GLOBAL ATTRACTOR FOR A COMPOSITE SYSTEM OF NONLINEAR WAVE AND PLATE EQUATIONS

FRANCESCA BUCCI
Università degli Studi di Firenze, Dipartimento di Matematica Applicata
Via S. Marta 3, 50139 Firenze, Italy

IGOR CHUESHOV
Kharkov University, Department of Mathematics and Mechanics
4 Svobody sq, 61077 Kharkov, Ukraine

IRENA LASIECKA
University of Virginia, Department of Mathematics
Charlottesville, VA 22901, USA

(Communicated by Alain Miranville)

Abstract. We prove the existence of a compact, finite dimensional, global attractor for a system of strongly coupled wave and plate equations with nonlinear dissipation and forces. This kind of models describes fluid-structure interactions. Though our main focus is on the composite system of two partial differential equations, the result achieved yields as well a new contribution to the asymptotic analysis of either (uncoupled) equation.

1. Introduction. The mathematical model under consideration consists of a semilinear wave equation defined on a bounded domain Ω, which is strongly coupled with Berger’s plate equation acting only on a part of the boundary of Ω. This kind of models, referred in the literature to as structural acoustic interactions, arise in the context of modeling gas pressure in an acoustic chamber which is surrounded by a combination of hard (rigid) and flexible walls. The pressure in the chamber is described by the solution to a wave equation, while vibrations of the flexible wall are described by the solution to a plate equation.

More precisely, let Ω ⊂ \mathbb{R}^n be a smooth bounded domain, n = 2, 3, with the boundary \partial Ω = \Gamma = \Gamma_0 \cup \Gamma_1 consisting of two open (in induced topology) connected disjoint parts \Gamma_0 and \Gamma_1 of positive measure. \Gamma_0 is flat and is referred to as the elastic wall, whose dynamics is described by the Berger plate (n = 3) or beam (n = 2) equation; for details on Berger model we refer to [15, Chap. 4] and to the literature quoted therein. The acoustic medium in the chamber Ω is described by...
a semilinear wave equation. Thus, we consider the following PDE system

\[
\begin{cases}
  z_{tt} + g(z_t) - \Delta z + f(z) = 0 & \text{in } \Omega \times (0, T) \\
  \frac{\partial z}{\partial \nu} = 0 & \text{on } \Gamma_1 \times (0, T) \\
  \frac{\partial z}{\partial \nu} = \alpha \kappa v_t & \text{on } \Gamma_0 \times (0, T) \\
  z(0, \cdot) = z^0, \quad z_t(0, \cdot) = z^1 & \text{in } \Omega \\
  v_{tt} + b(v_t) + \Delta^2 v + \left[ Q - \int_{\Gamma_0} |\nabla v(x, t)|^2 dx \right] \Delta v + \beta \kappa z_t|_{\Gamma_0} = p_0 & \text{in } \Gamma_0 \times (0, T) \\
  v = \Delta v = 0 & \text{on } \partial \Gamma_0 \times (0, T) \\
  v(0, \cdot) = v^0, \quad v_t(0, \cdot) = v^1 & \text{in } \Gamma_0.
\end{cases}
\]

In the above system, \( g(s) \) and \( b(s) \) are non-decreasing functions describing the dissipation effects in the model, while the term \( f(z) \) represents a nonlinear force acting on the wave component; \( \nu \) is the outer normal vector, \( \alpha \) and \( \beta \) are positive constants; the parameter \( \kappa \geq 0 \) has been introduced in order to cover also the case of non-interacting wave and plate equations (\( \kappa = 0 \)). The part \( \Gamma_1 \) of the boundary describes a rigid (hard) wall, while \( \Gamma_0 \) is a flexible wall where the coupling with the plate equation takes place. The boundary term \( \beta \kappa z_t|_{\Gamma_0} \) describes back pressure exercised by the acoustic medium on the wall. The real parameter \( Q \) describes in-plane forces applied to the plate, while \( p_0 \in L^2(\Omega) \) accounts for transversal forces. The described interactive system involves a coupling between \( n \)- and \( (n-1) \)-dimensional manifolds, and as such is of hybrid type [35].

Structural acoustic models are well known in both the physical and mathematical literature and go back to the canonical models considered in [6, 36, 28]. More recently, these models were studied in the context of control theory, where problems such as the active control of pressure and vibrations by means of actuators placed on the flexible wall, or stabilization and controllability of the overall structure become issues of focal interest. There is a very large literature devoted to this topic; the reader is referred to the monograph [32], which also provides a rather comprehensive overview of related works. More recent contributions include [2, 3, 13, 26, 30], where questions of exact controllability or uniform stability are dealt with for interactions of wave/Kirchhoff plates [3], wave/Reissner-Mindlin plates [26] or wave/shell models [13], respectively. We also mention the papers [7, 8, 9, 14] which consider the coupled system of a linear wave equation (in \( \Omega = \mathbb{R}^3_+ := \{ (x_1, x_2, x_3) : x_3 > 0 \} \)) and von Karman equations (in \( \Gamma_0 \)). This case corresponds to nonlinear aeroelastic plate problem in a flow of gas.

In contrast, in the present paper we study the long-time behavior of structures subject to nonlinear external or internal excitations. This type of issues are dealt with within the framework of dynamical systems. Hence, central questions to be discussed are the existence of global attractors and their properties. Our choice of a semilinear wave equation and Berger’s plate model, both with nonlinear damping, beside being of great physical relevance, is also suggested by mathematical considerations. These two models provide canonical examples of relatively simple
hyperbolic-like PDE systems, yet they exhibit the main technical difficulties encountered in proving existence of attractors, or asserting their regularity or finite-dimensionality in the presence of a nonlinear damping. In fact, the intrinsic hyperbolic (hyperbolic-like) character of the dynamics involved makes the study of long time behavior of the corresponding models somewhat challenging. This is related to the fact that the unstable part of the free models is infinite-dimensional (unlike the case of parabolic-like equations). This puts a strong demand on the damping which must control an “infinite dimensional spectrum”. In the case of linear damping, this has been dealt with rather successfully [4, 15, 27, 40]. However, the presence of nonlinear damping along with the intrinsic non-differentiability of the flow on the phase space has been recognized in the literature as a source of main difficulties.

In recent years there has been a steady progress in this area, particularly in the context of wave equations [25, 33, 37], and novel techniques have been developed with an eye to hyperbolic-like dynamics exemplified by second order evolutions. The recent work by the second and third authors [18, 19, 20] provides a comprehensive account of new abstract results, along with the analysis of relevant PDE examples such as waves and plates equations with nonlinear damping and critical (i.e. non-compact) nonlinear terms.

The present work takes advantage of the new methods developed or reported in [20], which are applied in the context of interactive structures. This way we not only achieve sharp results for structural acoustic interactions, but also improve some of the previous results available for the uncoupled models.

In this paper we aim to show the existence and to study the properties of a global attractor for problem (1).

Our first main result, Theorem 3.2, states the existence of a global attractor for problem (1) under rather general conditions on the nonlinear functions $g$, $f$ and $b$; see Assumption 3.1. In particular, it is not assumed that the damping functions $g$ and $b$ are (i) differentiable, nor (ii) strictly increasing. The implications of this result for the uncoupled wave and plate equations are also new.

Our second main result, Theorem 3.4, deals with the dimension and smoothness of the global attractor. It requires additional (structural) hypotheses concerning the damping functions $g$ and $b$ and the nonlinear force $f$; see Assumption 3.3. In particular, strong monotonicity of $g$ and $b$ will be assumed. In the case $\kappa = 0$, this result is in accordance with the results previously established in [18], [19, Chap. 5] and [22] for the wave equation, and in [20, Chap. 7] for Berger model. The statements pertaining to either uncoupled equation are given explicitly in Corollary 3.5 and Corollary 3.6, respectively.

The paper is organized as follows. Section 2 contains some background material: we give here the abstract formulation of problem (1), we then state Theorem 2.3 on well-posedness of the PDE problem (whose proof is left to an Appendix), and describe the properties of suitable energy functionals and stationary solutions. In Section 3 we state our main results, namely Theorem 3.2 on the existence of a compact global attractor for system (1), and Theorem 3.4 on the attractor’s dimension and smoothness. Their Corollaries 3.5 and 3.6, followed by detailed remarks, are also given here. In Section 4 we develop some preliminary tool for the proofs of Theorems 3.2 and 3.4. Section 5 contains the proof of asymptotic smoothness of the dynamical system generated by (1): this property is crucial in order to show
existence of a global attractor established in Theorem 3.2. In Section 6 we establish a so-called stabilizability estimate, which will play a key role in the proof of Theorem 3.4 in Section 7.

2. Preliminaries. The notation below is largely standard within the literature. For the reader’s convenience, we just recall that the symbols $||\cdot||_\sigma$ and $(\cdot,\cdot)_\sigma$ denote the norm and the inner product in $L_2(\Omega)$. The subscripts in $(\cdot,\cdot)_\sigma$ and $||\cdot||_\sigma$ will be often omitted when apparent from the context. We denote $||\cdot||_{\sigma,\sigma}$ the norm in the $L_2$-based Sobolev space $H^\sigma(\Omega)$. Here we will have either $\Omega = \Omega$ or $\Omega = \Gamma_0$. We also denote by $H^\sigma_0(\Omega)$ the closure of $C^\infty_0(\Omega)$ in $H^\sigma(\Omega)$.

2.1. Basic assumption. We shall impose the following basic assumptions on the nonlinear functions $g$ and $f$ which affect the wave component of the system and on the damping function $b$ in the plate equation.

**Assumption 2.1.**  
• $g \in C(\mathbb{R})$ is a non-decreasing function, $g(0) = 0$, and there exists a constant $C > 0$ such that
  \[ |g(s)| \leq C (1 + |s|^p), \quad s \in \mathbb{R}, \]
  where $1 \leq p \leq 5$ when $n = 3$, while $1 \leq p < \infty$ for $n = 2$.
• $f \in \text{Lip}_\text{loc}(\mathbb{R})$, and there exists a positive constant $M$ such that
  \[ |f(s_1) - f(s_2)| \leq M (1 + |s_1|^q + |s_2|^q) \, |s_1 - s_2| \quad \text{for } s_1, s_2 \in \mathbb{R}, \]
  where $q \leq 2$ when $n = 3$, and $q < \infty$ for $n = 2$. Moreover, the following dissipativity condition holds true:
  \[ \mu := \frac{1}{2} \liminf_{|s| \to \infty} \frac{f(s)}{s} > 0. \]
• $b \in C(\mathbb{R})$ is a non-decreasing function such that $b(0) = 0$.
• $p_0 \in L_2(\Gamma_0)$.

2.2. Abstract formulation. We find convenient to represent the PDE system (1) in an abstract-semigroup form. In fact, we shall see that the resulting formulation is a special case of a general nonlinear structural acoustic model given in [32, Sect. 2.6]. In order to accomplish this we introduce the following spaces and operators.

Let $A : \mathcal{D}(A) \subset L_2(\Omega) \to L_2(\Omega)$ be the positive self-adjoint operator defined by
\[ Ah = -\Delta h + \mu h, \quad \mathcal{D}(A) = \{ h \in H^2(\Omega) : \frac{\partial h}{\partial \nu}|_{\Gamma_0} = 0 \}; \]
where $\mu > 0$ is given by (4). Next, let $N_0$ be the Neumann map from $L_2(\Gamma_0)$ to $L_2(\Omega)$, defined by
\[ \psi = N_0 \varphi \iff \left\{ (-\Delta + \mu) \psi = 0 \text{ in } \Omega; \left. \frac{\partial \psi}{\partial \nu} \right|_{\Gamma_0} = \varphi, \left. \frac{\partial \psi}{\partial \nu} \right|_{\Gamma_1} = 0 \right\}. \]
It is well known (see, e.g., [34, Chap.3]) that $N_0$ continuous : $L_2(\Gamma_0) \to H^{3/2}(\Omega) \subset \mathcal{D}(A^{3/4-\varepsilon})$, $\varepsilon > 0$.

In particular, we have that
\[ A^{3/4-\varepsilon}N_0 \text{ continuous : } L^2(\Gamma_0) \to L^2(\Omega). \] (5)
It is also well known that by Green’s second theorem, the following trace result holds true (see, e.g., [34]):
\[ N_0^* Ah = h|_{\Gamma_0} \quad \text{for } h \in \mathcal{D}(A), \] (6)
where $N_0^* : L_2(\Omega) \to L_2(\Gamma_0)$ is the adjoint of $N_0$. The validity of (6) may be extended to all $h \in H^1(\Omega) \equiv D(A^{1/2})$, as $D(A)$ is dense in $D(A^{1/2})$.

