We study the collective modes of gluons in an anisotropic thermal medium in the presence of a constant background magnetic field using the hard-thermal loop (HTL) perturbation theory. The momentum space anisotropy of the medium has been incorporated through the generalized ‘Romatschke-Strickland’ form of the distribution function, whereas, the magnetic modification arising from the quark loop contribution has been taken into account in the lowest Landau level approximation. We consider two special cases: (i) a spheroidal anisotropy with the anisotropy vector orthogonal to the external magnetic field and (ii) an ellipsoidal anisotropy with two mutually orthogonal vectors describing anisotropies along and orthogonal to the field direction. The general structure of the polarization tensor in both cases are equivalent and consists of six independent basis tensors. We find that the introduction of momentum anisotropy ingrains azimuthal angular dependence in the thermo-magnetic collective modes. Our study suggests that the presence of a strong background magnetic field can significantly reduce the growth rate of the unstable modes which may have important implications in the equilibration of magnetized quark-gluon plasma.

I. INTRODUCTION

The incredible success of hydrodynamical approaches in describing the enormous data of heavy ion collision experiments [1, 2], including the nuclear suppression factor, radial flow and elliptic flow measurements provides compelling evidence to believe that the produced hot and dense matter thermalizes within about 0.6 fm/c after the initial impact [3–5]. However, in the seminal work by Baier, Mueller, Schiff and Son in Ref. [6], the thermalization time via scattering processes in the weak-coupling limit has been estimated theoretically to be 2.5 fm/c or above. Recent studies [7–9] in weak coupling limit have improved our understanding of quark-gluon-plasma (QGP) equilibration to a great extent. On the other hand, several attempts also has been made to study the thermalization at strong coupling limit within AdS/CFT formulation [10–14]. The modern formulations of relativistic fluid dynamics suggests that neither local near-equilibrium nor near-isotropy is required in order to have a successful hydrodynamical description of the experimental results [15]. Several efforts have been made over the years in the development of relativistic viscous hydrodynamics [16] which systematically incorporates the dissipative effects [17–20]. The viscous hydrodynamics concludes that at time $\sim 2$ fm/c, the QGP created in ultra relativistic heavy-ion collisions (URHIC) has different longitudinal and transverse pressures [10]. This occurs due to the rapid expansion of the QCD matter along the longitudinal direction (beam direction) which gives rise to a large local rest frame momentum space anisotropy [10, 21, 22] in the $p_T - p_L$ plane. This anisotropic momentum distribution can cause plasma instabilities in the system which contribute in the thermalization and isotropization process of the QCD plasma [23–26]. It is found that the exponential growth of the unstable modes plays an important role in the dynamics of the system in weak coupling limit [27]. In the hydrodynamic side, the anisotropic hydrodynamics (aHydro) framework is formulated to efficiently take into account the large momentum space anisotropy of the system [28, 29]. On the other hand, the hard-thermal-loop perturbation theory [30, 31] has been employed [22, 32] to systematically study the properties of anisotropic QCD plasma. Typically, one uses a specific distribution function of light quarks and gluons which is widely known as the ‘Romatschke-Strickland’ (RS) form [22, 33]. There has been a concerted effort to study the effect of the momentum-space anisotropies on the heavy-quark potential [32, 34, 35], bottomonia suppression [36–38], photon and dilepton production rates [39, 40], wake potential [41] and so on. The generalized RS form of the distribution function [42] that takes into account the azimuthal momentum-space anisotropy has been recently investigated in Refs. [43–45].

On the other hand, the production of strong magnetic fields [46, 47] at early stages of the non-central heavy-ion collisions has triggered enormous research interest in the theoretical, phenomenological and experimental understanding.
of the strongly interacting matter under extreme conditions [48, 49]. The time dependence of the produced magnetic field has remained a subject of debate for a long period of time in the heavy-ion collision community [48, 50, 51]. At the early stages of the collision, the system is dominated mainly by the gluons. Subsequently, a large number of quarks and anti-quarks are produced and the system evolves towards the equilibrium. Therefore, the system is believed to be much less conducting in the early times. Considering the Pb+Pb collision at \( \sqrt{s} = 2.76 \text{ TeV} \), it is found in Ref. [51] that for an insulating medium, a magnetic field of strength \( \sim 100 m_\gamma^2 \) rapidly decays to a very low value within around 0.1 fm/c after the initial impact [48]. The rapid decrease in the field strength follows the 1/\( t^3 \) behavior. However, the electrical conductivity of the medium significantly influences the time evolution of the electromagnetic fields in the late stage when the system reaches near the equilibrium state. It is shown in the Refs. [52, 53] that the electrical conductivity of the medium can resist the decay of the magnetic field, at least to some extent. Intense research works have been performed to study the properties of the QCD matter in presence of such strong magnetic background which resulted in several interesting findings like chiral magnetic effect [46, 54], magnetic catalysis [55], inverse magnetic catalysis [56, 57], non-trivial magnetic modifications of chiral symmetry broken/restored phases [58, 59], photon and dilepton production rate [60–65], thermodynamic properties [56, 66–68], heavy quark potential [69], transport coefficients [70, 71] and so on.

The production of strong magnetic field at early stages of collision naturally motivates one to investigate the magnetic field effects on anisotropic QGP. In presence of external magnetic field ( with intensity \( B \) ), one can define a hierarchy of energy scales as \( \sqrt{eB} \gg T \gg g_s T \) which essentially determines the regime of validity of the strong magnetic field approximation. Here \( e \) denotes the electric charge of proton and \( g_s \) is the strong coupling constant. In this regime, the quarks occupy only the lowest Landau level and the dynamics becomes 1+1 dimensional. In this article, we restrict ourselves to the lowest Landau level approximation and investigate the gluon collective modes in presence of anisotropic momentum distribution. For this purpose, the one loop gluon self energy is obtained in the HTL approximation using the real time formalism of thermal field theory. We note here that the general structure of the polarization tensor plays an important role in the determination of the effective propagator and the collective modes. The thermo-magnetic collective modes has been studied recently in Refs. [72, 73]. The direction of the external magnetic field brings in an anisotropy in the system and naturally breaks the spherical symmetry. It also appears among the available four vectors that has to be taken into account for the construction of the general structure. The situation is similar to the spherialoid momentum space anisotropy. Thus, it is interesting to compare the two scenarios: one is the anisotropy due to the background field and the other is the anisotropy that arises due to the modeling of the non-equilibrium distribution function from the equilibrium distribution by suitable stretching or squeezing.

In the present study we systematically address this issue. Throughout the article, we use the following convention: \( \eta_{\mu\nu} = \text{diag}(1,0,0,-1) \) and \( \eta^\mu_\nu = \text{diag}(0,-1,-1,0) \) with \( \eta_{\mu\nu} = \eta^\mu_\nu + \eta^\nu_\mu \) where the Lorentz indices \( \{ \mu, \nu \} \in \{ 0, 1, 2, 3 \} \).

For a generic four vector \( a^\mu \), we define \( a^\mu_0 = (a^0, 0, 0, a^z) = (a_0, 0, a_z) \) and \( a^\mu_1 = (0, a^1, a^2, 0) = (0, a_x, a_y, 0) \). The corresponding scalar products are defined as \( (a_0 \cdot b_0) = a^0 b^0 - a^3 b^3 \) and \( (a_1 \cdot b_1) = -a^1 b^1 - a^2 b^2 \).

