Online and Random-order Load Balancing Simultaneously

Marco Molinaro
PUC-Rio, Brazil

Abstract

We consider the problem of online load balancing under $\ell_p$-norms: sequential jobs need to be assigned to one of the machines and the goal is to minimize the $\ell_p$-norm of the machine loads. This generalizes the classical problem of scheduling for makespan minimization (case $\ell_\infty$) and has been thoroughly studied. However, despite the recent push for beyond worst-case analyses, no such results are known for this problem.

In this paper we provide algorithms with simultaneous guarantees for the worst-case model as well as for the random-order (i.e. secretary) model, where an arbitrary set of jobs comes in random order. First, we show that the greedy algorithm (with restart), known to have optimal $O(p)$ worst-case guarantee, also has a (typically) improved random-order guarantee. However, the behavior of this algorithm in the random-order model degrades with $p$. We then propose algorithm \textsc{Simultaneous LB} that has simultaneously optimal guarantees (within constants) in both worst-case and random-order models. In particular, the random-order guarantee of \textsc{Simultaneous LB} improves as $p$ increases.

One of the main components is a new algorithm with improved regret for Online Linear Optimization (OLO) over the non-negative vectors in the $\ell_q$ ball. Interestingly, this OLO algorithm is also used to prove a purely probabilistic inequality that controls the correlations arising in the random-order model, a common source of difficulty for the analysis. Another important component used in both \textsc{Simultaneous LB} and our OLO algorithm is a smoothing of the $\ell_p$-norm that may be of independent interest. This smoothness property allows us to see algorithm \textsc{Simultaneous LB} as essentially a greedy one in the worst-case model and as a primal-dual one in the random-order model, which is instrumental for its simultaneous guarantees.

*Email: mmolinaro@inf.puc-rio.br
1 Introduction

We study the following classical online $\ell_p$-GENERALIZED LOAD-BALANCE (GLB$_p$) problem: There are $m$ machines, and $n$ jobs come one-by-one. Each job can be processed in the machines in $k$ different ways, so the $t$-th job has an $m \times k$ matrix $A^t$ with entries in $[0, 1]$ whose column $j$ gives the load $(A^t_{1j}, A^t_{2j}, \ldots, A^t_{mj})$ the machines incur if the job is processed with option $j$. When the $t$-th job arrives, the algorithm needs to select a processing option for it (namely a vector $x^t \in \{0, 1\}^k$ with exactly one 1) based only on the jobs seen thus far, and the goal is to minimize the $\ell_p$-norm of the total load incurred in the machines $\|\sum_{t=1}^n A^t x^t\|_p$, where $\|u\|_p := \left(\sum_i u_i^p\right)^{1/p}$. The performance of the algorithm is compared against the offline optimal solution $\text{Opt} := \min \|\sum_t A^t x^t\|_p$.

This generalizes the fundamental problem of scheduling on unrelated machines to minimize makespan, which corresponds to the case $\ell_{\infty}$ (and diagonal matrices $A^t$’s). The generalization to the $\ell_p$-norm has been studied since the 70’s [CW75, CC76], since in some applications they better capture how well-balanced an allocation is [AAG+95].

Optimal (within constants) guarantees for this problem are well-known. Awerbuch et al. [AAG+95] showed that the greedy algorithm that chooses the processing option that least increases the $\ell_p$-load has $O(p)\cdot \text{Opt}$. They also provided the following matching lower bound (this is a slightly more general statement, but the proof is basically the same; we present it in Appendix A for completeness).

**Theorem 1.1 (Extension of [AAG+95])** Consider GLB$_p$ in the worst-case model. Then for any positive integer $M$, there is an instance where $\text{Opt} = M m^{1/p}$ but any (possibly randomized) online algorithm has expected load at least $\frac{1}{2^{2+\sqrt{7p}/p}} \cdot p M m^{1/p}$.

For the makespan case of $\ell_{\infty}$, one can apply the greedy algorithm with the $\ell_{\log m}$-norm to obtain an optimal $O(\log m)$-approximation (see also [AAF+93]); this uses the fact that $\ell_p$ for large $p$ approximates $\ell_{\infty}$, see Section 1.2. Special cases with improved guarantees [AAS01, AERW04, BCK00, CFK+11, SDJ13], as well as more general version of GLB$_p$ [KKP15, ACP14, BCG+14], have also been studied.

However, despite all these results, the GLB$_p$ problem has been mostly overlooked in non-worse-case models. Such models have received considerable attention recently, since avoiding worst-case instances often allows one to give algorithms with stronger guarantees that can be more representative of the behavior found in practice. A popular non-worse-case model is the random-order (i.e. secretary) model, where in this context the set of jobs is arbitrary but they come one-by-one in uniformly random order (see [Mey01, BIKK08, DH09, KTRV14] for a few examples).

Even better are algorithms that have simultaneously a worst-case guarantee and an improved random-order guarantee. There only seems to be a few examples of such strong guarantees for different problems in the literature, most of them obtained quite recently [Mey01, MGZ12, KMZ15].

Our main contribution is to provide algorithms for the GLB$_p$ problem that attain simultaneously optimal worst-case competitive ratio as well as stronger guarantees in the random-order model (see Table 1). In fact, we provide algorithm SIMULTANEOUSLB that has optimal guarantees (within constants) for both worst-case and random-order models. These are also the first random-order guarantees for this general problem (such results were not known even for the non-generalized load balancing problem where the matrices $A^t$’s are diagonal).

1.1 Our results

**Simultaneous guarantee for the greedy algorithm.** Our first result shows that a small modification of the greedy algorithm, namely restarting it at time $n/2$, maintains optimal $O(p)$-approximation in the worst-case model while having improved approximation guarantee for the random-order model.

**Theorem 1.2** For all $p \in [2, \infty)$, the greedy algorithm with restart GREEDYWR has the following guarantees:

(a) In the worst-case model is $O(p)$-competitive
Moreover, for \( p = \infty \), \textsc{GreedyWR} with \( p = \Theta\left(\frac{\log m}{\varepsilon}\right) \) has worst-case competitive ratio \( O\left(\frac{\log m}{\varepsilon}\right) \) and random-order guarantee \((1 + \varepsilon)\text{Opt} + O\left(\frac{m\log m}{\varepsilon}\right)\).

Note that the lower bound from Theorem 1.1 shows that in the worst-case model no algorithm can have guarantee of the form \( c\text{st} \cdot \text{Opt} + \alpha \) with \( \alpha \) depending only on \( m \) and \( p \), and hence the random-order guarantee of Theorem 1.2 does not hold in the worst-case model. Moreover, typically \( \text{Opt} \) grows with the number of jobs \( n \); in this case, the guarantee becomes \((1 + \varepsilon)\text{Opt} + o(\text{Opt})\), asymptotically giving arbitrarily close approximations, a big improvement over the best possible \( O(\text{Opt}) \) worst-case guarantee.

A main ingredient for proving the random-order guarantee is the optimal modulus of strong smoothness of \( \|\cdot\|^2_p \) proved recently in the context of inequalities for the \( \ell_p \)-norm of random vectors \([LD10]\). Also, as in \([GM14]\), restarting the algorithm reduces the correlations that arise in the random-order model: at each step, the current state now depends on at most \( \frac{m}{2} - 1 \) jobs, so the next job has “enough randomness” for the analysis to go through.

**Improved simultaneous guarantee and Online Linear Optimization.** While the above algorithm typically asymptotically gives arbitrarily close approximations in the random-order model, notice the guarantee degrades as \( p \) increases, as it happens in the worst-case model. The following simple extreme example illustrates this.

**Example 1.3** Consider an instance for \( p = \infty \) with \( m \) machines and \( m \) jobs, with 2 processing options each, where job \( i \)'s processing options have load vectors \((1 - \varepsilon, 1 - \varepsilon, \ldots, 1 - \varepsilon)\), for \( \varepsilon \in (0, 1) \), and \( e^i \) (the \( i \)th canonical vector). It is easy to see that regardless of the order of the jobs, \textsc{GreedyWR} always chooses processing option \((1 - \varepsilon, \ldots, 1 - \varepsilon)\), incurring total load \( m(1 - \varepsilon) \). On the other hand, \( \text{Opt} = 1 \). This gives a \( \Omega(m) \) additive/multiplicative gap even in the random-order model. (This example still holds for finite \( p \gg m, \frac{1}{\varepsilon} \).)

However, we provide a new algorithm, \textsc{SimultaneousLB}, that has simultaneously optimal guarantees (within constants) in both worst-case and random order models. In particular, its random-order guarantee improves with \( p \).

**Theorem 1.4** For all \( p \in [2, \infty) \) algorithm \textsc{SimultaneousLB} has the following guarantees:

(a) In the worst-case model is \( O(p) \)-competitive

(b) In the random-order model has expected load at most \((1 + 4\varepsilon)(\text{Opt} + \frac{6p(m^{1/p} - 1)}{\varepsilon})\) for any \( \varepsilon \in (0, 1] \).

Moreover, in the case \( p = \infty \), \textsc{SimultaneousLB} with \( p = \Theta\left(\frac{\log m}{\varepsilon}\right) \) has worst-case competitive ratio \( O\left(\frac{\log m}{\varepsilon}\right)\text{Opt} \) and random-order guarantee \((1 + \varepsilon)\text{Opt} + O\left(\frac{m\log m}{\varepsilon}\right)\).

