Binary mixture of pseudospin-\(\frac{1}{2}\) Bose gases with interspecies spin exchange: from classical fixed points and ground states to quantum ground states

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We consider the effective spin Hamiltonian describing a mixture of two species of pseudo-spin-\(\frac{1}{2}\) Bose gases with interspecies spin exchange. First we analyze the stability of the fixed points of the corresponding classical dynamics, of which the signature is found in quantum dynamics with a disentangled initial state. Focusing on the case without an external potential, we find the nature of entanglement and its relation with classical bifurcation is investigated. When the total spins of the two species are unequal, the maximal entanglement at the parameter point of classical bifurcation is possessed by the excited state corresponding to the classical fixed point which bifurcates, rather than by the ground state.

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I. INTRODUCTION

A remarkable discovery in recent years is that bifurcation in classical dynamics is related to quantum entanglement in the ground state of the corresponding quantum Hamiltonian. In addition to its theoretical demonstration in the Dicke model \([1,2]\), in a model of two coupled giant spins describing magnetic clusters \([3]\), and in an integrable dimer model \([4]\), this association has also been studied in the area of Bose-Einstein condensation (BEC), including two-component BEC \([5,6]\) and two-mode atomic-molecular BEC \([7,8]\). More recently, a classical bifurcation has been observed in an experiment realizing an internal Josephson effect in a spinor Bose-Einstein condensate, as an important step toward entanglement generation close to critical points \([9]\). Moreover, in a laser-cooled atom, experimental evidence has been observed for entanglement being a quantum signature of chaos \([10]\).

On the other hand, a novel kind of BEC, the so-called EBEC, that is, BEC of interspecies spin singlet pairs, was found to be the ground state of a Bose system composed of two species of pseudo-spin-\(\frac{1}{2}\) Bose atoms with both intraspecies and interspecies spin-exchange interactions in a considerable parameter regime \([11,12]\). Under the usual single orbital-mode approximation, the Hamiltonian of this system can be transformed into that of two coupled giant spins. Alas, the ground states in all parameter regimes have not yet worked out.

In this paper, we make each of the above two lines of research useful for the other. We focus on the case in the absence of an external potential. First, we study the bifurcation of the classical dynamics corresponding to the Hamiltonian of this Bose mixture, by analyzing the stability of each fixed point. Quantum dynamics displays some features similar to the classical dynamics if the initial state is a disentangled state, which, however, is not an energy eigenstate. When the numbers of the atoms of the two species are equal, a bifurcation of the fixed points indeed corresponds to a quantum phase transition to a maximally entangled ground state. When they are unequal, the quantum state corresponding to the classical fixed point which bifurcates is also maximally entangled at the bifurcation point. However, it is not the ground state. Finally, we analytically obtain all the quantum ground states by considering quantum fluctuations around the classical ground state in each parameter regime. The analytical results fit the numerical results very well.

The rest of this paper is organized as follows. The model is introduced in Sec. II. The fixed points and bifurcations are studied in Sec. III with the detailed calculation reported in the Appendix. The classical ground states are described in Sec. IV and the classical evolution is described in Sec. V, whose quantum analog is described in Sec. VI. Section VII describes the absence of the correspondence between classical bifurcation and maximal entanglement in the case of unequal populations of the two species. In Sec. VIII we find the quantum ground state in each parameter regime by approximating the Hamiltonian around the classical ground state there. Finally the paper is summarized in Sec. IX.

II. THE MODEL

Consider a dilute gas composed of two distinct species of Bose atoms with the following property \([12,13]\). Each atom has an internal degree of freedom represented as a pseudospin-\(\frac{1}{2}\), with \(z\)-component basis states \(\uparrow\) and \(\downarrow\). Under the usual single orbital-mode approximation, for each species \(\alpha (\alpha = a, b)\) and pseudospin \(\sigma (\sigma = \uparrow, \downarrow)\), only the single-particle orbital ground state \(\phi_{\alpha\sigma}(\mathbf{r})\) is occupied, then the Hamiltonian can be transformed into that
of two coupled giant spins with spin quantum numbers \( S_a = N_a/2 \) and \( S_b = N_b/2 \). Here we focus on the uniform case 14,15, for which

\[
\mathcal{H} = J_\perp (\hat{S}_{ax} \hat{S}_{bx} + \hat{S}_{ay} \hat{S}_{by}) + J_z \hat{S}_{az} \hat{S}_{bz} \tag{1}
\]

where \( J_\perp = 4\pi \hbar^2 \xi_c/(m_{ab} \Omega \hbar) \) while \( J_z = 4\pi \hbar^2 (\xi_s - \xi_d)/(m_{ab} \Omega \hbar) \), with \( m_{ab} \) being the reduced mass of an \( a \) atom and a \( b \) atom, \( \xi_c \) is the scattering length for the scattering in which an \( a \) atom and a \( b \) atom exchange pseudospins, \( \xi_s \) being the scattering length for the forward scattering in which an \( a \) atom and a \( b \) atom have different pseudospins without spin exchange, \( \xi_d \) being the scattering length for the forward scattering in which an \( a \) atom and a \( b \) atom have the same pseudospin without spin exchange, \( \Omega \) being the volume of the system, \( \hat{S}_a = \hat{S}_{ax} \hat{a}_s \hat{a}_s' \), \( \hat{S}_{az} \) is the single spin operator, \( a_s \) denotes the annihilation operator associated with \( \phi_{a_s}(r) \) of species \( a \). Without loss of generality, suppose \( S_a \geq S_b \).

The corresponding classical Hamiltonian is obtained from (1) by treating the spin operators as the classical spin variables,

\[
\mathcal{H}_{cl} = J_\perp (S_{ax} S_{bx} + S_{ay} S_{by}) + J_z S_{az} S_{bz} \tag{2}
\]

\[= J_\perp \sqrt{(S_a^2 - S_{ax}^2)(S_b^2 - S_{bx}^2)} \cos(\varphi_a - \varphi_b) + J_z S_{az} S_{bz} \tag{3}\]

\[
\mathcal{J} = \begin{pmatrix}
0 & -J_z S_{bz} & J_\perp S_{by} & 0 & J_\perp S_{az} & -J_z S_{ay} \\
J_z S_{bz} & 0 & -J_\perp S_{bx} & -J_\perp S_{az} & 0 & J_z S_{ax} \\
-J_\perp S_{by} & J_\perp S_{bx} & 0 & J_\perp S_{ay} & -J_\perp S_{ax} & 0 \\
0 & J_\perp S_{bz} & -J_z S_{by} & 0 & -J_z S_{az} & -J_z S_{ay} \\
-J_\perp S_{by} & 0 & J_z S_{bx} & -J_\perp S_{az} & 0 & J_z S_{ax} \\
J_\perp S_{ay} & J_\perp S_{ax} & -J_z S_{az} & 0 & J_z S_{ax} & 0
\end{pmatrix}
\]

From the Hamiltonian (1), one obtains the equations of motion

\[\frac{d\hat{S}_{ax}}{dt} = J_\perp \hat{S}_{ay} \hat{S}_{bz} - J_z \hat{S}_{by} \hat{S}_{az}, \]

\[\frac{d\hat{S}_{ay}}{dt} = J_\perp \hat{S}_{az} \hat{S}_{bx} - J_z \hat{S}_{by} \hat{S}_{az}, \]

\[\frac{d\hat{S}_{az}}{dt} = J_\perp (\hat{S}_{ay} \hat{S}_{bx} - \hat{S}_{ax} \hat{S}_{by}). \tag{4}\]

The corresponding classical equations of motion, obtained either from the classical Hamiltonian (2) or from the quantum equations of motion by replacing the spin operators as the classical spin variables, can be written as

\[\frac{dA}{dt} = JA, \tag{5}\]

where \( A \equiv (S_{ax}, S_{ay}, S_{az}, S_{bx}, S_{by}, S_{bz})^T \).

In studying the stability of a fixed point, \( \mathcal{J} \) becomes the Jacobian matrix when the spin variables adopt the values at this fixed point.

The classical Hamiltonian in the form of (2) suggests that the classical state of the system is completely determined by the variables \( S_{az}, S_{bz}, \) and \( \varphi_a - \varphi_b \).

We shall use (3) studying the evolution in Sec. V while using (2) in analyzing the fixed points in Sec. III because there is arbitrariness of angles \( \varphi_a \) and \( \varphi_b \) in some fixed points.

