Real Kähler Submanifolds in Codimension 6

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Abstract

We show that a real Kähler submanifold in codimension 6 is essentially a holomorphic submanifold of another real Kähler submanifold in lower codimension if the second fundamental form is not sufficiently degenerated. We also give a shorter proof of this result when the real Kähler submanifold is minimal, using recent results about isometric rigidity.

1 Introduction

Let $M^{2n}$ be a Kähler manifold of real dimension $2n$. A real Kähler submanifold of codimension $p$ is an isometric immersion $f : M^{2n} \to \mathbb{R}^{2n+p}$. The structure of this submanifolds are sensitive to its codimension. For instance, by [9], when the codimension is one and $M^{2n}$ is complete then $f$ must be the product of an isometric embedding of a complete surface $g : \Sigma^2 \to \mathbb{R}^3$ with the identity map of $\mathbb{C}^{n-1}$. In other words, surfaces in $\mathbb{R}^3$ are essentially the only real Kähler submanifolds of codimension one. In codimension two, the situation is also well-understood. Namely, the minimal case was analyzed in details in [3], and the non-minimal case was classified in [10]. In codimension three, it was proved in [4] that unless the submanifold $M^{2n}$ is a holomorphic hypersurface of a real Kähler submanifold of codimension one, the complex codimension of the invariant part of the relative nullity, under the almost complex structure of the Kähler manifold, has to be less than or equal to 3. Recently, this result was extended in [12] for codimension 4. They also conjectured that the result holds for higher codimensions.

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Our goal in this paper is twofold. First we give a simple proof of this conjecture when the immersion is minimal and $p \leq 6$ using the modern tools developed in [2]. More precisely, we are going to prove the following result.

**Theorem A.** Let $f : M^{2n} \to \mathbb{R}^{2n+p}$ be a real Kähler minimal submanifold with $p \leq 6$ and $\nu_f$ be the index of relative nullity of $f$. Then, along each connected component $U$ of an open dense subset of $M^{2n}$, one of the following holds:

(i) $\nu_f(x) \geq 2n - 2p$ for all $x \in U$;

(ii) $f$ extends to a real Kähler submanifold, that is, there exist a real Kähler submanifold $g : N^{2n+2s} \to \mathbb{R}^{2n+p}$ and a holomorphic embedding $h : U \to N^{n+s}$ such that $f|_U = g \circ h$. Moreover, the extension $g$ is also minimal.

**Remark 1.** A real Kähler submanifold is holomorphic if and only if admits a real Kähler extension of codimension zero.

The second main goal is to remove the minimal hypothesis of Theorem A. For this, we use similar arguments from the ones used in [12], that is, we use the complexified second fundamental of $f$ in order to avoid the necessity of the existence of a conjugate immersion of the minimal real Kähler submanifold. The goal is to prove the following theorem.

**Theorem B.** Let $f : M^{2n} \to \mathbb{R}^{2n+p}$ be a real Kähler submanifold with $p \leq 6$ and $\nu_f$ be the index of relative nullity of $f$. Then, along each connected component $U$ of an open dense subset of $M^{2n}$, one of the following holds:

(i) $\dim \Delta \cap J\Delta \geq 2n - 2p$ for all $x \in U$, where $\Delta$ is the relative nullity and $J$ is the almost complex structure of $f$. In particular, $\nu_f(x) \geq 2n - 2p$ for all $x \in U$;

(ii) $f$ extends to a real Kähler submanifold, that is, there exist a real Kähler submanifold $g : N^{2n+2s} \to \mathbb{R}^{2n+p}$ and a holomorphic embedding $h : U \to N^{n+s}$ such that $f|_U = g \circ h$.

Even though Theorem B implies Theorem A, we have that the proof of Theorem A is quite simpler and is almost a direct consequence of Theorem 14 in [2]. On the other hand, the proof of Theorem B will require a structure that is similar to the one constructed in [2].

This paper is organized as follows. In Section 2 we establish the fundamental concepts and notations. In Section 3 we prove Theorem A and, in Section 4 we construct specific subbundles in order to obtain Theorem B.
2 Preliminaries

In this section we fix the notation and discuss the main tools that will be used in the work. We divide this section in three parts. First, we define the flat bilinear forms that will be used to identify the subbundles necessary to construct the extensions in Theorem A and Theorem B. Second, we study the structure of the tangent and normal bundles of a pair of isometric immersions and their extensions, both given by [2]. Finally, we review some properties about the complexification of the second fundamental form that will be used to obtain the main algebraic lemma required for Theorem B.

Throughout this section, we assume that $f : M^{2n} \to \mathbb{R}^{2n+p}$ is a real Kähler submanifold with almost complex structure $J$.

