P-V criticality of AdS black holes in a general framework

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Interpreting the cosmological constant as pressure, it has been observed that AdS black holes behave as van der Waals system. The critical exponents for the phase transition for all AdS black holes are exactly same as those for the van der Waals system. Till now this has been observed case by case. Here, without using any specific form of the black hole metric, we present a general framework based on just two universal inputs. These are the general forms of the Smarr formula and the first law of thermodynamics. We find that the same values of the critical exponents can be obtained by this general analysis. Therefore there is no need to investigate for a particular metric.

The importance of our analysis is that it highlights the observed universality, as well as reveals the reason for such universality.

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I. INTRODUCTION

Starting from the pioneering works of Bekenstein \cite{Bekenstein1973} and Hawking \cite{Hawking1975}, it is now well established that black hole horizons have thermodynamic structure, irrespective of gravitational theories, which is quite similar to the usual thermodynamics (For more, see \cite{Cai2008}). Within this semiclassical approach, such a similarity provides us ways to explore different aspects of gravity. Several investigations strongly suggest that it can be one of the fruitful methods to deal gravity.

In conventional thermodynamics, phase transition is an important subject – it explains how a substance undergoes from one phase to another. For black holes, in the same spirit like the standard one, the investigation of phase transition started long ago by the work of Davis \cite{Davis1980} and later in a more rigorous way by Hawking and Page \cite{Hawking1987}. Subsequently, different other ways have been found to investigate different phases of a black hole. See \cite{Bhattacharyya2009, Cardoso2009, Cardoso2010, Cardoso2011, Cardoso2013, Cardoso2014} for a complete list of them. Among them, recently two approaches attracted a lot of attentions. In one approach, one mainly looks for the divergence of specific heat and later in a more rigorous way by Hawking and Page \cite{Hawking1987}. Subsequently, different other ways have been found to investigate different phases of a black hole. See \cite{Bhattacharyya2009, Cardoso2009, Cardoso2010, Cardoso2011, Cardoso2013, Cardoso2014} for a complete list of them. Among them, recently two approaches attracted a lot of attentions. In one approach, one mainly looks for the divergence of specific heat and inverse of the isothermal compressibility \cite{Cardoso2009, Cardoso2010, Cardoso2011, Cardoso2013, Cardoso2014}. Other approach is confined to the AdS black holes. Here the cosmological constant $\Lambda$ has been regarded as pressure term. As a result, the obtained Smarr formula and the first law of thermodynamics are in “non-standard” form \cite{Majhi2010}. An interesting feature of this approach is that thermodynamical variables of black hole satisfy van der Waal like equation of state. Moreover, the critical exponents are identical to those of standard van der Waals system \cite{Majhi2010}. Recently, this idea has been used in many research papers. See \cite{Bhattacharyya2009, Cardoso2009, Cardoso2010, Cardoso2011, Cardoso2013, Cardoso2014} for an extensive (not exhaustive) list. For a review, see \cite{Bhattacharyya2013, Cardoso2015}.

Clearly, in the last few years people put lots of effort and time to explore phase transition of different solutions of black holes, mainly in the context of above mentioned two approaches. The common feature of these studies came out that values of the critical exponents are same for all metrics. Of course, the values found in one approach do not match with those in the other. Nevertheless, the universality within its own domain is very interesting and may have some valuable implications. Naturally one question arises: what makes them universal? The answer to this is very important as it will explain the underlying reason of such universality and provides better understanding of phase transition. Moreover, it will help us to gain the knowledge about the critical exponents without going into the details of the black holes and hence one does to need to study case by case.

Very recently we provided a general approach of exploring the critical exponents in the context of first approach \cite{Majhi2015}. We found that, general method is much more easy and transparent compared to the metric specific approach. In this paper our goal is the same. Here we want to look for a similar general discussion for the van der Waal’s type method without invoking any specific AdS black hole metric. Reassuringly, a successful methodology has been found. This is the main result of our paper. It will be discussed in detail in the next section.

