Entropy of Black-Branes System and T-Duality

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A general black-branes system under the T-duality transformation will become another smeared system with different dimensional black branes. We first use some simple examples to see that both systems have a same value of entropy and then present a rigorous method to prove this general property. Using the property we could easily know the entropy of some complex black-brane systems.

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1 Introduction

T-duality is a symmetry of string theory relating small and large distances. It does not shown in point particle theory, indicating that the extended object of strings will experience the spacetime in dramatically different way from that in the point object of particle. Especially, it relates different string theories to each other, which preceded the second superstring revolution [1].

T-duality could also relate a $D_p$ brane to another $D_{p+1}$ brane. The property had been used to find many supergravity backgrounds which are dual to different kind of field theories. For example, Maldacena and Russo [2] had found the supergravity background which duals to the non-commutative gauge theory through T-duality. They also discussed the thermodynamics of near-extremal D3-branes with B fields and found that the entropy and other thermodynamic quantities are the same as those of the corresponding D3-branes without B fields. This means that the thermodynamics of non-commutative gauge theory is the same as that in the commutative gauge theory [2,3].

Using the T-duality and twist one can construct the supergravity backgrounds which dual to the finite temperature non-commutative dipole field theories [4,5]. The spacetime found for the case of $N = 2$ theory is [5]

$$ds^2_{10} = f(r)^{-1/2} \left[ -h(r)dt^2 + dx^2 + dy^2 + \frac{dz^2}{1 + B^2U^2 \sin^2 \theta} \right] + f(r)^{1/2} \left[ h(r)^{-1}dr^2 + r^2 \left( d\theta^2 + \cos^2 \theta d\phi^2 + \sin^2 \theta (d\chi_1^2 + \cos^2 \chi_1 d\chi_2^2 + \sin^2 \chi_1 d\chi_3^2) \right) \right]$$

$$B_{z\chi_i} = \frac{r^2 B \sin^2 \theta (\cos^2 \chi_1 d\chi_2 + \sin^2 \chi_1 d\chi_3)}{1 + r^2 B^2 \sin^2 \theta}, \quad e^{2\Phi} = \frac{1}{1 + r^2 B^2 \sin^2 \theta},$$

in which $f(r) = 1 + \frac{N^4}{r^4}$ and $h(r) = 1 - \frac{N^4}{r^4}$. Using the resulting metric and dilaton field we can find the entropy $S$ of the solution with a help of symbolic algebra calculation in a computer. The result shows that it is the same as that without dipole field strength $B$ fields. The spacetime for the case of $N = 1$ and $N = 0$ have also been found by us in [5] (which is too lengthy to be cited in here). After a computer calculation we also find that the system has a same entropy as that without dipole field strength.

It is known that a black-branes system under the T-duality transformation will become another smeared system with different dimensional black branes. Above property leads us to suspect that both systems have a same value of entropy in the general case and we will prove this property in this short paper.

Historically, Suzuki [6] had shown the property in the system of the non-extremal intersecting D-branes. In this paper we present a more straightforward method to prove this property in a general black-brane system which has non-zero NS-NS B field.
2 Entropy of Black-Brane Systems and T-duality

To begin with, we quote the formula of T-duality [7]. After the T-duality on z coordinate the metric and dilaton field become:

\[
\tilde{g}_{zz} = 1/g_{zz}, \quad \tilde{g}_{\mu z} = B_{\mu z}/g_{zz}, \quad \tilde{g}_{\mu \nu} = g_{\mu \nu} - (g_{\mu z}g_{\nu z} - B_{\mu z}B_{\nu z})/g_{zz} \quad e^{-2\tilde{\phi}} = g_{zz} e^{-2\phi}. \tag{2}
\]

**Step 1**: First, let us consider the simplest case of black D3 brane which has the metric

\[
ds^2_D = f(r)^{-1/2} \left[ -h(r)dt^2 + dx^2 + dy^2 + dz^2 \right] + f(r)^{1/2} \left[ h(r)^{-1}dr^2 + r^2d\Omega_5^2 \right], \tag{3}
\]

in which D=10. It has zero value of dilaton field. After the T-duality on z coordinate the metric becomes

\[
ds^2_{10} = f(r)^{-1/2} \left[ -h(r)dt^2 + dx^2 + dy^2 \right] + f(r)^{1/2} \left[ dz^2 + h(r)^{-1}dr^2 + r^2d\Omega_5^2 \right], e^{-2\phi} = f(r)^{-1/2}, \tag{4}
\]

which describes black D2-brane smeared along z-direction. It is interesting to see that the metric of a black D0-brane which is uniformly smeared along x, y, z is described by [8]

\[
ds^2_{10} = f(r)^{-1/2} \left[ -h(r)dt^2 \right] + f(r)^{1/2} \left[ dx^2 + dy^2 + dz^2 + h(r)^{-1}dr^2 + r^2d\Omega_5^2 \right], e^{-2\phi} = f(r)^{-3/2}. \tag{5}
\]

With a T-duality transformation on the directions x, y we deduce the solution for black D2-brane smeared along the z-direction, which is also described by (4).

