Same-diff?
Part I: Conceptual similarities (and one difference) between gauge transformations and diffeomorphisms

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Abstract

The following questions are germane to our understanding of gauge-(in)variant quantities and physical possibility: in which ways are gauge transformations and spacetime diffeomorphisms similar, and in which are they different? To what extent are we justified in endorsing different attitudes—sophistication, quidditism/haecceitism, or full elimination—towards each? In a companion paper, I assess new and old contrasts between the two types of symmetries. In this one, I propose a new contrast: whether the symmetry changes pointwise the dynamical properties of a given field. This contrast distinguishes states that are related by a gauge-symmetry from states related by generic spacetime diffeomorphisms, as being ‘pointwise dynamically indiscernible’. Only the rigid isometries of homogeneous spacetimes fall in the same category, but they are neither local nor modally robust, in the way that gauge transformations are. In spite of this difference, I argue that for both gauge transformations and spacetime diffeomorphisms, symmetry-related models are best understood through the doctrine of ‘sophistication’.

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1 Introduction

Same-diff [noun]: an oxymoron, used to describe something as being the same as something else. Often used as an excuse for being wrong. (Urban dictionary).

Diff: A common abbreviation for “diffeomorphism”. E.g. Diff(M) is the group of diffeomorphisms of the (differentiable) manifold M.

1.1 Motivation

Gauge theories lie at the heart of modern physics: in particular, they constitute the standard model of particle physics. Philosophers of physics generally accept as the leading idea of a gauge theory—or as the main connotation of the phrase ‘gauge theory’—that it involves a formalism that uses more variables than there are physical degrees of freedom in the system.
described; and thereby more variables that one strictly speaking needs to use. Hence the common soubriquets: ‘descriptive redundancy’, ‘surplus structure’, and more controversially, ‘descriptive fluff’ (e.g. Earman (2002, 2004)).

Although the main idea and connotation of descriptive redundancy is undoubtedly correct—and endorsed by countless presentations in the physics literature—some celebrated philosophers, such as Healey (2007) and Earman (2002) among others, have gone beyond this connotation, and defended a stronger, eliminativist view that gauge symmetry must be eliminated before determining which models of a theory represent distinct physical possibilities, on pain of radical indeterminism. For them, the connotation of ‘fluff’ is that it can have no purpose. But radical indeterminism also threatens theories such as general relativity, embodying diffeomorphism symmetry; a threat revealed by the famous hole argument. In that context, the most convincing—and popular—way to defuse the threat is called sophisticated substantivalism. It is not eliminativist: it is a form of structuralism, labeled anti-haecceitism, which takes spacetime points to have no metaphysically robust identity across possibilities. According to this doctrine, points can only acquire identity through their complex web of properties and relations, as encoded in fields.\(^1\)

A similar resolution is available for gauge symmetry, in the form of ‘anti-quidditism’; but it is there much less popular.\(^2\) In the case of gauge symmetry, attempting to eliminate the symmetry-related models is considered a more viable alternative. But is this alternative really more justified in the case of gauge symmetry? If so, why?

1.2 My position

Symmetries Dynamical symmetries are sets of transformations acting on the models of a given theory such that the symmetry-related models are empirically indiscernible according to that theory. This is the general gloss on dynamical symmetries, and it provides important intuition, but a precise definition is far from straightforward. For instance, defining a dynamical symmetry as any transformation that takes each solution of the equations of motion of a theory to another solution is too weak: such a definition would imply that any solution is related by a symmetry to any other. And there are other problems. For instance: models which we would intuitively take to depict physically distinct situations may nonetheless be symmetry-related, depending on the notion of symmetry; and it is also false that empirically identical situations are always symmetry-related according to every account of symmetry. Belot (2013) gives an exposition of the obstacles to a general definition.

Let \(\mathcal{M}\) be the space of models of the theory. Each of the models and also \(\mathcal{M}\) are endowed with some mathematical structures (e.g. topological, differential, vector space, set-theoretic, etc). The mathematical structures relevant for the models and for \(\mathcal{M}\) need not be the same.\(^3\) And let \(S\) be some quantity on \(\mathcal{M}\) that respects these structures (e.g. is continuous, smooth, linear, bijective, etc). Then a transformation \(\Phi : \mathcal{M} \rightarrow \mathcal{M}\) is an \(S\)-symmetry iff \(\Phi\):

(i) respects the structure of \(\mathcal{M}\) (e.g. is continuous, smooth, linear, bijective, etc);

(ii) is definable without fixed parameters from \(\mathcal{M}\), i.e. all models enter as free variables in the transformation \(\Phi\); and

\(^1\)See (Pooley, 2013) for a thorough exposition.

\(^2\)Though recently the position has garnered support, starting with Dewar (2017) and followed by Jacobs (2020, 2021); Martens & Read (2020).

\(^3\)For example, in non-relativistic mechanics, we could have each model be a configuration of \(N\) point particles in Euclidean space, \(\mathbb{R}^3\). The space of models is configuration space, which is isomorphic to \(\mathbb{R}^{3N}\). The linear and smooth structure of \(\mathbb{R}^3\) is part of each model, and we use it for important operations, such as taking derivatives. And we also require e.g. the smooth structure of configuration space to do variational calculus. In field theories, the space of models \(\mathcal{M}\) is usually infinite-dimensional, but nonetheless has the mathematical structure to allow definitions of neighborhoods of models, differentiable one-parameter families of models, etc.
(iii) $\Phi$ preserves the values of $S$: for any model $m$, $S(\Phi(m)) = S(m)$.

An $S$-symmetry relates empirically indistinguishable models if $S$ captures all the empirically accessible quantities. Theories are their own arbiters of empirical (in)discernibility (cf. (Martens & Read, 2020)), so different theories may have different $S$’s being sufficient for empirical indiscernibility. But most, if not all, theories of modern physics take either an action functional or a Hamiltonian to capture all empirically accessible quantities, and so taking $S$ as the action functional will be enough for our purposes.⁴

Since item (iii) implies that symmetries can be composed, we could, specializing further to the language of category theory, demand that symmetries form a groupoid, with the objects of the category being the models, i.e. the elements of $\mathcal{M}$, and the maps, or arrows, being isomorphisms in the category-theoretic sense.⁵ Here it will prove useful to make a further assumption: that symmetries are represented as groups (which could be infinite-dimensional), labeled $\mathcal{G}$, such that, given the space of models of a theory, $\mathcal{M}$, there is an action of $\mathcal{G}$ on $\mathcal{M}$, a map $\Phi : \mathcal{G} \times \mathcal{M} \to \mathcal{M}$, that preserves the action functional.⁶

Eliminativism An ardent supporter of Leibniz’s metaphysical Principle of the Identity of Indiscernibles might be motivated by empirical indiscernibility to construct a formalism that equates, or identifies, the symmetry-related models. Such a theory would not have the tools to articulate any physical difference between symmetry-related models and thus would, in this sense, have a more straightforward interpretation. This broad position is eliminativism: it favors the complete elimination of symmetry-related models of the theory. A weaker doctrine, also associated with the name of Leibniz, is Leibniz equivalence:⁷ this regards all symmetry related models as representing the same physical possibility. This doctrine is compatible with sophisticated substantivalism (and anti-haecceitism, as introduced in Section 1.1).

In his influential book, Healey (2007) endorses Leibniz equivalence for quantities that are related by diffeomorphism, while denying it for quantities that are related by gauge-symmetry, for which eliminativism is his preferred attitude. And indeed, there is a philosophical folklore that redundancy due to gauge symmetry is superfluous, or eliminable, in a way that diffeomorphism symmetry is not.

The contrasts In this and the accompanying paper Gomes (2021), I will contrast diffeomorphisms and gauge transformations in various different respects. In this paper, I proceed at a more formal, high-altitude level; and in the accompanying paper, in more technical, low-altitude detail. The only non-trivial distinction that survives this thorough analysis is parallel to a well-known distinction between Abelian and non-Abelian theories. That distinction runs as follows.

In electromagnetism—an Abelian theory—the field-strength tensor is a gauge-invariant variable; but in the non-Abelian theory the field-strength is gauge-variant. In the standard formulation of general relativity, the Riemann curvature tensor, which is usually taken as the variable dynamically analogous to the electromagnetic field-tensor, likewise varies under

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⁴Boundary conditions are here taken as features of $\mathcal{M}$, jointly with the other mathematical structure delineated above.

⁵A groupoid is a category in which every arrow is an isomorphism, in the abstract category-theoretic sense of ‘isomorphism’, i.e. every arrow has an inverse.

⁶Such an assumption—that symmetries are represented by the action of an infinite-dimensional group—holds for the covariant Lagrangian version of both Yang-Mills theories and general relativity, and it holds for the Hamiltonian version (in which $\mathcal{M}$ is phase space) of Yang-Mills theory, but it does not hold for the Hamiltonian version of general relativity; there we have only a groupoid structure (see Blohmann et al. (2013)).

⁷The term was introduced for spacetime diffeomorphisms by Earman & Norton (1987), but nowadays has a wider use, as defined here.
diffeomorphisms. For example, it varies under the infinitesimal version of the transformation, and this variation is given very simply by the Lie derivative of the tensor.\(^8\)

However, there is a more advanced mathematical formalism for non-Abelian Yang-Mills theory in which its field-strength tensor is \textit{invariant}. Such a mathematical formulation is available for Yang-Mills theories, but not for general relativity. That is because while both the Riemann curvature and non-Abelian Yang-Mills curvature are gauge-variant, in the \textit{usual} formalisms their gauge variance is conceptually distinct. Under an isomorphism of non-Abelian Yang-Mills theory, the curvature tensor still transforms in a prescribed, algebraic manner (i.e. without any derivative); while in general relativity the Riemann curvature transforms, as mentioned above, like any other tensor, through a Lie derivative. This conceptual distinctness is reflected in the more mathematically sophisticated formalism for gauge theory—the Atiyah-Lie algebroid—in which the non-Abelian field strength is fully gauge-invariant, just like it is in the Abelian case.\(^9\)

We also need to allow for the fact that—as the Aharonov-Bohm effect shows—there are degrees of freedom of the gauge potential that are ‘non-locally possessed’, as Healey (2007, p.192) puts it. This means that the Yang-Mills curvature can only represent the locally possessed degrees of freedom of the fields.

So here is my proposal for the main difference between the symmetries of Yang-Mills theory and of general relativity:

\[ \Delta: \text{Yang-Mills theory, but not general relativity, admits a formalism in which the local, dynamical content of the theory is fully invariant under the appropriate symmetry transformations.} \]

Here, as just explained: for non-Abelian Yang-Mills one has to advance to the Atiyah-Lie algebroid to obtain such a formalism. I will call this distinction, \( \Delta \), for ‘distinction’, or ‘difference’.

Can \( \Delta \) be cashed out in terms of a reduced formalism? That is: does the distinction support the folklore that for gauge theories we should adopt such a formalism, i.e. be eliminativist? In the concluding Section 5 I will buttress the formal considerations of Sections 2 and 3 and argue ‘no’: Yang-Mills theory no less than general relativity should resist eliminativism.

1.3 Roadmap and Summary of this paper

This is the first of two papers analysing the similarities and distinctions between the gauge symmetries of Yang-Mills theory and the spacetime diffeomorphisms of general relativity. The first will analyse more formal aspects while the second will analyse more detailed aspects of this comparison.

My argument requires brief expositions of symmetries, for both Yang-Mills theory and general relativity: the theories that best represent the importance of gauge and diffeomorphism symmetry, respectively. I undertake this analysis in Section 2 for general relativity and in Section 3 for Yang-Mills theory. Since there are many good references for the foundations of spacetime physics (e.g. Earman (1989); Maudlin (2015)), I will concentrate on developing the conceptual foundations of Yang-Mills theories, such as the theory of principal fiber bundles; and so Section 3 is much longer and complete than Section 2.

But both Sections 2 and 3 close with the interpretation of symmetries that I endorse: \textit{sophisticated substantivalism}. I take this to be a \textit{structural} interpretation of the theories: anti-haecceitist for general relativity diffeomorphisms and anti-quiddistic for the gauge symmetries.

\(^8\)Beware: that a quantity such as the Riemann curvature varies under diffeomorphisms tends to be forgotten in the hole argument literature’s emphasis on dragging along metric fields. This forgetfulness bears on the philosophical morals of the hole argument Gomes & Butterfield (2021).

\(^9\)The Atiyah-Lie algebroid is a more recent nomenclature arising from the influential work of ?. The construction involved is also known as the \textit{bundle of connections}, as it was named in the contemporary paper by Kobayaschi (1957); see also (Kolar et al., 1993, Ch. 17.4).
of Yang-Mills theory. This jargon can be quickly summarized: haecceitistic possibilities involve individuals being “swapped” or “exchanged” without any qualitative difference, and quidditiastic possibilities involve properties being “swapped” or “exchanged” without any qualitative difference. Anti-haecceitists about spacetime points thus deny that there are worlds that instantiate the same distribution of qualitative properties and relations over spacetime points, yet differ only over which spacetime points play which qualitative roles. Similarly, the anti-quidditist will insist that there are no two worlds worlds that instantiate the same nomological structure, and yet differ only over which properties play which nomological roles. (Black, 2000) is a standard example of the anti-quidditist position, while (Lewis, 2009) is an example of the quidditist one.\(^{10}\)

As stated, sophistication is a metaphysical thesis: symmetries reveal an underlying invariant structure, which is what has ontic significance. And indeed, the relation between symmetries and structure is familiar: the more symmetries there are, the less structure remains invariant under their action; and the fewer symmetries there are, the more structure that remains invariant (cf. ? for a more thorough discussion about this relationship). But does any symmetry, even the arbitrarily defined, reveal a compelling, or ‘metaphysically perspicuous’ underlying structure? This question is contentious (cf. e.g. Dewar (2017); Jacobs (2020); Martens & Read (2020)). We will design our own criterion to single out those symmetries that should be interpreted as revealing metaphysically perspicuous underlying structure. That criterion is whether the symmetry in question can be construed as simply a change of notation in the formalism. That is, whether active symmetries have passive counterparts. If they do, we can reveal, in each chart, the common structure of the symmetry-related models indirectly, as expressible quantities that are coordinate-invariant. We find that both gauge transformations and diffeomorphisms, in suitable formalisms, satisfy this criterion.

Thus Sections 2 and 3 will adjudicate whether we can find salient differences between the symmetries of Yang-Mills and general relativity at a broad, formal, or high-altitude, level. The verdict will be that we cannot: both theories find a natural expression with a structural, or ‘sophisticated’ attitude towards symmetries.

The following section, Section 4, deals with the attempt to distinguish gauge and diffeomorphism in terms of the labels: ‘external’ and ‘internal’; and this will lead me to my promised real difference, labelled ∆ at the end of Section 1.2.

