Polymer deformation and particle tunneling from Schwarzschild black hole

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Abstract

In this paper, we investigate a tunneling mechanism of massless particles from the Schwarzschild black hole in the framework of polymer quantum mechanics. According to the corresponding invariant Liouville volume, we determine the tunneling rate from Schwarzschild black hole by the polymeric quantization procedure. In this regard, we show that the temperature and tunneling radiation of the black hole receive new corrections in such a way that the exact radiant spectrum is no longer precisely thermal.

Keywords: Quantum tunneling; Schwarzschild black hole; Polymer quantization

1 Introduction

From the time that for the first time the minimal length idea in presence of quantum gravity was propounded, lots of endeavors have been made in this field and remarkable works has been introduced [1]. On this basis, the uncertainty principle of Heisenberg was generalized by fundamental measurable length which today it is known as generalized uncertainty principles (GUP) [2]. Some of the phenomenological concepts of GUP which have been studied for example in [3, 4, 5], show a meaningful compatibility with the other approaches of quantum gravity [6]. Especially in [5] a method is proposed to test the effects of quantum gravity on the thermodynamical properties of a sonic black hole or dumb hole. By this method the usual logarithmic correction to the entropy is generated with correct coefficient. As an inspiration from the primitive attempts of Snyder in 1947 who formulated the Lorentz invariant discrete space-time, a symplectic structure of phase space in none-canonical framework is formulated in the non-relativistic limit [7]. The results of the deformed phase space is very similar to the string theory which leads us to a new definition of GUP [8]. From a geometrical point of view, the authors in [9] have shown that GUP may be extracted from quantum geometry in which the quantum effects are encoded in the space-time geometry. They did this by considering an upper limit on the acceleration of massive particles and concluded that it affects the canonical commutation relations, deform them in the form of GUP.

Between the useful notions that also use a minimal length scale in their formalism, we can mention the polymer quantization [10], which uses the methods very similar to the effective theories of loop quantum gravity. In polymer approach of quantum mechanics a polymer length scale, $\lambda$, which shows the scale of the segments of the granular space enters into the Hamiltonian of the system to deform its functional form into a so-called polymeric Hamiltonian. This means that in a polymeric quantized system in addition of a quantum parameter $\hbar$, which is responsible to canonical quantization of...
the system, there is also another quantum parameter \( \lambda \), that labels the granular properties of the underlying space. This approach then opened a new window for the theories which are dealing with the quantum gravitational effects in physical systems such as quantum cosmology and black hole physics, see for instance [11] and [12] and the references therein.

Recently, Parikh and Wilczek have shown a smooth model for performing the semi-classical tunneling process from the event horizon of black holes (BH) by correcting the radiation of self-gravitation and incorporating the modified background geometry as a dynamical system [13]. Following this idea, though a number of recent papers have tried to extend this semi-classical tunneling method for various types of BHs such as rotating and charged BHs [14], less research is focused in high energy scales.

In this letter, incited by the above expressed, we want to investigate how the tunneling mechanism from the Schwarzschild black hole (S-BH) gets modifications due to the polymerization. Since the temperature of a BH can be written in terms of the tunneling rate, the corrections to the BH’s tunneling rate yield naturally modifications to its thermodynamics. To do this, we begin with a general form of a S-BH space-time. We then follow the Parikh-Wilczek procedure to evaluate the polymer corrected tunneling probability. So, after a brief review of the polymer representation and classical polymerization in section 2, we shall deal with the tunneling mechanism in polymer phase space in section 3. We summarize the results in section 4.

