Approximation properties of the $q$-Balázs-Szabados operators in the case $q \geq 1$

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Abstract
This paper deals with approximation properties of the newly defined $q$-generalization of the Balázs-Szabados operators in the case $q \geq 1$. Quantitative estimates of the convergence and Voronovskaja type theorem are given. In particular, it is proved that the rate of approximation by the $q$-Balázs-Szabados ($q > 1$) is of order $q^{-n}$ versus $1/n$ for the classical Balázs-Szabados ($q = 1$) operators. The results are new even for the classical case $q = 1$.

1 Introduction
The goal of the paper is to define a $q$-analogue and study approximation properties for the rational complex Balázs-Szabados operators given by

$$R_n(f; z) = \sum_{k=0}^{n} f(k) \binom{n}{k} (a_n z)^k,$$

where $a_n = n^\beta - 1$, $b_n = n^\beta$, $0 < \beta \leq \frac{3}{2}$, $n \in \mathbb{N}$ and $z \in \mathbb{C}$, $z \neq -\frac{1}{a_n}$.

In the real form rational operators were introduced and studied in Balázs [1] and Balázs-Szabados [2]. Totik [7] settled the saturation properties of $R_n(f)$. Further studies on these operators in the case of real variable can be found in the paper Abel-Della Vecchia [3]. They studied the complete asymptotic expansion for operators $R_n(f)$ as $n \to \infty$. Approximation properties of the complex Balázs-Szabados operators were studied in Gal [5]. The $q$-analogue of these operator was given by Doğru who investigated statistical approximation properties of $q$-Balázs-Szabados operators [4]. The approximation properties of the complex $q$-Balázs-Szabados operators is studied in [6].

We introduce some notations and definitions of $q$-calculus, see [9], [?]. Let $q > 0$. For any $n \in \mathbb{N} \cup \{0\}$, the $q$-integer $[n]_q$ is defined by

$$[n]_q := \begin{cases} 
(1 - q^n) / (1 - q), & q \neq 1 \\
n, & q = 1 
\end{cases}, \quad [0]_q := 0;$$

and the $q$-factorial $[n]_q!$ by

$$[n]_q! := [1]_q [2]_q \ldots [n]_q, \quad [0]_q! := 1.$$

For integers $0 \leq k \leq n$, the $q$-binomial coefficients are defined by

$$\binom{n}{k}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$
For fixed $q > 0$, $q \neq 1$, we denote the $q$-derivative $D_q f(z)$ of $f$ by

\[
D_q f(z) = \begin{cases} \frac{f(qz)-f(z)}{(q-1)z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases}
\]

Let us introduce a $q$-Balázs-Szabados operator.

**Definition 1** Let $q > 0$. For $f : [0, \infty) \to \mathbb{R}$ we define the Balázs-Szabados operator based on the $q$-integers as follows.

\[
R_{n,q}(f;x) = \frac{1}{(1+a_n x)^n} \sum_{k=0}^{n} f\left(\frac{[k]_q}{b_n}\right) \begin{pmatrix} n \\ k \end{pmatrix}_q (a_n x)^k \prod_{s=0}^{n-k-1} \left(1 + (1-q) [s]_q a_n x\right),
\]

where $a_n = [n]_q^{\beta-1}$, $b_n = [n]_q^\beta$, $0 < \beta \leq \frac{2}{3}$, $n \in \mathbb{N}$ and $x \neq -\frac{1}{a_n}$.

In the case $q = 1$ these polynomials coincide with the classical ones. For $q \neq 1$ one gets a new class of polynomials having interesting properties. It should be mentioned that in the case $q \in (0,1)$ $q$-Balázs-Szabados operators generate positive linear operators $R_{n,q} : f \to R_{n,q}(f;x)$. In the case $q > 1$ positivity fails, however, the results of this paper show that in this case approximating properties of $q$-Balázs-Szabados operators may be better than the case $0 < q \leq 1$.

Throughout this paper, let $D_R := \{z \in \mathbb{C} : |z| < R\}$ denote the disk of radius $R$ centered at $0$. Moreover, it is assumed that $f(z) = \sum_{m=0}^{\infty} c_m z^m$, for $z \in D_R$.

Assuming $f : D_R \cup [R, +\infty) \to \mathbb{C}$ and simply replacing $x$ by $z$ in (1) we obtain the complex form of the $q$-Balázs-Szabados operator:

\[
R_{n,q}(f;z) = \frac{1}{(1+a_n z)^n} \sum_{k=0}^{n} f\left(\frac{[k]_q}{b_n}\right) \begin{pmatrix} n \\ k \end{pmatrix}_q (a_n z)^k \prod_{s=0}^{n-k-1} \left(1 + (1-q) [s]_q a_n z\right),
\]

where again $a_n = [n]_q^{\beta-1}$, $b_n = [n]_q^\beta$, $0 < \beta \leq \frac{2}{3}$, $n \in \mathbb{N}$, $z \in \mathbb{C}$ and $z \neq -\frac{1}{a_n}$.

