CONTRACTIBLE OPEN MANIFOLDS WHICH EMBED IN NO
COMPACT, LOCALLY CONNECTED AND LOCALLY
1-CONNECTED METRIC SPACE

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Abstract. This paper pays a visit to a famous contractible open 3-manifold $W^3$
proposed by R. H. Bing in 1950’s. By the finiteness theorem [Hak68], Haken proved
that $W^3$ can embed in no compact 3-manifold. However, until now, the question
about whether $W^3$ can embed in a more general compact space such as a compact,
locally connected and locally 1-connected metric 3-space was unknown. Using the
techniques developed in Sternfeld’s 1977 PhD thesis [Ste77], we answer the above
question in negative. Furthermore, it is shown that $W^3$ can be utilized to pro-
duce counterexamples for every contractible open $n$-manifold ($n \geq 4$) embeds in a
compact, locally connected and locally 1-connected metric $n$-space.

1. Introduction

Counterexamples for every open 3-manifold embeds in a compact 3-manifold have
been discovered for over 60 years. Indeed, there are plenty of such examples even for
open manifolds which are algebraically very simple (e.g., contractible). A rudimen-
tary version of such examples can be traced back to [Whi35] (the first stage of the
construction is depicted in Figure 9) where Whitehead surprisingly found the first ex-
ample of a contractible open 3-manifold different from $\mathbb{R}^3$. However, the Whitehead
manifold does embed in $S^3$. In 1962, Kister and McMillan noticed the first counterex-
ample in [KM62] where they proved that an example proposed by Bing (see Figure
1) doesn’t embed in $S^3$ although every compact subset of it does. In the meantime,
they conjectured that Bing’s example is a desired counterexample, i.e., such example
embeds in no compact 3-manifold. This conjecture was confirmed later by Haken
using his famous finiteness theorem [Hak68] stating that there is an upper bound on
the number of incompressible nonparallel surfaces in a compact 3-manifold. Similar
examples can readily derive from Haken’s finiteness theorem (or see [MW79, Thm.
2.3]). In 1977, an interesting example (see Figure 10) was given in Sternfeld’s PhD
dissertation [Ste77]. Instead of using Haken’s finiteness theorem, Sternfeld applied

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covering space theory to produce a contractible open $n$-manifold ($n \geq 3$) that embeds in no compact $n$-manifold\footnote{It doesn’t appear that Haken’s finiteness theorem can be used to produce high-dimensional examples.}. His constructions can be viewed as a modification of Bing’s\footnote{A connection between Bing’s and Sternfeld’s examples are illustrated in §7.}, but he claimed that his examples cannot embed as an open subset in any compact, locally connected and locally $1$-connected metric space, which is much more general than a compact manifold. More importantly, at the time of writing, Sternfeld’s constructions are the only known examples of such phenomenon in high dimensions.

**Remark 1.** There is an error in Sternfeld’s dissertation which directly affects his whole argument. In the process of proving our main theorem, we correct this error, thereby, confirming the validity of his example (see Remark 2 in §4 for details).

It is natural to ask if Bing’s example can embed in a more general compact space, say, a compact absolute neighborhood retract or compact, locally connected and locally $1$-connected 3-dimensional metric space. Here we answer the above question in negative.

**Theorem 1.1.** $W^3$ embeds as an open subset in no compact, locally connected, locally $1$-connected metric space. In particular, $W^3$ embeds in no compact 3-manifold. 

Making use of the high-dimensional construction developed in [Ste77], we extend Theorem 1.1 to all finite dimensions.

**Theorem 1.2.** There exists a contractible open $n$-manifold $W^n$ ($n \geq 4$) which embeds as an open subset in no compact, locally connected, locally 1-connected metric $n$-space. Hence, $W^n$ embeds in no compact $n$-manifold.

The strategy of our proof heavily relies on the techniques and results from Sternfeld’s dissertation [Ste77]. Succinctly speaking, the key is to show that the union of $W^3$ and a 3-ball (advertised as a knot complement $K_j$) has a finite cover which contains infinitely many pairwise disjoint incompressible surfaces. Many results from [Ste77] will not be re-proved here, but we will take shortcuts afforded by knot theory and software GAP [GAP18] in this work.

The outline of this paper is: §2 gives a detailed review of the construction of Bing’s example and discusses its crucial connection with a knot space $K_j$. That is, showing Bing’s example can embed in no compact, locally connected and locally 1-connected metric space is equivalent to showing $\pi_1(K_j)$ is not finitely generated. Towards that goal, in §3 we find the Wirtinger presentation of $\pi_1(K_j)$ and in §4, we define an important surjection of $\pi_1(K_j)$ onto $A_5$. Meanwhile, we fix an error in Sternfeld’s dissertation. §5 paves the road for §6 by showing that the key ingredient is to focus on an object called a cube with a trefoil-knotted hole. §6 proves Theorem 1.1 by using results obtained from §2-§5. The proof of Theorem 1.2 is presented at the end of this section. In §7, we discuss some related questions of this work.
2. The construction of a 3-dimensional example

First, we reproduce the example originally proposed by Bing, i.e., a 3-dimensional contractible open manifold $W^3$. Let $\{T_l\mid l = 0, 1, 2, \ldots\}$ be a collection of disjoint solid tori standardly embedded in $S^3$. Let the solid torus $T'_l$ be embedded in $\text{Int} T_l$ as in Figure 1.\(^3\) Let the oriented simple closed curve $\alpha_l$, $\beta_l$, $\gamma_l$, and $\delta_l$ be as shown in Figure 1. The curves $\alpha_l$ and $\beta_l$ are transverse in $\partial T_l$, and meet at the point $q_l \in \partial T_l$. In a similar fashion, the curves $\gamma_l$ and $\delta_l$ are transverse in $\partial T'_l$, and meet at the point $p_l \in \partial T'_l$. For $l \geq 1$, let $L_l = T_l \setminus \text{Int} T'_l$. Define an embedding $h_{l+1}^l : T_l \rightarrow T'_{l+1}$ so that $T_l$ is carried onto $T'_{l+1}$ with $h_{l+1}^l(\alpha_l) = \delta_{l+1}$ and $h_{l+1}^l(\beta_l) = \gamma_{l+1}$. $W^3$ is the direct limit of the $T_l$’s and denoted as $W^3 = \lim_{l \rightarrow \infty} (T_l, h_{l+1}^l)$. That is equivalent to view $W^3$ as the quotient space: $\sqcup_l T_l \xrightarrow{\partial} W^3$, where $\sqcup_l T_l$ is the disjoint union of the $T_l$’s and $q$ is the quotient map induced by the relation $\sim$ on $\sqcup_l T_l$. If $x \in T_l$ and $y \in T_j$, then $x \sim y$ if there exists a $k$ larger than $i$ and $j$ such that $h_k^l(x) = h_k^j(y)$, where $h_k^l = h_l^{t-1} \circ h_{l-1}^{t-2} \circ \ldots \circ h_{s+2}^1 \circ h_s^1$ for $t > s$. Let $\iota_l : T_l \hookrightarrow \sqcup_l T_l$ be the obvious inclusion map. The composition $\iota_l \circ \iota_l$ embeds $T_l$ in $W^3$ as a closed subset. The injectivity follows from the injectivity of $h_k^l$. It is closed since for $j > l$ the set $h_j^l(T_l)$ is closed in $T_j$. Let $T_l^* = \iota_l(T_l)$, $T_l^*$ is embedded in $T_{l+1}^*$ just as the way $h_{l+1}^l(T_l)$ ($= T_{l+1}^*$) is embedded in $T_{l+1}$. Hence, Figure 1 can be viewed as a picture of the embedding of $T_l^*$ in $T_{l+1}^*$. In general, for $k > l$, $T_l^*$ is embedded in $T_k^*$ just as $h_k^l(T_l)$ is embedded in $T_k$.

