Quantifying Resilience and the Risk of Regime Shifts Under Realistic Noise Conditions

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April 8, 2022

Abstract

Commonly proposed statistical early warning measures are far away from realistic applications in which limited data availability, coarse-grained sampling and strong correlated noise are typical. Even under favourable simulation conditions the measures are of limited capacity due to their qualitative nature and sometimes ambiguous trend-to-noise ratio. In order to solve these shortcomings, we analyse the stability of the system via the slope of the deterministic term of a Langevin equation, which is hypothesized to underlie the system dynamics close to the fixed point. The open-source available method is applied to a previously studied seasonal ecological model under realistic noise level and correlation scenarios. We compare the results to autocorrelation, standard deviation, skewness and kurtosis as leading indicator candidates by a Bayesian model comparison with a linear and a constant model. We can show that the slope of the deterministic term is a promising alternative due to its quantitative nature and high robustness against noise levels and types. The commonly computed indicators apart from the autocorrelation with deseasonalization fail to provide reliable insights into the stability of the system in contrast to a previously performed study in which the standard deviation was found to perform best. In addition, we discuss the significant influence of the seasonal nature of the data to the robust computation of the various indicators, before we determine approximately the minimal amount of data per time window that leads to significant trends for the drift slope estimations.

Keywords ecology · regime shift · early warning signals · leading indicator · critical transition

1 Introduction

Even if the idea of universal early warning indicators (Dakos et al. [2012, 2009], Scheffer et al. [2015], Liang et al. [2017]) for critical transitions is a fascinating and attractive vision throughout the fields of ecology, climate research, biology and much more (Veraart et al. [2011], Drake and Griffen [2010], Dakos et al. [2017], Livina et al. [2010, 2015], Lenton [2012], Izrailtyan et al. [2000], Chadaelx [2014], Coutura-Sanchez et al. [2012], van de Leemput et al. [2013]), the research done over the years in this field has discovered plenty of problems, drawbacks and limitations of the proposed leading indicators (Clements et al. [2015], Hastings and Wysham [2010], Ditlevsen and Johnsen [2010], Wilkat et al. [2019]). The difficulties and limitations result from the sometimes mentioned problematic claim of “universality” which is hard or impossible to achieve. Just by definition the mentioned universality is already limited to special cases of regime shifts as bifurcation induced tipping events (Scheffer et al. [2009], Ritchie and Sieber [2017], Ashwin et al. [2012]), because the leading indicators are a consequence of the commonly observed phenomenon of critical slowing down prior to a bifurcation or flickering in noisy bi-stable systems (Scheffer et al. [2012], Wissel [1984], Schröder et al. [2005]). Critical slowing down is the increased relaxation time of perturbations near a bifurcation whereas flickering determines jumps of a system between two alternative stable states. Furthermore, a successful
The detection of a critical transition depends on the eigen-direction in which the transition takes place and the time series at hand (Boerlijst et al. [2013]). Apart from that it remains difficult to get an impression of the leading indicators’ quality applied to real-world systems because the tests are often performed with historical test data for which it is known that a transition is present (Boettiger and Hastings [2012]).

Following this argumentation it is proposed to design specialized indicators in specific fields of research or systems that are known at least in part (Perretti and Munch [2012], Gsell et al. [2016], Dablander et al. [2020]). One of those research areas is the field of ecology in which standard leading indicators as autocorrelation at a lag of one (AR1), the standard deviation (std) θ, the skewness Γ or the kurtosis ω are often very limited in their applicability due to high correlated noise contributions and low sampled short time series that are characteristic of the limits imposed by the experimental and funding resources as stated in (Bissonnette [1999], Perretti and Munch [2012]).

Furthermore, even in simulations in which the afore-mentioned practical limitations do not play a role, the inherent design of the indicators raises problems. As discussed in Biggs et al. [2009] the standard leading indicator candidates are difficult to interpret because of their qualitative nature: They are designed upon trend changes which can be too gradual and ambiguous to rely on for decision-makers. And in addition, unfortunately these changes are often realized too late for policymakers to adapt management and avoid uprising transitions. Therefore, in the case that a developed early warning measure should be applicable in practice the authors of Biggs et al. [2009] claim that it would rely on: (i) defining critical levels of the regime shift indicators, (ii) linking these critical levels to long-term sustainable impact levels, and (iii) finding or developing indicators that have critical levels that are relatively transferable across different ecosystem types.

Based on these demands (Biggs et al. [2009]) and the poor performance of standard leading indicator candidates under strong correlated noise found in Perretti and Munch [2012], we want to introduce the alternative drift slope estimation (Heßler and Kamps [2021], Heßler [2021a,b]) to tackle the problem of anticipating an ecological regime shift and compare it to the above mentioned indicators. Similar to Carpenter and Brock [2011] the alternative approach considers the data to be generated by a stochastic differential equation of the Langevin form (Kloeden and Platen [1992])

\[
\dot{x}(x,t) = h(x(t), t) + g(x(t), t)\Gamma(t),
\]

where the drift \( h(x(t), t) \) captures the deterministic part of the system dynamics under the stochastic influence of a Gaussian and \( \delta \)-correlated noise process \( \Gamma(t) \) that scales with the diffusion \( g(x(t), t) \). The method estimates the parameterized drift and diffusion terms via Markov Chain Monte Carlo sampling (MCMC) and calculates the drift slope \( \zeta \) in the fixed point \( x^* \) in rolling windows as a resilience measure of the system. The drift slope \( \zeta \) is negative for stable systems and increases with proceeding destabilization. A zero crossing of the drift slope corresponds to a regime shift (Heßler and Kamps [2021]).

