ISOPERIMETRIC INEQUALITIES FOR MINIMAL
SUBMANIFOLDS IN RIEMANNIAN MANIFOLDS:
A COUNTEREXAMPLE IN HIGHER CODIMENSION

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Abstract.
For compact Riemannian manifolds with convex boundary, B. White proved
the following alternative: Either there is an isoperimetric inequality for mini-
mal hypersurfaces or there exists a closed minimal hypersurface, possibly with
a small singular set. There is the natural question if a similar result is true for
submanifolds of higher codimension. Specifically, B. White asked if the non–
existence of an isoperimetric inequality for \( k \)-varifolds implies the existence of
a nonzero, stationary, integral \( k \)-varifold. We present examples showing that
this is not true in codimension greater than two. The key step is the construc-
tion of a Riemannian metric on the closed four–dimensional ball \( B^4 \) with the
following properties: (1) \( B^4 \) has strictly convex boundary. (2) There exists a
complete nonconstant geodesic \( c : \mathbb{R} \to B^4 \). (3) There does not exist a closed
geodesic in \( B^4 \).

1. Introduction

If \( D \) is a two-dimensional Riemannian disc with locally convex boundary \( \partial D \) and
if there is no closed geodesic in \( D \), then there is a constant \( C > 0 \) such that every
geodesic segment in \( D \) has length at most \( C \). An equivalent formulation of this
fact is: If there exists a nonconstant geodesic \( c : \mathbb{R} \to D \), then \( D \) contains a closed
geodesic. This fact is due to Birkhoff, cf. [2, VI. 10], and played a role in the proof
that there exist infinitely many closed geodesics on every Riemannian 2-sphere, cf.
[1] and [5].

In arbitrary dimensions, an analogous result has been proven by B. White [8, Theorem 2.1] in the codimension one situation, i.e. when geodesics are replaced
by minimal hypersurfaces. As part of the opening colloquium of the collaborative
research center SFB/Transregio 71 in Freiburg, April 2009, B. White lectured on
this result, and posed the question if there could be a version of the result that is
not restricted to the codimension one case, see also [8, Remark 2.8].

Here, we construct a Riemannian metric \( g \) on the closed four-dimensional ball
\( B^4 \) such that \( \partial B^4 \) is strictly convex and such that \( B^4 \) carries a complete geodesic,
but no closed geodesic. Actually one would expect that such an example exists
already on the closed 3-ball. We believe that this is the case, but our construction
would be considerably more complicated.
Now we explain how this can be used to answer B. White’s question [8, Remark 2.8], that explicitly asks:

Let $N$ be a compact, $k$-convex Riemannian manifold containing a nonzero, stationary $k$-varifold. Does this imply that $N$ contains an integral stationary $k$-varifold?

For more details on this question see Section 4.

Taking the Riemannian product of an arbitrary closed Riemannian manifold $M$ with our example $(B^4, g)$ we obtain a compact manifold $\tilde{M}$ of dimension $\tilde{m} \geq 4$. This $\tilde{M}$ has convex boundary. So $\tilde{M}$ is $k$-konvex for every $k < \tilde{m}$. The product of $M$ with a complete geodesic in $B^4$ gives an $(\tilde{m} - 3)$-dimensional minimal submanifolds in $\tilde{M}$. Hence, from [8, Theorem 2.3] we know that there exists a nonzero, stationary $(\tilde{m} - 3)$-varifold in $\tilde{M}$. Indeed, we can describe explicitly such a varifold $V_0$ in $M$, and prove that up to scale $-V_0$ is the only stationary $(\tilde{m} - 3)$-varifold in $\tilde{M}$.

From the explicit description of $V_0$ we conclude that $V_0$ is not rectifiable and, hence, not integral. This gives a negative answer to B. White’s question for the case of varifolds of arbitrary dimension, and codimension at least three.

Finally we sketch the idea underlying the construction of the metric $g$ on $B^4$. First, we deform the standard metric $g_0$ on the ball $B^4 \subset \mathbb{R}^4$ of radius 2 so that all the spheres $S^3(\rho) \subset B^4$ of radius $\rho \in [0, 2]$ remain strictly convex, except for $S^3(1)$ whose second fundamental form vanishes precisely on the vectors tangent to an irrational geodesic foliation $\mathcal{F}$ of the Clifford torus $T^2 \subset S^3(1)$. This implies that there are no closed geodesics in $B^4$ with respect to this metric. Moreover, we achieve that also the second fundamental form of the Clifford torus $T^2$ vanishes in the direction of $\mathcal{F}$. Then the leaves of $\mathcal{F}$ are complete geodesics not only in $T^2$ but also with respect to the metric on $B^4$.

## Contents

1. Introduction 1
2. Convex Distance Functions 2
3. The Example 3
4. An answer to a question by Brian White 6
5. Appendix 10
6. References 11

## 2. Convex Distance Functions

In this section we will recall some well known facts about geodesics and distance functions. Let $(M, g)$ denote a Riemannian manifold and $i : N \hookrightarrow M$ a submanifold. We will denote the induced metric on $N$ by $g^N$. Then a curve $c : I \subset \mathbb{R} \rightarrow N$ is a $g$-geodesic if and only if $c$ is a $g^N$-geodesic and the second fundamental form of $N$ vanishes on its tangent vectors.

