Some Estimates of the Maximum Modulus for Polynomials with Gaps

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Abstract. Let \( p(z) \) be a polynomial of degree \( n \) having some zeros at a point \( z_0 \in \mathbb{C} \) with \( |z_0| < 1 \) and the rest of the zeros lying on or outside the boundary of a prescribed disk. In this brief note, we consider this class of polynomials and obtain some bounds for \( \left( \max_{|z|=R} |p(z)| \right)^s \) in terms of \( \left( \max_{|z|=1} |p(z)| \right)^s \) for any \( R \geq 1 \) and \( s \in \mathbb{N} \).

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1 Introduction

Let \( p(z) \) be a polynomial of degree \( n \). For effective management of space, we shall adopt the following notations:

\[
\begin{align*}
D(0,k) &:= \{z : |z| < k\}, & S(0,k) &:= \{z : |z| = k\}, & M(p,R) &:= \max_{|z|=R} |p(z)|, \\
m(p,k) &:= \min_{|z|=k} |p(z)|, & ||p|| &:= \max_{|z|=1} |p(z)|,
\end{align*}
\]

where \( k \) and \( R \) are positive real numbers.

By using the maximum modulus principle, one obtains that for \( R \geq 1 \),

\[
M(p,R) \geq ||p||.
\]

The general problem of interest, however, is the following:

\[\text{(P)}: \quad \text{Find a factor } (\ast) \text{ such that } M(p,R) \leq (\ast) ||p|| \text{ for any } R \geq 1.\]

In view of (P), S. Bernstein [6, pp. 442] observed that for \( R \geq 1 \),

\[
M(p,R) \leq R^n||p||. \tag{1.1}
\]

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The above result is best possible with equality holding for $p(z) = λz^n$, $λ$ being a complex number. Since the extremal polynomial $p(z) = λz^n$ in (1.1) has all its zeros at the origin. It should be possible to improve upon the bound in (1.1) for polynomial not vanishing at the origin. For this, Ankeny and Rivlin [1] proved that if $p(z)$ has no zero in $D(0, 1)$, then for $R \geq 1$,

$$M(p, R) \leq \frac{R^n + 1}{2}||p||.$$  \hspace{1cm} (1.2)

As a sharpening of the above result, Aziz and Dawood [3] proved that for $R \geq 1$,

$$M(p, R) \leq \frac{R^n + 1}{2}||p|| - \frac{R^n - 1}{2}m(p, 1).$$  \hspace{1cm} (1.3)

Now, for the class of polynomials not vanishing in the disk $D(0, k), k \geq 1$, Shah [10] proved that if $p(z)$ is a polynomial of degree $n$ having no zero in $D(0, k), k \geq 1$, then for every real number $R > k$,

$$M(p, R) \leq \frac{R^n + 1}{1 + k}||p|| - \frac{R^n - 1}{1 + k}m(p, k).$$ \hspace{1cm} (1.4)

Several research articles have been written on this subject of inequalities (see for example Govil and Mohapatra [4], Rahman and Schmeisser [9], and recent article of Govil and Nwaeze [5].)

Inspired by the work in [8], we consider polynomials having some zeros at a point $z_0 \in \mathbb{C}$ with $|z_0| < 1$ and the rest of the zeros lying on or outside the boundary of a prescribed disk. For this, we estimate $(M(p, R)/||p||)^s$ for any $R \geq 1$ and any natural number $s$. The paper is organized as follows: we present two lemmas in Section 2 which will be used in the proof of our results. In Section 3, the results are formulated and proved and then followed by a short conclusion in Section 4.

## 2 Lemmas

For the proof of our theorems, we will need the following lemmas due to Nakprasit and Somsuwan [7].

**Lemma 2.1.** Let

$$p(z) = (z - z_0)^m \left(a_0 + \sum_{j=\mu}^{n-m} a_j z^j\right), \hspace{0.5cm} 1 \leq \mu \leq n - m, \hspace{0.5cm} 0 \leq m \leq n - 1,$$

be a polynomial of degree $n$ having zero of order $m$ at $z_0$ with $|z_0| < 1$ and the remaining $n - m$ zeros are outside $D(0, k), k \geq 1$. Then

$$\max_{|z|=1} |p'(z)| \leq \left[ \frac{m}{(1 - |z_0|)} + \frac{A}{(1 - |z_0|)^m} \right] ||p|| - \frac{A}{(k + |z_0|)^m}m(p, k),$$
where

\[ A = \frac{(1 + |z_0|)^{m+1}(n - m)}{(1 + k^\mu)(1 - |z_0|)}. \]

