Critical behavior of a general $O(n)$-symmetric model of two $n$-vector fields in $D = 4 - 2\epsilon$

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Abstract

The critical behavior of the $O(n)$-symmetric model with two $n$-vector fields is studied within the field-theoretical renormalization group approach in a $D = 4 - 2\epsilon$ expansion. Depending on the coupling constants, the $\beta$-functions, fixed points and critical exponents are calculated up to the one- and two-loop order, respectively ($\eta$ in two- and three-loop order). Both continuous lines of fixed points and $O(n) \times O(2)$ invariant discrete solutions were found. Apart from already known fixed points two new ones arise. One agrees in one-loop order with a known fixed point, but differs from it in two-loop order.

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1. Introduction

The renormalization group approach provides a natural framework for the understanding of critical properties of phase transitions. A very large variety of critical phenomena can be described by the so-called $\phi^4$ models. The simple $O(n)$-symmetric one-field model

$$ S_{O(n)}(\phi) = \frac{1}{4}[(\nabla \phi)^2 + \tau \phi^2] + \frac{1}{4!}g(\phi^2), $$

where $\phi = (\phi_1, \ldots, \phi_n)$ is a real $n$-component vector field, while $\tau$ is a temperature-like parameter and $g > 0$, was extended in [1] to the interplay of two vector fields under the $O(n) + O(m)$ symmetry

$$ S_{O(n) + O(m)}(\phi_1, \phi_2) = \frac{1}{4}[(\nabla \phi_1)^2 + (\nabla \phi_2)^2 + \tau_1 \phi_1^2 + \tau_2 \phi_2^2] $$

$$ + \frac{1}{4!}[g_1(\phi_1^2)^2 + g_2(\phi_2^2)^2 + g_3(\phi_1^2)(\phi_2^2)]. $$

Six different fixed points were found. Three of them are always unstable and the stability of three others depends on $n$ and $m$. The $O(n) + O(m)$ model has been used to describe multicritical phenomena. We mention the critical behavior of uniaxial antiferromagnets in a magnetic field parallel to the field direction [1] and the $SO(5)$ theory of high-$T_c$ superconductors [2–4]. Also interesting phenomena of inverse symmetry breaking, symmetry
nonrestoration and re-entrant phase transitions were reported [5, 6]. This model as well as model (3) presented below has also been investigated in [7].

Recently, frustrated spin systems with noncollinear or canted spin ordering have been the object of intensive research [8–11]. Examples are helical magnets and layered triangular Heisenberg antiferromagnets [12]. In the corresponding action

\[ S_{O(n)×SO(n)}(\phi_1, \phi_2) = \frac{1}{2}\left( (\nabla \phi_1)^2 + (\nabla \phi_2)^2 + \tau (\phi_1^2 + \phi_2^2) \right) \]

the scalar product \( \phi_1 \phi_2 \) is present [13–15]. Both fields have \( n \) components and the model possesses the \( O(n) \times O(2) \) symmetry. In the \( 4 - 2\epsilon \) expansion, the number of fix points (FP) and their stability depend on \( n \), however different theoretical methods lead to contradictory results [4].

The results based on three-loop renormalization group calculations [16–18] show that in three-dimensional chiral magnets with \( n = 2, 3 \) critical fluctuations destroy continuous phase transitions converting them into the first-order ones, i.e. the chiral class of universality does not exist. On the other hand, the analysis of the higher-order—five-loop and six-loop—RG expansions reveals a new stable fixed point for physical values of \( n \) [19]. This new fixed point turns out to be a focus [20] that governs the critical behavior of the system in a somewhat unusual way. It was found to exist only for \( n < 6 \) [21] having no generic relation to the stable chiral fixed point seen at small \( \epsilon \) and large \( n \), since for small \( \epsilon \) one obtains complex \( \omega \)'s in the region of complex fixed point couplings, whereas the analysis [19–21] predicts real fixed point couplings and complex \( \omega \)'s. This indicates that such fixed points are not found by simple continuation in \( \epsilon \). The situation in two dimensions seems to be similar [22].

The major part of the results obtained within other approaches (‘exact’ renormalization group, Monte Carlo simulations, etc) may be considered as favoring the fluctuation-induced first-order chiral transitions for \( n = 2, 3 \) [23–29]. Such transitions are characterized by effective critical exponents that are non-universal and depend on the magnet or antiferromagnet studied. Arguments were presented [30] that the new chiral fixed point found in [19] may be an artifact produced by rather long RG expansions. For detailed discussion and most recent results see, e.g. [31, 32].

The purpose of this paper is to investigate the critical behavior of the general \( O(n) \) symmetric theory

\[ S_{O(n)} (\phi_1, \phi_2) = S_0(\phi_1, \phi_2, \tau) + S_{int}(\phi_1, \phi_2, g), \]

\[ S_0(\phi_1, \phi_2, \tau) = \frac{1}{2} \left( \sum_{k=1}^{3} (\nabla \phi_k)^2 + \sum_{k=1}^{3} \tau_k I_k \right), \]

\[ S_{int}(\phi_1, \phi_2, g) = \frac{1}{8} \sum_{k,l=1}^{3} g_{kl} I_k I_l = \frac{1}{8} g I^2 \]

of two classical fields with \( n \) components respectively, with

\[ \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} \phi_1^2 \\ \phi_2^2 \\ \sqrt{2} \phi_1 \phi_2 \end{pmatrix}. \]

\( g \) is assumed to be symmetric, \( g_{ij} = g_{ji} \). Whenever possible we use only \( g_{ij} \) with \( i \leq j \). The model (4) becomes \( O(n)+O(n) \) symmetric when \( \tau_3 = g_{13} = g_{23} = g_{33} = 0 \). On the other hand, setting

\[ \tau_1 = \tau_2, \quad \tau_3 = g_{13} = g_{23} = 0, \quad g_{11} = g_{22}, \quad g_{12} = g_{11} - g_{33} \]

leads to the \( O(n) \times O(2) \) model of frustrated spins.
As a function of $n$ we find 10 FPs in total. To our knowledge the FPs we denote by RS 2.1b and RS 2.3 are new ones. The FP RS 2.1b is remarkable, since in one-loop order it coincides with the FP RS 2.1a, which describes two decoupled isotropic systems. RS 2.1b shows in order $\epsilon^{3/2}$ a coupling between both systems for general $n$.

In the following section we give the expression of the $\beta$-function and of various anomalous dimensions, which allow the determination of the critical exponents $\eta$, $\nu$ and $\omega$ and the cross-over exponents in one-loop order for model (4). In section 3, we consider orthogonal transformations between the two fields $\phi_1$ and $\phi_2$. As a consequence there will be discrete FPs (invariant under this transformation) and lines of FPs. Then we classify the solutions according to the behavior in the large-$n$ limit. In section 5, the various fixed points are determined and the corresponding critical exponents are given for finite $n$. If in some range of $n$ the FP becomes complex, we determine in order $\epsilon$ the limit $n_c$, where it becomes complex (for positive $n$ only).

Comparison is made with the known models (1)–(3) in section 6. A summary concludes the paper.

2. The $4 - 2\epsilon$ expansion

The expression for the critical exponents can be taken from the review article by Brézin et al [33]. Writing

$$S_{\text{int}} = \frac{1}{4!} g_{i\alpha,j\beta,k\kappa,l\lambda} \phi_i\alpha \phi_j\beta \phi_k\kappa \phi_l\lambda,$$

one obtains

$$g_{i\alpha,j\beta,k\kappa,l\lambda} = \frac{1}{\upsilon_{i,j} \upsilon_{k,l}} g_{\rho_{i,j},\rho_{k,l}} \delta_{\alpha\beta} \delta_{\kappa\lambda} + \frac{1}{\upsilon_{i,k} \upsilon_{i,j}} g_{\rho_{i,k},\rho_{i,j}} \delta_{\alpha\kappa} \delta_{\beta\lambda} + \frac{1}{\upsilon_{i,l} \upsilon_{i,j}} g_{\rho_{i,l},\rho_{i,j}} \delta_{\alpha\lambda} \delta_{\beta\kappa},$$

$$\upsilon_{1,1} = \upsilon_{2,2} = 1, \quad \upsilon_{1,2} = \upsilon_{2,1} = \sqrt{2}, \quad \rho_{1,1} = 1, \quad \rho_{2,2} = 2, \quad \rho_{1,2} = \rho_{2,1} = 3.$$  

The six $\beta$ functions $\beta_{ij} = \mu \partial_\mu g_{ij}$, where $\mu$ is an auxiliar parameter with the critical dimension 1, can be written in one-loop order

$$\beta_{ij} = -2\epsilon g_{ij} + \frac{1}{2} (n + 8) g_{ik} g_{kl} + \frac{1}{2} C_{ij,kl,mm} g_{kl} g_{mn}$$

with

$$\begin{align*}
1, 1 & \quad -8g_{12}^2 + 2g_{12}g_{33} + g_{33}^2 \\
1, 2 & \quad -6g_{11}g_{12} - 6g_{12}g_{23} - 4g_{13}g_{23} + g_{11}g_{33} + 4g_{12}^2 + 4g_{13}^2 + 4g_{23}g_{33} + 2g_{23}^2 + 8g_{33}^2 \\
1, 3 & \quad -6g_{12}g_{23} - 3g_{13}g_{33} + 6g_{12}g_{13} + 3g_{23}g_{33} \\
2, 2 & \quad -8g_{12}^2 + 2g_{12}g_{33} + g_{33}^2 \\
2, 3 & \quad -6g_{12}g_{13} - 3g_{23}g_{33} + 6g_{12}g_{23} + 3g_{13}g_{33} \\
3, 3 & \quad -2g_{13}^2 - 2g_{23}^2 - 6g_{33}^2 + 2g_{11}g_{33} + 8g_{12}g_{33} + 4g_{13}g_{23} + 2g_{22}g_{33}.
\end{align*}$$

We have rescaled the couplings by a factor $8\pi^2$ as usual.

The FPs $g^*$ are the solutions of $\beta_{ij}(g^*) = 0$. We observe that (4) is symmetric under the simultaneous interchange of $g_{11}$ with $g_{22}$ and $g_{13}$ with $g_{23}$. The simultaneous change of signs of $g_{13}$ and $g_{23}$ leaves the solution to (10) invariant.
The stability matrix
\[ \omega_{ij,kl} = \frac{\partial \beta_{ij}(g)}{\partial g_{kl}} \bigg|_{g = g^*} \] (12)
is easily obtained. The eigenvalues of (12) are the critical exponents \( \omega \).

