A new technique for solving singular IVPs of Emden-Fowler type

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Abstract

A new analytic approximate technique for addressing nonlinear problems, namely the optimal perturbation iteration method, is introduced and implemented to singular initial value Lane-Emden type problems to test the effectiveness and performance of the method. This technique provides us to adjust the convergence regions when necessary. Comparing different methods reveals that the proposed method is highly accurate and has great potential to be a new kind of powerful analytical tool for nonlinear differential equations.

1 Introduction

Many problems of science and engineering lead to different types of differential equations and it is still very hard to solve them in the presence of strong nonlinearity. In order to cope with this issue, there has been much attention devoted to investigate better and more efficient solution techniques such as homotopy decomposition method [1, 2], auxiliary equation method [3, 4], homotopy perturbation method [5, 6], Taylor collocation method [7-9], Sumudu transform method [10] and Adomian decomposition method [11-14].

Emden-Fowler equation is one of the most important differential equations of mathematical physics. It distinctively characterizes many scientific phenomena. The generalized Emden-Fowler equation is defined in the following form

\[ y'' + \alpha(x)y' + \beta(x)\gamma(y) = 0 \]  

(1.1)

where \( \alpha(x), \beta(x), \gamma(y) \) are some arbitrary functions. For different \( \gamma(y) \), the Eq. (1.1) has been subject of many studies in the literature such as the theory of stellar structure, thermionic currents and isothermal gas spheres. When \( \alpha(x) = \frac{k}{x} \), \( \beta(x) = \beta_0 x^r \), \( \gamma(y) = y^s \) (\( k \) and \( \beta_0 \) are constants, \( s \) and \( r \) are real numbers), Eq. (1.1) reduces to the classic Emden-Fowler equation:

\[ y'' + \frac{k}{x}y' + \beta_0 x^r y^s = 0. \]  

(1.2)

Furthermore, by choosing \( r = 0 \) and \( k = 2 \), we get the standard Lane-Emden equation

\[ y'' + \frac{2}{x}y' + \beta_0 y^s = 0 \]  

(1.3)

which arises in astrophysics. Eq. (1.2) is also used to model the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules. Many analytical techniques have been considered by various researchers to obtain the approximate solutions for these types of equations [15-18].

Perturbation iteration method has been developed recently by Pakdemirli et al and it has been successfully implemented to some strongly nonlinear systems [19-22]. In this paper, new optimal perturbation iteration algorithms (OPIA) are constructed based on perturbation iteration method. Then the simplest OPIA is applied to obtain reliable approximate solutions to the Lane-Emden equations

\[ y'' + \frac{2}{x}y' + \beta(x)\gamma(y) = 0 \]  

(1.4)

with different choices of \( \beta(x), \gamma(y) \). It is also proved that this method enables us to control the convergence of solution series for the given illustrations.

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Key words: Optimal perturbation iteration method, singular initial value problems, Lane-Emden equation.

Mathematics Subject Classification: 34E20, 34A12, 85A04
2 Optimal Perturbation Iteration Algorithms

In this section, the following formulation is given to explain the basic concept of OPIAs for second order differential equations.

(a) Write the governing differential equation as:

\[ F(y'', y', y, \varepsilon) = A(y) + g(x) = 0 \tag{2.1} \]

where \( \varepsilon \) is the auxiliary perturbation parameter, \( y = y(x) \) is the unknown function and \( g(x) \) is the source term. Furthermore, Eq. \( 2.1 \) can be decomposed into \( A(y) = L(y) + N(y) \) where \( L, N \) are linear and nonlinear parts respectively. Here we have a great freedom to choose linear part \( L \).

(b) Approximate solution is taken as

\[ y_{n+1} = y_n + \varepsilon (y_c)_n \tag{2.2} \]

with one correction term in the perturbation expansion. Substituting the Eq. \( 2.2 \) into \( 2.1 \) and expanding Taylor series will create the algorithms. OPIAs are classified with respect to the degrees of derivatives in the Taylor expansions \( m \). Briefly, this process is represented as OPIA-\( m \). OPIA-1 and OPIA-2 are constructed below.

