ON HOMALOIDAL POLYNOMIALS

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Let $\mathbb{P}^n$ be the projective space over a field of characteristic zero. If $F$ is a homogeneous polynomial we say that $F$ is homaloidal if the polar map $\partial F$ defined by the partial derivatives of $F$ is a birational selfmap of $\mathbb{P}^n$. Although the problem of determining homaloidal polynomials has a classical flavour the theme only recently was raised in an algebro-geometric context by Dolgachev ([Do]), following suggestions stemming from the theory of prehomogeneous varieties: the relative invariants of prehomogeneous spaces are in fact homaloidal polynomials ([KiSa],[EKP]). Dolgachev classifies homaloidal polynomials in $\mathbb{P}^2$ (see also [Di]) and characterizes homaloidal polynomials in $\mathbb{P}^3$ which are products of linear forms as products of four independent linear forms. Dolgachev also raises the question if it is true that a non square free product of linear forms is homaloidal if and only the product of its factors with multiplicity one is. This question has been given a positive answer in a specific case (see [KS]) and in full generality ([DiPa]) in a topological context.

We will give an algebraic proof of the following:

**Theorem A.** Suppose $F = \prod_{i=0}^{r} L_i$ is a square free homaloidal polynomial which is the product of linear forms. Then $r = n$ and the linear forms $L_0, \ldots, L_n$ are independent, so that $\partial F$ is a composition of a projectivity and of a standard Cremona transformation.

**Theorem B.** Suppose $F = \prod_{i=0}^{r} L_i^{n_i}$ is a homogeneous polynomial with $L_0, \ldots, L_r$ linear forms. Then $F$ is homaloidal if and only if the polynomial $F_{\text{red}} = \prod_{i=0}^{r} L_i$ is homaloidal. In particular, if $F$ is homaloidal, then $r = n$, the linear forms $L_0, \ldots, L_n$ are independent and the map $\partial F$ is a composition of projectivities and of a standard Cremona transformation.

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1. Preliminaries We start with the following classical:

**Definition 1** If $F \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ is a homogeneous polynomial, the *polar map defined by $F$* is the rational map defined by

$$
\partial F : \mathbb{P}^n \rightarrow \mathbb{P}^{n*}
$$

$$
\partial F(p) = [\partial F/\partial X_0(p), \ldots, \partial F/\partial X_n(p)],
$$

while the *polar system defined by $F$* is the linear system

$$
|\partial F| := |<\partial F/\partial X_0, \ldots, \partial F/\partial X_n>| \subset |\mathcal{O}_{\mathbb{P}^n}(d−1)|.
$$

It follows from the definition that, if $Z_F \subset \mathbb{P}^n$ is the hypersurface defined by $F$, the base locus of the polar map defined by $F$ is the singular locus of $Z_F$. Moreover, since $Z_F$ is an hypersurface, $Z_F$ is not reduced if and only if $F$ is not square free. In case $F$ is square free the polar map $\partial F$ is then free of base divisors and if $Z_F$ is smooth the polar map $\partial F$ is a morphism whose image is the dual variety $Z_{F^*} \subset \mathbb{P}^{n*}$ of $Z_F$.

If $F$ is not square free, let us write

$$
F = \prod_{i=0}^{r} G_i^{n_i},
$$

with $d = \sum_{i=0}^{r} n_i \deg G_i$. Then the base components of the polar system $|\partial F|$ are given by the hypersurface defined by the polynomial

$$
F' = \prod_{i=0}^{r} G_i^{n_i−1}.
$$

We will indicate by $F_{\text{red}}$ the polynomial $F/F'$.

The polar system defined by $F$ is naturally split in a fixed and in a moving part:

**Definition 2** The *moving part* of the polar system defined by a homogeneous polynomial $F$ is the linear system $|\mathcal{M}(\partial F)|$ obtained by removing all base components from the polar system $|\partial F|$.

