Next-to-leading-logarithm $k_T$ resummation for $B_c \to J/\psi$ decays

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We derive the $k_T$ resummation for a transverse-momentum-dependent charmonium wave function, which involves the bottom quark mass $m_b$, the charm quark mass $m_c$, and the charm quark transverse momentum $k_T$, up to the next-to-leading-logarithm (NLL) accuracy under the hierarchy $m_b \gg m_c \gg k_T$. The resultant Sudakov factor is employed in the perturbative QCD (PQCD) approach to the $B_c \to J/\psi$ transition form factor $A_{B_c \to J/\psi}^{(0)}(0)$ and the $B_c^+ \to J/\psi \pi^+$ decay. We compare the NLL resummation effect on these processes with the leading-logarithm one in the literature, and find a $(5-10)\%$ enhancement to the form factor $A_{B_c \to J/\psi}^{(0)}(0)$ and a $(10-20)\%$ enhancement to the decay rate $\text{BR}(B_c^+ \to J/\psi \pi^+)$. The improved $k_T$ resummation formalism is applicable to the PQCD analysis of heavy meson decays to other charmonia.

$B_c$ meson decays contain rich heavy quark dynamics in both perturbative and nonperturbative regimes, that is worth of thorough exploration with high precision. It is thus crucial to develop an appropriate theoretical framework for analyzing $B_c$ meson decays, for which data have been accumulated rapidly. A framework available in the literature is the perturbative QCD (PQCD) approach, which basically follows the conventional one for $B$ meson decays: the dependence on the finite charm quark mass is included in hard decay kernels but neglected in the $k_T$ resummation formula for meson wave functions [1−10]. Hence, a theoretical challenge from these decays is to derive the $k_T$ resummation associated with energetic charm quarks with a finite mass. Such a rigorous $k_T$ resummation formalism for a typical transition $B_c \to J/\psi$ was first attempted in [11]. The derivation relies on the power counting for the involved multiple scales, the bottom (charm) quark mass $m_b$ ($m_c$) and the parton transverse momentum $k_T$. We have adopted the power counting rule proposed in [12], which obeys the hierarchy $m_b \gg m_c \gg k_T$. An intermediate impact of this hierarchy is that the large infrared logarithms $\ln(m_b/k_T)$, in addition to the ordinary ones $\ln(m_b/k_T)$, appear in the PQCD evaluation of $B_c$ meson decays, and need to be resummed.

To proceed the $k_T$ resummation, we considered the $B_c \to J/\psi$ transition process, constructed the transverse-momentum-dependent (TMD) $B_c$ and $J/\psi$ meson wave functions in the $k_T$ factorization theorem [13, 14], and then performed the one-loop evaluation according to the wave-function definition as a nonlocal hadronic matrix element. The large logarithms attributed to the overlap of the collinear and soft radiative corrections were found to differ from those in $B$ meson decays into light mesons [15], because of the additional charm quark scale. However, only the leading double logarithms from the correction to the quark-Wilson-line vertex in meson wave functions were captured in [11], namely, the $k_T$ resummation for the $B_c \to J/\psi$ decays was achieved at the leading-logarithm (LL) accuracy so far. How the charm quark mass dependence in the LL $k_T$ resummation affects the $B_c \to J/\psi$ transition form factor and the $B_c^+ \to J/\psi \pi^+$ branching ratio was then investigated [11].

In this letter we will complete the $k_T$ resummation for the $B_c$ and $J/\psi$ meson wave functions up to the next-to-leading-logarithm (NLL) accuracy. Since the analysis involves the convolution with the corresponding hard decay kernel at the NLL accuracy, it is more convenient to perform the resummation in the impact parameter $b$ space, which is conjugate to the transverse momentum $k_T$. We start with the one-loop calculation for the $J/\psi$ meson wave function, from which all important logarithms are identified. It is found that these logarithms are grouped into two sets, $\ln(m_b b)$ and $\ln(m_c b)$, with their coefficients being identical but opposite in sign. It hints that the resummation technique can be applied to these two sets of logarithms separately: the $k_T$ resummation is constructed for the first set, that for the second set can be inferred trivially via the replacement of the argument $m_b$ by $m_c$, and the final result is given by the difference between them. Moreover, the resummation technique applied to the first set of logarithms is the same as the one applied to a light meson case [16] without the intermediate scale $m_c$. We emphasize that the matching to the one-loop $J/\psi$ meson wave function is crucial for achieving the NLL accuracy. The NLL $k_T$ resummation for the $B_c$ meson wave function is then done in a similar way. At last, the NLL resummation effect is employed.
in the PQCD evaluation of the $B_c \to J/\psi$ transition form factor and the $B_c^+ \to J/\psi \pi^+$ branching ratio, and compared with the LL effect observed in [11].

