Light Deflection and Gauss–Bonnet Theorem: Definition of Total Deflection Angle and its Applications

Abstract In this paper, we re-examine the light deflection in the Schwarzschild and the Schwarzschild–de Sitter spacetime. First, supposing a static and spherically symmetric spacetime, we propose the definition of the total deflection angle $\alpha$ of the light ray by constructing a quadrilateral $\Sigma^4$ on the optical reference geometry $\mathcal{M}^{opt}$ determined by the optical metric $\tilde{g}_{ij}$. On the basis of the definition of the total deflection angle $\alpha$ and the Gauss–Bonnet theorem, we derive two formulas to calculate the total deflection angle $\alpha$: (i) the angular formula that uses four angles determined on the optical reference geometry $\mathcal{M}^{opt}$ or the curved $(r, \phi)$ subspace $\mathcal{M}^{sub}$ being a slice of constant time $t$ and (ii) the integral formula on the optical reference geometry $\mathcal{M}^{opt}$ which is the areal integral of the Gaussian curvature $K$ in the area of a quadrilateral $\Sigma^4$ and the line integral of the geodesic curvature $\kappa$ along the curve $C_{\Gamma}$. As the curve $C_{\Gamma}$, we introduce the unperturbed reference line that is the null geodesic $\Gamma$ on the background spacetime such as the Minkowski or the de Sitter spacetime, and is obtained by projecting $\Gamma$ vertically onto the curved $(r, \phi)$ subspace $\mathcal{M}^{sub}$. We demonstrate that the two formulas give the same total deflection angle $\alpha$ for the Schwarzschild and the Schwarzschild–de Sitter spacetime. In particular, in the Schwarzschild case, the result coincides with Epstein–Shapiro’s formula when the source $S$ and the receiver $R$ of the light ray are located at infinity. In addition, in the Schwarzschild–de Sitter case, there appear order $O(\Lambda m)$ terms in addition to the Schwarzschild-like part, while order $O(\Lambda)$ terms disappear.

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1 Introduction

In 1919, light deflection by the mass of the sun was detected during the total solar eclipse in Sobral and Principe [1]. These observations provided the first evidence of the validity of the general theory of relativity, as well as the stunning explanation of the perihelion advance of Mercury. Thereafter, measurement of the bending of light played an important role in the verification of the general theory of relativity [2]. Further, gravitational lensing which is based on light deflection is widely used as a powerful tool to investigate various astronomical/astrophysical phenomena such as dark matter exploration and extrasolar planet search, see e.g., [3, 4] and the references therein.

Because the method for calculating the total deflection angle $\alpha$ is currently described in many textbooks and literature, one may think that computing the total deflection angle is now an elementary problem and essential issues concerning the deflection of light have already been understood and solved completely. However, there still exist some basic issues to be settled that were highlighted when considering the contribution of cosmological constant $\Lambda$ to the bending of light.

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Islam [5] showed that the trajectory of the light ray is not related to the cosmological constant \( \Lambda \) because the second-order differential equation of the photon (a null geodesic) does not contain \( \Lambda \). Therefore, it was considered for a long time that \( \Lambda \) does not contribute to the light deflection. However, in 2007, Rindler and Ishak [6] pointed out that \( \Lambda \) affects the bending angle of the light ray by using the invariant cosine formula under the Schwarzschild–de Sitter/Kottler solution. Starting with this paper [6], many authors intensively discussed its appearance in diverse ways; see [7] for a review article, and also see e.g., [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19] and the references therein. However, despite various discussions and approaches, a definitive conclusion has not yet emerged. The main reasons are as follows:

1. If there exists a cosmological constant \( \Lambda \) such as the Schwarzschild–de Sitter solution, the spacetime is not asymptotically flat unlike the Schwarzschild spacetime, and we cannot then apply the standard procedure described in many textbooks and literature.

2. In the first place, it is ambiguous and not clear what is the total deflection angle and how it should be defined.

The application of the Gauss–Bonnet theorem has the potential to solve above problems and settle the arguments by examining the total deflection angle correctly in the curved spacetime such as the Schwarzschild–de Sitter/Kottler solution. The Gauss–Bonnet theorem has already been adopted in previous papers e.g., [20, 21, 22, 23]. Although these pioneering works, especially [22], have been successful in showing effectiveness of the Gauss–Bonnet theorem, it seems that they have not reached the stage of giving a definite definition of the total deflection angle and some points remain to be clarified.

Not only from a theoretical viewpoint but also from the standpoint of an observational evidence, the cosmological constant problem is important (See reviews by e.g., [24, 25]). Recent cosmological observations strongly suggest the acceleration of cosmic expansion. Today, it is widely considered that the cosmological constant \( \Lambda \), or the dark energy in a more general sense, is the most promising candidate to explain the observed accelerating expansion of the universe although the details are completely unknown so far. As one way to tackle this problem from another perspective, it is natural to investigate the effect of the cosmological constant \( \Lambda \) on the classical tests of the general theory of relativity, especially on the deflection of the light ray.

In this paper, first we will propose a definition for total deflection angle \( \alpha \) of the light ray presupposing the static and spherically symmetric spacetime. After introducing a definition for total deflection angle we will develop two methods for calculating the total deflection angle on the basis of the Gauss–Bonnet theorem. These two formulas will be applied to the Schwarzschild and the Schwarzschild–de Sitter spacetime, and we will demonstrate that they give equivalent results. In particular, in the Schwarzschild case, the result coincides with Epstein–Shapiro’s formula when the source \( S \) and the receiver \( R \) of the light ray are located at infinity. In addition, in the case of the Schwarzschild–de Sitter spacetime, our results indicate that the cosmological constant \( \Lambda \) contributes to the total deflection angle as the form of the \( \Theta(\Lambda m) \) terms (here \( m \) is the mass of the central body), while the \( \Theta(\Lambda) \) terms do not appear; this is connected with the fact that the light ray does not bend in the de Sitter spacetime.

This paper is organized as follows; first, the optical metric is introduced in section 2 and the outline of the Gauss–Bonnet theorem is presented in section 3. The total deflection angle is defined, and on the basis of the definition of the total deflection angle we construct two formulas for calculating the total deflection angle in section 4. The two formulas are applied to an asymptotically flat spacetime in section 5 and to a non-asymptotically flat spacetime in section 6. Finally, section 7 concludes the paper.

2 Optical Metric

We assume that the spacetime is expressed in the static and spherically symmetric form

\[
d s^2 = g_{\mu\nu}dx^\mu dx^\nu = -f(r)dr^2 + \frac{1}{f(r)}d\theta^2 + r^2(d\phi^2 + \sin^2 \theta d\phi^2),
\]

(1)

where \( f(r) \) is a function of the radial coordinate \( r \), a Greek subscript such as \( \mu, \nu \) runs from 0 to 3, and we choose the geometrical unit \( c = G = 1 \). Because of the spherical symmetry, without losing generality, it is possible to take the equatorial plane \( \theta = \pi/2, d\theta = 0 \) as the orbital plane of the light rays

\[
d s^2 = -f(r)dr^2 + \frac{1}{f(r)}d\theta^2 + r^2(d\phi^2).
\]

(2)
We have two constants of motion, that is, the energy \( E \) and the angular momentum \( L \),

\[
E = f(r) \frac{dt}{d\lambda}, \quad L = r^2 \frac{d\phi}{d\lambda},
\]

in which \( \lambda \) is an affine parameter. From two constants of motion \( E \) and \( L \), another constant, the so-called the impact parameter \( b \), is determined by

\[
b \equiv \frac{L}{E}.
\]

Further, we derive the following relation from Eqs. (3) and (4),

\[
\frac{d\phi}{dt} = b f(r) r^2.
\]

The light ray satisfies the null condition \( ds^2 = 0 \), and from this null condition let us introduce the optical metric \( \bar{g}_{ij} \) which can be regarded as the Riemannian geometry experienced by the light rays:

\[
ds^2 = \bar{g}_{rr} dr^2 + \bar{g}_{\phi\phi} d\phi^2 = \int_1^2 \sqrt{f(r)^2 \bar{g}_{rr}(k^r)^2 + \bar{g}_{\phi\phi}(k^\phi)^2} dt = t_2 - t_1.
\]

We mention that Eqs. (6) and (7) are connected by the conformal transformation

\[
\bar{g}_{ij} = \omega^2(x) g_{ij},
\]

or more generally,

\[
\bar{g}_{\mu\nu} = \omega^2(x) g_{\mu\nu},
\]

where \( \omega^2(x) \) is called the conformal factor which in our case is

\[
\omega^2(x) = \frac{1}{f(r)}.
\]

It is noteworthy that the conformal transformation preserves the angle of the point at which the two curves intersect, and rescales the coordinate value. Furthermore the null geodesic does not change its form by the conformal transformation because of the null condition; see e.g., Appendix G in [27].

Setting the unit tangent vector \( k^i \) of light ray path on \( \mathcal{M}^{opt} \) as

\[
k^i = \frac{dx^i}{dt},
\]

and from Eq. (3), \( k^i \) is actually unit vector:

\[
1 = \bar{g}_{ij}k^ik^j.
\]

It should be noticed that on the optical reference geometry \( \mathcal{M}^{opt} \), \( t \) plays the role of an arc length parameter because

\[
\int_{t_1}^{t_2} dt = \int_{t_1}^{t_2} \sqrt{\bar{g}_{rr}(k^r)^2 + \bar{g}_{\phi\phi}(k^\phi)^2} dt = t_2 - t_1.
\]
Therefore, the optical reference geometry $M^{\text{opt}}$ is suitable for application to the Gauss–Bonnet theorem later.

Finally, in accordance with [22], we prepare the radial and tangential unit vector $e_r^i$, $e_\phi^i$ as

$$e_r^i = \left( \frac{1}{\sqrt{\bar{g}_{rr}}}, 0 \right), \quad e_\phi^i = \left( 0, \frac{1}{\sqrt{\bar{g}_{\phi\phi}}} \right).$$

(14)

It is easy to check that these two vectors are certainly the unit vectors because of

$$\bar{g}_{ij} e_r^i e_r^j = 1, \quad \bar{g}_{ij} e_\phi^i e_\phi^j = 1.$$  

(15)

Because we are working in the optical reference geometry $M^{\text{opt}}$, the inner product of the two vectors is defined by using the optical metric $\bar{g}_{ij}$ instead of $g_{ij}$.