Similarly the critical exponents \( \eta \) are obtained from the eigenvalues \( \gamma_{\Phi}^* \) of the symmetric \( 2 \times 2 \) matrix \( \gamma_{\Phi} \) at \( g = g^* \),

\[
\{ \gamma_{\Phi} \}_{11} = \frac{1}{16} \left( 2(n + 2)g_{11}^2 + (n + 2)g_{23}^2 + (n + 1)g_{33}^2 + 2ng_{12}^2 + 4g_{12}g_{33} + 3(n + 2)g_{13}^2 \right), \\
\{ \gamma_{\Phi} \}_{21} = \frac{\sqrt{2}(n + 2)}{16} \left( (g_{11} + g_{12} + g_{33})g_{13} + (g_{22} + g_{12} + g_{33})g_{23} \right), \\
\{ \gamma_{\Phi} \}_{22} = \frac{1}{16} \left( (n + 2)g_{13}^2 + 2(n + 2)g_{23}^2 + 2ng_{13}^2 + 3(n + 2)g_{23}^2 + (n + 1)g_{33}^2 + 4g_{12}g_{33} \right),
\] (13)
calculated at the specific FP, with respect to \( \eta_i = 2\gamma_{\Phi,i}^* \).

The critical behavior of perturbations bilinear in the fields \( \phi \) is governed by the expression for \( 1/\nu - 2 \) given by Brézin et al which as a function of the \( n \) components of the fields can be written as

\[
\left( \frac{1}{\nu} - 2 \right)_{\alpha,\beta;\kappa,\lambda} = d_{\alpha,\beta;\kappa,\lambda}^{(1)} + d_{\alpha,\beta;\kappa,\lambda}^{(2)} + d_{\alpha,\beta;\kappa,\lambda}^{(2)},
\] (14)

Eigenfunctions of this matrix are of three types:

(i) They may be \( O(n) \) symmetric corresponding to the variation of the \( \tau_i \). Thus one applies eigenfunctions of type \( \hat{d}_{\alpha,\beta;\kappa,\lambda}^{(1)} \) to (14) and with

\[
\hat{d}_{\alpha,\beta;\kappa,\lambda}^{(1)} = \eta_i \nu_{\alpha,\beta;\kappa,\lambda}, \quad \hat{d}_{\alpha,\beta;\kappa,\lambda}^{(2)} = \eta_i \nu_{\alpha,\beta;\kappa,\lambda} \left( d_{\alpha,\beta;\kappa,\lambda}^{(2)} \right)
\] (15)

the eigenvalues are those of the \( 3 \times 3 \) matrix

\[
\gamma_{\tau} = nd^{(1)} + d^{(2)},
\] (16)

which in one-loop order reads

\[
\gamma_{\tau} = \frac{1}{2} \begin{pmatrix}
(n + 2)g_{11} & ng_{12} + g_{33} & (n + 2)g_{13} \\
g_{12} + g_{33} & (n + 2)g_{22} & (n + 2)g_{23} \\
(n + 2)g_{13} & (n + 2)g_{23} & 2g_{12} + (n + 1)g_{33}
\end{pmatrix}_{g = g^*}. \] (17)

(ii) They may be of type \( a_{\alpha,\beta}b_{\kappa,\lambda} \) with \( a \) and \( b \) symmetric in the indices, and \( b_{\kappa,\kappa} = 0 \). They yield cross-over exponents which are obtained from the eigenvalues of the \( 3 \times 3 \) matrix

\[
\gamma_{\tau, a} = d^{(2)}_{a_{\alpha,\beta}b_{\kappa,\lambda}},
\] (18)

which in one-loop order reads

\[
\gamma_{\tau, a} = \frac{1}{2} \begin{pmatrix}
2g_{11} & g_{33} & 2g_{13} \\
g_{33} & 2g_{22} & 2g_{23} \\
2g_{13} & 2g_{23} & 2g_{12} + g_{33}
\end{pmatrix}_{g = g^*}. \] (19)

(iii) Finally they may be of type \( a_{\alpha,\beta}b_{\kappa,\lambda} \), but now with both \( a \) and \( b \) antisymmetric in their indices. They are obtained from

\[
\gamma_{\tau, a} = 2 \left( d^{(2)}_{a_{12,12}} - d^{(2)}_{a_{12,21}} \right),
\] (20)

which in one-loop order reads

\[
\gamma_{\tau, a} = -g_{12} + \frac{1}{2}g_{33}.
\] (21)
The various $\gamma$’s given here are the anomalous dimensions in terms of the length scale. The full dimension $y$ is written as

$$y_i = D - \frac{D - 2}{2} N \pm \gamma_i$$

for perturbations homogeneous in $\phi$ of order $N$. For $\gamma^\ast_i/\phi_1$ the minus sign applies, whereas for the other exponents the plus sign has to be taken. The first two contributions in the last expression are the bare exponents valid for the trivial fixed point, whereas the last term constitutes the anomalous contribution. If one singles out a linear combination of the scalar products $I$ as multiplied by the temperature difference $\tau$ from the critical point, then the singular part of the free energy shows the scaling behavior

$$F_{\text{sing}}(\tau, \{\mu_i\}) = |\tau|^{D\nu} F_{\text{sing}, \pm}(\{\mu_i/|\tau|^{1/\Delta_i}\}),$$

near criticality, where $\tau$ and $\mu_i$ are multiplied by scaling operators. $\nu$ obeys $y_\tau = 1/\nu$ and the gap exponents $\Delta_i$ are related to the $y_i$ by

$$\Delta_i = y_i y_\tau = \nu y_i.$$  

In the special case of operators bilinear in $\phi$ the exponents $\Delta_i$ are cross-over exponents.

### 3. Field rotations

One may perform a rotation between the fields $\phi_1$ and $\phi_2$ in the model (4),

$$\begin{pmatrix} \phi'_1 \\ \phi'_2 \end{pmatrix} = \begin{pmatrix} \cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \quad (25)$$

Performing the rotation (25) yields

$$\begin{pmatrix} I'_1 \\ I'_2 \\ I'_3 \end{pmatrix} = M \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix},$$

$$M = \begin{pmatrix} \frac{1}{2} - \frac{1}{2} \cos(2\varphi) & \frac{1}{2} - \frac{1}{2} \cos(2\varphi) & \sqrt{\frac{1}{2}} \sin(2\varphi) \\ \sqrt{\frac{1}{2}} \sin(2\varphi) & \sqrt{\frac{1}{2}} \sin(2\varphi) & \cos(2\varphi) \end{pmatrix}. \quad (26)$$

The matrix $M$ is orthogonal and the interaction transforms according to

$$S_{\text{int}}(\phi'_1, \phi'_2, g') = \frac{1}{2} I^{T} g' I', \quad g' = M g M^{T}. \quad (27)$$

Obviously both sets of couplings describe the same critical behavior. One finds that

$$a_1 = g_{11} + g_{22} + 2g_{12}, \quad a_2 = g_{11} + g_{22} + g_{33} \quad (28)$$

are invariant under the rotations, whereas

$$a_{31} = g_{11} - g_{22}, \quad a_{32} = \sqrt{2}(g_{13} + g_{23}),$$

$$a_{41} = -g_{11} + 2g_{12} - g_{22} + 2g_{33}, \quad a_{42} = -\sqrt{2}(g_{13} - g_{23}) \quad (29)$$

transform according to

$$\begin{pmatrix} a'_{11} \\ a'_{12} \end{pmatrix} = \begin{pmatrix} \cos(2\varphi) & \sin(2\varphi) \\ -\sin(2\varphi) & \cos(2\varphi) \end{pmatrix} \begin{pmatrix} a_{31} \\ a_{32} \end{pmatrix} \quad (31)$$

and

$$\begin{pmatrix} a'_{41} \\ a'_{42} \end{pmatrix} = \begin{pmatrix} \cos(4\varphi) & \sin(4\varphi) \\ -\sin(4\varphi) & \cos(4\varphi) \end{pmatrix} \begin{pmatrix} a_{41} \\ a_{42} \end{pmatrix}. \quad (32)$$

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For the interactions invariant under $O(n) \times O(2)$ the amplitudes $a_{31}, a_{32}, a_{41}, a_{42}$ have to vanish. For all other interactions we may choose $\varphi$. We will choose it so that

$$a_{42} = 0,$$

that is $g_{23} = g_{13}$. \hfill (33)

In the following section we will derive the FPs of (10) with the condition (33), from which all other fixed points can be obtained by means of the transformations (31) and (32) leaving the expressions (28) invariant.

4. The classification of the fixed points in the large $n$ limit

4.1. The form of the projectors

In the large-$n$ limit we may neglect the last term in (10). We express $g$ in terms of the matrix $p$,

$$g = 4\epsilon p/(n + 8).$$ \hfill (34)

We see that at criticality ($\beta_{ij} \equiv 0$) and in the limit $n \to \infty$ the matrix $p$ becomes idempotent: $p = p^2$. The only eigenvalues of idempotent matrices are 0 and 1. Thus depending on the number $k$ of eigenvalues 1 there are four types of symmetric $(3 \times 3)$ idempotent matrices $p^{(k)}$

$$p_{ij}^{(0)} = 0, \quad p_{ij}^{(1)} = \delta_{ij} - z_iz_j, \quad p_{ij}^{(2)} = \delta_{ij}; \quad i, j = 1, 2, 3,$$

with the restriction

$$z_1^2 + z_2^2 + z_3^2 = 1.$$ \hfill (35)

Next the solution to (10) in the limit $n \to \infty$ is calculated by considering the first two orders in $1/(n + 8)$ to $g^*$. This yields further conditions on $z$ for the classes $p^{(1,2)}$.

4.2. The class $p^{(0)}$

This class consists of the trivial FP $g^* = 4\epsilon p^{(0)}/(n + 8) = 0$ only. The stability matrix

$$\omega_{ij} = -(2\epsilon)\delta_{ij}$$ \hfill (37)

is diagonal as we can see from (10). All its eigenvalues are negative and the FP is unstable. This FP is exact and remains invariant under the orthogonal transformations.

