Dragging spin–orbit-coupled solitons by a moving optical lattice

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Abstract
It is known that the interplay of the spin–orbit-coupling (SOC) and mean-field self-attraction creates stable two-dimensional (2D) solitons (ground states) in spinor Bose–Einstein condensates. However, SOC destroys the system’s Galilean invariance, therefore moving solitons exist only in a narrow interval of velocities, outside of which the solitons suffer delocalization. We demonstrate that the application of a relatively weak moving optical lattice (OL), with the 2D or quasi-1D structure, makes it possible to greatly expand the velocity interval for stable motion of the solitons. The stability domain in the system’s parameter space is identified by means of numerical methods. In particular, the quasi-1D OL produces a stronger stabilizing effect than its full 2D counterpart. Some features of the domain are explained analytically.

Keywords: soliton, Bose–Einstein condensates, spin–orbit coupling, moving optical lattice

1. Introduction
The spin–orbit coupling (SOC) is a fundamental effect in physics of semiconductors, induced by the interaction of the electron’s spin with the magnetic field produced by the Lorentz transform of the electrostatic field of the crystalline lattice in the reference frame moving along with the electron [1–3]. While in the solid-state settings SOC is a complex phenomenon, it has been demonstrated that it may be emulated in a much simpler form in binary atomic Bose–Einstein condensates (BECs). The experiments have realized the SOC emulation in BEC by mapping the spinor wave function of electrons into the two-component (pseudo-spinor) wave function of the atomic condensate [5–7]. In terms of coupled Gross–Pitaevskii equations (GPEs), which provide a very accurate dynamical model for BEC in the mean-field approximation [4], SOC, i.e., the coupling of the momentum and pseudospin of the matter waves, is represented by linear terms with first spatial derivatives which mix two components of the spinor wave function [8, 9].

Many theoretical works addressed the interplay of SOC with the intrinsic nonlinearity of BEC, which represents, in the mean-field approximation, effects of collisions between atoms in the dilute quantum gas. In the case of the self-attractive sign of the nonlinearity, the analysis has predicted the modulational instability [10] and various species of one-dimensional (1D) solitons [11–21, 24] under the action of SOC. In the case of the repulsive nonlinearity, the use of spatially periodic optical-lattice (OL) potentials has made it possible to predict 1D gap solitons [22, 23, 25]. Otherwise, the interplay of SOC with self-repulsion gives rise to families of dark solitons [26–30].

While most experimental realizations of SOC were reported in the effectively 1D geometry, SOC has also been created in the 2D setting [31]. Theoretical analyses of 2D setups, including SOC and the intrinsic repulsive nonlinearity, addressed vortices [32–35], monopoles [36], skyrmions [37, 38], and other delocalized states. In the presence of the OL potential and self-repulsion, 2D gap solitons were predicted too [39].
The lattice potential can also stabilize 2D SOC solitons in the case of self-attraction [40]. Another possibility to create stable 2D solitons is offered by the higher-order (beyond-mean-field) self-repulsion in BEC components [41], or by long-range dipole–dipole interactions [42, 43].

As concerns settings based on GPEs with the mean-field cubic self-attraction in the free 2D space, originally it was believed that all self-trapped states generated by these models, such as Townes solitons [44], are completely unstable, as the same setting gives rise to the critical collapse which leads to destruction of solitons by perturbations [45–47]. Nevertheless, it has been found that the addition of the usual linear SOC of the Rashba [2] type is sufficient to suppress the collapse and create otherwise missing ground states in the linearly coupled system of GPEs with the cubic attractive terms [48]. Then, the same result was produced [49] by the consideration of the binary system linearly coupled by a combination of the Rashba and Dresselhaus [1] SOC terms. This possibility to stabilize 2D solitons was further elaborated in references [50–53], see also a brief review in [54].

There are two different species of 2D solitons supported by the attractive nonlinearity in the two-component SOC system, in cases when ratio $\gamma$ of the strength of the attraction between the components to the strength of the self-attraction in each component is $\gamma < 1$ or $\gamma > 1$. In the former case, the 2D system produces stable semi-vortex (SV) solitons as the ground state, in which one component has zero vorticity, and the other one carries vorticity 1. On the other hand, the ground state of the system with $\gamma > 1$ is represented by mixed modes (MMs), which are composed of terms with zero and nonzero vorticities in both components (therefore they called ‘mixed’) [48]. Simultaneously, SVs and MMs exist but are unstable at $\gamma > 1$ and $\gamma < 1$, respectively.

