Self-gravitating warped discs around supermassive black holes

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ABSTRACT
We consider warped equilibrium configurations for stellar and gaseous discs in the Keplerian force field of a supermassive black hole, assuming that the self-gravity of the disc provides the only acting torques. Modelling the disc as a collection of concentric circular rings and computing the torques in the non-linear regime, we show that stable, strongly warped precessing equilibria are possible. These solutions exist for a wide range of disc-to-black-hole mass ratios $M_d/M_{bh}$, can span large warp angles of up to $\pm 120^\circ$, have inner and outer boundaries, and extend over a radial range of a factor of typically two to four. These equilibrium configurations obey a scaling relation such that in good approximation $\phi/\Omega \propto M_d/M_{bh}$ where $\phi$ is the (retrograde) precession frequency and $\Omega$ is a characteristic orbital frequency in the disc. Stability was determined using linear perturbation theory and, in a few cases, confirmed by numerical integration of the equations of motion. Most of the precessing equilibria are found to be stable, but some are unstable. The main result of this study is that highly warped discs near black holes can persist for long times without any persistent forcing other than their self-gravity. The possible relevance of this to galactic nuclei is briefly discussed.

Key words: stellar dynamics – Galaxy: centre – galaxies: active – galaxies: nuclei – galaxies: Seyfert.

1 INTRODUCTION

The increasing power and spatial resolution of modern observations has provided evidence that warps are not unique to galactic discs, but appear also on much smaller scales. These include nuclear and accretion discs surrounding supermassive black holes in galactic nuclei (nuclear discs hereafter). The pioneering example is the maser disc of NGC 4258. The high-velocity masers mapped by Miyoshi et al. (1995) are best explained by the existence of a mildly warped disc extending from 0.13 to 0.26 pc (Herrnstein, Greenhill & Moran 1996). The nearby Seyfert galaxies NGC 1068 and Circinus also harbour warped discs in their centres (Greenhill et al. 2003a; Gallimore, Baum & O’Dea 2004), again traced by the maser emission. Of the ~100 massive young stars in the centre of our Galaxy about half form a well-defined, warped disc, and some of the others are on a counterrotating structure which may be a dissolving disc (Genzel et al. 2003; Paumard et al. 2006; Bartko et al. 2009; Lu et al. 2009).

Nuclear discs can develop a warped shape through several mechanisms. Very close to the centre, the dragging of inertial frames by a rotating black hole (Lense & Thirring 1918) causes precession of a planar disc, if it is inclined to the plane perpendicular to the black hole’s spin. Internal viscous torques try to align the disc angular momentum with the black hole angular momentum. Beyond a transition radius, the disc does not feel the effect of the black hole and remains at its initial inclination, while inside this radius the alignment proceeds. Hence, the disc becomes warped (the Bardeen & Petterson 1975 effect). Natarajan & Armitage (1999) showed that for black holes with masses of the order of $10^6 M_\odot$ and accretion rates close to the Eddington limit the alignment time-scale is short ($t \lesssim 10^6$ yr). Application of this effect to the warped discs of NGC 4258 and NGC 1068 shows that the alignment radius lies well inside the observed positions of the maser spots, and models can be constructed that fit the observed warps rather well (Caproni et al. 2007; Martin 2008).

When a warped disc is exposed to radiation from a central source, or from its own inner portions, it is not illuminated isotropically. If it is also optically thick, the emission received at each position and reradiated perpendicular to the local disc plane induces a torque on the disc, and the warp is modified (Petterson 1977). Perturbations to planar discs can, therefore, cause radiation driven warping (Pringle 1996). Assuming a radiative efficiency $\epsilon \sim 10^{-2}$ and a black hole mass of $10^6 M_\odot$, an initially flat disc is prone to warping beyond $r \geq 0.1$ pc, when the vertical and radial viscosity coefficients are comparable (Pringle 1997). Maloney, Begelman & Pringle (1996) studied the stable and unstable modes of radiatively excited linear warps and found that the warp in NGC 4258 may be explained by this mechanism only if the radiative efficiency is high.

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Warp discs generated by gravitational interactions have been investigated mainly in the galactic context. Hunter & Toomre (1969) studied the linear bending waves of a self-gravitating, isolated, thin disc. They showed that such a disc permits long-lasting bending modes only when its surface density near the outer radius is truncated sufficiently fast, but not when realistic smooth edges are considered. This suggested interactions with nearby companion galaxies as a likely cause of warp excitation. Later, as the evidence for dark matter haloes around galaxies became stronger, modellers developed scenarios in which the disc assumes the shape of a normal mode in the potential of a flattened dark halo (Spárke 1984; Kuijken 1991). However, subsequent work showed that these modes are damped quickly when the internal dynamical response of the halo is taken into account (Nelson & Tremaine 1995; Binney, Jiang & Dutta 1998). Today, it seems most plausible that the galactic warps result from interactions and from accretion of material with misaligned angular momentum (Jiang & Binney 1999).

On nuclear disc scales, Papaloizou, Terquem & Lin (1998) studied in linear theory the evolution of a thin self-gravitating viscous disc interacting with a massive object orbiting the central mass, with application to NGC 4258. They concluded that the warp in the maser disc of NGC 4258 might have been excited by a binary companion with a mass comparable to or higher than that of the maser disc. Their model also suggests a small twist (i.e. varying line-of-nodes) due to viscosity. Nayakshin (2005) considered the case of a non-self-gravitating disc perturbed by a massive ring. Employing the gravitational torques in the linear regime, he evaluated the precession induced by the ring on the disc elements. When the self-gravity of the disc is not taken into account, the rings precess differentially, which tends to destroy the disc structure.

Can models of warped nuclear discs be generalized to the fully non-linear regime? And assuming that the observed warps in galactic nuclei have been excited by one of the mechanisms discussed above, can the disc self-gravity maintain the warp even after the exciting torque has ceased to exist? As a first step towards answering these questions, the goal of the present paper is to investigate the possibility of steadily precessing, stable, non-linearly warped self-gravitating discs in the (Keplerian) gravitational potential of a massive black hole. In the following sections, we use a simple circular orbit ring model to find stable warped equilibria for systems with 2, 3, and many rings, assuming that the self-gravity of the rings provides the only acting torques.

2 STEADILY PRECESSING WARPED DISCS AND THEIR SCALING RELATION

2.1 Cold disc model and equations of motion

We consider a cold disc in which stars or gas are assumed to move on very nearly circular orbits. Following similar analysis of galactic warps (e.g. Toomre 1983; Sparke 1984; Kuijken 1991), we model such a disc as a collection of concentric circular rings. The orbital motion in the disc is maintained by the central black hole, and the self-gravity of the disc causes the rings to precess around the total angular momentum direction. Each ring may represent a set of stars or gas elements uniformly spread around their circular orbit. Moreover, when the precession frequency arising from the self-gravity of the disc is small compared to the orbital frequency of motion, the orbital parameters of single stars change only slowly and so one can average over the orbital motion. In this case, also the force exerted by a single star or mass element on the rest of the disc can be replaced by the force due to a ring of material spread over the orbit (Goldreich 1966).

