Cayley Hamilton theorem with sandwich coefficients for $n \times n$ matrices over a ring satisfying $[x,y][u,v] = 0$

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Abstract. If $A$ is an $n \times n$ matrix over a ring $R$ satisfying the polynomial identity $[x,y][u,v] = 0$, then an invariant Cayley-Hamilton identity of the form

$$\sum_{0 \leq i,j \leq n} A^i c_{i,j} A^j = 0$$

with $c_{i,j} \in R$ and $c_{n,n} = (n!)^2$ holds for $A$.

1. INTRODUCTION

The Cayley-Hamilton theorem and the corresponding trace identity play a fundamental role in proving classical results about the polynomial and trace identities of the $n \times n$ matrix algebra $M_n(K)$ over a field $K$ (see [2] and [3]).

In case of $\text{char}(K) = 0$, Kemer’s pioneering work (see [5]) on the T-ideals of associative algebras revealed the importance of the identities satisfied by the $n \times n$ matrices over the Grassmann (exterior) algebra $E = K \langle v_1, v_2, ..., v_r, ... \mid v_i v_j + v_j v_i = 0 \text{ for all } 1 \leq i \leq j \rangle$ generated by the infinite sequence of anticommutative indeterminates $(v_i)_{i \geq 1}$. Accordingly, the importance of matrices over non-commutative rings is an evidence in the theory of PI-rings, nevertheless this fact has been obvious for a long time in other branches of algebra (e.g. in the structure theory of semisimple rings). Thus a Cayley-Hamilton type identity for such matrices seems to be of general interest.

In the general case (when $R$ is an arbitrary non-commutative ring with 1) Paré and Schelter proved (see [8]) that any matrix $A \in M_n(R)$ satisfies a monic identity in which the leading term is $A^k$ for some integer $k \geq 2^{n-1}$ and the other summands are of the form $r_0 A r_1 A r_2 \cdots r_{l-1} A r_l$ with left scalar coefficient $r_0 \in R$, right scalar coefficient $r_l \in R$ and sandwich scalar coefficients $r_2, \ldots, r_{l-1} \in R$. An explicit monic identity for $2 \times 2$ matrices arising from the argument of [8] was given by Robson in [11]. Further results in this direction can be found in [9] and [10].

Obviously, imposing extra algebraic conditions on the base ring $R$, we can expect “stronger” identities in $M_n(R)$. A number of examples show that certain

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polynomial identities satisfied by $R$ can lead to “canonical” constructions providing an invariant Cayley-Hamilton identity for $A$ of much lower degree than $2^{2n-1}$.

If $R$ satisfies the polynomial identity
\[
[[[[...[[x_1, x_2], x_3], ...], x_m], x_{m+1}] = 0
\]
of Lie-nilpotency (with $[x, y] = xy - yx$), then a left (and right) Cayley-Hamilton identity of degree $n^m$ was constructed in [12]. Since $E$ is Lie-nilpotent of index $m = 2$, this identity for a matrix $A \in M_n(E)$ is of degree $n^2$.

In [1] Domokos considered a slightly modified version of the mentioned identity in which the left (as well as the right) coefficients are invariant under the conjugate action of $GL_n(K)$ on $M_n(E)$. For a $2 \times 2$ matrix $A \in M_2(E)$ the left scalar coefficients of his Cayley-Hamilton identity are expressed as polynomials (over $K$) of the traces $\text{tr}(A)$, $\text{tr}(A^2)$ and $\text{tr}(A^3)$.

If $\frac{1}{2} \in R$ and $R$ satisfies the so called weak Lie-solvability
\[
[[x, y], [x, z]] = 0,
\]
then for a $2 \times 2$ matrix $A \in M_2(R)$ a Cayley-Hamilton trace identity (of degree 4 in $A$) with sandwich coefficients was exhibited in [7]. If $R$ satisfies the identity
\[
[x_1, x_2, ..., x_{2^t}]_{\text{solv}} = 0
\]
of general Lie-solvability, then a recursive construction (also in [7]) gives a similar Cayley-Hamilton trace identity for $A \in M_2(R)$ (its degree depends on $t$).

In the present paper we consider an $n \times n$ matrix $A \in M_n(R)$ over a ring $R$ (with 1) satisfying the identity $[x, y][u, v] = 0$ and construct an invariant Cayley-Hamilton identity of the form
\[
\sum_{0 \leq i, j \leq n} A^i c_{i,j} A^j = 0,
\]
where $c_{i,j} \in R$ are the sandwich coefficients and $c_{n,n} = (n!)^2$ is the (central) leading coefficient.

