SHARP CONTINUITY RESULTS FOR THE SHORT-TIME FOURIER TRANSFORM AND FOR LOCALIZATION OPERATORS

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Abstract. We completely characterize the boundedness on $L^p$ spaces and on Wiener amalgam spaces of the short-time Fourier transform (STFT) and of a special class of pseudodifferential operators, called localization operators. Precisely, a well-known STFT boundedness result on $L^p$ spaces is proved to be sharp. Then, sufficient conditions for the STFT to be bounded on the Wiener amalgam spaces $W(L^p, L^q)$ are given and their sharpness is shown. Localization operators are treated similarly. Using different techniques from those employed in the literature, we relax the known sufficient boundedness conditions for localization operators on $L^p$ spaces and prove the optimality of our results. More generally, we prove sufficient and necessary conditions for such operators to be bounded on Wiener amalgam spaces.

1. Introduction

A fundamental issue in Time-frequency analysis is the so-called short-time Fourier transform (STFT). Precisely, consider the linear operators of translation and modulation (so-called time-frequency shifts) given by

$$T_x f(t) = f(t - x) \quad \text{and} \quad M_\omega f(t) = e^{2\pi i \omega t} f(t).$$

For a non-zero window function $\varphi$ in $L^2(\mathbb{R}^d)$, the short-time Fourier transform (STFT) of a signal $f \in L^2(\mathbb{R}^d)$ with respect to the window $\varphi$ is given by

$$V_\varphi f(x, \omega) = \langle f, M_\omega T_x \varphi \rangle = \int_{\mathbb{R}^d} f(t) \overline{\varphi(t - x)} e^{-2\pi i \omega t} dt.$$

This definition can be extended to pairs of dual topological vector spaces and Banach spaces, whose duality, denoted by $\langle \cdot, \cdot \rangle$, extends the inner product on $L^2(\mathbb{R}^d)$.

The intuitive meaning of the previous “time-frequency” representation can be summarized as follows. If $f(t)$ represents a signal varying in time, its Fourier
transform $\hat{f}(\omega)$ shows the distribution of its frequency $\omega$, without any additional information about “when” these frequencies appear. To overcome this problem, one may choose a non-negative window function $\varphi$ well-localized around the origin. Then, the information of the signal $f$ at the instant $x$ can be obtained by shifting the window $\varphi$ till the instant $x$ under consideration, and by computing the Fourier transform of the product $f(x)\varphi(t-x)$, that localizes $f$ around the instant time $x$. The decay and smoothness of the STFT have been widely investigated in the framework of $L^p$ spaces [26] and in certain classes of Wiener amalgam and modulation spaces [5, 8, 24]. A basic result, proved in Lieb [26], is the following one:

For every $p \geq 2$, $f \in L^r(\mathbb{R}^d)$, $\varphi \in L^{r'}(\mathbb{R}^d)$, with $p' \leq \min\{r, r'\}$, it turns out $V_{\varphi}f \in L^p(\mathbb{R}^{2d})$ and

$$
\|V_{\varphi}f\|_p \leq C\|\varphi\|_{r'}\|f\|_r.
$$

(As usual, $p'$ and $r'$ are the conjugate exponents of $p$ and $r$, respectively). The previous estimate is sharp (see the subsequent Proposition 3.3).

A new contribution of the present paper is the study of the boundedness of the STFT on the Wiener amalgam spaces $W(L^p, L^q)$, $1 \leq p, q \leq \infty$. We recall that a measurable function $f$ belongs to $W(L^p, L^q)$ if the following norm

\begin{equation}
\|f\|_{W(L^p, L^q)} = \left( \sum_{n \in \mathbb{Z}^d} \left( \int_{\mathbb{R}^d} |f(x)T_n\chi_Q(x)|^p \right)^{\frac{2}{pq}} \right)^{\frac{1}{2}},
\end{equation}

where $Q = [0, 1]^d$ (with the usual adjustments if $p = \infty$ or $q = \infty$) is finite (see [25] and Section 2 below). In particular, $W(L^p, L^p) = L^p$. For heuristic purposes, functions in $W(L^p, L^q)$ may be regarded as functions which are locally in $L^p$ and decay at infinity like a function in $L^q$.

A simplified version of our results (see Proposition 3.4) can be formulated as follows.

Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $f \in W(L^p, L^{q'})(\mathbb{R}^d)$, $1 \leq q' \leq p \leq \infty$, $q \geq 2$. Then $V_{\varphi}f \in W(L^p, L^q)(\mathbb{R}^{2d})$, with the uniform estimate

\begin{equation}
\|V_{\varphi}f\|_{W(L^p, L^q)} \leq C_{\varphi}\|f\|_{W(L^p, L^{q'})}.
\end{equation}

The previous estimate is optimal. Indeed, if (1.4) holds for a given non-zero window function $\varphi \in \mathcal{S}(\mathbb{R}^d)$, then $p \geq q'$ and $q \geq 2$ (see Proposition 3.5).

The other related topic of the present paper is the study of localization operators. The name “localization operator” first appeared in 1988, when Daubechies [12] used these operators as a mathematical tool to localize a signal on the time-frequency plane. But localization operators with Gaussian windows were already known in physics: they were introduced as a quantization rule by Berezin [1] in 1971 (the so-called Wick operators). Since their first appearance, they have been extensively
studied as an important mathematical tool in signal analysis and other applications (see [5, 6, 9, 28, 36] and references therein). We also recall their employment as approximation of pseudodifferential operators (wave packets) [11, 22]. Localization operators are also called Toeplitz operators (see, e.g., [13]) or short-time Fourier transform multipliers [20, 33]. Their definition can be given by means of the STFT as follows.

**Definition 1.1.** The localization operator \( A_{\phi_1,\phi_2}^a \) with symbol \( a \in \mathcal{S}(\mathbb{R}^d) \) and windows \( \phi_1, \phi_2 \in \mathcal{S}(\mathbb{R}^d) \) is defined to be

\[
A_{\phi_1,\phi_2}^a f(t) = \int_{\mathbb{R}^{2d}} a(x, \omega) V_{\phi_1} f(x, \omega) M_{\omega, T_x} \phi_2(t) \, dx \, d\omega, \quad f \in L^2(\mathbb{R}^d).
\]

The definition extends to more general classes of symbols \( a \), windows \( \phi_1, \phi_2 \), and functions \( f \) in natural way. If \( a = \chi_\Omega \) for some compact set \( \Omega \subseteq \mathbb{R}^d \) and \( \phi_1 = \phi_2 \), then \( A_{\phi_1,\phi_2}^a \) is interpreted as the part of \( f \) that “lives on the set \( \Omega \)” in the time-frequency plane. This is why \( A_{\phi_1,\phi_2}^a \) is called a localization operator. If \( a \in \mathcal{S}'(\mathbb{R}^d) \) and \( \phi_1, \phi_2 \in \mathcal{S}(\mathbb{R}^d) \), then (1.5) is a well-defined continuous operator from \( \mathcal{S}(\mathbb{R}^d) \) to \( \mathcal{S}'(\mathbb{R}^d) \). If \( \phi_1(t) = \phi_2(t) = e^{-\pi t^2} \), then \( A_a = A_{\phi_1,\phi_2}^1 \) is the classical anti-Wick operator and the mapping \( a \rightarrow A_{\phi_1,\phi_2}^a \) is interpreted as a quantization rule [1, 29, 36].

In [5, 6] localization operators are viewed as a multilinear mapping

\[
(a, \phi_1, \phi_2) \mapsto A_{\phi_1,\phi_2}^a,
\]

acting on products of symbol and windows spaces. The dependence of the localization operator \( A_{\phi_1,\phi_2}^a \) on all three parameters has been studied there in different functional frameworks. The results in [5] enlarge the ones in the literature, concerning \( L^p \) spaces [36], potential and Sobolev spaces [2], modulation spaces [20, 30, 31]. Other boundedness results for STFT multipliers on \( L^p \), modulation, and Wiener amalgam spaces are contained in [33].

On the footprints of [5], the study of localization operators can be carried to Gelfand-Shilov spaces and spaces of ultra-distributions [10]. Finally, the results in [9] widen [5, 6] interpreting the definition of \( A_{\phi_1,\phi_2}^a \) in a weak sense, that is

\[
\langle A_{\phi_1,\phi_2}^a f, g \rangle = \langle a V_{\phi_1} f, V_{\phi_2} g \rangle = \langle a, \overline{V_{\phi_1} f} V_{\phi_2} g \rangle, \quad f, g \in \mathcal{S}(\mathbb{R}^d).
\]

Here we study the action of the operators \( A_{\phi_1,\phi_2}^a \) on the Lebesgue spaces \( L^p \) and on Wiener amalgam spaces \( W(L^p, L^q) \), when \( a \) is also assumed to belong to these spaces. For example, one wonders what is the full range of exponents \((q, r)\) such that for every \( a \in L^q \) the operator \( A_{\phi_1,\phi_2}^a \) turns out to be bounded on \( L^r \), for all windows \( \phi_1, \phi_2 \) in reasonable classes. Partial results in this connection have been obtained in [3, 4, 35]. We will give a complete answer to this question and, more generally, to a similar question in the framework of Wiener amalgam spaces.
Our present approach differs from the preceding ones. In particular, the techniques employed not use Weyl operators results as in [3, 10]. Indeed, we rewrite \( A_{\varphi_1, \varphi_2}^a \) as an integral operator. Next, we prove a Schur-type test for the boundedness of integral operators on Wiener amalgam spaces (see Proposition 4.2) and we use it to obtain sufficient continuity results for \( A_{\varphi_1, \varphi_2}^a \) with symbol \( a \) in Wiener amalgam spaces and acting either on \( L^p \) or on \( W(L^p, L^q) \) (see Theorems 4.1, 4.5 and 6.2). For instance, the result for Lebesgue spaces can be simplified as follows (see Theorem 6.2 for more general windows):

Let \( a \in L^q(\mathbb{R}^{2d}), \varphi_1, \varphi_2 \in S(\mathbb{R}^d) \). Then, \( A_{\varphi_1, \varphi_2}^a \) is bounded on \( L^r(\mathbb{R}^d) \), for all \( 1 \leq p \leq \infty, \frac{1}{q} \geq \left| \frac{1}{r} - \frac{1}{2} \right| \), with the uniform estimate

\[
\| A_{\varphi_1, \varphi_2}^a f \|_r \leq C_{\varphi_1, \varphi_2} \| a \|_q \| f \|_r.
\]

Moreover, the boundedness results obtained on both Lebesgue and Wiener amalgam spaces reveal to be sharp: see Theorems 5.3 and 5.4. In particular, the optimality for Lebesgue spaces reads:

Let \( \varphi_1, \varphi_2 \in C_0^\infty(\mathbb{R}^d) \), with \( \varphi_1(0) = \varphi_2(0) = 1 \), \( \varphi_1 \geq 0, \ \varphi_2 \geq 0 \). Assume that for some \( 1 \leq q, r \leq \infty \) and every \( a \in L^q(\mathbb{R}^{2d}) \) the operator \( A_{\varphi_1, \varphi_2}^a \) is bounded on \( L^r(\mathbb{R}^d) \). Then

\[
\frac{1}{q} \geq \left| \frac{1}{r} - \frac{1}{2} \right|.
\]

We end up underlining that similar methods can be applied to study the boundedness of localization operators on weighted \( L^p \) and Wiener amalgam spaces as well.

