UNIQUENESS OF VISCOSITY SOLUTIONS OF LOCAL CAHN-HILLIARD-NAVIER-STOKES SYSTEM

SHEETAL DHARMATTI\textsuperscript{1} AND PERISETTI LAKSHMI NAGA MAHENDRANATH\textsuperscript{2}

\textbf{Abstract.} In this work, we consider the local Cahn-Hilliard-Navier-Stokes equation with regular potential in two dimensional bounded domain. We formulate distributed optimal control problem as the minimization of a suitable cost functional subject to the controlled local Cahn-Hilliard-Navier-Stokes system and define the associated value function. We prove the Dynamic Programming Principle satisfied by the value function. Due to the lack of smoothness properties for the value function, we use the method of viscosity solutions to obtain the corresponding solution of the infinite dimensional Hamilton-Jacobi-Bellman equation. We show that the value function is the unique viscosity solution of the Hamilton-Jacobi-Bellman equation. The uniqueness of the viscosity solution is established via comparison principle.

1. Introduction

The famous Navier-Stokes equations govern the complex motions of a single-phase fluid and is studied in the literature extensively by physicists, engineers and mathematicians. For comprehensive mathematical study of these equations one can refer to [10, 12, 28, 38] and references there in. The mathematical study of binary or multi-phase mixture flows has garnered interest in the last few decades. J.W. Cahn and J.E. Hilliard were the first to formulate the mathematical equations of this problem who studied the spinodal decomposition of binary alloys (see [7, 8]). Similar phenomena occur in the phase separation of binary fluids, that is, fluids composed by either two phases of the same chemical species or phases of different composition. In this case, however, the phenomenology is much more complicated because of the interplay between the phase separation stage and the fluid dynamics. The mathematical analysis of these phenomena is far from being well understood. Different phase field models can be developed by coupling Cahn-Hilliard equations with equations describing dynamics of the flow. Thus the equations are not just non-linear but are also coupled and hence the mathematical study is challenging as well as difficult.

For the coupled Cahn-Hilliard-Navier-Stokes system, (CHNS system), the chemical interactions between two phases at the interface are governed by the Cahn-Hilliard system, and the Navier-Stokes equations with surface tension terms acting at the interface gives the hydrodynamic properties of the mixture. When the two fluids have the same constant density, the temperature differences are negligible and the diffusive interface between the two phases has a small but non-zero
thickness, a well-known model is the so-called “model H” (see [26]). The coupled Cahn-Hilliard-
Navier-Stokes system (model H) is described as follows:

\[
\begin{align*}
\frac{\partial \varphi}{\partial t} + \mathbf{u} \cdot \nabla \varphi &= m \Delta \mu, \quad \text{in } \Omega \times (0, T), \\
\mu &= -\Delta \varphi + f(\varphi), \quad \text{in } \Omega \times (0, T), \\
\mathbf{u}_t - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla \pi &= \mathcal{K} \mu \nabla \varphi + \mathbf{U}, \quad \text{in } \Omega \times (0, T), \\
\text{div} \mathbf{u} &= 0, \quad \text{in } \Omega \times (0, T), \\
\mathbf{u}(0) &= \mathbf{u}_0, \varphi(0) &= \varphi_0, \quad \text{in } \Omega, 
\end{align*}
\]

where \(\Omega \subset \mathbb{R}^n, n = 2, 3\), is a bounded subset with smooth boundary \(\partial \Omega\), \(\mathbf{U}\) is an external volume forcing and we have assumed the density to be equal to one. Here, \(\mathbf{u} = (u_1, \ldots, u_n)\), \(n = 2, 3\) represents the mean velocity field and \(\varphi\) is the order parameter which represents the relative concentration of one of the fluids. The quantities \(\nu, m, \mathcal{K}\) are viscosity, mobility and capillary coefficient respectively, which are positive constants. We assume that the boundary conditions for \(\varphi\) are the natural no-flux condition

\[
\frac{\partial \varphi}{\partial \mathbf{n}} = 0, \quad \text{on } \partial \Omega \times (0, T), 
\]

where \(\mathbf{n}\) is the outward normal to \(\partial \Omega\). Note that (1.6) implies that

\[
\frac{\partial \mu}{\partial \mathbf{n}} = 0, \quad \text{on } \partial \Omega \times (0, T), 
\]

where \(\mu\) is the chemical potential of the binary mixture. It is given by the first variation of the following Landau-Ginzburg energy functional

\[
E(\varphi) = \int_{\Omega} \left( \frac{1}{2} |\nabla \varphi(x)|^2 + F(\varphi(x)) \right) dx, 
\]

where \(F(s) = \int_0^s f(\tau) d\tau\) is a suitable double-well potential. A typical example of potential \(F\) is a logarithmic potential. However, this potential is very often replaced by a polynomial approximation of the regular potential (eg: \(F(s) = s^2(s^2 - 1)\)).

From (1.1) and (1.7), we deduce the conservation of the average of \(\varphi\) denoted by

\[
\langle \varphi(t) \rangle = \frac{1}{|\Omega|} \int_{\Omega} \varphi(x, t) dx, 
\]

where \(|\Omega|\) is the Lebesgue measure of \(\Omega\). More precisely, we have

\[
\langle \varphi(t) \rangle = \langle \varphi(0) \rangle, \quad \forall t \geq 0.
\]

A different form of the free energy has been proposed in [20, 21] and rigorously justified as a macroscopic limit of microscopic phase segregation models with particle conserving dynamics. In this case the gradient term in (1.8) is replaced by a non-local spatial operator

\[
\mathcal{E}(\varphi) = \frac{1}{4} \int_{\Omega} \int_{\Omega} J(x - y)(\varphi(x) - \varphi(y))^2 dx dy + \int_{\Omega} F(\varphi(x)) dx, 
\]

where \(J : \mathbb{R}^n \to \mathbb{R}\) is a smooth function such that \(J(x) = J(-x)\). The system with the chemical potential

\[
\mu = a \varphi - J \ast \varphi + f(\varphi), 
\]

which is first variation of \(\mathcal{E}\), is called nonlocal Cahn-Hilliard-Navier-Stokes system. The well-
posedness for nonlocal CHNS system has been well studied in the literature by several authors
(see [16, 13, 9, 14, 15, 4]). For optimal distributed control problems for the same see [18, 17, 3].

For the local system (1.1)-(1.7), the existence and uniqueness of a weak solution has been obtained in [5] in the case of regular potential and also the existence of a strong solution. In the
same work author also studies the case of singular potential, for the existence of a weak solution by approximating the singular potential with sequence of regular potentials and passing to the limit of the corresponding solutions. Certain stability results have also been established. In \([19]\), authors analyse the asymptotic behaviour of the solution of local Cahn-Hilliard-Navier-Stokes system. In fact, they have proved the existence of global and exponential attractors.

Optimal control theory of fluid dynamics models has been an important research area of applied mathematics with many applications in the fields like fluid mechanics, geophysics, engineering and technology. In \([29]\), authors have considered the distributed optimal control problem as the minimisation of the total energy and dissipation of energy of the flow with local Cahn-Hilliard-Navier-Stokes system where the controls appear in the form of volume force densities. The existence of an optimal control as well as the first order necessary optimality conditions is established, and the optimal control is characterized in terms of adjoint variable. The optimal control problems with state constraint and robust control for the same system are investigated in \([30, 31]\), respectively. Optimal control problems of semi discrete Cahn-Hilliard-Navier-Stokes system for various cases like distributed and boundary control, with non smooth Landau-Ginzburg energies and with non matched fluid densities are studied in \([25, 24, 23]\). These works considered the local Cahn-Hilliard-Navier-Stokes equations for their numerical studies. All these works consider the optimal control problem using Pontryagin’s maximum principle. The dynamic programming principle approach is completely open for such problems. In this work our main aim is to study the dynamic programming principle for a control problem governed by local CHNS system and derive the corresponding Hamilton-Jacobi-Bellman equation satisfied by the value functional. Further we want to show that value function is the unique viscosity solution of the corresponding Hamilton-Jacobi-Bellman equation.

The viscosity solution method, a notion of generalised solution, well suited for the first order fully non linear partial differential equations typically of the Hamilton-Jacobi type was first introduced by Crandall and Lions in \([11]\). They have further studied Hamilton-Jacobi equations in infinite dimensions using viscosity solution in a series of eight papers during 1984-1992. The value function of an optimal control problem subjected to ordinary differential equations satisfies the Hamilton-Jacobi-Bellman (HJB) equations whenever the value function is smooth. However seldom these value functions are \(C^1\). The viscosity solution theory helps to tackle this issue by showing that value function is the unique viscosity solution of the corresponding HJB equations. For comprehensive treatment of these ideas one can look at \([2]\) and references therein. In \([36, 37]\) authors have generalised the ideas of viscosity solution theory to infinite dimensional problems in general Banach spaces. This allowed to treat value functions corresponding to the control problems constrained by partial differential equations using viscosity solution theory. Various works in these directions can be found in \([27, 35]\). The optimal control problems governed by non-linear PDE’s and coupled non-linear equations is well studied using Pontryagin’s maximum principle approach however not much work is done using viscosity solution theory. For the famous fluid flow equations of Navier-Stokes’ equations, an optimal control problem is treated in \([22, 34]\) using viscosity solution method. The authors prove the dynamic programming principle satisfied by the corresponding value function and existence of a unique viscosity solutions. Recently, \([32]\) treats tidal dynamics equations, a coupled non-linear system to prove the existence of viscosity solution using Dynamic Programming Principle though the uniqueness question is left open.

In the current work, we propose the use of viscosity solution technique to study the optimal control problems governed by local CHNS system. As per our knowledge this is the first attempt to study the optimal control problem of coupled non linear system using viscosity solution method; which proves the existence and uniqueness of solution of the corresponding HJB equations satisfied by the value function.
The structure of the paper is as follows. In the next section, we describe the mathematical setting to study the local Cahn-Hilliard-Navier-Stokes equation. We recall some existence and uniqueness results available in the literature. We also present some convergence results of the solution with respect to the initial data. In section 3 we define the value function and derive it’s continuity properties. We also state Dynamic Programming Principle (DPP) (see Theorem 3.3) and prove that the value function satisfies the DPP. In Section 4 we state the Hamilton-Jacobi- Bellman (HJB) equations satisfied by the value function and prove that the value function satisfies the HJB equation in the viscosity sense (see Theorem 4.3). In the last section we show that the value function is the unique viscosity solution of the corresponding HJB equation via comparison principle (see Theorem 5.1).

2. Mathematical setting

In this section, we introduce the necessary function spaces needed throughout the paper. We define some operators to write (1.1)-(1.5) in the abstract form. We also state the existence, uniqueness and strong solution results of the system (1.1)-(1.5).

Hereafter, we assume that the domain $\Omega$ is bounded subset of $\mathbb{R}^2$ with a smooth boundary $\partial \Omega$. We denote by $W^{m,p}(\Omega; \mathbb{R}^2) = [W^{m,p}(\Omega; \mathbb{R})]^2$ the Sobolev space of order $m \in [0, \infty)$ and power $p \geq 0$ of functions with values in $\mathbb{R}^2$. The norm of $u \in W^{m,p}(\Omega; \mathbb{R}^2)$ will be denoted by $\|u\|_{m,p}$. We denote $L^p = W^{0,p}$. Moreover, we will write $H^m = W^{m,2}(\Omega; \mathbb{R}^2)$ and $H = W^{0,2}(\Omega; \mathbb{R}^2)$ for vector valued functions, and $H^m = W^{m,2}(\Omega; \mathbb{R})$, $H = W^{0,2}(\Omega; \mathbb{R})$ for scalar valued functions.

Let us set

$$D = \{u \in C_0^\infty(\Omega) \mid \text{div}(u) = 0\}. $$

Then we define

$$\mathcal{G}_{\text{div}} = \text{closure of } D \text{ in } L^2(\Omega),$$

$$\mathcal{V}_{\text{div}} = \text{closure of } D \text{ in } H^1_0(\Omega).$$

We define the operator $A$ by

$$Au = -\mathcal{P}\Delta u, \forall u \in D(A) = H^2 \cap \mathcal{V}_{\text{div}},$$

where $\mathcal{P}$ is the Leray-Helmholtz projector or the Stokes operator in $H$ onto $\mathcal{G}_{\text{div}}$. We know that $A$ is self-adjoint and positive definite, $A^{-1}$ is compact and $A$ generates an analytic semigroup. For $\alpha \geq 0$ we denote by $\mathcal{V}_{\alpha}$ the domain of $A^\frac{\alpha}{2}, \ D(A^\frac{\alpha}{2}),$ equipped with the norm

$$\|u\|_{\alpha} \leq \|A^\frac{\alpha}{2}u\|. \quad (2.1)$$

We introduce the non-negative linear unbounded operator on $L^2(\Omega)$

$$A_N \phi = -\Delta \phi, \quad \forall \phi \in D(A_N) = \{\phi \in H^2(\Omega) : \frac{\partial \phi}{\partial n} = 0 \text{ on } \partial \Omega\} \quad (2.2)$$

and we endow $D(A_N)$ with the norm $\|A_N(\cdot)\| + \|\phi\|$, which is equivalent to the $H^2$-norm. We also define linear positive unbounded operator on the Hilbert space $L^2_0(\Omega)$ of the $L^2$-functions with zero mean value

$$B_N \phi = -\Delta \phi, \quad \forall \phi \in D(B_N) = D(A_N) \cap L^2_0(\Omega)$$

Note that $B_N^{-1}$ is a compact linear operator on $L^2_0(\Omega)$. More generally, we can define $B_N^s$ for any $s \in \mathbb{R}$ noting that $\|B_N^s\|_{s > 0}$, is an equivalent norm to the canonical $H^s$-norm on $D(B_N^s) \subset H^s(\Omega) \cap L^2_0(\Omega)$. Also, note that $A_N = B_N$ on $D(B_N)$. If $\phi$ is such that $\phi - \langle \phi \rangle \in D(B_N^s)$, we have
Lemma 2.5 (Agmon’s inequality, Theorem 13.2, [1]). Let \( \Omega \subset \mathbb{R}^n \), \( u \in W^{m,p}(\Omega; \mathbb{R}^n) \), \( p \geq 1 \) and fix \( 1 \leq r, q \leq \infty \) and a natural number \( m \). Suppose also that a real number \( \alpha \) and a natural number \( j \) are such that

\[
\alpha = \left( \frac{j}{n} + 1 - \frac{1}{r} \right) \left( \frac{m}{n} - \frac{1}{p} + \frac{1}{q} \right)^{-1}
\]

and \( \frac{m}{r} \leq \alpha \leq 1 \). Then for any \( u \in W^{m,p}(\Omega; \mathbb{R}^n) \), we have

\[
\|\nabla^j u\|_{L^r} \leq C \left( \|\nabla^m u\|_{L^p}^\alpha \|u\|_{L^q}^{1-\alpha} + \|u\|_{L^r} \right),
\]

where \( s > 0 \) is arbitrary and the constant \( C \) depends upon the domain \( \Omega, m, n \).

We state some useful and known estimates as lemmas below.

**Lemma 2.1** (Gagliardo-Nirenberg interpolation inequality, Theorem 1, [33]). Let \( \Omega \subset \mathbb{R}^n \), \( u \in W^{m,p}(\Omega; \mathbb{R}^n) \), \( p \geq 1 \) and fix \( 1 \leq r, q \leq \infty \) and a natural number \( m \). Suppose also that a real number \( \theta \) and a natural number \( j \) are such that

\[
\theta = \left( \frac{j}{n} + 1 - \frac{1}{r} \right) \left( \frac{m}{n} - \frac{1}{p} + \frac{1}{q} \right)^{-1}
\]

and \( \frac{m}{r} \leq \theta \leq 1 \). Then for any \( u \in W^{m,p}(\Omega; \mathbb{R}^n) \), we have

\[
\|\nabla^j u\|_{L^r} \leq C \left( \|\nabla^m u\|_{L^p}^\theta \|u\|_{L^q}^{1-\theta} + \|u\|_{L^r} \right),
\]

where \( s > 0 \) is arbitrary and the constant \( C \) depends upon the domain \( \Omega, m, n \).

**Lemma 2.2** (Ladyzhenskaya’s inequality). For \( u \in C_0^\infty(\Omega; \mathbb{R}^n), n = 2, 3 \), there exists a constant \( C \) such that

\[
\|u\|_{L^4} \leq C^{1/4}\|u\|^{1-\frac{4}{n}}\|\nabla u\|^{\frac{4}{n}}, \text{ for } n = 2, 3,
\]

where \( C = 2, 4 \), for \( n = 2, 3 \) respectively.

