Decoherence and dephasing in multilevel systems interacting with thermal environment

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We examine the effect of multilevels on decoherence and dephasing properties of a quantum system consisting of a non-ideal two level subspace, identified as the qubit and a finite set of higher energy levels above this qubit subspace. The whole system is under interaction with an environmental bath through a Caldeira-Leggett type coupling. The model interaction we use can generate nonnegligible couplings between the qubit states and the higher levels up to $N \sim 10$. In contrast to the pure two-level system, in a multilevel system the quantum information may leak out of the qubit subspace through nonresonant as well as resonant excitations induced by the environment. The decoherence properties of the qubit subspace is examined numerically using the master equation formalism of the system’s reduced density matrix. We numerically examine the relaxation and dephasing times as the environmental frequency spectrum, the environmental temperature, and the multilevel system parameters are varied. We observe the influence of all energy scales in the noise spectrum on the short time dynamics implying the dominance of nonresonant transitions at short times. The relaxation and dephasing times calculated, strongly depend on $N$ for $4 < N < 10$ and saturate for $10 < N$. We also examine double degenerate systems with $4 \leq N$ and observe a strong suppression (almost by two orders of magnitude) of the low temperature relaxation and dephasing rates.

An important observation for $4 \leq N$ in doubly degenerate energy configuration is that, we find a strong suppression of the RD rates for such systems relative to the singly degenerate ones. These results are also compared qualitatively with the relaxation rates found from the Fermi Golden Rule.

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I. INTRODUCTION

Currently a large number of model approaches are present for formulating the decoherence phenomena in the literature. The original Caldeira-Leggett model is based on a quantum system under the influence of a double well tunneling potential with a linear coupling to an infinite bath of harmonic oscillators. If the potential is sufficiently smooth and the high energy levels are sufficiently above the tunneling barrier this original model is normally represented as a two level system (2LS) interacting with a bosonic environmental bath (spin-boson model). An incomplete list of this wide literature is provided in \[1-3\]. Another popular model of decoherence is the central spin system in which central 2LS couples to large number of environmental two level systems. The pros and cons of these two rival models have been extensively studied.

Realistically, and aside from the genuine 2LS, a large majority of physical systems suggested for qubit is far from being ideal. In a quantum computational environment, the parameters of the physical systems are manipulated to perform the gate operations. For instance, in a multilevel system (MLS), short time pulses used in the manipulation of the states in the qubit subspace induce nonresonant transitions out of the qubit subspace. That nonresonant transitions contribute to the decoherence of the quantum system was recently addressed by Tian and Lloyd. They suggest that after these transitions are induced an optimized sequence of controlled pulses can be applied to cancel the nonresonant effects at arbitrary precision. The idea being physically correct, requires an additional fine knitting of error correction which undoubtedly costs computational time. On the other hand, one may address the same issue by seeking for an alternative solution: Can one understand the effect of the higher levels on decoherence in a well-parameterized MLS coupled to an environment?

The MLS can itself be manifestly $N$-levelled or a truncated approximation of a larger system with much higher number of levels. Well known examples of both cases have been known. For the former, organic molecules with certain discrete rotational symmetries and ground state low temperature configurations of single polymerized chains are good examples. The vibrational energy spectra of atoms and molecules is a good example for the latter.

These type of realistic MLS can be found for instance in already well-examined superconducting systems such as the rf-SQUID in the charge, flux or phase regimes. We remark however, that a concise treatment of the decoherence effects in MLS has not been fully developed. This work is planned to be a modest step forward in that direction.

In section II we give an introduction of the model MLS used in the present work. Here we merely concentrate on the properties of the environmentally induced dipole matrix elements. Section III recalls the reduced density matrix (RDM) master equation formalism and adapts it for the coupling of the MLS to the environment. The noise correlator, which is considered to be in thermal equilibrium, and the system-noise kernel, for which no Markovian assumption is made, are defined in section III.A. The results are presented together with the car-
lier observations of the 2LS (in section III.B) to allow a
comparison with some of the established facts. The MLS
with three or higher levels are examined in section III.C
separately for \( N = 3, N = 4 \) and \( 4 < N \). The section
III.C and the following section IV comprise most of the
original results of the manuscript. In section IV, the re-
relaxation times for MLS, which we produced numerically
in section III, are compared with the estimates found by
using the semianalytic Fermi Golden Rule (FGR).

## II. THE MULTILEVEL SYSTEM

In principle the majority of the well established meth-
ods (particularly the influence functional) used in the
literature automatically accomodate multilevel dynam-
ics. The results are generally hoped to be true for 2LS in
the WKB limit at sufficiently low temperatures. Such
techniques are also often preferable since they allow ex-

cplicit analytic expressions for the decoherence times as
functions of the system’s parameters. On the other hand,

exact methods are also available on pure 2LS.

However, we need to accomodate in our parameters explicit
degeneracy factors as well as the Hilbert space dimensions.

For the latter, the influence functional formalism has no
parametric dependence. In this work, in order to retain in
our calculations the dependence on the system’s Hilbert
space dimension, we resort to the system’s eigenenergy
basis representation.