Now, let $F : N \times (-\varepsilon, \varepsilon) \rightarrow M$ be a normal variation with variational vector field $V = \frac{dF}{dt}_{|t=0}$ along $i = F(\cdot, 0)$. Then, for any tangent vectors $v_1, v_2 \in T_pN$, one
calculates, cf. \cite{3} (1.33)],
\[
\frac{d}{dt}_{t=0} (F^*_t g)(v_1, v_2) = g(\nabla_{v_1} V, v_2) + g(v_1, \nabla_{v_2} V),
\]
(1)
where $F_t : N \to M$ is defined by $F_t(\cdot) = F(\cdot, t)$.

If, additionally, $|V| = 1$, it follows from equation (1), that the second fundamental form $h^N(\cdot, \cdot)$ of $N$ with respect to $V$ is given by
\[
h^N(v_1, v_2) = \frac{1}{2} \frac{d}{dt}_{t=0} (F^*_t g)(v_1, v_2).
\]
(2)

We will use this fact in the special case where $N$ is a level set of a $C^\infty$-function $d$ with $|\text{grad } d| = 1$. These functions will be called distance functions, cf. \cite{3} 2.3.1]. Then the restriction of the gradient flow $\Phi_t$ to $N$ is a normal variation with variational vector field $V = \text{grad } d$. The gradient of $d$ is contained in the null space of the Hessian $\nabla^2 d$ and for any $v_1, v_2 \in T_p N$ one obtains
\[
\nabla^2 d(v_1, v_2) = g(\nabla_{v_1} V, v_2) = -h^N(v_1, v_2).
\]
(3)

Hence a distance function is a convex function if the second fundamental form (with respect to grad $d$) of any of its level sets is everywhere negative semidefinite. Recall that a $C^2$-function $f : M \to \mathbb{R}$ is convex if one of the following equivalent conditions is satisfied:

- For any geodesic segment $c : I \to M$ the composition $f \circ c : I \to \mathbb{R}$ is convex.
- The Hessian $\nabla^2 f$ is everywhere positive semidefinite.

In particular, we have:

\textbf{Fact 1.} Let $f : M \to \mathbb{R}$ be a convex function. Then any closed geodesic in $M$ is contained in one of the level sets of $f$. If the second fundamental form of a smooth level set of $f$ is definite at some point, then there is no closed geodesic passing through this point. \hfill $\square$

Therefore there are no closed geodesics on a manifold that is equipped with a convex distance function, if its Hessian restricted to the tangent spaces of the level sets is everywhere definite.

3. The Example

Consider the closed standard 4-ball $(B^4, g_0)$ with radius 2 and the Clifford torus $(T^2, g_0^{T^2})$ given by \{\sqrt{2} (\sin \varphi, \cos \varphi, \sin \theta, \cos \theta) \mid \varphi, \theta \in [0, 2\pi]\}. The Clifford torus is a flat torus that is isometrically embedded in the standard sphere $S^3 \subset B^4$. The map $(\varphi, \theta) \in \mathbb{R}^2 \to \frac{1}{\sqrt{2}}(\sin \varphi, \cos \varphi, \sin \theta, \cos \theta)$ from euclidean $\mathbb{R}^2$ to the Clifford torus $T^2$ is a homothetic covering map. The projection to $T^2$ of a family of parallel lines in $\mathbb{R}^2$ will be called a geodesic foliation of $T^2$. A geodesic foliation of $T^2$ is called rational if the corresponding family of parallels has rational slope and irrational otherwise. The geodesics of a rational foliation of $T^2$ are all closed, while the geodesics of an irrational geodesic foliation of $T^2$ are all dense on $T^2$.

The metric $g$ that we will define on $B^4$ will have the following properties:

(G1) The induced metric $g^{T^2}$ on $T^2$ is the flat one induced by $g_0$.
(G2) The function $d : B^4 \to [0, 2]$ given by the euclidean distance to zero is a convex distance function with respect to the metric $g$.,
(G3) There exists an irrational geodesic foliation $\mathcal{F}$ of $T^2$ such that the following holds for the hessian $\nabla^2(d^2)$ with respect to $g$: $\nabla^2(d^2)|_x$ is positive definite for all $x \in B^4 \setminus T^2$, and for $x \in T^2$ the nullspace of $\nabla^2(d^2)|_x$ coincides with the tangent line to $\mathcal{F}$ at $x$.

(G4) The second fundamental form of the Clifford torus $T^2$ as a submanifold of $S^3$ vanishes on the vectors tangent to the irrational geodesic foliation $\mathcal{F}$ of $T^2 \subset S^3$.

From (G2), (G3) and equation (3) we conclude

(G3') For any sphere $S^3(\rho) = d^{-1}(\rho)$, $\rho \in [0, 2] \setminus \{1\}$, the second fundamental form $h^{S^3}(\rho)$ with respect to grad $d$ is negative definite, and on $S^3 = S^3(1)$ the zero directions of $h^{S^3}$ are precisely the vectors tangent to the irrational geodesic foliation $\mathcal{F}$.

