Resilient Interval Observer for Simultaneous Estimation of States, Modes and Attack Policies

Mohammad Khajenejad  Zeyuan Jin  Sze Zheng Yong

Abstract—This paper considers the problem of designing interval observers for hidden mode switched nonlinear systems with bounded noise signals that are compromised by false data injection and switching attacks. The proposed observer consists of three components: i) a bank of mode-matched observers, which simultaneously estimates the corresponding mode-matched continuous states and discrete states (modes), as well as learns a model of the unknown attack policy, ii) a mode observer that eliminates the incompatible modes based on a residual-based set-membership criterion, and iii) a global fusion observer that combines the outputs of i) and ii). Moreover, in addition to showing the correctness, stability and convergence of the mode-matched estimates, we provide sufficient conditions to guarantee that all false modes will be eliminated after sufficiently large finite time steps, i.e., the system is mode-detectable under the proposed observer.

I. INTRODUCTION

Computation and communication constituents are tightly intertwined in Cyber-Physical Systems (CPS). While this coupling can enhance the functionality of control systems and improve their performance, it might also become a source of vulnerability to faults or attacks. On the other hand, given various sources of real world uncertainties, complete information/direct knowledge of the decisions and intentions of other systems/agents, is not available to autonomous decision makers, e.g., self-driving cars or robots. These safety-critical systems can be studied using a general framework of hidden mode hybrid/switched systems (HMHS, see, e.g., [1] and references therein). The ability to estimate the continuous states, attacks/unknown inputs and modes/discrete states of such systems is important for monitoring them as well as for designing safe and secure (optimal) feedback controllers.

Literature review. There has been a relatively large body of literature on the problem of designing filters/observers for hidden mode systems without considering unknown inputs/faults/data injection attacks, e.g., in [2] and references therein. For stochastic settings, extensions were proposed, e.g., in [1], to obtain state and unknown input point estimates, i.e., the most likely or best single estimates. However, especially when hard guarantees or bounds are important, it might be preferable to consider set-valued uncertainties, e.g., bounded-norm noise. Moreover, probabilistic distributions/stochastic characteristics of uncertainty are often unavailable in real world applications. Consequently, to estimate the “set” of compatible states, set-valued or set-membership observers, e.g., [3], have been proposed. Later, the study in [4] extended this framework to include estimation of unknown inputs/attacks. Nonetheless, these approaches are not directly applicable to systems with hidden modes that are considered in this paper.

A common approach to consider hidden modes for representing attack or fault models is to construct residual signals (see, e.g., [1], [2], [5]), where to distinguish between consistent and inconsistent modes, some residual-based criteria/thresholds are used. The work in [6] presented a robust control-inspired approach for linear systems with bounded-norm noise that consists of local estimators, residual detectors, and a global fusion detector for resilient state estimation against sparse data injection attacks. Similar residual-based approaches have been proposed for uniformly observable nonlinear systems in [7] and some classes of nonlinear systems in [8], where only sensors were compromised by sparse attacks, which is a special case of hidden mode switched systems discussed in our previous works [1], [9].

On the other hand, when the system model is not exactly known, in order to find a set of dynamics that frame/bracket the unknown system dynamics [10], set-valued data-driven approaches have been developed to use input-output data to abstract or over-approximate unknown dynamics or functions [10], [11], under the assumption that the unknown dynamics is continuous, e.g., [11]. In our previous work [12], we leveraged interval observers for such data-driven models, for resilient state and data injection attack estimation, assuming that the attack signal has an unknown dynamics. In this work, we assume mode/switching attacks in addition to data injection attacks, where the attack signals are governed by an unknown and to-be-learned attack policy.

Contributions. To tackle this problem, leveraging a multiple-model framework proposed in our previous works [9], [13], we first design a bank of mode-matched set-valued observers, where we combine a model-based interval observer approach used in [12], [14], with our previously introduced set-membership learning technique [15], to derive set-valued mode-matched estimates for the states and attack signal values, as well as to learn model abstractions/over-approximations for the attack policy, where we derive several desired properties for the mode-matched estimates, such as correctness, stability and convergence. Then, we introduce a novel elimination-based mode observer, based on a set-membership criterion, to eliminate inconsistent modes from the bank of observers. Furthermore, we provide sufficient conditions for mode-detectability, i.e., all false modes will be eventually ruled out under some reasonable assumptions. Finally, we illustrate the performance of our proposed ap-
proach by applying it on a power system example.

II. PRELIMINARIES

Notation. \( \mathbb{R}^n \), \( \mathbb{R}^{n \times m} \) and \( \mathcal{D}_n \) denote the \( n \)-dimensional Euclidean space, the space of \( n \) by \( m \) matrices and the set of all diagonal matrices in \( \mathbb{R}^{n \times n} \) with their diagonal arguments being 0 or 1. For vectors \( v, w \in \mathbb{R}^n \) and a matrix \( M \in \mathbb{R}^{n \times n} \), \( |v| \equiv \sqrt{v^\top v} \) and \( |M| \) denote their (induced) 2-norm, and \( v \leq w \) is an element-inequality. The transpose, Moore-Penrose pseudoinverse, \((i,j)\)th element and rank of \( M \) are given by \( M^\top \), \( M^{ij} \), \( \text{rk}(M) \), while \( M_{(r,s)} \) is a submatrix of \( M \), consisting of its \( r \)th through \( s \)th rows, and its row support is \( r = \text{rowsupp}(M) \in \mathbb{R}^p \), where \( r_i = 0 \) if the \( i \)th row of \( M \) is zero and \( r_i = 1 \) otherwise, \( \forall i \in \{1, \ldots, p\} \). Also, \( M^+ \equiv \max(M, 0) \), \( M^{-} \equiv M^{+} - M \) and \( |M| \equiv M^{+} + M^{-} \). \( M \) is a non-negative matrix, if \( M_{ij} \geq 0 \), \( \forall (i,j) \in \{1, \ldots, p\} \times \{1, \ldots, q\} \).

Next, we introduce some useful definitions and results.

Definition 1 (Interval, Maximal and Minimal Elements, Interval Width). An (multi-dimensional) interval \( \mathcal{I} \subset \mathbb{R}^n \) is the set of all real vectors \( x \in \mathbb{R}^n \) that satisfies \( \underline{x} \leq x \leq \bar{x} \), where \( \underline{x} \) and \( |\bar{x} - \underline{x}| \) are called minimal vector, maximal vector and width of \( \mathcal{I} \), respectively.

