Projective Curvature Tensor on $N(\kappa)$–Contact Metric Manifold Admitting Semi-Symmetric Non-Metric Connection

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Abstract

The object of the present paper is to classify $N(\kappa)$-contact metric manifolds admitting the semi-symmetric non-metric connection with certain curvature conditions the projectively curvature tensor. We studied projective flat, $\xi$–projectively flat, $\phi$–projectively flat $N(\kappa)$-contact metric manifolds admitting the semi-symmetric non-metric connection. Also, we examine such manifolds under some local symmetry conditions related to projective curvature tensor.

1. Introduction

An almost contact metric manifold is a $(2n + 1)$–dimensional differentiable manifold with a structure $(\phi, \xi, \eta, g)$ such as

$$\phi^2(W_1) = -W_1 + \eta(W_1)\xi, \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta(\phi(W_1)) = 0, g(\phi(W_1), \phi(W_2)) = g(W_1, W_2) - \eta((W_1))\eta((W_2))$$

for any vector fields $W_1, W_2 \in \chi(M)$, where $g$ is Riemannian metric, $\phi$ is a $(1,1)$–tensor field, $\xi$ is a vector field and $\eta$ is a 1–form on $M$ [1]. Blair, et al. [2] introduced the $(\kappa, \mu)$-nullity distribution of an almost contact metric manifold $M$ that is defined by

$$N(\kappa, \mu): p \longrightarrow N_p(\kappa, \mu)$$

$$N_p(\kappa, \mu) = \{ W_3 \in \Gamma(T_pM) : R(W_1, W_2)W_3 = (\kappa I + \mu h)[g(W_2, W_3)W_1 - g(W_1, W_3)W_2] \}$$

for all $W_1, W_2 \in \Gamma(TM)$, where $\kappa$ and $\mu$ are real constants and $p \in M$. If $\xi \in N(\kappa, \mu)$, then $M$ is called $(\kappa, \mu)$–contact metric manifold. If $\mu = 0$, the $(\kappa, \mu)$-nullity distribution reduces to $\kappa$-nullity distribution.

The idea of $\kappa$-nullity distribution on a contact metric manifold was firstly presented by Tanno in 1988 [3]. $\kappa$-nullity distribution of an almost contact metric manifold $(M, \phi, \xi, \eta, g)$ is a distribution defined as

$$N(\kappa): p \longrightarrow N_p(\kappa) = \{ W_3 \in \Gamma(T_pM) : R(W_1, W_2)W_3 = \kappa [g(W_2, W_3)W_1 - g(W_1, W_3)W_2] \}$$

for any $W_1, W_2 \in \Gamma(TM)$ and $\kappa \in \mathbb{R}$, where $R$ is the Riemannian curvature tensor of $M$. If $\xi$ belongs to $\kappa$–nullity distribution then $M$ is called $N(\kappa)$–contact metric manifold. Thus on a $N(\kappa)$ contact metric manifold, we have

$$R(W_1, W_2)\xi = \kappa[\eta(W_2)W_1 - \eta(W_1)W_2].$$
A $N(\kappa)$-contact metric manifold is Sasakian if and only if $k = 1$. Also, if $k = 0$, then the manifold is locally isometric to the product $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$ [4]. The Riemannian geometry of $N(\kappa)$–contact metric manifolds is studied in [2]. [5]-[9]. Levi-Civita connection $\nabla$ is a torsion free, i.e has zero torsion, and a metric connection, i.e $\nabla g = 0$. There are some kinds of linear connections except for Levi-Civita connection which is not need to be torsion free or metric. One of them is semi-symmetric non-metric connection [10]. Manifolds with semi-symmetric non-metric connection have been studied by many researchers [11]-[15]. In the Riemannian geometry of contact manifolds curvature tensors-such as conformal, concircular, projective curvature tensor etc.-have important applications. Some of geometric properties of structure on manifolds have been examined by the certain conditions on these curvature tensors. Many works on contact manifolds are stated in [16]-[23].

