Examining the validity of Schatten-$p$-norm-based functionals as coherence measures

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I. INTRODUCTION

Quantum coherence is a fundamental property of quantum mechanics and describes the capability of a quantum state to exhibit quantum interference phenomena. It provides an important resource for various quantum information processing tasks, such as quantum algorithms, quantum cryptography, nanoscale thermodynamics, quantum metrology, and quantum biology. The resource theory of coherence has attracted a growing interest due to the rapid development of quantum information science.

All quantum resource theories have two fundamental ingredients: free states and free operations. For the resource theory of coherence, the free states are the quantum states that are diagonal in a prefixed reference basis. However, there is no general consensus on the definition of free operations. With different physical and mathematical considerations, researchers have proposed a number of free operations, such as maximally incoherent operations, incoherent operations, strictly incoherent operations, and genuinely incoherent operations, respectively, where $I$ is the set of incoherent states and $\Delta \rho$ is the diagonal part of density operator $\rho$. Of these questions, all we know is that $C_1(\rho)$ is not a valid coherence measure under incoherent operations and strictly incoherent operations, but all other aspects remain open. In this paper, we prove that (1) $C_1(\rho)$ is a valid coherence measure under both strictly incoherent operations and genuinely incoherent operations but not a valid coherence measure under incoherent operations, (2) $C_1(\rho)$ is not a valid coherence measure even under genuinely incoherent operations, and (3) neither $C_{p,1}(\rho)$ nor $\tilde{C}_{p,1}(\rho)$ is a valid coherence measure under any of the three sets of operations. This paper not only provides a thorough examination on the validity of taking $C_p(\rho)$ and $\tilde{C}_p(\rho)$ as coherence measures, but also finds an example that fulfills the monotonicity under strictly incoherent operations but violates it under incoherent operations.

measurements on average, and (C4) the nonincreasing of coherence under mixing of quantum states. A functional of density operators can be taken as a valid coherence measure if and only if it satisfies the four conditions.

Based on the notions of incoherent states and free operations, frameworks for quantifying coherence can be established. A framework for the quantification of coherence usually consists of four conditions: (C1) the coherence being zero (positive) for incoherent states (all other states), (C2) the monotonicity of coherence under free operations of coherence, (C3) the monotonicity of coherence under selective

\[ C_p(\rho) = \min_{\sigma \in I} \| \rho - \sigma \|_p \]  

(1)

and

\[ \tilde{C}_p(\rho) = \| \rho - \Delta \rho \|_p \]  

(2)

with $p \geq 1$, which were proposed in Refs. [8] and [10], re-
It has monotonicity under free operations:

\[ \sum_{i=0}^{d-1} |i⟩⟨i| \] is the diagonal part of density operator \( \rho \).

\[ C_ρ(\rho) \]

represents the set of incoherent states, and \( \Delta \rho = \sum_{i=0}^{d-1} |i⟩⟨i| \] is the diagonal part of density operator \( \rho \).

It has been asked by different authors whether the Schatten-\( p \)-norm-based functional are valid coherence measures under incoherent operations, strictly incoherent operations, and genuinely incoherent operations, respectively. Several previous papers have addressed these questions \[ \{8, 10, 46, 51, 52\} \].

However, it is a nontrivial work to answer them. For example, \( C_1(\rho) \), i.e., \( C_p(\rho) \) for \( p = 1 \), had been expected as a valid coherence measure under incoherent operations, but it finally proved invalid by the authors in Ref. \[ 46 \] after several efforts made by others \[ \{51, 52\} \]. Also, \( C_{p>1}(\rho) \) has been proved not to be a valid coherence measure under both incoherent operations and strictly incoherent operations in Ref. \[ 52 \] but it is unclear whether \( C_{p>1}(\rho) \) is a valid coherence measure or not under genuinely incoherent operations. \( C_{p>1}(\rho) \) has been proved to satisfy conditions \( (C_1), (C_2), \) and \( (C_4) \) under genuinely incoherent operations \[ 10 \] but has not been proved to satisfy condition \( (C_3) \) or not, and therefore it is not clear whether \( C_{p>1}(\rho) \) is a valid coherence measure under genuinely incoherent operations, not to mention it under incoherent operations or strictly incoherent operations. So far, about whether the two classes of Schatten-\( p \)-norm-based functionals can be taken as valid coherence measures under incoherent operations, strictly incoherent operations, and genuinely incoherent operations, all we know is that \( C_p(\rho) \) is not a valid coherence measure under incoherent operations and strictly incoherent operations, but all other aspects remain open, as shown in Table I. In this paper, we will resolve these open questions, filling up the gaps in the table.

