Tensorial q-number commutator function for abelian gauge fields in quantum Einstein gravity

Ritsu Yoshida

Department of Business Administration, Asia University Junior College, Tokyo 180-8629, Japan
∗E-mail: rysd@gol.com

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A tensorial version of the gravitational Pauli–Jordan D function is introduced in the manifestly covariant canonical formalism of quantum gravity. This function is defined as a bilocal operator that relates to the 4D commutation relation between an abelian gauge field and itself. Some properties of this function and the gravitational Pauli–Jordan D function are investigated.

Subject Index B05, B39, E05

1. Introduction

In 1982, Nakanishi [1] extended the Pauli–Jordan D function to a q-number and introduced it in the manifestly covariant canonical formalism of quantum gravity. This is called the quantum-gravity D function [2], and is defined by the following q-number version of a Cauchy problem [1–3]:

\[ \partial_{\mu}^x [h(x)g^{\mu\nu}(x)\partial_{\nu}^y D(x, y)] = 0, \quad (1.1) \]

\[ D(x, y)|_0 = 0, \quad (1.2) \]

\[ \partial_0^x D(x, y)|_0 = -\frac{\delta^3}{h(x)g^{00}(x)}. \quad (1.3) \]

Here, \( g^{\mu\nu}(x) \) is the gravitational field, \( h \equiv \sqrt{-\det g_{\mu\nu}}, \) \( \delta^3 \) denotes the spatial delta function \( \Pi_{k=1}^3 \delta(x^k - y^k), \) a symbol \( |_0 \) means to set \( x^0 = y^0, \) and \( D(x, y) \) the quantum-gravity D function; we use Greek small letters for \( GL(4) \) indexes, and raise or lower them by \( g^{\mu\nu} \) or \( g_{\mu\nu}. \) Arguments of space-time functions are often omitted for simplicity.

The quantum-gravity D function \( D(x, y) \) is a bilocal operator and is not a function of \( x - y \) alone, since it is not translationally invariant [2,4]. We must strictly distinguish \( \partial^y \) from \( -\partial^x. \) Hence Kanno and Nakanishi [3] proved

\[ [D(x, y)\hat{\partial}_{\mu}^y, h(y)g^{\mu\nu}(y)]\hat{\partial}_{\nu}^y = 0, \quad (1.4) \]

\[ D(x, y)|_0 = \frac{\delta^3}{h(y)g^{00}(y)}. \quad (1.5) \]

on the basis of Eqs. (1.1)–(1.3). The operator \( D(x, y) \) does not commute with the gravitational field.

Using \( D(x, y) \), we have an integral representation for massless fields that satisfies the quantum-gravitational d’Alembert equation [1,3]. For example [2,4], the field equation for the electromagnetic…
B-field,
\[ \partial_\mu (h g^{\mu \nu} \partial_\nu B) = 0, \] (1.6)
gives us
\[ B(x) = \int d^3 z [D(x, z) \bar{\partial}_\nu \cdot h(z) g^{0 \nu}(z) B(z) - D(x, z) h(z) g^{0 \nu}(z) \partial_\nu B(z)], \] (1.7)
in the quantum gravi-electrodynamics [5]. This is a quantum-gravitational version of an integral representation in QED [2],
\[ B(x) = \int d^3 z [D(x - z) \bar{\partial}_\nu \cdot B(z) - D(x - z) \partial_\nu B(z)], \] (1.8)
where \( D(x - z) \) is the Pauli–Jordan invariant D function. Both the right-hand sides of Eqs. (1.7) and (1.8) are independent of \( z^0 \).

The 4D commutation relation between the B-field \( B(x) \) in (1.7) and the electromagnetic field \( A_\nu(y) \) is given by
\[ [B(x), A_\nu(y)] = -i \mathcal{D}(x, y) \bar{\partial}_\nu + \mathcal{C}_\nu(x, y; B). \] (1.9)
Here, \( \mathcal{C}_\nu(x, y; B) \) is a linear functional [1] of the B-field and contains commutation relations between \( A_\nu \) and \( \mathcal{D} \) or \( \partial \mathcal{D} \). Equation (1.9) is a quantum-gravitational version of the 4D commutation relation in QED [2],
\[ [B(x), A_\nu(y)] = -i D(x - y) \partial_\nu , \] (1.10)
on the correspondence between (1.7) and (1.8).

Comparing (1.9) with (1.10), we find that the 4D commutation relation between \( A_\mu(x) \) and \( A_\nu(y) \) is given by a form of
\[ [A_\mu(x), A_\nu(y)] = -i \mathcal{D}_{\mu \nu}(x, y) + \mathcal{C}_{\mu \nu}(x, y; \lambda, B). \] (1.11)
The first term on the right-hand side should be reduced to the 4D commutation relation in the free QED [2],
\[ [A_\mu(x), A_\nu(y)] = -i \eta_{\mu \nu} D(x - y) + i (1 - \alpha) \partial_\mu \partial_\nu E(x - y), \] (1.12)
where \( \eta_{\mu \nu} \) is the Minkowski metric, \( \alpha \) a gauge parameter, and \( E(x - y) \) a dipole-ghost invariant D-function. The second term, \( \mathcal{C}_{\mu \nu}(x, y; \lambda, B) \), is a linear functional of \( \lambda \) and of \( B \) that contains commutation relations between \( A_\mu \) and \( \mathcal{D}_{\mu \nu}, \mathcal{D}_{\mu \nu}, \mathcal{D}, \) or \( \partial \mathcal{D} \).

The purpose of the present paper is to investigate \( \mathcal{D}_{\mu \nu}(x, y) \) in (1.11). For this purpose, we analogously use the properties of the quantum-gravity Pauli–Jordan D function.