**III. FIXED POINTS AND BIFURCATIONS IN CLASSICAL DYNAMICS**

The fixed points of the classical dynamics are obtained from

\[\frac{dA}{dt} = 0. \tag{6}\]

The stability of each fixed point can be examined first by studying the eigenvalues of Jacobian matrix \( \mathcal{J} \) at this point: It is stable if every eigenvalue has a negative real part, while it is unstable if any eigenvalue has a positive real part. Otherwise, one cannot judge whether the fixed point is stable from the eigenvalues of \( \mathcal{J} \), but it is certainly stable if a Lyapunov function can be constructed. A Lyapunov function \( \mathcal{F} \) is such that in a neighborhood of the fixed point, \( \mathcal{L} \) is minimal (or maximal) at the fixed point, and \( d\mathcal{L}/dt \leq 0 \) (or \( d\mathcal{L}/dt \geq 0 \)).

There exist eight fixed points in our system. Their stability is analyzed in the Appendix. The stable parameter regimes of these fixed points are shown in Fig. I. We specify the fixed point in terms of direction \( n_{ax} \) of the spin vector \( S_a = S_{ax} n_{ax} \), that is, \( n_{ax} = (\sin \theta_a \cos \varphi_a, \sin \theta_a \sin \varphi_a, \cos \theta_a) \), with \( 0 \leq \theta_a \leq \pi, 0 \leq \varphi_a < 2\pi, a = a, b \).

As shown in Fig. I, the fixed points and their stable regimes are the following.
The two spins are opposite. This fixed point only exists when $\eta_1|J_z| > |J_{\perp}|$, where $\eta_1 \equiv \frac{1}{2} \left( \sqrt{\frac{S_z}{S_a}} + \sqrt{\frac{S_z}{S_a}} \right)$.

(2) $n_a = -n_b = (0,0,1)$. One spin is the parallel to the $z$ direction while the other is antiparallel to the $z$ direction. This fixed point is stable when $\eta_1|J_z| > |J_{\perp}|$, where $\eta_1 \equiv \frac{1}{2} \left( \sqrt{\frac{S_z}{S_a}} + \sqrt{\frac{S_z}{S_a}} \right)$.

The two spins are antiparallel and are both on the $x-y$ plane. This fixed point is stable if $J_{\perp} > 0$ and $J_{\perp} > \eta_2 J_z$, or $J_{\perp} < 0$ and $J_{\perp} < \eta_2 J_z$, where $\eta_2 \equiv \frac{S_a}{S_z}$.

(3) $n_a = n_b = (0,0,1)$. The two spins are parallel and are on the $x-y$ plane. This fixed point is stable if $J_{\perp} > 0$ and $J_{\perp} > \eta_2 J_z$, or $J_{\perp} < 0$ and $J_{\perp} < \eta_2 J_z$.

(4) $n_a = n_b = (0,0,\pm 1)$. The two spins are both parallel or antiparallel to the $z$ direction. This fixed point is always stable.

(5) $n_a = -n_b$. The two spins are always antiparallel. This fixed point only exists at $J_{\perp} = J_z$ and is stable.

(6) $n_a = n_b$. The two spins are always parallel. This fixed point only exists at $J_{\perp} = J_z$ and is stable.

(7) $n_a = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$ while $n_b = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, -\cos \theta)$.

The $z$ components of the two spins are opposite. This fixed point only exists at $J_{\perp} = -J_z$ and is stable.

(8) $n_a = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$ while $n_b = (-\sin \theta \cos \varphi, -\sin \theta \sin \varphi, \cos \theta)$. The $xy$ components of the two spins are opposite. This fixed point only exists at $J_{\perp} = -J_z$ and is stable.

It can be seen that $J_{\perp} = \pm \eta_1 J_z$ and $J_{\perp} = \pm \eta_2 J_z$ are bifurcation points.

**IV. CLASSICAL GROUND STATES**

As depicted in Fig. 2, we can find that classically the energy is minimal at fixed point (1) when $J_z > |J_{\perp}|$; at fixed point (2) when $J_\perp > |J_z|$; at fixed point (3) when $J_z < -|J_{\perp}|$; at fixed point (4) when $J_z < -|J_{\perp}|$; at fixed point (5) when $J_z = J_{\perp} > 0$; at fixed point (6) when $J_z = J_{\perp} < 0$; at fixed point (7) when $J_{\perp} = -J_z > 0$; at fixed point (8) when $J_{\perp} = -J_z < 0$.

If $S_a = S_b$, we have $\eta_1 = \eta_2 = 1$; therefore the parameter regimes of the bifurcation points are completely the same as those of the classical ground states, respectively. But if $S_a \neq S_b$, there are differences although there are overlap regimes.

**V. CLASSICAL EVOLUTION**

Because the Hamiltonian conserves $S_{az} + S_{bz}$, we study the dynamical evolution of $S_{az} - S_{bz}$ and $(\varphi_a - \varphi_b)/2$ for some given values of $S_{az} + S_{bz}$. When $S_{az} = S_b$ while $S_{bz} = 0$, or $S_{az} = -S_b$ while $S_{bz} = -S_b$, the system is at the fixed point (1). When $S_{az} = 0$, $S_{bz} = 0$ while $(\varphi_a - \varphi_b)/2 = \pi/2$, the system is at the fixed point (2).

When $S_{az} = 0$, $S_{bz} = 0$ while $(\varphi_a - \varphi_b)/2 = 0$ or $\pi$, the system is at the fixed point (3).

We have studied the evolution trajectories near fixed points (1), (2) and (3) under various values of $S_a = S_b$. As shown in Fig. 3 when $J_{\perp}/J_z < 1$, fixed points (1)
FIG. 3: (Color online) The evolution trajectories on the plane of $(S_{a±} - S_{b±})/S_b$ and $(\varphi_a - \varphi_b)/2$, for various values of $S_a = S_b$. Here $\eta \equiv J_1/J_2$, $S_{a±} + S_{b±} = 0$. The evolution trajectories are consistent with the above theoretical analysis.

FIG. 4: (Color online) The evolution trajectories on the plane of $(S_{a±} - S_{b±})/S_b$ and $(\varphi_a - \varphi_b)/2$, for various values of $S_a = 2S_b$. Here $\eta \equiv J_1/J_2$. The solid lines describe the dynamics for $S_{a±} + S_{b±} = 0$ and the dot lines describe the dynamics for $S_{a±} + S_{b±} = \pm (S_a - S_b)$.

and (3) are stable while fixed point (2) is unstable; when $J_1/J_2 > 1$, fixed points (2) and (3) are stable while fixed point (1) is unstable. This conclusion is reached by considering that a fixed point is stable if the evolution trajectories are loops around a fixed point, otherwise it is unstable.

We have also studied the case of $S_a \neq S_b$. As shown in Fig. 4 for $S_a = 2S_b$, the evolution trajectories are different from the case of $S_a = S_b$. For $J_1/J_2 = 0.9$ and for $J_1/J_2 = 1.03$, three fixed points are all stable.

Note that all the results of the numerical simulation are consistent with the above theoretical analysis.

VI. QUANTUM EVOLUTION WITH DISENTANGLED INITIAL STATE

To simulate a quantum process closest to classical evolution, we choose as the initial state a disentangled state, which can always be written as

$$|\psi\rangle = (e^{-i\varphi_a/2}\cos\frac{\theta_a}{2}|\uparrow\rangle_a + e^{i\varphi_a/2}\sin\frac{\theta_a}{2}|\downarrow\rangle_a)^{N_a} \otimes (e^{-i\varphi_b/2}\cos\frac{\theta_b}{2}|\uparrow\rangle_b + e^{i\varphi_b/2}\sin\frac{\theta_b}{2}|\downarrow\rangle_b)^{N_b} \quad (7)$$

where $|\uparrow\rangle_a$ denotes the spin state of a single atom of species $a$, while $|S_a n_a\rangle$ represents the state of the total spin of species $a$. In this state, the spin components of each species are similar to classical spins; that is, $\langle S_{ax}\rangle = S_a\sin\theta_a\cos\varphi_a$, $\langle S_{ay}\rangle = S_a\sin\theta_a\sin\varphi_a$, and $\langle S_{az}\rangle = S_a\cos\theta_a$ ($a = a, b$). Moreover, we choose $\theta_a$ and $\varphi_a$ in such a way that $\langle S_a\rangle$ corresponds to a fixed point in classical dynamics. For fixed point (1), $n_a = -n_b = (0, 0, 1)$; thus the initial state is $|\psi(1)\rangle = |\uparrow\rangle_a^{\otimes N_a} |\downarrow\rangle_b^{\otimes N_b}$. For fixed point (2), $n_a = -n_b = (\cos\varphi, \sin\varphi, 0)$; thus the initial state is $|\psi(2)\rangle = (\frac{\sqrt{2}}{2}e^{-i\varphi/2}|\uparrow\rangle_a^{\otimes N_a} + \frac{\sqrt{2}}{2}e^{i\varphi/2}|\downarrow\rangle_a^{\otimes N_a})^{\otimes N_a}$ and $|\psi_a\rangle = (\frac{\sqrt{2}}{2}e^{-i\varphi/2}|\uparrow\rangle_a^{\otimes N_a} + \frac{\sqrt{2}}{2}e^{i\varphi/2}|\down\rangle_a^{\otimes N_a})^{\otimes N_a}$. This is so because $\hat{S}_{az} = N_\sigma \hat{\sigma}_z$, where $N_\sigma$ is the number of atoms of species $\alpha$ with spin $\sigma$ ($\alpha = a, b; \sigma = \uparrow, \downarrow$).