**Proposition 2.1.** $W^3$ is a contractible open connected 3-manifold.

**Proof.** By the construction described above, $W^3$ is an expanding union of $T_l^*$’s, hence, connected. The interior of each $h_k^l(T_l)$ is open in $T_j$, so $\text{Int} T_l^*$ is open in $W^3$. Since $T_l^*$ is contained in $\text{Int} T_{l+1}^*$, $W^3$ is an open 3-manifold.

To show the contractibility of $W^3$, we first triangulate $W^3$ by choosing for each $T_l$ ($l \geq 0$), a simplicial subdivision such that each embedding $h_k^l$ ($k \geq 0$) is simplicial with respect to the chosen subdivision of its domain and range. Let $H : W^3 \times [0, 1] \rightarrow W^3$ be the contraction to be constructed. Define $H$ inductively on the skeleton of $W^3 \times [0, 1]$. Pick $p \in W^3$ to be the point to which we want to contract. Map each vertex cross $[0, 1]$ to a path beginning at the vertex and ending at $p$. Let $\Delta^{(1)}$ be a 1-simplex of $W^3$. Define the restrictions $H|_{\Delta^{(1)} \times \{0\}}$ to be the identity and $H|_{\Delta^{(1)} \times \{1\}}$ to be the constant map taking all points to $p$. Note that $\partial \Delta^{(1)}$ lies in the 0-skeleta of $W^3$. $H$ has already been defined on $\partial \Delta^{(1)} \times [0, 1] = \partial \Delta^{(1)} \times [0, 1])$. Note that $T^*_l$ contracts in $T_{l+1}^*$ (see Figure 1). $H$ can be extended to the rest of $\Delta^{(1)} \times [0, 1]$ by the fact that $H|_{\partial \Delta^{(1)} \times [0, 1]}$ contracts in $W$. Doing this for all 1-simplexes so $H$ is well-defined on the 1-skeleta cross $[0, 1]$. One can do this for 2- and 3-skeleta cross $[0, 1]$ inductively. □

\(^3\)Changing the cube with a trefoil-knotted hole $C_l$ as shown in Figure 1 can result in different contractible open manifold. For instance, one can replace $C_l$ by a cube with a square-knotted hole. Proposition 2.1 is true for all contractible manifolds constructed in such fashion.
Definition 2.1. A topological space $X$ is locally 1-connected at the point $x \in X$ if for each neighborhood $U$ of $x$ there is a neighborhood $V$ of $x$, $V \subset U$, such that every loop in $V$ contracts in $U$. We say that $X$ is locally 1-connected if $X$ is locally 1-connected at each of its points.

The approach of proving Theorem 1.1 does not rely on Haken’s finiteness theorem [Hak68]. Instead, we take advantage of the covering space argument in [Ste77].

Suppose there is a compact, locally connected, locally 1-connected metric space $U$ such that $U$ contains $W^3$ as an open subset. By taking the component of $U$ containing $W^3$ we may assume that $U$ is connected. Then the following result assures that $\pi_1(U \setminus \text{Int} T^*_t)$ must be finitely generated.

Lemma 2.2. [Ste77, Lemma 1.1, P.7] If $X$ is a compact, connected, locally connected, locally 1-connected metric space, then $\pi_1(X)$ is finitely generated.

Instead of working on $\pi_1(U \setminus \text{Int} T^*_t)$ directly, it is easier to focus on a knot space $K_j = S^3 \setminus \text{Int} h^0_j(T_0)$ ($j \geq 1$).\footnote{In [Ste77], $K_i$ (instead of our $K_j$) denotes the knot space corresponding to his 3-dimensional example $W$. In addition, $K_i$ is homeomorphic to an amalgamation $A_i$ in his thesis. At the end of this section, we also decompose $K_j$ into an amalgamation (see (2.1)).} Combining with Claim 2, we have an observation as follows.
Claim 1. \( \pi_1(K_j) \) is a homomorphic image of \( \pi_1(U \setminus \text{Int } T_0^*) \).

Proof. Let \( p_j \) and \( p'_j \) be quotient maps in the commutative diagram (see Figure 2). The inclusion, \( \iota_j \), followed by \( p_j \) induces the map \( g_j \) since the restriction of \( p_j \) on \( T_j^* \setminus \text{Int } T_0^* \) is to collapse \( \partial T_j^* \) to a point. It’s not hard to see that \( g_j \) is actually a homeomorphism. Since \( \partial T_j^* \) is collared in \( T_j^* \setminus \text{Int } T_0^* \), Lemma 5.4 implies that \( p'_j \) induces a surjection on fundamental groups. By the commutativity of the diagram 2, \( p'_j = g_j^{-1}p_j \iota_j \), where \( p'_j \), \( g_j \), \( p_j \) and \( \iota_j \) are the homomorphisms induced by maps \( p'_j \), \( g_j \), \( p_j \) and \( \iota_j \) respectively. Since \( p'_j \) is a surjection, \( g_j^{-1}p_j \) is also a surjection. Hence, \( \pi_1((T_j \setminus \text{Int } T_0^*)/\partial T_j^*) \) is a homomorphic image of \( \pi_1(U \setminus \text{Int } T_0^*) \). According to the construction of \( W^3 \), the pair \((T_j^* \setminus \text{Int } T_0^*) \) is homeomorphic to the pair \((T_j, h_j^0(T_0))\). Then the claim follows from Claim 2. \( \Box \)

Since the rank\(^5\) of a group must be at least as large as that of any homomorphic image, it suffices to show that the rank of \( \pi_1(K_j) \) is unbounded.

The space \( K_j \) is advertised as “knot space” because it can be viewed as a knot complement. To see that, we need the construction based on two important tools in producing knots. The first one is

Definition 2.2. Let \( K_P \) be a non-trivial knot in \( S^3 \) and \( V_P \) an unknotted solid torus in \( S^3 \) with \( K_P \subset V_P \subset S^3 \). Let \( K_C \subset S^3 \) be another knot and let \( V_C \) be a tubular neighborhood of \( K_C \) in \( S^3 \). Let \( h : V_P \to V_C \) be a homeomorphism and let \( K_W \) be \( h(K_P) \). We say \( K_C \) is a companion of any knot \( K_W \) constructed (up to knot type) in this manner. If \( h \) is faithful, meaning that \( h \) takes the preferred longitude\(^6\) and meridian of \( V_P \) respectively to the preferred longitude and meridian of \( V_C \), We say \( K_W \) is an untwisted Whitehead double of \( K_C \). Otherwise, \( K_W \) is a twisted Whitehead double. For instance, Figure 3 is a 3-twisted Whitehead double of a trefoil knot. The pair \((V_P, K_P)\) is the pattern of \( K_W \).

The second tool is based on a type of connected sum of a pair of manifolds \((M_1^m, N_1^n) \#(M_2^m, N_2^n)\), where \( N_i^n \) is a locally flat submanifold of \( M_i^m \). Treat the above pair as \((S^3, k_1) \#(S^3, k_2)\) where \( k_i \) are tame knots. Removing a standard

\(^5\)When we say the rank of a group \( G \), denoted by \( \text{Rank } G \), it means the smallest cardinality of a generating set for \( G \).

\(^6\)"Preferred longitude" means that \( K_W \) has writhe number zero.
ball pair \((B_3^1, B_1^1)\) from \((S^3, k_1)\) and gluing the resulting pairs by a homeomorphism
\(h : (\partial B_3^2, \partial B_1^2) \to (\partial B_3^1, \partial B_1^1)\) to form the pair connected sum. For convenience, we use \(k_1 \# k_2\) other than pairs of manifolds. See [Rol76] for details.

