Specialization of $F$-Zips

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Abstract. In [MW], B. Moonen and the author defined a new invariant, called $F$-Zips, of certain varieties in positive characteristics. We showed that the isomorphism classes of these invariants can be interpreted as orbits of a certain variety $Z$ with an action of a reductive group $G$. In loc. cit. we gave a combinatorial description of the set of these orbits.

In this manuscript we give an explicit combinatorial recipe to decide which orbits are in the closure of a given orbit. We do this by relating $Z$ to a semi-linear variant of the wonderful compactification of $G$ constructed by de Concini and Procesi. As an application we give an explicit criterion of the closure relation for Ekedahl-Oort strata in the moduli space of principally polarized abelian varieties.

Introduction

Let $k$ be a field of characteristic $p > 0$ and let $X$ be a smooth, proper scheme over $k$ such that the Hodge spectral sequence degenerates. Examples are abelian varieties, complete intersections in a projective space, K3-surfaces, toric varieties, and curves. In [MW] it was shown that for each $i$ the $i$-th de Rham cohomology of $X$ is endowed with the structure of an $F$-zip (see loc. cit., Definition 2.1, for the definition of an $F$-zip). Moreover, $F$-zips also arise as mod $p$ reductions of completely divisible lattices in filtered isocrystals (see loc. cit., Example 7.3) and from the $p$-torsion of a Barsotti-Tate group. The main result in loc. cit. then is the classification of isomorphism classes of $F$-zips in terms of combinatorial data associated to certain Coxeter groups.

This result was obtained by showing that isomorphism classes of $F$-zips correspond to $G$-orbits of a certain variety $Z_{G,J}$. Here $G = GL_n$ for some $n$ and $J$ is a fixed set of simple reflections in the Weyl group of $G$ (see loc. cit., Section 4.8, or [22] below for the definition of $Z_{G,J}$). In fact, the varieties $Z_{G,J}$ can be defined for an arbitrary reductive group $G$, defined over a finite field, and for any set $J$ of simple reflections in the Weyl group $W$ of $G$. On $W$ there is a natural partial order, namely the Bruhat order, which we denote by $\leq$. Then in loc. cit. it was shown that there is a natural bijection

$$\{G\text{-orbits of } Z_{G,J}\} \leftrightarrow ^JW,$$
where \( J^W = \{ w \in W \mid w < sw \text{ for all } s \in J \} \). In particular, there are only finitely many \( G \)-orbits in \( Z_{G,J} \).

In this paper we will prove how these orbits specialize into each other: For \( w \in J^W \) denote the corresponding \( G \)-orbit of \( Z_{G,J} \) by \( Z_{G,J}^w \). Let \( x = w_0^J \) be the maximal element in \( W^J = \{ w \in W \mid w < ws \text{ for all } s \in J \} \), and let \( W_J \) be the subgroup of \( W \) generated by \( J \). We denote by \( F : W \xrightarrow{\sim} W \) the automorphism of \( W \) induced by the Frobenius. Then the main result is (see (5.4)):

**Theorem.** For any \( w, w' \in J^W \) the following two assertions are equivalent:

1. \( Z_{G,J}^w \subset Z_{G,J}^{w'} \).
2. There exists \( u \in W_J \) such that \( u^{-1}w\delta(u) \leq w' \) with \( \delta(u) = xF(u)x^{-1} \).

We obtain the following application to the Ekedahl-Oort stratiﬁcation of the moduli space \( A_g \) of \( g \)-dimensional principally polarized abelian varieties in characteristic \( p \) (or more generally to good reductions of Shimura varieties of PEL-type): One corollary of the results in [MW] is that the Ekedahl-Oort strata of \( A_g \) (e.g., deﬁned in [Oo]) can be parametrized by \( J^W \). Here \( W \) is the Weyl group of the symplectic group \( \text{Sp}_{2g} \) and \( J \) is the type of the Siegel parabolic in \( \text{Sp}_{2g} \), see (6.2) for the precise deﬁnition of \( W \) and \( J \). This parametrization had been obtained beforehand by F. Oort in [Oo] with a different formulation and, for arbitrary good reductions of Shimura varieties of PEL-type, by B. Moonen in [Mo]. Moreover, F. Oort ([Oo]) and the author ([Wd1]) both have shown by different methods that the Ekedahl-Oort strata are locally closed and equidimensional and that the closure of an Ekedahl-Oort stratum is a union of Ekedahl-Oort strata. But it remained an open question which strata are contained in the closure of a given one (see [EMO], Problem 11). Now the theorem above implies:

**Corollary.** For \( w \in J^W \) denote by \( A_g^w \) the corresponding Ekedahl-Oort stratum in \( A_g \). For \( w, w' \in J^W \) the following two assertions are equivalent:

1. \( A_g^w \subset A_g^{w'} \).
2. There exists \( u \in W_J \) such that \( u^{-1}w\delta(u) \leq w' \) with \( \delta(u) = w_{0,J}uw_{0,J} \).

Here \( w_{0,J} \) denotes the maximal element in \( W_J \).

The varieties \( Z_{G,J} \) are a Frobenius-linear variant of varieties \( Z_{G,J}^w \) deﬁned by Lusztig in [Lu1] and [Lu2]. In this linear setting he also obtains \( G \)-stable subvarieties \( Z_{G,J}^{w^w} \) which are parametrized (after renormalization) by \( J^W \). Also, X. He shows in [He], that the closure of \( Z_{G,J}^{w^w} \) is a union of such \( G \)-stable subvarieties. Although the constructions in [Lu1], [Lu2], and [He] do not carry over to the Frobenius-linear setting, the proof of the Theorem above uses similar methods.

We introduce notations in Section 1 and recall some deﬁnitions and results from [MW] in Section 2. Then it is shown that \( Z_{G,J} \) carries a transitive \((G \times G)\)-action. Moreover there is a \((G \times G)\)-equivariant smooth surjective
morphism from $Z_{G,J}$ to a Frobenius-linear variant of a $(G \times G)$-orbit $X_{G^{ad},J}$ in the wonderful compactification by de Concini and Procesi of the adjoint group $G^{ad}$. This morphism induces a bijection of $(B \times B)$-orbits, where $B \subset G$ is a Borel subgroup. As the closure relation of the $(B \times B)$-orbits in the wonderful compactification is known by the work of Springer [Sp], one obtains the closure relation of the $(B \times B)$-orbits in $Z_{G,J}$. Moreover, for every $w \in JW$ we find a $(B \times B)$-orbit which is contained in $Z_{wG,J}$, see Section 3.

In Section 4 the partial order on $JW$ is introduced which corresponds to taking closures of the $G$-orbits $Z_{wG,J}$. Although we have to work in a more general setting than He in [He], the proofs in this section are easy modifications of the proofs given by He. Finally in Section 5, we prove the main theorem making use of the connection of $G$-orbits and $(B \times B)$-orbits.

In the last section we apply these results to the Ekedahl-Oort stratification and we obtain the corollary above.

1 Refinement of parabolic subgroups

(1.1) In the sequel $p$ is a prime number and $q$ is a fixed power of $p$. For a scheme $S$ of characteristic $p$ we denote by $\text{Fr}_S: S \to S$ the morphism which is the identity on the underlying topological space and the homomorphism $x \mapsto x^q$ on the sheaves of rings. For an $\mathcal{O}_S$-module $M$ we set $M^{(q)} = \text{Fr}_S^*M$.

Let $\overline{F}$ be an algebraic closure of $\mathbb{F}_q$. All varieties we will consider are reduced schemes of finite type over $\overline{F}$, although sometimes they will have fixed $\mathbb{F}_q$-rational structures. We will also systematically confuse a variety $X$ and its $\overline{F}$-valued points.

(1.2) Let $G$ be a connected reductive group over $\mathbb{F}_q$. We fix a maximal torus $T$ and a Borel subgroup $B$ of $G$ containing $T$ which are both defined over $\mathbb{F}_q$. We denote by $G^{ad}$ the adjoint group of $G$ and by $T^{ad}$ and $B^{ad}$ the image of $T$ and $B$ in $G^{ad}$. More general, we write $H^{ad}$ for the image in $G^{ad}$ of an algebraic subgroup $H$ of $G$.

Let $W$ be the Weyl group of $G$ associated to $T$ and let $I \subset W$ be the set of simple reflections corresponding to $B$. We endow $W$ and any subset of $W$ with the Bruhat order which will be denoted by $\leq$ and denote by $w \mapsto \ell(w)$ the length function on $W$.

If there is no risk of confusion, we simply write $F: G \to G$ for the Frobenius $F_{G/\mathbb{F}_q}: G \to G^{(q)} = G$, which is the map $x \mapsto x^q$ on local sections. It is an endomorphism of $G$ which induces an automorphism of the Weyl group $W$, again denoted by $F$. As $B$ is defined over $\mathbb{F}_q$, we have $F(I) = I$.

For any subset $J \subset I$ we denote by $W_J$ the subgroup of $W$ generated by $J$. We denote by $w_{0,J}$ the longest element in $W_J$ and set $w_0 = w_{0,I}$. Finally let $w_0^J = w_0w_{0,J}$ be the element of maximal length in $W^J$. Similarly we set $J^Jw_0 = w_{0,J}w_0$. Finally we write $J^{opp} = w_0Jw_0 \subset I$. 

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We denote by \( P_J \supset B \) the standard parabolic subgroup corresponding to \( J \) and by \( L_J \subset P_J \) the unique Levi subgroup of \( P_J \) with \( L_J \supset T \). A parabolic subgroup \( P \) of \( G \) is called of type \( J \) if \( P \) is conjugated to \( P_J \).

