Stability of the phase separation state for compressible Navier-Stokes/Allen-Cahn system

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Abstract

This paper is concerned with the large time behavior of the Cauchy problem for Navier-Stokes/Allen-Cahn system describing the interface motion of immiscible two-phase flow in 3-D. The existence and uniqueness of global solutions and the stability of the phase separation state is proved under the small initial perturbations. Moreover, the optimal time decay rates are obtained for higher-order spatial derivatives of density, velocity and phase. Our results implies that if the immiscible two-phase flow is initially located near the phase separation state, then under small perturbation conditions, the solution exists globally and decays algebraically to the complete separation state of the two-phase flow, that is, there will be no interface fracture, vacuum, shock wave, mass concentration at any time, and the interface thickness tends to zero as the time $t \to + \infty$.

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1 Introduction

Two-phase flows or multi-phase flows are important in many industrial applications, for instance, in aerospace, chemical engineering, micro-technology and so on. They have attracted studies from many engineers, geophysicists and astrophysicists. In this paper, we study a diffusive interface model which is coupled with the compressible barotropic Navier-Stokes equations and Allen-Cahn equation, called the compressible barotropic Navier-Stokes/Allen-Cahn system as follows (see [16]):

\[
\begin{align*}
\rho_t + \text{div}(\rho \mathbf{u}) &= 0, \\
(\rho \mathbf{u})_t + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) &= \text{div} \mathbf{T}, \\
(\rho \phi)_t + \text{div}(\rho \mathbf{u} \phi) &= -\mu, \\
\rho \mu &= \frac{\partial f}{\partial \phi} - \text{div} \left( \rho \frac{\partial f}{\partial \nabla \phi} \right),
\end{align*}
\]

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where $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ represents the spatial variable, $t > 0$ represents the time variable, $\text{div}$ and $\nabla$ are the divergence operator and gradient operator respectively. $\rho = \rho(\mathbf{x}, t)$, $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) = (u_1, u_2, u_3)(\mathbf{x}, t)$ and $\phi = \phi(\mathbf{x}, t)$ denote the density, the velocity and the concentration difference of the mixture fluids respectively, and while $\mu$ is the chemical potential, $f$ is the phase-phase interfacial free energy density, here we consider its common form as following (see Lowengrub-Truskinovsky [23], Heida-Málek-Rajagopal [16])

$$f(\rho, \phi, \nabla \phi) = \frac{1}{4\epsilon} \left( \phi^2 - 1 \right)^2 + \frac{\epsilon}{2\rho} |\nabla \phi|^2, \quad (1.2)$$

where $\epsilon > 0$ is the thickness of the interface between the phases. The Cauchy stress-tensor $\mathbb{T}$ is represented by

$$\mathbb{T} = \nu \left( \nabla \mathbf{u} + \nabla^\top \mathbf{u} \right) + \lambda \text{div} \mathbb{I} - \rho \nabla \phi \otimes \frac{\partial f}{\partial \nabla \phi}, \quad (1.3)$$

where $\mathbb{I}$ is the unit matrix, $\nabla$ represents the transpose of a matrix, and $\nu, \lambda$ are viscosity coefficients satisfying

$$\nu > 0, \quad \lambda + \frac{2}{3} \nu \geq 0. \quad (1.4)$$

The total pressure is

$$P = p(\rho) - \frac{\epsilon}{2} |\nabla \phi|^2, \quad (1.5)$$

which implies the sum of fluid pressure and capillary pressure. In this paper, we assume that the pressure $p$ in (1.5) is the smooth function of $\rho$, and it holds that

$$p'(\rho) > 0 \quad \text{for any} \quad \rho > 0. \quad (1.6)$$

Substituting (1.2), (1.3) and (1.5) into (1.1), then (1.1) is simplified as

$$\begin{cases} 
\rho_t + \text{div}(\rho \mathbf{u}) = 0, \\
\rho \mathbf{u}_t + \rho (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p(\rho) = \nu \Delta \mathbf{u} + (\nu + \lambda) \nabla \text{div} \mathbf{u} \\
- \epsilon \text{div} (\nabla \phi \otimes \nabla \phi - \frac{1}{2} |\nabla \phi|^2 \mathbb{I}), \\
\rho \phi_t + \rho \mathbf{u} \cdot \nabla \phi = -\mu, \\
\rho \mu = \frac{\rho}{\epsilon} \left( \phi^3 - \phi \right) - \epsilon \Delta \phi. 
\end{cases} \quad (1.7)$$

**Remark 1.1** The phase function $\phi$ is introduced to identify the two fluids ($\{\mathbf{x} : \phi(\mathbf{x}, t) = 1\}$ is occupied by fluid 1 and $\{\mathbf{x} : \phi(\mathbf{x}, t) = -1\}$ by fluid 2). Physically, the function $\phi$ and the total density are constructed as follows: for compressible immiscible two-phase flow, Take any volume element $V$ in the flow, $M_i$ is assumed to be the mass of the components in the representative material volume $V$, $\phi_i = \frac{M_i}{\rho}$ the mass concentration, $\rho_i = \frac{M_i}{\rho}$ the apparent mass density of the fluid $i$ ($i = 1, 2$). The total density is given by $\rho = \rho_1 + \rho_2$, and the difference of the two components for the fluid mixture $\phi = \phi_1 - \phi_2$. We also call $\phi$ the phase function or phase field. Obviously, $\phi$ describes the distribution of the interface.

During the past decade, the mathematical study for the Navier-Stokes/Allen-Cahn system has been extensively studied. For a diffuse interface model of two viscous fluids which
lead to the incompressible Navier-Stokes/Allen-Cahn system, a first result on existence of
axisymmetric solutions was obtained by Xu-Zhao-Liu [33]. In the case with matched density,
Zhao-Guo-Huang [35] studied the existence of weak solution in 3D, well-posedness of
strong solution in 2D and the vanishing viscosity limit. Also, we refer to [13, 14, 25] for the
asymptotic behavior and attractors. Recently, Favre-Schimperna [9] showed the existence
of weak solution in 3D and well-posedness of strong solution in 2D on an incompressible
model with inertial effects. For the incompressible model with different densities, Li-Ding-
Huang [22] established a blow-up criterion for strong solutions and Li-Huang [21] studied
the existence and uniqueness of local strong solutions in 3D case. Moreover, by using
an energetic variational approach, Jiang-Li-Liu [17] derived a different model of Navier-
Stokes/Allen-Cahn, then proved the existence of weak solutions in 3D, the well-posedness
of strong solutions in 2D, and studied the long time behavior of the strong solutions.

For the initial boundary value problem of the compressible barotropic model (1.7) in 3-D
bounded domain, Feireisl-Petzeltová-Rocca-Schimperna [10] developed a rigorous existence
theory based on the concept of weak solution for the compressible Navier-Stokes system
introduced by Lions for the adiabatic exponent of pressure \( \gamma > 6 \). This result was recently
extended to \( \gamma > 2 \) by Chen-Wen-Zhu [6]. Freistühr [11] showed that the possibility of
traveling waves corresponding to phase boundaries arises during a phase transition at a
critical temperature under natural assumptions. For the problem obtained by linearizing
the system (1.7) around a traveling waves, Kotschote [19] proved the results on local well-
posedness and a detailed description of the point and essential spectrum. For the Cauchy
problem of the 3-D compressible Navier-Stokes/Allen-Cahn system, Zhao [36] studied the
global well-posedness and time-decay rates of solutions via a refined pure energy method,
in which the equilibrium of the concentration difference \( \phi \) takes the value 0. For the 1-D
problem of the system (1.7) in bounded interval, Ding-Li-Luo [7] proved the existence and
uniqueness of global classical solution, the existence of weak solutions and the existence
of unique strong solution for initial data \( \rho_0 \) without vacuum states. Later, Chen-Guo [2]
established the global existence and uniqueness of strong and classical solutions by using the
energy estimates, when the initial vacuum is allowed. Recently, Ding-Li-Tang [8] obtained
the similar result as in [7] for the problem with free boundary. Also, Luo-Yin [24] and
Yin-Zhu [34] proved the asymptotic stability toward the rarefaction wave of Cauchy problem
in 1-D, and the combination of stationary solution and rarefaction wave for the initial
boundary value problem in half line, respectively.

On the other hand, for the full compressible model, Kotschote [18] derived the more
general model, and proved the existence and uniqueness of local strong solutions on a
problem with a mixed boundary condition in bounded domain. Recently, Chen-He-Huang-Shi
[3, 4] studied the global strong solutions for the 1-D Cauchy problem and initial boundary
value problem, respectively.

In this paper, we focus on the existence of the global smooth solutions and the stability
of the phase separation state for the Cauchy problem of the system (1.7) in \( \mathbb{R}^3 \) with the
initial condition

\[
(\rho, u, \phi)(x, 0) = (\rho_0, u_0, \phi_0)(x), \quad x \in \mathbb{R}^3.
\]

As the space variable tends to infinity, we assume

\[
\lim_{|x| \to \infty} (\rho_0, u_0)(x) = (\bar{\rho}, 0), \quad \text{and} \quad \lim_{|x| \to \infty} |\phi_0(x)| = 1,
\]

where \( \bar{\rho} > 0 \) is the given positive constant.
Remark 1.2 Compared with the results in [36] about the global well-posedness and time-decay rates of solutions near the equilibrium state of the concentration difference of the mixture fluids $\phi = 0$, we give the global solution and the similar time-decay rates near the equilibrium state of $|\phi| = 1$, which represents the state of phase separation. More precisely, the condition (1.3) can be used to describe immiscible two-phase flows in different phase fields when $x \to \pm \infty$, so it can be used to describe the phenomenon of the phase separation in immiscible two-phase flow. From this perspective, the condition (1.3) can be seen as an essential improvement on the condition (13) in [24] and (1.15) in [34] which can be only described the disturbance near single-phase flow.

Finally, we point out there are some mathematical results for the compressible Navier-Stokes-Cahn-Hilliard system which is another diffusive interface model describing the motion of a mixture of two compressible viscous fluids (see [1, 20, 5] and references therein).

**Notation.** In this paper, $L^p(\mathbb{R}^3)$ and $W^k_p(\mathbb{R}^3)$ denote the usual Lebesgue and Sobolev spaces on $\mathbb{R}^3$, with norms $\| \cdot \|_{L^p}$ and $\| \cdot \|_{W^k_p}$, respectively. When $p = 2$, we denote $W^k_p(\mathbb{R}^3)$ by $H^k(\mathbb{R}^3)$ with the norm $\| \cdot \|_{H^k}$ and $\| \cdot \|_{H^0} = \| \cdot \|$ will be used to denote the usual $L^2$-norm. The notation $\|(A_1, A_2, \cdots, A_l)\|_{H^k}$ means the summation of $\|A_i\|_{H^k}$ from $i = 1$ to $i = l$. For an integer $m$, the symbol $\nabla^m$ denotes the summation of all terms $D^\alpha$ with the multi-index $\alpha$ satisfying $|\alpha| = m$. We use $C, c$ to denote the constants which are independent of $x, t$ and may change from line to line. We also omit the spatial domain $\mathbb{R}^3$ in integrals for convenience. For $d \times d$-matrices $F, H$, denote $F : H = \sum_{i,j=1}^d F_{ij} H_{ij}$, $|F| \equiv (F : F)^{1/2}$. For vectors $a$ and $b$, we denote their tensor product by $a \otimes b := (a_i b_j)_{d \times d}$. We will employ the notation $a \lesssim b$ to mean that $a \leq C b$ for a universal constant $C > 0$ that only depends on the parameters coming from the problem.

In order to establish the negative Sobolev estimates, we should review the following useful results. To this end, let us first introduce the following necessary definition.

**Definition 1.1** For $s \in \mathbb{R}$, $H^s(\mathbb{R}^3)$ is defined as the homogeneous Sobolev space of $f$, with the following norm:

$$\|f\|_{H^s} = \|\Lambda^s f\|,$$

where $\Lambda^s$ is defined by

$$(\Lambda^s f)(x) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \xi^s \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi,$$

where $\hat{f}(\xi)$ is the Fourier transform of $f$, $\hat{f}(\xi) \overset{\text{def}}{=} \int_{\mathbb{R}^3} f(x) e^{-2\pi i x \cdot \xi} dx$.

Moreover, we need the following standard results which will be used extensively in our estimates.

**Lemma 1.1** (Gagliardo-Nirenberg inequality, [26] or [32, Lemma A.1]) Let $l, s$ and $k$ be any real numbers satisfying $0 \leq l, s < k$, and let $p, r, q \in [1, \infty]$ and $\frac{1}{k} \leq \theta \leq 1$ such that

$$\frac{l}{3} - \frac{1}{p} = \left(\frac{s}{3} - \frac{1}{r}\right)(1 - \theta) + \left(\frac{k}{3} - \frac{1}{q}\right) \theta.$$
Then, for any $u \in W^k_q(\mathbb{R}^3)$, we have
\[
\|\nabla^l u\|_{L^p} \lesssim \|\nabla^s u\|_{L^p}^{1-\theta} \|\nabla^k u\|_{L^p}^\theta.
\] (1.10)

**Lemma 1.2** ([28, Lemma 2.5]) Let $f(\varphi)$ and $f(\sigma, w)$ be smooth functions of $\varphi$ and $(\varphi, w)$, respectively, with bounded derivatives of any order, and $\|\varphi\|_{L^\infty(\mathbb{R}^3)} \leq 1$. Then for any integer $m \geq 1$, we have
\[
\|\nabla^m f(\varphi)\|_{L^p} \leq C\|\nabla^m \varphi\|_{L^p},
\]
\[
\|\nabla^m f(\varphi, w)\|_{L^p} \leq C\|\nabla^m(\varphi, w)\|_{L^p},
\] (1.11)
for any $1 \leq p \leq \infty$, where $C$ may depend on $f$ and $m$.

By the Parseval theorem and Hölder’s inequality, it is easy to check the following result (see [32]):

**Lemma 1.3** Let $s \geq 0$ and $l \geq 0$. Then, we have
\[
\|\nabla^l f\| \lesssim \|\nabla^{l+1} f\|_{H^{-s}}^{1-\theta} \|f\|_{H^s}^\theta, \quad \text{with} \quad \theta = \frac{1}{l+s+1}.
\] (1.12)

If $s \in (0, 3)$, $\Lambda^{-s}q$ is the Riesz potential. Then, we have the following $L^p$ type inequality by the Hardy-Littlewood-Sobolev theorem (see [30], pp. 119, Theorem 1):

**Lemma 1.4** Let $0 < s < 3, 1 < p < q < \infty$ and $\frac{1}{q} + \frac{s}{3} = \frac{1}{p}$. Then, we have
\[
\|\Lambda^{-s} f\|_{L^q} \lesssim \|f\|_{L^p}.
\] (1.13)

Now our main results are given as following:

**Theorem 1.1** Assume that (1.4), (1.6), and
\[
(\rho_0 - \bar{\rho}, u_0) \in H^3(\mathbb{R}^3), \quad \inf_{x \in \mathbb{R}^3} \rho_0(x) > 0,
\] (1.14)
\[
\nabla \phi_0 \in H^2(\mathbb{R}^3), \quad \phi_0^2 - 1 \in L^2(\mathbb{R}^3).
\] (1.15)

Then, there exists a positive constant $\delta > 0$ such that if
\[
\| (\rho_0 - \bar{\rho}, u_0) \|_{H^3} + \|\nabla \phi_0\|_{H^2} + \|\phi_0^2 - 1\| \leq \delta,
\] (1.16)
then the Cauchy problem (1.7)-(1.9) admits a unique solution $(\rho, u, \phi)$ on $[0, \infty)$ satisfying
\[
(\rho - \bar{\rho}, u) \in C([0, \infty), H^3(\mathbb{R}^3)), \quad \phi^2 - 1 \in C([0, \infty), L^2(\mathbb{R}^3)), \quad \nabla \phi \in C([0, \infty), H^2(\mathbb{R}^3)),
\]
\[
\nabla \rho \in L^2(0, \infty; H^2(\mathbb{R}^3)), \quad \nabla \phi \in L^2(0, \infty; H^3(\mathbb{R}^3)), \quad \nabla u \in L^2(0, \infty; H^3(\mathbb{R}^3)),
\] and
\[
\sup_{t \in \mathbb{R}_+} \left\{ \| (\rho_0 - \bar{\rho}, u(t)) \|_{H^3}^2 + \|\nabla \phi(t)\|_{H^2}^2 + \|\phi^2(t) - 1\|^2 \right\}
\]
\[
+ \int_0^{+\infty} \left( \|\nabla \rho\|_{H^2}^2 + \|\nabla u, \nabla \phi\|_{H^3}^2 \right) dt
\]
\[
\leq C \left( \|\rho_0 - \bar{\rho}\|_{H^3}^2 + \|u_0\|_{H^3}^2 + \|\nabla \phi_0\|_{H^2}^2 + \|\phi_0^2 - 1\|^2 \right).
\] (1.17)
where $C$ is the positive constant independent of $x, t$ and $\delta$.