Mobility of solitons in SOC systems is an issue of straightforward physical interest [20]. It is a nontrivial property because SOC terms break the Galilean invariance [11, 48, 55]. By means of numerical methods, it was found [48, 52] that SVs and MMs may stably move only in one direction in the 2D plane (note that SOC destroys the system’s isotropy too), with velocity $v_0$ taking values in a finite interval,

$$0 < v_0 < (v_{\text{max}})_{\text{SV,MM}}.$$  

At $v_0 = v_{\text{max}}$, the soliton disappears through delocalization. The critical velocity $v_{\text{max}}$ takes moderate values for MMs, being very small for SVs. This observation is explained by the fact the SV’s structure is actually incompatible with the mobility, see details below.

A possibility to enhance mobility of solitons is to drag them by means of a moving OL potential, which is an experimentally available tool [56, 57]. In this work we aim to elaborate this option and demonstrate that the moving OL with a relatively small amplitude is able to stabilize the motion of the 2D solitons up to much higher values of the velocity than in the free space. The model and a relevant analytical framework are presented in section 2. Systematic results, obtained, chiefly, by means of numerical methods are reported in section 3. In particular, for the dragged SVs, the increase of $v_0$ leads, first, to their transformation into MMs, while the delocalization takes place at much higher velocities. Considering both the full 2D lattice and its quasi-1D counterpart, we conclude that the latter one produces an essentially stronger stabilizing effect on the moving solitons than the full 2D lattice. Moreover, a surprising result is that the strongest stabilization is provided by the quasi-1D lattice with the wave vector directed perpendicular to the velocity. This finding is explained by the fact that such a lattice suppresses delocalization of the moving soliton in the transverse direction. The paper is concluded by section 4.

2. The model and analytical framework

2.1. The basic equations

Following reference [48], we consider the system of coupled GPEs for two components $\phi_{\pm}(x, y, t)$ of the BEC spinor wave function written in the laboratory reference frame. The equations include, as the first option, a square-shaped OL potential $U_{2D}(x, y, t)$ with amplitude $U_0$ and wavenumbers $q$, moving at velocity $v_0$ along the $y$ direction in the $(x, y)$ plane:

$$i \frac{\partial \phi_+}{\partial t} = -\frac{1}{2} \nabla^2 \phi_+ - \left( |\phi_+|^2 + \gamma |\phi_-|^2 \right) \phi_+ - U_{2D}(x, y; t) \phi_+ + \lambda \left( \frac{\partial \phi_-}{\partial x} - i \frac{\partial \phi_-}{\partial y} \right),$$  

$$i \frac{\partial \phi_-}{\partial t} = -\frac{1}{2} \nabla^2 \phi_- - \left( |\phi_-|^2 + \gamma |\phi_+|^2 \right) \phi_- - U_{2D}(x, y; t) \phi_- - \lambda \left( \frac{\partial \phi_+}{\partial x} + i \frac{\partial \phi_+}{\partial y} \right),$$  

$$U_{2D}(x, y; t) = U_0 \cos(qx) \cos(q(y - v_0 t)).$$

In this notation, $\hbar$ and the atomic mass, as well as the effective coefficient of the self-attraction in each component, are scaled to be 1, while $\gamma \geq 0$ is the above-mentioned ratio of the cross/self interaction strengths, and $\lambda > 0$ is the real coefficient of SOC of the Rashba type.

Parallel to the full 2D potential (4), we consider its quasi-1D variants, with the wave vector oriented parallel or perpendicular to the velocity:

$$U_{1D}(y; t) = U_0 \cos(q(y - v_0 t)); \quad U_{1D}(x) = U_0 \cos(qx).$$

In the latter case, the quasi-1D OL is not actually moving (but the solitons will move in the $y$ direction), therefore equations (2) and (3) with potential $U_{1D}(x)$ do not explicitly depend on time. Using the remaining scaling invariance of equations (2) and (3) (which includes a freedom of rescaling the coordinated by an arbitrary factor), in most cases we fix

$$q = 2\pi/3,$$

in potentials (4) and (5), which is a value convenient for numerical simulations. Nevertheless, some results for other values of $q$ are presented below too, see equations (40) and (42). Including results in this form is relevant because in the experiment
it is possible to change the period of the OL, keeping other parameters fixed [58].

It is relevant to mention that, in the framework of the usual GPE with cubic self-attraction, 2D solitons may be stabilized not only by the full 2D spatially periodic potential [59–62], but also by its quasi-1D version [63], see also reference [64]. Furthermore, in the free 2D space, solitons can be stabilized by the SOC terms taken in an essentially quasi-1D form [53]. Somewhat surprisingly, the present analysis reveals, in the next section, that the quasi-1D potentials (5), especially $U_{1D}(x)$, provide an essentially stronger stabilizing effect for moving solitons than the full 2D potential.

