Super-Matrix KdV and
Super-Generalized NS Equations
from
Self-Dual Yang-Mills Systems with Supergauge Groups

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Abstract

Super-matrix KdV and super-generalized non-linear Schrödinger equations are shown to arise from a symmetry reduction of ordinary self-dual Yang-Mills equations with supergauge groups. PACS: 11.15.-q, 11.30.Pb
1 Introduction

It has been known for some time that many completely integrable systems in 1 + 1 dimensions can be obtained by symmetry reduction of self-dual Yang-Mills (SDYM) field equations in the four-dimensional Euclidean space $\mathbb{E}^4$ or the (2, 2) pseudo-Euclidean space $\mathbb{E}^{(2,2)}$, which corresponds to $\mathbb{R}^4$ endowed with the diagonal metric: $\text{diag}(+1, +1, -1, -1)$ [1, 2]. Super-extensions of known integrable systems in 1+1 dimensions have also been obtained from supersymmetric SDYM field equations by means of symmetry reduction [3, 4].

Some insight into this relationship can be achieved by examining the linear system associated with the non-linear SDYM equations. Both the SDYM and its associated linear system are invariant under conformal as well as gauge transformations. Therefore, the integrability conditions of the linear system reduced by a subgroup of the conformal group coincide with the similarly reduced SDYM field equations. This fact makes the linear system invaluable in the study of the reduction of the SDYM field equations to known integrable systems by means of symmetry. Indeed, to show that the SDYM field equations give rise to a known integrable system, one only needs to find the appropriate reduction symmetry, gauge group, and linear system. Accordingly, a linear system for an integrable system can be revealed if the reduction steps from the SDYM equations to the integrable system are applied to the SDYM linear system.

Linear systems known as Lax pairs have also been useful in the study of (super-) integrable systems in 1 + 1 dimensions and their associated hierarchies [1]. Lax pairs constructed from general techniques (e.g. refs. [5, 6, 7]) can
sometimes be utilized to help find the symmetry reductions of SDYM equations which lead to known (super-) integrable systems. In fact, ref. [8] showed that a super-generalized non-linear Schrödinger (super-GNS) equations can be obtained from symmetry reduction of SDYM systems with gauge group \( SL(2/1) \) using the methods of ref. [6].

However, in general the Lax pairs may involve a (spectral) parameter, \( \lambda \), of order \( O(\lambda^3) \) or higher. This will preclude application of these Lax pairs to SDYM systems reduced by translation because the latter permit a spectral parameter of order at most \( O(\lambda^2) \). For instance, ref. [7] gives a Lax pair for the super-KdV equation, but it contains a spectral parameter of order \( O(\lambda^3) \).

There is an alternate Lax pair formulation for the ordinary (matrix) KdV equation with spectral parameter of order \( O(\lambda^2) \) [9, 10, 11]. This suggests that it might be possible to find an order \( O(\lambda^2) \) linear system for super-(matrix) KdV as well. It is indeed possible, and in subsection 3.1 we will exhibit linear systems with a spectral parameter \( O(\lambda^2) \) which give rise to super-KdV and super-matrix KdV equations.

Consequently, the super-(matrix) KdV as well as the super-GNS equations can be obtained from a symmetry reduction of SDYM systems with a super-gauge group. This could be seen as a supersymmetric extension of the SDYM reductions to the KdV and NS equations presented in [12]. It is noteworthy that after symmetry reduction the ordinary SDYM equations acquire a supersymmetry associated with spatial translations.

A theory-based procedure for finding linear systems associated with the
super-matrix KdV is not shown here. Our results were obtained by considering
the most general possible supergauge fields along with some guidance from the
non-supersymmetric case.

2 SDYM Linear System

We begin with a principal bundle $P(B, \mathcal{G})$. The base space $B$ is $\mathbb{R}^{(2,2)}$, and the
structure group $\mathcal{G}$ is a supergroup of even dimension $m$ and odd dimension $n$. The gauge potential $A_\mu$, where $\mu \in \{1, \ldots, 4\}$, is Lie superalgebra (cf: [13, 14])
valued:

$$A_\mu(x) = A_\mu^a(x)M_a + \xi_\alpha^\mu(x)N_\alpha. \quad (2.1)$$

Here $x \in B$, and $\{M_a, a \in \{1, \ldots, m\}\}$ represent the even and $\{N_\alpha, \alpha \in \{1, \ldots, n\}\}$ the odd basis elements of the Lie superalgebra. We will restrict
our attention to gauge potentials which are pure, i.e., either anti-commuting
($a$-type) or commuting ($c$-type).