OPIA-1

Inserting \( 2.2 \) into the nonlinear part of the \( 2.1 \) and expanding in a Taylor series with first derivatives yields

\[ N(y_n'', y_n', y_n, 0) + N_y(y_c)_n \varepsilon + N_{y'}(y_c')_n \varepsilon + N_{y''}(y_c'')_n \varepsilon + N_c \varepsilon = 0 \tag{2.3} \]

or by reorganizing

\[ (y_c'')_n + \frac{N_{y''}}{N_{y''}} (y_c')_n + \frac{N_y}{N_{y''}} (y_c)_n = -\frac{N_c}{N_{y''}}. \tag{2.4} \]

OPIA-2

Using the second order derivatives, we get

\[ N(y_n'', y_n', y_n, 0) + N_y(y_c)_n \varepsilon + N_{y'}(y_c')_n \varepsilon + N_{y''}(y_c'')_n \varepsilon + N_c \varepsilon + \frac{1}{2} \varepsilon^2 N_{y''y'}(y_c')_n^2 + \frac{1}{2} \varepsilon^2 N_{y'y''}(y_c)_n^2 + \varepsilon^2 N_{y''y'}(y_c')_n (y_c')_n + \varepsilon^2 N_{y'y''}(y_c')_n (y_c)_n + N_{y''y'}(y_c')_n \varepsilon^2 + N_{y'y''}(y_c')_n \varepsilon^2 + \varepsilon^2 N_{y''y''}(y_c)_n \varepsilon^2 + \varepsilon^2 N_{y'y''}(y_c)_n \varepsilon^2 = 0 \tag{2.5} \]

or by rearranging we obtain

\[ (y_c'')_n \left( \varepsilon N_{y''} + \varepsilon^2 N_{y''y'} \right) + (y_c')_n \left( \varepsilon N_{y'} + \varepsilon^2 N_{y'y''} \right) + (y_c)_n \left( \varepsilon N_y + \varepsilon^2 N_{y'y''} \right) + (y_c')_n \left( \frac{\varepsilon^2 N_{y'y''}}{2} \right) + (y_c)_n \left( \frac{\varepsilon^2 N_{y''y'}}{2} \right) \tag{2.6} \]

It should be emphasized that all derivatives and functions are calculated at \( \varepsilon = 0 \). To describe the iterative scheme, first correction terms \( (y_c)_0 \) can be computed from \( 2.4 \) and \( 2.6 \) by using an initial guess \( y_0 \).

The Eqs. \( 2.4 \) and \( 2.6 \) may seem very complicated at first, but it should not be forgotten that we use the general form of the differential equations of second order to illustrate the proposed method. Actually, most differential equations in literature contain only some of the nonlinear terms \( y, y', y'' \). So, the Eqs. \( 2.4 \) and \( 2.6 \) reduce to some simple mathematical expressions in many cases.

c) After obtaining \( (y_c)_0 \) for each OPIA-\( m \), we use the following equation

\[ y_{n+1} = y_n + S_n(\varepsilon)(y_c)_n \tag{2.7} \]

to increase the accuracy of the results and effectiveness of the method. Here \( S_n(\varepsilon) \) is an auxiliary function which provides us to adjust the convergence. The choices of functions \( S_n(\varepsilon) \) could be exponential, trigonometric, polynomial, etc. In this study, we select auxiliary function in the form

\[ S_n(\varepsilon) = C_0 + \varepsilon C_1 + \varepsilon^2 C_2 + \varepsilon^3 C_3 + \cdots = \sum_{i=0}^{n} \varepsilon^i C_i \tag{2.8} \]
where $C_1, C_2, \ldots$ are constants which are to be determined later. Substituting $(y_c)_0$ into Eq. (2.7) yields:

$$y_1 = y_0 + S_0(\varepsilon)(y_c)_0 = y_0 + C_0(y_c)_0.$$ (2.9)

By using initial and boundary conditions, the first approximate result is obtained:

$$y_1 = y(x, C_0).$$ (2.10)

