14
STATUS OF A SELF-BOUND EQUATIONS OF STATE AND ANALYTIC SOLUTIONS IN GENERAL RELATIVITY

P.S. Negi¹ and M.C. Durgapal²

Department of Physics, Kumaun University, Nainital-263 002, India

Received 23 December 1999

We have obtained a criterion for spherically symmetric and static structures under hydrostatic equilibrium in general relativity (GR), which states that for a given value of \( \sigma \equiv (P_0/E_0) \equiv \text{the ratio of central pressure to central energy-density} \), the compaction parameter \( u \equiv (M/a) \), where \( M \) is the total mass and \( a \) is the radius of the configuration, or the surface redshift of any regular configuration cannot exceed that of the corresponding homogeneous density sphere, that is, \( u \leq u_h \), where \( u_h \) is the compaction parameter of the homogeneous density sphere. By examining various exact solutions and equations of state available in the literature, we find that this criterion is fulfilled only by those configurations in which the surface density vanishes along with the pressure (meaning thereby that the pressure, energy density, metric parameters and their first derivatives are continuous on the surface). On the other hand, configurations having a finite density on the surface (that is, the self-bound structures) do not fulfill this criterion. This criterion puts a severe restriction on the static structures based upon the general relativistic field equations and consequently on the upper limit of mass, surface and central redshift and other physical parameters of spherically symmetric and static configurations.

1. Introduction

The exact solution for an incompressible fluid sphere of uniform energy density, \( E \), in GR was first obtained by Schwarzschild [1]. In spite of its nonphysical behaviour (the adiabatic speed of sound \( v = \sqrt{dP/dE} \) is infinite throughout the structure, and the pressure vanishes at finite surface density), it has important features characterizing configurations compatible with the structure of general relativity: (i) it gives an absolute upper limit for the compaction parameter, \( u \equiv (M/a) \), mass to size ratio of the entire configuration in geometrized units \( \leq (4/9) \) for any regular static solution in hydrostatic equilibrium [2], and (ii) for an assigned value of the compaction parameter \( u \) and the radius \( a \), the minimum central pressure \( P_0 \) corresponds to the homogeneous density solution (see, e.g., [3]).

In this paper, by the use of important feature (ii) mentioned above, we extract the following statement: for a given value of \( \sigma \equiv (P_0/E_0) \) ratio of the central pressure and central density \( \), the maximum value of the compaction parameter corresponds to the homogeneous density sphere.

Chandrasekhar [4] discussed the hydrostatic equilibrium condition under small adiabatic perturbations and showed that for a compressible homogeneous density distribution, as long as the ratio of the specific heats \( \gamma \) remains finite, a dynamical instability intervenes before the Schwarzschild limit \( u = 4/9 \) is reached. And for \( \gamma \to \infty \), \( u \to (4/9) \). Thus for any physically viable solution one may expect a finite value

¹ e-mail: psnegi_nainital@yahoo.com
of $\gamma$, and the dynamical instability would intervene well before the compaction parameter $u$ approaches the limiting value of (4/9).

Exploring various physical properties of a superdense object like a neutron star, one may expect to have some physically viable equation of state (EOS). However, for such objects the equations of state are not well known (empirically) because of the lack of knowledge of nuclear interactions beyond the density $\gtrsim 10^{14}$ g/cm$^3$ [5], and the only way of obtaining an EOS far beyond this density range is extrapolation. Various such extrapolated equations are available in the literature [6]. As a way out, one can impose some restrictions upon the known physical quantities, such that the speed of sound inside the configuration, $v/\sqrt{|dP/dE|}$, does not exceed the speed of light in vacuo, i.e., $v \leq c = 1$ (in geometrized units), and obtain an upper bound on stable neutron star masses [7–9]. Haensel and Zdunik [10] have shown that the EOS represents an “abnormal” state of matter in the sense that the pressure vanishes at densities of the order of nuclear density or even higher.

Alternatively, one may assume an expression for one of the parameters, $P, E, \nu, \lambda$, and $\lambda$ in terms of the radial coordinate $r$ (or an algebraic relation between the parameters) and obtain an exact solution of the Einstein field equations. Many exact solutions are available in the literature [11–16], which may be used to obtain various physical properties of spherical and static compact object (provided they are physically realistic).

