Deciding the Computability of Regular Functions over Infinite Words

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Abstract

The class of regular functions from infinite words to infinite words is characterised by MSO-transducers, streaming $\omega$-string transducers as well as deterministic two-way transducers with look-ahead. In their one-way restriction, the latter transducers define the class of rational functions. This paper proposes a decision procedure for the fundamental question: given a regular function $f$, is $f$ computable (by a Turing machine with infinite input)? For regular functions, we show that computability is equivalent to continuity, and therefore the problem boils down to deciding continuity. We establish a generic characterisation of continuity for functions preserving regular languages under inverse image (such as regular functions). We exploit this characterisation to show the decidability of continuity (and hence computability) of rational functions in $\text{NLogSpace}$ (it was already known to be in $\text{PTime}$ by Prieur), and of regular functions.

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1 Introduction

The notions of computability and continuity have been central in computability theory, as well as in real and functional analysis. Computability for discrete sets like natural numbers, finite words, finite graphs and so on have been extensively studied over the last seven to eight decades, through several models of computation including Turing machines working on finite words. Computability notions have been extended to infinite objects, like infinite sequences of natural numbers, motivated by real analysis, or computation of functions of real numbers. An infinite word $\alpha$ over a finite alphabet $\Sigma$ is a function $\alpha : \mathbb{N} \rightarrow \Sigma$ and is written as $\alpha = \alpha(0)\alpha(1)\ldots$. The set of infinite words over $\Sigma$ is denoted by $\Sigma^\omega$.

Computability of functions over infinite words In this paper, we are interested in functions from infinite words to infinite words. The model of computation we consider for infinite words is a deterministic multitape machine with 3 tapes: a read-only one-way tape holding the input, a two-way working tape with no restrictions and a write-only one-way output tape. All three tapes hold infinite words. A function $f$ is computable if there exists such a machine $M$ such that, if its input tape is fed with an infinite word $u$ in the domain of $f$, then $M$ outputs longer and longer prefixes of $f(x)$ when reading longer and longer prefixes of $x$. This machine model has been defined for instance in [17]. Not all functions are computable. For instance, assuming an effective enumeration $M_1, M_2, \ldots$ of Turing machines (on finite word inputs), the function $f_H$ defined as $f_H(\alpha^\omega) = b_1b_2b_3\ldots$ where $b_i \in \{0, 1\}$ is
such that $b_i = 1$ iff $M_i$ halts on input $\epsilon$, is not computable, otherwise the halting problem would be decidable.

This raises a natural decision problem: given a (finite) specification of a function $f$ from infinite words to infinite words, is $f$ computable? In the domain of program synthesis, this could be rephrased as “Is $f$ implementable?” since the notion of computability gives a natural answer to the question of what it means to be an implementation for such a function. Even though the notion of computability for functions of infinite words require “physically unrealisable” infinite input, it makes sense, for instance, in a streaming scenario where the input is received as a non-terminating stream of symbols. In this context, it is important to have the guarantee that infinitely often, one can output some finite part of the output.

**Computability and continuity** It turns out that there is some connection between computability and continuity. It is known that computable functions are continuous for the Cantor topology, that is when words are close to each other if they share a long common prefix. We give an intuition for this here. A function $f : \Sigma^\omega \to \Gamma^\omega$ is continuous if whenever a sequence $x_1, x_2, \ldots$ of infinite words in the domain of $f$ converges to an infinite word $x$ in the domain of $f$, the sequence $(x_1), (x_2), \ldots$ converges. The notion of convergence is quantified using a metric between two infinite words. For $p, q \in \Sigma^\omega$, the distance between $p, q$ is defined as 0 if $p = q$, or as $2^{-n}$, where $n$ is the length of the longest common prefix of $p, q$. Assume $x_1, x_2, \ldots$ in the domain of $f$ converges to $x$, then, up to taking a subsequence, we can assume that the $x_i$ share longer and longer common prefixes with $x$. As $f$ is computable by a deterministic machine, it behaves the same on these prefixes, and therefore it outputs words on $x_i$ which gets closer as $i$ tends to infinity, hence they converge as well. This shows that computable functions are continuous. However, in the reverse direction, not all continuous functions are computable, a counter-example being the function $f_H$ defined before, which is continuous since it is defined on a single input.

**Regular functions** In this paper, we study regular functions, which form a very well-behaved class. Regular functions on infinite words are captured by streaming $\omega$-string transducers (SST), deterministic two-way Muller transducers with look around ($2\text{DMT}_\text{la}$), and also by MSO-transducers à la Courcelle [1]. We propose the model of deterministic two-way transducers with a prophetic Büchi look-ahead ($2\text{DFT}_\text{pla}$) and show that they are equivalent to $2\text{DMT}_\text{la}$. This kind of transducers is defined by a deterministic two-way automaton without accepting states, extended with output words on the transitions, and which can consult another automaton, called the look-ahead automaton, to check whether an infinite suffix satisfies some regular property. We assume this automaton to be a prophetic Büchi automaton [4], because this class has interesting properties while capturing all regular languages of infinite words. The look-ahead is necessary to capture regular functions: the regular function $j(wab^\omega) = wab^\omega$, and $j(b^\omega) = b^\omega$ cannot be captured by a two-way deterministic transducer without look-around, not even a non-deterministic transducer. The reason is that one needs a non-deterministic choice to identify the last occurrence of $a$ and after this choice, check that there are only $b$ symbols on the input. However, two passes over $w$ are necessary, and so those non-deterministic choices must be done at the same input position, which is impossible to ensure with finite memory.

Consider the function defined for all $w \in \{a, b\}^\omega$ by $f(w) = a^\omega$ if there are infinitely many as in $w$, by $f(w) = b^\omega$ otherwise. This function can be realised by a non-deterministic
two-way (even one-way) transducer without look-ahead which guesses whether there are infinitely many $a$ or not. The restriction of $f$ to the regular language of words which contain infinitely many $a$’s is however definable by a deterministic one-way transducer. Another, less trivial example, is given by the function $g(u_1\#u_2\ldots\#u_n\ldots) = u_1u_2u_2\ldots$ defined for all inputs which contain infinitely many $\#$, and realisable by a deterministic two-way transducer without look-ahead, which performs two passes for each factor $u_i$ in between two $\#$ symbols.

**Contributions** We first show that for regular functions, computability and continuity coincide. To the best of our knowledge, this connection was not made before. We then prove that continuity (and hence computability) is decidable for regular functions given by deterministic Büchi two-way transducers with look-ahead. Using our techniques, we also get almost for free the decidability of uniform continuity, although we do not connect it to any notion of computability. For rational functions (functions defined by non-deterministic one-way Büchi transducers), we also get that continuity and uniform continuity are decidable in NLOGSPACE. Deciding continuity and uniform continuity for rational functions was already known to be decidable in PTIME, from Prieur [12]. Our proof technique relies on a characterisation of non-continuity by the existence of pairs of sequences of words which have a nice regular structure. This characterisation works for any function $f$ which preserves regular languages of infinite words under inverse image. In particular, it applies to regular functions, which are known to have this property. Using this characterisation, we derive a decidability test for continuity of rational functions based on a transducer structural pattern which, using the pattern logic of [7], is decidable in NLOGSPACE. For regular functions, the decidability test is based on a thorough study of the form of output words produced by idempotent loops in two-way transducers, which was done in [2] and used in another context. We leave as open the question of which transducer model captures continuous regular functions and conjecture it is the model of deterministic two-way transducers without look-ahead.

**Related work** To the best of our knowledge, our results are new and the notion of continuity has not been extensively studied in the transducers literature. We have mentioned the work by Prieur [12] which is the closest to ours. Another related result in the context of reactive synthesis has been obtained in [8]: they show that for any binary relation $R$ of infinite words defined by a one-way letter-to-letter Büchi transducer, it is decidable whether there exists a total continuous function $f$ such that for all input $x \in \Sigma^\omega$, $(x, f(x)) \in R$. Notions of continuity have been defined for rational functions in [3] but they are different from the classical notion we take in this paper and have been defined for finite words.

**Structure of the Paper** Section 2 introduces the model of transducers used in the paper. Section 3 establishes a connection between computability and continuity for regular functions. Section 4 proves a characterisation of continuity of functions preserving regular languages under inverse image. Finally, Section 5 studies the decidability of continuity. *Most of the proofs have been omitted but can be found in Appendix.*

## 2 Preliminaries

Given a finite set $\Sigma$, we denote by $\Sigma^*$ (resp. $\Sigma^\omega$) the set of finite (resp. infinite) words over $\Sigma$, and by $\Sigma^\infty$ the set of finite and infinite words. We denote by $|u| \in \mathbb{N} \cup \{\infty\}$ the length of $u \in \Sigma^\infty$ (in particular $|u| = \infty$ if $u \in \Sigma^\omega$). For a word $w = a_1a_2a_3\ldots$, $w[j]$ denotes the prefix $a_1a_2\ldots a_j$ of $w$. $w[i]$ denotes $a_i$, the $i$th symbol of $w$. $w[j]$ denotes the suffix $a_{j+1}a_{j+2}\ldots$ of $w$. For a word $w$ and $i \leq j$, $w[i:j]$ denotes the factor of $w$ with positions
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We say that \( u \preceq v \) (resp. \( u < v \)) denotes that \( u \) is a prefix (resp. strict prefix) of \( v \) (in particular if \( u, v \in \Sigma^\infty \), \( u \preceq v \) iff \( u = v \)). For \( u \in \Sigma^* \), let \( \uparrow u \) denote the set of words \( w \in \Sigma^\infty \) having \( u \) as prefix i.e. \( u \preceq w \). Given two words \( u, v \in \Sigma^\infty \), we say that there exists a mismatch, denoted \( \text{mismatch}(u, v) \), between \( u \) and \( v \), if there exists a position \( i \leq |u|, |v| \) such that \( u[i] \neq v[i] \).

A Büchi automaton is a tuple \( B = (Q, \Sigma, \delta, Q_0, F) \) consisting of a finite set of states \( Q \), a finite alphabet \( \Sigma \), a set \( Q_0 \subseteq Q \) of initial states, a set \( F \subseteq Q \) of accepting states, and a transition function \( \delta : Q \times \Sigma \rightarrow 2^Q \). A run \( \rho \) on a word \( w = a_1a_2\cdots \in \Sigma^\omega \) starting in a state \( q_i \) in \( B \) is an infinite sequence \( q_i \xrightarrow{a_i} q_{i+1} \xrightarrow{a_{i+1}} \cdots \) such that \( q_{i+1} \in \delta(q_i, a_i) \). Let \( \text{Inf}(\rho) \) denote the set of states visited infinitely often along \( \rho \). The run \( \rho \) is a final run iff \( \text{Inf}(\rho) \cap F \neq \emptyset \). A run is accepting if it is final and starts from an initial state. A word \( w \in \Sigma^\omega \) is accepted (\( w \in L(B) \)) if it has an accepting run. A language of \( \omega \)-words \( L \) is called regular if \( L = L(B) \) for some Büchi automaton \( B \).

An automaton is co-deterministic if any two final runs on any word \( w \) are the same \[^1\]. Likewise, an automaton is co-complete if every word has at least one final run. A prophetic automaton \( P = (Q_p, \Sigma, \delta_p, Q_0, F_P) \) is a Büchi automaton which is co-deterministic and co-complete. Equivalently, a Büchi automaton is prophetic if each word admits a unique final run. The states of the prophetic automaton partition \( \Sigma^\omega : \) each state \( q \) defines a set of words \( w \) such that \( w \) has a final run starting from \( q \). For any state \( q \), let \( L(P, q) \) be the set of words having a final run starting at \( q \). Then \( \Sigma^\omega = \bigcup_{q \in Q_p} L(P, q) \). It is known \[^2\] that prophetic automata capture \( \omega \)-regular languages.

Transducers

We recall the definitions of one-way and two-way transducers over infinite words. A one-way transducer \( \mathcal{A} \) is a tuple \((Q, \Sigma, \Gamma, \delta, Q_0, F)\) where \( Q \) is a finite set of states, \( Q_0, F \) respectively are sets of initial and accepting states; \( \Sigma, \Gamma \) respectively are the input and output alphabets; \( \delta \subseteq (Q \times \Sigma \times \Gamma \times F^* \times 0^*) \) is the transition relation. \( \mathcal{A} \) has the Büchi acceptance condition. A transition in \( \delta \) of the form \((q, a, q', \gamma)\) represents that from state \( q \), on reading a symbol \( a \), the transducer moves to state \( q' \), producing the output \( \gamma \). Runs, final runs and accepting runs are defined exactly as in Büchi automata, with the addition that each transition produces some output \( \in \Gamma^* \).

The output produced by an accepting run \( \rho \), denoted \( \text{out}(\rho) \), is obtained by concatenating the outputs generated by transitions along \( \rho \). Let \( \text{dom}(\mathcal{A}) \) represent the language accepted by the underlying automaton of \( \mathcal{A} \), ignoring the outputs. The relation computed by \( \mathcal{A} \) is defined as \( [\mathcal{A}] = \{ (u, v) \in \Sigma^\omega \times \Gamma^* | u \in \text{dom}(\mathcal{A}), v = \text{out}(\rho) \} \).

We say that \( \mathcal{A} \) is functional if \([\mathcal{A}] \) is a function. A relation (function) is rational if it is recognized by a one-way (functional) transducer.

Two-way transducers extend one-way transducers and two-way finite state automata. A two-way transducer is a two-way automaton with outputs. Let \( \Sigma_\omega = \Sigma \cup \{ \uparrow \} \). A deterministic Büchi two-way transducer \( (2\text{DBT}) \) is given as \( \mathcal{B} = (Q, \Sigma, \Gamma, \delta, q_0, F) \) where \( Q \) is a finite set of states, \( q_0 \) is the unique initial state, and \( F \subseteq Q \) is a set of accepting states, and \( \Sigma \) and \( \Gamma \) are finite input and output alphabets respectively, and the transition function has type \( \delta : Q \times \Sigma \rightarrow Q \times \Gamma^* \times \{1, -1\} \). A two-way transducer stores its input \( \uparrow a_1, a_2, \ldots \) on a two-way tape, and each index of the input can be read multiple times. A configuration of a two-way transducer is a tuple \((q, i) \in Q \times \mathbb{N} \) where \( q \in Q \) is a state and \( i \in \mathbb{N} \) is the current position on the input tape. The position is an integer representing the gap between consecutive symbols. Thus, at \( \uparrow \), the position is 0, between \( \uparrow \) and \( a_1 \), the

\[^1\] \text{We assume that final runs always produce infinite words, which can be enforced by a Büchi condition.}
position is 1, between \(a_i\) and \(a_{i+1}\), the position is \(i+1\) and so on. Given \(w = a_1a_2 \ldots\), from a configuration \((q, i)\), on a transition \(\delta(q, a_j) = (q', \gamma, d)\), \(d \in \{1, -1\}\), we obtain the configuration \((q', i + d)\) and the output \(\gamma\) is appended to the output produced so far. This transition is denoted as \((q, i) \overset{a_j/\gamma}{\longrightarrow} (q', i + d)\). A run \(\rho\) of a 2DBT is a sequence of transitions \((q_0, i_0 = 0) \overset{a_{i_0}/\gamma_1}{\longrightarrow} (q_1, i_1) \overset{a_{i_1}/\gamma_2}{\longrightarrow} \ldots\). The output of \(\rho\), denoted \(\text{out}(\rho)\), is then \(\gamma_1\gamma_2\ldots\). The run \(\rho\) reads the whole word \(w\) if \(\sup\{i_n | 0 \leq n < \|\rho\|\} = \infty\). The output \(B[w](w)\) of a word \(w\) on run \(\rho\) is defined only when \(\sup\{i_n | 0 \leq n < \|\rho\|\} = \infty\), \(\inf(\rho) \cap F \neq \emptyset\), and equals \(\text{out}(\rho)\).