Now we will prove

**Proposition 1.** Suppose $g$ is a Riemannian metric on $B^4$ satisfying conditions (C1)-(C4). Then there exists a complete (non-constant) $g$-geodesic $c : \mathbb{R} \to B^4$, but no closed $g$-geodesic in $B^4$. Moreover, $\partial B^4 = S^3(2)$ is strictly convex.

**Proof.** Note first that by conditions (G3) and (H) the geodesics of the irrational foliation are complete $g$-geodesics contained in $T^2 \subset B^4$, cf. the discussion at the beginning of Section 2. Next we will show that there are no closed $g$-geodesics in $B^4$. So, let us assume that there exists a closed $g$-geodesic $c : S^1 \to B^4$. Using properties (C2), (G3) and Fact 1 we conclude that $c$ lies in the euclidean sphere $S^3$ and that $c$ is a leaf of the irrational foliation $\mathcal{F}$ of $T^2 \subset S^3$. This contradicts our assumption that $c$ is closed.

Now we describe how one can construct a Riemannian metric $g$ on $B^4$ that satisfies properties (C1)-(C4). We consider the coordinate system

$$F : [0, 2[ \times [0, \pi/2[ \times \mathbb{R}^2 \longrightarrow B^4$$

$$(\rho, \psi, \varphi, \theta) \longrightarrow \begin{pmatrix} \rho \cos \psi \sin \varphi \\ \rho \cos \psi \cos \varphi \\ \rho \sin \psi \sin \theta \\ \rho \sin \psi \cos \theta \end{pmatrix}. \quad (4)$$

For $\rho = 1$ and $\psi = \pi/4$ the coordinates $\varphi$ and $\theta$ describe the Clifford torus, i.e. $T^2 = F([1] \times \{\pi/4\} \times \mathbb{R}^2)$. We denote the induced coordinate vectors on $\text{im}(F) := F([1] \times [0, \pi/2[ \times \mathbb{R}^2)$ by $\partial_\rho$, $\partial_\psi$, $\partial_\varphi$, $\partial_\theta$. They form a $g_0$-orthogonal frame on $\text{im}(F)$, and the metric $g_0$ is given in these coordinates by the diagonal matrix

$$\text{diag}(1, \rho^2, \rho^2 \cos^2 \psi, \rho^2 \sin^2 \psi).$$

This shows, in particular, that $F|_{[1] \times \{\pi/4\} \times \mathbb{R}^2}$ is - up to the constant factor $1/\sqrt{2}$ - an isometric covering map with group of deck transformation $2\pi \mathbb{Z} \times 2\pi \mathbb{Z}$. For fixed $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ we consider the vectorfield $Y = \partial_\varphi + \alpha \partial_\theta$. The restriction of $Y$ to the torus $T^2$ is tangent to an irrational geodesic foliation and the vector field $Z := \alpha \tan \psi \partial_\varphi - \cot \psi \partial_\theta$ completes $\partial_\rho$, $\partial_\psi$ and $Y$ to an orthogonal frame on $\text{im}(F)$. We define a new metric $g$ on $\text{im}(F)$ by requiring that the vectorfields $\partial_\rho$,
\( \partial_{\psi}, Y \) and \( Z \) are pairwise \( g \)-orthogonal and by setting for \( x = F(\rho, \psi, \varphi, \theta) \)
\[
g(\partial_{\rho}, \partial_{\rho}) \mid x = g_0(\partial_{\rho}, \partial_{\rho}) \mid x = 1,
\]
\[
g(\partial_{\psi}, \partial_{\psi}) \mid x = g_0(\partial_{\psi}, \partial_{\psi}) \mid x = \rho^2,
\]
\[
g(Y,Y) \mid x = R(\rho, \psi),
\]
\[
g(Z,Z) \mid x = g_0(Z,Z) \mid x = \rho^2(\cos^2 \psi + \alpha^2 \sin^2 \psi),
\]
where the function \( R \in C^\infty([0,2[ \times ]0, \pi/2[; \mathbb{R}^+) \) is chosen such that the following conditions are fulfilled:

(R1) \( R(\rho, \psi) = \rho^2(\cos^2 \psi + \alpha^2 \sin^2 \psi) \),

if \((\rho, \psi) = (1, \pi/4)\) or \((\rho, \psi) \in ([0,2[ \times ]0, \pi/2[) \setminus ([1/2,3/2] \times [\pi/8,3\pi/8])\)

(R2) \( \frac{\partial}{\partial \rho} R(\rho, \psi) > 0 \) if \((\rho, \psi) \neq (1, \pi/4)\)

(R3) \( \frac{\partial}{\partial \rho} R(1, \pi/4) = 0 \)

For completeness, we will construct such a function \( R \) in the appendix. First note that condition (R1) ensures that \( g \) coincides with the standard metric \( g_0 \) outside the tubular neighborhood of \( T^2 \) given by the image of \([1/2,3/2] \times [\pi/8,3\pi/8] \times \mathbb{R}^2 \) under \( F \). Therefore, the standard metric extends \( g \) to a smooth metric on all of \( B^4 \).