The paper is organized as follows. In Sec. II, we review some notions related to coherence measures. In Sec. III, we present one by one our main results on whether \( C_p(\rho) \) and \( C_{p>1}(\rho) \) can be taken as coherence measures. Section IV presents our conclusions.

## II. PRELIMINARIES

We recapitulate some notions related to coherence measures, such as incoherent states, incoherent operations, strictly incoherent operations, genuinely incoherent operations, and frameworks of quantifying coherence.

Let \( \mathcal{H} \) represent the Hilbert space of a \( d \)-dimensional quantum system. A particular basis of \( \mathcal{H} \) is denoted as \([|i⟩, i = 0, 1, \ldots, d-1]\), known as the incoherent basis, which is chosen according to the physical problem under consideration. The coherence of a state is then measured based on the basis chosen. We use \( \rho = \sum_{i,j} \rho_{ij} |i⟩⟨j| \) to denote a general density operator in the basis, where \( \rho_{ij} \) are the elements of the density matrix. A state is called an incoherent state, specially denoted as \( \sigma \), if its density operator is diagonal in the basis, and the set of all incoherent states is denoted by \( \mathcal{I} \). It follows that a density operator \( \rho \) belonging to \( \mathcal{I} \) is of the form

\[ \sigma = \sum_{i=0}^{d-1} \sigma_{ii} |i⟩⟨i|. \]

All other states which cannot be written as diagonal matrices in the basis are called coherent states.

An incoherent operation \[ \{8\} \] is defined by a completely positive and trace preserving map,

\[ \Lambda(\rho) = \sum_n K_n \rho K_n^\dagger \]

with the Kraus operators fulfilling not only \( \sum_n K_n^\dagger K_n = 1 \) but also

\[ K_n^\dagger K_n \subset \mathcal{I}, \]

i.e., each \( K_n \) maps an incoherent state to an incoherent state.

A strictly incoherent operation \[ \{10\} \] is defined by a completely positive and trace preserving map, \( \Lambda(\rho) = \sum_n K_n \rho K_n^\dagger \) with the Kraus operators fulfilling not only \( \sum_n K_n^\dagger K_n = 1 \) but also

\[ K_n^\dagger K_n \subset \mathcal{I}, \]

i.e., each \( K_n \) as well \( K_n^\dagger \) maps an incoherent state to an incoherent state.

A genuinely incoherent operation \[ \{10\} \] is defined by a completely positive and trace preserving map, \( \Lambda(\rho) = \sum_n K_n \rho K_n^\dagger \) with the Kraus operators fulfilling not only \( \sum_n K_n^\dagger K_n = 1 \) but also

\[ \Lambda_G(\sigma) = \sum_n K_n \sigma K_n^\dagger = \sigma, \]

i.e., all incoherent states are fixed points and therefore each \( K_n \) must be diagonal. Obviously, the Kraus operators of a genuinely incoherent operation naturally fulfill Eqs. \[ 6 \] and \[ 7 \] too.

If we use \( S_{\mathcal{I}0}, S_{\mathcal{S}10}, \) and \( S_{\mathcal{G}10} \) to represent the sets of incoherent operations, strictly incoherent operations, and genuinely incoherent operations, respectively, they have the inclusion relationships

\[ S_{\mathcal{G}10} \subset S_{\mathcal{S}10} \subset S_{\mathcal{I}0}. \]
(C4) It is nonincreasing under mixing of quantum states, i.e., convexity: \[ \sum_n q_n C(\rho_n) \geq C(\sum_n q_n \rho_n) \] for any set of states \( \{\rho_n\} \) and any probability distribution \( \{q_n\} \).

An alternative statement of the framework for quantifying coherence consists of three conditions \cite{40}, instead of the above four conditions. A functional \( C \) can be taken as a valid coherence measure, if it satisfies the following three conditions:

(A1) \( C(\rho) \geq 0 \) for all states, and \( C(\rho) = 0 \) if and only if \( \rho \) are incoherent states;

(A2) \( C(\rho) \geq C(\Lambda(\rho)) \) if \( \Lambda \) is a free operation;

(A3) For all block-diagonal states \( \rho \) in the incoherent basis, there is \( C(p_1 \rho_1 \oplus p_2 \rho_2) = p_1 C(\rho_1) + p_2 C(\rho_2) \).

The above two statements are exactly equivalent under incoherent operations and strictly incoherent operations, and hence one can use any of them to examine whether a functional \( C(\rho) \) can be taken as a coherence measure.