Before going into the main analysis, let us mention the underlying inputs to achieve the goal. Since the general structure of the Smarr formula and the first law of thermodynamics are very much universal, they are considered to be valid in all black holes. On top of that we assume that there is a van der Waal like critical point about which two phases of black holes exist. With these simple ingredients it will be shown that all the goals can be successfully achieved. Moreover in this way the reason for this universality will be unfolded. We shall see that the unnecessary coefficients, which actually depends on the specific structure of the black holes, do not have any contribution to the values of the critical exponents.

Notations: $C_V$ and $K_T$ denote the specific heat at constant volume and isothermal compressibility at constant

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temperature respectively. The critical values of pressure, temperature and volume are specified by $P_c$, $T_c$ and $V_c$. We mark order parameter, the difference between the volumes of two phases, by $\eta$.

II. CRITICAL PHENOMENA: A UNIFIED PICTURE

The critical exponents for a van der Waals thermodynamic system are defined as

$$C_V \sim |t|^{-\alpha} ;$$
$$\eta \sim V_l - V_s \sim |t|^\beta ;$$
$$K_T = -\frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_T \sim |t|^{-\gamma} ;$$
$$P - P_c \sim |V - V_c|^\delta ;$$

where $t = T/T_c - 1$. The critical point is a point of inflection, denoted by two conditions:

$$\frac{\partial P}{\partial V} = 0 = \frac{\partial^2 P}{\partial T^2} .$$

We shall use these conditions together with the basic definitions to find the values of the critical exponents for a AdS black hole.

To start with, let us consider the two general results which are universally established: one is the general form of Smarr formula and the other one is the first law of thermodynamics. The Smarr formula for a $D$-dimensional AdS black hole, in general, can be expressed as

$$M = f_1(D)TS - f_2(D)PV + f_3(D)XY ,$$

where $M$, $T$ and $S$ are mass, temperature and entropy of the horizon while $P$ and $V$ are thermodynamic pressure and volume, respectively. The values of the unknown functions $f_1, f_2, f_3$ depend on the spacetime dimensions only. For example, in $D = 4$, one has $f_1 = 2$, $f_2 = 2$ and $f_3 = 1$. We shall see that the explicit forms of them are not required for our main purpose. In the above equation, $X$ stands for electric potential $\Phi$ or angular velocity $\Omega$ or both whereas $Y$ refers to charge $Q$ or angular momentum $J$ or both for a black hole. The corresponding first law of thermodynamics is given by

$$dM = TdS + XdY + VdP .$$

Note that, in the above expression $M$ is not the energy and the present form is not identical to the standard form of first law. But this can be expressed in the standard form:

$$dE = TdS + XdY + PdV$$

if one identifies the energy of the black hole as

$$E = M - PV .$$

Thus $M$ is the enthalpy of black hole. Using (3), the Helmholtz free energy ($F$) of the system is found to be

$$F = E - TS = (f_1 - 1)TS - (f_2 + 1)PV + f_3XY .$$

Now for a black hole, if one keeps $Y$ constant then mass ($M$) has to be a function of pressure ($P$) and location of the horizon ($r_+$); i.e. $M = M(r_+, P)$. The reason is as follows. The location of the horizon is defined by the vanishing of the metric coefficient and in general this coefficient is a function of $M$, $Y$ and $P$ (as $P = -\Lambda/(8\pi)$ for a AdS black hole. Therefore, $r_+ = r_+(M, Y, P)$ and hence for a fixed value of $Y$, $M$ is a function of $r_+$ and $P$. On the other hand, we have $S = S(r_+), V = V(r_+)$ and in general $X = X(r_+)$. Therefore the Smarr formula (3) shows that for a fixed value of $Y$, one finds the solution for pressure from the equation