The function \( p(m^{1/p} - 1) \) is decreasing in \( p \), hence the random-order bound of algorithm \textsc{SimultaneousLB} is always better (within constants) than that of \textsc{GreedyWR}. Moreover, this function converges to \( \ln m \) as \( p \)

| \( p \in [2, \infty) \) | Algorithm | Worst-case | Random-order |
|---|---|---|---|
| | \textsc{GreedyWR} | \( O(p) \) \[AAG'95\] | \( (1 + \varepsilon)\text{Opt} + O\left(\frac{pm^{1-1/p}}{\varepsilon}\right) \) |
| | \textsc{SimultaneousLB} | \( O(p) \) | \( (1 + \varepsilon)\text{Opt} + O\left(\frac{p(m^{1/p} - 1)}{\varepsilon}\right) \) |
| \( p = \infty \) | \textsc{GreedyWR}, using \( p \approx \frac{\log m}{\varepsilon} \) | \( O\left(\frac{\log m}{\varepsilon}\right) \) \[AAG'95\] | \( (1 + \varepsilon)\text{Opt} + O\left(\frac{m\log m}{\varepsilon}\right) \) |
| | \textsc{ExpertLB} \[GM14\] | \( O\left(\frac{\log m}{\varepsilon}\right) \) | \( (1 + \varepsilon)\text{Opt} + O\left(\frac{m\log m}{\varepsilon}\right) \) |
| | \textsc{SimultaneousLB}, using \( p \approx \frac{\log m}{\varepsilon} \) | \( O\left(\frac{\log m}{\varepsilon}\right) \) | \( (1 + \varepsilon)\text{Opt} + O\left(\frac{m\log m}{\varepsilon}\right) \) |

Table 1: Worst-case competitive ratio and random-order guarantees for \( \ell_p \)-\textsc{Generalized Load-Balance}. New results are shown in bold.

\(^1\)Recall that we have assume all loads to be in the interval \([0, 1]\); in general, the additive term in this expression scales with max load.
goes to infinity $\text{[Wik16]}$. Thus, this guarantee matches the only known result for $\text{GLB}_p$ in the random-order model, the $(1 + \varepsilon)\text{Opt} + O(\log m/\varepsilon)$ bound given for the special case $p = \infty$ in $\text{[GM14]}$. Moreover, setting in hindsight the approximately optimal value $\varepsilon = \sqrt{\frac{p(m^{1/p}-1)}{\text{Opt}}}$ shows that $\text{SIMULTANEOUSLB}$’s solution has load at most $\text{Opt} + O(\sqrt{\text{Opt} \cdot pm^{1/p}})$. The following result, whose proof is presented in Appendix $\text{[B]}$, shows that this guarantee cannot be significantly improved.

**Theorem 1.5** For every even $p \geq 2$, there is an instance of $\text{GLB}_p$ with $m = 2^p$ and $\text{Opt} = \frac{pm^{1/p}}{2}$ such that any algorithm incurs expected total load at least $\text{Opt} + \text{cst} \cdot \sqrt{\text{Opt} \cdot pm^{1/p}}$ for constant $\text{cst} = 1/(100\sqrt{2})$.

The main idea behind algorithm $\text{SIMULTANEOUSLB}$, or more precisely its precursor $\text{SMOOTHGREEDY}$, is that we can see it simultaneously as approximately greedy and as an approximately primal-dual algorithm; from the “greedy” part we get the worst-case guarantee, and from the “primal-dual” part the random-order guarantee. Moreover, in the approximately primal-dual view, the dual variables are set according to a new algorithm for Online Linear Optimization (OLO) over the non-negative vectors in the $\ell_q$-ball (the dual of the $\ell_p$-ball). In this game, in each round the player needs to choose a non-negative vector $v^t \in \mathbb{R}^m$ with $\ell_q$-norm at most 1, and then the adversary chooses a non-negative vector $w^t \in [0, 1]^m$, giving reward $\langle v^t, w^t \rangle$ to the algorithm. The goal of the algorithm is to maximize the sum of the rewards obtained. As usual, the reward of the algorithm is measured against the optimal fixed solution $v^*$ in hindsight. This is a generalization of classical Prediction with Experts Problem $\text{[EBL06]}$, which corresponds to the case $q = 1$.

A general connection between guarantees in the random-order (or the weaker i.i.d) model and OLO games has been recently shown in $\text{[GM14, AD13]}$. However, a crucial point is that since we simultaneously want worst-case guarantee as well, it is not clear that we can employ an OLO algorithm in a black-box fashion. Interestingly, our new OLO algorithm has better regret than what is available in the literature, which is needed for the optimal random-order guarantee of $\text{SIMULTANEOUSLB}$. We are interested in OLO algorithms with multiplicative/additive regret of the form $\text{Algo} \geq (1 - \varepsilon)\text{OptFixed} - R$, where $\text{OptFixed}$ denotes the reward of the best fixed solution in hindsight. To the best of our knowledge, the best such bound for this OLO game is $\text{Algo} \geq (1 - \varepsilon)\text{OptFixed} - O(\frac{p^{1/p} \log m}{\varepsilon})$, obtained in the seminal paper of Kalai and Vempala $\text{[KV05]}$. Our OLO algorithm has regret $\text{Algo} \geq (1 - \varepsilon)\text{OptFixed} - O(\frac{p^{1/p} \log m}{\varepsilon^2})$, see Theorem $\text{2.2}$; this gives a log $m$ factor reduction in the additive term for small $p$, and dominates the Kalai-Vempala bound for all $p$.

Another interesting connection is that we use our OLO algorithm to prove a purely probabilistic inequality (Lemma $\text{3.1}$) that controls the correlations arising in the random-order model, a common source of difficulty for the analysis in this model. In $\text{[GM14]}$, such control was obtained via a maximal inequality and union bound for the special case of the $\ell_\infty$-norm. However, for general $\ell_p$-norms a straightforward union bound gives a weaker bound than the OLO-based approach, leading to suboptimal guarantees.

An important component used in both $\text{SIMULTANEOUSLB}$ and our OLO algorithm is a smoothened version $\psi_{\varepsilon, \delta}$ of the $\ell_p$-norm; in particular, this is what allows us to see $\text{SIMULTANEOUSLB}$ simultaneously as both an approximately greedy algorithm and an approximately primal-dual algorithm, as mentioned before. This smoothened function can be seen as a generalization of the exp-sum function $\expSum(u) = \frac{1}{\varepsilon} \ln \sum_i e^{\varepsilon u_i}$, a much used smoothing of the $\ell_\infty$-norm. Given the host of applications of exp-sum, we hope that the smoothings $\psi_{\varepsilon, \delta}$ will find use in other contexts.

### 1.2 Roadmap and notation

In Section $\text{3}$ we present and analyze our OLO algorithm $\text{SMOOTHBASELINE}$, and also define the smoothing $\psi_{\varepsilon, \delta}$ used throughout. In the following section, we use this OLO result to prove the correlation inequality that is needed for the random-order analyses of all the algorithms considered. In Section $\text{4}$, we analyze the greedy algorithm with restart $\text{GREEDYWR}$. In Section $\text{5}$ we present algorithm $\text{SMOOTHGREEDY}$, which has improved
random-order guarantee but has a spurious term in its worst-case guarantee. Finally, in Section 6 we combine this algorithm with the greedy one to remove this spurious term, obtaining algorithm SIMULTANEOUSLB.

We now define some notation. We use $\ell_q^+$ to denote the set of non-negative vectors $\mathbb{R}^m_+$ with $\ell_q$ norm at most 1. Given $p \in (1, \infty)$, its Hölder conjugate $q$ is the number that satisfies $\frac{1}{p} + \frac{1}{q} = 1$. It follows from norm duality that if $p$ and $q$ are Hölder conjugate, then for every vector $x \geq 0$

$$\forall y \in \ell_q^+ \quad \langle x, y \rangle \leq \| x \|_p, \quad \text{and} \quad \| x \|_p = \max_{y \in \ell_q^+} \langle x, y \rangle.$$  

(1.1)

Also, we will use the well-known comparison between norms: if $p \geq p'$, then for every vector $x \in \mathbb{R}^m$ we have $\| x \|_p \leq \| x \|_{p'} \leq m^{\frac{1}{p} - \frac{1}{p'}} \| x \|_{p'}$. Finally, we use bold letters for random variables.

2 The $\ell_q^+$ OLO problem and the Smoothened Baseline Gradient algorithm

$\ell_q^+$ OLO problem. Recall that the $\ell_q^+$ OLO problem proceeds in $n$ rounds. In round $t$, first the algorithm chooses a vector $v^t \in \ell_q^+$ based on the adversary’s previous vectors $w^1, \ldots, w^{t-1}$. Then the adversary chooses a vector $w^t \in [0, 1]^m$, and the algorithm obtains reward $\langle w^t, v^t \rangle$. The goal of the algorithm is to maximize the sum of the rewards $\sum_{t=1}^n \langle w^t, v^t \rangle$. The regret of the algorithm is obtained by comparing against the best fixed decision $v \in \ell_q^+$ in hindsight. We say that an algorithm has $(\varepsilon, R)$-regret if

$$\sum_{t=1}^n \langle w^t, v^t \rangle \geq e^{-\varepsilon} \left( \max_{v \in \ell_q^+} \sum_{t=1}^n \langle w^t, v \rangle - R \right).$$  

(2.2)

Recall that $e^{\pm \varepsilon}$ is approximately $(1 \pm \varepsilon)$ for small values of $\varepsilon$.

Smoothened Baseline Gradient Algorithm. To obtain an intuition about algorithms for this problem, we can see the right-hand side of the regret expression in a different way. Let $p$ be the Hölder conjugate of $q$. Then duality of norms (equation (1.1)) gives that $\max_{v \in \ell_q^+} \sum_t \langle w^t, v \rangle = \| \sum_t w^t \|_p$, hence the regret expression becomes $\sum_t \langle w^t, v^t \rangle \geq e^{-\varepsilon} (\| \sum_t w^t \|_p - R)$. Thus, we can interpret the algorithm’s decision $v^t$ as trying to locally approximate the baseline potential $\| \|_p$ at $\sum_{t=1}^n v^t$ to capture the increase in norm caused by the unknown $w^t$.

