UNIVERSAL REGULAR CONTROL FOR GENERIC SEMILINEAR SYSTEMS

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Abstract. We consider discrete-time projective semilinear control systems
\[ \xi_{t+1} = A(u_t) \cdot \xi_t, \]
where the states \( \xi_t \) are in projective space \( \mathbb{R}P^{d-1} \), inputs \( u_t \) are in a manifold \( \mathcal{U} \) of arbitrary finite dimension, and \( A: \mathcal{U} \rightarrow GL(d, \mathbb{R}) \) is a differentiable mapping.

An input sequence \( (u_0, \ldots, u_{N-1}) \) is called universally regular if for any initial state \( \xi_0 \in \mathbb{R}P^{d-1} \), the derivative of the time-\( N \) state with respect to the inputs is onto.

In this paper we deal with the universal regularity of constant input sequences \( (u_0, \ldots, u_0) \). Our main result states that generically in the space of such systems, for sufficiently large \( N \), all constant inputs of length \( N \) are universally regular, with the exception of a discrete set. More precisely, the conclusion holds for a \( C^2 \)-open and \( C^\infty \)-dense set of maps \( A \), and \( N \) only depends on \( d \) and on the dimension of \( \mathcal{U} \). We also show that the inputs on that discrete set are nearly universally regular; indeed there is a unique non-regular initial state, and its corank is 1.

In order to establish the result, we study the spaces of bilinear control systems. We show that the codimension of the set of systems for which the zero input is not universally regular coincides with the dimension of the control space. The proof is based on careful matrix analysis and some elementary algebraic geometry. Then the main result follows by applying standard transversality theorems.

1. Introduction

1.1. Basic definitions and some questions. Consider discrete-time control systems of the form:
\[ x_{t+1} = F(x_t, u_t), \quad (t = 0, 1, 2, \ldots) \]
where \( F: \mathcal{X} \times \mathcal{U} \rightarrow \mathcal{X} \) is a map. We will always assume that the space \( \mathcal{X} \) of states and the space \( \mathcal{U} \) of controls are manifolds, and that the map \( F \) is continuously differentiable.

A sequence \( (x_0, \ldots, x_N; u_0, \ldots, u_{N-1}) \) satisfying (1.1) is called a trajectory of length \( N \); it is uniquely determined by the initial state \( x_0 \) and the input \( (u_0, \ldots, u_{N-1}) \). Let \( \phi_N \) denote the time-\( N \) transition map, which gives the final state as a function of the initial state and the input:
\[ x_N = \phi_N(x_0; u_0, \ldots, u_{N-1}). \]

We say that the system (1.1) is accessible from \( x_0 \) in time \( N \) if the set \( \phi_N(\{x_0\} \times \mathcal{U}^N) \) of final states that can be reached from the initial state \( x_0 \) has nonempty interior.
The implicit function theorem gives a sufficient condition for accessibility. If the derivative of the map \( \phi_N(x_0; :t) \) at input \((u_0, \ldots, u_{N-1})\) is an onto linear map then we say that the trajectory determined by \((x_0; u_0, \ldots, u_{N-1})\) is regular. So the existence of such a regular trajectory implies that the system is accessible from \(x_0\) in time \(N\).

Let us call an input \((u_0, \ldots, u_{N-1})\) universally regular if for every \(x_0 \in \mathcal{X}\), the trajectory determined by \((x_0; u_0, \ldots, u_{N-1})\) is regular; otherwise the input is called singular.

The concept of universal regularity is central in this paper; it was introduced by Sontag and Wirth in [SW]. They showed that if the system (1.1) is accessible from every initial condition \(x_0\) in uniform time \(N\) then universally regular inputs do exist, provided one assumes the map \(F\) to be analytic. In fact, under those hypotheses they showed that universally regular inputs are abundant: in the space of inputs of sufficiently large length, singular ones form a set of positive codimension.

In this paper, we are interested in control systems (1.1) where the next state \(x_{t+1}\) depends linearly on the previous state \(x_t\) (but non-linearly on \(u_t\), in general). This means that the state space is \(\mathbb{K}^d\), where \(\mathbb{K}\) is either \(\mathbb{R}\) or \(\mathbb{C}\), and that (1.1) now takes the form:

\[
x_{t+1} = A(u_t) \cdot x_t, \quad \text{where } A: \mathcal{U} \to \text{Mat}_{d \times d}(\mathbb{K}).
\]

Following [CK1], we call this a semilinear control system.

In the case that the map \(A\) above takes values in the set \(\text{GL}(d, \mathbb{K})\) of invertible matrices of size \(d \geq 2\), we consider the corresponding projectivized control system:

\[
\xi_{t+1} = A(u_t) \cdot \xi_t,
\]

where the states \(\xi_t\) take value in the projective space \(\mathbb{P}^{d-1} = \mathbb{K}^d / \mathbb{K}_*\). We call this a projective semilinear control system. The projectivized system is also a useful tool for the study of the original system (1.1); see e.g. [Wi], [CK2].

Universally regular inputs for projective semilinear control systems were first considered by Wirth in [Wi]. Under his working hypotheses, the existence and abundance of such inputs is guaranteed by the aforementioned result of [SW]; then he uses universally regular inputs to obtain global controllability properties.

The purpose of this paper is to establish results on the existence and abundance of universally regular inputs for projective semilinear control systems. Differently from [SW], [Wi], we will not necessarily assume our systems to be analytic. Let us consider systems (1.4) with \(\mathbb{K} = \mathbb{R}\) and \(A: \mathcal{U} \to \text{GL}(d, \mathbb{R})\) a map of class \(C^r\), for some fixed \(r \geq 1\). To compensate for less rigidity, we do not try to obtain results that work for all \(C^r\) maps \(A\), but only for generic ones, i.e., those maps in a residual (dense \(G_d\) subset, or, even better, in an open dense subset.

To make things more precise, assume \(\mathcal{U}\) is a \(C^\infty\) (real) manifold without boundary. All manifolds are assumed to be Hausdorff paracompact with a countable base of open sets, and of finite dimension. We will always consider the space \(C^r(\mathcal{U}, \text{GL}(d, \mathbb{R}))\) endowed with the strong \(C^r\) topology (which coincides with the usual uniform \(C^r\) topology in the case that \(\mathcal{U}\) is compact).

Hence the first question we pose is this:

Taking \(N\) sufficiently large, is it true that for \(C^r\)-generic maps \(A\), the set of universally regular inputs in \(\mathcal{U}^N\) is itself generic?

It turns out that this question has a positive answer. Actually, in a work in preparation we show that for generic maps \(A\), all inputs in \(\mathcal{U}^N\) are universally regular,
except for those in a stratified closed set of positive codimension. So another natural question is this:

Fixed parameters $d$, $\dim U$, $N$, and $r$, what is the minimum codimension of the set of singular inputs in $U^N$ that can occur for $C^r$-generic maps $A: U \to GL(d, \mathbb{R})$?

In full generality, this question seems to be very difficult. A simpler setting would be to restrict to non-resonant inputs, namely those inputs $(u_0, \ldots, u_{N-1})$ such that $u_i \neq u_j$ whenever $i \neq j$. In this paper we consider the most resonant case. Define a constant input of length $N$ as an element of $U^N$ of the form $(u_0, u_0, \ldots, u_0)$. We propose ourselves to study universal regularity of inputs of this form.

1.2. The main result. We prove that generically the singular constant inputs form a very small set:

**Theorem 1.1.** Given $d \geq 2$ and $m \geq 1$, there exists an integer $N$ with $1 \leq N \leq d^2$ such that the following properties hold. Let $U$ be a smooth $m$-dimensional manifold without boundary. Then there exists a $C^2$-open $C^\infty$-dense subset $\mathcal{O}$ of $C^2(U, GL(d, \mathbb{R}))$ such that for every system (1.4) with $A \in \mathcal{O}$, all constant inputs of length $N$ are universally regular, except for those in a zero-dimensional (i.e., discrete) set.

By saying that a subset $\mathcal{O}$ of $C^2(U, GL(d, \mathbb{R}))$ is $C^\infty$-dense, we mean that for all $r \geq 2$, the intersection of $\mathcal{O}$ with $C^r(U, GL(d, \mathbb{R}))$ is dense in $C^r(U, GL(d, \mathbb{R}))$.

It is remarkable that the generic dimension of the set of singular constant inputs (namely, 0) does not depend on the dimension $m$ of the control space $U$, neither on the dimension $d-1$ of the state space. A partial explanation for this phenomenon is the following: First, the obstruction to universal regularity of the input $(u, u, \ldots, u)$ is the combined degeneracy of the matrix $A(u)$ and of the derivatives of $A$ at $u$. If $m$ is small then the image of the generic map $A$ will avoid too degenerate matrices, which increases the chances of obtaining universal regularity. If $m$ is large then more degenerate matrices $A(u)$ will inevitably appear; however the large number of control parameters compensates, so universal control is still likely.

The singular inputs that appear in Theorem 1.1 are not only rare; we also show that they are “almost” universally regular:

**Theorem 1.2** (Addendum to Theorem 1.1). The set $\mathcal{O} \subset C^2(U, GL(d, \mathbb{R}))$ in Theorem 1.1 can be taken with the following additional properties: If $A \in \mathcal{O}$ and a constant input $(u, \ldots, u)$ of length $N$ is singular then:

1. There is a single direction $\xi_0 \in \mathbb{RP}^{d-1}$ for which the corresponding trajectory of system (1.4) is not regular.
2. The derivative of the map $\phi_N(\xi_0; \cdot)$ at input $(u, \ldots, u)$ has corank 1.

To sum up, for generic systems (1.4), the universal regularity of constant inputs can fail only in the weakest possible way: there is at most one non-regular state, which can be moved in all directions but one.

We actually describe precisely in Appendix E the singular inputs that appear in Theorem 1.2. We show that these singular inputs can be unremovable by perturbations, and therefore Theorem 1.1 is optimal in the sense that there are $C^2$-open (actually even $C^1$-open) sets of maps $A$ for which the set of singular constant inputs is nonempty. Also, by $C^1$-perturbing any $A$ in those $C^2$-open sets, one can obtain an infinite number of singular constant inputs. In particular, the set $\mathcal{O}$ in the statement of the Theorem 1.1 is not $C^1$-open in general.
1.3. Reduction to the study of the set of poor data. The bulk of the proof of Theorem 1.1 consists on the computation of the dimension of certain canonical sets, as we now explain.

We fix \( A: \mathcal{U} \to \text{GL}(d, \mathbb{K}) \) and consider the projective semilinear system (1.4). By the chain rule, the universal regularity of an input \((u_0, u_1, \ldots, u_{N-1})\) depends only on the 1-jets of \( A \) at points \( u_0, \ldots, u_{N-1} \), i.e., on the first order Taylor approximations of \( A \) around those points.

Let us discuss the case of constant inputs \((u_0, \ldots, u_0)\). If we take local coordinates such that \( u_0 = 0 \) and replace the matrix map \( A: \mathcal{U} \to \text{GL}(d, \mathbb{K}) \) by its linear approximation, system (1.4) becomes:

\[
\xi_{t+1} = \left(A + \sum_{j=1}^{m} u_t C_j \right) \xi_t, \quad (t = 0, 1, 2, \ldots),
\]

where \( A = A(u_0) \) and \( C_1, \ldots, C_m \) are the partial derivatives at \( u_0 = 0 \). This is the projectivization of a bilinear control system (see [El]). For these systems, the zero input is a distinguished one and the focus of more attention.

To study system (1.5) it is actually more convenient to consider normalized derivatives \( B_j = C_j A^{-1} \), which intrinsically take values in the Lie algebra \( \mathfrak{gl}(d, \mathbb{K}) \). Consider the matrix datum \( A = (A, B_1, \ldots, B_m) \). We will explain how the universal regularity of the zero input is expressed in linear algebraic terms. Recall that the adjoint operator of \( A \) acts on \( \mathfrak{gl}(d, \mathbb{K}) \) by the formula \( \text{Ad}_A(B) = ABA^{-1} \). Consider the linear subspace \( \Lambda_N(A) \) of \( \mathfrak{gl}(d, \mathbb{K}) \) spanned by the matrices

\[
\text{Id} \quad \text{and} \quad (\text{Ad}_A)^i(B_j), \quad (i = 0, \ldots, n-1, \ j = 1, \ldots, m).
\]

(The identity matrix appears because of the projectivization.) This is nothing but the reachable set from 0 for the linear control system \((\text{Ad}_A, \text{Id}, B_1, \ldots, B_m)\). Then:

**Proposition 1.3.** The constant input \((0, \ldots, 0)\) of length \( N \) is universally regular for system (1.5) if and only if the space \( \Lambda_N(A) \) is transitive.

Here we say that a subspace of \( d \times d \) matrices with entries in the field \( \mathbb{K} \) is transitive if it acts transitively in the set \( \mathbb{K}^d \) of nonzero vectors.

Clearly, the spaces \( \Lambda_N(A) \) form a nested sequence that stabilizes to a space \( \Lambda(A) \) at some time \( N \leq d^2 \). If \( \Lambda(A) \) is transitive then the datum \( A \) is called rich; otherwise it is called poor. Let \( \mathcal{P}_m^{(\mathbb{K})} = \mathcal{P}_m^{(\mathbb{K})} \) denote the set of poor data. A major part of our work is to study these sets. We prove:

**Theorem 1.4.** The set \( \mathcal{P}_m^{(\mathbb{R})} \) is closed and semialgebraic, and its codimension in \( \text{GL}(d, \mathbb{R}) \times (\mathfrak{gl}(d, \mathbb{R}))^m \) is \( m \).

**Theorem 1.5.** The set \( \mathcal{P}_m^{(\mathbb{C})} \) is algebraic, and its (complex) codimension in \( \text{GL}(d, \mathbb{C}) \times (\mathfrak{gl}(d, \mathbb{C}))^m \) is \( m \).

So Theorems 1.4 and 1.5 say how frequent universal regularity of the zero input is in the space of projective bilinear control systems (1.5).

1.4. Overview of the proofs. Theorem 1.1 follows rather directly from Theorem 1.4 by applying standard results from transversality theory. More precisely, the fact that the set \( \mathcal{P}_m^{(\mathbb{R})} \) is semialgebraic implies that it has a canonical stratification. This permits us to apply Thom’s jet transversality theorem and obtain Theorem 1.1.

On the other hand, Theorem 1.4 follows from its complex version Theorem 1.5 by simple abstract arguments.
Thus everything is based on Theorem 1.5. One part of the result is easily obtained: we give examples of small disks of codimension \( m \) formed by poor data, so concluding that the codimension of \( \mathcal{P}_m^{(C)} \) is at most \( m \).

To prove the other inequality, one could try to exhibit an explicit codimension \( m \) set containing all poor data. For \( m = 1 \) this task is feasible (and we actually perform it, because with these conditions we can actually check universal regularity in concrete examples). However, for \( m = 2 \) already the task would be very laborious, and to expect to find a general solution seems unrealistic.

Our actual approach to prove the lower bound on the codimension of \( \mathcal{P}_m^{(C)} \) is indirect. Crude speaking, after careful matrix computations, we find some sets in the complement of \( \mathcal{P}_m^{(C)} \) that are reasonably “large” (basically in terms of dimension). Then, by using some abstract results of algebraic geometry, we are able to show that \( \mathcal{P}_m^{(C)} \) is “small”, thus proving the other half of Theorem 1.5.

Let us give more detail about this strategy. We decompose the set \( \mathcal{P}_m = \mathcal{P}_m^{(C)} \) into fibers:

\[
\mathcal{P}_m = \bigcup_{A \in \text{GL}(d, \mathbb{C})} \{ A \} \times \mathcal{P}_m(A), \quad \mathcal{P}_m(A) \subset [\mathfrak{gl}(d, \mathbb{C})]^m.
\]

It is not very difficult to show that for generic \( A \) in \( \text{GL}(d, \mathbb{C}) \), the fiber \( \mathcal{P}_m(A) \) has precisely the wanted codimension \( m \). However, for degenerate matrices \( A \), the fiber \( \mathcal{P}_m(A) \) may be much bigger. (For example, one can show that if \( A \) is an homothecy and \( m \leq 2d - 3 \) then \( \mathcal{P}_m(A) \) is the whole \( [\mathfrak{gl}(d, \mathbb{C})]^m \).) In order to show that codim \( \mathcal{P}_m \geq m \), we need to make sure that those degenerate matrices do not form a large set. More precisely, we show that:

\[
(1.6) \quad \forall k \in \{0, \ldots, m\}, \ \text{codim} \{ A \in \text{GL}(d, \mathbb{C}); \ \text{codim} \mathcal{P}_m(A) \leq m - k \} \geq k.
\]

Let us explain how we prove (1.6). In order to estimate the dimension of \( \mathcal{P}_m(A) \) for any matrix \( A \in \text{GL}(d, \mathbb{C}) \), we consider a quantity \( r = r(A) \) which is the least number such that a rich datum of the form \( (A, C_1, \ldots, C_r) \) exists. In particular, if \( r = r(A) \leq m \) then the following affine space

\[
(1.7) \quad \{(C_1, C_2, \ldots, C_r, B_{r+1}, \ldots, B_m); \ B_j \in [\mathfrak{gl}(d, \mathbb{C})] \}
\]

is contained in the complement of \( \mathcal{P}_m(A) \).

In certain situations, if two algebraic subsets have large enough dimensions then they necessarily intersect; for example, two algebraic curves in the complex projective plane \( \mathbb{CP}^2 \) always intersect. This kind of phenomenon happens here: the dimension of the affine space (1.7) forces a lower bound for the codimension of \( \mathcal{P}_m(A) \), namely:

\[
(1.8) \quad \text{codim} \mathcal{P}_m(A) \geq m + 1 - r(A).
\]

So we need to show that matrices \( A \) with large \( r(A) \) are rare. A careful matrix analysis provides an upper bound to \( r(A) \) based on the numbers and sizes of the Jordan blocks of \( A \), and on the occasional algebraic relations between the eigenvalues. This bound together with (1.8) implies (1.6) and therefore concludes the proof of Theorem 1.5.

