A regularity criterion of 3D incompressible MHD system with mixed pressure-velocity-magnetic field

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Abstract

This work focuses on the 3D incompressible magnetohydrodynamic (MHD) equations with mixed pressure-velocity-magnetic field in view of Lorentz spaces. Our main result shows the weak solution is regular, provided that

\[
\pi \left( e^{-|x|^2} + |u| + |b| \right)^\theta \in L^p(0, T; L^{q, \infty}(\mathbb{R}^3)), \quad \text{where} \quad \frac{2}{p} + \frac{3}{q} = 2 - \theta \quad \text{and} \quad 0 \leq \theta \leq 1.
\]

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1 Introduction

We are interested in the regularity of weak solutions to the viscous incompressible magnetohydrodynamics (MHD) equations in $\mathbb{R}^3$

\[
\begin{aligned}
\partial_t u + (u \cdot \nabla) u - (b \cdot \nabla) b - \Delta u + \nabla \pi &= 0, \\
\partial_t b + (u \cdot \nabla) b - (b \cdot \nabla) u - \Delta b &= 0, \\
\nabla \cdot u &= \nabla \cdot b = 0, \\
u(x, 0) &= u_0(x), \quad b(x, 0) = b_0(x),
\end{aligned}
\tag{1.1}
\]

where $u = (u_1, u_2, u_3)$ is the velocity field, $b = (b_1, b_2, b_3)$ is the magnetic field, and $\pi$ is the scalar pressure, while $u_0$ and $b_0$ are the corresponding initial data satisfying $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$ in the sense of distribution.

Local existence and uniqueness theories of solutions to the MHD equations have been studied by many mathematicians and physicists (see, e.g., [2, 5, 18]). But due to the presence of Navier-Stokes equations in the system (1.1) whether this unique local solution can exist globally is an outstanding challenge problem. For this reason, there are many regularity criteria of weak solutions for the MHD equations has been investigated by many authors over past years (see e.g., [3, 4, 6, 7, 9, 10, 11, 16, 17, 21, 22] and references therein). Note that the literatures listed here are far from being complete, we refer the readers to see for example [8, 12, 13, 14, 15] for expositions and more references.

More recently, Beirão and Yang [1] proved the following regularity criterion for the mixed pressure-velocity in Lorentz spaces for Leray-Hopf weak solutions to 3D Navier-Stokes equations

\[
\frac{\pi}{(e^{-|x|^2} + |u|)^\theta} \in L^p(0, T; L^{q, \infty}(\mathbb{R}^3)), \quad 0 \leq \theta \leq 1 \quad \text{and} \quad \frac{2}{p} + \frac{3}{q} = 2 - \theta, \quad (1.2)
\]

where $L^{q, \infty}(\mathbb{R}^3)$ denotes the Lorentz space (c.f. [20]).

Motivated by the recent work of [1], the purpose of this note is to establish the regularity for the MHD equations (1.1) with the mixed pressure-velocity-magnetic in Lorentz spaces. Our main result can be stated as follows:

**Theorem 1.1** Suppose that $(u_0, b_0) \in L^2(\mathbb{R}^3) \cap L^4(\mathbb{R}^3)$ with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$ in the sense of distribution. Let $(u, b)$ be a weak solution to the MHD equations on some interval $[0, T]$ with $0 < T \leq \infty$. Assume that $0 \leq \theta \leq 1$ and that

\[
\frac{\pi}{(e^{-|x|^2} + |u| + |b|)^\theta} \in L^p(0, T; L^{q, \infty}(\mathbb{R}^3)), \quad \text{where} \quad \frac{2}{p} + \frac{3}{q} = 2 - \theta, \quad (1.3)
\]

then the weak $(u, b)$ is regular on $(0, T]$.

**Remark 1.1** A special consequence of Theorem 1.1 and its proof is the regularity criterion of the 3D Navier-Stokes equations with the mixed pressure-velocity in Lorentz spaces. This generalizes those of [1].
In order to derive the regularity criterion of weak solutions to the MHD equations (1.1), we introduce the definition of weak solution.

Next, let us writing
\[ w^\pm = u \pm b, \quad w^\pm_0 = u_0 \pm b_0. \]

We reformulate equation (1.1) as follows. Formally, if the first equation of MHD equations (1.1) plus and minus the second one, respectively, then MHD equations (1.1) can be re-written as:
\[
\begin{align*}
\partial_t w^+ - \Delta w^+ + (w^- \cdot \nabla) w^+ + \nabla \pi &= 0, \\
\partial_t w^- - \Delta w^- + (w^+ \cdot \nabla) w^- + \nabla \pi &= 0, \\
\text{div } w^+ &= 0, \quad \text{div } w^- = 0, \\
w^+(x, 0) &= w^+_0(x), \quad w^-(x, 0) = w^-_0(x).
\end{align*}
\]

The advantage is that the equations becomes symmetric.

