Differential Stability of Convex Optimization Problems with Possibly Empty Solution Sets

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Abstract
This paper studies differential stability of infinite-dimensional convex optimization problems, whose solution sets may be empty. By using suitable sum rules for \( \varepsilon \)-subdifferentials, we obtain exact formulas for computing the \( \varepsilon \)-subdifferential of the optimal value function. Several illustrative examples are also given.

Keywords Parametric convex programming · Optimal value function · Conjugate function · \( \varepsilon \)-Subdifferentials · \( \varepsilon \)-Normal directions

Mathematics Subject Classification 49J53 · 49Q12 · 90C25 · 90C31

1 Introduction

Studying differential stability of optimization problems usually means to study differentiability properties of the optimal value function in parametric mathematical programming. We refer to [1–8] and the references therein for some old and new results in this direction.

According to Penot [6, Chapter 3], the class of convex functions is an important class that enjoys striking and useful properties. The consideration of directional derivative makes it possible to reduce this class to the subclass of sublinear functions. This subclass is next to the family of linear functions in terms of simplicity: The epigraph...
of a sublinear function is a convex cone, a notion almost as simple and useful as the notion of linear subspace.

Differential properties of convex functions have been studied intensively in the last five decades. The fundamental contributions of J. -J. Moreau and R. T. Rockafellar have been widely recognized. Their results led to the beautiful theory of convex analysis [7]. The derivative-like structure for convex functions, called subdifferential, is one of the main concepts in this theory. Subdifferentials generalize the derivatives to nonsmooth functions, which make them one of the most useful instruments in nonsmooth optimization.

The concept of the $\varepsilon$-subdifferential or approximate subdifferential was first introduced by Brøndsted and Rockafellar [9]. It has become an essential tool in convex analysis. For example, approximate minima and approximate subdifferentials are linked together by Legendre–Fenchel transforms (see, e.g., [10]). Like for the subdifferential, calculus rules on the $\varepsilon$-subdifferential are of importance and attract the attention of many researchers; see, e.g., [8,10–18], and the references therein.

In [1], An and Yen presented formulas for computing the subdifferential of the optimal value function of convex optimization problems under inclusion constraints in a Hausdorff locally convex topological vector space setting. Afterward, An and Yao [2] obtained new results on subdifferential of the just-mentioned function for problems under geometrical and functional constraints in Banach spaces. In both papers, the authors assumed that the original convex program has a nonempty solution set. A natural question arises: Is there any analogous version of the formulas given in [1,2] for the case where the solution set can be empty?

By using sum rules for the $\varepsilon$-subdifferentials from [11] and appropriate regularity conditions, this paper presents formulas for the $\varepsilon$-subdifferential of the optimal value function of convex optimization problems under inclusion constraints in Hausdorff locally convex topological vector spaces.

The contents of the paper are as follows: Section 2 recalls several definitions and elementary results related to $\varepsilon$-subdifferentials of convex functions. Section 3 is devoted to a detailed analysis of several sum rules for $\varepsilon$-subdifferentials. Differential stability results of unconstrained and constrained convex optimization problems are established in Sect. 4. Several illustrative examples are also presented in this section.

## 2 Preliminaries

Let $X$ and $Y$ be Hausdorff locally convex topological vector spaces whose topological duals are denoted, respectively, by $X^*$ and $Y^*$. Let a function $f : X \to \overline{\mathbb{R}}$, where $\overline{\mathbb{R}} := [−\infty, +\infty] = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$. One says that $f$ is proper if the domain $\text{dom} \ f := \{x \in X : f(x) < +\infty\}$ is nonempty, and if $f(x) > -\infty$ for all $x \in X$.

It is well known that if $\text{epi} \ f$ of $f$ is convex, then $f$ is said to be a convex function, where $\text{epi} \ f := \{(x, \alpha) \in X \times \mathbb{R} : \alpha \geq f(x)\}$.

If $\text{epi} \ f$ is a closed subset of $X \times \mathbb{R}$, $f$ is said to be a closed function. Denoting the set of all the neighborhoods of $x$ by $\mathcal{N}(x)$, one says that $f$ is lower semicontinuous (l.s.c.) at $x \in X$ if for every $\varepsilon > 0$ there exists $U \in \mathcal{N}(x)$ such that $f(x') \geq f(x) - \varepsilon$
for any $x' \in U$. If $f$ is l.s.c. at every $x \in X$, $f$ is said to be l.s.c. on $X$. It is easy to show that $f$ is l.s.c. on $X$, if and only if both $f$ and $\text{dom } f$ are closed.

It is convenient to denote the set of all proper lower semicontinuous convex functions on $X$ by $\Gamma_0(X)$.

**Definition 2.1** Let $f$ be a convex function defined on $X$, $\bar{x} \in \text{dom } f$, and $\epsilon \geq 0$. The $\epsilon$-subdifferential of $f$ at $\bar{x}$ is the set

$$
\partial \epsilon f(\bar{x}) := \{x^* \in X^* : \langle x^*, x - \bar{x} \rangle \leq f(x) - f(\bar{x}) + \epsilon, \ \forall x \in X\}.
$$

The set $\partial \epsilon f(\bar{x})$ reduces to the subdifferential $\partial f(\bar{x})$ when $\epsilon = 0$. From the definition, it follows that $\partial \epsilon f(\bar{x})$ is a weakly*-closed, convex set. In addition, for any nonnegative values $\epsilon_1, \epsilon_2$ with $\epsilon_1 \leq \epsilon_2$, one has $\partial \epsilon_1 f(\bar{x}) \subset \partial \epsilon_2 f(\bar{x})$. Moreover,

$$
\partial f(\bar{x}) = \partial_0 f(\bar{x}) = \bigcap_{\epsilon > 0} \partial \epsilon f(\bar{x}).
$$

If $f \in \Gamma_0(X)$, then $\partial \epsilon f(\bar{x})$ is nonempty for all $\bar{x} \in \text{dom } f$ and $\epsilon > 0$ (see, e.g., [11]).

The following example shows that the traditional subdifferential $\partial f(\bar{x})$ may be empty, while $\partial \epsilon f(\bar{x}) \neq \emptyset$ for all $\epsilon > 0$.

**Example 2.1** Let $X = IR$ and $\bar{x} = 0$. Clearly, the function $f : X \to IR$ given by

$$
f(x) = \begin{cases} 
-\sqrt{x} & \text{if } x \geq 0, \\
+\infty & \text{otherwise}
\end{cases}
$$

belongs to $\Gamma_0(X)$ and $\bar{x} \in \text{dom } f$. For every $\epsilon > 0$, one has

$$
\partial \epsilon f(\bar{x}) = \{x^* \in X^* : \langle x^*, x - \bar{x} \rangle \leq f(x) - f(\bar{x}) + \epsilon, \ \forall x \in X\}
= \{x^* \in IR : x^* x \leq -\sqrt{x} + \epsilon, \ \forall x \geq 0\}
= \left[-\infty, -\frac{1}{4\epsilon}\right].
$$

Meanwhile, it is easy to verify that $\partial f(\bar{x}) = \emptyset$.

