Additivity of on-line decision complexity is violated by a linear term in the length of a binary string DRAFT

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Abstract

We show that there are infinitely many binary strings \( z \), such that the sum of the on-line decision complexity of predicting the even bits of \( z \) given the previous uneven bits, and the decision complexity of predicting the uneven bits given the previous even bits, exceeds the Kolmogorov complexity of \( z \) by a linear term in the length of \( z \).

KEYWORDS: Decision complexity – Kolmogorov complexity – Decompositions of Kolmogorov complexity

1 Introduction

On-line decision complexity has been introduced and investigated in [?, ?]. It also naturally appears in the definition of ideal influence tests [?, ?]. A natural question is whether algorithmic mutual information of two time series \( x, y \), can be decomposed into an information flow going from \( x \) to \( y \), a flow going from \( y \) to \( x \), and an information flow instantaneously present in both strings. It turns out [?] that this question is related to the question of defining a decomposition of \( K(x, y) \) with \( l(x) = l(y) \) as the sum of the complexity of predicting \( x_{i+1} \) given \( x_1 \ldots x_i \) and \( y_1 \ldots y_i \), \( i \leq n \), and the complexity of predicting \( y_{i+1} \) given \( x_1 \ldots x_{i+1} \) and \( y_1 \ldots y_i \). It will be shown that using on-line decision complexity for this complexity,

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this sum exceeds $K(x, y)$ by a linear constant in $l(x)$. A modification of this definition of on-line decision complexity will be shown to have an approximate decomposition \cite{?}, \cite{?}.

Non-additivity of decision complexity was also shown in \cite{?}, in the context of randomness defined by supermartingales. Using natural definitions for randomness a paradox is shown: if the even bits of $z$ given the past uneven bits of $z$ are random, and also the uneven bits of $z$ given the past even bits of $z$ are random, than it is possible that $z$ is not random. The proof of this result implies that additivity of on-line decision complexity is violated by a logarithmic term.

2 Definitions and notation

For excellent introductions to Kolmogorov complexity we refer to \cite{?}, \cite{?}. Let $\omega$, $\omega^{<\omega}$, $\mathbb{N}$ and $2^{<\omega}$ denote the set of the Natural numbers, the set of finite sequences of Natural numbers, the binary strings of length $N$, and the binary strings of finite length. Other definitions are analogue. Let $\epsilon$ denote the empty sequence. Remark that there is a natural bijection between $\omega$ and $2^{<\omega}$, defined by:

$$\epsilon \rightarrow 0, 0 \rightarrow 1, 1 \rightarrow 2, 00 \rightarrow 3, 01 \rightarrow 4, ...$$

$[\omega]$ is the set of nested sequences of Natural numbers, with finite depth. Mathematically, it is the closure of $\omega$ under the mapping $f(S) = S^{<\omega}$. Remark that there is a computable bijection between $\omega$ and $[\omega]$, therefore most complexity and computability results in $\omega$ also hold in $[\omega]$.

An interpreter $\Phi$ is a partial computable function from $2^{<\omega} \times [\omega] \rightarrow [\omega]$. An interpreter is prefix-free if for any $x$, the set $D_x$ of all $p$ where $\Phi(p|x)$ is defined, is prefix-free. Let $\Phi$ be some fixed optimal universal prefix-free interpreter.

For any $x \in 2^{<\omega}$, $l(x)$ denotes the length of $x$. For any $x \in \omega^{<\omega}$, $l(\overline{x})$ corresponds to the length of some prefix-free encoding of $x$ on a binary tape:

$$l(\overline{x}) = \sum_{i=1}^{l(x)} 2 \log x_i.$$ 

For $x, y \in [\omega]$, the Kolmogorov complexity $K(x|y)$, is defined as:

$$K(x|y) = \min\{l(p) : \Phi(p|y) \downarrow = x\}.$$ 

The Kolmogorov complexity of elements in $2^{<\omega}$ is defined by using the computable bijection mentioned in the beginning of this section.
For $Z \in [\omega], Q, A \in \omega^n$, $Q^i$ denotes $Q_1...Q_i$. The on-line decision complexity is defined by:

$$K(Q_1 \to A_1;...;Q_n \to A_n | Z) = \min \{ l(p) : \forall i < n [\Phi(p|Q^i, Z) \downarrow = A_i] \}.$$ 

