UNIFIED PRODUCTS AND SPLIT EXTENSIONS OF HOPF ALGEBRAS

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Abstract. The unified product was defined in [1] related to the restricted extending structure problem for Hopf algebras: a Hopf algebra $E$ factorizes through a Hopf subalgebra $A$ and a subcoalgebra $H$ such that $1 \in H$ if and only if $E$ is isomorphic to a unified product $A \ltimes H$. Using the concept of normality of a morphism of coalgebras in the sense of Andruskiewitsch and Devoto we prove an equivalent description for the unified product from the point of view of split morphisms of Hopf algebras. A Hopf algebra $E$ is isomorphic to a unified product $A \ltimes H$ if and only if there exists a morphism of Hopf algebras $i : A \to E$ which has a retraction $\pi : E \to A$ that is a normal left $A$-module coalgebra morphism. A necessary and sufficient condition for the canonical morphism $i : A \to A \ltimes H$ to be a split monomorphism of bialgebras is proved, i.e. a condition for the unified product $A \ltimes H$ to be isomorphic to a Radford biproduct $L \ast A$, for some bialgebra $L$ in the category $A \YD$ of Yetter-Drinfel’d modules. As a consequence, we present a general method to construct unified products arising from an unitary not necessarily associative bialgebra $H$ that is a right $A$-module coalgebra and a unitary coalgebra map $\gamma : H \to A$ satisfying four compatibility conditions. Such an example is worked out in detail for a group $G$, a pointed right $G$-set $(X, \cdot, \triangleright)$ and a map $\gamma : G \to X$.

Introduction

A morphism $i : C \to D$ in a category $C$ is a split monomorphism if there exists $p : D \to C$ a morphism in $C$ such that $p \circ i = \text{Id}_C$. Fundamental constructions like the semidirect product of groups or Lie algebras, the smash product of Hopf algebras, Radford’s biproduct, etc can be viewed as tools to answer the following problem: describe, when is possible, split monomorphisms in a given category $C$. The basic example is the following: a group $E$ is isomorphic to a semidirect product $G \ltimes A$ of groups if and only if there exists $i : A \to E$ a split monomorphism of groups. In this case $E \cong G \ltimes A$, where $G = \text{Ker}(p)$, for a splitting morphism $p : E \to A$. The generalization of this elementary result at the level of Hopf algebras was done in two steps. The first one was made by Molnar in [3] Theorem 4.1]: if $i : A \to E$ is a split monomorphism of Hopf algebras having a splitting map $p : E \to A$ which is a normal morphism of Hopf algebras then $E$ is isomorphic as a Hopf algebra to a smash product $G \ast A$ of Hopf algebras, where $G := \text{KER}(p) = \{ x \in E \mid x(1) \otimes p(x(2)) \otimes x(3) = x(1) \otimes 1_A \otimes x(2) \}$, is the kernel of $p$ in the

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category of Hopf algebras (the original statement of Molnar’s theorem, as well as the one of Radford below, is restated and updated according to the present development of the theory). The general case was done by Radford in [9, Theorem 3]: if \( i : A \to E \) is a split monomorphism of Hopf algebras then \( E \) is isomorphic as a Hopf algebra to a biproduct \( G \rtimes A \), for a bialgebra \( G \) in the pre-braided category \( \mathcal{YD} \) of left-left Yetter-Drinfel’d modules. The price paid for leaving aside the normality assumption of the splitting map is reasonable: the algebra structure of the biproduct \( G \rtimes A \) is the one of the smash product algebras and the coalgebra structure of \( G \rtimes A \) is given by the smash coproduct of coalgebras. In the last decade, the biproduct (also named bosonization) was intensively used in the classification of finite dimensional pointed Hopf algebras (see [3] and the references therein).

The great impact generated by Radford’s theorem made his result the subject of many generalizations and recent developments ([4], [5], [7], [10]). The first step was made by Schauenburg, who considered the splitting map \( p \) to be only a coalgebra map. Although it is formulated in a different manner, [10, Theorem 5.1] proves that if a morphism of Hopf algebras \( i : A \to E \) has a retraction \( \pi : E \to A \) which is a left \( A \)-module coalgebra morphism then \( E \) is isomorphic to a new product \( G \Leftrightarrow A \), that generalizes the biproduct, in the construction of which \( G \) and \( A \) are connected by four maps satisfying seventeen compatibility conditions. As a vector space \( G \Leftrightarrow A = G \otimes A \) with the multiplication given by a very laborious formula. The special case in which the splitting map \( p \) is a coalgebra \( A \)-bimodule map was highlighted briefly in [10, Section 6.1] and in full detail in [6, Theorem 3.64].

The unified product was constructed in [1] from a completely different reason, related to what we have called the extending structures problem. Given a Hopf algebra \( A \) and a coalgebra \( E \) such that \( A \) is a subcoalgebra of \( E \), the extending structures problem asks for the description and classification of all Hopf algebra structures that can be defined on the coalgebra \( E \) such that \( A \) becomes a Hopf subalgebra of \( E \). The unified product is the tool for solving the restricted version of the ES-problem: a Hopf algebra \( E \) factorizes through a Hopf subalgebra \( A \) and a subcoalgebra \( H \) such that \( 1 \in H \) if and only if \( E \) is isomorphic to a unified product \( A \ltimes H \). In the construction of a unified product \( A \ltimes H \) the Hopf algebra \( A \) and the coalgebra \( H \) are connected by three coalgebra maps: two actions \( \triangleleft : H \otimes A \to H \), \( \triangleright : H \otimes A \to A \) and a generalized cocycle \( f : H \otimes H \to A \) satisfying seven compatibility conditions.

In this paper we shall prove an equivalent description for the unified product from the view point of split extensions of Hopf algebras. Let \( A \ltimes H \) be a unified product associated to a bialgebra extending structure \( \Omega(A) = (H, \triangleleft, \triangleright, f) \) of a Hopf algebra \( A \) (see Section [1]). Then we have an extension of bialgebras \( i_A : A \to A \ltimes H \). This extension is split in the sense of [10]: there exists \( \pi_A : A \ltimes H \to A \) a left \( A \)-module coalgebra morphism such that \( \pi_A \circ i_A = \text{Id}_A \). Thus the unified product \( A \ltimes H \) appears as a special case of the Schauenburg’s product \( A \rtimes H \) but is a much more malleable version of it. Furthermore, there is more to be said and this makes a substantial difference: \( \pi_A \) is also a normal morphism of coalgebras in the sense of Andruskiewitsch and Devoto [2]. This context fully characterizes unified products: we prove that a Hopf algebra \( E \) is isomorphic to a unified product \( A \ltimes H \) if and only if there exists a morphism of Hopf algebras
i : A → E which has a retraction π : E → A that is a normal left A-module coalgebra morphism (Theorem 2.3). The proof of Theorem 2.3 is different from the one in [10] and is based on the factorization theorem [1, Theorem 2.7]. As a bonus, a simplified and more transparent version of [8, Theorem 4.1] is obtained in Corollary 2.5: if π : E → A is a normal split epimorphism of Hopf algebras, then E ∼= A#H, where A#H is the right version of the smash product of bialgebras for some right A-module bialgebra H.

