HAMILTONIAN DYNAMICS OF A SPACESHIP IN ALCUBIERRE AND GÖDEL METRICS: RECURSION OPERATORS AND UNDERLYING MASTER SYMMETRIES

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We study the Hamiltonian dynamics of a spaceship in the background of Alcubierre and Gödel metrics. We derive the Hamiltonian vector fields governing the system evolution, and construct and discuss the associated recursion operators generating the constants of motion. We also characterize relevant master symmetries.

Keywords: Hamiltonian dynamics, Alcubierre metric, Poisson bracket, Gödel metric, recursion operator, master symmetry

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1. Introduction

In 1949, Gödel [1] found a solution of the Einstein equations corresponding to a homogeneous mass distribution that rotates at each point of space [2]. This distribution of matter causes unusual effects, such as the existence of closed timelike curves (CTCs). However, Gödel’s metric has the advantage that it is of a rather compact form and most calculations can be carried out analytically [3]. The Gödel solution makes it apparent that general relativity permits solutions with closed time-like worldlines, even when the metric possesses a local Lorentzian character that ensures an inherited regular chronology and therefore the local validity of the causality principle [4]. In recent decades, the study of CTCs attracted the attention of several authors (see, e.g., [2]–[6]). In particular, in 2004, Kajari et al. [3] presented exact expressions for the Sagnac effect of a Gödel Universe. In their work, they proposed a formulation of the Sagnac time delay in terms of invariant physical quantities and showed that this result is very close to the analogous formula for the Sagnac time delay of a rotating coordinate system in Minkowski spacetime.
Moreover, it is known that in general relativity, faster-than-light (FTL) speed is only forbidden locally [2], and is not as exotic as it might seem at first glance. For instance, the expansion of the Universe can make two distant galaxies move at an FTL relative speed, while each one is moving locally inside its light cone. The opposite might also be possible: if the spacetime were contracting fast enough, each of the two galaxies would be moving near the speed of light locally (inside its light cone) in opposite directions, but globally they would be approaching each other. With these considerations in mind, Alcubierre in 1994 introduced [7] the so-called warp drive metric, within the framework of general relativity, which in principle allows superluminal motion, that is, FTL travel [2], [8]. Alcubierre’s idea was to create spacetime contraction in front of an object (a spaceship, for example), and spacetime dilation behind it. The contraction then pulls the object forward, and the dilation pushes the object forward as well [2]. Locally, the object is inside its light cone, but due to this spacetime manipulation, it moves at a speed higher than the speed of light \( c \) in a flat-spacetime vacuum. The object is within the so-called warp bubble. In this way, the object can travel at arbitrarily high speeds, without violating the laws of special and general relativity, or other known physical laws [8]. Following Alcubierre’s idea, many investigations were done (see, e.g. [2], [8]).

In addition, in the last few decades, there was a renewed interest in completely integrable Hamiltonian systems (IHSs), the concept of which goes back to Liouville in 1897 [9] and Poincaré in 1899 [10]. In short, IHSs are defined as nonlinear differential equations admitting a Hamiltonian description and possessing sufficiently many constants of motion that allow them to be integrated by quadratures [11]. Many of these systems obey Hamiltonian dynamics with respect to two compatible symplectic structures [12]–[14], permitting a geometric interpretation of the so-called recursion operator [15]. A description of integrability working both for systems with finitely many degrees of freedom and for field theory can be given in terms of an invariant, diagonalizable mixed \((1,1)\)-tensor field with bidimensional eigenspaces and the vanishing Nijenhuis torsion.

A powerful method for describing IHSs with involutive Hamiltonian functions or constants of motion involves the recursion operator admitting a vanishing Nijenhuis torsion. In 2015, Takeuchi constructed recursion operators of Hamiltonian vector fields of geodesic flows for some Riemannian and Minkowski metrics [16] and obtained the corresponding constants of motion. In his work, he used five particular solutions of the Einstein equation in the Schwarzschild, Reissner–Nordström, Kerr, Kerr–Newman, and FLRW metrics, and constructed recursion operators ensuring the complete integrability of the Hamiltonian functions. In 2019, we investigated the same problem in a noncommutative Minkowski phase space and the Kepler problem in a deformed phase space, and obtained the associated constants of motion [17], [18].

Since the work by Magri [13], the integrability associated with bi-Hamiltonian structures [12] became one of the most efficient methods used in studying evolution equations in both finite- and infinite-dimensional dynamical systems [15], [19]. When a completely integrable Hamiltonian system admits a bi-Hamiltonian construction, infinite hierarchies of conserved quantities can be generated using the construction by Oevel [20] based on scaling invariances and master symmetries [21], [22]. In 1997 and 1999, Smirnov [19], [22] formulated a constructive method for transforming a completely integrable Hamiltonian system, in Liouville’s sense, into the Magri–Morosi–Gel’fand–Dorfman’s (MMGD) bi-Hamiltonian form. In 2005, Rañada [23] proved the existence of a bi-Hamiltonian structure arising from a nonsymplectic symmetry as well as the existence of master symmetries and additional integrals of motion (weak superintegrability) in certain particular cases. Recently, in 2021 [24], we also constructed a hierarchy of bi-Hamiltonian structures for the Kepler problem, and computed conserved quantities using related master symmetries.

In this paper, we address the Hamiltonian dynamics of a spaceship in the Alcubierre and Gödel metrics. We derive the related recursion operators and discuss their relevant master symmetries. We prove that the two models satisfy the same dynamics and exhibit a set of similar master symmetries.
The paper is organized as follows. In Sec. 2, we give the main tools used in this work. In Sec. 3, we give the Hamiltonian function, the symplectic form, and the vector field describing the Hamiltonian dynamics of a spaceship in the Alcubierre metric and construct the associated recursion operators. In Sec. 4, we do the same for the Gödel metric. In Sec. 5, we introduce bi-Hamiltonian structures, define the hierarchy of master symmetries, and compute the corresponding conserved quantities. We conclude in Sec. 6.

2. Recursion operator and master symmetry

A characterization of IHSs is given by De Filippo et al. in the following theorem [11].

**Theorem 1.** Let $X$ be a dynamical vector field on a $2n$-dimensional manifold $M$. If the vector field $X$ admits a diagonalizable mixed $(1,1)$-tensor field $T$ that is invariant under $X$, and has a vanishing Nijenhuis torsion and doubly degenerate eigenvalues with nowhere vanishing differentials, then there exists a symplectic structure and a Hamiltonian function $H$ such that the vector field $X$ is a separable Hamiltonian vector field of $H$, and $H$ is completely integrable with respect to the symplectic structure. Such a $(1, 1)$-tensor field $T$ is called a recursion operator of $X$.

