Online Bayesian Recommendation with No Regret

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Abstract

We introduce and study the online Bayesian recommendation problem for a platform, who can observe a utility-relevant state of a product, repeatedly interacting with a population of myopic users through an online recommendation mechanism. This paradigm is common in a wide range of scenarios in the current Internet economy. For each user with her own private preference and belief, the platform commits to a recommendation strategy to utilize his information advantage on the product state to persuade the self-interested user to follow the recommendation. The platform does not know user’s preferences and beliefs, and has to use an adaptive recommendation strategy to persuade with gradually learning user’s preferences and beliefs in the process.

We aim to design online learning policies with no Stackelberg regret for the platform, i.e., against the optimum policy in hindsight under the assumption that users will correspondingly adapt their behaviors to the benchmark policy. Our first result is an online policy that achieves double logarithm regret dependence on the number of rounds. We then present a hardness result showing that no adaptive online policy can achieve regret with better dependency on the number of rounds. Finally, by formulating the platform’s problem as optimizing a linear program with membership oracle access, we present our second online policy that achieves regret with polynomial dependence on the number of states but logarithm dependence on the number of rounds.

1 Introduction

Thanks to the rapid growth of modern technology, social platforms have become a major component of today’s economy. By the end of 2021, there are at least 30 platforms with at least 100 million monthly active users, and seven of them have more than 1 billion users. Based on a recent report [Knowledge Sourcing Intelligence LLP, 2021], the global networking platforms market is evaluated at 192 billion U.S. dollar for the year 2019 and is projected to reach a market size of 940 billion U.S. dollar by the year 2026. Numerous important algorithmic problems rise in this expanding industry.

A prominent example of this sort, which is a major motivating application of this work, is the video recommendation in short-video platforms such as TikTok, Instagram Reels and YouTube Shorts. Taking TikTok as an example, it has a trademark feature – “For You”. It is a feed of videos that are recommended to a user in a real-time manner. In particular, there will be one video

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displayed per time, and the user can decide either to watch it or skip (i.e., not watch) it. If the user starts watching the video, the profit is generated for the platform (e.g., through in-stream ads, sponsored videos). We highlight two features in this application. First, this system not only recommends videos that the user is familiar with, but also intersperses diverse types of videos which may be potentially interesting to the user. Second, the recommendation decision is adaptively formed based on the user's interaction history (which reveals information about the user’s personal preference and belief) and the information of the video (e.g., captions, sounds, hashtags). Similar real-time recommendation applications also appear in other fields (e.g., e-commerce platforms such as Amazon Live, Taobao Live and Xiaohongshu).

Motivated by the above applications, we introduce and study the online Bayesian recommendation problem. Here we describe the problem in the language of video recommendation. Consider a sequential interaction between a video platform and a population of users with the same private preference and belief. At each time, there is a video displayed by the platform to an incoming user. To capture the uncertain characteristics of the video, we study a Bayesian model, in which the payoff-relevant characteristics of the video is captured by a (random) state of the video. The platform and user each have their own preferences over the video states, which are captured by their utility functions respectively. We assume a natural information asymmetry between the platform and users — only the platform can privately observe the realized state of each video, whereas all users only have a prior belief about the video state. Notably, the platform also has its own prior belief over the video state, which is allowed to be different from the users’ belief (after all, they form such beliefs from completely different sources). The platform designs and commits to a recommendation strategy which makes different levels of recommendation (e.g., “not recommend”, “standard”, “recommended”, “highly recommended”) based on his private information about the video, i.e., its realized state. After observing the recommendation level, together with her initial belief, the user forms a posterior belief about the video and decides either to watch this video or skip it.

In the idealized situation where the platform knew both the user’s preferences and prior beliefs, this sequential Bayesian recommendation problem turns out to be a standard Bayesian persuasion problem and thus can be solved by a linear problem (Kamenica and Gentzkow, 2011; Alonso and Camara, 2016; Dughmi and Xu, 2019). This paper, however, addresses the more realistic yet challenging situation in which the platform does not know user’s preferences neither user’s prior beliefs. Therefore, the platform has to adaptively update his recommendation strategy based on user’s past behaviors, so as to maximize its own accumulated utility. The goal of this paper is to design online learning policies with no Stackelberg regret for the platform. Notably, the Stackelberg regret is a new regret notion recently developed for strategic settings (Dong et al., 2018; Chen et al., 2020), which compares to the optimal policy in hindsight, assuming users will correspondingly adapt their behaviors to the benchmark policy (thus the “Stackelberg” in its name). While previous works have demonstrated the difficulty of obtaining sublinear Stackelberg regret in online classification problems (Chen et al., 2020), we surprisingly show that our problem admits efficient online learning algorithms with Stackelberg regret that has logarithmic dependence on the number of rounds $T$. 

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1In lots of applications such as TikTok, there is no interface for the user to directly report her preference, belief or manually customize her recommendation policy.

2Or equivalently, a sequential interaction between the platform and a myopic user.

3In practice, platforms make joint decisions on which video to display and how to recommend. Here we decouple them and focus on the recommendation problem, by assuming that the displaying decision is made exogenously.
Main results and techniques. In this paper, we introduce and analyze two online policies (Algorithm 2 and Algorithm 3) with no regret for the online Bayesian recommendation problem. Algorithm 2 achieves the double logarithm dependence on the number of rounds $T$. This regret dependence on $T$ is optimal, that is, no online policy can obtain a better regret dependence than double logarithm even in the special case where the number of possible states $m$ is two. However, there is one caveat in Algorithm 2: its regret dependence on $m$ is exponential. For a wide range of applications, it is reasonable to focus on problem instances with constant $m$.

Nonetheless, to also shed lights for problem instances with large $m$, we introduce Algorithm 3, whose regret dependence is polynomial on $m$ and logarithm on $T$. Below, we present more details on our results.

One of the features in the online Bayesian recommendation problem is that the platform’s feedback is limited and probabilistic. Specifically, when a recommendation strategy (a.k.a., signaling scheme) which maps each video state to a possibly random recommendation level (a.k.a., signal) is deployed, the platform only observes the user’s reaction against the realized recommendation level but nothing about other recommendation levels. The feedback is also probabilistic, since the realized recommendation level depends on the realized (video) state, which is drawn from the prior distribution exogenously. As a consequence, the platform may bear a large regret in order to learn the user’s reaction to a specific signal realization, or her preference and belief for a specific state. This aforementioned feedback feature becomes one of the major issues that Algorithm 2 and Algorithm 3 need to overcome or bypass, which distinguishes our problem from other online learning problems with logarithm regret.

In the optimum policy in hindsight that has the complete knowledge of users’ preference and belief, the signaling schemes in all rounds are identical and can be solved separately as the classic Bayesian persuasion problem (Kamenica and Gentzkow, 2011; Alonso and Camara, 2016; Dughmi and Xu, 2019). By the revelation principle, this optimum signaling scheme in hindsight is a direct signaling scheme which has binary recommendation level (e.g., “not recommended” and “recommended”). In particular, it specifies an order (based on users’ preference and belief) over all states and recommends every state above a threshold state in this order. The threshold state is selected such that whenever the signaling scheme recommends a video, the user is indifferent between watching and skipping it.

When the platform has no knowledge of users’ preference or belief, the order as well as the threshold state specified in the optimum signaling scheme in hindsight remains unknown. Due to the limited and probabilistic feedback feature, designing an online policy to pin down this order with logarithm regret seems implausible. On the other hand, suppose this order is given, a specific binary search over the threshold state can be accomplished with the double logarithm regret dependence on $T$, under a careful treatment due to the feedback feature. We formalize this idea and design Algorithm 2.

To overcome the uncertainty of the aforementioned order, Algorithm 2 enumerates over all possible orders over all states, which leads to an exponential regret dependence on the number of states $m$.

**Theorem 1.1.** Algorithm 2 achieves $O((m!) \cdot \log \log T)$ regret.

Similar to the optimum policy in hindsight, Algorithm 2 and Algorithm 3 (discussed below) only use direct signaling schemes with binary recommendation level. Note that direct signaling schemes are prevalent in a vast number of real-world applications such as For You in TikTok. However, since the platform does not know users’ preference and belief, the revelation principle fails and thus

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4In our recommendation problem, two videos should be considered as having different states if (a) the platform has enough information to distinguish them, and (b) the user’s utility for watching them are different.
it is unclear whether restricting to direct signaling schemes with binary recommendation level is without loss of generality. In this paper, we give a hardness result showing that introducing more recommendation levels cannot improve the regret dependence on $T$.

**Theorem 1.2.** No online policy can achieve a regret better than $\Omega(\log \log T)$ even for problem instances with binary state.

To show this hardness result, we first construct a reduction from the single-item dynamic pricing problem (cf. Kleinberg and Leighton, 2003) to a special case of the online Bayesian recommendation problem, where the state space is restricted to be binary, and the signaling schemes are restricted to have binary signal space. Then, we argue that in the online Bayesian recommendation problem, when the state space is binary, every online policy can be converted into an online policy which only uses direct signaling scheme with the same regret.

We obtain Algorithm 3 with polynomial regret dependence on $m$ and logarithm on $T$ by phrasing the problem as optimizing a linear program with membership oracle access. In particular, the optimum signaling scheme in hindsight can be formulated as the optimal solution of a linear program as follows. Every feasible solution corresponds to a signaling scheme. The objective is the platform’s utility. The constraints are the feasibility constraint and the persuasiveness constraint. Here the feasibility constraint ensures that every feasible solution of the linear program is indeed a signaling scheme, and the persuasiveness constraint ensures that the user prefers to follow the recommendation. When the platform has no knowledge of users’ preference or belief, the persuasiveness constraint remains unknown. Nonetheless, the platform may check the persuasiveness of a given signaling scheme by deploying this signaling scheme to users. In this sense, the platform obtains a membership oracle for the aforementioned linear program.

**Theorem 1.3.** Algorithm 3 achieves $O(\text{poly}(m \log T))$ regret.

Similar approach (i.e., formulating the learning problems as optimizing linear programs) has been applied in other online learning problem such as contextual dynamic pricing (e.g., Leme and Schneider, 2018), security game (e.g., Blum et al., 2014). Similar to Algorithm 2, Algorithm 3 requires particular handle to overcome the challenge due to our feedback feature. Moreover, comparing with separation oracle, linear optimization with membership oracle access is harder. First, there is still gap between lower bound and upper bound of the number of queries required to the membership oracle. Second, when only membership oracle is given, an interior point of the linear program is required to be constructed, which becomes one major technical hurdle in the design and analysis of Algorithm 3.

**Related work.** Our work connects to several strands of existing literature. First, when user’s preference is known and shares the same prior belief with the platform, the one-shot instantiation of our problem exactly follows the formulation of the canonical Bayesian persuasion problem (Kamenica and Gentzkow, 2011). Bayesian persuasion concerns the problem that an informed sender (i.e., platform) designs an information structure (i.e., signaling scheme) to influence the behavior of a receiver (i.e., user). There is a growing literature, including our work, on studying the relaxation of one fundamental assumption in the Bayesian persuasion model – The sender perfectly knows receiver’s preference and her prior belief. There are generally two approaches to deal

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5Loosely speaking, this reduction suggests that our problem is harder than the dynamic pricing problem.
with such sender’s uncertainty: the robust approach (Dworczak and Pavan, 2020; Babichenko et al., 2021; Kosterina, 2018; Hu and Weng, 2021) which tries to design signaling schemes that perform robustly well for all possible receiver’s utilities; the online learning approach (Castiglioni et al., 2020, 2021; Zu et al., 2021) which studies the regret minimization when the sender repeatedly interacts with receivers. Our work falls into the second approach. In particular, Castiglioni et al. (2020) concerns the sender interacting with receivers who have the unknown type. They provide an algorithm with regret guarantee $O(T^{4/5})$ but has exponential running-time over the number of states. Zu et al. (2021) studies a setting where the sender has unknown prior distribution, and they require sender to make persuasive signaling schemes at each round. They provide an algorithm with $O(\sqrt{T})$ regret bound, and also demonstrate that it is tight whenever the receiver has five (or more) actions.

Our work differs from the above works in many ways. First, instead of assuming unknown types, our setting directly relaxes the knowledge on user’s utilities. Second, we do not require platform’s singling scheme to be persuasive at each round. Third, we achieve logarithmic regrets over the time horizon.

Second, our work also relates to research on Bayesian exploration in multi-armed bandit (Kremer et al., 2014; Mansour et al., 2015, 2021). In both our Bayesian recommendation and their Bayesian exploration, the platform utilizes his information advantage to persuade the user to take the desired action, and the user observes the platform’s message and forms her posterior which will be used to make her optimal action. But different from theirs, the goal of the platform is different, and our problem do not require the platform to make incentive-compatible action recommendation at each round. Thus, the analysis and the technique of this work are quite different from theirs.

Third, our analysis is related to literature on (contextual) dynamic pricing (Kleinberg and Leighton 2003; Leme and Schneider 2018; Liu et al., 2021). In particular, we prove our lower bound via a non-trivial reduction to the single-item dynamic pricing problem. Though it is seemingly that our problem for multiple states shares the similarity to the contextual dynamic pricing (e.g., we both need to learn an unknown vector: in our setting, it is the user’s preference of product state, and in contextual pricing, it is buyer’s preference of product features), we note that there are significant differences in our problem structure like the platform’s actions, and the probabilistic feedback from users (see the end of Section 3.2 for the detailed comparisons).

2 Preliminary

2.1 Basic Setup

Motivated by the applications of short-video platform, this paper considers the following Bayesian recommendation problem. We begin with describing a static model and then introduce the online model studied in this work.

In the static model, there are two players: a platform and a user. The platform wants to recommend a video to the user. The video is associated with a private state $\theta$ drawn from a finite set $[m] = \{1, \ldots, m\}$ according to a prior distribution $\lambda \in \Delta([m])$, which is common knowledge among both players. We use notation $\theta$ to denote the state as a random variable, and $i, j \in [m]$ as its possible

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6We refer the reader to the work by Dworczak and Pavan (2020); Babichenko et al. (2021) for a comprehensive overview on different methods in the robust approach.

7In this paper, we use “he” to denote the platform and “she” to denote the user.
realizations. The user has a binary-action set $A = \{0, 1\}$ (i.e., not watch or watch), and a utility function $\rho : [m] \times A \to \mathbb{R}$ mapping from the state of the video and her action to her utility. The platform has a state-independent utility function $\xi : A \to \mathbb{R}$ that only depends on the user’s action $a$. Without loss of generality, we normalize $\xi(a) = a$. The platform has the ability to recommend the video in different levels based on its private state.

In particular, the platform can design a finite signal space $\Sigma$ where each signal $\sigma \in \Sigma$ represents a recommendation level for the video (e.g., “recommended”, “highly recommended”, “best of today”). A signaling scheme $\pi : [m] \to \Delta(\Sigma)$ is a mapping from video (based on its state) into probability distributions over signals. We denote by $\pi(i, \sigma)$ the probability of sending signal $\sigma \in \Sigma$ at state $i$.

