On the eigenvalue asymptotics of Zonal Schrödinger operators in even metric and non-even metric

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Abstract: We discuss a spectral asymptotics theory of an even zonal metric and a Schrödinger operator with zonal potentials on a sphere. We decompose the eigenvalue problem into a series of one-dimensional problems. We consider the individual behavior of this series of one-dimensional problems. We find certain Weyl’s type of asymptotics on the eigenvalues.

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1. Introduction

In this paper, we compute the eigenvalue asymptotics of the operator $\Delta_g$ with a metric $g$ that satisfies the following even zonal metric assumption: Let $(x_1, \ldots, x_{N+1}, z)$ be the standard coordinate on $\mathbb{R}^{N+1,2}$. We consider the hypersurface defined by the equation

$$\sum_{n=1}^{N+1} x_n^2 = r(z)^2, \quad -1 \leq z \leq 1,$$

in which we assume $r$ is an even function of $z$, $r(-1) = r(1) = 0$ and $0 < r(z) < \infty$ for $-1 \leq z \leq 1$. Following the framework of Carlson (1997), we let $s(z)$ denote the arc length

$$s(z) = \int_{-1}^{z} \sqrt{1 + \left( \frac{dr(t)}{dt} \right)^2} dt,$$

ABOUT THE AUTHOR

The author grew up in Taiwan, and received his PhD in mathematics at Purdue University, 2007. He held a postdoctoral position at National Taiwan University in 2008. Currently, he is teaching at National Chung-Cheng University in Chia-Yi. The author's research interests are scattering theory and spectral analysis of differential operators. Recently, he has been involved with the inverse problems concerning wave propagation. It sounds a bit cliched, but when he is not with math, he's with his family and wine tasting. He is an audiophile, and has collected a few thousands of music CDs across all genres.

PUBLIC INTEREST STATEMENT

In this paper, we describe a connection between the eigenvalue distribution theorems and the geometric characteristics for a class of manifolds: eigenvalues are frequencies. It is believed that one can figure out some of the physical characteristics of a wave-reflecting object by hitting it with a band of frequencies of testing waves. It is a research interest among many disciplines, e.g. acoustics, optics, remote sensing, medical imaging, national defense, astrophysics, and quantum mechanics. Wherever there is a wave propagating through the media, we ask if one can analyze the perturbed wave and figure out the perturbation. As asked by Mark Kac, “Can you hear the shape of a drum?”.
in which we set $L: = s(1)$. Hence, a metric is induced on the hypersurface. Now, we calculate $\Delta_g$ using the method provided in Shubin (1987, p. 157), and then we deduce that

$$\Delta_g = r^{-N}(s)\partial_s r^N(s)\partial_s + r^{-2}(s)\Delta_s,$$

(3)

in which $S$ is the $N$ sphere. We put the operator in the Liouville form with respect to the variable $s$:

$$r^{N/2}(s)\Delta_g r^{-N/2} = \partial_s^2 - \frac{N^2}{4} \left(\frac{r'}{r}\right)^2 - \frac{N}{2} \left(\frac{r'}{r}\right)' + r^{-2}(s)\Delta_s,$$

(4)

where we observe that the function $-\frac{N^2}{4} \left(\frac{r'}{r}\right)^2 - \frac{N}{2} \left(\frac{r'}{r}\right)'$ is again an even function about the midpoint of $[0, L]$. For the standard $N + 1$ sphere, we note that $r(s) = \sin(s)$. Accordingly, we are required to assume that $r(s) \in C^2[0, L]$ and for some $p_0 \in C^2[0, L/2]$, we have the following properties:

$$r(s) = s[1 + p_0(s)], \lim_{s \to 0} p_0(s) = 0, \lim_{s \to 0} p_0'(s)/s \text{ exists.}$$

(5)

Here, (1.5) is to assure the hypersurface behaves like $N + 1$ sphere near $z = -1$. Then, for $0 < s < L/2$,

$$r^{-1}(s) = s^{-1} \left(1 + \frac{1}{p_0(s)}\right),$$

and from which we deduce that

$$r^{-2}(s) = s^{-2} + p_1(s), p_1(s) \in C[0, L/2].$$

(6)

Accordingly,

$$-\frac{N^2}{4} \left(\frac{r'}{r}\right)^2 - \frac{N}{2} \left(\frac{r'}{r}\right)' = \frac{-N(N - 2)}{4s^2} + p_2(s), p_2(s) \in C[0, L/2].$$

(7)

The function $p_1, p_2$ depends on $p_0$ and its derivatives.