Moreover, if $(\rho_0 - \bar{\rho}, u_0, \nabla \phi_0, \phi_0^2 - 1) \in \dot{H}^{-s}(\mathbb{R}^3)$ for some $s \in [0, \frac{3}{2})$, then

$$
\| (\rho - \bar{\rho}, u, \nabla \phi, \phi^2 - 1) (t) \|^2_{H^{-s}} \leq C_0,
$$

and

$$
\| \nabla^l (\rho - \bar{\rho}, u) (t) \|^2_{H^{-l-1}} + \| \nabla^l (\nabla \phi, \phi^2 - 1) (t) \|^2_{H^{-l+1-1}} \leq C_0 (1 + t)^{-l+s},
$$

for $l = 0, 1, 2$, where $\dot{H}^{-s}(\mathbb{R}^3)$ denotes the homogeneous negative Sobolev space.

If the initial value has a higher regularity, the above Theorem 1.1 can be generalized as follows.

**Corollary 1.1** assuming that $N \geq 3$, $(\rho_0 - \bar{\rho}, u_0) \in H^3(\mathbb{R}^3)$, $\nabla \phi_0 \in H^2(\mathbb{R}^3)$, $\phi_0^2 - 1 \in L^2(\mathbb{R}^3)$, and all the other assumptions in the Theorem 1.2 remain unchanged, then, the regularity of the solution which obtained in Theorem 1.2 can also be improved correspondingly, i.e.

$(\rho - \bar{\rho}, u) \in C([0, \infty), H^N(\mathbb{R}^3))$, $\phi^2 - 1 \in C([0, \infty), L^2(\mathbb{R}^3))$, $\nabla \phi \in C([0, \infty), H^{N-1}(\mathbb{R}^3))$,

$$
\nabla \rho \in L^2(0, \infty; H^{N-1}(\mathbb{R}^3))$, $\nabla \phi \in L^2(0, \infty; H^N(\mathbb{R}^3))$, $\nabla u \in L^2(0, \infty; H^N(\mathbb{R}^3)),$

and

$$
\| \nabla^l (\rho - \bar{\rho}, u) (t) \|^2_{H^{-l-1}} + \| \nabla^l (\nabla \phi, \phi^2 - 1) (t) \|^2_{H^{-l+1-1}} \leq C_0 (1 + t)^{-l+s}, \quad l = 0, \cdots, N - 1.
$$

By Lemma 1.3, we obtain that for $p \in (1, 2)$, $L^p(\mathbb{R}^3) \subset \dot{H}^{-s}(\mathbb{R}^3)$ with $s = 3 \left( \frac{1}{p} - \frac{1}{2} \right) \in [0, \frac{3}{2})$. Then by Theorem 1.1, we have the following corollary of the usual $L^p - L^2$ type of optimal decay results:

**Corollary 1.2** Under the assumptions of Corollary 1.1 except that we replace the $\dot{H}^{-s}$ assumption by that $(\rho_0 - \bar{\rho}, u_0, \nabla \phi_0, \phi_0^2 - 1) \in L^p(\mathbb{R}^3)$ for some $p \in (1, 2)$, then the following decay results hold:

$$
\| \nabla^l (\rho - \bar{\rho}, u) (t) \|^2_{H^{-l-1}} + \| \nabla^l (\nabla \phi, \phi^2 - 1) (t) \|^2_{H^{-l+1-1}} \lesssim (1 + t)^{-l-3\left( \frac{1}{p} - \frac{1}{2} \right)},
$$

for $l = 0, 1, \cdots, N - 1$.

The followings are several remarks for Theorem 1.1 and Corollary 1.1-1.2.

**Remark 1.3** 1) For the global existence of the solution, we only assume that the $H^3$-norm of initial data is small, while the higher-order Sobolev norms can be arbitrarily large, as in several works for compressible fluid flows (see [12, 31, 13]).

2) Notice that the decay rate (1.20) of global solution is optimal in the sense that it coincides with the decay rate of solutions to the linearized system

$$
\begin{align*}
\sigma_t + \bar{\rho} \text{div} u &= 0, \\
\mathbf{u}_t - \nu \rho^{-1} \Delta u - (\nu + \lambda) \bar{\rho}^{-1} \nabla \sigma + \frac{p'(\bar{\rho})}{\bar{\rho}} \nabla \sigma + \epsilon \nabla \phi \Delta \phi &= 0, \\
\phi_t - \frac{\epsilon}{\rho^2} \Delta \phi &= 0.
\end{align*}
$$

(cf. [12, Remark 1.2]).
Remark 1.4 1) Note that we obtain the $L^2$ optimal decay rate of the higher-order spatial derivatives of the solution in Corollary 1.2. Then the general optimal $L^q(p > 2)$ decay rates of the solution follow by the Gagliardo-Nirenberg inequality (see Lemma 1.1). For instance, it follows from (1.20) that

$$\|(\rho - \bar{\rho}, u, \nabla \phi, \phi^2 - 1)(t)\|^2_{L^q} \lesssim (1 + t)^{-3\left(\frac{1}{p} - \frac{1}{q}\right)}, \text{ for } 2 < q \leq \infty. $$

2) The constraint $s < \frac{3}{2}$ in Theorem 1.1 comes from applying Lemma 1.4 to estimate the nonlinear terms when doing the negative Sobolev estimates. This in turn restricts $p > 1$ in Corollary 1.2 by our method. Note that the nonlinear estimates would not work for $s \geq \frac{3}{2}$.

3) Using the Green function, a priori estimates and the Fourier splitting method developed by Schonb"{e}k [29], we can prove Corollary 1.2 with $p = 1$ by the same lines as in [12, Section 4].

Remark 1.5 By using the Sobolev embedding theorem, it can be obtained that the initial data $\inf_{x \in \mathbb{R}^3} \phi_0 > \frac{1}{\delta}$ when $\delta$ in (1.13) is small enough. From a physical point of view, we know that $\phi_0 = \pm \sqrt{2}$ are the critical points of the phase-phase interfacial free energy density $f$, $\inf_{x \in \mathbb{R}^3} \phi_0 > \frac{1}{\delta}$ means that the initial phase field is in the stable region, that is, in a state near phase separation. Theorem 1.1 implies that if the immiscible two-phase flow is initially located near the phase separation state, then under small perturbation conditions, the solution exists globally and decays algebraically to the complete separation state of the two-phase flow, that is, the interface thickness tends to zero as the time $t \to +\infty$.

As we know, for the Cauchy problem of the compressible fluid flows in $\mathbb{R}^3$, there exist several works to study the global well-posedness and algebraic decay estimates of smooth solutions: we refer to [15] for Navier-Stokes system, [32] for Navier-Stokes-Poisson system, [31] for Navier-Stokes-Kortweg system and [12] for nematic liquid crystal flows. To prove Theorem 1.1, we will use the energy method developed by [15], which relies essentially on the following two main steps:

Step 1. Energy estimates at $l$–th level:

$$\frac{d}{dt} \mathcal{E}_l(t) + \|\nabla^{l+1} \rho(t)\|_{H^{-l-1}}^2 + \|\nabla^{l+1} u(t)\|_{H^{-l-1}}^2 + \|\nabla^{l+1} \phi(t)\|_{H^{-l-1}}^2 \leq 0, \quad (1.21)$$

for any $0 \leq l \leq 3$, where

$$\mathcal{E}_l(t) \simeq \|\nabla^{l}(\rho - \bar{\rho}_0, u(t)\|_{H^{-l-1}}^2 + \|\nabla^{l+1} \phi(t)\|_{H^{-l-1}}^2 + \|\phi^2 - 1\|_{L^2}^2,$$

and $A \simeq B$ means $CA \leq B \leq \frac{1}{C}A$ for a generic constant $C > 0$.

Step 2. Negative Sobolev norm estimate:

$$\frac{d}{dt} \mathcal{E}_{-s}(t) \lesssim \left(\|\nabla \rho\|_{H^2}^2 + \|\nabla (u, \nabla \phi)\|_{H^3}^2 + \|\nabla (\phi^2 - 1)\|_{L^2}^2\right) \mathcal{E}_{-s}(t), \quad (1.22)$$

for $0 < s \leq \frac{1}{2}$, and

$$\frac{d}{dt} \mathcal{E}_{-s}(t) \lesssim \|\rho - \bar{\rho}_0, u, \phi^2 - 1, \nabla \phi\|_{L^2}^{\frac{s}{2}} \left(\|\nabla (\rho, u, \nabla \phi)\|_{H^3}^2 + \|\nabla (\phi^2 - 1)\|_{L^2}^2\right)^{\frac{s}{2} - s} \mathcal{E}_{-s}(t), \quad (1.23)$$
for $\frac{1}{2} < s < \frac{3}{2}$, where
\[ E_s = \| \Lambda^{-s}(\rho - \bar{\rho}) \|^2 + \| \Lambda^{-s} u \|^2 + \| \Lambda^{-s} \nabla \phi \|^2 + \| \Lambda^{-s}(\phi^2 - 1) \|^2. \]

If we prove (1.21), then it is easy to show that there exists a solution of the system (1.7)-(1.9) satisfying (1.17) by the continuation argument of local solution (Subsection 3.1). Moreover, by using (1.22) or (1.23), and Lemma 1.3 and the following estimate on differential inequality
\[ \frac{df(t)}{dt} + c_0(f(t))^{1+\frac{1}{p}} \leq 0 \Rightarrow f(t) \leq \left( f(0)^{-\frac{1}{1+s}} + \frac{c_0 t}{l+s} \right)^{-(l+s)} \lesssim (1 + t)^{-(l+s)}, \quad (1.24) \]
we can get the decay estimates (1.19) (Subsection 3.2). Therefore, the estimates (1.21)-(1.23) are essential in the proof of Theorem 1.1.

Here, we briefly review some difficulties and key analytical techniques in deriving (1.21)-(1.23), compared with previous works in [32, 31, 12]. The main difficulty comes from the Allen-Cahn equation (1.7), rewritten it as a second order nonlinear parabolic equation
\[ \phi_t - \frac{c}{\rho^2} \Delta \phi - \frac{1}{\epsilon \rho} (1 - \phi^2) \phi = -u \cdot \nabla \phi, \quad (1.25) \]
where the strong nonlinear term $- (1 - \phi^2) \phi$ makes a trouble for desired estimates because $\| \phi(t) \|_{L^\infty(\mathbb{R}^3)}$ is not small and $\phi \notin L^p(\mathbb{R}^3)$ for any $1 \leq p < \infty$ due to (1.9). Moreover, the antiderivative of this nonlinear term, that is the phase-phase interfacial free energy density, is not a convex function, this makes it difficult to obtain the upper bound estimate for $L^2([0, T])$ norm of $\| \nabla \phi(t) \|$. On the other hand, the coupling between the Navier-Stokes equations (1.7),(1.2) and the Allen-Cahn equation (1.25) also bring trouble to get the density estimation. In order to overcome these difficulties, we first find that $\int_0^T \| \nabla \phi(\tau) \|^2 d\tau$ can be controlled by the small perturbations for the initial energy. Further, we obtained the upper bound estimate of $\| \phi \|_{L^\infty(\mathbb{R}^3)}$ by using the method of energy estimates, (see Lemma 2.1). Based on these facts, we complete the estimate (2.18) for $\nabla \phi$ at $k$–th level, where a new $L^p$–estimate (1.11) for nonlinear functions is used essentially together with Gagliardo-Nirenberg inequality (1.10) (Lemma 2.2). By the same lines as in [32, 31, 12], we derive the estimate (2.25) and (2.30) for $(\rho - \bar{\rho}, u)$ at $k$–th level (Lemma 2.3 and Lemma 2.4). Combining Lemmas 2.1,2.4 we could obtain (1.21) (see Subsection 3.1). Next, multiplying (1.25) by $2\phi$ and applying $\Lambda^{-s}$ to the resulting equality (2.37), we could obtain the following type of inequality
\[ \frac{d}{dt} \| \Lambda^{-s} (\phi^2 - 1, \nabla \phi) \|^2 + \| \Lambda^{-s} \Delta \phi \|^2 + \| \Lambda^{-s} \nabla (\phi^2 - 1) \|^2 \lesssim \cdots \cdots, \]
which is a key point in completing the negative Sobolev norm estimates (1.22) and (1.23) (see 2.38 and Section 2).

2 The local existence and a series of energy estimates

This section is devoted to establish the local existence for the solutions of the Cauchy problem (1.7)-(1.9), and to derive the a series of energy estimates which play a key role
in the process of extending the local solution to the global solution. For any interval \( I \subset [0, \infty) \), and \( \forall m > 0, M > 0 \), we suppose that \((\sigma, u, \phi) \in X_{m,M}(I)\) is the solution to the system (1.7)-(1.9), where the solution space \( X_{m,M}(I) \) is defined as follows

\[
X_{m,M}(I) = \{(\sigma, u, \phi) \mid (\sigma, u) \in C(I; H^3(\mathbb{R}^3)), \nabla \phi \in C(I; H^2(\mathbb{R}^3)), \\
\phi^2 - 1 \in C(I; L^2(\mathbb{R}^3)), \nabla \sigma \in L^2(I; H^2(\mathbb{R}^3)), \\
\nabla \phi \in L^2(I; H^3(\mathbb{R}^3)), \nabla u \in L^2(I; H^3(\mathbb{R}^3)) \}
\]

(2.1)

We are now in a position to establish the existence and uniqueness for the local strong solutions \((\rho, u, \phi)\) of the Cauchy problem (1.7)-(1.9).