To help readers get a better feeling about group \(\pi_1(K_j)\), we show that \(\pi_1(K_j)\) is isomorphic to \(\pi_1((T_j \setminus \text{Int } h_j^0(T_0))/\partial T_j)\). Geometrically, \(K_j\) is the space obtained by sewing the solid torus \(S^3 \setminus \text{Int } T_j\) to \(T_j \setminus \text{Int } h_j^0(T_0)\) along \(\partial T_j\). We decompose \(S^3 \setminus \text{Int } T_j\) into two 3-cells \(B_1\) and \(B_2\), i.e., \(S^3 \setminus \text{Int } T_j = B_1 \cup B_2\), where \(B_1\) is the thickened meridional disk \(D\) in \(S^3 \setminus \text{Int } T_j\) with \(\partial D = \alpha_j\) (see Figure 4) and \(B_2\) is the closure of the complement of \(B_1\) in \(S^3 \setminus \text{Int } T_j\). Sewing \(B_1\) to \(T_j \setminus \text{Int } h_j^0(T_0)\) along an annular neighborhood of \(\alpha_j\) in \(\partial T_j\). By Seifert-van Kampen, the inclusion \(T_j \setminus \text{Int } h_j^0(T_0) \hookrightarrow (T_j \setminus \text{Int } h_j^0(T_0)) \cup B_1\) induces a surjection on fundamental groups whose kernel is the normal closure of the curve \(\alpha_j\) in \(\pi_1(T_j \setminus \text{Int } h_j^0(T_0))\).

Adding \(B_2\) to \((T_j \setminus \text{Int } h_j^0(T_0)) \cup B_1\) to form the knot complement \(K_j\) does not affect the fundamental group. This follows readily from Seifert-van Kampen. Hence, the inclusion \(T_j \setminus \text{Int } h_j^0(T_0) \hookrightarrow K_j\) induces a surjection on fundamental groups whose kernel is the normal closure of the curve \(\alpha_j\) in \(\pi_1(T_j \setminus \text{Int } h_j^0(T_0))\).

Claim 2. \(\pi_1(K_j)\) is isomorphic to \(\pi_1((T_j \setminus \text{Int } h_j^0(T_0))/\partial T_j)\).

Proof. It’s sufficient to show that the meridian \(\beta_j\) of \(T_j\) is trivial in \(\pi_1(K_j)\). In other words, we will show that \(\beta_j\) contracts in the complement of \(h_j^0(T_0)\). Consider Figure 4. \(h_j^0(T_0)\) (not pictured) is contained in \(h_j^{2-1}(T_{j-1})\), which is also contained in the solid torus \(A\). Since \(A\) is an unknotted solid torus, \(\beta_j\) bounds a 2-chain in \(S^3 \setminus A\). □

It’s clear that \(\pi(K_1)\) is isomorphic to a trefoil knot group.

Claim 3. \(\pi_1(K_2)\) is isomorphic to the knot group of the connected sum of a trefoil knot and a 3-twisted Whitehead double of a trefoil knot.

Proof. By the construction of \(W^3, T_1^*\) embeds in \(T_2^*\) just as the way \(T_0^*\) embeds in \(T_1^*\) (as shown in Figure 1). Note that the space \(K_2 = S^3 \setminus \text{Int } h_2^0(T_0)\) can be decomposed.

![Figure 3. A 3-twisted Whitehead double of a trefoil knot](image-url)
Figure 4. $\beta_j$ contracts in $S^3 \setminus \text{Int} h^0_j(T_0)$, where $h^0_j(T_0)$ is not pictured.

Let $K_1$ be a trefoil knot corresponding to the knot space $K_1$. Denote a knot $K_2$ by $K_1^{Wh} \# K_1$ such that $\pi_1(S^3 \setminus K_2) \cong \pi_1(K_2)$. Similarly, one can further find a knot $K_3$ on the 3rd stage which is a connected sum of a twisted Whitehead double of $K_2$ and $K_1$. By iteration, a knot $K_j$ can be viewed as $K_j^{Wh} \# K_1$.

Let $G_3$ and $G_j^{Wh}$ be the knot group of $K_1$ and $K_j^{Wh}$ respectively. By the definition of connected sum, there is a tame 2-sphere $S^2$ dividing $S^3$ into two balls $B_1$ containing $K_j^{Wh}$ and $K_1$ respectively. The intersection of $K_j^{Wh}$ and $K_1$ is an arc $\zeta$ lying in $S^2$. View $K_j = K_j^{Wh} \# K_1$ as the union of $K_j^{Wh}$ and $K_1$ minus $\text{Int} \zeta$ (see Figure 5). Then we have the following diagram “pushout” commutative diagram 6.
Figure 5. The connected sum of a twisted Whitehead double of $K_1$ and $K_1$ ($\approx$ trefoil knot). Here $\approx$ stands for homeomorphic.

\[
\begin{align*}
\pi_1(S^2 \setminus K_j) & \cong \mathbb{Z} \\
\pi_1(B_{Wh} \setminus K_{j-1}^{Wh}) & \cong G_{j-1}^{Wh} \\
\pi_1(B_1 \setminus K_1) & \cong G_3 \\
\pi_1(S^3 \setminus K_j) & \\
\end{align*}
\]

Figure 6. “Pushout” commutative diagram

Clearly, the two upper homomorphisms in Figure 6 are injective. By the Seifert-van Kampen theorem, the other two homomorphisms $\iota_1, \iota_2$ are also injective. That means

\[G_j = \pi_1(S^3 \setminus K_j) = G_{j-1}^{Wh} * \langle \lambda \rangle G_3\]

is a free product with amalgamation along an infinite cyclic group, where $[\lambda]$ corresponds to the loop class in $\pi_1(S^2 \setminus K_j)$. According to this set-up, $G_{j-1}^{Wh}$ and $G_3$ are two subgroups of $G_j$ and $\langle \lambda \rangle$ is a subgroup of both $G_{j-1}^{Wh}$ and $G_3$. Since both $G_{j-1}^{Wh}$ and $G_3$ are abelianized to $\langle \lambda \rangle \cong \mathbb{Z}$, $G_j$ is a split amalgamated free product.
Although the work in [Wei99] guarantees a lower bound for $\text{Rank} G_j *_{\langle \lambda \rangle} G_3$, i.e., $\text{Rank} G_j *_{\langle \lambda \rangle} G_3 \geq 2$, the ultimate goal is to show that $\text{Rank} G_j *_{\langle \lambda \rangle} G_3$ has no upper bound as $j \to \infty$. At the time of writing, we don’t know whether there is a direct knot theoretical approach to this. So, we use the covering space theory as developed by Sternfeld in [Ste77].

We start by constructing a surjective homomorphism $\Phi_j : G_{j-1}^{Wh} *_{\langle \lambda \rangle} G_3 \twoheadrightarrow A_5$, where $A_5$ is an alternating group on 5 letters. To that end, by the definition of $W^3$, we decompose $K_j$ into an amalgamation of $L_j$’s. That is, for $j \geq 1$,

\begin{equation}
K_j \approx (S^3 \setminus \text{Int} T_j) \cup_{t_0} L_j \cup_{h_j} L_{j-1} \cup_{h_j^{-1}} L_{j-2} \cdots \cup_{h_1} L_1,
\end{equation}

where the sewing homeomorphism $h_{t+1}$ identifies the boundary component $\partial T_l$ of $L_l$. It’s clear that $\pi_1(K_j) \cong G_j$. So, we convert the problem to finding a surjection from $\pi_1(K_j) \twoheadrightarrow A_5$ which will be discussed in the following two sections.

