Large and very singular solutions to semilinear elliptic equations

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Abstract
We consider equation $-\Delta u + f(x, u) = 0$ in smooth bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, with $f(x, r) > 0$ in $\Omega \times \mathbb{R}_+$ and $f(x, r) = 0$ on $\partial \Omega$. We find the condition on the order of degeneracy of $f(x, r)$ near $\partial \Omega$, which is a criterion of the existence-nonexistence of a very singular solution with a strong point singularity on $\partial \Omega$. Moreover, we prove that the mentioned condition is a sufficient condition for the uniqueness of a large solution and conjecture that this condition is also a necessary condition of the uniqueness.

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1 Introduction and main results

This paper deals with two problems:
(1) the uniqueness of large solutions,
(2) the existence of very singular solutions
to a semilinear elliptic equation of the form:

$$-\Delta u + f(x, u) = 0 \quad \text{in } \Omega \subset \mathbb{R}^N, \ N > 1, \tag{1.1}$$

where nonlinear absorption term $f(x, s) > 0 \ \forall \ x \in \Omega, \ \forall \ s > 0$, degenerates on $\partial \Omega$: $f(x, s) = 0 \ \forall \ x \in \partial \Omega, \ f(x, 0) = 0 \ \forall \ x \in \Omega. \tag{1.2}$

When $f(s)$ is monotonic the existence of the large solution, i.e. a solution of equation (1.1) satisfying boundary condition:

$$\lim_{d(x) \to 0} u(x) = \infty, \quad d(x) := \text{dist}(x, \partial \Omega), \tag{1.3}$$

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is associated with a well known Keller–Osserman [9, 21] condition on the growth of \( f(s) \) as \( s \to \infty \). An adaptation of the KO-condition to nonmonotonic \( f(s) \) was realized in [4], to general nonlinearities \( f(x, s) \) — in [12]. A generalization of the KO-condition for higher order semilinear equations and inequalities was introduced in [10]. The uniqueness of large solutions was firstly proved by C. Loewner and L. Nirenberg in [11] for smooth domain \( \Omega \) and \( f(s) = s^p, p = \frac{N+2}{N-2} \). The first general result about the uniqueness was obtained by C. Bandle and M. Marcus [1] for smooth bounded domain \( \Omega \) and \( f(s) = s^p, p > 1 \). Asymptotic methods, introduced in [1], was applied to different classes of nonlinearities \( f(x, s) \) by many authors (see [13] and references therein).

It appears clear that the uniqueness of the large solution mostly depends on the order of degeneracy of nonlinearity \( f(x, s) \) on the boundary of \( \Omega \). So, in [19] the uniqueness was proved for \( C^2 \)-smooth bounded domain when:

\[
f(x, s) \geq c_0 d(x)^\alpha s^p \quad \forall x \in \Omega, \; \forall s \geq 0, \; p > 1, \; \alpha > 0, \; c_0 = \text{const} > 0,
\]

where \( d(x) \) is from (1.3). In [13] authors conjectured the uniqueness under the following condition:

\[
f(x, s) \geq c_0 \exp \left( -\frac{c_1}{d(x)^\alpha} \right) s^p \quad \forall x \in \Omega, \; \forall s \geq 0, \; 0 < \alpha < 1, \; c_1 = \text{const} \geq 0.
\]

In the present paper, we prove the validity of this hypothesis. Moreover, we prove even more general results. Namely, in bounded domain \( \Omega \in \mathbb{R}^N \) with \( C^2 \)-smooth boundary \( \partial \Omega \) we consider the following semilinear equation:

\[
Lu + H(x)u^p := - \sum_{i, j=1}^N (a_{ij}(x)u_{x_i})_{x_j} + H(x)u^p = 0 \quad \text{in} \; \Omega, \; p > 1, \tag{1.4}
\]

where \( C^{1,\lambda} \)-smooth functions \( a_{ij}(\cdot) \) satisfy the ellipticity condition:

\[
d_1 |\xi|^2 \geq \sum_{i, j=1}^N a_{ij}(x)\xi_i\xi_j \geq d_0 |\xi|^2 \quad \forall \xi \in \mathbb{R}^N, \; \forall x \in \overline{\Omega}, \; d_1 < \infty, \; d_0 > 0, \tag{1.5}
\]

and the absorption potential \( H(\cdot) \) satisfies

\[
H(x) \geq h_\omega(d(x)) \quad \forall x \in \overline{\Omega}, \; h_\omega(s) := \exp \left( -\frac{\omega(s)}{s} \right) \quad \forall s \in (0, \rho_0). \tag{1.6}
\]

**Theorem 1.1** Let potential \( H(x) \) satisfy estimate (1.6), where nondecreasing continuous function \( \omega(\cdot) \) satisfies the technical condition:

\[
s^{\gamma_1} \leq \omega(s) < \omega_0 = \text{const} < \infty \quad s \in (0, \rho_0), \; 0 < \gamma_1 < 1 \tag{1.7}
\]

and the Dini condition

\[
\int_0^c \frac{\omega(s)}{s} ds < \infty; \tag{1.8}
\]

Then equation (1.4) admits only one large solution in the mentioned domain \( \Omega \).

We conjecture that the Dini condition (1.8) is also a necessary condition for the uniqueness of the large solution. As an indirect confirmation of the validity of this conjecture we consider the second main result of this paper about the necessity of the Dini condition (1.8) for the existence of a very singular (v.s.) solution to equations of structure (1.4). Let us remind
that a v.s. solution was first discovered as a nonnegative solution of the semilinear parabolic equation:

$$u_t - \Delta u + hu^p = 0 \quad \text{in } \mathbb{R}_+^1 \times \mathbb{R}^N, \quad 1 < p < 1 + 2N^{-1}, \quad h = \text{const} > 0,$$

(1.9)

satisfying the initial condition:

$$u(0, x) = \infty \delta(x), \quad \delta(x) \text{ is Dirac measure},$$

(1.10)

in the following sense: $$u(0, x) = 0 \quad \forall \ x \mid x \neq 0,$$ and

$$\lim_{t \to 0} \int_{\mathbb{R}^N} u_\infty(t, x)dx = \infty.$$ 

Moreover, v.s. solution $$u_\infty(t, x)$$ can be obtained as $$\lim_{k \to \infty} u_k(t, x)$$, where $$u_k(t, x)$$ is a solution of (1.9) satisfying the initial condition $$u_k(0, x) = k\delta(x)$$ (see [3, 5, 8] and references therein). The next step was to study v.s. solutions to problem (1.9), (1.10) with variable absorption potential $$h = h(t, x) \geq 0$$, degenerating on initial hyperplane:

$$h(0, x) = 0 \quad \forall \ x \in \mathbb{R}^N.$$ 

(1.11)

A new phenomenon was observed in [20]: if $$h = h(t) = \exp\left(-\omega t^{-1}\right), \quad \omega = \text{const} > 0,$$ then $$u_\infty(t, x) = \lim_{k \to \infty} u_k(t, x)$$ is not a v.s. solution, but is a large solution, namely, $$u_\infty(0, x) = \infty \quad \forall \ x \in \mathbb{R}^N.$$ In [24] it was found a sharp condition on the degeneracy of $$h(t, x)$$ which guarantees the existence of v.s. solution $$u_\infty(t, x)$$ with strong point singularity: $$h(t) \geq \exp\left(-\omega(t)t^{-1}\right),$$ where $$\omega(\cdot)$$ is a continuous nondecreasing function, satisfying the following Dini condition:

$$\int_0^c s^{-1} \omega(s)\frac{1}{s} < \infty.$$ 

(1.12)

So far, we haven’t known whether this condition is also a necessary condition for the existence of a v.s. solution to semilinear parabolic equations of the structure (1.9). But in the case of the semilinear elliptic equation (1.4) the role of the Dini-type condition (1.8) for the existence of the corresponding v.s. solution has been studied more fully by now. Particularly, in [23] the following result about the sufficiency was proved. Let $$\{u_k(x)\}$$ be a sequence of solutions of equation (1.4), (1.5), (1.6), satisfying the boundary condition:

$$u_k = k\delta_a(x), \quad \text{on } \partial \Omega, \quad a \in \partial \Omega, \quad k = 1, 2, \ldots$$ 

(1.13)

Let the potential $$H(x)$$ satisfy estimate (1.6), where nonnegative function $$\omega(s)$$ satisfies all conditions of Theorem 1.1. Then $$u_\infty(x) = \lim_{k \to \infty} u_k(x)$$ is a v.s. solution of (1.4), i.e. a solution with a strong (more strong than the corresponding Poisson kernel) boundary singularity at $$a \in \partial \Omega$$ and $$\lim_{x \to y} u(x) = 0 \quad \forall \ y \in \partial \Omega \setminus \{a\}.$$

Let us consider the following model problem:

$$-\Delta u + h_\omega(|x'|)u^p = 0 \quad \text{in } \Omega := \mathbb{R}_+^N = \{x \in \mathbb{R}_+^N : x_N > 0\},$$

(1.14)

$$u \mid_{x_N = 0} = K\delta_a(x), \quad a \in L \subset \partial \Omega, \quad K \in \mathbb{R}_+^1,$$

(1.15)

where $$N \geq 2, \quad p > 1, \quad x' = (x_2, \ldots, x_N), \quad L$$ is a straight line $$\{x = (x_1, 0, \ldots, 0)\};$$

$$h_\omega(s) = \exp\left(-\frac{\omega(s)}{s}\right) \quad \forall s \geq 0.$$ 

(1.16)
Here function $\omega(\cdot)$ satisfies the following conditions:

(i) $\omega \in C(0, \infty)$ is a positive nondecreasing function,

(ii) $s \rightarrow \mu(s) := \frac{\omega(s)}{s}$ is monotonically decreasing on $\mathbb{R}_{+}^1$, (1.17)

(iii) $\lim_{s \rightarrow 0} \mu(s) = \infty$.

Thus $h_{\omega(\cdot)}(\cdot)$ is the absorption potential of equation (1.14) which degenerates on the line $L$ from (1.15). If

$$P_0(x, z) = c_N x_N | x - z |^{-N}, \quad c_N = \pi^{-\frac{N}{2}} \Gamma \left( \frac{N}{2} \right),$$

(1.18)

is the Poisson kernel for $-\Delta$ in $\mathbb{R}_{+}^N$, then (see [16]) inequality

$$\int_{|x| < R, x_N > 0} h_{\omega(\cdot)}(\cdot) P_0(x, a)^p x_N dx < \infty \quad \forall \ R : 0 < R < \infty \quad (1.19)$$

guarantees the existence of a unique solution of the problem (1.14), (1.15) dominated by the supersolution $K P_0(x, a)$. Thus, if condition (1.19) holds, then for an arbitrary monotonically increasing sequence

$$\{ K_j \}, \ K_j \rightarrow \infty \quad \text{as} \quad j \rightarrow \infty \quad (1.20)$$

due to the comparison principle) sequence of solutions $u_j(x)$ of the problem (1.14), (1.15) with $K = K_j$. Moreover, since $h_{\omega(\cdot)}(\cdot)$ is a positive function in $\Omega \setminus L$, equation (1.14) possesses a maximal solution $U$ in $\Omega$, which is a large solution (see [22]):

$$\lim_{x_N \rightarrow 0, |x| < M} U(x) = \infty \quad \forall \ M > 0. \quad (1.21)$$

Since $u_j(x) \leq U \quad \forall \ x \in \Omega \ \forall \ j \in \mathbb{N}$, the mentioned sequence converges to some function $u_{\infty}$, which is a positive solution of (1.14).

**Theorem 1.2** Let the parameter $p$ in equation (1.14) additionally satisfy

$$1 < p < p_0 := 1 + \frac{2}{N - 1} \quad (1.22)$$

and $\{ u_j(x) \}$ be a sequence of solutions of problem (1.14), (1.15), corresponding to $K = K_j$ from (1.20). Assume that functions $\omega(s), \mu(s)$ satisfy conditions (1.17) and

$$\lim_{j \rightarrow \infty} \mu \left( 2^{-j+1} \right) \mu \left( 2^{-j} \right)^{-1} < 1; \quad \omega(s) \rightarrow 0 \quad \text{as} \quad s \rightarrow 0. \quad (1.23)$$

Assume also that Dini condition (1.8) is not satisfied, namely:

$$\int_0^1 s^{-1} \omega(s) ds = \infty. \quad (1.24)$$

Then $u_{\infty}(x) := \lim_{j \rightarrow \infty} u_j(x)$ is a solution of (1.14) which satisfies

$$u_{\infty}|_{\partial\Omega \setminus L} = 0, \quad u_{\infty}|_{\partial\Omega \cap L} = \infty. \quad (1.25)$$
Remark 1 It is clear that the problem:

\[-\Delta u + h_\omega(\text{dist}(x, L))u^p = 0 \quad \text{in } \Omega = \mathbb{R}^N_+,
\]

\[u|_{\partial\Omega = \{x_N = 0\}} = K_\delta_\alpha(x), \quad a \in L,\]

where \(L\) is an arbitrary straight line in \(\partial\Omega = \mathbb{R}^{N-1}\), can be transformed into a problem of the form (1.14), (1.15), using corresponding linear orthonormal change of variables \((x_1, \ldots, x_N)\). Therefore, the conclusion of Theorem 1.2 is true for solutions \(u_j(x)\) of the mentioned problem with \(K = K_j\) too.