In the sequel, we will also need the notion of conjugate function. By definition, the function $f^* : X^* \to IR$ given by

$$
f^*(x^*) = \sup_{x \in X} \langle x^*, x \rangle - f(x), \ x^* \in X^*,
$$

is said to be the conjugate function (also called the Young–Fenchel transform, the Legendre–Fenchel conjugate) of $f : X \to IR$. The conjugate function of $f^*$, denoted by $f^{**}$, is a function defined on $X$ and has values in $IR$:

$$
f^{**}(x) = \sup_{x^* \in X^*} \left[\langle x^*, x \rangle - f^*(x^*)\right] \ (x \in X).
$$
Clearly, the function $f^{**}$ is convex and closed. (In the sense that epi $f^{**}$ is closed in the weak topology of $X \times \mathbb{R}$; in other words, $f^{**}$ is lower semicontinuous w.r.t. the weak topology of $X$.) According to the Fenchel–Moreau theorem (see [19, Theorem 1, p. 175]), if $f$ is a function on $X$ everywhere greater than $-\infty$, then $f = f^{**}$ if and only if $f$ is closed and convex.

According to [11], there are two basic ways to describe $\partial_\varepsilon f(\bar{x})$:

(a) Via the conjugate function $f^*$ of $f$;
(b) Via the support function $\delta^*(x; \partial_\varepsilon f(\bar{x})) := \sup \{ \langle x^*, x \rangle : x^* \in \partial_\varepsilon f(\bar{x}) \}$ of $\partial_\varepsilon f(\bar{x})$.

**Proposition 2.1** (See [11, Propositions 1.1 and 1.2]) The following holds:

(i) If $\bar{x} \in \text{dom } f$ and $\varepsilon \geq 0$, then

$$x^* \in \partial_\varepsilon f(\bar{x}) \iff f^*(x^*) + f(\bar{x}) \leq \langle x^*, \bar{x} \rangle + \varepsilon.$$ 

(ii) If $f \in \Gamma_0(X)$, $\bar{x} \in \text{dom } f$ and $\varepsilon \geq 0$, then

$$\delta^*(v; \partial_\varepsilon f(\bar{x})) := \inf_{t > 0} \frac{f(\bar{x} + tv) - f(\bar{x}) + \varepsilon}{t} \quad (v \in X).$$

To deal with constrained optimization problems, we will need some results on $\varepsilon$-normal directions from [12]. Let $C$ be a nonempty convex set in a Hausdorff locally convex topological vector space $X$.

**Definition 2.2** The set $N_\varepsilon(\bar{x}; C)$ of $\varepsilon$-normal directions to $C$ at $\bar{x} \in C$ is defined by

$$N_\varepsilon(\bar{x}; C) = \{ x^* \in X^* : \langle x^*, x - \bar{x} \rangle \leq \varepsilon, \forall x \in C \}.$$

As usual, the indicator function $\delta(\cdot; C)$ of $C$ is defined by setting $\delta(x; C) = 0$ if $x \in C$ and $\delta(x; C) = +\infty$ if $x \notin C$. It is easy to see that $N_\varepsilon(\bar{x}; C) = \partial_\varepsilon \delta(\bar{x}; C)$ for every $\varepsilon \geq 0$. Moreover, when $\varepsilon = 0$, $N_\varepsilon(\bar{x}; C)$ reduces to the normal cone of $C$ at $\bar{x}$, which is denoted by $N(\bar{x}; C)$. However, as a general rule, $N_\varepsilon(\bar{x}; C)$ is not a cone when $\varepsilon > 0$.

The polar set of $A \subset X$ is defined by

$$A^0 = \{ x^* \in X^* : \langle x^*, x \rangle \leq 1, \forall x \in A \}.$$ 

**Proposition 2.2** (See [12, p. 222]) The following properties of $\varepsilon$-normal directions are valid:

(i) $N_\varepsilon(x; C) = \varepsilon(C - x)^0$ for any $x \in C$ and $\varepsilon > 0$;

(ii) $N(x; C) = \bigcap_{\eta > 0} \eta N_\varepsilon(x; C)$ for any $x \in C$ and $\varepsilon \geq 0$.

The first assertion of Proposition 2.2 shows that the set of the $\varepsilon$-normal directions $N_\varepsilon(x; C)$ can be computed via the polar set of a set containing 0. Provided that the set $N_\varepsilon(x; C)$ has been found, by using the second assertion of Proposition 2.2, one can compute the normal cone $N(x; C)$. Due to the importance of the polar sets of sets containing the origin, it is reasonable to consider an illustrative example. Let $X = \mathbb{R}^2$ and $\overline{B}_{\mathbb{R}^2}$ be the unit closed ball in $\mathbb{R}^2$. 

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Example 2.2 Consider the set
\[ A = \overline{B}((0, 1); 1) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + (x_2 - 1)^2 \leq 1\}, \]
we have \( A^0 = \{x^* = (x_1^*, x_2^*) \in \mathbb{R}^2 : x_1^* + ||x^*|| \leq 1\}, \) where \( ||x^*|| = \sqrt{x_1^{*2} + x_2^{*2}}. \)
Indeed, since \( A = (1, 0) + \overline{B}_{\mathbb{R}^2}, \) one has
\[
A^0 = \{x^* = (x_1^*, x_2^*) \in \mathbb{R}^2 : \langle (x_1^*, x_2^*), (1, 0) \rangle + v \leq 1, \ \forall v \in \overline{B}_{\mathbb{R}^2}\}
= \{x^* \in \mathbb{R}^2 : x_1^* + ||x^*|| \leq 1\}.
\]
Now, consider a proper convex function \( f : X \to \overline{\mathbb{R}} \) and suppose that \( \bar{x} \in \text{dom } f. \)
The relationship between \( \partial_{\varepsilon} f(\bar{x}) \) and \( N_{\varepsilon}((\bar{x}, f(\bar{x})); \text{epi } f) \) is described in [12, p. 224] as follows:
\[
\partial_{\varepsilon} f(\bar{x}) = \{x^* \in X^* : (x^*, -1) \in N_{\varepsilon}((\bar{x}, f(\bar{x})); \text{epi } f)\} \quad (\varepsilon \geq 0). \quad (1)
\]
Taking \( \varepsilon = 0, \) from (1) we recover the following fundamental formula in convex analysis, which relates subdifferentials of a given convex function to the normal cone of its epigraph:
\[
\partial f(x) = \{x^* \in X^* : (x^*, -1) \in N((x, f(x)); \text{epi } f)\} \quad (\forall x \in \text{dom } f).
\]

3 Sum Rules for \( \varepsilon \)-Subdifferentials

In convex analysis and optimization, summing two functions is a key operation. The Moreau–Rockafellar Theorem can be viewed as a well-known result, which describes the subdifferential of the sum of two subdifferentially functions. Invoking a result on the infimal convolution of two functions, one gets a sum rule for \( \varepsilon \)-subdifferentials.