For each initial state $|\psi\rangle$ in the form of (7), we evaluate

$$\langle \hat{S}_{az}(t) \rangle = \langle \psi | e^{i\hat{H}t} \hat{S}_{az} e^{-i\hat{H}t} | \psi \rangle, \quad (8)$$

whose evolution actually represents the change of the distribution of the atoms of species $\alpha$ in the two pseudospin states.

We choose the same initial conditions as in the classical case in last section to start the quantum dynamics. It is found that the classification of the stability of the classical dynamics still applies. The result is shown in Fig. 5 for the case of $S_a = S_b$, and in Fig. 6 for the case of $S_a = 2S_b$. 

FIG. 5: (Color online) Quantum dynamics of $\langle S_{a\sigma}/S_a \rangle$ for $S_a = S_b = 300$ and various values of $\eta \equiv J_1/J_2$. The figures on the upper line exhibit the stability of fixed point (1). The figures on the second line exhibit the stability of fixed point (2). The unit of $t$ is $1/J_z$. 

$$\langle S_{a\sigma}/S_a \rangle = \frac{1}{2} + \frac{1}{2} \cos \left( \frac{\theta_a}{2} \right) \cos \left( \frac{\theta_b}{2} \right) + \sin \left( \frac{\theta_a}{2} \right) \cos \left( \frac{\theta_b}{2} \right) \sin \left( \frac{\varphi_a - \varphi_b}{2} \right)$$

$$\langle S_{b\sigma}/S_b \rangle = \frac{1}{2} - \frac{1}{2} \cos \left( \frac{\theta_a}{2} \right) \cos \left( \frac{\theta_b}{2} \right) + \sin \left( \frac{\theta_a}{2} \right) \cos \left( \frac{\theta_b}{2} \right) \sin \left( \frac{\varphi_a - \varphi_b}{2} \right)$$

where $\theta_a$ and $\varphi_a$ are defined in such a way that $\langle S_a \rangle$ corresponds to a fixed point in classical dynamics.
The reason why quantum dynamics under the disentangled initial state is so close to classical one is the following. In Heisenberg picture, the quantum equations of motion (14) reduce to the classical ones (15), with \( S_i = (\sum_j S_{ij}) / S_a \), \( (i = x, y, z) \), substituting the corresponding classical spin variable.

VII. BIFURCATION AND ENTANGLEMENT

The ground state can always be written as \( |G_{S_a} \rangle = \sum f(m, S_z)|S_a, m, S_z - m, b \rangle \), where the interspecies entanglement can be quantified as \( \sum_m f^2(m, S_z) \log_2 f(m, S_z) \). It has been shown that when \( S_a = S_b = S \), the entanglement of the ground state is maximal at \( J_1 = J_z \), where the ground state is \( (\sqrt{2S+1})^{-1} \sum_{m=-S}^{S} (-1)^m |S, m, a \rangle |S, -m, b \rangle \). Using the transformation \( U = e^{iS_m} \), we can obtain the ground state at \( J_1 = -J_z \) as \( (\sqrt{2S+1})^{-1} \sum_{m=-S}^{S} |S_a, m, a \rangle |S_b, -m, b \rangle \), which is also maximally entangled.

When \( S_a = S_b \), there is only one bifurcation point at \( J_1 = J_z \) between the fixed points (1) and (2); there is also only one bifurcation point at \( J_1 = -J_z \) between the fixed points (1) and (3). Each of these bifurcation points corresponds to a maximally entangled quantum ground state.

However, when \( S_a \neq S_b \), \( \eta_1 \neq \eta_2 \), there are two bifurcation points \( J_1 = \eta_1 J_z \) and \( J_1 = \eta_2 J_z \) between the fixed points (1) and (2). Similarly, there are two bifurcation points \( J_1 = -\eta_1 J_z \) and \( J_1 = -\eta_2 J_z \) between the fixed points (1) and (3).

Numerical calculations indicate that in consistency with the classical ground states, the total \( z \)-component spin exhibits the following features. When \( J_z > 0 \) while \( \eta < 1 \), \( S_z = \pm (S_a - S_b) \). When \( J_z > 0 \) while \( \eta > 1 \), \( S_z = p \), with \( p = 0 \) if \( S_a - S_b \) is an integer while \( p = \pm 1/2 \) if \( S_a - S_b \) is a half integer.

Numerical results of the entanglement entropy of the states with \( S_z = S_a - S_b \) and \( S_z = 0 \), varying with \( \eta \), are shown in Fig. 7 for some integer values of \( S_b \) and \( S_a = 3S_b \). For \( \eta < 1 \), the ground state is the one with \( S_z = S_a - S_b \), whose entanglement values are plotted as empty triangles. For \( \eta > 1 \), the ground state is the one with \( S_z = 0 \), whose entanglement values are plotted as filled triangles. Therefore, there is a discontinuity of entanglement in passing \( \eta = 1 \), where both states are degenerate ground states.

According to the bifurcation analysis discussed above, when \( S_a = 3S_b \), the two bifurcation points are \( \eta_1 = \frac{1}{2} (\sqrt{1/3} + \sqrt{3}) \approx 1.1547 \) and \( \eta_2 = 0.6 \). As indicated in Fig. 7, the entanglement entropy of the states with \( S_z = 0 \) and \( S_z = S_a - S_b \) is maximal at \( \eta_1 \) and \( \eta_2 \), respectively, and decreases rapidly in deviating from each of them, with the decrease more rapid for \( \eta \) larger than the maximal point.

Therefore, when \( S_a \neq S_b \), the quantum state corresponding to each classical fixed point still possesses maximal entanglement at the parameter point where the fixed point bifurcates. However, this quantum state is not the quantum ground state. In other words, the entanglement of the quantum ground state at each bifurcation point is not maximal anymore.

VIII. ANALYTICAL SOLUTIONS OF THE QUANTUM GROUND STATES

We now proceed to analytically find out the quantum ground states on all \( J_z \) parameter regimes, by using effective Hamiltonians which describe deviations from the classical ground state in each parameter regime. All the ground states are summarized in Fig. 8. Regimes A \( (J_z > |J_{\perp}|) \) and B \( (J_z < -|J_{\perp}|) \) both correspond to \( |\xi_a - \xi_d| > |\xi_c| \); i.e., the interspecies spin exchange scattering is quite weak, with A and B differing.
in whether the equal-spin forward scattering length is larger or smaller than the unequal-spin forward scattering length. Regimes C ($J_{\perp} > |J_z|$) and D ($J_{\perp} < -|J_z|$) both correspond to $|\xi_c| > |\xi_d|$; i.e., the interspecies spin exchange scattering is quite strong, with C and D differing in whether the spin-exchange scattering length is positive or negative.

Regime A ($J_z > J_{\perp}$) corresponds to $\xi_c - \xi_d > |\xi_c|$; i.e., the interspecies spin exchange scattering is quite weak.

**A. $J_z > J_{\perp}$**

As shown in Fig. 2, in this parameter regime, the classical ground state is fixed point (1), i.e., $|S_a, S_a\rangle$ or $|S_a, -S_a\rangle$.