**Lemma 2.3** (Agmon’s inequality, Lemma 13.2, [1]). Let \( u \in H^2 \cap H_0^1(\Omega) \). Then, there exists a constant \( C > 0 \) such that

\[
\|u\|_{L^\infty} \leq C\|u\|^{1/2}\|u\|_{H^2}^{1/2}.
\]

**Lemma 2.4** (Poincare-Wirtinger inequality, [6]). Let \( \Omega \) be a connected open set of class \( C^1 \) and let \( 1 \leq p \leq \infty \). Then there exists a constant \( C > 0 \) such that

\[
\|u - \langle u \rangle\| \leq C\|\nabla u\|_{L^p}, \quad \forall u \in W^{1,p}(\Omega).
\]

**Lemma 2.5** ([22]). If \( m \geq 0, mp \leq 2 \) and \( p \leq q \leq \frac{2p}{2-mp} \) then \( W^{m,p}(\Omega) \hookrightarrow L^q(\Omega) \), i.e.,

\[
\|u\|_{0,q} \leq C\|u\|_{m,p} \quad \text{for } u \in W^{m,p}(\Omega),
\]

(note that when \( mp = 2 \) the embedding holds for all \( q < \infty \)). Combining the above with the equivalence of norms of \( \nabla \alpha \) and \( H^\alpha \), we find that for \( \alpha \in (0, 1) \) and \( q \in [2, \frac{2}{1-\alpha}] \) \((q \in [2, \infty) \text{ if } \alpha = 1) \nabla \alpha \leftrightarrow L^q(\Omega)\), i.e.,

\[
\|u\|_{0,q} \leq C\|u\|_\alpha \quad \text{for } u \in \nabla \alpha.
\]

In particular we have for \( u = A^{-1}u \) and \( \alpha = 1 \),

\[
\|A^{-1}u\|_{L^q} \leq \|A^{-1/2}u\|, \quad 2 \leq q < \infty.
\]

Thorough out the paper we assume that \( f \) appearing in (1.1) - (1.5) satisfies following properties:

(A1) We assume that \( f \in C^2(\mathbb{R}) \) satisfies

\[
\begin{align*}
\lim_{|r| \to \infty} f'(r) &> 0, \\
|f''(r)| &\leq C_f(1 + |r|^{m-1}), \quad \forall r \in \mathbb{R},
\end{align*}
\]

where \( C_f \) is some positive constant and \( m \in [1, \infty) \) is fixed.
From (2.9) it follows that
\[ |f'(r)| \leq C_f(1 + |r|^m), \quad |f(r)| \leq C_f(1 + |r|^{m+1}), \quad \forall r \in \mathbb{R}. \] (2.10)

Let us define the following operators,

\[ b(u, v, w) = \langle B(u, v), w \rangle = \int_\Omega (u \cdot \nabla)v \cdot w, \quad \forall u, v, w \in D(A), \]

\[ b_1(u, \varphi, \psi) = \langle B_1(u, \varphi), \psi \rangle = \int_\Omega (u \cdot \nabla \varphi) \psi, \quad \forall u \in D(A), \varphi, \psi \in D(A_N), \]

\[ b_2(\mu, \varphi, w) = \langle B_2(\mu, \varphi), w \rangle = \int_\Omega \mu(\nabla \varphi \cdot w) \quad \forall \mu \in L^2(\Omega), \varphi \in D(A_N) \cap H^3(\Omega), \quad w \in D(A). \]

Also recall that using the properties of these operators and standard inequalities mentioned above we can deduce the following estimates for these operators, [19].

\[ \|B(u, v)\|_{\mathcal{V}_{\text{div}}} \leq C\|u\|^{1/2}\|\nabla u\|^{1/2}\|\nabla v\|, \]

\[ \|B(u, v)\| \leq C\|u\|^{1/2}\|\nabla u\|^{1/2}\|Av\|^{1/2}, \]

\[ |B_1(u, \varphi)|_{D(B_N^{1/2})} \leq C\|u\|^{1/2}\|\nabla u\|^{1/2}\|\nabla \varphi\|, \]

\[ \|B_1(u, \varphi)\| \leq C\|u\|^{1/2}\|\nabla u\|^{1/2}\|\nabla \varphi\|^{1/2}\|A_N\varphi\|^{1/2}, \]

\[ |B_2(A_N\varphi, \psi)|_{\mathcal{V}_{\text{div}}} \leq C\|A_N\varphi\|_{H^3}^{1/2}\|\nabla \psi\|, \]

\[ \|B_2(A_N\varphi, \psi)\| \leq C\|A_N\varphi\|_{H^3}^{1/2}\|\nabla \psi\|^{1/2}\|A_N\psi\|^{1/2}. \]

Using above defined operators and assuming that the external forcing term acts as a control we write the controlled system (1.1)-(1.5) in the abstract form as follows

\[ \frac{d\varphi}{dt} + B_1(u, \varphi) + A_N\mu = 0, \] (2.11)

\[ \mu = A_N\varphi + f(\varphi), \] (2.12)

\[ \frac{d\mathbf{u}}{dt} + A\mathbf{u} + B(u, u) - B_2(A_N\varphi, \varphi) = \mathbf{u}, \] (2.13)

\[ (\varphi, \mathbf{u})(\tau) = (\rho, \mathbf{v}). \] (2.14)

**Definition 2.6.** [30] Let \( \rho \in D(B_N^{1/2}) \) and \( \mathbf{v} \in \mathcal{G}_{\text{div}} \). Let \( \mathbf{U} \in L^2(\tau, T; \mathcal{G}_{\text{div}}) \). Then, a pair \((\varphi, \mathbf{u})\) is called a weak solution of (2.11)-(2.14) on \([\tau, T]\) if it satisfies (2.11)-(2.14) in a weak sense and

\[ \varphi \in C([\tau, T]; D(B_N^{1/2})) \cap L^2(\tau, T; D(B_N)), \quad \varphi_t \in L^2(\tau, T; D(B_N^{1/2})), \]

\[ \mathbf{u} \in C([\tau, T]; \mathcal{G}_{\text{div}}) \cap L^2(\tau, T; \mathcal{V}_{\text{div}}), \quad \mathbf{u}_t \in L^2(\tau, T; \mathcal{V}_{\text{div}}). \]

**Definition 2.7.** [30] If \( \rho \in D(B_N) \) and \( \mathbf{v} \in \mathcal{V}_{\text{div}} \), then a weak solution \((\varphi, \mathbf{u})\) is called a strong solution of the system (2.11)-(2.14) and it satisfies

\[ \varphi \in C([\tau, T]; D(B_N)) \cap L^2(\tau, T; D(B_N) \cap H^3(\Omega)) \]

\[ \mathbf{u} \in C([\tau, T]; \mathcal{V}_{\text{div}}) \cap L^2(\tau, T; D(A)) \]

**Theorem 2.8 (Proposition 2.1, [30]).** For \( \rho \in D(B_N^{1/2}), \mathbf{v} \in \mathcal{G}_{\text{div}}, F \in C^2(\mathbb{R}) \) and \( \mathbf{U} \in L^2(\tau, T; \mathcal{G}_{\text{div}}) \), the system (2.11)-(2.14) has a unique weak solution and the following estimates hold for all \( t \in [\tau, T] \)

\[ \|\varphi(t)\|_{D(B_N^{1/2})}^2 + \|\mathbf{u}(t)\|^2 + \int_\tau^t \|\mu(s)\|^2_{H^1} + \|\nabla \mathbf{u}(s)\|^2 ds \leq Q_0(\|\rho\|^2 + \|\mathbf{v}\|^2) + C \int_\tau^t \|\mathbf{U}(s)\|^2_{\mathcal{V}_{\text{div}}} ds, \]

\[ \int_\tau^t \|A_N\varphi(s)\|^2 + \|\varphi(s)\|^2_{H^3} ds \leq Q_0(\|\rho\|^2 + \|\mathbf{v}\|^2) + C \int_\tau^t \|\mathbf{U}(s)\|^2_{\mathcal{V}_{\text{div}}} ds, \]
\[ \int_{\tau}^{t} \left( \| \varphi(t) \|_{D(B(N))}^2 + \| u(t) \|_{V_{\text{div}}}^2 \right) dt \leq Q_0(\| \rho \|^2 + \| v \|^2) + C \int_{\tau}^{t} \| U(s) \|_{V_{\text{div}}}^2 ds. \]

where, \( Q_0 \) denotes a monotone non-decreasing function independent of time and the initial data.

**Theorem 2.9** (Proposition 2.2, [30]). If \( \rho \in D(B(N)), v \in V_{\text{div}}, \) then the system \((2.11)-(2.14)\) admits a unique strong solution and the solution satisfies

\[ \| \varphi(t) \|_{D(B(N))}^2 + \| u(t) \|_{V_{\text{div}}}^2 \leq C(\| \rho \|_{D(B(N))}^2 + \| u \|_{V_{\text{div}}}^2). \]

Using the techniques in Lemma 3.3 [19] we can prove the following result.

**Theorem 2.10.** Let \((\varphi_1, u_1)\) be the solution of \((2.11)-(2.14)\) corresponding to the initial data \((\varphi_i(\tau), u_i(\tau)) = (\rho_i, v_i), i = 1, 2.\) Then the following estimate holds

\[ \| \nabla(\varphi_1 - \varphi_2)(t) \|^2 + \| (u_1 - u_2)(t) \|_{V_{\text{div}}}^2 + \int_{\tau}^{t} \| \nabla(u_1 - u_2)(s) \|^2 ds + \int_{\tau}^{t} \| (\varphi_1 - \varphi_2)(s) \|_{H^2} ds \leq C e^{Lt}(\| \nabla(\rho_1 - \rho_2) \|^2 + \| v_1 - v_2 \|_{V_{\text{div}}}^2), \]

where \( C \) and \( L \) are positive constants depending only on the norms of the initial data, on \( \Omega \) and the parameters of the problem, but are both independent of time.

Now we establish continuous dependence results which will be used in the later sections.

**Theorem 2.11.** Let \((\varphi_1, u_1)\) and \((\varphi_2, u_2)\) be weak solutions of the system \((2.11)-(2.14)\) corresponding to initial data \((\rho_1, v_1) \) and \((\rho_2, v_2), \) respectively, where \((\rho_i, v_i) \in D(B^{1/2}_N) \times G_{\text{div}}, i = 1, 2.\) Then

\[ \| (\varphi_1 - \varphi_2)(t) \|^2 + \| (u_1 - u_2)(t) \|_{V_{\text{div}}}^2 + \int_{\tau}^{t} \| B_N(\varphi_1 - \varphi_2)(s) \|^2 ds + \int_{\tau}^{t} \| u_1(s) - u_2(s) \|^2 ds \]

\[ \leq C(\| \rho_1 - \rho_2 \|^2 + \| v_1 - v_2 \|_{V_{\text{div}}}^2). \] (2.15)

**Proof.** Let us denote by \( \varphi = \varphi_1 - \varphi_2 \) and \( u = u_1 - u_2. \) Then \((\varphi, u)\) satisfies the following system.

\[ \varphi_t + B_1(u, \varphi_1) + B_1(u_2, \varphi) = -A_N \mu, \] (2.16)

\[ \mu = B_N \varphi + (f(\varphi_1) - f(\varphi_2)), \] (2.17)

\[ u_t + A u + B(u_1, u_1) - B(u_2, u_2) = B_2(B_N \varphi, \varphi_1) + B_2(B_N \varphi_2, \varphi). \] (2.18)

Taking inner product of (2.16) with \( \varphi \) and (2.18) with \( A^{-1} u, \) using the properties of operators \( b, b_1 \) and \( b_2, \) and adding we get, for \( t \in [\tau, T], \)

\[ \frac{1}{2} \frac{d}{dt}(\| \varphi \|^2 + \| u \|_{V_{\text{div}}}^2) + \| u \|^2 = -b_1(u, \varphi_1, \varphi) - (A_N \mu, \varphi) - b(u, u_1, A^{-1} u) \]

\[ - b(u_2, u, A^{-1} u) + b_2(B_N \varphi, \varphi_1, A^{-1} u) + b_2(B_N \varphi_2, \varphi, A^{-1} u). \] (2.19)

We estimate the terms in (2.19) as follows. Observe that

\[ -(A_N \mu, \varphi) = -(A_N(B_N \varphi + f(\varphi_1) - f(\varphi_2)), \varphi) = -\| B_N \varphi \|^2 - (A_N(f(\varphi_1) - f(\varphi_2)), \varphi). \] (2.20)
and

\[
\|(A_N(f(\varphi_1) - f(\varphi_2)), \varphi) \| \leq \| f(\varphi_1) - f(\varphi_2) \| |A_N\varphi| \\
\leq \frac{1}{4} \| B_N \varphi \|^2 + C_f \| \varphi \|^2 \tag{2.21}
\]

Using Hölder's inequality and Agmon's inequality \[(2.6), we get \]

\[
|b_1(u, \varphi_1, \varphi)| \leq \|u\| \|\nabla \varphi_1\|_{L^\infty} \|\varphi\| \\
\leq \|u\| \|\nabla \varphi_1\|^{1/2} \|\nabla \varphi_1\|^{1/2}_{H^2} \|\varphi\| \\
\leq \frac{1}{6} \|u\|^2 + C \|\nabla \varphi_1\| \|\varphi_1\|_{H^3} \|\varphi\|^2 \\
\leq \frac{1}{6} \|u\|^2 + C \|\varphi\|^2 (\|\nabla \varphi_1\|^2 + \|\varphi_1\|_{H^3}^2), \tag{2.22}
\]

and

\[
|b(u, u_1, A^{-1}u)| = |b(u, A^{-1}u, u_1)| \leq \|u\| \|A^{-1/2}u\|_{L^4} \|u_1\|_{L^4} \\
\leq \|u\| \|A^{-1/2}u\|^{1/2} \|u\|^{1/2} \|\nabla u_1\|^{1/2} \|u_1\|^{1/2} \\
\leq \frac{1}{6} \|u\|^2 + C \|u\|^2 \|\nabla u_1\|^2 \|u_1\|^2. \tag{2.23}
\]

Similarly,

\[
|b_2(B_N\varphi, \varphi_1, A^{-1}u)| \leq \|B_N\varphi\| \|\varphi_1\|_{L^\infty} \|A^{-1}u\| \\
\leq \frac{1}{4} \|B_N\varphi\|^2 + C \|\varphi_1\|_{L^\infty}^2 \|u\|^2_{H^2}. \tag{2.25}
\]

By Poincare inequality since, \(\langle \varphi \rangle = 0\), and \[(2.8), we can estimate the following \]

\[
|b_2(B_N\varphi_2, \varphi, A^{-1}u)| = \|B_N\varphi_2\|_{L^4} \|\nabla \varphi\| \|A^{-1}u\|_{L^4} \\
\leq \|B_N\varphi_2\|^{1/2} \|B_N\varphi_2\|_{H^2}^{1/2} \|B_N\varphi\| \|u\|^2_{H^2} \\
\leq \frac{1}{4} \|B_N\varphi\|^2 + C \|u\|^2 \|\varphi_2\|_{H^2}^2. \tag{2.26}
\]

Substituting \[(2.20)-(2.26) in \(2.19), we get \]

\[
\frac{d}{dt}(\|\varphi(t)\|^2 + \|u(t)\|_{H^2}^2) + \frac{1}{2} \|u(t)\|^2 + \frac{1}{2} \|B_N\varphi(t)\|^2 \leq \alpha_1(t) \|\varphi(t)\|^2 + \beta_1(t) \|u(t)\|^2_{H^2} \tag{2.27}
\]

where \(\alpha_1(t) = C(1 + \|\nabla \varphi(t)\|^2 + \|\varphi(t)\|_{H^2}^2)\) \(\text{and} \beta_1(t) = C(\|\nabla u_1(t)\|^2 + \|u_1(t)\|^2 + \|\nabla u_2(t)\|^2 + \|u_2(t)\|^2 + \|\varphi_1(t)\|_{H^2}^2 + \|\varphi_2(t)\|_{H^2}^2).\)

Observe that, since \((\varphi_1, u_1)\) and \((\varphi_2, u_2)\) are weak solutions, from Theorem \[(2.8\text{ we have } \alpha_1(\cdot), \beta_1(\cdot) \in L^1(0, T).\ By applying Gronwall's lemma, we get required result, namely \[(2.15).\]