Our MLS model is an rf-SQUID operating in the quan-
tum coherence regime given by the Hamiltonian

\[
H_s/(\hbar \Omega_0) = \frac{1}{2} \left[ -\partial_z^2 + (z - \phi_{\text{bias}})^2 \right] + \beta \cos(\gamma z) \tag{1}
\]

where \( \Omega_0 = 2\pi/\sqrt{LC} \) is the harmonic energy with \( L \) be-
ing the inductance of the SQUID loop and \( C \) is the effec-
tive capacitance of the Josephson junction, \( \beta = E_J/\hbar \Omega_0 \)
the dimensionless ratio of the Josephson energy \( E_J \)
to the harmonic energy, \( \gamma = \hbar \sqrt{\frac{2}{\Phi_0}} \) is a dimen-
sionless scale parameter, \( \phi_{\text{bias}} = 2\pi \Phi_{\text{bias}}/\Phi_0 \) is the
effective bias in the flux which is applicable in a current
bias junction, and \( z = 2\pi \Phi/(\gamma \Phi_0) \) is the flux \( (\Phi) \)
dependent dimensionless dynamical variable \( (\Phi_0 = \hbar c/2e \)
is the superconducting flux quantum). Truncating the
Hilbert space dimension at \( N \) in the energy eigenbasis,
the system Hamiltonian in (1) is written in the diagonal
form

\[
H_s = \sum_{n=0}^{N-1} E_n(\{\zeta\}) |\{\zeta\}, n\rangle \langle \{\zeta\}, n|
\tag{2}
\]

where \( \{\zeta\} \) describes the set of system parameters
\( \Omega_0, \beta, \gamma, \phi_{\text{bias}} \) where \( E_n(\{\zeta\}) \), and
\( |\{\zeta\}, n\rangle \) are respectively the parameter dependent
eigenenergies and eigenvectors of the MLS. This set of parameters is sufficiently
general to accomodate for all possible effects including
the degeneracy in the qubit subspace, the symme-
try of the wavefunctions etc. [we define the degener-
cy factor by \( \eta = (E_2 - E_1)/(E_1 - E_0) \) for MLS and
\( \eta_2 = (E_1 + E_0)/(E_1 - E_0) \) for 2LS]. These three pa-
rameters \( \Omega_0, \beta, \phi_{\text{bias}} \) control the high energy harmonic
spectrum, the low energy anharmonic spectrum and the
reflection symmetry of the rf-SQUID potential respective-
ly. At low energies, a simple numerical diagonalization
(1) reveals that there are low lying eigenenergy
configurations within the double well regime in which the
SQUID potential is strongly anharmonic. The param-
eters of the potential can therefore be manipulated to
search within this regime for those configurations satisfi-
ing optimal qubit conditions. An interesting case here is
to find highly degenerate levels corresponding to the
first two eigenstates for the symmetric double well po-
tential (i.e. \( \phi_{\text{bias}} = 0 \)). This particular case has been
extensively examined previously for 2LS using semi-
classical methods with an arbitrarily weak tunneling be-
 tween the wells. Another configuration that turns out to be
important in our calculations is the doubly degenerate
(DD) configuration for systems with \( 4 \leq N \) in which
the first four levels are pairwise degenerate with large
degeneracy factors. The double-well potential and ener-
gies corresponding to both SD and DD configurations are
shown in Fig’s (1) and (2) respectively.

We numerically find that, the relaxation/dephasing
(RD) times for the MLS can be controlled by the degener-
acy \( \eta \). Normally, degeneracy is also crucial in control-
ning the dynamical tunneling rates. In our calculations how-
ever we directly use the system eigenstates. Therefore for
the highly degenerate configurations bare tunneling be-
tween symmetric and anti-symmetric parts of the wave-
functions can be neglected to a large extend. This turns

![FIG. 1: The double-well potential and the eigenenergy config-
urations corresponding to the singly degenerate (SD) case. Here
the numerical values of the dimensionless parameters for
this SD configuration are \( \beta \approx 1.616 \) and \( \gamma \approx 1.753 \). The
harmonic energy scale and the degeneracy factors are respecti-
vely \( h\Omega_0 = 10^{-5}eV \) and \( \eta_2 = 10^6 \). Here the numerical values
of the dimensionless parameters are \( \beta \approx 1.616 \) and \( \gamma \approx 1.753 \).](image)
out to be especially important for quantum computation in the sense that once the computation is finished the wavefunction can be maximally localized in one of the double wells before any measurement or read-out process.

Although we use the truncated rf-SQUID as the N-level model quantum system to study decoherence effects, our treatment is not at the microscopic level. The rf-SQUID is shown to be an ideal model to study multilevel effects due to the fact that, the transitional dipole couplings between the low lying energy states and the high levels are nonnegligible [see Fig.(3)]. Any other physical Hamiltonian with sufficient number of adjustable parameters as well as nonnegligible dipole couplings would be suitable for the calculations presented here.

In the rest of the paper the harmonic energy \( \Omega_0 = 10^{-3} \) eV, \( \eta_2 \approx 10^5 \), \( \eta \approx 3 \times 10^6 \).