**Lemma 1.** Consider vector fields $X_l = -\frac{\partial}{\partial x_{n+l}}, \quad l = 1, \ldots, n,$ on $\mathbb{R}^{2n}$ and let $T$ be a $(1,1)$-tensor field on $\mathbb{R}^{2n}$ given by

$$T = \sum_{i=1}^{n} x_i \left( \frac{\partial}{\partial x_i} \otimes dx_i + \frac{\partial}{\partial x_{n+i}} \otimes dx_{n+i} \right).$$

Then the Nijenhuis torsion $N_T$ and the Lie derivative $\mathcal{L}_{X_l}$ of $T$ vanish,

$$(N_T)_{ij}^h := T_i^h \frac{\partial T_j^h}{\partial x^i} - T_j^h \frac{\partial T_i^h}{\partial x^j} + T_k^h \frac{\partial T_i^k}{\partial x^j} - T_k^h \frac{\partial T_j^k}{\partial x^i} = 0, \quad \mathcal{L}_{X_l} T = 0,$$

and hence the $(1,1)$-tensor field $T$ is a recursion operator of $X_l$ ($l = 1, \ldots, n$).

Given a general dynamical system defined on a $2n$-dimensional manifold $Q$ [19],

$$\dot{x}(t) = X(x), \quad x \in Q, \quad X \in TQ,$$

(1)

where $TQ$ is the tangent bundle of $Q$. If this system (1) admits two different Hamiltonian representations

$$\dot{x}(t) = X_{H_1, H_2} = P_1 \, dH_1 = P_2 \, dH_2,$$

(2)

its integrability and many other properties can be studied within Magri’s approach: a bi-Hamiltonian vector field $X_{H_1, H_2}$ is defined by two pairs of Poisson bivectors $P_1$ and $P_2$ and Hamiltonian functions $H_1$ and $H_2$. The Poisson bivectors $P_1$ and $P_2$ are compatible, having a vanishing Schouten–Nijenhuis bracket [25]: $[P_1, P_2]_{SN} = 0$. Such a manifold $Q$ equipped with two Poisson bivectors is called a double Poisson manifold and the quadruple $(Q, P_1, P_2, X_{H_1, H_2})$ is called a bi-Hamiltonian system.

In differential geometric terms, a vector field $Y$ on the cotangent bundle $T^*Q$ that satisfies

$$[X_{H'}, Y] \neq 0, \quad [X_{H'}, X] = 0, \quad [X_{H'}, Y] = X,$$

is called a master symmetry or a generator of symmetries of degree $m = 1$ for the Hamiltonian vector field $X_{H'}$ [21], [23], [26]–[28].
3. Recursion operator of a Hamiltonian vector field in the Alcubierre metric

In this paper, without loss of generality, we consider the particular Alcubierre metric [7]

\[ ds^2 = -dt^2 + \left( dx - v_s f(r_s) \, dt \right)^2 + dy^2 + dz^2 \]

describing the motion of a spaceship along the \( x \) axis of a Cartesian coordinate system,

\[ \alpha = 1, \quad \beta_2 = -v_s f(r_s), \quad \beta_3 = \beta_4 = 0, \quad \gamma_{ij} = \delta_{ij} \quad (\delta_{ij} \text{ is the Kronecker symbol}), \]

where

\[ v_s = \frac{dx_s(t)}{dt}, \quad r_s(t) = \left( (x - x_s(t))^2 + y^2 + z^2 \right)^{1/2}, \]

\[ f(r_s) = \frac{\tanh(\sigma(r_s + R)) - \tanh(\sigma(r_s - R))}{2\tanh(\sigma R)} \]

where \( \sigma > 0, \, R > 0 \) are arbitrary parameters, and

\[ \lim_{\sigma \to \infty} f(r_s) = \begin{cases} 
1 & \text{for } r_s \in [-R, R[,
1/2 & \text{for } r_s \in (-R, R],
0 & \text{otherwise}.
\end{cases} \]

In the limit as \( \sigma \to \infty \), with \( r_s \in [-R, R[ \), this particular Alcubierre metric becomes

\[ ds^2 = -dt^2 + (dx - v_s \, dt)^2 + dy^2 + dz^2, \quad (3) \]

where the tensor metric and its inverse are given by

\[ g_{\nu\mu} = \begin{pmatrix} -(1 - v_s^2) & -v_s & 0 & 0 \\ -v_s & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad g^{\nu\mu} = \begin{pmatrix} -1 & -v_s & 0 & 0 \\ -v_s & (1 - v_s^2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]

Geometrically, this spacetime can be interpreted as follows [7]. First, because \( \gamma_{ij} = \delta_{ij} \), the 3-geometry of the hypersurfaces is always flat. Second, the fact that the lapse is given by \( \alpha = 1 \) implies that the time-like curves normal to these hypersurfaces are geodesics, i.e., the Eulerian observers are in free fall. We note that the spacetime is nevertheless not flat due to the presence of a nonuniform shift. Finally, because the shift vector vanishes for \( r_s \gg R \), at any time \( t \), the spacetime is essentially flat everywhere except within a region with a radius of the order of \( R \), centered at the point \( (x_s(t), 0, 0) \).

Let now \( Q = \mathbb{R}^4 := \{ q^1 = t, q^2 = x, q^3 = y, q^4 = z \} \) be the manifold describing the configuration space and \( T^*Q = Q \times \mathbb{R}^4 \) be the cotangent bundle with local coordinates \( (q, p) \), and the natural symplectic structure \( \omega_A : TQ \to T^*Q \) given by

\[ \omega_A = \sum_{\nu=1}^{4} dp_\nu \wedge dq_\nu, \]

where \( TQ \) is the tangent bundle. By definition, \( \omega_A \) is nondegenerate. It induces a map \( \mathcal{P}_A : T^*Q \to TQ \), called a bivector field, defined by

\[ \mathcal{P}_A = \sum_{\nu=1}^{4} \frac{\partial}{\partial p_\nu} \wedge \frac{\partial}{\partial q_\nu}. \]
which is the inverse map of \(\omega_A\), i.e., \(\omega_A \circ \mathcal{P}_A = \mathcal{P}_A \circ \omega_A = 1\) [29]. In this case, the Hamiltonian vector field \(X_f\) of a Hamiltonian function \(f\) is given by \(X_f = \mathcal{P}_A df\). On the cotangent bundle \(T^*Q\), (3) takes the form

\[
ds^2 = -(dq^1)^2 + (dq^2 - v_s dq^1)^2 + (dq^3)^2 + (dq^4)^2.
\]

In our framework, the Hamiltonian function \(\mathcal{H}_A\) describing the dynamics of a spaceship in the Alcubierre metric and its corresponding 1-form \(d\mathcal{H}_A \in T^*Q\) are given by

\[
\mathcal{H}_A := \frac{1}{2} \sum_{\nu,\mu=1}^4 g^{\nu\mu} p_\nu p_\mu = \frac{1}{2} (-p_1^2 - v_s p_1 p_2 + (1 - v_s^2) p_2^2 + p_3^2 + p_4^2),
\]

and

\[
d\mathcal{H}_A = -(p_1 + v_s p_2) dp_1 + (-v_s p_1 + (1 - v_s^2) p_2) dp_2 + p_3 dp_3 + p_4 dp_4 - v_s(p_1 + v_s p_2) p_2 dq^1.
\]

Then the Hamiltonian vector field of \(\mathcal{H}_A\) with respect to the symplectic structure \(\omega_A\) is derived as

\[
X_{\mathcal{H}_A} := \{\mathcal{H}_A, \cdot\} = -(p_1 + v_s p_2) \frac{\partial}{\partial q^1} + (-v_s p_1 + (1 - v_s^2) p_2) \frac{\partial}{\partial q^2} + p_3 \frac{\partial}{\partial q^3} + p_4 \frac{\partial}{\partial q^4} + v_s(p_1 + v_s p_2) \frac{\partial}{\partial p_1}.
\]

This Hamiltonian vector field satisfies the required condition for a Hamiltonian system, i.e.,

\[
\iota_{X_{\mathcal{H}_A}} \omega_A = -d\mathcal{H}_A,
\]

where \(\iota_{X_{\mathcal{H}_A}} \omega_A\) is the inner product of \(\omega_A\) with respect to the Hamiltonian vector field \(X_{\mathcal{H}_A}\). Hence, the triplet \((T^*Q, \omega_A, \mathcal{H}_A)\) is a Hamiltonian system.

