Natural Generalization of Bosonic String Amplitudes

Makoto Natsuume
Institute for Theoretical Physics
University of California
Santa Barbara, California 93106-4030
and
Theory Group
Department of Physics
University of Texas
Austin, Texas 78712

ABSTRACT

The similarity between tree-level string theory scalar amplitudes, the Koba-Nielsen form ($S^1$) and the Virasoro-Shapiro form ($S^2$) suggests a natural $S^n$ generalization for a scalar amplitude. It is shown that the $S^n$ amplitude shares many essential properties of the string theory amplitudes, including $SO(n + 1, 1)$ conformal symmetry and linear Regge trajectories for the mass spectrum. We also discuss factorization and the critical dimension for the amplitude, which are the necessary conditions for the quantum mechanical consistency (unitarity) of the amplitude.

*makoto@sbitp.ucsb.edu
1 Introduction

In bosonic string theory, the tree-level tachyon amplitudes are given by the Veneziano amplitude for open strings (and the Koba-Nielsen $M$-point generalization; KN hereafter) and the Virasoro-Shapiro (VS) amplitude for closed strings. These describe the scattering process by $M$ identical tachyonic scalars. In integral representation, KN is

$$A_{KN} = \int_{S^1} \prod_{i=1}^{M} dz_i \prod_{i<j} |z_i - z_j|^{2k_ik_j}$$

and VS is

$$A_{VS} = \int_{S^2} \prod_{i=1}^{M} d^2 z_i \prod_{i<j} |z_i - z_j|^{2k_ik_j}.$$ 

Note the remarkable resemblance between KN and VS amplitudes. The only difference is the domain of integration: the domains are $S^1$ (KN) and $S^2$ (VS). The similarity naturally suggests that one should examine the following possible formula for an amplitude by extending the integration domain into $S^n$.

$$A_M = \int_{S^n} \prod_{i=1}^{M} d^n z_i \prod_{i<j} |\vec{z}_i - \vec{z}_j|^{2k_ik_j}. \quad (1)$$

The variables $\vec{z}_i$ are $n$-dimensional vectors integrated over the sphere $S^n$ and $|\vec{z}|^2$ should be understood as a norm of the vector $\vec{z}$. As shown in section 3, $A_M$ expresses a $M$-point scalar amplitude like the KN and VS amplitudes.

The main purpose of this paper is to study the symmetry and the unitarity of the amplitude, which are the first obvious issues to be investigated.

1 Strictly, what we mean by ‘tachyon’ is the scalar which is the ground state of the mass spectrum. The distinction between tachyon and the ground state scalar is necessary since the amplitude considered in this paper allows the higher spin tachyons in its excitation spectrum.

2 We adopt a space-time metric with signature $(-,+,+,...,+)$ in accord with standard string theory convention. The slope parameter $\alpha'$ will be chosen to be $\alpha' = 1$ for convenience.
It turns out that $A_M$ has a natural generalization of the conformal symmetry for the string amplitudes; the algebra of the symmetry is isomorphic to $SO(n + 1, 1)$ if and only if external scalars satisfy a mass-shell condition (Recall that $SL(2, R) \sim SO(2, 1)$ and $SL(2, C) \sim SO(3, 1).$). The unitarity analysis is not completely conclusive due to the limitation the analysis has, but our results do not contradict unitarity; in particular, $A_M$ satisfies the factorization condition, which is a necessary condition for unitarity. Our proposal of $A_M$ does not have a physical motivation, but the simplicity of the generalization and its potential relevance to physics are reasons enough that the amplitude be taken seriously.