## II. FORMALISM

In this section we obtain the one loop gluon self energy in presence of anisotropic thermo-magnetic medium within HTL approximation. For this purpose we follow the real-time Schwinger-Keldysh formalism [26, 74–77] based on contour Green’s functions which is applicable for non-equilibrium field theories. The basic formalism to obtain the retarded, advanced and the Feynman self-energies is reviewed in [32, 77] in a self-contained manner. Here we briefly recall the essential steps to obtain the retarded part of the gluon self-energy in an anisotropic background. In the real time Keldysh formalism, the Green’s functions for the quark field of a given flavour \( \psi_\alpha \) and gluon field \( A_\mu^a \) can be expressed as

\[
\begin{align*}
    i [S(x,y)]^{ab}_{\alpha\beta} &= \langle \hat{T} [\psi_\alpha^a(x)\overline{\psi}_\beta^b(y)] \rangle , \\
    i [D(x,y)]^{ab}_{\mu\nu} &= \langle \hat{T} [A_\mu^a(x)A^b_{\nu}(y)] \rangle .
\end{align*}
\]

where the spinor indices are represented by the set \( \{ \alpha, \beta \} \in \{ 1, 2, 3, 4 \} \) and the color indices in fundamental and adjoint representations of \( SU(N_c) \) group with \( N_c = 3 \) are represented by the sets \( \{ i,j \} \in \{ 1, 2, 3 \} \) and \( \{ a, b \} \in \{ 1, 2, 3, 4 \} \) respectively. Here the angular bracket notation \( \langle \cdots \rangle \) denotes the expectation value, and the time ordering \( \hat{T} \) of two generic fields \( \Phi_1 \) and \( \Phi_2 \) has the usual meaning

\[
\hat{T} [\Phi_1(x)\Phi_2(y)] = \Theta(x^0 - y^0)\Phi_1(x)\Phi_2(y) \pm \Theta(y^0 - x^0)\Phi_2(y)\Phi_1(x) ,
\]

where \( \Theta \) denotes the Heaviside step function and the upper (lower) sign corresponds to the bosonic (fermionic) nature of the \( \Phi \) fields. At one loop level, the gluon polarization function has three different contributions arising namely from

\[
\begin{align*}
    \hat{T} [\Phi_1(x)\Phi_2(y)] &= \Theta(x^0 - y^0)\Phi_1(x)\Phi_2(y) \pm \Theta(y^0 - x^0)\Phi_2(y)\Phi_1(x) ,
\end{align*}
\]
the gluon tadpole and loop diagrams, the ghost loop diagram and the quark loop diagram. In presence of external magnetic field, the contributions from gluon and ghost remain unmodified whereas corrections appear in the quark loop contribution. Moreover, in the HTL approximation, the expressions of the photon and gluon self-energy differ only in the definition of the Debye mass. Thus, to find the net contribution of the gluon and ghost loops in presence of anisotropic momentum distribution, it is convenient to obtain the photon polarization function first without the external magnetic field, and then replace the QED Debye mass by the corresponding QCD expression for pure glue [32]. The retarded self energy so obtained, is given by [43, 44]:

\[
\tilde{\Pi}_{ab}^{\mu\nu}(\omega, p, \xi) = \delta_{ab} \bar{m}_D^2 \int \frac{d\Omega_w}{4\pi} \frac{v^\mu + \xi_1 (v \cdot a_1) a_1^\mu + \xi_2 (v \cdot a_2) a_2^\mu}{(1 + \xi_1 (v \cdot a_1)^2 + \xi_2 (v \cdot a_2)^2)^2} \left[ \eta^{\nu\ell} - \frac{v^\nu p^\ell}{\omega - p \cdot v + i0^+} \right] \bigg|_{\xi \in \{1, 2, 3\}},
\]

where the strong coupling constant \(g_s\) appears explicitly in the expression of \(\bar{m}_D^2 = \frac{g_s^2}{4\pi} N_c\) which corresponds to the QCD Debye mass with \(N_f = 0\), and the scale \(\Lambda_T\) in the equilibrium limit corresponds to the temperature. It should be mentioned here that the dependence on the parameters \(\Lambda_T, \xi_1\) and \(\xi_2\), originates from the modeling of the non-equilibrium distribution function following the ‘Romatschev-Strickland’ procedure. The anisotropic distribution function for the gluons and ghosts is constructed from the bosonic equilibrium distribution function as [78]

\[
f_{\text{aniso}}^B(k) \equiv f_{\text{iso}}^B \left( \frac{1}{\Lambda_T} \sqrt{k^2 + \xi_1 (k \cdot \hat{x})^2 + \xi_2 (k \cdot \hat{y})^2 + \xi_3 (k \cdot \hat{z})^2} \right).
\]

In the conformal limit, with a simple rearrangement of the parameters given by

\[
\frac{1 + \xi_x}{1 + \xi_y} \rightarrow 1 + \xi_1, \quad \frac{1 + \xi_x}{1 + \xi_y} \rightarrow 1 + \xi_2, \quad \Lambda_T \rightarrow \frac{\Lambda_T}{\sqrt{1 + \xi_y}},
\]

one can characterize the ellipsoidal anisotropic distribution function in terms of the anisotropy tuple \(\xi = (\xi_1, \xi_2)\) and the scale \(\Lambda_T\). Thus, the nonequilibrium bosonic distribution relevant for the present scenario is given by [32, 44]

\[
f_{\text{aniso}}^B(k) \equiv f_{\text{iso}}^B \left( \frac{\sqrt{k^2 + \xi_1 (k \cdot \hat{a}_1)^2 + \xi_2 (k \cdot \hat{a}_2)^2}}{\Lambda_T} \right).
\]

In this work, the spatial anisotropy vectors \(\hat{a}_1, \hat{a}_2\) are chosen along \(\hat{x} = (1, 0, 0)\) and \(\hat{z} = (0, 0, 1)\) directions respectively whereas the spatial components \(v^\mu\) of the parton four velocity \(v^\mu = (1, v)\) as well as the external momentum vector \(p\) are chosen in the spherical polar coordinates with angles \((\theta_k, \phi_k)\) and \((\theta_p, \phi_p)\) respectively. To obtain the quark loop contribution of the retarded self energy in real time, here we recall the required definitions of the four Green’s functions based on the propagation along the contour:

\[
\begin{align*}
&i \left[ S^\text{>}^\text{a}(x, y) \right]_{\alpha\beta}^{ij} = \left\langle \psi_\alpha^i(x) \overline{\psi}_\beta^j(y) \right\rangle, \\
&i \left[ S^\text{<}^\text{a}(x, y) \right]_{\alpha\beta}^{ij} = -\left\langle \overline{\psi}_\beta^j(y) \psi_\alpha^i(x) \right\rangle, \\
&i \left[ S^\text{<}^\text{c}(x, y) \right]_{\alpha\beta}^{ij} = \left\langle \hat{T}^c \left\langle \psi_\alpha^i(x) \overline{\psi}_\beta^j(y) \right\rangle \right\rangle, \\
&i \left[ S^\text{>}^\text{c}(x, y) \right]_{\alpha\beta}^{ij} = \left\langle \hat{T}^c \left\langle \overline{\psi}_\beta^j(y) \psi_\alpha^i(x) \right\rangle \right\rangle.
\end{align*}
\]

