_{j=0}^{\nu} k_{2j+1} = \sum_{l=1}^{\nu} k_{2l}, \quad \sum_{j=0}^{\nu} |k_{2j+1}|^2 = \sum_{l=1}^{\nu} |k_{2l}|^2.
$$

This is also the key ingredient in the proof of the previous results [6, 7], and hence the restriction on the range of $s$ in Proposition 1 is the same as that in [6, 7].

A complete characterization of the resonant set

$$
\mathcal{R}_{d,\nu}(k) := \{(k_m)_{m=1}^{2\nu+1} \in (\mathbb{Z}^d)^{2\nu+1} \mid k = \sum_{m=1}^{2\nu+1} (-1)^{m+1} k_m, \ |k|^2 = \sum_{m=1}^{2\nu+1} (-1)^{m+1} |k_m|^2\}
$$

(for $k \in \mathbb{Z}^d$ given) is easily obtained in the $\nu = 1$ case; see [7, Proposition 4.1] for instance. In Proposition 2 below, we will provide a complete characterization of the set $\mathcal{R}_{1,2}(0)$, which may be of interest in itself. Since $(k_m)_{m=1}^{2\nu+1} \in \mathcal{R}_{1,2}(k)$ if and only if $(k_m - k)_{m=1}^{2\nu+1} \in \mathcal{R}_{1,2}(0)$, we have a characterization of $\mathcal{R}_{1,2}(k)$ for any $k \in \mathbb{Z}$ as well. However, in the proof of Proposition 1 we only need the fact that $\mathcal{R}_{d,\nu}(0)$ has an element consisting of non-zero vectors in $\mathbb{Z}^d$, except for $(d, \nu) = (1, 1)$.

**Proof of Proposition 1.** We follow the proof of Theorem 1.1 but take different initial data to show infinite loss of regularity.

Let $N \gg 1$ be a large positive integer and define $\phi \in H^{\infty}(\mathbb{Z})$ by

$$
\hat{\phi} : = r N^{-s} \chi_{\Sigma + Q_1},
$$

where $\delta > 0$ exists on $[0, T]$ such that $\|\phi\|_{H^\sigma} < \delta$ and $0 < T < \delta$ such that the corresponding smooth solution $u$ to (2) exists on $[0, T]$ and $\|u(T)\|_{H^\sigma} > \delta^{-1}$ for all $\sigma \in \mathbb{R}$.

\[\text{More precisely, we show } \|\hat{\phi}(T)\|_{L^2(|\xi| \leq 1)} > \delta^{-1}. \] This implies the claim if we define the Sobolev norm of negative indices $\sigma$ as $\|f\|_{H^\sigma} := \min(1, |\xi|^\sigma) \hat{f}(\xi) \|_{L^2}$.
where $r = r(N) > 0$ is a constant to be chosen later, $Q_1 := [-\frac{1}{2}, \frac{1}{2}]^d$, and

$$\Sigma := \begin{cases} 
\{N, 2N\} & \text{if } d = \nu = 1, \\
\{N e_d, N e_{d-1} + e_d\} & \text{if } d \geq 2, \nu = 1, \\
\{N e_d, 3N e_d, 4N e_d\} & \text{if } d \geq 1, \nu \geq 2,
\end{cases}$$

$$e_d := (0, \ldots, 0, 1), \quad e_{d-1} := (0, \ldots, 0, 1, 0).$$

The argument in Section 3 (with $A = 1$) shows the following:

- The unique solution $u = u[\phi]$ to (9) exists on $[0, T]$ and has the power series expansion $u = \sum_{k=1}^{\infty} U_k[\phi]$ if $\rho := rN^{-s}T^{\frac{1}{2}} \ll 1$.
- $\|U_1[\phi](T)\|_{H^s} = \|\phi\|_{H^s} \sim r$ for any $T \geq 0$.
- $\|U_k[\phi](T)\|_{H^s} \leq C \rho^{k-1}rN^{-s}$ for any $T \geq 0$ and $k \geq 2$.