The paper is organized as follows. Section 2 is devoted to preliminary definitions and properties of the involved function spaces. In Section 3 we study the boundedness of the STFT whereas Section 4 and 5 are devoted to sufficient and necessary conditions, respectively, for boundedness of localization operators. Finally, in Section 6 we present further refinements.

**Notation.** To be definite, let us fix some notation we shall use later on (and have already used in this Introduction). We define \( xy = x \cdot y \), the scalar product on \( \mathbb{R}^d \). We define by \( C_0^\infty(\mathbb{R}^d) \) the space of smooth functions on \( \mathbb{R}^d \) with compact support. The Schwartz class is denoted by \( S(\mathbb{R}^d) \), the space of tempered distributions by \( S'(\mathbb{R}^d) \). We use the brackets \( \langle f, g \rangle \) to denote the extension to \( S(\mathbb{R}^d) \times S'(\mathbb{R}^d) \) of the inner product \( \langle f, g \rangle = \int f(t)g(t)dt \) on \( L^2(\mathbb{R}^d) \). The Fourier transform is normalized to be \( \hat{f}(\omega) = \mathcal{F}f(\omega) = \int f(t)e^{-2\pi i t \omega}dt \). Throughout the paper, we shall use the notation \( A \lesssim B, A \gtrsim B \) to indicate \( A \leq cB, A \geq cB \) respectively, for a suitable constant \( c > 0 \), whereas \( A \asymp B \) if \( A \leq cB \) and \( B \leq kA \), for suitable \( c,k > 0 \).
2. Time-Frequency Methods

First we summarize some concepts and tools of time-frequency analysis, for an extended exposition we refer to the textbooks [22, 24].

2.1. STFT properties. The short-time Fourier transform (STFT) is defined in (1.2). The STFT \( \mathcal{V}_g f \) is defined for \( f, g \) in many possible pairs of Banach spaces or topological vector spaces. For instance, it maps \( L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \) into \( L^2(\mathbb{R}^{2d}) \) and \( \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \) into \( \mathcal{S}(\mathbb{R}^{2d}) \). Furthermore, it can be extended to a map from \( \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d) \) into \( \mathcal{S}'(\mathbb{R}^{2d}) \). The crucial properties of the STFT (for proofs, see [24] and [26]) we shall use in the sequel are the following.

**Lemma 2.1.** Let \( f, g, \in L^2(\mathbb{R}^d) \), then we have

(i) (STFT of time-frequency shifts) For \( y, \xi, \in \mathbb{R}^d \), we have

\[
(2.1) \quad \mathcal{V}_g(M_\xi T_y f)(x, \omega) = e^{-2\pi i(\omega - \xi)y} (\mathcal{V}_g f)(x - y, \omega - \xi).
\]

(ii) (Orthogonality relations for STFT)

\[
(2.2) \quad \| \mathcal{V}_g f \|_{L^2(\mathbb{R}^{2d})} = \| f \|_{L^2(\mathbb{R}^d)} \| g \|_{L^2(\mathbb{R}^d)}.
\]

(iii) (Switching \( f \) and \( g \))

\[
(2.3) \quad (\mathcal{V}_f g)(x, \omega) = e^{-2\pi i x \omega} \mathcal{V}_f (x, -\omega).
\]

(iv) (Fourier transform of a product of STFTs)

\[
(2.4) \quad (\widehat{\mathcal{V}_f g})(x, \omega) = \mathcal{V}_f(\widehat{g}(\omega, -x)).
\]

(v) (STFT of the Fourier transforms)

\[
(2.5) \quad \mathcal{V}_g f(x, \omega) = e^{-2\pi i x \omega} \mathcal{V}_g f(\omega, -x).
\]

**Proposition 2.2.** Suppose \( p \geq 2, f \in L^r(\mathbb{R}^d), g \in L^{r'}(\mathbb{R}^d) \), with \( p' \leq \min\{r, r'\} \). Then \( \mathcal{V}_g f \in L^p(\mathbb{R}^{2d}) \) and

\[
(2.6) \quad \| \mathcal{V}_g f \|_p \leq \left( \frac{p'}{p} \right)^{\frac{1}{2}} \| g \|_{r'} \| f \|_r.
\]

The following inequality is proved in [24, Lemma 11.3.3]:

**Lemma 2.3.** Let \( g_0, g, \gamma \in \mathcal{S}(\mathbb{R}^d) \) such that \( \langle \gamma, g \rangle \neq 0 \) and let \( f \in \mathcal{S}'(\mathbb{R}^d) \). Then

\[
|\mathcal{V}_{g_0} f(x, \omega)| \leq \frac{1}{|\langle \gamma, g \rangle|} (|\mathcal{V}_g f| * |\mathcal{V}_{g_0} \gamma|)(x, \omega),
\]

for all \((x, \omega) \in \mathbb{R}^{2d}\).
2.2. Function Spaces. For $1 \leq p \leq \infty$, recall the $\mathcal{F}L^p$ spaces, defined by
$$\mathcal{F}L^p(\mathbb{R}^d) = \{ f \in \mathcal{S}'(\mathbb{R}^d) : \exists h \in L^p(\mathbb{R}^d), \hat{h} = f \};$$
they are Banach spaces equipped with the norm
\begin{equation}
\| f \|_{\mathcal{F}L^p} = \| h \|_{L^p}, \quad \text{with} \; \hat{h} = f. \tag{2.7}
\end{equation}

The mixed-norm space $L^{p,q}(\mathbb{R}^d)$, $1 \leq p, q \leq \infty$, consists of all measurable functions on $\mathbb{R}^d$ such that the norm
\begin{equation}
\| F \|_{L^{p,q}} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |F(x, \omega)|^p dx \right)^{\frac{q}{p}} d\omega \right)^{\frac{1}{q}} \tag{2.8}
\end{equation}
(with obvious modifications when $p = \infty$ or $q = \infty$) is finite.

The function spaces $L^p L^q(\mathbb{R}^d)$, $1 \leq p, q \leq \infty$, consists of all measurable functions on $\mathbb{R}^d$ such that the norm
\begin{equation}
\| F \|_{L^p L^q} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |F(x, \omega)|^q d\omega \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} \tag{2.9}
\end{equation}
(with obvious modifications when $p = \infty$ or $q = \infty$) is finite. Notice that, for $p = q$, we have $L^p L^q(\mathbb{R}^d) = L^{p,p}(\mathbb{R}^d) = L^p(\mathbb{R}^d)$.

**Wiener amalgam spaces.** We briefly recall the definition and the main properties of Wiener amalgam spaces. We refer to [14, 16, 18, 19, 21, 25] for details.

Let $g \in C_0^\infty$ be a test function that satisfies $\| g \|_{L^2} = 1$. We will refer to $g$ as a window function. Let $B$ one of the following Banach spaces: $L^p, \mathcal{F}L^p, L^{p,q}, L^p L^q$, $1 \leq p, q \leq \infty$. Let $C$ be one of the following Banach spaces: $L^p, L^{p,q}, L^p L^q$, $1 \leq p, q \leq \infty$. For any given temperate distribution $f$ which is locally in $B$ (i.e. $gf \in B$, $\forall g \in C_0^\infty$), we set $f_B(x) = \| f T_x g \|_B$.

The **Wiener amalgam space** $W(B, C)$ with local component $B$ and global component $C$ is defined as the space of all temperate distributions $f$ locally in $B$ such that $f_B \in C$. Endowed with the norm $\| f \|_{W(B, C)} = \| f_B \|_C$, $W(B, C)$ is a Banach space. Moreover, different choices of $g \in C_0^\infty$ generate the same space and yield equivalent norms.

If $B = \mathcal{F}L^1$ (the Fourier algebra), the space of admissible windows for the Wiener amalgam spaces $W(\mathcal{F}L^1, C)$ can be enlarged to the so-called Feichtinger algebra $W(\mathcal{F}L^1, L^1)$. Recall that the Schwartz class $\mathcal{S}$ is dense in $W(\mathcal{F}L^1, L^1)$.

An equivalent norm for the space $W(L^p, L^q)$ is provided by (1.3).

We use the following definition of mixed Wiener amalgam norms. Given a measurable function $F$ on $\mathbb{R}^{2d}$ we set
$$\| F \|_{W(L^{p_1} L^{q_1}), W(L^{p_2} L^{q_2})} = \| F(\cdot, \cdot) \|_{W(L^{p_2} L^{q_2})} \| F(\cdot, \cdot) \|_{W(L^{p_1} L^{q_1})}.$$ 

The following properties of Wiener amalgam spaces will be frequently used in the sequel.
Lemma 2.4. Let $B_i, C_i, i = 1, 2, 3$, be Banach spaces such that $W(B_i, C_i)$ are well defined. Then,

(i) Convolution. If $B_1 * B_2 \hookrightarrow B_3$ and $C_1 * C_2 \hookrightarrow C_3$, we have

$$W(B_1, C_1) * W(B_2, C_2) \hookrightarrow W(B_3, C_3).$$ (2.10)

(ii) Inclusions. If $B_1 \hookrightarrow B_2$ and $C_1 \hookrightarrow C_2$,

$$W(B_1, C_1) \hookrightarrow W(B_2, C_2).$$

Moreover, the inclusion of $B_1$ into $B_2$ need only hold “locally” and the inclusion of $C_1$ into $C_2$ “globally”. In particular, for $1 \leq p_i, q_i \leq \infty$, $i = 1, 2$, we have

$$p_1 \geq p_2 \text{ and } q_1 \leq q_2 \implies W(L^{p_1}, L^{q_1}) \hookrightarrow W(L^{p_2}, L^{q_2}).$$ (2.11)

(iii) Complex interpolation. For $0 < \theta < 1$, we have

$$[W(B_1, C_1), W(B_2, C_2)]_{\theta} = W([B_1, B_2]_{\theta}, [C_1, C_2]_{\theta}),$$

if $C_1$ or $C_2$ has absolutely continuous norm.