Remark 2 Let us consider additionally the following problem:

\[-\Delta u + h_\omega(x_N)u^p = 0 \quad \text{in } \mathbb{R}^N,
\]

\[u|_{x_N = 0} = K_j\delta_\alpha(x), \quad a \in \mathbb{R}^{N-1}, \quad K_j \to \infty \quad \text{as } j \to \infty,\]

where \(h_\omega(s)\) is the same as in (1.16). Since \(h_\omega(x_N) \leq h_\omega(\text{dist}(x, L))\) and \(h_\omega(x_N)\) degenerates on the whole hyperplane \(\{x : x_N = 0\}\), then due to Theorem 1.2 and the comparison principle, solution \(u_\infty(x) := \lim_{j \to \infty} u_j(x)\) satisfies: \(u_\infty(x'', 0) = \infty \quad \forall x'' \in \mathbb{R}^{N-1}\). Moreover, in [23] it was proved that if \(\omega(s)\) from (1.16) satisfies condition (1.8) instead of (1.24), then

\[u_\infty(x'', 0) = 0 \quad \forall x'' \in \mathbb{R}^{N-1} : x'' \neq a.\]

Thus, the Dini condition (1.8) is a necessary and sufficient condition for the existence of the very singular solution \(u_\infty(x)\) with point singularity.

Remark 3 Condition \(\omega(s) \to 0\) as \(s \to 0\) is technical for our proof of Theorem 1.2 and can be omitted by simple arguments. Let \(\omega(s) \geq \omega_0 = \text{const} > 0 \quad \forall s > 0\). Then we can find a continuous nondecreasing function \(\tilde{\omega}(s) \geq 0\):

\[\tilde{\omega}(s) > 0 \quad \forall s > 0, \quad \tilde{\omega}(s) \to 0 \quad \text{as } s \to 0, \quad \tilde{\omega}(s) \leq \omega_0 \quad \forall s > 0,
\]

which satisfies condition (1.24). Let now \(\tilde{u}_j(x)\) be a sequence of solutions to problem (1.26), (1.27) with absorption potential \(h_\tilde{\omega}(x_N) := \exp\left(-\frac{\tilde{\omega}(x_N)}{x_N}\right)\) instead of \(h_\omega(x_N)\). Then due to Th.1.2 \(\tilde{u}_\infty(x'', 0) = \infty\) for an arbitrary \(x'' \in \mathbb{R}^{N-1}\). If now \(u_j^{(0)}(x)\) be a sequence of solutions of problem (1.26), (1.27) with \(h_{\omega_0}(x_N) := \exp\left(-\frac{\omega_0}{x_N}\right)\) instead of \(h_\omega(x_N)\), then by comparison principle

\[u_j(x) \geq u_j^{(0)}(x) \geq \tilde{u}_j(x) \quad \forall j \in \mathbb{N}, \quad \forall x \in \Omega.
\]

Therefore \(\infty = \tilde{u}_\infty(x'', 0) \leq u_j^{(0)}(x'', 0) \leq u_\infty(x'', 0) = \infty\) and, as consequence, \(u_j^{(0)}(x'', 0) = u_\infty(x'', 0) = \infty\). Notice that this last property of propagation of the strong point singularity of solution \(u_j^{(0)}(x)\) along the whole boundary of the domain, when \(\omega = \omega_0 > 0\), was firstly discovered by M. Marcus, L. Veron [17].

The paper is organized as follows. Section 2 is devoted to the proof of the main auxiliary Theorem 2.1, where our variant of the local energy estimate method is applied for the study of the asymptotic behavior of solutions to semilinear elliptic equations of diffusion-absorption type near the singularity set. In Sect. 3 the technique, elaborated in Sect. 2, is adapted to the proof of Theorem 1.1. Finally, in Sect. 4 Theorem 1.2 about the necessity of the Dini condition is proved.
2 Local energy estimates near the boundary singularity set

Let $\Omega \subset \mathbb{R}^N_+$ be a bounded domain with $C^2$–boundary $\partial \Omega$, such that

$$\Gamma_{\bar{R}+\rho_0} := \{(x'',0) : |x''| \leq \bar{R} + \rho_0\} \subset \partial \Omega, \quad \Gamma_{\bar{R}+\rho_0} \times (0, \rho_0) \subset \Omega, \quad (2.1)$$

where $\bar{R} > 0$, $\rho_0 > 0$. Let $G_i$, $i = 1, 2, \ldots, l$, be bounded subdomains of hyperplane $\{x_N = 0\}$ with $C^2$–boundaries $\partial G_i$, such that

$$G_i \subset \{ |x''| < \bar{R} \} \quad \forall i \leq l, \quad (2.2)$$

$$\text{dist}(G_i, G_j) := \inf_{x \in G_i, y \in G_j} |x - y| > \rho_0 \quad \forall i \neq j, \quad (2.3)$$

In this domain $\Omega$ we consider the following boundary Dirichlet problem:

$$u |_{\partial G_i} = K^{(i)} = \text{const} > 0, \quad i = 1, 2, \ldots, l; \quad u = 0 \text{ on } \partial \Omega \setminus \bigcup_{i=1}^{l} \bar{G}_i, \quad (2.4)$$

for equation (1.4). Introduce now $l$ sequences

$$\{K^{(i)}_j\}, \quad i \leq l, \quad j = 1, 2, \ldots : \quad K^{(i)}_j \rightarrow \infty \text{ as } j \rightarrow \infty \quad \forall i \leq l, \quad (2.5)$$

and let $\{u_j\}, \quad j = 1, 2, \ldots$, be an infinite sequence of solutions of equation (1.4) satisfying the boundary condition

$$u_j |_{\partial G_i} = K^{(i)}_j, \quad u_j = 0 \text{ on } \partial \Omega \setminus \bigcup_{i=1}^{l} \bar{G}_i, \quad (2.6)$$

**Theorem 2.1** Let functions $h_\omega(\cdot)$ and $H(\cdot)$ satisfy relation (1.6) and let $\omega$ from (1.6) be a nondecreasing continuous function satisfying technical condition (1.7) and Dini condition (1.8). If $u_j$ is a solution of problem (1.4), (2.6), then $u_\infty := \lim_{j \rightarrow \infty} u_j$ is a solution of equation (1.4), satisfying the boundary conditions

$$\lim_{x \to y} u_\infty(x) = 0 \quad \forall y \in \partial \Omega \setminus \bigcup_{i \geq j} \bar{G}_i, \quad \lim_{x \to y} u_\infty(x) = \infty \quad \forall y \in \bigcup_{i \geq j} \bar{G}_i \quad (2.7)$$

**Proof** Let us introduce the following families of subdomains of $\Omega$ from (2.1)–(2.3):

$$\Omega_s := \{ x \in \Omega : d(x) > s \} \quad \forall s \in \mathbb{R}^+_1, \quad (2.8)$$

$$\Omega^s := \{ x \in \Omega : 0 < d(x) < s \} \quad \forall s \in \mathbb{R}^+_1$$

Due to the smoothness of $\partial \Omega$ there exists $\bar{s} > 0$, such that $\partial \Omega^s \cap \Omega = \partial \Omega_s$ is $C^2$–smooth for any $s : 0 < s < \bar{s}$. Moreover, we can assume that

$$d(x) = x_N \quad \forall x \in \Gamma_{\bar{R}+\rho_0} \times (0, \rho_0). \quad (2.9)$$

Let $u$ be a nonnegative solution of equation (1.4) in $\Omega$. Introduce the following energy function, connected with $u$:

$$I(s) := \int_{\Omega_s} | \nabla_x u |^2 + h_\omega(d(x)) u^{p+1} \, dx, \quad s > 0. \quad (2.10)$$

**Lemma 2.2** The function $I(\cdot)$ from (2.10) satisfies the estimate:

$$I(s) \leq d_3 \left[ \int_0^s h_\omega(r)^\frac{2}{p+2} \, dr \right]^{\frac{p+3}{p+1}}, \quad \forall s : 0 < s < \bar{s}, \quad (2.11)$$

where constant $d_3 < \infty$ does not depend on $u$. 

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Proof Multiplying equation (1.4) by \( u \) and integrating it over \( \Omega_s, s > 0 \), we obtain:

\[
\int_{\Omega_s} \left( \sum_{i,j=1}^{N} a_{ij}(x) u_{x_i} u_{x_j} + H(x) u^{p+1} \right) dx = \int_{\partial \Omega_s} \sum_{i,j=1}^{N} a_{ij}(x) u_{x_i} u_{x_j} d\sigma \leq \int_{\partial \Omega_s} \sum_{i,j=1}^{N} a_{ij}(x) u_{x_i} u_{x_j} d\sigma
\]

\[
\leq \left( \int_{\partial \Omega_s} \sum_{i,j=1}^{N} a_{ij}(x) u_{x_i} u_{x_j} d\sigma \right)^{\frac{1}{2}} \left( \int_{\partial \Omega_s} \sum_{i,j=1}^{N} a_{ij}(x) v_i v_j u^2 d\sigma \right)^{\frac{1}{2}},
\]

(2.12)

where \( v(x) = (v_1, ..., v_N) \) is an outward normal unit vector to \( \partial \Omega \). By (1.5), (1.6) and Hölder’s inequality, we have:

\[
\left( \int_{\partial \Omega_s} \sum_{i,j=1}^{N} a_{ij}(x) v_i v_j u^2 d\sigma \right)^{\frac{1}{2}} \leq c(\text{meas } \partial \Omega_s)^{\frac{q-1}{2(q+p+1)}} h_\omega(s)^{-\frac{1}{q+p+1}} \left( \int_{\partial \Omega_s} h_\omega(s) u^{q+1} d\sigma \right)^{\frac{1}{q+p+1}}.
\]

Substituting this estimate into (2.12) and using Young’s inequality we obtain:

\[
I(s) \leq c_1 h_\omega(s)^{-\frac{1}{p+1}} \left( \int_{\partial \Omega_s} (|\nabla u|^2 + h_\omega(d(x)) u^{p+1}) d\sigma \right)^{1-\frac{p-1}{2(p+1)}}.
\]

(2.13)

It is easy to see that

\[
\frac{dI(s)}{ds} = - \int_{\partial \Omega_s} (|\nabla u|^2 + h_\omega(d(x)) u^{p+1}) d\sigma.
\]

Substituting this relation into (2.13) we derive the following differential inequality:

\[
I(s) \leq c_2 h_\omega(s)^{-\frac{1}{p+1}} \left( -I'(s) \right)^{1-\frac{p-1}{2(p+1)}}.
\]

Solving this inequality we obtain (2.11). \( \square \)

Now we derive the global upper a priori estimates for solutions \( u_j \) of the problem (1.4), (2.6) when \( j \to \infty \). For an arbitrary small \( \delta > 0 \) we introduce \( C^1 \)-smooth function \( \xi_\delta(x'') \) with \( \text{supp } \xi_\delta \subset \{ x'' \in \mathbb{R}^{N-1} : |x''| \leq R + \delta \}, \delta < 2^{-1} \rho_0 \), such that:

\[
\xi_\delta(x'') = 1 \text{ if } x'' \in \overline{G}_i \quad \forall i \leq l,
\]

(2.14)

\[
\xi_\delta(x'') = 0 \text{ if } \text{dist}(x'', \overline{G}_i) := \min_{y \in \overline{G}_i} |x'' - y| \geq \delta \quad \forall i \leq l,
\]

(2.15)

\[
0 \leq \xi_\delta(x'') \leq 1 \quad \forall x'' \in \mathbb{R}^{N-1} \setminus \bigcup_{i \leq l} \overline{G}_i : \text{min } \text{dist}(x'', \overline{G}_i) < \delta.
\]

(2.16)

It is clear that

\[
|\nabla \xi_\delta| \leq c \delta^{-1} \quad \forall \delta : 0 < \delta < 2^{-1} \rho_0,
\]

(2.17)

where \( c < \infty \) does not depend on \( \delta \). Let \( u_{j,\delta}, j = 1, 2, ..., \) be a solution of equation (1.4) satisfying the regularized boundary condition:

\[
u_{j,\delta}(x) = K_j \xi_\delta \text{ on } \partial \Omega, \quad K_j := \max_{i \leq l} K^{(i)}_j.
\]

(2.18)

By the comparison principle we have:

\[
u_{j,\delta}(x) \geq u_j(x) \quad \forall x \in \overline{\Omega}, \forall j \in \mathbb{N}, \forall \delta : 0 < \delta < 2^{-1} \rho_0.
\]

(2.19)
Therefore, to prove Theorem 2.1 it is sufficient to investigate and estimate from above the solution \( u_{j,\delta} \) with an arbitrary small \( \delta > 0 \). For the sake of simplicity of the notations we omit \( \delta \) in \( u_{j,\delta} \) and denote \( u_{j,\delta} \) by \( u_j \).