We note that $[x, y][u, v] = 0$ is the generating identity of the algebra $U_2(K)$ of $2 \times 2$ upper triangular matrices (see [6]). The identity $[x, y][x, z] = 0$ (as well as $[[x, y], [x, z]] = 0$) is a consequence of the Lie-nilpotency $[[x, y], z] = 0$ (see [4]). Clearly, the algebra $E$ shows that $[x, y][u, v] = 0$ is not a consequence of $[[x, y], z] = 0$ and the algebra $U_2(K)$ shows that $[[x, y], z] = 0$ is not a consequence of $[x, y][u, v] = 0$. Results about the logical relationships among the identities
\[
[x, y][u, v] = 0, \quad [[x, y], z] = 0 \quad \text{and} \quad [[x, y], [u, v]] = 0
\]
can be found in [7].

We shall make extensive use of the so called symmetric characteristic polynomial and the results in [12] and [13]. In order to provide a self contained treatment in Section 2 we present all the necessary prerequisites.

2. CAYLEY-HAMILTON IDENTITY WITH MATRIX COEFFICIENTS

Let $R$ be an arbitrary ring with 1. The preadjoint of a matrix $A = [a_{i,j}]$ in $M_n(R)$ was defined in [12] as $A^* = [a^*_{r,s}]$, where
\[
a^*_{r,s} = \sum_{\tau, \rho} \text{sgn}(\rho) a_{\tau(1), \rho(\tau(1))} \cdots a_{\tau(s-1), \rho(\tau(s-1))} a_{\tau(s+1), \rho(\tau(s+1))} \cdots a_{\tau(n), \rho(\tau(n))}
\]
and the sum is taken over all permutations $\tau, \rho \in S_n$ of the set $\{1, 2, \ldots, n\}$ with $\tau(s) = s$ and $\rho(s) = r$. The left and right determinants of $A$ were defined in [13] as follows:

$$\text{idet}(A) = \text{tr}(A^*A) \text{ and } \text{rdet}(A) = \text{tr}(AA^*).$$

If the base ring $R$ is commutative, then $\text{tr}(AB) = \text{tr}(BA)$ for all $A, B \in M_n(R)$. In spite of the fact that this well known trace identity is no longer valid for matrices over a non-commutative ring, the left and right determinants of $A$ coincide (it was not recognized in [13]).

**Proposition 2.1.** The traces of the product matrices $A^*A$ and $AA^*$ are equal: $\text{tr}(A^*A) = \text{tr}(AA^*)$.

**Proof.** The trace of a matrix is the sum of the diagonal entries, hence

$$\text{tr}(A^*A) = \sum_{1 \leq r, s \leq n} a_{r,s}^* a_{s,r} = \sum_{\rho \in S_n} \text{sgn}(\rho) u(\rho, \tau, s),$$

where $S_n^* = \{ (\tau, s) \mid \tau \in S_n, 1 \leq s \leq n \text{ and } \tau(s) = s \}$ and

$$u(\rho, \tau, s) = a_{\tau(1), \rho(\tau(1))} \cdots a_{\tau(s), \rho(\tau(s))} a_{\tau(s+1), \rho(\tau(s+1))} \cdots a_{\tau(n), \rho(\tau(n))}.$$

Similarly,

$$\text{tr}(AA^*) = \sum_{1 \leq r, p \leq n} a_{p,r} a_{r,p} = \sum_{\rho \in S_n} \text{sgn}(\rho) v(\rho, \alpha, p),$$

where

$$v(\rho, \alpha, p) = a_{\alpha(p), \rho(\alpha(p))} a_{\alpha(1), \rho(\alpha(1))} \cdots a_{\alpha(p-1), \rho(\alpha(p-1))} a_{\alpha(p+1), \rho(\alpha(p+1))} \cdots a_{\alpha(n), \rho(\alpha(n))}.$$

Consider the following $S_n^* \rightarrow S_n^*$ maps

$$\Theta(\tau, s) \longmapsto (\Theta(\tau, s), \tau(1)) \text{ and } (\alpha, p) \longmapsto (\Delta(\alpha, p), \alpha(n)),$$

where the permutations $\Theta(\tau, s)$ and $\Delta(\alpha, p)$ in $S_n$ are defined by

$$(\tau(1), 2, \ldots, \tau(1) - 1, \tau(1) + 1, \ldots, n - 1, n) \overset{\Theta(\tau, s)}{\longrightarrow} (\tau(1), \tau(2), \ldots, \tau(s-1), \tau(s+1), \ldots, \tau(n), s)$$

and

$$(1, 2, \ldots, \alpha(n) - 1, \alpha(n) + 1, \ldots, n - 1, n) \overset{\Delta(\alpha, p)}{\longrightarrow} (\alpha(1), \ldots, \alpha(p-1), \alpha(p+1), \ldots, \alpha(n)),$$

respectively. It is straightforward to see that the above maps are mutual inverses of each other:

$$\Delta(\Theta(\tau, s), \tau(1)) = \tau, \Theta(\tau, s)(n) = s \text{ and } \Theta(\Delta(\alpha, p), \alpha(n)) = \alpha, \Delta(\alpha, p)(1) = p.$$