Repeating the similar steps by using the Eq. (2.10) and setting $\varepsilon = 1$, other approximate results are found as:

$$y_2(x, C_0, C_1) = y_1 + S_1(\varepsilon)(y_c)_0 = y_1 + (C_0 + C_1)(y_c)_1$$
$$y_3(x, C_0, C_1, C_2) = y_2 + (C_0 + C_1 + C_2)(y_c)_2$$
$$\vdots$$
$$y_m(x, C_0, \ldots, C_{m-1}) = y_{m-1} + (C_0 + \cdots + C_{m-1})(y_c)_{m-1}$$ (2.11)

d) Substitute $y_m$ into the Eq. (2.1) and the general problem results in the following residual:

$$R(x, C_1, \ldots, C_{m-1}) = L(y_m(x, C_1, \ldots, C_{m-1})) + N(y_m(x, C_1, \ldots, C_{m-1})) + g(x).$$ (2.12)

Obviously, when $R(x, C_1, \ldots, C_{m-1}) = 0$ then the approximation $y_m(x, C_1, \ldots, C_{m-1})$ will be the exact solution. Generally it doesn’t occur in nonlinear equations. To obtain the optimum values of $C_1, C_2, \ldots$; one may use the method of least squares:

$$J(C_1, \ldots, C_{m-1}) = \int_a^b R^2(x, C_1, \ldots, C_{m-1})dx$$ (2.13)

where $a$ and $b$ are selected from the domain of the problem. The constants $C_1, C_2, \ldots$ can also be defined from

$$R(x_1, C_i) = R(x_2, C_i) = \cdots = R(x_m, C_i) = 0, i = 1, 2, \ldots, m$$ (2.14)

where $x_i \in (a, b)$. Putting these constants into the last one of the Eqs. (2.11), the approximate solution of order $m$ is determined.

### 3 Applications

In this section, we only consider OPIA-1 to solve the problems. Obtained results show that even the simplest algorithm gives better solutions when compared with many other methods.

**Example 3.1** Consider the following homogeneous Lane-Emden type equation [23, 24]:

$$y'' + \frac{2}{x}y' - (4x^2 + 6)y = 0, y(0) = 1, y'(0) = 0, 0 \leq x \leq 1.$$ (3.1)

Rewrite the Eq. (3.1) as:

$$F(y'', y', y, \varepsilon) = y'' + \varepsilon \left(\frac{2}{x}y' - (4x^2 + 6)y\right) = L(y) + N(y) = 0$$ (3.2)

where $L = y''$ and $N(y) = \varepsilon \left(\frac{2}{x}y' - (4x^2 + 6)y\right)$. Eq. (2.3) simplifies to:

$$N(y_n, 0) + N_y(y_c)_n \varepsilon + N_{y'}(y_c')_n \varepsilon + N_{\varepsilon} \varepsilon = 0$$ (3.3)

Using the linear terms with Eqs. (2.2), (3.3) and setting $\varepsilon = 1$ yields

$$(y_c')'' = -(y_n)'' - \frac{2}{x}(y_n)' + (4x^2 + 6)y_n.$$ (3.4)

One may select $y_0 = 1$ as a starting guess which satisfies the given initial conditions. Using the Eqs. (2.7) and (3.4), we have
It should be emphasized that $y_1$ does not represent the first correction term; rather it is the approximate solution after the first iteration. By using the procedure mentioned in Section 2, one obtains the following approximate solutions:

$$y_1 = 1 + C_0 \left( \frac{x^4}{3} + 3x^2 \right).$$  \hspace{1cm} (3.5)

for the third order approximation. Using the Eq. (2.14) with $x = 0.3, 0.6, 0.9$ yields

$$C_0 = 0.3342343984217452, \quad C_1 = 0.31859877627965916, \quad C_2 = 0.20764389922289617$$

Substituting these constants into the Eq. (3.7), the approximate solution of the third order is obtained as:

$$y_3 = 1 + C_0 \left( \frac{x^4}{3} + 3x^2 \right) + (C_0 + C_1) \left[ \frac{x^2}{630} \left( 15C_0x^6 + 294C_0x^4 + 595C_0x^2 - 5670C_0 + 210x^2 + 1890 \right) \right] + \frac{(C_0 + C_1 + C_2)x^2}{127600} \times \left[ 525C_0^2x^{10} + 525C_0C_1x^{10} + 16247C_0^2x^8 + 16247C_0C_1x^8 + 63195C_0^2x^6 + 34650C_0x^6 + 63195C_0C_1x^6 + 17325C_1x^6 - 121113C_0^2x^4 + 679140C_0x^4 - 121113C_0C_1x^4 + 1374450C_0x^2 - 4419800C_0x^2 - 4419800C_0C_1x^2 + 687225C_1x^2 + 19646550C_0C_1 - 6548850C_1 + 242550x^2 + 2182950 \right]$$

