ANALOG OF THE MEAN-VALUE THEOREM FOR POLYNOMIALS OF SPECIAL FORM

O.D. TROFYENKO

ABSTRACT. In the present paper a new mean value theorem for polynomials of special form is obtained. The case of sums on vertices of a regular polygon is studied. A criterion for a certain equation to be satisfied is obtained.

1. INTRODUCTION

The classical Gaussian theorem characterizes the class of harmonic functions by the mean value formula. This theorem has received further development and elaboration in many papers (see, for example, reviews [1], [2] and monographs [3], [4] with extensive bibliography). One of the main ways in this study is a description for classes of functions. This classes satisfy given integral equations, that have certain geometric meaning. There are mean value theorems that characterize harmonic polynomials [5], bianalytic functions [6], a solution of convolution equations with finite convolver and others (see [7]). In addition, similar results are very important in integral geometry and various applications (see [3]).

In this paper the mean value theorem for polyanalytic polynomials of a special form is obtained. A feature of it is function’s value on vertices of a regular polygon in the mean value formula (see below). Also there is a value of some differential operator in the center of this polygon.

We need the following notation for accurate formulation of the main result.

Assume that $s, h \in \mathbb{Z}$, $n, m \in \mathbb{N}$, $n \geq 3$, $0 \leq h < n - s$, $0 \leq s \leq m - 1$, $q = \min\{h + s, m - 1\}$.

Let $B_R$ be the disk in $\mathbb{C}$ centered at origin with radius $R$. Denote by $\zeta_\nu = R e^{\frac{2\pi i}{n}} (\nu = 1, ..., n)$ the vertices of a regular $n$-gon with circumradius $R$ and inscribed radius $r$.

Let $R_*(n, R, r) = \begin{cases} \sqrt{5R^2 + 4rR} & \text{where } n \text{ is odd,} \\ \sqrt{8R^2 + R^4/r^2} & \text{where } n \text{ is even.} \end{cases}$

2. FORMULATION OF THE MAIN RESULT

Theorem 1. Let $R > \frac{1}{2}R_*(n, R, r)$ and $f \in C^q(B_R)$. Then the next statements are equivalent.

1) For all $0 \leq \alpha \leq 2\pi$, $z + \zeta_\nu e^{i\alpha} \in B_R$, $\nu \in \{1, ..., n\}$ there holds

$$\sum_{\nu=1}^{n} (\zeta_\nu e^{i\alpha})^s f(\zeta_\nu e^{i\alpha} + z) = \sum_{p=s}^{q} \frac{n R^{2p}}{(p-s)!p!} \left( \frac{\partial}{\partial z} \right)^{p-s} \left( \frac{\partial}{\partial \bar{z}} \right)^p f(z).$$

2) The function $f$ has the following form

$$f(z) = \sum_{k=0}^{h} \sum_{l=0}^{m-1} c_{k,l} z^k \bar{z}^l,$$

where $c_{k,l}$ are some constants.
Lemma 1. Let a function \( f(z) \) have the form (2).

Then the following equality holds

\[
\sum_{\nu=1}^{n} (\zeta_{\nu} e^{i\alpha})^{s} f(\zeta_{\nu} e^{i(\alpha+\beta)} + ze^{i\beta}) = \sum_{p=s}^{q} \frac{nR_{2p}}{(p-s)!p!} \left( \frac{\partial}{\partial z} \right)^{p-s} \left( \frac{\partial}{\partial \overline{z}} \right)^{p} f(ze^{i\beta}),
\]

where \( 0 \leq \alpha \leq 2\pi, 0 \leq \beta \leq 2\pi, ze^{i\beta} + \zeta_{\nu} e^{i(\alpha+\beta)} \in B_{\mathcal{R}}. \)

Proof. We have the following

\[
e^{i\alpha s} \sum_{\nu=1}^{n} e^{i\beta s} f(\zeta_{\nu} e^{i(\alpha+\beta)} + ze^{i\beta}) = \sum_{h=0}^{m-1} \sum_{p=0}^{m-1} \sum_{k=0}^{l} \sum_{j=0}^{k} \sum_{l=0}^{j} c_{k,h}^{j} C_{i}^{p} e^{i\alpha s} e^{i\beta s} \zeta_{\nu} C_{i}^{p} e^{i\alpha(i-j-p)} e^{i\beta k} e^{-i\beta l}, \]

