Entropy of holographic dark energy and the generalized second law

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Abstract
In this paper we have considered holographic dark energy and studied its cosmology and thermodynamics. We have analyzed the generalized second law (GSL) of thermodynamics in a flat universe consisting of interacting dark energy and dark matter. We performed the analysis under both thermal equilibrium and nonequilibrium conditions. If the apparent horizon is taken as the boundary of the universe, we have shown that the rate of change of the total entropy of the universe is proportional to $(1 + q)^2$, which in fact shows that the GSL is valid at the apparent horizon, irrespective of the sign of the deceleration parameter, $q$. Hence, for any form of dark energy, the apparent horizon can be considered as a perfect thermodynamic boundary of the universe. We confirmed this conclusion by using the holographic dark energy model. When the event horizon is taken as the boundary, we found that the GSL is only partially satisfied. The analysis under nonequilibrium conditions revealed that the GSL is satisfied if the temperature of the dark energy is greater than the temperature of the dark matter.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The observational evidence has proven beyond doubt that the present universe is accelerating [1, 2]. The exotic matter—the dark energy—causing the accelerated expansion of the universe is assumed to have a negative pressure. Various models were proposed in an attempt to understand the properties of dark energy. The cosmological constant is the most prominent of
these models, but it faces problems with fine tuning, and it also does not explain the coincidence of dark energy density and dark matter density. This leads to consideration of dynamic dark energy models, among which the holographic dark energy model \[3–7\], which is based on the holographic principle, is one of the most frequently discussed in the recent literature. According to the holographic principle \[8\], the vacuum energy density is bounded. The significance of the principle lies in the constraint that the total energy inside a region of size \(L\) should not exceed the mass of a black hole of the same size. From effective quantum field theory, an effective infra-red (IR) cutoff can saturate the length scale, so that the dark energy density (the vacuum energy density) can be written as

\[
\rho_A = 3c^2(M_p^2)L^{-2},
\]  

(1)

where \(c\) is a dimensionless numerical factor. The possible choices for the IR cutoff are the Hubble horizon distance, particle horizon distance, event horizon distance, and a generalized IR cutoff \[9–11\]. However, the first two options could not support the accelerated expansion of the universe, while the event horizon posed the problem of causality. Thus, a new model was introduced by Granda and Oliveros \[12\] which takes the Ricci scalar as the IR cutoff; it was later studied by many others \[13\]. This model seemed efficient in solving the coincidence problem, the causality problem, and the fine tuning problem. A modified form of this model was studied in interaction with dark matter, assuming a hidden nongravitational coupling exists between them \[14–16\]. In this paper, we will explore the thermodynamics, especially the status of the generalized second law of thermodynamics in a flat universe containing holographic dark energy interacting with the dark matter.

The thermodynamics of cosmological models attracted great interest after Bekenstein and Hawking’s discovery of black hole thermodynamics \[18–20\], which says that the entropy of a black hole is proportional to the area of its event horizon. Through the Hawking radiation, the black hole could lose its thermal energy. This loss results in the decrease of its entropy, which is not allowed by the second law of thermodynamics if the black hole is assumed as an isolated system. To elaborate further, even though the entropy of the black hole decreases, the entropy loss is offset by the increase in the entropy of the matter outside the black hole boundary; eventually the total entropy of the black hole and that of the surrounding area increases, which is defined as the generalized second law (GSL) \[18, 19\]. The black hole thermodynamics, which says that the maximum entropy of a bounded region is proportional to its area rather than its volume, was applied in the cosmological holographic model by Susskind \[8\] and ’t Hooft \[21\]. Gibbons and Hawking \[22\] studied the de Sitter model of the universe, where the apparent horizon and the cosmological event horizon coincide with each other, and they found that the GSL is satisfied. Assuming the whole universe is a thermodynamic system in which the validity of the first law of thermodynamics is undoubtedly assured, this paper attempts to validate the GSL in a universe filled with holographic dark energy interacting with dark matter. Similar types of analysis were carried out by others, for other forms of dark energy. For example, the first law of thermodynamics and GSL are found to be satisfied for the apparent horizon, but violated in the case of the event horizon for dark energy with a constant equation of state and in a universe with generalized Chaplygin gas \[17\]. The event horizon describes a global concept of space–time; hence, it is difficult to understand the thermodynamics and Hawking radiation clearly \[23\]. Thus, there are arguments in favor of using the apparent horizon as the real physical boundary \[17\]. In order to discern the real physical boundary of the universe from the point of view of thermodynamics, we consider whether the GSL is satisfied by the apparent horizon and event horizon in the realm of a holographic dark energy model.
2. Interacting dark energy model

