Some essential bi-ideals and essential fuzzy bi-ideals in a semigroup

Nattapon Panpetch, Thanathip Muangngao, Thiti Gaketem*

Abstract

In this paper, we give the concepts of essential bi-ideals and essential fuzzy bi-ideals in semigroups. In the main results, we characterized regular, left regular, intra-regular, semisimple semigroups in terms of essential fuzzy ideals and essential fuzzy bi-ideals in semigroups.

Keywords: Essential bi-ideals, minimal bi-ideals, essential minimal bi-ideals, essential fuzzy minimal bi-ideals.

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1. Introduction

The concept of fuzzy sets was proposed by Zadeh in 1965 [8]. These concepts were applied in many areas such as medical science, theoretical physics, robotics, computer science, control engineering, information science, measure theory, logic, set theory, topology etc. In 1979, Kuroki [3] defined a fuzzy semigroup and various kinds of fuzzy ideals in semigroups and characterized them.

Essential fuzzy ideals of ring studied by Medhi et al. in 1971 [4]. Later in 2013, Medhi and Saikia [5] studied concept T-fuzzy essential ideals and proved properties of T-fuzzy essential ideals.

Recently in 2020, Baupradist et al. [1] studied essential ideals and essential fuzzy ideals in semigroups. Together 0-essential ideals and 0-essential fuzzy ideals in semigroups.

In this paper, we give the concepts of essential bi-ideals and essential fuzzy bi-ideals in semigroups. In the main results, we characterized regular, left regular, intra-regular, semisimple semigroups in terms of essential fuzzy ideals and essential fuzzy bi-ideals in semigroups.

2. Preliminaries

In this section, we give some basic definitions and theorems that we need.

A non-empty subset $I$ of a semigroup $S$ is called a subsemigroup of $S$ if $I^2 \subseteq I$. A non-empty subset $I$ of a semigroup $S$ is called a left (right) ideal of $S$ if $SI \subseteq I$ ($IS \subseteq I$). An ideal $I$ of $S$ is a non-empty subset...
which is both a left ideal and a right ideal of \( S \). A subsemigroup \( I \) of a semigroup \( S \) is called a bi-ideal of \( S \) if \( I \) is both a left ideal and a right ideal of \( S \). It well-know, every ideal of a semigroup \( S \) is a bi-ideal of \( S \). For any \( a, b \in [0, 1] \), we have

\[
    a \lor b = \max\{a, b\}, \quad \text{and} \quad a \land b = \min\{a, b\}.
\]

A fuzzy set of a non-empty set \( T \) is function from \( T \) into unit closed interval \([0, 1]\) of real numbers, i.e., \( f : T \rightarrow [0, 1] \).

For any two fuzzy sets of \( f \) and \( g \) of a non-empty of \( T \), we defined the support of \( f \) instead of

\[
    \supp(f) = \{u \in T \mid f(u) \neq 0\}, \quad f \subseteq g \text{ if } f(u) \leq g(u), \quad (f \lor g)(u) = \max\{f(u), g(u)\} = f(u) \lor g(u) \quad \text{and} \quad (f \land g)(u) = \min\{f(u), g(u)\} = f(u) \land g(u)
\]

for all \( u \in T \).

For two fuzzy sets \( f \) and \( g \) in a semigroup \( S \), define the product \( f \circ g \) as follows: for all \( u \in S \),

\[
    (f \circ g)(u) = \begin{cases} \bigvee \{(y, z) \in F_u \mid (y, z) \in \supp(f)\}, & \text{if } F_u \neq \emptyset, \\ 0, & \text{if } F_u = \emptyset, \end{cases}
\]

where \( F_u := \{(y, z) \in S \times S \mid u = yz\} \).

A fuzzy subsemigroup of a semigroup \( S \) if \( f(uv) \geq f(u) \land f(v) \) for all \( u, v \in S \). A fuzzy left (right) ideal of a semigroup \( S \) if \( f(uv) \geq f(v) \) \( f(uv) \geq f(u) \) for all \( u, v \in S \). A fuzzy bi-ideal of a semigroup \( S \) if \( f \) is a fuzzy subsemigroup of \( S \) and \( f(uvw) \geq f(u) \land f(w) \) for all \( u, v, w \in S \). It well-know, every fuzzy ideal of a semigroup \( S \) is a fuzzy bi-ideal of \( S \).

The characteristic fuzzy set \( \chi_I \) of a non-empty set is defined as follows:

\[
    \chi_I : T \rightarrow [0, 1], u \mapsto \begin{cases} 1, & \text{if } u \in I, \\ 0, & \text{if } u \not\in I. \end{cases}
\]

The following of theorems are true.

