Disorder Induced Depinning Transition

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Abstract

The competition in the pinning of a directed polymer by a columnar pin and a background of random point impurities is investigated systematically using the renormalization group method. With the aid of the mapping to the noisy-Burgers’ equation and the use of the mode-coupling method, the directed polymer is shown to be marginally localized to an arbitrary weak columnar pin in 1+1 dimensions. This weak localization effect is attributed to the existence of large scale, nearly degenerate optimal paths of the randomly pinned directed polymer. The critical behavior of the depinning transition above 1+1 dimensions is obtained via an $\epsilon$-expansion.
I. INTRODUCTION

The statistical mechanics of an elastic manifold embedded in a medium of random point defects has been the subject of many studies in the past decade [1–3]. Such systems are encountered in a variety of contexts, ranging from the fluctuations of domain walls in random magnets [4–7], to the pinning of magnetic flux lines in dirty superconductors [8–10]. Over the years, many theoretical methods have been developed to understand these systems [9–17]. In particular, through a mapping to stochastic hydrodynamics [15–17], we now know many properties of one-dimensional manifolds (directed polymers) in random media. There have also been numerous numerical simulations; some recent studies can be found in Refs. [18–20].

Recently, strong flux pinning effects [21–23] exhibited by samples of high temperature superconductor with extended defects such as columnar faults and twin planes lead naturally to the investigation of competition between extended and point defects [24,25]. It has been argued in the case of many interacting flux lines that pinning by extended defects are weakened by point defects [24]. There have also been a number of studies on the competing effects between extended and point defects on a single flux line or a directed polymer [26–33]. An early study of this type was done by Kardar [26], who investigated numerically the pinning of a directed polymer to a line defect in the presence of a background of point defects in 1+1 dimensions. The polymer was found to depin from the line defect, if the pinning potential of the line defect is smaller than a certain threshold value. Critical behavior associated with the depinning transition was later investigated in more detail by Zapatocky and Halpin-Healy [27]. The results of Refs. [26] and [27] have been challenged by Tang and Lyuksyutov [28], who argued against the existence of a depinning transition.
in two dimensions, based on large scale simulations and approximate renormalization-group analysis on a hierarchical lattice. Tang and Lyuksyutov proposed instead that a directed polymer is *always* localized, although weakly, to the line defect in 1+1 dimensions, and that the depinning transition only exists above 1+1 dimensions. This conclusion is supported by a recent study of Balents and Kardar [32], who performed numerical simulations, and also developed a functional renormalization group analysis by describing the directed polymer as a generalized $D$-dimensional manifold and studying the depinning transition using a *double* expansion (in the dimensionality of the manifold $D = 4 - \delta$ and the dimensionality of the embedding space $d = d_c(D) + \epsilon$.) On the other hand, analysis of the directed polymer itself by Kolomeisky and Straley have led to different conclusions when different renormalization group Ansatz were used [30, 31, 33].

In this paper, we give a *systematic* analysis of the competition in pinning between point and line defects for a directed polymer. By exploiting known knowledge of the randomly pinned directed polymer in 1+1 dimensions in the absence of any extended defects [17], we construct a renormalization-group analysis *directly* in 1+1 dimensions, the critical dimension for this problem. Our results prove the conclusion of Ref. [28], that the polymer is always pinned at and below 1+1 dimensions, and that the depinning transition only occurs above two dimensions. The existence of weak localization at the critical dimension is understood in terms of the anomolous large scale excitations of the directed polymer, known as the “droplet excitations” [17]. Critical scaling behaviors at the depinning transition are then obtained using a $1 + 1 + \epsilon$ dimensional expansion.

The paper is organized as follows: We first introduce the known properties of the randomly pinned directed polymer in Section II. We consider the effect of pinning by a line
defect using phenomenological scaling arguments, which establishes 1+1 dimensions to be
the critical dimension. In Section III, we attempt to solve the problem using the replica
method at the critical dimension. Despite the existence of an exact solution via the Bethe
Ansatz \([11,12]\) when the extended defect is absent, we failed to develop a systematic and con-
trolled way of incorporating a weak extended defect. We next use an uncontrolled Hartree
approximation, which contains the right physical ingredients, but overestimates the effect of
point disorder. A \textit{systematic} analysis taking advantage of known mapping of the directed
polymer to the noisy-Burgers’ equation \([17]\) is presented in Section IV. Through explicit
calculations, facilitated by the use of the mode coupling approximation in 1 + 1 dimensions,
we construct the renormalization-group recursion relation and derive various scaling behav-
iors near the critical dimension. We show that the results can be understood naturally in
terms of the anomalous droplet excitations \([17]\). In Section V, we generalize our method to
describe the directed polymer pinned by a variety of extended and point defects, including
columnar pin of extended range and trajectory, and point defects with long range corre-
lations. Some useful relations about the randomly pinned directed polymer, in particular,
the mode coupling approximation, is summarized in Appendix A, and a number of detail
calculations are relegated to Appendix B.
We describe the statistical mechanics of a randomly pinned directed polymer of length $t$ by the Hamiltonian
\begin{equation}
H_0 = \int_0^t dz \left\{ \frac{\kappa}{2} \left( \frac{d\xi}{dz} \right)^2 + \eta[\xi(z), z] \right\},
\end{equation}
where $\xi(z) \in \mathbb{R}^{d_\perp}$ denotes the transverse displacement of the polymer in $d = d_\perp + 1$ embedding dimensions (with $d_\perp$ being the co-dimension), $\kappa$ is the polymer line tension, and $\eta(x, z)$ describes a background medium of uncorrelated point defects. The random potential is assumed to be Gaussian distributed with mean zero and the variance
\begin{equation}
\overline{\eta(x, z)\eta(x', z')} = 2\Delta \delta^{d_\perp}(x - x')\delta(z - z'),
\end{equation}
where the overbar denotes disorder average.

The randomly pinned directed polymer has been the topic of detailed investigations during the last decade \cite{9}–\cite{20}. The emerging qualitative picture for a long polymer ($t \to \infty$) is the following: At low temperatures, the static properties of the polymer are dominated by the random potential and are controlled by one or a few optimal path(s) which minimize the total free energy \cite{14,17}. To take advantage of fluctuations in the random potential $\eta$, the optimal path executes large transverse wandering [solid line of Fig. 1(a)]. If we fix one end of the polymer, then the root-mean-square displacement of the other end of an optimal path is
\begin{equation}
X(t) = Bt^{\zeta},
\end{equation}
where $3/4 > \zeta > 1/2$ is an universal exponent which depends on $d$. The result $\zeta = 2/3$ in $d = 1 + 1$ has been obtained exactly by a number of methods \cite{11,15}. Numerical calculations
give $\zeta \approx 5/8$ in $d = 2 + 1$. Displacement of the fixed end of the polymer by a distance $r \ll X(t)$ typically result in a rearrangement of the optimal path within a segment of length $\tau \sim (r/B)^{1/\zeta}$ from the fixed end, as shown in Fig. 1(a). For sufficiently large displacement, i.e., for $r \gg B t^\zeta$, the optimal path is completely changed [dotted line of Fig. 1(a)], with a completely different free energy [Fig. 1(b)]. The typical free energy difference $\Delta F_0$ between such independent paths is

$$\Delta F_0 \approx A t^\theta,$$  \hspace{1cm} (2.4)  

with an identity

$$\theta = 2\zeta - 1$$  \hspace{1cm} (2.5)  

relating the two exponents. $\Delta F_0$ also sets the scale of the free energy difference between the optimal path and a typical path.

Thermal fluctuations do not change this picture qualitatively. They merely wipe out the fine structure of the random potential, leading to temperature dependent amplitudes. For instance, $B \propto (\Delta/kT)^{1/3}$ and $A \propto k B^2$ in $d = 1 + 1$. In $d \leq 2 + 1$ dimensions, this picture is in fact correct for all temperatures. In $d > 2 + 1$, the polymer undergoes a continuous phase transition and becomes dominated by thermal fluctuations rather than disorder at sufficiently high temperatures. We shall however be focused on the more interesting low temperature phase throughout this paper.

Let us consider the influence of an additional pinning potential, $U_p(x)$, in the form of a line defect with a short range $a$, located at the origin. We model the defect by the Hamiltonian

$$\mathcal{H}_1 = \int_0^t \, dz \, U_p[\xi(z)],$$  \hspace{1cm} (2.6)
with \( U_p(x) = -U \) for \( |x| < a \), and \( U_p(x) = 0 \) for \( |x| > a \). Clearly, such an attractive potential favors configurations of the polymer close to the origin, i.e., it attempts to localize the polymer. On the other hand, the polymer will try to wander away from the defect in order to take advantage of fluctuations in the background random potential \( \eta(x, z) \). Depending on the outcome of this competition, the pinning potential \( U_p \) may have a number of possible effects: It may always or never localize the polymer for any finite pinning strengths \( U \), or it may localize the polymer only above a certain critical pinning strength \( U_c \), thus giving rise to a depinning transition.

To gain an understanding of the competing effects, we first consider the pure problem with \( \eta(x, z) = 0 \). Let

\[
W^{(0)}(x, t) = \int_{(0,0)}^{(x,t)} D[\xi] e^{-(H_0^{(0)} + H_1)/T}
\]

be the Boltzmann weight of propagating the polymer from \((0,0)\) to \((x,t)\), with \( H_0^{(0)} = H_0[\eta = 0] \) and the superscript \((0)\) indicating the absence of the random potential, then \( W^{(0)} \) satisfies the diffusion equation

\[
T \frac{\partial}{\partial t} W^{(0)}(x, t) = \frac{T^2}{2\kappa} \nabla^2 W^{(0)}(x, t) - U_p(x) W^{(0)}(x, t).
\]

Our problem is equivalent to that of a \( d_\perp \)-dimensional (imaginary-time) quantum mechanical particle with a pinning well \( U_p \) at the origin. In the thermodynamic limit \( t \to \infty \), the statistical mechanics of the polymer is given by the ground state of the quantum problem whose solution is well known: There is always a bound state for arbitrarily weak pinning potential in \( d_\perp \leq 2 \), where the polymer is always bounded to the line defect. For \( d_\perp > 2 \), a critical pinning strength of the order \( U_0^{(0)} \equiv T^2/(2\kappa a^2) \) is necessary to have a bound state. A depinning transition occurs at \( U = U_c^{(0)} \sim U_0^{(0)} \). Fig. 2 summarizes the phase diagram of
the pure system in various dimensions.