In what follows, we consider the Hamilton–Jacobi equation with respect to Hamiltonian function (4) and introduce a generating function \(W\) satisfying the canonical transformations [30], [31]

\[
p = \frac{\partial W}{\partial q}, \quad P = -\frac{\partial W}{\partial Q}.
\]

Because the Hamiltonian function \(\mathcal{H}_A\) does not explicitly depend on the time \(t\), it follows that setting \(V = W - Et\), we can find an additive separable solution:

\[
W = W_1(q^1) + W_2(q^2) + W_3(q^3) + W_4(q^4).
\]

The Hamilton–Jacobi equation [31]

\[
\frac{\partial V}{\partial t} + \mathcal{H}_A \left( \frac{\partial V}{\partial q}, q, t \right) = 0
\]

then reduces to the nonlinear equation

\[
E = \frac{1}{2} \left\{ \left( \frac{\partial W}{\partial q^1} \right)^2 - 2v_s \frac{\partial W}{\partial q^1} \frac{\partial W}{\partial q^2} + (1 - v_s^2) \left( \frac{\partial W}{\partial q^2} \right)^2 + \left( \frac{\partial W}{\partial q^3} \right)^2 + \left( \frac{\partial W}{\partial q^4} \right)^2 \right\},
\]

where \(E\) is a constant.
We note that the Hamiltonian function does not include \( q^2, q^3, \) and \( q^4 \). Then, putting
\[
\frac{dW_2}{dq^2}(q^2) = \alpha_0, \quad \frac{dW_3}{dq^3}(q^3) = \beta_0, \quad \frac{dW_4}{dq^4}(q^4) = \gamma_0,
\]
where \( \alpha_0, \beta_0, \gamma_0 \) are constants and \( 2E \leq \beta_0^2 + \gamma_0^2 + \alpha_0^2 \), we bring (5) to the form
\[
\left( \frac{dW_1}{dq^1} \right)^2 + 2\alpha_0 v_s \frac{dW_1}{dq^1} + K = 0,
\]
where \( K = 2E - (\beta_0^2 + \gamma_0^2 + (1 - v_s^2)\alpha_0^2) \). Now, setting \( \Psi = dW_1/dq^1 \), with \( W_1(0) = 0 \), we obtain the quadratic equation
\[
\Psi^2 + 2\alpha_0 v_s \Psi + K = 0,
\]
with \( \Delta_A = -8E + 4(\beta_0^2 + \gamma_0^2 + \alpha_0^2) \geq 0 \). We then have two possible cases \( \Delta_A > 0 \) or \( \Delta_A = 0 \).

**Case 1.** For \( \Delta_A > 0 \), we obtain
\[
\Psi_1 = -v_s \alpha_0 + \sqrt{(\beta_0^2 + \gamma_0^2 + \alpha_0^2) - 2E}, \quad \Psi_2 = -v_s \alpha_0 - \sqrt{(\beta_0^2 + \gamma_0^2 + \alpha_0^2) - 2E},
\]
leading to solutions for the generating function \( W \)
\[
W_a = -\alpha_0 q_s - \sqrt{(\beta_0^2 + \gamma_0^2 + \alpha_0^2) - 2E} \cdot q^1 + \alpha_0 q^2 + \beta_0 q^3 + \gamma_0 q^4,
\]
\[
W_b = -\alpha_0 q_s + \sqrt{(\beta_0^2 + \gamma_0^2 + \alpha_0^2) - 2E} \cdot q^1 + \alpha_0 q^2 + \beta_0 q^3 + \gamma_0 q^4.
\]
which can be expressed in terms of \( q^i \) and \( Q^i \) as
\[
W_a = -Q^2 q_s - \sqrt{\sum_{k=2}^{4} (Q^k)^2 - 2Q^1} - v_s Q^2, \quad W_b = -Q^2 q_s + \sqrt{\sum_{k=2}^{4} (Q^k)^2 - 2Q^1} - v_s Q^2,
\]
\[
q^1 = -P_1 \sqrt{\sum_{k=2}^{4} (Q^k)^2 - 2Q^1}, \quad q^2 = q_s - P_2 - Q^2 P_1, \quad q^3 = -P_3 - Q^3 P_1, \quad q^4 = -P_4 - Q^4 P_1;
\]
\[
P_1 = \frac{P_1 + v_s P_2}{P_1 + v_s P_2} q^1 - q^2; \quad P_2 = \frac{-P_2 q_s}{P_1 + v_s P_2} + q_s - q^2, \quad P_3 = \frac{-P_3 q_s}{P_1 + v_s P_2} - q^3, \quad P_4 = \frac{-P_4 q_s}{P_1 + v_s P_2} - q^4,
\]
where \( Q^1 = H, Q^2 = p_2, Q^3 = p_3, Q^4 = p_4 \).

- For \( W = W_a \), we obtain the relations

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• For \( W = W_b \), we have

\[
\begin{align*}
  p_1 &= \sqrt{\sum_{k=2}^{4} (Q^k)^2 - 2Q^1 - v_q Q^2}, \\
  p_2 &= Q^2, \\
  p_3 &= Q^3, \\
  p_4 &= Q^4, \\
  q_1 &= P_1 \sqrt{\sum_{k=2}^{4} (Q^k)^2 - 2Q^1}, \\
  q_2 &= q_s - P_2 - Q^2 P_1, \\
  q_3 &= -P_3 - Q^3 P_1, \\
  q_4 &= -P_4 - Q^4 P_1.
\end{align*}
\]

(9)

Defined in the coordinate system \((Q, P)\), the Alcubierre symplectic form and the vector field are respectively given by

\[
\omega_A = \sum_{\nu=1}^{4} dP_\nu \wedge dQ^\nu, \quad X_{\mathcal{H}_A} := \{\mathcal{H}_A, \cdot\} = -\frac{\partial}{\partial P_1}.
\]

In this condition, a tensor field \( T_A \) of \((1, 1)\) type can be expressed as

\[
T_A = \sum_{\nu=1}^{4} Q^\nu \left( \frac{\partial}{\partial P_\nu} \otimes dP_\nu + \frac{\partial}{\partial Q^\nu} \otimes dQ^\nu \right).
\]