**Theorem 2.1** ([6]). Let \( S \) be a semigroup. Then \( I \) is a subsemigroup (left ideal right ideal, bi-ideal) of \( S \) if and only if characteristic function \( \chi_I \) is a fuzzy subsemigroup (left ideal right ideal, bi-ideal) of \( S \).

**Theorem 2.2** ([6]). Let \( I \) and \( J \) be subsets of a non-empty set \( S \). Then \( \chi_{I \cap J} = \chi_I \land \chi_J \) and \( \chi_I \circ \chi_J = \chi_{I \circ J} \).

**Theorem 2.3** ([6]). Let \( f \) be a nonzero fuzzy set of a semigroup \( S \). Then \( f \) is a fuzzy subsemigroup (ideal, bi-ideal) of \( S \) if and only if \( \supp(f) \) is a subsemigroup (ideal, bi-ideal) of \( S \).

Next, we will review of essential ideals and fuzzy essential ideals in a semigroup and properties of those.

**Definition 2.4.** An essential left (right) ideal \( I \) of a semigroup \( S \) if \( I \) is a left (right) ideal of \( S \) and \( I \cap J \neq \emptyset \) for every left (right) ideal \( J \) of \( S \).

**Definition 2.5** ([1]). An essential ideal \( I \) of a semigroup \( S \) if \( I \) is an ideal of \( S \) and \( I \cap J \neq \emptyset \) for every ideal \( J \) of \( S \).

**Theorem 2.6** ([1]). Let \( I \) be an essential ideal of a semigroup \( S \). If \( I \) is an ideal of \( S \) containing \( I \), then \( I \) is also an essential ideal of \( S \).

**Definition 2.8** ([1]). An essential fuzzy ideal \( f \) of a semigroup \( S \) if \( f \) is a nonzero fuzzy ideal of \( S \) and \( f \cap g \neq \emptyset \) for every nonzero fuzzy ideal \( g \) of \( S \).

**Theorem 2.9** ([1]). Let \( I \) be an ideal of a semigroup \( S \). Then \( I \) is an essential ideal of \( S \) if and only if \( \chi_I \) is an essential fuzzy ideal of \( S \).

**Theorem 2.10** ([1]). Let \( f \) be a nonzero fuzzy ideal of a semigroup \( S \). Then \( f \) is an essential fuzzy ideal of \( S \) if and only if \( \supp(f) \) is an essential ideal of \( S \).
3. Essential subsemigroups and essential fuzzy subsemigroups

In this section, we will study concepts of essential subsemigroups in a semigroup and fuzzy essential subsemigroups in a semigroup and their properties.

Definition 3.1. An essential subsemigroup I of a semigroup S if I is a subsemigroup of S and I ∩ J ≠ ∅ for every subsemigroup J of S.

Example 3.2.

(1) Let \( E \) be set of all even integers. Then \((E, +)\) and \((\mathbb{N}, +)\) are subsemigroups of \((\mathbb{Z}, +)\). Thus \((E, +) \cap (\mathbb{N}, +) \neq \emptyset\). Hence, \((E, +)\) is an essential subsemigroup of \((\mathbb{Z}, +)\).

(2) Let \( A = \{2n \mid n \in \mathbb{Z}\} \) and \( B = \{3n \mid n \in \mathbb{Z}\} \). Then \((A, \cdot)\) and \((B, \cdot)\) are subsemigroups of \((\mathbb{Z}, \cdot)\). Thus \((A, \cdot) \cap (B, \cdot) \neq \emptyset\). Hence \((A, \cdot)\) is an essential subsemigroup.

Theorem 3.3. Let I be an essential subsemigroup of a semigroup S. If \( I_1 \) is an ideal of S containing I, then \( I_1 \) is also an essential subsemigroup of S.

Proof. Suppose that \( I_1 \) is a subsemigroup of S such that \( I_1 \subseteq I \) and let \( J \) be any subsemigroup of S. Thus, \( I \cap J \neq \emptyset \). Hence, \( I_1 \cap J \neq \emptyset \). Therefore \( I_1 \) is an essential subsemigroup of S.

Theorem 3.4. Let I and J be essential subsemigroups of a semigroup S. Then \( I \cup J \) and \( I \cap J \) are essential subsemigroups of S.

Proof. Since \( I \subseteq I \cup J \) and I is an essential subsemigroup, we have \( I \cup J \) is an essential subsemigroup of S, by Theorem 3.3.

Since I and J are essential subsemigroups of S we have I and J are subsemigroups of S. Thus \( I \cap J \) is a subsemigroup of S.

Let \( K \) be a subsemigroup of S. Then \( I \cap K \neq \emptyset \). Thus there exists \( u, v \in I \cap K \). Let \( u, v \in J \). Then \( uv \in (I \cap J) \cap K \). Thus \((I \cap J) \cap K \neq \emptyset \). Hence \( I \cap J \) is an essential subsemigroup of S.