The next aim of the paper is to make the connection between the unified product and Radford’s biproduct: for a Hopf algebra A, Proposition 2.6 gives necessary and sufficient conditions for i_A : A → A ⊗ H to be a split monomorphism of bialgebras. In this case the unified product A ⊗ H is isomorphic as a bialgebra to a biproduct L * A, and the structure of L as a bialgebra in the category $^H\mathcal{YD}$ of Yetter-Drinfel’d modules is explicitly described. Furthermore, in this context a new equivalent description of the unified product A ⊗ H, as well as of the associated biproduct L * A, is given in Proposition 2.7: both of them are isomorphic to a new product A ⊗ H, which is a deformation of the smash product of bialgebras A#H using a coalgebra map γ : H → A. Finally, Theorem 2.8 gives a general method for constructing unified products, as well as bidecompositions arising from a right A-module coalgebra (H, <·>) and a unitary coalgebra map γ : H → A. Example 2.9 gives an explicit example of such a product starting with a group G, a pointed right G-set (X, ·, <·>) and a map γ : G → X satisfying two compatibility conditions.

1. Preliminaries

Throughout this paper, k will be a field. Unless specified otherwise, all modules, algebras, coalgebras, bialgebras, tensor products, homomorphisms and so on are over k. For a coalgebra C, we use Sweedler’s Σ-notation: Δ(c) = c_{(1)} ⊗ c_{(2)}, (I ⊗ Δ)Δ(c) = c_{(1)} ⊗ c_{(2)} ⊗ c_{(3)}, etc (summation understood). We also use the Sweedler notation for left C-comodules: ρ(m) = m_{<−1>} ⊗ m_{<0>}, for any m ∈ M if (M, ρ) ∈ C M is a left C-comodule. Let A be a bialgebra and H an algebra and a coalgebra. A k-linear map f : H ⊗ H → A will be denoted by f(g, h) = f(g ⊗ h); f is the trivial map if f(g, h) = ε_H(g)ε_H(h)1_A, for all g, h ∈ H. Similarly, the k-linear maps <· : H ⊗ A → H, > : H ⊗ A → A are the trivial actions if h · a = ε_A(a)h and respectively h > a = ε_H(h)a, for all a ∈ A and h ∈ H.

For a Hopf algebra A we denote by $^A\mathcal{M}$ the category of left-left A-Hopf modules: the objects are triples (M, ·, ρ), where (M, ·) ∈ A M is a left A-module, (M, ρ) ∈ $^A\mathcal{M}$ is a left A-comodule such that
\[ ρ(a · m) = a_{(1)}m_{<−1>} ⊗ a_{(2)} · m_{<0>} \]
for all a ∈ A and m ∈ M. For (M, ·, ρ) ∈ $^A\mathcal{M}$ we denote by $M^{co(A)} = \{ m ∈ M | ρ(m) = 1_A ⊗ m \}$ the subspace of coinvariants. The fundamental theorem for Hopf modules states that for any A-Hopf module M the canonical map
\[ ϕ : A ⊗ M^{co(A)} → M, \quad ϕ(a ⊗ m) := a · m \]
for all a ∈ A and m ∈ M is bijective with the inverse given by
\[ ϕ^{-1} : M → A ⊗ M^{co(A)}, \quad ϕ^{-1}(m) := m_{<−2>} ⊗ S(m_{<−1>}) · m_{<0>} \]
for all \( m \in M \).

\( A \mathcal{Y} D \) will denote the pre-braided monoidal category of left-left Yetter-Drinfel’d modules over \( A \): the objects are triples \( (M, \cdot, \rho) \), where \( (M, \cdot) \in A \mathcal{M} \) is a left \( A \)-module, \( (M, \rho) \in A \mathcal{M} \) is a left \( A \)-comodule such that

\[
\rho(a \cdot m) = a_{(1)}m_{<-1>}S(a_{(3)}) \otimes a_{(2)} \cdot m_{<0>}
\]

for all \( a \in A \) and \( m \in M \). If \((L, \cdot, \rho)\) is a bialgebra in the monoidal category \( A \mathcal{Y} D \) then the Radford biproduct \( L * A \) is the vector space \( L \otimes A \) with the bialgebra structure given by

\[
(l * a)(m * b) := l(a_{(1)} \cdot m) * a_{(2)} b
\]

\[
\Delta(l * a) := l_{(1)} * l_{(2)} < -1 > a_{(1)} \otimes l_{(2)} < 0 > * a_{(2)}
\]

for all \( l, m \in L \) and \( a, b \in A \), where we denoted \( l \otimes a \in L \otimes A \) by \( l * a \). \( L * A \) is a bialgebra with the unit \( 1_L * 1_A \) and the counit \( \varepsilon_{L * A}(l * a) = \varepsilon_L(l)\varepsilon_A(a) \), for all \( l \in L \) and \( a \in A \).

We recall from [1] the construction of the unified product. An extending datum of a bialgebra \( A \) is a system \( \Omega(A) = (H, \triangleleft, 
\triangleright, f) \), where \( H = (H, \Delta_H, \varepsilon_H, 1_H, \cdot) \) is a \( k \)-module such that \((H, \Delta_H, \varepsilon_H)\) is a coalgebra, \((H, 1_H, \cdot)\) is an unitary not necessarily associative \( k \)-algebra, the \( k \)-linear maps \( \triangleleft : H \otimes A \rightarrow H, \triangleright : H \otimes A \rightarrow A, f : H \otimes H \rightarrow A \) are coalgebra maps such that the following normalization conditions hold:

\[
\begin{align*}
    h \triangleright 1_A &= \varepsilon_H(h)1_A, & 1_H \triangleright a &= a, & 1_H \triangleleft a &= \varepsilon_A(a)1_H, & h \triangleleft 1_A &= h \\
    \Delta_H(1_H) &= 1_H \otimes 1_H, & f(h, 1_H) &= f(1_H, h) &= \varepsilon_H(h)1_A
\end{align*}
\]

for all \( h \in H \), \( a \in A \).

Let \( \Omega(A) = (H, \triangleleft, \triangleright, f) \) be an extending datum of \( A \). We denote by \( A \ltimes \Omega(A) H = A \ltimes H \) the \( k \)-module \( A \otimes H \) together with the multiplication:

\[
(a \ltimes h) \cdot (c \ltimes g) := a(h_{(1)} \triangleleft c_{(1)})f(h_{(2)} \triangleleft c_{(2)}, g_{(1)}) \ltimes (h_{(3)} \triangleleft c_{(3)}) \cdot g_{(2)}
\]

for all \( a, c \in A \) and \( h, g \in H \), where we denoted \( a \otimes h \in A \otimes H \) by \( a \ltimes h \). The object \( A \ltimes H \) is called the unified product of \( A \) and \( \Omega(A) \) if \( A \ltimes H \) is a bialgebra with the multiplication given by (3), the unit \( 1_A \ltimes 1_H \) and the coalgebra structure given by the tensor product of coalgebras, i.e.:

\[
\begin{align*}
    \Delta_{A \ltimes H}(a \ltimes h) &= a_{(1)} \ltimes h_{(1)} \otimes a_{(2)} \ltimes h_{(2)} \\
    \varepsilon_{A \ltimes H}(a \ltimes h) &= \varepsilon_A(a)\varepsilon_H(h)
\end{align*}
\]

for all \( h \in H \), \( a \in A \). We have proved in [1] Theorem 2.4] that \( A \ltimes H \) is an unified product if and only if \( \Delta_H : H \rightarrow H \otimes H \) and \( \varepsilon_H : H \rightarrow k \) are \( k \)-algebra maps, \((H, \triangleleft)\) is a right \( A \)-module structure and the following compatibilities hold:

\[
(\text{BE1}) \quad (g \cdot h) \cdot l = (g \triangleleft f(h_{(1)}, l_{(1)})) \cdot (h_{(2)} \cdot l_{(2)})
\]

\[
(\text{BE2}) \quad g \triangleright (ab) = (g_{(1)} \triangleright a_{(1)})[(g_{(2)} \triangleleft a_{(2)}) \triangleright b]
\]

\[
(\text{BE3}) \quad (g \cdot h) \triangleleft a = [g \triangleleft (h_{(1)} \triangleright a_{(1)})] \cdot (h_{(2)} \triangleleft a_{(2)})
\]
If one of the components $f$ or $\prec$ of a bialgebra extending structure $(H, \prec, \triangleright, f)$ is trivial, the unified product contains as special cases the bicrossed product of bialgebras or the crossed product. If $\triangleright$ is the trivial action, then a new product that will be used later on appears as a special case of the unified product.