In the moving reference frame with coordinate $\tilde{y} = y - v_0 t$, equations (2) and (3) are rewritten as

$$\dot{\phi}_+ - iv_0 \frac{\partial \phi_+}{\partial \tilde{y}} = -\frac{1}{2} \nabla^2 \phi_+ - \left( |\phi_+|^2 + \gamma |\phi_-|^2 \right) \phi_+ - U_{2D}(x, \tilde{y}; t) \phi_+ + \lambda \left( \frac{\partial \phi_+}{\partial x} - i \frac{\partial \phi_+}{\partial \tilde{y}} \right),$$

(7)

$$\dot{\phi}_- - iv_0 \frac{\partial \phi_-}{\partial \tilde{y}} = -\frac{1}{2} \nabla^2 \phi_- - \left( |\phi_-|^2 + \gamma |\phi_+|^2 \right) \phi_- - U_{2D}(x, \tilde{y}; t) \phi_- - \lambda \left( \frac{\partial \phi_-}{\partial x} + i \frac{\partial \phi_-}{\partial \tilde{y}} \right),$$

(8)

$$U_{2D}(x, \tilde{y}) = U_0 \cos(q x) \cos(q \tilde{y}); \quad U_{1D}(\tilde{y}) = U_0 \cos(q \tilde{y}).$$

(9)

(10)

In particular, equations (7) and (8) with $U_{2D} = 0$ admit a family of continuous-wave (CW) solutions with arbitrary amplitude $A$ and wavenumber $k_x$ (for the definiteness’ sake, we here assume $k_x > 0$):

$$(\phi_{\pm}(\tilde{y}, t))_{CW} = \pm A \exp \left( ik_x \tilde{y} - i \mu_{CW} t \right),$$

$$\mu_{CW} = \frac{k_x^2}{2} - k_y (\lambda + v_0) - \Lambda^2.$$  

(11)

Signs $\pm$ in front of the components of this solution are chosen to select the CW branch with lower energy.

Soliton solutions to equations (7) and (8) with the OL potential (9) or (10) and chemical potential $\mu$ are looked for in the usual form,

$$\phi_{\pm}(x, y; t) = u(x, y) \exp(-i \mu t).$$  

(12)

These solutions, along with the respective values of $\mu$, were obtained by means of the imaginary-time evolution method [65–67] applied to equations (7) and (8) for a fixed value of the total norm,

$$N = \iint \left( |\phi_+(x, y)|^2 + |\phi_-(x, y)|^2 \right) \, dx \, dy,$$

(13)

as the method is adjusted for finding solutions under this condition.

Even if the above equations are not invariant with respect to the Galilean transform, for analytical considerations it is useful to rewrite them in the moving reference frame, applying the formal Galilean boost to equations (7) and (8):

$$\phi_{\pm} \equiv \exp \left( i v_0 \tilde{y} + \frac{i}{2} v_0^2 t \right) \tilde{\phi}(x, \tilde{y}, t).$$  

(14)

The accordingly transformed system is

$$\dot{\tilde{\phi}}_+ = -\frac{1}{2} \nabla^2 \tilde{\phi}_+ - \left( |\tilde{\phi}_+|^2 + \gamma |\tilde{\phi}_-|^2 \right) \tilde{\phi}_+ - U_{2D}(x, \tilde{y}; t) \tilde{\phi}_+ + \lambda \left( \frac{\partial \tilde{\phi}_+}{\partial x} - i \frac{\partial \tilde{\phi}_+}{\partial \tilde{y}} \right) + \lambda v_0 \tilde{\phi}_-, $$

(15)

$$\dot{\tilde{\phi}}_- = -\frac{1}{2} \nabla^2 \tilde{\phi}_- - \left( |\tilde{\phi}_-|^2 + \gamma |\tilde{\phi}_+|^2 \right) \tilde{\phi}_- - U_{2D}(x, \tilde{y}; t) \tilde{\phi}_- - \lambda \left( \frac{\partial \tilde{\phi}_-}{\partial x} + i \frac{\partial \tilde{\phi}_-}{\partial \tilde{y}} \right) + \lambda v_0 \tilde{\phi}_+. $$

(16)

Unlike equations (7) and (8), this system includes direct inter-component mixing with coefficient $\lambda v_0$, but does not include the group-velocity terms, $-iv_0 \partial \phi_{\pm} / \partial \tilde{y}$.