Here \(\hat{T}^c\) is same as the usual time-ordering operator \(\hat{T}\) defined earlier whereas the anti-time-ordering operator \(\hat{T}^a\) is defined as

\[
\hat{T}^a [\Phi_1(x) \Phi_2(y)] = \Theta(y^0 - x^0) \Phi_1(x) \Phi_2(y) \pm \Theta(x^0 - y^0) \Phi_2(y) \Phi_1(x),
\]

where the upper (lower) sign corresponds to the bosonic (fermionic) nature of the generic \(\Phi\) fields. The Green’s function \(S^{\text{>}/\text{<}}(x, y)\) is the same as the time ordered propagator \(S(x, y)\) with both \(x^0\) and \(y^0\) chosen on the upper/lower branch of the contour where the contour runs along the forward/backward time direction. On the other hand \(S^{\text{<}/\text{>}}(x, y)\) and \(S^{\text{<}/\text{>}}(x, y)\) is same as \(S(x, y)\) with \(x^0\) on the upper and \(y^0\) on the lower branch and vice versa. To avoid clutter in the notations, let us first consider the photon self-energy in presence of magnetic background which is given by

\[
i\Pi^{\mu\nu}(x, y) = -e^2 \text{Tr} \left[ \gamma^\mu S(x, y) \gamma^\nu S(y, x) \right]
\]
where $e$ is the magnitude of the electron charge and $S(x, y)$ represents the electron propagator in presence of magnetic field. From the similar definitions as given in (7), one can easily express the polarization tensor as a sum of $\Pi^\mu_\nu$ and $\Pi^\mu_\nu$, where

\[
i \Pi^\mu_\nu(x, y) = -e^2 \text{Tr} \left[ \gamma^\mu S^>(x, y) \gamma^\nu S^<(y, x) \right],
\]

\[
i \Pi^\nu_\mu(x, y) = -e^2 \text{Tr} \left[ \gamma^\nu S^>(x, y) \gamma^\mu S^<(y, x) \right].
\]

(10)

Now, the retarded self energy is defined as

\[
\Pi^\mu_\nu_R(x, y) = \theta(x^0 - y^0) \left[ \Pi^\mu_\nu(x, y) - \Pi^\nu_\mu(x, y) \right].
\]

(11)

It should be noted here that the fermion propagator in presence of a background magnetic field possess a multiplicative phase factor which spoils translational invariance [79]. However, in the one loop photon polarization, the phase factor arising from the two fermion propagator cancels each other and only the translationally invariant parts of the propagators contribute. The same argument also applies for the quark loop in the gluon polarization tensor that we are interested in. Thus, from here on out, it is useful to decompose the fermion propagator as [80] $S(x, y) = e^{i \Phi(x, y)} \mathcal{S}(x - y)$ and consider only the invariant $\mathcal{S}(x - y)$ part in the self energy. In that case, we are free to choose $y = 0$ because of translationally invariant and obtain the retarded self-energy as

\[
i \Pi^\mu_\nu_R(x) = -\frac{e^2}{2} \text{Tr} \left[ \gamma^\mu \mathcal{S}_F(x) \gamma^\nu \mathcal{S}_A(-x) + \gamma^\nu \mathcal{S}_R(x) \gamma^\mu \mathcal{S}_F(-x) \right].
\]

(12)

Note that, in the above expression, the $S^>$ and $S^<$ propagators that arise from Eq.(10) and Eq.(11), have been expressed in terms of the Feynman, advanced and retarded propagators which are defined respectively as

\[
S_F(x, y) = S^>(x, y) + S^<(x, y),
\]

\[
S_A(x, y) = -\theta(y^0 - x^0) \left[ S^>(x, y) - S^<(x, y) \right],
\]

\[
S_R(x, y) = \theta(x^0 - y^0) \left[ S^>(x, y) - S^<(x, y) \right].
\]

(13)

In the momentum space one obtains

\[
i \Pi^\mu_\nu_R(p) = -\frac{e^2}{2} \int \frac{d^4k}{(2\pi)^2} \text{Tr} \left[ \gamma^\mu \mathcal{S}_F(k) \gamma^\nu \mathcal{S}_A(q) + \gamma^\nu \mathcal{S}_R(k) \gamma^\mu \mathcal{S}_F(q) \right],
\]

(14)

where $q = k - p$. In the mass-less limit, the invariant part of the propagators with lowest Landau level approximation are given by [80]

\[
\mathcal{S}_R(k) = \bar{k}_\parallel (1 + s_\perp i\gamma^1 \gamma^2) \Delta_R(k) = \bar{k}_\parallel (1 + s_\perp i\gamma^1 \gamma^2) \exp \left( -\frac{k^2}{k_B T} \right),
\]

\[
\mathcal{S}_A(k) = \bar{k}_\parallel (1 + s_\perp i\gamma^1 \gamma^2) \Delta_A(k) = \bar{k}_\parallel (1 + s_\perp i\gamma^1 \gamma^2) \exp \left( -\frac{k^2}{k_B T} \right),
\]

\[
\mathcal{S}_F(k) = \bar{k}_\parallel (1 + s_\perp i\gamma^1 \gamma^2) \Delta_F(k) = \bar{k}_\parallel (1 + s_\perp i\gamma^1 \gamma^2) \left[(2\pi i) \left[ -1 + 2 f_F(k_\parallel) \right] \delta(k_\parallel^2) \exp \left( -\frac{k^2}{k_B T} \right) \right],
\]

(15)

where $s_\perp = \text{sgn}(e_f B)$ with ‘sgn’ representing sign function and the electric charge of the fermion is denoted as $e_f = q e$ which is equal to $-e$ for the electron. Also we note that the expressions of the fermion propagator used here is derived for a constant magnetic field intensity $B$ along the $\hat{z}$ direction which is same as the direction of the anisotropy vector $\mathbf{a}_2$. It should be noted that in presence of a background magnetic field, the energy eigenvalue for the free fermion only depends on the longitudinal momentum (say $k_\parallel$) and the Landau level index (say $n$) as these are the conserved quantum numbers independent of the gauge choice. On the other hand, the transverse momentum, which appears in the expression of the propagators, should be considered only as a conjugate variable arising from the Fourier transform of the translationally invariant part and it does not appear in the energy eigenvalue. Hence, in the lowest landau level approximation ($n = 0$), we construct the nonequilibrium fermion distribution function from the equilibrium Fermi-Dirac distribution function $f_F(k_\parallel)$ as

\[
f^F_{\text{aniso}}(k_\parallel) \equiv f^F_{\text{iso}} \left( \sqrt{k_\parallel^2 + \xi_\perp^2 (k \cdot \hat{z})^2 / \Lambda_T^2} \right) = f^F_{\text{iso}} \left( |k_\parallel| / \Lambda_T \right),
\]

(16)
As in the thermal case, the above expression can further be simplified by expressing the magnitude and the angular structure of the propagator, the longitudinal and the transverse part of the integrals gets separated and one can easily perform the integral over the transverse momentum as

$$\int \frac{d^2 k_z}{(2\pi)^2} \exp \left(-\frac{k_z^2}{|eB|}\right) \exp \left(-\frac{q_z^2}{8\pi}\right) = \frac{|eB|}{8\pi} \exp \left(-\frac{p_z^2}{2|eB|}\right).$$  \hfill (18)

The polarization function now becomes

$$\Pi^{\mu\nu}_R(p) = -4e^2|eB| \exp \left(-\frac{p_z^2}{2|eB|}\right) \int \frac{d^2 k_z}{(2\pi)^2} f_F(k_z) \left[ \frac{2k_z^2 k_{z,\mu}^\nu - (k_{1z}^\mu p_z^\nu + k_{2z}^\mu p_z^\nu) + \eta_{1z}^{\mu\nu}(k_1 \cdot p_\perp)}{p_z^2 - 2(k_1 \cdot p_\perp) - i\epsilon \text{sgn}(k_0 - p_0)} \right] \delta(k_z^2).$$  \hfill (19)