The Friedmann equations for a flat universe with a Friedmann–Robertson–Walker (FRW) metric can be written as

\[ 3H^2 = \rho_m + \rho_{de}, \tag{2} \]

where \( H \) is the Hubble parameter, \( \rho_m \) is the energy density of the dark matter, and \( \rho_{de} \) is that of the holographic dark energy. The expression for the holographic dark energy with a modified Ricci radius as the IR cutoff is [14]

\[ \rho_{de} = 2\left(\dot{H} + (3/2)aH^2\right)/\Delta, \tag{3} \]

where \( \dot{H} \) represents the derivative of \( H \) with respect to the cosmic time, \( t \), and \( \Delta = \alpha - \beta \) where \( \alpha \) and \( \beta \) are parameters of the model. The interaction between the dark energy and dark matter is studied using the following conservation equations:

\[ \dot{\rho}_{de} + 3H(\rho_{de} + p_{de}) = -Q \tag{4} \]
\[ \dot{\rho}_m + 3Hp_m = Q, \tag{5} \]

where \( \rho_{de} \) is the pressure of the dark energy, \( Q \) is the interaction term, and the dot represents the derivative with respect to time. This type of nongravitational interaction was originally introduced by Chimento et al [24]. From the equations above, one should note that when one considers both dark matter and dark energy together, the conservation equation will be satisfied without the interaction term, \( Q \). Dark matter is assumed to be pressureless. The term \( Q \) can have mainly three forms: \( Q = 3bH(\rho_{de} + \rho_m), Q = 3bH\rho_m, \) and \( Q = 3bH\rho_{de} \). For this study, we have considered the form \( Q = 3bH(\rho_m + \rho_{de}) \).

Substituting the dark energy density given by equation (3) in the Friedmann equation (2), changing the variable cosmic time \( t \) to the variable \( x = \log t \), and differentiating the equation once more leads to the second-order differential equation

\[ \frac{d^2 h^2}{dx^2} + 3(\beta + 1) \frac{dh^2}{dx} + 9(\beta + b\Delta) h^2 = 0, \tag{6} \]

where \( h = H/H_0, H_0 \) is the current value of the Hubble parameter. The general solution for this can be obtained as

\[ h^2 = c_1 e^{2m_1x} + c_2 e^{2m_2x}, \tag{7} \]

where the constants evaluated using the initial conditions, \( h^2|_{x=0} = 1 \) and \( \frac{dh}{dx}|_{x=0} = 3\Omega_{de} - 3\alpha \), are obtained as

\[ c_1 = \frac{2(\Omega_{de} - \alpha) - m_2}{m_1 - m_2}, \quad c_2 = 1 - c_1 \tag{8} \]

and \( m_{1,2} \) are obtained as

\[ m_{1,2} = -1 - \beta \pm \sqrt{1 - 4b\alpha - 2\beta + 4b\beta + \beta^2}. \tag{9} \]

Using the Friedmann equation, the dark energy density parameter can be obtained as

\[ \Omega_{de} = c_1 e^{2m_1x} + c_2 e^{2m_2x} - \Omega_{m0} e^{-3x}. \tag{10} \]
The pressure of the dark energy and its equation of state parameter are calculated as

\[
P_{\text{de}} = -\left[ c_1 \left( 1 + \frac{m_1}{2} \right) e^{\frac{3m_1}{2}x} + c_2 \left( 1 + \frac{m_2}{2} \right) e^{\frac{3m_2}{2}x} \right]
\]

(11)

and

\[
\omega_{\text{de}} = -1 - \frac{1}{2} \left( \frac{c_1 m_1 e^{\frac{3m_1}{2}x} + c_2 m_2 e^{\frac{3m_2}{2}x} + 2\Omega_{m0} e^{-3x}}{c_1 e^{\frac{3m_1}{2}x} + c_2 e^{\frac{3m_2}{2}x} - \Omega_{m0} e^{-3x}} \right).
\]