Theorem 3.1 (See [11, Theorem 2.1]) Suppose that \( f_1, f_2 : X \to \overline{\mathbb{R}} \) are two proper convex functions on a Hausdorff locally convex topological vector space \( X \) and the qualification condition
\[
(f_1 + f_2)^*(x^*) = \min \{ f_1^*(x_1^*) + f_2^*(x_2^*) : x_1^*, x_2^* \in X^*, x_1^* + x_2^* = x^* \} \quad (\forall x^* \in X^*)
\]
(2)
holds. Then, for every \( \bar{x} \in \text{dom } f_1 \cap \text{dom } f_2 \) and \( \varepsilon > 0, \) one has
\[
\partial_{\varepsilon}(f_1 + f_2)(\bar{x}) = \bigcup_{\varepsilon_1, \varepsilon_2 \geq 0, \varepsilon_1 + \varepsilon_2 = \varepsilon} \left\{ \partial_{\varepsilon_1} f_1(\bar{x}) + \partial_{\varepsilon_2} f_2(\bar{x}) \right\}. \quad (3)
\]
Condition (2) means that, for every \( x^* \in X^*, \) one has
\[
(f_1 + f_2)^*(x^*) = \inf \{ f_1^*(x_1^*) + f_2^*(x_2^*) : x_1^*, x_2^* \in X^*, x_1^* + x_2^* = x^*\}, \quad (4)
\]
and the infimum is attained, i.e., there exist \( \bar{x}_1^*, \bar{x}_2^* \) from \( X^* \) with \( \bar{x}_1^* + \bar{x}_2^* = x^* \) such that

\[
    f_1^*(\bar{x}_1^*) + f_2^*(\bar{x}_2^*) = \inf \{ f_1^*(x_1^*) + f_2^*(x_2^*) : x_1^* + x_2^* = x^* \}. \tag{5}
\]

A deeper understanding of condition (2) is achieved via the notion of infimal convolution [19, p. 168] of convex functions.

The infimal convolution \( f_1 \oplus f_2 \) of two proper convex functions \( f_1 : X \to \overline{\mathbb{R}} \) and \( f_2 : X \to \overline{\mathbb{R}} \) is defined by

\[
    (f_1 \oplus f_2)(x) := \inf \{ f_1(x_1) + f_2(x_2) : x_1 + x_2 = x \} \quad (x \in X).
\]

Applying this construction to the functions \( f_1^* : X^* \to \overline{\mathbb{R}} \) and \( f_2^* : X^* \to \overline{\mathbb{R}} \), we have

\[
    (f_1^* \oplus f_2^*)(x^*) = \inf \{ f_1^*(x_1^*) + f_2^*(x_2^*) : x_1^* + x_2^* = x^* \}. \tag{6}
\]

The attainment of the infimum on the right-hand side of (6) at a point \( x^* \) is a kind of qualification on the functions \( f_1, f_2 \) in a dual space setting. The writing

\[
    (f_1^* \oplus f_2^*)(x^*) = \min \{ f_1^*(x_1^*) + f_2^*(x_2^*) : x_1^* + x_2^* = x^* \}
\]

means that there exist \( \bar{x}_1^*, \bar{x}_2^* \) from \( X^* \) with \( x^* = \bar{x}_1^* + \bar{x}_2^* \) and

\[
    (f_1^* \oplus f_2^*)(x^*) = f_1^*(\bar{x}_1^*) + f_2^*(\bar{x}_2^*).
\]

According to [19, p. 168], the infimal convolution of proper convex functions is a convex function. However, the latter can fail to be proper. For example, if \( f_1 \) and \( f_2 \) are linear functions not equal to one another, then their infimal convolution is identically \( -\infty \).

By the definition of conjugate function, we have

\[
    (f_1 + f_2)^*(x^*) = \sup_{x \in X} \{ (x^*, x) - (f_1 + f_2)(x) \}.
\]

So, substituting \( x^* = x_1^* + x_2^* \) with \( x_1^* \in X^* \) and \( x_2^* \in X^* \) yields

\[
    (f_1 + f_2)^*(x^*) = \sup_{x \in X} \{ (x_1^* + x_2^*, x) - f_1(x) - f_2(x) \}
    = \sup_{x \in X} \{ (x_1^*, x) - f_1(x) + (x_2^*, x) - f_2(x) \}
    \leq \sup_{x \in X} \{ (x_1^*, x) - f_1(x) \} + \sup_{x \in X} \{ (x_2^*, x) - f_2(x) \}.
\]

Thus, the inequality

\[
    (f_1 + f_2)^*(x^*) \leq f_1^*(x_1^*) + f_2^*(x_2^*) \tag{7}
\]
holds for all \( x^*, x_1^*, x_2^* \in X^* \) satisfying \( x^* = x_1^* + x_2^* \). For any \( x^* \in X^* \), taking infimum of both sides of (7) on the set of all \( (x_1^*, x_2^*) \) with \( x_1^* + x_2^* = x^* \), we get

\[
(f_1 + f_2)^*(x^*) \leq (f_1^* + f_2^*)(x^*); \tag{8}
\]

see [19, p. 181]. Since (4) can be rewritten as

\[
(f_1 + f_2)^*(x^*) = (f_1^* + f_2^*)(x^*), \tag{9}
\]

condition (2) requires that, for the functions \( f_1 \) and \( f_2 \) in question, the inequality in (8) holds as equality for all \( x^* \in X^* \). Luckily, this requirement is satisfied under some verifiable regularity conditions. The following theorem describes a condition of this type.

**Theorem 3.2** (See [19, Theorem 1, p. 178]) Suppose that \( f_1, f_2 \) are proper convex functions. If

\[
\quad \left\{ \begin{array}{l}
\text{one of the functions } f_1, f_2 \text{ is continuous at a point belonging to the effective domain of the other,}
\end{array} \right. \tag{10}
\]

then the equality \((f_1 + f_2)^*(x^*) = (f_1^* + f_2^*)(x^*)\) holds for every \( x^* \in X^* \). Moreover, for every \( x^* \in \text{dom} (f_1 + f_2)^* \), there exist points \( \bar{x}_i^* \in \text{dom} f_i^*, i = 1, 2 \), such that

\[
f_1^*(\bar{x}_1^*) + f_2^*(\bar{x}_2^*) = (f_1 + f_2)^*(x^*).
\]

**Remark 3.1** Under the assumptions of Theorem 3.2, qualification condition (2) is satisfied. Indeed, suppose that one of the proper convex functions \( f_1, f_2 \) is continuous at a point \( x^0 \) belonging to the effective domain of the other. Then, one has \( x^0 \in \text{dom} (f_1 + f_2) \). It follows that \((f_1 + f_2)^*(x^*)\) is everywhere greater than \(-\infty\) for every \( x^* \in X^* \). If \( x^* \notin \text{dom} (f_1 + f_2)^* \), then \((f_1 + f_2)^*(x^*) = +\infty\). Choose \( \bar{x}_1^*, \bar{x}_2^* \in X^* \) such that \( x^* = \bar{x}_1^* + \bar{x}_2^* \). From (7),

\[
+\infty = (f_1 + f_2)^*(x^*) \leq f_1^*(\bar{x}_1^*) + f_2^*(\bar{x}_2^*).
\]

Noting that \( f_1^*(\bar{x}_1^*) > -\infty \) and \( f_2^*(\bar{x}_2^*) > -\infty \) because \( f_1, f_2 \) are two proper functions, from this we infer that at least one of the values \( f_1^*(\bar{x}_1^*) \) and \( f_2^*(\bar{x}_2^*) \) must be \(+\infty\). Combining this with (7) yields (5). Since (4) is equivalent to (9), and the latter is fulfilled. Thanks to Theorem 3.2, we have thus proved that the equality in (2) is satisfied for every \( x^* \notin \text{dom} (f_1 + f_2)^* \). If \( x^* \in \text{dom} (f_1 + f_2)^* \), then the equality in (2) follows immediately from Theorem 3.2.