In the HTL approximation we consider the external momentum to be ‘soft’ that is \(p \sim eA_T\) and the internal momentum is ‘hard’ that is \(k \sim A_T\). With this hierarchy, a Taylor series expansion of the terms inside the square brackets can be performed which up to second order is given as

$$\frac{2k_z^2 k_{z,\mu}^\nu - (k_{1z}^\mu p_z^\nu + k_{2z}^\mu p_z^\nu) + \eta_{1z}^{\mu\nu}(k_1 \cdot p_\perp)}{p_z^2 - 2(k_1 \cdot p_\perp) - i\epsilon \text{sgn}(k_0 - p_0)} \approx \frac{2k_z^2 k_{z,\mu}^\nu}{-2(k_1 \cdot p_\perp) - i\epsilon \text{sgn}(k_0)} - \frac{\eta_{1z}^{\mu\nu}(k_1 \cdot p_\perp)}{2} + \frac{2k_z^2 k_{z,\mu}^\nu}{2(k_1 \cdot p_\perp) + i\epsilon \text{sgn}(k_0)} - \frac{p_z^2}{2(k_1 \cdot p_\perp) + i\epsilon \text{sgn}(k_0)}.$$  \hfill (20)

As in the thermal case, the first term in the expansion does not contribute. Integrating over the \(k_0\) variable using the delta function property

$$\delta(k_z^2) = \frac{\delta(k_0 - |k_z|) + \delta(k_0 + |k_z|)}{2|k_z|},$$  \hfill (21)

one obtains

$$\Pi^{\mu\nu}_R(p) = e^2|eB| \exp \left(-\frac{p_z^2}{2|eB|}\right) \int \frac{dk_z}{2\pi} \frac{f_F(k_z)}{|k_z|} \left[ \eta_{1z}^{\mu\nu} - k_{1z}^\mu p_z^\nu + k_{2z}^\mu p_z^\nu + \eta_2^{\mu\nu}(k_2 \cdot p_\perp) + i\epsilon \text{sgn}(k_0 - p_0) \right] \bigg|\bigg|_{k_0=|k_z|}.$$  \hfill (22)

The term in the square braces can be related to a total derivative term as

$$\left[ \eta_{1z}^{\mu\nu} - k_{1z}^\mu p_z^\nu + k_{2z}^\mu p_z^\nu + \frac{p_z^2 k_{1z}^\mu k_{2z}^\nu}{(k_1 \cdot p_\perp) + i\epsilon} \right]_{k_0=|k_z|} = -|k_z| \frac{\partial}{\partial k_z} \left[ p_z |k_z| \frac{k_{1z}^\mu k_{2z}^\nu}{(k_1 \cdot p_\perp) + i\epsilon} \right]_{k_0=|k_z|}.$$  \hfill (23)

After performing an integration by parts with the assumption \(\lim_{k_z \to \pm\infty} f(k_z) = 0\) one obtains

$$\Pi^{\mu\nu}_R(p) = e^2|eB| \exp \left(-\frac{p_z^2}{2|eB|}\right) \int \frac{dk_z}{2\pi} \frac{\partial f_F(k_z)}{\partial k_z} \left[ p_z |k_z| \frac{k_{1z}^\mu k_{2z}^\nu}{(k_1 \cdot p_\perp) + i\epsilon} - \frac{k_{2z}^\mu \eta_{2z}^{\nu\mu}}{|k_z|} \right]_{k_0=|k_z|}. $$  \hfill (24)

As in the thermal case, the above expression can further be simplified by expressing the magnitude and the angular integrals separately. Considering the anisotropic distribution function as given in Eq. (16) one obtains

$$\Pi^{\mu\nu}_R(p) = \frac{m_B^2 e}{2} \exp \left(-\frac{p_z^2}{2|eB|}\right) \sum_{\text{sgn}(k_z) = \pm 1} \frac{v_{1z}^\mu v_{1z}^\nu}{1 + |\xi_2|} \left[ \eta_{1z}^{\mu\nu} - \frac{v_{1z}^\mu p_z^\nu}{(v_z \cdot p_\perp) + i\epsilon} \right]_{l=3},$$  \hfill (25)
where the Debye mass is defined as

\[ m_D^2 = -\frac{e^2}{\pi^2} |eB| \int d|k_z| \left| \frac{\partial f^{(0)} (|k_z|)}{\partial |k_z|} \right| = \frac{e^2 |eB|}{2\pi^2}. \]  

(26)

One can observe that, as a consequence of dimensional reduction in the strong field approximation, the solid angle integral with the 4π angular average in Eq. (3), now reduces to a summation along with an average over two possible directions. It should be noticed that unlike the thermal case, the self-energy is independent of the momentum tensor can be obtained from Eq. (25) as [81]

However, the implicit dependence on the momentum scale is present due to the running of the coupling constant. Now, incorporating the flavor sum and the color factor, the quark loop contribution in the retarded gluon polarization tensor can be obtained from Eq. (25) as [81]

\[ \Pi_{ab}^{\mu\nu}(p) = \delta_{ab} \sum_f g_f^2 \frac{|e_f B|}{8\pi^2} \exp \left( -\frac{p^2_T}{2|e_f B|} \right) \sum_{\text{sgn}(k_z) = \pm 1} \frac{u^\mu_{||} v^\nu_{||}}{1 + \xi_2} \left[ \eta^{\rho\theta} - \frac{v^\rho_{||} p^\theta_{||} (v_{||} \cdot p_{||} + i\epsilon)}{\pi} \right] \right|_{l=3}. \]  

(27)

In the static limit (\( \omega = 0, p \to 0 \)), the temporal component \( \Pi_{ab}^{\mu\nu} \) with \( \xi_2 = 0 \) becomes [73]

\[ \tilde{m}_D^2 = \sum_f g_f^2 \frac{|e_f B|}{4\pi^2}, \]

which, together with the magnetic field independent contribution \( \tilde{m}_D^2 \), defines the Debye screening mass \( \tilde{m}_D = \sqrt{\tilde{m}_D^2 + m_D^2} \). Finally, the retarded gluon polarization function is obtained from the individual contributions given in Eq. (3) and Eq. (27) as

\[ \Pi_{ab}^{\mu\nu}(p, eB, \xi, \Lambda_T) = \tilde{\Pi}_{ab}^{\mu\nu}(p, \xi_2, \Lambda_T) + \Pi_{ab}^{\mu\nu}(p, eB, \xi_2, \Lambda_T), \]

(29)

where the dependence on the external parameters \( p, eB, \xi \) and \( \Lambda_T \) has been shown explicitly in each case. The polarization function is symmetric in the Lorentz indices (\( \Pi^{\mu\nu} = \Pi^{\nu\mu} \)) and satisfies the transversality condition \( p_\mu \Pi^{\mu\nu} = 0 \). Incorporating these constraint relations, the general structure of the polarization function can be constructed from the available basis tensors. A suitable choice in this regard is the basis set constructed for the ellipsoidal momentum anisotropy in Ref. [44]. A list of the required basis tensors is provided in the Appendix A for completeness. In that basis, the gluon polarization tensor can be expressed as

\[ \Pi^{\mu\nu} = \alpha A^{\mu\nu} + \beta B^{\mu\nu} + \gamma C^{\mu\nu} + \delta D^{\mu\nu} + \sigma E^{\mu\nu} + \lambda F^{\mu\nu}, \]

(30)

and the corresponding form factors can be extracted from Eq. (29) through suitable projections. Here we note that, all of the projection tensors are symmetric and transverse to the external momentum. Thus, the decomposition of the polarization function trivially satisfies the symmetry constraint as well as the transversality condition. Now, the effective gluon propagator can be obtained from the Dyson–Schwinger equation

\[ D = D_0 - D_0 \Pi D. \]

Here \( D_0 \) is the bare propagator and its inverse, with the gauge fixing parameter \( \zeta \), is given by

\[ (D_0^{-1})^{\mu\nu} = -p^2 \eta^{\mu\nu} - \frac{1 - \zeta}{\zeta} p^\mu p^\nu. \]  