In this paper we study projective curvature tensor on $N(\kappa)$-contact metric manifolds with semi-symmetric non metric connection. In [24], Barman gave the curvature relations on such as manifolds. We use these properties and we examine flatness conditions of projective curvature tensor. Specifically, we given results for $\xi$-projectively flat, pseudo-quasi-projectively flat and $\phi$–projectively flat on $N(\kappa)$-contact metric manifolds with semi-symmetric non metric connection. After we investigate $\phi$–projectively semi-symmetric on $N(\kappa)$-contact manifolds admitting the semi-symmetric non-metric connection, we characterize this manifolds satisfying $Q.P = 0$ and $S.P = 0$, where $P, Q, Ric$ are projective curvature tensor, Ricci tensor, Ricci curvature tensor, with a semi-symmetric non metric connection, respectively.

2. Preliminaries

Let $(M, \phi, \xi, \eta, g)$ be an almost contact metric manifold. The $h = \frac{1}{2} \mathcal{L}_\xi \phi$, $\mathcal{L}_\xi$ denotes the Lie derivative along vector field $\xi$. For any $W_1 \in \Gamma(TM)$, we have

$$\nabla_{W_1} \xi = -\phi W_1 - \phi h W_1$$

An almost contact metric manifold $M$ is called $K$–contact if $\xi$ is Killing vector field. $M$ is called normal contact metric manifold if $N_\phi + 2d\eta \otimes \xi = 0$, where $N_\phi$ is the Nijenhuis tensor of $\phi$. A normal contact metric manifold is called Sasakian.

On the other hand a contact metric manifold is Sasakian if and only if

$$R(W_1, W_2)\xi = [\eta(W_2)W_1 - \eta(W_1)W_2]$$

for all $W_1, W_2 \in \Gamma(TM)$. On a $K$–contact and Sasakian manifold $h = 0$.

A $N(\kappa)$-contact metric manifold is Sasakian if $\kappa = 1$. $N(\kappa)$-contact metric manifolds are characterized the different values of $\kappa$. As we mentioned in the introduction when $\kappa = 0$ then the manifold $M$ is locally isometric to $E^{(n+1)}(0) \times S^n(4)$. On a $N(\kappa)$-contact metric manifold $M^{2n+1}$, we have following relations (for details see [1]):

$$(\nabla_{W_1} \phi) W_2 = g(W_1 + h W_2, W_2)\xi - \eta(W_2)(W_1 + h W_1),$$
$$(\nabla_{W_1} \eta) W_2 = g(W_1 + h W_1, \phi W_2).$$

The Riemannian curvature $R$ of a $N(\kappa)$–contact metric manifold has following properties:

$$R(W_1, W_2)\xi = \kappa[\eta(W_2)W_1 - \eta(W_1)W_2]$$

$$(\xi, W_1)W_2 = \kappa[g(W_1, W_2)\xi - \eta(W_2)W_1]$$

for all $W_1, W_2 \in \Gamma(TM)$. On the other hand the Ricci curvature of $M$ is stated as [1]:

$$Ric(W_1, W_2) = 2(n - 1)g(W_1, W_2) + 2(n - 1)g(hW_1, W_2) + 2(n\kappa - (n - 1))\eta(W_1)\eta(W_2)$$

$$Ric(\phi W_1, \phi W_2) = Ric(W_1, W_2) - 2n\kappa\eta(W_1)\eta(W_2) - 4(n - 1)g(hW_1, W_2)$$

$$Ric(W_1, \xi) = 2\kappa\eta(W_1)$$

and the scalar curvature is given by

$$r = 2n(2n + \kappa - 2).$$

Example 2.1. E. Boeckx [25] gave a classification for non-Sasakian $(\kappa, \mu)$–spaces. The number $I_M = \frac{1 - \frac{n}{2}}{1 - \frac{\mu}{2}}$ is called by Boeckx invariant. D.E. Blair, et al. [26] gave an example of $N(\kappa)$–contact metric manifolds by using Boeckx invariant. They constructed $(2n + 1)$-dimensional $N(1 - \frac{1}{n})$-contact metric manifold, $n > 1$. For details see [26].