Separating the concerns of the \( z \) and \( \zeta_{\nu} \), we get

\[
e^{i\alpha s} \sum_{\nu=1}^{n} e^{i\beta s} \frac{1}{j!p!} \left( \frac{\partial}{\partial z} \right)^{j} \left( \frac{\partial}{\partial \overline{z}} \right)^{p} f(ze^{i\beta}) \sum_{\nu=1}^{n} R_{s+j+p} e^{i(s+j-p)\frac{2\pi}{n}} = 0,
\]

except the case, where \( s + j - p = q_{1}n, q_{1} \in \mathbb{Z}. \)

Let us estimate \( q_{1}n: \)

\[1 - m \leq q_{1}n \leq m - 1 + h \leq m - 1 + n - s \leq m - 1 + n.\]

This yields: \( q_{1}n = 0. \)

So, \( s + j - p = 0. \) Then

\[
e^{i\alpha s} \sum_{\nu=1}^{n} e^{i\beta s} \frac{1}{j!p!} \left( \frac{\partial}{\partial z} \right)^{j} \left( \frac{\partial}{\partial \overline{z}} \right)^{p} f(ze^{i\beta}) \sum_{\nu=1}^{n} R_{s+j+p} e^{i(s+j-p)\frac{2\pi}{n}} = \]

\[= \sum_{p=s}^{q} \frac{nR_{2p}}{(p-s)!p!} \left( \frac{\partial}{\partial z} \right)^{p-s} \left( \frac{\partial}{\partial \overline{z}} \right)^{p} f(ze^{i\beta}).\]

Then it is obvious that equality (3) holds. \( \square \)

Let us introduce the corresponding Fourier series for a function \( f \in C^{q}(B_{\mathcal{R}}) \)

\[
f(z) = \sum_{k=-\infty}^{\infty} f_{k}(\rho) e^{i\varphi k},
\]

where \( f_{k}(\rho) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\rho e^{i\varphi}) e^{-i\varphi k} d\varphi. \)

Lemma 2. Assume that \( f \in C^{q}(B_{\mathcal{R}}) \) and that for this function equality (3) holds. Then this equality holds for every term of the Fourier series of this function. The converse is also true.

Proof. Necessity.

We multiply the left and the right sides of \( (3) \) by \( e^{-i\beta k} \) and integrate over \( \beta \) from \(-\pi\) to \( \pi\). We have

\[
\int_{-\pi}^{\pi} e^{-i\beta k} \sum_{\nu=1}^{n} (\zeta_{\nu} e^{i\alpha})^{s} \sum_{k=-\infty}^{\infty} f_{k}(\rho') e^{i(\varphi - \frac{2\pi}{n} - \alpha)k} e^{i\beta k} d\beta =
\]

3. Auxiliary Statements
where \( \rho' = \sqrt{R^2 + \rho^2 + 2R\rho \cos(\varphi - \frac{2\pi\nu}{n} - \alpha)}. \)

Next,

\[
\sum_{\nu=1}^{n} (\zeta e^{i\alpha})^s f_k(\rho') e^{i(\varphi - \frac{2\pi\nu}{n} - \alpha)k} = \sum_{p=s}^{q} \frac{n R^{2p}}{(p - s)! p!} \left( \frac{\partial}{\partial \overline{z}} \right)^{p-s} \left( \frac{\partial}{\partial \overline{z}} \right)^p f_k(\rho) e^{i\varphi_k},
\]

which concludes the proof.

**Sufficiency.**

Let

\[
\lambda(\alpha) = \sum_{\nu=1}^{n} (\zeta e^{i\alpha})^s f(\rho' \cos(\varphi - \frac{2\pi\nu}{n} - \alpha + \beta), \rho' \sin(\varphi - \frac{2\pi\nu}{n} - \alpha + \beta)) - \sum_{p=s}^{q} \frac{n R^{2p}}{(p - s)! p!} \left( \frac{\partial}{\partial \overline{z}} \right)^{p-s} \left( \frac{\partial}{\partial \overline{z}} \right)^p f(\rho \cos(\varphi + \beta), \rho \sin(\varphi + \beta)).
\]