Let $\{\beta_k\}$ be the eigenvalues of $\Delta_s$. Then, the eigenvalue problem (4) is reduced to be

$$\partial_s^2 y + \left[-\frac{N^2}{4} \left(\frac{r'}{r}\right)^2 - \frac{N}{2} \left(\frac{r'}{r}\right)' + r^{-2}(s)\beta_k\right] y = \lambda y,$$

(8)

For a Schrödinger operator with a zonal potential, we are dealing with an equation in the following form:

$$\partial_s^2 y + \left[-\frac{N^2}{4} \left(\frac{r'}{r}\right)^2 - \frac{N}{2} \left(\frac{r'}{r}\right)' + r^{-2}(s)\beta_k\right] y + p(s)y = \lambda y,$$

(9)

where $p \in L^2[0, L]$ is assumed to be an even function with midpoint $L/2$, and $r(s) = \sin(s)$. We say $p(s)$ is zonal because $p$ is a function of $z$ on the hypersurfaces (1.1) and (1.2) (Gurarie, 1988, 1990). A potential of this type has various applications in mathematical physics. We refer more introduction on potentials of this class to (Gurarie, 1990, p. 567). The differential equations (1.8) and (1.9) require the following regularization conditions:

$$\lim_{s \to 0} \frac{y(s)}{s^{m+1}} < \infty,$$

(10)

$$\lim_{s \to L} \frac{y(s)}{s^{m+1}} < \infty.$$

(11)

Moreover, (1.6) and (1.7) imply that

$$-\frac{N^2}{4} \left(\frac{r'}{r}\right)^2 - \frac{N}{2} \left(\frac{r'}{r}\right)' + r^{-2}(s)\beta_k = \frac{4\beta_k - N(N - 2)}{4s^2} + p(s), 0 < s < L/2,$$

(12)
in which \( r(s) \) is even, \( p(s) = p_\gamma(s) + \beta_k p_\gamma(s) \in C[0,L] \) by its construction and its solution satisfies the initial condition \( r'(L/2) = 0 \). Theorem 1.4 in Carlson (1993) says that \( p(s) \) is uniquely determined by the spectral data of \( \Delta_g \). According to (1.8) and (1.9), we set that

\[
m(m + 1) = \frac{N^2 - 2N - 4\beta_k}{4}. \tag{13}
\]

Now, we are studying the eigenvalue asymptotics of the equation of the following form:

\[
\frac{d^2 y}{d s^2} + \left[ -\frac{m(m + 1)}{s^2} + p(s) \right] y = z^2 y, \quad z \in \mathbb{C}, \tag{14}
\]

in which the asymptotic expansion of \( y(s; z) \) is analyzed in Carlson (1993, 1997, 1994). Setting the solution \( y = y(s; z) \), \( y(s; z) \) is an entire function of exponential type (Carlson, 1993; Pöschel & Trubowitz, 1987). The union of all eigenvalues of (1.14) over \( \{ \beta_k \} \) gives the collection of the total eigenvalues of \( \Delta_g \) in (1.4) and vice versa. The cluster structure of the eigenvalues for each \( \beta_k \) is known among the work in Gurarie (1988, 1990) and many others. Most important of all (Carlson, 1993, p. 23), due to the evenness assumption on \( \Delta_g \) and \( \Delta_g + p \), the eigenvalues of (1.14) is split into two kinds for each \( \beta_k \): the zeros of \( y(L/2; z) \) and \( y'(L/2; z) \). The zeros of \( y(L/2; z) \) and of \( y'(L/2; z) \) correspond to the Dirichlet and Neumann spectral data of (1.14) at \( r = L/2 \), respectively. Hence, in the first part of this paper, we collect all zeros of \( y(L/2; z) \) and \( y'(L/2; z) \) for each \( \beta_k \).

In this paper, we consider the Weyl’s type of eigenvalue asymptotics of (1.8) and (1.9) on surfaces of type (1.4).