The SDYM field equations, $F = *F$, are a set of nonlinear partial differential
equations in terms of the gauge fields. Here $F = d\omega + \omega \wedge \omega$ is the curvature 2-
form associated with the 1-form $\omega = A_\mu \theta^\mu$ where $\{\theta^\mu\}$ is a basis of $T^*B$ and $*$ is
the Hodge operator or duality transformation associated with the $(2, 2)$ pseudo-
Euclidean metric. It is recalled that the self-dual field equations considered
involve gauge fields with both bosonic and fermionic variables coupled by the
field equations.

There exists a linear system of partial differential equations whose integra-
bility condition reproduces the non-linear SDYM equations. The linear system can be written

\[ \begin{align*}
(D_1 + iD_2 + \lambda(D_3 - iD_4))\Psi(x, \lambda, \bar{\lambda}) &= 0 \\
(D_3 + iD_4 + \lambda(D_1 - iD_2))\Psi(x, \lambda, \bar{\lambda}) &= 0
\end{align*} \] (2.2)

where \( D_\mu := \partial_\mu + A_\mu(x), \lambda \in \mathbb{H}^2\)-sheet, and \( \partial_{\bar{\lambda}}\Psi = 0 \). The nature of \( \Psi \), which can be inferred from the twistor construction of the linear system \[15\], has no direct bearing on the developments in this letter. In other words, for our purposes the specific nature of \( \Psi \), which could be a multiplet of scalar fields with values in the adjoint representation of the Lie (super)algebra, does not matter and (2.2) can be viewed as operator equations.

Following \[10, 11\], we introduce null coordinates on \( \mathbb{E}^{2,2} \) defined by

\[ \begin{align*}
t &= 2^{-\frac{1}{2}}(x_2 - x_4) \\
y &= 2^{-\frac{1}{2}}(x_1 - x_3) \\
u &= 2^{-\frac{1}{2}}(x_2 + x_4) \\
z &= 2^{-\frac{1}{2}}(x_1 + x_3).
\end{align*} \] (2.3)

In these coordinates, the covariant derivatives become

\[ \begin{align*}
D_t &= 2^{-\frac{1}{2}}(D_2 - D_4) \\
D_y &= 2^{-\frac{1}{2}}(D_1 - D_3) \\
D_u &= 2^{-\frac{1}{2}}(D_2 + D_4) \\
D_z &= 2^{-\frac{1}{2}}(D_1 + D_3).
\end{align*} \] (2.4)

Consequently, the linear system (2.2) on \( B = \mathbb{E}^{(2,2)} \) decouples:

\[ \begin{align*}
(D_z + \omega D_u)\Psi(x, \omega, \bar{\omega}) &= 0 \\
(D_t - \omega D_y)\Psi(x, \omega, \bar{\omega}) &= 0
\end{align*} \] (2.5)

where \( \omega := \frac{i(1-\lambda)}{1+\lambda} \). The holomorphy condition on \( \Psi \) becomes simply: \( \partial_{\bar{\omega}}\Psi = 0 \).
3 Reduced Linear System

The SDYM field equations and their associated linear system are invariant under both conformal transformations on the base space $B$ as well as supergauge transformations. The conformal invariance can be used to reduce the SDYM and its associated linear system with respect to any subgroup, $G$, of the conformal group. Loosely speaking, it entails rewriting (2.5) in terms of orbit and invariant coordinates of the $G$-action and inserting $G$-invariant gauge fields and a $G$-invariant $\Psi$.

