Secure Nested Codes for Type II Wiretap Channels

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Abstract—This paper considers the problem of secure coding design for a type II wiretap channel, where the main channel is noiseless and the eavesdropper channel is a general binary-input symmetric-output memoryless channel. The proposed secure error-correcting code has a nested code structure. Two secure nested coding schemes are studied for a type II Gaussian wiretap channel. The nesting is based on cosets of a good code sequence for the first scheme and on cosets of the dual of a good code sequence for the second scheme. In each case, the corresponding achievable rate-equivocation pair is derived based on the threshold behavior of good code sequences. The two secure coding schemes together establish an achievable rate-equivocation region, which almost covers the secrecy capacity-equivocation region in this case study. The proposed secure coding scheme is extended to a type II binary symmetric wiretap channel. A new achievable perfect secrecy rate, which improves upon the previously reported result by Thangaraj et al., is derived for this channel.

I. INTRODUCTION

Fostered by the rapid proliferation of wireless communication devices, technologies, and applications, the need for reliable and secure data communication over wireless networks is more important than ever before. Due to its broadcast nature, wireless communication is particularly susceptible to eavesdropping. Security and privacy systems have become critical for wireless providers and enterprise networks. The aim of this paper is to study practical secure coding schemes for wireless communication systems.

![Wiretap channel model](image)

Fig. 1. Wiretap channel model

Shannon provided the first truly scientific treatment of secrecy in [1], where a secret key is considered to protect confidential messages. The ingenuity of his remarkable work lies not only in the method used therein but also in the incisive formulation that Shannon made of the secrecy problem based on information-theoretic concepts. Later, Wyner proposed an alternative approach to secure communication schemes in his seminal paper [2], where he introduced the so-called wiretap channel model. As shown in Fig. 1, the confidential communication via a discrete, memoryless main channel is eavesdropped upon by a wiretapper, who has access to the degraded channel output. Wyner demonstrated that secure communication is possible without sharing a secret key and determined the secrecy capacity for a wiretap channel. Construction of explicit and practical secure encoders and decoders whose performance is as good as promised by Wyner is still an unsolved problem in the general case, except for the binary erasure wiretap channel [3]–[5].

We note that channel coding and secrecy coding are closely related. Roughly speaking, the goal of channel coding is to send a message with sufficient redundancy so that it can be understood by the receiver, whereas the goal of secrecy coding is to provide sufficient randomness so that the message cannot be understood by anyone else. In modern communication networks, error-correcting codes have traditionally been designed to ensure communication reliability. Various coding techniques have been thoroughly developed and tested for ensuring reliability of virtually all current single-user, point-to-point physical channels. However, only very limited work has considered ways of using error-correcting codes to also ensure security. In [3], Ozarow and Wyner considered error-correcting code design for a type II binary erasure wiretap channel based on a coset coding scheme. More recently, low-density parity-check (LDPC) based coding design has been studied for binary erasure wiretap channels in [4], where the authors have also presented code constructions for a type II binary symmetric wiretap channel based on error-detection codes. In another line of recent related work, secret key agreement protocols based on powerful LDPC codes have been studied by several authors [6]–[8]. Designing practical secure coding schemes for additive white Gaussian noise (AWGN) wiretap channels, for example, is still an open problem.

In this work, we focus on secure coding schemes for a type II wiretap channel, where the main channel is noiseless and the eavesdropper channel is a binary-input symmetric-output memoryless (BISOM) channel. We first review and summarize the prior results of [2]–[4]. Inspired by [9], we propose a more general secure nested code structure. Next, we consider a type II AWGN wiretap channel and describe two secure coding schemes, both of which have a nested structure. The nesting is based on cosets of a good code sequence for the first scheme and on cosets of the dual of a good code sequence for the second scheme. In each case,
we derive the corresponding achievable rate-equivocation pair based on the threshold behavior of good code sequences [10], [11]. By combining the two secure coding schemes, we establish an achievable rate-equivocation region, which almost covers the secrecy capacity-equivocation region for the described case study. Finally, we extend the secure coding to a type II binary symmetric wiretap channel and derive a new achievable (perfect) secrecy rate, which improves upon the result previously reported in [4].