2.1 Flat bilinear forms

Let $W^{p,q}$ be a $(p+q)$-dimensional vector space endowed with a possibly indefinite inner product of signature $(p,q)$, where $q \geq 0$ is the maximal dimension of the subspaces in which the inner product is negative definite. Let $U$ and $V$ be finite dimensional vector spaces. A bilinear form $\gamma : U \times V \to W^{p,q}$ is said to be flat if

$$\langle \gamma(X, Z), \gamma(Y, W) \rangle - \langle \gamma(Y, Z), \gamma(X, W) \rangle = 0,$$

for all $X, Y \in U$ and $Z, W \in V$. Denote the (left) nullity of $\gamma$ by

$$N(\gamma) = \{X \in U : \gamma(X, Z) = 0 \text{ for all } Z \in V\}$$

and the span of $\gamma$ by

$$S(\gamma) = \operatorname{span}\{\gamma(X, Z) : X \in U, Z \in V\}.$$ 

A subset $S \subset W^{p,q}$ is called null if $\langle \xi, \eta \rangle = 0$ for every $\xi, \eta \in S$. We call $X \in U$ a (left) regular element of $\gamma$ if

$$\dim \gamma^X(V) = \max_{Y \in U} \{\dim \gamma^Y(V)\},$$

where $\gamma^X(Z) = \gamma(Y, Z)$. We will use such elements to obtain the estimates of Theorem B.

Flat bilinear forms were introduced by Moore in [11] to study isometric immersions of the round sphere in Euclidean space in low codimension, and can be used to obtain several results about isometric rigidity (see [6] for more details).
2.2 The genuine structure

The work [2] provides, in addition to the main results on genuine rigidity, a strong local geometric structure of any pair of isometric immersions of a given Riemannian manifold. We will make use of such structure in the minimal case. Assume that \( f \) is a minimal isometric immersion and define

\[
f_\theta(x) = \int_{x_0}^x f_\ast \circ J_\theta
\]

(2)

to be the well-known one-parameter associated family to \( f \), where \( x_0 \) is a fixed point in \( M^{2n} \) and \( J_\theta = \cos \theta I + \sin \theta J \) (see [3] or [6] for more details). We represent by \( p f \) the conjugated immersion of \( f \) and, by (2), we can identify \( T K f M \) with \( T K p f M \).

Applying Theorem 11 from [2] for the pair \( t f, p f \) we have a special triple \( p \tau, L, L^\perp \), where \( \tau : L \subset T^0 f M \rightarrow \hat{L} \subset T^1 f M \) is a vector bundle isometry and \( D^d = \mathcal{N}(\alpha L^\perp) \cap \mathcal{N}(\hat{\alpha L^\perp}) \subset TM \) satisfies the following conditions.

\[
\begin{align*}
(C_1) & \quad \text{The isometry } \tau \text{ is parallel and preserves second fundamental forms, that is, } (\hat{\nabla}_X \tau \xi)_L = \tau (\hat{\nabla}_X \xi)_L \text{ for all } \xi \in L, X \in TM, \text{ and } \\
& \quad \tau \circ \alpha f = \alpha \hat{f}; \\
(C_2) & \quad \text{The subbundles } L \text{ and } \hat{L} \text{ are parallel along } D^d \text{ in the normal connections.}
\end{align*}
\]

(3)

As in [2], the bilinear form \( \phi_\tau : (TM \oplus L) \times TM \rightarrow L^\perp \times \hat{L}^\perp \) given by

\[
\phi_\tau(Y + \xi, X) = \left( (\hat{\nabla}_X (Y + \xi))_{L^\perp}, (\hat{\nabla}_X (JY + \tau \xi))_{\hat{L}^\perp} \right),
\]

(4)

plays a key role in the construction of the isometric extensions of \( f \) and \( \hat{f} \). Observe that condition \((C_1)\) is equivalent to the flatness of \( \phi_\tau \).

We give an idea on how the extensions of \( f \) and \( \hat{f} \) are constructed. Choose any smooth rank \( s \) subbundle \( \Lambda \subset TM \oplus L \), and define the maps \( F_{\Lambda, f} : \Lambda \rightarrow \mathbb{R}^{2n+p} \) and \( \hat{F}_{\Lambda, f} : \Lambda \rightarrow \mathbb{R}^{2n+p} \) by

\[
F(v) = f(x) + v, \quad \hat{F}(v) = \hat{f}(x) + (J \oplus \tau)v, \quad v \in \Lambda_x, x \in M^{2n}.
\]

(5)

We say that \( F_{\Lambda, f} \) is a extension of \( f \) if it is an immersion in some open neighborhood of the 0-section of \( \Lambda \). In particular, if \( \Lambda \) is transversal to \( TM \) then \( F_{\Lambda, f} \) is an extension. By Proposition 12 in [7], we have that the two maps \( F_{\Lambda, f} \) and \( \hat{F}_{\Lambda, f} \) are isometric if and only if \( \phi_\tau(\Lambda, TM) \) is a null subspace.
By Proposition 9 in [2], we have that $f$ and $\hat{f}$ are mutually $D^d$-ruled when there is no such extensions. By $D^d$-ruled (or just $d$-ruled) we mean that the leaves of the distribution $D^d$ (a $d$-dimensional distribution) are mapped diffeomorphically by $f$ to (open subsets of) affine subspaces of $\mathbb{R}^{n+p}$.