Lemma 2.2. Let

\[ p(z) = (z - z_0)^m\left(a_0 + \sum_{j=\mu}^{n-m} a_jz^j\right), \quad 1 \leq \mu \leq n - m, \quad 0 \leq m \leq n - 1, \]

be a polynomial of degree \( n \) having zero of order \( m \) at \( z_0 \) with \( |z_0| < 1 \) and the remaining \( n - m \) zeros are on \( S(0,k) \), \( k \geq 1 \). Then

\[ \max_{|z|=1} |p'(z)| \leq \left[ \frac{m}{(1 - |z_0|)} + \frac{(1 + |z_0|)^{m+1}(n - m)}{(k^{n-m-2\mu} + k^{n-m-\mu+1})(1 - |z_0|)^{m+1}} \right] ||p||. \]

3 Main results

Theorem 3.1. Let

\[ p(z) = z^m\left(a_0 + \sum_{j=\mu}^{n-m} a_jz^j\right), \quad 1 \leq \mu \leq n - m, \quad 0 \leq m \leq n - 1, \]

be a polynomial of degree \( n \) having zero of order \( m \) at the origin and the remaining \( n - m \) zeros are outside \( D(0,k) \), \( k \geq 1 \). Then for \( R \geq 1 \) and every natural number \( s \),

\[ [M(p, R)]^s \leq \left[ \frac{n + nk^\mu + (n + mk^\mu)(R^{ns} - 1)}{n(1 + k^\mu)} \right] ||p||^s - \left[ \frac{(n - m)(R^{ns} - 1)m(p,k)}{nk^m(1 + k^\mu)} \right] ||p||^{s-1}. \]

If we take \( m = 0 \) and \( s = 1 \), we obtain

Corollary 3.1. Let

\[ p(z) = a_0 + \sum_{j=\mu}^{n} a_jz^j, \quad 1 \leq \mu \leq n, \]

be a polynomial of degree \( n \) having all its zeros outside \( D(0,k) \), \( k \geq 1 \). Then for \( R \geq 1 \),

\[ M(p, R) \leq \frac{k^\mu + R^n}{1 + k^\mu} ||p|| - \frac{R^n - 1}{1 + k^\mu} m(p,k). \]

The above corollary is a generalization of a result due to Aziz [2, Theorem 4]. Setting \( k = 1 \) in Corollary 3.1, we obtain the result of Aziz and Dawood given in inequality (1.3). Instead of proving Theorem 3.1, we will prove the following more general result.
Theorem 3.2. Let

\[ p(z) = (z - z_0)^m \left( a_0 + \sum_{j=\mu}^{n-m} a_j z^j \right), \quad 1 \leq \mu \leq n - m, \quad 0 \leq m \leq n - 1, \]

be a polynomial of degree \( n \) having zero of order \( m \) at \( z_0 \) with \( |z_0| < 1 \) and the remaining \( n - m \) zeros are outside \( D(0, k), k \geq 1 \). Then for \( R \geq 1 \) and every natural number \( s \),

\[ |M(p, R)|^s \leq \left[ 1 + \frac{m(R^{ns} - 1)}{n(1 - |z_0|)} + \frac{A(R^{ns} - 1)}{n(1 - |z_0|)^m} \right] |p|^s \]

\[ - \left[ \frac{A(R^{ns} - 1)m(p, k)}{n(k + |z_0|)^m} \right] |p|^{s-1}, \]

where

\[ A = \frac{(1 + |z_0|)^m + (n - m)}{(1 + k^m)(1 - |z_0|)}. \]

Proof. Applying inequality (1.1) and Lemma 2.1 to the polynomial \( p'(z) \) which is of degree \( n - 1 \), it follows that for \( R \geq 1 \) and \( \theta \in [0, 2\pi] \), we have

\[ |p'(Re^{i\theta})| \leq \max_{|z| = R} |p'(z)| \leq R^{n-1} \max_{|z| = 1} |p'(z)| \]

\[ \leq R^{n-1} \left[ \frac{m}{(1 - |z_0|)} + \frac{A}{(1 - |z_0|)^m} \right] |p| - \frac{AR^{n-1}}{(k + |z_0|)^m} m(p, k). \quad (3.1) \]

From the fundamental principle of calculus, we obtain that

\[ |p(Re^{i\theta})|^s - |p(e^{i\theta})|^s = \int_1^R \frac{d[p(Re^{i\theta})]^s}{dt} \, dt = \int_1^R s[p(Re^{i\theta})]^{s-1} p'(Re^{i\theta}) e^{i\theta} \, dt. \quad (3.2) \]

