Quantum Revivals in Periodically Driven Systems close to nonlinear resonance

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We calculate the quantum revival time for a wave-packet initially well localized in a one-dimensional potential in the presence of an external periodic modulating field. The dependence of the revival time on various parameters of the driven system is shown analytically. As an example of application of our approach, we compare the analytically obtained values of the revival time for various modulation strengths with the numerically computed ones in the case of a driven gravitational cavity. We show that they are in very good agreement.

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In nature interference phenomena lead to recurrences [1, 2]. In quantum mechanical evolution, for instance, interference plays an important role and manifests itself in quantum recurrences [3]. A quantum wave-packet spreads all over the available space after a few classical periods following wave mechanics and collapses. However, due to quantum dynamics it rebuilds itself after a certain evolution time which is larger than the classical period. In one-dimensional systems this phenomenon is well studied and is known as quantum revivals and fractional revivals [4, 5, 6, 7, 8, 9]. In higher dimensional systems, which exhibit classical and quantum chaos, this phenomenon has been numerically observed earlier [10, 11, 12, 13, 14]. In particular, the quantum revivals occurring in driven gravitational cavities [15, 16] have been investigated analytically and numerically.

In this Letter we provide a general analytical prescription to the revival phenomenon occurring in any periodically-driven time-dependent systems. By using semi-classical secular theory [17, 18, 19], we derive a simple relation which allows us to calculate the quantum revival time in the presence of the external periodic modulation. For the sake of concreteness, we apply our general result to the dynamics of atoms in a modulated gravitational cavity [20] which is accessible to laboratory experiments [21]. Hence we argue that dynamical revivals are generic to all periodically-driven explicitly time-dependent systems, so far as resonances prevail.

Let us consider a one-dimensional system driven by an external periodic field. In order to calculate its quasi-energy, we consider the nonlinear reso-
nances of the time-dependent system [17, 18]. We denote the eigenstates and eigenvalues of the corresponding time independent system by $|n\rangle$ and $E_n$, respectively, so that $H_0|n\rangle = E_n|n\rangle$, where $H_0$ is the unperturbed Hamiltonian. We consider the evolution of a wave-packet in an arbitrary one-dimensional potential in the presence of an external periodic field. The Hamiltonian of the driven system in dimensionless form can be expressed as

$$H = H_0 + \lambda V(x) \sin t ,$$  

(1)

where $\lambda$ is the dimensionless modulation strength and $V(x)$ defines the coupling.

In order to study the quantum nonlinear resonances of the system we make the ansatz [17, 18] that the solution of the Schrödinger equation corresponding to the Hamiltonian (1) can be written in the form

$$|\psi(t)\rangle = \sum_n C_n(t)|n\rangle \exp\left\{-i\left[E_m + (n - r)\frac{k}{N}\right]t\right\} ,$$  

(2)

where $k$ denotes the effective Planck’s constant, and $E_r$ is the average energy of the wave-packet in the $N$th resonance.

We substitute Eq. (2) into the Schrödinger equation and project the result onto the state vector $\langle m |$. This leads to

$$i k \dot{C}_m = \left[ E_m - E_r + (m - r)\frac{k}{N} \right] C_m(t) + \frac{\lambda}{2i} \sum_n V_{m,n}$$

$$\times \left[ e^{i(n-m+N)t/N} - e^{i(n-m-N)t/N} \right] C_n(t) ,$$  

(3)

where $V_{n,m} = \langle n | V(x) | m \rangle$ are the matrix elements of $V(x)$. In Eq. (3) we drop the fast oscillating terms and keep only the resonant ones, that is, $n = m \pm N$. 
Moreover, for large $m$ we may take $[17, 18]$ $V_{m,m+N} \approx V_{m,m-N} = V$. Hence, Eq. (3) reduces to

$$i\hbar \dot{C}_m = \left[ E_m - E_r + (m - r) \frac{k}{N} \right] C_m(t)$$

$$+ \frac{\lambda V}{2i} (C_{m+N} - C_{m-N}) . \quad (4)$$

In view of the slow dependence of the energy $E_m$ on the quantum number $m$ around the initially excited level $r$ in a nonlinear resonance, we expand the energy $E_m$ up to second order. Hence, the equation of motion for the probability amplitude $C_m$ takes the form

$$i\hbar \dot{C}_m = (m - r) \left( E'_r - \frac{k}{N} \right) C_m$$

$$+ \frac{1}{2} (m - r)^2 E''_r C_m + \frac{\lambda V}{2i} (C_{m+N} - C_{m-N}) . \quad (5)$$