We are particularly interested in the subgroup $\tilde{G} = \{P_u, P_y - P_z\}$, where $P_X$ denotes the generator of translations along the $X$ coordinates, e.g. $P_X = \frac{\partial}{\partial X}$ as a (vector field) representation. Reducing (2.5) by $\tilde{G}$ leads to the $O(\omega^2)$ linear system \[\begin{align*}
(\partial_t + A_t(t,x) + \omega[A_z(t,x) - A_y(t,x)]) + \omega^2 A_u(t,x)\Psi &= 0 \\
(\partial_x + A_z(t,x) + \omega A_u(t,x))\Psi &= 0
\end{align*}\]

where $x := (y + z)$. The integrability condition of (3.1) yields

\[
\begin{align*}
\partial_x A_u + [A_y, A_u] &= 0 & O(\omega^2) \\
\partial_t A_u - \partial_x (A_z - A_y) + [A_z, A_y] + [A_t, A_u] &= 0 & O(\omega^1) \\
\partial_t A_z - \partial_x A_t + [A_t, A_z] &= 0 & O(\omega^0).
\end{align*}
\]

Equations (3.2) coincide with the similarly reduced SDYM field equations.

\(^1\)One may consult refs. \[16, 17, 18, 19, 20, 21, 22, 23\] for a complete description of the general symmetry reduction method including details concerning general invariance conditions on gauge fields and the holomorphic nature of $\Psi$. 

5
3.1 Example: Super-Matrix KdV Equations

The compatibility condition (3.2) for the reduced linear system can be used to exhibit a super-extension of the matrix KdV equation. We consider a SDYM theory with supergauge group \( \mathcal{G} = GL(2n/n) \) and choose the following gauge fields:

\[
A_u(t, x) = \begin{pmatrix}
0_n & 0_n & 0_n \\
-1_n & 0_n & 0_n \\
0_n & 0_n & 0_n 
\end{pmatrix}, \quad A_z(t, x) = \begin{pmatrix}
0_n & 0_n & 0_n \\
u_n(t, x) & 0_n & 0_n \\
0_n & 0_n & \theta \varphi_n(t, x) 
\end{pmatrix},
\]

(3.3)

\[
A_{z-y}(t, x) = \begin{pmatrix}
0_n & 0_n & 0_n \\
0_n & 0_n & \theta 1_n \\
\theta 1_n & 0_n & 0_n 
\end{pmatrix}, \quad A_t(t, x) = \begin{pmatrix}
0_n & 0_n & 0_n \\
a_{21}(t, x) & 0_n & 0_n \\
0_n & 0_n & \theta a_{33}(t, x) 
\end{pmatrix}.
\]

Substituting (3.3) into (3.2) yields super-matrix KdV equations:

\[
u_n,t = \nu_{n,xxx} + 3(\nu_n \nu_n, x + \nu_n, x \nu_n) - 3(\varphi_n, x \varphi_n, x + \varphi_n \varphi_n, xx) \quad (3.4)
\]

\[\varphi_n,t = \varphi_{n,xxx} + 3(\varphi_n \nu_n, x + \varphi_n, x \nu_n).
\]

These super-matrix KdV equations reduce to the ordinary matrix KdV equation [24] when \( \varphi_n = 0 \), and they are invariant under the supersymmetry transformations:

\[
\delta _{\varepsilon } \nu_n = \varepsilon \varphi_{n,x}, \quad \delta _{\varepsilon } \varphi_n = \varepsilon \nu_n. \quad (3.5)
\]
Moreover, when \( n = 1 \), (3.4) reduces to the super-KdV equations [25]:

\[
\begin{align*}
    u_t &= u_{xxx} + 6uu_x - 3\varphi\varphi_{xx} \\
    \varphi_t &= \varphi_{xxx} + 3(u_x\varphi + u\varphi_x).
\end{align*}
\]

Linear systems for the super-matrix KdV equations can also be constructed for the gauge groups \( SL(2n/n) \) and \( SL(n/n) \). Noticing that the right-hand sides of (3.4) are total partial derivatives, the construction is rather simple. For \( SL(2n/n) \) we have

\[
A_u(t, x) = \begin{pmatrix} 0_n & 0_n & 0_n \\ -1_n & 0_n & 0_n \\ 0_n & 0_n & 0_n \end{pmatrix}, \quad A_z(t, x) = \begin{pmatrix} 0_n & 0_n & 0_n \\ u_n(t, x) + \theta\varphi_n(t, x) & 0_n & 0_n \\ 0_n & 0_n & 0_n \end{pmatrix},
\]

\[
A_{z-y}(t, x) = \begin{pmatrix} 0_n & 0_n & 0_n \\ 0_n & 0_n & 0_n \\ 0_n & 0_n & 0_n \end{pmatrix}, \quad A_t(t, x) = \begin{pmatrix} 0_n & 0_n & 0_n \\ a_{21}(t, x) + \theta a_{33}(t, x) & 0_n & 0_n \\ 0_n & 0_n & 0_n \end{pmatrix}.
\]

The case \( SL(n/n) \) is obtained from (3.7) by deleting the third rows and columns, and adjusting \( A_z \) and \( A_t \) by moving the odd parameter \( \theta \) over to the even components. When \( n = 1 \), one can delete the first rows and second columns to obtain a linear system for the super-KdV equations.