**Proposition 2.** The metric \( g \) fulfills conditions (C1)-(C4).

**Proof.** First note that condition (C1) follows from condition (R1). Next, our definition of \( g \) directly implies that \( \partial_{\rho} \) is the \( \rho \)-gradient of the euclidean distance \( d \) from zero. Hence \( d \) is a distance function also with respect to \( g \). Thus, by the discussion in Section 2, we can calculate its \( g \)-Hessian on \( \text{im}(F) \) with equations (2) and (3). As \( g \mid_{\text{im}(F)} = \partial_{\rho} \) commutes with \( \partial_{\psi}, Y \) and \( Z \), we obtain for \( V, W \in \{\partial_{\psi}, Y, Z\} \)
\[
\frac{d}{dt}_{|t=0} (\Phi_t^* g)_{|p}(V,W) = \frac{d}{dt}_{|t=0} g_{\Phi_t(p)}(V_{|\Phi_t(p)}, W_{|\Phi_t(p)}),
\]
where \( \Phi_t \) denotes the gradient flow of \( d \). Remember that on \( \text{im}(F) \) the flow lines of \( \Phi \) are the \( \rho \)-coordinate lines. Using the preceding equation and equations (2) and (3) we see that on \( \text{im}(F) \) the matrix of the \( g \)-Hessian of \( d \) with respect to the frame \( \partial_{\rho}, \partial_{\psi}, Y, Z \) is the diagonal matrix given by

\[
\text{diag}
\left(0, \rho, \frac{1}{2} \frac{\partial}{\partial \rho} R(\rho, \psi), \rho(\cos^2 \psi + \alpha^2 \sin^2 \psi)\right).
\]

Now, condition (C3) follows immediately from (R2) and (R3). Since the metric coincides with the standard metric in a neighborhood of \( B^4 \setminus \text{im}(F) \) and the Hessian of \( d \) is positive semidefinite on \( \text{im}(F) \), the function \( d \) is convex everywhere. So, also condition (C2) is proven. Finally, to prove (C4), we consider the projection \( \pi_{\psi} : S^3 \cap \text{im}(F) \to ]0, \pi/2[; \quad F(1, \psi, \varphi, \theta) \mapsto \psi \). This provides a distance function with gradient \( \partial_{\psi} \) whose gradient flowlines are given by the coordinate lines of \( \psi \). Now we calculate the second fundamental form \( h^S \) of \( T^2 \) in \( S^3 \) with respect to \( \partial_{\psi} \), using equation (2). Then \( Y, \partial_{\psi} \mid 0 = 0 \) and condition (R3) imply:
\[
h^S(Y, Y) = -\frac{1}{2} \frac{\partial}{\partial \psi} R(1, \pi/4) = 0.
\]
This completes the proof. \( \square \)
Remark 1. The construction above can easily be generalized to balls $B$ of dimension $n \geq 5$. The construction yields a Riemannian metric on $B$ fulfilling properties (G1)-(G4) with the obvious modifications of the dimension.

4. An answer to a question by Brian White

As mentioned in the introduction, our example is related to isoperimetric inequalities in Riemannian manifolds. Brian White [8] showed that an isoperimetric inequality holds for minimal hypersurfaces (or -more generally- for codimension one varifolds) in a compact, connected Riemannian manifold $\tilde{M}$ with mean-convex boundary if $\dim(\tilde{M}) < 7$ and if there does not exist a smooth, closed, embedded minimal hypersurface $N \subset \tilde{M}$ (The same conclusion is true if $\dim(\tilde{M}) \geq 7$, provided one replaces “smooth” by “smooth except for a singular set of Hausdorff dimension at most $\dim(\tilde{M}) - 7$”).

An isoperimetric inequality in higher codimension is obtained in [8, Theorem 2.3] under the stronger condition, that there does not exist any nonzero, stationary $k$-varifold in a compact, $k$-convex Riemannian manifold $N$ implies the existence of a nonzero, stationary, integral $k$-varifold in $N$. For a brief introduction to varifolds on Riemannian manifolds see [8, Appendix].

In the following Proposition we answer this question in the negative for codimension larger than 2. Starting with an arbitrary closed, connected, $m$-dimensional Riemannian manifold $(M, g')$ we consider the product metric $\tilde{g} = g' \oplus g$ on $\tilde{M} = M \times B$, where $B$ is a closed ball of dimension $n \geq 4$ and $g$ a Riemannian metric on $B$ fulfilling (G1)-(G4), cf. Section 3. Then $\partial \tilde{M} = M \times \partial B$ has the following convexity property. The second fundamental form of $\partial \tilde{M}$ with respect to the inward pointing unit normal is positive semi-definite, and its kernel consists of the vectors tangent to the factor $M$. In Proposition 3 we will show that $(\tilde{M}, \tilde{g})$ contains a unique stationary, $(m+1)$-dimensional varifold $V_0$ of unit mass, and, in Fact 2, that $V_0$ is not rectifiable and, hence, not integral. This provides a negative answer to the question posed in [8, Remark 2.8]. It is easy to see (and follows from Proposition 3) that in our case there is a unique limit varifold and that this is equal to $V_0$.