**Proposition 2.1 (local existence).** Assume that (1.4), (1.6), (1.14)-(1.15). Let \( \|(\rho_0 - \bar{\rho}, u_0)\|_{H^3} + \|\nabla \phi_0\|_{H^2} + \|\phi_0^2 - 1\| \leq M, \inf_{x \in \mathbb{R}^3} \phi_0^2(x) - \frac{1}{3} > m > 0 \) and \( \inf_{x \in \mathbb{R}^3} \rho_0(x) > m > 0 \), then there exists \( T^* \) small enough, such that, the Cauchy problem (1.7)-(1.9) admits a unique solution \((\rho, u, \phi) \in X_{m,M}^\sharp([0, T^*])\) satisfying

\[
(\rho - \bar{\rho}, u) \in C([0, T^*]; H^3(\mathbb{R}^3)), \nabla \phi \in C([0, T^*]; H^2(\mathbb{R}^3)), \phi^2 - 1 \in C([0, T^*]; L^2(\mathbb{R}^3)), \\
\nabla \rho \in L^2([0, T^*]; H^2(\mathbb{R}^3)), \nabla \phi \in L^2([0, T^*]; H^3(\mathbb{R}^3)), \nabla u \in L^2([0, T^*]; H^3(\mathbb{R}^3)).
\]

Proposition (2.1) can be obtained by the Schauder's fixed point method. The proof is standard and we omit here. Now, in order to get the global solution, on the basis of the existence and uniqueness of local solutions, we will give the a priori estimate by the following lemmas in this section. Setting \( \sigma = \rho - \bar{\rho} \), and using

\[
\text{div}(\nabla \phi \otimes \nabla \phi) = \nabla \left( \frac{|
abla \phi|^2}{2} \right) + \nabla \Delta \phi,
\]

we reformulate the system (1.7) as

\[
\begin{cases}
\sigma_t + \bar{\rho} \text{div} u = g_1, \\
u_t - \nu \bar{\rho}^{-1} \Delta u - (\nu + \lambda) \bar{\rho}^{-1} \nabla \text{div} u + \frac{\nu'(\bar{\rho})}{\bar{\rho}} \nabla \sigma + \frac{\mu}{\bar{\rho}} \nabla \phi \Delta \phi = g_2, \\
\rho \phi_t + \rho \mu u \cdot \nabla \phi = -\mu, \quad \rho \mu = \frac{\mu}{\epsilon} (\phi^3 - \phi) - \epsilon \Delta \phi,
\end{cases}
\]

(2.2)

where \( g_1 \) and \( g_2 \) are defined respectively by

\[
g_1 = -\text{div}(\sigma u),
\]

\[
g_2 = -(u \cdot \nabla) u + h_1(\sigma) \nabla \sigma - h_2(\sigma) \left( \nu \Delta u + (\nu + \lambda) \nabla \text{div} u - \epsilon \nabla \phi \Delta \phi \right),
\]

(2.3)

here \( h_1(\sigma) = \frac{\nu'(\bar{\rho}) - \nu'(\rho)}{\rho} \) and \( h_2(\sigma) = \bar{\rho}^{-1} - \rho^{-1} \). Considering the definition (2.1), combining with Sobolev embedding theorem \( H^2 \hookrightarrow C^0 \), we can choose \( M_0 > 0 \), such that, \( \forall 0 < M < M_0 \),

\[
\frac{\bar{\rho}}{2} \leq \rho(x, t) \leq 2\bar{\rho}, \quad 3\phi^2 - 1 > m_0 \overset{\text{def}}{=} \inf_{x \in \mathbb{R}^3} (3\phi_0^2 - 1).
\]

(2.4)
The following lemma 3.1 is the basic energy inequality and the gradient estimation about the phase field \( \phi \).

**Lemma 2.1** Under the assumption (2.1), it holds that

\[
\| (\sigma, u, \phi^2 - 1, \nabla \phi)(t) \|^2 + \int_0^t \| (\mu, \nabla u, \Delta \phi, \nabla \phi) \|^2 d\tau \leq C_0 \| (\sigma, u, \phi^2 - 1, \nabla \phi)(0) \|^2.
\] (2.5)

and

\[
\| \phi(t) \|_{L^\infty} \lesssim 1,
\] (2.6)

where

\[
G(\rho) = \rho \int_\rho^\infty \frac{p(z) - p(\bar{\rho})}{z^2} dz, \quad \rho > 0.
\] (2.7)

**Proof:** Noticing that

\[
\rho G'(\rho) = G(\rho) + (p(\rho) - p(\bar{\rho})), \quad \rho G''(\rho) = p'(\rho),
\]

\[
G(\rho) + \text{div}(G(\rho)u) + (p(\rho) - p(\bar{\rho}))\text{div}u = 0,
\]

and using (1.7), we have

\[
\frac{1}{2} \rho u^2 + G(\rho) + \frac{\nu}{2} \int |\nabla u|^2 dx + \rho \int |\phi^2 - 1|^2 dx + \int |\mu|^2 dx = -\epsilon \int u \cdot \nabla \phi \Delta \phi dx = 0.
\] (2.8)

Multiplying (2.2)3 by \( \mu \) and using (2.2)4 yields that

\[
\frac{1}{4} \int (\phi^2 - 1)^2 \text{div}(\rho u) dx = -\int \rho u \cdot \nabla \phi (\phi^3 - \phi) dx.
\] (2.9)

Adding (2.8) and (2.9), and using (1.4), we get

\[
\frac{d}{dt} \int \left( \frac{1}{2} \rho u^2 + G(\rho) + \frac{\epsilon}{2} |\nabla \phi|^2 + \frac{\rho}{4\epsilon} (\phi^2 - 1)^2 \right) dx + \nu \| \nabla u \|^2 + \| \mu \|^2 \leq 0.
\] (2.10)

Using (2.7), (2.1) and (2.4), we have

\[
c_\rho (\rho - \bar{\rho})^2 \leq G(\rho) \leq C_\rho (\rho - \bar{\rho})^2.
\] (2.11)

Therefore, by (2.10), (2.11) and (2.4), we obtain from (2.10) that

\[
\| (\sigma, u, \phi^2 - 1, \nabla \phi)(t) \|^2 + \int_0^t \| (\mu, \nabla u) \|^2 d\tau \leq C_0 \| (\sigma, u, \phi^2 - 1, \nabla \phi)(0) \|^2.
\] (2.12)

We rewrite (2.2)3,4 as

\[
\rho^2 \phi_t + \rho^2 u \cdot \nabla \phi = \epsilon \Delta \phi - \frac{\rho}{\epsilon} (\phi^3 - \phi).
\] (2.13)
Multiplying it by $2\phi$, we have
\[
\rho^2(\phi^2 - 1) + \rho^2 u \cdot \nabla (\phi^2 - 1) - \epsilon \Delta (\phi^2 - 1) + \frac{2\rho}{\epsilon}(\phi^2 - 1) = -2\epsilon |\nabla \phi|^2 - \frac{2\rho}{\epsilon}(\phi^2 - 1)^2 \leq 0. \tag{2.14}
\]
By using the maximum principle for parabolic equation (see Lemma 2.1 in [27] and (2.4), we obtain
\[
\phi^2 - 1 \leq 0, \tag{2.15}
\]
which yields (2.6). On the other hand, multiplying (2.2) by $-\Delta \phi$, we have
\[
\epsilon \|\Delta \phi\|^2 + \frac{1}{\rho^2} \int \rho(3\phi^2 - 1)|\nabla \phi|^2 \, dx
\]
\[
= -\int t_2 \rho \mu \Delta \phi \, dx - \frac{1}{\epsilon} \int t_3 (\phi^2 - 1) \phi \nabla \phi \cdot \nabla \sigma \, dx. \tag{2.16}
\]
For the estimates on $I_i (i = 1, 2, 3)$, we have
\[
I_1 \geq \frac{\rho \epsilon}{4\epsilon} \int \|\nabla \phi\|^2 \, dx, \quad I_2 \leq \frac{2\rho \mu}{\epsilon} \|\nabla \phi\|^2 \leq \frac{\epsilon}{4} \|\Delta \phi\|^2 + \frac{4\rho^2}{\epsilon} \|\mu\|^2, \tag{2.17}
\]
\[
I_3 \lesssim \|\phi^2 - 1\|_{L^6} \|\phi \nabla \phi\| \|\nabla \sigma\|_{L^3} \lesssim M \|\nabla \phi\|^2. \tag{2.18}
\]
Substituting the estimates on $I_i (i = 1, 2, 3)$ into (2.16), and using (2.4), for $M$ suitable small, we get
\[
\epsilon^2 \|\Delta \phi\|^2 + \|\nabla \phi\|^2 \leq \|\mu\|^2. \tag{2.19}
\]
By (2.10) and (2.17), we get (2.5). The proof of Lemma 2.1 is completed.

The following lemma is a higher-order estimate of the gradient of the phase field $\phi$.

**Lemma 2.2** Under the the assumption (2.1), it holds that
\[
\frac{d}{dt} \|\nabla^{k+1} \phi\|^2 + \|\nabla^{k+2} \phi\|^2 \lesssim M \left( \|\nabla^{k+1} \sigma\|^2 + \|\nabla^{k+1} \phi\|^2 + \|\nabla^{k+2} u\|^2 \right), \tag{2.20}
\]
for $k = 1, 2$.

**Proof:** We rewrite (2.2) as
\[
\phi_t + u \cdot \nabla \phi - \frac{\epsilon}{\rho^2} \Delta \phi + \frac{\phi^2 - 1}{\epsilon \rho} \phi = 0. \tag{2.21}
\]
Applying $\nabla^k$ to (2.21) and multiplying it by $-\Delta \nabla^k \phi$, we have
\[
\frac{1}{2} \frac{d}{dt} \|\nabla^{k+1} \phi\|^2 + \int \frac{\epsilon}{\rho^2} \|\nabla^k \Delta \phi\|^2 \, dx
\]
\[
= \int \nabla^k (u \cdot \nabla \phi) \Delta \nabla^k \phi \, dx - \epsilon \sum_{l \leq k} C_k^l \int t_4 \nabla^l \left( \frac{1}{\rho^2} \right) \Delta \nabla^{k-l} \phi \nabla^k \Delta \phi \, dx \tag{2.22}
\]
\[
+ \frac{1}{\epsilon} \sum_{0 \leq l \leq k} C_k^l \int t_6 \left( \frac{\phi^2 - 1}{\rho} \right) \nabla^{k-l} \phi \nabla^k \Delta \phi \, dx. \tag{2.23}
\]
We estimate $I_i (i = 4, 5, 6)$. For $I_4$, we have

$$I_4 = \sum_{0 \leq l \leq k} C^l_k \int \nabla^l u \cdot \nabla^{k+l+1} \phi \Delta \nabla^k \phi \, dx \lesssim \sum_{0 \leq l \leq k} \| \nabla^l u \cdot \nabla^{k+l+1} \phi \| \| \nabla^{k+2} \phi \|.$$  

If $l \leq \left[ \frac{k+1}{2} \right]$, we get

$$\| \nabla^l u \cdot \nabla^{k-l+1} \phi \| \lesssim \| \nabla^l u \|_{L^3} \| \nabla^{k-l+1} \phi \|_{L^3} \lesssim \| u \|^{1-\frac{l+1}{l+2}} \| \nabla^{k+2} u \|^{\frac{l+1}{l+2}} \| \nabla^\alpha \phi \|^{\frac{l+1}{l+2}} \| \nabla^{k+2} \phi \|^{1-\frac{1}{l+2}} \lesssim M \left( \| \nabla^{k+2} u \| + \| \nabla^{k+2} \phi \| \right),$$

where $\alpha$ is defined by

$$\frac{l-1}{3} = \left( \frac{\alpha}{3} - \frac{1}{2} \right) \left( 1 - \frac{l}{k+1} \right) + \left( \frac{k+2}{3} - \frac{1}{2} \right) \frac{l}{k+1} \quad \Rightarrow \quad \alpha = \frac{3}{2} + \frac{l}{2(k+1-l)} \in \left[ \frac{3}{2}, \frac{5}{2} \right].$$

If $\left[ \frac{k+1}{2} \right] + 1 \leq l \leq k$ (if $k \leq \left[ \frac{k+1}{2} \right] + 1$, then it’s nothing in this case, and hereafter, etc.), we get

$$\| \nabla^l u \cdot \nabla^{k-l+1} \phi \| \lesssim \| \nabla^l u \|_{L^3} \| \nabla^{k-l+1} \phi \|_{L^3} \lesssim \| u \|^{1-\frac{l+1}{l+2}} \| \nabla^{k+2} u \|^{\frac{l+1}{l+2}} \| \nabla^\alpha \phi \|^{\frac{l+1}{l+2}} \| \nabla^{k+2} \phi \|^{1-\frac{1}{l+2}} \lesssim M \left( \| \nabla^{k+2} u \| + \| \nabla^{k+2} \phi \| \right),$$

where $\alpha$ is defined by

$$\frac{k-l}{3} = \left( \frac{\alpha}{3} - \frac{1}{2} \right) \frac{l+1}{k+2} + \left( \frac{k+2}{3} - \frac{1}{2} \right) \left( 1 - \frac{l+1}{k+2} \right) \quad \Rightarrow \quad \alpha = \frac{3(k+2)}{2(l+1)} \in \left[ \frac{3}{2}, \frac{3}{2} \right].$$

Therefore, we obtain

$$|I_4| \lesssim M \left( \| \nabla^{k+2} u \|^2 + \| \nabla^{k+2} \phi \|^2 \right). \quad (2.21)$$

Also, for $I_5$, we have

$$I_5 = \sum_{1 \leq l \leq k} C^l_k \int \nabla^l \left( \frac{1}{\rho^2} \right) \nabla^{k-l} \Delta \phi \nabla^k \Delta \phi \, dx \lesssim \sum_{1 \leq l \leq k} \| \nabla^l \left( \frac{1}{\rho^2} \right) \nabla^{k-l} \Delta \phi \| \| \nabla^{k+2} \phi \|. \quad (2.21)$$
If $1 \leq l \leq \left[ \frac{k+1}{2} \right]$, we get

$$\|\nabla^l \left( \frac{1}{\rho^2} \right) \nabla^{k-l} \Delta \phi \| \lesssim \|\nabla^l \sigma\|_{L^6} \|\nabla^{k-l+2} \phi\|_{L^6} \tag{1.11}$$

$$\lesssim \|\nabla^l \sigma\|_{L^6} \|\nabla^{k-l+1} \sigma\|_{L^6} \|\nabla^{k+l} \phi\|_{L^6} \|\nabla^{k+2} \phi\|_{L^6} \tag{1.10}$$

$$\lesssim M \|\nabla^{k+l} \sigma\|_{L^6} \|\nabla^{k+2} \phi\|_{L^6} \tag{2.3}$$

where $\alpha$ is defined by

$$\frac{l-1}{3} = \left( \frac{\alpha}{3} - \frac{1}{2} \right) \left( 1 - \frac{l-1}{k+1} \right) + \left( \frac{k+1}{3} - \frac{1}{2} \right) \frac{l-1}{k+1}$$

$$\implies \alpha = \frac{3}{2} + \frac{l-1}{2(k+2-l)} \in \left[ \frac{3}{2}, \frac{5}{2} \right].$$

If $\left[ \frac{k+1}{2} \right] + 1 \leq l \leq k$, we get

$$\|\nabla^l \left( \frac{1}{\rho^2} \right) \nabla^{k-l} \Delta \phi \| \lesssim \|\nabla^l \sigma\|_{L^6} \|\nabla^{k-l+2} \phi\|_{L^6} \tag{1.11}$$

$$\lesssim \|\sigma\|_{L^6} \|\nabla^{k+1} \sigma\|_{L^6} \|\nabla^\alpha \phi\|_{L^6} \|\nabla^{k+2} \phi\|_{L^6} \tag{1.10}$$

$$\lesssim M \|\nabla^{k+1} \sigma\|_{L^6} \|\nabla^{k+2} \phi\|_{L^6} \tag{2.3}$$

where $\alpha$ is defined by

$$\frac{k-l+1}{3} = \left( \frac{\alpha}{3} - \frac{1}{2} \right) \frac{l+1}{k+1} + \left( \frac{k+1}{3} - \frac{1}{2} \right) \left( 1 - \frac{l+1}{k+1} \right)$$

$$\implies \alpha = 1 + \frac{k+1}{2(l+1)} \in \left[ \frac{3}{2}, 3 \right].$$

Therefore, we obtain

$$|I_5| \lesssim M \left( \|\nabla^{k+1} \sigma\|^2 + \|\nabla^{k+2} \phi\|^2 \right). \tag{2.22}$$

To estimate $I_6$, we rewrite it as

$$I_6 = -\int_{t^6} \left( \frac{\phi^2 - 1}{\rho} \right) |\nabla^{k+1} \phi|^2 dx - \int_{t^6} \nabla \left( \frac{\phi^2 - 1}{\rho} \right) \nabla \phi \cdot \nabla^{k+1} \phi dx$$

$$\quad + \sum_{1 \leq l \leq k} C^l \int_{t^6} \left( \frac{\phi^2 - 1}{\rho} \right) \nabla^{k-l} \phi \nabla^l \Delta \phi \, dx \tag{2.23}$$