In a Banach space setting, one has the following analogue of Theorem 3.2, where \( f_1 \) and \( f_2 \) must be assumed closed. Recall that

\[
\text{IR}_+(A) := \{ta \in X : t \in \text{IR}_+, a \in A\}
\]
and int A, respectively, are the cone generated by the set A and the interior of A.

**Theorem 3.3** (See [20, Theorem 1.1], [4, Theorem 4.2 (iii)]) Let functions $f_1, f_2 : X \to \mathbb{R}$ be proper, closed, convex functions defined on a Banach space X. Suppose that

$$
IR_+(\text{dom } f_1 - \text{dom } f_2) \text{ is a nonempty, closed subspace of } X.
$$

(11)

Then, for every $x^* \in X^*$, one has $(f_1 + f_2)^*(x^*) = (f_1^* \oplus f_2^*)(x^*)$. Moreover, for any $x^* \in \text{dom } (f_1 + f_2)^*$ there are $x_1^*, x_2^* \in X^*$ such that $x^* = x_1^* + x_2^*$ and

$$(f_1 + f_2)^*(x^*) = f_1^*(x_1^*) + f_2^*(x_2^*).$$

Later we will also need another version of Theorem 3.2, where a geometrical regularity condition is employed.

**Theorem 3.4** (See [21, Theorem 2.171]) Let $f_1, f_2 : X \to \mathbb{R}$ be proper, closed, convex functions defined on a Banach space X. If the regularity condition

$$
0 \in \text{int } (\text{dom } f_1 - \text{dom } f_2)
$$

(12)

is satisfied, then the equality $(f_1 + f_2)^*(x^*) = (f_1^* \oplus f_2^*)(x^*)$ holds for every $x^* \in X^*$. Moreover, if $x^*$ is such that the value $(f_1 + f_2)^*(x^*)$ is finite, then the set of $x_1^*$ satisfying $(f_1^* \oplus f_2^*)(x^*) = f_1^*(x_1^*) + f_2^*(x^* - x_1^*)$ is nonempty and weakly*-compact.

**Remark 3.2** Under the assumptions of Theorem 3.3 (respectively, of Theorem 3.4), qualification condition (2) is satisfied. Indeed, suppose that $f_1, f_2 : X \to \mathbb{R}$ are proper, closed, convex functions defined on a Banach space X, and (11) (resp., (12)) is fulfilled. We have $0 \in \text{dom } f_1 - \text{dom } f_2$. So, there is $x^0 \in X$ with $x^0 \in \text{dom } f_1 \cap \text{dom } f_2$. Then $x^0 \in \text{dom } (f_1 + f_2)$. Applying Theorem 3.3 (resp., Theorem 3.4) and the arguments already used in Remark 3.1, we obtain (2).

We now show that assumption (2) is essential for Theorem 3.1.

**Example 3.1** Let $X = \mathbb{R}$, $f_1(x) = 0$ for $x = 0$, and $f_1(x) = +\infty$ for $x \neq 0$. Define $f_2$ by setting $f_2(x) = -\sqrt{x}$ for $x \geq 0$, and $f_2(x) = +\infty$ for $x < 0$. By a simple computation, we obtain $f_1^*(x^*) = 0$ for all $x^* \in \mathbb{R}$, and

$$
f_2^*(x^*) = \begin{cases} 
-\frac{1}{4x^*} & \text{if } x^* < 0, \\
+\infty & \text{if } x^* \geq 0.
\end{cases}
$$

Since $(f_1 + f_2)(x) = 0$ for $x = 0$ and $(f_1 + f_2)(x) = +\infty$ for all $x \neq 0$, the equality $(f_1 + f_2)^*(x^*) = 0$ holds for every $x^* \in \mathbb{R}$. So, for $x^* = 0$, (4) holds, but the infimum on the right-hand side is not attained. This means that condition (2) is not satisfied. For $\bar{x} = 0$ and $\varepsilon > 0$, equality (3) holds because $\partial_{\varepsilon}^{-1}(f_1 + f_2)(\bar{x}) = \mathbb{R}$, $\partial_{\varepsilon_1}^{-1} f_1(\bar{x}) = \mathbb{R}$ for every $\varepsilon_1 \geq 0$, $\partial f_2(\bar{x}) = \emptyset$, and $\partial_{\varepsilon_2} f_2(\bar{x}) = \left[-\infty, -\frac{1}{4\epsilon_2^2}\right]$ for every $\varepsilon_2 > 0$ (see Example 2.1). Nevertheless, for $\bar{x} = 0$ and $\varepsilon = 0$, equality (3) is violated because the left-hand side is $\mathbb{R}$, while the right-hand side is the empty set.
The sum rule (3) requires the fulfillment of condition (2), which is implied by the regularity conditions (10), (11), and (12) and the corresponding assumptions of Theorems 3.2, 3.3, and 3.4. We now clarify the relationships between the regularity conditions (10), (11), and (12).

**Proposition 3.1** (See also [1, Subsection 6.1]) Let \( f_1, f_2 : X \to \text{IR} \) be proper, closed, convex functions defined on a Hausdorff locally convex topological vector space \( X \). Then, (10) implies (11) and (12).

**Proof** Without loss of generality, suppose that \( f_1 \) is continuous at a point \( \bar{x} \in \text{dom} f_2 \). Then, there exists a neighborhood \( U \) of 0 such that \( \bar{x} + U \subset \text{dom} f_1 \). So, \( U = (\bar{x} + U) - \bar{x} \subset \text{dom} f_1 - \text{dom} f_2 \). This yields (12) and the equality

\[
\text{IR}_+(\text{dom} f_1 - \text{dom} f_2) = X,
\]

which justifies (11).

Implication (12) \( \Rightarrow \) (11) is obvious. Let us present two simple examples to show that the converse implication and assertion (12) \( \Rightarrow \) (10) are not true.

**Example 3.2** Let \( X = \text{IR}^2 \), \( f_1(x) = x_1^2 \) for all \( x = (x_1, 0) \), \( f_1(x) = +\infty \) for all \( x = (x_1, x_2) \) with \( x_1 \neq 0 \), and \( f_2 \equiv f_1 \). Then,

\[
\text{IR}_+(\text{dom} f_1 - \text{dom} f_2) = \text{dom} f_1 - \text{dom} f_2 = \text{IR} \times \{0\}
\]

is a closed subspace of \( X \). However, both conditions (10) and (12) are violated.

**Example 3.3** Let \( X \) and \( f_1 \) be the same as in Example 3.2. Put \( f_2(x) = x_2^2 \) for all \( x = (0, x_2) \), \( f_2(x) = +\infty \) for all \( x = (x_1, x_2) \) with \( x_2 \neq 0 \). Then (12) is satisfied, but (10) fails to hold.

### 4 Main Results

Differential stability of convex optimization problems with the possibly empty solution set in infinite-dimensional spaces is studied in this section. To make the presentation as clear as possible, we distinguish two cases:

(a) unconstrained problems;
(b) constrained problems.