II. Preliminaries

We review here some definitions and results from [2]–[4] and propose a secure nested coding structure, which serves as preliminary material for the rest of the paper.

A. General Wiretap Channel Model

We consider the classic wiretap channel [2] illustrated in Fig. 1, where the transmitter sends a confidential message to a legitimate receiver via the main channel in the presence of an eavesdropper, who listens to the message through its own channel. Both the main and the eavesdropper channels are discrete memoryless, and in particular, the eavesdropper channel is a degraded version of the main channel. A confidential message \( w \in \mathcal{W} \) is mapped into a channel input sequence \( x = [x_1, x_2, \ldots, x_n] \) of length \( n \), where \( \mathcal{W} = \{1, \ldots, M\} \) and \( M \) is the number of distinct confidential messages that may be transmitted. The outputs from the main channel and the eavesdropper channel are \( y \) and \( z \), respectively. The level of ignorance of the eavesdropper with respect to the confidential message is measured by the equivocation \( H(W|Z) \). A rate-equivocation pair \((R, R_e)\) is achievable if there exists a rate \( R \) code sequence with the average probability of error \( P_e \to 0 \) as the code length \( n \) goes to infinity and with the equivocation rate \( R_e \) satisfying

\[
R_e \leq \lim_{n \to \infty} H(W|Z)/n.
\]

Perfect secrecy requires that, for any \( \epsilon_0 > 0 \) there exists a sufficiently large \( n \) so that the normalized equivocation satisfies

\[
H(W|Z)/n \geq H(W)/n - \epsilon_0.
\]

Hence, perfect secrecy happens when \( R_e = R \), i.e., all the information transmitted over the main channel is secret. The capacity-equivocation region of the wiretap channel \( X \rightarrow (Y, Z) \) [2] contains rate-equivocation pairs \((R, R_e)\) that satisfy

\[
\begin{align*}
R_e &\leq R \leq \max_{p(x)} I(X;Y) \\
0 &\leq R_e \leq \max_{p(x)} [I(X;Y) - I(X;Z)].
\end{align*}
\]

B. Wyner Codes and Secrecy Bins

It is instructive to review first the problem of unstructured secure code design in terms of the stochastic encoding scheme introduced by Wyner [2]. As demonstrated in [2] the secrecy capacity of the wiretap channel is achieved by using a stochastic encoder, where a mother codebook \( \mathcal{C}_0(n) \) of length \( n \) is randomly partitioned into “secret bins” or sub-codes \( \{\mathcal{C}_2(n), \mathcal{C}_2(n), \ldots, \mathcal{C}_M(n)\} \). A message \( w \) is associated with a sub-code \( \mathcal{C}_w(n) \) and the transmitted codeword is randomly selected within the sub-code. Such codebook allows for decomposing the twofold objective of achieving both reliability and secrecy into two separate objectives. The mother code \( \mathcal{C}_0(n) \) provides enough redundancy so that the legitimate receiver can decode the message reliably, whereas each sub-code is sufficiently large and, hence, introduces enough randomness so that the eavesdropper’s uncertainty about the transmitted message can be guaranteed.

Even though [2] does not describe a structured coding scheme, it does suggest that encoding for reliability and confidentiality would be to partition the mother code into sub-codes. This idea has been extended to structured or semi-structured codes by using coset codes in [3], [4].

C. Secure Nested Codes

In the following, we construct secure error-correcting codes with the nested code structure [9].