4.3. The class $p^{(1)}$

Here, the ansatz

$$g_{ij}^* = \frac{4\epsilon}{(n + 8)}z_iz_j + \frac{4\epsilon}{(n + 8)^2}h_{ij} + O\left(\frac{1}{(n + 8)^3}\right)$$ \hfill (38)

with the symmetric matrix $h$ is put into the $\beta$-functions (10). We neglect the terms of higher order in $1/(n + 8)$ and obtain

$$-\frac{\epsilon^2 z_iz_j}{n + 8} - \frac{\epsilon^2 h_{ij}}{(n + 8)^2} + \frac{\epsilon^2}{n + 8} \left(z_iz_k + \frac{h_{ik}}{n + 8}\right) \left(z_jz_k + \frac{h_{jk}}{n + 8}\right) + \frac{\epsilon^2 c_{ij}}{(n + 8)^2} \equiv 0,$$ \hfill (39)

where

$$c_{ij} \equiv C_{ij,kl,mn}z_kz_lz_mz_n.$$ \hfill (40)

The equation for the terms of first order in $1/(n + 8)$ gives the already known condition (36).
From the equation for the terms of second order we obtain
\[-h_{ij} + z_i(z_k h_{kj}) + z_j(z_k h_{ki}) + c_{ij} = 0.\] (41)

Two of these six equations fix \(z\), the remaining four can be used to determine \(h\). We multiply (41) by \(z_i\) and sum over \(i\)
\[-z_i h_{ij} + z_k h_{kj} + z_j(z_k h_{ki} z_i) + z_i c_{ij} = 0.\] (42)

With \(c_j \equiv z_i c_{ij}\) we obtain \(z_j(z_k h_{ki} z_i) = -c_j\) (43)
or \(z_i c_j - z_j c_i = 0.\) (44)

The constants \(c_{ij} = c_{ji}\) in (39) can be calculated with (10) and (11). The constants \(c_i = c_{ij} z_j\) then read
\[
c_1 = (z_1^2 - 2 z_1 z_2) \left( 7 z_1^2 z_2 + 3 z_2^3 + 4 z_2 z_1 z_3 - 2 z_1 (z_2^2 + z_3^2) \right),
\]
\[
c_2 = (z_3^2 - 2 z_1 z_2) \left( z_1 (3 z_2^2 - 2 z_1 z_2 + 2 z_2^2) \right) + 2 (2 z_1 z_2 - z_2 z_3^2),
\]
\[
c_3 = 3 z_3 (2 z_1 z_2 - z_3^2) \left( (z_1 - z_2)^2 + 2 z_3^2 \right).
\]

Two of the three equations (44) turn out to be identical. With (36) we obtain the following conditions on \(z\):
\[
(1 - z_1^2) (4 - z_1^2) z_{12} (z_1 - z_2) = 0,
\]
\[
(1 - z_1^2) (4 - z_1^2) z_{12} z_3^2 = 0,
\]
\[
z_{12} := z_1 + z_2.
\]

Thus solutions are given by
\[
z_{12} = 0, \pm 1, \pm 2, \pm \sqrt{2},
\]
where the first solutions can be read off immediately from equations (48), whereas the last pair of solutions follows from \(z_1 - z_2 = 0, z_3 = 0\) and equation (36). This last solution describes an \(O(n) \times O(2)\)-invariant interaction. Due to the ansatz (38) a change of the sign of the \(z's\) does not alter the fixed point. Thus \(z_{12}\) and \(-z_{12}\) yield the same class of fixed points. The interaction can be written as
\[
S_{\text{int}}^{(i)} = \frac{\epsilon}{2(n+8)} (z_i I_i)^2
\]
in this large-\(n\) limit. One realizes that the rotation (26) of \(I\) can be rewritten as
\[
I_1' + I_2' = I_1 + I_2, \quad \frac{I_1' - I_2'}{\sqrt{2} I_3'} = \begin{pmatrix} \cos(2\varphi) & \sin(2\varphi) \\ -\sin(2\varphi) & \cos(2\varphi) \end{pmatrix} \frac{I_1 - I_2}{\sqrt{2} I_3}. \]

Thus \(z_{12}\) in
\[
z_i I_i = \frac{1}{2} z_{12} (I_1 + I_2) + \frac{1}{2} (z_1 - z_2) (I_1 - I_2) + z_3 I_3
\]
stays constant, whereas \(z_1 - z_2\) and \(z_3\) vary under rotation with
\[
(z_1 - z_2)^2 + 2 z_3^2 = 2 - z_{12}^2.
\]
Thus for \(z_{12} \neq \pm \sqrt{2}\) one obtains a whole continuum of solutions.
The eigenvalues of the stability matrix are determined in appendix A. In leading order they are independent of $z_{12}$. Similarly one can determine the other exponents from equations (13)–(21) and obtain in the limit of large $n$

$$\omega = \{(2\epsilon), 0(2\times), -(2\epsilon)(3\times)\}, \quad \gamma^*_\sigma = \{-(2\epsilon), 0(2\times)\},$$

$$\gamma^*_\Phi = \left\{ \frac{(2\epsilon)^2}{8n}(1 \pm z_{12}\sqrt{2 - z_{12}^2}) \right\}.$$  \hfill (55)

Here and in the following exponents appearing several times are indicated by $(\ldots \times)$. If $a \pm$ appears in an exponent, then exponents with both signs contribute. The exponent $\gamma^*_{\sigma,a}$ is always the last one of $\gamma^*_\sigma$.

### 4.4. The class $p^{(2)}$

Here the ansatz

$$g^*_{ij} = \frac{4\epsilon}{n+8}(\delta_{ij} - z_i z_j) + \frac{4\epsilon}{(n+8)^2} h_{ij} + O\left(\frac{1}{(n+8)^3}\right),$$

with a symmetric matrix $h$ is put into (10). This leads to

$$-\frac{\epsilon^2}{n+8}(\delta_{ij} - z_i z_j) - \frac{\epsilon^2 h_{ij}}{(n+8)^2} + \frac{\epsilon^2}{n+8} \left( \delta_{ik} - z_i z_k + \frac{h_{ik}}{n+8} \right) \left( \delta_{kj} - z_k z_j + \frac{h_{kj}}{n+8} \right)$$

$$+ \frac{\epsilon^2 c_{ij}}{(n+8)^2} = O\left(\frac{1}{(n+8)^3}\right).$$  \hfill (57)

with

$$c_{ij} := C_{ij,kl,\ldots,n} (\delta_{ik} - z_i z_k) (\delta_{mn} - z_m z_n).$$  \hfill (58)

The equation for the first-order terms in $1/(n+8)$ gives (36) again. The equation for the second-order terms is

$$-h_{kj} z_i z_k + h_{ij} - h_{ik} z_k z_j + c_{ij} = 0.$$  \hfill (59)

With the same arguments which led from (41) to (48) we now deduce conditions on $z$ corresponding to (48):

$$\left( z_{12}^2 + 1 \right) z_{12}^2 (z_1 - z_2) = 0,$$

$$\left( z_{12}^2 + 1 \right) z_{12} z_3^2 = 0.$$  \hfill (60, 61)

Thus solutions are given by

$$z_{12} = 0, \pm i, \pm \sqrt{2},$$  \hfill (62)

where the first two solutions are immediately obvious from equations (60) and the last one follows from $z_1 = z_2, z_3 = 0$, and equation (36). This last solution represents an $O(n) \times O(2)$-invariant model. The interaction can be written as

$$S^{(2)}_{\text{int}} = \frac{\epsilon}{2(n+8)} (I_i I_i - (z_i I_i)^2)$$  \hfill (63)

in the large-$n$ limit. Note that $I_i I_i$ is invariant under rotations (26). Thus the same argument concerning the invariance of $z_{12}$ under rotations as for $p^{(1)}$ applies here. Again for $z_{12} \neq \pm \sqrt{2}$ one obtains a continuous set of models related by the transformation (51)–(54).
The stability matrix $\omega$ yields in this limit eigenvalues opposite in sign to those of $p^{(1)}$ (appendix A). Similarly one determines the other exponents from equations (13)–(21) and obtains in the limit of large $n$

$$\omega = [(2\epsilon) (3\times), 0 (2\times), -(2\epsilon)], \quad \gamma^*_f = \{-\epsilon(2\times), 0\},$$

$$\gamma^*_c = \left\{ \frac{(2\epsilon)}{n} (-2 + z_{12}^2), \frac{(2\epsilon)}{n} \left(-1 \pm \sqrt{1 + 2z_{12}^2 - z_{12}^4}\right), \frac{(2\epsilon)}{n} z_{12}^2 \right\}, \quad \gamma^*_\Phi = \left\{ \frac{(2\epsilon)^2}{8n} (2 \pm z_{12}\sqrt{2 - z_{12}^2}) \right\}. \quad (64)$$

4.5. The class $p^{(3)}$

In the large-$n$ limit one obtains $g^* = 4\epsilon p^{(3)}/(n + 8)$, which yields the exponents in leading order

$$\omega = [(2\epsilon) (6\times)], \quad \gamma^*_f = \{-\epsilon(3\times)\},$$

$$\gamma^*_c = \left\{ \frac{-3(2\epsilon)}{n}, \frac{-\epsilon}{n} (2\times), \frac{(2\epsilon)}{n} \right\}, \quad \gamma^*_\Phi = \left\{ \frac{3(2\epsilon)^2}{8n} (2 \times) \right\}. \quad (65)$$

5. Solutions for finite $n$

5.1. Fixed points

In order to solve equations (10) for the couplings $g^*$ for finite $n$, we observe that the ‘gauge’ condition $a_{12} = g_{13} - g_{23} = 0$ yields

$$\beta_{11} - \beta_{22} = -\frac{1}{2} (g_{11} - g_{22})(4\epsilon - (n + 8)(g_{11} + g_{22})) = 0, \quad (66)$$

$$\beta_{13} - \beta_{23} = \frac{n + 8}{2} (g_{11} - g_{22})g_{13} = 0, \quad (67)$$

$$\beta_{13} = -\frac{1}{2} g_{13}(4\epsilon - (n + 8)(g_{11} + g_{12} + g_{13})) = 0. \quad (68)$$

Thus we have to solve any of the two equations

$$g_{13} = 0, \quad g_{11} = g_{22} \quad (69)$$

$$g_{13} = 0, \quad 4\epsilon - (n + 8)(g_{11} + g_{22}) = 0 \quad (70)$$

$$g_{11} = g_{22}, \quad 4\epsilon - (n + 8)(g_{11} + g_{12} + g_{33}) = 0 \quad (71)$$

together with the three equations

$$\beta_{11} = \beta_{12} = 0, \quad (72)$$

$$\beta_{33} = 0. \quad (73)$$

If $g_{13} = 0$, then $\beta_{33}$ factors

$$\beta_{33} = -\frac{1}{2} g_{33}(4\epsilon - (n + 2)g_{33} + 2g_{11} + 2g_{22} + 8g_{12}). \quad (74)$$
Then we distinguish the two cases

\[
g_{33} = 0, \quad (75)
\]

\[
4\epsilon - (n + 2)g_{33} + 2g_{11} + 2g_{22} + 8g_{12} = 0.
\]

(76)

One obtains the following solutions from (69), (72) and (75):

\[
g_{11} = g_{22}, \quad g_{13} = g_{23} = g_{33} = 0
\]

\[
RS \quad \frac{g_{11}}{\epsilon} = \frac{g_{12}}{\epsilon} = \frac{g_{23}}{\epsilon} = \frac{g_{33}}{\epsilon}
\]

\[
0, 0, 0, 0, 0, 0 
\]

\[
1. \quad \frac{4}{n + 8}, 0, 0, \frac{8}{n + 8}, \frac{8}{n + 8}, -\frac{8}{n + 8}, 0
\]

(77)

