THE COLLISIONLESS MAGNETOVISCOUS-THERMAL INSTABILITY

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ABSTRACT

It is likely that nearly all central galactic massive and supermassive black holes are nonradiative: their accretion luminosities are orders of magnitude below what can be explained by efficient black hole accretion within their ambient environments. These objects, of which Sagittarius A* is the best-known example, are also dilute (mildly collisional to highly collisionless) and optically thin. In order for accretion to occur, magnetohydrodynamic (MHD) instabilities must develop that not only transport angular momentum, but also gravitational energy generated through matter infall, outward. A class of new magnetohydrodynamical fluid instabilities—the magnetoviscous-thermal instability (MVTI)—was found to transport angular momentum and energy along magnetic field lines through large (fluid) viscosities and thermal conductivities. This paper describes the analog to the MVTI, the collisionless MVTI (CMVTI), that similarly transports energy and angular momentum outward, expected to be important in describing the flow properties of hot, dilute, and radiatively inefficient accretion flows around black holes. We construct a local equilibrium for MHD stability analysis in this differentially rotating disk. We then find and characterize specific instabilities expected to be important in describing their flow properties, and show their qualitative similarities to instabilities derived using the fluid formalism. We conclude with further work needed in modeling this class of accretion flow.

Key words: accretion, accretion disk – Galaxy: center – instabilities – magnetic fields – magnetohydrodynamics (MHD) – plasmas

1. INTRODUCTION

Within the recent past, much progress has been made in characterizing the important dynamics of accretion flows. The magnetorotational instability (MRI; Velikhov 1959; Chandrasekhar 1960) has been applied to accretion disks (Balbus & Hawley 1991) and been shown to drive magnetohydrodynamic (MHD) fluid turbulence that can provide an outward angular momentum flux and mass accretion rate consistent with astrophysical observations, as demonstrated in a variety of numerical simulations (Hawley et al. 1996; Wardle 1999; Sano & Stone 2002; de Villiers & Hawley 2003; Fromang et al. 2004). However, there exists observational evidence of hot dilute flows, in accretion about dim mass-starved supermassive black holes, for which the mean free path is of the order of the system scale or larger. Chandra X-ray observations by Baganoff et al. (2003) have resolved the inner 1′′ around the Sagittarius A central black hole and demonstrated that the ion mean free path at its capture radius is only a few times smaller than the system scale. The unambiguous detection of Faraday rotation in the high-frequency radio emission about Sagittarius A (Aitken et al. 2000; Bower et al. 2003; Marrone et al. 2005) implies that the magnetic field is very easily strong enough to result in a gyrokinetic reduction in plasma dynamics. Estimates of mass accretion from the ambient conditions about this object overestimate its bolometric luminosity by approximately five orders of magnitude over radiatively efficient accretion (Narayan 2002), implying that very little of the gravitational energy produced by mass accretion is radiated. However, recent nonlinear local simulations, with limits on electron pressure anisotropy due to gyrokinetic electron instabilities, show that the nonlinear development of the collisionless MRI can turbulently heat electrons sufficiently to allow these flows to become radiative (Sharma et al. 2007); accretion in collisionless environments, such as those around Sag. A*, may naturally be radiative enough that the accretion rate must remain orders of magnitude below the Bondi rate in order to explain their low luminosity. Regardless of whether this accretion is radiatively inefficient, it is very likely that in the steady state these plasmas are dilute, optically thin, and the bulk of their thermal energy lies with the protons. Furthermore, MHD plasma turbulence that transports energy generated from accretion may play an important role even in high-energy radiative, but collisionless, accretion flows.

The fact that, very plausibly, these systems may be radiatively inefficient points to the fact that these high energy, dilute plasmas are at least partially pressure supported: this is in contrast to a large class of models of radiatively efficient classical accretion disks, in which the accreting disk of matter remains geometrically thin and rotationally supported due to the efficient radiation of energy perpendicular to the disk. Numerical simulations of the MRI in a canonical black hole accretion flow (de Villiers & Hawley 2003; de Villiers et al. 2003) tend to stabilize into thick disks. The large aspect ratio of these disks invites an analysis of these disks with vertical disk structure included, or as a beginning a local analysis in which dynamically important gradients of temperature and pressure govern the nature of local instabilities.

A formulation of magnetized plasma dynamics that is especially well-suited for collisionless or mildly collisional MHD plasma equilibrium and dynamics is that of Kulsrud’s drift-kinetic approximation to the Boltzmann equation (Kulsrud 1983, 2005). To lowest order the particle distribution function is characterized by dynamics only along magnetic field lines, MHD conditions of quasi-neutrality with ions and electrons moving together, and conservation of magnetic moment for particle distributions. Furthermore, it is expected that additional dynamics that cannot be modeled through the Kulsrud formalism, such as momentum and energy transfer processes resulting in temperature equilibration or electric resistivity, may not be dynamically important to a first approximation.

For this problem we consider the following hierarchy of scales appropriate to lowest-order gyro kinetic expansion:
\[ \frac{1}{T} < \omega_{pi} \ll \Omega, \quad 1/L < \omega_{ni}/c \ll \Omega, \quad \text{where} \ \omega_{pi} \ \text{is the ion plasma frequency,} \ \Omega \ \text{is the gyrofrequency,} \ \rho_i \ \text{is the ion gyroradius,} \ \omega_{ni}/c \ \text{is the inverse internal depth, and} \ L \ \text{and} \ T \ \text{are the shortest length and fastest timescales associated with} \ \text{this system. Densities are large enough that Alfven velocities are smaller than the speed of light, therefore relativistic MHD effects may be ignored. The gravitational acceleration is purely due to that of the central object. We consider a plasma equilibrium where pressures parallel and perpendicular to the magnetic field are equal, hence the equilibrium particle distribution for electrons and ions has one temperature. We formulate the problem in a cylindrical geometry, where the axis of rotational lies along the vertical axis,} \ \hat{R}, \ \hat{\phi}, \ \text{and} \ \hat{z} \ \text{are unit vectors in the radial, azimuthal, and vertical directions, respectively.}\]

The organization of this paper is as follows. In Section 2 we discuss the variables and nomenclature used in this paper. In Section 3, we use a form of the drift kinetic equation that represents particle dynamics in a co-rotating frame, explicitly state the equilibrium we choose in our local analysis, and include total MHD force balance and MHD induction equations in a co-rotating frame. In Section 4 we justify and modify turbulent and average wave quantities appropriate to characterize accretion (see, e.g., Balbus & Hawley 1998; Balbus 2004) in dilute and radiatively inefficient magnetized flows. In Section 5 we consider the stability of hot dilute rotating plasmas to a new instability, the collisionless analog to the magnetoviscous-thermal instability (MVTI; Islam 2012), the collisionless MVTI or CMVTI. We also demonstrate that quadratic estimates of heat fluxes and Reynolds stress are of the right form to drive accretion in this dilute thick flow. In Section 6 we summarize our main results as well as describe directions for further research.