### 2.2. Analytical estimates

Knowledge of the spectrum of the linearized set of equations (15) and (16) without the potential ($U_0 = 0$) helps one to predict the existence region for solitons. A straightforward calculation for small-amplitude excitations, taken as

$$\tilde{\phi}_{\pm} \sim \exp \left( ik_x x + ik_y y - i \tilde{\mu} t \right),$$

(17)

yields two branches of the dispersion relation between chemical potential $\mu$ and wave vector $(k_x, k_y)$,

$$\tilde{\mu} = \frac{1}{2} k_x^2 \pm \lambda \sqrt{k_y^2 + (k_x + v_0)^2}. $$

(18)

Expression (18) takes values in the propagation band,

$$\tilde{\mu} \geq \tilde{\mu}_{\text{min}} \equiv -\lambda^2 / 2 - |\lambda v_0|,$$

(19)

while solitons may populate the remaining semi-infinite bandgap, $\tilde{\mu} < \tilde{\mu}_{\text{min}}$. Note that the increase of the velocity pushes $\tilde{\mu}_{\text{min}}$ down, thus reducing the bandgap. All the soliton solutions produced indeed satisfy the condition $\tilde{\mu} < \tilde{\mu}_{\text{min}}$.

It is relevant to mention that the 1D limit of equations (15), (16) and (10), corresponding to no $x$ dependence and potential $U_0 \cos(q \tilde{y})$, admits an obvious substitution,

$$\tilde{\phi}_{\pm}(\tilde{y}, t) = \exp \left( -i \tilde{\mu} \tilde{y} + i \left( \lambda^2 / 2 - \lambda v_0 \right) t \right) \phi(\tilde{y}, t),$$

(20)

with which the system reduces to the single equation:

$$\dot{\phi} = -\frac{1}{2} \frac{\partial^2 \phi}{\partial \tilde{y}^2} - (1 + \gamma) |\phi|^2 \phi - U_0 \cos(q \tilde{y}) \phi,$$

(21)
Evidently, equation (21) is the usual 1D GPE with the standard OL potential, which always has soliton solutions [58], hence this 1D limit, unlike the full 2D system, admits the motion of solitons with an unlimited velocity.

Coming back to the 2D system, a crude explanation for the existence of the limit value of the velocity, \( v_{\text{max}} \) (see equation (1)), may be proposed, based on the numerical observation that, close to the delocalization transition at \( v_0 = v_{\text{max}} \), the solitons are, quite naturally, very broad in the x direction, keeping weakly separated maxima in the two components (see figure 1 below), i.e., they feature splitting of the components. This observation suggests to address a possibility of the splitting in terms of the stationary version of equations (15) and (16) for constant-amplitude solutions with chemical potential \( \mu < 0 \):

\[
\frac{\partial \bar{u}_\pm}{\partial t} = \pm \bar{u}_\pm \exp(-i\hat{\mu}t), \tag{22}
\]

where amplitudes \( \bar{u}_\pm \) may be real, and (close to the center) \( U_{2D}(x,\bar{y}) \) is replaced by \( U_0 \):

\[
\begin{align*}
(\hat{\mu} + U_0) \bar{u}_+ + \left( \bar{u}_+^2 + \gamma \bar{u}_-^2 \right) \bar{u}_+ - \lambda \bar{v}_0 \bar{u}_- &= 0, \tag{23} \\
(\hat{\mu} + U_0) \bar{u}_- + \left( \bar{u}_-^2 + \gamma \bar{u}_+^2 \right) \bar{u}_- - \lambda \bar{v}_0 \bar{u}_+ &= 0. \tag{24}
\end{align*}
\]

Then, it is relevant to look for a critical point at which a solution with an infinitesimal splitting between the components, \( \Delta \bar{u} \equiv \bar{u}_- - \bar{u}_+ \), appears on top of the obvious solution to equations (23) and (24) with identical (unsplit) components,

\[
\bar{u}_\pm^2 = \mp (\hat{\mu} + U_0 - \lambda \bar{v}_0) / (1 + \gamma). \tag{25}
\]

A simple calculation, based on equations (23) and (24) linearized with respect to \( \Delta \bar{u} \), yields the value of \( v_0 \) at the critical point:

\[
v_{\text{max}} = (1 - \gamma) (\hat{\mu} + U_0) / (2\lambda), \tag{26}
\]

the splitting being impossible at \( v_0 > v_{\text{max}} \). Finally, the substitution of value (26) in equation (25) yields the background amplitude at the critical point, \( \bar{u}_\pm^2 = (\hat{\mu} + U_0) / 2 \). Because the delocalization transition proceeds via small-amplitude solitons (see figure 1 below), the present consideration is relevant for small \( |\hat{\mu} + U_0| \) and, naturally, for small values of the SOC strength, \( \lambda \), to justify the use of the constant-amplitude solution. The prediction of the delocalization point, given by equation (26), makes sense at \( \gamma < 1 \), and it does not apply at \( \gamma > 1 \), when the formal expression predicts a negative velocity.

This consideration reveals the possibility of the splitting between the components which is only qualitatively similar to what occurs in the solitons of the MM type. Therefore, it is relevant to compare dependence \( v_{\text{max}} = \text{const} \cdot \lambda^{-1} \), predicted by equation (26), with numerical results (see the dashed hyperbola in figure 5(b) below), fitting const to the numerical data, rather than using the coefficient from equation (26).