Definition 3.5. An essential fuzzy subsemigroup \( f \) of a semigroup S if \( f \) is a nonzero fuzzy subsemigroup of S and \( f \cap g \neq \emptyset \) for every nonzero fuzzy subsemigroup \( g \) of S.

Theorem 3.6. Let I be a subsemigroup of a semigroup S. Then I is an essential subsemigroup of S if and only if \( \chi_I \) is an essential fuzzy subsemigroup of S.

Proof. Suppose that I is an essential subsemigroup of S and let \( g \) be a nonzero fuzzy subsemigroup of S. Then \( \text{supp}(g) \) is subsemigroup of S. By assumption we have I is a subsemigroup of S. Thus \( I \cap \text{supp}(g) \neq \emptyset \). So there exists \( u \in I \cap \text{supp}(g) \). It implies that \((\chi_I \cap g)(u) \neq 0\). Hence, \( \chi_I \cap g \neq 0 \). Therefore, \( \chi_I \) is an essential fuzzy subsemigroup of S.

Conversely, assume that \( \chi_I \) is an essential fuzzy subsemigroup of S and let J be a subsemigroup of S. Then \( \chi_J \) is a nonzero fuzzy subsemigroup of S. Since \( \chi_J \) is an essential fuzzy subsemigroup of S we have \( \chi_I \) is a fuzzy subsemigroup of S. Thus, \( \chi_I \cap \chi_J \neq 0 \). So by Theorem 2.2, \( \chi_I \cap J \neq 0 \). Hence, \( I \cap J \neq \emptyset \). Therefore I is an essential subsemigroup of S.

Theorem 3.7. Let \( f \) be a nonzero fuzzy subsemigroup of a semigroup S. Then \( f \) is an essential fuzzy subsemigroup of S if and only if \( \text{supp}(f) \) is an essential subsemigroup of S.

Proof. Assume that \( f \) is an essential fuzzy subsemigroup of S. Then \( \text{supp}(f) \) is a subsemigroup of S. Let I be a subsemigroup of S. Then by Theorem 2.1, \( \chi_I \) is a subsemigroup of S. Since \( f \) is an essential fuzzy subsemigroup of S we have \( f \) is a fuzzy subsemigroup of S. Thus \( f \cap \chi_I \neq 0 \). So there exists \( u \in S \) such that \( (f \cap \chi_I)(u) \neq 0 \). It implies that \( f(u) \neq 0 \) and \( \chi_I \neq 0 \). Hence, \( u \in \text{supp}(f) \cap I \) so \( \text{supp}(f) \cap I \neq \emptyset \). It implies that \( \text{supp}(f) \) is an essential subsemigroup of S.
Conversely, assume that supp(f) is an essential ideal of S and let g be a nonzero fuzzy subsemigroup of S. Then supp(g) is a subsemigroup of S. Thus supp(f) ∩ supp(g) ≠ ∅. So there exists

u ∈ supp(f) ∩ supp(g).

This implies that f(u) ≠ 0 and g(u) ≠ 0 for all u ∈ S. Hence, (f ∧ g)(u) ≠ 0 for all u ∈ S. Therefore, f ∧ g ≠ 0. We conclude that f is an essential fuzzy subsemigroup of S.

Theorem 3.8. Let f be an essential fuzzy subsemigroup of a semigroup S. If f₁ is a fuzzy subsemigroup of S such that f ⊆ f₁, then f₁ is also an essential fuzzy subsemigroup of S.

Proof. Let f₁ be a fuzzy subsemigroup of S such that f ⊆ f₁ and let g be any fuzzy subsemigroup of S. Thus, f ∧ g ≠ 0. So f₁ ∧ g ≠ 0. Hence f₁ is an essential fuzzy subsemigroup of S.

Theorem 3.9. Let f₁ and f₂ be essential fuzzy subsemigroups of a semigroup S. Then f₁ ∨ f₂ and f₁ ∧ f₂ are essential fuzzy subsemigroups of S.