This paper is organized as follows. In the next section, we review the manifestly covariant canonical quantum theory of the interacting system of an abelian gauge field and the gravitational one. In Sect. 3, we introduce \( \mathcal{D}_{\mu \nu} \) and investigate its time derivatives. In Sect. 4, we show the transformation properties of \( \mathcal{D}_{\mu \nu} \) and \( \mathcal{D} \). Some remarks are made in the last section.

2. Covariant operator formalism for abelian gauge fields

The Lagrangian density [5] for an abelian gauge field \( A_\mu(x) \) interacting with the gravitational one \( g_{\mu \nu}(x) \) is given by
\[ \mathcal{L}_A = -\frac{\hbar}{4} g^{\kappa \mu} g^{\lambda \nu} F_{\kappa \lambda} F_{\mu \nu} - h g^{\mu \nu} A_\mu \partial_\nu B + \alpha \frac{\hbar}{2} B^2 - i \hbar g^{\mu \nu} \partial_\mu \overline{C} \cdot \partial_\nu C, \] (2.1)
where \( F_{\kappa \lambda} \equiv \partial_\kappa A_\lambda - \partial_\lambda A_\kappa, B \) is an auxiliary scalar field, \( C \) and \( \overline{C} \) are the Faddeev–Popov ghost scalar fields, and \( \alpha \) denotes a gauge parameter. We treat \( A_\mu \) as the electromagnetic field without any currents in this paper.
In order to take $A_\mu$, $C$, and $\overline{C}$ as the canonical variables, we replace $\mathcal{L}_A$ by $\tilde{\mathcal{L}}_A$ [5] as follows:

$$
\tilde{\mathcal{L}}_A = -\frac{\hbar}{4} g^{\kappa \mu} g^{\lambda \nu} F_{\kappa \lambda} F_{\mu \nu} + \partial_\mu (hg^{\mu \nu} A_\nu) \cdot B + \frac{\alpha}{2} B^2 - i h g^{\mu \nu} \partial_\mu \overline{C} \cdot \partial_\nu C.
$$

(2.2)

Here the discarded term is a total divergence that is given by

$$
\mathcal{L}_A - \tilde{\mathcal{L}}_A = \partial_\lambda (-hg^{\lambda \mu} A_\mu B).
$$

(2.3)

The field equations related to the electromagnetic field are

1. $\partial_\kappa [h(g^{\kappa \mu} g^{\lambda \nu} - g^{\kappa \nu} g^{\lambda \mu}) \partial_\mu A_\nu] - hg^{\lambda \nu} \partial_\nu B = 0$,  
2. $\partial_\lambda (hg^{\lambda \mu} A_\mu) + \alpha h B = 0$, 
3. $\partial_\mu (hg^{\mu \nu} \partial_\nu C) = 0$, 
4. $\partial_\mu (hg^{\mu \nu} \partial_\nu \overline{C}) = 0$.

(2.4) - (2.7)

The field equation (1.6) for $B(x)$ follows from (2.4).

The Lagrangian density (2.2) is invariant under the electromagnetic BRST transformations [5],

$$
\delta_B(A_\mu) = \partial_\mu C,
$$

(2.8)

$$
\delta_B(B) = 0,
$$

(2.9)

$$
\delta_B(C) = 0,
$$

(2.10)

$$
\delta_B(\overline{C}) = i B,
$$

(2.11)

except for the total divergence in (2.3). The electromagnetic BRST charge [5] is defined by

$$
Q_B \equiv \int d^3 x h g^{0 \nu} (B \partial_\nu C - \partial_\nu B \cdot C).
$$

(2.12)

The action integral of $\tilde{\mathcal{L}}_A$ in (2.2) is invariant under the gravitational BRST transformations [2],

$$
\delta_G(A_\mu) = -\kappa \partial_\mu c^\lambda \cdot A_\lambda,
$$

(2.13)

$$
\delta_G(B) = 0,
$$

(2.14)

$$
\delta_G(C) = 0,
$$

(2.15)

$$
\delta_G(\overline{C}) = 0,
$$

(2.16)

where $\kappa$ is Einstein’s gravitational constant, and $c^\lambda$ is the gravitational Faddeev–Popov ghost field. The gravitational BRST charge [2] is defined by

$$
Q_G \equiv \int d^3 x h g^{0 \nu} (b_\rho \partial_\nu c^\rho - \partial_\nu b_\rho \cdot c^\rho),
$$

(2.17)

with the gravitational B-field $b_\rho$. 

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$$

(2.17)

with the gravitational B-field $b_\rho$. 

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The canonical conjugates of $A_\lambda, C$, and $\overline{C}$ are defined by

\[
\pi_A^\lambda \equiv \frac{\partial \tilde{L}_A}{\partial \dot{A}_\lambda} = -h(g^{0\mu}g^{\lambda\nu} - g^{0\nu}g^{\lambda\mu})\partial_\mu A_\nu + hg^{0\lambda} B,
\]

\[
\pi_C \equiv \frac{\partial \tilde{L}_A}{\partial \dot{C}} = ihg^{0\mu}\partial_\mu C,
\]

\[
\pi_{\overline{C}} \equiv \frac{\partial \tilde{L}_A}{\partial \dot{\overline{C}}} = -ihg^{0\nu}\partial_\nu C,
\]

respectively. Here, the functional derivative with respect to $C$ or $\overline{C}$ is made from the left of each operand. The equal-time canonical commutation and anti-commutation relations are set as follows:

\[
[A^\lambda_\mu, A'_\nu] = -i\delta^\lambda_\mu \delta^3, \quad (2.21)
\]

\[
\{\pi_C, C'\} = -i\delta^3, \quad (2.22)
\]

\[
\{\pi_{\overline{C}}, \overline{C}'\} = -i\delta^3. \quad (2.23)
\]

In the above and subsequently, a prime attached to a space-time function means that its argument is not $x^\lambda$ but $y^\lambda$ where it is understood that $x^0 = y^0$.