In [1], regular functions are shown to be those definable by a two-way deterministic transducer with Muller acceptance condition, along with a regular look-around (2DMT). Formally, a 2DMT is a tuple \((\mathcal{T}, A, B)\) where \(\mathcal{T}\) is a deterministic two-way automaton with outputs, equipped with Muller acceptance condition. \(A\) is a look-ahead automaton and \(B\) is a look-behind automaton. The look-ahead is a one-way automaton with the Muller acceptance condition and the look-behind is a DFA. In this paper, we propose an alternative machine model for regular functions, namely, 2DFT$_{\text{pl}}$. A 2DFT$_{\text{pl}}$ is a deterministic two-way automaton with outputs, along with a look-ahead given by a prophetic automaton [4].

Formally, a 2DFT$_{\text{pl}}$ is a pair \((\mathcal{T}, A)\) where \(A = (Q_A, \Sigma, \delta_A, S_A, F_A)\) is a prophetic look-ahead Büchi automaton, and \(\mathcal{T} = (Q, \Sigma, \Gamma, \delta, q_0)\) is a two-way transducer s.t. \(\Sigma\) and \(\Gamma\) are finite input and output alphabets, \(Q\) is a finite set of states, \(q_0 \in Q\) is a unique initial state, \(\delta : Q \times \Sigma \times Q_A \rightarrow Q \times \Gamma \times \{-1, +1\}\) is a partial transition function. \(\mathcal{T}\) has no acceptance condition: every infinite run in \(\mathcal{T}\) is a final run. The 2DFT$_{\text{pl}}$ is deterministic in the sense that for every word \(w = a_1a_2a_3 \ldots \in \Sigma^\omega\), every input position \(i \in \mathbb{N}\), and state \(q \in Q\), there is a unique state \(p \in Q_A\) such that \(a_ia_{i+1} \ldots \in L(A, p)\). Given \(w = a_1a_2 \ldots\), from a configuration \((q, i)\), on a transition \(\delta(q, a_i, p) = (q', \gamma, d)\), \(d \in \{1, -1\}\), such that \(a_ia_{i+1} \ldots \in L(A, p)\), we obtain the configuration \((q', i + d)\) and the output \(\gamma\) is appended to the output produced so far. This transition is denoted as \((q, i) \overset{a_i/p/\gamma}{\longrightarrow} (q', i + d)\). A run \(\rho\) of a 2DFT$_{\text{pl}}$ \((\mathcal{T}, A)\) is a sequence of transitions \((q_0, i_0 = 0) \overset{a_{i_0}/p_1/\gamma_1}{\longrightarrow} (q_1, i_1) \overset{a_{i_1}/p_2/\gamma_2}{\longrightarrow} \ldots\). The output of \(\rho\), denoted \(\text{out}(\rho)\), is then \(\gamma_1\gamma_2\ldots\). The run \(\rho\) reads the whole word \(w\) if \(\sup\{i_n | 0 \leq n < \|\rho\|\} = \infty\). The output \(\mathcal{T}[w](w)\) of a word \(w\) on run \(\rho\) is defined as in 2DMT, \(\sup\{i_n | 0 \leq n < \|\rho\|\} = \infty\), \(\inf(\rho) \cap F_A \neq \emptyset\), and is equal to \(\text{out}(\rho)\). It is known that 2DFT$_{\text{pl}}$ is strictly more expressive than 2DBT [1], moreover, 2DFT$_{\text{pl}}$ are equivalent to 2DMT, and capture all regular functions.

**Theorem 1.** A function \(f : \Sigma^\omega \rightarrow \Gamma^\omega\) is regular iff it is 2DFT$_{\text{pl}}$ definable.

Consider the function \(j : \Sigma^\omega \rightarrow \Gamma^\omega\) such that \(j(\omega b\omega) = \omega b\omega\) for \(u \in \Sigma^\ast\) and \(j(\omega a\omega) = \omega a\omega\), depicted in the figure. The 2DFT$_{\text{pl}}$ is on the left, \(P\) is on the right. The transitions are decorated as \(\alpha, p | \gamma, d\) where \(\alpha \in \{a, b\}\), \(p\) is a state of \(P\), \(\gamma\) is the output and \(d\) is the direction. In transitions not using the look-ahead information, the decoration is simply \(\alpha | \gamma, d\). The runs are then \(\rho \in \Sigma^\ast a\omega, \rho \in \Sigma^\ast a\omega, \rho \in \Sigma^\ast b\omega, \rho \in \Sigma^\ast ab\omega, \rho \in \Sigma^\ast ab\omega, \rho \in \Sigma^\ast ab\omega\). Recall that, as mentioned in the introduction, \(j\) cannot be realised by a non-deterministic two-way transducer w/o look-ahead.

\[L(P, p_1) = \Sigma^\ast ab\omega, L(P, p_2) = \Sigma^\ast b\omega, L(P, p_3) = \Sigma^\ast ab\omega, L(P, p_4) = \Sigma^\ast ab\omega.\]
3 Computability and Continuity for Regular functions

We first define the notions of continuity and computability for functions of ω-words. Given two words u, v ∈ Σω, their distance is defined as d(u, v) = 0 if u = v, and 2−|u|−|v| if u ≠ v. u ∧ v is the longest common prefix of u and v. Next, we define the notion of continuity. We interchangeably use the following two equivalent notions for continuity.

- **Definition 2 (Continuity).** A function f : Σω → Γω is continuous at x ∈ dom(f) if (equivalently)
  
  (a) for all (xn)_{n∈N} converging to x, where x_i ∈ dom(f) for all i ∈ N, (f(x_n))_{n∈N} converges.
  
  (b) ∀i ≥ 0 ∃j ≥ 0 ∀y ∈ dom(f), |x ∧ y| ≥ j ⇒ |f(x) ∧ f(y)| ≥ i

  A function is continuous if it is continuous at any x ∈ dom(f).

- **Definition 3 (Computability).** A function f : Σω → Γω is computable if there exists a deterministic multitape machine M computing it in the following sense. M has a read-only one-way input tape, a two-way working tape, and a write-only one-way output tape. All tapes have a left delimiter † and are infinite to the right. Let x ∈ dom(f). For any j ∈ N, let M(x, j) denote the output produced by M till the time it moves to the right of position j, onto position j + 1 (or ε if this move never happens). f is computable by M if for all x ∈ dom(f) such that y = f(x), ∀i ≥ 0, ∃j ≥ 0 such that y[i; j] ≤ M(x, j).

**Example.** As a first example, consider g : {a, b}ω → {c, d}ω defined by g(aω) = aω, g(a^nbω) = a^{2n}cω, and g(a^ndω) = a^ndω for all n ≥ 0. It is easy to verify that g is continuous. g is also computable via the machine M described as follows. As long as M reads an a, it outputs an a and writes an a on its working tape, and keeps moving right. When it sees a d, it continues moving right, with output d. It is easy to see that ∀j ≥ 0, ∃i = j such that y[i; j] ≤ M(x, i). If it sees a c after the a’s, then it outputs an c, writes a c on the working tape, and moves back on the working tape till the beginning. Then it reads each a on the working tape, outputs an a, and moves right, till it reads c on the working tape. Then on, it keeps on reading and producing c.

Consider now the function f defined as f(a^ω) = c^ω, f(a^nbω) = d^ω for all n ≥ 0. f is not continuous: for x = a^ω, i = 1, for all j, ∃y_j = a^jb^ω such that |x ∧ y_j| ≥ j, but |f(x) ∧ f(y_j)| < i. f is not computable as well. Indeed, when reading a sequence of as, if the machine outputs c, then maybe after that sequence there is an infinite sequence of b and this output was wrong. The machine would have to know if a b occurs in the future.

As announced, continuity and computability coincide for regular functions:

- **Theorem 4.** A regular function f : Σω → Γω is computable if and only if it is continuous.

**Sketch of Proof.** If f is computable by some machine M, then it is not difficult to see that it is continuous. Intuitively, the longer the prefix of input x ∈ dom(f) is processed by M, the longer the output M produces on that prefix, which converges to f(x), according to the definition of continuity.

The converse direction is less trivial. Suppose that f is continuous. We design the machine M_f, represented as Algorithm 1 which is shown to compute f. This machine processes longer and longer prefixes x[i] of its input x (for loop at line 2), and tests (line 3) whether a symbol γ can be safely appended to the output. This is the case if for any accepting continuation x’ of x[i], i.e., y = x[i];x’ ∈ dom(f), the word out.γ, where out is the output produced so far, is a prefix of f(y). Since f(y) is infinite, this is equivalent to saying there is no mismatch between f(y) and out.γ. Since f is continuous, it can be shown that infinitely often the test at line 3 holds true and since out is invariantly a prefix of f(x), we
get the result. If \( f \) is given by a \( 2\text{DFT}_{\text{pla}} \), we show that the test at line 3, which we call in the following the mismatch problem is decidable (Lemma 5 below), concluding the proof. See Appendix B.1 for details.

\[
\text{Algorithm 1: Algorithm describing } M_f.
\]

\begin{algorithm}
\begin{algorithmic}[1]
\State \textbf{Input:} \( x \in \Sigma^\omega \)
\State \texttt{out} := \( \epsilon \); \text{ this is written on the working tape }
\For{\( i = 0 \) to \(+\infty\) do }
\If{\( \exists \gamma \in \Gamma \text{ s.t } \forall y \in \downarrow x[i] \cap \text{dom}(f), \text{ mismatch(out, } \gamma, f(y) \) \)}
\State \texttt{out} := \( \text{out, } \gamma \); \text{ append to the working tape }
\State \texttt{output } \gamma ; \text{ this is written on the output tape }
\EndIf
\EndFor
\end{algorithmic}
\end{algorithm}

The mismatch problem for \( 2\text{DFT}_{\text{pla}} \). Given \( u \in \Sigma^* \) and \( v \in \Gamma^* \), and a \( 2\text{DFT}_{\text{pla}} (T, P) \) realising \( f \), is there \( y \in \Sigma^\omega \) such that \( uy \in \text{dom}(f) \) and \( \text{mismatch}(v, f(uy)) \)?

\begin{lemma}
The mismatch problem for \( 2\text{DFT}_{\text{pla}} \) is \text{PSPACE}-complete.
\end{lemma}

\begin{proof}[Sketch of proof] This lemma is proved in two steps. First, we show that the problem can be reduced to the same problem, but for a transducer without look-ahead, modulo annotating input words with look-ahead states (i.e. words over alphabet \( \Sigma \times Q_P \)). In particular for any \( 2\text{DFT}_{\text{pla}} \), one can construct in \text{PTIME}, an equivalent \( 2\text{DBT} \) working on valid annotated words, where valid means that the annotation is correct (position \( i \) is annotated with state \( p \) iff the suffix of the input starting at position \( i \) belongs to the look-ahead language \( L(P, p) \)).

In a second step, we show how to decide the mismatch problem on \( 2\text{DBT} A \). Since the output \( v \) is given as input for the mismatch problem, we know that the mismatch, if it exists, occurs within the first \(|v|\) positions of the output of \( f(y) \). Therefore, we are able to construct a two-way automaton \( A_{u,v,f} \) working on \( \omega \)-words whose language is non-empty iff there exists \( y \in \Sigma^\omega \) such that \( uy \in \text{dom}(f) \) and there is a mismatch between \( v \) and \( f(uy) \). The automaton \( A_{u,v,f} \) simulates the behaviour of \( A \) by counting, up to \(|v|\), the length of the prefixes of \( f(y) \) produced by \( A \), and accepts whenever it finds a position \( i \leq |v| \) such that \( v[i] \neq f(y)[i] \), and rejects otherwise. The hardness is obtained by reduction of the intersection problem for \( n \) DFAs. Appendix B.2 has the full proof.
\end{proof}

\section{Continuity for Functions Preserving Rational Languages}

\textbf{Topology preliminaries.} A regular word (sometimes called ultimately periodic) over \( \Sigma \) is a word of the form \( uv^\omega \) with \( u \in \Sigma^* \) and \( v \in \Sigma^+ \). The set of regular words is denoted by \( \text{Rat}(\Sigma) \). The topological closure of a language \( L \subseteq \Sigma^\omega \), denoted by \( \bar{L} \), is the smallest language containing it and closed under taking the limit of converging sequences, i.e. \( \bar{L} = \{ x \mid \forall u < x, \exists y, uy \in L \} \). A language is closed if it is equal to its closure. Let \( L \subseteq D \subseteq \Sigma^\omega \). The language \( L \) is closed for \( D \) if \( L = \bar{L} \cap D \). A sequence of words \( (x_n)_{n \in \mathbb{N}} \) is called regular if there exists \( u, v, w \in \Sigma^* \), \( z \in \Sigma^+ \) such that for all \( n \in \mathbb{N} \), \( x_n = u v^n w z^\omega \). The following results are folklore or easy-to-get results shown in Appendix C.1.

\begin{proposition}
Let \( L \subseteq D \subseteq \Sigma^\omega \) be regular languages. Then
\begin{enumerate}
\item \( \bar{L} \) is regular,
\end{enumerate}
\end{proposition}
2. \( L \subseteq L \cap \text{Rat}(\Sigma) \) (i.e. the regular words are dense in a regular language),
3. any regular word of \( L \) is the limit of a regular sequence of \( L \),
4. \( L \) is closed for \( D \) iff \( L \cap \text{Rat}(\Sigma) \) is closed for \( D \cap \text{Rat}(\Sigma) \).

Characterizations of continuity and uniform continuity for regularity-preserving functions

We now give a characterisation of continuity and uniform continuity for functions preserving regular languages under inverse image, called regularity-preserving functions [10]. This characterisation will be useful later on to get decidability of continuity and uniform continuity for rational and regular functions. We recall the definition of uniform continuity:

- **Definition 7** (Uniform continuity). A function \( f : \Sigma^\omega \rightarrow \Gamma^\omega \) is uniformly continuous if:
  \[
  \forall i \geq 0 \ \exists j \geq 0 \text{ s.t. } \forall x, y \in \text{dom}(f), \ |x \land y| \geq j \Rightarrow |f(x) \land f(y)| \geq i.
  \]

  For totally bounded metric spaces, uniform continuity coincides with another notion of continuity, Cauchy continuity, which is usually weaker. Cauchy continuity is a more local notion than uniform continuity and will suit us more in the following. A Cauchy continuous function is a function which maps Cauchy sequences (here converging sequences since we deal with complete spaces) to Cauchy sequences.

- **Definition 8** (Cauchy continuity). Let \( f : \Sigma^\omega \rightarrow \Gamma^\omega \) be a function. We say that \( f \) is Cauchy continuous at \( x \) if for any sequence \( (x_n)_{n \in \mathbb{N}} \) of \( \text{dom}(f)^\omega \) converging to \( x \in \Sigma^\omega \), the sequence \( (f(x_n))_{n \in \mathbb{N}} \) converges. Moreover, \( f \) is Cauchy continuous if and only if it is at any point.

  The following is a standard result shown in Appendix [C.2].

- **Proposition 9.** \( f : \Sigma^\omega \rightarrow \Gamma^\omega \) is uniformly continuous iff it is Cauchy continuous.

- **Remark 10.** Notice that a function is continuous if and only if it is Cauchy continuous at any point of its domain. Similarly, a function is Cauchy continuous if and only if it is Cauchy continuous at any point of the topological closure of its domain.

  The following notions define particular sequences and pairs of sequences of words:

- **Definition 11.** Let \( f : \Sigma^\omega \rightarrow \Gamma^\omega \) be a partial function.