Proof. Let f₁ and f₂ be essential fuzzy subsemigroups of S. Then by Theorem 3.8, f₁ ∨ f₂ is an essential fuzzy subsemigroup of S. Since f₁ and f₂ are essential fuzzy subsemigroups of S we have f₁ ∩ f₂ is a fuzzy subsemigroup of S. Let g be a nonzero fuzzy subsemigroup of S. Then f₁ ∧ g ≠ 0. Thus there exists u ∈ S such that f₁(u) ≠ 0 and g(u) ≠ 0. Since f₁ ≠ 0 and let v ∈ S such that f₂(v) ≠ 0. Since f₁ and f₂ are fuzzy subsemigroups of S we have f₁(uv) ≥ f₁(u) ∧ f₁(v) > 0 and f₂(uv) ≥ f₂(u) ∧ f₂(v) > 0. Thus (f₁ ∧ f₂)(uv) = f₁(uv) ∧ f₂(uv) ≠ 0. Since g is a fuzzy subsemigroup of S and g(u) ≠ 0 we have g(uv) ≠ 0 for all u, v ∈ S. Thus [(f₁ ∧ f₂) ∧ g](uv) ≠ 0. Hence [(f₁ ∧ f₂) ∧ g] ≠ 0. Therefore f₁ ∧ f₂ is an essential fuzzy subsemigroup of S.

4. Essential bi-ideals and essential fuzzy bi-ideals

In this section, we defined essential bi-ideals and essential fuzzy bi-ideal in semigroup and its integrated properties.

Definition 4.1. An essential bi-ideal I of a semigroup S if I is a bi-ideal of S and I ∩ J ≠ ∅ for every bi-ideal J of S.

Example 4.2. Let S = {Ψ, Ω, Υ, Π} be semigroup with the following Cayley table.

|       | Ψ | Ω | Υ | Π |
|-------|---|---|---|---|
| Ψ     | Ψ | Ψ | Ψ | Ψ |
| Ω     | Ψ | Ψ | Ψ | Ψ |
| Υ     | Ψ | Ψ | Ω | Ψ |
| Π     | Ψ | Ψ | Ω | Ω |

Then {Ψ}, {Ψ, Ω}, {Ψ, Ω, Υ}, {Ψ, Ω, Π}, and {Ψ, Ω, Υ, Π} are bi-ideal of S. Thus {Ψ} ∩ {Ψ, Ω} ≠ ∅ and

{Ψ, Ω, Π} ∩ {Ψ, Ω, Υ, Π} ≠ ∅.

Hence {Ψ} and {Ψ, Ω, Π} are essential bi-ideals of S.

Theorem 4.3. Let I be an essential bi-ideal of a semigroup S. If I₁ is an ideal of S containing I, then I₁ is also an essential bi-ideal of S.

Proof. Suppose that I₁ is a bi-ideal of S such that I₁ ⊆ I and let J be any bi-ideal of S. Thus, I ∩ J ≠ ∅. Hence, I₁ ∩ J ≠ ∅. Therefore I₁ is an essential bi-ideal of S.

Theorem 4.4. Let I and J be essential bi-ideals of a semigroup S. Then I ∪ J and I ∩ J are essential bi-ideals of S.
Proof. Since I and J are essential bi-ideals of a semigroup S we have I and J are essential subsemigroups of a semigroup S. Thus by Theorem 3.4, I \cup J and I \cap J are essential subsemigroups of S. Since I \subseteq I \cup J and I is an essential bi-ideal we have I \cup J is an essential bi-ideal of \( S \).

Let K be a bi-ideal of S. Then I \cap K \neq \emptyset. Thus there exists u, v and w \in I \cap K. Let u, v and w \in J. Then uvw \in (I \cap J) \cap K. Thus (I \cap J) \cap K \neq \emptyset. Hence I \cap J is an essential bi-ideal of S. \( \square \)

Definition 4.5. An essential fuzzy bi-ideal \( f \) of a semigroup \( S \) if \( f \) is a nonzero fuzzy bi-ideal of \( S \) and \( f \cap g \neq 0 \) for every nonzero fuzzy bi-ideal \( g \) of \( S \).

Theorem 4.6. Let \( I \) be a bi-ideal of a semigroup \( S \). Then \( I \) is an essential bi-ideal of \( S \) if and only if \( \chi_I \) is an essential fuzzy bi-ideal of \( S \).

Proof. Suppose that I is an essential bi-ideal of \( S \) and let \( g \) be a nonzero fuzzy bi-ideal of \( S \). Then by Theorem 3.6, supp(\( g \)) is subsemigroup of \( S \) and \( \chi_I \) is an essential fuzzy subsemigroup of \( S \). Thus there exists \( u, v, w \in I \cap \text{supp}(g) \) such that \( (f \cap I) \cap (uvw) \neq 0 \). It implies that \( \chi_I \cap g \neq 0 \). Therefore, \( \chi_I \) is an essential fuzzy bi-ideal of \( S \).