Proposition 1 (Slight Generalization of [16, Lemma 2]). Let \( B \in \mathbb{R}^{n \times p} \) be an interval matrix satisfying \( \underline{B} \leq B \leq \bar{B} \).

i) if \( A \in \mathbb{R}^{m \times n} \) is a constant matrix, then \( A + \underline{B} - A + \bar{B} \leq AB \leq A + \underline{B} - A + \bar{B} \)

ii) if \( A \in \mathbb{R}^{m \times n} \) is an interval matrix satisfying \( A \leq \bar{A} \), then \( A + \underline{B} - A + \bar{B} - A + \bar{B}^\top + A - \bar{B} \leq AB \leq A + \underline{B} - A + \bar{B} + A - \bar{B}^\top + A - \bar{B} \)

Proof. The results follow from defining \( x_i \) as the \( i \)th column of \( B \), applying [16, Lemma 2] on \( A \) and \( x_i \), for all \( i \in \{1, \ldots, p\} \) and then stacking the resulting inequalities.

Proposition 2 (Parallel Affine Abstractions [12]). Let the entire space be defined as \( \mathcal{X} \) and suppose \( \mathcal{X} \) is bounded. Consider the vector fields \( \psi(\cdot), \psi(\cdot) : \mathcal{X} \subset \mathbb{R}^n \to \mathbb{R}^m \) satisfying \( \psi(x) \leq \overline{\psi}(x) \), \( \forall x \in \mathcal{X} \), a (given) global parallel affine abstraction with known \( (A^\psi, \bar{B}^\psi, \underline{B}^\psi) \) on \( \mathcal{X} \), i.e.,

\[
A^\psi x + \underline{B}^\psi \leq \psi(x) \leq A^\psi x + \bar{B}^\psi, \forall x \in \mathcal{X}.
\]

and the following Linear Program (LP):

\[
\min_{\theta_B} \theta_B^{\underline{B}^\psi} \tag{2a}
\]

s.t. \( A^\psi_B x + \underline{B}^\psi + \alpha^\psi \leq \psi(x) \leq A^\psi_B x + \bar{B}^\psi - \alpha^\psi \),

\[
\bar{B} - 2\alpha^\psi \leq \theta_B \leq 0, \tag{2b}
\]

where \( \theta_B = [x, \underline{x}] \subset \mathcal{X} \) is a local interval domain and \( \mathcal{V}_B \) being its maximal, minimal and set of vertices, respectively, \( \mathcal{V}_m \) \( \subseteq \mathbb{R}^m \) is a vector of ones, \( \alpha^\psi \) is given in [17, Proposition 1 and (8)] for different classes of continuous vector fields. Then, \( (A^\psi_B, \bar{B}^\psi, \underline{B}^\psi) \) are the local parallel affine abstraction matrices for the pair of functions \( \underline{\psi}(\cdot), \overline{\psi}(\cdot) \) on \( \mathcal{B} \), i.e.,

\[
A^\psi_B x + \underline{B}^\psi \leq \psi(x) \leq A^\psi_B x + \bar{B}^\psi, \forall x \in \mathcal{B}.
\]

Definition 2 (Mixed-Monotone Mappings and Decomposition Functions). [18, Definition 4] A mapping \( f : \mathcal{X} \subset \mathbb{R}^n \to \mathcal{T} \subseteq \mathbb{R}^m \) is mixed-monotone if there exists a decomposition function \( f_d : \mathcal{X} \times \mathcal{X} \to \mathcal{T} \) satisfying: i) \( f_d(x, x) = f(x) \), ii) \( x_1 \geq x_2 \Rightarrow f_d(x_1, y) \geq f_d(x_2, y) \) and iii) \( y_1 \geq y_2 \Rightarrow f_d(x, y_1) \leq f_d(x, y_2) \).

Proposition 3. [19, Theorem 1] Let \( f : \mathcal{X} \subset \mathbb{R}^n \to \mathcal{T} \subseteq \mathbb{R}^m \) be a mixed-monotone mapping with decomposition function \( f_d : \mathcal{X} \times \mathcal{X} \to \mathcal{T} \) and \( \underline{x} \leq x \leq \bar{x}, \forall x, \underline{x}, \bar{x} \in \mathcal{X} \). Then \( f_d(x, \underline{x}) \leq f_d(x, x) \leq f_d(x, \bar{x}) \).

Corollary 1 (Nonlinear Bounding). Let \( f : \mathcal{X} \subset \mathbb{R}^n \to \mathcal{T} \subseteq \mathbb{R}^m \) satisfies the assumptions in Propositions 2 and 3.

Then, for all \( \underline{x}, \bar{x} \in \mathcal{X} \) satisfying \( \underline{x} \leq x \leq \bar{x} \), the following inequality holds: \( f \leq f(x) \leq \mathcal{T} \), where

\[
\mathcal{T} = \min(f_d(x, \underline{x}), f_d(x, \bar{x})), \tag{4}
\]

\[
f_d = \max(f_d(x, \underline{x}), f_d(x, \bar{x})).
\]

Proof. The results directly follow from Propositions 1-3.

Note that the decomposition function of a vector field is not unique and a specific one is given in [18, Theorem 2]: If a vector field \( q = [q_1, \ldots, q_m]^\top : \mathcal{X} \subset \mathbb{R}^n \to \mathbb{R}^m \) is differentiable and its partial derivatives are bounded with known bounds, i.e., \( \frac{\partial q}{\partial x_i} \in \mathcal{B}(a^q_{ij}, b^q_{ij}), \forall x \in \mathcal{X} \subset \mathbb{R}^n \), where \( a^q_{ij}, b^q_{ij} \in \mathbb{R} \), then \( q \) is mixed-monotone with a decomposition function \( q_d = [q_{d1}, \ldots, q_{dn}]^\top \), where \( q_d(x, y) = q_i(z) + (\alpha^q_{ij} - \beta^q_{ij})^\top (x - y), \forall i \in \{1, \ldots, n\} \), and \( z, \alpha^q_{ij}, \beta^q_{ij} \in \mathbb{R}^n \) can be computed in terms of \( x, y, a^q_{ij}, b^q_{ij} \) as given in [18, (10)–(13)]. Consequently, for \( x = [x_1, \ldots, x_n]^\top, y = [y_1, \ldots, y_n]^\top \), we have