In this study we show that in contrast to the common qualitative leading indicator candidates, the method provides a quantitative and easy-to-interpret resilience measure which is able to fulfill the requirements stated in Biggs et al. [2009] for the system discussed there under realistic ecological conditions, i.e. correlated strong noise influence (Perretti and Munch [2012]). In addition, we discuss the important role of the seasonal nature of the data that affects the trend quality of the time series analysis methods. The performance of the early warning signals is tested by comparing the probability the data might be explained by a linear trend or a constant model with a Bayesian model comparison. The drift slope \( \zeta \) and - if the seasonality is taken into account - the AR1 provide reliable results in our study. Interestingly, in contrast to previous results regarding the same system (Perretti and Munch [2012]), the AR1 seems to be preferred to the insignificant standard deviation. Apart from this we can reproduce the findings of a generally poor performance of the standard leading indicator candidates (Perretti and Munch [2012]). In the end, the drift slope seems to be a promising alternative to common leading indicators because of its quantitative nature, easy interpretation and robustness to realistic noise contributions. However, its applicability remains limited to situations in which it is possible to generate the necessary amount of data which is around 50 data points per year for the investigated ecological model.

The ecological system is presented in section 2. The results of the applied leading indicators are discussed in section 3 which is divided into three parts: First, the drift slope results are presented in subsection 3.1. Second, the drift slope performance as leading indicator is compared to established candidates via a Bayesian model comparison in subsection 3.2 before the needed minimum amount of data per window for the drift slope estimation for the model at hand is defined in subsection 3.3. Finally, we summarize our findings in section 4.

## 2 Ecological model

In order to investigate the performance of the Bayesian stability analysis tool under rather realistic conditions in the field of ecology the multi-species model derived in Carpenter and Brock [2004], described in detail in Biggs et al. [2009] and used as a basis of leading indicator performance tests in Perretti and Munch [2012] is simulated via the

2
Figure 1: A scheme of the considered foodweb model. In the predation area the adult piscivores $A$ hunt the juvenile piscivores $J$ and the planktivores $F$ which only hunt juvenile piscivores $J$. Both, the juvenile piscivores $J$ and the planktivores $F$ can hide themselves in a refuge area in order to retire. External white or colored stochastic influence $Z$ is added to the planktivore population with the noise level $\sigma$. We discuss the possibility of regime shifts due to high angling pressure represented by the harvest rate $qE$ which is given as the product of catchability $q$ and the effort $E$. Here, the model is restricted to fish, but in general other animals, as e.g. some seabirds, are included in the term “piscivores”.

Euler-Maruyama scheme. The ecological system consists of three parties: juvenile piscivores ($J$), adult piscivores ($A$) and planktivores ($F$). The model contains a continuous “monitoring interval”

$$\frac{dA}{dt} = -qEA$$

$$\frac{dF}{dt} = DF(\frac{FR - F}{h + \nu + cRF}) - cFAA + \sigma Z$$

$$\frac{dJ}{dt} = -cJA - cJFA + \nu + cJFJ$$

and a discrete annual “maturation interval” realized as the map equations

$$A_{y+1} = s(A_{y;t=1} + J_{y;t=1})$$

$$F_{y+1} = F_y$$

$$J_{y+1} = fA_{y+1}$$

where the index $y$; $t = 1$ means the abundance of each party at the end of the monitoring interval (i.e. $t = 1$) of the corresponding year $y$. In the map $s$ determines the survivorship between maturation intervals and $f$ the fecundity rate of the adult piscivores $A$. The harvest rate of the adult piscivores is determined via the product of the catchability $q$ and the effort $E$. The planktivores exchange between a protected area, the so-called refuge reservoir $FR$ and the foraging arena $DF$. The parameters $c_{i,j}$ with $i,j = \{A,F,J\}$ model the consumption rates of $i$ by $j$. Besides, the piscivores
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| parameter, value, short definition |
|------------------------------------|
| \( qE_{\text{init}} \) | 1 | initial harvest rate |
| \( \Delta(qE) \) | 0.013 | change of harvest rate per year |
| \( F_R \) | 100 | refuge reservoir for planktivores |
| \( D_F \) | 0.1 | foraging arena |
| \( c_{FA} \) | 0.3 | rate at which adult piscivores consume planktivores |
| \( c_{JA} \) | 0.001 | Control of juvenile piscivores by adult piscivores |
| \( c_{JF} \) | 0.5 | rate at which planktivores consume juvenile piscivores |
| \( \nu \) | 1 | rate at which juvenile piscivores become vulnerable against planktivores |
| \( h \) | 8 | rate at which juvenile planktivores enter the refuge |
| \( f \) | 2 | fecundity rate of adult piscivores |
| \( s \) | 0.5 | survival rate of adult and juvenile piscivores over the winter period |

Table 1: The parameter values of the ecological model with short definitions.