In fact, the results of this analysis are even better, and we conclude that the codimension inequality (1.6) is strict when \( k \geq 1 \). This implies that poor data \( (A, B_1, \ldots, B_m) \) for which the matrix \( A \) is degenerate form a subset of \( \mathcal{P}_m^{(C)} \) with strictly bigger codimension. Thus we can show that the poor data that appear generically are well-behaved, which leads to Theorem 1.2.
1.5. Holomorphic setting. In the case of complex matrices (i.e., $K = \mathbb{C}$), we have a corresponding version of Theorem 1.1 where the maps $A$ are holomorphic. Given an open subset $U \subset \mathbb{C}^m$, we denote by $\mathcal{H}(U, GL(d, \mathbb{C}))$ the set of holomorphic mappings $A: U \to GL(d, \mathbb{C})$ endowed with the usual topology of uniform convergence on compact sets.

**Theorem 1.6.** Given integers $d \geq 2$ and $m \geq 1$, there exists an integer $N \geq 1$ with the following properties. Let $U \subset \mathbb{C}^m$ be open, and let $K \subset U$ be compact. Then there exists an open and dense subset $\mathcal{O}$ of $\mathcal{H}(U, GL(d, \mathbb{C}))$ such that for any $A \in \mathcal{O}$ the constant inputs in $K^N$ are all universally regular for the system (1.4), except for a finite subset.

We have the straightforward corollary:

**Corollary 1.7.** Given integers $d \geq 2$ and $m \geq 1$, there exists an integer $N \geq 1$ with the following properties. Let $U \subset \mathbb{C}^m$ be an open subset. There exists a residual subset $R$ of $\mathcal{H}(U, GL(d, \mathbb{C}))$ such that for any $A \in R$ the constant inputs in $U^N$ are all universally regular for the system (1.4), except for a discrete subset.

1.6. Directions for future research. One can also study uniform regularity of periodic inputs of higher period. Using our results for constant inputs, it is not difficult to derive some (non-sharp) codimension bounds for singular periodic inputs for generic systems. However, for highly resonant non-periodic inputs, we have no idea on how to obtain reasonable dimension estimates.

To obtain good estimates for the codimension of non-resonant singular inputs for generic systems is relatively simpler from the point of view of matrix computations, but needs more sophisticated transversality theorems (e.g., multijet transversality). Since highly resonant inputs have large codimension themselves, it seems possible to obtain reasonably good codimension estimates for general inputs for generic systems.

Another interesting direction of research is to consider other Lie groups of matrices.

1.7. Organization of the paper. Section 2 contains some basic results about transitivity of spaces of matrices and its relation with universal regularity. We also obtain the easy parts of Theorems 1.4 and 1.5, namely (semi)algebraicity and the upper codimension inequalities.

In Section 3 we introduce the concept of rigidity, which is related to the quantity $r(A)$ mentioned above. We state the central rigidity estimates (Theorem 3.6), which consist into two parts. The first and easier part is proved in the same Section 3, while the whole Section 4 is devoted to the proof of the second part.

Section 5 starts with some preliminaries in elementary algebraic geometry. Then we use the rigidity estimates to prove Theorem 1.3, following the strategy outlined above (§1.4). Theorem 1.4 follows easily. We also obtain a lemma that is needed for the proof of Theorem 1.2

In Section 6 we deduce Theorem 1.1 from previous results and standard theorems on stratifications and transversality.

The paper also has some appendices:

Appendix A basically reobtains the major results in the special case $m = 1$, where we actually gain additional information of practical value: as mentioned in §1.4, it is possible to describe explicitly what 1-jets the map $A$ should avoid in order to satisfy the conclusions of Theorems 1.4 and 1.2. The arguments necessary for the $m = 1$ case are much simpler and more elementary than those in Sections 3 to 6. Therefore the appendix is also useful to give the reader some intuition about the general problem, and as a source of examples. Appendix A is written in a
slightly informal way, and it can be read after Section 2 (though the final part requires Lemma 3.1).

Appendix B contains the proofs of necessary algebraic-geometric results, especially the one that allows us to obtain estimate (1.3).

Appendix C reviews the necessary concepts and results on stratifications, and proves a prerequisite transversality proposition.

In Appendix D we apply Theorem 1.3 to prove a version of Theorem 1.1 for holomorphic mappings.

In Appendix E we study the singular constant inputs of generic type, proving Theorem 1.2 and the other assertions made at the end of § 1.2 concerning the sharpness of Theorem 1.1. We also discuss the generic validity of some control-theoretic properties related to accessibility and regularity.

2. Preliminary facts on the poor data

In this section, we review some basic properties related to poorness, and prove the easy inequalities in Theorems 1.4 and 1.5.

2.1. Transitive spaces. Let $E$ and $F$ be finite-dimensional vector spaces over the field $\mathbb{K}$. Let $\mathcal{L}(E, F)$ be the space of linear maps from $E$ to $F$. A vector subspace $\Lambda$ of $\mathcal{L}(E, F)$ is called transitive if for every $v \in E \setminus \{0\}$, we have $\Lambda \cdot v = F$, where $\Lambda \cdot v = \{L(v); L \in \Lambda\}$.

Under the identification $\mathcal{L}(\mathbb{K}^n, \mathbb{K}^m) = \text{Mat}_{m \times n}(\mathbb{K})$, we may also speak of transitive spaces of matrices.

The following examples illustrate the concept; they will also be needed in later considerations.

Example 2.1. Recall that a Toeplitz matrix, resp. a Hankel matrix, is a matrix of the form

\[
\begin{pmatrix}
0 & t_1 & \cdots & t_{d-1} \\
-1 & 0 & \cdots & t_{d-2} \\
\vdots & \vdots & \ddots & \vdots \\
t_{d-1} & \cdots & 0 & t_0
\end{pmatrix}, \quad \text{resp.} \quad \begin{pmatrix}
h_0 & h_1 & \cdots & h_{d-1} \\
h_1 & h_0 & \cdots & h_{d-2} \\
\vdots & \vdots & \ddots & \vdots \\
h_{d-1} & \cdots & h_0 & h_d
\end{pmatrix},
\]

The set of Toeplitz matrices and the set of complex Hankel matrices constitute examples transitive subspaces of $\text{gl}(d, \mathbb{K})$. Transitivity of the Toeplitz space is a particular case of Example 2.2, and transitivity of Hankel space follows from Remark 2.2. For $\mathbb{K} = \mathbb{C}$, these spaces are optimal, in the sense that they have the least possible dimension; see [Az].

Example 2.2. A generalized Toeplitz space is a subspace $\Lambda$ of $\text{Mat}_{d \times d}(\mathbb{K})$ (where $d \geq 2$) with the following property: For any two matrix entries $(i_1, j_1)$ and $(i_2, j_2)$ which are not in the same diagonal (i.e., $i_1 - j_1 \neq i_2 - j_2$), the linear map $(b_{i,j})_{i,j} \in \Lambda \mapsto (b_{i_1,j_1}, b_{i_2,j_2}) \in \mathbb{C}^2$ is onto. Equivalently, a space is generalized Toeplitz if it can be defined by a number of linear relations between the matrix coefficients so that each relation involves only the entries on a same diagonal, and so that the relations do not force any relation entry to be zero. We will prove later (see § 3.3) that every generalized Toeplitz space is transitive.

Remark 2.3. If $\Lambda$ is a transitive subspace of $\mathcal{L}(E, F)$ and $P \in \mathcal{L}(E, E)$, $Q \in \mathcal{L}(F, F)$ are invertible operators then $P \cdot \Lambda \cdot Q := \{PLQ; L \in \Lambda\}$ is a transitive subspace of $\mathcal{L}(E, F)$.

Let us see that transitivity is a semialgebraic or algebraic property, according to the field. Recall that:

- A subset of $\mathbb{K}^n$ is called algebraic if it is expressed by polynomial equations with coefficients in $\mathbb{K}$. 

A subset of $\mathbb{R}^n$ is called semialgebraic if it is the union of finitely many sets, each of them defined by finitely many real polynomial equations and inequalities (see [BR], [BCR]).

**Proposition 2.4.** Let $\mathcal{N}_{m,n,k}^{(R)}$ be the set of $(B_1, \ldots, B_k) \in [\text{Mat}_{m \times n}^{(\mathbb{R})}]^k$ such that $\text{span}\{B_1, \ldots, B_k\}$ is not transitive. Then:

1. The set $\mathcal{N}_{m,n,k}^{(R)}$ is semialgebraic.
2. The set $\mathcal{N}_{m,n,k}^{(C)}$ is algebraic.

**Proof.** Consider the set of $(B_1, \ldots, B_k, v) \in [\text{Mat}_{m \times n}^{(\mathbb{R})}]^k \times \mathbb{R}^n$ such that $\text{span}\{B_1, \ldots, B_k\} \cdot v \neq \mathbb{R}^m$.

For $K = \mathbb{R}$, this is a semialgebraic set, because it is expressed by the vanishing of certain determinants plus the condition $v \neq 0$. Projecting this set along the $\mathbb{R}_u$ fiber we obtain $\mathcal{N}_{m,n,k}^{(R)}$; so, by the Tarski–Seidenberg theorem (see [BR] p. 60 or [BCR] p. 26), this set is semialgebraic, proving part 1.

To see part 2, we take $K = \mathbb{C}$ and projectivize the $\mathbb{C}^n_u$ fiber, obtaining an algebraic subset $[\text{Mat}_{m \times n}^{(\mathbb{C})}]^k \times \mathbb{CP}^{n-1}$ whose projection along the $\mathbb{CP}^{n-1}$ fiber is $\mathcal{N}_{m,n,k}^{(C)}$. So part 2 follows from the fact that projections along projective fibers take algebraic sets to algebraic sets (see [Sh] p. 58).

Complex transitivity of real matrices is a stronger property than real transitivity:

**Proposition 2.5.** The real part of $\mathcal{N}_{m,n,k}^{(C)}$ (that is, its intersection with $[\text{Mat}_{m \times n}^{(\mathbb{R})}]^k$) contains $\mathcal{N}_{m,n,k}^{(R)}$.

The proof is an easy exercise.

### 2.2. Universal regularity for constant inputs and richness

In this subsection we prove Proposition 1.3; in fact we prove a more precise result, and also fix some notation.

Given a linear operator $H : E \to E$, where $E$ is a finite-dimensional vector space over the field $K$, and vectors $v_1, \ldots, v_m \in E$, we denote by $\mathcal{R}_H^N(v_1, \ldots, v_m)$ the space spanned by the family of vectors $H^i(v_1)$, where $1 \leq i \leq m$ and $0 \leq t < N$. In other words, $\mathcal{R}_H^N(v_1, \ldots, v_m)$ is the reachable set from 0 of the linear control system

$$\xi_{t+1} = H \xi_t + \sum_i u_{t,i} v_i.$$

The sequence of spaces $\mathcal{R}_H^N(v_1, \ldots, v_m)$ is nested nondecreasing, and thus stabilize to a space $\mathcal{R}_H(v_1, \ldots, v_m)$ after $N \leq \dim H$ steps.

If $A : \mathcal{U} \to \text{GL}(d, \mathbb{C})$ is a differentiable map then the normalized derivative of $A$ at a point $u$ is the linear map $T_u \mathcal{U} \to \mathfrak{gl}(d, \mathbb{R})$ given by $h \mapsto (DA(u) \cdot h) \circ A^{-1}(u)$.

Let $\phi_N(\xi_0, \hat{u})$ be the state $\xi_N \in \mathbb{K}^p$ of the system (1.3) determined by the initial state $\xi_0$ and the input sequence $\hat{u} \in \mathcal{U}^N$. Let $\partial_2 \phi_N(\xi_0, \hat{u})$ be the derivative of the map $\phi_N(\xi_0, \cdot)$ at $\hat{u}$.

Fix a constant input $\hat{u} = (u, \ldots, u) \in \mathcal{U}^N$, and local coordinates on $\mathcal{U}$ around $u$. Let $B_j$ be the normalized partial derivatives of the map $A$ at $u$ with respect to the $i$th coordinate. Consider the datum $A = (A, B_1, \ldots, B_m)$, where $A = A(u)$. Define the following subspace of $\mathfrak{gl}(d, \mathbb{K})$:

$$A_N(A) = \mathcal{R}_N^{Ad_A}(\text{Id}, B_1, \ldots, B_m),$$

where $Ad_A(B) = ABA^{-1}$.
Proposition 2.6. For all $\xi_0 \in \mathbb{K}^{d-1}$ and any $x_0 \in \mathbb{K}^d \setminus \{0\}$ representing $\xi_0$, 
\[
\text{rank } \phi_N(\xi_0, \hat{u}) = \dim \left[ \Lambda_N(A) \cdot (A^N x_0) \right] - 1.
\]

In particular (since $A = A(u)$ is invertible), the input $\hat{u}$ is universally regular if and only if $\Lambda_N(A)$ is a transitive space, which is the statement of Proposition 1.3.

Proof of Proposition 2.6. Let $\xi_0 = [x_0]$, where $x_0 \in \mathbb{K}^d$. Let $\psi_N(x_0, \hat{u})$ be the final state of the non-projectivized system determined by the initial state $x_0$ and by the sequence of controls $\hat{u} \in U_N$. Using local coordinates with $u$ in the origin, we have the following first order approximation for $\hat{u} \approx 0$:
\[
\psi_N(x_0, \hat{u}) \approx A^N x_0 + \sum_{1 \leq i,j \leq m \atop 0 < k < N} u_{t,k} A^{N-t-1} B_i A^{t+1} x_0
\]

where $x_N = \psi_N(x_0, 0) = A^N x_0$. Therefore the image of $\phi_N(x_0, \hat{u})$ is the following subspace of $T_{A^N x_0} \mathbb{K}^d$:
\[
V = \left( \text{span} \left\{ \text{Ad}_A(B_j) \right\} \right) \cdot x_N,
\]

The image of $\phi_N(\xi_0, \hat{u})$ equals $D\pi(x_N)(V)$, where $\pi : \mathbb{K}^d \to \mathbb{K}^{d-1}$ is the canonical projection. Notice that $\text{Ker } D\pi(x) = \mathbb{K} x$ for any $x \in \mathbb{K}^d$. It follows that
\[
\text{rank } \phi_N(\xi_0, \hat{u}) = \dim \left[ \pi(x_N)(V) \right] = \dim \left[ \pi(x_N)(\mathbb{K} x_N + V) \right] = \dim [\mathbb{K} x_N + V] - 1
\]

Since $\mathbb{K} x_N + V = \Lambda_N(A) \cdot x_N$, the proposition is proved.

2.3. The sets of poor data. For emphasis, we repeat the definition already given in the introduction: The datum $A = (A, B_1, \ldots, B_m) \in \text{GL}(d, \mathbb{K}) \times [\mathfrak{gl}(d, \mathbb{K})]^m$ is rich if the space $\Lambda(A) = \Lambda_{d, \mathbb{K}}(A)$ is transitive, and poor otherwise. The concept in fact depends on the field under consideration. The set of such poor data is denoted by $\mathcal{P}(d)_{m, d}$.

It follows immediately from Proposition 2.4 that $\mathcal{P}^{(\mathbb{R})}_{m, d}$ is a closed and semialgebraic subset of $\text{GL}(d, \mathbb{R}) \times [\mathfrak{gl}(d, \mathbb{R})]^m$ and $\mathcal{P}^{(\mathbb{C})}_{m, d}$ is an algebraic subset of $\text{GL}(d, \mathbb{C}) \times [\mathfrak{gl}(d, \mathbb{C})]^m$. This proves part of Theorems 1.3.

Also, by Proposition 2.3, the real poor data are contained in the real part of the complex poor data, i.e.,
\[
\mathcal{P}^{(\mathbb{R})}_{m, d} \subset \mathcal{P}^{(\mathbb{C})}_{m, d} \cap \left[ \text{GL}(d, \mathbb{R}) \times [\mathfrak{gl}(d, \mathbb{R})]^m \right].
\]

For later use, we note that the sets of poor data are saturated in the sense of the following definition: A set $\mathcal{Z} \subset [\text{Mat}_{m,d}(\mathbb{K})]^{1+m}$ will be called saturated if $A, B_1, \ldots, B_m) \in \mathcal{Z}$ implies that: $A, B_1, \ldots, B_m) \in \mathcal{Z}$ implies that:

- for all $P \in \text{GL}(d, \mathbb{K})$ we have $(P^{-1} A P, P^{-1} B_1 P, \ldots, P^{-1} B_m P) \in \mathcal{Z}$;
- for all $Q = (q_{ij}) \in \text{GL}(m, \mathbb{K})$, letting $B' = \sum_{ij} q_{ij} B_j$, we have $(A, B'_1, \ldots, B'_m) \in \mathcal{Z}$. 