  Let \( (x_n)_{n \in \mathbb{N}} \) be a sequence of \( \text{dom}(f)^\omega \) converging to \( x \in \Sigma^\omega \), such that \((f(x_n))_{n \in \mathbb{N}}\) is not convergent. Such a sequence is called a bad sequence at \( x \) for \( f \).

  Let \( (x_n)_{n \in \mathbb{N}} \) and \( (x'_n)_{n \in \mathbb{N}} \) be two sequences of \( \text{dom}(f)^\omega \) both converging to \( x \in \Sigma^\omega \), such that either \((f(x_n))_{n \in \mathbb{N}}\) is not convergent, \((f(x'_n))_{n \in \mathbb{N}}\) is not convergent, or \( \lim_n f(x_n) \neq \lim_n f(x'_n) \). Such a pair of sequences is called a bad pair of sequences at \( x \) for \( f \).

  A bad pair is called regular if both its sequences are regular sequences.

- **Remark 12.** A function is Cauchy continuous if and only if it has no bad sequence. A function is continuous if it has no bad sequence at any point of its domain. Similarly a function has a bad sequence at a point \( x \) if and only if it has a bad pair at the same point.

  We now show that for regularity-preserving functions, continuity and Cauchy continuity can be characterized by the behavior of regular sequences only.

- **Lemma 13.** Let \( f : \Sigma^\omega \rightarrow \Gamma^\omega \) be a regularity-preserving function. If \( f \) has a bad sequence at some point \( x \) then it has a regular bad pair at some point \( z \). Moreover \( x \in \text{dom}(f) \iff z \in \text{dom}(f) \).
Proof. Let \( f : \Sigma^\omega \rightarrow \Gamma^\omega \) be a function preserving regularity. Let us assume that \( f \) has a bad sequence \( (x_n)_{n \in \mathbb{N}} \) at some point \( x \). By compactness of \( \Gamma^\omega \) we can extract two sub-sequences \( (x_{n_i})_{n_i \in \mathbb{N}} \) and \( (x_{n'_i})_{n'_i \in \mathbb{N}} \) of \( \text{dom}(f)^\omega \) both converging to \( x \), such that \( y = \lim_{n} f(x_n) \neq \lim_{n} f(x'_n) = y' \).

Let \( i = |y \wedge y'| \), and let \( B_y = \{ z \mid |y \wedge z| > i \} = \uparrow y'[i+1] \) and \( B_{y'} = \{ z \mid |y' \wedge z| > i \} = \uparrow y'[i+1] \). By definition we have \( B_y \cap B_{y'} = \emptyset \), and moreover both sets \( B_y, B_{y'} \) are regular. Up to extracting subsequences, we can assume that for all \( n, x_n \in f^{-1}(B_y) \) and \( x'_n \in f^{-1}(B_{y'}) \). This means that \( x \in f^{-1}(B_y) \cap f^{-1}(B_{y'}) \). Since \( f \) is regularity-preserving, and from Proposition [6], the set \( f^{-1}(B_y) \cap f^{-1}(B_{y'}) \) is regular, and non-empty. Hence there exists a regular word \( z \in f^{-1}(B_y) \cap f^{-1}(B_{y'}) \). Moreover, since \( \text{dom}(f) \) is also regular, we can choose \( z \) so that \( x \in \text{dom}(f) \Leftrightarrow z \in \text{dom}(f) \). Since \( z \in f^{-1}(B_y) \), there is a sequence \( (z_n)_{n \in \mathbb{N}} \) of words in \( f^{-1}(B_y) \) which converges to \( z \). Furthermore, since \( f^{-1}(B_y) \) is regular and since \( z \) is a regular word, we can assume, from Proposition [6], that the sequence \( (z_n)_{n \in \mathbb{N}} \) is regular. Similarly, there is a regular sequence \( (z'_n)_{n \in \mathbb{N}} \) in \( f^{-1}(B_{y'}) \) which converges to \( z \).

If either sequence \( (f(z_n))_{n \in \mathbb{N}} \) or \( (f(z'_n))_{n \in \mathbb{N}} \) is not convergent then we are done. If both sequences are convergent, then \( \lim_{n} f(z_n) \in B_y \) and \( \lim_{n} f(z'_n) \in B_{y'} \) (because \( B_y \) and \( B_{y'} \) are both closed), which means that \( |\lim_{n} f(z_n) \wedge \lim_{n} f(z'_n)| \leq i \), hence the pair \( ((f(z_n))_{n \in \mathbb{N}},(f(z'_n))_{n \in \mathbb{N}}) \) is bad and regular.

Let us introduce a notion that will make dealing with regular bad pairs a bit easier. We say that a pair of sequences is synchronized if it is of the form: \((u^nwz\omega)_n,(uw^nzw'\omega)_n\). Note that a synchronized pair is in particular regular. By taking subsequences, it is not difficult to turn a regular bad pair into a synchronized bad pair (shown in Appendix C.3):

**Lemma 14.** Let \( f : \Sigma^\omega \rightarrow \Gamma^\omega \) be a function with a regular bad pair at some point \( x \). Then \( f \) has a synchronized bad pair at \( x \).

Finally, as a consequence of the previous lemmas, we get the following characterisation:

**Corollary 15.** A function \( f : \Sigma^\omega \rightarrow \Gamma^\omega \) preserving regularity (by inverse) is continuous (resp. uniformly continuous) iff it has no synchronized bad pair at any point of its domain (resp. it has no synchronized bad pair).

## 5 Deciding Continuity and Uniform Continuity

In this section, we show how to decide continuity and uniform continuity for rational and regular functions.

**Rational case** We exhibit structural patterns which are shown to be satisfied by a one-way Büchi transducer iff the rational function it defines is not continuous (resp. not uniformly continuous). We express those patterns in the pattern logic for transducers defined in [7], which is based on existential run quantifiers of the form \( \exists \pi : p \xrightarrow{u|v} q \) where \( \pi \) is a run variable, \( p, q \) are state variables and \( u, v \) word variables, and which intuitively means that there exists a run \( \pi \) from state \( p \) to state \( q \) on input \( u \), producing output \( v \). The two patterns are given in Figure 1. A one-way transducer is called trim if all its states appear in some accepting run. Any one-way Büchi transducer can be trimmed in polynomial time.

**Lemma 16.** A trim one-way Büchi transducer defines a non continuous (resp. non uniformly continuous) function if and only if it satisfies the formula \( \phi_{\text{cont}} \) (resp. \( \phi_{u\text{-cont}} \)).
Deciding the Computability of Regular Functions over Infinite Words

Figure 1 Patterns characterising non-continuity and non-uniform continuity of rational functions

Sketch of Proof. Showing that the patterns of Figure 1 induce non-continuity and non-uniform continuity, respectively is quite simple. Indeed, the first pattern is a witness that $(uv^nzw)_n \in \mathbb{N}$ is a bad sequence at a point $uv^\omega$ of its domain, for $z$ a word with a final run from $r_2$, which entails non-continuity by Remark 12. Similarly, the pattern of $\phi_{u\text{-cont}}$ witnesses that the pair $(uv^nzw)_n \in \mathbb{N}$, $(uv^nzw')_{n \in \mathbb{N}}$ is synchronised and bad (with $z,z'$ words that have a final run from $r_1,r_2$, respectively), which entails non-uniform continuity by Coro. 15.

In order to show the other direction, we make use of the characterization of Coro. 15. From a synchronized bad pair, we are able to find a pair of runs with a synchronized loop, such that iterating the loop does not affect the existing mismatch between the outputs of the two runs, which is in essence what the pattern formulas of Figure 1 state. The full proof is available in Appendix D.1.

Theorem 17. Deciding if a one way Büchi transducer defines a continuous (resp. uniformly continuous) function can be done in NLogSpace.

Proof. From Lemma 16, non continuity (resp. non uniform continuity) is equivalent to the existence of some patterns. According to [7], such patterns are NLogSpace decidable.

Regular case The case of regular functions is more intricate. To get decidability, we have to exploit the form of the output words produced by particular loops of any run of a two-way transducer, called idempotent loops. Idempotent loops always exist for sufficiently long inputs and indeed have a nice structure which allows one to characterise the form of the output words produced when iterating such loops [2]. The definition of idempotent loops is quite technical and we refer the reader to [2] for a detailed definition. Moreover, we have abstracted the main property of idempotent loops, which is a key result in our context, and for which it is not necessary to know the precise definition of idempotency. So, given a deterministic two-way transducer $T$ on finite words (we need the notion only for finite words) and an input word $u_1u_2u_3$, we will say that $u_2$ is idempotent in $(u_1,u_2,u_3)$ (or just idempotent when $u_1,u_3$ are clear from the context), if in the run $r$ of $T$ on $u_1u_2u_3$, the restriction of $r$ to $u_2$ (which is a sequence of possibly disconnected runs on $u_2$) is idempotent in the sense of [2].

Given a language of $\omega$-words $L \subseteq \Sigma^\omega$, we denote by $\text{Pref}(L)$ the set of finite prefixes of words in $L$, i.e. $\text{Pref}(L) = \{ u \in \Sigma^* \mid \exists v \in L : u \preceq v \}$. In order to deal with look-aheads more easily, we remove look-aheads by considering words annotated with look-ahead information. Given a 2DFT$_{pla}$ $(T,P)$ over alphabet $\Sigma$ and with a set of look-ahead states $Q_P$ realizing a function $f$, we define $\tilde{T}$, a 2DBT over $\Sigma \times Q_P$ which simulates $(T,P)$ over words annotated
with look-ahead states, and which accepts only words with a correct look-ahead annotation with respect to $P$ (the formal definition can be found in Appendix B.2). We denote by $f$ the function it realises, in particular for all words $u \in \text{dom}(f)$, there exists a unique annotated word $\tilde{u} \in \text{dom}(f)$ such that $\tilde{f}(\tilde{u}) = f(u)$. For any annotated word $\tilde{u}$, $\pi(\tilde{u}) = u$ stands for its $\Sigma$-projection.

From $\overline{T}$, we define $T_*$, a deterministic two-way transducer over $(\Sigma \times Q_P)^*$, which just simulates $\overline{T}$ and accepts words in $\text{Pref}(\text{dom}(\overline{T}))$. In particular, $\overline{T}$ behaves as $T$ until it reaches the right border of its input. Let $f_*$ be the function realized by $T_*$. We have that, for any infinite word $x \in \text{dom}(\overline{T})$, $\overline{T}(x) = \lim_{n \to \infty} f_*(u_n)$.

The following lemma is a first characterisation of non-continuity which we can get by exploiting the existence of synchronised bad pairs.

Lemma 18. Let $f : \Sigma^\omega \to \Gamma^\omega$ be a regular function defined by some deterministic transducer $T$ with look-ahead and let $Q_P$ be the set of look-ahead states. Then $f$ is not continuous (resp. uniformly continuous) iff there exist finite words $u_1, u_1', u_2, u_2', u_3, u_3' \in (\Sigma \times Q_P)^*$ such that $u_1u_2u_3, u_1'u_2'u_3' \in \text{dom}(f_*)$ and

1. $\pi(u_1) = \pi(u_1')$, $\pi(u_2) = \pi(u_2')$, and $x = \pi(u_1)\pi(u_2)^\omega \in \text{dom}(f)$ (resp. $x \in \Sigma^\omega$)
2. $u_2$ and $u_2'$ are idempotent in $(u_1, u_2, u_3)$ and $(u_1', u_2', u_3')$ respectively (for $T_*$)
3. there exists $i$ such that for all $n \geq 1$, $f_*(u_1u_2u_3)[n] \neq f_*(u_1'u_2'u_3')[n]$. 

Sketch of Proof. The proof of this result is similar to the one of Lemma 16. The easy direction is to show that the existence of words $u_1, u_1', u_2, u_2', u_3, u_3'$ as above is enough to exhibit non-continuity (resp. non uniform continuity).

In the other direction, as for the rational case, we start from the result of Lemma 15 which states that it suffices to check for synchronized bad pair to decide continuity/uniform continuity. Like in the rational case, we successively extract subsequences of the synchronized bad pair and at each step we need to preserve synchronicity as well as badness. The main idea is that if we iterate enough times the loop in the synchronized bad pair, we will end up with synchronized idempotent loops. The more detailed version is available in Appendix D.2.

Given a deterministic two-way transducer $T$ and words $u_1, u_2, u_3 \in \Sigma^*$ such that $u_1u_2u_3 \in \text{Pref}(\text{dom}(T))$ and $u_2$ is idempotent for $T$, we say that $u_2$ is producing in $(u_1, u_2, u_3)$ if the run of $T$ on $u_1u_2u_3$ produces something when reading at least one symbol of $u_2$, at some point in the run. If $u_2$ is producing, then $|f_*(u_1u_2u_3)| < |f_*(u_1u_2u_3')|$ for all $i \geq 1$.

Our goal is now to give another characterisation of (non) continuity, which replaces the quantification on $n$ in Lemma 18 (property 3) by a property which does not need iteration, and therefore which is more amenable to an algorithmic check. It is based on the following key result.

Lemma 19. Let $\Sigma$ be an alphabet such that $\# \notin \Sigma$. Let $f : \Sigma^\omega \to \Gamma^\omega$ be a regular function defined by some deterministic two-way transducer $T$. There exists a function $\rho_T : (\Sigma^*)^3 \to \Gamma^*$ defined on all tuples $(u_1, u_2, u_3)$ such that $u_2$ is idempotent and $u_1u_2u_3 \in \text{Pref}(\text{dom}(f))$, and which satisfies the following conditions:

1. if $u_2$ is producing in $(u_1, u_2, u_3)$, then $\rho_T(u_1, u_2, u_3) \prec \rho_T(u_1u_2, u_2, u_2u_3)$
2. for all $n \geq 1$, $\rho_T(u_1, u_2, u_3) \preceq f_*(u_1u_2u_3)$
3. for all $n \geq 1$, $\rho_T(u_1, u_2, u_3) = f_*(u_1u_2u_3)$ if $u_2$ is not producing in $(u_1, u_2, u_3)$
4. the finite word function $\rho_T' : u_1\#u_2\#u_3 \mapsto \rho_T(u_1, u_2, u_3)$ is effectively regular.

Proof. The proof of Lemma 19 is based on a thorough study of the form of the output words produced by idempotent loops, which heavily relies on results from [2]. The whole proof, which requires technical notions, can be found in Appendix D.3.
We give a new characterisation of continuity based on the function $\rho_T$. In contrast to Lemma 18, this characterisation states that we do not need to iterate the loop to check the existence of a mismatch for all iterations, as we just need to inspect $\rho_T(u_1, u_2, u_3)$ as defined in Lemma 19.

**Lemma 20.** Let $f : \Sigma^\omega \to \Gamma^\omega$ be a function defined by some deterministic two-way transducer $T$ with look-ahead and let $Q_F$ be the set of look-ahead states. $f$ is not continuous (resp. not uniformly continuous) iff there exist $u_1, u_1', u_2, u_2', u_3, u_3' \in (\Sigma \times Q_F)^*$ s.t.

1. $\pi(u_1) = \pi(u_1')$, $\pi(u_2) = \pi(u_2')$, and $x = \pi(u_1)\pi(u_2) \omega \in \text{dom}(f)$ (resp. $x \in \Sigma^\omega$)
2. $u_2$ and $u_2'$ are idempotent in $(u_1, u_2, u_3)$ and $(u_1', u_2', u_3')$ respectively (for $T$)
3. there is a mismatch between $\rho_T(u_1, u_2, u_3)$ and $\rho_T(u_1', u_2', u_3')$.