Now, one may enquire whether the amplitude has any relevance to string theory or the amplitude suggests some generalization of bosonic string theory. Let us briefly consider the possible physical interpretation of the amplitude suggested by this particular representaion. This formula is defined on $S^n$, not on the world-sheet $S^2$ as string theories. It is then plausible to think $A_M$ expresses the dynamics of relativistic membranes or $p$-branes and it is our conjecture that the $S^n$ amplitude has some relevance to $p$-brane study. However, it is not a trivial issue to make sense of this formula on $p$-branes since the integrand $|\vec{z}_i - \vec{z}_j|^{2k_i k_j}$ may be a specific characteristic of a 2-dimensional world-sheet. In string theories, such a polynomial behavior was the direct consequence of the facts that the Green’s function in 2-dimensions is given by

$$\ln(\mu |z_i - z_j|),$$

and that the vertex operator transforms as the wave function under Poincaré transformations i.e.

$$V_{tachyon} = \int d^2 z e^{i k \cdot X}.$$
Since Green’s functions do not have logarithmic behavior except in 2-dimensions, one needs a more complicated vertex operator in order to interpret the scalar amplitude as the $p$-brane one. In view of such difficulty, we first focus our attention only on unitarity analysis to decide whether $A_M$ is physically sensible or not; delaying its physical interpretation.

In general, unitarity has two possible consequences: factorization of $M$-point amplitudes, and no negative norm states in intermediate processes. The second consequence reduces to the problem of obtaining the critical dimension of the theory. These issues are discussed in Section 3 and 4. In order to carry out the critical dimension analysis, we calculate the 4-point amplitude $A_4$ in Section 2. As a by-product, the symmetry of $A_M$ is shown. Also, we find that in gamma function representation, $A_4$ coincides with a 4-point scalar amplitude proposed by Virasoro\cite{1} (which we call Virasoro amplitude.) through a dual resonance model study.\cite{3} In Section 4, it is also shown that the original Virasoro amplitude is not unitary for non-positive $n$. This result with the fact that the parameter $n$ originates in $S^n$ imply that the $S^n$ is really a physically important object.

## 2 Symmetry and the Virasoro amplitude

The calculation of $A_4$ can be accomplished in a completely analogous way as the KN and VS formulas. The first step is to identify the symmetry of $A_M$ in order to fix the gauge.

$A_M$ is invariant under the following infinitesimal conformal transforma-
translations: \( z'^\mu = z^\mu + \alpha^\mu \)

dilatation: \( z'^\mu = \beta z^\mu \)

\( O(n) \) rotations: \( z'^\mu = \epsilon^\mu_\nu z_\nu \)

special conformal transformations: \( z'^\mu = -2(\gamma \cdot z)z^\mu + \gamma^\mu |z|^2 \).

Its finite form is
\[
\delta z^\mu = a^\mu + b z^\mu + e^\mu_\nu z_\nu + \frac{z^\mu + e^\mu |z|^2}{1 + 2 c \cdot z + |c|^2 |z|^2}.
\] (2)

The only nontrivial symmetries are dilatation and special conformal transformations (SCT). Like the KN and VS amplitudes, the invariance under those symmetries is guaranteed once we impose a mass-shell condition on external scalars.

A SCT is an inversion followed by a translation and another inversion;
\[
\vec{z} \xrightarrow{I} \frac{\vec{z}}{|z|^2} \xrightarrow{T} \frac{\vec{z} + \vec{c} |z|^2}{|z|^2} \xrightarrow{I} \frac{\vec{z} + \vec{c} |z|^2}{1 + 2 c \cdot z + |c|^2 |z|^2}
\]
(Some vector symbols are omitted in order to simplify the expressions.). Hence, one has to only verify the invariance of \( A_M \) under the inversion instead of the general SCT. Under an inversion \( \vec{z}_i \rightarrow \vec{z}_i/|z_i|^2 \),
\[
|\vec{z}_i - \vec{z}_j|^2 \rightarrow \frac{1}{|z_i|^2 |z_j|^2} |\vec{z}_i - \vec{z}_j|^2,
\]
which yields
\[
\prod_{i<j} |\vec{z}_i - \vec{z}_j|^{2k_i k_j} \rightarrow \prod_i |\vec{z}_i|^{-2m_i^2} \prod_{i<j} |\vec{z}_i - \vec{z}_j|^{2k_i k_j},
\] (3)
where \( m_i \) is the mass for the \( i \)th external particles. The measure transforms as
\[
d^n z_i \rightarrow |\vec{z}_i|^{2n} d^n z_i.
\] (4)
Consequently,

\[ A_M \rightarrow \int \prod_i d^n z_i \, |z_i|^{-2(m_i^2+n)} \prod_{i<j} |z_i - z_j|^{2k_i k_j}, \]  

(5)

which shows \( A_M \) is invariant under the inversion if and only if \( m_i^2 = -n \) for all \( i \).