Let define a map $V$ on a Riemann manifold $M$ as

$$\nabla \eta W_2 = \nabla W_1 + \eta(W_2)W_1$$
where $\nabla^*$ is Levi-Civita connection on $M$. This map is a linear connection. The torsion of $\nabla^*$ is given by

$$
\mathbf{T}^*(W_1, W_2) = \eta(W_2)W_1 - \eta(W_1)W_2
$$

for all $W_1, W_2 \in \Gamma(TM)$. Also we have

$$
(\mathbf{\nabla}^*_{U, g})(W_1, W_2) = -\eta(W_1)g(W_2, U) - \eta(W_2)g(W_1, U) \neq 0.
$$

Thus $\nabla^*$ is not symmetric and not metric connection. This type of connection is called by semi-symmetric non-metric connection [10].

$N(\kappa)$–contact metric manifolds with a semi-symmetric non-metric connection were studied by Barman [24]. For the sake of brevity we denote $(M, \nabla^*)$ by a $N(\kappa)$–contact metric manifolds with a semi-symmetric non-metric connection. Barman gave the curvature of $(M, \nabla^*)$ as follow:

$$
\mathbf{\nabla}^*\mathbf{R}(W_1, W_2)W_3 = \mathbf{R}(W_1, W_2)W_3 + g(W_1, \phi W_3)W_2 + g(h W_1, \phi W_3)W_2 - \eta(W_1)\eta(W_3)W_2 - g(W_2, \phi W_3)W_1
$$

for all $W_1, W_2, W_3 \in \Gamma(TM)$. Also we have

$$
(\mathbf{\nabla}^*_{U, g})(W_1, W_2) = -\eta(W_1)g(W_2, U) - \eta(W_2)g(W_1, U) \neq 0.
$$

Thus, we have following curvature properties [24]:

$$
\mathbf{\nabla}^*\mathbf{R}(\xi, W_2)W_3 = \kappa g(W_2, W_3)\xi - (\kappa + 1)\eta(W_2)W_3 - g(W_2, \phi W_3)\xi - g(h W_2, \phi W_3)\xi + \eta(W_1)\eta(W_2)\xi
$$

$$
\mathbf{\nabla}^*\mathbf{R}(\xi, W_2)\xi = (\kappa + 1)(\eta(W_2)\xi - W_2)
$$

$$
\mathbf{\nabla}^*\mathbf{R}(W_1, W_2)\xi = (\kappa + 1)(\eta(W_2)W_1 - \eta(W_1)W_2).
$$

The Ricci curvature of a $(M, \nabla^*)$ is given by

$$
\mathbf{R}ic^*(W_2, W_3) = \mathbf{R}ic(W_2, W_3) - 2ng(W_2, \phi W_3) - 2ng(h W_2, \phi W_3) + 2n\eta(W_3)\eta(W_2).
$$

Thus, we have

$$
\mathbf{R}ic^*(W_2, \xi) = 2n(\kappa + 1)\eta(W_2)
$$

$$
\mathbf{\nabla}^*\mathbf{R}ic(W_2, W_3) = 2n(\kappa + 1)\eta(W_2)
$$

$$
\mathbf{\nabla}^*\mathbf{R}ic(W_2, \xi) = 2n(\kappa + 1)\eta(W_2)
$$

where $\mathbf{R}ic$, $\mathbf{R}$ and $\mathbf{\nabla}^*\mathbf{R}$ are the Ricci tensor, the Riemann curvature tensor and scalar curvature admitting the semi-symmetric non-metric connection respectively [24].

The projective curvature tensor $P$ admitting the semi-symmetric non-metric connection is defined by

$$
P^*(W_1, W_2)W_3 = \mathbf{R}^*(W_1, W_2)W_3 - \frac{1}{2n}\left(\mathbf{R}ic^*(W_2, W_3)W_1 - \mathbf{R}ic(W_2, W_3)W_1\right),
$$

for all $W_1, W_2, W_3 \in \Gamma(TM)$.