**Theorem 2.12.** Let \((\rho_1, v_1), (\rho_2, v_2) \in D(B_N) \times \mathbb{V}_{\text{div}}\) and \((\varphi_1, u_1), (\varphi_2, u_2)\) are corresponding strong solutions of the system \[(2.11)-(2.14), \text{ respectively. Then, } \varphi = \varphi_1 - \varphi_2, u = u_1 - u_2 \text{ satisfies:} \]

\[
\|\varphi(t)\|^2_{D(B_N)} + \|u(t)\|^2_{H^2} + \frac{\nu}{2} \int_t^\tau \|Au(s)\|^2 ds \\
+ \int_t^\tau \left(\frac{1}{2} \|B_N^2 \varphi(s)\|^2 + \frac{1}{2} \|B_N \bar{\mu}(s)\|^2\right) ds \leq C(\|\rho_1 - \rho_2\|^2_{D(B_N)} + \|v_1 - v_2\|^2). \tag{2.28}
\]
Proof. As in the previous proof \((\varphi, u)\) satisfies
\begin{align*}
\varphi_t + B_1(u, \varphi_1) + B_1(u_2, \varphi) &= -B_N \bar{\mu}, \\
\bar{\mu} &= B_N \varphi + (f(\varphi_1) - f(\varphi_2)) - \langle \mu \rangle, \\
u_t + Au + B(u, u_1) + B(u_2, u) &= B_2(B_N \varphi, \varphi_1) + B_2(B_N \varphi_2, \varphi).
\end{align*}
(2.29) (2.30) (2.31)
Taking \(L^2\) inner product of (2.29) with \(B_N \varphi\), (2.30) with \(B_N \bar{\mu}\) and \(B_N^2 \varphi\), and (2.31) with \(Au\), and adding we get
\begin{align*}
\frac{1}{2} \frac{d}{dt} (\|B_N \varphi\|^2 + \|\nabla u\|^2) + \|B_{N}^{3/2} \varphi\|^2 + \|B_N \bar{\mu}\|^2 + \nu \|Au\|^2
&= -b(u, u_1, Au) - b(u_2, u, Au) + b(B_N \varphi, \varphi_1, Au) + b(B_N \varphi_2, \varphi, Au) - b_1(u, \varphi_1, B_N^2 \varphi) \\
&- b_1(u_2, \varphi, B_N^2 \varphi) + (f(\varphi_1) - f(\varphi_2), B_N^2 \bar{\mu}) + (B_N \varphi, B_N \bar{\mu}) - (f(\varphi_1 - f(\varphi_2), B_N^2 \varphi)).
\end{align*}
(2.32)
We estimate first two terms on the right hand side of (2.32) using the properties of \(b\) and Young’s inequality and Agmon’s inequality as follows
\begin{align*}
|b(u, u_1, Au)| &\leq \|u\|_{L^1} \|u_1\|_{L^1} \|Au\| \\
&\leq \|u\|^{1/2} \|\nabla u\|^{1/2} \|u_1\|^{1/2} \|Au\|^{1/2} \|\nabla u\|^2, \\

|b(u_2, u, Au)| &\leq \|u_2\| \|\nabla u\| \|Au\| \\
&\leq \frac{\nu}{10} \|Au\|^2 + C \|\nabla u\| \|u_2\| \|\nabla u\|^2.
\end{align*}
(2.33) (2.34)
Using the Gagliardo-Nirenberg inequality
\begin{align*}
|b_2(B_N \varphi, \varphi, Au)| &\leq \|B_N \varphi\| \|\nabla \varphi\| \|Au\| \\
&\leq \frac{\nu}{10} \|Au\|^2 + C \|B_N \varphi\| \|B_N \varphi\| \|\nabla \varphi\| \|B_N \varphi\| \\
&\leq \frac{\nu}{10} \|Au\|^2 + C \|\varphi\|_{H^3} \|B_N \varphi\|^2.
\end{align*}
(2.35)
Using Hölder’s inequality and Agmon’s inequality we get
\begin{align*}
|b_2(B_N \varphi, \varphi_1, Au)| &\leq \|B_N \varphi\| \|\nabla \varphi_1\| \|Au\| \\
&\leq \frac{\nu}{10} \|Au\|^2 + C \|B_N \varphi\|^2 \|\varphi_1\|_{H^1} \|\varphi_1\|_{H^3}.
\end{align*}
(2.36)
Using integration by parts and Hölder’s inequality with exponents (2, 2) and \((4, \frac{4}{3})\) we estimate the following
\begin{align*}
|b_1(u_2, \varphi, B_N^2 \varphi)| &\leq C \|B_N^{1/2} B_1(u_2, \varphi)\| \|B_N^{3/2} \varphi\| \\
&\leq C (\|\nabla u_2\|_{L^1} \|\nabla \varphi\|_{L^1} + \|u_2\|_{L^1} \|B_N \varphi\|_{L^1}) \|B_N^{3/2} \varphi\| \\
&\leq C \|\nabla u_2\|^{1/2} \|Au_2\|^{1/2} \|\nabla \varphi\|^{1/2} \|B_N \varphi\|^{1/2} \|B_N^{3/2} \varphi\| \\
&+ C \|u_2\|^{1/2} \|\nabla u_2\|^{1/2} \|B_N \varphi\|^{1/2} \|B_N^{3/2} \varphi\|^3/2 \\
&\leq \frac{1}{8} \|B_N^{3/2} \varphi\|^2 + C \|\nabla u_2\| \|Au_2\| \|B_N \varphi\|^2 + \frac{1}{8} \|B_N^{3/2} \varphi\|^2 + C \|u_2\|^2 \|\nabla u_2\|^2 \|B_N \varphi\|^2 \\
&\leq \frac{1}{4} \|B_N^{3/2} \varphi\|^2 + C (\|\nabla u_2\| \|Au_2\| + \|u_2\|^2 \|\nabla u_2\|^2) \|B_N \varphi\|^2.
\end{align*}
(2.37)
Similarly,
\[
|b_1(u, \varphi_1, B_N^2 \varphi)| \leq C\|B_N^{1/2} B_1(u, \varphi_1)\| \|B_N^{3/2} \varphi\|
\]
\[
\leq \frac{1}{4} \|B_N^{3/2} \varphi\|^2 + C(\|\nabla u\|_{L^4} \|\nabla \varphi_1\|_{L^4} + \|u\|_{L^4} \|B_N \varphi_1\|_{L^4})^2
\]
\[
\leq \frac{1}{4} \|B_N^{3/2} \varphi\|^2 + C(\|\nabla u\| \|A u\| \|\varphi_1\|_{H^1} \|\varphi_1\|_{H^2} + \|u\| \|\nabla u\| \|\varphi_1\|_{H^2} \|\varphi_1\|_{H^3})
\]
\[
\leq \frac{1}{4} \|B_N^{3/2} \varphi\|^2 + \frac{\nu}{10} \|A u\|^2 + \|\nabla u\|^2 (\|\varphi_1\|_{H^1}^2 \|\varphi_1\|_{H^2}^2 + \|\varphi_1\|_{H^2} \|\varphi_1\|_{H^3}). \tag{2.38}
\]

Rest of the terms are estimated as follows
\[
|(B_N \varphi, B_N \bar{\mu})| \leq \|B_N \varphi\| \|B_N \bar{\mu}\| \leq \frac{1}{4} \|B_N \bar{\mu}\|^2 + C \|B_N \varphi\|^2, \tag{2.39}
\]
\[
|(f(\varphi_1) - f(\varphi_2), B_N^2 \bar{\mu})| = |(A_N (f(\varphi_1) - f(\varphi_2)), B_N \bar{\mu})|
\]
\[
\leq \frac{1}{4} \|B_N \bar{\mu}\|^2 + C \|B_N \varphi\|^2, \tag{2.40}
\]
\[
|(f(\varphi_1) - f(\varphi_2), B_N^2 \varphi)| = (A_N (f(\varphi_1) - f(\varphi_2)), B_N \varphi)
\]
\[
\leq C \|A_N (f(\varphi_1) - f(\varphi_2))\| \|B_N \varphi\|
\]
\[
\leq C \|B_N \varphi\|^2. \tag{2.41}
\]

Substituting (2.33)-(2.41) in (2.32), we get
\[
\frac{1}{2} \frac{d}{dt} (\|B_N \varphi(t)\|^2 + \|\nabla u(t)\|^2) + \frac{1}{2} \|B_N^{3/2} \varphi(t)\|^2 + \frac{1}{2} \|B_N \bar{\mu}(t)\|^2 + \frac{\nu}{2} \|A u(t)\|^2
\]
\[
\leq \alpha_2(t) \|\nabla u(t)\|^2 + \beta_2(t) \|B_N \varphi(t)\|^2,
\]
where \(\alpha_2(t) = C(\|\nabla u_1\| \|A u_1\| + \|u_2\| \|u_2\|_{H^2} + \|\varphi_1\|_{H^1}^2 \|\varphi_1\|_{H^2}^2 + \|\varphi_1\|_{H^2} \|\varphi_1\|_{H^3})\) and \(\beta_2(t) = C(\|\varphi_2\|_{H^3}^2 + \|\varphi_1\|_{H^1} \|\varphi_1\|_{H^3} + \|\nabla u_2\| \|A u_2\| + \|u_2\|^2 \|\nabla u_2\|^2 + 1)\). Integrating (2.42) from \(\tau\) to \(t\), we get
\[
\|B_N \varphi(t)\|^2 + \|\nabla u(t)\|^2 + \int_\tau^t \left( \frac{1}{2} \|B_N^{3/2} \varphi(s)\|^2 + \frac{1}{2} \|B_N \bar{\mu}(s)\|^2 + \frac{\nu}{2} \|A u(s)\|^2 \right) \, ds
\]
\[
\leq \|B_N \varphi(\tau)\|^2 + \|\nabla u(\tau)\|^2 + \int_\tau^t (\alpha_2(s) \|\nabla u(s)\|^2 + \beta_2(s) \|B_N \varphi(s)\|^2) \, ds \tag{2.43}
\]

Observe that from Theorem 2.8 we have that \(\alpha_2(\cdot), \beta_2(\cdot) \in L^1(\tau, T)\). By applying Gronwall’s lemma, we deduce (2.28). \(\square\)

The following Proposition is useful in proving smoothness properties of the value function.

**Proposition 2.13.** Let \((\rho, v) \in (D(B_N) \cap H^3) \times \mathbb{V}_{\text{div}}^N\) and \(U \in L^2(\tau, T; \mathcal{U}_R)\). Let \((\varphi, u)\) be the strong solution corresponding to the initial data \((\varphi(\tau), u(\tau)) = (\rho, v)\). Then the following holds:
\[
\|\nabla (\varphi(t) - \rho)\|^2 + \|u(t) - v\|^2 + \int_\tau^t (\|B_N \xi(s)\|^2 + \nu \|\nabla z(s)\|^2 + \|\nabla \bar{\mu}(s)\|^2) \, ds \leq C(t - \tau) \tag{2.44}
\]

**Proof.** Let \((\varphi, u)\) be the strong solution of the system (2.11)-(2.14) with with control \(U\) and initial data \((\varphi(\tau), u(\tau)) = (\rho, v)\). Let us denote \(\xi = \varphi(t) - \rho\) and \(z = u(t) - v\). Then \((\xi, z)\) satisfies
\[
\frac{d\xi}{dt} + B_1(z, \varphi) + B_1(v, \varphi) + A_N \mu = 0. \tag{2.45}
\]
\[
\mu = B_N \xi + f(\varphi) + B_N \rho \tag{2.46}
\]
\[
\frac{dz}{dt} + \nu Az = -\nu Av - B(z, u) - B(v, z) - B(v, v) + B_2(B_N\xi, \varphi) + B_2(B_N\rho, \varphi) + U \tag{2.47}
\]

Equivalently,
\[
\frac{d\xi}{dt} + B_1(z, \varphi) + B_1(v, \varphi) + B_N\bar{\mu} = 0. \tag{2.48}
\]
\[
\bar{\mu} = B_N\xi + f(\varphi) + B_N\rho - \langle \mu \rangle \tag{2.49}
\]
\[
\frac{dz}{dt} + \nu Az = -\nu Av - B(z, u) - B(v, z) - B(v, v) + B_2(B_N\xi, \varphi) + B_2(B_N\rho, \varphi) + U. \tag{2.50}
\]

Now take inner product of \((2.48)\) with \(B_N\xi\), \((2.49)\) with \(B_N\bar{\mu} - B_N\xi\) and \((2.50)\) with \(z\). By adding we get
\[
\frac{1}{2} \frac{d}{dt}(\|\nabla \xi\|^2 + \|z\|^2) + \|B_N\xi\|^2 + \|\nabla z\|^2 + \|\nabla \bar{\mu}\|^2
\]
\[
= -(B(v, \varphi), B_N\xi) + (\mu, B_N\xi) + (f(\varphi), B_N\mu) + (B_N\rho, B_N\mu) + (f(\varphi), B_N\xi) + (B_N\rho, B_N\xi)
\]
\[
- (Av, z) - (B(z, u), z) - (B(v, v), z) + (B_2(B_N\xi, \varphi), z) + (B_2(B_N\rho, \varphi), z) + (U, z). \tag{2.51}
\]

Now we estimate the right hand side by one by one using Hölder’s, Young’s and Gagliardo-Nirenberg inequalities
\[
|(B(v, \varphi), B_N\xi)| \leq C\|v\|_{L^4}\|\nabla \varphi\|_{L^4}\|B_N\xi\|
\]
\[
\leq \frac{1}{8}\|B_N\xi\|^2 + C\|\nabla v\|^2\|\nabla \varphi\|^2_{L^4}
\]
\[
\leq \frac{1}{8}\|B_N\xi\|^2 + C\|\nabla v\|^2\|\varphi\|^2_{H^2}, \tag{2.52}
\]
\[
|(\bar{\mu}, B_N\xi)| = |(\nabla \bar{\mu}, \nabla \xi)| \leq \frac{1}{4}\|\nabla \bar{\mu}\|^2 + C\|\nabla \xi\|^2, \tag{2.53}
\]
\[
|(f(\varphi), B_N\bar{\mu})| = |(\nabla f(\varphi), \nabla \bar{\mu})| \leq \frac{1}{4}\|\nabla \bar{\mu}\|^2 + C\|\nabla \varphi\|^2, \tag{2.54}
\]
\[
|(B_N\rho, B_N\bar{\mu})| \leq \|(B_N^{3/2}\rho, \nabla \bar{\mu})\| \leq \frac{1}{4}\|\nabla \bar{\mu}\|^2 + C\|\rho\|^2_{H^3}, \tag{2.55}
\]
\[
|(f(\varphi), B_N\xi)| \leq \|f(\varphi)\|\|B_N\xi\| \leq \frac{1}{8}\|B_N\xi\|^2 + C\|f(\varphi)\|^2, \tag{2.56}
\]
\[
|(B_N\rho, B_N\xi)| \leq \|B_N\rho\|\|B_N\xi\| \leq \frac{1}{8}\|B_N\xi\|^2 + C\|B_N\rho\|^2, \tag{2.57}
\]
\[
\nu|(Av, z)| \leq \nu\|\nabla v\|\|\nabla z\| \leq \frac{\nu}{10}\|\nabla z\|^2 + C\|\nabla v\|^2, \tag{2.58}
\]
\[
|(B(z, u), z)| \leq C\|z\|\|\nabla u\|\|\nabla z\| \leq \frac{\nu}{10}\|\nabla z\|^2 + C\|z\|^2\|\nabla u\|^2, \tag{2.59}
\]
\[
|(B(v, v), z)| \leq C\|v\|\|\nabla v\|\|\nabla z\| \leq \frac{\nu}{10}\|\nabla z\|^2 + C\|v\|^2\|\nabla v\|^2, \tag{2.60}
\]
\[
|(B_2(B_N\xi, \varphi), z)| \leq \|B_N\xi\|\|\nabla \varphi\|_{L^4}\|z\|_{L^4}
\]
\[
\leq C\|B_N\xi\|\|\nabla \varphi\|^{1/2}\|B_N\varphi\|^{1/2}\|z\|^{1/2}\|\nabla z\|^{1/2}
\]
\[
\leq \frac{1}{8}\|B_N\xi\|^2 + C\|\nabla \varphi\|\|B_N\varphi\|\|z\|\|\nabla z\|
\]
\[
\leq \frac{1}{8}\|B_N\xi\|^2 + \frac{\nu}{10}\|\nabla z\|^2 + C\|\nabla \varphi\|^2\|B_N\varphi\|^2\|z\|^2 \tag{2.61}
\]
Theorem 2.14. Substituting (2.52)-(2.64) in (2.51), we get
\begin{align*}
|\langle B_2(B_N \rho, \varphi), z \rangle| & \leq \|B_N \rho\|_{L^4} \|\nabla \varphi\| \|z\|_{L^4} \\
& \leq C\|\rho\|_{H^3} \|\nabla \varphi\| \|z\| \\
& \leq \frac{\nu}{10} \|\nabla z\|^2 + C\|\rho\|^2_{H^3} \|\nabla \varphi\|^2, \quad (2.62)
\end{align*}
\begin{align*}
|\langle B_2(B_N \rho, \xi), z \rangle| & \leq C\|B_N \rho\| \|\nabla \xi\|_{L^4} \|z\|_{L^4} \\
& \leq C\|B_N \rho\| \|\nabla \xi\| \|B_N \xi\| \|\nabla z\| \\
& \leq \frac{\nu}{10} \|\nabla z\|^2 + C\|B_N \rho\|^2 \|\nabla \xi\| \|B_N \xi\| \\
& \leq \frac{\nu}{10} \|\nabla z\|^2 + \frac{\delta}{8} \|B_N \xi\|^2 + C\|B_N \rho\|^4 \|\nabla \xi\|^2, \quad (2.63)
\end{align*}
\begin{align*}
|\langle U, z \rangle| & \leq \|U\| \|z\| \leq C\|U\| \|\nabla z\| \leq \frac{\nu}{10} \|\nabla z\|^2 + C\|U\|^2, \quad (2.64)
\end{align*}
Substituting (2.52)–(2.64) in (2.51), we get
\begin{align*}
\frac{1}{2} \frac{d}{dt} (\|\nabla \xi\|^2 + \|z\|^2) + \frac{1}{2} \|B_N \xi\|^2 + \frac{\nu}{2} \|\nabla z\|^2 + \frac{1}{2} \|\nabla \mu\|^2 & \leq \alpha_3(t) \|\nabla \xi\|^2 + \beta_3(t) \|z\|^2 + h_1(t) \quad (2.65)
\end{align*}
where \(\alpha_3(t) = C(1 + \|B_N \rho\|_4)^4, \beta_3(t) = C(\|\nabla u\|^2 + \|\nabla \varphi\|^2 \|B_N \varphi\|^2)\) and \(h_1(t) = C(\|\nabla v\|^2 \|\varphi\|^2_{H^2} + \|\nabla \varphi\|^2 + \|\rho\|^2_{H^3} + f(\varphi) + \|B_N \rho\|^2 + \|\nabla v\|^2 + \|v\|^2 \|\nabla v\|^2 + \|\rho\|^2_{H^3} \|\nabla \varphi\|^2 + \|U\|^2)\). Integrating (2.65) from \(t\) to \(\tau\) we get
\begin{align*}
\|\nabla \xi(t)\|^2 + \|z(t)\|^2 + \int_\tau^t (\|B_N \xi(s)\|^2 + \nu \|\nabla z(s)\|^2 + \|\nabla \mu(s)\|^2)ds \\
& \leq \int_\tau^t (\alpha_3(s) \|\nabla \xi(s)\|^2 + \beta_3(s) \|z(s)\|^2)ds + \int_\tau^t h_1(s)ds
\end{align*}
Now, applying Gronwall’s lemma and using Theorem 2.18 we get (2.44) since \(\alpha_3(\cdot), \beta_3(\cdot), h_1(\cdot) \in L^1(\tau, T)\).