Let \( P_J^{\text{opp}} \) be the unique parabolic subgroup of \( G \) such that \( P_J \cap P_J^{\text{opp}} = L_J \). It is a parabolic subgroup of type \( J^{\text{opp}} \) such that \( w_0 P_J^{\text{opp}} w_0 = P_J^{\text{opp}} \). We denote by \( \pi_J \) and \( \pi_J^{\text{opp}} \) the projections of \( P_J \) (resp. \( P_J^{\text{opp}} \)) onto \( L_J \).

For any parabolic subgroup \( P \) of \( G \) we denote by \( U_P \) its unipotent radical. We set \( U = U_B \) and \( U^{\text{opp}} = U_B^{\text{opp}} \). For each subset \( J \) of \( I \) we set \( U_J = L_J \cap U \) and \( U_J^{\text{opp}} = L_J \cap U^{\text{opp}} \). Then multiplication induces isomorphisms

\[
U_J \times U_{P_J} \overset{\sim}{\longrightarrow} U \overset{\sim}{\longleftarrow} U_{P_J} \times U_J. \tag{1.2.1}
\]

Let \( \Phi \subset X^*(T) \) be the set of roots of \( (G \otimes_{\mathbb{F}_q} \mathcal{F}, T \otimes_{\mathbb{F}_q} \mathcal{F}) \). For each \( i \in I \) let \( \alpha_i \) be the corresponding simple root in \( \Phi \). We denote by \( \Phi_J \) the set of roots which are in the \( \mathbb{Z} \)-span of \( \{ \alpha_j \mid j \in J \} \) and set \( \Phi_J^+ = \Phi^+ \cap \Phi_J \) where \( \Phi^+ \) is the set of positive roots.

(1.3) We denote by \( J^W \) (resp. \( W^J \)) the set of minimal length coset representatives of \( W_J \backslash W \) (resp. \( W \backslash W_J \)). For \( J, K \subset I \) we write \( J^W K = J^W \cap W^K \). This is a system of representatives in \( W \) for \( W_J \backslash W / W_K \).

We have the following descriptions for \( J^W \):

\[
J^W = \{ w \in W \mid w \leq w' \text{ for all } w' \in W_J w \} = \{ w \in W \mid \ell(w) < \ell(sw) \text{ for all } s \in J \} = \{ w \in W \mid \ell(uw) = \ell(u) + \ell(w) \text{ for all } u \in W_J \} = \{ w \in W \mid w^{-1}(\Phi_J^+) \subset \Phi^+ \}.
\]

For every element \( x \in W \) there exist unique elements \( u \in W_J \) and \( w \in J^W \) such that \( x = uw \). We call \( u \) the \( W_J \)-part of \( x \) and \( w \) the \( J^W \)-part of \( x \).

If \( J' \subset J \subset I \) are two subsets, we clearly have \( J^W \subset J'^W \). Conversely, we also have a canonical surjective map

\[
J'^W \leftrightarrow W_{J'} \backslash W \rightarrow W_J \backslash W \leftrightarrow J^W. \tag{1.3.2}
\]

(1.4) Let \( w \in J^W \) and \( s \in I \). By [Bo], §1, n° 1.7, there are three possibilities:

1. \( ws > w \) and \( ws \in J^W \).
2. \( ws > w \) and \( ws = tw \) for some \( t \in J \).
3. \( ws < w \) and \( ws \in J^W \).

(1.5) We have the following easy lemma.

**Lemma.** Let \( J \subset I \) and set \( K = J^{\text{opp}} = w_0 J w_0 \).

1. The map \( x \mapsto x^{-1} \) defines a bijection \( J^W \leftrightarrow W^J \) which preserves Bruhat order and length.
(2) The map \( x \mapsto w_0 x w_0 \) defines a bijection \( W^J \leftrightarrow W^K \) which preserves Bruhat order and length.

(3) The map \( x \mapsto x^{-1} w_0^K \) defines a bijection \( J W \to W^K \) which reverses Bruhat order. Moreover \( \ell(x^{-1} w_0^K) = \ell(w_0^K) - \ell(x) \).

(1.6) For any subgroup \( H \) of \( G \) and for any element \( g \in G \) we set \( g^H = gHg^{-1} \).

For any two parabolic subgroups \( P \) and \( Q \) of type \( J \) and \( K \), respectively, we denote by \( \text{relpos}(P,Q) \in J W^K \subset W \) their relative position. Recall that the relative position of \( P \) and \( Q \) can be defined as follows: Denote by \( W(P,Q) \) the set of elements \( w \in W \) such that there exists a \( g \in G \) such that \( g^P = P \) and \( g^Q = w^P K \). Then \( W_J \) acts from the left and \( W_K \) acts from the right on \( W(P,Q) \), and \( W(P,Q) \) is a single orbit for this action. Hence there exists a unique element \( \text{relpos}(P,Q) \in W(P,Q) \cap J W^K \).

In particular we have

\begin{align*}
(1.6.1) \quad & \text{relpos}(g^P, g^Q) = \text{relpos}(P,Q) \quad & \text{for all } g \in G, \\
(1.6.2) \quad & \text{relpos}(P,Q) = \text{relpos}(Q,P)^{-1}, \\
(1.6.3) \quad & \text{relpos}(P_J, P_J^{pp}) = w_0^J, \\
(1.6.4) \quad & \text{relpos}(P_J, w^P K) = w \quad & \text{for all } w \in J W^K.
\end{align*}

(1.7) If \( P \) and \( Q \) are two parabolic subgroups of \( G \), we set

\[ \text{Ref}_Q(P) = (P \cap Q)U_P, \]

the refinement of \( P \) by \( Q \) (cf. [MW], Section 3.7 and Example 3.3). This is a parabolic subgroup of \( G \) which is contained in \( P \).

If \( J \) and \( K \) are the types of \( P \) and \( Q \), respectively, \( \text{Ref}_Q(P) \) is of type \( J \cap \text{relpos}(P,Q) \).

(1.8) Lemma. Let \( J \) and \( K \) be two subsets of \( I \), and let \( w \in W \). Then \( \text{Ref}_{w^P K}(P_J) \) contains \( B \) if and only if

\[ w^{-1} \Phi_J^+ \subset \Phi_K \cup \Phi^+. \]

In particular, this is the case if \( w \in J W \).

Proof. For any root \( \alpha \) let \( \mathfrak{g}_\alpha \subset \text{Lie}(G) \) be the subspace where \( T \) acts via \( \alpha \). As \( P := \text{Ref}_{w^P K}(P_J) \) contains the chosen maximal torus \( T \), we can write \( \text{Lie}(P) = \text{Lie}(T) \oplus \bigoplus_{\alpha \in \Phi_P} \mathfrak{g}_\alpha \) for a certain subset \( \Phi_P \) of \( \Phi \). Then \( B \) is contained in \( P \) if and only if \( \Phi^+ \subset \Phi_P \). By definition of \( P \) we have

\[ \Phi_P = (\Phi_J \cup \Phi^+) \cap (w \Phi_K \cup w \Phi^+) \cup (\Phi^+ \setminus \Phi_J) \]

and therefore \( \Phi^+ \subset \Phi_P \) if and only if \( \Phi_J^+ \subset w \Phi_K \cup w \Phi^+ \). The last claim follows from (1.3.1). \( \Box \)
(1.9) We have the following result by Howlett (see [Ca], Proposition 2.7.5):

**Lemma.** Let \( J, K \subset I \) and \( \bar{w} \in JWK \). Set \( K' := K \cap w^{-1}J \). Then each element \( w \in W_J \bar{w}WK \) can be uniquely expressed in the form \( w = u \bar{w}v \) with \( u \in W_J \) and \( v \in WK \cap K'W \). Moreover, we have \( \bar{w}v \in JW \) and \( \ell(w) = \ell(u) + \ell(\bar{w}) + \ell(v) \).

(1.10) In a special case, we have also the following more precise version of Howlett’s lemma, proved in [He], Lemma 3.6:

**Lemma.** Let \( J, K \subset I \) and \( \bar{w} \in JWK \). Set \( K' := K \cap w^{-1}J \) and \( J' := J \cap wK \). Let \( w \in \bar{w}WK \). Then the unique decomposition of \( w \) in Howlett’s lemma is of the form \( w = u \bar{w}v \) with \( v \in WK \cap K'W \) and \( u \in W_{J'} \).

2 The classifying variety

(2.1) From now on we fix a subset \( J \subset I \) which is defined over \( \mathbb{F}_q \), in particular we have \( F(P_J) = P_J \) and \( F(J) = J \). We set \( K = w_0Jw_0 \subset I \) which is also defined over \( \mathbb{F}_q \).

(2.2) Let \( Z_J = ZG,J \) be the variety of triples \( (P, Q, [g]) \) where \( P \) and \( Q \) are parabolic subgroups of types \( J \) and \( K \), respectively, and where \([g]\) is a double coset in \( U_Q \setminus G/F(U_P) \) such that \( \text{relpos}(Q, gF(P)) = w_0^J = Kw_0 \). This is a special case of the variety \( Z_J \) defined in [MW], Section 4.8.