### 3. A PRESENTATION OF $\pi_1(K_j)$

First we spell out a Wirtinger presentation similar to what Sternfeld did in [Ste77, P.20–26] for $\pi_1(L_l)$, where $l \geq 1$. Let $\Sigma_l$ and $\Omega_l$ be polyhedral simple closed curves contained in $S^3$ such that $S^3 \setminus (\Sigma_l \cup \Omega_l)$ deformation retracts onto $L_l$. $\Sigma_l$ and $\Omega_l$ can be viewed as cores of the solid tori $T'_l$ and $S^3 \setminus \text{Int} T_l$ respectively (see Figures 1 and 7). Let the arc $\mu_l$ in Figure 7 run from one end point $p_l \in \partial T_l$ and to the other end point $q_l \in \partial T'_l$. $\mu_l$ is properly embedded in $L_l$.

Hence, the presentation of $\pi_1(S^3 \setminus (\Sigma_l \cup \Omega_l), p_l)$ is

\begin{align}
\text{Generators: } & a, b, c, \ldots, i \\
\text{Relators: } & \\
& \begin{cases}
R_{l,1} : b = c^{-1}ac \\
R_{l,2} : c = a^{-1}ba \\
R_{l,3} : d = b^{-1}cb \\
R_{l,4} : e = gdg^{-1} \\
R_{l,5} : f = heh^{-1} \\
R_{l,6} : g = efe^{-1} \\
R_{l,7} : a = h^{-1}gh \\
R_{l,8} : h = g^{-1}ig \\
R_{l,9} : i = fhf^{-1},
\end{cases}
\end{align}

where the subscripts $l$’s are surpressed.
Write loop classes $[\alpha_l], [\beta_l], [\gamma_l]$ and $[\delta_l]$ as words in the generators $a_l, b_l, \ldots, i_l$ of (3.1):

\[
\begin{align*}
[\alpha_l] &= h_l \\
[\beta_l] &= f_l^{-1} g_l \\
[\gamma_l] &= a_l \\
[\delta_l] &= c_l a_l b_l g_l^{-1} h_l^{-1} e_l^{-1} h_l
\end{align*}
\]

where $[\alpha_l]$ is determined by the oriented simple closed curve $\alpha_l$ lying in $\partial L_l$ (see Figures 1 and 7) and the arc $\mu_l$ connecting $\alpha_l$ to the base point $p_l$. Likewise, $[\beta_l], [\gamma_l]$ and $[\delta_l]$ are defined in the same manner. Deformation retract $S^3 \setminus (\Sigma_l \cup \Omega_l)$ onto $L_l$. It’s clear that Presentation (3.1) is a presentation of $\pi_1(L_l, p_l)$. Consider the loop classes $a_l, b_l, \ldots, i_l$ in $\pi_1(L_l, p_l)$ (represented by the same loops as before) as loops in $L_l$. At the same time, $[\alpha_l], [\beta_l], [\gamma_l]$ and $[\delta_l]$ may be written as the same words (3.2) in the generators of $\pi_1(L_l, p_l)$.

Recall in the previous section, we have the following knot space

\[ K_j \approx (S^3 \setminus \text{Int } T_j) \cup \text{Id } L_j \cup h_j^{-1} L_{j-1} \cup h_{j-2}^{-1} \cdots \cup h_2^{-1} L_1, \]

where the sewing homeomorphism $h_{t+1}^{l_t}$ identifies the boundary component $\partial T_l$ of $L_l$ to the boundary component $\partial T_{l+1}$ of $L_{l+1}$ such that the transverse oriented simple closed curves $\alpha_l$ and $\beta_l$ of $\partial T_l$ are mapped in an orientation preserving manner to the
transverse oriented simple closed curves $\delta_{l+1}$ and $\gamma_{l+1}$ respectively in $\partial T'_{l+1}$. Using the words (3.2), this can be described by the following relators

\[
\begin{aligned}
\text{Relators:} & \quad \left\{
S_{l,1} : h_{l-1} = c_1 a_l b_l g^{-1}_l h^{-1}_l e^{-1}_l h_l \text{ for } j \geq l \geq 2 \\
S_{l,2} : f_{l-1} g_{l-1} = a_l \text{ for } j \geq l \geq 2.
\right\}
\end{aligned}
\]

Combine the words (3.1) and (3.3), we obtain

**Proposition 3.1.** $\pi_1(K_j, p_1)$, $j \geq 1$, has the following presentation

\[
\begin{aligned}
\text{(3.4) Generators:} & \quad a_l, b_l, c_l, \ldots, i_l \text{ for } j \geq l \geq 1 \\
\text{Relators:} & \quad \left\{ R_{l,k} \text{ for } j \geq l \geq 1 \text{ and } 9 \geq k \geq 1 \\
& \quad S_{l,1} \text{ for } j \geq l \geq 2 \\
& \quad S_{l,2} \text{ for } j \geq l \geq 2 \\
& \quad h_j = 1,
\right\}
\end{aligned}
\]

where the generators $a_l, \ldots, i_l$ of Presentation (3.4) correspond to those of Presentation (3.1) conjugated by the path $\mu_l$.

**Proof.** The proof is an easy modification of the proof of Proposition 4.1 in [Ste77]. \(\square\)

### 4. The surjection of $\pi_1(K_j, p_1)$ onto $A_5$

Here we shall define a homomorphism $\Phi_j : \pi_1(K_j, p_1) \to A_5$, where $j \geq 1$. It suffices to define $\Phi_j$ on the generators of Presentation (3.4) of $\pi_1(K_j, p_1)$ and check that the definition is compatible with the relators of the presentation. That is, if the following words

\[
\begin{aligned}
w(a_1, b_1, \ldots, i_1, \ldots, a_j, b_j, \ldots, i_j) = w'(a_1, b_1, \ldots, i_1, \ldots, a_j, b_j, \ldots, i_j)
\end{aligned}
\]

is a relator of the presentation, then

\[
\begin{aligned}
w(\Phi_1(a_1), \ldots, \Phi_1(i_1), \ldots; \Phi_j(a_j), \ldots; \Phi_j(i_j)) = w'(\Phi_1(a_1), \ldots, \Phi_1(i_1), \ldots; \Phi_j(a_j), \ldots; \Phi_j(i_j))
\end{aligned}
\]

must hold for $A_5$.

Consider an extreme case by “unknotting” every small trefoil knot in the link (corresponding to $L_l$) as shown in Figure 7. The link in Figure 7 can be viewed as a connected sum of a Whitehead link and a trefoil knot. Thus, we can abelianize the trefoil knot group to a connected sum of a Whitehead link and a trefoil knot. In other words, the new knot space is a concatenation of Whitehead links with 3 half-twists to $T_l$ due to the writhe of trefoil knot (before abelianization) in $T'_{l+1}$ is 3.
half-twists. Denote the corresponding knot space by \( K_j^{**} \). By the above procedure, \( \pi_1(K_j^{**}) \) can be obtained by adding relators \( a_l = b_l, b_l = c_l, c_l = d_l \) to the presentation of \( \pi_1(K_j) \) in Proposition 3.1

\[
\begin{align*}
\text{Generators: } & a_l, b_l, c_l, \ldots, i_l & & \text{for } j \geq l \geq 1 \\
\text{Relators: } & \begin{cases} 
R_{l,k} \text{ for } j \geq l \geq 1 \text{ and } 9 \geq k \geq 1 \\
S_{l,1} \text{ for } j \geq l \geq 2 \\
S_{l,2} \text{ for } j \geq l \geq 2 \\
h_j = 1.
\end{cases}
\end{align*}
\]

Clearly, there is a surjection of \( \psi_j : \pi_1(K_j) \to \pi_1(K_j^{**}) \) by sending \( a_l, \ldots, d_l \) in Presentation (3.4) to \( a_l \) in Presentation (4.1). So, it suffices to find a surjection \( \phi_j \) of \( \pi_1(K_j^{**}) \) onto \( A_5 \).