Although we can extend the pair \{f, $\hat{f}$\}, we are just interested in the extensions of $f$ and its almost complex structure to obtain Kähler extensions. For the sake of clarity, we will enunciate a version of Proposition 9 in [2] for a real Kahler submanifold. Let $T : L' \rightarrow L'$ be a vector bundle isometry that satisfies conditions ($C_1$) and ($C_2$) for $D = \mathcal{N}(\alpha_{L'})$. By Lemma 8 in [2], the distribution $\mathcal{D}$ is integrable and $\mathcal{N}(\phi_T) \cap TM = D$ holds. Define $F = F_\Lambda$, where $\Lambda$ is given by $\mathcal{N}(\phi_T) = D \oplus \Lambda$. Then, we have the following proposition.

**Proposition 2.** The isometric immersion $F$ is a $\mathcal{N}(\phi_T)$-ruled real Kähler extension of $f$. Moreover, there is a smooth orthogonal splittings $T_F^\perp N = \mathcal{L} \oplus L'$ and a vector bundle isometry $T : \mathcal{L} \rightarrow \mathcal{L}$ such that $\mathcal{N}(\phi_T) = \mathcal{N}(\alpha_{T})$, and the triple $(T, \mathcal{L}, \mathcal{N}(\phi_T))$ satisfies conditions ($C_1$) and ($C_2$) in [2].

**Proof.** It is a corollary from Proposition 9 in [2] once observed that $\Lambda$ is $T$-invariant and the almost complex structure of $\Lambda$ is the parallel isometry $(J \oplus T)|_\Lambda : \Lambda \rightarrow \Lambda$ (in the induced connection). ■

### 2.3 The structure of the complexified second fundamental form

Our objective is to show that, at a generic point $x \in M^{2n}$, the second fundamental form takes a rather special form. We shall begin with the following definitions.

We denote by $\Delta(x)$ the *relative nullity* of $f$ at $x$, that is,

$$\Delta(x) = \{ Z \in T_x M : \alpha(Z, Y) = 0 \text{ for all } Y \in TM \},$$

and by $\nu_f(x)$ the index of the relative nullity of $f$ at $x$, i.e., $\nu_f(x) = \dim_{\mathbb{R}} \Delta(x)$. Let us define the complex subspace $\Delta_0 \subset T_x M$ by $\Delta_0(x) = \Delta(x) \cap J\Delta(x)$, and denote its complex dimension by $\nu'(x) = \dim_{\mathbb{C}} \Delta_0(x)$. Define the *pluriharmonic nullity* $\Delta_J$ of $f$ by

$$\Delta_J := \{ Z \in TM : \alpha(Z, JY) = \alpha(JZ, Y), \text{ for all } Y \in TM \},$$

and observe that $\Delta_0 = \Delta \cap \Delta_J$.

We can naturally extend $J$ to a linear operator on the complexified tangent space $T_x M \otimes \mathbb{C}$ then let $V$ be the eigenspaces associated to the
eigenvalue $i$ of $J$, thus $V$ is the complex subspace of $T_x M \otimes \mathbb{C}$ defined as $V = \{X - iJX : X \in T_x M\}$. Denote by $W$ the complex linear subspace of $V$ perpendicular to $\Delta_0$, that is, $W \oplus \overline{W} = \Delta_0 \otimes \mathbb{C}$.

Let us now recall the following decomposition of the second fundamental form $\alpha$ of $f$ at $x \in M$. Extend $\alpha$ bilinearly over $\mathbb{C}$ and the inner product in $T_x M$ bilinearly over $\mathbb{C}$, and still denote it by $\alpha$,

$$\alpha : T_x M \otimes \mathbb{C} \times T_x M \otimes \mathbb{C} \rightarrow T_x^2 M \otimes \mathbb{C}.$$  

Using that $T_x M \otimes \mathbb{C} = V \oplus \overline{V}$, we can write

$$H = \alpha|_{V \times V} \quad \text{and} \quad S = \alpha|_{V \times V}$$

for the $(1, 1)$ and $(2, 0)$ parts of $\alpha$, respectively. Let $W' \subset V$ be the complex linear subspace given $W' = \ker H \cap \ker S$. Hence, $\Delta_0 \otimes \mathbb{C} = W' \oplus \overline{W'}$ and $V = W \oplus W'$. Notice that $\ker H = \Delta_J$ and, by Lemma 7 in [8], we have

$$\dim \Delta_J \geq 2n - 2p.$$  

As observed in [8], the Kählerness of $M$ implies that the Hermitian bilinear form $H$ and the symmetric bilinear form $S$ satisfy the following symmetry conditions

$$\langle H(X, \overline{Y}), H(Z, \overline{W}) \rangle = \langle H(Z, \overline{Y}), H(X, \overline{W}) \rangle$$  

(2.6)  

$$\langle H(X, \overline{Y}), S(Z, W) \rangle = \langle H(Z, \overline{Y}), S(X, W) \rangle$$  

(2.7)  

$$\langle S(X, Y), S(Z, W) \rangle = \langle S(Z, Y), S(X, W) \rangle$$  

(2.8)

for any $X, Y, Z, W \in V$.