For $I_6^1$, we have

$$I_6^1 \lesssim \|\phi^2 - 1\|_{L^6} \|\nabla^{k+1} \phi\| \|\nabla^{k+1} \phi\|_{L^6} \tag{1.10}$$

$$\lesssim \|\nabla^{k+1} \phi\|^2 + \frac{1}{2} \|\nabla (\phi^2 - 1)\|_{L^6} \|\nabla^{k+1} \phi\| \|\nabla^{k+2} \phi\| \tag{2.3}$$

$$\lesssim M \left( \|\nabla^{k+1} \phi\|^2 + \|\nabla^{k+2} \phi\|^2 \right).$$
For $I_6^2$, we have
\[ I_6^2 \lesssim \left\| \nabla \left( \frac{\phi^2 - 1}{\rho} \right) \right\| \left\| \nabla^k \phi \right\|_{L^6} \left\| \nabla^{k+1} \phi \right\|_{L^3} \leq \left( \left\| \nabla \phi \right\| + \left\| \nabla \sigma \right\| \right) \left\| \nabla^k \phi \right\|^{\frac{1}{2}} \left\| \nabla^{k+1} \phi \right\|^{\frac{1}{2}} \lesssim M \left( \left\| \nabla^{k+1} \phi \right\|^2 + \left\| \nabla^{k+2} \phi \right\|^2 \right). \]

For $I_6^3$, we have
\[ I_6^3 \lesssim \sum_{1 \leq l \leq k} \left\| \nabla^l \left( \frac{\phi^2 - 1}{\rho} \right) \right\| \left\| \nabla^{k-l} \phi \right\| \left\| \nabla^{k+2} \phi \right\|. \]

If $1 \leq l \leq \left\lfloor \frac{k}{2} \right\rfloor$, we get
\[ \left\| \nabla^l \left( \frac{\phi^2 - 1}{\rho} \right) \right\| \lesssim \left( \left\| \nabla' \sigma \right\|_{L^6} + \left\| \nabla' \phi \right\|_{L^6} \right) \left\| \nabla^{k-l} \phi \right\|_{L^3} \lesssim \left( \left\| \nabla^\alpha \sigma \right\|^{1 - \frac{k}{l+1}} \left\| \nabla^k \sigma \right\|^\frac{k}{l+1} + \left\| \nabla^\alpha \phi \right\|^{1 - \frac{k}{l+1}} \left\| \nabla^k \phi \right\|^\frac{k}{l+1} \right) \left\| \nabla \phi \right\|^\frac{k}{l+1} \left\| \nabla^{k+1} \phi \right\|^{1 - \frac{k}{l+1}} \lesssim M \left( \left\| \nabla^{k+1} \sigma \right\| + \left\| \nabla^{k+1} \phi \right\| \right), \]

where $\alpha$ is defined by
\[ \frac{l - 1}{3} = \left( \frac{\alpha - 1}{3} - \frac{1}{2} \right) \left( 1 - \frac{l}{k} \right) + \left( \frac{k + 1}{3} - \frac{1}{2} \right) \frac{l}{k} \implies \alpha = \frac{1}{2} + \frac{l}{2k} \in \left[ \frac{1}{2}, 1 \right]. \]

If $\left\lfloor \frac{k}{2} \right\rfloor + 1 \leq l \leq k$, we get
\[ \left\| \nabla^l \left( \frac{\phi^2 - 1}{\rho} \right) \right\| \lesssim \left( \left\| \nabla' \sigma \right\|_{L^6} + \left\| \nabla' \phi \right\|_{L^6} \right) \left\| \nabla^{k-l} \phi \right\|_{L^3} \lesssim \left( \left\| \sigma \right\|^{1 - \frac{k}{l+1}} \left\| \nabla^k \sigma \right\|^\frac{k}{l+1} + \left\| \phi \right\|^{1 - \frac{k}{l+1}} \left\| \nabla^k \phi \right\|^\frac{k}{l+1} \right) \left\| \nabla^\alpha \phi \right\|^\frac{k}{l+1} \left\| \nabla^{k+1} \phi \right\|^{1 - \frac{k}{l+1}} \lesssim M \left( \left\| \nabla^{k+1} \sigma \right\| + \left\| \nabla^{k+1} \phi \right\| \right), \]

where $\alpha$ is defined by
\[ \frac{k - l - 1}{3} = \left( \frac{\alpha - 1}{3} - \frac{1}{2} \right) \frac{l + 1}{k + 1} + \left( \frac{k + 1}{3} - \frac{1}{2} \right) \left( 1 - \frac{l + 1}{k + 1} \right) \implies \alpha = \frac{k + 1}{2l + 1} \in \left[ \frac{1}{2}, 2 \right]. \]

Therefore, we obtain
\[ I_6^3 \lesssim M \left( \left\| \nabla^{k+1} \sigma \right\|^2 + \left\| \nabla^{k+2} \phi \right\|^2 + \left\| \nabla^{k+1} \sigma \right\|^2 \right). \]

Therefore, we obtain from (2.24) and the estimates on $I_6^i (i = 1, 2, 3)$ that
\[ |I_6| \lesssim M \left( \left\| \nabla^{k+1} \sigma \right\|^2 + \left\| \nabla^{k+2} \phi \right\|^2 + \left\| \nabla^{k+2} \phi \right\|^2 \right). \]
Substituting (2.21), (2.22) and (2.24) into (2.20), we have (2.18). The proof of Lemma 2.2 is completed.

The following lemma 3.3 is a higher-order estimate of \((\sigma, u)\).

**Lemma 2.3** Under the assumption (2.1), it holds that, for \(k = 0, 1, 2, 3\),

\[
\frac{d}{dt} \left( \|\nabla^k u\|^2 + \|\nabla^k \sigma\|^2 \right) + \|\nabla^{k+1} u\|^2 \lesssim M \left( \|\nabla^k \sigma\|^2 + \|\nabla^{k+1} \phi\|^2 \right).
\]  

(2.25)

**Proof:** Applying \(\nabla^k\) to (2.21) and (2.22) yields respectively that

\[
\nabla^k \sigma_t + \rho \text{div}\nabla^k u = \nabla^k g_1,
\]

(2.26)

\[
\nabla^k u_t - \frac{\nu}{\rho} \Delta \nabla^k u - \frac{(\nu + \lambda)}{\rho} \nabla^{k+1} \text{div} u + \frac{\rho'(\rho)}{\rho} \nabla^{k+1} \sigma + \frac{\kappa}{\rho} \nabla^k (\nabla \phi \Delta \phi) = \nabla^k g_2.
\]

Multiplying (2.26) \(_2\) by \(\nabla^k u\), and using (2.26) \(_1\) and (2.3), we have

\[
\frac{1}{2} \frac{d}{dt} \left( \|\nabla^k u\|^2 + \frac{\rho'}{\rho^2} \|\nabla^k \sigma\|^2 \right) + \int \left( \frac{\nu}{\rho} \|\nabla^{k+1} u\|^2 + \frac{(\nu + \lambda)}{\rho} \|\text{div}\nabla^k u\|^2 \right) dx = I_7,
\]

(2.27)

where

\[
I_7 = \int_{t_2}^{t_1} \nabla^k \sigma \nabla^k \text{div}(\sigma u) dx - \int_{t_2}^{t_1} \nabla^k [(\sigma, \nabla) u - h_1(\sigma) \nabla \sigma \cdot \nabla^k u] dx - \int_{t_2}^{t_1} \nabla^k [h_2(\sigma) (\nu \Delta u + (\nu + \lambda) \nabla \text{div} u - \epsilon \nabla \phi \Delta \phi)] \cdot \nabla^k u dx - \int_{t_2}^{t_1} \frac{\nu}{\rho} \nabla^k (\nabla \phi \Delta \phi) \cdot \nabla^k u dx + \epsilon \int_{t_2}^{t_1} \nabla^k [h_2(\sigma) \nabla \phi \Delta \phi] \cdot \nabla^k u dx.
\]

(2.28)

We will give the energy estimate of \(I_7\) below. By the same lines as in [32, Lemma 2.1], we can derive that

\[
I_7^i \lesssim M \left( \|\nabla^k \sigma\|^2 + \|\nabla^{k+1} u\|^2 \right) \quad \text{for} \; i = 1, 2, 3.
\]

For \(I_7^1\), it is easy to check that

\[
I_7^1 \lesssim \|\nabla \phi\| \|\nabla^2 \phi\|_{L^\infty} \|\nabla u\| \lesssim M \left( \|\nabla \phi\|^2 + \|\nabla u\|^2 \right), \quad \text{for} \; k = 0.
\]

and

\[
I_7^1 \lesssim \|\nabla \phi\|_{L^\infty} \|\nabla^2 \phi\| \|\nabla^2 u\| \lesssim M \left( \|\nabla^2 \phi\|^2 + \|\nabla^2 u\|^2 \right), \quad \text{for} \; k = 1.
\]

When \(k \geq 2\), using Leibniz formula yields that

\[
I_7^1 = - \int \nabla^{k-1} [\nabla \phi \Delta \phi] \cdot \nabla^{k+1} u dx \lesssim \sum_{0 \leq l \leq k-1} \|\nabla^{l+1} \phi \nabla^{k-l+1} \phi\| \|\nabla^{k+1} u\|.
\]
\[ \frac{l}{3} = \left( \frac{\alpha}{3} - \frac{1}{2} \right) \left( 1 - \frac{l-1}{k-1} \right) + \left( \frac{k+1}{3} - \frac{1}{2} \right) \frac{l-1}{k-1} \rightarrow \alpha = 2 - \frac{k-1}{2(k-1)} \in [1, 2]. \]

If \( \frac{k+1}{2} \leq l \leq k-1 \), we get
\[ \| \nabla^{l+1} \phi \nabla^{k-l+1} \phi \| \lesssim \| \nabla^{l+1} \phi \|_{L^6} \| \nabla^{k-l+1} \phi \| \]
\[ \lesssim \| \nabla^\alpha \phi \|^{1 - \frac{l-1}{k-1}} \| \nabla^{k+1} \phi \| \| \nabla^2 \phi \|^{\frac{l-1}{k-1}} \| \nabla^{k+1} \phi \|^{1 - \frac{l-1}{k-1}} \]
\[ \lesssim M \| \nabla^{k+1} \phi \|, \]
where \( \alpha \) is defined by
\[ \frac{k-l}{3} = \left( \frac{\alpha}{3} - \frac{1}{2} \right) \frac{l+1}{k} + \left( \frac{k+1}{3} - \frac{1}{2} \right) \left( 1 - \frac{l+1}{k} \right) \rightarrow \alpha = 1 + \frac{3k}{2(l+1)} \in [2, 4]. \]

Therefore, we have
\[ I_7^i \lesssim M \left( \| \nabla^{k+1} \phi \|^2 + \| \nabla^{k+1} u \|^2 \right). \]

In a similar way, we obtain
\[ I_7^5 \lesssim M \left( \| \nabla^{k+1} \phi \|^2 + \| \nabla^{k+1} u \|^2 \right). \]

Substituting the estimates on \( I_7^i (i = 1, \cdots, 5) \) into (2.28), we have
\[ I_7 \lesssim M \left( \| \nabla^k \sigma \|^2 + \| \nabla^{k+1} u \|^2 + \| \nabla^{k+1} \phi \|^2 \right). \]

Substituting (2.29) into (2.27), we have (2.25). The proof of Lemma 2.3 is completed. ■

**Lemma 2.4** Under the the assumption (2.1), it holds that
\[ \frac{d}{dt} \int \nabla^k u \cdot \nabla^{k+1} \sigma \, dx + \| \nabla^{k+1} \sigma \|^2 \]
\[ \lesssim M \left( \| \nabla^{k+2} u \|^2 + \| \nabla^{k+2} \phi \|^2 \right) + \| \nabla^{k+1} u \|^2, \quad \text{for } k = 0, 1, 2. \]

**Proof:** Multiplying \( (2.26)_2 \) by \( \nabla^{k+1} \sigma \), and using \( (2.26)_1 \) and (2.3), we have
\[ \frac{d}{dt} \int \nabla^k u \cdot \nabla^{k+1} \sigma \, dx + \frac{p'(\bar{\rho})}{\bar{\rho}} \| \nabla^{k+1} \sigma \|^2 - \bar{\rho} \int (\text{div} \nabla^k u)^2 \, dx = I_8, \]
where

\[ I_8 = \int \nabla^k \text{div}(\sigma u) \text{div} \nabla^k u dx - \int \nabla^k [(u, \nabla)u - h_1(\sigma)\nabla \sigma \cdot \nabla^{k+1} \sigma dx \]
\[ - \int \nabla^k [h_2(\sigma) (\nu \Delta u + (\nu + \lambda) \nabla \text{div} u)] \cdot \nabla^{k+1} \sigma dx \]
\[ - \epsilon \int \nabla^k [\nabla \phi \Delta \phi] \cdot \nabla^{k+1} \sigma dx + \epsilon \int \nabla^k [h_2(\sigma) \nabla \phi \Delta \phi] \cdot \nabla^k \sigma dx . \]  

\hfill (2.32)

We estimate \( I_8 \). By the same lines as in [32, Lemma 2.2], we can derive that

\[ I_8^1 \lesssim M \left( \| \nabla^{k+1} \sigma \|^2 + \| \nabla^{k+1} u \|^2 \right) , \]
\[ I_8^2 + I_8^3 \lesssim M \left( \| \nabla^{k+1} \sigma \|^2 + \| \nabla^{k+2} u \|^2 \right) . \]

For \( I_8^4 \), it is easy to check that

\[ I_8^4 \lesssim \| \nabla \phi \|_{L^\infty} \| \nabla^2 \phi \| \| \nabla \sigma \| \lesssim M \left( \| \nabla^2 \phi \|^2 + \| \nabla \sigma \|^2 \right) , \]  

for \( k = 0 \). When \( k \geq 1 \), using Leibniz formula yields that

\[ I_8^4 = \int \nabla^k [\nabla \phi \Delta \phi] \cdot \nabla^{k+1} \sigma dx \lesssim \sum_{0 \leq l \leq k} \| \nabla^{l+1} \phi \| \| \nabla^{k-l+1} \phi \| \| \nabla^{k+1} \sigma \| . \]

If \( l \leq \left[ \frac{k+1}{2} \right] \), we get

\[ \| \nabla^{l+1} \phi \| \| \nabla^{k-l+1} \phi \| \lesssim \| \nabla^{l+1} \phi \|_{L^5} \| \nabla^{k-l+1} \phi \|_{L^6} \]
\[ \lesssim \| \nabla^\alpha \phi \| \| \nabla^{k+2} \phi \|_{L^2} \| \nabla^3 \phi \|_{L^2} \| \nabla^{k+2} \phi \|_{L^3} \lesssim M \| \nabla^{k+2} \phi \| , \]

where \( \alpha \) is defined by

\[ \frac{l}{3} = \left( \frac{\alpha}{3} - \frac{1}{2} \right) \left( 1 - \frac{l - 1}{k - 1} \right) + \left( \frac{k + 2}{3} - \frac{1}{2} \right) \frac{l - 1}{k - 1} \rightarrow \alpha = 3 - \frac{k - 1}{2(k-l)} \in [2, 3] . \]

If \( \left[ \frac{k+1}{2} \right] + 1 \leq l \leq k \), we get

\[ \| \nabla^{l+1} \phi \| \| \nabla^{k-l+1} \phi \| \lesssim \| \nabla^{l+1} \phi \|_{L^5} \| \nabla^{k-l+1} \phi \|_{L^3} \]
\[ \lesssim \| \nabla^2 \phi \|_{L^\infty} \| \nabla^{k+2} \phi \|_{L^2} \| \nabla^3 \phi \|_{L^2} \| \nabla^{k+2} \phi \|_{L^3} \lesssim M \| \nabla^{k+2} \phi \| , \]

where \( \alpha \) is defined by

\[ \frac{k - l}{3} = \left( \frac{\alpha}{3} - \frac{1}{2} \right) \frac{l + 1}{k} + \left( \frac{k + 2}{3} - \frac{1}{2} \right) \left( 1 - \frac{l + 1}{k} \right) \]
\[ \Rightarrow \alpha = 2 - \frac{k}{2(l+1)} \in [1, 2] . \]
Lemma 2.5 establish the evolution of negative Sobolev norms on solutions to the system (1.7)-(1.9).