### 3 Gauss–Bonnet Theorem

We suppose that the line element is written in a diagonal form determined by the optical metric $\bar{g}_{ij}$; see Eq. (6). Due to Eq. (13), the time $t$ plays the role of an arc length parameter in the optical reference geometry $M^{\text{opt}}$, and hereafter we denote an arc length parameter by $t$ instead of $s$ which is usually used.

Consider a polygon $\Sigma^n$ with $n$ vertices on $M^{\text{opt}}$ which is orientable and bounded by $n$ smooth piecewise regular curves $C_p$ ($p = 1, 2, \cdots, n$); see Fig. 1. The (local) Gauss–Bonnet theorem is expressed as, e.g., on p. 139 in [28], p. 170 in [29], and p. 272 in [30]

$$\int_{\Sigma^n} K d\sigma + \sum_{p=1}^{n} \int_{C_p} \kappa_p dt + \sum_{p=1}^{n} \theta_p = 2\pi,$$

(16)

in which an arc length parameter $t$ moves along the curve $C_p$ in such a sense that a polygon $\Sigma^n$ stays on the left side, $d\sigma = \sqrt{\det |\bar{g}|} dx^1 dx^2 = \sqrt{\det |\bar{g}|} dr d\phi$ is an areal element, and $\theta_p$ is the external angle at the $p$-th vertex which is described as the sense leaving the internal angle on the left. $K$ is the Gaussian curvature defined as,

![Figure 1](image-url)  

**Fig. 1** Schematic diagram of Gauss–Bonnet theorem. A polygon $\Sigma^n$ is bounded by the curves $C_1, C_2, \cdots, C_n$, and the external angles of a polygon $\Sigma^n$ are denoted by $\theta_p$ ($p = 1, 2, \cdots, n$).

E.g., on p. 147 in [29]

$$K = -\frac{1}{\sqrt{\bar{g}_{rr}\bar{g}_{\phi\phi}}} \left[ \frac{\partial}{\partial r} \left( \frac{1}{\sqrt{\bar{g}_{rr}}} \frac{\partial}{\partial r} \sqrt{\bar{g}_{\phi\phi}} \right) + \frac{\partial}{\partial \phi} \left( \frac{1}{\sqrt{\bar{g}_{\phi\phi}}} \frac{\partial}{\partial \phi} \sqrt{\bar{g}_{rr}} \right) \right].$$

(17)
The Gaussian curvature $K$ represents how surface is curved. $\kappa_g$ is the geodesic curvature along the curve $C_p$, e.g., on p. 256 in \[30\]

$$\kappa_g = \frac{1}{2\sqrt{g_{rr}g_{\phi\phi}}} \left( d\tilde{g}_{\phi\phi} \frac{d\phi}{dt} - d\tilde{g}_{r\phi} \frac{dr}{dt} \right) + \frac{d\Phi}{dt}, \quad (18)$$

where $\Phi$ is the angle between the radial unit vector $e'_{r}$ and the tangent vector of the curve $C_p$ in our case. The geodesic curvature $\kappa_g$ indicates how far the curve $C_p$ deviates from the geodesic. It should be mentioned if the curve $C_p$ is the geodesic, then $\kappa_g = 0$. We note that because we are working in the optical reference geometry $\mathcal{M}_{\text{opt}}$, $K$ and $\kappa_g$ are determined in terms of the optical metric $\tilde{g}_{ij}$ instead of $g_{ij}$.

4 Definition of Total Deflection Angle

4.1 Unperturbed Reference Line

In order to define the total deflection angle on the basis of the Gauss–Bonnet theorem, we must prepare a polygon $\Sigma^4$ on the optical reference geometry $\mathcal{M}_{\text{opt}}$. The most natural way to do this is to construct a triangle $\Sigma^3$ with three points; the center $O$ where the massive star or galaxy with mass $m$ is located, the source of light ray $S$ and the receiver of the light ray $R$. However, in the case of the Schwarzschild and the Schwarzschild–de Sitter spacetime, the center $O$ is a singularity, so that it is not relevant to constructing a triangle $\Sigma^3$ on $\mathcal{M}_{\text{opt}}$. Therefore, instead of a triangle $\Sigma^3$, let us configure a quadrilateral $\Sigma^4$ on $\mathcal{M}_{\text{opt}}$ that is bounded by the trajectory of the light rays, $\gamma_1$ two radial null geodesics $\gamma_S$ connecting the center $O$ and the source of light ray $S$ and $\gamma_R$ connecting the center $O$ and the receiver of light ray $R$, and the curve $C$. So far, it is not clear what is the curve $C$; however if the curve $C$ is determined appropriately, then we can lay a quadrilateral $\Sigma^4$ on $\mathcal{M}_{\text{opt}}$.

To seek the curve $C$, we start from the investigation of the curved $(r, \phi)$ subspace $\mathcal{M}_{\text{sub}}$ determined by Eq. (7) before considering the optical reference geometry $\mathcal{M}_{\text{opt}}$ determined by Eq. (6). We recall the fact that if the curved $(r, \phi)$ subspace $\mathcal{M}_{\text{sub}}$ is restricted in the spherically symmetric form as Eq. (7), two curved $(r_1, \phi_1)$ and $(r_2, \phi_2)$ subspaces $\mathcal{M}_{\text{sub}}^1$ and $\mathcal{M}_{\text{sub}}^2$, characterized by $f_1(r)$ and $f_2(r)$, respectively, have the same coordinate values $r_1 = r_2$ and $\phi_1 = \phi_2$ because $r_1$ and $r_2$ are determined as the circumference radius $\ell = 2\pi r_1 = 2\pi r_2$ for the constant $r$ (then $dr = 0$). Further, the same relation also holds between the curved $(r, \phi)$ subspace $\mathcal{M}_{\text{sub}}$ and the flat $(\rho, \varphi)$ plane of the three-dimensional flat space with cylindrical coordinates $(\rho, \varphi, z)$, $r = \rho$ and $\phi = \varphi$ because $\ell = 2\pi r = 2\pi \rho$ for constant $r$ ($dr = 0$) and $\rho$ ($d\rho = 0$); see Appendix A and also section 11.3 in \[31\] and section II in \[17\].

First, we examine the relation between the Schwarzschild and the Minkowski spacetime. The Minkowski spacetime can be considered as the background spacetime of the Schwarzschild one. Because the Minkowski space coincides with the flat $(\rho, \varphi)$ plane, we describe the Minkowski space by using $(\rho, \varphi)$ coordinates. Fig. 6 shows the relation between the Minkowski $(\rho, \varphi)$ flat space and the Schwarzschild curved $(r, \phi)$ subspace $\mathcal{M}_{\text{sub}}$. The differential equation of light ray $\Gamma$ in the Minkowski $(\rho, \varphi)$ flat space is

$$\left( \frac{d\rho}{d\varphi} \right)^2 = \rho^2 \left( \frac{\rho^2}{b^2} - 1 \right), \quad (19)$$

and it yields the equation of the trajectory of the light ray:

$$\rho = \frac{\sin \varphi}{b}, \quad (20)$$

Footnote 1: In General, $\Phi$ is determined as an angle between the tangent vector of the curve $C_p$ and the vector $\partial S^i(u, v)/\partial u_i$, where $S^i$ means the point on the surface $S$. In our case, $u = r, v = \phi$ then $\partial S^i(r, \phi)/\partial r$ can be understood as the vector pointing in the $r$ direction. The unit vector,

$$\frac{\partial S^i(r, \phi)}{\partial r}$$

corresponds to $e'_r$ in our notation.
where \( b \) is the impact parameter defined by Eq. (4). The form of the trajectory of the light ray, Eq. (20), is a straight line in the Euclidean sense. Whereas, in the Schwarzschild case, the differential equation of light ray \( \gamma \) on the curved \((r, \phi)\) subspace \( \mathcal{M}^{\text{sub}} \) is

\[
\left( \frac{dr}{d\phi} \right)^2 = r^2 \left( \frac{\ell^2}{b^2} + \frac{2m}{r^2} - 1 \right),
\]

(21)

and \( r_\gamma \) can be expressed by means of \( \phi \) as the form

\[
r_\gamma = r_\gamma(\phi;m;b).
\]

(22)

For the concrete form of \( r_\gamma \), see e.g., Eq. (39). Eq. (22) is a null geodesic \( \gamma \) on the curved \((r, \phi)\) subspace \( \mathcal{M}^{\text{sub}} \). However, because of the relation

\[
r = \rho, \quad \phi = \varphi,
\]

(23)

the form of Eq. (22) does not change on the Minkowski \((\rho, \varphi)\) flat space; however it is not geodesic \( \gamma \) but just a curve \( C_\gamma \) which is vertical projection of \( \gamma \) onto the Minkowski \((\rho, \varphi)\) flat space.

We mention that the two geodesics \( \Gamma \) on the Minkowski \((\rho, \varphi)\) flat space and \( \gamma \) on the curved \((r, \phi)\) subspace \( \mathcal{M}^{\text{sub}} \) (the Schwarzschild space) are the straight line in such a sense that according to Fermat’s principle, they take the shortest path between two points in the shortest time. Therefore, both \( \Gamma \) on the Minkowski \((\rho, \varphi)\) flat space and \( \gamma \) on the curved \((r, \phi)\) subspace \( \mathcal{M}^{\text{sub}} \) (the Schwarzschild space) do not bend; in fact, the line integral of geodesic curvature \( \kappa_\gamma \) along the null geodesics \( \Gamma \) and \( \gamma \) vanish (please remind the meaning of the geodesic and the geodesic curvature \( \kappa_\gamma \)).

In order to obtain the total deflection angle \( \alpha \), we must compare the null geodesics \( \Gamma \) and \( \gamma \) existing in the distinct spaces. For this purpose, we usually project \( \gamma \) vertically onto the Minkowski \((\rho, \varphi)\) flat space and regard the difference in the direction of light trajectory between the null geodesic \( \Gamma \) and the curve \( C_\gamma \) as the total deflection angle \( \alpha \); see Fig. 2. However, this approach works only when the source \( S \) and the receiver \( R \) of light ray are located at infinity because the Schwarzschild curved \((r, \phi)\) subspace becomes asymptotically flat, corresponding to the Minkowski \((\rho, \varphi)\) flat space. Hence, if the source \( S \) and/or the receiver \( R \) of light ray are placed within a finite region on the curved \((r, \phi)\) subspace \( \mathcal{M}^{\text{sub}} \), we cannot determine the total deflection angle on the Minkowski \((\rho, \varphi)\) flat space anymore.

