Free Energy of a Hot Gluon Plasma and Hard-thermal-loop Resummation

Jens O. Andersen, a

aPhysics Department, Ohio State University, Columbus OH 43210, USA

In this talk I briefly discuss the thermodynamics of the quark-gluon plasma. The calculation of the free energy of a hot gluon plasma to leading order in hard-thermal-loop perturbation theory [1] is outlined. The HTL free energy is compared with the weak-coupling expansion and lattice results.

1. Introduction

Relativistic heavy-ion collisions will soon allow the experimental study of hadronic matter at energy densities that will probably exceed that required to create a quark-gluon plasma. A quantitative understanding of the properties of a quark-gluon plasma is essential in order to determine whether it has been created. Because QCD is asymptotically free, the running coupling constant \( \alpha_s \) becomes small at sufficiently high temperatures. It therefore seems plausible that one can understand the quark-gluon plasma using perturbative methods. The free energy of QCD has been calculated to fifth order in the weak-coupling expansion [2–4]. For a pure-glue plasma the result is

\[
F_{\text{QCD}} = F_{\text{ideal}} \left[ 1 - \frac{15}{4} \alpha_s \pi + 30 \left( \frac{\alpha_s}{\pi} \right)^{3/2} + \frac{135}{2} \left( \log \frac{\alpha_s}{\pi} - \frac{11}{36} \log \frac{\mu_4}{2\pi T} + 3.51 \right) \left( \frac{\alpha_s}{\pi} \right)^2 
+ \frac{495}{2} \left( \log \frac{\mu_4}{2\pi T} - 3.23 \right) \left( \frac{\alpha_s}{\pi} \right)^{5/2} + \mathcal{O}(\alpha_s^3 \log \alpha_s) \right],
\]

(1)

where \( F_{\text{ideal}} = -(8\pi^2/45)T^4 \) is the free energy of an ideal gas of massless gluons and \( \alpha_s = \alpha_s(\mu_4) \) is the running coupling constant in the \( \overline{\text{MS}} \) scheme. In Fig. [1], the free energy is shown as a function of \( T/T_c \), where \( T_c \) is the critical temperature for the deconfinement phase transition. The weak-coupling expansions through orders \( \alpha_s, \alpha_s^{3/2}, \alpha_s^2, \) and \( \alpha_s^{5/2} \) are shown as bands that correspond to varying the renormalization scale \( \mu_4 \) by a factor of two from the central value \( \mu_4 = 2\pi T \). The successive approximations to the free energy alternate in sign and the perturbative expansion does not converge for temperatures that are relevant for heavy-ion collisions. The \( \alpha_s^{3/2} \) contribution is smaller than the \( \alpha_s \) contribution only if \( T \) is larger than \( 10^5 \) GeV, while the temperatures at RHIC are expected not to exceed \( T = 0.5 \) GeV. The poor convergence of the perturbative series is somewhat surprising since very accurate lattice results [5] indicate that the quark-gluon plasma can be approximated quite well by an ideal gas unless the temperature is very close to \( T_c \).

*Talk given at 15th International Conference on Particle and Nuclei (PANIC 99), Uppsala, Sweden, 10-16 June 1999.
The lattice results for pure-glue QCD by Boyd et al. [5] are also shown in Fig. 1 as diamonds. The free energy approaches that of an ideal gas from below as $T/T_c$ increases. There are several ways of reorganizing the perturbative series to improve its convergence. One of the most successful approaches for scalar field theories is “screened perturbation theory” which is an expansion around an ideal gas of massive quasiparticles. The convergence for the free energy is very good even for large values of the coupling constant.

Evidence that such a reorganization might be useful in QCD is provided by the success of quasi-particle models. The analyses in [7] indicate that lattice results for thermodynamic functions can be fit very well by a gas of massive quarks and gluons with temperature dependent masses of order $T$. The quasi-particle models, however, suffer from several theoretical inconsistencies because introducing thermal masses by hand violates gauge invariance.