Consider the $B_c(P_1) \to J/\psi (P_2)$ transition at the maximal recoil, where

$$P_1 = \frac{m_{B_c}}{\sqrt{2}} (1, 1, 0_T), \quad P_2 = \frac{m_{B_c}}{\sqrt{2}} (1, r_{J/\psi}^2, 0_T),$$

(1)

in the light-cone coordinates label the $B_c$ and $J/\psi$ meson momenta, respectively, with $r_{J/\psi} = m_{J/\psi}/m_{B_c}, m_{B_c} (m_{J/\psi})$ being the $B_c (J/\psi)$ meson mass. The $J/\psi$ meson wave function $\Phi_{J/\psi}$ factorized out of the above transition process depend on two external vectors, the momentum $P_2$ and the direction $n$ of the gauge link [11]. Because the Feynman rule for the gauge link is scale-invariant in $n$, $\Phi_{J/\psi}$ must depend on $n$ through the ratio of the Lorentz invariants, $\xi^2 = 4(P_2 \cdot n)^2/n^2$ [17, 18]. According to [13], a parton, carrying a longitudinal momentum initially, gains transverse momenta by radiating gluons, which generate the $k_T$ dependence of a TMD wave function. We then write the $J/\psi$ meson wave function $\Phi_{J/\psi}(x, b, \xi^2, m_c)$ in the impact parameter space, $x$ being the momentum fraction carried by the spectator charm quark.

![FIG. 1. One-loop vertex corrections to the $J/\psi$ meson wave function.](image)

To identify the important logarithms in the $J/\psi$ meson wave function, we calculate the one-loop effective diagrams displayed in Fig. 1 with an on-shell charm quark. Assume that the spectator charm quark carries the momentum $k = x P_2$, and the upper charm quark carries $k \equiv P_2 - k = (1 - x) P_2$. Figure 1(a), which does not induce a transverse momentum of the charm quark, gives the loop integral

$$\Phi_a^{(1)} = -\frac{i}{4} g^2 \frac{\alpha_s}{4 \pi} \frac{\mu_F^2}{m_c^2} 2 \epsilon \int \frac{d^4 l}{(2\pi)^4} \text{tr} \left[ \gamma_5 \not{\psi} + \frac{k + l + m_c}{(k + l)^2 - m_c^2} \gamma_\nu \not{x} - \gamma_5 \right] \frac{1}{l^2 - m_g^2 n \cdot l}$$

(2)

with the color factor $C_F = 4/3$, the on-shell condition $k^2 \approx m_c^2$, the factorization scale $\mu_f$, the gluon momentum $l$, and the gluon mass $m_g$ as an infrared regulator. The projectors $\gamma_5 \not{x} + \not{y} - \gamma_5$ select the leading twist contribution, and $1/ε \to \epsilon \cdot l$ represents the Feynman rule associated with the gauge link. A straightforward computation yields

$$\Phi_a^{(1)} = \frac{\alpha_s}{4 \pi} C_F \left[ \frac{1}{\epsilon} + \ln \frac{4 \pi \mu_f^2}{m_c^2} - 2 \ln \frac{(1 - x)^2 \xi^2}{m_c^2} \ln \frac{(1 - x)^2 \xi^2}{m_c^2} + 2 \ln \frac{(1 - x)^2 \xi^2}{m_c^2} + 2 - \frac{\pi^2}{3} \right],$$

(3)

where $1/ε$ denotes an ultraviolet divergence and $\gamma_E$ is the Euler constant. It is clear that the collinear divergence regularized by the charm quark mass $m_c$ and the soft divergence regularized by the gluon mass $m_g$ have overlapped to produce the double logarithm in the above expression.