## 2 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. In order to do it, we first recall the following estimates for the pressure in terms of \( u \) and \( b \) (see e.g., [8]):
\[
\| \pi \|_{L^q} \leq C \left( \| u \|_{L^2}^2 + \| b \|_{L^2}^2 \right), \quad \text{with} \quad 1 < q < \infty. \tag{2.1}
\]

We are now in position to prove our main result.

**Proof:** Multiplying the first and the second equations of (1.4) by \( |w^+|^2 \) \( w^+ \) and \( |w^-|^2 \) \( w^- \), respectively, integrating by parts and summing up, we have
\[
\frac{1}{4} \frac{d}{dt} \left( \| w^+ \|_{L^4}^4 + \| w^- \|_{L^4}^4 \right) + \int_{\mathbb{R}^3} \left( \| \nabla w^+ \|^2 \| w^+ \|^2 + \| \nabla w^- \|^2 \| w^- \|^2 \right) dx + \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla |w^+|^2|^2 + |\nabla |w^-|^2|^2) dx
\]
\[
= - \int_{\mathbb{R}^3} \nabla \pi \cdot (w^+ |w^+|^2 + w^- |w^-|^2) dx
\]
\[
= \int_{\mathbb{R}^3} \pi \cdot \text{div}(w^+ |w^+|^2 + w^- |w^-|^2) dx
\]
\[
\leq \int_{\mathbb{R}^3} \| \pi \| (|w^+| + |w^-|)(|\nabla |w^+|^2 + |\nabla |w^-|^2) dx
\]
\[
\leq C \int_{\mathbb{R}^3} |\pi|^2 (|w^+| + |w^-|)^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} (|\nabla |w^+|^2|^2 + |\nabla |w^-|^2|^2) dx.
\]

Notice that \( u = \frac{1}{2}(w^+ + w^-) \) and \( b = \frac{1}{2}(w^+ - w^-) \), then the above inequality means that
\[
\frac{d}{dt} (\| u \|_{L^4}^4 + \| b \|_{L^4}^4) + 2 \left( \| \nabla |u|^2 \|_{L^2}^2 + 2 \| \nabla |b|^2 \|_{L^2}^2 \right)
\]
\[
+ 2 \| |u| \|_{L^2}^2 + 2 \| |b| \|_{L^2}^2 + 2 \| |u| |b| \|_{L^2}^2 + 2 \| |b| |u| \|_{L^2}^2
\]
\[
\leq C \int_{\mathbb{R}^3} |\pi|^2 (|u| + |b|)^2 dx = K, \tag{2.2}
\]
where we have used
\[
|w^+| + |w^-| \leq |w^+ + w^-| + |w^+ - w^-|.
\]
For $K$, borrowing the arguments in \[1\], we set
\[
V = e^{-|x|^2} + |u| + |b| \quad \text{and} \quad \bar{\pi} = \frac{\pi}{(e^{-|x|^2} + |u| + |b|)^\gamma}.
\]
By the Hölder inequality and the following interpolation in Lorentz space (see \[20\])
\[
\|f^\alpha\|_{L^p,q(\mathbb{R}^3)} \leq C \|f\|_{L^{p,q}(\mathbb{R}^3)}^\alpha \quad \text{for} \quad \alpha > 0, \quad p > 0, \quad q > 0,
\]
we have
\[
K = \int_{\mathbb{R}^3} |\bar{\pi}|^\lambda V^{-\lambda \theta} |\pi|^{2-\lambda} V^\lambda (|u| + |b|)^2 dx
\]
\[
\leq \int_{\mathbb{R}^3} |\bar{\pi}|^\lambda |\pi|^{2-\lambda} V^{2-\lambda \theta} dx
\]
\[
\leq \left( \|\bar{\pi}\|^\lambda_{L^{q,\infty}} \right)^{\frac{1}{q}} \left( \|\pi\|^\lambda_{L^{q,\infty}} \right)^{\frac{1}{q}} \|V\|^2_{L^{q,\infty}} \|V\|_{L^{r,2}}^2
\]
where
\[
\frac{\lambda}{q} + \frac{1}{s} + \frac{1}{r} = 1 \quad \text{and} \quad \lambda = \frac{2}{2 - \theta}.
\]
By \[2.1\], we have
\[
K \leq \|\bar{\pi}\|_{L^{q,\infty}}^{\lambda} \left( \|u\|_{L^{(2-\lambda),2}} + \|b\|_{L^{(2-\lambda),2}} \right)^{2-\lambda} \|V\|_{L^{r,2}}^\lambda
\]
\[
\leq C \|\bar{\pi}\|_{L^{q,\infty}}^{\lambda} \|V\|_{L^{(2-\lambda),2}} \|V\|_{L^{r,2}}^\lambda.
\]
By the interpolation and Sobolev inequalities in Lorentz spaces, it follows that
\[
\begin{align*}
\left\{ \begin{array}{l}
\|V^2\|_{L^{(2-\lambda),2}} \leq C \|V\|_{L^{2}}^{1-\delta_1} \|V^2\|_{L^{6,2}}^{\delta_1} \leq C \|V\|_{L^{2}}^{1-\delta_1} \|\nabla V\|_{L^{2}}^{\delta_1}, \\
\|V^2\|_{L^{r,2}} \leq C \|V\|_{L^{2}}^{1-\delta_2} \|V^2\|_{L^{6,2}}^{\delta_2} \leq C \|V\|_{L^{2}}^{1-\delta_2} \|\nabla V\|_{L^{2}}^{\delta_2},
\end{array} \right.
\end{align*}
\]
(2.3)
where $0 < \delta_1, \delta_2 < 1$ and
\[
\frac{1}{2} = \frac{1 - \delta_1}{6} + \frac{\delta_1}{6}, \quad \frac{1}{2} = \frac{1 - \delta_2}{6} + \frac{\delta_2}{6}.
\]
Hence from \[2.3\] and Young inequality, it follows that
\[
K \leq C \|\bar{\pi}\|_{L^{q,\infty}}^{\lambda} \|V^2\|_{L^{2}}^{(2-\lambda)(1-\delta_1) + \lambda(1-\delta_2)} \|\nabla V^2\|_{L^{2}}^{(2-\lambda)\delta_1 + \lambda\delta_2}
\]
\[
\leq C \|\bar{\pi}\|_{L^{q,\infty}}^{\lambda} \|V^2\|_{L^{2}}^{\frac{2 - \lambda}{2 - \lambda + \delta_1 - \delta_2}} \|\nabla V^2\|_{L^{2}}^{\frac{\delta_1}{2}} + \frac{1}{2} \|\nabla V^2\|_{L^{2}}^{\frac{\delta_2}{2}}.
\]
Due to the definition of $V$, we see that
\[
\|V^2\|_{L^{2}}^{2} \leq C(1 + |u| + |b|)^2 + \|u\|_{L^{2}}^2 + \|b\|_{L^{2}}^2,
\]
and
\[
\|\nabla V^2\|_{L^{2}}^{2} \leq C(1 + |u| + |b|)^2 + \|\nabla(|u| + |b|)^2 + \|\nabla(|u| + |b|)^2\|_{L^{2}}^2).
\]
and
\[
\|\nabla V^2\|_{L^{2}}^{2} \leq C(1 + |u| + |b|)^2 + \|\nabla(|u| + |b|)^2 + \|\nabla(|u| + |b|)^2\|_{L^{2}}^2).
\]
Consequently, we get
\[ K \leq C \|\nabla \|^\frac{2}{2-n} \|u\|_{L^4} \|b\|_{L^4} (1 + \|u\| + \|b\|^2_{L^2} + \|u\|^2_{L^4} + \|b\|^2_{L^4}) \]
\[ + C(1 + \|u\| + \|b\|^2_{L^2} + \|\nabla (|u| + |b|)\|_{L^2}) + \frac{1}{2} \|\nabla (|u|^2 + |b|^2)\|_{L^2} \]
\[ \leq C \|\nabla \|^\frac{2}{2-n} \|u\|_{L^4} \|b\|_{L^4} (1 + \|u\|^2_{L^2} + \|b\|^2_{L^2} + \|u\|^2_{L^4} + \|b\|^2_{L^4}) \]
\[ + C(1 + \|u\|^2_{L^2} + \|b\|^2_{L^2} + \|\nabla u\|^2_{L^2} + \|\nabla b\|^2_{L^2}) + \frac{1}{2} \|\nabla |u|^2\|_{L^2} + \frac{1}{2} \|\nabla |b|^2\|_{L^2}. \]

