Asymptotic behavior of solutions to a dissipative nonlinear Schrödinger equation with time dependent harmonic potentials

Masaki KAWAMOTO
Department of Engineering for Production, Graduate School of Science and Engineering, Ehime University, 3 Bunkyo-cho Matsuyama, Ehime, 790-8577, Japan
Email: kawamoto.masaki.zs@ehime-u.ac.jp

Takuya SATO
Mathematical Institute, Tohoku University, Sendai, Miyagi, 980-8578, Japan
E-mail: takuya.sato.b1@tohoku.ac.jp

Abstract

We consider the Cauchy problem of a dissipative nonlinear Schrödinger equation with a time dependent harmonic potential. We find a critical situation that the $L^2$-norm of dissipative solutions decays or not and which is decided by a nonlinear power and time decay order of harmonic potential.

Mathematical Subject Classification:
Primary: 35Q55, Secondly: 35Q40.

1 Introduction

We consider the Cauchy problem of the nonlinear Schrödinger equation with a time dependent harmonic potential:

\[
\begin{aligned}
&i\partial_t u - H_0(t)u = \lambda |u|^{p-1}u, \ t \in \mathbb{R}, \ x \in \mathbb{R}^n, \\
u(0, x) = u_0(x), \ x \in \mathbb{R}^n, 
\end{aligned}
\] (1.1)

where $\lambda \in \mathbb{C} \setminus \{0\}$ with $\text{Im} \lambda < 0$, $p > 1$, $u = u(t, x) : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{C}$ is the unknown function and $u_0 : \mathbb{R}^n \rightarrow \mathbb{C}$ is a given initial data. Here, $H_0(t) = -\frac{1}{2}\Delta + \sigma(t)|x|^2$, $\sigma(t)$ is a real valued function.

In case with $\sigma \equiv 0$, i.e., the Cauchy problem of the nonlinear Schrödinger equation;

\[
\begin{aligned}
&i\partial_t u + \frac{1}{2}\Delta u = \lambda |u|^{p-1}u, \ t \in \mathbb{R}, \ x \in \mathbb{R}^n, \\
u(0, x) = u_0(x), \ x \in \mathbb{R}^n, 
\end{aligned}
\] (1.2)

the local well-posedness of the problem (1.2) was established by Ginibre-Velo [14], Kato [25], Y. Tsutsumi [53] and Yajima [55] (see also [5], [10]). For $\lambda \in \mathbb{R} \setminus \{0\}$, it is well known that the exponent $p = 1 + 2/n$ is a borderline to indicate that solutions scatter or not and this exponent is called as the Barab-Ozawa exponent (see [2], [47]). For $\lambda \in \mathbb{C}$ with $\text{Im} \lambda < 0$, the
asymptotic behavior of the solution to (1.2) have been studied by many authors recently. In this setting, the solution to (1.2) satisfies the dissipative energy identity:

$$\|u(t)\|_{L^2}^2 + |\text{Im } \lambda| \int_0^t \|u(\tau)\|_{L^{p+1}}^{p+1} \, d\tau = \|u_0\|_{L^2}^2, \quad t \geq 0.$$  \hfill (1.3)

Hence, the $L^2$-norm of the corresponding solution is monotone decreasing in $t \geq 0$, however it is whether the $L^2$-norm decays or not as $t$ goes to infinity. In recent works [17], [21], [37], [45], [48], [49], [52], it is known that $p = 1 + 2/n$ is the critical exponent to exhibit the $L^2$-decay of dissipative solutions to (1.2). The $L^2$-lower bound of the dissipative solution is proved when $p > 1 + 2/n$ in [49]. For $p \leq 1 + 2/n$, Kita-Shimomura [37] and Shimomura [52] observed the dissipative structure of (1.2) under $\text{Im } \lambda < 0$ and proved the $L^2$-decay of dissipative solutions (cf. Sunagawa [51] for the case of the nonlinear Klein-Gordon equation). Kita-Shimomura [38] removed the restriction of the size of initial data and Hayashi-Li-Naumkin [17] showed the decay rate of the dissipative solution with large data by imposing the strong dissipative condition

$$\frac{p-1}{2\sqrt{p}} |\text{Re } \lambda| \leq |\text{Im } \lambda|,$$  \hfill (1.4)

where $p > 1$ is the exponent of the power type nonlinearity in the problem (1.2). The condition (1.4) was considered by Liskevich-Perelmuter [40]. Okazawa-Yokota [44] applied (1.4) to the theory of the complex Ginzburg-Landau equation, where they showed the global existence and smoothing property of the solution. Hayashi-Li-Naumkin [17] obtained the $L^2$-decay rate of the dissipative solution under the condition (1.4). Hoshino [21] showed the $L^2$-decay order of dissipative solutions in the Sobolev space with a low differential order. Ogawa-sato [45] and Sato [48] showed the $L^2$-decay order of solutions which has analytic or Gevrey regularity in spatial variable.

Under the existence of $\sigma(t)$, Carles [3] and Carles-Silva [4] considered the existence of global solution and blow-up issues for (1.1) with more generalized time-dependent harmonic potentials such that $H_0(t) = -\Delta/2 + V(t, x)$ with condition $|V(t, x)| \leq C(1+|x|^2)$ as $|x| \gg 1$. In the paper [3], the time-in-local Strichartz estimates for (1.1) with $\lambda = 0$ also have been shown and which are applied in some proofs. On the other hand, Kawamoto-Yoneyama [30] and Kawamoto [27] focussed on the case where the harmonic potential can be written as $\sigma(t)|x|^2$ with explicit expression of $\sigma(t) = c_1 t^{-2}$, $c_1 \in [0, 1/4)$ for $|t| \gg 1$. In that case, the linear solution of (1.1) has weak dispersive estimates compared with the linear solution of (1.2) and that possible to show global-in-time Strichartz estimates. By using such Strichartz estimates, the asymptotic behavior of solutions have been concerned in Kawamoto [29]. In [29], the critical case $c_1 = 1/4$ is also investigated and found that the threshold of scattering of solutions change by the situation of a time decaying harmonic oscillator (see also [28], [31]).

Our interest in this paper is to find the $L^2$-decay (mass decay) criterion for the problem (1.1) with a dissipative nonlinear setting and a time dependent harmonic potential. If $\sigma$ is positive constant, Antonelli-Carles-Sparber [1] showed that the mass of dissipative solutions
decays as $t \to \infty$ whenever $p > 1$, namely the critical situation for the mass decay is no longer appeared in the dissipative nonlinear Schrödinger equation. In this paper, we consider the problem (1.1) which contains the time dependent harmonic potential and we reveal a relation between the nonlinear power, an effect of the time decaying potential and the mass decay of dissipative solutions.

Let $\mathcal{S}(\mathbb{R}^n)$ be the set of smooth and rapidly decreasing functions. For any $f \in \mathcal{S}(\mathbb{R}^n)$, we define the Fourier transform and the Fourier inverse transform, respectively.

$$
\mathcal{F}[f](\xi) = \hat{f}(\xi) \equiv \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) \, dx,
$$

$$
\mathcal{F}^{-1}[f](x) \equiv \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi) \, d\xi.
$$

For $s > 0$ and $r > 0$, let

$$
H^{s,r}(\mathbb{R}^n) \equiv \left\{ f \in H^s(\mathbb{R}^n); \|f\|_{H^{s,r}} \equiv \|\langle x \rangle^r f\|_{L^2} + \|\langle \nabla \rangle^s f\|_{L^2} < \infty \right\},
$$

where $\langle x \rangle \equiv (1 + |x|^2)^{1/2}$ and $\langle \nabla \rangle^s f \equiv \mathcal{F}^{-1}[\langle \xi \rangle^s \hat{f}]$ and we denote $H^{s,0}(\mathbb{R}^n) = H^s(\mathbb{R}^n)$.

Let us introduce the setting of a time dependent harmonic potential in the linear part:

$$
H_0(t) = -\frac{1}{2}\Delta + \frac{\sigma(t)}{2} |x|^2.
$$

We define $U_0(t, s)$ as a propagator for $H_0(t)$, that is, a family of unitary operators on $L^2(\mathbb{R}^n)$ satisfying

$$
i\partial_t U_0(t, s) = H_0(t)U_0(t, s), \quad i\partial_s U_0(t, s) = -U_0(t, s)H_0(s),
$$

$$
U_0(t, \tau)U_0(\tau, s) = U_0(t, s), \quad U(s, s) = \text{Id}_{L^2}
$$
on $\mathcal{D}(-\Delta + |x|^2)$. Let $y_j(t)$, $j = 1, 2$ be a solution to

$$
y''(t) + \sigma(t)y(t) = 0, \quad \begin{cases} y_1(0) = 1, \quad y_1'(0) = 0, \\ y_2(0) = 0, \quad y_2'(0) = 1. \end{cases}
$$

We assume that $y_j(t)$ satisfies following conditions;

(A). There exist $c_0 > 0$ and $T_0 \geq 0$ such that for any $t \geq T_0$, it follows that $|y_2(t)| \geq c_0$.

(B). For all $t \geq T_0$, $y_j(t)$ and $y_j'(t)$, $j = 1, 2$ are continuous.

(C). There exist $C > 0$ and $\delta > 0$ such that for all $t \geq T_0$, an asymptotic behavior is given by

$$
\left| \frac{y_1(t)}{y_2(t)} \right| \leq Ct^{-\delta}.
$$
Here, we do not assume the some asymptotic conditions to $\sigma(t)$ but to $y_j(t)$, $j = 1, 2$. The reason why is the dispersive estimates for linear equations is characterized through $y_j(t)$, $j = 1, 2$, which was found by Korotyaev;

**Lemma 1.1** (Korotyaev [39]). For $\phi \in L^1(\mathbb{R}^n)$, the following pointwise estimate holds: For any $t, s \in \mathbb{R}$,

$$\|U_0(t, s)\phi\|_{L^\infty} \leq C |y_1(t)y_2(s) - y_1(s)y_2(t)|^{-\frac{n}{2}} \|\phi\|_{L^1}.$$  

Particularly, it holds in case with $s = 0$ that

$$\|U_0(t, 0)\phi\|_{L^\infty} \leq C |y_2(t)|^{-\frac{n}{2}} \|\phi\|_{L^1}.$$  

Physically, the classical position $x(t)$ of a quantum particle governed by $H_0(t)$ can be express as $x(t) = y_1(t)x_0 + y_2(t)p_0/m$, where $x_0$, $p_0$ and $m$ are the initial position, the initial momentum and the mass of the quantum particle, respectively. Indeed, if $\sigma(t) \equiv 0$, i.e., the linear Schrödinger operator, we have $y_1(t) = 1$ and $y_2(t) = t$, and get $x(t) = x_0 + t(p_0/m)$ is the uniform linear motion. Considering corresponding $\sigma(t)$'s, it appears some important physical models (see [22], [29]).

We say $p > 1$ is the critical in the sense of $y_2(t)$, if there exists $c_+ > 0$ such that

$$\lim_{t \to \infty} t |y_2(t)|^{-\frac{n}{2}(p-1)} = c_+$$  

and say the sub-critical in the sense of $y_2(t)$, if there exist $c_+ > 0$ and $\delta_+ > 0$ such that

$$\lim_{t \to \infty} t^{1-\delta_+} |y_2(t)|^{-\frac{n}{2}(p-1)} = c_+.$$  

We also say the super-critical in the sense of $y_2(t)$, if there exist $c_+ > 0$ and $\delta_+ > 0$ such that

$$\lim_{t \to \infty} t^{1+\delta_+} |y_2(t)|^{-\frac{n}{2}(p-1)} = c_+.$$  

In the case where $\sigma(t) \equiv 0$, we have $y_1(t) = 1$ and $y_2(t) = t$, i.e., $p = 1 + 2/n$ is the critical. In the case where $\sigma(t) = \sigma_0 t^{-2}$, $\sigma_0 \in [0, 1/4)$, we have that for $t \geq T_0$, the functions $t^\lambda$ and $t^{1-\lambda}$ solve $y''(t) + \sigma(t)y(t) = 0$, where $\lambda = (1 - \sqrt{1 - 4\sigma_0})/2 \in [0, 1/2)$. Choosing the suitable $\sigma(t)$ in $t \in [0, T_0)$, see, e.g. [29], we get $y_1(t) = c_1 t^\lambda$ and $y_2(t) = c_2 t^{1-\lambda}$ where $c_1, c_2 \in \mathbb{R}\{0\}$. In this case, it holds that $\lim t |y_2(t)|^{-\frac{n}{2}(p-1)} = |c_2|^{-\frac{n}{2}(p-1)}$, which yields $p = 1 + 2/(n(1 - \lambda))$ is the critical. In the appendix, we introduce some other models. We state that the asymptotic behavior of dissipative solutions to (1.1) vary by the situation of $p$, namely, the critical exponent for the $L^2$-decay of dissipative solutions depends on the decay order of the time decaying potential $\sigma$.