Equations (69), (72) and (76) yield for \(g_{33} \neq 0\)

\[
g_{11} = g_{22}, \quad g_{13} = g_{23} = 0
\]

\[
RS \quad \frac{g_{11}}{\epsilon} = \frac{g_{12}}{\epsilon} = \frac{g_{33}}{\epsilon}
\]

\[
2.1 \quad \frac{2}{\frac{n}{8}} \quad \frac{2}{\frac{n}{8}} \quad \frac{4}{\frac{n}{8}} \quad \frac{4}{\frac{n}{8}} \quad \frac{4}{\frac{n}{8}} \quad \frac{4}{\frac{n}{8}} 
\]

\[
1.2 \quad \frac{3(n^2 - 2n + 24)(n - 6)\sqrt{n^2 - 24n + 48}}{n^4 + 4n^2 - 24n + 144} \quad \frac{3(n^2 - 2n + 24)(n - 6)\sqrt{n^2 - 24n + 48}}{n^4 + 4n^2 - 24n + 144} \quad \frac{4(n^2 - 12 - 3n^2 + 2n^2 - 24n + 48)}{n^4 + 4n^2 - 24n + 144} 
\]

(78)

\[
a_1/\epsilon \quad a_2/\epsilon \quad a_{41}/\epsilon
\]

\[
2.1 \quad \frac{8}{\frac{n}{8}} \quad \frac{8}{\frac{n}{8}} \quad \frac{8}{\frac{n}{8}} 
\]

\[
1.2 \quad \frac{16}{\frac{n}{8}} \quad \frac{4}{\frac{n}{8}} \quad \frac{8}{\frac{n}{8}} - \frac{8}{\frac{n}{8}} 
\]

Equations (70), (72) and (73) with \(g_{11} \neq g_{22} \) yield

\[
g_{13} = g_{23} = 0
\]

\[
RS \quad \frac{g_{11}}{\epsilon} = -\frac{2}{\frac{n}{8}} \quad \frac{g_{12}}{\epsilon} = \frac{g_{33}}{\epsilon}
\]

\[
1.1 \quad \pm \frac{2}{\frac{n}{8}} \quad 0 \quad 0 
\]

\[
1.4 \quad \pm \frac{2}{\frac{n}{8}} \quad \frac{6}{\frac{n}{8}} \quad 0 
\]

(79)

\[
a_1/\epsilon \quad a_2/\epsilon \quad a_{41}/\epsilon
\]

\[
2.1 \quad \frac{4}{\frac{n}{8}} \quad \frac{4}{\frac{n}{8}} \quad \frac{4}{\frac{n}{8}} 
\]

\[
1.4 \quad \frac{8}{\frac{n}{8}} \quad \frac{8}{\frac{n}{8}} \quad \frac{8}{\frac{n}{8}} 
\]

\[
2.3 \quad \frac{4(3n^2 + 24n + 68)}{n^4 + 4n^2 + 20n} \quad \frac{8(n^2 + 4n - 2)}{n^4 + 4n^2 + 20n} \quad \frac{4(n^2 + 36)}{n^4 + 4n^2 + 20n} 
\]

(80)
Equations (71), (72) and (73) yield for $g_{13} \neq 0$

$$g_{11} = g_{22}, \quad g_{13} = g_{23}$$

| $RS$ | $g_{11}/\epsilon$ | $g_{12}/\epsilon$ | $g_{33}/\epsilon$ | $(g_{13}/\epsilon)^2$ |
|------|--------------------|--------------------|--------------------|---------------------|
| 1.1  | $\frac{1}{n+8}$   | $\frac{1}{n+8}$   | $\frac{2}{n+8}$   | $\frac{2}{(n+8)^2}$ |
| 1.4  | $\frac{4}{n+8}$   | $\frac{4}{n+8}$   | $\frac{4}{n+8}$   | $\frac{16(1-\epsilon)}{(n+8)^4}$ |
| 2.3  | $\frac{5n^2+24n-4}{(n+8)(n^2+4n+20)}$ | $\frac{(n+10)(n+14)}{(n+8)(n^2+4n+20)}$ | $\frac{-2(n+2)(n+14)}{(n+8)(n^2+4n+20)}$ | $\frac{-2(3n+22)(n-2(n+2)(n+4)(n+14)}{(n+8)^2(n^2+4n+20)^2}$ |

$\partial \beta_{ij}/\partial g_{kl}$

All three sets of exponents contain two degenerate exponents. The first exponent since the perturbation is obtained from an infinitesimal rotation between $\phi$ creates lines of fixed points. This exponent is not a true scaling exponent, but a redundant one, since the perturbation is obtained from an infinitesimal rotation between $\phi_1$ and $\phi_2$. Therefore all solutions for which

(i) the solutions, which are invariant under $O(n) \times O(2)$. There is one solution for each $k$, $RS 0.1, 1.3, 2.2,$ and $3.1$;

(ii) solutions for which $a_{31} = a_{32} = 0$, $RS 1.2, 2.1$;

(iii) solutions for which $a$’s can be different from 0, $RS 1.1, 1.4,$ and $2.3$. The solutions can be seen in both (79) and (80). They are obtained from one another by a rotation by $\varphi = \pi/4$.

All solutions with the exception of the trivial fixed point $RS 0.1$ have an exponent $\omega = 2\epsilon$ independent of $n$ in one-loop order, since $\beta_{ij} = -2\epsilon g_{ij}$ + term bilinear in the $g$’s and thus $\partial \beta_{ij}/\partial g_{kl}|_{\epsilon=g}g_{kl} = 2\epsilon g_{ij}^*$. For the solutions (i) of symmetry $O(n) \times O(2)$ equation (6) holds. Then equations (13), (17) and (19) yield the eigenvalues

$$\gamma_\Phi^* = \left\{ \frac{n+1}{4}g_{11}^2 + \frac{3(n-1)}{16}g_{33}^2 - \frac{n-1}{4}g_{11}^*g_{33}^* (2\times) \right\},$$

$$\gamma_\tau^* = \left\{ -(n+1)g_{11}^* + \frac{n-1}{2}g_{33}^* - g_{11}^* - \frac{n-1}{2}g_{33}^* (2\times) \right\},$$

$$\gamma_{\tau,s}^* = \left\{ -g_{11}^* - \frac{1}{2}g_{33}^*, -g_{11}^* + \frac{1}{2}g_{33}^* (2\times) \right\}.$$
5.2. Critical exponents

In the following, we give the critical exponents of the various fixed points.

RS 0.1. This is the trivial (interaction free) fixed point. All anomalous exponents \( \gamma^* \) vanish

\[
\gamma^*_0 = \{0 \times 2\}, \quad \gamma^*_1 = \{0 \times 3\}, \quad \gamma^*_\omega = \{0 \times 4\}, \quad \omega = \{-(2\varepsilon) \times 6\}.
\] (82)

RS 1.1. Representatives of these solutions are given in (79) and (80). The critical exponents are given by

\[
\gamma^*_\tau = \left\{ -\frac{(n + 2)(2\varepsilon)}{n + 8}, 0 \times 2 \right\}, \quad \gamma^*_\omega = \left\{ -\frac{2(2\varepsilon)}{n + 8}, 0 \times 3 \right\},
\]

\[
\gamma^*_\tau = \left\{ \frac{(n + 2)(2\varepsilon)^2}{4(n + 8)^2}, 0 \right\}, \quad \omega = \left\{ (2\varepsilon), -(2\varepsilon) \times 2, \frac{(n + 6)(2\varepsilon)}{n + 8}, 0\right\}. \] (83)

RS 1.2. Representatives are given in (77) and (78). The critical exponents are

\[
\gamma^*_e = \left\{ \frac{n(n^2 - 3n + 8)(2\varepsilon)^2}{8(n^2 + 8)^2}, 0 \times 2 \right\}, \quad \gamma^*_\tau = \left\{ -\frac{3n(2\varepsilon)}{n^2 + 8}, \frac{(1 - n)n(2\varepsilon)}{n^2 + 8}, \frac{(n - 4)(2\varepsilon)}{n^2 + 8} \right\},
\]

\[
\gamma^*_\tau = \left\{ -\frac{n(2\varepsilon)}{n^2 + 8}, \frac{(n - 4)(2\varepsilon)}{n^2 + 8} \right\}, \quad \omega = \left\{ 0, (2\varepsilon), \frac{8(n - 1)(2\varepsilon)}{n^2 + 8}, \frac{(4 - n)(2 + n)(2\varepsilon)}{n^2 + 8}, \frac{(4 - n)(n - 2)(2\varepsilon)}{n^2 + 8}, \frac{(2 - n)(4 + n)(2\varepsilon)}{n^2 + 8} \right\}. \] (84)

In the representation (77) \( g_{13} = g_{23} = 0 \) holds and the \( \gamma_1 \) matrix (17) becomes a block matrix and has the eigenvalues \( \gamma^*_1 = \{\gamma^*_\tau, \gamma^*_\omega, \gamma^*_\tau\} \) which in the case of \( g^*_{11} = g^*_{22} \) belong to the eigenvectors \((1, 1, 0), (1, -1, 0)\) and \((0, 0, 1)\) respectively in our convention. Thus the first entry represents an ordinary critical exponent when \( \tau_1 = \tau_2 \), the third entry is the critical exponent of \( \tau_3 \), and the second entry as well as the exponents \( \gamma^*_\omega \) are related to the crossover.

RS 1.3. This solution is not only invariant under \( O(n) \times O(2) \), but even under \( O(2n) \). \( g^* \) is given in (77). Its critical exponents are

\[
\gamma^*_e = \left\{ \frac{(2n + 2)(2\varepsilon)^2}{4(2n)^2 + 8}, 0 \times 2 \right\}, \quad \gamma^*_\tau = \left\{ -\frac{2(2n + 2)(2\varepsilon)}{2n + 8}, \frac{2(2\varepsilon)}{2n + 8} \times 2 \right\},
\]

\[
\gamma^*_\tau = \left\{ -\frac{2(2\varepsilon)}{2n + 8}, 4 \times 4 \right\}, \quad \omega = \left\{ (2\varepsilon), \frac{8(2\varepsilon)}{2n + 8}, 2 \times 2, \frac{(4 - 2n)(2\varepsilon)}{2n + 8}, 3 \times 3 \right\}. \] (85)

The last two exponents of \( \gamma^*_\tau \) belong to cross-over exponents (discussion after (81)). These exponents and all exponents \( \gamma^*_\omega \) are equal.