In a similar way, we obtain

\[ I^2_s \lesssim M \left( \| \nabla^{k+2} \phi \|^2 + \| \nabla^{k+1} \sigma \|^2 \right). \]

Substituting the estimates on \( I^i_s \) (i = 1, · · · , 5) into (2.32), we have

\[ I^8_s \lesssim M \left( \| \nabla^{k+1} \sigma \|^2 + \| \nabla^{k+1} \sigma \|^2 \right). \]

Substituting (2.33) into (2.31), we have (2.30). The proof of Lemma 2.4 is completed. ■

Now in order to obtain the estimate of the decay of the solution over time, we will establish the evolution of negative Sobolev norms on solutions to the system (1.7)-(1.9).

Lemma 2.5 Under the assumption (2.1), it holds that for \( s \in (0, \frac{1}{2}] \), we have

\[
\frac{d}{dt} \left( \frac{p'(\rho)}{\rho^2} \| \Lambda^{-s} \sigma \|^2 + \| \Lambda^{-s} u \|^2 + \| \Lambda^{-s} \nabla \phi \|^2 + \| \Lambda^{-s} (\phi^2 - 1) \|^2 \right) \\
\lesssim \left( \| \nabla \sigma \|^2_{H^s} + \| \nabla (u, \nabla \phi) \|^2_{H^1} + \| \nabla (\phi^2 - 1) \|^2 \right) \times \\
\left( \| \Lambda^{-s} \sigma \| + \| \Lambda^{-s} u \| + \| \Lambda^{-s} \nabla \phi \| + \| \Lambda^{-s} (\phi^2 - 1) \| \right)
\] (2.34)

and for \( s \in (\frac{1}{2}, \frac{3}{2}) \), we have

\[
\frac{d}{dt} \left( \frac{p'(\rho)}{\rho^2} \| \Lambda^{-s} \sigma \|^2 + \| \Lambda^{-s} u \|^2 + \| \Lambda^{-s} \nabla \phi \|^2 + \| \Lambda^{-s} (\phi^2 - 1) \|^2 \right) \\
\lesssim \| (\sigma, u, \phi^2 - 1, \nabla \phi) \|_{H^{s-\frac{1}{2}}} \times \left( \| \nabla (\sigma, u, \nabla \phi) \|_{H^1} + \| \nabla (\phi^2 - 1) \| \right)^{\frac{3-s}{2}} \times \\
\left( \| \Lambda^{-s} \sigma \| + \| \Lambda^{-s} u \| + \| \Lambda^{-s} \nabla \phi \| + \| \Lambda^{-s} (\phi^2 - 1) \| \right).
\] (2.35)

Proof: Applying \( \Lambda^{-s} \) to (2.21) and (2.22), multiplying the resulting identities by \( \frac{p'(\rho)}{\rho^2} \Lambda^{-s} \sigma \) and \( \Lambda^{-s} u \), respectively, summing up and using (2.3), we deduce that

\[
\frac{1}{2} \frac{d}{dt} \left( \frac{p'(\rho)}{\rho^2} \| \Lambda^{-s} \sigma \|^2 + \| \Lambda^{-s} u \|^2 \right) + \frac{\nu}{\rho} \| \Lambda^{-s} \nabla u \|^2 + \frac{\nu + \lambda}{\rho} \| \Lambda^{-s} \text{div} u \|^2 \\
= -\epsilon \int \Lambda^{-s} (\nabla \phi \Delta \phi) \cdot \Lambda^{-s} u dx - \frac{p'(\rho)}{\rho^2} \int \Lambda^{-s} (\text{div} u + \nabla \sigma \cdot u) \Lambda^{-s} \sigma dx \tag{I_{10}} \\
- \int \Lambda^{-s} [(u, \nabla) u - h_1(\sigma) \nabla \sigma - h_2(\sigma) (\nu \Delta u + (\nu + \lambda) \nabla \text{div} u) - \epsilon \nabla \phi \Delta \phi \cdot \Lambda^{-s} u] dx \tag{I_{11}}.
\]

Also, we rewrite (2.19) as

\[
(\phi^2 - 1)_t + u \cdot \nabla (\phi^2 - 1) - \frac{\epsilon}{\rho^2} \Delta (\phi^2 - 1) + \frac{2\epsilon}{\rho^2} |\nabla \phi|^2 + 2 \frac{\phi^2 - 1}{\rho} = 0. \tag{2.37}
\]
Then, applying $\Lambda^{-s}$ to (2.19) and (2.37), multiplying the resulting identities by $-\Lambda^{-s}\Delta\phi$ and $\Lambda^{-s}(\phi^2-1)$, respectively, and summing up the resulting equations, we deduce that

$$
\frac{1}{2} \frac{d}{dt} \left( \|\Lambda^{-s}\nabla\phi\|^2 + \Lambda^{-s}(\phi^2-1) \right) + \frac{\epsilon}{\rho^2} \|\Lambda^{-s}\Delta\phi\|^2 + \frac{2\epsilon}{\rho^2} \|\Lambda^{-s}\nabla(\phi^2-1)\|^2
= -\int \Lambda^{-s}\nabla \left( \frac{\phi^3-\phi}{\epsilon\rho} \right) \Lambda^{-s}\nabla\phi \, dx -2\int \Lambda^{-s} \left( \frac{\phi^2-1}{\epsilon\rho^2} \phi^2 \right) \Lambda^{-s}(\phi^2-1) \, dx \\
+ \int \Lambda^{-s} \nabla (-u \cdot \nabla\phi + h_3(\sigma)\Delta\phi) \Lambda^{-s}\nabla\phi \, dx \\
+ \int \Lambda^{-s} \left[ -u \cdot \nabla(\phi^2-1) + 2h_3(\sigma)\Delta(\phi^2-1) + \frac{\epsilon}{\rho^2} \nabla\phi \right]^2 \Lambda^{-s}(\phi^2-1) \, dx 
$$

(2.38)

where $h_3(\sigma) = \frac{\sigma}{\rho^2} - \frac{\sigma}{\rho^2}$. In order to estimate the nonlinear terms in the right-hand side of (2.36) and (2.38), we shall use the estimate (1.13). This forces us to require that $s \in (0, \frac{3}{2})$. If $s \in (0, \frac{1}{2})$, then $\frac{1}{2} + s < 1$ and $\frac{3}{2} \geq 6$. Then, we have

$$
I_9 \lesssim \|\Lambda^{-s}(\nabla\phi\Delta\phi)\| \|\Lambda^{-s}u\|
\lesssim \|\nabla\phi\Delta\phi\| \|\Lambda^{-s}u\| \lesssim \|\nabla\phi\| \|\nabla^2\phi\| \|\Lambda^{-s}u\|
\lesssim \|\nabla^2\phi\|^{\frac{1}{2}+s} \|\nabla^3\phi\|^{\frac{1}{2}-s} \|\nabla^2\phi\| \|\Lambda^{-s}u\| \lesssim (\|\nabla^2\phi\|^2 + \|\nabla^3\phi\|^2) \|\Lambda^{-s}u\|.
$$

Further from this, by the same arguments as above and in (2.3), we have

$$
I_{10} + I_{11} \lesssim \left( \|\nabla\sigma\|^2_{H^1} + \|\nabla u\|^2_{H^1} \right) \left( \|\Lambda^{-s}\sigma\| + \|\Lambda^{-s}u\| \right), \text{ for } s \in (0, \frac{1}{2}).
$$

The estimate on $I_{12}$ is more subtle. Next, we rewrite it as

$$
I_{12} = -\frac{2}{\epsilon\rho} \int |\nabla^{-s}\nabla\phi|^2 \, dx + \frac{2}{\epsilon} \int \Lambda^{-s} \left[ \left( \frac{1}{\rho} - \frac{1}{\rho} \right) \nabla\phi \right] \Lambda^{-s}\nabla\phi \, dx \\
- \frac{3}{\epsilon} \int \Lambda^{-s} \left( \frac{\phi^2-1}{\rho} \nabla\phi \right) \Lambda^{-s}\nabla\phi \, dx \\
- \frac{1}{\epsilon} \int \Lambda^{-s} \left[ (\phi^3 - \phi) \nabla \left( \frac{1}{\rho} \right) \right] \Lambda^{-s}\nabla\phi \, dx
$$

(2.39)

Then, we have

$$
I_{12} \lesssim \|\Lambda^{-s} \left[ \left( \frac{1}{\rho} - \frac{1}{\rho} \right) \nabla\phi \right] \| \|\Lambda^{-s}\nabla\phi\| \lesssim \|\nabla\phi\| \|\Lambda^{-s}\nabla\phi\| \lesssim (\|\nabla\sigma\|^{\frac{1}{2}+s} \|\nabla^2\sigma\|^{\frac{1}{2}-s} \|\nabla^2\phi\|^{1-\theta} \|\Lambda^{-s}\nabla\phi\|^{1+\theta})
\lesssim \left( \|\nabla\sigma\|^2_{H^1} + \|\nabla^2\phi\|^2 \right) \|\Lambda^{-s}\nabla\phi\| + M \|\Lambda^{-s}\nabla\phi\|^2 \text{ with } \theta = \frac{1}{2} + s.
$$
\[ I_{12}^3 \lesssim \left\| \Lambda^{-s} \left( \frac{\phi^2 - 1}{\rho} \right) \right\| \left\| \Lambda^{-s} \nabla \phi \right\| \lesssim \left\| \frac{\phi^2 - 1}{\rho} \right\|_{L^{\frac{1}{2 + \theta}}} \left\| \Lambda^{-s} \nabla \phi \right\| \]

\[ \lesssim \left\| \frac{\phi^2 - 1}{\rho} \right\|_{L^{\frac{1}{2 + \theta}}} \left\| \Lambda^{-s} \nabla \phi \right\| \]

\[ \lesssim \| \nabla (\phi^2 - 1) \|_{L^{\frac{1}{2 + \theta}}} \| \nabla^2 (\phi^2 - 1) \|_{L^{\frac{1}{2 + \theta}}} \left\| \Lambda^{-s} \nabla \phi \right\| \]

\[ \lesssim \left( \| \nabla (\phi^2 - 1) \| + \| \nabla^2 (\phi^2 - 1) \| \right) \left( \| \nabla^2 \phi \| + \| \Lambda^{-s} \nabla \phi \| \right) \]

\[ \lesssim \left( \| \nabla (\phi^2 - 1) \|^2 + \| \nabla^2 \phi \|^2 \right) \left\| \Lambda^{-s} \nabla \phi \right\| + M \left\| \Lambda^{-s} \nabla \phi \right\|^2 \text{ with } \theta = \frac{1}{2 + s}, \]

and

\[ I_{12}^3 \lesssim \left\| \Lambda^{-s} \left( \frac{\phi^2 - 1}{\rho} \right) \right\| \left\| \Lambda^{-s} \nabla \phi \right\| \lesssim \left\| \frac{\phi^2 - 1}{\rho} \right\|_{L^{\frac{1}{2 + \theta}}} \left\| \Lambda^{-s} \nabla \phi \right\| \]

\[ \lesssim \| \nabla (\phi^2 - 1) \|_{L^{\frac{1}{2 + \theta}}} \| \nabla^2 (\phi^2 - 1) \|_{L^{\frac{1}{2 + \theta}}} \left\| \Lambda^{-s} \nabla \phi \right\| \]

\[ \lesssim \left( \| \nabla (\phi^2 - 1) \| + \| \nabla^2 (\phi^2 - 1) \| \right) \left\| \nabla^2 \phi \| + \left\| \Lambda^{-s} \nabla \phi \right\| \]

\[ \lesssim \left( \| \nabla (\phi^2 - 1) \|^2 + \| \nabla^2 \phi \|^2 \right) \left\| \Lambda^{-s} \nabla \phi \right\|. \]

Therefore, we obtain from (2.39) that

\[ I_{12} + \frac{1}{\epsilon \rho} \| \Lambda^{-s} \nabla \phi \|^2 \lesssim \left( \| \nabla \sigma \|^2_{H^1} + \| \nabla (\phi^2 - 1) \|^2 + \| \nabla^2 \phi \|^2 \right) \left\| \Lambda^{-s} \nabla \phi \right\|. \]

Similarly, for \( I_{13} \), we rewrite it as

\[ I_{13} = -\frac{2}{\epsilon \rho} \int |\Lambda^{-s} (\phi^2 - 1)|^2 dx + \frac{2}{\epsilon} \int \Lambda^{-s} \left[ \left( \frac{1}{\rho} - \frac{1}{\rho} \right) (\phi^2 - 1) \right] \Lambda^{-s} (\phi^2 - 1) dx \]

\[ -\frac{2}{\epsilon} \int \Lambda^{-s} \left( \frac{\phi^2 - 1}{\rho} \right) \Lambda^{-s} (\phi^2 - 1) dx \]

(2.40)

Then, we have

\[ I_{13}^2 \lesssim \left\| \Lambda^{-s} \left[ \left( \frac{1}{\rho} - \frac{1}{\rho} \right) (\phi^2 - 1) \right] \right\| \left\| \Lambda^{-s} \nabla \phi \right\| \]

\[ \lesssim \| \nabla \sigma \|^2_{H^1} \| \phi^2 - 1 \| \left\| \Lambda^{-s} \nabla \phi \right\| \]

\[ \lesssim \| \nabla \sigma \|^2_{H^1} \| \phi^2 - 1 \| \left\| \Lambda^{-s} \nabla \phi \right\| \]

\[ \lesssim \left( \| \nabla \sigma \|^2_{H^1} + \| \nabla (\phi^2 - 1) \|^2 + \| \nabla^2 \phi \|^2 \right) \left\| \Lambda^{-s} \nabla \phi \right\|. \]
and

\[ I_{13}^2 \lesssim \left| \Delta^{-s} \left( \frac{(\phi^2 - 1)^2}{\rho} \right) \right| \right| \Delta^{-s} (\phi^2 - 1) \right| \lesssim \|\phi^2 - 1\|_{L^{2+s}} \|\phi^2 - 1\| \|\Delta^{-s} (\phi^2 - 1) \|
\]

\[ \lesssim \|\nabla (\phi^2 - 1)\|^{1/2+s} \|\nabla^2 (\phi^2 - 1)\|^{1/2-s} \|\nabla (\phi^2 - 1)\|^{1-\theta} \|\Delta^{-s} (\phi^2 - 1)\|^{1+\theta}
\]

\[ \lesssim (\|\nabla (\phi^2 - 1)\| + \|\nabla^2 (\phi^2 - 1)\|) \|\nabla (\phi^2 - 1)\| + \|\Delta^{-s} (\phi^2 - 1)\| \|\Delta^{-s} (\phi^2 - 1)\|\]

\[ \lesssim (\|\nabla (\phi^2 - 1)\|^{2} + \|\nabla^2 \phi \|^{2}) \|\Delta^{-s} (\phi^2 - 1)\| + M \|\Delta^{-s} (\phi^2 - 1)\|^{2}, \text{ with } \theta = \frac{1}{2+s}.
\]

Therefore, we obtain from (2.40) that

\[ I_{13} + \frac{2}{\epsilon \rho} \|\Delta^{-s} (\phi^2 - 1)\|^{2} \lesssim \left( \|\nabla \sigma \|_{H^{1}} + \|\nabla (\phi^2 - 1)\|^{2} + \|\nabla^2 \phi \|^{2} \right) \|\Delta^{-s} (\phi^2 - 1)\|.
\]