Regarding the plate model, let $\mathcal{A} : D(\mathcal{A}) \subset L_2(\Gamma_0) \to L_2(\Gamma_0)$ be the positive, self-adjoint operator defined by
\[
\mathcal{A}w = \Delta^2 w, \quad D(\mathcal{A}) = \{ w \in H^4(\Gamma_0) \cap H^1_0(\Gamma_0) : \Delta w = 0 \text{ on } \partial \Gamma_0 \}.
\]
The fractional powers of $\mathcal{A}$ are well defined and we have, in particular,
\[
\mathcal{A}^{1/2}v = -\Delta v \quad \text{on the domain } D(\mathcal{A}^{1/2}) = H^2(\Gamma_0) \cap H^1_0(\Gamma_0),
\]
\[
\|\mathcal{A}^{1/2}v\|_{\Gamma_0} = \|\nabla v\|_{\Gamma_0} \quad \text{for any } v \in D(\mathcal{A}^{1/2}) = H^1_0(\Gamma_0).
\]
By using the above dynamic operators, the coupled PDE problem (1) can be rewritten as the following abstract second order system, which is a particular case of the one studied in [32, Sect. 2.6]:
\[
\begin{align*}
&z_{tt} + A(z - \alpha \kappa N_0 v_t) + D(z_t) + F_1(z) = 0, \quad (7a) \\
v_{tt} + \mathcal{A}v + B(v_t) + \beta \kappa N_0^* A v_t + F_2(v) = 0, \quad (7b) \\
z(0) = z^0, \quad z_t(0) = z^1; \quad v(0) = v^0, \quad v_t(0) = v^1, \quad (7c)
\end{align*}
\]
where we have introduced the operators
\[
D(h) := g(h), \quad F_1(z) = f(z) - \mu z,
\]
in (7a), whereas
\[
B(w) := b(w), \quad F_2(v) = -\left(Q - \|\mathcal{A}^{1/2}v\|_{L^2(\Gamma_0)}^2\right)\mathcal{A}^{1/2}v - p_0
\]
in (7b). Regarding the nonlinear force terms we have that
\[
F_1(z) = \Pi'(z) \quad \text{with} \quad \Pi(z) = \int_{\Omega} \int_0^{2(\xi)} (f(\xi) - \mu \xi) \, d\xi \, dx,
\]

where $'$ stands for the Fréchet derivative in an appropriate space. It readily follows from (4) that
\[
\Pi(z) \geq \delta_f \|z\|_{\Omega}^2 - M_f, \quad z \in H^1(\Omega), \quad (9)
\]
for some positive constants $\delta_f$ and $M_f$. Similarly, we have that
\[
F_2(v) = \Phi'(v) \quad \text{with} \quad \Phi(v) = \frac{1}{4} \|\mathcal{A}^{1/4}v\|_{L^2(\Gamma_0)}^4 - \frac{Q}{2} \|\mathcal{A}^{1/4}v\|_{\Gamma_0}^2 - (p_0, v). \quad (10)
\]
The phase spaces $Y_1$ for the wave component $[z, z_t]$ and $Y_2$ for the plate component $[v, v_t]$ of system (7) are given, respectively, by:
\[
Y_1 := D(A^{1/2}) \times L_2(\Omega) \equiv H^1(\Omega) \times L_2(\Omega); \quad Y_2 := D(A^{1/2}) \times L_2(\Gamma_0) \equiv [H^2(\Gamma_0) \cap H^1_0(\Gamma_0)] \times L_2(\Gamma_0).
\]
The state space for problem (7) is then
\[
Y = Y_1 \times Y_2 = D(A^{1/2}) \times L^2(\Omega) \times D(A^{1/2}) \times L^2(\Gamma_0), \quad (11)
\]
which will be supplemented with the following norm
\[
\|y\|_Y^2 = \|\{z_1, z_2, v_1, v_2\}\|_Y^2 := \beta \|\{z_1, z_2\}\|_{Y_1}^2 + \alpha \|\{v_1, v_2\}\|_{Y_2}^2 \quad (12)
\]
and with the corresponding inner product.
Notice that an important consequence of Assumption 2.1 and of criticality of the parameter \( q \) is that the nonlinear operators \( F_1 \) and \( F_2 \) are locally Lipschitz continuous from \( H^1(\Omega) \) into \( L_2(\Omega) \) and from \( H^2(\Gamma_0) \cap H_0^1(\Gamma_0) \) into \( L_2(\Gamma_0) \), respectively, that is
\[
\|F_1(z_1) - F_1(z_2)\|_{\Omega} \leq C(\rho)\|z_1 - z_2\|_{1,\Omega}, \quad \|z_1\|_{1,\Omega} \leq \rho < \infty, \tag{13}
\]
\[
\|F_2(v_1) - F_1(v_2)\|_{\Gamma_0} \leq C(\rho)\|v_1 - v_2\|_{2,\Gamma_0}, \quad \|v_1\|_{2,\Gamma_0} \leq \rho < \infty,
\]
where \( C(\rho) \) denotes a function which is bounded for bounded arguments. It is important to emphasize that while the operators are bounded on the respective spaces, they are not compact. This fact justifies the notion of “criticality” for the parameter \( q \) and for the nonlinear terms \( F_1 \) and \( F_2 \).

The natural (nonlinear) energy functions associated with the solutions to the uncoupled wave and plate models are given, respectively, by
\[
\mathcal{E}_z(z(t), z_\ell(t)) := E^0_z(z(t), z_\ell(t)) + \Pi(z(t)) \tag{14a},
\]
\[
\mathcal{E}_v(v(t), v_\ell(t)) := E^0_v(v(t), v_\ell(t)) + \Phi(v(t)) \tag{14b},
\]
where we have set
\[
E^0_z(z(t), z_\ell(t)) := \frac{1}{2} \left\{ \|A^{1/2}z(t)\|_{\Omega}^2 + \|z_\ell(t)\|_{\Omega}^2 \right\} := E^0_\ell(t), \tag{15a}
\]
\[
E^0_v(v(t), v_\ell(t)) := \frac{1}{2} \left\{ \|A^{1/2}v(t)\|_{\Gamma_0}^2 + \|v_\ell(t)\|_{\Gamma_0}^2 \right\} := E^0_v(t). \tag{15b}
\]
Since both energy functionals in (14) may be negative, it is convenient to introduce the following positive energy functions
\[
E_z(z, z_\ell) := E^0_z(z, z_\ell) + \Pi(z) + M_f = \mathcal{E}_z(z, z_\ell) + M_f, \n\]
\[
E_v(v, v_\ell) := E^0_v(v, v_\ell) + \frac{1}{4} \|A^{1/4}v\|^4 = \mathcal{E}_v(v, v_\ell) + \frac{Q}{2} \|A^{1/4}v\|^2 + (p_0, v),
\]
where \( M_f \) is the constant in (9).

Finally, we introduce the total energy \( \mathcal{E}(t) = \mathcal{E}(z(t), z_\ell(t), v(t), v_\ell(t)) \) of the system, namely
\[
\mathcal{E}(t) = \mathcal{E}(z(t), z_\ell(t), v(t), v_\ell(t)) := \beta \mathcal{E}_z(z, z_\ell) + \alpha \mathcal{E}_v(v, v_\ell), \tag{16}
\]
whose positive part is given by
\[
E(t) = E(z, z_\ell, v, v_\ell) := \beta E_z(z, z_\ell) + \alpha E_v(v, v_\ell). \tag{17}
\]
It is easy to see from the structure of the energy functionals and in view of (9) and (10) that for any \( \alpha, \beta > 0 \) there exist positive constants \( c, C, \) and \( M_0 \) such that
\[
cE(z, z_\ell, v, v_\ell) - M_0 \leq E(z, z_\ell, v, v_\ell) \leq CE(z, z_\ell, v, v_\ell) + M_0, \tag{18}
\]
where \( \mathcal{E} \) and \( E \) are the energies defined in (16) and (17).

2.3. Well-posedness. In this subsection we study well-posedness of problem (1). Since the corresponding abstract system (7) is a special case of a general abstract model studied in [32, Sect. 2.6], local and global existence results for the solutions can be deduced from Theorem 2.6.1 and Theorem 2.6.2 in [32]. These results are based on the theory of monotone operators; see [5], [11], or [38].

In order to make our statements precise, we need to introduce the concepts of strong and generalized solutions.
Definition 2.2. A pair of functions

\((z(t), v(t)) \in C([0, T], D(A^{1/2}) \times D(A^{1/2})) \cap C^1([0, T], L^2(\Omega) \times L^2(\Gamma_0))\)

which satisfy the initial conditions (7c) is said to be

(S) a strong solution to problem (7) on the interval \([0, T]\), iff

- for any \(0 < a < b < T\) one has

\(z(t), v(t)) \in L^1([a, b], D(A^{1/2}) \times D(A^{1/2}))\)

and

\(z(t), v(t)) \in L^1([a, b], L^2(\Omega) \times L^2(\Gamma_0));\)

- \(A[z(t) - \alpha N_0v(t)] + D(z_t(t)) \in L^2(\Omega)\) and \(Av(t) \in L^2(\Gamma_0)\) for almost all \(t \in [0, T];\)

- equations (7a) and (7b) are satisfied in \(L^2(\Omega) \times L^2(\Gamma_0)\) for almost all \(t \in [0, T];\)

(G) a generalized solution to problem (7) on the interval \([0, T]\), iff there exists a sequence of strong solutions \((z_n(t), v_n(t))\) to problem (7) with initial data \((z_n^0, z_n^1, v_n^0, v_n^1)\) (in place of \((z_0, z^1, v_0, v^1)\)) such that

\[\lim_{n \to \infty} \max_{t \in [0, T]} \left\{ \|\partial_t z(t) - \partial_t z_n(t)\|_\Omega + \|A^{1/2}(z(t) - z_n(t))\|_\Omega \right\} = 0\]

and

\[\lim_{n \to \infty} \max_{t \in [0, T]} \left\{ \|\partial_t v(t) - \partial_t v_n(t)\|_{\Gamma_0} + \|A^{1/2}(v(t) - v_n(t))\|_{\Gamma_0} \right\} = 0.\]

The main result concerning well-posedness of problem (1) is the following assertion.

Theorem 2.3. Under Assumption 2.1 the PDE system (1) is well-posed on

\(Y = H^1(\Omega) \times L^2(\Omega) \times [H^2(\Gamma_0) \cap H^0_0(\Gamma_0)] \times L^2(\Gamma_0),\)

i.e. for any \((z^0, z^1, v^0, v^1) =: y_0 \in Y\) there exists a unique generalized solution \(y(t) = (z(t), z_t(t), v(t), v_t(t))\) which depends continuously on initial data. This solution satisfies the energy inequality

\[\mathcal{E}(t) + \beta \int_s^t (D(z_t), z_t(\Omega) d\tau + \alpha \int_s^t (B(v_t), v_t(\Gamma_0) d\tau \leq \mathcal{E}(s), \quad 0 \leq s \leq t, \quad (19)\]

with the total energy \(\mathcal{E}(t)\) given by (16). Moreover, if initial data \(y_0\) are such that

\(z^0, z^1 \in D(A^{1/2}), \quad v^0 \in D(A), \quad v^1 \in D(A^{1/2}),\)

and

\(A[z^0 - \alpha N_0v^1] + D(z^1) \in L^2(\Omega),\)

then there exists a unique strong solution \(y(t)\) satisfying the energy identity:

\[\mathcal{E}(t) + \beta \int_s^t (D(z_t), z_t(\Omega) d\tau + \alpha \int_s^t (B(v_t), v_t(\Gamma_0) d\tau = \mathcal{E}(s), \quad 0 \leq s \leq t. \quad (20)\]

Both strong and generalized solutions satisfy the inequality

\[\mathcal{E}(t) \equiv \mathcal{E}(z(t), z_t(t), v(t), v_t(t)) \leq \mathcal{E}(z(s), z_t(s), v(s), v_t(s)) \equiv \mathcal{E}(s) \quad \text{for} \quad t \geq s, \quad (21)\]

which implies, in particular, (see (18))

\[E(z(t), z_t(t), v(t), v_t(t)) \leq C (1 + E(z^0, z^1, v^0, v^1)), \quad \text{for} \quad t \geq 0. \quad (22)\]
The existence and uniqueness of generalized solutions asserted in Theorem 2.3 follows from Theorem 2.6.2 in [32]. To see this, it is sufficient to notice that the conditions in Assumption 2.1 imply that (i) the operators $F_1$ and $F_2$ are locally Lipschitz continuous in the respective spaces (see (13)), with appropriate a-priori bounds, and (ii) the damping functions modeled by $g$ and $b$ satisfy the monotonicity and continuity properties required in [32, Sect. 2.6]. However, for the sake of completeness and since it is focused on nonlinear boundary damping, which requires additional technicalities, we shall provide in the Appendix a self-contained proof, tailored for the specific problem under investigation.

**Remark 1.** The existence of generalized solutions established in Theorem 2.3 is obtained by using the theory of nonlinear semigroups. As such, these solutions are defined as strong limits of regular (strong) solutions, as in the part (G) of Definition 2.2. This does not automatically imply that generalized solutions satisfy a variational equality. However, in the present paper, the regularity of $g$ and $f$ enable us to compute appropriate limits and to obtain the variational form stated below. Indeed, using the same argument as in [21] one can prove that any generalized solution $y(t) = (z(t), z_0(t), v(t), v_1(t))$ to problem (7) is also weak, i.e. it satisfies the following (variational) relations:

\[
\frac{d}{dt}(z_0, \phi)_{\Omega} + (\nabla z, \nabla \phi)_{\Omega} + (g(z_0), \phi)_{\Omega} - \alpha \kappa(v_t, \phi)_{\Gamma_0} + (f(z), \phi)_{\Omega} = 0, \quad (23a)
\]

\[
\frac{d}{dt}(v_t + \beta \kappa z, \psi)_{\Gamma_0} + (\Delta v, \Delta \psi)_{\Gamma_0} + (b(v_t), \psi)_{\Gamma_0} + [Q - \|\nabla v\|^2_{L^2}(\Omega)] (\Delta v, \psi)_{\Gamma_0} = (p_0, \psi)_{\Gamma_0}, \quad (23b)
\]

for any $\phi \in H^1(\Omega)$ and $\psi \in H^2(\Gamma_0) \cap H^1_0(\Gamma_0)$ in the sense of distributions.