This definition differs slightly with the definition of $\Phi$, with respect that $A \in \omega^i$ is chosen, in stead of $A \in 2^n$. Also a shorter notation $\Phi$ will be used:

$$K(x|y \uparrow) = K(0 \to x_1;...;y_{n-1} \to x_n),$$

$$K(y|x \uparrow^+) = K(x_1 \to y_1;...;x_n \to y_n).$$

### 3 Main result and proof tactic

**Proposition 3.1.**

$$\exists c > 0 \exists x, y \in \omega^{<\omega} \left[ K(x|y \uparrow) + K(y|x \uparrow^+) - K(x, y) > c(l(x) + l(y)) \right].$$

In $\Phi$ and repeated in $\Phi$, it is proven that for any $n$ there is an $x \in 2^n$ such that:

$$K(K(x)|x) \geq^+ \log n - \log \log n.$$

Let $y$ be the binary expansion of $K(x)$. From this and equation (11) it can be shown that

$$K(x) + K(y|x) - K(x, y) \geq^+ \log n - \log \log n.$$

By inserting zeros at the right places in $x, y$, it can be shown that there exists infinity many $x, y$ with $l(x) = l(y)$:

$$K(x|y \uparrow) + K(y|x \uparrow^+) - K(x, y) > O(\log l(x)).$$

This shows proposition 3.1 for a logarithmic term in $l(x)$. It seems natural to think that such a result can be improved to a linear term, by concatenating such strings. This is what eventually will happen in the proof, at equation (12). However, to be able to add up these differences, conditional complexities must add up in some way to on-line decision complexity, in what extend this is possible is still an open problem. Happily, Lemma 4.4 can circumvent this, if some extra information is available. This information is stored in sequences $u$ and $v$ and is added to $x$ and $y$. Adding this information requires, some more bounds to make the proof work: (10), (11). The proof below provides all technical details.
4 Proof

First some definitions and lemmas are given. \(f(x) \leq^+ g(x)\) is short for \(f(x) \leq g(x) + O(1)\), and \(f(x) =^+ g(x)\) is short for \(f(x) = g(x) \pm O(1)\).

For any \(a, b \in [\omega]\), \(a \rightarrow b\) means that there is a fixed \(p \in 2^{<\omega}\) with \(l(p) \leq O(1)\), such that \(\Phi(p|a) \downarrow = b\). Remark that if \(a \rightarrow b\), then \(K(a) \geq^+ K(b)\). The shortest program witnessing \(K(a|b)\) is denoted by:

\[
a^*[b] = \min\{p : \Phi(p|b) \downarrow = a\}.
\]

\(a^*\) is short for \(a^*[\epsilon]\). Remark that:

\[
a^*[b], b \leftrightarrow (a^*[b])^*[b], b. \tag{1}
\]

Lemmas 4.1, 4.2, and 4.3 provide observations, known within the community, and stated here explicitly for later reference.

**Lemma 4.1.** For \(A \in \omega^{<\omega}\),

\[
\sum_{i \leq n} K(A_i|A_{i-1}) \geq K(A) - O(n).
\]

**Proof.** For \(U, V \in \omega\), prefix-free complexity satisfies additivity [?]:

\[
K(U, V|W) =^+ K(U|W) + K(V|U^*[W]). \tag{2}
\]

Since there is a computable bijection between \(\omega\) and \([\omega]\), this result also applies to \([\omega]\). Let \(U, V \in [\omega]\), since \(U^*[W], W \rightarrow U,\)

\[
K(V|U, W) \geq^+ K(V|U^*[W], W).
\]

Inductive application of both equations above on \(A^i\) proves the lemma.