2 Polymerization

In polymeric formulation of quantum mechanics, the position space (labeled by the coordinate \( q \)) is presumed to be discrete. Indeed, this space consists of small segments with discreteness parameter \( \lambda \). This makes it impossible to define the generator of the displacement in a natural way and thus the momentum operator associated to \( q \) does not exist [15]. Nevertheless, it is possible to find an effective momentum in the semi-classical picture by means of the Weyl exponential operator (shift operator). To do this, let us consider a function \( f(q) \) which its derivative with respect to the discrete coordinate \( q \) may be estimated by utilizing the Weyl operator as [16]

\[
\partial_q f(q) \approx \frac{1}{2\lambda}[f(q + \lambda) - f(q - \lambda)] = \frac{1}{2\lambda}(e^{ip\lambda} - e^{-ip\lambda}) f(q) = \frac{i}{\lambda} \sin(\lambda p) f(q),
\]

and in a similar way, the second derivative will be

\[
\partial_q^2 f(q) \approx \frac{1}{\lambda^2}[f(q + \lambda) - 2f(q) + f(q - \lambda)] = \frac{2}{\lambda^2}(\cos(\lambda p) - 1) f(q).
\]

Having the above approximations at hand, we define the polymerization process for the finite values of the parameter \( \lambda \) as

\[
\hat{p} \to \frac{1}{\lambda} \sin(\lambda p), \quad \hat{p}^2 \to \frac{2}{\lambda^2}(1 - \cos(\lambda p)).
\]

One may extend these substitutions to the classical dynamical variables so that a classical theory is achieved through above mechanism but now without any attribution to the Weyl operator. This is what usually is called classical polymerization in the literature of quantum gravity [10], [16]

\[
q \to q, \quad p \to \frac{\sin(\lambda p)}{\lambda}, \quad p^2 \to \frac{2}{\lambda^2}[1 - \cos(\lambda p)].
\]
where the pair \((p, q)\) are variables of classical phase space. By means of this approach, a polymerized
classical system can be described by the Hamiltonian

\[
H_\lambda = \frac{1}{m\lambda^2} (1 - \cos(\lambda p)) + U(q).
\]  
(5)

The first outcome of the classical polymerization \([4]\) and its associated Hamiltonian \([5]\) is that the
momentum is periodic and its changing range should be according to \(p \in \left[-\frac{\pi}{\lambda}, +\frac{\pi}{\lambda}\right]\) which in the limit
\(\lambda \to 0\) recovers the usual range for the canonical momentum \(p \in (-\infty, +\infty)\).

2.1 Darboux chart

We may consider the phase space with coordinates \((q, p)\) as a two-dimensional symplectic manifold \(\mathcal{M}\) equipped with a closed nondegenerate 2-form \(\omega\) as its symplectic structure. According to the
Darboux theorem \([17]\), a local chart always exists in which this 2-form takes the canonical form

\[
\omega = dq \wedge dp.
\]  
(6)

By means of the polymer corrected Hamiltonian given by the relation \([5]\), one can write the time
evolution of the system with the help of the Hamiltonian vector field \(x_H\) which satisfies the equation

\[
i_x \omega = dH_\lambda.
\]  
(7)

Considering the effective Hamiltonian \([5]\) by solving above equation we reach to the equation

\[
x_H = \frac{\sin(\lambda p)}{m\lambda} \frac{\partial}{\partial q} - \frac{\partial U}{\partial q} \frac{\partial}{\partial p}.
\]  
(8)

Now, we are going to consider the system as a many particle system and to apply the above formalism
we need a statistical mechanics point of view. In this regard, we recall the Liouville theorem according
to which the volume of a 2D-dimensional symplectic manifold is given by

\[
\omega^D = (-1)^{D(D-1)/2} \frac{D!}{D \text{ times}} \underbrace{\omega \wedge ... \wedge \omega}_{D \text{ times}}.
\]  
(9)

In the special case of two-dimension manifolds, this volume coincides with the symplectic 2-form.
Therefore, for a one dimensional space with spatial volume \(L\), the total volume will be

\[
\text{Vol}(\omega^1) = \int \omega^1 = \int_L dq \times \int_{-\frac{\pi}{\lambda}}^{\frac{\pi}{\lambda}} dp = 2\pi \left(\frac{L}{\lambda}\right).
\]  
(10)

It is seen that in spite of the standard classical mechanics in which the total volume of the phase
space is infinite due to no restriction on the momentum of the test particles, here the total volume
of the phase space is finite because of an upper bound for the momenta in the classical polymeric
systems. In other word, a circle of radius \(\lambda^{-1}\) determines the topology of the momentum part of the
polymeric symplectic manifold \([18]\).