It is useful to understand these results by simple physical considerations: Assume first that the polymer is completely localized within the pinning well. Then the energy gained (compared to a free polymer) is $U$ per length, and the entropy cost (or the kinetic energy cost of localizing a quantum particle) is of the order $U_0^{(0)}$ per length. Thus if $U \gg U_0^{(0)}$, the total free energy of the polymer is $F^{(0)} = -U + U_0^{(0)} < 0$, and the polymer is localized. However, if $U \lesssim U_0^{(0)}$, then $F^{(0)} > 0$ and the polymer will not be completely localized to the well since the free energy exceeds that of a free polymer in this case. An alternative scenario when $U \lesssim U_0^{(0)}$ is to have the polymer completely delocalized, ignoring the existence of the weak pinning potential. This is self-consistent for $d_\perp > 2$, since for a polymer with one end fixed at the origin, the accumulated probability of a long free polymer returning to the origin is $t^{(2-d_\perp)/2} \to 0$. Thus we see that for $d_\perp > 2$, the polymer is essentially free for $U \ll U_0^{(0)}$, and is completely localized if $U \gg U_0^{(0)}$. The simplest phase diagram is then to have a depinning transition separating the pinned and free phases, occurring at $U = U_c^{(0)} \sim U_0^{(0)}$. On the other hand, for $d_\perp \leq 2$, the probability of a long free polymer returning to the origin is of order 1. Thus the completely delocalized phase is not self consistent here, and the effect of a weak pinning potential can never be ignored in the limit $t \to \infty$.

The above can be stated more quantitatively by comparing the average pinning energy $\langle \mathcal{H}_1 \rangle^{(0)}$ with the available thermal energy $T$. A useful parameter to focus on is the dimensionless ratio $g^{(0)} = \langle \mathcal{H}_1 \rangle^{(0)}/T$. The polymer is free if $g^{(0)} \ll 1$ and is completely localized if $g^{(0)} \gg 1$ in the limit $t \to \infty$. If we just perform the thermal average $\langle \ldots \rangle^{(0)}$ by the Wiener measure of a free polymer, $e^{-\mathcal{H}_0^{(0)}/T}$, then we obtain
where $\epsilon^{(0)} = (2 - d_\perp)/2$ and $a^{(0)}_\parallel = (\kappa/T)a^2$ is the short distance cutoff along the length of the polymer. Thus the free polymer phase is obtained for $d_\perp > 2$ (i.e., $\epsilon^{(0)} < 0$) if $U < U_c^{(0)} \propto |\epsilon^{(0)}|U_0^{(0)}$. For $U > U_c^{(0)}$, the polymer is free only for

$$t < l^{(0)}_\parallel = a^{(0)}_\parallel \left[ \frac{U}{U_c^{(0)}} - 1 \right]^{-\frac{1}{|\epsilon^{(0)}|}}. \quad (2.10)$$

$l^{(0)}_\parallel$ is a crossover length above which the polymer becomes localized to the pinning potential. From it, we can define a localization length

$$l^{(0)}_\perp = \left[ \frac{T l^{(0)}_\parallel}{\kappa} \right]^{1/2} = a \left[ \frac{U}{U_c^{(0)}} - 1 \right]^{-\frac{1}{2|\epsilon^{(0)}|}}, \quad d_\perp > 2 \quad (2.11)$$

which characterize the typical transverse excursion of the polymer. On the other hand, for $d_\perp < 2$ where $\epsilon^{(0)} > 0$, $g^{(0)}_0 \gg 1$ for arbitrary small $U$ in the limit of long polymer $t \to \infty$.

Defining the crossover and localization lengths in the same way, we have

$$l^{(0)}_\perp = a[U/U_c^{(0)}]^{-\frac{1}{2|\epsilon^{(0)}|}}, \quad \epsilon^{(0)} > 0 \quad (2.12)$$

with the polymer delocalized only in the limit $U \to 0$. Qualitatively similar behavior is obtained in $d_\perp = 2$, with

$$l^{(0)}_\perp = ae^{t^{(0)}_\perp}/U. \quad (2.13)$$

Eqs. (2.11) – (2.13) are valid as long as $l^{(0)}_\perp \gg a$. Throughout this paper, we shall only be interested in this critical regime, where the scaling behavior of localization length is insensitive to the detail shape of $U_p$.

Let us now return to the problem of a randomly pinned directed polymer, i.e., with $\eta(x, z) \neq 0$, and estimate the competition between the pinning and the random potential that
occurs in this case. Just as the localization of a polymer costs entropy in the pure problem, here, localization prevents the polymer from seeking out favorable regions of the random potential $\eta$ far from the origin, and therefore leads to a loss in the random component of the energy that could otherwise be gained even as $T \to 0$. If a polymer is localized within a distance $l_\perp$ about the origin, then it is consisted of a number of uncorrelated segments of length $l_\parallel \approx (l_\perp/B)^{1/\zeta}$, each of which having a free energy of order $A l_\parallel^\theta$ higher than that of the delocalized, optimal path (see Fig. 3). Thus the free energy cost of localization is of the order $A(l_\perp/B)^{(\theta-1)/\zeta}$ per length. This plays the role of the entropy cost in the localization of a free polymer. For a very strong pinning potential, the polymer is again always localized completely within the pinning well because the energy per length gained, $U$, always exceeds the random energy cost of localization, which is of the order $U_0 = A(a/B)^{(\theta-1)/\zeta}$ per length.

The effect of a weak pinning potential can be estimated perturbatively from $\langle \mathcal{H}_1 \rangle$ as just described for the pure problem, except that it now depends on the realization of the random potential $\eta$. The natural quantity to examine is the average energy gained by the polymer in the presence of the pinning potential,

$$\delta F = \langle \mathcal{H}_1 \rangle,$$

given that one end is fixed at the origin (as shown in Fig. 3). As in the case of the pure problem, this is just the average of the accumulated return probability of the polymer to the origin. Since the rms displacement is $B t^\zeta$ in the absence of the pinning potential $U_p$, then $\delta F_0 \propto U (a/B t^\zeta)^{\perp t}$ to leading order in $U$. To determine the effect of the pinning potential, it is necessary to compare $\delta F_0$ to $\Delta F_0 \approx A t^\theta$, the intrinsic variations in the free energy [see Fig. 1(b)]. The dimensionless ratio that characterizes the strength of pinning is now
to leading order in $U/U_0$, with $a_\parallel = (a/B)^{1/\zeta}$ being the short distance cutoff along the length of the polymer, and

$$
\epsilon(d) = 1 - d_\perp \zeta - \theta = 2 - (d + 1)\zeta(d).
$$

The critical dimension is $d = 1 + 1$ since $\zeta(2) = 2/3$ exactly \[11,15\]. If $g_0 \ll 1$, which is the case when $\epsilon < 0$, then the weak pinning potential is irrelevant in the thermodynamic limit. If $\epsilon > 0$, then $g_0 \gg 1$ and even a small $U$ will completely change the energy landscape of the randomly pinned directed polymer. For example, the free energy of the pinned state in Fig. 3 now becomes much lower than that of the optimal path of the unperturbed system. The apparent difference between the problem with point disorder and the pure problem is the behavior of $g_0$ at the critical dimension where $\epsilon = 0$: For the pure problem, $g_0^{(0)}$ diverges logarithmically according to Eq. (2.9), indicating the marginal relevance of the pinning potential. But with point disorder,

$$
g_0 \propto U/U_0 \propto U a/(A B)
$$

remains finite. This naively suggests the irrelevance of a small pinning potential at the critical dimension. If this is true, then there will have to be a depinning transition at the critical dimension $d = 1 + 1$, since a strong enough pinning potential still produces a localized state. The simplest phase diagram in this case [Fig. 4(a)] will look quite different from that of the pure problem (Fig. 2). Fig. 4(a) is actually consistent with early numerical results of Kardar [26] and Zapatocky and Halpin-Healy [27], who find a depinning transition in $d = 1 + 1$. However, through a systematic renormalization-group analysis presented
in Sec. IV, we will find that $U$ itself is renormalized and diverges logarithmically, while all other parameters such as $A$ and $B$ only suffer finite renormalization. These results lead to a logarithmically diverging $g$ and hence the marginal relevance of pinning at the critical dimension. Thus the phase diagram for the randomly pinned directed polymer is like that of the pure problem (Fig. 2), but with a shifted critical dimension and $\epsilon$, as shown in Fig. 4(b). This result is supported by the recent large scale numerical studies of Refs. [28,32]. Renormalization group analysis can then be used to obtain the critical behavior at the depinning transition, in particular, the divergence of the correlation length,

$$l_{\perp} = a \left( \frac{U}{U_c} - 1 \right)^{\nu_{\perp}},$$

(2.18)

where $\nu_{\perp}$ is the liberation exponent.
III. ANALYSIS IN REPLICA SPACE

A convenient way to treat one dimensional objects like the directed polymer is the transfer matrix approach described in Sec. II. The full Boltzmann weight (or the restricted partition function) \( W(x, t) \) of a polymer propagating from \((0, 0)\) to \((x, t)\) in a random medium \( \eta(x, t) \) is described by an equation analogous to Eq. (2.8), except with \( U_p(x) \) replaced by \( U_p(x) + \eta(x, t) \). All physical properties follow from disorder averages of the free energy \( F(x, t) = -T \ln W(x, t) \) and derivatives thereof. One approach to performing this average is the replica method, which exploits the identity \( \ln W(x, t) = \lim_{n \to 0} \frac{1}{n} (W^n(x, t) - 1) \). Up to an exchange of the \( n \to 0 \) and the thermodynamic limit (see below), \( F(x, t) \) is given by the \( n^{th} \) moment of \( W \),

\[
W(x_1, \ldots, x_n; t) = \prod_{\alpha=1}^{n} \int_{(0,0)}^{(x_{\alpha}, t)} D[\xi_{\alpha}] e^{-\mathcal{H}_n/T},
\]