Any of the rings is characterized by its mass $m_i$, radius $r_i$, inclination angle $\theta_i$ with respect to the reference plane, and azimuthal angle $\phi_i$ where the line-of-nodes cuts this plane. Later, we will identify the reference plane as the plane perpendicular to the total angular momentum vector. The Lagrangian $\mathcal{L}_i$ of ring $i$ is given by (Goldstein, Poole & Safko 2002)

$$\mathcal{L}_i = \frac{m_i r_i^2}{4} \left( \dot{\theta}_i^2 + \dot{\phi}_i^2 \sin^2 \theta_i \right) + \frac{m_i r_i^2}{2} \left( \dot{\psi} + \phi_i \cos \theta_i \right)^2$$

$$- \frac{2 G m_i M_{bh}}{r_i}$$

(1)

The first two terms in equation (1) represent the kinetic energy of the motion $T_i$, $V(r_i, \theta_i, \phi_i)$ represents the gravitational potential energy, the Lagrangian is $\mathcal{L}_i = T_i - V_i$, and the energy of a ring is $E_i = T_i + V_i$. $\psi$ is the position of a point on the ring, measured from the ascending node; $(\theta, \phi, \psi)$ are Euler angles. The angular momentum of the motion along the ring

$$p_{\psi} = m_i r_i^2 \dot{\psi}(r_i) = m_i r_i^2 (\dot{\psi} + \phi_i \cos \theta_i)$$

(2)

is conserved since $\mathcal{L}_i$ does not depend on the coordinate $\psi$. The other momenta are the $p_\theta$, the angular momentum around the z-direction, and $p_\phi$, the angular momentum around the line-of-nodes. The equations of motion are

$$p_\theta = \frac{m_i r_i^2}{2} \dot{\theta}_i$$

(3)

$$p_\phi = \frac{m_i r_i^2}{2} \phi_i \sin^2 \theta_i + p_\psi \cos \theta_i$$

(4)

$$p_\phi = \frac{m_i r_i^2}{2} \phi_i^2 \sin \theta_i \cos \theta_i - \dot{\phi}_i p_\psi \sin \theta_i - \frac{\partial V(r_i, \theta_i, \phi_i)}{\partial \theta}$$

(5a)

$$= - \frac{\partial V(r_i, \theta_i, \phi_i)}{\partial \theta} - 2 \left( \frac{p_\phi - p_\theta \cos \theta_i}{m_i r_i^2 \sin^2 \theta_i} \right)$$

(5b)

$$p_\phi = - \frac{\partial V(r_i, \theta_i, \phi_i)}{\partial \phi}$$

(6)

and the Hamiltonian is

$$\mathcal{H}_i = \frac{p_\psi^2}{m_i r_i^2} + \frac{p_\theta^2}{2 m_i r_i^2} + \frac{(p_\phi - p_\theta \cos \theta_i)^2}{m_i r_i^2 \sin^2 \theta_i} + V(r_i, \theta_i, \phi_i).$$

(7)

2.2 Components of $V(r, \theta, \phi)$, and evaluation of the torques

The gravitational potential energy, $V(r, \theta, \phi)$, has two components. One arises due to the central black hole, and is simply

$$V_{bh} = - \frac{G m_i M_{bh}}{r_i}$$

(8)

at the position of the ring. The other component is the potential term $V_{aw}$ arising from the interaction of the ring under consideration with all other rings. We follow the description of Arnaboldi & Sparke (1994), using the derivation of Binney & Tremaine (1987) (section 2.6.2), to evaluate the torque arising from the ring interactions.

The gravitational potential due to a circular ring of mass $m_i$ and radius $r_i$ in the $(\tilde{x}, \tilde{y})$ plane is

$$\Phi(\tilde{x}, \tilde{y}, \tilde{z}) = - \frac{2 G m_i K(k)}{\pi} \sqrt{1 - k^2/2} \sqrt{r^2 + r_i^2}.$$

(9)
The mutual torque between two neighboring concentric rings, whose radii $r_{\text{out}}$ and $r_{\text{in}}$ are in the ratio $v$ as specified on the plot. The linear regime is limited to the left of the peak amplitude in these curves.

where

$$k^2 = \frac{4Rr_i}{(r_i^2 + r_j^2 + 2Kr_i)}. \quad (10)$$

Here, $K(k)$ is the complete elliptic integral of the first kind and $R$ is the cylindrical radius $R^2 = x^2 + y^2$, so that $R^2 = r^2 - z^2$. A second ring of radius $r_j$ at an angle $\alpha_j$ to the first ring follows a curve $\hat{z} = r_j \sin \alpha_j \sin \psi$, where $\psi$ runs between 0 and $2\pi$. The mutual potential energy is

$$V_i(\alpha_j) = - \frac{GM_i m_j}{\pi^2 (r_i^2 + r_j^2)^{3/2}} \int_0^{2\pi} K(k) \sqrt{1 - k^2/2} d\psi, \quad (11)$$

where $m_j$ is the mass of the second ring and $k$ depends on $r_i/r_j$, $\sin \alpha_j$ and $\psi$. The angle $\alpha_j$ between the two rings is given by

$$\cos \alpha_j = \cos \theta_j \cos \phi_j + \sin \theta_j \sin \phi_j \cos (\phi_j - \phi_i), \quad (12)$$

which reduces to $\cos \alpha_j = \cos (\theta_j - \theta_i)$ when the line-of-nodes are aligned ($\phi = \phi_i$). The torque between the two rings $(i, j)$ is

$$I_{ij} = \frac{4}{\pi^2} \int_0^{\pi/2} \left[ \frac{E(k)(1 - k^2/2)}{(1 - k^2)} - K(k) \right] \times \frac{\sin^2 \psi d\psi}{k^2 \sqrt{1 - \sin^2 \alpha_j \sin^2 \psi}} \quad (13a)$$

We use the numerical program of Arnaboldi & Sparke (1994) for evaluating the integrals in this expression. The torques with respect to the angles $(\theta_i, \phi_i)$ follow from multiplying equation (13a) by $\partial \alpha_i/\partial \theta_i$ or $\partial \alpha_i/\partial \phi_i$. In the following, we will write $V_{m,i} \equiv \sum_{j \neq i} V_{ij}$ for the potential energy of ring $i$ due to the other rings, so that its total potential energy becomes $V(r_i, \theta_i, \phi_i) = V_{bh}(r_i) + V_{m,i}$. For further reference, we also define $M_{ij} \equiv -\partial V_{ij}/\partial \theta_i$ and $M_{G,i} \equiv -\partial V_{m,i}/\partial \theta_i$ for the total gravitational torque on ring $i$ around its line-of-nodes.

Fig. 1 shows the torque between two rings with radii in the ratio $v \equiv r_{\text{out}}/r_{\text{in}}$ as a function of their mutual inclination $\alpha$. The maximum of the torque occurs at very small angles, as noted previously by Kuijken (1991) who gives the approximation $\alpha_{\text{max}} \simeq 1.2|v - 1|$. Only for $\alpha < \alpha_{\text{max}}$ can the mutual torque be approximated as a linear function of $\alpha$. Thus, solutions $\theta(r)$ for the warp shape in linear theory can be scaled by a constant multiplicative factor only so long as the local gradient $d\theta/dr < 1.2/r$. Otherwise, the local self-gravity torques of the disc are no longer able to maintain the linear theory warp shape, the non-linear equations must be used, and the shape of the warp must change.