(12)

The evolution of the equation of state parameter of the interacting holographic dark energy is plotted against redshift \( z \) for the model parameters, \((\alpha, \beta) = (1.2, 0.1)\), and is shown in figure 1. The figure shows that in the past stage of the universe, the equation of state was nearly zero. As the universe evolves, the equation of state decreases and stabilizes as \( z \rightarrow -1 \). In a dark-energy-dominated universe, \( \Omega_{\text{de0}} \sim 1 \) and \( \Omega_m \sim 0 \), the coefficients in the above equation become \( m_1 = -2, m_2 = -2\beta, c_1 = 0, \) and \( c_2 = 1 \). Then the equation of state parameter becomes \( \omega_{\text{de}} = -1 + \beta \), which shows that for positive values of \( \beta \), the equation of state \( \omega_{\text{de}} > -1 \) corresponds to quintessence-type behavior, and for \( \beta < 0 \), the equation of state \( \omega_{\text{de}} < -1 \) corresponds to phantom behavior. At this point it is not out of place to compare these data with the results on equation of state in [27], which are among the first to consider the holographic dark energy. In the work of Wang et al, the authors considered the interacting holographic dark energy with the event horizon as the IR cutoff, and showed that the equation of state increases first, and then decreases and stabilizes at a value near \(-1\) with an interaction coupling, \( b^2 = 0.08 \). In contrast, in our model, \( \omega \) uniformly decreases and stabilizes at values greater than \(-1\) for \( \beta > 0 \), and it asymptotically tends to values less than \(-1\) for \( \beta < 0 \).

The deceleration parameter, \( q \), for the dark energy considered is obtained as

\[
q = -\frac{3 \left( c_1 m_1 e^{\frac{3m_1}{2}x} + c_2 m_2 e^{\frac{3m_2}{2}x} \right)}{4 \left( c_1 e^{\frac{3m_1}{2}x} + c_2 e^{\frac{3m_2}{2}x} \right)} - 1.
\]

(13)

The nature of the deceleration parameter is studied by plotting \( q \) against the redshift \( z \), and the result is shown in figure 2. It is evident from the figure that the \( q \) parameter was positive during the initial evolution of the universe, and this fact implies deceleration in the expansion.
As the universe expanded, the \( q \) parameter entered the negative value region in the recent past, implying that the current acceleration began in the recent past. It is also seen that the \( q \) parameter saturates as \( z \to -1 \). For a dark-energy-dominated universe with \( \Omega_m \sim 0 \), the deceleration parameter becomes \( q = -1 + (3/2)\beta \). This implies that for \( \beta > 0 \), the deceleration parameter \( q > -1 \) corresponds to the quintessence phase, and for \( \beta < 0 \), \( q < -1 \) corresponds to phantom behavior. In comparison, in the case of holographic dark energy with the event horizon as the IR cutoff [27], the authors have shown that, in the early phase of the matter-dominated era, the \( q \) parameter is positive and the universe is in a deceleration phase. However, in the later evolution, the dark energy dominates over the matter; subsequently the \( q \) parameter becomes negative, which implies acceleration in the expansion.