### A. Coupling to Noise

The system-noise interaction is considered to be a Caldeira-Leggett type inductive coupling between the SQUID’s macroscopic flux coordinate \( z \) and the environmental flux-like coordinate \( \hat{\varphi}_c \) expanded in harmonic environmental modes as \( \hat{\varphi}_c = \sum \eta_k (b^\dagger_k + b_k) \). The system noise interaction is simply \( \mathcal{H}_{int} = \frac{\alpha}{2} z \hat{\varphi}_c \) where \( \alpha \) is some number representing the strength of the inductive coupling (\( \alpha \) is to be normalized by \( \Omega_0 \) for a dimensionless coupling). In the diagonal basis \( |\{\zeta\},n\rangle \) of the model system the interaction Hamiltonian is given by

\[
\mathcal{H}_{int} = \frac{\alpha}{2} \sum_{r,s=0}^{N-1} (z)_{rs} |s(\zeta)\rangle \langle r(\zeta)| \hat{\varphi}_c , \tag{3}
\]

Here \( (z)_{rs} = \langle \{\zeta\} | r| \{\zeta\} s \rangle \) are the noise induced perturbative dipole matrix elements of the macroscopic system coordinate \( z \) in the MLS's diagonal basis in (2). For the model MLS described by (1), the dipole matrix elements are real and symmetric.

The rf-SQUID poses a general example in which the multilevelledness of the master system manifests itself by finite dipole transition matrix elements for both the symmetric (i.e. \( \varphi_{bias} = 0 \)) and asymmetric (i.e. \( \varphi_{bias} \neq 0 \)) potential configurations. In the symmetric contribution even parity transitions vanish which results in manifestly off-diagonal system-noise coupling. Physically, this is in contrast to the most popular models used in the literature. On the other hand, when the potential is tilted, the parity selection no more holds by which finite diagonal couplings are also created.

In Fig.(3) the noise induced couplings for an asymmetric potential is plotted as a function of the truncated Hilbert space dimension. Data indicate that the induced dipole strengths between the qubit and the higher energy states are comparable to those among the qubit states. Therefore, the high energy transitions cannot be trivially ignored. The high energy transitions normally appear as a result of resonant interactions with the high energy sector of the noise spectrum under long interaction times. However, in the reduced system these transitions appear in the short time dynamics as well, and the short time dynamics is dominated by the nonresonant processes. Considering that decoherence is dominantly affected by the short time behaviour, the nonresonant
processes are expected to have observable effects in the decoherence properties of the RDM. Indeed we observe these effects in the solution of the master equation for the MLS (section III.B and C).

The next is to consider the environmental spectrum and the availability of the bath frequencies for these excitations. Regarding this, we consider a thermal Gaussian environment spectrum

\[ I(\omega) = \omega^{1+\nu} e^{-\omega^2/4\Lambda^2} \coth(\omega/2T) \quad (4) \]

where \( \Lambda \) is the effective noise cutoff frequency and \( \nu \) describes the subohmic (i.e. \( \nu < 0 \)), superohmic (i.e. \( \nu > 0 \)), and ohmic (i.e. \( \nu = 0 \)) character of the spectrum. The three environmental parameters \( \nu, \Lambda, T \) determine the sectors of the spectrum where the system-noise coupling is most effective. For \(-1 \lesssim \nu \) (extreme subohmic), two regions are of particular importance: a) at sufficiently low temperatures and high cutoff corresponding to \( T \ll \omega < \Lambda \), the dominant mechanism of relaxation is through spontaneous deexcitations. We call this region region-I; b) at high temperatures and high cutoff the region \( 0 \leq \omega \ll \min(\Lambda, T) \) provides a wider range of strong environmental couplings which we call as region-II. If the character of the spectrum is more like ohmic or superohmic, i.e. \( \nu \approx 0 \) or \( \nu \approx 1 \) respectively, there is lesser room for deexcitations as the availability of the low energy modes is suppressed. Therefore, in the ohmic and superohmic regimes, the region-II dominates the RDM phenomena.

Another feature of (4) is related to the majority of critical crossover behaviour in the vicinity of \( \nu = 0 \) as predicted earlier by Leggett et al. and depicted in Fig.4. In this figure, the \( \nu \approx 0 \) is a critical vicinity in the Ohmic region separating the subohmic \( \nu < 0 \) regime from that \( 0 < \nu \). In the subohmic regime \( I(\omega)/(2T)^{1+\nu} \) is very small except for vanishingly small frequencies (\( \omega/2T \ll 1 \)). Whereas, in the regime \( 0 < \nu \) the maximum value of \( I(\omega) \) is observed at higher frequencies \( \omega \approx 2\Lambda \sqrt{(1+\nu)/2} \) with an intensity proportional to \( (\Lambda/T)^{1+\nu} \).

III. MASTER EQUATION AND THE REDUCED DENSITY MATRIX FOR THE MLS

In the study of decoherence effects due to the weak environmental influences, one conventional way is to calculate the time dependent RDM elements by solving the master equation. This formalism has been known since the independent works of Bloch, Redfield and Fano (BRF) on spin magnetic resonance and widely applied to the spin-boson systems for which many standard references exist. The standard BRF formalism assumes Markov conditions for the solution of the master equations, which leads to exactly solvable results for 2LS. However, the Markovian assumption is not free of drawbacks and that was questioned originally in and lately in as well as in the context of spin magnetic resonance and relaxation.