The invariance under the dilatation now follows automatically; under a dilatation \( \vec{z}_i \rightarrow b \vec{z}_i \), the measure and the integrand pick up a factor

\[ |b|^{nM+2 \sum_{i<j} k_i k_j} = |b|^{M(n+m^2)}, \]  

(6)

which is just unity by the mass-shell condition.

The algebra of the conformal group is isomorphic to \( SO(n+1,1) \). In differential operator forms, \((n+1)(n+2)/2\) generators have the following representations:

\[
\begin{align*}
L^\mu_+ &= \frac{\partial}{\partial z_\mu} \\
L_0 &= z^\nu \frac{\partial}{\partial z^\nu} \\
L^\mu^\nu_0 &= z^\mu \frac{\partial}{\partial z^\nu} - z^\nu \frac{\partial}{\partial z^\mu} \\
L^\mu_- &= (2z^\mu z^\nu + \delta^{\mu\nu} |z|^2) \frac{\partial}{\partial z^\nu}.
\end{align*}
\]

Define \( L^{\alpha\beta} = -L^{\beta\alpha} \) (Letters from the beginning of the Greek characters run from 1 to \( n+2 \).) such that

\[
\begin{align*}
L^{\mu\nu} &= L^{\mu\nu}_0 \\
L^{\mu \, n+1} &= \frac{1}{2}(L^\mu_+ - L^\mu_-) \\
L^{\mu \, n+2} &= \frac{1}{2}(L^\mu_+ + L^\mu_-) \\
L^{n+1 \, n+2} &= L_0.
\end{align*}
\]
The generators $L^{\alpha \beta}$ satisfy the commutation relations

$$[L^{\alpha \beta}, L^{\gamma \delta}] = -(\eta^{\alpha \gamma} L^{\beta \delta} - \eta^{\beta \gamma} L^{\alpha \delta} - \eta^{\alpha \delta} L^{\beta \gamma} + \eta^{\beta \delta} L^{\alpha \gamma}),$$

(7)

where $\eta = \text{diag}(+, +, \ldots, +, -)$ showing the algebra of the conformal transformations is isomorphic to $SO(n + 1, 1)$.

Since the translations, dilatation, and SCT consist of a non-compact quotient group, the integral $A_4$ has to be divided by the volume factor of the quotient group by a standard gauge fixing procedure. Corresponding to the dimensionality of the non-compact space, $2n + 1$ coordinates can be fixed, which we take $z_1 = 0$, $z_2$ so that $|z_2| = 1$, and $z_3 = \infty$. Using the residual rotational symmetry of the amplitude to further fix the gauge $z_2 = \hat{e}_1 = (1, 0, \ldots, 0)$, one obtains the Jacobian of the transformation:

$$\frac{\partial(z_1, z_2^1, z_3)}{\partial(\alpha, \beta, \gamma)} = |z_3|^{2n},$$

(8)

which therefore yields

$$A_4 = \int d^n z_4 |\vec{z}_4|^{2k_1 k_4} |\hat{e}_1 - \vec{z}_4|^{2k_2 k_4}$$

(9)

$$= \int dw \cdots dy (w^2 + \cdots + y^2)^{k_1 k_4} \left\{ (1-w)^2 + x^2 + \cdots + y^2 \right\}^{k_2 k_4},$$

(10)

where $\vec{z}_4 = (w, x, \ldots, y)$. This integral is evaluated by a standard trick and the result is the Virasoro amplitude for positive integer $n$:

$$A_4 = \pi^{n/2} \frac{\Gamma\left(-\frac{1}{2}\alpha(s)\right)\Gamma\left(-\frac{1}{2}\alpha(t)\right)\Gamma\left(-\frac{1}{2}\alpha(u)\right)}{\Gamma\left(-\frac{1}{2}\alpha(s) - \frac{1}{2}\alpha(t)\right)\Gamma\left(-\frac{1}{2}\alpha(t) - \frac{1}{2}\alpha(u)\right)\Gamma\left(-\frac{1}{2}\alpha(u) - \frac{1}{2}\alpha(s)\right)}.$$ \(\text{(11)}\)