3. Flatness conditions of projective curvature tensor on $(M, \nabla^*)$

In this section, we examine that a $(M, \nabla^*)$ is $\xi$-projectively flat, pseudo-quasi-projectively flat and $\phi$ – projectively flat.

**Definition 3.1.** A $(M, \nabla^*)$ is called

- $\xi$-projectively flat if we have $P^*(W_1, W_2)\xi = 0$ for all $W_1, W_2 \in \Gamma(TM)$,
- pseudo-quasi-projectively flat if we have $g(P^*(\phi W_1, W_2)W_3, \phi W_4) = 0$ for all $W_1, W_2, W_3 \in \Gamma(TM)$,
- $\phi$-projectively flat if we have $g(P^*(\phi W_1, \phi W_2)W_3, \phi W_4) = 0$ for all $W_1, W_2, W_3 \in \Gamma(TM)$.

**Theorem 3.2.** A $(M, \nabla^*)$ is always $\xi$-projectively flat.
Proof. By putting $W_3 = \xi$ in (2.10), we obtain
\[
\hat{P}(W_1, W_2)\xi = \hat{R}(W_1, W_2)\xi - \frac{1}{2n} \left( \hat{R}(W_2, \xi)W_1 - \hat{R}(W_1, \xi)W_2 \right).
\]
Also from (2.7) and (2.9), we get
\[
\hat{P}(W_1, W_2)\xi = R(W_1, W_2)\xi - \eta(W_1)W_2 + \eta(W_2)W_1 - \frac{1}{2n} (2n(k + 1)\eta(W_2)W_1 - 2n(k + 1)\eta(W_1)W_2).
\]
and take into account (2.1), we have
\[
\hat{P}(W_1, W_2)\xi = 0 \tag{3.1}
\]
for all $W_1, W_2 \in \Gamma(TM)$.

**Theorem 3.3.** If $(M, \nabla)$ is pseudo-quasi-projectively flat, then $M$ is an Einstein manifold admitting Levi-Civita connection.

Proof. Using (2.10), we have
\[
g(\hat{P}(\phi W_1, W_2)W_3, \phi W_4) = \hat{R}(\phi W_1, W_2, W_3, \phi W_4) - \frac{1}{2n} [\hat{R}(\phi W_2, W_3)g(\phi W_1, \phi W_4) - \hat{R}(\phi W_1, W_3)g(W_2, \phi W_4)]. \tag{3.2}
\]
Let $(M, \nabla)$ be a pseudo-quasi-projectively flat. Then, by using (2.8) in (3.2), it follows that
\[
\hat{R}(\phi W_1, W_2, W_3, \phi W_4) = \frac{1}{2n} [(Ric(W_2, W_3) - 2ng(W_2, \phi W_3) - 2ng(hW_2, \phi W_3) + 2n\eta(W_2)\eta(W_3))g(W_1, \phi W_4)
- (Ric(\phi W_1, W_3) - 2ng(\phi W_1, \phi W_3) - 2ng(h\phi W_1, \phi W_3))g(W_2, \phi W_4)]
\]
and from (2.6), we get
\[
R(\phi W_1, W_2, W_3, \phi W_4) = \frac{1}{2n} [Ric(W_2, W_3)g(\phi W_1, \phi W_4) - Ric(\phi W_1, W_3)g(W_2, \phi W_4)]. \tag{3.3}
\]
Take a local orthonormal basis set of $M$ as $\{e_1, e_2, ..., e_{2n}, \xi\}$, then $\{\phi e_1, \phi e_2, ..., \phi e_{2n}, \phi \xi\}$ is also a local orthonormal basis. Putting $W_1 = W_4 = e_i$ in (3.3) and summing over $i = 1$ to $2n$, we get
\[
\sum_{i=1}^{2n} R(\phi e_i, W_2, W_3, \phi e_i) = \frac{1}{2n} \left[ \sum_{i=1}^{2n} (Ric(W_2, W_3)g(\phi e_i, \phi e_i) - Ric(\phi e_i, W_3)g(W_2, \phi e_i)) \right].
\]
From (2.2) and (2.5), we obtain
\[
Ric(W_2, W_3) = 2n\kappa g(W_2, W_3).
\]

**Theorem 3.4.** Let $(M, \nabla)$ be $\phi$-projectively flat. If $\xi$ is Killing vector field, then the manifold is an Einstein manifold.