I begin by considering one obvious reason to distinguish gauge symmetries from spacetime diffeomorphisms: that the former acts ‘internally’—shuffling around properties at each spacetime point—whereas the former acts ‘externally’: shuffling around the spacetime points themselves. But before we attribute too much significance to this distinction, we need to be sure it does not just spring from our more everyday acquaintance with the ‘medium-sized dry goods’ of spacetime—where diffeomorphisms act—than with the ‘internal spaces’, where the gauge-transformations act. And I argue that we cannot be sure of this. For think of how most macroscopic bodies are electrically neutral, so that electromagnetic forces are not easy to perceive; and the other non-gravitational forces are confined to subatomic length scales. But this difference between the forces described by gauge theories and by gravitational physics does not necessarily provide a significant distinction between diffeomorphisms and gauge symmetries. In other words, the obvious distinction between external and internal symmetries may not be a fundamental one; and indeed, it has been challenged by ‘bundle substantivalists’, such as

\(^{10}\)An example may help visualise these concepts. For anti-hacceitism, picture a connected graph, in which the vertices do not have an identity beyond their connectivity, or at least no such identity playing a nomological role. So a permutation of the vertices yields a duplicate of the original graph. It is important to note here that although a point’s intrinsic identity may have no nomological role, they are not easily expunged from our representation, for they are required in order to describe the graph’s connectivity. An example for anti-quidditism is similarly straightforward: e.g. construe the edges as being dyadic relations of the vertices. Again, permutation of the edges will not alter connectivity and so will “give the same graph again”.)
(Arntzenius, 2012, Ch. 6.3). To better probe the everyday distinction between external and internal, I will briefly summarize the interpretation of gauge theories via the empirically equivalent Kaluza-Klein formalism. The formalism employs only external, or spacetime directions, and by so doing casts doubt on whether we can really thus distinguish external and internal transformations.

But one conceptual distinction between gauge theories and general relativity survives in the Kaluza-Klein formalism, and it is present also in the principal fiber bundle formalism. For, even in the Kaluza-Klein framework, internal directions are ‘background’ structures: they do not respond to the distribution of matter and energy. This rigidity corresponds to the distinction between the pointwise actions of gauge symmetries and diffeomorphisms on the dynamical quantities of Yang-Mills and general relativity respectively, that I called ∆ in Section 1.2. That is: the gauge and Riemann curvature tensors transform, pointwise, in qualitatively different ways (even if both are in their own sense, covariant). The gauge curvature transforms homogeneously and the Riemann curvature, generically, does not.

The Atiyah-Lie algebroid, introduced in Section 4, allows us to polish this distinction. In this formalism, the curvature of the gauge potential is fully gauge-invariant, in the non-Abelian as well as in the Abelian or electromagnetic case (even though in both cases the gauge potential is still gauge variant). In this formalism, the curvature, or field-strength tensor, exhausts the local gauge-invariant degrees of freedom, in the same way that the familiar electromagnetic field-strength tensor does, in the Abelian version of Yang-Mills theory. The contrast then is that the Riemann curvature tensor in general relativity is not pointwise invariant if we drag it along directions that generate the infinitesimal spacetime diffeomorphisms; but the gauge field strength is invariant if we drag it along directions that generate the infinitesimal gauge symmetries.

Therefore, taking a cue from the Atiyah-Lie formalism, one could say that gauge transformations are isomorphisms of the theory that relate states that are dynamically locally indiscernible. In contrast, spacetime diffeomorphisms are those isomorphisms of the theory that implement locally discernible changes in the dynamical quantities. The distinction applies whether we take a bundle substantivalist view, like Kaluza and Klein, or not. That is, we identify the curvature, or field strength tensor, with the local dynamical part of the gauge field; and these quantities are invariant under the action of gauge transformations, but they are not invariant under the action of diffeomorphisms. This is ∆.

In the last section, Section 5, I will first provide another overview of what we have achieved.

2 Diffeomorphisms in general relativity

This Section will be briefer than the following one, on gauge symmetry, since the interpretation of redundancy in general relativity is less controversial than in gauge theory. Nonetheless, I would like to give it a non-standard treatment, that gives due attention to the definition of smooth structure through charts and atlases.

In Section 2.1, I will introduce the isomorphisms and symmetries that occur in general relativity. The definition of the smooth structure of the manifold through atlases, introduced in Section 2.2, will then help us understand the origin and significance of the symmetries discussed in the previous subsection.

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11Even if the models of the theory as a whole may have the same physical (structural) content Gomes & Butterfield (2021).
2.1 Isomorphism and symmetries in general relativity, briefly introduced

I will take general relativity in the metric formalism, where the most general models of the theory, sometimes labeled *kinematically possible models* (KPMs) (so as to avoid confusion with those models that satisfy the equations of motion, which are labeled *dynamically possible models* (DPMs)), are given by the tuples: $\langle M, g_{ab}, \nabla, \psi \rangle$. Here $M$ is a smooth manifold, $g_{ab}$ is a Lorentzian metric (a $(0, 2)$-rank tensor with signature $(-, +, +, +)$; $\nabla$ is a covariant derivative operator (here taken to be the Levi-Civita one, i.e. obeying $\nabla g_{ab} = 0$), and $\psi$ represents some distribution of matter and radiation. I will call the space of these KPMs $M$, and if we simplify to fixing $M$ and consider the theory in vacuo, i.e. setting $\psi = 0$, then $M = \text{Lor}(M)$, the space of Lorentzian metrics over $M$.

In terms of the category-theoretic language introduced at the start of Section 1.2: the category of smooth manifolds has as objects the smooth manifolds, and diffeomorphisms as the isomorphisms; diffeomorphisms are those maps that preserve the smooth global structure of manifolds.

The matter and gravitational fields are maps from points of the manifold to some other value space; we will look at this definition in detail when we discuss vector bundles in Section 3.2.2. The dependence of the fields on spacetime points implies that an action by a diffeomorphism on this base set will lift to an action on the fields. We can represent such an action of the diffeomorphisms of $M$ on $\langle g_{ab}, \nabla, \psi \rangle$, by the pull-backs, $(f^* g_{ab}, f^* \nabla, f^* \psi)$. It is also useful to represent the local, infinitesimal action of diffeomorphisms. Namely, for a one-parameter family of diffeomorphisms $f_t \in \text{Diff}(M)$, such that $f_0 = \text{Id}$, we write the flow of $f_t$ at $t = 0$ as the vector field $X^a$. Then, infinitesimally we obtain:

$$\left. \frac{d}{dt}\right|_{t=0} f_t^* g_{ab} \equiv L_X g_{ab} = \nabla_{(a} X_{b)}, \quad (2.1)$$

where $L_X$ denotes the Lie derivative along $X^a$.

For simplicity, let us assume that the covariant derivative $\nabla$ is the Levi-Civita one, and thus implicitly defined by the metric. Now, what are the ‘natural’ isomorphisms of the composite objects $(M, g_{ab}, \psi)$? Standard mathematical practice takes isomorphisms in this category to be just those induced by the diffeomorphisms of the base set $M$, and in Section 2.2.1, we will give a brief argument for this; for now, we accept it. Then, in vacuo, two models $\langle M, g_{ab} \rangle$ and $\langle M, h_{ab} \rangle$ are isomorphic if and only if there is a diffeomorphism of $M$, $f \in \text{Diff}(M)$, such that $f^* g_{ab} = h_{ab}$. If matter and radiation fields are included, an isomorphism would require the same map to similarly relate their distributions in the two models.

Thus we have described the isomorphisms of this space of KPMs. Spacetime physical theories usually assume that isomorphisms are symmetries of the theory, in the sense that a large, salient set of quantities, and their values, will be physically represented equally well by any isomorphism-related model. Indeed, if one model satisfies the Einstein equations, an isomorphic model will also satisfy them.
Therefore, in vacuo, we will say that \( \langle M, g_{ab} \rangle \) and \( \langle M, h_{ab} \rangle \) are both isomorphic and symmetry-related iff there is an \( f \in \text{Diff}(M) \), such that \( h_{ab} = f^* g_{ab} \). We write this as:

\[
\langle M, g_{ab} \rangle \sim \langle M, f^* g_{ab} \rangle.
\] (2.2)

Thus, in the notation introduced in Section 1.2, we identify the symmetry group as \( \mathcal{G} := \text{Diff}(M) \), which acts on the space of Lorentzian metrics over \( M \), namely, \( \mathcal{M} = \text{Lor}(M) \).

Another point to note: diffeomorphisms act transitively on \( M \); any point can be carried to any other point. This means that there is no non-trivial orbit for \( \text{Diff}(M) \) defined as a subset of \( M \). Of course, \( \text{Diff}(M) \) does not act transitively on \( \text{Lor}(M) \): there we can easily identify the orbits of \( \mathcal{G} \) by (2.2), and one orbit does not cover the entire space of models.

As we will see, this means that diffeomorphisms and gauge-symmetries are indiscernible at the level of entire models; to discern them we must zoom in on their action on the base set, or what we will call the pointwise action of the symmetries.

2.2 Sophistication for diffeomorphisms

In Subsection 2.2.1, I will define the smooth structure of \( M \) through charts and atlases, in what I will call a chart-nominalist interpretation. This will help understand the reason why the symmetries of general relativity should at least contain the (induced action of the) diffeomorphisms. It will also help articulate an important correspondence between active and passive diffeomorphisms. For physical theories, I take this correspondence to deflate the ontic significance of the multiplicity of symmetry-related models, since they can be conceptually glossed as notational variants of each other. In Subsection 2.2.2, I use this gloss to give a succinct defence of anti-haecceitism or sophistication.

2.2.1 The correspondence between active and passive diffeomorphisms

First, it is important to note that smooth manifolds cannot be defined without invoking charts: these are bijective maps from subsets \( U \) of \( M \) (whose union covers \( M \)), to \( \mathbb{R}^n \), that have smooth transition functions wherever they overlap. That is, given \( \phi_1, \phi_2 : U \rightarrow \mathbb{R}^n \), so \( U \) is the intersection of the domains of \( \phi_1, \phi_2 \), then \( \phi_2 \circ \phi_1^{-1} \) is a smooth bijective function from a subset \( \phi_1(U) \) of \( \mathbb{R}^n \) to \( \phi_2(U) \).\(^{14}\) Any such complete collection of charts is called an atlas for \( M \), and any two compatible atlases—whose transition functions are smooth and with smooth inverses—are equivalent. The smooth structure of the manifold is defined as the equivalence class of atlases; or equivalently, as the maximal atlas, including all compatible charts. A maximal atlas can be taken to simply define the smooth and topological structure of the manifold. In particular, one does not need to remain faithful to some prior topological or smooth structure of \( M \): the topology, as well as the differentiable structure, are bequeathed to \( M \) by the charts of a maximal atlas.\(^{15}\) I will call this understanding of \( M \) chart-nominalism.

In the chart-nominalist spirit, an important—and largely neglected—conceptual point about interpreting the symmetries of general relativity is that they have a well-understood correspondence to passive transformations. The central idea for relating diffeomorphisms of \( M \) with fields can be identified as the flow—the infinitesimal versions—of the maps \( (g_{ab}, \nabla, \psi) \rightarrow (f^* g_{ab}, f^* \nabla, f^* \psi) \). Namely, these directions are given by \( \mathcal{L}_x g_{ab} \) of (2.1), and they generate the isomorphisms induced by the diffeomorphisms of \( M \).

\(^{14}\)The very notion of smoothness invokes the use of charts: \( k \)-smoothness is defined, for a function \( f : M \rightarrow \mathbb{R} \), as differentiability up to \( k \)-th order of the representative functions of \( f \) on each chart \( \phi \), namely, as \( k \)-th differentiability of \( \tilde{f} := f \circ \phi^{-1} : U \rightarrow \mathbb{R} \).

\(^{15}\)The set of all domains of charts in the atlas forms a topological base for the manifold: it is closed under finite intersections and its union is the whole manifold. With respect to this topology all charts are homeomorphisms by construction. Cf. (Lang, 2012, p. 22-23) for a textbook definition of smooth structure in this manner, and Wallace (2019) for a conceptual treatment.
passive coordinate transformations is that any chart is dragged by a diffeomorphism to another chart. So, for \( f \in \text{Diff}(M) \) and a given tensor field \( T := T_{b_1,\ldots,b_k}^{a_1,\ldots,a_l} \), we obtain a transformed field \( \tilde{T} := f^*T \). Suppose that, under a chart \( \phi_1 : U_1 \to \mathbb{R}^n \), the components of \( T \) that lie in \( \phi_1 \)'s domain are given by \( T_{b_1,\ldots,b_k}^{a_1,\ldots,a_l} \). Then, there will be a second, compatible chart, \( \phi_2 : U_2 \to \mathbb{R}^n \), for which the components of the transformed field, \( \tilde{T} \), are also given by the untilded \( T_{b_1,\ldots,b_k}^{a_1,\ldots,a_l} \). The relation between \( \phi_1 \) and \( \phi_2 \) is, of course, just \( \phi_1 = \phi_2 \circ f \), where \( U_2 = f(U_1) \). Thus, given the joint description of \( T \) by the charts of a given atlas for \( M \), there will be a second atlas for which the different tensor, \( \tilde{T} = f^*T \), has that same description. In equations: \( \phi_1 \ast(T) = \phi_2 \ast(\tilde{T}) \).

In words, the images (i.e. the values of components) of the transformed tensor under the new charts are the same as the images of the untransformed tensor under the old charts. The fact that the domains of these charts will differ seems inconsequential, since, in the chart-nominalist interpretation, the manifold structure (topological, smooth, etc) is defined by the charts.

Now we can address the question posed in Section 2.1 (after (2.1)), namely, why it is safe to assume that the isomorphisms of the n-tuple \((M, g_{ab}, \psi)\) is not a smaller set than the isomorphisms of \( M \). The reason is that any stronger notion of isomorphism would imply, through the implicit construction through atlases and the passive-active correspondence above, that the composite objects would not be fully covariant under all possible (i.e. smooth) coordinate transformations, thus allowing only a subset of coordinate systems to describe the system.\(^{17}\)

### 2.2.2 Sophistication as anti-haecceitism

First, I would like to make a broader point, tangential to the topic of sophistication, about this paper’s focus on general relativity. For the other theories, based on fields other than the metric, would also, by the arguments of Section 2.2.1 (see last paragraph) carry diffeomorphism-invariant structure. So the reader would be right to ask: why link the interpretation of spacetime points to the symmetries of \( \text{Lor}(M) \)? Or, equivalently, why the focus on general relativity when discussing spacetime diffeomorphisms? There is a historical reason and a physical reason for this focus. The historical reason is that general covariance and the equivalence principle played a very important role in Einstein’s discovery of the theory. The physical reason is that given some mild assumptions about physical theories written in terms of spacetime, every matter field must couple to the metric. Since the metric is nowhere vanishing (or at least, the values of the given field will only contribute to the action functional where the metric does not vanish), the transformation properties of each field under diffeomorphisms are dictated by their coupling to the metric and the transformation properties of the metric.\(^{18}\) Thus, in physical

\(^{16}\)Since maps and their inverses are both smooth, we can mostly ignore the distinction between push-forward and pull-back.