In this work, we consider the following repeated interaction between the platform and a population of users. All users share the same utility function $\rho(\cdot, \cdot)$ which is unknown to the platform. All players (including the platform) have the same prior $\lambda$. The setting proceeds for $T$ rounds. For each round $t = 1, \ldots, T$:

1. The platform commits to a signal space $\Sigma_t$ and a signaling scheme $\pi_t : [m] \to \Delta(\Sigma_t)$.
2. A video with state $\theta_t \sim \lambda$ is realized according to the prior $\lambda$ and a signal $\sigma_t$ is realized according to $\{\pi_t(\theta_t, \sigma)\}_{\sigma \in \Sigma_t}$.
3. Upon seeing the signal $\sigma_t$, user $t$ updates her belief given prior $\lambda$ and signaling scheme $\pi_t$. In particular, she forms a posterior distribution $\mu_t : \Sigma_t \to \Delta([m])$ that maps realized signal $\sigma_t$ into probability distribution over state space $[m]$. We assume that users are Bayesian, i.e., $\mu_t(\sigma_t, i) \triangleq \frac{\lambda(i)\pi_t(i, \sigma_t)}{\sum_{j \in [m]} \lambda(j)\pi_t(j, \sigma_t)}$.
4. With the posterior $\mu_t$, user $t$ chooses an action $a_t$ that maximizes her expected utility, i.e., $a_t = \arg\max_{a \in A} \mathbb{E}_{\theta \sim \mu_t}[\rho(\theta, a)]$.
5. The platform then derives the utility $U_t$.

Given a signaling scheme $\pi$, let $U(\pi)$ denote the platform’s expected payoff. The goal of the platform is to design an online policy which constructs signaling schemes $\{\pi_t\}_{t \in [T]}$ to maximize his long-term expected utility $\sum_{t \in [T]} U(\pi_t)$.

### 2.2 Stackelberg Regret and Benchmark

We evaluate the performance of an online policy by its Stackelberg regret (Chen et al., 2020) against the optimum policy in hindsight. The optimum policy in hindsight knows users’ utility function $\rho(\cdot, \cdot)$, and maximizes the platform’s long-term expected utility. Since users are all identical, the

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8 For ease of presentation, our main context will focus on a stylized setup where (i) the platform and the user share the same prior belief $\lambda$ over $[m]$; (ii) the platform has state-independent utility function and only benefits from the user’s action 1 (i.e., click). In Section 3, we illustrate how our algorithms and results can be easily extended to general settings with different prior beliefs and arbitrary utility functions of the platform.

9 Namely, the platform gains 1 unit of profit if the user watches the video.

10 For ease of presentation, our main context restricts the signal space to be finite. Our main result can be extended to continuous signal space.

11 In practice, the platform may not have such commitment power, and cannot adapt his signaling scheme daily. Instead, the platform sticks to one signaling scheme for a longer time cycle, and a user is able to learn her posterior as well as her best response under this signaling scheme. Thus, we can interpret one round in our model as one time cycle in practice.
optimum policy in hindsight commits to the same signaling scheme \( \pi^* \) (see its characterization in program \( P_{\text{opt}} \) and Lemma 2.1) for every round \( t \in [T] \).

**Definition 2.1.** Given user’s utility function \( \rho \), let \( \pi^* \) be the optimal signaling scheme. The Stackelberg regret of online policy \( \text{ALG} \) is

\[
\text{REG}[\text{ALG}] \triangleq \sum_{t \in [T]} U(\pi^*) - \mathbb{E}_{\pi_1, \ldots, \pi_T} \left[ \sum_{t \in [T]} U(\pi_t) \right]
\]

where \( \pi_t \) is the signaling scheme committed by \( \text{ALG} \) in each round \( t \in [T] \).

Different from the regret notation (e.g., external regret) in classic single-agent no regret learning literature (cf. Blum and Mansour, 2007), the Stackelberg regret compares to the optimum policy in hindsight, where users have the opportunity to re-generate a different history by best-responding to the new signaling scheme \( \pi^* \). In the remaining of the paper, we simplify the terminology Stackelberg regret as regret.

When users’ utility function \( \rho(\cdot, \cdot) \) is known to the platform, the optimal signaling scheme in hindsight. By the revelation principle (Kamenica and Gentzkow, 2011), there always exists an optimal signaling scheme with binary signal space \( \Sigma = \{0, 1\} \equiv A \) that corresponds to action recommendations. In particular, it can be solved by a linear program as follows,

\[
\pi^* = \arg \max_{\pi} \sum_{i \in [m]} \lambda(i) \pi(i, 1) \quad \text{s.t.} \quad (\text{IC}) \quad \sum_{i \in [m]} (\rho(i, 1) - \rho(i, 0)) \lambda(i) \pi(i, 1) \geq 0
\]

Here constraint (IC) ensures persuasiveness of the signaling schemes, i.e., taking action 1 is indeed user’s optimal action given her posterior when action 1 is recommended\(^{12}\) For ease of presentation, with slight abuse of notation, we use \( \pi^*(i) \triangleq \pi^*(i, 1) \) and thus \( \pi^*(i, 0) \equiv 1 - \pi^*(i) \). Additionally, we introduce auxiliary variable \( \omega(i) \triangleq (\rho(i, 1) - \rho(i, 0)) \lambda(i) \). To make the problem non-trivial, we make the following two assumptions on user’s utility function throughout this paper\(^{13}\)

**Assumption 1.** For user’s utility function, there exists at least one state \( i \in [m] \) such that \( \omega(i) > 0 \).

**Assumption 2.** For user’s utility function, \( \sum_{i \in [m]} \omega(i) < 0 \).

Program \( P_{\text{opt}} \) can be interpreted as a fractional knapsack problem, where the budget is zero, and each state \( i \) corresponds to an item with value \( \lambda(i) \) and (possibly negative) cost \( \omega(i) \). Thus, its optimal solution \( \pi^* \) has the following characterization.

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\(^{12}\) In program \( P_{\text{opt}} \) the persuasive constraint for action 0 is omitted, since two programs have the same optimal solution. This is due to our assumption that the platform’s utility of action 1 is higher than his utility of action 0.

\(^{13}\) When Assumption 1 is violated, the problem becomes trivial since user takes action 0 regardless of the signaling scheme and thus any online policy achieves zero regret. Assumption 2 can be verified in round 1 by committed to a no-information-revealing signaling scheme, which induces regret at most 1. If \( \sum_{i \in [m]} \omega(i) \geq 0 \), then committing to no-information-revealing signaling scheme in the remaining \( T - 1 \) rounds attains optimal utility for the platform.
Lemma 2.1 (See for example Renault et al., 2017). The optimum signaling scheme \( \pi^* \) in hindsight is the optimal solution of linear program \( P_{opt} \). There exists a threshold state \( i^\dagger \in [m] \) such that (a) for every state \( i \neq i^\dagger \), \( \pi^*(i) = 1 \left[ \frac{\omega(i)}{\lambda(i)} \geq \frac{\omega(i^\dagger)}{\lambda(i^\dagger)} \right] \), and (b) \( \pi^*(i^\dagger) = -\sum_{i \neq i^\dagger} \frac{\omega(i)\pi^*(i)}{\omega(i^\dagger)} \).

In words, Lemma 2.1 states that the signaling scheme \( \pi^* \) reveals whether the state is above or below a threshold state \( i^\dagger \), with possibly randomization at state \( i^\dagger \).

2.3 A Useful Subroutine for Checking Persuasiveness

When user’s utility function \( \rho(\cdot, \cdot) \) is unknown, the standard revelation principle fails. As a consequence, restricting to binary signal space (e.g., \{“recommended”, “not recommended”\}) is not without loss of generality. Nonetheless, as we formally show later, restricting to the subclass of signaling schemes with binary signal space does not hurt the optimal regret. We now formally define such signaling schemes as follows.

**Definition 2.2.** A direct signaling scheme \( \pi : [m] \rightarrow \Delta(A) \) is a mapping from state into probability distributions over action recommended to user.

With slight abuse of notation, for every direct signaling scheme \( \pi \), we use \( \pi(i) \triangleq \pi(i, 1) \) and thus \( \pi(i, 0) \equiv 1 - \pi(i) \). When facing a direct signaling scheme \( \pi \), user takes the action that maximizes her expected utility given her posterior. We say \( \pi \) is persuasive if the user takes action 1 as long as action 1 is recommended by signaling scheme \( \pi \). The proofs of Lemma 2.2, Lemma 2.3 and Lemma 2.4 are deferred to Appendix A.

**Lemma 2.2.** A direct signaling scheme \( \pi \) is persuasive if and only if \( \sum_{i \in [m]} \omega(i)\pi(i) \geq 0 \).

Before we finish the preliminary section, we provide Procedure 1 as a useful subroutine which will be used in our online policies. Procedure 1 takes a direct signaling scheme as input, and determines whether this direct signaling scheme is persuasive. Its correctness guarantee is given in Lemma 2.3 and the regret guarantee is given in Lemma 2.4.

**Procedure 1: CheckPersu**

**Input:** direct signaling scheme \( \pi \)

**Output:** True/False – whether \( \pi \) is persuasive; or round-exhausted if there is no round left

1 while there are rounds remaining do
2    /* suppose now is round \( t \) */
3    Commit to signaling scheme \( \pi \) to user \( t \).
4    if \( \sigma_t = 1 \) and \( a_t = 1 \) then
5        return True
6    else if \( \sigma_t = 1 \) and \( a_t = 0 \) then
7        return False
8    else if \( \sigma_t = 0 \) and \( a_t = 1 \) then
9        return False
10   move to next round, i.e., \( t \leftarrow t + 1 \)
11 return round-exhausted

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14 Throughout this paper, we say a state \( i \) is above (resp. below) state \( j \) if it satisfies that \( \frac{\omega(i)}{\lambda(i)} \geq \frac{\omega(j)}{\lambda(j)} \) (resp. \( \frac{\omega(i)}{\lambda(i)} < \frac{\omega(j)}{\lambda(j)} \)).
Lemma 2.3. Given a direct signaling scheme $\pi$, Procedure \ref{alg:checkpersuasiveness} returns True only if $\pi$ is persuasive, and returns False only if $\pi$ is not persuasive.

Lemma 2.4. Given a direct signaling scheme $\pi$, the expected regret of Procedure \ref{alg:checkpersuasiveness} is at most

$$\frac{U(\pi^*)}{\sum_i \lambda(i) \pi(i)} - 1[\pi \text{ is persuasive}].$$

The intuition behind the Lemma 2.4 is as follows. Given a direct signaling scheme $\pi$, as long as its probability (i.e., the value $\sum_i \lambda(i) \pi(i)$) for recommending action 1 is a constant approximation to the optimal payoff $U(\pi^*)$, then the expected regret of Procedure \ref{alg:checkpersuasiveness} for checking its persuasiveness can be upper bounded by this constant. However, if this probability is small compared to $U(\pi^*)$, then the incurred expected regret can be very large, regardless of the persuasiveness of $\pi$ or the magnitude of $U(\pi^*)$.

3 An Algorithm with $O(\log \log T)$ Regret

In this section, we provide our first result – an online policy with $O(\log \log T)$ regret, and show its regret dependency on the number of rounds $T$ is optimal. Specifically, in Section 3.1, we first introduce Algorithm \ref{alg:threshold} with $O(\log \log T)$ regret. Then in Section 3.2 we provide a matching regret lower bound $\Omega(\log \log T)$, yielding the tightness of the result.

Before diving into our result, let us highlight one of the main challenges in the design of a good online policy. In our problem, the platform’s feedback is limited and probabilistic. Specifically, when signaling scheme $\pi_t$ is used in round $t$ and signal $\sigma_t \sim \pi_t(\theta_t)$ is realized, the platform only observes user’s action under signal $\sigma_t$ and learns his corresponding payoff, but nothing about his payoff under other signals. Meanwhile, this feedback is also probabilistic, since the realized signal $\sigma_t$ depends on the realized state $\theta_t$. Because of these two features of the feedback, some natural tasks towards learning user’s utility may not be completed easily. Here are two illustrative examples.

**Identifying the signs of $\{\omega(i)\}$.** Recall that the optimum in hindsight signaling scheme $\pi^*$ follows from a threshold signaling scheme − it recommends action 1 deterministically for all states above a threshold state $i^\dagger$, recommends action 1 randomly at threshold state, and recommends action 0 deterministically for all states below $i^\dagger$. Follow the same logic, a natural attempt to design a good online policy is trying to identify the threshold state and the states that are above the threshold state. However, it is unclear on how to identify the threshold state. In fact, it is even challenge to identify the sign of $\omega(i)$ of a state $i$. To see this, ideally, identifying the sign of $\omega(i)$ needs to solicit the user’s action when the user’s posterior belief is concentrated on state $i$ when a particular signal is realized. A signaling scheme $\pi$ with $\pi(j) = 1[j = i], \forall j \in [m]$ can shape user’s posterior belief to be concentrated on state $i$ when a signal 1 is realized. However, since the feedback is limited, such signaling scheme cannot collect useful information whenever other signal is realized. Consequently, it bears a large regret if it happens to be the case when $\lambda(i)$ is small.

**Determining if $U(\pi^*) \geq C$.** Consider a problem instance with $m = 2$ states. Suppose the platform knows that $\omega(1) > 0$, and $\omega(2) < 0$. This implies that the threshold state $i^\dagger = 2$, and state 1 is above threshold state 2. Note that a good online policy should be able to approximately identify the value of $U(\pi^*)$. Now, suppose the platform only wants to determine whether $U(\pi^*) \geq C$. If platform can determine whether this following natural signaling scheme $\pi(1) = 1$ and $\pi(2) = (C - \lambda(1))/\lambda(2)$,
is persuasive or not, then the platform can determine whether $U(\pi^*) \geq C$. However, since the feedback is probabilistic, it takes $1/C$ rounds (in expectation) to learn the persuasiveness of $\pi$. Thus, even if $U(\pi^*)$ is small (i.e., $U(\pi^*) = o(1)$), by Lemma 2.4, the aforementioned attempt bears a superconstant regret as long as $C = o(U(\pi^*))$.

3.1 Towards $O(\log \log T)$ Regret

Despite the above mentioned challenges, in this subsection, we present an algorithm that can have $O(\log \log T)$ regret guarantee. The details of our algorithm are in Algorithm 2.

Overview of the algorithm. In our online policy, the whole $T$ rounds are divided into the exploring phase and the exploiting phase. The exploring phase has two subphases. The first subphase (i.e., exploring phase I) identifies a lower bound and an upper bound of $U(\pi^*)$, i.e., it identifies a persuasive signaling scheme $\pi$ with $U := U(\pi)$ such that $U \leq U(\pi^*) \leq 2U$. Note that once we narrow down the value of $U(\pi^*)$ to be in the interval $[U, 2U]$, with the persuasive signaling scheme $\pi$, we can ensure that expected regret to check the persuasiveness of the signaling schemes in the later rounds is at most a constant, which addresses the second challenge (i.e., determining if $U(\pi^*) \geq C$) we just mentioned before. We will show that the expected cumulative regret in exploring phase I is $O(1)$. The second subphase (i.e., exploring phase II) identifies a signaling scheme $\pi^\dagger$ whose per-round expected regret is $1/T$, and we will show that its expected cumulative regret is at most $O(\log \log T)$. The identified signaling scheme $\pi^\dagger$ from the exploring phase II is used in the remaining rounds considered as the exploiting phase, which induces $O(1)$ expected cumulative regret. See Algorithm 2 for a formal description.