Theorem 2.14. Let \((\rho, v) \in (D(B_N) \cap H^4) \times (V_{d\nabla} \times \mathbb{R}^3)\) and \(U \in L^2(\tau, T; U_R)\). Let \((\varphi, u)\) be a solution of the system with the initial data \((\varphi(\tau), u(\tau)) = (\rho, v)\). Then the following estimate holds:
\begin{align*}
\|B_N (\varphi(t) - \rho)\|^2 + \|\nabla (u(t) - v)\|^2 + \int_\tau^t (\|B_N^{3/2} \xi(s)\|^2 + \|B_N \mu(s)\|^2 + \nu \|A z(s)\|^2)ds & \leq C(t - \tau) \quad (2.66)
\end{align*}
Proof. Taking inner product of (2.48) with \(B_N^2 \xi\) and (2.49) with \(B_N^2 \mu\) + \(B_N^2 \xi\), and (2.50) with \(Az\), we arrive at
\begin{align*}
\frac{1}{2} \frac{d}{dt} (\|B_N \xi\|^2 + \|\nabla z\|^2) + \|B_N^{3/2} \xi\|^2 + \|B_N \mu\|^2 + \nu \|A z\|^2 & = -(B_1(z, \varphi), B_N^2 \xi) - (B_1(v, \varphi), B_N \xi) \\
& \quad + (B_1(\rho, \mu), B_N \xi) + (f(\varphi) + B_N \rho, B_N^2 \mu) - (f(\varphi) + A_N \rho, B_N^2 \xi) - \nu (Av, Az) - (B(v, z), Az) \\
& \quad - (B(\varphi, z), Az) - (B(v, v), Az) + (B_2(B_N \xi, \varphi), Az) + (B_2(B_N \rho, \varphi), Az) + (U, Az) \quad (2.67)
\end{align*}
Now we estimate the right hand side terms in (2.67) as follows. Using Gagliardo-Nirenberg and Hölder’s inequality we estimate the following
\begin{align*}
|\langle B_1(\rho, \varphi), B_N^{3/2} \xi \rangle| & \leq C \|B_N^{1/2} B_1(\rho, \varphi)\| \|B_N^{3/2} \xi\| \\
& \leq C \|\nabla z\|_{L^4} \|\nabla \varphi\|_{L^4} \|B_N^{3/2} \xi\| + C \|z\|_{L^4} \|\Delta \varphi\|_{L^4} \|B_N^{3/2} \xi\| \\
& \leq C \|B_N^{3/2} \xi\| (\|\nabla z\|_{L^4} \|A z\| \|\varphi\|_{L^4} \|\varphi\|_{L^4} + \|z\| \|\nabla z\| \|\varphi\|_{L^4} \|\varphi\|_{L^4}) \\
& \leq \frac{\delta}{10} \|B_N^{3/2} \xi\|^2 + \|\nabla z\| \|A z\| \|\varphi\|_{H^2} \|\varphi\|_{H^2} + \|z\| \|\nabla z\| \|\varphi\|_{H^2} \|\varphi\|_{H^2}
\end{align*}
\[
\leq \frac{\delta}{10} \|B_N^{3/2} \xi\|^2 + \frac{\nu}{16} \| Az\|^2 + \| \nabla z\|^2 (\| \varphi\|^2_{H^1} + \| \varphi\|_{H^2} + \| \varphi\|_{H^3}^2), \tag{2.68}
\]
\[
| (B_1(v, \varphi), B_N^2 \xi)| \leq C \| B_N^{1/2} B_1(v, \varphi) \| B_N^{3/2} \xi\|
\leq \| B_N^{3/2} \xi\| (\| v\|_{L^2} \| \nabla \varphi\|_{L^2} + \| \nabla v\|_{L^2} \| \nabla \varphi\|_{L^2})
\leq \| B_N^{3/2} \xi\| (\| v\|_{L^2} \| \nabla \varphi\|_{L^2} + \| \nabla v\|_{L^2} \| \varphi\|_{H^1}^2 + \| \nabla v\|_{L^2} \| \varphi\|_{H^2} + \| \nabla v\|_{L^2} \| \varphi\|_{H^3}^2)
\leq \frac{\delta}{10} \| B_N^{3/2} \xi\|^2 + \| v\| \| \nabla v\| \| \varphi\|_{H^1} \| \varphi\|_{H^2} + \| \nabla v\| \| \varphi\|_{H^3}^2, \tag{2.69}
\]
\[
| (B_N \bar{\mu}, B_N \xi)| \leq \| B_N \bar{\mu}\| \| B_N \xi\|
\leq \frac{1}{4} \| B_N \bar{\mu}\|^2 + C \| B_N \xi\|^2, \tag{2.70}
\]
\[
| (f(\varphi) + B_N \rho, B_N^2 \bar{\mu})| \leq | (f(\varphi), B_N^2 \bar{\mu})| + | (B_N \rho, B_N^2 \bar{\mu})|
\leq | (A_N f(\varphi), B_N \bar{\mu})| + \| B_N^2 \rho\| \| B_N \bar{\mu}\|
\leq \frac{1}{8} \| B_N \bar{\mu}\|^2 + C \| \varphi\|_{H^2}^2 + \frac{1}{8} \| B_N \bar{\mu}\|^2 + C \| B_N^2 \rho\|^2,
\leq \frac{1}{4} \| B_N \bar{\mu}\|^2 + C (\| \varphi\|_{H^2}^2 + \| B_N^2 \rho\|^2), \tag{2.71}
\]
\[
| (f(\varphi) + B_N \rho, B_N^2 \xi)| = | (B_N^{1/2} (f(\varphi) + B_N \rho), B_N^{3/2} \xi)|
\leq \frac{1}{10} \| B_N^{3/2} \xi\|^2 + C \| \varphi\|_{H^1}^2 + \| \rho\|_{H^3}^2, \tag{2.72}
\]
where we used the fact that \( f \in C^2(\mathbb{R}) \) in the above two estimates. Using Hölder’s and Ladyzhenskaya’s inequalities at appropriate places we estimate the following
\[
| \nu(Av, Az) | \leq \nu \| Av\| \| Az\| \leq \frac{\nu}{16} \| Az\|^2 + C \| Av\|^2, \tag{2.73}
\]
\[
| (B(z, u), Az) | \leq \| z\|^{1/2} \| \nabla z\|^{1/2} \| u\|^{1/2} \| u\|^{1/2} \| \nabla u\| \| Az\|
\leq \frac{\nu}{16} \| Az\|^2 + \| \nabla z\|^2 \| \nabla u\| \| Az\|, \tag{2.74}
\]
\[
| (B(v, z), Az) | \leq \| v\|_{L^4} \| \nabla z\|_{L^4} \| Az\|
\leq C \| v\|_{L^4} \| \nabla v\|^{1/2} \| \nabla z\|^{1/2} \| Az\|^{1/2} \| Az\|
\leq \frac{\nu}{16} \| Az\|^2 + \| v\|^2 \| \nabla v\|^2 \| \nabla z\|^2, \tag{2.75}
\]
\[
| (B(v, v), Az) | \leq \| v\|_{L^4} \| \nabla v\|_{L^4} \| Az\|
\leq \frac{\nu}{16} \| Az\|^2 + \| v\| \| \nabla v\|^2 \| Az\|, \tag{2.76}
\]
\[
| (B_2(B_N \xi, \varphi), Az) | \leq \| B_2(B_N \xi, \varphi) \| \| Az\|
\leq \| B_N \xi\|_{L^4} \| \nabla \varphi\|_{L^\infty} \| Az\|
\leq \frac{\nu}{16} \| Az\|^2 + C \| \varphi\|_{H^1} \| \varphi\|_{H^3} \| B_N \xi\|^2, \tag{2.77}
\]
\[
| (B_2(B_N \rho, \varphi), Az) | \leq \| B_2(B_N \rho, \varphi) \| \| Az\|
\leq \| B_N \rho\|^{1/2} \| B_N^{3/2} \rho\|^{1/2} \| \nabla \varphi\|_{H^2} \| \varphi\|_{H^3}^{1/2} \| Az\|. \tag{2.78}
\]
\[ \leq \frac{\nu}{16} \| Az \|^2 + C \| \varphi \|_{H^1} \| \varphi \|_{H^2} \| B_N \rho \|_{H^3}, \]  

(2.78)

and

\[ |(U, Az)| \leq \| U \| \| Az \| \leq \frac{\nu}{16} \| Az \|^2 + C \| U \|^2. \]  

(2.79)

Substituting (2.68)-(2.79) in (2.67), we get

\[ \frac{1}{2} \int_0^T (\| B_N \xi \|^2 + \| \nabla z \|^2) + \frac{1}{2} \| B_N^2 \xi \|^2 + \frac{1}{2} \| B_N \bar{u} \|^2 + \frac{\nu}{2} \| Az \|^2 \leq \alpha_4(t) \| B_N \xi \|^2 + \beta_4(t) \| \nabla z \|^2 + h_2(t) \]

where \( \alpha_4(t) = C(1 + \| \varphi \|_{H^1} \| \varphi \|_{H^2} + \| \varphi \|_{H^1} \| \varphi \|_{H^3}, \beta_4(t) = C(\| \varphi \|_{H^1}^2 \| \varphi \|_{H^2}^2 + \| \varphi \|_{H^2} \| \varphi \|_{H^3} + \| \nabla u \|_{H^1} \| \nabla u \|_{H^2}) \) and \( h_2(t) = \| \nabla v \|_{H^2} \| \varphi \|_{H^1} + \| \nabla v \|_{H^2} \| \varphi \|_{H^2} + \| \nabla v \|_{H^1} \| \varphi \|_{H^2} + \| B_N^2 \rho \|^2 + \| \varphi \|_{H^1}^2 + \| \rho \|_{H^3}^2 + \| A v \|^2 + \| \nabla v \|^2 \| A v \| + \| \varphi \|_{H^1} \| \varphi \|_{H^2} \| B_N \rho \|_{H^3} + \| U \|^2). \]

Since \( \alpha_4(\cdot), \beta_4(\cdot), h_2(\cdot) \in L^1(\tau, T) \) from Theorem 2.8 and Theorem 2.9 (2.66) follows from Gronwall's lemma.

\[ \Box \]

3. Dynamic Programming Principle

In this section, we formulate the optimal control problem as minimization of a suitable cost functional subject to the controlled Cahn-Hilliard-Navier-Stokes system and describe the dynamic programming principle. Let us define the cost functional

\[ J(\tau, \rho, v, U) = \frac{1}{2} \int_\tau^T (\| \varphi(t) \|^2 + \| u(t) \|^2 + \| U(t) \|^2) dt + \frac{\nu}{2} (\| \varphi(T) \|^2 + \| u(T) \|^2), \]  

(3.1)

where \( U \in U_{ad} \) and \( U_{ad} \) is a closed and convex subset of \( L^2(\tau, T; \mathbb{G}_{\text{div}}) \) containing 0. Observe that \( U_{ad} \) is non-empty. We formulate the optimal control problem as

\[ \inf_{U \in U_{ad}} J(\tau, \rho, v, U) \]  

(3.2)

A solution to the problem (3.2) is called an optimal control and \((\varphi^*, u^*, U^*)\) is called optimal triplet where \((\varphi^*, u^*)\) is the strong solution of the system (2.11)-(2.14) with optimal control \( U^* \) and initial data \((\varphi(\tau), u(\tau)) = (\rho, v)\). The following theorem gives the existence of the optimal control problem.

**Theorem 3.1 (Theorem 3.1, [29])**. Let \((\rho, v) \in D(B_N^{1/2}) \times \mathbb{G}_{\text{div}}\). The optimal control problem (3.2) admits a solution.

We consider the optimal control problem (3.2) with \( U_{ad} = L^2(\tau, T; U_R) \), where

\[ U_R := \{ U \in \mathbb{G}_{\text{div}} : \| U \| \leq R \}. \]

Moreover, we take \((\varphi(\tau), u(\tau)) \in \big( D(B_N^{1/2}) \cap H^1 \big) \times \big( \mathbb{G}_{\text{div}} \cap H^2 \big)\). Let us define a value function

\[ W : [0, T] \times D(B_N^{1/2}) \times \mathbb{G}_{\text{div}} \rightarrow \mathbb{R} \]  

as

\[ W(\tau, \rho, v) = \inf_{U \in U_{ad}} J(\tau, \rho, v, U). \]  

(3.3)

**3.1. Continuity properties of a value function.** In the following theorem we present continuity properties of the value function (3.3).

**Theorem 3.2.** For every \( K > 0 \) there exists a constant \( C_K \) such that the value function defined in (3.3) satisfies

\[ |W(t_1, \rho_1, v_1) - W(t_2, \rho_2, v_2)| \leq C_K \left( |t_1 - t_2|^{1/2} + \| \rho_1 - \rho_2 \| + \| v_1 - v_2 \|_{v_1'} \right) \]  

(3.4)

for \( t_1, t_2 \in [0, T] \) and \( \| \rho_1 \|_{D(B_N^{1/2})}, \| \rho_2 \|_{D(B_N^{1/2})}, \| v_1 \|, \| v_2 \| \leq K. \)
Proof. Let \((\varphi_2, u_2, U)\) be an optimal triplet for the initial data \((\tau, \rho_2, v_2)\) which solves (3.2). For the control \(U\), let \((\varphi_1, u_1)\) be corresponding solution with initial data \((\tau, \rho_1, v_1)\). Note that \(U\) need not be an optimal control for the system with the initial data \((\tau, \rho_1, v_1)\). For \(\|\rho_1\|_{D(B_H^{1/2})}, \|\rho_2\|_{D(B_H^{1/2})}, \|v_1\|, \|v_2\| \leq K\) we have,
\[
W(t, \rho_1, v_1) - W(t, \rho_2, v_2) = \inf_{U \in U_{ad}} J(t, \rho_1, v_1, U) - \inf_{U \in U_{ad}} J(t, \rho_2, v_2, U)
\]
\[
\leq \frac{1}{2} \int_{\tau}^{T} (\|\varphi_1(t)\|^2 - \|\varphi_2(t)\|^2) + \frac{1}{2} \int_{\tau}^{T} (\|u_1(t)\|^2 - \|u_2(t)\|^2)
\]
\[
+ \frac{1}{2} (\|\varphi_1(T)\|^2 - \|\varphi_2(T)\|^2) + \frac{1}{2} (\|u_1(T)\|^2 - \|u_2(T)\|^2).
\]
Now consider
\[
\int_{\tau}^{T} (\|\varphi_1(t)\|^2 - \|\varphi_2(t)\|^2) + \int_{\tau}^{T} (\|u_1(t)\|^2 - \|u_2(t)\|^2)
\]
\[
\leq \int_{\tau}^{T} (\|\varphi_1(t)\| - \|\varphi_2(t)\|)(\|\varphi_1(t)\| + \|\varphi_2(t)\|) + \int_{\tau}^{T} (\|u_1(t)\| - \|u_2(t)\|)(\|u_1(t)\| + \|u_2(t)\|)
\]
\[
\leq \int_{\tau}^{T} (\|\varphi_1(t) - \varphi_2(t)\|)(\|\varphi_1(t)\| + \|\varphi_2(t)\|)ds + \int_{\tau}^{T} (\|u_1(t) - u_2(t)\|)(\|u_1(t)\| + \|u_2(t)\|)ds
\]
\[
\leq (T - \tau) \sup_{t \in [\tau, T]} (\|\varphi_1(t) - \varphi_2(t)\|) \sup_{t \in [\tau, T]} (\|\varphi_1(t)\| + \|\varphi_2(t)\|)
\]
\[
+ \left(\int_{\tau}^{T} (\|u_1(t) - u_2(t)\|^2)ds\right)^{1/2} \left(\int_{\tau}^{T} (\|u_1(t)\|^2 + \|u_2(t)\|^2)\right)^{1/2}
\]
Hence, from Theorem 2.8 and Theorem 2.11 we get
\[
\int_{\tau}^{T} (\|\varphi_1(t)\|^2 - \|\varphi_2(t)\|^2) + \int_{\tau}^{T} (\|u_1(t)\|^2 - \|u_2(t)\|^2)
\]
\[
\leq C (\|\nabla \rho_1\|, \|\nabla \rho_2\|, \|v_1\|, \|v_2\|, \tau, T)(\|\rho_1 - \rho_2\| + \|v_1 - v_2\|_{V'_{\text{div}}}). \quad (3.5)
\]
Similarly we have
\[
\|\varphi_1(T)\|^2 - \|\varphi_2(T)\|^2 + \|u_1(T)\|^2 - \|u_2(T)\|^2
\]
\[
\leq (\|\varphi_1(T)\| - \|\varphi_2(T)\|)(\|\varphi_1(T)\| + \|\varphi_2(T)\|) + (u_1(T) - u_2(T), u_1(T))_H + (u_1(T) - u_2(T), u_2(T))_H
\]
\[
\leq (\|\varphi_1(T) - \varphi_2(T)\|)(\|\varphi_1(T)\| + \|\varphi_2(T)\|) + (u_1(T) - u_2(T))_{V'_{\text{div}}}(\|u_1(T)\|_{V'_{\text{div}}} + \|u_2(T)\|_{V'_{\text{div}}})
\]
\[
\leq C (\|\nabla \rho_1\|, \|\nabla \rho_2\|, \|v_1\|, \|v_2\|, \tau, T)(\|\rho_1 - \rho_2\| + \|v_1 - v_2\|_{V'_{\text{div}}}). \quad (3.6)
\]
Combining (3.5) and (3.6) we get
\[
|W(t, \rho_1, v_1) - W(t, \rho_2, v_2)| \leq C (\|\nabla \rho_1\|, \|\nabla \rho_2\|, \|v_1\|, \|v_2\|, \tau, T)(\|\rho_1 - \rho_2\| + \|v_1 - v_2\|_{V'_{\text{div}}}). \quad (3.7)
\]
To establish continuity of \(W\) in time, we consider two systems which evolve from same initial data \((\rho, v)\) at time \(t_1\) and \(t_2\). Let \((\varphi_2, u_2, U_2)\) be an optimal triplet corresponding to initial data \((t_2, \rho, v)\). Fix \(U_0 \in U_{R}\). We define a control in the following way
\[
U_1(s) = \begin{cases} U_0(s), & s \in (t_1, t_2), \\ U_2(s), & s \in (t_2, T). \end{cases}
\]
with \(0 \leq t_1 \leq t_2 \leq T\). Let \((\varphi_1, u_1)\) be the solution of the system (2.10)-(2.14) with initial data \((\varphi_1(t_1), u_1(t_1)) = (\rho, v)\), and control \(U_1\). Let \((\rho, v) \in D(B_H^{1/2}) \times \mathbb{G}_{\text{div}}\). Then
\[
W(t_1, \rho, v) - W(t_2, \rho, v) \leq J(t_1, \rho, v, U_1) - J(t_2, \rho, v, U_2)
\]
\[ \frac{1}{2} \int_{t_1}^{T} \left[ \left\| \varphi_1(t) \right\|^2 + \left\| u_1(t) \right\|^2 + \left\| U_1(t) \right\|^2 \right] dt - \frac{1}{2} \left( \left\| \varphi_1(T) \right\|^2 + \left\| u_1(T) \right\|^2 \right) \\
- \left( \frac{1}{2} \int_{t_2}^{T} \left[ \left\| \varphi_2(t) \right\|^2 + \left\| u_2(t) \right\|^2 + \left\| U_2(t) \right\|^2 \right] dt + \frac{1}{2} \left( \left\| \varphi_2(T) \right\|^2 + \left\| u_2(T) \right\|^2 \right) \right) \\
\leq \frac{1}{2} \int_{t_2}^{T} \left( \left\| \varphi_1(t) \right\|^2 - \left\| \varphi_2(t) \right\|^2 + \left\| u_1(t) \right\|^2 - \left\| u_2(t) \right\|^2 \right) dt + \frac{1}{2} \int_{t_1}^{t_2} \left[ \left\| \varphi_1(t) \right\|^2 + \left\| u_1(t) \right\|^2 + \left\| U_0 \right\|^2 \right] dt \\
+ \frac{1}{2} \left( \left\| \varphi_1(T) \right\|^2 - \left\| \varphi_2(T) \right\|^2 + \left\| u_1(T) \right\|^2 - \left\| u_2(T) \right\|^2 \right).
\]