In this work, the system noise kernel is treated with its most general non-Markovian character. The time evolution of the RDM is obtained in the interaction representation by

\[ -i\hbar \frac{d}{dt} \hat{\rho}(t) = [\hat{\rho}(t), \hat{H}_{int}(t)] \quad (5) \]

where \( \hat{\rho}(t) \) denotes the interaction picture. In the context of decoherence, we will give more emphasis on short observational times in the solution of (5). A convenient way to proceed is then to apply the Born approximation in which the full density matrix is initially a product of the system and environmental ones (i.e. \( \hat{\rho}(0) = \hat{\rho}^{(S)}(0) \otimes \hat{\rho}^{(n)}(0) \)) and at any later and short time approximately separates as \( \hat{\rho}(t) \approx \hat{\rho}^{(S)}(t) \otimes \hat{\rho}^{(n)}(0) \).

The exact iterative solution of (5) including the second order in the interaction with the partial trace performed over the environmental degrees of freedom yields for the RDM the integro-differential equation

\[ \frac{d}{dt} \hat{\rho}_{nm}^{(S)}(t) = -\int_0^t dt' \sum_{r,s} K_{rs}^{nm}(t,t') \hat{\rho}_{rs}^{(S)}(t') \]

in which we adopt the model interaction Hamiltonian \( \hat{H}_{int} \) for the system-noise kernel which is found to be

\[ K_{rs}^{nm}(t,t') = \frac{2}{\pi} \left\{ F(t-t')(|\tilde{z}_t|^2)_{s,m} - (\tilde{z}_t)_{s,r} (\tilde{z}_t)_{s,m} \right\} \]

Note the the kernel depends on two times as a signature of the non Markovian treatment and it is not time translationally invariant. Here \( F(t-t') = F^*(t'-t) \) is the

\[ F(t-t') = F^*(t'-t) \]
complex noise correlation function

\[ F(t-t') = \mathcal{F}_E \{ \frac{\tilde{\varphi}^{(n)}(t) \tilde{\varphi}^{(n)}(t')}{\varphi^{(n)}(0)} \} \]

and

\[ \tilde{z}_t = \sum_{k,\ell=0}^{N-1} (z)_{k\ell} e^{i(E_k-E_\ell)\frac{t}{\tau_R}} \langle k | \ell \rangle \]

is the time dependent dipole operator in the interaction picture where \( E_k, |k\rangle \) comprise the eigensolution of the model system. Expanding the noise field \( \tilde{\varphi}_\alpha \) in the independent harmonic modes and calculating \( F \) in thermal equilibrium one obtains the standard thermal noise correlator

\[ F(t-t') = 2 \sum_k \eta_2^2 \left[ \coth(\omega_k/2T) \cos \omega_k(t-t') - \sin \omega_k(t-t') \right] \]

The Markovian versus non Markovian character of the solution of (8) is determined in the weak system-noise interaction limit by the competition of three time scales: \( \tau_B \), noise correlation time scale, \( \tau_R \) and \( \tau_{dep} \), the relaxation and dephasing time scales of the reduced system respectively. The noise correlation time scale is found roughly from the thermal Gaussian bath spectral width as \( \tau_B \approx 1/\Lambda \). At the Markovian limit, the environmental correlation time \( \tau_B \approx \Lambda^{-1} \) is much smaller than the RD times. For two level systems this condition can be met provided that the system noise-coupling is sufficiently small. However, for MLS, the question of whether the Markovian condition is satisfied is more nontrivial. The basic reason is that in the MLS there is a larger number of time scales and decay channels of which presence may considerably reduce the effective decoherence times.

In this work, the numerical solution of (8) is performed by discretizing time in steps \( \Delta t = 10^{-2} \Lambda^{-1} \). The Hermiticity and the normalization of the RDM at each time step is maintained within an accuracy of 10^{-25}.

A. The system-noise Kernel

In the model Hamiltonian defined in (11) all energies and time scales (particularly the RD times) are given in units of \( \hbar \Omega_0 \) and \( \Omega_0^{-1} \) respectively. The parameters affecting the numerical results are, \( \nu, \Lambda, T \) for the thermal noise, \( \alpha \) for the system-noise bare coupling and the dipole matrix elements \( (z)_{nm} \) for the pure MLS. The noise spectrum is assumed to be continuous of which the real part is responsible for RD effects and is given by the spectral density in (9). In the Markovian limit the imaginary part of \( F(t-t') \) is vanishingly small and the resulting Lamb-type energy renormalization effects are negligible.

In the numerical calculations however we include the full complex noise correlation function as \( \tilde{z}_t \).