### 3. GSL under thermal equilibrium conditions

In this section, we study the validity of the GSL of dark energy under thermal equilibrium. By thermal equilibrium, we mean that the temperatures of the dark sectors and the horizon are equal. The temperature of the horizon is shown to be inversely proportional to the horizon radius in [28]. For the apparent horizon, the horizon temperature is proportional to \( H \). It is natural to suppose that the temperature of the dark sector is proportional to the temperature of the horizon. In the simplest case, one can assume that the temperatures of the dark sectors were equal to that of the horizon [29–31] under local thermal equilibrium. In general, one can assume that systems interacting with each other for some length of time will soon come to equilibrium. Therefore, the interacting dark sectors will attain thermal equilibrium in due course of their evolution, but the equilibrium between the dark sectors and the horizon is hard to understand. One can expect that, since the dark sectors and the horizon are the integral parts of the universe, ultimately any equilibrium of the entire universe means the equality of the temperatures of the dark sectors and the horizon. The above scenario of equilibrium can also be understood in light of the Le Chatelier–Braun principle, which states that when a system is disturbed out of its equilibrium state, it reacts and tries to restore a new equilibrium condition [32–35]. At sufficiently early times, the matter temperature is higher than the dark energy temperature, and as the universe expands further, both systems attain a common
equilibrium temperature. Generally the temperature of the horizon (for example, the apparent horizon) is proportional to $H$, and the temperatures of the dark sectors were proportional to their respective equations of state. The interaction among dark sectors will soon bring them to a common temperature. Energy will flow from the dark sectors sectors to the horizon until the difference in temperature between them is equalized. One expects that this transfer of energy will subsequently lead to an equilibrium between the horizon and dark sectors, so that both have the same temperature [24, 27, 36]. For a stable equilibrium configuration, the dissipative signals will propagate with velocity less than light [37–41]. To check the stability of the equilibrium between the dark sectors and the horizon, one should actually verify the velocity of the instabilities during this accelerated expansion of the universe, which is not practical in this study. However, one could expect that during the evolution of the universe, the dark sectors might soon come to equilibrium with the horizon, such that the temperature of the dark sectors becomes equal to the horizon temperature. Otherwise, the energy flow between the dark sectors and the horizon would deform the geometry [28, 42]. The validity of the GSL implies that the sum of the change in the entropy of the dark sectors added with the entropy change of the cosmological horizon is greater than or equal to zero [7]. That is,

$$\dot{S} + \dot{S}_h \geq 0,$$

(14)

where $\dot{S}$ denotes the change in entropy of the dark sectors with respect to cosmic time, and $\dot{S}_h$ is the change in the horizon entropy with respect to cosmic time. The study is performed by considering that the dark sectors and the horizon were in thermal equilibrium, so the temperature of the dark sectors is equal to the temperature of the horizon. In the following section, we will analyze separately the GSL for the universe with the apparent horizon as boundary and with the event horizon as the boundary.

3.1. GSL validity inside the apparent horizon

Here, the validity of the GSL is that the sum of the entropy of the dark energy, dark matter, and the apparent horizon must increase with time [43, 44]. Apparent horizons do exist for all kinds of FRW universes. The apparent horizon is a causal horizon for dynamic space–time and is associated with gravitational entropy and surface gravity [45]. For a universe with an FRW metric, the apparent horizon radius can be obtained as

$$r_a = \frac{1}{\sqrt{H^2 + k/a^2}}.$$ 

(15)

For a flat FRW universe ($k = 0$), the apparent horizon becomes $r_a = 1/H$, which is the same as the radius of the Hubble horizon. The temperature on the apparent horizon can be defined as $T_a = \kappa / 2 \pi$, where $\kappa$ is the surface gravity [46]. For a flat universe, the surface gravity depends on the horizon radius, and simple considerations lead to the temperature as

$$T_a = \frac{H}{2 \pi}.$$ 

(16)

In [47, 48], the authors argued that the dark energy and dark matter inside the apparent horizon were in equilibrium with the Hawking temperature, which is equal to the temperature of the apparent horizon under local thermal equilibrium if there is no interaction between the dark sectors. Wang et al [27] argued that energy from the fluid content of the universe can be added to the horizon until their temperatures equilibrate with each other. Therefore, we assume that the temperatures of the fluids within the horizon are equal to the horizon temperature under equilibrium conditions.
The entropy of the apparent horizon is \( S = \frac{A}{4G} \) with \( 8\pi G = 1 \) [22, 28, 49], where the area \( A = 4\pi r^2 \). Thus, the entropy of the apparent horizon can be obtained as [50, 51]

\[
S_a = \frac{8\pi^2}{H^2}.
\]  

(17)

The entropy of the dark sectors can be found using Gibbs’ equation,

\[
TdS = dE + PdV,
\]

(18)

where \( V = \frac{4}{3}\pi r^3 \) is the volume occupied by the dark entities, and \( E = \frac{4}{3}\pi r^3 (\rho_{de} + \rho_m) \) and the pressure, \( P \), have contributions only from the dark energy, since dark matter is non-relativistic, and hence pressureless. Thus, using the above equations and the conservation equation of the dark sectors given in the previous section, the total entropy variation with respect to \( x = \log a \) is obtained as

\[
S' = \frac{16\pi^2}{H^2} + \frac{16\pi^2}{H^2} \left( 1 + \frac{3}{2} \left( 1 + \omega_{de} \Omega_{de} \right) \right) q.
\]