2.3 Steadily precessing equilibria

A configuration of inclined rings precessing as a rigid body with constant $\phi$ will in the following be denoted as a steadily precessing equilibrium, or equilibrium for short. In earlier works by Hunter & Toomre (1969), Sparke (1984) and Sparke & Casertano (1988), it was found that the eigenfrequencies of linear $m = 1$ warp modes are purely real. Papaloizou et al. (1998) showed that this is a consequence of the self-adjointness of the operator in the tilt equation when there are no viscous or non-conservative forces. Then, the eigenvectors are also real and thus the warp has a straight line-of-nodes and no spirality.\(^1\) In the light of these linear theory results, our effort will also be to find equilibria where all the rings have the same azimuth $\phi$. The condition that all rings maintain constant inclination, $\dot{\theta} = 0$, implies also $p_\theta = 0$, and for a given precession rate $\dot{\phi}$, the simultaneous solution of this equation for each of the rings determines the inclination angles, i.e. the equilibrium shape corresponding to this value of $\dot{\phi}$. We note that for $\dot{\phi} = 0$ equations (5a) admit a trivial tilt solution $\theta_i = \text{const}$, but will assume $\dot{\phi} \neq 0$ in what follows.

From equations (2) and (5a), we can solve for the precession rate of ring $i$:

$$\dot{\phi}_i = \frac{\Omega_i}{\cos \theta_i} \pm \sqrt{\frac{\Omega_{\theta,i}^2}{\cos^2 \theta_i} + 2 \frac{m_i}{r_i^2} \cos \theta_i \sin \theta_i \sum_j \frac{\partial V_{ij}}{\partial \theta_i}}. \quad (14)$$

when $\theta_i \neq 0$. Here, $\Omega_i = \sqrt{GM_{bh}/r_i}$ is the angular velocity of particles on the ring around the black hole, and the term $\sum_j \partial V_{ij}/\partial \theta_i$ is the torque on ring $i$ caused by all other rings $j$. The precession rate can, therefore, be fast or slow, corresponding to the plus and minus signs in this expression. When the interaction potential $V_{ij}$ increases away from the plane $\theta = 0$, the second term in the square root is positive, so that the slow precession is retrograde ($\dot{\phi} < 0$). In the remainder of this paper, we focus on such slow retrograde precession.

The components of angular momentum along the original $(x, y, z)$ axes for a single ring read as

$$I_{x,i} = p_{\phi,i} \cos \phi_i - \frac{\sin \phi_i (p_{\phi_i} - p_{\phi_i} \cos \theta_i)}{\sin \theta_i}, \quad (15)$$

$$I_{y,i} = p_{\phi,i} \sin \phi_i - \frac{\cos \phi_i (p_{\phi_i} - p_{\phi_i} \cos \theta_i)}{\sin \theta_i}, \quad (16)$$

$$I_{z,i} = p_{\theta,i}. \quad (17)$$

Let us assume that we have found a precessing equilibrium from solving equations (3)–(6), with $p_{\phi,i} = 0$, $p_{\theta_i} = 0$ and $p_{\phi_i} = \text{const}$, $\phi = \text{const}$, $\theta_i = \text{const}$. Inserting equations (4) and (5a) into the expression for $I_{x,i}$, simplifying and summing over all rings gives the total angular momentum

$$I_x = \sum_i I_{x,i} = -\frac{\sin \phi}{\dot{\phi}} \sum_{i,j} \frac{\partial}{\partial \theta_i} V_{ij} = 0, \quad (18)$$

which sums to zero because for each pair of rings with interaction potential $V_{ij}$, the torques are equal and opposite. Similarly, the total $I_y = 0$. Thus, the total angular momentum of such a precessing...
equilibrium configuration is parallel to the z-axis. By construction, the angular momentum of the precession alone is also along the z-axis, i.e. the disc precesses around the total angular momentum vector axis.

For a uniformly precessing configuration, additional insight may be obtained by moving to a coordinate system which rotates around the angular momentum axis with the disc’s precession frequency $\phi$ (Kuijken 1991). In this reference frame, the shape of the precessing disc is stationary, but the particles in the different rings still spin about their rings’ symmetry axes. If the particles in ring $i$ rotate with velocity $\Omega(r)\,r_i$ in the positive sense, they experience a Coriolis force in the rotating system which, integrated over the ring, results in a Coriolis torque on ring $i$ along the $p_\theta$-axis (line-of-nodes), given by

$$M_{\phi i} = -m_i r_i^2 \Omega(r_i) \dot{\phi} \sin \theta_i.$$  

(19)

For $0 < \theta < \pi/2$ and negative $\dot{\phi}$ this torque is along the positive $p_\theta$-axis, i.e. is trying to retard the ring relative to the rotating frame. Because the retrograde precession speeds are small, we can neglect the centrifugal force terms. In this case, a stationary precessing configuration is obtained when the forward gravity torques and the retarding Coriolis torque balance in the rotating frame.

2.4 Two- and three-ring cases

The argument just described suggests that there should exist steadily precessing two-ring configurations in which one ring is tilted above the plane $\theta = 0$ and a second ring is tilted below this plane. Both rings are pulled towards $\theta = 0$ by the gravitational force from the other ring. The resulting gravity torques cause the angular momentum vectors of the two rings to precess in the same sense, and are balanced by the Coriolis torques in the precessing frame. To find such configurations, we need to solve $p \equiv 0$ using equation (5a) for both rings simultaneously. Assuming $\dot{\phi} \ll \Omega$, we can neglect terms of order $\dot{\phi}^2$; then using equation (2), the equation for the inner ring at $r_1$ becomes

$$\sin \theta_1 \simeq -\frac{V_{12}}{\partial \theta_1} \frac{m_1 r_1^2 \Omega(r_1)}{1 + v_1^2} \dot{\phi} \sin \theta_1,$$

(20)

and the ratio of the two equations is

$$\sin \theta_1 / \sin \theta_2 \simeq -m_2 r_2^2 \Omega(r_2)/m_1 r_1^2 \Omega(r_1),$$

(21)

where $m_1, m_2$ are the two ring masses, $r_1, r_2$ their radii, $\theta_1, \theta_2$ their inclinations, and $V_{12}$ the interaction potential. Given the ring masses and radii and $\theta_1, \theta_2$, say, we can determine from these equations $\theta_2$, the interaction potential, and thus finally the precession rate $\dot{\phi}$ required for steady precession.

Using the expression in (13a) for the torque between the rings, equation (20) can be cast into a more useful form:

$$\dot{\phi} = \frac{\sin 2\alpha}{\sin \theta_1} \frac{I_{12}(\alpha, v)}{1 + v^2} \frac{m_1}{M_{\text{bh}}} \dot{\phi},$$

(22)

where the angles $\alpha = \theta_1 - \theta_2$, $v = r_2/r_1$, $\mu = m_2/m_1$ and $I_{12}$ denote the integral expression of equation (13b).