Environmental stochasticity of the lower level of the food web is incorporated via \( Z \). Following the simplified autocorrelated noise implementation of Perretti and Munch [2012] the noise range from white over pink to red noise is defined by

\[
dZ = -\phi Z dt + \sqrt{2\phi} dW
\]

with a standard Wiener process \( W \). The adjusted \( \phi \) values for white, pink and red noise in Perretti and Munch [2012] are given as 0, 0.53, 0.92, respectively. For each of the three noise types the model is evaluated for three different noise intensities, explicitly

\[
\sigma dt = 0.002 \\
\sigma dt = 0.044 \\
\sigma dt = 0.09
\]

depending on the harvest rate the system settles into a piscivore- or planktivore-dominated state. In the first mentioned scenario the planktivore abundance is kept at a low level because of a large occurrence of adult piscivores whereas in the second scenario the large population of planktivores hinders the piscivore population to grow because the planktivores’ predation of the juvenile group.

Numerical method

Starting with the Langevin equation 1 we parameterize the drift and diffusion as \( h(x(t), t) \equiv h(x(t)) \) and \( g(x(t), t) \equiv \text{const.} =: \sigma \). Since we assume to be in a fixed point and close to a bifurcation we develop \( h(x, t) \) into a Taylor series up to order three which is sufficient to describe the normal forms of simple bifurcation scenarios (Strogatz [2015]). This results in

\[
h(x(t), t) = \alpha_0(t) + \alpha_1(t)(x-x^*) + \alpha_2(t)(x-x^*)^2 + \alpha_3(t)(x-x^*)^3 + O((x-x^*)^4),
\]

so that the information on the linear stability is incorporated in \( \alpha_1 \). For practical reasons equation 12 is used in the form

\[
h_{MC}(x(t), t) = \theta_0(t; x^*) + \theta_1(t; x^*) \cdot x + \theta_2(t; x^*) \cdot x^2 + \theta_3(t; x^*) \cdot x^3 + O(x^4)
\]
in the numerical approach, where an arbitrary fixed point \( x^* \) is incorporated in the coefficients \( \theta \) by algebraic transformation and comparison of coefficients. A change of the negative sign of the slope

\[
\zeta = \left. \frac{dh(x)}{dx} \right|_{x=x^*}
\]  

(14)

of the nonlinear drift at the fixed point \( x^* \) which is estimated to be the data mean corresponds to a loss of stability via the formalism of linear stability analysis (Heßler and Kamps [2021]).

The task is now to estimate the parameters \( \theta \). Their posterior distribution is given by applying Bayes’ theorem

\[
p(\theta, \sigma | d, I) = \frac{p(d| \theta, \sigma, I) \cdot p(\theta, \sigma | I)}{p(d| I)}
\]  

(15)

The likelihood \( p(d| \theta, \sigma, I) \) is given as the transition probability of the process defined by equation (1) (see Heßler and Kamps [2021]) and the prior knowledge is incorporated in \( p(\theta, \sigma | I) \). The evidence \( p(d| I) \) normalizes the posterior probability density function (pdf) \( p(\theta, \sigma | d, I) \). One advantage of this procedure is the consistent definition of credibility bands of the estimated parameters based on the posterior pdf. The posterior distribution of the parameters can be estimated via MCMC sampling with the flat Jeffreys’ priors

\[
p\text{prior}(\theta_0, \theta_1) = \frac{1}{2\pi(1 + \theta_i^2)^{\frac{1}{2}}}
\]  

(16)

and

\[
p\text{prior}(\sigma) = \frac{1}{\sigma}
\]  

(17)

for the scale variable \( \sigma \) (von der Linden et al. [2014]). Gaussian priors

\[
p\text{prior}(\theta_2) = N(\mu, \sigma_{\theta_2}),
\]  

\[
p\text{prior}(\theta_3) = N(\mu, \sigma_{\theta_3})
\]  

(18)

centred around the mean \( \mu = 0 \) with standard deviations \( \sigma_\theta \), in an adequate range are used for the rest of the parameters. We use the MCMC sampling algorithm implemented in the python package emcee (Foreman-Mackey et al. [2013]). The method is applied in rolling windows in order to resolve the time evolution of the drift slope. A detailed description of the presented algorithm and its implementation steps can be found in Heßler and Kamps [2021].