### 3 Minimal Case

The aim of this section is to prove Theorem A. Since $f$ is minimal, as discussed in Section 2, there is a pair of isometric immersions $\{f, \tilde{f}\}$ and a triple $(\tau, L, D)$ that satisfies the conditions $(C_1)$ and $(C_2)$. We have the following result.

**Lemma 3.** Let $E \subset L$ be the maximal invariant subspace by $\tau$. Then, the dimension of $E$ is even and $D = \mathcal{N}(\alpha_{L^\perp}) \cap \mathcal{N}(\tilde{\alpha}_{L^\perp}) = \mathcal{N}(\alpha_{E^\perp})$.

**Proof.** It follows directly from the involution propriety of $\tau$ that the dimension of $E$ is even. Since $E \subset L \cap \tilde{\hat{L}}$, we have that $(L \cap \tilde{\hat{L}})^\perp \subset E^\perp$. So, it
is clear that, $\mathcal{N}(\alpha_{E^\perp}) \subset \mathcal{N}(\alpha_{L^\perp}) \cap \mathcal{N}(\tilde{\alpha}_{L^\perp})$. Consider the orthogonal decomposition $E^\perp = L^\perp \oplus V$, where $V$ is the orthogonal complement of $E$ in $L$. Take an unit vector $\xi \in E^\perp$ and define the following sequence $\xi = \xi_0^V + \xi_0^{L^\perp}$ and $\tau(\xi_i^V) = \xi_{i+1}^V + \xi_{i+1}^{L^\perp}$, where $\xi_{i+1}^V$ is the $V$-component and $\xi_{i+1}^{L^\perp}$ is the $L^\perp$-component of $\tau(\xi_i^V)$.

Note that the norm of the sequence $\xi_i^V$ satisfies

$$0 \leq \ldots \leq \|\xi_{i+1}^V\| \leq \|\xi_i^V\| \leq \ldots \leq \|\xi_0^V\| \leq 1.$$ 

Therefore, any convergent subsequence of $\xi_i^V$ converges to a vector with norm equal to $\lim_{i \to \infty} \|\xi_i^V\|$. Then, up to a subsequence, $\xi_i^V$ converges to $w \in V$ and so $\tau(\xi_i^V) = \xi_{i+1}^V + \xi_{i+1}^{L^\perp}$ converges to $w' \in V$. In fact, since $\tau$ is an isometry and $\xi_i^V, \xi_{i+1}^{L^\perp}$ converges to vectors of the same norm then the $L^\perp$-component of $\tau(\xi_i^V)$ converges to 0. Then it is easy to see that $w = w' = 0$, otherwise $E \oplus \text{span}\{w, w'\}$ is invariant by $\tau$ which contradicts the maximality of $E$.

Let $X \in \mathcal{N}(\alpha_{L^\perp}) \cap \mathcal{N}(\tilde{\alpha}_{L^\perp})$ be an unit vector. By definition of the sequence we have that

$$A_{\xi} X = A_{\xi_0}X = -JA_{\tau\xi}X = -JA_{\tau\xi_i^V}X = \ldots = \pm -JA_{\xi_i^V}X,$$

for all $i \in \mathbb{N}$. By taking the limit it follows that $A_{\xi} X = 0$ for all $\xi \in E^\perp$ and $X \in \mathcal{N}(\alpha_{L^\perp}) \cap \mathcal{N}(\tilde{\alpha}_{L^\perp})$ and therefore $\mathcal{N}(\alpha_{L^\perp}) \cap \mathcal{N}(\tilde{\alpha}_{L^\perp}) \subset \mathcal{N}(\alpha_{E^\perp})$ which finishes the proof of the lemma.

\[\blacksquare\]

**Remark 4.** Note that the maximal invariant $E$ is given by $L \cap \tau(L)$ and, in particular, it has dimension at least $2l - p$.