2.2 Noncanonical chart

In this subsection, to study the statistical aspects of the polymeric systems we adopt another approach
by introducing the noncanonical transformation
\[(q, p) \rightarrow (q', p') = \left( q, \frac{2}{\lambda} \sin \left( \frac{\lambda p}{2} \right) \right), \quad (11)\]

which transforms the Hamiltonian \( H_\lambda(q, p) \rightarrow H_\lambda(q', p') \) with

\[H_\lambda(q', p') = \frac{p'^2}{2m} + U(q'). \quad (12)\]

Although the Hamiltonian (12) has the standard form like the Hamiltonians in classical mechanics, the new momentum \( p' \) is bounded as \( p' \in [-\frac{2}{\lambda}, \frac{2}{\lambda}] \) due to the above transformation. In this sense the Hamiltonian (12) is different from the standard classical Hamiltonians. It is easy to see that the corresponding 2-form in this noncanonical chart takes the form

\[\omega = \frac{dq' \wedge dp'}{\sqrt{1 - (\lambda p'/2)^2}}. \quad (13)\]

Now, the Poisson bracket between the variables \( p', q' \) is given by

\[\{q', p'\} = \sqrt{1 - (\lambda p'/2)^2}, \quad (14)\]

which turns to its nondeformed counterpart \( \{q, p\} = 1 \) in the limit \( \lambda \rightarrow 0 \). Calculating again the Liouville volume based on the above approach yields

\[\text{Vol}(\omega^1) = \int_L dq' \times \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{dp'}{\sqrt{1 - (\lambda p'/2)^2}} = 2\pi \left( \frac{L}{\lambda} \right), \quad (15)\]

which is nothing but the result obtained before in Darboux chart (10). This is not an unexpected result since, the total volume of the phase space should be invariant over the symplectic manifold. In summary, to deal with the polymeric symplectic manifold, one may adopt two equivalent pictures: (i) the effective deformed Hamiltonian (5), the canonical symplectic structure in Darboux chart (6) and the standard canonical Poisson algebra; (ii) the Hamiltonian in the standard form (12), the noncanonical symplectic structure (13) and the nonstandard Poisson algebra (14). Though the trajectories on the polymeric phase space are the same in these two pictures, as we will see in the next section, working in the framework of the noncanonical chart is more admissible from the statistical point of view.

### 3 S-BH and the polymeric tunneling mechanism

According to Birkhoff’s theorem, the Schwarzschild metric is the most spherically symmetric, vacuum solution of the Einstein field equations. A Schwarzschild BH has no electric charge, \( Q \), or angular momentum, \( J \). In Schwarzschild coordinates, the line element for the Schwarzschild metric has the form

\[ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2 d\Omega_2^2, \quad F(r) = 1 - \frac{2M}{r}, \quad (16)\]

where \( d\Omega_2^2 \) is the line element on the 2-dimensional unit sphere. As is well known from the standard BH thermodynamics, its entropy \( S \) is given in terms of its horizon area \( A = 4\pi r^2 \) by \( S = \frac{A}{4} = \pi r^2 \).