\[ q_d(x, y) = q(z) + C^q(x - y), \]
where \( x_k \in \mathbb{R}^n \) is the continuous system state and \( q \in Q = \{1, 2, \ldots, Q\} \) is the hidden discrete state or mode. For each (fixed) mode \( q \), \( u_k^q \in U_k^{q} \subset \mathbb{R}^m \) is the known input, \( d_k^q \in \mathbb{R}^p \) is the unknown but sparse input, i.e., every vector \( d_k^q \) has precisely \( \rho \in \mathbb{N} \) nonzero elements where \( \rho \) is a known parameter and \( y_k \in \mathbb{R}^l \) is the measured output. The unknown input signal \( d_k^q \) is considered as the realization of an attacker’s unknown policy \( \mu^q(x, y, z, v) : \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^s \rightarrow \mathbb{R}^p \), which is an unknown mapping from state and known input to the set of attack signals. Moreover, \( u_k \in U \triangleq \{ \mu, \nu \} \subset \mathbb{R}^n \) and \( y_k \in \mathcal{V} \triangleq [\nu, \tau] \subset \mathbb{R}^l \) are bounded process and measurement disturbances with known minimal and maximal values \( \mu, \nu, \tau, \tau \), respectively. Further, the mappings \( f, g \), as well as the matrices \( G_k^q \in \mathbb{R}^{n \times p} \) and \( H_k^q \in \mathbb{R}^{l \times p} \) are known.

More precisely, \( G_k^q \) and \( H_k^q \) represent the different hypotheses for each mode \( q \in Q \), about the sparsity pattern of the unknown inputs, which in the context of sparse attacks corresponds to which actuators and sensors are attacked or not attacked. In other words, we assume that \( G_k^q = G_k^{q'} \) and \( H_k^q = H_k^{q'} \) for some input matrices \( G \in \mathbb{R}^{n \times n} \) and \( H \in \mathbb{R}^{l \times l} \), where \( t_a \) and \( t_s \) are the number of vulnerable actuator and sensor signals respectively. Note that \( \rho^s_a \leq t_a \leq m \) and \( \rho^s_s \leq t_s \leq l \), where \( \rho^s_a (\rho^s_s) \) is the number of attacked actuator (sensor) signals and clearly cannot exceed the number of vulnerable actuator (sensor) signals, which in turn cannot exceed the total number of actuators (sensors). Further, we assume that the maximum number of unknown inputs/attacks in each mode is known and equals \( \rho = \rho_a + \rho_s \) (sparsity assumption). Moreover, the index matrix \( T_G^q \in \mathbb{R}^{t_a \times p} (T_H^q \in \mathbb{R}^{t_s \times p}) \) represents the sub-vector of \( d_k^q \in \mathbb{R}^p \) that indicates signal magnitude attacks on the actuators (sensors).

We are interested in estimating the state trajectories, as well as the unknown mode and the attack policy mapping in the system in (6), when they are initialized in a given interval \( \mathcal{X}_0 \subset \mathcal{X} \subset \mathbb{R}^n \). Furthermore, when they are initialized in a given interval \( \mathcal{X}_0 \subset \mathcal{X} \subset \mathbb{R}^n \).

**Assumption 1.** The vector fields \( f, g \) are known, Lipschitz continuous and mixed-monotone. Moreover, the values of the unknown input signal \( d_k^q \) is considered as the realization of an attacker’s unknown policy \( \mu^q(x, y, z, v) : \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^s \rightarrow \mathbb{R}^p \), which is an unknown mapping from state and known input to the set of attack signals. Moreover, \( u_k \in U \triangleq \{ \mu, \nu \} \subset \mathbb{R}^n \) and \( y_k \in \mathcal{V} \triangleq [\nu, \tau] \subset \mathbb{R}^l \) are bounded process and measurement disturbances with known minimal and maximal values \( \mu, \nu, \tau, \tau \), respectively. Further, the mappings \( f, g \), as well as the matrices \( G_k^q \in \mathbb{R}^{n \times p} \) and \( H_k^q \in \mathbb{R}^{l \times p} \) are known.

**Assumption 2.** Given mode \( q \), the attacker’s policy mapping \( \mu^q(\cdot) = [\mu_1^q(\cdot), \ldots, \mu_p^q(\cdot)]^\top \) is unknown, but each \( \mu_j^q(\cdot), \forall j \in \{1, \ldots, p\} \) is known to be Lipschitz continuous. Moreover, for simplicity and without loss of generality we assume that the Lipschitz constants \( L_j^q, \forall j \in \{1, \ldots, p\} \) are known, otherwise, they can be estimated with any desired precision using the approach in [15, Equation (12) and Proposition 3].

**Assumption 3.** There is only one “true” mode, i.e. the true mode \( q^* \in Q \) is constant over time.

Note that the approach in our paper can be easily extended to handle mode-dependent \( f, g, \mu, \nu, \tau \) and \( \rho \), but is omitted to simplify the notations. Further, we formally define the notions of framers, correctness and stability that are used throughout the paper.

**Definition 3** (Framers and Correct Interval Observers). Given a hidden mode switched nonlinear system (6), let us define the augmented state \( z_k = [x_k^q \ y_k^q \ z_k^q]^\top \), for all \( k \in \mathbb{K} \triangleq \mathbb{N} \cup \{0\} \), where \( d_k = d_k^q \) is the true attack signal. The sequences \( \{z_k, z_k^q\}_{k=0}^\infty \) are called upper and lower framers for the augmented states of system (6), if \( \forall k \in \mathbb{K} \), \( z_k^q \leq z_k \leq z_k^q \). In other words, starting from the initial interval \( z_0 \in [z_0, z_0^q] \), the true augmented state of the system in (6), \( z_k \), is guaranteed to evolve within the interval flow-pipe \([z_k^q, z_k]\), for all \( k \in \mathbb{K} \). Finally, any algorithm that returns framers for the states of system (6) is called a correct interval observer for system (6).

**Definition 4** (Stability). The mode-matched observer (11) is stable, if the sequence of interval widths \( \{\|z_k - z_k^q\|\}_{k=0}^\infty \) is uniformly bounded, and consequently, the sequence of estimation errors \( \{\|z_k - z_k^q\|\}_{k=0}^\infty \) is also uniformly bounded.

Using the modeling framework above, the simultaneous state, hidden mode and policy estimation problem is three-fold and can be stated as follows:

**Problem 1.** Given a discrete-time bounded-error hidden mode switched nonlinear system with unknown inputs (6) and assuming that Assumptions 2 hold,

i) Design a bank of mode-matched observers that for each mode, conditioned on the mode being the true mode, finds uniformly bounded set estimates of compatible (augmented) states and learns a guaranteed model abstraction of the attacker’s policy.

ii) Develop a mode observer via elimination and the corresponding criteria to eliminate false modes.

iii) Find sufficient conditions for eliminating all false modes.