Conversely, assume that \( \chi_I \) is an essential fuzzy bi-ideal of \( S \) and let \( J \) be a bi-ideal of \( S \). Then \( \chi_J \) is an essential fuzzy subsemigroup of \( S \) and \( J \) is a subsemigroup of \( S \). Thus by Theorem 3.6, I is an essential subsemigroup of \( S \). Since \( J \) be a bi-ideal of \( S \) we have \( \chi_J \) is a nonzero fuzzy bi-ideal of \( S \). Then, \( \chi_I \cap J \neq 0 \). Thus, \( \chi_I \cap J \neq \emptyset \). Hence \( I \cap J \neq \emptyset \). Therefore \( I \) is an essential bi-ideal of \( S \). \( \square \)

Theorem 4.7. Let \( f \) be a nonzero fuzzy bi-ideal of a semigroup \( S \). Then \( f \) is an essential fuzzy bi-ideal of \( S \) if and only if \( \text{supp}(f) \) is an essential bi-ideal of \( S \).

Proof. Assume that \( f \) is an essential fuzzy bi-ideal of \( S \). Then \( f \) is an essential fuzzy subsemigroup of \( S \). Thus by Theorem 3.7, \( \text{supp}(f) \) is an essential subsemigroup of \( S \). Let \( I \) be a bi-ideal of \( S \). Then by Theorem 2.1, \( \chi_I \) is a bi-ideal of \( S \). Thus \( f \cap \chi_I \neq 0 \). Thus there exists \( u \in S \) such that \( (f \cap \chi_I)(u) \neq 0 \). It implies that \( f(u) \neq 0 \) and \( \chi_I \neq 0 \). Hence, \( u \in \text{supp}(f) \cap I \) so \( \text{supp}(f) \cap I \neq \emptyset \) it implies that \( \text{supp}(f) \) is an essential bi-ideal of \( S \).

Conversely, assume that \( \text{supp}(f) \) is an essential bi-ideal of \( S \) and let \( g \) be a nonzero fuzzy bi-ideal of \( S \). Then \( \text{supp}(f) \) is an essential bi-ideal of \( S \). Since \( g \) be a nonzero fuzzy bi-ideal of \( S \) we have \( f \) is an essential fuzzy subsemigroup of \( S \) and \( \text{supp}(g) \) is a subsemigroup of \( S \), by Theorem 3.7. This implies that \( \text{supp}(f) \cap \text{supp}(g) \neq \emptyset \). So there exists \( u \in \text{supp}(f) \cap \text{supp}(g) \), this implies that \( f(u) \neq 0 \) and \( g(u) \neq 0 \). Hence, \( (f \cap g)(u) \neq 0 \). Therefore, \( f \cap g \neq 0 \). We conclude that \( f \) is an essential fuzzy bi-ideal of \( S \). \( \square \)

Theorem 4.8. Let \( f \) be an essential fuzzy bi-ideal of a semigroup \( S \). If \( f_1 \) is a fuzzy bi-ideal of \( S \) such that \( f \subseteq f_1 \), then \( f_1 \) is also an essential fuzzy bi-ideal of \( S \).

Proof. Let \( f_1 \) be a fuzzy bi-ideal of \( S \) such that \( f \subseteq f_1 \) and let \( g \) be any fuzzy bi-ideal of \( S \). Thus \( f \cap g \neq 0 \). So \( f_1 \cap g \neq 0 \). Hence, \( f_1 \) is an essential fuzzy bi-ideal of \( S \). \( \square \)

Theorem 4.9. Let \( f_1 \) and \( f_2 \) be essential fuzzy bi-ideals of a semigroup \( S \). Then \( f_1 \lor f_2 \) and \( f_1 \land f_2 \) are essential fuzzy bi-ideals of \( S \).

Proof. Let \( f_1 \) and \( f_2 \) be essential fuzzy bi-ideal of \( S \). Then by Theorem 4.8, \( f_1 \lor f_2 \) is an essential fuzzy bi-ideal of \( S \). Since \( f_1 \) and \( f_2 \) are essential fuzzy bi-ideals of \( S \) we have \( f_1 \) and \( f_2 \) is an essential fuzzy subsemigroup of \( S \). Thus \( f_1 \land f_2 \) is an essential fuzzy subsemigroup of \( S \). Let \( g \) be a nonzero fuzzy bi-ideal of \( S \). Then \( f_1 \land g \neq 0 \). Thus there exists \( u, w \in S \) such that \( f_1(uw) \neq 0 \) and \( (g)(uw) \neq 0 \). Since \( f_2 \neq 0 \) and let \( v \in S \) such that \( f_2(v) \neq 0 \). Since \( f_1 \) and \( f_2 \) are fuzzy subsemigroups of \( S \) we have

\[
\text{f}_1(\text{u}vw) \geq \text{f}_1(\text{u}) \land \text{f}_1(\text{w}) > 0,
\]

and

\[
\text{f}_2(\text{u}vw) \geq \text{f}_2(\text{u}) \land \text{f}_2(\text{w}) > 0.
\]
Thus \((f_1 \land f_2)(uvw) = f_1(uvw) \land f_2(uvw) \neq 0\). Since \(g\) is a fuzzy subsemigroup of \(S\) and \(g(v) \neq 0\) we have \(g(uvw) \neq 0\) for all \(u, v \in S\). Thus \([(f_1 \land f_2) \land g](uvw) \neq 0\). Hence \([(f_1 \land f_2) \land g] \neq 0\). Therefore \(f_1 \land f_2\) is an essential fuzzy bi-ideal of \(S\).