Next we describe the $(m+1)$-varifold $V_0$ in $\tilde{M}$: A general $(m+1)$-varifold in $\tilde{M}$ is a finite Borel measure on the total space of the Grassmann bundle $\pi : G_{m+1}(\tilde{M}) \to \tilde{M}$. The support of $V_0$ is the subset $\tilde{\mathcal{F}}$ of $G_{m+1}(\tilde{M})$ given by

$$\tilde{\mathcal{F}} = \{ T_p M \times T_q \mathcal{F} \mid (p, q) \in M \times \mathbb{T}^2 \},$$

Remark 2. According to B. White’s proof of [8, Theorem 2.3] any limit of the varifolds induced by the $M_n$, normalized so as to have mass one, is a non-zero, stationary, $(m+1)$-dimensional varifold. It is easy to see (and follows from Proposition 3) that in our case there is a unique limit varifold and that this is equal to $V_0$. 

Next we describe the $(m+1)$-varifold $V_0$ in $\tilde{M}$: A general $(m+1)$-varifold in $\tilde{M}$ is a finite Borel measure on the total space of the Grassmann bundle $\pi : G_{m+1}(\tilde{M}) \to \tilde{M}$. The support of $V_0$ is the subset $\tilde{\mathcal{F}}$ of $G_{m+1}(\tilde{M})$ given by

$$\tilde{\mathcal{F}} = \{ T_p M \times T_q \mathcal{F} \mid (p, q) \in M \times \mathbb{T}^2 \},$$
where $\mathcal{F}$ is the foliation of the Clifford torus $T^2 \subset B$ defined in (G3). In particular, $\pi|_F$ is one-to-one. Now $V_0$ is the pushforward of the normalized Riemannian volume of $M \times T^2$, i.e. $V_0 = (\pi|_F^{-1})# \text{vol}_{M \times T^2}$.

In particular, the weight measure $\mu_{V_0}$ of $V_0$ is the normalized Riemannian volume $\text{vol}_{M \times T^2}$ of the $(m+2)$-dimensional submanifold $M \times T^2$. This implies that the $(m+1)$-density of $\mu_{V_0}$ is identically zero.

For rectifiable $(m+1)$-varifolds $V$ the weight measure $\mu_V$ has an approximate tangent space for $\mu_V$ almost every point and hence its $(m+1)$-density is positive $\mu_V$-almost everywhere, cf. [7] §15. Since the $(m+1)$-density of $\mu_{V_0}$ vanishes, we conclude

**Fact 2.** The $(m+1)$-varifold $V_0$ is not rectifiable.

Here is the main result of this section.

**Proposition 3.** Let $(\hat{M}, \hat{g})$ and $V_0$ be as above. Then $V_0$ is stationary, and $V_0$ is the only stationary $(m+1)$-varifold of mass one in $(\hat{M}, \hat{g})$.

**Remark 3.** Statement and proof of Proposition 3 include the case $\dim(M) = m = 0$. In this case the only stationary, unit mass 1-varifold in $B$ is the stationary, non-rectifiable 1-varifold $V_0$ with support on the tangent vectors to the irrational geodesic foliation $\mathcal{F}$ of $T^2$ (see the description of $V_0$ above).

We first recall the following well known fact from ergodic theory:

**Fact 3.** (cf. [4], p. 69) Suppose $T_t$ is the one-parameter group of translations on the standard torus $\mathbb{R}^2/(2\pi\mathbb{Z})^2$ given by $[(x_1, x_2)] \mapsto [(x_1+\alpha_1 t, x_2+\alpha_2 t)]$ with $\alpha_1$ and $\alpha_2$ rationally independent. Then the flow $T_t$ is uniquely ergodic, i.e. the Lebesgue measure $\mu$ on $\mathbb{R}^2/(2\pi\mathbb{Z})^2$ is the – up to scale – unique $T_t$-invariant Borel measure on $\mathbb{R}^2/(2\pi\mathbb{Z})^2$.

**Corollary 1.** Let $\bar{Y}$ be the unit vector field on $T^2$ tangent to $\mathcal{F}$, that is given by the normalisation of $Y|_{T^2}$, cf. Section 3, and denote by $\varphi^Y_t$ its flow. Then the Riemannian area $\text{vol}_{T^2}$ is the – up to scale – unique Borel measure on $T^2$ that is invariant under $\varphi^Y_t$.