For the first nonlinear term $U_{2\nu+1}[\phi]$, we observe that

$$|U_{2\nu+1}[\phi](T, \xi)| = c(rN^{-s})^{2\nu+1} \left| \int_{\Gamma} \prod_{m=1}^{2\nu+1} \chi_{Q_1}(\xi_m) \left( \int_0^T e^{it\Phi} dt \right) d\xi_1 \ldots d\xi_{2\nu+1} \right|,$$

where

$$\Gamma := \{ (\xi_1, \ldots, \xi_{2\nu+1}) | \sum_{j=0}^{\nu} \xi_{2j+1} - \sum_{l=1}^{\nu} \xi_{2l} = \xi \},$$

$$\Phi := |\xi|^2 - \sum_{j=0}^{\nu} |\xi_{2j+1}|^2 + \sum_{l=1}^{\nu} |\xi_{2l}|^2.$$ 

Now, we restrict $\xi$ to the low-frequency region $Q_{1/2}$. If $d = \nu = 1$, then we have

$$\chi_{Q_{1/2}}(\xi) \int_{\Gamma} \prod_{m=1}^{2\nu+1} \chi_{Q_1}(\xi_m) \int_0^T e^{it\Phi} dt$$

$$= 2\chi_{Q_{1/2}}(\xi) \int_{\Gamma} \chi_{Q_1}(\xi_1) \chi_{2N+Q_2}(\xi_2) \chi_{N+Q_1}(\xi_3) \int_0^T e^{it\Phi} dt,$$

and $\Phi = O(N^2)$ in the integral. If $d \geq 2$ and $\nu = 1$, we have

$$\chi_{Q_{1/2}}(\xi) \int_{\Gamma} \prod_{m=1}^{2\nu+1} \chi_{Q_1}(\xi_m) \int_0^T e^{it\Phi} dt$$

$$= 2\chi_{Q_{1/2}}(\xi) \int_{\Gamma} \chi_{N e_d-1 + Q_1}(\xi_1) \chi_{N e_d + Q_1}(\xi_2) \chi_{N e_d + Q_1}(\xi_3) \int_0^T e^{it\Phi} dt,$$

and the resonant property implies that

$$\Phi = \begin{cases} 
O(N) & \text{if } d_2 = 0, 1, \\
O(1) & \text{if } d_2 \geq 2
\end{cases}$$

in the integral. Therefore, in these cases we have the following lower bound:

$$\|U_{2\nu+1}[\phi](T)\|_{L^2(Q_{1/2})} \geq cT(rN^{-s})^{2\nu+1} = c\rho^{2\nu}rN^{-s}$$

(10)

for any $0 < T \ll \begin{cases} 
N^{-2} & \text{if } d = \nu = 1, \\
N^{-1} & \text{if } d \geq 2, \nu = 1, d_2 = 0, 1, \\
1 & \text{if } d \geq 2, \nu = 1, d_2 \geq 2.
\end{cases}$
The quintic and higher cases are slightly different. On one hand, there are “almost resonant” interactions such as
\[ \prod_{j=1,3} \chi_{N_{2d}+Q_1}(\xi_j) \prod_{l=2,4} \chi_{3N_{2d}+Q_1}(\xi_l) \prod_{m=5}^{2\nu+1} \chi_{4N_{2d}+Q_1}(\xi_m), \]
for which it holds
\[ \Phi = \begin{cases} O(N) & \text{if } d_2 = 0, \\ O(1) & \text{if } d_2 \geq 1 \end{cases} \]
in the integral. On the other hand, some non-resonant interactions such as
\[ \chi_{3N_{2d}+Q_1}(\xi_1) \chi_{4N_{2d}+Q_1}(\xi_2) \prod_{m=3}^{2\nu+1} \chi_{N_{2d}+Q_1}(\xi_m) \]
also create low-frequency modes, with \(|\Phi| \sim N^2\) in the integral. Hence, if we choose \(T > 0\) as
\[ N^{-2} \ll T \ll \begin{cases} N^{-1} & \text{if } d \geq 1, \nu \geq 2, d_2 = 0, \\ 1 & \text{if } d \geq 1, \nu \geq 2, d_2 \geq 1, \end{cases} \]
then
\[ \Re \left( \int_0^T e^{i\Phi \, dt} \right) \geq \frac{1}{2} T \quad \text{for “almost resonant” interactions,} \]
\[ \left| \int_0^T e^{i\Phi \, dt} \right| \leq C N^{-2} \ll T \quad \text{for non-resonant interactions,} \]
so that no cancellation occurs among “almost resonant” interactions, which dominate the non-resonant interactions. Therefore, we have (10) for such \(T\) as above.