Proof. Firstly, putting $W_2 = \phi W_2$ and $W_3 = \phi W_3$ in (3.2), we get
\[
g(\hat{P}(\phi W_1, \phi W_2)\phi W_3, \phi W_4) = \hat{R}(\phi W_1, \phi W_2, \phi W_3, \phi W_4) - \frac{1}{2n} \left( \hat{R}(\phi W_2, \phi W_3)g(\phi W_1, \phi W_4) - \hat{R}(\phi W_1, \phi W_3)g(W_2, \phi W_4) \right). \tag{3.4}
\]
Now, by using (2.8) in (3.4) and from definition of $\phi$-projectively flat, it follows that
\[
\hat{R}(\phi W_1, W_2, W_3, \phi W_4) = \frac{1}{2n} [(Ric(\phi W_2, \phi W_3) - 2ng(\phi W_2, \phi^2 W_3) - 2ng(h\phi W_2, \phi^2 W_3))g(W_1, \phi W_4)
- (Ric(\phi W_1, \phi W_3) - 2ng(\phi W_1, \phi^2 W_3) - 2ng(h\phi W_1, \phi^2 W_3))g(W_2, \phi W_4)]
\]
and from (1.1), we get
\[
R(\phi W_1, \phi W_2, \phi W_3, \phi W_4) = \frac{1}{2n} (Ric(\phi W_2, \phi W_3)g(W_1, \phi W_4) - Ric(\phi W_1, \phi W_3)g(W_2, \phi W_4)). \tag{3.5}
\]
For local orthonormal basis $\{e_1, e_2, ..., e_{2n}, \phi e_1, \phi e_2, ..., \phi e_{2n}, \xi\}$ of $M$ by putting $W_1 = W_4 = e_i$ in (3.5) and summing over $i = 1$ to $2n$, we get
\[
\sum_{i=1}^{2n} R(\phi e_i, W_2, W_3, \phi e_i) = \frac{1}{2n} \left[ \sum_{i=1}^{2n} (Ric(W_2, W_3)g(\phi e_i, \phi e_i) - Ric(\phi e_i, W_3)g(W_2, \phi e_i)) \right]
\]
From (2.2) and (2.5), we obtain
\[
Ric(\phi W_2, \phi W_3) = 2n\kappa g(\phi W_2, \phi W_3).
\]
Also, from (2.4) we have
\[
Ric(W_2, W_3) = 2n\kappa g(W_2, W_3) + 4(n - 1)g(hW_2, W_3)
\]
If $\xi$ is Killing vector field then $M$ is an Einstein manifold.
4. Symmetry conditions admitting projective curvature tensor on \((M, \hat{\nabla})\)

In this section, we study on a \((M, \hat{\nabla})\) under certain symmetry conditions. We firstly examine \(\phi\)–projectively semi-symmetric \((M, \hat{\nabla})\) and then we characterize this manifolds satisfying \(\hat{\nabla} \cdot \hat{P} = 0\) and \(\hat{Ric} \cdot \hat{P} = 0\), where \(\hat{Q}\) is the Ricci operator defined by \(\hat{Ric}(W_1, W_2) = g(\hat{Q}W_1, W_2)\).

**Definition 4.1.** A \((M, \hat{\nabla})\) is said to be \(\phi\)–projectively semisymmetric if \(\hat{P}(W_1, W_2)\phi = 0\) for all \(W_1, W_2 \in \Gamma(M)\).