(19)

where \( \omega_{de} \) is the equation of state parameter of the dark energy, \( \Omega_{de} \) is the dark energy density parameter, and \( q \) is the deceleration parameter. The GSL is valid if \( S' > 0 \). Since \( H^2 > 0 \), the validity of the GSL requires that \( q \geq -1/[1 + (3/2)(1 + \omega_{de} \Omega_{de})] \). This condition can approximately be translated as \( H \geq 1/t(1 - 1/c) \), where \( c \sim (3/2)(1 + \omega_{de} \Omega_{de}) \) (a comparatively smaller value). In the current case where \( c \) is comparatively small, \( H \geq 1/t \). This is a reasonable boundary for \( H \) in this context and will lead to an increase in the change of entropy, and therefore to the validity of the GSL.

Equation (19) can be further simplified using the relation for the \( q \) parameter,

\[
q = -1 - \frac{H}{H^2} = \frac{1}{2} \left( 1 + 3\omega_{de} \Omega_{de} \right).
\]

(20)

Equation (19) can be rewritten as

\[
S' = \frac{16\pi^2}{H^2} (1 + q)^2 = \frac{36\pi^2}{H^2} (1 + \omega_{de} \Omega_{de})^2.
\]

(21)

This equation shows that, as far as \( H > 0 \), the GSL is satisfied for almost all kinds of dark energy, irrespective of the sign of \( q \), which is positive for decelerated expansion (radiation- or matter-dominated era) and negative for accelerated expansion of the universe. Deceleration parameter \( q > -1 \) (corresponds to \( \omega_{de} > -1 \)) in the quintessence phase and \( q < -1 \) (corresponds to \( \omega_{de} < -1 \)) in the phantom phase of expansion. Therefore, in both the quintessence phase and the phantom phase of expansion, the GSL is satisfied with the apparent horizon as the boundary of the universe. Current cosmological observations hint that \( \omega_{de} \) may be as low as \(-1.5 \) [52]. In the dark energy model, we consider that equation (13) implies that, in the dark-energy-dominated case, the universe is in the quintessence phase when \( \beta > 0 \), and when \( \beta < 0 \) the universe enters the phantom phase of expansion as \( z \to -1 \). Equation (21) shows that the GSL is satisfied for both positive and negative values of the \( \beta \) parameter. Here, our conclusion agrees with one of the early results reported by Wang et al [27]. In that work, by considering interacting holographic dark energy with the event horizon as the IR cutoff, the authors proved that the entropy of matter and fluids inside the apparent horizon plus the entropy of the apparent horizon do not decrease with time. Hence, the GSL is satisfied.

We have studied the evolution of \( S' \) with \( x \) using the equation of \( q \) derived in the last section to check these conclusions. Figures 3 and 4 show the behavior of the \( S' \) using
equation (19) for various choices of the model parameters $\alpha, \beta$, and also show that the GSL is always satisfied in confirmation with equation (21).

3.2. GSL validity inside the event horizon

In this section, we analyze the GSL with the event horizon as the boundary under thermal equilibrium conditions. The event horizon distance can be evaluated using the standard relation,

$$R_h = \frac{1}{1 + z} \int_{-1}^{-1} \frac{dz}{H}.$$  \hspace{1cm} (22)

On integration, the event horizon turns out to be

$$R_h = \frac{K_1 (1 + z)^{3n+1}}{\sqrt{c_1 (1 + z)^{3/2}}} \times \left[ g_+ 0.5, 1 + g_+ \frac{c_1}{c_2} (1 + z)^{3/2} \right]$$ \hspace{1cm} (23)

for positive $\beta$ values and

$$R_h = \frac{K_2 (1 + z)^{3n+1}}{\sqrt{c_2}} \times \left[ g_- 0.5, 1 + g_- \frac{c_1}{c_2} (1 + z)^{3/2} \right]$$ \hspace{1cm} (24)