\( \alpha \in \{1, \ldots, n\} \) is the replica index, and \( \mathcal{H}_n \) denotes the replica Hamiltonian

\[
\mathcal{H}_n = \sum_{\alpha=1}^{n} \int_0^t dz \left\{ \frac{\kappa}{2} \left( \frac{d\xi_{\alpha}}{dz} \right)^2 + U_p[\xi_{\alpha}(z)] - \sum_{\beta \neq \alpha} \frac{\Delta}{T} \delta^{d\perp} (\xi_{\alpha}(z) - \xi_{\beta}(z)) \right\}.
\]

For analytic simplicity, we approximate the columnar pinning potential by a \( \delta \)-function, i.e., \( U_p(x) = -u\delta^{d\perp}(x) \) with \( u \approx U a^{d\perp} \), from here on. Applying the transfer matrix approach directly to Eq. (3.1) leads to the Schrödinger-like equation

\[
T \frac{\partial}{\partial t} W(x_1, \ldots, x_n; t) = -\hat{\mathcal{H}}(n) W(x_1, \ldots, x_n; t),
\]

where

\[
\hat{\mathcal{H}}(n) = \sum_{\alpha=1}^{n} \left\{ -\frac{T^2}{2\kappa} \nabla^2_{\alpha} - u\delta^{d\perp}(x_{\alpha}) - \sum_{\beta \neq \alpha} \frac{\Delta}{T} \delta^{d\perp}(x_{\alpha} - x_{\beta}) \right\},
\]

As usual, Eq. (3.3) can be solved with the Ansatz...
and $\hat{H}(n)\Phi_j = E_j(n)\Phi_j$. In $d_\perp = 1$ and for $U_p = 0$, the ground state wave function is easily found by using the Bethe-Ansatz \[11\], $\Phi_0(x_1, ..., x_n) \propto \exp\{-\frac{\Delta}{2}\kappa \sum_{\alpha<\beta} |x_\alpha - x_\beta|\}$. A similar solution follows for the ground state of a randomly pinned directed polymer confined to the semi-infinite plane $x \geq 0$ and subject to an attractive potential $U_p$ at $x = 0$ \[11\]. Upon changing the strength of the attractive potential, one obtains a depinning transition for the half-plane problem when the ground state energy becomes larger than that for $U_p = 0$. However, searches for a similar solution to the Hamiltonian (3.4) have not been successful.

A naive application of the perturbation theory for small $U_p$ starting from the wave function $\Phi_0(x_1, ..., x_n)$ fails likewise, because the low energy excited states of $\hat{H}(n)$ are not uniquely defined in the limit the number of particles $n \to 0$ \[12\]. This ambiguity, arising from the exchange of the $n \to 0$ and the thermodynamic limits, is a well-known “trouble spot” for the replica method, and often calls for elaborate schemes with broken replica symmetry \[13\].

For the problem at hand, the ground state itself does not involve replica symmetry breaking \[11,12\]. However, it is demonstrated in Ref. \[12\] that states with broken replica symmetry could have energies arbitrary close to the ground state energy $E_0(n)$. These will dominate in any perturbative calculations.

We shall circumvent the problem of replica symmetry breaking by introducing a completely different method in Sec. IV. For now, we consider another limit, $\Delta \to 0$, where the system almost decouples into $n$ one-particle problems. Here an attractive potential $U_p$ always gives a bounded ground state in $d_\perp = 1$. In term of the reduced length $r = x/x_0$ where $x_0 = T^2/(2u\kappa)$, the reduced Hamiltonian $\hat{h}(n) = \hat{H}(n)/T^2 x_0^2$ becomes
\[ \hat{h}(n) = -\sum_{\alpha} \left\{ \frac{\partial^2}{\partial r^2_{\alpha}} + 2\delta(r_{\alpha}) + \frac{2\Delta}{uT} \sum_{\beta \neq \alpha} \delta(r_{\alpha} - r_{\beta}) \right\}. \quad (3.6) \]

We now attempt to treat the case of weak inter-replica interaction \( \Delta \ll uT \) by using the Hartree approximation \([34]\). Making the Ansatz for the wave function \( \Phi(r_1, r_2, \ldots, r_n) = \phi_1(r_1)\phi_2(r_2)\ldots\phi_n(r_n) \) and minimizing the reduced energy \( E(n) = \int dr_1 \ldots dr_n \Phi \hat{h}(n) \Phi \) under the conditions \( \int dr \phi^2_{\alpha}(r) = 1 \) which we impose by Lagrangian multipliers \( \varepsilon_{\alpha} \), we get a set of Hartree equations \([34]\). These simplify to a single equation if we impose symmetry between the replicas \( \phi_{\alpha}(r) = \phi(r) \) and \( \varepsilon_{\alpha} = \varepsilon \) for all \( \alpha \)'s,

\[ \left[ \frac{\partial^2}{\partial r^2} + 2\delta(r) + 4(n-1)g_0^{-1}\phi^2(r) - \varepsilon \right] \phi(r) = 0, \quad (3.7) \]

where

\[ g_0^{-1} = \frac{\Delta}{uT}. \quad (3.8) \]

This equation can be solved easily with the boundary condition \( \phi(r) = \phi'(r) = 0 \) for \( |r| \to \infty \), giving a localized wave function

\[ \phi(r) = e^{-r\sqrt{\varepsilon}} \left[ \sqrt{\varepsilon} + \sqrt{\varepsilon - 2(n-1)g_0^{-1}\phi^2(r)} \right] \sqrt{1 + \varepsilon} \quad (3.9) \]

with a negative ground state energy

\[ E_0(n) = -\frac{n}{3}(1 + \sqrt{\varepsilon} + \varepsilon) = -n \left[ 1 + (n-1)g_0^{-1} + \frac{1}{3}(n-1)^2g_0^{-2} \right] \quad (3.10) \]

where \( \varepsilon = [1 + (n-1)g_0^{-1}]^2 \), and the localization length is

\[ l_\perp = x_0/\sqrt{\varepsilon} \propto \left( 1 + (n-1)g_0^{-1} \right)^{-1}. \quad (3.11) \]

Note that the Hartree results Eqs. \((3.9) - (3.11)\) are in fact the exact solutions of the Hamiltonian \([34]\) in the limit \( g_0^{-1} \to 0 \), where the very strong pinning potential always
localizes the directed polymer within $l_\perp$ of the origin. However, we see that in the limit $n \to 0$, the random potential characterized by $g_0^{-1}$ tends to increase the localization length. If we extrapolate the Hartree result Eq. (3.11) to finite value of $g_0^{-1}$, then we find an instability, i.e., $l_\perp \to \infty$ as $g_0^{-1} \to 1^-$. An obvious interpretation of this instability is the occurrence of a depinning transition.

On the other hand, an analogous Hartree calculation for the problem of an attractive potential in the half plane yields again the solution (3.9), except with $\varepsilon = (1 + 2(n - 1)g_0^{-1})^2$. This result suggests a delocalization transition at $g_c^{-1} = 1/2$ with $l_\perp \propto (g_0 - g_c)^{-1}$, or $\nu_\perp = 1$ for the half-plane problem. However, the exact Bethe-Ansatz calculation [11] gave $g_c^{-1} = 1$ and $\nu_\perp = 2$. Thus we see that while the Hartree solution gives the qualitative effect of a background random potential — a tendency to depin the directed polymer from the origin — it overestimates the influence of randomness and cannot be used reliably to characterize the depinning transition quantitatively. Indeed, we will show in the following section that the directed polymer is always pinned to the origin for all finite values of $g_0$ in $d_\perp = 1$ dimensions.

[ Note: Although our Hartree equation (3.7) agrees with that considered previously by Zhang [33] if we set $u = 0$, the physics of our solution is completely different: In the limit $n \to 0$ which we consider, the disorder leads to a repulsive interaction $g_0^{-1}\delta(x_\alpha - x_\beta)$ between the replicas, which makes a change of the roughness exponent $\zeta$ from $1/2$ for free polymers to the larger value $2/3$ possible. It is the pinning potential $U_p$ that confines the solution close to the origin. In contrast, Zhang considered the large-$n$ limit where the interaction between the replicas is attractive, leading to the unphysical solution of an exponentially decreasing wave function even if $u = 0$. ]
As we have seen in the Sections II and III, both naive scaling arguments and the uncontrolled Hartree solution of the replicated system suggest that the polymer undergoes a depinning transition in the critical dimension \(d = 1 + 1\), consistent with the phase diagram sketched in Fig. 4(a). However, we have not been able to construct a \textit{controlled} perturbative study for weak pinning potential \(U_p\) within the replica formalism, despite the knowledge of the exact ground state. The technical problem encountered is the occurrence of replica symmetry breaking for the low energy excited states. In this section, we shall formulate a perturbative renormalization group study of the depinning transition in \(1 + 1\) dimensions without ever introducing the notion of replica. We shall take advantage of the mapping of the directed polymer to the hydrodynamics of the noisy-Burgers’ equation \([15–17]\). Many details of the mapping have been discussed in Ref. \([17]\) and some are summarized in Appendix A. Here, we will outline our approach and quote the results. A number of detailed calculations are given in Appendix B.

\textbf{A. Formalism}

It is useful to introduce first a formal language to characterize the properties of the directed polymer in the absence of the pinning potential \(U_p\). For convenience, we shall fix one end \(\xi(t)\) of the polymer at some arbitrary point \(x\). This is implemented by introducing the one-point restricted partition function,

\[
Z_0(x, t) = \int \mathcal{D}[\xi] \delta^d(\xi(t) - x) e^{-\mathcal{H}_0/T},
\]

(4.1)
where the subscript 0 is used to indicate \( U_p = 0 \). All thermal averages taken with respect to \( Z_0(x, t) \) will be denoted by \( \langle \ldots \rangle_{x,t} \). The disorder averaged probability of finding a segment \( \xi(t_0) \) of the polymer at position \( y \) given that the other end is fixed at \((x, t)\) is

\[
G^{(1,1)}_t(x - y, t - t_0) \equiv \langle \delta^d(\xi(t_0) - y) \rangle_{x,t},
\]

(4.2)

where we use the subscript \( t \) to emphasize the dependence of \( G^{(1,1)} \) on the total polymer length \( t \). The functional form of \( G^{(1,1)} \) has been investigated in details elsewhere [17] and is summarized in Appendix A. Here, we just mention the qualitative behavior, \( G^{(1,1)}_t(r, \tau) = (B\tau^\zeta)^{-d_\perp} \) for \( r \ll B\tau^\zeta \) and vanishes rapidly as \( r \gg B\tau^\zeta \).