For Fig. 1(b), the transverse loop momentum $l_T$ flows through the hard decay kernel, and this $l_T$ dependence is negligible in the $k_T$ factorization [13]. To facilitate the loop calculation, we perform the Fourier transformation to turn the convolution between the hard decay kernel and the $J/\psi$ meson wave function into a product, and write the integral for the latter in the impact parameter $b$ space as

$$\Phi_b^{(1)} = \frac{i}{4} g^2 C_F \int \frac{d^4 l_T}{(2\pi)^4} \exp(i l_T \cdot b) \text{tr} \left[ \gamma_5 \not{x} + \frac{k + l + m_c}{(k + l)^2 - m_c^2} \gamma_\nu \not{y} - \gamma_5 \right] \frac{1}{l_T^2 - m_g^2 n \cdot l}. \quad (4)

After performing the integration, we find

$$\Phi_b^{(1)} = \frac{\alpha_s}{4 \pi} C_F \left[ \frac{1}{2} \ln \frac{(1 - x)^2 \xi^2}{m_c^2} + 2 \ln \frac{(1 - x)^2 \xi^2}{m_c^2} \ln \frac{2(1 - x) \xi}{m_g^2 e^{\gamma_E}} \right]. \quad (5)

Compared to Eq. (3), the above expression is free of an ultraviolet divergence due to the Fourier factor $\exp(i l_T \cdot b)$. Note that the integration over the transverse momentum $l_T$ in the presence of $\exp(i l_T \cdot b)$ generates a Bessel function $K_0$, which can be approximated by a logarithmic function as its argument approaches to zero. Hence, Eq. (5) is valid only up to the logarithmic term, strictly speaking. This approximation works well enough for the matching between the NLL resummation and the one-loop result.
The sum of Eqs. (3) and (5) gives

$$\Phi_{a+b}^{(1)} = \frac{\alpha_s}{4\pi} C_F \left[ \frac{1}{\epsilon} + \ln \frac{4\pi\mu^2}{m_c^2\epsilon} + \frac{1}{2} \ln 2 \frac{(1-x)^2\xi^2}{m_c^2} - \ln \frac{1}{m_c^2} \ln \frac{(1-x)\epsilon\gamma E}{2} + \ln \frac{(1-x)^2\xi^2}{m_c^2} + 2 - \frac{\pi^2}{3} \right].$$

It is seen in the first line that the infrared regulator \( m_g \) has been cancelled as expected, because Eqs. (2) and (4) become identical except for a sign difference in the soft limit of the loop momentum \( l \to 0 \), and that the soft scale has been replaced by \( 1/b \). In the second line we have reorganized the sum into the desired form: the logarithms can be grouped into two sets, one containing \( \ln(\xi b) \) and another containing \( \ln(m_c b) \), as postulated in [11]. The difference between them arises only from the arguments \( (1-x)\xi \) and \( m_c \), and from the opposite sign. The ultraviolet logarithm can be removed by choosing the factorization scale \( \mu_f = m_c \) which serves as the characteristic scale of the \( J/\psi \) meson wave function.

A remark is in order. It has been elaborated recently [19] that the coefficient of a double logarithm associated with an on-shell parton is half of the coefficient in the off-shell case. Taking Fig. 1(a) as an example, we have evaluated its contribution for an energetic charm quark off-shell by \(-k_T^2\), and obtained [11]

$$\Phi_{a}^{(1)} = \frac{\alpha_s}{4\pi} C_F \left[ \frac{1}{\epsilon} + \ln \frac{4\pi\mu^2}{m_c^2\epsilon} - \ln^2 \frac{(1-x)^2\xi^2}{k_T^2} + \ln^2 \frac{m_c^2}{k_T^2} + \ln \frac{(1-x)^2\xi^2}{m_c^2} + 2 - \frac{2\pi^2}{3} \right].$$

Comparing Eq. (7) with Eq. (3), we indeed find that the coefficients of the double logarithms have been reduced to half in the on-shell case. We explain that the fixed-order calculation with an off-shell quark is required for the proof of the \( k_T \) factorization [13], in which the common parton virtuality \(-k_T^2\) is adopted to regularize the infrared divergences in both QCD and effective diagrams. The \( k_T \) factorization holds, if the infrared logarithms \( \ln k_T^2 \) could be shown to cancel between these two sets of diagrams [13]. As deriving the \( k_T \) resummation formula, we consider an on-shell initial parton, which becomes virtual by transverse momenta through radiations. The resummation technique aims at collecting these radiations to all orders.