Then we have the next equality

\[
\int_{-\pi}^{\pi} \lambda(\rho) e^{-i\beta k} d\rho = \sum_{\nu=1}^{n} \left( \int_{-\pi}^{\pi} (\zeta e^{i\alpha})^s f(\rho' \cos \beta, \rho' \sin \beta) e^{-i\beta k} d\beta \right) e^{i(\varphi - \frac{2\pi\nu}{n} - \alpha)k} - \sum_{p=s}^{q} \frac{n R^{2p}}{(p - s)! p!} \left( \frac{\partial}{\partial \overline{z}} \right)^{p-s} \left( \frac{\partial}{\partial \overline{z}} \right)^p \int_{-\pi}^{\pi} f(\rho \cos \beta, \rho \sin \beta) e^{-i\beta k} d\beta e^{i\varphi} = 0.
\]

Thus, equality \( \lambda(\alpha) = 0 \) completes the proof of Lemma. \( \square \)

**Lemma 3.** Let \( f(z) = c N_0(\lambda|z|) \), where \( N_0(\lambda|z|) \) is the Neumann function, \( \lambda \neq 0 \), \( c \) is a constant, and \( f(z) \) satisfies (1) in \( B_R \). Then \( c = 0 \).

**Proof.** The function \( N_0(\lambda|z|) \) is real-analytic, so from [3, Part 1] we have

\[
N_0(\lambda|z|) = \frac{2}{\pi} J_0(\lambda|z|) \left( \log \frac{|z|}{2} + \gamma \right) - \frac{2}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m (\lambda|z|)^{2m}}{(m!)^2} \sum_{k=1}^{m} \frac{1}{k},
\]

where \( J_0(\lambda|z|) \) is the Bessel function, \( \gamma = \lim_{N \to +\infty} \left( \sum_{k=1}^{N} \frac{1}{k} - \log N \right) \). Using the Taylor expansion of the Bessel functions, we obtain

\[
N_0(\lambda|z|) = \frac{2}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m (\lambda|z|)^{2m}}{(m!)^2} \left( \log \frac{\lambda|z|}{2} + \gamma - \sum_{k=1}^{m} \frac{1}{k} \right).
\]

Let us substitute \( f(z) = c N_0(\lambda|z|) \) into equality (1).

We have

\[
\sum_{\nu=1}^{n} (\zeta e^{i\alpha})^s c N_0(\lambda|\zeta e^{i\alpha} + z|) = \sum_{p=s}^{q} c \frac{n R^{2p}}{(p - s)! p!} \left( \frac{\partial}{\partial \overline{z}} \right)^{p-s} \left( \frac{\partial}{\partial \overline{z}} \right)^p N_0(\lambda|z|),
\]

where the function \((\zeta e^{i\alpha})^s N_0(\lambda|\zeta e^{i\alpha} + z|)\) is real-analytic.
Denote the derivation of the Neumann function in the last equation by $DN_0(\lambda|z|)$, which is a real-analytic function. According to [3, Part 1, Prop. 7.1] we have $D = (\Delta + \lambda^2)P(\partial)$, where $P(\partial)$ is a differential operator in $\mathbb{R}^2$. Then $DN_0(\lambda|z|) = 0$ everywhere except $z = 0$.

On the other hand, using the obtained form for $N_0(\lambda|z|)$, we have

$$\sum_{\nu=1}^{n} (\zeta_\nu e^{i\nu})^*cN_0(\lambda|\zeta_\nu e^{i\nu} + z|) \equiv 0$$

in a neighborhood of the point $z = 0$.

Hence, the constant $c$ is zero in the general form of the function $f(z)$. □

**Lemma 4.** Let $\gamma \in \mathbb{R}^1$, $z = x + iy$, $\lambda \in \mathbb{C} \setminus \{0\}$, $c \in \mathbb{C}$ and $f^*(z) = ce^{i(x\cos \gamma + y\sin \gamma)\lambda}$. Then

$$f^*(z) = \sum_{k=-\infty}^{\infty} c_k J_k(\lambda \rho)e^{ik\varphi},$$

where $c_k$ are constants.

**Proof.** At first let us show that the initial function satisfies the following equation

$$\Delta f^*(z) + \lambda^2 f^*(z) = 0.$$ (5)

We have

$$-c\lambda^2 e^{i(x\cos \gamma + y\sin \gamma)\lambda} + \lambda^2 ce^{i(x\cos \gamma + y\sin \gamma)\lambda} = 0.$$ 

Now we check that each term $f_k(\rho)e^{ik\varphi}$ of the expansion of $f^*(z)$ satisfies equation (5) too.