**Lemma 2.3** Solution \( u_j = u_{j,\delta} \) of problem (1.4), (2.18) satisfies the following estimate:

\[
\int_{\Omega} \left( | \nabla u_j |^2 + h_\omega (d(x)) u_j^{p+1} \right) dx \leq \bar{K}_j := \bar{c} (K_j^{p+1} + \delta^{-1} K_j^2),
\]

where constant \( \bar{c} < \infty \) does not depend on \( j \in \mathbb{N}, \delta \in (0, 2^{-1} \rho_0) \).

**Proof** Let us introduce \( C^2 \)-cut–off function \( \zeta = \zeta_\delta(s) \), such that \( \zeta_\delta(s) = 1 \) if \( s \leq \delta, \zeta_\delta(s) = 0 \) if \( s > 2\delta, 0 \leq \zeta_\delta(s) \leq 1, | \nabla \zeta_\delta | < c\delta^{-1} \). Multiplying (1.4) by

\[
v_j(x) = u_j(x) - K_j \zeta_\delta(x'') \zeta_\delta(d(x)), \quad K_j \text{ is from (2.18)},
\]

and integrating it over \( \Omega \), due to \( v_j = 0 \) on \( \partial \Omega \) we obtain:

\[
\int_{\Omega} \left( \sum_{i,k=1}^N a_{ik}(x) u_{j,i} u_{j,k} + H(x) u_j^{p+1} \right) dx = \int_{\Omega} \sum_{i,k=1}^N a_{ik}(x) u_{j,i} \left( \zeta_\delta(x'') \zeta_\delta(d(x)) \right)_{x_k} K_j dx + \int_{\Omega} K_j H(x) u_j^p \zeta_\delta(x'') \zeta_\delta(d(x)) dx := A_1 + A_2.
\]

By Young’s inequality and properties (2.14)–(2.17) we get:

\[
| A_1 | \leq 2^{-1} \int_{\Omega} \sum_{i,k=1}^N a_{ik}(x) u_{j,i} u_{j,k} dx + c\delta^{-1} K_j^2,
\]

\[
| A_2 | \leq 2^{-1} \int_{\Omega} H(x) u_j^{p+1} dx + c' K_j^{p+1},
\]

where constants \( c, c' \) do not depend on \( \delta, j \). By (2.23) and (2.22) we have:

\[
\int_{\Omega} \left( \sum_{i,k=1}^N a_{ik}(x) u_{j,i} u_{j,k} + H(x) u_j^{p+1} \right) dx \leq \bar{c} (K_j^{p+1} + \delta^{-1} K_j^2), \quad \bar{c} = \max(c, c'),
\]

which yields the estimate (2.20) due to properties (1.6), (1.5).

Introduce now the following family of subdomains of the domain \( \Omega^s \) with an arbitrary \( s \in (0, \rho_0) \):

\[
\Omega^s(\tau) := \Omega^s \setminus \{ x = (x'', x^N) \in \Omega^s : r(x'') := \min_{l \leq l} dist(x'', G_i) < \tau \} \quad \forall \tau \in (0, \frac{\rho_0}{2}),
\]

(2.25)

where \( \rho_0 > 0 \) is from (2.9). Introduce also another family of energy functions for the solution \( u_j = u_{j,\delta} \) under consideration:

\[
J_j(s, \tau) := \int_{\Omega^s(\tau)} \left( | \nabla u_j |^2 + h_\omega (d(x)) u_j^{p+1} \right) \zeta_s(d(x)) dx,
\]

(2.26)

where \( \zeta_s(\cdot) \) is a function from (2.21): \( \zeta_s(d) = 1 \) if \( d \leq s, \zeta_s(d) = 0 \) if \( d > 2s, 0 \leq \zeta_s(d) \leq 1, | \nabla \zeta_s | \leq cs^{-1} \).

\( \square \) Springer
Lemma 2.4 The energy function $J_j(s, \tau)$ from (2.26) satisfies the following differential inequality

$$J_j(s, \tau) \leq cs \left( -\frac{d}{d\tau} J_j(s, \tau) \right) + Ch_\omega(s)^{-\frac{2}{p-1}} - \nu \quad \forall \tau \in (\delta, \rho_0), \forall j \in \mathbb{N},$$

$$\forall s \in \left(0, \frac{\rho_0}{2}\right), \forall \nu > 0, C = C(\nu) \to \infty \text{ as } \nu \to 0,$$  

where constants $c, C$ do not depend on $j$.

Proof We multiply equation (1.4) for the solution $u_j(x)$ by $u_j(x)\xi_s(d(x))$ and integrate it over $\Omega^{2s}(\tau), \tau > \delta$. As a result we obtain the following relation:

$$\mathcal{J}_j(s, \tau) := \int_{\Omega^{2s}(\tau)} \left( \sum_{i,k=1}^N a_{ik}(x)u_{jxi}u_{jxk} + H(x)u_j^{p+1} \right) \xi_s(d(x))dx$$

$$= R_1 + R_2 := \int_{\Gamma^{2s}(\tau)} \sum_{i,k=1}^N a_{ik}(x)u_{jxi}u_j v_k(x)\xi_s(d(x))d\sigma$$

$$- \int_{\Omega^{2s}(\tau) \setminus \Omega^s(\tau)} \sum_{i,k=1}^N a_{ik}(x)u_{jxi} \xi_s(d(x))x_k u_j dx,$$

where $\Gamma^{2s}(\tau) := \bigcup_{i \leq j} \Gamma^{2s}_i(\tau), \Gamma^{2s}_i(\tau) := \{ x = (x'', x_N) : x_N < 2s, dist(x'', G_i) = \tau \}$. Notice that due to (2.9) and (2.2), (2.3) we have: $\Gamma^{2s}_i(\tau) \cap \Gamma^{2s}_j(\tau) = \emptyset \forall i \neq j, \forall \tau < 2^{-1}\rho_0$.

Now using Hölder’s inequality we estimate $R_1$ from above:

$$|R_1| \leq \left( \int_{\Gamma^{2s}(\tau)} \sum_{i,k=1}^N a_{ik} u_{jxi}u_{jxk} \xi_s(d(x))d\sigma \right)^\frac{1}{2}$$

$$\times \left( \int_{\Gamma^{2s}(\tau)} \sum_{i,k=1}^N a_{ik} v_i v_k u_j^2 \xi_s(d(x))d\sigma \right)^\frac{1}{2} (R_1^{(1)})^\frac{1}{2} (R_1^{(2)})^\frac{1}{2}. \quad (2.29)$$

By (1.5) we estimate $R_1^{(2)}$:

$$R_1^{(2)} \leq d_1 \int_{\Gamma^{2s}(\tau)} u_j^2 \xi_s(d(x))d\sigma = d_1 \left( \int_{\Gamma^{2s}(\tau) \setminus \Gamma^s(\tau)} u_j^2 \xi_s(d(x))d\sigma + \int_{\Gamma^s(\tau)} u_j^2 d\sigma \right)$$

$$= d_1 \left( R^{(2)}_{1,1} + R^{(2)}_{1,2} \right).$$

Since $u_j(x'', 0) = 0 \forall x'' \in \Gamma^s(\tau) : \delta < \tau < \rho_0$, we derive by the Poincare’s inequality:

$$R^{(2)}_{1,2} = \int_{\Gamma^s(\tau)} u_j^2 d\sigma \leq d_2 s^2 \int_{\Gamma^s(\tau)} |\frac{\partial u_j}{\partial x_N}|^2 d\sigma \leq d_2 s^2 \int_{\Gamma^s(\tau)} |\nabla u_j|^2 d\sigma$$

$$\forall \tau : \delta < \tau < \rho_0/2.$$ 

(2.30)
We estimate the term $R_{1,1}^{(2)}$ by the standard trace interpolation inequality (see e.g. [6]):

$$
\int_{\Gamma_{i,N}(\tau)} u_j(x'', x^N)^2 \, d\sigma'' \leq c_1 \left( \int_{\tau < |x''| < \rho_0} |\nabla_{x''} u_j(x'', x^N)|^2 \, dx'' \right)^{\frac{1}{2}}
\times \left( \int_{\tau < |x''| < \rho_0} u_j(x'', x_N)^2 \, dx'' \right)^{\frac{1}{2}} + c_2 \int_{\tau < |x''| < \rho_0} u_j(x'', x_N)^2 \, dx''
\quad \forall \tau : \delta < \tau < \frac{\rho_0}{2}, \forall x_N \in (s, 2s), \forall i \leq l,
$$

where $\Gamma_{i,N}(\tau) := \{x = (x'', x_N) : \text{dist}(x'', G_i) = \tau, x_N = \text{const}\}$, constants $c_1, c_2$ do not depend on $\tau, s$. Integrating the last inequality with respect to $x_N$ over the interval $(s, 2s)$ and summing obtained inequalities from $i = 1$ up to $i = l$, we obtain after simple computations:

$$
R_{1,1}^{(2)} \leq c_1 \left( \int_{\Omega^2(\tau) \setminus \Omega^1(\tau)} |\nabla_{x''} u_j |^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega^2(\tau) \setminus \Omega^1(\tau)} u_j(x)^2 \, dx \right)^{\frac{1}{2}}
+ c_2 \int_{\Omega^2(\tau) \setminus \Omega^1(\tau)} u_j(x)^2 \, dx := c_1 (R_{1,1,1}^{(2)})^{\frac{1}{2}} (R_{1,1,2}^{(2)})^{\frac{1}{2}} + c_2 R_{1,1,2}^{(2)}
\quad \forall \tau \in (\delta, 2^{-1} \rho_0), \forall s \in (0, 2^{-1} \rho_0).
$$

By Hölder’s inequality we get:

$$
R_{1,1,2}^{(2)} \leq c_{3s}^{p-1} \frac{p-1}{p+1} h_\omega(s)^{-\frac{2}{p+1}} \left( \int_{\Omega^2(\tau) \setminus \Omega^1(\tau)} h_\omega(d(x)) u_j(x)^{p+1} \, dx \right)^{\frac{2}{p+1}}.
$$

It follows from (2.32), (2.33) that

$$
R_{1,1}^{(2)} \leq c_{4s}^{p-1} \frac{p-1}{p+1} h_\omega(s)^{-\frac{2}{p+1}} \left( \int_{\Omega^2(\tau) \setminus \Omega^1(\tau)} h_\omega(d(x)) u_j(x)^{p+1} \, dx \right)^{\frac{2}{p+1}}
+ c_{5s}^{p-1} \frac{p-1}{2p+11} h_\omega(s)^{-\frac{1}{p+1}} \left( \int_{\Omega^2(\tau) \setminus \Omega^1(\tau)} |\nabla_{x''} u_j |^2 \, dx \right)^{\frac{1}{2}}
\times \left( \int_{\Omega^2(\tau)} h_\omega(d(x)) u_j(x)^{p+1} \, dx \right)^{\frac{1}{p+1}}
\leq c_{4s}^{p-1} \frac{p-1}{p+1} h_\omega(s)^{-\frac{2}{p+1}} (I_j(s) - I_j(2s))^{1-\frac{p-1}{p+1}}
+ c_{5s}^{p-1} \frac{p-1}{2p+11} h_\omega(s)^{-\frac{1}{p+1}} (I_j(s) - I_j(2s))^{1-\frac{p-1}{2p+11}},
$$

where $I_j(s) = \int_{\Omega^2(\tau)} (|\nabla_{x''} u_j |^2 + h_\omega(d(x)) u_j(x)^{p+1}) \, dx$. Plugging estimates (2.30) and (2.34) into (2.29) and using Young’s inequality we obtain:
The term $R_1$ is easy to check that it is bounded:

$$| R_1 | \leq c_6 \left( \int_{\Gamma^{2i}(\tau)} | \nabla u_j |^2 \zeta_s d\sigma \right)^{\frac{1}{2}} \left[ \int_{\Gamma^{1s}(\tau)} | \nabla u_j |^2 d\sigma \right]^{\frac{1}{2}} \left[ s^{\frac{p-1}{2(\rho+1)}} h_\omega(s) - \frac{1}{m+1} \right] \left( I_j(s) - I_j(2s) \right)^{1-\frac{p-1}{2m+1}}$$

$$+ s^{\frac{p-1}{2(\rho+1)}} h_\omega(s) - \frac{1}{m+1} \left( I_j(s) - I_j(2s) \right)^{1-\frac{p-1}{2m+1}}$$