3 Results

3.1 Drift slope analysis

The drift slope estimation method that is shortly summarized in section 2 and described in detail in Heßler and Kamps [2021] is applied to time series simulations of the seasonal ecological model with white, pink and red noise each of which is realised for three noise levels \( \sigma = \{0.1, 2.3, 4.5\} \). The data is evaluated in windows of 750 data points that are shifted by 30 points per step and analysed in two scenarios: First, without pre-processing the data by deseasonalization and detrending at the same time. However, the trend component of the data is weak and does not affect the conclusions. The flat Jeffreys’ priors are chosen broadly as \([-50, 50]\) for \( \theta_{0,1} \) and \([0, 50]\) for \( \theta_4 \) except for the analysis of the deseasonalized versions of the correlated models. In these cases (red lines in D-I) the prior range is chosen even broader as \([-70, 70]\) for \( \theta_{0,1} \) and \([0, 70]\) for \( \theta_4 \) to make sure that the available data determines the posterior distribution. The Gaussian priors for \( \theta_{2,3} \) are implemented with \( \sigma = \{4, 8\} \), respectively. The analysis results are presented in figure 2. The results of the first approach are marked in blue with orange credibility bands defined as the 16% to 84% and 1% to 99% percentile of the drift slope posterior modelled by a kernel density estimate of the sampled parameters (Pedregosa et al. [2011]). The second approach is shown in red with the corresponding green credibility bands. The green dotted and orange solid vertical lines are defined equivalently to Perretti and Munch [2012] as the attractor switch point and the “point of no return”, respectively, which is defined as the year in which even a reduction of the harvest rate to \( qE = 0.1 \) does not inhibit the destabilization process of the ecological system. The
beginning of the grey shaded area is a subjectively defined time at which the previously small planktivore population exceeds 21 individuals and serves as an orientation for the ongoing destabilization process. Each column from left to right belongs to one of the three noise levels \( \sigma = \{0.1, 2.3, 4.5\}\). The first (A-C), second (D-F) and third row (G-I) contain the drift slope results of the realisations of the model with additional white, pink and red noise, respectively. By comparing the results of the analyses with and without deseasonalization over various noise environments of the model we gain valuable insights in the capacities an limits of the methodological concept: In the figures 2(A-C) the deseasonalized cases perform rather similar to the cases without deseasonalization apart from the small noise case (A) with \( \sigma = 0.1\). This leads to the conclusion that in the small noise case (A) the seasonal effects in the data are interpreted by the model probably in terms of noise fluctuations because the parameterization cannot capture the predominant seasonality. With increasing noise the seasonal effects become insignificant as visible in the figures 2(B, C), because the noise level covers and hides the seasonal component of the data. The drift slope indicator seems to be suitable to provide information about the resilience of this ecological model with white noise whereas seasonal aspects should be treated carefully for small noise levels. Technically, the drift slope estimation is not designed in order to deal with correlated noise and thus, with Non-Markovianity, but the results in (D-I) show that nevertheless it can be a helpful tool even in highly correlated and noisy situations. Similar to the results in the small white noise case (A) the results reach the critical zero line around the attractor switch point and exhibit less clear trends as their deseasonalized counterparts that reach the critical zero around the actual transition that is approximately marked by the beginning of the grey shaded area. In contrast to the white noise cases the seasonality in the correlated noise cases influences the results for all noise levels in a similar way: Without deseasonalization the drift slope reaches zero around the attractor switch point whereas it reaches zero around the point of no return in the absence of seasonality. The critical zero crossing of the drift slope in the deseasonalized versions of the correlated noise cases seems to be a bit earlier than the crossings of the white noise counterparts. The high impact of the seasonality in the strong correlated noise cases compared to the strong white noise cases (B, C) is due to the correlation of the noise itself: The noise correlation tends to amplify or weaken the annual amplitudes, whereby the seasonal component of the time series is not hidden by the noise, but more or less preserved. Note also, that there is no clear formal reason for the zero crossing of the blue drift slopes at the attractor switch point or for the red drift slopes reaching the critical zero around the point of no return in the correlated noise cases. Besides, the strong fluctuating slope estimates after the transition time in figure 2(D, G) without deseasonalization are numerical artefacts probably caused by the small correlated noise contributions in the new stable state. The drift slope trends are rather robust in the presented model cases and provide reliable information about the resilience and destabilization of the ecological system. The method is relatively complicated to implement in contrast to leading indicator candidates as the AR1 or the std \( \hat{\sigma} \). Anyhow, its performance and robustness could be important advantages in the field of ecology as outlined in the next subsection 3.2 in which the performance of the drift slope in this dynamical rolling window setting is compared to common leading indicator candidates.

### 3.2 Comparison of leading indicators’ performance

In order to compare the performance of the drift slope indicator with established early warning candidates as the autocorrelation at lag-1 (AR1), the standard deviation (std) \( \hat{\sigma} \), the skewness \( \Gamma \) or the kurtosis \( \omega \) we use a Bayesian model comparison in which we compute the Bayes factors \( BF_{ij} \) with \( i, j \in \{1, 2\} \) and \( i \neq j \) that are defined as the ratio

\[
BF_{ij} = \frac{p(\mathcal{L}|M_i)}{p(\mathcal{L}|M_j)}
\]