The sum of the contributions form Figs. 1(c) and 1(d) can be obtained simply by substituting the momentum fraction \( x \) for \( (1-x) \) in Eq. (6),

$$\Phi_{c+d}^{(1)} = \frac{\alpha_s}{4\pi} C_F \left[ \frac{1}{\epsilon} + \ln \frac{4\pi\mu^2}{m_c^2\epsilon} - \frac{1}{2} \ln 2 \frac{x^2\xi^2b^2\epsilon\gamma E-1}{4} + \frac{1}{2} \ln^2 \frac{m_c^2b^2\epsilon^2\gamma E-1}{4} + 2 - \frac{\pi^2}{3} \right].$$

\[\text{FIG. 2. Graphic representation of the derivative } \xi \frac{d}{d\xi} \Phi_{J/\psi}/d\xi.\]

We then proceed with the NLL \( k_T \) resummation for the \( J/\psi \) meson wave function under the hierarchy \( \xi \gg m_c \gg 1/b \), based on the above complete one-loop results. The strategy is to focus only on the first set of logarithms \( \ln(\xi b) \), whose treatment is similar to that of a light meson case, and then infer the resummation formula for the second set via the replacement of \( \xi \) by \( m_c \).

We study the variation of the \( J/\psi \) meson wave function with the gauge link direction \( n \), which is equivalent to the variation with the dominant component \( P^+_2 \) of the \( J/\psi \) momentum via the scale \( \xi \).

$$P^+_2 \frac{d}{dP^+_2} \Phi_{J/\psi} = \xi \frac{d}{d\xi} \Phi_{J/\psi} = -\frac{n^2}{P^+_2 \cdot n} P^+_2 \frac{d}{dn^\alpha} \cdot \Phi_{J/\psi}.$$  

The technique of varying gauge links has been applied to the resummation of various types of logarithms, such as the rapidity logarithms in the \( B \) meson wave function [20], and the joint logarithms in the pion wave function [21]. Moreover, this technique is applicable to the definitions of TMD wave functions involving various designs of gauge links, including the non-dipolar links proposed in [22]. The differentiation of each eikonal vertex and of its associated eikonal propagator on the gauge link with respect to \( n_\alpha \),

$$-\frac{n^2}{P^+_2 \cdot n} P^+_2 \frac{d}{dn^\alpha} \frac{n^\mu}{P^2 / n \cdot l \cdot l} = \frac{n^2}{P^+_2 \cdot n} \left( \frac{P^+_2 \cdot l}{n \cdot l} n^\mu - P^\mu_2 / 2 \right) \frac{1}{n \cdot l} \equiv \tilde{n}^\mu / n \cdot l.$$  

(10)
leads to the derivative $\xi d\Phi_{J/\psi}/d\xi$ depicted in Fig. 2. The summation in Fig. 2 is performed over different attachments of the new vertex $\hat{n}^\mu$ defined by the last expression in Eq. (10), and represented by the symbol “•”.

As stated before, terms of order $m_c^2$ can be neglected for the resummation of the first set of logarithms. If the loop momentum $l$ is parallel to $P_2$, the factor $P_2 \cdot l$, being of order $m_c^2$, is negligible. When the second term $P_2^\mu$ in $\hat{n}^\mu$ is contracted with a vertex in $\Phi_{J/\psi}$, where all momenta are mainly parallel to $P_2$, the contribution from this collinear region is also of order $m_c^2$, and negligible. That is, the leading regions of $l$ are soft and hard. According to [17], as the loop momentum flowing through the new vertex is soft, only the diagram with the new vertex being located at the outer most end of the gauge link is important, and gives large single logarithms. In this soft region the subdiagram containing the new vertex can be factorized using the eikonal approximation, and the remainder is assigned to $\Phi_{J/\psi}$. This subdiagram is absorbed into a soft function $K$, whose $O(\alpha_s)$ contribution is displayed in Fig. 3(a). As the loop momentum flowing through the new vertex is hard, only the diagram with the new vertex being located at the inner most end of the gauge link is important. In this region the subdiagram containing the new vertex is factored into a hard function $G$, whose $O(\alpha_s)$ contribution is displayed in Fig. 3(b), and the remainder is identified to be $\Phi_{J/\psi}$.