We consider a nested linear code pair \((\mathcal{C}_0(n), \mathcal{C}_1(n))\), where \( \mathcal{C}_0(n) \) is a fine code of rate \( R_0 \), and \( \mathcal{C}_1(n) \) a coarse code of rate \( R_1 \). We use the fine code \( \mathcal{C}_0(n) \) as the mother code, which is partitioned into \( M \) sub-codes consisting of the coarse code \( \mathcal{C}_1(n) \) and its cosets. Each coset corresponds to a confidential message. The transmitter encodes a message \( w \in \mathcal{W} \) into an \( n \)-tuple of coded symbols randomly selected within the corresponding coset \( \mathcal{C}_w(n) \). By determining the coset of the transmitted codeword, the legitimate receiver can retrieve the confidential message \( w \). The redundancies provided by each coset are used to confuse the eavesdropper who has full knowledge about the code and its cosets. We refer to a code structured in this manner as a secure nested code. We note that the code \( \mathcal{C}_1(n) \) and its cosets have the same (Hamming) distance properties. Hence, the secure coding design problem is to find a suitable nested code pair \((\mathcal{C}_0(n), \mathcal{C}_1(n))\) that satisfies both confidentiality and reliability requirements. Denote by \( \{\mathcal{C}(n)\} \) a sequence of binary linear codes, where \( \mathcal{C}(n) \) is an \((n, k_n)\) code having a common rate \( R_e = k_n/n \). Now, we define the secure nested code sequence as follows.

Definition 1 (secure code sequence): \( \{\mathcal{C}_0(n), \mathcal{C}_1(n)\} \) is a secure nested code sequence if \( \mathcal{C}_0(n) \) is a (mother) fine code of rate \( R_0 \), and \( \mathcal{C}_1(n) \) is a coarse code of rate \( R_1 \) so that \( \mathcal{C}_1(n) \subseteq \mathcal{C}_0(n) \) and \( R_1 \leq R_0 \). The information rate of this code sequence is \( R_0 - R_1 \).

D. Good Code and Its Noise Threshold

Following MacKay [10], we say that a code sequence \( \{\mathcal{C}(n)\} \) is good if it achieves arbitrarily small word (bit) error probability when transmitted over a noisy channel at a nonzero rate \( R_e \). Capacity-achieving codes are good codes whose rate \( R_e \) is equal to the channel capacity. The class of good codes includes, for example, turbo, LDPC, and repeat-accumulate codes.

1 In this paper, we consider binary-input wiretap channels and nested linear codes. This idea can be extended to nested lattice codes for channels with continuous inputs.
codes, whose performance is characterized by a threshold behavior in a single channel model [11].

**Definition 2 (noise threshold):** For a (single) channel model described by a single parameter, the noise threshold of a code sequence \( \{ C(n) \} \) is defined as the worst case channel parameter value at which the word (bit) error probability decays to zero as the codeword length \( n \) increases.

For example, the noise threshold is described in terms of the erasure rate threshold \( \delta^* \) for a binary erasure channel (BEC) and the SNR threshold \( \lambda^* \) for a binary-input AWGN (BI-AWGN) channel. Noise thresholds associated with good codes and the corresponding maximum-likelihood (ML), "typical pair", and iterative decoding algorithms have been studied in [12]–[14].

**E. Type II Wiretap Channel**

The type II wiretap channel was introduced by Ozarow and Wyner in [3] as a special binary-input wiretap channel with a noiseless main channel. Throughout the paper, we focus on type II wiretap channels associated with different eavesdropper channels.

**Example 1 (BEC-WT):** Let BEC-WT(\( c \)) denote a binary-input wiretap channel where the main channel is noiseless and the eavesdropper channel is a BEC with erasure rate \( c \). We refer to such a channel as the type II binary erasure wiretap channel. The secrecy capacity of BEC-WT(\( c \)), \( C_s,BEC(\epsilon) \), equals \( c \).

Let \( \{ C^\perp(n) \} \) be a sequence of dual codes, where

\[
C^\perp(n) = \{ x \in \{ 0,1 \}^n | x \cdot y = 0, \forall y \in C(n) \}
\]

is the dual code of \( C(n) \). By employing the dual code as the coarse code in the secure nested code structure, we reorganize the results of [3], [4] in the following lemma.