Using the field equations, the canonical conjugates, and the equal-time canonical (anti-)commutation relations, we obtain various commutation relations [5], e.g.,

\[
[A^\mu_\mu, B'] = i\delta^0_\mu \frac{\delta^3}{hg^{00}}, \quad (2.24)
\]

\[
[\dot{A}_{\mu}, A'_\nu] = i \left[ g_{\mu\nu} - (1 - \alpha)\frac{\delta^0_\mu \delta^0_\nu}{g^{00}} \right] \frac{\delta^3}{hg^{00}}, \quad (2.25)
\]

\[
[\dot{B}, A'_\nu] = i \partial_\nu \left( \frac{1}{hg^{00}} \right) \cdot \delta^3 + i \left( \frac{2g^{0k}g^{00} - \delta^k}{g^{00} \partial_\nu} - \delta^k \right) \partial_\nu \left( \frac{\delta^3}{hg^{00}} \right), \quad (2.26)
\]

\[
[B, \dot{B}'] = 0. \quad (2.27)
\]

Note that we use the de Donder condition [2],

\[
\partial_\mu (hg^{\mu\nu}) = 0, \quad (2.28)
\]

in Eqs. (2.25) and (2.26).

The (anti-)commutation relations [5] between the electromagnetic BRST charge $Q_B$, and the fields $A_\mu, B, C, \text{and } \overline{C}$ are given by

\[
[i Q_B, A_\mu] = \partial_\mu C, \quad (2.29)
\]

\[
[i Q_B, B] = 0, \quad (2.30)
\]

\[
[i Q_B, C] = 0, \quad (2.31)
\]

\[
[i Q_B, \overline{C}] = iB. \quad (2.32)
\]

The charge $Q_B$ commutes with $g_{\mu\nu}$ and $b_\rho$, and anti-commutes with the gravitational Faddeev–Popov ghost fields.
The (anti-)commutation relations [2] between the gravitational BRST charge $Q_G$, and the matter fields $A_\mu$, $B$, $C$, and $\overline{C}$ are given by

\[
[ i Q_G, A_\mu ] = -\kappa \partial_\mu c^\rho \cdot A_\rho - \kappa c^\rho \partial_\rho A_\mu , \tag{2.33}
\]
\[
[ i Q_G, B ] = -\kappa c^\rho \partial_\rho B , \tag{2.34}
\]
\[
[ i Q_G, C ] = -\kappa c^\rho \partial_\rho C , \tag{2.35}
\]
\[
[ i Q_G, \overline{C} ] = -\kappa c^\rho \partial_\rho \overline{C} . \tag{2.36}
\]

Here note the distinction between the BRST transformation in Eqs. (2.13)–(2.16) and that in Eqs. (2.33)–(2.36). The former is the intrinsic BRST transformation [2], which is not commutative with $\partial_\mu$. The latter is the total BRST transformation [2], which is realizable as an algebraic transformation at the operator level.

The above BRST charges give us the subsidiary conditions,

\[
Q_B|_{\text{phys}} = 0 , \tag{2.37}
\]
\[
Q_G|_{\text{phys}} = 0 , \tag{2.38}
\]

to define the physical subspace of the indefinite-metric Hilbert space.

3. Introducing $D_{\mu \nu}(x, y)$

In order to define $D_{\mu \nu}(x, y)$ in the right-hand side of (1.11), we form a q-number Cauchy problem for it, which is analogous to Eqs. (1.1)–(1.3) for $D(x, y)$. Equation (1.1) relates to the gravitational d’Alembert equation, e.g., Eq. (1.6) for the electromagnetic B-field. Equations (1.2) and (1.3) are the quantum-gravitational version of the conditions for the Pauli–Jordan D function,

\[
D(x - y)|_0 = 0 , \tag{3.1}
\]
\[
\partial^x_0 D(x - y)|_0 = -\delta^3(x - y) , \tag{3.2}
\]

respectively.

Comparing (1.9) with (1.11), we see a correspondence

\[
\begin{align*}
B(x) &\rightarrow D(x, y) \hat{\partial}^\Sigma_\nu \\
A_\mu(x) &\rightarrow D_{\mu \nu}(x, y)
\end{align*} \tag{3.3}
\]

This implies that $D_{\mu \nu}(x, y)$ should satisfy two equations similar to (2.4) and (2.5) for $A_\mu(x)$ with respect to $x$, as $D(x, y) \hat{\partial}^\nu_\tau$ satisfies (1.1) similar to (1.6) for $B(x)$. We thus have the following equations:

\[
\partial^x_\tau \left[ h(g^{k\lambda} g^{\sigma \mu} - g^{e \mu} g^{\sigma \lambda}) \partial^x_\lambda D_{\mu \nu}(x, y) \right] - h g^{\sigma \tau} \cdot \hat{\partial}^x_\nu D(x, y) \hat{\partial}^\nu_\tau = 0 , \tag{3.4}
\]
\[
\partial^x_\tau \left[ h g^{\lambda \mu} D_{\mu \nu}(x, y) \right] + \alpha h \cdot D(x, y) \hat{\partial}^\tau_\nu = 0 , \tag{3.5}
\]

where the argument of $g_{\mu \nu}$ or $h$ is $x$ and is omitted for simplicity. Equation (3.4) corresponds to the field equation (2.4); of course, its four-divergence with respect to $x$ is consistent with (1.1). Equation (3.5) corresponds to the gauge-fixing condition (2.5).
Next, we set the following conditions for \( D_{\mu\nu}(x, y) \) at \( x^0 = y^0 \):

\[
D_{\mu\nu}(x, y)|_0 = 0, \tag{3.6}
\]

\[
\partial_0^3 D_{\mu\nu}(x, y)|_0 = - \left[ g_{\mu\nu} - (1 - \alpha) \frac{\delta^0_\mu \delta^0_\nu}{g^{00}} \right] \frac{\delta^3}{hg^{00}}, \tag{3.7}
\]

Equation (3.6) relates to the commutability between \( A_\mu(x) \) and \( A_\nu(y) \) in (1.11) at \( x^0 = y^0 \). Equation (3.7) relates to the fact that the differentiation of (1.11) with respect to \( x^0 \) should coincide with (2.25) at \( x^0 = y^0 \).