In particular, we have that

$$
|\mathcal{M}(\partial F)| \subset |\mathcal{O}_{\mathbb{P}^n}(d−1−\deg F')|.
$$

Notice that the above definition makes perfectly sense in case $F$ is square free, in which case $F'$ is a constant.
**Definition 3** A homaloidal polynomial of degree $d$ is a homogeneous polynomial $F \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ such that the moving part $|M(\partial F)|$ of the polar system defined by $F$ induces a birational map.

Well known examples of homaloidal polynomials in $\mathbb{P}^n$ are those defining smooth quadrics. A remarkable result in [EKP] is the classification of homaloidal polynomials of degree $d = 3$: the irreducible ones define the secant varieties of the four Severi varieties and a classification seems at hand at least in degree $d = 4$. Probably the most known and important example of homaloidal polynomial is the polynomial $F = X_0 \cdots X_n$, which has degree $d = n + 1$ and whose associated polar map is a standard Cremona transformation. Another class of examples of homaloidal polynomials is given by the polynomials $F(m_0, \ldots, m_n) := X_0^{m_0} \cdots X_n^{m_n}$, with $m_i \geq 1$, for all $i = 0, \ldots, n$, of arbitrary large degree $d = \sum_{i=0}^{n} m_i$. The base locus of $\partial F$ has a divisorial component defined by the polynomial $F' = \prod_{i=0}^{n} X_i^{m_i-1}$. Once we remove it, we simply compute that $|M(\partial F)| = |< m_0 \prod_{i=1}^{n} X_i, \ldots, m_j \prod_{i \neq j}^{n} X_i, \ldots, m_n \prod_{i=0}^{n-1} X_i >|,$ so that $\partial F$, induces the same map as a composition of a projectivity (a homothety) and $\partial F_{red} = \partial \prod_{i=0}^{n} X_i$, so that it is homaloidal.

To say that $F$ is homaloidal is equivalent to say that $\partial F$ is dominant and that there exists a resolution of singularities $\xymatrix{ X \ar@{-->}[r]^f & \mathbb{P}^n \ar@{-->}[l]_g \ar[l]_{\partial F}}$, such that if $Y$ is a general member of $|M(\partial F)|$ and if $\overline{Y}$ denotes its strict transform on $X$, we have $(\overline{Y})^n = 1$, because in fact $\overline{Y} \simeq g^*\mathcal{O}_{\mathbb{P}^n}(1))$, where $\simeq$ denotes linear equivalence as Cartier divisors and $g$ is a birational morphism.

An important property of homaloidal polynomials is the following:

**Proposition 4.** If $F$ is a homaloidal polynomial, $Z_F$ is not a cone. In particular if $F$ is the product of linear forms $F = \prod_{i=0}^{r} L_i^{m_i}$, then $< L_0, \ldots, L_r > = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, so that in particular $r \geq n$. 


Proof. If \( Z_F \) is a cone the image of the polar map defined by \( F \) is contained in the linear space dual to the vertex of the cone of \( Z_F \). If \( F \) is a product of linear forms \( Z_F \) is a cone if and only if \( \langle L_0, \ldots, L_r \rangle \neq H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \). □

In fact (see [R]) even more is true: \( Z_F \) is a cone if and only if the image of the polar map associated to \( F \) lies in a hyperplane.

2. Products of linear forms
In this section we will prove Theorems A and B. We will always assume that \( F \) is a homaloidal polynomial which splits in the product of linear forms. Suppose first that \( F \) is square free so that \( \partial F \) is a linear system free of base components. We will fix a minimal resolution of singularities:

\[
X \\
\begin{array}{c}
\text{f} \\
\text{g}
\end{array} \\
\mathbb{P}^n \rightarrow \partial F \rightarrow \mathbb{P}^n^*.
\]

By definition, if \( Y \) is a general member of the polar system \( |\partial F| \), and if \( \overline{Y} \) is its strict transform on \( X \), we have \( \overline{Y} \simeq g^* \mathcal{O}_{\mathbb{P}^n^*}(1) \).