We shall define \( \phi_j \) inductively on the generators of Presentation (4.1). If \( j = 1 \), we use GAP [GAP18] to define a surjection \( \phi_1 \) on \( a_1, \ldots, i_1 \) by Table 1a. This definition is compatible with the relators \( R_{1,k} \) and \( h_1 = 1 \), where \( 1 \leq k \leq 9 \). If \( j = 2 \), both Tables 1a and 1b are used. Besides relators \( R_{1,k}, R_{2,k} \) and \( h_2 = 1 \), relators \( S_{2,1}, S_{2,2} \) are also compatible. Similarly, if \( j = 3 \) (resp. \( j = 4 \)), Tables 1a-1c (resp. 1a-2a) are applied. When \( j \geq 5 \), Tables 1a-2b will be applied periodically. That is, extend \( \phi_j \) to the generators \( a_l, \ldots, i_l \) according to Table 1a if \( l = j \), according to Table 1b if \( l = j - 1 - 4T \), according to Table 1c if \( l = j - 2 - 4T \), according to Table 2a if \( l = j - 3 - 4T \) and according to Table 2b if \( l = j - 4 - 4T \), where \( T \in \mathbb{N} \) and \( 0 \leq T \leq (j-1)/4 \). One can either use GAP [GAP18] or simply by hand to check such extension is compatible with relators in Presentation (3.4). Hence, the composition \( \Phi_j = \phi_j \circ \psi_j \) is the desired surjection.

### Table 1

| (A) \( l = j \) | (B) \( l = j - 1 - 4T \) | (C) \( l = j - 2 - 4T \) |
|-----------------|-----------------|-----------------|
| **Generators**  | **Image**       | **Generators**  | **Image**       | **Generators**  | **Image**       |
| \( a_l \)       | (1,2)(3,4)      | \( a_l \)       | (1,2,3)         | \( a_l \)       | (1,3)(4,5)      |
| \( b_l \)       | (1,2)(3,4)      | \( b_l \)       | (1,2,3)         | \( b_l \)       | (1,3)(4,5)      |
| \( c_l \)       | (1,2)(3,4)      | \( c_l \)       | (1,2,3)         | \( c_l \)       | (1,3)(4,5)      |
| \( d_l \)       | (1,2)(3,4)      | \( d_l \)       | (1,2,3)         | \( d_l \)       | (1,3)(4,5)      |
| \( e_l \)       | (1,2)(3,4)      | \( e_l \)       | (2,4,3)         | \( e_l \)       | (1,2,4)         |
| \( f_l \)       | (1,2)(3,4)      | \( f_l \)       | (1,3,4)         | \( f_l \)       | (1,3,4)         |
| \( g_l \)       | (1,2)(3,4)      | \( g_l \)       | (1,4,2)         | \( g_l \)       | (2,3,4)         |
| \( h_l \)       | ()              | \( h_l \)       | (1,2)(3,4)      | \( h_l \)       | (1,2,3)         |
| \( i_l \)       | ()              | \( i_l \)       | (1,3)(2,4)      | \( i_l \)       | (1,3,2)         |

**Remark 2.** In line 16 [Ste77, P.28], the author claims that the definition of \( \Phi_i : \pi_1(A_i) \to A \) given in Table 1 [Ste77, P.29] is compatible with the relators \( S_{j,1}, S_{j,2} \) for
Table 2

| Generators | Image    | Generators | Image    |
|------------|----------|------------|----------|
| \(a_l\)   | \((3,4,5)\) | \(a_l\)   | \((1,2)(3,4)\) |
| \(b_l\)   | \((3,4,5)\) | \(b_l\)   | \((1,2)(3,4)\) |
| \(c_l\)   | \((3,4,5)\) | \(c_l\)   | \((1,2)(3,4)\) |
| \(d_l\)   | \((3,4,5)\) | \(d_l\)   | \((1,2)(3,4)\) |
| \(e_l\)   | \((1,3,5)\) | \(e_l\)   | \((1,2)(4,5)\) |
| \(f_l\)   | \((1,4,3)\) | \(f_l\)   | \((1,2)(3,4)\) |
| \(g_l\)   | \((1,5,4)\) | \(g_l\)   | \((1,2)(3,5)\) |
| \(h_l\)   | \((1,3)(4,5)\) | \(h_l\)   | \((3,4,5)\) |
| \(i_l\)   | \((1,5)(3,4)\) | \(i_l\)   | \((3,5,4)\) |

\(l \geq j \geq 2\), where \(A\) is an alternating group on 5 letters \(v, w, x, y\) and \(z\). However, for \(l < i\), \(\Phi(o_{l-1}^{-1}h_{l-1}f_{l-1}^{-1}q_{l-1})\) is not equal to \(\Phi(a_l)\). That is, using Table 1 [Ste77, P.29], \(\Phi(o_{l-1}) = (vy)(wz), \Phi(h_{l-1}) = (vy)(xz), \Phi(f_{l-1}) = (wx)(yz)\) and \(\Phi(q_{l-1}) = (vw)(yz)\). Hence, \(\Phi(o_{l-1}^{-1}h_{l-1}f_{l-1}^{-1}q_{l-1}) = (vw)(xz) \neq \Phi(a_l) = (vw)(xy)\). That means the definition of the so claimed \(\Phi_i\) is not compatible with the relators \(S_{j,1}, S_{j,2}\) for \(l \geq j \geq 2\). This error directly affects the following statement [Ste77, P.52]: “The composition \(\pi_1(C_j, x_j) \xrightarrow{k} \pi_1(A_i, x_j) \xrightarrow{M_j} \pi_1(A_i, x_i) \xrightarrow{\Phi_i} A\) has image isomorphic to \(\mathbb{Z}_2\) in \(A\) since \(\Phi_i\) maps \(a_j\) and \(b_j\) to the same element of order 2 in \(A\). Thus, the kernel of \(\Phi_i \circ M_j \circ k\) has index 2 in \(\pi_1(C_j, x_j)\).” To fix this error, we provide a series of correct tables here.

We have to use at least 3 tables (instead of 2 tables) such that the definition of \(\Phi_i\) is compatible with all the relators. Similar to how we define a surjection of \(\pi_1(K_j, p_1) \to A_5\) in the beginning of this section, with the assistance of GAP [GAP18], the following tables provide a surjection of \(\Phi_i : \pi_1(A_i, x_1) \to A_5\). If \(i = 1\), we defined \(\Phi_i\) on \(a_1, \ldots, u_1\) by Table 3a. If \(i = 2\), then Tables 3a and 3b are used. Otherwise, when \(i \geq 3\), Tables 3a, 3b and 3c are applied. That is, extend \(\Phi_i\) to the generators \(a_l, \ldots, u_l\) according to Table 3a if \(l = i\), according to Table 3b at \(l = i - 1 - 2T\) and according to Table 3c at \(l = i - 2 - 2T\), where \(T \in \mathbb{N}\) and \(0 \leq T \leq (i - 1)/2\).
5. Properties of a cube with a trefoil-knotted hole

One of the key ingredients in proving Theorem 1.1 is to understand the covering space of a cube with a trefoil-knotted hole as shown in Figure 1. In this section, we collect a number of important properties about cubes with a trefoil-knotted hole. Let \( C \) be the cube with a trefoil-knotted hole as shown in Figure 8. Here \( C \) is the complement in \( S^3 \) of the interior of a regular neighborhood of the polyhedral simple closed curve \( \Gamma \). There is a deformation retract of \( S^3 \setminus \Gamma \) onto \( C \). The presentation of \( \pi_1(S^3 \setminus \Gamma) \) (i.e., trefoil knot group) is a presentation of \( \pi_1(C, p_0) \), where \( p_0 \) is a base point. Hence, one can use the Wirtinger presentation of \( \pi_1(S^3 \setminus \Gamma) \) to obtain the following proposition.