For \( I_{14} \), we can prove that

\[ I_{14} \lesssim \|\Delta^{-s} \nabla (-u \cdot \nabla \phi + h_{3}(\sigma) \Delta \phi) \| \|\Delta^{-s} \nabla \phi\|
\]

\[ \lesssim \|\nabla (-u \cdot \nabla \phi + h_{3}(\sigma) \Delta \phi) \|_{L^{2+s}} \|\Delta^{-s} \nabla \phi\|
\]

\[ \lesssim (\|u\|_{L^{2+s}} \|\nabla \phi\|_{L^{2+s}} + \|u\|_{L^{2+s}} \|\nabla \phi\|_{L^{2+s}}) \|\Delta^{-s} \nabla \phi\|
\]

\[ \lesssim (\|u\|_{L^{2+s}} + \|\nabla \phi\|_{L^{2+s}} + \|\sigma\|_{L^{2+s}}) \|\Delta^{-s} \nabla \phi\|
\]

\[ \lesssim (\|\nabla \phi\|_{L^{2+s}} + \|\nabla^2 \phi\|_{L^{2+s}} + \|\sigma\|_{L^{2+s}}) \|\Delta^{-s} \nabla \phi\|
\]

Last, for \( I_{15} \), we get

\[ I_{15} \lesssim \|\Delta^{-s} \left[ -u \cdot \nabla (\phi^2 - 1) + 2h_{3}(\sigma) \Delta (\phi^2 - 1) + \frac{2\epsilon}{\rho^{2}} \|\nabla \phi\|^{2} \right] \|\Delta^{-s} (\phi^2 - 1)\|
\]

\[ \lesssim \|\nabla (-u \cdot \nabla (\phi^2 - 1) + 2h_{3}(\sigma) \Delta (\phi^2 - 1) + \frac{2\epsilon}{\rho^{2}} \|\nabla \phi\|^{2}) \|_{L^{2+s}} \|\Delta^{-s} (\phi^2 - 1)\|
\]

\[ \lesssim (\|u\|_{L^{2+s}} + \|\nabla (\phi^2 - 1)\| + \|\sigma\|_{L^{2+s}}) \|\Delta^{-s} (\phi^2 - 1)\|
\]

\[ \lesssim \left( \|\nabla \phi\|_{L^{2+s}} + \|\nabla^2 \phi\|_{L^{2+s}} + \|\sigma\|_{L^{2+s}} \right) \|\Delta^{-s} (\phi^2 - 1)\|
\]

\[ \lesssim (\|\nabla \phi\|_{L^{2+s}} + \|\nabla^2 \phi\|_{L^{2+s}} + \|\sigma\|_{L^{2+s}}) \|\Delta^{-s} (\phi^2 - 1)\|
\]

Substituting the estimates on \( I_{i}(i = 9, \ldots, 15) \) into (2.36) and (2.38), respectively, we obtain (2.34).
Next, we derive (2.39). To this end, for \( s \in (\frac{1}{2}, \frac{3}{2}) \), we shall estimate the right-hand sides of (2.36) and (2.38) in a different way. Since \( s \in (\frac{1}{2}, \frac{3}{2}) \), we have that \( \frac{1}{2} + \frac{\theta}{3} < 1 \) and \( 2 < \frac{3}{s} < 6 \). Then, we have

\[
I_9 \lesssim \cdots \lesssim \| \nabla \phi \|_{L^\frac{2}{3}} \| \nabla^2 \phi \| \| \Lambda^{-s} u \|.
\]

Also, for \( s \in (\frac{1}{2}, \frac{3}{2}) \), by the same arguments as in [32, Section 3], we have

\[
I_{10} + I_{11} \lesssim \|(\sigma, u)\|^{s-\frac{1}{2}} \| \nabla(\sigma, u) \| \frac{3}{2} \| \Lambda^{-s} \sigma \| + \| \Lambda^{-s} u \|.
\]

For the estimate on \( I_{12} \), using (2.39), we have

\[
I_{12} \lesssim \cdots \lesssim \| \sigma \|_{L^\frac{2}{3}} \| \nabla \phi \| \| \Lambda^{-s} \nabla \phi \|
\]

\[
\lesssim \| \sigma \|^{s-\frac{1}{2}} \| \nabla \sigma \| \| \nabla^2 \phi \|^{1-\theta} \| \Lambda^{-s} \nabla \phi \|^{1+\theta}
\]

\[
\lesssim \| \sigma \|^{s-\frac{1}{2}} \| \nabla \sigma \| \| \nabla^2 \phi \| \| \Lambda^{-s} \nabla \phi \|^{1-\theta} \| \Lambda^{-s} \nabla \phi \|^{1+\theta}
\]

\[
\lesssim \| \sigma \|^{s-\frac{1}{2}} \| \nabla \sigma \| \| \nabla^2 \phi \| \| \Lambda^{-s} \nabla \phi \|^{1-\theta} \| \Lambda^{-s} \nabla \phi \|^{1+\theta}
\]

\[
\lesssim \| \sigma \|^{s-\frac{1}{2}} \| \nabla \sigma \| \| \nabla^2 \phi \| \| \Lambda^{-s} \nabla \phi \| + M \| \Lambda^{-s} \nabla \phi \|^2, \text{ with } \theta = \frac{1}{2} + s.
\]

\[
I_{12} \lesssim \cdots \lesssim \| \phi^2 - 1 \|_{L^\frac{2}{3}} \| \nabla \phi \| \| \Lambda^{-s} \nabla \phi \|
\]

\[
\lesssim \| \phi^2 - 1 \|^{s-\frac{1}{2}} \| \nabla (\phi^2 - 1) \| \| \nabla^2 \phi \|^{1-\theta} \| \Lambda^{-s} \nabla \phi \|^{1+\theta}
\]

\[
\lesssim \| \phi^2 - 1 \|^{s-\frac{1}{2}} \| \nabla (\phi^2 - 1) \| \| \nabla^2 \phi \| \| \Lambda^{-s} \nabla \phi \|^{1-\theta} \| \Lambda^{-s} \nabla \phi \|^{1+\theta}
\]

\[
\lesssim \| \phi^2 - 1 \|^{s-\frac{1}{2}} \| \nabla (\phi^2 - 1) \| \| \nabla^2 \phi \| \| \Lambda^{-s} \nabla \phi \| + M \| \Lambda^{-s} \nabla \phi \|^{2}, \text{ with } \theta = \frac{1}{2} + s,
\]

and

\[
I_{13} \lesssim \cdots \lesssim \| \phi^2 - 1 \|_{L^\frac{2}{3}} \| \nabla \sigma \| \| \Lambda^{-s} \nabla \phi \|
\]

\[
\lesssim \| \phi^2 - 1 \|^{s-\frac{1}{2}} \| \nabla (\phi^2 - 1) \| \| \nabla \sigma \| \| \Lambda^{-s} \nabla \phi \|.
\]

Therefore, we obtain from (2.39) that

\[
I_{12} + \frac{1}{\rho} \| \Lambda^{-s} \nabla \phi \|^2 \leq \|(\sigma, \phi^2 - 1)\|^{s-\frac{1}{2}} \| \nabla (\sigma, \phi^2 - 1, \nabla \phi)\| \| \Lambda^{-s} \nabla \phi \|.
\]

Similarly, for the estimate on \( I_{13} \), using (2.40), we have

\[
I_{13} \lesssim \cdots \lesssim \| \sigma \|_{L^\frac{2}{3}} \| \phi^2 - 1 \| \| \Lambda^{-s} (\phi^2 - 1) \|
\]

\[
\lesssim \| \sigma \|^{s-\frac{1}{2}} \| \nabla \sigma \| \| \nabla (\phi^2 - 1) \| \| \Lambda^{-s} (\phi^2 - 1) \|^{1+\theta}
\]

\[
\lesssim \| \sigma \|^{s-\frac{1}{2}} \| \nabla \sigma \| \| \nabla (\phi^2 - 1) \| \| \Lambda^{-s} (\phi^2 - 1) \|^{1+\theta}
\]

\[
\lesssim \| \sigma \|^{s-\frac{1}{2}} \| \nabla \sigma \| \| \Lambda^{-s} (\phi^2 - 1) \| \| \Lambda^{-s} (\phi^2 - 1) \| + M \| \Lambda^{-s} (\phi^2 - 1) \|^2,
\]

\[
\| \Lambda^{-s} \nabla \phi \|^2
\]

\[
\| \Lambda^{-s} (\phi^2 - 1) \| + M \| \Lambda^{-s} (\phi^2 - 1) \|^2,
\]

22
and

\[ I_{13} \lesssim \cdots \lesssim \| \phi^2 - 1 \|_{L^2} \| \phi^2 - 1 \| \Lambda^{-s} (\phi^2 - 1) \| \]

\[(1.10) \quad \lesssim \| \phi^2 - 1 \|^s \frac{1}{2} \| \nabla (\phi^2 - 1) \|^s \| \nabla (\phi^2 - 1) \|^{-1} \| \Lambda^{-s} (\phi^2 - 1) \|^{-1+\theta} \]

\[ \lesssim \| \phi^2 - 1 \|^s \frac{1}{2} \| \nabla (\phi^2 - 1) \|^s \| \nabla (\phi^2 - 1) \| + \| \Lambda^{-s} (\phi^2 - 1) \| \| \Lambda^{-s} (\phi^2 - 1) \| \]

\[ \lesssim \| \phi^2 - 1 \|^s \frac{1}{2} \| \nabla (\phi^2 - 1) \|^s \| \nabla (\phi^2 - 1) \| \| M \| \Lambda^{-s} (\phi^2 - 1) \| \|^2 , \text{ with } \theta = \frac{1}{2 + s} . \]

Therefore, we obtain from (2.40) that

\[ I_{13} + \frac{2}{\epsilon \rho} \| \Lambda^{-s} (\phi^2 - 1) \|^2 \lesssim \| (\sigma, \phi^2 - 1) \|^s \frac{1}{2} \| \nabla (\sigma, \phi^2 - 1) \| \| \Lambda^{-s} (\phi^2 - 1) \|. \]

For \( I_{14} \), we can prove that

\[ I_{14} \lesssim \cdots \]

\[ \lesssim \left( \| u \|_{L^2} + \| \nabla \phi \|_{L^2} + \| \sigma \|_{L^2} + \| \nabla \sigma \|_{L^2} \right) \left( \| \nabla u \| + \| \nabla^2 \phi \|_{H^1} \right) \| \Lambda^{-s} \nabla \phi \|

\[(1.11) \quad \lesssim \| u \|^s \frac{1}{2} \| \nabla u \|^s \frac{1}{2} + \| \nabla \phi \| s \frac{1}{2} \| \nabla^2 \phi \| \| \frac{1}{2} - s \| \nabla \sigma \| \| \frac{1}{2} - s \|

+ \| \sigma \| s \frac{1}{2} \| \nabla^2 \phi \| \| \frac{1}{2} - s \| \| \nabla \sigma \| \| \frac{1}{2} - s \|

\[ \lesssim \| (\sigma, u, \nabla \phi) \| s \frac{1}{2} \| \nabla (\sigma, u, \nabla \phi) \| \| \frac{1}{2} - s \| \| \nabla u \| + \| \nabla^2 \phi \|_{H^1} \| \Lambda^{-s} \nabla \phi \|

+ \| \nabla \sigma \|_{H^1}^2 + \| \nabla u \|^2 + \| \nabla^2 \phi \|_{H^1}^2 \| \Lambda^{-s} \nabla \phi \| . \]

Last, for \( I_{15} \), we get

\[ I_{15} \lesssim \cdots \]

\[ \lesssim \left( \| u \|_{L^2} \| \nabla (\phi^2 - 1) \| + \| \sigma \|_{L^2} \| \nabla^2 (\phi^2 - 1) \| + \| \nabla \|_{L^2} \| \nabla \phi \| \right) \| \Lambda^{-s} (\phi^2 - 1) \|

\[(1.10) \quad \lesssim \left( \| u \|^s \frac{1}{2} \| \nabla u \|^s \frac{1}{2} \| \nabla (\phi^2 - 1) \| + \| \sigma \|^s \frac{1}{2} \| \nabla \sigma \|^s \frac{1}{2} \| \nabla^2 \phi \| \right) \| \Lambda^{-s} (\phi^2 - 1) \|

\[(1.12) \quad + \| \nabla \phi \|^s \frac{1}{2} \| \nabla^2 \phi \|^s \frac{1}{2} \| \nabla^2 \phi \|^s \frac{1}{2} \| \Lambda^{-s} \nabla \phi \|^s \frac{1}{2} - \theta \| \Lambda^{-s} \nabla \phi \|^s \frac{1}{2} \| (\phi^2 - 1) \|

\[ \lesssim \left( \| u \|^s \frac{1}{2} \| \nabla u \|^s \frac{1}{2} + \| \sigma \|^s \frac{1}{2} \| \nabla \sigma \|^s \frac{1}{2} + \| \nabla \phi \|^s \frac{1}{2} \| \nabla^2 \phi \|^s \frac{1}{2} \right) \times \]

\[ \times \left( \| \nabla (\phi^2 - 1) \| + \| \nabla^2 \phi \| \right) \| \Lambda^{-s} (\phi^2 - 1) \| + M \left( \| \Lambda^{-s} \nabla \phi \|^2 + \| \Lambda^{-s} (\phi^2 - 1) \|^2 \right) , \]

with \( \theta = \frac{1}{2 + s} \). Substituting the estimates on \( I_i (i = 9, \cdots, 15) \) into (2.36) and (2.38), respectively, we obtain (2.35). The proof of Lemma 2.1 is completed.

3 The proof of Theorem 1.1

In this section, we shall combine all the estimates that we have derived in the previous section and the Sobolev interpolation to prove Theorem 1.1.
3.1 The a priori estimate and the existence for global solutions

In order to extend the local solution to global solution continuously, we first give a priori estimate by using the series Lemmas in section 2.

