The comparing FRW and Gödel background with Finsler and Riemannian geometries

Z Nekouee 1, J Sadeghi 2 and A Behzadi 1

1 Department of Mathematics, Faculty of Basic Science, University of Mazandaran, P. O. Box 47416-95447, Babolsar, Iran
2 Department of Physics, Faculty of Basic Science, University of Mazandaran, P. O. Box 47416-95447, Babolsar, Iran
E-mail: z.nekouee@stu.umz.ac.ir

Abstract. In this paper, we employ two geometries as Riemannian and Finsler geometry. In that case, we take two metrics background as FRW and Gödel metrics. They play important roles in explanation of several phenomena in cosmology. And then, we calculate the Killing vectors that correspond with two metrics by using two geometries. Also, we obtain the generators of algebra and compare two geometries from Killing vector point of view. Finally, we achieve the commutation relations of such geometries from two metrics background.

1. Introduction
As we know the Finslerian geometry describes very well modified theories as viscous, non-homogeneity, fractals and complicity system with breaking some symmetry. This breaking symmetry come from Finslerian geometry, it changes the system to another physical system. As an example, one can say that, in the Clifford-Finslerian mathematical structure, spontaneous symmetry breaking is automatically embedded in fractal branches. Because such geometry changes the general system as a fractal phenomena. The symmetry breaking leads us to have the following explanation. In General Relativity (GR) as a classical theory of gravitation describes our universe in large scales and there are some observational evidences to prove it. In cosmological scenarios, we deal with solving the Einstein field equation to obtain dynamical equations. The main solution for the Einstein equation is based on a famous background metric that is named Friedmann-Roberson-Walker (FRW) and it leads to the Friedmann equations as a dynamical equations of the universe. The FRW metric plays a great role in the cosmological situations because it is almost isotropic and homogenous in large scale structure. Likewise, this metric with different curvature constants ($\kappa = -1, 0, +1$) can predict different ends for our universe. Even for the modified theories of gravity, this metric is applied to explain the universe. One of the most important solution of Einstein equation is Gödel cosmological model. [1] This solution will be interesting from physics and mathematical point of view. Such solutions sometimes are unstable with respect to quantum fluctuations because it has closed time-like curves. [2,3] The FRW and Gödel metrics have some similar applications because they can describe some phenomena in cosmology with different topologies and mathematics. But also the Gödel background has some black hole solutions which has important properties for the candidate of moving particle in the
near of such corresponding particles. Maybe such solution sometimes candidates for AdS/CFT because of existence of thermal properties of corresponding black hole. On the other hand this solution has many unusual properties in particular, the existence of closed time-like curves that would allow time travel in a universe at least in theory. Another remarkable feature for Gödel metric is that it considers rotation for the universe and because of this fact, there are some papers that are specified for application of Gödel metric to investigate rotating Black Holes. [4] Although the Gödel solution does not describe our universe as well as the FRW metric but it demonstrates the kind of phenomena that we cannot easily dismiss in GR.

All above information about two metrics helps us to discuss and investigate the Killing vector from two geometries. So, it is better to introduce two geometries which are Riemannian and Finsler geometries. So, the Riemannian geometry as foundation of GR has various applications in cosmology and also the mentioned metrics are based upon it. Despite these successes, the Riemannian geometry has an alternative that is called Finsler geometry. [5] In fact, Finsler geometry is a generalized form of Riemannian geometry without some limitations. In Finsler geometry, the symmetry of space-time is described by isotropic group and the generators of isometric group are directly connected to Killing vectors. In other words, the isometric group is a Lie group in the Riemannian manifold. This fact also is valid in the Finsler manifold. Generally, the Finsler space-time admits Killing vectors less than the Riemannian space-time. The number of independent Killing vectors for an \(n\)-dimensional Finslerian space-time should not be more than \(\frac{n(n-1)}{2} + 1\). [6] Therefore, the Finsler space-time breaks symmetry of space-time which naturally involves preferred directions. In Finsler context, some people have attempted to make a gravitational theory on Finsler geometry instead of GR but it isn’t enough yet. [7–9] Hence, the majority of people prefer to engage Finsler metrics on the Riemannian geometry to explain cosmological scenarios such as Cosmic Inflation, Dark Energy, Black Holes, Very Special Relativity (VSR) Ref. [10,11] and so on.