### III. MAIN RESULTS

With these preliminaries, we may now present our results. We will first show that \( \tilde{C}_1(\rho) \) is a valid coherence measure under strictly incoherent operations and genuinely incoherent operations but is not a valid coherence measure under incoherent operations, and we then demonstrate \( \tilde{C}_1(\rho) \), \( \tilde{C}_{p>1}(\rho) \), and \( \tilde{C}_{p>1}(\rho) \) cannot be taken as coherence measures even under genuinely incoherent operations.

#### A. \( \tilde{C}_1(\rho) \) a coherence measure under strictly incoherent operations and genuinely incoherent operations

We first show that \( \tilde{C}_1(\rho) = \|\rho - \Delta \rho\|_1 \) is a valid coherence measure under strictly incoherent operations. For this, we only need to prove that it fulfills the three conditions (A1), (A2), and (A3).

First, it is straightforward to see \( \tilde{C}_1(\rho) = 0 \) for \( \rho \) being an incoherent state, since \( \rho = \Delta \rho \). For \( \rho \) being a coherent state, \( \sqrt{(\rho - \Delta \rho)^2} \) is a positive semidefinite matrix and there is \( \tilde{C}_1(\rho) = \|\rho - \Delta \rho\|_1 = \text{Tr} \sqrt{(\rho - \Delta \rho)^2} = \sum \lambda_i > 0 \), where \( \lambda_i \) are eigenvalues of \( \sqrt{(\rho - \Delta \rho)^2} \). That is, condition (A1) is valid for \( \tilde{C}_1(\rho) \).

Second, since \( \Lambda(\Delta \rho) = \Delta(\Lambda(\rho)) \) for any strictly incoherent operation, there is

\[
\tilde{C}_1(\Lambda(\rho)) = \|\Lambda(\rho) - \Delta(\Lambda(\rho))\|_1 = \|\Lambda(\rho) - \Lambda(\Delta \rho)\|_1.
\]

Noting that \( \|\rho - \sigma\|_1 \), as a distance functional, is contractive under trace preserving quantum operations \cite{1}, satisfying \( \|\Lambda(\rho) - \Lambda(\sigma)\|_1 \leq \|\rho - \sigma\|_1 \), we then have

\[
\tilde{C}_1(\Lambda(\rho)) \leq \|\rho - \Delta \rho\|_1 = \tilde{C}_1(\rho),
\]

i.e., condition (A2) is valid for \( \tilde{C}_1(\rho) \).

Third, we show that \( \tilde{C}_1(\rho) \) satisfies condition (A3) too. By the definitions, \( \tilde{C}_1(\rho) = \|\rho - \Delta \rho\|_1 \) and \( \|M\|_1 = \text{Tr} \sqrt{M^\dagger M} \), we have

\[
\tilde{C}_1(p_1 \rho_1 \oplus p_2 \rho_2) = \text{Tr} \sqrt{p_1(\rho_1 - \Delta \rho_1) + p_2(\rho_2 - \Delta \rho_2))^2} = \text{Tr} \sqrt{p_1^2(\rho_1 - \Delta \rho_1)^2 + p_2^2(\rho_2 - \Delta \rho_2)^2}.
\]

To make further calculations, we need to rewrite \( \sqrt{p_1^2(\rho_1 - \Delta \rho_1)^2 + p_2^2(\rho_2 - \Delta \rho_2)^2} \) as \( \sqrt{p_1^2(\rho_1 - \Delta \rho_1)^2 \oplus p_2^2(\rho_2 - \Delta \rho_2)^2} \). The validity of this rewriting can be proved by using the decomposition expression of a Hermitian operator with its eigenvalues and eigenvectors. Indeed, \( p_1^2(\rho_1 - \Delta \rho_1)^2 \) and \( p_2^2(\rho_2 - \Delta \rho_2)^2 \) are two Hermitian operators, and they can be expressed as \( \sum \lambda_i |\psi_i\rangle \langle \psi_i| \) and \( \sum \lambda_j |\psi'_j\rangle \langle \psi'_j| \), respectively, where \( \lambda_i \) and \( |\psi_i\rangle \) are the eigenvalues and eigenvectors of the first operator, and \( \lambda_j \) and \( |\psi'_j\rangle \) are the eigenvalues and eigenvectors of the second operator. Then, \( \lambda_i \) and \( \lambda_j \) will be the eigenvalues of \( p_1^2(\rho_1 - \Delta \rho_1)^2 \oplus p_2^2(\rho_2 - \Delta \rho_2)^2 \), and \( |\Psi_i\rangle = |\psi_i\rangle + |0\rangle \) and \( |\Psi'_j\rangle = |\psi'_j\rangle + |0\rangle \) will be the corresponding eigenvectors of \( p_1^2(\rho_1 - \Delta \rho_1)^2 \oplus p_2^2(\rho_2 - \Delta \rho_2)^2 \), where \( |0\rangle \) and \( |0\rangle \) are null vectors. Thus, there is