Thus, a natural strategy is to choose $v^t$ as a (sub-)gradient $\nabla \| u \|_{\ell_q^+}$ belonging to $\ell_q^+$. However, one can show that this strategy has too high regret. The issue is that the gradient can quickly vary from one point to another, so approximating the value $\| u + v \|_p = \| u \|_p + \int_0^1 \nabla \| u + xv \|_p \, dx$ by the first order expression $\| u \|_p + \langle \nabla \| u \|_p, v \rangle$ is not good enough. To avoid this problem, we will replace the norm $\| \cdot \|_p$ by a smoother function $\psi$ satisfying the following:

(a) (additive error) For all $u \in \mathbb{R}^m_+$, $\| u \|_p \leq \psi(u) \leq \| u \|_p + R$  

(2.3)

(b) (stability) For all $u \in \mathbb{R}^m_+$ and $v \in [0, 1]^m$, $\nabla \psi(u + v) \ |_{\text{pointwise}} = e^{\pm \varepsilon} \cdot \nabla \psi(u)$  

(2.4)

To obtain such smoothing, we notice that $\| \cdot \|_p$ is a generalized $f$-mean, namely $\| w \|_p = f^{-1}(\sum_i f(w_i))$ for the function $f(x) = x^p$. We then define the smoothened function

$$\psi_{\varepsilon, p}(u) = f_{\varepsilon, p}^{-1} \left( \sum_i f_{\varepsilon, p}(u_i) \right), \quad \text{where} \quad f_{\varepsilon, p}(x) = \left( 1 + \frac{\varepsilon x}{p} \right)^p,$$  

(2.5)

which written explicitly is $\psi_{\varepsilon, p}(u) = \frac{L}{p} \ln | 1 + \varepsilon u^p | - \frac{L}{p} u$. Notice that as $p$ goes to infinity, $f_{\varepsilon, p}(x)$ converges to $e^{\varepsilon x}$, and so $\psi_{\varepsilon, p}$ converges to the exp-sum function $\text{ExpSum}(w) = \frac{1}{p} \ln \sum_i e^{\varepsilon w_i}$, a commonly used smoothing of $\ell_\infty$.

One of the main properties that motivate our definition of $f_{\varepsilon, p}(x) = (1 + \frac{\varepsilon x}{p})^p$ is that its derivative is much more stable than that of $x^p$ for $\pm 1$ perturbations: for example, $(x^p)'(0) = 0$ but $(x^p)'(1) = 1$, while $f_{\varepsilon, p}'(0) = \varepsilon$.
and \( f^{\varepsilon}_P(1) = \varepsilon (1 + \frac{\varepsilon}{p})^{p-1} \leq \varepsilon \varepsilon \leq \varepsilon (1 + \varepsilon) \). Such functions are also used for obtaining sharp estimates of moments of sums of random variables (see Section 1.5 of [PnG99]).

Once we have the “right” definition of the smoothed function \( \psi_{\varepsilon, P} \), it is not hard to prove that it satisfies properties (2.3)-(2.4); the proof is found in Appendix C.

**Lemma 2.1** Function \( \psi_{\varepsilon, P} \) satisfies properties (2.3)-(2.4) with \( R = \frac{p(m^{1/p} - 1)}{\varepsilon} \).

Now we formally state the \( \psi \)-based SMOOTHBASELINE algorithm for the \( \ell^+_q \) OLO problem.

**Algorithm 2.1 SMOOTHBASELINE**

Let \( p \) be such that \( \frac{1}{p} + \frac{1}{q} = 1 \), define \( w^0 = 0 \).

**for each time** \( t \) **do**

- Play vector \( v^t \triangleq \nabla \psi_{\varepsilon, P}(w^1 + \ldots + w^{t-1}) \)
- Observe adversarial vector \( w^t \in [0, 1]^p \)

We show that this algorithm indeed outputs a solution to the \( \ell^+_q \) OLO problem (i.e. \( v^t \in \ell^+_q \)) with low regret.

**Theorem 2.2** For every \( p \in (1, \infty) \) the SMOOTHBASELINE algorithm outputs a solution to the \( \ell^+_q \) OLO problem with \( (\varepsilon, \frac{p(m^{1/p} - 1)}{\varepsilon}) \)-regret.

**Proof.** The fact that the actions \( v^t \) played belong to \( \ell^+_q \) follows directly from the expression of the gradient \( \nabla \psi_{\varepsilon, P} \) (see equation (C.12) in the appendix, and notice \( q = \frac{p}{p-1} \)). So we just bound the regret of the algorithm; to simplify the notation we drop the subscripts from \( \psi_{\varepsilon, P} \) and use \( s^t = w^1 + \ldots + w^t \).

We need to show \( \varepsilon \cdot \sum_{t=1}^n \langle w^t, v^t \rangle \geq \|s^n\| - \frac{p(m^{1/p} - 1)}{\varepsilon} \). The main idea is to relate the value obtained by the algorithm to the smoothed function \( \psi \), showing

\[
\varepsilon \cdot \sum_{t=1}^n \langle w^t, v^t \rangle \geq \psi(s^n) - \psi(0), \tag{2.6}
\]

First, the convexity of \( \|s\|_p \) directly implies that \( \psi \) is convex. Thus, for every time step \( t \) we have \( \psi(s^{t-1}) \geq \psi(s^t) + \langle \nabla \psi(s^t), -w^t \rangle \), or equivalently \( \psi(s^t) - \psi(s^{t-1}) \leq \langle \nabla \psi(s^t), w^t \rangle \). Since Lemma 2.1 guarantees that \( \psi \) satisfies the gradient stability property (2.4), we can upper bound the right-hand side of this expression to obtain \( \psi(s^t) - \psi(s^{t-1}) \leq \varepsilon \langle \nabla \psi(s^t), w^t \rangle = \varepsilon \langle v^t, w^t \rangle \). Adding over all \( t \)'s then gives inequality (2.6).

From Lemma 2.1 we have the comparison \( \psi(s^n) \geq \|s^n\|_p \), and notice that \( \psi(0) = \frac{p(m^{1/p} - 1)}{\varepsilon} \); employing these observation to inequality (2.6) gives \( \varepsilon \cdot \sum_{t=1}^n \langle w^t, v^t \rangle \geq \|s^n\|_p - \frac{p(m^{1/p} - 1)}{\varepsilon} \), thus concluding the proof.

We remark that the idea of using the gradient of a smoothed baseline to obtain a low regret OLO algorithm was already used in [ALST14]. However, our notion of smoothness is different from the ones they used (partially because we are interested in multiplicative/additive regret), and their results cannot be directly applied to obtain the regret of Theorem 2.6.

### 3 Handling Correlations of the Random-order Model

Informally, one of the difficulties of analyzing algorithms in the random-order model is that, unlike in the i.i.d. model, there are correlations between jobs in different time steps because they are being sampled without replacement from the underlying collection of jobs. In this section we control the correlations of vectors in the random-order model, which will be crucial for analyzing algorithms for the GLB \( \ell^+_p \) problem. Interestingly, we use the OLO algorithm SMOOTHBASELINE to prove this purely probabilistic inequality (see [RS15] for another connection between OLO algorithms and martingale concentration inequalities).
Lemma 3.1  Consider a set of vectors \( \{y_1, \ldots, y^n\} \in [0, 1]^m \) and let \( Y^1, \ldots, Y^t \) be sampled without replacement from this set. Let \( Z \) be a random vector in \( \ell_q^+ \) that depends only on \( Y^1, \ldots, Y^{t-1} \). Then for all \( \varepsilon > 0 \),

\[
\mathbb{E}(Y^t, Z) \leq e^\varepsilon \|\mathbb{E}Y^t\|_p + \frac{1}{n - (t - 1)} \cdot \frac{p(m^{1/p} - 1)}{\varepsilon}.
\]

(Recall \( \mathbb{E}Y^t \) denotes the vector obtained by taking component-wise expectation.) To understand the meaning of this lemma, notice that if \( z \) is a fixed vector in \( \ell_q^+ \) (or simply independent of \( Y^t \)), then \( \mathbb{E}\langle Y^t, z \rangle = \langle \mathbb{E}Y^t, z \rangle \leq \|\mathbb{E}Y^t\|_p \). On the other hand, if \( Z \) is highly correlated to \( Y^t \), say \( Z = \frac{Y^t}{\|Y^t\|_q} \), then we only have \( \mathbb{E}(Y^t, Z) = \mathbb{E}\|Y^t\|_p \), which in general can be arbitrarily larger than \( \|\mathbb{E}Y^t\|_p \) (e.g. if \( \mathbb{E}Y^t = 0 \)).

The main element for proving Lemma 3.1 is to show that because \( Y^t \) is bounded and non-negative, actually \( \mathbb{E}\|Y^t\|_p \approx \|\mathbb{E}Y^t\|_p \); more precisely, we show \( \mathbb{E}\|Y^1 + \ldots + Y^n\|_p \leq \|\mathbb{E}Y^1 + \ldots + \mathbb{E}Y^n\|_p \). This was proved in [GM14] for the special case \( p = \infty \) using a maximal inequality, but can also be proved using Bernstein’s inequality to obtain concentration for each coordinate of the sum \( Y^1 + \ldots + Y^n \), taking a union bound to obtain concentration of the norm \( \|Y^1 + \ldots + Y^n\|_p \), and then integrating its tail. However, for general \( p \) the union bound is loose and bound obtained has an extra \( \log m \) factor. We use the OLO algorithm SMOOTHBASELINE and Hoeffding’s Comparison Lemma to quickly provide a bound without such extra factor.

Lemma 3.2  Consider a set of vectors \( \{y_1, \ldots, y^n\} \in [0, 1]^m \) and let \( Y^1, \ldots, Y^\kappa \) be sampled without replacement from this set. Then for all \( \varepsilon > 0 \)

\[
\mathbb{E}\|Y^1 + \ldots + Y^\kappa\|_p \leq e^\varepsilon \|\mathbb{E}Y^1 + \ldots + \mathbb{E}Y^\kappa\|_p + \frac{p(m^{1/p} - 1)}{\varepsilon}.
\]

Proof. We show it suffices to prove the result for i.i.d vectors. Let \( Y^1, \ldots, Y^\kappa \) be i.i.d sampled with replacement from \( \{y_1, \ldots, y^n\} \). Then \( \tilde{Y}^t \) has the same expectation as \( Y^t \) and hence \( \|\mathbb{E}Y^1 + \ldots + \mathbb{E}Y^n\|_p = \|\mathbb{E}Y^1 + \ldots + \mathbb{E}\tilde{Y}^\kappa\|_p \). Moreover, Hoeffding’s Comparison Lemma [Hoe63, GM10] gives that for every continuous convex function \( f, \mathbb{E}f(Y^1 + \ldots + Y^n) \leq \mathbb{E}f(\tilde{Y}^1 + \ldots + \tilde{Y}^\kappa) \); since the norm \( \|\cdot\|_p \) satisfies these properties, \( \mathbb{E}\|Y^1 + \ldots + Y^n\|_p \leq \mathbb{E}\|\tilde{Y}^1 + \ldots + \tilde{Y}^\kappa\|_p \). Thus, it suffices to prove the lemma for the i.i.d variables.