Finally, we set
\[ \begin{aligned} r := N^{s+\frac{1}{2}} \log N, & \quad T := N^{-2}(\log N)^{-1} \quad \text{if } d = \nu = 1, \\ r := N^{s+\frac{1}{2}} \log N, & \quad T := N^{-1}(\log N)^{-1} \quad \text{if } d \geq 2, \nu = 1, d_2 = 0, \\ r := N^s \log N, & \quad T := (\log N)^{-\left(2\nu+\frac{1}{2}\right)} \quad \text{otherwise.} \end{aligned} \]

We see that, under the assumption on \(s\), \(||\phi||_{H^s} \sim r \ll 1, T \ll 1, \rho \ll 1, \) and
\[ \|\hat{\phi}(T)\|_{L^2(Q_{1/2})} \geq c \|U_{2\nu+1}\hat{\phi}(T)\|_{L^2(Q_{1/2})} - C \left( \|U_1\hat{\phi}(T)\|_{H^s} + \sum_{l \geq 2} \|U_{2l+1}\hat{\phi}(T)\|_{H^s} \right) \]
\[ \geq c \|U_{2\nu+1}\hat{\phi}(T)\|_{L^2(Q_{1/2})} \gg 1. \]
We conclude the proof by letting \(N \to \infty.\)

At the end of this section, we give a characterization of resonant interactions creating the zero mode in the one-dimensional quintic case.

**Proposition 2.** A quintuplet \((k_1, \ldots, k_5) \in \mathbb{Z}^5\) satisfies
\[ k_1 + k_3 + k_5 = k_2 + k_4, \quad k_1^2 + k_3^2 + k_5^2 = k_2^2 + k_4^2 \quad (11) \]
have the same parity. To see this, we notice that the four integers
\[ n_1, n_2, n_3, n_4 \]
are of the same parity, since all of
\[ k_1, k_2, k_3, k_4 \]
and that
\[ (k_1, k_2, k_3, k_4) = (a, b, p, q) \]
from (11) we see that
\[ u(k) \]
for some \( a, b, p, q \in \mathbb{Z} \).

Example 1. (i) Taking \( a = p = b = q = 1 \) in (12), we have the quintuplet
\( (1, 3, 1, 3, 4) \) which has appeared in the proof of Proposition 1 above. Also, with
\( (a, b, p, q) = (-1, 2, -2, 1) \) we have \( (2, 3, 2, 0, -1) \), which gives a resonant interaction
for quartic nonlinearities \( u^3\dot{u}, u\dot{u}^3 \) exploited in the proof of Lemma 3.8 (iv) above.

(ii) The quintuplets \( (pq, -q^2, -pq,p^2, p^2 - q^2) \) given in [7, Lemma 4.2] can be
obtained by setting \( a = -q, b = p \) in (12).

Proof of Proposition 2. The if part is verified by a direct computation, so we show
the only if part.