\[
\sqrt{p_1^2(\rho_1 - \Delta \rho_1)^2 \oplus p_2^2(\rho_2 - \Delta \rho_2)^2} = \sqrt{\sum \lambda_i |\Psi_i\rangle \langle \Psi_i| \oplus \sum \lambda_j |\Psi'_j\rangle \langle \Psi'_j|}.
\]
Substituting Eq. (14) into Eq. (13), we finally obtain
\begin{align*}
\bar{C}_1(p_1, p_2) &= \text{Tr} \left( p_1^3 (\rho_1 - \Delta p_1)^2 \otimes p_2^3 (\rho_2 - \Delta p_2)^2 \right) \\
&= \text{Tr} \left( p_1 (\rho_1 - \Delta p_1)^2 \otimes p_2 (\rho_2 - \Delta p_2)^2 \right) \\
&= p_1 \bar{C}_1(p_1) + p_2 \bar{C}_1(p_2),
\end{align*}
(15)
i.e., \( \bar{C}_1(\rho) \) satisfies condition (A3).

The above discussion shows that \( \bar{C}_1(\rho) \) fulfills all the three conditions (A1), (A2), and (A3), and hence it is a valid coherence measure under strictly incoherent operations. Certainly, it is also a valid coherence measure under genuinely incoherent operations, since the set of genuinely incoherent operations is a subset of strictly incoherent operations.

**B. \( \bar{C}_1(\rho) \) is not a coherence measure under incoherent operations**

Although \( \bar{C}_1(\rho) \) is a valid coherence measure under strictly incoherent operations and genuinely incoherent operations, it is not a valid coherence measure under incoherent operations since it does not satisfy condition (C3) under incoherent operations. We now demonstrate this point by giving a counter example, which is found by trial and error.

The state in the counter example is
\[
\rho = a \begin{pmatrix}
797 & 166 & 141 & 210 & 208 \\
166 & 8961 & 3829 & 349 & 4411 \\
141 & 3829 & 8961 & 106 & 312 \\
210 & 321 & 224 & 441 & 401 \\
208 & 158 & 158 & 461 & 401 \\
4411 & 2091 & 2327 & 1658 & 2509
\end{pmatrix}
\]
with \( a = \frac{31496 \times 17124 \times 91780}{31496 \times 18253 \times 15173} \), and the incoherent operation is
\[
\Lambda(\rho) = K_1 \rho K_1^\dagger + K_2 \rho K_2^\dagger
\]
with
\[
K_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad K_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

In this case, there are
\[
\rho_1 = \frac{K_1 \rho K_1^\dagger}{\text{Tr}(K_1 \rho K_1^\dagger)} = \frac{1}{p_1} \begin{pmatrix}
561 & 2046268 \sqrt{3} & 461 & 401 \\
2046268 \sqrt{3} & 5018 & 461 & 3316 \\
461 & 461 & 3120635 & 112 \\
2046268 \sqrt{3} & 461 & 3316 & 1928 \\
401 & 3316 & 112 & 1928
\end{pmatrix}
\]
and
\[
\rho_2 = \frac{K_2 \rho K_2^\dagger}{\text{Tr}(K_2 \rho K_2^\dagger)} = \frac{a}{p_2} \begin{pmatrix}
1537235 & -58517 \sqrt{3} & 14168369681 & -461 & 401 \\
-58517 \sqrt{3} & 55817 & -461 & 3316 \\
-1537235 & 461 & 379081 \sqrt{3} & 112 \\
14168369681 & -461 & 379081 \sqrt{3} & 1928 \\
-461 & 3316 & 112 & 1928
\end{pmatrix}
\]
where
\[
p_1 = \text{Tr}(K_1 \rho K_1^\dagger) = \frac{8279592399562440949}{15679843920215781000},
\]
and
\[
p_2 = \text{Tr}(K_2 \rho K_2^\dagger) = \frac{22200071412133764703}{47039531760647343000},
\]
Substituting \( \rho_1 \) and \( \rho_2 \) into \( \bar{C}_1(\rho) = \| \rho - \Delta \rho \|_1 \), \( \bar{C}_1(p_1) = \| \rho_1 - \Delta \rho_1 \|_1 \), and \( \bar{C}_1(p_2) = \| \rho_2 - \Delta \rho_2 \|_1 \), respectively, we can calculate each of them, and finally we obtain, by numerical calculation,
\[
\sum_n p_n \bar{C}_1(p_n) - \bar{C}_1(\rho) = 0.0152,
\]
which violates condition (C3). Therefore, \( \bar{C}_1(\rho) \) cannot be taken as a coherence measure under incoherent operations.

