Universality of the second mixed moment of the characteristic polynomials of the 1D band matrices: real symmetric case

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Abstract

We prove that the asymptotic behavior of the second mixed moment of the characteristic polynomials of the 1D Gaussian real symmetric band matrices coincides with those for the Gaussian Orthogonal Ensemble (GOE). Here we adapt the approach of [18], where the case of 1D Hermitian random band matrices was considered.

1 Introduction

In [18] we proved that the asymptotic behavior of the second mixed moment of the characteristic polynomials of the 1D Gaussian Hermitian band matrices coincides with those for the Hermitian random matrices with i.i.d. (modulo symmetry) Gaussian random entries (GUE). The convenient integral representation for the second correlation function of the characteristic polynomials was obtained there by using the supersymmetry techniques (SUSY). The SUSY method is widely used in the physics literature (see, e.g., [8,15]) and is potentially very powerful but the rigorous control of the integral representations, which can be obtained by this method, is difficult. So far the most of rigorous results obtained by using the SUSY approach concern the case of the Hermitian matrices (i.e., the case of unitary symmetry). The goal of this paper is to show that the SUSY approach can be applied to the real symmetric matrices (i.e., to the case of the orthogonal symmetry) as well, as to the Hermitian case.

As in [18], we consider the following model of the real symmetric Gaussian random band matrices (RBM). Let $H_N$ be real symmetric $N \times N$ matrices (we enumerate indices of entries by $i,j \in \mathcal{L}$, where $\mathcal{L} = [-n,n]^d \cap \mathbb{Z}^d$, $N = (2n+1)^d$) whose entries $H_{ij}$ are random real Gaussian variables with mean zero such that

$$\mathbb{E}\{H_{ij}H_{lk}\} = \delta_{ik}\delta_{jl}J_{ij}, \quad (1.1)$$

where

$$J_{ij} = \left(-W^2\Delta + 1\right)^{-1}_{ij}, \quad (1.2)$$

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and \( \Delta \) is the discrete Laplacian on \( \mathcal{L} \) with Neumann boundary conditions. Note that
\[
J_{ij} \approx C_1 W^{-1} \exp\{-C_2 |i-j|/W\} \quad \text{for } J \text{ of (1.2) with } d = 1,
\]
and so the variance of matrix elements is exponentially small when \( |i-j| \gg W \). Hence \( W \) can be considered as the width of the band.

The probability law of \( H_N \) can be written in the form
\[
P_n(dH_N) = \prod_{-n \leq i < j \leq n} \frac{dH_{ij}}{2\pi J_{ij}} e^{-\frac{\pi^2}{4|J_{ij}|}} e^{-\frac{n^2}{4J_{ii}}}.
\]

The density of states \( \rho \) of the ensemble is given by the well-known Wigner semicircle law (see [2, 16]):
\[
\rho(\lambda) = \frac{1}{\sqrt{4-\lambda^2}}, \quad \lambda \in [-2, 2].
\]

The key physical parameter of the model is the localization length, which describes the typical length scale of the eigenvectors of random matrices. The system is called delocalized if the localization length \( \ell \) is comparable with the matrix size, and it is called localized otherwise. Delocalized systems correspond to electric conductors, and localized systems are insulators.

In the case of 1D RBM there is a physical conjecture (see [5, 12]) stating that \( \ell \) is of order \( W^2 \) (for the energy in the bulk of the spectrum), which means that varying \( W \) we can see the crossover: for \( W \gg \sqrt{N} \) the eigenvectors are expected to be delocalized and for \( W \ll \sqrt{N} \) they are localized. In terms of eigenvalues this means that the local eigenvalue statistics in the bulk of the spectrum changes from Poisson, for \( W \ll \sqrt{N} \), to GUE/GOE (Hermitian/real symmetric matrices with i.i.d Gaussian elements), for \( W \gg \sqrt{N} \). At the present time only some upper and lower bounds for \( \ell \) are proven rigorously. It is known from the paper [17] that \( \ell \leq W^8 \). On the other side, in the resent papers [9, 10] it was proven first that \( \ell \gg W^{7/6} \), and then that \( \ell \gg W^{5/4} \).

The questions of the order of the localization length are closely related to the universality conjecture of the bulk local regime of the random matrix theory. The main objects of the local regime are \( k \)-point correlation functions \( R_k \ (k = 1, 2, \ldots) \), which can be defined by the equalities:
\[
E \left\{ \sum_{j_1 \neq \ldots \neq j_k} \varphi_k(\lambda_{j_1}^{(N)}, \ldots, \lambda_{j_k}^{(N)}) \right\} = \int_{\mathbb{R}^k} \varphi_k(\lambda_1^{(N)}, \ldots, \lambda_k^{(N)}) R_k(\lambda_1^{(N)}, \ldots, \lambda_k^{(N)}) d\lambda_1^{(N)} \ldots d\lambda_k^{(N)},
\]
where \( \varphi_k : \mathbb{R}^k \to \mathbb{C} \) is bounded, continuous and symmetric in its arguments and the summation is over all \( k \)-tuples of distinct integers \( j_1, \ldots, j_k \in \{1, \ldots, N\} \).

The bulk local regime deals with the behavior of eigenvalues of \( N \times N \) random matrices on the intervals whose length is of the order \( O(N^{-1}) \). According to the Wigner – Dyson universality conjecture (see, e.g., [14]), this local behavior does not depend on the matrix probability law (ensemble) and is determined only by the symmetry type of matrices (real symmetric, Hermitian, or quaternion real in the case of real eigenvalues and orthogonal, unitary or symplectic in the case of eigenvalues on the unit circle).

In this language the conjecture about the crossover for 1D RBM states that we get the same behavior of \( R_k \) as for GUE/GOE for \( W \gg \sqrt{N} \) (which corresponds to delocalized...
states), and we get another behavior, which is determined by the Poisson statistics, for $W \ll \sqrt{N}$ (and corresponds to localized states). For the general Wigner matrices (i.e., $W = N$) the bulk universality has been proved in [11, 21]. However, in the general case of RBM the question of bulk universality of local spectral statistics is still open even for $d = 1$.

Other more simple objects of the local regime of the random matrix theory are the correlation functions (or the mixed moments) of characteristic polynomials. The correlation function of the characteristic polynomials is

$$F_{2k}(\Lambda) = \int \prod_{s=1}^{2k} \det(\lambda_s - H_N) P_n(dH_N),$$

(1.6)

where $P_n(dH_N)$ is defined in (1.3), and $\Lambda = \text{diag} \{\lambda_1, \ldots, \lambda_{2k}\}$ are real or complex parameters that may depend on $N$.

As was mentioned before, an additional source of motivation for the current work is the development of the SUSY approach in the context of random operators with non-trivial spatial structures. Although in the case of RBM (and some related types of the Wegner models) the SUSY method has been applied rigorously so far mostly to the density of states (see [6], [7]), the result of [19] for the second correlation function of the block-band matrices gives hope that the method can be applied also for $R_k$. From the SUSY point of view characteristic polynomials correspond to the so-called fermionic sector of the supersymmetric full model, which describes the correlation functions $R_k$. So the analysis of the local regime of correlation functions of the characteristic polynomial is an important step towards the proof of the universality of the correlation functions $R_k$ for the case of real symmetric 1D RBM.

The asymptotic local behavior in the bulk of the spectrum of the 2-point mixed moment for GOE is well-enough studied. It was proved for $k = 1$ by Brézan and Hikami [3], who used the SUSY approach, and for general $k$ by Borodin and Strahov [4], who used different techniques, that

$$F_{2k}(\Lambda_0 + \hat{\xi}/N\rho(\lambda_0)) = C_N \text{Pf}\{dS(\pi(\xi_i - \xi_j))\}^{2k}_{i,j=1} \frac{\triangle(\xi_1, \ldots, \xi_{2k})}{(1 + o(1))},$$

(1.7)

where

$$dS(x) = -3 \frac{d}{dx} \frac{\sin x}{x} = 3 \left( \frac{\sin x}{x^3} - \frac{\cos x}{x^2} \right),$$

$\triangle(\xi_1, \ldots, \xi_k)$ is the Vandermonde determinant of $\xi_1, \ldots, \xi_k$, and

$$\hat{\xi} = \text{diag} \{\xi_1, \ldots, \xi_{2k}\}, \quad \Lambda_0 = \lambda_0 \cdot I.$$

The same is valid for $k = 1$ for real symmetric Wigner and general sample covariance matrices (see [13]).

In this paper we obtain the same result for $k = 1$ for matrices (1.1) as $N, W \to \infty$, $W^2 = N^{1+\theta}$, $0 < \theta \leq 1$ (i.e., $W \gg \sqrt{N}$). Set

$$\lambda_j = \lambda_0 + \frac{\xi_j}{N\rho(\lambda_0)}, \quad j = 1, 2,$$
where \( N = 2n + 1 \), \( \lambda_0 \in (-2, 2) \), \( \rho \) is defined in (1.4), and \( \{\xi_1, \xi_2\} \) are real parameters varying in any compact set \( K \subset \mathbb{R} \), and define

\[
D_2 = \prod_{i=1}^{2} F_2^{1/2} \left( \lambda_0 + \frac{\xi_i}{N\rho(\lambda_0)}, \lambda_0 + \frac{\xi_i}{N\rho(\lambda_0)} \right). \tag{1.8}
\]

The main result of the paper is the following theorem:

**Theorem 1.** Consider the random matrices (1.1) – (1.3) with \( W^2 = N^{1+\theta} \), where \( 0 < \theta \leq 1 \). Define the second mixed moment \( F_2 \) of the characteristic polynomials as in (1.6). Then we have

\[
\lim_{n \to \infty} D_2^{-1} F_2 \left( \Lambda_0 + \hat{\xi} / (N\rho(\lambda_0)) \right) = 3 \left( \frac{\sin \pi (\xi_1 - \xi_2)}{\pi^3 (\xi_1 - \xi_2)^3} - \frac{\cos \pi (\xi_1 - \xi_2)}{\pi^2 (\xi_1 - \xi_2)^2} \right), \tag{1.9}
\]

and the limit is uniform in \( \xi_1, \xi_2 \) varying in any compact set \( K \subset \mathbb{R} \). Here \( \rho(\lambda) \) and \( D_2 \) are defined in (1.4) and (1.8), \( \Lambda_0 = \text{diag} \{\lambda_0, \lambda_0\} \), \( \lambda_0 \in (-2, 2) \), \( \hat{\xi} = \text{diag} \{\xi_1, \xi_2\} \).

In the case \( W \ll \sqrt{N} \) the limit is expected to be different from (1.9), but we will not discuss it in this paper.

The paper is organized as follows. In Sec. 2 we obtain a convenient integral representation for \( F_2 \), using the integration over the Grassmann variables. In Sec. 3 we give the sketch of the proof of Theorem 1. Sec. 4 deals with the most important preliminary results needed for the proof. In Sec. 5 we prove Theorem 1 applying the steepest descent method to the integral representation. Sec. 6 is devoted to the proofs of the auxiliary statements.

1.1 Notation

We denote by \( C, C_1, \) etc. various \( W \) and \( N \)-independent quantities below, which can be different in different formulas. Integrals without limits denote the integration (or the multiple integration) over the whole real axis, or over the Grassmann variables.

Moreover,

- \( N = 2n + 1 \);
- \( J = (-W^2 \Delta + 1)^{-1} \);
- \( E\{ \ldots \} \) is an expectation with respect to the measure (1.3);
- \( U_\varepsilon(x) = (x - \varepsilon, x + \varepsilon) \subset \mathbb{R} \);
- \( a_\pm = \pm \sqrt{4 - \lambda_0^2} = \pm \pi \rho(\lambda_0), \quad \bar{a}_\pm = (a_\pm, \ldots, a_\pm) \in \mathbb{R}^N \), \tag{1.10}

where \( \rho \) is defined in (1.4);
- \( \sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) \tag{1.11}
\[ \Lambda_0 = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \xi = \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix}, \quad L = \begin{pmatrix} a_+ & 0 \\ 0 & a_- \end{pmatrix}; \]
\[ \Lambda_{0,4} = \begin{pmatrix} \Lambda_0 & 0 \\ 0 & \Lambda_0 \end{pmatrix}, \quad \hat{\xi}_4 = \begin{pmatrix} \hat{\xi} & 0 \\ 0 & \hat{\xi} \end{pmatrix}, \quad L_4 = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}; \] (1.12)
\[ \hat{U}(2) = U(2)/U(1) \times U(1), \quad \hat{Sp}(2) = Sp(2)/Sp(1) \times Sp(1) \]
\[ d\mu \text{ is the Haar measure on } \hat{U}(2), \text{ } d\nu \text{ is the Haar measure on } \hat{Sp}(2); \]
\[ f(x) = (x + i\lambda_0/2)^2/2 - \log(x - i\lambda_0/2), \]
\[ f_\pm(x) = \Re(f(x) - f(a_\pm)) = (x^2 - \lambda_0^2/4 - \log(x^2 + \lambda_0^2/4))/2 - \Re f(a_\pm); \] (1.13)
\[ \Omega_\delta \text{ is a union of } \]
\[ \Omega_\delta^+ = \{\{a_j\}, \{b_j\} : a_j, b_j \in U_\delta(a_+) \forall j\}, \]
\[ \Omega^\delta = \{\{a_j\}, \{b_j\} : a_j, b_j \in U_\delta(a_-) \forall j\}, \]
\[ \Omega^{\pm}_\delta = \{\{a_j\}, \{b_j\} : (a_j \in U_\delta(a_+), \ b_j \in U_\delta(a_-)) \]
\[ \text{or } (a_j \in U_\delta(a_-), \ b_j \in U_\delta(a_+)) \forall j\}, \] (1.14)
\[ \text{where } \delta = W^{-\kappa} \text{ and } \kappa < \theta/8. \]
\[ c_\pm = 1 - \frac{\lambda_0^2}{4} \pm \frac{i\lambda_0}{2} \sqrt{1 - \lambda_0^2/4}, \quad c_0 = \Re f(a_+) = \frac{2 - \lambda_0^2}{4}; \] (1.15)
\[ \mu_\gamma(x) = \exp \left\{ -\frac{1}{2} \sum_{j=-n+1}^{n} (x_j - x_{j-1})^2 - \frac{\gamma}{W^2} \sum_{j=-n}^{n} x_j^2 \right\}; \] (1.16)
\[ \langle \ldots \rangle_0 = Z_{\delta, \gamma}^{-1} \int_{-\delta W}^{\delta W} (\ldots) \cdot \mu_\gamma(x) \prod_{q=-n}^{n} dx_q, \quad Z_{\delta, \gamma} = \int_{-\delta W}^{\delta W} \mu_\gamma(x) \prod_{q=-n}^{n} dx_q, \] (1.17)
\[ \langle \ldots \rangle = Z_{\gamma}^{-1} \int_{-\delta W}^{\delta W} (\ldots) \cdot \mu_\gamma(x) \prod_{q=-n}^{n} dx_q, \quad Z_{\gamma} = \int \mu_\gamma(x) \prod_{q=-n}^{n} dx_q, \]
\[ \text{where } \delta > 0 \text{ and } \gamma \in \mathbb{C}, \Re \gamma > 0; \]
\[ \langle \ldots \rangle_\ast \text{ (and } \langle \ldots \rangle_{0, \ast} \text{ is (1.17) with } \mu_{\Re \gamma}(x) \text{ instead of } \mu_\gamma(x). \]

2 Integral representation

In this section we obtain an integral representation for $F_2$ of (1.6) by using rather standard SUSY techniques, i.e., integrals over the Grassmann variables. Integration over the Grassmann variables has been introduced by Berezin and is widely used in the physics literature. For the reader’s convenience we give a brief outline of the techniques in Appendix.