\(^{17}\)This subset would possibly correspond to another mathematical structure on \( M \), which would then be sufficient to serve as the base set for the theory in question. This point is in the spirit of Earman (1989, p. 45-47).

\(^{18}\)This is also why we can associate the spacetime diffeomorphisms with the vacuum constraints of the Hamiltonian formalism of general relativity (see Gomes (2021)); and why, in extending the vacuum constraints to encompass other sources of energy-momentum, we are assured that the generated symmetries will also apply to other fields (Teitelboim, 1973). But there are many ways to see the universality property of the metric vis à vis covariance, in particular using the action functional formalism (cf. footnote 13). For instance, a dynamically non-trivial theory will contain terms that are non-linear in the given tensor field (i.e. the field appears at least quadratically). Since the metric is used to contract indices, it will necessarily couple to these fields, and the covariance of the metric will determine the covariance of the fields. (Indeed, using this covariance property of tensor fields, one can derive the universality of the metric as a close cousin of the equivalence principle from the assumption of Lorentz invariance in perturbative quantum field theory (Weinberg, 1964).) More generally, the integral involved in the action functional contains the metric in its measure: in a chart, it is the volume element \( \sqrt{\sqrt{g}} \) that is appropriately invariant under coordinate transformations. All diffeomorphism-invariant quantities take the form of integrals of scalar densities, and thus involve the metric in at least this way.
theories, the individuation of spacetime points is best thought of as associated to the metric, i.e. to chronogeometric structure.

Following a nomenclature suggested in (Belot, 2003, p. 220) and adopted by Møller-Nielsen (2017) (see also Dewar (2017); Martens & Read (2020)) we can say that a chronogeometric interpretation of the models of general relativity is perspicuously sophisticated. It takes mathematical objects to represent physically significant quantities if it relates them to quantities about coincidences of material point-particles, elapsed proper times along a particle worldline, etc. For example, if one makes a journey from one planet to another, all empirically measurable quantities about the trip will be represented as diffeomorphism-invariant functions. These include: the time elapsed along the journey, whether the spaceship is accelerating or not as it passes some asteroid, all operations involved in signaling with particles or light pulses, etc. This is an anti-haecceitist position, since the points of the manifold are not taken to have an intrinsic identity across physical possibilities: they are only places in a structure. Anti-haecceitism is thus classified as a structuralist position.

More specifically, the position labeled sophisticated substantivalism illustrates structuralism in the context of general relativity. It was introduced in the context of the hole argument, and endorses Leibniz equivalence by allowing symmetry-related models to ‘peacefully co-exist’, without reduction. Thus it is anti-haecceitist, as it must also reject the view that spacetime points have primitive identities which persist across possibilities (cf. Pooley (2013) for details).

While, by keeping symmetry-related models on a par, the representation of chronogeometry in general relativity retains some redundancy, this is no blemish on the structural interpretation of the formalism. For representations of theories that are based on spacetime inevitably bring redundancy in their wake: mathematics endemically identifies points other than by their qualitative relations and it does so by stipulating the identity of points, in any of number of ways; this arbitrariness is how mathematics shows its indifference to identity, or its commitment to structure. The same is true in formal semantics, i.e. model theory.¹⁹

Although the identities of points play no nomological role, it may well be impossible to represent the theory without them. As an illustrative example (cf. the beginning of Section 1.3, footnote 10): consider a connected graph, and suppose there are some laws that depend only on the connectivity of the graph, i.e. are independent of which vertex plays which role. Here we can clearly resort to a sophisticated view of permutation symmetry without fear; the ensuing permutation redundancy does not obscure our understanding of the physical structure.²⁰

Thus it should be no surprise that there is no known, non-trivial way to ‘get rid’ of diffeomorphism symmetry. Any reformulation of general relativity whose variables distill only the structural qualities of the metric, or that employs only diffeomorphism-invariant syntax, incurs significant pragmatic and explanatory disadvantages, as it must jointly reject the underlying spacetime picture.²¹

But, to emphasise: we need not mourn the loss of unique representation; I will have more to say about this in Gomes (2021). At a more “pedestrian level”, as the passive-active cor-

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¹⁹But this practice of stipulation is not, I maintain, a commitment to the philosophical doctrine that trans-world identity of points (or more generally: of objects) should always be treated as a matter of free stipulation—a doctrine like that of Kripke (1982). That doctrine is contentious; and duly criticized by e.g. (Lewis, 1986, p. 222-227) as elusive. (Cf. Gomes & Butterfield (2021) for more on this topic, in the context of the hole argument).

²⁰Such an understanding of points within spacetime structuralism is developed in Esfeld & Lam (2008), and it is compatible with the sense of sophistication that I am advocating here.

²¹For instance, Earman, in a series of publications (see (Earman, 1989, Ch. 9.9) for the consolidated view), argued that a physically possible world is captured by what is called an Einstein algebra; an algebra that does not involve space or time in the ordinary sense of a manifold of points. But Rynasiewicz (1992) observed that one can nevertheless define points in algebraic approaches to topological spaces, and he argued that this meant one could recover precisely the isometries used in the hole argument as isomorphisms between Einstein algebras. ‘Keep the points’ and you will keep the symmetries. (cf. footnote 47).
respondence of Section 2.2.1 shows in the case of smooth manifolds, the ensuing redundancy of representation should be no more surprising than that due to the multiplicity of equivalent choices of charts or coordinate systems used for representing quantities; a multiplicity most would consider harmless or transparent. So, even though spacetime representations are inevitably redundant, according to structuralism, this redundancy is not conceptually opaque.

In sum, in general relativity the interpretation of the diffeomorphism-invariant structure is easy to state in words: it is chronogeometric; it is about how the spacetime points stand in relation to each other in a network. As Einstein already realized, “Space-time does not claim existence on its own, but only as a structural quality of the field.” (Einstein, 1920, pp. 155), [my italics]. The sophisticationist identifies the structural content of the theory with the set of symmetry-invariant quantities, and he takes these, and only these quantities to denote, or to have ontic significance.

As we will see in the next section, Section 3, in the case of Yang-Mills theories the same considerations apply, mutatis mutandis, with quidditism in place of haecceitism and properties in place of objects (or points). And, as I will argue more fully (particularly in Section 3.4), there I also take the passive-active correspondence to be central to a conceptually transparent extension of sophistication to the case of gauge theories.

3 Gauge transformations in Yang-Mills theories

This section will explore details of symmetries in gauge theories: more specifically, of Yang-Mills theories.

Speaking metaphysically, the previous Section construed the symmetries of general relativity as isomorphisms of a natural geometric structure. And a natural misgiving is that the symmetries of gauge theory are less natural, and thus have a less natural structural interpretation than general relativity.

I believe that the concern is indeed justified in the case of gauge transformations in the gauge-potential formalism for electromagnetism, which we discuss in Section 3.1. But that formalism is not the last word in the theoretical development of Yang-Mills theories. In Section 3.2 I motivate the need for a more complete, geometric understanding of what the fields and gauge symmetries of modern physics are about. Section 3.3 presents the mathematical formalism that we will need going forward in more detail. In Section 3.4, I will use this geometric understanding of Yang-Mills theory to re-assess the question of whether there is a clear sense in which the gauge symmetries are as natural as the symmetries of general relativity. I will then adjudicate whether we can formulate for Yang-Mills theories a position analogous to sophisticated substantivalism in general relativity. With this presentation of these theories, I hope to undermine several philosophers’ strong intuitions that there are strong conceptual differences from general relativity.

3.1 Symmetries need not be isomorphisms: an example from gauge theory

In electromagnetism, the basic dynamical variable is the electromagnetic field tensor, $F_{ab}$. Upon choosing a spacetime split into spatial and time directions, the components of the electromagnetic tensor become the familiar electric and magnetic fields: $F_{i0} = E_i$, and $F_{ij} \epsilon^{jk} = B_i$ (where we used the three-dimensional totally-antisymmetric tensor, $\epsilon$, or the spatial Hodge star, to obtain a 1-form).

The Maxwell equations are written, in terms of $F_{ab}$, as:

$$\partial^a F_{ab} = j_b, \quad \text{and} \quad \partial_{[a} F_{bcd]} = 0,$$

(3.1)
where \( j \) is the current and square brackets denote anti-symmetrization of indices. The second equation of (3.1) is called ‘the Bianchi identity’, and it is read as a constraint on the field tensor. A geometric explanation for this constraint is that \( F_{ab} = \partial_a A_b \), or, in exterior calculus notation, \( dA = F \), where \( A_a \) is called the gauge-potential.

Gauge-potentials for electromagnetism are locally just smooth one-forms on the manifold, and the natural notion of isomorphism here is just the one inherited from differential geometry: again, pull-backs by diffeomorphisms. That is, the KPMs of the theory are given by \( \langle M, A \rangle \), where \( A = A_a dx^a \), i.e. the potentials are sections of the cotangent bundle—real-valued one-forms over each topologically trivial patch—on the manifold \( M \). Since they are differential forms, we could rehearse the argument of Section 2.1 and conclude that the isomorphisms of the space of models are again pull-back by diffeomorphisms.

But the dynamics of the theory are another matter. The equations of motion of this theory—now assuming \textit{in vacuo}, i.e. \( j = 0 \), for simplicity—are:

\[
\partial^a \partial_b A_a - \partial^a \partial_a A_b = 0.
\]

These equations are obtained from the action functional:

\[
S[A] := \int \partial_a A_b \partial^a A^b = \int *F \wedge F,
\]

where \( * \) is the Hodge-star operator (which takes an argument to its dual) and \( \wedge \) is the exterior (wedge) product between forms. (The action functional provides a more complete characterization of the theory, since it can be used as a starting-point for quantization within either the Lagrangian or Hamiltonian formalisms.) If we then follow the definition of symmetries given at the beginning of Section 1.2, we arrive at the standard gauge transformations.\(^{22}\)

Namely, it is easy to see that, since \( \partial_a \partial_b = 0 \), the transformations that preserve the value of (3.3) for any \( A \) (and that take \textit{any} solution of (3.2) to another solution), consist in adding the gradient of a smooth function to the gauge-potential one-form: \( A \to A + d\xi \), for \( \xi \in C^\infty(M) \), and where \( d \) is the exterior derivative.\(^{23}\) The dynamical symmetries are therefore ‘larger’ than those expected from the geometric properties of the fields.

But as we will see in the next section, there is a formulation of gauge theory that articulates its symmetries in a more ‘organic’ fashion.

### 3.2 Fiber bundles as the mathematical representation of fields and symmetries

The modern mathematical formalism of gauge theories relies on the theory of principal and associated fibre bundles. We will not give a comprehensive account here (cf. (Kobayashi & Nomizu, 1963)), but only introduce the necessary ideas.

Our intuitive idea of a field over space is something like temperature. A temperature field can be written as a map from space to the real numbers, \( T : M \to \mathbb{R} \). Being told that there are fields that have a more complicated ‘internal structure’ than temperature—for instance, vector fields that over each point of spacetime can point in different directions—we may want to generalize the scalar map above to \( \rho : M \to F \), a map from spacetime to some internal vector space \( F \).

\(^{22}\)See also footnote 13 for a more thorough account of how we would go about defining the symmetries also in this case.

\(^{23}\)This conclusion could be reached following essentially the same procedure advocated in footnote 13. Note that the symmetries involve only differential geometric operations—such as exterior differentiation—and thus composition with diffeomorphisms is well-defined. Indeed, the two operations commute, since the exterior derivative commutes with the pull-back: for \( f \in \text{Diff}(M) \), the object and arrow \((A,\xi)\) gets mapped to \((f^* A, f^* \xi)\).
For tensor bundles, made up of tensor products of tangent and cotangent vectors, $F$ is “soldered” onto spacetime, $M$. But the fields employed in modern theoretical physics live in more general vector bundles, $F$, which are not thus soldered to spacetime. Generically, those fields have many components at each point, which are not associated to spacetime directions.

The worry might arise that to examine the symmetry structure of a certain gauge group we would have to examine its action for each matter field separately: how it acts on electrons, on neutrinos, on quarks, etc; and these actions could, in principle, differ in their general features. But nature is kind: the symmetry group acts similarly, though perhaps with different representations on the various matter fields, meaning that the parallel transport of internal quantities is compatible for all the fields. This ‘coincidence’ is conveniently described if we encode the symmetries through the formalism of principal fiber bundles (PFBs): they contain the essential symmetry structure of each type of interaction—e.g. electromagnetic—indeed of the individual matter fields that are susceptible to this interaction.

The first subsection below, Section 3.2.1, will present the main idea of principal fiber bundles. The aim of this section is to convince the reader through non-mathematical arguments that a principal fiber bundle admits a structural interpretation of its relevant quantities, to the same extent that the metric admits a structural interpretation of its relevant quantities. The second subsection, Section 3.2.2, will show, through a more familiar example, how it is that principal bundles can orchestrate the interaction of a single given force with all the various matter fields.

3.2.1 Principal fiber bundles: the main idea

States of different species of matter are represented in (as sections of) different vector bundles: one vector bundle per field. A principal fiber bundle ‘orchestrates’ the symmetry properties of all these matter fields. As articulated convincingly by Weatherall (2016): even if vector bundles represent possible local states of matter, the connection of a principal bundle orchestrates the symmetry properties of all the fields that interact through some given force. Charged scalar fields, electron fields, quark fields, etc., all interact electromagnetically; and indeed they respond to the same electromagnetic fields (mutatis mutandis, for other interactions, e.g. replacing ‘electromagnetism’ by the ‘strong force’). This means that the covariant derivative operators on the vector bundles in which these fields are valued have the same parallel transport and curvature properties. Such universality is mathematically enforced because these vector bundles are associated to the same connection on that principal bundle, and this means they have their covariant derivative operators defined by that connection.

We can thus, with a clear conscience, focus our efforts on understanding symmetry as it is mathematically manifested in a principal fiber bundle formalism. And the main idea underlying the physical significance of this symmetry structure is perhaps best summarized in the original paper by Yang & Mills (1954):

The conservation of isotopic spin is identical with the requirement of invariance of all interactions under isotopic spin rotation. This means that when electromagnetic interactions can be neglected, as we shall hereafter assume to be the case, the orientation of the isotopic spin is of no physical significance. The differentiation between a neutron and a proton is then a purely arbitrary process. As usually conceived, however, this arbitrariness is subject to the following limitation: once one chooses what to call a proton, what a neutron, at one space-time point, one is then not free to make any choices at other space-time points.