In passing, we point out that \( \bar{C}_1(\rho) \) must also violate condition (C2), i.e., (A2), under incoherent operations. This can be inferred from the fact that \( \bar{C}_1(\rho) \) satisfies conditions (A1) and (A3) (see the last subsection) but not a valid coherence measure under incoherent operations.

It is worth noting that \( \bar{C}_1(\rho) \) is the first functional that is a valid measure under strictly incoherent operations but violates monotonicity under incoherent operations, found so far. It fills up the gap mentioned in Ref. [9].

**C. \( \bar{C}_1(\rho) \) is not a coherence measure under genuinely incoherent operations**

It has been shown that \( C_1(\rho) = \min_{\rho_\sigma} \| \rho - \sigma \|_1 \) is not a valid coherence measure under both incoherent operations and strictly incoherent operations [46]. We here show that \( C_1(\rho) \) is not a valid coherence measure under genuinely incoherent operations too. To this end, we give a counter example.

Let us consider the state
\[
\rho = a \begin{pmatrix}
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6}
\end{pmatrix}
\]
and the operation $\Lambda(\cdot) = K_1(\cdot)K_1^\dagger + K_2(\cdot)K_2^\dagger$ with

$$K_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad K_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$ 

It is easy to see that $\Lambda(\cdot)$ is a genuinely incoherent operation. In this case, there are

$$\rho_1 = \frac{K_1\rho K_1^\dagger}{\text{Tr}(K_1\rho K_1^\dagger)} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\rho_2 = \frac{K_2\rho K_2^\dagger}{\text{Tr}(K_2\rho K_2^\dagger)} = \frac{1}{3} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

with $\rho_1 = \text{Tr}(K_1\rho K_1^\dagger) = \frac{1}{4}$ and $\rho_2 = \text{Tr}(K_2\rho K_2^\dagger) = \frac{1}{2}$.

By the definition of $C_1(\rho)$, we have

$$C_1(\rho) = \min_{\sigma \in \mathcal{F}} \|\rho - \sigma\|_1 \leq \|\rho - \sigma_0\|_1 = 1,$n

where $\sigma_0 = \text{diag}(\frac{1}{2}, \frac{1}{2}, 0, 0, 0)$. To calculate $C_1(\rho_1) = \min_{\sigma \in \mathcal{F}} \|\rho_1 - \sigma\|_1$, we explicitly write $\sigma$ as

$$\sigma = \begin{pmatrix} \sigma_{00} & 0 & 0 & 0 & 0 \\ 0 & \sigma_{11} & 0 & 0 & 0 \\ 0 & 0 & \sigma_{22} & 0 & 0 \\ 0 & 0 & 0 & \sigma_{33} & 0 \\ 0 & 0 & 0 & 0 & \sigma_{44} \end{pmatrix}$$

with its diagonal elements being $\sigma_{ii} \geq 0$ and satisfying $\sum_{i=0}^{4} \sigma_{ii} = 1$, and rewrite $\rho_1$ as

$$\rho_1 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By using the fact that $\|M_1 \oplus M_2\|_1 = \|M_1\|_1 + \|M_2\|_1$, we then have

$$\|\rho_1 - \sigma\|_1 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \sigma_{00} & 0 & 0 \\ 0 & \sigma_{11} & 0 \\ 0 & 0 & \sigma_{22} \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_{44} \end{pmatrix}$$

and we finally obtain

$$C_1(\rho_1) = \min_{\sigma \in \mathcal{F}} \left[ \sqrt{1 + (\sigma_{00} - \sigma_{11})^2} + (1 - \sigma_{00} - \sigma_{11}) \right] = 1.$$ 

To calculate $C_1(\rho_2) = \min_{\sigma \in \mathcal{F}} \|\rho_2 - \sigma\|_1$, we rewrite $\rho_2$ as

$$\rho_2 = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

We then have

$$\|\rho_2 - \sigma\|_1 = \left\| \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} - \begin{pmatrix} \sigma_{00} & 0 & 0 & 0 & 0 \\ 0 & \sigma_{11} & 0 & 0 & 0 \\ 0 & 0 & \sigma_{22} & 0 & 0 \\ 0 & 0 & 0 & \sigma_{33} & 0 \\ 0 & 0 & 0 & 0 & \sigma_{44} \end{pmatrix} \right\|_1.$$