Since

$$u(\rho, \tau, s) = v(\rho, \Theta(\tau, s), \tau(1)) \text{ and } v(\rho, \alpha, p) = u(\rho, \Delta(\alpha, p), \alpha(n)),$$

our claim is proved. $\square$

In view of Proposition 2.1,

$$\text{idet}(A) = \text{tr}(A^*A) = \text{tr}(AA^*)$$

can be called the symmetric determinant of $A$. Let $R[x]$ denote the ring of polynomials of the single commuting indeterminate $x$, with coefficients in $R$. Let $[R, R]$ denote the additive subgroup of $R$ generated by all commutators $[r, s] = rs - sr$.
with \( r, s \in R \). Using the unit matrix \( I \in M_n(R) \), the symmetric characteristic polynomial of \( A \) is the symmetric determinant of the \( n \times n \) matrix \( xI - A \) in \( M_n(R[x]) \):

\[
p(x) = \text{sdet}(xI - A) = \text{tr}((xI - A)(xI - A)^*) = \text{tr}((xI - A)^*(xI - A)).
\]

The proof of Theorem 3.1 is based on the use of the following result from [13].

Theorem 2.2. The symmetric characteristic polynomial \( p(x) \in R[x] \) of a matrix \( A \in M_n(R) \) is of the form

\[
p(x) = \lambda_0 + \lambda_1 x + \cdots + \lambda_{n-1} x^{n-1} + \lambda_n x^n
\]

with \( \lambda_0, \lambda_1, \ldots, \lambda_n \in R \) and \( \lambda_n = n! \). The product matrices \( n(xI - A)(xI - A)^* \) and \( n(xI - A)^*(xI - A) \) can be written as

\[
n(xI - A)(xI - A)^* = p(x)I + C_0 + C_1 x + \cdots + C_n x^n
\]

and

\[
n(xI - A)^*(xI - A) = p(x)I + D_0 + D_1 x + \cdots + D_n x^n,
\]

where the matrices \( C_i, D_i \in M_n(R) \), \( 0 \leq i \leq n \) are uniquely determined by \( A \). The entries of the matrices \( C_i, D_i \) are in \([R, R]\), i.e. \( C_i, D_i \in M_n([R, R]) \) for all \( 0 \leq i \leq n \). The right

\[
(\lambda_0 I + C_0) + A(\lambda_1 I + C_1) + \cdots + A^{n-1}(\lambda_{n-1} I + C_{n-1}) + A^n n! I + C_n = 0
\]

and the left

\[
(\lambda_0 I + D_0) + (\lambda_1 I + D_1) A + \cdots + (\lambda_{n-1} I + D_{n-1}) A^{n-1} + (n! I + D_n) A^n = 0
\]

Cayley-Hamilton identities hold for \( A \).

3. MATRICES OVER A RING SATISFYING \([x, y][u, v] = 0\)

Theorem 3.1. If \( A \in M_n(R) \) is a matrix over a ring \( R \) satisfying the polynomial identity \([x, y][u, v] = 0\), then an invariant Cayley-Hamilton identity of the form

\[
\sum_{0 \leq i,j \leq n} A^i c_{i,j} A^j = 0
\]

holds for \( A \). The sandwich coefficients can be obtained as \( c_{i,j} = \lambda_i \lambda_j \), where \( p(x) = \lambda_0 + \lambda_1 x + \cdots + \lambda_n x^n \) (with \( \lambda_n = n! \)) is the symmetric characteristic polynomial of \( A \).

Proof. Rearranging the left and the right Cayley Hamilton identities in Theorem 2.2, we obtain

\[
\lambda_0 I + A \lambda_1 + \cdots + A^{n-1} \lambda_{n-1} + A^n \lambda_n = -(C_0 + AC_1 + \cdots + A^{n-1} C_{n-1} + A^n C_n)
\]

and

\[
\lambda_0 I + \lambda_1 A + \cdots + \lambda_{n-1} A^{n-1} + \lambda_n A^n = -(D_0 + D_1 A + \cdots + D_{n-1} A^{n-1} + D_n A^n).
\]

The multiplication of the above identities gives that

\[
\sum_{0 \leq i, j \leq n} A^i \lambda_i \lambda_j A^j = \sum_{0 \leq i, j \leq n} A^i C_i D_j A^j.
\]

Now \( C_i D_j = 0 \) is a consequence of \( C_i, D_j \in M_n([R, R]) \) and of \([x, y][u, v] = 0\) in \( R \). To complete the proof it is enough to note that the coefficients \( \lambda_i \), \( 0 \leq i \leq n - 1 \) of the symmetric characteristic polynomial of \( A \) are invariant under the conjugate.
action of $\text{GL}_n(Z(R))$ on $M_n(R)$, where $Z(R)$ denotes the centre of $R$ (see [1] and [13]). □

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