**Sketch of proof.** We show how to replace condition 3 of Lemma 18 by condition 3 of this lemma. One direction is easy: if $\rho_T(u_1, u_2, u_3)[i] \neq \rho_T(u_1', u_2', u_3')[i]$ for some $i$, then by Condition 2 of Lemma 19, we get the result. Conversely, assume there is $i$ such that $f_s(u_1 u_2 u_3)[i] \neq f_s(u_1' u_2' u_3')[i]$ for all $n \geq 1$ and $u_2, u_2'$ are both producing (the other cases are similar and done in Appendix). By Condition 1 of Lemma 19, $\rho_T(u_1, u_2, u_3) \prec \rho_T(u_1 u_2, u_2, u_3) \prec \cdots \prec \rho_T(u_1 u_2^k, u_2, u_3)$ for all $k \geq 1$, and similarly for the $u_i'$. Therefore, by taking $k$ large enough, $\rho_T(u_1 u_2^k, u_2, u_3)$ and $\rho_T(u_1', u_2^k, u_2', u_3')$ have length at least $i$. By Condition 2, $x = \rho_T(u_1 u_2^k, u_2, u_3) \preceq f_s(u_1 u_2 u_3)$ and $x' = \rho_T(u_1' u_2^k, u_2, u_3') \preceq f_s(u_1' u_2^k u_3')$ for all $n \geq 2k + 1$, from which get $x[i] \neq x'[i]$.

Finally, we show how to decide continuity by reduction to the emptiness problem of finite-visit two-way Parikh automata [3, 5].

**Theorem 21.** Continuity and uniform continuity are decidable for regular functions.

**Sketch.** The proof is based on Lemma 20. First, we encode words $u_1, u_1', u_2, u_2'$ as words over the alphabet $(\Sigma \times Q_F^2)^*$ to hard-code condition 1 of the lemma. In particular, we define the language $L$ of words of the form $w_1 \# w_2 \# w_3 \# w_4$ such that $w_1, w_2 \in (\Sigma \times P^2)^*$ represent $u_1, u_1', u_2, u_2'$ and such that conditions 1-2-3 of the lemma are satisfied. Condition 2 and condition $\pi(u_1)\pi(u_2) \omega \in \text{dom}(f)$ are simple because they are regular properties of words, the domain of $f$ being regular. For condition 4, we need counters to identify positions $i$ and $j$ such that $\rho_T(u_1, u_2, u_3)[i] \neq \rho_T(u_1, u_2, u_3)[j]$, and later on check that $i = j$. In particular, we rely on the model of two-way Parikh automata which extend two-way automata with counters which can be only incremented, and an accepting semi-linear condition on the counters. If such automata visit any input position a bounded number of times, their emptiness is decidable [3, 5]. We show that $L$ is definable by such an automaton which simulates the transducer obtained by Lemma 19(4) and which is finite-visit.

**6 Discussion and Future Work**

Although we also study the notion of uniform continuity, we do not make any connection with a computational model. The notion of *effectively uniformly continuous* functions $f$, well known in the field of computable analysis, seems to be a good candidate notion. Additionally to being computable in the sense of this paper, it also requires the existence of a function $m : N \to N$, called a *modulus of continuity*, which is computable and such that for any words $x, y$, we have $|f(x) - f(y)| \geq m(|x - y|)$. Effective uniform continuity is arguably a more useful notion than simple computability, since it tells you how far into the input you should
look in order to produce a close enough approximation of the output. In contrast, for a computable function, one might need to look arbitrarily far into the input in order to produce a single letter of the output, which does not sound very practical.

Another interesting direction we already mentioned is finding a transducer model which captures exactly the computable regular functions. We conjecture that 2DFT characterize computable regular functions but we have no proof of it yet.

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A  Section 2: Preliminaries

Proof of Theorem 1. We show that a function is $2\text{DFT}_{la}$ definable iff it is $2\text{DFT}_{pla}$ definable. Given a $2\text{DFT}_{pla}$ $(A, P)$, where $A = (Q, \Sigma, \delta_A, q_0)$ and $P = (Q, \Sigma, \delta_P, Q_0, F)$, we construct a $2\text{DFT}_{la} (T, A, B)$ as follows: For every state $p \in Q_P$, we construct an equivalent Muller automaton $A_p$ with initial state $s_p$ s.t. $L(p, p) = L(A_p)$. The Muller look-ahead automaton $A$ used in the $2\text{DFT}_{la} (T, A, B)$ is the disjoint union of the Muller automata $A_p$ for all $p \in Q_P$. $T = (Q, \Sigma, \delta_T, q_0, 2^\varnothing \setminus \emptyset)$, and $\delta_T$ is obtained by modifying the transition function $\delta_A(q, a, p) = (q', \gamma, d)$ as $\delta_T(q, a, s_p) = (q', \gamma, d)$. Since the language accepted by two distinct states of a prophetic automaton are disjoint, the language accepted by $A_p$ and $A'_p$ are disjoint for $p \neq p'$. The look-behind automaton $B$ accepts all of $\Sigma^*$. It is easy to see that $(T, A, B)$ is deterministic : on any position $i$ of the input word $a_1a_2 \ldots$, and any state $q \in Q_T$, there is a unique $A_p$ accepting $a_{i+1}a_{i+2} \ldots$. The domain of $(T, A, B)$ is the same as that of $(A, P)$, since the accepting states of $A$ are the union of the accepting states of all the $A_p, p \in Q_P$. Since all the transitions $\delta_T$ have the same outputs as in $\delta_A$ for each $(q, a, p)$, the function computed by $(A, P)$ is the same as that computed by $(T, A, B)$.

For the other direction, given a $2\text{DFT}_{la}$, it is easy to remove the look-behind by a product construction [6]. Assume we start with such a modified $2\text{DFT}_{la} (T, A)$ with no look-behind. Let $T = (Q_T, \Sigma, \delta_T, s_T, F_T)$ and $A = (Q_A, \Sigma, \delta_A, s_A, F_A)$ where $F_T, F_A$ respectively are the Muller sets corresponding to $T$ and $A$. We describe how to obtain a corresponding $2\text{DFT}_{pla} (A, P)$ with $P$, a prophetic Büchi look-ahead automaton. $A$ is described as $(Q_T, \Sigma, \delta_T, s_T)$. The prophetic look-ahead automaton is described as follows. Corresponding to each state $q \in Q_A$, let $P_q$ be a prophetic automaton such that $L(A, q) = L(P_q)$ (this is possible since prophetic automata capture $\omega$-regular languages [4]). Since $A$ has no accepting condition, we also have a prophetic look-ahead automaton to capture $\text{dom}(T)$. The Muller acceptance of $T$ can be translated to a Büchi acceptance condition, and let $P_{\text{dom}(T)}$ represent the prophetic automaton such that $L(P_{\text{dom}(T)}) = \text{dom}(T)$. Thanks to the fact that prophetic automata are closed under synchronized product, the prophetic automaton $P$ we need, is the product of $P_q$ for all $q \in Q_A$ and $P_{\text{dom}(T)}$. Assuming an enumeration $q_1, \ldots, q_n$ of $Q_A$, the states of $P$ are $|Q_A| + 1$ tuples where the first $|Q_A|$ entries correspond to states of $P_{q_1}, \ldots, P_{q_n}$, and the last entry is a state of $P_{\text{dom}(T)}$. Using this prophetic automaton $P$ and transitions $\delta_T$, we obtain the transitions $\delta_A$ of $A$ as follows. Consider a transition $\delta_T(p, a, q_i) = (q', \gamma, d)$. Correspondingly in $A$, we have $\delta_A(p, a, \kappa) = (q', \gamma, d)$ where $\kappa$ is a $|Q_A| + 1$ tuple of states such that the $i$th entry of $\kappa$ is an initial state of $P_{q_i}$. From the initial state $s_T$, on reading $\top$, if we have $\delta_T(p_0, \top, q_i) = (q', \gamma, d)$, then $\delta_A(p_0, \top, \kappa) = (q', \gamma, d)$ such that the $i$th entry of $\kappa$ is an initial state of $P_{q_i}$ and the last entry of $\kappa$ is an initial state of $P_{\text{dom}(T)}$.

To see why $(A, P)$ is deterministic. For each state $q \in Q_T$, for each position $i$ in the input word $a_1a_2 \ldots a_i \ldots$, there is a unique state $p \in Q_A$ such that $a_{i+1}a_{i+2} \ldots$ is accepted by $A$. By our construction, the language accepted from each $p \in Q_A$ is captured by the prophetic automaton $P$; by the property of the prophetic automaton $P$, we know that for any $q_1, q_2 \in Q_A$ with $q_1 \neq q_2$, $L(P_{q_1}) \cap L(P_{q_2}) = \emptyset$. Thus, in $(A, P)$, for each state $q$ of $A$, there is a unique state $p \in P$ such that the suffix is accepted by $L(P)$; further, from the initial state of $A$, from $\top$, there is a unique state $p \in P$ which accepts $\text{dom}((T, A))$. Hence $\text{dom}((A, P))$ is exactly same as $\text{dom}((T, A))$. For each $\delta_T(q, a, q_i) = (q', \gamma, d)$, we have the transition $\delta_A(q, a, \kappa) = (q', \gamma, d)$ with the $i$th entry of $\kappa$ equal to the initial state of $P_{q_i}$, which preserves the outputs on checking that the suffix of the input from the present position is in $L(P_{q_i})$. Notice that the entries $j \neq i$ of $\kappa$ are decided uniquely, since there is a unique state in each $P_{q_i}$ from where each word has an accepting run. Hence, $(A, P)$ and $(T, A)$ capture
the same function.

\section{Section 3}

\subsection{Proof of Theorem 4}

The two directions of this equivalence are given by Lemma 22 and Lemma 23 respectively.

\begin{lemma}
If a function $f$ is computable, then it is continuous.
\end{lemma}

\begin{proof}
Assume that $f$ is computable. We prove the continuity of $f$. Let $M$ be the machine computing $f$. Let $x \in \text{dom}(f)$ be on the input tape of $M$. For all $i \geq 0$, define $\text{Pref}_M(x,i)$ to be the smallest $j \geq 0$ such that, when $M$ moves to the right of $x[j]$ into cell $j + 1$, it has output at least the first $i$ symbols of $f(x)$. For any $i \geq 0$, choose $j = \text{Pref}_M(x,i)$. Consider any $z \in \text{dom}(f)$ such that $|x \land z| \geq j$. After reading $j$ symbols of $x$, the machine $M$ outputs at least $i$ symbols of $f(x)$. Since $|x \land z| \geq j$, the first $j$ symbols of $x$ and $z$ are the same, and $M$ being deterministic, outputs the same $i$ symbols on reading the first $j$ symbols of $z$ as well. These $i$ symbols form the prefix for both $f(x)$, $f(z)$, and hence, $|f(x) \land f(z)| \geq i$.

Thus, for every $x \in \text{dom}(f)$ and for all $i$, there exists $j = \text{Pref}_M(x,i)$ such that, for all $z \in \text{dom}(f)$, if $|x \land z| \geq j$, then $|f(x) \land f(z)| \geq i$ implying continuity of $f$.
\end{proof}

\begin{lemma}
If a regular function is continuous, then it is computable.
\end{lemma}

\begin{proof}
Let $f$ be a continuous regular function. We define a machine $M_f$ to compute $f$. The working of $M_f$ is described in Algorithm 1. For two words $x, y$, let $\text{mismatch}(x, y)$ denote that there is some position $i \geq 1$ such that $x[i] \neq y[i]$. To argue the termination of algorithm 1 on all inputs, we have to decide the test in line number 3. We first show (Lemma 24) the soundness of the algorithm (that is, $M_f$ indeed computes $f(x)$) assuming the decidability of the test in line number 3. Then we show the decidability of the test (Lemma 5).

The following Lemma proves the soundness of Algorithm 1 and thanks to that, the limit of $\text{out}$ converges to $f(x)$.

\begin{lemma}
For a continuous function $f$, $x \in \text{dom}(f)$, $\text{out}$ is updated infinitely often in Algorithm 1. Moreover, machine $M_f$ computes $f(x)$ as defined in Definition 5.
\end{lemma}

\begin{proof}
Assume that $\text{out}$ is not updated infinitely often. Then, line 4 is not executed after some iteration $m$. Let $\text{out}_m$ represent the value of $\text{out}$ after $m$ iterations, and let the length of $\text{out}_m$ be $\ell$. Then, for all $k > m$, and for all $\gamma \in \Gamma$, there is an extension $y_k$ of $x[k]$, $y_k \in \text{dom}(f)$, for which $\text{mismatch}(\text{out}_m, \gamma, f(y_k))$. This violates the continuity of $f$ as seen below. For $x \in \text{dom}(f)$, choose $i = \ell + 1$.

\begin{itemize}
\item For all $j > m$, the extension $y_j$ is s.t. $|x \land y_j| \geq j$ and $|f(x) \land f(y_j)| < i$.
\item For all $j \leq m$, the extension $y_{m+1}$ is s.t. $|x \land y_{m+1}| \geq j$ and $|f(x) \land f(y_{m+1})| < i$.
\end{itemize}

This contradicts the continuity of $f$, proving that $\text{out}$ is updated infinitely often.

Next we show that, $M_f$, as described in the algorithm indeed computes $f(x) = y$. Observe that, in each iteration $i$, $\text{out}$ is appended with a symbol $\gamma$ if $\gamma \text{out} \gamma$ has no mismatch with $f(z)$, for all possible extensions $z$ of $x[i]$. This gives the invariant that, in each iteration $i$, $\text{out}_i \leq f(x)$. In the Algorithm 1 $M_f(x,i)$ denotes $\text{out}_i$. Assume there exists a position $j$ of $y = f(x)$ such that for all positions $i \geq 0$ of $x$, $y[j] \neq \text{out}_i$. This implies that for all $i \geq 0$, either $\text{mismatch}(y[j], \text{out}_i)$ or $\text{out}_i < y[j]$. However, as observed already, we have the invariant $\text{out}_i \leq f(x)$ for all $i \geq 0$. Hence, $\text{mismatch}(y[j], \text{out}_i)$ is not possible.

Since $\text{out}$ is updated infinitely often, there is a strictly increasing sequence $i_1 < i_2 < \cdots \in \mathbb{N}$ such that $\forall i < i_k$ $\text{out}_i < \text{out}_k \cdots$. For all $j \geq 0$, there exists $i_k$ s.t. $i_k < \text{out}_k$, and $y[j] \leq \text{out}_{i_k} = M_f(x, i_k)$. Thus, the machine $M_f$ described in Algorithm 1 computes $f$.
\end{proof}
B.2 The mismatch problem: Proof of Lemma 5

Before we discuss the proof of the decidability of the mismatch problem, we set up some notations. Given a 2DFT (\(T, P\)) over input alphabet \(\Sigma\), and \(P = (Q_P, \Sigma, \delta_P, S_P, F_P)\), let \(u \in (\Sigma \times Q_P)^\omega\) be an annotated word over the extended alphabet \(\Sigma \times Q_P\). Given \(u = (i, p_0)(a_1, p_1)(a_2, p_2)\ldots\), let \(u[i]\) represent \((a_i, p_i)\) and \(\pi(u) \in \Sigma^\omega\) and \(\pi_P(u) \in Q_P^\omega\) respectively denote the projections of \(u\) to its first and second components respectively. An annotated word \(u\) is good if \(\forall i \geq 1, \pi_u(u[i-1]) \in L(P, \pi_P(u[i-1]))\). That is, the suffix \(a_i a_{i+1} a_{i+2} \ldots\) of \(\pi(u)\) has a final run in \(P\) starting from the state \(p_{i-1} = \pi_P(u[i-1])\).