Let us write

\[ F = \prod_{i=0}^r L_i. \]

Up to a projectivity we can assume that \( L_0 = X_0 \). We will denote by \( H_0 \) the hyperplane defined by \( X_0 = 0 \). Let us define the polynomial

\[ G := \frac{F}{X_0} = \prod_{i=1}^r L_i. \]

With this choice, a basis of the polar system \( |\partial F| \) defined by \( F \) is given by:

\[ |\partial F| = | < G + X_0 \frac{\partial G}{\partial X_0}, X_0 \frac{\partial G}{\partial X_1}, \ldots, X_0 \frac{\partial G}{\partial X_n} > |. \]

We first observe that, if \( D \subset X \) is the strict transform of the hyperplane \( H_0 \), looking at the equations for \( \partial F \), the map \( g \) contracts \( D \), because \( \partial F \) contracts \( H_0 \) to its dual point \( U_0 = [1 : 0 : \cdots : 0] \).

Let us define \( G_0 \in H^0(H_0, \mathcal{O}_{H_0}(d-1)) \) as the restriction of \( G \) to \( H_0 \); notice that \( G_0 \) doesn’t need to be reduced.

Lemma 5. The irreducible divisor \( D \subset X \) is the unique divisor contracting to the point \( U_0 \in \mathbb{P}^n^* \) and the map \( g : X \rightarrow \mathbb{P}^n^* \) factors through the blow up \( h' : Z \rightarrow \mathbb{P}^n^* \) of \( \mathbb{P}^n^* \) at \( U_0 \).
Proof. Suppose that $W \neq D$ is a divisor in $X$ which is $g$–exceptional and such that $g(W) = g(D) = U_0$. By minimality of the resolution of the rational map $\partial F$, $W$ is not $f$–exceptional, so that it corresponds to a hypersurface $f(W) \subset \mathbb{P}^n$ which is distinct from $H_0$. Let $J$ be an equation of $f(W)$. Looking at the equations of $\partial F$, it follows that $J$ must divide $\frac{\partial G}{\partial X_i}$ for all $i \geq 1$. If we restrict the system $| < \frac{\partial G}{\partial X_1}, \ldots, \frac{\partial G}{\partial X_n} > |$ to the hyperplane $H_0$, by analyticity of polynomials, we obtain nothing but the polar system defined by $G_0$ on $H_0$. This implies that $J$ restricted to $H_0$ is a base component of $|\partial G_0|$ and this means that $f(W) \cap H_0 \subset \bigcup \{L_i \mid$ there exists $L_j \in \langle H_0, L_i \rangle \}$.

But the polar map $\partial F$ contracts each one the hyperplanes defined by the linear forms $L_i$ to their dual points in $\mathbb{P}^n^*$, so that it cannot be $g(W) = g(H_0)$.

The irreducible divisor $D$ corresponds then to the extraction of a valuation centered at $U_0 \in \mathbb{P}^n^*$ and we need to show that this valuation corresponds to the whole maximal ideal $m_{U_0}$. We have already noticed en passant that the system $| < \frac{\partial G}{\partial X_1}, \ldots, \frac{\partial G}{\partial X_n} > |$ corresponds on $X$ to the system $|g^* \mathcal{O}_{\mathbb{P}^n^*}(1) - D|$ and that it is of codimension one in $|g^* \mathcal{O}_{\mathbb{P}^n^*}(1)|$. Suppose that $g^* \mathcal{O}_X(-D) = m'$ with $\sqrt{m'} = m_{U_0}$; since

$$g^* g^* \mathcal{O}_{\mathbb{P}^n^*}(1) \otimes \mathcal{O}_X(-D) = m' \otimes \mathcal{O}_{\mathbb{P}^n^*}(1),$$

we have that $H^0(\mathbb{P}^n^*, m' \otimes \mathcal{O}_{\mathbb{P}^n^*}(1)) = n$ and hence that

$$m_{U_0} = m'.$$

Hence $D$ is the strict transform of the exceptional divisor under the blow up of $\mathbb{P}^n^*$ at $U_0$, $h' : Z \rightarrow \mathbb{P}^n^*$ and the result follows $\Box$.