In the limit of large \( \lambda \), opposite to one considered above, \( v_{\text{max}} \) can be estimated using the variational approximation (VA). To simplify the matters, one may apply it to the 1D limit of equations (15) and (16), in which the y derivatives are dropped (this limit case is opposite to one considered above in the form of equations (20) and (21), where the x derivatives were omitted). The stationary version of these equations for real functions \( \bar{u}_\pm(x) \), defined as in equation (22), is

\[
\begin{align*}
\mu + U_{1D}(x) + \frac{1}{2} \frac{d^2}{dx^2} + \alpha \bar{u}_-^2 + \gamma \bar{u}_+^2 \right) \bar{u}_+ + \lambda \left( v_0 - \frac{d}{dx} \right) \bar{u}_- &= 0, \tag{27} \\
\mu + U_{1D}(x) + \frac{1}{2} \frac{d^2}{dx^2} + \alpha \bar{u}_+^2 + \gamma \bar{u}_-^2 \right) \bar{u}_- + \lambda \left( v_0 + \frac{d}{dx} \right) \bar{u}_+ &= 0. \tag{28}
\end{align*}
\]

Because, in the case of large \( \lambda \), the delocalization always happens with states of the MM type, one may use the following Gaussian ansatz for mirror-symmetric components of the wave function:

\[
(\bar{u}_\pm(x))_{\text{ans}} = A \exp \left( -\frac{(x - \xi)^2}{2W^2} \right), \tag{29}
\]

where \( A \) and \( W \) are the amplitude and width, \( 2\xi \) being the separation between the split peaks of the components. The full form of VA turns out to be cumbersome, but, in the limit of large \( \lambda \), the prediction for \( (v_\lambda)_{\text{max}} \), as the critical value at which the solution for \( W \), with a fixed value of the norm, becomes impossible, is simple: \( (v_\lambda)_{\text{max}} = \lambda \) (this approximation neglects the presence of the potential). The particular coefficient in this expression depends on the assumptions adopted to apply VA, but the linear form of the dependence,

\[
(v_\lambda)_{\text{max}} = \text{const} \cdot \lambda, \tag{30}
\]

is a corollary of scaling properties of equations (15) and (16), in the absence of the external potential (the scaling leaves the total norm (13) of the 2D system invariant). The scaling is corroborated in detail below by a typical numerical solution displayed in figure 4. Equation (30) explains, approximately or exactly, numerical findings presented below in figures 5(a) and 8.

We note, in passing, that the linearized version of equations (27) and (28) with the 1D potential (10) admits a parametric resonance, accounted for by solutions in the form of

\[
\bar{u}_\pm(x) = a_\pm \cos \left( \frac{q \lambda v_0}{2} \right) + b_\pm \sin \left( \frac{q \lambda v_0}{2} \right). \tag{31}
\]

Straightforward consideration of equations (27) and (28) demonstrates that the parametric resonance takes place at

\[
\hat{\mu} = \frac{q^2}{2} \pm \sqrt{\left( \frac{q}{2} \right)^2 + \left( \lambda \bar{v}_0 \pm \frac{U_0}{2} \right)^2}, \tag{32}
\]

where the two signs \( \pm \) are mutually independent. However, the parametric resonance does not play an essential role in this work.

3. Results

As mentioned above, stationary 2D soliton solutions were produced by means of the imaginary-time integration of equations (7) and (8), performed for a fixed total norm (13) of the solitons. It was then verified by systematic simulations of perturbed evolution of the solitons in real time that they are
stable in the respective existence intervals (1). The identification of $v_{\text{max}}$ is the main objective of the numerical analysis. For the system of equations (7) and (8) with $\gamma = 0$ and $\gamma = 2$, we report the results with $N = 5$ and 3, respectively, as these values make it possible to present generic results. Numerical computations were performed in the 2D domain of size $12 \times 12$, which is sufficient to display all details of the 2D soliton profiles.

3.1. Dragging 2D solitons by the square-shaped OL in the absence of the nonlinear cross-interaction ($\gamma = 0$)

Results for the system with $\gamma = 0$ in equations (2) and (3), when, as mentioned above, only SVs are relevant solutions at $v_0 = 0$, are presented in figures 1–6. These results are produced for the full 2D lattice, defined as per equation (4), with amplitude $U_0$ (some figures display the results for $U_0 = 0$).