**Proposition 5.1.** \( \pi_1(C, p_0) \) has presentation
\[
\langle a, b | b^{-1}a^{-1}b^{-1}aba = 1 \rangle,
\]
where \( a = [A] \) and \( b = [B] \) as shown in Figure 8.

**Corollary 5.2.** \( \pi_1(C, p_0) \) has Rank 2.
Proof. Obviously, $\text{Rank} \, \pi_1(C, p_0) \leq 2$. By the classification of finite simple groups, $\text{Rank} \, A_5 = 2$. Using GAP [GAP18], one can find a surjection of $\pi_1(C, p_0)$ onto $A_5$ by $(a, b) \mapsto ((1, 3, 5, 4, 2), (1, 2, 3, 4, 5))$. That means $\text{Rank} \, \pi_1(C, p_0)$ has to be greater or equal to 2. Hence, $\text{Rank} \, \pi_1(C, p_0) = 2$. \hfill \Box

**Proposition 5.3.** [Ste77, Prop.6.3] $C$ has a unique 2-fold cover, $\tilde{C}^2$, the boundary $\partial \tilde{C}^2$ is connected and the quotient map

$$ Q : \tilde{C}^2 \to \tilde{C}^2 / \partial \tilde{C}^2 $$

induces a surjection on fundamental groups.

**Lemma 5.4.** [Ste77, Lemma 1.3] Let $B$ be a subspace of $X$. Let $B$ and $X$ be path connected. If $B$ is collared in $X$, then the quotient map $q : X \to X/B$ induces a surjection of fundamental groups whose kernel is the normal closure in $\pi_1(X)$ of $i_* \pi_1(B)$, where $i_*$ denotes the inclusion induced homomorphism.
The following result generalizes Proposition 5.3 for the \( k \)-fold cyclic cover of \( C \).

**Proposition 5.5.** Let \( \tilde{C}^k \) be the \( k \)-fold cyclic cover of \( C \). Then \( \partial \tilde{C}^k \) is connected and the quotient map

\[
Q : \tilde{C}^k \to \tilde{C}^k / \partial \tilde{C}^k
\]

induces a surjection on fundamental groups.

**Proof.** First, we show \( \partial \tilde{C}^k \) is connected. Let \( \tilde{P} : \tilde{C}^k \to C \) be the \( k \)-fold cyclic cover. The restriction of \( \tilde{P} \) to each component of \( \tilde{P}^{-1}(\partial C) \) is a covering map of \( \partial C \). Note that the \( k \)-fold cyclic cover is defined to be the one which corresponds to the kernel of the composite

\[
\pi_1(C) \xrightarrow{\text{abelianization}} \mathbb{Z} \xrightarrow{\text{projection}} \mathbb{Z}_k.
\]

The uniqueness of the abelianization and the projection assures that the simple closed curve \( A \) (see Figure 8) in \( \partial C \) based at a point \( p_0 \) has a lift \( \tilde{A} \) which is not a loop since the loop \([A]\) corresponding to the generator \( a \) in Proposition 5.1 is not in the kernel. Therefore, the component of \( \partial \tilde{C}^k \) that contains \( \tilde{A} \) must be a least a double cover of \( \partial C \) since the two end points of \( \tilde{A} \) cover \( p_0 \). Since each point of \( C \) has precisely \( k \) preimages in \( \tilde{C}^k \), the component of \( \partial \tilde{C}^k \) that contains \( \tilde{A} \) must be all of \( \partial \tilde{C}^k \). Thus \( \partial \tilde{C}^k \) is (path) connected.

Applying Lemma 5.4 finishes the proof. \( \square \)

**Proposition 5.6.** \( \pi_1(\tilde{C}_2 / \partial \tilde{C}_2) \cong \mathbb{Z}_3 \).

**Proof.** The proof is a standard covering space argument. See the proof of Prop.6.4 in [Ste77, P.39-46]. \( \square \)

**Proposition 5.7.** Let \( \tilde{C}^3 \) be the 3-fold cyclic cover of \( C \). Then \( \text{Rank} \pi_1(\tilde{C}^3 / \partial \tilde{C}^3) \geq 1 \).

**Proof.** Standard cyclic cover argument [Rol76, Ch.6] assures the first homology group \( H_1(\tilde{C}^3) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z} \). “Modulo out” the generators corresponding the boundary \( \tilde{C}^3 \) can at most reduce the rank by 2, hence, \( \text{Rank} \pi_1(\tilde{C}^3 / \partial \tilde{C}^3) \geq 3 - 2 = 1 \). \( \square \)

6. **Proof of Theorem 1.1**

Recall in Section 2 we pointed out the key in proving Theorem 1.1 is to show that \( \text{Rank} \pi_1(K_j, p_1) \) is not bounded. Since \( A_5 \) has order 60 and \( \Phi_j : \pi_1(K_j, p_1) \to A_5 \) is onto, \( \ker \Phi_j \) has index 60 in \( \pi_1(K_j, p_1) \). Then the following formula guarantees that it suffices to show that \( \text{Rank} \ker \Phi_j \) is not bounded.

The formula can be viewed as a corollary of the Schreier index theorem. A detailed proof by utilizing covering space theory can be found in [Ste77, Lemma 1.4].

**Lemma 6.1.** Let \( G \) be a group and \( H \) be a subgroup of index \( i \). If \( \text{Rank} \ H \geq m \), then \( \text{Rank} \ G \geq \frac{m-1}{i} + 1 \).

Let \( P_j : (\tilde{K}_j, \tilde{p}_1) \to (K_j, p_1) \) be the covering map such that the induced map \( P_j^* : \pi_1(\tilde{K}_j, \tilde{p}_1) \to \pi_1(K_j, p_1) \) is an isomorphism onto \( \ker \Phi_j \). By Lemma 6.1, it remains to show that \( \text{Rank} \ker \Phi_j \) is not bounded above as \( j \to \infty \), which is equivalent
to showing that \( \text{Rank} \pi_1(\tilde{K}_j, \hat{p}_1) \geq 25j \) (resp. \( 5(5j + 1) \)) when \( j \) is even (resp. odd). The key is the fact that \( K_j \) contains \( j \) pairwise disjoint incompressible cubes with trefoil-knotted hole. Figure 1 shows that each \( L_l, l \geq 1 \) contains a cube with trefoil-knotted hole \( C_l \). Recall

\[
K_j \approx (S^3 \setminus \text{Int } T_j) \cup_{\text{id}} L_j \cup_{h_{j-1}} L_{j-1} \cup_{h_{j-2}} \cdots \cup_{h_0} L_1,
\]

\( K_j \) contains \( C_1, C_2, \ldots, C_j \), pairwise disjoint cubes with trefoil-knotted hole. The disjointness follows from that each \( C_l \) lies in its own \( L_l \) and touches only the “inner” boundary of its \( L_l \). In \( K_j \), when we sew two adjacent \( L_l \)’s together, only the “outer” boundary of one is glued to the “inner” boundary of the next.

Next, we shall show that \( C_l \) in \( K_j \) has preimage under the restriction of the covering map \( P_j \) has 30 disjoint double covers and 20 disjoint triple covers. The proof heavily relies on the argument given in [Ste77, P.50-55]. For the convenience of readers, we spell out the proof in details.