Let \( (k_1, \ldots, k_5) \in \mathbb{Z}^5 \) satisfy (11). We start with observing that at least one of
\( k_1, k_3, k_5 \) is an even integer; otherwise, we would have
\[ k_1^2 + k_3^2 + k_5^2 \equiv 3 \neq 1 \equiv k_2^2 + k_4^2 \mod 4, \]
contradicting (11). Without loss of generality, we assume \( k_5 \) to be even and set
\[ n_j := k_j - \frac{1}{2}k_5 \in \mathbb{Z} \quad (j = 1, \ldots, 5), \quad n_6 := -\frac{1}{2}k_5 \in \mathbb{Z}. \]
From (11) we see that
\[ n_1 + n_3 + n_5 = n_2 + n_4 + n_6, \quad n_1^2 + n_3^2 = n_2^2 + n_4^2, \quad n_5 = -n_6. \]
The second equality implies that two vectors \( (n_1-n_2, n_3-n_4), (n_1+n_2, n_3+n_4) \in \mathbb{Z}^2 \)
are orthogonal to each other (unless one of them is zero), which allows us to write
\[ (n_1-n_2, n_3-n_4) = \alpha(q, p), \quad (n_1+n_2, n_3+n_4) = \beta(-p, q) \]  
with \( \alpha, \beta, p, q \in \mathbb{Z} \). Note that \( n_1, \ldots, n_4 \) are then written as
\[ n_1 = \frac{1}{2}(\alpha q - \beta p), \quad n_2 = -\frac{1}{2}(\alpha q + \beta p), \]
\[ n_3 = \frac{1}{2}(\alpha p + \beta q), \quad n_4 = -\frac{1}{2}(\alpha p - \beta q), \]
and that
\[ n_5 = -n_6 = \frac{1}{2}(n_5 - n_6) = -\frac{1}{2}\{(n_1-n_2) + (n_3-n_4)\} = -\frac{1}{2}\alpha(p + q). \]
Recalling \( k_j = n_j - n_6 \) (\( j = 1, \ldots, 5 \)), we have
\[ k_1 = -\frac{1}{2}(\alpha + \beta)p, \quad k_3 = -\frac{1}{2}(\alpha - \beta)q, \quad k_5 = -\alpha(p + q), \]
\[ k_2 = -\frac{1}{2}(\alpha + \beta)p - \alpha q, \quad k_4 = -\frac{1}{2}(\alpha - \beta)q - \alpha p. \]  
We next claim that the integers \( \alpha, \beta, p, q \) can be chosen in (13) so that \( \alpha \) and \( \beta \)
have the same parity. To see this, we notice that the four integers \( n_1 \pm n_2, n_3 \pm n_4 \)
are of the same parity, since all of
\[ (n_1 + n_2) + (n_1 - n_2) = 2n_1, \quad (n_3 + n_4) + (n_3 - n_4) = 2n_3, \]
\[ (n_1 - n_2) + (n_3 - n_4) = n_6 - n_5 = 2n_6 \]
are even. If \( n_1 \pm n_2, n_3 \pm n_4 \) are odd integers, then by (13) \( \alpha \) and \( \beta \) must be odd.
So, we assume that they are all even. If one of \( p, q \) is odd, then both \( \alpha \) and \( \beta \) must
be even. If both \( p \) and \( q \) are even, we replace \( (\alpha, \beta, p, q) \) with \( (2\alpha, 2\beta, p/2, q/2) \) to
obtain another expression (13) with both $\alpha$ and $\beta$ being even. Hence, the claim is proved.

Finally, we set $a := -\frac{1}{2}(\alpha + \beta)$, $b := -\frac{1}{2}(\alpha - \beta)$, both of which are integers. Inserting them into (14), we find the expression (12).

\section*{Appendix B. Norm inflation for 1D cubic NLS at the critical regularity.}

In this section, we consider the particular equation

\begin{equation}
\begin{aligned}
\{ & i\partial_t u + \partial_x^2 u = \pm |u|^2 u, \\
& u(0, x) = \phi(x).
\end{aligned}
\tag{15}
\end{equation}

We will show the inflation of the Besov-type scale-critical Sobolev and Fourier-Lebesgue norms with an additional logarithmic factor:

\begin{definition}
For $1 \leq p < \infty$, $1 \leq q \leq \infty$ and $\alpha \in \mathbb{R}$, define the $D^{[\alpha]}_{p,q}$-norm by

$$
\| f \|_{D^{[\alpha]}_{p,q}} := \left\| N^{-\frac{\alpha}{2}} (\log N)^{\alpha} \left\| \hat{f} \right\|_{L^p_t((N \leq \xi) < 2N)} \right\|_{L^q_x(\mathbb{R}^2)}.
$$

