A New Modified Newton Method use of Haar wavelet for solving Nonlinear equations

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Abstract

In this paper, we present a new modified Newton method a use of Haar wavelet formula for solving non-linear equations. This new method do not require the use of the second-order derivative. It is shown that the new method has third-order of convergent. Furthermore, some numerical experiments are conducted which confirm our theoretical findings.

Keywords: Newton method; Haar wavelet; Iterative Method; Third-order convergence; Non-linear equations; Root-finding

1 Introduction

In numerical analysis, finding a solution of non-linear equation is one of the most attractive problem. In this paper, we emphasize on an iterative method to find a simple root \( \alpha \) of a non-linear equation \( f(x) = 0 \), i.e., \( f(\alpha) = 0 \) and \( f'(\alpha) \neq 0 \). Here, we less concern about multiple roots. Newtons method [1],[6] is the well known algorithm to solve nonlinear equation. It is given by

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad n = 0, 1, 2, \cdots
\]

and it converges quadratically.

Earlier, [3]-[5] and [8]-[13] derived third-order convergence methods based on integral interpretation of Newton’s method. Where, Newton’s method derived from different quadrature formulas for the indefinite integral arising from Newton’s theorem [2]

\[
f(x) = f(x_0) + \int_{x_n}^{x} f'(t) dt.
\]
Weerakoon et al. [3] have approximated the integral part of (1.2) by trapezoidal rules and derived a variant of Newton’s method. It is, further, shown that this method converges cubically. Subsequently, Frontini et al. [4] have proposed a third order convergent method by approximate the integral by the midpoint rule. In [10], Homeier has developed a cubically convergent iteration scheme by considering Newton’s theorem for the inverse function. Further, in [11], [12] modified Newton methods are derived for multivariate case. Kou et al. in [13] have applied a new interval of integration on Newton’s theorem and arrived a third-order convergent iterative scheme.

Recently, Islam et al. [7] have applied Haar wavelet function to derived quadrature rules for indefinite integration. In this paper, we modified Newton’s theorem by using the quadrature rule proposed by Islam in [7]. It is shown that the new method has third order convergent. Further, the new method did not evaluate second derivative of \( f \). The efficiency of the new method is demonstrated by numerical examples.

This paper is organized as follows. In Section 2, we discuss a modified Newton’s method. Section 3, we establish convergence analysis for the new method. finally in section 4, various numerical experiments conduct to confirm our theoretical finding.

## 2 A Modified Newton’s Method

To derive the new method, we consider Newtons theorem

\[
f(x) = f(x_n) + \int_{x_n}^{x} f'(\tau) d\tau
\]  
(2.1)

We use the Haar wavelet function to approximate the integral term of (2.1) as

\[
\int_{x_n}^{x} f'(\tau) d\tau = \frac{(x - x_n)}{2M} \sum_{k=1}^{2M} f'(x_n + \frac{(x - x_n)(k - 0.5)}{2M}).
\]  
(2.2)

where \( M = 2^{J_1} \) and \( J_1 \) is the maximum level of resolution of Haar wavelets, see [7]. Substitute (2.2) in (2.1) to obtain

\[
f(x) = f(x_n) + \frac{(x - x_n)}{2M} \sum_{k=1}^{2M} f'(x_n + \frac{(x - x_n)(k - 0.5)}{2M}).
\]  
(2.3)

Now, looking for \( f(x) = 0 \) we arrive at

\[
x_{n+1} = x_n - \frac{2M(f(x_n))}{\sum_{k=1}^{2M} f'(x_n + \frac{(x - x_n)(k - 0.5)}{2M})} \]  
(2.4)

Further, substitute \( x_{n+1} = x \) in (1.1) and replace \( x - x_n \) in (2.4) we obtain the new method as

\[
x_{n+1} = x_n - \frac{2M(f(x_n))}{\sum_{k=1}^{2M} f'(x_n - \frac{f(x_n)(k - 0.5)}{2M})} \]  
(2.5)
3 Convergence Analysis

**Theorem 3.1.** Let the function \( f : I \subset \mathbb{R} \rightarrow \mathbb{R} \) has a simple root \( \alpha \in I \), where \( I \) is an open interval. Assume \( f(x) \) has first, second and third derivatives in the interval \( I \). If the initial guess \( x_0 \) is closed to \( \alpha \), then the method defined by (2.5) converges cubically to \( \alpha \).