We arrive at the differential equation in the impact parameter space

$$P_2^+ \frac{d}{dP_2^+} \Phi_{J/\psi} = 2 \left[ K(b\mu, \alpha_s(\mu)) + G(P_2^+/\mu, \alpha_s(\mu)) \right] \Phi_{J/\psi},$$

where the arguments of $K$ and $G$ specify their characteristic scales. Figure 3(a) contributes

$$K = -ig^2 C_F \mu^\epsilon \int \frac{d^4-l}{(2\pi)^4} \frac{\hat{n}^\mu}{\hat{l}^2} \frac{g_{\mu\nu}}{P_2^\nu} \frac{P_2^\nu}{(P_2 + l)^2} \left[ 1 - \exp(i1_F \cdot b) \right] - \delta K,$$

with $\delta K$ being an additive counterterm. The Fourier factor $\exp(i1_F \cdot b)$ appears in the second diagram of Fig. 3(a), because the loop momentum flows through $\Phi_{J/\psi}$, such that the scale $1/b$ serves as an infrared cutoff of the loop integral in Eq. (12). The $O(\alpha_s)$ contribution to $G$ from Fig. 3(b), where the soft subtraction is to avoid double counting of the soft contribution, is written as

$$G = -ig^2 C_F \mu^\epsilon \int \frac{d^4-l}{(2\pi)^4} \frac{\hat{n}^\mu}{\hat{l}^2} \frac{g_{\mu\nu}}{P_2^\nu} \left[ \frac{P_2^\nu + l}{(P_2 + l)^2} \right] \gamma^\rho - \frac{P_2^\nu}{P_2^2} \right] - \delta G,$$

where the charm quark mass $m_c$ has been dropped as explained before, and $\delta G$ is an additive counterterm. Choosing the subtraction scheme

$$\delta K = -\frac{\alpha_s}{2\pi} C_F \left[ \frac{2}{\epsilon} + \ln(\pi \epsilon) \right] = -\delta G,$$

we get the soft and hard functions

$$K = -\frac{\alpha_s}{2\pi} C_F \ln(b^2 \mu^2),$$

$$G = -\frac{\alpha_s}{2\pi} C_F \ln \left[ \frac{\xi^2 e^{2\gamma_E} - 1}{4\mu^2} \right].$$

Since $K$ and $G$ contain only single soft and ultraviolet logarithms, respectively, they can be treated by RG methods:

$$\mu \frac{d}{d\mu} K = -\lambda_K = -\mu \frac{d}{d\mu} G,$$

in which the anomalous dimension of $K$, $\lambda_K = \mu d\delta K/d\mu$, is given, up to two loops, by [23]

$$\lambda_K(\alpha_s) = \frac{\alpha_s}{\pi} C_F + \left( \frac{\alpha_s}{\pi} \right)^2 C_F \left[ C_A \left( \frac{67}{36} - \frac{\pi^2}{12} \right) - \frac{5}{18} n_f \right],$$

where $C_A = 3$ for SU(3) color.
with the number of quark flavors \( n_f \) and the color factor \( C_A = 3 \). As solving Eq. (16), we allow the scale \( \mu \) to evolve to the infrared cutoff \( 1/b \) in \( K \) and to \( P_2^+ \) in \( G \), and obtain the RG solution

\[
K(b\mu, \alpha_s(\mu)) + G(P_2^+ / \mu, \alpha_s(\mu)) = K(1, \alpha_s(1/b)) + G(1, \alpha_s(P_2^+)) = -\frac{\alpha_s(P_2^+)}{2\pi} C_F \ln \frac{e^{2\gamma_E-1}}{2} - \int_{1/b}^{P_2^+} \frac{d\bar{\mu}}{\bar{\mu}} \lambda_K(\alpha_s(\bar{\mu})).
\]

(18)