**Proposition 3.1 (a priori estimate).** There is positive constant $m_1 > 0$ and $M_1 > 0$, then, if $(\rho, u, \phi) \in X_{m_1, M_1}([0, T^*])$

\[
\sup_{t \in [0, T]} \{ \| (\rho - \bar{\rho}, u) (t) \|_{H^2}^2 + \| \nabla \phi (t) \|_{H^2}^2 + \| \phi^2 (t) - 1 \|^2 \} \\
+ \int_0^T \| \nabla \rho \|_{H^2}^2 \, d\tau + \int_0^{+\infty} \| \nabla u, \nabla \phi \|_{H^2}^2 \, d\tau \\
\lesssim \| \rho_0 - \bar{\rho} \|_{H^3}^2 + \| u_0 \|_{H^3}^2 + \| \nabla \phi_0 \|_{H^2}^2 + \| \phi_0^2 - 1 \|^2.
\]

**Proof:** We first close the energy estimates at each $l$-th level to prove (1.17). Let $0 \leq l \leq 2$. Summing up the estimate (2.18) of Lemma 2.2 for from $k = l$ to $2$, we obtain

\[
\frac{d}{dt} \sum_{1 \leq l \leq k \leq 2} \| \nabla^{k+1} \phi \|^2 + \sum_{1 \leq l \leq k \leq 2} \| \nabla^{k+2} \phi \|^2 \\
\lesssim M \sum_{1 \leq l \leq k \leq 2} \left( \| \nabla^{k+1} \sigma \|^2 + \| \nabla^{k+1} \phi \|^2 + \| \nabla^{k+2} u \|^2 \right).
\]

Also, summing up the estimate (2.25) of Lemma 2.3 for from $k = l$ to $3$, we obtain

\[
\frac{d}{dt} \sum_{0 \leq l \leq k \leq 3} \left( \| \nabla^k u \|^2 + \frac{p'(\hat{\rho})}{\hat{\rho}} \| \nabla^k \sigma \|^2 \right) + \sum_{0 \leq l \leq k \leq 3} \| \nabla^{k+1} u \|^2 \\
\lesssim M \sum_{0 \leq l \leq k \leq 3} \left( \| \nabla^k \sigma \|^2 + \| \nabla^{k+1} \phi \|^2 \right).
\]

Last, summing up the estimate (2.30) of Lemma 2.4 for from $k = l$ to $2$, we obtain

\[
\frac{d}{dt} \sum_{0 \leq l \leq k \leq 2} \int \nabla^k u \cdot \nabla^{k+1} \sigma \, dx + \sum_{0 \leq l \leq k \leq 2} \| \nabla^{k+1} \sigma \|^2 \\
\leq M \sum_{0 \leq l \leq k \leq 2} \left( \| \nabla^{k+2} u \|^2 + \| \nabla^{k+2} \phi \|^2 \right) + \sum_{0 \leq l \leq k \leq 2} \| \nabla^{k+1} u \|^2.
\]

Let $\eta \in (0, 1]$ be suitably small. Then, summing (2.5), (3.2), (3.3) and $\eta \cdot (3.4)$, and choosing $M > 0$ small enough, we obtain

\[
\frac{d}{dt} E_t (t) + \Lambda_t (t) \leq 0,
\]

where

\[
E_t (t) \overset{def}{=} \sum_{0 \leq l \leq k \leq 3} \left( \| \nabla^k u \|^2 + \frac{p'(\hat{\rho})}{\hat{\rho}} \| \nabla^k \sigma \|^2 \right) + \eta \sum_{0 \leq l \leq k \leq 2} \int \nabla^k u \cdot \nabla^{k+1} \sigma \, dx \\
+ \sum_{1 \leq l \leq k \leq 2} \| \nabla^{k+1} \phi \|^2 + \int \left( \rho u^2 + |\rho - \bar{\rho}|^2 + |\nabla \phi|^2 + (\phi^2 - 1)^2 \right) \, dx,
\]
and

\[
\Lambda_t(t) \overset{\text{def}}{=} \eta \sum_{1 \leq l+1 \leq k \leq 3} \|\nabla^k \sigma\|^2 + (1 - M) \sum_{2 \leq l+1 \leq k \leq 3} \|\nabla^{k+1} \phi\|^2 \\
+ (1 - \eta) \sum_{0 \leq l \leq k \leq 3} \|\nabla^{k+1} u\|^2 + \|\nabla u\|^2 + \|\mu\|^2
\]

(3.7)

Notice that since \(\eta \in (0, 1]\) and \(M > 0\) are suitably small, we obtain from (3.6) and (3.7) that

\[
\mathcal{E}_t^3(t) \leq \|\nabla^l(\sigma, u)(t)\|^2_{H^2} + \|\nabla^{l+1} \phi(t)\|^2_{H^{2-l}} + \|\phi^2 - 1\|^2, \\
\Lambda_t^3(t) \leq \|\nabla^{l+1} \sigma(t)\|^2_{H^{2-l}} + \|\nabla^{l+1} u(t)\|^2_{H^{2-l}} + \|\nabla^{l+1} \phi(t)\|^2_{H^{2-l}} + \|\mu\|^2
\]

uniformly for all \(t \geq 0\). Now taking \(l = 0\) and \(n = 3\) in (3.5), and then using (3.8), we get

\[
\|\sigma(t)\|^2_{H^3} + \|u(t)\|^2_{H^3} + \|\nabla \phi(t)\|^2_{H^3} + \|\phi^2(t) - 1\|^2 \\
\lesssim \|\rho_0 - \bar{\rho}\|^2_{H^3} + \|u_0\|^2_{H^3} + \|\nabla \phi_0\|^2_{H^3} + \|\phi^2_0 - 1\|^2.
\]

(3.9)

Using (1.16) and (3.9), by a standard continuity argument, we can close the a priori estimate (2.1). This in turn allows us to take \(l = 0\) in (3.5), and then integrate it directly in time to obtain (5.1).

From the a priori estimate (3.1) and the local existence of the solution \((\rho, u, \phi)\) for the Cauchy problem (1.7)-(1.9) (see Proposition (2.1)), we can construct a solution for \(t \in [0, T^*]\), which satisfies (2.4) in \(\mathbb{R}^3 \times [0, T^*]\), \(T^*\) only depends on the initial data of the Cauchy problem (1.7)-(1.9). From (3.9), one can start again from \(T^*\), by the same way, one can find a solution in \([T^*, 2T^*]\), and so on. Thus the existence and uniqueness of the global solution is obtained. Meanwhile, by using maximum principle, \(-1 \leq \phi \leq 1\).

### 3.2 Decay rate

We first prove (1.18)-(1.19) for \(s \in (0, \frac{1}{2}]\). Define

\[
\mathcal{E}_{-s} := \frac{p'(\rho)}{\rho^2} \|\Lambda^{-s} \sigma\|^2 + \|\Lambda^{-s} u\|^2 + \|\Lambda^{-s} \nabla \phi\|^2 + \|\Lambda^{-s} (\phi^2 - 1)\|^2.
\]

Then, integrating in time (2.33) and using (1.17), we obtain that

\[
\mathcal{E}_{-s}(t) \lesssim \mathcal{E}_{-s}(0) + \int_0^t \left(\|\nabla \sigma\|^2_{H^2} + \|\nabla (u, \nabla \phi)\|^2_{H^1} + \|\nabla (\phi^2 - 1)\|^2\right) \sqrt{\mathcal{E}_{-s}(\tau)} d\tau \\
\lesssim 1 + \sup_{0 \leq \tau \leq t} \mathcal{E}_{-s}(\tau), \text{ for } s \in (0, \frac{1}{2}],
\]

which implies (1.18). For the proof of (1.19), we need the following lemma.

**Lemma 3.1** Under the the assumption (2.1), it holds that

\[
\frac{1}{2} \frac{d}{dt} \|\nabla^k (\phi^2 - 1)\|^2 + \frac{\epsilon}{8\rho^2} \|\nabla^{k+1} (\phi^2 - 1)\|^2 + \frac{1}{2\rho^2} \|\nabla^{k+1} (\phi^2 - 1)\|^2 \\
\lesssim M \left(\|\nabla^{k+1} \sigma\|^2 + \|\nabla^{k+2} \phi\|^2 + \|\nabla^{k+1} u\|^2\right), \text{ for } k = 1, 2.
\]

(3.10)
Proof: Applying $\nabla^k$ to (2.37), and multiplying it by $\nabla^k (\phi^2 - 1)$, we have
\[
\frac{1}{2} \frac{d}{dt} \| \nabla^k (\phi^2 - 1) \|_2^2 + \int \frac{\epsilon}{\mu^2} |\nabla^{k+1} (\phi^2 - 1) |^2 \, dx + \int \frac{2}{\rho} |\nabla^k (\phi^2 - 1) |^2 \, dx
= \epsilon \sum_{1 \leq l \leq k} C^l_k \int \nabla^l \left( \frac{1}{\rho^2} \right) \nabla^k \Delta (\phi^2 - 1) \, dx \tag{3.11}
- \epsilon \int \nabla \left( \frac{1}{\rho^2} \right) \nabla^{k+1} (\phi^2 - 1) \, dx \tag{3.17}
- \frac{2}{\epsilon} \sum_{1 \leq l \leq k} C^l_k \int \nabla^l \left( \frac{1}{\rho^2} \right) \nabla^{k-l} (\phi^2 - 1) \, dx \tag{3.18}
- \int \nabla^{k-l} \left[ \nabla \cdot \nabla (\phi^2 - 1) \right] \, dx - 2\epsilon \int \nabla^k \left( \frac{\nabla \phi^2}{\rho} \right) \, dx \tag{3.19}
- \frac{2}{\epsilon} \sum_{0 \leq l \leq k} C^l_k \int \nabla^l \left( \frac{\phi^2 - 1}{\rho} \right) \nabla^{k-l} (\phi^2 - 1) \, dx . \tag{3.20}
\]
We estimate $I_i (i = 16, \ldots, 21)$. For $I_{16}$, we have
\[
I_{16} \lesssim \sum_{1 \leq l \leq k} \| \nabla^l \left( \frac{1}{\rho^2} \right) \nabla^{k-l} \Delta (\phi^2 - 1) \|_{L^6} \| \nabla^k (\phi^2 - 1) \|_{L^6} \tag{3.10}
\lesssim \sum_{1 \leq l \leq k} \| \nabla^l \left( \frac{1}{\rho^2} \right) \nabla^{k-l} \Delta (\phi^2 - 1) \|_{L^6} \| \nabla^{k+1} (\phi^2 - 1) \|. \tag{3.11}
\]
If $l \leq \left[ \frac{k}{2} \right]$, we get
\[
\| \nabla^l \left( \frac{1}{\rho^2} \right) \nabla^{k-l} \Delta (\phi^2 - 1) \|_{L^6} \lesssim \| \nabla^l \sigma \|_{L^3} \| \nabla^{k-l+2} (\phi^2 - 1) \|
\lesssim \| \nabla^l \sigma \|_{L^3} \| \nabla^{k-l+2} (\phi^2 - 1) \|
\lesssim M \left( \| \nabla^{k+1} \sigma \| + \| \nabla^{k+1} (\phi^2 - 1) \| \right) ,
\]
where $\alpha$ is defined by
\[
\frac{l - 1}{3} = \left( \frac{\alpha}{3} - \frac{1}{2} \right) \left( 1 - \frac{l - 1}{k + 1} \right) + \left( \frac{k + 1}{3} - \frac{1}{2} \right) \frac{l - 1}{k + 1}
\rightarrow \alpha = \frac{3}{2} \left( 1 + \frac{l - 1}{k + 2 - l} \right) \in \left[ \frac{3}{2}, 3 \right].
\]
If $\left[ \frac{k}{2} \right] + 1 \leq l \leq k$, then we get
\[
\| \nabla^l \left( \frac{1}{\rho^2} \right) \nabla^{k-l} \Delta (\phi^2 - 1) \|_{L^6} \lesssim \| \nabla^l \sigma \| \| \nabla^{k-l+2} (\phi^2 - 1) \|_{L^3}
\lesssim \| \sigma \|_{L^3} \| \nabla^{k+1} \sigma \|_{L^3} \| \nabla^l (\phi^2 - 1) \|_{L^3} \| \nabla^{k+1} (\phi^2 - 1) \|_{L^3}
\lesssim M \left( \| \nabla^{k+1} \sigma \| + \| \nabla^{k+1} (\phi^2 - 1) \| \right) ,
\]
where \( \alpha \) is defined by
\[
\frac{k + 1 - l}{3} = \left( \frac{\alpha}{3} - \frac{1}{2} \right) \frac{l}{k + 1} + \left( \frac{k + 1}{3} - \frac{1}{2} \right) \left( 1 - \frac{l}{k + 1} \right)
\]
\[\Rightarrow \alpha = \frac{3(k + 1)}{2l} \in \left[ \frac{3}{2}, 3 \right].\]

Therefore, we obtain
\[
|I_{16}| \lesssim M \left( \|\nabla^{k+1} \sigma\|^2 + \|\nabla^{k+1} (\phi^2 - 1)\|^2 \right).
\]

It is easy to check that
\[
I_{17} \lesssim \|\nabla \left( \frac{1}{\rho^2} \right) \|_{L^3} \|\nabla^{k+1} (\phi^2 - 1)\| \|\nabla^{k} (\phi^2 - 1)\|_{L^6}
\]
\[\lesssim \|\nabla \sigma\|_{L^3} \|\nabla^{k+1} (\phi^2 - 1)\| \lesssim M \|\nabla^{k+1} (\phi^2 - 1)\|^2.
\]

By the similar arguments, we have
\[
I_{18} \lesssim \sum_{1 \leq |l| \leq k} \|\nabla^l \left( \frac{1}{\rho} \right) \nabla^{k-l} (\phi^2 - 1)\| \|\nabla^{k} (\phi^2 - 1)\|.
\]

If \( l \leq \left\lfloor \frac{k}{2} \right\rfloor \), we get
\[
\|\nabla^l \left( \frac{1}{\rho} \right) \nabla^{k-l} (\phi^2 - 1)\| \lesssim \|\nabla^l \sigma\|_{L^3} \|\nabla^{k-l} (\phi^2 - 1)\|_{L^6}
\]
\[\lesssim \|\nabla^\alpha \sigma\|^{1-\frac{1}{k}} \|\nabla^{k+1} \sigma\|^{\frac{1}{k}} \|\nabla (\phi^2 - 1)\|^{\frac{1}{k}} \|\nabla^{k+1} (\phi^2 - 1)\|^{1-\frac{1}{k}}
\]
\[\lesssim M \left( \|\nabla^{k+1} \sigma\| + \|\nabla^{k+1} (\phi^2 - 1)\| \right),
\]
where \( \alpha \) is defined by
\[
\frac{l - 1}{3} = \left( \frac{\alpha}{3} - \frac{1}{2} \right) \left( 1 - \frac{l}{k} \right) + \left( \frac{k + 1}{3} - \frac{1}{2} \right) \frac{l}{k}
\]
\[\Rightarrow \alpha = \frac{1}{2} - \frac{l}{2(k - l)} \in \left[ 0, \frac{1}{2} \right]. \tag{3.12}
\]

If \( \left\lceil \frac{k}{2} \right\rceil + 1 \leq l \leq k \), then we get
\[
\|\nabla^l \left( \frac{1}{\rho} \right) \nabla^{k-l} (\phi^2 - 1)\| \lesssim \|\nabla^l \sigma\|_{L^3} \|\nabla^{k-l} (\phi^2 - 1)\|_{L^3}
\]
\[\lesssim \|\nabla^\alpha \sigma\|^{1-\frac{1}{k}} \|\nabla^{k+1} \sigma\|^{\frac{1}{k}} \|\nabla^\alpha (\phi^2 - 1)\|^{\frac{1}{k}} \|\nabla^{k+1} (\phi^2 - 1)\|^{1-\frac{1}{k}}
\]
\[\lesssim M \left( \|\nabla^{k+1} \sigma\| + \|\nabla^{k+1} (\phi^2 - 1)\| \right),
\]
where \( \alpha \) is defined by
\[
\frac{k - l - 1}{3} = \left( \frac{\alpha}{3} - \frac{1}{2} \right) \frac{l}{k} + \left( \frac{k + 1}{3} - \frac{1}{2} \right) \left( 1 - \frac{l}{k} \right) \Rightarrow \alpha = 1 - \frac{k}{2l} \in [0, 1].
\] (3.13)

Therefore, we obtain
\[
|I_{18}| \lesssim M \left( \| \nabla^{k+1} \sigma \|^2 + \| \nabla^k (\phi^2 - 1) \|^2 + \| \nabla^{k+1} (\phi^2 - 1) \|^2 \right).
\]

Also, for \( I_{19} \), we have
\[
I_{19} = \int \mathbf{u} \cdot \nabla^{k+1} (\phi^2 - 1) \nabla^k (\phi^2 - 1) \, d\mathbf{x}
+ \sum_{1 \leq l \leq k} C^l_1 \int \nabla^l \mathbf{u} \cdot \nabla^{k+1-l} (\phi^2 - 1) \nabla^k (\phi^2 - 1) \, d\mathbf{x}
\lesssim M \left( \| \nabla^k (\phi^2 - 1) \|^2 + \| \nabla^{k+1} (\phi^2 - 1) \|^2 \right)
+ \sum_{1 \leq l \leq k} \| \nabla^l \mathbf{u} \cdot \nabla^{k-l+1} (\phi^2 - 1) \| \| \nabla^k (\phi^2 - 1) \|.
\]