A subclass of direct signaling schemes $\{\pi^{(r,u)}\}$. Our online policy will repeatedly consider a subclass of direct signaling schemes. Recall program $\mathcal{D}^{opt}$ indicates that the optimum signaling scheme in hindsight $\pi^*$ can be thought as the optimal solution of a fractional knapsack problem, where each state $i$ corresponds to an item with value $\lambda(i)$ and cost $\omega(i)$. This observation implies that there must exist a total order $r^*$ over all states with respect to true their bang-per-buck $\lambda(i)/\omega(i)$. Since user’s utility function is unknown, the bang-per-buck as well as total order $r^*$ are unknown to the platform. In the exploring phase, our online policy maintains a subset $\mathcal{P}$ of total orders over $[m]$ that contains $r^*$. Given an arbitrary order $r$, we define $\pi^{(r,u)}$ to be the direct signaling scheme as follows: there exists a threshold state $i^\dagger$ such that (a) for every state $i \neq i^\dagger$, $\pi^{(r,u)}(i) = 1[r(i) > r(i^\dagger)]$ and (b) $\pi^{(r,u)}(i^\dagger) = u - \frac{\sum_{i \in [m]} \lambda(i)^{\pi^{(r,u)}}(i)}{\lambda(i^\dagger)}$. As a sanity check, observe that the signaling scheme $\pi^{(r^*,U(\pi^*))}$ is exactly the optimum signaling scheme in hindsight $\pi^*$. It is also guaranteed that $\sum_{i \in [m]} \lambda(i)^{\pi^{(r,u)}}(i) = u$ by construction. We note that by maintaining the set $\mathcal{P}$ that contains the optimal total order $r^*$, we bypass the challenge on identifying the value, order or even signs of $\{\omega(i)\}$. Indeed, for general problem instances, our Algorithm 2 does not explicitly learn those quantities, nor they can be inferred from the outcome of Algorithm 2.\footnote{Under this signaling scheme $\pi$, if $\pi$ is persuasive, then we have $U(\pi^*) \geq U(\pi) \geq C$, otherwise $U(\pi^*) < U(\pi) < C$.}

\footnote{Given an order $r$, we denote $r(i)$ by the rank of state $i$.}
Algorithm 2: \(O(\log \log T)\) Search

**Input:** number of rounds \(T\), number of states \(m\), prior distribution \(\lambda\)

1. Initialize \(\mathcal{P}\) to be all total orders over \([m]\)

   /* exploring phase I - identify \(U\) such that \(U \leq U(\pi^*) \leq 2U\)

2. Initialize \(U \leftarrow \frac{1}{2}\)

3. while True do

4.   if there exists \(r \in \mathcal{P}\) such that \(\text{CheckPersu}(\pi(r,U)) = \text{True}\) then

5.     \(\mathcal{P} \leftarrow \{r \in \mathcal{P} : \text{CheckPersu}(\pi(r,U)) = \text{True}\}\)

6.     break

7.   else

8.     \(U \leftarrow \frac{U}{2}\)

   /* exploring phase II - identify a signaling scheme \(\pi^\dagger\) such that \(U(\pi^\dagger) \geq U(\pi^*) - \frac{1}{T}\)

9. Initialize \(R \leftarrow 2U, L \leftarrow U, \delta \leftarrow 1\)

10. while \(R - L \geq \frac{1}{T}\) do

11.    \(\varepsilon \leftarrow \frac{\delta}{2}, S \leftarrow \lceil \frac{R - L}{\varepsilon L} \rceil\)

12.    for \(\ell = 1, 2, \ldots, S\) do

13.       if there exists \(r \in \mathcal{P}\) such that \(\text{CheckPersu}(\pi(r,L + \ell \varepsilon L)) = \text{True}\) then

14.          \(\mathcal{P} \leftarrow \{r \in \mathcal{P} : \text{CheckPersu}(\pi(r,L + \ell \varepsilon L)) = \text{True}\}\)

15.       else

16.          \(R \leftarrow L + (\ell - 1) \varepsilon L, L \leftarrow L + \ell \varepsilon L, \delta \leftarrow \sqrt{\varepsilon}\)

17.          break

18. Set \(\pi^\dagger \leftarrow \pi(r,L)\) for an arbitrary \(r \in \mathcal{P}\)

   /* exploiting phase */

19. Use signaling scheme \(\pi^\dagger\) for all remaining rounds.

Remark 3.1. In both exploring phase I and II, Algorithm 2 checks whether there exists \(r \in \mathcal{P}\) such that \(\text{CheckPersu}(\pi(r,u)) = \text{True}\) for some \(u\). Our regret bound has an \((m!)\) dependence due to brute-force searching over \(r\). We would like to note that (i) in many practical applications, the number of states \(m\) is small or even constant, and thus our main focus in this section is the optimal dependence on the number of rounds \(T\), and (ii) when \(N (\leq m!)\) identical problem instances are allowed to run in parallel, the regret dependence on \(m\) becomes \(m!/N\). Finally, we provide another online policy with \(O(\text{poly}(m \log T))\) regret in Section 4.

Remark 3.2. The exploring phase II is similar to the single-item dynamic pricing mechanism in Kleinberg and Leighton (2003). As we will see in Theorem 3.2, the Bayesian recommendation problem is not only harder than the single-item dynamic pricing problem, but also its generalization - contextual dynamic pricing problem.

We are now ready to describe the main result of this section.

Theorem 3.1. The expected regret of Algorithm 2 is at most \(O((m!) \cdot \log \log T)\).

Proof. We analyze the expected regret in exploring phase I, exploring phase II, and exploiting phase separately. We first assume that Algorithm 2 finishes exploring phase I and II before \(T\) rounds are exhausted. Similar argument follows for the other case where exploring phase I or exploring phase...
We finish this part by showing \( \Omega(\log(\log T)) \). By definition, \( \text{CheckPersu}(\pi^{(r, 2^{-k})}) = \text{False} \) for all \( r \in \mathcal{P} \) and \( k \in [K - 1] \), and \( \text{CheckPersu}(\pi^{(r, 2^{-K})}) = \text{True} \). Thus, in the end of exploring phase I, \( U \) is \( 2^{-K} \), and there are \( K \) iterations in the while loop. For each iteration \( k \in [K] \), \( \text{CheckPersu}(\pi^{(r, 2^{-k})}) \) is called for every \( r \in \mathcal{P} \). By Lemma 2.4, the total expected regret is

\[
\sum_{k \in [K]} \sum_{r \in \mathcal{P}} \sum_{i \in [m]} \frac{U(\pi^*)}{\lambda(i)\pi^{(r, 2^{-k})}(i)} \leq \sum_{k \in [K]} \sum_{r \in \mathcal{P}} \frac{2^{-(K-1)}}{2^{-k}} = |\mathcal{P}| \sum_{k \in [K]} 2^{-(K-k-1)} \leq 4(m!)
\]

where the denominator in the right-hand side of inequality (a) is due to the construction of \( \pi^{(r, 2^{-k})} \).

**Exploring phase II.** By construction, there are \( O(\log \log T) \) iterations in the while loop. Thus, it is sufficient to show the expected regret in each iteration is \( O(m!) \).

In each iteration \( k \), for every \( r \in \mathcal{P} \), let \( l^\dagger \in [S] \) be the smallest index that the signaling scheme \( \pi^{(r, L+\ell \varepsilon L)} \) is not persuasive. The expected regret in iteration \( k \) for \( r \in \mathcal{P} \) is at most

\[
\sum_{\ell = 1}^{\ell^\dagger - 1} \left( \frac{U(\pi^*)}{\lambda(i)\pi^{(r, L+\ell \varepsilon L)}} - 1 \right) + \sum_{i \in [m]} \frac{U(\pi^*)}{\lambda(i)\pi^{(r, L+\ell \varepsilon L)}} \\
= \sum_{\ell = 1}^{\ell^\dagger - 1} \left( \frac{U(\pi^*)}{L + \ell \varepsilon L} - 1 \right) + \sum_{\ell = 1}^{\ell^\dagger - 1} \left( \frac{R}{L - 1} - 1 \right) + \frac{R}{L} \leq (S - 1) \frac{R - L}{L} + 2 \leq \frac{(R - L)^2}{\varepsilon L^2} + 2
\]

where equality (a) holds due to the construction of \( \pi^{(r, L+\ell \varepsilon L)} \) and \( \pi^{(r, L+\ell \varepsilon L)} \); inequality (b) holds since \( U(\pi^*) \leq R \); inequality (c) holds since \( \ell \leq S \) and \( R \leq 2L \); and inequality (d) holds since \( S = \lfloor \frac{R - L}{\varepsilon L} \rfloor \).

We finish this part by showing \( R - L \leq \sqrt{2\varepsilon L} \) by induction. Let \( L^{(k)}, R^{(k)}, \delta^{(k)} \) and \( \varepsilon^{(k)} \) be the value of \( L, R, \delta, \varepsilon \) in each iteration \( k \). The claim is satisfied for iteration \( k = 1 \), since \( R^{(1)} - L^{(1)} = 2U - U = L^{(1)} \) and \( \varepsilon^{(1)} = 1/2 \). Suppose the claim holds for iteration \( k - 1 \). Now, for iteration \( k \), we know that \( R^{(k)} - L^{(k)} = \varepsilon^{(k-1)} L^{(k-1)} \leq \varepsilon^{(k-1)} L^{(k-1)} L^{(k)} = \sqrt{\delta^{(k)}} L^{(k)} = \sqrt{2\varepsilon^{(k)}} L^{(k)} \), which finishes the induction.

**Exploiting phase.** By construction, \( \mathcal{P} \) is not empty in the end of exploring phase II, and thus signaling scheme \( r^\dagger \) is well-defined. Additionally, we know that \( \pi^\dagger \) is persuasive and \( U(\pi^\dagger) \geq U(\pi^*) - 1/r \), which concludes the proof.

### 3.2 \( \Omega(\log \log T) \) Regret Lower Bound

We now show a tighter lower bound of \( \Omega(\log \log T) \) regret of any online policy, even when the number of states is 2. Here we allow the online policy to be randomized (i.e., can commit to different signaling schemes at random) and have non-binary (but finite) signal spaces (i.e., can have multiple recommendation levels).

**Theorem 3.2.** No online policy can achieve an expected regret better than \( \Omega(\log \log T) \).
Overview of the proof. To show Theorem 3.2, we focus on problem instances with binary state. Our proof mainly consists of two steps. In the first step, we show that for problem instances with binary state, any online policy can be transformed into a randomized online policy that only uses signaling schemes with binary signal space. This statement is no longer true for general problem instances with non-binary state, since the classic revelation principle fails. The key technical ingredient (Lemma 3.4) is to show that any posterior distribution of binary state can be induced by a convex combination of signaling schemes with binary signal space, which may be independent of interest. In the second step, we show a reduction from the single-item dynamic pricing problem to our online Bayesian recommendation problem with binary state. Thus, the $\Omega(\log \log T)$ regret lower bound known in dynamic pricing problem [Kleinberg and Leighton, 2003] can be extended to our problem.

Below we provide detailed discussion and related lemmas for the above mentioned two steps. In the end of this subsection, we combine all pieces together to conclude the proof of Theorem 3.2.

Step 1: Binary signals suffice. Our first step is to show that every online policy can be transformed into a randomized online policy with binary signal space. While this might appear obvious at first as binary signal suffices in the optimal signaling scheme in hindsight, it is not a-priori clear whether restricting binary signal is without loss in an online policy without knowing user’s utility.

**Lemma 3.3.** Given any problem instance with binary state, for any online policy $\text{ALG}$, there exists an online policy $\text{ALG}^\dagger$ which only uses signaling schemes with binary signal space and has regret $\text{REG}[\text{ALG}^\dagger] = \text{REG}[\text{ALG}]$.

We now first sketch the intuition behind Lemma 3.3. Fix an arbitrary online policy $\text{ALG}$, we construct a randomized online policy $\text{ALG}^\dagger$ with binary signal space that uses the original policy $\text{ALG}$ as a blackbox. Briefly speaking, in each round $t$, policy $\text{ALG}^\dagger$ first asks which signaling scheme $\pi_t$ is used by $\text{ALG}$ in this round. Then, $\text{ALG}^\dagger$ uses a signaling scheme $\pi_t^\dagger$ with binary signal space at random such that the distribution of user $t$’s posterior belief induced in $\pi_t^\dagger$ (over the randomness of state, signaling scheme $\pi_t$ used by $\text{ALG}$, and $\pi_t^\dagger$ itself) is the same as the one induced by $\pi_t$. Note that from user $t$’s perspective, her best response is uniquely determined by her posterior belief. Thus, the distribution of user $t$’s action is the same in both $\text{ALG}$ and $\text{ALG}^\dagger$. Finally, $\text{ALG}^\dagger$ sends user $t$’s action $a_t$ as the feedback to $\text{ALG}$, and moves to the next round.

The following lemma guarantees that for any distribution $\mu$ of posterior belief over binary state, there exists a distribution of signaling schemes with binary signal space that implements $\mu$.

**Lemma 3.4.** Let $\pi : [2] \to \Delta(\Sigma)$ be a signaling scheme that maps binary state into probability distributions over finite signal space $\Sigma$, and $\mu : \Sigma \to \Delta([2])$ be the distribution of posterior belief induced by $\pi$. There exists a positive integer $K$, and a finite set $\{\pi^{(k)}\}_{k \in [K]}$ where each $\pi^{(k)} : [2] \to \Delta([0,1])$ is a signaling scheme with binary signal space. Let $\mu^{(k)}$ be the distribution of posterior belief induced by $\pi^{(k)}$ for each $k \in [K]$. Then, there exists a distribution $F$ over $[K]$ such that for every possible posterior belief realization $x \in \text{supp}(\mu)$, $\Pr[\mu = x] = \mathbb{E}_{k \sim F}[\Pr[\mu^{(k)} = x]]$.

The proof of Lemma 3.4 relies on Lemma 3.5 and Lemma 3.6 as follows.

---

17 Namely, $\text{ALG}^\dagger$ randomly picks a signaling scheme $\pi_t^\dagger$ and commits to it in round $t$. 
Lemma 3.5 (Kamenica and Gentzkow, 2011). Let $\lambda \in \Delta([2])$ be a prior distribution over binary state space $[2]$. A distribution of posterior belief $\mu \in \Delta(\Delta([2]))$ is implementable (i.e., can be induced by some signaling scheme) if and only if $\Pr_{x \sim \mu, \theta \sim \lambda}[\theta = 1] = \lambda(1)$.