**Examples 1.1.** (1) Let $A$ be a bialgebra and $\Omega(A) = (H, \prec, \triangleright, f)$ an extending datum of $A$ such that the action $\triangleright$ is trivial, that is $h \triangleright a = \varepsilon_H(h)a$, for all $h \in H$ and $a \in A$. Then $\Omega(A) = (H, \prec, \triangleright, f)$ is a bialgebra extending structure of $A$ and the following compatibilities hold:

\[(g \cdot h) \cdot l = (g \prec f(h(1), l_{(1)})) \cdot (h(2) \cdot l_{(2)})\]

\[a(1) f(g \prec a(2), h \prec a(3)) = f(g, h)a\]

\[f(h(1), l_{(1)}) f(g \prec f(h(2), l_{(2)}), h(3) \cdot l_{(3)}) = f(g(1), h(1)) f(g(2) \cdot h(2), l)\]

\[g \prec a(1) \otimes a(2) = g \prec a(2) \otimes a(1)\]

\[g(1) \cdot h(1) \otimes f(g(2), h(2)) = g(2) \cdot h(2) \otimes f(g(1), h(1))\]

for all $g, h, l \in H$ and $a \in A$. The unified product $A \otimes H$ associated to the bialgebra extending structure $\Omega(A) = (H, \prec, \triangleright, f)$ with $\triangleright$ the trivial action will be denoted by $A \hat{\otimes} H$ and will be called the **twisted product** of $A$ and $\Omega(A)$. In this bialgebra extending datum we shall omit the action $\triangleright$ and denote it by $\Omega(A) = (H, \prec, f)$. The bialgebra extending structure associated to $\Omega(A) = (H, \prec, f)$ will be called **twisted bialgebra extending structure** of $A$. Thus $A \hat{\otimes} H = A \otimes H$ as a $k$-module with the multiplication given by:

\[(a \hat{\otimes} h) \cdot (c \hat{\otimes} g) := ac(1) f(h(1) \prec c(2), g(1)) \hat{\otimes} (h(2) \prec c(3)) \cdot g(2)\]

for all $a, c \in A$, $h$ and $g \in H$, where we denoted $a \otimes h \in A \otimes H$ by $a \hat{\otimes} h$. The twisted product $A \hat{\otimes} H$ is a bialgebra with the tensor product of coalgebras.

(2) The right version of the smash product of Hopf algebras as defined in [8] is a special case of the twisted product $A \hat{\otimes} H$, corresponding to the trivial cocycle $f$. We explain this briefly in what follows. Let $\Omega(A) = (H, \prec, \triangleright, f)$ be an extending datum of a bialgebra $A$ such that the action $\triangleright$ and the cocycle $f$ are trivial. Then $\Omega(A) = (H, \prec, \triangleright, f)$ is
a bialgebra extending structure of $A$ if and only if $H$ is a bialgebra, $(H, \triangleleft)$ is a right $A$-module bialgebra such that

$$g \triangleleft a_{(1)} \otimes a_{(2)} = g \triangleleft a_{(2)} \otimes a_{(1)}$$

for all $g \in H$ and $a \in A$. In this case, the associated unified product $A \ltimes H = A \# H$, is the right version of the smash product of bialgebras introduced in [8] in the cocommutative case. Thus $A \# H = A \otimes H$ as a $k$-module with the multiplication given by:

$$(a \# h) \cdot (c \# g) := ac_{(1)} \# (h_{(2)} \triangleleft c_{(2)})g_{(2)}$$

for all $a, c \in A$, $h$ and $g \in H$, where we denoted $a \otimes h \in A \otimes H$ by $a \# h$.

For a future use we shall recall the main characterization of a unified product given in [1, Theorem 2.7].

**Theorem 1.2.** Let $E$ be a bialgebra, $A \subseteq E$ a subbialgebra, $H \subseteq E$ a subcoalgebra such that $1_E \in H$ and the multiplication map $u : A \otimes H \to E$, $u(a \otimes h) = ah$, for all $a \in A$, $h \in H$ is bijective. Then, there exists $\Omega(A) = (H, \triangleleft, \triangleright, f)$ a bialgebra extending structure of $A$ such that $u : A \ltimes H \to E$, $u(a \ltimes h) = ah$, for all $a \in A$ and $h \in H$ is an isomorphism of bialgebras.

Explicitly, the actions, the cocycle and the multiplication of $\Omega(A)$ are given by the formulas:

$$\triangleright : H \otimes A \to A, \quad \triangleright := (Id \otimes \varepsilon_H) \circ \mu$$

$$\triangleleft : H \otimes A \to H, \quad \triangleleft := (\varepsilon_A \otimes Id) \circ \mu$$

$$f : H \otimes H \to A, \quad f := (Id \otimes \varepsilon_H) \circ \nu$$

$$\cdot : H \otimes H \to H, \quad \cdot := (\varepsilon_A \otimes Id) \circ \nu$$

where

$$\mu : H \otimes A \to A \otimes H, \quad \mu(h \otimes a) := u^{-1}(ha)$$

$$\nu : H \otimes H \to A \otimes H, \quad \nu(h \otimes g) := u^{-1}(hg)$$

for all $h, g \in H$ and $a \in A$.

2. **Unifying products versus split extensions of Hopf algebras**

For any bialgebra map $i : A \to E$, $E$ will be viewed as a left $A$-module via $i$, that is $a \cdot x = i(a)x$, for all $a \in A$ and $x \in E$. We shall adopt a definition from [2] due in the context of Hopf algebra maps.

**Definition 2.1.** Let $A$ and $E$ be two bialgebras. A coalgebra map $\pi : E \to A$ is called normal if the space

$$\{x \in E | \pi(x_{(1)}) \otimes x_{(2)} = 1_A \otimes x\}$$

is a subcoalgebra of $E$.

Let $G$ be a finite group, $H \leq G$ a subgroup of $G$ and $k[G]^*$ the Hopf algebra of functions on $G$. Then the restriction morphism $k[G]^* \to k[H]^*$ is a normal morphism if and only if $H$ is a normal subgroup of $G$ ([2]).