Finally, we estimate $R_2$. Using Hölder’s inequality and property (1.5) we get:

$$| R_2 | \leq c_s^{-1} \left( \int_{\Omega^{2i}(\tau) \setminus \Omega^{1s}(\tau)} | \nabla u_j |^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega^{2i}(\tau) \setminus \Omega^{1s}(\tau)} u_j^2 dx \right)^{\frac{1}{2}} =: c_s^{-1} (R_2^{(1)})^{\frac{1}{2}} (R_2^{(2)})^{\frac{1}{2}}.$$

The term $R_2^{(2)}$ coincides with $R_1^{(2)}$ and it can be estimated as in (2.33). Therefore, by Young’s inequality we get from (2.36) that

$$| R_2 | \leq c_s^{-1} \left( \frac{p-1}{2m+1} \right) h_\omega(s) - \frac{1}{m+1} \left( \int_{\Omega^{2i}(\tau) \setminus \Omega^{1s}(\tau)} \left( | \nabla u_j |^2 + h_\omega(d(x))u_j^{p+1} \right) dx \right)^{1-\frac{p-1}{2m+1}}.$$ (2.37)

Thus, due to estimates (2.35) and (2.37) it follows from (2.28) that

$$J_j(s, \tau) \leq d_0^{-1} J_j(s, \tau) \leq c_s \int_{\Gamma^{2i}(\tau)} | \nabla u_j |^2 \zeta_s(d(x)) d\sigma + c_1 s^{-\frac{2}{m+1}} h_\omega(s)^{-\frac{2}{m+1}} \times$$

$$\times \left( I_j(s) - I_j(2s) \right)^{1-\frac{p-1}{2m+1}} + c_2 s^{-\frac{p+3}{2m+1}} h_\omega(s)^{-\frac{1}{m+1}} \left( I_j(s) - I_j(2s) \right)^{1-\frac{p-1}{2m+1}}.$$ (2.38)

It is easy to check that

$$\int_{\Gamma^{2i}(\tau)} \left( | \nabla u_j |^2 + h_\omega(d(x))u_j^{p+1} \right) \zeta_s(d(x)) d\sigma \leq -\bar{c} \frac{d}{d\tau} J_j(s, \tau),$$ (2.39)

where $\bar{c} = const > 0$ does not depend on $\tau, s, j$. Substituting (2.39) into (2.38) we obtain:

$$J_j(s, \tau) \leq \bar{c} s \left( -\frac{d}{d\tau} J_j(s, \tau) \right) + c_1 F_j(s) \forall \tau \in \left( \delta, \frac{\rho_0}{2} \right), \forall s \in \left( 0, \frac{\rho_0}{2} \right),$$

$$F_j(s) := \frac{(I_j(s) - I_j(2s))^{1-\frac{p-1}{2m+1}}}{s^{\frac{p-1}{2m+1}} h_\omega(s)^{\frac{1}{m+1}}} + \frac{(I_j(s) - I_j(2s))^{1-\frac{p-1}{2m+1}}}{s^{\frac{p+3}{2m+1}} h_\omega(s)^{\frac{1}{m+1}}}.$$ (2.40)

It only remains to estimate $F_j(s)$ from above. By lemma 2.2 we have the following uniform, with respect to $j \in \mathbb{N}$, upper estimate for the energy functions $I_j$:

$$I_j(s) \leq d_3 \left[ \int_0^s h_\omega(r)^{\frac{2}{m+1}} dr \right]^{-\frac{p+3}{2m+1}} \forall s: 0 < s < \bar{s}.$$ (2.41)
Since \( \omega(\cdot) \) is a nondecreasing function it is easy to check (see lemma 2.4 from [23]) that
\[
\int_0^s \exp \left( -\frac{b \omega(t)}{t} \right) \, dt \geq \frac{s^2}{2s + b \omega(s)} \exp \left( -\frac{b \omega(s)}{s} \right) \quad \forall \ b > 0. \quad (2.42)
\]
Therefore, by (2.42) it follows from (2.41) that
\[
I_j(s) \leq d_3 \left( \frac{2s + \frac{2}{p+5} \omega(s)}{s^2} \right)^{\frac{p+3}{p-1}} \exp \left( \frac{2}{p-1} \frac{\omega(s)}{s} \right) := d_3 \Phi_1(s) h_\omega(s)^{-\frac{2}{p-1}}. \quad (2.43)
\]
Substituting estimate (2.43) into the definition of the function \( F_j(s) \) we obtain:
\[
F_j(s) \leq d_4 \left( \frac{\Phi_1(s)}{s} \right)^{\frac{p+3}{p+1}} + \left( \frac{\Phi_1(s)}{s} \right)^{\frac{p+3}{p+1}} h_\omega(s)^{-\frac{2}{p-1}}. \quad (2.44)
\]
By condition (1.7) for the function \( \omega(s) \) we have:
\[
h_\omega(s)^\nu \leq \exp \left( -\nu s^{-(1-\gamma)} \right) \quad \forall \ s \in (0, \rho_0), \ \forall \ \nu > 0,
\]
which yields the upper estimate for \( F_j(s) \):
\[
F_j(s) \leq C(\nu) h_\omega(s)^{-\frac{2}{p-1} - \nu} \quad \forall \ \nu > 0, \ C(\nu) \to \infty \text{ as } \nu \to 0. \quad (2.45)
\]
Now we get down to the main step of the proof of the theorem, which consists of a careful analysis of the vanishing properties of the energy functions \( J_j(s, \tau) \), satisfying inequalities (2.27) for all \( j \in \mathbb{N} \). Notice that due to global estimate (2.20) function \( J_j(s, \tau) \) satisfies the following "initial" condition:
\[
J_j(s, \delta) \leq \bar{K}_j := \bar{\alpha}(K_j^{p+1} + \delta^{-1} K_j^2) \quad \forall \ j \in \mathbb{N}, \quad (2.46)
\]
where \( \delta > 0 \) and \( K_j \) are from boundary condition (2.18). Let us fix \( j \) large enough and \( \nu > 0 \) small enough. Next we define \( s_j > 0 \) by the following relation:
\[
C(\nu) h_\omega(s_j)^{-\frac{2}{p-1} - \nu} = \bar{K}_j^{\theta}, \quad C(\nu) \text{ from (2.45)}, \quad (2.47)
\]
where \( 0 < \theta < 1 \) will be defined later. It follows from (2.27), (2.46) that \( J_j(s_j, \tau) \) satisfies the following differential inequality:
\[
J_j(s_j, \tau) \leq \bar{c} s_j \left( -\frac{d}{d\tau} J_j(s_j, \tau) \right) + \bar{K}_j^{\theta} \quad \forall \ \tau : \ \delta < \tau < 2^{-1} \rho_0, \quad (2.48)
\]
Let us define now value \( \tau_j \) by the equality:
\[
J_j(s_j, \delta + \tau_j) = 2\bar{K}_j^{\theta}, \quad (2.49)
\]
where \( \theta \) is from (2.47). To find an upper estimate for \( \tau_j \), we notice that
\[
J_j(s_j, \tau) > 2\bar{K}_j^{\theta} \quad \forall \ \tau \in (\delta, \delta + \tau_j).
\]
Hence (2.48) yields:
\[
J_j(s_j, \tau) \leq 2\bar{c} s_j \left( -\frac{d}{d\tau} J_j(s_j, \tau) \right) \quad \forall \ \tau \in (\delta, \delta + \tau_j). \quad (2.50)
\]
Solving this differential inequality and taking into account the initial condition in (2.48), we obtain:
\[ J_j(s_j, \tau) \leq \overline{K}_j \exp \left( -\frac{\tau - \delta}{2\tilde{c}s_j} \right) \quad \forall \tau \in (\delta, \delta + \tau_j). \tag{2.51} \]

By (2.49) and (2.51) we get: 
\[ 2\overline{K}_j^\theta \leq \overline{K}_j \exp \left( -\frac{\tau_j}{2\tilde{c}s_j} \right), \]
where \( \tilde{c} \) is a constant from (2.48). Hence, \( \tau_j \) satisfies:
\[ 0 < \tau_j \leq 2\tilde{c}s_j (\ln 2 + (1 - \theta) \ln \overline{K}_j). \tag{2.52} \]

Next, notice that by definitions (2.10), (2.26) we have:
\[ \int_{\Omega^0(\delta + \tau_j)} \left( |\nabla u_j|^2 + h_\omega(d(x))u_j^{p+1} \right) dx \leq I_j(s_j) + J_j(s_j, \delta + \tau_j), \quad \text{if} \quad \delta + \tau_j < 2^{-1} \rho_0. \tag{2.53} \]

Due to estimate (2.43) and condition (1.7) on \( \omega(s) \), analogously to (2.45), we have:
\[ I_j(s_j) \leq C_1(v)h_\omega(s_j)^{-\frac{2}{p-1}}v. \quad \forall v > 0, \quad C_1(v) \to \infty \quad \text{as} \quad v \to 0. \tag{2.54} \]

Using now definition (2.47) of \( s_j \) and (2.49) of \( \tau_j \), we deduce from (2.53) and (2.54):
\[ \int_{\Omega^0(\delta + \tau_j)} \left( |\nabla u_j|^2 + h_\omega(d(x))u_j^{p+1} \right) dx \leq \left( 2 + \frac{C_1(v)}{C(v)} \right) \overline{K}_j^\theta. \tag{2.55} \]

Now we define sequences \( \{K_i\} \) and \( \{\overline{K}_i\} \), \( i = 1, 2, \ldots \), which are connected by the relation (2.46). Firstly introduce
\[ \overline{K}_i := \exp \exp i, \quad i \in \mathbb{N}. \tag{2.56} \]

Then define \( \{K_i\} = \{K_i(\delta, \tilde{c})\} \) as solutions of algebraic equation (2.46). It is easy to see that \( K_i = K_i(\delta, \tilde{c}) \to \infty \) as \( i \to \infty \). Now we have to fix parameter \( \theta \) from definition (2.47) of \( s_j \), namely, we have to guarantee the validity of the following inequality:
\[ \left( 2 + \frac{C_1(v)}{C(v)} \right) \overline{K}_j \leq \overline{K}_{j-1}. \tag{2.57} \]

Due to (2.56) inequality (2.57) is equivalent to:
\[ \ln \left( 2 + \frac{C_1(v)}{C(v)} \right) + \theta \exp(1) \leq \exp(j - 1) = e^{-1} \exp j. \tag{2.58} \]

It is easy to see that (2.58) is satisfied by
\[ \theta = (2e)^{-1} \quad \text{if} \quad j \geq j_0 := 1 + \ln 2 + \ln \ln \left( 2 + \frac{C_1(v)}{C(v)} \right). \tag{2.59} \]

With such \( \theta \) inequality (2.55) yields:
\[ \int_{\Omega^0(\delta + \tau_j)} \left( |\nabla u_j|^2 + h_\omega(d(x))u_j^{p+1} \right) dx \leq \overline{K}_{j-1}. \tag{2.60} \]

Now we obtain explicit upper estimates of \( \tau_j, s_j \), defined by (2.47), (2.49). Firstly, (2.47) yields:
\[ C(v) \exp \left( \left( \frac{2}{p - 1} + v \right) \frac{\omega(s_j)}{s_j} \right) = \overline{K}_j \Rightarrow \theta \ln \overline{K}_j \leq \frac{\theta}{2} \ln \overline{K}_j \leq \left( \frac{2}{p - 1} + v \right) \frac{\omega(s_j)}{s_j} \leq \theta \ln \overline{K}_j \]
\[ \forall j \geq j' = j'(v) = \ln \ln C(v) + \ln \theta^{-1} + \ln 2. \tag{2.61} \]
By (2.61), (1.7) and (2.56) we have:
\[
    s_j \leq 2 \left( \frac{2}{p-1} + v \right) \theta^{-1} (\ln K_j)^{-1} \omega(s_j) \leq 2 \left( \frac{2}{p-1} + v \right) \theta^{-1} \omega_0 \exp(-j).
\] (2.62)