The following theorem we will use the basic knowledge of ideal and bi-ideal in semigroups to prove essential bi-ideal in semigroup.

**Theorem 4.10.** Every essential ideal of semigroup \(S\) is an essential bi-ideal of \(S\).

**Proof.** The proof is obvious. 

**Theorem 4.11.** Every essential fuzzy ideal of semigroup \(S\) is an essential fuzzy bi-ideal of \(S\).

**Proof.** The proof is obvious. 

5. Characterizing some semigroups by using essential fuzzy ideals and essential fuzzy bi-ideals

In this section, we will characterize regular, left regular, intra-regular, semisimple semigroups by using essential fuzzy ideals and essential fuzzy bi-ideals in semigroups. The following lemmas will be used to prove Theorem 5.3.

**Lemma 5.1.** Let \(S\) be a semigroup. If \(f\) is an essential fuzzy right ideal and \(g\) is an essential fuzzy left ideal of \(S\) then \(f \circ g \subseteq f \land g\).

**Proof.** Assume that \(f\) and \(g\) is an essential fuzzy right ideal and an essential fuzzy left ideal of \(S\) respectively. Then \(f\) and \(g\) is a fuzzy right ideal and a fuzzy left ideal of \(S\) respectively. Let \(u \in S\). If \(F_u = \emptyset\), then \((f \circ g)(u) = 0 \leq ((f \land g)(u)) = (f \land g)(u)\). If \(F_u \neq \emptyset\), then

\[
(f \circ g)(u) = \bigvee_{(i,j) \in F_u} \{f(i) \land g(j)\} \leq \bigvee_{(i,j) \in F_u} \{(f(ij) \land g(ij))\}
\]

\[
= (f(u) \land g(u)) = (f \land g)(u).
\]

Hence, \((f \circ g)(u) \leq (f \land g)(u)\). Therefore, \(f \circ g \subseteq f \land g\). 

**Lemma 5.2 ([6]).** A semigroup \(S\) is regular if and only if \(RL = R \cap L\) for every right ideal \(R\) and left ideal \(L\) of \(S\).

The following theorem show an equivalent conditional statement for a regular semigroup.

**Theorem 5.3.** A semigroup \(S\) is regular if and only if \((f \circ g = f \land g)\) for every essential fuzzy right ideal \(f\) and essential fuzzy left ideal \(g\) of \(S\).

**Proof.** \((\Rightarrow)\): Let \(f\) and \(g\) be an essential fuzzy right ideal and an essential fuzzy left ideal of \(S\) respectively. Then \(f\) and \(g\) is a fuzzy right ideal and a fuzzy left ideal of \(S\) respectively. Then by Lemma 5.1, \(f \circ g \subseteq f \land g\). Let \(u \in S\). Then there exists \(x \in S\) such that \(u = u xu\). Thus

\[
(f \circ g)(u) = \bigvee_{(y,z) \in F_u} \{f(y) \land g(z)\} = \bigvee_{(y,z) \in F_{uxu}} \{f(y) \land g(z)\}
\]

\[
\leq f(u) \land g(u) = (f \land g)(u).
\]

Hence, \((f \land g)(u) \geq (f \circ g)(u)\), and so \((f \land g)(u) \subseteq (f \circ g)(u)\). Therefore, \((f \circ g = f \land g)\).

\((\Leftarrow)\): Let \(R\) and \(L\) be a right ideal and a left ideal of \(S\) respectively. Then by Theorem 2.1, \(x_R\) and \(x_L\) is an essential fuzzy right ideal and an essential fuzzy left ideal of \(S\) respectively. By supposition and Theorem 2.2, we have

\[
x_{RL}(u) = (x_R \circ x_L)(u) = (x_R \land x_L)(u) = x_{R \land L}(u) = 1.
\]

Thus \(u \in RL\), and so \(RL = R \cap L\). It follows that by Lemma 5.2, \(S\) is regular. 


Lemma 5.4 ([6]). A semigroup \( S \) is regular if and only if \( R_1 \cap R_2 \cap B \subseteq R_1 R_2 B \), for every ideal \( R_1, R_2 \) and every bi-ideal of \( B \).

Theorem 5.5. A semigroup \( S \) is regular if and only if \( f \wedge g \wedge h \subseteq f \circ g \circ h \), for every essential fuzzy right ideals \( f, g \) and every essential fuzzy bi-ideal \( h \) of \( S \).