It is also useful to consider the free energy of the one-point restricted polymer,

\[
F_0(x, t) = -T \log Z_0(x, t),
\]

(4.3)

which satisfies the noisy-Burgers’ equation [36–38]

\[
\partial_t F_0(x, t) = \frac{T}{2\kappa} \nabla^2 F_0 - \frac{1}{2\kappa} (\nabla F_0)^2 + \eta(x, t).
\]

(4.4)

The free energy correlation function is

\[
C_t(x - y) = [F_0(x, t) - F_0(y, t)]^2,
\]

(4.5)

with

\[
C_t(r) = 2A^2t^{2\theta} \quad \text{for} \quad r \gg Bt^\zeta,
\]

(4.6)

\[
C_t(r) \propto A^2(r/B)^{2\theta/\zeta} \quad \text{for} \quad r \ll Bt^\zeta.
\]

(4.7)

The sample-to-sample free energy variation introduced in Eq. (2.4) is just

\[
\Delta F_0 = \sqrt{C_t(\infty)/2} = At^\theta.
\]

(4.8)
Again, the scaling form for $C_t$ is summarized in Appendix A. As we shall see, the functions $G, C$, together with a number of other distribution functions, will allow us to compute the effect of an additional pinning potential perturbatively.

In terms of $Z_0$, the full partition function $Z(x, t)$ of the polymer in the presence of the pinning potential $U_p(\xi(z)) = -u\delta^{d-1}(\xi(z))$ is just

$$Z(x, t) = Z_0(x, t)\langle e^{-H_1/T}\rangle_{x,t}, \quad (4.9)$$

where $H_1 = \int_0^t dz \, U_p$ as in Eq. (2.6). For convenience, we again use the continuum delta function supplemented by a short distance cutoff $a$ (with $u = Ua^{d-1}$) to model the pin.

**B. The Renormalization Group Analysis**

We now investigate the effect of $H_1$ on the “bare” system $Z_0$ perturbatively. Clearly the analysis is complicated by the fact that the “bare” system itself is glassy and thus highly nontrivial. However, to understand whether the phase diagram belongs to that of Fig. 4(a) or Fig. 4(b), we only need to compute the marginal relevancy of the pinning potential in 1 + 1 dimensions. Fortunately, the randomly pinned directed polymer in $d = 1 + 1$ is one of the very few glassy systems for which a great deal is known. In particular we note that there are two pieces of exact information: (i) The disorder averaged thermal displacement is the same as that of the pure system, i.e.,

$$\langle[\xi(t) - \xi(0)]^2\rangle_{x,t} = \frac{T}{\kappa} t, \quad (4.10)$$
due to a statistical tilt symmetry $[17, 39]$ preserved by the point disorder. (ii) A fluctuation-dissipation theorem for the noisy-Burgers’ equation $[30, 58]$ which states that
\[
\frac{\partial_x F_0(x, t) \partial_y F_0(y, t)}{\Delta_T} \delta(x - y) = \frac{\Delta\kappa}{T}\delta(x - y) \quad (4.11)
\]

in the limit of large \(t\) (see Appendix A). Eq. (4.11) implies that \(C_t(r) = (\Delta\kappa/T)|r|\). Comparison with Eq. (4.1) yields \(2\theta/\zeta = 1\), and \(A^2/B \propto \Delta\kappa/T\). Along with the exponent identity Eq. (2.3), we have \(\theta = 1/3\), \(\zeta = 2/3\), and dimensional analysis gives \(B \propto (\Delta/\kappa T)^{1/3}\). We see that our “bare” problem, the fixed point of the randomly pinned directed polymer, is a fixed plane spanned by the axis \(\kappa\) and \(\Delta\kappa/T\), or alternatively by the parameters \(A \propto \kappa B^2\) and \(B\). Hence all disorder averaged functions depend only on these two parameters in the limit of large \(t\). For example, the bare coupling constant of the problem is defined as \(g_0 = \delta F_0/\Delta F_0\) where \(\Delta F_0 = At^{1/3}\) and

\[
\delta F_0 = \langle H_1 \rangle_{0,t} = -u \int_0^t dz \langle \delta[\xi(z)] \rangle_{0,t}
= -u \int_0^t dz \ G_i^{(1,1)} (0, t - z). \quad (4.12)
\]

Scaling form for \(G^{(1,1)}\) (see Appendix A) yields \(\delta F_0 \propto -(u/B)t^{1/3}\), hence \(g_0 \propto u/(AB) \propto uT/\Delta\) as in Eqs. (2.17) and (3.8). In subsequent calculations, we shall use \(g_0 \equiv u/(\kappa B^3)\).

Following standard renormalization group treatment, we now consider the renormalization of the parameters \(\kappa\), \(B\), and \(u\) due to the perturbation \(H_1\). [Note that the effect of the pinning potential will obviously depend on the position of the fixed end \(\xi(t)\). If \(\xi(t)\) is sufficiently far away from the origin, the polymer will never feel the effect of \(U_p\). In our calculations, we shall fix \(\xi(t)\) at the origin (as shown in Fig. 3) to evaluate the maximal effect of the pinning potential.]

We begin with the renormalization of the stiffness \(\kappa\), since the presence of a pinning potential breaks the statistical tilt symmetry and hence the exact relation Eq. (4.10). Adding a term \(-h \int_0^t dz (d\xi/dz)\) to \(H\) and using the transformation \(\xi(z) \rightarrow \xi(z) + h(t - z)/\kappa\) and
the property $\eta(\xi(z) + h(t - z)/\kappa, z) = \eta(\xi(z), z)$ in the statistical sense, we obtain the renormalized free energy (with one end fixed at the origin)

$$F[0, t; h] = F_0(0, t) - \frac{h^2}{2 \kappa} t - u \int_0^t dz \langle \delta[\xi(z) + h(t - z)/\kappa] \rangle_{0, t}$$

(4.13)

to the lowest order in $u$. Note that $F_0$ does not depend on $h$ due to the statistical symmetry.

Remembering that $\langle \langle [\xi(t) - \xi(0)]^2 \rangle \rangle_{x, t} = -T \partial^2 h F|_{h=0}$ where $\langle \langle . . . \rangle \rangle$ denotes thermal average using the full Hamiltonian $\mathcal{H}$, and defining the renormalized stiffness constant from

$$\langle \langle [\xi(t) - \xi(0)]^2 \rangle \rangle_{0, t} = \frac{T t}{\kappa},$$

(4.14)

we have

$$\bar{\kappa}^{-1} = \kappa^{-1} - u \frac{\partial^2}{\partial h^2} \int_0^t dz \frac{G_t^{(1,1)}(-h(t - z)/\kappa, t - z)|_{h=0}}{t}$$

(4.15)

$$= \kappa^{-1} (1 - C_{\kappa} g_0).$$

(4.16)

In Appendix B, we find $C_{\kappa}$ to be a finite constant given by Eq. (B2). Thus, there is only a finite renormalization of $\kappa$ which could be absorbed in its redefinition.

Next, we consider the renormalization of the amplitude of the mean square displacement of the polymer

$$\bar{X}^2 \equiv \int dr \ r^2 G_t^{(1,1)}(r, t).$$

(4.17)

Expansion to the lowest order in $u$ yields

$$\bar{X}^2 \equiv (\bar{B} t^{\zeta})^2 = \int_{-\infty}^{\infty} dy \ y^2 \left[ G_t^{(1,1)}(y, t) + \frac{1}{T} \langle \delta[\xi(0) - y] \mathcal{H}_1 \rangle_{x, t} \right]$$

$$= (B t^{\zeta})^2 + u \int_{-\infty}^{\infty} dy y^2 \int_0^t dz G_t^{(2,1)}(0, t - z; y, t)$$

(4.18)

where
\begin{equation}
G^{(2,1)}_t(x - y_1, t - z_1; x - y_2, t - z_2) = -\frac{1}{T} \frac{\delta (\xi (z_1) - y_1) \delta (\xi (z_2) - y_2)}{x,t}
\end{equation}

is a nonlinear response function of the noisy-Burgers' equation. In Appendix A, we obtain an expression for \(G^{(2,1)}\) in term of \(G^{(1,1)}\) using the mode coupling approximation, the validity of which will be discussed shortly. Using the relations (A4) and (A21), we obtain from Appendix B

\begin{equation}
\tilde{B} = B(1 - C_B g_0),
\end{equation}

where \(C_B\) is another finite constant given in Eq. (B6). As for \(\kappa\), the small correction to \(B\) can be absorbed in its definition.

To check the above result in a different way, we calculated also corrections to \(\Delta F_0\), or the relation (4.11), due to the perturbation \(\mathcal{H}_1\). Expanding the total free energy to the lowest order in \(u\) gives

\begin{equation}
\frac{\partial}{\partial x} F(x,t) \frac{\partial}{\partial y} F(y,t) = \frac{\Delta \kappa}{T} \delta(x - y) - u \frac{\partial^2}{\partial x \partial y} \int_0^t dz G^{(1,2)}_t(x - y, 0; \frac{x + y}{2}, t)
\end{equation}

where

\begin{equation}
G^{(1,2)}_t(x - y, 0; \frac{x + y}{2}, t) = F_0(x,t)(\delta(\xi(z)))_{y,t} + F_0(y,t)(\delta(\xi(z)))_{x,t}
\end{equation}

is another three-point response function of the noisy-Burgers equation. Again, we use the mode-coupling approximation to relate \(G^{(1,2)}\) to the elementary functions \(G^{(1,1)}\) and \(C\) in a simple way. Using the relations (A4), (A13), and (A22), we obtain

\begin{equation}
\frac{\partial}{\partial x} F(x,t) \frac{\partial}{\partial y} F(y,t) = \frac{\Delta \kappa}{T} [\delta(x - y) - g_0 f(x,y)],
\end{equation}

with \(f(x, y) \sim |x - y|^{-1}\) if \(x + y = 0\), and a faster decay for \(x + y \neq 0\) [10]. Thus, the perturbation \(\mathcal{H}_1\) changes the form of the correlation function. However, it does not lead to
a *singular* renormalization of $\Delta F_0$, which is obtained by integrating Eq. (4.23) over both the $x$ and $y$ coordinates.