This estimate due to the monotonicity of \( \omega(\cdot) \) yields:
\[
    \omega(s_j) \leq \omega(C_3 \exp(-j)), \quad C_3 = 2 \left( \frac{2}{p-1} + v \right) \theta^{-1} \omega_0.
\] (2.63)

As to \( \tau_j \), we get from (2.52) and (2.62) that
\[
    \tau_j \leq 2c s_j (1 - \theta) \ln K_j \leq 4c (1 - \theta) \left( \frac{2}{p-1} + v \right) \theta^{-1} \omega(s_j) \leq C_2 \omega(s_j),
\] (2.64)

where \( C_2 = 4 \theta^{-1} (1 - \theta) \tilde{c} \left( \frac{2}{p-1} + v \right) \). Substituting (2.63) into (2.64) we obtain:
\[
    \tau_j \leq C_2 \omega(C_3 \exp(-j)).
\] (2.65)

So, estimates (2.60), (2.62), (2.64) are the results of the first circle of the computation and a starting point for the second circle. Similar to (2.47), we define value \( s_{j-1} \):
\[
    C(v) h_{\omega}(s_{j-1}) - \frac{2}{\rho - 1} - v = \overline{K}_{j-1}^\theta, \quad \theta = (2\epsilon)^{-1}, \quad C(v) \text{ is from (2.45).}
\] (2.66)

Then the energy function \( J_j(s_{j-1}, \tau) \) satisfies the following differential inequality:
\[
    J_j(s_{j-1}, \tau) \leq \tilde{c} s_{j-1} \left( -\frac{d}{d\tau} J_j(s_{j-1}, \tau) \right) + \overline{K}_{j-1}^\theta \quad \forall \tau \in (\delta + \tau_j, 2^{-1} \rho_0)
\] (2.67)

instead of (2.48), and the following ”initial” condition:
\[
    J_j(s_{j-1}, \delta + \tau_j) \leq \overline{K}_{j-1}.
\] (2.68)

which is a consequence of inequality (2.60) from the first circle of the computations. Next we define \( \tau_{j-1} \) by the analog of (2.49):
\[
    J_j(s_{j-1}, \delta + \tau_j + \tau_{j-1}) = 2 \overline{K}_{j-1}^\theta.
\] (2.69)

Similar to (2.50), by (2.67), (2.68), (2.69) we have the following relation:
\[
    J_j(s_{j-1}, \tau) \leq 2 \tilde{c} s_{j-1} \left( -\frac{d}{d\tau} J_j(s_{j-1}, \tau) \right) \quad \forall \tau \in (\delta + \tau_j, \delta + \tau_j + \tau_{j-1}).
\] (2.70)

Solving this differential inequality by the ”initial” condition (2.68), we obtain:
\[
    J_j(s_{j-1}, \tau) \leq \overline{K}_{j-1} \exp \left( -\frac{\tau - \delta - \tau_j}{2 \tilde{c} s_{j-1}} \right) \quad \forall \tau \in (\delta + \tau_j, \delta + \tau_j + \tau_{j-1}).
\] (2.71)

Definition (2.69) of \( \tau_{j-1} \) and estimate (2.71) lead to the explicit estimate of \( \tau_{j-1} \):
\[
    \tau_{j-1} \leq 2 \tilde{c} s_{j-1} \left( -\ln 2 + (1 - \theta) \ln \overline{K}_{j-1} \right),
\]
and, finally, to the following analogs of estimates (2.62), (2.65):
\[
    s_{j-1} \leq C_3 \exp(-j + 1), \quad \tau_{j-1} \leq C_2 \omega(C_3 \exp(-j + 1)).
\] (2.72)

Inequality (2.43) yields the analog of (2.54):
\[
    I_j(s_{j-1}) \leq \frac{C_1(v)}{C(v)} \left( C(v) h_{\omega}(s_{j-1}) - \frac{2}{\rho - 1} - v \right).
\] (2.73)
Summing estimates (2.69) and (2.73), using definition (2.66) of $s_{j-1}$ and keeping in mind the validity of property (2.57) for all $j \geq j_0$ with $j_0$ from (2.59), we get:

$$\int_{\Omega_0^2(\delta + \tau_j + \delta_{j-1})} \left( | \nabla u_j |^2 + h_\omega(d(x))u_j^{p+1} \right) dx \leq \left( 2 + \frac{C_1(v)}{C(v)} \right) \overline{K}_{j-1} \leq \overline{K}_{j-2}, \text{ if } j > j_0 + 1.$$  

(2.74)

This estimate is the result of the second circle of computations and a starting point for the next circle. Realizing $i$ such circles, we obtain the following analog of estimates (2.74), (2.72):

$$\int_{\Omega_0^2(\delta + \sum_{k=0}^{i-1} \tau_{j-k})} \left( | \nabla u_j |^2 + h_\omega(d(x))u_j^{p+1} \right) dx \leq \overline{K}_{j-i}.$$  

(2.75)

$$\tau_{j-k} \leq C_2\omega(C_3 \exp(-j + k)), \quad s_{j-k} \leq C_3 \exp(-j + k), \quad \forall k \leq i - 1.$$  

(2.76)

There are two restrictions on value $i$. First of them follows from (2.58), (2.59):

$$j - i \geq j_0 := 1 + \ln 2 + \ln \ln(2 + C_1(v)C(v)^{-1}).$$  

(2.77)

The second restriction follows from the analog of (2.50), (2.70), namely, estimates of the interval, where differential inequality for energy function $J_j(s_{j-i}, \tau)$ has to be satisfied:

$$\delta + \sum_{k=0}^{i-1} \tau_{j-k} \leq 2^{-1}\rho_0.$$  

(2.78)

Due to estimate (2.76) and monotonicity of $\omega(\cdot)$ we have:

$$\sum_{k=0}^{i-1} \tau_{j-k} \leq C_2 \sum_{k=0}^{i-1} \omega(C_3 \exp(-j + k)) \leq C_2C_3^{-1} \int_{C_3 \exp(-j)}^{C_3 \exp(-j+i)} r^{-1}\omega(r) dr \leq C_2C_3^{-1} \int_0^{C_3 \exp(-i)} r^{-1}\omega(r) dr =: C_2C_3^{-1}\Phi(j - i),$$  

(2.79)

where $\Phi(s) \to 0$ as $s \to \infty$ due to the Dini condition (1.8). Therefore for arbitrary $\delta > 0$, $\rho_0 > 2\delta$ there exists finite $j^{(0)} = j^{(0)}(\delta, \rho_0)$ such that $\delta + C_2C_3^{-1}\Phi(j^{(0)}) \leq 2^{-1}\rho_0$ and, hence, condition (2.78) is satisfied if $j - i \geq j^{(0)}$. Thus, due to (2.79) estimate (2.75) leads to:

$$\int_{\Omega_0^2(\delta + C_3^{-1}\Phi(j - i))} \left( | \nabla u_j |^2 + h_\omega(d(x))u_j^{p+1} \right) dx \leq \overline{K}_{j-i} \quad \forall j: j - i \geq \tilde{j},$$  

(2.80)

where $\tilde{j} := \max\{j_0, j^{(0)}, j^*\}, j^*$ is from (2.61), $u_j(x) = u_j,\delta(x)$. Let us notice that for arbitrary small $\delta > 0$ we can find finite number $j^{(1)} = j^{(1)}(\delta)$ such that:

$$j^{(1)} = \min(j \in \mathbb{N} : \delta \geq K_j^{-\rho^{-1}}).$$  

(2.81)

By definition (2.46) of $\overline{K}_j$ we have:

$$\overline{K}_j \leq 2\tilde{c}K_j^{\rho+1} \quad \forall j \geq j^{(1)},$$  

(2.82)

which means that $\overline{K}_j$ does not depend on $\delta$ if $j \geq j^{(1)}$. Now by condition (1.8) we can find finite $j^{(2)} = j^{(2)}(\delta) \geq j^{(1)}(\delta)$, such that

$$C_2C_3^{-1}\Phi(j - i) \leq \delta \quad \forall j: j - i > j^{(2)}$$
and, by (2.80) we get:
\[
\int_{\Omega(2\delta)} \left| \nabla u_j \right|^2 + h_\omega(d(x)) u_j^{p+1} \right) dx \leq K_{j''} \text{ if } j > j'' := \max(j, j^{(2)}).
\] (2.83)

This estimate yields the following uniform with respect to \( j \in \mathbb{N} \) a priori estimate:
\[
|u_{j, \delta}|_{H^1(\Omega(2\delta), \partial \Omega(2\delta) \cap \partial \Omega)} \leq C = C(\delta) < \infty \quad \forall \ j \in \mathbb{N},
\] (2.84)

where for an arbitrary set \( S \subset \Omega \) by \( H^1(\Omega, S) \) we define the closure in the norm of \( H^1(\Omega) \) of set \( C^1(\Omega, S) := \{ f \in C^1(\Omega) : f|_S = 0 \} \). Since \( h_\omega(d(x)) \geq 0 \) in \( \overline{\Omega} \), all functions \( u_{j, \delta} \) are subsolutions of the corresponding linear elliptic equation. Therefore, by Harnack inequality, for subsolutions of linear elliptic equations (see, e.g., [6]) we get:
\[
\left( \sup_{\Omega(3\delta)} u_{j, \delta} \right)^2 \leq c(\delta, \rho_0) \int_{\Omega(2\delta)} |u_{j, \delta}(x)|^2 dx \quad \forall j \in \mathbb{N}, \ \forall \delta > 0, \ \rho_1 = \frac{\rho_0}{2}.
\] (2.85)

By (2.85) and (2.83) we have:
\[
\sup_{\Omega(3\delta)} u_{j, \delta} \leq c_1(\delta) \quad \forall j \in \mathbb{N}, \ \forall \delta > 0.
\] (2.86)

In virtue of (2.19) the last inequality yields:
\[
\sup_{\Omega(3\delta)} u_j \leq c_1(\delta) \quad \forall j \in \mathbb{N}, \ \forall \delta > 0,
\] (2.87)

where \( u_j \) is a solution to the problem (1.4), (2.6). It is easy to see that \( u_j(x) \) is a solution to the following problem:
\[
- Lu_j = g_j(x) := H(x) u_j(x)^p \quad \text{in } \Omega(3\delta)
\] (2.88)
\[
u_j \mid_{\partial \Omega(3\delta) \cap \partial \Omega} = 0 \quad \forall j \in \mathbb{N},
\] (2.89)

where, \( |g_j|_{L^q(\Omega(3\delta))} \leq c_2(\delta) \forall j \in \mathbb{N}, \forall q > 1 \) due to (2.87). Hence, by the classical \( L^q \) a priori estimates for solutions of linear elliptic problems (see, for example, [6]) we get:
\[
|u_j|_{W^{2,q}(\Omega(4\delta))} \leq c_3(\delta) \quad \forall j \in \mathbb{N}, \ \forall q > 1, \ \rho_2 = \frac{\rho_1}{2}.
\] (2.90)

By the comparison principle, sequence \( \{u_j\}, j = 1, 2, \ldots \), is monotonically nondecreasing in \( \Omega \) and, hence, \( u_j(x) \to u_\infty(x) \) pointwise for all \( x \in \overline{\Omega} \). Then by the uniform estimate (2.90) and the compact embedding of the space \( W^{2,q}(\Omega(4\delta)) \) into \( C^{1,\lambda}(\overline{\Omega}(4\delta)) \), \( 0 < \lambda < 1 - \frac{N}{q} \), we have:
\[
|u_j - u_\infty|_{C^{1,\lambda}(\overline{\Omega}(4\delta))} \to 0 \text{ as } j \to \infty.
\] (2.91)

Since \( \delta \) is an arbitrary positive number and \( u_j \to 0 \) on \( \partial \Omega \setminus \{ \cup_{i \leq l} \overline{G}_i \} \) it follows from (2.91) that \( u_\infty(x) = 0 \ \forall x \in \partial \Omega \setminus \{ \cup_{i \leq l} \overline{G}_i \} \). Theorem 2.1 is proved.

3 About the uniqueness of large solution

Here we prove Theorem 1.1. Our proof consists in verifying that the equation under consideration has the so-called strong barrier property. This property was introduced in [18, 19], where the sufficiency of this property for the uniqueness of the large solution was proved for the equation under consideration.