To overcome this situation, let us try to determine the total deflection angle \( \alpha \) on the Schwarzschild curved \((r, \phi)\) subspace \( \mathcal{M}^{\text{sub}} \) because the actual light trajectory lies on the curved \((r, \phi)\) subspace \( \mathcal{M}^{\text{sub}} \). Due to the relation \( r = \rho \) and \( \phi = \varphi \), Eq. (20) holds in the same form on the Schwarzschild curved \((r, \phi)\) subspace \( \mathcal{M}^{\text{sub}} \).
In this case, Eq. (20) is no longer the geodesic $\Gamma$ and is just a curve $C_\Gamma$ which is the vertical projection of $\Gamma$ onto the Schwarzschild curved $(r, \phi)$ subspace:

$$\frac{1}{r_{C_\Gamma}} = \frac{\sin \phi}{b}. \tag{24}$$

It can be said that to evaluate $C_\Gamma$ on the curved $(r, \phi)$ subspace, $\mathcal{M}^{\text{sub}}$ means to measure $C_\Gamma$ with a curved metric (ruler) instead of with a flat metric (ruler). By using Eqs. (22) and (24), we can construct a quadrilateral $\Sigma_4$ and determine the total deflection angle $\alpha$ on the curved $(r, \phi)$ subspace $\mathcal{M}^{\text{sub}}$.

Here, let us name the curve $C_\Gamma$, e.g., Eq. (24), “the unperturbed reference line” which is the vertical projection of null geodesic $\Gamma$ onto the curved $(r, \phi)$ subspace $\mathcal{M}^{\text{sub}}$. The reason is that it is originally a null geodesic $\Gamma$ on the Minkowski $(\rho, \varphi)$ flat space (the background space) and it does not change its form on the curved $(r, \phi)$ subspace $\mathcal{M}^{\text{sub}}$ by vertical projection; therefore it seems to be reliable to regard $C_\Gamma$ as the reference with respect to the actual null geodesic $\gamma$ on the curved $(r, \phi)$ subspace $\mathcal{M}^{\text{sub}}$.

Next, in the case of the Schwarzschild–de Sitter spacetime, the spacetime does not become asymptotically flat, and so the Minkowski $(\rho, \varphi)$ flat space cannot be used as the background spacetime. Hence, we assume the de Sitter spacetime to be the background spacetime of the Schwarzschild–de Sitter spacetime, see Fig. 3. In fact we can think of the Schwarzschild–de Sitter spacetime as the de Sitter spacetime distorted by the central mass $m$ (See section 14.4 in [31]). From the considerations in Appendix A the coordinate value of the curved $(r, \phi)$ subspace of the Schwarzschild–de Sitter space is the same as that of the de Sitter spacetime

$$r_{\text{SdS}} = r_{\text{dS}}, \quad \phi_{\text{SdS}} = \phi_{\text{dS}}. \tag{25}$$

because $r_{\text{dS}}$ and $r_{\text{SdS}}$ are determined as the circumference radius $\ell = 2\pi r_{\text{dS}} = 2\pi r_{\text{SdS}}$. Furthermore, as will be seen later, the form of the null geodesic $\Gamma$ on the de Sitter space is similar to that on the Minkowski $(\rho, \varphi)$ flat space; see Eqs. (60) and (62). Therefore, in the Schwarzschild–de Sitter case, we adopt Eq. (62) as the unperturbed reference line $C_\Gamma$ which is vertically projected from the de Sitter space to the Schwarzschild–de Sitter space.

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Fig. 3 The relation between the de Sitter curved $(r_{\text{dS}}, \phi_{\text{dS}})$ subspace $\mathcal{M}^{\text{dS}}$ and the Schwarzschild–de Sitter curved $(r_{\text{SdS}}, \phi_{\text{SdS}})$ subspace $\mathcal{M}^{\text{SdS}}$. As in Fig. 2 the null geodesics $\Gamma$, $\gamma$ are represented by solid lines, the curves $C_\Gamma$, $C_\gamma$ are represented by dotted lines, and the two circumferences $\ell$ are represented by dashed lines. The points $P_\Gamma$ on $\Gamma$ and $P_{C_\gamma}$ on $C_\gamma$ have the same coordinate values and the points $P_\gamma$ on $\gamma$ and $P_{C_\Gamma}$ on $C_\Gamma$ also have the same coordinate values because the circumference radius $\ell = 2\pi r_{\text{SdS}} = 2\pi r_{\text{dS}}$ and the vertical projection.

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2 The authors of [20, 22] employed the circular arc with a constant coordinate radius as the curve $\gamma$ [20], and $C_\gamma$ [22] instead of $C_\Gamma$ in our case. However this seems to be inadequate because the circular arc is not related to the null geodesic in the background spacetime.
So far, we have discussed the relation between the geodesic and the curve on the curved \((r, \phi)\) subspaces \(M^{opt}\) and its background space such as the Minkowski \((p, \phi)\) flat space and the de Sitter subspace. However, in the present paper, we will be working in the optical reference geometry \(M^{opt}\) to consider the behavior of the light ray and to apply the Gauss–Bonnet theorem. As observed in section 4 the null geodesic (in our case, \(\gamma, \gamma_R\) and \(\gamma_S\); see Fig. 4 below) can be written in the same form both on the optical reference geometry \(M^{opt}\) and the curved \((r, \phi)\) subspace \(M^{sub}\) due to the property of conformal transformation. Moreover, the conformal transformation preserves the angle between two vectors such as between \(\gamma\) and \(\gamma_S\) (namely \(E_S\)) and so on; see also Fig. 4. It is sufficient then to deduce the unperturbed reference line \(C_T\) does not change its form both on the curved \((r, \phi)\) subspace \(M^{sub}\) and the optical reference geometry \(M^{opt}\) as well as the null geodesics \(\gamma, \gamma_S\) and \(\gamma_R\). Therefore in this paper, we take the stance that the unperturbed reference line \(C_T\) on the optical reference geometry \(M^{opt}\) obeys the same equation on the curved \((r, \phi)\) subspace \(M^{sub}\).

### 4.2 Total Deflection Angle

Let us define the total deflection angle. To this end, we construct a quadrilateral \(\Sigma^4\) on the optical reference geometry \(M^{opt}\) that is bounded by the null geodesic of light ray \(\gamma\), two radial null geodesics \(\gamma_S\) connecting the center \(O\) and the source \(S\) of light ray and \(\gamma_R\) connecting the center \(O\) and the receiver \(R\) of light ray, and the unperturbed reference line \(C_T\) introduced in the previous subsection 4.1; see Fig. 4.

Let the internal angles of a quadrilateral \(\Sigma^4\) be \(\beta_p\) \((p = 1, 2, 3, 4)\). According to the value of the Gaussian curvature \(K\), the sum of the internal angles \(\beta_p\) of a quadrilateral \(\Sigma^4\) on \(M^{opt}\) becomes

\[
\sum_{p=1}^{4} \beta_p = \begin{cases} 
> 2\pi & \text{for } K > 0 \\
= 2\pi & \text{for } K = 0 \\
< 2\pi & \text{for } K < 0 
\end{cases}.
\]

(26)

Then, let us suppose the total deflection angle \(\alpha\) takes the positive value, and we define the total deflection angle \(\alpha\) as follows

\[
\alpha = \sum_{p=1}^{4} \beta_p - 2\pi.
\]

(27)

Next, we introduce four angles \(\epsilon_S, \epsilon_R, E_S\), and \(E_R\) which pertain to the quadrilateral \(\Sigma^4\) on \(M^{opt}\). \(\epsilon_S\) and \(\epsilon_R\) are angles between the trajectory of the light ray \(\gamma\) and the radial null geodesics \(\gamma_S\) and \(\gamma_R\), respectively. \(E_S\) and

![Fig. 4 Schematic diagram of a quadrilateral \(\Sigma^4\) on the optical reference geometry \(M^{opt}\). A quadrilateral \(\Sigma^4\) is bounded by the null geodesics \(\gamma, \gamma_S, \gamma_R\) and the unperturbed reference line \(C_T\). The internal angles and the external angles are \(\beta_p\) and \(\theta_p\) \((p = 1, 2, 3, 4)\), respectively. \(E_S\) and \(E_R\) are angles between the geodesics \(\gamma\) and \(\gamma_S\) at \(S\) and the angles between \(\gamma\) and \(\gamma_R\) at \(R\), respectively. \(E_S\) and \(E_R\) are angles between the unperturbed reference line \(C_T\) and the geodesic \(\gamma_S\) and the angles between the unperturbed reference line \(C_T\) and \(\gamma_R\), respectively. \(\Phi_S\) and \(\Phi_R\) were introduced in Eq. (25).](image)
\( E_r \) are angles between the unperturbed reference line \( C_k \) and the radial null geodesics \( \gamma_k \) and \( \gamma_r \), respectively. We select these four angles in such a way that \( E_S \) and \( E_R \) are corresponding angles, and as are \( E_R \) and \( E_r \), see Fig. 4. We note that \( E_S \) and \( E_R \) are a function of the angular coordinate value \( \phi_S \), and \( E_R \) and \( E_r \) are a function of \( \phi_R \). The internal angles \( \beta_i \) are then expressed as follows:

\[
\beta_1 = \pi - E_S, \quad \beta_2 = E_S, \quad \beta_3 = \pi - E_R, \quad \beta_4 = E_R.
\]

(28)

Therefore, from Eqs (27) and (28), the total deflection angle \( \alpha \) can be rewritten by as

\[
\alpha = |(E_R - E_R) - (E_S - E_S)|.
\]

(29)

Because we are working in the optical reference geometry \( \mathcal{M}_{opt} \) described by the optical metric \( \tilde{g}_{ij} \), the angles \( E_S \) and \( E_R \) are computed by using Eq. (5) as

\[
\tan \varepsilon = \sqrt{\frac{\tilde{g}_{\phi\phi}(r_{\gamma})}{\tilde{g}_{rr}(r_{\gamma})}} \frac{d\phi}{d\gamma} = \sqrt{f(r_{\gamma})r_{\gamma}^2} \frac{d\phi}{d\gamma},
\]

(30)

where subscript \( \gamma \) means \( r \) and \( d\phi/dr \) are the trajectory of the light ray \( \gamma \) and its differential equation, respectively, and the optical metric \( \tilde{g}_{ij} \) is a function of \( r_{\gamma} \). On the other hand, \( E_S \) and \( E_R \) are similarly obtained as

\[
\tan E = \sqrt{\frac{\tilde{g}_{\phi\phi}(r_{C_k})}{\tilde{g}_{rr}(r_{C_k})}} \frac{d\phi}{d\gamma} = \sqrt{f(r_{C_k})r_{C_k}^2} \frac{d\phi}{d\gamma},
\]

(31)

in which subscript \( C_k \) indicates \( r \) and \( d\phi/dr \) are the unperturbed reference line \( C_k \) and its differential equation, respectively. In this case, the optical metric \( \tilde{g}_{ij} \) is a function of \( r_{C_k} \). From the practical point of view, the angles \( \varepsilon \) and \( E \) should be computed by the tangent formula as Eqs. (30) and (31) instead of the sine and cosine formulas; see Appendix B.