We consequently obtain a q-number Cauchy problem (3.4)–(3.7) for \( D_{\mu\nu}(x, y) \). The bilocal operator \( D_{\mu\nu}(x, y) \) is not a function of \( x - y \) alone, because of the same reason for the quantum-gravity D function \( D(x, y) \). In addition, the index \( \mu \) of \( D_{\mu\nu}(x, y) \) relates to the vector property at the point \( x \) while the index \( \nu \) does to that at the point \( y \). These facts come from the form of the left-hand side of (1.11).

We have

\[
[D_{\mu\nu}(x, y)]^\dagger = -D_{\nu\mu}(y, x) \tag{3.8}
\]

via the Hermitian conjugate of (1.11). Using this relation and

\[
[D(x, y)]^\dagger = -D(y, x) \tag{3.9}
\]

in Ref. [3], we obtain

\[
[D_{\mu\nu}(x, y)\hat{\partial}_\rho^\dagger \cdot h(g^{\rho\sigma} g^{\nu\kappa} - g^{\nu\sigma} g^{\rho\kappa})]\hat{\partial}_\sigma^\dagger - \hat{\partial}_\mu^\dagger D(x, y)\hat{\partial}_\tau^\dagger \cdot hg^{\tau\kappa} = 0, \tag{3.10}
\]

\[
[D_{\mu\nu}(x, y)hg^{\nu\rho}]\hat{\partial}_\rho^\dagger + \alpha \hat{\partial}_\mu^\dagger D(x, y) \cdot h = 0, \tag{3.11}
\]

\[
D_{\mu\nu}(x, y)\hat{\partial}_0^\dagger |_0 = \left[ g_{\mu\nu} - (1 - \alpha) \frac{\delta^0_\mu \delta^0_\nu}{g^{00}} \right] \frac{\delta^3}{hg^{00}}, \tag{3.12}
\]

via the Hermitian conjugates of Eqs. (3.4), (3.5), and (3.7). Here the argument of \( g_{\mu\nu} \) or \( h \) is \( y \) and is omitted for simplicity.

Now, we investigate some higher derivatives of \( D_{\mu\nu}(x, y) \). In the manner of Ref. [3], we use the following notation:

\[
t = \frac{x^0 + y^0}{2}, \quad \tau = \frac{x^0 - y^0}{2}, \tag{3.13}
\]

\[
D_{\mu\nu}(x, y) \equiv D^0_{\mu\nu}(t, \tau), \tag{3.14}
\]

\[
\partial_0^3 D_{\mu\nu}(x, y) \equiv D^1_{\mu\nu}(t, \tau), \tag{3.15}
\]

where the spatial arguments of \( D^0_{\mu\nu} \) and \( D^1_{\mu\nu} \) are suppressed. Then we rewrite (3.6) and (3.7) by

\[
D^0_{\mu\nu}(t, 0) = 0, \tag{3.16}
\]

\[
D^1_{\mu\nu}(t, 0) = - \left[ g_{\mu\nu} - (1 - \alpha) \frac{\delta^0_\mu \delta^0_\nu}{g^{00}} \right] \frac{\delta^3}{hg^{00}}, \tag{3.17}
\]

respectively. From (3.16), we have

\[
(\partial^\nu)^\alpha D^0_{\mu\nu}(t, 0) = 0, \tag{3.18}
\]