FIG. 5: Time dependence of the RDM in units of \( \Omega_0^{-1} \) for various representative \( \alpha, \nu \) parameter sets at \( T = 0 \) and \( \Lambda = 0.1 \). For the model system the potential is symmetric and the bare TLS is in a SD configuration.

\[ F(t-t') = 2 \int_0^\infty \rho e^{i \omega t} \left[ \coth(\omega/2T) \cos \omega(t-t') - \sin \omega(t-t') \right] \]

Inserting (11) in (7) we obtain the system-noise kernel for our model. An upper frequency cutoff of \( \omega_{max} = 5\Lambda \) is used in the numerical integral in (11).

B. Overview of the 2LS results

We now examine the time behaviour of the RDM in (12) for a 2LS. The solution is shown in Fig 5 on the logarithmic scale for a few representative parameters and for the SD configuration. The degeneracy parameter is \( \eta_2 \sim 10^6 \). We also fixed \( \alpha = 0.01 \) in the rest of the work unless otherwise stated.

In Fig. 4 the first observation is that, exponential RD is effective immediately in the short time regime \( t < 20\Omega_0^{-1} \). We also confirmed numerically that the asymptotic time behaviour as well as the RD rates are independent of the initially prepared state. As the asymptotic time behaviour is concerned, for symmetric configurations (pure \( \sigma_z \) coupling), the density matrix converges to the maximum entropy (informationless) limit \( I/2 \), where \( I \) is the unit matrix, irrespectively from the spectral properties of the noise or the system-noise coupling. The results also indicate that the relaxation time scale \( \tau_B \) (read from the filled symbols) and the dephasing time scale \( \tau_{dep} \) (read from the hollow symbols) are compatible. This result is in agreement particularly with the recent exact 2LS calculations using the path integral influence functional formalism 5.

We also confirm that all regions in the noise spectrum have strong influence on RD. For this observation,
one has to compare Fig’s. correponding to different spectral properties and temperatures. For instance, for \( \alpha = 0.01, \Lambda = 0.1 \) and \( T = 0 \) [see Fig.], we recover the under damped and weak dephasing limit of \( \alpha \) for all \( \nu \). In addition, no oscillations are observed in the SD configuration.

Larger RD rates are observed as \( \nu \) is made to be more negative towards \( \nu = -1 \). We identify this behaviour as the manifestation of the region-II in the noise spectrum (see the end of section II.A). More data are also shown in the same figure for indicating the effect of various \( \alpha \) values. In Fig. the nondegenerate case with an energy difference between the levels \( \Delta E = 1.2 h \Omega_0 \) is shown for \( \alpha = 0.01, \nu = -1, \Lambda = 0.1 \) and for various temperatures. The \( \nu = 0 \) and \( \nu = 1 \) curves again yield weaker RD rates within the indicated temperature ranges. Another noticeable feature in Fig. is the weakly oscillating behaviour. The weak oscillations are more prominent at high temperatures and short times. In this case for \( \omega \ll T \) the system-noise coupling is large due to the large number of thermally activated environmental modes. This behaviour, which is characterized by weakly damped Rabi-like oscillations, was predicted in the analytic calculations of Leggett et al. In order to examine the influence of the spectral width \( \Lambda \) a similar zero temperature plot as in Fig. is made in Fig. for a wider spectral width \( \Lambda = 1 \). Comparing this figure with Fig. a crossover in the time dependence of the RDM can be observed in the \( \Lambda - \nu \) plane. (The crossover can also be activated thermally as to be seen in the following Fig.’s and)

There are a number of ways to increase the effective system-noise coupling. The following Fig. gives a sample from the \( \alpha - T \) behaviour for the asymmetric potential configuration. Increasing the temperature increases the coupling by filling the available photon modes for \( \omega < T \). The sample data is shown for \( T = 0, 1, 5 \) at \( \alpha = 0.01, \Lambda = 5 \) and \( \nu = -1 \) (indicated by the filled and opaque symbols connected with solid lines). The second way to increase the system-noise coupling is to directly increase \( \alpha \) (indicated by the dotted dashed lines). In the small coupling regime, for which the sample data is shown for \( \alpha = 5 \times 10^{-4}, 10^{-2}, 5 \times 10^{-2} \) at \( T = 0 \) \( \nu = -1 \) and for an increased \( \Lambda (\Lambda = 5) \), the RDM experiences stronger damping. The weakly damped oscillations survive at short times at finite and small temperatures. The larger the temperature the larger the amplitude of the oscillations and the faster they diminish.

For completeness we also add in Fig. the behaviour in the \( \Lambda - \nu \) plane at zero temperature. A comparison between the Fig.’s and reveals a temperature modulated crossover (confirm, for instance, a similar decay of the sets for \( \Lambda = 5, \nu = -1 \) at \( T = 5 \) in Fig. with \( \Lambda = 5, \nu = 1 \) at \( T = 0 \) in Fig.).