Consider now the following diagram of maps:

$$\begin{array}{ccc}
X & \xrightarrow{f} & \mathbb{P}^n^* \\
\downarrow h & & \downarrow \pi \\
Z & \xrightarrow{h'} & \mathbb{P}^n^* \\
\end{array}$$

where $\pi$ is the projection from the point $U_0$ to the hyperplane $P$.

We can use Lemma 5 in order to factorize the morphism $g$ through the blow up $Z$ of $\mathbb{P}^n^*$ at $U_0$. We have the following diagram:

$$\begin{array}{ccc}
X & \xrightarrow{h} & \mathbb{P}^n^* \\
\downarrow & & \downarrow \pi \\
Z & \xrightarrow{h'} & \mathbb{P}^n^* \\
\end{array}$$

where $\pi$ is the projection from the point $U_0$ to the hyperplane $P$. 

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with \( g = h' h \). Let us denote by \( Y' \) a general member of the polar system \( |\partial G| \) and recall that we denote by \( Y \) a general element in \( \partial F \) and by \( D \) the strict transform of \( H_0 \) in \( X \).

**Lemma 6.** With the above notations:

1. the composition \( th : X \to P \) is a morphism and \( (th)^* O_P(1) \simeq Y' \simeq Y - D \),
2. the restriction of the linear system \( |(th)^* O_P(1)| \) to \( D \) induces a morphism \( m : D \to P \) which is a resolution of singularities of the polar map associated to \( G_0 \) on \( H_0 \), i.e. \( |(Y - D)|_D = |M(\partial G_0)| \),
3. the polynomial \( G_0 \) is homaloidal in \( H_0 \),
4. \( G_0 \) is square free if and only there don’t exist \( L_i \) and \( L_j \), with \( i \neq j \), such that \( X_0 \in < L_i, L_j > \)

**Proof.**

It follows from the definitions that the morphism \( th \) is set theoretically the same as the rational map \( \pi g \), so that in fact \( th \) is a resolution of singularities of \( \pi g \): they define the same morphism up to removing base components from the linear system defining \( \pi g \), which is in fact \( D \). Now, the linear system defining \( th \) is \( |(Y - D)| = |(th)^* O_P(1)| = |g^* O_{\mathbb{P}^n}(1) - D| \); by the above argument, this is also the moving part of \( |\partial G| \), so that in fact \( Y' \simeq Y - D \). It follows from the explicit equations that if we restrict to \( H_0 \) the map \( \pi g \), the resulting restriction is the rational map \( \partial G_0 \), whose resolution of singularities is the map \( th \) restricted to \( D \). We just notice here that the linear system inducing \( m := th|_D \) is \( |(Y - D)|_D = |M(\partial G_0)| \).

The map \( \partial G_0 \) is surjective because a composition of surjections. In order to show that \( G_0 \) is homaloidal on \( H_0 \) it suffices to show that

\[
D \cdot ((th)^* O_P(1) - D)^{n-1} = 1.
\]

This follows from the fact that \( D \) is the strict transform of the exceptional divisor under the blow up \( h' : Z \to \mathbb{P}^{n*} \). Finally, since the base components of \( |\partial G_0| \) on \( H_0 \) are non reduced components of \( Z_{G_0} \), and since \( G \) is square free, the last part of the thesis is proved. \( \square \)

**Remark** As a result of our Theorems A and B it will follow that \( Z_G \) is a cone. If one is able to prove this directly, the proof of Theorem A follows easily and directly from Lemma 6 (see Theorem 8). It is very easy to show that \( Z_G \) is a cone if and only if \( th \) is in fact the full polar map \( \partial G \) if and only if in \( \mathbb{P}^{n*} \) the point corresponding to \( H_0 \) is not on a line connecting \( Z_{L_i}^* \) and \( Z_{L_j}^* \) for \( i \neq j \neq 0 \) if and only if the polynomial \( G_0 \) is square free. This is the hard part of the problem, connected with cohomological properties of the corresponding arrangement of hyperplanes in \( \mathbb{P}^n \) and with syzygies of the base locus of \( \partial F \). It is in fact remarkable that Dolgachev proves it directly in case \( n = 3 \).