The variety \( Z_J \) carries a left \( G \times G \)-action by

\[(h, h') \cdot (P, Q, [g]) = (h'P, hQ, [hgF(h')^{-1}]).\]

(2.3) For any point \( (P, Q, [g]) \in Z_J \) we define a sequence of pairs of parabolic subgroups \( (P_n, Q_n)_{n \geq 0} \) inductively:

\[
\begin{align*}
P_0 &:= P, & Q_0 &:= Q, \\
P_n &:= \text{Ref}_{Q_n-1}(P_{n-1}), & Q_n &:= \text{Ref}_{F(P_n)}(Q_{n-1}).
\end{align*}
\]

Clearly \( P_n \subset P_{n-1} \) and \( Q_n \subset Q_{n-1} \). Therefore there exists an \( N \geq 0 \) such that \( (P_n, Q_n) = (P_{n+1}, Q_{n+1}) \) for all \( n \geq N \). We set \( (P_\infty, Q_\infty) = (P_N, Q_N) \) and obtain

\[\text{relpos}(P_\infty, Q_\infty) \in JWK_\infty\]

where \( JWK_\infty \subset J \) and \( KWK_\infty \subset K \) are the types of \( P_\infty \) and \( Q_\infty \), respectively. It follows from [MW], Theorem 4.11 and Section 4.6, that \( \text{relpos}(P_\infty, Q_\infty) \in JW \) and we set \( \alpha([g, g', z]) := \text{relpos}(P_\infty, Q_\infty) \). In this way we obtain a map

\[\alpha: Z_J \to JW.\]
(2.4) Now Theorem 4.11 of [MW] describes the $G$-orbits of $Z_J$ where we consider $G$ embedded diagonally in $G \times G$. We get:

**Theorem.** The map $\alpha$ induces a bijection of the set of $G$-orbits of $Z_J$ and the set $J^W$.

We denote the $G$-orbit of $Z_J$ corresponding to $w \in J^W$ by $Z_J^w$.

(2.5) Let $z \in Z_J$ and let $(P_n, Q_n)_{n \geq 0}$ be the associated sequence of pairs of parabolic subgroups (2.4) and let $J_{\infty}$ (resp. $K_{\infty}$) be the type of $P_{\infty}$ (resp. $Q_{\infty}$). Assume that $z \in Z_J^w$ for some $w \in J^W$. It follows from [MW], Section 4.6, that $J_{\infty}$ can be described as the largest subset $J'$ of $J$ such that $w_w^J J' = J'$ and that $K_{\infty} = w^{-1} J_{\infty} = w_0^\prime J_{\infty}$.

(2.6) For any $w \in W$ we choose an element $\bar{w} \in N_G(T)$ which maps to $w$. We set

$$\delta: W_J \longrightarrow W_K, \quad u \mapsto w_0^J F(u)(w_0^J)^{-1}.$$  

**Lemma.** Let $w \in W$ and $b \in B$ and set

$$z = z(w, b) = (P_J, w P_K, [\bar{w} w_0^J F(b)]) \in Z_J.$$

We write $w = u w'$ with $u \in W_J$ and $w' \in J^W$. Then there exists an element $v \in W_K$ such that $w' v \in J^W$ and $z \in Z_J^{w v}$ and $\delta^{-1}(v) \leq u$.

(2.7) **Corollary.** Let $w \in J^W$ and $b \in B$ and set

$$z = z(w, b) = (P_J, w P_K, [\bar{w} w_0^J F(b)]) \in Z_J.$$

Then $z \in Z_J^w$.

**Proof.** With the notations of (2.6), we have $u = 1$ and hence $v = 1$. \hfill $\square$

(2.8) **Proof of (2.6).** Let $(P_n, Q_n)_{n \geq 0}$ be the sequence of pairs of parabolics associated to $z$ (2.3). Let $J_n$ be the type of $P_n$ and $K_n$ be the type of $Q_n$. We set $y_n = \text{relpos}(P_n, Q_n)$. Let $\alpha \geq 0$ be the smallest integer such that $P_{\infty} = P_{\alpha+1}$. Then we have for all $n \geq \alpha$, $P_n = P_{\infty}$, $Q_n = Q_{\infty}$, $J_n = J_{\infty}$, $K_n = K_{\infty}$ and $y_n = y_{\infty}$.

By definition $z \in Z_J^{y_{\infty}}$ and $y_{\infty} \in J^{W_{\infty}}$. We have to show that $y_{\infty} = w' v$ for some $v \in W_K$ with $\delta^{-1}(v) \leq u$. For all $n \geq 0$,

$$J_{n+1} = J_n \cap y_n K_n, \quad K_{n+1} = w_0^J J_{n+1}$$

and $K_{\infty} = w_0^J J_{\infty} = y_{\infty}^{-1} J_{\infty}$ by [MW], Section 4.6.

We first consider two special cases. The first one is that $\infty = 0$. Then $y_{\infty} = \text{relpos}(P, Q)$. By (2.9) we can write $w' = y_{\infty} v$ with $v \in W_K \cap K_{\infty}^{y_{\infty}} J^W$. \hfill 7
As \( J_1 = J_0 \), we have \( K \cap w_\infty^{-1} J = K \) and therefore \( v = 1 \). This proves the lemma in this case.

The second special case is the case \( u = 1 \), i.e., we prove the Corollary \((2.9)\). Then we have to show that \( z \in Z^y_\infty \). We claim that \( B \subset P_n \cap w^{-1} Q_n \) for all \( n \geq 0 \), in other words, \( P_n = P_{J_n} \) and \( Q_n = wP_{K_n} \). The claim is shown by induction on \( n \).

It certainly holds for \( P_0 = P_J \) and \( Q_0 = wP_K \). Assume that it holds for \( n \geq 0 \). Then we have

\[
P_{n+1} = \text{Ref}_{wP_{K_n}}(P_{J_n}) \supset B
\]

by \((1.8)\), as \( w \in JW \subset J_n W \). Moreover

\[
w^{-1}Q_n + 1 = w^{-1}\text{Ref}_{w_0JF(P_{J_n+1})}(Q_n) = \text{Ref}_{w_0JF(P_{J_n+1})}(P_{K_n}) \supset B
\]

again by \((1.8)\), as \( w_0J \in KJW \subset K_nW \). This proves the claim.

Therefore we have seen that \( w \in W_{J_nJ}/K_{J_nJ} \cap J_nW \). By \((1.9)\) this implies \( w = y_\infty^v \) for some \( v \in W_{J_nJ}/K_{J_nJ} \cap J_nW = W_{J_nJ}/K_{J_nJ} \). Hence \( v = 1 \) and \( w = y_\infty^1 \), which proves the case \( u = 1 \).

In general we define \( Z_{J_n,w_0^J} \) as the variety of triples \((P,Q,[g])\), where \( P \) is a parabolic subgroup of type \( J_n \), \( Q \) is a parabolic subgroup of type \( K_n = w_0^J J_n \), and \([g] \in U_Q\setminus G/U_F(P)\) with \( \text{relpos}(Q,gF(P)) = w_0^J \). This variety carries a \( G \times G \)-action as in \((2.2)\), and in particular a \( G \)-action where we consider \( G \) embedded diagonally in \( G \times G \). As in \((2.4)\), the \( G \)-orbits of \( Z_{J_n,w_0^J} \) are parametrized by \( J_nW \) \((MW, \text{Theorem } 4.11)\). In particular, we can consider the \( G \)-orbit \( Z_{J_n,w_0^J}^{y_\infty} \) corresponding to the element \( y_\infty^1 \in JW \subset J_nW \). By \((MW)\), Lemma 4.6 and Lemma 4.7, for any \( n \geq 0 \) the map

\[
\vartheta: Z_{J_n,w_0^J}^{y_\infty} \to Z_{J_n+1,w_0^J}^{y_\infty}, \\
(P,Q,[g]) \mapsto (\text{Ref}_{Q}(P), \text{Ref}_{F(\text{Ref}_{Q}(P))}(Q), [g])
\]

is a well defined surjective smooth morphism which induces an isomorphism of the fppf-quotients

\[
G \setminus Z_{J_n,w_0^J}^{y_\infty} \simto G \setminus Z_{J_n+1,w_0^J}^{y_\infty}.
\]

We define inductively elements \( u_n \in W_{J_nJ} \cap W^{-J_{n+1}} \) and \( u'_n \in W_{J_nJ} \) as follows: For \( n = 0 \) we set \( u'_0 := u \) and write \( u = u_0 = u_0u_1 \) with \( u_0 \in W_{J} \cap W^{-J_1} \) and \( u_1 \in W_{J_1} \). For \( n \geq 1 \) we write \( u'_n = u_nu'_{n+1} \) with \( u_n \in W_{J_nJ} \cap W^{-J_{n+1}} \) and \( u'_{n+1} \in W_{J_{n+1}} \). This completes the inductive definition. Then \( u_n = 1 \) for \( n \geq \infty \) and

\[
(2.8.1) \quad u = u_0u_1u_2 \cdots u_{n-1}, \quad \ell(u) = \ell(u_0) + \ell(u_1) + \cdots + \ell(u_{n-1}).
\]

For the proof of the lemma we can replace \( z \) by

\[
z_0 = (P_J, w'_P K, [w'_wF(b)F(u)]).
\]
as both are in the same $G$-orbit. In general, let $z_n \in Z_{J_n,w_0'}$ be an element of the form

$$z_n = (P_{J_n}, w_n'P_{K_n}, [h_n w_n''_0 F(b_n)F(u_n)])$$

where $h_n \in w'U_K$, $w_n''_0 \in J_n W \cap w'W_K$, $b_n \in B$. By definition, we can write $u_n' := u_n u_{n+1}'$. By (2.9) below there exist $v_n \in W_K$, $h_{n+1} \in w'U_K$, and $b_{n+1} \in B$ such that $\delta^{-1}(v_n) \leq u_n$, $u_{n+1}' := w_n'v_n \in J_{n+1}W$, and such that $\vartheta(z_n)$ is in the same $G$-orbit as

$$z_{n+1} := (P_{J_{n+1}}, w_{n+1}'P_{K_{n+1}}, [h_{n+1} w_{n+1}' w_n''_0 F(b_{n+1})F(u_{n+1}')]).$$