Finally, let us relate the definition of the total deflection angle, Eq. (27), to the Gauss–Bonnet theorem, Eq. (16). For this purpose, we prepare the external angles \( \theta_p \) in such a way that

\[
\theta_1 = E_S, \quad \theta_2 = \pi - E_S, \quad \theta_3 = E_R, \quad \theta_4 = \pi - E_R.
\]

(32)

It should be emphasized here that due to the conformal transformation, all the angles, \( \beta_p, \theta_p, E_S, E_R \) and \( E_R \), are given the same value both on the optical reference geometry \( \mathcal{M}_{opt} \) and the curved \((r, \phi)\) subspace \( \mathcal{M}_{sub} \). The sum of the external angles becomes

\[
\sum_{p=1}^{4} \theta_p = 2\pi + (E_S - E_S) - (E_R - E_R).
\]

(33)

Because \( \gamma, \gamma_S \) and \( \gamma_R \) are null geodesics, it is enough to integrate the line integral of the geodesic curvature \( \kappa_\gamma \) along the curve \( C_k \), then, according to Eqs. (16), (29) and (33), we find

\[
\alpha = |(E_R - E_R) - (E_S - E_S)|
\]

(34)

\[
= \left| \int_{C_k} K d\sigma + \int_{C_k} \kappa_\gamma dt \right|,
\]

(35)

noting that the Gaussian curvature \( K \) and the geodesic curvature \( \kappa_\gamma \) are determined by the optical metric \( \tilde{g}_{ij} \) instead of \( g_{ij} \).

Let us call Eq. (34) the angular formula of the total deflection angle \( \alpha \) and Eq. (35) the integral formula of the total deflection angle \( \alpha \). It should be pointed out that due to the conformal transformation, the angular formula, Eq. (34), holds not only on the optical reference geometry \( \mathcal{M}_{opt} \) but also on the curved \((r, \phi)\) subspace \( \mathcal{M}_{sub} \). The angular formula, Eq. (34), is directly related to the definition of the total deflection angle, Eq. (27), whose meaning is geometrically clear. By contrast, in the case of the integral formula, the results of the integration differ for the optical reference geometry \( \mathcal{M}_{opt} \) and the curved \((r, \phi)\) subspace \( \mathcal{M}_{sub} \) because the conformal transformation rescales the coordinate values under the transformation \( g_{ij} \leftrightarrow \tilde{g}_{ij} \).

\footnote{The angle \( \Psi \) in (22) should be considered as \( E \) instead of \( \varepsilon \) in our case because the authors of (22) seem to regard \( 1/r - \sin \phi/b \) as the equation of the light ray and inserted it into e.g., Eq. (32) in (22) to calculate \( \Psi \). See also FIG. 4 in (22).}

\footnote{It is clear that the one-sided deflection angle \( e = \Psi - \phi \) in (2) is rewritten as \( e = \varepsilon - \phi \) in our notation. This should then be corrected as \( e = \Psi - E \), replacing the angular coordinate value \( \phi \) by \( E \). Our one-sided deflection angle \( e = E - E \) seems to be equivalent to Eq. (140), \( \beta_{M} = \beta_{M2} - \beta_{M1} \), in (17).}
5 Asymptotically Flat Spacetime

In the beginning, let us consider the asymptotically flat spacetime, namely the Schwarzschild spacetime. We apply the formulas Eqs. (34) and (35) to calculate the total deflection angle $\alpha$, and show that the two formulas give the same results.

5.1 Schwarzschild Spacetime

In the case of the Schwarzschild spacetime, the form of $f(r)$ is

$$f_{\text{Sch}}(r) = 1 - \frac{2m}{r},$$

in which $m$ is the mass of the central object. We assume that the source $S$ and the receiver $R$ of light rays are located at a finite distance from the center on the curved region of $\mathcal{M}^{\text{opt}}$, and we obtain the total deflection angle up to the order $O(m^2)$.

5.1.1 Angular Formula

First, we calculate the total deflection angle $\alpha$ by using angular formula, Eq. (34). $\varepsilon_S$ and $\varepsilon_R$ are obtained by Eq. (30):

$$\tan \varepsilon = \sqrt{1 - \frac{2m}{r} r_{\gamma} \frac{d\phi}{dr} r_{\gamma}},$$

where we used the differential equation of the light ray $\gamma$ on $\mathcal{M}^{\text{opt}}$:

$$\left( \frac{dr}{d\phi} \right)^2 = r_{\gamma}^2 \left( \frac{r_{\gamma}}{b^2} + \frac{2m}{r_{\gamma}} - 1 \right).$$

From Eq. (38), the trajectory of the light ray $\gamma$ is given by up to the order $O(m^2)$; see Eq. (7) in [15]:

$$\frac{1}{r_{\gamma}} = \frac{\sin \phi}{b} + \frac{m}{2b^2} \left( 3 + \cos 2\phi \right)$$

$$+ \frac{m^2}{16b^3} \left[ 37 \sin \phi + 30(\pi - 2\phi) \cos \phi - 3 \sin 3\phi \right] + O(m^3).$$

Inserting Eq. (39) into Eq. (37) and expanding up to the order $O(m^2)$, $\tan \varepsilon$ is written as

$$\tan \varepsilon = \tan \phi + \frac{2m}{b} \sec \phi + \frac{m^2}{8b^2} \left[ 15 \sec^2 \phi (\pi - 2\phi + \sin 2\phi) \right] + O(m^3).$$

Thus, $\varepsilon$ is

$$\varepsilon = \arctan \left( \tan \phi + \frac{2m}{b} \sec \phi + \frac{m^2}{8b^2} [15 \sec^2 \phi (\pi - 2\phi + \sin 2\phi)] \right)$$

$$= \phi + \frac{2m}{b} \cos \phi + \frac{m^2}{8b^2} [15(\pi - 2\phi) - \sin 2\phi] + O(m^3).$$

Angle $E$ is obtained by using Eq. (31):

$$\tan E = \sqrt{1 - \frac{2m}{r_{\gamma}} r_{C}\frac{d\phi}{dr} |_{C\gamma}} = \sqrt{1 - \frac{2m}{r_{C\gamma}} \tan \phi},$$
which is obtained from the relation
\[
\frac{1}{r_{c_{T}}} \left( \frac{dr}{d\phi} \right)^2 \bigg|_{c_{T}} = \left( \frac{r_{c_{T}}^2}{b^2} - 1 \right) = \frac{1}{\tan^2 \phi}, \tag{43}
\]
noting that \(1/r_{c_{T}} = \sin \phi / b\). In a similar way to the derivation of Eq. (41), \(E\) becomes
\[
E = \phi - \frac{m}{b} \sin^2 \phi \cos \phi - \frac{m^2}{2b^2} \sin^3 \phi \cos (2 - \cos 2\phi) + O(m^4). \tag{44}
\]
Therefore, from Eqs. (41), (44), and (35), the total deflection angle in the Schwarzschild spacetime is obtained as
\[
\alpha_{Sch} = (\epsilon_S - E_S) - (\epsilon_R - E_R) \nonumber
\]
\[
= -\frac{m}{b} \left[ 2(\cos \phi_R - \cos \phi_S) + \sin^2 \phi_R \cos \phi_R - \sin^2 \phi_S \cos \phi_S \right] \nonumber
\]
\[
+ \frac{m^2}{8b^2} \left[ 30(\phi_R - \phi_S) + \sin 2\phi_R - \sin 2\phi_S \right] \nonumber
\]
\[
- 4 \left[ \sin^3 \phi_R \cos \phi_R (2 - \cos 2\phi_R) - \sin^3 \phi_S \cos \phi_S (2 - \cos 2\phi_S) \right] \nonumber
\]
\[
+ O(m^3), \tag{45}
\]
where the sign is taken in such a sense that the total deflection angle \(\alpha\) is positive.

### 5.1.2 Integral Formula

Next, let us show that the integral formula, Eq. (35), also gives the same result as Eq. (45). At first, we integrate the areal integral of the Gaussian curvature \(K_{Sch}\). On the basis of the optical metric Eq. (6) and Eq. (36), the Gaussian curvature Eq. (17) is given by
\[
K_{Sch} = -\frac{2m}{r^3} \left( 1 - \frac{3m}{2r} \right), \tag{46}
\]
and the areal element \(d\sigma = \sqrt{\det [g]} dr d\phi\) is
\[
d\sigma = r \left( 1 - \frac{2m}{r} \right)^{-\frac{3}{2}} dr d\phi. \tag{47}
\]

The quadrilateral \(\Sigma^4\) is bounded by the geodesic of the light ray \(\gamma\), the two radial null geodesics \(\gamma_S\) and \(\gamma_R\) and the unperturbed reference line \(C_T\). \(\gamma_S\) and \(\gamma_R\) are parametrized by the angular coordinate values \(\phi_S\) and \(\phi_R\), respectively. Hence, using Eqs. (46) and (47), and expanding up to the order \(O(m^2)\), we obtain
\[
\int_{\Sigma^4} K_{Sch} d\sigma = -\int_{\phi_S}^{\phi_R} \int_{r_{c_{T}}}^{r_{c_{R}}} \left( \frac{2m}{r^2} + \frac{3m^3}{8r^3} \right) dr d\phi + O(m^3) \nonumber
\]
\[
= -\frac{m^2}{8b^2} \left[ 24(\phi_R - \phi_S) + 4(\sin 2\phi_R - \sin 2\phi_S) \right] + O(m^3), \tag{48}
\]
where \(1/r_{c_{T}} = \sin \phi / b\) and Eq. (39) is used as the trajectory of the light ray \(r_{c_{T}}\). However in practice, it is enough to insert the order \(O(m)\) solution of \(r_{c_{T}}\) to integrate the areal integral of \(K_{Sch}\) because the leading order of the Gaussian curvature \(K_{Sch}\) is \(O(m)\).