With \( x_\nu = Q^\nu \) and \( x_{\nu+4} = P_\nu \) in Lemma 1, where \( \nu = 1, 2, 3, 4 \), the tensor field \( T_A \) takes the form

\[
T_A = \sum_{\nu=1}^{4} Q^\nu \left( \frac{\partial}{\partial P_\nu} \otimes dP_\nu + \frac{\partial}{\partial Q^\nu} \otimes dQ^\nu \right) = \sum_{i,j=1}^{2n} (T_A)^i_j \frac{\partial}{\partial x^i} \otimes dx^j,
\]

where \( x \equiv (Q^1, \ldots, Q^4, P_1, \ldots, P_4) \). The matrix \((T_A)^i_j\) is given by

\[
(T_A)^i_j = \begin{pmatrix} G^t & O \\ O & G \end{pmatrix}, \quad G = \begin{pmatrix} Q^1 & 0 & 0 & 0 \\ 0 & Q^2 & 0 & 0 \\ 0 & 0 & Q^3 & 0 \\ 0 & 0 & 0 & Q^4 \end{pmatrix}.
\]

The tensor \( T_A \) satisfies \( \mathcal{L}_{X_{\mathcal{H}_A}} T_A = 0 \), \( N_{T_A} = 0 \) and \( \deg Q^\nu = 2 \), proving that \( T_A \) is a recursion operator of \( X_{\mathcal{H}_A} \).

The constants of motion are

\[
\text{Tr} T_A^h = 2((Q^1)^h + (Q^2)^h + (Q^3)^h + (Q^4)^h), \quad h \in \mathbb{N}.
\]

Returning to the original coordinate system \((q, p)\), the generating functions \( W_a \) and \( W_b \) lead to the following result for the Alcubierre metric:

\[
ds^2 = -(dq^1)^2 + (dq^2 - v_s dq^1)^2 + (dq^3)^2 + (dq^4)^2.
\]

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Proposition 1. Under the conditions

\[
\dot{v}_s - \frac{1}{q^t},
\]

(10)

\[
\dot{v}_s v^{h-1} = \left( \frac{p_1}{p_2} \right)^{h-1} (p_1 + v_s p_2)^{h-1}, \quad h \in \mathbb{N},
\]

(11)

the Hamiltonian vector field has a recursion operator given by

\[
T_\Lambda = \sum_{\mu, \nu=1}^{4} \left( \tilde{M}_\mu^\nu \frac{\partial}{\partial q^\mu} \otimes dq^\nu + \tilde{N}_\mu^\nu \frac{\partial}{\partial p^\nu} \otimes dp^\mu + \tilde{L}_\mu^\nu \frac{\partial}{\partial q^\mu} \otimes dq^\nu + \tilde{R}_\mu^\nu \frac{\partial}{\partial p^\nu} \otimes dq^\mu \right),
\]

(12)

with the corresponding constants of motion

\[
\text{Tr} T^h_\Lambda = \mathcal{H}^h + 2(p_2^h + p_3^h + p_4^h) + \left( \frac{\mathcal{H} p_1}{p_1 + v_s p_2} \right)^h + \left( \frac{v_s p_2 q_1 (\mathcal{H} - p_2)}{(p_1 + v_s p_2)^2} \right)^h + (\dot{v}_s p_2 \mathcal{H})^h,
\]

\[h \in \mathbb{N},\]

where the coordinate-dependent quantities \(\tilde{M}_\mu^\nu, \tilde{N}_\mu^\nu, \tilde{L}_\mu^\nu,\) and \(\tilde{R}_\mu^\nu\) are expressed as

\[
\begin{align*}
\tilde{M}_1^1 &= J p_1 \mathcal{H}_\Lambda, \\
\tilde{M}_2^1 &= p_2 [J(p_2 - \mathcal{H}_\Lambda) - v_s], \\
\tilde{M}_k^1 &= J p_k (p_k - \mathcal{H}_\Lambda), \quad k = 3, 4, \\
\tilde{M}_j^1 &= p_j, \quad j = 2, 3, 4, \\
\tilde{M}_n^1 &= 0 \text{ otherwise}, \\
\tilde{L}_k^1 &= -\tilde{L}_k^1 = J^2 p_k q_1^1 (p_k - \mathcal{H}_\Lambda), \quad k = 2, 3, 4, \\
\tilde{L}_j^1 &= J^2 v_s p_k q_1^1 (\mathcal{H}_\Lambda - p_k), \quad k = 2, 3, 4, \\
\tilde{L}_j^1 &= 0, \quad j = 1, 3, 4, \\
\tilde{L}_n^1 &= 0 \text{ otherwise}, \\
\tilde{N}_1^1 &= \mathcal{H}_\Lambda, \\
\tilde{N}_2^1 &= (p_2 - \mathcal{H}_\Lambda)(J p_2 - v_s), \\
\tilde{N}_k^1 &= J p_k (p_k - \mathcal{H}_\Lambda), \quad k = 3, 4, \\
\tilde{N}_j^1 &= p_j, \quad j = 2, 3, 4, \\
\tilde{N}_n^1 &= 0 \text{ otherwise}, \\
\tilde{R}_1^1 &= \dot{v}_s p_2 \mathcal{H}_\Lambda, \\
\tilde{R}_n^1 &= 0 \text{ otherwise}.
\end{align*}
\]

Here, \(n, m = 1, 2, 3, 4, J = 1/(p_1 + v_s p_2)\) (and \(p_1 + v_s p_2 > 0\)).

Proof. Using (8) or (9), it is straightforward to obtain (12). Furthermore, using conditions (10) and (11), we obtain \(\mathcal{L}_{X_{\mathcal{H}_\Lambda}} (\text{Tr} T^h_\Lambda) = 0\), proving that \(\text{Tr} T^h_\Lambda\) are constants of motion.

Case 2. For \(\Delta_\Lambda = 0\), we obtain the double root \(\Psi = -v_s \alpha_0\), yielding

\[
W = -\alpha_0 q_s + \alpha_0 q^2 + \beta_0 q^3 + \gamma_0 q^4,
\]

or, equivalently,

\[
W = -Q^2 q_s + \sum_{k=2}^{4} Q^k q^k
\]

in terms of \(q^i\) and \(Q^i\), \(Q^2 = \alpha_0, Q^3 = \beta_0,\) and \(Q^4 = \gamma_0\), inducing the following relation between the canonical coordinate systems \((Q, P)\) and \((q, p)\):

\[
\begin{align*}
p_1 &= -v_s Q^2, \\
p_2 &= Q^2, \\
p_3 &= Q^3, \\
p_4 &= Q^4, \\
q^2 &= q_s - P_2, \\
q^3 &= -P_3, \\
q^4 &= -P_4; \\
P_2 &= q_s - q^2, \\
P_3 &= -q^3, \\
P_4 &= -q^4, \\
Q^2 &= p_2, \\
Q^3 &= p_3, \\
Q^4 &= p_4.
\end{align*}
\]
The Hamiltonian function

\[ \mathcal{H}_A = \frac{1}{2} \sum_{k=2}^{4} (Q^k)^2 \]

describing the dynamics of a free particle system in the coordinate system \((Q, P)\), and the associated Hamiltonian vector field

\[ X_{\mathcal{H}_A} = -\sum_{k=2}^{4} Q^k \frac{\partial}{\partial P_k}. \]

