Abstract. In this paper we continue the investigation of finding the max/min polygons which can be inscribed in a given triangle. We are concerned here with equilateral triangles. This may seem uninteresting or benign at first, but there are some surprises later.

An inscribed polygon in a given triangle is a polygon which has all of its vertices on the triangle. In a previous paper [3] we considered the inscribed polygons: parallelograms, rectangles, and squares. In this paper we investigate inscribed equilateral triangles in a given triangle. Figure 1 shows two examples of inscribed equilateral triangles in $\Delta ABC$.

The first Puzzle is a quick investigation of properties of equilateral triangles ($ET$ for short). All graphs were made with Sketchpad*.

Puzzle 1. If the sides of the $ET \Delta XYZ$ have known length $s$, determine the height $h$, and the area of $\Delta XYZ$ (denoted by $(XYZ)$). If the height $h$ of $ET \Delta XYZ$ is known, determine the length of the side $s$, and the area $(XYZ)$ of $\Delta XYZ$.

Tip: The left and right hand halves of $\Delta XYZ$ are 30°-60°-90° triangles with sides in the ratios $s/2$, $s$, and $h = s\sqrt{3}/2$, Fig. 2.

Thus $(XYZ) = hs/2 = s^2\sqrt{3}/4$.

On the other hand, given $h$, the equality of ratios $h\sqrt{3} = s/2$ holds.

Hence $s = 2h\sqrt{3}/3$, and $(XYZ) = h^2\sqrt{3}/3$.

Puzzle 2. Show that an inscribed $ET$ of largest area in a given $\Delta ABC$, always shares at least one vertex with $\Delta ABC$.

By trial and error it can be shown that there are only two ways an $ET$ of maximum area will fit in a given $\Delta ABC$. Either with its base on one side of the triangle and its upper vertex coincides with

†Some of this material has appeared previously [4].

*All graphs in this paper were made with Sketchpad v5.10 BETA.
the upper vertex of the given triangle, Fig. 3 Left, or it’s base on one side of the triangle and at least one of it’s base vertices coincides with one of the base vertices of the given triangle, Fig. 3 Right. That one side of the largest ET must be on a side of the triangle is well known [5], [6].

![Figure 3](image)

Observe that if the base angles of a side of ΔABC are both greater than 60°, then there is no inscribed ET possible on that side (look at examples, or see Fig. 13).

In Fig. 3 Left, notice that both base angles of ΔABC are less than 60°, so the area of the largest ET for this side is determined by the height h of ΔABC. From Puzzle 1, the area of the ET is \(h\sqrt{3}/3\).

In Fig. 3 Right, base angle B of ΔABC is greater than 60°, and base angle A is less than 60°, so it will take some additional calculating to compute the area of the largest ET for that side of ΔABC (see Puzzles 3 & 4).

Since we are looking for the **inscribed ET of maximum area**, if we can determine the largest inscribed ET for each side of the given triangle, then the largest of these three ETs is called the **max ET** of the triangle, and the smallest of these 3 is called the **min ET** of the triangle.

By the results in Puzzle 2 we know exactly where to look for the max and min ETs. We will see below that this makes our search much easier.

We consider an example in Puzzle 3 which illustrates Fig. 3 Right.

The angle notation is standard, angle CAB = α, and so on.

**Puzzle 3.** *Let ΔABC be the given triangle with α = 45°, β = 75°, γ = 60°, and side a of length 2, Fig. 4. Let ΔBDE be the largest ET with base EB on side AB, and D on side AC. Compute s, the length of the side of ΔBDE, and the exact area of ΔBDE. Is ΔBD the max ET in ΔABC?*

Hint: Drop a perpendicular from B to AC at M.

Then ΔMBC is a 30°-60°-90° triangle with |BM| = \(\sqrt{3}\).

This means that \(c = \sqrt{6}\), since ΔABM is a 45°-45°-90° triangle.

By the Law of Sines, \(s = \sqrt{6}(\sin 45°/\sin 75°) \approx 1.793\), since angle BDA = 75°. Hence (BDE) = \(s^2\sqrt{3}/4 \approx 1.392\).

For side b construct a point F on AC such that |FC| = 2.

Then ΔBCF is an ET which is the largest ET on both sides a and b, and (BCF) = \(\sqrt{3} \approx 1.732\), by Puzzle 1.

Thus ΔBCF is the max ET in ΔABC.

![Figure 4](image)
The key to solving Puzzle 3 was knowing that $\gamma = 60^\circ$. But what if $\gamma \neq 60^\circ$? How do we compute the area of $\triangle ABDE$ in that case? Here’s a way to compute the area of such an ET.

**Puzzle 4.** Compute the area of the largest inscribed ET on side c of $\triangle ABC$, for $\alpha < 60^\circ$, $\beta > 60^\circ$.