Theorem 2.3 makes it possible to define a dynamical system $(Y, S_t)$ with the phase space $Y$ given by (11) and with the evolution operator $S_t : Y \rightarrow Y$ given by the relation

\[S_t y = (z(t), z_0(t), v(t), v_1(t)), \quad y = (z^0, z^1, v^0, v^1),\]

where $(z(t), v(t))$ is a generalized solution to (7). Moreover, the monotonicity of the damping operators $D$ and $B$, combined with the estimate in (13) and the boundedness property given by (22) imply (by a pretty routine argument) that the semi-flow $S_t$ is locally Lipschitz on $Y$. Moreover, there exists $a > 0$ and $\omega(\rho) > 0$ such that

\[\|S_t y_1 - S_t y_2\|_Y \leq a e^{\omega(\rho)t} \|y_1 - y_2\|_Y, \quad \|y_t\|_Y \leq \rho, \quad t \geq 0. \quad (24)\]

### 2.4. Energy functionals and stationary solutions.

We conclude this section by discussing several properties of the energy functionals and stationary solutions. It follows from (18) that the energy $E(z^0, z^1, v^0, v^1)$ is bounded from below on $Y$ and $E(z^0, z^1, v^0, v^1) \rightarrow +\infty$ when $\|(z^0, z^1, v^0, v^1)\|_Y \rightarrow +\infty$. This, in turn, implies that there exists $R_* > 0$ such that the set

\[\mathcal{W}_R = \{y = (z^0, z^1, v^0, v^1) \in Y : E(z^0, z^1, v^0, v^1) \leq R\} \quad (25)\]

is a non-empty bounded set in $Y$ for all $R \geq R_*$. Moreover any bounded set $B \subset Y$ is contained in $\mathcal{W}_R$ for some $R$ and, as it follows from (21), the set $\mathcal{W}_R$ is invariant with respect to the semi-flow $S_t$, i.e. $S_t \mathcal{W}_R \subset \mathcal{W}_R$ for all $t > 0$. Thus we can consider the restriction $(\mathcal{W}_R, S_t)$ of the dynamical system $(Y, S_t)$ on $\mathcal{W}_R$, $R \geq R_*$.  

We introduce next the set of stationary points of $S_t$ denoted by $\mathcal{N}$,

\[\mathcal{N} = \{V \in Y : S_t V = V \text{ for all } t \geq 0\}.\]
Every stationary point \( V \) has the form \( V = (z, 0, v, 0) \), where \( z \in H^1(\Omega) \) and \( v \in H^2(\Gamma_0) \cap H_0^0(\Gamma_0) \) are weak (variational) solution to the problems
\[
-\Delta z + f(z) = 0 \quad \text{in } \Omega, \quad \frac{\partial z}{\partial \nu} = 0 \quad \text{on } \Gamma,
\]
and
\[
\begin{aligned}
\Delta^2 v + \left[ Q - \int_{\Gamma_0} |\nabla v(x, t)|^2 \, dx \right] \Delta v &= p_0 \quad \text{on } \Gamma_0 \\
v = \Delta v &= 0 \quad \text{on } \partial \Gamma_0.
\end{aligned}
\]
Using the properties of the potentials \( \Pi \) and \( \Phi \) given by (8) and (10) one can easily prove the following assertion.

**Lemma 2.4.** Under Assumption 2.1 the set \( N \) of stationary points for the semiflow \( S_t \) generated by equations (7) is a closed bounded set in \( Y \), and hence there exists \( R^* \geq R^* \) such that \( N \subset W^r_R \) for every \( R \geq R^* \).

Later we will also need the notion of unstable manifold \( M^u(N) \) emanating from the set \( N \), which is defined as the set of all \( W \in Y \) such that there exists a full trajectory \( \gamma = \{ W(t) : t \in \mathbb{R} \} \) with the properties
\[
W(0) = W \quad \text{and} \quad \lim_{t \to -\infty} \text{dist}_Y(W(t), N) = 0.
\]
We finally recall that a continuous curve \( \gamma = \{ W(t) : t \in \mathbb{R} \} \subset Y \) is said to be a full trajectory if \( S_tW(\tau) = W(t + \tau) \) for any \( t \geq 0 \) and \( \tau \in \mathbb{R} \).

3. **The statement of main results.** The goal of the present paper is to show the existence of a global attractor for the dynamical system generated by problem (1), and to study its properties.

Let us recall (cf. [4, 15, 27, 40]) that a global attractor for a dynamical system \((X, S_t)\) on a complete metric space \( X \) is a closed bounded set \( \mathfrak{A} \) in \( X \) which is invariant (i.e. \( S_t\mathfrak{A} = \mathfrak{A} \) for any \( t > 0 \)) and uniformly attracting, i.e.
\[
\lim_{t \to +\infty} \sup_{y \in B} \text{dist}_X(S_t y, \mathfrak{A}) = 0 \quad \text{for any bounded set } B \subset X.
\]
The **fractal dimension** \( \dim_f M \) of a compact set \( M \) is defined by
\[
\dim_f M = \limsup_{\varepsilon \to 0} \frac{\ln N(M, \varepsilon)}{\ln(1/\varepsilon)},
\]
where \( N(M, \varepsilon) \) is the minimal number of closed sets of diameter \( 2\varepsilon \) which cover the set \( M \).

To prove the existence of a global attractor for problem (7) we need additional hypotheses concerning the damping functions \( g \) and \( b \).

**Assumption 3.1.** Besides to Assumption 2.1, suppose that for any \( \varepsilon > 0 \) there exists \( c_\varepsilon > 0 \) such that
\[
s^2 \leq \varepsilon + c_\varepsilon sg(s) \quad \text{for } s \in \mathbb{R}, \quad (26)
\]
and
\[
s^2 \leq \varepsilon + c_\varepsilon sb(s) \quad \text{for } s \in \mathbb{R}. \quad (27)
\]
Remark 2. One can see that the inequalities (26) and (27) are equivalent to the following property: for any $\delta > 0$ there exists $m_\delta > 0$ such that
\[ |g(s)| \geq m_\delta |s| \quad \text{and} \quad |b(s)| \geq m_\delta |s| \quad \text{for} \quad |s| \geq \delta . \tag{28} \]
Alternatively, (26) and (27) are readily satisfied provided that (i) both functions $g(s)$ and $b(s)$ are non-decreasing on $\mathbb{R}$ and strictly increasing in some (small) neighbourhood of 0, and (ii) we have that
\[ \liminf_{|s| \to \infty} \frac{g(s)}{s} > 0 \quad \text{and} \quad \liminf_{|s| \to \infty} \frac{b(s)}{s} > 0 . \]
In particular, this means that Assumption 3.1 allows the damping functions $g$ and $b$ to be constants on some closed finite intervals which are away from zero.

Our first main result is the following theorem.

**Theorem 3.2.** Under Assumption 3.1 the dynamical system $(Y,S_t)$ generated by problem (7) has a compact global attractor $\mathfrak{A}$ which coincides with the unstable manifold $M^u(N)$ emanating from the set $N$ of stationary points for $S_t$, $\mathfrak{A} \equiv M^u(N)$. Moreover,
\[ \lim_{t \to +\infty} \text{dist}_Y(S_tW,N) = 0 \quad \text{for any} \quad W \in Y. \tag{29} \]
To state our second main result we need additional hypotheses.

**Assumption 3.3.** Besides to Assumption 3.1, let the following conditions hold:

- there exist positive constants $m$ and $M$ such that
\[ m \leq \frac{g(s_1) - g(s_2)}{s_1 - s_2} \leq M \left( 1 + s_1 g(s_1) + s_2 g(s_2) \right) \sigma , \quad s_1, s_2 \in \mathbb{R} , \quad s_1 \neq s_2 , \tag{30} \]
where $0 \leq \sigma < 1$ is arbitrary in the case $n = 2$ and $\sigma = 2/3$ in the case $n = 3$;

- there exist positive constants $m_1$ and $M_1$ such that
\[ m_1 \leq \frac{b(s_1) - b(s_2)}{s_1 - s_2} \leq M_1 \left( 1 + s_1 b(s_1) + s_2 b(s_2) \right) , \quad s_1, s_2 \in \mathbb{R} , \quad s_1 \neq s_2 ; \tag{31} \]

- $f \in C^2(\mathbb{R})$ and $|f''(s)| \leq C(1 + |s|^{q-1})$ for $s \in \mathbb{R}$, where $q = 2$ for $n = 3$ and $1 \leq q < \infty$ for $n = 2$.

Remark 3. If $g$, $b \in C^1(\mathbb{R})$, then (30) and (31) are equivalent to the requirements
\[ m' \leq g'(s) \leq M' \left( 1 + sg(s) \right) \sigma \quad \text{for all} \quad s \in \mathbb{R} , \tag{32} \]
and
\[ m'_1 \leq b'(s) \leq M'_1 \left( 1 + sb(s) \right) \quad \text{for all} \quad s \in \mathbb{R} , \tag{33} \]
for some constants $m'$, $M' > 0$ and $m'_1$, $M'_1 > 0$. Moreover, one can see that the inequality on the right hand side of (32) holds true if we assume that
\[ m' < g'(s) \leq M' \left( 1 + |s|^{q-1} \right) \quad \text{and} \quad sg(s) \geq m'' |s|^{(p-1)/\sigma} - M'' \quad \text{for all} \quad s \in \mathbb{R} , \tag{34} \]
for some $m'' > 0$. We also note that the second requirement in (34) follows from the first one if $1 \leq p \leq 1 + 2\sigma$. The same is true in the case of (33). In addition, the condition (33) allows also an exponential behavior for $b(s)$, e.g., $|b(s)| \sim e^{\alpha|s|}$ as $|s| \to \infty$ for some $\alpha > 0$.

Our second main result is the following theorem.

**Theorem 3.4.** Let Assumption 3.3 hold. Then the compact global attractor $\mathfrak{A}$ given by Theorem 3.2 possesses the following properties:
1. The attractor $A$ has a finite fractal dimension.
2. The attractor $A$ is a bounded set in the space
   
   $$Y_* = W^{2}_{6/p}(\Omega) \times \mathcal{D}(A^{1/2}) \times \mathcal{D}(A) \times \mathcal{D}(A^{1/2})$$

   in the case $n = 3$ and $3 < p \leq 5$, where $W^{2}_{6/p}(\Omega)$ is the $L_{6/p}$-based second order Sobolev space, and in the space
   
   $$Y_{**} = H^{2}(\Omega) \times \mathcal{D}(A^{1/2}) \times \mathcal{D}(A) \times \mathcal{D}(A^{1/2})$$

   in other cases.
3. There exists a fractal exponential attractor $A_{exp}$ for $(Y, S_t)$ (whose dimension is finite in some extended space) provided that $b(s)$ is polynomially bounded at infinity.

   The concept of fractal exponential attractor has been introduced in [23]. Let us recall from [23] that a compact set $A_{exp} \subset X$ is said to be an inertial set (or a fractal exponential attractor) for a dynamical system $(X, S_t)$ iff (i) $A_{exp}$ is a positively invariant set of finite fractal dimension and (ii) for every bounded set $D \subset X$ there exist positive constants $C_D$ and $\gamma_D$ such that
   
   $$d_X\{S_t D \mid A_{exp}\} \equiv \sup_{x \in D} \text{dist}_X(S_t x, A_{exp}) \leq C_D e^{-\gamma_D t}, \quad t \geq 0.$$  

   Fractal exponential attractors have been intensively studied by many authors. We refer to [24] for a recent survey with focus on second order parabolic equations in the case of linear damping, and to [19, 20] for results for nonlinearly damped models.

   In the case $\kappa = 0$ the above theorems result in the following two assertions.

   **Corollary 3.5.** Suppose that $f$ and $g$ satisfy the conditions in Assumption 3.1. Then the dynamical system $(Y_1, S_t^1)$ generated by problem
   
   $$\begin{cases}
   z_{tt} + g(z_t) - \Delta z + f(z) = 0 \quad \text{in } \Omega \times (0, T) \\
   \frac{\partial z}{\partial \nu} = 0 \quad \text{on } \Gamma \times (0, T)
   \end{cases} \tag{35}$$

   possesses a compact global attractor $A_1 \equiv M^u(N^1_1)$, where $N^1_1$ is the set of equilibria for (35). If $f$ and $g$ satisfy the conditions in Assumption 3.3, then (i) the attractor $A_1$ has a finite fractal dimension; (ii) $A_1$ is a bounded set in the space $W^{2}_{6/p}(\Omega) \times \mathcal{D}(A^{1/2})$ in the case $n = 3$ and $3 < p \leq 5$, and in the space $\mathcal{D}(A) \times \mathcal{D}(A^{1/2})$ in other cases; moreover (iii) there exists a fractal exponential attractor $A_{1 exp}^1$ for $(Y_1, S_t^1)$.

   In contrast to most results available in the literature (see, e.g., [18], [19, Chap. 5], [39] and the references therein) our Corollary 3.5 deals with the wave dynamics subject to Neumann boundary conditions and—which is more important—does not require differentiability and strict monotonicity of the damping $g$ for the existence of a global attractor. Furthermore, the present result does not require either subcriticality (unlike [39]) or superlinearity (unlike [19, Chap. 5]) of the damping function $g$. Since the same method can be also applied in the case of Dirichlet boundary conditions, we achieve as well a generalization of the results given in [19, Chap. 5] and [39]. We finally stress that, in contrast to [18] and [19, Chap. 5], our result on the existence of a fractal exponential attractor requires neither additional geometrical assumptions concerning the set $N^1_1$ of equilibria, nor large damping hypothesis.