Fig. 2 shows the difference $\theta_1 - \theta_2$ between the inclination angles of the two rings versus the square root of the ratio of their radii, $r_1/r_2$, for different precession frequencies, expressed in units of the Keplerian frequency at $r_1$. The combined mass of the two rings is chosen to be 0.5 per cent of the central mass, approximately as inferred for the system of two stellar rings in the Galactic centre (Genzel et al. 2003). $\theta_1 - \theta_2$ increases with decreasing precession speed when the mass ratio is fixed. Equation (22) shows that the same precessing equilibrium configuration can be obtained by changing $\dot{\phi} \propto M_\text{a} = m_1 + m_2$ and leaving all other parameters unchanged. More massive rings must precess faster for the same inclinations. Thus, the sequence of curves in Fig. 2 can also be interpreted as a sequence of fixed precession frequency but with mass ratio $M_\text{a}/M_{\text{bh}}$ increasing from bottom right to upper left.

Next consider three rings. In this case, each of the rings precesses in the potential of the other two rings, and the reference frame is defined by the common plane of precession perpendicular to the total angular momentum vector. Again $p_\theta$ (equation 5a) should be zero at equilibrium for each of the rings. We can sum these three equations as

$$\sum_{i=1}^3 \frac{1}{p_{\theta i}} = \sum_{i=1}^3 \left( \frac{m_i r_i^2}{2} \frac{\dot{\phi}^2}{\sin \theta_i \cos \theta_i - \frac{\dot{\phi}}{p_{\theta i}} \sin \theta_i} \right),$$

(23)

The $V_{ij}$ terms cancel since $\partial V_{ij}/\partial \theta_i = -\partial V_{ij}/\partial \theta_j$. The remaining terms can be rewritten as

$$\dot{\phi} \left( \sum_{i=1}^3 m_i r_i^2 \sin \theta_i \left[ -\Omega(r_i) + \frac{1}{2} \dot{\phi} \cos \theta_i \right] \right) = 0,$$

(24)

making use of equation (2). This shows that, apart from the non-precession solution, a steadily precessing equilibrium is possible only when at least one of the rings lies on the opposite side of the equator with respect to the others, i.e. has $\dot{\theta}_i < 0$. Likewise, the two rings of a precessing two-ring system must lie on opposite sides of the equator. Fig. 3 shows as an example the three-dimensional (3D) view of a three-ring system with mass $M_\text{a} = 0.05M_{\text{bh}}$ and $\dot{\phi} = -0.0021 \Omega(r_2)$.

2.5 Approximation of a disc with n-rings

We now consider a disc represented as a collection of $n$ concentric rings. To find a precessing equilibrium, we solve $p_\theta = 0$ (equation 5a) for all rings simultaneously, summing over the torques from all other rings (equations 13a). These are $n$ equations for $n + 1$ unknowns, the $n$ inclinations $\theta_i$ and $\dot{\phi}$, which we solve keeping $\dot{\phi}$ fixed.
Figure 3. 3D view of a three-ring system. The middle ring lies close to the equator, while the others are distributed almost symmetrically around it.

Figure 4. Inclination of a disc of constant surface density at different radii. The model consists of 35 rings precessing at a rate of \( \dot{\phi} = -10^{-3} \) units, so that \( \dot{\phi}/\Omega_{\text{ref}} = -0.0021 \) on its middle (reference) ring. Each curve corresponds to a different \( M_{i}/M_{\text{bh}} \) mass fraction. The warp becomes more pronounced when the disc mass is increased. The smallest and highest masses correspond to the limits for stability (see Section 2.7).

(Arnaboldi & Sparke 1994).\(^2\) Fig. 4 shows a sequence of equilibria obtained for a constant surface density disc consisting of 35 rings. On each curve, the extent of the disc (i.e. \( \Delta r = r_{\text{out}} - r_{\text{in}} \)) is fixed at 2.2 units, and the precession rate is \( \dot{\phi}/\Omega_{\text{ref}} = -0.0021 \) where \( \Omega_{\text{ref}}(r_{\text{ref}}) \) is the circular frequency on the middle (reference) ring. The disc mass fraction \( M_{i}/M_{\text{bh}} \) varies from 0.16 to 21 per cent. As the mass of the disc increases, the degree of warping increases dramatically so that the Coriolis torques can keep the balance of the gravity torques. The basic shape of the disc is similar to that of the system of three rings in Fig. 3. The middle rings lie closest to the equator, while the inner and outer rings are almost symmetrically distributed around it.

Obviously, the larger the number of rings the better the approximation to a continuous disc. Fig. 5 shows the convergence of the total torques (upper panel), and of the inclination angles obtained for a constant surface density disc (Fig. 4). Torques converge. The inclination of the outer and inner rings have fixed. One sees that quite a number of rings are needed before the disc is increased but the extent and the mass of the disc are kept fixed.

2.6 Scaling the solutions

Now we go back to the equilibria themselves, in particular to the question of their scaling properties. When the torque on ring \( i \) from all other rings is decomposed as

\[
M_{\text{G},i} = -\sum_{j} \frac{\partial V_{i}}{\partial \theta_{i}} = -\sum_{j} \frac{\partial V_{j}}{\partial \alpha_{ij}} \frac{\partial \alpha_{ij}}{\partial \theta_{i}}. \tag{25}
\]

the mass- and radius-dependent part is the first derivative on the right-hand side (rhs). The second factor in each term of this sum depends from equation (12) only on the two sets of angles \( \theta_{i}, \phi_{i}, \theta_{j}, \phi_{j} \). For equilibria with a common precessing line-of-nodes, \( \alpha_{ij} = (\theta_{j} - \theta_{i}) \), so the derivative is always unity. For the potential derivative terms, we use equation (13a) and express all ring masses and radii in terms of the mass and radius of a reference ring, i.e. we write \( \mu_{i} \equiv m_{i}/m_{\text{ref}}, \nu_{i} \equiv r_{i}/r_{\text{ref}} \) and similar for \( j \). Then, equation (25) takes the form

\[
M_{\text{G},i} = -\frac{Gm_{\text{ref}}^{2}}{r_{\text{ref}}^{2}} \sum_{j} \frac{\mu_{i} \mu_{j} \nu_{i} \nu_{j} \sin \alpha_{ij}}{(\nu_{i}^{2} + \nu_{j}^{2})^{3/2}} I_{ij}(\alpha_{ij}, \nu_{i}, \nu_{j}) \tag{26}
\]

where we denote the expression over the brace as \( I_{ij} \). For equilibria with a common precessing disc mass configuration (i.e. fixed ring masses, radii and inclinations, hence fixed \( D_{i}/2A_{i} \)), the precession rate \( \dot{\phi}_{i} \) scales proportional to the Keplerian frequency at some reference radius in the disc and proportional to the disc-to-black-hole mass ratio. Vice versa, equation (28) can be interpreted as a...
scaling relation which says that a precessing equilibrium solution remains unchanged in shape ($\theta_i$) under changes of the disc mass, disc radius, black hole mass and precession rate, provided the ratio ($\phi_i/\Omega_{ref})/(M_{bh}/M_{bh})$ is held constant.