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24For instance, we can identify elements of the tangent bundle with tangent vectors of curves on the base manifold.
What is a proton and what is a neutron at a given point is essentially a *relational* or, more broadly, a *structural* property in $P$.  

The only physically relevant information seems to be sameness across different points of spacetime: thus, once we label a given particle as e.g. a proton at one point of spacetime, the structure of the bundle specifies what would also count as a proton at a neighbouring spacetime point. These constraints are imposed by a *connection-form*: the main geometric structure of the bundle. A connection-form $\omega$ allows us to define which points of neighbouring fibres can be taken as equivalent to an arbitrary starting-off point in an initial fibre. In this framework, *curvature* acquires meaning as non-holonomicity. Let $p$ be a given point in the bundle; and take its projection onto spacetime, $x$ to be the starting point of two spacetime curves that later reconverge to another spacetime point, $y$. These two curves have a unique type of ‘lift’ to curves in the bundle passing through $p$, called a *horizontal* lift: such lifts represent parallel transport. Even though the projected paths in $M$ close-off at $y$, the endpoints of their horizontal lifts will in general differ. It is this disagreement that carries physical consequences. That is, the bundle encodes structural, or relational, properties, that arise from comparisons: and which, at least infinitesimally, are captured by certain function(al)s of the connection, namely, the curvature. And yet, globally, or non-infinitesimally, these comparisons may still carry information that is not captured by the curvature.

### 3.2.2 PFBs from tangent spaces

To gather intuition about principal fiber bundles (PFBs) as the ‘organizers’ of symmetry principles, it is worthwhile to introduce them in the context of the familiar tangent vector fields on $M$.

I begin with the main idea of a fibre bundle and then consider the tangent bundle. The main idea of fiber bundles is that they are spaces that locally look like a product, i.e. a fiber ‘bundle’. So the many fields of nature would be represented as maps that take each point of spacetime (or space) into its respective value space, or fiber.

We denote fiber bundles by $E$; they are smooth manifolds that admit the action of a surjective projection $\pi : E \to M$ so that locally $E$ is of the form $\pi^{-1}(U) \simeq U \times F$, for $U \subset M$ and $F$ is some ‘fiber’: a space that ‘inhabits’ each point of $M$ and in which the fields take their values.

But the decomposition $\pi^{-1}(U) \simeq U \times F$ is not unique, and will depend on what is called ‘a trivialization’ of the bundle, which is basically a coordinate system that makes the local product structure explicit. Thus, in principle there is no unique identification of an element of $F$ at a point $x \in M$ with an element of $F$ at a point $y \in M$. In principle, there is no identification of a vector, or even of a scalar quantity, like temperature, as possessed at different points of spacetime.

So, to be explicit: $F$ is some space where we can have quantities in spacetime take their value; for instance, a scalar field could take values in $\mathbb{R}$ or $\mathbb{C}$, whereas a more complicated field such as a vector field or a spinor field, could take values in $\mathbb{R}^4, \mathbb{C}^4$, etc. A choice of *section* of the bundle represents fields taking values in $F$: e.g. a spinor field, or a quark field, etc, which are all vector bundles, in that $F$ is a vector space. A field-configuration for $E$ is called *(confusingly, see Section 3.3 and footnote 33)* a *section*, and it is a map $\kappa : M \to E$ such that $\pi \circ \kappa = \text{Id}_M$. Sections replace the functions $\tilde{\kappa} : M \to F$, that we would employ if the fields that physics used had a fixed, or “absolute”—i.e. spacetime independent—value space. We denote smooth sections like this by $\kappa \in C^\infty(E)$. 

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25Of course this example, which originally motivated Yang and Mills, applies only in the context of the (approximate) isospin symmetry. Otherwise, the electric charge tells protons and neutron apart in an intrinsic manner.
A useful example of a vector bundle is the tangent bundle, $TM$. A smooth tangent vector field is a smooth assignment of elements of $TM$ over $M$, denoted $X \in C^\infty(TM)$, with $\pi : TM \to M$, mapping $X \in T_xM \to x \in M$. The tangent bundle $TM$ locally has the form of a product space, $U \times F$, with $F \simeq \mathbb{R}^4$. But even if $TM$ was globally trivializable, so that a product structure could be found for its totality, this would not mean we could identify an element $v \in \mathbb{R}^4$ at different points of $M$. Differential geometry teaches us to attach a vector space to each point of $M$ and to have vectors at different points objectively related only according to some definition of parallel transport along paths in $M$.

This example is also useful to articulate what we mean by a principal fiber bundle that ‘orchestrates the parallel transport’ of the other fields. Here the principal bundle that orchestrates parallel transport of tangent vectors (and tensor bundles in general) can be taken to be the bundle of linear frames of $TM$, called ‘the frame bundle’ (where ‘frame’ means ‘basis of the tangent space $T_xM$’), written $L(TM)$. The fibre over each point of the base space $M$ consists of all of the linear frames of the tangent space there, i.e. all choices \( \{e_I(x)\}_{I=1,\ldots,4} \in L(TM) \), of sets of spanning and linearly independent vectors (here the index $I$ enumerates the basis elements).

So each point $p \in P$ of the frame bundle above a point $x \in M$ (i.e. such that $x = \pi(p)$) is just a basis for the tangent space $T_xM$; and there is a one-to-one map between the group $GL(\mathbb{R}^4)$ and the fibre: we can use the group to go from any frame to any other (at that same point), but there is no basis that canonically corresponds to the identity element of the group. This example illustrates a feature of principal fiber bundles that distinguishes them from vector bundles: in the former, the fibers are isomorphic to some Lie group $G$; and there is no “zero” or identity element on each fibre, as there is in a vector bundle.

![Figure 1: A principal bundle over spacetime, with $G$ as a structural group. $[\gamma]$ is a curve on spacetime, that is horizontally lifted to $\gamma$, in $P$.](image)

If we imagine the orbits of the group, or the fibers, as being in the vertical direction, as in Figure 1, directions transversal to the fiber will connect frames over neighbouring points of $M$. We thus dub as horizontal those directions by which a connection identifies—or ‘links’ and takes as identical—frames on neighbouring fibers.\(^{27}\) That is: to link fibres, we need to postulate more structure: a connection (cf. end of Section 3.2.1).

\(^{26}\)Depending on the theory, we will take different subsets of the linear frames, and of the corresponding structure group. For instance, for general relativity, we take the structure group as $O(4)$ (or $SO(3,1)$) acting on the orthonormal bases.

\(^{27}\)In general relativity, we could take this to be a torsion-free connection-form on $P$ by $de^I = \omega^I_J e^J$, where $\omega$ here satisfies the expected equations (3.5) (and we used the one-forms algebraically dual to the vector basis: $e^I(e_j) = \delta^I_j$). This equation translates to one using the covariant derivative $\nabla$ as: $\nabla e_I = \omega^I_J e_J$. 

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To see how these horizontal directions encode parallel transport of vectors, we need to return to the tangent bundle \( TM \), from the frame bundle, \( L(TM) \). We proceed as follows:

- Take a point of \( TM \), i.e. a vector at a given point \( x \in M \), \( X_x \in F \) as an element of the fiber \( F = \mathbb{T}_xM \cong \mathbb{R}^4 \), where the ordered quadruplet are the components of \( X_x \) according to a frame, \( \{ e_I(x) \} \in L(TM) \). So, we write \( X_x = a^I e_I \in T_xM \) as the ordered quadruplet \((a^1, \cdots, a^4) \in \mathbb{R}^4 \). 
  - Of course, if we rotate the frame by an element of the group in question, i.e. \( GL(\mathbb{R}^4) \), say by a matrix \( g^{IJ} = \rho(g) \), where \( \rho: G \to GL(\mathbb{R}^4) \) is the matrix representative of the abstract group, then, as long as we undo that rotation on the components, we obtain the same vector, in the original frame. That is, \( a^K g^{-1}_K g^{LI} e_I = a^I e_I \). Thus, if we write a doublet \((p,v)\) as, respectively, the frame and the components, we want to identify \((gp, vg^{-1})\) (where we have simplified the notation for the action of the group to be just juxtaposition). This is a standard construction of an associated bundle, denoted by \( TM \cong L(TM) \times \rho \mathbb{R}^4 \). 

Once we have constructed associated bundles in this way, parallel transport, for any vector bundle comes naturally from a notion of horizontality in the principal bundle. To find the parallel transport of the vector \( X_x \) along \( Y_x \), we:

(i) choose one frame \( p \) at \( x \), and find the corresponding—parallel transported—frames as one moves horizontally along (a direction \( \tilde{Y}_p \) that projects to) \( Y_x \),
(ii) write out the component of \( X \) at \( x \) in that frame. If, along \( Y \), \( X \) were equal to its parallel transport, these components would remain numerically constant, since the frame is assumed to be the same, or parallel transported, i.e. identified across points of spacetime. So we can
(iii) compare the parallel transported components of \( X \) with the actual components of \( X \); their non-constancy corresponds to the failure of \( X \) to be parallel transported, and to the non-vanishing covariant derivative of \( X \). In this way a covariant derivative is just the standard derivative of the components in the horizontal—or parallel transported—frame. This is, in words, the description of the covariant derivative of \( X \) along \( Y \) at \( x \in M \).

The picture is useful in that it applies to any vector bundle on which the structure group \( G \) in question acts. For instance, in the standard model of particle physics, the fundamental forces are associated to Lie groups, and each field that interacts via such a force lives in a vector bundle that admits an action of the corresponding group. Thus for a given vector bundle with typical fiber \( F \), we have a linear representation of the Lie group in question, \( G \), \( \rho: G \to GL(F) \), and we can take the principal connection—the notion of horizontality in the PFB with structure group \( G \)—to induce a notion of parallel transport in the bundle \( E \) with fiber \( F \). Indeed, we can take the same procedure as above, building a linear frame for \( F \) at each point; parallel-transport then encodes an appropriate \( G \)-covariant way to identify vector values along paths in the base space \( M \).

### 3.3 Principal fibre bundles: interpreting the formalism

We have introduced the the main idea of a principal fiber bundle in Section 3.2.1, and its function, i.e. as determining parallel transport, in Section 3.2.2. Now we give the formal definitions. In Section 3.3.1 I will briefly introduce the general formalism for the principal bundles, including the idea of connection forms and gauge potentials. In Section 3.3.2 is parallel to Section 2.2.1: there I will discuss the relationship between active and passive gauge transformations, and the relation between passive transformations and changes of bases of a given frame bundle. In Section 3.3.3 I introduce the Atiyah-Lie algebroid, as a global, spacetime representative of the connection-form. Finally, in Section 3.3.4 I provide what I judge to be a perspicuous physical interpretation of the formalism.
3.3.1 The general construction

A principal fibre bundle is, in short, just a manifold where some group acts. In detail: it is a smooth manifold \( P \) that admits a smooth free action of a (path-connected, semi-simple) Lie group, \( G \): i.e. there is a map \( G \times P \to P \) with \((g,p) \mapsto gp\) for some left action · and such that for each \( p \in P \), the isotropy group is the identity (i.e. \( G_p := \{ g \in G \mid g \cdot p = p \} = \{ e \} \)). Naturally, we construct a projection \( \pi : P \to M \) onto equivalence classes, given by \( p \sim q \iff p = g \cdot q \) for some \( g \in G \). That is: the base space \( M \) is the orbit space of \( P \), \( M = P/G \), with the quotient topology, i.e. it is characterized by an open and continuous \( \pi : P \to M \). By definition, \( G \) acts transitively on each fibre, i.e. orbit. The automorphism group of \( P \)—those transformations that preserve the structures—are fibre-preserving diffeomorphisms \( \tau : P \to P \), i.e. such that \( \tau(g \cdot p) = g \cdot \tau(p) \). Purely internal, or gauge transformations can be identified as those for which \( \pi \circ \tau \circ \pi^{-1} = \text{Id}_M \); that is, as purely ‘vertical’ automorphisms of the bundle; (the orbits are usually drawn going up the page, as in Figure 1, hence ‘vertical’).

- The Ehresmann connection-form. On \( P \), we consider an Ehresmann connection \( \omega \), which is a 1-form on \( P \) valued in the Lie algebra \( \mathfrak{g} \) of \( G \) that satisfies appropriate compatibility properties with respect to the fibre structure and the group action of \( G \) on \( P \). We will first see how such a \( \mathfrak{g} \)-valued 1-form on \( P \) selects a “vertical” subspace of the tangent space \( TP \) at \( p \in P \), which “points in the direction of the fiber”, and how it selects a “horizontal” subspace—which gives the notion of parallel transport linking nearby fibres, which we introduced in Section 3.2.1.

Given an element \( \xi \) of the Lie-algebra \( \mathfrak{g} \), we define the vertical space \( V_p \) at a point \( p \in P \), as the linear span of vectors of the form

\[
v_\xi(p) := \frac{d}{dt} \vert_{t=0} (\exp(t\xi) \cdot p), \quad \text{for} \quad \xi \in \mathfrak{g}.
\]

And then the conditions on \( \omega \) are:

\[
\omega(v_\xi) = \xi \quad \text{and} \quad L_g^* \omega = g^{-1} \omega g,
\]

where \( L_g^* \omega_p(v) = \omega_{gp}(L_g v) \) and where \( L_g \) is the push-forward of the tangent space for the left-action \( g : P \to P \). Thus, we can only characterize the action of \( \omega \) on vector fields on \( P \), i.e. on sections of the vector bundle \( TP \), say \( \zeta \in C^\infty(TP) \), if they are left-invariant, i.e. if \( \zeta_{gp} = L_{g\tau} \zeta_p \). Such vector fields generate the automorphisms of \( P \).

At each orbit, we obtain the infinitesimal transformation:

\[
\mathcal{L}_{v_\xi} \omega = [\xi, \omega].
\]

But if the vector field \( \zeta \) as above is the generator of a vertical automorphism, we obtain, instead of (3.6),

\[
\mathcal{L}_{\zeta} \omega = [\omega(\zeta), \omega] + d_v \zeta,
\]

where \( d_v \) is here the exterior derivative on the smooth manifold \( P \).