(32)

From the pole of the effective propagator, one can obtain the gluon collective modes by solving

\[ p^2 - \Omega_{0,\pm}(p) = 0. \]  

(33)

Any deviation from the light-like dispersion is encoded in the mode functions \( \Omega_{0,\pm} \) which are given by [44]

\[ \Omega_0 = \frac{1}{3} (\alpha + \beta + \delta) - \frac{1}{3} \left( \frac{\varpi}{\chi + \sqrt{4 \varpi^2 + \chi^2}} \right)^{\frac{1}{\gamma}} + \frac{1}{3} \left( \frac{\chi + \sqrt{4 \varpi^2 + \chi^2}}{2} \right)^{\frac{1}{\gamma}}, \]

(34)

\[ \Omega_{\pm} = \frac{1}{3} (\alpha + \beta + \delta) + \frac{1}{6} \sqrt{3} \left( \frac{\varpi}{\chi + \sqrt{4 \varpi^2 + \chi^2}} \right)^{\frac{1}{\gamma}} - \frac{1}{6} \sqrt{3} \left( \frac{\chi + \sqrt{4 \varpi^2 + \chi^2}}{2} \right)^{\frac{1}{\gamma}}, \]

(35)
where, the \( \varpi \) and \( \chi \) in the expression are defined in terms of the form factors as

\[
\varpi = \alpha (\beta - \alpha) + \beta (\delta - \beta) + \delta (\alpha - \delta) - 3(\gamma^2 + \lambda^2 + \sigma^2),
\]

\( (36) \)

\[
\chi = (2\alpha - \beta - \delta)(2\beta - \delta - \alpha)(2\delta - \alpha - \beta) + 54\gamma\lambda\sigma - 9\left[\alpha(2\lambda^2 - \sigma^2 - \gamma^2) + \beta(2\sigma^2 - \gamma^2 - \lambda^2) + \delta(2\gamma^2 - \lambda^2 - \sigma^2)\right].
\]

\( (37) \)

It should be noted here that among the six form factors, only the \( \alpha, \beta \) and \( \gamma \) gets modified in presence of the external magnetic field. However, all of the form factors depend on the external anisotropy parameter \( \xi \). Now, each of the mode functions being a non-trivial combination of all six form factors, it is expected that, in addition to the anisotropy induced effects, all the gluon collective modes will possess magnetic modifications. We explore such anisotropic gluon collective modes in the following section.

**III. RESULTS**

**FIG. 1.** A convenient choice of the reference frame is schematically shown for the following three cases: (a) the ellipsoidal momentum anisotropy without the external magnetic field, (b) the external magnetic field without the momentum space anisotropy and (c) the external magnetic field along with the momentum anisotropies as analyzed in the present study.

As mentioned earlier, for a fixed set of external parameters, the gluon collective modes can be obtained by solving Eq. (33) which only requires the knowledge of the momentum dependence of the form factors. Choosing a particular orientation of the reference frame, this momentum dependence can be obtained from the components of the polarization tensor. Regarding the choice of the reference frame, it should be noted that usually in the studies concerned with a single momentum space anisotropy (say \( \hat{a}_1 \)) without the background field, the anisotropy vector is chosen along the \( \hat{z} \) direction which refers to the beam direction. In the ellipsoidal generalization (see for example Ref. [44]), a convenient way to introduce another anisotropy direction (say \( \hat{a}_2 \)) is to consider it to be orthogonal to the previous anisotropy vector \( \hat{a}_1 \). Without any loss of generality, one can reorient the reference frame to consider the \( \hat{a}_2 \) direction along the \( \hat{x} \) axis as schematically shown in Fig. 1(a). On the other hand, in the studies concerned with the external magnetic field (see for example Refs. [72, 73]), the \( \hat{z} \) direction usually refers to the direction of the constant magnetic field which is orthogonal to the reaction plane. A schematic representation of the corresponding geometry is shown in Fig. 1(b). Now, to incorporate the two scenarios simultaneously, which is the main motivation of the present study, it is important to take into account the relative orientations of the anisotropy directions with respect to the beam axis. A convenient choice of the reference frame for this purpose is schematically shown in Fig. 1(c). In this geometry, the reaction plane is considered to be the X-Y plane, the polar angle \( \theta_p \) is measured with respect to the external magnetic field direction (\( \hat{z} \)) whereas the azimuthal angle \( \phi_p \) represents the angle between the anisotropy direction \( \hat{a}_1 \) chosen along the beam axis (\( \hat{x} \)) and the projection of the momentum vector \( \hat{p} \) in the reaction plane. For the following numerical analysis, we fix the orientation of the reference frame according to Fig. 1(c). Note that, the second anisotropy vector \( \hat{a}_2 \) is assumed to be parallel to the external field direction. As a consequence, when the two momentum anisotropies are switched off, the geometry reduces to the scenario shown in Fig. 1(b). Also it is evident from Fig. 1(c) that even without the second anisotropy vector \( \hat{a}_2 \), one has to incorporate two anisotropy directions for the analysis of the collective modes. Consequently, the general structure of the polarization tensor in this case, as
mentioned earlier, becomes similar to the ellipsoidal momentum anisotropy scenario. Here we note that, without the background field, \( \Lambda_T \) being the only available energy scale for the anisotropic plasma, the tensor components (and consequently the form factors and the mode functions) are proportional to the square of the thermal Debye screening mass given by

\[
m_D = \sqrt{\frac{g_s^2 \Lambda_T^2}{3}} \left( N_c + \frac{N_f}{2} \right),
\]

and one can do away with the \( \Lambda_T \) dependence by simply expressing the dispersion in terms of the scaled variables \( \omega/m_D \) and \( |p|/m_D \). When the external magnetic field is turned on, the gluon and the quark loop contribution respectively becomes proportional to \( \bar{m}_D^2 \) and \( \bar{m}_D^2 \). However, to compare with the thermal case, here we consider the same \( m_D \) for scaling instead of the thermo-magnetic Debye mass \( \bar{m}_D \). With \( N_c = 3 \), the ratio of the square of the Debye masses arising from the gluon and the quark contribution can be expressed as

\[
\frac{\bar{m}_D^2}{m_D^2} = R^2 \sum_f |q_f|,
\]

where the ratio of the two energy scales is set by \( 2\pi R = \sqrt{eB}/\Lambda_T \). In the present study, we consider \( eB = 30m^2 \) and \( \Lambda_T = 0.2 \text{ GeV} \) which gives \( 2\pi R \sim 3 \). The value of the coupling \( g_3 \) at the fixed \( \Lambda_T \) is determined considering the one loop running. For this purpose, the \( \overline{\text{MS}} \) renormalization scale is set at 0.176 GeV by fixing the QCD fine structure constant \( \alpha_s(1.5 \text{ GeV}, N_f = 3) = 0.326 \) [82, 83]. With these fixed set of external parameters, we now obtain the three stable gluon collective modes characterized by the corresponding mode functions given in Eqs. (34) and (35). At first we consider the case with one anisotropy direction. This can arise either due to the presence of the external magnetic field or by the expansion of the medium resulting in an anisotropic momentum distribution of the partons. Now, irrespective of the origin of the anisotropy, the gluon polarization tensor can be expressed in terms of four basis tensors and the pole of the effective propagator gives rise to the same mode functions given by [44, 73]

\[
\Omega_0 = \frac{1}{2} \left( \alpha + \beta + \sqrt{(\alpha - \beta)^2 + 4\gamma^2} \right),
\]

\[
\Omega_+ = \frac{1}{2} \left( \alpha + \beta - \sqrt{(\alpha - \beta)^2 + 4\gamma^2} \right),
\]

\[
\Omega_- = \delta.
\]