Proof. \((\Rightarrow):\) Let \( f, g \) be two essential fuzzy right ideals, \( h \) be an essential fuzzy bi-ideal of \( S \). Then \( f, g \) be two fuzzy right ideals, \( h \) is a fuzzy bi-ideal of \( S \). Let \( u \in S \) Since \( S \) is regular, there exists \( x \in S \) such that \( u = xu^2 \). Thus

\[
(f \circ g \circ h)(u) = \bigvee_{(i,j) \in F_u} \{f(i) \wedge (g \circ h)(j)\} = \bigvee_{(i,j) \in F_{uxu}} \{f(i) \wedge (g \circ h)(j)\} \\
\geq (f(ux) \wedge (g \circ h)(u)) = f(ux) \wedge (\bigvee_{(p,q) \in F_u} \{g(p) \wedge h(q)\}) \\
= f(ux) \wedge (\bigvee_{(p,q) \in F_{uxu}} \{g(p) \wedge h(q)\}) \geq f(ux) \wedge (g(ux) \wedge h(u)) \\
= f(u) \wedge (g(u) \wedge h(u)) = (f \wedge g \wedge h)(u).
\]

Hence, \( f \wedge g \wedge h \subseteq f \circ g \circ h \). Therefore, \( f \wedge g \wedge h \subseteq f \circ g \circ h \).

\((\Leftarrow):\) Let \( R_1, R_2 \) be two right ideals and let \( B \) be a bi-ideal of \( S \). Then by Theorem 2.1, \( \chi_{R_1} \) and \( \chi_{R_2} \) are essential fuzzy right ideals and \( \chi_B \) is an essential fuzzy bi-ideal of \( S \). Thus \( \chi_{R_1} \) and \( \chi_{R_2} \) are fuzzy right ideals and \( \chi_B \) is a fuzzy bi-ideal of \( S \). By supposition and Lemma 2.2, we have

\[
1 = (\chi_{R_1} \cap \chi_{R_2} \cap B)(u) = (\chi_{R_1}) \wedge (\chi_{R_2}) \wedge (\chi_B)(u) \\
\subseteq (\chi_{R_1} \circ (\chi_{R_2}) \circ \chi_B)(u) = \chi_{R_1} \circ \chi_{R_2} \circ \chi_B(u).
\]

Thus, \( u \in R_1 R_2 B \) and so, \( R_1 \cap R_2 \cap B \subseteq R_1 R_2 B \). It follows that by Lemma 5.4, \( S \) is regular.

Definition 5.6 ([6]). A semigroup \( S \) called left regular if for each element \( u \in S \), there exists an element \( x \in S \) such that \( u = xu^2 \).

Lemma 5.7 ([6]). A semigroup \( S \) is left regular if and only if \( 1 \cap B \subseteq IB \), for every ideal \( I \) of \( S \) and every bi-ideal \( B \) of \( S \).

Theorem 5.8. A semigroup \( S \) is left regular if and only if \( f \wedge g \subseteq f \circ g \), for every essential fuzzy ideal \( f \) and every essential fuzzy bi-ideal \( g \) of \( S \).

Proof. \((\Rightarrow):\) Assume that \( f \) and \( g \) is an essential fuzzy ideals and an essential fuzzy bi-ideal of \( S \) respectively. Then \( f \) and \( g \) is a fuzzy ideals and a fuzzy bi-ideal of \( S \) respectively. Let \( u \in S \). Since \( S \) is left regular, there exist \( x \in S \) such that \( u = xu^2 \). Thus

\[
(f \circ g)(u) = \bigvee_{(i,j) \in F_u} \{f(i) \wedge g(j)\} = \bigvee_{(i,j) \in F_{uxu}} \{f(i) \wedge g(j)\} \\
\geq f(ux) \wedge g(u) \geq f(u) \wedge g(u) = (f \wedge g)(u).
\]

Hence, \( f \wedge g \subseteq f \circ g \). Therefore, \( f \wedge g \subseteq f \circ g \).

\((\Leftarrow):\) Let \( I \) and \( B \) be an ideal and a bi-ideal of \( S \) respectively. Then by Theorem 2.1, \( \succ I \) and \( \succ J \) is an essential fuzzy ideal and an essential fuzzy bi-ideal of \( S \) respectively. Thus \( \succ I \) and \( \succ J \) is a fuzzy ideal and a fuzzy bi-ideal of \( S \) respectively. By supposition and Lemma 2.2, we have

\[
\chi_{I \cap B}(u) = (\chi_I \wedge \chi_B)(u) \subseteq (\chi_I \circ \chi_B)(u) = \chi_{I B}(u) = 1.
\]

Thus, \( u \in IB \) and so, \( IB \subseteq I \cap B \). It follows that by Lemma 5.8, \( S \) is left regular.
The following definition and lemma will be used to prove in Theorem 5.11.