(iv) Duality. If $B', C'$ are the topological dual spaces of the Banach spaces $B, C$ respectively, and the space of test functions $C_0^\infty$ is dense in both $B$ and $C$, then

$$W(B, C)' = W(B', C').$$ (2.12)

(v) Hausdorff-Young. If $1 \leq p, q \leq 2$ then

$$\mathcal{F}(W(L^p, L^q)) \hookrightarrow W(L^{q'}, L^{p'})$$  

(local and global properties are interchanged on the Fourier side).

(vi) Pointwise products. If $B_1 \cdot B_2 \hookrightarrow B_3$ and $C_1 \cdot C_2 \hookrightarrow C_3$, we have

$$W(B_1, C_1) \cdot W(B_2, C_2) \hookrightarrow W(B_3, C_3).$$ (2.14)

The following result will be useful in the sequel.

Lemma 2.5 ([8, Lemma 5.4]). Let $\varphi_{a+ib}(x) = e^{-\pi(a+ib)|x|^2}$, $a > 0, b \in \mathbb{R}$, $x \in \mathbb{R}^d$. Then,

$$\|\varphi_{a+ib}\|_{W(L^p, L^q)} \asymp a^{-\frac{d}{p}}(a + 1)^{\frac{d}{2}(\frac{1}{q} - \frac{1}{p})}. \quad (2.15)$$

Finally we establish a useful estimate for Wiener amalgam spaces, based on the classical Bernstein’s inequality (see, e.g., [34]), that we are going to recall first. Let $B(x_0, R)$ be the ball of center $x_0 \in \mathbb{R}^d$ and radius $R > 0$ in $\mathbb{R}^d$.

Lemma 2.6 (Bernstein’s inequality). Let $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $\hat{f}$ is supported in $B(x_0, R)$, and let $1 \leq p \leq q \leq \infty$. Then, there exists a positive constant $C$, independent of $f, x_0, R, p, q$, such that

$$\|f\|_q \leq CR^d(\frac{1}{p} - \frac{1}{q})\|f\|_p. \quad (2.16)$$
Proposition 2.7. Let $1 \leq p \leq q \leq \infty$. For every $R > 0$, there exists a constant $C_R > 0$ such that, for every $f \in \mathcal{S}'(\mathbb{R}^d)$ whose Fourier transform is supported in any ball of radius $R$, it turns out

$$
\|f\|_{W(L^p,L^p)} \leq C_R \|f\|_p.
$$

Proof. Choose a Schwartz function $g$ whose Fourier transform $\hat{g}$ has compact support in $B(0,1)$, as window function arising in the definition of the norm in $W(L^q,L^p)$. Then, the function $(T_x g)f$, $x \in \mathbb{R}^d$, has Fourier transform supported in a ball of radius $R+1$. Therefore it follows from Bernstein’s inequality (Lemma 2.6) that

$$
\|(T_x g)f\|_q \leq C_R \|(T_x g)f\|_p.
$$

Taking the $L^p$-norm with respect to $x$ gives the conclusion. \qed

Modulation spaces. For their basic properties we refer to [15, 24].

Given a non-zero window $g \in \mathcal{S}(\mathbb{R}^d)$ and $1 \leq p, q \leq \infty$, the modulation space $M^{p,q}(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that the STFT, defined in (1.2), fulfills $V_g f \in L^{p,q}(\mathbb{R}^{2d})$. The norm on $M^{p,q}$ is

$$
\|f\|_{M^{p,q}} = \|V_g f\|_{L^{p,q}} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \left| V_g f(x,\omega) \right|^p dx \right)^{q/p} d\omega \right)^{1/p}.
$$

If $p = q$, we write $M^p$ instead of $M^{p,p}$.

$M^{p,q}$ is a Banach space whose definition is independent of the choice of the window $g$. Moreover, if $g \in M^1 \setminus \{0\}$, then $\|V_g f\|_{L^{p,q}}$ is an equivalent norm for $M^{p,q}(\mathbb{R}^d)$.

Among the properties of modulation spaces, we record that $M^{2,2} = L^2$, $M^{p_1,q_1} \hookrightarrow M^{p_2,q_2}$, if $p_1 \leq p_2$ and $q_1 \leq q_2$. If $1 \leq p, q < \infty$, then $(M^{p,q})' = M^{p',q'}$.

Modulation spaces and Wiener amalgam spaces are closely related: for $p = q$, we have

$$
(2.17) \quad \|f\|_{W(\mathcal{F}L^p,L^p)} = \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| V_g f(x,\omega) \right|^p m(x,\omega)^p dx d\omega \right)^{1/p} \asymp \|f\|_{M^p}.
$$

Finally, the next results will be useful in the sequel.

Proposition 2.8 ([3, Proposition 3.4]). Let $g \in M^1(\mathbb{R}^d)$ and $1 \leq p \leq \infty$ be given. Then $f \in M^p(\mathbb{R}^d)$ if and only if $V_g f \in M^p(\mathbb{R}^{2d})$ with

$$
(2.18) \quad \|V_g f\|_{M^p} \asymp \|g\|_{M^p} \|f\|_{M^p}.
$$

Lemma 2.9 ([3, Lemma 4.1]). Let $1 \leq p, q \leq \infty$. If $f \in M^{p,q}(\mathbb{R}^d)$ and $g \in M^1(\mathbb{R}^d)$, then $V_g f \in W(\mathcal{F}L^1,L^{p,q})(\mathbb{R}^{2d})$ with norm estimate

$$
(2.19) \quad \|V_g f\|_{W(\mathcal{F}L^1,L^{p,q})} \lesssim \|f\|_{M^{p,q}} \|g\|_{M^1}.
$$
We finally recall a characterization of modulation spaces. Let $\psi \in \mathcal{S}(\mathbb{R}^d)$ be such that
\begin{equation}
\text{supp } \psi \subset (-1,1)^d \quad \text{and} \quad \sum_{k \in \mathbb{Z}^d} \psi(\omega - k) = 1 \quad \text{for all } \omega \in \mathbb{R}^d.
\end{equation}

**Proposition 2.10.** (32) Let $1 \leq p, q \leq \infty$. An equivalent norm in $M^{p,q}$ is given by
\[
\|f\|_{M^{p,q}} \asymp \left( \sum_{k \in \mathbb{Z}^d} \|\psi(D - k)f\|_{L^p}^q \right)^{1/q},
\]
where $\psi(D - k)f := \mathcal{F}^{-1}(\hat{f}T_k\psi)$.

### 3. Boundedness of the STFT

To prove sharp results for the STFT, we need the following lemmata.

**Lemma 3.1.** Let $h \in C_0^\infty(\mathbb{R}^d)$, and consider the family of functions
\[
h_\lambda(x) = h(x)e^{-\pi i \lambda |x|^2}, \quad \lambda \geq 1.
\]
Then, if $q \geq 2$,
\begin{equation}
\|\mathbf{\hat{h}}_\lambda\|_q \lesssim \lambda^{\frac{d}{q} - \frac{d}{2}},
\end{equation}
whereas, if $1 \leq q < 2$,
\begin{equation}
\|\mathbf{\hat{h}}_\lambda\|_q \gtrsim \lambda^{\frac{d}{q} - \frac{d}{2}}.
\end{equation}

**Proof.** The upper bound (3.1) is shown, e.g., in [34, page 28]. The lower bound (3.2) follows from (3.1) since, for $q < 2$,
\[
0 \neq C = \|h_\lambda\|_2^2 = \|\mathbf{\hat{h}}_\lambda\|_2^2 \leq \|\mathbf{\hat{h}}_\lambda\|_q \|\mathbf{\hat{h}}_\lambda\|_q \lesssim \lambda^{\frac{d}{q} - \frac{d}{2}} \|\mathbf{\hat{h}}_\lambda\|_q.
\]

**Lemma 3.2.** Let $\varphi(t) = e^{-\pi |t|^2}$ and consider the family of functions $\varphi_\lambda(t) = e^{-\pi \lambda |t|^2}$, $\lambda > 0$. We have
\[
V_\varphi \varphi_\lambda(x, \omega) = (\lambda + 1)^{-\frac{d}{2}} e^{-\frac{\pi (\lambda |x|^2 + |\omega|^2)}{\lambda + 1}} e^{-\frac{\pi i |x| |\omega|}{\lambda + 1}}.
\]

**Proof.** We have
\[
V_\varphi \varphi_\lambda(x, \omega) = \int e^{-\pi \lambda |t|^2} e^{-\pi |t-x|^2} e^{-2\pi i \omega t} dt = \int e^{-\pi (\lambda + 1) |t-x|^2} e^{-\frac{\pi \lambda |t|^2}{\lambda + 1}} e^{-2\pi i \omega t} dt
\]
\[
= e^{-\frac{\pi \lambda |x|^2}{\lambda + 1}} \mathcal{F} \left( T_{\lambda + 1} e^{-\pi (\lambda + 1) |t|^2} \right)(\omega) = (\lambda + 1)^{-\frac{d}{2}} e^{-\frac{\pi \lambda |x|^2}{\lambda + 1}} M_{\lambda + 1} e^{-\frac{\pi |\omega|^2}{\lambda + 1}}
\]
\[
= (\lambda + 1)^{-\frac{d}{2}} e^{-\frac{\pi (\lambda |x|^2 + |\omega|^2)}{\lambda + 1}} e^{-\frac{\pi i |x| |\omega|}{\lambda + 1}}.
\]
as desired.

Then, the result of Proposition 2.2 is sharp, as explained below.

**Proposition 3.3.** Suppose that, for some $1 \leq p, r \leq \infty$, the following inequality holds

\begin{equation}
\|Vg f\|_p \leq C\|g\|_r \|f\|_r, \quad \forall f, g \in \mathcal{S}(\mathbb{R}^d).
\end{equation}

Then $p' \leq \min\{r, r'\}$ (in particular, $p \geq 2$).

**Proof.** We start by proving the constraint $p' \leq r$. Let $\phi(t) = e^{-\pi|t|^2}$ and $\varphi_\lambda(t) = e^{-\pi \lambda |t|^2}$. Then, by Lemma 3.2, we have

$$|V\varphi_\lambda(x, \omega)| = (\lambda + 1)^{-\frac{d}{2}} e^{-\frac{\pi(\lambda^2 + |\omega|^2)}{\lambda + 1}}.$$ 

Hence,

\begin{equation}
\|V\varphi_\lambda\|_p \simeq (\lambda + 1)^{-\frac{d}{2}} \left(\frac{\lambda}{\lambda + 1}\right)^{-\frac{d}{2}} \left(\frac{1}{\lambda + 1}\right)^{-\frac{d}{2}}.
\end{equation}

Since $\|\varphi_\lambda\|_r \simeq \lambda^{-d/(2r)}$, writing (3.3) for $f = \varphi_\lambda, g = \phi$ and letting $\lambda \to +\infty$, we get $p' \leq r$, as desired.