**Theorem 1.1** Let \( N(v) \) be the eigenvalue counting function in an interval of length \( v \) starting at the origin. Then, the following asymptotics holds:

\[
N(v) \sim \frac{2Lv^{\frac{\Delta_g}{2}}}{\pi N^{\frac{\Delta_g}{2}}}. \tag{15}
\]

A Weyl’s theorem of this kind is classic in many perturbations (Chen, 2015a; Gurarie, 1988, 1990; Shubin, 1987). We provide an extra information on the arc length \( l \). The new ingredient in this paper is an analysis in entire function theory and its extension to non-even metrics.

**2. Zeros of \( y(L/2; z) \) and \( y'(L/2; z) \)**

We apply the entire function in complex analysis (Koosis, 1997; Levin, 1972, 1996) to study the distribution of its zeros.

**Definition 2.1** Let \( f(z) \) be an entire function. Let \( M_f(r) = \max_{|z|=r} |f(z)| \). An entire function of \( f(z) \) is said to be a function of finite order if there exists a positive constant \( k \) such that the inequality

\[
M_f(r) < e^{kr}, \tag{16}
\]

is valid for all sufficiently large values of \( r \). The greatest lower bound of such numbers \( k \) is called the order of the entire function \( f(z) \). By the type \( \sigma \) of an entire function \( f(z) \) of order \( \rho \), we mean the greatest lower bound of positive number \( A \) for which asymptotically we have

\[
M_f(r) < e^{Ar}, \tag{17}
\]

That is,

\[
\sigma = \limsup_{r \to \infty} \frac{\ln M_f(r)}{r^\rho} \tag{18}
\]

If \( 0 < \sigma < \infty \), then we say \( f(z) \) is of normal type or mean type.
We note that
\[
e^{(a-r)r} < M_I(r) < e^{(a+r)r},
\]
where we mean the first inequality holds for some sequence going to infinity and the second one holds asymptotically.

**Definition 2.2** If an entire function \(f(z)\) is of order one and of normal type, then we say it is an entire function of exponential type \(\sigma\).

**Lemma 2.3** Let \(f\) and \(g\) be two entire functions. Then, the following two inequalities hold.
\[
h_g(\theta) \leq h_f(\theta) + h_g(\theta), \text{ if one limit exists;}
\]
\[
h_f+g(\theta) \leq \max \{h_f(\theta), h_g(\theta)\},
\]
where if the indicator of the two summands is not equal at some \(\theta_n\), then the equality holds in (2.6).

The equality in (2.5) holds if one function is of completely regular growth. This is classic and we refer these to Levin (1972, p. 51, 159).

\[\square\]

**Definition 2.4** Let \(f(z)\) be an integral function of finite order \(\rho\) in the angle \([\theta_1, \theta_2]\). We call the following quantity the indicator of the function \(f(z)\).
\[
h_f(\theta) : = \limsup_{r \to \infty} \frac{\ln |f(re^{i\theta})|}{r}, \ \theta_1 \leq \theta \leq \theta_2.
\]

**Definition 2.5** The following quantity is called the width of the indicator diagram of the entire function \(f\):
\[
d_f : = h_f\left(\frac{\pi}{2}\right) + h_f\left(-\frac{\pi}{2}\right).
\]

The order and the type of an integral function in an angle can be defined similarly. The connection between the indicator \(h_f(\theta)\) and its type \(\sigma\) is specified by the following theorem.

**Definition 2.6** Let \(f(z)\) be an entire function of order \(\rho\). We use \(N_f([\alpha, \beta], r)\) to denote the number of the zeros of \(f(z)\) inside \([\alpha, \beta]\) and \(|z| \leq r\); we define the density function
\[
\Delta_f(\alpha, \beta) : = \limsup_{r \to \infty} \frac{N_f([\alpha, \beta], r)}{r^\rho}
\]
and
\[
\Delta(\beta) : = \Delta(\alpha_0, \beta),
\]
with fixed \(\alpha_0 \in E\), with \(E\) as an at most countable set.

The two definitions above are necessary vocabularies to apply Cartwright–Levinson theory (Levin, 1972, p. 251) in complex analysis.