A choice of connection is equivalent to a choice of covariant ‘horizontal’ complements to the vertical spaces, i.e. \( H_p \oplus V_p = TP \), with \( H \) compatible with the group action. That is, since \( \omega \) is \( \mathfrak{g} \)-valued and gives an isomorphism between \( V_p \) and \( \mathfrak{g} \), the first condition of (3.5) means that: i) the kernel \( \text{Ker}(\omega_p) = H_p \), and ii) since \( V_p = \text{Ker}(\pi_*) \), \( H_p \) will be 1-1 projected by \( \pi_* \) onto the tangent space \( T_{\pi(p)} M \). Thus the vectors spanning \( \text{Ker}(\omega_p) \) are the so-called horizontal vectors in the bundle, and each represents a unique ‘horizontal lift’ at \( p \) of a direction at \( T_{\pi(p)} M \). This

\[28\text{We could also write this non-infinitesimally, in the more traditional notation: } \tau^* \omega = \Psi \omega \Psi^{-1} + \Psi^{-1} d_v \Psi \text{ where } \tau(p) = \Psi(p) \cdot p, \text{ as introduced below.}\]
The forces and interactions “communicated” to all the
frame commute with propagation. The curvature of the connection.
– The curvature of the connection. The forces and interactions “communicated” to all the
vector bundles by \( \omega \) are encoded by the curvature of \( \omega \), a Lie-algebra-valued 2-form on \( P \):

\[
\Omega = d_p \omega + \omega \wedge_p \omega, \tag{3.8}
\]

where \( \wedge_p \) is the exterior product on \( \Lambda(P) \); it gives anti-symmetrized tensor products of dif-
ferential forms. Using the decomposition of the tangent space \( H_p \oplus V_p = T_p P \), we have associated
orthogonal projectors, \( \hat{H}_p \) and \( \hat{V}_p \), with \( \hat{H} : p \mapsto \hat{H}_p : T_p P \to H_p \), we can rewrite the curvature
(3.8), using (3.4), as:

\[
v_{\Omega(\bullet, \bullet)} = \hat{V}(\hat{H}(\bullet), \hat{H}(\bullet)|_{TP}), \tag{3.9}
\]

where \( \bullet \) is used as the open slot of a differential form, and the square brackets here denotes
the commutator of vector fields on \( P \). The intuitive idea is as before: one goes around an
infinitesimal horizontal parallelogram and finds a certain displacement along the orbit.

The analogue of (3.7) for the curvature is:

\[
\mathcal{L}_\zeta \Omega = [\omega(\zeta), \Omega]. \tag{3.10}
\]

In other words, the curvature is fully left-invariant: there is no inhomogeneous term in its
transformation. This is the crucial property that will, in Section 4, distinguish the gauge
symmetries from the diffeomorphisms, yielding the distinction that I labelled \( \Delta \) at end of
Section 1.2.

– The gauge and curvature potentials. Given local sections \( s \) on each chart \( U_\alpha \), i.e. maps
\( s : U_\alpha \to P \) such that \( \pi \circ s = \text{id} \), we define a local spacetime representative \( A \) of \( \omega \), as the
pullback of the connection, \( A^s := s^* \omega \in \Lambda^1(U_\alpha, \mathfrak{g}) \); (here \( s \) is not a spacetime index; we keep it
in the notation as a reminder of the reliance on a choice of section).\(^{30}\) We will expand on
the significance of these sections in Section 3.3.2 below.

In a basis for a given chart on \( U \subset M \), we write: \( A = A^I_a \, dx^a \tau_I \), \( \tau_I \in \mathfrak{g} \) is a Lie-algebra
basis, and \( A^I_a \in C^\infty(U) \).\(^{31}\) As in (3.18), vertical automorphisms are represented as gauge
transformations, which, infinitesimally, for a Lie-algebra valued function \( \xi^a \in C^\infty(U, \mathfrak{g}) \), act as

\[
\delta_\xi A^I_a = \partial_a \xi^I + [A_a, \xi^I] = D_a \xi^I, \tag{3.11}
\]

\(^{29}\)We could also write this analogue non-infinitesimally, in the more traditional notation: \( \tau^* \Omega = \Psi \Omega \Phi^{-1} \)
where \( \tau(p) = \Phi(p) \cdot p \) (cf. the previous footnote, 28).

\(^{30}\)Note that \( A \) only captures the content of \( \omega \) in directions that lie along the section \( s \). The vertical component
of \( \omega \)—which is dynamically inert, as per the first equation of (3.5)—can be seen (in a suitable interpretation of
differential forms, cf. Bonora & Cotta-Ramusino (1983)) as the BRST ghosts. This interpretation geometrically
encodes gauge transformations through the BRST differential Thierry-Mieg (1980). Although interesting in its
own right, we will not explore this topic here. See Gomes (2019); Gomes & Riello (2017) for more about the
relationship between ghosts and the gluing of regions.

\(^{31}\)Clearly, \( I \) are Lie-algebra indices and \( a \) are spacetime indices. We take \( \{dx \otimes \tau\} \) to stand in for the frame
discussed in Section 3.2.2, as the basis for a vector bundle \( T^* U \otimes \mathfrak{g} \).
where $D_a(\bullet) = \partial_a(\bullet) + [A_a, \bullet]$, the gauge-covariant derivative, is defined to act on Lie-algebra valued functions.

Since the exterior derivative and the pullback operation commute, we also have, from (3.8) for the spacetime representative of the curvature:

$$F^s := s^*\Omega = dA^s + A^s \wedge A^s$$

(3.12)

where now $d$ and $\wedge$ are the familiar exterior derivative and products in $\Lambda(M)$. But, unlike the gauge potential (cf. (3.11)), the curvature transforms homogeneously under a gauge transformation:

$$\delta_\epsilon F^s_{ab} = [F^s_{ab}, \epsilon] = [A^s_{ab}, \epsilon].$$

(3.13)

Later, in Section 3.3.3, we will see how the Atiyah-Lie-algebroid enables us to define global spacetime representatives of $\omega$ (and $\Omega$) in coordinate-independent ways.

### 3.3.2 Active and passive correspondence

As with the definition of a manifold using an atlas (cf. Section 2.2), here too, the intrinsic construction of bundles above “hides under the hood” the explicit formulation via local trivializations. Namely, we use local trivializations and conditions on the transition functions between charts to define the bundle structure. Then, as in Section 2.2.1, we can find a straightforward correspondence between active and passive gauge transformations.

- **Local sections** 
  Locally over $M$, it is possible to choose a smooth embedding of the group identity into the fibres of $P$. That is, for $U \subset M$, there is a map $s : U \to P$ such that $P$ is locally of the form $U \times G$. Namely, $s$ induces a diffeomorphism $U \times G \simeq \pi^{-1}(U)$, given by $\pi : U \times G \to P$, such that:

$$\pi : (x, g) \mapsto g \cdot s(x),$$

whose inverse is $\pi^{-1} : p \mapsto (\pi(p), g_s(p)^{-1})$

(3.14)

where $g_s : \pi^{-1}(U) \to G$ gives $g_s(p)$ as the unique group element taking $p$ to the local section, i.e. $g_s(p)$ is the group element such that $g_s(p) \cdot p = s(\pi(p))$.\footnote{The precise form of $g_s$ will of course depend on $s$.} Thus we have a condition:

$$g_s(g^p) = g_s(p)g^{-1}.$$  

(3.15)

Call this **equivariance** of $g_s$ between the given action of $G$ on $P$ and $G$’s action on itself by conjugation.

The maps $s$ are called **local sections** of $P$.\footnote{It is somewhat confusing that a section of a vector bundle is an entirely different object: it is a vector field. So, for instance two different choices of the electron field are two different sections of its vector bundle, and thus are not counted as ‘equivalent’ in the way that two sections of a principal bundle are; it is for this reason that we chose different notations for them ($s$ as opposed to $\kappa$).} We can also define local sections without reference to spacetime: as submanifolds of $P$ that intersect each orbit in an open set only once (and thereby transversally). This definition is more useful when we only look at geometric quantities that are intrinsic to $P$.

A transition between the charts implied by $s$ and $s'$ over the same $U$ takes an $(x, g)$ in the domain of $s$ to an element in $U \times G$ as the domain of $s'$ by first taking $(x, g) \mapsto p = g \cdot s(x)$ and then using the inverse $p \mapsto (\pi(p), g_{s'}(p)^{-1})$. Namely:

$$(x, g) \mapsto \left(\pi(g \cdot s(x)), (g_{s'}(g \cdot s(x)))^{-1}\right) = \left(x, (g_{s'}(s(x))g^{-1})^{-1}\right) = \left(x, g(g_{s'}(s(x)))^{-1}\right).$$

(3.16)
then such transition functions satisfy the cocycle conditions: $\psi_{ss} = \text{Id}$ and $\psi_{ss'}\psi_{s's'}\psi_{ss'}^{-1} = \text{Id}$, which are, from the perspective of an atlas, just conditions that the transitions between charts must satisfy. Thus the intrinsic bundle structure is defined through charts and their allowed transition functions, and we can again call this type of definition chart-nominalism, as we did in Section 2.2.1 for the smooth manifold structure.

And a vertical automorphism locally induces a diffeomorphism between two sections $s$ and $s'$, i.e. a map $\psi_{ss'}$ as above. For vertical automorphisms $\tau$ can be represented with $\Psi : P \to G$, where $\tau(p) = \Psi(p) \cdot p$ with $\Psi(g \cdot p) = g\Psi(p)g^{-1}$, which is $\Psi$'s equivariance condition.

Then any vertical automorphism $\tau$ induces a diffeomorphism of $U \times G$, as follows. Let $\tau(p) := \Psi(p) \cdot p$, as above. Then, for a section $s$ and a general $p \in \pi^{-1}(U)$ we write $p = \bar{s}(x, g) = g \cdot s(x)$ and therefore $\tau(p) = \tau(\bar{s}(x, g))$ gives:

\[
\tau(p) = \Psi(g \cdot s(x)) \cdot (g \cdot s(x)) = (\Psi(g \cdot s(x))g) \cdot s(x);
\]

(3.17)

(where we only use the fact that the group acts on the left on $P$). As expected, $\tau$ just takes $s$ to a different section, $s' := \Psi(s) \cdot s$. Moreover, since $\bar{s}^{-1}(g \cdot s(x)) = (x, g)$, we obtain that $\bar{s}^{-1} \circ \bar{s}$ is a ‘coordinate transformation’, or diffeomorphism of $U \times G$, in analogy to (3.16):

\[
(x, g) \mapsto (x, \Psi(g \cdot s(x))g) = (x, g\Psi(s(x))),
\]

(3.18)

where we used the equivariance property of $\Psi$.

In sum, over each patch, vertical automorphisms are in 1-1 correspondence with elements $\psi_s := \Psi \circ s : U \to G$, that is, $\psi_s \in G$, which we call gauge transformations; these are the local, passive counterparts of the active $\Psi : P \to G$, described above (and, to be defined, they require a trivialization).

We can also characterize $\psi_s$ by using a second section $s'$, as above, for which $\psi_s \cdot s = s'$, and thus label it $\psi_{s's}$. This second section $s'$ is, of course such that $\tau(s(x)) = s'(x)$ for $x \in U$. This means that, for any function $f : \pi^{-1}(U) \to \mathbb{R}$, if we represent it in a trivialization, i.e. such that $f_s : U \times G \to \mathbb{R}$ (with $f(p) = f_s(x, g)$, where $p = g \cdot s(x)$) we will find that $\tau^* f = f_s$, under a different trivialization $s'$, has the same coordinate representation as the original function had under $s$, i.e. $f_s = f_{s'}$. This is the correspondence between active and passive transformations on each trivialization patch, that we saw in Section 2 for diffeomorphisms of spacetime.

-- Passive gauge transformations as changes of bases. In the construction of Section 3.2.2 that we used as a motivation, the principal bundle $P$ was originally identified with $L(TM)$ and the vector bundle $E$ was identified with $TM$, and we took $G = GL(\mathbb{R}^4)$. In other words, $P$ was the space of frames of the tangent bundle, with no preferred frame and in which the group acted transitively on frames over each point $x \in M$. This elucidating construction can naturally extended to a more general setting, and to a general understanding of principal fiber bundles, as discussed in Weatherall (2016).

Given some general vector space $F$ and structure group $G$ and $\rho : G \to GL(F)$, and $P$ a $G$-principal bundle over $M$, we can find the associated vector bundle over $M$, which is denoted $E := P \times_\rho F$. Conversely, the frame bundle for a given vector bundle $E$, $L(E)$ (formed by the bases of $E_x$ for each $x \in M$) is a principal bundle $P'$ with structure group $GL(F)$. But we can form another principal bundle $P$, as a sub-bundle of $P'$ as follows. Since $P' \simeq L(E) = L(P \times_\rho F)$ we can see $P$ as a sub-bundle of $L(P \times_\rho F)$ corresponding to a subset of frames of $L(P \times_\rho F)$.

\[\text{Since the map } \psi_s : U \times G \to G \text{ given by } (x, g) \mapsto g\Psi(s(x)) \text{ is smooth (since } s \text{ and } \Psi \text{ are), and, for fixed } x, \text{ the } \psi_{s(x)} : G \to G \text{ given by } g \mapsto g\Psi(s(x)) \text{ is clearly a diffeomorphism of } G \text{ (since it is just the action of } G \text{ on the element } \Psi(s(x))). \text{ The inverse is of course just } (x, g) \mapsto (x, g\Psi(s(x))^{-1}), \text{ which enjoys the same properties.}\]
related by $\rho(G)$.\footnote{Of course, this raises a puzzle: if the principal bundle is construed as just a bundle of linear frames, how can we justify the restriction of $\rho(G)$ to a subset of the most general group of transformations between frames? As discussed by (Weatherall, 2016, Sec. 4), the restriction corresponds to the preservation of some added structure to $F$. In other words, when $F$ is not just a bare vector space, but e.g. a normed vector space, we would like changes of basis to preserve this structure, e.g. the orthonormality of the basis vectors, and this restricts the bundle of linear frames to the appropriate sub-bundle. To see this, define $P \times_\rho F$ as the equivalence class for the doublet $(p,v) \in P \times F$ with $(p,v) \sim (g \cdot p, \rho(g^{-1})v)$. Suppose that $F$ is a Riemannian vector space, with metric $(\cdot,\cdot)$. We can induce a metric in $P_F = P \times_\rho F$ defining, for any $p$ and $v, v' \in F$: $\langle [p,v],[p,v'] \rangle := \langle v,v' \rangle$. To be well-defined, we must have:}

As emphasized by Weatherall (2016, p. 2401), the conceptual advantage of this construal of $P$ is that, assuming the action of $\rho : G \to GL(F)$ is faithful, we can interpret passive gauge transformations as just point-dependent changes of bases of the value space $F$ (i.e. the allowed changes of frames for $E$). In other words, a section $s : U \to P$ may be understood as a frame field for a certain vector bundle $(P \times_\rho F)$, and changes of section may be understood as the allowed change of basis at each point. Weatherall (2016, p. 2404) writes:

> We are thus led to a picture on which we represent matter by sections of certain vector bundles (with additional structure), and the principal bundles of Yang–Mills theory represent various possible bases for those vector bundles.