2. VARIABLES AND NOMENCLATURE

Our coordinate system for the rotating disk is a cylindrical system located about the central mass. \( \hat{R}, \ \hat{\phi}, \ \text{and} \ \hat{z} \ \text{are unit vectors in the radial, azimuthal, and vertical directions, respectively. For field variables of temperature} \ T, \ \text{pressure} \ p, \ \text{density} \ \rho, \ \text{electric and magnetic fields} \ E \ \text{and} \ B, \ \text{and pressure} \ p, \ \text{we use the following notation.}\]

1. Equilibrium value of, say density: \( \rho_0. \)
2. Perturbed density: \( \delta \rho. \)
3. Total density (equilibrium + perturbed): \( \rho = \rho_0 + \delta \rho. \)

For velocity, we use the following notation.

1. Primary equilibrium flow velocity, which is azimuthal: \( \mathbf{v}_0 = R\Omega(\hat{R}\hat{\phi}), \ \text{where} \ \Omega(R) \ \text{is the orbital angular velocity.}\)
2. Perturbed flow velocity: \( \delta \mathbf{u}. \)
3. Total flow velocity: \( \mathbf{v} = R\Omega(\hat{R}\hat{\phi}) + \mathbf{u}. \)

Components of an equilibrium vector quantity, such as the radial component of the equilibrium magnetic field, are written as \( B_0. \)

3. THE DRIFT KINETIC AND CONSTITUENT EQUATIONS IN ROTATING FRAME

In this section, we state the equations and disk equilibrium used in the eigenmodal analysis and the demonstration of quadratic heat flux of the CMVTI. Without derivation (see, e.g., Hinton & Hazeltine 1976; Sharma et al. 2003; Sharma & Hammett 2006), the collisionless drift kinetic equation in a rotating frame can be shown to be of the following form

\[
\begin{align*}
\left( \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi} \right) & \left( f_s \mathbf{B} \right) + \nabla \cdot \left( \left[ \mathbf{v}_s \mathbf{b} + \mathbf{u}_s \right] f_s \mathbf{B} \right) \\
+ \ & \left( \frac{\partial}{\partial v_1} \right) f_s \left[ \left( \frac{Z_e}{m_e} \mathbf{E}_1 + \frac{1}{m_i n_0} \mathbf{b} \cdot \nabla p_0 \right) \right] \mathbf{B} \\
+ \ & \left( \frac{\partial}{\partial v_2} \right) f_s \left[ -\mathbf{b} \cdot \left( \frac{\partial}{\partial \phi} \Omega \frac{\partial}{\partial \phi} \mathbf{u}_s + [\mathbf{v}_s + \mathbf{u}_s] \cdot \nabla \mathbf{u}_s \right) \right] \\
+ \ & \mu B \mathbf{V} \cdot \mathbf{b} + 2 \Omega \mathbf{z} \cdot (\mathbf{b} \times \mathbf{u}) - b_\rho R(\mathbf{u}_s + \mathbf{v}_s) \cdot \nabla \mathbf{u}_s = 0,
\end{align*}
\]

where \( f_s \) is the species particle distribution function, and \( m_s \) and \( Z_s \) is the mass and charge of a particle of species \( s \). \( v_s \) is the component of corotating velocity along the magnetic field, and \( \mu \) is the magnetic moment (\( m v_s^2 / (2B) \)). Additional terms appear in the formulation of Equation (1) that do not appear explicitly in the normal drift-kinetic equation of Kulsrud (1983): non-inertial rotational accelerations along the magnetic field, \( 2 \Omega \mathbf{z} \cdot (\mathbf{b} \times \mathbf{u}) - b_\rho R(\mathbf{u}_s + \mathbf{v}_s) \cdot \nabla \mathbf{u}_s \), and accelerations along the magnetic field associated with large thermal energies, \( 1/(m_s n_0) \mathbf{b} \cdot \nabla p_0 \).

Next, the form of the full MHD force balance and induction equations in a co-rotating frame are given by

\[
\begin{align*}
\rho \left( \left[ \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi} \right] \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - 2 \mathbf{u} \times \mathbf{z} + R \mathbf{u} \cdot \nabla \Omega \right) \\
= \frac{1}{c} \mathbf{J} \times \mathbf{B} + \frac{n_e}{n_0} \nabla p - \nabla \cdot \mathbf{P},
\end{align*}
\]

\[
\frac{\partial \mathbf{B}}{\partial t} = -\mathbf{u} \cdot \nabla \mathbf{B} - \mathbf{B} (\mathbf{v} \cdot \mathbf{u}) + \mathbf{B} \cdot \nabla \mathbf{u} + \mathbf{R} \mathbf{B} \cdot \nabla \mathbf{u} - \Omega \frac{\partial}{\partial \phi} \mathbf{B},
\]

where \( p = p_e + p_i, \ \mathbf{P} = p_{\perp} \mathbf{I} + (p_{||} - p_{\perp}) \mathbf{b} \mathbf{b}, \ p_{||} = p_{||0} + p_{\perp}, \) and \( p_{\perp} = p_{\perp0} + p_{\perp}. \) Parallel, perpendicular, and total pressures are given by their standard forms

\[
\begin{align*}
p_{||0} &= 2\pi \int m_i (v_i - u_i^2) f_i(B d\mu dv_i), \\
p_{\perp} &= 2\pi \int m_i B f_i(B d\mu dv_i), \\
p_s &= \frac{2}{3} p_{\perp} + \frac{1}{3} p_{||}.
\end{align*}
\]

In this paper we analyze the stability and quadratic transport of an equilibrium geometrically thin nonradiative, collisionless disk at its midplane. For simplicity, temperature is independent
of height above the disk. To lowest order there is no net current, the plasma velocity is purely azimuthal, and the equilibrium magnetic field is nonradial and axisymmetric. Therefore, the electron and ion pressure and density as a function of vertical coordinate \( z \) goes as

\[
n_i(z, \Omega) = n_i z \exp \left( -\frac{z^2}{2H^2} \right).
\]