Because \( W \) is independent of \( Q^1 \) and \( P_1 \), the \((1,1)\)-tensor field \( T_A \) can be given as

\[ T_A = \sum_{\nu=2}^{4} Q^\nu \left( \frac{\partial}{\partial P_\nu} \otimes dP_\nu + \frac{\partial}{\partial Q_\nu} \otimes dQ_\nu \right). \]

Then \( T_A \) satisfies \( \mathcal{L}_{X_{\mathcal{H}_A}} T_A = 0, \mathcal{N}_{T_A} = 0 \) and \( \deg Q^\nu = 2 \), proving by Theorem 1 that \( T_A \) is a recursion operator of \( X_{\mathcal{H}_A} \), with the constants of motion

\[ \text{Tr} T_A^h = 2((Q^2)^h + (Q^3)^h + (Q^4)^h), \quad h \in \mathbb{N}. \]

In the original coordinate system \((q, p)\), \( T_A \) becomes

\[ T_A = \sum_{\mu, \nu=1}^{4} \left( A^\nu_\mu \frac{\partial}{\partial q^\nu} \otimes dq^\mu + B^\nu_\mu \frac{\partial}{\partial p^\nu} \otimes dp^\mu \right), \]

where

\[ A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ v_s p_2 & p_2 & 0 & 0 \\ 0 & 0 & p_3 & 0 \\ 0 & 0 & 0 & p_4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -v_s p_2 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & p_3 & 0 \\ 0 & 0 & 0 & p_4 \end{pmatrix}, \]

and the constants of motion become \( \text{Tr} T_A^h = 2(p_2^h + p_3^h + p_4^h), \quad h \in \mathbb{N}. \)

4. Recursion operator of a Hamiltonian vector field in the Gödel metric

To compare the results, we also consider the Gödel line element \( ds^2 \) in dimensionless cylindrical coordinates [3]

\[ ds^2 = c^2 \, dt^2 - \frac{1}{1 + (r/2a)^2} \, dr^2 - r^2 \left( 1 - \left( \frac{r}{2a} \right)^2 \right) d\phi^2 - dz^2 + \frac{2r^2 c^2}{a \sqrt{2}} \, dt \, d\phi, \]

where \( a \) is a parameter with units of length, which represents a characteristic distance. In particular, \( r = 2a \) represents the critical radius at which CTCs can exist [15]. The corresponding tensor metric and its inverse
are given by
\[
g_{\nu\mu} = \begin{pmatrix}
1 & 0 & \frac{r^2}{a\sqrt{2}} & 0 \\
0 & 1 + (r/2a)^2 & 0 & 0 \\
\frac{r^2}{a\sqrt{2}} & 0 & -r^2(1 - \left(\frac{r}{2a}\right)^2) & 0 \\
0 & 0 & 0 & -1
\end{pmatrix},
\]
\[
g^{\nu\mu} = \begin{pmatrix}
(2a)^2 - r^2 & 0 & \frac{2a\sqrt{2}}{(2a)^2 + r^2} & 0 \\
0 & (2a)^2 + r^2 & 0 & 0 \\
\frac{2a\sqrt{2}}{(2a)^2 + r^2} & 0 & -\frac{(2a)^2}{r^2((2a)^2 + r^2)} & 0 \\
0 & 0 & 0 & -1
\end{pmatrix},
\]
where we put \( c = 1 \).

Now, let the manifold
\[
Q = \mathbb{R}^4 := \{q^1 = t, q^2 = r, q^3 = \phi q^4 = z\},
\]
\( t \in (-\infty, +\infty), \quad r \in (0, \infty), \quad \phi \in (0, 2\pi), \quad z \in (-\infty, +\infty), \)
describe the configuration space, and \( T^*Q = Q \times \mathbb{R}^4 \) be the cotangent bundle with the local coordinates \((q, p)\). The natural symplectic form and its corresponding Poisson bivector are given by
\[
\omega_G = \sum_{\nu=1}^{4} dp_\nu \wedge dq^\nu, \quad \mathcal{P}_G = \sum_{\nu=1}^{4} \frac{\partial}{\partial p_\nu} \wedge \frac{\partial}{\partial q^\nu},
\]
where \( TQ \) is the tangent bundle.

In the cotangent bundle \( T^*Q \), the Gödel metric takes the form
\[
ds^2 = (c dq^1)^2 - \frac{1}{1 + (q^2/2a)^2} (dq^2)^2 - (q^2)^2 \left(1 - \left(\frac{q^2}{2a}\right)^2\right)(dq^3)^2 -
\]
\[- (dq^4)^2 + \frac{2(cq^2)^2}{a\sqrt{2}} dq^1 dq^3. \tag{13}\]
Assuming that \((q^2)^3/2a \ll 1\), line element (13) is approximately given by
\[
ds^2 \simeq (c dq^1)^2 - (dq^2)^2 - (q^2)^2 (dq^3)^2 - 2(q^2)^2 \Omega_G dq^1 dq^2 - (dq^4)^2 + O(\Omega_G^2),
\]
where \( \Omega_G = c/\sqrt{2a} \) and \( q^2/2a \ll 1 \). Setting \( c = 1 \) leads to
\[
ds^2 \simeq (dq^1)^2 - (dq^2)^2 - (q^2)^2 (dq^3)^2 - 2(q^2)^2 \Omega_G dq^1 dq^2 - (dq^4)^2 + O(\Omega_G^2), \tag{14}\]
and the Hamiltonian function
\[
\mathcal{H}_G = \frac{1}{2((q^2)^2 \Omega_G + 1)} p_1^2 - \frac{1}{2} p_2^2 - \frac{1}{2(q^2)^2((q^2)^2 \Omega_G^2 + 1)} p_3^2 + \frac{\Omega_G}{(q^2)^2 \Omega_G + 1} p_1 p_3 - \frac{1}{2} p_4^2
\]
with the associated Hamiltonian vector field given by

\[ X_{\mathcal{H}_G} = \sum_{\mu=1}^{4} \left( U_\mu' \frac{\partial}{\partial q^\mu} - V_\mu' \frac{\partial}{\partial p_\mu} \right), \]

where

\[ U_1' = \frac{1}{(q^2)^2 \Omega_G^2 + 1} p_1 + \frac{\Omega_G}{(q^2)^2 \Omega_G^2 + 1} p_3, \quad U_2' = -p_2, \]

\[ U_3' = \frac{\Omega_G}{(q^2)^2 \Omega_G^2 + 1} p_1 - \frac{1}{(q^2)^2((q^2)^2 \Omega_G^2 + 1)} p_3, \quad U_4' = -p_4, \]

\[ V_1' = V_2' = V_3' = 0, \quad V_2 = \frac{(q^2)^2 \Omega_G^2 (2p_3^3 - (q^2)^2 p_1^3) + p_3 (p_3 - \Omega_G^3 (q^2)^2 p_1)}{(q^2)^2((q^2)^2 \Omega_G^2 + 1)^2}. \]

The vector field \( X_{\mathcal{H}_G} \) satisfies the required condition for a Hamiltonian system, i.e., \( \iota_{X_{\mathcal{H}_G}} \omega_G = -d\mathcal{H}_G \).