**IV. Proposed Observer Design**

Leveraging a multiple-model approach similar to [9], [13], for simultaneous mode, state and attack policy (SMSP) estimation, our goal in this section is to propose an observer to find set estimates \( \mathcal{X}_k, \mathcal{D}_k \) and \( \mathcal{Q}_k \) for the states \( x_k \), attacks \( d_k \) and modes \( q \in \mathbb{Q} \) at time step \( k \), respectively, as well as to compute a model abstraction \( \{\mathcal{P}_k, \mathcal{L}_k\}_{k \in \mathbb{K}} \) for the attack policy, such that \( \mu_k(x_k) \leq \mu(x) \leq \mathcal{P}_k(x_k) \) for all \( k \in \mathbb{K} \).

**A. Multiple-Model Approach: An Overview**

Similar to the approach in [13], we propose a three-step multiple-model design consisting of: (i) a bank of mode-matched interval observers to obtain mode-matched state and attack estimates, as well as mode-matched policy abstractions/over-approximations, (ii) a mode estimation algorithm to eliminate incompatible modes using residual detectors, and (iii) a global fusion observer that outputs the desired set-valued mode, attack (policy) and state estimates.

1) Mode-Matched Set-Valued State and Attack Policy Observer: First, we design a bank of mode-matched observers, which consists of \( \mathcal{Q} \triangleq \{Q\} \) simultaneous state, attack and policy mode-matched interval observers, designed in a similar manner as our approach in [12], with the difference
that in [12], the unknown input (i.e., attack) signal is treated as a state with unknown and to-be-learned dynamics, whereas in the current work, the attack signal is governed by an unknown policy/state feedback law, i.e., an unknown function of the actual state, that should be learned/approximated. With that in mind, given mode $q$, each mode-matched interval observer at time step $k \in \mathbb{N}$, returns
\[
\hat{X}_k = \{x_k^q, \bar{x}_k^q\}, \quad \hat{D}_k = \{d_k^q, \bar{d}_k^q\}, \quad \{\mu_k^q, \nu_k^q\},
\]
such that $x_k \in \hat{X}_k$, $d_k \in \hat{D}_k$, and $\mu(x_k) \in \{\mu_k^q(x_k), \bar{\mu}_k^q(x_k)\}$ through the following steps (with the augmented state $z_k \equiv [x_k^q \quad d_k^q]^T$ and known $x_0$ and $\bar{x}_0$ such that $x_0 \leq \bar{x}_0$):

**State Propagation:**
\[
\begin{align}
\hat{z}_p^{q,p} &= \left[\begin{array}{c}
\hat{z}_k^{q,p} - \hat{z}_k^{q,p} \\
\hat{z}_k^{q,p} + \hat{z}_k^{q,p} \\
\end{array}\right], \\
\hat{z}_p^{q,p} &= \left[\begin{array}{c}
\hat{z}_k^{q,p} \\
\hat{z}_k^{q,p} \\
\end{array}\right], \\
\end{align}
\]

**Measurement Update:**
\[
\begin{align}
\hat{z}_p^{q,p} &= \left[\begin{array}{c}
\hat{z}_k^{q,p} \\
\hat{z}_k^{q,p} \\
\end{array}\right], \\
\hat{z}_p^{q,p} &= \left[\begin{array}{c}
\hat{z}_k^{q,p} \\
\hat{z}_k^{q,p} \\
\end{array}\right], \\
\end{align}
\]

**Unknown Input Estimation:**
\[
\begin{align}
\hat{\theta}_k^{q,p} &= \min_{\theta \in \Theta} \left[\begin{array}{c}
\hat{z}_k^{q,p} \\
\hat{z}_k^{q,p} \\
\end{array}\right], \\
\hat{\theta}_k^{q,p} &= \left[\begin{array}{c}
\hat{z}_k^{q,p} \\
\hat{z}_k^{q,p} \\
\end{array}\right], \\
\end{align}
\]

**Attack Policy Learning:**
\[
\begin{align}
\mu_k^{q,j}(x_k) &= \min_{t \in \{0(\ldots,-1\)} \left[\begin{array}{c}
\hat{d}_k^{q,j} - \hat{d}_k^{q,j} \\
\hat{d}_k^{q,j} + \hat{d}_k^{q,j} \\
\end{array}\right], \\
\end{align}
\]

**unknown Input Estimation:**
\[
\begin{align}
\hat{\theta}_k^{q,p} &= \min_{\theta \in \Theta} \left[\begin{array}{c}
\hat{z}_k^{q,p} \\
\hat{z}_k^{q,p} \\
\end{array}\right], \\
\hat{\theta}_k^{q,p} &= \left[\begin{array}{c}
\hat{z}_k^{q,p} \\
\hat{z}_k^{q,p} \\
\end{array}\right], \\
\end{align}
\]

**Residual Evaluation:**
\[
\begin{align}
\hat{r}_k^q &= \sum_{i=1}^{p} \left(\hat{z}_k^{q,i} - \hat{z}_k^{q,i}\right), \\
\hat{r}_k^q &= \left[\begin{array}{c}
\hat{z}_k^{q,1} \\
\hat{z}_k^{q,2} \\
\end{array}\right], \\
\end{align}
\]

**Mode Estimation Observer:** To estimate the set of compatible modes, we consider a membership-based elimination approach that checks if residual signals are within some compatible intervals. We first define the mode-matched residual signal $r_k^q$ as follows.

**Definition 5:** (Residuals). For each mode $q$ at time step $k$, the residual signal $r_k^q$ is defined as:
\[
\begin{align}
r_k^q &= \hat{r}_k^q - \frac{1}{2}(\hat{g}_k^q + \hat{q}_k^q), \\
\end{align}
\]

where $\hat{g}_k^q, \hat{q}_k^q$ are the bounding signals based on (4) applied on the mapping $g^q(\cdot)$.

Then, we eliminate a specific mode $q$, if its corresponding residual signal $r_k^q$ violates to be within an interval given in the following proposition.