**Lemma 2.7** (Levin, 1972, p. 72) The maximal value of the indicator \(h_f(\theta)\) of the function \(f(z)\) on the interval \(\alpha \leq \theta \leq \beta\) is equal to the type \(\sigma\) of this function inside the angle \(\alpha \leq \arg z \leq \beta\).

**Lemma 2.8** (Levin, 1972) Let \(\alpha\) and \(\beta\) be real constants.
\[
h_{\sin(\alpha \theta)}(\theta) = |\alpha \sin \theta|.
\]
Let \( y(s; z) \) be the solution of the following problem in Carlson (1997):

\[
\begin{align*}
-\gamma''(s; z) + \frac{m(m+1)(s^2)}{2^{m+1}/s!} + p(s)y(s; z) &= z^2 \gamma(s; z); \\
\lim_{s \to 0} \left( \frac{y(s; z)}{s} - j_m(sz) \right) &= 0.
\end{align*}
\]

(26)

For the initial conditions, we actually have

\[
\gamma_0(0) = 0, \quad \gamma_0'(0) = 1; \quad \gamma(0) = 0, \quad \gamma'(0) = 0, \quad m > 0.
\]

(27)

The following result is well known for the classic case (Pöschel & Trubowitz, 1987, p. 14) and for singular potentials without a lower bound (Faddeev, 1960, (14.14), (14.15)). However, we give the more precise first-order asymptotics from a point of view from Carlson (1993,1997).

**Proposition 2.9** \( y(L/2; z) \) and \( y'(L/2; z) \) are entire functions of exponential type \( L/2 \).

Firstly, we apply the estimates in Carlson (1993,1997,1994).

\[
|y(s; z) - \frac{\sin sz}{z}| \leq K \log(1 + |z|) \exp\{|3z|s\}/|z|^2,
\]

(28)

so we have

\[
y(s; z) = \frac{\sin sz}{z} + O(\frac{\log(1 + |z|) \exp\{|3z|s\} / |z|^2}{|z|^2}).
\]

(29)

The classic result (Pöschel & Trubowitz, 1987, p. 27) shows that there exists a constant \( C \), depending on the distance to the zeros of \( \sin sz \), such that the following inequality holds away from the zeros of \( \sin sz \):

\[
\exp\{|s3z|\} < C |\sin sz|.
\]

(30)

Hence,

\[
y(s; z) = \frac{\sin sz}{z} \left[ 1 + \frac{z}{\sin sz} O\left(\frac{\log(1 + |z|) \exp\{|3z|s\}}{|z|^2}\right) \right]
\]

\[
= \frac{\sin sz}{z} \left[ 1 + O\left(\frac{1}{|z|^{1-\epsilon}}\right)\right], \quad z \notin \frac{\pi Z}{s}.
\]

(31)

Thus, the indicator function of \( y(s; z) \) is

\[
h_{y(sz)}(\theta) = s \sin \theta, \quad \theta \neq 0, \pi.
\]

(32)

Because \( y(s; z) \) is entire in \( z \) for a fixed \( s \), \( h_{y(sz)}(\theta) \) is a continuous function of \( \theta \) (Levin, 1972). Thus,

\[
h_{y(sz)}(\theta) = s \sin \theta.
\]

(33)

Lemma 2.7 implies that \( y(s; z) \) is an entire function of exponential type \( s \). A similar argument holds for \( y'(s; z) \). That is,

\[
y'(s; z) = \cos sz\left[ 1 + O\left(\frac{1}{|z|^{1-\epsilon}}\right)\right], \quad z \notin \frac{\pi Z + \frac{s}{2}}{z};
\]

(34)

\[
h_{y'(sz)}(\theta) = s \sin \theta.
\]

(35)
This proves the lemma setting \( s = L / 2 \). □

**Lemma 2.10**  Let \( z_j \) be the zeros of \( y(s; z) \) and \( z'_j \) be the zeros of \( y'(s; z) \). Then,

\[
\begin{align*}
z_j &= \frac{j \pi}{s} + O\left(\frac{1}{j}\right), \quad j \in \mathbb{Z}; \\
z'_j &= \frac{j - \frac{1}{2} \pi}{s} + O\left(\frac{1}{j}\right), \quad j \in \mathbb{Z}.
\end{align*}
\]

(36)