i.e.,

\[
(\partial_0^\tau + \partial_0^\mu)^\nu D^1_{\mu\nu}(x, y)|_0 = 0, \tag{3.19}
\]
for any \( n = 0, 1, 2, \ldots \). Equation (3.17) yields
\[
(\partial^f)^n \mathcal{D}^1_{\mu\nu}(t, 0) = -(\partial^f)^n \left\{ \left[ g_{\mu\nu}(t) - (1 - \alpha) \frac{\delta^0_{\mu} \delta^0_{\nu}}{g^{00}(t)} \right] \frac{\delta^3}{h(t) g^{00}(t)} \right\},
\]
(3.20)
i.e.,
\[
(\partial^f + \partial^y)^n \partial^x_{\nu} \mathcal{D}_{\mu\nu}(x, y)|_0 = -(\partial_0)^n \left\{ \left[ g_{\mu\nu} - (1 - \alpha) \frac{\delta^0_{\mu} \delta^0_{\nu}}{g^{00}} \right] \frac{\delta^3}{h g^{00}} \right\},
\]
(3.21)
for any \( n = 0, 1, 2, \ldots \).
Furthermore, we derive \((\partial^f_0)^2 \mathcal{D}_{\mu\nu}(x, y)\) from Eqs. (3.4)–(3.6) as follows. Firstly, modifying (3.4) and using (3.6), we have
\[
h(\delta^0_{\mu} g^{0\mu} - \delta^0_{\nu} g^{0\nu})(\partial^f_0)^2 \mathcal{D}_{\mu\nu}(x, y)|_0 = \partial^x_0 \left\{ h \left[ (\delta^0_{\mu} g^{0\mu} - \delta^0_{\nu} g^{0\nu}) \frac{\delta^3}{h g^{00}} \right] \right\}
\]
\[
- h(2 \delta^0_{\mu} g^{0\mu} - \delta^0_{\nu} g^{0\nu} - \delta^0_{\mu} \delta^0_{\nu}) \frac{\delta^3}{h g^{00}} \mathcal{D}_{\mu\nu}(x, y)|_0 + h g^{0\mu} \cdot \partial^x_0 \mathcal{D}(x, y) \partial^x_0 \mathcal{D}(x, y)|_0.
\]
(3.22)
Secondly, differentiating (3.5) with respect to \( x^0 \), and using (3.6), we have
\[
hg^{0\mu}(\partial^f_0)^2 \mathcal{D}_{\mu\nu}(x, y)|_0
= -hg^{0\mu}(\partial^x_0 \partial^y_0 \mathcal{D}_{\mu\nu}(x, y)|_0 - \partial^0_0 \left( \delta^0_{\nu} \partial^x_0 \mathcal{D}_{\mu\nu}(x, y)|_0 \right) - \alpha \left[ \partial^0_0 \mathcal{D}(x, y) \partial^x_0 \mathcal{D}(x, y)|_0 - \partial^0_0 \mathcal{D}(x, y) \partial^x_0 \mathcal{D}(x, y)|_0 \right].
\]
(3.23)
Consequently, combining these equations, we obtain
\[
(\partial^f_0)^2 \mathcal{D}_{\mu\nu}(x, y)|_0
= \partial^0_0 \left[ \partial^x_0 \mathcal{D}_{\mu\nu}(x, y)|_0 \right]
- 2 \left[ \frac{g^{0k}(x)}{g^{00}(x)} \partial^x_0 \partial^x_0 \mathcal{D}_{\mu\nu}(x, y)|_0 - \partial^x_0 \left[ \frac{h(x) g^{0k}(x)}{h(x) g^{00}(x)} \partial^x_0 \mathcal{D}_{\mu\nu}(x, y)|_0 \right] \right]
- \left[ (1 - \alpha) \frac{\delta^0_{\mu} \delta^0_{\nu}}{g^{00}(x)} \left\{ 2 \frac{g^{0k}(x)}{g^{00}(x)} \partial^x_0 \left[ \frac{\delta^3}{h(x) g^{00}(x)} \right] + \partial^x_0 \left[ \frac{h(x) g^{0k}(x)}{h(x) g^{00}(x)} \right] \right\} \right]
+ \frac{g^{0\mu}(x)}{g^{00}(x)} \partial^y_0 \partial^y_0 \mathcal{D}_{\mu\nu}(x, y)|_0 + \partial^\mu \left[ \frac{\delta^3}{h(x) g^{00}(x)} \right]
- \frac{g^{00}(y)}{g^{00}(y)} [\partial^y_0 \mathcal{D}_{\mu\nu}(x, y)|_0 \partial^y_0]
- \left[ \partial^y_0 \mathcal{D}_{\mu\nu}(x, y)|_0 \partial^y_0 \right] \frac{\delta^0_{\mu}}{g^{00}(y)} \frac{\delta^0_{\nu}}{g^{00}(y)} \frac{\delta^3}{h(y) g^{00}(y)}. \tag{3.24}
\]
Note that Eq. (3.21) for \( n = 1 \) relates to (3.24):
\[
\partial^x_0 \mathcal{D}_{\mu\nu}(x, y) \partial^y_0 |_0 = \partial_0 \left[ \partial^x_0 \mathcal{D}_{\mu\nu}(x, y)|_0 \right] - (\partial^f_0)^2 \mathcal{D}_{\mu\nu}(x, y)|_0.
\]
(3.25)
The bilocal operators \( \mathcal{D}_{\mu\nu}(x, y) \) and \( \mathcal{D}(x, y) \) yield a bilocal “current”,
\[
\mathcal{J}_\mu^\lambda(x, y) \equiv \mathcal{D}_{\mu\nu}(x, y) \partial^y_0 \cdot h(y) [g^{0\rho}(y)g^{0\sigma}(y) - g^{0\nu}(y)g^{0\sigma}(y)] A_\sigma(y) - \mathcal{D}_{\mu\nu}(x, y) h(y) [g^{0\rho}(y)g^{0\sigma}(y) - g^{0\nu}(y)g^{0\sigma}(y)] \partial^y_0 A_\sigma(y)
+ \mathcal{D}_{\mu\nu}(x, y) h(y) g^{0\nu}(y) B(y) - \partial^x_0 \mathcal{D}(x, y) \cdot h(y) g^{0\sigma}(y) A_\sigma(y).
\]
(3.26)
which is conserved with respect to \( y \) by virtue of Eqs. (2.4), (2.5), (3.10), and (3.11). Hence, an integral representation for \( A_\mu \),

\[
A_\mu(x) = \int d^3 y \, J_\mu^0(x, y),
\]

(3.27)
is independent of \( y^0 \); of course, the right-hand side is reduced to \( A_\mu(x) \) at \( y^0 = x^0 \).

4. Transformation properties

In the quantum gravi-electrodynamics [5], there are the affine, the electromagnetic BRST, and the gravitational BRST symmetry. We investigate the transformation properties of \( D_{\mu\nu}(x, y) \) and \( D(x, y) \) with respect to these three symmetries.

4.1. Affine transformation

Let \( \hat{\mathcal{P}}_\lambda \) and \( \hat{M}^\kappa_\lambda \) be the translation generator and the \( GL(4) \) one [2], respectively. The transformation properties [1] of the quantum-gravity D function are given by

\[
[i \hat{\mathcal{P}}_\lambda, D(x, y)] = (\partial^\kappa_\lambda + \hat{\mathcal{P}}^\kappa_\lambda)D(x, y),
\]

(4.1)

\[
[i \hat{M}^\kappa_\lambda, D(x, y)] = (x^\kappa \partial^\lambda_\kappa + y^\kappa \hat{\mathcal{P}}^\lambda_\kappa)D(x, y).
\]

(4.2)
The proof is done by showing that Eqs. (4.1) and (4.2) are consistent with Eqs. (1.1)–(1.3), and (3.9).