**Proof.** The norm of the vectorfield $Y = \partial_\phi + \alpha \partial_\theta$ is constant on $T^2$, and we denote it by $a = |Y|_{T^2} = \frac{1}{\sqrt{2}} \sqrt{1+\alpha^2}$. Now, the covering map $\rho : \mathbb{R}^2 \to T^2$, $\rho(x_1, x_2) = F(1, \frac{x_1}{\sqrt{2}}, x_2, x_2)$ induces a diffeomorphism $\hat{\rho} : \mathbb{R}^2/(2\pi\mathbb{Z})^2 \to T^2$ conjugating the irrational linear flow $T_t$ from Fact 3 with $\alpha_1 = \frac{1}{\sqrt{2}}$ and $\alpha_2 = \frac{\sqrt{2}}{\alpha}$ to the flow $\varphi^Y_t$. So, by Fact 3 the push-forward $\hat{\rho}_#\mu$ of the Lebesgue measure $\mu$ on $\mathbb{R}^2/(2\pi\mathbb{Z})^2$ is the – up to scale – unique $\varphi^Y_t$-invariant Borel measure on $T^2$. On the other hand, $\hat{\rho}_#\mu$ equals $\text{vol}_{T^2}$ up to a factor since $\hat{\rho}$ is a homothety. \hfill \Box

We first give a short outline of the proof of Proposition 3. In Step 1 we calculate that $V_0$ is indeed stationary, see also Remark 3. In Step 2 and 3 we consider an arbitrary nonzero, stationary $(m+1)$-varifold $V$ in $\hat{M}$. In Step 2 we show that its support is contained in the set $\tilde{\mathcal{F}} \subset G_{m+1}(\hat{M})$. This relies on the convexity properties of the spheres $S^3(\rho) \subset B$, cf. Section 3. In the last step, we use the Constancy Theorem [7, 41.2(3)] to prove that the weight measure $\mu_V$ of $V$ has a product structure. Then the unique ergodicity of the flow $\varphi^Y_t$ can be used to show that $\mu_V$ is indeed proportional to the product measure $\text{vol}_M \otimes \text{vol}_{T^2}$. This proves that $V = \lambda V_0$ for some $\lambda > 0$. 


The preceding discussion shows that trace \(S\) is parallel, and spans \(T_qF\) at every point \(q\) of \(T^2\). We decompose any vectorfield \(X\) on \(M\) as a sum \(X(p, q) = X_1(p) + X_2(q)\), where \(X_1(p) \in T_pM\) and \(X_2(q) \in T_qB\). So, by the special character of the Levi Civita connection of a Riemannian product, we obtain for every \((p, q) \in M \times T^2:\)

\[
\begin{align*}
\text{div}_{T_pM \times T_qF} X &= \text{div}_M(X_1^1)_{p} + g(\nabla_{\bar{Y}} X_2^1, \bar{Y})_{q} \\
&= \text{div}_M(X_1^1)_{p} + \frac{d}{dt}_{t=0} g(X_2^1, \bar{Y}) \circ \varphi^\bar{Y}_t(q),
\end{align*}
\]

where \(\varphi^\bar{Y}_t\) denotes the flow of \(\bar{Y}\). Now the Gauss Theorem and the invariance of the volume of the flat torus under \(\varphi^\bar{Y}_t\), cf. Corollary 1 imply that

\[
\delta V_0(X) = \int_{M \times T^2} \text{div}_{T_pM \times T_qF} X \, d\mu_0(p, q)
= \int_{T^2} \int_M \text{div}_M(X_1^1)_{p} \, d\nu_1(M) \, d\nu_2(q)
+ \int_M \int_{T^2} \frac{d}{dt}_{t=0} g(X_2^1, \bar{Y}) \circ \varphi^\bar{Y}_t(q) \, d\nu_1(M) \, d\nu_2(q)
= 0 + \int_M \int_{T^2} \frac{d}{dt}_{t=0} \left( \int_{T^2} g(X_2^1, \bar{Y})_{q} \, d(\varphi^\bar{Y}_t)^\# \nu_2(q) \right) \, d\nu_1(M)
= 0.
\]

So \(V_0\) is stationary.

Now, we consider an arbitrary nonzero, stationary \((m + 1)\)-varifold \(V\) in \(\tilde{M}\).

Step 2: First, we prove that the varifold \(V\) has support in \(\tilde{F}\).
We consider \(f : \tilde{M} \to \mathbb{R}_{\geq 0}, (p, q) \mapsto d^2(q)\), where \(d(q)\) denotes the (euclidean) distance from \(q \in B\) to \(0 \in B\), cf. Section 3. Note that (C2) and (C3) imply the following: If \((v, w) \in T_pM \times T_qB\) then \(\nabla^2 f((v, w), (v, w)) > 0\) except in the following two cases

- \(w = 0\), or
- \(q \in \mathbb{T}^2\) and \(w \in T_qF\).

Now suppose \(V\) is a stationary \((m + 1)\)-varifold in \(\tilde{M}\). We test \(V\) against the vectorfield \(X = \text{grad} \, f\). Then we have

\[
0 = \delta V(X) = \int_{\mathbb{A}(\tilde{M})} \text{div}_S X \, dV(S) = \int_{G_{m+1}(\tilde{M})} \text{trace}_S(\nabla^2 f) \, dV(S).
\]

The preceding discussion shows that \(\text{trace}_S(\nabla^2 f) > 0\) except if \(S \in \tilde{F}\). Hence \(\text{spt}(V) \subset \tilde{F}\).