**Definition 5.9 ([6]).** A semigroup $S$ is called *intra-regular* if for each $u \in S$, there exist $a, b \in S$ such that $u = au^2b$.

**Lemma 5.10 ([6]).** A semigroup $S$ is intra-regular if and only if $L \cap R \subseteq LR$, for every left ideal $L$ and every right ideal $R$ of $S$.

**Theorem 5.11.** A semigroup $S$ is intra-regular if and only if $f \cap g \subseteq f \circ g$, for every essential left ideal $f$ and essential right ideal $g$ of $S$.

**Proof.** ($\Rightarrow$): Assume that $f$ and $g$ is an essential fuzzy left ideal and an essential right ideal of $S$ respectively. Then $f$ and $g$ is a left ideal and a right ideal of $S$ respectively. Let $u \in S$. Since $S$ is intra-regular, there exist $a, b \in S$ such that $u = au^2b$. Thus

$$
(f \circ g)(u) = \bigvee_{(l_j) \in F_u} \{f(\cdot) \cap g(j)\} = \bigvee_{(l_j) \in F_{auub}} \{f(\cdot) \cap g(j)\} \\
\geq f(au) \cap g(ub) \geq f(u) \cap g(u) = (f \cap g)(u).
$$

It implies that, $(f \cap g)(u) \subseteq (f \circ g)(u)$. Hence, $f \cap g \subseteq f \circ g$.

($\Leftarrow$): Let $R$ and $L$ be a right ideal and a left ideal of $S$ respectively. Then by Theorem 2.1, $\chi_R$ and $\chi_L$ is an essential fuzzy right ideal and an essential fuzzy left ideal of $S$ respectively. Thus $\chi_R$ and $\chi_L$ is a fuzzy right ideal and a fuzzy left ideal of $S$. By supposition and Lemma 2.2, we have

$$
\chi_{R \cap L}(u) = (\chi_R \cap \chi_L)(u) \geq \chi_{RL}(u) = (\chi_R \circ \chi_L)(u) = 1.
$$

Thus $u \in LR$, and so $L \cap R \subseteq LR$. It follows that by Lemma 5.10, $S$ is intra-regular. \qed

The following definition and lemma will be used to prove in Theorem 5.15.

**Definition 5.12 ([6]).** A semigroup $S$ is called *semisimple* if every ideal of $S$ is idempotent.

**Remark 5.13.** A semigroup $S$ is semisimple if and only if $u \in (SuS)(SuS)$ for every $u \in S$, that is there exist $w, y, z \in S$ such that $u = wuyuz$.

**Lemma 5.14 ([6]).** A semigroup $S$ is semisimple if and only if $I \cap J = IJ$, for every ideals $I$ and $J$ of $S$.

**Theorem 5.15.** A semigroup $S$ is semisimple if and only if $f \cap g = f \circ g$, for every essential fuzzy ideals $f$ and $g$ of $S$.

**Proof.** ($\Rightarrow$) Assume that $f$ and $g$ are essential fuzzy ideals of $S$. Then $f$ and $g$ are fuzzy ideals of $S$. Then by Theorem 5.1, $f \circ g \subseteq f \cap g$. Let $u \in S$. Since $S$ is semisimple, there exist $w, x, y, z \in S$ such that $u = (xuy)(wuz)$. Thus

$$
(f \circ g)(u) = \bigvee_{(l_j) \in F_u} \{f(\cdot) \cap g(j)\} = \bigvee_{(l_j) \in F_{(xuy)(wuz)}} \{f(\cdot) \cap g(j)\} \\
\geq f(xuy) \cap g(wuz) \geq f(xu) \cap g(uz) \\
\geq f(u) \cap g(u) = (f \cap g)(u).
$$

Hence, $(f \cap g)(u) \subseteq (f \circ g)(u)$, and so $f \cap g \subseteq f \circ g$. Therefore, $f \cap g = f \circ g$.

($\Leftarrow$): Let $I$ and $J$ be ideals of $S$. Then by Theorem 2.1, $\chi_I$ and $\chi_J$ are essential fuzzy ideals of $S$. Thus $\chi_I$ and $\chi_J$ are fuzzy ideals of $S$. By supposition and Lemma 2.2, we have

$$
\chi_{IJ}(u) = (\chi_I \circ \chi_J)(u) = (\chi_I \cap \chi_J)(u) = \chi_{I \cap J}(u) = 1.
$$

Thus $u \in IJ$, and so $IJ = I \cap J$. It follows that by Lemma 5.14, $S$ is semisimple. \qed
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