Let

$$f_k(\rho)e^{ik\varphi} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f^*(x\cos t - y\sin t, x\sin t + y\cos t)e^{-ikt}dt.$$ (6)

We denote $h(x, y, t) = f^*(x\cos t - y\sin t, x\sin t + y\cos t)$.

Then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \Delta h(x, y, t)e^{-ikt}dt + \lambda^2 \frac{1}{2\pi} \int_{-\pi}^{\pi} h(x, y, t)e^{-ikt}dt = 0.$$ 

Now it is clear that

$$\Delta(f_k(\rho)e^{ik\varphi}) + \lambda^2 f_k(\rho)e^{ik\varphi} = 0$$

It is known that

$$\frac{\partial}{\partial x} (f_k(\rho)e^{ik\varphi}) = \frac{1}{2} \left( f'_k + \frac{k}{\rho} f_k \right) e^{i(k-1)\varphi} + \frac{1}{2} \left( f'_k - \frac{k}{\rho} f_k \right) e^{i(k+1)\varphi}$$

and

$$\frac{\partial}{\partial y} (f_k(\rho)e^{ik\varphi}) = \frac{i}{2} \left( f'_k + \frac{k}{\rho} f_k \right) e^{i(k-1)\varphi} - \frac{i}{2} \left( f'_k - \frac{k}{\rho} f_k \right) e^{i(k+1)\varphi}.$$ 

Now we can find the following

$$\frac{\partial^2}{\partial x^2} (f_k(\rho)e^{ik\varphi}) = \frac{1}{4} \left( \left( f'_k + \frac{k}{\rho} f_k \right)' + \frac{k - 1}{\rho} \left( f'_k + \frac{k}{\rho} f_k \right) \right) e^{i(k-2)\varphi} + \frac{1}{4} \left( 2f''_k + 2f'_k \rho - \frac{k^2 f_k}{\rho^2} \right) e^{ik\varphi} + \frac{1}{4} \left( \left( f'_k - \frac{k}{\rho} f_k \right)' - \frac{k + 1}{\rho} \left( f'_k - \frac{k}{\rho} f_k \right) \right) e^{i(k+2)\varphi}.$$ 

Similarly,

$$\frac{\partial^2}{\partial y^2} (f_k(\rho)e^{ik\varphi}) = -\frac{1}{4} \left( \left( f'_k + \frac{k}{\rho} f_k \right)' + \frac{k - 1}{\rho} \left( f'_k + \frac{k}{\rho} f_k \right) \right) e^{i(k-2)\varphi} + \frac{1}{4} \left( 2f''_k + 2f'_k \rho - \frac{k^2 f_k}{\rho^2} \right) e^{ik\varphi} - \frac{1}{4} \left( \left( f'_k - \frac{k}{\rho} f_k \right)' - \frac{k + 1}{\rho} \left( f'_k - \frac{k}{\rho} f_k \right) \right) e^{i(k+2)\varphi}.$$
Considering the above, we have the Bessel equation for the function \( f_k(\rho) \)

\[
\frac{1}{4} \left( \left( f_k' - \frac{k}{\rho} f_k \right)' - \frac{k+1}{\rho} \left( f_k' - \frac{k}{\rho} f_k \right) \right) e^{i(k+2)\varphi}.
\]

(7)

Since \( f \in C^\infty(\mathbb{C}) \), it follows from (6) that the function \( f_k(\rho) \) is continuous on \([0, +\infty]\). Hence from equality (7) we obtain \( f_k(\rho) = c_k J_k(\lambda \rho) \). \( \square \)

Lemma 5. Let \( f(z) = c J_0(\lambda |z|) \) (\( \lambda \neq 0 \), \( c \in \mathbb{C} \)) satisfy (7) in \( B_{R_c} \). Then \( c = 0 \).

Proof. From Lemma 4, \( f(z) = ce^{i(x \cos \gamma + y \sin \gamma)\lambda} \).

Substitute \( f(z) \) in equation (1).

\[
\sum_{\nu=1}^{n} (\zeta_{\nu} e^{i\alpha})^s ce^{i(x \cos \gamma + y \sin \gamma)\lambda} e^{i(x \cos \gamma + y \sin \gamma)\lambda} = \sum_{p=s}^{q} \frac{nR^{2p}}{(p-s)!p!} c^{2p-s} e^{i\gamma s} e^{i(x \cos \gamma + y \sin \gamma)\lambda}.
\]

It is obvious that \( c = 0 \). \( \square \)

Denote by \( C_z^q \) a class of \( q \) times differentiated radial functions.