We also define the $D^s_{p,q}$-norm for $s \in \mathbb{R}$ by

$$
\| f \|_{D^s_{p,q}} := \left\| |N|^s \left\| \hat{f} \right\|_{L^p_t((N \leq \xi) < 2N)} \right\|_{L^q_x(\mathbb{R}^2)}.
$$

\end{definition}

\begin{remark}
(i) We see that $D^{[0]}_{2,2} = D^{\frac{1}{2}}_{2,2} = B^{\frac{1}{2}}_{2,2}$ (Besov norm) and $D^{[0]}_{p,p} = FL^{\frac{1}{p},p}$ (Fourier-Lebesgue norm). In the case of $Z = \mathbb{R}$, the homogeneous version of $D^{[0]}_{p,q}$ is scale invariant for any $p,q$.

(ii) We have the embeddings $D^{[\alpha]}_{p,q} \hookrightarrow D^{[\alpha]}_{p',q'}$ if $p_1 \leq p_2$, $D^{[\alpha]}_{p,q} \hookrightarrow D^{[\alpha]}_{p',q'}$ if $q_1 \leq q_2$.

(iii) We will not consider the space $D^{[\alpha]}_{p,q}$ with $p = \infty$ here, since our argument seems valid only in the space of negative regularity.

\end{remark}

\begin{proposition}
For the Cauchy problem (15), norm inflation occurs in the following cases:

(i) In $D^{[\alpha]}_{p,q}$ for any $1 \leq q \leq \infty$ and $\alpha < \frac{1}{2q}$, if $\frac{3}{2} \leq p < \infty$.

(ii) In $D^{[\alpha]}_{p,q}$ and $D^s_{p,q}$, for any $1 \leq q \leq \infty$, $\alpha \in \mathbb{R}$ and $s < -\frac{3}{4}$, if $1 \leq p < \frac{3}{2}$.

\end{proposition}

\begin{remark}
(i) If $\frac{3}{2} \leq p < \infty$ and $1 \leq q < \infty$, Proposition 3 shows inflation of a “logarithmically subcritical” norm (i.e., $D^{[\alpha]}_{p,q}$ with $\alpha > 0$). Moreover, if $1 \leq p < \frac{3}{2}$ we show norm inflation in $D^{s}_{p,q}$ for subcritical regularities $-\frac{3}{4} > s > -\frac{1}{p}$. However, for $q = \infty$ and $p \geq \frac{3}{2}$, inflation is not detected even in the critical norm $D^{[0]}_{p,\infty}$.

(ii) In [24, Theorem 4.7] global-in-time a priori bound was established in $D^{[\frac{3}{2}]}_{2,2}$ and $D^{[\frac{3}{2}]}_{2,\infty}$. Recently, Oh and Wang [36] proved global-in-time bound in $FL^{0,p}$ for $Z = \mathbb{T}$ and $2 \leq p < \infty$. There are still some gaps between these results and ours. In fact, Proposition 3 shows inflation of $D^{[\frac{3}{2}]}_{2,2}$ and $D^{[0]}_{2,\infty}$ norms, as well as in a norm only logarithmically stronger than $FL^{-\frac{1}{p},p}$ for $p \geq 2$.

(iii) Guo [16] also studied (15) on $\mathbb{R}$ in “almost critical” spaces. It would be interesting to compare our result with [16, Theorem 1.8], where he showed well-posedness (and hence a priori bound) in some Orlicz-type generalized modulation spaces which are barely smaller than the critical one $M^{1}_{2,\infty}$. There is no conflict between these results, because the function spaces for which norm inflation is claimed in Proposition 3 are not included in $M^{1}_{2,\infty}$ due to negative regularity. Note also that the function spaces in [16, Theorem 1.8] admit the initial data $\phi$ of the form
\( \hat{\phi}(\xi) = [\log(2 + |\xi|)]^{-\gamma} \) only for \( \gamma > 2 \) (see [16, Remark 1.9]), while it belongs to \( D^{[\alpha]}_{p,q} \) if \( \gamma > \alpha + \frac{1}{\#} \).