Hint: By the Exterior Angle Theorem, angle $CDB = 60^\circ + \alpha$.

By the Law of Sines, $s = \frac{asin\gamma}{sin(60^\circ + \alpha)}$.

But if $h_b$ is the altitude of $B$ to $AC$ at $M$, $h_b = BM$, as in Fig. 4, then $sin\gamma = h_b/a$, hence $s = h_b/sin(60^\circ + \alpha)$.

So, by the solution to Puzzle 1, $(BDE) = \left(\frac{h_b}{sin(60^\circ + \alpha)}\right)^2 \sqrt{3}/4$.

We will now determine the location of the max and min ETs for all triangles.

Also, we assume from here on that $\triangle ABC$ is scalene, unless noted otherwise, and recall in particular that $\alpha > \beta > \gamma$ iff $a > b > c$, [2, Vol.1, Props. 18, 19].

By symmetry, any result we get for $\triangle ABC$ under these assumptions will also hold for a triangle reflected about the vertical line through the mid-point of the base $AB$. So we need only consider those triangles with upper vertex $C$ to the left of this vertical line. If $C$ is on this line, then $\triangle ABC$ is an isosceles triangle, which is a simpler case to work with.

In the chart in Fig. 6 there is a list of examples generated on Sketchpad. The ratio of the area of the ET to the area of $\triangle ABC$ is listed in the right-hand column, marked *Ratios*. This makes it easier to compare the sizes of the areas of the ETs. The bold $x$ marks the side $a$, $b$, $c$, where the max ET occurs, and the italic $x$ marks the side where the min ET is found. Several patterns can be observed here.

The general results are given in Puzzle 5. In each part of Puzzle 5 an example(s) of triangles from this chart is considered.

**Puzzle 5.** Given $\triangle ABC$, suppose the sides $a$, $b$, $c$ satisfy $a > b > c$. Show that the max ET is always on the long side $a$ ($a$ and $c$, when $\beta = 60^\circ$), and the min ET is on:

- **A.** side $c$, when $120^\circ < \alpha$; and sides $b$ and $c$, when $\alpha=120^\circ$;
- **B.** side $b$, when $60^\circ < \alpha < 120^\circ$, and $\beta < 60^\circ$;
- **C.** side $b$, when $60^\circ \leq \beta$. 
In Figures 7 - 10, notice that all 3 largest ETs (2 ETs coincide in Fig.10) have vertex A in common. Then ET ΔGHA, on the long side a, is the max ET, since JA and FA are interior to it (a consequence of Euclid Prop. 19 [2]). Since ΔGHA and ΔAEF coincide when β = 60°, Fig. 10, the max ET is on both the long side a and the short side c for this case. If β > 60°, Fig. 10, then α > β > 60°, and m∠JAB > m∠IBA. So |HI| > |JA|, by Euclid Prop. 19 [2], and ΔGHI is the max ET on the long side a.

These results agree with what is observed in the chart in Fig. 6.

**A.** Fig. 7 is a 130°-30°-20° triangle with ET ΔGHA on the long side a, which is clearly the max ET, since |AG| > |AJ| and |AG| > |AF|. If we compare |AJ| and |AF| we discover that ΔAEF is the min ET.

For the general case for α > 120°, since m∠CAJ = 60° = m∠FAB, it follows that m∠AJC > m∠AFB. So, in ΔAJF, s_b = |JA| > |FA| = s_c, and the min ET is on the short side c.

Figure 8 is a 120°-35°-25° triangle with the max ET, ΔGHA, on the long side a. From the first of part A it follows that ΔGHA is larger than the other two ETs, and ΔAEF and ΔJKA are congruent. For a general argument, if α = 120°, then s_b = |JA| = |FA| = s_c, so ΔAEF and ΔJKA are congruent, and the min ET is on both the short side c, and the middle side b.

**B.** In the 110°-40°-30° triangle, Fig. 9, the area of the min ET, ΔAJK on side b, is less in area than ET ΔAEF on side c, which is less in area than the max ET ΔGHA on side a, by comparing sides AG, AF, and AJ.

For a formal argument, the situation is similar to part A, since m∠CAJ = 60° = m∠FAB, so we have m∠AJC > m∠AFB. In ΔAJF it follows that s_b = |JA| < |FA| = s_c, and ΔAJK is the min ET on the middle side b.

**C.** In the 75°-60°-45° triangle Fig. 10, the ETs on sides c and a coincide, and have sides of length c, while the ET on side b, ΔAJK, has side s_b < c.

In general, if β = 60°, then ΔAEF and ΔGHA coincide. Also AJ is interior to ΔAEF, s_b = |JA| < |AE| = s_c = c. This means the min ET is ΔAJK on the middle side b.

In the 70°-65°-45° triangle Fig.11, the length of the side of the min ET, ΔAJK on side b, is less than that of the max ET, ΔGHI on side a, by comparing |AJ| and |IB|.