The relation \( \xi^2 = 2P_2^{x^2}n^-/n_+ = 2P_2^{x^2} \) for \( n^+ = n^- \) has been inserted to get the initial condition \( G(1, \alpha_s(P_2^+)) \). Substituting Eq. (18) into Eq. (11), we derive

\[
\Phi_{J/\psi} = \exp \left[ -\int_{1/b}^{(1-x)P_2^+} \frac{d\bar{\mu}}{\bar{\mu}} \left( \int_{1/b}^{\bar{\mu}} \frac{d\bar{\mu}}{\bar{\mu}} \lambda_K(\alpha_s(\bar{\mu})) + \frac{\alpha_s(\bar{\mu})}{2\pi} C_F \ln \frac{e^{2\gamma_E-1}}{2} \right) \right] \times \exp \left[ -\int_{1/b}^{xP_2^+} \frac{d\bar{\mu}}{\bar{\mu}} \left( \int_{1/b}^{\bar{\mu}} \frac{d\bar{\mu}}{\bar{\mu}} \lambda_K(\alpha_s(\bar{\mu})) + \frac{\alpha_s(\bar{\mu})}{2\pi} C_F \ln \frac{e^{2\gamma_E-1}}{2} \right) \right] \Phi_{J/\psi}(x, b).
\]

(19)

We have set the lower bound of the variable \( \bar{\mu} \) to \( 1/b \), and the upper bounds to \( (1-x)P_2^+ \) and \( xP_2^+ \) for the integrals associated with Figs. 1(a) and 1(b), and Figs. 1(c) and 1(d), respectively, so that the initial condition \( \Phi_{J/\psi}(x, b) \) depends on \( x \) and \( b \). As pointed out before, the \( k_T \) resummation formula for the second set of important logarithms can be inferred from Eq. (19) by substituting \( m_c \) for \( (1-x) \xi \) and \( x \xi \), namely, \( m_c/\sqrt{2} \) for the upper bounds of \( \bar{\mu} \), and flipping the signs of the integrands. Combining the two resummation formulas, we get the final result

\[
\Phi_{J/\psi}(x, b, \xi, m_c) = \exp \left[ -\int_{m_c/\sqrt{2}}^{(1-x)P_2^+} \frac{d\bar{\mu}}{\bar{\mu}} \left( \int_{1/b}^{\bar{\mu}} \frac{d\bar{\mu}}{\bar{\mu}} \lambda_K(\alpha_s(\bar{\mu})) + \frac{\alpha_s(\bar{\mu})}{2\pi} C_F \ln \frac{e^{2\gamma_E-1}}{2} \right) \right] \times \exp \left[ -\int_{m_c/\sqrt{2}}^{xP_2^+} \frac{d\bar{\mu}}{\bar{\mu}} \left( \int_{1/b}^{\bar{\mu}} \frac{d\bar{\mu}}{\bar{\mu}} \lambda_K(\alpha_s(\bar{\mu})) + \frac{\alpha_s(\bar{\mu})}{2\pi} C_F \ln \frac{e^{2\gamma_E-1}}{2} \right) \right] \Phi_{J/\psi}(x, b, m_c),
\]

(20)

where the initial condition \( \Phi_{J/\psi}(x, b, m_c) \) further depends on \( m_c \). Expanding Eq. (20) to \( O(\alpha_s) \) for a constant \( \alpha_s \), we reproduce all the logarithms in Eqs. (6) and (8). The remaining constant pieces will go into the \( O(\alpha_s) \) hard decay kernel, when the one-loop \( J/\psi \) meson wave function and the one-loop decay amplitude are matched.

The above expression represents the complete NLL \( k_T \) resummation for the \( J/\psi \) meson wave function, which involves the three scale \( m_b, m_c \) and \( k_T \). Compared to [11], we have included the so-called \( B \) term, i.e., the second terms in the exponents in Eq. (20), and determined the order-unity coefficient associated with the lower bound of the variable \( \bar{\mu} \) to be \( 1/\sqrt{2} \), both of which correspond to NLL effects. The inclusion of these NLL pieces requires a complete one-loop calculation of the \( J/\psi \) meson wave function in the impact parameter \( b \) space. The \( k_T \) resummation formula for the spectator charm quark in the \( B_c \) meson then takes the form

\[
\Phi_{B_c}(x, b, \xi, m_c) = \exp \left[ -\int_{m_c/\sqrt{2}}^{xP_2^+} \frac{d\bar{\mu}}{\bar{\mu}} \left( \int_{1/b}^{\bar{\mu}} \frac{d\bar{\mu}}{\bar{\mu}} \lambda_K(\alpha_s(\bar{\mu})) + \frac{\alpha_s(\bar{\mu})}{2\pi} C_F \ln \frac{e^{2\gamma_E-1}}{2} \right) \right] \Phi_{B_c}(x, b, m_c),
\]

(21)

for which the relevant large longitudinal component of the spectator momentum is \( xP_2^- \).