The disk scale height \( H \) is given by

\[
H^2 = \frac{k_B (T_i + T_e)}{(m_i + m_e) \Omega^2}.
\]

The equilibrium solution to Equation (1) is

\[
f_{s0} = \frac{n_0(z = 0)}{(2\pi k_B T_{s0}/m_i)^{1/2}} \exp \left( -\frac{z^2}{2H^2} - \frac{m_i v_1^2}{2kB T_{s0}} - \frac{m_e B}{k_B T_{s0}} \right).
\]

The equilibrium magnetic field \( B_0 \) and its vector normal \( \mathbf{b}_0 \) are

\[
\mathbf{B}_0 = B_0 (\hat{\phi} \sin \chi + \hat{\zeta} \cos \chi), \\
\mathbf{b}_0 = \hat{\phi} \sin \chi + \hat{\zeta} \cos \chi.
\]

Global equilibria of axisymmetric, and at least partially rotationally supported, plasmas (Hinton & Hazeltine 1976; Bisnovatyi-Kogan & Seidov 1985; Ogilvie 1997) are characterized by a complicated global geometry due to the requirements of centrifugal force balance and equilibrium along axisymmetric magnetic surfaces. Local analysis away from the disk midplane, or global analysis of the longer wavelength CMVTI in a high-aspect ratio collisionless accretion disk, is beyond the scope of this paper.

4. TURBULENT AND WAVE FLUXES FOR DILUTE ROTATING PLASMAS

The evolution equation for the total energy within a disk, using methods outlined in Balbus & Hawley (1998), is given by the following (see, e.g., Sharma & Hammett 2006):

\[
\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi} \left( \frac{1}{2} \rho u^2 + \frac{3}{2} p + \frac{B^2}{8\pi} \right) + \nabla \cdot \mathbf{F}_E - \rho u \frac{1}{p_0} \nabla p_0 = -\frac{\partial}{\partial \Omega} W_{R\phi} = -R \frac{\partial}{\partial \phi} W_{\phi\phi} - Q, \tag{12}
\]

A fuller derivation of Equation (12) can be found in, e.g., Islam (2007). \( \mathbf{F}_E \) is the heat flux arising from local fluctuations, \( W_{R\phi} \) is the azimuthal stress, \( W_{\phi\phi} \) is the vertical-azimuthal stress, \( Q \) is a radiative loss term. One may look to Sharma & Hammett (2006) and Islam (2007) for fuller derivations of the energy balance term including the pressure expression term. In the context of disk accretion theory, the above expresses the fact that energy is generated by azimuthal stresses that couple to the free energy available from radial and vertical angular velocity gradients. This energy can then be accounted for in various ways: in a classical accretion disk, the energy flux is almost wholly radiated away; in a geometrically thick accretion disk, turbulent heat fluxes are large enough to transport at least some of this viscously generated energy (Balbus & Hawley 1998; Balbus 2003). Even in collisionless accretion, turbulent energy generation may heat electrons until they become radiatively efficient at locally dissipating energy (Sharma et al. 2007). However, in nonradiative flows (Narayan et al. 1998), viscously generated energy must be carried away by a turbulent heat flux (Balbus 2004).

The energy flux is given by

\[
\mathcal{F}_E = \mathbf{u} \left( \frac{1}{2} \rho u^2 + \frac{3}{2} p \right) + \frac{1}{4\pi} \mathbf{B} \times (\mathbf{u} \times \mathbf{B}) + \mathbf{b} q + p_v \left[ (\mathbf{u} \cdot \mathbf{b}) \mathbf{b} - \frac{1}{3} \mathbf{u} \right].
\]

The first term in \( \mathcal{F}_E \) corresponds to flux of gas kinetic energy, the second to the enthalpy, and the third term corresponds to Poynting MHD flux. The heat flux \( q = q_h + q_e \) and pressure difference \( p_v = p_v^* + p_v^e \) are defined in, e.g., Chang & Callen (1992a, 1992b) in the context of heat flux expressions to model collisionless transport due to specific instabilities into a fluid formalism, and shown here

\[
q_h = \frac{1}{2} q_{\|} + q_{\perp},
\]

\[
q_{\|} = 2\pi \int m_i (v_1 - u_1)^3 (B d\mu v_\|) f_\|, \tag{14}
\]

\[
q_{\perp} = 2\pi \int m_i (v_1 - u_1)^4 (\mu B) (B d\mu, d\nu) f_\|, \tag{15}
\]

\[
p_v^* = p_{\|} - p_{\perp}. \tag{16}
\]

The fourth and fifth terms of Equation (13) correspond to contributions due to heat fluxes along the magnetic field and the viscous stress. \( W_{R\phi} \) and \( W_{\phi\phi} \) are given by

\[
W_{R\phi} = \rho u R u_\phi - \frac{B_R B_\phi}{4\pi} + p_v b R b_\phi, \tag{17}
\]

\[
W_{\phi\phi} = \rho u u_\phi - \frac{B_\phi B_\phi}{4\pi} + p_v b_\phi b_\phi. \tag{18}
\]

The angular momentum flux can be derived from MHD force balance and continuity, and for an accretion disk is given by (Balbus & Hawley 1998; Islam 2007)

\[
\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi} \left( \rho R [u_\phi + R\Omega] \right) + \nabla \cdot \left( \rho u [u_\phi + R\Omega] - \frac{B_\phi B_\phi}{4\pi} + p_v b_\phi b_\phi \right) + \left[ p_{\|} + \frac{B^2}{8\pi} \right] \frac{\partial}{\partial \phi} \phi = 0. \tag{19}
\]

To understand how local fluctuations about mean quantities of the form \( A = A_0 + \delta A \), whether waves or turbulence, can tap into sources of energy within this rotating system, it is easiest to consider the truncated dynamics of this system by averaging vertically and azimuthally. Define the following averaged quantity:

\[
\langle A \rangle = \frac{1}{H} \int_0^{2\pi} \int_{z=-\infty}^{z=\infty} A \, dz \, d\phi.
\]
and consider fluctuations which spatially average to zero, i.e., \( \langle \delta A \rangle = 0 \). Contributions of fluctuations appear at second order. Since in equilibrium \( u_0 = 0 \), \( p_{i0} = p_{e0} = \rho_0 \), \( q_{e0} = 0 \), and \( q_{v,e0} = 0 \), the energy and angular momentum equations are

\[
\frac{\partial (\rho u_r)}{\partial t} + \frac{1}{R} \frac{\partial}{\partial R} \left( R^2 \Omega (\rho u_r) + R (W_{\phi R}) \right) = 0, \tag{20}
\]

\[
\frac{\partial (\rho v_\phi)}{\partial t} + \frac{1}{R} \frac{\partial}{\partial R} \left( R (F_{ER}) - \langle \rho u_r \rangle \right) + \frac{\partial p_0}{\rho_0} = - \frac{\partial \Omega}{\partial \ln R} (W_{\phi R}) - O_. \tag{21}
\]