Here $s$, $t$, and $u$ are the conventional Mandelstam variables. And $\alpha(s)$, etc. are the Regge trajectory functions satisfying

$$\alpha(s) = s + \alpha(0)$$

(12)
with the intercept of the Regge trajectory $\alpha(0) = n$. Even though the original expression $A_M$ is defined only for positive integer $n$, we can now analytically continue $n$ to be any real numbers once we obtain the above expression. $A_4$ therefore reduces to the Virasoro amplitude.

The amplitude exhibits a pole at $s = 2r - n$ in the $s$ channel, where $r$ is a non-negative integer. The mass spectrum is, therefore, given by $m^2 = 2r - n$, which means all particle poles lie on linear Regge trajectories\(^1\) (For the leading trajectory, spin $J = 2r$. See equation (19).). In addition to the linear Regge trajectories, the Virasoro amplitude shares the following physical features with KN and VS amplitudes\(^1\):

1. Crossing symmetry.

2. Superconvergence sum rules.

3. Regge behavior at asymptotic energies.

Although the Virasoro formula itself has been known for over 20 years, the amplitude has been little explored so far. One problem which has limited the study of the Virasoro amplitude was the lack of the $M$-point scalar generalization, since the 4-point scalar amplitude itself can not be a complete solution for the most general $S$-matrix elements. On the other hand, the $M$-point scalar amplitude does contain the complete solution as we will mention in the next section. Also, the lack of the integral representation was another serious problem since it gives clues about the symmetry of the underlying theory and about the elementary quantity of the theory (i.e. particle, string, etc.). Thus, the $S^n$ amplitude may shed light on the Virasoro amplitude.
The residue of the $12 \cdots m \rightarrow (m+1) \cdots M$ channel should be the product of $12 \cdots mP$ and $Pm+1 \cdots M$ tree amplitudes.

## 3 Factorization

We now examine the unitarity of the amplitude. Since our knowledge is limited only to amplitude formulas and the underlying theory is unknown, we lack the systematic analyses using operator formalism or path integral approach, which proved their powers in string theory to show factorization and critical dimension. In order to carry out these analyses, we employ the methods which do not rely on these formalisms but rely only on the amplitude formula.

We employ the same method discussed by Mandelstam\[4\] to show factorization.

In tree-level unitarity, factorization requires the following. Consider a tree-level process with $M$ scalars. The residue of a pole associated with $12 \cdots m \rightarrow (m+1) \cdots M$ channel should be the product of three factors.
The first and the second factors are tree-level subprocess amplitudes with $12 \cdots mP$ and $P(m+1) \cdots M$ respectively. The last factor is an angular factor depending on the angular momentum of the intermediate state (Equation (19) gives a simple example of factorization.). For a scalar pole, the angular factor is a numerical factor, therefore the residue must be the product of two tree-amplitude formulas, which are $m+1$ and $M-m+1$ scalar amplitudes.

The analysis is of course important by itself for unitarity, but there is another reason why we would like to stress this study. The factorization of $A_M$ implies only tree-level unitarity if $A_M$ is regarded as a Born term in a perturbation expansion as KN and VS amplitudes[3]. But once one proves this, factorization enables one to formulate the loop amplitudes required for full unitarity. Moreover, the general $S$-matrix elements are now constructed by the repeated factorization from the $M$-point scalar amplitude[4].

The fixed variables for the gauge fixing are chosen to be $\vec{z}_1 = 0$, $\vec{z}_{m+1} = \hat{e}_1$, and $\vec{z}_M = \infty$. We also introduce the polar coordinate for $\vec{z}_m$, $\vec{z}_m = (\rho, \phi, \theta_1, \cdots, \theta_{n-2})$ in which $0 < \rho < \infty$, $0 < \phi < 2\pi$, and $0 < \theta_k < \pi$.