However, it should be observed that the parameter dependences of the form factors in the two cases are completely different and it is interesting to compare the two scenarios. In Fig. 2, we consider the dispersion corresponding to the mode function \( \Omega_0 \) for two different values of \( \theta_p = \left\{ \pi/2, \pi/4 \right\} \) which represents the angle between the anisotropy vector and the external momentum. One can notice that the angular dependence is weak in both cases. In contrast to the magnetic field case, the mode corresponding to the spheroidal anisotropy shows more prominent angular dependence in the low momentum regime. Also it can be noticed that the plasma frequency in presence of the external magnetic field is significantly larger compared to the spheroidal anisotropy scenario. The introduction of the magnetic field enhances the plasma frequency for this mode compared to the isotropic case (also shown in the figure) whereas a spheroidal anisotropy decreases it. The scenario is quite different in case of the collective mode corresponding to \( \Omega_+ \) as shown in Fig. 3. In presence of a background magnetic field one can observe a prominent angular dependence in the dispersion shown in Fig. 3(a). In the two limiting cases when the propagation angle \( \theta_p \) is zero and \( \pi/2 \), the collective mode becomes identical respectively to the transverse (\( \Pi_T \)) and the longitudinal (\( \Pi_L \)) mode of the isotropic gluonic medium [72, 73]. It should be noted here that as \( \gamma \) vanishes in the isotropic case and \( \beta \) and \( \delta = \Pi_T \) become degenerate, one obtains two distinct dispersive modes (also shown in the figure) corresponding to the mode functions [84]

\[
\Omega_0 = \Pi_L = -\bar{m}_D^2 \frac{\omega^2 - p^2}{p^2} \left[ 1 - \frac{\omega}{2p} \ln \frac{\omega + p}{\omega - p} \right],
\]

and

\[
\Omega_\pm = \Pi_T = \frac{\bar{m}_D^2}{2} \omega^2 \left[ 1 - \omega^2 - \frac{p^2}{2\omega p} \ln \frac{\omega + p}{\omega - p} \right].
\]

For the intermediate angles (shown for \( \theta_p = \pi/12, \pi/6 \)), the mode lies within the isotropic dispersion curves of the pure gluonic medium. On the other hand, in case of spheroidal momentum space anisotropy, the angular dependence
FIG. 2. The collective mode of gluon corresponding to the mode function $\Omega_0$ is shown at fixed momentum scale $\Lambda_T = 0.2$ GeV and propagation angles $\theta_p = \pi/2$ (shown in solid style) and $\pi/4$ (shown in dotted style) for two different cases: (i) with external magnetic field $eB = 30m_D^2$ (shown in red) and (ii) with spheroidal anisotropy (shown in blue). The light cone (magenta) and the isotropic collective modes (green and cyan) are also shown for comparison.

of the collective mode is quite different from the magnetic field case as shown in Fig. 3(b). Here we consider anisotropy tuple $\xi = (0,10)$. One can notice that the angular dependence is weaker. Moreover, the isotropic dispersions can not be recovered by simply varying the propagation angle. It should be noted here that in this case the isotropic mode functions are same as Eq. (43) and Eq. (44) however with the replacement of $\tilde{m}^2_D$ by $m^2_D$. Comparing the modes in Fig. 3(a) and Fig. 3(b), one can observe that in both cases the plasma frequency decreases compared to the isotropic value with $N_f = 3$ and for the external magnetic field, it becomes equal to the plasma frequency of the isotropic pure gluonic medium ($N_f = 0$). Interestingly, due to the similar decomposition of the basis tensor, in both cases the mode characterized by $\Omega_{-}$ becomes identical to the corresponding isotropic transverse mode (see Eq. (42) where $\delta$ is respectively proportional to $\tilde{m}^2_D$ and $m^2_D$ for the magnetic and spheroidal anisotropy case) and consequently becomes independent of the propagation angle.

FIG. 3. Angular variation of the collective mode of gluon corresponding to the mode function $\Omega_{\pm}$ is shown at fixed momentum scale $\Lambda_T = 0.2$ GeV for two different cases: (a) with external magnetic field $eB = 30m_D^2$ and (b) with spheroidal anisotropy. The light cone is shown in continuous magenta style and the isotropic collective modes (with (a) $N_f = 0$ and (b) $N_f = 3$) are shown in continuous green and cyan style for comparison.

Finally, the dispersion relation for the three stable modes in presence of momentum space anisotropy as well as external magnetic field is shown in Fig. 4. In the left panel, we consider the momentum anisotropy along $\hat{x}$ which is orthogonal to the magnetic field direction (along $\hat{z}$) and fix the anisotropy tuple at $\xi = (10,0)$, whereas, in the right panel, the dispersion is shown for ellipsoidal momentum anisotropy with two anisotropy directions : one along the magnetic field (i.e. along $\hat{z}$) and the other orthogonal to it (i.e. along $\hat{x}$). In this case the anisotropy tuple is set at $\xi = (10,5)$. It should be noted that in the presence of either magnetic field or spheroidal momentum anisotropy (say along $\hat{z}$), the rotational symmetry of the system is broken and the dispersive modes depend on the direction of propagation of the gluons which is characterized by the polar angle $\theta_p$. However, when the two anisotropy directions are considered together, as long as they are not parallel to each other, the azimuthal symmetry of the system is also broken and consequently, the collective modes show azimuthal angular dependence. Here we consider
a fixed propagation direction now characterized by \( \theta_p = \pi/4 \) and \( \phi_p = \pi/6 \). Unlike the magnetic field case discussed earlier (where the plasma frequencies of \( \omega^+ \) and \( \omega^- \) were degenerate), one can observe from Fig. 4(a), that all the collective modes possess different plasma frequencies. Moreover, an overall decrease in the magnitude is observed compared to the thermo-magnetic modes (shown in Figs. 2 and 3(a)). Once the ellipsoidal anisotropy is considered, the plasma frequencies further decreases for all the modes as can be seen from Fig. 4(b). This is in fact expected from Eq. (27) as the anisotropy parameter \( \xi_2 \) essentially suppresses the quark loop contribution thereby decreasing the overall magnitude.

Let us now consider the influence of the magnetic field on the unstable modes of the anisotropic medium. As in the case of spheroidal [22] and ellipsoidal momentum anisotropy [44], in the limit \( \omega \to 0 \), one can define three mass scales (\( m_0 \) and \( m_{\pm} \)) corresponding to the mode functions \( \Omega_0 \) and \( \Omega_{\pm} \). A negative value of a given squared mass indicates the existence of an unstable mode. It should be mentioned here that instead of considering \( N_f = 3 \), if one considers a two flavour plasma, all the qualitative features remain the same and in the following, we study the mass scales and the instability growth rate considering \( N_f = 2 \).

In Fig. 5 we show the variation of the squared mass with the propagation angle of the gluon with respect to the magnetic field direction. For a fixed \( \phi_p = \pi/12 \), we consider two scenarios: one with \( \xi = (10,0) \) and the other with \( \xi = (10,5) \). In the former case, as we increase \( \theta_p \), \( m_0^2 \) and \( m_{\pm}^2 \) gradually become negative. However, a positive value is observed for \( m_0^2 \) throughout the \( \theta_p \) range. One should note that, at small \( \phi_p \) (as considered here), the higher
values of $\theta_p$ indicates proximity to the anisotropy axis and the observed angular dependence of the mass scales is similar to the spheroidal anisotropy case [22]. When the momentum anisotropy along the magnetic field direction is turned on, all the mass scales become nearly independent of $\theta_p$. In this case, a prominent negative value for $m^2_{2+}$ is observed for the entire range of the polar angle. It is interesting to compare the scenario with the ellipsoidal anisotropy results as obtained in Ref. [44]. For this purpose, in Fig. 6, we show the directional dependence of the square mass scales with and without the external magnetic field. Here we consider $\xi = (10, 5)$. One can notice that the angular dependence of the mass scales are similar to the ellipsoidal anisotropy scenario showing a positive $m^2_0$ throughout the considered range of $\theta_p$ and $\phi_p$ along with instability windows for $m^2_{\pm}$.