We have ignored the flux of gas kinetic energy, that appears at third order in fluctuating quantities, and the Poynting flux, which is subdominant to the other terms in the energy flux. We have taken \( W_{\phi R} \) to be an even function of height. From Equation (11), the equilibrium azimuthal component of the magnetic normal vector is \( \cos \chi \).

\[
(L) = \langle \rho R (u_\phi + R \Omega) \rangle, \tag{22}
\]

\[
\langle \mathcal{E} \rangle = \left\{ \frac{1}{2} \rho u_r^2 + \frac{1}{2} p_r + p_\perp + \frac{B^2}{8\pi} \right\}, \tag{23}
\]

\[
\langle W_{\phi R} \rangle = \left\{ \rho_0 \delta u_r \delta u_\phi - \frac{\delta B R \delta B_\phi}{4\pi} + \delta p_\perp \delta b_\perp \sin \chi \right\}, \tag{24}
\]

\[
\langle F_{ER} \rangle = \frac{5}{2} \rho_0 \delta u_r \delta \theta - \frac{1}{3} \left( \delta p_\perp \delta u_r \right). \tag{25}
\]

Note that the radial mass flux term \( \langle \rho u_r \rangle = \langle \delta \rho u_r \rangle + \rho_0 u_{r2} \), where \( u_{r2} \) is a second order steady bulk radial flow of matter with magnitude of the order of \( \langle \delta \rho / \rho_0 \rangle^2 \); as noted by Balbus (2003), in a steady-state geometrically thin disk, the net radial matter flux has a magnitude given by \( |\langle \rho u_r \rangle| \sim \rho |\langle u \rangle|^2/(R \Omega) \).

5. STABILITY ANALYSIS AND QUADRATIC HEAT FLUXES

The discussion of the CMVTI is divided into the following subsections. Sections 5.1 and 5.2 describe the eigenmodal equations, and collisionless pressure expressions, used to derive the full dispersion relation for the CMVTI, which is not shown in this work. In the limit of zero equilibrium pressure and temperature gradients, the CMVTI reduces to the collisionless MRI. Section 5.3 estimates quadratic modal expressions for heat fluxes and Reynolds stress from the CMVTI. We find it useful to use the following variables and scalings:

\[
\begin{align*}
\theta_0 &= \frac{k_B T_0}{m_i} \\
\nu_A &= \frac{B_0^2}{4\pi \rho_0} \\
x &= k_1 v_A / \Omega \\
\kappa &= ku_A / \Omega \\
\gamma &= \Gamma / \Omega \\
\alpha_r &= - \left( \theta_0^{1/2} / \Omega \right) \frac{\partial \ln \rho_0}{\partial R} \\
\alpha_T &= - \left( \theta_0^{1/2} / \Omega \right) \frac{\partial \ln T_0}{\partial R} \\
\beta &= \theta_0 / \nu_A^2. \tag{26}
\end{align*}
\]

Our expression for the Alfvén speed \( v_A \) differs by a factor of \( \sqrt{2} \) from the standard definition. We also explore the stability of stratified media that are convectively stable, hence one in which \( \alpha_r < 0 \) or equivalently \( \alpha_T < (2/5) \alpha_r \). All plots of dispersion relations, heat fluxes, and Reynolds stresses use a plasma equilibrium with \( \chi = \pi / 4 \) (equal equilibrium toroidal and vertical magnetic field components), plasma \( \beta = 10^2 \). Keplerian rotation profile \( \Omega \propto R^{-3/2} \), and purely vertical wavenumbers.

5.1. Perturbed Axisymmetric Distribution

Function at the Mid-plane

Here we consider an equilibrium density and temperature distribution given in Section 2. Assume axisymmetric perturbations to equilibrium quantities of the form \( \delta a(t) \propto \exp(ik_R R + ik_z z + \Gamma t) \), and define \( k_0 = k \cdot b_0 \). Equation (1) then reduces to the following form for ions and electrons, where we assume equal scale heights of radial and vertical ion and electron temperature gradients:

\[
\begin{align*}
\delta f_{i,e} / f_{i,e} &= m_{i,e} v_{i,e} \left[ \frac{\Gamma + ik_{||} v_{i,e}}{k_B T_{i,e}} \right] \\
&- \frac{2(\Omega + \Omega R) \Gamma + ik_{||} v_{i,e} \Omega R}{ik_{||} (\Gamma + ik_{||} v_{i,e})} \hat{B}_R \sin \chi \\
&- \frac{\hat{B}_R}{ik_z} \left( \frac{\partial \ln n_0}{\partial R} - 3 \frac{\partial \ln T_0}{2 \partial R} + \left[ \frac{m_i B_0}{k_B T_0} + \frac{m_i v_{i,e}^2}{2 k_B T_0} \right] \frac{\partial \ln T_0}{\partial R} \right) \\
&+ \frac{\hat{B}_R v_{i,e} \partial \ln p_0 / \partial R}{\Gamma + ik_{||} v_{i,e}}. \tag{27}
\end{align*}
\]

\[
\begin{align*}
\delta f_{i,e} / f_{i,e} &= m_{i,e} v_{i,e} \left[ \frac{\Gamma + ik_{||} v_{i,e}}{k_B T_{i,e}} \right] \\
&- \frac{2(\Omega + \Omega R) \Gamma + ik_{||} v_{i,e} \Omega R}{ik_{||} (\Gamma + ik_{||} v_{i,e})} \hat{B}_R \sin \chi \\
&- \frac{\hat{B}_R}{ik_z} \left( \frac{\partial \ln n_0}{\partial R} - 3 \frac{\partial \ln T_0}{2 \partial R} + \left[ \frac{m_i B_0}{k_B T_0} + \frac{m_i v_{i,e}^2}{2 k_B T_0} \right] \frac{\partial \ln T_0}{\partial R} \right) \\
&+ \frac{\hat{B}_R v_{i,e} \partial \ln p_0 / \partial R}{\Gamma + ik_{||} v_{i,e}}. \tag{28}
\end{align*}
\]