Lemma 3.6. Let $X$ be a random variable with discrete support $\text{supp}(X)$. There exists a positive integer $K$, a finite set of $K$ random variables $\{X_k\}_{k \in [K]}$, and convex combination coefficients $f \in [0,1]^K$ with $\sum_{k \in [K]} f_k = 1$ such that:

1. **Bayesian-plausibility**: for each $k \in [K]$, $\mathbb{E}[X_k] = \mathbb{E}[X]$;
2. **Binary-support**: for each $k \in [K]$, the size of $X_k$’s support is at most 2, i.e., $|\text{supp}(X_k)| \leq 2$;
3. **Consistency**: for each $x \in \text{supp}(X)$, $Pr[X = x] = \sum_{k \in [K]} f_k \cdot Pr[X_k = x]$.

Proof. For notation simplicity, we first introduce the following notations. We denote $\mathbb{E}[X]$ as $\lambda$, and $\text{supp}(X)$ as $\mathcal{S}$. We partition $\mathcal{S}$ into $\mathcal{S}_+ \equiv \{x_+^{(1)}, \ldots, x_+^{(n_1)}\}$ and $\mathcal{S}_- \equiv \{x_-^{(1)}, \ldots, x_-^{(n_2)}\}$ where $\forall i \in [n_1]$, $x_+^{(i)} \geq \lambda$; and $\forall j \in [n_2]$, $x_-^{(j)} < \lambda$. We first show the statement holds if $\lambda \not\in \mathcal{S}$. A similar argument holds for the other case where $\lambda \in \mathcal{S}$.

Now suppose $\lambda \not\in \mathcal{S}$. Let $q_+^{(i)}$ denote $\Pr\left[X = x_+^{(i)}\right]$ and $q_-^{(j)}$ denote $\Pr\left[X = x_-^{(j)}\right]$. Consider the following linear system with variable $\{f_{ij}\}_{i \in [n_1], j \in [n_2]}$:

$$
\begin{align*}
\sum_{j \in [n_2]} \frac{x_+^{(i)} - x_-^{(j)}}{x_+^{(i)} - x_-^{(j)}} f_{ij} &= q_+^{(i)} & \forall i \in [n_1] \\
\sum_{i \in [n_1]} \frac{x_+^{(i)} - \lambda}{x_+^{(i)} - x_-^{(j)}} f_{ij} &= q_-^{(j)} & \forall j \in [n_2] \\
f_{ij} &\geq 0 & \forall i \in [n_1], j \in [n_2]
\end{align*}
$$

Below we first show how to construct random variable $\{X_k\}_{k \in [K]}$ required in lemma statement with any feasible solution in linear system (1) as the convex combination coefficients. Then we show the existence of the feasible solution in linear system (1).

Fix a feasible solution $\{f_{ij}\}_{i \in [n_1], j \in [n_2]}$ in linear system (1). Let $K = n_1 \cdot n_2$. Consider the set of random variables $\{X_{ij}\}_{i \in [n_1], j \in [n_2]}$ as follows,

$$
X_{ij} = \begin{cases} 
  x_+^{(i)} & \text{with probability } \frac{x_+^{(i)} - \lambda}{x_+^{(i)} - x_-^{(j)}} \\
  x_-^{(j)} & \text{otherwise}
\end{cases}
$$

Note that $\{f_{ij}\}$ is valid convex combination coefficient for $\{X_{ij}\}_{i \in [n_1], j \in [n_2]}$. In particular,\[
\sum_{i \in [n_1]} \sum_{j \in [n_2]} f_{ij} = \sum_{i \in [n_1]} \sum_{j \in [n_2]} f_{ij} \left( \frac{x_+^{(i)} - x_-^{(j)}}{x_+^{(i)} - x_-^{(j)}} + \frac{x_+^{(i)} - \lambda}{x_+^{(i)} - x_-^{(j)}} \right) = \sum_{i \in [n_1]} q_+^{(i)} + \sum_{j \in [n_2]} q_-^{(j)} = 1\]

To see why random variables $\{X_{ij}\}_{i \in [n_1], j \in [n_2]}$ with convex combination coefficient $\{f_{ij}\}_{i \in [n_1], j \in [n_2]}$ satisfy the statement requirement, note that “Bayesian-plausibility” property and “Binary-support”
property are satisfied by construction. To verify “Consistency” property, consider each \( x_+^{(i)} \in S_+ \),

\[
\sum_{i \in [n_1]} \sum_{j \in [n_2]} f_{ij} \cdot \Pr[X_{ij} = x_+^{(i)}] = \sum_{j \in [n_2]} f_{ij} \cdot \Pr[X_{ij} = x_+^{(i)}] = \sum_{j \in [n_2]} f_{ij} \cdot \frac{\lambda - x_-^{(j)}}{x_+^{(i)} - x_-^{(j)}} = q_+^{(i)} = \Pr[X = x_+^{(i)}]
\]

Similarly, for each \( x_-^{(j)} \in S_- \),

\[
\sum_{i \in [n_1]} \sum_{j \in [n_2]} f_{ij} \cdot \Pr[X_{ij} = x_-^{(j)}] = \sum_{i \in [n_1]} f_{ij} \cdot \Pr[X_{ij} = x_-^{(j)}] = \sum_{i \in [n_1]} f_{ij} \cdot \frac{x_+^{(i)} - \lambda}{x_+^{(i)} - x_-^{(j)}} = q_-^{(j)} = \Pr[X = x_-^{(j)}]
\]

Hence, we conclude that random variables \( \{X_{ij}\}_{i \in [n_1], j \in [n_2]} \) with convex combination coefficient \( \{f_{ij}\}_{i \in [n_1], j \in [n_2]} \) satisfy the statement requirement.

Next, we show the existence of feasible solution \( \{f_{ij}\}_{i \in [n_1], j \in [n_2]} \) in linear system (I). Let \( \hat{f}_{ij} = \frac{f_{ij}}{x_+^{(i)} - x_-^{(j)}} \). It is equivalent to show that

\[
\begin{align*}
\sum_{j \in [n_2]} (\lambda - x_-^{(j)}) \hat{f}_{ij} &= q_+^{(i)} \quad \forall i \in [n_1] \\
\sum_{i \in [n_1]} (x_+^{(i)} - \lambda) \hat{f}_{ij} &= q_-^{(j)} \quad \forall j \in [n_2] \\
f_{ij} &\geq 0 \quad \forall i \in [n_1], j \in [n_2]
\end{align*}
\]

has a feasible solution. We show this by an induction argument.

**Induction Hypothesis.** Fix any positive integer \( n_1, n_2 \in \mathbb{N}_{\geq 1} \), and arbitrary non-negative numbers \( \{x_+^{(i)}, q_+^{(i)}\}_{i \in [n_1]}, \{x_-^{(j)}, q_-^{(j)}\}_{j \in [n_2]} \), and \( \lambda \). If \( x_+^{(i)} > \lambda \) for all \( i \in [n_1] \), \( x_-^{(j)} < \lambda \) for all \( j \in [n_2] \), and \( \sum_{i \in [n_1]} x_+^{(i)} q_+^{(i)} + \sum_{j \in [n_2]} x_-^{(j)} q_-^{(j)} = \left( \sum_{i \in [n_1]} q_+^{(i)} + \sum_{j \in [n_2]} q_-^{(j)} \right) \lambda \); then linear system (I) has a feasible solution.

**Base Case** \((n_1 = 1 \text{ or } n_2 = 1)\). Here we show the induction hypothesis holds for \( n_1 = 1 \). A similar argument holds for \( n_2 = 1 \). Consider a solution \( \{\hat{f}_{1j}\}_{j \in [n_2]} \) constructed as follows,

\[
\hat{f}_{1j} = \frac{q_-^{(j)}}{x_+^{(1)} - \lambda}
\]

It is obvious that \( \{\hat{f}_{1j}\} \) is non-negative and the equality for every \( j \in [n_2] \) is satisfied. Now consider the equality for \( i = 1 \). Note that

\[
\sum_{j \in [n_2]} (\lambda - x_-^{(j)}) \hat{f}_{ij} = \sum_{j \in [n_2]} \frac{(\lambda - x_-^{(j)}) q_-^{(j)}}{x_+^{(1)} - \lambda} = \sum_{j \in [n_2]} (\lambda - x_-^{(j)}) q_-^{(j)} \frac{x_+^{(1)} - \lambda}{x_+^{(1)} - \lambda} = (x_+^{(1)} - \lambda) q_+^{(1)} = q_+^{(1)}
\]

\(\text{Here we do not assume that } \sum_{i \in [n_1]} q_+^{(i)} + \sum_{j \in [n_2]} q_-^{(j)} = 1.\)
where equality (a) uses the assumption that \( x_+^{(1)} q_+^{(1)} + \sum_{j \in [n_2]} x_-^{(j)} q_-^{(j)} = (q_+^{(1)} + \sum_{j \in [n_2]} q_-^{(j)}) \lambda \).

**Inductive Step** (\( n_1 \geq 2 \) and \( n_2 \geq 2 \)). Here we show the induction hypothesis holds for \( n_1 \geq 2 \) and \( n_2 \geq 2 \). In addition, we assume that \( x_+^{(n_1)} q_+^{(n_1)} + x_-^{(n_2)} q_-^{(n_2)} \geq (q_+^{(n_1)} + q_-^{(n_2)}) \lambda \). A similar argument holds for \( x_+^{(n_1)} q_+^{(n_1)} + x_-^{(n_2)} q_-^{(n_2)} < (q_+^{(n_1)} + q_-^{(n_2)}) \lambda \). Below we show that there exists a feasible solution where we fix

\[
\forall i \in [n_1 - 1] : \hat{f}_{i n_2} = 0 \quad \text{and} \quad \hat{f}_{n_1 n_2} = \frac{q_-^{(n_2)}}{x_+^{(n_1)}} - \lambda
\]

To see this, observe that the equality for \( j = n_2 \) is satisfied. Next, we invoke the induction hypothesis on instance \((n_1, n_2 - 1, x_+^{(1)}, q_+^{(1)}, \ldots, x_+^{(n_1 - 1)}, x_+^{(n_1)}, q_+^{(n_1)}, x_+^{(1)}, q_+^{(1)}, x_-^{(1)}, \ldots, x_-^{(n_2 - 1)}, q_-^{(n_2 - 1)}, \lambda)\) where \( \hat{q}_+^{(n_1)} = q_+^{(n_1)} - \hat{f}_{n_1 n_2} (\lambda - x_-^{(n_2)}) \). It is sufficient to show that this instance satisfies the assumption in the induction hypothesis. In particular, we can verify that

\[
\hat{q}_+^{(n_1)} = q_+^{(n_1)} - \hat{f}_{n_1 n_2} (\lambda - x_-^{(n_2)}) = q_+^{(n_1)} - \frac{q_-^{(n_2)} (\lambda - x_-^{(n_2)})}{x_+^{(n_1)} - \lambda} \geq 0
\]

since we assume \( x_+^{(n_1)} q_+^{(n_1)} + x_-^{(n_2)} q_-^{(n_2)} \geq (q_+^{(n_1)} + q_-^{(n_2)}) \lambda \); and

\[
\sum_{i \in [n_1 - 1]} x_+^{(i)} q_+^{(i)} + \sum_{j \in [n_2 - 1]} x_-^{(j)} q_-^{(j)} = \left( x_+^{(n_1)} q_+^{(n_1)} - \hat{q}_+^{(n_1)} + x_-^{(n_2)} q_-^{(n_2)} \right)
\]

\[
= \sum_{i \in [n_1]} x_+^{(i)} q_+^{(i)} + \sum_{j \in [n_2]} x_-^{(j)} q_-^{(j)} - \left( x_+^{(n_1)} q_+^{(n_1)} - \hat{q}_+^{(n_1)} + x_-^{(n_2)} q_-^{(n_2)} \right)
\]

\[
= \sum_{i \in [n_1]} x_+^{(i)} q_+^{(i)} + \sum_{j \in [n_2]} x_-^{(j)} q_-^{(j)} - \left( x_+^{(n_1)} q_+^{(n_1)} (\lambda - x_-^{(n_2)}) / x_+^{(n_1)} \lambda + x_-^{(n_2)} q_-^{(n_2)} \right)
\]

\[
(a) = \left( \sum_{i \in [n_1]} q_+^{(i)} + \sum_{j \in [n_2]} q_-^{(j)} \right) \lambda - \left( \frac{x_+^{(n_1)} q_+^{(n_1)} (\lambda - x_-^{(n_2)}) / x_+^{(n_1)} \lambda + x_-^{(n_2)} q_-^{(n_2)} \right)
\]

\[
(b) = \left( \sum_{i \in [n_1]} q_+^{(i)} + \sum_{j \in [n_2]} q_-^{(j)} \right) \lambda - \left( \frac{q_-^{(n_2)} (\lambda - x_-^{(n_2)}) / x_+^{(n_1)} \lambda + q_-^{(n_2)} \right) \lambda
\]

\[
= \left( \sum_{i \in [n_1]} q_+^{(i)} + \hat{q}_+^{(n_1)} + \sum_{j \in [n_2 - 1]} q_-^{(j)} \right) \lambda
\]

where equality (a) uses the assumption that

\[
\sum_{i \in [n_1]} x_+^{(i)} q_+^{(i)} + \sum_{j \in [n_2]} x_-^{(j)} q_-^{(j)} = \left( \sum_{i \in [n_1]} q_+^{(i)} + \sum_{j \in [n_2]} q_-^{(j)} \right) \lambda
\]

and equality (b) is by algebra.

Now we are ready to show Lemma 3.4.
Proof of Lemma 3.4. Let $\lambda \in \Delta([2])$ be the prior distribution over binary state space $[2]$, and $\theta$ be the state drawn from $\lambda$. Fix an arbitrary signaling scheme $\pi$ and let $\mu$ be the distribution of posterior belief induced by $\pi$. Let $\sigma$ be the signal issued by signaling scheme $\pi$, and set random variable $X = \Pr[\theta = 1 \mid \sigma]$. By Lemma 3.5, $E_{\sigma}[X] = \lambda(1)$.

Lemma 3.6 ensures that there exists a positive integer $K$, a finite set of $K$ random variable $\{X_k\}$, and convex combination coefficients $f$ that satisfy “Bayesian-plausibility” property, “binary-support” property, and “consistency” property. Invoking Lemma 3.5, we know that each random variable $X_k$ can be thought as a distribution of posterior belief $\mu^{(k)}$ which can be induced by some signaling scheme $\pi^{(k)}$ due to the “Bayesian-plausibility” property. The “binary-support” property ensures that $\pi^{(k)}$ has binary signal space. Let $F$ be the distribution over $[K]$ such that $\Pr_{k \sim F}[k = \ell] = f_{\ell}$. The “consistency” property guarantees that for every possible posterior belief realization $x \in \text{supp}(\mu)$, $\Pr[\mu = x] = E_{k \sim F}[^{k}(\mu^{(k)}) = x]$.

Now with Lemma 3.4, we present the proof for Lemma 3.3.

Proof of Lemma 3.3. Fix an arbitrary problem instance $I$ with binary state, and an arbitrary online policy $ALG$. Below we construct a randomized online policy $ALG^\dagger$ that only uses signaling scheme with binary signal. Then, through a coupling argument, we show that users’ actions under $ALG$ and users’ actions under $ALG^\dagger$ are the same for each sample path, which finishes the proof.