of the evidences \( p(\mathcal{L}|M_{1,2}) \) that a linear trend model (model \( M_1 \)) or a constant model (model \( M_2 \)) explain the leading indicator datasets \( \mathcal{L} \) up to the “point of no return”. The \( BF_{ij} \) are calculated for each of the above mentioned noise levels, noise types and the datasets without and with deseasonalization. A Bayes factor is declared to be significant for \( BF_{ij} > 100 \) to take into account the fact that most of the Bayes factors lie in the range \( 10 < BF_{ij} < 100 \) or are significantly bigger than 100. The results of the comparison without deseasonalizing the data are summarized in table 2 where the color code follows [Heßler and Kamps 2021] with a significant \( BF_{12} \) or \( BF_{21} \) marked by green and orange tiles, respectively, and grey tiles denote cases in which none of the models is favourable. The results of the kurtosis \( \omega \) are excluded from further discussion, because of the ambiguous, non-monotone and very noisy trends with jumps which cannot be reliably interpreted by eye or captured by the linear model \( M_1 \) of the Bayes model comparison. In some cases the constant model was erroneously preferred or the results were not significant. The corresponding curves of the leading indicators of each case can be found in the supplementary material [Heßler 2022]. Bayes factor pairs with infinite and zero entries correspond to one of the two models with evidence of zero and thus, the model with infinite evidence is preferred.

The biased autocorrelation at lag-1 is computed via `statsmodels.tsa.stattools.acf` [Seabold and Perktold 2010] and the biased standard deviation \( \hat{\sigma} \) via `numpy.std` [Harris et al. 2020]. The skewness \( \gamma \) and kurtosis \( \omega \) calculations are performed with the biased uncorrected estimators of the python package `scipy.stats` [Virtanen et al. 2020]. The biased versions are used, because of the large sample sizes which provide sufficient accuracy. The skewness definition follows
Figure 2: Results of the drift slope analysis for the ecological model with white (A-C), pink (D-F) and red noise (G-I). The columns from left to right correspond to the noise levels $\sigma = \{0.1, 2.3, 4.5\}$. The computations are performed on the time series without deseasonalization (blue lines with orange credibility bands) and with preparation by deseasonalizing the data (red lines with green credibility bands). The green dotted and the orange solid vertical lines indicate the attractor switch point of the deterministic system and the point of no return, respectively, that is defined as the time at which the destabilization cannot be stopped by reducing the harvest rate to $qE = 0.1$. The deseasonalized versions exhibit clear trends and reach the critical zero marked by the red dotted horizontal line around the transition time that is approximately signed by the beginning of the grey shaded area that is defined as the time at which the small planktivore population counts more than 21 individuals. Although, the method is not designed to deal with correlated noise and non-Markovian time series the seasonality of the data has much more influence than the correlated noise. The seasonality reduces clearness of the trends and leads to an earlier zero crossing of the drift slope for small white and all correlated noise scenarios. In the small white noise case the numerical method seems to interpret the seasonal effects incorrectly probably as noise influence. For bigger noise levels the seasonal effects become insignificant for the white noise cases, but not for the correlated noise scenarios. The strong fluctuation of the drift slope estimates in the post transition region of the subfigures (D, G) are probably due to the small correlated noise contributions in the new stable state.
the not-adjusted Fisher-Pearson estimator and the kurtosis is defined via the Pearson estimator corresponding to a kurtosis \( \omega = 3 \) for a Gaussian distribution. Without deseasonalization the common leading indicators AR1, std \( \hat{\sigma} \) and the skewness \( \Gamma \) do not exhibit a significant slope following the Bayesian model comparison in most of the cases, although the time series resolution is relatively high (Perretti and Munch \[2012\]) and the time windows are chosen as big as in the last subsection 3.1. Without deseasonalization the AR1 just performs well in the white noise cases with \( \sigma = \{2.2, 4.5\} \) whereas the skewness \( \gamma \) does not exhibit any reliable pattern of applicability. Note, that these results remain unchanged if the data is only detrended, but not deseasonalized. The corresponding analysis can be found in the supplementary material Heßler \[2022\]. If the results are compared to the deseasonalized counterparts of table 3 the green tiles of std \( \hat{\sigma} \) and the significant white noise cases of the skewness \( \gamma \) turn out to be artefacts due to the seasonal nature of the time series. Interestingly, the deseasonalization leads to a consistent significance pattern of the skewness \( \gamma \) if only the correlated noise cases are considered. Therefore, the general applicability of the skewness \( \gamma \) as leading indicator is ill-advised since it is rather sensitive to noise types, seasonality and e.g. bistability of the system. Nevertheless, it could be useful under specific conditions as the correlated noise cases considered here or in flickering regimes of bistable systems. Only the recently proposed drift slope as leading indicator and the AR1 with deseasonalization seem to yield reliable results. Without deseasonalization the AR1 just performs well as leading indicator in the white noise cases with \( \sigma = \{2.2, 4.5\} \) whereas the performance is significantly improved by deseasonalization that leads to significant trends in all cases apart from the white noise case with \( \sigma = 0.1 \) as suggested by a comparison of the tables 3 and 2. Under the same conditions the drift slope \( \zeta \) turns out to be not very sensitive to the seasonal character of the data apart from the early plateaus discussed in subsection 3.1. The drift slope \( \zeta \) leads to significant positive trends in all considered cases without distinction of non-deseasonalized and deseasonalized data. The results confirm in most instances the results of Perretti and Munch \[2012\] where a very poor applicability of the standard leading indicator candidates to the ecological test dataset is observed. The most robust leading indicator under strong noise was found to be the variance or std \( \hat{\sigma} \) in this study. The Bayes factor analysis proposes AR1 to be the most reliable indicator of the standard measures and rejects the std \( \hat{\sigma} \) as a robust indicator. Following the results of this study the drift slope \( \zeta \) is a possible leading indicator candidate also in very noisy situations, provided that a suitable sampling rate of the time series is guaranteed. In the next subsection 3.3 the limitations of the drift slope estimates \( \zeta \) and their sensitivity to small window sizes are investigated because, as stated in Perretti and Munch \[2012\], ecological time series are often short and possible window sizes are strongly limited by that fact.
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| Noise Level | White Noise | Pink Noise ($\phi = 0.53$) | Red Noise ($\phi = 0.92$) |
|-------------|-------------|----------------------------|--------------------------|
|             | $\sigma = 0.1$ | $\sigma = 2.2$ | $\sigma = 4.5$ | $\sigma = 0.1$ | $\sigma = 2.2$ | $\sigma = 4.5$ | $\sigma = 0.1$ | $\sigma = 2.2$ | $\sigma = 4.5$ |
| Indicator $I$ | $BF_{12}$ | $BF_{12}$ | $BF_{12}$ | $BF_{12}$ | $BF_{12}$ | $BF_{12}$ | $BF_{12}$ | $BF_{12}$ | $BF_{12}$ |
| Slope $\zeta$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $3.2 \cdot 10^{-99}$ | $2.9 \cdot 10^{-99}$ | $\infty$ | $\infty$ | $2.1 \cdot 10^{129}$ |
|              | 0 | 0 | 0 | 0 | $3.1 \cdot 10^{-100}$ | $3.4 \cdot 10^{-99}$ | 0 | 0 | $4.3 \cdot 10^{130}$ |
| AR1 | 12 | $8.8 \cdot 10^7$ | $3.8 \cdot 10^7$ | 9.5 | 3.1 | 1.2 | 9.5 | 4.3 | 1.7 |
|              | $8.3 \cdot 10^{-2}$ | $1.1 \cdot 10^{-8}$ | $2.6 \cdot 10^{-8}$ | 0.11 | 0.32 | 0.85 | 0.11 | 0.23 | 0.60 |
| Std $\sigma$ | 19 | 18 | 57 | 19 | 10 | $8.3 \cdot 10^4$ | 18 | 76 | 16 |
|              | $5.3 \cdot 10^{-2}$ | $5.6 \cdot 10^{-2}$ | $1.8 \cdot 10^{-2}$ | 5.4 | $1.2 \cdot 10^{-5}$ | $5.5 \cdot 10^{-2}$ | 1.3 | $10^{-2}$ | $6.1 \cdot 10^{-2}$ |
| Skewness $\gamma$ | $6.1 \cdot 10^6$ | $1.3 \cdot 10^6$ | 0.91 | $1.1 \cdot 10^7$ | 61 | $3.4 \cdot 10^4$ | $1.6 \cdot 10^7$ | 14 | 0.24 |
|              | $1.6 \cdot 10^{-7}$ | $7.9 \cdot 10^{-7}$ | 1.1 | $8.8 \cdot 10^{-8}$ | $1.6 \cdot 10^{-2}$ | $3.0 \cdot 10^{-5}$ | $6.3 \cdot 10^{-8}$ | 7.0 | $10^{-2}$ | 4.1 |