Lemma 3 implies that $(\tau = \tau|_E, E^{2k}, D^d)$ also satisfies the conditions (C1) and (C2) in [3]. Then by Proposition 9 in [2] we have an isometric extension $F_{\Lambda,f}$ (possibly trivial), where $\Lambda$ is given by the orthogonal splitting $\mathcal{N}(\phi) = D \oplus \Lambda$ and is trasversal to $TM$. Note that $J \oplus \tau : TM \oplus E \to TM \oplus E$ is an isometry and parallel in the induced connection. Thus, if $f$ extend isometrically then the extension is Kähler. Otherwise, $f$ is $D^d$-ruled and from remark 20 in [2] we have $d \geq 2n - 2p + 2l$. Moreover, if $E = L$ and $p \leq 6$ (See Remark 5) it follows from Theorem 14 in [2] a better estimative, namely, $d \geq 2n - 2p + 3l$.

**Remark 5.** Theorem 14 in [2] and Lemma 3 implies that if $p = 6$ and $l = 0$ then $D^d$ is the relative nullity and $d \geq 2n - 2p - 1$. Furthermore, since $D^d$ is $J$-invariant then $d$ is even and so, in fact, $d \geq 2n - 2p$.

The discussion above yields the following proposition.
Proposition 6. Let \( f : M^{2n} \to \mathbb{R}^{2n+p} \) be a real Kähler minimal submanifold with \( p \leq 6 \) and \( \hat{f} = f_{\pi/2} : M^{2n} \to \mathbb{R}^{2n+p} \) its conjugate. Then, the pair \( \{f, \hat{f}\} \) are either mutually \( D^d \)-ruled with \( d \geq 2n - 2p + 2l \) or extends to a real Kähler submanifold (locally). Moreover, if \( E = L \) and the pair \( \{f, \hat{f}\} \) does not extend, then \( d \geq 2n - 2p + 3l \).

We now give a complete proof of Theorem A using the triple \( (\tau, E^{2k}, D^d) \) above.

Proof of theorem A. It follows from Proposition 6 that, along each connected component of an open dense subset of \( M^{2n} \), either \( f \) and \( \hat{f} \) extend isometrically to a real Kähler submanifold, or \( \{f, \hat{f}\} \) are mutually \( d \)-ruled, with \( d \geq 2n - 2p + 2l \). If the first case occurs the conclusion of the theorem is immediate. So we assume that \( \{f, \hat{f}\} \) are mutually \( D^d \)-ruled.

Since \( D^d \) is an asymptotic space we have that \( \\text{ker} A_\xi \cap D^d \subset D^\bot \) for any \( \xi \in T_f^\bot \) and therefore

\[
\dim(\text{ker} A_\xi \cap D^d) \geq 2n + 4l - 4p, \tag{3.1}
\]

for all \( \xi \in T_f^\bot \). Also, from condition \((C_1)\) we have that \( \text{ker} A_\xi = \text{ker} A_\tau \xi \) for all \( \xi \in E^{2k} \), where \( E^{2k} \) is the maximal invariant space of \( \tau \) as in Lemma 3. Thus, the relative nullity \( \Delta \) of \( f \) is the intersection of \( D^d \) and \( \bigcap_{i=1}^k \text{ker} A_\xi_i \) for some basis \( \{\xi_1, \cdots, \xi_k, \tau \xi_1, \cdots, \tau \xi_k\} \) of \( E^{2k} \). Combining the inequalities (3.1) for \( \xi_1, \cdots, \xi_k \), we have that

\[
\nu_f \geq d - k(2p - 2l) \geq (2n - 2p) + 2l(k + 1) - 2pk.
\]

Thus, the theorem follows if we prove that

\[
l(k + 1) - pk \geq 0.
\]

This easily holds for \( p \leq 4 \). For \( p \in \{5, 6\} \) the inequality fails if \( l = 2k = 2 \). In this case we can apply the estimate \( d \geq 2n - 2p + 3l \) since \( L = E \) (see remark 5), and conclude in an analogous way that

\[
\nu_f \geq d - k(2p - 3l) \geq (2n - 2p) + 3l(k + 1) - 2pk \geq 2n - 2p.
\]

\[\blacksquare\]

Remark 7. If the real Kähler submanifold in Theorem A is complete then the rules obtained by Theorem 11 are holomorphic distributions and they are naturally related with the appearance of the complex ruled submanifolds as in [3] and [9] (see [6] for more details).
4 Non-minimal case

For the non-minimal case we are not able to apply directly the main result in [2]. However we are still able to obtain the whole structure of bundles constructed there.