Let \( X, Y \) be Hausdorff locally convex topological vector spaces and a function \( \varphi : X \times Y \to \text{IR} \) an extended real-valued function.

**4.1 Unconstrained Convex Optimization Problems**

Consider the parametric unconstrained convex optimization problem

\[
\min \{ \varphi(x, y) : y \in Y \} \tag{13}
\]
depending on the parameter $x$. The function $\varphi$ is called the **objective function** of (13). The **optimal value function** $\mu : X \to \overline{\mathbb{R}}$ of (13) is

$$\mu(x) := \inf \{ \varphi(x, y) : y \in Y \}. \quad (14)$$

The solution set of (13) is defined by $M(\bar{x}) := \{ y \in Y : \mu(\bar{x}) = \varphi(\bar{x}, y) \}$. For $\eta > 0$, one calls $M_\eta(\bar{x}) := \{ y \in Y : \varphi(\bar{x}, y) \leq \mu(\bar{x}) + \eta \}$ the **approximate solution set** of (13).

We now obtain formulas for the $\varepsilon$-subdifferential of $\mu(.)$. Since the following result was given in [16, Corollary 5] as a consequence of a more general result and in [8, Theorem 2.6.2] with a brief proof, we will present a detailed, direct proof to make the presentation as clear as possible. Our arguments are based on a proof scheme of [16].

**Theorem 4.1** (See [16, Corollary 5] and [8, Theorem 2.6.2, p. 109]) Suppose that $\varphi : X \times Y \to \overline{\mathbb{R}}$ is a proper convex function and $\mu(\cdot)$ is finite at $\bar{x} \in X$. Then, for every $\varepsilon \geq 0$, one has

$$\partial_\varepsilon \mu(\bar{x}) = \bigcap_{\eta > 0} \bigcap_{y \in M_\eta(\bar{x})} \left\{ x^* \in X^* : (x^*, 0) \in \partial_{\varepsilon + \eta} \varphi(\bar{x}, y) \right\} = \bigcap_{\eta > 0} \bigcup_{y \in Y} \left\{ x^* \in X^* : (x^*, 0) \in \partial_{\varepsilon + \eta} \varphi(\bar{x}, y) \right\}. \quad (15)$$

In particular,

$$\partial \mu(\bar{x}) = \bigcap_{\eta > 0} \bigcap_{y \in M_\eta(\bar{x})} \left\{ x^* \in X^* : (x^*, 0) \in \partial \varphi(\bar{x}, y) \right\} = \bigcap_{\eta > 0} \bigcup_{y \in Y} \left\{ x^* \in X^* : (x^*, 0) \in \partial \varphi(\bar{x}, y) \right\}. \quad (16)$$

Moreover, if $M(\bar{x}) \neq \emptyset$, then for every $\varepsilon \geq 0$, one has

$$\partial_\varepsilon \mu(\bar{x}) = \left\{ x^* \in X^* : (x^*, 0) \in \partial_\varepsilon \varphi(\bar{x}, y) \right\}, \quad (17)$$

for all $y \in M(\bar{x})$.

**Proof** We put

$$M_\eta(\bar{x}) = \bigcap_{y \in M_\eta(\bar{x})} \left\{ x^* \in X^* : (x^*, 0) \in \partial_{\varepsilon + \eta} \varphi(\bar{x}, y) \right\},$$

$$N_\eta(\bar{x}) = \bigcup_{y \in Y} \left\{ x^* \in X^* : (x^*, 0) \in \partial_{\varepsilon + \eta} \varphi(\bar{x}, y) \right\}.$$

Since $\mu(\bar{x}) = \inf_{y \in Y} \varphi(\bar{x}, y)$ by (14), the set $M_\eta(\bar{x})$ is nonempty for every $\eta > 0$. Thus, one has $M_\eta(\bar{x}) \subset N_\eta(\bar{x})$ for all $\eta > 0$. Hence, $\bigcap_{\eta > 0} M_\eta(\bar{x}) \subset \bigcap_{\eta > 0} N_\eta(\bar{x})$. So, the equalities in (15) will be proved, if we can show that

$$\partial_\varepsilon \mu(\bar{x}) = \bigcap_{\eta > 0} \bigcap_{y \in M_\eta(\bar{x})} \left\{ x^* \in X^* : (x^*, 0) \in \partial_{\varepsilon + \eta} \varphi(\bar{x}, y) \right\} = \bigcap_{\eta > 0} \bigcup_{y \in Y} \left\{ x^* \in X^* : (x^*, 0) \in \partial_{\varepsilon + \eta} \varphi(\bar{x}, y) \right\}. \quad (18)$$
\begin{align}
\partial_\varepsilon \mu(\bar{x}) &\subset \bigcap_{\eta > 0} \mathcal{M}_\eta(\bar{x}) \tag{18} \\
\text{and} \\
\bigcap_{\eta > 0} \mathcal{N}_\eta(\bar{x}) &\subset \partial_\varepsilon \mu(\bar{x}). \tag{19}
\end{align}

To prove (18), take any \( x^* \in \partial_\varepsilon \mu(\bar{x}), \eta > 0, \) and \( y \in M_\eta(\bar{x}). \) Thanks to the first assertion of Proposition 2.1, we know that \( x^* \in \partial_\varepsilon \mu(\bar{x}) \) if and only if

\[
\mu(\bar{x}) + \mu^*(x^*) \leq \langle x^*, \bar{x} \rangle + \varepsilon. \tag{20}
\]

Adding \( \eta > 0 \) to both sides of (20) yields

\[
\mu(\bar{x}) + \mu^*(x^*) + \eta \leq \langle x^*, \bar{x} \rangle + \varepsilon + \eta. \tag{21}
\]

Since \( y \in M_\eta(\bar{x}), \) one has \( \varphi(\bar{x}, y) \leq \mu(\bar{x}) + \eta. \) So, (21) gives

\[
\varphi(\bar{x}, y) + \mu^*(x^*) \leq \langle x^*, \bar{x} \rangle + \varepsilon + \eta. \tag{22}
\]

For every \( v^* \in X^*, \) we have \( \mu^*(v^*) = \varphi^*(v^*, 0). \) Indeed, by the definition of conjugate function,

\[
\mu^*(v^*) = \sup_{x \in X} \left\{ \langle v^*, x \rangle - \mu(x) \right\} \\
= \sup_{x \in X} \left\{ \langle v^*, x \rangle - \inf_{y \in Y} \varphi(x, y) \right\} \\
= \sup_{(x, y) \in X \times Y} \left\{ \langle v^*, x \rangle - \varphi(x, y) \right\} \\
= \sup_{(x, y) \in X \times Y} \left\{ \langle v^*, 0 \rangle, (x, y) \rangle - \varphi(x, y) \right\} \\
= \varphi^*(v^*, 0).
\]

Substituting \( \mu^*(x^*) = \varphi^*(x^*, 0) \) into (22), one obtains

\[
\varphi(\bar{x}, y) + \varphi^*(x^*, 0) \leq \langle x^*, \bar{x} \rangle + \varepsilon + \eta. \tag{23}
\]

According to Proposition 2.1, inequality (23) yields \( (x^*, 0) \in \partial_{\varepsilon+\eta} \varphi(\bar{x}, y) \) for all \( \eta > 0 \) and \( y \in M_\eta(\bar{x}). \) This means that \( x^* \in \bigcap_{\eta > 0} \mathcal{M}_\eta(\bar{x}), \) so (18) is valid.