In this paper, we try to obtain Killing equations and Killing vectors for the FRW and Gödel metrics in Finsler geometry separately and we will compare the results with the Killing vectors of those in Riemannian geometry. Also, it seems that the geodesic for two metrics in Riemannian geometry is the same with Finsler geometry. [12,13] Eventually, we will compare the obtained results for two cosmological metrics from the mentioned geometries point of view.

The above information gives motivations us to arrange the paper with following form. In Section 2, the Killing vectors for the FRW metric are introduced in the Riemannian geometry and also they are calculated in the Finsler geometry. In section 3, the strategy is the same with the previous section for Gödel metric. Also, we compare two metrics and express some physical results in section 4. Eventually in section 5, we have some conclusions and also progresses for the future research.

In this section, we consider the FRW metric as the main cosmological metric. The Killing vectors calculate in Riemannian and Finsler geometries and they will compare together.

2. Killing vectors in Riemannian and Finsler geometries for FRW metric

Now, we are going to obtain the Killing vector of FRW background from Riemannian manifold. For this reason first we explain such manifold directly. Let \((M, g)\) be a Riemannian manifold (pseudo), and also let \(f : M \rightarrow M\) be a diffeomorphism map. In that case, \(f\) is called isometry if we have

\[
g_p = f^*g_{f(p)}. \tag{1}\]

For \(X, Y \in T_pM\), if \(g_{f(p)}(f_*X, f_*Y) = g_p(X, Y)\), the expression (1) can be rewritten by the following equation

\[
\frac{\partial g^\alpha}{\partial x^\mu} \cdot \frac{\partial g^\beta}{\partial x^\nu} \cdot g_{\alpha\beta}(f(p)) = g_{\mu\nu}(p). \tag{2}\]
On the other hand, a vector field $X$ is a Killing vector field ($K_V$) if $\mathcal{L}_X g = 0$. \cite{14} We can translate this equation as

$$g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0.$$ \hspace{1cm} (3)

For the local coordinates, the Killing equation $\nabla \mu X^\nu + \nabla \nu X^\mu = 0$ can be easily rewritten by the following form

$$(\nabla \mu X^\nu + \nabla \nu X^\mu) = X^\lambda \frac{\partial}{\partial \lambda} g_{\mu \nu} + \partial_\nu X^\kappa g_{\mu \kappa} + \partial_\mu X^\kappa g_{\nu \kappa}.$$ \hspace{1cm} \hspace{1cm} (4)

Now, we focus on FRW metric to describe the universe with the line element

$$ds^2 = dt^2 - R^2(t)\left(\frac{dr^2}{1 - k r^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2\right),$$ \hspace{1cm} (5)

we use Eq. (4) and (5) one can obtain \cite{15},

$$X_1 = \sqrt{1 - k r^2}(\sin \phi \sin \theta \frac{\partial}{\partial r} + \frac{1}{r} \sin \phi \cos \theta \frac{\partial}{\partial \theta} + \frac{1}{r} \cos \phi \frac{\partial}{\partial \phi}),$$ \hspace{1cm} (6)

$$X_2 = \sqrt{1 - k r^2}(- \cos \phi \sin \theta \frac{\partial}{\partial r} - \frac{1}{r} \cos \phi \cos \theta \frac{\partial}{\partial \theta} + \frac{1}{r} \sin \phi \frac{\partial}{\partial \phi}),$$ \hspace{1cm} (7)

$$X_3 = \sqrt{1 - k r^2}(\cos \theta \frac{\partial}{\partial r} - \frac{1}{r} \sin \theta \frac{\partial}{\partial \theta}),$$ \hspace{1cm} (8)

$$X_4 = \cos \phi \frac{\partial}{\partial \theta} - \cot \theta \sin \phi \frac{\partial}{\partial \phi},$$ \hspace{1cm} (9)