For the exact resonance case $E'_r = k / N$, and as a result the first term on the right hand side vanishes. We may write this equation in angle representation by introducing the Fourier representation of $C_m$ as

$$C_m = \frac{1}{2\pi} \int_{0}^{2\pi} g(\varphi) e^{-i(m-r)\varphi} \, d\varphi$$

$$= \frac{1}{2N\pi} \int_{0}^{2N\pi} g(\theta) e^{-i(m-r)\theta/N} \, d\theta . \quad (6)$$

This particular choice of the Fourier representation of $C_m$ yields a Schrödinger-like equation, $i\hbar \dot{g}(\theta, t) = \mathcal{H} g(\theta, t)$, where the Hamiltonian $\mathcal{H}(\theta)$ reads

$$\mathcal{H}(\theta) = -\frac{N^2 E''_r}{2} \frac{\partial^2}{\partial \theta^2} + \lambda V \sin(\theta) , \quad (7)$$
and is independent of time. To obtain this equation, we have assumed that the function $g(\theta, t)$ has a $2N\pi$-periodicity in $\theta$-coordinate.

Due to the time-independent behavior of $\mathcal{H}(\theta)$, we can write the time evolution of $g$ as $g(\theta, t) = \tilde{g}(\theta)e^{-i\mathcal{E}t/k}$. On substituting this representation in the equation of motion for $g(\theta, t)$, we are left with the rather simple equation

$$
\left[-\frac{N^2 E''_r}{2} \frac{\partial^2}{\partial \theta^2} - \mathcal{E} + \lambda V \sin(\theta)\right] \tilde{g}(\theta) = 0 .
$$

(8)

With the change of variable $\theta = 2z + \pi/2$, we can reduce it to the standard Mathieu equation

$$
\left[\frac{\partial^2}{\partial z^2} + a - 2q \cos(2z)\right] \tilde{g}(z) = 0 .
$$

(9)

Here, $a = 8\mathcal{E}/N^2E''_r$, $q = 4\lambda V/N^2E''_r$, and $\tilde{g}(\varphi)$ is a $\pi$-periodic function.

Through this procedure we have mapped our initially unsolvable time-dependent Schrödinger equation onto the known Mathieu equation [17, 18] under the assumption of exact resonance. The Floquet solution of the Mathieu equation [19] can be written as $\tilde{g}_\nu(z) = e^{iz\nu}P_\nu(z)$, where $P_\nu$ are the Matthieu functions [22]. Since we require a $\pi$-periodicity of $\tilde{g}$, we are forced to take only the even values of the index, i.e., $\nu = 2k/N$.

Hence, the Floquet quasi-energy eigenvectors [19] can be written as

$$
|\psi_k(t)\rangle = e^{-i\mathcal{E}_k t/k} |u_k(t)\rangle ,
$$

(10)

where $\mathcal{E}_k$ and $|u_k(t)\rangle$ are defined as

$$
\mathcal{E}_k \equiv \frac{N^2 E''_r}{8} a_\nu(q) ,
$$

(11)
\[ |u_k(t) \rangle \equiv \frac{1}{2\pi} \sum_n e^{int/N} \int_0^{2\pi} d\varphi e^{i\nu N\varphi/2} e^{-i(n-r)\varphi/2} \times P_{\nu(k)} |n \rangle. \] (12)

In this way, we have obtained an approximate solution for a nonlinear resonance of our explicitly time-dependent system.

In order to check the correctness of our result, we study the case of zero modulation strength, that is \( \lambda = 0 \), for which \( q = 0 \). The corresponding value for the Mathieu characteristic parameter becomes \( a_{\nu(n)}(q = 0) = 4(n - r)^2/N^2 \). This reduces the quasi-energy given in Eq. (11) to

\[ \mathcal{E}_n(q = 0) = \frac{E''}{2} (n - r)^2, \] (13)

which is indeed the energy of the un-modulated system expanded around \( n = r \) for the exact resonance case. This is obtained by considering \( k = n - r \), and therefore we may express \( \nu_n(q = 0) = 2(n - r)/N \).

The eigenfunctions given in Eq. (10) form a complete set of basis vectors. Therefore, we can write the propagated wave-packet at any later time in terms of the quasi-energy eigenstates, such that

\[ |\psi(z, t) \rangle = \sum_n \xi_n e^{-i\mathcal{E}_n t/k} |u_n(t) \rangle, \] (14)

where \( \xi_n \) describes the probability amplitude in the \( n \)th state.