For that, run the OLO algorithm SMOOTHBASELINE over the input sequence \( \tilde{Y}^1, \ldots, \tilde{Y}^\kappa \), letting \( Z^1, \ldots, Z^\kappa \) be the vectors played by the algorithm. Using the guarantee of this algorithm (Theorem 2.2) for every scenario and taking expectations, we have

\[
\sum_{t \leq \kappa} \mathbb{E}\langle \tilde{Y}^t, Z^t \rangle \geq e^{-\varepsilon} \left( \mathbb{E}\|\sum_{t \leq \kappa} \tilde{Y}^t\|_p - \frac{p(m^{1/p} - 1)}{\varepsilon} \right).
\]

(3.7)

Since \( Z^t \) only depends on \( \tilde{Y}^1, \ldots, \tilde{Y}^{t-1} \) and \( \tilde{Y}^t \) is independent from these variables, we have \( \mathbb{E}\langle \tilde{Y}^t, Z^t \rangle = \langle \mathbb{E}\tilde{Y}^t, \mathbb{E}Z^t \rangle \). Moreover, the \( \tilde{Y}^t \)'s are identical, so their expectations equal \( \frac{1}{n} \) of \( \mu := \mathbb{E}\tilde{Y}^1 + \ldots + \mathbb{E}\tilde{Y}^\kappa \), hence

\[
\sum_{t \leq \kappa} \mathbb{E}\langle \tilde{Y}^t, Z^t \rangle = \sum_{t \leq \kappa} \left( \frac{1}{n} \mathbb{E}Z^t \right) = \left( \frac{1}{n} \right) \sum_{t \leq \kappa} \mathbb{E}Z^t \leq \|\mu\|_p,
\]

where the last inequality follows from the fact that \( \frac{1}{n} \sum_{t \leq \kappa} Z^t \|_q \leq 1 \) (since each \( \|Z^t\|_q \leq 1 \) and \( \|\cdot\|_q \) is convex) and inequality (1.1). Employing this on inequality (3.7) gives the desired inequality and concludes the proof.

Proof. [of Lemma 3.1] Let \( E_{t-1} \) denote the expectation conditioned on \( Y^1, \ldots, Y^{t-1} \). We break the expectation as \( \mathbb{E}\langle Y^t, Z \rangle = \mathbb{E} E_{t-1} \langle Y^t, Z \rangle \). Again, since \( Z \) is determined by \( Y^1, \ldots, Y^{t-1} \) and belongs to \( \ell_q^+ \), we get \( \mathbb{E}_{t-1} \langle Y^t, Z \rangle = \langle \mathbb{E}_{t-1} Y^t, Z \rangle \leq \|\mathbb{E}_{t-1} Y^t\|_p \); thus it suffices to upper bound this quantity in expectation.
Let $\mu = \frac{1}{n} \sum_{i \neq t} y_i'$ be the average of the vectors. Since $Y^t$ is uniformly sampled from the vectors that have not appeared in the samples $Y^1, \ldots, Y^{t-1}$, we have that its conditional expectation is

$$\mathbb{E}_{t-1} Y^t = \frac{n \mu - (Y^1 + \ldots + Y^{t-1})}{n - (t - 1)}.$$  

Moreover, notice that $n \mu - (Y^1 + \ldots + Y^{t-1})$ (i.e., the sum of the remaining $n - (t - 1)$ vectors) has the same distribution as $Y^1 + \ldots + Y^{n-(t-1)}$, so $\mathbb{E}_{t-1} Y^t$ has the same distribution as $\frac{Y^1 + \ldots + Y^{n-(t-1)}}{n-(t-1)}$. Then using Lemma 3.2, we can upper bound the expected value of $\|\mathbb{E}_{t-1} Y^t\|$ as

$$\mathbb{E} \|\mathbb{E}_{t-1} Y^t\|_p \leq \frac{1}{n - (t - 1)} \left[ e^\varepsilon \|(n - (t - 1))\|_p + \frac{p(m^{1/p} - 1)}{\varepsilon} \right]$$  

for all $\varepsilon \in (0, 1]$. Reorganizing this expression concludes the proof.

\section{Greedy algorithm for the GLB\textsubscript{p} problem}

Now we return to our main problem of interest, the $\ell_p$-\textsc{Generalized Load-Balance} problem, defined in the introduction. In this section we consider the greedy algorithm with restart at time $n/2$, which can be more formally described as follows:

\textbf{Algorithm 4.1 GreedyWR}

\begin{algorithmic}
  \FOR{$t = 1, \ldots, \frac{n}{2}$}
    \STATE Select $\bar{x}_t$ to minimize the load $\|\sum_{t=1}^t A^T \bar{x}_t\|_p$
  \ENDFOR
  \FOR{$t = \frac{n}{2} + 1, \ldots, n$}
    \STATE Select $\bar{x}_t$ to minimize the load $\|\sum_{t=\frac{n}{2}+1}^t A^T \bar{x}_t\|_p$
  \ENDFOR
\end{algorithmic}

Also recall that Theorem 1.2 presented in the introduction states the worst-case and random-order guarantees of this algorithm; in the remainder of this section we prove this theorem.

Since the greedy algorithm without restart is $O(p)$-competitive in the worst-case, it is straightforward to show that GreedyWR also inherits this guarantee: by triangle inequality, the load of the algorithm is $\|\sum_{t} A^T \bar{x}_t\|_p \leq \|\sum_{t \leq n/2} A^T \bar{x}_t\|_p + \|\sum_{t > n/2} A^T \bar{x}_t\|_p$; but these terms are respectively at most $O(p)$ times the optimal load for the first and second half of the instance, each of which is at most Opt, thus concluding the argument. Therefore, it suffices to analyze the random-order behavior of the algorithm, proving part (b) of the theorem.

So we use $A^t$ to denote the random matrix that arrives at time $t$ and $\bar{x}_t$ to denote the random fractional assignment output by GreedyWR. Also let $\bar{x}^t$ be the optimal offline decision for time $t$.

Because of the restart, and random order, the load vectors obtained by GreedyWR in the first and second half of the process, namely $\sum_{t \leq n/2} A^t \bar{x}_t$ and $\sum_{t > n/2} A^t \bar{x}_t$, have the same distribution. Again due to triangle inequality, it then suffices to analyze the first half and show that

$$\mathbb{E} \left\| \sum_{t \leq n/2} A^t \bar{x}_t \right\|_p \leq \left( \frac{e^\varepsilon}{2} + \varepsilon \right) \text{Opt} + m^{1/p} \left( \frac{p-1}{2e} + \frac{p m^{1/p} - 1}{\varepsilon} \right);$$

\begin{equation} \tag{4.8} \end{equation}

notice that this implies the bound in Theorem 1.3 because $e^\varepsilon \leq 1 + 2 \varepsilon$ for $\varepsilon \in [0, 1]$ and $m^{1/p} \leq m^{1-1/p}$ for $p \geq 2$. To simplify the notation, let $S' = \sum_{t' \leq t} A^{t'} \bar{x}_{t'}$ be the random load vector of GreedyWR up to time $t$.\footnote{More formally, let $(\bar{x}^t)_t$ be an optimal solution for the offline instance, and let $\sigma$ be the random permutation of $[n]$ such that $A^t = A^{\sigma(t)}$; then define $\bar{x}^t := \bar{x}^{\sigma(t)}$.}
The main tool for analyzing the load increments $\|S^t\|_p - \|S^{t-1}\|_p$ incurred by the algorithm is the following estimate for the $\ell_p$-norm. One of its crucial features is that it shows that the linearization of the $\ell_p$-norm is increasingly better as we move away from the origin. It is a quick corollary of the optimal modulus of strong smoothness of the square of the $\ell_p$-norm recently proved in [LD10], and is proved in Appendix B.

**Lemma 4.1** Consider $p \in [2, \infty)$ and let $q$ be its Hölder conjugate. Then for every non-negative vectors $u \in \mathbb{R}^n_+ \setminus \{0\}$ and $v \in \mathbb{R}^n_+$, there is a vector $g(u) \in \mathbb{R}^n_+$ with $\|g(u)\|_q \leq 1$ such that

$$\|u + v\|_p \leq \|u\|_p + \|g(u), v\| + \frac{(p-1)\|v\|_p^2}{2\|u\|_p^p}. $$

Now we analyze algorithm GREEDYWR. We handle separately the initial time steps where the load is small, so define the stopping time $\tau = \min\{t \leq n/2 : \|S^t\|_p > \varepsilon \text{Opt}\}$ (set $\tau = n/2$ for the scenarios with $\|S^{n/2}\|_p \leq \varepsilon \text{Opt}$), load of the algorithm up to time $n/2$ can be written as $\|S^{n/2}\|_p = \|S^\tau\|_p + \sum_{t=\tau+1}^{n/2} (\|S^t\|_p - \|S^{t-1}\|_p)$. From the greedy property we have the load $\|S^t\|_p$ is at most $\|S^{t-1}\|_p + \hat{\ell}^t_p$, where $\hat{\ell}^t = A^t \hat{x}^t$ is the load incurred by Opt at time $t$. Thus, employing the estimate from Lemma 4.1 we get

$$\|S^{n/2}\|_p \leq \|S^\tau\|_p + \sum_{t=\tau+1}^{n/2} \langle g(S^{t-1}), \hat{\ell}^t \rangle + \sum_{t=\tau+1}^{n/2} \frac{(p-1)\|\hat{\ell}^t\|_p^2}{2\|S^{t-1}\|_p}. $$

We upper bound each term of the right-hand side separately.

**First term of RHS of (4.9).** Since $\tau$ is the first time $\|S^\tau\|_p$ goes above $\varepsilon \text{Opt}$, and the load does not increase by more than $m^{1/p}$ per time step (which uses the fact that the entries of the matrices $A^t$ are in $[0, 1]$), we have $\|S^\tau\|_p \leq \|S^{t-1}\|_p + \|A^\tau \hat{x}^\tau\|_p \leq \varepsilon \text{Opt} + m^{1/p}$.