Various EOS and exact solutions discussed in the literature, in fact, fulfill the criterion (i), that is, the equilibrium configurations pertaining to these EOS and analytic solutions correspond to the compaction parameter $u$ which is always smaller than the value 4/9. However, we show in the present paper that the EOS or analytic solutions, corresponding to a finite surface density (where the pressure vanishes at finite surface density, the so-called “self-bound” state of matter), in fact, are not compatible with the structure of GR, as they do not fulfill the criterion (ii) (namely, for an assigned value of $\sigma$ the compaction parameter $u$ of any regular configuration should not be greater than that of the homogeneous density configuration). We have shown this inconsistency particularly for the EOS $(dP/dE) = 1$ (as it represents the most successful EOS for obtaining various extreme characteristics of neutron stars), and for the exact solution put forward by Durgapal and Fuloria [15] (as it represents various characteristics expected for a physically realistic superdense object [17]). Furthermore, it can be shown that this inconsistency always exists for any EOS or analytic solution with finite surface density. On the other hand, we have found that only those EOS or exact solutions which correspond to vanishing density at the surface of the configuration (i.e., gravitationally bound structures) are compatible with the structure of GR. We show that the above criterion is fulfilled for polytropic equations of state, $P = KE^2$ (where $K$ and $\Gamma$ are constants [18]), and $P = K\rho^{\Gamma_1}$ (where $K$ and $\Gamma_1$ are constants and $\rho$ is the rest mass density [19]), and Tolman’s type VII exact solution [20], $E = E_0(1 - r^2/a^2)$, where $E_0$ is the central energy density and $a$ is the radius of the configuration.

Thus, an important consequence of the EOS or analytic solutions with vanishing surface density lies in the fact that the “abnormality” in the sense discussed in the literature disappears. Or, in other words, the EOS or analytic solutions, compatible with the structure of general relativity, would always represent configurations corresponding to the normal state of matter.

2. Criteria for static spherical configurations to be consistent with the structure of GR

Let us consider homogeneous sphere of uniform energy-density $E$. The equations for isotropic pressure $P$ and density $\rho$ can be written in terms of the compaction parameter $u$ and the radial coordinate measured in units of configuration size, $y = r/a$, as

$$8\pi Ea^2 = 6u,$$  

$$8\pi Pa^2 = 6u \left(1 - \frac{2uy^2}{3(1 - 2u)}\right)^{1/2} - \left(1 - \frac{2uy^2}{3(1 - 2u)}\right)^{1/2}.$$  

Consider a regular variable density sphere (with some given EOS or analytic solution) with central energy density $E_0$ and central pressure $P_0$, corresponding to the compaction parameter $u = u_v$. Now, we can always construct a homogeneous density sphere with the same value of the compaction parameter $u_v$ and energy-density $E_0$, because if $P_{0h}$ corresponds to the central pressure of this sphere, the ratio $\sigma_h (\equiv P_{0h}/E_0)$ depends only on the assigned value of the compaction parameter $u_v$. And, $P_{0h}$ is given by

$$P_{0h} = \frac{6u}{8\pi a^2} \frac{1 - (1 - 2u)^{1/2}}{3(1 - 2u)^{1/2} - 1}.$$  

Now, according to the feature (ii) [3], we may write

$$P_0 \geq P_{0h},$$  

or

$$P_0/E_0 \geq P_{0h}/E_0.$$  

Hence for a given value of $u = u_v$, we obtain

$$\sigma_v \geq \sigma_h$$  

where $\sigma_v$ is defined as the ratio $P_0/E_0$. 