The main result of the section is the following proposition
Proposition 1. The second correlation function of the characteristic polynomials for 1D real symmetric Gaussian band matrices, defined in (1.1), can be represented as follows:

\[ F_2(\Lambda_0 + \frac{\hat{\xi}}{N\rho(\lambda_0)}) = -(2\pi^3)^{-N} \det^{-3} J \int \exp \left\{ -\frac{W^2}{4} \sum_{j=-n+1}^n \text{Tr} \left( F_j - F_{j-1} \right)^2 \right\} \] (2.1)

\[ \times \exp \left\{ -\frac{1}{4} \sum_{j=-n}^n \text{Tr} \left( F_j + \frac{i\Lambda_{0,4}}{2} + \frac{i\hat{\xi}_4}{N\rho(\lambda_0)} \right)^2 \right\} \prod_{j=-n}^n \det^{1/2}(F_j - i\Lambda_{0,4}/2) \prod_{j=-n}^n dF_j, \]

where \( \Lambda_{0,4} \) and \( \hat{\xi}_4 \) are defined in (1.1), and

\[ F_j = \begin{pmatrix} x_j & w_{1j} & 0 & w_{2j} \\ \overline{w}_{1j} & y_j & -w_{2j} & 0 \\ 0 & -\overline{w}_{2j} & x_j & \overline{w}_{1j} \\ w_{2j} & 0 & \overline{w}_{1j} & y_j \end{pmatrix}, \quad dF_j = dx_j dy_j d\Re w_{1j} d\Im w_{1j} d\Re w_{2j} d\Im w_{2j}. \] (2.2)

Moreover, (2.1) can be rewritten in the form

\[ F_2(\Lambda_0 + \frac{\hat{\xi}}{N\rho(\lambda_0)}) = -\frac{C(\xi) \det^{-3} J}{(24\pi)^N} \int \exp \left\{ -\frac{W^2}{4} \sum_{j=-n+1}^n \text{Tr} \left( Q_j^* A_{j,4} Q_j - A_{j-1,4} \right)^2 \right\} \] \[ \times \exp \left\{ -\sum_{j=-n}^n (f(a_j) + f(b_j)) - \frac{i}{2N\rho(\lambda_0)} \sum_{j=-n}^n \text{Tr} \left( R_j P_{-n}^* A_{j,4} (R_j P_{-n}) \hat{\xi}_4 \right) \right\} \] \[ \times \prod_{l=-n}^n (a_l - b_l)^4 d\nu(P_{-n}) \overline{d\alpha} d\overline{b} \prod_{p=-n+1}^n d\nu(Q_p), \] (2.3)

where \( f \) is defined in (1.13), \( A_{j,4} = \text{diag}(a_j, b_j, a_j, b_j) \), \( \{R_j\} \) and \( P_{-n} \) are \( 4 \times 4 \) symplectic matrices, \( d\nu(P) \) is the Haar measure on \( \text{Sp}(2) \), and

\[ R_k = \prod_{s=k}^{-n+1} Q_s, \quad C(\xi) = \exp \left\{ \frac{\lambda_0(\xi_1 + \xi_2)}{2\rho(\lambda_0)} + \frac{\xi_1^2 + \xi_2^2}{2N\rho(\lambda_0)^2} \right\}. \] (2.4)

Remark 1. Formula (2.1) is valid for any dimension if we change the sum \( \sum \text{Tr} (F_j - F_{j-1})^2 \) to \( \sum \text{Tr} (F_j - F_{j'})^2 \), where the last sum runs over all pairs of nearest neighbor \( j, j' \) in the volume \( \mathcal{L} \subset \mathbb{Z}^d \) (see the definition of RBM (1.1) – (1.2)).

Proof. Representing determinants as integrals over Grassmann variables (see (1.7)), we obtain

\[ F_2(\Lambda) = \mathbb{E} \left\{ \int e^{-\frac{2}{\alpha=1} \sum_{j=-n}^n (\lambda_j - \overline{H}_{j,k}) \overline{\psi}_{j\alpha} \psi_{k\alpha} \sum_{j=-n}^n \prod_{j=-n}^n d\overline{\psi}_{j\alpha} d\psi_{j\alpha} \right\} \]

\[ = \mathbb{E} \left\{ \int e^{-\frac{2}{\alpha=1} \lambda_j \sum_{j=-n}^n \overline{\psi}_{j\alpha} \psi_{j\alpha}} \exp \left\{ \sum_{j<k} H_{jk} \sum_{\alpha=1}^2 (\overline{\psi}_{j\alpha} \psi_{k\alpha} + \overline{\psi}_{k\alpha} \psi_{j\alpha}) \right\} \right\}. \]
where \( \{ \psi_{j\alpha} \}, j = -n, \ldots, n, \alpha = 1, 2 \) are the Grassmann variables \((2n + 1)\) variables for each determinant in (1.3). Here and below we use Greek letters like \( \alpha, \beta \) etc. for the field index and Latin letters \( j, k \) etc. for the position index.

Integrating over the measure (1.3), we get

\[
F_2(\Lambda) = \int \prod_{\alpha=1}^{2} \prod_{q=-n}^{n} d\overline{\psi}_{q\alpha} d\psi_{q\alpha} \exp \left\{ -\sum_{\alpha=1}^{2} \lambda_\alpha \sum_{p=-n}^{n} \overline{\psi}_{p\alpha} \psi_{p\alpha} \right\} \tag{2.5}
\]

\[
\times \exp \left\{ \frac{1}{2} \sum_{j<k} J_{jk}(\overline{\psi}_{j1}\psi_{k1} + \overline{\psi}_{j2}\psi_{k2} + \overline{\psi}_{k1}\psi_{j1} + \overline{\psi}_{k2}\psi_{j2})^2 \right\}.
\]

Applying a couple of times the Hubbard-Stratonovich transform (see (7.8)), we can write:

\[
\int \exp \left\{ -\left( J^{-1} x, x \right)/2 + i \sum_{j=-n}^{n} x_j \overline{\psi}_{j1}\psi_{j1} \right\} \prod_{j=-n}^{n} dx_j \tag{2.6}
\]

\[
= (2\pi)^{N/2} \cdot \det^{1/2} J \cdot \exp \left\{ -\frac{1}{2} \sum_{j,k=-n}^{n} J_{jk}\overline{\psi}_{j1}\psi_{k1} \right\},
\]

\[
\int \exp \left\{ -\left( J^{-1} y, y \right)/2 + i \sum_{j=-n}^{n} y_j \overline{\psi}_{j2}\psi_{j2} \right\} \prod_{j=-n}^{n} dy_j \tag{2.7}
\]

\[
= (2\pi)^{N/2} \cdot \det^{1/2} J \cdot \exp \left\{ -\frac{1}{2} \sum_{j,k=-n}^{n} J_{jk}\overline{\psi}_{j2}\psi_{k2} \right\},
\]

\[
\int \exp \left\{ -\left( J^{-1} \Re w_1, \Re w_1 \right) - \left( J^{-1} \Im w_1, \Im w_1 \right) \right\} \tag{2.8}
\]

\[
\times \exp \left\{ i \sum_{j=-n}^{n} w_{j1} \overline{\psi}_{j1}\psi_{j1} + i \sum_{j=-n}^{n} \overline{w}_{j1}\psi_{j2}\psi_{j1} \right\} \prod_{q=-n}^{n} d\Re w_{q1} d\Im w_{q1}
\]

\[
= \pi^N \cdot \det J \cdot \exp \left\{ -\sum_{j \neq k} J_{jk}\overline{\psi}_{j1}\psi_{k1} - \sum_{j=-n}^{n} J_{jj}\overline{\psi}_{j2}\psi_{j2} \right\},
\]

\[
\int \exp \left\{ -\left( J^{-1} \Re w_2, \Re w_2 \right) - \left( J^{-1} \Im w_2, \Im w_2 \right) \right\} \tag{2.9}
\]

\[
\times \exp \left\{ i \sum_{j=-n}^{n} w_{j2} \overline{\psi}_{j1}\psi_{j2} + i \sum_{j=-n}^{n} \overline{w}_{j2}\psi_{j1}\psi_{j2} \right\} \prod_{q=-n}^{n} d\Re w_{q2} d\Im w_{q2}
\]

\[
= \pi^N \cdot \det J \cdot \exp \left\{ -\sum_{j \neq k} J_{jk}\overline{\psi}_{j1}\psi_{k1} - \sum_{j=-n}^{n} J_{jj}\overline{\psi}_{j2}\psi_{j2} \right\}.
\]

Substituting this and (1.2) for \( J_{jk}^{-1} \) into (2.5), putting \( \Lambda = \Lambda_0 + \xi/N\rho(\lambda_0) \), and using (7.7)
to integrate over the Grassmann variables, we obtain
\[
F_2 \left( \Lambda_0 + \frac{\hat{\xi}}{N\rho(\lambda_0)} \right) = -(2\pi^3)^{-N} \det^{-3} J \int \exp \left\{ - \frac{W^2}{4} \sum_{j=-n+1}^{n} \Tr (F_j - F_{j-1})^2 \right\}
\times \exp \left\{ - \frac{1}{4} \sum_{j=-n}^{n} \Tr F_j^2 \right\} \prod_{j=-n}^{n} \det^{1/2} \left( F_j - i\Lambda_{0,4} - i\hat{\xi}_4/N\rho(\lambda_0) \right) \prod_{j=-n}^{n} dF_j
\]

with \( F_j \) of (2.2) and \( \Lambda_{0,4}, \hat{\xi}_4 \) of (1.1). This gives (2.1) after shifting \( F_j \to F_j + i\Lambda_{0,4}/2 + i\hat{\xi}_4/N\rho(\lambda_0) \). The reason of such a shift is that we need to have saddle-points lying on the contour of the integration (see (1.1)).

The matrices of the form (2.2) have two eigenvalues \( a_j, b_j \) of the multiplicity two and can be considered as quaternion \( 2 \times 2 \) matrices. At this language \( F \) is a quaternion self-dual Hermitian matrix, and it can be diagonalized by the quaternion unitary \( 2 \times 2 \) matrices \( Sp(2) \) (see, e.g., [14], Chapter 2.4), i.e., unitary \( 4 \times 4 \) matrices \( P \) which admit the relation
\[
P \left( \begin{array}{cc} 0 & I_2 \\ -I_2 & 0 \end{array} \right) P^* = \left( \begin{array}{cc} 0 & I_2 \\ -I_2 & 0 \end{array} \right).
\]

Change the variables to \( F_j = P_j^* A_{j,4} P_j \), where \( P_j \in Sp(2) \) and \( A_{j,4} = \text{diag} \{ a_j, b_j, a_j, b_j \} \). Then \( dF_j \) of (2.2) becomes (see, e.g., [14])
\[
\frac{\pi^2}{12} (a_j - b_j)^4 da_j db_j d\nu(P_j),
\]

where \( d\nu(P_j) \) is the normalized to unity Haar measure on the symplectic group \( Sp(2) \).

Thus, we have
\[
F_2 \left( \Lambda_0 + \frac{\hat{\xi}}{N\rho(\lambda_0)} \right) = - \frac{C(\xi) \det^{-3} J}{(2\pi)^N} \int d\tilde{a} d\tilde{b} \int_{Sp(2)^N} \prod_{j=-n}^{n} d\nu(P_j)
\times \exp \left\{ - \frac{W^2}{4} \sum_{j=-n+1}^{n} \Tr (P_j^* A_{j,4} P_j - P_{j-1}^* A_{j-1,4} P_{j-1})^2 \right\}
\times \exp \left\{ - \frac{1}{4} \sum_{j=-n}^{n} \Tr \left( A_{j,4} + \frac{i\Lambda_{0,4}}{2} \right)^2 - \frac{i}{2N\rho(\lambda_0)} \sum_{j=-n}^{n} \Tr P_j^* A_{j,4} P_j \hat{\xi}_4 \right\}
\times \prod_{k=-n}^{n} (a_k - i\lambda_0/2)(b_k - i\lambda_0/2) \prod_{k=-n}^{n} (a_k - b_k)^4,
\]

where
\[
d\tilde{a} = \prod_{j=-n}^{n} da_j, \quad d\tilde{b} = \prod_{j=-n}^{n} db_j, \quad C(\xi) = \exp\{\lambda_0(\xi_1 + \xi_2)/2\rho(\lambda_0)\}.
\]

Now changing the “angle variables” \( P_j \) to \( Q_j = P_j P_j^* \), \( j = -n+1, \ldots, n \) (i.e., the new variables are \( P_{-n}, Q_{-n+1}, Q_{-n+2}, \ldots, Q_n \)), we get (2.3). \( \square \)
3 Sketch of the proof of Theorem 1

The strategy of the proof is the same as in [18].

First we note that the main integrations over $a_j, b_j$ are the same as in [18], eq.(2.11), and so the expected saddle-points for each $a_j$ and $b_j$ are still $a_\pm$ (see (1.1)). Moreover, we can use the results of [18], Sec. 4.1 – 4.2, where the properties of the function $f$ and of the complex Gaussian distribution $\mu_\gamma$ of (1.16) were studied (see Sec. 4.1).

The second step is to prove that the main contribution to the integral (2.3) is given by $\Sigma$, i.e., by the integral over $\Omega$ (see (1.14)). More precisely, we are going to prove that

$$F_2 \left( \Lambda_0 + \frac{\hat{\xi}}{N \rho(\lambda_0)} \right) = -\frac{C(\xi) \det^{-3} J}{(24\pi)^N} \cdot \Sigma \cdot (1 + o(1)), \quad W \to \infty. \quad (3.1)$$

The bound for the complement $|\Sigma_c|$ can be obtained by inserting the absolute value inside the integral and by performing exactly the integral over the symplectic groups. After this, since we are far from the saddle-points of $f$, one can control the integral in the same way as in [18] (see Lemma 7).

The next step is the calculation of $\Sigma$ (see Sec. 5.2, Lemma 8). We are going to show that the main contribution to $\Sigma$ is given by $\Sigma_\pm$, i.e., the integral over $\Omega_\delta$ (see (1.14)). First note that shifting $P_j \to \left( \begin{array}{cc} \sigma' & 0 \\ 0 & \sigma' \end{array} \right) \cdot P_j$ for some $j$ ($\sigma'$ is defined in (1.1)), we can rotate each domain of type

$$\{\{a_j\}, \{b_j\} : (a_j \in U_\delta(a_+), b_j \in U_\delta(a_-)) \text{ or } (a_j \in U_\delta(a_-), b_j \in U_\delta(a_+)) \forall j \}$$

to the $\delta$-neighborhood of the point $(\bar{a}_+, \bar{a}_-)$ with $\bar{a}_\pm$ of (1.1). Thus, we can consider the contribution over $\Omega_\delta^\pm$ as $2^N$ contributions of the $\delta$-neighborhood of the point $(\bar{a}_+, \bar{a}_-)$. Consider this neighborhood, and change the variables as

$$a_j \to a_j + \tilde{a}_j/W, \quad |\tilde{a}_j| \leq \delta W,$$
$$b_j \to b_j + \tilde{b}_j/W, \quad |\tilde{b}_j| \leq \delta W,$$

and set $\tilde{A}_{j,4} = \text{diag}\{\tilde{a}_j, \tilde{b}_j, \tilde{a}_j, \tilde{b}_j\}$. To compute $\Sigma_\pm$, one has to perform first the integral over the symplectic groups. This integral is some analytic in $\{a_j/W\}, \{\tilde{b}_j/W\}$ function $\mathcal{F}$. As in [18], the main idea is to prove that the leading part of this function can be obtained by replacing all $Q_s$ in the “bad” term

$$\exp \left\{ -\frac{i}{2N \rho(\lambda_0)} \sum_{j=-n}^{n} \text{Tr} \left( \prod_{s=j}^{n} Q_s \cdot P_{-n})^* (L_4 + \tilde{A}_{s,4}/W) \prod_{s=j}^{n} Q_s \cdot P_{-n}) \hat{\xi}_4 \right\}$$

with $I$. To this end, we expand the “bad” term into the series and for each summand, which is analytic in $\{a_j/W\}, \{\tilde{b}_j/W\}$, find the bound for its Taylor coefficients (see Lemma 10). This means that we obtain the proper majorant for $\mathcal{F}$ in the sense of [18] (i.e., some function whose Taylor expansion's coefficients are at least the absolute value of the corresponding coefficient of the Taylor expansion of $\mathcal{F}$), which helps to change the
averaging over the complex measure by the averaging of the majorant over the positive one (see Lemma 9). Then, similarly to [18], we will show that the leading term of $\Sigma_\pm$ is the integral over the Gaussian measures $\mu_{c_\pm}$ in $\{a_j\}$ and $\{b_j\}$ variables, and the integral over the symplectic group $d\nu(P_{-\eta})$ which gives the kernel (1.7). This yields an asymptotic expression for $\Sigma_\pm$ (see Lemma 9).