   In the case of Berger’s model, Theorems 3.2 and 3.4 yield the following result.
Corollary 3.6. Suppose that $b$ and $p_0$ satisfy the conditions in Assumption 2.1 and the inequality (27). Then the dynamical system $(Y_2, S^2_t)$ generated by problem
\[
\begin{cases}
    v_{tt} + b(v_t) + \Delta^2 v + \left[Q - \int_0^T |\nabla v(x, t)|^2 dx \right] \Delta v = p_0 & \text{on } \Gamma_0 \times (0, T) \\
    v = \Delta v = 0 & \text{on } \partial \Gamma_0 \times (0, T)
\end{cases}
\] (36)
possesses a compact global attractor $\mathfrak{A}_2 \equiv M^u(\mathcal{N}_2)$, where $\mathcal{N}_2$ is the set of equilibria for (36). If, in addition, condition (31) holds true, then the attractor $\mathfrak{A}_2$ is a bounded set in $\mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}^{1/2})$ and has a finite fractal dimension. Moreover, the system $(Y_2, S^2_t)$ possesses a fractal exponential attractor $\mathfrak{A}^{\text{exp}}$ provided that $b(s)$ is polynomially bounded at infinity.

In the case when $b \in C^1(\mathbb{R})$, the existence of a global attractor for (36) was established in [19, Chap. 7] under the additional hypotheses that $b(s)$ is strictly increasing and $b'(s) \geq m > 0$ for $|s| \geq 1$. As for dimension and smoothness of the attractor, in the special case when $b \in C^3(\mathbb{R})$, Corollary 3.6 requires the same hypotheses as in [20, Chap. 7]. Instead, the issue of existence of a fractal exponential attractor for Berger’s model with nonlinear damping had not been explored before.

4. Main inequality. The key ingredient for the proofs of both Theorems 3.2 and 3.4 is the following Proposition.

Proposition 4.1. Let Assumption 2.1 hold. Assume that $y_1, y_2 \in \mathcal{W}_R$ for some $R > R_*$, where $\mathcal{W}_R$ is defined by (25) and denote
\[
(h(t), h_t(t), u(t), u_t(t)) := S_t y_1, \quad (\zeta(t), \zeta_t(t), w(t), w_t(t)) := S_t y_2.
\]
Let $z(t) := h(t) - \zeta(t)$ and $v(t) := u(t) - w(t)$. Then, there exists $T_0 > 0$ and positive constants $c_0$, $c_1$ and $c_2(R)$ independent of $T$ such that for every $T \geq T_0$
\[
TE^0(T) + \int_0^T E^0(t) dt \leq c_0 \left[ \int_0^T \left( \|z_t\|^2 + \|v_t\|^2 \right) dt + \beta G^T_0(z) + \alpha G^T_0(v) \right]
\]
\[+ c_1 \left[ H^T_0(z) + H^T_0(v) + \Psi_T(z, v) \right] + c_2(R) \int_0^T (\|z\|^2 + \|v\|^2) dt, \tag{37}
\]
where $E^0(t) = \beta E^0_0(t) + \alpha E^v(t)$ with $E^0_0(t)$ and $E^v_0(t)$ given by (15). We also used the notations
\[
G^T_s(z) = \int_s^t \left( D(\zeta_t + z_t) - D(\zeta_t), z_t \right)_\Omega d\tau, \quad G^T_s(v) = \int_s^t \left( B(w_t + v_t) - B(w_t), v_t \right)_{\Gamma_0} d\tau,
\]
\[
H^T_s(z) = \int_s^t \left| \left( D(\zeta_t + z_t) - D(\zeta_t), z_t \right)_\Omega \right| d\tau, \quad H^T_s(v) = \int_s^t \left| \left( B(w_t + v_t) - B(w_t), v_t \right)_{\Gamma_0} \right| d\tau \tag{38}
\]
and
\[
\Psi_T(z, v) = \left| \int_0^T (\mathcal{F}_1(z), z_t) dt \right| + \left| \int_0^T \int_t^T (\mathcal{F}_1(z), z_t) d\tau dt \right|
\]
\[+ \left| \int_0^T (\mathcal{F}_2(v), v_t) dt \right| + \left| \int_0^T \int_t^T (\mathcal{F}_2(v), v_t) d\tau dt \right| \tag{40}
\]
with
\[
\mathcal{F}_1(z) = F_1(\zeta + z) - F_1(\zeta) \quad \text{and} \quad \mathcal{F}_2(v) = F_2(w + v) - F_2(w). \tag{41}
\]
As we shall see, the inequality (37) established in Proposition 4.1 provides a common first step for the proof of all the statements of Theorem 3.2 and Theorem 3.4. This inequality represents equipartition of the energy: the potential energy is reconstructed from the kinetic energy and the nonlinear quantities entering the equation. Eventually, these quantities will need to be absorbed (‘modulo’ lower order terms) by the dampings. The realization of this step depends heavily on the assumptions imposed on the model, and hence the argument used in the proof of compactness will be different from the one given for the proof of finite-dimensionality and/or regularity.

Proof. Step 1 (Energy identity). Without loss of generality, we can assume that \((h(t), u(t))\) and \((\zeta(t), w(t))\) are strong solutions. By the invariance of \(W_R\) and in view of relation (18) there exists a constant \(C_R > 0\) such that

\[
E_h^0(h(t), h_t(t)) + E^0_\zeta(\zeta(t), \zeta_t(t)) + E^0_u(u(t), u_t(t)) + E^0_w(w(t), w_t(t)) \leq C_R, \quad t \geq 0,
\]

where \(E^0_\zeta(t)\) and \(E^0_u(t)\) denote the corresponding (free) energies as in (15).

We establish first an energy type equality regarding \((0)\). Next, by standard energy methods we obtain

\[
E^0(T) + \beta G^T(t) + \alpha G^T_\zeta(v) = E^0(t) - \beta \int_t^T (F_1(z), z_t) d\tau - \alpha \int_t^T (F_2(v), v_t) d\tau, \quad (43)
\]

where \(G^T_t(z)\) and \(G^T_\zeta(v)\) are given by (38), while \(F_1(z)\) and \(F_2(z)\) are defined by (41).

Proof. Writing down the equations satisfied by \(h = \zeta + z, u = w + v, \zeta\) and \(w\) (see (7)), it is elementary to derive the following system of coupled equations:

\[
\begin{align*}
\dot{z} + Az - \alpha \kappa N_0 v_t + D(\zeta + z_t) - D(\zeta_t) + F_1(z) &= 0, \quad (44a) \\
\dot{v} + Av + B(w_t + v_t) - B(v_t) + \beta \kappa N_0^* Az_t + F_2(v) &= 0, \quad (44b)
\end{align*}
\]

with \(F_i\) defined in (41). Next, by standard energy methods we obtain

\[
E^0_\zeta(T) + G^T_t(z) = E^0(T) + \alpha \int_t^T (AN_0 v_t, z_t) d\tau - \int_t^T (F_1(z), z_t) d\tau; \quad (45)
\]

\[
E^0_\zeta(T) + G^T_\zeta(v) = E^0_\zeta(T) - \beta \kappa \int_t^T (N_0^* Az_t, v_t) d\tau - \int_t^T (F_2(v), v_t) d\tau. \quad (46)
\]

Combining (45) with (46) we immediately see that (43) holds true. \(\square\)

Step 2. (Reconstruction of the energy integral) We return to the coupled system (44) satisfied by \((z, v)\). We multiply equation (44a) by \(z\), and integrate between 0 and \(T\), thereby obtaining

\[
\int_0^T \|A^{1/2}z\|^2 dt \leq c_0 \left( E^0_\zeta(T) + E^0_\zeta(0) \right) + \int_0^T \|z_t\|^2 dt
\]

\[
+ H^T_t(z) + \alpha \int_0^T \|v_t, N_0^* Az\| dt + \int_0^T \|z_t, \zeta_t\| dt,
\]

where \(H^T_t(z)\) is defined in (39). It is clear from (13) and (42) that

\[
\|\zeta_t\| \leq C_R \|A^{1/2}z\| \|z\|.
\]
Using (5) with $\varepsilon = 1/4$ we have that
\[
|\langle v_t, N^*_0 A z \rangle| \leq \|v_t\|_{\Gamma_0} \|N^*_0 A^{1/2}\| \|A^{1/2} z\|_\Omega \leq C \|v_t\|_{\Gamma_0} \|A^{1/2} z\|_\Omega,
\]
so that
\[
\int_0^T \|A^{1/2} z\|^2 dt \leq C_0 \left( E^0_v(T) + E^0_v(0) \right) + C_1 \int_0^T \left( \|z_t\|^2 + \|v_t\|_{\Gamma_0}^2 \right) dt
\]
\[+ C_2 H^T_0(z) + C_3(R) \int_0^T \|z\|^2 dt. \tag{47}\]
Regarding the plate component, by using the bounds in (13) and (42) we obtain
\[
|\langle F_2(v), v \rangle| \leq C_R \|A^{1/2} v\| \|v\|,
\]
so that
\[
\int_0^T \|A^{1/2} v\|^2 dt \leq c_0 \left( E^0_v(T) + E^0_v(0) \right) + 2 \int_0^T \|v_t\|^2 dt + 2H^T_0(v)
\]
\[+ 2\beta \varepsilon \int_0^T (z_t, v)_{\Gamma_0} dt + C(R) \int_0^T \|v\|^2 dt, \tag{48}\]
where $H^T_0(v)$ is defined in (39). Integrating by parts in time and using the standard form of the trace theorem it is easy to see that
\[
\left| \int_0^T (z_t, v)_{\Gamma_0} dt \right| \leq C_1 \left( E^0_v(T) + E^0_v(0) \right) + \varepsilon \int_0^T \|A^{1/2} z\|^2 dt + C_2 \int_0^T \|v_t\|_{\Gamma_0}^2 dt \tag{49}\]
for every $\varepsilon > 0$. Consequently, summing up (47) with (48) and using (49), we get
\[
\int_0^T E^0(t) dt \leq c_0 \left( E^0_v(T) + E^0_v(0) \right) + c_1 \int_0^T \left( \|z_t\|^2 + \|v_t\|^2 \right) dt
\]
\[+ c_2 \left[ H^T_0(z) + H^T_0(v) \right] + c_3(R) \int_0^T \left( \|z\|^2 + \|v\|^2 \right) dt. \tag{50}\]
On the other hand, it follows from Lemma 4.2 that
\[
E^0(0) = E^0(T) + \beta G^T_0(z) + \alpha G^T_0(v) + \beta \int_0^T (F_1(z), z_t) d\tau + \alpha \int_0^T (F_2(v), v_t) d\tau, \tag{51}\]
and
\[
TE^0(T) \leq \int_0^T E^0(t) dt - \beta \int_0^T \int_t^T (F_1(z), z_t) d\tau dt - \alpha \int_0^T \int_t^T (F_2(v), v_t) d\tau dt. \tag{52}\]
Therefore, combining (52) with (50) and (51), it is readily shown that (37) holds true, provided that $T$ is sufficiently large. This concludes the proof of Proposition 4.1.

5. **Asymptotic smoothness and proof of Theorem 3.2.** In this section we will show that the semi-flow $S_t$ generated by the PDE system (1) is asymptotically smooth. This property is critical for proving existence of global attractors (see, e.g., [4, 15, 27, 40]). We recall (see, e.g., [27]) that a dynamical system $(X, S_t)$ is said to be asymptotically smooth if for any bounded set $B$ in $X$ such that $S_t B \subset B$ for $t > 0$ there exists a compact set $K$ in the closure $\overline{B}$ of $B$, such that
\[
\lim_{t \to +\infty} \sup_{y \in B} \text{dist}_X \{S_t y, K\} = 0.
\]
Our main result in this section is the following assertion.

**Theorem 5.1.** Let Assumption 3.1 hold. Then the dynamical system \( (Y, S) \) generated by the PDE problem (1) is asymptotically smooth.

In order to prove Theorem 5.1 we shall invoke a compactness criterion due to [29, Thm. 2], which is recalled below in an abstract version formulated and used in [21] (see also [20, Chap. 2]).

**Proposition 5.2** ([21]). Let \( (X, S) \) be a dynamical system on a complete metric space \( X \) endowed with a metric \( d \). Assume that for any bounded positively invariant set \( B \) in \( X \) and for any \( \varepsilon > 0 \) there exists \( T = T(\varepsilon, B) \) such that

\[
d(S_T y_1, S_T y_2) \leq \varepsilon + \Psi_{\varepsilon, B, T}(y_1, y_2), \quad y_1 \in B,
\]

where \( \Psi_{\varepsilon, B, T}(y_1, y_2) \) is a function defined on \( B \times B \) such that

\[
\lim_{m \to \infty} \liminf_{n \to \infty} \Psi_{\varepsilon, B, T}(y_n, y_m) = 0 \tag{53}
\]

for every sequence \( \{y_n\}_n \) in \( B \). Then \( (X, S) \) is an asymptotically smooth dynamical system.

In the proof of Theorem 5.1 we shall make use of further inequalities which are a standard tool for proving the absorption property; see [20].