As a first step, we show that, given \(f\) specified as a 2DFT (\(T, P\)) with input alphabet \(\Sigma\), we can construct a function \(T\) specified as a 2DBT \(T\) over the input alphabet \(\Sigma \times Q_P\), such that \(\bar{f}(u) = f(\pi_u(u))\) for all good annotated words \(u\).

Elimination of look-ahead, construction of \(T\). Let \(f\) be specified as a 2DFT (\(T, P\)), with \(T = (Q_T, \Sigma, \Gamma, \delta_T, s_T, 2Q)^\\emptyset\) and \(P = (Q_P, \Sigma, \delta_P, S_P, F_P)\). The state space of \(T\) is \(Q_T \cup (Q_T \times \Sigma \times Q_P) \cup (Q_T' \times \Sigma \times Q_P)\), and has initial state \(s_T\). Given a word \((i, p_0)(a_1, p_1)(a_2, p_2)\ldots\), we start in state \(s_T\), reading \((i, p_0)\), move to the right in state \(s_T\), reading \((i, p_0)\), and output \(\epsilon\). The states of \(T\), \((Q_T \times \Sigma \times Q_P)\) behave in a deterministic manner: from any state \(r \in Q_T\), on reading some \((a_i, p_i)\), we move right, in state \((r', a_i, p_i)\), and output \(\epsilon\) if \(p \in \delta_P(p_i, a_i)\). This step checks the consistency of the annotation: if \((a_i, p_i)\) and \((a_{i+1}, p_{i+1})\) appear consecutively in the annotated word, then it must be that \(p_{i+1} = \delta_P(p_i, a_i)\). From a state \((q, a, p)\) in \((Q_T \times \Sigma \times Q_P)\), on reading \((a_i, p)\), we mimic the transitions of \((T, P): \delta_T((q, a, p), (a_i, p)) = (r, \gamma, d)\) if \(a = a_i, p = p_i\), and \(\delta_T(q, a, p) = (r, \gamma, d)\). The Büchi acceptance condition is given by the set of states \(Q_T \times \Sigma \times Q_P\).

It is easy to see that \(T\) is deterministic: \(\bar{T}\) has all the transitions of \(T\) from states of the form \(Q_T \times \Sigma \times Q_P\). It also has the transitions from \(Q_T \cup (Q_T' \times \Sigma \times Q_P)\) which behave deterministically as described above. The determinism of \(T\) follows from the determinism of \((T, P)\). Now we show that \(\bar{f}(u) = f(\pi_u(u))\).

Consider any word \(w = \overline{a_1 a_2 \cdots} \in \text{dom}(f)\). \(w\) has a unique accepting run in \((T, P)\). By the property of \(P\), at each position \(i\), there is a unique state \(p_i\) of \(P\) such that \(a_{i+1} a_{i+2} \cdots \in L(P, p_i)\). Consider the good annotation \((i, p_0)(a_1, p_1)\ldots\) of \(w\). This word is accepted by \(\bar{T}\): we check the consistency of the annotation of every two consecutive symbols, using states from \(Q_T \cup (Q_T' \times \Sigma \times Q_P)\) without producing any outputs, and states \((q, a, p)\) in \(Q_T \times \Sigma \times Q_P\) mimicking the transition \(\delta_T(q, a, p)\), producing the same outputs and moving in the same direction. Since \(w\) is accepted in \((T, P)\), we know that \(w \in L(P)\). The Büchi acceptance condition of \(\bar{T}\) checks the same condition for acceptance of the good annotated word; hence \(w \in \text{dom}((T, P))\) iff the good annotation of \(w\) is in \(\text{dom}(\bar{T})\). By construction, \(\bar{f}(\bar{w}) = f(w)\), where \(\bar{w}\) is the good annotation of \(w\).

The converse direction is done in a similar way, starting from an annotated word \(u = (i, p_0)(a_1, p_1)\ldots\) accepted by \(\bar{T}\). The transitions in \(\bar{T}\) ensure that (i) the annotation is consistent, (ii) the outputs produced are same at each position \(i\), and (iii) the acceptance condition checks that the annotation is good. If there were two consecutive symbols \((a_i, p_i)(a_{i+1}, p_{i+1})\) such that \(p_{i+1} \in \delta_P(a_i, p_i)\), and \(a_i a_{i+1} a_{i+2} \cdots \notin L(P, p_{i+1})\), then we will not see a final state of \(P\) infinitely often, since all subsequent states of \(P\) appearing in the annotation will witness the non-acceptance of \(a_i a_{i+1} a_{i+2} \ldots\). Hence, whenever \(u\) is accepted in \(\bar{T}\), producing \(f(u)\), \(\pi_u(u)\) is accepted in \((T, P)\), such that \(f(\pi_u(u)) = f(u)\).

We work on \(T\) rather than \((T, P)\) to decide the mismatch problem. By the above construction of \(\bar{T}\), for a given \(u \in \Sigma^*, v \in \Gamma^*\), there exists a \(y \in \Sigma^*, \) s.t. \(\text{mismatch}(v, f(uy))\)
iff there is a good annotation $\tilde{uy} \in \text{dom}(T)$ for which $\text{mismatch}(v, f(\tilde{uy}))$. Hence the mismatch problem for $2\text{DFT}_{\text{pla}}(T, P)$ reduces to the mismatch problem for $2\text{DBT} \ T$.

The mismatch problem for $2\text{DBT}$ is PSpace-complete

Proof. We first show the PSPACE-membership. Let $f$ be a function specified as a 2DBT $T = (Q, \Sigma, \Gamma, \delta, q_0, F)$. Without loss of generality, assume that $T$ produces at most one output symbol in each transition. Given $u \in \Sigma^*$, $v \in \Gamma^*$, define $L_{\text{mis}} = \{uy \in \Sigma^* | uy \in \text{dom}(f), \text{mismatch}(v, f(uy))\}$. Given $u, v$ as above, we construct a two-way Büchi automaton $A$ such that $L(A) = L_{\text{mis}} \neq \emptyset$ iff there exists $y \in \Sigma^*$ s.t. $\text{mismatch}(v, f(uy))$ and $uy \in \text{dom}(T)$.

The state space of $A$ is $Q \cup \{1, 2, \ldots, |u|\} \cup (Q \times \{1, 2, \ldots, |v|\}) \cup \{\bot\}$. The initial state of $A$ is 1, and $F$ is the set of accepting states. The transitions are defined as follows.

1. Given an input $w \in \Sigma^*$, $A$ ensures that the first $|u|$ symbols of $w$ satisfy $u = w[|u|]$.
   Since $u = a_1 \ldots a_k$ is an input to the mismatch problem, this is done by starting in state 1, reading the first symbol of $w$, checking if it is same as $a_1$, and if so, move to the right in state 2, and continue this till we reach the last symbol of $u$ in state $|u|$. Anytime we find a symbol not in $u$, $A$ enters the state $\bot$. On successfully reading the first $|u|$ symbols of $w$ and checking it if be $u$, $A$ comes all the way back to $\top$, and enters the state $(q_0, 1)$.

2. From state $(q_0, 1)$, $A$ mimics $T$, and checks if a mismatch with $v$ is detected in the first $|v|$ output symbols produced. The second component of the state grows till at most $|v|$ while checking for the mismatch. To begin, if the first symbol of $v$ is the same as the first symbol produced by $T$, then, the second component of the state is incremented from 1 to 2. In general, if the second component is $i$, and if the $i$th symbol of $v$ is the same as the $i$th output symbol produced by $T$, then, the second component increments to $i + 1$. If no mismatch is detected, and the second component is already $|v|$, then the trap state $\bot$ is entered. Formally, for $\gamma \in \Gamma$,
   \begin{align*}
   \delta((q, i), a) &= ((q', i + 1), \gamma, d) \text{ if } \delta_T(q, a) = (q', \gamma, d) \text{ and } \gamma \neq v[i], \\
   \delta((q, i), a) &= (q', \gamma, d) \text{ if } \delta_T(q, a) = (q', \gamma, d) \text{ and } \gamma = v[i], \\
   \delta((q, |v|), a) &= \bot \text{ if } \delta_T(q, a) = (q', \gamma, d) \text{ and } \gamma = v[|v|],
   \end{align*}

Once the mismatch is detected, $A$ behaves just like $T$, and all transitions of $T$ are also present. If $T$ accepts $w$, so does $A$.

The size of $A$ is polynomial in the size of $T$. From $[13, 11]$, we know that given a two-way Büchi automata with $n$ states, we can construct an equivalent NBA with $O(2^{2^n})$ states. This, along with the NLLogSpace complexity of emptiness checking of NBA $[10]$, gives us a PSPACE procedure to test the mismatch for $2\text{DBT}$.

Next, we show PSPACE hardness. We reduce the emptiness problem of the intersection of $n$ DFAs to the mismatch problem for $2\text{DBT}$. Given $n$ DFAs $A_1, \ldots, A_n$ over $\Sigma = \{a, b\}$, checking if $\bigcap_{i=1}^n L(A_i)$ is empty is PSPACE-complete. Consider a function $f : \Sigma^* \Rightarrow \#^* \cup \$^*$ defined as follows.

$$f(w) = \begin{cases} 
\#^*, & \text{if } w \in \bigcap_{i=1}^n L(A_i) \\
\$, & \text{if } w \notin \bigcap_{i=1}^n L(A_i)
\end{cases}$$

Let $u = \epsilon, v = \$$. Then there exists $y \in \Sigma^* \Rightarrow \#^*$ such that $\text{mismatch}(v, f(y))$ iff $w \in \bigcap_{i=1}^n L(A_i)$. $f$ can be specified as a $2\text{DBT} \ A$ whose size is polynomial in $A_1, \ldots, A_n$ as follows. Given an input $\top \Rightarrow w \Rightarrow \$, $A$ starts in the initial state of $A_1$, and checks if $w \in L(A_1)$, and if so, moves all the way back to $\top$. Then from the initial state of $A_2$, it checks if $w \in L(A_2)$, comes back to $\top$ and so on, until it has checked if $w \in L(A_n)$. Nothing is output till the checks are complete. If the check on $A_n$ is successful, from the final state of $A_n$, $A$ keeps moving right,
and outputs # on each input symbol. If $w \notin L(A_j)$ for some $A_j$, then from the rejecting state of $A_j$, $A$ continues moving right, and outputs a $\$ on each input symbol. Clearly, the description of $A$ is polynomial in the sizes of $A_1, \ldots, A_n$. ▶

To summarize, the mismatch problem for $2\text{DFT}_{\text{pla}}$ is solved as follows. Given $u \in \Sigma^*$ and $v \in \Gamma^*$, guess a good annotation $\bar{u}$ of $u$. Let $A$ be the 2DBT that checks the goodness of the annotation. The size of $A$ is polynomial in the size of the $2\text{DFT}_{\text{pla}}$. Check the mismatch w.r.t $\bar{u}$ and $v$. Thanks to Lemma 5 we obtain the $\text{PSPACE}$-completeness of the mismatch problem for $2\text{DFT}_{\text{pla}}$.

C Section 4 Continuity for Functions Preserving Rational Languages

C.1 Proof of Proposition 6

Proof. Let $L \subseteq \Sigma^\omega$ be regular. Let $uv^\omega$ be a regular word in $L$. By regularity of $L$, there is a power of $v$, $v^k$ such that for any words, $w, x, wv^kx \in L \Leftrightarrow wv^{k+i}x \in L$. Let us consider the language $K = \{ x \mid wv^kx \in L \}$ which is non-empty since $uv^\omega \in L$. Moreover, $K$ is regular, and thus contains a regular word $wz^\omega$.

Hence we have that $wz^\omega$ is the limit of the sequence $(wv^knwz^\omega)_{n\in\mathbb{N}}$ of $L$. ▶

C.2 Proof of Proposition 9

Proof. Uniform continuity implies Cauchy continuity for any metric space. Let $\Gamma^\omega$ be the uniform completion of $\Gamma$. Let $f : \Sigma^\omega \rightarrow \Gamma^\omega$ be a uniformly continuous function and let $(x_n)_{n \in \mathbb{N}}$ of $\text{dom}(f)^\omega$ be converging to $x$. Let $i \geq 0$, and let $j$ be such that $\forall x, y \in \text{dom}(f), |x \land y| \geq j \Rightarrow |f(x) \land f(y)| \geq i$. Since $(x_n)_{n \in \mathbb{N}}$ converges, there is an integer $N$ such that for any $m, n \geq N$, $|x_m \land x_n| \geq j$, and thus $|f(x_m) \land f(x_n)| \geq i$. Hence $(f(x_n))_{n \in \mathbb{N}}$ converges (it is a Cauchy sequence and $\Gamma^\omega$ is complete).

Let us now assume that $f$ is not uniformly continuous. Then: $\exists i \geq 0 \forall j \geq 0, \exists x_j, y_j \in \text{dom}(f), |x_j \land y_j| \geq j \land |f(x_j) \land f(y_j)| < i$. Let $(x_{j_n})_{n \in \mathbb{N}}$ be convergent subsequences of $(x_j)_{j \in \mathbb{N}}$. We have that $|x_{j_n} \land y_{j_n}| \geq n$ and $|f(x_{j_n}) \land f(y_{j_n})| < i$ for any $n \in \mathbb{N}$. Let $x'_n = x_{j_n}$ and let $y'_n = y_{j_n}$. We can now consider $(y'_{n_m})_{m \in \mathbb{N}}$ a convergent subsequence of $(y_n)_{n \in \mathbb{N}}$. Let $x''_m = x'_{n_m}$ and let $y''_m = y'_{n_m}$. We still have that $|x''_m \land y''_m| \geq m$ and $|f(x''_m) \land f(y''_m)| < i$. Since $(x''_m)_{m \in \mathbb{N}}$ and $(y''_m)_{m \in \mathbb{N}}$ are both convergent, then they converge both to the same limit $x$. Let $\lim_n f(x''_n) = z$, $\lim_n f(y''_n) = t$ and for all $n$, $|f(x''_n) \land f(y''_n)| < i$, which means that $|z \land t| < i$. Now the sequence alternating between $x''_n$ and $y''_n$ converges to $x$ but its image is divergent, hence $f$ is not Cauchy continuous at $x$. ▶
C.3 Proof of Lemma 14

Proof. Let \( f : \Sigma^\omega \to \Gamma^\omega \) be a function with a regular bad pair \( ((x_n)_{n \in \mathbb{N}}, (x'_n)_{n \in \mathbb{N}}) \) at some point \( x \). If one of the image sequences is divergent, let us say \( (f(x_n), u_{n})_{n \in \mathbb{N}} \), then \( ((x_n)_{n \in \mathbb{N}}, (x'_n)_{n \in \mathbb{N}}) \) is a synchronized bad pair at \( x \). Let us assume that both image sequences converge.

Let \( x_n = u^n w z^n \) and let \( x'_n = u^n w^m z^{m} \).

Let us first assume that \( x'_n \) is constant equal to \( x \). Let \( x' = u^{-1} x \), we have that \( u^n x' = x' \) and \( u v^n x' = x \) for any \( n \). Thus the pair \( ((x_n)_{n \in \mathbb{N}}, (x')_{n \in \mathbb{N}}) \) is a synchronized bad pair.

Let us now assume that neither sequence is constant, which means that \( |v|, |v'| > 0 \).

Without loss of generality, let us assume that \( |u| \geq |u'| \), let \( k \in \mathbb{N} \), let \( p < |v'| \) be such that \( |u'| + k|v'| + p = |u| \), let \( v'' = v'[p + 1 : |v'|]v'[1 : p] \) and let \( u'' = v''[p + 1 : |v'|]u' \). Then we can write \( x''_n = x'_{n+k+1} = u''(v'')^k v'[1 : p] \).

Note that \( v'' = v'(v')^k \), which means that \( v''[v'] = v''[v] \).