Also it will be shown in Sec. 5.2.2 that the integrals $\Sigma_+^\pm$ and $\Sigma_-^\pm$ over $\Omega_\delta^+$ and $\Omega_\delta^-$ have smaller orders than $\Sigma_\pm$ (see Lemma 12).

### 4 Preliminary results

In this section we restated the results of [18], Sec. 4.2., where the properties of the complex Gaussian distribution $\mu_\gamma$ of (1.16) were studied. Since we will need to modify some of the results in Sec. 5.2.2, we give also the proofs of the most important of them.

First note that the straightforward calculation gives in the small neighborhood of $a_\pm$

$$f(x) - f(a_\pm) = c_\pm(x - a_\pm)^2 + s_3(x - a_\pm)^3 + \ldots =: c_\pm(x - a_\pm)^2 + \varphi_\pm(x - a_\pm), \quad (4.1)$$

where $c_\pm$ is defined in (1.15) and $|\varphi_\pm(x - a_\pm)| = O(|x - a_\pm|^3)$.

Now let us study the properties of the complex Gaussian distribution $\mu_\gamma$ defined in (1.16). Set

$$\mu_\gamma(x) = \exp \left\{-\frac{1}{2} \sum_{j=2}^m (x_j - x_{j-1})^2 - \frac{\gamma}{W^2} \sum_{j=1}^m x_j^2 \right\}. \quad (4.2)$$

**Lemma 1.** We have for any $\gamma \in \mathbb{C}$, $\Re \gamma > 0$

1. $$Z_\gamma^{(m)} := \int \mu_\gamma(x) \prod_{q=1}^m \, dx_q = (2\pi)^{m/2} \det^{-1/2}(-\Delta + 2\gamma/W^2) \quad (4.3)$$
   $$= (2\pi)^{m/2} \left( \frac{\sqrt{2\gamma}}{W} \sinh \frac{m\sqrt{2\gamma}}{W} \right)^{-1/2} (1 + o(1))$$

Moreover, if we set

$$G^{(m)}(\gamma) = \left( -\Delta + \frac{2\gamma}{W^2} \right)^{-1}, \quad (4.4)$$

then

$$|G_{ii}^{(m)}(\gamma)| \leq C_3 \frac{W}{\sqrt{2\gamma}} \coth \frac{m\sqrt{2\gamma}}{W} (1 + o(1)). \quad (4.5)$$

2. $$\frac{|Z_\gamma^{(m)} - Z_{\delta,\gamma}^{(m)}|}{|Z_\gamma^{(m)}|} = |Z_\gamma^{(m)}|^{-1} \left| \int_{|x| > \delta W} \mu_\gamma(x) \prod_{q=1}^m \, dx_q \right| \leq C_1 e^{-C_2 \delta^2 W}, \ \ W \to \infty,$$

where $m > CW$, $\delta = W^{-\kappa}$ for sufficiently small $\kappa < \theta/8$, and

$$Z_{\delta,\gamma}^{(m)} = \int_{-\delta W}^{\delta W} \mu_\gamma(x) \prod_{q=1}^m \, dx_q.$$
In addition, for any \( m \)

\[
|Z^{(m)}_{\gamma}|^{-1} \int_{|x_k - x_1| > \delta W} \mu^{(m)}_{\gamma}(x) \prod_{q=1}^{m} dx_q \leq C_1 e^{-C_2 \delta^2 W}, \quad W \to \infty, 
\]

and for \( m > CW \) and any \( \gamma_1, \gamma_2 \in \mathbb{C}, \Re \gamma_1, \Re \gamma_2 > 0 \)

\[
\frac{|Z^{(m)}_{\gamma_1}|}{|Z^{(m)}_{\gamma_2}|} \leq e^{C_1 m/W}, \quad W \to \infty. \tag{4.6}
\]

(3) Let \( m > C_1 W, k \leq C m/W, S = \{i_1, \ldots, i_s\} \subset \{1, \ldots, m\} \), and \( \sum_{i=1}^{s} k_i = k \), where \( k_i \in \{1, \ldots, k\} \). Then

\[
|Z^{(m)}_{\gamma}|^{-1} \int_{\max|x_i| > \delta W} \prod_{j \in S} (x_j/W)^{k_j} \cdot \mu^{(m)}_{\gamma}(x) \prod_{q=1}^{m} dx_q \leq e^{-C_1 \delta^2 W}, \quad W \to \infty,
\]

where \( \delta = W^{-\kappa} \) for sufficiently small \( \kappa < \theta/8 \).

The proof of the lemma is rather standard and can be found in [18] (see Lemma 3).

Let us study the properties of the averages of (1.17), where \( \delta = W^{-\kappa}, \kappa < \theta/8 \).

We will use below the following form of the Wick theorem:

**Lemma 2.** (i) For any smooth function \( f \)

\[
\langle x_{i_1} f(x_{i_1}, \ldots, x_{i_p}) \rangle = \sum_{j=1}^{p} \langle x_{i_1} x_{i_j} \rangle \langle \partial f(x_{i_1}, \ldots, x_{i_p})/\partial x_{i_j} \rangle. \tag{4.7}
\]

The same is valid for \( \langle \ldots \rangle_{*} \), where \( \langle \ldots \rangle, \langle \ldots \rangle_{*} \) are defined in (1.17).

(ii)

\[
\langle x_{i_1}^{k_1} \ldots x_{i_j}^{k_j} \rangle \leq \langle x_{i_1}^{k_1} \ldots x_{i_j}^{k_j} \rangle_{*}. \tag{4.8}
\]

**Proof.** The first part of the lemma is well-known Wick’s theorem, which can be easily proven using the integration by parts.

To prove the second part set

\[
M = -\Delta + \gamma/W^2 = (2 + \gamma/W^2)I - \tilde{M}, \quad M_{*} = -\Delta + \Re \gamma/W^2 = (2 + \Re \gamma/W^2)I - \tilde{M}, \tag{4.9}
\]

where \( \tilde{M} = \Delta + 2I \). Then

\[
\langle x_{i} x_{j} \rangle = (M^{-1})_{ij}, \quad \langle x_{i} x_{j} \rangle_{*} = (M_{*}^{-1})_{ij}. 
\]

Besides, since all entries of \( \tilde{M} \) are positive and \( \Re \gamma > 0 \),

\[
\left|(M^{-1})_{ij}\right| = \sum_{k=0}^{\infty} \frac{(\tilde{M}^k)_{ij}}{(2 + \gamma/W^2)^{k+1}} \leq \sum_{k=0}^{\infty} \frac{(\tilde{M}^k)_{ij}}{\left(2 + \gamma/W^2\right)^{k+1}} \leq \sum_{k=0}^{\infty} \frac{(\tilde{M}^k)_{ij}}{\left(2 + \Re \gamma/W^2\right)^{k+1}} = \left((M_{*}^{-1})_{ij}\right).
\]

This and (4.7) yield (ii).
Define
\[ E_n[g] := \exp \left\{ - \sum_{j=-n}^{n} g(x_j/W) \right\} \] (4.10)
for any function \( g : \mathbb{R} \to \mathbb{C} \).

We need

**Lemma 3.** For \( E_n \) of (4.10) we have
\[ \left| \langle E_n[\varphi_{\pm}] \rangle_0 - 1 \right| = o(1), \] (4.11)
where \( \varphi_{\pm} \) are defined in (4.1).

The key point in the proof of Lemma 3 is

**Lemma 4.** Let \( g \) be a polynomial of degree \( q \) with real coefficients starting from the third power, i.e., \( g(x) = \sum_{j=3}^{q} c_j x^j, \ c_j \in \mathbb{R} \). Then we have
\[ \left| \langle E_n[g] \rangle_{0,*} - 1 \right| = o(1), \quad n \to \infty. \] (4.12)

**Proof.** The lower bound.
Since \( e^x - 1 \geq x \), we have
\[ \langle E_n[g] \rangle_{0,*} - 1 \geq \left( \sum_{j=-n}^{n} g(x_j/W) \right)_{0,*} = \left( \sum_{j=-n}^{n} g(x_j/W) \right)_{*} + o(1), \]
where we use the third assertion of Lemma 1 in the last equality. Using Wick’s theorem (4.7) and \((M^{-1}_{*})_{ii} = CW\) (see the assertion (1) of Lemma 1), we can write
\[ \langle (x_j/W)^{2l} \rangle_{*} = O(W^{-l}), \]
and hence
\[ \langle \sum_{j=-n}^{n} g(x_j/W) \rangle_{*} = O(N/W^2) = o(1). \]

**The upper bound.**
Let us prove that
\[ \langle E_n[g] \rangle_{0,*} - 1 \leq \varepsilon_{1,n} \langle E_n[g] \rangle_{0,*}, \] (4.13)
which implies
\[ \langle E_n[g] \rangle_{0,*} - 1 \leq 2\varepsilon_{1,n}, \]
where \( \varepsilon_{1,n} = o(1) \), as \( n \to \infty \).

**Step 1. Replacing \( \langle \ldots \rangle_{0,*} \) with \( \langle \ldots \rangle_{*} \)**
Note that if we choose \( s_\kappa > 3 \) such that (recall that \( \delta = W^{-\kappa}, \ \kappa < \theta/8 \))
\[ W^{-\kappa s_\kappa} \leq W^{-2}, \] (4.14)
then for any \( p > s_\kappa/2 \) and for \( x_j \in (-\delta W, \delta W) \)

\[
\sum_{j=-n}^{n} (x_j/W)^{2p} < N/W^2 = o(1),
\]

and thus if we replace \( g(x) \) by \( g(x) + Cx^{2p} \) with any \( C \), then \( E_n[g] \) will be changed by \( E_n[g](1 + o(1)) \). Since it is easy to see that we can choose \( C \) such that \( c_0 x^2/2 + g(x) + Cx^{2p} \) has only one minimum \( x = 0 \) in \( \mathbb{R} \), without loss of the generality we can assume that \( c_0 x^2/2 + g(x) \geq c_0 x^2/4 \). Moreover, \( c_0 x^2/2 + g(x) \leq c_0 x^2 \) for \( x \in (-\delta, \delta) \). This and assertions (1), (2) of Lemma 1 give

\[
\frac{\int_{\max |x| > \delta W} E_n[g] \mu_{c_0}(x) \, dx}{\int_{\max |x| \leq \delta W} E_n[g] \mu_{c_0}(x) \, dx} \leq \frac{\int_{\max |x| > \delta W} \mu_{c_0/2}(x) \, dx}{\int_{\max |x| \leq \delta W} \mu_{2c_0}(x) \, dx} \leq e^{C_n/W - W\delta^2} = o(1),
\]

because \( \delta = W^{-\kappa} \) with \( \kappa < \theta/8 \). Thus,

\[
\langle E_n[g] \rangle_{0,*} = \langle E_n[g] \rangle_\ast + o(1). \tag{4.15}
\]

**Step 2. Application of Wick’s theorem (Lemma 2 (i))**

Since for \( x \in \mathbb{R} \)

\[
e^x \leq 1 + xe^x,
\]

we can write using Wick’s theorem (4.7)

\[
\langle E_n[g] \rangle_\ast - 1 \leq \sum_{i_1} \langle g(x_{i_1}/W) \cdot E_n[g] \rangle_\ast = \sum_{i_1} \sum_{l=3}^{q} \frac{c_l x_{i_1}^l}{W^l} \cdot \langle E_n[g] \rangle_\ast
\]

\[
\leq \sum_{i_1} \sum_{l=3}^{q} \frac{(l-1)!c_l |x_{i_1}^2|}{W^2} \left| \left( \frac{x_{i_1}^{l-2}}{W^{l-2}} \right) \cdot \langle E_n[g] \rangle_\ast \right|
\]

\[
+ \sum_{i_1,i_2} \sum_{l=3}^{q} \frac{|c_l| |x_{i_1} x_{i_2}|}{W^2} \left| \left( \frac{x_{i_1} x_{i_2}}{W} \right) \cdot g' \left( \frac{x_{i_2}}{W} \right) \cdot \langle E_n[g] \rangle_\ast \right|
\]

\[
\leq \sum_{i_1} \sum_{l=4}^{q} \frac{(l-1)(l-3)!c_l |x_{i_1}^2|}{W^4} \left| \left( \frac{x_{i_1}^{l-4}}{W^{l-4}} \right) \cdot \langle E_n[g] \rangle_\ast \right|
\]

\[
+ \sum_{i_1,i_2} \sum_{l=3}^{q} \frac{(2l-3)!c_l |x_{i_1} x_{i_2}|}{W^4} \left| \left( \frac{x_{i_1} x_{i_2}}{W^{l-3}} \right) \cdot g' \left( \frac{x_{i_2}}{W} \right) \cdot \langle E_n[g] \rangle_\ast \right|
\]

\[
+ \sum_{i_1,i_2} \sum_{l=3}^{q} \frac{|c_l| |x_{i_1} x_{i_2}^2|}{W^4} \left| \left( \frac{x_{i_1} x_{i_2}}{W^{l-2}} \right) \cdot g'' \left( \frac{x_{i_2}}{W} \right) \cdot \langle E_n[g] \rangle_\ast \right|
\]

\[
+ \sum_{i_1,i_2,i_3} \sum_{l=3}^{q} \frac{|c_l| |x_{i_1} x_{i_2} x_{i_3}|}{W^4} \left| \left( \frac{x_{i_1} x_{i_2} x_{i_3}}{W^{l-2}} \right) \cdot g' \left( \frac{x_{i_2}}{W} \right) g' \left( \frac{x_{i_3}}{W} \right) \cdot \langle E_n[g] \rangle_\ast \right| = \ldots
\]
For every term $\langle x_{i_1}^{m_1} \cdots x_{i_k}^{m_k} E_n[g] \rangle_x$, we take $x_{i_l}$ with the smallest index $l$ and find its pair according to (4.7). We repeat this procedure until we get $\langle E_n[g] \rangle_x$ or until the number of steps becomes bigger than $s_\kappa$, where $s_\kappa$ is defined in (4.14). All terms have the form

$$\sum_{i_1, \ldots, i_{p+l}} G(x_{i_1}, \ldots, x_{i_{p+l}}) \left( \left\langle \frac{x_{i_{p+1}}^{\alpha_{p+1}} x_{i_{p+2}}^{\alpha_{p+2}} \cdots x_{i_{p+l}}^{\alpha_{p+l}}}{W\alpha} \right\rangle_\kappa \right)$$

where $\alpha_{p+1}, \ldots, \alpha_{p+l} \in \mathbb{N}$ are bounded by some absolute constant (since in any case we make a finite number of steps), $\alpha = \alpha_{p+1} + \cdots + \alpha_{p+l}$. Here $G(x_{i_1}, \ldots, i_{p+l})$ is the product of the expectations of some pairing $x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_{p+l}}^{k_{p+l}}$ with $k_j \geq 1$, $j = 1, \ldots, p$ and $k_j \geq 1$, $j = p + 1, \ldots, p + l$ (all $\{k_j\}$ are bounded by some absolute constant) divided by $W^{k_1 + \cdots + k_{p+l}}$, with some bounded positive coefficient.

We can visualize these pairings as connected multigraphs (i.e., graphs which may contain multiple edges and loops) with vertices $i_1, \ldots, i_{p+l}$, where $p + l \leq s_\kappa$. The degree of $i_j$ is at least 3 for $j \leq p$ and is at least 1 for $j = p + 1, \ldots, p + l$. The multigraphs are connected, since one can proceed to a different connected component only if we obtained $\langle E_n[g] \rangle_x$.

Let $H$ be one of such multigraphs. Any $\langle x_i x_j \rangle_x$ gives $(M_*^{-1})_{ij}$. Thus, any loop gives a factor $(M_*^{-1})_{ii} = CW(1 + o(1))$ (see the assertion (1) of Lemma 4). Moreover, according to the Cauchy-Schwarz inequality, we have

$$(M_*^{-1})_{ij} \leq (M_*^{-1})_{ii}^{1/2} (M_*^{-1})_{jj}^{1/2}. $$

Hence, we can remove the edge $(j_1, j_2)$ from any cycle $(j_1, j_2, \ldots, j_r, j_1)$ ($r \neq 1$) and replace it with two semiloops $(j_1, j_1)$, $(j_2, j_2)$ (a “semiloop” is a loop counted with the coefficient 1/2, i.e., the contribution of a semiloop is $[(M_*^{-1})_{ii}]^{1/2}$ instead of $(M_*^{-1})_{ii}$; one semiloop adds 1 to the degree of the vertex, and two semiloops add up to one loop.) In this way we transform the multigraph $H$ to a tree $H_0$ with some loops and semiloops (the degree of each vertex is still the same as in $H$).