Now, define new variables by

$$\vec{z}_i' = \frac{1}{\rho} R^T \vec{z}_i \quad (13)$$

for particles $i = 1, 2, \cdots, m$. Here, the rotation matrix $R^T(\phi, \theta_k)$ is defined so that $R^T \vec{z}_m = \rho \hat{e}_1$. Note $\vec{z}_m' = \hat{e}_1$.

In terms of the new variables, $A_M$ is expressed as

$$A_M = \int_0^\infty d\rho \rho^{-s+m^2+1} \int d\phi \prod_{k=1}^{n-2} \sin^k \theta_k d\theta_k \int d^n z_2' \cdots d^n z_{m-1}' d^n z_{m+2} \cdots d^n z_{M-1} \prod_{i<j\leq m} |\vec{z}_i' - \vec{z}_j'|^{2k_i k_j} \prod_{i>j\geq m} |\vec{z}_i' - \vec{z}_j'|^{2k_i k_j} \prod_{i>m, j\leq m} |\vec{z}_i' - \rho R^T \vec{z}_j'|^{2k_i k_j}, \quad (14)$$
where \( s = -(k_1 + \cdots + k_m)^2 \).

We expand the last factor in a Taylor series around \( \rho = 0 \):

\[
\prod_{i>m,j\leq m} |\vec{z}_i - \rho \vec{R} \vec{z}'_j|^{2k_i k_j} = \sum_r \frac{1}{r!} \left( \frac{\partial}{\partial \rho} \right)^r \prod_{i>m,j\leq m} |\vec{z}_i - \rho \vec{R} \vec{z}'_j|^{2k_i k_j} \bigg|_{\rho=0} \rho^r. \tag{15}
\]

The \( \rho \) integration in (14) gives a pole at \( s = 2r - n \) for \( n > 1 \) or at \( s = r - n \) for \( n = 1 \). This follows by observing that

\[
|\vec{z}_i - \rho \vec{R} \vec{z}'_j|^2 = |\vec{z}_i|^2 - 2 \rho \vec{z}_i \cdot (\vec{R} \vec{z}'_j) + \rho^2 |\vec{z}'_j|^2. \tag{16}
\]

Consider the terms in (15) which contain the odd power of \( r \); those are proportional to the second term in (16) which vanishes by the angular part of the integral in (14). On the other hand, the even \( r \) terms always contain the terms which are angle independent, hence nonvanishing by the integral. Obviously, this is the case except \( n = 1 \).

The residue \( R_{2r} \) for the pole at \( s = 2r - n \) is

\[
R_{2r} \propto \int d^n z'_2 \cdots d^n z'_{m-1} \prod_{i<j\leq m} |\vec{z}'_i - \vec{z}'_j|^{2k_i k_j} \\
\times \int d^n z_{m+2} \cdots d^n z_{M-1} \prod_{i>j\geq m} |\vec{z}_i - \vec{z}_j|^{2k_i k_j} \prod_{i>m} |\vec{z}_i|^{-2k_i \sum_{j>m} k_j} \\
\times F_{2r}(\vec{z}_i, \vec{z}'_j, k_i k_j), \tag{17}
\]

where \( F_{2r} \) is the angular factor. This factor can be decomposed into a sum of terms, each consisting of two factors which depend only on \( \vec{z} \) and \( \vec{z}' \) respectively; therefore implies factorization. Each term in the sum expresses the resonance with spin ranging from 0 to \( 2r \).

As an explicit example, the residue \( R_0 \) associated with the first pole, \( s = -n \), is

\[
R_0 \propto \int d^n z'_2 \cdots d^n z'_{m-1} \prod_{i<j\leq m} |\vec{z}'_i - \vec{z}'_j|^{2k_i k_j}
\]
\[ \times \int d^n z_{m+2} \cdots d^n z_M \prod_{i>j \geq m} |\vec{z}_i - \vec{z}_j|^{2k_i k_j} \prod_{i>m} |\vec{z}_i|^{-2k_i} \prod_{j>m} k_j \cdot \prod_{i>m} |\vec{z}_i|^{-2k_i} \cdot \prod_{j>m} k_j. \] (18)