But I think Weatherall misses an important distinction between passive and active gauge transformations when he continues:\footnote{Having said that, there are hints further along the paper that Weatherall (2016, Sec. 5) distinguishes the two types of invariance, and highlights the lack of invariance of the connection-form under the active transformations.}

> [...] these considerations lead to a deflationary attitude towards notions related to “gauge”: a choice of gauge is just a choice of frame field relative to which some geometrically invariant objects [...] may be represented, analogously to how geometrical objects may be represented in local coordinates.

As I remarked in Section 2.2.1, and as we will return to in Section 3.4.2, invariance under coordinate change can only play this deflationary role once an active-passive correspondence for the symmetries of the theory is established, as it was above.

But this construction leaves a remaining puzzle: once we establish a (structured) vector space that will function as the typical fiber over $M$, we can characterize sections of an associated bundle in a frame-invariant manner, and so, by the correspondence discussed here, in a manner that is invariant under passive gauge transformations. So what happens when we apply this rationale to the typical fibers of the gauge potential, $A$?

### 3.3.3 The Atiyah-Lie algebroid

At first sight, we face one difficulty: $A$ is an object that mixes tensorial indices with internal indices. The natural principal bundle for the tensorial part, as discussed above (see (Weatherall, 2016, Sec. 3)), would be a sub-bundle of the frame bundle $L(TM)$. The internal part, corresponding to $g$, would require a sub-bundle of $L(P \times_\rho g)$.\footnote{Here $\rho = Ad : G \to GL(g)$, where $Ad_g v = g^{-1}vg$ is the natural, adjoint action of $G$ on $g$, appearing in (3.5) and (3.11)).}
To work this out, one would need to splice bundles of such different characters together (see e.g. (Bleecker, 1981, Ch. 7.1)). Although this is possible, it would involve the introduction of yet more formalism. But there is an alternative way, that leads to the same answer (see the Proposition in (Kolar et al., 1993, Ch. 17.5)) and makes explicit use of the purported distinction, \( \Delta \), between diffeomorphism and gauge symmetries, as announced in Section 1.2.

Parallel transport is determined by horizontal directions in the bundle, as we saw in Section 3, and we know that the horizontal bundle \( H \subset TP \), is left-invariant (see text preceding equation (3.6)). So, if we know what parallel transport is at \( p \), we know what it is at \( g \cdot p \). By getting rid of this redundancy, we can find a global spacetime representation of the connection \( \omega \). To do that, we first note that there is a 1-1 relation between connection-forms and left-invariant sections of \( TP \) (see (Kobayashi & Nomizu, 1963, Ch. 4)).

Left-invariant vector fields are not unconstrained sections of the vector bundle \( TP \), i.e. \( C^\infty(TP) \). But they are unconstrained sections of \( TP/G \), the so-called Atiyah-Lie algebroid (see e.g. (Ciambelli & Leigh, 2021, Sec. 3.2); (de León & Zajac, 2020, p.9); (Sardanashvily, 2009, p.60); (Kolar et al., 1993, Ch. 17.4) and (Jacobs, 2020, Ch. 7)). In other words, the difference between sections of \( TP \) and \( TP/G \) is that, while both can be seen as sections over \( TM \) (with \( \pi \), the projection), the latter—\( TP/G \)—is more constrained, since it can only encode left-equivariant objects defined on the first, \( TP \).

The main idea in the construction of this bundle is to take the projection map \( \pi_s : TP \to TM \), and make it ‘forget’ at which point or “height” of the orbit before it was applied. The formalism represents parallel transport of internal quantities for the directions in spacetime, rather than for directions in the bundle \( P \). Thus \( TP/G \) is most naturally a vector bundle over \( TM \) rather than over \( M \) or \( P \). But since \( TM \) is itself a bundle over \( M \), \( TP/G \) can also be construed as a bundle over \( M \).

To define the fiber of \( TP/G \), recall that a point in \( TP \) is locally described by \( (p, v_p) \) with \( v_p \in T_p P \), and the group \( G \) acts (freely and transitively) as \( (p, v_p) \mapsto (g \cdot p, L_{g_\ast}(v_p)) \), which is the relation by which we define the left-invariant vector fields. Thus \( TP/G \) is defined by identifying

\[
(p, v_p) \sim (g \cdot p, L_{g_\ast}(v_p)), \quad \text{for all } g \in G.
\]

Since locally (i.e. given some trivialization of the tangent bundle) for \( x = \pi(p) \) and \( \xi \in \mathfrak{g} \), we can represent \( p = (x, g) := g \cdot s(x) \) and \( v_p = (X_x, \xi) := \xi + s_\ast(X_x) \) we have, locally, \( (p, v_p) = (x, g, X_x, \xi) \). If we take the quotient, we obtain that the elements of the new vector bundle will be locally of the form \( (x, X_x, \xi) \).

Given a point on \( M \), and a tangent direction on \( M \), and a local trivialization of the bundle, an element of the vector bundle \( T^\ast P/G \) spits out a Lie-algebra element. Thus, as in the standard manner of obtaining \( A^\ast \) from \( \omega \), here we also locally recover, in a trivialization, that the representative of the connection, call it \( \Gamma \), is the \( \mathfrak{g} \)-valued 1-form on \( M \); \( \Gamma \) is global, but in a local trivialization, it would be represented by \( A_i^\mu \), where the indices refer to a Lie-algebra and a tangent bundle basis. This is just like the way \( g_{\mu\nu} \) would be locally represented by \( g_{\mu\nu} \), where the indices refer to the tangent bundle basis. The values of \( \Gamma \) according to different trivializations are related by the transformation (3.11), just as the values of \( g_{\mu\nu} \) are related by coordinate transformations. These are correlates of the passive transformations we saw in Sections 2.2.1 and 3.3. Thus we find, as announced in the introduction to this Section, the appropriate analogy, comparing a section of \( TP/G \) with a global vector field, \( \mathbf{X} \), which we can write locally with coordinates, \( X^\mu \partial_\mu \), where \( A^\ast \) stands in analogy to the components \( X^\mu \). Thus, the sections of the bundle \( TP/G \) will be frame-invariant, and therefore, invariant under passive gauge transformations.

We can sum up this section as follows: a section of \( T^\ast P/G \) should be seen as the global, coordinate-independent generalization of \( A^\ast \); the advantage of a section of \( T^\ast P/G \) over the
standard gauge potential and connection-form is that it is globally defined, with $T^*M$ as its base space, and it is independent of internal coordinates (coordinates for the Lie algebra, and tangent bundle); the disadvantage is that it is highly abstract. This formulation allows a strong analogy between the basic kinematical variables of the gauge theory and the metric, in a coordinate-independent manner.

### 3.3.4 The unifying power of the principal connection

To finish this Section, let us briefly focus once again on the geometrical meaning of $\omega$. The unifying power of the principal connection is that it defines compatible parallel transport for any field/particle that interacts with the force associated to $G$, even for the as-of-yet undiscovered forces and groups.

We can think of $\Gamma$, the section of the vector bundle $T^*P/G$, as one more physical field on spacetime. Since it is a section of a certain vector bundle, upon introducing coordinates (or frames) it admits changes of bases with which it is described, and these can be construed as passive gauge transformations. But just as the connection $\omega$ is invariant with respect to these passive transformations, so will be $\Gamma$. Nonetheless, it remains variant under the active transformations. This is analogous to how geometric objects on differentiable manifold can be invariant with respect to passive coordinate transformations, but are not invariant under active transformations.

In physical terms, we can associate fundamental forces to structure groups. We associate each structure group to a field $\Gamma$, as above. Then any field that interacts with this force will couple to the appropriate $\Gamma$. As we move from one point of spacetime to another, it is this coupling that will provide a standard of constancy for the field with respect to that interaction. $\Gamma$ represents a structure: that of covariant differentiation, or parallel transport, of the internal quantities that are sensitive to the given force.

So here we also find a tight analogy to the gravitational case: just as the transformation properties of the connection in a principal bundle dictate the transformation properties of every matter field under the corresponding structure group, so the transformation properties of the metric dictate those of all the other fields under diffeomorphisms. That is, the metric has a unique compatible notion of covariant differentiation, or parallel transport, of the tensor fields on spacetime.

### 3.4 Sophistication for gauge theories

Now we are ready to discuss sophistication for gauge theories. In Section 3.4.1 I will briefly introduce the debates about extending sophistication from general relativity to other theories. Although something like sophistication has been suggested for gauge theories since its inception in terms of principal bundles, in Yang & Mills (1954), only recently has the extension of the position been more thoroughly conceptually analysed (see Dewar (2017)). When can it be applied, and to what interpretative advantage? In Section 3.4.2 I assess how a straightforward formalization of sophistication fares in the case of general relativity, as summarized in Section 2.2.2. I point out that this straightforward formal criterion for a ‘metaphysically perspicuous’ interpretation of the symmetry-related models, namely, that their core ontology be based on a definition of a symmetry-invariant structure, is either too strict or conceptually opaque. By thus reassessing the case of general relativity, I then present a resolution in terms of a correspondence between passive and active symmetry transformations, as elaborated in Section

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38Cf. Lyre (2001) for some comparisons between the equivalence principle and this universality property of the connection (see also (Healey, 2007, Ch. 6.3)).
2.2.1. In Section 3.4.3 I apply this resolution to the case of Yang-Mills theory, and provide the corresponding metaphysically perspicuous interpretation of the theory.

3.4.1 Sophistication on the cheap?

For general relativity, the symmetries of the theory, given in equation (2.2), are induced by the action of the diffeomorphisms on the spacetime manifold. In vacuum, these symmetries coincide with the isomorphisms of a well-understood mathematical structure: (semi-)Riemannian geometry. This coincidence, perhaps aided by our everyday acquaintance with space and time, ease us into accepting the accompanying anti-haecceitist, structural interpretation of diffeomorphism-invariant quantities as being "metaphysically perspicuous", as elaborated in Section 2.2.2.

Now, should we also accept the anti-quidditist, structural interpretation of gauge-invariant quantities as being similarly metaphysically perspicuous? Dewar (2017), in a paper that kicked off considerable recent literature, suggests we should.

To clarify the extension of sophistication to a wider class of theories beyond general relativity, Dewar (2017, p. 502) contrasts it with reduction: while reduction advocates altering the syntax (i.e. the variables) of the theory in order to articulate the common content of the symmetry-related models, sophistication advocates altering the semantics of that theory, such that "we obtain [the new semantics] by taking the old objects, and declaring, by fiat, that the symmetry transformations are now going to "count" as isomorphisms".

We are here in the vicinity of two related philosophical debates about symmetry. The first is about whether symmetry-related models of a given theory should invariably be regarded as being physically equivalent even in the absence of a clear, or metaphysically perspicuous, understanding of the common ontology of the models. Møller-Nielsen (2017) labels the ‘yes’ answer—symmetry-related models are physically equivalent—as the interpretational approach to symmetries, and contrasts it with the motivational approach, which requires a characterisation of the common ontology of symmetry-related models before acknowledging physical equivalence, which is the answer he endorses.

But of course, the decision between those approaches turns on just what we can accept as an explication of the common ontology. This brings us to the second, related debate, about whether sophistication can be attained too easily. For sophistication by brute force, as advocated by Dewar (2017), may not satisfy the motivationalist. As argued by Martens & Read (2020), iff there is no ‘metaphysically perspicuous package” accompanying the understanding of the symmetry-related models, ‘sophistication’ seems to be gotten on the cheap. In their words (Martens & Read, 2020, p. 9): “the traditional sophisticationist methodology must be applied in order to regard those models—interpreted as being isomorphic—as in fact representing the same physical states of affairs.”

And whereas both parties accept that finding a reduced theory, in which the basic variables of the theory no longer admit a non-trivial action of the symmetries, would provide sufficient explication to satisfy the motivationalist, neither gauge theory nor general relativity can be formulated in this manner; at least not without significant explanatory and pragmatic deficits (cf. (Gomes, 2021, Section 4.2) and Section 2.2.2). Thus the straightforward criterion is too strict, and we must give an answer to the second debate—about when sophistication clarifies underlying structure—in order to calibrate our answer to the first debate: about when we can stop searching for a metaphysically perspicuous package for the symmetry-related models.

Martens & Read (2020) argue, in line with a motivational attitude towards symmetries, that sophistication should be adopted only in cases where the symmetries coincide with isomorphisms of some structure. But there is an obvious difficulty about applying this criterion: given a space of models and a symmetry transformation acting on this space, one could appar-
ently just announce that structure is defined implicitly, as ‘whatever is left invariant’ by the action of the symmetries. In so doing, the symmetries become isomorphisms of the implicitly defined structure. Thus the criterion is no longer strict, but it is opaque.

Both Jacobs (2020) and Martens & Read (2020) argue that an implicit definition does not satisfactorily explicate the common ontology of the symmetry-related models. Following Jacobs (2020, Ch. 4.1), here it is useful to label the two broad different approaches to sophistication:

- **The symmetry-first approach:** finding a structure implicitly as ‘what stays constant across the symmetry-related models’;
- **The structure-first approach:** finding the symmetry-related models as those that possess the same structure.

On Jacobs (2020)’s favoured structure-first approach, the aim is to give an ‘intrinsic’ characterisation of a structure in terms of relations defined over its domain, such that this structure is invariant under the dynamical symmetries of the theory. This broad idea works out beautifully for the example that he focuses on (scaling of masses in Newtonian gravity). In that case, one successfully characterizes the mathematical structure of the theory first, and then deduces the isomorphisms that preserve that structure. At least in that example, it seems that sophistication is apparently not condemned as cheap. In sum, we can escape cheapness in two steps: (1) insisting symmetries coincide with isomorphisms of some structure, and (2) first defining the invariant structure, and then finding the symmetries that preserve it.

### 3.4.2 Obstacles to Sophistication: the case of general relativity reassessed

But I would argue that even in the case of general relativity, the precise identification of the structure that remains invariant is highly abstract. For the main idea in defining structure through symmetry-invariance is that the models of the theory are structured sets, \( D = (D, R) \), consisting of a base (unstructured) set \( D \) and relations among the elements of this set, \( R \). For example, in general relativity, a model “consists of a base manifold \( M \) over which we have defined some (geometrical) structure in the form of the tensor fields”. But what exactly are the relations that stay invariant under the symmetries, which are given by the (pullback of) active diffeomorphisms (cf. Section 2.2.1)? Tensor fields certainly do not remain invariant. Do distances between points? No, at least not the distances between the points of \( M \) seen as an unstructured base set, since the distance between \( x \) and \( y \) according to \( g_{ab} \) is not the same distance as according to \( f^*g_{ab} \) (see footnote 47 for more on this topic). The same reasoning would of course apply to angles, Riemann curvature scalars, etc. The structure that remains invariant is the abstract set of diffeomorphism-invariant quantities, just as the symmetry-first approach—not the structure-first approach—would suggest.