Let \( y_n = x''_{n}|v| = u''v''[v]w z^{n} \) and let \( y''_n = x''_{n}|v| = u''v''[v]w z^{n} \). Then the pair \( ((y_n)_{n \in \mathbb{N}}, (y''_n)_{n \in \mathbb{N}}) \) is a synchronized bad pair at \( x \).

\( \square \)

D Section 5: Deciding Continuity and Uniform Continuity

D.1 Proof of Lemma 16

Proof. Let us consider the continuous case. The uniformly continuous case is similar. Let \( f \) be a function realized by a trim transducer \( T \).

Let us first show that if the pattern of \( \phi_{\text{cont}} \) appears, then we can exhibit a bad pair at a point of the domain. Let us assume:

\[ \exists \pi_1 : p_1 \xrightarrow{u_1} q_1, \exists \pi'_1 : q_1 \xrightarrow{v_1} q_1 \]
\[ \exists \pi_2 : p_2 \xrightarrow{u_2} q_2, \exists \pi'_2 : q_2 \xrightarrow{v_2} q_2, \exists \pi''_2 : q_2 \xrightarrow{w_2} r_2 \]
\[ (\text{init}(p_1) \wedge \text{init}(p_2)) \wedge \text{final}(q_1) \wedge \text{mismatch}(u_1, u_2) \vee (v_2 = \epsilon \wedge \text{mismatch}(u_1, u_2)) \]

Then the word \( u v w \) is in the domain of the function realized by the transducer since it has an accepting run. Let \( z t \) be a word accepted from state \( r_2 \), which exists by trimness, and let us consider the pair \( ((u v w z t)_{n \in \mathbb{N}}, (u v w z t)_{n \in \mathbb{N}}) \). We have \( f(u v w z t) = u_1 t_1 \) and \( u_2 \leq \lim_n f(u v w z t) \). If \( \text{mismatch}(u_1, u_2) \) then the pair is bad, otherwise, we must have \( u_2 = \epsilon \) and thus \( u_2 w_2 \leq \lim_n f(u v w z t) \). Since \( \text{mismatch}(u_1, u_2) \) we again have that the pair is bad.

Let us assume that the function \( f \) is not continuous at some point \( x \in \text{dom}(f) \). Then according to Corollary 15 there is a synchronized bad pair \( ((u v w z)_{n \in \mathbb{N}}, (u v w z)_{n \in \mathbb{N}}) \)

converging to some point of \( \text{dom}(f) \).

Our goal is to exhibit a pattern as in \( \phi_{\text{cont}} \). If \( v \) is empty, then the sequences are constant which contradicts the functionality of \( T \). Then either \( ((u v w z)_{n \in \mathbb{N}}, (u v w z)_{n \in \mathbb{N}}) \) is a bad pair or \( ((u v w z)_{n \in \mathbb{N}}, (u v w z)_{n \in \mathbb{N}}) \) is a bad pair (or both). Without loss of generality, let \( ((u v w z)_{n \in \mathbb{N}}, (u v w z)_{n \in \mathbb{N}}) \) be a bad pair.

Let us consider an accepting run \( \rho \) of \( T \) over \( u v w \). Since the run is accepting, there is a final state \( q_1 \) visited infinitely often. Let \( l \in \{1, \ldots, |v|\} \) be such that, the state reached after reading \( u v [1 : l] \) in the run \( \rho \) is \( q_1 \) for infinitely many \( n \). Let \( v' = v[l + 1 : |v|]v[1 : l] \), let \( u' = u v[1 : l] \) and let \( u'' = v[l + 1 : |v|] \), we have that \( ((u v w z)_{n \in \mathbb{N}}, (u v w z)_{n \in \mathbb{N}}) \) is a bad pair. To simplify notations, we write this pair again as \( ((u v w z)_{n \in \mathbb{N}}, (u v w z)_{n \in \mathbb{N}}) \).
We have gained that on the run over \( uv^\omega \), the final state \( q_1 \) is reached infinitely often after reading \( v \) factors, and let \( k, l \) be integers such that \( p_1 \xrightarrow{uv} q_1 \xrightarrow{v^k} q_1 \), with \( p_1 \) an initial state. We consider \( l \) different sequences, for \( r \in \{0, \ldots, l - 1\} \) we define the sequence \( s_r = (uv^{l+n+r}wz^\omega)_{n \in \mathbb{N}} \). Since \( ((uv^n w^\omega)_{n \in \mathbb{N}}, (uv^\omega)_{n \in \mathbb{N}}) \) is a bad pair, there must be a value \( r \) such that \( ((uv^k(v^{n+r})w^\omega)_{n \in \mathbb{N}}, ((uv^k(v^{n+r})w^\omega)_{n \in \mathbb{N}}) \) is a bad pair. To simplify notations, again, we rename this pair \( ((uv^n w^\omega)_{n \in \mathbb{N}}, (uv^\omega)_{n \in \mathbb{N}}) \), and we know that \( p_1 \xrightarrow{u} q_1 \xrightarrow{v} q_1 \), with \( p_1 \) initial and \( q_1 \) final.

For any \( m \) larger than the number of states \( |Q| \) of \( T \), let us consider the run of \( T \) over \( uv^m w^\omega \). Let \( i_m, j_m, k_m \) be such that we have the accepting run \( p_m \xrightarrow{uv^m} q_m \xrightarrow{v^m} q_m \xrightarrow{s^m w^\omega} \), with \( i_m + j_m + k_m = m \) and \( 0 < j_m \leq |Q| \). And let \( r_m = i_m + k_m \mod j_m \). Let us consider for some \( m \), the sequence \( ((uv^m)^{(v^m)^n}w^\omega)_{n \in \mathbb{N}} \). By the presence of these loops, each of these sequences has a convergent image. There is actually a finite number of such sequences, since \( j_m \) and \( r_m \) take bounded values. Since the original pair is bad, this means that there must exist \( m \) such that the sequence \( f(((uv^m)^{(v^m)^n}w^\omega))_{n \in \mathbb{N}} \) does not converge to \( f(u v^\omega) \). Let \( u' = uv^m, \ v' = v^m \) and \( u'' = v'^k w \), then \( ((u'v''v^m)^n(w^\omega)_{n \in \mathbb{N}}, (u'v''v^\omega)_{n \in \mathbb{N}}) \) is a bad pair. Once again we rename the pair \( ((uv^n w^\omega)_{n \in \mathbb{N}}, (uv^\omega)_{n \in \mathbb{N}}) \) and now we have both \( p_1 \xrightarrow{u} q_1 \xrightarrow{v} q_1 \) and \( p_2 \xrightarrow{u} q_2 \xrightarrow{v} q_2 \), such that \( p_1, p_2 \) are initial, \( q_1 \) is final and \( w^\omega \) has a final run from \( q_2 \).

Now we have established the shape of the pattern, we only have left to show the mismatch properties.

\[
\begin{align*}
p_1 &\xrightarrow{u|u_1} q_1 \xrightarrow{v|v_1} q_1 \\
p_2 &\xrightarrow{u|u_2} q_2 \xrightarrow{v|v_2} q_2 \xrightarrow{w^\omega|\omega} \\
\end{align*}
\]

Let us first assume that \( v_2 \neq \epsilon \). Then \( \lim_n f(u v^n w^\omega) = u v^\omega \). Since the pair is bad, there exists \( k \) such that \( \text{mismatch}(u_1 v_1^k, u_2 v_2^k) \). Hence, up to taking \( u' = uv^k \), we have established the pattern:

\[
\exists \pi_1 : p_1 \xrightarrow{u'|u_1'} q_1, \quad \exists \pi_1' : q_1 \xrightarrow{v'|v_1} q_1
\]

\[
\exists \pi_2 : p_2 \xrightarrow{u'|u_2'} q_2, \quad \exists \pi_2' : q_2 \xrightarrow{v'|v_2} q_2
\]

\[
\text{(init}(p_1) \land \text{init}(p_2) \land \text{final}(q_1)) \land (\text{mismatch}(u_1', u_2'))
\]

Let us now assume that \( v_2 = \epsilon \). Then \( \lim_n f(u v^n w^\omega) = u v^\omega \). There is then a prefix \( w_2 \) of \( w \) and an integer \( k \) such that \( \text{mismatch}(u_1 v_1^k, u_2 v_2^k) \). Hence, up to taking \( u' = uv^k \), \( w' \) a sufficiently long prefix of \( w^\omega \) we have established the pattern:

\[
\exists \pi_1 : p_1 \xrightarrow{u|u_1} q_1, \quad \exists \pi_1' : q_1 \xrightarrow{v|v_1} q_1
\]

\[
\exists \pi_2 : p_2 \xrightarrow{u|u_2} q_2, \quad \exists \pi_2' : q_2 \xrightarrow{v|v_2} q_2
\]

\[
(\text{init}(p_1) \land \text{init}(p_2) \land \text{final}(q_1)) \land (v_2 = \epsilon \land \text{mismatch}(u_1, u_2 v_2^k))
\]

which concludes the proof.

\[\square\]

D.2 Proof of Lemma 18

Proof. Let us assume there are such words satisfying the above properties. Since \( f_* \) is defined over \( \text{Pref(dom}(f)) \), for any \( n \) there are some \( u_4, n, u_4' n \) such that \( u_1 u_2^3 u_4, n, u_1' u_2' u_4, n \in \text{dom}(f) \). Moreover, the sequences \((\pi(u_1 u_2^3 u_4, n))_{n \in \mathbb{N}}\) and \((\pi(u_1' u_2' u_4, n))_{n \in \mathbb{N}}\) both converge to \( x \). However, since there is a mismatch between \( f_4(u_1 u_2^3 u_4) \) and \( f_4(u_1' u_2' u_4) \) at position \( i \), there is also one between \( f(u_1 u_2^3 u_4, n) \) and \( f(u_1' u_2' u_4, n) \) at position \( i \). Thus the pair
(\(\pi(u_1u_2u_4n)\))_{n \in \mathbb{N}}$, \((\pi(u_1'w_2u_4'n))_{n \in \mathbb{N}}\) is a bad pair. Hence \(f\) is not Cauchy continuous at \(x\). Thus if \(x \in \text{dom}(f)\), \(f\) is not continuous.

Let us assume that \(f\) is not Cauchy continuous. Then according to Lemma \[3\] there exists synchronized bad pair \((u^nv^nz)n \in \mathbb{N}\) at some point \(x\). Moreover, if \(f\) is not continuous, we can even assume \(x \in \text{dom}(f)\).

Let \(m\) be larger than \(|QP|\) and let us consider the run of \(v^nw^z\) of the look-ahead automaton \(P\) of \((T, P)\). If we look at the sequence of states reached after reading powers of \(v\), we have \(p_1 \xrightarrow{v^m} q_1 \xrightarrow{v^m} q_1 \xrightarrow{v^m} w^z\), with \(j_m + k_m + l_m = m\) and \(0 < k_m \leq |QP|\). Let \(r_m = j_m + l_m \mod k_m\). There exists only a finite number of sequences \((u^nv^m(v^m)w^z)n \in \mathbb{N}\) since \(j_m, r_m\) take bounded values. Since the original pair is bad, there exists some \(m\) such that \((u^nv^{2m}(v^m)w^z)n \in \mathbb{N}\) is a bad pair. Thus we have that for some \(0 < r < k_m\), the synchronized pair \((u^nv^m(v^m)w^z)n \in \mathbb{N}\) is bad. To simplify notations, we rename this pair \((u^nv^m(w^z)w^z)n \in \mathbb{N}\). Moreover we have obtained that \(p \xrightarrow{u} q_1 \xrightarrow{v} q_1 \xrightarrow{w^z} q_1 \xrightarrow{w^z} \). Doing the same construction for the other sequence we can also assume that \(p' \xrightarrow{u} q_1' \xrightarrow{v} q_1' \xrightarrow{w^z} \) is a final run.

Thus there exists \(u_1, u_2, u_3\) and \(u_1', u_2', u_3'\) the respective labelings of \(u, v, w^z\) and \(u', v', w^z\), in their corresponding runs. Since \(v\) loops over a state in both runs, we have for any \(m\), the labelings corresponding to the runs of \(u^nv^mz\) and \(u^nv^mz\) are \(u_1u_2^nu_3\) and \(u_1'u_2'^nu_3'\), respectively.

Let us first consider the case where one of the two sequences, \((\tilde{f}(u_1u_2u_3))_{n \in \mathbb{N}}\) \((\tilde{f}(u_1u_2'^nu_3'))_{n \in \mathbb{N}}\), is not converging. Without loss of generality, we assume it is the former. There exists a large enough number \(K\), such that any sequence crossing sequences larger than \(K\) contains an idempotent loop. Let us now consider the (infinite) run of \(T_x\) over \(u_1u_2^{2m}u_3\), for \(m\) larger than \(K\). Then we have for any \(m \geq K\), the run \(C_1 \xrightarrow{u_1u_2^{2m}} C_2 \xrightarrow{u_2^{2m}u_3} \) with \(j_m + k_m + l_m = m\) and \(0 < k_m \leq K\) such that \(u_2^{2m}\) is idempotent in \((u_1u_2^{2m}, u_2^{2m}, u_2^{2m})u_3\). Let \(r_m = j_m + l_m \mod k_m\). First we remark that since we have a loop in the run over \(u_1u_2^{2m}u_3\) the sequences \((\tilde{f}(u_1u_2^{2m}(u_2^{2m})^{r_m}u_3))_{n \in \mathbb{N}}\) are all converging. (see \[2\]). By assumption, we can find \(m, m'\) such that \((\tilde{f}(u_1u_2^{2m}(u_2^{2m})^{r_m}u_3))_{n \in \mathbb{N}}\) and \((\tilde{f}(u_1u_2'^{2m'}(u_2'^{2m'})^{r_{m'}}u_3))_{n \in \mathbb{N}}\) are converging to different limits, then we extract subsequences and we consider the pair \((u_1(u_2^{k_{m-k_m}}u_2^{2m}u_3))_{n \in \mathbb{N}}, (u_1(u_2^{k_{m-k_m}}u_2'^{2m'}u_3))_{n \in \mathbb{N}}\) which is synchronized and whose images both converge but to different limits. Once more we rename the pair \((u_1(u_2)^nu_3)_{n \in \mathbb{N}}, (u_1(u_2')^nu_3')_{n \in \mathbb{N}}\).

Now we only have left to treat the case when the two images sequences converge to different limits. Up to considering a high enough power of \(u_2, u_2'\), and adding some factors in the prefix and suffix, we can assume that the loops are idempotent. Since the images of the two sequences converge, to different sequences there exists some \(i\) such that there is a mismatch between \(\lim_n \tilde{f}(u_1u_2)^nu_3\) and \(\lim_n \tilde{f}(u_1'u_2')^nu_3'\) at position \(i\). More over since the sequences are converging, there is an integer \(N\) such that for \(n \geq N\), all images \(\tilde{f}(u_1'u_2')^nu_3'\) agree up to position \(i\), included; and all \(\tilde{f}(u_1'u_2')^nu_3'\) also agree up to position \(i\), included.
D.3 Proof of Lemma 19

We first establish Lemma 19 for regular functions of finite words, i.e. we show the following lemma:

Lemma 25. Let \( \Sigma \) be an alphabet such that \( \# \not\in \Sigma \). Let \( f : \Sigma^* \to \Gamma^* \) be a regular function defined by some deterministic two-way transducer \( T_\cdot \). There exists a function \( \rho_T : (\Sigma^*)^3 \to \Gamma^* \) defined on all tuples \( (u_1, u_2, u_3) \) such that \( u_2 \) is idempotent and \( u_1u_2u_3 \in \text{dom}(f) \), and which satisfies the following conditions:

1. if \( u_2 \) is producing in \( (u_1, u_2, u_3) \), then \( \rho_T(1, u_2, u_3) \prec \rho_T(u_1u_2, u_2, u_2u_3) \)
2. for all \( n \geq 1 \), \( \rho_T(u_1, u_2, u_3) \leq f(u_1u_2^nu_3) \)
3. for all \( n \geq 1 \), \( \rho_T(u_1, u_2, u_3) = f(u_1u_2^nu_3) \) if \( u_2 \) is not producing in \( (u_1, u_2, u_3) \)
4. the finite word function \( \rho_T : u_1\#u_2\#u_3 \mapsto \rho_T(u_1, u_2, u_3) \) is effectively regular.