We can consider online policy $ALG : [T] \times \mathcal{H}_a \times \mathcal{H} \rightarrow \Pi$ as a mapping from the round index $t$, users’ action history $h_a := (a_1, \ldots, a_{t-1})$ in the previous $t - 1$ rounds, and the other user-irrelevant history, $h$ to the signaling scheme $\pi$ used by $ALG$ in round $t$. Here $\mathcal{H}_a$ is the set of all possible users’ action history, $\mathcal{H}$ is the set of all possible user-irrelevant history, and $\Pi$ is the set of all signaling schemes.

Now we describe the construction of $ALG^\dagger$ which uses mapping $ALG$ as a blackbox. We also need to define a coupling between the sample path under $ALG$ and the sample path under $ALG^\dagger$. We construct $ALG^\dagger$ and its coupling with $ALG$ inductively (i.e., round by round). Under the construction of $ALG^\dagger$, together with its coupling, each user $t$ forms the same posterior belief and takes the same action on both the sample path under $ALG$ and the sample path under $ALG^\dagger$.

We start with round 1. Let $h$ be the user-irrelevant history under $ALG$. Since $h$ is user-irrelevant, it can be simulated in $ALG^\dagger$. $ALG^\dagger$ first determines the signaling scheme $\pi_1 \triangleq ALG(1, \emptyset, h)$ that $ALG$ uses in round 1 given users’ action history $\emptyset$ (which is empty in the beginning of round 1), and user-irrelevant history $h$. By Lemma 3.4, there exists a distribution $F_1^\dagger$ over signaling schemes with binary signal space such that the distribution of posterior belief is the same as $\pi_1$. Then $ALG^\dagger$ randomly draws a signaling scheme $\pi_1^\dagger$ from distribution $F_1^\dagger$, and commits to it in round 1. By coupling state $\theta_1$ with state $\theta_1^\dagger$, and properly coupling signal $\sigma_1$ from $\pi_1$ with signal $\sigma_1^\dagger \sim \pi_1^\dagger$ ($\sim F_1^\dagger$), we can ensure that the realized posterior belief $\mu_1$ under $ALG$ is the same as the realized posterior belief $\mu_1^\dagger$ under $ALG^\dagger$, and thus user 1’s action $a_1$ under $ALG$ is the same as her $a_1^\dagger$ under $ALG^\dagger$.

Suppose we have constructed $ALG^\dagger$ together with its coupling for the first $t - 1$ rounds. In round $t$, let $h_a$ be the users’ action history under $ALG$, and $h_a^\dagger$ be the users’ action history under $ALG^\dagger$.
Because of the coupling in the first $t - 1$ rounds, we have $h_{a} = h_{a}^{†}$. Let $h$ be the user-irrelevant history under $\text{ALG}$. Again, since $h$ is user-irrelevant, $\text{ALG}^{†}$ can compute the distribution of $h$ that is consistent with the users' action history $h_{a}^{†} = h_{a}$, and sample $h^{†}$ from this distribution. Here we couple the user-irrelevant history $h$ under $\text{ALG}$ with the simulated $h^{†}$ in $\text{ALG}^{†}$, so that $h^{†} = h$. $\text{ALG}^{†}$ first determines the signaling scheme $\pi_{t}^{†} \triangleq \text{ALG}(t, h_{a}^{†}, h^{†})$ that $\text{ALG}$ uses in this round $t$ given users' action history $h_{a}^{†}$, and user-irrelevant history $h^{†}$. The remaining construction of distribution $F_{t}^{†}$ over signaling schemes with binary signal space, realized signaling scheme $\pi_{t}^{†}$ and their coupling (so that $a_{t}^{†} = a_{t}$) are the same as what we do in round 1. We omit them to avoid redundancy.

Given the construction of $\text{ALG}^{†}$ and its coupling described above, we conclude that users' actions are the same under $\text{ALG}^{†}$ and $\text{ALG}$, which finishes the proof. \hfill \Box

**Step 2: Reduction from dynamic pricing.** The second step in the proof of Theorem 3.2 is a reduction from the single-item dynamic pricing problem to our online Bayesian recommendation problem. The definition of single-item dynamic pricing problem is as follows.

**Definition 3.3.** In the single-item dynamic pricing problem, there is a seller with unlimited units of a single item and $T$ buyers. In each round $t \in [T]$, the seller wants to sell a new unit of the item (by setting a price $p_{t}$) to buyer $t$. Buyer $t$ has a private value $v^{*}$ that is unknown to the seller, and will buy the item (and pay $p_{t}$) if and only if $v^{*} \geq p_{t}$. The regret of a dynamic pricing mechanism $\text{ALG}$ is

\[
\text{REG}[^{\text{ALG}}] \triangleq T \cdot v^{*} - \mathbb{E}_{p_{1}, \ldots, p_{T}} \left[ \sum_{t \in [T]} p_{t} \cdot 1[p_{t} \leq v^{*}] \right]
\]

where $p_{t}$ is the price posted by $\text{ALG}$ in each round $t \in [T]$.

**Theorem 3.7** [Kleinberg and Leighton, 2003]. In single-item dynamic pricing problem, no randomized dynamic pricing mechanism can achieve an expected regret better than $\Omega(\log \log T)$.

The following lemma formally states the reduction from the single-item dynamic pricing problem to our online Bayesian recommendation problem.

**Lemma 3.8.** For every single-item dynamic pricing problem instance $I$, there exists an online Bayesian recommendation problem instance $I^{†}$ with binary state. For every online policy $\text{ALG}^{†}$ with binary signal space and regret $\text{REG}[^{\text{ALG}^{†}}]_{I^{†}}$ on online Bayesian recommendation instance $I^{†}$, there exists a dynamic pricing mechanism $\text{ALG}$ with regret $\text{REG}[^{\text{ALG}}]_{I} \leq \text{REG}[^{\text{ALG}^{†}}]_{I^{†}} + 1$ on dynamic pricing instance $I$.

**Proof.** Fix an arbitrary single-item dynamic pricing problem instance $I = (T, v^{*})$ such that there are $T$ rounds and each buyer has private value $v^{*}$. Without loss of generality, we assume that $v^{*} \leq \frac{1}{2}$ and $T \geq 2$. We first present the construction of the Bayesian recommendation problem instance $I^{†}$. Then, given any online policy $\text{ALG}^{†}$ with binary signal space for the Bayesian recommendation instance $I^{†}$, we present the construction of dynamic pricing mechanism $\text{ALG}$ for the original dynamic pricing instance $I$ with regret $\text{REG}[^{\text{ALG}}]_{I} \leq \text{REG}[^{\text{ALG}^{†}}]_{I^{†}} + 1$.

**Construction of the Bayesian recommendation instance.** Consider the following Bayesian recommendation instance $I^{†}$. There are $m^{†} = 2$ states, and $T^{†} = T$ rounds. Let $\epsilon = \frac{1}{T^{†}}$. State

\[\text{REG}[^{\text{ALG}}] \triangleq T \cdot v^{*} - \mathbb{E}_{p_{1}, \ldots, p_{T}} \left[ \sum_{t \in [T]} p_{t} \cdot 1[p_{t} \leq v^{*}] \right]\]

\[\text{REG}[^{\text{ALG}^{†}}]_{I^{†}} \leq \text{REG}[^{\text{ALG}^{†}}]_{I^{†}} + 1.\]

\[21\text{We use notation } ^{†}\text{ and } ^{‡}\text{ to denote the Bayesian recommendation instance.}\]
1 is realized with probability $\lambda^\dagger(1) = \epsilon$ and state 2 is realized with probability $\lambda^\dagger(2) = 1 - \epsilon$. The users’ utility is defined as follow,

- for state 1: $\rho^\dagger(1, a^\dagger) = 1[a^\dagger = 1]$
- for state 2: $\rho^\dagger(2, a^\dagger) = -\frac{\epsilon}{v^*} \cdot 1[a^\dagger = 1]$

By construction, $\omega^\dagger(1) = \epsilon$, $\omega^\dagger(2) = -\frac{\epsilon(1-\epsilon)}{v^*}$, and the optimal signaling in hindsight $\pi^*\dagger$ satisfies that $\pi^*(1) = 1$, $\pi^*(2) = \frac{v^*}{1-\epsilon}$, and $U(\pi^*) = v^* + \epsilon$.

**Construction of the dynamic pricing mechanism.** Fix an arbitrary online policy $\text{ALG}^\dagger$ with binary signal space for the Bayesian recommendation instance $I^\dagger$. Here we show that there exists a dynamic pricing mechanism $\text{ALG}$ with regret $\text{REG}_{I}[\text{ALG}] \leq \text{REG}_{I^\dagger}[\text{ALG}^\dagger] + 1$. Our argument contains two steps. First, we show that every online policy $\text{ALG}^\dagger$ can be converted into an online policy $\text{ALG}^\dagger$ within a subclass, which has weakly smaller regret. Then, we show how to construct a dynamic pricing mechanism $\text{ALG}$ based on online policy $\text{ALG}^\dagger$.

- **[Step I]** Notice that in the construction of the Bayesian recommendation instance $I^\dagger$, users prefer action 1 in state 1 and action 0 in state 2. Thus, in the optimum signaling scheme in hindsight $\pi^\star\dagger$, the threshold state is state 2 and state 1 is above state 2. Intuitively speaking, it is natural to consider a subclass of signaling schemes $\Pi^\dagger$ such that for every signaling scheme $\pi^\dagger \in \Pi^\dagger$, it issues signal $\sigma^\dagger = 1$ (i.e., recommends action 1) deterministically (i.e., $\pi^\dagger(1) = 1$) when the state is 1, and issues signal $\sigma^\dagger = 1$ with probability $\pi^\dagger(2)$ when the state is 2. Following this intuition, below we show that for any online policy $\text{ALG}^\dagger$ (with binary signal space), there exists an online policy $\text{ALG}^\dagger$ which only uses signaling schemes in $\Pi^\dagger$ and achieves weakly smaller regret.

Suppose in round $t$, signaling scheme $\pi^\dagger_t$ is used in $\text{ALG}^\dagger$. Recall $\mu^\dagger_{t}(\sigma^\dagger, i)$ is the posterior probability $\Pr[\theta^\dagger_t = i \mid \sigma^\dagger_t]$ under $\pi^\dagger_t$. Without loss of generality, we assume that $\mu^\dagger_{t}(1, 2) \leq \lambda^\dagger_{t}(2) \leq \mu^\dagger_{t}(0, 2)$. Since $\omega^\dagger_{t}(1) + \omega^\dagger_{t}(2) < 0$, user $t$ must take action 0 when the realized signal $\sigma^\dagger_t = 0$. Thus, the regret under $\pi^\dagger_t$ is

$$\text{REG}_{I^\dagger}[\pi^\dagger_t] = U(\pi^\star\dagger) - (\lambda^\dagger_{t}(1)\pi^\dagger_{t}(1) + \lambda^\dagger_{t}(2)\pi^\dagger_{t}(2)) \cdot 1[\text{user } t \text{ takes action } 1 \mid \sigma^\dagger_t = 1]$$

$$= U(\pi^\star\dagger) - (\lambda^\dagger_{t}(1)\pi^\dagger_{t}(1) + \lambda^\dagger_{t}(2)\pi^\dagger_{t}(2)) \cdot 1[\text{user } t \text{ takes action } 1 \mid \text{posterior belief is } \mu^\dagger_{t}(1, \cdot)]$$

Thus, in round $t$, $\text{ALG}^\dagger$ can use signaling scheme $\pi^\dagger_t$ such that $\pi^\dagger_{t}(1) = 1$ and $\pi^\dagger_{t}(2) \neq \frac{\pi^\dagger_{t}(1)}{\pi^\dagger_{t}(2)}$. Since $\mu^\dagger_{t}(1, 2) \leq \lambda^\dagger_{t}(2)$ and $\mu^\dagger_{t}(1, 2) = (\lambda^\dagger_{t}(2)\pi^\dagger_{t}(2))/(\lambda^\dagger_{t}(1)\pi^\dagger_{t}(1) + \lambda^\dagger_{t}(2)\pi^\dagger_{t}(2))$, we have $\pi^\dagger_{t}(2) \neq \frac{\pi^\dagger_{t}(1)}{\pi^\dagger_{t}(1)} \leq 1$ and thus $\pi^\dagger_t$ is well-defined. Additionally, by construction, posterior belief $\mu^\dagger_{t}(1, \cdot)$ under signal 1 in $\pi^\dagger_t$ is the same as posterior belief $\mu^\dagger_{t}(1, \cdot)$ under signal 1 in $\pi^\dagger_t$. Thus,

$$\text{REG}_{I^\dagger}[\pi^\dagger_t] = U(\pi^\star\dagger) - (\lambda^\dagger_{t}(1)\pi^\dagger_{t}(1) + \lambda^\dagger_{t}(2)\pi^\dagger_{t}(2)) \cdot 1[\text{user } t \text{ takes action } 1 \mid \sigma^\dagger_t = 1]$$

$$= U(\pi^\star\dagger) - (\lambda^\dagger_{t}(1)\pi^\dagger_{t}(1) + \lambda^\dagger_{t}(2)\pi^\dagger_{t}(2)) \cdot 1[\text{user } t \text{ takes action } 1 \mid \text{posterior belief is } \mu^\dagger_{t}(1, \cdot)]$$

$$\leq \text{REG}_{I^\dagger}[\pi^\dagger_t]$$

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Therefore, suppose $\text{ALG}^\dagger$ uses signaling scheme $\pi^\dagger_t$ in round $t$. $\text{ALG}^\dagger$ can mimic $\text{ALG}^\ddagger$ by using signaling scheme $\pi^\ddagger_t$ defined above and suffers a weakly smaller expected regret. Though posterior belief $\mu^\dagger_t(0, \cdot)$ under signal 0 in $\pi^\dagger_t$ may not equal to posterior belief $\mu^\ddagger_t(0, \cdot)$ under signal 0 in $\pi^\ddagger_t$, user $t$ must take action 0 under both $\mu^\dagger_t(0, \cdot)$ and $\mu^\ddagger_t(0, \cdot)$. Thus, $\text{ALG}^\ddagger$ has more information than $\text{ALG}^\dagger$ and can continue mimicking $\text{ALG}^\ddagger$ in the future rounds.

- **[Step II]** Here we show that given any online policy $\text{ALG}^\ddagger$ which only uses signaling scheme in $\Pi^\ddagger$ defined in [Step I], we can construct a dynamic pricing mechanism $\text{ALG}$ with regret $\text{REG}_I[\text{ALG}] \leq \text{REG}_I[\text{ALG}^\ddagger] + 1$.