First we consider the quantum ground state near $|S_a, S_a\rangle$ or $|S_a, -S_a\rangle$. One can make the Holstein-Primackoff transformation 

$$\tilde{S}_a^+ = \hat{f}_a^\dagger \sqrt{2S_a - \hat{f}_a^\dagger \hat{f}_a}, \quad \tilde{S}_a^- = S_a - \hat{f}_a^\dagger \hat{f}_a,$$

$$\tilde{S}_b^+ = \sqrt{2S_b - \hat{f}_b^\dagger \hat{f}_b}, \quad \tilde{S}_b^- = \hat{f}_b^\dagger \hat{f}_b - S_b,$$  

(9)

with $\tilde{S}_{a\pm} \equiv \tilde{S}_{a\alpha} \pm i\tilde{S}_{a\beta}$, $\alpha = a, b$, $\tilde{f}_a$ and $\tilde{f}_b$ being bosonic operators satisfying $\hat{f}_a|n_a\rangle = \sqrt{n_a}|n_a - 1\rangle$, $\hat{f}_b^\dagger|n_b\rangle = \sqrt{n_b + 1}|n_b + 1\rangle$, $[\hat{f}_a, \hat{f}_b] = 0$, $[\hat{f}_{a\alpha}, \hat{f}_{b\beta}] = \delta_{a\beta}$, where

$$|n_a\rangle \equiv |S_a, S_a - n_a\rangle,$$

$$|n_b\rangle \equiv |S_b, -S_b + n_b\rangle,$$  

(10)

(11)

then the Hamiltonian (1) can be approximated as

$$\tilde{H} \approx -J_zS_aS_b + J_z(S_b\hat{f}_a\hat{f}_a + S_a\hat{f}_b\hat{f}_b) + J_{\perp}\sqrt{S_aS_b}(\hat{f}_a^\dagger \hat{f}_b^\dagger + \hat{f}_b\hat{f}_a),$$

(12)

Then we make the Bogoliubov transformation

$$\hat{f}_c = \sqrt{\frac{\Delta_1 + 1}{2}} \hat{f}_a + \text{sgn}(J_{\perp})\sqrt{\frac{\Delta_1 - 1}{2}} \hat{f}_b,$$

$$\hat{f}_d = \text{sgn}(J_{\perp})\sqrt{\frac{\Delta_1 - 1}{2}} \hat{f}_c + \sqrt{\frac{\Delta_1 + 1}{2}} \hat{f}_b,$$  

(13)

where $\text{sgn}(J_{\perp})$ is the sign of $J_{\perp}$, $\Delta_1 \equiv \sqrt{J_z^2(S_a + S_b)^2 - 4J_{\perp}^2S_aS_b}$. When $S_a = S_b$ and $J_z = J_{\perp}$, $\Delta_1 \pm 1$ should be 1. Hamiltonian (12) becomes

$$\tilde{H}_A = \epsilon_{1c}\hat{f}_c^\dagger \hat{f}_c + \epsilon_{1d}\hat{f}_d^\dagger \hat{f}_d + E_{10},$$

(14)

where $E_{10} \equiv -J_zS_aS_b - J_{\perp}\sqrt{(\Delta_1^2 - 1)S_aS_b}$, $\epsilon_{1c} = \frac{J_{\perp}(\Delta_1 + 1)}{2}S_a + J_z(\Delta_1 + 1)S_b - J_z\sqrt{(\Delta_1^2 - 1)S_aS_b}$, $\epsilon_{1d} = \frac{J_z(\Delta_1 + 1)}{2}S_a + J_{\perp}(\Delta_1 + 1)S_b - J_{\perp}\sqrt{(\Delta_1^2 - 1)S_aS_b}$. Thus the energy spectrum is $E_A(n_c, n_d) = \epsilon_{1c}n_c + \epsilon_{1d}n_d + E_{10}$, where $n_c$ and $n_d$ are nonnegative integer numbers. For $J_z > |J_{\perp}|$, $\epsilon_{1c}$ and $\epsilon_{1d}$ are always positive; therefore the ground-state energy is $E_A(0, 0) = E_{10}$. When $J_z \rightarrow |J_{\perp}|$, $E_A(0, 0)$ approaches $-S_b(S_a + 1)$, which is the exact ground-state energy at $J_z = |J_{\perp}|$.

Like the original Hamilton (1), the effective Hamiltonian (12) also conserves the $z$ component of the total spin. Therefore any of its eigenstates can be written as

$$|\psi_1(n_c, n_d)\rangle = \sum_m g_1(n_c, n_d, m)|S_a, m\rangle_a|S_b, S_z - m\rangle_b,$$  

(15)

where $\max(-S_a, -S_z - S_b) \leq m \leq \min(S_a, S_z + S_b)$, $g_1(n_c, n_d, m)$ is the expansion coefficient, and $S_z$ is the total $z$ component of the spin system. Using (13) and considering that $|\psi_1(n_c, n_d)\rangle$ is an eigenstate of both $\hat{f}_c^\dagger \hat{f}_c$ and $\hat{f}_d^\dagger \hat{f}_d$ with eigenvalues $n_c$ and $n_d$ respectively, we obtain

$$n_c - n_d = S_a - S_b - S_z.$$  

(16)

For the ground state $|\psi_1(0, 0)\rangle$, $S_z = S_a - S_b$.

It is easy to find the ground state $|\psi_1(0, 0)\rangle$ of (12) from $\hat{f}_c|\psi_1(0, 0)\rangle = 0$.
|ψ_1(0,0)⟩ = D \sum_{m=-S_b}^{S_a} \left[ -sgn(J_\perp) \sqrt{\frac{\Delta_1+1}{\Delta_1-1}} \right]^m |S_a, m_a|S_b, S_a - S_b - m_b⟩, \quad (17)
|ψ_1(n_c,0)⟩ = D_c \exp(-iζπS_{a_\perp}) \sum_{n=0}^{n_c} \sum_{m=-S_a}^{S_a} (-1)^{m+n} \left( \frac{\Delta_1+1}{\Delta_1-1} \right)^m \sqrt{C_{n_m}^{n_c}} |S_a, m_a⟩|S_b, S_a - S_b - n_c - m_b⟩, \quad (19)
|ψ_1(0,n_d)⟩ = D_d \exp(-iζπS_{a_\perp}) \sum_{m=-S_b}^{S_a} (-\sqrt{\frac{\Delta_1+1}{\Delta_1-1}})^m \sqrt{C_{n_d}^{n_m}} |S_a, m_a⟩|S_b, S_a - S_b + n_d - m_b⟩, \quad (20)

where D ≡ \left[ \frac{(\Delta_1+1)^{S_a-S_b}(\Delta_1+1)^{2S_b+1}-(\Delta_1-1)^{2S_b+1}}{2(\Delta_1-1)^{S_b}} \right]^{-1/2} is the normalization coefficient. When J_\perp → 0, \{G_A\} → |S_a, S_a⟩|S_b, -S_b⟩, which is an exact ground state of the Hamiltonian (1) with J_\perp = 0 and J_\parallel > 0.

The excited states of (12) can be obtained by the action of $f_c^\dagger$ and $f_d^\dagger$ on the ground state $|ψ_1(0,0)⟩$. With $S_a > S_b$, $\epsilon_{1_c} < \epsilon_{1_d}$, for a given $S_z$, the lowest excited state is $|ψ_1(n_c,0)⟩$ if $S_z < S_a - S_b$ and is $|ψ_1(0,n_d)⟩$ if $S_z > S_a - S_b$. These two excited states can be written as

$$
|ψ_1(n_c,0)⟩ = D_c \exp(-iζπS_{a_\perp}) \sum_{n=0}^{n_c} \sum_{m=-S_a}^{S_a} (-1)^{m+n} \left( \frac{\Delta_1+1}{\Delta_1-1} \right)^m \sqrt{C_{n_m}^{n_c}} |S_a, m_a⟩|S_b, S_a - S_b - n_c - m_b⟩, \quad (19)
|ψ_1(0,n_d)⟩ = D_d \exp(-iζπS_{a_\perp}) \sum_{m=-S_b}^{S_a} (-\sqrt{\frac{\Delta_1+1}{\Delta_1-1}})^m \sqrt{C_{n_d}^{n_m}} |S_a, m_a⟩|S_b, S_a - S_b + n_d - m_b⟩, \quad (20)
$$

where $ζ = 0$ if $J_\parallel > 0$ while $ζ = 1$ if $J_\parallel < 0$; $D_c$ and $D_d$ are the normalization constants, and $C_{n_m}^{n_c}$ is the binomial coefficient.