**Theorem 1.2** (Small data case). Let $1 \leq n \leq 3$, $n/2 < s < p$, $\lambda \in \mathbb{C}$ with $\text{Im} \lambda < 0$, and $\sigma$ be the real function such that there exists fundamental solutions $y_1, y_2$ of the equation (1.6) satisfying (A)-(C). Assume that $p > 1$ is the critical, namely $p$ satisfies (1.8). Then, there exists small $\delta_0 > 0$ such that for any $u_0 \in H^{s,s}(\mathbb{R}^n)$ with $\|u_0\|_{H^{s,s}} \leq \delta_0$, the Cauchy problem
(1.1) has a unique global solution \( u \in C([0, \infty); H^s(\mathbb{R}^n)) \) such that \( U_0(t, 0)|x|^s U_0(t, 0)^{-1} u \in C([0, \infty); L^2(\mathbb{R}^n)) \) and

\[
\sup_{t \geq 0} \|u(t)\|_{H^s} < \infty. \tag{1.11}
\]

Moreover, the solution satisfies following decay estimates: For some \( T_0 \) and any \( t \geq T_0 \),

\[
\|u(t)\|_{L^\infty} \leq C[y_2(t)]^{-\frac{n}{2}} \left( \int_{T_0}^t |y_2(\tau)|^{-\frac{n}{2}(p-1)} d\tau \right)^{-\frac{1}{p-1}}, \tag{1.12}
\]

\[
\|u(t)\|_{L^2} \leq C \left( \int_{T_0}^t |y_2(\tau)|^{-\frac{n}{2}(p-1)} d\tau \right)^{-\frac{1}{p-1}(s+n)}, \tag{1.13}
\]

On the other hand, assume that \( p \) is the super-critical, namely \( p \) satisfies (1.10). Then, for any non-trivial \( u_0 \in H^{s,s}(\mathbb{R}^n) \) with \( \|u_0\|_{H^{s,s}} \leq \delta_0 \), the corresponding solution to (1.1) has a uniformly lower and upper bounds such that for any \( t \geq T_0 \),

\[
e^{-C_{T_0} |\text{Im } \lambda||u_0||_{H^{s,s}}^p} \|u_0\|_{L^2} \leq \|u(t)\|_{L^2} \leq \|u_0\|_{L^2}, \tag{1.14}
\]

where \( C_{T_0} > 0 \) is independent of \( t \geq T_0 \).

Even for the case \( \sigma(t) \equiv 0 \), the uniform estimate (1.11) extends the result [45] to higher dimensions \( n = 2, 3 \) and fractional derivatives \( s \in (1, 2) \). The key of such extension is uniform \( H^s \) estimate (1.11) which enable us to deal with the high-frequency parts for higher dimensions (see, section 4). Besides our results includes more generalized models of \( \sigma(t) \) and if \( \sigma(t) \) appears, then the critical exponent for \( L^2 \)-decay of solutions is different depending on the time decay order of \( \sigma(t) \). Hayashi-Li-Naumkin [17] and Hoshino [21] showed an analogous uniform bound of solutions to (1.2) with \( \sigma \equiv 0 \) by imposing special condition (1.4). Theorem 1.2 relax the dissipative restriction (1.4) and associated estimates (1.11) and (1.13) hold by only assuming \( \text{Im } \lambda < 0 \) even if \( \sigma(t) \) is appearance.

The pointwise estimate (1.12) corresponds to the decay estimate of the dissipative problem (1.2) under the critical or sub-critical setting. Hence, (1.12) extends the results [52], [37] and [38] to the problem (1.1) which contains the time decaying harmonic potential.

If we assume further that \( \lambda \in \mathbb{C} \setminus \{0\} \) satisfies (1.4), then the solution to (1.1) exists globally in time without any restriction of the size of initial data and its \( L^2 \)-norm decays for all dimensions \( n \geq 1 \). In the followings, we let \( n \geq 1 \) and \( p > 1 \) for \( n = 1, 2 \) and \( 1 < p < \frac{n}{n-2} \), \( n \geq 3 \), and define \( \gamma = \gamma(n) \) so that \( \gamma(1) = \frac{1}{2} \) and \( \gamma(n) = 1 \) for \( n \geq 2 \).

**Theorem 1.3** (Large data case). Let \( \lambda \in \mathbb{C} \setminus \{0\} \) satisfies (1.4) and \( \sigma \) be the real function such that there exists fundamental solutions \( y_1, y_2 \) of the equation (1.6) satisfying (A)-(C). Assume that \( p \) is the critical or sub-critical, namely \( p \) satisfies (1.8) or (1.9), and that \( \delta_0 < \delta \theta \) with some \( 0 < \theta < \frac{n}{2}(1-p) + \gamma p \). Then, for any \( u_0 \in H^{1,1}(\mathbb{R}^n) \), the Cauchy problem (1.1) has a unique global solution \( u \in C([0, \infty); H^{1,1}(\mathbb{R}^n)) \) such that \( U_0(t, 0)|x|U_0(t, 0)^{-1} u \in C([0, \infty); L^2(\mathbb{R}^n)) \) and it satisfies for any \( t \geq 0 \),

\[
\|\nabla u(t)\|_{L^2} \leq C\|\nabla u_0\|_{L^2}, \quad \|U_0(t, 0)|x|U_0(t, 0)^{-1} u(t)\|_{L^2} \leq C\|x|u_0\|_{L^2}, \tag{1.15}
\]
where $C > 0$ is independent of $t \geq 0$. Moreover, there exists $C > 0$ such that for any $t \geq T_0$, if $p$ is subcritical,

$$\|u(t)\|_{L^2} \leq C \max \left\{ t^\frac{-2\delta_+}{(p-1)(2+n)}, t^{\delta_- - \frac{n}{2}} \right\},$$  \hspace{1cm} (1.16)$$

where $\delta_+$ is given by (1.9) and if $p$ is critical,

$$\|u(t)\|_{L^2} \leq C \left( \int_{T_0}^{t} |y_2(\tau)|^{-\frac{n}{2}} d\tau \right)^{\frac{2}{(p-1)(2+n)}}.$$  \hspace{1cm} (1.9)

Now we consider the case where $\sigma(t) \equiv 0$. Then we see that $\delta = 1$ and $\delta_+ = 1 - \frac{n}{2}(p-1)$ since $y_1(t) = 1$ and $y_2(t) = t$. In that case, letting

$$p(n) = \begin{cases} 3 + \sqrt{n^2 + 2n + 9}, & n \geq 2, \\ 1 + \sqrt{2}, & n = 1, \end{cases}$$

Hayashi-Li-Naumkin [17] found the decay estimate such that for $p(n) < p < 1 + \frac{2}{n}$,

$$\|u(t)\|_{L^2} \leq Ct^{-\frac{2\delta_+}{(p-1)(2+n)}}, \quad \left( \frac{2\delta_+}{(p-1)(2+n)} = \left( \frac{1}{p-1} - \frac{n}{2} \right) \frac{2}{n+2} \right)$$

and as far as we know, the case where $p < p(n)$ has not been known yet. Here in theorem 1.3 with $\sigma \equiv 0$, the necessary condition for $L^2$-decay is $\delta_+ < \frac{4\theta}{2}$ and, together with $\delta = 1$ and $\delta_+ = 1 - \frac{n}{2}(p-1)$, that yields the lower bounds for $p$ such as $p > p_{**}(n)$, where $p_{**}(n)$ is given by

$$p_{**}(n) = \begin{cases} 2, & n = 1, \\ 1, & n = 2, \\ \frac{4+n}{2+n}, & n \geq 3. \end{cases}$$

We notice that $p_{**}(n) < p(n)$ for $n = 1, 2$, and also $\frac{4+n}{2+n} = \frac{3+\sqrt{n^2+2n+1}}{2+n} < p(n)$ for $n \geq 3$. Moreover, setting

$$p_*(n) \equiv \frac{n^2 + 3n + 6 + \sqrt{9n^2 + 28n + 36}}{(n+2)^2},$$

we find that $p_{**}(n) < p_*(n) < p(n)$. For $p_*(n) \leq p < p(n)$, we have $\left( \frac{1}{p-1} - \frac{n}{2} \right) \frac{2}{n+2} < \frac{2\delta_+}{(p-1)(2+n)}$, which implies a natural extension of [17] in the sense of lower bounds for $p$:

**Theorem 1.4.** Let $n \geq 1$, $\sigma(t) \equiv 0$ and that $p(n)$, $p_*(n)$ and $p_{**}(n)$ are equivalent to that of (1.17), (1.19) and (1.18), respectively. Suppose that all conditions in Theorem 1.3 with $\sigma(t) \equiv 0$ are fulfilled. Then the following decay estimate holds:

$$\|u(t)\|_{L^2} \leq \begin{cases} Ct^{-\frac{2\delta_+}{(p-1)(2+n)}}, & p_*(n) \leq p < 1 + \frac{2}{n}, \\ Ct^{\delta_+ - \frac{n}{2}}, & p_{**}(n) < p < p_*(n). \end{cases}$$  \hspace{1cm} (1.20)
The decay estimate (1.20) contains the result [17] for $p > p(n)$ and also implies that the dissipative solution to (1.2) shows the same $L^2$-decay in [17] for $p_+(n) \leq p \leq p(n)$. Even if $p_+(n) < p \leq p_+(n)$, dissipative solutions show the $L^2$-decay and $p_+(n)$ is appeared by estimating approximate solutions to (1.2) as $t \to \infty$. Namely, the estimate (1.20) is improvement for the lower bound of $p$ to exhibit the $L^2$-decay of solutions to (1.2) under the sub-critical case.

Due to Theorem 1.2 and Theorem 1.3, the integrability of $|y_2(t)|^{-n(p-1)/2}$ give a criterion for the asymptotic behavior of dissipative solutions to (1.1).

The proof of Theorem relies on the reduction of (1.1) to the following problem:

\[
\begin{cases}
i\partial_t v + \frac{1}{2y_1(t)^2} \Delta v = \lambda |y_1(t)|^{-\frac{n(p-1)}{2}} |v|^{p-1} v, & t > T_0, \quad x \in \mathbb{R}^n, \\
v(T_0, x) = v_0, & x \in \mathbb{R}^n,
\end{cases}
\]

(1.21)

where $v_0 = e^{-i y_1(T_0) y_1(T_0)|x|^2/2 \log |y_1(T_0)| A u_0(x)$, $A = (x \cdot (-i \nabla) + (-i \nabla) \cdot x)/2$ is the generator of the dilation group. In virtue of this transformation, one enable to use the decomposition of the associated semigroup to observe the asymptotic behavior of dissipative solutions. Namely, the solution to (1.21) is well approximated by

\[\mathcal{F}[U_Y(t)^{-1}v](t, \xi), \quad U_Y(t) = e^{i \frac{y_2(t)}{y_1(t)} \Delta} \]

(1.22)

for large $t$ and any $\xi \in \mathbb{R}^n$. Here $Y(t) = y_2(t)/2y_1(t)$ and the approximated solution (1.22) satisfies

\[
\frac{1}{2} \partial_t \left| \mathcal{F}[U_Y(t)^{-1}v](\xi) \right|^2 = - \frac{|\text{Im} \lambda|}{|y_2(t)|^{n(p-1)/2}} \left| \mathcal{F}[U_Y(t)^{-1}v](\xi) \right|^{p+1}
\]

\[+ O\left(|y_2(t)|^{-n(p-1)/2} Y(t)^{-\frac{q}{2}}, \quad t \to \infty, \right. \]

(1.23)

for $0 < \theta < 1$ (cf. [18], [26], [34]). If a top term (first term) is dominant compared with a remainder term (second term) in (1.23), by solving (1.23), we formally see that

\[
\left| \mathcal{F}[U_Y(t)^{-1}v](\xi) \right|^2 \simeq \left( \frac{\left| \mathcal{F}[U_Y(T_0)^{-1}v](\xi) \right|^{p-1}}{1 + |\text{Im} \lambda| \left| \mathcal{F}[U_Y(T_0)^{-1}v](\xi) \right|^{p-1} \int_{T_0}^t |y_2(\tau)|^{-\frac{q}{2} (p-1)} d\tau} \right)^{\frac{2}{p-1}}
\]

(1.24)

as $t \to \infty$. If the remainder term is dominant, we does not have the expression (1.24). However one can obtain the $L^2$-decay property of dissipative solutions by extracting a time decay from the remainder term.

If $|y_2(t)|^{-\frac{q}{2} (p-1)}$ in the denominator of the right side of (1.24) is integrable, the approximated solution (1.22) does not decay and hence the solution itself, while the solution decays if $|y_2(t)|^{-\frac{q}{2} (p-1)}$ is not integrable with $t = \infty$.

2 Preliminaries

We start to compare the nonlinear powers $p(n)$ and $p_+(n)$ defined by (1.17) and (1.19).
Lemma 2.1. For $n \geq 1$, let $p(n)$ and $p_*(n)$ be defined by (1.17) and (1.19), respectively. Then $p(n) > p_*(n)$.

Proof of Lemma 2.1. In the case where $n = 3, 4$, they are calculated as

$$p(3) = \frac{3 + \sqrt{24}}{5} > \frac{24 + \sqrt{201}}{25} = p_*(3), \quad p(4) = \frac{3 + \sqrt{33}}{6} > \frac{34 + \sqrt{252}}{36} = p_*(4).$$

As for $n \geq 5$, by employing the Bernoulli inequality; $\sqrt{1 + x} \leq 1 + \frac{1}{2}x$ for any $x \geq 0$, we compute

$$(n + 2)^2(p(n) - p_*(n)) = (n + 2)\sqrt{n^2 + 2n + 9} - n^2 - 3n\sqrt{1 + \frac{28}{9n} + \frac{4}{n^2}}$$

$$\geq (n + 2)\sqrt{n^2 + 2n + 9} - \left(n^2 + 3n + \frac{14}{3} + \frac{6}{n}\right).$$

For $n \geq 5$, we have that $n^2 + 3n + \frac{14}{3} + \frac{6}{n} \leq n^2 + 3n + 6$ and which implies

$$(n + 2)^2(p(n) - p_*(n)) \geq \sqrt{(n^2 + 3n + 6)^2 + 8n} - (n^2 + 3n + 6) > 0.$$

Thus we show $p_*(n) < p(n)$. \hfill \square

Before considering the $L^2$-decay, it is necessary to show the existence of global-in-time solutions to (1.1) and hence we first divide the equation (1.1) into two parts for the time interval; in the case where $t \in [0, T_1), T_1 > 2T_0$, we obtain the time local solution to (1.1) via the approach due to Kawamoto-Muramatsu [31].