Subsequently, we calculate the line integral of the geodesic curvature \(\kappa_{g}^{Sch}\). The geodesic curvature Eq. (18) is expressed as
\[
\kappa_{g}^{Sch} = \frac{b}{r^2} \left( 1 - \frac{3m}{2r} \right) \sqrt{1 - \frac{2m}{r}} + \frac{d\Phi}{dt}. \tag{49}
\]
Noting that the integration of $\kappa_g$ is carried out along the unperturbed reference line $1/r_{C'} = \sin \phi / b$, we have

$$\int_{C'}^{r} \kappa_{g}^{Sch} dt = \int_{\phi_S}^{\phi_R} \left( 1 - \frac{2m}{r_{C'}} \right) \frac{3m^2}{2r_{C'}^2} d\phi + \int_{S}^{R} \frac{d\Phi}{dt} dt + \mathcal{O}(m^3)$$

$$= \phi_R - \phi_S + \frac{2m}{b} (\cos \phi_R - \cos \phi_S)$$

$$- \frac{m^2}{8b^2} \left[ 6(\phi_R - \phi_S) - 3(\sin 2\phi_R - \sin 2\phi_S) \right]$$

$$+ \Phi_R - \Phi_S + \mathcal{O}(m^3) \quad (50)$$

where we changed the integration variable of the first term from $t$ to $\phi$ using

$$dt = \frac{r^2}{b} \left( 1 - \frac{2m}{r} \right)^{-1} d\phi, \quad (51)$$

which is derived from Eq. (5). Because the arc length parameter $t$ moves along the curve $C'$ in the counterclockwise direction, the direction of the tangent vector of curve $C'$ is also counterclockwise; see Fig. 4. Then $\Phi_S$ and $\Phi_R$ should be given by

$$\Phi = \pi - E. \quad (52)$$

Substituting Eqs. (44) and (52) into (50), we have

$$\int_{C'}^{r} \kappa_{g}^{Sch} dt = \frac{m}{b} \left[ 2(\cos \phi_R - \cos \phi_S) + \sin^2 \phi_R \cos \phi_R - \sin^2 \phi_S \cos \phi_S \right]$$

$$- \frac{m^2}{8b^2} \left[ 6(\phi_R - \phi_S) - 3(\sin 2\phi_R - \sin 2\phi_S) \right]$$

$$- 4 \left[ \sin^3 \phi_R \cos \phi_R (2 - \cos 2\phi_R) - \sin^3 \phi_S \cos \phi_S (2 - \cos 2\phi_S) \right]$$

$$+ \mathcal{O}(m^3). \quad (53)$$

From Eqs. (48), (53), and (55), the total deflection angle $\alpha$ becomes

$$\alpha_{Sch} = - \iint_{\Sigma^4} K^{Sch} d\sigma - \int_{C'}^{r} \kappa_{g}^{Sch} dt$$

$$= \frac{m}{b} \left[ 2(\cos \phi_R - \cos \phi_S) + \sin^2 \phi_R \cos \phi_R - \sin^2 \phi_S \cos \phi_S \right]$$

$$+ \frac{m^2}{8b^2} \left[ 30(\phi_R - \phi_S) + \sin 2\phi_R - \sin 2\phi_S \right.$$

$$- 4 \left[ \sin^3 \phi_R \cos \phi_R (2 - \cos 2\phi_R) - \sin^3 \phi_S \cos \phi_S (2 - \cos 2\phi_S) \right]$$

$$+ \mathcal{O}(m^3). \quad (54)$$

We find that Eq. (54) is in perfect agreement with Eq. (45).

5.2 Limit of Infinite Distance Source and Receiver

Let us confirm that Eqs (45) and (54) coincide with Epstein–Shapiro’s formula [32] in the limit of the infinite distance source $S$ and receiver $R$. To this end, we put $\phi_R \to \pi$ and $\phi_S \to 0$; then

$$\alpha_{Sch} \to \frac{4m}{b} + \frac{15\pi m^2}{4b^2} + \mathcal{O}(m^3). \quad (55)$$

Two points are noteworthy; first, we observe the breakdown of Eq. (55) in terms of the integral formula, Eq. (45). The areal integral part is

$$- \iint_{\Sigma^4} K^{Sch} d\sigma = \frac{12\pi m^2}{4b^3}. \quad (56)$$
Thus, the areal integral does not contribute to the first-order in \( m \) because the leading order of the Gaussian curvature is \( O(m) \), which is enough to insert the zero-th order trajectory of light rays \( 1/r_\gamma = \sin\phi/b \) when calculating the total deflection angle \( \alpha \). As a consequence, the upper limit \( r_C \) of integration in \( r \) coincides with the lower limit \( r_\gamma \), and the areal integral vanishes. On the other hand, the line integral part becomes

\[
- \int_{C_T} \kappa_{\gamma}^{\text{Sch}} \, dt = \frac{4m}{b} + \frac{3\pi m^2}{4b^2}.
\]

(57)

Second, in terms of the angular formula, Eq. (58), it is found for \( \phi_S \to 0, \phi_R \to \pi \) that \( E_S \to 0, E_R \to \pi \); see Eq. (44). Hence:

\[
\alpha \to E_S(\phi_S \to 0) - E_R(\phi_R \to \pi) + \pi,
\]

(58)

which has a different sign for \( \pi \) compared to the standard treatment found in textbooks and the literature. This difference is due to the fact that in the standard treatment, the total deflection angle is calculated on the Minkowskian flat space then \( E_R(\phi_R \to \pi) - E_S(\phi_S \to 0) > \pi \), whereas, in the optical reference geometry \( \mathcal{M}_{\text{opt}} \) or the curved \((r, \phi)\) subspace \( \mathcal{M}_{\text{sub}} \), the Gaussian curvature \( K^{\text{Sch}} \) takes the negative value \( K^{\text{Sch}} < 0 \) so that the total deflection angle is given by Eq. (58).

6 Non-asymptotically Flat Spacetime

The goal of this section is to investigate and reveal the contribution of the cosmological constant \( \Lambda \) to the bending of light in terms of the Schwarzschild–de Sitter spacetime. As is the case with the Schwarzschild spacetime in the previous section, we demonstrate that the angular formula Eq. (34) and the integral formula Eq. (35) lead to the same result.

6.1 de Sitter Spacetime

Before discussing the total deflection angle in the Schwarzschild–de Sitter spacetime, let us examine the light rays in the de Sitter spacetime which can be considered as the background spacetime of the Schwarzschild–de Sitter spacetime. The conclusion arrived at here may be trivial; however it provides an important suggestion when considering the total deflection angle in the Schwarzschild–de Sitter spacetime.

In the case of the de Sitter spacetime, \( f(r) \) is given by

\[
f_{\text{dS}}(r) = 1 - \frac{\Lambda}{3}r^2,
\]

(59)

where \( \Lambda \) is the cosmological constant, and the differential equation of the light ray \( \gamma \) on the optical reference geometry \( \mathcal{M}_{\text{opt}} \) is

\[
\left( \frac{dr}{d\phi} \right)^2 = r_\gamma^2 \left[ \frac{1}{b^2} + \frac{\Lambda}{3} \right] - 1.
\]

(60)

If we reduce the two constants \( 1/b^2 \) and \( \Lambda/3 \) to a single constant,

\[
\frac{1}{B^2} \equiv \frac{1}{b^2} + \frac{\Lambda}{3},
\]

(61)

and Eq. (60) gives the similar form of equation as Eq. (24):

\[
\frac{1}{r_\gamma} = \frac{\sin\phi}{B}.
\]

(62)

We notice that from Eqs. (60) and (61), \( B \) can be expressed by \( r_0 \) which is the radial coordinate at the point of closest approach of light trajectory,

\[
\frac{1}{B^2} = \frac{1}{r_0^2} + \frac{\Lambda}{3} = \frac{1}{r_0^2},
\]

(63)
because $dr/d\phi|_{r_0} = 0$. Then $B$ is obtained without any information of $\Lambda$. Accordingly, the equation of the light trajectory $\gamma$ corresponds to that of the unperturbed reference line $C\Gamma$ (please imagine that in Fig. 5 below, for $m \to 0$, the Schwarzschild–de Sitter space reduces to de Sitter space); then $r_\gamma = r_{C\Gamma}$, and conversely $C\Gamma$ is now the null geodesic $\gamma$ itself. In this case, $\varepsilon_S = E_S$ and $\varepsilon_R = E_R$, and $\gamma$ coincides with $C\Gamma$, see Fig. 5. From

![Optical reference geometry](image)

**Fig. 5** Relation between $\gamma$ (solid line) and $C\Gamma$ (dotted line) on de Sitter space. Because $\gamma$ and $C\Gamma$ coincide, the area of quadrilateral $\Sigma^4$ is zero.

the angular formula, Eq. (64):

$$\alpha_{\text{DS}} = |(\varepsilon_R - E_R) - (\varepsilon_S - E_S)| = 0. \quad (64)$$

Because $C\Gamma$ is now the null geodesic $\gamma$, $\kappa^\text{DS}_S = 0$, and we have:

$$\int_{C\Gamma} \kappa^\text{DS}_S \dd t = \int_\gamma \kappa^\text{DS}_S \dd t = 0. \quad (65)$$

The Gaussian curvature of the de Sitter spacetime is

$$K^\text{DS} = -\frac{\Lambda}{3} \neq 0, \quad (66)$$

but the upper limit $r_{C\Gamma}$ and lower limit $r_\gamma$ of integration in $r$ are the same $r_\gamma = r_{C\Gamma}$ (see Fig. 5); consequently:

$$\iint_{\Sigma^4} K^\text{DS} \dd s = 0. \quad (67)$$

Hence, we also obtain by means of the integral formula:

$$\alpha_{\text{DS}} = -\iint_{\Sigma^4} K^\text{DS} \dd s - \int_{C\Gamma} \kappa^\text{DS}_S \dd t = 0. \quad (68)$$

We can conclude that in the de Sitter spacetime the total deflection angle of the light ray is zero, $\alpha_{\text{DS}} = 0$ then the light rays do not bend.