**Lemma 5.3.** Under Assumption 2.1 there exist constants \( 0 < \delta < 1/8 \) and \( C_0 > 0 \) such that

\[
|\langle D(\xi) - D(\eta), h \rangle| \leq C_0 \left[ |\langle D(\xi), \eta \rangle + \langle D(\eta), \xi \rangle| \right] \|A^{1/2} h\| + C_0 \|h\| \tag{54}
\]

for any \( \xi, \eta \in \mathcal{D}(A^{1/2}) \), and

\[
|\langle B(w + u) - B(w), v \rangle| \leq C_0 \left( 1 + |\langle B(w), w \rangle + |\langle B(w + u), w + u \rangle| \right) \|A^{1/2 - \delta} v\| \tag{55}
\]

for any \( w, u, v \in \mathcal{D}(A^{1/2}) \).

**Proof.** The inequality in (54) can be easily derived by using similar arguments as in [20, Chap. 5]. However, for the sake of completeness we sketch the proof in the case \( n = 3 \). We obviously have that

\[
\int_\Omega |D(\xi) - D(\eta)| |h| \, dx \leq \int_\Omega |g(\xi + z)| |h| \, dx + \int_\Omega |g(\xi)| |h| \, dx.
\]

Let \( \Omega_1 = \{ x \in \Omega : |\xi(x)| \geq 1 \} \) and \( \Omega_2 = \Omega \setminus \Omega_1 \). Then,

\[
\int_\Omega |g(\xi)| |h| \, dx \leq \int_{\Omega_1} |g(\xi)| |h| \, dx + C \int_{\Omega_2} |h| \, dx \\
\leq \left[ \int_{\Omega_1} |g(\xi)|^{6/5} \, dx \right]^{5/6} \|h\|_{L_6(\Omega)} + C \|h\|.
\]

Next, by (2) it follows that \( |g(\xi)|^{6/5} = |g(\xi)| |g(\xi)|^{1/5} \leq C \xi g(\xi) \) on \( \Omega_1 \). Therefore, using the embedding \( H^{1}(\Omega) \subset L_6(\Omega) \) and the fact that \( |g(\xi)| \geq \xi_0 > 0 \) on \( \Omega_1 \), we obtain

\[
\int_\Omega |g(\xi)| |h| \, dx \leq C \|A^{1/2} h\| \int_\Omega \xi g(\xi) \, dx + C \|h\| \leq C \left[ (D(\xi), \xi) \|A^{1/2} h\| + \|h\| \right].
\]

With similar computations for \( \int_\Omega |g(\xi + z)| |h| \, dx \), we readily obtain (54).

The second statement follows trivially from the Sobolev’s embedding

\[
\mathcal{D}(A^{1/2 - \delta}) \subset H^{2 - 4\delta}(\Gamma_0) \subset C(\overline{\Gamma_0}), \quad \delta < 1/8,
\]

for any \( \xi \in \mathcal{D}(A^{1/2 - \delta}) \).
and the obvious inequality

\[ |(B(w), v)| \leq C|v|_{C([0,T])} \left( 1 + \int_0^T b(w) \, dx \right) ; \]

for further details we refer to \([18, 19]\)).

**Proof of Theorem 5.1.** Since any bounded positively invariant set belongs to \(\mathcal{W}_R\) for some \(R > R_*\), where \(\mathcal{W}_R\) is defined by (25), it is sufficient to consider the case \(B = \mathcal{W}_R\) for every \(R > R_*\) only.

Let \(y_1, y_2 \in \mathcal{W}_R\). Below we use the same notations as in Proposition 4.1. Namely, we denote the solutions corresponding to initial data \(y_1\) and \(y_2\), respectively, by

\[(h(t), h_1(t), u(t), u_1(t)) := S_t y_1, \quad (\zeta(t), \zeta_1(t), w(t), w_1(t)) := S_t y_2, \]

and set \(z(t) := h(t) - \zeta(t)\) and \(v(t) := u(t) - w(t)\).

Now we establish the key estimate for the proof of Theorem 5.1.

**Proposition 5.4.** Let the assumptions of Theorem 5.1 be in force. Then, given \(\varepsilon > 0\) and \(T > T_0\) there exists constants \(C_\varepsilon(R)\) and \(C(R,T)\) such that

\[ E^0(T) \leq \varepsilon + \frac{1}{T} \left[ C_\varepsilon(R) + \Psi_T(z, v) \right] + C(R,T) \text{lot}(z, v), \]

where \(E^0(t) := \beta E^0_0(t) + \alpha E^0_0(t)\) with \(E^0_0(t)\) and \(E^0(t)\) given in (15), the functional \(\Psi_T(z, v)\) is given by (40) while \(\text{lot}(z, v)\) is defined by

\[ \text{lot}(z, v) := \sup_{[0,T]} \|A^{1/2 - \delta} z(t)\| + \sup_{[0,T]} \|A^{1/2 - \delta} v(t)\|, \quad \text{for some} \quad 0 < \delta \leq \frac{1}{2}. \]

**Proof.** It follows from the energy inequality (19) that

\[ \beta \int_0^T (D(h_t), h_t) \, dt + \alpha \int_0^T (B(u_t), u_t) \, dt \leq C_R, \tag{57a} \]

\[ \beta \int_0^T (D(\zeta_t), \zeta_t) \, dt + \alpha \int_0^T (B(w_t), w_t) \, dt \leq C_R, \tag{57b} \]

where crucially \(C_R\) does not depend on \(t\).

Let \(H^0_t(z)\) and \(H^0_t(v)\) be given by (39). By using Lemma 5.3, the estimates (57) and the fact that \(\|A^{1/2} z\| \leq C_R\), we obtain

\[ H^0_t(z) \leq C_R + C \text{lot}(z, v) \quad \text{and} \quad H^0_t(v) \leq (1 + T) C_R \text{lot}(z, v). \tag{58} \]

Using now the inequalities in Assumption 3.1 and once again the uniform estimates (57), one can see that

\[ \int_0^T \left( \|z_t\|^2 + \|v_t\|^2 \right) \, dt \leq \varepsilon T + C_\varepsilon(R) \quad \text{for every} \quad \varepsilon > 0. \tag{59} \]

Taking first \(t = 0\) in (43) and using the fact that \(E^0(0) \leq C_R\), we get

\[ \beta G^T_0(z) + \alpha \varphi^T_0(v) \leq C_R + \beta \int_0^T (F_1(z), z_t) \, d\tau \quad + \alpha \int_0^T (F_2(v), v_t) \, d\tau. \tag{60} \]

Therefore, (56) follows from Proposition 4.1 and the estimates (58), (59) and (60). \(\square\)
We are now in a position to complete the proof of Theorem 5.1. It follows from Proposition 5.4 that given \( \varepsilon > 0 \) there exists \( T = T(\varepsilon) > 1 \) such that for initial data \( y_1, y_2 \in B \) we have

\[
\|S_T y_1 - S_T y_2\|_Y = \|(z(T), z_t(T), v(T), v_t(T))\|_Y \\
\leq C|E^0(T)|^{1/2} \leq \varepsilon + \Psi_{\varepsilon,B,T}(y_1, y_2),
\]

with

\[
\Psi_{\varepsilon,B,T}(y_1, y_2) = C_{B,\varepsilon,T} \{ \Psi_{T}(z, v) + \text{lot}(z, v) \}^{1/2},
\]

(61)

where \( \Psi_{T}(z, v) \) is given by (40). Thus, in order to establish asymptotic smoothness for the dynamical system under investigation, we aim to invoke Proposition 5.2, which will allow us to conclude the proof of Theorem 5.1. Hence, what we need to prove is the validity of the sequential limits (53) for \( \Psi_{\varepsilon,B,T} \) defined by (61). To do that, we shall use similar arguments as in the completion of the proof of Theorem 3.1 in [21]. Some computations are given for the reader’s convenience.

Let \( (h^n, u^n)_n \) be a sequence of solutions to the PDE system (1) corresponding to initial data \( y^{0,n} := (h^{0,n}, h^{1,n}, u^{0,n}, u^{1,n}) \) in \( W_R \subset Y \). Since the compactness condition in (53) deals with lower limits, it is sufficient to establish (53) for some subsequence of \( (y^{0,n})_n \). Therefore we can assume that

\[
y^n(t) := (h^n, h^n_t, u^n, u^n_t) \rightarrow (h, h_t, u, u_t) := y(t) \quad \text{weakly}^* \text{ in } L_\infty(0,T;Y)
\]

(62)

for some solution \( (h, h_t, u, u_t) \in L_\infty(0,T;Y) \); in addition, by Aubin’s Lemma,

\[
\sup_{[0,T]} \{ ||z^{n,m}(t)||_{1-\delta,\Omega} + ||v^{n,m}(t)||_{2-\eta,\Gamma} \} \rightarrow 0,
\]

(63)

as \( n,m \rightarrow \infty \), for some \( \delta, \eta > 0 \), where we have set \( z^{n,m}(t) = h^n(t) - h^m(t) \) and \( v^{n,m}(t) = u^n(t) - u^m(t) \). The above convergence statement implies that \( \text{lot}(z^{n,m}, v^{n,m}) \rightarrow 0 \). Therefore, in view of (61), we must show that

\[
\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \Psi_T(z^{n,m}, v^{n,m}) = 0.
\]

(64)

Let us begin with the analysis of the integral \( \int_t^T (F_1(z^{n,m}), z_t^{n,m}) \, d\tau \), which explicitly reads as follows:

\[
\int_t^T (F_1(z^{n,m}), z_t^{n,m}) \, d\tau = \int_t^T (F_1(h^n(\tau)) - F_1(h^m(\tau)), h^n_t(\tau) - h^m_t(\tau)) \, d\tau \\
= \int_t^T (\Pi(h^n(\tau)), h^n_t(\tau)) \, d\tau - \int_t^T (F_1(h^n(\tau)), h^n_t(\tau)) \, d\tau \\
- \int_t^T (F_1(h^m(\tau)), h^m_t(\tau)) \, d\tau + \int_t^T (\Pi(h^m(\tau)), h^m_t(\tau)) \, d\tau.
\]

This implies that

\[
\int_t^T (F_1(z^{n,m}), z_t^{n,m}) \, d\tau = \underbrace{\Pi(h^n(T)) - \Pi(h^m(T))}_{A_{n,m}} + \underbrace{\Pi(h^n(T)) - \Pi(h^m(T))}_{B_{n,m}} \\
- \underbrace{\int_t^T (F_1(h^n(\tau)), h^n_t(\tau)) \, d\tau - \int_t^T (F_1(h^m(\tau)), h^m_t(\tau)) \, d\tau}_{C_{n,m}, D_{n,m}}.
\]

(65)
Let us observe that by (3) the Nemytski operator
\[ z(x) \mapsto \int_0^{z(x)} (f(\xi) - \mu \xi) \, d\xi \]
is continuous from \( L_{q+2}^r(\Omega) \) into \( L_1(\Omega) \). Therefore, since \( H^{1-\delta}(\Omega) \subset L_{q+2}^r(\Omega) \) for some \( \delta > 0 \), the potential \( \Pi \) defined in (8) is continuous on \( \mathcal{D}(A^{(1-\delta)/2}) \). Consequently, (63) implies that
\[ \Pi(h^n(T)) \to \Pi(h(T)), \quad \Pi(h^n(t)) \to \Pi(h(t)) \quad \text{as} \quad n \to \infty, \]
so that
\[ \lim_{n \to \infty} \lim_{m \to \infty} (A_{n,m} + B_{n,m}) = 2 \left[ \Pi(h(T)) - \Pi(h(t)) \right]. \tag{66} \]
In order to estimate the term \( C_{n,m} \) in (65) we first note that by (13)
\[ \sup_{t \in [0,T]} \| F_1(h^n(t)) \| \leq C \quad \text{for all} \quad n, \]
hence
\[ F_1(h^n) \to F_0 \quad \text{weakly}^* \quad \text{in} \quad L_{\infty}(0,T;L_2(\Omega)). \tag{67} \]
From (62) and (63) we also have that \( h^n \to h \) strongly in \( L_2(0,T;L_2(\Omega)) \), and hence almost everywhere (along a subsequence). Since \( F_1 \) is continuous, we also have that \( F_1(h^n) \to F_1(h) \), a.e. in \( (0,T) \times \Omega \). This allows to recover the limit in (67) and claim that \( F_0 = F_1(h) \). Thus, letting \( m \to \infty \) first and \( n \to \infty \) next, we obtain for term \( C_{n,m} \):
\[ \lim_{n \to \infty} \lim_{m \to \infty} C_{n,m} = \lim_{n \to \infty} \lim_{m \to \infty} \int_0^T (F_1(h^n(\tau)), h^n_i(\tau)) \, d\tau \]
\[ = \lim_{n \to \infty} \int_0^T (F_1(h^n(\tau)), h_i(\tau)) \, d\tau \]
\[ = \int_0^T (F_1(h(\tau)), h_i(\tau)) \, d\tau = \Pi(h(T)) - \Pi(h(t)). \tag{68} \]
The same argument applies to the term \( D_{n,m} \) giving
\[ \lim_{n \to \infty} \lim_{m \to \infty} D_{n,m} = \Pi(h(T)) - \Pi(h(t)). \tag{69} \]
Combining (66) with (68), (69), we see from (65) that
\[ \lim_{n \to \infty} \lim_{m \to \infty} \int_0^T (F_1(z^{n,m}_t), z^{n,m}_t) \, d\tau = 0. \tag{70} \]
Similar computations performed on \( \int_0^T (F_2(v^{n,m}_t), v^{n,m}_t) \, d\tau \) yield, as well,
\[ \lim_{n \to \infty} \lim_{m \to \infty} \int_0^T (F_2(v^{n,m}_t), v^{n,m}_t) \, d\tau = 0. \tag{71} \]
Recalling (40), and taking into account (70) and (71) we finally conclude that (64) holds true, as desired. Thus Theorem 5.1 is proved.