RS 1.4. Its representative couplings are given in (79) and (80). In one-loop order one obtains the exponents
\[
\gamma_\phi^+ = \left\{ \frac{(n^2 + 37n + 16)(2\epsilon)^2}{8(n + 8)^3} \pm \frac{(n + 2)\sqrt{2(1-n)(2\epsilon)^2}}{2(n + 8)^{3/2}} \right\},
\]
\[
\gamma_w^+ = -\left\{ \frac{(2 + n)(2\epsilon)}{2(n + 8)} \pm \frac{\sqrt{n^3 + 48n^2 + 32(2\epsilon)} - 3(2\epsilon)}{2(n + 8)^{3/2}} \right\},
\]
\[
\gamma_g^+ = \left\{ -\frac{(2\epsilon)}{n + 8} \pm \frac{2\sqrt{2(1-n)(2\epsilon^2)}}{(n + 8)^{3/2}} - \frac{3(2\epsilon)}{n + 8} (2\times) \right\},
\]
\[
\omega = \left\{ 0, \frac{(6 - n)(2\epsilon)}{n + 8}, \frac{(10 - n)(2\epsilon)}{n + 8}, \frac{(n + 2)(2\epsilon)}{2(n + 8)} \pm \frac{\sqrt{n^3 - 188n + 196(2\epsilon)}}{2(n + 8)} \right\}.
\]

We consider the coupling in two-loop order, since it yields in order \(\epsilon\) the region in which the couplings are real. Using the representation \(79\) the couplings may be written as

\[
g_{11,22}^{1,2} = u^{(1)}(\epsilon) + u^{(2)}(\epsilon)^2 \pm \sqrt{V + O(\epsilon^3)},
\]
\[
V = v^{(1)}(\epsilon) \pm \frac{v^{(2)}(\epsilon)^2}{\sqrt{w}}.
\]

with

\[
u^{(1)} = \frac{2}{n + 8},
\]
\[
u^{(2)} = -\frac{2(n^3 + 24n^2 - 27n - 160)}{(n + 8)^4},
\]
\[
u^{(1)} = \frac{4}{(n + 8)^2},
\]
\[
u^{(2)} = \frac{n^4 + 80n^3 - 2004n^2 - 880n + 2560}{2(n + 8)^4},
\]
\[
w = 2(1 - n)(8 + n),
\]
\[
g_{12}^* = \frac{6}{n + 8} \pm \frac{-n^3 - 66n^2 + 450n + 832}{(n + 8)^4} \epsilon^2,
\]
\[
g^*_{13} = g^*_{23} = g^*_{33} = 0.
\]

Now \(V\) can be rewritten as

\[
V = v^{(1)}(\epsilon) \sqrt{w + \frac{2v^{(2)}(\epsilon)^2}{v^{(1)}}} \epsilon.
\]

Thus with \(w(n_0) = 0\) the limit of real couplings is given by

\[
n_c = n_0 - \frac{2u^{(2)}(n_0)}{v^{(1)}(n_0)w^{(0)}(\epsilon)},
\]

which in our case yields \(n_c = 1 - (2\epsilon)/48 + O(\epsilon^2)^2\).

**RS 2.1.** Representatives in one-loop order are given in \(77\) and \(78\). Two of the exponents \(\omega\) equal 0 for any \(n\) in one-loop order; one is due to the invariance under rotations between the fields \(\phi\). The other one indicates that there may branch off a second line of FPs. Indeed one finds besides the FP of two decoupled systems \(g^{11}_{11} = g^{22}_{12} = g^{33}_{13} = g^{13}_{11} = g^{23}_{23} = 0\) (which we denote \(RS 2.1a\)) another solution with \(g^{11}_{11} = g^{22}_{23}, g^{12}_{12} = g^{13}_{13} = O(\epsilon^2), g^{23}_{23} = g^{33}_{33} = O(\epsilon^{3/2}),\) which we denote \(RS 2.1b\). Both types of FPs agree in one-loop order, but differ in the next...
order. Note that the first FP has \(a_{31} = a_{32} = 0\), whereas the second does not show this symmetry. In the following we give the FPs and critical exponents in two-loop order (for \(\gamma^*_q\) in three-loop order).

First the general scheme to obtain the FPs beyond first order is explained. Let the \(\beta\)-function up to two-loop order read

\[
\beta_i = -(2\epsilon)g_i + \sum_{pq} k_{ipq} g_p g_q + \sum_{pq} l_{ipq} g_p g_q g_r + \cdots, \tag{92}
\]

where the indices \(i, p, q, r\) replace the double indices \(ij\) and expand the contributions in one-loop order

\[
k_{ipq} = k_{ipq}^0 + \epsilon k_{ipq}^1 + \cdots \tag{93}
\]

and similarly the higher-loop orders. With

\[
g_i = \epsilon g_{1,i} + \epsilon^2 g_{2,i} + \cdots \tag{94}
\]

one obtains from \(\beta_i(g^*_r)\) order \(\epsilon^r, r > 2\) the equation

\[
B_{ij} g^*_{r-1,j} = r_{\text{h.s.}} \tag{95}
\]

where the rhs of equation (95) contains only \(g^*_{r',q}\) with \(r' < r - 1\). The matrix \(B\) is the matrix \(\omega\) in one-loop order. If none of the eigenvalues of this matrix vanishes, then equation (95) can be used to calculate \(g^*_{r,i}\) in increasing order \(r\). If due to the rotation invariance one of the eigenvalues vanishes, then the condition (33) reduces the number of independent couplings by 1 and eliminates the vanishing eigenvalue. If, however, a second eigenvalue vanishes, then the calculation has to be modified. For this RS 2.1 we assume \(g_{22} = g_{11}\) and expand \(g_{11}, g_{12}, g_{33}\) as in equation (94), but denote \(g_u = g_{13} = g_{23}\) and expand

\[
g_u = \epsilon^{3/2} g_{1,u} + \epsilon^{5/2} g_{2,u} + \cdots \tag{97}
\]

From now on the indices \(i, p, q, \ldots\) stand only for the double indices 11, 12, 33, but not for 13.

Order \(\epsilon^2\) of \(\beta_i(g^*_r) = 0\) is fulfilled by the solutions of RS 1.2

\[
ge^*_{1,11} = \frac{4}{n + 8}, \quad g^*_{1,12} = g^*_{1,33} = 0. \tag{98}
\]

Order \(\epsilon^{5/2}\) of \(\beta_u = 0\) yields

\[
B_{uu} g^*_{1,u} = 0. \tag{99}
\]

Since for the FP RS 2.1 \(B_{uu} = 0\), this is automatically fulfilled. Next \(\beta_i(g^*_r) = 0\) in order \(\epsilon^3\) yields

\[
\sum_q B_{iq} g^*_{2,q} + k^0_{ipq} g^*_{1,p} g^*_{1,q} + \sum_{pq} l_{ipq} g^*_{1,p} g^*_{1,q} g^*_{1,r} = 0, \tag{100}
\]

from which one calculates \(g^*_{2,i}\). Note that it depends on the yet unknown \(g^*_{1,u}\). Now \(\beta(g^*_r) = 0\) in order \(\epsilon^{7/2}\) yields

\[
B_{uu} g^*_{2,u} + C_u g^*_{1,u} = 0, \quad C_u := 2 \sum_p k^0_{upu} g^*_{2,u} + 2 \sum_p k^1_{upu} g^*_{1,u} + 3 \sum_{p,q} l^0_{upu} g^*_{1,p} g^*_{1,q}. \tag{101}
\]
Since $B_{uu} = 0$, we have either $g^*_{1,u} = 0$ (RS 2.1a) or $C_u = 0$, which constitutes a quadratic equation in $g^*_{1,u}$ yielding the FP (RS 2.1b).

Higher orders in $\epsilon$ are determined uniquely. Order $\epsilon^{r}, r > 3$ of $\beta_i = 0$ yields

$$
\sum_q B_{uq} g^*_{r-1,q} + 2k^0_{1uu} g^*_{1,u} g^*_{r-2,u} = \text{rhs},
$$

where the right-hand side of the equation contains $g^*_{r,q}$ with $r' < r - 1$ and $g^*_{r',u}$ with $r' < r - 2.$ Order $\epsilon^{r+1/2}$ of $\beta_u = 0$ yields

$$
B_{uu} g^*_{r-1,u} + C_u g^*_{r-2,u} + 2 \sum_p k^0_{upu} g^*_{1,u} g^*_{r-1,p} = \text{rhs}.
$$

The rhs contains $g^*_{r,q}$ with $r' < r - 1$ and $g^*_{r',u}$ with $r' < r - 2.$ In all cases $B_{uu} = 0.$ For RS 2.1a one has $g^*_{1,u} = 0$ and $C_u \neq 0$, which allows a unique determination of $g^*_{r-2,u}.$ Since each term of the rhs contains at least one factor $g^*_{r',u}$, one obtains $g^*_{r-2,u} = 0.$ For RS 2.1b both $B_u = C_u = 0$ vanish. However the sum $\sum_p k^0_{upu} g^*_{1,u} g^*_{r-1,p}$ depends via $g^*_{r-1,p}$ on $g^*_{r-2,u}.$ As a result one obtains from this equation $g^*_{r-2,u}$.

**RS 2.1a**

$$g^*_{11} = g^*_{22} = \frac{4}{n + 8} \epsilon - \frac{4(n^2 - 2n - 20)}{(n + 8)^3} \epsilon^2,$$

$$g^*_{12} = g^*_{23} = g^*_{13} = g^*_{23} = 0.$$

This solution describes two independent $O(n)$ models:

$$
\gamma^*_{\phi} = \left\{ \frac{(n + 2)}{4(n + 8)^2} (2\epsilon)^2 - \frac{(n + 2)(n^2 - 56n - 272)}{16(n + 8)^4} (2\epsilon)^3 (2\times) \right\},
$$

$$
\gamma^*_{t} = \left\{ -\frac{n + 2}{2(n + 8)^2} (2\epsilon)^2, -\frac{n + 2}{n + 8} (2\epsilon) - \frac{(n + 2)(13n + 44)}{2(n + 8)^3} (2\epsilon)^2 (2\times) \right\},
$$

$$
\gamma^*_{c} = \left\{ -\frac{2}{n + 8} (2\epsilon) + \frac{(n + 4)(n - 22)}{2(n + 8)^3} (2\epsilon)^2 (2\times), -\frac{n + 2}{2(n + 8)^2} (2\epsilon)^2 (2\times) \right\},
$$

$$\omega = \left\{ (2\epsilon) - \frac{3(3n + 14)}{(n + 8)^2} (2\epsilon)^2 (2\times), \frac{n - 4}{n + 8} (2\epsilon) + \frac{(n + 2)(13n + 44)}{(n + 8)^3} (2\epsilon)^2, \right.\right.$$

$$\left. -\frac{n + 4}{n + 8} (2\epsilon) - \frac{(n + 4)(n - 22)}{(n + 8)^3} (2\epsilon)^2, \frac{n + 2}{2(n + 8)^2} (2\epsilon)^2, 0 \right\}.$$