The main properties of the bialgebra extension $A \subset A \ltimes H$ are given by the following:
Proposition 2.2. Let $A$ be a bialgebra, $\Omega(A) = (H, k, \cdot, f)$ a bialgebra extending structure of $A$ and the $k$-linear maps:
$$i_A : A \rightarrow A \times H, \quad i_A(a) = a \ltimes 1_H, \quad \pi_A : A \times H \rightarrow A, \quad \pi_A(a \ltimes h) = \varepsilon_H(h)a$$
for all $a \in A, h \in H$. Then:

1. $i_A$ is a bialgebra map, $\pi_A$ is a normal left $A$-module coalgebra morphism and $\pi_A \circ i_A = \text{Id}_A$.
2. $\pi_A$ is a right $A$-module map if and only if $\triangleright$ is the trivial action.
3. $\pi_A$ is a bialgebra map if and only if $\triangleright$ and $f$ are the trivial maps, i.e. the unified product $A \ltimes H = A\# H$, the right version of the smash product of bialgebras.

Proof. (1) The fact that $i_A$ is a bialgebra map and $\pi_A$ is a coalgebra map is straightforward. We show that $\pi_A$ is normal. Indeed, the subspace
$$\left\{ \sum_i a_i \ltimes h_i \in A \ltimes H \mid \sum_i a_{i(1)} \varepsilon_H(h_{i(1)}) \otimes a_{i(2)} \ltimes h_{i(2)} = \sum_i 1_A \otimes a_i \ltimes h_i \right\}$$
is a subcoalgebra in $A \ltimes H$ since it can be identified with $1_A \ltimes H$: more precisely, applying $\varepsilon_A$ on the second position in the equality from the above subspace we obtain that $\sum_i a_i \ltimes h_i = 1_A \ltimes \sum_i \varepsilon_A(a_i) h_i \in 1_A \ltimes H$, as needed. On the other hand
$$\pi_A(a \cdot (c \ltimes h)) = \pi_A(i_A(a) \cdot (c \ltimes h)) = \pi_A(ac \ltimes h) = ac \varepsilon_H(h) = a \pi_A(c \ltimes h)$$
for all $a, c \in A$ and $h \in H$, i.e. $\pi_A$ is a left $A$-module map.

(2) and (3) are proven by a straightforward computation. \qed

The next Theorem gives the converse of Proposition 2.2 (1) and the generalization of [8, Theorem 4.1]. Our proof is based on the factorization Theorem 1.2 and the fundamental theorem of Hopf-modules.

Theorem 2.3. Let $i : A \rightarrow E$ be a Hopf algebra morphism such that there exists $\pi : E \rightarrow A$ a normal left $A$-module coalgebra morphism for which $\pi \circ i = \text{Id}_A$. Let
$$H := \left\{ x \in E \mid \pi(x_{(1)}) \otimes x_{(2)} = 1_A \otimes x \right\}$$
Then there exists a bialgebra extending structure $\Omega(A) = (H, k, \cdot, f)$ of $A$, where the multiplication on $H$, the actions $\triangleright$, $\triangleleft$ and the cocycle $f$ are given by the formulas:

$$h \cdot g := i\left(S_A(\pi(h_{(1)}g_{(1)}))g_{(2)}\right), \quad f(h, g) := \pi(hg)$$
$$h \triangleleft a := i\left(S_A(\pi(h_{(1)}i(a_{(1)})))\right)a_{(2)}, \quad h \triangleright a := \pi(hi(a))$$

for all $h, g \in H, a \in A$ such that
$$\varphi : A \ltimes H \rightarrow E, \quad \varphi(a \ltimes h) = i(a)h$$
for all $a \in A$ and $h \in H$ is an isomorphism of Hopf algebras.

Proof. There are two possible ways to prove the above theorem: the first one is to give a standard proof in the way this type of theorems are usually proved in Hopf algebra theory, by showing that all compatibility conditions $(BE1) - (BE7)$ hold and then to prove that $\varphi$ is an isomorphism of Hopf algebras. We prefer however a more direct
approach which relies on Theorem 1.2, it has the advantage of making more transparent to the reader the way we obtained the formulas which define the bialgebra extending structure $\Omega(A)$. First we observe that $1_E \in H$ since $\pi(1_E) = \pi(i(1_A)) = 1_A$ and $E$ is a subcoalgebra of $E$ as $\pi$ is normal. $E$ has a structure of left-left $A$-Hopf module via the left $A$-action and the left $A$-coaction given by

$$a \cdot x := i(a)x, \quad \rho(x) = x_{< -1>} \otimes x_{<0>} := \pi(x_{(1)}) \otimes x_{(2)}$$

for all $a \in A$ and $x \in E$. Indeed let us prove the Hopf module compatibility condition:

$$\rho(a \cdot x) = \rho(i(a)x) = \pi(i(a_1)x_{(1)}) \otimes i(a_2)x_{(2)} = \pi(a_1)x_{(1)} \otimes i(a_2)x_{(2)} = a_1x_{< -1>} \otimes a_2 \cdot x_{<0>}$$

for all $a \in A$ and $x \in E$, i.e. $E$ is a left-left $A$-Hopf module. We also note that $H = E^{co(A)}$. It follows from the fundamental theorem of Hopf modules that the map

$$\varphi : A \otimes H \rightarrow E, \quad \varphi(a \otimes x) := a \cdot x = i(a)x$$

for all $a \in A$ and $h \in H$ is an isomorphism of vector spaces with the inverse given by

$$\varphi^{-1}(x) := x_{< -2>} \otimes S_A(x_{< -1>}) \cdot x_{<0>} = \pi(x_{(1)}) \otimes i\left(S_A(\pi(x_{(2)}))\right)x_{(3)} \tag{12}$$

for all $x \in E$. Thus $E$ is a Hopf algebra that factorizes through $i(A) \cong A$ and the subcoalgebra $H$. Now, from Theorem 1.2 we obtain that there exists a bialgebra extending structure $\Omega(A) = (H, \langle, >, f\rangle)$ of $A$ such that $\varphi : A \triangleright H \rightarrow E$, $\varphi(a \triangleright h) = i(a)h$, for all $a \in A$ and $h \in H$ is an isomorphism of Hopf algebras. Using (12) the actions $\triangleright, \langle$ given by the formulas (5), (9) take the explicit form:

$$h \triangleright a = (Id_A \otimes \varepsilon_H)\varphi^{-1}(hi(a))$$

$$= \pi(h_{(1)}i(a_{(1)}))\varepsilon_H(h_{(2)})\varepsilon_A(a_{(2)})\varepsilon_H(h_{(3)})\varepsilon_A(a_{(3)}) = \pi(hi(a))$$

and

$$h \triangleleft a = (\varepsilon_A \otimes Id_E)\varphi^{-1}(hi(a))$$

$$= i\left(S_A(\pi(h_{(1)}i(a_{(1)})))\right)h_{(2)}i(a_{(2)})$$

for all $h \in H$ and $a \in A$. Finally, using once again (12), the multiplication $\cdot$ on $H$ and the cocycle $f$ given by (10), (11) take the form

$$h \cdot g := i\left(S_A(\pi(h_{(1)}g_{(1)}))\right)h_{(2)}g_{(2)}, \quad f(h, g) := \pi(hg)$$

for all $h, g \in H$. The proof is now finished by Theorem 1.2. \hfill \Box

The next corollary covers the case in which the splitting map $\pi$ is also a right $A$-module map. The version of Corollary 2.3 below in which the normality condition of the splitting map $\pi$ is dropped was proved in [6 Theorem 3.64]. In this case, the input data of the construction of the product that is used is called a dual Yetter-Drinfel’d quadruple [6 Definition 3.59], and consists of a system of objects and maps satisfying eleven compatibility conditions.
Corollary 2.4. Let \( i : A \to E \) be a Hopf algebra morphism such that there exists \( \pi : E \to A \) a normal \( A \)-bimodule coalgebra morphism such that \( \pi \circ i = \text{Id}_A \). Then there exists \( \Omega(A) = (H, \cdot, \circ, f) \) a twisted bialgebra extending structure of \( A \) such that
\[
\varphi : A \hat{\otimes} H \to E, \quad \varphi(a \hat{\otimes} h) = i(a)h
\]
for all \( a \in A \) and \( h \in H \) is an isomorphism of Hopf algebras.