Proof of Lemma 19. Instead of \( f \), consider the function \( f_* : \text{Pref}(\text{dom}(f)) \to \Gamma^* \) on finite words given by the deterministic two-way transducer \( T_* \) (which stops whenever it reaches the right border of its input and otherwise behaves as \( T \)). By Lemma 25, we get the existence of a function \( \rho_T \) defined for all \( u_1, u_2, u_3 \) such that \( u_2 \) is idempotent for \( T \) and \( u_1u_2u_3 \in \text{dom}(f_*) \), satisfying:

1. if \( u_2 \) is producing in \( (u_1, u_2, u_3) \), then \( \rho_T(u_1, u_2, u_3) \prec \rho_T(u_1u_2, u_2, u_2u_3) \)
2. for all \( n \geq 1 \), \( \rho_T(u_1, u_2, u_3) \leq f_*(u_1u_2^nu_3) \)
3. for all \( n \geq 1 \), \( \rho_T(u_1, u_2, u_3) = f_*(u_1u_2^nu_3) \) if \( u_2 \) is not producing in \( (u_1, u_2, u_3) \)
4. the finite word function \( u_1\#u_2\#u_3 \mapsto \rho_T(u_1, u_2, u_3) \) is effectively regular.

Note that \( u_1u_2u_3 \in \text{dom}(f_*) \) iff \( u_1u_2u_3 \in \text{Pref}(\text{dom}(f)) \). Also note that \( u_2 \) is idempotent for \( T \) iff it is idempotent for \( T_* \), by definition of \( T_* \). Moreover, \( u_2 \) is producing in \( (u_1, u_2, u_3) \) for \( T \) iff it is for \( T_* \). Hence, by taking \( \rho_T = \rho_{T_*} \), we get a function satisfying the condition of Lemma 19, concluding the proof.

We now turn to the proof of Lemma 25. We let \( f : \Sigma^* \to \Gamma^* \) a regular function of finite words given by a deterministic two-way transducer \( T \). In order to define the function \( \rho_T \), one needs results about the form of the output words produced when iterating an idempotent \( u_2 \). Such a study has fortunately been made in [2] however, to introduce the results of [2], one needs a few notations and new notions which we will try to present not too formally but precisely enough. The reader who wants a formal definition of those notions is referred to [2].

Let \( r \) be a run of \( T \) on \( u_1u_2u_3 \). It turns out that the run \( r \), when restricted to \( u_2 \), defines a sequence of factors of \( r \) which follows a particular structure. We denote by \( r|_{u_2} \) this sequence. As an example, consider Figure 2 inspired from an example given in [2]. This figure represents the run \( r \) decomposed into \( r = r_1r_2 \ldots r_{15} \) and for all \( i \), \( \alpha_i = \text{out}(r_i) \). Intersection on \( u_2 \), the run \( r \) defines the sequence of runs \( r|_{u_2} = r_2, r_4, r_6, r_8, r_{10}, r_{12}, r_{14} \). Factors on \( u_2 \) can be classified according to four categories: an LR-run is a factor of \( r \) entering \( u \) from the left and leaving it from the right. An RR-run is a run entering and leaving \( u \) from the right. We define RL- and LL-run symmetrically. A traversal is either an LR-, RL-, LL- or RR-run.

Those factors can themselves be grouped into what is called components in [2]. Let us define what a component is. An LR-component \( C \) on \( u_2 \) (for \( r \)) is a sequence \( l_1, l_2, \ldots, l_p, r_1, \ldots, r_p \) such that for all \( i \in \{1, \ldots, k\} \), \( l_i \) is an LL-run on \( u_2 \), \( r_i \) is an RR-run on \( u_2 \), and \( t \) is an LR-run on \( u_2 \), such that the last state of \( l_i \) is equal to the initial state of \( r_i \), the initial state of \( t \) is equal to the last state of \( l_p \), and the last state of \( t \) is equal to the initial state of \( l_1 \).

An RL-component \( C \) is defined symmetrically as a sequence \( r_1, \ldots, r_p, t, l_1, \ldots, l_p \) where the \( r_i \), ...
A run on \( u_1 u_2 u_3 \) where \( u_2 \) is idempotent, and its three components are RR-runs, \( t \) is an RL-run and the \( l_i \) are LL-runs. As an example, on Figure 2 we have highlighted the three components of the idempotent loop on \( u_2 \), which are respectively given by the sequences \( C_1 = r_2 r_4 r_6 \), \( C_2 = r_8 r_{10} r_{12} \) and \( C_3 = r_{14} \). Note that \( C_1 \) and \( C_3 \) are LR while \( C_2 \) is RL.

We define the trace of \( C \) as \( \text{tr}(C) = \pi_1 \pi_2 \pi_3 \ldots \pi_p \) and the output of \( C \) as \( \text{out}(C) = \text{out}(\text{tr}(C)) \). For example, on Figure 2 we have \( \text{tr}(C_1) = r_4 r_2 r_6 \) and \( \text{out}(C_1) = \alpha_4 \alpha_4 \alpha_6 \), \( \text{tr}(C_2) = r_{10} r_8 r_{12} \) and \( \text{tr}(C_3) = r_{14} \).

An anchor point is a position in \( r \) which is the initial position of either an LR- or an RL-run in \( u_2 \). On Figure 2, the anchor point are represented by black dots. Note that if there are \( k \) components in \( r_1 u_2 u_3 \), there are \( k \) anchor points, because each component contains exactly one RL- or LR-run.

An idempotent loop in a run has the following nice structure and property, proved in [2]:

\[ \pi_0 \text{tr}(C_1)^n \pi_1 \text{tr}(C_2)^n \ldots \text{tr}(C_k)^n \pi_k \]

where:

- \( \pi_0 \) is the prefix of \( r \) up to the first anchor point,
- for all \( 1 \leq i < k \), \( \pi_i \) is the factor of \( r \) between the \( i \)th and \((i + 1)\)th anchor point,
- \( \pi_k \) is the suffix of \( r \) starting from the last anchor point.

As an illustration, the runs \( \pi_i \) are depicted on Figure 3 for our running example. In particular we have \( \pi_0 = r_1 r_2 r_3 \), \( \pi_1 = r_4 \ldots r_9 \), \( \pi_2 = r_{10} r_{11} r_{12} r_{13} \) and \( \pi_3 = r_{14} r_{15} \).

On Figure 4 we have iterated the idempotent three times and shown the decomposition of the run of \( T \) on \( u_1 u_2^3 u_3 \) into the factors \( \pi_i \) and \( \text{tr}(C_i) \).
As an example, on Figure 3, assuming that we let run
\[ \rho \]
the first non-empty component, and we have run \( \rho \)
components are empty, or first anchor point of a non-empty component (if it exists). If all components are empty, then we let run \( \rho \).

We explain why 1.
\[ |T_1| \ldots \alpha \]
\[ u \]
(2) is empty if \( u \) is the prefix of \( \pi_0 \cdot \pi_1 \).

Since \( C \) is non-empty, we have \( \text{out}(\text{tr}(C)) \neq \epsilon \). If one applies Lemma 26 to \( (u_1 u_2) u_2 (u_3 u_3) \) where the second occurrence of \( u_2 \) is considered as the idempotent loop, we get the existence of runs \( \pi_0 \cdot \pi_1 \ldots \pi_j \) such that the run \( r' \) of \( T \) on \( u_1 u_2^2 u_3 \) is of the form
\[ r' = \pi_0 \cdot \pi_1 \ldots \pi_k \]
where \( \pi_0' \) is the prefix of \( r' \) up to the first anchor point of the idempotent loop (second occurrence of \( u_2 \) in \( u_1 u_2^2 u_3 \)), for all \( i \in \{1, \ldots, k - 1\} \), \( \pi_i' \) is the factor of \( r' \) in between
two consecutive anchor points, and $\pi'_k$ is the suffix of $r'$ from the last anchor point. On Figure 5, we have illustrated the $\pi'_i$ on our running example.

A close inspection of the proof of Lemma 26 shows that we have the following relationship:

$$\pi'_0 = \pi_0 \text{tr}(C_0) \quad \pi'_i = \text{tr}(C_i)\pi_{i+1} \quad \pi'_k = \text{tr}(C_k)\pi_k$$

for all $i \in \{1, \ldots, k-1\}$. We also have by definition of $\rho_T$, $\rho_T(u_1 u_2, u_2, u_3) = \text{out}(\pi'_0 \ldots \pi'_j) = \text{out}(\pi_0 \ldots \pi_j \text{tr}(C_{j+1})) = \rho_T(u_1, u_2, u_3)\text{out}(\text{tr}(C_{j+1}))$, from which we can conclude since $\text{out}(\text{tr}(C_{j+1})) \neq \epsilon$.

2. Property 2 is quite easy to obtain. Assume that $u_2$ is producing (the case where it is not producing is a consequence of property 3). Let $j+1$ be the first non-empty component. For all $n \geq 0$, the run of $T$ on $u_1 u_2^{n+1} u_3$ is of the form

$$f(u_1 u_2^{n+1} u_3) = \text{out}(\pi_0 \pi_1 \ldots \pi_j \text{tr}(C_{j+1})^n \pi_{j+1} \ldots \text{tr}(C_k)^n \pi_k)$$

by Lemma 26. Since all components $C_i$ for $i < j + 1$ are empty, one gets, for all $n \geq 0$:

$$f(u_1 u_2^{n+1} u_3) = \rho_T(u_1, u_2, u_3)\text{out}(\text{tr}(C_{j+1})^n \pi_{j+1} \ldots \text{tr}(C_k)^n \pi_k)$$
concluding the proof of this property.

3. If \(u_2\) is not producing, then all components are empty and for all \(n \geq 0\), we get \(f(u_1u_2^n u_3) = \text{out}(\pi_0 \ldots \pi_k) = \rho_T(u_1, u_2, u_3)\).

4. Finally, we show that the function \(u_1 \# u_2 \# u_3 \mapsto \rho_T(u_1, u_2, u_3)\) is computable by a non-deterministic two-way transducer. Its domain is regular: the set of idempotent elements for a deterministic two-way transducer being regular (Lemma 28), and the domain of a regular function a finite words being regular as well.

A skeleton is a word over the alphabet \(\{x_{RR}, x_{RL}, x_{LR}, x_{LL}\}\). An LR-skeleton component is a word in \(\bigcup_k x_{LL}^k x_{LR} x_{RR}^k\). An RL-skeleton component is a word in \(\bigcup_k x_{RR}^k x_{RL} x_{LL}^k\). A skeleton \(s\) is valid if it can be decomposed into skeleton components \(s = c_1 \ldots c_n\) alternating between LR-skeleton components and RL-skeleton components, such that \(c_1\) and \(c_n\) are LR-skeleton components. A run \(r\) on \(u_1u_2u_3\) with \(u_2\) idempotent satisfies a valid skeleton \(s\) decomposed into skeleton components \(c_1 \ldots c_k\), written \(r \models s\), if the sequence of components \(C_1 \ldots C_{k'}\) on \(u_2\) satisfies \(k = k'\) and if \(c_i\) is LR (resp. RL), then \(C_i\) is LR (resp. RL).

Let \(N\) be the number of states of \(T\). Since \(T\) is deterministic, any of its accepting run visit any input position at most \(N\) times. Therefore, any accepting run of \(T\), on an
idempotent $u_2$, has at most $N$ components, and each component has at most $N$ elements. Therefore, we consider only valid skeletons of length $N^2$, they are finitely many.

For each such valid skeleton $s$, we consider the language $D_s$ of words $u_1 \# u_2 \# u_3$ such that $u_1 u_2 u_3 \in \text{dom}(f)$, $u_3$ is idempotent and the accepting run $r$ of $T$ on $u_1 u_2 u_3$ satisfy $s$. The language $D_s$ is regular. Indeed, it can be defined by a deterministic two-way automaton $A_s$ which simulates $T$ (w/o producing anything). In a first pass, $A_s$ checks that the input is valid, i.e. that $u_1 u_2 u_3 \in \text{dom}(f)$ and $u_2$ is idempotent. As said before, the set of such words $u_1 \# u_2 \# u_3$ is regular so there is a one-way automaton defining this language: $A_s$ first starts by running this automaton. If it eventually reaches the end of its input in some accepting state, then $A_s$ comes back to the first position of the word and runs another deterministic two-way automaton $B_s$ that we now explain.

At any point, using the $\#$ symbols, $B_s$ can know where it is, either in $u_1$, in $u_2$ or $u_3$. It keeps this information in its state. It also keeps a suffix of $s$ in its state. So, states of $B_s$ are in the set $Q \times \{1, 2, 3\} \times \{s' \mid \exists s'' . s''s' = s\}$, where $Q$ is the set of states of $T$. The initial state of $B_s$ is $(q_0, 1, s)$ where $q_0$ is the initial state of $T$. On the first component, $B_s$ behaves as $T$ (ignoring the output) and changes its second component according to whether it is in $u_1$, $u_2$ or $u_3$ (using the $\#$ symbols). When $B_s$ is in some state $(q, p, xy Z')$ where $xy Z' \in \{xRR, xLR, xLL\}$ and $B_s$ enters $u_2$, it checks that $Y = R$ if it enters $u_2$ from the right (otherwise it rejects), or $Y = L$ if it enters from the left. The next time $B_s$ leaves $u_2$, if it is from the left, it checks that $Z = L$ and if it is from the right, it checks that $Z = R$. When leaving $u_2$, it moves to state $(q', p', s')$ for some $q' \in Q$, $p' \in \{1, 3\}$. Its accepting states are states $(q_f, p, \epsilon)$ where $q_f$ is accepting for $T$.

Given a valid skeleton $s$ and its decomposition into skeleton components $c_1 c_2 \ldots c_k$, a marking of $s$ is a word $s^*$ of the form $c_1 c_2 \ldots c_{j-1} \bullet c_j c_{j+1} \ldots c_k$ for all $j \in \{1, \ldots, k\}$ where $\bullet$ is a new symbol which can be placed only before some skeleton component. Therefore, there are $k$ possible markings of $s$. The meaning of this marking is to mark the first non-empty component $c_j$. Given a marked word $s^*$, it is not difficult to modify $B_s$ into $B_s'$, which behaves as $B_s$, but with the additional features that $B_s'$ also checks that the components before $\bullet$ are empty, and that the first component after $\bullet$ is non-empty. Finally, as $B_s'$ also simulates $T$ (w/o considering the output), we can easily turn it into a two-way transducer $T_s'$ which also outputs everything $T$ outputs before the first LR- or RL-run of the first non-empty component (which is marked by $\bullet$).

The final transducer $T_s$ is obtained as a disjoint union of all transducers $T_s'$ for all valid skeleton $s$ of length at most $N^2$ and all possible markings $s^*$ of $s$. Note that the transducer $T_s$ still defines a function. Indeed, since $T$ is deterministic, for any input $u_1 \# u_2 \# u_3$ such that $u_1 u_2 u_3 \in \text{dom}(T)$ and $u_2$ is idempotent, there is only one possible skeleton $s$ for the run of $T$ on $u_1 u_2 u_3$ and one possible marking.