Since \((u, b)\) is a weak solution to (1.1), then \((u, b)\) satisfies
\[ (u, b) \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3)). \]

Inserting the above estimates into (2.2), we obtain
\[ \frac{d}{dt} (\|u\|^4_{L^4} + \|b\|^4_{L^4}) + \|\nabla |u|^2\|^2_{L^2} + \|\nabla |b|^2\|^2_{L^2} \]
\[ + 2 \|u\| |\nabla u| + \|b\| |\nabla b| + 2 \|u\| |\nabla u| |\nabla b| + 2 \|b\| |\nabla b| |\nabla u| \]
\[ \leq C \|\nabla \|^\frac{2}{2-n} \|u\|_{L^4} \|b\|_{L^4} (1 + \|u\|^2_{L^2} + \|b\|^2_{L^2} + \|u\|^2_{L^4} + \|b\|^2_{L^4}) \]
\[ + C(1 + \|u\|^2_{L^2} + \|b\|^2_{L^2} + \|\nabla u\|^2_{L^2} + \|\nabla b\|^2_{L^2}) \]
\[ \leq C \|\nabla \|^\frac{2}{2-n} \|u\|_{L^4} \|b\|_{L^4} (1 + \|u\|^4_{L^4} + \|b\|^4_{L^4}) + C(1 + \|\nabla u\|^2_{L^2} + \|\nabla b\|^2_{L^2}), \]

Using Gronwall’s inequality with the assumption (1.3), we deduce that
\[ (u, b) \in L^\infty(0, T; L^4(\mathbb{R}^3)) \subset L^8(0, T; L^4(\mathbb{R}^3)). \]

We complete the proof of Theorem 1.1. \(\Box\)

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