Since we make a finite number of steps, there is only a finite number of graphs $H$ such that corresponding graphs $H_0$ are equal to each other. Hence, we can consider the sum over $H_0$ instead of $H$. Let $G_0(x_{i_1}, \ldots, x_{i_{p+l}})$ be the function, which corresponds to the new graph $H_0$.

Note that, according to (4.15),

$$\left\langle \frac{x_{i_{p+1}}^{\alpha_{p+1}} x_{i_{p+2}}^{\alpha_{p+2}} \cdots x_{i_{p+l}}^{\alpha_{p+l}}}{W\alpha} \right\rangle_\kappa \leq \left\langle \frac{x_{i_{p+1}}^{\alpha_{p+1}} x_{i_{p+2}}^{\alpha_{p+2}} \cdots x_{i_{p+l}}^{\alpha_{p+l}}}{W\alpha} \right\rangle_0 + o(1)$$

$$\leq \frac{1}{W^\kappa} \langle E_n[g] \rangle_0 + o(1) = \frac{1}{W^\kappa} \langle E_n[g] \rangle_x + o(1).$$

Therefore, we are left to prove

**Lemma 5.** Let $H_0$ be a tree with loops and semiloops, whose vertices $i_1, \ldots, i_{p+l}$ admit the following condition: the degree of each vertex $i_j$ is at least 3 for $j \leq p$ and is at least 1 for $j = p + 1, \ldots, p + l$. Denote by $m$ the sum of degrees of all vertices and let also $G_0(x_{i_1}, \ldots, x_{i_n})$ be the function of the pairing, which corresponds to $H_0$. Then we have

$$\sum_{i_1, \ldots, i_{p+l}} G_0(x_{i_1}, \ldots, x_{i_{p+l}}) \leq N/W^{m/2-(p+l)+1}. \quad (4.16)$$
Moreover,
\[ \sum_{i_1, \ldots, i_{p+l}} W^{-\kappa a} G_0(x_{i_1}, \ldots, x_{i_{p+l}}) = o(1). \] (4.17)

Proof. Since $(1, \ldots, 1)$ is an eigenvector for $M_*$ of (4.9) with eigenvalue $W^2 / \Re \gamma$, we have
\[ \sum_j (M_*^{-1})_{ij} = \frac{W^2}{\Re \gamma}, \quad i = -n, \ldots, n. \] (4.18)

Let us consider the sum over $i_1, \ldots, i_{p+l}$. Any loop or semiloop gives $W$ or $W^{1/2}$ respectively. Thus, since the tree has $p + l - 1$ edges, all loops and semiloops give the contribution $W^{m/2-(p+l)+1}$. Using (4.18), we obtain that the contribution of the tree edges is $N/W m/2 - (p+l) + 1$. Therefore, we get that the sum over $i_1, \ldots, i_{p+l}$ is bounded by $N/W m/2 - (p+l) + 1$. Evidently $m$ is even, and hence $m/2 - p + 1$ is integer.

Now let us prove (4.17). Consider two cases

(1) The case $l = 0$.

First consider the case when we get $\langle E_n [g] \rangle_*$ at some step. Then $l = 0$ and $H_0$ is a tree with $p$ vertices and some loops and semi loops, where the degree of each vertex is at least 3. Hence, $m > 3p$, where $m$ is the sum of all degrees, and thus $m/2 - p + 1 = m/6 + 1 > 1$, which means $m/2 - p + 1 \geq 2$. Therefore, $N/W m/2 - k + 1 = N/W^2 = o(1)$, and hence (4.16) implies (4.17).

(2) The case $l > 0$.

Let now $l > 0$. Set $k = p + l$. Using (4.16), we get that the sum over all vertices is not greater than $N/W m/2 - k + 1$. Since $H_0$ is a connected graph, we have $m > 2(k-1)$, i.e., $m/2 - k + 1 > 0$. The sum in (4.17) has a factor $W^{-\kappa a}$. In addition, $m + \alpha \geq 3k$ and $m = 2s\kappa$, since we did $s\kappa$ steps and thus obtained $s\kappa$ edges. If $m/2 - k + 1 \geq 2$, then the sum can be bounded by $W^{-\kappa a} N/W^2 = o(1)$. Hence, we are left to consider the case $0 \leq m/2 - k + 1 \leq 1$, i.e., $2(k-1) \leq m \leq 2k$. Therefore, we get $k \geq s\kappa$ (because $m = 2s\kappa$), thus
\[ 2s\kappa + \alpha = m + \alpha \geq 3k \geq 3s\kappa, \]
which implies $\alpha > s\kappa$. Hence, the sum in (4.17) is bounded by
\[ W^{-\kappa a} N/W m/2 - k + 1 \leq N/W^\kappa a = o(1), \]
which gives (4.17).

Now (4.17) implies (4.13) with $\langle \ldots \rangle_*$ and hence with $\langle \ldots \rangle_{0,*}$ (see (4.15)).

Proof of Lemma 3

We can write for $x \in (-\delta, \delta)$
\[ \tilde{\varphi}_\pm(x) := \exp\{-\varphi_\pm(x)\} - 1 = \sum_{l=3}^{\infty} \phi_l x^l, \]
where \(|\phi| \leq (C_0)^{t}\). Thus,

\[
|\langle E_n[\varphi_{\pm}]\rangle_0 - 1| = \left| \left\langle \prod_{j=-n}^{n} \left( 1 + \sum_{l=3}^{\infty} \phi_l x^l \right) \right\rangle_0 - 1 \right| = \left| \sum_{k=3}^{\infty} \Sigma_k^0 \right|,
\]

where \(\Sigma_k^0, \Sigma_k\) are the sums of all terms \(\langle \prod_{l=1}^{s} (\phi_{k_1} x_{i_1}^{k_1}/W^{k_1}) \rangle_0\) and \(\langle \prod_{l=1}^{s} (\phi_{k_1} x_{i_1}^{k_1}/W^{k_1}) \rangle\) respectively with \(k_1 + \ldots + k_s = k, k \in \{3, \ldots, k\}\) (\(\Sigma_k^0, \Sigma_k\) are defined by the same way with \(\langle \ldots \rangle_0\) instead of \(\langle \ldots \rangle\) and with \(|\phi|\) instead of \(\phi_l\)).

Also denote

\[
S_s^0 = \sum_{i_1 < \ldots < i_s} \left\langle \prod_{l=1}^{s} |x_{i_l}/W|^3 \right\rangle_{0,*}.
\]

According to (4.6), we have

\[
|\langle (\phi_{k_1} x_{i_1}^{k_1}/W^{k_1}) \ldots (\phi_{k_s} x_{i_s}^{k_s}/W^{k_s}) \rangle_0| \leq (C_0)^k \delta^{k-3s} e^{C_n/W} \langle \prod_{l=1}^{s} |x_{i_l}/W|^3 \rangle_{0,*}.
\]

Hence, since the number of partitions of \(k\) to \(s\) non-zero summands is not grater than \(\binom{k}{s}\), we obtain

\[
|\Sigma_k^0| \leq e^{C_n/W} (C_0)^k \sum_{s=1}^{k/3} \binom{k}{s} \delta^{k-3s} S_s^0 \leq e^{C_n/W} (2C_0)^k \sum_{s=1}^{k/3} \delta^{k-3s} S_s^0.
\]

Note now that

\[
|x|^3 \leq \frac{p^{-1}x^2 + px^4}{2},
\]

and hence, again according to (4.6), we get for any \(p > 0\)

\[
S_s^0 \leq \sum_{i_1 < \ldots < i_s} \left\langle \prod_{l=1}^{s} \frac{p^{-1}x_{i_l}^2/W^2 + px_{i_l}^4/W^4}{2} \right\rangle_{0,*} := \tilde{S}_s^0.
\]

Besides,

\[
1 + q \cdot \frac{p^{-1}x^2 + px^4}{2} \leq (1 + \frac{qx^2}{2p})(1 + \frac{pqx^4}{2}) \leq e^{qx^2/2p}(1 + \frac{pqx^4}{2}),
\]

and thus, taking in account (4.6), we have for any \(p, q > 0\) such that \(q/p < c_0\) with \(c_0\) of (1.15)

\[
1 + \sum_{k=1}^{2n+1} q^k \tilde{S}_k^0 = \left\langle \prod_{j=-n}^{n} \left( 1 + q \cdot \frac{p^{-1}x_{j}^2/W^2 + px_{j}^4/W^4}{2} \right) \right\rangle_{0,*} \leq \left\langle e^{q/2p \Sigma_{j} x_{j}^2/W^2} \prod_{j=-n}^{n} \left( 1 + \frac{pqx_{j}^4/W^4}{2} \right) \right\rangle_{0,*} \leq e^{C_p,qn/W} \left\langle \prod_{j=-n}^{n} \left( 1 + \frac{pqx_{j}^4/W^4}{2} \right) \right\rangle_{0,c_0-q/p} \leq C e^{C_p,qn/W},
\]
where the last inequality holds in view of Lemma 4 \((\ldots)_{0,c_0-q/p} means \(1.17\) with \(\gamma = c_0 - q/p\)). This gives
\[
\tilde{S}_k^0 \leq e^{C_{p,q}n/W / q^k},
\]
and we have from (1.23) for \(k > Cn/W\) with sufficiently big \(C\)
\[
S_k^0 \leq e^{(C_{p,q}+C_1)n/W q^{-k}}. \tag{4.24}\]
Take \(q > (2|C_0|e)^3\). Then (4.25) and (4.24) yield for \(k > C_1 n/W\)
\[
|\Sigma_k^0| \leq e^{Cn/W} \sum_{s=1}^{k/3} (2C_0)^k \delta^{k-3s} q^{-s} \leq e^{Cn/W} \frac{(2C_0)^k}{q^{k/3}} \sum_{s=1}^{k/3} (\delta^{3} q)^{k/3-s} \leq 2e^{Cn/W-k}. \tag{4.25}\]
This and (4.19) imply
\[
|\langle E_n[\varphi_{\pm}]\rangle_0 - 1| \leq \left| \sum_{k=3}^{Cn/W} \Sigma_k^0 \right| + e^{-C_1n/W}. \tag{4.26}\]
Taking into account that the number of distributions of \(k\) items into \(n\) boxes is \(\binom{n+k-1}{k}\) and using the assertion (3) of Lemma 1, we get
\[
|Z_{\gamma}|^{-1} \int \sum_{\max \{|x_i| > \delta W\}} \sum_{s=1}^{k/3} \sum_{k_1,\ldots,k_s} |x_{i_1}^{k_1} \cdots x_{i_s}^{k_s}| W^k \mu_\gamma(x) dx \leq e^{-C_2 \delta W/n+k-1} \leq e^{2k \log(n/k)-C_2 \delta W} \leq e^{-C_2 \delta W/4},
\]
where the second sum in the first line is over all collections \(\{k_i\}_{i=1}^s, \sum k_i = k, k_i \in \{3,\ldots,k\}\). This yields
\[
\Sigma_k = \Sigma_k^0 + e^{-C_2 \delta W/4}, \quad \Sigma_{k,\ast} = \Sigma_{k,\ast}^0 + e^{-C_2 \delta W/4}, \quad k \leq Cn/W,
\]
and thus by (4.8) we have
\[
\left| \sum_{k=1}^{Cn/W} \Sigma_k^0 \right| = \left| \sum_{k=1}^{Cn/W} \Sigma_k \right| + e^{-C_2 \delta W/4} \leq \sum_{k=1}^{Cn/W} \Sigma_{k,\ast} + e^{-C_2 \delta W/4} \tag{4.27}\]
\[
\leq \sum_{k=1}^{Cn/W} \Sigma_{k,\ast}^0 + 2e^{-C_2 \delta W/4} \leq \left( \prod_{i=-n}^{n} (1 + \sum_{l=3}^{\infty} |\phi_l| x_i^l / W^l) - 1 \right)_{0,\ast} + 2e^{-C_2 \delta W/4}.
\]
Since \(|\phi_l| \leq (C_0)^l\), there exists \(C\) such that
\[
1 + \sum_{l=3}^{\infty} |\phi_l| x^l / W^l \leq e^{C(x^3/W^3 + x^4/W^4)}, \quad x \in (-\delta W, \delta W). \tag{4.28}\]
This, Lemma 1 and (4.26) yield (1.11).
Introduce the following partial ordering. Let $\Phi_1(x_1, \ldots, x_n), \Phi_2(x_1, \ldots, x_n)$ be two analytic functions in some ball centered at 0, and let the coefficients of the Taylor expansion of $\Phi_2$ be non-negative. Then we write
\[ \Phi_1 \prec \Phi_2 \] (4.29)
if the absolute value of each coefficient of the Taylor expansion of $\Phi_1$ does not exceed the corresponding coefficient of $\Phi_2$.

It is easy to see that
\[ \Phi_3 \prec \Phi_1, \quad \Phi_4 \prec \Phi_2 \Rightarrow \Phi_3 \Phi_4 \prec \Phi_1 \Phi_2. \] (4.30)

We will need

**Lemma 6.** (i) Let $|\phi_1| \leq CW^{-1}, |\phi_2| = o(1)$ and $|\phi_k| \leq C^k$ for some absolute constant $C > 0$. Then
\[ \langle \prod_{i=-n}^{n} (1 + \sum_{l=1}^{\infty} |\phi_l|x_i^l/W^l)\rangle_{0,*} \leq \exp\{C|\phi_2|n/W\}. \] (4.31)

(ii) If $\Phi_1(s_1, \ldots, s_n) - \Phi_1(0, \ldots, 0) \prec \prod_{j=1}^{n} (1 + q(s_i)) - 1$,
where $s_i = s(a_i/W, a_{i+1}/W, \ldots, \tilde{a}_{i+k}/W, \tilde{b}_i/W, \tilde{b}_{i+1}/W, \ldots, \tilde{b}_{i+k}/W)$ is a polynomial with $s(0, \ldots, 0) = 0$, $k$ is an $n$-independent constant, and $q(s) = \sum_{j=1}^{\infty} |c_j|s^j$ with $|c_1| \leq CW^{-1}, |c_2| = o(1), |c_l| \leq (C_0)^l, l \geq 3$, then
\[ \langle \Phi_1(s_1, \ldots, s_n) - \Phi_1(0, \ldots, 0) \rangle_0 \leq \langle \prod_{j=1}^{n} (1 + q(s_i^*)) - 1 \rangle_{0,*} + e^{-Cn/W}, \]
where $s_i^*$ is obtained from $s_i$ by replacing the coefficients of $s$ with their absolute values.

For the proof of the lemma see [18], Lemma 8.

### 4.1 Integration over the symplectic group $\hat{Sp}(2)$

**Proposition 2.** (i) Let $C$ be a normal $2 \times 2$ matrix with distinct eigenvalues $c_1, c_2$ and $D = \text{diag}\{d_1, d_2\}, d_i \in \mathbb{C}$. Then
\[ \int_{U(2)} \exp\{t \text{Tr} CU^* DU\} d\mu(U) = \frac{e^{t(c_1d_1 + c_2d_2)} - e^{t(c_1d_2 + c_2d_1)}}{t(c_1 - c_2)(d_1 - d_2)}, \] (4.32)
where $t \in \mathbb{C}$ is some constant.