for negative $\beta$ values. The functions $\text{F}_1[...]$ are the hypergeometric functions. The constants, $K_1, K_2, g_+$, and $g_-$, are different for different values of $\beta$, and the suffix +(-) refers to positive (or negative) values of $\beta$. The values of the various constants are found to be $g_0 = 0.185, 0.238, 0.238$ for $(\alpha, \beta) = (1.2, 0.1), (1.2, 0.3), (4/3, 0.3)$, respectively, and $K_1 = 8.484 \times 10^{17}$ for positive $\beta$ parameters. For negative values of $\beta$, the constants are found to be $g_0 = 0.345, 0.349, 0.349$ for $(\alpha, \beta) = (1.01, -0.01), (1.2, -0.1), (4/3, -0.1)$, respectively, and $K_2 = -4.227 \times 10^{17}$.

In this case, the temperature is $T = 1/4\pi R_h$ and the area of the horizon is $A = 4\pi R_h^2$. Using the Gibb’s relation, the rate of change of the entropy of dark energy plus matter is...
Substituting the temperature, \( T \), and using the relation \( \rho_R \) from equation (25), the above equation becomes

\[
T \left( S'_\text{de} + S'_\text{m} \right) = H^{-1} \left( \rho_{\text{de}} + \rho_{\text{m}} + p_{\text{de}} \right) 4\pi R_h^2 \left( \dot{R}_h - H R_h \right).
\]  

(25)



Considering that the fluid inside the horizon satisfies the dominant energy condition \( \rho + p > 0 \) [50, 51], the above equation shows that the rate of change of entropy of dark energy plus dark matter within the event horizon decreases as far as \( R_h > 0 \). Adding the rate of change of horizon entropy to the above equation, we get the rate of change of the total entropy as

\[
S' = H^{-1} \left[ 16\pi^2 R_h \left( \dot{R}_h - \frac{R_h^2}{2} \left( \rho_{\text{de}} + \rho_{\text{m}} + p_{\text{de}} \right) \right) \right].
\]  

(27)

The above equation can be modified by using \( \dot{H} = - (1/2)(\rho_{\text{de}} + \rho_{\text{m}} + p_{\text{de}}) \); we get

\[
S' = H^{-1} \left[ 16\pi^2 R_h \left( \dot{R}_h + H R_h^2 \right) \right].
\]  

(28)

For \( H > 0 \) and \( R_h > 0 \), the GSL is satisfied (i.e., \( S' \geq 0 \) if \( \dot{R}_h + H R_h^2 \geq 0 \)). This can be integrated as \( H R_h^2 \geq 1 \), avoiding the constant of integration. The exact condition for the validity of GSL becomes

\[
\dot{R}_h \geq \frac{1}{2} (\rho + p) R_h^2.
\]  

(29)

This shows that the GSL is valid only if the universe satisfies the dominant energy condition, \( \rho + p \geq 0 \), [50, 51], in the case of increasing horizon radius (i.e., in the quintessence phase).

Using the Friedmann equation and the conservation equation, a more exact relation for the validity of GSL from the condition \( \dot{R}_h + H R_h^2 \geq 0 \) can be obtained as
\[ HR_h \geq 1 + \frac{3}{2} (1 + \omega_{de}\Omega_{de}) H^2 R_h^2. \]  

(30)

In a de Sitter universe in which the apparent horizon and event horizons coincide such that \( R_B = H^{-1} \), the equation of state parameter \( \omega_{de} = -1 \) and \( \Omega_{de} = 1 \); the above condition reduces to \( HR_h \geq 1 \).

Using the relation for \( q \) in terms of equation of state and the dark energy mass parameter, the above equation can be translated into the form

\[ q \leq -1 - \frac{R_h}{H^2 R_h^2}. \]  

(31)

For de Sitter \( \dot{R}_h = 0 \), the above condition is critically satisfied with \( q = -1 \), and the GSL is satisfied. In such a universe, the equation of state \( \omega_{de} = -1 \).

For the holographic dark energy described in section 2, it was found that for positive values of \( \beta \), the equation of state \( \omega_{de} > -1 \), and for negative values of \( \beta \) the equation of state \( \omega_{de} \leq -1 \). We made a numerical analysis on the evolution of condition (30) by plotting \( HR_h - 1 - (3/2)(1 + \omega_{de}\Omega_{de}) H^2 R_h^2 \) with the redshift \( z \). The plot shows that for positive values of the \( \beta \) parameter (see figure 5), the GSL is partially satisfied at the event horizon. On the other hand, for negative values (see figure 6) the GSL is completely violated at the event horizon. This result is in agreement with the results in [27], in which the authors argued that with the event horizon as the boundary, the GSL is not satisfied for holographic dark energy with the event horizon as the IR cutoff.