$$X_5 = - \sin \phi \frac{\partial}{\partial \theta} - \cot \theta \cos \phi \frac{\partial}{\partial \phi},$$ \hspace{1cm} (10)

and

$$X_6 = \frac{\partial}{\partial \phi}. \hspace{1cm} (11)$$

In second step, we are going to obtain the Killing vector and generators of algebra of FRW background from Finsler manifold. So, a Finsler space $F^n = (M, F)$ is an n-dimensional manifold $M$ equipped with a Finsler metric $F : TM \to \mathbb{R}$, $(x, y) \longmapsto F(x, y)$, $x \in M$, $y \in TM$, holding conditions

(1) regularity: $F$ is a $C^\infty$ function on $TM \setminus \{0\}$,

(2) positive homogeneity (of degree 1): $F(x, \lambda y) = \lambda F(x, y)$, $\lambda \in \mathbb{R}$,

(3) strong convexity: $g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}(x, y)$ is positively defined.

Let $\alpha(x, y) = \sqrt{g_{ij}(x, y)} y^i y^j$ is a Riemannian metric and $\beta$ is a differential one-form on $M$ with $\|\beta\|_\alpha := \sqrt{g^{ij} b_i b_j} < 1$ where $g^{ij}$ is the inverse matrix of $g_{ij}$ and the Einstein notation is used. Then $F(x, y) = \alpha \phi(s)$, $s = \frac{\alpha}{\beta}$ defines a $(\alpha, \beta)$ metric where $\phi(s)$ is a smooth function. If $\phi(s) = 1 + s$, then $F(x, y) = \alpha + \beta$ is called Randers metric.
Finding the Killing vectors for the general Finsler manifold has some difficulties, so we concentrate on the \((\alpha, \beta)\) space. In Randers space, the Killing equation is expressed as

\[ K_V(\alpha) + K_V(\beta) = 0, \]  

(12)

where \(K_V(\alpha)\) is the Riemannian Killing equation and \(K_V(\beta)\) is defined as

\[ K_V(\beta) = (X^i \frac{\partial b_j}{\partial x^i} + b_i \frac{\partial X^j}{\partial x^i})y^j. \]  

(13)

Since \(K_V(\alpha)\) contains irrational term of \(y^i\) and \(K_V(\beta)\) only contains rational term of \(y^i\), so the Eq. (12) is valid if \(K_V(\alpha) = K_V(\beta) = 0\). [16, 17]

In the present work, we limit ourselves to the following form of the Randers metric

\[ F(x, y) = \alpha(x, y) + b_i(x)y^i, \quad \alpha(x, y) = \sqrt{g_{\kappa\lambda}(x)y^\kappa y^\lambda} \]  

(14)

where \(g_{\kappa\lambda}(x)\) is the FRW metric and \(b_i = 0\) for \(i = 1, 2, 3\) and \(b_0 = b_0(t)\). [18]

By combination of the Killing equations of Riemannian sector and following Killing equations

\[ X^t \frac{\partial b_0}{\partial t} + b_0 \frac{\partial X^t}{\partial t} = 0, \]  

(15)

\[ b_0 \frac{\partial X^t}{\partial r} = 0, \]  

(16)

\[ b_0 \frac{\partial X^t}{\partial \theta} = 0, \]  

(17)

and

\[ b_0 \frac{\partial X^t}{\partial \phi} = 0, \]  

(18)

we have the same components of the Killing vector \((X^i, i = t, r, \theta, \phi)\) in Riemannian space. We note that according to the calculations in the previous section, we do not need to restrict \(t\). The generators of algebra from Finsler manifold in case of FRW are given by Ref.[19]

\[
\dot{X}_1 = \sqrt{1 - kr^2}((\sin \phi \sin \theta \frac{\partial}{\partial r} + \frac{1}{r} \sin \phi \cos \theta \frac{\partial}{\partial \theta}) + (1 \cos \phi \frac{\partial}{\partial \phi}) + \left(\frac{-kr}{1 - kr^2}y^r \sin \theta \sin \phi + y^\theta \cos \theta \sin \phi + y^\phi \sin \theta \cos \phi\right)\frac{\partial}{\partial y^r} + \left(\frac{-1}{r^2(1 - kr^2)}y^r \cos \theta \sin \phi - \frac{1}{r} y^\theta \sin \theta \sin \phi + \frac{1}{r} y^\phi \cos \phi \cos \theta - \frac{1}{r^2(1 - kr^2)}y^r \cos \phi \sin \theta \sin \phi - \frac{1}{r^2 \sin^2 \theta}y^r \sin \theta \cos \phi\right)\frac{\partial}{\partial y^\theta})
\]