(iv) In contrast to the results in [24, 36], complete integrability of the equation will play no role in our argument. In particular, Proposition 3 still holds if we replace the nonlinearity in (15) with any of the other cubic terms \( u^3, \bar{u}^3, u\bar{u}^2 \) or any linear combination of them with complex coefficients.

**Proof of Proposition 3.** We follow the argument in Section 3. For \( 1 \leq \rho < \infty \) and \( A > 0 \), let \( M^\rho_A \) be the rescaled modulation space defined by the norm

\[
\|f\|_{M^\rho_A} := \sum_{\xi \in AZ} \|\hat{f}\|_{L^\rho(\xi + I_A)}, \quad I_A := \left[ -\frac{A}{2}, \frac{A}{2} \right].
\]

It is easy to see that \( M^\rho_A \) is a Banach algebra with a product estimate:

\[
\|fg\|_{M^\rho_A} \leq CA^{1 - \frac{1}{\#}} \|f\|_{M^\rho_A} \|g\|_{M^\rho_A}.
\]

Mimicking the proof of Lemma 3.4, we see that the operators \( U_k \) defined as in Definition 3.1 satisfy

\[
\|U_k[\phi](t)\|_{M^\rho_A} \leq t^{\frac{1}{\#}} (CA^{1 - \frac{1}{\#}} \|\phi\|_{M^\rho_A})^{k-1} \|\phi\|_{M^\rho_A}, \quad t \geq 0, \quad k \geq 1.
\]

We also recall that from Corollary 1, the power series expansion of the solution map \( u[\phi] = \sum_{k \geq 1} U_k[\phi] \) is verified in \( C([0, T]; M^\rho_A) \) whenever

\[
0 < T \ll (A^{1 - \frac{1}{\#}} \|\phi\|_{M^\rho_A})^{-2}.
\]

For the proof of norm inflation in \( D^{[\alpha]}_{P,q} \), we restrict the initial data \( \phi \) to those of the form (7); for given \( N \gg 1 \), we set

\[
\hat{\phi} := rA^{-\frac{1}{\#}} N^{\frac{\#}{4}} \chi(N + I_A), \quad \xi \in (2N + I_A),
\]

where \( r > 0 \) and \( 1 \ll A \ll N \) will be specified later according to \( N \). Then, since \( \|\phi\|_{M^\rho_A} \sim rA^{\frac{1}{\#}} N^{\frac{\#}{4}} \), the condition (17) is equivalent to

\[
0 < r(TN^2)^{\frac{1}{\#}} \left( \frac{A}{N} \right)^{1 - \frac{1}{\#}} \ll 1.
\]

Moreover, it holds that

\[
\|U_1[\phi](T)\|_{D^{[\alpha]}_{P,q}} = \|\phi\|_{D^{[\alpha]}_{P,q}} \sim r(\log N)^{\alpha}, \quad T \geq 0,
\]

and similarly to Lemma 3.8 (i), that

\[
\|U_3[\phi](T)\|_{D^{[\alpha]}_{P,q}} \geq cT \left( rA^{-\frac{1}{\#}} N^{\frac{\#}{4}} \right)^{\#} A^2 \|\mathcal{F}^{-1} \chi_{I_{A/2}}\|_{D^{[\alpha]}_{P,q}}, \quad 0 < T \leq \frac{1}{100} N^{-2},
\]

\[
= c \left( r(TN^2)^{\frac{1}{\#}} \left( \frac{A}{N} \right)^{1 - \frac{1}{\#}} \right)^2 r \left( \frac{A}{N} \right)^{-\frac{1}{\#}} f_{P,q}(A),
\]

where

\[
f_{P,q}(A) := \|\mathcal{F}^{-1} \chi_{I_{A/2}}\|_{D^{[\alpha]}_{P,q}} \sim \begin{cases} (\log A)^{\alpha + \frac{1}{\#}}, & \alpha > -\frac{1}{\#}, \\ (\log \log A)^{\frac{1}{\#}}, & \alpha = -\frac{1}{\#}, \\ 1, & \alpha < -\frac{1}{\#}. \end{cases}
\]