**Theorem 4.2.** A \(\phi\)–projectively \((M, \hat{\nabla})\) is isometric to Example 2.1.

**Proof.** Suppose \((M, \hat{\nabla})\) be a \(\phi\)–projectively. Then, we have

\[
\hat{P}(W_1, W_2)\phi W_3 - \phi(\hat{P}(W_1, W_2) W_3) = 0.
\]

From (2.10), it follows that

\[
\hat{P}(W_1, W_2)\phi W_3 = \hat{R}(W_1, W_2)\phi W_3 - \frac{1}{2n} \left( \hat{Ric}(W_2, \phi W_3) W_1 - \hat{Ric}(W_1, \phi W_3) W_2 \right).
\]

Using (2.8) in (4.2), we obtain

\[
\hat{P}(W_1, W_2)\phi W_3 = \hat{R}(W_1, W_2)\phi W_3 - \frac{1}{2n} \left( \hat{Ric}(W_2, \phi W_3) W_1 - \hat{Ric}(W_1, \phi W_3) W_2 \right).
\]

From (2.1), (2.2) and (2.6), we have

\[
\hat{P}(W_1, W_2)\phi W_3 = \kappa g(W_2, \phi W_3) W_1 - \kappa g(W_1, \phi W_3) W_2 - \frac{1}{2n} \hat{Ric}(W_2, \phi W_3) W_1 + \frac{1}{2n} \hat{Ric}(W_1, \phi W_3) W_2.
\]

Also, by applying \(\phi\) to \(\hat{P}\), we get

\[
\phi(\hat{P}(W_1, W_2) W_3) = \phi(\hat{R}(W_1, W_2) W_3) - \frac{1}{2n} \phi \left[ \hat{Ric}(W_2, \phi W_3) W_1 - \hat{Ric}(W_1, \phi W_3) W_2 \right],
\]

and using (2.8) in (4.4) yields

\[
\phi(\hat{P}(W_1, W_2) W_3) = \phi(\hat{R}(W_1, W_2) W_3) - \frac{1}{2n} \left( \hat{Ric}(W_2, W_3) W_1 - \hat{Ric}(W_1, W_3) W_2 \right).
\]

Thus from (4.6) and (2.6), we have

\[
\phi(\hat{P}(W_1, W_2) W_3) = \kappa g(W_2, W_3) W_1 - \kappa g(W_1, W_3) W_2 - \frac{1}{2n} \hat{Ric}(W_2, W_3) W_1 + \frac{1}{2n} \hat{Ric}(W_1, W_3) W_2.
\]

Putting (2.3), (4.3) and (4.5) in (4.1), we have

\[
\hat{P}(W_1, W_2)\phi W_3 - \phi(\hat{P}(W_1, W_2) W_3) = \kappa g(W_2, \phi W_3) W_1 - \kappa g(W_1, \phi W_3) W_2 - \frac{2(n-1)}{2n} \left[ g(W_2, W_3) g(W_1, \phi W_3) \right] W_1
\]

\[
+ \frac{2}{2n} \left[ g(W_1, W_3) g(W_2, \phi W_3) \right] W_2 - \kappa g(W_2, W_3) W_1 + \kappa g(W_1, W_3) W_2
\]

\[
+ \frac{1}{2n} [2(n-1)(g(W_2, W_3) + g(W_1, W_3)] W_1 + 2(n-k) \eta(W_2) \eta(W_3)] \phi W_1
\]

\[
- \frac{1}{2n} [2(n-1)(g(W_1, W_3) + g(W_2, W_3)] W_1 + 2(n-k) \eta(W_1) \eta(W_3)] \phi W_2
\]

Let take inner product with \(W_4\) of (4.6) and then to contract \(W_2\) and \(W_4\), we obtain

\[
\left\{ 2\kappa(1-n) + \frac{n^2 - 2n + 1}{n} \right\} g(W_1, W_5) + \left\{ 2(n-1) \right\} g(W_1, W_5) W_5 = 0.
\]
Now, putting $W_3 = \phi W_3$ in (4.7) and from (1.1), we get
$$
\left\{ 2\kappa(1 - n) + 2\left(\frac{n^2 - 2n + 1}{n}\right) \right\} g(\phi W_1, \phi W_3) + \{2(n - 1)\} g(hW_1, W_3) = 0.
$$
(4.8)
Taking trace in both sides of (4.8) and using $trh = 0$, we obtain
$$
\kappa = \frac{n - 1}{n}.
$$
Thus $M$ is isometric to Example 2.1.