**Proposition 4** (Mode Elimination Criterion). Mode $q$ is not a true mode if
\[
\begin{align}
r_k^q \notin \mathcal{R}_k^q \equiv \left\{ r_k^q \mid \max_{q \in \mathcal{Q}_k^q} \left[\begin{array}{c}
\hat{r}_k^q - \hat{r}_k^q \\
\hat{r}_k^q + \hat{r}_k^q \\
\end{array}\right] \right\} \\
\end{align}
\]

**Proof.** If $q$ is the true mode, then $y_k = g^q(x_k, d_k, v_k)$ by (6). Consequently, $y_k \in [\hat{q}_k^q, \hat{q}_k^q]$ which is equivalent to $r_k^q \in \mathcal{R}_k^q$, given the definition of $r_k^q$ in (17), and with $\hat{g}_k^q, \hat{q}_k^q$ obtained from (4) in Corollary [1].

By Proposition [4] if the residual signal of a particular mode $q$ is not within the given interval in (18) conditioned on this mode being true, then $q$ can be ruled out as incompatible.

3) Global Fusion Observer: Finally, combining the outputs of both components above, our proposed global fusion observer will provide mode, attack and state set-valued estimates, as well as attack policy abstractions, at each time step $k$ as:
\[
\hat{Q}_k = \{q \in \mathcal{Q}_k^q \mid r_k^q \in \mathcal{R}_k^q\}, \\
\hat{X}_k = \bigcup_{q \in \hat{Q}_k} \mathcal{X}_k^q, \\
\hat{D}_k = \bigcup_{q \in \hat{Q}_k} \mathcal{D}_k^q, \\
\hat{\mu}_k(\cdot) = \min_{q \in \hat{Q}_k} \hat{\mu}_k^q(\cdot), \\
\hat{\nu}_k(\cdot) = \min_{q \in \hat{Q}_k} \hat{\nu}_k^q(\cdot).
\]

The simultaneous mode, state and attack policy (MSMP) estimation approach is summarized in Algorithm [1].

**B. Properties of Mode-Matched Observers**

In this section, following a similar approach to our previous work [12], we show that each of the mode-matched...
Algorithm 1 Simultaneous Mode, State and Attack Policy (SMSP) Estimation

1: \( \hat{Q}_0 = Q \); 
2: for \( k = 1 \) to \( N \) do 
   3:     for \( q \in \hat{Q}_{k-1} \) do
   4:         Mode-Matched State and Attack Policy Set-Valued Estimates
   5:         Compute \( \hat{z}_k^{(q)} = \hat{z}_k^{(q)} \) through (8a)–(11b).
   6:         Mode Observer via Elimination
   7:         Compute \( \hat{z}_k^{(q)} \) via Definition [3].
   8:         if \((13)\) holds then \( \hat{Q}_k = \hat{Q}_k \backslash \{q\} \); 
   9:     end if
   10:   end for
   11: end for

observers is correct (cf. Definition [3]) and stable (cf. Definition [3]) under some sufficient conditions. Moreover, the sequence of mode-matched interval widths is convergent to some computable steady state values.

**Lemma 1** (Correctness). Consider System (6) and suppose Assumptions [7][8] hold. Then, for all mode \( q \in Q \), the dynamical system in (8a)–(11b) constructs a correct mode-matched interval observer for System (6), conditioned on the mode being the true mode, i.e., \( q = q^* \). In other words, \( \forall k \in K \setminus (N \cup \{0\}) \), \( \hat{z}_k^{(q)} \leq z_k^{(q)} \leq z_k^{(q)} \), where \( z_k^{(q)} = [x^T \ d^T] \) and \( [z_k^{(q)}] \) are the augmented vectors of state and unknown inputs in the dynamical systems in (6) and the augmented estimates from (13) at time \( k \in K \), respectively.

**Proof**. Using induction, the proof follows similar lines to the proof of [12, Theorem 1].

Next, we address the stability of each mode-matched observer. Note that similar to [12], our goal is to obtain sufficient stability conditions that can be checked a priori instead of for each time step \( k \). For the other hand, for the implementation of the update step, we iteratively find new mode-matched local parallel abstraction slopes \( A^q_i, k \) by iteratively solving the LP (13) for \( g^q_i \) on the intervals obtained in the previous iteration, \( B^q_i,k = [z^q_i, k \ z^q_i, k] \), to find local framers \( z^q_i, k \) (cf. (13)–(15)), with additional constraints given in (2b) in the optimization problems, which guarantees that the iteratively updated local intervals obtained using the local abstraction slopes are inside the global interval, i.e.,
\[
\Delta_{s}^q = \Delta_{s}^q(A^q_i, k) \triangleq \Delta_{s}^q(A^q_i, k) \leq \cdots \leq \Delta_{s}^q(A^q_1, k) \leq \Delta_{s}^q(A^q_0, k)
\]

With that in mind, we next show through the following proposition that the sequence of the widths of the interval-valued estimates are upper bounded by a difference equation, i.e., a discrete-time dynamical system, for each mode.

**Proposition 5** (Interval Widths Upper System). Consider System (6) along with the observer in (8a)–(11b) and suppose that all the assumptions in Lemma [7] hold and the decomposition function \( f_d \) is constructed using (5). Let us define the mode-matched width of the interval-valued estimate \( [\hat{z}_k^{(q)}, \hat{z}_k^{(q)}] \), at time \( k \), as \( \Delta_k^{q} = \Delta_k^{q}(A^q_i, k) \Delta_k^{q}(D^q_i, D^q_i) \Delta_k^{q-1} + \hat{A}_k(q, D^q_i) \). Then, for each mode \( q \in Q \), the following inequality holds for \( k \in \mathbb{N} \):
\[
\Delta_k^{q} \leq A^q_i(D^q_i, D^q_i)A^q_i(D^q_i, D^q_i) + [I - D^q_i] + (I - D^q_i) \cdot [I - D^q_i] + (I - D^q_i) + \cdots + 2\kappa D^q_i r^q,
\]
where
\[
\begin{align*}
A^q_i(D^q_i, D^q_i) & \triangleq D^q_i(A^q_i \mid D^q_i \mid D^q_i) + [I - D^q_i] + (I - D^q_i) + (I - D^q_i) + \cdots + 2\kappa D^q_i r^q, \\
\hat{A}_k(q, D^q_i, D^q_i) & \triangleq \Delta_k^{q}(A^q_i, k) \Delta_k^{q}(D^q_i, D^q_i) + (I - D^q_i) + (I - D^q_i) + \cdots + 2\kappa D^q_i r^q,
\end{align*}
\]
while \( r^q \triangleq \text{rowsupp}(I - A^q_i A^q_i)^{-1} \), \( C^q_i \triangleq [C^q_i, C^q_i, C^q_i] \) from (5), \( \kappa \) is a very large positive real number (infinity) and \( \Delta_k^{q} \leq \Delta_k^{q} - \Delta_k^{q} \leq \Delta_k^{q} \leq \Delta_k^{q} - \Delta_k^{q} \leq \Delta_k^{q} \leq \Delta_k^{q} - \Delta_k^{q} \leq \Delta_k^{q} \). On the other hand, for \( q = q^* \). Comparing Lemma in a similar manner to [12, Theorem 2], through the following lemma.