4. GSL under thermal nonequilibrium condition

For further analysis, we consider the universe with the apparent horizon as the boundary. A complete thermal nonequilibrium condition means that there is no interaction between the dark sectors. Hence, they have different temperatures; that is, \( T_{de} \neq T_m \neq T_h \) where \( T_{de} \) is the temperature of the dark energy, \( T_m \) is the temperature of the dark matter, and \( T_h \) is the temperature of the horizon. There are arguments in the literature that, even under local thermal equilibrium, the temperatures of the different components may be different [28, 53, 54]. Also it was argued in [24] that, owing to the discrepancies between the numerical simulations of the noninteracting cold dark matter (CDM) halo models with observations at the galactic scale, it is reasonable to expect that the dark matter is not out of thermodynamic equilibrium. Therefore, it is also worth studying the cases under thermal nonequilibrium. Using the Gibb’s relation, the rate of change of the total entropy comprising that of entropy of the dark energy, dark matter, and horizon, can be written as

\[ S' = H^{-1} \left[ \frac{4\pi}{H^2} \left( \frac{\rho_{de} + P_{de}}{T_{de}} + \frac{\rho_m}{T_m} \right) q + \frac{8\pi}{T_h} (1 + q) \right], \]  

(32)

which can be simplified using the basic definition of \( q \) and Friedmann equations to

\[ S' = H^{-1} \left[ 8\pi (1 + q) \left( \frac{q}{T_{de}} + \frac{1}{T_h} \right) + 12\pi q \Omega_m \left( \frac{1}{T_m} - \frac{1}{T_{de}} \right) \right]. \]  

(33)

As a first case, we will consider the validity of the GSL in which the dark sectors were in equilibrium, \( T_{de} = T_m \). Then, the second term in the above equation will vanish. It is clear from the remaining equation that the validity of the GSL demands that \( T_{de}/T_h \geq -q \), provided that \( (q + 1) \geq 0 \). This implies that for an accelerating universe with \( q \geq -1 \), the dark energy
temperature is greater than the horizon temperature, \( T_{\text{de}} > T_h \). The special case, \( q = -1 \), corresponds to a de-Sitter-type universe in which \( H \propto \sqrt{\Lambda} \), where \( \Lambda \) is the cosmological constant, the change in entropy is zero, and as a result, the total entropy of the universe remains constant. For \( (1 + q) < 0 \), the GSL is satisfied provided that \( T_{\text{de}}/T_h < -q \). The condition \( (1 + q) < 0 \) implies that the universe is in the phantom phase [55], and for such phases of expansion the above equation shows that the dark energy temperature is less than the horizon temperature.

As a further case, let us now avoid the dark matter contribution to the total entropy of the universe. Equation (33) now becomes
From this relation, it can be shown that the GSL is satisfied if
\[
\omega_\text{de} > \left( \frac{1 + \omega_\text{de}}{1 + q} \right) \frac{3}{2} \frac{T_\text{de}}{T_\text{h}} + \frac{q}{1 + q}.
\]  
(35)

In an accelerated expanding universe (i.e., \( q < 0 \)), if the expansion is such that the universe will not enter the phantom behavior so that \( 1 + \omega_\text{de} > 0 \) and \( 1 + q > 0 \), then the above ratio implies that \( T_\text{de}/T_\text{h} > 0 \); that is, the dark energy temperature is greater than that of the horizon. The cases of \((1 + \omega_\text{de}) < 0\) and \((1 + q) < 0\) correspond to the phantom phase, and also the \( T_\text{de} > T_\text{h} \). Thus, in a dark-energy-dominated universe, the temperature of the dark energy must be greater than the temperature of the horizon for the GSL to be valid.