Now we consider the ground state close to the other classical degenerate ground state $|S_a, -S_a⟩|S_b, S_b⟩$, in a way similar to the above. The Holstein-Primakoff transformation is

$$
S_{b_-}′ = f_b^\dagger \sqrt{2S_b - f_b^\dagger f_b}, \quad S_{b_+}′ = \sqrt{2S_b - f_b^\dagger f_b},
S_{a_-}′ = S_b - f_b^\dagger f_b,
S_{a_+}′ = \sqrt{2S_a - f_a^\dagger f_a}, \quad S_{a_0}′ = f_a^\dagger f_a - S_a,
\quad (21)
$$

where the bosonic operators $f_a^\dagger$ and $f_b^\dagger$ act on

$$
|n_{a_0}⟩ \equiv |S_a, -S_a + n_{a_0}⟩, \quad (22)
|n_{b_0}⟩ \equiv |S_b, S_b - n_{b_0}⟩. \quad (23)
$$

Thus one obtains a Hamiltonian

$$
\hat{H}_A ≈ -J_z S_a S_b + J_\perp (S_b f_a^\dagger f_a + S_a f_b^\dagger f_b)
+ J_\parallel \sqrt{S_a} f_b^\dagger f_b + \sqrt{S_b} f_a^\dagger f_a,
\quad (24)
$$

where

$$
\hat{f}_c = \sqrt{\frac{\Delta_1+1}{2}} f_a + sgn(J_\perp) \sqrt{\frac{\Delta_1-1}{2}} f_b^\dagger,
\hat{f}_d = sgn(J_\perp) \sqrt{\frac{\Delta_1-1}{2}} f_a^\dagger + \sqrt{\frac{\Delta_1+1}{2}} f_b.
$$

Therefore the eigenstates can be written as

$$
|ψ_1(n_c',n_d')⟩ = \sum_{m} g_1(n_{c'}, n_{d'}, m)|S_a, m⟩|S_b, S_a - m⟩, \quad (25)
$$

with the constraint

$$
n_{c'} - n_{d'} = S_a - S_b + S_z. \quad (26)
$$

For the ground state $|ψ_1'(0,0)⟩$, $S_z = S_b - S_a$. 
\[ |\psi'(0,0)\rangle = D' \sum_{m=-S_b}^{S_b} \left[ -\text{sgn}(J_\perp) \sqrt{\frac{\Delta_1 + 1}{\Delta_1 - 1}}^m |S_a, S_b - m\rangle_a |S_b, m\rangle_b, \right] \]

where \( D' = \frac{[\Delta_1 + 1]^2 S_a + (\Delta_1 - 1)^2 S_b + 1}{2[\Delta_1 - 1]^2} \) when \( J_\perp \to 0 \), \( |G_A\rangle \to |S_a, -S_a\rangle |S_b, S_b\rangle \), which is an exact ground state of the Hamiltonian \( \text{II} \) with \( J_\perp = 0 \) and \( J_z > 0 \).

The excited states of \( \text{II} \) can be obtained by the action of \( f'_a \) and \( f'_b \) on the ground state \( |\psi'(0,0)\rangle \). With \( S_a > S_b \), \( \epsilon_{1c} < \epsilon_{1d} \), for a given \( S_z \), the lowest excited state is \( |\psi'(n_c',0)\rangle \) if \( S_z > S_a - S_b \) and is \( |\psi'(0,n_d')\rangle \) if \( S_z < S_b - S_a \). The explicit expressions of \( |\psi'(n_c,0)\rangle \) and \( |\psi'(0,n_d)\rangle \) are like (11) and (20) for \( |\psi'(n_c,0)\rangle \) and \( |\psi'(0,n_d)\rangle \), with \( |S_a, m\rangle_a, |S_b, S_a - S_b - n_c - m\rangle_b \) and \( |S_a, S_b - nd + m\rangle_b \) replaced as \( |S_a, -m\rangle_a, |S_b - S_a + n_c + m\rangle_b \) and \( |S_a - S_b + nd + m\rangle_b \), respectively.

It is important to note that \( |\psi'(0,0)\rangle \) and \( |\psi'(0,0)\rangle \) are orthogonal unless \( S_a = S_b \). Therefore, when \( S_a \neq S_b \), the ground states are doubly degenerate ones \( |\psi'(0,0)\rangle \) and \( |\psi'(0,0)\rangle \) at each parameter point in this regime.

When \( S_a = S_b, \gamma' \equiv |\langle 0|0\rangle \rangle |\psi'(0,0)\rangle = \langle 0|0\rangle \rangle |\psi'(0,0)\rangle = \langle 0|0\rangle \rangle |\psi'(0,0)\rangle = \langle 0|0\rangle \rangle |\psi'(0,0)\rangle \rangle \); hence we must find the ground state in their two-dimensional subspace. Clearly \( \langle 0|0\rangle \rangle |\psi'(0,0)\rangle = \langle 0|0\rangle \rangle |\psi'(0,0)\rangle \rangle \approx E_{10} \), \( \langle 0|0\rangle \rangle \rangle |\psi'(0,0)\rangle = \langle 0|0\rangle \rangle |\psi'(0,0)\rangle \rangle \approx E_{10} \). Consequently, the ground state is found to be

\[ |G_A(S_a = S_b)\rangle = \frac{1}{\sqrt{2}}(|\psi'(0,0)\rangle + |\psi'(0,0)\rangle) \]

with energy \( E_{10}(1 + \gamma) \). The energy of \( \frac{1}{\sqrt{2}}(|\psi'(0,0)\rangle - |\psi'(0,0)\rangle) \) is \( E_{10}(1 - \gamma) \).

When \( J_\perp \) and \( J_z \) approach the boundary \( J_z = -J_\perp > 0 \) from the regime of \( |G_A\rangle \), \( |G_A\rangle \) approaches \( e^{i\pi S_z} |S_a - S_b, \pm (S_a - S_b)\rangle \). When \( J_z \) and \( J_z \) approach the boundary \( J_z = J_\perp > 0 \) from the regime of \( |G_A\rangle \), \( |G_A\rangle \) approaches \( |S_a - S_b, \pm (S_a - S_b)\rangle \).

**B. \( J_z < -|J_\perp| \)**

In this parameter regime, the classical ground states are \( |S_a, S_a\rangle |S_b, S_b\rangle \), in which the two spins are both along the \( z \) direction, and \( |S_a, -S_a\rangle |S_b, -S_b\rangle \), in which the two spins are both along the \(-z\) direction.

\[ |\psi(0,0)\rangle = D_{2c} \exp(-i\pi S_{ax}) \sum_{m=\max(n_c, -2S_b, 0)}^{n_c} (-\sqrt{1 - \Delta_2}/\sqrt{1 + \Delta_2})^m \sqrt{C_n^m} |S_a, S_a - m\rangle_a |S_b, S_b - n_c + m\rangle_b, \]

where \( \zeta = 0 \) if \( J_\perp > 0 \) while \( \zeta = 1 \) if \( J_\perp < 0 \).

Consider the ground state close to \( |S_a, S_a\rangle |S_b, S_b\rangle \). We make the Holstein-Primakoff transformation \( \hat{S}_a = \hat{h}_a \sqrt{2S_a - \hat{h}_a^\dagger \hat{h}_a}, \hat{S}_b = \sqrt{2S_a - \hat{h}_a^\dagger \hat{h}_a}, \hat{S}_a = S_a - \hat{b}_a^\dagger \hat{b}_a \), where \( \hat{h}_a \) and \( \hat{h}_b \) are bosonic operators, now with \( |n_a\rangle \equiv |S_a, S_a - n_a\rangle \), where \( n_a = 0, 1, \ldots, 2S_a \). When \( S_a \) is very large, \( \langle b_\alpha^\dagger h_\alpha \rangle \ll 2S_a, \hat{S}_a \approx (2S_a)^{1/2} \hat{h}_a^\dagger \hat{h}_a \), \( \hat{S}_a \hat{S}_b \approx S_a \hat{b}_a^\dagger \hat{h}_a - S_a \hat{h}_a^\dagger \hat{b}_a \). Then the Hamiltonian \( \text{II} \) becomes

\[ \hat{H}_B = J_z S_a S_b - J_z (S_b \hat{h}_a + S_a \hat{b}_a^\dagger \hat{h}_a) + J_\perp \sqrt{S_a S_b} (\hat{h}_a^\dagger \hat{h}_b + \hat{h}_b^\dagger \hat{h}_a) \]