**Lemma 2** (Stability). Consider the hidden mode switched system (6) along with the mode-matched observer in (8a)–(11b). Suppose that all the assumptions in Proposition [5] hold. Then, for each mode \( q \in Q \), the mode-matched observer in (8a)–(11b) is stable in the sense of Definition [2] if there exist \( D^q_i \in \mathbb{D}_{n+p} \), \( D^q_i \in \mathbb{D}_i \), \( D^q_i \in \mathbb{D}_n \) that satisfy \( D^q_i, D^q_i \in \mathbb{D}_{n+p} \times \mathbb{D}_i \times \mathbb{D}_n \nabla \{D^q_i, D^q_i, D^q_i\} \in \mathbb{D}_{n+p} \times \mathbb{D}_i \times \mathbb{D}_n \nabla \{D^q_i, D^q_i, D^q_i\} \) such that
\[
\mathcal{L}^q_i(D^q_i, D^q_i, D^q_i) \leq \mathcal{A}_q_i(D^q_i, D^q_i, D^q_i) \leq 1, \quad (20)
\]
with \( \mathcal{A}_q_i(D^q_i, D^q_i, D^q_i) \) and \( \mathcal{A}_q_i(D^q_i, D^q_i, D^q_i) \) defined in Proposition [5].

**Proof**. Our goal is to show that our specific choices for \( D^q_i, D^q_i, D^q_i \), make the right hand side of (19) finite in finite time. To do this, since \( k \) can be infinitely large, we choose \( D^q_i \in \mathbb{D}_{n+p} \) such that \( r^q = 0 \), i.e., \( D^q_i, D^q_i \in \mathbb{D}_n + \mathbb{D}_n \nabla \{D^q_i, D^q_i, D^q_i\} \), it is sufficient for uniform boundedness of \( \Delta_k^{q} \) such that the following system:
\[
\Delta_k^{q} = \mathcal{A}_q_i(D^q_i, D^q_i, D^q_i) \Delta_k^{q} + \hat{A}_k(q, D^q_i, D^q_i), \quad (21)
\]
is stable, where \( \hat{A}_k(q, D^q_i, D^q_i) \triangleq \Delta_k^{q}(D^q_i, D^q_i) + \mathcal{A}_q_i(D^q_i, D^q_i, D^q_i) \) is a bounded disturbance. This implies that the system is stable (in the sense of uniform stability of the interval sequences) if and only if the matrix \( \mathcal{A}_q_i(D^q_i, D^q_i, D^q_i) \) is stable.

We will provide a proof of [12, Theorem 2], with some minor modifications, by replacing the unknown mapping \( h \) with the unknown policy \( \mu \) and making all variable maps model-dependent.
(non-strictly) stable for at least one choice of \((D^q_1, D^q_2, D^q_3)\), and equivalently, (20) should hold.

Finally, the mode-matched interval widths are upper bounded and convergent to steady-state values, as follows.

**Proposition 6** (Upper Bounds of the Interval Widths and their Convergence). Consider the system (6) and the observer (19–11b) and suppose all the assumptions in Lemma 2 hold. Then, for each mode \(q \in Q\), the sequence of \(\{\Delta^q_k \triangleq D^q_k - D^q_{k-1}\}_{k=0}^{\infty}\) is uniformly upper bounded by a convergent sequence, as
\[
\Delta^q_k \leq \Delta^q_0 + \sum_{j=0}^{k-1} \Delta^q_j \xi^q, \quad k \to \infty,
\]
where \(\Delta^q_0 \triangleq \mathcal{A}_q(D^q_1, D^q_2, D^q_3) \triangleq A - q^\delta(D^q_1, D^q_2)\mathcal{L}^\epsilon_1(D^q_3), \Delta_q = \Delta^q_0(D^q_1, D^q_2) + A^q(D^q_1, D^q_2)\mathcal{L}^\epsilon_1(D^q_3)\) and \((D^q_1, D^q_2, D^q_3)\) is a solution of the following problem:
\[
\min_{D_1, D_2, D_3} \|e^{\mathcal{A}_q(D_1, D_2, D_3)}(\Delta^q_0(D_1, D_2) + A^q(D_1, D_2)\Delta^q_k(D_3))\|_{s.t. (D_1, D_2, D_3) \in \{D_1, D_2, D_3 \in \mathbb{D}^3 \mathcal{L}^\epsilon_1(D_1, D_2, D_3) < 1\}}.
\]

Consequently, the interval widths \(\{\Delta^q_k\}_{k=0}^{\infty}\) are uniformly upper bounded by a convergent sequence, i.e.,
\[
\|\Delta^q_k\| < \delta^q_0 \triangleq \|\Delta^q_0\| + \sum_{j=0}^{\infty} \frac{1}{\xi^q} \|\Delta^q_{j+1}\| \xi^q < \infty, \quad e^{\Delta^q_k} \xi^q,
\]

Proof. The proof is straightforward by applying [22, Lemma 1], computing (19) iteratively, using triangle inequality and the fact that by Theorem 2, \(\mathcal{A}_q(D^q_1, D^q_2, D^q_3)\) is a stable matrix and \((D^q_1, D^q_2, D^q_3)\) is a solution of (20).

---

**V. MODE-DETECTABILITY**

In addition to the nice properties regarding the correctness, stability and convergence of the mode-matched interval estimates of states and inputs, as discussed in the previous section, we now provide some sufficient conditions for the system dynamics and attack policies, which guarantee that regardless of the observations, after some large enough time steps, all the (false, i.e., not true) modes can be eliminated, when applying Algorithm 1. To do so, first, we define the concept of mode-detectability as well as some assumptions for deriving our sufficient conditions for mode-detectability.

**Definition 6** (Mode-Detectability). System (6) is called mode-detectable under Algorithm 1 if there exists a natural number \(K \in \mathbb{N}\), such that for all time steps \(k \geq K\), all false modes are eliminated.