Table 2: Summary of the Bayes factors comparing a linear model $M_1$ with positive slope to a constant model $M_2$ for the drift slope $\zeta$, the AR1, the std $\sigma$ and the skewness $\gamma$ for various noise types and levels without deseasonalization of the data. The kurtosis $\omega$ is excluded because of non-monotone behaviour. Green tiles mark a $BF_{12} > 100$ which is the threshold for a significant leading indicator trend. Grey tiles mark insignificant results. The constant model $M_2$ is never preferred in the analysis. Infinite Bayes factors result from one model with evidence zero which leads to preferring the finite evidence model. Only the drift slope $\zeta$ performs well in the considered cases. The AR1 has a very limited applicability and the skewness $\gamma$ is not reliable over all cases. The green tile of the std $\sigma$ is an artefact of the seasonality which is confirmed by a comparison with table 3. For completeness, the same analysis is performed for the skewness with a linear model $M_1$ with negative slope in the supplementary material Heßler [2022].
### Table 3: Summary of the Bayes factors comparing a linear model $M_1$ with positive slope to a constant model $M_2$ for the drift slope $\zeta$, the AR1, the std $\hat{\sigma}$ and the skewness $\gamma$ for various noise types and levels with deseasonalization of the data. The kurtosis $\omega$ is excluded because of non-monotone behaviour. Green tiles indicate cases in which model $M_1$ is preferred upon the threshold $BF_{12} > 100$. Grey tiles mark insignificant results. The constant model $M_2$ is never preferred. A pair of infinite and zero Bayes factors mirrors the situation that one model has a finite evidence and is possible whereas the other one has a numerical zero evidence and thus, cannot explain the data. The drift slope $\zeta$ applies as before to all test cases. However, deseasonalization improves the performance of the AR1 as leading indicator $\hat{\sigma}$ significantly as it also works for all deseasonalized cases, whereas the std $\hat{\sigma}$ does not work. This leads to the conclusion that the fragmentary applicability of the std $\hat{\sigma}$ in table 2 is an artefact due to a misinterpretation of the seasonal character of the time series. The skewness becomes a reliable indicator for the considered correlated noise cases, whereas its positive trends disappear in the white noise cases due to the deseasonalization procedure. The same analysis for the skewness with a linear model $M_1$ with negative slope can be found in the supplementary material Heßler [2022].