Finally, we consider the renormalization of $u$ itself. This follows from the expansion of the disorder averaged free energy up to second order in $u$:

$$
\delta F(x, t) \equiv F(x, t) - F_0(x, t) = \langle H_1 \rangle_{x,t} - \frac{1}{2T} \langle H_1^2 \rangle_{x,t} .
$$

(4.24)

It will be convenient to integrate over the fixed point $x$. Using the normalization of $G^{(1,1)}$, $\int_{-\infty}^{\infty} dx G^{(1,1)}_t(x, \tau) = 1$, we have

$$
\int dx \delta F(x, t) = -ut + \frac{u^2}{2} \int_0^t dz_1 dz_2 \int_{-\infty}^{\infty} dx G^{(2,1)}_t(x, t - z_1; x, t - z_2) .
$$

(4.25)

Using Eqs. (A4) and (A21), we get the following correction to $u$

$$
\tilde{u} = u[1 + C_u g_0 \log(t/a_{\parallel})] 
$$

(4.26)

where $C_u$ is a finite positive constant given by Eq. (B12), and $a_{\parallel} \approx (a/B)^{3/2}$ is the short distance cutoff along the length of the directed polymer (see Appendix B). The logarithmic divergence in Eq. (4.26) indicates the breakdown of the small-$u$ perturbation for $g_0 \log(t/a_{\parallel}) \gg 1$. Thus the pinning potential is *strongly relevant* beyond the localization length

$$
l_{\parallel} = a_{\parallel} e^{1/g_0} ,
$$

(4.27)

and the directed polymer becomes localized to the line defect.

Since the only quantity that suffers nontrivial renormalization is the pinning strength $u$, it is straightforward to form a renormalization-group recursion relation which allows one to
obtain information about the polymer in dimensions $d > d_c$ using the calculations performed at the critical dimensionality $d_c = 1 + 1$. From Eq. (4.26) and with the rescaling $t' = bt$, we obtain the recursion relation

$$b \frac{dg}{db} = \epsilon(d) g + C_u g^2$$

(4.28)

where $\epsilon(d) = 2 - \zeta(d)(d + 1)$ as given in Eq. (2.16). Thus the coupling constant $g(b)$, which is a dimensionless measure of the pinning strength at scale $b$, flows to large values from any non-zero initial value, leading to the pinning of the directed polymer to the line defect at large scales if $\epsilon \geq 0$, i.e., if $d \leq 1 + 1$. But for $\epsilon < 0$ or $d > 1 + 1$, the pinning potential is only effective if its strength exceeds a critical value, $g_c = \epsilon/C_u$. Thus the phase diagram of the directed polymer is in fact given by Fig. 4(b), like that of the pure problem (Fig. 2), rather than Fig. 4(a) as suggested by naive scaling arguments and the uncontrolled Hartree calculation.

C. The Depinning Transition

We now investigate the critical behavior of the directed polymer close to the depinning transition. Right at the depinning point $g_c$, the wandering exponent $\zeta_c$ and the energy exponent $\theta_c$ are simply

$$\zeta_c = \zeta(d) \quad \text{and} \quad \theta_c = \theta(d)$$

(4.29)

to $O(\epsilon)$ since none of the parameters $\kappa$, $A$ and $B$ pick up divergent renormalization. The divergence of the correlation lengths, as one approaches the depinning transition from the pinned side, can be read off from Eq. (4.28). We obtain
\[ l_\parallel = a_\parallel \left( \frac{g_0}{g_c} - 1 \right)^{-\nu_\parallel} \quad \text{and} \quad l_\perp = a \left( \frac{g_0}{g_c} - 1 \right)^{-\nu_\perp} \quad (4.30) \]

with the liberation exponents

\[ \nu_\parallel = 1/|\epsilon|, \quad \text{and} \quad \nu_\perp = \zeta_\parallel \nu_\parallel = \zeta(d)/|\epsilon|, \quad (4.31) \]

again valid to \( O(\epsilon) \). In \( d = 2 + 1 \) where \( \zeta \approx 5/8 \) and \( \epsilon = 1/2 \), Eq. \( (4.31) \) gives \( \nu_\perp \approx 1.25 \) which is comparable to the results of numerical simulations: \( \nu_\perp = 1.3 \pm 0.6 \) by Balents and Kardar \[29\], and \( \nu_\perp = 1.8 \pm 0.6 \) by Tang and Lyuksutov \[28\]. It will be interesting to see whether Eqs. \( (4.29) \) and \( (4.31) \) might be valid to all orders in \( \epsilon \) (up to some upper critical dimensions), as they do in the case of the pure problem without point disorders \[31\]. This can be directly probed numerically in \( d = 1 + 1 \) by using correlated point disorder or by modifying the form of \( U_p \) (see Sec. V).

The results Eqs. \( (4.29) \) and \( (4.31) \) have been conjectured earlier by Kolomeisky and Sta-ley \[30,31\], who combined the renormalization-group flow equations for the pure depinning problem with \( \eta = 0 \) and the one for only point disorder with \( U_p = 0 \) in an \textit{ad hoc} way. In particular, Kolomeisky and Straley neglected to consider the renormalization of the stiffness \( \kappa \), \( B \), and the amplitude of the disorder potential \( \Delta \) due to the pinning potential \( U_p \). It is the lack of any divergent renormalization of these quantities, as obtained through explicit calculations in this study, that ensures the validity of Eq. \( (4.29) \), at least to \( O(\epsilon) \). The lack of such divergent renormalization follows from the large transverse wandering of the polymer at the depinning transition, thus diminishing the effect of the pinning potential. (Similar behaviors have been found for the renormalization of the surface tension of a wetting layer at the wetting transition \[11\].)

Balents and Kardar \[32\] obtained similar conclusions, Eqs. \( (4.29) \) and \( (4.31) \), by general-
izing the directed polymer to a $D$-dimensional manifold, and then computing the analogy of the coefficients $C_\kappa$, $C_B$ and $C_u$ in a functional renormalization group (FRG) analysis to first order in $\delta = 4 - D$. Since the $O(\epsilon)$ results (4.29) and (4.31) do not depend on the numerical values of the $C$'s, the FRG approach is effective despite the large expansion parameter ($\delta = 3$) for the directed polymer.

Finally, we comment on the validity of the mode coupling approximation used in the evaluation of the expressions leading to Eqs. (4.15), (4.20), (4.23) and (4.26). Due to the combination of a statistical tilt symmetry and a fluctuation dissipation theorem in $d = 1+1$, the mode coupling approach gives the correct scaling behavior for the functions $G^{(m,n)}$ (see Refs. [46,47] and Appendix A). This is all that is needed here since, as explained above, the results (4.29) and (4.31) are independent of the numerical values of the coefficients $C$'s. On the other hand, the mode coupling method is known to give very good quantitative results even for the scaling functions [20,46,47]. Thus the coefficient $C$'s computed in this way should be quantitatively accurate.

[ Note added: Very recently, Kolomeisky and Staley [33] modified their renormalization-group analysis and obtained a power-law rather than exponential divergence of the localization length at the critical dimension $1 + 1$. We disagree with their conclusions and point out what we believe to be the key difference between Ref. [33] and the present work: The analysis of Ref. [33] is a one-loop calculation with respect to the pure directed polymer. The effect of point disorder is taken into account by an Ansatz which makes appropriate rescaling of the coefficients. On the other hand, the analysis presented in this section starts from the low-temperature fixed point of the randomly-pinned directed polymer, and takes into account of the extended pinning potential in a systematic treatment. The two approaches... ]
should eventually lead to the same scaling behavior (see Sec. IV.D). However, the approach of Ref. [33] requires a careful, consistent formulation. In particular, the choice of parameters [Eq. (2.4)] used in Ref. [33] implies that not all “temperatures” renormalize in the same way (since Galilean invariance must be respected). This introduces some ambiguities in the scaling ansatz used there. ]

D. Physical Interpretations

As shown by the renormalization group analysis, the logarithmic divergence of the effective pinning strength $\tilde{u}$ at the critical dimension is the single most important element leading to the phase diagram Fig. 4(b) and the exponents (4.29) and (4.31). Discussions of the preceding paragraphs also illustrates the robustness of these results — the same results are obtained from a variety of methods, some with drastic unjustified approximations, as long as the scaling behaviors are properly included. We shall now provide a physical picture of the weak localization using a phenomenological scaling theory, and argue that the logarithmic divergence of $\tilde{u}$ should indeed be expected at the critical dimension.

In Sec. II, we established the energy gained by the directed polymer (due to the presence of a weak pinning potential) to be $\delta F_0 \propto u \int_0^t d\tau (B\tau)^{-d+1}\zeta \sim t^{1/3}$ in $1 + 1$ dimensions. This corresponds to describing the optimal path of the directed polymer as a generalized random walk (with the wandering exponent $\zeta$ rather than $1/2$), and then equating the energy gained to the accumulated return probability of the random walk. However, such a treatment of the optimal path is oversimplified. As discussed in detail in Ref. [17], while the optimal path in a typical sample (or a typical region of a very large sample) is unique, there is a nonzero probability that there exists a different path whose free energy is within $O(1)$ of that of the
optimal path. The probability of finding two such paths a distance $\Delta$ apart is $p(\Delta) \sim \Delta^{-3/2}$ in 1+1 dimension [17], and the “droplet” formed (i.e., the difference between the two paths) typically have a length $\tau \sim \Delta^{1/\zeta}$, which is of $O(\Delta^{3/2})$ in 1+1 dimension. Thus, if one of the optimal path encounters the origin, then the probability of having a nearly degenerate optimal path also encountering the origin is $p(\tau^{2/3}) \sim \tau^{-1}$ in 1+1 dimensions. In this way, we see that the accumulated effect of the statistically droplets existing at different scales leads to a logarithmic divergence, i.e., $\int dt \tau p(\tau^{2/3}) \sim \log t$. This makes the columnar pin marginally relevant at the critical dimension, and results in a phase diagram of the type depicted in Fig. 4(b) rather than that in Fig. 4(a). Note that the important ingredients of the above argument are only the probability of droplet formation and the shape of the droplets. It therefore suggests the marginal relevance of an extended defect at the critical dimension to be a general consequence of the droplet scaling theory.