First, a set of cross sections of the moving solitons, corresponding to gradually increasing velocities, are displayed in figure 1. Panel (a) represents, for the sake of comparison, the results for the free space ($U_0 = 0$), which corresponds to reference [48]. This set of profiles demonstrates that, with the increase of $v_0$, the SV, which exists at $v_0 = 0$, is gradually transformed into a soliton with a quasi-MM structure. Indeed, linear-mixing terms $\sim \lambda v_0$ in the system written in the form of equations (15) and (16) make the existence of pure SVs, whose components carry different vorticities, 0 and 1, impossible. The growth of the mixing terms, with the increase of $v_0$, tends to make the two components mutually mirror-symmetric. The transition to this shape, which is the signature of the MM structure, occurs in figure 1(a) at

$$v_0 = v_{\text{SV-MM}} (U_0 = 0) \approx 0.25. \tag{33}$$

Simultaneously, the soliton expands in the $x$ direction, and eventually disappears, through complete delocalization, at

$$v_{\text{max}} (U_0 = 0) \approx 0.65. \tag{34}$$

Figure 1(b) displays similar results, produced by the imaginary-time-simulation method in the moving reference frame in the presence of the square-shaped OL potential (9) with a relatively small amplitude, $U_0 = 0.5$. In this case, the SV $\to$ MM and delocalization transitions occur at, respectively,

$$v_{\text{SV-MM}} (U_0 = 0.5) \approx 0.32, \tag{35}$$

$$v_{\text{max}} (U_0 = 0.5) \approx 1.75. \tag{36}$$

cf equations (33) and (34). It is seen that the effect of the OL potential is small in terms of the former transition, and quite strong for the expansion of the existence region of the moving solitons. In addition, figure 2 shows the full 2D shape of the solitons by means of contour plots of $|\phi_+ (x, y)|$ at $v_0 = 0.2$ (a) and 0.4 (b) for $U_0 = 0$. In particular, the plots clearly show the presence of a vortex–antivortex pair at $v_0 = 0.4$, which is a characteristic feature of patterns of the MM type, and is not possible in SVs.

The results for $U_0 = 0$ and 0.5 are further detailed in figure 3. Panel (a) shows the peak value $A$ of squared component $|\phi_+|^2$ as a function of the velocity. In the delocalized state, the peak amplitude does not fall to zero, because of the finite system’s size. Further, the SV $\to$ MM transition is quantified in panel (b) by plots of the parameter characterizing the asymmetry between the two components,

$$R \equiv N^{-1} \int \int |\phi_+ (x, y)|^2 dx dy - 1/2, \tag{37}$$
and (8) with rescaled profiles.

Figure 3. (a) Peak values $A$ of squared component $|\phi_+|^2$ of the moving solitons, as functions of the velocity, $v$, in the free space ($U_0 = 0$, shown by rhombuses), and under the action of the 2D lattice potential (9) with $U_0 = 0.5$, shown by crosses. (b) The asymmetry parameter (37) vs $v$ for the same soliton families. (c) The chemical potential vs $v_0$ for families. The dashed line, plotted as per equation (11) with $k_x = 2\pi \cdot 7/L$, represents the fully delocalized CW state. The solutions are obtained with $\lambda = 3$ and $\gamma = 0$ in equations (2) and (3). The fixed norm of the solitons is $N = 5$.

Figure 4. (a) Profiles of cross sections $|\phi_+(x)|$ and $|\phi_-(x)|$ (solid and dashed lines, respectively) of the stationary solutions of equations (7) and (8) with $\lambda = 3$, $v_0 = 0.2$, and $U_{2D} = 0$. (b) The same at $\lambda = 1.5$ and $v_0 = 0.1$. (c) Juxtaposition of $|\phi_+(x)|$ from (a) (solid lines) with rescaled profiles $2|\phi^{(s)}(x')|$ from (b) (dashed lines), plotted in rescaled coordinates $x' = 0.5x$, $y' = 0.5y$. The superimposed profiles completely overlap.

vs the velocity for $U_0 = 0$ and 0.5 (here $N$ is the total norm defined as per equation (13)). For MM solitons, whose components are mirror images of each other, $R = 0$, while one has $R > 0$ for SVs. Figure 3(b) clearly demonstrates the SV $\rightarrow$ MM transition at points (33) and (35) for $U_0 = 0$ and 0.5, respectively. Further, figure 3(c) shows the relationship between chemical potential $\mu$ of the solitons (see equation (12)) and velocity $v_0$ for the same soliton families. The dashed line is the chemical potential of the delocalized CW state at $U_0 = 0$, as given by equation (11), in which constant $A^2$ is expressed in terms of $N$, and $k_x = 2\pi \cdot 7/L$ is chosen. The chain of rhombuses coinciding with the CW line at $v_0 > v_{\text{max}}$ ($U_0 = 0$) $\approx 0.65$ (see equation (34)) represents fully delocalized states.

