On Monotone Drawings of Trees

Philipp Kindermann
Chair of Computer Science I
Universität Würzburg

Joint work with
André Schulz, Joachim Spoerhase & Alexander Wolff
Monotone Drawings

A path is *monotone*: \( \exists \) direction \( d \) such that vertex-order in \( d \) = vertex-order along the path.
Monotone Drawings

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A graph is monotone: \( \exists \) monotone path for every vertex-pair.
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A \( u-v \) path is *strongly monotone* if \( d = \overrightarrow{uv} \).
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Monotone Drawings

A path is monotone: \( \exists \) direction \( d \) such that vertex-order in \( d = \) vertex-order along the path.

A graph is monotone: \( \exists \) monotone path for every vertex-pair. A \( u-v \) path is strongly monotone if \( d = \overrightarrow{uv} \).
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A \( u-v \) path is *strongly monotone* if \( d = \overrightarrow{uv} \).
Known Results

[Angelini et al. JGAA’12]

Any $n$-vertex tree admits a straight-line monotone drawing on a grid of size $O(n^{1.6}) \times O(n^{1.6})$ or $O(n) \times O(n^2)$.
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- Any connected outerplane graph admits a straight-line monotone drawing on a grid of size $O(n) \times O(n^2)$.
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[Angelini et al. Algorithmica’13]
Any connected outerplane graph admits a straight-line monotone drawing on a grid of size $O(n) \times O(n^2)$.

[Hossain and Rahman FAW’14]
Any connected planar graph admits a straight-line monotone drawing on a grid of size $O(n) \times O(n^2)$. 
Convex Drawings

Convex drawing: Every face is convex.
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[Carlson & Eppstein GD’06]
Can compute convex drawings of trees, with optimal angular resolution.
Our Main Tool: Primitive Vectors

Let \( P_d = \{(x, y) \mid \gcd(x, y) = 1, \ 0 \leq x \leq y \leq d\} \).
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Lemma. Any two vectors of $P_d$ are separated by an angle of $\Omega(1/|P_d|)$. 
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**Lemma.** Any two vectors of $P_d$ are separated by an angle of $\Omega(1/|P_d|)$. For $|P_d| \geq n - 1$, choose $d \approx 4\sqrt{n}$. 
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![Farey sequence diagram](image)
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Farey sequence

$F_1: \frac{0}{1}, \frac{1}{1}$

$F_2: \frac{0}{1}, \frac{1}{1}, \frac{1}{2}$

$F_3: \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}$

$F_4: \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}$

Diagram: 

$\begin{array}{c}
\frac{a}{b} \\
\frac{a}{b} \\
\frac{c}{b+d} \\
\frac{c}{d}
\end{array}$
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For \( |P_d| \geq n - 1 \), choose \( d \approx 4\sqrt{n} \).
Step I: Rank Edges
Step I: Rank Edges
Step I: Rank Edges
Step I: Rank Edges
Step I: Rank Edges
Step I: Rank Edges
Step I: Rank Edges
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Step I: Rank Edges
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Step I: Rank Edges
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Step I: Rank Edges
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Step II: Primitive Vectors
Step I: Rank Edges

Step II: Primitive Vectors
Step I: Rank Edges

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Step I: Rank Edges

Step II: Primitive Vectors

Step III: Draw Tree
Step I: Rank Edges

Step II: Primitive Vectors

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Step III: Draw Tree
Step I: Rank Edges

Step II: Primitive Vectors

Step III: Draw Tree

Theorem.
Every tree has a monotone and convex drawing on a grid of size $O(n^{1.5}) \times O(n^{1.5})$. 
Strongly Monotone Drawings

*Proper Binary Trees*: No degree-2 vertex
Strongly Monotone Drawings

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dummy
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All angles $< \pi \Rightarrow$ strictly convex ✓
Strongly Monotone Drawings

Proper Binary Trees: No degree-2 vertex

All angles $< \pi \Rightarrow$ strictly convex

Strongly monotone?
Observation.
A $u$-$v$-path is not strongly monotone $\iff \exists$ an edge $e$ with $\angle(\vec{e}, \vec{u}v) > \pi/2$. 
Properties

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A $u$-$v$-path is *not* strongly monotone $\iff \exists$ an edge $e$ with $\angle(\vec{e}, \vec{uv}) > \pi/2$. 
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\[ d > \pi/2 \]
Properties

**Observation.**
A $u$-$v$-path is *not* strongly monotone
$\iff \exists$ an edge $e$ with $\angle(\vec{e}, \vec{uv}) > \pi/2$.