Terms with \( \Omega \) arise due to the fact that the plasma is rotating; terms with equilibrium gradients of temperature, density, or pressure may drive convective and free energy gradient instabilities. \( \delta E_i \) is the electric field that ensures quasineutrality, i.e., \( \int \delta f_i^0 B d\mu = \int \delta f_e^0 B d\mu \). One can demonstrate that in the limit of dominating ion thermal energy \( T_{i0} \gg T_{e0} \) that the electric field \( \delta E_i \) and electron dynamic terms (such as \( \delta u_{r,e} \)) become unimportant in describing the plasma dynamics. This is the simplification employed by Quataert et al. (2002) and Sharma et al. (2003). However, with equilibrium electron temperatures up to one-tenth of that of the ion temperatures, as implied by local nonlinear simulations of the collisionless MRI (Sharma et al. 2007), the CMVTI dispersion relation is not substantially altered. Figure 1 shows that the dispersion relation of the CMVTI is not significantly different between cases where the electron temperature is negligible \( (T_{e0} = 10^{-2} T_{i0}) \) and where the electron temperature equals the ion temperature.

Using the induction equation (Equation (3)) and the continuity equation, the total force balance equation, Equation (2),
is represented by the following in terms of Equation (26):

\[
\gamma^2 \mathbf{B} - \gamma^2 \mathbf{b_0} \left( \frac{\delta \rho}{\rho} - \frac{\alpha_p - \alpha_T}{i \chi} \mathbf{B_R} \right) - 2 \frac{d \ln \Omega}{d \ln R} \hat{B_R} \hat{R} + 2 \gamma \sin \chi \left( \frac{\delta \rho}{\rho} - \frac{\alpha_p - \alpha_T}{i \chi} \mathbf{B_R} \right) \hat{R}
\]

\[
+ 2 \gamma \hat{x} \times \mathbf{B} = \hat{k}_x \beta \frac{\delta p_{\|}}{p_0} + x^2 \beta \frac{\delta p_{\perp}}{p_0} - \delta p_{\perp} \mathbf{b_0}
\]

\[
- i x \beta^{1/2} \frac{\delta \rho}{\rho} \hat{R} = x^2 \hat{B} + \hat{k}_x \frac{\delta B}{B}.
\]

(29)

\[
\delta B/B = \hat{B_0} \sin \chi - (k_R/k_Z) \hat{B}_R \cos \chi, \quad \delta p_{\|} = \delta p_{\|} + \delta p_{\perp},
\]

and \( \delta p_{\perp} = \delta p_{\perp} + \delta p_{\perp} \). Contributions due to \( \delta \rho/\rho - (\alpha_p - \alpha_T)(i x \beta^{1/2}) \hat{B_R} \) arise from finite plasma compressibility; in the Boussinesq limit these terms are set to zero. The eigenvalue problem consists of three equations for solving \( \hat{B_R}, \hat{B_0}, \) and \( \delta \rho/\rho \): radial force balance, azimuthal force balance, and force balance along \( \mathbf{b_0} \).

\[
\left( \gamma^2 + x^2 \left[ 1 + \frac{k_R^2}{k_Z^2} \right] + 2 \frac{d \ln \Omega}{d \ln R} - 2 \gamma \sin \chi \left( \frac{\alpha_p - \alpha_T}{i \chi} \right) \right) \hat{B_R}
\]

\[
- \left( 2 \gamma + x^2 \tan \chi \frac{k_R}{k_Z} \right) \hat{B_0} + \frac{\delta \rho}{\rho} \left( 2 \gamma \sin \chi + i x \beta^{1/2} \alpha_p \right)
\]

\[
= \frac{k_R}{k_Z \cos \chi} \frac{x^2 \beta}{p_0} \frac{\delta p_{\perp}}{\rho_0}.
\]

(30)

\[
\left( \gamma^2 \sin \chi \left( \frac{\alpha_p - \alpha_T}{i \chi} \right) + 2 \gamma \right) \hat{B_R} + \left( \gamma^2 + x^2 \right) \hat{B_0} - \gamma^2 \sin \chi \frac{\delta \rho}{\rho}
\]

\[
= x^2 \beta \frac{\delta p_{\|} - \delta p_{\perp}}{p_0} \sin \chi.
\]

(31)

\[
\delta \rho / \rho = \frac{\delta \rho}{\rho} - \frac{\delta B}{B} \left( R(i \xi) - 1 \right) + \frac{\hat{B_R}}{i k_Z} \left( \frac{\partial \ln n_0}{\partial R} - \frac{\partial \ln p_0}{\partial R} \right).
\]

(33)

\[
\delta p_{\perp} / p_0 = \left( 1 - 2 \xi^2 R(i \xi) \right) \frac{\delta \rho}{\rho} - \left( \frac{1 - [1 + 2 \xi^2] R(i \xi)}{R(i \xi)} \right) \frac{\delta B}{B}
\]

\[
+ \frac{\hat{B_R}}{i k_Z} \left( 1 - 2 \xi^2 R(i \xi) \right) \frac{\partial \ln n_0}{\partial R} - \frac{\partial \ln p_0}{\partial R}.
\]

(34)

\[
\beta \phi - \beta \phi = 5, \quad \text{and} \quad \beta \phi = 2 - \text{marginal convective stability.}
\]

This figure, and a more comprehensive plasma response incorporating electron pressure dynamics and finite equilibrium electron temperature, is taken from Islam (2007).

**Figure 1.** Plot of the real part of the growth rate of the CMVTI, for the case where the ion temperature is much larger than the electron temperature \( T_{i0}/T_{e0} = 10^2 \), and the case where they are equal. \( \alpha_p = 5 \), and \( \alpha_T = 2 \) – marginal convective stability. Figure 2, and a more comprehensive plasma response incorporating electron pressures. Therefore, subsequent expressions for perturbed and equilibrium pressure will refer to the ionic component (e.g., \( \delta p_{\perp} \rightarrow \delta p_{\perp}, p_{i0} \rightarrow p_0, p_{e0} \rightarrow p_0, T_{i0} \rightarrow T_i \)).
\( \zeta = \Gamma / (k_{||} \theta_0 \sqrt{2}) \) and \( R(\zeta) \) is the plasma response function

\[
R(\zeta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} xe^{-x^2} \, dx. \tag{35}
\]

Since the phase velocity of the modes are at best of the order of the sound speed, i.e., \( |\zeta| \lesssim 1 \), these perturbations are not adiabatic and the opposite, slow wave (\( |\zeta| \ll 1 \)) limit, holds for most unstable wavenumbers. The plasma response function in the slow wave limit is

\[
R(i\zeta) = 1 - \zeta \sqrt{\pi} + O(\zeta^2). \tag{36}
\]