We treat only (4.4), since \( \hat{\mathcal{P}}_\lambda \) can formally be regarded as \( \hat{M}^5_\lambda \) with \( x^5 = y^5 = 1 \), and \( \delta^5_\mu = \delta^5_\nu = 0 \). We define the difference between both sides of (4.4) as follows:

\[
\mathcal{A}^\kappa_{\lambda\mu\nu}(x, y) \equiv [i \hat{M}^\kappa_\lambda, D_{\mu\nu}(x, y)] - (x^\kappa \partial^\lambda_\kappa + y^\kappa \hat{\mathcal{P}}^\lambda_\kappa)D_{\mu\nu}(x, y) - \delta^\kappa_\mu D_{\lambda\nu}(x, y) - \delta^\kappa_\nu D_{\lambda\mu}(x, y).
\]

(4.5)

By virtue of Eqs. (3.4)–(3.7), (3.12), (3.24), (3.25), and

\[
[i \hat{M}^\kappa_\lambda, h] = x^\kappa \partial^\lambda_\kappa h + \delta^\kappa_\lambda h,
\]

(4.6)

\[
[i \hat{M}^\kappa_\lambda, g^\rho^\sigma] = x^\kappa \partial^\rho_\kappa g^\sigma - \delta^\rho_\lambda g^\sigma - \delta^\rho_\kappa g^\lambda,
\]

(4.7)

we obtain

\[
\partial^\rho_\kappa [h(g^\rho^\sigma g^{\tau^\mu} - g^\rho^\mu g^{\tau^\sigma})\partial^\tau_\sigma \mathcal{A}^\kappa_{\lambda\mu\nu}(x, y)] = 0,
\]

(4.8)

\[
\partial^\rho_\kappa [h(g^\rho^\mu \mathcal{A}^\kappa_{\lambda\mu\nu}(x, y))] = 0,
\]

(4.9)

\[
\mathcal{A}^\kappa_{\lambda\mu\nu}(x, y)|_0 = 0,
\]

(4.10)

\[
\partial^\rho_\kappa \mathcal{A}^\kappa_{\lambda\mu\nu}(x, y)|_0 = 0.
\]

(4.11)

We thus find

\[
\mathcal{A}^\kappa_{\lambda\mu\nu}(x, y) = 0.
\]

(4.12)

Equation (3.8) and the Hermitian conjugate of (4.12) yield

\[
[i \hat{M}^\kappa_\lambda, D_{\mu\nu}(x, y)]^\dagger = -(x^\kappa \partial^\lambda_\kappa + y^\kappa \hat{\mathcal{P}}^\lambda_\kappa)D_{\nu\mu}(y, x) - \delta^\kappa_\mu D_{\nu\lambda}(y, x) - \delta^\kappa_\nu D_{\lambda\mu}(y, x).
\]

(4.13)

Hence, Eqs. (4.3) and (4.4) are proved.
On the basis of Eqs. (4.1)–(4.4), (4.6), and (4.7), the affine transformation of the integral representation (3.27) for $A_\mu$ is given by

$$[i \hat{\mathcal{P}}_\lambda, A_\mu(x)] = \partial_\lambda^x A_\mu(x) + \int d^3y \, \partial_\lambda^y \mathcal{J}_\mu(x, y),$$  \hspace{1cm} (4.14)

$$[i \hat{\mathcal{M}}_\lambda, A_\mu(x)] = x^\lambda \partial_\lambda^x A_\mu(x) + \delta^\lambda_\mu A_\lambda(x) + \int d^3y \{ \partial_\lambda^y \mathcal{J}_{\mu,0}^0(x, y) - \delta^0_\lambda \mathcal{J}_\mu^x(x, y) \}. \hspace{1cm} (4.15)$$

In the right-hand sides of these equations, the integral terms vanish since the “current” $\mathcal{J}_{\mu,0}^0(x, y)$ is conserved with respect to $y$.

### 4.2. Electromagnetic BRST transformation

Let us denote the electromagnetic BRST transformation of the quantum-gravity $D$ function by $B(x, y)$:

$$[i Q_B, D(x, y)] = B(x, y).$$  \hspace{1cm} (4.16)

The bilocal operator $B(x, y)$ should satisfy the electromagnetic BRST transformations of Eqs. (1.1), (1.2), and (1.3):

$$\partial_\lambda^x [ h g^{\lambda \mu} \partial_\lambda^y B(x, y) ] = 0,$$  \hspace{1cm} (4.17)

$$B(x, y)|_0 = 0,$$  \hspace{1cm} (4.18)

$$\partial_0^x B(x, y)|_0 = 0.$$  \hspace{1cm} (4.19)

We thus find

$$B(x, y) = 0.$$  \hspace{1cm} (4.20)

Of course, this is consistent with the electromagnetic BRST transformation of the integral representation (1.7) for $B$.

In a similar way, we denote the electromagnetic BRST transformation of $D_{\mu \nu}(x, y)$ by $B_{\mu \nu}(x, y)$:

$$[i Q_B, D_{\mu \nu}(x, y)] = B_{\mu \nu}(x, y).$$  \hspace{1cm} (4.21)

The bilocal operator $B_{\mu \nu}(x, y)$ should satisfy the electromagnetic BRST transformations of Eqs. (3.4)–(3.7):

$$\partial_\lambda^y [ h ( g^{\lambda \kappa} g_{\sigma \mu} - g^{\kappa \mu} g^{\sigma \lambda} ) \partial_\lambda^x B_{\mu \nu}(x, y) ] = 0,$$  \hspace{1cm} (4.22)

$$\partial_\lambda^x [ h g^{\lambda \mu} B_{\mu \nu}(x, y) ] = 0,$$  \hspace{1cm} (4.23)

$$B_{\mu \nu}(x, y)|_0 = 0,$$  \hspace{1cm} (4.24)

$$\partial_0^x B_{\mu \nu}(x, y)|_0 = 0.$$  \hspace{1cm} (4.25)