To verify the above-mentioned scaling which links different soliton solutions of equations (7) and (8) with $U_0 = 0$, we note that, if a spinor wave function $\phi_\pm(x, y, t)$ is a solution for parameters $\{\lambda, v_0, \gamma\}$, then a solution for the set of $\{s\lambda, s\gamma, \gamma\}$ is given by

$$\phi^{(s)}_\pm = s\phi_\pm(sx, s\gamma y, s^2 t; s\lambda, s\gamma v_0),$$

where $s$ is an arbitrary scaling factor. To check this property, figure 4 shows $|\phi_+|$ and $|\phi_-|$ (solid and dashed lines, respectively) for the numerically found soliton solutions with (a) $\{\lambda = 3, v_0 = 0.2\}$ and (b) $\{\lambda = 1.5, v_0 = 0.1\}$, which corresponds to $s = 0.5$ in equation (38). To check relation (38) in detail, figure 4(c) compares $|\phi_+|$ (with both components drawn by solid lines) for the former solution and the rescaled version of the latter one, $2|\phi^{(s)}(x')|$ (dashed lines), plotted in rescaled coordinates $x' = 0.5x$, $y' = 0.5y$. The overlap of the profiles confirms scaling relation (38) and, consequently, the linear relation,

$$(v_0)_{\text{max}}(s\lambda) = s(v_0)_{\text{max}}(\lambda),$$

for $U_0 = 0$.

As seen below in figures 5(b) and 8(b), the presence of the moving lattice breaks the exact linearity of equation (39), but keeps it as an approximate dependence between $(v_0)_{\text{max}}$ and $\lambda$. The slope of the approximately linear dependence is strongly affected by the lattice—actually, helping to expand the existence domain of the moving solitons.
The findings produced by the numerical solution for the solitons dragged by the 2D lattice are summarized in figure 5 by diagrams which display existence regions of the 2D solitons of the SV and MM types in the parameter plane of $(\lambda, v_0)$ for $U_0 = 0.5$ (a), and in the plane of $(U_0, v_0)$ for a fixed value of the SOC strength, $\lambda = 1.5$ (b). In these plots, symbol 0 implies the delocalization (nonexistence of solitons). The MM area appears in figure 5(a) at $\lambda > 0.75$. Again, these plots demonstrate that the effect of the OL potential is weak for the SV $\rightarrow$ MM transition, and strong for the expansion of the solitons’ existence range.

The shape of the left boundary between the SV and 0 areas in figure 5(a) is qualitatively explained by equation (26), as shown by the dashed hyperbola, drawn with a fitting coefficient $0.16$. As said above, this approximation is relevant only for small values of the SOC strength, $\lambda$, therefore it does not apply to other boundaries in figures 5(a) and (b). On the other hand, the roughly linear shape of the MM-delocalization boundaries is explained, as mentioned above, by the scaling relation (30).

To verify robustness of solutions for the solitons pulled by the moving OL, we have also performed direct real-time simulations of equations (2) and (3) written in the laboratory reference frame. As initial conditions, we used stationary solutions which were obtained, as above, by means of the imaginary-time integration in the coordinates moving at a certain velocity, $(v_0)_{\text{init}}$, while equations (2) and (3) were simulated in real time with the 2D potential moving at a higher (final) velocity, $(v_0)_{\text{fin}} > (v_0)_{\text{init}}$.

As a result, one might expect, in principle, to observe non-stationary solitons traveling at some mean speed $(\bar{v}) < (v_0)_{\text{fin}}$, so that they lag behind the dragging OL. However, our simulations have not produced such solutions. Instead, in all cases when $(v_0)_{\text{fin}}$ belongs to the stability areas shown in figure 5, the initial solitons either pick up the speed $(v_0)_{\text{fin}}$, moving with some internal vibrations, or suffer destruction. Characteristic examples are displayed in figure 6. As expected according to figure 5(b), the initial soliton, corresponding to $(v_0)_{\text{init}} = 0.1$, and the established one, with $(v_0)_{\text{fin}} = 0.2$ in panel (a), belong to the SV type, while the soliton eventually traveling at $(v_0)_{\text{fin}} = 0.3$ in (b) is of the MM type. Finally, setting $(v_0)_{\text{fin}} = 0.5$ in (c) leads to destruction of the soliton (delocalization), due the large mismatch between $(v_0)_{\text{fin}}$ and $(v_0)_{\text{init}}$. Residual intrinsic vibrations of the established solitons are illustrated by oscillations of asymmetry factor $R(t)$ (defined above in equation (37)), which are displayed in figure 6(d). The
oscillations are nearly regular, keeping $R > 0$ (i.e., the soliton keeps the asymmetry between its components, which is a signature of SVs) for $(v_0)_{\text{fin}} = 0.2$, or irregular, oscillating around $R = 0$, for the MM observed at $(v_0)_{\text{fin}} = 0.3$. The regular oscillations in the case of the SV are, essentially, performed by the ‘lighter’ vortex component of the SV moving around the ‘heavier’ zero-vorticity one. The compound soliton of the MM type actually has a larger number of effective degrees of freedom, as its both components have equal ‘masses’ (norms), and each component features sub-units with zero and nonzero vorticities. Therefore, the structure of the MM soliton opens a way to observe more complex internal dynamics.