In order to investigate the dynamical revivals, we calculate the autocorrelation function between the initially excited wave-packet and the wave-packet \( |\psi(z, t) \rangle \) after a certain evolution time \( t \), viz.

\[ \langle \psi(0)|\psi(t) \rangle = \sum_n |\xi_n|^2 e^{-i\mathcal{E}_n t/k}. \] (15)
We take into account that the wave-packet is initially centered around $n = r$ in the energy representation. Therefore, by expanding the quasi-energy of the system around the resonant level $r$, we can write

$$
\langle \psi(0)|\psi(t)\rangle = \sum_n |\xi_n|^2 \exp \left\{ -i \left[ \omega^{(0)} + (n - r)\omega^{(1)} + (n - r)^2\omega^{(2)} + \cdots \right] t \right\},
$$

(16)

where $\omega^{(j)} = (j! k)^{-1}\mathcal{E}_n^{(j)}|_{n=r}$ and $\mathcal{E}_n^{(j)}|_{n=r}$ denotes the $j$th derivative of $\mathcal{E}_n$ calculated at $n = r$.

We then compare the auto-correlation function for the driven system, as given in Eq. (16), with the auto-correlation function for the corresponding one-dimensional unperturbed system [5, 15]. This comparison helps us to identify the classical period for the driven system as

$$
T_{cl} \equiv \frac{2\pi}{\omega^{(1)}} = \frac{2\pi k}{\mathcal{E}_n^{(1)}|_{n=r}},
$$

(17)

and the quantum revival time as

$$
T_{\lambda} \equiv \frac{2\pi}{\omega^{(2)}} = \frac{4\pi k}{\mathcal{E}_n^{(2)}|_{n=r}}.
$$

(18)

Let us now consider the case of small perturbations, i.e. $\lambda \ll 1$. We may therefore use the expansion for the Mathieu characteristic parameter $a_\nu$ [22] only up to second order in $q$. Then Eq. (11) becomes

$$
\mathcal{E}_n = \frac{E''}{8} \left( \nu_n^2 + \frac{q^2}{2} \frac{1}{\nu_n^2} - 1 \right) + O(q^4),
$$

(19)

and a simple expression for the revival time in the driven potential can be derived from Eqs. (18) and (19).
In general, the initially excited wave-packet is away from the center of the resonance and the condition $E'_r = k/N$ is not satisfied. For the sake of clarity, in the above discussion we have only considered the case of the exact resonance. However, taking into account the non-resonant situation, the revival time for a time-dependent system can be calculated in the general case and reads [15]

$$T_\lambda = T_0 \left[ 1 - \frac{1}{2} \left( \frac{\lambda V}{E''_r} \right)^2 \left( \frac{2}{N} \right)^4 \frac{3\nu_n^2 + 1}{(\nu_n^2 - 1)^3} \right],$$

(20)

Here, $T_0 = 4\pi k/E''_r$ denotes the revival time of the initial wave-packet in the un-driven potential and the index $\nu_n$ is defined as

$$\nu_n = \frac{2}{N}(n-r) + \frac{2(E'_r - k/N)}{N E''_r},$$

(21)

where the second term on the right hand side is arising due to the non-resonant case. Substituting the value of $\nu_n$ from Eq. (21) calculated at $n = r$ in Eq. (20), we obtain the general expression for the quantum revival time in a periodically driven system as

$$T_\lambda = T_0 \left[ 1 - \frac{1}{2} \left( \frac{\lambda V}{E''_r} \right)^2 \frac{3\mu^2 + N^2/4}{(\mu^2 - N^2/4)^3} \right],$$

(22)

where $\mu = \frac{(E'_r - k/N)}{E''_r}$. Equation (22) constitutes our main result: it expresses the modification of the quantum revival time in the presence of an external periodic driving field as a function of the modulation strength $\lambda$ and of the other characteristic parameters of the system.
The present approach is valid for small modulation strengths. In the weak binding potentials, for which the level spacing decreases as we increase the quantum number, the nonlinear resonances disappear after small modulations, as observed numerically for gravitational cavity [23, 24] and also for hydrogen atoms. As a consequence, this technique is reasonably good for these kind of potentials. For tight binding potentials, for which the level spacing increases as we go up in the energy, we find nonlinear resonances for higher modulation as well. In this kind of potential we need to consider higher-order terms in the expansion of the Mathieu characteristic parameter in order to correctly calculate the quantum recurrence time.