**Assumption 4** (Destabilizing Attack Policy). For all \(q \in Q\), the vector fields \(f^d(x, \varphi^d(x), w)\) and \(\varphi^d(x)\) satisfy the following bounds on their Jacobians: \(v_j \in \{x, d\}, \forall v(x, w) \in \mathcal{X} \times \mathcal{W}, J^d_{v}(x, \varphi^d(x), w) \in \{J^d_{vf}, J^d_{vf}, J^{\epsilon_1}_v\}\) and \(J^{\epsilon_1}_v(x) \in \{J^{\epsilon_1}_{vf}, J^{\epsilon_1}_{vf}, J^{\epsilon_1}_{v}\}\), with known \(J^d_{vf} \triangleq J^d_v + J^\epsilon_1_v\), \(J^d_{vg} \triangleq J^d_g + J^{\epsilon_1}_g\), a priori, where \(J^d_{vf}, J^d_{vg}\) are Jacobians of \(f^d\), with respect to its first argument, x, and second argument, \(d, \epsilon, v\), respectively. Moreover, \(J^{\epsilon_1}_v \triangleq \frac{1}{2}(J^{\epsilon_1}_v + J^{\epsilon_1}_v + J^d_{vf} + J^d_{vg})\) is strictly Schur unstable, where \(J^{\epsilon_1}_v \triangleq J^{\epsilon_1}_v + J^{\epsilon_1}_v - J^d_v - J^d_v - J^d_v - J^d_v\).}

1A strictly Schur unstable matrix is a square matrix that has at least one eigenvalue with its real part being strictly outside the interval \([0, 1]\).

\[
J^{\epsilon_1}_v \triangleq J^{\epsilon_1}_v + J^{\epsilon_1}_v - J^d_v - J^d_v - J^d_v - J^d_v = J^d_{vf} - J^d_{vg} + J^{\epsilon_1}_{vf} - J^{\epsilon_1}_{vg}.
\]

**Corollary 2.** Assumption 4 implies that the attack policy \(\mu(\cdot) \equiv \mu^d(\cdot)\) destabilizes the system in (6), and hence, the true state trajectory of (6) becomes unbounded.

Proof. Defining \(f \triangleq f^d\), \(\mu(x) \equiv \mu^d(x)\) and \(f(x, w) \equiv f(x, \mu(x), w)\), as well as using chain rule, we have \(J^d_{vf} = J^d_{vf}(x, \mu(x), w) + J^d_{vf}(x, \mu(x), w)J^d_{vg}(x)\). Combining this and Assumption 2 as well as applying Proposition 1 returns \(J^d_{vf} = J^d_{vf}(x, \mu(x), w) + J^d_{vf}(x, \mu(x), w)J^d_{vg}(x)\). Note, now that since \(J^d_{vf}\) is strictly Schur unstable by Assumption 4 then the interval matrix \(J^d_{vf}\) is strictly Schur unstable by [23, Lemma 2b], and hence, the linearized form of the system in (6) is strictly Schur unstable. Consequently, the nonlinear system in (6) is unstable by the Chetaev instability theorem [24], i.e., the attack policy \(\mu\) is a destabilizing policy.

Now, we are ready to state our main result on mode-detectability, through the following theorem.

**Theorem 1** (Sufficient Conditions for Mode-Detectability). Suppose Assumption 2 and all the assumptions in Lemma 2 hold for all \(q \in Q\). Then, using Algorithm 7 System (6) is mode-detectable in the sense of Definition 6.

Proof. We need to show that there exists \(K \in \mathbb{N}\), such that (18) holds for all \(k \geq K, \forall q \neq q^* \in Q\), where \(q^*\) is the true mode. Given the definition of the residual signal in (17) and since \(q^*\) is unknown, a sufficient condition for (18) to hold is that \(\forall q_1 \neq q_2 \in Q, \exists K \in \mathbb{N}, \forall k \geq K, g^{\epsilon_2}(\xi^q_k) \notin [g^{q_1}, g^{q_2}]\), where \(\forall q \in Q, \xi^q_k \triangleq [x^k d^k T^k]^T\). Equivalently, there should exist a dimension \(i \in \mathbb{N}_i\), such that \(g^{\epsilon_2}(\xi^q_k) < g^{q_1}_{ik}\) or \(g^{\epsilon_2}(\xi^q_k) > g^{q_2}_{ik}\). Since \(q_1 \neq q_2\) can be any two arbitrary modes, then without loss of generality, we only consider the former inequality, for which to hold, a sufficient condition is that \(\mathcal{F}^{k}_{ik} < g^{q_1}_{ik}\), which by defining \(\Delta^q_{ki} \triangleq g^{q_1}_{ik} - g^{q_2}_{ik} \geq 0\) and \(\Delta^q_{ki} \triangleq g^{q_1}_{ik} - g^{q_2}_{ik} \geq 0\), is equivalent to \(g^{2}_{ik}(\xi^q_k) + \Delta^q_{ki} > g^{q_1}_{ik}(\xi^q_k) - \Delta^q_{ki}\), that can be rewritten as:
\[
\Delta^q_{ki} \leq g^{q_1}_{ik} - g^{q_2}_{ik}, \quad (22)
\]
Now, note that the left hand side of (22) can be verified to be bounded as follows: 0 \(\leq g^{q_1}_{ik} - g^{q_2}_{ik} \leq \Delta^q_{ki} + \Delta^q_{ki}\), where \(\forall q \in Q, \Delta^q_{ki} \triangleq g^{q_1}_{ik} - g^{q_2}_{ik}\) is bounded by the the Lipschitz-like property of the decomposition functions (cf. [12, Lemma 2]), as well as the stability of each of the mode-matched observers (cf. Lemma 2). Now that the left hand side of (22) is proven to be bounded, if we show that the right hand side grows unboundedly, then (22) must always hold after some sufficiently large time step \(k \in \mathbb{N}\). To do so, we consider some \(x_0 \in \mathcal{X}\) and apply the mean value theorem on both \(g^{q_1}_{ik}\) and \(g^{q_2}_{ik}\) to obtain \(g^{q_1}_{ik}(\xi^q_k) - g^{q_2}_{ik}(\xi^q_k) = (J^{q_1}_{g_{ik}}(\xi^q_k) -J^{q_2}_{g_{ik}}(\xi^q_k))(x_k - x_0)\), where \(\xi^q_k, x_k \in \mathcal{X} \times \mathbb{R}^q \times \mathbb{V}\) and for all \(q \in Q, J_{g_{ik}}(\xi^q_k)\) and \(J_{g_{ik}}(\xi^q_k)\) denote the Jacobian matrix of \(g^{q_1}\) and its \(i^{th}\) row, evaluated at \(\xi^q_k\). Finally, the right hand side of the above equality eventually becomes unbounded.
since the Jacobian matrix of $J^q$ is bounded for all $q \in \mathcal{Q}$ by Lipschitz continuity of $g^q$ (cf. Assumption 1), and $x_k$, i.e., the true state trajectory becomes unbounded by Assumption 4 and Corollary 2. Hence, the left hand side of the above equality also becomes eventually unbounded, which returns the desired result.