Let us now assume that the temperature of the dark energy \( T_\text{de} \propto T_\text{h} \), and let us especially assume that \( T_\text{de} = kT_\text{h} \) with \( k > 1 \). Then the entropy of the dark energy is
\[
S_\text{de} = \frac{8\pi^2}{kH^2} \Omega_\text{de} \left( 1 + \omega_\text{de} \right),
\]  
(36)

where we have used the standard relation for entropy, \( S = (\rho + P)V/T \) [56]. The evolution of \( S_\text{de} \) with \( x \) is shown in figure 7, and it is clear that the entropy of the dark energy is decreasing as the universe expands. For the total entropy of the universe, which can be obtained by adding the horizon entropy to the above equation, we get (avoiding the matter contribution)
\[
S = \frac{8\pi^2}{kH^2} \Omega_\text{de} \left( 1 + \omega_\text{de} \right) + \frac{8\pi^2}{H^2}.
\]  
(37)

The evolution of this total entropy, plotted in figure 8, shows that the total entropy is increasing as the universe expands, thus satisfying the GSL. This shows that the decrease in the entropy of the dark energy is offset by the increase in the horizon entropy. This confirms our previous analysis that if the temperature of the dark energy is greater than the horizon temperature, the total entropy of the universe will always increase which guarantees the validity of the GSL. In [57], the authors argued that the total entropy is increasing and seems to attain a maximum in the far future of the evolution of the universe.
Conclusions

In this paper, we analyzed the GSL in a flat universe with holographic dark energy and nonrelativistic dark matter. The analysis was performed under both thermal equilibrium and nonequilibrium conditions. In equilibrium conditions, the following are our main conclusions. In the case of taking the apparent horizon as the boundary of the universe, we obtained that the rate of change of total entropy is proportional to $(1 + q)^2$. Hence, the GSL is in fact valid for any kind of dark energy. We verified it with holographic dark energy with the Ricci scalar as the IR cutoff. Therefore, one can conclude that the apparent horizon is a perfect thermodynamic boundary. This conclusion is in line with the results in [27, 36], where the authors concluded that the GSL is satisfied at the apparent horizon. We have also analyzed the status of the GSL with event horizon as the boundary of the universe, and found that in the presence of holographic dark energy, the GSL is only partially satisfied for positive values of the model parameter $\beta$, but completely unsatisfied for negative values of the parameter. In the existing literature, there are studies regarding the status of the GSL at the apparent horizon. By considering viscous dark energy in a nonflat universe, it was found in [43] that the GSL is valid at the apparent horizon. In [58], the authors analyzed the validity of the GSL at the apparent horizon with a logarithmically corrected entropy relation for the horizon, and showed that the GSL is satisfied at the boundary if the model parameter $\alpha = 0$; it also holds for $\alpha < 0$ if $H < 0$. Thus, the validity of the GSL at the apparent horizon seems to be a general fact, as advocated by our analysis. There are other studies in the literature regarding the status of the GSL at the event horizon. For example in [16, 27, 36, 58], the authors argued that the GSL is only partially satisfied at the event horizon of the universe.

We have extended our analysis of the validity of the GSL with thermal nonequilibrium conditions. If there is partial nonequilibrium, such that the temperatures of the dark sectors are equal and are different from the horizon temperature, the validity of the GSL demands that the temperature of the dark sector is greater than the horizon temperature if the expansion is in the quintessence phase. On the other hand, if the universe is in the phantom phase of expansion, the dark sector temperature is less than the horizon temperature. However, in a dark-energy-dominated universe (avoiding the dark matter contribution), equivalent to a condition of full nonequilibrium where all the components have different temperatures, the temperature of the

![Figure 8. Variation of the sum of the entropy of the holographic Ricci dark energy and the apparent horizon, along with the entropy of the horizon alone, against $x$ for $k = 1.25$ under thermal nonequilibrium conditions.](image)
dark energy is greater than the horizon temperature, both in the quintessence and phantom phases of expansion. There have been attempts to study the thermodynamics of the horizon under nonequilibrium conditions. For example, in [58], the authors analyzed the conditions for the validity of the GSL under nonequilibrium conditions, but no specific conclusions or criteria were obtained for the validity of the GSL. Therefore, it seems difficult to analyze the status of GSL under nonequilibrium conditions. However, we carried out the analysis by assuming the validity of the GSL at the apparent horizon, and we showed that the GSL is valid if the dark energy temperature is greater than the horizon temperature.

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