As an application of our method, we consider the dynamics of atoms in a gravitational cavity [20] in the presence of an external periodic field [25]. In this case the atoms move in a potential \( x + V_0 e^{-\kappa x} \), where the linear term is due to the gravitational potential and the exponential term comes from the evanescent-wave field which can be obtained by total internal reflection of a laser light field on the surface of a glass prism [20, 21, 27, 28, 29, 30]. In the approximation of a triangular well potential, the coupling constant for the \( N \)th resonance is given by

\[
V \equiv \langle m | x | m \pm N \rangle = -\frac{2E_m}{N^2 \pi^2 [1 \mp N/6m + O(m^{-2})]^2},
\]

which in the limit of large \( m \) becomes \( V \approx -2E_N/N^2 \pi^2 \). Substituting this expression into Eq. (22), it is possible to calculate the time of revival for a
wave packet initially excited with average energy $E_r$ around the $N$th resonance, which reads

$$T_\lambda = T_0 \left\{ 1 - \frac{1}{8} \left( \frac{\lambda}{E_r} \right)^2 \frac{3(1 - \alpha)^2 + a^2}{[(1 - \alpha)^2 - a^2]^3} \right\}, \quad (24)$$

where $\alpha \equiv (E_N/E_r)^{1/2}$ and $a \equiv \alpha^2 k / 4E_r$. If the initial energy is large enough, that is $E_r \gg 1$, we may neglect $a^2$ with respect to $(1 - \alpha)^2$ in Eq. (24), obtaining the simpler formula

$$T_\lambda = T_0 \left[ 1 - \frac{3}{8} \left( \frac{\lambda}{E_r} \right)^2 \frac{1}{(1 - \alpha)^4} \right]. \quad (25)$$

This expression allows us to compare the analytically calculated revival time with the numerically computed ones.

We may prepare a wave packet by trapping and cooling cesium atoms in a MOT [20, 21, 28]. In laboratory experiments the driving frequency can be changed from 0 to 2MHz, see Ref. [21]. Hence if we select a frequency of $\omega = 2\pi \times 0.93 KHz$ out of this domain, for cesium atoms of mass $M = 2.2 \times 10^{-25} Kg$, the dimensionless Plank’s constant is $\hbar = 1$. Moreover, by using effective Rabi frequency $\Omega_{eff} = 2\pi \times 3.72 KHz$ and the steepness $0.57 \mu m$, we have $V = 1$ and $\kappa = 1$. In our numerical calculations we use these values of the dimensionless parameters. In order to measure the quantum recurrence time, we place the atomic wave packet at $z_0 = 29.8 \mu m$ for which $E_r = 104.1$, and $z_0 = 20.1 \mu m$, for which $E_r = 70.28$ above the atomic mirror.
Our numerical results show a complete qualitative and quantitative agreement with the analytical results. In fact, from Eq. (25) we learn that (i) the revival time changes quadratically as a function of the strength \( \lambda \) of the external modulation field, and (ii) the revival time depends inversely on the square of the initial average energy of the wave-packet. Our numerical investigation confirms both these analytical results.

In Fig. 1 we compare the quantum revival times calculated from Eq. (25) (dashed lines) with those obtained numerically (solid lines) for an atomic wave-packet bouncing in a modulated gravitational cavity. We see from the numerical data that the revival time displays a quadratic dependence on the strength \( \lambda \) of the external modulation. Moreover, the change in the revival time is just about 8\% for \( \lambda \) ranging from 0 to 0.25 when the initial average energy \( E_r = 104.1 \), (solid line with circles), whereas it increases to almost 40\% for the same range of \( \lambda \) when the initial average energy \( E_r = 70.28 \) (solid line with squares), in agreement with the inverse square dependence of \( T_\lambda \) on \( E_r \).

In summary, we have presented a general approach to the investigation of the phenomenon of quantum revivals in periodically driven systems. We have derived a simple relation which provides the quantum revival time as a function of the modulation strength. We have finally applied our theory to the case of the revival time arising in the dynamics of atoms bouncing in a modulated gravitational cavity, obtaining an excellent agreement with the exact numerical data. For the set of parameters which we have used in
our calculations a modulated gravitational cavity can be realized within the framework of presently available technology [15]. Therefore we suggest an experiment, designed with the help of these parameters, to test the quantum recurrences in the driven gravitational cavity.

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Figure 1: A comparison between the numerically computed ratio $T_h/T_0$ for two different initial conditions, $E_r = 104.1$ (solid line with circles) and 
$E_r = 70.28$ (solid line with squares), and the corresponding [Eq. (25)] analytical results (dashed lines) in the case of an atom bouncing in a modulated 
gravitational cavity. The other parameters are $k = 1$, $V_0 = 1$, and $\kappa = 1$.

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