Step 3: We show that \(\mu_V\) equals \(\nu_{M \times T^2}\) up to a constant.
First, we prove that for any Borel set \(A \subset B\) there exists \(c_A > 0\) such that the Borel measure \(\mu^A\) on \(M\) defined by \(\mu^A(\cdot) := \mu_V(\cdot \times A)\) is given by \(c_A \cdot \nu_{M}\).
Note that $\mu^A$ can be considered as an $m$-varifold on the $m$-dimensional manifold $M$. We will show that $\mu^A$ is a stationary $m$-varifold, and then the Constancy Theorem [7, 41.2(3)] implies that $\mu^A$ is a multiple of the Riemannian volume measure $\text{vol}_M$ as claimed. Denote the measure $(\pi_2)_# \mu_V$ on $B$ by $\mu_{V,2}$, where $\pi_2 : M \times B \to B$ denotes the usual projection to the second component. We choose a sequence $f_n \in C^\infty(B)$ converging to the indicator function $\chi_A$ in $L^1(\mu_{V,2})$. This implies that $f_n \circ \pi_2$ converges to $\chi_{M \times A}$ in $L^1(\mu_V)$. Denoting the projection $M \times B \to M$ by $\pi_1$ we calculate for every vectorfield $X$ on $M$

$$\int_M \text{div}_M X \, d\mu^A = \int_{M \times A} \text{div}_M X \circ \pi_1 \, d\mu_V$$

$$= \lim_{n \to \infty} \int_{M \times B} (f_n \circ \pi_2) \cdot (\text{div}_M X \circ \pi_1) \, d\mu_V$$

$$= \lim_{n \to \infty} \int_{M \times B} \text{div}_{T_p M \times T_q F}( (f_n \circ \pi_2) \cdot (X \circ \pi_1)) \, d\mu_V (p, q),$$

since it follows from equation (6) that

$$\text{div}_{T_p M \times T_q F}( (f_n \circ \pi_2) \cdot (X \circ \pi_1)) = f_n(q) \cdot \text{div}_M X|_p.$$  

Since $V$ is stationary, we know from Step 2 that $\text{spt}(V) \subset \tilde{\mathcal{F}}$. Hence

$$\delta V((f_n \circ \pi_2) \cdot (X \circ \pi_1)) = \int_{M \times \mathbb{R}^2} \text{div}_{T_p M \times T_q F}( (f_n \circ \pi_2) \cdot (X \circ \pi_1)) \, d\mu_V (p, q) = 0$$

for all $n \in \mathbb{N}$. Thus $\int_M \text{div}_M X \, d\mu^A = 0$ for every vectorfield $X$ on $M$, i.e. the $m$-varifold defined by $\mu^A$ is stationary, and hence a multiple of $\text{vol}_M$, see [7, 41.2(3)]. Using the abbreviation $\mu_{V,2} = (\pi_2)_# \mu_V$ introduced above, the constant $c_A$ can be calculated as follows

$$c_A = \frac{1}{\text{vol}_M(M)} \mu^A(M) = \frac{1}{\text{vol}_M(M)} \mu_{V,2}(A).$$

Hence, $\mu_V$ is given as a product of $\text{vol}_M$ and $\mu_{V,2}$. Next, we prove that – up to scale – $\mu_{V,2}$ coincides with the Riemannian area $\text{vol}_2$.

The idea is to show invariance of $\mu_{V,2}$ under the flow $\varphi_t^{\tilde{Y}}$ of $\tilde{Y}$. Then the unique ergodicity of $\varphi_t^{\tilde{Y}}$ implies that $\mu_{V,2}$ is a multiple of $\text{vol}_2$, cf. Corollary [11].

We consider $f \in C^1(B)$ and $\tilde{X} = (f\tilde{Y}) \circ \pi_2$. Since $\tilde{X}$ is defined in a neighborhood of $\text{spt}(\mu_V)$ and $V$ is stationary we have

$$0 = \delta V(\tilde{X}) = \int \text{div}_S \tilde{X} \, dV(S).$$
Since \( \text{spt}(V) \subset \tilde{F} \), equation 6 implies
\[
0 = \int_{M \times \mathbb{T}^2} \text{div}_{T_pM \times T_qF} \tilde{X} \, d\mu_V(p,q) = \int_{M \times \mathbb{T}^2} g(\nabla_Y f \tilde{Y}, \tilde{Y}) \circ \pi_2 \, d\mu_V \\
= \int_{M \times \mathbb{T}^2} \left( df(\tilde{Y}) + fg(\nabla_Y \tilde{Y}, \tilde{Y}) \right) \circ \pi_2 \, d\mu_V \\
= \int_{\mathbb{T}^2} df(\tilde{Y}) \, d\mu_{V;2}.
\]
Since every function \( f \in C^1(\mathbb{T}^2) \) can be extended to a \( C^1 \)-function on \( B \) we conclude that
\[
\int_{\mathbb{T}^2} df(\tilde{Y}) \, d\mu_{V;2} = 0
\]
for all \( f \in C^1(\mathbb{T}^2) \). This implies that \( \mu_{V;2} \) is \( \varphi^\tilde{Y} \)-invariant. For convenience, we include the simple proof. Since \((d(f \circ \varphi^\tilde{Y}_t))(\tilde{Y}_p) = \left. \frac{d}{dt} f \circ \varphi^\tilde{Y}_t \right|_t (p)\), we have for all \( t > 0 \)
\[
0 = \int_0^t \int_{\mathbb{T}^2} d(f \circ \varphi^\tilde{Y}_t)(\tilde{Y}) \, d\mu_{V;2} \, dt \\
= \int_{\mathbb{T}^2} f \, d(\varphi^\tilde{Y}_t)(\mu_{V;2})) - \int_{\mathbb{T}^2} f \, d\mu_{V;2}.
\]
This together with the Borel regularity of \( \mu_{V;2} \) implies the \( \varphi^\tilde{Y}_t \)-invariance of \( \mu_{V;2} \). Now the unique ergodicity of \( \varphi^\tilde{Y}_t \) implies our claim, cf. Corollary 4.