Proposition 2.2 ([31]). Let $n/2 < s < p$. Then for any $T_1$, there exists $\varepsilon_{0,0} = \varepsilon_{0,0}(T_1) > 0$ such that for any $u_0$ satisfying $\|u_0\|_{H^{s,s}} = \varepsilon_{0,0}$, there uniquely exists a solution to (1.1) and $0 < \varepsilon_0 \ll 1$ such that $u \in C([0, T_1]; H^{s,s})$ and

$$\sup_{t \in [0, T_1]} \|u(t, \cdot)\|_{H^{s,s}} \leq \varepsilon_0 \|u_0\|_{H^{s,s}}.$$

On the other hand, the case where $t \in [T_1, \infty)$, to consider the mass decay property and the existence of the global-in-time solution at the same time, we reduce the equation by using unitary transform. Thanks to this reduction, one enable to use the approaches in some previous works. By the sub-consequence of the decomposition scheme by Korotyaev [39], we have the following;

Lemma 2.3. Let $T_0 > 0$, $u(t, x)$ and $y_0(t)$ be a solution to (1.1) and a solution to $y''(t) + \sigma(t)y(t) = 0$, respectively. Assume that there exist $c > 0$ and $T_0 \geq 0$ such that $|y_0(t)| \geq c$ and $y_0(t), y_0'(t)$ are continuous for all $t \geq T_0$. Define $v(t, x)$ as

$$v(t, x) = e^{-iy_0(t)y_0'(t)/4}e^{i(\log |y_0(t)|A)}u(t, x), \quad A = \frac{i}{2}x \cdot \nabla + \nabla \cdot x$$
with the unitary operators $e^{-iy_0(t)y^0_0(t)x^2/2}$ and $e^{i\log |y_0(t)|A}$ on $L^2(\mathbb{R}^n)$. Then $v(t, x)$ satisfies (1.21) with

$$v_0 = e^{-iy_0(T_0)y^0_0(T_0)x^2/2}e^{i\log |y_0(T_0)|A}u_0(x).$$

Moreover, the mass conserves

$$\|u(t)\|_{L^2} = \|v(t)\|_{L^2}, \quad t \geq T_0 \tag{2.1}$$

and it follows the $L^\infty$-identity:

$$\|u(t)\|_{L^\infty} = |y_0(t)|^{1/2} \|v(t)\|_{L^\infty}, \quad t \geq T_0. \tag{2.2}$$

**Proof.** Let $y_0(t)$ be a solution to $y''(t) + \sigma(t)y(t) = 0$ and $D_x = -i\nabla$. Straightforward calculation shows

$$i\frac{\partial}{\partial t}v(t, x) = A_1(t) + A_2(t)$$

with

$$A_1(t) \equiv i\frac{\partial}{\partial t} \left( e^{-iy_0(t)y^0_0(t)x^2/2}e^{i\log |y_0(t)|A}u(t, x) + e^{-iy_0(t)y^0_0(t)x^2/2}e^{i\log |y_0(t)|A}H_0(t)u(t, x) \right)$$

and

$$A_2(t) = \mu e^{-iy_0(t)y^0_0(t)x^2/2}e^{i\log |y_0(t)|A}|u(t, x)|^2 u(t, x).$$

At first, we calculate $A_1(t)$. Let $J_0(t) := \left( e^{-iy_0(t)y^0_0(t)x^2/2}e^{i\log |y_0(t)|A} \right)$. On $S(\mathbb{R}^n)$, we have

$$i\frac{\partial}{\partial t}J_0(t) = \frac{(y_0')^2 + y_0y''_0}{2}J_0(t) - \frac{y_0'}{y_0} \left( e^{-iy_0(t)y^0_0(t)x^2/2}Ae^{i\log |y_0(t)|A} \right)$$

$$= \left( \frac{(y_0')^2 + y_0y''_0}{2} - \frac{y_0'}{y_0} \frac{x \cdot (D_x + y_0y'_0x) + (D_x + y_0y'_0x) \cdot x}{2y_0} \right) J_0(t)$$

$$= \left( \frac{y_0y''_0 - (y_0')^2}{2} \right) |x|^2 - \frac{y_0'}{y_0} A J_0(t)$$

On the other hand, commutator calculations on $S(\mathbb{R}^n)$ yields (e.g., see, Ishida-Kawamoto [23] 4) yields

$$e^{i/3A}x^2e^{-i/3A} = e^{2\beta}x^2, \quad e^{i/3A}|D_x|^2e^{-i/3A} = e^{2\beta}|D_x|^2$$

and by using this equation, we have

$$J_0(t)|D_x|^2 = \frac{1}{y_0^2} e^{-iy_0(t)y^0_0(t)x^2/2} |D_x|^2 e^{i\log |y_0(t)|A}$$

$$= \frac{1}{y_0^2} (D_x + y_0y'_0x)^2 J_0(t)$$

and

$$J_0(t)|x|^2 = y_0^2 |x|^2 J(t).$$
Therefore we have

\[ J_0(t)H_0(t) = \left( \frac{|D_x|^2}{2y_0} + \frac{y_0'}{y_0} A + \frac{2}{y_0^2} |x|^2 + \frac{y_0^2}{2} |x|^2 \right) J_0(t) \]

and

\[ A_1(t) = \left( \frac{|D_x|^2}{2y_0(t)^2} + \frac{y_0(t)}{2} (y_0''(t) + \sigma(t)y_0(t)) |x|^2 \right) J_0(t)u(t, x) \]

\[ = -\frac{\Delta}{2y_0(t)^2} v(t, x), \]

where we use the fact that \( y_0 \) solves (1.6). As for \( A_2(t) \), we employ the equation on \( \phi \in S(\mathbb{R}^n) \) that

\[ (e^{i\log |y_0(t)|} A \phi) (x) = |y_0(t)|^{n/2} \phi (y_0(t)x), \]

see e.g., Kawamoto-Yoneyama [30] 2. By this equation, it holds that

\[ J_0(t)|u(t, x)|^{p-1} u(t, x) = |y_0(t)|^{n/2} e^{-i\log |y_0(t)|} |y_0(t)x|^{p-1} u(t, y_0(t)x) \]

\[ = e^{-i\log |y_0(t)|} |y_0(t)|^{n/2} |y_0(t)|^{-n/2} e^{i\log |y_0(t)|} A u(t, x) |^{p-1} e^{i\log |y_0(t)|} A u(t, x) \]

\[ = |y_0(t)|^{-n(p-1)/2} |v(t, x)|^{p-1} v(t, x). \]

for all \( t \geq T_0 \) and \( x \in \mathbb{R}^n \). \( \square \)

**Remark 2.4.** The Lemma 2.3 is true for any \( y_0 \) satisfying the conditions in Lemma 2.3. On the other hand, the \( w \) solution to the linear part of reduced equation

\[ i\partial_tw + \frac{1}{2y_0(t)^2} \Delta w = 0 \]

can be written as

\[ w(t, \cdot) = e^{i\frac{\Delta}{2} \int_{T_0}^{t} \frac{1}{y_0(s)^2} ds} w(T_0, \cdot) \]

and to characterize the mass decay/conserve properties with using \( \int_{T_0}^{t} y_0(s)^{-2} ds \) is complicated. Since the arbitrariness of the choice of \( y_0(t) \), we replace \( y_0 \) as \( y_1 \). Then one finds

\[ \frac{d}{dt} \frac{y_2(t)}{y_1(t)} = \frac{1}{y_1(t)^2} \]

and hence \( w(t) \) with replacement \( y_0 \) to \( y_1 \) can be written as \( w(t, \cdot) = e^{i\Delta(Y(t) - Y(T_0))} w(T_0, \cdot) \), where \( Y(t) = (y_2(t)/(2y_1(t))) \). This expression of \( w \) enable us to characterize the mass decay/conserve properties with only using the asymptotic behavior of \( y_1(t) \) and \( y_2(t) \) in \( t \).
In the following, we set
\[ U(t, s) = e^{i(Y(t) - Y(s))\Delta} =: U_Y(t)U_Y(s)^{-1}, \quad Y(t) = \frac{y_2(t)}{2y_1(t)}, \quad J(t) = e^{-iy_1(t)y'_1(t)|x|^2/2}e^{i\log|y_1(t)|A}. \]

Then, we have the following lemma;

**Lemma 2.5.** Let \( T_0 > 0 \) and \( \tilde{v}(t, x) \equiv U(t, T_0)\tilde{v}_0 \). Then \( \tilde{v} = \tilde{v}(t, x) \) solves
\[
\begin{aligned}
& i\partial_t \tilde{v} + \frac{1}{2y_1(t)^2}\Delta \tilde{v} = \lambda |y_1(t)|^{-\frac{n(y_2(t))}{2}}|v|^{p-1}v, \quad t > T_0, \ x \in \mathbb{R}^n, \\
& \tilde{v}(T_0, x) = \tilde{v}_0(x), \quad x \in \mathbb{R}^n
\end{aligned}
\tag{2.3}
\]
and an operator \( U_Y(t) \) is decomposed as
\[
U_Y(t)\tilde{v}_0 = e^{\frac{i\tilde{v}(t)}{2y_1(t)^2}\Delta} \tilde{v}_0 = \mathcal{M} \left( \frac{y_2(t)}{2y_1(t)} \right) \mathcal{D} \left( \frac{y_2(t)}{2y_1(t)} \right) \mathcal{F}\mathcal{M} \left( \frac{y_2(t)}{2y_1(t)} \right) \tilde{v}_0 \tag{2.4}
\]
for all \( t \geq T_0 \).

This lemma can be shown by combining the identity \( U(T_0, T_0) = \text{Id}_{L^2} \), the uniqueness of the propagator \( U(t, T_0) \) and the well-known decomposition formula that for any \( s \in \mathbb{R}, \ e^{is\Delta/2} = \mathcal{M}(s)\mathcal{D}(s)\mathcal{F}\mathcal{M}(s). \)

Noting these, we consider the nonlinear equation associated the linear problem (2.3).
\[
\begin{aligned}
& i\partial_t v + \frac{1}{2y_1(t)^2}\Delta v = \lambda |y_1(t)|^{-\frac{n(y_2(t))}{2}}|v|^{p-1}v, \quad t > T_0, \ x \in \mathbb{R}^n, \\
& v(T_0, x) = v_0, \quad x \in \mathbb{R}^n
\end{aligned}
\tag{2.5}
\]
where \( y_1 \) and \( y_2 \) are fundamental solutions to (1.6). Here we remark that \( 2y_1(t)Y(t) = y_2(t) \).

Let \( u \) be the solution to the problem (1.1) on \([0, T_1] \), \( T_1 > 2T_0 \). Take \( T_0 \) so that Assumption (A)–(C) hold and fix it. Then thanks to Proposition 2.2, there exist a solution to (1.1) satisfying \( u(t, \cdot) \in C([0, T_1]; H^{s,s}(\mathbb{R}^n)) \) and a sufficient small constant \( \varepsilon_0 > 0 \) so that
\[
\sup_{t \in [T_0, T_1]} \| u(t, \cdot) \|_{H^{s,s}} \leq \varepsilon_0.
\]
Here for all \( \phi \in H^{s,s} \) and \( t \in [T_0, T_1] \),
\[
\| J(t)\phi \|_{H^{s,s}} \leq C \langle \| -\Delta \|^{s/2} + |x|^s + 1 \rangle J(t)|\phi\|_{L^2} = C \langle \| -\Delta \|^{s/2} + |x|^s + 1 \rangle e^{-iy_1(t)y'_1(t)|x|^2/2}e^{i\log|y_1(t)|A}|\phi\|_{L^2} = C \langle \| -i\nabla y_1(t) - y'_1(t)x|^{s} + |y_1(t)|^{-s}|x|^s + 1 \rangle |\phi\|_{L^2} \leq \tilde{C}_0(T_1) \| \phi \|_{H^{s,s}}
\tag{2.6}
\]
holds, where we use \( |y_1(t)| \geq c_0 \) for \( t \geq T_0 \) and interpolation between
\[
\| (-a_2|x|+b)^{s} \cdot 1 \cdot (\langle -\Delta + |x|^2 + 1 \rangle^{s} \| \leq \tilde{C}_0(T_1)
\]
and
\[ \left\| -ia\nabla + bx \cdot 1 \cdot (-\Delta + |x|^2 + 1)^{-1} \right\| \leq \hat{C}_1(T_1), \]
see, e.g., Kato [24], where \( a, b \in \mathbb{R} \). With regard to the above inequality, (2.1) and (2.2), we also have the local-in-time solution \( v(t) \) on \( t \in [T_0, T_1] \) to (1.21) and that has the properties for \( t \in [T_0, T_1] \);
\[
(Y(t))^{n/2} \|v(t)\|_{L^\infty} = (Y(t))^{n/2} \|J(t)u(t)\|_{L^\infty}
\]
\[
= (Y(t))^{n/2} |y_1(t)|^{n/2} \|u(t)\|_{L^\infty}
\]
\[
\leq \hat{C}_1(T_1) \|u(t)\|_{L^\infty}, \tag{2.7}
\]
\[
\|v(t)\|_{H^{s,s}} \leq \|J(t)u(t)\|_{H^{s,s}} \leq \hat{C}_0(T_1) \|u(t)\|_{H^{s,s}} \tag{2.8}
\]
and
\[
\|J^s(t)v(t)\|_{L^2} \leq C(T_1) \|J(t)u(t)\|_{H^{s,s}} \leq \hat{C}_2(T_1) \|u(t)\|_{H^{s,s}}. \tag{2.9}
\]
Rough calculation demands the \( T_1 \) dependence for constants \( \hat{C}_j(T_1), \ j = 0, 1, 2 \) and hence in the followings we remove the \( T_1 \) dependence of which by using the energy estimate for the associated problem (1.1).