Since the solution of the system (2.11)-(2.14) with controls $U_1$ and $U_2$ is unique, using the semi-group property, we have

\[ X_1(s; t_1, \rho, v, U_1) = X_2(s; t_2, t_1, \rho, v, U_1), \text{ for } t_1 \leq t_2 \leq s, \quad (3.8) \]

where $X_1 = (\varphi_1, u_1)$ and $X_2 = (\varphi_2, u_2)$. Now consider

\[ \int_{t_2}^{T} \left| \left\| \varphi_1(t) \right\|^2 - \left\| \varphi_2(t) \right\|^2 \right| + \left| \left\| u_1(t) \right\|^2 - \left\| u_2(t) \right\|^2 \right| dt \\
\leq \int_{t_2}^{T} \left\| \varphi_1(t) - \varphi_2(t) \right\| \left( \left\| \varphi_1(t) \right\| + \left\| \varphi_2(t) \right\| \right) + \left\| u_1(t) - u_2(t) \right\| \left( \left\| u_1(t) \right\| + \left\| u_2(t) \right\| \right) dt \\
\leq \left( \sup_{t \in [t_2, T]} \left\| \varphi_1(t) - \varphi_2(t) \right\| \sup_{t \in [t_2, T]} \left( \left\| \varphi_1(t) \right\| + \left\| \varphi_2(t) \right\| \right) \\
+ \sup_{t \in [t_2, T]} \left\| u_1(t) - u_2(t) \right\| \sup_{t \in [t_2, T]} \left( \left\| u_1(t) \right\| + \left\| u_2(t) \right\| \right) \right) (T - t_2). \\
\]

where we used the Poincaré-Wirtinger’s inequality, (3.8), and Theorem 2.8 to estimate $\left( \left\| \varphi_1(t) \right\| + \left\| \varphi_2(t) \right\| \right)$ term and Theorem 2.8 and Theorem 2.10 to estimate difference terms. Using Proposition 2.13 in (3.9) we deduce

\[ \int_{t_2}^{T} \left| \left\| \varphi_1(t) \right\|^2 - \left\| \varphi_2(t) \right\|^2 \right| + \left| \left\| u_1(t) \right\|^2 - \left\| u_2(t) \right\|^2 \right| dt \leq C(t_2 - t_1)^{1/2}. \quad (3.10) \]

Similarly, we have

\[ \frac{1}{2} \left( \left\| \varphi_1(T) \right\|^2 - \left\| \varphi_2(T) \right\|^2 + \left\| u_1(T) \right\|^2 - \left\| u_2(T) \right\|^2 \right) \\
\leq \left( \left\| \varphi_1(T) - \varphi_2(T) \right\| \left( \left\| \varphi_1(T) \right\| + \left\| \varphi_2(T) \right\| \right) \right) + \left( \left\| u_1(T) - u_2(T) \right\| \left( \left\| u_1(T) \right\| + \left\| u_2(T) \right\| \right) \right) \\
\leq C \left( \left\| \nabla \rho \right\|, \left\| v \right\| \right) \left( \left\| \varphi_1(t_2) - \varphi_2(t_2) \right\| \right) + \left\| u_1(t_2) - u_2(t_2) \right\| \right) \right) \\
\leq C(t_2 - t_1)^{1/2}. \quad (3.11) \]

Finally, using Hölder’s inequality and Poincaré-Wirtinger inequality (since mean value of $\varphi_1$ is zero), and Theorem 2.8 we get

\[ \frac{1}{2} \int_{t_1}^{t_2} \left[ \left\| \varphi_1(t) \right\|^2 + \left\| u_1(t) \right\|^2 + \left\| U_0 \right\|^2 \right] dt \leq C(t_2 - t_1)^{1/2} \quad (3.12) \]

Adding (3.10)-(3.12) we get

\[ \left| W(t_2, \rho, v) - W(t_1, \rho, v) \right| \leq C \left| t_2 - t_1 \right|^{1/2}. \quad (3.13) \]

Combining (3.7) and (3.13) we get (3.4)
3.2. Dynamic Programming Principle (DPP). Now we prove the main theorem of this section namely, dynamic programming principle.

**Theorem 3.3 (Bellman’s principle of optimality).** Let $W$ be as defined in (3.3). Then for $0 \leq \tau \leq t_0 \leq T$, we have

$$W(\tau, \rho, v) = \inf_{U \in \mathcal{U}_{ad}} \left\{ \frac{1}{2} \int_{\tau}^{t_0} \left[ \|\varphi(t)\|^2 + \|u(t)\|^2 + \|U(t)\|^2 \right] dt + W(t_0, \varphi(t_0), u(t_0)) \right\}.$$  

**Proof.** Let $U$ be an arbitrary control. Let $(\varphi, u)$ be the corresponding solution of the system (2.11)-(2.14) with initial data $(\varphi(\tau), u(\tau)) = (\rho, v)$. Then, for $\tau \leq t_0 \leq T$

$$J(\tau, \rho, v, U) = \frac{1}{2} \int_{\tau}^{T} \left[ \|\varphi(t)\|^2 + \|u(t)\|^2 + \|U(t)\|^2 \right] dt + \frac{1}{2} (\|\varphi(T)\|^2 + \|u(T)\|^2)$$

$$= \frac{1}{2} \int_{\tau}^{t_0} \left[ \|\varphi(t)\|^2 + \|u(t)\|^2 + \|U(t)\|^2 \right] dt + \int_{t_0}^{T} \left[ \|\varphi(t)\|^2 + \|u(t)\|^2 + \|U(t)\|^2 \right] dt$$

$$+ \frac{1}{2} (\|\varphi(T)\|^2 + \|u(T)\|^2)$$

$$\geq \frac{1}{2} \int_{\tau}^{t_0} \left[ \|\varphi(t)\|^2 + \|u(t)\|^2 + \|U(t)\|^2 \right] dt$$

$$+ \inf_{U \in \mathcal{U}_{R}} \left\{ \int_{t_0}^{T} \left[ \|\varphi(t)\|^2 + \|u(t)\|^2 + \|\bar{U}(t)\|^2 \right] dt + \frac{1}{2} (\|\varphi(T)\|^2 + \|u(T)\|^2) \right\}$$

where $(\bar{\varphi}, \bar{u})$ is the solution of (2.11)-(2.14) with initial conditions $(\varphi(t_0), u(t_0))$ and control $\bar{U} \in \mathcal{U}_R$.

Then

$$J(\tau, \rho, v, U) \geq \frac{1}{2} \int_{\tau}^{t_0} \left[ \|\varphi(t)\|^2 + \|u(t)\|^2 + \|U(t)\|^2 \right] dt + W(t_0, \varphi(t_0), u(t_0)).$$

(3.14)

Since $U$ is arbitrary, taking infimum on both sides of (3.14) over $U$ we arrive at

$$W(\tau, \rho, v) \geq \inf_{U \in \mathcal{U}_{ad}} \left\{ \frac{1}{2} \int_{\tau}^{t_0} \left[ \|\varphi(t)\|^2 + \|u(t)\|^2 + \|U(t)\|^2 \right] dt + W(t_0, \varphi(t_0), u(t_0)) \right\}.$$  

For the case " $\leq $", Let $U_1$ be a control and $(\varphi_1, u_1)$ be corresponding solution of the system (2.11)-(2.14) with initial data $(\tau, \rho, v)$. For $\varepsilon > 0$, let $U_2$ be such that $W(t_0, \varphi_1(t_0), u_1(t_0)) + \varepsilon \geq J(t_0, \varphi_2(t_0), u_2(t_0), U_2)$, where $(\varphi_2, u_2)$ is the solution of the system (2.11)-(2.14) with control $U_2$. Now we define a new control

$$U(t) = \begin{cases} U_1(t), & \tau \leq t \leq t_0, \\ U_2(t), & t > t_0. \end{cases}$$

Let $(\varphi, u)$ be solution of the system (2.11)-(2.14) with the above control $U$ and initial data $(\tau, \rho, v)$, then

$$W(\tau, \rho, v) \leq \frac{1}{2} \int_{\tau}^{t_0} \left[ \|\varphi(t)\|^2 + \|u(t)\|^2 + \|U(t)\|^2 \right] dt + \int_{t_0}^{T} \left[ \|\varphi(t)\|^2 + \|u(t)\|^2 + \|U(t)\|^2 \right] dt$$

$$+ \frac{1}{2} (\|\varphi(T)\|^2 + \|u(T)\|^2)$$

$$\leq \int_{\tau}^{t_0} \left[ \|\varphi_1(t)\|^2 + \|u_1(t)\|^2 + \|U_1(t)\|^2 \right] dt + \int_{t_0}^{T} \left[ \|\varphi_2(t)\|^2 + \|u_2(t)\|^2 + \|U_2(t)\|^2 \right] dt$$

$$+ \frac{1}{2} (\|\varphi_2(T)\|^2 + \|u_2(T)\|^2)$$

$$\leq \int_{\tau}^{t_0} \left[ \|\varphi_1(t)\|^2 + \|u_1(t)\|^2 + \|U_1(t)\|^2 \right] dt + J(t_0, \varphi_1(t_0), u_1(t_0))$$
where \((\varphi_2, u_2)\) is the solution of (1.1)-(1.5) corresponding to control \(U_2\). Since \(\epsilon\) and \(U_1\) are arbitrary we conclude that

\[
W(\tau, \rho, v) \leq \inf_U \left\{ \int_\tau^{t_0} \left[ \|\varphi(t)\|^2 + \|u(t)\|^2 + \|U(t)\|^2 \right] dt + W(t_0, \varphi(t_0), u(t_0)) \right\}
\]

where \((\varphi, u)\) is the solution with control \(U\).

\[\square\]

4. HJB equations

In this section, our main aim is to show that the value function \(W\) defined in (3.3) is a viscosity solution of an infinite dimensional Hamilton-Jacobi equation:

\[
-\frac{\partial W}{\partial t} + H(\rho, v, \partial_\rho W, \partial_v W) = 0,
\]

where

\[
H(\rho, v, \partial_\rho W, \partial_v W) = (B_1(v, \rho) + A_N^2 \rho + A_N f(\rho), \partial_\rho W) + (A v + B(v, v) - B_2(A_N \rho, \rho), \partial_v W)
\]

\[
- \frac{1}{2} (\|\rho\|^2 + \|v\|^2) + \sup_{U} \left( -(U, \partial_v W) - \frac{1}{2} \|U\|^2 \right)
\]

and \((\tau, \rho, v) \in (0, T) \times (D(B_N) \cap H^4 \times (V_{\text{div}} \times \mathbb{H}^2))\).

Assuming that the value function \(W\) defined in (3.3), is smooth enough and using dynamic programming principle one can derive the Hamilton-Jacobi-Bellman equation. We skip this derivation here as the proof is similar to the proof of theorem [4.3] below and can be obtained by replacing the test function \(\Psi\) in the proof by \(W\).

**Definition 4.1.** A function \(\Psi\) is a test function of class \(D\) if \(\Psi \in C^1((0, T) \times D(B_N^{1/2}) \times \mathbb{G}_{\text{div}})\) and if \(A_N^2 \partial_v \Psi\) and \(A_N \partial_\rho \Psi\) are continuous, and \(\Psi\) is weakly sequentially lower semi continuous.

**Definition 4.2.** A function \(V\) is called a viscosity subsolution (viscosity supersolution) to the HJB equation (4.1)-(4.2) if for every test functions \(\Psi \in D\), if \(V - \Psi\) attains local maximum at \((t_0, \rho_0, v_0)\) then

\[
- \partial_t \Psi(t_0, \rho_0, v_0) + H(\rho_0, v_0, \partial_\rho \Psi(t_0, \rho_0, v_0), \partial_v \Psi(t_0, \rho_0, v_0)) \leq 0,
\]

(respectively, if \(V - \Psi\) attains local minimum at \((t_0, \rho_0, v_0)\) then

\[
- \partial_t \Psi(t_0, \rho_0, v_0) + H(\rho_0, v_0, \partial_\rho \Psi(t_0, \rho_0, v_0), \partial_v \Psi(t_0, \rho_0, v_0)) \geq 0.)
\]

**Theorem 4.3.** The value function \(W\) defined in (3.3) is a viscosity solution of the HJB equation (4.1)-(4.2).

**Proof.** We first show that \(W\) satisfies (4.3). Let \(W - \Psi\) attains maximum at \((t_0, \rho_0, v_0)\). Then

\[
\Psi(t_0, \rho_0, v_0) - \Psi(t, \rho, v) \leq W(t_0, \rho_0, v_0) - W(t, \rho, v) \quad \forall (t, \rho, v).
\]

Let \(U\) be any constant control and \((\varphi, u)\) be the solution of the system (2.11)-(2.14) with initial data \((\rho_0, v_0)\) at \(t_0\). Then

\[
\Psi(t_0, \rho_0, v_0) - \Psi(t_0 + \epsilon, \varphi(t_0 + \epsilon), u(t_0 + \epsilon)) \leq W(t_0, \rho_0, v_0) - W(t_0 + \epsilon, \varphi(t_0 + \epsilon), u(t_0 + \epsilon))
\]