(19)

\[
\dot{X}_2 = \sqrt{1 - kr^2}((1 - \cos \phi \sin \theta \frac{\partial}{\partial r} - \frac{1}{r} \cos \phi \cos \theta \frac{\partial}{\partial \theta}) + \left(\frac{kr}{1 - kr^2}y^r \sin \theta \cos \phi + \frac{1}{r} \sin \phi \frac{\partial}{\partial \phi}\right) + \left(\frac{-kr}{1 - kr^2}y^r \cos \theta \sin \phi - \frac{1}{r} y^\theta \sin \theta \sin \phi + \frac{1}{r} y^\phi \cos \phi \cos \theta - \frac{1}{r^2 \sin^2 \theta}y^r \sin \theta \cos \phi\right)\frac{\partial}{\partial y^\theta})
\]
\[-y^\theta \cos \theta \cos \phi + y^\phi \sin \theta \sin \phi \frac{\partial}{\partial y^\phi} + \frac{1}{r^2(1 - kr^2)} y^r \cos \theta \cos \phi + \frac{1}{r} y^\theta \sin \theta \cos \phi + \frac{1}{r} y^\phi \cos \theta \sin \phi - \frac{1}{r} y^\phi \sin \theta \sin \phi \frac{\partial}{\partial y^\phi}, \tag{20}\]

\[\hat{X}_3 = \sqrt{1 - kr^2} \left( (\cos \theta \frac{\partial}{\partial r} - \frac{1}{r} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{r} y^\theta \cos \phi \frac{\partial}{\partial y^\phi} \right), \tag{21}\]

\[\hat{X}_4 = \cos \phi \frac{\partial}{\partial \theta} - \cot \theta \sin \phi \frac{\partial}{\partial \phi} - y^\phi \sin \phi \frac{\partial}{\partial y^\phi} + \left( \frac{y^\phi \sin \phi}{\sin^2 \theta} - y^\phi \cot \theta \cos \phi \right) \frac{\partial}{\partial y^\phi}, \tag{22}\]

\[\hat{X}_5 = -\sin \phi \frac{\partial}{\partial \theta} - \cot \theta \cos \phi \frac{\partial}{\partial \phi} - y^\phi \cos \phi \frac{\partial}{\partial y^\phi} + \left( \frac{y^\phi \cos \phi}{\sin^2 \theta} + y^\phi \cot \theta \sin \phi \right) \frac{\partial}{\partial y^\phi}, \tag{23}\]

and

\[\hat{X}_6 = \frac{\partial}{\partial \phi}, \tag{24}\]

the generators can be rewritten as \(\hat{X}_i = X_i + Y_i\) where \(X_i\) is generator in Riemannian case. So, generally one can say that the form and number of generators for the FRW background will be same in two manifolds of point of view.

3. Killing vectors in Riemannian and Finsler geometries for Gödel metric

Second example will be Gödel background. Also here we calculate two groups of Killing vectors from Riemannian and Finsler geometries. So, we consider Gödel metric as another solution of the Einstein equation. And here the strategy is the same with the previous section. Now, we consider the general form of Gödel metric background, which is given by [1]

\[ds^2 = a^2(dx_0^2 - dx_1^2 + \frac{e^{2x_1}}{2}dx_2^2 - dx_3^2 + 2e^{x_1}dx_0dx_2), \tag{25}\]

the Killing vector fields take the following form [20]

\[X_1 = -2e^{-x_1} \frac{\partial}{\partial x_0} + x_2 \frac{\partial}{\partial x_1} + \left( e^{-2x_1} - \frac{x_2}{2} \right) \frac{\partial}{\partial x_2}. \tag{26}\]
\[ X_2 = \frac{\partial}{\partial x_0}, \quad (27) \]
\[ X_3 = \frac{\partial}{\partial x_2}, \quad (28) \]
\[ X_4 = \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2}, \quad (29) \]
and
\[ X_5 = \frac{\partial}{\partial x_3}. \quad (30) \]