$$P = \frac{f_1TS(r_+)}{f_2V(r_+)} - \frac{M(r_+, P)}{f_2V(r_+)} + \frac{f_3X(r_+)Y}{f_2V(r_+)} ,$$

which has to be function of both temperature and horizon radius; i.e. $P = P(T, r_+)$. Hence the free energy (7) is in general a function of both $T$ and $r_+$. Since entropy can be calculated from the relation $S = -(\partial F/\partial T)_V \equiv -(\partial F/\partial T)_{r_+}$ as $V = V(r_+)$, we find from (7)

$$\left( \frac{\partial F}{\partial T} \right)_{r_+} = (f_1 - 1)S(r_+) - (f_2 + 1)V(r_+) \left( \frac{\partial P}{\partial T} \right)_{r_+} .$$

The last term on the right hand side can be evaluated as follows. Use of (8) yields

$$\left( \frac{\partial P}{\partial T} \right)_{r_+} = \frac{f_1S}{f_2V} - \frac{1}{f_2V} \left( \frac{\partial M}{\partial T} \right)_{r_+} = \frac{f_1S}{f_2V} - \frac{1}{f_2V} \left( \frac{\partial M}{\partial P} \right)_{r_+} \left( \frac{\partial P}{\partial T} \right)_{r_+} .$$

In the last step we have used the fact that for fixed $r_+$, $T$ is a function of $P$ only. On the other hand, from the first law of thermodynamics (4) it is easy to verify that $(\partial M/\partial P)_{r_+} = V$. Therefore the above reduces to $(\partial P/\partial T)_{r_+} = (f_1S)/(f_2V) - (1/f_2)(\partial P/\partial T)_{r_+}$. The solution turns out to be

$$\left( \frac{\partial P}{\partial T} \right)_{r_+} = \frac{f_1}{1 + f_2V} .$$

Substitution of this in (9) gives us

$$S = -\left( \frac{\partial F}{\partial T} \right)_{r_+} = S(r_+) ;$$

i.e. we obtain the original expression for horizon entropy which is independent $T$. Therefore the specific heat at constant volume $C_V = T(\partial S/\partial T)_V = 0$ and hence by (1) one finds the value of the critical exponent $\alpha = 0$.

Next we shall find the other critical exponents. Remember that the pressure here is in general a function of both $T$ and $r_+$. Now since thermodynamic volume $V$ is a function of $r_+$ only, we take $P$ as function of $T$ and $V$ for our purpose. Let us expand $P(T, V)$ around the
critical values $T_c$ and $V_c$:

\[ P = P_c + \left[ \left( \frac{\partial P}{\partial T} \right)_{V_c} \right] (T - T_c) \]

\[ + \frac{1}{2!} \left[ \left( \frac{\partial^2 P}{\partial T^2} \right)_{V_c} \right] (T - T_c)^2 \]

\[ + \left[ \left( \frac{\partial^2 P}{\partial T \partial V} \right)_{V_c} \right] (T - T_c)(V - V_c) \]

\[ + \frac{1}{3!} \left[ \left( \frac{\partial^3 P}{\partial V^3} \right)_{T_c} \right] (V - V_c)^3 + \ldots \]  \hspace{1cm} (13)

In the above expression, we have used the fact that at the critical point $(\partial P/\partial V)_c = 0 = (\partial^2 P/\partial V^2)_c$. Defining two new variables $t = T/T_c - 1$ and $\omega = V/V_c - 1$, and above expression is written as,

\[ P = P_c + Bt + Bt\omega + D\omega^3; \]  \hspace{1cm} (14)

where $R, B, D,$ etc. are constants calculated from the derivatives at the critical point. Here we have ignored the other higher order terms since they are very small.