---

5 From the result of Eq. (65), we may immediately say that the light ray in the de Sitter spacetime does not bend.
6.2 Schwarzschild–de Sitter Spacetime

From now on, we examine the total deflection angle $\alpha$ in the Schwarzschild–de Sitter spacetime. The total deflection angle is calculated up to the order $O(m^2, \Lambda, \Lambda m)$. However, as will be clarified later, the total deflection angle does not contain $O(\Lambda)$ terms though they appear in the intermediate steps of the calculation.

The Schwarzschild–de Sitter/Kottler spacetime is characterized by \[ f_{SdS}(r) = 1 - \frac{2m}{r} - \frac{\Lambda}{3} r^2. \] (69)

where $m$ is the mass of the central body and $\Lambda$ is the cosmological constant. As in the case of the Schwarzschild spacetime, the total deflection angle is calculated in such a way that the source $S$ and the receiver $R$ of the light ray are located at a finite distance from the center.

6.2.1 Angular Formula

The angle $\varepsilon$ is obtained from Eq. (30):

$$\tan \varepsilon = \sqrt{1 - \frac{2m}{r_\gamma} - \frac{\Lambda}{3} r_\gamma^2} \left| \frac{d\phi}{dr} \right|_{r_\gamma} = \sqrt{1 - \frac{2m}{r_\gamma} - \frac{\Lambda}{3} r_\gamma^2} \left( 1 + \frac{2mB^2}{r_\gamma^2} - \frac{B^2}{r_\gamma^2} \right)^{-\frac{1}{2}},$$

(70)

because the differential equation of the light ray in the Schwarzschild–de Sitter spacetime is

$$\left( \frac{dr}{d\phi} \right)^2 = r_\gamma^2 \left( \frac{r_\gamma^2}{B^2} + \frac{2m}{r_\gamma} - 1 \right),$$

(71)

where we used Eq. (61). The equation of the light trajectory $\gamma$ on the optical reference geometry $M_{opt}$ has the same form as Eq. (39):

$$\frac{1}{r_\gamma} = \frac{\sin \phi}{B} + \frac{m}{2B^2} (3 + \cos 2\phi)$$

$$+ \frac{m^2}{16B^2} [37 \sin \phi + 30(\pi - 2\phi) \cos \phi - 3 \sin 3\phi] + O(m^3),$$

(72)

except that the constant $b$ is replaced by $B$ (see Eq. (61)). We mention that from Eqs. (71) and (61), $B$ can be written by $r_0$ being the radial coordinate at the point of closest approach of light trajectory

$$\frac{1}{B^2} = \frac{1}{b^2} + \frac{\Lambda}{3} = \frac{1}{r_0^2} - \frac{2m}{r_0^3},$$

(73)

because of $dr/d\phi|_{r_0} = 0$, then $B$ is obtained without any information of $\Lambda$.

Inserting Eq. (72) into Eq. (70) and expanding up to the order $O(m^2, \Lambda, \Lambda m)$, one obtains,

$$\tan \varepsilon = \tan \phi + \frac{2m}{B} \sec \phi + \frac{m^2}{8B^2} [15 \sec^2 \phi (\pi - 2\phi + \sin 2\phi)]$$

$$- \frac{\Lambda B^2}{6} \csc \phi \sec \phi - \frac{\Lambda Bm}{3} (1 - \cot^2 \phi) \sec \phi + O(m^3, \Lambda^2, \Lambda m^2).$$

(74)

As with Eq. (41), $\varepsilon$ is given by

$$\varepsilon = \phi + \frac{2m}{B} \cos \phi + \frac{m^2}{8B^2} [15(\pi - 2\phi) - \sin 2\phi]$$

$$- \frac{\Lambda B^2}{6} \cot \phi + \frac{\Lambda Bm}{3} \cot \phi \csc \phi + O(m^3, \Lambda^2, \Lambda m^2).$$

(75)

The first line of Eq (75) corresponds to Eq. (41), with $b$ being replaced by $B$ in the Schwarzschild part, and the second line is due to the contribution of the cosmological constant $\Lambda$. 
On the other hand, $E$ is calculated with Eq. \((75)\):

$$\tan E = \sqrt{1 - \frac{2m}{r_C} - \frac{\Lambda}{3}} r_C^{2} r_C \left. \frac{d\phi}{dr} \right|_{r_C} = \sqrt{1 - \frac{2m}{r_C} - \frac{\Lambda}{3}} r_C \tan \phi,$$

(76)

because $1/r_C = \sin \phi / B$; then

$$\frac{1}{r_C} \left( \frac{dr}{d\phi} \right)^2 \left|_{r_C} = \left( \frac{r_C^2}{B^2} - 1 \right) = \frac{1}{\tan^2 \phi}. \right. \tag{77}$$

It follows that

$$E = \phi - \frac{m}{B} \sin^2 \phi \cos \phi - \frac{m^2}{2B^2} \sin^3 \phi \cos \phi \cos(2\cos 2\phi)$$

$$- \frac{\Lambda B^2}{6} \cot \phi - \frac{\Lambda Bm}{6} \cos \phi (2 - \cos 2\phi) + \mathcal{O}(m^3, \Lambda^2, Am^2). \tag{78}$$

As in Eq. \((75)\), the first line of Eq. \((78)\) corresponds to Eq. \((44)\), the Schwarzschild part with replaced $b$ by $B$, and the second line is due to the contribution of the cosmological constant $\Lambda$.

As a result, the total deflection angle $\alpha$ becomes, using Eqs. \((75)\), \((76)\), and \((74)\),

$$\alpha_{SdS} = (\epsilon_S - E_S) - (\epsilon_R - E_R)$$

$$= -\frac{m}{B} \left[ 2(\cos \phi_R - \cos \phi_S) + \sin^2 \phi_R \cos \phi_R - \sin^2 \phi_S \cos \phi_S \right]$$

$$+ \frac{m^2}{2B^2} \left[ 30(\phi_R - \phi_S) + \cos 2\phi_R - \cos 2\phi_S ight.$$

$$- \frac{\Lambda B^2}{6} \left[ \cot \phi_R \csc \phi_R - \cot \phi_S \csc \phi_S \right.

$$+ \cos \phi_R(2 - \cos 2\phi_R) - \cos \phi_S(2 - \cos 2\phi_S) + \mathcal{O}(m^3, \Lambda^2, Am^2). \tag{79}$$

We mention here that although four angles $\epsilon_S$, $\epsilon_R$, $E_S$ and $E_R$ include the $\mathcal{O}(\Lambda)$ terms, the $\mathcal{O}(\Lambda)$ terms do not appear in Eq. \((79)\). Further, among the four angles $\epsilon_S$, $\epsilon_R$, $E_S$ and $E_R$, $\epsilon_S$ and $\epsilon_R$ can be considered as the observable angle on the curved space $\mathbb{S}^2$. However comparing Eq. \((75)\) to Eq. \((79)\), twice of $\epsilon_S$, $\epsilon_R$ or $\epsilon_S - \epsilon_R$ does not coincide simply with $\alpha_{SdS}$ due to the presence of $E_S$ and $E_R$.

6.2.2 Integral Formula

We now compute the total deflection angle $\alpha$ by using the integral formula, Eq. \((55)\). In the case of the Schwarzschild–de Sitter spacetime, the Gaussian curvature $K_{SdS}$ becomes

$$K_{SdS} = -\frac{2m}{r^3} \left( 1 - \frac{3m}{2r} + \frac{\Lambda}{6} r^3 - \Lambda r^2 \right), \tag{80}$$

and the areal element $d\sigma$ is

$$d\sigma = r \left( 1 - \frac{2m}{r} - \frac{\Lambda}{3} r^2 \right)^{-2} dr d\phi, \tag{81}$$
In the same way that Eq. (48) was integrated, we construct the quadrilateral \( \Sigma^4 \) bounded by three geodesics \( \gamma \), \( \gamma_s \) and \( \gamma_h \) and the unperturbed reference line \( C_R \) which is the null geodesic \( \Gamma \) on the de Sitter spacetime. We carry out the integration up to the order \( O(\ell) \). In the same way that Eq. (48) was integrated, we construct the quadrilateral \( \Sigma^4 \) bounded by three geodesics \( \gamma \), \( \gamma_s \) and \( \gamma_h \) and the unperturbed reference line \( C_R \) which is the null geodesic \( \Gamma \) on the de Sitter spacetime. We carry out the integration up to the order \( O(\ell) \).

We notice again that the constant in Eq. (85) is not \( \ell \). We carry out the integration up to the order \( \ell^0 \).