**Lemma 4.2.** For \(a, b \in \omega\) and \(c \in \omega^{<\omega}\):

\[
K(a, b|c) =^+ K(a, b, K(b|a^*[c], c)|c).
\]

**Proof.** The proof below, shows the unconditioned version of the lemma, since the proof of the conditioned version is the same. In [?] and exercise 3.3.7 in [?] it is stated that for every \(w \in \omega\), and \(n \geq K(w)\):

\[
\log |\{p \in 2^n : \Phi(p) \downarrow = w\}| \leq^+ n - K(w, n), \tag{3}
\]

and

\[
K(w, K(w)) =^+ K(w). \tag{4}
\]
Therefore, for $c$ constant, there are an $O(1)$ number of programs that compute $a, b$ and have length $K(a, b) + c$. Let $S$ be the set of these programs. Remark that the elements of $S$ can be enumerated given $a, b, K(a, b)$ and therefore, for any $p \in S$, using (4), we have:

$$K(a, b) =^+ K(a, b, K(a, b)) =^+ K(p). \tag{5}$$

By equation (2), we have:

$$K(a, b) =^+ K(a) + K(b|a*).$$

The programs $a^*$ and $b^*[a^*]$, can be combined into a program $p$ computing $a, b$. This program $p$ can be constructed such that $p \rightarrow a^*, b^*[a^*]$, and it has a length below $K(a, b) + c$, for $c$ constant and large enough. Therefore $p \in S$, and since $b^*[a^*] \rightarrow K(b|a^*) = l(b^*[a^*]):$

$$K(p) \geq^+ K(a, b, K(b|a^*)).$$

Combining with equation (5), finishes the proof. \qed

**Lemma 4.3.** For $b \in 2^{<\omega}, a, c \in [\omega]:$

$$K(a, b|c) \geq^+ K(a, b^*[a, c]|c) - 2 \log l(b).$$

**Proof.** The unconditioned version of the lemma is proven, since the conditioned proof is essentially the same. It suffices to show that:

$$K(b^*[a]|b) \leq^+ 2 \log l(b).$$

Again the proof of the unconditioned version of this equation is the same as the conditioned one:

$$K(b^*|b) \leq^+ 2 \log l(b).$$

Given $b$ and $K(b)$ all programs of length $K(b)$ that output $b$ can be enumerated. By equation (3), there are maximally a constant such programs, therefore:

$$K(b^*|b) =^+ K(K(b)|b).$$

Remark that by the prefix-free code $b_10b_20...b_{l(b)}1$ we have:

$$K(b) \leq^+ 2l(b).$$

Using the natural bijection between $\omega$ and $2^{<\omega}$, this shows that for $n \in \omega, K(n) \leq^+ 2 \log n.$

$$K(K(b)|b) \leq^+ K(K(b)) \leq^+ 2 \log K(b) \leq^+ 2 \log l(b).$$
Let \( Z \in \omega \), \( A, Q \in \omega^n \) for some \( n \), and \( N \in \omega \). For \( i < n \), let \( T_i = (Q_i|A_i) \) and \( T = (T_1, ..., T_n) \).

\[
K(T) = K(A|Q \uparrow, N) \\
K(T_i|Z) = K(A_i|A^{i-1}, Q^{i-1}, N, Z).
\]

For some fixed \( N \), and for all \( i \leq n \), we define the sets \( S_i \) and the numbers \( L_i \):

\[
S_0(T) = 2^N \\
S_i(T) = S_{i-1} \cap \{ p : \Phi(p|Q^i, N) \downarrow = A_i \} \\
L_i(T) = \begin{cases} 
-1 & \text{if } |S_i(T)| = 0 \\
\lceil \log |S_i(T)| \rceil & \text{otherwise.}
\end{cases}
\]

A lower bound for \( K(T) \) is now proven.