**Proof.** Let \( \alpha \) is the simple root of \( f(x) \) and \( x_n = \alpha + \epsilon_n \). A use of Taylor expansion with \( f(\alpha) = 0 \), we have

\[
f(x_n) = f'(\alpha)(\epsilon_n + C_2\epsilon_n^2 + C_3\epsilon_n^3 + O(\epsilon_n^4)),
\]

(3.6)

where \( C_k = \frac{1}{k!} f^{(k)}(\alpha) \). Again

\[
f'(x_n) = f'(\alpha)(1 + 2C_2\epsilon_n + 3C_3\epsilon_n^2 + 4C_4\epsilon_n^3 + O(\epsilon_n^4)).
\]

(3.7)

Further, dividing (3.9) by (3.7) yields

\[
\frac{f(x_n)}{f'(x_n)} = (\epsilon_n - C_2\epsilon_n^2 + 2(C_2^2 - C_3)\epsilon_n^3 + O(\epsilon_n^4)).
\]

(3.8)

Now,

\[
x_n - M_k\frac{f(x_n)}{f'(x_n)} = x_n - M_k(\epsilon_n - C_2\epsilon_n^2 + 2(C_2^2 - C_3)\epsilon_n^3 + O(\epsilon_n^4)),
\]

(3.9)

where \( M_k = \frac{(k-0.5)}{2M} \).

Equation (3.9) rewrite as

\[
x_n - M_k\frac{f(x_n)}{f'(x_n)} = x_n - M_k(\epsilon_n - C_2\epsilon_n^2 + 2(C_2^2 - C_3)\epsilon_n^3 + O(\epsilon_n^4)),
\]

\[
= x_n - M_k\epsilon_n + M_kC_2\epsilon_n^2 - 2(C_2^2 - C_3)M_k\epsilon_n^3 + O(\epsilon_n^4),
\]

\[
= x_n - (1 - 1 + M_k)\epsilon_n + M_kC_2\epsilon_n^2 - 2(C_2^2 - C_3)M_k\epsilon_n^3 + O(\epsilon_n^4),
\]

\[
= x_n - \epsilon_n + M_kC_2\epsilon_n^2 - 2(C_2^2 - C_3)M_k\epsilon_n^3 + O(\epsilon_n^4),
\]

\[
= \alpha + (1 - M_k)\epsilon_n + M_kC_2\epsilon_n^2 - 2(C_2^2 - C_3)M_k\epsilon_n^3 + O(\epsilon_n^4).
\]

(3.10)

From (3.10) we can easily find that

\[
f'(x_n - M_k\frac{f(x_n)}{f'(x_n)}) = f'(\alpha)(1 + 2C_2(1 - M_k)\epsilon_n + (2C_2^2M_k + 3C_3(1 - M_k)^2)e_n^2 + O(\epsilon_n^3)).
\]

(3.11)
Hence,
\[
\sum_{k=1}^{N} f'(x_n - M_k) \frac{f(x_n)}{f'(x_n)} = \sum_{k=1}^{N} f'(\alpha) \left(1 + 2C_2(1 - M_k)e_n + (2C_2^2M_k + 3C_3(1 - M_k)^2)e_n^2 + O(e_n^3)\right),
\]
\[
= f'(\alpha) \sum_{k=1}^{N} (1 + 2C_2(1 - M_k)e_n + (2C_2^2M_k + 3C_3(1 - M_k)^2)e_n^2 + O(e_n^3)),
\]
\[
= f'(\alpha) \left(\sum_{k=1}^{N} 1 + 2C_2e_n \sum_{k=1}^{N} (1 - M_k) + 2C_2^2e_n \sum_{k=1}^{N} M_k + 3C_3e_n \sum_{k=1}^{N} (1 - M_k)^2 + O(e_n^3)\right),
\]
\[
= f'(\alpha) \left(N + 2C_2Ne_n - 2C_2e_nN/2 + 2C_2^2e_nN/2 + 3C_3e_nN + 3C_3e_n^2\left(\frac{N}{3} - \frac{1}{12N}\right) - 6C_3e_n^2N/2 + O(e_n^3)\right)
\]
\[
= f'(\alpha)(N + 2C_2Ne_n + (NC_2^2 + NC_3 - \frac{C_3}{4N})e_n^2 + O(e_n^3))\quad (3.12)
\]

Substitute (3.12) and (3.6) in (2.5) we obtain
\[
x_{n+1} = x_n - \frac{\left(e_n + C_2e_n^2 + C_3e_n^3 + O(e_n^4)\right)}{\left(N + 2C_2Ne_n + (NC_2^2 + NC_3 - \frac{C_3}{4N})e_n^2 + O(e_n^3)\right)}.
\]
\[
x_{n+1} = x_n - \left(e_n + \left(-C_2^2 + \frac{C_3}{4N^2}\right)e_n^2 + O(e_n^3)\right)\quad (3.13)
\]

Subtract \(\alpha\) from both side of (3.13), then we have
\[
e_{n+1} = (C_2^2 - \frac{C_3}{4N^2})e_n^3 + O(e_n^4)\quad (3.14)
\]
this completes the rest of the proof. \(\square\)

4 Numerical Examples

In this section, we present some numerical results for various third order convergent iterative methods. The following methods were compared:

Modified Newton’s Method in Weerakoon and Fernando \[3\] \(x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\)

Modified Newton’s Method proposed by Frontini et al. \[4\] \(x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} + f'(x_n)\)

Modified Newton’s Method derived by Ozbal et al. \[5\] \(x_{n+1} = x_n - \frac{f(x_n)}{2f'(x_n)}\left(1 + \frac{1}{f'(x_n)}\right)\)

Modified Newton’s Method derived by Kou et al. \[13\] \(x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - f'(x_n)\)

and method proposed by (2.5) with \(M = 1\), \(x_{n+1} = x_n - \frac{2f(x_n)}{\sum_{k=1}^{N} f'(x_n) \left(x_n - f(x_n) \frac{k-0.5}{f'(x_n)}\right)}\).