This completes the proof of Step 3. Together, Step 2 and Step 3 prove the claimed uniqueness of \( V_0 \). \( \square \)

**Remark 4.** Actually, the calculation in Step 1 can be replaced by the following more involved argument showing that \( V_0 \) is stationary. Since \( \tilde{M} \) does not satisfy an isoperimetric inequality for \((m+1)\)-varifolds, B. White’s Theorem 2.3 from 8 implies that \( \tilde{M} \) contains a nonzero, stationary \((m+1)\)-varifold \( V \). But now the preceding two steps show that this \( V \) is a nonzero multiple of \( V_0 \). Hence \( V_0 \) is stationary.

5. Appendix

**Lemma 1.** There is a function \( R \in C^\infty([0,2[ \times ]0,\pi/2[, \mathbb{R}^+) \) that fulfills conditions (H1)-(H3).

**Proof.** It is easy to find a function \( k \in C^\infty(\mathbb{R}^2, \mathbb{R}) \) that meets conditions (R2) and (R4), and the following weakening of condition (R1)

\[
R1' \quad k(1, \pi/4) = \cos^2(\pi/4) + \alpha^2 \sin^2(\pi/4).
\]

For example \( k(\rho, \psi) = (\psi - \pi/4)^2 \rho + (\rho - 1)^3 + \cos^2(\pi/4) + \alpha^2 \sin^2(\pi/4) \) has these properties, but we do not need the explicit formula. In addition, we define the function \( l \in C^\infty(\mathbb{R}^2, \mathbb{R}^+) \) by \( l(\rho, \psi) = \rho^2(\cos^2 \psi + \alpha^2 \sin^2 \psi) \). Then
\[
(k - l)(1, \pi/4) = 0 \quad \text{and} \quad \frac{\partial}{\partial \rho}(k - l)(1, \pi/4) = -2 \left( \cos^2 (\pi/4) + \alpha^2 \sin^2 (\pi/4) \right) < 0.
\]
Therefore we can find \( \frac{1}{2} < \rho_1 < \rho_2 < 1 < \rho_3 < \rho_4 < \frac{3}{2} \) and \( \pi/8 < \psi_1 < \pi/4 < \psi_2 < \frac{3\pi}{8} \) such that for any \( \psi \in [\psi_1, \psi_2] \)

\[
(k - l)(\rho, \psi) > 0 \quad \text{if } \rho \in [\rho_1, \rho_2], \quad (k - l)(\rho, \psi) < 0 \quad \text{if } \rho \in [\rho_3, \rho_4].
\]

(7)

Now choose a bump function \( \beta \in C^\infty(\mathbb{R}^2, [0, 1]) \) with support in \([\rho_1, \rho_4] \times [\psi_1, \psi_2] \), that is constantly equal to 1 in a neighbourhood of \((1, \pi/4)\), and has the following property for any \( \psi \in [\psi_1, \psi_2] \)

\[
\frac{\partial}{\partial \rho} \beta(\rho, \psi) \begin{cases} 
\geq 0 & \text{for } \rho \in [\rho_1, \rho_2] \\
= 0 & \text{for } \rho \in [\rho_2, \rho_3] \\
\leq 0 & \text{for } \rho \in [\rho_3, \rho_4].
\end{cases}
\]

(8)

Since \((k - l)(1, \pi/4) = 0\) we can choose the parameters \( \rho_1, \rho_2, \rho_3, \rho_4, \psi_1 \) and \( \psi_2 \) in such a way that the function

\[
R := (1 - \beta)l + \beta k = l + \beta(k - l)
\]

is positive on the open set \([0, 2] \times [0, \pi/2] \). Obviously the restriction of \( R \) to \([0, 2] \times [0, \pi/2] \) meets conditions \((R1)\) and \((R3)\). To finish the proof we check the monotonicity condition \((R2)\):

\[
\frac{\partial}{\partial \rho} R = (1 - \beta) \frac{\partial}{\partial \rho} l + \beta \frac{\partial}{\partial \rho} k + (k - l) \frac{\partial}{\partial \rho} \beta,
\]

where the sum of the first two terms is positive if \((\rho, \psi) \neq (1, \pi/4)\) and the last term is nonnegative as \((7)\) and \((8)\) show.

\[\square\]

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