We thus find

$$B_{\mu \nu}(x, y) = 0.$$  \hspace{1cm} (4.26)

By virtue of Eqs. (4.20) and (4.26), the electromagnetic BRST transformation of the integral representation (3.27) for $A_\mu$ is given by

$$[i Q_B, A_\mu(x)] = \int d^3y \{ D_{\mu \nu}(x, y) \partial_\nu^x h(y)[ g^{0 \rho}(y) g^{\nu \sigma}(y) - g^{0 \nu}(y) g^{\rho \sigma}(y) ] \partial_\sigma^y C(y)$$

$$- \partial_\mu^x D(x, y) \cdot h(y) g^{0 \nu}(y) \partial_\nu^y C(y) \}. \hspace{1cm} (4.27)$$

Since the first term of the integrand in (4.27) does not involve $\partial_0 C$, we integrate it by parts. Hence we can reduce (4.27) to (2.29) via (3.10).
4.3. Gravitational BRST transformation

We show that the gravitational BRST transformation of the quantum-gravity D function is given by

\[ [i Q_G, D(x, y)] = -\kappa [c^\rho (x) \partial_\rho D(x, y) + D(x, y) \overrightarrow{\partial_\rho} \cdot c^\rho(y)]. \]  \hspace{1cm} (4.28)

We define the difference between both sides of (4.28) as follows:

\[ G(x, y) \equiv [i Q_G, D(x, y)] + \kappa [c^\rho (x) \partial_\rho D(x, y) + D(x, y) \overrightarrow{\partial_\rho} \cdot c^\rho(y)]. \]  \hspace{1cm} (4.29)

By virtue of Eqs. (1.1)–(1.3), (1.5), and

\[ [i Q_G, h] = -\kappa \partial_\mu (hc^\mu), \]  \hspace{1cm} (4.30)

\[ [i Q_G, g^{\lambda \mu}] = \kappa (g^{\lambda \nu} \partial_\nu c^\mu + g^{\mu \nu} \partial_\nu c^\lambda - c^\nu \partial_\nu g^{\lambda \mu}), \]  \hspace{1cm} (4.31)

we obtain

\[ \partial_\mu [h g^{\mu \nu} \partial_\nu G(x, y)] = 0, \]  \hspace{1cm} (4.32)

\[ G(x, y)|_0 = 0, \]  \hspace{1cm} (4.33)

\[ \partial_0 \overrightarrow{G}(x, y)|_0 = 0. \]  \hspace{1cm} (4.34)

In (4.34), we also use the expressions for \((\partial_0^*)^2 D(x, y)|_0\) and \(\partial_0^* D(x, y) \overrightarrow{\partial_0}|_0\) given in Ref. [3].

From Eqs. (4.32)–(4.34), we thus find

\[ G(x, y) = 0. \]  \hspace{1cm} (4.35)

Equation (3.9) and the Hermitian conjugate of (4.35) yield

\[ [i Q_G, D(x, y)]^\dagger = \kappa [c^\rho (y) \partial_\rho D(y, x) + D(y, x) \overrightarrow{\partial_\rho} \cdot c^\rho(x)]. \]  \hspace{1cm} (4.36)

Hence, Eq. (4.28) is proved.

Note that the gravitational BRST transformation of the integral representation (1.7) for \(B\) reproduces (2.34) on the basis of Eqs. (4.28), (4.30), and (4.31).

In a similar way, we show that the gravitational BRST transformation of \(D_{\mu \nu}(x, y)\) is given by

\[ [i Q_G, D_{\mu \nu}(x, y)] = -\kappa [\partial_\nu c^\rho (x) \cdot D_{\rho \nu}(x, y) + c^\rho(x) \partial_\rho D_{\mu \nu}(x, y) \]  \hspace{1cm} (4.37)

\[ + D_{\mu \sigma}(x, y) \partial_\nu c^\sigma (y) + D_{\mu \nu}(x, y) \overrightarrow{\partial_\sigma} \cdot c^\sigma(y)]. \]

We define the difference between both sides of (4.37) as follows:

\[ G_{\mu \nu}(x, y) \equiv [i Q_G, D_{\mu \nu}(x, y)] + \kappa [\partial_\nu^T c^\rho (x) \cdot D_{\rho \nu}(x, y) + c^\rho(x) \partial_\rho D_{\mu \nu}(x, y) \]  \hspace{1cm} (4.38)

\[ + D_{\mu \sigma}(x, y) \partial_\nu c^\sigma (y) + D_{\mu \nu}(x, y) \overrightarrow{\partial_\sigma} \cdot c^\sigma(y)]. \]

By virtue of Eqs. (3.4)–(3.7), (3.12), (3.24), (3.25), (4.28), (4.30), and (4.31), we obtain

\[ \partial_\rho [h (g^{\sigma \rho} g^{\tau \mu} - g^{\rho \mu} g^{\sigma \tau}) \partial_\nu G_{\mu \nu}(x, y)] = 0, \]  \hspace{1cm} (4.39)

\[ \partial_\rho [h g^{\rho \mu} G_{\mu \nu}(x, y)] = 0, \]  \hspace{1cm} (4.40)

\[ G_{\mu \nu}(x, y)|_0 = 0, \]  \hspace{1cm} (4.41)

\[ \partial_0 G_{\mu \nu}(x, y)|_0 = 0. \]  \hspace{1cm} (4.42)
We thus find

\[ G_{\mu\nu}(x, y) = 0. \]  

(4.43)

Equation (3.8) and the Hermitian conjugate of (4.43) yield

\[
[i Q_G, D_{\mu\nu}(x, y)]^\dagger = \kappa [\partial_\rho^\gamma c^\rho(y) \cdot D_{\rho\mu}(y, x) + c^\rho(y) \partial_\rho D_{\nu\mu}(y, x) \\
+ D_{\nu\sigma}(y, x) \partial_\sigma^\gamma c^\sigma(x) + D_{\nu\mu}(y, x) \overline{\partial_\sigma^\gamma \cdot c^\sigma(x)}].
\]

(4.44)

Hence, Eq. (4.37) is proved.