**Lemma.**
If a path is monotone to $\vec{v}_1$ and $\vec{v}_2$,
then it is monotone to $\vec{v}_3$ between $\vec{v}_1$ and $\vec{v}_2$. 
Observation.
A $u$-$v$-path is *not* strongly monotone
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Lemma.
If a path is monotone to $\vec{v}_1$ and $\vec{v}_2$,
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**Observation.**
A $u$-$v$-path is *not* strongly monotone $\iff \exists$ an edge $e$ with $\angle(\vec{e}, \vec{u} \vec{v}) > \pi/2$.

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If a path is monotone to $\vec{v}_1$ and $\vec{v}_2$, then it is monotone to $\vec{v}_3$ *between* $\vec{v}_1$ and $\vec{v}_2$. 
Properties

**Observation.**
A $u$-$v$-path is *not* strongly monotone if and only if there exists an edge $e$ with $\angle(\vec{e}, \vec{u} \vec{v}) > \pi/2$.

**Lemma.**
If a path is monotone to $\vec{v}_1$ and $\vec{v}_2$, then it is monotone to $\vec{v}_3$ between $\vec{v}_1$ and $\vec{v}_2$. 
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A $u$-$v$-path is not strongly monotone if there exists an edge $e$ with $\angle(\vec{e}, \vec{uv}) > \pi/2$.

Lemma.
If a path is monotone to $\vec{v}_1$ and $\vec{v}_2$, then it is monotone to $\vec{v}_3$ between $\vec{v}_1$ and $\vec{v}_2$. 

Diagram: 
- $\vec{v}_1$ to $\vec{v}_2$ with $\vec{v}_3$ and $\vec{p}$ in between.
Proper Binary Trees

Proper Binary Trees: No degree-2 vertex

- All angles $< \pi \Rightarrow$ strictly convex ✓
- Strongly Monotone?
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Proper Binary Trees

Proper Binary Trees: No degree-2 vertex
- All angles $< \pi \Rightarrow$ strictly convex ✓
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W.l.o.g. assume $a$ lies bottom-left
Proper Binary Trees

Proper Binary Trees: No degree-2 vertex

- All angles $< \pi \Rightarrow$ strictly convex ✓
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W.l.o.g. assume $a$ lies bottom-left

Case 1: $a$ and $b$ on common root-leaf path
Proper Binary Trees

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W.l.o.g. assume a lies bottom-left

Case 1: $a$ and $b$ on common root-leaf path

$\Rightarrow \vec{ba} \in \mathbb{R}^2$
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**Case 1:** $a$ and $b$ on common root-leaf path

$\implies \overrightarrow{ba} \in \downarrow$, path from $b$ to $a \in \downarrow$
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$\Rightarrow \overrightarrow{ba} \in \nabla$, path from $b$ to $a \in \nabla$

$\Rightarrow$ path from $b$ to $a$ is strongly monotone
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$\Rightarrow \overrightarrow{ba} \in \leftarrow$, path from $b$ to $a \in \leftarrow$

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W.l.o.g. assume a lies bottom-left

Case 1: a and b on common root-leaf path ✓

Case 2: a and b in opposite sectors
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W.l.o.g. assume $a$ lies bottom-left

Case 1: $a$ and $b$ on common root-leaf path ✔

Case 2: $a$ and $b$ in opposite sectors ✔

$\Rightarrow \vec{ba} \in \text{sector}$, path from $b$ to $a \in \text{sector}$

$\Rightarrow$ path from $b$ to $a$ is strongly monotone
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Case 2: $a$ and $b$ in opposite sectors ✓