Expressions for perturbed pressure reduce to those of first order in \( \zeta \):

\[
\frac{\delta p_{||}}{p_0} \rightarrow \frac{\delta \rho}{\rho} + \sqrt{\pi} \zeta \frac{\delta B}{B} - \xi_R \frac{\partial \ln T_0}{\partial R}, \tag{37}
\]

\[
\frac{\delta p_{\perp}}{p_0} \rightarrow \frac{\delta \rho}{\rho} - \sqrt{\pi} \zeta \frac{\delta B}{B} + \xi_R \left( 3 \frac{\partial \ln n_0}{\partial R} - \frac{\partial \ln p_0}{\partial R} \right). \tag{38}
\]

From the radial component of Equation (3), \( B_R = i k_1 \xi_R \) where \( \xi_R \) is the radial fluid displacement. Dispersion relations for the CMVTI are displayed in Figure 2. One feature of the plasma response vis-a-vis the CMVTI is that of relatively strong collisionless Barnes damping of MHD modes along the magnetic fields response via the CMVTI is that of relatively strong collisionless damping of MHD modes along the magnetic fields.

For the MVTI (Islam 2012), the range of unstable wavenumbers match between fluid and collisionless analogs:

\[
0 \leq k^2 v_A^2 / \Omega \leq 2 \frac{d \ln \Omega}{d \ln R} + \alpha_p \alpha_T. \tag{39}
\]

Instead of collisionless damping in the case of the instabilities analyzed within this paper, in fluid treatments it is finite (but dynamically important) viscosity and thermal conductivity that plays this role. Figure 1 from Islam (2012), which is reproduced as Figure 3 here, shows the real part of the dispersion relation for the MVTI, the form of the relative perturbed pressures (Equations (33) and (34)) and heat fluxes given the following:

\[
W_{\phi R} = \text{Re} \left( p_0 \delta u_{R R} \delta u_\phi - v_A^2 B_{RR}^* \delta u_{\phi R} + \sin \chi B_{RR}^* \delta p_v \right), \tag{40}
\]

\[ F_{\phi R} = \text{Re} \left( \frac{5}{2} \delta u_{\phi R} \delta \theta - \delta q B_{RR}^* + \frac{1}{3} \delta p_v B_{RR}^* \right). \tag{41} \]

One can employ expressions for the total pressure (Equation (6)), pressure difference (Equation (15)), and total heat flux (Equation (14)) with expressions for the perturbed pressures (Equations (33) and (34)) and heat fluxes given (Equations (40) and (41)). The form of the relative perturbed density and toroidal magnetic field \( \delta \rho / \rho \) and \( B_{\phi R} \) are described in the eigenvalue equations (Equations (30)–(32)). Using variable scalings as given by Equation (26), expressions for \( \delta u_{\phi R}, B_{RR}, \delta q_{\phi R}, \delta \rho_{\|}, \delta \rho_{\perp}, \delta \rho_\perp, \delta \theta, \) and \( \delta q \) in terms of \( \xi_R \) are

\[
\delta u_{\phi R} = \gamma (\Omega \xi_R), \tag{42}
\]

\[ B_{RR} = i x \left( \frac{\Omega}{v_A} \xi_R \right), \tag{43} \]

\[
\delta \rho_{\|} = -\frac{2 \gamma (\cos^2 \chi + R \left( \frac{i \gamma}{x^2 \sqrt{2} \beta} \right) \delta \rho_\phi)}{x \sqrt{2} \beta} \sin \chi \left( \frac{v_A^2}{x^2 \beta} \right) \sin \left( \frac{\gamma \theta_0}{x \beta^{1/2}} \right) \times i x \left( \frac{\Omega}{v_A} \xi_R \right), \tag{44}
\]

\[
\delta \rho_{\perp} = -\frac{\gamma v_A B_{\phi}}{i x} \left( \frac{\Omega}{v_A} \xi_R \right) \times \left( \frac{i \gamma}{x^2 \sqrt{2} \beta} \right) \sin \chi \left( \frac{v_A^2}{x^2 \beta} \right) \sin \left( \frac{\gamma \theta_0}{x \beta^{1/2}} \right) \times i x \left( \frac{\Omega}{v_A} \xi_R \right), \tag{45}
\]

\[
\delta \rho_{\perp} = \left( p_0 H^{-1} \xi_R \right) \left( \frac{v_A^2}{x^2 \beta} - 1 \right) \left( \frac{i \gamma}{x \sqrt{2} \beta} \right) + 1 \right) \times \left( \frac{1 + \frac{\gamma^2}{2 x^2 \beta}}{x \sqrt{2} \beta} \right) \left( \frac{i \gamma}{x \sqrt{2} \beta} \right) - 1 \right), \tag{46}
\]

### 5.3. Quadratic Fluxes

Here, we determine the normalized quadratic heat flux, Equation (25), and the radial azimuthal stress, Equation (24), associated with a given mode of purely vertical wavenumber \( k_z \). We normalize these fluxes as a function of fixed Lagrangian radial displacement \( \xi_R = \delta u_{R R} / \Gamma \). From Equation (27), we have the following expressions for \( \delta u_{\|}, \delta q_{\|}, \) and \( \delta q_{\perp} \):

\[
\frac{\delta u_{\|}}{\theta_0^2} = -i \zeta \sqrt{2} R (i \zeta) \left( \frac{\delta B}{B} - \frac{2 \Omega \chi}{k_{||}^2 \theta_0} B_R \sin \chi \right) + i \frac{R}{k_{||} \theta_0} \left( \frac{\partial \ln p_0}{\partial R} \right), \tag{47}
\]

\[
\frac{\delta q_{\|}}{p_0 \theta_0^2} = i \left( \frac{\sqrt{2} \Omega \chi}{k_{||}^2 \theta_0} R \sin \chi \right) \left( (2\zeta^2 + 3) R(\zeta) - 1 \right), \tag{48}
\]

\[
\frac{\delta q_{\perp}}{p_0 \theta_0^2} = -i \zeta \sqrt{2} \left( \frac{\delta B}{B} \right) R (i \zeta). \tag{49}
\]
Figure 2. Plot of the real part of the growth rate for the CMVTI and different equilibrium radial temperature gradients. Here $\alpha_P = 5$ and different $\alpha_T = 0$, such that $0 < \alpha_T < (2/5)\alpha_P$, so that the plasma remains convectively stable.