Fig. 6 depicts the values of $-D_i/2A_i$ for a system of five rings. The radii of the rings are calculated such that $r_i = k^{i-1} \times r_1$, with $i = 1, 2, \ldots, n$, $k = 1.07$, $r_1 = 5.75$ and $n = 5$, so if the third ring is the reference ring, $v_i = k^{i-3}$. The ring masses are assumed to have the same value, 0.5516, so $\mu_1 = 1$, the black hole has a mass of 51.16, and $M_{bh}/M_{bh} = 0.054$. On each curve in Fig. 6, the precession rate $\dot{\phi}$ increases with steps of $-5 \times 10^{-3}(\Delta \phi_i/\Omega(r_1) = -1.18 \times 10^{-4})$, starting from a value of $-1 \times 10^{-4}$ (i.e. $\phi_i/\Omega(r_1) = -2.4 \times 10^{-4}$) at the top. We checked the accuracy of the scaling and of our calculations by computing $-D_i/2A_i$ values for different parameter pairs of the system that should give the same $-D_i/2A_i$ according to equation (28). We overlay the results for the first ring and precession speed $\dot{\phi}/\Omega(r_1) = -0.0021$, in the lower panel of Fig. 6, zooming into the parameter region $-2.3 < D_i/2A_i < -1.9$ where the different curves deviate from each other the most. In the worst case, due to the scaling of the ring radii, the deviations of the $\theta_i$ from their values for the original five ring system are still less than $1^\circ$. Changes in radii cause the largest deviations from the scaling relation because of the way in which they enter in the quantity $D_i$ (equation 27). The scaling results for the other rings are similar.

2.7 Stability

In this section, we investigate the stability of the precessing equilibrium solutions found above. Hunter & Toomre (1969) proved that isolated thin self-gravitating discs are stable to all $m = 1$ warp perturbations and this carries over to discs embedded in spherical or oblate potentials (e.g. Sparke & Casertano 1988). We show here that the non-linearly warped precessing discs can be both stable and unstable to general ring-like perturbations. We describe small perturbations of the precessing disc solutions by the linearized equations of motion around equilibrium:

$$\Delta \dot{\theta}_i = \frac{\partial^2 T_i}{\partial p_{\theta_i}^2} \Delta p_{\theta_i},$$

$$\Delta \dot{\phi}_i = \frac{\partial^2 T_i}{\partial p_{\theta_i} \partial \theta_i} \Delta \theta_i + \frac{\partial^2 T_i}{\partial \theta_i^2} \Delta \theta_i,$$

$$\Delta p_{\theta_i} = -\frac{\partial^2 V_{m,i}}{\partial \theta_i^2} \Delta \theta_i - \frac{\partial^2 V_{m,i}}{\partial \phi_i \partial \theta_i} \Delta \phi_i - \frac{\partial^2 V_{m,i}}{\partial \phi_i^2} \Delta \phi_i,$$

$$\Delta p_{\phi_i} = -\frac{\partial^2 V_{m,i}}{\partial \phi_i \partial \theta_i} \Delta \theta_i - \frac{\partial^2 V_{m,i}}{\partial \phi_i^2} \Delta \phi_i - \sum_j \frac{\partial^2 V_{m,i}}{\partial \phi_i \partial \phi_j} \Delta \theta_j - \sum_j \frac{\partial^2 V_{m,i}}{\partial \phi_i \partial \phi_j} \Delta \phi_j.$$  

(29)

Here, $T_i$ and $V_{m,i}$ are the kinetic and potential energy terms in the Hamiltonian (7) for ring $i$, respectively, and the partial derivatives are evaluated at the equilibrium solution ($\theta_i, \phi_i = \text{const.}$, $p_{\theta_i} = 0, p_{\phi_i} = 0$). These linear equations have solutions of the form $e^{\lambda \Delta t} \Delta \theta_i, \ldots, \text{etc.}$, where $\lambda = \lambda_{\text{exc}} \pm i\lambda_{\text{rot}}$ with its real and imaginary parts. The (constant) coefficients of the $\Delta t$ terms in equation (29) form a matrix $H$ which carries the information on stability. When the real parts of the eigenvalues of the matrix $H$, $\lambda_{\text{exc}} = 0$, the imaginary parts of the eigenvalues, $\lambda_{\text{rot}}$, constitute a rotation matrix through which the solutions oscillate around the precessing equilibrium with frequencies $\lambda_{\text{rot}}$, and the equilibrium is said to be stable. When $\lambda_{\text{exc}} < 0$, the solutions spiral towards the unperturbed equilibrium positions, leading to asymptotic stability of the equilibrium. If, however, any of the eigenvalues have a non-zero real part, $\lambda_{\text{exc}} > 0$, the system moves away from equilibrium exponentially, and is unstable.

For determining the stability of any of our precessing $n$-ring solutions, we compute the $4n \times 4n$ stability matrix $H$, using the equilibrium ($\theta_i, \phi_i = \text{const.}$, $p_{\theta_i} = 0, p_{\phi_i} = 0$). We then evaluate the eigenvalues of the matrix $H$, using routine F02EBF of the Numerical Algorithms Group (NAG). This routine is suitable for computing eigenvalues and optionally eigenvectors of real matrices.

First, we briefly discuss one example for the convergence of the linear stability results. In Section 2.5, we had already discussed the convergence of the gravitational torques, and of the inclination angles obtained for the precessing equilibrium solutions, as a function of the number of rings used to represent the disc (see Fig. 5). Fig. 7 shows how the shape of a solution near the lower stable boundary of Fig. 4 and its stability changes with the number of rings used to represent the disc. The transition from unstable to stable occurs at a disc mass fraction of $M_{bh}/M_{bh} = 0.0018, 0.0013, 0.0011, 0.0011$ for $n = 15, 25, 45, 75$ rings. This shows that the transition has approximately converged when $n \approx 45$.

Next, we consider the issue of scaling. We have already seen that the equilibrium solutions can be scaled in radius, mass and...
3 STEADILY PRECESSING, NON-LINEARLY WARPED KEPLERIAN DISCS: RESULTS

3.1 Warp shapes and warp angles of stable precessing discs

We have already seen in the discussion of three-ring and $n$-ring systems in Section 2 that self-gravitating precessing discs in a Keplerian potential can be strongly warped. In fact, some of the discs shown in Fig. 4 are so strongly warped that they would obscure the central black hole from most lines of sight.

In this section, we discuss these results in more detail. Fig. 10 shows the warped stable equilibrium solutions obtained for a sequence of discs with varying precession frequency. These discs have constant surface density between fixed inner and outer radii, and a total disc-to-black-hole mass ratio of $M_d = 0.0016M_{bh}$. The solutions shown in Fig. 10 are all linearly stable, according to the analysis described in Section 2.7. Outside the range of models bounded by the upper and lower curves one can find further equilibria, but these are unstable.

By construction, these discs have a fixed line-of-nodes at all radii, and their shapes are given in terms of the inclination angle $\theta$ relative to the plane defined by the total angular momentum vector. In all cases, there is a middle section of the disc which lies approximately in this plane, whereas the inner and outer parts warp away from this plane in opposite directions. For the most strongly warped stable solution in Fig. 10, the inner warp is by $\sim 180^\circ$ and the outer warp by $\sim 120^\circ$. This is obtained for the lowest stable pattern speed, in accordance with the balance between gravitational and Coriolis torques (see Fig. 1 and equation 19): the torques are weakest for the
Figure 10. Inclination of a disc of 35 rings at different radii for $M_d/M_{bh} = 0.0016$. On each curve the precession frequency $\dot{\phi}$ has a different value, given on the figure in terms of the orbital frequency at the position of the middle ring. The upper and lower curves show the boundaries of stable solutions. See Section 2.7 and Fig. 9.