**RS 2.1b.** The second FP to RS 2.1 is given by

$$g^*_{11} = g^*_{22} = \frac{4}{n + 8} \epsilon - \frac{9n^3 + 98n^2 - 400n - 2272}{2(n + 8)^3(n + 14)} \epsilon^2,$$

$$g^*_{13} = g^*_{23} = \frac{\sqrt{2(n + 4)(n + 2)(n - 4)}}{(n + 8)^2 \sqrt{n + 14}} \epsilon^{3/2},$$

$$g^*_{12} = -\frac{n + 2}{2(n + 8)(n + 14)} \epsilon^2,$$

$$g^*_{23} = \frac{(n + 2)(n - 4)}{(n + 8)^2(n + 14)} \epsilon^2.$$
In the limit $D = 4$ it is real for $n \geq 4$. Its critical exponents are

\[
\gamma_0^\pm = \left\{ \frac{(n+2)(2\epsilon)^2}{4(n+8)^2} \pm \frac{(n+2)^2}{16(n+8)^3} \left( \frac{2\epsilon}{\sqrt{n+4}} \right)^{5/2} \right\},
\]

\[
y_i^* = \left\{ -\frac{n+2}{n+8}(2\epsilon) - \frac{(n+2)(29n^2 + 470n + 1256)}{4(n+14)(n+8)}(2\epsilon)^2, \right\},
\]

\[
y_\infty^* = \left\{ 2(n+8)(2\epsilon) + \frac{n^3 - 12n^2 - 660n - 2416}{4(n+8)^3(n+14)}(2\epsilon)^2, \right\},
\]

\[
\omega = \left\{ (2\epsilon)^3 - \frac{3(3n + 14)(2\epsilon)^2 (2\times)}{(n+8)^2}, \frac{-n+2}{(n+8)^3}(2\epsilon)^2, 0, \frac{n-4}{n+8}, \right\}.
\]

\[
\gamma_\infty^\pm = \left\{ \frac{(2n^6 + 37n^5 + 348n^4 + 2360n^3 + 9376n^2 + 13904n - 9152)(2\epsilon)^2}{8(n+8)^3(n^2 + 4n + 20)^2} \right\},
\]

\[
\gamma_\infty^* = \left\{ \frac{(2\epsilon)(n-1)(n-2)(n+6)}{(n+8)(n^2 + 4n + 20)}, \frac{-n+2}{2(n+8)}, \right\},
\]

\[
\omega = \left\{ 0, \frac{(2\epsilon)(n^3 + 10n^2 - 4n - 232)}{(n+8)(n^2 + 4n + 20)}, 2(n+8)(n^2 + 4n + 20)^{1/2} \right\}.
\]
where $\lambda'$ is solution of the equation

$$\lambda'^3 + 16(n^2 + 4n + 20)\lambda'^2 - 4(n + 4)(n^3 - 18n^4 - 392n^5 - 1648n^6 - 496n + 8928)\lambda' - 16(3n + 22)(n - 2)(n + 6)(n - 6)(n + 4)(n + 2)(n + 14)^2 = 0.$$  

(109)

In an expansion in $1/(n + 8)$ one obtains the $\omega$'s

$$u^{(2)} = \begin{pmatrix} -6 \over (n + 8) - 296 \over (n + 8)^2 - 11272 \over (n + 8)^3 + O \left( \frac{1}{(n + 8)^4} \right) \end{pmatrix},$$

$$v^{(2)} = \begin{pmatrix} -20 \over (n + 8) - 78 \over (n + 8)^2 + 906 \over (n + 8)^3 + O \left( \frac{1}{(n + 8)^4} \right) \end{pmatrix},$$

(110)

$$w = -(3n + 22)(n - 2)(n + 2)(n + 4)(n + 8)(n + 14).$$

Similarly as for RS 1.4 we consider the coupling in two-loop order, since it yields in order $\epsilon$ the region in which the couplings are real. Using the representation (79) the couplings may be written in the form (87) and (88) with

$$u^{(1)} = \frac{2}{n + 8},$$

$$u^{(2)} = -\frac{2(n^2 + 21n^6 + 249n^7 + 1564n^8 + 2312n^9 - 13808n^2 - 53104n - 55360)}{(n^2 + 4n + 20)^2(n + 8)^4},$$

$$v^{(1)} = \frac{2}{(n + 8)^2(n^2 + 4n + 20)},$$

$$v^{(2)} = \frac{-2(n + 2)(n + 4) + 2(n^2 + 4n + 20)}{(n + 14)(n + 8)^4(n^2 + 4n + 20)^2},$$

$$w = -(3n + 22)(n - 2)(n + 2)(n + 4)(n + 8)(n + 14),$$

$$g_{12}^* = \frac{4(n + 4)(n + 6)}{(n + 8)(n^2 + 4n + 20)} - \frac{2n^2}{(n + 14)(n + 8)^4(n^2 + 4n + 20)^3} - \frac{2}{n^2 + 3n + 9n^2 + 13222n - 189636n - 1087512n^3} - \frac{2}{n^2 + 3n + 9n^2 + 13222n - 189636n - 1087512n^3}$$

$$g_{33}^* = \frac{4(n^2 - 36)}{(n + 8)(n^2 + 4n + 20)} - \frac{4}{(n + 14)(n + 8)^4(n^2 + 4n + 20)^3} - \frac{4}{n^2 + 3n + 9n^2 + 13222n - 189636n - 1087512n^3} - \frac{4}{n^2 + 3n + 9n^2 + 13222n - 189636n - 1087512n^3}$$

$$g_{13}^* = g_{23}^* = 0.$$  

From (90) and (91) we obtain $n_c = 2 - (2\epsilon)/140 + O(2\epsilon)^2$.  

RS 2.2 and 3.1. These two fixed points are solutions of one and the same quadratic equation. Both fixed points are $O(n) \times O(2)$ invariant. In two-loop order the solutions $g^*$ can be written as

$$g_{ij}^* = u_{ij}^{(1)} \epsilon + u_{ij}^{(2)} \epsilon^2 + sV_{ij},$$

$$V_{ij} = v_{ij}^{(1)} \sqrt{w} \epsilon + v_{ij}^{(2)} \sqrt{w} \epsilon^2,$$

$$g_{11}^* = g_{22}^*, \quad g_{13}^* = g_{23}^* = 0,$$

$$u_{11}^{(1)} = \frac{3n^2 - 2n + 24}{N}, \quad v_{11}^{(1)} = \frac{n - 6}{N},$$

$$u_{12}^{(1)} = -\frac{(n + 12)(n - 6)}{N}, \quad v_{12}^{(1)} = \frac{n + 6}{N}.$$
\[ u_{33}^{(1)} = \frac{4(n-3)(n-4)}{N}, \quad v_{33}^{(1)} = \frac{-12}{N}, \]
\[ N = n^3 + 4n^2 - 24n + 144, \]
\[ w = n^2 - 24n + 48, \]
\[ (112) \]
where \( s = 1 \) corresponds to RS 3.1 called chiral FP, and \( s = -1 \) to RS 2.2 called antichiral. Close to \( D = 4 \) they are real only for \( n \geq 22 \) and \( n \leq 2 \). The critical exponents read
\[ \gamma^{*}_s = \begin{cases} \frac{(5n^5 - 3n^4 - 16n^3 - 656n^2 + 3072n - 1152 + s(n - 3)(n + 4)w^{3/2}) (2\epsilon)^2}{16N^2}, \\ \frac{-(n(48 + n + n^2) + s(n - 3)(4 + n)\sqrt{w})(2\epsilon)}{2N}, \\ \frac{(-2n^3 - 3n^2 + 28n - 48 + 5sn\sqrt{w})(2\epsilon)}{2N} \end{cases} (2\times), \]
\[ \gamma^{*}_c = \begin{cases} \frac{(-5n^2 - s(n - 12)\sqrt{w})(2\epsilon)}{2N}, \\ \frac{(-n^2 + 4n - 48 - sn\sqrt{w})(2\epsilon)}{2N}, \\ \frac{(3n^2 + 8n - 96 - s(n + 12)\sqrt{w})(2\epsilon)}{2N} \end{cases} (2\times). \]
The exponent \( \gamma^{*}_s \) determines \( \nu \), whereas the two degenerate ones yield cross-over exponents.

The other cross-over exponents are obtained from
\[ \gamma^{*}_c = \begin{cases} \frac{(n + 4)((n + 4)(n - 3) - 3s\sqrt{w})(2\epsilon)}{N}, \quad (2\times), \\ \frac{(n^3 + 14n^2 + 56n - 96 + s(n + 8)(n - 6)\sqrt{w})(2\epsilon)}{2N}, \\ \frac{(-3(n^2 - 24n + 48) + s(n + 4)(n - 3)\sqrt{w})(2\epsilon)}{2N}, \end{cases} (2\epsilon). \]
\[ \omega = \begin{cases} \frac{(n + 4)((n + 4)(n - 3) - 3s\sqrt{w})(2\epsilon)}{N}, \quad (2\times), \\ \frac{(n^3 + 14n^2 + 56n - 96 + s(n + 8)(n - 6)\sqrt{w})(2\epsilon)}{2N}, \\ \frac{(-3(n^2 - 24n + 48) + s(n + 4)(n - 3)\sqrt{w})(2\epsilon)}{2N}, \end{cases} \]
\[ \gamma^{*}_s = \begin{cases} \frac{(n^3 - 3n^2 - 12n - 24 + s(n^2 + 8n - 96 - 8n\sqrt{w})(2\epsilon)}{N^3}, \\ \frac{(-3(n^2 - 24n + 48) + s(n + 4)(n - 3)\sqrt{w})(2\epsilon)}{2N}, \end{cases} \]
\[ (116) \]
In two-loop order one can write
\[ u_{11}^{(2)} = \frac{1}{N^3} (-3n^8 + 10n^7 - 432n^6 + 1710n^5 + 7480n^4 + 20976n^3 + 3456n^2 - 411 264n + 456 192), \]
\[ u_{12}^{(2)} = \frac{1}{N^3} (n^8 + 20n^7 - 286n^6 - 3550n^5 - 11 960n^4 + 32 208n^3 + 165 888n^2 + 148 608n - 290 304), \]
\[ u_{33}^{(2)} = \frac{-2(n - 3)(n + 4)}{N^3} (2n^6 + 3n^5 + 94n^4 - 2688n^3 - 5904n^2 - 20 736n + 31 104), \]
\[ v_{ij}^{(2)} = \frac{F_{ij}w}{N^3} + \frac{v_{ij}^{(1)}L}{N^2}, \]
\[ F_{11} = -n^7 + 50n^6 + 552n^5 + 11 726n^4 + 230 912n^3 + 5022 864n^2 + 109 907 904n - 910 069 632, \]
\[ F_{12} = -n^7 + 4n^6 + 658n^5 + 19 546n^4 + 416 192n^3 + 8896 560n^2 + 193 441 728n + 909 986 688, \]
18
We mention the results for the exponents $\gamma_{nc}$ which yields
\begin{equation}
3 \text{ Equation (4.5) in [9] is misprinted. The correct result is found in (3.10) of [12].}
\end{equation}

As a result one obtains
\begin{equation}
n_c = n_0 - \frac{2L(n_0)}{N^2(n_0)w'(n_0)}
\end{equation}
which yields $n_c = 12 \pm 4\sqrt{6} - (12 \pm 14\sqrt{6}/3)(2\varepsilon) + O(2\varepsilon)^2$ in agreement with [12].