The residue is separated into two integrals, the integrals with \( i < j \leq m \) variables and the integrals with \( i > j \geq m \) variables. Thus, we can identify these as \( A_{m+1} \) and \( A_{M-m+1} \) respectively with the sets of fixed variables:

\[
\left\{ \begin{array}{l}
A_{m+1} : \quad \vec{z}'_1 = 0, \quad \vec{z}'_m = \hat{e}_1, \quad \vec{z}'_P = \infty \\
A_{M-m+1} : \quad \vec{z}_P = 0, \quad \vec{z}_{m+1} = \hat{e}_1, \quad \vec{z}_M = \infty.
\end{array} \right.
\]

Here, \( \vec{z}_P \) and \( \vec{z}'_P \) are the new variables corresponding to the particle in the intermediate states.

This example gives a proof that the external tachyons are scalars, and the same scalars as the ground state on the leading Regge trajectory.

### 4 Critical Dimension

Since \( A_M \) describes a spinless scattering process as shown above, factorization and partial wave analysis demand that the residue \( R \) of the 4-point amplitude \( A_4 \) is expressed as

\[ R = \sum_{l=0}^{\infty} G_l^2 P_l(z) \] (19)

for the incoming and outgoing states which differ by a relative angle. Here, \( z = \cos \theta \), where \( \theta \) is center-of-mass scattering angle. \( P_l \) are Legendre polynomials in \( d \)-dimensional spacetime\(^5\). Up to numerical factors, \( G_l \)'s are the coupling constants of the external scalars with intermediate spin-\( l \) particles. In general, the hermitity of a Lagrangian, which requires the couplings to be real, also implies \( G_l^2 \geq 0 \). This requirement strongly constrains a given amplitude formula, so that the formula is valid only for a small interval of \( d \).

This is a nice trick to get the magic number 26 for open string\(^6\). Consider the second pole in the Veneziano amplitude, namely the \( \alpha(s) = 2 \) pole. The
Residue is given by
\[ R_2(z) \propto (z^2 - \frac{1}{25}). \] (20)

The corresponding partial wave analysis formula gives
\[ R_2(z) = G_2^2(z^2 - \frac{1}{d-1}) + G_0^2. \] (21)

Comparing these two formulas, one concludes the coupling of the scalar, \( G_0^2 \), is negative for \( d > 26 \). The scalar completely decouples and the intermediate state becomes pure spin-2 when \( d = 26 \).

The problem of the method is that this does not work even for closed strings. The first massive level for closed string is \( \alpha(s) = 4 \) pole with \( s = 2 \). A similar calculation gives \( d = 72 \), which is certainly wrong.

The origin of the problem is not difficult to see. A closed string state is formed by a tensor product of an open string state with itself. The open string state for \( \alpha(s) = 2 \) contained a physical spin-2 and an unphysical spin-0 state. So, the corresponding closed string state \( \alpha(s) = 4 \) contains a physical spin-2 state in addition to two unphysical spin-2 states. Thus, the amplitude with the incoming and outgoing states \( |s\rangle \) and \( |s'\rangle \), is written as
\[ \langle s'|s \rangle = \cdots + \left\{ (26 - d)|\langle s|u_1\rangle|^2 + (26 - d)|\langle s|u_2\rangle|^2 + |\langle s|p\rangle|^2 \right\} P_2(z) + \cdots, \]
where \( |u_i\rangle \) and \( |p\rangle \) are unphysical and physical spin-2 states respectively. Also, we extract the angular dependence from the amplitude. Since \( |\langle s|p\rangle|^2 \) is positive definite, the value within the braces can be positive even when \( d > 26 \). In other words, the problem is that there exist several distinct states at a given spin level, so that the amplitude must be described not by a single coupling \( G_2^2 \), but by several \( (G_2^2)^2 \), which have to be all positive in order to satisfy unitarity.
Supposing such physical states also 'contaminate' $A_4$ for general $n$, one gets the necessary condition for unitarity, but not the sufficient condition. For a $\alpha(s) = 2r$ pole, where $r$ is a positive integer, we obtain