We define another two bosonic operators,

\[ \hat{h}_c = -\text{sgn}(J_\perp) \sqrt{1 - \Delta_2} \hat{h}_a + \sqrt{1 + \Delta_2} \hat{h}_b, \]

\[ \hat{h}_d = \text{sgn}(J_\perp) \sqrt{1 + \Delta_2} \hat{h}_a - \sqrt{1 - \Delta_2} \hat{h}_b, \]

where \( \Delta_2 \equiv \frac{J_z(S_a - S_b)}{\sqrt{J_z^2(S_a - S_b)^2 + J_\perp^2 S_a S_b}} \). Then the Hamiltonian \( \text{II} \) becomes

\[ \hat{H}_B = \epsilon_{2c} \hat{h}_c^\dagger \hat{h}_c + \epsilon_{2d} \hat{h}_d^\dagger \hat{h}_d + E_{20}, \]

where \( E_{20} = J_z S_a S_b, \epsilon_{2c} = -\frac{J_z(1 - \Delta_2)}{2} S_a + \frac{J_z(1 + \Delta_2)}{2} S_b + |J_\perp| \sqrt{(1 - \Delta_2^2) S_a S_b}, \epsilon_{2d} = \frac{J_z(1 - \Delta_2)}{2} S_a + \frac{J_z(1 + \Delta_2)}{2} S_b - |J_\perp| \sqrt{(1 - \Delta_2^2) S_a S_b}. \) The energy spectrum is \( E_{20}(n_c, n_d) \approx \epsilon_{2c} n_c + \epsilon_{2d} n_d + E_{20} \). Thus the ground state is \( |S_a, S_a\rangle |S_b, S_b\rangle \).

The excited state \( |\psi(0, n_c)\rangle \) of \( \text{II} \) can be obtained by the action of \( \hat{h}_c^\dagger \) and \( \hat{h}_d^\dagger \) on the ground state \( |S_a, S_a\rangle |S_b, S_b\rangle \). It is obvious that \( \epsilon_{2c} \) is always larger than \( \epsilon_{2d} \), for a given \( S_z \), the lowest excited state is \( |\psi(2c, 0)\rangle \), with \( n_c = S_a + S_b - S_z \).

\[ D_{2c} = \left( \frac{(1 + \Delta_2)^{n_c - (1 - \Delta_2)^{n_c}} - (1 + \Delta_2)^{n_c - (1 - \Delta_2)^{n_c + 1}}}{2\Delta_2(1 + \Delta_2)^{n_c - (1 - \Delta_2)^{n_c + 1}} - 1} \right)^{-1/2}, \]
where \( n_0 = \max(n_c - 2S_b, 0) \).

Using the same method, we obtain the approximate Hamiltonian \( \hat{H}_B \) close to the other classical ground state \(|S_a, -S_a⟩|S_b, -S_b⟩\), which turns out to be the quantum ground state. The Holstein-Primakoff transformation is now \( \hat{S}_{a+} = \hat{h}_a\sqrt{2S_a - \hat{h}_a^\dagger \hat{h}_a}, \hat{S}_{a-} = \sqrt{2S_a - \hat{h}_a^\dagger \hat{h}_a}, \hat{S}_{bz} = \hat{h}_b^\dagger \hat{h}_a - S_a, \) where \( \hat{h}_a^\dagger \) and \( \hat{h}_a \) are bosonic operators, with \( \{|n'_a⟩\} ≡ |S_a, n'_a - S_a⟩\). Thus we have Eqs. (29) to (31) with primed operators. In this set of eigenstates, the lowest excited states for a given \( S_z \) are \(|\psi'_2(n'_c, 0)⟩\), with

\[
n'_c = S_z + S_a + S_b.
\]

(34)

\(|\psi'_2(n'_c, 0)⟩\) are like (33), with \(|S_a, S_a - m_a⟩\) and \(|S_a, S_b - n_c + m_b⟩\) replaced by \(|S_a, -S_a + m_a⟩\) and \(|S_b, -S_b - n_c - m_b⟩\).

\(|\psi_2(0, 0)⟩\) and \(|\psi'_2(0, 0)⟩\) are orthogonal, hence are just the doubly degenerate ground states in this regime, and can be written as

\[
|G_B⟩ = |S_a + S_b, ±(S_a + S_b)⟩.
\]

(35)

When \( J_{z} \) and \( J_z \) approach the boundary \( J_z = J_{z} < 0 \) from the regime of \(|G_b⟩\), \(|G_B⟩\) approaches \(|S_a + S_b, ±(S_a + S_b)⟩\). When \( J_z \) and \( J_z \) approach the boundary \( J_{z} = -J_z > 0 \) from the regime of \(|G_B⟩\), \(|G_B⟩\) approaches \(|S_a + S_b, ±(S_a + S_b)⟩\).

C. \( J_z > |J_z| \)

In this parameter regime, it is convenient to rewrite the Hamiltonian as

\[
\mathcal{H} = J_z\sqrt{(S_a^2 - S_{az}^2)/(S_b^2 - S_{bz}^2)} \cos(\varphi_a - \varphi_b) + J_z S_{az} S_{bz},
\]

(36)

where \( \varphi_a (\alpha = a, b) \) is the azimuthal angle.

In the vicinity of the classical ground state, \( S_{az} \sim 0, S_{bz} \sim 0, \varphi_a - \varphi_b \sim \pi \), for simplicity we define \( \varphi_a - \varphi_b = \varphi_{ab} + \pi \). Therefore

\[
\mathcal{H}_C ≈ -J_z\sqrt{S_a S_b (S_a + 1)(S_b + 1)} + \frac{J_{z}^2 - J_z^2}{4(J_{z} + J_z)} S_z^2
\]

\[
+ \frac{J_{z} + J_z}{4} (S_{2z} - \frac{J_z}{J_{z} + J_z} S_{1z}^2)^2
\]

\[
+ 2J_z \sqrt{S_a S_b (S_a + 1)(S_b + 1)(\varphi_{ab})^2},
\]

(37)

where \( S_{2z} = S_a + S_b, \varphi_z ≡ \frac{1}{2} (\frac{S_a + S_b}{S_a - S_b}) \).

\( S_z \) commutes with \( \mathcal{H} \) and is thus a constant of motion. Then \( \{P_z, S_{2z} - \frac{J_{z} + J_z}{J_{z} - J_z} S_z\} \) and \( \{\dot{S}_z, \dot{P}_z\} = \varphi_{ab}/2 \) are conjugate variables, as \( S_{2z} \) and \( \varphi_z \) are canonically conjugate variables. The Hamiltonian is then similar to that of a harmonic oscillator. The energy spectrum is thus

\[
E_\delta(n, S_z) = -J_z\sqrt{S_a S_b (S_a + 1)(S_b + 1)}
\]

\[
+ \frac{J_{z}^2 - J_z^2}{4(J_{z} + J_z)} S_z^2
\]

\[
+ (n + 1)\frac{1}{2} \sqrt{2J_z (J_z - S_z)} S_a S_b (S_a + 1)(S_b + 1)
\]

\[
≈ -J_z\sqrt{S_a S_b (S_a + 1)(S_b + 1)} + \frac{J_{z}^2 - J_z^2}{4(J_{z} + J_z)} S_z^2
\]

\[
+ (n + 1)\frac{1}{2} \sqrt{2J_z (J_z - S_z)} S_a S_b,
\]

(38)

where \( n \) is the quantum number of the harmonic oscillator. The eigenstate for \( n = 0 \) can be written as

\[
|\psi_3(0, S_z)⟩ = Z(S_z) \sum_m f(m, S_z)|S_a, m_a⟩|S_b, S_b - m_b⟩,
\]

(39)

where \( f(m, S_z) = (-1)^m \exp[-\frac{J_z}{2S_a S_b} m (m - 1)] \), with \( m = \min(S_a, S_b + S_z) \leq m \leq \max(S_a, S_z) \).

The ground state is thus

\[
|G_C⟩ = |\psi_3(0, p)⟩ = Z(p) \sum_m f(m, p)|S_a, m_a⟩|S_b, p - m_b⟩,
\]

(40)

where \( p = 0 \) if \( S_a - S_b \) is an integer, while \( p = \pm 1/2 \) if \( S_a - S_b \) is a half integer.

For \( S_a = S_b = S \) and \( J_{z} \gg J_z \), we have calculated the entanglement entropy of \( |G_C⟩ \),

\[
\mathcal{E}(|G_C⟩) = -\sum_m [Z(p) f(m, p)]^2 \log_2 [Z(p) f(m, p)]^2.
\]

(41)

It is evaluated that when \( S \to ∞, \mathcal{E}(|G_C⟩) \approx 1/2 \), which is very large.