(4.5)
From dynamic programming principle Theorem 3.3 we have
\[
W(t_0, \rho_0, v_0) \leq \frac{1}{2} \int_{t_0}^{t_0+\epsilon} (\|\varphi(t)\|^2 + \|u(t)\|^2 + \|U\|^2) dt + W(t_0 + \epsilon, \varphi(t_0 + \epsilon), u(t_0 + \epsilon)) \tag{4.6}
\]
Substituting (4.6) in (4.5) we get
\[
\Psi(t_0, \rho_0, v_0) - \Psi(t_0 + \epsilon, \varphi(t_0 + \epsilon), u(t_0 + \epsilon)) \leq \frac{1}{2} \int_{t_0}^{t_0+\epsilon} (\|\varphi(t)\|^2 + \|u(t)\|^2 + \|U\|^2) dt \tag{4.7}
\]
Dividing (4.7) by \(\epsilon\), as \(\epsilon \to 0\) and using (2.11)-(2.14), we get
\[
- \partial_t \Psi(t_0, \rho_0, v_0) - (\partial_\rho \Psi, B_1(v_0, \rho_0) + A_N^2 \rho_0 + A_N f(\rho_0)) - (\partial_\varphi \Psi, A v_0 + B(v_0, v_0) - B_2(A N \rho_0, \rho_0) - U) - \frac{1}{2} (\|\rho_0\|^2 + \|v_0\|^2 + \|U\|^2) \leq 0
\]
Since \(U\) is arbitrary we conclude that
\[
- \partial_t \Psi(t_0, \rho_0, v_0) + H(\rho_0, v_0, \partial_\rho \Psi(t_0, \rho_0, v_0), \partial_\varphi \Psi(t_0, \rho_0, v_0)) \leq 0, \tag{4.8}
\]
which shows that \(W\) is viscosity subsolution. Now we prove (4.4). For, let \((t_0, \rho_0, v_0)\) be a local minimum for \(W - \Psi\). Then
\[
\Psi(t_0, \rho_0, v_0) - \Psi(t, \rho, v) \geq W(t_0, \rho_0, v_0) - W(t, \rho, v) \quad \forall (t, \rho, v). \tag{4.9}
\]
From Theorem 3.3 for \(\epsilon > 0\) there exists \(U_\epsilon\) such that
\[
W(t_0, \rho_0, v_0) + \epsilon^2 > \int_{t_0}^{t_0+\epsilon} (\|\varphi_\epsilon\|^2 + \|u_\epsilon\|^2 + \|U_\epsilon\|^2) dt + W(t_0 + \epsilon, \varphi_\epsilon(t_0 + \epsilon), u_\epsilon(t_0 + \epsilon))
\]
where \((\varphi_\epsilon, u_\epsilon)\) is the solution with control \(U_\epsilon\) and initial data \((\rho_0, v_0)\) at \(t_0\). Then by (4.9),
\[
\Psi(t_0, \rho_0, v_0) - \Psi(t_0 + \epsilon, \varphi_\epsilon(t_0 + \epsilon), u_\epsilon(t_0 + \epsilon)) \geq - \epsilon^2 + \int_{t_0}^{t_0+\epsilon} (\|\varphi_\epsilon(t)\|^2 + \|u_\epsilon(t)\|^2 + \|U_\epsilon(t)\|^2) dt.
\]
Using chain rule and then dividing by epsilon on both sides, we get
\[
- \frac{1}{\epsilon} \int_{t_0}^{t_0+\epsilon} (\partial_t \Psi(t, \varphi_\epsilon(t), u_\epsilon(t))) + ((\varphi_\epsilon)_t, \partial_\rho \Psi(t, \varphi_\epsilon(t), u_\epsilon(t))) + ((u_\epsilon)_t, \partial_\varphi \Psi(t, \varphi_\epsilon(t), u_\epsilon(t))) \geq - \epsilon + \frac{1}{\epsilon} \int_{t_0}^{t_0+\epsilon} (\|\varphi_\epsilon(t)\|^2 + \|u_\epsilon(t)\|^2 + \|U_\epsilon(t)\|^2) dt.
\]
Using (2.11)-(2.14) we obtain
\[
- \frac{1}{\epsilon} \int_{t_0}^{t_0+\epsilon} \partial_t \Psi(t, \varphi_\epsilon(t), u_\epsilon(t)) dt
\]
\[
+ \frac{1}{\epsilon} \int_{t_0}^{t_0+\epsilon} (B_1(u_\epsilon(t), \varphi_\epsilon(t)) + A_N^2 \varphi_\epsilon(t) + A_N f(\varphi_\epsilon(t)), \partial_\rho \Psi(t, \varphi_\epsilon(t), u_\epsilon(t))) dt
\]
\[
+ \frac{1}{\epsilon} \int_{t_0}^{t_0+\epsilon} (A u_\epsilon(t) + B(u_\epsilon(t), \varphi_\epsilon(t)) - B_2(A N \varphi_\epsilon(t), \varphi_\epsilon(t)) - U_\epsilon(t), \partial_\varphi \Psi(t, \varphi_\epsilon(t), u_\epsilon(t))) dt
\]
\[
\geq - \epsilon + \frac{1}{\epsilon} \int_{t_0}^{t_0+\epsilon} (\|\varphi_\epsilon(t)\|^2 + \|u_\epsilon(t)\|^2 + \|U_\epsilon(t)\|^2) dt.
\]
(4.10)
We rewrite (4.10) by adding few terms on both sides and rearranging as follows

\[
- \partial_t \Psi(t, t_0, v_0) + (B_1(v_0, t_0) + A_N^2 t_0 + A_N f(t_0, v_0)) + (A v_0 + B(v_0, v_0))
- B_2(A_N v_0, v_0, \Psi(t, t_0, v_0)) - (||t_0||^2 + ||v_0||^2) - \frac{1}{\epsilon} \int_{t_0}^{t_0+\epsilon} (||v(t)||^2 + (U(t), \partial v \Psi(t, v(t), u(t))) dt
\geq -\epsilon + \frac{1}{\epsilon} \int_{t_0}^{t_0+\epsilon} (||v(t)||^2 - ||t_0||^2) + (||u(t)||^2 - ||v_0||^2) dt
+ \frac{1}{\epsilon} \int_{t_0}^{t_0+\epsilon} (\partial_t \Psi(t, v(t), u(t)) - \partial_t \Psi(t_0, v_0, v_0)) dt - \frac{1}{\epsilon} \int_{t_0}^{t_0+\epsilon} [(B_1(u(t), v(t)) + A_N^2 t_0) + A_N f(t_0, v_0, v_0))] dt
- \frac{1}{\epsilon} \int_{t_0}^{t_0+\epsilon} [(A u(t) + B(u(t), v(t)) - B_2(A_N v(t), v(t))) - \partial v \Psi(t, v(t), u(t))] dt - (A v_0 + B(v_0, v_0) - B_2(A_N v_0, v_0, \Psi(t_0, v_0, v_0))) dt.
\]

Now we estimate right hand side terms. Note that, from the properties of \(\Psi\) we get

\[
||A^{1/2} \partial v \Psi(s, v(t), u(t))|| \leq C(||v(s)||, ||v_0||),
||A_N \partial v \Psi(s, v(t), u(t))|| \leq C(||v(s)||, ||v_0||).
\]

Let \(g(\epsilon)\) denote a generic modulus of continuity function such that \(g(\epsilon) \to 0\) as \(\epsilon \to 0\), which may vary within calculation. Using (2.44) and continuity property of \(\Psi\) we get

\[
\left| \frac{1}{\epsilon} \int_{t_0}^{t_0+\epsilon} (\partial_t \Psi(t, v(t), u(t)) - \partial_t \Psi(t_0, v_0, v_0)) dt \right| \leq \frac{1}{\epsilon} \int_{t_0}^{t_0+\epsilon} (|t - t_0| + ||v(t) - t_0|| + ||u(t) - v_0||) dt \leq g(\epsilon),
\]

where \(g(\epsilon) \to 0\) as \(\epsilon \to 0\).

Using Hölder’s and Gagliardo-Nirenberg inequalities we get

\[
||B_1(u(t) - v_0, v(t))\partial v \Psi(t, v(t), u(t)))|| \leq ||u(t) - v_0|| \sqrt{||v(t)||} ||\partial v \Psi(t, v(t), u(t))||
\leq C ||\nabla (u(t) - v_0)|| ||\nabla (v(t))|| ||\partial v \Psi(t, v(t), u(t))||,
||B_1(v_0, v(t) - t_0)\partial v \Psi(t, v(t), u(t)))|| \leq ||v_0|| \sqrt{||v(t)||} ||\nabla (v(t) - t_0)|| ||\partial v \Psi(t, v(t), u(t))||
\leq C ||\nabla v_0|| ||\nabla (v(t) - t_0)|| ||\partial v \Psi(t, v(t), u(t))||,
||B_1(v_0, v(t) - t_0)\partial v \Psi(t, v(t), u(t))) - \partial v \Psi(t_0, v_0, v_0)|| \leq ||v_0|| ||\nabla v_0|| ||\partial v \Psi(t, v(t), u(t)) - \partial v \Psi(t_0, v_0, v_0)||
\leq C ||\nabla v_0|| ||\nabla v_0|| ||\partial v \Psi(t, v(t), u(t)) - \partial v \Psi(t_0, v_0, v_0)||.
\]

Using (2.44), (2.66) and continuity properties of \(\partial v \Psi\) (Definition 4.1), we get

\[
\left| \frac{1}{\epsilon} \int_{t_0}^{t_0+\epsilon} (B_1(u(t), v(t))\partial v \Psi(t, v(t), u(t))) - (B_1(v_0, v_0)\partial v \Psi(t_0, v_0, v_0)) \right| \leq C g(\epsilon).
\]
Similarly,
\[
|\langle A_N^2 \varphi(t), \partial_\rho \Psi(t, \varphi(t), u(t)) - A_N^2 \rho_0, \partial_\rho \Psi(t_0, \rho_0, v_0) \rangle| \\
\leq |\langle A_N^2 \varphi(t) - A_N^2 \rho_0, \partial_\rho \Psi(t, \varphi(t), u(t)) \rangle| + |\langle A_N^2 \rho_0, \partial_\rho \Psi(t, \varphi(t), u(t)) - \partial_\rho \Psi(t_0, \rho_0, v_0) \rangle|,
\]
we can estimate as
\[
|\langle A_N^2 \varphi(t) - A_N^2 \rho_0, \partial_\rho \Psi(t, \varphi(t), u(t)) \rangle| = |\langle A_N(\varphi(t) - \rho_0), A_N \partial_\rho \Psi(t, \varphi(t), u(t)) \rangle| \\
\leq \|A_N(\varphi(t) - \rho_0)\| \|A_N \partial_\rho \Psi(t, \varphi(t), u(t))\|,
\]
and
\[
|\langle A_N^2 \rho_0, \partial_\rho \Psi(t, \varphi(t), u(t)) - \partial_\rho \Psi(t_0, \rho_0, v_0) \rangle| = |\langle A_N^{3/2} \rho_0, A_N^{1/2} \partial_\rho \Psi(t, \varphi(t), u(t)) - \partial_\rho \Psi(t_0, \rho_0, v_0) \rangle| \\
\leq \|A_N^{3/2} \rho_0\| \|A_N^{1/2} \partial_\rho \Psi(t, \varphi(t), u(t)) - \partial_\rho \Psi(t_0, \rho_0, v_0) \|.
\]

Then using properties of \(\Psi\), we get
\[
\frac{1}{\epsilon} \int_{t_0}^{t_0 + \epsilon} |\langle A_N^2 \varphi(t), \partial_\rho \Psi(t, \varphi(t), u(t)) - A_N^2 \rho_0, \partial_\rho \Psi(t_0, \rho_0, v_0) \rangle| \leq g(\epsilon). \tag{4.14}
\]

Similarly,
\[
|\langle A_N f(\varphi(t)), \partial_\rho \Psi(t, \varphi(t), u(t)) \rangle - \langle A_N f(\rho_0), \partial_\rho \Psi(t_0, \rho_0, v_0) \rangle| \\
\leq |\langle A_N f(\varphi(t)) - A_N f(\rho_0), \partial_\rho \Psi(t, \varphi(t), u(t)) \rangle| + |\langle A_N f(\rho_0), \partial_\rho \Psi(t, \varphi(t), u(t)) - \partial_\rho \Psi(t_0, \rho_0, v_0) \rangle| \\
|\langle A_N f(\varphi(t)) - A_N f(\rho_0), \partial_\rho \Psi(t, \varphi(t), u(t)) \rangle| \\
= |\langle A_N^{3/2} f(\varphi(t)) - A_N^{1/2} f(\rho_0), A_N^{3/2} \partial_\rho \Psi(t, \varphi(t), u(t)) \rangle| \\
\leq \|A_N^{3/2} f(\varphi(t)) - A_N^{1/2} f(\rho_0)\| \|A_N^{3/2} \partial_\rho \Psi(t, \varphi(t), u(t))\| \\
\leq C(\varphi(t) - \rho_0) \|A_N^{3/2} \partial_\rho \Psi(t, \varphi(t), u(t))\|,
\]
and
\[
|\langle A_N f(\rho_0), \partial_\rho \Psi(t, \varphi(t), u(t)) - \partial_\rho \Psi(t_0, \rho_0, v_0) \rangle| \\
= |\langle A_N^{3/2} f(\rho_0), A_N^{3/2} \partial_\rho \Psi(t, \varphi(t), u(t)) - \partial_\rho \Psi(t_0, \rho_0, v_0) \rangle| \\
\leq \|A_N^{3/2} f(\rho_0)\| \|A_N^{3/2} \partial_\rho \Psi(t, \varphi(t), u(t)) - \partial_\rho \Psi(t_0, \rho_0, v_0) \| \\
\leq C(\partial_\rho \Psi(t_0, \rho_0, v_0)) \|A_N^{3/2} \partial_\rho \Psi(t, \varphi(t), u(t)) - \partial_\rho \Psi(t_0, \rho_0, v_0) \|.
\]

From the properties of \(f\) and \(\Psi\) we get
\[
\frac{1}{\epsilon} \int_{t_0}^{t_0 + \epsilon} |\langle A_N f(\varphi(t)), \partial_\rho \Psi(t, \varphi(t), u(t)) - \langle A_N f(\rho_0), \partial_\rho \Psi(t_0, \rho_0, v_0) \rangle \rangle dt \leq Cg(\epsilon) \tag{4.16}
\]

Using continuity of \(A_N^{1/2} \partial_\nu \Psi\), we estimate the following,
\[
|\langle A u(t), \partial_\nu \Psi(t, \varphi(t), u(t)) \rangle - \langle A v_0, \partial_\nu \Psi(t_0, \rho_0, v_0) \rangle| \\
\leq |\langle A(u(t) - v_0), \partial_\nu \Psi(t, \varphi(t), u(t)) \rangle| + |\langle A v_0, \partial_\nu \Psi(t, \varphi(t), u(t)) - \partial_\nu \Psi(t_0, \rho_0, v_0) \rangle| \\
\leq |\langle A^{1/2} (u(t) - v_0), A^{1/2} \partial_\nu \Psi(t, \varphi(t), u(t)) \rangle| + |\langle A^{1/2} v_0, A^{1/2} \partial_\nu \Psi(t, \varphi(t), u(t)) - \partial_\nu \Psi(t_0, \rho_0, v_0) \rangle| \\
\leq \|A^{1/2} (u(t) - v_0)\| \|A^{1/2} \partial_\nu \Psi(t, \varphi(t), u(t))\| + \|v_0\| \|A^{1/2} \partial_\nu \Psi(t, \varphi(t), u(t)) - \partial_\nu \Psi(t_0, \rho_0, v_0) \| \\
\leq C(\partial_\nu \Psi(t_0, \rho_0, v_0)) \|A^{1/2} \partial_\nu \Psi(t, \varphi(t), u(t))\| + \|v_0\| \|A^{1/2} \partial_\nu \Psi(t, \varphi(t), u(t)) - \partial_\nu \Psi(t_0, \rho_0, v_0) \|.\]
which, using (2.44), (2.66), implies
\[
\left| \frac{1}{\epsilon} \int_{t_0}^{t_0 + \epsilon} \left( \langle A u_\epsilon(t), \partial_v \Psi(t, \varphi_\epsilon(t), u_\epsilon(t) \rangle - \langle A v_0, \partial_v \Psi(t_0, \rho_0, v_0) \rangle \right) dt \right| \leq g(\epsilon). \quad (4.17)
\]

Now consider the trilinear form
\[
\begin{align*}
&\left| \langle B(u_\epsilon(t), u_\epsilon(t)), \partial_v \Psi(t, \varphi_\epsilon(t), u_\epsilon(t)) \rangle - \langle B(v_0, v_0), \partial_v \Psi(t_0, \rho_0, v_0) \rangle \right| \\
&\leq \left| \langle B(u_\epsilon(t) - v_0, u_\epsilon(t)), \partial_v \Psi(t, \varphi_\epsilon(t), u_\epsilon(t)) \rangle \right| + \left| \langle B(v_0, u_\epsilon(t) - v_0), \partial_v \Psi(t, \varphi_\epsilon(t), u_\epsilon(t)) \rangle \right| \\
&\quad + \left| \langle B(v_0, v_0), \partial_v \Psi(t, \varphi_\epsilon(t), u_\epsilon(t)) \rangle - \partial_v \Psi(t_0, \rho_0, v_0) \right| \\
&\leq C \left\| \nabla(u_\epsilon(t) - v_0) \right\| \left\| \nabla u_\epsilon(t) \right\| \left\| \nabla \partial_v \Psi(t, \varphi_\epsilon(t), u_\epsilon(t)) \right\|,
\end{align*}
\]

We observe that
\[
\begin{align*}
&\left| \langle B(u_\epsilon(t) - v_0, u_\epsilon(t)), \partial_v \Psi(t, \varphi_\epsilon(t), u_\epsilon(t)) \rangle \right| \leq C \left\| \nabla(u_\epsilon(t) - v_0) \right\| \left\| \nabla u_\epsilon(t) \right\| \left\| \nabla \partial_v \Psi(t, \varphi_\epsilon(t), u_\epsilon(t)) \right\|,
\end{align*}
\]
and
\[
\begin{align*}
&\left| \langle B(v_0, v_0), \partial_v \Psi(t, \varphi_\epsilon(t), u_\epsilon(t)) \rangle - \partial_v \Psi(t_0, \rho_0, v_0) \right| \\
&\leq C \left\| \nabla v_0 \right\| \left\| \nabla \partial_v \Psi(t, \varphi_\epsilon(t), u_\epsilon(t)) \right\| \left\| \nabla \partial_v \Psi(t_0, \rho_0, v_0) \right\|.
\end{align*}
\]

It follows from (2.44) and (2.66) that
\[
\left| \frac{1}{\epsilon} \int_{t_0}^{t_0 + \epsilon} \left( \langle B(u_\epsilon(t), u_\epsilon(t)), \partial_v \Psi(t, \varphi_\epsilon(t), u_\epsilon(t)) \rangle - \langle B(v_0, v_0), \partial_v \Psi(t_0, \rho_0, v_0) \rangle \right) dt \right| \leq g(\epsilon). \quad (4.18)
\]