To simplify the last line in Eq. (26), we let $U_k = \Sigma_{i=0}^{2} |i+k\rangle \langle i|$ with $(i + k)$ being mod(3) and $k = 0, 1, 2$, and use the relation $\|UMU^\dagger\|_1 = \|M\|_1$ for any unitary operator $U$. We have

$$\|\rho_2 - \sigma\|_1 = \frac{1}{3} \sum_{k=0}^{2} \|U_k \left[ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} - \begin{pmatrix} \sigma_{22} & 0 & 0 \\ 0 & \sigma_{33} & 0 \\ 0 & 0 & \sigma_{44} \end{pmatrix} \right] U_k^\dagger\|_1 = \frac{1}{3} \sum_{k=0}^{2} \|U_k \left[ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} - \begin{pmatrix} \sigma_{22} & 0 & 0 \\ 0 & \sigma_{33} & 0 \\ 0 & 0 & \sigma_{44} \end{pmatrix} \right] U_k^\dagger\|_1 = \|\frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} - \frac{\sigma_{22} + \sigma_{33} + \sigma_{44}}{3} \|_1 = 1 + \frac{\sigma_{22} + \sigma_{33} + \sigma_{44}}{3}.$$ 

From Eqs. (26) and (27), we obtain

$$\|\rho_2 - \sigma\|_1 \geq \frac{4}{3} \left( \frac{2(\sigma_{00} + \sigma_{11})}{3} \right).$$

We then have

$$C_1(\rho_2) = \min_{\sigma \in \mathcal{F}} \|\rho - \sigma\|_1 \geq \frac{4}{3}.$$ 

On the other hand,

$$C_1(\rho_2) = \min_{\sigma \in \mathcal{F}} \|\rho_2 - \sigma\|_1 \leq \|\rho_2 - \Delta \rho_2\|_1 = \frac{4}{3}. \quad (30)$$

From Eqs. (29) and (30), we finally obtain

$$C_1(\rho_2) = \frac{4}{3}. \quad (31)$$
By using the results in Eqs. (20), (23), and (31) and noting that \( p_1 = p_2 = \frac{1}{2} \), we immediately obtain

\[
\sum_n p_n C_1(\rho_n) - C_1(\rho) \geq \frac{1}{2} \left( \frac{2}{3} - 1 \right) = \frac{1}{6} > 0,
\]  

(32)

which violates condition (C3).

Therefore, \( C_1(\rho) \) cannot be taken as a coherence measure under genuinely incoherent operations.

D. Neither \( C_{p>1}(\rho) \) nor \( \tilde{C}_{p>1}(\rho) \) is a coherence measure under genuinely incoherent operations

Finally, we show that neither \( \tilde{C}_{p>1}(\rho) \) nor \( C_{p>1}(\rho) \) fulfills condition (C3) even under genuinely incoherent operations.

Let us consider the state

\[
\rho = \begin{pmatrix}
\frac{1}{4} & 0 & \frac{1}{4} & 0 \\
0 & \frac{1}{4} & 0 & \frac{1}{4} \\
\frac{1}{8} & 0 & \frac{1}{4} & 0 \\
0 & \frac{1}{8} & 0 & \frac{1}{4}
\end{pmatrix}
\]  

(33)

and the genuinely incoherent operation

\[
\Lambda(\cdot) = K_1(\cdot)K_1^\dagger + K_2(\cdot)K_2^\dagger + K_3(\cdot)K_3^\dagger + K_4(\cdot)K_4^\dagger
\]  

(34)

with

\[
K_1 = \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad K_2 = \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
K_3 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & 0
\end{pmatrix}, \quad K_4 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{2}}
\end{pmatrix}.
\]

In this case, there are

\[
\rho_1 = \frac{K_1\rho K_1^\dagger}{\text{Tr}(K_1\rho K_1^\dagger)} = \begin{pmatrix}
\frac{1}{4} & 0 & \frac{1}{4} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{4} & 0 \\
0 & 0 & 0 & \frac{1}{4}
\end{pmatrix},
\]

\[
\rho_2 = \frac{K_2\rho K_2^\dagger}{\text{Tr}(K_2\rho K_2^\dagger)} = \begin{pmatrix}
\frac{1}{4} & 0 & 0 & 0 \\
0 & \frac{1}{4} & 0 & 0 \\
0 & 0 & \frac{1}{4} & 0 \\
0 & 0 & 0 & \frac{1}{4}
\end{pmatrix},
\]