$$d \leq 5 - 4r + \frac{r(2r - 1)(3n + 2r)^2}{\sum_{i=0}^{r-1}(-n + 2r - 4i)^2}$$

by demanding $(G_{2r-2})^2 \geq 0$. The lower spin state equations, $(G_{2r-2i})^2 \geq 0$, where $1 < i \leq r$, gives the weaker conditions on $d$ in general. For closed string case, this is because the lower spin levels contain more and more physical states. Eq. (22) gives $d < 57$ and $83$ for $S^2$ and $S^3$ respectively, at their minima. In spite of the limitation the analysis has, eq. (22) indicates that the Virasoro amplitude for non-positive $n$ does not lead to sensible quantum theories. $n = 0$ gives $d < 2$ and negative $n$ have negative critical dimensions. This exclusion of the non-positive region for $n$ might imply $S^n$ is not simply a convenient integral representation but really a physical object such that non-positive $n$ are ill-defined.

5 Comments

In this paper, we explored some aspects of the $S^n$ amplitude putting stress on unitarity. In the future, the question of critical dimension must be regarded as the first and foremost problem to be solved.

There is one possibility which may improve our calculation of critical dimension. Consider more general configurations than 2-body scattering, such as 3-body scattering. The residues can be evaluated in the same manner as in Section 3, but now the residue of the poles contain the particle four-momenta as free parameters which we can vary. Then, one might find the region in parameter space in which the physical states decouple.
Another problem we must solve is to derive $A_M$ as a $p$-brane amplitude. We have noticed this is a non-trivial issue, but it becomes more evident if one notes that $A_M$ has the linear Regge trajectories. This clearly contradicts the relation given by Kikkawa and Yamasaki\cite{footnote}. The difference comes from the fact that (1) needs a dimensionful constant $\alpha'$ in the exponent; therefore, the ‘tension’ has the unit $M/L$ whereas the $p$-brane tension has the unit $M/L^p$.

Unfortunately, this amplitude is ‘worse’ than the conventional string theories: the mass spectrum is more and more ‘tachyonic’ as $n$ increases since the Virasoro amplitude gives $m^2 = J - n$. On the other hand, this means one gets massless states with higher spins. For example, the $S^4$ case contains a spin-4 massless state. This is a point worth making since no quantum mechanically consistent theories with massless high spin particles are known. Our amplitude therefore may provide a theory with massless high spin particles just as string theories provided consistent theories with massless spin-2 particles for the first time in physics history.

**Note added**

After the completion of this paper, I learned there exist papers which cover some of this work. For the earliest work, see R. C. Brower and P. Goddard, Lett. Nuovo Cimento 1, 1075 (1971). A recent work on this idea is M. B. Green and C. B. Thorn, Nucl. Phys. B367, 462 (1991). I wish to thank D. Fairlie and M. B. Green for comments about these early literatures.

**Acknowledgements**

I am grateful to J. Polchinski for having suggested the problem and for his continuous assistance throughout the work. I also thank J. LaChapelle
for critical reading of the manuscript. This research was supported in part by the Robert A. Welch Foundation, NSF Grant PHY 8904035 and 9009850, and the Texas Advanced Research Program Grant 476.
References

[1] M. A. Virasoro, Phys. Rev. 177, 2309 (1969); and independently by Rubinstein. See G. Altarelli and H. R. Rubinstein, Phys. Rev. 178, 2165 (1969).

[2] J. H. Schwarz, Phys. Rep. 8, 269 (1973).

[3] K. Kikkawa, B. Sakita, and M. A. Virasoro, Phys. Rev. 184, 1701 (1969).

[4] S. Mandelstam, Phys. Rep. 13, 259 (1974); K. Bardakci and S. Mandelstam, Phys. Rev. 184, 1640 (1969).

[5] See, for instance, H. Hochstadt, The Functions of Mathematical Physics (Dover Publications, New York, 1986).

[6] P. H. Frampton, Dual Resonance Models (W. A. Benjamin Inc., Reading, Mass., 1974); G. Veneziano, Phys. Rep. 9, 199 (1974).

[7] K. Kikkawa and M. Yamasaki, Prog. Theor. Phys. 76, 1379 (1986).