Figure 3. Plot of the real portion of the growth rate for the MVTI various $\alpha_T$. $\alpha_P = 10$, viscous diffusion coefficient $v\Omega/\nu_A^2 = 10^7$, Prandtl number $Pr = 1/101$, and $\alpha_S = 5\alpha_T/3 - 2\alpha_P/3$. Rollover occurs at wavenumbers $k \sim \sqrt{\nu_A/\nu} \ll \Omega/\nu_A$. 
\[
\frac{\delta \theta}{\theta_0} = \frac{\delta p}{p_0} - \frac{\delta \rho}{\rho} = (\xi_R H^{-1}) \\
\times \left( \alpha_T + \alpha_p \left( \frac{5}{3} \cdot \frac{\gamma^2}{3 \alpha^2 \beta} \right) R \left( \frac{i \gamma}{x \sqrt{2 \beta}} \right) - \frac{5}{3} \right) \\
+ \frac{1}{3} \delta \theta_0 \sin \chi \left( \frac{\gamma^2}{2 \alpha^2 \beta} - 1 \right) R \left( \frac{i \gamma}{x \sqrt{2 \beta}} \right) + 1 \\
- \frac{2}{3} i \beta \sin \chi \left( \frac{\Omega}{v_A} \xi_R \right) \\
\times \left[ \left( 1 + \frac{\gamma^2}{2 \alpha^2 \beta} \right) \left( \frac{i \gamma}{x \sqrt{2 \beta}} \right) - 1 \right],
\] (50)

\[
\delta q = (p_0 \Omega \xi_R) \sin \chi \left( i \alpha_p \frac{\gamma}{x \beta^{1/2}} + \frac{2 \gamma^2}{x \alpha^2 \beta} \right) \\
\times \left[ \left( \frac{3}{2} \cdot \frac{\gamma^2}{2 \alpha^2 \beta} \right) R \left( \frac{i \gamma}{x \sqrt{2 \beta}} \right) - \frac{1}{2} \right] \\
+ i \left( p_0 \Omega \xi_R \right) \sin \chi \left( \frac{p_0 \Omega H^{-1}}{v_A} \xi_R \right) \\
\times \left[ \left( 1 + \frac{\gamma^2}{2 \alpha^2 \beta} \right) \left( \frac{i \gamma}{x \sqrt{2 \beta}} \right) - 1 \right].
\] (51)

The azimuthal stress is normalized in units of \( p_0 \Omega^2 |\xi_R|^2 \) and the heat flux in terms of \( \rho_0 \Omega^1/2 \xi_R^2 \), \( \rho_0 \delta \Omega H^{-1} |\xi_R|^2 \). The relatively involved quadratic expressions for angular momentum and heat flux are not shown. In Figures (4) and (5) are plots of the heat flux and azimuthal stress for the CMVTI for different \( \alpha_T \). One can easily demonstrate, by setting \( \alpha_p = \alpha_T = 0 \), that the heat flux for the collisionless MRI is zero. There are no equilibrium radial gradients of temperature or density, the growth rate is purely real, so that for a given mode the temperature and viscous pressure perturbations are out of phase with the perturbed radial velocity, and the perturbed heat flux is out of phase with the perturbed radial magnetic field. The salient features of these instabilities is that they produce the right type of azimuthal stress that can drive accretion. The general sense of the Reynolds stress is outward for all unstable wavenumbers for the CMVTI; however, Islam (2012) demonstrates that the MVTI can have a generally small range of small wavenumbers to the requirement of geometrically thick disks to efficiently transport angular momentum without radiative losses, a global stability analysis with realistic disk structure is needed.

6. SUMMARY OF RESULTS AND FURTHER WORK

In this paper we have derived the drift kinetic equation explicitly in a rotating frame with possible significant gas pressures and only mild collisionality, with application to hot, dilute, weakly-magnetized (in the sense that magnetic forces are subdominant in equilibrium), at best mildly relativistic systems such as dim accretion about supermassive black holes. Section 6.1 describes the main results of this paper. Section 6.2 elaborates on the main directions for future work.

6.1. Summary of Results

We see physical terms explicitly associated with disk stratification as well as rotation. We also see that one may rather easily derive modifications of the azimuthal stress and heat flux due to fluctuations or waves in accreting systems (Balbus & Hawley 1998; Balbus 2003) due to dilute plasmas, as demonstrated in Section 4, in order to characterize how or whether instabilities may create the right type of turbulence that drives accretion.

We have analyzed the CMVTI, which have been demonstrated (Islam & Balbus 2005; Islam 2012) from a fluid treatment to destabilize a plasma, through anisotropic viscosities and thermal conductivities, that possesses adverse angular velocity or temperature gradients. We demonstrate the congruence in the dispersion relation for the CMVTI with the MVTI. Heat fluxes and azimuthal stresses associated with this instability have the right sense (i.e., positive), to drive accretion in fat dilute nonradiative rotating plasmas, and roughly match their respective fluid counterparts. Furthermore, we note that expressions for the normalized pressure and heat gradients, \( \alpha_p \) and \( \alpha_T \), go as \( H/R \) if we assume that equilibrium temperature and pressure radial scale heights are of order the disk radius. Therefore, we expect only geometrically thick disks to efficiently transport angular momentum in nonradiative accretion flows.

6.2. Future Work

Although we have applied the drift-kinetic equation to a single but important class of instability in Keplerian-like rotating systems, its representation as given in Equation (1) lends itself to much richer studies of these types of dilute plasmas. Immediate analytic work can enhance our understanding of the stability of a collisionless nonradiative accretion disk to the CMVTI. Due to the requirement of geometrically thick disks to efficiently transport angular momentum without radiative losses, a global stability analysis with realistic disk structure is needed.

Fluid MHD models of local nonlinear evolution in collisionless astrophysical plasmas have employed prescriptions to model collisionless and fast, small-length scale isotropizing phenomena. First, Landau fluid expressions of heat flux and viscosity represent, as practical as is possible, the collisionless momentum and heat transport driven by the instabilities of interest. Second, a hard wall on relative ion pressure anisotropies reflects observations of marginal pressure anisotropy in the solar wind (Hellinger et al. 2006; Bale et al. 2009), due to unresolvable fast (on the order of the ion gyroperiod) and short wavelength (on the order of the ion gyroradius) instabilities driven by pressure anisotropy. Similar pressure anisotropies are found to develop for the CMVTI, as shown by a more comprehensive stability analysis (Islam 2007). Although these prescriptions have been fruitfully applied to local simulations of the collisionless
Figure 4. Outward normalized azimuthal stress for the CMVTI and various convectively stable equilibrium profiles with $\alpha_p = 5$ and $0 \leq \alpha_T \leq 2$.