The constraint $p' \leq r'$ follows similarly by switching the role of $f$ and $g$, i.e., by using (2.3).

Of course, $p' \leq \min\{r, r'\}$ implies $p \geq 2$. Here is an alternative direct proof. Assume, by contradiction, that $p \leq 2$, and test the estimate (3.3) on the functions $f = h_\lambda, \lambda \geq 1$, of Lemma 3.1. It is well-known (see, e.g., [17, 18, 23, 27]) that, for distributions supported in a fixed compact subset, the norm in $M^p$ is equivalent to the norm in $\mathcal{F}L^p$. Hence, using (3.2),

$$\|V\varphi h_\lambda\|_p \simeq \|h_\lambda\|_{M^p} \simeq \|\hat{h}_\lambda\|_p \gtrsim \lambda^{\frac{d}{2} - \frac{d}{2}}.$$ 

Since $\|h_\lambda\|_r = \|h\|_r = C$, letting $\lambda \to +\infty$ in (3.3), the claim $p \geq 2$ follows.

We now focus on the boundedness of the STFT on the Wiener amalgam spaces $W(L^p, L^q)$. We will show that, if the windows are in $M^1$, the optimal range of admissible pairs $(1/q, 1/p)$ is the shadowed region of Figure 1.
Figure 1: Estimate (3.5) holds for all pairs $(1/q, 1/p)$ in the shadowed region.

3.1. **Sufficient boundedness conditions for the STFT on $W(L^p, L^q)$.**

**Proposition 3.4.** Let $\varphi \in M^1(\mathbb{R}^d)$, $f \in W(L^p, L^q)(\mathbb{R}^d)$, $1 \leq q' \leq p \leq \infty$, $q \geq 2$. Then $V_\varphi f \in W(L^p, L^q)(\mathbb{R}^{2d})$, with the uniform estimate

\[
\|V_\varphi f\|_{W(L^p, L^q)} \lesssim \|f\|_{W(L^p, L^q)} \|\varphi\|_{M^1}.
\]

**Proof.** By interpolation it suffices to prove the desired result when $p = q'$, $2 \leq q \leq \infty$, and when $p = \infty$, $2 \leq q \leq \infty$.

Let $p = q'$, $2 \leq q \leq \infty$. We have

\[
\|V_\varphi f\|_{W(L^{q'}, L^q)} \lesssim \|V_\varphi f\|_{q'},
\]

because of the embedding $L^q \hookrightarrow W(L^{q'}, L^q)$ (recall, $q \geq 2$). By Lemma 2.3, Young’s Inequality and the equality $M^q = W(\mathcal{F}L^{q}, L^q)$, we obtain

\[
\|V_\varphi f\|_{q} \lesssim \|\varphi\|_{M^1} \|f\|_{W(\mathcal{F}L^{q}, L^q)} \lesssim \|\varphi\|_{M^1} \|f\|_{W(L^{q'}, L^q)},
\]

where the last inequality is due to the embedding $W(L^{q'}, L^q) \hookrightarrow W(\mathcal{F}L^{q}, L^q)$ (that is a consequence of the Hausdorff-Young Inequality).

Let now $p = \infty$, $2 \leq q \leq \infty$. We have

\[
\|V_\varphi f\|_{W(L^\infty, L^q)} \lesssim \|V_\varphi f\|_{W(\mathcal{F}L^1, L^q)} \lesssim \|f\|_{M^q} \|\varphi\|_{M^1},
\]
where the second inequality is due to \(2.19\). The last expression is

\[
\|f\|_{W(\mathcal{F}L^p, L^q)} \|\varphi\|_{M^1} \lesssim \|f\|_{W(L^{p'}, L^q')} \|\varphi\|_{M^1} \lesssim \|f\|_{W(L^\infty, L^q)} \|\varphi\|_{M^1}.
\]

This concludes the proof. \(\square\)

3.2. Necessary boundedness conditions for the STFT on \(W(L^p, L^q)\). In what follows, we shall prove the sharpness of the results obtained above.

Proposition 3.5. Let \(g \in M^1 \setminus \{0\}\). Suppose that, for some \(1 \leq p, q \leq \infty\), \(C > 0\), the estimate

\[
\|V_g f\|_{W(L^p, L^q)} \leq C \|f\|_{W(L^p, L^q)}, \quad \forall f \in \mathcal{S}(\mathbb{R}^d),
\]

holds. Then \(p \geq q'\) and \(q \geq 2\).

Proof. We claim that we may just consider the case of the window \(\varphi(t) = e^{-\pi|t|^2}\). Indeed, if \(g\) is a non-zero window function in \(M^1\), Lemma 2.3 and (2.10) give

\[
\|V_g f\|_{W(L^p, L^q)} \leq \frac{1}{\|\varphi\|_2^2} \|V_{\varphi} f\|_{W(L^p, L^q)} \|V_g \varphi\|_1,
\]

and

\[
\|V_{\varphi} f\|_{W(L^p, L^q)} \leq \frac{1}{\|g\|_2^2} \|V_g f\|_{W(L^p, L^q)} \|V_{\varphi} g\|_1,
\]

so that \(\|V_g f\|_{W(L^p, L^q)} \asymp \|V_{\varphi} f\|_{W(L^p, L^q)}\). Hence, from now on, we assume \(g = \varphi\).

First we prove that \(p \geq q'\). Let \(\varphi_\lambda(t) = e^{-\pi|t|^2}\). Then, by Lemma 3.2, we have

\[
|V_{\varphi} \varphi_\lambda(x, \omega)| = (\lambda + 1)^{-\frac{d}{2}} e^{-\frac{\pi\lambda^2}{\lambda + 1}|x|^2/2}.\]

By taking \(g(x, \omega) = (g_1 \otimes g_2)(x, \omega)\) as window function in the definition of the Wiener amalgam norm, one sees that

\[
\|V_{\varphi} \varphi_\lambda\|_{W(L^p, L^q)} \asymp (\lambda + 1)^{-\frac{d}{2}} e^{-\frac{\pi\lambda^2}{\lambda + 1} \|x\|^2/2} \cdot \|e^{-\frac{\pi\lambda^2}{\lambda + 1} |x|^2/2}\|_{W(L^p, L^q)}. \tag{3.7}
\]

By the estimate (2.15), it turns out, for \(\lambda \geq 1\),

\[
\|\varphi_\lambda\|_{W(L^p, L^q)} \asymp \lambda^{-\frac{d}{2\varphi}} (\lambda + 1)^{\frac{d}{2} \left(\frac{1}{\varphi} - \frac{1}{p}\right)} \asymp \lambda^{-\frac{d}{2\varphi}}, \tag{3.8}
\]

and

\[
\|e^{-\frac{\pi\lambda^2}{\lambda + 1} |x|^2/2}\|_{W(L^p, L^q)} \asymp \left(\frac{\lambda}{\lambda + 1}\right)^{-\frac{d}{2\varphi}} \left(\frac{\lambda}{\lambda + 1} + 1\right)^{\frac{d}{2} \left(\frac{1}{\varphi} - \frac{1}{p}\right)},
\]

\[
\|e^{-\frac{\pi\lambda^2}{\lambda + 1} |x|^2/2}\|_{W(L^p, L^q)} \asymp \left(\frac{1}{\lambda + 1}\right)^{-\frac{d}{2\varphi}} \left(\frac{1}{\lambda + 1} + 1\right)^{\frac{d}{2} \left(\frac{1}{\varphi} - \frac{1}{p}\right)},
\]

which, by (3.7), give

\[
\|V_{\varphi} \varphi_\lambda\|_{W(L^p, L^q)} \asymp \lambda^{-\frac{d}{2\varphi}}. \tag{3.9}
\]

Letting \(\lambda \to +\infty\), the previous estimate and (3.8) yield \(p \geq q'\).
Let us now prove the condition $q \geq 2$. If $p \leq q$ this follows from the condition $p \geq q'$, so that we can suppose $p > q$. Then $W(L^p, L^q) \hookrightarrow L^q$, and from (3.6) we deduce

$$\|f\|_{M^q} \approx \|V_\varphi f\|_{L^q} \lesssim \|f\|_{W(L^p, L^q)}.$$  

We now argue as in the proof of Proposition 3.3 and test this estimate on the functions $f = h_\lambda$ in Lemma 3.1. This implies

$$\|V_\varphi h_\lambda\|_q \approx \|h_\lambda\|_{M^q} \approx \|\hat{h}_\lambda\|_{L^q} \lesssim \|h_\lambda\|_{W(L^p, L^q)} = C_{p,q}.$$  

Since the constant $C_{p,q}$ is independent of $\lambda$, it follows from (3.2), letting $\lambda \to +\infty$, that $q \geq 2$.  

4. SUFFICIENT BOUNDEDNESS CONDITIONS FOR LOCALIZATION OPERATORS

In what follows, we study the boundedness of localization operators on $L^p$ and on Wiener amalgam spaces.

First of all, we prove sufficient boundedness results for localization operators acting on Lebesgue spaces. This is in fact a particular case of Theorem 4.5, but we establish it separately for the benefit of the reader.

**Theorem 4.1.** Let $a \in W(L^p, L^q)(\mathbb{R}^d)$, $\varphi_1, \varphi_2 \in M^1(\mathbb{R}^d)$. Then $A_{a,\varphi_1}^{\varphi_2}$ is bounded on $L^r(\mathbb{R}^d)$, for all $1 \leq p, q, r \leq \infty$, $\frac{1}{q} \geq |\frac{1}{r} - \frac{1}{2}|$, with the uniform estimate

$$\|A_{a,\varphi_1}^{\varphi_2}\|_{B(L^r)} \lesssim \|a\|_{W(L^p, L^q)}\|\varphi_1\|_{M^1}\|\varphi_2\|_{M^1}. \tag{4.1}$$  

Figure 2 illustrates the range of exponents $(1/r, 1/q)$ for the boundedness of $A_{a,\varphi_1}^{\varphi_2}$.  


Proof. By the inclusion relations for Wiener amalgam spaces, it suffices to show the claim for $p = 1$. For $q = \infty$, $r = 2$, the result was already proved in [5, Theorem 1.1]. Indeed, using the inclusion $L^1 \subset \mathcal{F}L^\infty$ and the inclusion relations for Wiener amalgam spaces, we have $W(L^1, L^\infty) \subset W(\mathcal{F}L^\infty, L^\infty) = M^\infty$. Hence, the symbol $a$ is in the modulation space $M^\infty$ and, consequently, the operator $A_{\varphi_1, \varphi_2}^a$ is bounded on $M^2 = L^2$.