Table 1. Various values of the compaction parameter \(u(\equiv M/a)\) corresponding to the exact solutions: \(E = E_0(1 - v^2/a^2)\) (indicated by \(u_{tvii}\)) and \(E = [8(9 + 2x + x^2)/7(1 + x)^3]\) (indicated as \(u_{dfn}\)), and equations of state: \(P = KE^\Gamma\) for \(\Gamma = 2\) (indicated as \(u_{rptr}\)), \(P = K\rho^{\Gamma_1}\) for \(\Gamma_1 = 2\) (indicated as \(u_{cptr}\)) and \(P = E - E_s\) (indicated by \(u_{sdf}\)), for different assigned values of \(\sigma \equiv P_0/E_0\). The compaction parameter of a homogeneous density distribution (Schwarzschild’s interior solution) is indicated as \(u_h\) for the same values of \(\sigma\). It is seen that \(u_{tvii}, u_{rptr}\) and \(u_{cptr} < u_h\), while \(u_{dfn} < u_{sdf} > u_h\). Thus configurations with vanishing surface density show compatibility with the structure of GR, while those with finite surface density do not show this compatibility.

| \(\sigma \equiv P_0/E_0\) | \(u_{rptr}\) | \(u_{cptr}\) | \(u_{tvii}\) | \(u_h\) | \(u_{dfn}\) | \(u_{sdf}\) |
|-----------------------------|-------------|-------------|-------------|---------|---------|---------|
| 0.1252                     | 0.1565      | 0.1559      | 0.1588      | 0.1654  | 0.1718  | 0.1683  |
| 0.1859                     | 0.1950      | 0.1926      | 0.1992      | 0.2102  | 0.2187  | 0.2150  |
| 0.2202                     | 0.2109      | 0.2076      | 0.2166      | 0.2301  | 0.2392  | 0.2354  |
| 0.2800                     | 0.2331      | 0.2265      | 0.2407      | 0.2580  | 0.2676  | 0.2646  |
| 0.3150                     | 0.2434      | 0.2346      | 0.2521      | 0.2714  | 0.2809  | 0.2793  |
| (1/3)                      | 0.2481      | 0.2379      | 0.2574      | 0.2778  | 0.2872  | 0.2858  |
| 0.3774                     | 0.2581      | 0.2441      | 0.2687      | 0.2914  | 0.3003  | 0.3000  |
| 0.4350                     | 0.2683      | 0.2484      | 0.2809      | 0.3062  | 0.3145  | 0.3153  |
| 0.4889                     | 0.2765      | 0.2489      | 0.2904      | 0.3178  | 0.3253  | 0.3271  |
| 0.5499                     | 0.2835      | 0.2466      | 0.2993      | 0.3289  | 0.3354  | 0.3383  |
| 0.6338                     | 0.2910      | 0.2375      | 0.3092      | 0.3415  | 0.3465  | 0.3501  |
| 0.6830                     | 0.2945      | 0.2292      | 0.3140      | 0.3476  | 0.3519  | 0.3554  |
| 0.7044                     | 0.2960      | 0.2250      | 0.3160      | 0.3501  | 0.3541  | 0.3573  |
| 0.7085                     | 0.2958      | 0.2241      | 0.3164      | 0.3506  | 0.3545  | 0.3577  |
| 0.7571                     | 0.2985      | 0.2132      | 0.3204      | 0.3558  | 0.3589  | 0.3611  |
| 0.8000                     | 0.3006      | 0.2033      | 0.3235      | 0.3599  | 0.3624  | 0.3631  |
| 0.8360                     | 0.3024      | 0.1959      | 0.3260      | 0.3630  | 0.3650  | 0.3638  |

Now, varying the compaction parameter \(u_v\) for the homogeneous density sphere from \(u_v\) to \(u_h\) (say) we should have

\[
\sigma_v = \sigma_h. \tag{7}
\]

For \(u = u_h\), the value of \(\sigma_h\) would become

\[
\sigma_h = \frac{(1 - 2u_h)^{1/2} - 1}{1 - 3(1 - 2u_h)^{1/2}}. \tag{8}
\]

Substituting Eq. (8) with the help of Eq. (7) into (6), we get

\[
\frac{(1 - 2u_h)^{1/2} - 1}{1 - 3(1 - 2u_h)^{1/2}} \geq \frac{(1 - 2u_v)^{1/2} - 1}{1 - 3(1 - 2u_v)^{1/2}}. \tag{9}
\]

It is clear from Eq. (9) that

\[
u_h \geq u_v. \tag{10}\]

That is, for an assigned value of the ratio of central pressure to central energy-density \(\sigma(\equiv \sigma_v)\), the compaction parameter of homogeneous density distribution \(u(\equiv u_h)\) should always be larger than or equal to the compaction parameter \(u(\equiv u_v)\) of any regular solution, compatible with the structure of GR. Or, in other words, for an assigned value of \(u\) the minimum value of \(\sigma\) corresponds to the homogeneous density sphere.