Figure 5. Same as Figure 4, except for quadratic heat flux.

MRI (Sharma & Hammett 2006; Sharma et al. 2007) and the buoyancy instability (Kunz et al. 2012), a more self-consistent numerical model is desired.

A more productive approach would be to use gyrokinetic or drift-kinetic MHD codes, such as Fokker–Planck (Grandgirard et al. 2006), ionic particle in cell (Kolesnikov et al. 2010; Chen & Parker 2009), or hybrid particle in cell (PIC; Brecht & Thomas 1988) modified such that ions move drift-kinetically, to simulate the dynamics of these plasmas. Recent work in modifying full particle in cell (Riquelme et al. 2012) and three-dimensional hybrid particle in cell (Kunz et al. 2014) for co-rotating local reference frames has found promise in the study of the collisionless differentially rotating plasmas, currently under situations in which the separation of length and timescales with ion gyromotion, disk rotational frequency, and the fastest growing wavelengths of the MRI are not too severe. These numerical models have shown promise in understanding the nonlinear development of initially weak-field (ion
gyroradius larger than the wavelength of the fastest growing mode) magnetototational instabilities (Krolik & Zweibel 2006; Ferraro 2007). Enhancements to these codes toward larger spatial and temporal separations between ion gyromotion and the slower, longer scale dynamics of collisionless MHD make them well suited toward understanding the nature of heat flux and angular momentum transport in the CMVTI.

The author thanks the referees, whose input has clarified and focused this paper into a generalization of the MVTI into the collisionless regime, for pointing out a crucial reference (Sharma et al. 2007) demonstrating that the collisionless MRI may heat electrons such that even collisionless accretions flows may become radiative, and for allowing a further iteration to repair mistakes in content.
REFERENCES

Aitken, D. K., Greaves, J., Chrysostomou, A., et al. 2000, ApJL, 534, L173
Baganoff, F. K., Maeda, Y., Morris, M. R., et al. 2003, ApJ, 591, 891
Balbus, S. A. 2003, ARA&A, 41, 555
Balbus, S. A. 2004, ApJ, 600, 865
Balbus, S. A., & Hawley, J. F. 1991, ApJ, 376, 214
Balbus, S. A., & Hawley, J. F. 1998, RvMP, 70, 1
Bale, S., Kasper, J., Howes, G., et al. 2009, PhRvL, 103, 211101
Bisnovatyi-Kogan, G. S., & Seidov, Z. F. 1985, Ap&SS, 115, 275
Bower, G., Wright, M. C. H., Falcke, H., & Backer, D. 2003, ApJ, 588, 331
Brecht, S. H., & Thomas, V. A. 1988, CoPhC, 48, 135
Chandrasekhar, S. 1960, PNAS, 46, 253
Chang, Z., & Callen, J. D. 1992a, PhFlB, 4, 1167
Chang, Z., & Callen, J. D. 1992b, PhFlB, 4, 1182
Chen, Y., & Parker, S. E. 2009, PhPl, 16, 052305
de Villiers, J.-P., & Hawley, J. F. 2003, ApJ, 592, 1060
de Villiers, J.-P., Hawley, J. F., & Krolik, J. 2003, ApJ, 599, 1238
Ferraro, N. M. 2007, ApJ, 662, 512
Fromang, S., de Villiers, J.-P., & Balbus, S. A. 2004, Ap&SS, 292, 439
Grandgirard, V., Brunetti, M., Bertrand, P., et al. 2006, JCoPh, 217, 395
Hawley, J. F., Gammie, C. F., & Balbus, S. A. 1996, ApJ, 464, 690
Hellinger, P., Trávníček, P., Kasper, J. C., & Lazarus, A. J. 2006, GeoRL, 33, 09101
Hinton, F. L., & Hazeltine, R. D. 1976, RvMP, 48, 239
Islam, T. 2007, PhD thesis, Univ. Virginia
Islam, T. 2012, ApJ, 746, 8
Islam, T., & Balbus, S. A. 2005, ApJ, 633, 328

Kolesnikov, R. A., Wang, W. X., Hinton, F. L., Rewoldt, G., & Tang, W. M. 2010, PhRvL, 117, 2506
Krolik, J. H., & Zweibel, E. G. 2006, ApJ, 644, 651
Kulsrud, R. M. 1983, in Basic Plasma Physics: Selected Chapters, Handbook of Plasma Physics, Vol. 1, ed. A. A. Galeev & R. N. Sudan (Amsterdam: North-Holland Publishing Company), 1
Kulsrud, R. M. 2005, Plasma Physics for Astrophysics (Princeton Series in Astrophysics; Princeton, NJ: Princeton Univ. Press)
Kunz, M. W., Bogdanović, T., Reynolds, C. S., & Stone, J. M. 2012, ApJ, 754, 122
Kunz, M. W., Stone, J. M., & Bai, X. N. 2014, JCoPh, 259, 154
Marrone, D. P., Moran, J. M., Zhao, J. H., & Rao, R. 2005, ApJ, 640, 308
Narayan, R. 2002, in Proc. MPA/ESO/MPE/USM Joint Astronomy Conference held in Garching, Lighthouses of the Universe: The Most Luminous Celestial Objects and Their Use for Cosmology, ed. M. Gilfanov, R. Sunyaev, & E. Churazov (Cambridge, MA: Harvard-Smithsonian Center for Astrophysics), 405
Narayan, R., Mahadevan, R., Grindlay, J. E., Popham, R. G., & Gammie, C. F. 1998, ApJ, 492, 554
Ogilvie, G. I. 1997, MNRAS, 288, 63
Quataert, E., Dorland, W. D., & Hammett, G. W. 2002, ApJ, 577, 524
Quataert, E., Sharma, P., & Spitkovsky, A. 2012, ApJ, 755, 50
Sano, T., & Stone, J. M. 2002, ApJ, 570, 314
Sharma, P., & Hammett, G. W. 2006, PhD thesis, Princeton Univ., Washington, DC
Sharma, P., Hammett, G. W., & Quataert, E. 2003, ApJ, 596, 1121
Sharma, P., Quataert, E., & Stone, J. M. 2007, ApJ, 671, 1696
Velikhov, E. P. 1959, ZhETF, 36, 1398
Wardle, M. 1999, MNRAS, 307, 849