**Remark 2** The complex operators $R_{n,q}(f;z)$ are well defined and analytic for all $n \geq n_0$ and $|z| \leq r < [n_0]_q^{1-\beta}$. Indeed, in this case we easily obtain that $z \neq -\frac{1}{a_n}$, for all $|z| \leq r < [n_0]_q^{1-\beta}$ and $n \geq n_0$, which implies that $1/ (1+a_n z)^n$ is analytic.

**Remark 3** There exists a close connection between $R_{n,q}(f;z)$ and the complex $q$-Bernstein polynomials given by

\[
B_{n,q}(f;z) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{b_n}\right) \begin{pmatrix} n \\ k \end{pmatrix}_q z^k \prod_{s=0}^{n-k-1} (1-q^s z).
\]

Indeed, denoting $F_n(z) = f\left(\frac{[n]_q}{b_n} z\right)$, we easily get

\[
R_{n,q}(f;z) = B_{n,q}\left(F_n; \frac{a_n z}{1+a_n z}\right),
\]

valid for all $n \geq n_0$ and $|z| \leq r < [n_0]_q^{1-\beta}$. This connection will be essential in our reasonings. For monomials $f(z) = e_m(z) = z^m$ it can be written as follows:

\[
R_{n,q}(e_m; z) = [n]_q^{m(1-\beta)} B_{n,q}\left(e_m; \frac{a_n z}{1+a_n z}\right).
\]
Remark 4 The lack of positivity makes the investigation of convergence in the case $q > 1$ essentially more difficult than for $0 < q < 1$. Notice that, the complex $q$-Bernstein type operators in the case $q > 1$ systematically are studied in \([10], [11], [12], [13], [14], \) and \([15]\).

Remark 5 Approximation properties of the complex Balázs-Szabados operators are studied in \([5]\). Notice that unlike to \([5]\) the growth conditions of exponential-type on $f$ is omitted. The only condition imposed to $f$ is to be uniformly continuous and bounded on $[0, +\infty)$. Therefore our results are new even for the classical Balázs-Szabados operators.

**Theorem 6** Let $n_0 \geq 2, 0 < \beta \leq \frac{2}{3}$. Assume that $f : \mathbb{D}_R \cup [R, +\infty) \to \mathbb{C}$ is uniformly continuous and bounded on $[0, +\infty)$, is analytic in $\mathbb{D}_R$. Then

$$|R_{n,q}(f;z) - f(z)| \leq \frac{1}{[n]_q^\beta} \sum_{m=2}^\infty |c_m| m (m-1) (4q^2 r)^m + \frac{2r}{[n]_q^{1-\beta}} \sum_{m=1}^\infty |c_m| (2r)^m,$$

$$q \geq 1, \quad \frac{1}{2} < r < \frac{R}{4q^2} \leq \frac{1}{2} [n_0]_{q}^{1-\beta}.$$

In \([7]\), Totik settled the saturation properties of $R_n$. Among other things he proved the Voronovskaja-type result for $0 < \beta \leq \frac{1}{2}, \beta \geq \frac{1}{2}$. The complete asymptotic expansion for $R_n$ is given by Abel and Della Vecchia \([3]\).

Next, we study Voronovskaja type formulas of the $q$-Balázs-Szabados operators of a function $f$ analytic in the disc $\mathbb{D}_R$. In order to formulate Voronovskaja type theorem we define the following function

$$L_q^\beta (f;z) := \begin{cases} \frac{D_q f(z) - f'(z)}{q-1}, & \text{if } |z| < R/q, R > q > 1, \ 0 < \beta < \frac{1}{2}, \\ -z^2 f'(z) + \frac{D_q f(z) - f'(z)}{q-1}, & \text{if } |z| < R/q, R > q > 1, \ \beta = \frac{1}{2}, \\ -z^2 f'(z), & \text{if } |z| < R/q, R > q > 1, \ \frac{1}{2} < \beta < 1, \end{cases}$$

and for $q = 1$,

$$L_q^\beta (f;z) := \begin{cases} \frac{z}{2} f''(z), & \text{if } |z| < R, \ 0 < \beta < \frac{1}{2}, \\ -z^2 f'(z) + \frac{z}{2} f''(z), & \text{if } |z| < R, \ \beta = \frac{1}{2}, \\ -z^2 f'(z), & \text{if } |z| < R, \ \frac{1}{2} < \beta < 1. \end{cases}$$

In the case of complex variable, the qualitative Voronovskaja-type result for $R_n$ is proved by Gal \([3]\). Note that the case $\beta = \frac{1}{2}$ remained open in \([3]\). We prove the following quantitative Voronovskaja type theorem for $R_{n,q}$, which covers the case $\beta = \frac{1}{2}$. Moreover, our results are new for the classical Balázs-Szabados operators ($q = 1$).