For estimating \( U_{2l+1}[\phi], l \geq 2 \) in \( D^{[\alpha]}_{P,q} \), we first observe that

\[
\|U_k[\phi](T)\|_{D^{[\alpha]}_{P,q}} \leq \|\mathcal{F}^{-1} \chi_{\text{supp} \hat{U}_k[\phi](T)}\|_{D^{[\alpha]}_{P,q}} \|\hat{U}_k[\phi](T)\|_{L^\infty}.
\]
A simple computation yields that
\[ \|F^{-1} \chi_\Omega\|_{D^{[\alpha]}_{p,q}} \leq C \|F^{-1} \chi_\Omega\|_{D^{[\alpha]}_{p,q}} \]
for any measurable set \( \Omega \subset \mathbb{R} \) of finite measure. From Lemma 3.6, we have
\[ \supp \hat{U}_k[\phi](T) \leq C^k A \quad T \geq 0, \quad k \geq 1, \]
and hence,
\[ \|F^{-1} \chi_{\supp \hat{U}_k[\phi](T)}\|_{D^{[\alpha]}_{p,q}} \leq C \|F^{-1} \chi_{C^k A}\|_{D^{[\alpha]}_{p,q}} \leq C^k f_{p,q}^\alpha(A). \]
Moreover, similarly to Lemma 3.7 (ii), we use Young’s inequality, (16) and Lemma 3.5 to obtain
\[ \|\hat{U}_k[\phi](T)\|_{L^\infty} \leq \sum_{k_1,k_2,k_3 \geq 1} \int_0^T \|\hat{U}_{k_1}[\phi](t)\|_{M^\alpha_{3A}} \|\hat{U}_{k_2}[\phi](t)\|_{M^{3,1-\frac{1}{p}}_A} \|\hat{U}_{k_3}[\phi](t)\|_{M^{3,1-\frac{1}{p}}_A} dt \]
\[ \leq \int_0^T t^{\frac{k-3}{2}} dt \cdot (Cr A^{1-\frac{1}{p}} N^\frac{1}{p})^{k-3} (Cr A^{\frac{2}{3}-\frac{1}{p}} N^\frac{1}{p})^3 \]
\[ \leq C (Cr T^\frac{3}{4} A^{1-\frac{1}{p}} N^\frac{1}{p})^{k-1} A^{-\frac{1}{p}} N^\frac{1}{p}, \quad T \geq 0, \quad k \geq 3. \]
Hence, we have
\[ \|U_k[\phi](T)\|_{D^{[\alpha]}_{p,q}} \leq C \left[ Cr (TN^2)^\frac{3}{4} \left( \frac{A}{N} \right)^{1-\frac{1}{p}} \right]^{k-1} r \left( \frac{A}{N} \right)^{-\frac{1}{p}} f_{p,q}^\alpha(A), \quad T \geq 0, \quad k \geq 3. \]
(21)

From (18)–(21), we only need to check if there exist \( r, A, T \) such that
\[ 1 \ll A \ll N, \quad r \ll (\log N)^{-\alpha}, \quad (TN^2) \leq \frac{1}{100}, \]
\[ \left[ r(TN^2)^\frac{3}{4} \left( \frac{A}{N} \right)^{1-\frac{1}{p}} \right]^2 \ll 1 \ll \left[ r(TN^2)^\frac{3}{4} \left( \frac{A}{N} \right)^{1-\frac{1}{p}} \right]^2 r \left( \frac{A}{N} \right)^{-\frac{1}{p}} f_{p,q}^\alpha(A). \]  
(22)