There is a way of incorporating plasma effects, such as quasi-particle masses, screening of interactions, and Landau damping, into perturbative calculations, while maintaining gauge invariance, and that is by using hard-thermal-loop (HTL) perturbation theory. HTL perturbation theory was originally developed to sum up all higher loop corrections that are leading order in $g$ for amplitudes having soft external lines with momenta of order $gT$ [8]. If it is applied to amplitudes with hard external lines with momenta of order $T$, it selectively resums corrections that are higher order in $g$. This resummation is a generalization of screened perturbation theory that respects gauge invariance. It corresponds to expanding around an ideal gas of massive quark and gluon quasiparticles.

A significant advantage of HTL perturbation theory over the approach using lattice gauge theory calculations is that it can be readily applied to the real-time processes that are the most promising signatures of a quark-gluon plasma. Due to the Euclidean formulation, lattice methods are restricted to calculating static quantities such as the free energy and the Debye mass [9].

2. HTL Free Energy

In the imaginary time formalism, the one-loop HTL free energy for an $SU(3)$ gauge theory is [1]

$$F_{HTL} = 8 [(d-1)F_T + F_L + \Delta F],$$

where

$$F_T = \sum_{n=-\infty}^{\infty} \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \log[k^2 + \omega_n^2 + \Pi_T(\omega_n, k)],$$

Figure 1: The free energy for pure-glue QCD as a function of $T/T_c$. See main text for details.
\[ \mathcal{F}_L = \frac{1}{2} \sum_{n=-\infty}^{\infty} \int \frac{d^d k}{(2\pi)^d} \log[k^2 - \Pi_L(\omega_n, k)], \] (4)

and \( \Delta \mathcal{F} \) is a counterterm. Dimensional regularization in \( d = 3 - 2\epsilon \) dimensions is used throughout. The HTL self-energy functions \( \Pi_T \) and \( \Pi_L \) are

\[ \Pi_T(\omega_n, k) = -\frac{3}{2} m_g^2 \omega_n^2 \left[ 1 + \frac{\omega_n^2 + k^2}{2i\omega_n k} \log \frac{i\omega_n + k}{i\omega_n - k} \right], \] (5)

\[ \Pi_L(\omega_n, k) = 3m_g^2 \left[ \frac{i\omega_n}{2k} \log \frac{i\omega_n + k}{i\omega_n - k} - 1 \right], \] (6)

and \( m_g \) is the gluon mass parameter. The sum over the Matsubara frequencies \( \omega_n = 2\pi n T \) can be rewritten as a contour integral around a contour \( C \) that encloses the points \( \omega = i\omega_n \). The integrand has branch cuts that start at \( \pm \omega_T(k) \) and \( \pm \omega_L(k) \), where \( \omega_T(k) \) and \( \omega_L(k) \) are the dispersion relations for transverse and longitudinal gluon quasiparticles, respectively. The integrand also has a branch cut running from \( \omega = -k \) to \( \omega = k \) due to the functions \( \Pi_T \) and \( \Pi_L \). The contour can be deformed to wrap around the quasiparticle and Landau-damping branch cuts. Some of the temperature-independent integrals over \( \omega \) can be calculated analytically, while others must be evaluated numerically. With dimensional regularization, the logarithmic ultraviolet divergences show up as poles in \( \epsilon \). Using the modified minimal subtraction (MS) renormalization prescription with the counterterm \( \Delta \mathcal{F} = 9m_g^4/64\pi^2 \epsilon \), we obtain

\[ \mathcal{F}_{HTL} = \frac{4T}{\pi^2} \int_0^\infty k^2 dk \left[ 2 \log(1 - e^{-\beta \omega_T}) + \log \frac{1 - e^{-\beta \omega_L}}{1 - e^{-\beta k}} \right] + \frac{4}{\pi^3} \int_0^\infty \frac{d\omega}{e^{\beta \omega} - 1} \int_0^\infty k^2 dk [\phi_L - 2\phi_T] + \frac{1}{2} m_g^2 T^2 + \frac{9 m_g^4}{8 \pi^2} \left[ \log \frac{m_g}{\mu_3} - 0.33 \right]. \] (7)

\( \mu_3 \) is the renormalization scale associated with dimensional regularization, and the angles \( \phi_T \) and \( \phi_L \) are complicated functions of \( \omega \) and \( k \).