For the general argument, suppose β > 60°, then α > β > 60°, and by subtracting 60° from α and β it follows that m∠JAB > m∠IBA. This means s_a = |HI| > |JA| = s_b, and ΔJKA is the min ET on
the middle side \( b \). There is no \( ET \) on side \( c \) in this case, as the base angles are greater than \( 60^\circ \).

In Fig. 12 the graphs of the regions which correspond to Puzzles 5A, 5B, and 5C, are shown. The regions are labeled with the puzzle letter. A typical triangle \( \Delta ABC \) is shown with vertex \( C \) in the region \( B \). Ignoring isosceles triangles for now, the dashed lines in red and green are off limits, as is the white circular area (recall it is assumed that \( a > b > c \)).

If \( ETs \) of largest area on side \( c \) are allowed, when vertex \( C \) is in region \( C \), there is a \textit{wedged} \( ET \) \((WET \text{ for short}) \) \cite{1,4}.

These are not inscribed \( ETs \) as defined above, since the \( ETs \) do not have all 3 vertices on \( \Delta ABC \), but they are the largest \( ETs \) on side \( c \). The term \textit{wedged} was first used by Calabi in \cite{1}.

The new results under this change will happen in Puzzle 5C, for then a \( WET \) on the base side \( c \) of the triangle can be included, see Figures 11 and 13.

We list these new results to Puzzle 5C here as 5C.1, 5C.2, and 5C.3.

**5C.1.** Let \( \Delta ABC \) satisfy \( 60^\circ \leq \beta < 80^\circ \), \( 60^\circ < \alpha < 90^\circ \), and \( \alpha + \beta/2 < 120^\circ \). Then the max \( WET \) is on the short side \( c \), and the min \( WET \) is on the middle side \( b \).

**5C.2.** In \( \Delta ABC \), if we assume:

i.) \( 60^\circ < \beta < 80^\circ < \alpha < 90^\circ \), \( \alpha/2 + \beta < 120^\circ \), and \( \alpha + \beta/2 \geq 120^\circ \), then the min \( WET \) is on the middle side \( b \), and the max \( WET \) is on both the short side \( c \) and the long side \( a \), when \( \alpha + \beta/2 = 120^\circ \), and on the long side \( a \), when \( \alpha + \beta/2 > 120^\circ \).

ii.) \( 75^\circ < \beta < 90^\circ \), \( 80^\circ < \alpha < 90^\circ \), and \( \alpha/2 + \beta \geq 120^\circ \), then the max \( WET \) is on the long side \( a \), and the min \( WET \) is on the short side \( c \), when \( \alpha/2 + \beta > 120^\circ \); and on the short side \( c \) and the middle side \( b \), when \( \alpha/2 + \beta = 120^\circ \).

**5C.3.** In \( \Delta ABC \), if \( 60^\circ < \beta < 90^\circ < \alpha \), then the max \( WET \) is on the long side \( a \), and the min \( WET \)

i.) is on the middle side \( b \), when \( \alpha/2 + \beta < 120^\circ \), and on the short side \( c \) and the middle side \( b \), when \( \alpha/2 + \beta = 120^\circ \);

ii.) is on the short side \( c \), when \( \alpha/2 + \beta > 120^\circ \).
As might be expected, the arguments for these results are a little more involved than those given for Puzzles 5A, B, and C above. For example, the following is an argument for result 5C.2.i.

Suppose $60^\circ < \beta < 80^\circ < \alpha < 90^\circ$, $\alpha/2 + \beta < 120^\circ$, $\alpha + \beta/2 = 120^\circ$, and $20^\circ < \gamma < 30^\circ$, by the boundaries of the subregion. This means $90^\circ < m\angle AEC$, and $m\angle BGC < 100^\circ$. Since $\beta < \alpha$, $m\angle EAB > m\angle GBA$, and $|GB| > |AE|$. So $\triangle BHG$ is a possible max WET on side $a$, and $\triangle AEF$ is a possible min WET on side $b$.

Let $BJ$ be the angle bisector of $\beta$, $J$ on $CA$, Fig. 13. Then $m\angle AJB = 60^\circ$, since $\alpha + \beta/2 = 120^\circ$. This also means $\gamma + \beta/2 = 60^\circ$, by the Ext. Angle Th., Figure 13, so $m\angle CJB = 120^\circ = m\angle CHG$, and $m\angle CGH = \beta/2$. But then $m\angle AGB = \alpha$, since $\alpha + \beta/2 = 120^\circ$, so $\triangle AGB$ is isosceles, and $c = |GB|$. Therefore $\triangle ABD$ and $\triangle BHG$ are equal max WETs on sides $c$ and $a$, resp., and $\triangle AEF$ is the min WET on side $b$.