\( \square \) Springer
Definition 1 (see [19], Def. 2.6) Let $z \in \partial \Omega$. We say that equation (1.4) possesses a strong barrier at point $z$ if there exists a number $R_z > 0$ such that for every $r \in (0, R_z)$ there exists a positive supersolution $u = u_{r,z} \in C(\overline{\Omega} \cap B_r(z))$ of equation (1.4) in $\Omega \cap B_r(z)$, such that
\[
\lim_{y \to x, \ y \in \Omega \cap B_r(z)} u_{r,z}(y) = \infty \text{ for all } x \in \Omega \cap \partial B_r(z).
\] (3.1)

If an equation has the mentioned property for an arbitrary point $z \in \partial \Omega$ then we say that the equation under consideration has the strong barrier property.

Without loss of generality we suppose that $\Omega \subset \mathbb{R}^N_+$, $z = 0 \in \partial \Omega$, $B_R(0) \cap \Omega = \{x = (x'', x_N) \in B_R(0) : x_N > 0\} := \Omega_R$ and $d(x) := \text{dist}(x, \partial \Omega) = x_N \ \forall \ x \in \Omega_R$. Introduce also a weight function
\[
\rho(x) := \text{dist}(x, \partial B_R(0)) = R - |x| = R - (|x''|^2 + x_N^2)^{1/2},
\] (3.2)
and a surface $\Gamma_R := \{x \in \Omega_R : \rho(x) = d(x)\}$. It is easy to check that $\Gamma_R$ is a paraboloid:
\[
\Gamma_R = \left\{x \in \Omega_R : x_N = \frac{R^2 - |x''|^2}{2R} \ \forall x'' : |x''| \leq R \right\}.
\] (3.3)

Now we are going to prove that equation (1.4) has the property from Definition 3.1 in the domain $\Omega \subset \mathbb{R}^N_+$. Namely, let $\{u_j(x)\}$ be an increasing sequence of solutions of equation (1.4) in the domain $\Omega_R$ satisfying the boundary conditions:
\[
u |_{\partial \Omega_R \cap \Omega} = K_j, \quad u |_{\partial \Omega \cap B_R(0)} = 0 \quad K_j \to \infty \text{ as } j \to \infty.
\] (3.4)

We will prove that $u_\infty = \lim_{j \to \infty} u_j(x)$ is a strong barrier at point $0 \in \partial \Omega$ for equation (1.4) in the sense of Definition 1. Firstly, we introduce a sequence $\{u_{j,\delta}(x)\}$ of solutions of equation (1.4), satisfying the following "regularized" boundary conditions:
\[
u |_{\partial \Omega_R \cap \Omega} = K_j, \quad u |_{\partial \Omega \cap B_R(0)} = K_j \xi_\delta(|x''|),
\] (3.5)
where $\xi_\delta(s) = \begin{cases} 0, & s \leq R - \delta, \\ 1 - \delta^{-1}(R - s), & s > R - \delta \end{cases}$
and $\delta > 0$ arbitrary small. By the comparison principle we have $u_{j,\delta}(x) \geq u_j(x) \ \forall j \in \mathbb{N}$, $\forall \delta > 0$ and, hence, $u_{\infty,\delta}(x) \geq u_\infty(x) \ \forall x \in \Omega_R$. The main part of our analysis consists in proving that
\[
u_{\infty,\delta}(x) = 0 \ \forall x = (x'', x_N) : x_N = 0, \ |x''| < R - c \delta,
\] (3.6)
where $c = \text{const} < \infty$ does not depend on $\delta$. This proof is some adaptation of the proof of Theorem 2.1. Similarly as in (2.8), introduce families of subdomains:
\[
\Omega_{R,s} = \{x \in \Omega_R : \tilde{\rho}(x) > s\} \ \forall s : 0 < s < \frac{R}{2},
\] (3.7)
\[
\Omega_R^3 = \{x \in \Omega_R : 0 < \tilde{\rho}(x) < s\} \ \forall s : 0 < s < \frac{R}{2},
\] (3.8)
where $\tilde{\rho}(x) := \min\{\rho(x), d(x)\}, d(x) = x_N, \rho(\cdot)$ is from (3.2). If $u$ is an arbitrary nonegative solution of equation (1.4) in $\Omega_R$ then we introduce the following energy function:
\[
I(s) := \int_{\Omega_{R,s}} (|\nabla u|^2 + h_\omega(d(x))u^{p+1})dx \ \forall s \in (0, 2^{-1}R).
\] (3.9)
Lemma 3.1  The function $I(\cdot)$ from (3.9) satisfies:

$$I(s) \leq c_1 \left[ \int_0^s h_\omega(r) \frac{2}{p+3} dr \right]^{\frac{p+3}{p-1}} \quad \forall s \in (0, 2^{-1} R),$$  \hspace{1cm} (3.10)

where constant $c_1 < \infty$ does not depend on $u$.

The proof is similar to the proof of lemma 2.2 with nonessential changes, so we omit it.

Lemma 3.2  Solution $u_j(x) := u_j,\delta$ of the problem (1.4), (3.5) in the domain $\Omega_R$ satisfies the following a priori estimate:

$$\int_{\Omega_R} \left( \left| \nabla u_j \right|^2 + h_\omega(d(x))u_j^{p+1} \right) dx \leq \overline{K}_j := c_2(K_j^{p+1} + \delta^{-1} K_j^2),$$  \hspace{1cm} (3.11)

where $c_2 < \infty$ does not depend on $j \in \mathbb{N}$.

Proof  It is easy to see that $v_j(x) := u_j(x) - K_j \xi_\delta(|x|) = 0$ on $\partial \Omega_R$.

Now multiplying equation (1.4) by $v_j$ and integrating it by parts we get the analog of relation (2.22):

$$\int_{\Omega_R} \left( \sum_{i,k=1}^{N} a_{ik} u_j x_i u_j x_k + H(x)u_j^{p+1} \right) dx = K_j \int_{\Omega_R} \sum_{i,k=1}^{N} a_{ik} u_j x_i \xi_\delta(|x|) x_k dx +$$

$$+ K_j \int_{\Omega_R} H(x)u_j^q \xi_\delta(|x|) dx.$$  \hspace{1cm} (3.12)

Further proof coincides with the proof of lemma 2.3. \hfill \Box

Similarly as in (2.25), introduce family of subdomains of $\Omega_R^s$:

$$\Omega_R^s(\tau) := \Omega_R^s \cap \left\{ (x'', x_N) : |x''| < \tau, x_N < s \right\} \quad \forall \tau \in (\delta, R), \ \forall s \in (0, s_\delta),$$

$$s_\delta := \delta - \frac{s^2}{2R} = \frac{R^2 - (R - \delta)^2}{2R} \quad \text{(see (3.3))},$$  \hspace{1cm} (3.13)

and, similarly as in (2.26), introduce energy function

$$J_j(s, \tau) := \int_{\Omega_R^{s_\delta}(\tau)} \left( \left| \nabla u_j \right|^2 + h_\omega(x_N)u_j^{p+1} \right) \xi_s(x_N) dx,$$  \hspace{1cm} (3.14)

where $\xi_s(\cdot)$ is a function from (2.21), (2.26), $u_j = u_j,\delta$.

Lemma 3.3  The energy function $J_j(s, \tau)$ satisfies the following relation:

$$J_j(s, \tau) \leq c_3 s \left( -\frac{d}{d\tau} J_j(s, \tau) \right) + Ch_\omega(s)^{-\frac{2}{p-1} - v} \quad \forall \tau \in (\delta, R), \ \forall j \in \mathbb{N},$$

$$\forall s \in (0, 2^{-1}s_\delta), \ \forall v > 0, \ C = C(v) \to \infty \ as \ v \to 0,$$  \hspace{1cm} (3.15)

where $c_3, C(v)$ do not depend on $j$; $s_\delta$ is from (3.13).
Proof Multiplying equation (1.4) by \( u_j(x) \xi_s(x_N) \), where \( u_j := u_{j, \delta} \), and integrating it over \( \Omega_2^{2s}(\tau), \tau \in (\delta, R), 2s \leq s_\delta = \delta - \frac{s^2}{2R} \), we obtain the following analog of (2.28):

\[
\tilde{J}_j(s, \tau) := \int_{\Omega_2^{2s}(\tau)} \left( \sum_{i,k=1}^N a_{ik}(x) u_{j,i} u_{j,k} + H(x) u_j^{p+1} \right) \xi_s(x_N) dx
\]

\[
= \int_{\partial'' \Omega_2^{2s}(\tau)} \left( \sum_{i,k=1}^N a_{ik}(x) u_{j,i} u_{j,k} \right) \xi_s(x_N) d\sigma
- \int_{\Omega_2^{2s}(\tau) \setminus \Omega_2^s(\tau)} \sum_{i,j=1}^N a_{ik} u_{j,i} \xi_s(x_N) u_j dx := R_1 + R_2 \quad \forall \tau \in (\delta, R), s < 2^{-1}s_\delta,
\]

where \( \partial'' \Omega_2^{2s}(\tau) = \{(x''', x_N) : |x''| = R - \tau, 0 < x_N < 2s\} \). Now notice the following important property of subdomains:

\[
\Omega_2^{2s}(\tau) \setminus \Omega_2^s(\tau) \subset \Omega_s(\tau) \setminus \Omega_2^s(\tau) \quad \forall \tau > \delta, \forall s : 0 < s < 2^{-1}s_\delta
\]

and, hence,

\[
\int_{\Omega_2^{2s}(\tau) \setminus \Omega_2^s(\tau)} (|\nabla u_j|^2 + h_\omega(x_N) u_j^{p+1}) dx \leq I_j(s) - I_j(2s) \quad \forall \tau > \delta, \forall s < 2^{-1}s_\delta.
\]

Estimating the term \( R_1 \) in (3.16) by the same way as in (2.29)–(2.35), we obtain due to properties (3.17), (3.18):

\[
| R_1 | \leq c_4 s \int_{\partial'' \Omega_2^{2s}(\tau)} |\nabla u_j|^2 \xi_s(x_N) d\sigma +
\]

\[
+ s^{-1+\frac{p-1}{p+1}} h_\omega(s) - \frac{1}{p+1} \left( I_j(s) - I_j(2s) \right) 1^{-\frac{p-1}{p+1}} +
\]

\[
+ s^{-1+\frac{p-1}{p+1}} h_\omega(s) - \frac{1}{p+1} \left( I_j(s) - I_j(2s) \right) 1^{-\frac{p-1}{p+1}} \quad \forall \tau \in (\delta, R), s < \frac{\delta}{2} - \frac{\delta^2}{4R}.
\]

Analogously to (2.37), we have:

\[
| R_2 | \leq c s^{-1\left(1-\frac{p-1}{p+1}\right)} h_\omega(s) - \frac{1}{p+1} \left( I_j(s) - I_j(2s) \right) 1^{-\frac{p-1}{p+1}} \quad \forall s < 2^{-1}s_\delta.
\]

Bearing in mind the following analog of relation (2.39):

\[
\int_{\partial'' \Omega_2^{2s}(\tau)} (|\nabla u_j|^2 + h_\omega(x_N) u_j^{p+1}) \xi_s(x_N) d\sigma \leq -c_0 \frac{d}{d\tau} J_j(s, \tau),
\]

and using estimates (3.19), (3.20) we get from (3.16) inequality (2.40) for all \( \tau \in (\delta, R) \) and \( s \in (0, 2^{-1}s_\delta) \). Estimating now \( F_j(s) \) in the same way as in (2.42)–(2.44), we obtain (2.45) and, consequently, necessary relation (3.15).

Further proof of Theorem 1.1 is similar to the corresponding part of the proof of Theorem 2.1 with obvious changes. By the arguments, similar to (2.46)–(2.83), we obtain the following analog of uniform estimate (2.83):

\[
\int_{\Omega_2^{2s}(2\delta)} (|\nabla u_j|^2 + h_\omega(x_N) u_j^{p+1}) dx \leq K_j'' \quad \forall j > j'' = j''(\delta),
\]
where \( j''(\delta) \to \infty \) as \( \delta \to 0 \). Since \( u_j(x) := u_{j,\delta}(x) = 0 \ \forall x = (x'', x_N) : x_N = 0, |x''| < R - \delta \), we deduce from (3.22) the following property by the same arguments as in (2.84)--(2.91):

\[ |u_{j,\delta} - u_{\infty,\delta}|_{C^{1,\alpha}(\Omega_R^{(\delta)})} \to 0 \text{ as } j \to \infty, \tag{3.23} \]

and, consequently, we get property (3.6) with \( c = 4 \). Now since \( u_{\infty,\delta}(x) \succeq u_{\infty}(x) \ \forall x \in \Omega_R \ \forall \delta > 0 \) relation (3.23) implies that \( u_{\infty}(x) = 0 \ \forall x = (x'', x_N) : x_N = 0, |x''| < R \). Moreover, \( u_{\infty}(x) = \infty \ \forall x = (x'', x_N) : x_N > 0 \). Thus \( u_{\infty} \) is the desired strong barrier for equation (1.4). Theorem 1.1 is proved.