If we start this process with $z_0$ and $w_0' = w'$, we get a sequence of elements $w_0', w_1', \ldots$ with $w_{n+1}' = w_n'v_n$. As in the proof of the case $u = 1$ it follows that $\text{relpos}(P_{J_\infty}, w'_\infty P_{K_\infty}) = w'_\infty$ and therefore $y_\infty = w'_\infty$ because $z_\infty \in Z_{J_\infty,w_0'}$. We have $w'_\infty = w_0'v_1 \cdots v_\infty - 1$ with $\delta^{-1}(v_n) \leq u_n$. If we set $v := v_0 v_1 \cdots v_\infty - 1$, we therefore have $\delta^{-1}(v) \leq u$ by (2.8.1) and this finishes the proof. \hfill \Box

(2.9) Lemma. Define $\delta: W_J \to W_K$ as in (2.6.1). Let $J' \subset J$, $w' \in J' W$, $b \in B$ and $u \in W_{J'}$. We set $K' = \delta(J') = w_0'F(J')(w_0')^{-1} \subset K$. Then there exists $v \in W_{K'}$ such that $v \leq \delta(u)$ and such that

$$w'w_0' F(b)F(u) \in w'U_{K'}w'vw_0'B.$$ 

Proof. We show the claim by induction on $\ell(u)$. If $u = 1$, nothing has to be shown. Now write $u = u_1 s$ with $\ell(u_1) < \ell(u)$ and $s \in J'$. Then by induction hypothesis there exists $v_1 \in W_{K'}$ such that $v_1 \leq \delta(u_1)$ and such that

$$w'w_0' F(b)F(u) = w'w_0' F(b)f(u_1)F(s) \in w'U_{K'}w'v_1 w_0' b'F(s)$$

for some $b' \in B$. Write $b' = b_2 b_1$ where $b_1 \in U_{P_{\{F(s)\}}} T$ and $b_2 \in U_{\{F(s)\}}$. Then $b'F(s) = b_2 F(s)b'_1$ for some $b'_1 \in U_{P_{\{s\}}} T \subset B$. Therefore, we can replace $b'$ by $b_2$ and hence assume $b' \in U_{\{F(s)\}}$.

Now we have $w_0' U_{\{F(s)\}} = U_{\{\delta(s)\}} \subset U_{K'}$ and therefore either $w'v_1 w_0' U_{\{F(s)\}}$ or $w'v_1 w_0' U_{\{F(s)\}}^{\text{app}}$ is contained in $w'U_{K'}$. If $w'v_1 w_0' U_{\{F(s)\}} \subset w'U_{K'}$, we have

$$w'U_{K'}w'v_1 w_0' b'F(s) \subset w'U_{K'}w'v_1 w_0' F(s) = w'U_{K'}w'v_1 \delta(s)w_0'.$$

In this case we set $v = v_1 \delta(s)$ and have $v \leq \delta(u_1) \delta(s) = \delta(u)$. Otherwise we note that $b'F(s) \in U_{F(s)}^{\text{app}} B$ and therefore

$$w'U_{K'}w'v_1 w_0' b'F(s) \subset w'U_{K'}w'v_1 w_0' b'F(s) B.$$ 

Then $v := v_1 \leq \delta(u_1) < \delta(u)$. Therefore the lemma is proved in both cases. \hfill \Box
3 Specialization of \((B \times B)\)-orbits

(3.1) Let \(\tilde{H}\) be any algebraic group over an algebraically closed extension \(k\) of \(\mathbb{F}_q\) and let \(X\) and \(Y\) be two varieties with \(\tilde{H}\)-action. Let \(\psi: X \to Y\) be a morphism of varieties which is a homeomorphism and which satisfies

\[\psi(\tilde{h} \cdot x) = F(\tilde{h})\psi(x)\]

for all \(x \in X\) and \(\tilde{h} \in \tilde{H}\). Then \(\psi\) induces a bijection of \(\tilde{H}\)-orbits which is compatible with taking closures.

If \(\tilde{H}\) is of the form \(H \times H\) and if the condition (3.1.1) is replaced by

\[\psi((h, h') \cdot x) = (h, F(h')) \cdot \psi(x),\]

the analogous statement holds.

(3.2) The group \(P_J^{opp} \times P_J\) acts on the right on \(G \times G \times L_J\) by

\[(g, g', z) \cdot (s, t) = (gs, g't, \pi_J^{opp}(s)^{-1} z F(\pi_J(t))).\]

We denote by

\[X_J := (G \times G \times L_J)/(P_J^{opp} \times P_J)\]

the fppf-quotient of this action.

For any element \((g, g', z) \in G \times G \times L_K\) we denote by \([g, g', z]\) its image in \(X_J\).

The variety \(X_J\) carries a left \(G \times G\)-action by

\[(h, h') \cdot [g, g', z] = [hg, h'g', z].\]

(3.3) Lemma. The map

\([g, g', z] \mapsto (g' P_J, g P_J^{opp}, [gzF(g')^{-1}])\]

defines a \(G \times G\)-equivariant isomorphism

\[\rho: X_J \cong Z_J\]

of varieties with transitive \(G \times G\)-action.

Proof. We have

\[\text{relpos}(g P_J^{opp}, gzF(g')^{-1} F(g' P_J)) = \text{relpos}(P_J^{opp}, z P_J) = \text{relpos}(P_J^{opp}, P_J) = w_0^J\]

and hence \(\rho\) lands in \(Z_J\).

It is straightforward to check that \(\rho\) is well-defined (use \(F(P_J) = P_J\)) and \(G \times G\)-equivariant. Now we show that \(\rho\) is a monomorphism. As \(\rho\) is
For any \( g \) there exist \((h,h') \in P_j^{\text{opp}} \times P_j \) such that relpos\((P_j^{\text{opp}},gP_j) = w_0' \) means that \( [h,gF(h')^{-1}] = [1] \). The relation relpos\((P_j^{\text{opp}},gP_j) = w_0' \) means that \( P_j^{\text{opp}} \) and \( gP_j \) are in opposition, that is, their intersection is a common Levi subgroup \( L \). There exists a (unique) \( u \in U_{P_j^{\text{opp}}} \) such that \( uL = L \) and therefore \( P_j^{\text{opp}} \cap \langle u \rangle P_j = L \) which implies \( ug \in P_j \). Now choose \( h' \in P_j \) such that \( F(h') = ug \) and set \( h = 1 \). Then \( [hgF(h')^{-1}] = [gg^{-1}u^{-1}] = [1] \). Note that this argument shows that the action for any affine \( k \)-scheme \( S \) the action of \( G(S) \times G(S) \) is transitive on \( Z_j(S) \).

As \( G \times G \) acts transitively on \( Z_j \), \( \rho \) is surjective. Moreover, as \( G(k[\varepsilon]) \times G(k[\varepsilon]) \) acts also transitively on \( Z_j(k[\varepsilon]) \) for \( k[\varepsilon] = k[T]/(T^2) \), \( \rho \) is also surjective on tangent spaces. As \( X_j \) and \( Z_j \) are smooth, we therefore have a smooth surjective monomorphism and therefore an isomorphism.

\begin{equation}
(3.4) \text{For } x, w \in W \text{ we denote by } \Sigma_{x,w} \text{ the } B \times B \text{-orbit of } [x,w,1] \text{ in } X_j.
\end{equation}

**Theorem.** Every \((B \times B)\)-orbit of \( X_j \) is of the form \( \Sigma_{x,w} \) for a pair \((x,w) \in W \times W \). We have \( \Sigma_{x,w} = \Sigma_{x',w'} \) if and only if there exists \( u \in W_j \) such that

\[
x' = xF(u)^{-1}, \quad u'u = w.
\]

For \( x, x' \in W_j \) and \( w, w' \in W \) the orbit \( \Sigma_{x,w} \) is contained in the closure of the orbit of \( \Sigma_{x',w'} \) if and only if there exists \( u \in W_j \) such that

\begin{equation}
(3.4.1) \quad xu^{-1} \leq x', \quad F(w')u \leq F(w).
\end{equation}

**Proof.** We set

\[
X_j^{\text{ad}} := \frac{(G^{\text{ad}} \times G^{\text{ad}} \times (L_j^{\text{ad}})^{\text{ad}}) / ((P_j^{\text{opp}})^{\text{ad}} \times P_j^{\text{ad}})}{(P_j^{\text{opp}})^{\text{ad}} \times P_j^{\text{ad}}}
\]

where the action of \((P_j^{\text{opp}})^{\text{ad}} \times P_j^{\text{ad}}\) is defined as in (3.2.1). Here \((L_j^{\text{ad}})^{\text{ad}}\) is the adjoint group of the image of \( L_j \) in \( G^{\text{ad}} \). The smooth surjective projection morphism \( X_j \rightarrow X_j^{\text{ad}} \) induces a bijection of \((B \times B)\)-orbits in \( X_j \) and \((B^{\text{ad}} \times B^{\text{ad}})\)-orbits in \( X_j^{\text{ad}} \) which preserves the closure of orbits. Hence it suffices to show the analogous assertion for \((B^{\text{ad}} \times B^{\text{ad}})\)-orbits in \( X_j^{\text{ad}} \). In particular we can assume that \( G = G^{\text{ad}} \) and therefore

\[
X_j^{\text{ad}} := \frac{(G \times G \times L_j^{\text{ad}}) / (P_j^{\text{opp}} \times P_j)}{(P_j^{\text{opp}} \times P_j)}
\]

with the action as in (3.2.1).