Proof. It follows from Theorem 2.3 that there exists a bialgebra extending structure \( \Omega(A) = (H, \cdot, \circ, f) \) of \( A \) such that
\[
\varphi : A \times H \to E, \quad \varphi(a \times h) = i(a)h
\]
for all \( a \in A \) and \( h \in H \) is an isomorphism of Hopf algebras. We observe that for any \( h \in H = \{ x \in E \mid \pi(x(1)) \otimes x(2) = 1_A \otimes x \} \), we have that \( \pi(h) = \varepsilon_H(h)1_A \). Thus, as \( \pi \) is also a right \( A \)-module map, the action \( \triangleright \) is given by:
\[
\pi(h) \triangleright a = \pi(hi(a)) = \pi(h \cdot a) = \pi(h)a = \varepsilon_H(h)a
\]
for all \( h \in H \) and \( a \in A \). So the action \( \triangleright \) is trivial and hence the unified product \( A \times H \)
is a twisted product \( A \hat{\otimes} H \). \( \square \)

The following is a simplified and more transparent version of [8, Theorem 4.1]. We use a minimal context: only the concept of normality of a morphism in the sense of [2] is used, contrary to [8, Definition 3.5] where it was taken as input data for [8, Theorem 4.1].

Corollary 2.5. Let \( \pi : E \to A \) be a normal split epimorphism of Hopf algebras. Then there exists an isomorphism of Hopf algebras \( E \cong A \# H \), for some right \( A \)-module bialgebra \( H \), where \( A \# H \) is the right version of the smash product of bialgebras.

Proof. Let \( i : A \to E \) be a Hopf algebra morphism such that \( \pi \circ i = \text{Id}_A \) and
\[
H = \{ x \in E \mid \pi(x(1)) \otimes x(2) = 1_A \otimes x \}
\]
which is a subcoalgebra in \( E \) as \( \pi \) is normal. In fact, as \( \pi \) is also an algebra map, \( H \) is a Hopf subalgebra of \( E \). We note that \( \pi(h) = \varepsilon_H(h)1_A \), for all \( h \in H \). \( \pi \) is also a \( A \)-bimodule map as \( \pi(a \cdot h) = \pi(i(a))h = \pi(i(a))\pi(h) = a\pi(h) \) and \( \pi(h \cdot a) = \pi(hi(a)) = \pi(h)a \), for all \( a \in A \) and \( h \in H \). Now we can apply Theorem 2.3. Thus, there exists a bialgebra extending structure \( \Omega(A) = (H, \cdot, \circ, f) \) of \( A \) such that
\[
\varphi : A \times H \to E, \quad \varphi(a \times h) = i(a)h
\]
is an isomorphism of bialgebras. Moreover the action \( \triangleright \) is the trivial one as \( \pi \) is a right \( A \)-module map (Corollary 2.4). On the other hand, the cocycle \( f \) as it was defined in Theorem 2.3 is also the trivial one: \( f(h, g) = \pi(hg) = \pi(h)\pi(g) = \varepsilon_H(h)\varepsilon_H(g)1_A \), for all \( h, g \in H \). Using once again the fact that \( \pi \) is an algebra map, the multiplication \( \cdot \) on \( H \) as it was defined in Theorem 2.3 is exactly the one of \( E \) (and this fits with the fact that \( H \) is a Hopf subalgebra of \( E \)). Finally, the right action as it was defined in Theorem 2.3 takes the simplified form: \( h \rhd a = i(S_A(a(1)))h a(1) \), for all \( h \in H \) and \( a \in A \). Thus, the unified product \( A \times H \) is the right version of the smash product of Hopf algebras from Example 1.1 and \( \varphi : A \# H \to E \) is an isomorphism of Hopf algebras. \( \square \)
From now on, $A$ will be a Hopf algebra and $\Omega(A) = (H, \triangle, \triangleright, f)$ a bialgebra extending structure of $A$. Let $i_A : A \to A \rtimes H$, $i_A(a) = a \rtimes 1_H$, for all $a \in A$ be the canonical bialgebra morphism.

**Proposition 2.6.** Let $A$ be a Hopf algebra and $\Omega(A) = (H, \triangle, \triangleright, f)$ a bialgebra extending structure of $A$. Then $i_A : A \to A \rtimes H$ is a split monomorphism in the category of bialgebras if and only if there exists $\gamma : H \to A$ a unitary coalgebra map such that

$$h \triangleright a = \gamma(h(1)) a(1) \gamma^{-1}(h(2)) \triangleright a(2)$$  \hspace{1cm} (13)

$$f(h, g) = \gamma(h(1)) \gamma(g(1)) \gamma^{-1}(h(2) \cdot g(2))$$  \hspace{1cm} (14)

for all $h, g \in H$ and $a \in A$, where $\gamma^{-1} = S_A \circ \gamma$.

**Proof.** $i_A : A \to A \rtimes H$ is a split monomorphism of bialgebras if and only if there exists a bialgebra map $p : A \rtimes H \to A$ such that $p(a \rtimes 1) = a$, for all $a \in A$. Such a Hopf algebra map $p$ is given by:

$$p(a \rtimes h) = p((a \rtimes 1_H) \bullet (1_A \rtimes h)) = a p(1_A \# h)$$

for all $a \in A$ and $h \in H$. We denote by $\gamma : H \to A$, $\gamma(h) := p(1 \rtimes h)$, for all $h \in H$. Hence, such a splitting map $p$ should be given by

$$p = p_\gamma : A \rtimes H \to A, \quad p_\gamma(a \rtimes h) = a \gamma(h)$$

for all $a \in A$ and $h \in H$. First we note that $p_\gamma$ is a coalgebra map if and only if $\gamma$ is a coalgebra map. Now, we prove that $p_\gamma$ is an algebra map if and only if $\gamma(1_H) = 1_A$ and the following two compatibilities are fulfilled:

$$\gamma(h)a = (h(1) \triangleright a(1)) \gamma(h(2) \triangleright a(2))$$  \hspace{1cm} (15)

$$\gamma(h)\gamma(g) = f(h(1), g(1)) \gamma(h(2) \cdot g(2))$$  \hspace{1cm} (16)

for all $h, g \in H$ and $a \in A$. Of course, $p_\gamma(1_A \rtimes 1_H) = 1_A$ if and only $\gamma(1_H) = 1_A$. We assume now that $\gamma(1_H) = 1_A$. Then, $p_\gamma$ is an algebra map if and only if $p_\gamma(xy) = p_\gamma(x)p_\gamma(y)$, for all $x, y \in T := \{a \rtimes 1_H \mid a \in A\} \cup \{1_A \rtimes g \mid g \in H\}$, the set of generators as an algebra of $A \rtimes H$ (see [1]). Now, for any $a, c \in A$ and $h \in H$ we have:

$$p_\gamma((a \rtimes 1_H) \bullet (c \rtimes 1_H)) = ac = p_\gamma(a \rtimes 1_H)p_\gamma(c \rtimes 1_H)$$

and

$$p_\gamma((a \rtimes 1_H) \bullet (1_A \rtimes h)) = a \gamma(h) = p_\gamma(a \rtimes 1_H)p_\gamma(1_A \rtimes h)$$