As already mentioned, the negative values in the square mass indicate the presence of unstable modes whose amplitude grows exponentially with time. The growth rate of such instabilities (that is the imaginary part of the mode frequency) can be obtained from the pole of the effective propagator. For this purpose, the mode frequency ($p^\mu = \omega$) in Eq. (33) is replaced by $i\Gamma_{0, \pm}$ and one looks for the solution of $\Gamma$ corresponding to each mode functions [43, 44]. The numerical solution for $\Gamma_+$ is shown in Fig. 7 for a fixed propagation direction ($\theta_p, \phi_p) = (\pi/3, \pi/12)$. In the left panel, we consider the spheroidal momentum anisotropy with $\xi = (10, 0)$ whereas in the right panel, we take $\xi = (10, 5)$ characterizing an ellipsoidal momentum space anisotropy. It can be observed that in both cases the amplitude of the growth rate significantly decreases in presence of the external magnetic field. It is also interesting to compare these modified growth rates with the pure gluon results. This is because, it provides information on the relative importance of the field dependent part compared to the field independent part of the self energy. In the left panel of Fig. 7, one can notice a prominent suppression compared to the pure gluon result where the anisotropy direction is considered orthogonal to the external field direction. On the other hand, with the same external parameter set, when the anisotropy along the magnetic field direction ($\xi_2$) is switched on (as shown in the right panel), the total $eB$ dependent quark loop contribution in the self energy gets suppressed which results in a growth rate similar to the pure gluonic plasma. It should be mentioned here that in the present analysis, the instability has been shown for some specific values of the external parameters namely the anisotropy tuple $\xi = (\xi_1, \xi_2)$, the external angles $\theta_p$ and $\phi_p$, the external field...
strength $eB$ and the momentum scale $\Lambda_T$. While the magnitudes of the field strength and the momentum scale are adjusted to justify the LLL approximation, the values of the anisotropy tuple $\xi$ are chosen similar to previous studies with the ellipsoidal anisotropic distribution \cite{43,44}. On the other hand, the propagation direction is set at $(\theta_p, \phi_p) = (\pi/3, \pi/12)$ where, at the fixed magnetic field value, the instability for the $\Omega_+$ mode is expected for both $\xi_2 = 0$ and $\xi_2 = 5$ (see Fig. 5 for the angular variation of the squared mass). The parameter set chosen for the present analysis is not unique and different parameter choices are possible. The growth rates being complicated functions of the angular variables, in general, a different propagation direction with unstable modes is expected to provide a different amount of suppression in the growth rate in the presence of the background magnetic field. However, based on the present analysis, a large value of $\xi_2$ is expected to significantly suppress the dimensionally reduced quark loop contribution thereby leading to instabilities similar to the pure glue scenario.

It is interesting to note that for the spheroidal and ellipsoidal anisotropy without any magnetic background, there exists a critical value of the momentum beyond which the growth rate becomes negative and the instability ceases to exist. When the external magnetic field is turned on, we observe a significant decrease in the critical momentum providing a smaller momentum window for the positive growth rate. The situation may be compared to the instabilities in collisional plasma \cite{85–87} where a critical collisional frequency exists beyond which the growth rate becomes negative for any value of external momentum. In a similar way, one may expect a critical magnetic field intensity beyond which no instabilities occur. Here we recall that in the present study we have considered the field intensity $\sqrt{eB}$ as high as three times the momentum scale $\Lambda_T$ to justify the lowest Landau level approximation. Now, for the anisotropic collisional plasma, a small change in the collisional frequency significantly reduces the growth rate \cite{85}. However, in the present study we find that, even if one increases the magnetic field to several times the considered value, the amplitude and the critical momentum corresponding to the growth rate hardly decreases. Thus, as long as the heavy ion collisions are concerned, a critical magnetic field intensity is unlikely to be present in the realistic scenario.

IV. SUMMARY AND CONCLUSION

In this article, the collective modes of gluon in the presence of momentum space anisotropy along with a constant background magnetic field have been studied using the hard-thermal loop perturbation theory. For this purpose, we have obtained the one loop gluon self energy in the real time Schwinger-Keldysh formalism. The contributions from the gluon and ghost loops remain unaffected by the external magnetic field whereas the entire modification arises from the quark loop contribution which has been evaluated in the lowest Landau level approximation. To extract the Lorentz invariant form factors from the polarization tensor, we implement the basis decomposition obtained in Ref. [44] which is originally constructed for describing the ellipsoidal momentum anisotropy. From the pole of the effective gluon propagator, we obtain three stable dispersive modes of gluon. At first we compare the collective modes of spheroidal anisotropy with that of isotropic thermal background along with external magnetic field. In both cases, the dispersion is governed by four non-vanishing form factors. Though the mode functions in terms of the form factors are identical in the two cases, the form factors themselves are different. Consequently, significant differences are observed in the angular dependence of the collective modes. When the external magnetic field is considered along with spheroidal or ellipsoidal momentum anisotropy, the azimuthal symmetry of the system is lost. As a result, the collective modes depend on the polar as well as on the azimuthal angles corresponding to the propagation direction. It is observed that due to the dimensional reduction in the LLL approximation, the parameter $\xi_2$ that characterizes the anisotropy along the magnetic field direction, appears in the quark loop only in an overall suppressing factor. Thus, the momentum anisotropy along the magnetic field direction essentially counterbalances the magnetic field effects. As the quark loop contribution is suppressed in this case, we observe smaller plasma frequencies for all the collective modes.

To investigate the unstable modes, we have studied the angular dependence of the squared mass scales corresponding to each mode functions. Depending upon the propagation direction, we have observed negative values in the squared masses corresponding to $\Omega_+$ indicating instability in the collective modes. Here we note that no unstable gluon mode exists in an isotropic medium even in the presence of a background magnetic field. It is the momentum space anisotropy that gives rise to the instability. However, the external magnetic field has a significant influence on the growth rate of the unstable modes. In particular, the amplitude as well as the critical momentum corresponding to the growth rate of the unstable mode is significantly reduced in presence of strong magnetic background. This observation is similar to the instability growth rate in anisotropic collisional plasma \cite{85} where larger collisional frequency suppresses the growth rate and eventually, no unstable mode exists beyond a critical frequency. However, it has been argued that the realistic collision frequencies usually lie within the critical value. Here also we find that for anisotropic thermal medium with realistic magnetic field intensity ( which is expected to be present in heavy ion collisions ), unstable collective modes do exist in certain propagation direction.

The present study has several interesting future directions. First of all, due to the lowest Landau level approxima-
tion, only a 1+1 dimensional quark dynamics is considered here. Consequently, the momentum anisotropy orthogonal to the magnetic field direction does not affect the quark loop contribution at all. However, a non-trivial influence of such momentum space anisotropy is expected in the weak field limit where the energy eigen value of the quarks have the usual three momentum dependence. Thus it is interesting to contrast such scenario with the strong field case as presented here. Also, the fermionic collective modes have recently been studied in presence of magnetic field [88] and also in case of ellipsoidal anisotropy [78]. Thus, the combined effect of the magnetic field and momentum space anisotropy on the fermionic collective modes deserves further investigation. Similar scenario also exists in the studies of heavy quark potential where the effect of the external magnetic field and the momentum anisotropy has been considered individually [32, 34, 69, 89, 90] and their mutual influence remains to be explored. We intend to pursue such exploration in future.