The FP (112) is stable for large $n$, where the sign $s$ in front of the root $\sqrt{w}$ is chosen positive. The stability of this FP in three dimensions is discussed on the basis of various calculation schemes in section 11.5.3 of [4], see also [16–21, 30–32].

The large-$n$ expansion of critical exponents for the FP (112) was performed in [12, 34], where $1/v_2 = 2 + \gamma_{c2}$. It yields a cross-over exponent $\Delta = v/v_2$, compare equations (23) and (24).

5.3. Stability in 3.99 dimensions

In this subsection a number of criteria for stability of the fixed points are summarized. Since the calculations are done in first order and in some cases in second order in $\varepsilon$, they apply only to dimensions close below 4.

In many cases one can easily read of the sign of the $\omega$’s from the expressions given above. In a number of cases it is not so easy. For these cases (the last two of RS 1.4, the last four of RS 2.3 and all of RS 2.2 and RS 3.1) we show in table 1 the regions of $n$ where $\omega$ is positive, negative or even complex. Table 2 gives the number of $\omega$’s with negative real part for the various FPs as function of $n$. Only the first order in $\varepsilon$ is taken into account with the exception of that $\omega$ in RS 2.1a/b which is identical to zero in first order and different from zero in second order due to the splitting of the fixed points.

In table 3 we show the regions of stable couplings $g$. We call couplings stable if the interaction $S_{int}$ (or its real part) is positive for any $\phi_1^2 + \phi_2^2 = 1$. If $S_{int}$ is positive for some directions $(\phi_1, \phi_2)$ but zero for special directions, then it is indicated as semistable ($\ast$). If it vanishes in all directions (RS 0.1), it is denoted by $i$. Otherwise the notation $u$ for unstable is used.

The stability is determined for representations with $g_{13} = g_{23} = 0$. In one-loop order there is such a representation for all fixed points in (77)–(79). For $g_{11} = g_{22}$ one obtains
\begin{equation}
g_{11}(\phi_1^2)^2 + g_{22}(\phi_2^2)^2 + 2g_{12}\phi_1^2\phi_2^2 + 2g_{33}(\phi_1\phi_2)^2 = (\phi_1^2)^2 + (\phi_2^2)^2 + g_{11} + g_{12} + q' g_{33})
\end{equation}
with $2g_{33}^2 = q((\phi_1^2)^2 + (\phi_2^2)^2)$. $0 \leq q \leq 1$ since $(\phi_1^2 - \phi_2^2)^2 \geq 0$, and $(\phi_1\phi_2)^2 = q'\phi_1^2\phi_2^2$, $0 \leq q' \leq 1$. Thus, the conditions for stability are $g_{11} > 0, \quad g_{11} + g_{12} > 0, \quad g_{11} + g_{12} + g_{33} > 0.$

\textit{Footnote:} Equation (4.5) in [9] is misprinted. The correct result is found in (3.10) of [12].
Table 1. Regions of \( n \) where \( \omega \) is either positive (\( \to^+ \)), negative (\( \to^- \)), complex with positive real part (\( \to^i \)) or complex with negative real part (\( \to^- \)). It is not shown if \( \omega = 0 \) only at a discrete value of \( n \).

| RS  | Mult. | From \( \to^\text{sign} \) down to |
|-----|-------|-------------------------------|
| 1.4 | 1     | \( \infty \to 186.95 \to 1.05 \to 1 \to -8 \to -\infty \) |
|     |       | \( \infty \to 186.95 \to 1.05 \to -8 \to -\infty \) |
| 2.3 | 1     | \( \infty \to 4.19 \to -8 \to -\infty \) |
|     |       | \( \infty \to 6 \to -8 \to -\infty \) |
|     |       | \( \infty \to 55.54 \to 2.02 \to -2 \to -2.17 \to -4 \to -8 \to -\infty \) |
|     |       | \( \infty \to 55.54 \to 2.02 \to 2 \to -2.17 \to -4 \to -6 \to -7.33 \to -8 \to -\infty \) |
| 2.2 | 2     | \( \infty \to 21.80 \to 3 \to 2.20 \to 2 \to -4 \to -8.68 \to -\infty \) |
|     |       | \( \infty \to 21.80 \to 2.20 \to -7.33 \to -8.68 \to -\infty \) |
|     |       | \( \infty \to 21.80 \to 2.20 \to 2 \to -8.68 \to -\infty \) |
| 3.1 | 2     | \( \infty \to 21.80 \to 3 \to 2.20 \to -4 \to -\infty \) |
|     |       | \( \infty \to 21.80 \to 2.20 \to -\infty \) |
|     |       | \( \infty \to 21.80 \to 2.20 \to -\infty \) |

Table 2. Number of \( \omega \)'s with negative real part given as function of \( n \).

| RS  | From \( \to^\text{neg,Re} \) down to |
|-----|--------------------------------|
| 0.1 | \( \infty \to 6 \to -\infty \) |
| 1.1 | \( \infty \to -6 \to -\infty \) |
| 1.2 | \( \infty \to 4 \to 1 \to -4 \to -8 \to -\infty \) |
| 1.3 | \( \infty \to 2 \to -4 \to -\infty \) |
| 1.4 | \( \infty \to 10 \to 6 \to 1 \to -8 \to -\infty \) |
| 2.1a| \( \infty \to 4 \to -4 \to -\infty \) |
| 2.1b| \( \infty \to 4 \to -2 \to -4 \to -\infty \) |
| 2.3 | \( \infty \to 6 \to 4.19 \to 2 \to -2 \to -4 \to 6 \to -7.33 \to -8 \to -\infty \) |
| 2.2 | \( \infty \to 21.80 \to 3 \to -4 \to -7.33 \to -8.68 \to -\infty \) |
| 3.1 | \( \infty \to 3 \to 2.20 \to 2 \to -4 \to -\infty \) |

If \( g_{11} \neq g_{22} \), one may set \( \phi_1^2 = w \phi_1^2 \) and \( \phi_2^2 = \phi_2^2 \) with \( w = \sqrt{g_{22}/g_{11}} \). Then \( g_{11} \) and \( g_{22} \) are replaced by \( \sqrt{g_{11}g_{22}} \). In that case the conditions of stability read

\[
\begin{align*}
g_{11} &> 0, \quad g_{22} > 0, \quad \sqrt{g_{11}g_{22}} + g_{12} > 0, \quad \sqrt{g_{11}g_{22}} + g_{12} + g_{33} > 0
\end{align*}
\]

If couplings are complex, they have to be replaced by their real parts.

The distinction between stable and unstable no longer applies for negative even \( n = -2r \) if the fields are expressed by \( r \) pairs of anticommuting (Grassmannian) components instead, whose integration is not subject to convergency conditions as in the case of real components. The integral over functions of scalar products of vectors with \( L \) real components and \( r \) pairs of
Here we review the FPs of our model (4) which also contain the actions (1)–(3). While the well-known subcases reduces to orthogonal transformations.

anticommuting variables depends only on $L - 2r$ [37–39]. The scalar product has the property that it is invariant under unitary orthosymplectic transformations, which in the case of $r = 0$ reduces to orthogonal transformations.)

### 6. The well-known subcases

Here we review the FPs of our model (4) which also contain the actions (1)–(3). While the trivial Gaussian FP is unstable in all models, their stable FPs, apart from the stable FP of (3), are found unstable in the general model (4).

The nontrivial $n$-Heisenberg FP of the simple $\phi^4$ model (1) is stable and corresponds to $RS \, 1.1$ if the second field is neglected. The quantities (83) reduce to

\[
\gamma_{\phi}^* = \frac{(n + 2)(2\epsilon)^2}{4(n + 8)^2}, \quad \gamma_{\tau}^* = -\frac{(n + 2)(2\epsilon)}{n + 8}, \quad \gamma_{\omega}^* = -\frac{2(2\epsilon)}{n + 8}, \quad \omega = (2\epsilon). \quad (121)
\]

The models (2) and (3) are special cases of model (4). Since the number of independent couplings $\tau$ and $g$ is less, the number of exponents ($\gamma_{\phi}^*$, $\gamma_{\tau}^*$, $\omega$) reduces to (2,2,3) for model (2) and to (1,1,2) for model (3). Those exponents $\gamma_{\tau}^*$ of (4), which are no longer $\gamma_{\phi}^*$ s of (2) and (3) belong now to the exponents $\gamma_{\omega}^*$.

The $O(n)+O(n)$ model (2) with $3g_{11} = g_1$, $3g_{22} = g_2$, $6g_{12} = g_3$, $g_{13} = 0$, $g_{23} = 0$, $g_{33} = 0$ has six nontrivial FPs. Three of them are decoupled ($g_1^* = 0$) and therefore represent tetracritical rather than bicritical behavior [1]: the $n$-Heisenberg–Gaussian FP with $g_1^* = 6(2\epsilon)/(n + 8)$ and $g_2^* = 0$, the Gaussian–$n$-Heisenberg FP with $g_1^* = 6(2\epsilon)/(n + 8)$ and $g_2^* = 0$ and the $n$-Heisenberg–$n$-Heisenberg FP with $g_1^* = g_2^* = 6(2\epsilon)/(n + 8)$. The critical exponents of the $n$-Heisenberg–Gaussian FP $RS \, 1.1$ in (79) are

\[
\gamma_{\phi}^* = \left\{ \frac{(n + 2)(2\epsilon)^2}{4(n + 8)^2}, \, 0 \right\}, \quad \gamma_{\tau}^* = \left\{ -\frac{(n + 2)(2\epsilon)}{n + 8}, \, 0 \right\}, \quad \omega = \left\{ (2\epsilon), \, -(2\epsilon), \, -\frac{6(2\epsilon)}{n + 8} \right\}. \quad (122)
\]
and the decoupled \( n \)-Heisenberg–\( n \)-Heisenberg FP RS 2.1 in (77) has

\[
\gamma_\Phi^* = \frac{(n + 2)(2\epsilon)^2}{4(n + 8)^2} \left( 2x \right), \quad \gamma_\tau^* = \frac{-(n + 2)(2\epsilon)}{n + 8} \left( 2x \right),
\]

\[
\omega = \left\{ \frac{(2\epsilon)}{2n + 8} \right\}.
\] (123)

The latter FP is clearly stable for \( n > 4 \).