This is a direct consequence of Rouché’s theorem on the boundary of a suitable sequence of neighborhoods containing the zeros of \( \sin sz \) or \( \cos sz \) under the estimates (2.9) and (2.12), respectively, by considering the following inequality:

\[
|y(s; z) - \frac{\sin(sz)}{z}| = O\left(\frac{1}{|z|^{1-s}}\right)\left|\frac{\sin(sz)}{z}\right|.
\]

(37)

We refer the detailed proof to Carlson (1993, 1997), Chen (2015a, 2015b), Pöschel and Trubowitz (1987). □

Therefore, there is an asymptotically uniform structure of eigenvalues of (1.3) for each \( \beta_k \)-eigenvalue. The first term in the asymptotics is independent of \( p(s) \). They overlay asymptotically periodically from one \( \beta_k \)-eigenvalue to another to give each cluster of eigenvalues of (1.3). Let \( N_k(v) \) be denoted as the counting function for eigenvalues in interval \([0, v]\) for each \( \beta_k \)-eigenvalues. We refer the structure of the eigenvalues of a zonal eigenvalues to (Gurarie, 1990, p. 576). We collect two kinds of spectra for each \( \beta_k \) by applying Lemmas 2.10 and (2.9):

\[
N_k(v) \sim \frac{Lv}{\pi^2}, \quad asv \to \infty.
\]

(38)

The locations of \( \{\beta_k\} \) of \( N \)-sphere are well known in Gurarie (1988) and Shubin (1987):

\[
\beta_k = k(k + N - 1), \quad k = 0, 1, \ldots,
\]

(39)

with increasing multiplicity \( d_k = \left(\begin{array}{c} N + k \\ N \end{array}\right) = O(k^{N-1}) \). Given an interval of length \( v \) starting at the origin, we have the quantity of

\[
1 - N + \sqrt{(N-1)^2 + 4v^2} / 2 \sum_{k=0}^{1-N} d_k \sim \frac{2v^{N/2}}{N!}
\]

(40)

of eigenvalues of \( \Delta_z \) (Shubin, 1987, p. 165). Hence,

\[
N(v) = \sum_{\beta_k} N_k(v) \sim \frac{2Lv^{N+1}}{\pi N!}, \quad asv \to \infty.
\]

(41)

This proves Theorem 1.1.

**3. Non-even zonal potentials**

Now, we drop the assumption that \( r \) and \( p \) are even functions in \( s \) in Theorem 1.1. We note that (1.7), (1.8), and (1.9) hold in \([0, L]\). Accordingly, we are considering a Schrödinger operator with a zonal potential \( p(s) \); we are dealing with the equation

\[
a^2 y + \left[ -\frac{N^2}{4} \left(\frac{d}{dr}\right)^2 - \frac{N}{2} \frac{d^2}{dr^2} + r^{-2} (s) \beta_k \right] y + p(s) y = z^2 y,
\]

(42)
where \( p \in L^2[0, L] \), \( r(-1) = r(1) = 0, 0 < r(z) < \infty \) for \(-1 \leq z \leq 1\) and \( r(s) = \sin(s \pi / L) \). Without the symmetry at \( s = L/2 \), we do not consider the zeros of \( y(I/z; z) \) and of \( y'(I/z; z) \) any more. After the linearization near \( s = 0 \) in (1.5) in Section 1, we consider a differential equation of the following form similar to (1.13) and (1.14).

\[
\frac{d^2_y}{s^2}y^{(l)} + \frac{-(l + 1)}{s^2} + p(s)y^{(l)} = z^2y^{(l)}, \quad l > 0,
\]

(43)

whose solutions are spanned by the Jost solutions \( \{f^{(l)}(s; z), f^{(l)}(s; -z)\} \) Faddeev (1960, (14.11)) and Reed and Simon (1979, p. 140), which satisfy the following integral equation

\[
f^{(l)}(s; z) = f^{(l)}_0(s; z) - \int_s^\infty J_1(s, t; z)p(t)f(t; z)\,dt,
\]

in which \( J_1(s, t; z) \) is defined as in (Faddeev, 1960, (14.12)). Thus,

\[
f^{(l)}(s; z) = f^{(l)}_0(s; z) = \hat{h}_1(sz),
\]