When \( J_{z} \) and \( J_z \) approach the boundary \( J_z = J_{z} > 0 \) from the regime of \( |G_C⟩ \), it approaches \(|S_a - S_b, p⟩\), where \( p = 0 \) if \( S_a - S_b \) is an integer while \( p = \pm 1/2 \) if \( S_a - S_b \) is a half integer. When \( J_{z} \) and \( J_z \) approach the boundary \( J_z = -J_z < 0 \) from the regime of \( |G_C⟩ \), it approaches \( e^{iπS_a}|S_a + S_b, p⟩ \).

D. \( J_z < |J_z| \)

The energy spectrum for \( J_z < 0 \) can be obtained by using \( H(J_{z}, J_z) = U H(−J_{z}, J_z) U^\dagger \), where \( U = e^{iπS_a} \).

Therefore, in the regime \( J_z < |J_z| \), the energy spectrum is also given by Eq. (38).

The ground state is thus

\[
|G_D⟩ = U|G_C⟩
\]

\[
= Z(p) \sum_m f(m, p)|S_a, m⟩|S_b, p - m⟩,
\]

(42)
with $\max(-S_a, S_z - S_b) \leq m \leq \min(S_a, S_z + S_b)$.

Obviously the entanglement entropy of $|G_D\rangle$ is the same as that of $|G_C\rangle$, with $J_\perp$ reversing its sign.

When $J_\perp$ and $J_z$ approach the boundary $J_z = -J_\perp > 0$ from the regime of $|G_D\rangle$, it approaches $\sum S_a |S_a - S_b, S_z\rangle$ with $S_z = S_b - S_a, \cdots, S_a - S_b$. Here $g(S_a - S_b, S_z, m)$ is the Clebsch-Gordan coefficient.

When $J_z = -J_\perp > 0$, the degenerate ground states are $e^{i\pi S_{az}} |S_a - S_b, S_z\rangle = \sum g(S_a - S_b, S_z, m)|S_a, S_z, m\rangle|S_b, S_z - m\rangle$, with $S_z = S_b - S_a, \cdots, S_a - S_b$.

When $J_z = J_\perp < 0$, the ground states are $|S_a + S_b, S_z\rangle = \sum g(S_a + S_b, S_z, m)|S_a, S_z, m\rangle|S_b, S_z + m\rangle$, with $S_z = -S_a - S_b, \cdots, S_a + S_b$.

The boundaries are where quantum phase transition take place. We have known that the ground states $|G_A\rangle, |G_B\rangle, |G_C\rangle, |G_D\rangle$, in the four regimes discussed in previous subsections, depend on the values of $J_z$ and $J_\perp$. Starting as a ground state in one of these regimes (see FIG. 8), when $J_z$ and $J_\perp$ adiabatically approach each boundary regime, the ground state always approaches one of the degenerate ground states on the boundary. In entering the other regime across the boundary, the ground state restarts from another one of the degenerate ground states on the boundary.

E. The ground states on the four parameter boundaries

When $J_z = J_\perp > 0$, the Hamiltonian $\mathcal{H} = J_z \hat{S}_a \cdot \hat{S}_b$, the degenerate ground states are $|S_a - S_b, S_z\rangle = \sum g(S_a - S_b, S_z, m)|S_a, S_z, m\rangle|S_b, S_z - m\rangle$, with $S_z = S_b - S_a, \cdots, S_a - S_b$. Here $g(S_a - S_b, S_z, m)$ is the Clebsch-Gordan coefficient.

When $J_z = -J_\perp > 0$, the degenerate ground states are $e^{i\pi S_{az}} |S_a - S_b, S_z\rangle = \sum g(S_a - S_b, S_z, m)|S_a, S_z, m\rangle|S_b, S_z - m\rangle$, with $S_z = S_b - S_a, \cdots, S_a - S_b$.

When $J_z = J_\perp < 0$, the ground states are $|S_a + S_b, S_z\rangle = \sum g(S_a + S_b, S_z, m)|S_a, S_z, m\rangle|S_b, S_z + m\rangle$, with $S_z = -S_a - S_b, \cdots, S_a + S_b$.

The boundaries are where quantum phase transition take place. We have known that the ground states $|G_A\rangle, |G_B\rangle, |G_C\rangle, |G_D\rangle$, in the four regimes discussed in previous subsections, depend on the values of $J_z$ and $J_\perp$. Starting as a ground state in one of these regimes (see FIG. 8), when $J_z$ and $J_\perp$ adiabatically approach each boundary regime, the ground state always approaches one of the degenerate ground states on the boundary. In entering the other regime across the boundary, the ground state restarts from another one of the degenerate ground states on the boundary.

F. Comparison with the numerical results

As each eigenstate for $J_\perp < 0$ can be obtained by acting $e^{i\pi S_{az}}$ on an eigenstate for $J_\perp = |J_\perp|$, we only need to consider the half of the parameter space with $J_\perp \geq 0$.

In this half parameter space, We have calculated the dependence of the entanglement on $1/\eta \equiv J_z/J_\perp$, using the ground states analytically obtained above in regimes $A$, $C$, and $B$. We compare these analytical results with the numerical results. The reason of choosing $1/\eta$ rather than $\eta$ is because $J_z = 0$ in the middle of the half parameter space. In this half parameter space, regime $B$ is $1/\eta < -1$, regime $C$ is $-1 < 1/\eta < -1$, while regime $A$ is $1/\eta > 1$.

Figure 9 shows the the entanglement in the ground states for different values of $S_a = S_b$ for $1/\eta > -1$. Neglected is regime $B$, i.e. $1/\eta < -1$, as the ground state is exactly $|S_a, S_a\rangle|S_b, S_b\rangle$ or $|S_a, -S_a\rangle|S_b, -S_b\rangle$, without entanglement. Figure 9 clearly indicates excellent fitting between the analytical results in this section and the numerical results.

Excellent fitting between our analytical results and the numerical results are also obtained for excited states. We have calculated the lowest energy states of different values of $S_z$ for $S_a = 12000$ and $S_b = 10000$. Figure 10 shows the the regime $1/\eta > -1$, i.e., regimes $C$ and $A$, while Fig. 11 shows the regime $1/\eta < -1$, i.e., regime $B$. The reason for this separation is that the low-energy excited state in regime $B$ is with large magnitudes of $S_z$, while those in regimes $C$ and $A$ are with small magnitudes of $S_z$.

In conclusion, our analytical results fit the numerical results very well.

IX. SUMMARY

In this paper, we considered a binary mixture of two species of pseudo-spin-$\frac{3}{2}$ atoms with interspecies spin exchange in the absence of an external potential, and extended the study of its ground states to the whole parameter space of the two effective spin coupling strengths. Meanwhile, this provides a model of studying the relation between the classical model and quantum ground states.

We first analyzed the corresponding classical Hamiltonian. We found the fixed points of the classical dynamics, and discussed their stability situation both analytically and numerically. The bifurcations were discussed.

The classical evolution can be reproduced in quantum dynamics if starting from an initial state which is disentangled between the two species, as we have demonstrated.

In the case that the atom numbers of the two species
near each classical ground state. Using entanglement
regimes, by obtaining an effective Hamiltonian
not maximal, while the state corresponding to the fixed
maximal entanglement in the quantum ground state.

Moreover, we find the result that when the two atom
are equal, we confirmed in our system the previous claim
quantum states as a function 1/η = J₂/J₁. Here
we show the regime η > −1, in which the low-energy excited
states are with small magnitudes of S₂. The filled symbols
describe the analytical results. The empty symbols describe
the numerical results.

A quantum ground state can be regarded as the clas-
sistency of Science and Technology of China (Grant No.
which a classical fixed point bifurcation corresponds to
results. The empty symbols describe the numerical results.

are equal, we confirmed in our system the previous claim
that a classical fixed point bifurcation corresponds to
maximal entanglement in the quantum ground state.
Moreover, we find the result that when the two atom
numbers are unequal, the entanglement of the quantum
ground state at the parameter point of the bifurcation is
not maximal, while the state corresponding to the fixed
point that bifurcates indeed possesses maximal entangle-
ment at that parameter point.

A quantum ground state can be regarded as the clas-
sical ground state with quantum fluctuations. This per-
pective leads to solutions of the ground states in all pa-
ter parameter regimes, by obtaining an effective Hamiltonian
near each classical ground state. Using entanglement en-
tropy as the quantity characterizing the ground states, we
find that the analytical results fit the numerical results
very well. We have made many detailed discussions.