Technically, the droplets manifest themselves in the expression describing the renormalization of the pinning strength in Eq. (4.25), a diagrammatic representation of which is shown in Fig. 6. The similarity of the droplet configuration in Fig. 5 and the loop diagram in Fig. 6 is striking. As explained in detail in Ref. [17], the distribution of droplets is given by the function $G^{(2,1)}$. However, the particular distribution discussed in Ref. [17] has to do with droplets with a given width $\Delta$ at the same vertical coordinate, whereas the distribution needed here is the one with a given length $\tau$ at the same transverse coordinate. Arguments leading to the logarithmic divergence in the preceding paragraphs assumes $\tau \sim \Delta^{1/\zeta}$. This is validated by the more detailed calculation given in Appendix B.

In some ways, the depinning transition here is similar to the de Almeida-Thouless [12] line in a spin glass. The columnar pin which breaks the statistical translational symmetry
of the directed polymer plays a role similar to the external magnetic field which breaks the statistical up-down symmetry of the Ising spin glass. It should be interesting to study the depinning transition in the replica formalism. The encounter of replica symmetry breaking alluded to in Sec. III is no longer mysterious now — it is simply how the droplet excitations are manifested within the replica formalism \cite{13,17}.
V. RELATED DEPINNING PROBLEMS

The results of Sec. IV can be easily generalized to long range correlated point disorders, other forms of pinning potentials, and higher dimensional elastic manifolds. Here we shall describe a few interesting cases to illustrate the method.

Following Nattermann \cite{43} and Medina et al \cite{38}, we consider a random potential with long-range correlations of the form

\[
\bar{\eta}(x, z)\bar{\eta}(x', z') = 2\Delta \delta(z - z')|x - x'|^{2\rho-d+1}.
\]

We shall be focus on the case \(d = 1 + 1\), where the model \((2.1)\) describes the domain wall of the low temperature phase of a random Ising model in two dimensions. The correlator \((5.1)\) extrapolates smoothly between the case of random bond \((\rho = 0)\) and random field \((\rho = 1)\) \cite{38,43}. It is known that the roughness exponent \(\zeta(\rho)\) stays at its value of \(2/3\) for short range correlated disorder as long as \(\rho \leq 1/4\). For \(1 \geq \rho \geq 1/4\) it takes on the Flory value of \(\zeta(\rho) = 3/(5 - 2\rho)\). From Eq. (4.28) and \(\epsilon = 2 - \zeta(d + 1)\), we see that \(\epsilon = 0\) for \(\rho \leq 1/4\), and \(\epsilon = (1 - 4\rho)/(5 - 2\rho) < 0\) for \(\rho > 1/4\). Thus for \(\rho > 1/4\), a small pinning potential is irrelevant. As one varies \(u\), there will be a depinning transition with exponents \(\nu_\parallel = 1/|\epsilon| = (5 - 2\rho)/(4\rho - 1)\) and \(\nu_\perp = \nu_\parallel \zeta_{\parallel} = 3/(4\rho - 1)\) to \(O(\epsilon)\). If the exponent \(\nu_\parallel = 1/|\epsilon|\) is indeed exact to all orders as in the pure case, then the above expression will be valid for all \(1 \geq \rho > 1/4\). In particular we should obtain a depinning transition with the exponents \(\nu_\perp = \nu_\parallel = 1\) for \(\rho = 1\). A numerical simulation of the depinning transition of the domain wall of the 2d random field Ising model should therefore be an efficient way of probing the “exactness” of the \(O(\epsilon)\) result.

We can generalize the methods described in this paper to study the pinning of a
directed polymer by other forms of the pinning potential $U_p(\mathbf{r})$. If we write $\mathcal{H}_1 = \int d^{d-1}\mathbf{r} U_p(\mathbf{r}) \int_0^t dz \delta^{d-1}[\mathbf{r} - \xi(z)]$, then the lowest order correction to the free energy is

$$\delta F_0(t) = \langle \mathcal{H}_1 \rangle = \int_0^t dz \int d^{d-1}\mathbf{r} U_p(\mathbf{r}) G^{(1,1)}(\mathbf{r}, t - z),$$

(5.2)

where $G^{(1,1)}$ is the one-point distribution function defined in Eq. (4.2), with the scaling properties described in Appendix A. Suppose $U_p(\mathbf{r})$ has a long tail, i.e., $U_p = u/r^s$, then $\delta F_0(t) \sim t^{1-s\zeta}$. Comparing this to $\Delta F_0 \sim t^{\theta}$ and recalling the exponent identity Eq. (2.5), we find the critical dimension of $U_p$ to be

$$2 = \zeta(d_c)(2 + s).$$

(5.3)

We expect a depinning transition at finite $u$ above the critical dimension $d_c$, with the liberation exponent $\nu_{\parallel} = 1/|2 - \zeta(d)(2 + s)|$.

We may also consider a pinning potential $U_p(\mathbf{r}) = -u \delta^{d-1}[\mathbf{r} - \mathbf{R}(z)]$, where the trajectory of the pin $\mathbf{R}(z)$ is itself an arbitrary (but fixed) function of $z$. For example, we may have $R(z) \sim z^{\zeta_R}$, describing the quenched defect trajectory of a superconductor subject to random collision by heavy ions. In this case, Eq. (5.2) becomes

$$\delta F_0(t) = -u \int_0^t dz \ G_t(\mathbf{R}(z), t - z).$$

(5.4)

If $R(z) < |t - z|^{\zeta}$ or $\zeta_R < \zeta$, i.e., if the transverse fluctuation of the defect trajectory is smaller than that of the randomly pinned directed polymer, then the $z$-dependence of the pinning potential is irrelevant and we have $\delta F_0(t) \sim ut^{1-d_{\perp}\zeta}$ as before, and the defect acts like a straight columnar pin. However, if $R(z) > |t - z|^{\zeta}$, as for example is the case for a misoriented columnar pin where $R(z) \propto |t - z|^d$ [14], then $G^{(1,1)}$ is sharply cutoff and $\delta F_0(t)$ becomes finite for large $t$. In this case, a weak pinning potential is irrelevant compared to
the random energy gain $\Delta F_0 \sim t^\theta$. This type of analysis can be extended to study a large
variety of pinning potentials, including the case where $R(z)$ itself is the trajectory of another
directed polymer [15]..

Another interesting ramification of our considerations pertains to $D$-dimensional oriented
manifolds in $d$ embedding dimensions, i.e., $z \in \mathbb{R}^D$ and $\xi(z) \in \mathbb{R}^{d-D}$. $D = 1$ describes a
directed polymer or a flux line, and $D = 2$ describes, for example, the domain wall of a three
dimensional ferromagnet. The free energy scales in this case with an exponent $\theta = 2\zeta + D - 2$,
which is the $D$-dimensional analog of the exponent identity (2.5). Following Ref. [29],
we also generalize the extended defect to be $n$-dimensional, i.e., $U_p(r) = -u\delta^{d-n}(r)$, and
$\mathcal{H}_1 = \int d^Dz U_p[\xi(z)]$. The columnar pinning potential discussed corresponds to $n = 1$, and
planar defects such as grain boundaries correspond to $n = 2$. Straightforward generalization
of the scaling arguments of Sec. II and Eq. (4.12) gives $\delta F_0(t) = \langle \mathcal{H}_1 \rangle \sim -ut^{D-\zeta(d-n)}$, where $t$ is now the linear size of the manifold. Comparing this to $\Delta F_0 \sim t^\theta$, we find the condition
for the relevance of $U_p$ to be given by

$$2 > \zeta(d)(2 + d - n). \quad (5.5)$$

An important application here is the pinning of an interface to a planar defect, say the
localization of a domain wall to a grain boundary in a random ferromagnet. In this case,
$D = n = 2$, $d = 3$, and Eq. (5.5) reads $\zeta(3) < 2/3$. For the interface of a random bond Ising
magnet, this is always fulfilled and the interface is always pinned by the planar defect. For
random field systems however, $\zeta(3) = 2/3$ exactly in 3 dimensions, and our criterion (5.5)
is inconclusive. Straightforward generalization of the scaling argument given in Sec. IV.D
shows that the interface is again weakly pinned by an arbitrary weak pinning potential.
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APPENDIX A: USEFUL RELATIONS FOR THE DIRECTED POLYMER

In order to make the paper more self-contained, we summarize here some useful results on the randomly pinned directed polymer obtained in earlier publications \[17,38,46\]. The restricted partition function of a directed polymer of length \(t\) with one end \(\xi(t)\) fixed at \(x\) is

\[
Z_0(x, t) = \int \mathcal{D}[\xi] \delta^{d-1} [x - \xi(t)] e^{-\mathcal{H}_0/T}.
\]  

(A1)

The free energy \(F_0(x, t) = -T \log Z_0(x, t)\) fulfills the noisy-Burgers’ equation \[36–38\]

\[
\partial_t F_0(x, t) = \frac{T}{2\kappa} \nabla^2 F_0 - \frac{1}{2\kappa} (\nabla F_0)^2 + \eta(x, t).
\]  

(A2)

In Sec. IV, we introduced various correlation and distribution functions to evaluate the effect of the pinning potential \(\mathcal{H}_1\). To begin with, the one-point distribution function is

\[
\langle \delta[\xi(z) - y] \rangle_{x, t} = G^{(1,1)}_t(x - y, t - z).
\]  

(A3)

Again, \(\langle \ldots \rangle_{x, t}\) denotes thermal average using the partition function \(Z_0(x, t)\) and the subscript \(t\) is used in \(G^{(1,1)}\) to denote the explicit \(t\)-dependence of the distribution function. By simple scaling and normalization requirements, we have

\[
G^{(1,1)}_t(r, \tau) = (B\tau^\zeta)^{-d-1} \tilde{g}_{t/\tau}(r/B\tau^\zeta),
\]  

(A4)

where \(\tilde{g}_{t/\tau}(0)\) is finite, \(\tilde{g}_{t/\tau}(s)\) decreases sharply for \(s \gg 1\), and \(\int d^{d-1}s \tilde{g}_{t/\tau}(s) = 1\) for all \(t/\tau\).