| noise level | white noise | pink noise ($\phi = 0.53$) | red noise ($\phi = 0.92$) |
|-------------|-------------|----------------------------|----------------------------|
| $\sigma = 0.1$ | BF$_{12}$ | BF$_{12}$ | BF$_{12}$ |
| $\sigma = 2.2$ | BF$_{21}$ | BF$_{12}$ | BF$_{12}$ |
| $\sigma = 4.5$ | BF$_{21}$ | BF$_{21}$ | BF$_{21}$ |

| indicator | $\hat{\sigma}$ | $\zeta$ | AR1 | std $\hat{\sigma}$ | skewness $\gamma$ |
|-----------|-----------------|---------|-----|--------------------|------------------|
|            | $\infty$       | $\infty$ | $\infty$ | $\infty$ | $\infty$ |
|            | $0$            | $0$     | $6.1 \cdot 10^6$ | $2.3 \cdot 10^9$ | $1.4 \cdot 10^{60}$ |
|            | $1.0$          | $1.1$   | $1.1$ | $0.89$ | $1.8 \cdot 10^{60}$ |
| $\hat{\sigma}$ | $14$          | $0.58$  | $0.44$ | $1.1$ | $7.1 \cdot 10^{-2}$ |

| $\sigma = 2.2$ | BF$_{12}$ | BF$_{12}$ | BF$_{12}$ |
| $\sigma = 4.5$ | BF$_{21}$ | BF$_{21}$ | BF$_{21}$ |

| $\sigma = 0.1$ | BF$_{12}$ | BF$_{12}$ | BF$_{12}$ |
| $\sigma = 2.2$ | BF$_{21}$ | BF$_{21}$ | BF$_{21}$ |
| $\sigma = 4.5$ | BF$_{21}$ | BF$_{21}$ | BF$_{21}$ |
3.3 Window size limits

In order to ensure comparability of the results to Perretti and Munch [2012], the drift slope estimates are calculated for comparable window sizes and the corresponding $BF_{12,21}$ are calculated to get an impression of the minimal necessary amount of data per window that yields significant results. In Perretti and Munch [2012] a low-sampled time series variant with one measurement per year and a high-sampled variant of the time series with 50 data points per year is investigated. Here, we will focus on the high-sampled variants because the discussed indicators including the proposed drift slope are only applicable if the information level in terms of available data is high enough to resolve the considered dynamics. This remains a common limitation of the discussed indicators.

However, focusing on the high-sampled datasets with 50 points per year the $BF_{12,21}$ are calculated for window sizes $\{150, 100, 50, 25\}$ in decreasing order until the $BF_{12}$ is no longer significant ($BF_{12} \leq 100$). The results for the discussed noise levels and types are summarized in table 4 without deseasonalization and in table 5 with deseasonalization. The color scheme is defined as in subsection 3.1. The tile is signed to be “inadequate” if both model evidences are numerically zero. A Bayes factor pair of infinite an zero indicates that one model has an evidence of zero and thus, does not fit the data at all. Without deseasonalization significant results are generated for windows sizes bigger than 50 and less or equal to 100 data points for the considered cases except for pink noise with $\sigma = 4.5$. Thus, the significant windows include a time interval of one up to two years which is mostly comparable to the computations in Perretti and Munch [2012] assuming windows of one year. Furthermore, a suitable deseasonalization is able to decrease the necessary window size for significant drift slope trends even below one year between more than 25 and less or equal to 50 data points for pink and red noise apart from the red noise case with $\sigma = 4.5$ where significance is reached between more than 50 and less than or equal to 100 data points. The performance for small windows tends to become worse for the case with strong white noise $\sigma = 4.5$. This is a sign for the difficulties of deseasonalization without removing valuable information for the drift slope estimation at the same time. It has to be mentioned that the drift slope trends for small window sizes as in this limit cases are volatile and thus, less appropriate for an on-line analysis approach.
Table 4: Summary of the Bayes factors comparing a linear model $M_1$ with a constant model $M_2$ for the small window sizes of $\{150, 100, 50\}$ for various noise types and levels without deseasonalization of the data. Model $M_1$ is preferred upon the threshold $BF_{12} > 100$ colored in green. Grey tiles are insignificant results. The constant model $M_2$ is never preferred. If both models had an evidence that resulted in a numerical zero, the tile is marked as “inadequate”, because none of the models was adequate to fit the data. In the case that one evidence was finite and one zero the Bayes factor ratio becomes infinite indicating that the model with an evidence of zero does not fit the data at all and thus, the other one is preferred. The results tend to be significant for more than 50 and less than or equal to 100 data points per window, apart from the noise level $\sigma = 4.5$ of the pink noise system. This corresponds to a period in time between one and two years of high-sampled observation of the ecological system.
Table 5: Summary of the Bayes factors comparing a linear model $M_1$ with a constant model $M_2$ for the small window sizes of $\{150, 100, 50, 25\}$ for various noise types and levels with deseasonalization of the data. The color scheme and notation is defined the same as before. A deseasonalization decreases the necessary data per window to generate significant results to less than or equal to 50 and more than 25 data points for all pink and red noise cases except for red noise with a high noise level $\sigma = 4.5$. This corresponds to time periods of a half year up to one year of observation in a high-sampled manner. The slightly worse results for the white noise cases give a hint that the method reacts sensitive to the deseasonalization in that noise case.
4 Summary and conclusion