Also here we want to apply the Finsler geometry in Gödel black hole which is completely different from FRW metric. By using Eq. (12) for the Gödel metric, the Killing generators in Finsler case are driven as
\[ \hat{X}_1 = \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} - y_2 \frac{\partial}{\partial y_2}, \quad (31) \]
\[ \hat{X}_2 = \frac{\partial}{\partial x_2}, \quad (32) \]
and
\[ \hat{X}_3 = \frac{\partial}{\partial x_3}. \quad (33) \]

4. The comparing of Killing vectors FRW and Gödel in two manifolds

As we know, the generators of algebra in the Riemannian and Finsler geometries for the FRW background will be the same. In means that, in this case we have six generators with so(3) algebra. So, the commutation relations for two manifolds with FRW background will be as shown in table 1. In Finsler geometry, we cannot exactly claim that the algebra has been extended. But generally we can see that the first terms of generators are the same for two geometries but in Finsler geometry we have some extra terms. So, we conclude that for the Finsler case the generators may be generalized or have some spectral flow. We know the spectral flow for any algebra plays an important role in physics specially in non-commutative geometry. We note that \( \hat{X}_1 \rightarrow X_1 + Y_1 \) where \( Y_1 \) is the shift of algebra generator. Here we can say that similar symmetry is preserved and the commutation relations as \([\hat{X}_1, \hat{X}_j]\) are similar to Riemannian geometry. But in Finsler case we have some extra terms as a spectral flow. This extra term give message to us in future to works on spectral flow and some exceptional algebra with central extension.

| \([X_i, X_j]\) | \(X_1\) | \(X_2\) | \(X_3\) | \(X_4\) | \(X_5\) | \(X_6\) |
|---------------|--------|--------|--------|--------|--------|--------|
| \(X_1\) & 0 & \(-kX_6\) & \(-kX_5\) & 0 & \(X_3\) & \(X_2\) |
| \(X_2\) & \(kX_6\) & 0 & \(-kX_4\) & \(X_3\) & 0 & \(-X_1\) |
| \(X_3\) & \(kX_5\) & \(kX_4\) & 0 & \(-X_2\) & \(-X_1\) & 0 |
| \(X_4\) & 0 & \(-X_3\) & \(X_2\) & 0 & \(X_6\) & \(-X_5\) |
| \(X_5\) & \(-X_3\) & 0 & \(X_1\) & \(-X_6\) & 0 & \(X_4\) |
| \(X_6\) & \(-X_2\) & \(X_1\) & 0 & \(X_5\) & \(-X_4\) & 0 |

If we look at to the Eqs. (26-30) one can rewrite the table 2 in Gödel background with Riemannian geometry. In this case we have five generators.
Table 2. The commutation relations algebra for Gödel in Riemannian geometry.

| \([X_i, X_j]\) | \(X_1\) | \(X_2\) | \(X_3\) | \(X_4\) | \(X_5\) |
|----------------|--------|--------|--------|--------|--------|
| \(X_1\)       | 0      | 0      | \(-X_4\) | \(X_1\) | 0      |
| \(X_2\)       | 0      | 0      | 0      | 0      | 0      |
| \(X_3\)       | \(X_4\) | 0      | 0      | \(-X_3\) | 0      |
| \(X_4\)       | \(-X_1\) | 0      | \(X_3\) | 0      | 0      |
| \(X_5\)       | 0      | 0      | 0      | 0      | 0      |

Here we will try to apply the Finslerian geometry in Gödel, there are three generators as Eqs. (31- 33) and we have table 3.