Hence, the triplet \((\mathcal{T}^* Q, \omega_G, \mathcal{H}_G)\) is a Hamiltonian system.

The Hamiltonian–Jacobi equation is given by

\[
E' = \frac{1}{2((q^2)^2 \Omega_G^2 + 1)} \left( \frac{\partial W'_1}{\partial q^1} \right)^2 - \frac{1}{2} \left( \frac{\partial W'_2}{\partial q^2} \right)^2 - \frac{1}{2((q^2)^2 \Omega_G^2 + 1)} \left( \frac{\partial W'_3}{\partial q^3} \right)^2 + \frac{\Omega_G}{(q^2)^2 \Omega_G^2 + 1} \frac{\partial W'_4}{\partial q^4},
\]

where \( E' \) is a constant and \( W' = \sum_{\mu=1}^{4} W'_\mu(q^\mu) \) is the generating function.

Because the Hamiltonian function \( \mathcal{H}_G \) does not include \( q^1, q^2, \) and \( q^3, \) we can set

\[
\frac{dW'_1}{dq^1} = \eta', \quad \frac{dW'_3}{dq^3} = \theta', \quad \frac{dW'_4}{dq^4} = \vartheta',
\]

yielding

\[
E' = \frac{1}{2((q^2)^2 \Omega_G^2 + 1)} \eta'^2 - \frac{1}{2} \left( \frac{dW'_2}{dq^2} \right)^2 - \frac{1}{2((q^2)^2 \Omega_G^2 + 1)} \theta'^2 + \frac{\Omega_G}{(q^2)^2 \Omega_G^2 + 1} \eta' \theta' - \frac{1}{2} \vartheta'^2,
\]

where \( \eta', \theta', \) and \( \vartheta' \) are constants such that the following conditions are satisfied:

\[
\frac{\eta'^2}{2E' + \vartheta'^2} \leq 1, \quad 2E' + \vartheta'^2 > 0, \quad (15a)
\]

\[
\frac{(\Omega_G \theta')^2}{2E' + \vartheta'^2} \leq \frac{1}{4}, \quad (15b)
\]

\[
\frac{\eta' \theta'}{2E' + \vartheta'^2} \leq \frac{1}{2 \Omega_G^2}. \quad (15c)
\]

We next obtain

\[
\left( \frac{dW'_2}{dq^2} \right)^2 = \frac{1}{(q^2)^2 \Omega_G^2 + 4} f(q^2), \quad (16)
\]

where

\[
f(q^2) = -(2E' + \vartheta'^2) \Omega_G^2 (q^2)^4 + (-2E + \theta'^2) + \eta'^2 + 2\eta' \theta' \Omega_G (q^2)^2 - \theta'^2.
\]

\[ \text{1011} \]
Putting $Z = (q^2)^2$ and considering the above condition (15a), we see that $f$ takes the form
\[ f(Z) = -(2E' + \vartheta'^2)\Omega^2_Z Z^2 + ((2E' + \vartheta'^2) + 2\eta'\Omega'_Z)Z - \theta'^2, \]
\[ \Delta_G = (2E' + \vartheta'^2)((2E' + \vartheta'^2) - 4\eta'\Phi - 4\Phi^2), \quad \Phi = \theta'\Omega_G. \]

After computation and using condition (15b), we obtain $\Delta_G = 16(2E' + \vartheta'^2) > 0$ and
\[ Z_1 = \frac{\eta'\Omega'}{(2E' + \vartheta'^2)\Omega_G} + \frac{1}{2\Omega^2_G}, \quad Z_2 = \frac{\eta'\Omega'}{(2E' + \vartheta'^2)\Omega_G} - \frac{3}{2\Omega^2_G}. \]

Using condition (15c), we obtain
\[ f(q^2) = \frac{(2E' + \vartheta'^2)}{\Omega^2_G}(1 - (q^2)^2\Omega^2_G)((q^2)^2\Omega^2_G + 1), \]

with $(q^2)^2\Omega^2_G \ll 1$. Thus, (16) becomes
\[ \frac{dW'_2}{dq^2} = \frac{\sqrt{2E' + \vartheta'^2}}{\Omega_G} \sqrt{(1 - (q^2)^2\Omega^2_G)} \simeq \frac{\sqrt{2E' + \vartheta'^2}}{\Omega_G} \left(1 - \frac{1}{2}(q^2)^2\Omega^2_G\right), \quad W'_2(0) = 0, \]

whence
\[ W'_2 \simeq \frac{\sqrt{2E' + \vartheta'^2}}{\Omega_G} q^2 \left(1 - \frac{1}{6}(q^2)^3\Omega^2_G\right) \simeq \frac{\sqrt{2E' + \vartheta'^2}}{\Omega_G} q^2. \]

Putting $Q^1 = E'$, $Q^2 = \eta'$, $Q^3 = \vartheta'$ and $Q^4 = \vartheta'$, we have
\[ W' \simeq Q^2 q^4 + \frac{\sqrt{2Q^1 + (Q^4)^2}}{\Omega_G} q^2 + Q^3 q^3 + Q^4 q^4. \]

We thus obtain the following relation between the canonical coordinate systems $(Q, P)$ and $(q, p)$:
\[
\begin{align*}
Q_1 &= p_2, \\
Q_2 &= \sqrt{2Q^1 + (Q^4)^2}/\Omega_G, \\
Q_3 &= p_3, \\
Q_4 &= p_4,
\end{align*}
\[
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p_2 &= \sqrt{2q^1 + (q^4)^2}/\Omega_G, \\
p_3 &= q_3, \\
p_4 &= q_4,
\end{align*}
\[
\begin{align*}
q_1 &= -p_2, \\
q_2 &= -p_1\Omega_G\sqrt{2Q^1 + (Q^4)^2}, \\
q_3 &= -p_3, \\
q_4 &= -p_4 + Q^4 p_1;
\end{align*}
\]

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q_1 &= -p_2, \\
q_2 &= -p_1\Omega_G\sqrt{2Q^1 + (Q^4)^2}, \\
q_3 &= -p_3, \\
q_4 &= -p_4 + Q^4 p_1;
\end{align*}
\]

In terms of the canonical coordinate system $(Q, P)$, the vector field $X_{\mathcal{H}_G}$ and the symplectic form $\omega_G$ are written as
\[ X_{\mathcal{H}_G} = \{\mathcal{H}_G, \cdot\} = -\frac{\partial}{\partial P_1}, \quad \omega_G = \sum_{\nu=1}^4 dp_\nu \wedge dQ_\nu. \]

By Lemma 1, the $(1,1)$-tensor field $T_G$ can be expressed as
\[ T_G = \sum_{\nu=1}^4 Q_\nu \left(\frac{\partial}{\partial P_\nu} \otimes dP_\nu + \frac{\partial}{\partial Q_\nu} \otimes dQ_\nu\right), \]

where the constants of motion are
\[ \text{Tr} T_G^h = 2((Q^1)^h + (Q^2)^h + (Q^3)^h + (Q^4)^h), \quad h \in \mathbb{N}. \]
We arrive at the following result.