(45)

whenever \( s \) is beyond the support of the perturbation. In our case, \([0, L]; \hat{h}_1(sz)\) is the spherical Bessel function of second kind. Therefore, we write \( y^{(l)}(s; z) \) as

\[
y^{(l)}(s; z) = \alpha_l(z)f^{(l)}(s; z) + \beta_l(z)f^{(l)}(s; -z).
\]

(46)

In general, we recall the scattering formula in Faddeev (1960, (14.17)) for \( z \in \mathbb{O} + \mathbb{R} \):

\[
y^{(l)}(s; z) = \frac{1}{2iz}[(1 - iz)f^{(l)}(s; z)M^{(l)}(z) - (1 + iz)f^{(l)}(s; -z)\bar{M}^{(l)}(z)],
\]

(47)

where \( S^{(l)}(z) = \frac{M^{(l)}(z)}{\bar{M}^{(l)}(z)} \) gives the scattering matrix to equation (3.2). For any \( z \) that solves (3.6) is an eigenvalue of (3.2). Most important of all, the \( M \)-function \( M^{(l)}(z) \) and \( S^{(l)}(z) \) are independent of \( l \) (Faddeev, 1960, Theorem 14.1). The proof is given in Faddeev (1960, p. 90) and carries to continuous \( l \). Moreover, \( S^{(l)}(z) \) can be meromorphically extended from the upper half plane to the complex \( z \)-plane or \( \log z \)-plane without poles on the real axis except for the origin (Melrose, 1995, p. 16) depending on dimension parity. For our case, the dimension is one. Therefore, \( \tilde{M}^{(l)}(z) \) and \( \tilde{S}^{(l)}(z) \) can be defined in

\[
\Lambda = \{z \in \mathbb{C} | -\epsilon \leq \arg z \leq \pi + \epsilon, \text{ for some } \epsilon > 0\}.
\]

(48)

For some extension and uniqueness theory of \( M(z) \), we refer to Faddeev (1960, p. 42). Furthermore, the constants \( \alpha_l(z) \) and \( \beta_l(z) \) are independent of the space variable \( s \) and can be solved by the scattering theory in half line Aktosun, Gintides, and Papanicolaou (2013, p. 13) and Freiling and Yurko (2001) as follows:

\[
\begin{bmatrix}
 y^{(l)}(s; z) \\
 y'^{(l)}(s; z)
\end{bmatrix} = \begin{bmatrix}
 f^{(l)}(s; z) & f^{(l)}(s; -z) \\
 f'^{(l)}(s; z) & f'^{(l)}(s; -z)
\end{bmatrix} \begin{bmatrix}
 \alpha_l(z) \\
 \beta_l(z)
\end{bmatrix}.
\]

(49)

Evaluating (3.7) at \( s = 0 \), (2.13) implies that

\[
\begin{bmatrix}
 0 \\
 0
\end{bmatrix} = \begin{bmatrix}
 f^{(l)}(0; z) & f^{(l)}(0; -z) \\
 f'^{(l)}(0; z) & f'^{(l)}(0; -z)
\end{bmatrix} \begin{bmatrix}
 \alpha_l(z) \\
 \beta_l(z)
\end{bmatrix}, \quad l > 0;
\]

(50)

\[
\begin{bmatrix}
 0 \\
 1
\end{bmatrix} = \begin{bmatrix}
 f^{(l)}(0; z) & f^{(l)}(0; -z) \\
 f'^{(l)}(0; z) & f'^{(l)}(0; -z)
\end{bmatrix} \begin{bmatrix}
 \alpha_l(z) \\
 \beta_l(z)
\end{bmatrix}, \quad l = 0.
\]

Evaluating (3.7) at \( s = L \), we have
Only the case \( l = 0 \) is solvable which is found in Aktosun et al. (2013, p. 13). Equations (3.9) and (3.10) imply that

\[
\begin{bmatrix}
    y_1^{(0)}(L;z) \\
    y_1^{(0)'}(L;z)
\end{bmatrix}
= \begin{bmatrix}
    f_0^{(0)}(L;z) & f_0^{(0)'}(L; - z) \\
    f_0^{(0)'}(L;z) & f_0^{(0)'}(L; z)
\end{bmatrix}
\begin{bmatrix}
    a_i(z) \\
    b_i(z)
\end{bmatrix},
\]

(51)

in which \( h_0(sz) = \exp\{isz\} \). Therefore, the eigenvalues of (14) are \( z \)-solutions to the equation (3.12).