**Theorem 7** Let $n_0 \geq 2, 0 < \beta \leq \frac{2}{3}$. Assume that $f : \mathbb{D}_R \cup [R, +\infty) \to \mathbb{C}$ is uniformly continuous and bounded on $[0, +\infty)$, is analytic in $\mathbb{D}_R$. Then

(i) For $0 < \beta < \frac{1}{2}, \frac{1}{2} < r < \frac{R}{\max (4q, 2q^2)} \leq \frac{R}{2} \ [n_0]_{q}^{1-\beta}, |z| \leq r$, we have

$$|R_{n,q}(f;z) - f(z) - \frac{z}{[n]_q^\beta} \frac{D_q f(z) - f'(z)}{q-1}| \leq \frac{4}{[n]_{q}^{2\beta}} \sum_{m=0}^\infty |c_m| (m-2) (4qr)^{m-2} + \frac{4}{[n]_{q}^{1-\beta}} \sum_{m=0}^\infty |c_m| m (m-1) (2q^2r)^{m+1};$$

(ii) For $\frac{1}{2} < \beta < 1, \frac{1}{2} < r < \frac{R}{\max (4q, 2q^2)} \leq \frac{R}{2} \ [n_0]_{q}^{1-\beta}, |z| \leq r$, we have

$$|R_{n,q}(f;z) - f(z) - \frac{z}{[n]_q^\beta} \frac{D_q f(z) - f'(z)}{q-1}| \leq \frac{4}{[n]_{q}^{2\beta}} \sum_{m=0}^\infty |c_m| (m-2) (4qr)^{m-2} + \frac{4}{[n]_{q}^{1-\beta}} \sum_{m=0}^\infty |c_m| m (m-1) (2q^2r)^{m+1};$$

(iii) For $\beta = 1, \frac{1}{2} < r < \frac{R}{\max (4q, 2q^2)} \leq \frac{R}{2} \ [n_0]_{q}^{1-\beta}, |z| \leq r$, we have

$$|R_{n,q}(f;z) - f(z) - \frac{z}{[n]_q^\beta} \frac{D_q f(z) - f'(z)}{q-1}| \leq \frac{4}{[n]_{q}^{2\beta}} \sum_{m=0}^\infty |c_m| (m-2) (4qr)^{m-2} + \frac{4}{[n]_{q}^{1-\beta}} \sum_{m=0}^\infty |c_m| m (m-1) (2q^2r)^{m+1};$$

(iv) For $\beta > 1, \frac{1}{2} < r < \frac{R}{\max (4q, 2q^2)} \leq \frac{R}{2} \ [n_0]_{q}^{1-\beta}, |z| \leq r$, we have

$$|R_{n,q}(f;z) - f(z) - \frac{z}{[n]_q^\beta} \frac{D_q f(z) - f'(z)}{q-1}| \leq \frac{4}{[n]_{q}^{2\beta}} \sum_{m=0}^\infty |c_m| (m-2) (4qr)^{m-2} + \frac{4}{[n]_{q}^{1-\beta}} \sum_{m=0}^\infty |c_m| m (m-1) (2q^2r)^{m+1};$$
(ii) For \( \frac{1}{2} < \beta \leq \frac{2}{3} \), \( \frac{1}{2} < r < \frac{R}{4q} < R \leq \frac{1}{2} \left[ n_0 \right]_{q}^{1-\beta} \), \( |z| \leq r \), we have

\[
\left| R_{n,q} (f; z) - f (z) + \frac{1}{[n]_q} z^2 f' (z) \right| \leq \frac{6}{[n]_q} \sum_{m=0}^{\infty} |c_m| m (m-1) (4qr)^m;
\]

(iii) For \( \beta = \frac{1}{2} \), \( \frac{1}{2} < r < \frac{R}{4q^2} < R \leq \frac{1}{2} \left[ n_0 \right]_{q}^{1-\beta} \), \( |z| \leq r \), we have

\[
\left| R_{n,q} (f; z) - f (z) + \frac{1}{\sqrt{[n]_q}} z \frac{z^2 f' (z)}{q-1} - \frac{1}{\sqrt{[n]_q}} \frac{z (D_q f (z) - f' (z))}{q-1} \right| \leq \frac{9}{[n]_q} \sum_{m=0}^{\infty} |c_m| m^2 (m-1)^2 (4q^2 r)^m.
\]

By using the above Voronovskaja’s theorem, we will obtain the exact order in approximation by the complex \( q \)-Balázs-Szabados operators. In this sense, we present the following results.