This implies that

\[ |p(Re^{i\theta})|^s \leq |p(e^{i\theta})|^s + s \int_1^R |p(Re^{i\theta})|^{s-1} |p'(Re^{i\theta})| \, dt. \quad (3.3) \]

Hence, using (2.1) and (3.1) together with the above inequality, we get

\[ |M(p, R)|^s \leq |p|^s + s \int_1^R t^{ns-n} |p|^{s-1} |p'(te^{i\theta})| \, dt \]

\[ \leq |p|^s + s |p|^s \left[ \frac{m}{(1 - |z_0|)} + \frac{A}{(1 - |z_0|)^m} \right] \int_1^R t^{ns-1} \, dt \]

\[ - s \frac{A |p|^{s-1}}{(k + |z_0|)^m} m(p, k) \int_1^R t^{ns-1} \, dt \]

\[ = \left[ 1 + \frac{m(R^{ns} - 1)}{n(1 - |z_0|)} + \frac{A(R^{ns} - 1)}{n(1 - |z_0|)^m} \right] |p|^s - \left[ \frac{A(R^{ns} - 1)m(p, k)}{n(k + |z_0|)^m} \right] |p|^{s-1}. \]
That proves our result. □

**Theorem 3.3.** Let

\[ p(z) = (z - z_0)^m \left( a_0 + \sum_{j=\mu}^{n-m} a_j z^j \right), \quad 1 \leq \mu \leq n - m, \quad 0 \leq m \leq n - 1, \]

be a polynomial of degree \( n \) having zero of order \( m \) at \( z_0 \) with \( |z_0| < 1 \) and the remaining \( n - m \) zeros are on \( S(0, k) \), \( k \geq 1 \). Then for \( R \geq 1 \) and every natural number \( s \),

\[
[M(p, R)]^s \leq \left[ 1 + \frac{m(R^n - 1)}{n(1 - |z_0|)} + \frac{(n - m)(1 + |z_0|)^{m+1}(R^n - 1)}{n(1 - |z_0|)^{m+1}(k^{n-m-2\mu+1} + k^{n-m-\mu+1})} \right] ||p||^s.
\]

By choosing \( s = 1 \) in Theorem 3.3 above, we obtain:

**Corollary 3.2.** Let

\[ p(z) = (z - z_0)^m \left( a_0 + \sum_{j=\mu}^{n-m} a_j z^j \right), \quad 1 \leq \mu \leq n - m, \quad 0 \leq m \leq n - 1, \]

be a polynomial of degree \( n \) having zero of order \( m \) at \( z_0 \) with \( |z_0| < 1 \) and the remaining \( n - m \) zeros are on \( S(0, k) \), \( k \geq 1 \). Then for \( R \geq 1 \),

\[
M(p, R) \leq \left[ 1 + \frac{m(R^n - 1)}{n(1 - |z_0|)} + \frac{(n - m)(1 + |z_0|)^{m+1}(R^n - 1)}{n(1 - |z_0|)^{m+1}(k^{n-m-2\mu+1} + k^{n-m-\mu+1})} \right] ||p||.
\]

The next corollary follows by setting \( z_0 = m = 0 \) in the above corollary.

**Corollary 3.3.** Let

\[ p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, \quad 1 \leq \mu \leq n, \]

be a polynomial of degree \( n \) having all its zeros on \( S(0, k) \), \( k \geq 1 \). Then for \( R \geq 1 \),

\[
M(p, R) \leq \left[ 1 + \frac{R^n - 1}{k^{n-2\mu+1} + k^{n-\mu+1}} \right] ||p||.
\]

**Proof.** We now present the proof of Theorem 3.3 by following a similar fashion as in the proof of Theorem 3.2. Using Lemma 2.2, we have that for \( R \geq 1 \), and \( \theta \in [0, 2\pi) \)

\[
[M(p, R)]^s \leq ||p||^s + s \int_1^R t^{ns-n}||p||^{s-1}|p'(te^{i\theta})| \, dt \\
\leq ||p||^s + s \left[ \frac{m}{(1 - |z_0|)} + \frac{(1 + |z_0|)^{m+1}(n - m)}{(k^{n-m-2\mu+1} + k^{n-m-\mu+1})(1 - |z_0|)^{m+1}} \right] \times ||p|| \int_1^R t^{ns-1} \, dt,
\]

hence, Theorem 3.3 follows. □
4 Conclusions

Much have not been done for polynomials having some of their zeros at a point and the rest on or outside a prescribed disk. This work investigates such class of polynomials to see how $[M(p, R)]^s$ compares with $||p||^s$ for any given $R \geq 1$ and $s \in \mathbb{N}$.

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