Using the property that the entropy is \( \frac{1}{4}A \) we can calculate the horizon area \( A \) from the integration of \( \sqrt{|g_{D-2}|} \) in which \( |g_{D-2}| \) is the corresponding determinate of the metric without the coordinate \( t \) and \( r \). Therefore, in the following we will investigate the invariant property of \( |g_{D-2}| \) itself. Now, it is a easy work to check that the metric in (3), (4) and (5) have the same value of \(|g_{D-2}|\) and thus these systems have a same entropy. **Notice that \(|g_{K}| \) is not an invariant quantity unless \( K = D - 2 \).**

How the intrinsic cancelation mechanism, which leads to the invariant property, occurs in the mathmatical calculation will be seen in the next example.

**Step 2**: Next, we consider the more general system with the D dimensional background described by the following metric and dilaton field

\[
ds^2_S = \sum_{i=1}^{D} A_i dx_i^2, \quad e^{-2\tilde{\phi}}, \quad \Rightarrow \quad ds^2_E = \left( e^{-2\phi} \right)^{1/(D-2)} ds^2_S, \tag{6}
\]

in which \( ds^2_S \) is the line element in the string frame metric and \( ds^2_E \) is that in the Einstein frame. It follows that

\[
|g_{D-2}| = \left[ \left( e^{-2\phi} \right)^{1/(D-2)} \prod_{i=1}^{D-2} A_i \right] = e^{-4\phi} \left( \prod_{i=1}^{D-2} A_i \right). \tag{7}
\]

After T-duality on the \( x_1 \) coordinate the metric and dilaton field become

\[
ds^2_S = \sum_{i \neq 1}^{D} A_i dx_i^2 + \frac{dx_1^2}{A_1}, \quad e^{-2\tilde{\phi}} = A_1 e^{-2\phi}, \quad \Rightarrow \quad ds^2_E = \left( A_1 e^{-2\phi} \right)^{1/(D-2)} ds^2_S. \tag{8}
\]
It follows that

\[ |\tilde{g}_{D-2}| = \left| \left( A_1 e^{-2\phi} \right)^{\frac{D-2}{2}} \right| \left( \prod_{i \neq 1} A_i \right) = \left( \frac{1}{A_1^2} \right)^{\frac{D-2}{2}} \left( \prod_{i = 1}^{D-2} A_i \right). \]  

(9)

Above calculations tell us that the extra factor in the dilaton term, i.e. \( A_1^2 \), just be canceled by the extra factor in the metric term, i.e. \( \frac{1}{A_1^2} \). The intrinsic cancelation mechanism therefore leads to the invariant property of \( |g_{D-2}| \) once \( |g_{D-2}| \) contains the coordinate which T-duality performs. Notice that \( |g_{D-2}| \) is the corresponding determinate of the metric without the coordinates \( x_2 \) and \( x_3 \) which can be the arbitrary coordinates except \( x_1 \).

**Step 3:** Finally, consider the most general black-branes system which is described by the metric \( g_{\mu\nu} \) and a dilaton field \( \phi \). After the T-dual transformation on \( z \) coordinate we have the new metric \( \tilde{g}_{\mu\nu} \) and a dilaton field \( \tilde{\phi} \). With the help of transformation relation (2) we now have to investigate the following determinate

\[
\begin{vmatrix}
\tilde{g}_{11} & \tilde{g}_{12} & \cdots & \tilde{g}_{1z} \\
\tilde{g}_{21} & \tilde{g}_{22} & \cdots & \tilde{g}_{2z} \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{g}_{1z} & \tilde{g}_{2z} & \cdots & \tilde{g}_{zz}
\end{vmatrix}
= 
\begin{vmatrix}
g_{11} - \frac{B_{1z} B_{1z}}{g_{zz}} & g_{12} - \frac{B_{1z} B_{2z}}{g_{zz}} & \cdots & \frac{B_{1z}}{g_{zz}} \\
g_{21} - \frac{B_{2z} B_{1z}}{g_{zz}} & g_{22} - \frac{B_{2z} B_{2z}}{g_{zz}} & \cdots & \frac{B_{2z}}{g_{zz}} \\
\vdots & \vdots & \ddots & \vdots \\
g_{1z} - \frac{B_{zz}}{g_{zz}} & g_{2z} - \frac{B_{zz}}{g_{zz}} & \cdots & \frac{1}{g_{zz}}
\end{vmatrix}
\]