Let us prove the thesis in the cases $1 \leq q \leq 2$ and $1 \leq r \leq \infty$, so that, using the interpolation diagram of Figure 1, we attain the desired result.

By the Schur’s test (see, e.g., [24, Lemma 6.2.1]), it is enough to show that the kernel of $A_{\varphi_1, \varphi_2}^a$ belongs to the spaces $L^1, \infty$ and $L^\infty L^1$, defined in (2.8) and (2.9), respectively. Let us start with the first case. $A_{\varphi_1, \varphi_2}^a$ is the integral operator with

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{Estimate (4.1) holds for all pairs $(1/r, 1/q)$ in the shadowed region.}
\end{figure}
kernel
\[ K(x, y) = \int_{\mathbb{R}^d} a(t, \omega) M_{\omega} T_{t} \varphi_2(x) M_{-\omega} T_{t} \varphi_1(y) dt d\omega \]
\[ = \int_{\mathbb{R}^d} \mathcal{F}_2 a(t, y-x) \overline{T_t \varphi_1(y) T_t \varphi_2(x)} dt \]
\[ = \int_{\mathbb{R}^d} \mathcal{F}(\varphi_1(y - \cdot) \varphi_2(x - \cdot))(\omega) \mathcal{F}_a(\omega, y-x) d\omega \]
\[ = \int_{\mathbb{R}^d} V_{\varphi_2}^*(T_y \varphi_1^*)(x, \omega) \mathcal{F}_a(\omega, y-x) d\omega, \]

with the notation \( f^*(t) = f(-t) \). Using the pointwise multiplication properties for Wiener amalgam spaces (2.14), and setting \( \psi_y(x, \omega) = (\omega, y-x) \) we obtain
\[ \int_{\mathbb{R}^d} |K(x, y)| dx \leq \| V_{\varphi_2}^*(T_y \varphi_1^*) \|_{W(\mathcal{L}^1, \mathcal{L}^1)} \| (\mathcal{F}_a) \circ \psi_y \|_{W(\mathcal{L}^q', \mathcal{L}^\infty)}, \]

On the other hand, it is easily seen that \( \| (\mathcal{F}_a) \circ \psi_y \|_{W(\mathcal{L}^q', \mathcal{L}^\infty)} \lesssim \| (\mathcal{F}_a) \|_{W(\mathcal{L}^q', \mathcal{L}^\infty)} \), uniformly with respect to \( y \). Hence, by the inclusion \( W(\mathcal{L}^1, \mathcal{L}^q) \subset \mathcal{F} W(\mathcal{L}^q', \mathcal{L}^\infty) \) (see the Hausdorff-Young property (2.13)), for \( 1 \leq q \leq 2 \), we deduce
\[ \int_{\mathbb{R}^d} |K(x, y)| dx \leq \| V_{\varphi_2}^*(T_y \varphi_1^*) \|_{W(\mathcal{L}^q, \mathcal{L}^1)} \| a \|_{W(\mathcal{L}^q, \mathcal{L}^q)} \lesssim \| V_{\varphi_2}^*(T_y \varphi_1^*) \|_{M^1} \| a \|_{W(\mathcal{L}^q, \mathcal{L}^q)} \]
\[ \lesssim \| T_y \varphi_1^* \|_{M^1} \| \varphi_2^* \|_{M^1} \| a \|_{W(\mathcal{L}^q, \mathcal{L}^q)} = \| \varphi_1 \|_{M^1} \| \varphi_2 \|_{M^1} \| a \|_{W(\mathcal{L}^q, \mathcal{L}^q)}, \]

where we have used \( M^1 = W(\mathcal{F} \mathcal{L}^1, \mathcal{L}^1) \subset W(\mathcal{L}^q, \mathcal{L}^1) \), by the local inclusion \( \mathcal{F} \mathcal{L}^1 \subset \mathcal{L}^q \) (see Lemma 2.4, item (ii)), and Proposition 2.8 with \( p = 1 \). Hence,
\[ \| K \|_{\mathcal{L}^1, \mathcal{L}^1} \lesssim \| a \|_{W(\mathcal{L}^q, \mathcal{L}^q)} \| \varphi_1 \|_{M^1} \| \varphi_2 \|_{M^1}. \]

Similarly, one obtains
\[ \| K \|_{\mathcal{L}^q, \mathcal{L}^1} \lesssim \| a \|_{W(\mathcal{L}^q, \mathcal{L}^q)} \| \varphi_1 \|_{M^1} \| \varphi_2 \|_{M^1} \]
and the estimate (4.1) follows.

To study the boundedness of localization operators on Wiener amalgam spaces, we rewrite them as integral operators and use the following Schur-type test for integral operators on Wiener amalgam spaces.

**Proposition 4.2.** Consider an integral operator defined by
\[ D_K f(x) = \int_{\mathbb{R}^d} K(x, y) f(y) dy. \]
If the kernel \( K \) belongs to the following spaces (see definitions (2.8) and (2.9)):
\[ K \in W(\mathcal{L}^1, \mathcal{L}^\infty, \mathcal{L}^\infty)(\mathbb{R}^{2d}) \cap W(\mathcal{L}^\infty \mathcal{L}^1, \mathcal{L}^1, \mathcal{L}^\infty)(\mathbb{R}^{2d}) \]

...
and
\begin{equation}
K \in L^\infty L^1(\mathbb{R}^d) \cap L^{1,\infty}(\mathbb{R}^d),
\end{equation}
then the operator $D_K$ is continuous on $W(L^p, L^q)(\mathbb{R}^d)$, for every $1 \leq p, q \leq \infty$.

**Proof.** Let $Q$ denote the unit cube $[0, 1]^d$. Then, we can write any function $f$ on $\mathbb{R}^d$ as $f(x) = \sum_{n \in \mathbb{Z}^d} f(x)T_n\chi_Q(x)$. Moreover, it is straightforward to show that $D_Kf(x) = \sum_{n \in \mathbb{Z}^d} D_K(fT_n\chi_Q)(x)$.

We first study the boundedness of $D_K$ on $W(L^1, L^\infty)$. Using the expression above, we have
\[
\|D_Kf\|_{W(L^1, L^\infty)} = \sup_{k \in \mathbb{Z}^d} \|D_K(\sum_{n \in \mathbb{Z}^d} fT_n\chi_Q)T_k\chi_Q\|_{L^1} \leq \sup_{k \in \mathbb{Z}^d} \|D_K(fT_n\chi_Q)T_k\chi_Q\|_{L^1}.
\]
The equality $T_n\chi_Q(y) = T_n\chi_Q^2(y) = (T_n\chi_Q(y))^2$, yields
\[
\|D_K(fT_n\chi_Q)T_k\chi_Q\|_{L^1} \leq \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} K(x, y)f(y)T_n\chi_Q(y)T_k\chi_Q(x)dxdy \right|
\leq \|T_n\chi_Qf\|_1 \sup_{y \in \mathbb{R}^d} \left( \int_{\mathbb{R}^d} T_n\chi_Q(y)|K(x, y)|T_k\chi_Q(x)dx \right).
\]
Hence,
\[
\sup_{k \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}^d} \|D_K(fT_n\chi_Q)T_k\chi_Q\|_{L^1, \infty} \leq \|f\|_{W(L^1, L^\infty)} \|K\|_{W(L^{1, \infty}, L^{\infty, L^1})}.
\]
Using similar arguments, one easily obtains
\[
\|D_Kf\|_{W(L^\infty, L^1)} \leq \|f\|_{W(L^\infty, L^1)} \|K\|_{W(L^{\infty, L^1}, L^{1, \infty})}.
\]
Since the statement holds for $p = q$ by the classical Schur’s test, the operator $D_K$ is bounded on $W(L^p, L^p) = L^p$. By complex interpolation between $(p, p)$ and $(p, q) = (1, \infty)$, we get the boundedness of $D_K$ on every $W(L^p, L^q)$, $1 \leq p < q < \infty$. The complex interpolation between $(p, p)$ and $(p, q) = (\infty, 1)$ yields the boundedness of $D_K$ on every $W(L^p, L^q)$, $1 \leq q < p \leq \infty$. It remains to study the cases $q = \infty$ and $1 < p < \infty$. For these cases we argue by duality: it suffices to verify that
\begin{equation}
|\langle D_Kf, g \rangle| \leq \|f\|_{W(L^p, L^\infty)} \|g\|_{W(L^{p', L^1})}, \quad \forall f \in W(L^p, L^\infty), \forall g \in W(L^{p', L^1}).
\end{equation}
Now
\[
|\langle D_Kf, g \rangle| \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |K(x, y)f(y)g(x)|dxdy = \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |K(x, y)g(x)|dx \right) |f(y)|dy
\leq \|D_Kg\|_{W(L^{p', L^1})} \|f\|_{W(L^p, L^\infty)},
\]
where $D_K$ is the operator with kernel $\tilde{K}(x, y) = |K(y, x)|$. Since it satisfies the same assumptions as $D_K$, it is continuous on $W(L^{p', L^1})$. This yields (4.8). \qed
Observe that the condition (4.7) is the assumption in the classical Schur’s test for the continuity of $D_K$ on $L^p$ spaces. Furthermore, the condition $K \in W(L^{1,\infty}, L^{\infty,1}) \cap W(L^{\infty,1}, L^{1,\infty})$ implies the assumption (4.6).