Define a linear version of \( X_j^{\text{ad}} \) as

\[
\tilde{G}_j = \frac{(G \times G \times L_j^{\text{ad}}) / ((P_j^{\text{opp}})^{\text{ad}} \times P_j)}{(P_j^{\text{opp}} \times P_j)},
\]

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now with the action given by
\[(g, g', z) \cdot (s, t) = (gs, g't, \pi_{opp}^{-1}(s)z\pi_J(t)).\]

This is the \((G \times G)\)-orbit of the wonderful compactification of de Concini and Procesi corresponding to \(J\) (cf. [Sp]).

The morphism of varieties
\[
\psi: X_J^{\text{ad}} \rightarrow \bar{G}_J,
\]
\[[g, g', z] \mapsto [g, F(g'), z]\]
is a homeomorphism such that for all \((h, h') \in G \times G\) and \(x \in X_J^{\text{ad}}\) the equality (3.1.2) holds. Therefore the theorem follows from the analogous result of Springer for the wonderful compactification ([Sp] 1.3 and 2.2).

(3.5) For \((x, w) \in W \times W\) we denote the image of \(\Sigma^{x,w}\) in \(Z_J\) under the isomorphism \(X_J \sim Z_J\) again by \(\Sigma^{x,w}\), i.e., as subvariety of \(Z_J\) we have
\[
\Sigma^{x,w} = (B \times B) \cdot (\pi_J^{opp}(wP_J, xP_J^{-1}, [xF(w)])].
\]

(3.6) To relate \((B \times B)\)-orbits and \(G_{\text{diag}}\)-orbits of \(Z_J\) we need the following lemma:

**Lemma.** For any \(w \in J^W\) we have
\[
Z_J^w = G_{\text{diag}} \cdot \Sigma^{w,w^d_0,1}.
\]

**Proof.** For any point \(z \in Z_J\) we have
\[
G_{\text{diag}} \cdot (B \times B) \cdot z = G_{\text{diag}} \cdot ((1) \times B) \cdot z.
\]

Therefore it suffices to show that
\[
z(w, b) := (P_J, wP_J^{opp}, [wP_J^{opp}(b)^{-1}], [xF(w)])
\]
is contained in \(Z_J^w\) for all \(b \in B\). As we have \(wP_J^{opp} = \pi_K\) and \(w \in J^W\), this follows from (2.7). \(\Box\)

(3.7) **Lemma.** Let \(w \in W\). Then
\[
\Sigma^{w,w^d_0,1} = \bigcup_{x \in W, x \leq w} \Sigma^{x,w^d_0,1}.
\]
Proof. We use the notations of the proof in (3.4). Again we can replace $G$ by $G^{\text{rad}}$, $X_J$ by $X_J^{\text{ad}}$, and every $(B \times B)$-orbit $\Sigma^{x,w}$ by its image in $X_J^{\text{ad}}$ which we denote again by $\Sigma^{x,w}$. The homeomorphism $\psi$ maps the orbit $\Sigma^{ww^0,1}$ to the $(B \times B)$-orbit of $[ww^0,1,1]$ in $\hat{G}_J$.

Set $K = w_0Jw_0$. By [Sp] 1.2 there exists a unique isomorphism

$$\sigma: \hat{G}_J \xrightarrow{\sim} \hat{G}_K$$

such that

$$\sigma((h,h') \cdot x) = (h',h) \cdot \sigma(x), \quad \text{for } h,h' \in G, \ x \in \hat{G}_J,$$

$$\sigma([1,1,1]) = [w_0,w_0,1].$$

The image of the $(B \times B)$-orbit of $[ww^0,1,1]$ under $\sigma$ is the $(B \times B)$-orbit of $[w_0,ww_0w_0Jw_0,1] = [w^K,w,1]$ in $\hat{G}_K$. Let $y \in W^K$ and $x \in W$ be such that the $(B \times B)$-orbit of $[y,x,1]$ in $G_K$ is contained in the closure of the $(B \times B)$-orbit of $[w^K_0,w,1]$. By [Sp] 2.2 this means that there exists a $v \in W^K$ such that

$$(3.7.1) \quad w^K_0v^{-1} \leq y, \quad xv \leq w.$$ 

But $w^K_0$ is the maximal element in $W^K$ and therefore $y \leq w^K_0$. Moreover, for any $v \in W_K$ we have $w^K_0 \leq w^K_0v^{-1}$. Hence the relations (3.7.1) are equivalent to $v = 1$, $y = w^K_0$, and $x \leq w$.

Now the $(B \times B)$-orbit of $[w^K_0,x,1]$ in $\hat{G}_K$ corresponds via $\sigma \circ \psi$ to $\Sigma^{ww^K_0,1}$ which shows the lemma. \qed

4 A partial order on $JW$

(4.1) In this chapter, we fix subsets $J$ and $K$ of $I$ and denote by $\delta: W_J \rightarrow W_K$ an automorphism such that $\delta(J) = K$. In particular we have $\delta(u) \leq \delta(u')$ if and only if $u \leq u'$ for $u,u' \in W_J$.

We will apply the results of this chapter with the automorphism $\delta(u) = w_0w_0,JF(u)w_0,Jw_0$.

(4.2) Definition. For elements $w,w' \in W$ we write $w \preceq_{J,\delta} w'$ or simply $w \preceq w'$ if there exists an element $u \in W_J$ such that $u^{-1}w\delta(u) \leq w'$. This order has also been examined in [He] in the case that the isomorphism $\delta$ is induced by an isomorphism $\hat{\delta}: W \xrightarrow{\sim} W$ with $\hat{\delta}(I) = I$. All of the following results in this paragraph are variants of the results in loc. cit., and the proofs are easy modifications of the arguments there.
By definition we have

\[(4.3.1)\quad w \leq w' \implies w \leq w'.\]

If \(w \in J\), we have \(\ell(u^{-1}w\delta(u)) \geq \ell(u^{-1}w) - \ell(\delta(u)) = \ell(u) + \ell(w) - \ell(u) = \ell(w)\). Therefore we see that for \(w \in J\) we have

\[(4.3.2)\quad w \leq w' \implies \ell(w) \leq \ell(w').\]

Now we give three lemmas on the Bruhat order which are all proved in [He].

**Lemma.** Let \(x, w \in W\). Then the subset

\[\{ y \in W \mid wy \leq x \}\]

of \(W\) contains a smallest element \(y_{\text{min}}\) and a largest element \(y_{\text{max}}\). Moreover we have

\[\ell(y_{\text{min}}) = \ell(w) - \ell(wy_{\text{min}}), \quad \ell(y_{\text{max}}) = \ell(w) + \ell(wy_{\text{max}}).\]

**Lemma.** Let \(x', w, w' \in W\) such that \(w \leq w'\).

1. There exists \(x \leq x'\) such that \(xw \leq x'w'\).
2. There exists \(x \leq x'\) such that \(x'w \leq xw'\).

**Lemma.** Let \(x \in J\), \(u \in W\) such that \(\ell(xu) = \ell(x) + \ell(u)\). We write \(xu = u'x'\) with \(u' \in W_J\) and \(x' \in J\). Then for any \(u_1' \leq u'\) there exists a \(u_1 \leq u\) such that \(xu_1 = u_1'x'\).

**Lemma.** Let \(w \in J\), \(u, v \in W_J\) such that \(v \leq u\). Then there exists \(x \leq v\) and a reduced decomposition \(x = s_1 \ldots s_r\) such that

\[(4.7.1)\quad \ell(s_i \ldots s_1w\delta(s_1)\ldots\delta(s_i)) = \ell(w), \quad \text{for all } i = 1, \ldots, r\]

and such that

\[x^{-1}w\delta(x) \leq u^{-1}w\delta(v).\]

**Proof.** The proof is by induction on \(#J\). Assume that the statement holds for all \(J' \subset I\) with \(#J' < #J\) and for the isomorphism \(\delta' = \delta|_{W_{J'}} : W_{J'} \to W_{\delta(J')}\). Then the statement is proven for \(J, K\), and \(\delta\) by induction on \(\ell(u)\).

Write \(w = w'b\) for \(w \in JWK\) and \(b \in WK \cap K'W\) where \(K' = K \cap w^{-1}J\) \([1.9]\). We also set \(J' = J \cap w^{-1}K = w^{-1}K'\).