Unknown constants can be obtained from the residual

$$R(x, C_0, C_1, C_2) = L(y_3(x, C_0, C_1, C_2)) + N(y_3(x, C_0, C_1, C_2))$$

Exact solution of the problem is $y = e^{x^2}$. This problem has been investigated by Ozis et al using variational iteration method (VIM) and homotopy perturbation method (HPM) [23, 24]. Figures 3.1, 3.2 and Table 1 give important information on the convergence of the absolute errors of OPIAs and other approximate solutions in literature. It is clear that the results obtained by OPIM are more accurate than those in [23, 24].

**Example 3.2** Consider the Lane-Emden type equation [12, 23, 26]:

$$y'' + \frac{2}{x}y' + e^y = 0, \quad y(0) = y'(0) = 0$$

which represents the isothermal gas spheres equation in the case that the temperature stays constant.

Reconsider the Eq. (3.13) as:

$$F(y'', y', y, \varepsilon) = y'' + \left( \frac{2e}{x}y' + e^y \right) = L(y) + N(y) = 0$$

where $L = y''$ and $N(y) = \left( \frac{2e}{x}y' + e^y \right)$. By using the Eqs. (2.2), (3.3), (3.14) and setting $\varepsilon = 1$, we have
Table 1: Comparison of absolute errors for Example 1 at different orders of approximations.

| $x$ | Errors for OPIA-1 |                  | Errors for VIM and HPM |                  |
|-----|------------------|------------------|------------------------|------------------|
|     | $|y_{exact} - y_4|$ | $|y_{exact} - y_5|$ | $|y_{exact} - y_4|$ | $|y_{exact} - y_5|$ |
| 0.1 | $3.08426 \times 10^{-13}$ | $2.22045 \times 10^{-16}$ | $1.11022 \times 10^{-15}$ | $1.00128 \times 10^{-16}$ |
| 0.2 | $3.85914 \times 10^{-13}$ | $2.22045 \times 10^{-16}$ | $5.72165 \times 10^{-12}$ | $3.26406 \times 10^{-14}$ |
| 0.3 | $5.62885 \times 10^{-13}$ | $1.00025 \times 10^{-17}$ | $7.47710 \times 10^{-10}$ | $9.59788 \times 10^{-12}$ |
| 0.4 | $9.64347 \times 10^{-13}$ | $2.44249 \times 10^{-15}$ | $2.38451 \times 10^{-8}$ | $5.43454 \times 10^{-10}$ |
| 0.5 | $1.95532 \times 10^{-12}$ | $1.50998 \times 10^{-14}$ | $3.51584 \times 10^{-7}$ | $1.24994 \times 10^{-8}$ |
| 0.6 | $4.77649 \times 10^{-12}$ | $1.01037 \times 10^{-13}$ | $3.18608 \times 10^{-6}$ | $1.62772 \times 10^{-7}$ |
| 0.7 | $5.45375 \times 10^{-11}$ | $5.02709 \times 10^{-13}$ | $0.0000206568$ | $1.43282 \times 10^{-6}$ |
| 0.8 | $2.78031 \times 10^{-11}$ | $2.05902 \times 10^{-12}$ | $0.000104921$ | $9.47740 \times 10^{-6}$ |
| 0.9 | $3.08426 \times 10^{-10}$ | $6.38223 \times 10^{-12}$ | $0.000042699$ | $0.000050436$ |
| 1   | $2.10201 \times 10^{-9}$ | $2.90235 \times 10^{-12}$ | $0.000161516$ | $0.000226273$ |

Figure 3.1: Comparison with the exact solution(–) for Example 1.

(a) OPIA-1 solution of second order. (b) VIM solution of second order.

Figure 3.2: Absolute errors for OPIA-1 and the variational iteration method for Example 1.