Fortunately, as discussed in Section 2.2.1 and 3.3, both general relativity and gauge theory (in the principal fiber bundle formulation) have a correspondence between active and passive symmetry transformations, in the sense that the global effect of an active symmetry corresponds at most to a re-labeling of the charts. These are the charts in which the passive transformations—coordinate or notational changes—are defined, and in each one, passive symmetries leave certain structures invariant. That is, even in general relativity, under a passive coordinate transformation many clearly defined quantities—distances between points, scalars involving arbitrary...

---

39 As Jacobs puts it:

Structure-first sophistication consists of: the stipulation of a set of relations over the theory’s subdomains, such that the bijections which induce dynamical symmetries of the theory’s models leave these relations invariant. If we agree that an interpretation of a theory provides a metaphysically perspicuous picture if it tells us plainly and clearly which entities the theory posits (ontology) and what the fundamental relations between these entities are (ideology), then structure-first sophistication is perspicuous in this sense. (Jacobs, 2020, p. 62)
tensors, and not so arbitrary ones, such as the Riemann curvature, etc—*are* invariant (again, cf. footnote 47), and have natural, or perspicuous, metaphysical interpretations.

The active-passive correspondence allows us to elevate the meaning of these structures; even though they are not fully invariant under active symmetries, the non-invariance corresponds to a certain notational change, or re-labeling of charts. Of course, one could still object that besides the re-labeling of the charts, we could still apply passive transformations to the charts, i.e. we could choose an entirely different set of charts to represent the same quantities. But this is no real challenge: it is easily met if we restrict physical significance to those quantities that are in each chart invariant with respect to a passive transformation. Yes, active transformations will still transform these quantities, but the effect will be a mere re-labeling of the charts through which we originally described the quantities, as we saw in Section 2.2.1.

Thus the metric formulation described in Section 2.1 can be perspicuously interpreted so as to identify physical significance as structural: it is represented not by quantities that are fully invariant under the symmetries of the theory, but by quantities that are invariant under passive coordinate transformations. And I defended, as others have defended, the idea that the concept of structure in this case is perspicuous: it is chronogeometric.

### 3.4.3 Sophistication as anti-quidditism in Yang-Mills theory

We can apply a very similar reasoning to gauge theories. Contrary to $\omega$ and $\Omega$, the spacetime representatives $A^s$ and $F^s$ are defined over charts of the spacetime $M$, rather than over the bundle $P$, or over all of $M$. In other words, although $\omega$ is globally defined on $P$, the $A^s$ are only defined on the respective patches $U_\alpha$ of $M$ through the choice of a local section $s$. At a fixed $\omega$, different choices of section give different $A^s$; the difference between the gauge potentials are solely due to different choices of trivialization, i.e. they are passive. And passive transformations do not change $\omega$.

Thus, according to this formalism and using the idea of an active-passive correspondence, we should think of $\omega$ (or the $\Gamma$ of Section 3.3.3), and not $A$, as encoding physical structure. And if we no longer need to worry about the active symmetries, we need not talk of equivalence-classes under the isomorphisms; a straightforward interpretation of the (passively-invariant) $\omega$ (or $\Gamma$) suffices. Thus we can extract the physical meaning of the fields in play directly, without reduction.

Here, as in Section 2.2.2, this property—that the active isomorphisms are locally equivalent to a passive transformation—gives a gloss of ‘notational redundancy’ to the symmetry in question, a type of redundancy most hands agree to be well understood. Indeed, invariance under different coordinate representations is usually *equated* with ‘physical status’. Thus, to take two examples at random: Nozick (2001, p. 82) explains: “Once we possess the covariant representation under which the equations stay the same for all coordinate systems, the quantities in the (covariant) equations are the real and objective quantities.” And, in the introduction to his magisterial book, Dirac (1930, p. vii) writes: “The important things in the world appear as the invariants [...] of these [coordinate] transformations.”

To a certain extent, this gloss vindicates some previous philosophical comments about the connection. For example, according to Maudlin in (Belot et al., 2009, p.6):

> The fact that there may be more than one way of representing geometric objects with numbers is a source of ‘gauge freedom’ that is totally unproblematic. There is more than one arithmetic way of representing a particular geometric point, but no one asserts that the geometric object in question is not unique. Similarly, there is more than one way of representing arithmetically the connection of a particular physical setup at each base space point, but this does not mean that there is more than one connection.
Maudlin’s comment applies to passive transformations, and, in relating them to active transformations, we can co-opt the intuitions it reports, which are widely shared.

We have therefore found a perspicuously sophisticated view of the symmetry-related models. Here the KPMs are given by \( \langle P, \omega \rangle \), which is both isomorphic and symmetry-related to \( \langle P, \omega' \rangle \) iff there is a fiber-preserving diffeomorphism \( \tau \in \text{Diff}(P) \), such that \( \omega = \tau^* \omega' \). In this formalism, just like in general relativity, the symmetries of the action match the isomorphisms of the underlying structured base set, \( P \), and these isomorphisms can be given a passive construal, which allows a perspicious interpretation of the structure that is common to all the symmetry-related models.

In line with our comments of Section 3.3.4, we can articulate the ontic commitments of gauge theory that ensue from this attitude as follows: each possible world, or physical possibility for the force-field—each particular choice of structure—is given by one way in which internal quantities are parallel transported over spacetime. The structural interpretation of Yang-Mills theory is conceptually similar to the structural interpretation of general relativity, discussed in Section 2.2. There, each possible world, or physical possibility—each particular choice of structure—is given by one way in which spacetime points are chronogeometrically related or distributed. Here, the structure represented by \( \omega \) refers to the parallel transport of quantities taking values in internal quantities.

Having used passive transformations to interpret the common structure of the symmetry-related models, we will now set them aside and focus entirely on the active symmetries of the theory. For it will be the properties of this type of transformation that distinguishes gauge from diffeomorphism symmetry. In short, we are now ready to evaluate the more concrete similarities and differences between the two theories, concerning especially their symmetries.

### 4 Distinguishing gauge and diffeomorphisms symmetries

At this point in the discussions of this paper, the reader is beginning to wonder when I will address the ‘elephant in the room’: I can argue all day about the similarities of the two types of theory, but surely there is a fundamental distinction between symmetries acting internally—as gauge symmetries do—and those acting ‘externally’, as spacetime diffeomorphisms do. And I agree that it is difficult to countenance the absence of such a fundamental distinction between internal and external dimensions of the universe. But if we had a more intimate acquaintance with the value space of these fields, for instance, if isotopic spin was macroscopically detectable, would we perhaps reach the same sort of intuition about the corresponding internal directions of a principal fiber bundle that we have for spacetime directions?

To probe our intuitions about these questions, in Section 4.1 I will introduce a formalism that unifies the gauge transformations and the diffeomorphisms. The unification articulates the different forces through Riemannian geometry, and it is obtained by enlarging spacetime in a particular manner.

In Section 4.2 I will show that internal and external symmetries are nonetheless qualitatively different.

In the principal bundle formalism of Section 3.3, the reason can be seen as follows: even though vertical automorphisms are diffeomorphisms, they act more like the isometries of a given metric, or like the change of basis for tangent vector fields, than like generic diffeomorphisms. This reflects both our proposal for a distinction, \( \Delta \), that first appeared at the end of Section 1.2, and the differences pointed to in Section 3.3.1 (text after (3.10)); and it will remain the most significant difference between the two types of symmetry to the end of the paper.
4.1 Finding common ground between diffeomorphisms and gauge transformations: The Kaluza-Klein framework

As we saw in Section 3.3.3 the Atiyah-Lie algebroid formulation of the gauge connection brings the metric formulation of general relativity and the formulation of gauge forces into close proximity.

In this Section we will expunge the distinction between internal and external by introducing a Kaluza-Klein formulation of gauge theory. This formulation geometrizes gauge interactions at the expense of adding extra dimensions to spacetime; it thus effaces the distinction between internal and external directions.

First, it is important to distinguish a position that takes internal dimensions to have ontic (and not just notational) status, such as (Arntzenius, 2012, p. 185)’s bundle substantivalism, from one that also ‘geometrizes’ these internal dimensions, such as the Kaluza-Klein framework (Kaluza, 1921) (a project that involves several issues that lie beyond the scope of what I am concerned with here).

The main idea of the Kaluza-Klein framework is remarkably simple (cf. (Bleecker, 1981, Ch. 9)): use an inner product \( \kappa \) on the Lie-algebra \( g \), the metric \( g_{ab} \) on \( M \), and the connection-form \( \omega \) on \( P \), to induce a Lorentzian inner product on \( P \) by “just summing the external and internal co-fibrations”, which we can then treat through the tools of general relativity/Riemannian geometry. The induced metric is given by:

\[
\eta(\bullet, \bullet) = g_{ab}(\pi_*\bullet, \pi_*\bullet) + \kappa(\omega(\bullet), \omega(\bullet)),
\]

or, more economically: \( \eta = \pi^* g_{ab} + \kappa \circ \omega \).

We could now compute the Ricci scalar for this metric, and the corresponding Einstein-Hilbert action. Upon variation of this action, we find both the Einstein equations for \( g_{ab} \) and the Yang-Mills equations for \( \omega \) as the extremum condition. Another surprising feature of the formalism is extracted from geodesic motion: since vertical directions are Killing directions of the metric (due to the covariance of \( \pi \) and \( \omega \)), vertical velocities are conserved during geodesic motion.\(^{42}\) Identifying a particle’s charge with its vertical velocity (i.e along the orbit of the group) then guarantees conservation of charge. Moreover, upon projection of the geodesic onto the base space, we get a dynamical trajectory that correctly captures the deviation from geodesic motion by the Lorentz force on the particle due to the curvature, \( F \). A mighty formalism indeed!\(^{43}\)

4.2 The real difference between gauge transformations and spacetime diffeomorphisms

This Section establishes the difference, \( \Delta \), announced in Section ??, between gauge and space-time diffeomorphism symmetries.

\(^{40}\)This is usually called the Killing form: in simple matrix representations of the Lie algebra, it is just the trace of the matrix product.

\(^{41}\)Cf. the discussion at the start of Section 4.2.1.

\(^{42}\)This is just an application of a basic theorem of (pseudo)-Riemannian geometry: the angle between a Killing direction and a geodesic remains constant.

\(^{43}\)This discussion briefly summarized what was achieved in the Kaluza-Klein formalism, circa 1920-1940. The main ideas of geometrizing electromagnetism appeared in Kaluza (1921), but that work left out the weak and strong nuclear forces. Klein extended the formalism in 1938, in a paper presented at, and published in the proceedings of, a Conference on New Theories in Physics held at Kazimierz (Poland) in 1938. The paper is reproduced in (Klein, 1986). See (O’Raifeartaigh, 1997, Ch. 3 and Ch. 6) for a historical account. It should be said that the formalism has immense scope: from string theory to the relational-absolutist debate, where it was employed to deal with rotations (Gomes & Gryb, 2021).
All symmetries will leave the equations of motion, or the action, or the symplectic form, or whatever structure that is dynamically relevant, invariant, according to our definition of symmetries, in Section 1. There is no distinction between symmetries to be made at that coarse dynamical level. But this leaves open the possibility that translating the representation of the fields along vertical directions is qualitatively and quantitatively different than translating them non-vertically: and the idea will give the real difference \( \Delta \). Translations along vertical directions correspond to isomorphisms of the theory that have a homogeneous effect on the dynamical quantities, like the curvature. The automorphisms that correspond to diffeomorphisms are those that fundamentally change the properties of the dynamical variables. Thus, according to this criterion, vertical symmetries could be more directly vulnerable to Leibniz’s Principle of the Identity of Indiscernibles (PII) than non-vertical ones.

In Section 4.2.1 I functionally characterize the differences between internal and external dimensions, and between gauge transformations and spacetime diffeomorphisms, in the Kaluza-Klein framework.

In Section 4.2.2, we use the Kaluza-Klein discussion as inspiration to characterize the distinction more broadly. We compare the diffeomorphisms with the gauge transformations, in a coordinate-free, manner, without charts, by using a proxy for the Atiyah-Lie algebroid.

4.2.1 The difference, from the viewpoint of Kaluza-Klein

We start with the Kaluza-Klein picture. Here it is easy to single out the vertical directions: they are the Killing directions of the Kaluza-Klein metric, \( \eta \) in (4.1). In other words, from the second property of (3.5) and since \( \kappa \) is taken to be invariant under the (adjoint) action of the Lie group, and \( L_{g_\zeta}^*\omega = g^{-1}\omega g \), we obtain:

\[
\kappa(\omega(L_{g_\zeta}\omega),\omega(L_{g_\zeta}\omega)) = \kappa(\omega(\omega),\omega(\omega));
\]

and since \( \pi_\ast \) is itself invariant under the action of the group:

\[
g_{ab}(\pi_\ast(L_{g_\zeta}\omega),\pi_\ast(L_{g_\zeta}\omega)) = g_{ab}(\pi_\ast(\omega(\omega)),\pi_\ast(\omega(\omega))).
\]

For some direction \( \zeta \) in \( P \) that is not along a gauge orbit, neither the first nor the second term in the metric (4.1) will be preserved. And even if the spacetime metric \( g_{ab} \) is preserved along some (projected) direction, i.e. such that \( L_{\pi_\ast(\omega)}g_{ab} = 0 \), it could still be the case that \( L_{\zeta}\kappa(\omega,\omega) \neq 0 \). Equality obtains iff the gauge curvature also vanishes along \( \zeta \). And thus the Kaluza-Klein metric will only be preserved non-vertically if the model is entirely ‘featureless’ along those directions.

And so, demanding some modal robustness from our definition—and thereby ignoring such highly homogeneous exceptions—we characterize gauge transformations to be those diffeomorphisms that are generated by Killing fields of the Kaluza-Klein metric. All Kaluza-Klein spaces will have these directions. And “Being a Killing direction” is a description that we could, with some allowance for vagueness, characterize as ‘filling a functional role’: it specifies a diffeomorphism-invariant property, ‘tracking’ the same directions in all diffeomorphism-related models.

In sum, in Kaluza-Klein the gauge directions are singled out by their rigidity: unlike the
fields along other directions, they are constrained to be of a certain form. That is, this split by functional roles corresponds to two sets of directions in $P$: those with a rigid structure—the vertical—and those with richer, more complex structure: the non-vertical.