We first consider the case that \(u \in W_{J'}\) and therefore \(v \in W_{J'}\). As \(J' \subset J\), we have \(w \in JW\). If \(J' \not\subset J\), we are done by induction hypothesis. If \(J' = J\) and hence \(K' = K\), we have \(b = 1\) and therefore \(w \in JWK\). But this implies \(u^{-1}w\delta(v) \geq w\), i.e., we can choose \(x = 1\).
Moreover, \(x' \leq v\). We also write \(v' = \ell(v) + \ell(v_1)\) such that \(v' \leq v\) and \(\ell(v) = \ell(u) + \ell(v_1)\). By induction hypothesis there exists \(x' \leq v'\) and a reduced decomposition \(x' = s_1 \ldots s_{r'}\) with \(\ell(s_1 \ldots s_{s1}w \delta(s_1) \ldots \delta(s_i)) = \ell(w')\) for all \(i = 1, \ldots, r'\) such that
\[
w' := x'^{-1}w \delta(x') \leq u'^{-1}w \delta(v').
\]

Now let \(x_1 \leq v_1\) be the element in \(W\) such that \(w' \delta(x_1)\) is the smallest element in \(\{w' \delta(y) \mid y \leq v_1\}\). Then
\[
\ell(x_1^{-1}w') = \ell(w') - \ell(x_1)
\]
and
\[
w' \delta(x_1) = w' \delta(v_1) = u'^{-1}w \delta(v).
\]

Now \(w \delta(v) \in \bar{w}W_K\) and therefore we can write \(w \delta(v) = a' \bar{w}b'\) with \(a \in W_J\) and \(b' \in W_K \cap W_J\) by (4.10). We now examine
\[
u^{-1}w \delta(v) = u_1^{-1}u'^{-1}a' \bar{w}b'.
\]

We have \(u_1^{-1} \in W_J\), \(u'^{-1}a' \in W_J\), \(\bar{w}b' \in \bar{J}W\) and \(u_1^{-1}u'^{-1}a' \in W_J\). Therefore
\[
\ell(u_1^{-1}w \delta(v)) = \ell(u_1^{-1}u'^{-1}a' \bar{w}b')
\]
\[
= \ell(u_1^{-1}u'^{-1}a') + \ell(\bar{w}b')
\]
\[
= \ell(u_1^{-1}) + \ell(u'^{-1}a') + \ell(\bar{w}b')
\]
\[
= \ell(u_1^{-1}) + \ell(u'^{-1}a' \bar{w}b')
\]
\[
= \ell(u_1^{-1}) + \ell(u'^{-1}w \delta(v)).
\]

Hence it follows from (4.7.3) and \(x_1 \leq u_1\) that
\[
(x'x_1)^{-1}w \delta(x'x_1) = x_1^{-1}w' \delta(x_1) \leq uw \delta(v)^{-1}.
\]

Moreover, \(x'x_1 \leq v\). Let \(x_1 = s_1 \ldots s_l\) be a reduced decomposition. For \(i = 0, \ldots, l\) set
\[
y_i = s_{i+1} \ldots s_l(x_1^{-1}w') \delta(s_1) \ldots \delta(s_i) = s_i \ldots s_lw' \delta(s_1) \ldots \delta(s_i).
\]

If we show that \(\ell(y_i) = \ell(w')\) for all \(i = 1, \ldots, r\), we are done. We have \(\ell(y_i) \leq l + \ell(x_1^{-1}w') = \ell(w')\) for all \(i\) by (4.7.2). On the other hand, each \(y_i\) is of the form \(a_i^{-1}w \delta(a_i)\) for some \(a_i \in W_J\) and therefore we have \(\ell(y_i) \geq \ell(a_i^{-1}w) - \ell(a_i) = \ell(w) = \ell(w')\) as \(w \in \bar{J}W\).

(4.8) Corollary. For \(w, w' \in \bar{J}W\), \(w \preceq w'\) if and only if there exists \(u \in W_J\) and a reduced decomposition \(u = s_1 \ldots s_r\) such that \(\ell(s_1 \ldots s_{s1}w \delta(s_1) \ldots \delta(s_i)) = \ell(w)\) for all \(i = 1, \ldots, r\) and such that \(u^{-1}w \delta(u) \leq w'\).
(4.9) Corollary. Let \( w, w' \in JW \), \( u, v \in W_J \) with \( v \leq u \). Assume that \( uw' \delta(v)^{-1} \leq w \). Then \( w' \leq w \).

Proof. By \( (4.7) \) we know that there exists \( x \in W_J \) such that \( x^{-1}w' \delta(x) \leq uw' \delta(v)^{-1} \leq w \), hence \( w' \leq w \).

(4.10) Lemma. Let \( w, w' \in JW \) such that \( w \leq_{J, \delta} w' \). Then there exist \( u, u' \in W_J \) such that
\[
uw \leq u'w' \delta(u')^{-1} \delta(u) =: w'_1
\]
and such that \( w'_1 \in JW \).

Proof. By definition there exists \( u_1 \in W_J \) such that \( u_1^{-1}w \delta(u_1) \leq w' \). By \( (4.5) \) there exists a \( v_1 \in \delta(u_1)^{-1} \) such that \( u_1^{-1}w \delta(u_1) \delta(u_1)^{-1} = u_1^{-1}w \leq w'v_1 \). Now \( v_1 \leq \delta(u_1)^{-1} \) implies \( \delta^{-1}(v_1) \leq u_1^{-1} \) and therefore \( \delta^{-1}(v_1)w \leq u_1^{-1}w \) as \( w \in JW \). Hence we see that there exists \( v_1 \in W_J \) such that
\[
\delta^{-1}(v_1)w \leq w'v_1.
\]
Let \( v_1 \in W_J \) be a minimal element such that \( (4.10.1) \) holds. Then \( \ell(w'v_1) = \ell(w') + \ell(v_1) \). Now write \( w'v_1 = v'w'_1 \) with \( v' \in W_J \) and \( w'_1 \in JW \). As \( v'w'_1 \geq \delta^{-1}(v_1)w \), there exists by \( (4.5) \) an element \( v'_1 \leq v' \) such that
\[
w'_1 \geq v'_1 \delta^{-1}(v_1)w.
\]
By \( (4.6) \) applied to \( w'v_1 = v'w'_1 \) and \( v'_1 \leq v' \), we see that there exists \( v_2 \leq v_1 \) such that \( v'_1 w'_1 = w'v_2 \). As we have \( \ell(v'_1x') = \ell(v'_1) + \ell(x') \), \( (4.10.2) \) implies \( v'_1 w'_1 \geq \delta^{-1}(v_1)w \). Therefore we have
\[
w'v_2 = v'_1w'_1 \geq \delta^{-1}(v_1)w \geq \delta^{-1}(v_2)w.
\]
By the minimality of \( v_1 \) we have \( v_1 = v_2 \) and hence \( v'_1 = v' \). Therefore \( w'_1 \geq v'^{-1}\delta^{-1}(v_1)w \). Now set \( u := v'^{-1}\delta^{-1}(v_1) \in W_J \) and \( u' := v'^{-1} \). Then we have \( w' = v'w'_1v_1^{-1} = u'^{-1}w'_1 \delta(u)^{-1}\delta(u) \).

(4.11) Lemma. Let \( w, w' \in JW \) such that \( w \leq w' \) and \( \ell(w) = \ell(w') \). Then \( w = w' \).

Proof. By \( (4.8) \) there exist \( u \in W_J \) and a reduced decomposition \( u = s_1 \ldots s_r \) such that \( \ell(s_i \ldots s_1w \delta(s_1) \ldots \delta(s_i)) = \ell(w) \) for all \( i = 1, \ldots, r \) and such that \( u^{-1}w \delta(u) \leq w' \). Moreover we have by \( (4.5.1) \)
\[
\ell(w) \leq \ell(u^{-1}w \delta(u)) \leq \ell(w') \leq \ell(w)
\]
and hence \( u^{-1}w \delta(u) = w' \in JW \). We show that \( w = w' \) by induction on \( \ell(u) \).

Assume that \( u = s \in J \) is a simple reflection. As \( \ell(w) = \ell(sw \delta(s)) \) and \( \ell(sw) = \ell(w) + 1 \), we have \( w \delta(s) \not\in JW \) and therefore \( w \delta(s) = s'w \) for some
The specialization order $s' \in J$ by (4.4). As $\ell(w) = \ell(sw\delta(s)) = \ell(ss'w) = \ell(ss') + \ell(w)$, we have $s = s'$ and therefore $sw\delta(s) = w$.

Now assume that $\ell(u) > 1$ and write $u = s_1u_1$ with $\ell(u_1) < \ell(u)$. Then

$$w' = u^{-1}w\delta(u) = u_1^{-1}s_1w\delta(s_1)\delta(u_1).$$

We claim that $s_1w\delta(s_1) \in J$. If this is shown, we are done by induction hypothesis and induction start.

Assume $s_1w\delta(s_1) \notin J$. As $s_1w\delta(s_1) = s_1s'w$ for some $s' \in J$. As $\ell(s_1w\delta(s_1)) = \ell(w)$, we have $s_1w\delta(s_1) = w \in J$, contradiction. Therefore $w\delta(s_1) \in J$ and $\ell(w\delta(s_1)) < \ell(w)$. As $w' \in J$, this implies

$$\ell(w) + \ell(u) = \ell(w') + \ell(u) = \ell(uu') = \ell(w\delta(s_1)\delta(u_1))$$
$$\leq \ell(w\delta(s_1)) + \ell(\delta(u_1)) < \ell(w) + \ell(u_1)$$
$$< \ell(w) + \ell(u)$$

and we obtain again a contradiction. \hfill \Box

(4.12) Proposition. The relation $\preceq_{J,\delta}$ is a partial order on $J$.

Proof. Let $w, w' \in J$ such that $w \preceq w'$ and $w' \preceq w$. By (4.3.2), we know that $\ell(w) = \ell(w')$ and hence $w = w'$ by (4.11). This proves the asymmetry of $\preceq_{J,\delta}$.