On the other hand, it is straightforward to see that

$$p_\gamma((1_A \rtimes h) \bullet (a \rtimes 1_H)) = p_\gamma(1_A \rtimes h)p_\gamma(a \rtimes 1_H)$$

if and only if (15) holds and

$$p_\gamma((1_A \rtimes h) \bullet (1_A \rtimes g)) = p_\gamma(1_A \rtimes h)p_\gamma(1_A \rtimes g)$$

if and only if (16) holds. Thus, we have proved that $p_\gamma$ is a bialgebra map if and only if $\gamma : H \to A$ is a unitary coalgebra map and the compatibility conditions (15) and (16) are fulfilled. Being a coalgebra map, $\gamma$ is invertible in convolution with the inverse given by $\gamma^{-1} = S_A \circ \gamma$. We observe that (15) is equivalent to (13) while (16) is equivalent to (14) and the proof is finished. \qed
Suppose now that we are in the setting of Proposition 2.6. The multiplication on the unified product $A \ltimes H$ given by (3) takes the following form

$$(a \ltimes h) \bullet (c \ltimes g) = a\gamma(h_{(1)})c_{(1)}\gamma(g_{(1)})\gamma^{-1}\left((h_{(2)} \triangleleft c_{(2)}) \cdot g_{(2)}\right) \ltimes (h_{(3)} \triangleleft c_{(3)}) \cdot g_{(3)} \quad (17)$$

for all $a, c \in A$ and $h, g \in H$, which is still very difficult to deal with. Next we shall give another equivalent description of the bialgebra structure on this special unified product in which the multiplication has a less complicated form. Let $A \otimes H = A \otimes H$, as a $k$-module with the unit $1A \otimes 1H$ and the following structures:

$$(a \otimes h) \cdot (c \otimes g) := ac_{(1)} \otimes \left(h \triangleleft (c_{(2)}\gamma^{-1}(g_{(1)}))\right) \cdot g_{(2)} \quad (18)$$

$$\Delta_{A \otimes H}(a \otimes h) := a_{(1)} \otimes h_{(2)} \otimes a_{(2)}\gamma^{-1}(h_{(1)})\gamma(h_{(3)}) \otimes h_{(4)} \quad (19)$$

$$\varepsilon_{A \otimes H}(a \otimes h) := \varepsilon_A(a)\varepsilon_H(h) \quad (20)$$

for all $a, c \in A$ and $h, g \in H$, where we denoted $a \otimes h \in A \otimes H$ by $a \ltimes h$.

The object $A \otimes H$ introduced above is an interesting deformation of the smash product of bialgebras $A \# H$ of Example 1.4 that can be recovered in the case that $\gamma : H \to A$ is the trivial map, that is $\gamma(h) = \varepsilon_H(h)1_A$, for all $h \in H$. If $H$ is cocommutative then the comultiplication given by (19) is just the tensor product of coalgebras.

**Proposition 2.7.** Let $\Omega(A) = (H, \triangleleft, \triangleright, f)$ be a bialgebra extending structure of a Hopf algebra $A$ and $\gamma : H \to A$ be a unitary coalgebra map such that (13) and (14) hold. Then $\varphi : A \ltimes H \to A \otimes H$, $\varphi(a \ltimes h) := a\gamma(h_{(1)}) \otimes h_{(2)}$ for all $a \in A$ and $h \in H$ is an isomorphism of bialgebras.

**Proof.** The map $\varphi : A \ltimes H \to A \otimes H$ is bijective with the inverse $\varphi^{-1} : A \otimes H \to A \ltimes H$ given by $\varphi^{-1}(a \otimes h) = a\gamma^{-1}(h_{(1)}) \ltimes h_{(2)}$, for all $a \in A$ and $h \in H$, where $\gamma^{-1} = S_A \circ \gamma$. Below we shall use the fact that $\gamma^{-1} : H \to A$ is an antimorphism of coalgebras, i.e. $\Delta(\gamma^{-1}(h)) = \gamma^{-1}(h_{(2)}) \otimes \gamma^{-1}(h_{(1)})$, for all $h \in H$. The unique algebra structure that can be defined on the $k$-module $A \otimes H$ such that $\varphi : A \ltimes H \to A \otimes H$ becomes an isomorphism of algebras is given by:

$$(a \otimes h) \cdot (c \otimes g) = \varphi^{-1}(a \otimes h) \bullet \varphi^{-1}(c \otimes h) \quad (17)$$

$$= \varphi\left((a\gamma^{-1}(h_{(1)}) \ltimes h_{(2)}) \bullet (c\gamma^{-1}(g_{(1)}) \ltimes g_{(2)})\right)$$

$$= \varphi\left(a\gamma^{-1}(h_{(1)})\gamma(h_{(2)})c_{(1)}\gamma^{-1}(g_{(3)})\gamma(g_{(4)})\gamma^{-1}[(h_{(3)} \triangleleft (c_{(2)}\gamma^{-1}(g_{(2)}))) \cdot g_{(5)}] \ltimes [(h_{(4)} \triangleleft (c_{(3)}\gamma^{-1}(g_{(1)}))) \cdot g_{(6)}]\right)$$

$$= \varphi\left(ac_{(1)}\gamma^{-1}[(h_{(1)} \triangleleft (c_{(2)}\gamma^{-1}(g_{(2)}))) \cdot g_{(3)}]\right)$$

$$\ltimes (h_{(2)} \triangleleft (c_{(3)}\gamma^{-1}(g_{(1)}))) \cdot g_{(4)}$$
\[= ac(1)\gamma^{-1}[(h(1) \triangleleft (c(2)\gamma^{-1}(g(3)))) \circ g(4)]\]
\[= \gamma[(h(2) \triangleleft (c(3)\gamma^{-1}(g(2)))) \circ (h(3) \triangleleft (c(4)\gamma^{-1}(g(1)))) \circ g(6)]\]
\[= ac(1) \circ \left(h \triangleleft (c(2)\gamma^{-1}(g(1)))\right) \circ g(2)\]

which is exactly the multiplication defined by (18) on \(A \oplus H = A \otimes H\). Thus, we have proved that \(\varphi : A \times H \rightarrow A \oplus H\) is an isomorphism of algebras.

It remains to prove that \(\varphi : A \times H \rightarrow A \oplus H\) is also a coalgebra map. For any \(a \in A\) and \(h \in H\) we have:

\[\Delta_{A \oplus H}(\varphi(a \times h)) = \Delta_{A \oplus H}(a \gamma(h(1)) \oplus h(2))\]
\[= a(1)\gamma(h(1)) \circ h(4) \circ a(2)\gamma(h(2))\gamma^{-1}(h(3)) \circ h(5) \circ h(6)\]
\[= \varphi(a(1) \times h(1)) \circ \varphi(a(2) \times h(2))\]

i.e. \(\varphi : A \times H \rightarrow A \oplus H\) is a coalgebra map, as needed. By assumption, \(A \times H\) is a bialgebra, thus \(A \oplus H\) is a bialgebra and \(\varphi\) is an isomorphism of bialgebras. \(\square\)

Proposition 2.6 provides a method to construct bialgebra extending structures, and thus unified products, starting only with a right action \(\triangleleft\) and a unitary coalgebra map \(\gamma : H \rightarrow A\).