**Proposition 4.3.** Let $a \in W(L^1, L^\infty)(\mathbb{R}^d)$, $\varphi_1, \varphi_2 \in M^1(\mathbb{R}^d)$. Then the operator $A_a^{\varphi_1,\varphi_2}$ is bounded on $W(L^2, L^s)(\mathbb{R}^d)$ for every $1 \leq s \leq \infty$, with the uniform estimate

\[ \|A_a^{\varphi_1,\varphi_2}\|_{B(W(L^2, L^1))} \lesssim \|a\|_{W(L^1, L^\infty)} \|\varphi_1\|_{M^1} \|\varphi_2\|_{M^1}. \]

**Proof.** We have to prove the estimate

\[ |\langle A_a^{\varphi_1,\varphi_2} f, g \rangle| \leq C \|a\|_{W(L^1, L^\infty)} \|f\|_{W(L^2, L^s)} \|g\|_{W(L^2, L^s')} \|\varphi_1\|_{M^1} \|\varphi_2\|_{M^1}, \]

for every $f \in W(L^2, L^s)$, $g \in W(L^2, L^s')$. Using the weak definition (4.7), we can write

\[ \langle A_a^{\varphi_1,\varphi_2} f, g \rangle = \int_{\mathbb{R}^d} a(x, \omega) V_{\varphi_1} f(x, \omega) \overline{V_{\varphi_2} g(x, \omega)} \, dx \, d\omega. \]

By Lemma 2.4 (ii),

\[ |\langle A_a^{\varphi_1,\varphi_2} f, g \rangle| \leq \|a\|_{W(L^1, L^\infty)} \|V_{\varphi_1} f \overline{V_{\varphi_2} g}\|_{W(L^1, L^\infty)}. \]

Write $f = \sum_{k \in \mathbb{Z}^d} f_k \chi_Q$ and $g = \sum_{h \in \mathbb{Z}^d} g_h \chi_Q$. Moreover, choose $\psi \in S(\mathbb{R}^d)$ satisfying (2.20) and write $\varphi_1 = \sum_{l \in \mathbb{Z}^d} \varphi_{1,l}$, $\varphi_2 = \sum_{m \in \mathbb{Z}^d} \varphi_{2,m}$, with $\varphi_{1,l} = \psi(D - l)\varphi_1$ and $\varphi_{2,m} = \psi(D - m)\varphi_2$ (with the notation in Proposition 2.11). We deduce (4.9)

\[ |\langle A_a^{\varphi_1,\varphi_2} f, g \rangle| \leq \|a\|_{W(L^1, L^\infty)} \sum_{m \in \mathbb{Z}^d} \sum_{l \in \mathbb{Z}^d} \|V_{\varphi_{1,l}}(f_k \chi_Q) \overline{V_{\varphi_{2,m}}(g_h \chi_Q)}\|_{W(L^1, L^\infty)}. \]

By applying Lemma 2.1 (iv) and (v) we see that

\[ |\mathcal{F} \left( V_{\varphi_{1,l}}(f_k \chi_Q) \overline{V_{\varphi_{2,m}}(g_h \chi_Q)} \right)(x, \omega)| \]

\[ \quad = |V_{g_h T_k \chi_Q}(f_k \chi_Q)(-\omega, x) \overline{V_{\varphi_{2,m}}(\varphi_{1,l})(-\omega, x)}| \]

\[ \quad = |V_{g_h T_k \chi_Q}(f_k \chi_Q)(-\omega, x) V_{\varphi_{2,m}}(\varphi_{1,l})(x, \omega)|. \]

The key observation is now the following one: if $\gamma_1, \gamma_2 \in L^2(\mathbb{R}^d)$ are supported in balls of radius, say, $R$, then the STFT $V_{\gamma_1} \gamma_2(x, \omega)$ has support in a strip of the type $B(y_0, 2R) \times \mathbb{R}^d$, for some $\gamma_0 \in \mathbb{R}^d$. This follows immediately from the definition of STFT. Hence the computation above shows that the expression

\[ V_{\varphi_{1,l}}(f_k \chi_Q) \overline{V_{\varphi_{2,m}}(g_h \chi_Q)} \]

has Fourier transform supported in a ball in $\mathbb{R}^{2d}$ whose radius is independent of $k, h, m, l$. Hence it follows from (4.9) and Proposition 2.7 that

\[ |\langle A_a^{\varphi_1,\varphi_2} f, g \rangle| \lesssim \|a\|_{W(L^1, L^\infty)} \sum_{m \in \mathbb{Z}^d} \sum_{l \in \mathbb{Z}^d} \|V_{\varphi_{1,l}}(f_k \chi_Q) \overline{V_{\varphi_{2,m}}(g_h \chi_Q)}\|_{1}. \]
As a consequence of Cauchy-Schwarz’s inequality and Parseval’s formula, this last expression is

\[
\leq \|a\|_{W(L^1,L^\infty)} \sum_{m,l \in \mathbb{Z}^d} \sum_{k,h \in \mathbb{Z}^d} \int_{\mathbb{R}^d} \|V_{\varphi,1} (fT_k \chi_Q) (x, \cdot)\|_2 \|V_{\varphi,2,m} (gT_h \chi_Q) (x, \cdot)\|_2 \, dx \\
= \|a\|_{W(L^1,L^\infty)} \sum_{m,l \in \mathbb{Z}^d} \sum_{k,h \in \mathbb{Z}^d} \int_{\mathbb{R}^d} \|fT_k \chi_Q T_x \varphi_{1,l}\|_2 \|gT_h \chi_Q T_x \varphi_{2,m}\|_2 \, dx \\
\leq \|a\|_{W(L^1,L^\infty)} \sum_{k,h \in \mathbb{Z}^d} \|fT_k \chi_Q\|_2 \|gT_h \chi_Q\|_2 \sum_{m,l \in \mathbb{Z}^d} \int_{\mathbb{R}^d} \|T_k \chi_Q T_x \varphi_{1,l}\|_\infty \|T_h \chi_Q T_x \varphi_{2,m}\|_\infty \, dx.
\]

We will prove that

\[
\int_{\mathbb{R}^d} \|T_k \chi_Q T_x \varphi_{1,l}\|_\infty \|T_h \chi_Q T_x \varphi_{2,m}\|_\infty \, dx = \int_{\mathbb{R}^d} \|\chi_Q T_x \varphi_{1,l}\|_\infty \|\chi_Q T_{x+k-h} \varphi_{2,m}\|_\infty \, dx = v_{k-h,l,m},
\]

for a sequence \( v = v_{k,l,m} \in l^1(\mathbb{Z}^d) \), satisfying

\[
(4.11) \quad \|v\|_{l^1(\mathbb{Z}^d)} \lesssim \|\varphi_1\|_{M^1} \|\varphi_2\|_{M^1}.
\]

Assuming (4.11), by Hölder’s and Young’s inequality,

\[
\langle A_{q}^{\varphi_1,\varphi_2} f, g \rangle \lesssim \|a\|_{W(L^1,L^\infty)} \sum_{m,l \in \mathbb{Z}^d} \sum_{k,h \in \mathbb{Z}^d} v_{k-h,l,m} \|gT_h \chi_Q\|_2 \|fT_k \chi_Q\|_2 \\
\lesssim \|a\|_{W(L^1,L^\infty)} \sum_{m,l \in \mathbb{Z}^d} \left( \sum_{k \in \mathbb{Z}^d} \left( \sum_{h \in \mathbb{Z}^d} v_{k-h,l,m} \|gT_h \chi_Q\|_2 \right)^s \right)^{1/s'} \left( \sum_{k \in \mathbb{Z}^d} \|fT_k \chi_Q\|_2 \right)^{1/s} \\
\leq \|a\|_{W(L^1,L^\infty)} \sum_{m,l \in \mathbb{Z}^d} \left( \sum_{k \in \mathbb{Z}^d} v_{k,l,m} \right) \left( \sum_{h \in \mathbb{Z}^d} \|gT_h \chi_Q\|_2 \right)^{1/s'} \left( \sum_{k \in \mathbb{Z}^d} \|fT_k \chi_Q\|_2 \right)^{1/s}
\]

and the desired estimate follows by (4.11).

Let us now prove (4.11). Indeed,

\[
(4.12) \quad \sum_{k \in \mathbb{Z}^d} v_{k,m,l} = \int_{\mathbb{R}^d} \|\chi_Q T_x \varphi_{1,l}\|_\infty \sum_{k \in \mathbb{Z}^d} \|\chi_Q T_{x+k-h} \varphi_{2,m}\|_\infty \, dx
\]

\[
(4.13) \quad \approx \|\varphi_{2,m}\|_{W(L^\infty,L^1)} \int_{\mathbb{R}^d} \|\chi_Q T_x \varphi_{1,l}\|_\infty \, dx \approx \|\varphi_{1,l}\|_{W(L^\infty,L^1)} \|\varphi_{2,m}\|_{W(L^\infty,L^1)}
\]

\[
(4.14) \quad \lesssim \|\varphi_{1,l}\|_1 \|\varphi_{2,m}\|_1.
\]
where the last estimate follows from Proposition 2.7. Then, by Proposition 2.10, we have

\[ \sum_{k,l,m} v_{k,m,n} \lesssim \sum_{l \in \mathbb{Z}^d} \| \varphi_{1,l} \|_1 \sum_{m \in \mathbb{Z}^d} \| \varphi_{1,m} \|_1 \asymp \| \varphi_{1} \|_{M^1} \| \varphi_{2} \|_{M^1}, \]

which concludes the proof.

\[ \square \]

**Remark 4.4.** An argument similar to that in the proof of Proposition 4.3 (but simpler) shows that, if \( a \in L^\infty(\mathbb{R}^d) \) and \( \varphi_1, \varphi_2 \in W(L^\infty, L^1)(\mathbb{R}^d) \), then \( A_{a}^{\varphi_1, \varphi_2} \) is bounded on \( W(L^2, L^s)(\mathbb{R}^d) \) for every \( 1 \leq s \leq \infty \).

Propositions 4.2 and 4.3 are the main ingredients to study the boundedness of localization operators on Wiener amalgam spaces.

**Theorem 4.5.** Let \( a \in W(L^p, L^q)(\mathbb{R}^d) \), \( \varphi_1, \varphi_2 \in M^1(\mathbb{R}^d) \). Then \( A_{a}^{\varphi_1, \varphi_2} \) is bounded on \( W(L^r, L^s)(\mathbb{R}^d) \), for all

\[ (4.15) \quad 1 \leq p, q, r, s \leq \infty, \quad \frac{1}{q} \geq \left| \frac{1}{r} - \frac{1}{2} \right|, \]

with the uniform estimate

\[ (4.16) \quad \| A_{a}^{\varphi_1, \varphi_2} \|_{B(W(L^r, L^s))} \lesssim \| a \|_{W(L^p, L^q)} \| \varphi_1 \|_{M^1} \| \varphi_2 \|_{M^1}. \]

**Proof.** We use similar arguments to those of Theorem 4.1. Again, it is enough to prove the claim for \( p = 1 \). We shall show that the cases \( 1 \leq q \leq 2 \) yield the continuity of \( A_{a}^{\varphi_1, \varphi_2} \) on \( W(L^r, L^s) \) for every \( 1 \leq r, s \leq \infty \), so that, by complex interpolation with the case \((q, r) = (\infty, 2)\), considered in Proposition 4.3 (see Lemma 2.4 (iii) and Figure 1) we obtain the desired boundedness of \( A_{a}^{\varphi_1, \varphi_2} \) under the conditions (4.15), if \( s < \infty \). The remaining cases, when \( s = \infty \) and \( q > 2 \) (and therefore \( r > 1 \)) follows by duality, for then \( W(L^r, L^s) = W(L^r, L^s)' \) (Proposition 2.4 (iv)), and \( (A_{a}^{\varphi_1, \varphi_2})^* = A_{a}^{\varphi_2, \varphi_1} \).