**Proof of Theorem 3.2.** Since \( \mathcal{W}_R \) is bounded and positively invariant, Theorem 5.1 implies the existence of a compact global attractor \( \mathfrak{A}_R \) for the dynamical system \( (\mathcal{W}_R, S_t) \), for each \( R \geq R_* \). We choose now \( R_0 \geq R_* + 1 \) such that the set \( \mathcal{N} \) of equilibria lies in \( \mathcal{W}_{R_0-1} \). It follows from (28) that \( s_q(s) > 0 \) and \( s_b(s) > 0 \) for \( s \neq 0 \). Therefore the energy inequality (19) implies that the energy \( E \) given by (16) is a strict Lyapunov function for \( (\mathcal{W}_R, S_t) \). This, in turn, implies (see, e.g., [4, Theorem 3.2.1] or [15, Theorem 1.6.1]) that \( \mathfrak{A}_R = M^s(\mathcal{N}) \) and by Lemma 2.4 \( \mathfrak{A}_R \)
does not depend on \( R \) for \( R \geq R_0 \); moreover, relation (29) holds. Thus Theorem 3.2 is proved.

6. Stabilizability estimate. In this section we aim to develop some analytic tool which will enable us to apply the abstract results presented in [18] and [19, 20] for the proof of Theorem 3.4. The crucial analytic tool will be a suitable so-called stabilizability estimate.

Let us preliminarily record two key estimates, whose proof can be found in [18, 19].

**Lemma 6.1 ([18, 19]).** Under Assumptions 3.3, the following statements are valid.

1. For every \( \varepsilon > 0 \) there exists \( C_\varepsilon > 0 \) such that
   \[
   |(D(\zeta + z) - D(\zeta), h)| \leq C_\varepsilon \left| (D(\zeta + z) - D(\zeta), \zeta + z) \right| A^{1/2} h^2
   \]
   for any \( \zeta, z, h \in D(A^{1/2}) \).

2. There exist a parameter \( \delta > 0 \) and constants \( C_1, C_2 > 0 \) such that
   \[
   |(B(w + v) - B(w), u)| \leq C_1 \left| (B(w + v) - B(w), v) + C_2 \left( 1 + (B(w), w) + (B(w + v), w + v) \right) \right| A^{1/2 - \delta} u^2
   \]
   for any \( u, v, w \in D(A^{1/2}) \).

**Proof.** We use (30) and apply the same argument as in [18] (or in [19, Chap. 5]) to obtain (72). Similarly, we can establish (73); see, e.g., [19, Chap. 6] for details.

The following Theorem provides a “stabilizability” inequality, which plays a key role in the proofs of both finite-dimensionality and regularity of attractors. This inequality shows that the difference of two trajectories can be exponentially stabilized to a compact (smooth) set.

**Theorem 6.2 (Stabilizability estimate).** Let Assumption 3.3 hold. Then there exist positive constants \( C_1, C_2 \) and \( \omega \) depending on \( R \) such that for any \( y_1, y_2 \in \mathcal{W}_R \) the following estimate holds true:

\[
\| S_t y_1 - S_t y_2 \|^2 \leq C_1 e^{-\omega t} \| y_1 - y_2 \|^2 + C_2 \text{lot}_t(h - \zeta, u - w), \quad t > 0,
\]

where

\[
\text{lot}_t(z, v) = \max_{[0, \delta]} \left( \| z(\tau) \|_{1-\delta}^2 + \| v(\tau) \|_{2-\delta}^2 \right).
\]

Above, we have used the notation

\[
(h(t), h(t), u(t), u(t)) := S_t y_1, \quad (\zeta(t), \zeta(t), w(t), w(t)) := S_t y_2.
\]

**Proof.** Let \( y_1, y_2 \in \mathcal{W}_R \) be given. We consider the solutions \( S_t y_1 \) and \( S_t y_2 \) and introduce, as previously in the paper, \( z = h - \zeta, v = u - w \). We recall that for these solutions the bounds (42) and (57) hold true.

We begin with the following critical assertion.
Lemma 6.3. Under Assumption 3.3, the following estimates hold true for some $\delta > 0$:

\[
\left| \int_t^T (F_1(z), z_t) \, dt \right| \leq C_{R,T} \max_{[0,T]} \|z\|^2_{1-\delta,\Omega} + \varepsilon \int_0^T \|A^{1/2}z\|^2_{\Omega} \, dt \\
+ C_\varepsilon (R) \int_0^T (\|h_t\|^2_{\Omega} + \|\zeta_t\|^2_{\Omega}) \|A^{1/2}z\|^2_{\Omega} \, dt, \quad (76a)
\]

\[
\left| \int_t^T (F_2(\nu), \nu_t) \, dt \right| \leq \varepsilon \int_0^T \|\nu_t\|^2 \, dt + C_\varepsilon (R, T) \max_{[0,T]} \|\nu\|^2_{2-\delta,\Gamma_0}, \quad (76b)
\]

for all $t \in [0,T]$, where $\varepsilon > 0$ can be taken arbitrarily small. Here, $F_1$ and $F_2$ are given by (41).

The idea behind the estimates in Lemma 6.3 is to exhibit explicitly the kinetic energy $\|h_t\|^2 + \|\zeta_t\|^2$, which is $L_1(\mathbb{R})$ (see (81) below). As such, this function may play a role of a small parameter for large $|t|$. As we shall see, this small “parameter” will allow to dispense with unnecessary assumptions such as subcritical damping parameter $\rho$, or large damping parameter in front of $g(z_t)$ (assumptions used in the previous treatments; see [18] and the references therein).

Proof. Let us prove the first estimate. We restrict ourselves to the case $n = 3$ (the case $n = 2$ is much simpler). One can see that

\[
\int_t^T (F_1(z), z_t) \, dt = \frac{1}{2} \int_{\Omega} \int_0^1 f'(\zeta + \lambda z) \, d\lambda |z|^2 \, dx \bigg|_t^T - \frac{\mu}{2} \|z(t)\|^2_{\Omega} - \frac{1}{2} \int_t^T \int_0^1 f''(\zeta + \lambda z) \cdot (\zeta_t + \lambda \zeta_t) \, d\lambda |z|^2 \, dx \, dt. \quad (77)
\]

Using the embedding $H^1(\Omega) \subset L_6(\Omega)$ and $H^{1/2}(\Omega) \subset L_3(\Omega)$, we obtain that

\[
\left| \int_0^1 f'(\zeta + \lambda z) \, d\lambda |z|^2 \, dx \right| \leq C \int_0^1 (1 + |z|^2 + |\zeta|^2) |z|^2 \, dx \\
\leq C(1 + \|\nu\|^2_{1,\Omega} + \|\zeta\|^2_{1,\Omega}) \|z\|^2_{L^2(\Omega)} \leq C_{R} \|z\|^2_{1/2,\Omega}. \quad (78)
\]

The second term on the right hand side of (77) can be estimated as follows:

\[
\left| \int_0^T \int_0^1 f''(\zeta + \lambda z) \cdot (\zeta_t + \lambda \zeta_t) \, d\lambda |z|^2 \, dx \, dt \right| \\
\leq C \int_0^T \left[ \int_\Omega (|\zeta_t| + |\zeta|)^2 \, dx \right]^{1/2} \left[ \int_\Omega (1 + |z| + |\zeta|)^2 |z|^4 \, dx \right]^{1/2} \, dt, \\
\leq C \int_0^T \left[ \int_\Omega (|\zeta_t| + |\zeta|)^2 \, dx \right]^{1/2} \left[ \int_\Omega (1 + |z| + |\zeta|)^6 \, dx \right]^{1/6} \|z\|^2_{1,\Omega}.
\]

Therefore

\[
\left| \int_0^T \int_\Omega f''(\zeta + \lambda z) \cdot (\zeta_t + \lambda \zeta_t) \, d\lambda |z|^2 \, dx \, dt \right| \\
\leq C_{R} \int_0^T (\|\zeta\|_{\Omega} + \|\zeta\|_{\Omega}) \|A^{1/2}z\|^2_{\Omega} \, dt, \quad (79)
\]

Thus, using (78) and (79) in (77), we finally obtain (76a).
Regarding the latter estimate (76b), one can readily check that
\[
\int_t^T \langle F_2(v), v_t \rangle \, dt = \frac{1}{2} \left( -Q + \|A^{1/4}w\|^2 \right) \|A^{1/4}v\|^2 \bigg|_t^T + \int_t^T \Sigma(\tau) \, d\tau ,
\] (80)
with
\[
\Sigma(\tau) := (\|A^{1/4}(w + v)\|^2 - \|A^{1/4}w\|^2)(A^{1/2}(w + v), v_t) - \|A^{1/4}v\|^2(A^{1/2}w, w_t).
\]
For the above term we have that
\[
|\Sigma(\tau)| \leq C_R \left( \|A^{1/4}v\| \|v_t\| + \|A^{1/4}v\|^2 \right),
\]
which inserted in (80) gives (76b), as desired.

By the lower bounds in (30) and (31) we have that
\[
\int_0^T \left( \|z_t\|^2 + \|v_t\|^2 \right) \, dt \leq C \left( G^T_0(z) + G^T_0(v) \right),
\] (81)
where \(G^T_0(z)\) and \(G^T_0(v)\) are given by (38), and also
\[
\|h_t(t)\|^2_1 + \|\zeta_t(t)\|^2 \leq D_{h,\zeta}(t) := m^{-1} \left[ (D(h_t(t)), h_t(t))_\Omega + (D(\zeta_t(t)), \zeta_t(t))_\Omega \right]
\] (82)
for all \(t \geq 0\).

Returning to Proposition 4.1 and using Lemma 6.3, we obtain for \(\Psi_T(z,v)\) in (40) the estimate
\[
\Psi_T(z,v) \leq \varepsilon \int_0^T E^0(t) \, dt + C_\varepsilon(T, R) \Xi_T(z,v)
\] (83)
for every \(\varepsilon > 0\), where \(E^0(t)\) is the same as in Proposition 4.1 and
\[
\Xi_T(z,v) = \text{lot}_T(z,v) + \int_0^T D_{h,\zeta}(\tau) \|A^{1/2}z(\tau)\|^2_\Omega \, d\tau ,
\] (84)
with \(\text{lot}_T(z,v)\) defined by (75) and \(D_{h,\zeta}(t)\) given in (82).

The next assertion is a direct consequence of Lemma 6.1.

**Lemma 6.4.** Under Assumption 3.3, the following estimate holds true with arbitrarily small \(\varepsilon > 0\):
\[
H^T_\varepsilon(z) + \mathcal{H}^T_0(v) \leq \varepsilon \int_0^T E^0(t) \, dt + C_\varepsilon(T, R) \left[ \Xi_T(z,v) + G^T_0(z) + G^T_0(v) \right],
\] (85)
where \(G^T_0(z), G^T_0(v)\) are defined in (38), and \(\Xi_T(z,v)\) is given by (84).

**Proof.** It is sufficient to apply the estimates recalled in Lemma 6.1. In fact, by using (72) we readily obtain that given \(\varepsilon > 0\), there exists a positive constant \(C_\varepsilon\) such that
\[
H^T_\varepsilon(z) \leq C_\varepsilon G^T_0(z) + \varepsilon \int_0^T E^0(t) \, dt + \varepsilon m \Xi_T(z,v).
\]
Similarly, using both (73) and the bounds (57) we have that
\[
\mathcal{H}^T_0(v) \leq C_1 G^T_0(v) + C_R \text{lot}_T(z,v).
\]
The two estimates above immediately give (85), as desired. \qed
Thus, combining the estimates in (81), (83) and (85), we see that from Proposition 4.1 it follows
\[ T E_0^0(T) + \int_0^T E_0^0(t) dt \leq C_1(T, R) \left[ \beta G_0^T(z) + \alpha G_0^T(v) \right] + C_2(T, R) \Xi_T(z, v) \quad (86) \]
for \( T \geq T_0 > 0 \). On the other hand, using the equality (43) and Lemma 6.3 we also have that
\[ \beta G_0^T(z) + \alpha G_0^T(v) \leq E_0^0(0) - E_0^0(T) + \varepsilon \int_0^T E_0^0(t) dt + C_\varepsilon(T, R) \Xi_T(z, v) \quad \text{for every} \ \varepsilon > 0. \]
Therefore (86) implies that there exists \( T > 1 \) such that
\[ E_0^0(T) \leq \gamma E_0^0(0) + C_{R,T} \Xi_T(z, v) \quad \text{with} \quad 0 < \gamma \equiv \gamma_{T,R} < 1. \quad (87) \]
We can now apply the same procedure described in [17] (see also [18] and [19, Chap. 3]). From (87) we have that
\[ E_0^0((m + 1)T) \leq \gamma E_0^0(mT) + c_T b_m, \quad m = 0, 1, 2, \ldots, \]
where
\[ b_m := \sup_{t \in [mT, (m+1)T]} \left( \| z(t) \|_{L^2}^2 + \| v(t) \|_{L^2}^2 \right) + \int_{mT}^{(m+1)T} D_{h,\zeta}(\tau) \| A^{1/2} z(\tau) \|_{L^2}^2 d\tau, \]
with \( D_{h,\zeta}(t) \) defined in (82). This yields
\[ E_0^0(mT) \leq \gamma^m E_0^0(0) + c \sum_{l=1}^m \gamma^{m-l} b_{l-1}. \]
Since \( \gamma < 1 \), there exists \( \omega > 0 \) such that
\[ E_0^0(mT) \leq C_1 e^{-\omega mT} E_0^0(0) \]
\[ + C_2 \left\{ \log mT \| z(v) \| + \int_{mT}^{(m+1)T} e^{-\omega(mT-\tau)} D_{h,\zeta}(\tau) \| A^{1/2} z(\tau) \|_{L^2}^2 d\tau \right\}, \]
which implies that
\[ E_0^0(t) \leq C_1 e^{-\omega t} E_0^0(0) + C_2 \left\{ \log t \| z(v) \| + \int_0^t e^{-\omega(t-\tau)} D_{h,\zeta}(\tau) E_0^0(\tau) d\tau \right\} \]
for all \( t \geq 0 \). Therefore, applying Gronwall’s lemma we find that
\[ E_0^0(t) \leq \left[ C_1 E_0^0(0) e^{-\omega t} + C_2 \log t \| z(v) \| \right] \exp \left\{ C_2 \int_0^t D_{h,\zeta}(\tau) d\tau \right\}. \]
Since by (57) we have that
\[ \int_0^t D_{h,\zeta}(\tau) d\tau = \frac{1}{m} \int_0^t ((D(h_t), h_t)_{\Omega} + (D(\zeta_t), \zeta_t)_{\Omega}) d\tau \leq C_R, \quad \text{for all} \ t \geq 0, \]
we obtain the estimate (74). This concludes the proof of Theorem 6.2. \( \square \)
7. **Proof of Theorem 3.4.**

1. **Finiteness of fractal dimension.** To prove finiteness of the fractal dimension \( \dim \mathfrak{A} \), we see that in view of Theorem 6.2 and local Lipschitz continuity (24) of the semi-flow \( S_t \), we can apply the abstract Theorem 3.11 in [18] with \( b(t) = e^{-\omega t} \).