Suppose in round $t$, signaling scheme $\pi^\ddagger_t$ is used in $\text{ALG}^\ddagger$. By the definition of $\text{ALG}^\ddagger$, we know that $\pi^\ddagger_t(1) = 1$. Thus, user $t$ takes action 1 if and only if the realized signal $\sigma^\ddagger = 1$ and her expected utility of taking action 1 is better than taking action 0 under her posterior belief, i.e.,

$$\omega^\dagger(1) \pi^\dagger_t(1) + \omega^\dagger(2) \pi^\dagger_t(2) \geq 0 \Rightarrow \pi^\dagger_t(2) \leq \frac{v^*}{1 - \epsilon}$$

Hence, the expected regret induced by signaling scheme $\pi^\dagger_t$ is

$$\text{REG}_I[\pi^\dagger_t] = U(\pi^\dagger_t) - (\lambda^\dagger(1) \pi^\dagger_t(1) + \lambda^\dagger(2) \pi^\dagger_t(2)) \cdot 1[u_t \text{ takes action } 1 \mid \sigma^\dagger = 1] = v^* + \epsilon - \left(\epsilon + (1 - \epsilon) \pi^\dagger_t(2)\right) \cdot 1[\pi^\dagger_t(2) \leq \frac{v^*}{1 - \epsilon}]$$

Therefore, suppose online policy $\text{ALG}^\ddagger$ uses signaling scheme $\pi^\ddagger_t$ in round $t$ for the Bayesian recommendation instance $I^\dagger$. We can construct the following dynamic pricing mechanism $\text{ALG}$ which posts price $p_t = (1 - \epsilon) \pi^\dagger_t(2)$ in round $t$ for the dynamic pricing instance $I$. The regret of posting price $p_t$ is

$$\text{REG}_I[p_t] = v^* - p_t \cdot 1[p_t \leq v^*] \leq \text{REG}_I[\pi^\dagger_t] + \epsilon$$

Since dynamic pricing mechanism $\text{ALG}$ has more information than online policy $\text{ALG}^\ddagger$, $\text{ALG}$ can simulate $\text{ALG}^\ddagger$ in the future rounds. The total regret is

$$\text{REG}_I[\text{ALG}] - \text{REG}_I[\text{ALG}^\ddagger] = \sum_{t \in [T]} \left(\text{REG}_I[p_t] - \text{REG}_I[\pi^\dagger_t]\right) \leq \epsilon \cdot T = 1$$

which concludes the proof.

Putting all pieces together, we are ready to proof Theorem 3.2.

**Proof of Theorem 3.2.** Combining Theorem 3.7 and Lemma 3.8. In the online Bayesian recommendation problem with binary state, no randomized online policy with binary signal space can achieve an expected regret better than $\Omega(\log \log T)$. Invoking Lemma 3.3 finishes the proof.

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22In particular, dynamic pricing mechanism deterministically learns whether $p_t \leq v^*$ (a.k.a., $1[\pi^\dagger_t(2) \leq \frac{v^*}{1 - \epsilon}]$), while online policy $\text{ALG}^\dagger$ only learns this information when signal 1 is realized.
Comparison with (contextual) dynamic pricing problem. In Lemma 3.8 we give a reduction from the single-item dynamic pricing problem to our online Bayesian recommendation problem with binary state. Roughly speaking, our problem with binary state can be interpreted as a dynamic pricing problem with probabilistic feedback – when price $p$ is posted, seller only learns whether buyer’s value is greater than price $p$ with probability $1/p$. Since the feedback is probabilistic, the classic dynamic pricing mechanism with $O(\log \log T)$ regret studied in [Kleinberg and Leighton (2003)] suffers significantly larger regret in our problem. In contrast, our Algorithm 2 uses exploring phase I to resolve this issue.

When the size of state space (i.e., $m$) is large, Algorithm 2 incurs an $O(m!)$ regret dependence, which may not be ideal. A natural question is whether we can improve the dependence on $m$ to $\text{poly}(m)$. To answer this question, one natural attempt is to revisit the multi-dimension generalization of the single-item dynamic pricing problem – contextual dynamic pricing problem, in which [Leme and Schneider (2018)] design a contextual dynamic pricing mechanism with $O(\text{poly}(m) \log \log T)$ regret. In the contextual dynamic pricing problem, the item has $m$ features, and buyers have private value $v^*(i)$ for each feature $i$. In each round $t \in [T]$, the nature selects a vector $\left(x_t(1), \ldots, x_t(m)\right) \in \mathbb{R}_{\geq 0}^m$, and the seller wants to sell a new unit of the item by setting a price $p_t$ to buyer $t$, who will buy the item (and pay $p_t$) if and only if $\sum_{i : v^*(i) \geq 0} x_t(i) \geq p_t$.

The contextual dynamic pricing problem shares some similarity to our problem with multiple states. Specifically, there is an unknown vector $\left\{v^*(i)\right\}$ (resp. $\left\{\omega(i)\right\}$), and the optimum in hindsight benchmarks can be formulated as similar linear programs depending on $\left\{v^*(i)\right\}$ (resp. $\left\{\omega(i)\right\}$). Nonetheless, there seems to be fundamental differences between the two problems besides the probabilistic and limited feedback feature mentioned before. In particular, in each round, the contextual dynamic pricing mechanism chooses a price $p_t$ which is a scalar, while the online Bayesian recommendation policy chooses a signaling scheme (i.e., a high-dimensional function). In [Leme and Schneider (2018)], authors obtain $O(\text{poly}(m) \log \log T)$ regret by formulating the contextual dynamic pricing as solving linear programs with a separation oracle. However, in our problem, it is unclear if such a simple separation oracle exists. In Section 4, we introduce an online policy with $O(\text{poly}(m \log T))$ regret by formulating our problem as solving linear programs with a membership oracle.

4 An Algorithm with $O(\text{poly}(m \log T))$ Regret

In this section, we provide our second result – an online policy with $O(\text{poly}(m \log T))$ regret.

**Theorem 4.1.** The expected regret of Algorithm 3 is at most $O\left(m^6 \log^{O(1)} (mT)\right)$.

Algorithm 3 uses a subroutine MembershipLP – an algorithm (e.g., [Lee et al., 2018]) to solve linear program with membership oracle access. We first formally introduce the linear program optimization with membership oracle access, and discuss its connection to our online Bayesian recommendation problem. Then we provide the formal description and the explanation of Algorithm 3 where we also present the proof of Theorem 4.1.

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[23] In particular, vector $\left(x_t(1), \ldots, x_t(m)\right)$ is served as the separating hyperplane for the oracle.

[24] Membership oracle is weaker than separation oracle. See more discussion between the two oracles in Section 4.

[25] Algorithm 3 uses MembershipLP as a blackbox. Namely, it can be replaced by other algorithms for linear program with membership oracle access.
Linear program optimization with membership oracle access. Optimizing a linear function \( f(\cdot) \) within an unknown convex set \( H \) has been studied extensively in the literature. There are two standard oracle assumptions: membership oracle and separation oracle. A membership oracle returns whether a queried point \( y \) is contained in convex set \( H \). In contrast, a separation oracle not only returns whether a queried point \( y \) is contained in convex set \( H \), but also returns a hyperplane that separates \( y \) from \( H \) if \( y \notin H \).

Recall that in our problem, the optimum signaling scheme in hindsight \( \pi^* \) is the optimal solution of linear program \( P_{\text{opt}} \). From the platform’s perspective, the only unknown component in this program is \( \{\omega(i)\} \) in the IC constraint. Nonetheless, using Procedure 1, the platform can determine the persuasiveness (i.e., whether the IC constraint is satisfied) of any direct signaling scheme. In other words, Procedure 1 works like a membership oracle for the convex set which contains all persuasive signaling schemes. Thus, finding the optimal signaling scheme \( \pi^* \) can be formulated as optimizing a linear program with membership oracle access. In particular, we leverage the algorithm introduced in [Lee et al. (2018)] with the following guarantee.

**Theorem 4.2** ([Lee et al. (2018)]). For any linear function \( f(\cdot) \), and convex set \( H \subseteq \mathbb{R}^m \), given an interior point \( x(0) \), a lower bound \( r \), an upper bound \( R \) such that \( B_2(x(0), r) \subseteq H \subseteq B_2(x(0), R) \) and given a membership oracle, there exists an algorithm MembershipLP that finds an \( \epsilon \)-approximate optimal solution for \( f(\cdot) \) in \( H \) with probability \( 1 - \delta \), using \( O(m^2 \log^{O(1)}(mR/\epsilon r)) \) queries to the oracle.

We note that when only membership oracle is given, the interior point \( x(0) \) as well as lower bound \( r \), and upper bound \( R \) such that \( B_2(x(0), r) \subseteq H \subseteq B_2(x(0), R) \) is necessary for any algorithms. Otherwise, there is an information-theoretic barrier (see [Grötschel et al. (2012)])

**Towards \( O(\text{poly}(m \log T)) \) Regret**

Before we describe our algorithm, let us highlight two major hurdles in applying the membership oracle approach to solve our Bayesian recommendation problem.

1. Though Theorem 4.2 upper bounds the total number of queries to the membership oracle (a.k.a., Procedure 1), as illustrated in our second example presented in Section 3, the regret from one execution of Procedure 1 may be superconstant.

2. Theorem 4.2 requires an interior point \( x \) as well as a lower bound radius \( r \), and an upper bound radius \( R \) such that \( B_2(x, r) \subseteq H \subseteq B_2(x, R) \), and the number of queries depends on the value of \( r \) and \( R \). However in our problem, the interior point is not given explicitly. How to find a proper interior point \( x \) with non-trivial lower bound radius \( r \) (without incurring too much regret) is not obvious in our problem.

**Overview of the algorithm.** We now sketch Algorithm 3. The high-level idea of this algorithm is to use MembershipLP as a subroutine to identify a signaling scheme \( \pi^\dagger \) whose per-round expected regret is \( 1/r \). In more detail, Algorithm 3 divides the whole \( T \) rounds into an exploring phase and an exploiting phase. In exploring phase, we use MembershipLP to identify a persuasive signaling scheme \( \pi^\dagger \), and exploiting phase uses \( \pi^\dagger \) until the rounds are exhausted. As mentioned in the above two hurdles, to use MembershipLP, we need to ensure that each query (i.e., a signaling scheme) to
the MembershipLP cannot incur too much regret, i.e., Procedure\(\mathbb{I}\) for checking the persuasiveness of a queried signaling scheme cannot be large; and we need to find a proper interior point with non-trivial lower bound radius. To achieve this, there are three subphases in the exploring phase:

Exploring phase I – Lowerbounding \(U(\pi^*)\): Similar to Algorithm \(2\) the first step of Algorithm \(3\) is to identify a lower bound and an upper bound of \(U(\pi^*)\). But different from Algorithm \(2\) here, we identify a set \(\hat{S}\) of persuasive direct signaling schemes such that for every signaling scheme \(\pi^I \in \hat{S}\), it has following two properties: (i) it has the same payoff \(U\) with other signaling schemes in set \(\hat{S}\), i.e., \(U \equiv U(\pi^I), \forall \pi^I \in \hat{S}\); and \(U\) is relatively good, i.e., \(U \geq \frac{U(\pi^*)}{m^2}\); (ii) signaling scheme \(\pi^I\) has a specific structure where it has non-zero probability for recommending action \(i\), such that \(\pi^I \in \mathcal{P}\) for every state \(i\) has non-zero probability for recommending action \(i\).

\[\text{Lemma 4.3 (informal). When exploring phase I terminates, } U \geq \frac{U(\pi^*)}{m^2} \text{ and } \hat{S} \text{ is not empty.}\]

At a high level, the property (i) implies that \(U \leq U(\pi^*) \leq m^2 U\), which can guarantee us whenever we use Procedure\(\mathbb{I}\) (as a membership oracle) to check the persuasiveness of a direct signaling scheme in the later rounds, the expected regret is at most \(O(m^2)\). The property (ii) can guarantee us to find an interior point with non-trivial lower bound radius \(r\) in the later subphase.

Exploring phase II – Excluding Degenerated States: To find the interior point for the program \(\mathcal{P}^{\text{opt}}\), however, we first note that it is possible the convex set in the program \(\mathcal{P}^{\text{opt}}\) is degenerated and thus no interior point exists. Nonetheless, those degenerated dimensions (i.e., states) must contribute little to \(U(\pi^*)\). Thus, in this exploring phase, we use the signaling schemes in \(\hat{S}\) obtained in Exploring phase I to exclude those states and obtain a set \(\hat{\Theta} \subseteq [m]\) that contains all relatively good states (i.e., the state whose \(\omega\) cannot be too negative).

\[\text{Lemma 4.4. When exploring phase II terminates,}\]

- for each state \(i \in \hat{\Theta}\): \(\omega(i) \geq -mT \cdot \max_{j \in [m]} \omega(j)\);
- for each state \(i \notin \hat{\Theta}\): \(\omega(i) < -\frac{mT}{3} \cdot \max_{j \in [m]} \omega(j)\).

With the obtained \(\hat{\Theta}\) at hand, we show that there exists a persuasive direct signaling scheme \(\hat{\pi}^{(0)}\) such that \(\frac{1}{s_{\mathcal{P}^{\text{opt}}}} \leq \hat{\pi}^{(0)}(i) \leq 1 - \frac{1}{mT}\) for every \(i \in \hat{\Theta}\), and \(\hat{\pi}^{(0)}(i) = 0\) for every \(i \notin \hat{\Theta}\). Furthermore, signaling scheme \(\hat{\pi}^{(0)}\) is an interior point\(^{27}\) of following linear program.

\[
\begin{align*}
\max_{\pi: \pi(i) = 0 \text{ \forall } i \notin \hat{\Theta}} & \sum_{i \in \hat{\Theta}} \lambda(i) \pi(i) \\
\text{s.t.} & \sum_{i \in \hat{\Theta}} \omega(i) \pi(i) \geq 0 \\
& \sum_{i \in \hat{\Theta}} \lambda(i) \pi(i) \geq \frac{1}{16} U \\
& \pi(i) \in [0, 1] \quad \forall i \in \hat{\Theta}
\end{align*}
\]

\((\mathcal{P}_{\hat{\Theta}}^{\text{opt}})\)

Because of Lemma 4.3 the optimal objective value of program \((\mathcal{P}_{\hat{\Theta}}^{\text{opt}})\) is close to \(U(\pi^*)\).

\[\text{Lemma 4.5. Let } \pi^I \text{ be the optimal solution in program } \mathcal{P}_{\hat{\Theta}}^{\text{opt}} \text{ i.e., } \pi^I = \arg\max \mathcal{P}_{\hat{\Theta}}^{\text{opt}}. \text{ Then } U(\pi^I) \geq U(\pi^*) - O\left(\frac{1}{T}\right).\]

\(^{27}\)Here we mean \(\hat{\pi}^{(0)}\) is an interior point of convex set in program \((\mathcal{P}_{\hat{\Theta}}^{\text{opt}})\) when we restrict to states in \(\hat{\Theta}\).
Exploring phase III – Executing MembershipLP with Interior Point Candidates: In this phase, we identify a direct signaling scheme $\pi^\dagger$ whose per-round expected regret is $O(\frac{1}{m})$ (i.e., $U(\pi^\dagger) \geq U(\pi^*) - O(\frac{1}{m})$) with probability $1 - \frac{1}{T}$. To do this, we solve a $\frac{1}{m}$-approximate solution in program $\mathcal{P}_{\hat{\Theta}}^{opt}$ by MembershipLP as a subroutine. However, we note that we cannot directly identify the interior point $\hat{x}^{(0)}$ to the program $\mathcal{P}_{\hat{\Theta}}^{opt}$ mentioned in exploring phase II. Instead, we introduce a specific modification for signaling schemes in $\hat{S}$ – for every signaling scheme $\pi^I \in \hat{S}$, its modification $\pi^{(0)}$ is an interior point candidate. In particular, because of Lemma 4.4, there exists a signaling scheme $\pi^I \in \hat{S}$ whose modification $\pi^{(0)}$ is indeed an interior point $\hat{x}^{(0)}$.