4 Necessity of the Dini condition for non-propagation of the point singularity

In this section we prove Theorem 1.2. First of all, we construct a family of subsolutions \( v_j(x) \), connected with solutions \( u_j(x), \ j = 1, 2, \ldots \) of the problem under consideration. Namely, introduce a family of subdomains \( \Omega_j \) of the domain \( \Omega \) from (1.14):

\[ \Omega_j := \{ x \in \Omega = \mathbb{R}_+^N : |x'|^2 := \sum_{i=2}^{N} x_i^2 < r_j^2, x_1 \in \mathbb{R}^1 \}, \quad r_j = 2^{-j}, \quad j = 1, 2, \ldots, \tag{4.1} \]

and numbers

\[ a_j = \exp(-\mu(r_j)) = \exp(-\frac{\omega(r_j)}{r_j}), \quad A_j = (a_j r_j^2)^{\frac{1}{p-1}} \ \forall j \in \mathbb{N}. \tag{4.2} \]

Consider now a family of auxiliary problems:

\[ -\Delta u + a_j u^p = 0 \text{ in } \Omega_j, \tag{4.3} \]

\[ v = 0 \text{ on } \partial \Omega_j \setminus \{ x : x_N = 0 \}, \tag{4.4} \]

\[ v \big|_{\partial \Omega_j \cap \{x_N = 0\}} = K_j a r_j \quad \forall a \in L = \{ x = (x_1, 0, \ldots, 0) \}. \tag{4.5} \]

Due to condition (1.22) on \( p \), inequality (1.19) is satisfied for equation (4.3), and, hence, problem (4.3)--(4.5) has solution \( v_j, \ j = 1, 2, \ldots \). Since \( u_j(x) \succeq v_j(x) \) on \( \partial \Omega_j \) then

\[ u_j(x) \succeq v_j(x) \ \forall j \in \mathbb{N}. \tag{4.6} \]

Next, we estimate \( v_j(x) \) from below. Let us perform the rescaling of the problem (4.3)--(4.5). Introducing new variables and an unknown function:

\[ y = r_j^{-1} x, \quad w_j(y) = A_j v_j(r_j y), \quad y \in G := \{ y \in \mathbb{R}_+^N : |y'| < 1, y_N > 0 \}, \tag{4.7} \]

where \( A_j \) is from (4.2), we obtain with respect to \( w_j(y) \) the following problem:

\[ -\Delta w_j + w_j^p = 0 \text{ in } G, \tag{4.8} \]

\[ w_j(y) = 0 \text{ on } \partial G \setminus \{ y : y_N = 0 \}, \tag{4.9} \]

\[ w_j \big|_{\partial G \cap \{y_N = 0\}} = K_j A_j r_j^{-(N-1)} s_{ar_j}^{-1}(y). \tag{4.10} \]

It is easy to see that without loss of generality we can suppose that \( a = 0 \). Let us specify now the choice of the sequence \( \{K_j\} \):

\[ K_j := A_j^{-1} r_j^{N-1} \ \forall j \in \mathbb{N}. \tag{4.11} \]
Then \( w_j(y) = w(y) \forall \, j \in \mathbb{N} \), where \( w(y) \) is a solution of the problem:

\[
-\Delta_y w + w^p = 0 \quad \text{in} \, G \tag{4.12}
\]

\[
w(y) = 0 \quad \text{on} \, \partial G \setminus \{ y : y_N = 0 \} \tag{4.13}
\]

\[
w |_{\partial G \cap \{ y_N = 0 \}} = \delta_0(y). \tag{4.14}
\]

It is obvious (due to comparison principle) that \( w(\cdot) \) satisfies the estimate:

\[
0 \leq w(y) \leq P_0(y, 0) = \frac{CN_{YN}}{(y_1^2 + y_2^2 + \ldots + y_N^2)^{N/2}} \quad \forall \, y \in G, \tag{4.15}
\]

where \( P_0(\cdot, \cdot) \) is the Poisson kernel from (1.18). Therefore, function \( w^{(y_1)}(y') := w(y_1, y') \) has the following properties:

\[
| w^{(y_1)}(\cdot) |_{L^\infty(B_{1,+}^{N-1})} < \infty \quad \forall \, y_1 \neq 0,
\]

\[
\varphi_w(y_1) := | w^{(y_1)}(\cdot) |_{L^\infty(B_{1,+}^{N-1})} \to 0 \quad \text{as} \quad | y_1 | \to \infty, \tag{4.16}
\]

where \( B_{1,+}^{N-1} = \{ y' = (y_2, \ldots, y_N) \in \mathbb{R}^{N-1} : | y' | < 1, \, y_N > 0 \} \). Let us fix an arbitrary value \( y_1^{(0)} > 0 \) and consider solution \( w \) as a solution of equation (4.12) in the infinite cylindrical domain \( G \cap \{ y : y_1 > y_1^{(0)} \} \) satisfying boundary condition:

\[
w(y) = 0 \quad \text{on} \, \partial G \cap \{ y : y_1 > y_1^{(0)} \}. \tag{4.17}
\]

Due to lemma 3.1 from [14, 15], for the solution \( w \) there exists a number \( \alpha > 0 \) such that

\[
\lim_{y_1 \to \infty} \exp \left( \sqrt{\lambda_1} \left( y_1 - y_1^{(0)} \right) \right) \quad w(y) = \alpha \psi_1(y'), \quad \max_{y' \in B_{1,+}^{N-1}} \psi_1(y') = \tilde{\psi}_1(\tilde{y}') = 1, \tag{4.18}
\]

uniformly in \( B_{1,+}^{N-1} \). Here \( \lambda_1 > 0 \) is the first eigenvalue and \( \psi_1 \) is the corresponding normalized eigenfunction of \( -\Delta \) in \( B_{1,+}^{N-1} \), constant \( \alpha = \alpha(y_1^{(0)}) \) satisfies estimate:

\[
0 < \alpha \leq c \sup_{y' \in B_{1,+}^{N-1}} w^{(y_1^{(0)})}(y') = c \varphi_w(y_1^{(0)}), \tag{4.19}
\]

where \( c < \infty \) doesn’t depend on solution \( w \), function \( \varphi_w(\cdot) \) is from (4.16).

**Remark 4** Lemma 3.1 is proved for the cylindrical domain \( G = \mathbb{R}^1 \times B_1^{N-1} \) in [14, 15]. But its proof is based on the results of §2 of the paper [2], which are true for a much more general class of cylindrical domains, particularly, for \( G = \mathbb{R}^1 \times B_1^{N-1} \).

Thus, due to (4.16), (4.11), (4.7), it follows from (4.18) the existence of a constant \( \beta : y_1^{(0)} < \beta < \infty \), which does not depend on \( j \), such that

\[
\frac{\alpha}{2A_j} \psi_1 \left( r_j^{-1}x' \right) \exp \left( -\sqrt{\lambda_1} \left( \frac{x_1}{r_j} - y_1^{(0)} \right) \right) \leq v_j(x) \leq \frac{2\alpha}{A_j} \psi_1 \left( r_j^{-1}x' \right) \exp \left( -\sqrt{\lambda_1} \left( \frac{x_1}{r_j} - y_1^{(0)} \right) \right) \quad \forall \, x \in \Omega_j : x_1 > \beta r_j, \, \forall \, j \in \mathbb{N}. \tag{4.20}
\]
Due to (4.6) inequality (4.20) yields the first rough estimate from below of the solution $u_j$:

$$u_j(x) \geq v_j(x) \geq B_j \psi_1 \left(r_j^{-1} x' \right) \exp \left(-\sqrt{\lambda_1} \frac{x_1}{r_j} \right) \forall x \in \Omega_j : x_1 > \beta r_j, \quad (4.21)$$

where $B_j = (2A_j)^{-1} \alpha (y_1^{(0)}) \exp \left(\sqrt{\lambda_1} y_1^{(0)} \right)$. Let us define number $\tau_j > 0$ by the following relation:

$$B_j \exp \left(-\sqrt{\lambda_1} \frac{\tau_j}{r_j} \right) = A_{j-1}^{-1} = \left(a_{j-1} r_{j-1}^2 \right)^{-\frac{1}{p-1}}, \quad (4.22)$$

which yields by simple computations:

$$\frac{\tau_j}{r_j} = \frac{\mu(r_j) - \mu(r_{j-1})}{\sqrt{\lambda_1} (p-1)} + c_1, \quad c_1 = y_1^{(0)} + \frac{\ln \alpha}{\sqrt{\lambda_1}} + \frac{(3-p) \ln 2}{(p-1) \sqrt{\lambda_1}}. \quad (4.23)$$

By condition (1.23), we have $(\mu(r_j) - \mu(r_{j-1})) \to \infty$ as $j \to \infty$. Hence, there exists a number $j'$, such that

$$\frac{r_j (\mu(r_j) - \mu(r_{j-1}))}{\sqrt{\lambda_1} (p-1)} \leq \tau_j \leq \frac{2r_j (\mu(r_j) - \mu(r_{j-1}))}{\sqrt{\lambda_1} (p-1)} \quad \forall j > j' \quad (4.24)$$

and, additionally,

$$\tau_j > \beta r_j \quad \forall j \geq j'. \quad (4.25)$$

As a consequence of definition (4.22), estimate (4.21) implies:

$$u_j(\tau_j, x') \geq v_j(\tau_j, x') \geq A_{j-1}^{-1} \psi_1 \left(r_j^{-1} x' \right) \quad \forall j > j'. \quad (4.26)$$

Let us fix arbitrary large $j > j'$ in (4.26) and consider a sequence $\{v_{j-k}(x)\}, k = 1, 2, ..., j-j'$ ($j'$ is from (4.24), (4.25)) of solutions of the following boundary value problems

$$-\Delta v_{j-k} + a_{j-k} v_{j-k}^p = 0 \quad \text{in} \quad \Omega_{j-k} \cap \{x_1 > \tau_j + ... + \tau_{j-k+1}\}, \quad (4.27)$$

$$v_{j-k} = 0 \quad \text{on} \quad \partial \Omega_{j-k} \cap \{x_1 > \sum_{i=0}^{k-1} \tau_{j-i}\}, \quad (4.28)$$

$$v_{j-k} \left(\sum_{i=0}^{k-1} \tau_{j-i}, x' \right) = \gamma_{j-k}(x') := \begin{cases} \frac{A_{j-k}^{-1} \psi_1 \left(x'/r_{j-k+1} \right)}{0} \text{ if } |x'| < r_{j-k+1}, \\ 0 \text{ if } r_{j-k+1} \leq |x'| \leq r_{j-k}. \end{cases} \quad (4.29)$$

where $\Omega_{j-k} = \{x \in \Omega : |x'| < r_{j-k}\}, A_{j-k} = \left(a_{j-k} r_{j-k}^2 \right)^{-\frac{1}{r-1}}, a_{j-k} = \exp(-\mu(r_{j-k})).$

To define the sequence $\{\tau_{j-k}\}, k = 1, 2, ...$, we need to transform problem (4.27)-(4.29) to new variables:

$$y = r_{j-k}^{-1} x, \quad w_{j-k}(y) := A_{j-k} v_{j-k} (r_{j-k} y). \quad (4.30)$$

It is easy to see that all these functions $w_{j-k}(y)$ can be obtained as a shift of the unique function $w(y)$:

$$w_{j-k}(y_1, y') := w(y_1 - \sum_{i=0}^{k-1} \tau_{j-i} r_{j-k}^{-1}, y'), \quad (4.31)$$
where \( w(y) \) is a solution of the problem:

\[
-\Delta_y w + w^p = 0 \text{ in } G \cap \{y_1 > 0\},
\]

\[
w = 0 \text{ on } \partial G \cap \{y_1 > 0\},
\]

\[
w(0, y') := \begin{cases} 
\psi_1(2y') & \text{if } |y'| < 2^{-1}, \\
0 & \text{if } 2^{-1} \leq |y'| \leq 1.
\end{cases}
\]

Due to lemma 3.1 from [14, 15] there exists a number \( \alpha_1 > 0 \), such that the function \( w \) has the following property:

\[
\lim_{y_1 \to \infty} \exp\left(\sqrt{\lambda_1} y_1\right) w(y) = \alpha_1 \psi_1(y').
\]