(ii) Let
\[ F = \begin{pmatrix} X & w_2^* \\ -\overline{w_2} & X^* \end{pmatrix}, \quad X = \begin{pmatrix} x & w_1 \\ \overline{w_1} & y \end{pmatrix}, \] (4.33)
where $\sigma$ is defined in (4.1), $x, y \in \mathbb{R}, w_1, w_2, d_1, d_2 \in \mathbb{C}$. The matrices of the form (4.33) can be diagonalized by $\hat{S}p(2)$ transformation $P$ and have two real eigenvalues $a, b$ of multiplicity two. Moreover, the measure

$$dF = dx \, dy \, d\Re w_1 \, d\Im w_1 \, d\Re w_2 \, d\Im w_2,$$

can be represented in the form

$$\frac{\pi^2}{12} (a - b)^4 d\nu(P)$$

with

$$d\nu(P) = 3(1 - 2|V_{12}|)^2 d\mu(U) d\mu(V). \quad (4.34)$$

Here $d\mu$ is a Haar measure over $\hat{U}(2),$

$$P = \begin{pmatrix} V & 0 \\ 0 & \overline{V} \end{pmatrix} \cdot \begin{pmatrix} \cos \varphi \cdot I & \sin \varphi \cdot e^{i\alpha} \cdot \sigma' \\ -\sin \varphi \cdot e^{-i\alpha} \cdot \sigma' & \cos \varphi \cdot I \end{pmatrix},$$

and

$$U = \begin{pmatrix} \cos \varphi & -\sin \varphi \cdot e^{i\alpha} \\ \sin \varphi \cdot e^{-i\alpha} & \cos \varphi \end{pmatrix}, \quad V = \begin{pmatrix} \cos \phi & \sin \phi \cdot e^{i\beta} \\ -\sin \phi \cdot e^{-i\beta} & \cos \phi \end{pmatrix}. $$

Moreover, if $\tilde{t} = t(c_1 - c_2)(d_1 - d_2)$, then

$$\int_{S^p(2)} \exp \left\{ t \text{Tr} \, G P^* H P/2 \right\} d\nu(P)$$

$$= \frac{6}{\tilde{t}^2} \left( e^{t(c_1 d_1 + c_2 d_2)} (1 - 2/\tilde{t}) + e^{t(c_1 d_2 + c_2 d_1)} (1 + 2/\tilde{t}) \right), \quad (4.35)$$

In addition,

$$\int_{\Omega} \exp \left\{ -\frac{t}{4} \text{Tr} \, (F - G)^2 \right\} \Phi(F) dF$$

$$= \frac{\pi^2}{t^2} \int_{\Omega} \exp \left\{ -\frac{t}{2} \text{Tr} \, (\hat{Y} - D)^2 \right\} \cdot \Phi(\hat{Y}) \cdot \frac{(y_1 - y_2)^2}{(d_1 - d_2)^2}$$

$$\times \left( 1 - \frac{2}{t(y_1 - y_2)(d_1 - d_2)} \right) \, dy_1 \, dy_2,$$

where $y_1, y_2$ are eigenvalues of $F, \hat{Y} = \text{diag} \{y_1, y_2\}$, and

$$dF = dx \, dy \, d\Re w_1 \, d\Im w_1 \, d\Re w_2 \, d\Im w_2.$$

Here $\Phi(F)$ is any function which is invariant over $\hat{S}p(2)$ transformation (i.e., depend only on $y_1, y_2$), $\Omega$ is any $\hat{S}p(2)$ invariant domain such that the eigenvalues of $F$ of the form (4.33) run over the symmetric domain $\hat{\Omega}$.

The proof of this proposition can be found in Sec.6.
5 Proof of the main theorem

In this section we will prove Theorem 1 applying the steepest descent method to the integral representation (2.3).

5.1 The bound for $\Sigma_c$

Lemma 7. Let $\Sigma_c$ be the part of the integral in (2.3) over the complement of the domain $\Omega_\delta$, which is defined in (1.14). Then

$$|\Sigma_c| \leq C_1 W^{-6N+4}(24\pi)^N e^{-2Nc_0} e^{-C_2 W^{1-2\kappa}},$$

where $\kappa < \theta/8$ and $c_0 = \Re f(a_\pm)$.

Proof. According to (2.3), we have

$$|\Sigma_c| \leq e^{-2Nc_0} \int_{\Omega_a^c} \exp \left\{ - \sum_{j=-n}^n (f_*(a_j) + f_*(b_j)) \right\}$$

$$\times \exp \left\{ - \frac{W^2}{4} \sum_{j=-n+1}^n \text{Tr} (Q_j A_j A_j Q_j - A_{j-1,4})^2 \right\}$$

$$\times \prod_{l=-n}^n (a_l - b_l)^4 d\nu(P_{-n}) da db \prod_{p=-n+1}^n d\nu(R_p),$$

where $f_*$ and $c_0$ are defined in (1.13) and (1.15). Here we insert the absolute value inside the integral and use that

$$\left| \exp \left\{ - \frac{i}{2N\rho(\lambda_0)} \sum_{j=-n}^n \text{Tr} (R_j P_{-n})^* A_{j,4} (R_j P_{-n}) \hat{\xi}_4 \right\} \right| = 1.$$

To simplify formulas below, set

$$I_0 = W^{-6N+4}(24\pi)^N e^{-2Nc_0} \cdot \left| \det^{-1} (-\Delta + 2c_+/W^2) \right|. \quad (5.1)$$

As we will see below, $I_0$ is an order of $\Sigma$ (see Lemma 8). Also recall that, according to Lemma 1, eq. (1.3),

$$e^{-C_1 N/W} \leq \left| \det^{-1} (-\Delta + 2c_+/W^2) \right| \leq e^{-C_2 N/W}, \quad (5.2)$$

and that $W^2 = N^{1+\theta}$, $\kappa < \theta/8$, and hence $CN/W \ll W^{1-2\kappa}$.

We are going to prove that

$$|\Sigma_c/I_0| \leq e^{-CW^{1-2\kappa}}. \quad (5.3)$$
Using (4.36), we get (recall that $A_{j} = \text{diag}\{a_{j}, b_{j}, a_{j}, b_{j}\}$, $j = -n, \ldots, n$ and $\Omega_{0}^{C}$ is still a symmetric domain)

$$I_{0}^{-1} \cdot |\Sigma_{c}| \leq \frac{12^{N-1}e^{-2Nc_{0}}}{W^{4(N-1)}I_{0}}\int_{\Omega_{0}^{C}} \exp \left\{ -\frac{W^{2}}{2} \sum_{j=-n+1}^{n} \left( (a_{j} - a_{j-1})^{2} + (b_{j} - b_{j-1})^{2} \right) \right\}$$

$$\times \exp \left\{ -\sum_{j=-n}^{n} \left( f_{*}(a_{j}) + f_{*}(b_{j}) \right) \right\} (a_{-n} - b_{-n})^{2}(a_{n} - b_{n})^{2} \times \prod_{j=-n}^{n} \left( 1 - \frac{2}{W^{2}(a_{j} - b_{j})(a_{j-1} - b_{j-1})} \right) \, da \, db$$

(5.4)

The first line here is obtained performing recursively the integral over $Q_{j}$ starting from $j = n$ and going backwards. At each step the integral can be written in the form (4.32), with a suitable choice of the function $f$. The last product of (5.4) can be bounded by $\exp\{CN/W^{2}\}$, thus

$$I_{0}^{-1} \cdot |\Sigma_{c}| \leq \frac{12^{N-1}e^{-2Nc_{0}} \cdot e^{CN/W^{2}}}{W^{4(N-1)}I_{0}}\int_{\Omega_{0}^{C}} \exp \left\{ -\frac{W^{2}}{2} \sum_{j=-n+1}^{n} \left( (a_{j} - a_{j-1})^{2} + (b_{j} - b_{j-1})^{2} \right) \right\}$$

$$\times \exp \left\{ -\sum_{j=-n}^{n} \left( f_{*}(a_{j}) + f_{*}(b_{j}) \right) \right\} (a_{-n} - b_{-n})^{2}(a_{n} - b_{n})^{2} \, da \, db$$

(5.5)

$$\leq C \cdot W^{4} \cdot (2\pi)^{-N} e^{CN/W} \int_{W\Omega_{0}^{C}} \exp \left\{ -\frac{1}{2} \sum_{j=-n+1}^{n} \left( (a_{j} - a_{j-1})^{2} + (b_{j} - b_{j-1})^{2} \right) \right\}$$

$$\times \exp \left\{ -\sum_{j=-n}^{n} \left( f_{*}(a_{j}/W) + f_{*}(b_{j}/W) \right) \right\} (a_{-n} - b_{-n})^{2}(a_{n} - b_{n})^{2} \, da \, db,$$

where $f_{*}$ and $c_{0}$ are defined in (1.13) and (1.15). Here in the third line we did the change $a_{j} \rightarrow a_{j}/W$, $b_{j} \rightarrow b_{j}/W$ and used (5.1) – (5.2).

Now the last integral in (5.5) is the same as in [18], eq. (5.5) and so can be bounded in the same way.

5.2 Calculation of $\Sigma$

Lemma 8. For the integral $\Sigma$ over the domain $\Omega_{0}$ (see [14]) we have

$$\Sigma = \frac{8\pi^{4}\rho(\lambda_{0})^{4}e^{-2Nc_{0}}(24\pi)^{N}}{3W^{6N-4}} \cdot dS(\pi(\xi_{1} - \xi_{2})) \cdot \det^{-1} \left( -\Delta + \frac{2c_{+}}{W^{2}} \right) \left( 1 + o(1) \right) \quad (5.6)$$

$$= 8(\pi\rho(\lambda_{0}))^{3/3} \cdot dS(\pi(\xi_{1} - \xi_{2})) \cdot I_{0}, \quad W \rightarrow \infty,$$

where $I_{0}$ is defined in (5.1).
Note that \((5.6)\) together with \((5.3)\) yield
\[
|\Sigma_\varepsilon| \leq e^{-C W^{1-2\varepsilon}} |\Sigma|,
\]
which gives \((3.1)\).

Now using \((3.1)\) and \((5.6)\) we get Theorem \([1]\)

Thus, we are left to compute \(\Sigma\). We are going to show that the leading term in \(\Sigma\) is given by \(\Sigma_{\pm}\), i.e., that the contributions of \(\Sigma_{+}\) and \(\Sigma_{-}\) are smaller.

### 5.2.1 Calculation of \(\Sigma_{\pm}\)

Consider the \(\delta\)-neighborhood of the point \((\bar{\omega}_{+}, \bar{\omega}_{-})\) with \(\bar{\omega}_{\pm}\) of \((1.1)\) and \(\delta = W^{-\kappa}\).

Let us show that

**Lemma 9.** For the integral \(\Sigma_{\pm}\) over the domain \(\Omega_{\delta}^{\pm}\) of \((1.14)\) we have, as \(W \to \infty\)
\[
\Sigma_{\pm} = \frac{8(\pi \rho(\lambda_{0}))^4 e^{-2Nc_0} (24\pi)^N}{3W^{6N-4}} \cdot dS(\pi(\xi_1 - \xi_2)) \cdot \left| \det^{-1} \left( -\Delta + \frac{2c_{\pm}}{W} \right) \right| (1 + o(1)).
\]

**Proof.** Performing the change \(a_j - a_+ = \tilde{a}_j/W\), \(b_j - a_- = \tilde{b}_j/W\) in \((2.3)\) and using \((4.1)\), we obtain (recall that \(a_{\pm} = \pm \pi \rho(\lambda_{0})\))
\[
\Sigma_{\pm} = \frac{2^N e^{-2Nc_0 - i\pi(\xi_1 - \xi_2)}}{W^{2N}} \int_{[\tilde{a}_j, \tilde{b}_j] \leq W^{1-\kappa}} \mu_{c_+}(a) \mu_{c_-}(b) \cdot e^{-\sum_{k=-n}^n (\phi_{\pm}(a_k/W) + \phi_{\pm}(-b_k/W))}
\]
\[
\times \int_{S(\bar{x})} e^{W^2/2} \sum_{j=-n+1}^n \text{Tr} (Q_j^*(L_4 + \tilde{A}_{j,4}/W)Q_j(L_4 + \tilde{A}_{j-1,4}/W) - (L_4 + \tilde{A}_{j,4}/W)(L_4 + \tilde{A}_{j-1,4}/W)\}
\]
\[
\times \exp \left\{ -\frac{i}{2N\rho(\lambda_{0})} \sum_{k=-n}^n (\text{Tr} (R_kP_{-n})^* (L_4 + \tilde{A}_{k,4}/W) (R_kP_{-n})\tilde{\xi}_4 - \text{Tr} L_4\tilde{\xi}_4) \right\}
\]
\[
\times \prod_{l=-n}^n (a_+ - a_- + (a_l - b_l)/W)^4 \sum_{q=-n+1}^n \nu(Q_q) \, d\tilde{a} \, d\tilde{b},
\]
where \(L_4 = \text{diag} \{a_+, a_-, a_+, a_-\}\), \(\tilde{A}_{j,4} = \text{diag} \{\tilde{a}_j, \tilde{b}_j, \tilde{a}_{j-1}, \tilde{b}_{j-1}\}\), and \(\mu_{\gamma}(a)\) is defined in \((1.16)\).

Now we are going to integrate over \(\{Q_j\}\). Introduce
\[
F(\tilde{a}, \tilde{b}, Q) = -\frac{i}{2\rho(\lambda_{0})} \sum_{k=-n}^n (\text{Tr} (R_kP_{-n})^* (L_4 + \tilde{A}_{k,4}/W) (R_kP_{-n})\tilde{\xi}_4 - \text{Tr} L_4\tilde{\xi}_4),
\]
\[
d\eta_j(Q_j, \tilde{A}_j) = e^{W^2/2} \text{Tr} (Q_j^*(L_4 + \tilde{A}_{j,4}/W)Q_j(L_4 + \tilde{A}_{j-1,4}/W) - (L_4 + \tilde{A}_{j,4}/W)(L_4 + \tilde{A}_{j-1,4}/W))d\nu(Q_j),
\]
\[
d\eta(Q, \tilde{A}) = \prod_{j=-n+1}^n d\eta_j(Q_j, \tilde{A}_j), \quad I_{\eta}(\tilde{A}) = \int d\eta(Q, \tilde{A}),
\]
\[
t_j = W^2(a_+ - a_- + (\tilde{a}_j - \tilde{b}_j)/W)(a_+ - a_- + (\tilde{a}_{j-1} - \tilde{b}_{j-1})/W), \quad q_j = 6/t_j.
\]
According to Proposition 2 we have
\[
I_n(\tilde{A}) = \prod_{j=-n+1}^{n} q_j \left[ 1 - \frac{2}{t_j} + e^{-t_j} \left( 1 + \frac{2}{t_j} \right) \right].
\] (5.9)

We want to integrate the r.h.s. of (5.7) over \( d\eta(Q, \tilde{A}) \). To this end, we expand \( F(\tilde{a}, \tilde{b}, Q) \) into a series with respect to the elements of \( Q_j \), \( j = -n + 1, \ldots, n \). We are going to show that the leading term of the integral is given by the summands without any elements of \( Q_j \).