Lemma 4.6 (informal). There exists a signaling scheme $\pi^I \in \hat{S}$ such that its modification $\pi^{(0)}$ is an interior point of program $\mathcal{P}_{\hat{\Theta}}^{opt}$. In particular, let $r = \frac{1}{16m^2T}$, then $B_2(\pi^{(0)}, r) \subseteq H(\mathcal{P}_{\hat{\Theta}}^{opt})$.

Finally, we run MembershipLP based on every interior point candidate $\pi^{(0)}$ (and in the end, we pick the best solution as $\pi^\dagger$), where we set the interior point $x^{(0)} \leftarrow \pi^{(0)}$, lower-bound radius $r \leftarrow \frac{1}{16m^2T}$, upper-bound radius $R \leftarrow \sqrt{m}$, precision $\epsilon \leftarrow \frac{1}{T}$, and success probability $\delta \leftarrow \frac{1}{T}$.

We present formal description of Algorithm 3 below.

Algorithm 3: $O(poly(m \log T))$ Search

Input: number of rounds $T$, number of states $m$, prior distribution $\lambda$, and linear program solver MembershipLP (Lee et al., 2013) with membership oracle access.

/* exploring phase I */
1 Initialize $U \leftarrow \frac{1}{2}$
2 while $U \geq \frac{1}{mT}$ do
3 if there exists $(i^\dagger, j^\dagger) \in [m] \times [m]$ such that $\text{CheckPersu} (\pi^I) = \text{True}$ then
4 $S \leftarrow \{(i^\dagger, j^\dagger) \in [m] \times [m] : \text{CheckPersu} (\pi^I) = \text{True}\}$
5 break
6 else
7 $U \leftarrow \frac{U}{2}$
8 if $S = \emptyset$ then
9 Set $\pi^\dagger : [m] \rightarrow \{0\}$ to be the signaling scheme which reveals no information.
10 Move to exploiting phase.
/* exploring phase II */
11 Initialize $\hat{\Theta} \leftarrow \emptyset$
12 for each state pair $(i^\dagger, j^\dagger) \in S$ do
13 $\hat{\Theta} \leftarrow \hat{\Theta} \cup \{i \in [m] : i = i^\dagger \text{ or } i = j^\dagger \text{ or } \text{CheckPersu} (\pi^{II}) = \text{True}\}$
/* exploring phase III */
14 for each pair $(i^\dagger, j^\dagger) \in S$ do
15 Solve $\pi^{(i^\dagger, j^\dagger)}$ by linear program solver MembershipLP for program $\mathcal{P}_{\hat{\Theta}}^{opt}$ set the interior point $x^{(0)} \leftarrow \pi^{(0)}$, lower-bound radius $r \leftarrow \frac{1}{16m^2T}$, upper-bound radius $R \leftarrow \sqrt{m}$, precision $\epsilon \leftarrow \frac{1}{T}$, and success probability $\delta \leftarrow \frac{1}{T}$.
16 Set $\pi^\dagger$ be best $\pi^{(i^\dagger, j^\dagger)}$ (i.e., maximizing $U((i^\dagger, j^\dagger))$) for all $(i^\dagger, j^\dagger) \in S$
/* exploiting phase */
17 Use signaling scheme $\pi^\dagger$ for all remaining rounds.

28 When an incorrect interior point is given, MembershipLP terminates with a suboptimal solution. The number of queries to the oracle is the same as the one in Theorem 4.2.
In this algorithm, three specific subclasses of direct signaling schemes \( \{ \pi^I \}, \{ \pi^{II} \}, \) and \( \{ \pi^{(0)} \} \) are used, whose constructions are as follows. We note that the aforementioned signaling scheme subset \( \tilde{S} \) in the algorithm overview is not explicitly defined in Algorithm \( \mathcal{B} \). Its formal definition is \( \tilde{S} \triangleq \{ \pi^I \} \) induced by \( (i^*, j^*) \in S \).

- Given \( (i^*, j^*, U) \), we let \( \pi^I \) denote a direct signaling scheme with \( \pi^I(i^*) = 1, \pi^I(j^*) = \frac{U}{\lambda(j^*)} \) if \( j^* \neq i^* \), and \( \pi^I(i) = 0 \) for every \( i \notin \{ i^*, j^* \} \).
- Given \( (i^*, j^*, U, i) \), we let \( \pi^{II} \) denote a direct signaling scheme with \( \pi^{II}(i^*) = 1, \pi^{II}(j^*) = \frac{1}{2m^2T} \) if \( j^* \neq i^* \), \( \pi^{II}(i) = \frac{3}{2m^2T} \), and \( \pi^{II}(j) = 0 \) for every \( j \notin \{ i^*, j^*, i \} \).
- Given \( (i^*, j^*, U, \tilde{\Theta}) \), we let \( \pi^{(0)} \) denote a direct signaling scheme with \( \pi^{(0)}(i^*) = 1 \), \( \pi^{(0)}(j^*) = \frac{1}{8m^2T} \) if \( j^* \neq i^* \), \( \pi^{(0)}(i) = \frac{1}{8m^2T} \) for every \( i \in \tilde{\Theta} \setminus \{ i^*, j^* \} \), and \( \pi^{(0)}(i) = 0 \) for every \( i \notin \tilde{\Theta} \).

**Formal Proofs of Algorithm \( \mathcal{B} \)**

Here we explain each phase of Algorithm \( \mathcal{B} \) in details. By combining the regret analysis in all phases, we prove Theorem \( \mathcal{H} \) in the end of this section.

**The analysis of exploring phase I.** We use the following lemma to characterize exploring phase I.

**Lemma 4.7** (restatement of Lemma \( \mathcal{I} \)). Suppose \( U(\pi^*) \geq \frac{1}{T} \). Let \( i^* = \arg \max_{i \in [m]} \omega(i) \). When exploring phase I terminates, \( U \geq \frac{U(\pi^*)}{m^2} \) and there exists a state \( j^* \in [m] \) such that \( (i^*, j^*) \in S \).

**Proof.** Let state \( j' = \arg \max_{j \in [m]} \lambda(j)\pi^*(j) \), namely, \( j' \) is the state that contributes the most to \( U(\pi^*) \). Consider a direct signaling scheme \( \pi \) with \( \pi(i^*) = 1, \pi(j^*) = \frac{\pi(j^*)}{m-1} \) if \( j' \neq i^* \), and \( \pi(i) = 0 \) for every \( i \notin \{ i^*, j^* \} \). We claim that \( \pi \) is persuasive. To see this, note that if \( j' \neq i^* \),

\[
\sum_{i \in [m]} \omega(i)\pi(i) = \omega(i^*)\pi(i^*) + \omega(j^*)\pi(j^*) = \omega(i^*) + \frac{1}{m-1}\omega(j^*)\pi^*(j^*)
\]

where inequality (a) holds since \( \omega(i^*) \geq \omega(i)\pi^*(i) \) for all \( i \in [m] \) by definition, and inequality (b) holds since \( \pi^* \) is persuasive. A similar argument holds for \( j' = i^* \). Moreover, we know that

\[
U(\pi) \geq \lambda(j^*)\pi(j^*) \geq \frac{1}{m-1}\lambda(j^*)\pi^*(j^*) \geq \frac{1}{m-1} \frac{1}{m} \sum_{i \in [m]} \lambda(i)\pi^*(i) = \frac{1}{m(m-1)}U(\pi^*)
\]

where inequality (a) holds due to the definition of state \( j' \).
The existence of the persuasive signaling scheme \( \pi \) constructed above implies that when the if-condition (line 3 in Algorithm 3) is satisfied if \( U \leq \frac{U(\pi^*)}{m(m-1)} \). Hence, if \( U \geq \frac{1}{m} \), when exploring phase I terminates, \( U \geq \frac{U(\pi^*)}{m(m-1)} \geq \frac{U(\pi^*)}{m^2} \).

Next, we argue the second part of lemma statement – when exploring phase I terminates, there exists a state \( j^t \in [m] \) such that \( (i^t, j^t) \in S \). For each pair of states \( (i'', j'') \) such that \( \omega(i'') \geq 0 \), consider a direct signaling scheme \( \pi(i'', j'') \) with

\[
\pi(i'', j'')(i') = 1 \quad \pi(i'', j'')(j'') = \lambda(j) \left( 1[\omega(j) \geq 0] + \min \left\{ \frac{-\omega(i^t)}{\omega(j)}, 1 \right\} \right) \cdot 1[\omega(j) < 0]
\]

\[
\pi(i'', j'')(i) = 0 \quad \text{for every state } i \not\in \{i'', j''\}
\]

Namely, \( \pi(i'', j'') \) is the persuasive direct signaling scheme that maximizes \( \lambda(j'')\pi(j'') \) when action 1 is only allowed to be recommended in state \( i'' \) or \( j'' \). By definition, among all pairs of states \( (i'', j'') \), the pair that maximizes \( \lambda(j'')\pi(j'') \) must be \( i'' = i^t \), which shows the second part of the lemma statement.

Due to Lemma 4.7, when exploring phase I terminates, \( U \geq \frac{U(\pi^*)}{m^2} \). This enables us to build the regret bound of exploring phase I as follows.

**Lemma 4.8.** In Algorithm 3, the expected regret in exploring phase I is at most \( O(m^4) \).

**Proof.** Let \( K_1 = -\log(U(\pi^*)) \), and \( K_2 = -[\log(U(\pi^*)/m^2)] \). By Lemma 4.7 when exploring phase I terminates, \( U \geq \frac{U(\pi^*)}{m^2} \), and thus there are at most \( K_2 \) iterations in the while loop (line 2 in Algorithm 3). For each iteration \( k \in [K] \), \( \text{CheckPersu}(\pi^I) \) is called for every pair of states \( (i, j) \) with \( U \leq \lambda(j) \). By Lemma 2.4, the total expected regret is at most

\[
m^2 \cdot \sum_{k \in [K_2]} \sum_{i \in [m]} \frac{U(\pi^*)}{\lambda(i)\pi^I(i)} \leq m^2 \cdot \sum_{k \in [K_2]} \frac{2^{-K_1}}{2^{-k}} = m^2 \cdot \sum_{k = K_1 - K_2}^{K_1} 2^{-k} = O(m^4)
\]

where the denominator in the right-hand side of inequality (a) is due to the construction of \( \pi^I \).

The analysis of exploring phase II.

The goal of exploring phase II is to identify an interior point \( \pi^{(0)} \) for the linear program solver MembershipLP with membership oracle access. To achieve this, Algorithm 3 excludes degenerated states which contributes little to \( U(\pi^*) \), and the remaining states forms the subset \( \hat{\Theta} \). We characterize \( \hat{\Theta} \) by the following lemma.

**Lemma 4.4.** When exploring phase II terminates,

- for each state \( i \in \hat{\Theta} \): \( \omega(i) \geq -mT \cdot \max_{j \in [m]} \omega(j) \);
- for each state \( i \not\in \hat{\Theta} \): \( \omega(i) < -\frac{mT}{3} \cdot \max_{j \in [m]} \omega(j) \).

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Proof. Let \( i^\dagger = \arg \max_{i \in [m]} \omega(i) \). For each state \( i \in \tilde{\Theta} \), suppose it is added into \( \tilde{\Theta} \) due to pair of state \((i', j') \in S\). Suppose \( i' \neq j' \) (A similar argument holds for \( i' = j' \)). In this case, we know that the signaling scheme \( \pi^{\Pi} \) corresponded to \((i', j', i, \bigcup)\) is persuasive, i.e.,

\[
0 \leq \sum_{j \in [m]} \omega(j)\pi^{\Pi}(j) = (a) \omega(i') + \omega(j') \frac{U}{2\lambda(j')} + \omega(i) \frac{3}{2mT} \leq \omega(i^\dagger) + \frac{1}{2}\omega(i^\dagger) + \omega(i) \frac{3}{2mT},
\]

which implies that \( \omega(i) \geq -mT \omega(i^\dagger) \). Here equality (a) holds due to the construction of \( \pi^{\Pi} \), and inequality (b) holds since \( \omega(i') \leq \omega(i^\dagger) \), \( \omega(j') \leq \omega(i^\dagger) \), and \( \bigcup \leq \lambda(j') \).

By Lemma 4.7 there exists a state \( j^\dagger \) such that \((i^\dagger, j^\dagger) \in S\). For each state \( i \notin \tilde{\Theta} \), we know that the signaling scheme \( \pi^{\Pi} \) corresponded to \((i^\dagger, j^\dagger, i, \bigcup)\) is not persuasive, i.e.,

\[
0 > \sum_{j \in [m]} \omega(j)\pi^{\Pi}(j) = (a) \omega(i^\dagger) + \omega(j^\dagger) \frac{U}{2\lambda(j^\dagger)} + \omega(i) \frac{3}{2mT} \geq \omega(i^\dagger) - \frac{1}{2}\omega(i^\dagger) + \omega(i) \frac{3}{2mT},
\]

which implies that \( \omega(i) < -\frac{mT}{2}\omega(i^\dagger) \). Here equality (a) holds due to the construction of \( \pi^{\Pi} \), and inequality (b) holds since \( \omega(i^\dagger) + \omega(j^\dagger) \frac{U}{2\lambda(j^\dagger)} \geq 0 \). \(\square\)

The first part of Lemma 4.4 guarantees that there exists a pair of state \((i^\dagger, j^\dagger) \in \tilde{\Theta}\) such that the corresponding \( P^{(0)} \) is an interior point of program \( P_{1\tilde{\Theta}}^{\text{opt}} \) (see Lemma 4.10). The second part of Lemma 4.4 guarantees that the optimum signaling scheme \( \pi^{\dagger} \) in program \( P_{1\tilde{\Theta}}^{\text{opt}} \) is close to the optimum signaling scheme \( \pi^{*} \) in program \( P_{\tilde{\Theta}}^{\text{opt}} \) (see Lemma 4.5).

Lemma 4.5. Let \( \pi^{\dagger} \) be the optimal solution in program \( P_{1\tilde{\Theta}}^{\text{opt}} \) i.e., \( \pi^{\dagger} = \arg \max_{\pi \in P_{1\tilde{\Theta}}^{\text{opt}}} P_{1\tilde{\Theta}}^{\text{opt}} \). Then \( U(\pi^{\dagger}) \geq U(\pi^{*}) - O(\frac{1}{T}) \).

Proof. By Lemma 2.1 in the optimum signaling scheme \( \pi^{*} \), there exists a threshold state \( i^\dagger \in [m] \).

For each state \( i \) above \( i^\dagger \), \( \pi^{*}(i) = 1 \); and for each state \( i \) below \( i^\dagger \), \( \pi^{*}(i) = 0 \).