Lemma 6. Let \( f \in C_z^q(B_{R_c}) \) and assume that (7) holds in \( B_{R_c} \). Then \( f(z) = \sum_{k=0}^{q} c_k |z|^{2k} \), where \( c_k \) are some constants.

Proof. From [3, Part 4, Th.3.2] we have the next statement.

Let \( f \in C^q(\mathbb{R}_c) \) and assume that there exists a polynomial \( Q: \mathbb{C} \rightarrow \mathbb{C} \) satisfying the equality

\[
\sum_{\nu=1}^{n} (\zeta_{\nu} e^{i\alpha})^s f(\zeta_{\nu} e^{i\alpha} + z) = \sum_{p=s}^{q} \frac{nR^{2p}}{(p-s)!p!} Q(\partial) f(z)
\]

Then there exists a polynomial \( P \) such that \( P(\Delta)f_0 = 0 \) in \( B_{R_c} \), where \( \Delta \) is the Laplacian.

Therefore, \( P(\Delta)f = \sum_{\nu=1}^{n} c_{\nu} \Delta^\nu f_0 = 0 \), where \( c_{\nu} \) are constants.

Thus for any \( i \in 1, ..., n \) we have the following

\[(\Delta - \lambda_i) F = 0, \]

where \( F = \prod_{j\neq i}^{n} (\Delta - \lambda_j) f_0 \), \( \lambda_i \neq 0 \) are solutions of \( P(z) = 0 \).

Let \( \lambda_i \neq 0 \).

We find \( F(z) \) as the solution of the differential Bessel equation

\[F(z) = c_1 J_0(\sqrt{\lambda_i}|z|) + c_2 N_0(\sqrt{\lambda_i}|z|),\]

where \( J_0, N_0 \) are the Bessel and Neumann functions respectively, and \( c_1, c_2 \) are some constants.

From Lemma 3 and Lemma 5 we see that \( c_1 = c_2 = 0 \).

Hence \( \prod_{j\neq i}^{n} (\Delta - \lambda_j) f_0 = 0 \).

Then \( (\Delta - \lambda_m) \prod_{j\neq m}^{n} (\Delta - \lambda_j) f_0 = 0 \), \( m \in \mathbb{N} \).

Similarly we obtain, that \( \Delta f_0 = 0 \), \( \lambda_j \neq 0 \).

If \( \lambda_j = 0 \), then let \( f = \sum_{k=\min\{h, m-1\}}^{N} c_k |z|^{2k}, N \in \mathbb{N} \)

Consider the case \( k = \min\{h, m-1\} + 1 \).
Function with the selected index does not satisfy equation (1). Now take the Laplacian of $f$. As a result, we have

$$\triangle \sum_{k>\min\{h,m-1\}}^{N} c_k |z|^{2k} = \sum_{k>\min\{h,m-1\}}^{N} \tilde{c}_k |z|^{2(k-1)},$$

where $\tilde{c}_k$ are some constants.

Now substitute the function in equation (1). The only case, when $k = \min\{h,m-1\} + 1$ fits for the initial equality. This means that $f = \tilde{c}_{\min\{h,m-1\}} |z|^{2(k-1)}$, where $\tilde{c}_{\min\{h,m-1\}}$ are constants.

Again, we take the Laplacian and obtain the following function

$$f = \sum_{k=0}^{q} c_k |z|^{2k}, \quad c_k = \text{const},$$

as contended. □

**Lemma 7.** Let $f_j(\rho)e^{ij\varphi}$ satisfy (1). Then we have the same equality for $f_{j+1}(\rho)e^{i(j+1)\varphi}$ and $f_{j-1}(\rho)e^{i(j-1)\varphi}$, $j = 0, 1, 2, \ldots$.

**Proof.** We obtain for $f_j(\rho)e^{ij\varphi}$ the following equality

$$\frac{\partial}{\partial x} \left( f_j(\rho)e^{ij\varphi} \right) = \left( f'_j + j \frac{f_j(\rho)}{\rho} \right) e^{i(j-1)\varphi} + \left( f'_j - j \frac{f_j(\rho)}{\rho} \right) e^{i(j+1)\varphi},$$

where

$$(8) \quad f'_j + j \frac{f_j(\rho)}{\rho} = f_{j-1}(\rho).$$

The function $f_j(\rho)e^{ij\varphi}$ satisfies (1) by the condition. Hence $\frac{\partial}{\partial x} (f_j(\rho)e^{ij\varphi})$ and $f_{j-1}(\rho)e^{i(j-1)\varphi}$ satisfy (1) too.