Let us now consider homaloidal polynomials which are products of linear forms with at least a square factor; notice that such polynomials arise naturally from
square free homaloidal polynomials which are products of linear forms: in Lemma 6 we have proven that the restriction of a homaloidal square free product of linear forms induces on each component of $Z_F$ a homaloidal product of linear forms. Quite surprisingly there is a priori no relation among $\partial F$ and $\partial F_{\text{red}}$.

Let us choose in fact $r+1$ distinct linear forms $L_0, \ldots, L_r$ in $\mathbb{P}^n$ together with an identification $X_0 = L_0$, and consider the following polynomials, where $H_0$ is the hyperplane of equation $X_0 = 0$ and $L_{i,0} = L_i \cap H_0$:

$$ F = X_0^{m_0} \prod_{i=1}^r L_i^{m_i}, \quad F' = X_0^{m_0-1} \prod_{i=1}^r L_i^{m_i-1}, \quad F_{\text{red}} = \frac{F}{F'}, $$

$$ G = \prod_{i=1}^r L_i^{m_i}, \quad G' = \prod_{i=1}^r L_i^{m_i-1}, \quad G_{\text{red}} = \frac{G}{G'}, $$

where

$$ G_0 = G \cap H_0, \quad G'_0 = G' \cap H_0. $$

We compute the moving parts of the polar systems defined by $F$ and $F_{\text{red}}$. We have:

$$ |M(\partial F)| = \langle m_0 G_{\text{red}} + X_0 \sum_{i=1}^r m_i \frac{\partial L_i}{\partial X_0} \prod_{j \neq i,0} L_j, \ldots, X_0 \sum_{i=0}^r m_i \frac{\partial L_i}{\partial X_n} \prod_{j \neq i,0} L_j \rangle, $$

$$ |M(\partial F_{\text{red}})| = |\partial F_{\text{red}}| = \langle G_{\text{red}} + X_0 \sum_{i=1}^r \frac{\partial L_i}{\partial X_0} \prod_{j \neq i,0} L_j, \ldots, X_0 \sum_{i=0}^r \frac{\partial L_i}{\partial X_n} \prod_{j \neq i,0} L_j \rangle. $$

We also compute an explicit basis of a system $|G_0''|$ which sits in a chain $|M(\partial G_0)| \subset |G_0''| \subset |\partial G_0|$ on $H_0$:

$$ |G_0''| = \langle \sum_{i=1}^r m_i \frac{\partial L_i}{\partial X_1} \prod_{j \neq i} L_{j,0}, \ldots, \sum_{i=1}^r m_i \frac{\partial L_i}{\partial X_n} \prod_{j \neq i} L_{j,0} \rangle, $$

Consider now the following diagram of maps, where $f$ and $g$ induce a minimal resolution of the morphism induced by $|M(\partial F)|$:

$$ \begin{array}{c}
X \\
\downarrow f \\
H_0 \subset \mathbb{P}^n \rightarrow M(\partial F) \rightarrow \mathbb{P}^n* \rightarrow \pi \rightarrow P.
\end{array} $$

We define $D$ to be the strict transform of $H_0$ in $X$, we denote by $Y$ a general member of $|M(\partial F')|$, by $Y'$ a general member of the system $|M(\partial G)|$, by $\overline{Y}$ and $\overline{Y}'$ their strict transforms on $X$. Quite surprisingly, all the arguments used in order to prove Lemma 5 and Lemma 6 apply verbatim and in particular it holds the following:
Lemma 7. With the above notations:

1. \( D \) is the strict transform on \( X \) of the exceptional divisor in the blow up \( h^* : Z \rightarrow \mathbb{P}^{n*} \) of \( \mathbb{P}^{n*} \) at \( U_0 \),
2. the restriction of the linear system \( (\text{th})^* \mathcal{O}_P(1) \) to \( D \) induces a morphism \( m : D \rightarrow P \) which is a resolution of singularities of the polar map defined by \( G_0 \) on \( H_0 \), i.e. \( |(\mathcal{N} - D)|_D = |M(\partial G_0)| \),
3. the polynomial \( G_0 \) is homaloidal in \( H_0 \).