Next, to prove (19), take any \( x^* \in \bigcap_{\eta > 0} \mathcal{N}_\eta(\bar{x}). \) Then, for every \( \eta > 0, \) there exists \( y \in Y \) such that \( (x^*, 0) \in \partial_{\varepsilon+\eta} \varphi(\bar{x}, y). \) By Proposition 2.1, this means that

\[
\varphi^*(x^*, 0) + \varphi(\bar{x}, y) - \langle (x^*, 0), (\bar{x}, y) \rangle \leq \varepsilon + \eta.
\]
The latter yields
\[ \phi^*(x^*, 0) + \phi(\bar{x}, y) - \langle x^*, \bar{x} \rangle \leq \varepsilon + \eta. \] (24)

Since \( \phi^*(x^*, 0) = \mu^*(x^*) \) and \( \mu(\bar{x}) \leq \phi(\bar{x}, y) \), (24) implies
\[ \mu^*(x^*) + \mu(\bar{x}) - \langle x^*, \bar{x} \rangle \leq \varepsilon + \eta. \] (25)

As (25) holds for every \( \eta > 0 \), letting \( \eta \to 0^+ \) yields
\[ \mu^*(x^*) + \mu(\bar{x}) - \langle x^*, \bar{x} \rangle \leq \varepsilon. \]

The last inequality shows that \( x^* \in \partial_\varepsilon \mu(\bar{x}) \). Therefore, (19) is fulfilled.

Next elementary property of the \( \varepsilon \)-subdifferential will be used later on.

**Proposition 4.1** Let \( \phi : X \times Y \to \mathbb{R} \) be a convex function. If the function \( \phi(x, y) = \varphi_1(x) + \varphi_2(y) \), where \( \varphi_1 : X \to \mathbb{R} \) and \( \varphi_2 : Y \to \mathbb{R} \) are convex functions, then for any \( \varepsilon \geq 0 \) and \((\bar{x}, \bar{y}) \in X \times Y\), one has
\[ \partial_\varepsilon \varphi(\bar{x}, \bar{y}) \subset \partial_\varepsilon \varphi_1(\bar{x}) \times \partial_\varepsilon \varphi_2(\bar{y}). \] (26)

**Proof** Suppose that \((x^*, y^*) \in \partial_\varepsilon \varphi(\bar{x}, \bar{y})\) for some \( \varepsilon \geq 0 \). Then, we have
\[ \langle (x^*, y^*), (x, y) - (\bar{x}, \bar{y}) \rangle \leq \varphi(x, y) - \varphi(\bar{x}, \bar{y}) + \varepsilon, \forall (x, y) \in X \times Y. \] (27)

By our assumption, (27) is equivalent to
\[ \langle x^*, x - \bar{x} \rangle + \langle y^*, y - \bar{y} \rangle \leq \varphi_1(x) - \varphi_1(\bar{x}) + \varphi_2(y) - \varphi_2(\bar{y}) + \varepsilon, \forall (x, y) \in X \times Y. \] (28)

On the one hand, substituting \( y = \bar{y} \) into (28), we get \( x^* \in \partial \varphi_1(\bar{x}) \). On the other hand, taking \( x = \bar{x} \), from (28) we have \( y^* \in \partial \varphi_2(\bar{y}) \). Therefore, for any \( \varepsilon \geq 0 \),
\[ \partial_\varepsilon \varphi(\bar{x}, \bar{y}) \subset \partial_\varepsilon \varphi_1(\bar{x}) \times \partial_\varepsilon \varphi_2(\bar{y}). \]

The second inclusion in (26) can be obtained easily by the definition of the \( \varepsilon \)-subdifferential. Thus, (26) is valid.

The following example is taken from [14, pp. 93–94].

**Example 4.1** Let \( f(x) = |x| \) for all \( x \in \mathbb{R} \) and \( \varepsilon \geq 0 \). We have
\[ \partial_\varepsilon f(x) = \begin{cases} \begin{bmatrix} -1, -1 - \frac{\varepsilon}{2} \end{bmatrix} & \text{if } x < -\frac{\varepsilon}{2}, \\ \begin{bmatrix} -1, 1 \end{bmatrix} & \text{if } -\frac{\varepsilon}{2} \leq x \leq \frac{\varepsilon}{2}, \\ \begin{bmatrix} 1 - \frac{\varepsilon}{x}, 1 \end{bmatrix} & \text{if } x > \frac{\varepsilon}{2}. \end{cases} \]
We now give an illustration for Theorem 4.1.

**Example 4.2** Choose \( X = Y = \mathbb{R} \), \( \varphi(x, y) = x^2 + |y| \), and \( \bar{x} = 0 \). Then the optimal value function (14) of the parametric problem (13) is \( \mu(x) = x^2 \). For any \( \varepsilon \geq 0 \), we have

\[
\partial_\varepsilon \mu(\bar{x}) = \{ x^* \in \mathbb{R} : x^* x \leq x^2 + \varepsilon, \ \forall x \in \mathbb{R} \}
\]

\[
= \{ x^* \in \mathbb{R} : -x^2 + x^* x - \varepsilon \leq 0, \ \forall x \in \mathbb{R} \}
\]

\[
= [-2\sqrt{\varepsilon}, 2\sqrt{\varepsilon}].
\]

In this case, \( \bar{y} = 0 \in M(\bar{x}) \), so we will clarify equality (17). By Proposition 4.1, one has \( \partial_\varepsilon \varphi(\bar{x}, y) \subseteq \partial_\varepsilon \varphi_1(\bar{x}) \times \partial_\varepsilon \varphi_2(y) \), where \( \varphi_1(x) = x^2 \) and \( \varphi_2(y) = |y| \). On the one hand, \( \partial_\varepsilon \varphi_1(\bar{x}) = [-2\sqrt{\varepsilon}, 2\sqrt{\varepsilon}] \). On the other hand, according to Example 4.1,

\[
\partial_\varepsilon \varphi_2(y) = \begin{cases} 
-1, & -1 - \frac{\varepsilon}{y} \text{ if } y < -\frac{\varepsilon}{2}, \\
-1, & 1 \text{ if } -\frac{\varepsilon}{2} \leq y \leq \frac{\varepsilon}{2}, \\
1 - \frac{\varepsilon}{y}, & 1 \text{ if } y > \frac{\varepsilon}{2}.
\end{cases}
\]

Then, the right-hand side of (17) can be computed as follows:

\[
RHS_{(17)} = \{ x^* \in \mathbb{R} : (x^*, 0) \in [-2\sqrt{\varepsilon}, 2\sqrt{\varepsilon}] \times [-1, 1] \}
\]

\[
= [-2\sqrt{\varepsilon}, 2\sqrt{\varepsilon}].
\]

Therefore, the conclusion of Theorem 4.1 is justified.