Figure 11. Variation with disc-to-black-hole mass ratio of the inclinations of the outermost ($\theta > 0$) and innermost ($\theta < 0$) rings of a 35-ring warped disc with $r_{out}/r_{in} = 1.44$, for two different values of the precession frequency. The figure shows how the warping increases with increasing disc mass fraction and with decreasing precession speed, and also illustrates the scaling relation of equation (28). Up to numerical errors, the minimum and maximum values of $\theta$ are the same on all two curves.

Figure 12. 3D views of several 15-ring discs similar to those shown in Fig. 4, with mass ratios given on the plots. The degree of warping increases with the disc mass fraction until (for disc mass fraction greater than 2 per cent) the central black hole is completely hidden behind the warped disc.

large inclinations. This can be seen already in the two-ring problem (see equations 20 and 22). The least strongly warped disc solution in this example has inner and outer warps $\sim 25$–30$^\circ$.

The variation of the maximum inner and outer warp angles with disc mass fraction is shown in Fig. 11 for fixed precession frequency and radial extent of the disc. The curves with $\theta > 0$ represent the outermost ring inclinations, and those with $\theta < 0$ show the innermost ring inclinations for different precession speeds. As we have already seen in Figs 4 and 10, these inclinations increase with increasing disc mass fraction and with decreasing precession speed. Fig. 11 also illustrates the scaling relation of equation (28). Up to numerical errors, the different curves can be scaled on top of each other.

Some 3D illustrations of warped discs from this family are shown in Fig. 12. From top to bottom, these plots shows warped discs with increasing amplitude of the warp, such that the disc in the bottom panel of Fig. 12 completely encloses the central black hole.

3.2 Comparison with linear theory solutions

Previous work on warped and twisted discs around black holes has often made use of the linear approximation, in which the inclination angles of all parts of the warped disc are assumed to be small. It is, therefore, interesting to briefly consider the linear limit of our analysis above.

For small warping angles $\alpha_{ij}$, the self-gravity torques simplify considerably. Because the leading term in equation (13a) is already $O(\alpha_{ij})$, only the $O(1)$ part of the $I_{ij}$ terms in this equation need to be included, while the next order, $O(\alpha_{ij}^2)$, can be neglected. Thus, in computing $I_{ij}$, the $k^2$ term for ring $j$ in equation (10) can be
approximated as
\[ k_{ij}^2 \approx \frac{4r_ir_j}{(r_i + r_j)^2} \] (30)
and the \( I_{ij} \) term can be integrated to give
\[ I_{ij} \approx \frac{1}{\pi} \left[ \frac{E(k_{ij})(1 - k_{ij}^2/2)}{1 - k_{ij}^2} - K(k_{ij}) \right] \frac{(1 - k_{ij}^2/2)^{3/2}}{k_{ij}^2}. \] (31)

The mutual torque on ring \( i \) from ring \( j \) becomes, to first order in \( \alpha_{ij} \),
\[ \frac{\partial V_{ij}}{\partial \theta_i} = \frac{\partial V_{ij}}{\partial \alpha_{ij}} \theta_i \theta_j \approx 2Gm_i m_j r_ir_j \alpha_{ij} \frac{\partial \alpha_{ij}}{\partial \theta_i} \] (32)
For a precessing equilibrium when \( \psi_i = \phi_j, \alpha_{ij} = \theta_i - \theta_j \), the equation \( p_{\psi} = 0 \) becomes \( O(\theta) \):
\[ p_{\psi} = \phi \frac{m_ir_i^2}{2} \theta_i - \phi p_{\psi} \theta_i - \sum_{j=1}^{n} 2Gm_i m_j r_ir_j I_i(\theta_i - \theta_j) = 0. \] (33)

Equation (33) is a quadratic eigenvalue problem for the precession frequencies. When linearized, it transforms into a generalized eigenvalue problem of dimensions \( 2n \times 2n \). We use the NAG routine F02BJF to find the eigenvalues and eigenvectors of equation (33).

The \( 2n \) eigenvalues constitute two distinct families in the frequency spectrum of the disc; the fast prograde and the slow retrograde frequencies. When sorted in decreasing order, the first retrograde frequency has a value of zero, and the associated eigenvector represents a tilt of the whole disc by the same angle, i.e. \( \theta_i = \) constant. The next eigenvalue corresponds to warps we have discussed so far where the disc has one radial node (modified tilt mode, Hunter & Toomre 1969; Sparke 1984; Sparke & Casertano 1988). In the following, we restrict our discussion to linear warp shapes of this kind.

In Fig. 13, we show the modified tilt mode in linear theory of a disc with \( M_d/M_{bh} = 0.005 \), \( r_i = 5 \) and \( r_{out} = 7.2 \) for 40 rings. This is obtained by solving equation (33) and is shown with the dashed line. We note that in linear theory the warp shape can be arbitrarily scaled as long as the local gradient of the tilt satisfies \( d\theta/dr < 2/3 \psi \) (see Section 2.2; if this condition is violated, the linear approximation to the self-gravity torques breaks down. Therefore, in Fig. 13, the linear mode is scaled to its maximum possible amplitude such that the condition is everywhere satisfied.

As mentioned above, the precession frequency of this mode is the first non-trivial eigenvalue in the retrograde family, and here it has a value of \( \phi/\Omega_{ref} = -0.0103 \) when normalized to the rotation frequency of the reference ring. For this frequency, we then solve equation (5a) to obtain the non-linear warp shape shown by the solid line in Fig. 13. The larger curvature of the non-linear warp near the inner and outer boundaries of the disc, with respect to the scaled linear mode, shows that the linear approximation overestimates the torques in these parts of the disc.

However, the main difference between linear modes and non-linear warps is that, for a given mass distribution of the disc (surface density profile, inner and outer boundaries), the precession frequency and shape of the modified tilt mode in linear theory is uniquely determined, whereas non-linear equilibrium warp solutions may exist for a range of precession frequencies and warp shapes or, e.g. for extended discs, may not exist at all. For the case shown in Fig. 13, non-linear warped equilibria are found for precession frequencies in the range \( \phi/\Omega_{ref} = -8.87 \times 10^{-5} \rightarrow -1.16 \times 10^{-2} \) and are stable in the range \( \phi/\Omega_{ref} = -9.24 \times 10^{-5} \rightarrow -9.63 \times 10^{-3} \). The particular non-linear warp shape obtained for the frequency of the linear mode and shown in Fig. 13 is unstable.

Alternatively, the warp shape may be parametrized by the inclination of the outermost ring, say, \( \theta_n \). Linear theory warps can in the previous example be considered valid up to \( \theta_n \sim 10^\circ \), and have all the same precession frequency. Non-linear warp modes are found in the range \( \theta_n = 19.1^\circ \rightarrow 130^\circ \), and are stable in the range \( \theta_n = 20.9^\circ \rightarrow 129^\circ \). They are disjunct from \( \theta_n \) from the linear modes, and their precession speeds decrease with \( \theta_n \) according to the balance of gravitational and Coriolis torques.