\[
\rho_3 = \frac{K_3\rho K_3^\dagger}{\text{Tr}(K_3\rho K_3^\dagger)} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \frac{1}{4} & 0 & 0 \\
0 & 0 & \frac{1}{4} & 0 \\
0 & 0 & 0 & \frac{1}{4}
\end{pmatrix},
\]

\[
\rho_4 = \frac{K_4\rho K_4^\dagger}{\text{Tr}(K_4\rho K_4^\dagger)} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{4} & 0 \\
0 & 0 & 0 & \frac{1}{4}
\end{pmatrix},
\]

and

\[
\rho_1 = \text{Tr}(K_1\rho K_1^\dagger) = \frac{1}{4}, \quad \rho_2 = \text{Tr}(K_2\rho K_2^\dagger) = \frac{1}{4},
\]

\[
\rho_3 = \text{Tr}(K_3\rho K_3^\dagger) = \frac{1}{4}, \quad \rho_4 = \text{Tr}(K_4\rho K_4^\dagger) = \frac{1}{4}.
\]

Substituting \( \rho \) and \( \rho_n \) into \( \tilde{C}_p(\rho) \) and \( C_p(\rho_n) \), respectively, we can obtain

\[
\tilde{C}_p(\rho) = 2\rho^{p-3}
\]

(36)

and

\[
\tilde{C}_p(\rho_1) = \tilde{C}_p(\rho_2) = \tilde{C}_p(\rho_3) = \tilde{C}_p(\rho_4) = 2\rho^{p-2},
\]

(37)

hence

\[
\sum_n p_n \tilde{C}_p(\rho_n) - \tilde{C}_p(\rho) = 2\rho^{p-2}(1 - 2\rho^{p-1}) > 0
\]

(38)

for \( p > 1 \). Therefore, \( \tilde{C}_{p>1}(\rho) \) violates condition (C3) under genuinely incoherent operations.

By using the same counter example defined by Eqs. (33)-(35), with the help of the result for \( \tilde{C}_{p>1}(\rho) \), we can demonstrate that \( C_{p>1}(\rho) \) also violates condition (C3) under genuinely incoherent operations.

To this end, we need to calculate \( C_p(\rho_n) = \min_{\sigma \in \mathcal{F}} \| \rho_n - \sigma \|_p \), for \( n = 1, 2, 3, 4 \). By letting \( U_1 = |0\rangle\langle 0| + |1\rangle\langle 2| + |2\rangle\langle 1| + |3\rangle\langle 3| \) and \( U_2 = |0\rangle\langle 1| + |1\rangle\langle 3| + |2\rangle\langle 2| + |3\rangle\langle 0| \), we can see that \( U_1\rho U_1^\dagger = U_2\rho U_2^\dagger = U_2\rho U_2^\dagger = \rho_0 \), where

\[
\rho_0 = \begin{pmatrix}
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \oplus \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

Since \( \min_{\sigma \in \mathcal{F}} \| \rho_0 - \sigma \|_p = \min_{\sigma \in \mathcal{F}} \| \rho_0 - \sigma \|_p \) for \( k = 1, 2 \), there is \( C_p(\rho_n) = C_p(\rho_0) = \min_{\sigma \in \mathcal{F}} \| \rho_0 - \sigma \|_p \), for \( n = 1, 2, 3, 4 \). By directly calculating the
TABLE II. $C_1(\rho)$ is a valid coherence measure under strictly incoherent operations and genuinely incoherent operations, while all others, including $C_1(\rho)$ under incoherent operations, $C_{p>1}(\rho)$ under any of the three sets of free operations, and $C_p(\rho)$ under any of the three sets of free operations, cannot be taken as a coherence measure. This fills up the gaps in Table III.

| Incoherent operations | Strictly incoherent operations | Genuinely incoherent operations |
|-----------------------|--------------------------------|--------------------------------|
| $C_1(\rho)$ | Not a coherence measure | Not a coherence measure | Not a coherence measure |
| $C_1(\rho)$ | Not a coherence measure | A coherence measure | A coherence measure |
| $C_{p>1}(\rho)$ | Not a coherence measure | Not a coherence measure | Not a coherence measure |
| $C_p(\rho)$ | Not a coherence measure | Not a coherence measure | Not a coherence measure |

\[ C_p(\rho_n) = \min_{\sigma \in F} ([|\lambda_1|^p + |\lambda_2|^p + |\lambda_3|^p + |\lambda_4|^p]^{\frac{1}{p}}, \] (39)

where $\lambda_1 = \frac{1}{2}[1 - \sigma_{00} - \sigma_{11} + \sqrt{1 + 4(\sigma_{00} - \sigma_{11})^2}]$, $\lambda_2 = \frac{1}{2}[1 - \sigma_{00} - \sigma_{11} - \sqrt{1 + 4(\sigma_{00} - \sigma_{11})^2}]$, $\lambda_3 = -\sigma_{22}$, and $\lambda_4 = -\sigma_{33}$.