### 4.2 Constrained Convex Optimization Problems

Let \( X, Y \) be Hausdorff locally convex topological vector spaces, \( \varphi : X \times Y \to \overline{\mathbb{R}} \) an extended real-valued function, and \( G : X \rightrightarrows Y \) a multifunction. Consider the parametric optimization problem under an inclusion constraint

\[
\min\{ \varphi(x, y) : y \in G(x) \}
\]  

(29)

depending on the parameter \( x \). The function \( \varphi \) (resp., the multifunction \( G \)) is called the objective function (resp., the constraint multifunction) of (29). The optimal value function \( \mu : X \to \overline{\mathbb{R}} \) of (29) is

\[
\mu(x) := \inf \{ \varphi(x, y) : y \in G(x) \}. 
\]  

(30)
The usual convention $\inf \emptyset = +\infty$ forces $\mu(x) = +\infty$ for every $x \notin \text{dom } G$. The solution map $M : \text{dom } G \rightrightarrows Y$ of (29) is defined by

$$M(x) = \{y \in G(x) : \mu(x) = \varphi(x, y)\}. $$

The approximate solution set of (29) is given by

$$M_\eta(\bar{x}) = \{y \in G(\bar{x}) : \varphi(\bar{x}, y) \leq \mu(\bar{x}) + \eta\}, \ \forall \eta > 0. \quad (31)$$

We are now in a position to formulate the first main result of this subsection. For any $\varepsilon \geq 0$ and $\eta \geq 0$, define by $\Gamma(\eta + \varepsilon)$ the set

$$\Gamma(\eta + \varepsilon) = \{(y_1, y_2) : y_1 \geq 0, y_2 \geq 0, y_1 + y_2 = \eta + \varepsilon\}. $$

\textbf{Theorem 4.2} Let $\varphi : X \times Y \rightarrow \overline{\mathbb{R}}$ be a proper convex function, $G : X \rightrightarrows Y$ a convex multifunction. Suppose that the optimal value function $\mu(\cdot)$ in (30) is finite at $\bar{x} \in X$.

If at least one of the following regularity conditions is satisfied:

(a) $\text{int}(\text{gph } G) \cap \text{dom } \varphi \neq \emptyset$,

(b) $\varphi$ is continuous at a point $(x^0, y^0) \in \text{gph } G$,

then, for every $\varepsilon \geq 0$, we have

$$\partial_{\varepsilon} \mu(\bar{x}) = \bigcap_{\eta > 0} \bigcup_{y \in Y \cap M_\eta(\bar{x})} \{x^* : (x^*, 0) \in \partial_{y_1} \varphi(\bar{x}, y) + N_{y_2}(\bar{x}, y); \text{gph } G\}, $$$$ \bigcup_{(y_1, y_2) \in \Gamma(\varepsilon)} \{x^* : (x^*, 0) \in \partial_{y_1} \varphi(\bar{x}, y) + N_{y_2}(\bar{x}, y); \text{gph } G\}, \quad (32)$$

where $M_\eta(\bar{x})$ is given in (31).

\textbf{Proof} (This proof is based on Theorems 3.1 and 4.1.) We apply Theorem 4.1 to the case where $\varphi(x, y)$ plays the role of $(\varphi + \delta(\cdot ; \text{gph } G))(x, y)$. Hence,

$$\partial_{\varepsilon} \mu(\bar{x}) = \bigcap_{\eta > 0} \bigcup_{y \in M_\eta(\bar{x})} \{x^* : (x^*, 0) \in \partial_{y_1} \varphi(\bar{x}, y) + N_{y_2}(\bar{x}, y); \text{gph } G\} \bigcup_{(y_1, y_2) \in \Gamma(\varepsilon)} \{x^* : (x^*, 0) \in \partial_{y_1} \varphi(\bar{x}, y) + N_{y_2}(\bar{x}, y); \text{gph } G\}, $$

$$= \bigcap_{\eta > 0} \bigcup_{y \in Y \cap M_\eta(\bar{x})} \{x^* : (x^*, 0) \in \partial_{y_1} (\varphi + \delta(\cdot ; \text{gph } G))(\bar{x}, y)\}, \quad (33)$$

We will show that

$$\partial_{\varepsilon + \eta} (\varphi + \delta(\cdot ; \text{gph } G))(\bar{x}, y) = \bigcup_{(y_1, y_2) \in \Gamma(\eta + \varepsilon)} \{\partial_{y_1} \varphi(\bar{x}, y) + N_{y_2}(\bar{x}, y); \text{gph } G\}, $$

$$\bigcup_{(y_1, y_2) \in \Gamma(\varepsilon)} \{x^* : (x^*, 0) \in \partial_{y_1} \varphi(\bar{x}, y) + N_{y_2}(\bar{x}, y); \text{gph } G\}, \quad (34)$$
where \( \Gamma(\eta + \varepsilon) = \{(\gamma_1, \gamma_2) : \gamma_1 \geq 0, \gamma_2 \geq 0, \gamma_1 + \gamma_2 = \varepsilon + \eta\} \). Indeed, suppose that at least one of the regularity conditions (a) or (b) is fulfilled. Since \( \text{gph} \, G \) is convex, \( \delta(\cdot; \text{gph} \, G) : X \times Y \to \overline{\mathbb{R}} \) is convex. Obviously, \( \delta(\cdot; \text{gph} \, G) \) is continuous at every point belonging to \( \text{int}(\text{gph} \, G) \). Hence, if the regularity condition (a) is satisfied, then \( \delta(\cdot; \text{gph} \, G) \) is continuous at a point in \( \text{dom} \, \varphi \). Consider the case where the regularity condition (b) is fulfilled. Since \( \text{dom} \, \delta(\cdot; \text{gph} \, G) = \text{gph} \, G \). From (b), it follows that \( \varphi \) is continuous at a point in \( \text{dom} \, \delta(\cdot; \text{gph} \, G) \). So, in the both cases, thanks to Theorem 3.2 and Remark 3.1, the qualification condition

\[
\left( \varphi + \delta(\cdot; \text{gph} \, G) \right)^* (x^*, y^*) = \min \left\{ \varphi^*(x_1^*, y_1^*) + \delta^*((x_2^*, y_2^*); \text{gph} \, G) : (x^*, y^*) = (x_1^*, y_1^*) + (x_2^*, y_2^*) \right\}
\]

(35)

holds for all \( (x^*, y^*) \in X^* \times Y^* \). So, all assumptions of Theorem 3.1 are satisfied. Therefore,

\[
\partial_{\varepsilon + \eta}(\varphi + \delta(\cdot; \text{gph} \, G))(\bar{x}, y) = \bigcup_{(\gamma_1, \gamma_2) \in \Gamma(\eta + \varepsilon)} \left\{ \partial_{\gamma_1} \varphi(\bar{x}, y) + \partial_{\gamma_2} \delta((\bar{x}, y); \text{gph} \, G) \right\},
\]

for any \( (\bar{x}, y) \in \text{dom} \, \varphi \cap \text{gph} \, G \). Moreover,

\[
\partial_{\gamma_2} \delta((\bar{x}, y); \text{gph} \, G) = N_{\gamma_2}((\bar{x}, y); \text{gph} \, G).
\]

Combining (33) with (34), we obtain the statement of the theorem.

The second main result of this section reads as follows.