2.4. The easy codimension inequality of Theorems 1.4 and 1.5. Here we will discuss the simplest examples of poor data.

To begin, notice that if \( A \in \text{GL}(d, \mathbb{C}) \) is diagonalizable then so is \( \text{Ad}_A \). Indeed, assume without loss of generality that \( A = \text{Diag}(\lambda_1, \ldots, \lambda_d) \). Consider the basis \( \{ E_{i,j}; i, j \in \{1, \ldots, d\} \} \) of \( \mathfrak{gl}(d, \mathbb{C}) \), where

\[
E_{i,j} \text{ is the matrix whose only nonzero entry is a 1 in the } (i, j) \text{ position.}
\]

Then \( \text{Ad}_A(E_{i,j}) = \lambda_i \lambda_j^{-1} E_{i,j} \). So if \( f \) is a polynomial and \( B = (b_{ij}) \) then

\[
f(\text{Ad}_A)(B) \text{ is } f(\lambda_i \lambda_j^{-1})b_{ij}.
\]

The datum \( A = (A, B_1, \ldots, B_m) \in \text{GL}(d, \mathbb{K}) \times \mathfrak{gl}(d, \mathbb{K})^m \) is called conspicuously poor if there exists a change of bases \( P \in \text{GL}(d, \mathbb{K}) \) such that:

- the matrix \( P^{-1}AP \) is diagonal;
- the matrices \( P^{-1}B_iP \) have a zero entry in a common off-diagonal position; more precisely, there are indices \( i_0, j_0 \in \{1, \ldots, d\} \) with \( i_0 \neq j_0 \) such that for each \( k \in \{1, \ldots, m\} \), the \((i_0, j_0)\) entry of the matrix \( P^{-1}B_kP \) vanishes.

(As in the definition of poorness, the concept depends on the field \( \mathbb{K} \).)

**Lemma 2.7.** Conspicuously poor data are poor.

**Proof.** Let \( A = (A, B_1, \ldots, B_m) \) be conspicuously poor. With a change of basis we can assume that \( A \) is diagonal. Let \( (e_1, \ldots, e_d) \) be the canonical basis of \( \mathbb{K}^d \). Let \((i, j)\) be the entry position where all \( B_i \)'s have a zero entry. By (2.3), all matrices in the space \( \Lambda(A) = \mathfrak{g}_\text{Ad}_A(\text{Id}, B_1, \ldots, B_m) \) have a zero entry in the \((i_0, j_0)\) position. In particular, there is no \( L \in \Lambda(A) \) such that \( L \cdot e_{j_0} = e_{i_0} \), showing that this space is not transitive. \( \square \)

The converse of this lemma is certainly false. (Many examples appear in Appendix A; see also Example 3.3.) However, we will see in Appendix A.1 that the converse holds for generic \( A \).

We will use Lemma 2.7 to prove the easy codimension inequalities for Theorems 1.4 and 1.5. First we need to recall the following:

**Proposition 2.8.** Suppose \( A \in \text{Mat}_{d \times d}(\mathbb{K}) \) is diagonalizable over \( \mathbb{K} \) and with simple eigenvalues only. Then there is a neighborhood of \( A \) where the eigenvalues vary smoothly, and where the eigenvectors can be chosen to vary smoothly.

**Proposition 2.9** (Easy half of Theorems 1.4 and 1.5). For both \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \), we have coding \( P_m^{(\mathbb{K})} \leq m \).

**Proof.** Using Proposition 2.8, we can exhibit smoothly embedded disks of codimension \( m \) inside \( \text{GL}(d, \mathbb{K}) \times \mathfrak{gl}(d, \mathbb{K})^m \) formed by conspicuously poor data. \( \square \)

3. Rigidity

The aim of this section is to state Theorem 3.4 and prove its first part. Along the way we will establish several lemmas which will be reused in the proof of the second part of the theorem in Section 4.

3.1. Acyclicity. Consider a linear operator \( H : E \to E \), where \( E \) is a finite-dimensional complex vector space. The acyclicity of \( H \) is defined as the least number \( n \) of vectors \( v_1, \ldots, v_n \in E \) such that \( \mathfrak{R}_H(v_1, \ldots, v_n) = E \). We denote \( n = \text{acyc} H \). If \( n = 1 \) then \( H \) is called a cyclic operator, and \( v_1 \) is called a cyclic vector.
Lemma 3.1. Let \( E \) be a finite-dimensional complex vector space and let \( H: E \to E \) be a linear operator. Assume that \( E_1, \ldots, E_k \subset E \) are \( H \)-invariant subspaces and that the spectra of \( A|E_i \) (\( 1 \leq i \leq k \)) are pairwise disjoint. If \( v_1 \in E_1, \ldots, v_k \in E_k \) then
\[
\mathcal{R}_H(v_1, \ldots, v_k) = \mathcal{R}_H(v_1 + \cdots + v_k).
\]

Proof. View \( E \) as a module over the ring of polynomials \( \mathbb{C}[x] \) by defining \( xv = H(v) \) for \( v \in E \). Then the lemma follows from [Ro, Theorem 6.4]. □

The geometric multiplicity of an eigenvalue \( \lambda \) of \( H \) is the dimension of the kernel of \( H - \lambda \text{id} \) (or, equivalently, the number of corresponding Jordan blocks).

Proposition 3.2. The acyclicity of an operator equals the maximum of the geometric multiplicities of its eigenvalues.

Proof. This follows from the Primary Cyclic Decomposition Theorem together with Lemma 3.1. □

Remark 3.3. The operators which interest us most are \( H = \text{Ad}_A \), where \( A \in \text{GL}(d, \mathbb{C}) \). It is useful to observe that the geometric multiplicity of \( 1 \) as an eigenvalue of \( \text{Ad}_A \) equals the codimension of the conjugacy class of \( A \) inside \( \text{GL}(d, \mathbb{C}) \). To prove this, consider the map \( \Psi_A: \text{GL}(d, \mathbb{C}) \to \text{GL}(d, \mathbb{C}) \) given by \( \Psi_A(X) = \text{Ad}_X(A) \). The derivative at \( X = \text{id} \) is \( H \mapsto HA - AH \); so \( \text{Ker} D\Psi_A(\text{id}) = \text{Ker}(\text{Ad}_A - \text{id}) \). Therefore when \( X = \text{id} \), the rank of \( D\Psi_A(X) \) equals the geometric multiplicity of \( 1 \) as an eigenvalue of \( \text{Ad}_A \). To see that this is true for any \( X \), notice that \( \Psi_A = \Psi_{\text{Ad}_X(A)} \circ R_{X^{-1}} \) (where \( R \) denotes a right-multiplication diffeomorphism of \( \text{GL}(d, \mathbb{C}) \)).

We will see later (Lemma 4.11) that \( 1 \) is the eigenvalue of \( \text{Ad}_A \) with the biggest geometric multiplicity. By Proposition 3.2, we conclude that \( \text{acyc} \text{Ad}_A \) equals the codimension of the conjugacy class of \( A \).

3.2. Definition of rigidity, and the main rigidity estimate. Let \( E \) and \( F \) be finite-dimensional complex vector spaces. Let \( H \) be a linear operator action on the space \( \mathcal{L}(E, F) \). We define the rigidity of \( H \), denoted \( \text{rig} H \), as the least \( n \) such that there exist \( L_1, \ldots, L_n \in \mathcal{L}(E, F) \) so that \( \mathcal{R}_H(L_1, \ldots, L_n) \) is transitive. Therefore
\[
1 \leq \text{rig} H \leq \text{acyc} H.
\]

For technical reasons, we also define a modified rigidity of \( H \), denoted \( \text{rig}_+ H \). The definition is the same, with the difference that if \( E = F \) then \( L_1 \) is required to be the identity map in \( \mathcal{L}(E, E) \). Of course,
\[
\text{rig} H \leq \text{rig}_+ H \leq \text{rig} H + 1.
\]

We want to give a reasonably good estimate of the modified rigidity of \( \text{Ad}_A \) for any fixed \( A \in \text{GL}(d, \mathbb{C}) \). (This will be achieved in Lemma 4.14.) We assume that \( d \geq 2 \); so \( \text{rig}_+ \text{Ad}_A \geq 2 \). The next example shows that “most” matrices \( A \) have the lowest possible \( \text{rig}_+ \text{Ad}_A \).

Example 3.4. If \( A \in \text{GL}(d, \mathbb{C}) \) is unconstrained (see Appendix A.1) then \( \text{rig}_+ \text{Ad}_A = 2 \). Indeed if we take a matrix \( B \in \mathfrak{gl}(d, \mathbb{C}) \) whose expression in the base that diagonalizes \( A \) has no zeros off the diagonal then, by Lemma A.3, \( \Lambda(A, B) = \mathcal{R}_{\text{Ad}_A}(\text{id}, B) \) is rich.

More generally, if \( A \in \text{GL}(d, \mathbb{C}) \) is little constrained (see Appendix A) then it follows from Proposition A.3 that \( \text{rig}_+ \text{Ad}_A = 2 \).

Example 3.5. Consider \( A = \text{Diag}(1, \alpha, \alpha^2) \) where \( \alpha = e^{2\pi i/3} \). (In the terminology of Appendix A.1 \( A \) has constraints of type 1.) Since \( \text{Ad}_A^1 \) is the identity, we have \( \dim \mathcal{R}_{\text{Ad}_A}(\text{id}, B) \leq 4 \) for any \( B \in \mathfrak{gl}(3, \mathbb{C}) \). By the result of Azoff [Az] already mentioned at Example 2.1, the minimum dimension of a transitive subspace of \( \mathfrak{gl}(3, \mathbb{C}) \) is 5. This shows that \( \text{rig}_+ \text{Ad}_A \geq 3 \). (Actually, equality holds, as we will see in Example 3.6 below.)
Let $T$ be the set of roots of unity. Define an equivalence relation $\equiv$ on the set $\mathbb{C}^*$ of nonzero complex numbers by:

$$\lambda \equiv \lambda' \iff \frac{\lambda}{\lambda'} \in T.$$  

We also say that $\lambda, \lambda'$ are equivalent mod $T$.

For $A \in \text{GL}(d, \mathbb{C})$, we denote

$$c(A) := \text{number of different classes mod } T \text{ of the eigenvalues of } A.$$  

We now state a technical result which has a central role in our proofs, as explained informally in §3.4.

**Theorem 3.6.** Let $d \geq 2$ and $A \in \text{GL}(d, \mathbb{C})$. Then:

1. If $c(A) = d$ then $\text{rig}_+ \text{ Ad}_A = 2$.
2. If $c(A) < d$ then $\text{rig}_+ \text{ Ad}_A \leq \text{acyc Ad}_A - c(A) + 1$.

**Remark 3.7.** When $c(A) = d$, we have $\text{acyc Ad}_A = d$ (this will follow from Lemma 4.11); so the conclusion of part 2 does not hold in this case.

**Remark 3.8.** The conditions of $A$ being unconstrained and $A$ having $c(A) = d$ both mean that $A$ is “non-degenerate”. Both of them imply small rigidity, according to Example 3.4 and part 1 of Theorem 3.6. It is important, however, not to confuse the two properties; in fact, none implies the other.

**Example 3.9.** Consider again $A$ as in Example 3.5. The eigenvalues of $\text{Ad}_A$ are $1, \alpha$, and $\alpha^2$, each with multiplicity 3; so Proposition 3.4 gives $\text{acyc Ad}_A = 3$. So Theorem 3.6 tells us that $\text{rig}_+ \text{ Ad}_A \leq 3$, which is actually sharp.

The proof of part 1 of Theorem 3.6 will be given in §3.5 after a few preliminaries (§§3.3 and 3.4). These preliminaries are also used in the proof of the harder part 2 which will be given in Section 4.

### 3.3. A criterion for transitivity

We will show the transitivity of certain spaces of matrices that remotely resemble Toeplitz matrices.

Let $t$ and $s$ be positive integers. Let $\mathcal{R}_1$ be a partition of the interval $[1, t] = \{1, \ldots, t\}$ into intervals, and let $\mathcal{R}_2$ be a partition of $[1, s]$ into intervals. Let $\mathcal{R}$ be the product partition. We will be interested in matrices of the following special form:

$$M = (m_{i,j})_{1 \leq i \leq t, 1 \leq j \leq s} = \begin{pmatrix}
\ast & 0 & 0 \\
0 & M_k & 0 \\
0 & 0 & \ast
\end{pmatrix},$$

where $\mathcal{R}$ is an element of the product partition $\mathcal{R}$, and $M_k$ is the submatrix $(m_{i,j})_{(i,j) \in \mathcal{R}}$.

Let $\Lambda$ be a vector space of $\ell \times s$ matrices. For each $\mathcal{R} \in \mathcal{R}$, say of size $k \times \ell$, we define the following space of matrices:

$$\Lambda^{[\mathcal{R}]} = \{N \in \text{Mat}_{k \times \ell}(\mathbb{C}) : \exists M \in \Lambda \text{ of the form (3.3)} \text{ with } M_k = N\}.$$  

We regard $\Lambda$ as a subspace of $\mathcal{L}(\mathbb{C}^s, \mathbb{C}^t)$. If the rectangle $\mathcal{R}$ is $[p, p+k-1] \times [q, q+\ell-1]$, we regard the space $\Lambda^{[\mathcal{R}]}$ as a subspace of

$$\mathcal{L}([0]^{q-1} \times \mathbb{C}^\ell \times [0]^{t-q-\ell+1}, [0]^{p-1} \times \mathbb{C}^k \times [0]^{t-p-k+1}).$$

**Lemma 3.10.** Assume that $\Lambda^{[\mathcal{R}]}$ is transitive for each $\mathcal{R} \in \mathcal{R}$. Then $\Lambda$ is transitive.
An interesting feature of the lemma which will be useful later is that it can be applied recursively. Before giving the proof of the lemma, we illustrate its usefulness by showing the transitivity of generalized Toeplitz spaces:

**Proof of Example 2.2.** Consider the partition of $[1, d]^2$ into $1 \times 1$ “rectangles”. If $\Lambda$ is a generalized Toeplitz space then $\Lambda^{[R]} = \text{Mat}_{1 \times 1}(\mathbb{C}) = \mathbb{C}$ for each rectangle $R$. These are transitive spaces, so Lemma 3.11 implies that $\Lambda$ is transitive. □

Before proving Lemma 3.10, notice the following dual characterization of transitivity, whose proof is immediate:

**Lemma 3.11.** A subspace $\Lambda \subset \mathcal{L}(\mathbb{C}^*, \mathbb{C}^\ell)$ is transitive iff for any non-zero vector $u \in \mathbb{C}^*$ and any non-zero linear functional $\phi \in (\mathbb{C}^\ell)^*$ there exists $M \in \Lambda$ such that $\phi(M \cdot u) \neq 0$.

**Proof of Lemma 3.11.** Take any non-zero vector $u = (u_1, \ldots, u_s)$ in $\mathbb{C}^*$ and a non-zero functional $\phi(v_1, \ldots, v_t) = \sum_{i=1}^t \phi_i v_i$ in $(\mathbb{C}^\ell)^*$. By Lemma 3.11 we need to show that there exists $M = (x_{ij}) \in \Lambda$ such that

\[
\phi(M \cdot u) = \sum_{i=1}^s \sum_{j=1}^t \phi_i x_{ij} u_j
\]

is non-zero.

Let $j_0$ be the least index such that $u_{j_0} \neq 0$, and let $i_0$ be the greatest index such that $\phi_{i_0} \neq 0$. Let $R$ be the element of $\mathcal{R}$ that contains $(i_0, j_0)$. Notice that if $M$ is of the form (3.3) then the $(i, j)$-entries of $M$ that are above left (resp. below right) of $R$ do not contribute to the sum (3.5), because $u_j$ (resp. $\phi_i$) vanishes. That is, $\phi(M \cdot u)$ depends only on $M_R$ and is given by $\sum_{(i, j) \in R} \phi_i x_{ij} u_j$; Since $\Lambda^{[R]}$ is transitive, by Lemma 3.11 there is a choice of a matrix $M \in \Lambda$ of the form (3.3) so that $\phi(M \cdot u) \neq 0$. So we are done. □

3.4. **Preorder in the complex plane.** We consider the set $\mathbb{C}_*/T$ of equivalence classes of the relation $\sim$. Since $T$ is the torsion subgroup of $\mathbb{C}_*$, the quotient $\mathbb{C}_*/T$ is an abelian torsion-free group.

**Proposition 3.12.** There exists a multiplication-invariant total order $\preceq$ on $\mathbb{C}_*/T$.

The proposition follows from a result of Levi [Le], but nevertheless let us give a direct proof:

**Proof.** There is an isomorphism between $\mathbb{R} \oplus (\mathbb{R}/\mathbb{Q})$ and $\mathbb{C}_*/T$, namely $(x, y) \mapsto \exp(x + 2\pi iy)$. So it suffices to find a multiplication-invariant order in $\mathbb{R}/\mathbb{Q}$ (and then take the lexicographic order). Take a Hamel basis $B$ of the $Q$-vector space $\mathbb{R}$ so that $1 \in B$. Then $\mathbb{R}/\mathbb{Q}$ is a direct sum of abelian groups $\bigoplus_{x \in B, x \neq 1} \mathbb{Q}$. Order each $x \mathbb{Q}$ in the usual way and take any total order on $B$. Then the induced lexicographic order on $\mathbb{R}/\mathbb{Q}$ is multiplication-invariant, and the proof is concluded. □

Let $[z] \in \mathbb{C}_*/T$ denote the equivalence class of $z \in \mathbb{C}_*$. Let us extend the notation, writing $z \preceq z'$ if $[z] \preceq [z']$. Then $\preceq$ becomes a multiplication-invariant total preorder on $\mathbb{C}_*$ that induces the equivalence relation $\sim$. In other words, for all $z, z', z'' \in \mathbb{C}_*$ we have:

- $z \preceq z'$ or $z' \preceq z$;
- $z \preceq z'$ and $z' \preceq z \iff z = z'$;
- $z \preceq z'$ and $z' \preceq z'' \iff z \preceq z''$;
- $z \preceq z' \implies zz'' \preceq z'z''$.

It follows that:

- $z \preceq z' \implies (z')^{-1} \preceq z^{-1}$. 

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We write $z < z'$ when $z \preceq z'$ and $z \not\succeq z'$.