2. **Smoothness of the global attractor.** It follows from Theorem 6.2, by using similar arguments as in [19, Sect. 4.2] (see also [33], where \( C^\infty \) regularity of the attractor is shown for a 2D wave equation with rather strong restrictions imposed on the damping).

Let \( \gamma = \{ y(t) = (z(t), z_t(t), v(t), v_t(t)) : t \in \mathbb{R} \} \subset \mathcal{Y} \) be a full trajectory from the attractor \( \mathfrak{A} \). Let \( |\sigma| < 1 \). Applying Theorem 6.2 with \( y_1 = y(s + \sigma), y_2 = y(s) \) (and, accordingly, the interval \([s, t] \) in place of \([0, t] \)), we have that

\[
\| y(t + \sigma) - y(t) \|^2 \leq C_1 e^{-\omega(t-\sigma)} \| y(s + \sigma) - y(s) \|^2 + C_2 \max_{\tau \in [s, t]} \left( \| z(\tau + \sigma) - z(\tau) \|^2 + \| v(\tau + \sigma) - v(\tau) \|^2 \right)
\]

(88)

for any \( t, s \in \mathbb{R} \) such that \( s \leq t \) and for any \( \sigma \) with \( |\sigma| < 1 \). Letting \( s \to -\infty \), (88) gives

\[
\| y(t + \sigma) - y(t) \|^2 \leq C_2 \max_{\tau \in [-\infty, t]} \left( \| z(\tau + \sigma) - z(\tau) \|^2 + \| v(\tau + \sigma) - v(\tau) \|^2 \right)
\]

for any \( t \in \mathbb{R} \) and \( |\sigma| < 1 \). By interpolation we have that

\[
\| z(\tau + \sigma) - z(\tau) \|^2 + \| v(\tau + \sigma) - v(\tau) \|^2 \leq \varepsilon \| y(t + \sigma) - y(t) \|^2 + C_\varepsilon \left( \| z(\tau + \sigma) - z(\tau) \|^2 + \| v(\tau + \sigma) - v(\tau) \|^2 \right)
\]

for every \( \varepsilon > 0 \). Therefore we obtain that

\[
\max_{\tau \in [-\infty, t]} \| y(\tau + \sigma) - y(\tau) \|^2 \leq C \max_{\tau \in [-\infty, t]} \left( \| z(\tau + \sigma) - z(\tau) \|^2 + \| v(\tau + \sigma) - v(\tau) \|^2 \right)
\]

(89)

for any \( t \in \mathbb{R} \) and \( |\sigma| < 1 \). On the attractor we obviously have that

\[
\frac{1}{\sigma} \| z(\tau + \sigma) - z(\tau) \| \leq \frac{1}{\sigma} \int_0^\sigma \| z_t(\tau + t) \| dt \leq C_R, \quad \tau \in \mathbb{R},
\]

and

\[
\frac{1}{\sigma} \| v(\tau + \sigma) - v(\tau) \| \leq \frac{1}{\sigma} \int_0^\sigma \| v_t(\tau + t) \| dt \leq C_R, \quad \tau \in \mathbb{R}.
\]

Therefore, by (89) we obtain that

\[
\max_{\tau \in \mathbb{R}} \left\| \frac{y(\tau + \sigma) - y(\tau)}{\sigma} \right\|_Y \leq C \quad \text{for} \quad |\sigma| < 1 .
\]

This implies that

\[
\| z_t(t) \|^2 + \| A^{1/2} z_t(t) \|^2 + \| v_t(t) \|^2 + \| A^{1/2} v_t(t) \|^2 \leq C , \quad t \in \mathbb{R},
\]

(90)

for any trajectory \( \gamma = \{ y(t) = (z(t), z_t(t), v(t), v_t(t)) : t \in \mathbb{R} \} \) from the attractor \( \mathfrak{A} \).

The estimate (90) enables us to establish the spatial smoothness of the attractor. Let us begin with the analysis of the plate variable \( v \). It follows from (23b) and (90) that on the attractor we have that \( Av(t) = f(t) \), where \( f(t) \) is a bounded function in \( L_2(\Gamma_0) \). Hence, we can conclude that \( t \mapsto v(t) \) is a bounded function in \( D(A) \).
As for the wave component \( z \), in the case \( n = 2 \) or \( \{ n = 3, 1 \leq p \leq 3 \} \) from (2) it follows that
\[
\| g(z_t) \| \leq C \left( 1 + \| z_t \|_{L^p_2(\Omega)}^p \right) \leq C \left( 1 + \| z_t \|_{L^1_1(\Omega)}^p \right).
\]
Therefore from (23a) and (90) we obtain that \( z(t) \) is solves the problem
\[
(-\Delta + \mu)z = f_1(t) \quad \text{in} \quad \Omega, \quad \frac{\partial z}{\partial \nu} = f_2(t) \quad \text{on} \quad \Gamma,
\]
where \( f_1 \in L_\infty(\mathbb{R}, L_2(\Omega)) \) and \( f_2 \in L_\infty(\mathbb{R}, H^s(\Gamma)) \) for any \( s < 3/2 \). By the elliptic regularity theory (see, e.g., [41, Chap. 5]) we can conclude that \( z(t) \) is a bounded function with values in \( H^2(\Omega) \).

In the case \( \{ n = 3, 3 < p \leq 5 \} \) we have that \( g(z_t) \) is bounded in \( L_{6/p}(\Omega) \) and therefore \( z \) solves (91) with \( f_1(t) \in L_\infty(\mathbb{R}, L_{6/p}(\Omega)) \). Again, the elliptic regularity theory gives us that \( z(t) \) is bounded in \( W_{6/p}^2(\Omega) \). Thus the second statement of Theorem 3.4 is proved.

3. Existence of a fractal exponential attractor. This is a direct consequence of Theorem 6.2, local Lipschitz continuity (24) of the semi-flow \( S_t \) and Theorem 4.43 in [20].

Appendix A. Proof of Theorem 2.3 (well-posedness). The abstract-operator version of the system given in (7) corresponds to a general form of a nonlinear structural acoustic model studied in [32]; see equations (2.6.41) and (2.6.42) (with \( M = I \)) in [32]. Hence, well-posedness for this model in the phase space \( Y \) would follow as a special case of the results given in Theorem 2.6.1 and Theorem 2.6.2 in [32]. Indeed, the nonlinear operators \( F_1 \) and \( F_2 \) are locally Lipschitz from \( D(A^{1/2}) \to L_2(\Omega) \) and \( D(A^{1/2}) \to L_2(\Gamma_0) \), respectively, with appropriate a-priori bounds (see (13)). The latter property guarantees the global (in time) existence of solutions. This allows to apply the well-posedness results established in [32]. However, since the model considered in [32] is more general and involves boundary dampings, which are more demanding from a technical point of view, we will give a specific proof of Theorem 2.3. As in [32] we shall use monotone operator theory; see, e.g., [5].

Step 1. Existence and uniqueness of a local solution. We consider the abstract second-order system (7) corresponding to the PDE model (1), and introduce the overall dynamics operator \( T : D(T) \subset Y \to Y \), i.e.
\[
T \begin{bmatrix} z_1 \\ z_2 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -z_2 \\ A z_1 + D(z_2) + z_2 - \alpha \kappa A N_0 v_2 \\ -v_2 \\ A v_1 + \beta \kappa N_0^* A z_2 + B(v_2) + v_2 \end{bmatrix},
\]
whose domain is
\[
D(T) = \left\{ [z_1, z_2, v_1, v_2] \in D(A^{1/2}) \times D(A^{1/2}) \times D(A) \times D(A^{1/2}) : A[z_1 - \alpha \kappa N_0 v_2] + D(z_2) \in L_2(\Omega) \right\},
\]
which is dense in \( Y \). Setting \( y = [z, z_t, v, v_t]^T \), it is elementary to derive from the second-order system (7) the first-order formulation of the original PDE problem (1), namely
\[
y' + Ty = C(y), \quad y(0) = y_0,
\]
with $T$ defined by (92) and $C$ given by

$$C[z_1, z_2, v_1, v_2]^T = [0, -F_1(z_1) + z_2, 0, -F_2(v_1) + v_2]^T.$$ 

As a consequence of (13), the nonlinear term $C$ is locally Lipschitz on the phase space $Y$. Thus, according to Theorem 7.2 in [16] a local existence result will follow once we prove that $T$ is a maximal accretive operator; see [11, 5].

**Accretivity.** If $y = [z_1, z_2, v_1, v_2]^T$, $\tilde{y}^T = [\tilde{z}_1, \tilde{z}_2, \tilde{v}_1, \tilde{v}_2] \in D(T)$, then

$$(Ty - T\tilde{y}, y - \tilde{y})_Y$$

$$= -\beta\alpha\kappa(AN_0(v_2 - \tilde{v}_2), z_2 - \tilde{z}_2) + \beta(g(z_2) - g(\tilde{z}_2) + z_2 - \tilde{z}_2, z_2 - \tilde{z}_2)$$

$$+ \alpha(b(v_2) - b(\tilde{v}_2) + v_2 - \tilde{v}_2, v_2 - \tilde{v}_2) + \alpha\beta\kappa\big(N_0^*A(z_2 - \tilde{z}_2), v_2 - \tilde{v}_2\big)$$

$$\geq \beta(g(z_2) - g(\tilde{z}_2), z_2 - \tilde{z}_2) + \alpha(b(v_2) - b(\tilde{v}_2), v_2 - \tilde{v}_2) \geq 0,$$

where we have used (11) and (12) to compute the inner product, and the monotonicity properties of $g$ and $b$ to obtain the latest inequality.

**Maximality.** As it is well known (see, e.g., [38, p. 18]), in order to show that $T$ is maximal, we only need to prove that $\mathcal{R}(I + T) = Y$. We can do it in the same way as in [1, 12].

Given $h = [\varphi_1, \varphi_2, \psi_1, \psi_2]^T \in Y$, we seek to solve the equation $(I + T)y = h$, which explicitly reads as

$$\begin{cases}
  z_1 - z_2 = \varphi_1 \\
  2z_2 + A\varphi_1 + g(z_2) + \alpha\kappa AN_0v_2 = \varphi_2 \\
  v_1 - v_2 = \psi_1 \\
  2v_2 + \beta\kappa N_0^*A z_2 + A\psi_1 + b(v_2) = \psi_2. 
\end{cases} \quad (95)$$

Eliminating $z_1$ and $v_1$ we arrive to the equation

$$(L + G) \begin{bmatrix} z_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \varphi_2 - A\varphi_1 \\ \psi_2 - A\psi_1 \end{bmatrix}, \quad (96)$$

with $L$ and $G$ given by

$$L = \begin{pmatrix} I + A & -\alpha\kappa AN_0 \\ \beta\kappa N_0^*A & I + A \end{pmatrix}, \quad G \begin{bmatrix} z_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} g(z_2) + z_2 \\ b(v_2) + v_2 \end{bmatrix}. \quad (97)$$

Let us observe that the right hand side of (96) belongs to the dual space $\mathcal{Y}'$ of $\mathcal{Y} := D(A^{1/2}) \times D(A^{1/2})$ with respect to the duality $\langle \cdot, \cdot \rangle_{\mathcal{Y}', \mathcal{Y}}$ given by

$$\langle (z, v), (h, w) \rangle_{\mathcal{Y}', \mathcal{Y}} = \beta(z, h) + \alpha(v, w).$$

Moreover, we have in particular that $L \in \mathcal{L}(\mathcal{Y}, \mathcal{Y}')$ while $G \colon \mathcal{Y} \to \mathcal{Y}'$ is a monotone hemicontinuous operator by Assumption 2.1.

The above decomposition is useful to show the following result.