If \( l \leq \left\lfloor \frac{k+1}{2} \right\rfloor \), we get
\[
\| \nabla^l \mathbf{u} \cdot \nabla^{k+1-l} (\phi^2 - 1) \| \lesssim \| \nabla^l \mathbf{u} \|_{L^3} \| \nabla^{k+1-l} (\phi^2 - 1) \|_{L^6}
\lesssim \| \nabla^\alpha \mathbf{u} \|^{1-l-1} \| \nabla^{k+1} \mathbf{u} \|^{\frac{l-1}{k}} \| \nabla (\phi^2 - 1) \|^{\frac{l-1}{k}} \| \nabla^{k+1} (\phi^2 - 1) \|^{1-\frac{l}{k}}
\lesssim M \left( \| \nabla^{k+1} \mathbf{u} \| + \| \nabla^{k+1} (\phi^2 - 1) \| \right),
\]
where \( \alpha \) is defined by
\[
\frac{l - 1}{3} = \left( \frac{\alpha}{3} - \frac{1}{2} \right) \frac{1 - l}{k} + \left( \frac{k + 1}{3} - \frac{1}{2} \right) \frac{l - 1}{k}
\Rightarrow \alpha = \frac{3}{2} + \frac{l - 1}{2(k + 1 - l)} \in \left[ \frac{3}{2}, \frac{5}{2} \right].
\]

If \( \left\lceil \frac{k+1}{2} \right\rceil + 1 \leq l \leq k \), then we get
\[
\| \nabla^l \mathbf{u} \nabla^{k+1-l} (\phi^2 - 1) \| \lesssim \| \nabla^l \mathbf{u} \|_{L^6} \| \nabla^{k+1-l} (\phi^2 - 1) \|_{L^3}
\lesssim \| \nabla \mathbf{u} \|^{1-\frac{l}{k}} \| \nabla^{k+1} \mathbf{u} \|^{\frac{l}{k}} \| \nabla^\alpha (\phi^2 - 1) \|^{\frac{l}{k}} \| \nabla^{k+1} (\phi^2 - 1) \|^{1-\frac{l}{k}}
\lesssim M \left( \| \nabla^{k+1} \mathbf{u} \| + \| \nabla^{k+1} (\phi^2 - 1) \| \right),
\]
where \( \alpha \) is defined by
\[
\frac{k - l}{3} = \left( \frac{\alpha}{3} - \frac{1}{2} \right) \frac{l}{k} + \left( \frac{k + 1}{3} - \frac{1}{2} \right) \left( 1 - \frac{l}{k} \right)
\Rightarrow \alpha = 1 + \frac{k}{2l} \in [1, 2].
\]
Therefore, we obtain

\[ |I_{19}| \lesssim M \left( \| \nabla^{k+1} u \|^2 + \| \nabla^k (\phi^2 - 1) \|^2 + \| \nabla^{k+1} (\phi^2 - 1) \|^2 \right). \]

Similarity, we have

\[
I_{20} = \int \frac{\nabla \phi}{\rho^2} \cdot \nabla^{k+2} \phi \nabla^k (\phi^2 - 1) \, dx \\
+ \sum_{1 \leq l \leq k} C_k \int \nabla^l \left( \frac{\nabla \phi}{\rho^2} \right) \cdot \nabla^{k+1-l} \phi \nabla^k (\phi^2 - 1) \, dx \\
\lesssim M \left( \| \nabla^k (\phi^2 - 1) \|^2 + \| \nabla^{k+1} \phi \|^2 \right) \\
+ \sum_{1 \leq l \leq k} \| \nabla^l \left( \frac{\nabla \phi}{\rho^2} \right) \cdot \nabla^{k-l+1} \phi \| \| \nabla^k (\phi^2 - 1) \|. 
\]

If \( l \leq \left[ \frac{k+1}{2} \right] \), we get

\[
\| \nabla^l \left( \frac{\nabla \phi}{\rho^2} \right) \cdot \nabla^{k+1-l} \phi \| \lesssim \left( \| \nabla^{l+1} \phi \|_{L^3} + \| \nabla^l \sigma \|_{L^3} \right) \| \nabla^{k+1-l} \phi \|_{L^6} \\
\lesssim \left( \| \nabla^{k+2} \phi \| \|| \nabla^{k+1} \sigma \| + \| \nabla^k \sigma \| \right) \| \nabla^{k+1-l} \phi \|
\]

where \( \alpha \) is defined by

\[
\frac{l - 1}{3} = \left( \frac{\alpha}{3} - \frac{1}{2} \right) \left( 1 - \frac{l}{k+1} \right) + \left( \frac{k+1}{3} - \frac{1}{2} \right) \frac{l}{k+1} \\
\Rightarrow \alpha = \frac{k+1}{2(k+1-l)} \in [0, 1].
\]

If \( \left[ \frac{k+1}{2} \right] + 1 \leq l \leq k \), then we get

\[
\| \nabla^l \left( \frac{\nabla \phi}{\rho^2} \right) \cdot \nabla^{k+1-l} \phi \| \lesssim \left( \| \nabla^{l+1} \phi \|_{L^6} + \| \nabla^l \sigma \|_{L^6} \right) \| \nabla^{k+1-l} \phi \|_{L^3} \\
\lesssim \left( \| \nabla^{k+2} \phi \| \| \nabla^{k+1} \sigma \| + \| \nabla^k \sigma \| \| \nabla^{k+1} \sigma \| \right) \| \nabla^{k+1-l} \phi \|
\]

where \( \alpha \) is defined by

\[
\frac{k-l}{3} = \left( \frac{\alpha}{3} - \frac{1}{2} \right) \frac{1}{k} + \left( \frac{k+2}{3} - \frac{1}{2} \right) \left( 1 - \frac{l}{k} \right) \Rightarrow \alpha = \frac{2 - \frac{k}{2l}}{2} \in [1, 2].
\]

Therefore, we obtain

\[
|I_{20}| \lesssim M \left( \| \nabla^{k+1} \sigma \|^2 + \| \nabla^k (\phi^2 - 1) \|^2 + \| \nabla^{k+2} \phi \|^2 \right). 
\]
Last, we have

\[ I_{21} \lesssim M \| \nabla^k (\phi^2 - 1) \|^2 + \sum_{1 \leq l \leq k} \| \nabla^l \left( \frac{\phi^2 - 1}{\rho} \right) \nabla^{k-l} (\phi^2 - 1) \| \nabla^k (\phi^2 - 1) \| . \]

If \( l \leq \left[ \frac{k}{2} \right] \), we get

\[
\| \nabla^l \left( \frac{\phi^2 - 1}{\rho} \right) \nabla^{k-l} (\phi^2 - 1) \| \lesssim \left( \| \nabla^l (\phi^2 - 1) \|_{L^2} + \| \nabla^l \sigma \|_{L^2} \right) \| \nabla^{k-l} (\phi^2 - 1) \|_{L^6}
\]

\[
\lesssim \left( \| \nabla^l (\phi^2 - 1) \|_{L^6} + \| \nabla^l \sigma \|_{L^6} \right) \| \nabla^{k-l} (\phi^2 - 1) \|_{L^6}
\]

\[
\lesssim \left( \| \nabla (\phi^2 - 1) \|_{L^6} \| \nabla^l (\phi^2 - 1) \|_{L^6} + \| \nabla \sigma \|_{L^6} \right) \times \| \nabla (\phi^2 - 1) \|_{L^6} \| \nabla^{k-l} (\phi^2 - 1) \|_{L^6}
\]

where \( \alpha \) is defined by (3.12). If \( \left[ \frac{k}{2} \right] + 1 \leq l \leq k \), then we get

\[
\| \nabla^l \left( \frac{\phi^2 - 1}{\rho} \right) \nabla^{k-l} (\phi^2 - 1) \| \lesssim \left( \| \nabla^l (\phi^2 - 1) \|_{L^6} + \| \nabla^l \sigma \|_{L^6} \right) \| \nabla^{k-l} (\phi^2 - 1) \|_{L^6}
\]

\[
\lesssim \left( \| \nabla (\phi^2 - 1) \|_{L^6} \| \nabla^l (\phi^2 - 1) \|_{L^6} + \| \nabla \sigma \|_{L^6} \right) \times \| \nabla^l (\phi^2 - 1) \|_{L^6} \| \nabla^{k-l} (\phi^2 - 1) \|_{L^6}
\]

where \( \alpha \) is defined by (3.13). Therefore, we obtain

\[ |I_{21}| \lesssim M \left( \| \nabla^{k+1} \sigma \|^2 + \| \nabla^k (\phi^2 - 1) \|^2 + \| \nabla^{k+2} \phi \|^2 \right). \]

Substituting the estimates on \( I_i(i = 16, \cdots, 21) \) into (3.22) and using (2.24), we obtain (3.10). The proof of Lemma 3.1 is completed.

We will continue the proof of (1.19) for \( s \in (0, \frac{1}{2}] \). Summing up the estimate (3.10) of Lemma 3.1 for from \( k = l \) to 2, we obtain

\[ \frac{1}{2} \frac{d}{dt} \| \nabla^l (\phi^2 - 1) \|^2_{H^{2-l}} + \frac{\epsilon}{8\rho^2} \| \nabla^{l+1} (\phi^2 - 1) \|^2_{H^{2-l}} \leq M \left( \| \nabla^{l+1} \sigma \|^2_{H^{2-l}} + \| \nabla^{l+1} u \|^2_{H^{2-l}} + \| \nabla^{l+2} \phi \|^2_{H^{2-l}} \right) \]

for \( l = 1, 2 \). Therefore, after summing (3.5) and (3.14), we have

\[ \frac{d}{dt} F_l(t) + G_l(t) \leq 0, \]

(3.15)
where
\[
F_l(t) := E_l(t) + \frac{1}{2} \| \nabla^l (\phi^2 - 1) \|_{H^{2-l}}^2,
\]
\[
G_l(t) := \Lambda_l(t) + \frac{\epsilon}{8p^2} \| \nabla^{l+1} (\phi^2 - 1) \|_{H^{2-l}}^2
- C_0 M \left( \| \nabla^{l+1} \sigma \|_{H^{2-l}}^2 + \| \nabla^{l+1} \mathbf{u} \|_{H^{3-l}}^2 + \| \nabla^{l+2} \phi \|_{H^{2-l}}^2 \right).
\]

Noticing that \( M > 0 \) is small, and using (3.6) and (3.7), we obtain from (3.10) that
\[
F_l(t) \leq \| \nabla^l (\sigma, \mathbf{u}) (t) \|_{H^{2-l}}^2 + \| \nabla^l (\nabla \phi, \phi^2 - 1) (t) \|_{H^{2-l}}^2,
\]
\[
G_l(t) \leq \| \nabla^{l+1} (\sigma, \nabla \phi, \phi^2 - 1) (t) \|_{H^{3-l}}^2 + \| \nabla^{l+1} \mathbf{u} (t) \|_{H^{3-l}}^2
\]
uniformly for all \( t \geq 0 \). If \( l = 0, 1, 2 \), we may use (1.12) to have
\[
\| \nabla^{l+1} f \| \gtrsim \| \Lambda^{-s} f \|_{\frac{1}{1+s}} \| \nabla^l f \|_{\frac{1}{1+s}}. \tag{3.18}
\]

By (3.18) and (1.18), we get
\[
\| \nabla^{l+1} (\sigma, \mathbf{u}, \nabla \phi, \phi^2 - 1) \| \gtrsim \| \nabla^l (\sigma, \mathbf{u}, \nabla \phi, \phi^2 - 1) \|_{\frac{1}{1+s}},
\]
which implies
\[
\| \nabla^{l+1} (\sigma, \nabla \phi, \phi^2 - 1) (t) \|_{H^{2-l}}^2 + \| \nabla^{l+1} \mathbf{u} (t) \|_{H^{3-l}}^2
\gtrsim \left( \| \nabla^l (\sigma, \nabla \phi, \phi^2 - 1) (t) \|_{H^{2-l}}^2 + \| \nabla^l \mathbf{u} (t) \|_{H^{3-l}}^2 \right)^{1+\frac{1}{1+s}}. \tag{3.19}
\]

Also, using \( \| \sigma(t) \|_{H^3} \leq 1 \) due to (1.17), we have
\[
\| \nabla^3 \sigma \| \gtrsim \| \nabla^3 \sigma \|_{1+\frac{1}{1+s}}. \tag{3.20}
\]

By (3.19) and (3.20), we obtain from (3.8) that
\[
G_l(t) \gtrsim (F_l(t))^{1+\frac{1}{1+s}}. \tag{3.21}
\]

Using (3.21), we deduce from (3.15) with \( m = 3 \) that
\[
\frac{d}{dt} F_l(t) + C_0 (F_l(t))^{1+\frac{1}{1+s}} \leq 0.
\]

Solving this inequality directly gives (see (1.24))
\[
F_l(t) \lesssim (1 + t)^{-(l+s)},
\]
which implies (1.19) for \( s \in \left[ 0, \frac{1}{2} \right] \), that is,
\[
\| \nabla^l (\sigma, \mathbf{u}) (t) \|_{H^{2-l}}^2 + \| \nabla^l (\nabla \phi, \phi^2 - 1) (t) \|_{H^{2-l}}^2 \lesssim (1 + t)^{-(l+s)},
\]
due to (3.17).

Next, we prove (1.18)-(1.19) for \( s \in \left( \frac{1}{2}, 1 \right) \). Notice that the arguments for the case \( s \in \left[ 0, \frac{1}{2} \right] \) can not be applied to this case. However, observing that we have \( (\sigma_0, \mathbf{u}_0, \nabla \phi_0) \in \)
\[ \dot{\mathcal{H}}^{\frac{1}{2}}(\mathbb{R}^3) \cap \mathcal{H}^{-s}(\mathbb{R}^3) \subset \dot{\mathcal{H}}^{-s}(\mathbb{R}^3) \] for any \( s' \in [0, s] \), we then deduce from what we have proved for (1.19) with \( s = \frac{1}{2} \) that the following decay result holds:

\[
\| \nabla^l (\sigma, u)(t) \|_{\mathcal{H}^{3-l}}^2 + \| \nabla^l (\nabla \phi, \phi^2 - 1)(t) \|_{\mathcal{H}^{2-l}}^2 \lesssim (1 + t)^{- (l + \frac{1}{2})},
\]

(3.22)

for \( l = 0, 1, 2 \). Hence, by (3.22), we deduce from (2.35) and (1.17) that for \( s \in (\frac{1}{2}, \frac{3}{2}) \),

\[
\mathcal{E}_{-s}(t) \lesssim \mathcal{E}_{-s}(0) + \int_0^t \| (\sigma, u, \phi^2 - 1, \nabla \phi) \|^{s - \frac{1}{2}} \left( \| \nabla (\sigma, u, \nabla \phi) \|_{L^2(\mathbb{R}^3)} + \| \nabla (\phi^2 - 1) \| \right)^{\frac{5}{2} - s} \sqrt{\mathcal{E}_{-s}(\tau)} d\tau
\]

\[
\lesssim 1 + \int_0^t (1 + \tau)^{- \alpha} \sqrt{\mathcal{E}_{-s}(\tau)} d\tau \lesssim 1 + \sup_{0 \leq \tau \leq t} \sqrt{\mathcal{E}_{-s}(\tau)},
\]

where \( \alpha \) is defined by

\[
\alpha = \frac{1}{4} \left( s - \frac{1}{2} \right) + \frac{1}{2} \left( 1 + \frac{1}{2} \right) \left( \frac{5}{2} - s \right) > 1, \quad \text{since} \quad s < \frac{3}{2},
\]

This implies (1.18) for \( s \in (\frac{1}{2}, \frac{3}{2}) \), that is

\[
\left\| (\rho - \bar{\rho}, u, \nabla \phi, \phi^2 - 1)(t) \right\|_{\mathcal{H}^{-s}} \lesssim 1, \quad \text{for} \quad s \in \left( \frac{1}{2}, \frac{3}{2} \right).
\]

(3.23)

Now that we have proved (3.23), we may repeat the arguments leading to (1.19) for \( s \in [0, \frac{1}{2}] \) to prove that they hold also for \( s \in (\frac{1}{2}, \frac{3}{2}) \). The proof of (1.18)-(1.19) for \( s \in [0, \frac{3}{2}] \) is completed.

**Conflict of Interests**

The authors declare that there is no conflict of interest regarding the publication of this paper.

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