Now let $w_1, w_2, w_3 \in J$ with $w_3 \preceq w_2$ and $w_2 \preceq w_1$. By (4.10), there exist $u, u' \in W_J$ such that $uw_3 \leq u'w_2\delta(u')^{-1}\delta(u) =: w'_2 \in J$. As $w_2 = u^{-1}w'_2\delta(u)^{-1}\delta(u') \leq w_1$, there exists $v \in W_J$ such that

$$v^{-1}w'_2\delta(u^{-1}v) \leq w_1.$$

As $w'_2 \in J$, the relation $uw_3 \preceq w'_2$ implies that (4.12.1)

$$v^{-1}uw_3 \preceq v^{-1}w'_2.$$

Applying (4.5) to (4.12.1), there exists $v' \leq u^{-1}v$ such that

$$v^{-1}uw_3\delta(v') \preceq v^{-1}w'_2\delta(u^{-1}v) \leq w_1.$$

Then (4.7) implies $w_3 \preceq w_1$, and therefore the relation is transitive. \hfill \Box

5 The specialization order

(5.1) From now on we are back in the situation of (2.2) and we set

$$\delta : W_J \xrightarrow{\sim} W_K, \quad u \mapsto w'_0F(u)(w'_0)^{-1}.$$
(5.2) We first need the following general lemma.

**Lemma.** Let $H$ be any algebraic group acting on a variety $Z$ and let $P \subset H$ be an algebraic subgroup such that $H/P$ is proper. Then for any $P$-invariant subvariety $Y \subset Z$ we have

$$H \cdot Y = \overline{H \cdot Y}.$$

**Proof.** Clearly we have

$$H \cdot Y \subset H \cdot Y \subset \overline{H \cdot Y}$$

and therefore it suffices to show that $H \cdot Y$ is closed in $Z$. We denote by $\pi: H \times Z \to Z$ the action of $H$ on $Z$. Define an action of $P$ on $H \times Z$ by $b \cdot (h, z) = (hb^{-1}, b \cdot z)$, and denote by $H \times^P Z$ the quotient. Then $\pi$ induces a morphism $\overline{\pi}: H \times^P Z \to Z$ which can be written as the composition

$$H \times^P Z \xrightarrow{\sim} H/P \times Z \to Z.$$

Here the first morphism is the isomorphism given by $[h, z] \mapsto (hP, h \cdot z)$ and the second morphism is the projection. As $H/P$ is proper, we see that $\overline{\pi}$ is proper. Now $Y$ is $P$-invariant and therefore $H \times^P Y$ is defined, and it is a closed subscheme of $H \times^P Z$. Therefore $\overline{\pi}(H \times^P Y) = H \cdot Y$ is closed in $Z$. \(\square\)

(5.3) **Lemma.** For $w \in ^JW$,

$$\overline{Z^J_w} = \bigcup_{x \in W \atop x \leq w} G \cdot \Sigma^{xw_0^d,1}.$$

**Proof.** We apply (5.2) to the action of $G$, embedded diagonally in $G \times G$, on $Z_J$ and to $Y = \Sigma^{w_0^d,1}$ which is invariant under the Borel subgroup $B \subset G$. Then we see by (3.6) and (3.7) that

$$\overline{Z^J_w} = G \cdot \Sigma^{w_0^d,1} = \bigcup_{x \in W \atop x \leq w} G \cdot \Sigma^{xw_0^d,1}.$$

\(\square\)

(5.4) **Theorem.** For $w \in ^JW$,

$$\overline{Z^J} = \bigcup_{w' \in ^JW \atop w' \leq w} Z^{w'}.$$

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Proof. Let \( w' \in \mathcal{J}W \) with \( w' \preceq w \), i.e., there exists \( u \in W_J \) such that \( u^{-1}w' \delta(u) \leq w \). By (5.3) we have
\[
G \cdot \Xi w^0(\delta(u)w^0)^{-1} \subset \mathcal{Z}^{w'}.
\]
By (3.6) we have
\[
Z_J^{w'} = G \cdot (P_J, w'^0P_{J^{opp}}, [w'w_0^J]) = G \cdot (P_J, w'^0P_K, [w'w_0^J]) = G \cdot (P_J, u^{-1}w'^0P_K, [u^{-1}w'w_0^JF(u)]) = G \cdot (P_J, u^{-1}w'^0P_K, [u^{-1}w'\delta(u)w_0^J])
\]
where the third equality holds as \( u^{-1}P_J = P_J \) and \( \delta(u)P_K = P_K \). Hence \( Z_J^{w'} \subset \mathcal{Z}^{w'} \).

Conversely, let \( z \in \mathcal{Z}^{w'} \), say \( z \in G \cdot \Xi xw_0^dP \) for some \( x \in W \) with \( x \leq w \) (5.3). We want to show that \( z \in Z_J^{w'} \) for some \( w' \in \mathcal{J}W \) with \( w' \preceq w \). For this we can replace \( z \) by some element in the same \( G \)-orbit and hence we can assume that \( z = (P_J, xP_K, [xw_0^dF(b)^{-1}]) \) for some \( b \in B \).

We write \( x = uw' \) with \( u \in W_J \) and \( x' \in \mathcal{J}W \). By (2.6) there exists \( v \in W_J \) with \( v \leq u \) such that \( z \in Z^{u^{-1}x\delta(v)} \). If we set \( w' = u^{-1}x\delta(v) \), we have \( uw'\delta(v)^{-1} = x \leq w \) and therefore \( w' \preceq w \) by (1.9).

\[\square\]

6 Applications to the Ekedahl-Oort stratification

(6.1) Let \( g \geq 1 \) be an integer. In this section we apply the results to the Ekedahl-Oort stratification of the moduli space \( \mathcal{A}_g \) of principally polarized abelian varieties of dimension \( g \) in characteristic \( p \). In fact all of the following results can also be applied to arbitrary good reductions to characteristic \( p > 2 \) of Shimura varieties of PEL-type (cf. MW, Sections 7.10-14).

(6.2) For this we consider the variety \( Z_{G,J} \) in a special case: Let \( \langle , \rangle \) be the standard symplectic pairing on \( V = \mathbb{F}^2_p \) given by the matrix \( J = \begin{pmatrix} 0 & J' \\ -J' & 0 \end{pmatrix} \) where \( J' \) is the matrix \( (a_{ij}) \) with \( a_{ij} = \delta_{i,g+1-j} \). Let \( G = \text{Sp}(V, \langle , \rangle) \) be the group of symplectic isomorphisms of \( (V, \langle , \rangle) \). Therefore the elements in \( G \) are of the form \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) where \( A, B, C, \) and \( D \) are \( (g \times g) \) matrices satisfying
\[
^{t}AJ'C - ^{t}CJ'A = ^{t}BJ'D - ^{t}DJ'B = 0, \quad ^{t}A(J')^D - ^{t}CJ'B = J'.
\]

The Frobenius \( F: G \to G \) is given by \( (a_{ij}) \mapsto (a_{ij}^p) \). As a maximal torus \( T \) we choose the group of diagonal matrices in \( G \) and as the Borel subgroup \( B \)
we choose the subgroup of matrices \((A \ B \ 0 \ D) \in G\), where \(A\) and \(D\) are upper triangular.

The normalizer of \(T\) in \(G\) is given by the subgroup of monomial matrices in \(GL_{2g}\) which are contained in \(G\) and therefore we can identify \(W\) with the group of permutations \(w \in S_{2g}\) such that

\[
(6.2.1) \quad w(i) + w(2g + 1 - i) = 2g + 1 \quad \text{for all } i = 1, \ldots, g.
\]

The induced action of the Frobenius \(F\) on \(W\) is trivial. An easy and elementary calculation shows that the set \(I\) of simple reflections with respect to \((T, B)\) consists of \(\{s_1, \ldots, s_g\}\) with

\[
s_i = \begin{cases} 
\tau_i \tau_{2g-i}, & \text{for } i = 1, \ldots, g-1; \\
\tau_g, & \text{for } i = g.
\end{cases}
\]

where \(\tau_j \in S_{2g}\) denotes the transposition of \(j\) and \(j + 1\). The longest element \(w_0 \in W\) is given by the permutation \(i \mapsto 2g + 1 - i\). It follows from (6.2.1) that \(w_0\) lies in the center of \(W\). In particular, \(K = w_0J = J\).

Let \(J = \{s_1, \ldots, s_{g-1}\} \subset I\). Then the standard parabolic subgroup \(P_J\) consists of the matrices \((A \ B \ C \ D)\) in \(G\) with \(C = 0\), and \(W_J\) is the subgroup of those permutation \(w \in W\) such that \(w(\{1, \ldots, g\}) = \{1, \ldots, g\}\). The map

\[
W_J \to S_g, \quad w \mapsto w|_{\{1, \ldots, g\}}
\]

is a group isomorphism. The maximal element \(w_{0,J}\) in \(W_J\) corresponds via this map to the permutation \(i \mapsto g + 1 - i\) in \(S_g\). The map \(\delta\) from (5.1) has therefore in this case the form

\[
(6.2.2) \quad \delta: W_J \cong W_J, \quad w \mapsto w_{0,J}ww_{0,J}.
\]

The set \(JW\) consists of those elements \(w \in W\) such that

\[
w^{-1}(1) < w^{-1}(2) < \cdots < w^{-1}(g).
\]

Of course, this implies \(w^{-1}(g + 1) < \cdots < w^{-1}(2g)\). For two permutations \(w\) and \(w'\) in \(JW\) we have \(w \leq w'\) if and only if \(w^{-1}(i) \leq w'^{-1}(i)\) for all \(i = 1, \ldots, g\).