The geodesic curvature \( \kappa_G \) is

\[
\kappa^{\text{SdS}}_g = \frac{b}{r^2} \left( 1 - \frac{3m}{r} \right) \sqrt{1 - \frac{2m}{r} - \frac{\Lambda}{3} r^2} \frac{d\phi}{dt},
\]

noting that the constant appearing in \( \kappa^{\text{SdS}}_g \) is \( b \) instead of \( B \). The line integral of \( \kappa^{\text{SdS}}_g \) is integrated along the curve \( C_R \); then we have

\[
\int_{C_R} \kappa^{\text{SdS}}_g \, dt = \int_{\phi_s}^{\phi_R} \left( 1 - \frac{2m}{r_{C_R}} + \frac{3m^2}{2r_{C_R}^2} + \frac{\Lambda}{6} \right) \, d\phi + \int_{S} \frac{d\Phi}{dt} \, dt + \mathcal{O}(m^3, \Lambda^2, \Lambda m^2)
\]

where, the line element of the first term, \( dt \), is replaced by \( d\phi \) as follows:

\[
dt = r^2 \left( 1 - \frac{2m}{r} - \frac{\Lambda}{3} r^2 \right)^{-1} \, d\phi,
\]

from Eq. (5). We notice again that the constant in Eq. (5) is not \( B \) but \( b \). Substituting Eqs. (52) and (78) into Eq. (84) yields

\[
\int_{C_R} \kappa^{\text{SdS}}_g \, dt = \frac{m}{b} \left[ 2 \cos \phi_R - \cos \phi_S \right] + \int_{\phi_s}^{\phi_R} \sqrt{1 - \frac{2m}{r} - \frac{\Lambda}{3} r^2} \, d\phi + \mathcal{O}(m^3, \Lambda^2, \Lambda m^2)
\]

where \( 1/r_{C_R} = \sin \phi/B \) and \( r_\gamma \) is given by Eq. (72); but in this case, it is enough to adopt the order \( \mathcal{O}(m) \) solution of \( r_\gamma \) as in Eq. (48).
We note that the order $\mathcal{O}(\Lambda)$ terms cancel out in Eq. (86). Then, from Eqs. (82), (86) and (35), the integral formula gives the same result as the angular formula:

$$\alpha_{\text{SdS}} = -\int_{\Sigma_s} K_{\text{SdS}}^s d\sigma - \int_{C_t} K_{\text{SdS}} \, dt$$

$$= -\frac{m}{B} \left[ 2(\cos \phi_R - \cos \phi_S) + \sin^2 \phi_R \cos \phi_R - \sin^2 \phi_S \cos \phi_S \right]$$

$$+ \frac{m^2}{8B^2} \left[ 30(\phi_R - \phi_S) + \sin 2\phi_R - \sin 2\phi_S 
- 4 \left[ \sin^3 \phi_R \cos \phi_R (2 - \cos 2\phi_R) - \sin^3 \phi_S \cos \phi_S (2 - \cos 2\phi_S) \right] \right]$$

$$+ \frac{\Lambda B m}{6} \left[ 2(\cot \phi_R \csc \phi_R - \cot \phi_S \csc \phi_S) \right.$$  

$$+ \cos \phi_R (2 - \cos 2\phi_R) - \cos \phi_S (2 - \cos 2\phi_S) \left. \right] + \mathcal{O}(m^3, \Lambda^2, \Lambda m^2).$$  

(87)

It should be pointed out here that Eqs. (79) and (87) do not contain the $\mathcal{O}(\Lambda)$ terms. This result can be regarded as reflecting the fact the light ray does not bend in the de Sitter spacetime.

Here, we comment on the difference between the present results, Eqs. (79) and (87), and our previous works, that is [15, 19]. In [15], we showed that the equation of light trajectory does not change its form in the Schwarzschild and Schwarzschild–de Sitter spacetime, and from the similarity of trajectory equation of light ray, we concluded that the total deflection angle of light ray $\alpha$ works, that is [15, 19]. In [15], we showed that the equation of light trajectory does not change even if $\Lambda \neq 0$, see Eq. (13) in [15]. However, because the equation of light trajectory and its differential equation give the relation of “coordinate values $r$ and $\phi$” and generally they do not have the meaning of length and angle in the curved spacetime. Then the total deflection angle must be determined by the metric in such a way that Eqs. (30) and (51) with Eq. (69). Therefore, the total deflection angle $\alpha$ should contain the contribution of $\Lambda$. On the other hand, in [19] the contribution of $\Lambda$ appears as the $\mathcal{O}(\Lambda)$ term (see Eq. (25) in [19]). But the appearance of the $\mathcal{O}(\Lambda)$ term means that the light ray bends in the de Sitter spacetime and it is contrary to the conclusion in 6.1. We can say that in [19] the background is the Minkowski spacetime instead of the de Sitter one.

### 6.3 Estimation of Observability

Before closing this section, let us investigate the observability of light deflection by the cosmological constant $\Lambda$ in terms of the Schwarzschild–de Sitter solution. From Eqs. (79) and (87), we extract only the part of the order $\mathcal{O}(\Lambda m)$ terms and put

$$\alpha_{\text{SdS}}^\Lambda = \frac{\Lambda B m}{6} \left[ 2(\cot \phi_R \csc \phi_R - \cot \phi_S \csc \phi_S) \right.$$  

$$+ \cos \phi_R (2 - \cos 2\phi_R) - \cos \phi_S (2 - \cos 2\phi_S) \left. \right].$$  

(88)

To evaluate $\alpha_{\text{SdS}}^\Lambda$, we assume $\Lambda \approx 10^{-52}$ m$^{-2}$. As an example, let us consider the galaxy as the lens object with mass $m \approx 10^{12} G M_\odot / c^2$ (where $G = 6.674 \times 10^{-11}$ m$^3$ kg$^{-1}$ s$^{-2}$ is a Newtonian gravitational constant, $c = 3.0 \times 10^8$ m s$^{-1}$ is the speed of light in vacuum, and $M_\odot = 2.0 \times 10^{30}$ kg is the mass of the sun) and we employ its radius as the impact parameter $b \approx B \approx R_{\text{galaxy}} \approx 5.0 \times 10^4$ ly $\approx 5 \times 10^{20}$ m. Using these values, we find

$$\frac{4m}{B} \approx 1.2 \times 10^{-5}, \quad \frac{\Lambda B m}{6} \approx 1.2 \times 10^{-17},$$

(89)

and therefore in general $\alpha_{\text{SdS}}^\Lambda$ is mostly 12 orders of magnitude smaller than the Schwarzschild part. However, due to the cot $\phi \csc \phi$ term, $\alpha_{\text{SdS}}^\Lambda$ increases sharply when the source $S$ and/or the receiver $R$ reaches the de Sitter horizon $r \to r_A = \sqrt{3/\Lambda} \approx 1.73 \times 10^{26}$ m. We notice that in terms of the angular formula, the cot $\phi \csc \phi$ terms come from the expression of $\epsilon$ which is the observable angle between the null geodesic $\gamma$ (actual path of light ray) and the radial null geodesic ($\gamma_S$ or $\gamma_R$) on the optical reference geometry $\mathcal{M}^{\text{opt}}$ or the curved $\phi$ subspace $\mathcal{M}^{\text{sub}}$ (see Eq. (75)). From $\sin \phi_A = B/r_A$, we have the angular coordinate of the de Sitter horizon as $\phi_A \approx 3.0 \times 10^{-6}$. Now, suppose the source $S$ and the receiver $R$ are placed in a symmetrical position near
the de Sitter horizon with respect to the lens plane, and set \( \phi_S = \phi_A \) and \( \phi_R = \pi - \phi_A \). Then \( \alpha_{\Lambda}^{\Lambda} \) becomes an extreme case:

\[
\alpha_{\Lambda}^{\Lambda} \approx -5.5 \times 10^{-6} \text{ rad}.
\]  

This value is almost half the value of the Schwarzschild part. Perhaps the estimation, Eq. (90), may largely exceed the value restricted by the current cosmological observations, nevertheless from Eq. (88), if both the source \( S \) and the receiver \( R \) are located near the de Sitter horizon, we may be able to detect the contribution of the cosmological constant \( \Lambda \) to light deflection, especially by the gravitational lensing. Fig. 6 shows \( \alpha_{\Lambda}^{\Lambda} \)

![Diagram](image)

**Fig. 6** \( \alpha_{\Lambda}^{\Lambda} \) as a function of \( \phi_S \) and \( \phi_R \). In this figure, \( \alpha_{\Lambda}^{\Lambda} \) is plotted for the domains \( 0 \leq \phi_S \leq \pi/2 \) and \( \pi/2 \leq \phi_R \leq \pi \).

As a function of \( \phi_S \) and \( \phi_R \). It is found that \( \alpha_{\Lambda}^{\Lambda} \) rapidly becomes a large value when \( \phi_S \to 0 \) and/or \( \phi_R \to \pi \).

7 Summary and Conclusions

In this paper, we re-examined the light deflection in the Schwarzschild and the Schwarzschild–de Sitter spacetime. First, we proposed a definition of the total deflection angle \( \alpha \) by constructing a quadrilateral \( \Sigma^4 \) on the optical reference geometry \( M_{\text{opt}} \) determined by the optical metric \( \bar{g}_{ij} \) under the static and spherically symmetric spacetime. To construct a quadrilateral \( \Sigma^4 \), we introduced the unperturbed reference line \( C_{\Gamma} \) which is the null geodesic \( \Gamma \) on a background spacetime such as the Minkowski or the de Sitter spacetime, and \( C_{\Gamma} \) is obtained by projecting \( \Gamma \) vertically onto the curved \((r, \phi)\) subspace \( M_{\text{sub}} \). After preparing the unperturbed reference line \( C_{\Gamma} \), we laid a quadrilateral \( \Sigma^4 \) bounded by the null geodesic of the light ray \( \gamma \), two radial null geodesics; \( \gamma_S \) connecting the center \( O \) and the source \( S \) of the light ray and \( \gamma_R \) connecting the center \( O \) and the receiver \( R \) of the light ray, and the unperturbed reference line \( C_{\Gamma} \). The total deflection angle \( \alpha \) is then defined as the sum of the four internal angles \( \beta_p \) minus \( 2\pi \); see Eq. (27).

On the basis of the definition of the total deflection angle \( \alpha \) and the Gauss–Bonnet theorem, we derived two formulas for calculating the total deflection angle \( \alpha \): (i) the angular formula Eq. (34) that uses four angles determined on the optical reference geometry \( M_{\text{opt}} \) or the curved \((r, \phi)\) subspace \( M_{\text{sub}} \) that is a slice of constant time \( t \) and (ii) the integral formula, Eq. (35), on the optical reference geometry \( M_{\text{opt}} \) is the area integral of the Gaussian curvature \( K \) in the area of the quadrilateral \( \Sigma^4 \) and the line integral of the geodesic curvature \( \kappa_\gamma \) along the curve \( C_{\Gamma} \). The angular formula, Eq. (34), is derived directly from the definition of the total deflection angle, Eq. (27), which gives the meaning of the total deflection angle \( \alpha \) geometrically clear.