3.5. Proof of the easy part of Theorem 3.6.

Proof of part 1 of Theorem 3.6. If $c(A) = d$ then in particular all eigenvalues are different and so the matrix $A$ is diagonalizable. So with a change of basis we can assume that $A = \text{Diag}(\lambda_1, \ldots, \lambda_d)$. We can also assume that the eigenvalues are increasing with respect to the preorder introduced in §3.4:

$$\lambda_1 < \lambda_2 < \cdots < \lambda_d.$$

Fix any matrix $B$ with only nonzero entries, and consider the space $\Lambda = R_{\text{Ad}_A}(B)$, which is described by (2.4). We will use Lemma 3.10 to show that $\Lambda$ is transitive.

Let $R$ be the partition of $[1, d]^2$ into $1 \times 1$ rectangles. Given a cell $R = ((i_0, j_0)) \in R$ and a coefficient $t \in \mathbb{C}$, there exists a polynomial $f$ such that $f(\lambda_i, \lambda_j^{-1})$ equals $t$ if $\lambda_i \lambda_j^{-1} = \lambda_{i_0} \lambda_{j_0}^{-1}$ and equals 0 otherwise. Because the eigenvalues are ordered, $M = f(\text{Ad}_A) \cdot B$ is a matrix in $\Lambda$ of the form (3.3). Also, $M_k = (t)$. So $\Lambda^{[h]} = \mathbb{C}$, which is transitive. This shows that rig Ad$_A = 1$, and rig$_+\text{Ad}_A \leq 2$. Thus, as $d \geq 2$, we have rig$_+\text{Ad}_A = 2$. □

4. Proof of the hard part of the rigidity estimate

This section is wholly devoted to proving part 2 of Theorem 3.6. In the course of the proof we need to introduce some terminology and to establish several intermediate results. None of these are used in the rest of the paper, apart from a simple consequence, which is Remark 4.12.

4.1. The normal form. Let $A \in \text{GL}(d, \mathbb{C})$. In order to describe the estimate on rig$_+\text{Ad}_A$, we need to put $A$ in a certain normal form, which we now explain. Fix a preorder $\preceq$ on $\mathbb{C}_*$ as in §3.4.

List the eigenvalues of $A$ without repetitions as

$$\lambda_1 \preceq \cdots \preceq \lambda_r.$$

Write each eigenvalue in polar coordinates:

$$\lambda_k = \rho_k \exp(\theta_k \sqrt{-1}), \quad \text{where } \rho_k > 0 \text{ and } 0 \leq \theta_k < 2\pi.$$

Up to reordering, we may assume

$$\lambda_k = \lambda_\ell \quad \text{if } k < \ell \quad \Rightarrow \quad \theta_k < \theta_\ell.$$

With a change of basis, we can assume that $A$ has Jordan form:

$$A = \begin{pmatrix} A_1 & & & \\ & \ddots & & \\ & & A_r \end{pmatrix}, \quad A_k = \begin{pmatrix} J_{t_{k,1}}(\lambda_k) & & \\ & \ddots & \\ & & J_{t_{k,\tau_k}}(\lambda_k) \end{pmatrix},$$

where $t_{k,1} + \cdots + t_{k,\tau_k} = s_k$ is the multiplicity of the eigenvalue $\lambda_k$, and $J_t(\lambda)$ is the following $t \times t$ Jordan block:

$$J_t(\lambda) = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}.$$

The matrix $A$ will be fixed from now on.
4.2. Rectangular partitions. This subsection contains several definitions that will be fundamental in all arguments until the end of the section. We will define certain subregions of the set \([1, \ldots, d]^2\) of matrix entry positions that depend on the normal form of the matrix \(A\). Later we will see they are related to \(\text{Ad}_A\)-invariant subspaces. Those regions will be c-rectangles, e-rectangles, and j-rectangles (where c stands for classes of eigenvalues, e for eigenvalues and j for Jordan blocks). Regions will have some numerical attributes (banners and weights) coming from their geometry and from the eigenvalues of \(A\) they will be associated to. Those attributes will be related to numerical invariants of \(\text{Ad}_A\) (eigenvalues and geometric multiplicities), but we use different names so that we remember their geometric meaning and so that they are not mistaken for the corresponding invariants of \(A\). We also introduce positional attributes of the regions (arguments and latitudes) which will be useful fundamental later in the proofs of our rigidity estimates.

Recall \(A\) is a matrix in normal form as explained in §4.1. Define three partitions \(P_c, P_e, P_j\) of the set \([1, d] = \{1, \ldots, d\}\) into intervals:

- The partition \(P_c\) corresponds to equivalence classes of eigenvalues under the relation \(\approx\), that is, the right endpoints of its atoms are the numbers \(s_1 + \cdots + s_k\) where \(k = r\) or \(k\) is such that \(\lambda_k < \lambda_{k+1}\).
- The partition \(P_e\) corresponds to eigenvalues: the right endpoints of its atoms are the numbers \(s_1 + \cdots + s_k\), where \(1 \leq k \leq r\). So \(P_e\) refines \(P_c\).
- The partition \(P_j\) corresponds to Jordan blocks: the right endpoints of its atoms are the numbers \(s_1 + \cdots + s_{k-1} + t_{k,1} + \cdots + t_{k,\ell}\), where \(1 \leq k \leq r\) and \(1 \leq \ell \leq \tau_k\). So \(P_j\) refines \(P_c\).

For \(e = c, e, j\), let \(P^e_\square\) be the partition of the square \([1, d]^2\) into rectangles that are products of atoms of \(P_e\). The elements of \(P^e_\square\) are called c-rectangles, the elements of \(P^e_\square\) are called e-rectangles, and elements of \(P^j_\square\) are called j-rectangles. Thus the square \([1, d]^2\) is a disjoint union c-rectangles, each of them is a disjoint union of e-rectangles, each of them is a disjoint union of j-rectangles.

**Example 4.1.** Suppose \(d = 17\), \(A\) has \(r = 5\) eigenvalues

\[
\lambda_1 = \exp \frac{\pi i}{10}, \quad \lambda_2 = \exp \frac{2\pi i}{10}, \quad \lambda_3 = \exp \frac{3\pi i}{10}, \quad \lambda_4 = \exp \frac{4\pi i}{10}, \quad \lambda_5 = \exp \frac{5\pi i}{10}
\]

with \(\lambda_1 \approx \lambda_2 = \lambda_3 < \lambda_4 = \lambda_5\) and respective Jordan blocks of sizes 4, 2, 1, 3, 2; 2; 1.

Then there are 4 c-rectangles, 25 e-rectangles, and 64 j-rectangles. See Fig. 1.

For each e-rectangle we define its row eigenvalue and its column eigenvalue in the obvious way: If an e-rectangle \(E\) equals \(I_k \times I_\ell\) where \(I_k\) and \(I_\ell\) are intervals with right endpoints \(s_1 + \cdots + s_k\) and \(s_1 + \cdots + s_\ell\), respectively, then the row eigenvalue of \(E\) is \(\lambda_k\) and the column eigenvalue of \(E\) is \(\lambda_\ell\). The row and column eigenvalues of a j-rectangle \(J\) are defined respectively as the row and column eigenvalues of the e-rectangle that contains it.

Let \(E\) be an e-rectangle with row eigenvalue \(\lambda_k\) and column eigenvalue \(\lambda_\ell\). The **banner** of \(E\) is defined by \(\lambda_{\ell-1}^{-1} \lambda_k\). The **argument** of the e-rectangle is the quantity \(\theta_\ell - \theta_k \in (-2\pi, 2\pi)\). It coincides modulo \(2\pi\) with the argument of the banner, but it contains more information than the argument of the banner.

Each j-rectangle \(J\) has an address of the type \(i\text{th row, } j\text{th column}, e\text{-rectangle } E^\bullet\); then the latitude of the j-rectangle \(J\) within the e-rectangle \(E\) is defined as \(j - i\). See an example in Fig. 1.

If two e-rectangles lie in the same c-rectangle then their banners are equivalent mod \(T\). Thus every c-rectangle has a well-defined **banner class** in \(C^*/T\).

If a j-rectangle, e-rectangle, or c-rectangle intersects the diagonal \(\{(1, 1), \ldots, (d, d)\}\) then we call it **equatorial**. Equatorial regions are always square. Thus every equatorial e-rectangle has banner 1.
The weight of a j-rectangle is defined as the minimum of its sides. The weight of a union $R$ of j-rectangles in $[1,d]^2$ is defined as the sum of the weights of those j-rectangles. We denote it by $\text{wgt } R$. We can in particular consider the weights of $e$ and $c$-rectangles, and of the complete square $[1,d]^2$.

Let us notice some facts on the location of the banners (which will be useful to apply Lemma 3.11):

**Lemma 4.2.** Let $E$ be an $e$-rectangle in a $c$-rectangle $C$. Consider the divisions of the square $[1,d]^2$ and the $c$-rectangle $C$ as in Fig. 2.

![Figure 2](image-url)
Let $\beta$ be the banner of the e-rectangle $E$, and let $[\beta]$ be the banner class of the c-rectangle $C$. Then:

1. All the c-rectangles with banner class $[\beta]$ are inside the rectangles marked with $\times$.
2. If the e-rectangle $E$ has nonnegative (resp. negative) argument then all the e-rectangles with nonnegative (resp. negative) argument and with same banner $\beta$ are inside the rectangles marked with $\ast$.

Proof. In view of the ordering of the eigenvalues \[1.1\], the banner class increases strictly (with respect to the order $\prec$, of course) when we move rightwards or upwards to another c-rectangle. So Claim (1) follows.

The argument of an e-rectangle takes values in the interval $(-2\pi, 2\pi)$. It increases strictly by moving rightwards or upwards inside $C$. If two e-rectangles in the same c-rectangle have both nonnegative or negative argument then they have the same banner if and only if they have the same argument. So Claim (2) follows. $\square$

4.3. The action of the adjoint of $A$. Given any $d \times d$ matrix $X = (x_{i,j})$ and a j-rectangle, e-rectangle or c-rectangle $R = [p, p + t - 1] \times [q, q + s - 1]$ we define the submatrix of $X$ corresponding to $R$ as $(x_{i,j})_{(i,j) \in R}$. We regard the space of $R$-submatrices as $L([0]^d \times [0]^{d-1}, [0]^{d-1} \times [0]^d)$ or, as the set of $d \times d$ matrices whose entries outside $R$ are all zero. Such spaces are denoted by $R^\circ$, and are invariant under $Ad_A$. Indeed, if $R = J$ is a j-rectangle then identifying $J^\circ$ with $Mat_{\times \times}(\mathbb{C})$, the action of $Ad_A|J^\circ$ is given by

$$X \mapsto J_1(\lambda_\ell) \cdot X \cdot J_1(\lambda_s)^{-1},$$

where $\lambda_\ell$ and $\lambda_s$ are respectively the row and the column eigenvalues of $J$ and $J$ denotes Jordan blocks as defined by \[1.3\].

Lemma 4.3. For each j-rectangle $J$, the only eigenvalue of $Ad_A|J^\circ$ is the banner of the e-rectangle that contains $J$. Moreover, the geometric multiplicity of the eigenvalue is the weight of the j-rectangle.

Proof. The matrix of the the linear operator $Ad_A|J^\circ$ can be described using the Kronecker product: see \[HJ\] Lemma 4.3.1. The Jordan form of this operator is then described by \[HJ\] Theorem 4.3.17(a). The assertions of the lemma follow. $\square$

Some immediate consequences are the following:

- The eigenvalues of $Ad_A$ are the banners of e-rectangles.
- The geometric multiplicity of the eigenvalue $\beta$ for $Ad_A$ is the total weight of e-rectangles of banner $\beta$.

If $R$ is an equatorial j-rectangle, e-rectangle, or c-rectangle we will refer to the $d \times d$-matrix in $R^\circ$ whose $R$-submatrix is the identity as the identity on $R^\circ$. The following observation will be useful:

Lemma 4.4. If $J$ is an equatorial j-rectangle then the identity on $J^\circ$ is an eigenvector of the operator $Ad_A|J^\circ$ corresponding to a Jordan block of size $1 \times 1$.

Proof. Suppose $J$ has size $t \times t$ and row (or column) eigenvalue $\lambda$. Assume that the claim is false. This means that there exists a matrix $X \in Mat_{t \times t}(\mathbb{C})$ such that $J_1(\lambda) X J_1(\lambda)^{-1} = X + \text{Id}$, which is impossible because $X$ and $X + \text{Id}$ have different spectra. $\square$
4.4. Rigidity estimates for $j$-rectangles and $e$-rectangles.

**Lemma 4.5.** For any $j$-rectangle $J$, we have $\text{rig}_+(\text{Ad}_A[J^0]) \leq n^0$. 

**Proof.** By Lemma 4.3 (and Proposition 3.2), $\text{Ad}_A[J^0]$ has acyclicity $n = \text{wgt} J$, that is, there are matrices $X_1, \ldots, X_n \in J^0$ such that $\mathcal{R}_{\text{Ad}_A}(X_1, \ldots, X_n)$ is the whole $J^0$ (and, in particular, is transitive in $J^0$). So $\text{rig}(\text{Ad}_A[J^0]) \leq n$, which proves the lemma for non-equatorial $j$-rectangles.

If $J$ is an equatorial $j$-rectangle then, by Lemma 4.4, $J^0$ splits invariantly into two subspaces, one of them spanned by the identity matrix on $J^0$. So we can choose the matrices $X_i$ above so that $X_1$ is the identity. This shows that $\text{rig}_+(\text{Ad}_A[J^0]) \leq n$. □

In all that follows, we adopt the convention $\max \emptyset = 0$.

**Lemma 4.6.** For any $e$-rectangle $E$,

$$\text{rig}_+(\text{Ad}_A[E^0]) \leq \sum_{\ell \text{ latitude}} \max_{J \text{ is a } e\text{-rectangle with latitude } \ell} \text{rig}_+(\text{Ad}_A[J^0]).$$

**Proof.** For each $j$-rectangle $J$ contained in $E$, let $r(J) = \text{rig}_+(\text{Ad}_A[J^0])$. Take matrices $X_{j,1}, \ldots, X_{j,r(J)}$ such that $\Lambda_j := \mathcal{R}_{\text{Ad}_A}(X_{j,1}, \ldots, X_{j,r(J)})$ is a transitive subspace of $J^0$, and $X_{j,1}$ is the identity matrix in $J^0$ if $J$ is an equatorial $j$-rectangle. Define $X_{j,i} = 0$ for $i > r(J)$. For each latitude $\ell$, let $n_{\ell}$ be the maximum of $r(J)$ over the $j$-rectangles $J$ of $E$ with latitude $\ell$, and let

$$Y_{\ell,i} = \sum_{J \text{ is a } j\text{-rectangle with latitude } \ell} X_{j,i}, \text{ for } 1 \leq i \leq n_{\ell}. $$

Notice that if $E$ is an equatorial $e$-rectangle then $Y_{0,1}$ is the identity matrix in $E^0$. Consider the space

$$\Delta = \mathcal{R}_{\text{Ad}_A}\{Y_{\ell,i} : \ell \text{ is a latitude, } 1 \leq i \leq n_{\ell}\}. $$

We claim that for every $j$-rectangle $J$ in $E$ and for every $M \in \Lambda_j$, we can find some $N \in \Delta$ with the following properties:

- the submatrix $N_j$ equals $M$;
- for every $j$-rectangle $J'$ in $E$ that has a different latitude than $J$, the submatrix $N_{J'}$ vanishes. 

Indeed, if $M = \sum_{i=1}^{r(J)} f_i(\text{Ad}_A)X_{j,i}$ for certain polynomials $f_i$, we simply take $N = \sum_{i=1}^{r(J)} f_i(\text{Ad}_A)Y_{\ell,i}$, where $\ell$ is the latitude of $J$.

In notation (3.4), the claim we have just proved means that $\Delta^{[J]} \supset \Lambda_j$. So we can apply Lemma 3.10 and conclude that $\Delta$ is a transitive subspace of $E^0$. Therefore $\text{rig}_+(\text{Ad}_A[E^0]) \leq \sum n_{\ell}$, as we wanted to show. □

**Example 4.7.** Using Lemmas 4.3 and 4.6, we see that the $e$-rectangle $E$ whose $j$-rectangle weights are indicated in Fig. 1 has $\text{rig}_+(\text{Ad}_A[E^0]) \leq 5$.

In fact, we will not use Lemmas 4.3 and 4.6 directly, but only the following immediate consequence:

**Lemma 4.8.** For every $e$-rectangle $E$ we have $\text{rig}_+(\text{Ad}_A[E^0]) \leq \text{wgt} E$. The inequality is strict if $E$ has more than one row of $j$-rectangles and more than one column of $j$-rectangles.
4.5. **Comparison of weights.** If $R$ is a j-rectangle, e-rectangle or c-rectangle, we define its row projection $\pi_r(R)$ as the unique equatorial j-rectangle, e-rectangle or c-rectangle (respectively) that is in the same row as $R$. Analogously, we define the column projection $\pi_c(R)$.

**Lemma 4.9.** For any e-rectangle $E$, we have

$$\text{wgt } E \leq \frac{\text{wgt } \pi_r(E) + \text{wgt } \pi_c(E)}{2}.$$  

Moreover, if equality holds then the number of rows of e-rectangles for $E$ equals the number of columns of j-rectangles.

This is a clear consequence of the abstract lemma below, taking $x_\alpha$, $\alpha \in F_0$ (resp. $\alpha \in F_1$) as the sequence of heights (resp. widths) of j-rectangles in $E$, counting repetitions.

**Lemma 4.10.** Let $F$ be a nonempty finite set, and let $x_\alpha$ be positive numbers indexed by $\alpha \in F$. Take any partition $F = F_0 \cup F_1$, where $\cup$ stands for disjoint union. For $\epsilon, \delta \in \{0, 1\}$, let

$$\Sigma_{\epsilon \delta} = \sum_{(\alpha, \beta) \in F_\epsilon \times F_\delta} \min(x_\alpha, x_\beta).$$

Then

$$\Sigma_{01} = \Sigma_{10} \leq \frac{\Sigma_{00} + \Sigma_{11}}{2}.$$  

Moreover, equality implies that $F_0$ and $F_1$ have the same cardinality.