We define an operator which provide a dispersive time decay of solutions to (1.1) by
\[
J(t)f = U_Y(t)xU_Y(t)^{-1}f = M(Y(t))iY(t)\nabla M^{-1}(Y(t))f = (x + iY(t)\nabla)f, \tag{2.10}
\]
and the fractional power of \( J(t) \) is defined that for \( 0 < \gamma < 1 \),
\[
|J|^{\gamma}f = U_Y(t)|x|^{\gamma}U_Y(t)^{-1}f = M(Y(t))|Y(t)|^{\gamma}D_x^{\gamma}M^{-1}f, \tag{2.11}
\]
where \( D_x^{\gamma}f = F^{-1}[\xi^{\gamma}F] \). Note that the operators \( J(t) \) and \( |J|^{\gamma}(t) \) commute with \( i\partial_t + \frac{1}{2y_1(t)^2} \partial_x^2 \). Then, the unitary operator \( U_Y(t) \) has the following decay property (cf. Ozawa [47], Hayashi-Naumkin [16], Kita-Shimomura [37], [38], see also [21], [34], [26], [52]).

**Lemma 2.6.** Let \( n \geq 1, s > n/2, s_1 = \min\{s - n/2, 1\} \) and \( p \) be the super-critical given by (1.10). Then, there exists \( C > 0 \) such that for any \( f \in \mathcal{S}(\mathbb{R}) \) and \( t \geq T_0 \), the following pointwise estimate holds:
\[
\left\| F[U_Y(t)^{-1}\langle f \rangle^{p-1}f] - |y_2(t)|^{-\frac{2}{p}(p-1)}|F[U_Y(t)^{-1}f]|^{p-1}|F[U_Y(t)^{-1}f]| \right\|_{L^\infty} \tag{2.12}
\]
\[
\leq C|y_2(t)|^{-\frac{2}{p}(p-1)}|Y(t)|^{-s_1}||f||_{L^2} + ||J^s(t)f||_{L^2}^p.
\]
Moreover, the following \( L^2 \)-decay property holds: Suppose \( 1 < p < \infty \), if \( n = 1, 2 \), and \( 1 < p < \frac{n}{n-2} \) if \( n \geq 3 \). Then we have
\[
\left\| F[U_Y(t)^{-1}\langle f \rangle^{p-1}f] - |y_2(t)|^{-\frac{2}{p}(p-1)}|F[U_Y(t)^{-1}f]|^{p-1}|F[U_Y(t)^{-1}f]| \right\|_{L^2} \tag{2.13}
\]
\[
\leq C|y_2(t)|^{-\frac{2}{p}(p-1)}|Y(t)|^{-\theta}||f||_{L^2} + ||J(t)f||_{L^2}^p,
\]
where \( \theta = 1 \) if \( n = 1, 0 < \theta < 1 \) if \( n = 2 \), and \( \theta = \frac{n}{2} - \frac{n-2}{2}p \) if \( n \geq 3 \). Here \( J(t) = x + iY(t)\nabla \) and \( Y(t) = y_2(t)/2y_1(t) \).
Proof of Lemma 2.6. The proof of (2.12) and (2.13) relies on the previous works [16], [26], [37], [38], [21]. We only prove (2.13) briefly (see [21] for details). Let the left side in (2.13) be \( R(t, \xi) \). Then, \( R \) is decomposed by \( R = R_1 + R_2 \), where

\[
R_1(t, \xi) = \frac{\lambda}{|y_2(t)|^{\frac{4}{p(p-1)}}} F(e^{-i \frac{|x|^2}{2Y(t)}} - 1) F^{-1} \left( |F[e^{-i \frac{|x|^2}{2Y(t)} U_Y^{-1} f}]|^{p-1} (F[e^{-i \frac{|x|^2}{2Y(t)} U_Y^{-1} f}]') \right),
\]

\[
R_2(t, \xi) = \frac{\lambda}{|y_2(t)|^{\frac{4}{p(p-1)}}} \left( |F[e^{-i \frac{|x|^2}{2Y(t)} U_Y^{-1} f}]|^{p-1} (F[e^{-i \frac{|x|^2}{2Y(t)} U_Y^{-1} f}]') - |FU_Y^{-1} f|^{p-1} (FU_Y^{-1} f)' \right).
\]

Let \( n \geq 3 \). By the inequality \( |e^{-i |x|^2 / 2Y(t)} - 1| \leq C(|x| / |Y(t)|)^{\theta / 2} \) for \( \theta = \frac{n}{2} - \frac{n-2}{2} p \in (0, 1) \), we have

\[
\|R_1(t)\|_L^2 \leq C|Y(t)|^{-\frac{n}{2}} \left\| \nabla^{\theta} \left( |F[e^{-i \frac{|x|^2}{2Y(t)} U_Y^{-1} f}]|^{p-1} (F[e^{-i \frac{|x|^2}{2Y(t)} U_Y^{-1} f}]') \right) \right\|_L^2 \\
\leq C|Y(t)|^{-\frac{n}{2}} \left\| |F[e^{-i \frac{|x|^2}{2Y(t)} U_Y^{-1} f}]|^{p-1} \right\|_{L^{\frac{n}{n-\theta}}} \left\| \nabla^{\theta} (F[e^{-i \frac{|x|^2}{2Y(t)} U_Y^{-1} f}]') \right\|_{L^q} \\
\leq C|Y(t)|^{-\frac{n}{2}} \left\| |F[e^{-i \frac{|x|^2}{2Y(t)} U_Y^{-1} f}]|^{p-1} \right\|_{L^{\frac{n(p-1)}{n(p-1)-\theta}}} \left\| \nabla (F[e^{-i \frac{|x|^2}{2Y(t)} U_Y^{-1} f}]') \right\|_{L^2},
\]

where we apply the fractional chain rule (see Christ-Weinstein [12] for details and Kenig-Ponce-Vega [32]) to the above second estimate and the Gagliardo-Nirenberg inequality and the Sobolev embedding with

\[
\frac{1}{q} - \frac{\theta}{n} = \frac{1}{2} - \frac{1}{n}, \quad \frac{n(p-1)}{n(p-1)-\theta} = \frac{2n}{n-2},
\]

respectively. Noting that \( J(t) f = U_Y |x| U_Y^{-1} f \), and using interpolation, we have (2.13).

We next estimate \( R_2(t) \) in the similar argument for \( R_1 \) that

\[
\|R_2(t)\|_L^2 \leq C \left( \left\| F[e^{-i \frac{|x|^2}{2Y(t)} U_Y^{-1} f}] \right\|_{L^{\frac{n(p-1)}{n(p-1)-\theta}}}^{p-1} + \left\| FU_Y^{-1} f \right\|_{L^{\frac{n(p-1)}{n(p-1)-\theta}}}^{p-1} \right) \left\| F(e^{-i \frac{|x|^2}{2Y(t)}} - 1) U_Y^{-1} f \right\|_{L^q} (2.14) \\
\leq C \|J(t) f\|_{L^2}^{p-1} \left\| \nabla^{1-\theta} F(e^{-i \frac{|x|^2}{2Y(t)}} - 1) U_Y^{-1} f \right\|_{L^2} \\
\leq C|Y(t)|^{-\frac{n}{2}} \|J(t) f\|_{L^2}^{p-1} \| |x| U_Y^{-1} f \|_{L^2},
\]

where \( \theta = \frac{n}{2} - \frac{n-2}{2} p \). By the interpolation, we obtain (2.13). Thus, the estimte (2.13) follows for any \( n \geq 3 \).

Let \( n = 2 \). By the fractional chain rule, we have that

\[
\|R_1(t)\|_L^2 \leq C|Y(t)|^{-\frac{n}{2}} \left\| \nabla^{\theta} \left( |F[e^{-i \frac{|x|^2}{2Y(t)} U_Y^{-1} f}]|^{p-1} (F[e^{-i \frac{|x|^2}{2Y(t)} U_Y^{-1} f}]') \right) \right\|_L^2 \\
\leq C|Y(t)|^{-\frac{n}{2}} \left\| |F[e^{-i \frac{|x|^2}{2Y(t)} U_Y^{-1} f}]|^{p-1} \right\|_{L^{2(p-1)}} \left\| \nabla (F[e^{-i \frac{|x|^2}{2Y(t)} U_Y^{-1} f}]') \right\|_{L^2} \\
\leq C|Y(t)|^{-\frac{n}{2}} \left\| F[e^{-i \frac{|x|^2}{2Y(t)} U_Y^{-1} f}] \right\|_{L^{2(p-1)}}^{p-1} \left\| \nabla (F[e^{-i \frac{|x|^2}{2Y(t)} U_Y^{-1} f}]') \right\|_{L^2},
\]

13
where we use the Hölder inequality, the Gagliardo-Nirenberg inequality with
\[
\frac{1}{2} = \frac{1 - \theta}{2} + \frac{1}{q}, \quad \frac{1 - \theta}{2(p - 1)} = \frac{1 - \theta_1}{2} + \theta_1 \left(\frac{1}{2} - \frac{1}{2}\right), \quad \frac{1 - \theta}{2} = \frac{1 - \theta_2}{2} + \theta_2 \left(\frac{1}{2} - \frac{1}{2}\right)
\]
for sufficiently small \(\varepsilon > 0\). We remark that \(\theta_2 = 1\) and \(\theta_1 = \frac{(p + \theta - 2)}{(p - 1)} < 1\) for \(\theta < 1\).

The similar argument of (2.14) yields \(\|R_2(t)\|_{L^2} \leq C|Y(t)|^{-\frac{1}{2}}\|J(t) f\|_{L^2}^p\) for any \(0 < \theta < 1\). Hence we obtain (2.13) for \(n = 2\).

Let \(n = 1\). Then the Leibnitz rule and the Sobolev embedding \(H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})\) implies
\[
\|R_1(t)\|_{L^2} \leq C|Y(t)|^{-\frac{1}{2}} \left\|\nabla \left(\left|\mathcal{F}[e^{-i\frac{|x|^2}{2}t}U_{Y,1} f]\right|^{p-1}\mathcal{F}[e^{-i\frac{|x|^2}{2}t}U_{Y,1} f]\right)\right\|_{L^2}
\leq C|Y(t)|^{-\frac{1}{2}} \left\|\mathcal{F}[e^{-i\frac{|x|^2}{2}t}U_{Y,1} f]\right\|_{L^\infty} \left\|\nabla \left(\left|\mathcal{F}[e^{-i\frac{|x|^2}{2}t}U_{Y,1} f]\right|^{p-1}\right)\right\|_{L^2}
\leq C|Y(t)|^{-\frac{1}{2}} (\|f\|_{L^2} + \|J(t) f\|_{L^2})^p,
\]
where we apply the a priori bound (3.2) to have the last estimate. Similarly we have \(\|R_2(t)\|_{L^2} \leq C|Y(t)|^{-\frac{1}{2}} (\|f\|_{L^2} + \|J(t) f\|_{L^2})^p\) and hence we obtain (2.13) for \(n = 1\).

\section{Global existence of dissipative solutions}

The argument of obtaining a global solution to (1.21) is similar to those of the previous works [16], [26] and [52]. For arbitrary small fixed \(\varepsilon_1 > 0\), we define for \(T_0 \leq T_1 \leq 2T_0\),
\[
\|v\|_{X_{T_1}} \equiv \sup_{T_0 \leq t \leq T_1} \left\{ \langle Y(t) \rangle^{\frac{\varepsilon}{2}} \|v(t)\|_{L^\infty} + \langle Y(t) \rangle^{-\varepsilon_1} \left(\|v(t)\|_{H^s} + \|\langle J \rangle^s(t) v\|_{L^2}\right)\right\}, \quad (3.1)
\]
where \(\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}\), \(Y(t) = \frac{\langle Y(t) \rangle}{\langle Y(t) \rangle}\) and \(\varepsilon_1 > 0\) is the small constant. We remark that \(Y(t)^{-1}\) has a time decay property by the assumption (C).

\begin{lemma}
Let \(v\) be the solution to the problem (1.21). Suppose that there exists \(0 < \varepsilon \ll 1\) such that \(\hat{C}_j(T_1) \varepsilon \leq \varepsilon / 3\), where \(\hat{C}_j(T_1), \ j = 0, 1, 2\) are the equivalent to those in (2.6)–(2.9). Then, there exists a constant \(C > 0\), independent of \(T_1\) such that
\[
\|v\|_{X_{T_1}} \leq C \|v_0\|_{H^{s,s}}, \quad (3.2)
\]
where \(\|v_0\|_{H^{s,s}} = \|v_0\|_{H^s} + \|\langle x \rangle^s u_0\|_{L^2}\) is sufficiently small and we denote \(\varepsilon_0 \equiv \|v_0\|_{H^{s,s}}\).

\begin{proof}
To prove the estimate (3.2), it suffices to show that there exists a constant \(C > 0\) such that
\[
\|v\|_{X_{T_1}} \leq C\|v_0\|_{H^{s,s}} + C\|v\|_{X_{T_1}}^p. \quad (3.3)
\]
The estimate (3.3) yields the uniform bound (3.2) if we restrict the size of initial data. The existence of the global solution to (1.21) is an immediate consequence of the a priori bound
The local existence theorem for the problem (1.1) proved by Kawamoto-Muramatsu [31] (cf. [14], [55], [10] and [33]). We estimate $\|\nabla^s v(t)\|_{L^2}$ and $\|J^s(t)u(t)\|_{L^2}$ by the energy method due to Hayashi-Naumkin [16] (see also [26], [34]). We prove the a priori bound in (3.3). Here we only consider the case where $p$ is critical in the sense of $y_2(t)$, because the super-critical case can be shown more easily. In the followings, we employ the Duhamel formula;

$$v(t) = U(t, T_0)v_0 - i\lambda \int_{T_0}^t U(t, \tau) |u(\tau)|^{p-1} u(\tau) d\tau$$

$$= U_Y(t) U_Y(T_0)^{-1} v_0 - i\lambda \int_{T_0}^t U_Y(t) U_Y(\tau)^{-1} |u(\tau)|^{p-1} u(\tau) d\tau.$$