**Proposition 2.** Under the condition

\[ V_2^h \simeq \left( \frac{p_2^2}{q_2^2} \right)^h \quad h \in \mathbb{N}, \]  

the Hamiltonian vector field \( X_{H_G} \) in the Gödel metric (14) has a recursion operator \( T_G \) in the original coordinate system \((q, p)\) given by

\[
T_G = \sum_{\mu, \nu=1}^{4} \left( \tilde{A}_\mu^\nu \frac{\partial}{\partial q^\nu} \otimes dq^\mu + \tilde{B}_\mu^\nu \frac{\partial}{\partial p^\nu} \otimes dp^\mu + \tilde{C}_\mu^\nu \frac{\partial}{\partial q^\nu} \otimes dp^\mu + \tilde{D}_\mu^\nu \frac{\partial}{\partial p^\nu} \otimes dq^\mu \right),
\]

where

\[
\begin{align*}
\tilde{A}_j & = p_j, & \tilde{B}_j & = p_j, & j = 1, 3, 4, \\
\tilde{A}_2 & = H_G(1 + q^2 V_2^2 S), & \tilde{B}_k & = H_G U_k^3 p_2 S, & k = 1, 2, 3, \\
\tilde{A}_4 & = -p_4 p_2 S (H_G + U_4'), & \tilde{B}_4 & = -p_4 p_2 S (H_G + U_4'), \\
\tilde{A}_m & = 0, & \tilde{B}_m & = 0, & \text{otherwise;}
\end{align*}
\]

\[
\begin{align*}
\tilde{C}_i & = H_G U_i' q^2 S, & i = 1, 3, \\
\tilde{C}_2 & = H_G q^2 \Omega_2^2 S (p_2 + U_2^3 S), & \tilde{D}_2 & = H_G V_2' p_2 S, \\
\tilde{C}_4 & = -p_4 S (H_G + U_4'), & \tilde{D}_4 & = 0, & \text{otherwise,} \\
\tilde{C}_m & = 0, & \tilde{D}_m & = 0, & \text{otherwise,}
\end{align*}
\]

where \( S = 1 / \Omega_G^2 p_2^2 \), \( \Omega_G^2 p_2^2 > 0 \), \( n, m = 1, 2, 3, 4 \). The constants of motion in the original coordinate system \((q, p)\) are

\[
\text{Tr} T_G^h = 2 (p_2^h + p_3^h + p_4^h) + H_G^h \left( 1 + \frac{q^2 V_2^2}{\Omega_G^2 p_2^2} \right)^h + \left( -\frac{H_G}{\Omega_G} \right)^h + \frac{H_G q^2}{p_2} \left( 1 + \frac{1}{\Omega_G^2} \right)^h + \left( \frac{H_G V_2'}{\Omega_G^2 p_2} \right)^h, \quad h \in \mathbb{N}.
\]

**Proof.** Using (17), after some computations we obtain (2). In addition, it follows from condition (18) that \( \mathcal{L}_{X_{H_G}} (\text{Tr} T_G^h) = 0 \), and hence

\[
2 (p_2^h + p_3^h + p_4^h) + H_G^h \left( 1 + \frac{q^2 V_2^2}{\Omega_G^2 p_2^2} \right)^h + \left( -\frac{H_G}{\Omega_G} \right)^h + \left( -\frac{H_G q^2}{p_2} \right)^h \left( 1 + \frac{1}{\Omega_G^2} \right)^h + \left( \frac{H_G V_2'}{\Omega_G^2 p_2} \right)^h
\]

are constants of motion.

It is worth noting that in the coordinate system \((Q, P)\), for \( \Delta_A > 0 \), the Alcubierre and Gödel metrics have the same Hamiltonian vector field \( X_H \) and the same recursion operator \( T \), which thus induce the same dynamics.
5. Master symmetries

Given the above common dynamical characteristics, we consider the Hamiltonian system \( (\mathcal{T}^*Q, \omega, Q^1) \) for which the Hamiltonian function \( H \), the vector field \( X_0 \), the symplectic form \( \omega \), and the bivector field in both the Alcubierre and Gödel metrics are given by

\[
H = Q^1, \quad X_0 = \{Q^1, \cdot\} = -\frac{\partial}{\partial P^1}, \quad \omega = \sum_{\nu=1}^{4} dP_\nu \wedge dQ^\nu, \quad \mathcal{P} = \sum_{\nu=1}^{4} \frac{\partial}{\partial P_\nu} \wedge \frac{\partial}{\partial Q^\nu}.
\]

We introduce the vector fields \( Y_j \in \mathcal{T}^*Q \),

\[
Y_j = \sum_{\nu=1}^{4} (Q^\nu)_j \left( (j+1)P_\nu \frac{\partial}{\partial P_\nu} - Q^\nu \frac{\partial}{\partial Q^\nu} \right), \quad j \in \mathbb{N},
\]

satisfying the relation

\[
\iota_{Y_j} \omega = -d\tilde{H}_j, \quad \text{with} \quad \tilde{H}_j = -\sum_{\nu=1}^{4} (Q^\nu)^{j+1} P_\nu.
\]

The symplectic structure \( \omega \) generates a set of Hamiltonian systems on the same manifold \( \mathcal{T}^*Q \). The Lie bracket between the vector fields \( X_i \) and \( Y_j \) obeys the relations

\[
[X_i, Y_j] = X_{i+j}, \quad [X_i, X_{i+j}] = 0, \quad \text{with} \quad X_{i+j} = -(j+1)(i+j+1)(Q^1)^{i+j+1} \frac{\partial}{\partial P^1}
\]

where \( i, j \in \mathbb{N} \). This is illustrated in Fig 1.

![Fig. 1. Diagrammatic illustration of Eq. (19).](image)

In differential geometric terms, \( Y_j \) and \( \tilde{H}_j \) are called the respective master symmetries for \( X_i \) and master integrals \([21], [23], [26]–[28]\).

From the master integrals \( \tilde{H}_j \), we can generate a family of Hamiltonian functions

\[
H_{i+j} := \{H_i, \tilde{H}_j\} = (i+1)(Q^1)^{i+j+1}, \quad H_0 = H, \quad i, j \in \mathbb{N}.
\]

The recursion operator

\[
T = \sum_{\nu=1}^{4} Q^\nu \left( \frac{\partial}{\partial P_\nu} \otimes dP_\nu + \frac{\partial}{\partial Q^\nu} \otimes dQ^\nu \right),
\]
We conclude that the vector field \( T = P_1 \circ P^{-1} \), where

\[
P_1 = \sum_{\nu=1}^{4} Q^\nu \frac{\partial}{\partial P_\nu} \wedge \frac{\partial}{\partial Q^\nu},
\]

and \( P, P_1 \) are two compatible Poisson bivectors with the vanishing Schouten–Nijenhuis bracket \([P, P_1]_{SN} = 0\).