**Lemma A.1.** Let $L$ and $G$ be the operators defined in (97). Then $\mathcal{R}(L + G) \equiv \mathcal{Y}'$.

**Proof.** By [5, Corollary 1.3, p. 48], it is necessary to verify the following properties: (i) $L$ is a monotone, hemicontinuous operator from $\mathcal{Y}$ to $\mathcal{Y}'$, (ii) $G$ is a maximal monotone operator from $\mathcal{Y}$ to $\mathcal{Y}'$, and (iii) $L + G$ is coercive.
We first compute
\[
\begin{pmatrix}
L_{z_2} z_2 \\
v_2
\end{pmatrix}_{\mathcal{Y}'} \begin{pmatrix}
z_2 \\
v_2
\end{pmatrix}_{\mathcal{Y}'}
= \begin{pmatrix}
\left[ z_2 + Az_2 - \alpha \kappa A_0 v_2 \right] \\
\beta \kappa A_0 A z_2 + v_2 + A v_2
\end{pmatrix}
= \beta \|z_2\|^2 + \beta \|A^{1/2} z_2\|^2 - \beta \alpha \kappa (A^{1/2} A_0 v_2, A^{1/2} z_2) + \alpha \beta \kappa (A_0 A z_2, v_2)
\]
\[
+ \alpha \|v_2\|^2 + \alpha \|A^{1/2} v_2\|^2 \geq \beta \|A^{1/2} z_2\|^2 + \alpha \|A^{1/2} v_2\|^2 = \|[z_2, v_2]\|_{\mathcal{Y}'}^2.
\]
This shows that $L$ is coercive, hence as a linear operator it is a monotone hemicontinuous operator, and thus (i) holds true. Moreover, since $G$ is monotone, then $L + G$ is coercive, as well, so that (iii) is satisfied. It remains to be shown that $G$ is maximal monotone as a mapping from $\mathcal{Y}$ to $\mathcal{Y}'$. Since $g_1(s) := g(s) + s$ is increasing, then $g_1 = \partial \Phi(\cdot)$, as a mapping from $\mathcal{D}(A^{1/2})$ to $[\mathcal{D}(A^{1/2})]'$, where $\Phi$ is some proper, convex, lower semi-continuous functional on $\mathcal{D}(A^{1/2})$ and $\partial \Phi$ denotes the subgradient of $\Phi$ (cf. [10, p. 37]). The same property holds for $b$ (in proper spaces). Thus, we can invoke [5, Theorem 2.1, p. 54] to obtain that $G$ is a maximal monotone operator from $\mathcal{Y}$ to $\mathcal{Y}'$, as desired.

Thus, solving the subsystem (96) yields $[z_2, v_2] \in \mathcal{Y}$, and returning to system (95) we then have $z_1 = z_2 + v_1 \in \mathcal{D}(A^{1/2})$, $v_1 = v_2 + \psi_1 \in \mathcal{D}(A^{1/2})$. Using the first and the third relations in (95) we can conclude that $y = [z_1, z_2, v_1, v_2]^T \in \mathcal{D}(T)$ and satisfies $(I + T)y = h$, hence $T$ is $m$-accretive.

We now see that equation (94) is a locally Lipschitz perturbation of an evolution equation with $m$-accretive generator. Therefore, by Theorem 7.2 in [16] for initial data $y^0 = (z^0, z^1, v^0, v^1) \in \mathcal{D}(T)$, there exists a unique strong solution $y(t) = (z(t), z_1(t), v(t), v_1(t))$ to (1) on an appropriate interval $(0, t_{\max})$. Instead, $y^0 \in \mathcal{Y}$ produces a unique generalized solution $y(t) \in C([0, t_{\max}), \mathcal{Y})$. In either case $t_{\max}$ depends on $\|y^0\|_{\mathcal{Y}}$; furthermore, $t_{\max} < \infty$ implies $\lim_{t \to t_{\max}} \|y(t)\|_{\mathcal{Y}} = +\infty$.

**Step 2. Energy inequalities/equalities.** To derive the energy identity (20) for strong solutions on the existence time interval a standard procedure applies (see, e.g., [12] or Lemma 4.2 above), provided boundedness of the damping operator $D(z_1)$ and $B(v_1)$ as a map from $\mathcal{D}(A^{1/2})$ (resp $\mathcal{D}(A^{1/2})'$) into the duals $[\mathcal{D}(A^{1/2})]'$ (resp $[\mathcal{D}(A^{1/2})]'$) is ascertained. The latter follows from criticality of the parameter $p$ (in the case of $D$) and from Sobolev’s embedding $\mathcal{D}(A^{1/2}) \subset C(\overline{T_0})$ (in the case of $B$). In fact, the damping $B$ is smoother than required, since $B(\mathcal{D}(A^{1/2})) \subset C(\overline{T_0})$. To establish the energy inequality (19) for generalized solutions, we only need to justify the limit transition (from strong to generalized solutions) in the damping terms. This can be done exploiting the properties of $L_2$-convergence and appropriate approximations of the damping functions $g$ and $b$ (the argument is the same as in [21]). Finally, the crucial property (21) that the energy is non-increasing immediately follows from the energy inequality (19). The upper bound (22) is obtained combining (21) with (18).

**Step 3. Global existence.** It follows from (22) that the solution cannot blow up in finite time. Therefore the same argument in Theorem 7.2 from [16] (see also [19, Chap.1]) shows global existence for both strong and generalized solutions.

**REFERENCES**

[1] G. Avalos and I. Lasiecka, *Uniform decay rates for solutions to a structural acoustic model with nonlinear dissipation*, Appl. Math. Comput. Sci., 8 (1998), no. 2, 287–312.
[2] G. Avalos and I. Lasiecka, *Exact controllability of structural acoustic interactions*, J. Math. Pures Appl., 82 (2003), 1047–1073.
[3] G. Avalos and I. Lasiecka, *Exact controllability of finite energy states for an acoustic wave/plate interaction under the influence of boundary and localized controls*, Adv. Differential Equations, 10 (2005), 901–930.
[4] A.V. Babin and M.I. Vishik, “Attractors of Evolution Equations,” Studies in Mathematics and its Applications 25, North-Holland Publishing Co., Amsterdam, 1992.
[5] V. Barbu, “Nonlinear Semigroups and Differential Equations in Banach Spaces,” Noordhoff, 1976.
[6] J.T. Beale, *Spectral properties of an acoustic boundary condition*, Indiana Univ. Math. J., 25 (1976), 895–917.
[7] A. Boutet de Monvel and I. Chueshov, *The problem on interaction of von Karman plate with subsonic flow of gas*, Math. Methods Appl. Sci., 22 (1999), 801–810.
[8] L. Boutet de Monvel and I. Chueshov, *Non-linear oscillations of a plate in a flow of gas*, C.R. Acad. Sci. Paris, Ser.I, 322 (1996), 1001–1006.
[9] L. Boutet de Monvel and I. Chueshov, *Oscillations of von Karman plate in a potential flow of gas*, Izvestiya: Mathematics, 63 (1999), 219–244.
[10] H. Brezis, *Problèmes unilatéraux*, J. Math. Pures Appl., 51 (1972), 1–168.
[11] H. Brezis, “Opérateurs Maximaux Monotones et semi-groupes des contractions dans les espaces de Hilbert,” North-Holland Mathematics Studies, Vol. 5, North-Holland Publishing Co., Amsterdam–London; American Elsevier Publishing Co., New York, 1973.
[12] F. Bucci, *Uniform decay rates of solutions to a system of coupled PDEs with nonlinear internal dissipation*, Differential Integral Equations, 16 (2003), no. 7, 865–896.
[13] J. Cagnol, I. Lasiecka and C. Lebiedzik, *Uniform stability in structural acoustic models with flexible curved walls*, J. Differential Equations, 186 (2002), 88–121.
[14] I. Chueshov, *Construction of solutions in a problem of the oscillations of a shell in a potential subsonic flow*, in “Operator Theory, Subharmonic Functions,” (ed. V.A. Marchenko), Naukova dumka, Kiev (1991), 147–154 (in Russian).
[15] I. Chueshov, “Introduction to the Theory of Infinite-Dimensional Dissipative Systems,” University Lectures in Contemporary Mathematics, Kharkov, 2002 (from the Russian edition (Acta, 1999)); see also http://www.emis.de/monographs/Chueshov/.
[16] I. Chueshov, M. Eller and I. Lasiecka, *On the attractor for a semilinear wave equation with critical exponent and nonlinear boundary dissipation*, Comm. Partial Differential Equations, 27 (2002), no. 9-10, 1901–1951.
[17] I. Chueshov, M. Eller and I. Lasiecka, *Finite dimensionality of the attractor for a semilinear wave equation with nonlinear boundary dissipation*, Comm. Partial Differential Equations, 29 (2004), no. 11-12, 1847–1876.
[18] I. Chueshov and I. Lasiecka, *Attractors for second-order evolution equations with a nonlinear damping*, J. Dynam. Differential Equations, 16 (2004), no. 2, 469–512.
[19] I. Chueshov and I. Lasiecka, *Long-time behaviour of second order evolution equations with nonlinear damping*, Scuola Normale Superiore of Pisa, Preprint no. 14, 2004; see http://math.sns.it/papers/chulas04.
[20] I. Chueshov and I. Lasiecka, “Long-time behaviour of second order evolution equations with nonlinear damping,” a version of [19] revised for Memoirs of AMS.
[21] I. Chueshov and I. Lasiecka, *Long-time dynamics of von Karman semi-flows with nonlinear boundary/interior damping*, J. Differential Equations (to appear).
[22] I. Chueshov and I. Lasiecka, *Long time dynamics of semilinear wave equation with nonlinear interior-boundary damping and sources of critical exponents*, AMS Contemporary Mathematics (to appear).
[23] A. Eden, C. Foias, B. Nicolaenko and R. Temam, “Exponential Attractors for Dissipative Evolution Equations,” Research in Appl. Math. 37, Masson, Paris, 1994.
[24] P. Fabrie, C. Galusinski, A. Miranville and S. Zelik, *Uniform exponential attractors for a singular perturbed damped wave equation*, Discrete Contin. Dyn. Syst., 10 (2004), 211–238.
[25] E. Feireisl, *Attractors for wave equations with nonlinear dissipation and critical exponents*, C.R. Acad. Sci. Paris, Ser.I, 315 (1992), 551–555.
[26] M. Grobbelaar-Van Dalsen, *On a structural acoustic model with interface a Reissner-Mindlin plate or a Timoshenko beam*, Journal of Mathematical Analysis and Applications, 320 (2006), no. 1, 121–144.
[27] J. K. Hale, “Asymptotic Behaviour of Dissipative Systems,” Mathematical Surveys and Monographs 25, American Mathematical Society, Providence, RI, 1988.
[28] M.S. Howe, “Acoustics of fluid-structure interactions,” Cambridge Monographs on Mechanics, Cambridge University Press, Cambridge, 1998.
[29] A. Kh. Khanmamedov, Global attractors for von Karman equations with nonlinear dissipation, *J. Math. Anal. Appl.*, **318** (2006), 92–101.
[30] I. Lasiecka, *Boundary stabilization of a three-dimensional structural acoustic model*, J. Math. Pures Appl., **78** (1999), 203–322.
[31] I. Lasiecka, *Existence and uniqueness of the solutions to second order abstract equations with nonlinear and nonmonotone boundary conditions*, Nonlinear Anal., **23** (1994), 797–823.
[32] I. Lasiecka, “Mathematical Control Theory of Coupled PDE’s,” CBMS-NSF Regional Conference Series in Applied Mathematics 75, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 2002.
[33] I. Lasiecka and A. Ruzmaikina, *Finite dimensionality and regularity of attractors for a 2-D semilinear wave equation with nonlinear dissipation*, J. Math. Anal. Appl., **270** (2002), 16–50.
[34] I. Lasiecka and R. Triggiani, “Control Theory for Partial Differential Equations: Continuous and Approximation Theories,” Vol. I: Abstract Parabolic Systems; Vol. II: Abstract Hyperbolic-like Systems over a Finite Time Horizon, Encyclopedia of Mathematics and its Applications, Vol. 74-75, Cambridge University Press 2000, 1067 pp.
[35] W. Littman and S. Markus, *Stabilization of a hybrid system of elasticity by feedback boundary damping*, Ann. Mat. Pura Appl., **152** (1988), no. 4, 281–330.
[36] P.M. Morse and K.U. Ingard, “Theoretical Acoustics,” McGraw-Hill, New York, 1968.
[37] D. Pražák, *On finite fractal dimension of the global attractor for the wave equation with nonlinear damping*, J. Dynam. Differential Equations, **14** (2002), 763–776.
[38] R. Showalter, “Monotone Operators in Banach Spaces and Nonlinear Partial Differential Equations”, Mathematical Surveys and Monographs 49, American Mathematical Society, Providence, RI, 1997.
[39] C. Sun, M. Yang and C. Zhong, *Global attractors for the wave equation with nonlinear damping*, J. Differential Equations, **227** (2006), 427-443.
[40] R. Temam, “Infinite-dimensional dynamical systems in mechanics and physics,” Applied Mathematical Sciences 68, Springer-Verlag, New York, 1997.
[41] H. Triebel, “Interpolation Theory, Function Spaces, Differential Operators,” North-Holland Mathematical Library 18, North-Holland Publishing Co., Amsterdam-New York, 1978.

Received March 2006; revised July 2006.

E-mail address: francesca.bucci@unifi.it; chueshov@univer.kharkov.ua; il2v@virginia.edu