Consider \( p_l \in C_l \). See Figures 7 and 8. From the Wirtinger presentation (3.4), a loop class with subscript \( l \) is the class of a loop formed by conjugation of a loop in \( L_l \) based at \( p_l \) by the path \( \mu_l^l \) running from \( p_l \) to \( p_l \) in \( K_j \). Define a change-basepoint isomorphism \( M_l : \pi_1(K_j, p_l) \to \pi_1(K_j, p_l) \) generated by conjugation by \( \mu_l^l \). By Figures 1 and 7, loop classes \( M_l^{-1}(a_l), M_l^{-1}(b_l) \) can be viewed as loop classes of \( \pi_1(C_l, p_l) \), where \( 1 \leq l \leq j \). Then Figures 7-8 and Proposition 5.1 assure that the set \( \{M_l^{-1}(a_l), M_l^{-1}(b_l) \} \) generates \( \pi_1(C_l, p_l) \).

Let \( \iota_* : \pi_1(C_l, p_l) \to \pi_1(K_j, p_l) \) be the inclusion induced homomorphism. Combine the results from §4 to obtain the following composition

\[
\pi_1(C_l, p_l) \xrightarrow{\iota_*} \pi_1(K_j, p_l) \xrightarrow{M_l} \pi_1(K_j, p_l) \xrightarrow{\Phi_j} \mathbb{A}_5,
\]

which has image isomorphic to \( \mathbb{Z}_2 \) (resp. \( \mathbb{Z}_3 \)) in \( \mathbb{A}_5 \) when \( l = j, j - 2 - 4T \) and \( j - 4 - 4T \) (resp. \( l = j - 1 - 4T \) and \( j - 3 - 4T \)). See Tables 1a, 1c and 2b (resp. 1b and 2a). That is because \( \Phi_j \) maps \( a_l \) and \( b_l \) of \( \pi_1(C_l, p_l) \) to the same element of order 2 (resp. 3) in \( \mathbb{A}_5 \). It follows that the kernel of \( \Phi_j \circ M_l \circ \iota_2 \) has index either 2 or 3 in \( \pi_1(C_l, p_l) \). Let \( q : (\tilde{C}_l^2, \tilde{p}_l) \to (C_l, p_l) \) be a 2-fold cover of \( (C_l, p_l) \) corresponding to the kernel.

**Claim 4.** Each \( \tilde{C}_l^2 \) embeds in \( \tilde{K}_j \).

**Proof.** Note that there exists a lift \( \tilde{p}_l \) of \( p_l \) in \( \tilde{K}_j \) so that \( P_j*(\pi_1(\tilde{K}_j, \tilde{p}_l)) = \ker(\Phi_j \circ M_l) \). The lift is obtained by lifting \( \mu_l^l \) to a path \( \tilde{\mu}_l^l \) so \( \tilde{\mu}_l^l(0) = \tilde{p}_l \) and the point \( \tilde{p}_l \) is defined to be \( \tilde{\mu}_l^l(1) \). Since \( \iota_*q_*(\pi_1(\tilde{C}_l^2, \tilde{p}_l)) \subseteq P_j*(\pi_1(\tilde{K}_j, \tilde{p}_l)) \), we have the following commutative diagram with \( \iota \) lifted to \( \tilde{\iota} \)

\[
\begin{array}{ccc}
(C_l, p_l) & \xrightarrow{\iota} & (K_j, p_l) \\
\downarrow q & & \downarrow P_j \\
(\tilde{C}_l^2, \tilde{p}_l) & \xrightarrow{\tilde{\iota}} & (\tilde{K}_j, \tilde{p}_l)
\end{array}
\]

We shall apply standard covering space theory to show \( \tilde{\iota} \) is an embedding. It suffices to prove that \( \tilde{\iota} \) is 1-1. Suppose \( x \) and \( y \) are two elements of \( \tilde{C}_l^2 \) such that \( \tilde{\iota}(x) = \tilde{\iota}(y) \).
The commutativity of the diagram above implies that \( q(x) = q(y) \). Connect \( x \) to \( y \) by a path \( \alpha \) and \( x \) to \( \tilde{p}_1 \) by a path \( \beta \) with \( \beta(0) = \tilde{p}_1 \) and \( \beta(1) = x \). Lift \( q(\beta) \) to \( \tilde{\beta} \) so that \( \tilde{\beta}(1) = y \). Suppose \( x \neq y \). Then \( \tilde{\beta} \) and \( \beta \) are distinct lifts of \( q(\beta) \). That means \( \beta(0) \neq \tilde{\beta}(0) \). So, \( \beta \alpha \tilde{\beta}^{-1} \) is not a loop. However, \( \tilde{\iota}(\beta \alpha \tilde{\beta}^{-1}) \) is a loop in \( \tilde{K}_j \). Since \( \tilde{\iota}(x) = \tilde{\iota}(y) \), \( \tilde{\iota} \beta \) and \( \iota \tilde{\beta} \) have to be the same lift of \( \iota q(\beta) \). By commutativity of the diagram, \( \iota q(\beta \alpha \tilde{\beta}^{-1}) = \iota P_j \iota (\beta \alpha \tilde{\beta}^{-1}) \). Hence, \( q(\beta \alpha \tilde{\beta}^{-1}) \) is a loop in \( \iota P_j (\iota(\tilde{K}_j, \tilde{p}_1)) \). Thus, \( q(\beta \alpha \tilde{\beta}^{-1}) \) must lift to a loop at \( \tilde{p}_1 \). Contradiction!

**Remark 3.** The above argument also works for the 3-fold cover \( \tilde{C}_i^3 \) which will soon be defined.

Since \( \iota \) is an embedding, \( l = j, j - 2 - 4T \) and \( j - 4 - 4T \), the restriction map \( P_j| : \iota(\tilde{C}_i^3) \rightarrow C_i \) is a 2-fold cover of \( C_i \). Since \( \ker \Phi_j \) has index 60 in \( \pi_1(K_j) \), the covering space \( P_j : \tilde{K}_j \rightarrow K_j \) has 60 covering translations. The components of \( P_j^{-1}(C_i) \) are the homeomorphic images of \( \iota(\tilde{C}_i^3) \) under the 60 covering translations of \( P_j \). Thus, every component of \( P_j^{-1}(C_i) \) is a 2-fold cover of \( C_i \) (i.e., a 2-fold cover of trefoil knot). By §2, each \( K_j \) contains \( j \) pairwise disjoint cuboids with trefoil-knotted hole \( C_i \), where \( 1 \leq l \leq j \). Hence, \( \tilde{K}_j \) must have \( 15j \) (resp. \( 15(j + 1) \)) when \( j \) is even (resp. odd) pairwise disjoint 2-fold covers of trefoil knot.

Likewise, let \( q' : (\tilde{C}_i^3, \tilde{p}_1) \rightarrow (C_i, p_1) \) be a 3-fold cover of \( (C_i, p_1) \) corresponding to the kernel of \( \Phi_j \circ M_1 \circ \tau_2 \). When \( l = j - 1 - 4T \) and \( j - 3 - 4T \), the restriction map \( P_j| : \iota(\tilde{C}_i^3) \rightarrow C_i \) is a 3-fold cover of \( C_i \).

**Claim 5.** \( P_j| : \iota(\tilde{C}_i^3) \rightarrow C_i \) yields a unique 3-fold (cyclic) cover of \( C_i \).

**Proof.** Since the 60-fold covering space of \( K_j \) is clearly regular, the restriction of the covering projection to each \( C_i \) is also a regular covering. Thus, the induced map \( P_j| : \iota(\pi_1(\tilde{C}_i^3)) \rightarrow \pi_1(C_i) \) goes onto an index 3 normal subgroup \( (\mathbb{Z}_3) \). Note that \( \pi_1(\tilde{C}_i^3) \) corresponds to the kernel of the composite \( \pi_1(C_i) \xrightarrow{\text{abelianization}} \mathbb{Z} \xrightarrow{\text{projection}} \mathbb{Z}_3 \). Then the claim follows immediately from the uniqueness of the abelianization and the projection.