Now, introducing the Poisson bracket \( \{ \cdot, \cdot \} \)

\[
\{ f, g \} \equiv \sum_{\nu=1}^{4} Q^\nu \left( \frac{\partial f}{\partial P_\nu} \frac{\partial g}{\partial Q^\nu} - \frac{\partial f}{\partial Q^\nu} \frac{\partial g}{\partial P_\nu} \right),
\]

with respect to the symplectic form \( \omega_1 = \sum_{\nu=1}^{4} (Q^\nu)^{-1} dP_\nu \wedge dQ^\nu \), we obtain

\[
X_i = \{ \bar{H}_i, \cdot \} = \{ \bar{H}_{i+1}, \cdot \}, \quad \bar{H}_0 = H, \quad \bar{H}_1 = \ln Q^1, \quad \bar{H}_j = -\frac{1}{j(Q^1)^j}, \quad Q^1 \neq 0,
\]

proving that \( X_i \) are bi-Hamiltonian vector fields defined by the two Poisson bivectors \( P \) and \( P_1 \). Then the quadruple \( (Q, P, P_1, X_i) \) is a bi-Hamiltonian system for each \( i \).

In addition, we have

\[
\mathcal{L}_{Y_0}(P) = 0 \quad (\alpha = 0), \quad \mathcal{L}_{Y_0}(P_1) = -\sum_{\nu=1}^{4} Q^\nu \frac{\partial}{\partial P_\nu} \wedge \frac{\partial}{\partial Q^\nu} \quad (\beta = -1),
\]

\[
\mathcal{L}_{Y_0}(H) = -Q^1 = -H \quad (\gamma = -1).
\]

We conclude that the vector field

\[
Y_0 = \sum_{\nu=1}^{4} \left( P_\nu \frac{\partial}{\partial P_\nu} - Q^\nu \frac{\partial}{\partial Q^\nu} \right),
\]

is a conformal symmetry for \( P, P_1 \) and \( H \) [21].

Defining the families of quantities \( X'_h, Y'_h, P'_h, \omega'_h, \) and \( dH'_h \) with \( h \in \mathbb{N} \) by

\[
X'_h := T^h X_0, \quad P'_h := T^h P, \quad \omega'_h := (T^*)^h \omega', \quad Y'_h := T^h Y_0, \quad dH'_h := (T^*)^h dH,
\]

where \( T^* := P^{-1} \circ P_1 \) is the adjoint of \( T \), we obtain

\[
P'_h = \sum_{\nu=1}^{4} (Q^\nu)^h \frac{\partial}{\partial P_\nu} \wedge \frac{\partial}{\partial Q^\nu},
\]

\[
Y'_h = \sum_{\nu=1}^{4} (Q^\nu)^h \left( P_\nu \frac{\partial}{\partial P_\nu} - Q^\nu \frac{\partial}{\partial Q^\nu} \right), \quad X'_h = -(Q^1)^h \frac{\partial}{\partial P_1},
\]

\[
\omega'_h = \sum_{\nu=1}^{4} (Q^\nu)^h dP_\nu \wedge dQ^\nu, \quad dH'_h = (Q^1)^h dQ^1 \quad \text{and} \quad H'_h = \frac{1}{h+1} (Q^1)^{h+1},
\]

leading to a plethora of conserved quantities:

\[
\mathcal{L}_{Y'^{l}}(Y'^{l}) = (h-l)Y'^{l+h}, \quad \mathcal{L}_{Y'^{l}}(X'^{l}) = -(l+1)X'^{l+h}, \quad \mathcal{L}_{Y'^{l}}(P'^{l}) = (h-l)P'^{l+h},
\]

\[
\mathcal{L}_{Y'^{l}}(\omega'^{l}) = -(l+h)\omega'^{l+h}, \quad \mathcal{L}_{Y'^{l}}(T) = -T^{l+h}, \quad \langle dH'^{l}, Y'^{l} \rangle = -(h+l+1)H'^{l+h}, \quad l \in \mathbb{N}.
\]
satisfying
\[ \mathcal{L}_{Y'_l}(Y'_h) = (\beta - \alpha)(l - h)Y'_{(l+h)}, \quad \mathcal{L}_{Y'_h}(X'_l) = (\beta + \gamma + (l - 1)(\beta - \alpha))X'_{l+h}, \]
\[ \mathcal{L}_{Y'_h}(P'_l) = (\beta + (l - h - 1)(\beta - \alpha))P'_{l+h}, \quad \mathcal{L}_{Y'_h}(\omega'_l) = (\beta + (l + h - 1)(\beta - \alpha))\omega'_{l+h}, \]
\[ \mathcal{L}_{Y'_h}(T) = (\beta - \alpha)T^{l+h}, \quad \langle dH'_l, Y'_h \rangle = (\beta + (l + h)(\beta - \alpha))H'_{l+h}, \]
similarly to Oevel’s formulas (see [19]–[22]).

6. Conclusion

We have analyzed the dynamics of a spaceship in the Alcubierre and Gödel metrics in detail. We have derived the Hamiltonian vector fields governing the system evolution, and constructed and discussed the associated recursion operators generating the constants of motion. Besides, we proved the existence of a bi-Hamiltonian structure in the considered canonical coordinate system and computed conserved quantities using the corresponding master symmetries.

This study has shown that Hamiltonian dynamics hints at a connection between the geometry of the physical system and conservation laws using the Poisson bracket. Our physical systems in the Alcubierre and Gödel metrics are symplectic manifolds equipped with Hamiltonian vector fields. In this connection, the spaceship positions on the manifolds are viewed as states and vector fields as laws governing how those states evolve.

We have observed that the spaceship obeys the same dynamics for particular choices of the Alcubierre and Gödel metrics. Indeed, using appropriate parameterizations, the Hamiltonian vector fields and the recursion operators have been expressed in identical ways for both metrics. The only difference between them is the relation between the original coordinates and the new coordinates. Further, we have noted that the Hamiltonian function of the spaceship remains constant along the trajectories (also called integral curves) for Hamiltonian vector fields.

We have used the recursion operator to compute the constants of motion, i.e., first integrals, which are an important step in the study of the dynamics of the spaceship. Each Hamiltonian vector field \( X_H \) is its own first integral, \( X_H(H) := \{H, H\} = 0 \) due to the antisymmetry of the Poisson bracket. This is characteristic of the physical principle of energy conservation.

Finally, from this study, we infer the formulation of a generalized Poisson bracket
\[
\{f, g\}_j := \sum_{\nu=1}^{4} (Q^\nu)^j \left( \frac{\partial f}{\partial P^\nu} \frac{\partial q}{\partial Q^\nu} - \frac{\partial f}{\partial Q^\nu} \frac{\partial q}{\partial P^\nu} \right), \quad j \in \mathbb{N},
\]
yielding a set of bi-Hamiltonian vector fields
\[ X_i = \{P_i, \cdot\} = \{P_{i+j}, \cdot\}_j, \quad i, j \in \mathbb{N}, \]
which can allow a straightforward extension of all previous results.

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