We first estimate the derivative term $\|v(t)\|_{H^s}$ such that for any $t \in [T_0, T_1]$,

$$\|v(t)\|_{H^s} \leq \|v(0)\|_{H^s} + |\lambda| \int_{T_0}^t |y_1(\tau)| \frac{n(p-1)}{2} \|v(\tau)\|^{p-1} u(\tau) \|_{H^s} d\tau$$

$$\leq C \|v_0\|_{H^s} + C \int_{T_0}^t |y_1(\tau)| \frac{n(p-1)}{2} \|v(\tau)\|^{p-1} \|v(\tau)\|_{H^s} d\tau$$

$$\leq C \|v_0\|_{H^s} + C \|v\|_{X_{T_1}} \int_{T_0}^t |y_1(\tau)| \frac{n(p-1)}{2} \|v(\tau)\| \|v(\tau)\|_{H^s} d\tau$$

$$\leq C \|v_0\|_{H^s} + C \|v\|_{X_{T_1}} \int_{T_0}^t |y_2(\tau)| \frac{n(p-1)}{2} \|v(\tau)\|_{H^s} d\tau,$$

where $Y(t) = y_2(t)/2 y_1(t)$ and we use $\|U_Y u\|_{H^s} = \|u\|_{H^s}$. The definition (3.1) and (1.8) yields

$$\|v(t)\|_{H^s} \leq \|v_0\|_{H^s} + \|v\|_{X_{T_1}} \langle Y(t) \rangle^{\varepsilon_1}. \quad (3.4)$$

We next estimate as $\|J^s(t)v(t)\|_{L^2}$. Using the commutative relation $[i\partial_x + \frac{1}{2q_1} \partial_y, J^s(t)] = 0$ and the expression of the operator $|J^s(t)|$ such that $|J^s(t)| = U_Y(t)|x|^s U_Y(t)^{-1}$, we also have

$$\|J^s(t)v(t)\|_{L^2} \leq \|x|^s U_Y(T_0)^{-1} v_0\|_{L^2} + |\lambda| \int_{T_0}^t |y_1(\tau)| \frac{n(p-1)}{2} \|x|^s U_Y(\tau)^{-1} |v(\tau)|^{p-1} v(\tau)\|_{L^2} d\tau$$

$$\leq C \|v_0\|_{H^s} + C \|v\|_{X_{T_1}} \int_{T_0}^t |y_1(\tau)| \frac{n(p-1)}{2} \langle Y(\tau) \rangle^{-\frac{n(p-1)}{2}} \|v(\tau)\|_{L^2} d\tau$$

$$\leq \|v_0\|_{H^s} + C \|v\|_{X_{T_1}} \int_{T_0}^t |y_2(\tau)| \frac{n(p-1)}{2} \|v(\tau)\|_{L^2} d\tau$$

The definition (3.1) and (1.8) yields

$$\|J^s(t)v(t)\|_{L^2} \leq \|v_0\|_{H^s} + \|v\|_{X_{T_1}} \langle Y(t) \rangle^{\varepsilon_1}. \quad (3.5)$$
Finally, we consider the $L^\infty$-bound of the solution to (1.21):
\[
\langle y_2(t) \rangle^{\frac{3}{2}} \|v(t)\|_{L^\infty} \leq C\|v_0\|_{H^{s,\ast}} + C\|v\|_{X_{T_1}^\ast}^p.
\]
(3.6)

To this end, we first show the uniform bound of
\[
\bar{v} \equiv \mathcal{F}[U_Y(t)^{-1}v].
\]

Namely, for any $t \in [T_0, T_1]$, we show
\[
\|\bar{v}(t)\|_{L^\infty} \leq C\|v_0\|_{H^{s,\ast}} + C\|v\|_{X_{T_1}^\ast}^p.
\]
(3.7)

If we show the uniform bound (3.7), we obtain the estimate (3.6) by applying the Fourier transform to (1.21) in such a way that
\[
i\partial_t \bar{v}(t, \xi) = \mathcal{F}[U_Y(t)^{-1}(i\partial_t v + \frac{1}{2y_1}\Delta v)] = \frac{\lambda}{|y_2(t)|^\frac{3}{2}(p-1)}\mathcal{F}[U_Y(t)^{-1}(|v|^{p-1}v)]
\]
(3.8)
\[
= \frac{\lambda}{|y_2(t)|^\frac{3}{2}(p-1)}|\bar{v}(t, \xi)|^{p-1}\bar{v}(t, \xi) + R(t, \xi),
\]

where $R$ is a remainder term given by
\[
R(t) = \frac{\lambda}{|y_2(t)|^\frac{3}{2}(p-1)}\mathcal{F}[U_Y(t)^{-1}(|v|^{p-1}v)] - \frac{\lambda}{|y_2(t)|^\frac{3}{2}(p-1)}|\bar{v}|^{p-1}\bar{v}.
\]
(3.9)

Multiplying the both sides of the equation (3.8) by $\bar{v}$ and taking the imaginary part, we obtain
\[
\frac{1}{2}\partial_t |\bar{v}(t, \xi)|^2 = \frac{\text{Im } \lambda}{|y_2(t)|^\frac{3}{2}(p-1)}|\bar{v}(t, \xi)|^{p+1} + \text{Im } R(t, \xi)(\bar{v}(t, \xi)).
\]
(3.10)

From the condition $\text{Im } \lambda < 0$, we see that the first term of the right hand side in (3.10) is non-positive. Therefore, by noting $\partial_t |f(t)|^2 = 2|f(t)|\partial_t |f(t)|$ for $f(t) \in C^1(\mathbb{R}; \mathbb{C})$, we obtain
\[
\frac{1}{2}\partial_t |\bar{v}(t, \xi)| \leq |R(t, \xi)|,
\]
$T_0 \leq t \leq T_1$, $\xi \in \mathbb{R}^n$.
(3.11)

Integrating the both sides of the inequality (3.11) over $[T_0, t]$, $t \leq T_1$, we see that
\[
|\bar{v}(t, \xi)| \leq |\bar{v}(T_0, \xi)| + 2\int_{T_0}^{T_1} |R(\tau, \xi)| d\tau.
\]
(3.12)

By applying the estimate (2.12) in Lemma 2.6 to the remainder term $R(t, \xi)$, we see that for any $t \in [T_0, T_1]$, $\xi \in \mathbb{R}^n$,
\[
|R(t, \xi)| \leq C|y_2(t)|^{-\frac{3}{2}(p-1)}|Y(t)|^{-s_1} (\|v(t)\|_{L^2} + \|J^s(t)v(t)\|_{L^2})^p,
\]
(3.13)
where $0 < s_1 < \min\{1, s - n/2\}$ and $Y(t) = y_2(t)/2y_1(t)$. Combining (3.13), (3.5), and the $L^2$-a priori bound:

$$
\|v(t)\|^2_{L^2} + 2|\Im \lambda| \int_0^t |y_1(\tau)|^{-\frac{n(p-1)}{2}} \|v(\tau)\|^{p+1}_{L^{p+1}} d\tau = \|v_0\|^2_{L^2}, \tag{3.14}
$$

we obtain that for arbitrary small $\varepsilon_1 > 0$ and $t \geq T_0$,

$$
\|R(t)\|_{L^\infty} \leq C|y_2(t)|^{-\frac{n}{2}(p-1)}|Y(t)|^{-s_1} (Y(t))^{p\varepsilon_1} \|v\|_{X_{T_1}}^p. \tag{3.15}
$$

There exists $\varepsilon_2 > 0$ such that for small $\varepsilon_1$, it follows $|Y(t)|^{-s_1} (Y(t))^{p\varepsilon_1} \leq Ct^{-\varepsilon_2}$, where we use (1.7), and hence the estimate (3.12) yields by (3.15) and the embedding $H^s(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$ for $s > n/2$ that

$$
\|\hat{v}(t)\|_{L^\infty} \leq C \|v(T_0)\|_{H^s} + C \|v\|_{X_{T_1}} \int_0^t |y_2(\tau)|^{-\frac{n}{2}(p-1)} \tau^{-\varepsilon_2} d\tau \tag{3.16}
$$

$$
\leq C \|v_0\|_{H^s} + C \|v\|_{X_{T_1}},
$$

where the constant $C > 0$ is not depend on $T_1$. Then we have by $U_Y(t) = \mathcal{M}(Y(t))\mathcal{D}(Y(t))\mathcal{F}\mathcal{M}(Y(t))$ in (2.4) that

$$
\|v(t)\|_{L^\infty} = \|U_Y(t)U_Y(t)^{-1}v(t)\|_{L^\infty} = |Y(t)|^{-n/2} \|\mathcal{F}[\mathcal{M}(Y(t))U_Y(t)^{-1}]v(t)\|_{L^\infty} \leq |Y(t)|^{-n/2} (\|\mathcal{F}U_Y(t)^{-1}v(t)\|_{L^\infty} + \|\mathcal{F}(\mathcal{M}(Y(t)) - 1)U_Y(t)^{-1}v(t)\|_{L^\infty}). \tag{3.17}
$$

Using $\|f(Y(t)) - 1\|_{L^1} \leq C|Y(t)|^{-s_1} \|x^s f\|_{L^2}$, we obtain

$$
\|v(t)\|_{L^\infty} \leq C|Y(t)|^{-n/2} \left(\|v_0\|_{H^s} + |Y(t)|^{-(s_1-p\varepsilon_1)} \|v\|_{X_{T_1}}^p\right). \tag{3.18}
$$

Therefore, we obtain the $L^\infty$-estimate (3.6).

From inequalities (3.4), (3.5) and (3.6), the a priori bound (3.3) holds and we obtain (3.2) for sufficiently small initial data.

**Lemma 3.2.** Let $u$ be the solution to (2.5) satisfying (3.2). Then, there exists $C > 0$ such that for any $t \geq T_0$,

$$
\|v(t)\|_{H^s} \leq C \|v_0\|_{H^s}. \tag{3.19}
$$

**Proof of Lemma 3.2.** Multiplying (3.10) by $|\xi|^s$ and integrating (3.10) over $[T_0, t]$, we have that

$$
\frac{1}{2} |\xi|^s \hat{v}(t, \xi)|^2 \leq \frac{1}{2} |\xi|^s \mathcal{F}U_Y(T_0)^{-1}v_0|^2 + \int_{T_0}^t \left|R(t, \xi)|\xi|^s \hat{v}(\tau, \xi)\right| d\tau
$$
and integrating it over $\mathbb{R}$ in $\xi$ and using Hölder’s inequality, we have

$$\|\xi|^{\frac{2}{p}} \tilde{v}(t)\|_{L^2}^2 \leq \|v_0\|_{H^{s,\rho}}^2 + 2 \int_{T_0}^t \|R(\tau)\|_{L^2} \|\xi|^{s} \tilde{v}(\tau)\|_{L^2} \, d\tau,$$

(3.20)

where we use

$$\|\xi|^{\frac{2}{p}} \tilde{v}(t, \xi)\|_{L^2}^2 = \left\| \mathrm{e}^{-\frac{\tau y(t)}{2}} |\xi|^{\frac{2}{p}} \mathcal{F}v(t, \xi)\right\|_{L^2}^2.$$

By the estimate (3.2), it follows

$$\|\xi|^{s} \tilde{v}(t)\|_{L^2} \leq C|Y(t)|^{\frac{s}{p}} \|v\|_{X_{T_1}} \leq C|Y(t)|^{\frac{1}{s}} \|v_0\|_{H^{s,\rho}},$$

(3.21)

and (2.13) yields the integrability of

$$\|R(\tau)\|_{L^2} \|\xi|^{s} \tilde{v}(\tau)\|_{L^2} \leq C|y_2(\tau)|^{-\frac{1}{p}} (Y(\tau))^{-\frac{s}{p} - \frac{1}{2}}, \|v_0\|_{H^{s,\rho}}.$$

Hence, by (3.21), we obtain the uniform bound (3.19). \qed

4 Proof of the main Theorems

In this section, we prove $L^2$-decay of dissipative solutions to (1.1). In our proof, we employ the previous approach by [17] besides the frequency dividing approach due to [45]. Such approach enable us to remove an extra exponent $\varepsilon > 0$ appeared in $L^2$-decay order in [17]. By applying Lemma 3.2, one can extract the $L^2$-decay of dissipative solutions to (1.1). We have that this $L^2$-decay holds for higher dimensions $n = 1, 2, 3$, even if we only assume $\text{Im} \lambda < 0$. Moreover this $L^2$-decay property can be seen for the problem (1.1) which contains the time dependent potential.