In general, by referring to Faddeev (1960, Lemma 1.5),

\[
f^{(0)}(s;z) = e^{isz} + o(e^{-\delta z}), \, \delta z \geq 0,
\]

(52)

uniformly for any \( s \geq 0 \). However, we need the behavior of \( M^{(0)}(z) \) slightly below the real axis (see Figure 1).

We recall the well-known integral equation (Faddeev, 1960, p. 38):

\[
M^{(0)}(k) = 1 + \int_0^\infty A(0, t)e^{ikt}dt, \, k \in \mathbb{C},
\]

(53)

where \( A(0, t) \) is compactly supported for our potential \( p(s) \). We refer the construction of \( A(0, t) \) to Faddeev (1960, p. 30). Thus, we observe that

\[
M^{(0)}(k - \delta i) = 1 + \int_0^\infty [A(0, t)e^{\delta t}]e^{ikt}dt, \, \delta > 0,
\]

(54)

to which we apply the Riemann–Lebesgue Lemma. Hence, the integral vanishes for large \(|k|\).

\[
M^{(0)}(k - \delta i) = 1 + o(1), \, \delta > 0, \, \text{as} |k| \to \infty.
\]

Moreover, \( e^{siz} \) are the exponential functions of type \( L \) by applying (2.4). We also infer from (2.14) and the complex analysis in section 2 that the right-hand side of (3.12) is an analytic function of order one and at most of type \( L \).

**Figure 1. Rouché’s Theorem.**
Because $M^{(l)} = M^{(l)}_0$ for any $l$, we take $f^{(l)}_0(0; z) = M^{(l)}_0(z)$ for any $l$ (Faddeev, 1960, Theorem 14.1). By (3.6) and (3.12), it suffices to study the zeros of the following analytic function

$$F(z) = y^{(l)}(L; z) - \frac{1}{2iz} \left( \frac{1}{iz} \right) f^{(l)}_0(0; -z) e^{iz} + \frac{1}{2iz} \left( \frac{1}{iz} \right) f^{(l)}_0(0; z) e^{-iz},$$

wherein (2.6) suggests that (3.17) is an analytic function of type at most $\alpha$. By (3.6) and (3.12), it suffices to study the zeros of the following analytic function

$$F(z) = \frac{\sin Lz}{z^{1+1}} \left( 1 + O\left( \frac{1}{|z|^{1+\alpha}} \right) \right) - \frac{1}{2iz} \left( \frac{1}{iz} \right) f^{(l)}_0(0; -z) e^{iz} + \frac{1}{2iz} \left( \frac{1}{iz} \right) f^{(l)}_0(0; z) e^{-iz},$$

for any $z \not\in \frac{\pi z}{S}$, $z \in \Lambda$. to which we use (2.16) again with (3.16) and deduce that

$$F(z) = \frac{\sin Lz}{z^{1+1}} \left( 1 + O\left( \frac{1}{|z|^{1+\alpha}} \right) \right) + o(1)$$

$$= \frac{\sin Lz}{z^{1+1}} \left(1 + o(1)\right), \quad z \not\in \frac{\pi z}{S}, \quad z \in \Lambda.$$

Hence, we apply Rouche’s theorem in a strip of width less than $\delta$ containing $0 + \mathbb{R} \setminus \{0\}$ in the interior of Figure 1. There are only finitely many zeros outside this strip when examined (3.19) by Rouche’s theorem. Let $N_\delta(v)$ be the eigenvalue counting function inside the strip $\{z: 0 < \Re z < v; 0 < |z| < \delta\}$ for each $\beta_\epsilon$-eigenvalue. Rouche’s theorem also suggests that the zeros of $F(z)$ asymptotically periodically approach to the ones of $\sin Lz$ with the eigenvalue density described by Lemma 2.10. Then,

$$N_\delta(v) \sim \frac{Lv}{\pi}, \quad as \quad v \to \infty.$$  

Previously, we have obtained this in (2.25) for even metrics. We repeat the same argument using (2.26), (2.27), and (2.28). Thus, (1.15) follows again for the asymptotics for non-even metrics.

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