**Lemma 1:** Consider a sequence of binary linear codes \( \{ C(n) \} \) of rate \( R_c \) and erasure rate threshold \( \delta^* \leq 1 - R_c \) (for the BEC). Let

\[
C_0(n) = \{ 0,1 \}^n \quad \text{and} \quad C_1(n) = C^\perp(n).
\]

Suppose that the secure nested code sequence \( \{ C_0(n), C_1(n) \} \) is transmitted over a BEC-WT(\( c \)). Then, if

\[
\epsilon \geq 1 - \delta^*,
\]

the achievable rate-equivocation pair \( (R, R_e) = (R_c, R_e) \) satisfies

\[
R_e \leq R \leq 1
\text{and} \quad 0 \leq R_e \leq 1 - C_{BI-AWGN}(\lambda)
\]

Then, the secure nested code sequence \( \{ C_0(n), C_1(n) \} \) achieves the secrecy capacity of BEC-WT(\( 1 - R_c \)).

**III. MAIN RESULTS**

In this section, we consider practical coding design for secure communication over a type II AWGN wiretap channel. As shown in Fig. 2, the eavesdropper channel is a BI-AWGN channel characterized by transition probabilities

\[
g(z|X = 1) = \frac{1}{\sqrt{2\pi}} \exp \left[ \frac{-(z + \sqrt{2\lambda})^2}{2} \right]
\]

and

\[
g(z|X = -1) = \frac{1}{\sqrt{2\pi}} \exp \left[ \frac{-(z - \sqrt{2\lambda})^2}{2} \right]
\]

where \( \lambda = E_s/N_0 \) is the ratio of the energy per coded symbol to the one-sided spectral noise density, which is referred to as the SNR of the eavesdropper channel. We denote this channel with AWGN-WT(\( \lambda \)). The capacity-equivocation region of AWGN-WT(\( \lambda \)) contains rate-equivocation pairs \( (R, R_e) \) that satisfy

\[
R_e \leq R \leq 1
\text{and} \quad 0 \leq R_e \leq 1 - C_{BI-AWGN}(\lambda)
\]

where

\[
C_{BI-AWGN}(\lambda) = 1 - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(y - \sqrt{2\lambda})^2} \log_2(1 + e^{-4y\sqrt{\lambda}}) dy
\]

is the channel capacity of BI-AWGN channel with SNR \( \lambda \).

In the following, we consider two approaches to designing secure codes, both of which have a nested structure. In each case, we derive the corresponding achievable rate-equivocation pair based on the threshold behavior of good codes [11].

We note that even for a general BI-AWGN channel (without a secrecy constraint), designing practical capacity-achieving codes is still an open problem. Hence, to allow secure codes to be implementable, we either loosen the perfect secrecy requirement (allow for a nonzero gap between the transmission rate and the equivalent rate) or reduce the transmission rate compared with the capacity. In the first approach, we construct practical codes ensuring an equivocation rate that is below the transmission rate; whereas, in the second approach, we design secure codes to achieve perfect secrecy with a transmission rate that is below the secrecy capacity. We summarize code designs and the corresponding achievable rate-equivocation pair as follows.
A. Approach I: Good Coarse Code

In Approach I, we use a good code as the coarse code $C_1(n)$. Theorem 1: Consider a sequence of secure nested codes $\{C_0(n), C_1(n)\}$, where $C_0(n) = \{0,1\}^n$ and $C_1(n)$ is a good binary linear code sequence of rate $R_1$ and SNR threshold $\lambda^*$ (for BI-AWGN channels). Suppose that the secure nested code sequence $\{C_0(n), C_1(n)\}$ is transmitted over AWGN-WT(λ). Then, if $\lambda \geq \lambda^*$, the rate-equivocation pair

$$ (R, R_e) = (1 - R_1, 1 - C_{BI-AWGN}(\lambda)) $$

is achievable.