**Proof.** We will in fact prove the stronger fact:

$$\Sigma_{00} - 2\Sigma_{01} + \Sigma_{11} \geq (|F_0| - |F_1|)^2 \min x_\alpha,$$

where $|\cdot|$ denotes set cardinality. The proof is by induction on $|F|$. It clearly holds for $|F| = 1$. Fix some $n$ and assume that $(4.4)$ always holds when $|F| = n$. Take a set $F$ with $|F| = n + 1$, and take positive numbers $x_\alpha$, $\alpha \in F$. We can assume that $F = \{1, \ldots, n + 1\}$ and that $x_1 \geq \cdots \geq x_{n + 1}$. Take any partition $F = F_0 \cup F_1$. Without loss of generality, assume that $n + 1 \in F_0$. Apply the induction hypothesis to $F' = \{1, \ldots, n\}$, obtaining

$$\Sigma_{00}' - 2\Sigma_{01}' + \Sigma_{11}' \geq (|F_0'| - 1 - |F_1'|)^2 x_n.$$

We have

$$\Sigma_{00} = \Sigma_{00}' + (2|F_0| - 1)x_{n + 1}, \quad \Sigma_{01} = \Sigma_{01}' + |F_1|x_{n + 1}, \quad \text{and} \quad \Sigma_{11} = \Sigma_{11}',$$

so $(4.4)$ follows. \(\square\)

If $R$ is a c-rectangle or the entire square $[1, d]^2$, let $\text{wgt}_1 R$ denote the sum of the weights of the e-rectangles in $R$ with banner 1.

Let us give the following useful consequence of Lemma 4.9.

**Lemma 4.11.** $\text{acyc } \text{Ad}_A = \text{wgt}_1 [1, d]^2$.

**Proof.** By Proposition 3.2, $\text{acyc } \text{Ad}_A$ is the maximum of the geometric multiplicities of the eigenvalues of $\text{Ad}_A$. Those eigenvalues are the banners $\beta$, and the geometric multiplicity of each $\beta$ is the total weight with banner $\beta$. Thus, to prove the lemma we have to show that banner 1 has biggest total weight.

Let $\beta$ be a banner. Then, using Lemma 4.4,

$$\sum_{E \text{ is an e-rectangle with banner } \beta} \text{wgt } E \leq \frac{1}{2} \sum_{E \text{ is an e-rectangle with banner } \beta} \text{wgt } \pi_r(E) + \frac{1}{2} \sum_{E \text{ is an e-rectangle with banner } \beta} \text{wgt } \pi_c(E).$$
Since no two e-rectangles in the same row (resp. column) can have the same banner, the restriction of \( \pi_r \) (resp. \( \pi_c \)) to the set of e-rectangles with banner \( \beta \) is a one-to-one map. This allows us to conclude.

**Remark 4.12.** The Jordan type of a matrix \( A \in \text{Mat}_{d \times d}(C) \) consists on the following data:

1. The number of different eigenvalues.
2. For each eigenvalue, the number of Jordan blocks and their sizes.

It follows from Lemma 4.11 that these data is sufficient to determine \( \text{acyc } \text{Ad}_A \).

### 4.6. Rigidity estimate for e-rectangles

**Lemma 4.13.** For any e-rectangle \( \mathcal{C} \),

\[
\text{rig}_+(\text{Ad}_A[\mathcal{C}^e]) \leq \frac{\omega_1 \pi_r(\mathcal{C}) + \omega_1 \pi_c(\mathcal{C})}{2}.
\]

In order to prove this lemma, it is convenient to consider separately the cases of non-equatorial and equatorial c-rectangles.

**Proof of Lemma 4.13 when \( \mathcal{C} \) is non-equatorial.** For each banner \( \beta \) in \( \mathcal{C} \), let \( n_\beta \) (resp. \( s_\beta \)) be the maximum of \( \text{rig}_+(\text{Ad}_A[\mathcal{E}^e]) \) over nonnegative (resp. negative) argument e-rectangles \( \mathcal{E} \) in \( \mathcal{C} \) with banner \( \beta \). For each e-rectangle \( \mathcal{E} \) with banner \( \beta \), choose matrices \( X_{E_1}, \ldots, X_{E_{n_\beta + s_\beta}} \in \mathcal{E}^e \) such that:

- \( \Lambda_E := \mathfrak{R}_{\text{Ad}_A}(X_{E_1}, \ldots, X_{E_m}) \) is a transitive subspace of \( \mathcal{E}^e \);
- if \( \mathcal{E} \) has negative argument then \( X_1 = X_2 = \cdots = X_{n_\beta} = 0 \);
- if \( \mathcal{E} \) has nonnegative argument then \( X_{n_\beta + 1} = \cdots = X_{n_\beta + s_\beta} = 0 \).

Also, let \( X_{E,j} = 0 \) for \( j > n_\beta + s_\beta \).

Next, define

\[
(4.5) \quad Y_{\beta,j} = \sum_{E \text{ is an e-rectangle of } \mathcal{C} \text{ with banner } \beta} X_{E,j}
\]

and

\[
(4.6) \quad Z_j = \sum_{\beta \text{ banner on } \mathcal{C}} Y_{\beta,j}
\]

Consider the space

\[
\Delta = \mathfrak{R}_{\text{Ad}_A}(Z_1, \ldots, Z_m), \quad \text{where} \quad m = \max_{\beta \text{ banner on } \mathcal{C}} (n_\beta + s_\beta)
\]

It follows from Lemma 3.10 that

\[
\Delta = \mathfrak{R}_{\text{Ad}_A}\{Y_{\beta,j}; \beta \text{ is a banner}, 1 \leq j \leq n_\beta + s_\beta\}.
\]

Recall notation (3.4). We claim that

\[
(4.7) \quad \Lambda_E \subset \Delta[6].
\]

Indeed, given \( M \in \Lambda_E \), write \( M = \sum_j f_j(\text{Ad}_A)X_{E,j} \), where the \( f_j \)'s are polynomials and \( f_j \equiv 0 \) whenever \( X_{E,j} = 0 \). Consider \( N = \sum_j f_j(\text{Ad}_A)Y_{\beta,j} \), where \( \beta \) is the banner of \( E \). Then it follows from Lemma 4.2 (part 2) that \( N \in \Delta[6] \). This shows (4.7). So, by Lemma 3.10, \( \Delta \) is a transitive subspace of \( \mathcal{C}^e \), showing that \( \text{rig}_+(\text{Ad}_A[\mathcal{C}^e]) \leq m \).

To complete the proof of the lemma in the non-equatorial case, we show that

\[
(4.8) \quad m \leq \frac{\omega_1 \pi_r(\mathcal{C}) + \omega_1 \pi_c(\mathcal{C})}{2}.
\]

Let \( \beta \) be the banner for which \( n_\beta + s_\beta \) attains the maximum \( m \). If \( n_\beta > 0 \), let \( \mathcal{E}_+ \) be a nonnegative argument e-rectangle in \( \mathcal{C} \) with banner \( \beta \) and \( \text{rig}_+(\text{Ad}_A[\mathcal{E}_+]^e) = n_\beta \). If \( s_\beta > 0 \), let \( \mathcal{E}_- \) be a negative argument e-rectangle in \( \mathcal{C} \) with banner \( \beta \) and...
rig_+(Ad_A|E^e_+) = s_\beta. Assume for the moment that both e-rectangles exist. Let E_1, E_2, E_3, E_4 be projected equatorial e-rectangles as in Fig. 3.

Figure 3. The case of non-equatorial c-rectangles: E_1 = \pi_+(E_1), E_2 = \pi_+(E_-), E_3 = \pi_-(E_+), E_4 = \pi_-(E_+).

Figure 4. The case of equatorial non-exceptional c-rectangles: E_1 = \pi_+(E_+), E_2 = \pi_+(E_-), E_3 = \pi_-(E_+), E_4 = \pi_-(E_-). It is possible that E_1 = E_2 or E_3 = E_4.

Then

\[ m = \text{rig}_+(\text{Ad}_A|E^e_+) + \text{rig}_+(\text{Ad}_A|E^e_-) \overset{(i)}{\leq} \text{wgt }E^e_+ + \text{wgt }E^e_- \]
\[ \overset{(ii)}{\leq} \frac{1}{2} (\text{wgt }E_1 + \cdots + \text{wgt }E_4) \leq \frac{1}{2} (\text{wgt }C_1 + \text{wgt }C_2), \]

where (i) and (ii) follow respectively from Lemmas 4.8 and 1.9. This proves (4.8) in this case. If there is no nonnegative argument e-rectangle or no negative argument e-rectangle within C with banner 1 then the proof of (4.8) is easier.

So the lemma is proved for non-equatorial C. □

We now consider equatorial c-rectangles. There is a special kind of c-rectangle for which the proof of the rigidity estimate has to follow a different strategy. A c-rectangle is called exceptional if it has only the banners 1 and \(-1\) (so it is equatorial and has 4 e-rectangles), each e-rectangle has a single j-rectangle, and all j-rectangles have the same weight.

**Proof of Lemma 4.14 when C is equatorial non-exceptional.** As in the previous case, let \(n_\beta\) (resp. \(s_\beta\)) be the maximum of \(\text{rig}_+(\text{Ad}_A|E^e)\) over the nonnegative (resp. negative) argument e-rectangles E in C with banner \(\beta\).

We claim that

\[ n_\beta + s_\beta < \text{wgt}_1 C \quad \text{for all banners } \beta \neq 1 \text{ in } C. \]

Let us postpone the proof of this inequality and see how to conclude.

Let \(M = \text{wgt}_1 C\). In view of Lemma 1.8 and relation (1.11), for each c-rectangle E we can take matrices \(X_{E,1}, \ldots, X_{E,M} \in E^e\) such that:

- \(\Lambda_E := \mathcal{R}_{\text{Ad}_A}(X_{E,1}, \ldots, X_{E,M})\) is a transitive subspace of \(E^e\);
- \(X_{E,M} = 0\) if E is non-equatorial;
- \(X_{E,M}\) is the identity in \(E^e\) if E is equatorial.

Then define matrices \(Z_j\) as before: by (1.10) and (1.11). Here we have that \(Z_M\) is the identity matrix in \(C^e\). As before, \(\mathcal{R}_{\text{Ad}_A}(Z_1, \ldots, Z_M)\) is a transitive subspace of \(C^e\). Hence \(\text{rig}_+(\text{Ad}_A|C^e) \leq M = \text{wgt}_1 C\), as desired.

Now let us prove (1.11). Consider a banner \(\beta \neq 1\) in C. Let \(E_+\) (resp. \(E_-\)) be a nonnegative (resp. negative) argument e-rectangle within C with banner \(\beta\) and of
maximal weight; assume for the moment that both e-rectangles exist. Let $E_1$, $E_2$, $E_3$, $E_4$ be projected equatorial e-rectangles as in Fig. 4. Then

$$n_\beta + s_\beta = \operatorname{rig}_+ (\operatorname{Ad}_{A}[E_+]) + \operatorname{rig}_+ (\operatorname{Ad}_{A}[E_-])$$

(4.10)

$$\leq \operatorname{wgt} E_+ + \operatorname{wgt} E_-$$

(4.11)

$$\leq \frac{1}{2} (\operatorname{wgt} E_1 + \cdots + \operatorname{wgt} E_4)$$

(4.12)

Inequality (4.10) follows from Lemma 4.8, inequality (4.11) follows from Lemma 4.9, and inequality (4.12) holds because the e-rectangles $E_1, \ldots, E_4$ are equatorial, and any e-rectangle can appear at most twice in this list. So

$$n_\beta + s_\beta \leq \operatorname{wgt}_1 \mathbb{C}.$$ 

(4.13)

In the case that there is no nonnegative argument e-rectangle or no negative argument e-rectangle with banner $\beta$ (i.e., $n_\beta$ or $s_\beta$ vanishes), a simpler argument shows that strict inequality holds in (4.13).

Now assume by contradiction that (4.13) does not hold. Then we must have equality in (4.13). By what we have just seen, both e-rectangles $E_1$ and $E_2$ above exist. Then the inequalities in (4.10)–(4.12) become equalities. Since (4.12) is equatorial, there must be exactly two equatorial e-rectangles in $\mathbb{C}$. So the non-equatorial banner $\beta$ satisfies $\beta^{-1} = \beta$, that is, $\beta = -1$. Since (4.11) is an equality, it follows from Lemma 4.9 that both non-equatorial e-rectangles have the same number of j-rectangles in each column and each row. So there is some $\ell$ such that all four e-rectangles in $\mathbb{C}$ have $\ell$ rows of j-rectangles and $\ell$ columns of j-rectangles. Since (4.10) is an equality, Lemma 4.8 implies that $\ell = 1$. That is, $\mathbb{C}$ is a exceptional c-rectangle, a situation which we excluded a priori. This contradiction proves (4.9) and Lemma 4.13 in the present case.

Let us now deal with exceptional c-rectangles. In all the previous cases, the transitive subspace we found had some vaguely Toeplitz form. For exceptional c-rectangles, however, this strategy is not efficient. What we are going to do is to find a transitive space of vaguely Hankel form, namely the following:

$$\Lambda_k = \left\{ \begin{pmatrix} P & M \\ M & N \end{pmatrix} : M, N, P \text{ are } k \times k \text{ matrices} \right\}.$$ 

(4.14)

Notice that $\Lambda_k = S_k \cdot \Gamma_k$, where

$$S_k = \begin{pmatrix} 0 & \operatorname{Id} \\ \operatorname{Id} & 0 \end{pmatrix} \quad \text{and} \quad \Gamma_k = \left\{ \begin{pmatrix} M \\ P & M \end{pmatrix} : M, N, P \text{ are } k \times k \text{ matrices} \right\}.$$ 

Since $\Gamma_k$ is a generalized Toeplitz space, it follows from Remark 2.3 that $\Lambda_k$ is transitive.

Proof of Lemma 4.13 when $\mathbb{C}$ is exceptional. If $\mathbb{C}$ is exceptional then it has size $2k \times 2k$ for some $k$, and the operator $\operatorname{Ad}_{A}[\mathbb{C}]$ is given by $X \mapsto \operatorname{Ad}_{L}(X)$, where

$$L = \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}, \quad \text{and} \quad J = J_k(1) \text{ is the Jordan block } (4.3).$$ 

Let $V$ be unique $\operatorname{Ad}_L$-invariant subspace of $\operatorname{Mat}_{2k \times 2k}(\mathbb{C})$ that has codimension 1 and does not contain the identity matrix (which exists by Lemma 4.4). Take matrices $X_1, \ldots, X_k \in \operatorname{Mat}_{2k \times 2k}(\mathbb{C})$ such that $X_1 = \operatorname{Id}$ and $V = \mathcal{R}_{\operatorname{Ad}_L}(X_2, \ldots, X_k)$. Define $Y_1, \ldots, Y_k \in \operatorname{Mat}_{2k \times 2k}(\mathbb{C})$ by

$$Y_1 = \begin{pmatrix} \operatorname{Id} & 0 \\ 0 & \operatorname{Id} \end{pmatrix}, \quad Y_j = \begin{pmatrix} X_j & 0 \\ 0 & 0 \end{pmatrix} \text{ for } 2 \leq j \leq k,$$
Then
\[ \mathcal{R}_{\text{Ad}}(Y_1, \ldots, Y_k) = \left\{ \begin{pmatrix} x\text{Id} + Z & 0 \\ 0 & x\text{Id} \end{pmatrix} : x \in \mathbb{C}, \ Z \in V \right\}. \]

For \( j = k + 1, \ldots, 2k \), define
\[ Y_j = \begin{pmatrix} 0 & X_{j-k} \\ X_{j-k} & 0 \end{pmatrix}. \]

Then, by Lemma 3.1,
\[ \mathcal{R}_{\text{Ad}}(Y_{k+1}, \ldots, Y_{2k}) = \left\{ \begin{pmatrix} 0 & M \\ M & N \end{pmatrix} : M, N \in \text{Mat}_{k \times k}(\mathbb{C}) \right\}. \]

Therefore \( \mathcal{R}_{\text{Ad}}(Y_1, \ldots, Y_{2k}) \) is the transitive space given by (4.14). Since \( Y_1 \) is the identity on \( \mathbb{C} \), this shows that \( \text{rig}_{+}(\text{Ad}_{\mathbb{A}}|\mathbb{C}) \leq 2k = \text{wgt}_{1}\mathbb{C} \), concluding the proof of Lemma 4.13. \( \square \)

4.7. The final rigidity estimate. Let \( c = c(A) \) be the number of equivalence classes mod \( T \) of eigenvalues of \( A \).

Lemma 4.14. If \( c < d \) then
\[ \text{rig}_{+} \text{Ad}_{\mathbb{A}} \leq \text{wgt}_{1}[1, d]^2 - c + 1. \]

Proof. Let \( m = \text{wgt}_{1}[1, d]^2 - c + 1 \). For each \( c \)-rectangle \( \mathbb{C} \), let
\[ r(\mathbb{C}) = \left\lfloor \frac{1}{2}(\text{wgt}_{1}\pi_{1}(\mathbb{C}) + \text{wgt}_{1}\pi_{c}(\mathbb{C})) \right\rfloor. \]

We claim that
\[ r(\mathbb{C}) \leq \begin{cases} m & \text{if } \mathbb{C} \text{ is an equatorial } c \text{-rectangle}, \\ m - 1 & \text{if } \mathbb{C} \text{ is a non-equatorial } c \text{-rectangle}. \end{cases} \tag{4.15} \]

Let us postpone the proof of this inequality and see how it implies the lemma.