The major difference of the model interaction Hamiltonian in from the standard (\( \sigma_z \)-type) spin-boson model is in the manipulation of the potential. In contrast to the standard spin-boson model, in our case only non-diagonal, \( \sigma_x \), type coupling is present under the symmetric potential [see Fig.]. As a result, dramatic differences in the time dependence of the reduced system are observed between the two models. For instance, the diagonal coupling is standardly considered for the study of pure dephasing. In this type of coupling the relaxation is manifestly forbidden and the initial states do not change their populations. The diagonal coupling also yields strongly temperature dependent dephasing rates with the rates vanishing at \( T = 0 \). On the other hand, when the system-noise coupling is not diagonal, the induced transitions between the system states
can probe the entire noise spectrum creating decoherence even at zero temperature. These induced transitions are nonresonant and they have observable effects particularly in the short time dynamics of the RDM. The RD times observed as the result of such system-noise coupling are expected to be nonzero even at zero temperature. This characteristic behaviour of the non-diagonal coupling is confirmed in our calculations both for the 2LS in Figs. 4 and for the MLS in the following sections. Recently, there are other claims using realistic models on decoherence effects in mesoscopic systems as well as a few experimental confirmations on the saturation of the RD rates at low temperatures.

A curious observation in Fig’s 5-9 is the strong dependence of RD time scales on the spectral width $\Lambda$. A naive expectation is that for $\Delta E \ll \Lambda$ and at very small temperatures the resonant transitions are unfavoured and there are no environmental modes available therefore the relaxation should be inhibited. The point that is often missed in this popular argument is the different role played by the short time nonresonant transitions. The resonant transitions are favoured when the system interacts with the environment at sufficiently large times. The system however relaxes differently at short times by preferring to stay off-resonant in its interaction with the noise field thereby sampling all regions of the noise spectrum. This causes the strong dependence on $\Lambda$ we observe at short times.

In summary, it is confirmed that the rich transient effects are observed usually in the short time behaviour in which all energy scales in the noise spectrum take part rendering the relaxation process sensitive to the relative magnitudes of those scales. We confirmed the several crossover regions that have been predicted in the path integral influence functional calculations.

The decoherence and dephasing dynamics is governed by all frequency regions in the noise spectrum. In particular, the short time behaviour is affected strongly by all frequency regions due to the nonresonant processes. For $\Lambda \ll 1$, Fig 4 indicates that as $\nu$ increases in the interval $-1 \leq \nu \leq 1$ the relaxation rates gradually increase still remaining in the weak relaxation regime. A crossover in the $\nu - \Lambda$ plane is observed [compare with Fig 7] as $\Lambda$ is increased. With this being the case for symmetric potentials, for asymmetric ones the additional feature of weak oscillations are present in the short time dynamics.

C. MLS with $3 \leq N$

The effect of multileveledness on decoherence has not yet received the attention that it deserves in the literature. This is, in part, due to the lack of practical analytic tools in solving the master equation. The complexity of the formal methods such as the noninteracting blip approximation increases at each time step as $N^2$ which renders the analytic sum over all virtual configurations in the path integral approach intractable. Usually, the common argument is that, for sufficiently low temperatures, a multilevel system, of which the first two levels (the qubit) are sufficiently well separated from the rest, behaves as a two state system. We have already observed that there are two major pitfalls in this assumption. Firstly, it excludes the very realistic case in which decohering effects are induced through interactions with a strongly fluctuating quantum field. In such a case, the fluctuations in the distribution of environmental modes in the noise spectrum is the major source of decoherence. Secondly, and more generally, the short time behaviour of a MLS -which is the most prominent regime in the quantum computational perspective- is affected by a large frequency region in the noise spectrum. These comprise the basic motiva-
and $\nu$ and the solids ones refer to $\rho$ $T$ $\rho$ $|_{\nu - 10}$ on the time dependence of the elements $\rho_{00}$, $\rho_{11}$ and $|\rho_{01}|$ between the 2LS and 3LS. The fixed parameters are $T = 0$, $\alpha = 0.01$ and $\nu = -1$. The open symbols refer to the case $N = 2$ and the solids ones refer to $N = 3$. Also the solid lines are the diagonal elements $\rho_{00}$ (with $\rho_{00}(0) = 0.1$) and $\rho_{11}$ (with $\rho_{11}(0) = 0.9$), the dotted-dashed lines are the non-diagonal ones $|\rho_{01}|$ (with $|\rho_{10}(0)| = 0.3$). More specifically, circle is $\Lambda = 1$, square is $\Lambda = 10$. The inlet is the case $\nu = 1$ and only $\Lambda = 10$ is displayed for simplicity.

![FIG. 10: Comparison of the effect of the spectral width Λ on the time dependence of the elements $\rho_{00}$, $\rho_{11}$ and $|\rho_{01}|$ between the 2LS and 3LS. The fixed parameters are $T = 0$, $\alpha = 0.01$ and $\nu = -1$. The open symbols refer to the case $N = 2$ and the solids ones refer to $N = 3$. Also the solid lines are the diagonal elements $\rho_{00}$ (with $\rho_{00}(0) = 0.1$) and $\rho_{11}$ (with $\rho_{11}(0) = 0.9$), the dotted-dashed lines are the non-diagonal ones $|\rho_{01}|$ (with $|\rho_{10}(0)| = 0.3$). More specifically, circle is $\Lambda = 1$, square is $\Lambda = 10$. The inlet is the case $\nu = 1$ and only $\Lambda = 10$ is displayed for simplicity.](image)

The asymptotically long time dynamics is independent of the system-noise parameters. In sufficiently long time the system looses all the information that is put in the initial state: $\rho_{kk}(t \to \infty) = 1/3, (k = 0, 1, 2)$ and $|\rho_{kj}(t \to \infty)| = 0, (k \neq j)$.