Theorem 1 is proved in Appendix A. Note that if the code sequence $C_1(n)$ is not a capacity-achieving sequence, then

$$ R_1 < C_{BI-AWGN}(\lambda^*) \leq C_{BI-AWGN}(\lambda) $$

The gap between the rate $R_1$ and the capacity $C_{BI-AWGN}(\lambda^*)$ implies $R_e \leq R$. Hence this approach cannot achieve perfect secrecy when using non capacity-achieving sequences.

Example 2: Consider a sequence of $(4,6)$ regular LDPC codes $\{C_{LDPC}(n)\}$ [16]. Let

$$ C_0(n) = \{0,1\}^n \quad \text{and} \quad C_1(n) = C_{LDPC}(n). $$

The design rate of $C_{LDPC}(n)$ is $R_1 = 1/3$. The SNR threshold of $\{C_{LDPC}(n)\}$ satisfies $\lambda^* \leq 0.302$ under typical pair decoding [14] (and hence, under ML decoding). Assume that the secure code $\{C_0(n), C_1(n)\}$ is transmitted over AWGN-WT($\lambda = 0.302$). The achievable rate-equivocation pair is given by

$$ (R, R_e) = (1 - R_1, 1 - C_{BI-AWGN}(0.302)) = (2/3, 0.663). $$

In this case, the gap between the transmission rate and the equivalent rate is less then 0.004. Moreover, Approach I can be extended to the general AWGN wiretap channel (the main channel is also a BI-AWGN channel) by constructing a nested LDPC code pair.

B. Approach II: Dual Good Code as Coarse Code

In Approach II, we use the dual code of a good code as the coarse code $C_1(n)$. Let

$$ Q(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz. $$

Theorem 2: Consider a sequence of good binary linear codes $\{C(n)\}$ of rate $R_e$ and erasure rate threshold $\delta^*$ (for BECs). Let

$$ C_0(n) = \{0,1\}^n \quad \text{and} \quad C_1(n) = C^\perp(n). $$

Suppose that the secure nested code sequence $\{C_0(n), C_1(n)\}$ is transmitted over an AWGN-WT($\lambda$). Then, if

$$ Q(\sqrt{2\lambda}) \geq (1 - \delta^*)/2, $$

the rate-equivocation pair $(R, R_e) = (R_e, R_e)$ is achievable.

We provide the proof in Appendix B. Theorem 2 implies that this code sequence can achieve perfect secrecy at the transmission rate $1/3$.

Corollary 2: Consider LDPC code sequences $\{C_T(n)\}$ and $\{C_R(n)\}$. Let $C_0(n) = \{0,1\}^n$ and

$$ C_1(n) = C^\perp_T(n) \quad \text{or} \quad C_1(n) = C^\perp_R(n). $$

Assume that the nested code sequence $\{C_0(n), C_1(n)\}$ is transmitted over AWGN-WT($\lambda$). Then, perfect secrecy can be achieved at (and below) the transmission rate $2Q(\sqrt{2\lambda})$. Corollary 2 implies that the gap between the secrecy capacity $6$ and the achievable (perfect) secrecy rate is

$$ \Delta = 1 - C_{BI-AWGN}(\lambda) - 2Q(\sqrt{2\lambda}). $$

The gap $\Delta$ can be reduced if one can find a tighter sufficient condition than (11).