Hence, let \( 1 \leq q \leq 2 \). In order to prove the boundedness of \( A_{a}^{\varphi_1, \varphi_2} \) on \( W(L^r, L^s) \) for every \( 1 \leq r, s \leq \infty \) we use the Schur’s test for Wiener amalgam spaces given by Proposition 4.2.

We already verified (4.7) for the integral kernel \( K \) of \( A_{a}^{\varphi_1, \varphi_2} \), which was computed in (4.5). Let us verify that \( K \in W(L^{1,\infty}, L^\infty L^1) \). To this end, we estimate the kernel as follows:

\[ |K(x, y)| \leq \int_{\mathbb{R}^d} |V_{\varphi_2} (T_y \varphi_1^*) (x, \omega) \mathcal{F} a(\omega, y - x)| d\omega \]

\[ = \int_{\mathbb{R}^d} |V_{\varphi_2} \varphi_1^* (x - y, \omega) \mathcal{F} a(\omega, y - x)| d\omega, \]
since the STFT fulfills \( V_g(T_y f)(x, \omega) = e^{-2\pi i y \omega} V_g f(x - y, \omega) \), see \((2.1)\). For sake of simplicity, we set \( \Phi := V_{\varphi_2}^{\infty} \). Moreover, we introduce the coordinate transformation \( \tau F(t, u) = F(-u, t) \). So that the kernel \( K \) can be controlled from above by

\[
|K(x, y)| \leq \int_{\mathbb{R}^d} |\tau \Phi(\omega, y - x)| |\mathcal{F} a(\omega, y - x)| d\omega.
\]

Let us estimate \( \|K\|_{W(L^1, \infty, L^1 \times L^1)} \). Setting \( B(t) := \int_{\mathbb{R}^d} |\tau \Phi(\omega, t)| |\mathcal{F} a(\omega, t)| d\omega \), we obtain

\[
\|K\|_{W(L^1, \infty, L^1 \times L^1)} = \sup_{k \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}^d} \sup_{x, y \in \mathbb{R}^d} \int_{\mathbb{R}^d} T_k \chi Q(x) T_n \chi Q(y) |K(x, y)| dx
\]

\[
\leq \sup_{k \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}^d} \sup_{x, y \in \mathbb{R}^d} \int_{\mathbb{R}^d} T_k \chi Q(x) T_n \chi Q(y) |B(y - x)| dx
\]

\[
\leq \sup_{k \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}^d} \sup_{x, y \in \mathbb{R}^d} T_n \chi Q(y) (T_k \chi Q \ast |B|)(y)
\]

\[
= \sup_{k \in \mathbb{Z}^d} \|T_k \chi Q \ast |B|\|_{W(L^\infty, \infty)}.
\]

Using \( L^\infty \ast L^1 \hookrightarrow L^\infty, L^1 \ast L^1 \hookrightarrow L^1 \) and the convolution relations for Wiener amalgam spaces in Lemma \(2.4\), Item (i), we obtain

\[
\|T_k \chi Q \ast |B|\|_{W(L^\infty, \infty)} \leq \|T_k \chi Q\|_{W(L^\infty, L^1)} \|B\|_{W(L^1, L^1)} \leq \|B\|_1,
\]

since \( \|T_k \chi Q\|_{W(L^\infty, L^1)} = 1 \) and \( W(L^1, L^1) = L^1 \). We are left to estimate \( \|B\|_1 \). Precisely,

\[
\|B\|_1 = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\tau \Phi(\omega, t)| |\mathcal{F} a(\omega, t)| d\omega dt
\]

\[
\leq \|\tau \Phi\|_{W(L^\infty, L^1)} \|\mathcal{F} a\|_{W(L^\infty, L^\infty)}
\]

\[
\leq \|\tau \Phi\|_{M^1} \|a\|_{W(L^1, L^\infty)}
\]

\[
= \|\Phi\|_{M^1} \|a\|_{W(L^1, L^\infty)}
\]

\[
\lesssim \|\varphi_1\|_{M^1} \|\varphi_2\|_{M^1} \|a\|_{W(L^1, L^\infty)},
\]

where we used the inclusion \( \mathcal{F} L^1 \subset L^\infty \), which holds locally and yields \( W(\mathcal{F} L^1, L^1) \subset W(L^q, L^1) \) (Lemma \(2.4\), item (ii)) and the equality \( W(\mathcal{F} L^1, L^1) = M^1 \) and, finally, \((2.18)\).

Similar arguments yield

\[
\|K\|_{W(L^\infty, L^1 \times L^\infty)} \lesssim \|\varphi_1\|_{M^1} \|\varphi_2\|_{M^1} \|a\|_{W(L^1, L^\infty)},
\]

and the desired result follows. \( \square \)
5. Necessary Boundedness Conditions for Localization Operators

We need the following version of Lemma 3.1 for Wiener amalgam spaces.

Lemma 5.1. With the notation of Lemma 3.1, we have, for $q \geq 2$,
\[
\|\widehat{h}_\lambda\|_{W(L^p, L^q)} \lesssim \lambda^{\frac{d}{q} - \frac{d}{p}}, \quad \lambda \geq 1.
\]

Proof. When $p \leq q$, the result follows at once from Lemma 3.1 using the embedding $L^q \hookrightarrow W(L^p, L^q)$.

When $p > q$ it suffices to apply Proposition 2.7 and Lemma 3.1 again. \qed

Proposition 5.2. Let $\varphi_1, \varphi_2 \in C_0^\infty(\mathbb{R}^d)$, $\chi \in C_0^\infty(\mathbb{R}^d)$, with $\varphi_1(0) = \varphi_2(0) = \chi(0) = 1$, $\varphi_1 \geq 0$, $\varphi_2 \geq 0$, $\chi \geq 0$. If the estimate
\[
\|\chi A_{a_\lambda}^{\varphi_1, \varphi_2} f\|_r \leq C\|a\|_{W(L^p, L^q)} \|f\|_{W(L^{r_1}, L^{r_2})}, \quad \forall f \in \mathcal{S}(\mathbb{R}^d), \forall a \in \mathcal{S}(\mathbb{R}^d),
\]
holds for some $1 \leq p, q, r, s_1, s_2 \leq \infty$, then
\[
\frac{1}{q} \geq \frac{1}{2} - \frac{1}{r}.
\]

Proof. We can assume $q \geq 2$, otherwise the conclusion is trivially true. Consider a real-valued function $h \in C_0^\infty(\mathbb{R}^d)$, with $h(0) = 1$, $h \geq 0$. Let
\[
h_\lambda(x) = h(x)e^{-\pi i \lambda |x|^2}, \quad \lambda \geq 1.
\]
We test the estimate (5.1) on $f = \overline{h}_\lambda$ and
\[
a(x, \omega) = a_\lambda(x, \omega) = h(x)(\mathcal{F}^{-1}h_\lambda)(\omega).
\]
Clearly, $\|h_\lambda\|_{W(L^{r_1}, L^{r_2})}$ is independent of $\lambda$. On the other hand, by Lemma 5.1 we have
\[
\|a_\lambda\|_{W(L^p, L^q)} = \|h\|_{W(L^p, L^q)} \|\mathcal{F}^{-1}h_\lambda\|_{W(L^p, L^q)} \lesssim \lambda^{\frac{d}{q} - \frac{d}{p}}.
\]
Hence, if we prove that
\[
\|\chi A_{a_\lambda}^{\varphi_1, \varphi_2} f\|_r \gtrsim \lambda^{-\frac{d}{r}},
\]
then (5.2) follows from (5.1), by letting $\lambda \to +\infty$.

Let us verify (5.3). It suffices to prove that
\[
|\chi(x)(A_{a_\lambda}^{\varphi_1, \varphi_2} \overline{h}_\lambda)(x)| \gtrsim 1, \quad \text{for } |x| \leq \lambda^{-1},
\]
and for all $\lambda \geq \lambda_0$ large enough. To this end, observe that, by (4.2), $A_{a_\lambda}^{\varphi_1, \varphi_2}$ is an integral operator with kernel
\[
K(x, y) = \int h(t)h(y - x)e^{-\pi i |y - x|^2} \varphi_1(y - t)\varphi_2(x - t) dt.
\]
Hence
\[
|\chi(x)(A_{a_\lambda}^{\varphi_1, \varphi_2} \overline{h}_\lambda)(x)| = \left| \chi(x) \int e^{2\pi i xy}h(t)h(y - x)\varphi_1(y - t)\varphi_2(x - t)h(y) dt dy \right|.
\]
Since $\varphi_1, \varphi_2$ have compact support, $K(x,y)$ is identically zero if $|x-y| \geq C$, for a convenient $C > 0$. Moreover, we are considering the last integral for $|x| \leq \lambda^{-1} \leq 1$ only, so that we can in fact integrate over $|y| \leq C + 1$ only. On the other hand, we have

$$\text{Re} e^{2\pi i \lambda xy} = \cos(2\pi \lambda xy) \geq \frac{1}{2}, \text{ if } |x| \leq \lambda^{-1}, \ |y| \leq C + 1,$$

for all $\lambda \geq \lambda_0$ large enough. It follows that

$$(5.4) \quad \left| \chi(x) \int \int e^{2\pi i \lambda xy} h(t) h(y-x) \varphi_1(y-t) \varphi_2(x-t) h(y) \ dt \ dy \right| \geq \frac{1}{2} \chi(x) \left| \int \int h(t) h(y-x) \varphi_1(y-t) \varphi_2(x-t) h(y) \ dt \ dy \right|.$$

Now, this last expression is clearly bounded from below by a constant $\delta > 0$, for $x$ in a neighbourhood of the origin. This follows by continuity, since for $x = 0$ the integral is strictly positive by our assumptions on $\varphi_1, \varphi_2, h,$ and $\chi(0) = 1$.

This concludes the proof. \hfill \Box

**Theorem 5.3.** Let $\varphi_1, \varphi_2 \in C_0^\infty(\mathbb{R}^d)$, with $\varphi_1(0) = \varphi_2(0) = 1$, $\varphi_1 \geq 0$, $\varphi_2 \geq 0$. If, for some $1 \leq p, q, r, s \leq \infty$, the estimate

$$(5.5) \quad \|A_a^{\varphi_1, \varphi_2} f\|_{W(L^r, L^s)} \leq C \|a\|_{W(L^p, L^q)} \|f\|_{W(L^r, L^s)}, \quad \forall f \in S(\mathbb{R}^d), \forall a \in S(\mathbb{R}^{2d}),$$

holds, then

$$(5.6) \quad \frac{1}{q} \geq \left| \frac{1}{r} - \frac{1}{2} \right|.$$

**Proof.** We can suppose $q \geq 2$ (otherwise (5.6) is trivially satisfied). We assume $r \geq 2$, the case $1 \leq r < 2$ follows by duality, for $(A_a^{\varphi_1, \varphi_2})^* = A_a^{\varphi_2, \varphi_1}$.