**Lemma 10.** In the notations of (5.8)
\[
\left| \left\langle \left( \exp\{F(\tilde{a}, \tilde{b}, Q) - F(0, 0, I)/N\} - 1 \right) \cdot \Pi_1 \cdot \Pi_2 \right\rangle \right| = o(1), \quad N \to \infty,
\] (5.10)
where \( \Pi_1, \Pi_2 \) are the products of the Taylor’s series for \( \exp\{\varphi_+(\tilde{a}_j/W)\} \) and for \( \exp\{\varphi_-(\tilde{b}_j/W)\} \) and
\[
\langle \ldots \rangle_\eta = \left( \prod_{j=-n+1}^{n} q_j \right)^{-1} \int (\ldots) d\eta(Q, \tilde{A}).
\] (5.11)

**Proof.** Since \( \tilde{\xi}_4 = \frac{\xi_1 + \xi_2}{2} I_4 + \frac{\xi_1 - \xi_2}{2} L_4 \) and \( a_+ = -a_- = \pi \rho(\lambda_0) \), we have
\[
\text{Tr} \left( R_k P_{-n}^* (a_+ L_4 + \tilde{A}_{k,4}/W) (R_k P_{-n}) \tilde{\xi}_4 \right) - \text{Tr} \left( a_+ L_4 + \tilde{A}_{k,4}/W \right) \tilde{\xi}_4
\]
\[
= \frac{\xi_1 - \xi_2}{2} \left[ \text{Tr} \left( (R_k P_{-n})^* (a_+ L_4 + \tilde{A}_{k,4}/W) (R_k P_{-n}) L_4 - (a_+ L_4 + \tilde{A}_{k,4}/W) L_4 \right) \right]
\]
\[
= 4\pi \rho(\lambda_0) (\xi_2 - \xi_1) (1 + (\tilde{a}_k - \tilde{b}_k)/2\pi \rho(\lambda_0) W) \cdot (|(R_k P_{-n})_{12}|^2 + |(R_k P_{-n})_{14}|^2).
\]

For any \( 4 \times 4 \) matrix \( P \) introduce
\[
S(P) = |P_{12}|^2 + |P_{14}|^2.
\] (5.12)

Then we can rewrite
\[
F(\tilde{a}, \tilde{b}, Q) - F(0, 0, I)
\]
\[
= 2i\pi (\xi_1 - \xi_2) \sum_{k=-n+1}^{n} (S(R_k P_{-n}) - S(P_{-n})) \cdot \left( 1 + \frac{\tilde{a}_k - \tilde{b}_k}{2\pi \rho(\lambda_0) W} \right).
\] (5.13)

Thus, we get
\[
\left\langle \exp \left\{ \frac{1}{N} \left( F(\tilde{a}, \tilde{b}, Q) - F(0, 0, I) \right) \right\} - 1 \right\rangle_\eta
\]
\[
= \sum_{p=1}^{\infty} \frac{C_p}{p! N^p} \sum_{k_1 \ldots k_p} \left\langle \prod_{j=1}^{p} \left[ (S(R_{k_j} P_{-n}) - S(P_{-n})) \cdot \left( 1 + \frac{\tilde{a}_{k_j} - \tilde{b}_{k_j}}{2\pi \rho(\lambda_0) W} \right) \right] \right\rangle_\eta,
\]
where \( \langle \ldots \rangle_\eta \) is defined in (5.11). Hence, we have to study
\[
\Phi_{k_1, \ldots, k_p}(\tilde{a}, \tilde{b}) = \left\langle \prod_{j=1}^{p} \left( S(R_{k_j} P_{-n}) - S(P_{-n}) \right) \right\rangle_\eta.
\] (5.14)
Let \( p < Cn/W \) for some constant \( C \). Introduce i.i.d vectors \( \{(x_j, y_j)\} \) such that the density of the distribution has the form

\[
\rho(x_j, y_j) = 24(a_+ - a_-)^4 x_j y_j \exp\{- (a_+ - a_-)^2 [x_j^2 + y_j^2]\} \cdot 1_{0 < x_j, y_j < W/2}.
\] (5.15)

Introduce matrices

\[
\tilde{Q}_j = V_j \cdot U_j,
\]

where

\[
V_j = \begin{pmatrix} \tilde{V}_j & 0 \\ 0 & \overline{\tilde{V}}_j \end{pmatrix}, \quad \overline{V}_j = \begin{pmatrix} \tilde{r}_j e^{i\tilde{\theta}_j} & \tilde{v}_j e^{i\sigma_j} \\ -\overline{\tilde{v}}_j e^{-i\sigma_j} & \overline{\tilde{r}}_j e^{-i\tilde{\theta}_j} \end{pmatrix},
\]

\[
U_j = \begin{pmatrix} \tilde{t}_j e^{i\tilde{\theta}_j} I & \tilde{u}_j e^{i\tilde{\sigma}_j} \sigma' \\ -\overline{\tilde{u}}_j e^{-i\tilde{\sigma}_j} \sigma' & \overline{\tilde{t}}_j e^{-i\tilde{\theta}_j} I \end{pmatrix}
\]

with

\[
\tilde{v}_j = x_j/p_j W, \quad \tilde{u}_j = y_j(1 - 2\tilde{v}_j^2)^{-1/2}/p_j W
\]

\[
p_j = \left(1 + \frac{\tilde{a}_j - \tilde{b}_j}{W(a_+ - a_-)}\right)^{1/2} \left(1 + \frac{\tilde{a}_{j-1} - \tilde{b}_{j-1}}{W(a_+ - a_-)}\right)^{1/2},
\]

\[
\tilde{r}_j = (1 - \tilde{v}_j^2)^{1/2}, \quad \tilde{t}_j = (1 - \tilde{u}_j^2)^{1/2}
\]

and \( \tilde{\theta}_j, \tilde{\theta}_j, \sigma_j, \tilde{\sigma}_j \in [-\pi, \pi) \). Define also

\[
d\tilde{\eta}_j = (2\pi)^{-4} \rho(x_j, y_j) dx_j dy_j d\tilde{\theta}_j d\tilde{\sigma}_j d\tilde{\sigma}_j, \quad d\tilde{\eta} = \prod_{j=-n+1}^n d\tilde{\eta}_j.
\] (5.17)

We need

**Lemma 11.**

\[
\Phi_{k_1, \ldots, k_p}(\tilde{a}, \tilde{b}) := \left\langle \prod_{j=1}^p \left(S(\tilde{R}_{k_j} \cdot P_{-n}) - S(P_{-n})\right) \cdot \prod_{i=-n+1}^n (1 - 2|\tilde{V}_i|_2^2) \right\rangle_{\tilde{\eta}} = \Phi_{k_1, \ldots, k_p}(\tilde{a}, \tilde{b}) + O(e^{-cW^2}),
\] (5.18)

where \( \langle \ldots \rangle_{\tilde{\eta}} \) means the expectation over \( d\tilde{\eta} \) and

\[
\tilde{R}_{k_j} = \prod_{l=k_j}^{-n+1} \tilde{Q}_l.
\]

The proof of the lemma can be found in Sec. 6.

Denote

\[
s_j = 1 - \left(1 + \frac{\tilde{a}_j - \tilde{b}_j}{W(a_+ - a_-)}\right) \left(1 + \frac{\tilde{a}_{j-1} - \tilde{b}_{j-1}}{W(a_+ - a_-)}\right).
\] (5.19)

Expanding \( V_j, U_j \) of (5.16) with respect to \( s_j \) we get

\[
V_j = \begin{pmatrix} \tilde{V}_j(0) & 0 \\ 0 & \overline{\tilde{V}}_j(0) \end{pmatrix} + \frac{x_j}{W} \cdot g_v(s_j) \cdot \begin{pmatrix} V_j^1 & 0 \\ 0 & \overline{V}_j^1 \end{pmatrix} + \frac{x_j^2}{W^2} \sum_{r=1}^{\infty} \begin{pmatrix} V_j^{(r)} & 0 \\ 0 & \overline{V}_j^{(r)} \end{pmatrix} s_j^r,
\]

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\[ U_j = U_j(0) + \frac{y_j}{W} \cdot h_u(s_j) \cdot \left( \begin{array}{cc} 0 & e^{i\theta_j} \sigma' \\ -e^{-i\theta_j} \sigma' & 0 \end{array} \right) + \frac{y_j^2}{W^2} \sum_{r=1}^{\infty} \left( U_j^{(r)} \begin{array}{c} 0 \\ U_j^{(r)} \end{array} \right) s_j^r, \]

where

\[ g_v(s_j) = (1-s_j)^{-1/2} - 1, \quad h_u(s_j) = (1-2x_j^2/W^2)^{-1/2} \left( \left( 1 - \frac{s_j}{1-2x_j^2/W^2} \right)^{-1/2} - 1 \right). \] (5.20)

Here \( \widetilde{V}_j(0), U_j(0) \) are unitary matrices (and hence \( \| \widetilde{V}_j(0) \| \leq 1, \| U_j(0) \| \leq 1 \)),

\[ \widetilde{V}_j = \left( \begin{array}{cc} 0 & e^{i\sigma_j} \\ -e^{-i\sigma_j} & 0 \end{array} \right), \quad \| \widetilde{V}_j^{(r)} \| \leq C^r, \quad \| U_j^{(r)} \| \leq C^r \quad (r = 1, 2, \ldots), \]

and \( \{ \widetilde{V}_j^{(r)} \}, \{ \widetilde{U}_j^{(r)} \} \) are diagonal matrices.

Since the integrals of \( e^{im\theta_j} \) equal 0 for \( m \neq 0 \) and \( 2\pi \) for \( m = 0 \), we conclude that if we replace the coefficients in front of \( e^{i\theta_j} \) and \( e^{-i\theta_j} \) with the bounds for their absolute values, then, after the averaging with respect to \( \theta_j \), the resulting coefficients in front of \( s_j^k \) will grow. The same is true for the integral with respect to \( \sigma_j \). Moreover, for \( x_j \in (0, W/2) \)

\[ g_v(s_j) \approx g_v^*(s_j^*) := \frac{C_1}{1 - C_2 s_j^*}, \]

\[ h_u(s_j) \approx h_u^*(s_j^*) := \frac{C_3}{1 - C_4 s_j^*} \]

where \( C_l, l = 1, \ldots, 4 \) are \( n \)-independent constant and

\[ s_j^* = \frac{\tilde{a}_j + \tilde{b}_j + \tilde{a}_{j-1} + \tilde{b}_{j-1}}{W(a_+ - a_-)} + \frac{(\tilde{a}_{j-1} + \tilde{b}_{j-1})(\tilde{a}_j + \tilde{b}_j)}{W^2(a_+ - a_-)^2}. \]

Hence,

\[ \Phi_{k_1, \ldots, k_p}(a, b) - \Phi_{k_1, \ldots, k_p}(0, 0) \]

\[ \approx \left\langle \left( \text{Prod}_p(x, \sigma) \text{Prod}_p(y, \theta) - 1 \right) \prod_{j=0}^{n} \left( 1 - \frac{2x_j^2}{W^2} + \frac{x_j^2}{W^2} \cdot \frac{s_j^*}{1 - s_j^*} \right) \right\rangle_{x_1, y_1, \sigma_1, \theta_1}, \]

where

\[ \text{Prod}_p(x, \sigma) = \prod \left| 1 + \frac{x_j}{W} e^{i\sigma_j} s_j^* g(s_j^*) + \frac{x_j^2}{W^2} s_j^* g(s_j^*) \right|^{2p}, \]

\[ \text{Prod}_p(y, \theta) = \prod \left| 1 + \frac{y_j}{W} e^{i\theta_j} s_j^* h(s_j^*) + \frac{y_j^2}{W^2} s_j^* h(s_j^*) \right|^{2p} \]

and \( g(t) \) and \( h(t) \) are the function of the form \( C_1/(1 - C_2 t) \) with positive \( n \)-independent \( C_1, C_2 \).

In addition,

\[ \left\langle \frac{x_j^{2k}}{W^{2k}} \right\rangle_{x_j} \leq \frac{k!}{(a_+ - a_-)^{2k}W^{2k}}, \quad \left\langle \frac{y_j^{2k}}{W^{2k}} \right\rangle_{y_j} \leq \frac{k!}{(a_+ - a_-)^{2k}W^{2k}} \]
and thus we conclude
\[
\langle \text{Prod}_p(x, \sigma) \cdot \prod_{j=-n}^{n} \left( 1 - \frac{2x_j^2}{W^2} + \frac{x_j^2}{W^2} \cdot \frac{s_j^*}{1-s_j^*} \right) \rangle_{x, \sigma} \\
\times \prod \left( 1 + \frac{p}{W^2} s_j^* g_1(s_j^*) + \frac{p^2}{W^2} (s_j^*)^2 g_1(s_j^*)^2 \right) \\
\langle \text{Prod}_p(y, \theta) \rangle_{y, \theta} < \prod \left( 1 + \frac{p}{W^2} s_j^* h_1(s_j^*) + \frac{p^2}{W^2} (s_j^*)^2 h_1(s_j^*)^2 \right)
\]

Hence
\[
\Phi_{k_1, \ldots, k_p}^p(a, b) - \Phi_{k_1, \ldots, k_p}^p(0, 0) < \prod \left( 1 + \frac{p}{W^2} s_j^* f_1(s_j^*) + \frac{p^2}{W^2} (s_j^*)^2 f_1(s_j^*)^2 - 1 \right) \\
\times \prod \left( 1 + \frac{2}{W^2} + \frac{1}{W^2} \frac{s_j^*}{1-s_j^*} - 1 \right)
\]

Set
\[
\Pi_3 = \prod_{j=1}^{p} \left( 1 + \frac{\tilde{a}_{k_j} - \tilde{b}_{k_j}}{(a_+ - a_-)W} \right), \quad \Pi_{3,*} = \prod_{j=1}^{p} \left( 1 + \frac{\tilde{a}_{k_j} + \tilde{b}_{k_j}}{(a_+ - a_-)W} \right).
\]

Thus, Lemma 6 yields
\[
\left| \langle \Phi_{k_1, \ldots, k_p}^p(a, b) - \Phi_{k_1, \ldots, k_p}^p(0, 0) \cdot \Pi_1 \cdot \Pi_2 \cdot \Pi_3 \rangle \right| \\
\leq \langle \left( \prod \left( 1 + \frac{2p}{W^2} s_j f_1(s_j) + \frac{p^2}{W^2} s_j^2 f_1(s_j)^2 - 1 \right) \cdot \Pi_{1,*} \cdot \Pi_{2,*} \cdot \Pi_{3,*} \right)_{0,*} \rangle \\
+ \langle \left( \prod \left( 1 + \frac{2}{W^2} + \frac{1}{W^2} \frac{s_j^*}{1-s_j^*} - 1 \right) \cdot \Pi_{1,*} \cdot \Pi_{2,*} \cdot \Pi_{3,*} \right)_{0,*} \rangle + e^{-Cn/W}
\]

Since \( p \leq Cn/W \), we have \( 2p/W^2 \leq W^{-1}, p^2/W^2 = o(1) \). In addition, \( \Pi_3 \) has degree \( p < Cn/W, |\Pi_3| \leq (1 + \delta)^p \). Hence, we can write
\[
\langle \left( \prod \left( 1 + \frac{2p}{W^2} s_j f_1(s_j) + \frac{p^2}{W^2} s_j^2 f_1(s_j)^2 - 1 \right) \cdot \Pi_{1,*} \cdot \Pi_{2,*} \cdot \Pi_{3,*} \right)_{0,*} \rangle \\
\leq (1 + \delta)^p \langle \left( \exp \left\{ \sum_{i=-n}^{n} \left( \frac{C_p}{W^2} \cdot \frac{\tilde{a}_i + \tilde{b}_i}{W} + \frac{p^2 c}{W^2} \cdot \frac{\tilde{a}_i^2 + \tilde{b}_i^2}{W^2} \right) \right\} - 1 \right) \cdot \Pi_{1,*} \cdot \Pi_{2,*} \rangle_{0,*} \\
\leq e^{\delta p} \langle \left( \exp \left\{ \sum_{i=-n}^{n} \left( \frac{C_p}{W^2} \cdot \frac{\tilde{a}_i + \tilde{b}_i}{W} + \frac{p^2 c}{W^2} \cdot \frac{\tilde{a}_i^2 + \tilde{b}_i^2}{W^2} \right) \right\} - 1 \right)_{0,*} \rangle^{1/2}_{0,*}
\]
where \( \Pi_1, \Pi_2 \) are the products of the Taylor's series for \( \exp \{ \nu_j(W) \} \) and for \( \exp \{ \nu_j(W) \} \), and \( \Pi_{1,*}, \Pi_{2,*} \) are obtained from \( \Pi_1, \Pi_2 \) by changing the coefficients to their absolute values.