Let $f : M^{2n} \to \mathbb{R}^{2n+p}$ be a real Kähler submanifold with second fundamental form $\alpha$ and endow the vector bundle $T_f^\perp M \oplus T_f^\perp M$ with the indefinite metric of type $(p, p)$ given by

$$\langle \langle \cdot, \cdot \rangle \rangle_{T_f^\perp M \oplus T_f^\perp M} = \langle \langle \cdot \rangle_{\pi_1 T_f^\perp M} - \langle \langle \cdot \rangle_{\pi_2 T_f^\perp M},$$

where $\pi_1$ and $\pi_2$ are the natural projections of $T_f^\perp M \oplus T_f^\perp M$. Set

$$\beta = \alpha(\cdot, \cdot) \oplus \alpha(J\cdot, \cdot) : TM \times TM \to S(\alpha) \oplus S(\alpha),$$

and let $\Omega \subset S(\alpha) \oplus S(\alpha) = S(\alpha) \oplus S(\alpha(J\cdot, \cdot))$ be the vector bundle with null fibers $\Omega = S(\beta) \cap S(\beta)^\perp$. Observe that $\Omega$ is an even dimensional vector bundle by the almost complex structure $J$. Indeed, if $\beta(X, Y) \in \Omega$ and is not zero for some $X, Y \in TM$ then $\beta(JX, Y) \in \Omega$ and $\beta(X, Y), \beta(JX, Y)$ are linearly independents. Accordingly, there is an orthogonal splitting

$$S(\alpha) = \Gamma \oplus \Gamma^\perp$$

where $\Gamma = S(\alpha) \cap \Omega^\perp$, and an isometry $T : \Gamma^\perp \to \Gamma^\perp$ such that

$$\Omega = \{ (\eta, T\eta) : \eta \in \Gamma^\perp \} \subset \Gamma^\perp \oplus \Gamma^\perp$$

and $\alpha_{\Gamma^\perp}(\cdot, \cdot) = T \circ \alpha_{\Gamma^\perp}(J\cdot, \cdot)$. Furthermore,

$$T^2 \circ \alpha_{\Gamma^\perp}(X, Y) = T \circ \alpha_{\Gamma^\perp}(JX, Y) = -\alpha_{\Gamma^\perp}(X, Y).$$

In particular, $T^2 = -I$.

Define $\beta^\perp_T : TM \times TM \to \Gamma \oplus \Gamma$ as the projection of $\beta$ over $\Gamma$ and a vector subbundle $\Theta \subset TM$ by $\Theta = \mathcal{N}(\beta_T)$. The vector subbundle $S \subset \Gamma^\perp$ defined by

$$S = S(\alpha)(\Theta_{TM})$$

is preserved by $T$, by the $J$-invariance of $\Theta$, and satisfies $\Theta = \mathcal{N}(\alpha_{S^\perp}) \cap \mathcal{N}(\alpha_{S^\perp}(J\cdot, \cdot))$. Now define a vector subbundle $S_0 \subset S$ by

$$S_0 = \ker \mathcal{K}(X),$$

where $\mathcal{K}(X) \in \Lambda^2(S)$, $X \in TM$ denotes the skew-symmetric tensor given by

$$\mathcal{K}(X)\eta = (\nabla^X_\eta)S - (\nabla^X_\tau \eta)S.$$
Then we define vector subbundles $L^l \subset S_0$ and $D^d \subset \Theta$ by

$$L^l = \{ \delta \in S_0 : \nabla_Y^l \delta \in S \text{ and } \nabla_Y^l \tau \delta \in S \text{ for all } Y \in \Theta \}$$

and $D^d = \mathcal{N}(\alpha_{L^l}) \cap \mathcal{N}(\alpha_{L^d}(J^\ast, \cdot))$, and let $\tau : L^l \to L^l$ be the induced vector bundle isometry given by $\tau = \tau|_{L^l}$. In particular, $\tau^2 = -I$, where $I$ is the identity map and $l = 2k$ for some integer $k \geq 0$. We can now state an Kähler version of Theorem 11 in [2].

**Theorem 8.** Let $f : M^{2n} \to \mathbb{R}^{2n+p}$ be a real Kähler submanifold. Then along each connected component of an open dense subset of $M^{2n}$ the triple $(\tau, L^l, D^d)$, satisfies $(C_1)$ and $(C_2)$ in [3]. In particular, $f$ has an isometric Kähler ruled extension $F : N^{2n+2k} \to \mathbb{R}^{2n+p}$ (possibly trivial) satisfying the conclusions of Proposition [2]. Moreover, if the extension is trivial then $f$ is $D^d$-ruled.

**Proof.** The proof is the same as the proof of Theorem 11 in [2] once you notice that there is no need for $\beta$ to be symmetric and Lemma 12 in [2] holds if we use the following propriety $R(X, Y) = R(JX, JY)$ for all $X, Y \in TM$. ■

Observe that, at first, we do not have any estimate for the dimension of $D^d$ and the theorem above holds for any codimension (the subbundles can be trivial). Even though we do not have the conjugated isometric immersion as in Theorem A, we have that the flat bilinear form $\beta$ is special. The reason is that the only lemma that requires symmetry in [2] holds for $\beta$, more precisely, we have the following algebraic lemma. In this lemma we will use the nomenclature of kernel and image (ker and Im) instead of nullity and span ($\mathcal{N}$ and $S$) to follow the same notation of [12].