VI. SIMULATION EXAMPLE

In this section, we illustrate the effectiveness of our proposed observer using a power network with multiple control areas. Specifically, we consider a 3-area system as shown in Figure 1 where each control area consists of generator and load buses. In addition, there are transmission lines between areas. The nonlinear model of bus $i$ is adopted from [25]:

$$
\dot{\theta}_i(t) = f_i(t) + w_{1,i}(t),
$$

$$
\dot{f}_i(t) = -\frac{D_if_i(t) + \Sigma P_M(t)(P_M(t) + d_i(t)) + P_L(t)}{m_i} + w_{2,i}(t),
$$

with the output model: $y_{i,k} = [\theta_{i,k}, f_{i,k}]^\top + [0, 1]^\top d_i + v_{i,k}$, where $\theta_i$ is phase angle, $f_i$ is the angular frequency, $P_M(t)$ denotes the mechanical power (the control input) and $P_L(t)$ is a known power demand. In our simulations, both $P_M(t)$ and $P_L(t)$ are set to be identically zero and the process noise $w_{1,i}$ and measurement noise $v_{i,k}$ are bounded by $[-0.1, 0.1]$ and $[-0.1, 0.1]$, respectively. When the circuit breakers are not engaged (or attacked), the power flow $P_{il}$ between areas $i$ and $l$ is as follows:

$$
P_{il}(t) = -P_{li}(t) = L_il \sin(\theta_i(t) - \theta_j(t)).
$$

A malicious agent is assumed to have access to circuit breakers that control the tie-lines, and is thus able to sever the connection between control areas. Two types of attack are considered based on the topology of the tie-line interconnection graph: 1) a node/vertex/bus attack (disconnection of a control area from all others); or 2) a link/edge/line attack (disabling of a specific tie-line between two control areas), i.e., the power flow across the tie lines is altered, if (1) the reenabling of a specific tie-line between two control areas) or (2) a link/edge/line attack graph: (1) a node/vertex/bus attack (disconnection of a bus between areas $q$ and $r$); or (2) a link/edge/line attack.

For the radial tie-line interconnection topology in Figure 1, the circuit breaker attacks result in $Q = 5$ possible modes of operation: all switches are safe ($q = 1$), only circuit breaker $i$ is attacked ($q = i + 1$, $i = 1, 2, 3$) and two or more circuit breakers are attacked ($q = 5$). Further, we denote the value of the variables at sampling time $t_k$ by adding subscript $k$, e.g., $f_i(t_k) = f_{i,k}$ and apply the Euler method to discretize the system: $\theta_{i,k+1} = \theta_{i,k} + f_{i,k}dt$, $f_{i,k+1} = f_{i,k} + f_{i,k}dt$, where the sampling time $dt$ is 0.01 s in our example. Moreover, we choose $d_i(t) = \theta_i(t)\sin(\theta_i(t))$ as to-be-learned unknown nonlinear attack policy and assume that we have 400 initial data points for each unknown attack policy $d_i$.

Due to space limitations, we only show the results for the case when the true operation mode is assumed to be $q_\text{true} = 1$ and provide figures for selected states and attack signals in Figure 2 and 3. Moreover, we compare our results with our previously developed simultaneous mode, state and unknown input (SMSP) observer in [13], where no unknown policy/feedback law was assumed to govern the attack signals, and hence, no learning step were included in the proposed observer design. As can be observed from Figures 2 and 3, the SMSP observer (proposed in this paper) returns tighter interval estimates than SMSI for both states and attack signals, when compared to SMSI. It is also worth mentioning that all the state interval widths converge to steady-state values by using SMSP with “learned” model for the attack policy, while the interval widths for some states do not converge when applying SMSI (not depicted for brevity), which highlights the effectiveness of the learning step.

Moreover, we compare the upper and lower learned model abstractions for $k = 0$ and $k = 1500$ in Figure 4 which showed tighter over-approximations with increasing number of data points. Further, as can be seen in Figure 5 all modes, except the true mode $q^\text{true} = 1$, are eliminated within 1500 time steps. Finally, the actual state and input estimation error sequence, as shown in Figure 6, is upper bounded by the interval widths and converges to steady-state values.

VII. CONCLUSION

This paper addresses the problem of designing interval observers for hidden mode switched nonlinear systems with bounded noise signals, that are compromised by false data injection and switching attacks. An interval observer with three constituents was proposed: i) a bank of mode-matched observers, where each of them simultaneously outputs the corresponding mode-matched state, mode and unknown input/attack estimates, as well as computes upper and lower abstractions/over-approximations for the attack policies, ii) a mode estimator that rules out the incorrect modes based on a residual-based set-membership criterion, and iii) a global fusion observer that returns the union of compatible state and
M. Milanese and C. Novara. Set membership identification of nonlinear systems. *Automatica*, 40:957–975, 2004.

Fig. 3: The true values of the attacks: $d_1, d_2$, and their upper and lower framers returned by the SMSP approach: $d_1^{\text{SMSP}}, d_2^{\text{SMSP}}, d_1^{\text{SMSI}}, d_2^{\text{SMSI}}$, as well as the SMSI approach: $d_1^{\text{SMSI}}, d_2^{\text{SMSI}}$.

Fig. 4: The true attack policy $\mu(\cdot)$, and its learned abstraction model $\{\hat{\mu}(\cdot), \bar{\mu}(\cdot)\}$ at time steps $k = 0$ and $k = 1500$.

attack estimates, as well as learned abstractions of the attack policy/state feedback law. Moreover, sufficient conditions for mode-detectability, i.e., for guaranteeing that all false modes will be eliminated after sufficiently large finite time steps, were provided. Finally, the effectiveness and performance of our proposed approach was demonstrated using a 3-area power network.

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