$N = 4$ case

We compare the 4LS with a 2LS in Fig. 11 for $\nu = -1$. Here, we have three sets of curves indicated by (a), (b) and (c). In Fig.11a the 4LS is compared to 2LS when both systems are highly degenerate. In Fig.11b the four level system is doubly degenerate and at zero temperature. The third set of curves are plotted in Fig.11c corresponding to the DD configuration at finite temperatures. For the singly degenerate (SD) case, the qualitative features between the 2LS and the 4LS are similar to the previously discussed case between 2LS and the 3LS. Here, we observe for the diagonal elements, a much higher relaxation rate (as well as leakage) out of the qubit subspace during the observed time although the dephasing rates are indistinguishable for the 2LS and the 4LS. The RDM asymptotically approaches to the informationless limit $\hat{\rho} = I/4$.

For $N = 4$ we have the chance to look at the DD configuration as depicted in Fig. 12. For the doubly degenerate configuration, the degeneracies are as high as $\eta_2 = (E_2 - E_1)/E_1 \approx 10^6$ and $\eta_1 = (E_3 - E_2)/E_2 \approx 10^6$. We surprisingly observe a tremendous suppression (by almost two orders of magnitude) at sufficiently small temperatures in the RD rates (Fig11b). The rates and the DD-suppression strongly depend on the temperature. A comparison of the $T = 0, \Lambda = 1$ in Fig11b and $T = 1, \Lambda = 1$ in Fig11c can reveal this strong dependence.

$4 < N$ case

However, these transitions are suppressed by the Gaussian cutoff. As a result, the 3LS is basically confined to its highly degenerate qubit subspace. This confinement can be observed all the way up to much higher spectral widths such as $\Lambda = 1$ as long as the Gaussian suppression is manifested. This behaviour is shown in Fig10 for $\Lambda = 1$ and $\Lambda = 10$. For considerably long duration (i.e. $\sim 100 \times \Omega_0^{-1}$) and for $\Lambda = 1$ the qubit subspace in the three level system has a negligible leakage into the third level. For much larger $\Lambda$ such as $\Lambda = 10$, the third level is allowed to participate in the transitions. The RD rates are therefore found to be significantly larger than that of the $N = 2$ case before.

We will continue by examining decoherence in a MLS. We will continue by examining decoherence in a MLS. We will continue by examining decoherence in a MLS. We will continue by examining decoherence in a MLS. We will continue by examining decoherence in a MLS.
For the spectral width $\Lambda$ on the time dependence of the elements $\rho_{00}, \rho_{11}$ and $|\rho_{01}|$ between the 2LS and 4LS. The symbols are the same as in Fig.10. The figure describes the singly degenerate case $\eta \approx 10^3$; (b) the rates at $T = 0$ and $\nu = -1$ for the doubly degenerate (denoted by DD in the figure title) configuration $\eta_2 = (E_2 - E_1)/E_2$ and $\eta_1 = (E_1 - E_5)/E_2 - E_1)\approx 10^6$.

FIG. 11: (a) Comparison, at $T = 0$ and $\nu = -1$, of the effect of the spectral width $\Lambda$ on the time dependence of the elements $\rho_{00}, \rho_{11}$ and $|\rho_{01}|$ between the 2LS and 4LS. The symbols are the same as in Fig.10. The figure describes the singly degenerate case $\eta \approx 10^3$; (b) the rates at $T = 0$ and $\nu = -1$ for the doubly degenerate (denoted by DD in the figure title) configuration $\eta_2 = (E_2 - E_1)/E_2$ and $\eta_1 = (E_1 - E_5)/E_2 - E_1)\approx 10^6$.

In order to extract some quantitative numbers for the RD times we made use of the numerical observation that for weak system-environment coupling the time dependence is approximately exponential at short times. We then follow and write for the time dependence of a general matrix element at short times

$$|\rho_{ij}(t)| \simeq |\rho_{ij}(\infty)| + |\rho_{ij}(0)| - |\rho_{ij}(\infty)| exp(-t/\tau_{ij}) .$$

The RD times are extracted from the time dependence of $\rho_{11}(t)$ and $|\rho_{10}(t)|$ respectively as

$$\tau_{ij}^{-1} \simeq \frac{1}{1 - |\rho_{ij}(\infty)/\rho_{ij}(0)|} \frac{d\ln|\rho_{ij}|}{dt} |_{t=0} \tag{13}$$

where $i = j = 1$ is used in the calculation of the relaxation rate and $i = 0, j = 1$ is for the qubit dephasing rate corresponding to the first excited level. For the RDM at asymptotic times we have $\rho_{11}(\infty) = 1/N$ and $|\rho_{01}(\infty)| = 0$. The equation \ref{eq:13} breaks down when $|\rho_{ij}(0)| = |\rho_{ij}(\infty)|$ which we stay away by appropriately choosing $\rho_{ij}(0)$. In Fig.12 the data are represented at zero temperature and $\nu = 0$. To be used in \ref{eq:13}, the initial conditions are set at $\rho_{00}(0) = 0.2, \rho_{11}(0) = 0.8, \rho_{10}(0) = 0.4i$ with all density matrix elements outside the qubit subspace zero. Three different curves stand for (bottom to top) $\Lambda = 0.1, 1, 10$ with the open symbols corresponding to dephasing and the solid ones to the relaxation rates. Each set of data is shown for SD as well as DD configurations separately. Also note that the vertical axis is logarithmic.