**Theorem 2.8.** Let \(A\) be a Hopf algebra, \(H\) a unitary not necessarily associative bialgebra such that \((H, \triangleleft)\) is a right \(A\)-module coalgebra with \(1_A \triangleleft a = \varepsilon_A(a)1_H\), for all \(a \in A\). Let \(\gamma : H \rightarrow A\) be a unitary coalgebra map and define

\[\triangleright : H \otimes A \rightarrow A, \quad h \triangleright a : = \gamma(h(1))a(1)\gamma^{-1}(h(2) \triangleleft a(2))\]  

\[f_{\gamma} : H \otimes H \rightarrow A, \quad f_{\gamma}(h, g) : = \gamma(h(1))\gamma(g(1))\gamma^{-1}(h(2) \circ g(2))\]

for all \(h, g \in H\) and \(a \in A\). Assume that the compatibility conditions (BE1), (BE3), (BE6) and (BE7) hold for \(\triangleright\) and \(f_{\gamma}\).

Then \(\Omega(A) = (H, \triangleleft, \triangleright, f_{\gamma})\) is a bialgebra extending structure of \(A\) and there exists an isomorphism of bialgebras \(A \times H \cong L \ast A\), where \(L \ast A\) is the Radford biproduct for a bialgebra \(L\) in the category \(\mathbb{YD}\) of Yetter-Drinfel’d modules.

**Proof.** First we have to prove that \(\triangleright\) and \(f_{\gamma}\) are coalgebra maps. Using the fact that \(\gamma^{-1} = S_A \circ \gamma\) is an antimorphism of coalgebras we have:

\[\Delta_A(h \triangleright a) = \gamma(h(1))a(1)\gamma^{-1}(h(4) \triangleleft a(4)) \circ \gamma(h(2))a(2)\gamma^{-1}(h(3) \triangleleft a(3))\]
\[= \gamma(h(1))a(1)\gamma^{-1}(h(3) \triangleleft a(3)) \circ h(2)\triangleright a(2)
\[= \gamma(h(1))a(1)\gamma^{-1}(h(2) \triangleleft a(2)) \circ h(3)\triangleright a(3)
\[= h(1)\triangleright a(1) \circ h(2)\triangleright a(2)\]
for all \( h \in H \) and \( a \in A \), i.e. \( \triangleright_{\gamma} \) is a coalgebra map. On the other hand, we have:

\[
\Delta_A(f_{\gamma}(h, g)) = \gamma(h_{(1)}) \gamma(g_{(1)}) \gamma^{-1}(h_{(4)} \cdot g_{(4)}) \otimes \gamma(h_{(2)}) \gamma(g_{(2)}) \gamma^{-1}(h_{(3)} \cdot g_{(3)}) \\
= \gamma(h_{(1)}) \gamma(g_{(1)}) \gamma^{-1}(h_{(3)} \cdot g_{(3)}) \otimes f_{\gamma}(h_{(2)}, g_{(2)})
\]

(\( BE_7 \))

\[
= f_{\gamma}(h_{(1)}, g_{(1)}) \otimes f_{\gamma}(h_{(2)}, g_{(2)}) \\
= (f_{\gamma} \otimes f_{\gamma}) \Delta_{H \otimes H}(h \otimes g)
\]

for all \( h, g \in H \), that is \( f_{\gamma} \) is a coalgebra map.

It remains to prove that the compatibility conditions \( BE_2 \), \( BE_4 \) and \( BE_5 \) hold for \( \triangleright_{\gamma} \) and \( f_{\gamma} \). For \( a, b \in A \) and \( g \in H \) we have:

\[
g \triangleright_{\gamma} (ab) = \gamma(g_{(1)})a_{(1)}b_{(1)}\gamma^{-1}(a_{(2)}b_{(2)}) \\
= \gamma(g_{(1)})a_{(1)}\gamma^{-1}(a_{(2)}) \cdot \gamma(g_{(3)} \triangleleft a_{(3)}) \cdot b_{(1)} \gamma^{-1}(a_{(4)} \triangleleft b_{(2)}) \\
= \gamma(g_{(1)})b_{\gamma}(a_{(1)})(g_{(2)} \triangleleft a_{(2)}) \triangleright_{\gamma} b
\]

i.e. \( BE_2 \) holds. We denote by LHS (resp. RHS) the left (resp. right) hand side of \( BE_4 \). We have:

\[
LHS = \gamma(g_{(1)})h_{(1)} \triangleright_{\gamma} a_{(1)} \gamma^{-1}(g_{(2)} \triangleleft (h_{(2)} \triangleright_{\gamma} a_{(2)})) \cdot \gamma(g_{(3)} \triangleleft (h_{(3)} \triangleright_{\gamma} a_{(3)})) \\
= \gamma(h_{(5)} \triangleleft a_{(5)}) \gamma^{-1}((g_{(4)} \triangleleft (h_{(4)} \triangleright_{\gamma} a_{(4)})) \cdot (h_{(6)} \triangleleft a_{(6)})) \\
= \gamma(g_{(1)})h_{(1)} \triangleright_{\gamma} a_{(1)} \gamma(h_{(3)} \triangleleft a_{(3)}) \gamma^{-1}((g_{(2)} \triangleleft (h_{(2)} \triangleright_{\gamma} a_{(2)})) \cdot (h_{(4)} \triangleleft a_{(4)}))
\]

\( BE_6 \)

\[
= \gamma(g_{(1)})h_{(1)} \triangleright_{\gamma} a_{(1)} \gamma(h_{(2)} \triangleleft a_{(2)}) \gamma^{-1}((g_{(2)} \triangleleft (h_{(3)} \triangleright_{\gamma} a_{(3)})) \cdot (h_{(4)} \triangleleft a_{(4)}))
\]

\( BE_3 \)

\[
= \gamma(g_{(1)})h_{(1)} \triangleright_{\gamma} a_{(1)} \gamma(h_{(2)} \triangleleft a_{(2)}) \gamma^{-1}((g_{(2)} \cdot h_{(3)}) \triangleleft a_{(3)})
\]

\( 21 \)

\[
= \gamma(g_{(1)})h_{(1)} \triangleright_{\gamma} a_{(1)} \gamma^{-1}(g_{(2)} \cdot h_{(2)} \triangleleft a_{(2)})
\]

\( = RHS \)

for all \( a \in A, h, g \in H \), i.e. \( BE_4 \) holds. Now, for \( g, h \) and \( l \in H \) the right hand side of \( BE_5 \) takes the following form:

\[
RHS = \gamma(g_{(1)})\gamma(h_{(1)})\gamma(l_{(1)})\gamma^{-1}((g_{(2)} \cdot h_{(2)}) \cdot l_{(2)})
\]

while the left hand side of \( BE_5 \) is

\[
LHS = \gamma(g_{(1)})f_{\gamma}(h_{(1)}, l_{(1)})\gamma(h_{(3)} \cdot l_{(3)}) \gamma^{-1}((g_{(2)} \triangleleft f_{\gamma}(h_{(2)}, l_{(2)})) \cdot (h_{(4)} \cdot l_{(4)}))
\]

\( BE_7 \)

\[
= \gamma(g_{(1)})f_{\gamma}(h_{(1)}, l_{(1)})\gamma(h_{(2)} \cdot l_{(2)}) \gamma^{-1}((g_{(2)} \triangleleft f_{\gamma}(h_{(3)}, l_{(3)})) \cdot (h_{(4)} \cdot l_{(4)}))
\]