Similarly, 

$$(9) \quad f'_j - j \frac{f_j(\rho)}{\rho} = f_{j+1}(\rho),$$

and $f_{j+1}(\rho)e^{i(j+1)\varphi}$ satisfies (1). □

4. **Proof of the Main Result.**

**Proof. Sufficiency.**

It is obviously that the equation (1) and (3) coincide for $\beta = 0$. Now Lemma 1 implies the sufficiency for the main theorem.

**Necessity.**

We have from equality (8)

$$f'_1(\rho) + \frac{f_1(\rho)}{\rho} = \sum_{k=0}^{q} c_k \rho^{2k}.$$ 

Then 

$$f_1(\rho) = \sum_{k=0}^{q} c_k \frac{\rho^{2k+1}}{2k+2} + \frac{c}{\rho},$$

where $c$ is a constant.
Substitute this function to equation (1), assuming that $c_{k,1} = \frac{c}{2k+2}$. We find

$$
\sum_{\nu=1}^{n} (\zeta_{\nu} e^{i\alpha})^s \left( \sum_{k=0}^{q} c_{k,1} (\zeta_{\nu} e^{i\alpha} + z)^{k+1} (\zeta_{\nu} e^{-i\alpha} + \bar{z})^k + \frac{c}{(\zeta_{\nu} e^{-i\alpha} + \bar{z})} \right) = 
$$

$$
= \sum_{p=s}^{q} \frac{mR^{2p}}{(p-s)!p!} \left( \frac{\partial}{\partial z} \right)^{p-s} \left( \frac{\partial}{\partial \bar{z}} \right)^p g(z) + \sum_{\nu=1}^{n} \frac{c}{(\zeta_{\nu} e^{-i\alpha} + \bar{z})},
$$

where $g(z)$ has the form (2). Hence it is clear that equality holds provided that $c = 0$ only. So, $f_1(\rho) = \sum_{k=0}^{q} c_{k,1} \rho^{2k+1}$.

Let $f_j(\rho) = \sum_{k=0}^{q} c_{k,j} \rho^{2k+j}$. Then

$$
f'_j(\rho) + (j + 1) \frac{f_{j+1}(\rho)}{\rho} = \sum_{k=0}^{q} c_j \rho^{2k+j},
$$

and $f_{j+1}(\rho) = \sum_{k=0}^{q} c_{k,j+1} \rho^{2k+j+1}$.

By induction we have $f_j(\rho) = \sum_{k=0}^{q} c_{k,j} \rho^{2k+j}$.

Now let us consider functions with negative indices. We start with $f_{-1}(\rho)$. From equality (9) we have the following

$$
f'_{-1}(\rho) + (-1) \frac{f_{-1}(\rho)}{\rho} = f_0(\rho).
$$

Hence, in the same way we get

$$
f_{-1}(\rho) = \sum_{k=0}^{q} c_{k,-1} \rho^{2k+1}.
$$

By induction we have again

$$
f_{-j}(\rho) = \sum_{k=0}^{q} c_{k,-j} \rho^{2k+j}, j \in \mathbb{N}.
$$

This implies that $f_{-j}(z) = (\sum_{k=0}^{q} c_{k,-j} \rho^{2k+j}) e^{-i\varphi_j}$. We should carefully consider the following two cases.

1. Let $h \leq m - 1$. Then

$$
f_{-j}(z) = \sum_{k=0}^{h} c_{k,-j} z^k \bar{z}^{k+j}.
$$

Hence

$$
\sum_{\nu=1}^{n} (\zeta_{\nu} e^{i\alpha})^s \sum_{k=0}^{h} c_{k,-j} (\zeta_{\nu} e^{i\alpha} + z)^{k+1} (\zeta_{\nu} e^{-i\alpha} + \bar{z})^k + \frac{c}{(\zeta_{\nu} e^{-i\alpha} + \bar{z})} = \sum_{p=s}^{\min\{h+j,h+s\}} \frac{nR^{2p}}{(p-s)!p!} \left( \frac{\partial}{\partial z} \right)^{p-s} \left( \frac{\partial}{\partial \bar{z}} \right)^p f(z).
$$

In this case $h + j = m - 1, h - (m - 1) = -j$.