**Theorem 4.3.** On a $(M, \tilde{\nabla})$, we have $\tilde{Q} \tilde{P} = 0$.

*Proof.* For all $W_1, W_2, W_3 \in \Gamma(TM)$, we have
$$
(\tilde{Q}(W_1) \tilde{P})(W_2, W_3) = \tilde{Q}(\tilde{P}(W_1, W_2)W_3) - \tilde{P}(\tilde{Q}W_1, W_2)W_3 - \tilde{P}(W_1, \tilde{Q}W_2)W_3 - \tilde{P}(W_1, W_2)\tilde{Q}W_3.
$$
(4.9)
From (2.8) and (2.9), we have
$$
\tilde{Q}W_2 = 2(n - 1)(W_2 + hW_2) + 2n(\phi W_2 + \phi hW_2) + 2(n\kappa + 1)\eta(W_2) \xi
$$
(4.10)
and so
$$
\tilde{Q} \xi = 2n(\kappa + 1) \xi.
$$
(4.11)
Thus, for $W_3 = \xi$ in (4.9) we get
$$
(\tilde{Q}(W_1) \tilde{P})(W_2, \xi) = \tilde{Q}(\tilde{P}(W_1, \xi)W_2) - \tilde{P}(\tilde{Q}W_1, W_2)\xi - \tilde{P}(W_1, \tilde{Q}W_2)\xi - \tilde{P}(W_1, W_2)\tilde{Q}\xi.
$$
From (3.1), (4.10) and (4.11), it follows that
$$
\tilde{Q} \tilde{P} = 0.
$$

**Theorem 4.4.** A $(M, \tilde{\nabla})$ satisfies $\tilde{P} \tilde{Ric} = 0$ if and only if $M$ is an Einstein manifold.

*Proof.* Let $\tilde{P} \tilde{Ric} = 0$ satisfies on $(M, \tilde{\nabla})$, then we get
$$
\tilde{Ric}(\tilde{P}(W_4, W_2)W_3, W_1) + \tilde{Ric}(\tilde{P}(W_4, W_2)W_1, W_3) = 0.
$$
(4.12)
Putting $W_1 = W_4 = \xi$ in (4.12), we have
$$
\tilde{Ric}(\tilde{P}(\xi, W_2)W_3, \xi) + \tilde{Ric}(\tilde{P}(\xi, W_2)W_1, \xi) = 0.
$$
(4.13)
Also, from (2.10), we get
$$
\tilde{P}(\xi, W_2)W_3 = \tilde{R}(\xi, W_2)W_3 - \frac{1}{2n} \left( \tilde{Ric}(W_2, W_3)\xi - \tilde{Ric}(\xi, W_3)W_2 \right),
$$
from (2.7), (2.8), (2.9), it follows that
$$
\tilde{P}(\xi, W_2)W_3 = \kappa g(W_2, W_3)\xi - \frac{1}{2n} \tilde{Ric}(W_2, W_3)\xi.
$$
(4.14)
Again putting $W_3 = \xi$ in (4.14) and using (2.5), we obtain
$$
\tilde{P}(\xi, W_2)\xi = 0.
$$
(4.15)
Using (2.9), (4.14) and (4.15) in (4.13), it follows that
$$
\tilde{Ric}(W_2, W_3) = 2n\kappa g(W_2, W_3).
$$
Conversely, let $M$ be an Einstein manifold, i.e $\tilde{Ric}(W_2, W_3) = 2n\kappa g(W_2, W_3)$. Then, we get
$$
\tilde{P}(W_1, W_2)W_3 = \kappa (g(W_2, W_3)W_1 - g(W_1, W_3)W_2) - \frac{1}{2n} (2n\kappa g(W_2, W_3)W_1 - 2n\kappa g(W_1, W_3)W_2).
$$
which implies $\tilde{P}(W_1, W_2)W_3 = 0$. This also give us $\tilde{P} \tilde{Ric} = 0$. 

\qed
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