Now for constant $t$, we find $dP = (Bt + 3D\omega^2) d\omega$ and so the Maxwell’s equal area law \[ \int \omega (Bt + 3D\omega^2) d\omega + \int \omega (Bt + 3D\omega^2) d\omega = 0 , \]  \hspace{1cm} (15)

where $\omega_l$ and $\omega_s$ denote the volumes for large and small black holes, respectively. Since the pressure does not change i.e. $P_l = P_s$, 

\[ \int dP = Bt(\omega_l - \omega_s) + D(\omega_l^3 - \omega_s^3) = 0 . \]  \hspace{1cm} (16)

So the second integral of (15) vanishes. Therefore it reduces to

\[ Bt(\omega_l^2 - \omega_s^2) + \frac{3D}{2}(\omega_l^4 - \omega_s^4) = 0 . \]  \hspace{1cm} (17)

One can now easily find the non-trivial solutions of the above two equations. These are $\omega_l = (-Bt/D)^{1/2}$ and $\omega_s = -(Bt/D)^{1/2}$. Therefore we find

\[ \eta \sim V_l - V_s = (\omega_l - \omega_s)V_c \sim |t|^{1/2} , \]  \hspace{1cm} (18)

which yields $\beta = 1/2$.

To find $\gamma$ we need to calculate $K_T$, given in (1). So we first find $(\partial P/\partial V)_T$ from (14). Upto first (leading) order it is given by

\[ \left( \frac{\partial P}{\partial V} \right)_T \simeq \frac{B}{V_c} t , \]  \hspace{1cm} (19)

where one needed to use $\partial\omega/\partial V = 1/V_c$. Therefore the value of $K_T$ near the critical point is

\[ K_T \simeq \frac{1}{Bt} \sim t^{-1} . \]  \hspace{1cm} (20)

This implies $\gamma = 1$. Next for $T = T_c$, the expression (14) for pressure yields 

\[ P - P_c \sim \omega^3 \sim (V - V_c)^3 ; \]  \hspace{1cm} (21)

i.e. the value of the critical exponent is $\delta = 3$. It must be pointed out that the derived critical exponents satisfy the following scaling laws:

\[ \alpha + 2\beta + \gamma = 2 ; \eta = \beta(\delta - 1) . \]  \hspace{1cm} (22)

As is well known, these scaling laws are universal in nature. Their thermodynamic analysis can be found in [51].

### III. CONCLUSIONS

Using the tools of thermodynamics, phase transition of completely dissimilar systems like chemical, magnetic, hydrodynamic etc. has been studied thoroughly. Black hole phase transition is a relatively new observation which deserves careful study. Though the first order phase transition from non-extremal to extremal black hole is known for some time, there are various other types of phase transition. A new type of phase transition has been found recently where cosmological constant ($\Lambda$) is treated as dynamical variable (instead of a constant) equivalent to pressure of hydrodynamic system [14]. In this interpretation, phase transition of different black holes has been shown to be quite analogous to van der Waal’s system. Interestingly critical exponents found from different metrics are same. This naturally deserves some explanation.

It is well known that critical exponents of quite different systems can be same. But for black holes, this point is not well appreciated. In our previous work [13] we showed that, starting from few very general assumptions critical exponents can be calculated without making any choice about specific black hole metric. In that work we did not treat $\Lambda$ as a dynamical variable. Present paper is a continuation of our previous work. Here we take a different interpretation of $\Lambda$. In this work we showed that, without taking any specific black hole, values of critical exponents can be calculated.

Till now calculations of critical exponents for this type of systems have been done case by case. It remained unexplained why these values are same for different AdS black holes. In this paper, by providing a general treatment we give a completely satisfactory answer to this question. Thus our present work fills an important gap in the understanding of black hole phase transition.

It may be mentioned that, the analysis shows that the values of the critical exponents are very much universal in nature and also they are independent of spacetime dimensions. In literature, different critical exponents have been found very recently [47], usually known as “nonstandard” critical exponents, for black hole solutions in higher curvature gravity theory. These are not due to the analysis around the standard critical point; rather around isolated one. In that respect our analysis is different from them. Of course it would be very much worthwhile to look at such non-standard phase transition.

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