**TABLE I.** Dependence on the shape parameter \( \beta_{B_c} \) of the quantities \( A_0^{B_c \rightarrow J/\psi}(0) \) and \( \text{BR}(B_c^+ \rightarrow J/\psi \pi^+) \) in the PQCD approach at the LL and NLL accuracy.

| Shape parameter | \( A_0^{B_c \rightarrow J/\psi}(0) \) | \( \text{BR}(B_c^+ \rightarrow J/\psi \pi^+) \) |
|-----------------|----------------------------------|----------------------------------|
| \( \beta_{B_c} \) | \( \text{LL} \) | \( \text{NLL} \) | \( \text{LL} \) | \( \text{NLL} \) |
| 0.8 GeV | 0.488 - i0.095 | 0.511 - i0.147 | 2.80 \times 10^{-3} | 3.10 \times 10^{-3} |
| 0.9 GeV | 0.434 - i0.070 | 0.460 - i0.114 | 2.10 \times 10^{-3} | 2.39 \times 10^{-3} |
| 1.0 GeV | 0.384 - i0.053 | 0.414 - i0.090 | 1.60 \times 10^{-3} | 1.87 \times 10^{-3} |
| 1.1 GeV | 0.341 - i0.039 | 0.373 - i0.071 | 1.23 \times 10^{-3} | 1.46 \times 10^{-3} |
| 1.2 GeV | 0.306 - i0.029 | 0.339 - i0.057 | 9.4 \times 10^{-3} | 1.16 \times 10^{-3} |
At last, we calculate the $B_c \to J/\psi$ transition form factor $A_{B_c \to J/\psi}^B(0)$ and the $B_c^+ \to J/\psi\pi^+$ branching ratio $\text{BR}(B_c^+ \to J/\psi\pi^+)$ in the PQCD approach, taking into account the NLL $k_T$ resummation effect from Eqs. (20) and (21). The explicit expressions for the above quantities, together with the input parameters and the models of the meson wave functions, can be found in [11]. The initial scale of the renormalization-group evolution for the meson wave functions, governed by the quark anomalous dimension [11], is modified from $m_c$ to $m_c/\sqrt{2}$ for consistency. We adopt the one-loop running formula for the strong coupling $\alpha_s$. It has been checked that the two-loop running causes only 1-2% reduction of the results from the one-loop running. The dependence of the quantities $A_{B_c \to J/\psi}^B(0)$ and $\text{BR}(B_c^+ \to J/\psi\pi^+)$ on the shape parameter $\beta_B$ of the $B_c$ meson wave function in the range $[0.8, 1.2]$ GeV is presented in Table I, and compared with that derived with the LL resummation effect [11]. The potential imaginary part of $A_{B_c \to J/\psi}^B(0)$, which is supposed to be a real object [24], increases a bit under the NLL resummation, but remains power suppressed. It is found that $A_{B_c \to J/\psi}^B(0)$ is enhanced by the NLL resummation effect by 5-10%, as $\beta_B$ varies from 0.8 GeV to 1.2 GeV, and thus $\text{BR}(B_c^+ \to J/\psi\pi^+)$ increases by about 10-20% accordingly. It implies that the NLL resummation effect is not negligible, and crucial for the determination of the $B_c$ meson wave function, when relevant data are available in the future. The values in Table I are consistent with those from other approaches in the literature, which have been summarized in [11].

In this letter we have improved the $k_T$ resummation for the $B_c \to J/\psi$ decays, which involve an additional intermediate charm scale compared with the $B \to \pi$ decays, to the NLL accuracy. We constructed the evolution equation for the TMD meson wave function by varying its associated Wilson link, performed the $k_T$ resummation by solving the evolution equation, and fixed the NLL pieces through the matching to the one-loop calculation. Our work represents the first NLL $k_T$ resummation for the $B_c$ meson decays to charmonia. Based on this work, we are ready to extend the $k_T$ resummation with multiple scales to energetic charmed mesons, for which the current formalism is still preliminary.

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