Case 3: else
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**Case 2:** $a$ and $b$ in opposite sectors ✓

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**Case 1**: $a$ and $b$ on common root-leaf path ✓

**Case 2**: $a$ and $b$ in opposite sectors ✓

**Case 3**: else

\[ a \rightarrow \rightarrow d \rightarrow \text{path monotone to } A \]
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W.l.o.g. assume $a$ lies bottom-left

Case 1: $a$ and $b$ on common root-leaf path

Case 2: $a$ and $b$ in opposite sectors

Case 3: else
   
   $a$-$d$-path monotone to $A$
   
   $d$-$b$-path monotone to $A$
Proper Binary Trees

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W.l.o.g. assume $a$ lies bottom-left

Case 1: $a$ and $b$ on common root-leaf path ✓

Case 2: $a$ and $b$ in opposite sectors ✓

Case 3: else

$a$-$d$-path monotone to $A$
$d$-$b$-path monotone to $A$
$\Rightarrow$ $a$-$b$-path monotone to $A$
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- $a$-$b$-path monotone to $A$
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Case 1: $a$ and $b$ on common root-leaf path ✓

Case 2: $a$ and $b$ in opposite sectors ✓

Case 3: else

- $a$-$b$-path monotone to $A$
- $a$-$b$-path monotone to $B$
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- $a$-$b$-path monotone to $A$
- $a$-$b$-path monotone to $B$
Proper Binary Trees

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- All angles \(< \pi \Rightarrow \) strictly convex ✓
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W.l.o.g. assume \(a\) lies bottom-left

Case 1: \(a\) and \(b\) on common root-leaf path ✓

Case 2: \(a\) and \(b\) in opposite sectors ✓

Case 3: else

\(a\)-\(b\)-path monotone to \(A\)
\(a\)-\(b\)-path monotone to \(B\)
\(a\)-\(b\)-path strongly monotone
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Case 1: $a$ and $b$ on common root-leaf path ✓

Case 2: $a$ and $b$ in opposite sectors ✓

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W.l.o.g. assume $a$ lies bottom-left

**Case 1:** $a$ and $b$ on common root-leaf path ✓

**Case 2:** $a$ and $b$ in opposite sectors ✓

**Case 3:** else ✓

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**Theorem.**

Any proper binary tree has a strongly monotone and strictly convex drawing.
General Trees

$w_k \quad w_i \quad w_1$
1. Substitute high-degree vertices by paths
General Trees

1. Substitute high-degree vertices by paths
2. Draw Proper Binary Tree
General Trees

1. Substitute high-degree vertices by paths
2. Draw Proper Binary Tree
3. Shortcut edges
General Trees

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General Trees

1. Substitute high-degree vertices by paths
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**Theorem.**
Any tree has a strongly monotone drawing.
**Theorem.**
Any biconnected outerplanar graph has a strongly monotone and strictly convex drawing.
Planar Graphs

**Theorem.**
Any biconnected outerplanar graph has a strongly monotone and strictly convex drawing.

**Theorem.**
There is an infinite family of connected planar graphs that do not have a strongly monotone drawing in any combinatorial embedding.
Does any tree have a strongly monotone drawing on a grid of polynomial size?
Open Problems

- Does any tree have a strongly monotone drawing on a grid of polynomial size?

- Is there a triconnected (or biconnected) planar graph that does not have any strongly monotone drawing?
Open Problems

- Does any tree have a strongly monotone drawing on a grid of polynomial size?

- Is there a triconnected (or biconnected) planar graph that does not have any strongly monotone drawing? If yes, can this be tested efficiently?
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- Are our drawings for general trees also convex?
Open Problems

Does any tree have a strongly monotone drawing on a grid of polynomial size?

Is there a triconnected (or biconnected) planar graph that does not have any strongly monotone drawing? If yes, can this be tested efficiently?

Are our drawings for general trees also convex? If yes, then all Halin graphs would automatically have convex and strictly monotone drawings, too.