**Algebraic Lemma.** Let $V \simeq \mathbb{C}^n$ and $N \simeq \mathbb{R}^p$ be equipped with inner products, and let $H$ and $S$ be (respectively) Hermitian and symmetric bilinear forms from $V$ into $N_{\mathbb{C}} = N \otimes \mathbb{C}$ satisfying symmetry conditions (2.6)-(2.8). Then, there exist a subspace $E$ of $(\text{Im } H)^\perp \subset N$ (possibly trivial) with real dimension $2k$, and an isometry $\tau$ of $E$ onto itself such that $\tau^2 = -I$ and $\langle S(\cdot, \cdot), \tau \eta \rangle = -i \langle S(\cdot, \cdot), \eta \rangle$ and dim$_{\mathbb{C}}$ ker $H \cap \ker S_{E^\perp} \geq n - p + k$.

**Proof.** When Im $H = \{0\}$, the lemma follows from Theorem 3 in [1] since $S$ is a flat symmetric bilinear form. In fact, this theorem say that $\langle S(\cdot, \cdot), \tau \eta \rangle = -i \langle S(\cdot, \cdot), \eta \rangle$ for all $\eta \in E$ and dim$_{\mathbb{C}}$ ker $S_{E^\perp} \geq n - p + k$ where $E \otimes \mathbb{C}$ is the null part of $S$. Notice that if $k = 0$ we actually have the inequality by the fact that $E$ is even dimensional (for $p = 6$, as discussed in Theorem A).

Since $H$ is Hermitian, its image space Im $H$ is a tensor product of some real linear subspace of $N$ which we denote also by Im $H$ and $\mathbb{C}$. Let $N_{\mathbb{C}} =$
(Im $H \oplus \text{Im } H^\perp) \otimes \mathbb{C}$ be the orthogonal (possibly trivial) decomposition and write $S = (S', S'')$ under this decomposition. Suppose that $\dim \text{Im } H = 1$, in this case it is easy to see that $S'$ is flat by equation (2.7). Since $S$ is already flat, it follows that $S''$ is also flat. Thus, applying Theorem 3 in [1] for $S''$ the lemma follows as above.

We can assume now that $6 \geq \dim \text{Im } H \geq 2$. By equation (2.7) we have that $S(\ker H, V) \subset (\text{Im } H)^\perp \otimes \mathbb{C}$. Define the flat bilinear form $\overline{S}$ by

$$\overline{S} = S|_{\ker H \times V} : \ker H \times V \to (\text{Im } H)^\perp \otimes \mathbb{C}.$$ 

The lemma will follows from Lemma 7 in [8] and the following claim.

**Claim.** $(\text{Im } H)^\perp$ possesses a direct sum decomposition $(\text{Im } H)^\perp = E \oplus E'$, such that $\overline{S}_E$ is null and $\overline{S}_{E'}$ is flat and satisfies $\dim \ker (\overline{S}_{E'}) \geq \dim \ker H - \dim E'$.

In fact, since $\dim_{\mathbb{C}} \ker H \geq n - \dim_{\mathbb{C}} \text{Im } H$, by Lemma 7 in [8], it follows from Claim that $\dim_{\mathbb{C}} \overline{S}_{E'} \geq \dim_{\mathbb{C}} \ker H - p + (\dim_{\mathbb{C}} \text{Im } H + k) \geq n - p + k$ and $E$ is trivial.

Now we will prove the claim. We can assume that $\dim_{\mathbb{C}} \ker \overline{S} < n - p$. Otherwise, by Lemma 7 in [8], it follows that $\dim_{\mathbb{C}} \ker H \cap \ker \overline{S} \geq n - p$ and there is nothing to do.

**Fact 1.** If $X \in \ker H$ is a regular element, then the restriction of $\langle \cdot, \cdot \rangle$ to $\overline{S}^X(V) = \text{Im } \overline{S}(X, \cdot)$ is degenerate.

Otherwise $\overline{S}^X(V) \cap (\overline{S}^X(V))^\perp = \{0\}$ and it follows that $\ker \overline{S}^X \subset \ker \overline{S}$. Since, by definition, $\ker \overline{S} \subset \ker \overline{S}^X$, we conclude $\dim_{\mathbb{C}} \ker \overline{S} = \dim_{\mathbb{C}} \ker \overline{S}^X \geq n - p + k$, which is a contradiction.

**Fact 2.** If $\overline{S}^X(V)$ is a null subspace of $(\text{Im } H)^\perp$ for all regular element $X \in \ker H$, i.e., $\langle \cdot, \cdot \rangle \equiv 0$ restricted to $\overline{S}^X(V) \times \overline{S}^X(V)$, for all regular element $X \in \ker H$. Then the Claim holds.