3. Examination of the criterion with some well-known equations of state and exact solutions

We have considered the EOS \(P = (E - E_s)\) (where \(E_s\) is the surface density of the configuration), \(P = KE^\Gamma\) (where \(K\) and \(\Gamma\) are constants), and \(P = K\rho^{\Gamma_1}\) (where \(K\) and \(\Gamma_1\) are constants and \(\rho\) is the rest-mass density). The former EOS pertains to nonvanishing the latter to vanishing surface density. Similarly, we have chosen the exact solutions

\[
(8\pi E/C) = \frac{8(9 + 2x + x^2)}{7(1 + x)^3}
\]

(where \(C\) is a constant and \(x = Cr^2\), which corresponds to a nonvanishing surface density), and

\[
E = E_0(1 - r^2/a^2)
\]

(where \(E_0\) is the central energy density and \(a\) is the size of the configuration; for \(r = a\), the surface density becomes zero).

Let us denote the compaction parameter corresponding to a homogeneous density configuration by \(u_h\) and that for the equations of state \(P = E - E_s\), \(P = KE^\Gamma\) and \(P = K\rho^{\Gamma_1}\) by \(u_{dfn}\), \(u_{rptr}\) and \(u_{cptr}\), respectively. For the exact solution corresponding to the density distribution \(8\pi E/C = \frac{8(9 + 2x + x^2)}{7(1 + x)^3}\) the compaction is denoted by \(u_{dfn}\), and for \(E = E_0(1 - r^2/a^2)\) by \(u_{tvii}\), respectively. Now,
for these equations of state and analytic solutions for various assigned values of \( \sigma \equiv (P_0/E_0) \) we obtain the corresponding values of the compaction parameters as shown in Table 1. It is seen that for every assigned value of \( \sigma \), \( u_{rpr} \) and \( u_{cpr} \leq u_h \leq u_{stff} \), and \( u_{tvii} \leq u_h \leq u_{dfn} \). Thus we conclude that the configurations defined by \( u_{stff} \) and \( u_{dfn} \) do not show compatibility with the structure of GR, while those defined by \( u_{rpr} \), \( u_{cpr} \) and \( u_{tvii} \) show such a compatibility. However, this type of characteristics (the value of the compaction parameter larger or smaller than \( u_h \) for some or all assigned values \( \sigma \)) can be seen for any EOS or exact solution having a finite surface density. On the other hand, it is seen that the compaction parameter value for EOS or analytic solution with vanishing surface density always remains smaller than \( u_l \) for all the assigned values of \( \sigma \). Therefore we conclude that to have an EOS or exact solution compatible with the structure of GR, it is necessary to assure the continuity of density (and the respective derivative of the metric parameter \( \chi' \)) along with other parameters at the surface of the configuration.

4. Results and conclusions

We have investigated various exact solutions and equations of state in hydrostatic equilibrium and obtained a criterion for the compatibility with the structure of GR in a physical sense which may be written as: for an assigned value of the ratio of central pressure to central energy density \( \sigma \) the compaction parameter \( u \) of any regular solution should not exceed the compaction parameter \( u_h \) of the homogeneous density distribution. This criterion should be fulfilled by any exact solution or equation of state, or core-envelope model, or core-mantle-envelope model, or any complicated distribution of matter, in order to have compatibility with the structure of the field equations.

Since this criterion imposes a severe restriction on static structures in GR, namely, to solutions of the Einstein equations corresponding to a vanishing density at the surface of the configuration, there must appear new restrictions on the upper limit on mass, surface redshift, central redshift and other physical properties of static, spherically symmetric configurations.

Acknowledgement

The authors acknowledge Uttar Pradesh State Observatory, Nainital for providing library facilities.

References

[1] K. Schwarzschild, *Sitz. Preuss. Akad. Wiss. Berlin*, p. 424 (1916).
[2] H.A. Buchdahl, *Phys. Rev.* 116, 1027 (1959).