We proved earlier (see Lemma 5(i)) that the second factor is \( 1 + o(1) \). Moreover, taking the Gaussian integral of the first factor, we obtain
\[
\langle \left( \prod \left( 1 + \frac{2p}{W^2} s_j f_1(s_j) + \frac{p^2}{W^2} s_j^2 f_1(s_j)^2 - 1 \right) \cdot \Pi_{1,*} \cdot \Pi_{2,*} \cdot \Pi_{3,*} \right)_{0,*} \rangle \\
\leq e^{\delta p} \langle \left( \exp \left\{ \frac{c^2 n}{W^3} \right\} - 1 \right) \rangle \leq e^{\delta p} \langle \left( \exp \left\{ \frac{cpn^2}{W^4} \right\} - 1 \right) \rangle
\]

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Similarly,

\[
\left\langle \left( \prod \left( 1 + \frac{2}{W^2} + \frac{1}{W^2} \frac{s_j^*}{1 - s_j^*} \right) - 1 \right) \cdot \Pi_{1,*} \cdot \Pi_{2,*} \cdot \Pi_{3,*} \right\rangle_{0,*} \leq e^{\delta p} \left( \exp \left\{ \frac{C_n}{W^4} \right\} - 1 \right).
\]

Thus, since \( p < Cn/W \), in view of (5.21), we get

\[
\sum_{p=1}^{Cn/W} \frac{(C_1)^p}{p!N_p} \sum_{k_1, \ldots, k_p} \left| \left\langle \left( \Phi_{k_1, \ldots, k_p} (a, b) - \Phi_{k_1, \ldots, k_p} (0, 0) \right) \cdot \Pi_1 \cdot \Pi_2 \cdot \Pi_3 \right\rangle \right| \leq \exp \left\{ C_1 e^{\delta + C_2 n^2/W^4} \right\} - e^{C_1 e^{\delta}} + \left( \exp \left\{ \frac{C_n}{W^4} \right\} - 1 \right) \cdot \left( e^{C_1 e^{\delta}} - 1 \right) = o(1).
\]

If \( p \gg n/W \), then \( 1/\sqrt{p!} \ll e^{-Cn/W} \), and hence we can replace \( \langle \ldots \rangle_0 \) with \( \langle \ldots \rangle_{0,*} \) (see Lemma [1]) and then take the absolute value under the integral and get the bound

\[
e^{C_1 n/W ([\sqrt{Cn/W}]!)^{-1}} \sum_{p=Cn/W}^{\infty} (C_2)^p/\sqrt{p!} = o(1).
\]

Let us prove now that

\[
\tilde{\Phi}_{k_1, \ldots, k_p} (0, 0) = \left\langle \prod_{j=1}^{p} \left( S(\tilde{R}_{k_j} (0) P_n) - S(P_n) \right) \cdot \prod_{i=-n+1}^{n} \left( 1 - 2|\tilde{V}_i (0)|^2 \right) \right\rangle_{\tilde{\eta}} = o(1).
\]

To this end, we write

\[
\left| \tilde{\Phi}_{k_1, \ldots, k_p} (0, 0) \right| \leq \left| \left| S(\tilde{R}_{k_1} (0) P_n) - S(P_n) \right| \right|_{\tilde{\eta}} = \left| \left| S(\tilde{R}_{k_1} (0) P_n) - S(P_n) \right| \right|_{\tilde{\eta}} = \left| \left| S(\tilde{R}_{k_1} (0) P_n) - S(P_n) \right| \right|_{\tilde{\eta}} + \left| \left| S(\tilde{Q}_{k_1} (0)) \right| \right|_{\tilde{\eta}} \cdot \left| \left| 1 - 2S(\tilde{R}_{k_1} (0) P_n) \right| \right|_{\tilde{\eta}} \leq \left| \left| S(\tilde{R}_{k_1} (0) P_n) - S(P_n) \right| \right|_{\tilde{\eta}} + \left| \left| S(\tilde{Q}_{k_1} (0)) \right| \right|_{\tilde{\eta}} \leq \left| \left| S(\tilde{R}_{k_1} (0) P_n) - S(P_n) \right| \right|_{\tilde{\eta}} + \left| \left| \frac{CN}{W^2} \right| \right|_{\tilde{\eta}} \leq \left| \left| \frac{CN}{W^2} \right| \right|_{\tilde{\eta}} \leq \frac{CN}{W^2} = o(1).
\]

Here we used that \( S(P) \in [0, 1] \) for \( P \in \tilde{S}_p(2) \).

This yields

\[
\sum_{p=1}^{Cn/W} \frac{(C_1)^p}{p!N_p} \sum_{k_1, \ldots, k_p} \left| \left\langle \tilde{\Phi}_{k_1, \ldots, k_p} (0, 0) \cdot \Pi_1 \cdot \Pi_2 \cdot \Pi_3 \right\rangle \right| \leq \frac{C_1 N}{W^2} \sum_{p=1}^{Cn/W} \frac{(C_1)^p(1 + \delta)^p}{p!} \leq C_1 N/W^2 = o(1),
\]

which together with (5.22) completes the proof of Lemma [10].
Thus, we can change $F(\bar{a}, b, Q)$ to $F(0, 0, I)$ in (5.7), and then integrate over $\eta$, according to (5.9). We obtain

$$\Sigma_{\pm} = 2^N 6^{N-1} W^{-6 N+4} e^{-2 N c_0} \int_{\tilde{S}^{(2)}} \mu c_+ (a) \mu c_- (b) \times \exp \left\{ - \sum_{j=-n}^{n} \varphi_+ (\tilde{a}_j / W) - \sum_{j=-n}^{n} \varphi_- (\tilde{b}_j / W) \right\} \prod_{j=-n+1}^{n} \left( 1 - \frac{2}{W^2 \Delta_j \Delta_{j-1}} \right)$$

(5.23)

Integrating over $P_-$ by the Itsykson-Zuber formula (see Proposition 2) and using Lemma 4 we get finally

$$\Sigma_{\pm} = 2^N 6^{N-1} e^{-2 N c_0} \frac{d S(\pi(\xi_1 - \xi_2))}{W^{6 N-4}} \int_{|\tilde{a}_j| \leq W^{1 - \kappa}} \prod_{q=-n}^{n} d \tilde{a}_q d \tilde{b}_q \cdot \mu c_+ (a) \mu c_- (b)$$

$$\times (a_+ - a_- + (\tilde{a}_n - \tilde{b}_n) / 2 W) (1 + o(1))$$

$$= \frac{8 \pi^4 \lambda_0^4 e^{-2 N c_0}}{3 W^{6 N-4}} d S(\pi(\xi_1 - \xi_2)) \left| \det^{-1} \left( - \Delta + \frac{2 c_+}{W^2} \right) \right| (1 + o(1)).$$

(5.24)

Thus, we can change $F(\bar{a}, b, Q)$ to $F(0, 0, I)$ in (5.7), and then integrate over $\eta$, according to (5.9). We obtain

$$\Sigma_{\pm} = 2^N 6^{N-1} W^{-6 N+4} e^{-2 N c_0} \int_{\tilde{S}^{(2)}} \mu c_+ (a) \mu c_- (b) \times \exp \left\{ - \sum_{j=-n}^{n} \varphi_+ (\tilde{a}_j / W) - \sum_{j=-n}^{n} \varphi_- (\tilde{b}_j / W) \right\} \prod_{j=-n+1}^{n} \left( 1 - \frac{2}{W^2 \Delta_j \Delta_{j-1}} \right)$$

(5.23)

Integrating over $P_-$ by the Itsykson-Zuber formula (see Proposition 2) and using Lemma 4 we get finally

$$\Sigma_{\pm} = 2^N 6^{N-1} e^{-2 N c_0} \frac{d S(\pi(\xi_1 - \xi_2))}{W^{6 N-4}} \int_{|\tilde{a}_j| \leq W^{1 - \kappa}} \prod_{q=-n}^{n} d \tilde{a}_q d \tilde{b}_q \cdot \mu c_+ (a) \mu c_- (b)$$

$$\times (a_+ - a_- + (\tilde{a}_n - \tilde{b}_n) / 2 W) (1 + o(1))$$

$$= \frac{8 \pi^4 \lambda_0^4 e^{-2 N c_0}}{3 W^{6 N-4}} d S(\pi(\xi_1 - \xi_2)) \left| \det^{-1} \left( - \Delta + \frac{2 c_+}{W^2} \right) \right| (1 + o(1)).$$

(5.24)

5.2.2 $\Sigma_+$ and $\Sigma_-$. 

In this section we prove that the integrals $\Sigma_+$ and $\Sigma_-$ over $\Omega_+^+$ and $\Omega_-^-$ have smaller orders than $\Sigma_{\pm}$.

Lemma 12. For the integral $\Sigma_+$ over the domain $\Omega_+^+$ of (1.14) we have, as $W \to \infty$

$$|\Sigma_+| \leq C W^{-2} |\Sigma_{\pm}|.$$

The same is valid for the integral $\Sigma_-$ over the domain $\Omega_-^-$.

Proof. Consider $\Omega_+^+$ ($\Omega_-^-$ is similar). Returning to $x_j, y_j, w_{1j}, w_{2j}$ coordinates (see (2.2)), we can write that $\Omega_+^+$ corresponds to the set

$$\tilde{\Omega}_+^+ = \{ x_j, y_j, w_{1j}, w_{2j} : x_j, y_j \in U_\delta(a_+), |w_{1j}| \leq \delta, |w_{2j}| \leq \delta \}.$$

Change variables as

$$x_j = a_+ + \frac{\tilde{x}_j}{W}, \quad w_{1j} = \frac{\tilde{w}_{1j}}{W},$$

$$y_j = a_+ + \frac{\tilde{y}_j}{W}, \quad w_{2j} = \frac{\tilde{w}_{2j}}{W}.$$
This yields

\[ \Sigma_+ = \frac{12^N C(\xi)^{-1}}{\pi^{2N} V^{6N}} \int_{|\tilde{x}_1|,|\tilde{y}_2| \leq W^{1-\kappa}} d\tilde{x} d\tilde{y} \int_{|\tilde{w}_{11}|,|\tilde{w}_{22}| \leq W^{1-\kappa}} d\tilde{R} \tilde{w}_1 d\tilde{R} \tilde{w}_2 d\tilde{R} \tilde{w}_2 \]

\[ \times \exp \left\{ - \sum_{j=-n+1}^{n} \left( (\tilde{x}_j - \tilde{x}_{j-1})^2/2 + (\tilde{y}_j - \tilde{y}_{j-1})^2/2 + |\tilde{w}_{1j} - \tilde{w}_{1,j-1}|^2 + |\tilde{w}_{2j} - \tilde{w}_{2,j-1}|^2 \right) \right\} \]

\[ \times \exp \left\{ - \frac{1}{2} \sum_{j=-n}^{n} \left( (a_+ + \frac{\tilde{x}_j}{W} + \frac{i\lambda_0}{2} + \frac{i\xi_1}{N} + \frac{i\xi_2}{N})^2 (a_+ + \frac{\tilde{y}_j}{W} + \frac{i\lambda_0}{2} + \frac{i\xi_1}{N} + \frac{i\xi_2}{N})^2 \right) \right\} \]

\[ \times \exp \left\{ - \sum_{j=-n}^{n} \left( |\tilde{w}_{1j}|^2/W^2 + |\tilde{w}_{2j}|^2/W^2 \right) \right\} \]

\[ \times \prod_{j=-n}^{n} \left( (a_+ + \frac{\tilde{x}_j}{W} - \frac{i\lambda_0}{2})(a_+ + \frac{\tilde{y}_j}{W} - \frac{i\lambda_0}{2}) - \frac{|\tilde{w}_{1j}|^2 + |\tilde{w}_{2j}|^2}{W^2} \right), \]

which gives after some transformations

\[ \Sigma_+ = \frac{12^N e^{-i\pi(\xi_1 + \xi_2)}}{\pi^{2N} V^{6N} e^{2N\rho}} \int_{|\tilde{x}_1|,|\tilde{y}_2| \leq W^{1-\kappa}} d\tilde{x} d\tilde{y} \int_{|\tilde{w}_{11}|,|\tilde{w}_{22}| \leq W^{1-\kappa}} d\tilde{R} \tilde{w}_1 d\tilde{R} \tilde{w}_2 d\tilde{R} \tilde{w}_2 \]

\[ \times \mu_{c_+}(\tilde{x}) \cdot \mu_{c_+}(\tilde{y}) \cdot \mu_{c_+}(\sqrt{2}\tilde{R} \tilde{w}_1) \cdot \mu_{c_+}(\sqrt{2}\tilde{R} \tilde{w}_2) \cdot \mu_{c_+}(\sqrt{2}\tilde{R} \tilde{w}_2) \]

\[ \times \exp \left\{ - \sum_{j=-n}^{n} \left( \frac{i\pi\xi_1}{N} \cdot \frac{\tilde{x}_j}{W} + \phi_+(\tilde{x}_j/W) + \frac{i\pi\xi_2}{N} \cdot \frac{\tilde{y}_j}{W} + \phi_+(\tilde{y}_j/W) \right) \right\} \]

\[ \times \exp \left\{ \sum_{j=-n}^{n} \Phi_+(\tilde{x}_j/W, \tilde{y}_j/W, \tilde{w}_{1j}/W, \tilde{w}_{2j}/W) \right\}, \]

where \(a_+ = a_+ - i\lambda_0/2\) and

\[ \Phi_+(x, y, w_1, w_2) = \log \left( 1 - \frac{|w_1|^2 + |w_2|^2}{(x + \tilde{a}_+)(y + \tilde{a}_+)} \right) + \frac{|w_1|^2 + |w_2|^2}{\tilde{a}_+^2}. \]  \hspace{1cm} (5.25)

Set

\[ d\mu_\gamma = \mu_{c_+}(\tilde{x}) \mu_\gamma(\tilde{y}) \mu_\gamma(\sqrt{2}\tilde{R} \tilde{w}_1) \mu_\gamma(\sqrt{2}\tilde{R} \tilde{w}_2) \]

\[ \times \mu_\gamma(\sqrt{2}\tilde{R} \tilde{w}_2) \mu_\gamma(\sqrt{2}\tilde{R} \tilde{w}_2) d\tilde{R} \tilde{w}_1 d\tilde{R} \tilde{w}_2 d\tilde{R} \tilde{w}_2, \]

and let \(\{ \ldots \} \mu_\gamma\) and \(\{ \ldots \} \int d\mu_\gamma\) be an expectation with respect to \(d\mu_\gamma\) over \(R^{6N}\) or over \([-W^{1-\kappa}, W^{1-\kappa}]^{6N}\) respectively. Computing the integral \(\int d\mu_{c_+}\) we get

\[ \Sigma_+ = \frac{(24\pi)^N e^{-i\pi(\xi_1 + \xi_2)} \det D}{W^{6N} e^{2N\rho}} \left\langle \prod_1 \cdot \prod_2 (y) \cdot \prod_3 \right\rangle_{0, \mu_{c_+}}, \]

where \(\prod_1(x)\) and \(\prod_2(y)\) are the products of Taylor’s series of \(\exp\{-i\pi \xi \tilde{x}_j/N \cdot W - \phi_+(\tilde{x}_j/W)\}\), \(l = 1, 2\) and \(\exp\{\Phi_+\}\) respectively, and

\[ D = -\Delta + \frac{2c_+}{W^2}. \]
Since according to Lemma 6 we have

$$\left\langle \text{Prod}_1(x) \cdot \text{Prod}_2(y) \right\rangle_{0, \tilde{\mu}_{c+}} = 1 + o(1),$$

and (see Lemma 1)

$$\det^{-1} D \leq CW,$$

we are left to prove that

$$\left\langle \text{Prod}_1(x) \cdot \text{Prod}_2(y) \cdot (\text{Prod}_3 - 1) \right\rangle_{0, \tilde{\mu}_{c+}} = o(1). \quad (5.26)$$

Note that the series for $\exp \{ \Phi_+ \}$ starts from the third order. Therefore, repeating almost literally the proof of Lemmas 4 and 3, we can prove that

$$\left\langle \text{exp} \left\{ \sum_{j=-n}^{n} \Phi_+ \left( \tilde{x}_j/W, \tilde{y}_j/W; \tilde{w}_{1j}/W, \tilde{w}_{2j}/W \right) - 1 \right\} \right\rangle_{0, \tilde{\mu}_{c+}} = o(1).$$

The only difference in the proof of Lemma 4 is that now we are integrating over $d\tilde{\mu}_{Rc+}$, i.e., over all variables together, and so each vertex of the multigraph $H$ corresponding to some site $j$ consists of six parts coming from the degree of each variables $\tilde{x}_j, \tilde{y}_j, \Re \tilde{w}_{1j}, \Im \tilde{w}_{1j}, \Re \tilde{w}_{2j}, \Im \tilde{w}_{2j}$ (see Step 2). This means that some pairing are forbidden (for example, between vertices corresponding to $\Re \tilde{w}_{1i}, \tilde{x}_i$ and $\Im \tilde{w}_{2i}, \tilde{y}_i$), and some different pairing can correspond to the same multigraph, but since the number of such pairing is finite (since we make the finite number of steps), it does not change the proof (recall that matrix $M_*$ are the same for each set of variables).

To prove Lemma 3 we should change $|x_j/W|^3$ in the bound of each addition of $\Sigma_k^0$ to $|s(w)_j^2 x_j/W^3|$ or $|s(w)_j^2 y_j/W^3|$, where $s(w)_j = \Re w_{1j}, \Im w_{1j}, \Re w_{2j}$ or $\Im w_{2j}$ (note that each summand in the Taylor’s series of $\exp \{ \Phi_+ \}$ has $s(w)_j^2/W^2$ and $x_j/W$ or $y_j/W$), and use

$$|s(w)_j^2 x/W^3| \leq \frac{p^{-1} x^2 + p s(w)^4/W^4}{2}$$

instead of (4.22).