Indeed, $\langle \overline{S}(X, Z), \overline{S}(X, W) \rangle = 0$, for all $Z, W \in V$. Since the set of regular elements is dense, we have by continuity that $\langle \overline{S}(X, Z), \overline{S}(X, W) \rangle = 0$, for all $Z, W \in V$ and $X \in \ker H$. By flatness

$$0 = \langle \overline{S}(X + Y, Z), \overline{S}(X + Y, W) \rangle = 2(\overline{S}(X, W), \overline{S}(Y, Z)),$$

for all $Z, W \in V$ and $X, Y \in \ker H$. Setting $E = (\text{Im } H)^\perp$, we obtain the conclusions of the affirmation in this case.
By Fact 2, we can assume that there exists a regular element \( X \in \ker H \) such that \( \overline{S}^X(V) \) is not a null subspace. Using Fact 1, we have that the subspace

\[
U^X = \overline{S}^X(V) \cap (\overline{S}^X(V))^\perp
\]

satisfies \( 1 \leq \dim_{\mathbb{C}} U^X \leq 3 \). Using the almost complex structure it is not hard to see that \( U^X \) is a complex vector space and we can write \( U^X = E \otimes \mathbb{C} \), where \( E \) is an even real dimensional subspace. By this discussion we have that \( \dim_{\mathbb{C}} U^X = 2 \).

Using the flatness of \( \overline{S} \) and that \( \dim U^X = 2 \) we can see that \( U^X \) is orthogonal to \( \text{Im} \overline{S} \). Thus, \( \text{Im} \overline{S} \cap \text{Im} \overline{S}^\perp = U^X \) and the projection of \( \overline{S} \) in \( E^\perp \oplus \mathbb{C} \) is non-degenerate. In particular, \( \dim_{\mathbb{C}} \overline{S}_{E^\perp} \geq \dim_{\mathbb{C}} \ker H - p + (\dim_{\mathbb{C}} \text{Im} H + 1) \) and the claim holds.

\[\blacksquare\]

**Remark 9.** We could have used Lemma 3 in [4] for \( \overline{S} \) when \( \dim H \geq 3 \) to shorten the proof of our Algebraic lemma. The proof of their lemma is not immediate as they mentioned and for this reason we prefer to make a more self-contained version.

**Remark 10.** This Algebraic Lemma makes strong use of the complexification of the second fundamental form and has already been proved up to codimension 4 in [12]. In general, Theorem 3 in [1] does not necessary holds for an arbitrary flat bilinear form.

Note that the \((0, 2)\) and \((1, 1)\) parts of the complexified second fundamental form of \( f \) contain the same algebraic information of \( \beta \). In fact, our Algebraic Lemma is an alternative version of Theorem 3 in [1] for \( \beta \) (remark 10) up to codimension 6.

Now we give a version of Theorem 14 in [2] that is a fundamental tool in the proof of Theorem B.

**Theorem 11.** Let \( f : M^{2n} \rightarrow \mathbb{R}^{2n+p} \) be a real Kähler submanifold with \( 2p < n \) and \( p \leq 6 \). Then either \( f \) is \( D^d \)-ruled with \( d \geq 2n - 2p + 3l \) or \( f \) extends to a real Kähler submanifold along each connected component of an open dense subset of \( M^{2n} \).

**Proof.** The proof is similar to the proof of Theorem 14 in [2] using Algebraic Lemma for \( \beta \) instead of Theorem 3 in [1] and using Theorem 11 in [2].

\[\blacksquare\]

We can now prove Theorem B.
Proof of Theorem B. It follows from Theorem 11 that along each connected component of an open dense subset of \( M^{2n} \), either \( f \) extend isometrically to a real Kähler submanifold, or \( f \) is \( D^d \)-ruled with \( d \geq 2n - 2p + 3l \). If the first case occurs the conclusion of the theorem is immediate. So we assume that \( f \) is \( D^d \)-ruled.

Since \( D^d \) is an asymptotic space we have that \( A_\xi|_D(D) \subset D^\perp \) for any \( \xi \in T_f^1 M \) and therefore

\[
\dim(\ker A_\xi \cap D) \geq 2n + 6l - 4p,
\]

for all \( \xi \in T_f^1 M \). Also, from condition \((C_1)\) we have that \( \ker A_\xi = \ker A_{\tau \xi} \) for all \( \xi \in L^{2k} \), where \( 2k = l \). By definition of \( D^d \), we have that \( \Delta \cap J\Delta \) is the intersection of \( D^d \) and \( \bigcap_{i=1}^k \ker A_{\xi_i} \) for some basis \( \{\xi_1, \cdots, \xi_k, \tau \xi_1, \cdots, \tau \xi_k\} \) of \( L^{2k} \). Combining the inequalities (4.1) for \( \xi_1, \cdots, \xi_k \), we have that

\[
\dim \Delta \cap J\Delta \geq d - k(2p - 3l) \geq (2n - 2p) + 3l(k + 1) - 2pk.
\]

Finally, observe that \( 3l(k + 1) - 2pk = 6k(k + 1) - 2pk \geq 0 \) for \( p \leq 6 \) and the theorem follows.

\[
\square
\]

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