The angular formula can be used not only on the optical reference geometry \( M_{\text{opt}} \) but also on the curved \((r, \phi)\) subspace \( M_{\text{sub}} \) but an order \( O(m^n) \) solution for \( r_\gamma \) is required when obtaining the total deflection angle \( \alpha \) in order \( O(m^n) \). On the other hand, the integral formula should be calculated on the optical reference geometry \( M_{\text{opt}} \); however, it is enough to prepare an order \( O(m^{n-1}) \) solution for \( r_\gamma \) when obtaining the total deflection angle \( \alpha \) in order \( O(m^n) \).
We demonstrated that the two formulas give the same total deflection angle $\alpha$ for the Schwarzschild and the Schwarzschild–de Sitter spacetime. In particular, in the Schwarzschild case, the result coincides with Epstein–Shapiro’s formula when the source $S$ and the receiver $R$ of the light ray are located at infinity, and in the Schwarzschild–de Sitter case, there appear order $\mathcal{O}(Am)$ terms as well as the Schwarzschild-like part, whereas order $\mathcal{O}(A)$ terms disappear because the light ray does not bend in the de Sitter spacetime.

In this paper, we took the stance that in order to obtain the total deflection angle $\alpha$, we need to compare the two null geodesics $\gamma$ on the actual curved space and $\Gamma$ on the background space, and for this purpose we vertically projected $\Gamma$ onto the curved space. Then, we expect that our approach is superior to that projecting $\gamma$ vertically onto the background space, e.g., the flat space for calculating the total deflection angle in the curved space.

8 acknowledgments

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A Relation of Coordinate Values among Two Dimensional Curved spaces and three Dimensional Flat Plane

The purpose of this appendix is to show the relation between coordinate values among the curved $(r, \phi)$ subspaces and the flat $(\rho, \phi)$ plane under the static and spherical symmetric spacetime.

First, let us observe that two curved $(r, \phi)$ subspaces $\mathcal{M}_{1}^{\text{sub}}$ and $\mathcal{M}_{2}^{\text{sub}}$ characterized by Eq. (7) which is a slice of constant time $t$ of Eq. (2),

$$d\ell^{2} = \frac{1}{f(r)}dr^{2} + r^{2}d\phi^{2},$$

have the same coordinate values. For the given curved $(r, \phi)$ subspaces $\mathcal{M}_{1}^{\text{sub}}$ and $\mathcal{M}_{2}^{\text{sub}}$, the form of $f(r)$ differs, i.e., $f_{1}(r_{1}) \neq f_{2}(r_{2})$. But in the case of the static and spherical symmetric spacetime, the radial coordinate $r$ is obtained as the circumference radius for $dr = 0$ (constant radius),

$$\ell = 2\pi r_{1} = 2\pi r_{2}. \quad (91)$$

Then the curved $(r_{1}, \phi_{1})$ subspace $\mathcal{M}_{1}^{\text{sub}}$ and the curved $(r_{2}, \phi_{2})$ subspace $\mathcal{M}_{2}^{\text{sub}}$ then take the same coordinate value:

$$r_{1} = r_{2}, \quad \phi_{1} = \phi_{2}. \quad (92)$$

Next, in order to investigate the relation between the curved $(r, \phi)$ subspace $\mathcal{M}^{\text{sub}}$ and the flat $(\rho, \phi)$ plane, let us recall that the curved $(r, \phi)$ subspace $\mathcal{M}^{\text{sub}}$ can be embedded into a three dimensional flat space with the cylindrical coordinates $(\rho, \phi, z)$:

$$d\ell^{2} = dp^{2} + \rho^{2}d\phi^{2} + dz^{2} = \left[1 + \left(\frac{dz}{d\rho}\right)^{2}\right]d\rho^{2} + \rho^{2}d\phi^{2}. \quad (93)$$

As the same way above, the circumference $\ell$ can be determined as,

$$\ell = 2\pi \rho = 2\pi r, \quad (94)$$

then we find

$$r = \rho, \quad \phi = \phi. \quad (95)$$

This means that the flat $(\rho, \phi)$ plane of the three-dimensional flat space and the curved $(r, \phi)$ subspace $\mathcal{M}^{\text{sub}}$ have the same coordinate values. We note that in the flat $(\rho, \phi)$ plane, $\rho$ and $\phi$ are directly related to the distance and angle, respectively, whereas in the curved $(r, \phi)$ subspace $\mathcal{M}^{\text{sub}}$, $r$ and $\phi$ are just coordinate values; then, in general they do not simply mean the distance and angle.

From this consideration, we find that for the circumference $\ell$, the flat $(\rho, \phi)$ plane and the two curved $(r, \phi)$ subspaces, namely curved $(r, \phi)$ subspaces $\mathcal{M}_{1}^{\text{sub}}$ and $\mathcal{M}_{2}^{\text{sub}}$, have "the same coordinate values" $r_{1} = r_{2} = \rho$ and $\phi_{1} = \phi_{2} = \phi$. This fact implies the following; consider an arbitrary curve $C$, which may or may not be the geodesic. If the curve $C$ is given on one of the curved $(r, \phi)$ subspaces or the flat $(\rho, \phi)$ plane, its form does not change on another curved $(r, \phi)$ subspace by the vertical projection, and vice versa.

For the purpose of reference, we obtain $z$ as a function of $r$ or $\rho$. Noting that $r = \rho$, $z$ can be obtained by solving the differential equation for the given $f(r)$:

$$1 + \left(\frac{dz}{dr}\right)^{2} = \frac{1}{f(r)}. \quad (96)$$
For the Schwarzschild case, $f(r)$ is

$$f_{\text{Sch}}(r) = 1 - \frac{2m}{r}, \quad (97)$$

where $m$ is the central mass and $z$ is expressed as

$$z = 2\sqrt{2m(r - 2m)} = 2\sqrt{2m(\rho - 2m)} \quad (98)$$

which is well known as Flamm’s paraboloid \[34\]. For the de Sitter case, the form of $f(r)$ is

$$f_{\text{dS}}(r) = 1 - \frac{\Lambda}{3}r^2, \quad (99)$$

in which $\Lambda$ is the cosmological constant, and it yields

$$z = \sqrt{\frac{3}{\Lambda} - r^2} = \sqrt{\frac{3}{\Lambda} - \rho^2}. \quad (100)$$

For the Schwarzschild-de Sitter case, $f(r)$ becomes

$$f_{\text{SdS}}(r) = 1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2, \quad (101)$$

however, it is not easy to obtain the relation $z$ with $r$ or $\rho$ explicitly. Nonetheless it is obvious that $z$ can be represented by a function of $r$ or $\rho$,

$$z = z(r) = z(\rho).$$

B Note on Calculating Angle

In this paper, we calculated the angles $\varepsilon_{S}, \varepsilon_{R}, E_{S}$ and $E_{R}$ with the tangent formulas, Eqs (30) and (31). However, it may seem to be easier to perform the computation by using sine formula such as Eq. (16) in [22]:

$$\sin \Psi = b \frac{\sqrt{f(r)}}{r} \quad (102)$$

which is equivalent to the cosine formula, see e.g., Eq. (15) in [22]:

$$\cos \Psi = \sqrt{\gamma_{rr} b A(r)} \frac{dr}{d\phi} = b \frac{dr}{r^2 d\phi}. \quad (103)$$

where we replaced $A(r)$ by $f(r)$ and $\gamma_{rr}$ by $1/f^2(r)$ appeared in [22]. Because of the appearance of $f(r)$ in Eq. (102), the angle $\Psi$ is measured on the curved $(r, \phi)$ subspace $\mathcal{M}_{\text{sub}}$ or the optical reference geometry $\mathcal{M}_{\text{opt}}$. In this appendix, we briefly show the formula, Eq. (102) diverges significantly when $\phi \to \pi/2$, whereas the tangent formulas, Eqs (30) and (31), avoid this divergence.

As an example, we calculate the angle $E$ of the Schwarzschild case and for simplicity we obtain $E$ up to the order $O(m)$. From the sine formula, Eq. (102),

$$\sin E = b \frac{\sqrt{f(r)}}{r_{C_f}} \sqrt{1 - \frac{2m}{r_{C_f}}} = \sin \phi - \frac{m}{b} \sin^2 \phi + O(m^2), \quad (104)$$

where $1/r_{C_f} = \sin \phi / b$ then

$$E = \arcsin \left( \sin \phi - \frac{m}{b} \sin^2 \phi \right) \quad (105)$$

$$= \phi - \frac{m}{b} \sin \phi \tan \phi + O(m^2). \quad (106)$$

Eq. (106) is equivalent to the expression of the second line of Eq. (32) in [22] after the following substitutions:

$$\arcsin(bu) = \phi, \quad \frac{1}{2} b r_{g} \frac{u^2}{\sqrt{1 - b^2 u^2}} = \frac{m}{r} \frac{b}{\sqrt{r^2 - b^2}} = \frac{m}{b} \sin \phi \tan \phi. \quad (107)$$

where $u = 1/r$ and $r_{g} = 2m$.

From the tangent formula, Eq. (31), $\tan E$ can be expressed as

$$\tan E = \tan \phi - \frac{m}{b} \sin \phi \tan \phi + O(m^2). \quad (108)$$
and

$$E^{\tan} = \arctan \left( \tan \phi - \frac{m}{b} \sin \phi \tan \phi \right) \tag{109}$$

$$= \phi - \frac{m}{b} \sin^2 \phi \cos \phi + O(m^2). \tag{110}$$

It is found that for $\phi \to 0$, Eqs. (106) and (110) approach 0, whereas for $\phi \to \pi/2$, Eq. (110) reaches $\pi/2$; on the other hand Eq. (106) becomes $-\infty$ due to the term $-(m/b) \sin \phi \tan \phi$. Therefore, the angle should be computed with the tangent formula rather than with the sine and cosine formulas. Fig. 7 plots the absolute value of the deviation between Eqs. (105) and (106), $\Delta E^{\sin}$, (solid line) and Eqs. (109) and (110), $\Delta E^{\tan}$, (dashed line). It is clear that as the coordinate value $\phi$ becomes larger, the result of Eq. (106) deviates from that of Eq. (105). On the other hand, the tangent formula does not produce such a divergence.

![Fig. 7 Deviations between Eqs. (105) and (106), $\Delta E^{\sin}$, (solid line), and Eqs. (109) and (110), $\Delta E^{\tan}$, (dashed line). In this plot, we set $m/b = 0.001$.](image)

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