Since $|\lambda_1|^p + |\lambda_2|^p \geq 2\left(\frac{1}{2} + \frac{\sqrt{1 + 4(\sigma_{00} - \sigma_{11})^2}}{p}\right) \geq 2^{1 - \frac{2}{p}}$. (40)

On the other hand,

\[ C_p(\rho_n) = \min_{\sigma \in F} ([|\lambda_1|^p + |\lambda_2|^p + |\lambda_3|^p + |\lambda_4|^p]^{\frac{1}{p}} \leq ([|\lambda_1|^p + |\lambda_2|^p + |\lambda_3|^p + |\lambda_4|^p]^{\frac{1}{p}})_{\sigma_{00} = \sigma_{11} = 0, \sigma_{22} = \sigma_{33} = 0} = 2^{1 - \frac{2}{p}}. \] (41)

From Eqs. (40) and (41), we immediately obtain

\[ C_p(\rho_n) = 2^{1 - \frac{2}{p}} = \bar{C}_p(\rho_n). \] (42)

Also, by the definitions of the two Schatten-$p$-norm-based functionals, there is $C_p(\rho) \leq \bar{C}_p(\rho)$. We then obtain

\[ \sum_n p_n C_p(\rho_n) - C_p(\rho) \geq \sum_n p_n \bar{C}_p(\rho_n) - \bar{C}_p(\rho) > 0, \] (43)

which implies that $C_{p>1}(\rho)$ violates condition (C3) since $\bar{C}_{p>1}(\rho)$ does. Hence, neither $C_{p>1}(\rho)$ nor $C_{p>1}(\rho)$ fulfills the monotonicity condition under genuinely incoherent operations, which resolves the open question raised in Ref. [10].

Since there is $S_{GIO} \subset S_{SI0} \subset S_{I0}$, we then immediately obtain that neither $C_{p>1}(\rho)$ nor $C_{p>1}(\rho)$ is a valid coherence measure under incoherent operations, strictly incoherent operations, and genuinely incoherent operations. Obviously, all the Schatten-$p$-norm-based functionals discussed in the paper are not valid coherence measures under maximally incoherent operations since the set of maximally incoherent operations contains incoherent operations as a subset.

IV. CONCLUSIONS

So far, we have resolved all the open questions on whether the two classes of Schatten-$p$-norm-based functionals $C_p(\rho) = \min_{\sigma \in F} \|\rho - \sigma\|_p$ and $\bar{C}_p(\rho) = \|\rho - \Delta\rho\|_p$ with $p \geq 1$ are valid coherence measures under incoherent operations, strictly incoherent operations, and genuinely incoherent operations, filling up the gaps in Table III. Our results show that only $\bar{C}_1(\rho)$ is a valid coherence measure under strictly incoherent operations and genuinely incoherent operations, while all others, including $\bar{C}_1(\rho)$ under incoherent operations, $\bar{C}_{p>1}(\rho)$ under any of the three sets of free operations and $C_p(\rho)$ under any of the three sets of free operations, cannot be taken as a coherence measure, as listed in Table II.

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53. For $p > 1$, there is always $|a + b|^p + |a - b|^p \geq 2a^p + 2b^p$ for any $a \geq b \geq 0$, since both $f(x) = x^p$ and $f(x) = px^{p-1}$ are increasing functions with respect to $x \geq 0$. Now, we let $a = \frac{1}{2}((1 - \sigma_{00} - \sigma_{11})$ and $b = \frac{1}{2}\sqrt{1 + (\sigma_{00} - \sigma_{11})^2}$ if $1 - \sigma_{00} - \sigma_{11} \geq \sqrt{1 + (\sigma_{00} - \sigma_{11})^2}$, and let $b = \frac{1}{2}(1 - \sigma_{00} - \sigma_{11})$ and $a = \frac{1}{2}\sqrt{1 + (\sigma_{00} - \sigma_{11})^2}$ if $1 - \sigma_{00} - \sigma_{11} \leq \sqrt{1 + (\sigma_{00} - \sigma_{11})^2}$. In any case, we can immediately obtain $|\lambda_1|^p + |\lambda_2|^p \geq 2\left(\frac{1}{2}\sqrt{1 + (\sigma_{00} - \sigma_{11})^2}\right)^p \geq 2^{1-2p}$.  