\( BE_1 \)

\[
= \gamma(g_{(1)})f_{\gamma}(h_{(1)}, l_{(1)})\gamma(h_{(2)} \cdot l_{(2)}) \gamma^{-1}((g_{(2)} \cdot h_{(3)}) \cdot l_{(3)})
\]

\( 22 \)

\[
= \gamma(g_{(1)})\gamma(h_{(1)})\gamma(l_{(1)}) \gamma^{-1}((g_{(2)} \cdot h_{(2)}) \cdot l_{(2)}) = RHS
\]

hence \( BE_5 \) also holds and thus \( \Omega(A) = (H, \triangleleft, \triangleright_{\gamma}, f_{\gamma}) \) is a bialgebra extending structure of \( A \).
For the final part we use Proposition 2.6 as \( p : A \times H \rightarrow A \), \( p(a \times h) = a\gamma(h) \) is a bialgebra map that splits \( i_A : A \rightarrow A \times H \). Thus, it follows from [9, Theorem 3] that there exists an isomorphism of bialgebras \( A \times H \cong L \ast A \), where \( (L, \rightarrow, \rho_L) \) is a bialgebra in \( A \mathcal{Y} D \) as follows:

\[
L = \left\{ \sum_i a_i \times h_i \in A \times H \mid \sum_i a_{i(1)} \times h_{i(1)} \otimes a_{i(2)} \gamma(h_{i(2)}) = \sum_i a_i \times h_i \otimes 1_A \right\}
\]

with the coalgebra structure described below:

\[
\Delta_L(\sum_i a_i \times h_i) = \sum_i (a_{i(1)} \times h_{i(1)}) \cdot \left( 1_A \times S_A(a_{i(2)} \gamma(h_{i(2)})) \right) \otimes a_{i(3)} \times h_{i(3)}
\]

\[
\varepsilon_L(\sum_i a_i \times h_i) = \sum_i \varepsilon_A(a_i) \varepsilon_H(h_i)
\]

for all \( \sum_i a_i \times h_i \in L \) and the structure of an object in \( A \mathcal{Y} D \) given by

\[
a \rightarrow \sum_i a_i \times h_i = \sum_i (a_{(1)} a_i \times h_i) \cdot (1_A \times S_A(a_{(2)}))
\]

\[
\rho_L(\sum_i a_i \times h_i) = \sum_i a_{i(1)} \gamma(h_{i(1)}) \otimes a_{i(2)} \times h_{i(2)}
\]

for all \( a \in A \) and \( \sum_i a_i \times h_i \in L \).

Using Theorem 2.8 we shall construct an example of an unified product, that is also isomorphic to a biproduct, starting with a minimal set of data.

**Example 2.9.** Let \( G \) be a group, \((X, 1_X)\) be a pointed set, such that there exists a binary operation \( \cdot : X \times X \rightarrow X \) having \( 1_X \) as a unit.

Let \( \triangleleft : X \times G \rightarrow X \) be a map such that \((X, \triangleleft)\) is a right \( G \)-set with \( 1_X \triangleleft g = 1_X \), for all \( g \in G \) and \( \gamma : X \rightarrow G \) a map with \( \gamma(1_X) = 1_G \) such that the following compatibilities hold

\[
(x \cdot y) \cdot z = \left( x \triangleleft \left( \gamma(y) \gamma(z) \gamma(y \cdot z)^{-1} \right) \right) \cdot (y \cdot z) \quad (23)
\]

\[
(x \cdot y) \triangleleft g = \left( x \triangleleft \left( \gamma(y) g \gamma(y \triangleleft g)^{-1} \right) \right) \cdot (y \triangleleft g) \quad (24)
\]

for all \( g \in G \), \( x, y, z \in X \).

Let \( A := k[G] \). Then \((H := k[X], \triangleleft, \triangleright, f_\gamma)\) is a bialgebra extending structure of \( k[G] \), where \( \triangleright, f_\gamma \) are given by [21], [22], that is

\[
x \triangleright \gamma g = \gamma(x) g \gamma(x \triangleleft g)^{-1}, \quad f_\gamma(x, y) = \gamma(x) \gamma(y) \gamma(x \cdot y)^{-1}
\]

for all \( x, y \in X \) and \( g \in G \).

Indeed, [23] and [21] show that the compatibility conditions \((BE1)\) and respectively \((BE3)\) hold. Now, the compatibilities \((BE6)\) and \((BE7)\) are trivially fulfilled as \( k[G] \) and \( k[X] \) are cocommutative coalgebras. Thus, it follows from Theorem 2.8 that \((k[X], \triangleleft, \triangleright, f_\gamma)\) is a bialgebra extending structure of \( k[G] \) and we can construct the unified product.
Given $E := k[G] \ltimes k[X]$ associated to $(k[X], \triangleleft, \triangleright, f_\gamma)$. As a vector space $E = k[G] \otimes k[X]$ and has the multiplication given by \eqref{eq:17} which in this case takes the form:

$$(g \otimes x) \bullet (h \otimes y) = g \gamma(x) h \gamma(y) \gamma((x \triangleleft h) \cdot y)^{-1} \triangleright (x \triangleleft h) \cdot y$$

for all $g, h \in G$ and $x, y \in X$. The coalgebra structure of $E$ is the tensor product of the group-like coalgebras $k[G]$ and $k[X]$.

The multiplication on the bialgebra $k[G] \ltimes k[X]$ can be simplified as follows: using Proposition \ref{prop:2.7}, we obtain that there exists an isomorphism of bialgebras

$$k[G] \ltimes k[X] \cong k[G] \otimes k[X]$$

where, the multiplication on $k[G] \otimes k[X]$ is given by \eqref{eq:18}:

$$(g \oplus x) \bullet (h \oplus y) = gh \oplus (x \triangleleft (h \gamma(y))^{-1}) \cdot y$$

for all $g, h \in G$ and $x, y \in X$.

**Example 2.10.** The construction of an explicit datum $(G, X, \cdot, \triangleleft, \gamma)$ as in Example \ref{ex:2.9} is a challenging problem considering the compatibilities \eqref{eq:23} - \eqref{eq:24} that need to be fulfilled by the map $\gamma: X \to G$. We give such an example below. Let $G$ be a group, $(X, 1_X)$ a pointed set such that $(X, \triangleleft)$ is a right $G$-set with $1_X \triangleleft g = 1_X$, for all $g \in G$. Let $\cdot : X \times X \to X$ be the following binary operation:

$$x \cdot y := \begin{cases} x, & \text{if } y = 1_X \\ y, & \text{if } y \neq 1_X \end{cases}$$

Then, by a straightforward computation it can be seen that the compatibility conditions \eqref{eq:23} and \eqref{eq:24} are trivially fulfilled for any map $\gamma: X \to G$ such that $\gamma(1_X) = 1_G$. Thus $(H := k[X], \triangleleft, \triangleright, f_\gamma)$ is a bialgebra extending structure of $k[G]$, where $\triangleright, f_\gamma$ are given by:

$$x \triangleright g = \gamma(x) g \gamma(x \triangleleft g)^{-1}, \quad f_\gamma(x, y) = \gamma(x) \gamma(y) \gamma(x \cdot y)^{-1} = \begin{cases} 1_G, & \text{if } y = 1_X \\ \gamma(x), & \text{if } y \neq 1_X \end{cases}$$

for all $x, y \in X$ and $g \in G$.

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