Let \( i^\dagger = \arg \max_{i \in [m]} \omega(i) \). We first show that for each state \( i \) above \( i^\dagger \), \( i \in \pi^{\dagger} \). To see this, note that \( \pi^{*} \) is persuasive, i.e.,

\[
0 = \sum_{j \in [m]} \omega(j)\pi^{*}(j) = \omega(i) + \sum_{j \in [m]\backslash\{i\}} \omega(j)\pi^{*}(j) \leq \omega(i) + (m-1)\omega(i^\dagger)
\]

which implies that \( \omega(i) \geq -(m-1)\omega(i^\dagger) \). By Lemma 4.4 we conclude that \( i \in \tilde{\Theta} \).\(^{29}\) Hence, we can now upperbound \( U(\pi^{*}) - U(\pi^{\dagger}) \) as follows,

\[
U(\pi^{*}) - U(\pi^{\dagger}) = (a) \lambda(i^\dagger)\pi^{*}(i^\dagger) \cdot 1[i^\dagger \notin \tilde{\Theta}] \leq \pi^{*}(i^\dagger) \cdot 1[i^\dagger \notin \tilde{\Theta}] \leq \frac{(m-1)\omega(i^\dagger)}{\omega(i^\dagger)} \cdot 1[i^\dagger \notin \tilde{\Theta}] \leq O\left(\frac{1}{T}\right)
\]

where inequality (a) holds since \( i \in \tilde{\Theta} \) for every \( i \) such that \( \pi^{*}(i) = 1 \); inequality (c) holds due to Lemma 4.3 and inequality (b) holds due to the persuasiveness of \( \pi^{*} \), i.e.,

\[
0 = \sum_{j \in [m]} \omega(j)\pi^{*}(j) = \omega(i^\dagger)\pi^{*}(i^\dagger) + \sum_{j \in [m]\backslash\{i^\dagger\}} \omega(j)\pi^{*}(j) \leq \omega(i^\dagger)\pi^{*}(i^\dagger) + (m-1)\omega(i^\dagger)
\]

and \( \omega(i^\dagger) < 0 \) if \( i^\dagger \notin \tilde{\Theta} \). \(\square\)

\(^{29}\)Here we assume \( T \geq 3 \).
Finally, we present the regret guarantee in exploring phase II.

**Lemma 4.9.** In Algorithm 3, the expected regret in exploring phase II is at most $O(m^5)$.

*Proof.* In exploring phase II, CheckPersu($\pi_{II}$) is called for every pair of states $(i^t, j^t) \in S$ and $i \in [m] \setminus \{i^t, j^t\}$. By Lemma 2.4, the total expected regret is at most

$$
\sum_{(i^t, j^t) \in S} \sum_{i \in [m] \setminus \{i^t, j^t\}} \frac{U(\pi)}{\sum_{j \in [m]} \lambda(j)\pi_{II}(j)} \leq m^3 \cdot \frac{U(\pi)}{\frac{1}{2} U} \leq O(m^5)
$$

where the denominator in the right-hand side of inequality (a) is due to the construction of $\pi_{II}$, and inequality (b) is due to Lemma 4.7. \qed

**The analysis of exploring phase III.** Let $H_{\hat{\Theta}^\mathrm{opt}}$ be the convex set in program $\hat{\Theta}^\mathrm{opt}$. Here we show that we can find an interior point $\pi^{(0)}$ for some pair of states $(i^t, j^t) \in S$.

**Lemma 4.10** (restatement of Lemma 4.6). There exists a pair of state $(i^t, j^t) \in S$ such that $\pi^{(0)}$ is an interior point of program $\hat{\Theta}^\mathrm{opt}$. In particular, let $r = \frac{1}{16m^2T}$, then $B_2(\pi^{(0)}, r) \subseteq H_{\hat{\Theta}^\mathrm{opt}}$.

*Proof.* Let $i^t = \arg \max_{i \in [m]} \omega(i)$. By Lemma 4.7, there exists a state $j^t$ such that $(i^t, j^t) \in S$. It is sufficient to show that the signaling scheme $\pi^{(0)}$ corresponds to $(i^t, j^t, \hat{\Theta})$ defined here satisfies the requirement. In particular, Fix an arbitrary $\pi \in B_2(\pi^{(0)}, r)$. Below, we show that every constraint in program $\hat{\Theta}^\mathrm{opt}$ is satisfied.

We first examine the feasibility constraint, i.e., $\pi(i) \in [0, 1]$ for every $i \in \hat{\Theta}$. For every state $i \neq j^t$, the feasibility constraint is satisfied obviously. For state $j^t$, note that $U \geq \frac{1}{m^2T}$ and thus $\pi^{(0)}(j^t) \geq \frac{1}{m^t}$, which guarantees the feasibility constraint.

We next examine the constraint that $\sum_{i \in \hat{\Theta}} \lambda(i)\pi(i) \geq \frac{1}{16} U$. To see this, note that

$$
\sum_{i \in \hat{\Theta}} \lambda(i)\pi(i) \geq \lambda(j^t)\pi(j^t) \geq \lambda(j^t) \left( \frac{U}{8\lambda(j^t)} - r \right) \geq \lambda(j^t) \frac{U}{16\lambda(j^t)} = \frac{1}{16} U
$$

Finally, we examine the persuasiveness constraint.

$$
\sum_{i \in [m]} \omega(i)\pi(i) \geq \frac{1}{2} \omega(i^t) + \omega(j^t)\pi(j^t) + \sum_{i \in \hat{\Theta} \setminus \{i^t, j^t\}} \omega(i)\pi(i) \\
\geq \frac{1}{4} \omega(i^t) - \lambda(j^t) \frac{1}{2} \omega(i^t)\pi(j^t) + \sum_{i \in \hat{\Theta} \setminus \{i^t, j^t\}} \left( \omega(i^t) \frac{1}{4m} - mT \cdot \omega(i^t)\pi(i) \right) \\
\geq \frac{1}{4} \omega(i^t) - \omega(i^t) \left( \frac{1}{8} + r \lambda(j^t) \frac{1}{4} \right) + \sum_{i \in \hat{\Theta} \setminus \{i^t, j^t\}} \left( \omega(i^t) \frac{1}{4m} - mT \cdot \omega(i^t) \left( \frac{1}{8m^2T} + r \right) \right) \\
\geq 0
$$

28
where inequality (a) holds since $\omega(i^\dagger) + \omega(j^\dagger) \frac{U(j^\dagger)}{\lambda(j^\dagger)} \geq 0$, and $\omega(i) \geq -mT \cdot \omega(i^\dagger)$ by Lemma 4.4 and inequality (b) holds since $r \lambda(j^\dagger) \leq \frac{1}{mT}$.

Next, we present the regret guarantee in exploring phase III.

**Lemma 4.11.** In Algorithm 3 the expected regret in exploring phase III is at most $O\left(m^6 \log^{O(1)}(mT) \right)$.

**Proof.** In exploring phase II, MembershipLP is executed for each $(i^\dagger, j^\dagger) \in S$ where $|S| \leq m^2$. Within each execution of MembershipLP, CheckPersu($\pi^{II}$) is called as the membership oracle. Note that we run Procedure II only if constraint $\sum \lambda(i) \pi(i) \geq \frac{1}{mT}$ is satisfied. Thus, by Lemma 2.3 and Lemma 4.7, the expected regret in exploring phase III is at most $O(m^2)$ times the total number of queries to the membership oracle in all executions. Invoking Theorem 4.2 finishes the proof.

The analysis of exploiting phase.

Here we present the regret guarantee in exploiting phase.

**Lemma 4.12.** In Algorithm 3 the expected regret in exploiting phase is at most $O(1)$.

**Proof.** There are three different cases. If exploring phase III terminates due to $U < \frac{1}{mT}$, by Lemma 4.7 we know that $U(\pi^*) < \frac{1}{T}$. Hence, using any signaling scheme (including $\pi^{\dagger}$) induces $O(1)$ regret.

If exploring phase III terminates with $S \neq \emptyset$, then linear program solver MembershipLP is executed. Recall that MembershipLP is a randomized algorithm with success probability $1 - \frac{1}{T}$. If it fails, the regret is at most $T$, which happens with probability $\frac{1}{T}$. If it successes, by Lemma 4.10 $U(\pi^{\dagger}) \geq U(\pi^*) - O(\frac{1}{T}) \geq U(\pi^*) - O(\frac{1}{T})$.

Combining the regret analysis in all phases, we can prove Theorem 4.1.

**Proof of Theorem 4.1** Invoking Lemma 4.8 Lemma 4.9 Lemma 4.11 and Lemma 4.12 finishes the proof.

5 Extensions

In this section, we briefly discuss two extensions: (i) platform with state-dependent utility function, and (ii) with misspecified beliefs. In both directions, the regret guarantees for both Algorithm 2 and Algorithm 3 continue to hold.

**Platform with state-dependent utility function.** Recall that our baseline model assumes that the platform’s utility function $\xi(\cdot)$ is state-independent, i.e., the platform gains one unit of profit if a user takes action 1. In this subsection, we relax this assumption and consider the platform’s utility function $\xi: [m] \times A \rightarrow \mathbb{R}$ as a mapping from both the realized state and the user’s action to the utility of the platform. Additionally, we assume that $\xi(i, 0) = 0$ and $\xi(i, 1) \in [0, 1]$ for every state $i \in [m]$. 

29
Under this variant model, it can be verified that the regret guarantees in Theorem 3.1 and Theorem 4.1 continue to hold for modified versions of Algorithm 2 and Algorithm 3 as follows.

In the modified version of Algorithm 2, variables $U, L, R$ now denote the lower bound and upper bound of $U(\pi^*) = \sum_{i \in [m]} \lambda(i) \xi(i, 1) \pi^*(i)$. Each direct signaling scheme $\pi^{(r,u)}(i, 1)$ used in the exploring phase is a signaling scheme such that there exists a threshold state $i^\dagger$ such that (a) for every state $i \neq i^\dagger$, $\pi^{(r,u)}(i) = 1$ if $r(i) > r(i^\dagger)$, and (b) $\pi^{(r,u)}(i^\dagger) = \frac{u - \sum_{i \neq i^\dagger} \lambda(i) \xi(i, 1) \pi^{(r,u)}(i)}{\lambda(i^\dagger) \xi(i^\dagger)}$. All other parts of Algorithm 2 remain the same.

The modification of Algorithm 3 is similar. Variable $U$ now denotes the lower bound of $U(\pi^*)$. In the construction of signaling scheme $\pi^I$, $\pi^\Pi$, and $\pi^{(0)}$, holding everything else the same as before, we modify $\pi^I(j^\dagger) = \frac{U}{\lambda(j^\dagger) \xi(j^\dagger, 1)}$, $\pi^\Pi(j^\dagger) = \frac{U}{2 \lambda(j^\dagger) \xi(j^\dagger, 1)}$, and $\pi^{(0)}(j^\dagger) = \frac{U}{8 \lambda(j^\dagger) \xi(j^\dagger, 1)}$. All other parts of Algorithm 3 remain the same.

**Users with misspecified beliefs.** Our algorithm and results can be directly extended to the setting where users share different prior beliefs with the platform (Alonso and Camara, 2016). In particular, we let $\lambda \in \Delta([m])$ denote the prior belief of the platform, and $\lambda^\dagger \in \Delta([m])$ denote the prior belief of users. In this setting, the optimum signaling scheme in hindsight $\pi^*$ can be solved by the following linear program,

$$
\pi^* = \arg \max_{\pi} \sum_{i \in [m]} \lambda(i) \pi(i, 1) \quad \text{s.t.} \quad \sum_{i \in [m]} (\rho(i, 1) - \rho(i, 0)) \lambda^\dagger(i) \pi(i, 1) \geq 0 \\
\pi(i, 1) + \pi(i, 0) = 1 \quad i \in [m] \\
\pi(i, 1) \geq 0, \pi(i, 0) \geq 0 \quad i \in [m]
$$

By rewriting $(\rho(i, 1) - \rho(i, 0)) \lambda^\dagger(i)$ as $\omega(i)$, we can observe that this is equivalent to the original Bayesian recommendation problem solved by Algorithm 2 and Algorithm 3 where the platform and users share the same beliefs. Moreover, it can be verified that the regret guarantees for both online policies continue to hold in this extension.

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A Omitted Proofs in Section 2

In this section, we present the omitted proofs of Lemma 2.2, Lemma 2.3, and Lemma 2.4 in Section 2.

**Lemma A.1.** A direct signaling scheme $\pi$ is persuasive if and only if $\sum_{i \in [m]} \omega(i)\pi(i) \geq 0$.

**Proof.** When action 1 is recommended, the posterior distribution is $\mu_t(1, i) = \frac{\lambda(i)\pi(i)}{\sum_{j \in [m]} \lambda(j)\pi(j)}$. Thus, user takes action 1 if and only if

$$\frac{1}{\sum_{j \in [m]} \lambda(j)\pi(j)} \sum_{i \in [m]} \rho(i, 1)\lambda(i)\pi(i) \geq \frac{1}{\sum_{j \in [m]} \lambda(j)\pi(j)} \sum_{i \in [m]} \rho(i, 0)\lambda(i)\pi(i)$$

Rearranging the terms finishes the proof. \qed

**Lemma A.2.** Given a direct signaling scheme $\pi$, Procedure 1 returns True only if $\pi$ is persuasive, and returns False only if $\pi$ is not persuasive.

**Proof.** By construction, Procedure 1 returns True if $\sigma_t = 1 = a_t$, which is exactly the definition of persuasiveness. Similarly, Procedure 1 returns False if $\sigma_t = 1 = 1 - a_t$ or $\sigma_t = 0 = 1 - a_t$. The correctness of the former case holds due to the definition of persuasiveness. To see the correctness of the latter case, note that when action 0 is recommended, user takes action 1 if and only if $\sum_{i \in [m]} \omega(i)(1 - \pi(i)) \geq 0$. Hence,

$$\sum_{i \in [m]} \omega(i)\pi(i) \leq \sum_{i \in [m]} \omega(i) < 0$$

where the last inequality holds due to Assumption 2. Invoking Lemma 2.2 finishes the proof. \qed

**Lemma A.3.** Given a direct signaling scheme $\pi$, the expected regret of Procedure 1 is at most

$$\frac{U(\pi^*)}{\sum_{i \in [m]} \lambda(i)\pi(i)} - 1[\pi \text{ is persuasive}].$$
Proof. Let $Q$ be the number of rounds used in Procedure $\Pi$. We start by upper bounding $\mathbb{E}[Q]$. Note that Procedure $\Pi$ returns if action 1 is recommended, which happens with probability $\sum_i \lambda(i)\pi(i)$ in each round. Thus, $\mathbb{E}[Q] \leq \frac{1}{\sum_i \lambda(i)\pi(i)}$, and the expected regret is at most

$$
\mathbb{E}[Q] \left( U(\pi^*) - U(\pi) \right) \leq \frac{U(\pi^*)}{\sum_i \lambda(i)\pi(i)} - \frac{U(\pi)}{\sum_i \lambda(i)\pi(i)} \leq \frac{U(\pi^*)}{\sum_i \lambda(i)\pi(i)} - \mathbb{E}[\pi \text{ is persuasive}]
$$

where the last inequality holds since $U(\pi) \geq (\sum_i \lambda(i)\pi(i)) \cdot \mathbb{1}[\pi \text{ is persuasive}]$. \qed