**Lemma 4.4.**

\[
K(T) \geq \min \{ N, \sum_i K(T_i|L_{i-1}) - O(n) \}.
\]

**Proof.** For each \( i \), a semimeasure \( P \) can be constructed using \( A^{i-1}, Q^i, L_{i-1}, N \):

\[
P(z) = 2^{-L_{i-1}}|\{ p \in S_{i-1} : \Phi(p|A^{i-1}, Q^i, N) \downarrow = z \}|.
\]

Remark that \( P \) defines a semimeasure and that \( P \) is enumerable. \( P(A_i) = 0 \), for some \( i \), implies that no program of length \( N \) can solve task \( T_i \), thus \( K(T) > N \). In this case the lemma is proven. Assume \( |S_i| \geq 1 \) and thus \( P(A_i) > 0 \). By applying the coding theorem \([?]\) on \( P \), it follows that:

\[
L_{i-1} - L_i \geq K(T_i|L_{i-1}) - O(1).
\]

Summing over \( i \), gives:

\[
L_0 - L_n \geq \sum_i K(T_i|L_{i-1}) - O(n). \tag{6}
\]

Let \( p \) be a program of length \( K(T) \), solving task \( T \). It possible to append \( 2^{N - K(T) - O(1)} \) different strings of length \( N - K(T) - O(1) \) to \( p \), in order to obtain elements from \( S_n \). Therefore:

\[
L_n \leq N - K(T). \tag{7}
\]

Observe that \( L_0 = N \). Combining equations (6) and (7) proves the lemma. \( \square \)
Proof of proposition 3.1 \ Let \( u, x, y, v \in \omega^n \) for some \( n \). Let 
\[
z = N, 0, 0, 0, u_1, x_1, y_1, v_1, ..., u_n, x_n, y_n, v_n.
\]

Define:
\[
T_{ux,i} = (u_i, x_i|z^{4i}) \\
T_{x,i} = (x_i|z^{4i+1}) \\
T_{yv,i} = (y_i, v_i|z^{4i+2}).
\]

For \( X = ux, yv \), let \( D_{X,1} = 0 \) and for \( i \geq 2 \) let:
\[
D_{X,i} = L_{i-1}(T_X) - L_i(T_X).
\]

Remark that:
\[
\sum_{j \leq i} D_{X,j} = N - L_i(X).
\]

Equations (8), (11), (10), and (12) are now derived.

- Let:
  \[
  u_i = D_{yv,i-1}[z^{4i}] \\
v_i = D_{ux,i-1}.
  \]

At the end of the proof \( u, x, y, v, N \) will be constructed such that equation (14) holds, and therefore, \( N \geq K(T_X) - O(n) \) for \( X = ux, yv \). Since
\[
z^{4i} \rightarrow u_i \rightarrow L_{i-1}(T_{ux}) \\
z^{4i+2} \rightarrow u_i \rightarrow L_{i-1}(T_{yv})
\]
we have by lemma 4.4
\[
K(T_X) \geq \sum_i K(T_{X,i}) - O(n). \tag{8}
\]

- Choose:
  \[
y_i = K(T_{x,i})^*[z^{4i+2}]. \tag{9}
  \]

By Lemma 4.2 it follows that:
\[
K(u_i, x_i|z^{4i}) = K(u_i, x_i, K(x_i|u_i^*[z^{4i}], z^{4i})|z^{4i}).
\]

By equation (11), we have that \( u_i^*[z^{4i}], z^{4i} \leftrightarrow u_i, z^{4i} \), and therefore:
\[
K(x_i|u_i^*[z^{4i}], z^{4i}) = K(x_i|u_i, z^{4i})
\]
\[
= K(x_i|z^{4i+1})
\]
\[
= K(T_{x,i})
\]
Therefore:

\[ K(u_i, x_i, K(x_i|u_i^*[z^{4i}], z^{4i})|z^{4i}) =^+ K(u_i, x_i, K(T_{x,i})|z^{4i}). \]

Remark that \( l(x_i) = m \), and therefore \( K(T_{x,i}) \leq + 2 \log m \). By Lemma 4.3, we have:

\[ K(u_i, x_i, K(T_{x,i})|z^{4i}) \geq K(u_i, x_i, K(T_{x,i})^*[z^{4i+2}]|z^{4i}) - O(\log \log m). \]

By definition of \( y_i \), (9), this shows that:

\[ K(u_i, x_i|z^{4i}) \geq K(u_i, x_i, y_i|z^{4i}) - O(\log \log m). \] (10)

• From equations (1) and (9), we have:

\[ y_i, z^{4i+2} \leftrightarrow y_i^*[z^{4i+2}], z^{4i+2}. \]