Now consider
\[
\begin{align*}
&\left| \langle B_2(A_N \varphi_\epsilon(t), \varphi_\epsilon(t)), \partial_v \Psi(t, \varphi_\epsilon(t), u_\epsilon(t)) \rangle - \langle B_2(A_N \rho_0, \rho_0), \partial_v \Psi(t_0, \rho_0, v_0) \rangle \right| \\
&\leq \left| \langle B_2(A_N \varphi_\epsilon(t) - \rho_0, \varphi_\epsilon(t)), \partial_v \Psi(t, \varphi_\epsilon(t), u_\epsilon(t)) \rangle \right| + \left| \langle B_2(A_N \rho_0, \varphi_\epsilon(t) - \rho_0), \partial_v \Psi(t, \varphi_\epsilon(t), u_\epsilon(t)) \rangle \right| \\
&\quad + \left| \langle B_2(A_N \rho_0, \rho_0), \partial_v \Psi(t, \varphi_\epsilon(t), u_\epsilon(t)) \rangle - \partial_v \Psi(t_0, \rho_0, v_0) \right| \\
&\leq C \left\| A_N \varphi_\epsilon(t) - \rho_0 \right\| \left\| \nabla \partial_v \Psi(t, \varphi_\epsilon(t), u_\epsilon(t)) \right\|,
\end{align*}
\]

From Hölder’s and Gagliardo-Nirenberg inequality we obtain
\[
\begin{align*}
&\left| \langle B_2(A_N \varphi_\epsilon(t) - \rho_0, \varphi_\epsilon(t)), \partial_v \Psi(t, \varphi_\epsilon(t), u_\epsilon(t)) \rangle \right| \\
&\leq \left\| A_N \varphi_\epsilon(t) - \rho_0 \right\| \left\| \nabla \varphi_\epsilon(t) \right\|_L^1 \left\| \partial_v \Psi(t, \varphi_\epsilon(t), u_\epsilon(t)) \right\|_L^1 \\
&\leq \left\| A_N \varphi_\epsilon(t) - \rho_0 \right\| \left\| \varphi_\epsilon(t) \right\|_L^{1/2} \left\| \partial_v \Psi(t, \varphi_\epsilon(t), u_\epsilon(t)) \right\|_H^{1/2},
\end{align*}
\]
and
\[
\begin{align*}
&\left| \langle B_2(A_N \rho_0, \varphi_\epsilon(t) - \rho_0), \partial_v \Psi(t, \varphi_\epsilon(t), u_\epsilon(t)) \rangle \right| \\
&\leq \left\| A_N \rho_0 \right\| \left\| \nabla \varphi_\epsilon(t) - \rho_0 \right\|_L^1 \left\| \partial_v \Psi(t, \varphi_\epsilon(t), u_\epsilon(t)) \right\|_L^1 \\
&\leq \left\| A_N \rho_0 \right\| \left\| \varphi_\epsilon(t) \right\|_L^{1/2} \left\| A_N \varphi_\epsilon(t) - \rho_0 \right\|_L^{1/2} \left\| \partial_v \Psi(t, \varphi_\epsilon(t), u_\epsilon(t)) \right\|_H^{1/2},
\end{align*}
\]

Combining above three estimates, we get
\[
\left| \frac{1}{\epsilon} \int_{t_0}^{t_0 + \epsilon} \langle B_2(A_N \varphi_\epsilon(t), \varphi_\epsilon(t)), \partial_v \Psi(t, \varphi_\epsilon(t), u_\epsilon(t)) \rangle - \langle B_2(A_N \rho_0, \rho_0), \partial_v \Psi(t_0, \rho_0, v_0) \rangle \rangle \leq C g(\epsilon). \quad (4.19)
\]
Using (4.12)–(4.19) in (4.11), we get
\[
- \partial_t \Psi(t_0, \rho_0, v_0) + (B_1(v_0, \rho_0) + A_N^2 \rho_0 + A_N f(\rho_0), \partial_\rho \Psi(t_0, \rho_0, v_0)) + (Av_0 + B(v_0, v_0)
- B_2(A_N \rho_0, \rho_0), \partial_\rho \Psi(t_0, \rho_0, v_0)) - \frac{1}{\epsilon} \int_{t_0}^{t_0+\epsilon} (\|U_\epsilon(t)\|^2 + (U_\epsilon(t), \partial \Psi(t, \varphi_\epsilon(t), u_\epsilon(t)))dt
\geq -g(\epsilon),
\]
(4.20)

By taking infimum over $U_\epsilon$ in the integral and letting $\epsilon \to 0$, we get
\[
- \partial_t \Psi(t_0, \rho_0, v_0) + H(\rho_0, v_0, \partial_\rho \Psi(t_0, \rho_0, v_0), \partial_\rho \Psi(t_0, \rho_0, v_0)) \geq 0.
\]
(4.21)

Hence, $W$ is a viscosity supersolution. Together with (4.8) we conclude that $W$ is a viscosity solution of (4.1)–(4.2).

\[
\Box
\]

5. Uniqueness

In this section we prove the comparison principle, which in turn implies the uniqueness of the viscosity solution under certain conditions. Viscosity solution theory intrinsically provides the well-posedness or uniqueness of solutions for non-linear Hamilton Jacobi type of equations satisfied in the viscosity sense.

Theorem 5.1. Let $W_1, W_2 : (0, T) \times D(\mathcal{B}_N^{1/2}) \times \mathbb{R} \to \mathbb{R}$ be a viscosity subsolution and a viscosity supersolution of the system (4.1)–(4.2), respectively. Assume that $W_1$ and $-W_2$ are bounded above. Then $W_1 \leq W_2$.

Proof. We employ ideas from [22, 27] to prove the Theorem. We prove the theorem by contradiction. Suppose $W_1 \leq W_2$ is not true. Then there exists a $(t, \bar{\rho}, \bar{v})$ such that $W_1(t, \bar{\rho}, \bar{v}) > W_2(t, \bar{\rho}, \bar{v})$. Hence, we can choose a $\mu > 0$ and define

\[
W_1^\mu(t, \bar{\rho}, \bar{v}) > W_2^\mu(t, \bar{\rho}, \bar{v})
\]

where

\[
W_1^\mu(t, \rho, v) = W_1(t, \rho, v) - \frac{\mu}{t}, \quad W_2^\mu(t, \rho, v) = W_2(t, \rho, v) + \frac{\mu}{t}.
\]

Observe that $W_1^\mu$ and $W_2^\mu$ satisfy
\[
- \partial_t W_1^\mu - (B_1(v, \rho) + A_N^2 \rho + A_N f(\rho), \partial_\rho W_1^\mu) - (Av + B(v, v) - B_2(A_N \rho, \rho), \partial_\rho W_1^\mu)
- \frac{1}{2}(\|\rho\|^2 + \|v\|^2) + \sup_U \left( (U, \partial_\rho W_1^\mu) - \frac{1}{2}\|U\|^2 \right) \leq -\frac{\mu}{T^2},
\]
and
\[
- \partial_t W_2^\mu - (B_1(v, \rho) + A_N^2 \rho + A_N f(\rho), \partial_\rho W_2^\mu) - (Av + B(v, v) - B_2(A_N \rho, \rho), \partial_\rho W_2^\mu)
- \frac{1}{2}(\|\rho\|^2 + \|v\|^2) + \sup_U \left( (U, \partial_\rho W_2^\mu) - \frac{1}{2}\|U\|^2 \right) \geq \frac{\mu}{T^2},
\]
respectively. Thus $W_1^\mu$ is a viscosity subsolution and $W_2^\mu$ is a viscosity supersolution of (4.1)–(4.2).

For $\epsilon, \delta, \gamma, \lambda > 0$ and $0 < T_\lambda < T$, consider the function $\Phi$
\[
\Phi(t, s, \rho, \bar{\rho}, v, \bar{v}) = W_1^\mu(t, \rho, v) - W_2^\mu(s, \bar{\rho}, \bar{v}) - \frac{1}{2\epsilon}(\|\rho - \bar{\rho}\|^2 + \|v - \bar{v}\|_{\bar{v}}^2).
\]

\[
- \delta(\|\nabla \rho\|^2 + \|v\|^2 + \|\nabla \bar{\rho}\|^2 + \|\bar{v}\|^2) - \frac{(t - s)^2}{2\gamma} - \lambda(t + s).
\]
(5.1)
where \( t,s > 0 \) and \( \rho, \bar{\rho}, \nu, \bar{\nu} \) are bounded independent of \( \epsilon, \gamma \) for fixed \( \delta, \lambda \). Since \( W_1 \) and \(-W_2\) are bounded above, the function \( \Phi \) has a global maximum, say at \((t_0, s_0, \rho_0, \bar{\rho}_0, \nu_0, \bar{\nu}_0)\). We will show that
\[
\lim_{\delta \to 0} \lim_{\lambda \to 0} \lim_{\epsilon \to 0} \lim_{\gamma \to 0} \sup \sup \sup \delta(\|\nabla \rho \|^2 + \|\nabla \tilde{\rho} \|^2 + \|\nu \|^2 + \|\tilde{\nu} \|^2) = 0, \tag{5.2}
\]
\[
\lim_{\lambda \to 0} \lim_{\epsilon \to 0} \lim_{\gamma \to 0} \sup \lambda(t_0 + s_0) = 0, \text{ for fixed } \delta, \tag{5.3}
\]
\[
\lim_{\epsilon \to 0} \lim_{\gamma \to 0} \frac{1}{2\epsilon}(\|\rho_0 - \tilde{\rho}_0\|^2 + \|\nu_0 - \tilde{\nu}_0\|^2_{\nabla \text{div}}) = 0 \text{ for fixed } \delta, \lambda, \tag{5.4}
\]
and
\[
\lim_{\gamma \to 0} \sup \frac{(t_0 - s_0)^2}{2\gamma} = 0 \text{ for fixed } \delta, \epsilon, \lambda. \tag{5.5}
\]

Since, \( W_1^\mu - W_2^\mu \) is positive at \((\rho, \bar{\rho}, \nu, \bar{\nu})\), \( \sup(W_1^\mu(t, \rho, \nu) - W_2^\mu(t, \rho, \nu)) > 0 \). Let,
\[
0 < m = \sup \{W_1^\mu(t, \rho, \nu) - W_2^\mu(s, \bar{\rho}, \bar{\nu})\},
\]
\[
m_1(\epsilon, \gamma, \delta, \lambda) = \sup \Phi(t, s, \rho, \bar{\rho}, \nu, \bar{\nu}),
\]
\[
m_2(\gamma, \delta, \lambda) = \sup \left\{W_1^\mu(t, \rho, \nu) - W_2^\mu(s, \bar{\rho}, \bar{\nu}) - \delta(\|\nabla \rho \|^2 + \|\nabla \bar{\rho} \|^2 + \|\nu \|^2 + \|\bar{\nu} \|^2) - \frac{(t - s)^2}{2\gamma} - \lambda(t + s)\right\},
\]
\[
m_3(\delta, \lambda) = \sup \left\{W_1^\mu(t, \rho, \nu) - W_2^\mu(s, \bar{\rho}, \bar{\nu}) - \delta(\|\nabla \rho \|^2 + \|\nabla \bar{\rho} \|^2 + \|\nu \|^2 + \|\bar{\nu} \|^2) - \lambda(t + s)\right\},
\]
\[
m_4(\lambda) = \sup \left\{W_1^\mu(t, \rho, \nu) - W_2^\mu(s, \bar{\rho}, \bar{\nu}) - \lambda(t + s)\right\},
\]
where supremum is taken over \( t, s \in (0, T_\delta), \rho, \bar{\rho} \in D(B_1^2), \nu, \bar{\nu} \in G_{\text{div}} \) as appropriate. Note that we have
\[
m = \lim_{\lambda \downarrow 0} m_4(\lambda), \quad m_4(\lambda) = \lim_{\delta \downarrow 0} m_3(\delta, \lambda), \quad m_3(\delta, \lambda) = \lim_{\gamma \downarrow 0} m_2(\gamma, \delta, \lambda), \quad m_2(\gamma, \delta, \lambda) = \lim_{\epsilon \downarrow 0} m_1(\epsilon, \gamma, \delta, \lambda). \tag{5.6}
\]
Now,
\[
m_1(\epsilon, \delta, \gamma, \lambda) = \Phi(t_0, s_0, \rho_0, \bar{\rho}_0, \nu_0, \bar{\nu}_0)
\]
\[
= W_1^\mu(t_0, \rho_0, \nu_0) - W_2^\mu(s_0, \bar{\rho}_0, \bar{\nu}_0) - \frac{1}{2\epsilon}(\|\rho_0 - \bar{\rho}_0\|^2 + \|\nu_0 - \bar{\nu}_0\|^2_{\nabla \text{div}})
\]
\[
- \delta(\|\nabla \rho \|^2 + \|\nabla \bar{\rho} \|^2 + \|\nu \|^2 + \|\bar{\nu} \|^2) - \frac{(t_0 - s_0)^2}{2\gamma} - \lambda(t_0 + s_0)
\]
For fixed \( \delta, \lambda \) we have
\[
m_1(\epsilon, \delta, \gamma, \lambda) + \frac{(t_0 - s_0)^2}{4\gamma} + \frac{1}{4\epsilon}(\|\rho_0 - \bar{\rho}_0\|^2 + \|\nu_0 - \bar{\nu}_0\|^2_{\nabla \text{div}})
\]
\[
\leq W_1^\mu(t_0, \rho_0, \nu_0) - W_2^\mu(s_0, \bar{\rho}_0, \bar{\nu}_0) - \frac{1}{4\epsilon}(\|\rho_0 - \bar{\rho}_0\|^2 + \|\nu_0 - \bar{\nu}_0\|^2_{\nabla \text{div}})
\]
\[
- \delta(\|\nabla \rho \|^2 + \|\nabla \bar{\rho} \|^2 + \|\nu \|^2 + \|\bar{\nu} \|^2) - \frac{(t_0 - s_0)^2}{4\gamma} - \lambda(t_0 + s_0) \leq m_1(2\epsilon, \delta, 2\gamma, \lambda).
\]
from which we get
\[
\frac{1}{4\epsilon}(\|\rho_0 - \bar{\rho}_0\|^2 + \|\nu_0 - \bar{\nu}_0\|^2_{\nabla \text{div}}) + \frac{(t_0 - s_0)^2}{4\gamma} \leq m_1(2\epsilon, \delta, 2\gamma, \lambda) - m_1(\epsilon, \delta, \gamma, \lambda).
\]
Then (5.4) and (5.5) follows from (5.6). Similarly,
\[
m_1(\epsilon, \delta, \gamma, \lambda) + \frac{\delta}{2}(\|\nabla \rho \|^2 + \|\nabla \bar{\rho} \|^2 + \|\nu \|^2 + \|\bar{\nu} \|^2) + \frac{\lambda}{2}(t_0 + s_0)
\]
\begin{equation}
W_1^\mu(t_0, \rho_0, \mathbf{v}_0) - W_2^\mu(s_0, \tilde{\rho}_0, \tilde{\mathbf{v}}_0) - \frac{1}{4\epsilon}(\|\rho_0 - \tilde{\rho}_0\|^2 + \|\mathbf{v}_0 - \tilde{\mathbf{v}}_0\|^2)_{\text{div}}
\end{equation}

\begin{equation}
- \frac{\delta}{2}(\|\nabla \rho_0\|^2 + \|\nabla \tilde{\rho}_0\|^2 + \|\mathbf{v}_0\|^2 + \|\tilde{\mathbf{v}}_0\|^2) - \frac{1}{2}(t_0 - s_0)^2 - \lambda(t_0 + s_0) \leq m_1(\epsilon, \frac{\delta}{2}, \gamma, \frac{\lambda}{2}),
\end{equation}

which gives

\begin{equation}
\frac{\delta}{2}(\|\nabla \rho_0\|^2 + \|\nabla \tilde{\rho}_0\|^2 + \|\mathbf{v}_0\|^2 + \|\tilde{\mathbf{v}}_0\|^2) + \frac{\lambda}{2}(t_0 + s_0) \leq m_1(\epsilon, \frac{\delta}{2}, \gamma, \frac{\lambda}{2}) - m_1(\epsilon, \delta, \gamma, \lambda).
\end{equation}

from which we obtain (5.2) and (5.3) using (5.6). Let us define

\begin{equation}
\Psi_1(t, \rho, \mathbf{v}) = W_2^\mu(s_0, \tilde{\rho}_0, \tilde{\mathbf{v}}_0) + \lambda(t + s_0) + \frac{1}{2\epsilon}(\|\rho - \tilde{\rho}_0\|^2 + \|\mathbf{v} - \tilde{\mathbf{v}}_0\|^2)_{\text{div}}
\end{equation}

\begin{equation}
+ \frac{\delta}{2}(\|\nabla \rho\|^2 + \|\nabla \tilde{\rho}_0\|^2 + \|\mathbf{v}\|^2 + \|\tilde{\mathbf{v}}_0\|^2) + \frac{1}{2}(t - s_0)^2.
\end{equation}

Then $W_1^\mu - \Psi_1$ has a local maximum at $(t_0, \rho_0, \mathbf{v}_0)$. By the definition of viscosity subsolution we have

\begin{equation}
- \lambda - \frac{t_0 - s_0}{\gamma} - (A_N^2 \rho_0 + \alpha A_N f(\rho_0), 2\delta A_N \rho_0) + \frac{1}{\epsilon}(B_1(\tilde{\mathbf{v}}_0, \tilde{\rho}_0) + A_N^2 \tilde{\rho}_0 + \alpha A_N f(\tilde{\rho}_0), \rho_0 - \tilde{\rho}_0)
\end{equation}

\begin{equation}
+ (A\mathbf{v}_0 + B(\mathbf{v}_0, \mathbf{v}_0) - B_2(A_N \rho_0, \rho_0), \frac{1}{\epsilon} A^{-1}(\mathbf{v}_0 - \tilde{\mathbf{v}}_0)) + (A\tilde{\mathbf{v}}_0 + B(\tilde{\mathbf{v}}_0, \tilde{\mathbf{v}}_0), 2\delta \tilde{\mathbf{v}}_0)
\end{equation}

\begin{equation}
- \frac{1}{2}(\|\rho\|^2 + \|\mathbf{v}\|^2) + \sum_U \left(- (U, \frac{1}{\epsilon} A^{-1}(\mathbf{v}_0 - \tilde{\mathbf{v}}_0) + 2\delta \tilde{\mathbf{v}}_0) - \frac{1}{2} \|U\|^2 \right) \leq -\frac{\mu}{T^2}.
\end{equation}