The three remaining FPs have a nonvanishing \( g_\Delta^* \) and therefore represent bicritical behavior. The first FP is the isotropic \( 2n \)-Heisenberg FP [35, 36] RS 1.3 in (77) with

\[
\gamma_\Phi^* = \frac{(2n + 2)(2\epsilon)^2}{4(n + 8)^2} \left( 2x \right), \quad \gamma_\tau^* = \frac{-(2n + 2)(2\epsilon)}{2n + 8} \left( 2x \right),
\]

\[
\omega = \left\{ \frac{8(2\epsilon)}{2n + 8} \right\}.
\] (124)

This FP is stable for \( n < 2 \). The first \( \gamma_\tau^* \sim O(\epsilon) \) is the true critical exponent for \( \tau \), the second \( \gamma_\tau^* \sim O(\epsilon/n) \) yields the cross-over exponent.

The second FP is the so-called biconical FP RS 1.2 in (77). Its critical exponents are

\[
\gamma_\Phi^* = \frac{\left( \frac{n}{2} + \frac{3}{2} + \frac{1}{2} \right)(2\epsilon)^2}{8(n^2 + 8)^2} \left( 2x \right), \quad \gamma_\tau^* = \frac{1-n}{n^2 + 8} \left( \frac{2n(2\epsilon)}{n^2 + 8} \right),
\]

\[
\omega = \left\{ \frac{8(n - 1)(2\epsilon)}{8 + n^2}, \frac{(4-n)(n-2)(2\epsilon)}{8 + n^2} \right\}.
\] (125)

The biconical FP is stable for \( n = 3 \) in our approximation.

The last FP is given by RS 1.4 in (79) and is complex for \( n > 1 \). Its critical exponents are given by \( \gamma_{\Phi,1,2}^*, \gamma_{\tau,1,2}^* \) and \( \omega_{2,5,6} \) of (86). This FP coincides with the biconical FP for \( n = 1 \).

The frustrated spin model (3) is invariant under \( O(n) \times O(2) \). It is obtained by \( \tau_2 = \tau_1, g_{11} = g_{22} = u/3, g_{12} = u/3 - v/6, g_{33} = v/6, g_{13} = g_{32} = 0 \). It has four FPs: the trivial Gaussian FP RS 0.1, the isotropic \( 2n \)-Heisenberg FP RS 1.3, and the fixed point RS 2.2 and RS 3.1. \( \gamma_\Phi^* \) is that of RS 2.2 and 3.1. \( \gamma_\tau \) is \( \gamma_{\tau,1}^* \) of (113), the other \( \gamma_\tau \) and the \( \gamma_{\tau,2} \) of (113) yield the cross-over exponents, and \( \omega \) equals \( \omega_{2,5,6} \) of (115).

The FPs 2.2 and 3.1 are complex for \( 2.2 < n < 21.8 \) close to \( D = 4 \). This region decreases with decreasing \( D \) [9, 17]. The question of the range of stability in \( D = 3 \) is under debate [4, 16–21, 30–32].

7. Summary and conclusion

We considered in detail the \( O(n) \)-model (4) of two fields.

We gave the expressions for the \( \beta \) functions (10) and (11) and the matrices \( \gamma_\Phi \) (13), \( \gamma_\tau \) (17), \( \gamma_{\tau,s} \) (18) and \( \omega \) (12), and \( \gamma_{\tau,a} \) (21) for the model (4) from which the critical exponents are obtained in one-loop order (for \( \eta \) in two-loop order).

Next we considered its properties under orthogonal transformations of the two fields. Two types of FPs emerge: four of them are invariant under \( O(n) \times O(2) \). The other FPs are not invariant under \( O(2) \) and yield lines of FPs. The transformation of the couplings under \( O(2) \) was given.

A classification of the FPs in the large-\( n \) limit was given, before they were determined for general \( n \). Under the numerous FPs the corresponding FPs of the well-known models were found. To our best knowledge the FPs RS 2.1b and 2.3 are new. RS 2.1b has the remarkable property that it agrees for arbitrary \( n \) with RS 2.1a, which describes two uncoupled systems, in one-loop order. For these FPs two of the exponents \( \omega \) vanish in one-loop order. For special
values of $n$ some of the FPs coincide or yield an extra vanishing $\omega$. This is left for further discussion.

The full description of the fixed-point structure and the values of the most essential critical exponents can be useful for analytical and numerical investigation of the critical features of the system near $D = 4$. In this way a better understanding of such interesting phenomena as inverse symmetry breaking, symmetry nonrestoration and reentrant phase transitions could be obtained. Our model generalizes the $O(n)+O(n)$ and the $O(n) \times O(2)$ model giving rise to a variety of multi-critical phenomena.

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Appendix A. Stability matrix in the large-$n$ limit

The stability matrix in one-loop order is given by

\[
\omega = \frac{\partial \beta}{\partial g} = -2\epsilon + \frac{n + 8}{2}\begin{pmatrix}
2g_{11} & 2g_{12} & 2g_{13} & 0 & 0 & 0 \\
g_{12} & g_{11} + g_{22} & g_{23} & g_{12} & g_{13} & 0 \\
g_{13} & g_{23} & g_{11} + g_{33} & 0 & g_{12} & g_{13} \\
0 & 2g_{12} & 0 & 2g_{22} & 0 & 2g_{23} \\
0 & g_{13} & g_{12} & g_{23} & g_{22} + g_{33} & g_{23} \\
0 & 0 & 2g_{13} & 0 & 2g_{23} & 2g_{33}
\end{pmatrix}
\]

(A.1)

and yields in the large-$n$ limit for $p^{(1)}$

\[
\omega = -2\epsilon + 2\epsilon B_1 B_2
\]

(A.2)

with

\[
B_1 = \begin{pmatrix}
2z_1 & 0 & 0 \\
z_2 & z_1 & 0 \\
z_3 & 0 & z_1 \\
0 & 2z_2 & 0 \\
0 & z_3 & z_2 \\
0 & 0 & 2z_3
\end{pmatrix}, \quad B_2 = \begin{pmatrix}
z_1 & z_2 & z_3 & 0 & 0 & 0 \\
0 & z_1 & 0 & z_2 & z_3 & 0 \\
0 & 0 & z_1 & 0 & z_2 & z_3
\end{pmatrix}.
\]

(A.3)

The $3 \times 3$-matrix $B_2 B_1$

\[
(B_2 B_1)_{ij} = \delta_{ij} + z_i z_j
\]

(A.4)

has one eigenvalue 2 and two eigenvalues 1. The matrix $B_1 B_2$ has the same eigenvalues and in addition three eigenvalues 0. As a consequence the stability matrix $\omega$ has three eigenvalues $-2\epsilon$, two eigenvalues 0 and one eigenvalue $2\epsilon$ independent of $z_{12}$.

For $p^{(2)}$ the stability matrix reads in this limit

\[
\omega = 2\epsilon + 2\epsilon B_1 B_2
\]

(A.5)

and thus the eigenvalues are the negative of those of $\omega$ for $p^{(1)}$. 

\[
\omega = 2\epsilon + 2\epsilon B_1 B_2
\]
References

[1] Nelson D R, Kosterlitz J M and Fisher M E 1976 Phys. Rev. B 13 412

[2] Zhang S-C 1997 Science 275 1089

[3] Aharony A 2003 J. Phys. Stat. 110 659

[4] Pelissetto A and Vicari E 2002 Phys. Rep. 368 549

[5] Weinberg S 1974 Phys. Rev. D 9 3357

[6] Pinto M B and Ramos R O 2006 J. Phys. A: Math. Gen. 39 6649

[7] Pelissetto A and Vicari E 2005 Condens. Matter Phys. (Ukraine) 8 87

[8] Collins M F and Petrenko O A 1997 Can. J. Phys. 75 605

[9] Kawamura H 1998 J. Phys. C: Solid State Phys. 10 4707

[10] Kawamura H 2001 Can. J. Phys. 79 1447

[11] Kawamura H 2002 arXiv:cond-mat/0202109

[12] Kawamura H 1998 Phys. Rev. B 38 4916

Kawamura H 1990 Phys. Rev. B 42 2610

[13] Kawamura H 1987 J. Appl. Phys. 61 3590

[14] Kawamura H 1986 J. Phys. Soc. Japan 55 2157

[15] Garel T and Pfeuty H 1976 J. Phys. C: Solid State Phys. 9 L245

[16] Antonenko S A and Sokolov A I 1994 Phys. Rev. B 49 15901

[17] Antonenko S A, Sokolov A I and Varnavski K B 1995 Phys. Lett. A 208 161

[18] Loison D, Sokolov A I, Delamotte B, Antonenko S A, Schotte K D and Diep H T 2000 Pis’ma v Zh. Eskp. Teor. Fiz. 72 487

Loison D, Sokolov A I, Delamotte B, Antonenko S A, Schotte K D and Diep H T 2000 JETP Lett. 72 337

[19] Pelissetto A, Rossi P and Vicari E 2001 Phys. Rev. B 63 140414

[20] Calabrese P, Parruccini P and Sokolov A I 2002 Phys. Rev. B 66 180403

[21] Calabrese P, Parruccini P and Sokolov A I 2003 Phys. Rev. B 68 094415

[22] Calabrese P, Orlov E V, Parruccini P and Sokolov A I 2003 Phys. Rev. B 67 024413

[23] Tissier M, Mouhanna D and Delamotte B 2000 Phys. Rev. B 61 15327

[24] Tissier M, Delamotte B and Mouhanna D 2000 Phys. Rev. Lett. 84 5208

[25] Tissier M, Delamotte B and Mouhanna D 2003 Phys. Rev. B 67 134422

[26] Delamotte B, Mouhanna D and Tissier M 2004 Phys. Rev. B 69 134413

[27] Iakura M 2003 J. Phys. Soc. Japan 72 74–82

[28] Peles A, Southern B W, Delamotte B, Mouhanna D and Tissier M 2004 Phys. Rev. B 69 220408

[29] Bekhechi S, Southern B W, Peles A and Mouhanna D 2006 Phys. Rev. E 74 016109

[30] Delamotte B, Holovatch Yu, Yvaneyko D, Mouhanna D and Tissier M 2008 J. Stat. Mech. P03014

[31] Loison D 2005 Frustrated Spin Systems ed H T Diep (Singapore: World Scientific) p 177

[32] Thanh Ngo V and Diep H T 2008 arXiv:0808.0520

[33] Brézin E, le Guillou J C and Zinn-Justin J 1976 Phase Transitions and Critical Phenomena vol 6 ed C Domb and M S Green p 125

[34] Weber A 2007 Diploma Thesis Heidelberg

[35] Wegner F 1972 Phys. Rev. B 6 1891

[36] Fisher M E and Pfeuty P 1972 Phys. Rev. B 6 1889

[37] Kieburg M, Kohler H and Guhr T 2008 arXiv:0809.2674

[38] Constantinescu F and de Groote H 1989 J. Math. Phys. 30 981

[39] Efetov K 1997 Supersymmetry in Disorder and Chaos (Cambridge: Cambridge University Press)