**Last term of RHS of (4.9).** First notice that since we are only adding terms after the stopping time $\tau$, each denominator will be at least $2\varepsilon \text{Opt}$; so we have the upper bound $\frac{(p-1)}{2\varepsilon \text{Opt}} \sum_{t=\tau+1}^{n/2} \|\hat{\ell}^t\|_p^2$. To bound this remaining sum, we will linearize it by passing to the $\ell_1$ norm and them back to $\ell_p$: Since all entries of the load vector $\hat{\ell}$ are in $[0, 1]$, we have that $\|\hat{\ell}^t\|_p \leq \|\hat{\ell}^t\|_1$:

$$\|\hat{\ell}^t\|_p^2 = \left( \sum_i (\hat{\ell}^t_i)^p \right)^{2/p} \leq \left( \sum_i (\hat{\ell}^t_i)^{p/2} \right)^{2/p} \leq \|\hat{\ell}^t\|_1^2. $$

Moreover, the non-negativity of these vectors give additivity for $\|\cdot\|_1$, namely $\sum_{t \leq n/2} \|\hat{\ell}^t\|_1 = \sum_{t \leq n/2} \|\hat{\ell}^t\|_1$. Finally, by comparison of norms this is at most $m^{1-1/p} \sum_{t \leq n/2} \|\hat{\ell}^t\|_p$. Thus, the last term of (4.9) can be upper bounded by $\frac{(p-1)m^{1-1/p}}{2\varepsilon \text{Opt}} \|\sum_{t \leq n/2} \|\hat{\ell}^t\|_p \leq \frac{(p-1)m^{1-1/p}}{2\varepsilon \text{Opt}}$.

**Second term of RHS of (4.9).** This is the main term in the RHS of (4.9), and we need to show that in expectation it is about at most $\varepsilon \text{Opt}$; this is the only place we use the random-order model and that the algorithm restarts at $n/2$. Since $g(S^{t-1})$ only depends on items seen up to time $t-1$, we can employ Lemma 3.3 to obtain that $\mathbb{E}(g(S^{t-1}), \hat{\ell}^t) \leq \frac{\varepsilon \|\hat{\ell}^t\|_p + \frac{1}{n} \frac{pm^{1-p-1}}{n(t-1)} \|\hat{\ell}^t\|_p}{\varepsilon \text{Opt}}$. Moreover, notice that Opt’s expected load $\mathbb{E}\hat{\ell}^t$ is the same in every time step, and so $\mathbb{E}\|\hat{\ell}^t\|_p = \frac{1}{n} \mathbb{E} \sum_{t=1}^n \|\hat{\ell}^t\|_p = \frac{1}{n} \varepsilon \text{Opt}$. Since we are only considering $t \leq n/2$, our expression can be further bounded as $\mathbb{E}(g(S^{t-1}), \hat{\ell}^t) \leq \frac{\varepsilon \|\hat{\ell}^t\|_p + \frac{2p}{\varepsilon n} \frac{m^{1-p-1}}{n(t-1)}}{\varepsilon \text{Opt}}$. Adding over all these time steps we get $\mathbb{E} \sum_{t=\tau+1}^{n/2} \langle g(S^{t-1}), \hat{\ell}^t \rangle \leq \frac{\varepsilon \|\hat{\ell}^t\|_p + \frac{2p}{\varepsilon n} \frac{m^{1-p-1}}{n(t-1)}}{\varepsilon \text{Opt}}$.

Employing all these bound in inequality (4.9) proves (4.8). This concludes the proof of Theorem 1.2.
5 Towards improved simultaneous guarantees: algorithm SMOOTHGREEDY

We now present the algorithm SMOOTHGREEDY that has improved random-order guarantee at the expense of a slightly suboptimal worst-case guarantee.

**Algorithm 5.1 SMOOTHGREEDY** $(p, \varepsilon)$

Let $\psi_{\varepsilon,p}(u) = \frac{\varepsilon}{p} \|1 + \frac{u}{p}\|_p - \frac{\varepsilon}{p}$ (as in equation (2.5)).

for time $t = 1, \ldots, \frac{n}{2}$ do
  Select $\bar{x}^t \in \Delta^m$ to minimize $\psi_{\varepsilon,p}(\sum_{t=1}^{\tau} A^T \bar{x}^t)$

for time $t = \frac{n}{2} + 1, \ldots, n$ do
  Select $\bar{x}^t \in \Delta^m$ to minimize $\psi_{\varepsilon,p}(\sum_{t=\tau+1}^{n} A^T \bar{x}^t)$

The motivation behind this algorithm is the following: First, since this is simply GREEDYWR on the modified function $\psi_{\varepsilon,p}(\cdot) = \frac{\varepsilon}{p} \|1 + \frac{\cdot}{p}\|_p - \frac{\varepsilon}{p}$, it is intuitive that it approximately inherits the worst-case guarantee of GREEDYWR. On the other hand, the smoothness of $\psi_{\varepsilon,p}$ (equation (2.4)), guarantees that its gradient captures well its behavior, so SMOOTHGREEDY is almost greedy on this gradient; this allow us to connect the algorithm with the SMOOTHBASELINE OLO algorithm to provide guarantees in the random-order model. Here is the formal guarantees of SMOOTHGREEDY.

**Theorem 5.1** For all $p \in [2, \infty)$ algorithm SMOOTHGREEDY has the following guarantees for any $\varepsilon > 0$

(a) In the worst-case model it has load at most $O(p) \text{Opt} + \frac{4p(m^{1/p} - 1)}{\varepsilon}$

(b) In the random-order model has expected load at most $e^{2p} \left( \text{Opt} + \frac{4p(m^{1/p} - 1)}{\varepsilon} \right)$.

**Analysis in the worst-case model.** We prove part (a) of Theorem 5.1, where the idea is connect with GREEDYWR by adding extra jobs to the input.

Assume the number of machines $m \geq 2$, otherwise it is easy to see that SMOOTHGREEDY is optimal. Consider an arbitrary input sequence $A^1, \ldots, A^n$ for SMOOTHGREEDY. Define $B^1 = \ldots = B^w$ to be the all 1’s $m \times k$ matrix, for $w = \frac{n}{2}$ (we assume for simplicity that $w$ is integral). Since $\psi_{\varepsilon,p}(u) = \|\frac{p}{\varepsilon} 1 + u\|_p - \frac{p}{\varepsilon} = \|(B^1 y^1 + \ldots + B^w y^w) + u\|_p - \frac{p}{\varepsilon}$ for any $y^t$’s, it is easy to see that behavior of SMOOTHGREEDY up to time $n/2$ is the same as that of GREEDY (without restart, over norm $\|\cdot\|_p$) on input $B^1, \ldots, B^w, A^1, \ldots, A^{\frac{n}{2}}$, and the same for time periods $n/2, \ldots, n$. Now one can directly employ the standard $O(p)$-guarantee for GREEDY to get that SMOOTHGREEDY has load at most $O(p) \left( \text{Opt} + \frac{p(m^{1/p} - 1)}{\varepsilon} \right)$; however this leads to an additive error that is quadratic in $p$. To obtain an improved bound, we use the following more refined guarantee for GREEDY, which can be obtained from the analysis in [Car08] (we present a proof in Appendix B).

**Lemma 5.2** Consider an arbitrary sequence of jobs $C^1, \ldots, C^n$, let $\{\bar{x}^t\}_t$ be the actions output by GREEDY over $\|\cdot\|_p$ and let $\{\bar{x}^t\}_t$ be the optimal solution. Then for all $\tau$

$$\left\| \sum_{t=\tau}^{n} C^t \bar{x}^t \right\|_p \leq O(p) \left( \sum_{t=\tau}^{n} C^t \bar{x}^t \right)$$

Then let $S^t$ be the total load vector obtained by SMOOTHGREEDY up to time $t$. Using triangle inequality and then Lemma 2.3 we decompose the load of the algorithm $\|S^n\|_p \leq \|S^{\tau}\|_p + \|S^n - S^{\tau}\|_p \leq \psi_{\varepsilon,p}(S^{\tau}) + \psi_{\varepsilon,p}(S^n - S^{\tau})$. To upper bound the term $\psi_{\varepsilon,p}(S^{\tau})$, we apply Lemma 5.2 with $\{B^t\}_t$ corresponding to $\{C^t\}_{t=\tau}$ and $\{A^t\}_t$ corresponding to $\{C^t\}_{t>\tau}$ to get

$$\psi_{\varepsilon,p}(S^{\tau}) = \left\| \frac{p}{\varepsilon} 1 + S^{\tau} \right\|_p - \frac{p}{\varepsilon} \leq O(p) \text{Opt} + 2^{1/p} \left\| \frac{p}{\varepsilon} 1 \right\|_p - \frac{p}{\varepsilon} \leq O(p) \text{Opt} + \frac{p((2m)^{1/p} - 1)}{\varepsilon},$$

9
where $\text{Opt}_1$ is the optimal load up to time $n/2$. Moreover, the function $x \mapsto x^{1/p} - 1$ is subadditive over $[0, \infty)$ (since it is non-negative, concave and has value 0 at the origin [HP57, Theorem 7.2.5]), and hence the last term of the right-hand side is at most $2\left((1/p^2) + (1/p^2)\right) = \frac{2p(m^{1/p}-1)}{\varepsilon}$. Similarly, for the second half we have $\psi_{e,p}(S^n - S^\frac{n}{2}) \leq O(p) \text{Opt} + \frac{2p(m^{1/p}-1)}{\varepsilon}$. Employing these bounds proves part (a) of Theorem 5.1.

**Analysis in the random-order model.** Now we analyze the algorithm SMOOTHGREEDY in the random-order model, proving part (b) of Theorem 5.1. As usual, let $\mathbf{A}_1, \ldots, \mathbf{A}_n$ be the sequence of jobs in random order, $\{\tilde{x}^t\}_t$ be the decisions output by SMOOTHGREEDY, and $\mathbf{S}^t = \sum_{t \leq t} \mathbf{A}^t \tilde{x}^t$ be the load vector up to time $t$.

The main idea for the analysis is that algorithm SMOOTHGREEDY is also “approximately greedy” with respect to the gradient $\nabla \psi_{e,p}$. This is due to the smoothness property (2.4), and a main reason for defining SMOOTHGREEDY as greedy over the smoothened function $\psi_{e,p}$ instead of the original one $\|\cdot\|_p$; this lemma follows directly by integrating property (2.4) (see Appendix F).