On the other hand, if $60^\circ < \beta < 80^\circ < \alpha < 90^\circ$, $\alpha/2 + \beta < 120^\circ$, and $\alpha + \beta/2 > 120^\circ$, then $15^\circ < \gamma < 30^\circ$ (check the conditions for $\alpha = 90^\circ$), and as above $\triangle CAE$ and $\triangle CBG$ are similar, so $m\angle AGB = m\angle AEB$. Then $\alpha > \beta$ implies $\alpha + \beta/2 > \alpha/2 + \beta$, $m\angle EAB > m\angle GBA$, and $m\angle AEB > \beta$, so $|GB| > c > |AE|$. This means that $\triangle BHG$ is the max WET on side $a$ and $\triangle AEF$ is the min WET on side $b$.

The argument for 5C.2.ii is analogous, but the angle $\alpha$ is bisected instead of angle $\beta$ in that case. Also, a similar argument can be used in 5C.3.

The other cases for isosceles triangles with central angle less than $90^\circ$ are contained in results 5C.1, and 5C.2.ii.

These new results for Puzzle 5C divide region C, Fig. 12, into subregions which correspond to each of these new results, Fig. 14.

The subregion C.2.i is of particular interest, as two of the boundaries are the graphs of cubic equations, determined by the conditions on the angles in that part.

In region C.2.ii, if $\alpha/2 + \beta = 120^\circ$, then $\gamma = 60^\circ - \alpha/2$, so by the law of sines, if $c = 1$, then $b = \sin \beta / \sin \gamma = \sin(120^\circ - \alpha/2) / \sin(60^\circ - \alpha/2) = \sin(60^\circ + \alpha/2) / \sin(60^\circ - \alpha/2)$, [2].

Converting to polar coordinates: $r = \sin(60^\circ + \theta/2) / \sin(60^\circ - \theta/2)$, for $80^\circ < \theta < 90^\circ$, $A = (0, 0)$ the polar pole, $B = (1, 0), \theta = \alpha$, and $AB$ on the polar axis.
Changing to rectangular coordinates, by the usual method, determines the associated cubic equation:

\[ \sqrt{3}x^3 + x^2(y - 2\sqrt{3}) + \sqrt{3}x(y^2 + 1) + y^3 - 2\sqrt{3}y^2 - y = 0. \]

The graph of this curve (green) is the lower bound of region **C.2.ii** (green) and the upper bound of region **C.2.i**.

Similarly, the lower bound of region **C.2.i** (blue) which is the upper bound of region **C.1** satisfies the condition \( \alpha + \beta/2 = 120^\circ \), so \( \gamma = 60^\circ - \beta/2 \).

By the law of sines, if \( c = 1 \), then \( b = \sin\beta/\sin\gamma = \sin(240^\circ - 2\alpha)/\sin(\alpha - 60^\circ) \).

The corresponding polar coordinates equation is: \( r = \sin(240^\circ - 2\theta)/\sin(\theta - 60^\circ) \), for \( 80^\circ < \theta < 90^\circ \).

In rectangular coordinates this corresponds to the cubic equation:

\[ \sqrt{3}x^3 - x^2(y + \sqrt{3}) + x(\sqrt{3}y^2 + 2y) + \sqrt{3}y^2 - y^3 = 0. \]

The graph of this curve (blue) is the lower bound of region **C.2.i**, and the upper bound of **C.1**.

The two cubic curves are symmetric about the vertical line through points \( D \) and \( I \). The point labeled \( I \) is a point where the 2 cubics intersect, and also the upper vertex of a triangle which is a WET analog of Calabi’s result for wedged squares in triangles, Puzzle 6. [1] [3].

It should be obvious that if \( \triangle ABC \) is itself an ET, then all inscribed ETs of largest area will coincide with \( \triangle ABC \). From our analysis above this is the only triangle where the 3 largest inscribed ETs are equal. However, if WETs are admitted, then a new possibility exists.

**Puzzle 6.** If \( \triangle ABC \) is isosceles with base angles of \( 80^\circ \), show that the largest WETs on sides \( a \), \( b \), and \( c \) are congruent.

**Hint:** Let \( \triangle BHG \), \( \triangle AEF \), and \( \triangle ABD \) be the largest WETs on sides \( a \), \( b \), and \( c \), respectively, Fig. 15.

Since \( \alpha = \beta = 80^\circ \), it follows that \( \gamma = 20^\circ \), and also \( m\angle EAB = m\angle GBA = 20^\circ \).

So, \( m\angle AEB = m\angle BGA = 80^\circ \).

Thus, it follows that triangles \( \triangle ABE \) and \( \triangle ABG \) are isosceles, so \( |GB| = |AB| = |AE| \).

Hence, all 3 largest WETs: \( \triangle ABD \), \( \triangle AEF \), and \( \triangle BHG \), are congruent.
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