In view of Lemma 4.13 and relation (4.15), for each \( c \)-rectangle \( \mathbb{C} \) we can take matrices \( X_{\mathbb{C},1}, \ldots, X_{\mathbb{C},m} \in \mathbb{C}^{d} \) such that:
- \( \Lambda_{\mathbb{C}} := \mathcal{R}_{\text{Ad}_{\mathbb{A}}}(X_{\mathbb{C},1}, \ldots, X_{\mathbb{C},m}) \) is a transitive subspace of \( \mathbb{C}^{d} \);
- \( X_{\mathbb{C},m} = 0 \) if \( \mathbb{C} \) is non-equatorial;
- \( X_{\mathbb{C},m} \) is the identity in \( \mathbb{C}^{d} \) if \( \mathbb{C} \) is equatorial.

Define matrices:
\[ Y_{\alpha,j} = \sum_{\mathbb{C} \text{ is a } c \text{-rectangle}} X_{\mathbb{C},j} \quad (\alpha \text{ is a banner class}, \ 1 \leq j \leq m), \]
\[ Z_{j} = \sum_{\alpha \text{ is a banner class}} Y_{\alpha,j} \quad (1 \leq j \leq m). \]

So \( Z_{m} \) is the \( d \times d \) identity matrix. Consider the space
\[ \Delta = \mathcal{R}_{\text{Ad}_{\mathbb{A}}}(Z_{1}, \ldots, Z_{m}). \]

It follows from Lemma 3.1 that
\[ \Delta = \mathcal{R}_{\text{Ad}_{\mathbb{A}}}(\{Y_{\alpha,j} : \alpha \text{ is a banner class}, 1 \leq j \leq m\}). \]

We claim that every \( c \)-rectangle \( \mathbb{C} \),
\[ \Lambda_{\mathbb{C}} \subset \Delta^{[\mathbb{C}]} \tag{4.16} \]

Indeed, if \( M \in \mathbb{C} \) then we can write \( M = \sum_{j} f_{j}(\text{Ad}_{\mathbb{A}})X_{\mathbb{C},j} \), where the \( f_{j} \)'s are polynomials. Consider \( N = \sum_{j} f_{j}(\text{Ad}_{\mathbb{A}})Y_{\alpha,j} \), where \( \alpha \) is the banner class of \( \mathbb{C} \). It follows Lemma 4.2 (part 11) that \( N \in \Delta^{[\mathbb{C}]} \). This proves (4.16). So, by Lemma 3.11 \( \Delta \) is a transitive subspace of \( \text{Mat}_{d \times d}(\mathbb{C}) \), showing that \( \text{rig}_{+} \text{Ad}_{\mathbb{A}} \leq m \).
To conclude the proof we have to show estimate (4.15). First consider a equatorial c-rectangle $C$. Since there are $c$ equatorial c-rectangles, and each of them has a nonzero $wgt_1$ value, we conclude that $r(C) \leq m$, as claimed.

Now take a non-equatorial $C$. Applying what we have just proved for the equatorial c-rectangles $\pi_r(C)$ and $\pi_c(C)$, we conclude that $r(C) \leq m$. Now assume that (4.15) does not hold for $C$, that is, $r(C) > m$. Then

$$wgt_1 \pi_r(C) = wgt_1 \pi_c(C) = m = wgt_1[1, d]^2 - c + 1.$$  

Since $wgt_1[1, d]^2 \geq wgt_1 \pi_t(C) + wgt_1 \pi_c(C) + c - 2$, we have $m = 1$ and $wgt_1[1, d]^2 = c$. This means that $wgt_1 C = 1$ for all equatorial c-rectangles $C$, which is only possible if $c = d$. However, this case was excluded by hypothesis.

This proves (4.15) and hence Lemma 4.14.

Example 4.15. If $A$ is the matrix of Example 4.1 then Lemma 4.14 gives the estimate $\text{rig}_+ \text{Ad}_A \leq 28$. A more careful analysis (going through the proofs of the lemmas) would give $\text{rig}_+ \text{Ad}_A \leq 7$ (see Example 4.7).

Proof of part 2 of Theorem 3.6. Apply Lemmas 4.11 and 4.14.

5. PROOF OF THE HARD PART OF THE CODIMENSION $m$ THEOREM

We showed in Proposition 2.9 that $\text{codim} P_m^{(X)} \leq m$. In this section, we will prove the reverse inequalities. More precisely, we will first prove Theorem 1.5 and then deduce Theorem 1.4 from it.

5.1. Preliminaries on elementary algebraic geometry.

5.1.1. Quasiprojective varieties. An algebraic subset of $\mathbb{C}^n$ is also called an affine variety. A projective variety is a subset of $\mathbb{CP}^n$ that can be expressed as the zero set of a family of homogeneous polynomials in $n + 1$ variables. The Zariski topology on an (affine or projective) variety $X$ is the topology whose closed sets are the (affine or projective) subvarieties of $X$.

An open subset $U$ of a projective variety $X$ is called a quasiprojective variety. We consider in $U$ the induced Zariski topology. The affine space $\mathbb{C}^n$ can be identified with a quasiprojective variety, namely its image under the embedding $(z_1, \ldots, z_n) \mapsto (1 : z_1 : \cdots : z_n)$.

If $X$ and $Y$ are quasi-projective varieties then the product $X \times Y$ can be identified with a quasiprojective variety, namely its image under the Segre embedding; see [Sh], § 5.1.

Recall the following property from [Sh] p. 58:

Proposition 5.1. If $X$ is a projective variety and $Y$ is a quasiprojective variety then the projection $p: X \times Y \rightarrow Y$ takes Zariski closed sets to Zariski closed sets.

A quasiprojective variety is called irreducible if it cannot be written as a nontrivial union of two quasiprojective varieties (that is, none contains the other).

5.1.2. Dimension. The dimension $\text{dim} X$ of an irreducible quasiprojective variety $X$ may be defined in various equivalent ways (see for instance [Ha] p. 133ff). It will be sufficient for us to know that there exists an (intrinsically defined) subvariety $Y$ of the singular points of $X$ such that in a neighborhood of each point of $X \setminus Y$, the set $X$ is a complex submanifold of dimension (in the classical sense of differential geometry) $\text{dim} X$; moreover, each irreducible component of $Y$ has dimension strictly less than $\text{dim} X$.

The dimension of a general quasiprojective variety is by definition the maximum of the dimensions of the irreducible components.
The following lemma is useful to estimate the codimension of an algebraic set $X$ from information about the fibers of a certain projection $\pi: X \to Y$.

**Lemma 5.2.** Let $Y$ be a quasiprojective variety. Let $X \subset Y \times \mathbb{CP}^n$ be a nonempty algebraically closed set. Let $\pi: X \to Y$ be the projection along $\mathbb{CP}^n$. Then:

1. For each $j \geq 0$, the set
   \[ C_j = \{ y \in \pi(X); \codim \pi^{-1}(y) \leq j \} \]
   is algebraically closed in $Y$.
2. The dimension of $X$ is given in terms of the dimensions of the $C_j$'s by:
   \[ \codim X = \min_{j: C_j \neq \emptyset} \left( j + \codim C_j \right). \] (1)

In the above, the codimensions of $\pi^{-1}(Y)$, $X$ and $C_j$ are taken with respect to $\mathbb{CP}^n, Y \times \mathbb{CP}^n$ and $Y$, respectively. The proof of the lemma is given in Appendix B.

**Remark 5.3.** Lemma 5.2 works with the same statement if $\mathbb{CP}^n$ is replaced by $\mathbb{C}^{n+1}$. Provided one assumes that $X \subset Y \times \mathbb{C}^{n+1}$ is homogeneous in the second factor (i.e., $(y, z) \in X$ implies $(y, tz) \in X$ for every $t \in \mathbb{C}$). Indeed, this follows from the fact that the projection $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ preserves codimension of homogeneous sets.

**5.1.3. Dimension estimates for sets of vector subspaces.** If $M \in \text{Mat}_{n \times m}(\mathbb{K})$, let $\text{col} M \subset \mathbb{K}^m$ denote the column space of $M$. A set $X \subset \text{Mat}_{n \times m}(\mathbb{K})$ is called column-invariant if
\[
\begin{align*}
M \in X \\
N \in \text{Mat}_{n \times m}(\mathbb{K}) \\
\text{col} M = \text{col} N
\end{align*}
\]
implies $N \in X$.

So a column-invariant set $X$ is characterized by its set of column spaces. We enlarge the latter set by including also subspaces, thus defining:
\[ [X] := \{ E \text{ subspace of } \mathbb{K}^m; E \subset \text{col} M \text{ for some } M \in X \}. \] (2)

In Appendix B we prove:

**Theorem 5.4.** Let $X \subset \text{Mat}_{n \times m}(\mathbb{C})$ be an algebraically closed, column-invariant set. Suppose $E$ is a vector subspace of $\mathbb{C}^n$ that does not belong to $[X]$. Then
\[ \codim X \geq m + 1 - \dim E. \]

**5.1.4. The real part of an algebraic set.** Let $X$ be an algebraically closed subset of $\mathbb{C}^n$. The real part of $X$ is defined as $X \cap \mathbb{R}^n$. This is an algebraically closed subset of $\mathbb{R}^n$. Indeed, generators of the corresponding ideal $f_1, \ldots, f_k$ in $\mathbb{C}[T_1, \ldots, T_n]$ can be replaced by the corresponding real and imaginary parts polynomials.

As in the complex case, there are many equivalent algebraic-geometric definitions of dimensions of real algebraic or semialgebraic sets. We just point out that a real algebraic or semialgebraic set admits a stratification into real manifolds such that the maximal differential-geometric dimension of the strata coincides with the algebraic-geometric dimension (see [BR § 3.4] or [BCR p. 50]).

The following is an immediate consequence of [BR Prop. 3.3.2]:

**Proposition 5.5.** If $X$ is an algebraically closed subset of $\mathbb{C}^n$ then $\dim_{\mathbb{R}}(X \cap \mathbb{R}^n) \leq \dim_{\mathbb{C}} X$.

**5.2. Rigidity and the dimension of the poor fibers.** For simplicity of notation, let us write $\mathcal{P}_m = \mathcal{P}_m(\mathbb{C})$. Also, for $A \in \text{GL}(d, \mathbb{C})$, write:
\[ r(A) := \text{rig} + \text{Ad} A - 1. \]

We decompose the set $\mathcal{P}_m$ of poor data in fibers:
\[ \mathcal{P}_m = \bigcup_{A \in \text{GL}(d, \mathbb{C})} \{ A \} \times \mathcal{P}_m(A), \quad \text{where } \mathcal{P}_m(A) \subset \mathfrak{gl}(d, \mathbb{C})^m. \] (3)
Lemma 5.6. For any $A \in \text{GL}(d, \mathbb{C})$, the codimension of $\mathcal{P}_m(A)$ in $\mathfrak{gl}(d, \mathbb{C})^m$ is at least $m + 1 - r(A)$.

The lemma follows easily from Theorem 5.4 above:

Proof. Fix $A \in \text{GL}(d, \mathbb{C})$, and write $r = r(A)$. We can assume that $r \leq m$, otherwise there is nothing to prove. By definition, there exists a $r$-dimensional subspace $E \subset \mathfrak{gl}(d, \mathbb{C})^m$ such that $\mathfrak{R}_{\text{Ad}_A}(\text{Id} \vee E)$ is transitive. Identify $\mathfrak{gl}(d, \mathbb{C})$ with $\mathbb{C}^d$ and thus regard $\mathcal{P}_m(A)$ as a subset of $\text{Mat}_{d^2 \times m}(\mathbb{C})$. Since the set $\mathcal{P}_m$ is algebraically closed and saturated (recall §2.3), the fiber $\mathcal{P}_m(A)$ is algebraically closed and column-invariant, as required by Theorem 5.4. In the notation (5.2), we have $E \notin [\mathcal{P}_m(A)]$. So Theorem 5.4 gives the desired codimension estimate. \hfill $\square$

5.3. How rare is high rigidity? For simplicity of notation, let us write:

$$a(A) := \text{acyc Ad}_A \quad \text{for } A \in \text{GL}(d, \mathbb{C}).$$

So Theorem 3.5 says that $r(A) \leq a(A) - c(A)$ provided $c(A) < d$.

Lemma 5.7. For any integer $k \geq 1$, the set

$$M_k = \{ A \in \text{GL}(d, \mathbb{C}); \ r(A) \geq k \};$$

is algebraically closed in $\text{GL}(d, \mathbb{C})$; moreover if $M_k \neq \emptyset$ then

$$\text{codim } M_k \begin{cases} = 0 & \text{if } k = 1, \\ \geq k & \text{if } k \geq 2. \end{cases}$$

Lemma 5.7 is basically a consequence of Theorem 3.6 using the following construction:

Lemma 5.8. There is a family $\mathcal{G}(A)$ of subsets of $\text{GL}(d, \mathbb{C})$, indexed by $A \in \text{GL}(d, \mathbb{C})$, such that the following properties hold:

1. Each $\mathcal{G}(A)$ contains $A$.
2. Each $\mathcal{G}(A)$ is an immersed manifold of codimension $a(A) - c(A)$.
3. There are only countably many different sets $\mathcal{G}(A)$.

Proof. Fix any $A \in \text{GL}(d, \mathbb{C})$. Then $A$ is conjugate to a matrix in Jordan form:

$$\tilde{A} = \begin{pmatrix} J_{t_1}(\lambda_1) & & \\ & \ddots & \\ & & J_{t_n}(\lambda_n) \end{pmatrix},$$

where $J_{\lambda}(t)$ denotes Jordan block as in (4.1). Let $U$ be the set of matrices of the form

$$\begin{pmatrix} J_{t_1}(\mu_1) & & \\ & \ddots & \\ & & J_{t_n}(\mu_n) \end{pmatrix},$$

where $\mu_1, \ldots, \mu_n$ are nonzero complex numbers such that

$$\lambda_i = \lambda_j \iff \mu_i = \mu_j \quad \text{and} \quad \lambda_i \neq \lambda_j \iff \frac{\lambda_i}{\lambda_j} = \frac{\mu_i}{\mu_j}.$$

Then $U$ is an embedded submanifold of $\text{GL}(d, \mathbb{C})$ of dimension $c(A)$. Every $Y \in U$ has the same Jordan type as $A$, and so, by Remark 4.12, $a(Y) = a(A)$. We define the set $\mathcal{G}(A)$ as the image of the map $\Psi = \Psi_A : \text{GL}(d, \mathbb{C}) \times U \to \text{GL}(d, \mathbb{C})$ given by $\Psi(X, Y) = \text{Ad}_X(Y)$. Notice that $\mathcal{G}(A)$ does not depend on the choice of $\tilde{A}$. Actually $\mathcal{G}(A)$ is characterized by the sizes of the Jordan blocks $t_1, \ldots, t_n$, the pairs $(i, j)$ such that $\lambda_i = \lambda_j$ and the corresponding roots of unity; in particular there are countably many such sets $\mathcal{G}(A)$.

Let us check that property (4) holds. Let $\partial_1 \Psi$ and $\partial_2 \Psi$ denote the partial derivatives with respect to $X$ and $Y$, respectively. As we have seen in Remark 3.3, the
rank of $\partial_1 \Psi(X,Y)$ is equal to $d^2 - a(Y) = d^2 - a(A)$ for every $(X,Y)$. On the other hand, $\partial_2 \Psi(X,Y)$ is one-to-one and therefore of rank $c(A)$. We claim that

\[(5.4) \quad (\text{image of } \partial_1 \Psi(X,Y)) \cap (\text{image of } \partial_2 \Psi(X,Y)) = \{0\};\]

To see this, consider the map $F: \text{Mat}_{d \times d}(\mathbb{C}) \to \mathbb{C}^d$ that associates to each matrix the coefficients of its characteristic polynomial. Then $\partial_1 (F \circ \Psi)(X,Y) = 0$, while $\partial_2 (F \circ \Psi)(X,Y)$ is one-to-one. So (5.4) follows. As a result, at every point the rank of the derivative of $\Psi$ is equal to the sum of the ranks of the partial derivatives, that is, $d^2 - a(A) + c(A)$. Therefore, by the Rank Theorem, the image of $\Psi$ is an immersed manifold of codimension $a(A) - c(A)$. □

Proof of Lemma 5.4. If $k = 1$ then $M_1 = \text{GL}(d, \mathbb{C})$ (since $d \geq 2$), so there is nothing to prove. Consider $k \geq 2$. We have already shown in §2.3 that $\mathcal{P}_k$ is algebraic. Since $M_k = \{ A \in \text{GL}(d, \mathbb{C}); \forall \tilde{X} \in \text{gl}(d, \mathbb{C})^k, (A, \tilde{X}) \in \mathcal{P}_k \}$, it is evident that $M_k$ is algebraically closed as well. We are left to estimate its dimension.