ACKNOWLEDGMENTS

A. M. would like to acknowledge fruitful discussions with Ashutosh Dash and Sunil Jaiswal. B. K. acknowledges HORIZON 2020 European research council (ERC) 2016 Consolidation grant, ERC-2016-COG: 725741: QGP TOMOGRAPHY (under contract with ERC). R. G. is funded by University Grants Commission (UGC). A. M. acknowledges Department of Science and Technology (DST), Government of India, for funding.

Appendix A: List of basis tensors

To obtain the general structure of the gluon polarization tensor in presence of spheroidal or ellipsoidal momentum space anisotropy along with background magnetic field, we require a set of six independent basis tensors. The basis tensors can be constructed using the metric \( \eta_{\mu\nu} \) and the available four vectors given by the fluid velocity \( u^\mu \), gluon four momentum \( p^\mu \) and the two anisotropy directions \( a_1^\mu \) and \( a_2^\mu \). In case of spheroidal anisotropy with external magnetic field, either \( a_1^\mu \) or \( a_2^\mu \) should be considered and the magnetic field vector \( b^\mu \) plays the role of the other anisotropy direction whereas for ellipsoidal scenario, \( b^\mu \) becomes redundant. A suitable choice for the independent basis tensors is given by the following set:

\[
A^{\mu\nu} = \frac{\tilde{u}^\mu \tilde{u}^\nu}{u^2},
\]

\[
B^{\mu\nu} = \frac{\tilde{a}_1^\mu \tilde{a}_2^\nu}{a_2^2},
\]

\[
C^{\mu\nu} = \frac{\tilde{u}^\mu \tilde{a}_1^\nu + \tilde{u}^\nu \tilde{a}_1^\mu}{\sqrt{u^2} \sqrt{a_1^2}},
\]

\[
D^{\mu\nu} = \frac{\tilde{a}_1^\mu \tilde{a}_1^\nu}{a_1^2},
\]

\[
E^{\mu\nu} = \frac{\tilde{u}^\mu \tilde{a}_2^\nu + \tilde{u}^\nu \tilde{a}_2^\mu}{\sqrt{u^2} \sqrt{a_2^2}},
\]

\[
F^{\mu\nu} = \frac{\tilde{a}_1^\mu \tilde{a}_2^\nu + \tilde{a}_2^\mu \tilde{a}_1^\nu}{\sqrt{a_1^2} \sqrt{a_2^2}},
\]

where, we have defined

\[
\tilde{u}^\mu = \left( \eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) u_\nu = V^{\mu\nu} u_\nu ,
\]

\[
\tilde{a}_1^\mu = (V_\nu - A_\nu^\mu) a_2^\nu = U_\nu^\mu a_2^\nu ,
\]

\[
\tilde{a}_2^\mu = (U_\nu^\mu - B_\nu^\mu) a_1^\nu = R_\nu^\mu a_1^\nu .
\]

Using the projection operator properties, one can obtain the form factors in terms of the self energy components. For example, it is easy to obtain the relations like

\[
\alpha = \frac{1}{u^2} u_\mu u_\nu \Pi^{\mu\nu},
\]

\[
\beta = \frac{1}{a_2^2} \left( (a_2)_\mu(a_2)_\nu - 2 \frac{a_2}{u^2} u_\mu (a_2)_\nu + \left( \frac{a_2}{u^2} \right)^2 u_\mu u_\nu \right) \Pi^{\mu\nu},
\]
Together with similar relations for the other three form factors, one can express the mode functions in terms of the self energy components. It is easy to show that due to the transversality condition and the symmetric nature of the polarization tensor, explicit evaluation of only six spatial components is sufficient to obtain all the form factors and hence, the mode functions. In the present analysis, the modified quark loop contribution in presence of the external magnetic field has been obtained analytically whereas the anisotropic pure gluon contribution or the anisotropic gluon self energy without $eB$ has been evaluated numerically. To obtain an analytic expression in the small-$\xi$ limit, it is convenient to rotate the lab reference axes to the parton frame [43, 45, 78]. In that case the rotated anisotropy and momentum directions are given by

$$ a^\mu_i = (0, 0, -\sin \theta_p, \cos \theta_p), $$

$$ a^\mu_i = (0, \sin \phi_p, \cos \phi_p, \sin \theta_p \cos \phi_p), $$

$$ \hat{p}^\mu = (\hat{p}_0, 0, 0, 1), $$

where $\hat{p}_0 = p_0/|p|$. Once the scaled form factors are expanded in $\xi_1$ and $\xi_2$, the solid angle integral can be analytically performed. For example, keeping only up to the leading order terms in the small-$\xi$ expansion, one obtains

$$ \alpha = \frac{1}{24} \left( \hat{p}_0^2 - 1 \right) \left( -3 \cos(2\theta_p)(\xi_1 - 2\xi_2) \left( 6\hat{p}_0^2 + (3\xi_0^2 - 2) \hat{p}_0 (\log(\hat{p}_0 - 1) - \log(\hat{p}_0 + 1)) - 2 \right) 
+ 6\xi_1 \sin^2 \theta_p \left( 6\hat{p}_0^2 + (3\xi_0^2 - 2) \hat{p}_0 (\log(\hat{p}_0 - 1) - \log(\hat{p}_0 + 1)) - 2 \right) \cos(2\phi_p) 
- 3\hat{p}_0 (\log(\hat{p}_0 - 1) - \log(\hat{p}_0 + 1)) \left( \xi_1 (\hat{p}_0^2 - 2) - 2\xi_2 \hat{p}_0^4 + 4 \right) + 3 \xi_1 (10 - 2\xi_0^2) + 4 \xi_2 (2\xi_2 - 6) \right) \right) $$

$$ \beta = \frac{1}{192} \left( 2 (\hat{p}_0^2 - 1) \cos(2\theta_p) (4\hat{p}_0^2 + 3 (7\hat{p}_0^2 - 5) \hat{p}_0 (\log(\hat{p}_0 - 1) - \log(\hat{p}_0 + 1)) - 16) \xi_1 \cos(2\phi_p) + 1 \right) $$

$$ - 3\hat{p}_0 (\log(\hat{p}_0 - 1) - \log(\hat{p}_0 + 1)) \left( 1 - \hat{p}_0^2 + 3 \hat{p}_0 (\hat{p}_0^2 + 1) + 8 - 3\xi_1 \hat{p}_0 \left( 3\hat{p}_0^2 - 1 \right) \log(\hat{p}_0 - 1) \right) $$

$$ + (1 - 3\hat{p}_0) \log(\hat{p}_0 - 1) + 6\hat{p}_0 \cos(2\phi_p) + 4\hat{p}_0 \left( -11\hat{p}_0 - 2\xi_2 + 6\xi_1 \hat{p}_0^2 - 6\xi_2 \hat{p}_0^4 + 24 \right)), $$

$$ \gamma = -\frac{1}{12} \sqrt{\hat{p}_0^2 - 1} \left( 11\hat{p}_0 - 12\hat{p}_0^3 + 3 (1 - 5\hat{p}_0^2 + 4\hat{p}_0^4) \coth^{-1}(\hat{p}_0) \right) \cos \theta_p (\xi_1 - 2\xi_2 + \xi_1 \cos(2\phi_p)) \sin \theta_p. $$

In the limit $\xi_1 = 0$, one can recover the small-$\xi$ expanded known results given in Ref. [22] once the differences in the basis construction (see for example Ref. [34] for the covariant formulation) is taken into account.

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