### 3.3 Dependence on surface density profile and radial extent

The warped discs presented in Figs. 4, 11 have constant surface density. For comparison, Fig. 14 shows the warping of an exponential disc, with surface density \( \Sigma(r) = \Sigma_0 \exp(-r/r_d) \) where \( \Sigma_0 \) denotes the central density and \( r_d \) is the scalelength, chosen to be 2.5 units in this example. The other parameters (relative ring radii, precession speed) are identical to those used in Fig. 4. The basic warp shapes are similar as for constant surface density, but the maximum outer warp angles are slightly larger. The range of stable disc masses is
also comparable to that for the constant surface density disc (for the same precession speed); see the curves showing the boundaries of stability in Figs 4 and 14.

Because the condition for a warped equilibrium is that the Coriolis and gravitational torques balance, clearly not only the mass fraction and mass distribution but also the radial extent of the disc must be important for determining the warp shape and its stability. To investigate this, we compiled a set of precessing equilibria with varying radius scaling factor $\kappa$ as follows (see also Section 2.5). After fixing the radius of the middle ring of the disc, $r_{mid}$, we determine the remaining ring radii such that

\[
r_j = r_{mid} \kappa^j, \quad \{ j = i - \frac{n+1}{2}, \quad n \text{ odd} \}
\]

\[
r_j = r_{mid} \kappa^j, \quad \{ j = i - \frac{n}{2}, \quad n \text{ even} \}
\]

\[
n = 1, 2, \ldots, n,
\]

\[
(34)
\]

where $n$ is the number of rings. For illustration, we consider a family of disc models with the same disc-to-black-hole mass fraction $M_d = 0.01 M_{bh}$, each with its own constant surface density given by $M_d$ and $\kappa$. All discs are made of $n = 35$ rings, and the middle ring radius is set to $r_{mid} = 6$ units.

Fig. 15 shows precessing equilibria for such discs for different $\kappa$. The upper and lower curves show the two disc shapes that bound the stable range of solutions in terms of the $\kappa$-factor. In the case where the rings have minimum possible separation from each other, the inner ring has a radius of $r_1 = 5.7$ units, and the outer ring has $r_n = 6.3$ units. On the other hand, for the most extended stable disc in this family, $r_1 = 3.9$ and $r_n = 9.1$. When the extent of the disc is increased, a slight decrease in the warping is observed in Fig. 15. This is due to the fact that the torque from a ring of constant mass decreases with distance to the ring, cf. equation (13a).

Fig. 16 shows the radial extent of the disc $r_{out}/r_{in}$ for which stable warped equilibria can be found, for different surface density profiles and as a function of disc-to-black-hole mass ratio. The most important result of these calculations is that stable non-linear warped discs can be maintained only for discs with inner and outer boundaries, for which $r_{out}/r_{in} \simeq 2-4$. This is reminiscent of the result of Hunter & Toomre (1969) that in linear theory only truncated discs permit long-lasting bending modes.

\section{3.4 Time-evolution of ring systems}

In this section, we consider the explicit time-evolution of a precessing system of self-gravitating rings in a massive black hole potential. By integrating the equations of motion, equations (3)–(6), starting from initial conditions corresponding to one of the precessing disc solutions found earlier, we can check the stability of this solution directly and compare with the linear stability analysis.

In these integrations, we use discs of 20 equal mass rings, equally spaced in radius. The ratio of the outermost ring radius to that of the innermost ring is 1.44. The initial $\theta$ are obtained from precessing equilibrium solutions; all rings have the same line-of-nodes, i.e. the same initial $\phi_i$. The equilibrium precession speed is given by $\dot{\phi}/\Omega_{ref} = -0.0021$.

In the following figures, symbols starting from the outer circle show the variation of inclination $\theta$ with ring radius, where the ring radii are shown as distances from the centre of the plot, with scale shown on the lower right. The symbols starting from the inner circle show how the azimuthal angle $\phi$ changes with the ring radius; for this part of the plot, the ring radii are scaled down to the half of their values to make the figure more easily readable. The elapsed time of the integration is shown on top of the figures, in terms of the number of orbital periods at the position of the outermost ring $n$ where $\dot{\phi}/\Omega_{ref} = -0.0027$.

Fig. 17 shows the time evolution of a disc of 20 rings with $M_d = 0.05 M_{bh}$. The disc stays in equilibrium for 12 orbital periods, consistent with its linear stability. Fig. 18 shows the evolution of a disc of 20 rings with $M_d = 0.1 M_{bh}$. This disc precesses as a unit for eight orbital periods, but then it starts to break into parts, hence the disc is unstable, as also predicted by linear stability analysis.

To strengthen the agreement between linear stability and time-evolution results, we integrate two of these ring systems for longer. Figs 19 and 20 show the time evolution of two discs with masses $M_d = 0.005 M_{bh}$ and $0.02 M_{bh}$, with 15 logarithmically spaced rings according to equation (34), over 50 orbital periods. In both cases, the discs are stable, as expected from the linear stability analysis.

\section{4 DISCUSSION}

\subsection{4.1 Theoretical issues}

In this work, we have considered warped discs around black holes for which the only acting force is gravity and the disc is approximated as a nested sequence of circular rings. We have focused on non-linearly warped, steadily precessing disc configurations,
contrary to most previous work in which small amplitude warps were considered, often of a transient nature. We have found that stable, steadily precessing, highly warped discs can be constructed, albeit only over a limited radial range, such that the typical ratio of the outer to the inner boundary radius is $\sim 2-4$.

In one illustrative case, we have compared with a linear theory warped disc. For a given disc mass configuration, the precession frequency of the linear, modified tilt mode is given as an eigenvalue, and the shape can be scaled up to the amplitude where the validity of the linear approximation to the gravitational torques breaks down. The corresponding non-linear warp with the same precession frequency is unstable. Stable non-linear warps for the same mass configuration exist for a disjunct range of precession speeds which are all slower than that of the linear mode. Their warp angles increase with decreasing precession speed, and the non-linear solutions are more strongly warped than the linear mode at the maximum scaling.

These warped discs obey a scaling relation in the sense that (i) they can be scaled to an arbitrary radius $r$, provided the precession speed is scaled to the circular frequency $\Omega(r)$ and (ii) they can be scaled in mass, provided the ratio of precession frequency to $\Omega(r)$ keeps in line with the ratio of disc mass to black hole mass.

In constructing these solutions, we have neglected the background potential generated by the surrounding nuclear star cluster, whose quadruple moment will often be important on scales of $\sim 0.1$ pc. Fig. 21 shows steadily precessing warped disc inclinations for one case including the background potential. For the parameters chosen, the solutions are qualitatively similar to those discussed earlier.
Stability was tested with respect to perturbations of the ring parameters, that is, the orbits of gas and stars were assumed to remain circular. We did not investigate instabilities by which the disc would become eccentric or lop-sided. Answering the question whether such instabilities are relevant for the warped discs considered here requires different techniques and must remain for future work (see e.g. Touma 2002).