Let us concentrate first on the SD configurations in Fig.12. In a large $\Lambda$ range $N = 4$ and $N \sim 10$ appear to be two crucial points. For $4 < N$ relaxation is approximately twice faster than dephasing and both rates rapidly saturate near $N \sim 10$ and they are independent of $N$ for $10 < N$. The onset of saturation is naturally model dependent. In our case this onset coincides with the range of strong dipole transition matrix elements of the model in (1) (see Fig.3). Turning to the DD configurations, we observe that for the same environmental parameters and for all $N$, decoherence rates for the DD case are strongly suppressed by nearly two orders of magnitude as compared to the SD configuration.

With this section we conclude the RD calculations for the interaction between the system and the thermally equilibrated noise. We now focus our attention on the calculations of the transition rates by a different approach, the Fermi Golden rule.

FIG. 12: Relaxation and dephasing rates against the number of levels for different spectral widths at $T = 0$ and $\nu = 0$. Note the logarithmic vertical axis. Small symbols refer to the singly degenerate MLS and the larger symbols refer to doubly degenerate one. The open and solid symbols refer to dephasing and relaxation times respectively.

IV. FERMI GOLDEN RULE

The Fermi Golden Rule (FGR) provides a simple and qualitative tool to reproduce many of the features of the relaxation times that we observe in Fig.12. Quantitative agreement should not be expected between the Fig.12 and the FGR results. This is mainly due to the fact that the data produced in Fig.12 reflects the effects of the short time dynamics whereas, the FGR gives more accurate results for the long time resonant interactions. In comparing the time scales found by directly solving the RDM and by FGR, we keep the absolute time scales arbitrary and only compare the qualitative behaviour.
FIG. 13: Relaxation rates against the number of levels for the SD (solid symbols) and DD (open symbols) cases. We assume that the MLS is prepared at $t = 0$ in the first excited state $|\{\zeta\}, 1\rangle$. The probability that the system stays in the same state after interacting with the environment for a duration $t$ is

$$p_{\psi}(t) = \left| \langle \psi(0) | \exp\left[ -i \frac{\bar{\hbar}}{\hbar} \int_0^t dt' \tilde{H}_{\text{int}}(t') \right] | \psi(0) \rangle \right|^2 \quad (14)$$

where for our case $|\psi(0)\rangle = |\{\zeta\}, 1\rangle$. Including second order perturbation in the dipole couplings with an environment in thermal equilibrium, (14) can be written as

\[ 1 - p_{\psi}(t) = \frac{1}{2} \sum_s \varphi_{s\zeta n_0}^2 \int_0^t dt' \int_0^{t''} dt''' e^{i(E_{n_0} - E_s)(t' - t'')} F(t' - t'') \quad (15) \]

where $F(t' - t'')$ is the environmental correlation function given by (10). The relaxation rate $\tau_{\text{FGR}}^{-1}$ corresponding to the first excited state is found by the same method that is described in (3.37) therein.

The relaxation times found by the FGR are summarized in Fig. 13 and 14. In Fig. 13, the relaxation times are plotted against the same parameters as in Fig. 12 with the same symbols. The FGR data reproduce many of the features in Fig. 12. The first observation is the same offset at $N = 4$ and the saturation of the rates slightly above this offset with a rapid increase for $4 < N$. The second observation is that by increasing the spectral width, the relaxation rates can be increased by as much as an order of magnitude.

When the temperature is varied, we observe the same trend as in the previous figure as depicted in Fig. 14.

V. CONCLUSIONS

We examined the RD properties of a multilevel system using non-Markovian master equation formalism. It is shown that the short time behaviour of the density matrix is influenced by nonresonant transitions in the MLS receiving contributions from all frequency regions in the noise spectrum. For the model interaction used, the dipole transitions are nonzero within a finite range of levels. The RD times calculated within this model show a saturation within the same range largely independent from the system-noise parameters.

It is generally found that the decoherence effects in MLS are more pronounced than those in the 2LS. We observe that a distinct counter example is posed by the doubly degenerate MLS with $4 \leq N$. The RD rates are found to be highly suppressed in comparison with the singly degenerate or non-degenerate systems for the same system and environment parameters. These result were also confirmed using the transition rates found from the Fermi-Golden rule. At the first glance, this curious suppression of decoherence reminds us the decoherence free subspaces. Nevertheless, the arbitrariness of the parameters of the DD model, and in particular of the dipole matrix elements rules out the possibility whether any set of invariant states can form a decoherence free subspace under the coupling with the environment. In order to understand the true nature of this strong suppression, more formal and analytic methods must be developed for the DD systems.
VI. ACKNOWLEDGEMENTS

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