**Lemma 5.3** For $u \in \mathbb{R}^m_+$ and $v, v' \in [0, 1]^m$, if $\psi_{e,p}(u+v) \leq \psi_{e,p}(u+v')$ then $\langle \nabla \psi_{e,p}(u), v \rangle \leq e^{2\varepsilon} \langle \nabla \psi_{e,p}(u), v' \rangle$.

Because of that, forgetting about the restart for now, algorithm SMOOTHGREEDY can be seen as an approximation to the primal-dual-type algorithms of [GM14, AD15]: Considering the expression $\sum_t \langle \nabla \psi_{e,p}(\mathbf{S}^{t-1}), \mathbf{A}^t \tilde{x}^t \rangle$, on the primal view the algorithm is choosing regret in Theorem 2.2. But since the gradient is playing the role of dual variables trying to maximize this expression, the last inequality follows from triangle inequality; thus, the load incurred by the algorithm satisfies

$$||S^n||_p \leq e^{\varepsilon} \left( \sum_{t} \langle A^{t} \tilde{x}^{t}, g^{t} \rangle + 2R \right).$$

(5.10)

Now we upper bound the right-hand side in expectation. Because of the restart of the algorithm, and the random order, the distribution of the first half ($\langle A^{t} \tilde{x}^{t}, g^{t} \rangle$)$_{t=1}^{\frac{n}{2}}$ is the same as that of the second half ($\langle A^{t} \tilde{x}^{t}, g^{t} \rangle$)$_{t=\frac{n}{2}+1}^{n}$, and so it suffices to bound the first half sum $E \sum_{t=1}^{\frac{n}{2}} \langle A^{t} \tilde{x}^{t}, g^{t} \rangle$.

Let $\tilde{x}_1, \ldots, \tilde{x}_n$ be the optimal offline solution. By the greedy criterion of SMOOTHGREEDY and Lemma 5.3, the algorithm has almost better load than the optimal solution when measured though the $g^{t}$’s: $\sum_{t} \langle A^{t} \tilde{x}^{t}, g^{t} \rangle = e^{2\varepsilon} \sum_{t} \langle A^{t} \tilde{x}^{t}, g^{t} \rangle$. But since the gradient $g^{t}$ is determined by $\mathbf{S}^{t-1}$ and belongs to $\ell_p^+$, Lemma 3.1 gives the upper bound $E \langle A^{t} \tilde{x}^{t}, g^{t} \rangle \leq e^{\varepsilon} ||E A^{t} \tilde{x}^{t}||_p + \frac{1}{\varepsilon} \cdot \frac{p(1/p-1)}{n(t-1)}$; also notice that $||E A^{t} \tilde{x}^{t}||_p = \frac{1}{n} \text{Opt}$. Adding these bounds over all $t \leq \frac{n}{2}$, we obtain

$$E \sum_{t \leq \frac{n}{2}} \langle A^{t} \tilde{x}^{t}, g^{t} \rangle \leq e^{\varepsilon} \left( \frac{\text{Opt}}{2} + \frac{p(1/p-1)}{\varepsilon} \right).$$

Plugging this bound in inequality (5.10) we get that the expected load of the algorithm is $E ||S^n||_p \leq e^{2\varepsilon} (\text{Opt} + 4R)$. This concludes the proof of part (b) of Theorem 5.1.
6 Algorithm SIMULTANEOUSLB

Since algorithm SMOOTHGREEDY incurs an additive error in the worst-case, if Opt is small it may not give the desired $O(p)$ multiplicative guarantee. Thus, the idea is to use the regular greedy algorithm until the accumulated load is large enough, and then switch to SMOOTHGREEDY.

Algorithm 6.1 SIMULTANEOUSLB $(p, \varepsilon)$

1. Run algorithm GREEDY, obtaining solution $\bar{x}^1, \ldots, \bar{x}^t$, until the first time $t$ the load $\|\sum_{t\leq \bar{t}} A^t \bar{x}^t\|_p$ obtained becomes larger than $\frac{p(m^{1/p} - 1)}{\varepsilon}$.

2. Run algorithm SMOOTHGREEDY over the remaining $n - \bar{t}$ time steps, obtaining $\bar{x}^{\bar{t} + 1}, \ldots, \bar{x}^n$.

The guarantees of the algorithm are given in Theorem 1.4, which we now prove. By triangle inequality the load of the algorithm is at most

$$\|\sum_{t<\bar{t}} A^t \bar{x}^t\|_p + \|A^t \bar{x}^t\|_p + \|\sum_{t>\bar{t}} A^t \bar{x}^t\|_p \leq \frac{p(m^{1/p} - 1)}{\varepsilon} + m^{1/p} + \|\sum_{t>\bar{t}} A^t \bar{x}^t\|_p \leq \frac{2p(m^{1/p} - 1)}{\varepsilon} + \|\sum_{t>\bar{t}} A^t \bar{x}^t\|_p,$$

where the last inequality uses $p \geq 2$ and $\varepsilon \leq 1$.

In the random-order model, notice that when we condition on the stopping time $\bar{t}$ and the items seen thus far, the items in the remaining times $\bar{t} + 1, \ldots, n$ are a random permutation of the remaining items, so we can apply the guarantee of SMOOTHGREEDY to bound the last term of the displayed inequality, giving part (b) of the theorem.

In the worst-case model, we can also apply the guarantee of SMOOTHGREEDY to bound this term, and further note that $\frac{m^{1/p} - 1}{\varepsilon} \leq O(\text{Opt})$: since the algorithm incurs load more than $\frac{p(m^{1/p} - 1)}{\varepsilon}$ up to time $\bar{t}$ and GREEDY is $O(p)$-approximate, the optimal load up to time $\bar{t}$ is at least $\Omega(\frac{m^{1/p} - 1}{\varepsilon})$, which lower bounds Opt. This gives part (a) of the theorem, and concludes the analysis of SIMULTANEOUSLB.

References

[AAF+93] James Aspnes, Yossi Azar, Amos Fiat, Serge Plotkin, and Orli Waarts. On-line load balancing with applications to machine scheduling and virtual circuit routing. In Proceedings of the Twenty-fifth Annual ACM Symposium on Theory of Computing, STOC ’93, pages 623–631, New York, NY, USA, 1993. ACM.

[AAG+95] B. Awerbuch, Y. Azar, E. F. Grove, Ming-Yang Kao, P. Krishnan, and J. S. Vitter. Load balancing in the lp norm. In Foundations of Computer Science, 1995. Proceedings., 36th Annual Symposium on, pages 383–391, Oct 1995.

[AAS01] A. Avidor, Y. Azar, and J. Sgall. Ancient and new algorithms for load balancing in the lp norm. Algorithmica, 29(3):422–441, 2001.

[ACP14] Yossi Azar, Ilan Reuven Cohen, and Debmalya Panigrahi. Online covering with convex objectives and applications. CoRR, abs/1412.3507, 2014.

[AD15] Shipra Agrawal and Nikhil R. Devanur. Fast algorithms for online stochastic convex programming. In Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2015, San Diego, CA, USA, January 4-6, 2015, pages 1405–1424, 2015.

[AERW04] Yossi Azar, Leah Epstein, Yossi Richter, and Gerhard J. Woeginger. All-norm approximation algorithms. J. Algorithms, 52(2):120–133, August 2004.

11
[ALST14] Jacob Abernethy, Chansoo Lee, Abhinav Sinha, and Ambuj Tewari. Online linear optimization via smoothing. In Proceedings of The 27th Conference on Learning Theory, COLT 2014, Barcelona, Spain, June 13-15, 2014, pages 807–823, 2014.

[BCG+14] Niv Buchbinder, Shahar Chen, Anupam Gupta, Viswanath Nagarajan, and Joseph Naor. Online packing and covering framework with convex objectives. CoRR, abs/1412.8347, 2014.

[BCK00] Piotr Berman, Moses Charikar, and Marek Karpinski. On-line load balancing for related machines. J. Algorithms, 35(1):108–121, April 2000.

[BIKK08] Moshe Babaioff, Nicole Immorlica, David Kempe, and Robert Kleinberg. Online auctions and generalized secretary problems. SIGecom Exchanges, 7(2), 2008.

[Car08] Ioannis Caragiannis. Better bounds for online load balancing on unrelated machines. In Proceedings of the Nineteenth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA ’08, pages 972–981, Philadelphia, PA, USA, 2008. Society for Industrial and Applied Mathematics.

[CBL06] Nicolo Cesa-Bianchi and Gabor Lugosi. Prediction, Learning, and Games. Cambridge University Press, New York, NY, USA, 2006.

[CC76] R. A. Cody and E. G. Coffman, Jr. Record allocation for minimizing expected retrieval costs on drum-like storage devices. J. ACM, 23(1):103–115, January 1976.

[CFK+11] Ioannis Caragiannis, Michele Flammini, Christos Kaklamanis, Panagiotis Kanellopoulos, and Luca Moscardelli. Tight bounds for selfish and greedy load balancing. Algorithmica, 61(3):606–637, 2011.

[CW75] Ashok K. Chandra and C. K. Wong. Worst-case analysis of a placement algorithm related to storage allocation. SIAM Journal on Computing, 4(3):249–263, 1975.

[DH09] Nikhil R. Devenur and Thomas P. Hayes. The adwords problem: online keyword matching with budgeted bidders under random permutations. In EC, 2009.

[GM14] Anupam Gupta and Marco Molinaro. How Experts Can Solve LPs Online, pages 517–529. Springer Berlin Heidelberg, Berlin, Heidelberg, 2014.

[GN10] D. Gross and V. Nesme. Note on sampling without replacing from a finite collection of matrices. ArXiv e-prints, January 2010. (http://arxiv.org/abs/1001.2738).

[Hoe63] Wassily Hoeffding. Probability inequalities for sums of bounded random variables. Journal of the American Statistical Association, 58(301):13–30, 1963.

[HP57] Einar Hille and Ralph Phillips. Functional Analysis and Semi-Groups. American Mathematical Society, 1957.