Table 3. The commutation relations algebra for Gödel in Finsler geometry.

| \([\hat{X}_i, \hat{X}_j]\) | \(\hat{X}_1\) | \(\hat{X}_2\) | \(\hat{X}_3\) |
|--------------------------|-------------|-------------|-------------|
| \(\hat{X}_1\)           | 0           | \(\hat{X}_2\) | 0           |
| \(\hat{X}_2\)           | \(-\hat{X}_2\) | 0           | 0           |
| \(\hat{X}_3\)           | 0           | 0           | 0           |

5. Conclusion
In this work, firstly we introduced the FRW metric and obtained the Killing vector equations for the corresponding background metric in Riemannian geometry. Also, we solved such equations and achieved the Killing vectors with maximal symmetry. Then, we introduced Finsler geometry as the generalized geometry with some new definitions. We took used from the mentioned geometry and applied in the FRW metric. In that case, we calculated the Killing vector equations with Finsler geometry. This case does not force any constraint for calculating the vector fields. Also, we see that the condition for the choice of generators is completely suitable with this formula \(\frac{n(n-1)}{2} + 1\). It means that in the Finsler geometry the number of generators for the corresponding metric is less than this expression. So, from comparing the commutation relations of generators for two geometries, we see some extra terms. These extra terms in Finsler geometry lead us to discuss the spectral flow for the corresponding algebra. Here, also we introduced the Gödel metric and obtained the Killing vectors from Riemannian and Finsler geometries. We compared two commutation relations and shown that in case of Finsler geometry, the generators will be less and some symmetries are broken. So, we have seen that in such metrics in case of Riemannian geometry we have maximal symmetry. The comparing two geometries for the calculating Killing vectors of FRW and Gödel background may give some new idea in VSR.

6. References
[1] Gödel K 1949 Rev. Mod. Phys 21 447
[2] Hawking S W and Ellis G F R 1973 The Large Scale Structure of Space-time (Cambridge University Press: Cambridge) chapter 5
[3] Hawking S W 1991 *Phys. Rev. D* **44** 3802
[4] Brecher D, Danielsson U H, Gregory J P and Olsson M E 2003 *JHEP* **11** 033
[5] Bao D, Chern S S and Shen Z 2000 *An Introduction to Riemannian-Finsler Geometry* (New York: Springer)
[6] Wang H C and Lond J 1947 *Math. Soc. S.* **1** 221 5
[7] Li X and Chang Z 2010 *Chinese Phys. C* **34** 28
[8] Huang X B 2007 Covariant theory of gravitation in the spacetime with Finsler structure Preprint gr-qc/0710.5803
[9] Pfeifer Ch and Wohlfarth M 2014 *Springer Proc. Phys.* **157** 305
[10] Gibbons G W, Gomis J and Pope C N 2007 *Phys. Rev. D* **76** 081701
[11] Zhang L and Xue X 2012 The Finsler type of space-time realization of deformed very special relativity Preprint math-ph/1205.1134
[12] Antonelli P L, Ingarden R S and Matsumoto M 1993 *The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology* (Dordrecht: Springer)
[13] Chandrasekhar S and Wright J P 1961 *Proc. Natl. Acad. Sci.* **47** 3 341
[14] Nakahara M 2003 *Geometry, Topology and Physics* (Institute of Physics Publ. Bristol and Philadelphia)
[15] Hari Dass N D and Desiraju H 2016 Killing vector of FLRW metric (in comoving coordinates) and zero modes of the scalar Laplacian Preprint gr-qc/1511.07142
[16] Li X, Wang S and Chang Z 2014 *Commun. Theor. Phys.* **61** 6 781
[17] Narasimhamurthy S K and Latha Kumari G N 2015 *Acta Mathematica Academiae Paedagogicae Nyiregyhaziensis* **31** 97
[18] Stavrinos P C, Kouretsis A P and Stathakopoulos M 2008 *General Relativity and Gravitation* **40** 1403
[19] Pfeifer C and Wohlfarth M N R 2012 *Phys. Rev. D* **85** 064009
[20] Rooman M 1998 *Class. Quant. Grav.* **15** 3241