**Theorem 4.3** Let \( G : X \Rightarrow Y \) be a convex multifunction between Banach spaces, whose graph is closed, and \( \varphi : X \times Y \to \overline{\mathbb{R}} \) a proper closed convex function. Suppose that the optimal value function \( \mu(\cdot) \) in (30) is finite at \( \bar{x} \in X \). Assume that either

(i) \( \mathbb{R}_+(\text{dom} \, \varphi - \text{gph} \, G) \) is a closed subspace of \( X \times Y \),

or

(ii) \( (0, 0) \in \text{int}(\text{dom} \, \varphi - \text{gph} \, G) \),

then (32) is valid.

**Proof** The proof is similar to that of Theorem 4.2. Having in hands the subdifferential representation for the optimal value function in Theorem 4.1, we apply therein the subdifferential sum rule from Theorem 3.1 under the corresponding conditions (i) and (ii). Namely, if the condition (i) (resp. (ii)) is satisfied, using Theorem 3.3 (resp. Theorem 3.4) and remembering Remark 3.2, then we obtain (35). In other words, all assumptions of Theorem 3.1 are satisfied. Thus, by the same manner as in Theorem 4.2, we can obtain the conclusion of the theorem.
4.3 An Application

In this section, we will present an illustrative example for the result in Sect. 4.2. This example is designed for the case graph of the constraint mapping \( G \) is a convex cone.

We have the following property about \( \varepsilon \)-normal directions of a convex cone.

**Proposition 4.2** (See [12, Example 2.1]) Let \( C \) be a convex cone with apex 0. Then one has for all \( \bar{x} \in C \) and all \( \varepsilon \geq 0 \) the equality

\[
N_{\varepsilon}(\bar{x}; C) = \{ x^* \in C^0 : \langle x^*, \bar{x} \rangle \geq -\varepsilon \}.
\]

In particular, \( N_{\varepsilon}(\bar{x}; C) = N(\bar{x}; C) \) for \( \bar{x} = 0 \).

**Proof** For all \( \varepsilon \geq 0 \), take any \( x^* \in N_{\varepsilon}(\bar{x}; C) \). By the definition of \( \varepsilon \)-normal directions, we have

\[
\langle x^*, x - \bar{x} \rangle \leq \varepsilon, \quad \forall x \in C.
\]  

(36)

Substituting \( x = 0 \), we get \( \langle x^*, \bar{x} \rangle \geq -\varepsilon \). Moreover, since \( C \) is a convex cone, \( \bar{x} + ty \in C \), for all \( t > 0, y \in C \). Now taking \( x = \bar{x} + ty \), (36) yields

\[
t\langle x^*, y \rangle \leq \varepsilon, \quad \forall y \in C.
\]  

(37)

Dividing two sides of (37) by \( t > 0 \) and letting \( t \to +\infty \), we obtain \( \langle x^*, y \rangle \leq 0 \), for all \( y \in C \). The latter means that \( x^* \in C^0 \).

Now suppose that \( x^* \in C^0 \) and \( \langle x^*, \bar{x} \rangle \geq -\varepsilon \) for every \( \varepsilon \geq 0 \). Given any \( x \in C \), we have \( \langle x^*, x \rangle \leq 0 \). Combining this with \( \langle x^*, \bar{x} \rangle \geq -\varepsilon \), we obtain \( x^* \in N_{\varepsilon}(\bar{x}; C) \).

We can easily get the following property of the \( \varepsilon \)-subdifferential.

**Proposition 4.3** Let \( f : X \to \overline{\mathbb{R}} \) be a proper convex function. Then, for any \( \varepsilon \geq 0 \) and \( \bar{x} \in \text{dom} \ f \) we have \( \partial_{\varepsilon}(\lambda f)(\bar{x}) = \lambda \partial_{\varepsilon/\lambda} f(\bar{x}) \) for every \( \lambda > 0 \).

Let us consider an illustrative example for Theorem 4.2.

**Example 4.3** Let \( X = Y = \mathbb{R} \) and \( \bar{x} = 0 \). Consider the optimal value function \( \mu(x) \) in (30) with \( \varphi(x, y) = |y| \) and \( G(x) = \{ y : y \geq \frac{1}{2}|x| \} \) for all \( x \in \mathbb{R} \). Then we have \( \mu(x) = \frac{1}{2}|x| \) for all \( x \in \mathbb{R} \). From Example 4.1 and Proposition 4.3, for any \( \varepsilon \geq 0 \) one has \( \partial_{\varepsilon} \mu(\bar{x}) = [0] \times \left[ -\frac{\gamma_1}{2}, 1 \right] \) if \( y < -\frac{\gamma_1}{2} \), \( \partial_{\varepsilon} \mu(\bar{x}) = [0] \times \left[ -\frac{1}{2}, 1 \right] \) if \( -\frac{\gamma_1}{2} \leq y \leq \frac{\gamma_1}{2} \), \( \partial_{\varepsilon} \mu(\bar{x}) = [0] \times \left[ 1 - \frac{\gamma_1}{y}, 1 \right] \) if \( y > \frac{\gamma_1}{2} \).
On the other hand, by Proposition 4.2, we have

\[ N_{\gamma_2}(\bar{x}, y; \text{gph } G) = \begin{cases} \{(0, 0)\} & \text{if } y > 0, \\ \{ (x^*, y^*) \in IR^2 : y^* \leq -2|x^*| \} & \text{if } y = 0, \\ \emptyset & \text{if } y < 0. \end{cases} \]

So, the right-hand side of (32) can be computed as follows

\[
\bigcap_{\eta > 0} \bigcup_{y \geq 0} \bigcup_{(\gamma_1, \gamma_2) \in \Gamma(\eta+\epsilon)} \left\{ x^* \in X^* : (x^*, 0) \in \partial_{\gamma_1} \varphi(\bar{x}, y) + N_{\gamma_2}(\bar{x}, y; \text{gph } G) \right\}
\]

\[
= \bigcap_{\eta > 0} \bigcup_{(\gamma_1, \gamma_2) \in \Gamma(\eta+\epsilon)} \left\{ x^* \in X^* : (x^*, 0) \in \partial_{\gamma_1} \varphi(\bar{x}, \bar{y}) + N_{\gamma_2}(\bar{x}, \bar{y}; \text{gph } G) \right\}
\]

\[
= \left\{ x^* \in IR : (x^*, 0) \in \{0\} \times [-1, 1] + \left\{ (x^*, y^*) \in IR^2 : y^* \leq -2|x^*| \right\} \right\}
\]

\[
= \left\{ x^* \in IR : \{x^*\} \times [-1, 1] \in \left\{ (x^*, y^*) \in IR^2 : y^* \leq -2|x^*| \right\} \right\}
\]

\[
= \left[ \frac{-1}{2}, \frac{1}{2} \right].
\]

This justifies the conclusion of Theorem 4.2.

5 Conclusions

We have obtained exact formulas for the $\epsilon$-subdifferential of the optimal value function of parametric convex programs under inclusion constraints in Hausdorff locally convex topological vector spaces. These formulas work even if the solution set of the problem under consideration is empty.

Relationships between various regularity conditions guaranteeing the validity of the sum rules for $\epsilon$-subdifferentials have been discussed in detail.

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