C. Achievable Rate-Equivocation Region

Now, we consider the achievable rate-equivocation region based on practical codes for AWGN-WT($\lambda$). For a given channel SNR $\lambda$, we choose a good code sequence $\{C(n)\}$ of rate $R^*$ so that its SNR threshold $\lambda^* \leq \lambda$. Let $C_0(n) = \{0,1\}^n$ and select $C_1(n)$ from

$$ \{0\}^n, \{0,1\}^n, C(n), \text{and } C^\perp_R(n) $$

corresponding to different equivocation rate requirements. By using a time-sharing strategy, we can show that the secure coding scheme achieves the rate-equivocation region

$$ R_{\text{AWT}} = \text{convex hull} \left\{ (0,0), \left(2Q(\sqrt{2\lambda}), 2Q(\sqrt{2\lambda}) \right), \left(1 - R_1, 1 - C_{BI-AWGN}(\lambda) \right), \left(1, 1 - C_{BI-AWGN}(\lambda) \right), (1,0) \right\}. $$

Example 4: Consider AWGN-WT($\lambda = 0.32$) and the good code sequence $\{C(n)\} = \{C_{LDPC}(n)\}$ described in Example 2 whose SNR threshold is bounded as

$$ \lambda^* \leq 0.302 < 0.32 = \lambda. $$

Fig. 3 depicts the region $R_{\text{AWT}}$ and compares it with the capacity-equivocation region for AWGN-WT(0.32).
IV. TYPE II BINARY SYMMETRIC WIRETAP CHANNEL

In this section, we study the type II binary symmetric wiretap channel. This channel was studied previously in [4] and an achievable secrecy rate based on error-detecting codes was given. In the following, we apply the coding technique in Approach II and obtain an improved secrecy rate with respect to the result in [4].

Let $BSC-WT(q)$ be a type II binary symmetric wiretap channel, where the eavesdropper channel is a binary symmetric channel (BSC) with crossover rate $q$. The secrecy capacity of $BSC-WT(q)$ is $C_{A,BSC}(q) = h(q)$, where $h(q)$ is a binary entropy function. We first summarize the result of [4] in the following lemma.

**Lemma 2**: Consider a sequence of error-detecting codes $\{C_D(n)\}$ of rate $R_1$, whose detection error rate is less than $2^{-nR_1}$. Let $C_0(n) = \{0,1\}^n$ and $C_1(n) = C_D(n)$. Assume that the nested code sequence $\{C_0(n), C_1(n)\}$ is transmitted over a $BSC-WT(q)$. The maximum possible secrecy rate that can be achieved by this construction is $-\log_2(1-q)$. The authors of [4] have also stated that error-detecting codes include Hamming codes and double-error-correcting BCH codes; however, most known classes of error-detecting codes have $R_1 = 0$. Hence, the implementation of such secure codes described in Lemma 2 is still an open problem.

Following Approach II, we construct implementable perfect secrecy nested codes for $BSC-WT(q)$ as follows.

**Theorem 3**: Consider a sequence of good binary linear codes $\{C(n)\}$ of rate $R_e$ and erasure rate threshold $\delta^*$ (for BECs). Let $C_0(n) = \{0,1\}^n$ and $C_1(n) = C^\perp(n)$. Suppose that the secure nested code sequence $\{C_0(n), C_1(n)\}$ is transmitted over an $BSC-WT(q)$. Then, if

$$q \geq (1 - \delta^*)/2,$$

the rate-equivocation pair $(R, R_e) = (R_e, R_e)$ is achievable.

**Proof**: The proof is similar to the one described in Appendix B by constructing an equivalent BSC channel as in Fig. 4.

By using the LDPC code sequence $\{C_R(n)\}$, i.e., setting $C_1(n) = C_R(n)$, the achievable (perfect) secrecy rate under this construction is $2q$, which is better than $-\log_2(1-q)$ derived in [4]. We compare the achievable (perfect) secrecy rate with the secrecy capacity for $BSC-WT(q)$ in Fig. 5.