Let $\chi \in C_0^\infty(\mathbb{R}^d)$, $\chi \geq 0$, $\chi(0) = 1$. Then, (5.5) implies that

$$\|\chi A_a^{\varphi_1, \varphi_2} f\|_{W(L^r, L^s)} \leq C \|a\|_{W(L^p, L^q)} \|f\|_{W(L^r, L^s)} \quad \forall f \in S(\mathbb{R}^d), \forall a \in S(\mathbb{R}^{2d}).$$

For functions $u$ supported in a fixed compact subset we have $\|u\|_{W(L^r, L^s)} \asymp \|u\|_r$, so that we have

$$\|\chi A_a^{\varphi_1, \varphi_2} f\|_r \leq C \|a\|_{W(L^p, L^q)} \|f\|_{W(L^r, L^s)}, \quad \forall f \in S(\mathbb{R}^d), \forall a \in S(\mathbb{R}^{2d}).$$

Then (5.6) follows from Proposition 5.2. \hfill \Box

**Theorem 5.4.** Let $\varphi_1, \varphi_2 \in C_0^\infty(\mathbb{R}^d)$, with $\varphi_1(0) = \varphi_2(0) = 1$, $\varphi_1 \geq 0$, $\varphi_2 \geq 0$. Assume that for some $1 \leq p, q, r, s \leq \infty$ and every $a \in W(L^p, L^q)(\mathbb{R}^{2d})$ the operator $A_a^{\varphi_1, \varphi_2}$ is bounded on $W(L^r, L^s)(\mathbb{R}^d)$. Then (5.6) must hold.
Proof. Consider first the case $1 \leq r, s < \infty$. By Theorem 5.3 and the Closed Graph Theorem it suffices to prove that the map

$$a \in W(L^p, L^q) \mapsto A_a^{\varphi_1, \varphi_2} \in B(W(L^r, L^s))$$

has closed graph. To this end, consider a sequence $a_n \to a$ in $W(L^p, L^q)$ and such that $A_{a_n}^{\varphi_1, \varphi_2} \to A$ in $B(W(L^r, L^s))$. We verify that $A_{a_n}^{\varphi_1, \varphi_2} = A$, that is $\langle A_{a_n}^{\varphi_1, \varphi_2} f, g \rangle = \langle Af, g \rangle$ for every $f, g \in \mathcal{S}(\mathbb{R}^d)$, because $\mathcal{S}(\mathbb{R}^d)$ is dense in $W(L^r, L^s)$. Clearly, $\langle A_{a_n}^{\varphi_1, \varphi_2} f, g \rangle \to \langle Af, g \rangle$. On the other hand, since $V_{\varphi_1} f, V_{\varphi_2} g$ are Schwartz functions,

$$\langle A_{a_n}^{\varphi_1, \varphi_2} f, g \rangle = \int a_n(x, \omega) V_{\varphi_1} f(x, \omega) \overline{V_{\varphi_2} g(x, \omega)} \, dx \, d\omega$$

tends to

$$\int a(x, \omega) V_{\varphi_1} f(x, \omega) \overline{V_{\varphi_2} g(x, \omega)} \, dx \, d\omega = \langle A_a^{\varphi_1, \varphi_2} f, g \rangle.$$

It remains to consider the cases $r = \infty$ or $s = \infty$. Here one can argue by contradiction. Namely, suppose that $A_a^{\varphi_1, \varphi_2}$ is bounded on $W(L^r, L^s)$ for every $a \in W(L^p, L^q)$, for some $q, r$ that do not satisfy (5.6). Hence, $\frac{1}{r} < \frac{1}{q} - \frac{1}{2}$. Suppose $r = \infty$, which implies $q > 2$. By interpolating with the case $(r, s) = (2, 2)$ (being fixed the pair $(p, q)$), one would obtain the boundedness of $A_a^{\varphi_1, \varphi_2}$ for every $a \in W(L^p, L^q)$, on certain $W(L^r, L^s)$, $\tilde{r}, \tilde{s} < \infty$, with $\tilde{r}$ arbitrarily large, so that (5.6) (with $\tilde{r}$ in place of $r$) fails. Similarly, for $s = \infty$ and interpolating with the case $(r, s) = (2, 2)$, one would obtain the boundedness of $A_a^{\varphi_1, \varphi_2}$ for every $a \in W(L^p, L^q)$, on certain $W(L^r, L^s)$, $\tilde{r}, \tilde{s} < \infty$, with $\frac{1}{q} < \frac{1}{\tilde{r}} - \frac{1}{2}$, so that (5.6) fails again. This contradicts what we showed in the first part of the present proof. 

6. Further results

In this section we discuss a slight improvement of Theorem 6.1 in the special case $p = q$, i.e., for symbols $a \in L^q$. Precisely, we consider windows in Lebesgue spaces only, rather than in $M^1$. To state the outcome of our study, we need the following definition. A measurable function $F$ on $\mathbb{R}^{2d}$ belongs to the space $W(L^{p_1}, L^{q_1})\mathcal{F}W(L^{p_2}, L^{q_2})(\mathbb{R}^{2d})$ if the mapping $x \mapsto \|F(x, \cdot)\|_{\mathcal{F}W(L^{p_2}, L^{q_2})}$ is in the Wiener amalgam space $W(L^{p_1}, L^{q_1})(\mathbb{R}^{d})$, with norm

$$\|F\|_{W(L^{p_1}, L^{q_1})\mathcal{F}W(L^{p_2}, L^{q_2})} = \|F(x, \cdot)\|_{\mathcal{F}W(L^{p_2}, L^{q_2})} \|_{W(L^{p_1}, L^{q_1})}. $$

Then our first result reads as follows.

Proposition 6.1. Let $1 \leq p_1, p_2, q_1, q_2, r \leq \infty$. Let

$$a \in W(L^{p_1}, L^{q_1})\mathcal{F}W(L^{p_2}, L^{q_2})(\mathbb{R}^{2d}),$$

and

$$\varphi_1, \varphi_2 \in \mathcal{N}(\mathbb{R}^d) := W(L^{p_1}, L^{q_1})(\mathbb{R}^{d}) \cap W(L^{p_2}, L^{q_2})(\mathbb{R}^{d}).$$
Then $A_a^{\varphi_1,\varphi_2}$ is bounded on $L^r(\mathbb{R}^d)$, with the uniform estimate
\begin{equation}
\|A_a^{\varphi_1,\varphi_2}\|_{L^r} \lesssim \|a\|_{W(L^{p_1},L^{q_1})} \|\varphi_1\|_{\mathcal{N}_r} \|\varphi_2\|_{\mathcal{N}_q}.
\end{equation}

**Proof.** The proof is similar to that of Theorem 4.1, so that we only present the main points. We apply Schur’s test. Hence we verify that the integral kernel
\[ K(x,y) = \int \phi_2(y) \phi_1(x) dt \]
of $A_a^{\varphi_1,\varphi_2}$ is in $L^\infty L^1 \cap L^{1,\infty}$.

Let us verify that $K \in L^\infty L^1$, the other condition is similar. By the kernel expression above and Hölder’s Inequality we have
\begin{align*}
\|K\|_{L^\infty L^1} &\leq \sup_{x \in \mathbb{R}^d} \int |T_i \varphi_2(x)| \int |T_i \varphi_1(y)| |\phi_2(a(t,y-x))| \, dy \, dt \\
&\leq \|\varphi_1\|_{W(L^{p_2},L^{q_2})} \sup_{x \in \mathbb{R}^d} \int |T_i \varphi_2(x)| \|\phi_2(a(t,y-x))\|_{F(W(L^{p_2},L^{q_2}))} \, dt \\
&\leq \|\varphi_1\|_{W(L^{p_2},L^{q_2})} \|\varphi_2\|_{W(L^{p_2},L^{q_2})} \|a\|_{W(L^{p_1},L^{q_1})} \|F(W(L^{p_2},L^{q_2}))\|_{L^r},
\end{align*}
which is the desired estimate. \[ \square \]

Finally, we can prove the boundedness result for symbols in $L^q$.

**Theorem 6.2.** Let $\mathcal{N}_q = L^q(\mathbb{R}^d) \cap L^{q'}(\mathbb{R}^d)$ if $1 \leq q \leq 2$ and $\mathcal{N}_q = L^2(\mathbb{R}^d)$ if $2 < q \leq \infty$. Let $a \in L^q(\mathbb{R}^{2d})$, $\varphi_1, \varphi_2 \in \mathcal{N}_q$. Then $A_a^{\varphi_1,\varphi_2}$ is bounded on $L^r(\mathbb{R}^d)$, for all $\frac{1}{q} \geq \left| \frac{1}{r} - \frac{1}{2} \right|$, with the uniform estimate
\begin{equation}
\|A_a^{\varphi_1,\varphi_2}\|_{L^r} \lesssim \|a\|_{\mathcal{N}_q} \|\varphi_1\|_{\mathcal{N}_q} \|\varphi_2\|_{\mathcal{N}_q}.
\end{equation}

**Proof.** By interpolation with the well-known case $q = \infty$, $r = 2$, it suffices to prove the desired result for $q \leq 2$. But this follows from Proposition 6.1 with $p_1 = q_1 = q$, $p_2 = q_2 = q'$ (recall, $W(L^q,L^{q'}) = L^q$, $W(L^{q'},L^q) = L^{q'}$), combined with the Hausdorff-Young inequality $L^q \hookrightarrow F L^{q'}$. \[ \square \]

Observe that Theorem 6.2 extends the result contained in Theorem 3.3 of [3] to larger index sets and this range is sharp. Indeed, for $s = r$ and $p = q$, Theorems 6.3 and 6.4 read as follows.

**Corollary 6.3.** Let $\varphi_1, \varphi_2 \in C_0^\infty(\mathbb{R}^d)$, with $\varphi_1(0) = \varphi_2(0) = 1$, $\varphi_1 \geq 0$, $\varphi_2 \geq 0$. Assume that for some $1 \leq q, r \leq \infty$ and every $a \in L^q(\mathbb{R}^{2d})$ the operator $A_a^{\varphi_1,\varphi_2}$ is bounded on $L^r(\mathbb{R}^d)$, or that the estimate
\begin{equation}
\|A_a^{\varphi_1,\varphi_2} f\|_r \leq C \|a\|_{\mathcal{N}_q} \|f\|_r, \quad \forall f \in \mathcal{S}(\mathbb{R}^d), \forall a \in \mathcal{S}(\mathbb{R}^{2d}),
\end{equation}
holds. Then
\begin{equation}
\frac{1}{q} \geq \left| \frac{1}{r} - \frac{1}{2} \right|.
\end{equation}
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