Proof. □

We are now able to prove Theorems A and B at once.

Theorem 8. Let \( L_0, \ldots, L_r \) be distinct linear forms and let \( F = \prod_{i=0}^r L_i^{m_i} \), with \( m_i \geq 1 \) for all \( i = 0, \ldots, r \). Then \( F \) is homaloidal if and only if \( F_{\text{red}} = \prod_{i=0}^r L_i \) is homaloidal and \( F_{\text{red}} \) is homaloidal if and only if \( r = n \) and the \( L_i \)'s are independent linear forms.

Proof. The proof is by induction on \( n \).

The starting point of the induction is the case \( n = 1 \) which is easy: if \( F = \prod_{i=0}^r L_i^{m_i} \) is homaloidal the base point free system \( |M(\partial F)| \) must be of degree one, from which it follows easily that \( r = 1 \) and that \( L_0 \) and \( L_1 \) are in linear general position (they are distinct by hypothesis). The converse is, up to a projectivity, been proved after Definition 3. The same argument works a fortiori if \( F \) is square free.

Let us then move to \( \mathbb{P}^n \), with \( n > 1 \) and consider first the case of a square free homaloidal polynomial \( F = \prod_{i=0}^r L_i \). We plug \( X_0 = L_0 \), and we apply Lemma 6 in order to get a homaloidal polynomial \( G_0 = \prod_{i=1}^r L_i,0 \) on \( H_0 \). If \( G_0 \) is reduced we have that by induction \( r = n \) and the \( L_i,0 \)'s are independent in \( H_0 \), from which it follows that \( X_0, L_1, \ldots, L_n \) are independent in \( \mathbb{P}^n \). Consider then the case in which \( G_0 \) is not reduced. \( G_0 \) is still homaloidal so that by induction we can reorder the \( L_i \)'s in such a way that \( L_1,0, \ldots, L_n,0 \) are independent and there exist \( m_1, \ldots, m_n \) for which \( \sum_{i=1}^n m_i = r \) and

\[
G_0 = \prod_{i=1}^n L_{i,0}^{m_i},
\]

in such a way that

1. \( L_{n+1}, \ldots, L_{n+m_1-1} \) are in \( < X_0, L_1 > \),
2. \( \ldots \),
3. \( L_{n+m_1+\ldots+m_{n-1}-n+2}, \ldots, L_{n+\sum_{i=1}^n m_i-n} \) are in \( < X_0, L_n > \).

In other words, looking at the dual points in \( \mathbb{P}^{n*} \), we must have that all \( Z_{L_i}^+ \)'s, with \( i \geq n + 1 \) must lie in the cone with vertex \( U_0 \) projecting \( Z_{L_1}^+, \ldots, Z_{L_n}^+ \). But we can apply the same reasoning we have applied to \( L_0 = X_0 \) to any other \( L_i = X_i \) for all \( i = 1, \ldots, n \) and the intersection of all these cones is empty, so that there will
exists some \( i \in \{0, \ldots, n\} \) for which the corresponding homaloidal polynomial \( G_i \) is reduced. We then proceed as if \( G_0 \) were reduced.

Suppose now that \( F = \prod_{i=0}^{r} L_i^{m_i} \) is nonreduced and homaloidal. We must prove that \( F_{\text{red}} \) is homaloidal, the converse being a consequence of the first part of this Proof and of the example after Definition 3. Plugging \( X_0 = L_0 \) and applying Lemma 7 we get that \(|M(\partial F)|\) induces on \( H_0 \) the homaloidal system defined by \( G_0 \). By induction and by the same argument as above, we get the thesis, i.e. that \( r = n \) and the linear forms \( L_0, \ldots, L_n \) are independent, so that \( F_{\text{red}} \) is homaloidal.

\[ \square \]

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