Similarly, define

\begin{equation}
\Psi_2(s, \tilde{\rho}, \tilde{\mathbf{v}}) = W_2^\mu(t_0, \rho_0, \mathbf{v}_0) - \lambda(t_0 + s) - \frac{1}{2\epsilon}(\|\rho - \tilde{\rho}\|^2 + \|\mathbf{v} - \tilde{\mathbf{v}}\|^2)_{\text{div}}
\end{equation}

\begin{equation}
- \frac{\delta}{2}(\|\nabla \rho\|^2 + \|\nabla \tilde{\rho}_0\|^2 + \|\mathbf{v}\|^2 + \|\tilde{\mathbf{v}}\|^2) - \frac{1}{2}(t_0 - s_0)^2.
\end{equation}

Then $W_2^\mu - \Psi_2$ attains a local minimum at $(s_0, \tilde{\rho}_0, \tilde{\mathbf{v}}_0)$ and by the definition of viscosity supersolution we have

\begin{equation}
\lambda - \frac{t_0 - s_0}{\gamma} - (A_N^2 \tilde{\rho}_0 + \alpha A_N f(\tilde{\rho}_0), 2\delta A_N \tilde{\rho}_0) + \frac{1}{\epsilon}(B_1(\tilde{\mathbf{v}}_0, \tilde{\rho}_0) + A_N^2 \tilde{\rho}_0 + \alpha A_N f(\tilde{\rho}_0), \rho_0 - \tilde{\rho}_0)
\end{equation}

\begin{equation}
+ (A\tilde{\mathbf{v}}_0 + B(\tilde{\mathbf{v}}_0, \tilde{\mathbf{v}}_0) - B_2(A_N \tilde{\rho}_0, \tilde{\rho}_0), \frac{1}{\epsilon} A^{-1}(\tilde{\mathbf{v}}_0 - \tilde{\mathbf{v}}_0)) - (A\tilde{\mathbf{v}}_0 + B(\tilde{\mathbf{v}}_0, \tilde{\mathbf{v}}_0), 2\delta \tilde{\mathbf{v}}_0)
\end{equation}

\begin{equation}
- \frac{1}{2}(\|\tilde{\rho}\|^2 + \|\tilde{\mathbf{v}}\|^2) + \sum_U \left(- (U, \frac{1}{\epsilon} A^{-1}(\mathbf{v}_0 - \tilde{\mathbf{v}}_0) - 2\delta \tilde{\mathbf{v}}_0) - \frac{1}{2} \|U\|^2 \right) \geq \frac{\mu}{T^2}.
\end{equation}

Combining the inequalities (5.7) and (5.8) we obtain

\begin{equation}
- 2\lambda + \frac{1}{\epsilon}(B_1(\mathbf{v}_0, \rho_0) - B_1(\tilde{\mathbf{v}}, \tilde{\rho}_0), \rho_0 - \tilde{\rho}_0) + \frac{1}{\epsilon} \|A_N(\rho_0 - \tilde{\rho}_0)\|^2 + \frac{1}{\epsilon} (A_N(\rho_0 + \rho_0) - A_N(\tilde{\rho}_0 + \tilde{\rho}_0))
\end{equation}

\begin{equation}
+ 2\delta(\|A_N^2 \rho_0\|^2 + \|A_N^2 \tilde{\rho}_0\|^2) + 2\delta((\alpha A_N f(\rho_0), A_N \rho_0) + (\alpha A_N f(\tilde{\rho}_0), A_N \tilde{\rho}_0)) + \frac{1}{\epsilon} \|\mathbf{v} - \tilde{\mathbf{v}}\|^2
\end{equation}

\begin{equation}
+ 2\delta(\|\nabla \mathbf{v}\|^2 + \|\nabla \tilde{\mathbf{v}}\|^2) + \frac{1}{\epsilon}(b(\mathbf{v}_0, \mathbf{v}_0, A^{-1}(\mathbf{v}_0 - \tilde{\mathbf{v}}_0)) - b(\tilde{\mathbf{v}}_0, \tilde{\mathbf{v}}_0, A^{-1}(\mathbf{v}_0 - \tilde{\mathbf{v}}_0)))
\end{equation}

\begin{equation}
+ \frac{1}{\epsilon}(B_2(A_N \rho_0, \rho_0) - B_2(A_N \tilde{\rho}_0, \tilde{\rho}_0), A^{-1}(\mathbf{v}_0 - \tilde{\mathbf{v}}_0)) + \frac{1}{2}(\|\rho\|^2 + \|\mathbf{v}\|^2 - \|\rho\|^2 - \|\mathbf{v}\|^2)
\end{equation}

\begin{equation}
+ \sum_U \left(- (U, \frac{1}{\epsilon} A^{-1}(\mathbf{v}_0 - \tilde{\mathbf{v}}_0) + 2\delta \tilde{\mathbf{v}}_0) - \frac{1}{2} \|U\|^2 \right) - \sum_U \left(- (U, \frac{1}{\epsilon} A^{-1}(\mathbf{v}_0 - \tilde{\mathbf{v}}_0) - 2\delta \tilde{\mathbf{v}}_0) - \frac{1}{2} \|U\|^2 \right) \leq -\frac{2\mu}{T^2}
\end{equation}

(5.9)
Now onwards, we denote $\sigma_i$ to be a local modulus of continuity and $C$ a generic constant. Now we estimate the terms in the left hand side of (5.9). First we write

$$\frac{1}{\epsilon} (B_1(v_0, \rho_0) - B_1(\tilde{v}_0, \tilde{\rho}_0), \rho_0 - \tilde{\rho}_0) = \frac{1}{\epsilon} (B_1(v_0 - \tilde{v}_0, \rho_0) - B_1(\tilde{v}_0, \rho_0 - \tilde{\rho}_0), \rho_0 - \tilde{\rho}_0).$$

Observe that $(B_1(\tilde{v}_0, \rho_0 - \tilde{\rho}_0), \rho_0 - \tilde{\rho}_0) = 0$.

$$\frac{1}{\epsilon}(B_1(v_0 - \tilde{v}_0, \rho_0), \rho_0 - \tilde{\rho}_0) \leq \frac{c}{\epsilon}\|v_0 - \tilde{v}_0\|\|\nabla \rho\|_{L^\infty}\|\rho_0 - \tilde{\rho}_0\|$$

$$\leq \frac{1}{6\epsilon}\|v_0 - \tilde{v}_0\|^2 + c\|\rho\|^2_{H^2}\frac{\|\rho_0 - \tilde{\rho}_0\|^2}{\epsilon}$$

$$\leq \frac{1}{6\epsilon}\|v_0 - \tilde{v}_0\|^2 + \sigma_1(\epsilon), \quad (5.10)$$

$$\frac{1}{\epsilon}(A_N^2\rho_0 - A_N^2\tilde{\rho}_0, \rho_0 - \tilde{\rho}_0) = \frac{1}{\epsilon}\|A_N(\rho_0 - \tilde{\rho}_0)\|^2. \quad (5.11)$$

Using the properties of $f$, we can estimate next two terms as follows,

$$\frac{1}{\epsilon}|(A_N f(\rho_0) - A_N f(\tilde{\rho}_0), \rho_0 - \tilde{\rho}_0)| = \frac{1}{\epsilon}|(f(\rho_0) - f(\tilde{\rho}_0), A_N(\rho_0 - \tilde{\rho}_0))|$$

$$\leq \frac{1}{\epsilon}C_f\|\rho_0 - \tilde{\rho}_0\|\|A_N(\rho_0 - \tilde{\rho}_0)\|$$

$$\leq \frac{1}{2\epsilon}\|A_N(\rho_0 - \tilde{\rho}_0)\|^2 + C\frac{\|\rho_0 - \tilde{\rho}_0\|^2}{\epsilon}$$

$$\leq \frac{1}{2\epsilon}\|A_N(\rho_0 - \tilde{\rho}_0)\|^2 + \sigma_2(\epsilon), \quad (5.12)$$

$$2\delta|((\alpha A_N f(\rho_0), A_N \rho_0) + (\alpha A_N f(\tilde{\rho}_0), A_N \tilde{\rho}_0))|$$

$$\leq 2\delta|(|\nabla f(\rho_0), A_N^{3/2}\rho_0|) + |(\nabla f(\tilde{\rho}_0), A_N^{3/2}\tilde{\rho}_0)|)$$

$$\leq 2\delta(|\nabla f(\rho_0)||A_N^{3/2}\rho_0| + |\nabla f(\tilde{\rho}_0)||A_N^{3/2}\tilde{\rho}_0|)$$

$$\leq 2\delta(|A_N^{3/2}\rho_0|^2 + |A_N^{3/2}\tilde{\rho}_0|^2) + 2\delta(|\nabla f(\rho_0)|^2 + |\nabla f(\tilde{\rho}_0)|^2)$$

$$\leq 2\delta(|A_N^{3/2}\rho_0|^2 + |A_N^{3/2}\tilde{\rho}_0|^2) + \sigma_3(\delta). \quad (5.13)$$

Now observe that

$$b_2(A_{N\rho_0}, \rho_0, A^{-1}(v_0 - \tilde{v}_0)) - b_2(A_{N\tilde{\rho}_0}, \rho_0, A^{-1}(v_0 - \tilde{v}_0))$$

$$= b_2(A_{N\rho_0}, \rho_0, A^{-1}(v_0 - \tilde{v}_0)) + b_2(A_{N\tilde{\rho}_0}, \rho_0 - \tilde{\rho}_0, A^{-1}(v_0 - \tilde{v}_0)).$$

Using Sobolev inequality and (2.8), we estimate

$$\frac{1}{\epsilon}|b_2(A_{N\rho_0}, \rho_0, A^{-1}(v_0 - \tilde{v}_0))| \leq \frac{c}{\epsilon}|A_N(\rho_0 - \tilde{\rho}_0)||\nabla \rho_0||L^4||A^{-1}(v_0 - \tilde{v}_0)||_{L^4}$$

$$\leq \frac{1}{2\epsilon}\|A_N(\rho_0 - \tilde{\rho}_0)\|^2 + C\|\rho_0\|^2_{H^2}\frac{\|v_0 - \tilde{v}_0\|^2_{H^2}}{\epsilon}$$

$$\leq \frac{1}{2\epsilon}\|A_N(\rho_0 - \tilde{\rho}_0)\|^2 + \sigma_4(\epsilon), \quad (5.14)$$

and

$$\frac{1}{\epsilon}|b_2(A_{N\tilde{\rho}_0}, \rho_0 - \tilde{\rho}_0, A^{-1}(v_0 - \tilde{v}_0))| \leq \frac{c}{\epsilon}|A_{N\tilde{\rho}_0}|_{L^4}\|\nabla (\rho_0 - \tilde{\rho}_0)||A^{-1}(v_0 - \tilde{v}_0)||_{L^4}$$

$$\leq C\|\tilde{\rho}_0\|_{H^2}\|\nabla (\rho_0 - \tilde{\rho}_0)||v_0 - \tilde{v}_0||_{V_{div}}.$$
We also have
\[
\frac{1}{2} \left| (\| \tilde{\rho}_0 \|^2 - \| \rho_0 \|^2) \right| = \frac{1}{2} (\| \tilde{\rho}_0 \| - \| \rho_0 \|)(\| \tilde{\rho}_0 \| + \| \rho_0 \|)
\leq \frac{1}{2} (\| \tilde{\rho}_0 - \rho_0 \|)(\| \tilde{\rho}_0 \| + \| \rho_0 \|)
\leq \frac{\| \tilde{\rho}_0 - \rho_0 \|^2}{\epsilon} + C \epsilon (\| \tilde{\rho}_0 \| + \| \rho_0 \|)^2
\leq \frac{\| \tilde{\rho}_0 - \rho_0 \|^2}{\epsilon} + \sigma_0(\epsilon),
\] (5.16)

and
\[
\frac{1}{2} (\| \tilde{v}_0 \|^2 - \| v_0 \|^2) = \frac{1}{2} (\| \tilde{v}_0 \| - \| v_0 \|)(\| \tilde{v}_0 \| + \| v_0 \|)
\leq \frac{1}{6\epsilon} \| \tilde{v}_0 - v_0 \|^2 + \sigma_7(\epsilon),
\] (5.17)

and
\[
\sup_{U \in U_R} \left| -(U, \frac{1}{\epsilon} A^{-1}(v_0 - \tilde{v}_0) + 2\delta v_0) - \frac{1}{2}\|U\|^2 \right| - \sup_{U \in U_R} \left| -(U, \frac{1}{\epsilon} A^{-1}(v_0 - \tilde{v}_0) - 2\delta \tilde{v}_0) - \frac{1}{2}\|U\|^2 \right|
\leq \sup_{U \in U_R} \left| -(U, \frac{1}{\epsilon} A^{-1}(v_0 - \tilde{v}_0) + 2\delta v_0) + (U, \frac{1}{\epsilon} A^{-1}(v_0 - \tilde{v}_0) - 2\delta \tilde{v}_0) \right|
\leq \sup_{U \in U_R} \left| (U, -2\delta \tilde{v}_0 - 2\delta v_0) \right| \leq 2\delta \sup_{U \in U_R} \|U, \tilde{v}_0 + v_0\|
\leq 2\delta R \| \tilde{v}_0 + v_0 \| \leq \delta (\| \tilde{v}_0 \|^2 + \| v_0 \|^2) + \sigma_8(\delta).
\] (5.18)

Substituting (5.10) in (5.9), we get
\[
-2\lambda + \frac{1}{2\epsilon} \| \tilde{v}_0 - v_0 \|^2 + \delta (\| v_0 \|^2 + \| v_0 \|^2) + b(v_0, v_0, A^{-1}(v_0 - \tilde{v}_0)) - b(\tilde{v}_0, \tilde{v}_0, A^{-1}(v_0 - \tilde{v}_0))
\leq - \frac{\mu}{T^2} + \frac{1}{\epsilon} \| \rho_0 - \tilde{\rho}_0 \|^2 + \sigma(\epsilon, \delta)
\] (5.19)

for some local modulus $\sigma$. Observe that
\[
b(v_0, v_0, A^{-1}(v_0 - \tilde{v}_0)) - b(\tilde{v}_0, \tilde{v}_0, A^{-1}(v_0 - \tilde{v}_0)) = b(v_0 - \tilde{v}_0, v_0, A^{-1}(v_0 - \tilde{v}_0)) + b(\tilde{v}_0, v_0 - \tilde{v}_0, A^{-1}(v_0 - \tilde{v}_0)).
\]

Using Hölder's inequality and (2.3) we get
\[
\frac{1}{\epsilon} |b(v_0 - \tilde{v}_0, v_0, A^{-1}(v_0 - \tilde{v}_0))| = \frac{1}{\epsilon} |b(v_0 - \tilde{v}_0, A^{-1}(v_0 - \tilde{v}_0), v_0)|
\leq \frac{1}{\epsilon} \| v_0 - \tilde{v}_0 \| \| A^{-1}(v_0 - \tilde{v}_0) \| \| v_0 \|_L^\infty
\leq \frac{1}{4\epsilon} \| v_0 - \tilde{v}_0 \|^2 + C \| v_0 \|^2 \| \tilde{v}_0 - v_0 \|^2 \| v_0 \|_v^2 \| v_0 \|_{v_0}^2.
\] (5.20)

Similarly,
\[
\frac{1}{\epsilon} |b(\tilde{v}_0, v_0 - \tilde{v}_0, A^{-1}(v_0 - \tilde{v}_0))| = \frac{1}{\epsilon} |b(\tilde{v}_0, A^{-1}(v_0 - \tilde{v}_0), v_0 - \tilde{v}_0)|
\leq \frac{1}{4\epsilon} \| \tilde{v}_0 - v_0 \|^2 + C \| \tilde{v}_0 \|^2 \| v_0 - \tilde{v}_0 \|^2 \| \tilde{v}_0 - v_0 \|^2 \| v_0 \|_{v_0}^2.
\]
\[
\begin{align*}
&\leq \frac{1}{\epsilon} \|\tilde{v}_0\|_{L^\infty} \|v_0 - \tilde{v}_0\| \|A^{-1}(v_0 - \tilde{v}_0)\| \\
&\leq \frac{1}{4\epsilon} \|v_0 - \tilde{v}_0\|^2 + C \|\tilde{v}_0\|_{H^2}^2 \frac{\|v_0 - \tilde{v}_0\|_{L^2}}{\epsilon}.
\end{align*}
\] (5.21)

Substituting (5.20) and (5.21) in (5.19) and sending limits \(\epsilon \to 0, \lambda \to 0, \delta \to 0\) for fixed \(\mu\), we will get \(0 < -\frac{\mu}{T^2}\) which is a contradiction to the assumption \(\mu > 0\). Thus \(W_1 \leq W_2\). This completes the proof of the theorem.

If \(W_1\) and \(W_2\) both are viscosity solutions of (4.1)-(4.2), then by interchanging the roles of \(W_1\) and \(W_2\) in the above proof, we will arrive at \(W_1 \geq W_2\). Thus \(W_1 = W_2\) and uniqueness follows.

\[\square\]

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