When \( j \) is even (resp. odd), let \( D \) be the complement of the interior of the 15\( j \) (resp. 15\( j + 1 \)) double covers and 10\( j \) (resp. 10\( (j - 1) \)) triple cover of trefoil knot in \( \tilde{K}_j \). Let \( Q_j : \tilde{K}_j \rightarrow \tilde{K}_j/D \) be quotient map. The quotient space \( \tilde{K}_j/D \) is 25\( j \) (resp. 5(5\( j + 1 \))) when \( j \) is even (resp. odd) pairwise disjoint 2-fold and 3-fold covers of trefoil knot modulo their boundaries, wedged at the point to which their boundaries are identified. By Propositions 5.6 and 5.7, \( \pi_1(\tilde{K}_j/D) \) has rank at least 25\( j \) (resp. 5(5\( j + 1 \))) when \( j \) is even (resp. odd). Then Propositions 5.3 and 5.5 assure that \( Q_j \) induces a surjection of \( \pi_1(\tilde{K}_j) \) onto \( \pi_1(\tilde{K}_j/D) \), hence, \( \text{Rank} \pi_1(\tilde{K}_j) \geq 25j \) (resp. 5(5\( j + 1 \))) when \( j \) is even (resp. odd).

This completes the proof of Theorem 1.1.

**Proof of Theorem 1.2.** Using our building block \( W^3 \), one can apply the standard “drilling tunnel” and “piping” to generate high-dimensional examples \( W^n \). We only
spell out an outline. A detailed proof described in [Ste77, P.56-62] can readily be applied.

Recall in §3 there is an arc $\mu^1_l$ connecting the base points $p_l \in \partial T_l'$ and $q_l \in \partial T_l$ (see Figure 7). The sewing homeomorphism $h^l_{i+1}$ identifies $q_l$ with $p_{l+1}$. By the construction of $W^3$, those arcs fit together to form a (base) ray $R$ in $W^3$. Then it suffices to show $\pi_1(U/p^{-1}(Int T_0^+))$ is not finitely generated just as how we prove Theorem 1.1. By definition of $N$, $T_0^+ = T_0^*$. Let $q$ be the quotient map

$$q : T_j^+ \setminus Int T_0^+ \to (T_j^+ \setminus Int T_0^+)/\partial T_j^+.$$ 

Extend $q$ to map $Q : U/p^{-1}(Int T_0^+) \to (T_j^+ \setminus Int T_0^+)/\partial T_j^+$. There should be no difficulty in doing so because $U/p^{-1}(Int T_0^+)$ can be decomposed into the union of $U/p^{-1}(Int T_j^+)$ and $p^{-1}(T_j^+ \setminus Int T_0^+)$. Then $Q$ can be defined as the union of the constant map $l : U/p^{-1}(Int T_j^+) \to (T_j^+ \setminus Int T_0^+)/\partial T_j^+$ and the restriction map $q \circ p|^{-1}(T_j^+ \setminus Int T_0^+)$. By Lemma 5.4, $q \circ p|_{^{-1}(T_j^+ \setminus Int T_0^+)}$ induces a surjection on fundamental groups, so does $Q$. Note that $(T_j^+ \setminus Int T_0^+)/\partial T_j^+$ and $(T_j^* \setminus Int T_0^+)/\partial T_j^*$ are homeomorphic. Thus, showing that $\text{Rank } \pi_1(U/p^{-1}(Int T_0^+))$ has no lower bound is equivalent to proving $\text{Rank } \pi_1((T_j^* \setminus Int T_0^+)/\partial T_j^*) = \text{Rank } \pi_1(K_j^*)$, which is just an application of Theorem 1.1.

7. Questions

Recall the construction of $W^3$ in §2

$$W^3 = \lim_{j \to \infty} L_j \cup_{h^{-1}_j} L_{j-1} \cup_{h^{-2}_j} \cdots \cup_{h^{-1}_j} L_1,$$

where the sewing homeomorphism $h^l_{i+1}$ identifies the boundary component $\partial T_l$ of $L_l$ to the boundary component $\partial T_l'_{i+1}$ of $L_{l+1}$. Unknotting the cube with trefoil-knotted hole as shown in Figure 1 results in a cobordism $L^*$, which is widely known as the first stage of constructing a Whitehead manifold. See Figure 9.

Consider a variation of $W^3$ by placing $L^*$ ahead of $L_j$ or inserting $L^*$ between adjacent $L_l$ and $L_{l+1}$ in (7.1)

$$W^* = \lim_{j \to \infty} L_j \cup_{H^*_j} L^* \cup_{H^*_{j-1}} L_{j-1} \cdots \cup_{h^*_j} L_1,$$

where the sewing homeomorphism $H^*_j$ identifies the boundary component $\partial T_l$ of $L_l$ to the boundary component $\partial T_l'$ of $L^*$ and the sewing homomorphism $H^*_{i+1}$ identifies
The “inner” boundary component of $L^*$ is $\partial T'$.
The “outer” boundary component of $L^*$ is $\partial T$.

the boundary component $\partial T$ of $L^*$ to the boundary component $\partial T'_{t+1}$ of $L_{t+1}$. Then we obtain an infinite collection $C$ by inserting $L^*$’s in (7.1).

The following result is an example of $C$.

**Proposition 7.1.** The 3-dimensional example $W$ constructed by Sternfeld belongs to the collection $C$.

**Proof.** The manifold $W$ constructed by Sternfeld is homeomorphic to $L^* \cup_{H^*_j} L_j \cup_{H^*_j} L^* \cdots$, i.e., inserting $L^*$ in (7.1) every other slot. See Figure 10. If one ignores the grey curves as shown in Figure 10, then the picture will be exactly the same picture given in [Ste77, P.4]. In other words, solid tori $T$ and $T'_{j-1}$ are the first stage of Sternfeld’s construction. □

**Remark 4.** Let $K_j$ and $K_i$ be the corresponding knot spaces of $W^3$ and $W$ respectively. Although both $W^3$ and $W$ contain a cube with a trefoil-knotted hole at each stage of the construction, the corresponding 60-fold covers of $K_j$ and $K_i$ are different. That is, the 60-fold cover of $K_j$ has both embedded 2-fold covers and embedded 3-fold covers of incompressible cube with a trefoil-knotted hole in $K_j$. However, the 60-fold cover of $K_i$ has only embedded 2-fold covers of incompressible cube with trefoil-knotted hole in $K_i$.

**Question 1.** Does $C$ contain an infinite subcollection of contractible open 3-manifolds $C'$ such that each manifold in $C'$ embeds in no compact, locally connected and locally 1-connected metric 3-space?
CONTRACTIBLE OPEN MANIFOLDS WHICH EMBED IN NO COMPACT, LC, 1-LC SPACE

Figure 10. The difference between solid torus $T$ (blue) and $T'$ (grey) is $L^*$. This $L_{j-1}$ is the area between $\partial T_{j-1}$ (which has been identified with $\partial T'$) and $\partial T_{j-1}'$.

**Question 2.** The cube with trefoil-knotted hole $C_l$ plays the key role in this paper. Let $K$ be an arbitrary (nontrivial) knot. Can $C_l$ be replaced by a cube with a $K$-knotted hole? More specifically, if we replace $C_l$ at each stage in the construction of $W^3$ by cube with a $K$-knotted hole, can the resulting contractible open manifold $W'$ embed in some compact, locally connected and locally 1-connected metric 3-space?

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