### 3.2 Example: Super-Generalized NS Equations

The supersymmetric version of the generalized non-linear Schrödinger equations related to the symmetric space \( SO(3)/SO(2) \) presented in refs. [6, 26] can also
be obtained by the same reduction with gauge group $SL(2/1)$. Let us recall the invariant gauge field components of the nonlinear system (3.1) [8]:

\[
A_u(t, x) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},
\]

\[
A_z(t, x) = A_{z-y}(t, x) = \begin{pmatrix} 0 & \Phi & 0 \\ -D\xi + (\partial_x^{-1}\Phi\xi)\xi & 0 & \xi \\ 0 & D\Phi + (\partial_x^{-1}\Phi\xi)\Phi & 0 \end{pmatrix},
\]

\[
A_t(t, x) = -\partial_x^{-1}([A_z, R(A_z)]) + R(A_z),
\]

where $R = ad_{A_z}(\partial_x - ad_{A_z} \partial_x^{-1} ad_{A_z})$, the bosonic field $\Phi = \phi_1 + \psi_1 \theta$, the fermionic variable $\xi = \psi_2 + \phi_2 \theta$, and $D = \partial_0 + \theta \partial_x$.

The corresponding reduced SDYM equations can be written as [6, 26]:

\[
\phi_{1,t} - \phi_{1,xx} + 2\phi_1 \psi_2 \psi_1 - 2\phi_1^2 \phi_2 = 0,
\]

\[
\psi_{1,t} - \psi_{1,xx} - 2\phi_1 \phi_{1,xx} \psi_2 - 2\phi_1^2 \psi_{2,xx} - 2\phi_1 \phi_2 \psi_1 = 0,
\]

\[
\psi_{2,t} + \psi_{2,xx} + 2\psi_2 \phi_1 \phi_2 = 0,
\]

\[
\phi_{2,t} + \phi_{2,xx} - 2\psi_2 \psi_1 \phi_2 + 2\phi_1 \phi_2^2 - 2\phi_1 \psi_2 \psi_{2,xx} = 0,
\]

They are left invariant under the supersymmetry transformations [6, 26]:

\[
\delta \epsilon \phi_1 = \epsilon \psi_1, \; \delta \epsilon \phi_2 = \epsilon \psi_2, \; \delta \epsilon \psi_1 = \epsilon \phi_{1,xx}, \; \delta \epsilon \psi_2 = \epsilon \phi_2.
\]
4 Summary

In this letter, the SDYM equations with $GL(m/n)$ supergauge group reduced under null and space translations have allowed to recover, with the addition of specific constraints on the gauge field components, a supersymmetric version of the matrix KdV equations. An outline of a similar SDYM reduction to a supersymmetric formulation of the GNS equations has been given for the gauge group $SL(2/1)$. A possible future research direction would be to investigate the reduction, by different subgroups of the conformal group acting on either $E^4$ or $E^{(2,2)}$, of the SDYM equations and their linear system in order to derive known (super-) integrable systems or new supersymmetric versions of integrable equations. Since the reduced systems obtained via the above procedure are not necessarily supersymmetric, a method allowing the determination of (all) the supersymmetries of the reduced equations would be useful. In view of recent results on cohomological quantum field theories (see for example : refs. [27, 28, 29]), a similar approach (symmetry reduction) could be applied to higher-dimensional (or generalized) self-dual Yang-Mills equations [30, 31, 32, 33, 34, 35, 36].

5 Note Added in Proof

After submission of this work, ref. [37], which uses a different formulation to obtain $N = 2$ supersymmetric matrix $(1, s)$-KdV hierarchies, was found on the hep-th archives.
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References

[1] M.J. Ablowitz and P.A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, (Cambridge University Press, Cambridge, 1991), section 6.5 and references therein.

[2] R.S. Ward, in: Twistors in Mathematics and Physics, eds. T.N. Bailey and R.J. Baston, London Mathematical Society Lecture Note Ser. 156, (Cambridge University press, Cambridge, 1990), p. 246.