For every problem an attempt made to find an approximation \(x_n\) of the simple root of equation \(f(x) = 0\) through \(n\) times iterations. The number of function evaluations (NFE) is counted as the sum of the number of evaluations of the function \(f\) and its first order derivative \(f'\). The computational results are displayed in Table 1.
The numerical experiments carried over the following equation:

\[
\begin{align*}
    f_1(x) &= x^5 - x + 1, \\
    f_2(x) &= \cos x - x, \\
    f_3(x) &= \arctan x, \\
    f_4(x) &= 10xe^{-x^2} - 1, \\
    f_5(x) &= e^{-x^2} \sin x + \log(x^2 + 1), \\
    f_6(x) &= x^3 - e^x, \\
    f_7(x) &= e^{-x} - \cos x,
\end{align*}
\]

(4.15)

The numerical results presented in Table 1 show that the proposed method has performed equally as compared with the other methods of the same order. Thus, the new methods can compete with other third-order methods in literature.

5 Conclusion

This article deals with a new modified Newton methods for solving nonlinear equations. In Theorem 3.1, it is proved that the new method has third order convergence. Further, the new methods can compete with other third-order methods. Finally, numerical experiments are conducted to confirm our theoretical findings. The new method has great practical utility.

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| Function | $x_0$ | Various Method | IT | NFE | $x_n$ |
|----------|-------|----------------|----|-----|-------|
| $f_1$    | 2     | MNM(3)         |     |     | Diverse |
|          |       | MNM(4)         |     |     | Diverse |
|          |       | MNM(5)         | 13  | 39  | -1.16730397826142 |
|          |       | MNM(13)        | 17  | 51  | -1.16730397826142 |
|          |       | MNM New        | 9   | 36  | -1.16730397826142 |
| $f_2$    | 1.2   | MNM(3)         | 4   | 12  | 0.739085133215161 |
|          |       | MNM(4)         |     |     | Diverse |
|          |       | MNM(5)         |     |     | Diverse |
|          |       | MNM(13)        | 4   | 12  | 0.739085133215161 |
|          |       | MNM New        | 4   | 16  | 0.739085133215161 |
| $f_3$    | 3     | MNM(3)         |     |     | Diverse |
|          |       | MNM(4)         |     |     | Diverse |
|          |       | MNM(5)         |     |     | Diverse |
|          |       | MNM(13)        | 4   | 12  | 0.000000000000015 |
|          |       | MNM New        | 4   | 16  | 0.0 |
| $f_4$    | 2.5   | MNM(3)         | 4   | 12  | 1.67963061042845 |
|          |       | MNM(4)         | 7   | 21  | 1.67963061042845 |
|          |       | MNM(5)         | 4   | 12  | 0.101025848315685 |
|          |       | MNM(13)        | 6   | 18  | 1.67963061042845 |
|          |       | MNM New        | 5   | 20  | 1.67963061042845 |
| $f_5$    | 1.3   | MNM(3)         | 3   | 9   | 7.68481808334733E-021 |
|          |       | MNM(4)         | 4   | 12  | 1.7197167733818E-028 |
|          |       | MNM(5)         | 4   | 12  | 5.12759588393657E-030 |
|          |       | MNM(13)        | 3   | 9   | 3.82180552357862E-019 |
|          |       | MNM New        | 3   | 12  | 4.53468286561001E-017 |
| $f_6$    | 2     | MNM(3)         | 5   | 15  | 0.77288295914921 |
|          |       | MNM(4)         | 4   | 12  | 0.77288295914921 |
|          |       | MNM(5)         | 4   | 12  | 0.77288295914921 |
|          |       | MNM(13)        | 5   | 15  | 0.77288295914921 |
|          |       | MNM New        | 4   | 16  | 0.77288295914921 |
| $f_7$    | 2     | MNM(3)         | 3   | 9   | 1.29269571937339 |
|          |       | MNM(4)         | 3   | 9   | 1.29269571937339 |
|          |       | MNM(5)         | 4   | 12  | 1.29269571937339 |
|          |       | MNM(13)        | 4   | 12  | 1.29269571937339 |
|          |       | MNM New        | 3   | 12  | 1.29269571937339 |

Table 1: Comparison of various third order convergent iterative methods and the New Newton method