Neglecting gas pressure and viscosity for our warped disc solutions is justified if the discs are cold and the viscous time-scale is much longer than the precession time-scale. Pringle (1992) has devised a system of equations for the evolution of the surface density and local angular momentum vector of a non-linearly warped, viscous disc. A logical next step is to add the gravitational torques to these equations and study the evolution of viscous, self-gravitating, non-linearly warped discs; this work is in progress.

4.2 Origin of warped discs

An important question is whether, and if so how, the warped nuclear discs we have considered can be set up in nature. Infall of gas clouds on inclined orbits has been discussed in the context of observations of the Galactic centre (see next section) as a possible model for generating a warped disc in the central parsec (Hobbs & Nayakshin 2009). If the potential of the nuclear star cluster is important, accretion of gas on to a plane inclined relative to its principal plane may lead to a warped disc. The combined quadrupole moment of the gas disc itself and of the background cluster potential would cause the orbits to precess and the disc to become warped. In both cases, the accreting gaseous material with misaligned angular momenta will not directly end up in a warped disc with the right density structure for steady state precession. However, the disc may settle into a warp mode if the energy associated with the transient response can be transported outwards by bending waves (Toomre 1983; Hofner & Sparke 1994), or in the case of gaseous disc, if it can be dissipated (see discussion in Papaloizou et al. 1998); this remains to be investigated.

Caproni et al. (2006) discuss four warping mechanisms for extragalactic accretion discs: tidal, radiative, magnetic and Bardeen–Petterson. If a planar disc has become warped by the radiation pressure instability discussed by Petterson (1977) and Pringle (1996) or through magnetic instabilities (Lai 2003), the gravitational torques might start to dominate once the source of the initial warping disappears. Highly warped discs have been reported before by Pringle (1997) in the context of the radiation pressure instability. We have done some simple time evolution calculations to show that initially highly warped discs often do not dissolve through self-gravity precession; the torques then cause wobbling but not break-up of the disc. The role of self-gravity in such models would be to ensure the long-term persistence of the warp. Future work along the lines discussed at the end of the last subsection may be able to clarify whether this is feasible.

4.3 Warped discs in galactic nuclei

Warped discs around central black holes have been inferred through observations of water maser emission in several nearby active galaxies such as NGC 4258, NGC 1068 and the Circinus galaxy. The maser discs in these galaxies extend radially between 0.16–0.28 pc (Herrnstein et al. 1999), 0.65–0.1 pc (Greenhill & Gwinn 1997), and 0.11–0.4 pc (Greenhill et al. 2003b), respectively. The most widely studied of these maser discs is in NGC 4258, where from the near-Keplerian rotation curve of the high-velocity masers, the black hole mass is deduced to be $3.8 \times 10^6 M_\odot$, and the dynamical upper limit to the mass of the disc is $<10^6 M_\odot$ (Herrnstein et al. 2005). Stationary, power-law accretion disc models constrained by theory and observations have mass fractions $10^{-4} - 10^{-3}$ of the central black hole, in which case the gravitational and viscous torques are comparable (Caproni et al. 2007; Martin 2008). Several explanations have been suggested for the observed warp in the disc (Caproni et al. 2006). In one model, the warp is caused by a binary companion orbiting outside the disc (Papaloizou et al. 1998); this would need a mass comparable to that of the disc. A second possibility is radiation pressure from the central source (Maloney et al. 1996; Pringle 1996), but Caproni et al. (2006) analysing several AGN discs find that these are stable against radiation warping. The most favoured explanation for the warp is the Bardeen–Petterson effect (Caproni et al. 2007; Martin 2008), but to reach a steady state the disc must be very long-lived. Gravitational torques have so far been mostly neglected; our results suggest that it may be worthwhile to consider models including both the gravity from the disc and possibly the quadrupole moment of the stellar cusp.

In the Galactic centre, near-infrared observations have identified one or possibly two discs of young stars at a distance of $\sim 0.04$ to 0.4 pc from the near black hole SgrA* (Genzel et al. 2003; Paumard et al. 2006; Bartko et al. 2009; Lu et al. 2009). These stellar discs are highly inclined both with respect to the Galactic plane, and with respect to each other. The total mass in the discs, as inferred from stellar number counts, is around $10^6 M_\odot$ (Paumard et al. 2006). This is a non-negligible fraction of the mass of SgrA*, $M_{SgrA}\sim 4 \times 10^6 M_\odot$ (Genzel et al. 2000; Ghez et al. 2005). The recent analysis of Bartko et al. (2009) shows that the clockwise rotating disc is warped, with angular momentum direction slewing over $\sim 60$' from the inner to the outer stars. We consider the precession of the warped disc in the Galactic centre elsewhere.

Warped discs could also have important implications for the unification of AGN (Phinney 1989). The unification theories rely on the obscuration along some lines of sight of the radiation from the central source by intervening matter. While this obscuring matter is usually depicted as a doughnut-like torus, an alternative possibility is that it could have the shape of a flared or warped disc. The highly warped solutions discussed above in principle provide the geometry to obscure the central engine from most lines of sight. The
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observing medium required for these unification scenarios must be clumpy (Nenkova, Ivezic & Elitzur 2002), perhaps suggesting fragmentation of the disc (Goodman 2003). Nayakshin (2005) studied the evolution of a highly inclined warped disc, where he showed that the disc indeed can conceal the central object for most of its lifetime. In the non-linear regime, warped discs can obscure a significant part of the solid angle of the source (see Fig. 12 in Section 3 above). Recently, Wu, Wang & Dong (2008) showed that because the outer parts of a warped disc receive a larger fraction of the central emission, the line ratios of the reprocessed Balmer emission lines can be successfully predicted by a warped disc model.

5 SUMMARY AND CONCLUSIONS

In this paper, we have investigated non-linearly warped disc solutions around black holes for which the only acting force is gravity. We used a simple model in which the disc is approximated as a nested sequence of circular rings. We have shown that with these approximations stable, steadily precessing, highly warped discs can be constructed.

These discs have a common line-of-nodes for all rings. In all cases, there is a middle section of the disc which lies approximately in this plane, whereas the inner and outer parts warp away from this plane in opposite directions. The warp angles of these solutions can be very large, up to ±120°, but they extend only over a limited radial range, such that the typical ratio of the outer to the inner boundary radius is ~2–4. Such precessing equilibria exist for a wide range of disc-to-black-hole mass ratios $M_d/M_{bh}$, including quite massive discs.

The stability of these precessing discs was determined using linear perturbation theory and, in a few cases, confirmed by numerical integration of the equations of motion. We found that over most of the parameter range investigated, the precessing equilibria are stable, but some are unstable.

These discs obey a scaling relation: they can be scaled to arbitrary radii $r$, provided the precession speed is scaled to the circular frequency $\Omega(r)$ and they can be scaled in mass, provided the ratio of precession frequency to $\Omega(r)$ is changed, in good approximation, proportionally to the ratio of disc mass to black hole mass.

The main result of this study is that persistent forcing of the disc other than by its own self-gravity is not necessarily required for maintaining a non-linearly warped disc in a Keplerian potential. Further work combining self-gravity with gas physics, etc. will show whether these self-gravitating warped disc solutions help to understand the observed warped discs in galactic nuclei.

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