Our work establishes BEC as a system manifesting
connections between classical dynamics and quantum be-

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Appendix: Classical fixed points

For our problem, the desired Lyapunov function will
always be found by defining

$$\mathcal{L} = \gamma_1 H + \gamma_2 J_z (S_{a_z} + S_{b_z})^2$$

where γ₁ and γ₂ are suitably chosen coefficients. It is
clear that \(\frac{d\mathcal{L}}{dt} = 0\).

We find the following fixed points specified by the values of \(S_a = S_a n_a\), where
\(n_a = (\sin \theta_a \cos \varphi_a, \sin \theta_a \sin \varphi_a, \cos \theta_a)\), \(0 \leq \theta_a \leq \pi, 0 \leq \varphi_a < 2\pi, a = a, b\).

1. \(n_a = n_b = (0, 0, \pm 1)\); that is, one spin is parallel to the z direction, the other is antiparallel
to the z direction. At these two point the eigenvalues
of \(J\) are \(\mu_1 = \mu_2 = 0, \mu_{3,4} = \pm \sqrt{J_2 - J_\perp^2}\), \(\mu_{5,6} =
\pm \sqrt{(S_{a_z}^2 + S_{b_z}^2)}\), \(\zeta_3 \equiv 2J_\perp^2 S_a S_b - J_z^2 (S_{a_z}^2 + S_{b_z}^2), \zeta_4 \equiv
J_z (S_a + S_b) \sqrt{J_\perp^2 (S_{a_z} + S_{b_z})^2 - 2J_\perp^2 S_a S_b}.\) If we define \(\eta_1 =
\frac{1}{2} \sqrt{\frac{S_{a_z}}{S_{a_z}^2 + S_{b_z}^2}}\), when \(\eta_1 |J_z| < |J_\perp|\), there are eigenval-
ues with positive real part, and these two fixed points are
unstable. Otherwise, the stabilization cannot be judged
by the eigenvalues. One finds the Lyapunov function
\(\mathcal{L} = H - \frac{1}{2} (S_{a_z} + S_{b_z})^2\), which is minimal for \(J_z > 0\) (or
maximal for \(J_z < 0\) at each of these two fixed points in the parameter region \(\eta_1 |J_z| > |J_\perp|\), where each of these
two fixed points is thus stable.

2. \(n_a = -n_b = (\cos \varphi, \sin \varphi, 0)\), where \(0 \leq \varphi < 2\pi\).
The two spins are antiparallel and both are on the x − y
plane. At this point, the eigenvalues of \(J\) are \(\mu_1 = \mu_2 =
\mu_3 = \mu_4 = 0, \mu_{5,6} = \pm \sqrt{2J_\perp J_z S_a S_b - J_z^2 (S_{a_z}^2 + S_{b_z}^2)}\). If
we define \(\eta_2 = \frac{2S_a S_b}{S_{a_z}^2 + S_{b_z}^2}\), when \(0 < J_\perp < \eta_2 J_z\) or \(\eta_2 J_z <
J_z < 0\), some eigenvalues have positive real part, hence
this fixed point is unstable. When \(J_z > \eta_2 J_z \geq 0\)
or \(J_z < \eta_2 J_z \leq 0\), one finds the Lyapunov function
\(\mathcal{L} = H - \frac{1}{2} (S_{a_z} + S_{b_z})^2\), which is minimal at the fixed point
as \(\gamma_2 \to \infty\). When \(J_z \gamma_2 < 0\), one finds that \(\mathcal{L} = -H +
\gamma_2 J_z (S_{a_z} + S_{b_z})^2\) is minimal at the fixed point as \(\gamma_2 \to \infty\).
Thus this fixed point is stable if $J_{\perp} > 0$, $J_{\perp} > \eta_2 J_z$, or $J_{\perp} < J_z < \eta_2 J_z$.

(3) $n_a = n_b = (\cos \varphi, \sin \varphi, 0)$; that is, the two spins are parallel and on the $x - y$ plane. At this point, the eigenvalues of $J$ are $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0$, $\mu_5, \mu_6 = \pm \sqrt{-J_1^2(S_a^2 + S_b^2) - 2J_1J_zS_aS_b}$. When $-\eta_2 J_z < J_z < 0$ or $0 < J_z < -\eta_2 J_z$, some eigenvalues have positive real parts, hence the fixed point is unstable. For $J_z > 0$, one finds $L = -\mathcal{H} + \gamma_2 J_z(S_a + S_b)^2$ is minimal at the fixed point as $\gamma_2 \to \infty$. For $J_z < 0$ one finds $\mathcal{L} = \mathcal{H} + \gamma_2 J_z(S_a + S_b)^2$, which is minimal at the fixed point as $\gamma_2 \to \infty$. Therefore the fixed point is stable when $J_z > 0$, $J_z > -\eta_2 J_z$ or $J_z < 0$, $J_z < J_z$.

(4) $n_a = n_b = (0, 0, 1)$; that is, the two spins are both parallel or antiparallel to the $z$ direction. At each of these two fixed points, the eigenvalues of $J$ are $\mu_1 = \mu_2 = 0$, $\mu_3, \mu_4 = \pm \sqrt{1 - \frac{4J_z^2}{2}}$, $\mu_5, \mu_6 = \pm \sqrt{\frac{6 - 2J_z^2}{2}}$; here $\zeta_1 \equiv -2J_2S_aS_b - J_2(S_a^2 + S_b^2)$, $\zeta_2 \equiv J_z(S_a + S_b)\sqrt{J_2(S_a - S_b)^2 + 4J_z^2S_aS_b}$. The stabilization cannot be judged by the eigenvalues. But one finds the Lyapunov function $\mathcal{L} = -(S_a + S_b)^2$, which is minimal at the fixed point. Hence these two fixed points are always stable.

(5) In case $J_z = J_z$, the solution $n_a = n_b$ with any possible is a fixed point; that is, the two spins are always parallel. The Lyapunov function $\mathcal{L} = -S_a \cdot S_b$ is minimal here, thus this fixed point is stable.

(6) In case $J_z = J_z$, the solution $n_a = -n_b$ with any possible is a fixed point; that is, the two spins are always antiparallel. The Lyapunov function $\mathcal{L} = -S_a \cdot S_b$ is minimal here, thus this fixed point is stable.

(7) In case $J_z = -J_z$, $n_a = (\cos \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$ while $n_b = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$ is a fixed point; that is, the $z$ components of the two spins are opposite. One finds a Lyapunov function $\mathcal{L} = S_a \cdot S_b$, where $S'_a = (\sin \theta \cos \varphi, S_b \sin \theta \sin \varphi, S_a \cos \theta)$, which is minimal at this fixed point. Thus this fixed point is stable.

(8) In case $J_z = -J_z$, $n_a = (\cos \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$ while $n_b = (\sin \theta \cos \varphi, -\sin \theta \sin \varphi, \cos \theta)$ is a fixed point; that is, the $x$ and $y$ components of the two spins are opposite. One finds a Lyapunov function $\mathcal{L} = S_a \cdot S_b'$, where $S'_a = (\sin \varphi \cos \theta, -S_b \sin \theta \sin \varphi, -S_a \cos \theta)$, which is minimal at this fixed point. Thus this fixed point is stable.

All of the fixed points and their stable regimes are listed in Table I.

### Table I: Stable regimes of the fixed points.

| No. | Fixed Points | Stable regions |
|-----|-------------|---------------|
| 1   | $n_a = -n_b = (0, 0, 1)$ | $\eta_1 | J_z | > | J_z | $ |
| 2   | $n_a = -n_b = (\cos \varphi, \sin \varphi, 0)$ | $J_z > 0$, $J_z > \eta_2 J_z$ or $J_z < 0$, $J_z < \eta_2 J_z$ |
| 3   | $n_a = n_b = (\cos \varphi, \sin \varphi, 0)$ | $J_z > 0$, $J_z > -\eta_2 J_z$ or $J_z < 0$, $J_z < -\eta_2 J_z$ |
| 4   | $n_a = n_b = (0, 0, 1)$ | $J_z = J_z$ |
| 5   | $n_a = -n_b = (0, 0, 1)$ | $J_z = J_z$ |
| 6   | $n_a = n_b = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$ | $J_z = -J_z$ |
| 7   | $n_a = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$ | $J_z = -J_z$ |
| 8   | $n_a = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$ | $J_z = -J_z$ |

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