Take a nonsingular point $A_0$ of $M_k$ where the local dimension is maximal. Let $D$ be the intersection of $M_k$ with a small neighborhood of $A_0$; it is an embedded disk. Each $A \in D$ has $r(A) \geq 2$; therefore by (both parts of) Theorem 3.6, we have $a(A) - c(A) \geq r(A) \geq k$. So, in terms of the sets from Lemma 5.8

$$D \subset \bigcup_{A \text{ s.t. } a(A) - c(A) \geq k} \mathcal{G}(A).$$

The right hand side is a countable union of immersed manifolds of codimension at least $k$. It follows (e.g. by Baire Theorem) that $D$ (and hence $M_k$) has codimension at least $k$. □

5.4. Proof of Theorems 1.4 and 1.5. Now we apply Lemmas 5.6 and 5.7 to prove one of our major results:

Proof of Theorem 1.5. The set $\mathcal{P}_m \subset \text{GL}(d, \mathbb{C}) \times [\text{gl}(d, \mathbb{C})]^m$ is homogeneous in the second factor. Using Lemma 5.2 together with Remark 5.3, we obtain that the sets

\[(5.5) \quad C_j = \{ A \in \text{GL}(d, \mathbb{C}); \text{ codim } \mathcal{P}_m(A) \leq j \}\]

are algebraically closed in $\text{GL}(d, \mathbb{C})$, and

$$\text{codim } \mathcal{P}_m = \min_{j: C_j \neq \emptyset} (j + \text{codim } C_j).$$

By Lemma 5.6, we have $C_j \subset M_{m+1-j}$. Therefore, by Lemma 5.7,

\[(5.6) \quad C_j \neq \emptyset \implies \text{codim } C_j \begin{cases} \geq 0 & \text{if } j = m, \\ \geq m - j + 1 & \text{if } j \leq m - 1. \end{cases}\]

So codim $\mathcal{P}_m \geq m$, as we wanted to show. □

The proof above only used that codim $C_j \geq m - j$. On the other hand, using the full power of (5.5) we obtain:

Scholium 5.9. The set of poor data in “fat fibers”, namely

$$\mathcal{F}_m := \{(A, B_1, \ldots, B_m) \in \mathcal{P}_m(\mathbb{C}); \text{ codim } \mathcal{P}_m(A) \leq m - 1\},$$

has codimension at least $m + 1$ in $\text{GL}(d, \mathbb{C}) \times [\text{gl}(d, \mathbb{C})]^m$.

Proof. The projection of $\mathcal{F}_m$ on $\text{GL}(d, \mathbb{C})$ is $C_{m-1}$. Use Lemma 5.2 (together with Remark 5.3) and 5.6. □

Next, let us consider the real case:
Indeed, a jet \( J^m \) of codimension \( m \) is a filtration by closed subsets of a smooth manifold \( X \)
\[ \Sigma = \Sigma_n \supset \Sigma_{n-1} \supset \cdots \supset \Sigma_0 \]
such that for each \( i \), the set \( \Gamma_i = \Sigma_i \setminus \Sigma_{i-1} \) (where \( \Sigma_{-1} := \emptyset \)) is a smooth submanifold of \( X \) without boundary, and the dimension of \( \Gamma_i \) decreases strictly with increasing \( i \).

We say that a \( C^1 \)-map is transverse to that stratification if it is transverse to each of the submanifolds \( \Gamma_i \). There are explicit, so-called Whitney conditions that guarantee that a stratification behaves nicely with respect to transversality, as the next proposition shows. A stratification satisfying those conditions is called a Whitney stratification. By the classical Theorem C.1 stated in Appendix C (see for instance [GWPL]), any semi-algebraic subset of an affine space admits a canonical Whitney stratification.

We refer the reader to Appendix C for the definitions of jets, jet extensions and Proposition 6.1.

**Proof of Theorem 1.4.** The real part of \( \mathcal{P}_m^{(R)} \) is a real algebraic set which, in view of Proposition 5.5, has codimension at least \( m \). Recall from § 2.2 that this set contains the semialgebraic set \( \mathcal{P}_m^{(R)} \), which therefore has codimension at least \( m \). Since we already knew from Proposition 2.4 that \( \text{codim} \mathcal{P}_m^{(R)} \leq m \), the theorem is proved.

6. Proof of the main result

We now use Theorem 1.4 and transversality theorems to prove our main result. For precise definitions and statements on the objects used in this section, see Appendix C.

A stratification is a filtration by closed subsets of a smooth manifold \( X \)
\[ \Sigma = \Sigma_n \supset \Sigma_{n-1} \supset \cdots \supset \Sigma_0 \]
such that for each \( i \), the set \( \Gamma_i = \Sigma_i \setminus \Sigma_{i-1} \) (where \( \Sigma_{-1} := \emptyset \)) is a smooth submanifold of \( X \) without boundary, and the dimension of \( \Gamma_i \) decreases strictly with increasing \( i \).

We say that a \( C^1 \)-map is transverse to that stratification if it is transverse to each of the submanifolds \( \Gamma_i \). There are explicit, so-called Whitney conditions that guarantee that a stratification behaves nicely with respect to transversality, as the next proposition shows. A stratification satisfying those conditions is called a Whitney stratification. By the classical Theorem C.1 stated in Appendix C (see for instance [GWPL]), any semi-algebraic subset of an affine space admits a canonical Whitney stratification.

We refer the reader to Appendix C for the definitions of jets, jet extensions and for a proof of the following:

**Proposition 6.1.** Let \( X, Y \) be \( C^{\infty} \)-manifolds without boundary. Let \( \Sigma \) be a Whitney stratified closed subset of the set of 1-jets from \( X \) to \( Y \). Then the set of maps \( f \in C^2(X, Y) \) whose 1-jet extension \( j^1 f \) is transverse to \( \Sigma \) is \( C^2 \)-open and \( C^r \)-dense in \( C^2(X, Y) \) (i.e., its intersection with \( C^r(X, Y) \) is \( C^r \)-dense, for every \( 2 \leq r \leq \infty \)).

By Theorem 1.4, \( \mathcal{P}_m^{(R)} \) is a closed semialgebraic subset of \( GL(d, \mathbb{R}) \times \text{gl}(d, \mathbb{R})^m \) of codimension \( m \). The closure \( \mathcal{P}_m^{(R)} \) of \( \mathcal{P}_m^{(R)} \) in \( [\text{Mat}_{d \times d}(\mathbb{R})]^{1+m} \) is a closed semialgebraic set of the affine space \( [\text{Mat}_{d \times d}(\mathbb{R})]^{1+m} \). As mentioned above, it admits a canonical Whitney stratification
\[ \mathcal{P}_m^{(R)} = \Gamma_n \supset \cdots \supset \Gamma_0 . \]

The differentiable codimension of that stratification is also \( m \). By locality of the Whitney conditions (see Proposition 6.2 of Appendix C), this stratification restricts to a Whitney stratification of codimension \( m \):
\[ \mathcal{P}_m^{(R)} = \Gamma_n \supset \cdots \supset \Gamma_0 . \]

Since that stratification of \( \mathcal{P}_m^{(R)} \) is canonical, the stratification (6.1) is invariant under polynomial automorphisms of \( GL(d, \mathbb{R}) \times \text{gl}(d, \mathbb{R})^m \) that preserve \( \mathcal{P}_m^{(R)} \).

**Proof of Theorem 1.7.** Let \( U \) be a smooth manifold without boundary and of dimension \( m \). Given local coordinates on an open set \( U \subset U \), the set \( J^1(U, GL(d, \mathbb{R})) \) of 1-jets from \( U \) to \( GL(d, \mathbb{R}) \) may be identified with the set
\[ U \times GL(d, \mathbb{R}) \times \text{gl}(d, \mathbb{R})^m . \]

Indeed, a jet \( J \) represented by a pair \( (u, A) \) can be identified with the point
\[ (u, A(u), B_1, \ldots, B_m) \in U \times GL(d, \mathbb{R}) \times \text{gl}(d, \mathbb{R})^m . \]
where \( B_i \in \text{Mat}_{d\times d}(\mathbb{R}) \) is the normalized derivative of \( A \) at \( u \), along the \( i \)th coordinate. Let us say that the 1-jet \( J \) is rich if the datum \( \mathbf{A} = (A(u), B_1, \ldots, B_m) \) is rich, or equivalently, if for sufficiently large \( N \), the input \( (u, \ldots, u) \in U^N \) is universally regular for the system \( (1.4) \). If the jet is not rich then it is called poor.

Define a filtration
\[
\Sigma_n \supset \cdots \supset \Sigma_0
\]
of the set of poor jets from \( U \) to \( GL(d, \mathbb{R}) \) as follows: a jet \( J \) represented as above in local coordinates by \((u, A(u), B_1, \ldots, B_m)\) belongs to \( \Sigma_i \) if and only if \((A(u), B_1, \ldots, B_m)\) belongs to the set \( \Gamma_i \) in \( (6.1) \). We need to check that this definition does not depend on the choice of the local coordinates. Indeed, this follows from \( \mathcal{P}_m \) being a saturated set (see \( \S \, 2.3 \)) and from the invariance of \( (6.1) \) by polynomial automorphisms.

We claim that the filtration \( (6.2) \) is a Whitney stratification of codimension \( m \). Indeed, the intersection of the filtration with the open subset \( J^1(U, GL(d, \mathbb{R})) \) of \( J^1(U, GL(d, \mathbb{R})) \) is identified (through a smooth diffeomorphism) with the filtration
\[
U \times \Gamma_n \supset \cdots \supset U \times \Gamma_0.
\]
Such a filtration is still a Whitney stratification (see Proposition \( 2.2 \) of Appendix \( A \)) of codimension \( m \) in \( J^1(U, GL(d, \mathbb{R})) = U \times GL(d, \mathbb{R}) \times \text{gl}(d, \mathbb{R})^m \). Covering \( U \) by open sets \( U \), we deduce that \( (6.2) \) is a Whitney stratification of codimension \( m \) in \( J^1(U, GL(d, \mathbb{R})) \).

Applying Proposition \( 6.1 \) we obtain a \( C^2 \)-open \( C^p \)-dense set \( \mathcal{O} \subset C^2(U, GL(d, \mathbb{C})) \) formed by maps \( A \) that are transverse to the stratification \( (6.2) \) of the set of poor jets. Since the codimension of the stratification equals the dimension of \( U \), if \( A \in \mathcal{O} \) then the points \( u \) for which \( j^1A(u) \) is poor form a 0-dimensional set. This proves Theorem \( 1.1 \). \( \square \)

**Appendix A. The Case of One-Dimensional Input**

As we explained in \( \S \, 1.4 \), this appendix contains a basically independent discussion of the case where \( m = \dim U \) equals 1. The prerequisites are all contained in Section \( 2 \) and \( \S \, 3.4 \).

**A.1. Elementary Constraints.** The material of this subsection is also used in Appendix \( E \).

An *elementary constraint* in the variables \( \lambda_1, \ldots, \lambda_d \) is a relation \( p = 0 \) where \( p \) is an irreducible factor of a polynomial of the form \( \lambda_i \lambda_j - \lambda_j \lambda_i \). Every elementary constraint can be written, after a permutation of the indices \( 1, \ldots, d \), as one of the following:
\[
(A.1) \quad \lambda_1 \lambda_3 = \lambda_2^2, \quad \lambda_1 \lambda_4 = \lambda_2 \lambda_3, \quad \lambda_1 = -\lambda_2, \quad \lambda_1 = \lambda_2,
\]
which will be called the *canonical constraints* respectively of type 1, 2, 3, 4. The *type* of elementary constrained is defined as the (unique) type of the associated canonical constraint.

We say that a matrix \( A \in GL(d, \mathbb{R}) \) is *unconstrained* if its eigenvalues, counted with multiplicity, satisfy no elementary constraint. (Equivalently, \( \text{Ad}_A \) has the maximal possible number of distinct eigenvalues, namely, \( d^2 - d + 1 \).

Let us see that the converse of Lemma \( 2.7 \) holds for unconstrained matrices:

**Lemma A.1.** Suppose that the datum \( \mathbf{A} = (A, B_1, \ldots, B_m) \in GL(d, \mathbb{K}) \times \text{gl}(d, \mathbb{K})^m \) is poor and that the matrix \( A \) is unconstrained. Then \( \mathbf{A} \) is conspicuously poor.
Proof. Suppose $A$ is unconstrained. In particular, $A$ has simple spectrum. With a change of basis we can assume that $A$ is diagonal.

Now suppose that $A = (A, B_1,\ldots, B_m)$ is not conspicuously poor. This means that for each off-diagonal position there is at least one of the matrices $B_k$ that has a non-zero entry in that position. (Notice that this fact does not depend on the change of basis chosen before.)

Since $A$ is unconstrained, the values $\lambda_i \lambda_j^{-1}$, where $(i,j)$ runs on the matrix positions outside the diagonal, are pairwise different, and all different from 1. Recall that one can always (using Lagrange formula) find a polynomial whose values at finitely many different points are prescribed. Restricting to polynomials $f$ such that $f(1) = 0$, it follows from (2.4) that the space $\Lambda(A)$ contains all matrices $(y_{ij})$ with only zeros in the diagonal. Since, by definition, $\Lambda(A)$ also contains the identity matrix, it contains all Toeplitz matrices. So $\Lambda(A)$ is transitive, i.e., $A$ is not poor. This proves the lemma.

□

A.2. Effective richness criteria for the case $m = 1$. We will describe an explicit set of rich data $(A, B)$ whose complement has codimension 1. In order to avoid technicalities, we will be sometimes informal, especially regarding questions of transversality.

Let us say that a matrix $A \in \text{GL}(d, \mathbb{R})$ is $(i)$-constrained, where $1 \leq i \leq 4$, if:

- its eigenvalues, counted with multiplicity, satisfy exactly one elementary constraint, which is a type $i$ constraint,
- if there is a type 4 constraint between the eigenvalues, then the matrix $A$ is not diagonalizable.

Suppose that there is no $i$ for which the matrix $A$ is $(i)$-constrained; then:

- either $A$ is unconstrained, i.e., its eigenvalues (with multiplicity) satisfy no elementary constraint;
- or the eigenvalues of $A$ satisfy at least two elementary constraints;
- or $A$ has a (multiple) eigenvalue corresponding to at least two Jordan blocks.

If either of the last two cases hold, we say that $A$ is multiconstrained.

Proposition A.2. 1. The complement of the set of unconstrained matrices has codimension 1 in $\text{GL}(d, \mathbb{R})$.

2. The set of multiconstrained matrices has codimension 2 in $\text{GL}(d, \mathbb{R})$.

Informal proof. Matrices that are not unconstrained have at least one constraint on their eigenvalues, so the corresponding set has codimension 1.

Matrices that are multiconstrained either have at least two constraints on their eigenvalues, or are derogatory, i.e., have an eigenvalue corresponding to at least two Jordan blocks. In both cases, the corresponding set has codimension 2. □

Let us define adapted bases for matrices $A$ that are not multiconstrained:

- If $A$ is unconstrained then an adapted basis is a basis of eigenvectors.
- If $A$ is $(i)$-constrained, for $i = 1, 2, 3$ then an adapted basis is an (ordered) basis of eigenvectors such that the corresponding eigenvectors $\lambda_1,\ldots,\lambda_d$ satisfy the canonical type $i$ constraint.
- If $A$ is $(4)$-constrained then an adapted basis for $A$ is a basis in which $A$ is written in the following modified Jordan form:

$$
\begin{pmatrix}
\lambda_1 & \lambda_1 & \cdots & \lambda_1 \\
0 & \lambda_1 & & \\
& \ddots & \ddots & \\
& & \ddots & \lambda_d
\end{pmatrix}.
$$
Obviously, such adapted bases always exist.

If a matrix \( A \) is (\( i \))-constrained then we say that \( d \times d \) matrix \( B \) is a good match for \( A \), if there is an adapted basis for \( A \) in which it writes as \( B = (b_{ij}) \), where all nondiagonal entries \( b_{ij} \) are nonzero and if \( b_{11} \neq b_{22} \), in the particular case where \( A \) is 3-constrained.

The usefulness of this definition is explained by the following Propositions \( \text{A.3} \) and \( \text{A.4} \). (Actually, the definition of a good match matrix is stronger than necessary for the validity of the propositions below. But in order to avoid complications, we chose a condition that works for all types of constraints.)

**Proposition A.3.** If \( A \) is not multiconstrained and \( B \) is a good match for \( A \) then the pair \((A, B)\) is rich.

In other words, \( \mathcal{P}_1^{(\mathbb{C})} \) is contained in the following set:

\[(A.2) \quad \mathcal{E} := \{(A, B) \in \text{GL}(d, \mathbb{C}) \times \mathfrak{gl}(d, \mathbb{C}); \text{ either } A \text{ is multiconstrained or } A \text{ is not multiconstrained but } B \text{ is not a good match for } A\}.

**Proposition A.4.**

1. The set \( \mathcal{E} \) has codimension 1.
2. The set \( \{(A, B) \in \mathcal{E}; \text{ } A \text{ is not unconstrained}\} \) has codimension 2.

**Informal proof.** Proposition \( \text{A.4} \) follows from Proposition \( \text{A.2} \) and from the fact that for each matrix \( A \) that is not multiconstrained, the set of \( B \)'s that are not good matches for \( A \) has positive codimension in \( \mathfrak{gl}(d, \mathbb{C}) \).

Theorem \( \text{1.5} \) in the case \( m = 1 \) follows from the propositions above. Therefore the other main results (Theorems \( \text{1.1}, \text{1.2}, \text{1.4} \) and \( \text{1.6} \)) in the \( m = 1 \) case also follow from the propositions. For any of these results, the propositions give extra information of practical value: with the explicit definition of the set \( \mathcal{E} \) in \( (A.2) \), we know which 1-jets should be avoided in Theorem \( \text{1.1} \) for example. The discussion given in Appendix \( \text{E} \) also applies: it gives explicit conditions on the 2-jet extension of the map \( A; U \to \text{GL}(d, \mathbb{R}) \) that ensure that \( A \) satisfies the conclusions of Theorems \( \text{1.1} \) and \( \text{1.2} \).

**Proof of Proposition \( \text{A.3} \).** Let \( A \) and \( B \) satisfy the hypotheses. We need to show that \( \Lambda(A, B) = \mathcal{R}_{\text{Ad}_A}(\text{Id}; B) \) is a transitive subspace of \( \mathfrak{gl}(d, \mathbb{C}) \). Let \( \Gamma = \mathcal{R}_{\text{Ad}_A}(B) \), so that \( \Lambda(A, B) = \{\text{Id}\} \vee \Gamma \).

The matrix \( A \) is not multiconstrained and so has an adapted basis as above. We change the basis so that \( A \) and \( B \) are “canonical”.

The proof is divided in cases according to the type of constraint. Except for the (4)-constrained case, the matrix \( A \) is diagonal, and so the space \( \Gamma \) is described by \( (2.4) \).