Therefore,

\[ K(v_i|z^{4i+3}) = K(v_i|y_i, z^{4i+2}) =^+ K(v_i|y_i^*[z^{4i+2}], z^{4i+2}) =^+ K(y_i, v_i|z^{4i+2}) - K(y_i|z^{4i+2}) \] (11)

• In [?, ?, ?] it is shown that for all \( m, w \) there is an \( x \in 2^m \) such that

\[ K(K(x|w)|x, w) \geq \log m - \log \log m - O(1). \]

Actually, the unconditioned version is shown, but this version has the same proof. Fix an \( m \) large enough and choose \( x_i \in 2^m \) such that by equation (9):

\[ K(y_i|z^{4i+2}) = K(K(T_{x,i})^*[z^{4i+2}]|z^{4i+2}) \geq^+ K(K(T_{x,i})|z^{4i+2}) = K(K(x_i|z^{4i+1})|x_i, z^{4i+1}) \geq^+ \log m - \log \log m. \] (12)

First using Lemma 4.1, then applying subsequently equations (10), (11), (12), and (8) gives:

\[
K(u, x, y, v) \\
\leq \sum_i K(u_i, x_i, y_i|z^{4i}) + \sum_i K(v_i|z^{4i+3}) + O(n) \\
\leq \sum_i K(u_i, x_i|z^{4i}) + \sum_i K(y_i, v_i|z^{4i+2}) - \sum_i K(y_i|z^{4i+2}) + O(n \log \log m) \\
\leq K(T_{ux}) + K(T_{yv}) - O(n \log m).
\]
Let $\langle \cdot, \cdot \rangle$ be a computable bijective pairing function such that for all $a, b \in \omega$, $l(\langle a, b \rangle) \leq l(a) + l(b)$. Let:

$$x'_i = \langle u_i, x_i \rangle$$

$$y'_i = \langle y_i, v_i \rangle.$$ 

To finish the proof it suffices to show that

$$l(x') + l(y') \leq N \leq O(mn). \quad (13)$$

Remark that because $x_i \in 2^m$, $l(x_i) \leq 2^m$ and because $y_i = K(T_{x,i})$, $l(y_i) \leq 2 \log m$:

$$l(x'_i) + l(y'_i) \leq l(x_i) + l(y_i) + l(v_i) \leq l(D_{ux,i}) + 2m + 2 \log m + l(D_{yv,i}).$$

Choose $N = 3mn$. For $X = ux, yv$, $\sum_i D_{X,i} \leq N + 1$, and therefore $\sum_i l(D_{X,i}) \leq 3n \log m$. This shows that for $m$ large enough:

$$l(x') + l(y') \leq 3mn = N. \quad (14)$$

This shows equation (13). \qed

**Corollary 4.5.** For some $c > 0$, for all but finitely many $n$, there exist a $z \in 2^{2n}$ such that:

$$K(0 \rightarrow z_1; \ldots; z_{2n-2} \rightarrow z_{2n-1}) + K(z_1 \rightarrow z_2; \ldots; z_{2n-1} \rightarrow z_{2n}) - K(z) \geq cn. \quad (15)$$

**Proof.** Let $x', y'$ be as constructed in the proof. Let $x'_i$ and $y'_i$ be binary prefix-free encodings corresponding to the definition of $l(x')$. of $x'_i$ and $y'_i$, $i \leq n$. Define $z$:

$$z = x_{1,1}, 0, \ldots, x_{1,l(x'_1)}, 0,$$

$$0, y_{1,1}, \ldots, 0, y_{1,l(y'_1)},$$

$$\ldots$$

$$x_{n,1}, 0, \ldots, x_{n,l(x'_n)}, 0,$$

$$0, y_{n,1}, \ldots, 0, y_{n,l(y'_n)}.$$ 

Since $\sum_{i\leq n} l(x'_i) + l(y'_i) \leq 3mn$, we have that $z \in 2^{6n}$. This shows that for all but finitely many $n$ a string of length maximally $6mn$ exists that satisfies the inequality of the lemma. By appending zeros to the end of $x'$ and $y'$, equality (15) can be satisfied for every $n$. \qed

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