Proof of Theorem 1.2. Let $1 \leq n \leq 3$, $p > 1$ be the critical or super-critical and $n/2 < s < p$. We first show the pointwise decay (1.12). By (2.12), the error term (3.9) is estimated by

$$\|R(t)\|_{L^\infty} \leq C|y_2(t)|^{-\frac{1}{p}} |Y(t)|^{-\left(s_1 - \rho \varepsilon_1\right)} \|v_0\|_{H^{s,\rho}}, \quad t \geq T_0,$$

(4.1)

where $Y(t) = y_2(t)/y_1(t)$. Let $Y_2(t) = \int_{T_0}^t |y_2(\tau)|^{-\frac{1}{p}} \, d\tau$ and $\tilde{v}(t, \xi) = \mathcal{F}[U_Y(t)^{-1}v](t, \xi)$. Then, the differential equation (3.10) and the inequality

$$\left|R(t)\tilde{v}(t, \xi)\right| \leq C|y_2(t)|^{-\frac{1}{p}} |Y(t)|^{-\left(s_1 - \rho \varepsilon_1\right)} \|v_0\|_{H^{s,\rho}} \left(\|v_0\|_{H^{s,\rho}} + \|v\|_{X_{T_1}}^p\right)
\leq C|y_2(t)|^{-\frac{1}{p}} |Y(t)|^{-\left(s_1 - \rho \varepsilon_1\right)} \|v_0\|_{H^{s,\rho}}^{p+1},$$

18
lead to
\[
\frac{d}{dt} \left( Y_2(t)^\frac{2}{p-1} |\tilde{v}(t, \xi)|^2 \right) \\
= \left( \frac{d}{dt} Y_2(t)^\frac{2}{p-1} \right) |\tilde{v}(t, \xi)|^2 + Y_2(t)^\frac{2}{p-1} \frac{d}{dt} |\tilde{v}(t, \xi)|^2 \\
\leq C |y_2(t)|^{-\frac{2}{p-1}} Y_2(t)^{\frac{2}{p-1}} |\tilde{v}(t, \xi)|^2 \\
+ Y_2(t)^{\frac{2}{p-1} + 1} \left( \frac{-|\text{Im} \lambda|}{|y_2(t)|^{\frac{2}{p-1}}} |\tilde{v}(t, \xi)|^{p+1} + C |y_2(t)|^{-\frac{n(p-1)}{2}} |Y(t)|^{-(s_1-p\varepsilon_1)} \|v_0\|_H^{p+1} \right).
\]

By the Young inequality with \( \frac{p-1}{p+1} + \frac{2}{p+1} = 1 \), there exists \( C > 0 \) such that for small \( \varepsilon > 0 \),
\[
|\tilde{v}(t, \xi)|^2 = (\varepsilon Y_2(t))^{-\frac{2}{p-1}} \cdot (\varepsilon Y_2(t))^{\frac{2}{p-1}} |\tilde{v}(t, \xi)|^2 \leq C \varepsilon^{-\frac{2}{p-1}} Y_2(t)^{-\frac{2}{p-1}} + C \varepsilon Y_2(t) |\tilde{v}(t, \xi)|^{p+1}.
\]
Hence, we have by taking \( \varepsilon = |\text{Im} \lambda| \varepsilon_3 \) with small \( \varepsilon_3 > 0 \) that
\[
\frac{d}{dt} \left( Y_2(t)^{\frac{2}{p-1} + 1} |\tilde{v}(t, \xi)|^2 \right) \leq C(\varepsilon_3 |\text{Im} \lambda|)^{-\frac{2}{p-1}} |y_2(t)|^{-\frac{n(p-1)}{2}} \\
+ CY_2(t)^{\frac{2}{p-1} + 1} |y_2(t)|^{-\frac{n(p-1)}{2}} |Y(t)|^{-(s_1-p\varepsilon_1)} \|v_0\|_H^{p+1} \quad (4.2)
\]
and integrating (4.2) over \([T_0, t] \), we deduce that
\[
Y_2(t)^{\frac{2}{p-1} + 1} |\tilde{v}(t, \xi)|^2 \leq C(\varepsilon_3 |\text{Im} \lambda|)^{-\frac{2}{p-1}} Y_2(t) \\
+ C \|v_0\|_H^{p+1} \int_{T_0}^{t} Y_2(\tau)^{\frac{2}{p-1} + 1} |y_2(\tau)|^{-\frac{n(p-1)}{2}} |Y(\tau)|^{-(s_1-p\varepsilon_1)} d\tau \\
\leq C(\varepsilon_3 |\text{Im} \lambda|)^{-\frac{2}{p-1}} Y_2(t) + C \|v_0\|_H^{p+1},
\]
where the second part is integrable since \( p \) is the critical and the assumption (C) holds. Thus, we have that for any \( t \geq T_0 \),
\[
\|\tilde{v}(t)\|_{L^2}^2 \leq C \left( (\varepsilon_3 |\text{Im} \lambda|)^{-\frac{2}{p-1}} + Y_2(t)^{-1} \|v_0\|_H^{p+1} \right) Y_2(t)^{-\frac{2}{p-1}} \quad (4.3)
\]
and we obtain by (3.17) and \( \|u(t)\|_{L^\infty} = |y_1|^{-\frac{n}{2}} \|v(t)\|_{L^\infty} \) that
\[
\|u(t)\|_{L^\infty} \leq C |y_1(t) Y(t)|^{-\frac{2}{p-1}} \left( \|\tilde{v}(t)\|_{L^\infty} + |Y(t)|^{-(s_1-p\varepsilon_1)} \|v\|_X^{p} \right) \\
\leq C |y_2(t)|^{-\frac{2}{p-1}} Y_2(t)^{-\frac{1}{p-1}} \left( (\varepsilon_3 |\text{Im} \lambda|)^{-\frac{1}{p-1}} + Y_2(t)^{-1} \|v_0\|_H^{p+1} \right)^{1/2} \\
+ |y_2(t)|^{-\frac{2}{p-1}} |Y(t)|^{-(s_1-p\varepsilon_1)} \|v_0\|_H^{p+1},
\]
where \( |Y(t)| = |y_2(t)/2y_1(t)| \leq Ct^{-\delta} \) and \( \varepsilon_2 > 0 \) is the same one in (3.16).

We prove the \( L^2 \)-decay of solutions to (1.1) with higher regular condition \( s > n/2 \). We decompose the low-frequency and high frequency part of the solution. The low frequency
part is controlled by the pointwise estimate (4.3) and high frequency part is estimated by the uniform estimate. Namely, the solution $u$ satisfies that for any $t \geq T_0$ and $r > 0$,

$$\|u(t)\|_{L^2} = \|\tilde{v}(t)\|_{L^2(\|\xi\| \leq r)} + \|\tilde{v}(t)\|_{L^2(\|\xi\| > r)} \leq C r^n \|\tilde{v}(t)\|_{L^\infty} + C r^{-s} \|\|\xi\|^n \tilde{v}(t)\|_{L^2(\|\xi\| > r)} \leq C r^n Y_2(t)^{-\frac{2}{p-1}} \left((\varepsilon_3|\text{Im}\lambda|)^{1-\frac{2}{p}} + Y_2(t)^{-1} \|v_0\|_{H^{s, r}}^{p+1} + C r^{-s} \|\|\xi\|^n \tilde{v}(t)\|_{L^2} \right) \leq C r^n Y_2(t)^{-\frac{2}{p-1}} \left((\varepsilon_3|\text{Im}\lambda|)^{1-\frac{2}{p}} + Y_2(t)^{-1} \|v_0\|_{H^{s, r}}^{p+1} + C r^{-s} \|v_0\|_{H^{s, r}}^{2} \right),$$

where $Y_2(t) = \int_{T_0}^t |y_2(\tau)|^{-\frac{2}{p} (p-1)} d\tau$ and we use Lemma 3.2. This inequality is optimized by taking $r = (|\text{Im}|Y_2(t)|)^{\frac{2}{p-1} (p+1)} \|v_0\|_{H^{s, r}}$.

Namely, we are valid that

$$\|u(t)\|_{L^2} \leq C (|\text{Im}|Y_2(t)|)^{\frac{2}{p-1} (p+1)} \|v_0\|_{H^{s, r}} + C |\text{Im}|Y_2(t)|^{\frac{2}{p-1} (p+1)} Y_2(t)^{-\frac{2}{p-1} (p+1)} \|v_0\|_{H^{s, r}}^{2}.$$ 

We next consider the uniform lower bound when $p$ is the super-critical case. Let $Y(t) = \frac{y_2(t)}{2y_1(t)}$, $J(t) f = (x + iY(t) \nabla) f_0$, $U(t, T_0) \equiv U_Y(t)^{-1} U_Y(T_0)$, $U_Y(t) = e^{iY(t)\Delta}$ and let $v$ be the solution to (1.21) with $v_0 \in H^{s, r}(\mathbb{R}^n)$ given via an a priori estimate (1.15). Since the pointwise estimate

$$\|v(t)\|_{L^\infty} \leq C \|v_0\|_{H^{s, r}} |Y(t)|^{-\frac{2}{p}} + C \|v_0\|_{H^{s, r}}^p |Y(t)|^{-\frac{2}{p} (p-1)} |Y(t)|^{-s_1} (1 + \varepsilon_1) \tag{4.4}$$

holds, one can apply the similar argument in [36]. We deduce by combining (4.4) and the $L^2$-dissipative identity:

$$\|v(t)\|_{L^2}^2 = \|v_0\|_{L^2}^2 - |\text{Im}| \lambda| \int_0^t |y_1(\tau)|^{-\frac{2}{p} (p-1)} \|v(\tau)\|_{L^p}^{p+1} d\tau, \quad t \geq 0, \tag{4.5}$$

that

$$\frac{d}{dt} \|v(t)\|_{L^2} \geq - |\text{Im}| \lambda| |y_1(t)|^{-\frac{2}{p} (p-1)} \|v(t)\|_{L^p}^{p-1} \|v(t)\|_{L^2}^2 \geq - C |\text{Im}| \lambda| |y_2(t)|^{-\frac{2}{p} (p-1)} \left(\|v_0\|_{H^{s, r}} + t^{-\varepsilon_2} \|v_0\|_{H^{s, r}} \right) \|v(t)\|_{L^2}^2. \tag{4.6}$$

By solving (4.6), one can obtain the $L^2$-lower bound for small solutions to (1.21) since $y_2$ satisfies (1.6) and since $\|v(t)\|_{L^2} = \|u(t)\|_{L^2}$, we obtain the $L^2$-lower bound (1.14) for the original solution to (1.1).

**Proof of Theorem 1.3.** We show the $L^2$-decay of solutions for $n \geq 1$ or under the low regularities for $u_0$. By the low regular condition $s = 1$, the pointwise estimate (4.3) does not hold for the higher spatial dimensions $n \geq 2$. Then, one can not prove the $L^2$-decay estimate in the same way of the proof of Theorem 1.2. Hence, we need to improve the estimate of the low-frequency part by considering the asymptotic form of the solution to (1.1). Let $n \geq 1$.
and $p > 1$, $\lambda$ satisfy (1.4) and $r > 0$. Under the assumption (1.4), we find as the consequence of [18] that for $t \geq T_0$,

$$\|v(t)\|_{L^2}^2 + \|v(t)\|_{H^1}^2 + \|J(t)v(t)\|_{L^2}^2 \leq \|v(T_0)\|_{L^2}^2 + \|v(T_0)\|_{H^1}^2 + \|J(T_0)v(T_0)\|_{L^2}^2. \quad (4.7)$$

Here the problem occurs when we consider the existence and boundedness of the time-in-local solution from $t = 0$ to $t = T_0$ since even the solution of the free equation $i\partial_t u = H_0(t)u$ does not have $H^1$ conservation law (conservation law of $(\zeta_2(t)D_x - \zeta'_2(t)x)^2$ is known, see, section 5 of [3] or Lemma 2.4 of [31]). Let $k \in \mathbb{N}$ and set the energy norm

$$\|u\|_{Y_{kT'}} \equiv \sup_{(k-1)T' \leq t \leq kT'} (\|u(t)\|_{H^1}^2 + \|u(t)\|_{H^{0,1}}^2)^{\frac{1}{2}},$$

where $T'$ is a small positive constant so that $0 < T' < 1/64$ and

$$T' \sup_{\tau \in \mathbb{R}} |\sigma(\tau)| < \frac{1}{64}$$

and show that for any $k \in \mathbb{N}$

$$\|u\|_{Y_{kT'}} \leq 4^{k-1} \|u_0\|_{H^{1,1}}. \quad (4.8)$$

Under the assumption (1.4), we can employ the approach of [18] and then the straightforward calculation shows

$$\frac{1}{2} \partial_t \|\nabla u(t)\|_{L^2}^2 = -2\sigma(t) \text{Im} \int_{\mathbb{R}^n} (x \cdot \nabla u) \bar{u} dx + \text{Im} \left( \lambda \int_{\mathbb{R}^n} \nabla (|u|^{p-1}u) \cdot \nabla \bar{u} dx \right)$$

and which deduces

$$\sup_{(k-1)T' \leq t \leq kT'} \|\nabla u(t)\|_{L^2}^2 \leq \|\nabla u((k-1)T')\|_{L^2}^2 + 4 \left( \int_{(k-1)T'}^{kT'} |\sigma(\tau)| d\tau \right) \|u\|_{Y_{kT'}}^2 \leq \|u\|_{Y_{(k-1)T'}}^2 + \frac{1}{16} \|u\|_{Y_{kT'}}^2.$$

By the similar calculations, we also get

$$\frac{1}{2} \partial_t \|xu(t)\|_{L^2}^2 = 2\text{Im} \int_{\mathbb{R}^n} (x \cdot \nabla u) \bar{u} dx + \text{Im} \left( \lambda \int_{\mathbb{R}^n} |x|^2 |u|^{p+1} dx \right) \leq 2\text{Im} \int_{\mathbb{R}^n} (x \cdot \nabla u) \bar{u} dx,$$

and that

$$\sup_{(k-1)T' \leq t \leq kT'} \|xu(t)\|_{L^2}^2 \leq \|u\|_{Y_{(k-1)T'}}^2 + \frac{1}{16} \|u\|_{Y_{kT'}}^2.$$

21
These deduce
\[
\|u\|_{Y_{kT'}}^2 \leq 4 \|u\|_{Y_{(k-1)T'}}^2. 
\] (4.9)
By the standard arguments in construction mappings, we have a unique solution \( u(t) \in C([0, T']; H^{1,1}) \) by taking \( k = 1 \), and using the induction (4.9), we have, for each \( k \), a unique solution \( u(t) \in C([(k-1)T', kT']; H^{1,1}) \). Using such solutions and (4.9), the desired result (4.8) can be obtained. Since we can choose \( k \) arbitrarily, one can also find the existence of a global-in-time unique solution \( u(t) \in C([0, \infty); H^{1,1}) \) without any restrictions of the size of \( u_0 \). By taking \( k_0 \in \mathbb{N} \) as the smallest integer so that \( k_0T' > T_0 \)
\[
\|u\|_{Y_{k_0T'}}^2 \leq 4^{k_0-1}\|u_0\|_{H^{1,1}} \leq 4^{T_0/T'}\|u_0\|_{H^{1,1}},
\]
where
\[
\|u\|_{Y_{k_0T'}} \equiv \sup_{0 \leq t \leq T_0} (\|u(t)\|_{H^1}^2 + \|u(t)\|_{H^{0,1}}^2)^{\frac{1}{2}}.
\]
After \( t \geq T_0 \), one gets the uniform bounds from (4.7) that
\[
\sup_{T_0 \leq t \leq T_1} \|v(t)\|_{H^1}^2 \leq \|v_0\|_{L^2}^2 + \|v(T_0)\|_{H^1}^2 + \|J(T_0)v(T_0)\|_{L^2}^2 \\
\leq C_{T_0} \|u(T_0)\|_{H^{1,1}}^2 \leq C_{T_0} 4^{T_0/T'}\|u_0\|_{H^{1,1}}^2
\]
and
\[
\sup_{T_0 \leq t \leq T_1} \|J(t)v(t)\|_{L^2}^2 \leq \|v_0\|_{L^2}^2 + \|v(T_0)\|_{H^1}^2 + \|J(T_0)v(T_0)\|_{L^2}^2 \\
\leq C_{T_0} \|u(T_0)\|_{H^{1,1}}^2 \leq C_{T_0} 4^{T_0/T'}\|u_0\|_{H^{1,1}}^2,
\]
where we can choose \( T_1 \) arbitrarily large. Piecing these time local arguments and (4.7), we have (1.15).