**Theorem 8** Let \( n_0 \geq 2 \), \( 0 < \beta \leq \frac{2}{3} \). Assume that \( f : \mathbb{D}_R \cup [R, +\infty) \to \mathbb{C} \) is uniformly continuous and bounded on \( [0, +\infty) \), is analytic in \( \mathbb{D}_R \).

(i) If \( 0 < \beta < \frac{1}{2} \), \( \frac{1}{2} < r < \frac{R}{\max(4q, 2q^2)} < R \leq \frac{1}{2} \left[ n_0 \right]_{q}^{1-\beta} \), and \( f \) is not a polynomial of degree \( \leq 1 \) in \( \mathbb{D}_R \), then

\[
\|R_{n,q} (f) - f\|_r \sim \frac{1}{[n]_q}, \quad n \in \mathbb{N}.
\]

(ii) If \( \frac{1}{2} < \beta \leq \frac{2}{3} \), \( \frac{1}{2} < r < \frac{R}{4q} < R \leq \frac{1}{2} \left[ n_0 \right]_{q}^{1-\beta} \), and \( f \) is not a constant function in \( \mathbb{D}_R \), then

\[
\|R_{n,q} (f) - f\|_r \sim \frac{1}{[n]_{q}^{1-\beta}}, \quad n \in \mathbb{N}.
\]

(iii) For \( \beta = \frac{1}{2} \), \( \frac{1}{2} < r < \frac{R}{4q^2} < R \leq \frac{1}{2} \left[ n_0 \right]_{q}^{1-\beta} \), and \( f \) is not a constant function in \( \mathbb{D}_R \), then

\[
\|R_{n,q} (f) - f\|_r \sim \frac{1}{\sqrt{[n]_q}}, \quad n \in \mathbb{N}.
\]

**References**

[1] K. Balázs, Approximation by Bernstein type rational functions, Acta Math. Acad. Sci. Hungar., 26, (1975) 123-134.

[2] K. Balázs and J. Szabados, Approximation by Bernstein type rational functions, II, Acta Math. Acad. Sci. Hungar., 40 (1982) 3-4, 331-337.

[3] U. Abel and B. Della Vecchia, Asymptotic approximation by the operators of K. Balázs and Szabados, Acta Sci. Math.(Szeged), 66, (2000) No. 1-2, 137{145}. 
[4] O. Dogru, On statistical approximation properties of Stancu type bivariate generalization of $q$-Balázs-Szabados operators, Proc. of Int. Conf. on Numer. Anal. and Approx. Th. Cluj-Napoca, Romanya, 2002.

[5] S.G. Gal, Approximation by Complex Bernstein and Convolution Type Operators. World Scientific Publishing, New Jersey, London, Singapore, Beijing, Shanghai, Hong Kong, Taipei, Chennai (2009)

[6] N. Ispir and Y. Özkan, Approximation properties of complex $q$-Balázs-Szabados operators in compact disks, Journal of Inequalities and Applications 2013, 2013:361

[7] V. Totik, Saturation for Bernstein-type rational functions, Acta Math. Hungar., 43 (1984), 219–250.

[8] R. P Agarwal and V. Gupta, On $q$-analogue of a complex summation-integral type operators in compact disks, Journal of Inequalities and Applications 2012, 2012:111.

[9] Andrews G E, Askey R, Roy R. Special functions. Cambridge: Cambridge University Press; 1999.

[10] S. Ostrovska: $q$-Bernstein polynomials and their iterates. J. Approximation Theory 123 (2003), 232–255.

[11] S. Ostrovska: The sharpness of convergence results for $q$-Bernstein polynomials in the case $q > 1$. Czech. Math. J. 58 (2008), 1195–1206.

[12] H. Wang, X. Wu: Saturation of convergence for $q$-Bernstein polynomials in the case $q > 1$. J. Math. Anal. Appl. 337 (2008), 744–750.

[13] Z. Wu: The saturation of convergence on the interval $[0,1]$ for the $q$-Bernstein polynomials in the case $q > 1$. J. Math. Anal. Appl. 357 (2009), 137–141.

[14] N. I. Mahmudov, Approximation by $q$-Durrmeyer type polynomials in compact disks in the case $q > 1$. Appl. Math. Comput. 237 (2014), 293–303

[15] N. I. Mahmudov, Approximation by genuine $q$-Bernstein-Durrmeyer polynomials in compact disks in the case $q > 1$. Abstr. Appl. Anal. 2014, Art. ID 959586, 11 pp.