But there is still an uncomfortable feature of the Kaluza-Klein metric, and in this definition of gauge transformations. General vertical vector fields (and not just fundamental vector fields), those that change from fiber to fiber, are not Killing vector fields. Are these not to count as gauge transformations? Surely they are. We will now move on to a more general definition of the distinction.

Essentially, the idea is that given the value of the connection on $p$, we know what its value on $g \cdot p$ must be: if we write $L_g^* \omega = \ldots$ (i.e. the pull-back of the connection by the left group action) we know what will appear on the right hand side, viz. $g^{-1} \omega g$. In contrast, note we do not know how to fill in the analogous equation, $f^* g_{ab} = \ldots$, unless, that is, $f$ is an isometry of $g_{ab}$, in which case $f^* g_{ab} = g_{ab}$.

4.2.2 The difference, from the viewpoint of the principal bundle

At the end of Section 3.3.3, we had found that the section of the Atiyah-Lie algebroid possessed many of the same properties as the metric: it could be written globally, on spacetime, in a coordinate-free manner. Moreover, were we to write down the field explicitly with a trivialization, or coordinate choice, the objects of both theories would be appropriately covariant under coordinate choice. Thus, for instance, if the gauge or the Riemann curvature vanishes at a point, it will vanish in every coordinate system that covers that point. But this is a statement about a passive transformation. If we are to instead consider active transformations—the automorphisms of the structure—the story is different in important ways.

More explicitly, two isomorphic connections $\omega$ and $\omega' := \tau^* \omega$—which correspond uniquely to the global sections of $TP/G$, $\Gamma, \Gamma'$—may be associated to different horizontal directions at a given point $p \in P$. There is an inhomogeneous term that enters the transformation of $\omega$ under a generic vertical automorphism (see equation 3.7 or footnote 28), and thus a horizontal vector for $\omega$ is not necessarily horizontal for $\omega'$.

Nonetheless, $\omega$ and $\omega'$ still encode a quantity, the curvature $\Omega$, whose transformations has no inhomogeneous term, since it is appropriately equivariant (see equation (3.10)). And upon quotienting $TP$ to $TP/G$ (as a vector bundle over $TM$ or $M$) by using the Atiyah-Lie algebroid, all right equivariant functions (like curvature), become gauge-invariant.

But the same cannot be said of the Riemann curvature: the vanishing of the gauge curvature at some point of the underlying set (be it $P$ or $M$) is physical, that of the Riemann curvature is not.\footnote{This long footnote can be seen in the context of Section 2.2.1. Of course, the Riemann curvature is covariant, and so ‘zero’ is a coordinate-independent value at a fixed point of $M$, but diffeomorphisms will shift the point, and thus the value (cf. Section 2.2.1). This does not imply the Riemann curvature has no structural interpretation: it does, in the same sense that the metric has a chronogeometric interpretation (cf. Section 2.2.2). One popular way to convey such interpretations, ‘wearing anti-haecceitism on our sleeves’ so to speak, is by rejecting the very idea of (non-trivial) active diffeomorphisms. This is accomplished by construing diffeomorphisms in terms of the ‘drag-along’ of properties and relations. Namely, the idea that points are individuated by their pattern of properties and relations, as encoded in the metric and matter fields, prompts the proposal that if we are given an isomorphism that sends the fields at $x$ to the fields at $y$, that is, $f$ sends the properties at point $x$ in one model to a point $y$ in another model, where $f(x) = y$, then we should “rebrand” $y$ in the codomain model as “really being” $x$; or “replace $y$ with $x$”. It is this proposal that Gomes & Butterfield (2021) call the drag-along response to the hole argument. Such an argument immediately renders any tensor on $M$ trivially diffeomorphism-invariant: it corresponds to an implicit definition of points through the places they fill in the patterns of instantiation of the properties and relations of the theory (as encoded by all the fields). The construal is employed e.g. in (Itin & Stachel, 2006; Weatherall, 2018) to disarm the under-determination implied by the hole argument (Earman & Norton, 1987). But Gomes & Butterfield (2021) claim, not only that the drag-along response is not mathematically mandatory, but also that: it is limited.}
Thus, we can locate or specify vertical automorphisms, among all the fiber-preserving diffeomorphisms of \( P \), by their roles: as those that transform the curvature homogeneously, or, equivalently, that leave the field-strength in the Atiyah-Lie algebroid invariant. Such transformations locally preserve the dynamical variables, and this characterization is independent of our underlying attitudes towards symmetry-related models: even if we take a substantivalist view of \( P \) (Arntzenius, 2012, p. 185), we can identify gauge transformations as those that relate indiscernible local, dynamical, states. This is the distinction, \( \Delta \), that we have been seeking. Namely, as stated at the end of Section 1.2: Yang-Mills, but not general relativity, admits a formalism in which the local, dynamical content of the theory is fully invariant under the appropriate symmetry transformations.

In \( \Delta \) the word ‘Local’ flags the pointwise validity of the distinction, and is crucial. For globally, two isomorphic models are indiscernible by definition, so spacetime diffeomorphisms also relate globally indiscernible states. To be concrete, suppose we define \( h_{ab} = f^* g_{ab} \). Then, at a given point \( x \in M \), \( h_{ab} \) and \( g_{ab} \) may disagree about many things, including the Riemann curvature. But there is no sense in which, globally, \( g_{ab} \) and \( h_{ab} \) disagree about the state of the Universe (see also Gomes & Butterfield (2021)). As presaged in the last paragraph of Section 2.1, any distinction to be found between the two kinds of symmetry had to be a local one; and it is.

And in \( \Delta \), the word ‘Dynamical’ refers to the use of the curvature, and not to the use of the dynamics (through the equations of motion, action functional or Hamiltonian). Indeed, this characterization is independent of the dynamics of the theory. And here we see the use of the Atiyah-Lie algebroid: In the standard formulation of the Abelian theory in a principal bundle, the curvature exhausted all the local gauge-invariant degrees of freedom; indeed the curvature itself can be seen to merely stand for the local gauge-invariant degrees of freedom of the theory. In the non-Abelian theory, things get much more complicated in the standard picture: the curvature is not invariant, and we must take traces of products of the curvature tensor to convey local (i.e. pointwise) gauge-invariant functions. And the theory does not have infinitely many physical degrees of freedom per spacetime point, and so we must choose a basis among all these invariant functions: a difficult task. The Atiyah-Lie algebroid cuts this Gordian knot by having a notion of field-strength that is invariant at each point, and which exhausts the number of local physical degrees of freedom of the theory, just like the field-strength does in the Abelian theory.\(^48\)

To further clarify the meaning of \( \Delta \), note that the definition applies to all models of the theory: i.e. it is modally robust. And indeed, for gauge theories, gauge transformations as vertical automorphisms of a principal bundle provide directions of local dynamical indiscernibility for all models of the theory. That is, irrespective of the particular state and at all points of the underlying domain. And of course, generically, the non-vertical generators of the automorphisms of \( P \) fail to meet such strict criteria, and are thus associated to shifts of the base manifold; that is, they generate spacetime diffeomorphisms. Under these more general shifts, local dynamical quantities transform non-trivially.\(^49\)

The main limitation being that general relativity, and our other spacetime theories, use means of identifying points other than by drag-along, as I said in Section 2.2.1. And they need to do so, on pain of trivialising important constructions—even elementary ones like the Lie derivative. Thus the qualification, in Section 2.2.1, that “there is no known, non-trivial way to get rid of diffeomorphism symmetry”. Barring the drag-along understanding of diffeomorphisms, the only diffeomorphism-invariant quantities are integrals of scalar densities over the manifold.

\(^48\)Of course, in neither the Abelian nor the non-Abelian theory, do the local gauge-invariant degrees of freedom exhaust the totality of physical degrees of freedom: there are global physical facts about parallel transport, facts that are not encoded in the curvature. (And the connection is not gauge-invariant in the Atiyah-Lie bundle, or, equivalently, it is not homogeneously equivariant in the principal bundle).

\(^49\)Subsets of diffeomorphisms could be put into a tighter comparison with gauge symmetries if we have
Thus, the functional distinction $\Delta$ is modally robust, in the sense that it is present for generic models of general relativity and Yang-Mills theory. The gauge transformations really do map between local quantities that are less discernible than those mapped by generic diffeomorphisms. In the Atiyah-Lie formalism, we can polish this distinction, since we find symmetries that do not change the local dynamical variables of the theory (e.g. the curvature of the Atiyah-Lie connection), irrespective of spacetime point or state of the field. This concludes the characterization of distinction $\Delta$, first proposed at the end of Section 1.2 (and subsuming the differences pointed to at the end of Section 3.3).

5 Summing up

I will divide this concluding section into two parts. I will first summarize and reflect on what has been argued thus far, in Section 5.1; a post-morten that reflects Section 1.3. In Section 5.2, I will briefly conclude.

5.1 Summary

The question driving this paper is whether we should endorse structuralism for spacetime diffeomorphisms, but not for gauge symmetries. As briefly mentioned in Section 1, there is an established jargon in modern metaphysics for the two structuralist construals in play here. The structuralist construal of points that is endorsed by Healey (and as he says: many others) is often called anti-haecceitism. And the structuralist construal of properties that, contra Healey, I recommend is often called anti-quidditism. In these terms, the question driving this paper is whether anti-haecceitism is right for spacetime symmetries, but some variant of reduction should be preferred to anti-quidditism about gauge symmetries.

Sections 2 and 3 rejected this claim at a formal level, and showed the interpretation of symmetry-related models in both types of theories find a home within sophistication. In these Sections I rejected the idea that accepting redundancy is accepting defeat; that it is a price we must pay. For the redundancy of gauge theory can be conceived structurally, in as perspicuous a manner as in general relativity. If the skeptic about redundancy is still not convinced, we can further mollify her by glossing redundancy as notational, as it is often done in the practice of physics and as we did in Sections 2.2.1 and 3.3 (see also footnote 47).

Of course I could not conclusively show that the structural representation is equally recommended in both cases; I can only show that it is equally conceptually transparent in both cases. Moving forward, in the accompanying paper I assess concrete attempts at drawing distinctions between general relativity and gauge theory, seeking to strengthen the license for sophistication of gauge symmetries to a recommendation. My final push for this recommendation will occur in the accompanying paper, (Gomes, 2021, Section 4).

The source of an intuitive conceptual distinction between gauge symmetries and spacetime diffeomorphisms is based on a genuine difficulty: they act in different spaces. But this difference can be overcome in two ways, as we saw in Section 4.

$L_X R^a_{bcd} = 0$, where $X$ are vector fields (the infinitesimal generators of diffeomorphisms) and $R^a_{bcd}$ is the curvature of the metric. It can be shown that $L_X R^a_{bcd} = 0$ occurs when $X$ is a Killing field. It follows from realizing that $L_X F(g_{ab})|_x = \int dy \delta F(x) \delta g_{ab}(y)$ where $\delta g_{ab} = L_X g_{ab}$. There are more complicated proofs that use the Bianchi identities and the algebraic properties of the Riemann tensor. But there are two problems with this: i) metrics with Killing directions form a meager set in the space of metrics (under any reasonable topology) and ii) the Killing vector fields are also a meager subset of the space of all vector fields (seen as generators of (small) diffeomorphisms).
First, in the Kaluza-Klein framework, internal dimensions become ‘just like’ external dimensions; and therefore gauge transformations ‘just like’ spacetime diffeomorphisms; and therefore, anti-haecceitism for general relativity implied a sort of anti-quidditism for gauge theories.

The representation with a Kaluza-Klein framework is suggestive, but it is not definitive: the framework has many other issues and features that we would not like to carry over to our understanding of gauge symmetry.\footnote{For example: a naïve interpretation of Kaluza-Klein as just general relativity in one more dimension cannot work, for the extra dimensions must have a fixed, non-dynamical geometry. Moreover, one cannot use Killing directions as the generators of all vertical automorphisms, and thus we must proceed to a more flexible framework.} Moreover, the geometrical interpretation of the Kaluza-Klein theory is very different than the interpretation of the connection on a principal bundle that I have been advocating. This brought us to the second way of comparing spacetime diffeomorphisms with gauge symmetries in the same domain: the Atiyah-Lie algebroid.

This formalism is entirely equivalent to the principal fiber bundle: it just enforces left-equivariance of all mathematical objects. But the Atiyah-Lie formalism is useful in two ways: (1) it expresses the gauge fields without coordinates, in the same way that we can express the metric without coordinates. And (2): within the formalism, the connection varies under vertical automorphisms, but its associated curvature does not. In this way the invariance of the gauge curvature in the Abelian case is reproduced globally in the non-Abelian case; and we can heuristically think of the formalism as representing a global, non-Abelian generalization of the Abelian gauge potential (whose curvature is gauge-invariant).\footnote{The Gribov problem here would still be present, in the same way that it is present for the metric field.}

The Atiyah-Lie formalism allows us to polish the technical and conceptual distinction between gauge symmetries and diffeomorphisms. But the formalism is not, strictly speaking necessary to glimpse this distinction: in all three formalisms—Atiyah-Lie, PFB, and Kaluza-Klein—gauge transformations can be identified among the active symmetries of the theory as being in some sense ‘more rigid’. They leave the curvature of the Atiyah-Lie section invariant, and equivalently, they transform the curvature of the bundle, $\Omega$, as $\mathcal{L}_\zeta \Omega = [\omega(\zeta), \Omega]$ homogeneously, or equivariantly. The (active) spacetime diffeomorphisms are, generically, not rigid in the same way. Unless the metric has non-trivial isometries (or conformal isometries), it admits no comparable transformations. Thus generically spacetime diffeomorphisms do not leave the Riemann curvature invariant, as gauge symmetries leave the curvature of the Atiyah-Lie algebroid invariant. Thus we have found a bona-fide distinction, that we labeled $\Delta$, between spacetime diffeomorphisms and gauge-symmetries.

5.2 Conclusion

The lesson of this paper is that gauge transformations and diffeomorphisms are structurally very similar, with the exception of one robust dissimilarity. Although we could find, at the end of the day, this conceptual difference between the two, the investigations of this and the accompanying paper Gomes (2021) have found no smoking gun to validate eliminativism for gauge while endorsing sophistication for diffeomorphisms.

Thus we can understand the ontological commitments of both theories as structural: one describes chronogeometric relations, in a well-understood sense, and the other describes the parallel transport of all sorts of charges that figure in the standard model, in a well-understood sense.

Therefore, we conclude that fiber bundle structuralism is a valid, explanatory perspective about the ontology of gauge theories. It suggests a form of anti-quidditism, as valid and explanatory as anti-haecceitism is for chronogeometric structure. So for us, the point here is of course: if anti-haecceitism is good for spacetime, why not also adopt anti-quidditism about...
gauge? Or, as they say in England: what is sauce for the goose can also be also sauce for the gander.

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