Our investigations are based on the destabilizing ecological model previously considered in [Perretti and Munch, 2012] with realistic white, correlated and weak up to strong noise. The simulations are almost comparable except for a slightly longer period of data sampling before the “point of no return”.

The main difficulties stated in [Perretti and Munch, 2012] concerning the applicability of established leading indicator candidates as AR1, std \( \hat{\sigma} \), skewness \( \Gamma \) and kurtosis \( \omega \) are given by the conditions of ecological data acquisition: Normally, just short time series with a low sampling rate and strong noise are available. Furthermore, the systems tend to be influenced by correlated pink or red noise and seasonality. The above mentioned early warning signals fail under these circumstances especially due to low data availability for their estimation and high noise levels. Besides, even under favourable simulation conditions the leading indicator candidates are not as reliable as necessary for management decisions [Biggs et al., 2009]. In the course of this work we have introduced an alternative leading indicator, the so-called “drift slope”, and evaluated its performance in comparison to the common leading indicators mentioned above.

The drift slope is derived from the MCMC-estimated parameters of the drift term of a stochastic differential equation whereas the drift term is approximated by a third-order Taylor polynomial. We could show that the drift slope gives reliable trends to estimate the resilience of the system almost regardless of the noise level and type and it fulfills the demands for an early warning signal stated by Biggs et al. (2009) which we cite in section 1:

(i) exhibits a clear threshold of destabilization at zero and the relative distance to zero measures the state of resilience,

(ii) provides trends which are easy-to-interpret regarding the necessity of management action,

(iii) is comparable across systems in similar contexts because of its parametric ansatz and quantitative nature.

The standard measures skewness \( \gamma \) and kurtosis \( \omega \) turn out to usually fail to predict the destabilization process which coincides with the observations in Perretti and Munch (2012). The kurtosis \( \omega \) exhibits non-monotone or ambiguous behaviour and is not suited to be applied as leading indicator in this study. Without deseasonalization the skewness \( \gamma \) shows only fragmentary significant results and thus, is not reliable over the range of the considered cases. With deseasonalization the skewness \( \gamma \) yields at least significant results under correlated noise conditions. In contrast to the results of Perretti and Munch (2012) the std \( \hat{\sigma} \) also fails to generate significant results whereas the AR1 seems to be the most robust of the standard measures. Nevertheless, the AR1 is very sensitive to the seasonality of the time series that seems to play an important role in the calculations of the leading indicators in general. deseasonalization has to be taken into account to achieve optimal results, if the noise intensity does not hide the seasonal component. Accordingly, the applicability of the AR1 is enlarged to correlated situations and the clearness of the drift slope trends could be improved. Furthermore, the minimum of necessary data per window for the drift slope estimation could be diminished due to a deseasonalization in the time series. The minimum of available data for the pink and red noise cases is decreased from between 50 and 100 to 25 – 50 data points except for the red noise case with \( \sigma = 4.5 \) and thus lie in the observation range of one year or less. The white noise cases do not benefit in that way from a deseasonalization.

In the end, the drift slope could be an interesting alternative in order to deal with very noisy correlated data under realistic circumstances in ecology and other fields, but it is limited due to the available amount of data. The low-sampled scenarios with one point per year are impossible to handle neither with the drift slope estimation nor with the standard measures. However, in some cases the opportunities of tracking resilience with the drift slope measure might be an attractive reason to improve sampling-rates and data collection wherever possible.

Data and software availability

The simulated data and Python codes are available on github via [https://github.com/MartinHessler/Quantifying_resilience_under_realistic_noise](https://github.com/MartinHessler/Quantifying_resilience_under_realistic_noise) under a GNU General Public License v3.0. The open source python-implementation of the described methods is named antiCPy and can be found at [https://github.com/MartinHessler/antiCPy](https://github.com/MartinHessler/antiCPy) under a GNU General Public License v3.0.

Acknowledgements

M. H. thanks the Studienstiftung des deutschen Volkes for a scholarship including financial support. We thank colleagues and friends for proofreading the manuscript.
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