The rms displacement \(X^2_t(\tau)\) is given by the second moment,

\[
X^2_t(\tau) = \int d^{d-1}r \, r^2 G^{(1,1)}_t(r, \tau)
= B^2 \tau^{2\zeta} \int d^{d-1}s \, s^2 \tilde{g}_{t/\tau}(s),
\]  

(A5)
which is a weak function of \(t/\tau\). It is found numerically \[19,20\] that \(X^2_\tau(t)/\tau^{2\zeta}\) is the same order of magnitude for \(t = \tau\) and \(t \gg \tau\). We fix \(B\) by the rms displacement of the free end, \(X^2(t) = B^2t^{2\zeta}\) (see Eq. (2.3)). This is accomplished by choosing

\[
\int d^d s \ s^2 \bar{g}_1(s) = 1. \tag{A6}
\]

Other than \(G^{(1,1)}\), we are also interested in the higher-order distribution functions,

\[
G^{(2,1)}(x - y_1, t - z_1; x - y_2, t - z_2) = -\frac{1}{T} \langle \delta[\xi(z_1) - y_1] \delta[\xi(z_2) - y_2] \rangle^{(r)}_{x,t}, \tag{A7}
\]

\[
G^{(1,2)}(x_1 - x_2, t_1 - t_2; \frac{x_1 + x_2}{2} - y, \frac{t_1 + t_2}{2} - z) = \left[ F_0(x_1, t_1) \langle \delta[\xi(z) - y] \rangle_{x_2, t_2} + F_0(x_2, t_2) \langle \delta[\xi(z) - y] \rangle_{x_1, t_1} \right], \tag{A8}
\]

as well as the free energy correlation function \[49\]

\[
C_{[t_1, t_2]}(x_1 - x_2, t_1 - t_2) = \left[ F_0(x_1, t_1) - F_0(x_2, t_2) \right]^2, \tag{A9}
\]

where the subscript \([t_1, t_2]\) denotes the larger of \(t_1\) and \(t_2\). In analogy to \(G^{(1,1)}\), the correlation function has a similar scaling form,

\[
C_t(r, \tau) = A^2 t^{2\theta} \bar{c}_{t/\tau} (r/B\tau^{\zeta}). \tag{A10}
\]

The scaling function \(\bar{c}\) gives the following scaling properties for \(C\),

\[
C_t(r, \tau) \begin{cases} = 2A^2 t^{2\theta} & B\tau^{\zeta} \ll Bt^{\zeta} \ll r, \\ \propto 2A^2 (r/B)^{2\theta/\zeta} & B\tau^{\zeta} \ll r \ll Bt^{\zeta}, \\ \propto 2A^2 r^{2\theta} & r \ll B\tau^{\zeta} \ll Bt^{\zeta}. \end{cases} \tag{A11}
\]

For convenience, we also write the correlation function as

\[
C_{[t_1, t_2]}(x_1 - x_2, t_1 - t_2) = G^{(0,2)}_{t_1}(0, 0) + G^{(0,2)}_{t_2}(0, 0) - 2G^{(0,2)}_{[t_1, t_2]}(x_1 - x_2, t_1 - t_2) \tag{A12}
\]
where

\[
G^{(0,2)}_{[t_1,t_2]}(x_1 - x_2, t_1 - t_2) = \overline{F_0(x_1, t_1)F_0(x_2, t_2)}. \tag{A13}
\]

The scaling properties of \(G^{(0,2)}_t(r, \tau)\) is easily obtained from Eq. (A11). For example, from Eq. (A12) and \(G^{(0,2)}_t(r \to \infty, \tau) \to 0\), we have \(G^{(0,2)}_t(0, 0) = \frac{1}{2}G_t(\infty) = A^2t^{2\theta}\). Below, we shall provide the approximate forms for all of the \(G^{(m,n)}\)’s.

It was shown in Ref. [17] that \(G^{(m,n)}\) can be obtained simply by adding a source term \(\tilde{J}(x, t)\) to right hand side of the equation of motion (A2) and then taking appropriate derivatives, i.e.,

\[
G^{(m,n)} = \frac{\delta}{\delta \tilde{J}(y_1, z_1)} \ldots \frac{\delta}{\delta \tilde{J}(y_m, z_m)} [F_0(x_1, t_1) \ldots F_0(x_n, t_n)]. \tag{A14}
\]

In the context of the stochastic hydrodynamics of the noisy-Burgers’ equation, the above is nothing but the generalized response function. In Ref. [17], it was shown that the nonlinear response function \(G^{(2,1)}\) gives the statistics of the rare but singular “droplet excitations”, which are connected to replica symmetry breaking in the replica formalism [12,18]. Here we encounter them again in perturbation theory, as we already did when using the replica formalism in Sec. III.

However, unlike Sec. III where we failed to develop a perturbative expansion due to the lack of knowledge of replica-symmetry broken excited states, here we can construct the forms of the nonlinear response functions \(G^{(m,n)}\) rather straightforwardly by exploiting a Fluctuation Dissipation Theorem (FDT), which the equation of motion (A2) satisfies in 1 + 1 dimensions [38,40]. For example, the FDT gives

\[
\frac{\partial}{\partial x} \frac{\partial}{\partial y} G^{(0,2)}_t(x - y, \tau) = \frac{\Delta \kappa}{T} G^{(1,1)}_t(x - y, \tau), \quad \tau > 0. \tag{A15}
\]
Taking the limit $\tau \to 0$ in Eq. (A13) and using the definition of $G^{(1,1)}$, we immediately obtain

$$\partial_x F_0(x, t) \partial_y F_0(y, t) = \frac{\Delta \kappa}{T} \delta(x - y).$$  \hspace{1cm} (A16)$$

Similarly, by integrating Eq. (A15) and using the scaling forms for $G^{(1,1)}$, we can recover the scaling properties of $C$ given in Eq. (A11).

The combination of the FDT and a Galilean invariance (corresponding to the statistical rotational symmetry) in $1 + 1$ dimensions allows one to use a mode-coupling scheme \[46–48\] to obtain the forms of the functions $G^{(m,n)}$. In particular, $G^{(1,1)}$ and $G^{(0,2)}$ are given, to a very good approximation, by the following set of self-consistent integral equations,

$$G^{(1,1)}_{t_1}(x - y, t - z) = \tilde{G}^{(1,1)}(x - y, t - z)$$

$$+ \frac{1}{\kappa^2} \int_{-\infty}^{\infty} dx' dy' \int_{0}^{t} dt' \int_{0}^{t'} dz' \tilde{G}^{(1,1)}(x - x', t - t') \frac{\partial}{\partial x'} G^{(1,1)}_{t'}(x' - y', t' - z') \frac{\partial}{\partial y'} G^{(1,1)}_{z'}(y' - y, z' - z),$$

$$G^{(0,2)}_{[t_1, t_2]}(x_1 - x_2, t_1 - t_2) = \tilde{G}^{(0,2)}_{[t_1, t_2]}(x_1 - x_2, t_1 - t_2)$$

$$+ \frac{1}{2\kappa^2} \int_{-\infty}^{\infty} dx'_1 dx'_2 \int_{0}^{t_1} dt'_1 \int_{0}^{t_2} dt'_2 G^{(1,1)}_{t_1}(x_1 - x'_1, t_1 - t'_1) \frac{\partial}{\partial x'_1} \frac{\partial}{\partial x'_2} G^{(0,2)}_{[t'_1, t'_2]}(x'_1 - x'_2, t'_1 - t'_2) \bigg[^2, \hspace{1cm} (A17)$$

where

$$\tilde{G}^{(1,1)}(r, \tau) = \sqrt{\frac{T}{2\pi\kappa\tau}} \exp \left[ -\frac{T r^2}{\kappa \tau} \right], \hspace{1cm} \tau > 0$$

is the “bare” response function, and

$$\tilde{G}^{(0,2)}_{[t_1, t_2]}(x_1 - x_2, t_1 - t_2) = 2\Delta \int_{-\infty}^{\infty} dx' \int_{0}^{t_2} dt' \tilde{G}^{(1,1)}(x_1 - x', t_1 - t') \tilde{G}^{(1,1)}(x_2 - x', t_2 - t')$$

$$\hspace{1cm} (A20)$$
is the “bare” correlation function. The mode-coupling equations (A17) and (A18) can be solved by using the scaling forms (A4) and (A11) for $G^{(1,1)}$ and $G^{(0,2)}$. The scaling functions $\tilde{g}$ and $\tilde{c}$ obtained in this way are in very good agreement with those from numerical simulations [19,20,46]. It is found that $\tilde{g}_\sigma(s)$ is approximately a Gaussian with a weak $\sigma$ dependence, and the width of the “Gaussian” is fixed by the condition (A6) to be 1.

The functions $G^{(1,1)}$ and $G^{(0,2)}$ can now be used to construct higher order response functions via the mode-coupling scheme. For instance,

$$G^{(2,1)}_t(x - y_1, t - z_1; \ x - y_2, t - z_2)$$

$$= -\frac{1}{\kappa} \int_{-\infty}^{\infty} dx' \int_0^t dt' G^{(1,1)}_t(x - x', t - t')$$

$$\frac{\partial}{\partial x'} G^{(1,1)}_{t'}(x' - y_1, t' - z_1) \frac{\partial}{\partial x'} G^{(1,1)}_{t'}(x' - y_2, t' - z_2)$$

(A21)

$$G^{(1,2)}_{[t_1, t_2]}(x_1 - x_2, t_1 - t_2; \ \frac{x_1 + x_2}{2} - y, \ \frac{t_1 + t_2}{2} - z)$$

$$= -\frac{1}{\kappa} \int_{-\infty}^{\infty} dx'_1 \int_0^{t_1'} dt'_1 G^{(1,1)}_{t'_1}(x_1 - x'_1, t_1 - t'_1)$$

$$\frac{\partial}{\partial x'_1} G^{(1,1)}_{t'_1}(x'_1 - y, t'_1 - z) \frac{\partial}{\partial x'_1} G^{(0,2)}_{[t'_1, t_2]}(x_2 - x'_1, t_2 - t'_1)$$

+ permutation of $[(x_1, t_1) \leftrightarrow (x_2, t_2)]$. (A22)

The validity of the above mode coupling approximation for $G^{(2,1)}$ was discussed in detail in Ref. [17]. Eqs. (A21) and (A22) should capture the key scaling properties but may not be quantitatively accurate. We shall nevertheless use the above expressions to evaluate the integrals obtained in the RG analysis of Sec. IV. As our main concern is the existence of logarithmic divergence in the renormalization of various parameters, rather than the numerical value of any particular integrals, the use of the mode coupling approximation should be adequate.
APPENDIX B: RESULTS OF CALCULATIONS

In this appendix, we compute the perturbative effect of $H_1$ on the parameters $\kappa$, $B$, and $u$, by using the expression for $G^{(m,n)}$’s obtained from the mode coupling approximation (see Appendix A). We are particularly interested in the $t$-dependence of the renormalized parameters $\tilde{\kappa}(t)$, $\tilde{B}(t)$ and $\tilde{u}(t)$ in the limit $t \to \infty$.