2. Let $h > m - 1$. Then

$$
f_{-j}(z) = \sum_{k=0}^{m-1} c_{k,-j} z^k \bar{z}^{k+j}.
$$
Now,
\[ \sum_{v=1}^{n} (\zeta_{v} e^{i\alpha})^{s} \sum_{k=0}^{m-1} c_{k,-j}(\zeta_{v} e^{i\alpha} + z)^{k}(\bar{z} + \zeta_{v} e^{-i\alpha})^{k+j} = \sum_{p=s}^{\text{min}\{m-1+j,m-1+s\}} \frac{nR^{2p}}{(p-s)!p!} \left( \frac{\partial}{\partial z} \right)^{p-s} \left( \frac{\partial}{\partial \bar{z}} \right)^{p} f(z). \]

Then \( m - 1 + j \neq m - 1 \) and condition of the second case does not fit.

Similarly we analyze functions with positive indices.

1. Again, \( h \leq m - 1 \), \( f_{j}(z) = \sum_{k=0}^{h} c_{k,j} z^{k+j} \bar{z}^{k}. \)

Then
\[ \sum_{v=1}^{n} (\zeta_{v} e^{i\alpha})^{s} \sum_{k=0}^{h} c_{k,j}(\zeta_{v} e^{i\alpha} + z)^{k+j}(\bar{z} + \zeta_{v} e^{-i\alpha})^{k} = \sum_{p=s}^{\text{min}\{h,h+j+s\}} \frac{nR^{2p}}{(p-s)!p!} \left( \frac{\partial}{\partial z} \right)^{p-s} \left( \frac{\partial}{\partial \bar{z}} \right)^{p} f(z). \]

Now \( h \neq h + j \) and condition of the first case does not fit.

2. Consider the case \( h > m - 1 \).

Then we have \( h - (m - 1) = j \).

Combining the argument on the functions \( f_{j}(z) \), we get
\[ f_{-j}(z) = \sum_{k=0}^{h} c_{k,-j} z^{k} \bar{z}^{k+j} \]
and
\[ f_{j}(z) = \sum_{k=0}^{m-1} c_{k,j} z^{k+j} \bar{z}^{k}. \]

Now, considering the above and Lemma 2 and 4, we have
\[ f(z) = f_{0}(z) + f_{+}(z) + f_{-}(z), \]
where
\[ f_{+}(z) = \sum_{j=0}^{h-(m-1)m-1} \sum_{k=0}^{h-1} c_{k,j} z^{k+j} \bar{z}^{k} + \sum_{l=0}^{m-1} \sum_{k=0}^{h-1} c_{k,l} z^{l} \bar{z}^{k} \]
\[ f_{-}(z) = \sum_{j=0}^{m-1-h} \sum_{k=0}^{m-1} c_{k,-j} z^{k+j} + \sum_{l=0}^{h} \sum_{k=0}^{m-1} c_{k,l} z^{l} \bar{z}^{k} \]

So, the desired function \( f(z) \) has the form \([2]\). \( \square \)

References

[1] I.Netuka and J.Vesely Mean value property and harmonic functions // Classical and Modern Potential Theory and Applications, (Conri Sankaran et al., ed.), (Kluwer acad.Publ.), 1994. – pp.359-398.
[2] L.Zalcman Mean values and differential equations // Israel, J.Math., 14, 1973. – pp.339-352.
[3] Volchkov V.V. Integral Geometry and Convolution Equations. – Kluwer Academic Publishers. Dordrecht/Boston/London, 2003. – 454 p.
[4] Volchkov V.V., Volchkov Vit.V. Harmonic Analysis of Mean Periodic Functions on Symmetric spaces and the Heisenberg Group. – Series: Springer Monographs in Mathematics, 2009.
[5] T.Ramsey and Y.Weit Mean values and classes of harmonic functions // Math.Proc.Camb.Dhil.Soc., 96, 1984. – pp.501-505.
[6] Maxwell O. Reade A theorem of Féderoff // Duke Math.J., 18, 1948. – pp.105-109.
[7] Trofymenko O.D. Some integral equations for special classes of polynomials. Donetsk: Transactions of the Institute of Applied Mathematics and Mechanics, 18, 2009. – pp.184-188.

Faculty of Mathematics and Information Technology, Donetsk National University, Universitetskaya str. 24, 83001, Donetsk, Ukraine

E-mail address: ol4anovskiy@gmail.com