3.2. Dragging and steering solitons by quasi-1D potentials

Proceeding to results obtained from equations (15) and (16) with $U_{2D}$ replaced by quasi-1D potentials (10), figure 7(a) shows the corresponding characteristics of the soliton families defined as in figure 3, i.e., the peak value $A$ of $|\phi_+|^2$ as a function of $v_0$. First, we notice that, if the 2D potential is replaced by $U_{1D}(y)$, with the same amplitude and wavenumber as above, the highest velocity admitting stable motion of the solitons (which are actually not dragged, but steered along the guiding channel, in such a case) is much larger. Indeed, figure 6(a) produces $v_{\text{max}} \approx 5$ for the solitons traveling under the action of potential $U_{1D}(x)$. This value is, roughly, three times larger than its counterpart (36) obtained above in the case of the 2D potential. The steep increase of $v_{\text{max}}$ in the latter case is explained by the fact that the potential $U_{1D}(x)$ tends to compress the 2D system of equations (15) and (16) into its 1D version. In turn, the 1D system may be reduced, as mentioned above, to the single GPE (21), which has no limitation for the existence of solitons at any velocity.

The increase of $v_{\text{max}}$ following the replacement of the 2D lattice by the quasi-1D ones has also been checked for the lattice wavenumbers different from value (6) which was fixed above. It was thus found that

$$q = 5\pi/6 : v_{\text{max}}^{(2D)} = 1.225, \quad v_{\text{max}}^{(1D_x)} = 1.525, \quad v_{\text{max}}^{(1D_y)} = 3.175,$$

$$q = 2\pi/3 : v_{\text{max}}^{(2D)} = 1.75, \quad v_{\text{max}}^{(1D_x)} = 2.25, \quad v_{\text{max}}^{(1D_y)} = 5,$$

respectively, for $U_{2D}(x,y)$, $U_{1D}(y)$, and $U_{1D}(x)$. Here, for the sake of comparison, equation (41) reproduces the above-mentioned results for $q = 2\pi/3$. It is seen that, in all the cases, $v_{\text{max}}$ decreases with the increase of $q$. This trend is naturally explained by the fact that the convolution of the soliton’s wave function with the rapidly oscillating OL potential produces a weaker effect.

3.3. Dragging MM solitons in the presence of the nonlinear cross-interaction ($\gamma > 1$)

As said above, the quiescent ($v_0 = 0$) ground-state solutions of equations (2) and (3) with $\gamma > 1$ in the free space ($V_0 = 0$) are MM solitons, while the SVs are unstable in this case. Then, the 2D potential (9) can drag the MM up to the respective limit velocity, $v_{\text{max}}$, above which the solitons suffer delocalization. First, the boundary between the moving MMs and
delocalized states in the free space ($U_0 = 0$) is plotted in figure 7(a) for the system with $\gamma = 2$. The linear shape of the boundary is rigorously explained by equation (30) which follows from the scaling properties of equations (2) and (3) with $U_0 = 0$ and a fixed norm. Next, the same boundary, but in the presence of the dragging potential, is plotted in figure 7(b). It is seen that even relatively weak potentials, with $U_0 = 0.2$ and $0.4$, help to strongly expand the stability area for the moving MM solitons.

4. Conclusion

In this work, we have demonstrated that mobility limits for two-component 2D matter-wave solitons, stabilized by the SOC effect, may be strongly expanded by means of relatively weak spatially periodic potentials moving at a desirable speed. Boundaries of the stable motion are identified by means of numerical methods, and the shape of some boundaries is explained analytically. If the stable quiescent solitons are SVs (semivortices), the motion converts them into MM, which suffer delocalization at much higher velocities. A remarkable finding is that quasi-1D potentials, especially the one with the wave vector directed perpendicular to the velocity, provide essentially stronger stabilization than the full 2D lattice.

The analysis reported here may be extended to develop schemes for the transfer of a soliton by a moving potential from an initial position to a predetermined final one, cf references [69–71]. Further, it may be interesting to develop the analysis for two- and multi-soliton complexes trapped in the lattice potential. A challenging option, suggested by reference [68], is to consider a possibility to stabilize 3D moving solitons, which are made metastable by SOC in the quiescent state.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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