We start with the renormalization of the stiffness $\kappa$. From Eq. (4.16) and the scaling form (A4) for $G^{(1,1)}$, we have

$$
\tilde{\kappa}^{-1} = \kappa^{-1} - u \int_0^t \frac{dz}{t} \frac{(t-z)^2}{\kappa^2 [B(t-z)]^\zeta} \frac{\partial^2}{\partial s^2} \tilde{g}_{t/(t-z)}(s) |_{s=0}
= \frac{1}{\kappa} - \frac{u}{\kappa^2 B^3} \int_0^1 d\sigma \tilde{g}''_{1/\sigma}(0),
$$

where ‘primes’ indicate derivatives of $\tilde{g}$ and $\zeta = 2/3$ in $d = 1 + 1$. Thus we obtain Eq. (4.15) with $g_0 = u/(\kappa B^3)$ and

$$
C_\kappa = \int_0^1 d\sigma \tilde{g}''_{1/\sigma}(0)
$$

which is finite.

Next we consider the renormalization of the transverse wandering coefficient $B$. From Eq. (4.18), we have

$$
\tilde{B}^2 t^{2\zeta} = B^2 t^{2\zeta} + I_B,
$$

with

$$
I_B = u \int_{-\infty}^{\infty} dy \ y^2 \int_0^t dz \ G^{(2,1)}_i(0,t-z;y,t)
= -\frac{u}{\kappa} \int_0^t dz \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy \ y^2 G^{(1,1)}_i(x',t-t')
\frac{\partial}{\partial x'} G^{(1,1)}_v(x'-y,t') \frac{\partial}{\partial x'} G^{(1,1)}_v(x',t'-z),
$$

39
where we used the mode-coupling approximation Eq. (A21) for $G^{(2,1)}$. Using the scaling form (A4) for $G^{(1,1)}$, and noting that $\tilde{g}(s)$ is symmetric in $s$, we find

$$I_B = -(B^2t^\xi)2g_0C_B \quad (B5)$$

where

$$C_B = \int_0^1 \frac{d\tau}{\tau^\xi} \int_0^1 d\sigma \int_{-\infty}^{\infty} ds (-s)\tilde{g}'_{(1-\tau)/\sigma}(s) \tilde{g}_{1/\tau}(s\sigma^\xi/\tau^\xi). \quad (B6)$$

Again $C_B$ is finite since $\tilde{g}$ are normalized and sharply cutoff for large argument. We thus obtain the result Eq. (4.20).

Finally, we consider the renormalization of $u$. The expression given by Eq. (4.25) can be described diagrammatically as in Fig. 6. If we use the mode-coupling approximation Eq. (A21) for $G^{(2,1)}$, and note the normalization condition $\int dx G^{(1,1)}_t(x, \tau) = 1$, we find

$$\delta F(t) = -ut + \frac{u^2}{2} \int_0^t dz_1dz_2 \int_{-\infty}^{\infty} dx G^{(2,1)}_t(x, t - z_1; x, t - z_2)$$

$$= -ut - \frac{u^2}{2\kappa} \int_0^t dt' \int_0^{t'} dz_1 \int_0^{t'} dz_2 \int_{-\infty}^{\infty} dx' \frac{\partial}{\partial x'} G^{(1,1)}_t(x', t' - z_1) \frac{\partial}{\partial x'} G^{(1,1)}_t(x', t' - z_2). \quad (B7)$$

Using the scaling form (A4) for $G^{(1,1)}$ again, we obtain

$$\delta F(t) = -ut - \frac{u^2}{2\kappa B^3} \int_0^t dt' I_u(t'), \quad (B8)$$

where

$$I_u(\tau) = \int_0^\tau \frac{dt_1 dt_2}{t_1^\xi t_2^\xi} \int_{-\infty}^{\infty} ds \tilde{g}'_{t_1/\tau}(s) \tilde{g}'_{t_2/\tau}(st_1^\xi/t_2^\xi). \quad (B9)$$

We shall see that $I_u$ is actually divergent. To regularize the integral, we insert a ultra-violet cutoff scale $a_\parallel \propto (a/B)^{1/\xi}$, since in our model, the columnar pin $U_p$ is really a potential well of finite size $a$; it is only approximated by a delta function at scales much larger than $a$. Eq. (B9) then becomes
\[ I_u(\tau) = \int_{a_1}^{\tau} \frac{dt_1}{t_1} F(t_1/\tau, a_1/\tau), \]  
\hspace{1cm} (B10)

where

\[ F(\hat{t}, \hat{a}) = \int_{1/\hat{a}}^{1/\hat{a}} d\sigma \sigma^{-2\zeta} \int_{-\infty}^{\infty} ds \ \tilde{g}_{1/\hat{a}}'(s) \ \tilde{g}_{1/(\sigma \hat{a})}'(s/\sigma^\zeta). \]  
\hspace{1cm} (B11)

Clearly the divergent part of \( I_u(\tau) \) comes from the limit \( \tau/a_1 \to 0 \) in Eq. (B10). So to leading order, we have \( I_u(\tau) = \log(\tau/a_1) C_u \), where

\[ C_u = \lim_{\hat{a} \to 0} F(\hat{a}, \hat{a}) \]
\[ = \lim_{\hat{a} \to 0} \int_{1/\hat{a}}^{1/\hat{a}} d\sigma \sigma^{-2\zeta} \int_{-\infty}^{\infty} ds \ \tilde{g}_{1/\hat{a}}'(s) \ \tilde{g}_{1/(\sigma \hat{a})}'(s/\sigma^\zeta) \]
\[ \approx \int_{1}^{\infty} \frac{d\sigma}{\sigma^{2\zeta}} \int_{-\infty}^{\infty} ds \ \tilde{g}_{\infty}'(s) \ \tilde{g}_{\infty}'(s/\sigma^\zeta), \]  
\hspace{1cm} (B12)

which is positive definite since \( 2\zeta = 4/3 > 1 \) and \( \tilde{g} \) is well behaved. The effective pinning potential \( \tilde{u} \) can now be defined as

\[ \tilde{u} \equiv -\frac{\partial}{\partial t} \delta F(t) = u[1 + g_0 C_u \log(t/a_1)], \]  
\hspace{1cm} (B13)

which is quoted in Eq. (4.26).
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FIGURES

FIG. 1. (a) The solid, dashed, and dotted paths are the optimal paths of a randomly pinned directed polymer with one end fixed at $x_1$, $x_2$, and $x_3$ respectively. The rms displacement of the free end is $X(t) \propto t^{\zeta}$, where $t$ is the polymer length. For small separation of the fixed end, $x_2 - x_1 \ll X(t)$, the two optimal paths typically merge at a length $\tau \propto |x_2 - x_1|^{1/\zeta}$ from the fixed end. For much larger separations, $x_3 - x_1 \gg X(t)$, the optimal paths become completely uncorrelated. (b) Variations in the directed polymer’s free energy as a function of the fixed end position, $x$. For large separation of the fixed end, typical free energy differences are $\Delta F_0 \propto t^\theta$.

FIG. 2. The phase diagram of a pure directed polymer pinned by a short-ranged columnar pinning well $U_p$. The arrows indicate the effective pinning strength $U$ in units of the entropy cost of localization, $U_0^{(0)} = T^2/(2\kappa a^2)$. The directed polymer is pinned in $d_\perp \leq 2$. In $d_\perp > 2$, a depinning transition at $U = U_c^{(0)} \propto (d_\perp - 2)U_0^{(0)}$ (indicated by the asterisk) separates the pinned and depinned phases.

FIG. 3. The solid line depicts a directed polymer localized to a columnar pin at the origin. The localization distance is $l_\perp$. The polymer is consisted of a number of uncorrelated segments of length $l_\parallel \propto l_\perp^{1/\zeta}$. An optimal path far away from the origin is depicted by the dashed line.

FIG. 4. (a) A possible phase diagram for the directed polymer pinned by both the columnar pin and a random background. The arrows indicate the effective strength $U$ of the columnar pin, in units of the random energy loss, $U_0 = A(a/B)^{(\theta-1)/\zeta}$. The critical dimension is $d_\perp = 1$. Unlike Fig. 2, this phase diagram contains a depinning transition in all dimensions. (b) An alternative phase diagram having the same structure as that of the pure one (Fig. 2), but with a shift in the critical dimension to $d_\perp = 1$. 
FIG. 5. The solid line is an optimal path which intersects the origin at \( z = z_1 \). The dashed line is a path that is nearly degenerate in free energy. It intersects the origin at \( z = z_2 \). The difference between the two paths is called a “droplet”. It has a length \( \tau \) and a width \( \Delta \sim \tau^\zeta \).

FIG. 6. A loop diagram describing the renormalization of the pinning strength \( u \) given in Eq. (4.25). The three-point function \( G^{(2,1)} \) is given by the convolution of a vertex (the black triangle) and three \( G^{(1,1)} \)'s (lines with arrows). The mode-coupling approach Eq. (A21) approximates the vertex by the “bare” vertex, i.e., two spatial derivatives (see Appendix A).