[IKKP15] S. Im, N. Kell, J. Kulkarni, and D. Panigrahi. Tight bounds for online vector scheduling. In Foundations of Computer Science (FOCS), 2015 IEEE 56th Annual Symposium on, pages 525–544, Oct 2015.

[KMZ15] Nitish Korula, Vahab Mirrokni, and Morteza Zadimoghaddam. Online submodular welfare maximization: Greedy beats 1/2 in random order. In Proceedings of the Forty-Seventh Annual ACM on Symposium on Theory of Computing, STOC ’15, pages 889–898, New York, NY, USA, 2015. ACM.
[KTRV14] Thomas Kesselheim, Andreas Tönnis, Klaus Radke, and Berthold Vöcking. Primal beats dual on online packing lps in the random-order model. In *Proceedings of the 46th Annual ACM Symposium on Theory of Computing*, STOC ’14, pages 303–312, New York, NY, USA, 2014. ACM.

[KV05] Adam Kalai and Santosh Vempala. Efficient algorithms for online decision problems. *J. Comput. Syst. Sci.*, 71(3):291–307, October 2005.

[LD10] Mark C. Veraar Jon A. Wellner Lutz Dmbgen, Sara A. van de Geer. Nemirovski’s inequalities revisited. *The American Mathematical Monthly*, 117(2):138–160, 2010.

[Mey01] A. Meyerson. Online facility location. In *Proceedings of the 42Nd IEEE Symposium on Foundations of Computer Science*, FOCS ’01, pages 426–, Washington, DC, USA, 2001. IEEE Computer Society.

[MGZ12] Vahab S. Mirrokni, Shayan Oveis Gharan, and Morteza Zadimoghaddam. Simultaneous approximations for adversarial and stochastic online budgeted allocation. In *Proceedings of the Twenty-third Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA ’12, pages 1690–1701, Philadelphia, PA, USA, 2012. Society for Industrial and Applied Mathematics.

[PnG99] Victor de la Peña and Evarist Giné. *Decoupling: From Dependence to Independence*. Springer-Verlag, New York, NY, USA, 1999.

[RS15] A. Rakhlin and K. Sridharan. On Equivalence of Martingale Tail Bounds and Deterministic Regret Inequalities. *ArXiv e-prints*, October 2015. [http://arxiv.org/abs/1510.03925](http://arxiv.org/abs/1510.03925).

[SDJ15] Tianping Shuai, Donglei Du, and Xiaoyue Jiang. On-line preemptive machine scheduling with \(\ell_p\)-norm on two uniform machines. *Journal of Scheduling*, 18(2):185–194, 2015.

[Wik16] Wikipedia. Natural logarithm — wikipedia, the free encyclopedia, 2016. [https://en.wikipedia.org/wiki/Natural_logarithm](https://en.wikipedia.org/wiki/Natural_logarithm).
Appendix

A  Proof of Theorem 1.1

The lower bound is essentially the same used in [AAG+95], we just need to work with the expected load incurred. Fix a positive integer $M$, let $m = 2^{p+1}$ (for simplicity we assume $p$ integral) and fix an algorithm. We will construct an instance where each job can be processed in a subset of the machines and incurs load 1 in the machine chosen to process it. So given a subset $S \subseteq [m]$ of machines, a type $S$ job is one that can be processed by the machines in $S$, again incurring load 1 in the chosen machine.

The instance is constructed in $\log m$ rounds. In round $i$, we have machines $\mathcal{U}_{i-1} \subseteq [m]$ “active”. These active machines are paired up into $|\mathcal{U}_{i-1}|/2$ disjoint pairs, and the adversary sends $M$ copies of jobs of types corresponding to each such pair $(a, b)$. Let $\ell^i$ be the (randomized) load vector incurred by the algorithm when processing these round-$i$ jobs. Then for each such pair $(a, b)$, the machine $j \in (a, b)$ with smallest load $\mathbb{E}\ell^i_j$ (ties broken arbitrarily) is deactivated, defining the next active set $\mathcal{U}_i$. Notice that for all machines $j \in \mathcal{U}_i$ we have $\mathbb{E}\ell^i_j \geq \frac{M}{2}$. This proceeds until round $\log m$.

We analyze this instance starting with $\text{Opt}$. Consider the following strategy: process all round-$i$ by spreading them uniformly over the machines $\mathcal{U}_{i-1} \setminus \mathcal{U}_i$, incurring load $M$ in each of them. By construction, the machines used in each round by this strategy are disjoint, and hence the final load vector is $(M, M, \ldots, M)$, with load $m^{1/p} M$; this provides an upper bound on $\text{Opt}$.

On the other hand, the algorithm considered has expected load $\mathbb{E} \| \sum_i \ell^i \|_p \geq \| \mathbb{E} \sum_i \ell^i \|_p$ (using Jensen’s inequality); to simplify the notation, let $\mu = \mathbb{E} \sum_i \ell^i$. So taking $p$-th powers

$$\left( \mathbb{E} \left\| \sum_{i=1}^m \ell^i \right\|_p \right)^p \geq \sum_j \mu_j^p \geq \sum_{j \in \mathcal{U}_i \setminus \mathcal{U}_{i+1}} \mu_j^p.$$  \hspace{1cm} (A.11)

By construction, $\mathcal{U}_i \setminus \mathcal{U}_{i+1}$ has $\frac{m}{2} - \frac{m}{2^{i+1}} = \frac{m}{2^{i+1}}$ machines. Moreover, for each such machine $j$, the algorithm incurs expected load at least $\frac{M}{2}$ in each of the rounds from 1, . . . , $i$, and hence $\mu_j \geq \frac{Mi}{2}$. Thus the right-hand side of (A.11) is at least

$$\sum_{i=1}^m \frac{m}{2^{i+1}} \left( \frac{Mi}{2} \right)^p \geq m \left( \frac{Mp}{2^{2+1/p}} \right)^p,$$

where the inequality is obtained by just using the term $i = \log m - 1 = p$. Thus, the load incurred by the algorithm is at least $m^{1/p} \frac{Mp}{2^{2+1/p}}$, thus proving the theorem.

B  Proof of Theorem 1.5

Let $m = 2^p$. Our instance is based on the following Walsh system. For $i = 1, \ldots, \log m$, define the vectors $v^i \in \{0, 1\}^m$ as follows: construct the $m \times \log m$ 0/1 matrix $M$ by letting its rows be all the $\log m$-bit strings; then the $v^i$’s are defined as the columns of this matrix. We use $(v^i)^c$ to denote 0/1 vector obtained by flipping all the coordinates of $v^i$.

The main motivation for this construction is the following intersection property.

Lemma B.1  Consider a subset $I \subseteq \lfloor \log m \rfloor$ and for each $i \in I$ let $v^i$ be either $v^i$ or $(v^i)^c$. Then the $v^i$’s intersect in $\frac{m}{2^{\lfloor \log m \rfloor}}$ coordinates, namely there is set of coordinates $J \subseteq [m]$ of size $\frac{m}{2^{\lfloor \log m \rfloor}}$ such that $u^i_j = 1$ for all $i \in I, j \in J$.

Proof. Let $j$ be uniformly random in $[m]$. Notice that the vector $(v^1_j, v^2_j, \ldots, v^\log m_j)$ is a random row of the matrix $M$, and hence it is a point uniformly distributed in $\{0, 1\}^\log m$. Due to the product structure of this set, this implies that the random variables $\{v^i_j\}_i$ are all independent, and each take value 1 with probability $1/2$.\[\]
Moreover, this is true if we complement some of these variables, i.e., replace $v^i$ for $(v^i)^c$ for some indices $i$ (notice we do now allow $v^i$ and $(v^i)^c$ to be simultaneously in the set). Therefore, this gives that the random variables $\{u_j^i\}_{i \in I}$ are independent, and thus the number of coordinates where they intersect is

$$m \cdot \Pr\left(\bigwedge_{i \in I} (u_j^i = 1)\right) = m \cdot \prod_{i \in I} \Pr(u_j^i = 1) = \frac{m}{2^{|I|}}.$$  

This concludes the proof.

The instance for $\text{GLB}_p$ is then constructed randomly as follows. There are $m$ machines and $\frac{p}{2} = \frac{\log m}{2}$ jobs. For $i = 1, \ldots, \frac{\log m}{2}$, let $u^i$ be a random vector that equals $v^i$ with probability $1/2$ and equals its complement $(v^i)^c$ with probability $1/2$. Then for each $i \in \left[\frac{\log m}{2}\right]$, we have one job with only one processing option of load vector $u^i$, and one job with $2$ processing options of load vectors $v^i$ and $(v^i)^c$. These jobs are then presented in random order.

Now we analyze this instance. The optimal offline solution can be upper bounded $\text{Opt} \leq \frac{p}{2} m^{1/p}$, since opt can process each job $\{v^i, (v^i)^c\}$ using the option that equals the complement of $u^i$, which gives total load vector $\frac{p}{2}1$, of $\ell_p$-norm $\frac{p}{2} m^{1/p}$. This is also tight, namely $\text{Opt} \geq \frac{p}{2} m^{1/p}$: by adding up all the loads of the jobs over all the machines, we see that any solution has total $\ell_1$ load exactly $\frac{mp}{2}$, and since $\|x\|_p \geq \frac{1}{p} \|x\|_1$ (see Section 1.2) the claimed lower bound follows.

However, it is hard for the online algorithm to “unmatch” the processing of $\{v^i, (v^i)^c\}$ with $u^i$, even in the random-order model. More precisely, consider any online algorithm and let $X_i$ be the indicator variable that the algorithm chose to process $\{v^i, (v^i)^c\}$ using the option that equals $u^i$. Since the instance is presented in random order, with probability $1/2$ the job $\{v^i, (v^i)^c\}$ comes before the job $u^i$; in this case, the random variable $X_i$ is independent from how the algorithm processes job $\{v^i, (v^i)^c\}$, and so with probability $1/2$ it equals the processing option that the algorithm chose for job $\{v^i, (v^i)^c\}$. Thus, $\mathbb{E}X_i \geq 1/4$, and hence $\mathbb{E}\sum_{i=1}^{p/2} X_i \geq \frac{p}{8}$ (i.e., on average the algorithm makes $\frac{p}{8}$ “mistakes”). Moreover, by employing Markov’s inequality to $\frac{p}{2} - \sum_i X_i$ we obtain that $\sum_i X_i \geq \frac{m}{16}$ with probability at least $1/7$.