V. CONCLUSION

In this paper, we have addressed the problem of secure coding design for a type II wiretap channel. A secure error-correcting code has been proposed in terms of a nested code structure. Two secure nested coding schemes have been studied for a type II AWGN wiretap and the corresponding achievable rate-equivocation pair has been derived based on the threshold behavior of good code sequences. Combining the two secure coding schemes, we have established an achievable rate-equivocation region, which almost covers the secrecy capacity-equivocation region in this case study. Furthermore, we have also applied the proposed secure coding scheme to a type II binary symmetric wiretap channel, and have obtained a new achievable (perfect) secrecy rate, which improves upon the previous result of [4].

APPENDIX

A. Proof of Theorem 7

The reliability at the desired receiver can be ensured since the main channel is noiseless. Now, we calculate only the
equivocation:

\[ H(W|Z) = H(W, Z) - H(Z) = H(W, X, Z) - H(X|W, Z) - H(Z) \geq H(X) - H(X|W, Z) - I(X; Z) \geq n - H(X|W, Z) - nC_{BL-AWGN}(\lambda). \] (14)

In order to calculate the conditional entropy \( H(X|W, Z) \), we consider the following situation. Let us fix \( W = w \) and assume that the transmitter sends a codeword \( x \in C_w(n) \). Given index \( W = w \), the eavesdropper decodes the codeword \( x \) based on the received sequence \( z \). Let \( P(C_w, n) \) denote the average probability of error under ML decoding at the eavesdropper incurred by using coset \( C_w(n) \). We note that the code \( C_1(n) \) and its coset \( C_w(n) \) have the same distance properties, and hence, have the same SNR threshold under ML decoding. Based on the threshold behavior of good codes [11] and the condition \( \lambda \geq \lambda^* \), we have \( \lim_{n \to \infty} P(C_w, n) = 0 \). Moreover, Fano’s inequality implies that

\[ \lim_{n \to \infty} H(X|W, Z)/n \leq \lim_{n \to \infty} [1/n + P(C_w, n)R_1] = 0. \] (15)

Combining (14) and (15), we have the desired result.

B. Proof of Theorem 2

To develop the achievable rate-equivocation pair, we consider an equivalent channel model illustrated in Fig. 6. We observe that the equivalent channel embeds a binary erasure wiretap channel \( X \to (Y, Z') \), where \( Z' \) is the BEC output with alphabet \( \{0, 1, -1\} \). The proof can be outlined as follows.

We first construct a BEC-WT(\( \epsilon \)) and an associated channel with transition probabilities \( f_{Z'|Z} \) so that the channel \( X \to Z \) is equivalent to the original BI-AWGN with SNR \( \lambda \). To this end, we choose the erasure rate \( \epsilon \) as follows

\[ \epsilon = \int_{-\infty}^{\infty} \min[g(z|X = -1), g(z|X = 1)] dz = 2Q(\sqrt{2}\lambda). \]

Let us define transition probabilities \( f_{Z'|Z} \) as

\[ f(z|Z') = \begin{cases} \frac{g(z|X = 1) - g(z|X = -1)}{4\epsilon} & z \geq 0 \\ 0 & z < 0 \end{cases} \]
\[ f(z|Z') = \begin{cases} g(z|X = -1)/\epsilon & z \geq 0 \\ g(z|X = 1)/\epsilon & z < 0 \end{cases} \]
\[ f(z|Z') = \begin{cases} 0 & z \geq 0 \\ \frac{g(z|X = 1) - g(z|X = -1)}{4\epsilon} & z < 0. \end{cases} \] (16)

We can easily verify that \( \sum_{z'} p(z'|x)f(z|z') = g(z|x) \). This implies that the designed concatenated channel is equivalent to the original AWGN-WT(\( \lambda \)).

Next, we design secure nested codes for the upgraded BEC-WT(\( \epsilon \)). Note that the confidential message \( W \), the BEC output \( Z' \), and the received signal at the eavesdropper \( Z \) satisfy the Markov chain \( W \to Z' \to Z \). The data processing inequality [18] implies that the normalized equivocation can be bounded as

\[ H(W|Z)/n \geq H(W|Z')/n. \]

Finally, we have the desired result by applying Lemma 1.