When \( 1 \leq p < \frac{3}{2} \), it holds that \( 2(1-\frac{1}{p}) \geq 0 > 2(1-\frac{1}{p}) - \frac{1}{p} \). Hence, we may choose
\[ r = (\log N)^{\min(-\alpha,0)-1}, \quad A = N^\frac{3}{4}, \quad T = \frac{1}{100} N^{-2}, \]
which clearly satisfies (22). (Note that \( f_{p,q}^\alpha(A) \gtrsim 1 \) for any \( p, q, \alpha \).)
If \( \frac{3}{2} \leq p < \infty \), (22) would imply that
\[ 1 \ll \left[ r(TN^2)^\frac{3}{4} \left( \frac{A}{N} \right)^{1-\frac{1}{p}} \right]^3 r \left( \frac{A}{N} \right)^{-\frac{1}{p}} f_{p,q}^\alpha(A) \ll r^3 f_{p,q}^\alpha(A) \ll (\log N)^{-3 \alpha} f_{p,q}^\alpha(A). \]
In particular, when \( \alpha > -\frac{1}{q} \) this condition requires
\[ (\log N)^{3 \alpha} \ll (\log N)^{\alpha + \frac{1}{q}}, \]
which shows the necessity of the restriction \( \alpha < \frac{1}{2q} \) in our argument. We now see the possibility of choosing \( r, A, T \) with the condition (22) in the following two cases separately: (a) If \( 1 \leq q < \infty \) and \( 0 \leq \alpha < \frac{1}{2q} \), we may take for instance
\[ r = (\log N)^{-\alpha} (\log \log N)^{-1}, \quad A = N(\log \log N)^{-1}, \quad T = \frac{1}{100} N^{-2}. \]
(Note that \( f_{p,q}^\alpha(A) \sim f_{p,q}^\alpha(N) \sim (\log N)^{\alpha + \frac{1}{q}} \)) (b) If \( \alpha < 0 \), we take
\[ r = (\log N)^{-\alpha} (\log \log N)^{-1}, \quad A = N(\log N)^{\alpha(1-\frac{1}{p})^{-1}}, \quad T = \frac{1}{100} N^{-2}. \]
In both cases we easily show (22).
Finally, we assume $1 \leq p < \frac{3}{2}$ and prove norm inflation in $D_{p,q}^s$ for $s < -\frac{2}{3}$. We use the initial data $\phi$ of the form
\[ \widehat{\phi} := rN^{-s} \chi_{[N,N+1]\cup[2N,2N+1]} \]
Then, the condition (17) with $A = 1$ is equivalent to
\[ 0 < T^\frac{1}{2} rN^{-s} = (TN^2)^\frac{1}{2} rN^{-s-1} \ll 1. \]
Repeating the argument above we also verify that
\[ \|U_1[\phi](T)\|_{D_{p,q}^s} = \|\phi\|_{D_{p,q}^s} \sim r, \]
\[ \|U_3[\phi](T)\|_{D_{p,q}^s} \geq c(T^\frac{1}{2} rN^{-s})^2 rN^{-s} = c(TN^2)^r N^{-3s-2} \quad \text{if } T \leq \frac{1}{100} N^{-2}, \]
\[ \|U_k[\phi](T)\|_{D_{p,q}^s} \leq C(CT^\frac{1}{2} rN^{-s})^{k-1} rN^{-s}, \quad T \geq 0, \quad k \geq 3. \]
Hence, we set
\[ r = N^{s+\frac{2}{3}} \log N, \quad T = \frac{1}{100} N^{-2}, \]
so that for $s < -\frac{2}{3}$ we have
\[ \|U_1[\phi](T)\|_{D_{p,q}^s} \sim N^{s+\frac{2}{3}} \log N \ll 1, \quad \|U_3[\phi](T)\|_{D_{p,q}^s} \gtrsim (\log N)^3 \gg 1, \]
\[ \sum_{l \geq 2} \|U_{2l+1}[\phi](T)\|_{D_{p,q}^s} \lesssim N^{-\frac{2}{3}} (\log N)^5 \ll 1, \]
from which norm inflation is detected by letting $N \to \infty$. \qed

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