Now, we obtain a detailed quantum tunneling calculation from the S-BH event horizon. According to [13][19], to describe the process of quantum tunneling where a particle moves in dynamical geometry and crosses through the horizon without singularity on its path, there should be a coordinates system...
in terms of which the metric is not singular at the horizon. In this sense, the above metric may be written as

\[ ds^2 = -(1 - \frac{2M}{r})dt^2 + 2\sqrt{2M} \frac{dt}{r} dr + dr^2 + r^2 d\Omega^2. \] (17)

The metric in these new coordinates is stationary, non-static and is regular at \( r = 2M \). The equation for the radial null geodesics for a (massless) particle is given by \( \dot{r} = \pm (1 - \sqrt{\frac{2M}{r}}) \), in which the outgoing and ingoing geodesics are identified by the upper and lower signs respectively.

Here, we are going to apply the Parikh-Wilczek method to evaluate temperature and tunneling radiation of BH in the framework of the polymer mechanism. So, assume the trajectory of a massless particle and consider the response of the background geometry to the radiated quantum of energy \( E \) with polymer space correction, i.e. \( \mathcal{E} \). Since the WKB approximation is valid near the horizon (19), for the classically prohibited region at the stationary phase, the tunneling probability as a function of the particle action takes the form

\[ \Gamma \sim \exp(-2\text{Im} \mathcal{I}) \approx \exp(-E/T). \] (18)

In order to compute \( \text{Im} \mathcal{I} \), we consider the radial null geodesics like an outgoing s-wave with positive energy which passes through the horizon outwards from \( r_{in} \) to \( r_{out} \). So, under condition: \( r_{in} > r_{out} \), where \( r_{in} = 2M \) and \( r_{out} = 2(M - \mathcal{E}) \), we have

\[ \text{Im} \mathcal{I} = \text{Im} \int_{r_{in}}^{r_{out}} p_r dr = \text{Im} \int_{r_{in}}^{r_{out}} \int_0^{p_r} dp_r' dr. \] (19)

After using the polymer-deformed Hamiltonian equation, we get

\[ \dot{r} = \{r, H\} = \{r, p_r\} \frac{dH}{dr} \bigg|_r. \] (20)

Since the Hamiltonian can be written as \( H = M - \mathcal{E}' \), we may use the approximation \( p^2 \simeq \mathcal{E}'^2 \Rightarrow p \simeq \mathcal{E}' \), so, we have

\[ \text{Im} \mathcal{I} = \text{Im} \int_{M}^{M - \mathcal{E}} r \int_{r_{in}}^{r_{out}} \frac{dH}{r} dr \int_{r_{in}}^{r_{out}} \frac{dr(-d\mathcal{E}')}{1 - \sqrt{2(M - \mathcal{E})}}. \] (21)

Now, we incorporate quantum gravity effects encoded in the presence of the polymer phase space which is derived from [7, 8]. To obtain the corresponding measure of integrals, one needs the Jacobian of the transformation (11) which is given by [20]

\[ \int dr d\mathcal{E} \rightarrow \int \frac{dr d\mathcal{E}}{J(r, \mathcal{E})} = \int \frac{dr d\mathcal{E}}{(\sqrt{1 - (\lambda \mathcal{E}/2)^2})}, \] (22)

where \( J(r, \mathcal{E}) \) is the Jacobian of the transformation. Therefore, the imaginary part of the action can be expressed as

\[ \text{Im} \mathcal{I} = \text{Im} \int_{0}^{\mathcal{E}} \int_{r_{in}}^{r_{out}} \frac{dr(-d\mathcal{E}')}{\sqrt{1 - (\lambda \mathcal{E}/2)^2} \left(1 - \sqrt{\frac{2(M - \mathcal{E}')}{r}}\right)} \]. (23)

\footnote{This form of the Schwarzschild metric can be obtained from the Schwarzschild coordinates by introducing the time coordinate

\[ t = t_s + 2\sqrt{2Mr} + 2M \ln \frac{\sqrt{r} - \sqrt{2M}}{\sqrt{r} + \sqrt{2M}}, \]

where \( t_s \) is Schwarzschild time, see the first of [13].}
Figure 1: The tunneling probabilities of S-BH in the presence of polymer space for \( \lambda = 1 \) and \( E = 0.1 \).