[3] S.J. Gates Jr. and Nishino H., Phys. Lett. B299 (1993), 255.

[4] C.R. Gilson, R. Martin, A. Restuccia, and J.G. Taylor, Commun. Math. Phys. 107 (1986), 377.

[5] V.G. Drinfel’d and V.V. Sokolov, J. Soviet Math. 30 (1985), 1975.

[6] H. Aratyn, A. Das, and C. Rasinariu, Mod. Phys. Lett. A12 (1997), 2623.

[7] C. Morosi and L. Pizzocchero, Commun. Math. Phys. 176 (1996), 353.
[8] M. Legaré, Reduction of self-dual Yang-Mills systems and super nonlinear Schrödinger equations, to appear in the Proceedings of the “Supersymmetry and Integrable Models Workshop”, Chicago, June 12-14, 1997.

[9] I. Bakas and D.A. Depireux, Mod. Phys. Lett. A6 (1991), 399.

[10] T.A. Ivanova and A.D. Popov, Phys. Lett. A170 (1992), 293.

[11] T.A. Ivanova and A.D. Popov, Theor. Math. Phys. 102 (1995), 280.

[12] L.J. Mason and G.A.J. Sparling, Phys. Lett. A137 (1989), 29.

[13] V.G. Kac, Commun. Math. Phys. 53 (1977), 31.

[14] J. Cornwell, Group Theory in Physics V. III, (Academic Press, N.Y., 1989).

[15] R.S. Ward and R.O. Wells Jr., Twistor Geometry and Field Theory, (Cambridge University Press, Cambridge, 1990).

[16] P. Forgács and N.S. Manton, Commun. Math. Phys. 72 (1980), 15.

[17] J. Harnad, S. Shnider, and L. Vinet, J. Math. Phys. 21 (1980), 2719.

[18] R. Jackiw and N.S. Manton, Ann. Phys. 127 (1980), 257.

[19] V. Hussin, J. Negro and M.A. del Olmo, Ann. Phys. 231 (1994), 211.

[20] P.J. Olver, Applications of Lie Groups to Differential Equations in Physics, (Springer-Verlag, N.Y., 1986).

[21] P. Winternitz, in: Partially Integrable evolution Equations in Physics, eds. R. Conte and N. Boccara, NATO ASI Ser. 310 (Kluwer Academic Publ., Dordrecht), 515.
[22] M. Legaré, Int. J. Mod. Phys. A12 (1997), 219.

[23] M. Legaré and A.D. Popov, Phys. Lett. A198 (1995), 195.

[24] M. Wadati and T. Kamijo, Progr. Theor. Phys. 52 (1974), 397.

[25] P. Mathieu, J. Math. Phys. 29 (1988), 2499.

[26] H. Aratyn and C. Rasinariu, Phys. Lett. B391 (1997), 99.

[27] L. Baulieu, H. Kanno, and I.M. Singer, Special Quantum Field Theories in Eight and Other Dimensions, preprint hep-th/9704167.

[28] M. Blau and G. Thompson, Phys. Lett. B415 (1997), 242.

[29] B.S. Acharya, J.M. Figueroa-O’Farrill, B. Spence and M. O’Loughlin, Euclidean D-branes and higher-dimensional gauge theory, preprint hep-th/9707118.

[30] E. Corrigan, C. Devchand, D.B. Fairlie, and J. Nuyts, Nucl. Phys. B214 (1983), 452.

[31] R.S. Ward, Nucl. Phys. B236 (1984), 381.

[32] T.A. Ivanova and A.D. Popov, Lett. Math. Phys. 24 (1992), 85.

[33] T.A. Ivanova, Phys. Lett. B315 (1993), 277.

[34] T.A. Ivanova and A.D. Popov, Theor. Math. Phys. 94 (1993), 225.

[35] B.S. Acharya and M. O’Loughlin, Phys. Rev. D55 (1997), 4521.
[36] L. Baulieu and C. Laroche, On Generalized Self-Duality Equations Towards Supersymmetric Quantum Field Theories of Forms, preprint hep-th/9801014.

[37] S. Krivonos and A. Sorin, Extended $N = 2$ supersymmetric matrix $(1, s)$-KdV hierarchies, preprint solv-int/9712002.