(a) OPIA-1 solution of fifth order. (b) VIM solution of fifth order.
(y_n)'' - \frac{2}{x}(y_n)' - y_n - 1. \tag{3.15}

By choosing \( y_0 = 0 \) as a starting guess, using the Eqs. (2.7), (3.14) with initial conditions, we get

\[ y_1 = \frac{-C_0}{2} (x^2) \tag{3.16} \]

\[ y_2 = \frac{-C_0}{2} (x^2) + \frac{1}{24} (C_0 + C_1) x^2 (C_0x^2 + 36C_0 - 12) \tag{3.17} \]

\[ y_3 = \frac{-C_0}{2} (x^2) + \frac{1}{24} (C_0 + C_1) x^2 (C_0x^2 + 36C_0 - 12) + \frac{C_0 + C_1 + C_2}{720} \left[ -C_0^2x^6 - 5C_0C_1x^6 - 140C_0^2x^4 + 60C_0x^4 - 140C_0C_1x^4 + 30C_1x^4 \right] \tag{3.18} \]

\[ y_4 = \frac{-C_0}{2} (x^2) + \frac{1}{24} (C_0 + C_1) x^2 (C_0x^2 + 36C_0 - 12) + \frac{C_0 + C_1 + C_2}{720} \left[ -C_0^2x^6 - C_0C_1x^6 - 140C_0^2x^4 + 60C_0x^4 - 140C_0C_1x^4 + 30C_1x^4 \right] + \frac{C_0 + C_1 + C_2 + C_3}{604800} \]

\[ \begin{array}{c}
15C_0^3x^8 + 15C_0C_1^2x^8 + 30C_0^2C_1x^8 + 15C_0^3C_2x^8 + 15C_0C_1C_2x^8 + 5096C_0^3x^6 - 2520C_0^2x^6 + 50C_1C_2x^6 - 840C_1^2x^6 + 10192C_2^3x^6 - 3360C_1x^6 + 5056C_2^3x^6 - 1680C_2x^6 + 5096C_0C_1C_2x^6 - 5096C_0C_1C_2x^6 - 422800C_0^2x^4 - 3252800C_0^2x^4 + 422800C_0C_1^2x^4 - 117600C_1^2x^4 + 75600C_0C_1x^4 + 845600C_0C_1x^4 - 470400C_1x^4 + 50400C_1^2x^4 + 422800C_0C_2^3x^4 - 2352C_2x^4 + 422800C_0C_1C_2x^4 - 117600C_1C_2x^4 + 25200C_2^3x^4 + 8164800C_0^3x^2 - 8164800C_2^3x^2 + 816C_0C_1C_2x^2 - 2721600C_2x^2 + 2721600C_0x^2 + 1639200C_0C_1x^2 - 10886400C_1x^2 + 1814400C_1x^2 - 5443200C_2x^2 + 8164800C_0C_1C_2x^2 - 2721600C_1C_2x^2 + 907200C_2x^2 - 302400x^2 \end{array} \tag{3.19} \]

Using the Eq. (2.14) with the residual

\[ R(x, C_0, C_1, C_2, C_3) = L \{ y_4(x, C_0, C_1, C_2, C_3) \} + N \{ y_4(x, C_0, C_1, C_2, C_3) \} \tag{3.20} \]

the unknown constants are obtained as

\[ C_0 = 2.0203622551, C_1 = -1.0201147822, C_2 = -0.9963202221, C_3 = 0.020789994 \tag{3.21} \]

for the fourth order approximation. Substituting the Eq. (3.21) into (3.19) yields:

\[ y_4(x) = -0.15989962328x^2 + 0.007920058473x^4 + -0.005608897215631x^6 + 0.000039055217456x^8. \tag{3.22} \]

Figure 3.3 is sketched for comparison of the higher order approximate solutions of OPIM and analytical results given by other methods in literature [25, 27]. It is also clear from the Figure 3.3 that OPIA-1 solutions are valid in larger region.

4 Conclusion

In this paper, OPIM is applied for the first time to investigate the new approximate solutions for Lane-Emden type differential equations. This new technique provides us to optimally control the convergence of solution series. Also, it gives a very good approximation even in a few terms to these kinds of nonlinear equations. The results obtained in this paper proves that the OPIM is a very effective technique for differential equations. It is worth mentioning that, a symbolic program is necessary for successive calculations after a few iterations. Mathematica 9 has been used to overcome the complicated calculations for this present research.

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Figure 3.3: Comparison between the results obtained by OPIM, ADM, DTM and the numerical results for Example 2.

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