On the basis of Eqs. (4.28), (4.30), (4.31), and (4.37), the gravitational BRST transformation of the integral representation (3.27) for \( A_\mu \) is given by

\[
[i Q_G, A_\mu(x)] = -\kappa \partial_\mu^\gamma c^\mu(x) \cdot A_\rho(x) - \kappa c^\mu(x) \partial_\rho^\gamma A_\mu(x) \\
- \kappa \int d^3y \{ \partial_\rho^\gamma [J_\mu^0(x, y) c^\rho(y)] - J_\mu^0(x, y) \partial_\rho^\gamma c^0(y) \}.
\]

(4.45)

In the right-hand side of this equation, the third term vanishes since the “current” \( J_\mu^\nu(x, y) \) is conserved with respect to \( y \).

5. Summary and remarks

In the present paper, we have defined \( D_{\mu\nu}(x, y) \) in (1.11) as a function that satisfies a q-number version of a Cauchy problem (3.4)–(3.7); our treatment of \( D_{\mu\nu}(x, y) \) has been basically analogous to that of \( D(x, y) \) in Refs. [1–3]. We have obtained \( (\partial_0^\mu)^2 D_{\mu\nu}(x, y)|_0 \) and the integral representation for \( A_\mu \). We have shown that the affine transformations, the electromagnetic BRST ones, and the gravitational BRST ones of \( D_{\mu\nu}(x, y) \) and \( D(x, y) \) are consistent with Eqs. (3.4)–(3.7), and (3.27).

Comparing (1.11) with (1.12), we find that \( D_{\mu\nu}(x, y) \) involves dipole contributions. In fact, we have

\[
\partial_\kappa \left\{ h g^{\kappa\lambda} \partial_\kappa \left[ \frac{1}{h} \partial_\mu (h g^{\mu\nu} D_{\nu\rho}) \right] \right\} = 0,
\]

(5.1)

\[
\partial_\kappa \left( h g^{\kappa\lambda} \partial_\kappa \left[ \frac{g_{\mu\nu}}{h} \partial_\rho [h (g^{\rho\sigma} g^{\nu\tau} - g^{\rho\tau} g^{\nu\sigma}) \partial_\sigma D_{\tau\omega}] \right] \right) \\
- \partial_\kappa \left( h g^{\kappa\lambda} \partial_\kappa \left[ \frac{g_{\lambda\nu}}{h} \partial_\rho [h (g^{\rho\sigma} g^{\nu\tau} - g^{\rho\tau} g^{\nu\sigma}) \partial_\sigma D_{\tau\omega}] \right] \right) = 0,
\]

(5.2)

on the combination of Eqs. (1.1), (3.4), and (3.5); these are independent of \( D(x, y) \) and \( \alpha \). We can express \( (\partial_0^\mu)^2 D_{\mu\nu}(x, y)|_0 \) or \( (\partial_0^\mu)^4 D_{\mu\nu}(x, y)|_0 \) in terms of lower time derivatives of \( D_{\mu\nu} \) and \( D \).

We also find that a nonlocal “current”

\[
J_{\mu\nu}(x, y, z) \equiv D_{\mu\nu}(x, y) \partial_\rho^\gamma \cdot h(y) [g^{\kappa\rho}(y) g^{\nu\sigma}(y) - g^{\kappa\nu}(y) g^{\rho\sigma}(y)] D_{\sigma\tau}(y, z) \\
- D_{\mu\nu}(x, y) h(y) [g^{\kappa\rho}(y) g^{\nu\sigma}(y) - g^{\kappa\nu}(y) g^{\rho\sigma}(y)] \partial_\rho^\gamma D_{\sigma\tau}(y, z) \\
+ D_{\mu\nu}(x, y) h(y) g^{\kappa\nu}(y) \cdot D(x, y) \partial_\tau^\gamma - \partial_\tau^\gamma D_{\mu\nu}(x, y) \cdot h(y) g^{\kappa\nu}(y) D_{\sigma\tau}(y, z),
\]

(5.3)

is conserved with respect to \( y \), by virtue of Eqs. (3.4), (3.5), (3.10), and (3.11). Thus, an integral representation for \( D_{\mu\nu}(x, z) \),

\[
D_{\mu\nu}(x, z) = \int d^3y J_{\mu\nu}^0(x, y, z),
\]

(5.4)

is independent of \( y^0 \).
Since both $D_{\mu\nu}(x, y)$ and $D(x, y)$ are not functions of $x - y$ alone, we must distinguish $\partial^y$ from $-\partial^x$. On the other hand, if a physical vacuum $|0\rangle$ is translationally invariant,

$$\hat{P}_\lambda |0\rangle = 0,$$  \hspace{1cm} (5.5)

then the vacuum expectation values of Eqs. (4.1) and (4.3) are given by

$$\langle 0| [i \hat{P}_\lambda, D(x, y)] |0\rangle = (\partial^x_\lambda + \partial^y_\lambda) \langle 0| D(x, y) |0\rangle = 0,$$  \hspace{1cm} (5.6)

$$\langle 0| [i \hat{P}_\lambda, D_{\mu\nu}(x, y)] |0\rangle = (\partial^x_\lambda + \partial^y_\lambda) \langle 0| D_{\mu\nu}(x, y) |0\rangle = 0.$$  \hspace{1cm} (5.7)

These mean that $\partial^y_\lambda$ is equivalent to $-\partial^x_\lambda$.

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