We next show the \( L^2 \)-decay of dissipative solutions to (1.1). By the Hölder inequality, it follows that for any \( r > 0 \),
\[
\|f\|_{L^2(|\xi| < r)} \leq C r^{\frac{m(p-1)}{2(p+1)}} \|f\|_{L^{p+1}(|\xi| < r)} \quad \text{for any } f \in L^{p+1}(\mathbb{R}^n), 
\] (4.10)
where \( p \geq 1 \) and \( \frac{p-1}{p+1} + \frac{2}{p+1} = 1 \). By applying Young’s inequality to the estimate (4.10) with \( \frac{p-1}{p+1} + \frac{2}{p+1} = 1 \), we see that there exists \( C > 0 \) such that for any \( \varepsilon > 0 \) and \( y > 0 \),
\[
\|f\|_{L^2(|\xi| < r)} \leq C \varepsilon y \|f\|_{L^{p+1}(|\xi| < r)}^{\frac{p+1}{p}} + C \varepsilon^{-\frac{2}{p-2}} y^{-\frac{2}{p-2}} r^n. 
\] (4.11)
Let \( v \) be the global solution to (1.21) with \( u_0 \in H^{1,1}(\mathbb{R}^n) \). Multiplying (3.8) by \( \bar{v} = \mathcal{F}[U^{-1}(t, 0)v] \) and taking imaginary part, we have
\[
\frac{1}{2} \partial_t |\bar{v}(t, \xi)|^2 = -\frac{|\text{Im} \lambda|}{|y_2(t)|^{\frac{m}{2}(p-1)}} |\bar{v}(t, \xi)|^{p+1} + \text{Im} \left\{ R\mathcal{F}[U^{-1}(t, 0)v] \right\}, 
\] (4.12)
where $R$ denotes the remainder term given by (3.9) and $R$ satisfies (3.13). Integrating (4.12) for $\xi$ over $\mathbb{R}^n$, we see that for any $0 < \theta < \frac{n}{2}(1 - p) + \gamma p$ with $\gamma = \frac{1}{2}$ for $n = 1$ and $\gamma = 1$ for $n \geq 2$, and $t \geq T_0$,

$$\frac{1}{2} \frac{d}{dt} \|\tilde{v}(t)\|^2_{L^2} \leq -\left|\text{Im} \lambda\right| \frac{1}{|y_1(t)|^{\frac{n}{2}(p-1)}} \|\tilde{v}(t)\|^2_{L^{p+1}} + C|y_2(t)| - \frac{n}{2}(p-1)|Y(t)|^{-\frac{\theta}{2}} \|u_0\|^{p+1}_{H^{1,1}}. \quad (4.13)$$

Let $\alpha > 0$ be sufficiently large and $Y_2(t) = \int_{t_0}^t |y_2(\tau)|^{-\frac{n}{2}(p-1)} d\tau$. From (4.10), (4.11) with $y = Y_2(t)$ and (4.13), we see that

$$\frac{d}{dt} \{Y_2(t)^{\alpha+1} \|\tilde{v}(t)\|^2_{L^2}\} \leq CY_2(t)^{\alpha} |y_2(t)|^{-\frac{n}{2}(p-1)} \left\{ \|\tilde{v}(t)\|^2_{L^2(|\xi|<r)} + \|\tilde{v}(t)\|^2_{L^2(|\xi| \geq r)} \right\}$$

$$+ CY_2(t)^{\alpha+1} |y_2(t)|^{-\frac{n}{2}(p-1)} \left\{ -2|\text{Im} \lambda| \|\tilde{v}(t)\|_{L^{p+1}} + C|Y(t)|^{-\frac{\theta}{2}} \|u_0\|^{p+1}_{H^{1,1}} \right\} \quad (4.14)$$

$$\leq CY_2(t)^{\alpha} |y_2(t)|^{-\frac{n}{2}(p-1)} \left\{ \left(\varepsilon Y_2(t) \|\tilde{v}(t)\|^2_{L^2(|\xi|<r)} + C|Y(t)|^{-\frac{\theta}{2}} \|u_0\|^{p+1}_{H^{1,1}} \right) \right\}$$

$$+ CY_2(t)^{\alpha+1} |y_2(t)|^{-\frac{n}{2}(p-1)} \left\{ -2|\text{Im} \lambda| \|\tilde{v}(t)\|_{L^{p+1}} + C|Y(t)|^{-\frac{\theta}{2}} \|u_0\|^{p+1}_{H^{1,1}} \right\}$$

By taking $\varepsilon = \varepsilon_5|\text{Im} \lambda|$ with small $\varepsilon_5 > 0$, the first term is absorbed by the fifth term in the above estimate (4.14), namely we have

$$\frac{d}{dt} \{Y_2(t)^{\alpha+1} \|\tilde{v}(t)\|^2_{L^2}\} \leq Y_2(t)^{\alpha} |y_2(t)|^{-\frac{n}{2}(p-1)} \left\{ \left(\varepsilon_5|\text{Im} \lambda|Y_2(t)\right)^{-\frac{\theta}{2}} r^n + \|\tilde{v}(t)\|^2_{L^2(|\xi| \geq r)} \right\}$$

$$+ CY_2(t)^{\alpha+1} |y_2(t)|^{-\frac{n}{2}(p-1)}|Y(t)|^{-\frac{\theta}{2}} \|u_0\|^{p+1}_{H^{1,1}}. \quad (4.15)$$

We estimate the high frequency part of the solution such that

$$\|\tilde{v}(t)\|^2_{L^2(|\xi| \geq r)} \leq r^{-2}\|\xi|\|\tilde{v}(t)\|^2_{L^2(|\xi| \geq r)}$$

$$\leq CR^{-2}\|\nabla v(t)\|^2_{L^2} \leq CR^{-2}\|u_0\|^2_{H^{1,1}}. \quad (4.15)$$

From (4.14) and (4.15), it holds that

$$\frac{d}{dt} \{Y_2(t)^{\alpha+1} \|\tilde{v}(t)\|^2_{L^2}\} \leq CY_2(t)^{\alpha} |y_2(t)|^{-\frac{n}{2}(p-1)} \left\{ \left(\varepsilon_5|\text{Im} \lambda|Y_2(t)\right)^{-\frac{\theta}{2}} r^n + r^{-2}\|u_0\|^2_{H^{1,1}} \right\}$$

$$+ CY_2(t)^{\alpha+1} |y_2(t)|^{-\frac{n}{2}(p-1)}|Y(t)|^{-\frac{\theta}{2}} \|u_0\|^{p+1}_{H^{1,1}}. \quad (4.16)$$

By taking $r = (|\text{Im} \lambda|Y_2(t))^{\frac{2}{(p-1)(2+n)}} \|u_0\|^{\frac{n}{H^{1,1}}}$, the inequality (4.16) is optimized for $r > 0$ and
we obtain that
\[
\frac{d}{dt} \left\{ Y_2(t)^{\alpha+1} \| \tilde{v}(t) \|_{L^2}^2 \right\} \\
\quad \leq C |y_2(t)|^{-\frac{2}{p-1}} \| Y_2(t)^{\alpha} \left| \text{Im} \lambda \right|^{-\frac{2}{(p-1)(2+n)}} \| u_0 \|_{H^{\frac{n+2}{2}}}^{2n} \\
\quad + CY_2(t)^{\alpha+1} |y_2(t)|^{-\frac{2}{p-1}} \| Y(t)^{\alpha} \left| \text{Im} \lambda \right|^{-\frac{2}{(p-1)(2+n)}} \| u_0 \|_{H^{\frac{n+2}{2}}}^{2n} \\
= CY_2(t) Y_2(t)^{\alpha} \left| \text{Im} \lambda \right|^{-\frac{2}{(p-1)(2+n)}} \| u_0 \|_{H^{\frac{n+2}{2}}}^{2n} \\
\quad + CY_2(t)^{\alpha+1} |y_2(t)|^{-\frac{2}{p-1}} \| Y(t)^{\alpha} \left| \text{Im} \lambda \right|^{-\frac{2}{(p-1)(2+n)}} \| u_0 \|_{H^{\frac{n+2}{2}}}^{2n} \tag{4.17}
\]
Since \( y_2(t) \) satisfies (1.8) or (1.9), integrating (4.17) over \([T_0, t]\), we see that
\[
Y_2(t)^{\alpha+1} \| \tilde{v}(t) \|_{L^2}^2 \leq CY_2(t)^{\alpha+1} \left| \text{Im} \lambda \right|^{-\frac{2}{(p-1)(2+n)}} \| u_0 \|_{H^{\frac{n+2}{2}}}^{2n} \\
\quad + C \| u_0 \|_{H^{\frac{n+2}{2}}}^{\alpha+1} \int_{T_0}^{t} \delta_\tau - (1 - \delta_\tau) - \frac{\delta \theta}{2} \, d\tau \\
\quad \leq CY_2(t)^{\alpha+1} \left| \text{Im} \lambda \right|^{-\frac{2}{(p-1)(2+n)}} \| u_0 \|_{H^{\frac{n+2}{2}}}^{2n} + C \| u_0 \|_{H^{\frac{n+2}{2}}}^{\alpha+1} Y_2(t)^{\alpha+1} t^{\delta_\tau - \frac{\delta \theta}{2}} \\
\quad \leq CY_2(t)^{\alpha+1} \left| \text{Im} \lambda \right|^{-\frac{2}{(p-1)(2+n)}} \| u_0 \|_{H^{\frac{n+2}{2}}}^{2n} + C \| u_0 \|_{H^{\frac{n+2}{2}}}^{\alpha+1} Y_2(t)^{\alpha+1} t^{\delta_\tau - \frac{\delta \theta}{2}}.
\]
Hence, under the assumption \( \delta > \frac{2 \delta_\tau}{\theta} \), one gets that
\[
Y_2(t)^{\alpha+1} \| \tilde{v}(t) \|_{L^2}^2 \leq CY_2(t)^{\alpha+1} \left| \text{Im} \lambda \right|^{-\frac{2}{(p-1)(2+n)}} \| u_0 \|_{H^{\frac{n+2}{2}}}^{2n} + C \| u_0 \|_{H^{\frac{n+2}{2}}}^{\alpha+1} Y_2(t)^{\alpha+1} t^{\delta_\tau - \frac{\delta \theta}{2}}.
\]
Therefore, we conclude by the Plancherel theorem that
\[
\| v(t) \|_{L^2} \leq C (Y_2(t) |\text{Im} \lambda|)^{-\frac{2}{(p-1)(2+n)}} \| u_0 \|_{H^{\frac{n+2}{2}}}^{2n} + C (Y_2(t)^{-\alpha-1} + t^{\delta_\tau - \frac{\delta \theta}{2}}) \| u_0 \|_{H^{\frac{n+2}{2}}}^{2n} 
\]
for any \( t \geq T_0 \). If \( p \) is critical, noting \( \delta_\tau = 0 \) and taking \( \alpha \) enough large, we get,
\[
\| v(t) \|_{L^2} \leq C (Y_2(t) |\text{Im} \lambda|)^{-\frac{2}{(p-1)(2+n)}} \| u_0 \|_{H^{\frac{n+2}{2}}}^{2n}.
\]
On the other hand, if \( p \) is sub-critical, we have
\[
Y_2(t) \geq C \int_{T_0}^{t} \tau^{-\delta_\tau} \, d\tau \geq C t^{\delta_\tau}
\]
and hence we get
\[
\| v(t) \|_{L^2} \leq C \| u_0 \|_{H^{\frac{n+2}{2}}} \max \{ (t^{\delta_\tau} |\text{Im} \lambda|)^{-\frac{2}{(p-1)(2+n)}}, t^{\delta_\tau - \frac{\delta \theta}{2}} \},
\]
where \( 0 < \theta < 1 \) is arbitrary.