Now we see how these mistakes factor into the load of the algorithm. When $X_i = 0$, the processing of $u^i$ and $\{v^i, (v^i)^c\}$ match, adding up to load vector $1$, and when $X_i = 1$ their load vector adds up to $2u^i$. Thus, the load of the algorithm is

$$\text{Algo} = \left\| \sum_i (1 - X_i)1 + \sum_i X_i2u^i \right\|_p.$$  

Using Lemma 3.1 above, we see that this total load vector has at least $\frac{m}{2}\sum_i X_i$ coordinates with value $\sum_i (1 - X_i) + 2 \sum_i X_i = \frac{p}{2} + \sum_i X_i$. Thus,

$$\text{Algo} \geq \left(\frac{m}{2\sum_i X_i} \left(\frac{p}{2} + \sum_i X_i\right)^{p} \right)^{1/p} = \frac{pm^{1/p}}{2} \left(1 + \frac{2\sum_i X_i}{p}\right) \frac{1}{2^{m/p}},$$  

Notice that always $\sum_i X_i \geq \frac{1}{16}$ and recall that with probability at least $1/7$ we have $\sum_i X_i \geq \frac{1}{16}$. Moreover, one can verify that the function $x \mapsto (1 + 2x)^{1/p}$ is increasing in the interval $[\frac{1}{16}, \frac{7}{16}]$, so its minimum is achieved at $x = \frac{1}{16}$ with value $\geq 1.07$. Thus, with probability at least $1/7$, the algorithm incurs load at least $1.07 \frac{pm^{1/p}}{2}$. With the remaining probability, the algorithm incurs load at least that of the offline $\text{Opt}$, which is at least $\frac{pm^{1/p}}{2}$; thus, the expected total load of the algorithm is at least

$$\frac{1.107 pm^{1/p}}{2} + \frac{6 pm^{1/p}}{7} = 1.01 \frac{pm^{1/p}}{2}.$$  

This concludes the proof of Theorem 1.5.
C Proof of Lemma 2.1

To simplify the notation we omit subscripts in $\psi_{\varepsilon,p}$.

Property (2.3). The upper bound follows directly from triangle inequality: $\psi(u) \leq \frac{p}{\varepsilon} (\|1\|_p + \frac{\|u\|_p}{\varepsilon}) - \frac{p}{\varepsilon} = \|u\|_p + \frac{p(m^{1/p - 1})}{\varepsilon}$. For the lower bound $\psi(u) \geq \|u\|_p$ for $u \in \mathbb{R}_+^m$, define $v = \frac{p}{\varepsilon} \frac{u}{\|u\|_p}$. Since $\frac{p}{\varepsilon} 1 + u$ is pointwise at least $v + u$, and the later is non-negative, we have

$$\|\frac{p}{\varepsilon} 1 + u\|_p \geq \|v + u\|_p = \left(\frac{p}{\varepsilon} \frac{1}{\|u\|_p} + 1\right) \|u\|_p = \|u\|_p + \frac{p}{\varepsilon} \|u\|_p \geq \|u\|_p + \frac{p}{\varepsilon}.$$

Since $\psi(u) = \|\frac{p}{\varepsilon} 1 + u\|_p - \frac{p}{\varepsilon}$ the result follows.

Property (2.4). Writing the partial derivatives:

$$\frac{\partial \psi}{\partial x_i}(u) = \frac{\left(1 + \frac{\varepsilon u_i}{p}\right)^{p-1}}{\left(\sum_i \left(1 + \frac{\varepsilon u_i}{p}\right)^p\right)^{\frac{1}{p}}}.$$ (C.12)

Since $u_i \geq 0$ and $v_i \in [0, 1]$, we have

$$\left(1 + \frac{\varepsilon u_i}{p}\right)^{p-1} \leq \left(1 + \frac{\varepsilon (u_i + v_i)}{p}\right)^{p-1} \leq \left(1 + \frac{\varepsilon v_i}{p}\right)^{p-1} \left(1 + \frac{\varepsilon u_i}{p}\right)^{p-1} \leq e^\varepsilon \left(1 + \frac{\varepsilon u_i}{p}\right)^{p-1},$$

where in the last inequality we use the fact that for all $x \geq 0$ we have $1 + x \leq e^x$. Moreover, the same holds if we change the powers from $p - 1$ to $p$. We can then use these inequalities in the numerator and denominator of (C.12) to obtain $\frac{\partial \psi}{\partial x_i}(u + v) \leq e^\varepsilon \frac{\partial \psi}{\partial x_i}(u)$ and $\frac{\partial \psi}{\partial x_i}(u + v) \geq \frac{1}{e^{\varepsilon(1/p)} \frac{\partial \psi}{\partial x_i}(u)} \geq e^{-\varepsilon} \frac{\partial \psi}{\partial x_i}(u)$. This concludes the proof.

D Proof of Lemma 4.1

We start with the following optimal bound on the modulus of strong smoothness of the square of the $\ell_p$ norm.

Lemma D.1 (LD10) For $p \in [2, \infty)$ and $x \in \mathbb{R}^m \setminus \{0\}$. Then for arbitrary $x, y \in \mathbb{R}^m$,

$$\|x\|_p^2 + \langle h(x), y \rangle \leq \|x + y\|_p^2 \leq \|x\|_p^2 + \langle h(x), y \rangle + (p - 1)\|y\|_p^2,$$

where the vector $h(x)$ is defined as $h(x) = 2\|x\|^{2-p} (|x_i|^{p-2} x_i)_i$.

Collecting the $\|x\|_p^2$ terms, the upper bound of this inequality becomes

$$\|x + y\|_p^2 \leq \|x\|_p^2 \left(1 + \frac{\langle h(x), y \rangle}{\|x\|_p^2} + \frac{(p - 1)\|y\|_p^2}{\|x\|_p^2}\right).$$ (D.13)

But since $\sqrt{\cdot}$ is concave, for any $\alpha$ we have (using linearization at 1) $\sqrt{1 + \alpha} \leq 1 + \frac{\alpha}{2}$ (this can also be checked directly by squaring both sides). Thus, taking square roots on both sides of (D.13) and employing this bound we get

$$\|x + y\|_p \leq \|x\|_p + \frac{\langle h(x), y \rangle}{2\|x\|_p^2} + \frac{(p - 1)\|y\|_p^2}{2\|x\|_p^2}.$$ (D.14)

Defining $g(x) = \frac{h(x)}{2\|x\|_p^2}$ we see that this expression is exactly the one in Lemma 4.1. To conclude the proof we show that for $x \geq 0$ we have $\|g(x)\|_q \leq 1$, or equivalently $\|h(x)\|_q \leq 2\|x\|_p$: noticing that $q = p/(p - 1)$,

$$\|h(x)\|_q = \frac{2}{\|x\|_p^{p-2}} \left(\sum_i x_i^{(p-1)q}\right)^{1/q} = \frac{2}{\|x\|_p^{p-2}} \left(\|x\|_p^{(p-1)/p}\right) = \frac{2\|x\|_p^{p-1}}{\|x\|_p^{p-2}} = 2\|x\|_p.$$

This concludes the proof of Lemma 4.1.
E  Proof of Lemma 5.2

We reproduce the proof of [Car08] for convenience. Let \( S^t = C^1 \bar{x}^1 + \ldots + C^t \bar{x}^t \) (and \( S^0 = 0 \)). By the greedy criterion, for each \( t \), we have \( \|S^t\|_p^p - \|S^{t-1}\|_p^p \leq \|S^{t-1} + C^t \bar{x}^t\|_p^p - \|S^{t-1}\|_p^p \). Adding all these inequalities for \( t \geq \tau \) we get

\[
\|S^n\|_p^p - \|S^{\tau-1}\|_p^p \leq \sum_{t \geq \tau} (\|S^{t-1} + C^t \bar{x}^t\|_p^p - \|S^{t-1}\|_p^p),
\]

(E.14)

where the last inequality follows from the fact that \( x \mapsto (x + a)^p - x^p \) is non-decreasing over \([0, \infty)\) for \( a \geq 0 \). Employing Lemma 3.1 of [Car08] we get that the right-hand side is at most

\[
\mathrm{RHS} \leq \left( \|S^n\|_p + \left\| \sum_{t \geq \tau} C^t \bar{x}^t \right\|_p \right)^p - \|S^n\|_p^p.
\]

(E.15)

If \( \|S^n\|_p^p \leq 2 \|S^{\tau-1}\|_p^p \) then Lemma 5.2 clearly holds. Otherwise \( \|S^{\tau-1}\|_p^p \leq \frac{\|S^n\|_p^p}{2} \), which used together with inequalities (E.14) and (E.15) gives

\[
\frac{3}{2} \|S^n\|_p^p \leq \left( \|S^n\|_p + \left\| \sum_{t \geq \tau} C^t \bar{x}^t \right\|_p \right)^p.
\]

Taking \( p \)-th roots and reorganizing we get \( \|S^n\|_p \leq \frac{1}{(\frac{3}{2})^{1/p} - \frac{1}{2}} \sum_{t \geq \tau} C^t \bar{x}^t \). Using the inequality \( e^x \geq 1 + x \), we have that \( \frac{1}{(\frac{3}{2})^{1/p} - \frac{1}{2}} \leq \frac{p}{\ln(3/2)} \), which gives the desired result. This concludes the proof of the lemma.

F  Proof of Lemma 5.3

Let \( \psi := \psi_{\varepsilon,p} \) to simplify the notation. Integrating and using Property (2.4) we get

\[
\psi(u + v) = \psi(u) + \int_0^1 \langle \nabla \psi(u + \lambda v), v \rangle d\lambda \leq \psi(u) + e^{\varepsilon} \langle \nabla \psi(u), v \rangle,
\]

and similarly for \( v' \). Thus,

\[
\langle \nabla \psi(u), v \rangle \leq e^\varepsilon [\psi(u + v) - \psi(u)] \leq e^\varepsilon [\psi(u + v') - \psi(u)] \leq e^{2\varepsilon} \langle \nabla \psi(u), v' \rangle.
\]

This concludes the proof.