(10)

Using the simple property of matrix determinate

\[
\begin{vmatrix}
a_1 + \alpha c_1 & b_1 + \beta c_1 & \cdots & c_1 \\
a_2 + \alpha c_2 & b_2 + \beta c_2 & \cdots & c_2 \\
\vdots & \vdots & \ddots & \vdots \\
a_k + \alpha c_k & b_k + \beta c_k & \cdots & c_k
\end{vmatrix}
= 
\begin{vmatrix}
a_1 & b_1 & \cdots & c_1 \\
a_2 & b_2 & \cdots & c_2 \\
\vdots & \vdots & \ddots & \vdots \\
a_k & b_k & \cdots & c_k
\end{vmatrix}
\]

(11)

relation (10) becomes

\[
\begin{vmatrix}
\tilde{g}_{11} & \tilde{g}_{12} & \cdots & \tilde{g}_{1z} \\
\tilde{g}_{21} & \tilde{g}_{22} & \cdots & \tilde{g}_{2z} \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{g}_{1z} & \tilde{g}_{2z} & \cdots & \tilde{g}_{zz}
\end{vmatrix}
= 
\begin{vmatrix}
g_{11} - \frac{g_{1z} g_{1z}}{g_{zz}} & g_{12} - \frac{g_{1z} g_{2z}}{g_{zz}} & \cdots & \frac{B_{1z}}{g_{zz}} \\
g_{21} - \frac{g_{2z} g_{1z}}{g_{zz}} & g_{22} - \frac{g_{2z} g_{2z}}{g_{zz}} & \cdots & \frac{B_{2z}}{g_{zz}} \\
\vdots & \vdots & \ddots & \vdots \\
g_{1z} - \frac{g_{zz}}{g_{zz}} & g_{2z} - \frac{g_{zz}}{g_{zz}} & \cdots & \frac{1}{g_{zz}}
\end{vmatrix}
\]

\[
= 
\begin{vmatrix}
g_{11} - \frac{g_{1z} g_{1z}}{g_{zz}} & g_{12} - \frac{g_{1z} g_{2z}}{g_{zz}} & \cdots & 0 \\
g_{21} - \frac{g_{2z} g_{1z}}{g_{zz}} & g_{22} - \frac{g_{2z} g_{2z}}{g_{zz}} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1/g_{zz}
\end{vmatrix}
\]
Notice that the above metric determinate transformation is performed in the string frame.
Now, the extra factor $\frac{1}{g_{zz}}$ in above result will be canceled by that from the dilation field if we consider the determinate $|g_{D-2}|$ in Einstein frame, as could be easily seen from (9). Thus we have shown that the entropy of the T-duality transformed black-branes system, which is a smeared higher or lower dimensional black-branes, is the same as the original system.

3 Discussions

Let us make following comments to conclude this paper.

1. As the entropy in statistics is related to the number of the microstate it is reasonable to believe that the number of the microstate does not be changed once the space was performed T-duality transformation. Thus the entropy of the T-duality transformed black-branes system is the same as the original system. However, this argument could only be directly applied to the p-branes system such as the Strominger-Vafa model [9], in which the entropy could be calculated directly by counting the degeneracy of BPS soliton bound states in the p-branes system.

2. The steps to construct the Maldacena and Russo [2] model are performing a T-dual, then a rotation and finally take a T-dual. As the rotation does not modify the volume the final value of $|g_{D-2}|$ is therefore invariant. Thus the entropy is invariant. When this procedure could be used to found the corresponding supergravity background for arbitrary gauge theory (that in [2] is the $N = 4$ theory) then we may conclude that the arbitrary gauge theory has a same entropy as its corresponding noncommutative part of gauge theory.

3. In constructing the dual gravity of noncommutative dipole theory [4,5] we perform a T-dual, then a twist and finally take a T-dual. As the twist does not modify the volume the entropy is also invariant. Using the property the entropy associated with (1) could be known without complex computer calculations.

4. Finally, the property derived in this paper could be the corresponding determinate of the metric without the two arbitrary coordinates, which in evaluating the black brane entropy are the time $t$ and radius $r$ coordinates. Thus, besides the entropy there are many other invariant quantities under the T-duality transformation. The physical meaning of these quantities are unknown as yet.

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