**Remark 4.1.** If \( \sigma(t) \equiv 0 \), it can be seen that, we do not need to assume the smallness for initial value \( u_0 \). In this case, we notice that \( \delta_\tau = 1 - \frac{\theta}{2} (p - 1) \) and \( \delta = 1 \). Hence we find if \( \delta_\tau < \frac{\delta \theta}{2} \), then \( p > p_\star(n) \), which is smaller than \( p(n) = \frac{3 + \sqrt{9 + \alpha^2 + 2n}}{n+2} \) which was found by [18].
Remark 4.2. Let \( n \geq 3 \) and \( \sigma(t) \equiv 0 \). When the decay order equilibriums, \( p \) can be written as \( p_*(n) = \frac{n^2 + 3n + 6 + \sqrt{9n^2 + 28n + 36}}{(n+2)^2} \) and we deduce \( p_*(n) < p_*(n) < p(n) \). Hence we see that if \( p > p(n) \), the candidate of decay order is \( t^{-\frac{2}{(p-1)(2+n)}} \) and which corresponds to the one in Theorem 1 in [18]. As for the case, \( p_*(n) < p < p_*(n) \), we have the weak decay

\[
\|u(t)\|_{L^2} \leq Ct^{\delta - \frac{44}{2}}.
\]

On the other hand, in \( p_*(n) < p < p(n) \), we have

\[
\|u(t)\|_{L^2} \leq C(t^{\delta_*} |\text{Im}\lambda|)^{-\frac{2}{(p-1)(2+n)}}
\]

and see that this estimate will be a natural extension of the result of [18] from the region \( p(n) < p < 1 + \frac{2}{n} \) to the region \( p_*(n) < p < 1 + \frac{2}{n} \).

5 Models of \( \sigma(t) \).

Here, we introduce some models of \( \sigma(t) \).

Example 1.: We consider the case where \( \sigma(t) = \sigma_0 t^{-2}, t \geq T_0, \sigma_0 \in [0, 1/4) \). Then employing \( \mu = (1 - \sqrt{1 - 4\sigma_0})/2 \in [0, 1/2) \), we have that

\[
y_0(t) = \alpha t^\mu + \beta t^{1-\mu}.
\]

Assuming the suitable condition on \( \sigma(t) \), one finds \( y_1(t) = c_1 t^\mu \) and \( y_2(t) = c_2 t^{1-\mu} + c_3 t^\mu \) with \( c_1, c_2 \neq 0 \) and \( c_3 \in \mathbb{R} \), (see, e.g., [30]). Then it follows that

\[
\left| \frac{y_1(t)}{y_2(t)} \right| \leq Ct^{1-2\mu}, \quad \lim_{t \to \infty} t|y_2(t)|^{-\frac{2}{p-1}} \sim \lim_{t \to \infty} t|t|^{-\frac{2}{p-1}(1-\mu)}
\]

i.e., \( \delta = 1 - 2\mu \) and \( p = 1 + \frac{2}{n(1-\mu)} \) is critical.

Example 2.: Let \( \sigma(t) = -\rho t^{-2}, t \geq T_0, \rho \geq 0 \). Then using \( \theta_- = (1 - \sqrt{1 + 4\rho})/2 < 0 \), we have

\[
y_0(t) = \alpha t^{1-\theta_-} + \beta t^{\theta_-}.
\]

By taking suitable \( \sigma(t) \), we find \( y_1(t) = c_1 t^{\theta_-} \) and \( y_2(t) = c_2 t^{1-\theta_-} + c_3 t^{\theta_-} \) with \( c_1, c_2 \neq 0 \) and \( c_3 \in \mathbb{R} \), which tells us

\[
\left| \frac{y_1(t)}{y_2(t)} \right| \leq Ct^{1-2\theta_-}, \quad \lim_{t \to \infty} t|y_2(t)|^{-\frac{2}{p-1}} \sim \lim_{t \to \infty} t|t|^{-\frac{2}{p-1}(1-\theta_-)}
\]

i.e., \( \delta = 1 - 2\theta_- \) and \( p = 1 + \frac{2}{n(1-\theta_-)} \) is critical.
Example 3.: Consider the case where \( \lim_{t \to \infty} t^2 \sigma(t) = 0 \). In that case, it was known that there exist \( c_1 \neq 0 \) and \( c_2 \neq 0 \) such that

\[
\lim_{t \to \infty} y_1(t) = c_1, \quad \lim_{t \to \infty} \frac{y_2(t)}{t} = c_2,
\]

see, e.g., Naito [42]. In this case, we have \( \delta = 1 \) and \( p = 1 + \frac{2}{n} \) is critical.

Acknowledgment. The first author is supported by JSPS grant-in-aid for Early-Career Scientists #20K14328 and the second author is supported by JSPS grant-in-aid for Early-Career Scientists #22K13937.

References

[1] Antonelli, P., Carles, R., Sparber, C., *On nonlinear the Schrödinger-type equations with nonlinear damping*, International Mathematics Research Notices 217 (2013), 23.

[2] Barab, J., *Nonexistence of asymptotically free solutions of a nonlinear Schrödinger equation*, J. Math. Phys. 25 (1984), 3270-3273.

[3] Carles, R., *Nonlinear Schrödinger equation with time dependent potential*, Comm. Math. Sci., 9 (2011), 937-964.

[4] Carles, R., Silva, J. D., *Large time behavior in nonlinear Schrödinger equation with time dependent potential*, Comm. Math. Sci., 13 (2015), 443-460.

[5] Cazenave, T., *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics vol. 10, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI.

[6] Cazenave, T., Han, Z., *Asymptotic behavior for a Schrödinger equation with nonlinear subcritical dissipation*, Descrete Contin. Dyn. Syst. 40 (2020), 4801-4819.

[7] Cazenave, T., Han, Z., Naumkin, I., *Asymptotic behavior for a dissipative nonlinear Schrödinger equation*, Nonlinear Anal. 205 (2021), 112243, 37pp.

[8] Cazenave, T., Naumkin, I., *Local existence, global existence, and scattering for the nonlinear Schrödinger equation*, Commun. Contemp. Math. 19 (2017), 1650038.

[9] Cazenave, T., Naumkin, I., *Modified scattering for the critical nonlinear Schrödinger equation*, J. Funct. Anal. 274 (2018), 402-432.

[10] Cazenave, T., Weissler, F., *The Cauchy problem for the nonlinear Schrödinger equation in \( H^1 \)*, Manuscripta Math. 61 (1988), 477-494.

[11] Cazenave, T., Weissler, F., *The Cauchy problem for the critical nonlinear Schrödinger equation in \( H^s \)*, Nonlinear Anal. 14 (1990), 807-836.
[12] Christ F. M., Weinstein, M. I., Dispersion of small amplitude solutions of the Generalized Korteweg-de Vries equation, J. Funct. Anal. 100 (1991), 87-109.

[13] Ginibre, J., Ozawa, T., Long range scattering for nonlinear Schrödinger and Hartree equations in space dimension $n \geq 2$, Comm. Math. Phys. 151 (1993), 619-645.

[14] Ginibre, J., Velo, G., On a class of nonlinear Schrödinger equations. I. The Cauchy problem, general case, J. Funct. Anal. 32 (1979), 1-31.

[15] Ginibre, J., Velo, G., On a class of nonlinear Schrödinger equations. II. Scattering theory, general case, J. Funct. Anal. 32 (1979), 33-71.

[16] Hayashi, N., Naumkin, P., Asymptotics for large time of solutions to nonlinear Schrödinger and Hartree equations, Amer. J. Math. 120 (1998), 369-389.

[17] Hayashi, N., Li, C., Naumkin, P., Time decay for nonlinear dissipative Schrödinger equations in optical fields, Adv. Math. Phys. Art. ID 3702738 (2016), 7.

[18] Hayashi, N., Li, C., Naumkin, P., Upper and lower time decay bounds for solutions of dissipative nonlinear Schrödinger equations, Commun. Pure Appl. Anal. 16 (2017), 2089-2104.

[19] Hayashi, N., Ozawa, T., Scattering theory in the weighted $L^2(\mathbb{R})$ spaces for some Schrödinger equation, Ann. Inst. H. Poincaré Phys. Théor. 48 (1988), 17-37.

[20] Hoshino, G., Scattering for solutions of a dissipative nonlinear Schrödinger equation, J. Differential Equations 266 (2019), 4997-5011.

[21] Hoshino, G, Asymptotic behavior for solutions to the dissipative nonlinear Schrödinger equations with the fractional Sobolev space, J. Math. Phys. 60 (2019), 111504, 11.

[22] Ishida, A., Kawamoto, M., Critical scattering in a time-dependent harmonic oscillator, J. Math. Anal. Appl. 492 (2020), 124475, 9.

[23] Ishida, A., Kawamoto, M., Existence and nonexistence of wave operators for time-decaying harmonic oscillators, Rep. Math. Phys. 85 (2020), 335-350.

[24] Kato, T., A generalization of the Heinz inequality, Proc. Jpn. Aca., 37 (1961), 305-308.

[25] Kato, T., On nonlinear Schrödinger equations, Ann. Inst. H. Poincaré Phys. Theor. 46 (1987), 113-129.

[26] Katayama, S., Li, C., Sunagawa, H., A remark on decay rates of solutions for a system of quadratic nonlinear Schrödinger equations in 2D, Differential Integral Equations 27 (2014), 301-312.

[27] Kawamoto, M., Strichartz estimates for Schrödinger operators with square potential with time-dependent coefficients, Diff. Eqn. Dyn. Sys., (2020).
[28] Kawamoto, M., *Final state problem for nonlinear Schrödinger equations with time-decaying harmonic oscillators*, J. Math. Anal. Appl. *503* (2021), 125292

[29] Kawamoto, M., *Asymptotic behavior for nonlinear Schrödinger equations with critical time-decaying harmonic potential*, J. Differential Equations *303*, (2021), 253-267.

[30] Kawamoto, M., Yoneyama, T., *Strichartz estimates for harmonic potential with time-decaying coefficient*, J. Evol. Equ. *18* (2018), 127-142.

[31] Kawamoto, M., Muramatsu, R., *Asymptotic behavior of solutions to nonlinear Schrödinger equations with time-dependent harmonic potentials*, J. Evol. Equ. *21* (2021), 699-723.

[32] Kenig, C., Ponce, G., Vega, L., *Well-posedness and scattering results for the generalized Kortweg-de Vries equation via the contraction principle* Comm. Pure. Appl. Math. *46* (1993), 527-620.

[33] Keel, M., Tao, T., *Endpoint Strichartz estimates*, Amer. J. Math. *120* (1998), 955-980.

[34] Kim, D., *A note on decay rates of solutions to a system of cubic nonlinear Schrödinger equations in one space dimension*, Asymptot. Anal. *98* (2016), 79-90.

[35] Kim, D., Sunagawa, H., *Remarks on decay of small solutions to systems of Klein-Gordon equations with dissipative nonlinearities*, Nonlinear. Anal. *97* (2014), 94-105.

[36] Kita, N., Sato, T., *Optimal L²-decay of solutions to a cubic dissipative nonlinear Schrödinger equation*, Asymptot. Anal. (2021), accepted.

[37] Kita, N., Shimomura, A., *Asymptotic behavior of solutions to Schrödinger equations with a subcritical dissipative nonlinearity*, J. Differential Equations *242* (2007), 192-210.

[38] Kita, N., Shimomura, A., *Large time behavior of solutions to Schrödinger equations with a dissipative nonlinearity for arbitrarily large initial data*, J. Math. Soc. Japan *61* (2009), 39-64.

[39] Korotyaev, E., L., *On scattering in an external, homogeneous, time-periodic magnetic field*, Math. USSR-Sb., *66* (1990), 499-522.

[40] Liskevich, V. A., Perelmuter, M. A., *Analyticity of submarkovian semigroups*, Proc. Amer. Math. Soc. *123* (1995), 1097-1104.

[41] Li, C., Sunagawa, H., *On Schrödinger systems with cubic dissipative nonlinearities of derivative type*, Nonlinearity *29* (2016), 1537-1563.

[42] Naito, M., *Asymptotic behavior of solutions of second order differential equations with integrable coefficients*, Tran. A.M.S., *282* (1984), 577-588.
[43] Nishii, Y., Sunagawa, H., *On Agemi-type structural conditions for a system of semilinear wave equations*, J. Hypabolic Differ. Equ. **17** (2020), 459-473.

[44] Okazawa, N., Yokota, T., *Global existence and smoothing effect for the complex Ginzburg-Landau equation with $p$-Laplacian*, J. Differential Equations **182** (2002), 541-576.

[45] Ogawa, T., Sato, T., *$L^2$-Decay rate for the critical nonlinear Schrödinger equation with a small smooth data*, Nonlinear Differ. Equ. Appl. **27** (2020), 18.

[46] Ogawa, T., Uriya, K., *Asymptotic behavior of solutions to a quadratic nonlinear Schrödinger system with mass resonance*, RIMS Kōkyūroku Bessatsu **B42** (2013), 153-170.

[47] Ozawa, T., *Long range scattering for nonlinear Schrödinger equations in one space dimension*, Comm. Math. Phys. **139** (1991), 479-493.

[48] Sato, T., *$L^2$-decay estimate for the dissipative nonlinear Schrödinger equation in the Gevrey class*, Arch. Math. **115** (2020), 575-588.

[49] Sato, T., *Lower bound estimate for the dissipative nonlinear Schrödinger equation*, SN Partial Differ. Equ. Appl. (2021), accepted.

[50] Strauss, W. A., *Nonlinear scattering theory at low energy*, J. Funct. Anal. **41** (1981), 110-133.

[51] Sunagawa, H., *Large time behavior of solutions to the Klein-Gordon equation with nonlinear dissipative terms*, J. Math. Soc. Japan **58** (2006), 379-400.

[52] Shimomura, A., *Asymptotic behavior of solutions for Schrödinger equations with dissipative nonlinearities*, Commun. Partial Differ. Equ. **31** (2006), 1407-1423.

[53] Tsutsumi, Y., *$L^2$ solution for nonlinear Schrödinger equation and nonlinear groups*, Funk. Ekva., **30**, (1987), 115-125.

[54] Tsutsumi, Y., Yajima, K., *The asymptotic behavior of nonlinear Schrödinger equations*, Bull. Amer. Math. Soc. **11** (1984), 186-188.

[55] Yajima, K., *Existence of solutions for Schrödinger evolution equations*, Comm. Math. Phys. **110** (1987), 415-426.