The leading-order HTL result for the pressure is shown in Fig. 2 as the shaded band that corresponds to varying the renormalization scales \( \mu_3 \) and \( \mu_4 \) by a factor of two around their central values \( \mu_3 = 0.717 m_g \) and \( \mu_4 = 2\pi T \). This value of \( \mu_3 \) is chosen in order to minimize the pathological behavior of \( \mathcal{F}_{HTL} \) at low temperatures [1]. We also show as dashed curves the weak-coupling expansions through order \( \alpha_s \), \( \alpha_s^{3/2} \), \( \alpha_s^2 \), and \( \alpha_s^{5/2} \) labelled 2, 3, 4, and 5. We have used a parametrization of the running coupling constant \( \alpha_s(\mu_4) \) that includes the effects of two-loop running.

![Figure 2: The HTL pressure for pure-glue QCD as a function of \( T/T_c \). See main text for details.](image_url)
of the renormalization scales, our leading-order result for the HTL free energy lies below the lattice results of Boyd et al. [5] (shown as diamonds) for $T > 2T_c$. However, the deviation from lattice QCD results has the correct sign and roughly the correct magnitude to be accounted for by next-to-leading order corrections in HTL perturbation theory [1]. This can be seen from the high-temperature expansion of the free energy, which is an expansion in powers of $m_g/T$. To order $m_g^4/T^4$ one obtains

$$F_{\text{HTL}} = F_{\text{ideal}} \left[ 1 - \frac{45}{4} \left( \frac{3m_g^2}{4\pi^2T^2} \right) + 30 \left( \frac{3m_g^2}{4\pi^2T^2} \right)^{3/2} + \frac{45}{8} \left( \log \frac{\mu^2}{4\pi^2T^2} - 1.232 \right) \left( \frac{3m_g^2}{4\pi^2T^2} \right)^2 + \mathcal{O}(m_g^6/T^6) \right]. \quad (8)$$

Comparing (8) with the weak-coupling expansion (1) and identifying $m_g^2$ with its weak-coupling limit $\frac{4\pi}{3} \alpha_s T^2$, we conclude that HTL perturbation theory overincludes the $\alpha_s$ contribution by a factor of three. The $\alpha_s^{3/2}$ contribution which is associated with Debye screening is included correctly. At next-to-leading order, HTL perturbation theory agrees with the weak-coupling expansion through order $\alpha_s^{3/2}$. Thus the next-to-leading order contribution to $F/F_{\text{ideal}}$ in HTL perturbation theory will be positive at large $T$ since it must approach $\frac{15}{2} \alpha_s/\pi$.

The talk is based upon work with Eric Braaten and Michael Strickland [1]. The author would like to thank the organizers of PANIC 99 for a stimulating meeting. This work was supported in part by a Faculty Development Grant from the Physics Department of the Ohio State University and by a fellowship from the Norwegian Research Council (project 124282/410).

REFERENCES

1. J.O. Andersen, E. Braaten and M. Strickland, [hep-ph/9902327, hep-ph/9905337]
2. P. Arnold and C. Zhai, Phys. Rev. D50 7603 (1994); Phys. Rev. D51 1906 (1995);
3. B. Kastening and C. Zhai, Phys. Rev. D51 7232 (1995).
4. E. Braaten and A. Nieto, Phys. Rev. Lett. 76 1417 (1996); Phys. Rev. D53 3421 (1996).
5. G. Boyd et al., Phys. Rev. Lett. 75 4169 (1995); Nucl. Phys. B469 419 (1996).
6. F. Karsch, A. Patkós, and P. Petreczky, Phys. Lett. B401 69 (1997).
7. A. Peshier, B. Kämpfer, O.P. Pavlenko, and G. Soff, Phys. Rev. D54 2399 (1996); P. Lévrier and U. Heinz, Phys. Rev. C57 1879 (1998).
8. E. Braaten and R.D. Pisarski, Phys. Rev. Lett. 64 1338 (1990); Nucl. Phys. B337 569 (1990).
9. K. Kajantie, M. Laine, J. Peisa, A. Rajantie, K. Rummukainen, and M. Shaposhnikov, Phys. Rev. Lett. 79 3130 (1997).