Then using Lemma 4 we can prove (5.26), thus Lemma 12.

This together with Lemma 9 yield Lemma 8.

6 Auxiliary result

Proof of the Proposition 2. Statement (i) is the well-known Harish Chandra/Itsykson-Zuber formula. Its proof can be found, e.g., in [14], Appendix 5.

To prove (4.36) note that one can diagonalize $X$ by unitary transformation and keep $Z$ and $T$ fixed. Indeed, consider any unitary matrix $U$ which diagonalize $X$. Since $U \in U(2)$, it has the form

$$U = \begin{pmatrix} \cos \varphi \cdot e^{i \theta_1} & \sin \varphi \cdot e^{i \theta_2} \\ -\sin \varphi \cdot e^{i \theta_3} & \cos \varphi \cdot e^{i (\theta_2 + \theta_3 - \theta_1)} \end{pmatrix}.$$  \quad (6.1)
Moreover, we can shift $U$ by any diagonal unitary matrix $U_1$. Choose $U_1$ such that

$$U_0 = UU_1 = \begin{pmatrix} \cos \varphi & \sin \varphi \cdot e^{i\alpha} \\ -\sin \varphi \cdot e^{-i\alpha} & \cos \varphi \end{pmatrix}.$$ 

Then

$$U_0\sigma U_0^\dagger = \sigma,$$

and thus

$$\left( \begin{array}{cc} U_0 & 0 \\ 0 & U_0 \end{array} \right) F \left( \begin{array}{cc} U_0 & 0 \\ 0 & U_0 \end{array} \right)^* = \left( \begin{array}{cc} U_0 XU_0^* w_2 U_0\sigma U_0^* & w_2 U_0 XU_0^* \\ -\overline{w_2} U_0\sigma U_0^* & U_0 XU_0^* \end{array} \right) = \left( \begin{array}{cc} \hat{X} & w_2 \sigma \\ -\overline{w_2} \sigma & \hat{X} \end{array} \right).$$

Hence, changing $X \to U_0^* \hat{X} U_0$ (the Jacobian is $\pi/2(x_1 - x_2)^2$) and using (i), we obtain

$$I_t(G) = \frac{\pi}{2} \int_{\Omega} \int_{U(2)} e^{-\frac{1}{2} \text{Tr}(U_0^* \hat{X} U_0 - D)^2 - t|w_2|^2} (x_1 - x_2)^2 \Phi(\hat{X}, w_2) d\hat{X} dw_2 d\overline{w_2} d\mu(U_0)$$

$$= \frac{\pi}{2t} \int_{\Omega_Y} e^{-\frac{1}{2} \text{Tr}(\hat{X} - D)^2 - t|w_2|^2} \frac{x_1 - x_2}{d_1 - d_2} \cdot (1 - e^{-\frac{1}{2}t(x_1 - x_2)(d_1 - d_2)}) \Phi(\hat{X}, w_2) d\hat{X} dw_2 d\overline{w_2}$$

$$= \frac{\pi}{2t} \int_{\Omega_Y} e^{-\frac{1}{2} \text{Tr}(Y - D)^2} \frac{\text{Tr} YL}{d_1 - d_2} \cdot (1 - e^{-\frac{1}{2}t \text{Tr} YL(1/d_1 - 1/d_2)}) \Phi(Y) dY,$$

where

$$Y = \begin{pmatrix} x_1 & w_2 \\ w_2 & x_2 \end{pmatrix}, \quad \mathcal{L} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad dY = d\hat{X} dw_2 d\overline{w_2}, \quad \Omega_Y = \{Y : F \in \Omega\}.$$ 

Now diagonalizing $Y$ by the unitary transformation $V$, writing

$$\text{Tr} V^* \hat{Y} V \mathcal{L} = (y_1 - y_2)(1 - 2|V_{12}|^2)$$

and again using (\ref{4.32}), we get finally

$$I_t(G) = \frac{\pi^2}{4t} \int \int \exp\left\{ -\frac{t}{2} \text{Tr}(V^* \hat{Y} V - D)^2 \right\} \cdot \frac{1 - 2|V_{12}|^2}{d_1 - d_2}$$

$$\times \left(1 - \exp\left\{ -t \text{Tr} V^* \hat{Y} V \mathcal{L} \cdot (d_1 - d_2)\right\}\right) (y_1 - y_2)^3 dy_1 dy_2 d\mu(V)$$

$$= \frac{\pi^2}{4t^2} \int dy_1 dy_2 \exp\left\{ -\frac{t}{2} \text{Tr}(\hat{Y}^2 + D^2)\right\} \cdot \Phi(\hat{Y}) \cdot \frac{(y_1 - y_2)^2}{(d_1 - d_2)^2}$$

$$\times \left[e^{t(y_1 d_1 + y_2 d_2)} \cdot \left(1 - \frac{2}{t(y_1 - y_2)(d_1 - d_2)}\right) + e^{t(y_1 d_2 + y_2 d_1)} \cdot \left(1 + \frac{2}{t(y_1 - y_2)(d_1 - d_2)}\right)\right],$$

which, taking into account the symmetry of $\hat{\Omega}$, yields (\ref{4.36}). Integral (\ref{4.35}) can be computed straightforward. □

**Proof of Lemma 11.** Note that all non-zero moments of measure $d\eta$ can be expressed via expectations of $v_j^{2s} := |(V_j)|_{12}^{2s}$, $u_j^{2l} := |(U_j)|_{12}^{2l}$. In addition, according to Proposition

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\[ \langle v_j^{2s} u_j^{2l} \rangle_{\eta_j} = 12q_j^{-1} \int_0^1 \int_0^{1/\sqrt{2}} v_j^{2s+1} u_j^{2l+1} e^{t_j(1-2v_j^2)(1-2u_j^2)} 2 - t_j/2 (1 - 2v_j^2)^2 d\nu_j d\eta_j \]

\[ = 24q_j^{-1} \int_0 \int_0^{1/\sqrt{2}} v_j^{2s+1} u_j^{2l+1} e^{t_j(1-2v_j^2)(1-2u_j^2)} 2 - t_j/2 (1 - 2v_j^2)^2 \]

\[ = \frac{24q_j^{-1}}{W^4 p_j^4} \int_0 p_j W^{1/2} \int_0 p_j W \sqrt{1 - 2v_j^2 / p_j^2} \]

\[ \times \exp \left\{ - (a_+ - a_-)^2(v_j^2 + u_j^2) \right\} \cdot \left( 1 - \frac{2v_j^2}{W^2 p_j^2} \right) \]

\[ = \langle \tilde{v}_j^{2s} \tilde{u}_j^{2l} \cdot (1 - 2\tilde{v}_j^2) \rangle_{\tilde{\eta}_j} + O(e^{-c_1W^2}), \]

where \( \eta_j, q_j \) and \( t_j \) are defined in (5.8), and in the third line we have changed \( t_j v_j^2 \to (a_+ - a_-)^2 v_j^2, t_j (1 - 2v_j^2) u_j^2 \to (a_+ - a_-)^2 u_j^2 \).

Now let \( E_k \) be the averaging with respect to the product of the measures \( d\tilde{\eta}_j \) for \( j \) from \( (-n + 1) \) to \( (-n + k) \) and the measures \( d\eta_j \) for \( j \) from \( (-n + k + 1) \) to \( n \). Thus, if

\[ \Psi_{k_1, ..., k_s} = \prod_{j=1}^s S(R_{k_j}, P_{-n}), \]

then it suffices to estimate

\[ \left| \tilde{\Psi}_{k_1, ..., k_s}^{0} - \tilde{\Psi}_{k_1, ..., k_s}^{2n} \right| \leq e^{-cW^2} \]

for \( s \leq p \), where

\[ \tilde{\Psi}_{k_1, ..., k_s}^{i} = E_i \left\{ \Psi_{k_1, ..., k_s}^{1} \prod_{j=-n+1}^{n+i} \left( 1 - 2\tilde{v}^2_j \right) \right\}. \]

Note that

\[ \left| \tilde{\Psi}_{k_1, ..., k_s}^{0} - \tilde{\Psi}_{k_1, ..., k_s}^{2n} \right| \leq \sum_{i=1}^{2n} \left| \tilde{\Psi}_{k_1, ..., k_s}^{i-1} - \tilde{\Psi}_{k_1, ..., k_s}^{i} \right|. \]

In each summand we write for \( \gamma = i - 1, i \) (we assume that all \( k_j \geq (-n + i) \))

\[ \tilde{\Psi}_{k_1, ..., k_s}^{\gamma} = E_\gamma \left\{ \prod_{j=1}^s S(R_{n+1+i}, R_{n+1+i}) P_{-n}^{-1} \prod_{j=-n+1}^{-n+\gamma} (1 - 2\tilde{v}_j^2) \right\} \]

\[ = E_\gamma \left\{ \prod_{j=1}^s \sum_{l_1=2,4} \sum_{\alpha, \alpha'=1,4} \left( R_{n+1+i} P_{-n}^{1} \right)_{\alpha, \alpha'} \left( R_{n+1+i} P_{-n}^{2} \right)_{\alpha' l} \prod_{j=-n+1}^{-n+\gamma} (1 - 2\tilde{v}_j^2) \right\} \]

\[ = \left\{ \sum_{k_{1,1}} C_{k,1} E_{\gamma} \left\{ \langle V_{n+1+i} \rangle_{12}^2 \langle U_{n+1+i} \rangle_{12}^2 \right\}, \right. \]

\[ \gamma = i - 1, \]

\[ \left. \sum_{k_{1,1}} C_{k,1} E_{\gamma} \left\{ \langle V_{n+i} \rangle_{12}^2 \langle U_{n+i} \rangle_{12}^2 (1 - 2\tilde{v}_{n+i}^2) \right\}, \right. \]

\[ \gamma = i, \]

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where the coefficients $C_{k,l}$ are the same for $\gamma = i$ and $\gamma = i - 1$ and can be bounded by $C^s$, since $|\langle R_{n-i-1} \rangle|_{1,0} \leq 1$ and $|\langle R_{n-i} R_{l} P_{n} \rangle|_{0,1} \leq 1$, $l = 2, 4$. Moreover, since

$$|\mathbf{E}_{i-1}\{(V_{-n+i})_{12}^{|2k|}(U_{-n+i})_{12}^{|2l|}\} - \mathbf{E}_{i}\{(V_{-n+i})_{12}^{|2k|}(U_{-n+i})_{12}^{|2l|}(1 - 2\bar{\gamma}_{-n+i}^2)|\} \leq C^s k! l! e^{-CW^2},$$

we obtain

$$\left|\bar{\Psi}^{0}_{k_1, \ldots, k_s} - \bar{\Psi}^{|2n|}_{k_1, \ldots, k_s}\right| \leq nC_1^p(p!)^2 e^{-CW^2}$$

Then the summation with respect to $s$ gives the bound $n! C^{2 n} \log n W e^{-CW^2} = O(e^{-cW^2})$. This yields Lemma [□], since the expression under the expectation in (5.14) has the same number of elements of $Q_j$ and $Q_j^*$. □

### 7 Appendix

#### 7.1 Grassmann integration

Let us consider two sets of formal variables $\{\psi_j\}_{j=1}^n$, $\{\bar{\psi}_j\}_{j=1}^n$, which satisfy the anticommutation conditions

$$\psi_j \psi_k + \psi_k \psi_j = \bar{\psi}_j \bar{\psi}_k + \bar{\psi}_k \bar{\psi}_j = 0, \quad j, k = 1, \ldots, n. \quad (7.1)$$

Note that this definition implies $\psi_j^2 = \bar{\psi}_j^2 = 0$. These two sets of variables $\{\psi_j\}_{j=1}^n$ and $\{\bar{\psi}_j\}_{j=1}^n$ generate the Grassmann algebra $\mathfrak{A}$. Taking into account that $\psi_j^2 = 0$, we have that all elements of $\mathfrak{A}$ are polynomials of $\{\psi_j\}_{j=1}^n$ and $\{\bar{\psi}_j\}_{j=1}^n$ of degree at most one in each variable. We can also define functions of the Grassmann variables. Let $\chi$ be an element of $\mathfrak{A}$, i.e.,

$$\chi = a + \sum_{j=1}^n (a_j \psi_j + b_j \bar{\psi}_j) + \sum_{j \neq k} (a_{j,k} \psi_j \psi_k + b_{j,k} \psi_j \bar{\psi}_k + c_{j,k} \bar{\psi}_j \bar{\psi}_k) + \ldots. \quad (7.2)$$

For any sufficiently smooth function $f$ we define by $f(\chi)$ the element of $\mathfrak{A}$ obtained by substituting $\chi - a$ in the Taylor series of $f$ at the point $a$. Since $\chi$ is a polynomial of $\{\psi_j\}_{j=1}^n$, $\{\bar{\psi}_j\}_{j=1}^n$ of the form (7.2), according to (7.1), there exists such $l$ that $(\chi - a)^l = 0$, and hence the series terminates after a finite number of terms, and so $f(\chi) \in \mathfrak{A}$.

For example, we have

$$\exp\{a \bar{\psi}_1 \psi_1\} = 1 + a \bar{\psi}_1 \psi_1 + (a \bar{\psi}_1 \psi_1)^2/2 + \ldots = 1 + a \bar{\psi}_1 \psi_1,$$

$$\exp\{a_1 \bar{\psi}_1 \psi_1 + a_2 \bar{\psi}_2 \psi_2 + a_2 \bar{\psi}_2 \psi_1 + a_2 \bar{\psi}_2 \psi_1\} = 1 + a_1 \bar{\psi}_1 \psi_1$$

$$+ a_2 \bar{\psi}_1 \psi_2 + a_2 \bar{\psi}_2 \psi_1 + a_2 \bar{\psi}_2 \psi_1 + (a_1 \bar{\psi}_1 \psi_1 + a_1 \bar{\psi}_1 \psi_1) \quad (7.3)$$

Following Berezin [1], we define the operation of integration with respect to the anticommuting variables in a formal way:

$$\int d\psi_j = \int d\bar{\psi}_j = 0, \quad \int \psi_j d\psi_j = \int \bar{\psi}_j d\bar{\psi}_j = 1, \quad (7.4)$$
and then extend the definition to the general element of $\mathfrak{A}$ by the linearity. A multiple integral is defined to be a repeated integral. Assume also that the “differentials” $d\psi_j$ and $d\overline{\psi}_k$ anticommute with each other and with the variables $\psi_j$ and $\overline{\psi}_k$.

Thus, according to the definition, if

$$f(\psi_1, \ldots, \psi_k) = p_0 + \sum_{j_1=1}^k p_{j_1} \psi_{j_1} + \sum_{j_1 < j_2} p_{j_1, j_2} \psi_{j_1} \psi_{j_2} + \ldots + p_{1,2,\ldots,k} \psi_1 \ldots \psi_k,$$

then

$$\int f(\psi_1, \ldots, \psi_k) d\psi_k \ldots d\psi_1 = p_{1,2,\ldots,k}. \quad (7.5)$$

Let $A$ be an ordinary matrix with a positive Hermitian part. The following Gaussian integral is well-known:

$$\int \exp \left\{ - \sum_{j,k=1}^n A_{j,k} z_j z_k \right\} \prod_{j=1}^n d\Re z_j d\Im z_j = \frac{1}{\det A}. \quad (7.6)$$

One of the important formulas of the Grassmann variables theory is the analog of (7.6) for the Grassmann variables (see [1]):

$$\int \exp \left\{ - \sum_{j,k=1}^n A_{j,k} \overline{\psi}_j \psi_k \right\} \prod_{j=1}^n d\overline{\psi}_j d\psi_j = \det A, \quad (7.7)$$

where $A$ now is any $n \times n$ matrix.

For $n = 1$ and $n = 2$ this formula follows immediately from (7.3) and (7.5).

Also we will need the Hubbard-Stratonovich transform (see, e.g., [20]). This is a well-known simple trick, which is just the Gaussian integration. In the simplest form it looks as following:

$$e^{a^2/2} = (2\pi)^{-1/2} \int e^{-x^2/2+ax} dx. \quad (7.8)$$

Here $a$ can be complex number or the sum of the products of even numbers of Grassmann variables.

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