Here constant \( \alpha_1 > 0 \) satisfies the estimate

\[
\alpha_1 \leq c \sup_{y' \in B_1^{N-1}} \psi_1(2y') = c
\]

with constant \( c \), the same as in (4.19). As in (4.20), by definition (4.31), property (4.35) implies the existence of a constant \( \beta_1 < \infty \), which does not depend on \( j \) and \( k \leq j - 1 \), such that

\[
\frac{\alpha_1}{2A_{j-k}} \psi_1(r_{j-k}^{-1} x') \exp\left(-\sqrt{\lambda_1} (x_1 - h_{j,k}) \right) \leq v_{j-k}(x) \leq \frac{2\alpha_1}{A_{j-k}} \psi_1(r_{j-k}^{-1} x') \times
\]

\[
\times \exp\left(-\sqrt{\lambda_1} (x_1 - h_{j,k}) \right) \quad \forall \ x \in \Omega_{j-k} : x_1 \geq h_{j,k} + r_{j-k} \beta_1; h_{j,k} := \sum_{i=0}^{k-1} t_{j-i}.
\]

Let us define value \( \tau_{j-k} \) by the relation:

\[
\frac{\alpha_1}{2A_{j-k}} \exp\left(-\sqrt{\lambda_1} \frac{\tau_{j-k}}{r_{j-k}} \right) = A_{j-k-1}^{-1}. \quad k = 1, 2, \ldots
\]

Using the nonnegativity of \( u_j \) in \( \Omega \), properties (4.26) and boundary condition (4.29) with \( k = 1 \), we get:

\[
u_j(\tau_j, x') \geq v_{j-1}(\tau_j, x') \quad \forall \ x' : |x'| < r_j.
\]

Using additionally that \( v_{j-1}(\tau_j, x') = 0 \) if \( r_j < |x'| < r_{j-1} \), we obtain

\[
u_j(\tau_j, x') \geq v_{j-1}(\tau_j, x') \quad \forall \ x' : |x'| < r_{j-1}.
\]

Now using the comparison principle for solution \( u_j \) and subsolution \( v_{j-1} \) of equation (1.14) in the domain \( \Omega_{j-1} \cap \{x_1 > \tau_j\} \), we obtain

\[
u_j(x) \geq v_{j-1}(x) \quad \forall \ x \in \Omega_{j-1} \cap \{x_1 > \tau_j\}.
\]

Next we will establish the main intermediate inequality:

\[
u_j(x) \geq v_{j-k}(x) \quad \forall \ x \in \Omega_{j-k} \cap \left\{x_1 \geq \sum_{i=0}^{k-1} t_{j-i}\right\} \forall \ k : 1 \leq k < j - j'','
\]
where $j'' < j$ does not depend on $j$. In virtue of (4.39) inequality (4.40) is true for $k = 1$. Let us suppose that (4.40) holds for some $k \geq 1$. Then we have to prove that

$$u_j(x) \geq v_{j-k-1}(x) \quad \forall x \in \Omega_{j-k-1} \cap \left\{ x_1 \geq \sum_{i=0}^{k} \tau_{j-i} \right\}. \tag{4.41}$$

To do this, it is sufficient, due to the comparison principle, to show that

$$u_j \left( \sum_{i=0}^{k} \tau_{j-i}, x' \right) \geq v_{j-k-1} \left( \sum_{i=0}^{k} \tau_{j-i}, x' \right). \tag{4.42}$$

From boundary condition (4.29) for the function $v_{j-k-1}(x)$ we have:

$$v_{j-k-1} \left( \sum_{i=0}^{k} \tau_{j-i}, x' \right) = \begin{cases} A_{j-k-1}^{-1} \psi_1 \left( \frac{x'}{r_{j-k}} \right), & \text{if } |x'| \leq r_{j-k} \\ 0, & \text{if } r_{j-k} < |x'| \leq r_{j-k-1}. \end{cases} \tag{4.43}$$

Now if number $\tau_{j-k}$, defined by relation (4.37), satisfies additionally the following inequality:

$$\tau_{j-k} \geq \beta_1 r_{j-k} \quad \text{with } \beta_1 \text{ from (4.36)}, \tag{4.44}$$

then relation (4.36) yields:

$$v_{j-k} \left( \sum_{i=0}^{k} \tau_{j-i}, x' \right) \geq A_{j-k-1}^{-1} \psi_1 \left( r_{j-k-1}^{-1} x' \right) \exp \left( -\sqrt{\lambda_{j-k}} \right). \tag{4.45}$$

In virtue of definition (4.37) of $\tau_{j-k}$ the last inequality leads to:

$$v_{j-k} \left( \sum_{i=0}^{k} \tau_{j-i}, x' \right) \geq A_{j-k-1}^{-1} \psi_1 \left( r_{j-k-1}^{-1} x' \right) \quad \forall x' : |x'| \leq r_{j-k}. \tag{4.46}$$

Since $u_j \left( \sum_{i=0}^{k} \tau_{j-i}, x' \right) \geq 0 = v_{j-k-1} \left( \sum_{i=0}^{k} \tau_{j-i}, x' \right)$ if $r_{j-k} \leq |x'| \leq r_{j-k-1}$, relations (4.43) and (4.45) lead to (4.42) under the comparison principle. Thus, inequality (4.40) is proved for all $k \geq 1$, such that estimate (4.44) holds for $\tau_{j-k}$, defined by (4.37). Using standard computations, we deduce from definition (4.37):

$$\frac{\tau_{j-k}}{r_{j-k}} = \frac{\mu(r_{j-k}) - \mu(r_{j-k-1})}{\sqrt{\lambda_1} (p-1)} + c_1, \quad c_1 = \ln \frac{\alpha}{\sqrt{\lambda_1}} + \frac{(3-p) \ln 2}{(p-1) \sqrt{\lambda_1}}. \tag{4.46}$$

Due to condition (1.23) on the function $\mu(\cdot)$ it follows from the last relation that there exists a constant $\tilde{j} < \infty$, which does not depend on $j$, such that the following inequalities hold:

$$\frac{r_{j-k} \left( \mu(r_{j-k}) - \mu(r_{j-k-1}) \right)}{\sqrt{\lambda_1} (p-1)} \leq \tau_{j-k} \leq \frac{2 \left( \mu(r_{j-k}) - \mu(r_{j-k-1}) \right) r_{j-k}}{\sqrt{\lambda_1} (p-1)}, \quad \forall \lambda < k < j - \tilde{j}. \tag{4.47}$$

Additionally, it follows from condition (1.23) the existence of a number $\tilde{j}_1 = \tilde{j}_1(\beta_1) < \infty$, such that

$$\frac{\mu(r_{j-k}) - \mu(r_{j-k-1})}{\sqrt{\lambda_1} (p-1)} \geq \beta_1 \quad \forall j \in \mathbb{N}, \forall k < j : j - k \geq \tilde{j}_1. \tag{4.48}$$

Therefore, the main intermediate inequality (4.40) holds with $j'' = \max\{j', j, \tilde{j}_1\}$. Moreover, condition (1.23) yields the existence of a constant $\tilde{\alpha} < 1$ and a number $\tilde{j} < \infty$, such that

$$\mu(r_{j-k-1}) \mu(r_{j-k})^{-1} < \tilde{\alpha} < 1 \quad \forall j \in \mathbb{N}, \forall k < j : j - k > \tilde{j}. \tag{4.49}$$
Therefore, it follows from (4.24), (4.46) that
\[
\frac{2\mu(r_{j-k})}{\sqrt{\lambda_1(p-1)}} \geq \frac{\tau_{j-k}}{r_{j-k}} \geq \frac{(1 - \varepsilon)\mu(r_{j-k})}{\sqrt{\lambda_1(p-1)}} \quad \forall j \in \mathbb{N}, \forall k < j : j-k > \max\{j'', \hat{j}\}. \tag{4.49}
\]

Hence, by definition (1.17) of \( \mu(\cdot) \), we have:
\[
\sum_{k=0}^{j-i} \tau_{j-k} \geq \varepsilon \sum_{k=0}^{j-i} \omega(r_{j-k}) \geq \varepsilon \sum_{k=0}^{j-i} \int_{r_{j-k}}^{r_j} \frac{\omega(s)}{s} \, ds = \varepsilon \int_{r_j}^{r_{j-i}} \frac{\omega(s)}{s} \, ds \\
\forall j, i : j > i > j^{(1)} := \max\{j'', \hat{j}\} = \max\{j', \tilde{j}, \tilde{j}_1, \hat{j}\}, \varepsilon_1 := \frac{1 - \varepsilon}{\sqrt{\lambda_1(p-1)}}. \tag{4.50}
\]

Additionally, the left inequality in (4.49) and assumption (1.23) on the function \( \omega(\cdot) \) imply that \( \tau_i \to 0 \) as \( i \to \infty \). Therefore, for an arbitrary fixed number \( g > 0 \) there exists a number \( i_g \), such that \( \tau_i < g \forall i > i_g \). Then, by virtue of condition (1.24) and inequalities (4.50), there exists a number \( \hat{j} = \hat{j}(i, g) < \infty \), such that
\[
g < \sum_{k=0}^{\tilde{j}-i} \tau_{j-k} = \sum_{k=0}^{\tilde{j}-i} \tau_{i+k}, \sum_{k=1}^{\tilde{j}-i} \tau_{i+k} \leq g \quad \forall i > j^{(2)} := \max\{j^{(1)}, i_g\}. \tag{4.51}
\]

Notice that \( \tilde{j}(i, g) - i \to \infty \) as \( i \to \infty \). Let us rewrite proved relation (4.40) in the following equivalent form:
\[
u_j(x) \geq v_i(x) \quad \forall x = (x_1, x') \in \Omega_i \cap \left\{x_1 \geq \sum_{k=1}^{j-i} \tau_{i+k}\right\} \forall j > i > j''. \tag{4.52}
\]

Let us define a sequence of points \( \{x^{(i)}\} \):
\[
x^{(i)} = \left(x^{(i)}_1, x^{(i)}\right) : x^{(i)} = r_{i+1} \tilde{y}', \tilde{y}' \text{ is from (4.18)},
\]
\[
x^{(i)}_1 = \sum_{k=0}^{\tilde{j}-i} \tau_{i+k} = \sum_{k=0}^{\tilde{j}-i} \tau_{j-k} \quad \forall i > j^{(2)}, \quad \tilde{j} = \tilde{j}(i, g) \text{ is from (4.51)}.
\]

We deduce from (4.29) after a simple transformation:
\[
u_i\left(\sum_{k=1}^{\tilde{j}-i} \tau_{i+k}, x^{(i)}\right) = A_i^{-1} \psi_1(r_{i+1}^{-1} x') \quad \forall x' : |x'| < r_{i+1}.
\]

Therefore using (4.51) and the definition of point \( x^{(i)} \) we get in virtue of (4.18):
\[
u_i \left(g + \lambda_i \tau_i, x^{(i)}\right) = A_i^{-1} \psi_1(r_{i+1}^{-1} x^{(i)}) = A_i^{-1} \quad \forall i > j^{(2)}, \quad 0 \leq \lambda_i < 1. \tag{4.53}
\]

Let us define sequence \( X^{(i)} = (X_1^{(i)}, x^{(i)}), X_1^{(i)} = g + \lambda_i \tau_i \). Then since \( A_i^{-1} \to \infty \) as \( i \to \infty \) we deduce from (4.52) and (4.53):
\[
u_i(X^{(i)}) \geq u_j(X^{(i)}) \geq v_i(X^{(i)}) = A_i^{-1} \to \infty \text{ as } i \to \infty, \tag{4.54}
\]
where \( \tilde{j} = \tilde{j}(i, g) \) is from (4.51). Since \( X_1^{(i)} \to g \) and \( x^{(i)} \to 0 \) as \( i \to \infty \), relation (4.54) yields \( u_\infty(g, 0, ..., 0) = \infty \). Since \( (g, 0, ..., 0) \) is an arbitrary point from \( L \cap [\mathbb{R}^1_+] = L_+ \) then
Now we return to the model problem (4.12)–(4.14) and consider its solution $w$ as a solution of equation (4.12) in the infinite cylindrical domain $G \cap \{ y : y_1 < y_1^0 < 0 \}$, satisfying boundary condition: $w(y) = 0$ on $\partial G \cap \{ y : y_1 < y_1^0 < 0 \}$. If we repeat all above analysis using this solution $w$ in the mentioned cylindrical domain, we obtain that $u_\infty \mid_{L \cap [\mathbb{R}^1_+]} = \infty$. Theorem 1.2 is proved.

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