**D.4 Proof of Lemma 20**

We first need some result:

> **Corollary 27.** For all deterministic two-way Büchi transducer $T$ defining a function $f$, for all $n \geq 2i + 1$, $\rho_T(u_1 u_2, u_2, u_3 u_3) \leq f_s(u_1 u_2, u_3)$. If $u_2$ is producing in $(u_1, u_2, u_3)$ then for all $i \geq 0$, $\rho_T(u_1 u_2, u_2, u_3) \preceq \rho_T(u_1 u_{2i+1}^+, u_2, u_{3i+1}^+)$.\footnote{Lemma 10.1}

**Proof.** First, note that for any triple $(u_1, u_2, u_3)$ such that $u_2$ is idempotent and $u_1 u_2 u_3 \in \text{Pref}(\text{dom}(f))$, we have $(u_1 u_2^2, u_2, u_3 u_3)$ which also satisfies those conditions, in particular \( u_1 u_{2i+1}^+ u_3 \in \text{Pref}(\text{dom}(f)) \). Hence, if we apply Lemma 10.1 on $(u_1 u_2, u_2, u_3 u_3)$, one gets the second statement of the corollary.
For the second statement, we again know by Lemma 19(2) that for all \( n \geq 1 \), \( \rho_T(u_1u_2^{n+1}u_3) \leq f_\ast(u_1u_2^{n+3}u_3) \), from which we get the first statement of the corollary.

**Proof of Lemma 20** Let \( f \) and \( u_1, u'_1, u_2, u'_2, u_3, u'_3 \in (\Sigma \times Q_P)^* \) satisfying the three conditions of the lemma. Let \( i \) such that \( \rho_T(u_1, u_2, u_3)[i] \neq \rho_T(u'_1, u'_2, u'_3)[i] \). Note that \( \overline{T} \) is a deterministic two-way transducer \( w/o \) look-ahead by definition. Let \( n \geq 0 \). We show that for all \( n \geq 1 \), \( f_s(u_1u_2^{n}u_3)[i] \neq f_s(u'_1u'_2^{n}u'_3)[i] \). This is due to the fact that \( \rho_T(u_1, u_2, u_3) \leq f_s(u_1u_2^{n}u_3) \) and \( \rho_T(u'_1, u'_2, u'_3) \leq f_s(u'_1u'_2^{n}u'_3) \), by Lemma 19. All the conditions of Lemma 18 are met, and therefore we get that \( f \) is not continuous.

Conversely, suppose that \( f \) is not continuous. By Lemma 18 we can assume the existence of words \( u_1, u_2, u_3, u'_1, u'_2, u'_3 \) satisfying all the conditions of Lemma 18. In particular, we know that for some \( i \geq 0 \) and for all \( n \geq 1 \), \( f_s(u_1u_2^{n}u_3)[i] \neq f_s(u'_1u'_2^{n}u'_3)[i] \). Suppose that \( u_2 \) is not producing in \( (u_1, u_2, u_3) \), then \( f_s(u_1u_2^{n}u_3) = f_s(u_1u_2u_3) \) for all \( n \geq 1 \). By Lemma 19(3) we also have \( f_s(u_1u_2u_3) = \rho_T(u_1, u_2, u_3) \), hence for all \( n \geq 1 \), \( f_s(u_1u_2^{n}u_3)[i] = \rho_T(u_1, u_2, u_3)[i] \). Similarly, if \( u'_2 \) is not producing in \( (u'_1, u'_2, u'_3) \) we get that for all \( n \geq 1 \), \( f_s(u'_1u'_2^{n}u'_3)[i] = \rho_T(u'_1, u'_2, u'_3)[i] \).

We now consider four cases:

1. \( u_2 \) is not producing in \( (u_1, u_2, u_3) \) and \( u'_2 \) is not producing in \( (u'_1, u'_2, u'_3) \), then by the latter observation, \( \rho_T(u_1, u_2, u_3)[i] \neq f_s(u_1u_2^{n}u_3)[i] \neq f_s(u'_1u'_2^{n}u'_3)[i] = \rho_T(u'_1, u'_2, u'_3)[i] \), hence all the conditions of Lemma 20 are met.

2. \( u_2 \) is producing in \( (u_1, u_2, u_3) \) but \( u'_2 \) is not producing in \( (u'_1, u'_2, u'_3) \). By Corollary 27 one can choose \( j \) large enough such that \( |\rho_T(u_1u_2^{j}u_3) - u_2^{j}u_3| \geq i \) and from the same corollary we also get that for all \( n \geq 2j + 1 \), \( \rho_T(u_1u_2^{n}u_3) \leq f_s(u_1u_2^{n}u_3) \), hence \( \rho_T(u_1u_2^{n}u_3)[i] \neq f_s(u_1u_2^{n}u_3)[i] \). On the other hand, Lemma 19(3) applied on \( (u_1u_2^{j}u_3, u'_2u'_2^{j}u'_3) \) yields \( \rho_T(u'_1u'_2^{j}u'_3) = f_s(u'_1u'_2^{j+1}u'_3) \) because \( u'_2 \) is not producing. By assumption we know that \( f_s(u'_1u'_2^{j+1}u'_3)[i] \neq f_s(u_1u_2^{n}u_3)[i] \) hence there is a mismatch between \( \rho_T(u'_1u'_2^{j}u'_3, u'_2u'_2^{j}u'_3) \) and \( \rho_T(u_1u_2^{n}u_3, u'_2u'_2^{j}u'_3) \). We can conclude since by taking the words \( u_1u_2^{j}u'_2, u'_1u'_2^{j}u'_3, u_2, u'_2u'_2^{j}u'_3 \) and \( u'_1u'_2^{j}u'_3 \) all conditions of Lemma 20 are met (we have just shown the last condition, but the other conditions are trivially satisfied, because \( u_1, u'_1, u_2, u'_2, u_3, u'_3 \) satisfy the respective conditions of Lemma 18).

3. \( u_2 \) is not producing and \( u'_2 \) is producing. This case is symmetric to the latter case.

4. \( u_2 \) is producing in \( (u_1, u_2, u_3) \) and \( u'_2 \) is producing in \( (u'_1, u'_2, u'_3) \). By Corollary 27 (applied on \( \rho_T \)), we can choose \( j \) large enough such that \( |\rho_T(u_1u_2^{j}u_3) - u_2^{j}u_3| \geq i \) and \( |\rho_T(u'_1u'_2^{j}u'_3) - u'_2^{j}u'_3| \geq i \). From the same corollary we also get that for all \( n \geq 2j + 1 \), \( \rho_T(u_1u_2^{n}u_3) \leq f_s(u_1u_2^{n}u_3) \) and \( \rho_T(u'_1u'_2^{n}u'_3) \leq f_s(u'_1u'_2^{n}u'_3) \). Hence, \( \rho_T(u_1u_2^{n}u_3)[i] \neq \rho_T(u'_1u'_2^{n}u'_3)[i] \). As before, we can conclude by taking the words \( u_1u_2^{j}u'_2, u'_1u'_2^{j}u'_3, u_2, u'_2u'_2^{j}u'_3 \) and \( u'_1u'_2^{j}u'_3 \) which satisfy all conditions of Lemma 20.

### D.5 Proof of Theorem 21

We now want to prove, based on the characterisation of Lemma 20 that continuity is decidable for regular functions. We need two simple lemmas first:

- **Lemma 28.** For any deterministic two-way transducer \( T \), the set of words \( u_1 \# u_2 \# u_3 \) such that \( u_2 \) is idempotent in \( (u_1, u_2, u_3) \) for \( T \) is regular.

- **Lemma 29.** For any regular function \( f: \Sigma^* \rightarrow T^* \), the set \( \text{Pref}(\text{dom}(f)) \) is regular. Moreover, one can construct a finite automaton recognising it from any transducer defining \( f \).
Proof. Let $T$ be some deterministic two-way transducer with look-ahead defining $f$. By ignoring the input, one gets a two-way automaton with look-ahead, $A$ recognising the domain of $f$. The look-ahead can be replaced by universal transitions: any transition of the form $\delta(q, \sigma, p) = (q', d)$ where $q, q'$ are states of $A$, $p$ is a state of the look-ahead automaton, $\sigma \in \Sigma$ and $d \in \{-1, +1\}$ can be replaced by the universal transition $\delta_p'(q, \sigma) = (p, +1) \land (q', d)$, and from any state $q$ of $A$ and $\sigma \in \Sigma$, we construct the alternating transition $\delta'(q, \sigma) = \bigvee_p \delta_p'(q, \sigma)$.

Hence, $A$ is equivalent to some alternating two-way Büchi automaton $A'$. Such automata are known to capture $\omega$-regular languages \[^{15}\]. Hence $\text{dom}(f)$ is $\omega$-regular, from which one easily gets that $\text{Pref}(\text{dom}(f))$ is regular as well. Briefly, $\text{dom}(f)$ being $\omega$-regular, it is definable by some non-deterministic Büchi automaton $B$. We remove from $B$ any state $q$ whose right language (the set of words $w$ such that there exists an accepting run on $w$ from state $q$) is empty. They can be computed in $\text{PTIME}$. Therefore, we get a subautomaton $B'$ of $B$ and set all its states to be accepting. Seen as an NFA (over finite words), we get $L(B') = \text{Pref}(\text{dom}(f))$. 

Proof of Theorem \[^{21}\] Let $f$ be a regular function given by some deterministic two-way transducer $\mathcal{T}$ with look-ahead, where $Q_p$ is the set of states of the look-ahead automaton. We show how to decide whether $f$ is not continuous by checking the existence of words $u_1, u_1', u_2, u_2', u_3, u_3' \in (\Sigma \times Q_p)^*$ satisfying the conditions of Lemma \[^{20}\]. To do that, we rely on some encoding of those six words as a single word, forming a language which is definable by a finite-visit two-way Parikh automaton, whose emptiness is known to be decidable \[^{21} \][^5].

For any two alphabets $A, B$ and any two words $w_1 \in A^*$ and $w_2 \in B^*$ of same length, we let $w_1 \otimes w_2 \in (A \times B)^*$ be their convolution, defined by $(w_1 \otimes w_2)[i] = (w_1[i], w_2[i])$ for any index $i$ of $w_1$.

We now define a language $L_{\mathcal{T}} \subseteq (\Sigma \times Q_p^2)^* \#(\Sigma \times Q_p)^* \#(\Sigma \times Q_p)^*$ which consists of words of the form $w = (w_1 \otimes a_1 \otimes a_1')#(w_2 \otimes a_2 \otimes a_2')#(w_3 \otimes a_3)\#(w_3' \otimes a_3')$ such that:

1. $w_1, w_2, w_3, w_3' \in \Sigma^*$, $a_1, a'_1, a_2, a'_2, a_3, a'_3 \in Q_p^*$
2. $w_2 \otimes a_2$ and $w_2' \otimes a_2'$ are idempotent for $\mathcal{T}$
3. $w_1w_3' \in \text{dom}(f)$ (in case we want to check for continuity)
4. $(w_1 \otimes a_1)(w_2 \otimes a_2)(w_3 \otimes a_3) \in \text{Pref}(\text{dom}(f))$ and $(w_1 \otimes a_1')(w_2 \otimes a_2')(w_3' \otimes a_3') \in \text{Pref}(\text{dom}(f))$

For a word $w$ of this form, we let $u_i(w) = w_i \otimes a_i$ for all $i = 1, 2, 3$ and $u'_i(w) = w_i \otimes a'_i$, $u_2(w) = w_2 \otimes a_2$ and $u_2'(w) = w_3' \otimes a_3'$. From Lemma \[^{28}\] and Lemma \[^{29}\] and the fact that the set of words $w_i \# w_2$ such that $w_1w_3' \in \text{dom}(f)$ is regular (as $\text{dom}(f)$ is regular), it is possible to show that $L_{\mathcal{T}}$ is effectively regular.

Finally, we restrict $L_{\mathcal{T}}$ to the language

$E_{\mathcal{T}} = \{ w \in L_{\mathcal{T}} | \rho_1(u_1(w), u_2(w), u_3(w))$ and $\rho_7(u_1'(w), u_2'(w), u_3'(w)) \text{ mismatch.} \}$

Clearly, $E_{\mathcal{T}} \neq \emptyset$ iff $f$ is not continuous by Lemma \[^{20}\].

We show that $E_{\mathcal{T}}$ is definable by two-way Parikh automaton which is additionally finite-visit. A two-way Parikh automaton $\mathcal{P}$ of dimension $d$ is a two-way automaton, running on finite words, extended with vectors of natural numbers of dimension $d$. A run on a finite word is accepting if it reaches some accepting state and the sum of the vectors met along the transitions belong to some given semi-linear set (or equivalently satisfy some given Presburger formula). The emptiness problem for two-way Parikh automata is undecidable.
but decidable when there exists some computable \( k \) such that in any accepting run of \( P \), any input position is visited at most \( k \) times by that run.

Our automaton \( P \) will be of dimension 2. After processing its input and ending in some accepting state, the sum \((x,y)\) of all the vectors met along the way will correspond to two output positions \( x \) and \( y \) of \( \tilde{T}(u_1(w),u_2(w),u_3(w)) \) and \( \tilde{T}(u'_1(w),u'_2(w),u'_3(w)) \) such that the label of \( x \) differs from the label of \( y \). The Parikh automaton will accept the run only if \( x = y \) (note that the latter equation defines a semi-linear set).

To do that, we know from Lemma 19 that there exist a deterministic two-way transducer \( T_{\tilde{T}} \) (over finite words) which, given any \( u_1 \# u_2 \# u_3 \) outputs \( \tilde{T}(u_1(w),u_2(w),u_3(w)) \) and \( \tilde{T}(u'_1(w),u'_2(w),u'_3(w)) \). The transducer \( H_{\tilde{T}} \) makes a first pass on its input to check that \( w \in L_T \) (since \( L_T \) is regular it is definable by some finite automaton), and then simulates \( T_{\tilde{T}} \) on the relevant components of the symbols composing the word (for instance, it ignores the third component of any symbol occurring in \( w \)). Similarly, one can construct \( H'_{\tilde{T}} \).

Finally, \( P \) reads inputs \( w \in L_T \) (it can check that indeed its input belongs to \( L_T \) during a first pass, as \( L_T \) is regular) and proceeds with two phases. In the first phase, it simulates \( H_{\tilde{T}} \) w/o producing anything but by summing vectors of the form \((\ell,0)\) where \( \ell \) is the length of output produced by the current simulated transition of \( H_{\tilde{T}} \) (wlog we assume that \( H_{\tilde{T}} \) outputs at most one symbol at a time, i.e. \( \ell \in \{0,1\} \)). Eventually, when \( H_{\tilde{T}} \) triggers a transition producing some \( \sigma \in \Gamma \), \( P \) non-deterministically decides to increment its first component by 1, one last time, and stores the symbol \( \sigma \) in its state. Then it proceeds to phase 2, which does exactly the same but on the second vector components (vectors of the form \((0,\ell)\)) and by simulating \( H'_{\tilde{T}} \) instead of \( H_{\tilde{T}} \). This is continued till a non-deterministic choice is made of stopping the increment of the second component, and storing in its state, the last symbol \( \beta \) output by \( H'_{\tilde{T}} \).

Finally, its set of accepting states are pairs \((\sigma,\beta)\) such that \( \sigma \neq \beta \) and the semi-linear accepting set is defined by the equation \( x = y \). Clearly, \( P \) is finite-visit because \( H_{\tilde{T}} \) and \( H'_{\tilde{T}} \), being deterministic two-way transducers, are finite-visit as well, concluding the proof. ▶