The integral over \( r \) can be evaluated by means of the residue method. To do, one may deform the contour around the pole at the horizon, where it lies along the line of integration and gives \((-\pi i)\) times the residue

\[
\text{Im} I = \text{Im} \int_0^E \frac{2(-\pi i)(M - \mathcal{E}')(d\mathcal{E}')}{\sqrt{1 - (\lambda \mathcal{E}/2)^2}}. \tag{24}
\]

Now, the imaginary part of the action takes the following form

\[
\text{Im} I = \frac{4\pi}{\lambda^2} \left( 2 - \sqrt{4 - \lambda^2 \mathcal{E}^2} - M \lambda \arcsin \left[ \frac{\lambda \mathcal{E}}{2} \right] \right) + ... \tag{25}
\]

After using a Taylor expansion the tunneling rate reads

\[
\Gamma \sim \exp \left[ \frac{8\pi}{\lambda^2} \left( \sqrt{4 - \lambda^2 \mathcal{E}^2} + M \lambda \arcsin \left[ \frac{\lambda \mathcal{E}}{2} \right] - 2 \right) \right] = \exp \left[ (-8\pi M \mathcal{E} + 4\pi \mathcal{E}^2) + \left( -\frac{2\pi}{3} M \mathcal{E}^3 + \frac{\pi \mathcal{E}^4}{2} \right) \lambda^2 \right] = \exp(\Delta S), \tag{26}
\]

where \( \Delta S \) is related to the change of Bekenstein Hawking entropy before and after emission \[19, 21, 22, 23, 24\]. In the string theory, it is expected that the the emission rates from excited D-branes on the Planck scales is related to differences between the counting of microstates in the canonical and microcanonical ensembles. The first term in the exponential function shows a spectrum of thermal Boltzmannian. In addition, the existence of extra terms explains that there is a polymer corrected non-thermal character of radiation. The expression above includes corrections of quantum gravity from polymer phase space which come from the deformed algebra. Figure 1 shows the tunneling probability of S-BH for different values of BH mass. It is interesting to note that the semiclassical tunneling probability for BH coincides with the super-Planckian results \[25\].

In this way, the Hawing temperature for BH is given by \[20\]

\[
\Gamma = \exp \left[ -\frac{\mathcal{E}}{T} \right] \rightarrow T_{BH} = \frac{\mathcal{E}}{2\text{Im} I}. \tag{27}
\]

Thus, we obtain the polymer corrected temperature as

\[
T_{\text{polymer}} = \frac{8\pi}{\lambda^2} \left( 2 - \sqrt{4 - \lambda^2 \mathcal{E}^2} - M \lambda \arcsin \left[ \frac{\lambda \mathcal{E}}{2} \right] \right). \tag{28}
\]
Figure 2: The figure shows that the polymer-corrected of the BH’s temperature coincides with its semi-classical counterpart for large values of the BH’s mass. However, it goes to zero as the BH’s mass vanishes. The figure is plotted for $\lambda = 1$ and $E = 0.1$.

Now, by using of the saturated form of the uncertainty principle $\mathcal{E}\Delta x = \frac{1}{2}$ and $\Delta x = 2M$ (which comes from the saturated form of the Heisenberg uncertainty principle $\Delta p \Delta x = \frac{\hbar}{2}$) in the above equation, its Taylor expansion takes the form

$$T_{Polymer} = T_{BH} \left( \frac{16M^2}{(8M^2 - 1)} + \frac{(16M^2 - 3)\lambda^2}{6(8M^2 - 1)^2} + \ldots \right),$$

(29)

where $T_{BH} = \frac{1}{8\pi M}$ is the classical temperature. In figure 2 we have plotted the temperature in terms of mass. As this figure shows, in the limit of large $M$, it coincides to the Hawking temperature. However, as the mass decreases the temperature goes to zero, a result which is in agreement, for example, with the one-loop corrected temperature obtained in [27].