If \(\Sigma = \{j_1 < \cdots < j_g\} \subset \{1, \ldots, 2g\}\) is a subset of \(g\) elements such that either \(i \in \Sigma\) or \(2g + 1 - i \in \Sigma\) for all \(i = 1, \ldots, g\), we get a corresponding element \(w_{\Sigma} \in JW\) by setting \(w^{-1}(i) = j_i\). The sets of these \(\Sigma\)'s is in bijection with \(\{0, 1\}^g\) by associating to \(\Sigma\) the tuple \((\epsilon_1, \ldots, \epsilon_g)\) with

\[
\epsilon_i = \begin{cases} 
0, & \text{if } i \in \Sigma; \\
1, & \text{otherwise}.
\end{cases}
\]

The length of such an element \((\epsilon_1, \ldots, \epsilon_g)\) is equal to

\[
\sum_{i=1}^{g} \epsilon_i(g + 1 - i).
\]

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(6.3) The moduli space \( \mathcal{A}_g \) is a Deligne-Mumford stack. Let \((\mathcal{X}, \lambda)\) be the universal abelian (relative) scheme over \( \mathcal{A}_g \) and \( f: \mathcal{X} \to \mathcal{A}_g \) its structure morphism. The first de Rham cohomology \( \mathcal{H} = H^1_{\text{DR}}(\mathcal{X}/\mathcal{A}_g) \) is endowed with the structure of an \( F \)-Zip with symplectic structure (cf. [MW], §7): \( \mathcal{H} \) is a locally free \( \mathcal{O}_{\mathcal{A}_g} \)-module of rank 2g and the principal polarization induces a symplectic pairing \( \beta \) on \( \mathcal{H} \). There are two canonical locally direct summands of rank \( g \), namely

\[
\mathcal{C}^1 := f_*\Omega^1_{\mathcal{X}/\mathcal{A}_g}, \quad D_0 := R^1f_*(\mathcal{H}^1_{\mathcal{X}/\mathcal{A}_g}).
\]

The Cartier isomorphism induces \( \mathcal{O}_{\mathcal{A}_g} \)-linear isomorphisms

\[
\varphi_0: (\mathcal{M}/\mathcal{C}^1)(p) \cong (R^1f_*\mathcal{O}_{\mathcal{X}})(p) \simto D_0
\]

\[
\varphi_1: (\mathcal{C}^1)(p) \simto f_*(\mathcal{H}^1_{\mathcal{X}/\mathcal{A}_g}) \cong M/D_0.
\]

Moreover, \( \mathcal{C}^1 \) and \( D_0 \) are both totally isotropic with respect to the symplectic pairing \( \beta \).

We obtain a morphism

\[
\pi: \mathcal{A}_g \longrightarrow [G\setminus Z_{G,J}]
\]

where \( G \) and \( J \) as defined in [BE]. Here the right hand side is the quotient in the sense of stacks. As the \( G \)-orbits of \( Z_{G,J} \) are parametrized by elements in \( J/W \), the underlying topological space of \([G\setminus Z_{G,J}]\) is in bijection with \( J/W \) and for \( w \in J/W \) we denote the corresponding point in \([G\setminus Z_{G,J}]\) by \( C(w) \). It is the image of the corresponding orbit \( Z^w_{G,J} \). Moreover, a point \( C(w) \) is contained in the closure of \( \{C(w')\} \) if and only if \( Z^w_{G,J} \subset Z^{w'}_{G,J} \) (see e.g. [Wd1] (4.4)). Each \( \{C(w)\} \) is locally closed in the underlying topological space of \([G\setminus Z_{G,J}]\) and there is a canonical structure of a locally closed substack on \( \{C(w)\} \) [MW], Section 5.6. Indeed, this is nothing but the unique structure of a reduced substack on \( \{C(w)\} \). We denote this substack again by \( C(w) \). As all \( w \in J/W \) are already defined over \( \mathbb{F}_p \), these substacks are also defined over \( \mathbb{F}_p \).

For all \( w \in J/W \) we define a locally closed substack \( \mathcal{A}_g^w \) of \( \mathcal{A}_g \) by the cartesian diagram

\[
\begin{array}{ccc}
\mathcal{A}_g^w & \longrightarrow & C(w) \\
\downarrow & \Box & \downarrow \\
\mathcal{A}_g & \xrightarrow{\pi} & [G\setminus Z_{G,J}].
\end{array}
\]

Now let \( k \) be an algebraically closed field and let \( x: \text{Spec}(k) \rightarrow \mathcal{A}_g \) be a \( k \)-valued point of \( \mathcal{A}_g \). Then \( x \) corresponds to a principally polarized abelian variety \((X, \lambda)\) of dimension \( g \) over \( k \). Its first de Rham cohomology \( \mathcal{H} = H^1_{\text{DR}}(X/k) \) carries the structure \((\mathcal{C}^1, D_0, \varphi_0, \varphi_1, \beta)\) of a symplectic \( F \)-zip as above. On the other hand, the Dieudonné module \((M', F, V)\) associated to the \( p \)-torsion \( X[p] \) of \( X \) carries a symplectic pairing \( \beta' \) and we can associate a symplectic \( F \)-zip to \((M', F, V, \beta')\) as follows: We set \( C'^1 = VM' = \ker(F) \)
and \( D'_0 = FM' = \text{Ker}(V) \) and we denote by \( \varphi'_0: (M'/VM')^{(p)} \cong FM' \) and \( \varphi'_1: (VM')^{(p)} \cong M'/FM' \) the \( k \)-linear isomorphisms induced by the Frobenius linear maps \( F \) and \( V^{-1} \), respectively. Then these two symplectic \( F \)-zips \((M, C^1, D_0, \varphi_0, \varphi_1, \beta)(x)\) and \((M', C'^1, D'_0, \varphi'_0, \varphi'_1, \beta')(x)\) associated to \( x \) are canonically isomorphic.

If \( x, x' \): Spec(\( k \)) \( \rightarrow \mathcal{A}_g \) are two \( k \)-valued points, corresponding to principally polarized abelian varieties \((X, \lambda)\) and \((X', \lambda')\), respectively, then they both factorize through the same locally closed substack \( \mathcal{A}_g^w \) if and only if the symplectic \( F \)-zip structures induced on the first de Rham cohomology are isomorphic. By the above this is equivalent to the fact that the principally quasi-polarized Dieudonné module of their \( p \)-torsions are isomorphic. Hence we see that the \( \mathcal{A}_g^w \) for \( w \in J^W \) are nothing but the Ekedahl-Oort strata, defined in [Oo].

(6.4) Moreover, by [Wd2] (for \( p > 2 \) this follows also from [Wd1]) we have:

**Theorem.** The morphism

\[
\pi: \mathcal{A}_g \rightarrow [G\backslash Z_{G,J}]
\]

is faithfully flat.

(6.5) In particular the theorem implies that for \( w, w' \in J^W \), we have \( \mathcal{A}_g^w \subset \mathcal{A}_g^{w'} \) if and only if \( Z^w_{G,J} \subset Z^{w'}_{G,J} \). Therefore (5.4) implies:

**Corollary.** The following two assertions are equivalent:

(1) \( \mathcal{A}_g^w \subset \mathcal{A}_g^{w'} \).

(2) There exists \( u \in W_J \) such that \( u^{-1}w\delta(u) \leq w' \) with \( \delta(u) = w_0,Juw_0,J \).

**References**

[Bo] N. Bourbaki: *Groupes et Algèbres de Lie*, chap. 4,5 et 6, Masson (1981).

[Ca] R.W. Carter: *Finite Groups of Lie type*, Wiley Classics Library, John Wiley & Sons, Ltd., Chichester (1993).

[EMO] S. Edixhoven, B. Moonen, F. Oort: *Open problems in algebraic geometry*, Bull. Sci. Math. 125 (2001), 1–22.

[He] X. He: *The G-stable pieces of the wonderful compactification*, math.RT/0412302 v2.

[Lu1] G. Lusztig: *Parabolic character sheaves I*, Mosc. Math. J. 4 (2004), no. 1, 153–179.
[Lu2] G. Lusztig: *Parabolic character sheaves II*, Mosc. Math. J. **4** (2004), no. 4, 869–896.

[Mo] B. Moonen: *Group schemes with additional structures and Weyl group cosets*, Moduli of abelian varieties (Texel Island, 1999), 255–298, Progr. Math. **195** (2001), Birkhäuser.

[MW] B. Moonen, T. Wedhorn: *Discrete invariants of varieties in positive characteristic*, Int. Math. Res. Not. **2004:72** (2004), pp. 3855 – 3903.

[Oo] F. Oort: *A stratification of a moduli space of abelian varieties*, Moduli of abelian varieties (Texel Island, 1999), 345–416, Progr. Math. **195** (2001), Birkhäuser.

[Sp] T.A. Springer: *Intersection cohomology of $B \times B$-orbit closures in group compactifications*, Journal of Algebra **258** (2002), pp. 71–111.

[Wd1] T. Wedhorn: *The dimension of Oort strata of Shimura varieties of PEL-type*, Moduli of abelian varieties (Texel Island, 1999), 441–471, Progr. Math. **195** (2001), Birkhäuser.

[Wd2] T. Wedhorn: *Flatness of the mod $p$ period morphism for the moduli space of principally polarized abelian varieties*, preprint Bonn, August 2004.