At this stage, let us compute the correlation between emitted particles. When there is some correlations between different emitted modes, some parts of the information coming out of the BH can be retained in these correlations. We come to conclusion that the probability of tunneling of two particles of energies $E_1$ and $E_2$ is not equal to the probability of tunneling of one particle with energy $E = E_1 + E_2$, i.e.

$$\Delta S_{E_1} + \Delta S_{E_2} \neq \Delta S_{E_1 + E_2} \Rightarrow \chi(E_1 + E_2; E_1, E_2) \neq 0,$$

(30)

where $\chi(E_1 + E_2; E_1, E_2)$ is the correlation function. To see how a nonzero correlation function may come into the play, let us compute the emission rate for a emitted quanta with energy $E_1$ as

$$\ln \Gamma_{E_1} = -8\pi ME_1 + 4\pi E_1^2 + \left( -\frac{2\pi}{3} M E_1^3 + \frac{\pi E_1^4}{2} \right) \lambda^2.$$

(31)

Therefore, the corresponding expression for a single quantum of energy $E_2$, should be written as

$$\ln \Gamma_{E_2} = -8\pi (M - E_1) E_2 + 4\pi E_2^2 + \left( -\frac{2\pi}{3} (M - E_1) E_2^3 + \frac{\pi E_2^4}{2} \right) \lambda^2.$$

(32)
On the other hand, if our single quantum has the energy $E = E_1 + E_2$, its emission rate reads
\[
\ln \Gamma_{(E_1+E_2)} = -8\pi M (E_1 + E_2) + 4\pi (E_1 + E_2)^2 \\
+ \left( -\frac{2\pi}{3} M (E_1 + E_2)^3 + \frac{\pi (E_1 + E_2)^4}{2} \right) \lambda^2.
\]
(33)

Now, it is easy to show that the statistical correlation function is non-zero and is equal to
\[
\chi(E_1 + E_2; E_1, E_2) = \frac{1}{3} \pi E_1 E_2 \left[ 6E_1^2 - 6M(E_1 + E_2) + 9E_1 E_2 + 4E_2^2 \right] \lambda^2,
\]
(34)
which includes the terms which depend on $M$ and $\lambda$. This means that these probabilities are actually correlated. One may compare the above mentioned correlation function with the similar expressions obtained in [4] and [19, 21] with using of other phenomenological approaches to quantum gravity. In each case the correlation is appeared due to incorporation of the characteristic parameter ($\lambda$ in our case) of the relevant theory. To understand the role of this non-thermal correlation, note that once the BH emits a quantum particle from its horizon, the emission of the next quantum is affected by aberrations (on the Planck scale) resulting from the propagation of the first quantum. Particularly, the effects of these aberrations are more important when the BH mass is of the order of Planck mass.

In this way, the information problem may be addressed by means of the back reaction effects come from the quantum gravity modifications. Indeed, one may suppose that information will leak out from the BH as a non-thermal correlations within the Hawking radiation.

4 Summary

In this letter, we have studied the effects of polymer phase space on the tunneling rate from the S-BH. After a brief review of the polymer representation of quantum mechanics, we have introduced the classical polymerization by means of which the Hamiltonian of the theory under consideration gets modification in such a way that a parameter $\lambda$, coming from polymer quantization, plays the role of a deformation parameter. In order to apply this mechanism on the S-BH, we first presented the tunneling probability as a function of the particle action. Then, we have applied the polymerization on the model to get the polymer-modified expression for the tunneling probability and compared the result with the standard formalism. We have shown that in the presence of polymer effects, tunneling mechanism of a particle is totally deviated from thermal emission. Finally, we evaluated the correlation function between emitted particles, and showed that the correlation is non-zero. This result leads us to the fact that within the Hawking radiation, there is also some information leak in the form of non-thermal emission.

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