**Unconstrained case:** It follows from Lemma \( \text{A.1} \) that if \( A \) is unconstrained and diagonal then the only way for the pair \((A, B)\) to be poor is that \( B \) has an off-diagonal zero entry. (The reader should review the proof of Lemma \( \text{A.1} \).)

(1)-constrained case: We see that the adjoint \( \text{Ad}_A \) has two eigenvalues (different from 1) of multiplicity 2, namely \( \lambda_1 \lambda_2^{-1} = \lambda_2 \lambda_1^{-1} \) and \( \lambda_2 \lambda_1^{-1} = \lambda_3 \lambda_2^{-1} \). By the same reasoning as in the unconstrained case, it follows that \( \{\text{Id}\} \vee \Gamma \) contains the space

\[ \{(y_{ij}) \in \mathfrak{gl}(d, \mathbb{C}); \ y_{11} = \cdots = y_{dd}, \ b_{12}^{-1}y_{12} = b_{23}^{-1}y_{23}, \ b_{21}^{-1}y_{21} = b_{32}^{-1}y_{32}\}. \]

This is a generalized Toeplitz space, and so by Example \( \text{2.2} \) it is transitive.

(2)-constrained case: The reasoning is very similar to that of the (1)-constrained case, but now the adjoint has four eigenvalues (different from 1) of multiplicity 2.


The space $\Lambda(A, B)$ contains the following subspace:
\[
\{ (y_{ij}) \in \mathfrak{gl}(d, \mathbb{C}) ; \ y_{11} = \cdots = y_{dd} , \ b_{13}^{-1} y_{13} = b_{24}^{-1} y_{24} , \ b_{12}^{-1} y_{12} = b_{34}^{-1} y_{34} , \ b_{21}^{-1} y_{21} = b_{43}^{-1} y_{43} , \ b_{31}^{-1} y_{31} = b_{41}^{-1} y_{41} \}.
\]
Again, this is a generalized Toeplitz space, and so it is transitive.

(3)-constrained case: This case is a little different from the two previous ones. The adjoint has an eigenvalue $-1$ of multiplicity 2. Recalling that $b_{11}$ and $b_{22}$ are different, and making use of the identity matrix, we see that $\Lambda(A, B)$ contains the following subspace:
\[
\hat{\Gamma} = \{ (y_{ij}) \in \mathfrak{gl}(d, \mathbb{C}) ; \ y_{33} = \cdots = y_{dd} , \ b_{12}^{-1} y_{12} = b_{21}^{-1} y_{21} \}.
\]
This is not a generalized Toeplitz space. However, consider the linear automorphism $S$ that swaps the first two elements of the canonical basis of $\mathbb{C}^n$, and fixes the others. Then
\[
S \cdot \hat{\Gamma} = \{ (z_{ij}) \in \mathfrak{gl}(d, \mathbb{C}) ; \ z_{33} = \cdots = z_{dd} , \ b_{12}^{-1} z_{22} = b_{21}^{-1} \}
\]
is a generalized Toeplitz space. By Remark 2.3, the space $S \cdot \hat{\Gamma}$ is transitive, and so are $\hat{\Gamma}$ and $\Lambda(A, B)$.

(4)-constrained case: This case is more involved because the operator $\text{Ad}_A$ is not diagonalizable. We will explain its Jordan form. Let us explain visually how $\text{Ad}_A$ acts: given any matrix, decompose it into blocks $C_{ij}$ as in the following picture
\[
\begin{pmatrix}
C_{22} & C_{23} & C_{24} & \cdots & C_{2d} \\
C_{32} & C_{33} & \cdots & & \\
C_{42} & & C_{44} & & \\
\vdots & & & \ddots & \\
C_{d2} & & & & C_{dd}
\end{pmatrix}
\]
where the block $C_{22}$ is a $2 \times 2$ matrix, the blocks $C_{2j}$ are $2 \times 1$, the blocks $C_{i2}$ are $1 \times 2$ and the others are $1 \times 1$. Then, the operator $\text{Ad}_A$ leaves invariant the space $\Gamma_{ij}$ of matrices whose nonzero coefficients lie inside the block $C_{ij}$.

Let us use notations $J_3(\lambda)$ from (4.3) and $E_{ij}$ from (2.3). It is easily computed that the operator $\text{Ad}_A$ has the following properties:

- the matrix of $\text{Ad}_A|\Gamma_{11}$ with respect to the basis formed by $M_1 = -2E_{12}$, $M_2 = E_{11} - E_{12} - E_{22}$, $M_3 = E_{21}$, $M_4 = E_{11} + E_{22}$ is $J_3(1) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.
- For any $j \geq 3$, the matrix of $\text{Ad}_A|\Gamma_{2j}$ with respect to the basis formed by $\lambda_2 \lambda_j^{-1} E_{1j}$ and $E_{2j}$ (where we use the notation $E_{i,j}$ from (2.3)) is $J_2(\lambda_2 \lambda_j^{-1})$.
- For any $i \geq 3$, the matrix of $\text{Ad}_A|\Gamma_{i2}$ with respect to the basis formed by $-\lambda_i \lambda_j^{-1} E_{i,1}$ and $E_{i,2}$ is $J_2(\lambda_i \lambda_j^{-1})$.
- For $3 \leq i,j \leq d$, the matrix of $\text{Ad}_A|\Gamma_{ij}$ with respect to the basis formed by the single vector $E_{ij}$ is $(\lambda_i \lambda_j^{-1})$.
- The spaces $\Gamma_{ij}$, for $2 \leq i,j \leq d$ have respective spectra $\{\lambda_i \lambda_j^{-1}\}$, which for $i \neq j$ are pairwise disjoint and different from $\{1\}$.

The concatenation of the bases described above gives a Jordan basis for $\text{Ad}_A$. Now take a matrix $B$ that is a good match for $A$, and consider its expression as a linear combination of the elements of that Jordan basis. One easily checks that all coefficients in this linear combination are nonzero, except possibly the coefficients of the vectors $M_1$, $M_2$, $M_4$ and the vectors $E_{ii}$, for all $3 \leq i \leq d$. Consider now the
splitting \( \text{Mat}_{d \times d}(\mathbb{C}) = V \oplus \Delta \), where \( \Delta \) is the subspace \( \mathbb{C}M_4 \oplus E_{33} \oplus \ldots \oplus E_{dd} \) of the space of diagonal matrices, and \( V \) is the space spanned by all other elements of the above Jordan basis. Note that

\[
V = (\mathbb{C}M_1 + \mathbb{C}M_2 + \mathbb{C}M_3) \oplus \left( \bigoplus_{2 \leq i \leq d, i \neq j} \Gamma_{ij} \right)
\]

is a decomposition of \( V \) into \( \text{Ad}_A \)-invariant subspaces with pairwise disjoint spectra. Let \( \pi \) be the projection onto \( V \) along \( \Delta \). Using Lemma 3.1 we see that \( \pi(B) \) is a cyclic vector for \( \text{Ad}_A[V] \). So, using the \( \text{Ad}_A \)-invariance of the spaces \( V \) and \( \Delta \), we have

\[
\pi(\Gamma) = \pi(\mathfrak{R}_{\text{Ad}_A}(B)) = \mathfrak{R}_{\text{Ad}_A}(\pi(B)) = V.
\]

Note that \( V \) contains the matrices \( E_{ij} \), for all \( i \neq j \), hence \( \{\text{Id}\} \vee V \) is a generalized Toeplitz space. As \( \pi \) projects along a subspace of diagonal matrices, \( \{\text{Id}\} \vee \Gamma \) is again a generalized Toeplitz space and in particular is a transitive space.

We have considered the four types, and Proposition A.3 is proved. \( \square \)

**Appendix B. Some general facts on dimensions of algebraic sets**

In this appendix we prove Lemma 5.2 and Theorem 5.4, which were used in Section 5. Lemma 5.2 is a simple consequence of standard theorems in algebraic geometry, but for the reader’s convenience let us spell out the details. Theorem 5.4 follows from intersection theory of the Grassmannians (“Schubert calculus”). We tried to make the exposition the least technical as possible, to make it accessible to non-experts (like ourselves).

**B.1. Fiberwise dimension estimate.**

Proof of Lemma 5.3. In what follows, all topologies are Zariski. We will prove the equivalent “dual form” of the lemma, namely, that the sets

\[ Y_k = \{ y \in \pi(X); \dim \pi^{-1}(y) \geq k \} \]

are algebraically closed in \( Y \), and

\[
\dim X = \max_{k \in \mathbb{N}} \dim Y_k.
\]

First, the sets \( X_k = \{ x \in X; \dim \pi^{-1}(\pi(x)) \geq k \} \) are closed. (see [Ha, Thrm. 11.12]). So, by Proposition 5.1, \( Y_k = \pi(X_k) \) is closed.

For each \( k \) with \( X_k \neq \emptyset \), let \( X_{k,i} \) denote the irreducible components of \( X_k \). Let

\[
\mu(k, i) = \min_{x \in X_{k,i}} \dim \pi^{-1}(\pi(x)).
\]

Then, by [Ha, Thrm. 11.12] (and the fact that taking closures does not affect dimension) we have

\[
\dim X_{k,i} = \mu(k, i) + \dim \pi(X_{k,i}).
\]

By definition, \( \mu(k, i) \geq k \); moreover equality holds unless \( X_{k,i} \subset X_{k+1} \). So

\[ X_{k,i} \subset X_{k+1} \Rightarrow \dim X_{k,i} = k + \dim \pi(X_{k,i}) \leq k + \dim Y_k. \]

Since \( X = \bigcup_{X_{k,i} \subset X_{k+1}} X_{k,i} \), this proves the \( \leq \) inequality in (B.1).

To prove the converse inequality, fix any \( k \) with \( Y_k \neq \emptyset \). Find \( i \) such that \( \dim \pi(X_{k,i}) = \dim Y_k \). Then

\[
\dim X \geq \dim X_{k,i} = \mu(k, i) + \dim Y_k \geq k + \dim Y_k.
\]

This proves (B.1) and hence the lemma. \( \square \)
B.2. A particular case of Theorem 5.4. Let us begin the proof of Theorem 5.4. For the reader’s convenience we recall the notations and the statement.

If $M \in \text{Mat}_{n \times m}(\mathbb{C})$, let $\text{col} M \subset \mathbb{C}^m$ denote the column space of $M$. A set $X \subset \text{Mat}_{n \times m}(\mathbb{C})$ is called column-invariant if

$$
\begin{align*}
M & \in X \\
N & \in \text{Mat}_{n \times m}(\mathbb{C}) \\
\text{col} M & = \text{col} N
\end{align*}
\Rightarrow \ N \in X.
$$

So a column-invariant set $X$ is characterized by its set of column spaces. We enlarge the latter set by including also subspaces, thus defining:

$$[X] := \{E \text{ subspace of } \mathbb{C}^n; \ E \subset \text{col} M \text{ for some } M \in X\}.$$ 

Then we have:

**Theorem 5.4.** Let $X \subset \text{Mat}_{n \times m}(\mathbb{C})$ be an algebraically closed, column-invariant set. Suppose $E$ is a vector subspace of $\mathbb{C}^n$ that does not belong to $[X]$. Then

$$\dim X \geq m + 1 - \dim E.$$ 

It is obvious that the algebraic hypothesis is indispensable.

Define

$$R_k := \{A \in \text{Mat}_{n \times m}(\mathbb{C}); \ \text{rank} \ A \leq k\}.$$ 

We recall (see [Ha, Prop. 12.2]) that this is an irreducible algebraically closed set of codimension

$$\dim R_k = (m-k)(n-k) \quad \text{if } 0 \leq k \leq \min(m,n).$$

**Proof of Theorem 5.4 in the case $E = \mathbb{C}^n$.** If $E = \mathbb{C}^n$ then the hypothesis $\mathbb{C}^n \notin [X]$ means that $X \subset R_{n-1}$. We can assume that $n - 1 < m$, otherwise the conclusion of the theorem is vacuous. Thus $\dim X \geq \dim R_{n-1} = m + 1 - n$, as we wanted to show. \qed

B.3. Reduction to a property of Grassmannians. As we will see, to prove Theorem 5.4 it is sufficient to prove a dimension estimate (Theorem B.1 below) for certain subvarieties of a Grassmannian.

B.3.1. Grassmannians. Given integers $n > k \geq 1$, the Grassmannian $G_k(\mathbb{C}^n)$ is the set of the vector subspaces of $\mathbb{C}^n$ of dimension $k$.

The Grassmannian can be interpreted as a subvariety of a higher dimensional complex projective space using the Plücker embedding $G_k(\mathbb{C}^n) \to P(\wedge^k \mathbb{C}^n)$, which maps each $V \in G_k(\mathbb{C}^n)$ to $[v_1 \wedge \cdots \wedge v_k]$, where $\{v_1, \ldots, v_k\}$ is any basis of $V$. This is clearly an one-to-one map. It can be shown (see e.g. [Ha, p. 61ff]) that the image is an algebraically closed subset of $P(\wedge^k \mathbb{C}^n)$. Its dimension is

$$\dim G_k(\mathbb{C}^n) = k(n-k).$$

If $E \subset \mathbb{C}^n$ is a vector space with $\dim E = e \leq k$ then we consider the following subset of $G_k(\mathbb{C}^n)$:

$$S_k(E) := \{V \in G_k(\mathbb{C}^n); \ V \supset E\}.$$ 

(This is a Schubert variety of a special type, as we will see later.) Since any $V \in S_k(E)$ can be written as $E \oplus W$ for some $V \subset W^\perp$, we see that $S_k(E)$ is homeomorphic to $G_{k-e}(\mathbb{C}^{n-e})$.

We will show that an algebraic set that avoids $S_k(E)$ cannot be too large:

**Theorem B.1.** Fix integers $1 \leq e \leq k < n$. Suppose that $Y$ is an algebraically closed subset of $G_k(\mathbb{C}^n)$ that is disjoint from $S_k(E)$, for some $e$-dimensional subspace $E \subset \mathbb{C}^n$. Then $\dim Y \geq k + 1 - e$. 
B.3.2. **Proof of Theorem B.4 assuming Theorem B.1.** Assuming Theorem B.1 for the while, let us see how it yields Theorem B.4.

Recalling notation B.2, define the quasiprojective variety

$$
\hat{R}_k := R_k \setminus R_{k-1}.
$$

We define a map $\pi_k : \hat{R}_k \to G_k(\mathbb{C}^n)$ by $A \mapsto \col A$.

**Lemma B.2.** If $X$ is an algebraically closed column-invariant subset of $\hat{R}_k$ then $Y = \pi_k(X)$ is algebraically closed subset of $G_k(\mathbb{C}^n)$, and the codimension of $Y$ inside $G_k(\mathbb{C}^n)$ is the same as the codimension of $X$ inside $\hat{R}_k$.

**Proof.** First, let us see that $\pi_k : \hat{R}_k \to G_k(\mathbb{C}^n)$ is a regular map. We identify $G_k(\mathbb{C}^n)$ with the image of the Plücker embedding. In a Zariski neighborhood of each matrix $A \in \hat{R}_k$, the map $\pi_k$ can be defined as $A \mapsto [a_{j_1} \wedge \cdots \wedge a_{j_k}]$ for some $j_1 < \cdots < j_k$, where $a_j$ is the $j$th column of $A$. This shows regularity.

Next, let us see that $Y = \pi_k(X)$ is closed with respect to the classical (not Zariski) topology. Consider the subset $K$ of $X$ formed by the matrices $A \in \hat{R}_k$ whose first $k$ columns form an orthonormal set, and whose $m-k$ remaining columns are zero. Then $K$ is compact (in the classical sense), and thus so is $\pi_k(K)$. But column-invariance of $X$ implies that $\pi_k(K) = Y$, so $Y$ is closed (in the classical sense).

It follows (see e.g. [Ha, p.39]) from regularity of $\pi_k$ is regular that the set $Y$ is constructible, i.e., it can be written as

$$
Y = \bigcup_{i=1}^p Z_i \setminus W_i,
$$

where $Z_i \supseteq W_i$ are algebraically closed subsets of $G_k(\mathbb{C}^n)$. We can assume that each $Z_i$ is irreducible. It follows from [Min, Thrm. 2.33] that $\overline{Z_i \setminus W_i} = Z_i$, where the bar denotes closure in the classical sense. In particular, $Y = \overline{Y} = \bigcup_{i=1}^p \overline{Z_i}$, showing that $Y$ is algebraically closed.

We are left to show the equality between codimensions. Since the codimension of an algebraically closed set equals the minimum of the codimensions of its components, we can assume that $X$ is irreducible.

By column-invariance of $X$, for each $y \in Y$, the whole fiber $\pi^{-1}(y)$ is contained in $X$. All those fibers have the same dimension $\mu = km$. By [Ha, Thrm. 11.12], $\dim X = \dim Y + km$. By B.3 and B.4, we have $\dim \hat{R}_k - \dim G_k = km$, so the claim about codimensions follows. \hfill \Box

**Proof of Theorem B.4.** Let $X \subset \Mat_{n \times m}(\mathbb{C})$ be a nonempty algebraically closed, column-invariant set. Suppose $E$ is a vector subspace of $\mathbb{C}^n$ that does not belong to $\|X\|$. Let $e = \dim E$. We can assume $e > 0$ (otherwise the result is vacuously true), and $e < n$ (because the case $e = n$ was already considered in § B.2).

Notice that $X \subset R_{n-1}$. Let

$$
X_k := X \cap \hat{R}_k \quad \text{and} \quad Y_k := \pi_k(X_k), \quad \text{for } 0 \leq k \leq \min(m, n-1).
$$

For every $k$ with $e \leq k < n$, the set $Y_k$ is disjoint from the set $S_k(E)$ defined by B.5. In view of Lemma B.2 and Theorem B.1, we have

$$
\text{codim}_{\hat{R}_k} X_k = \text{codim} Y_k \geq k + 1 - e.
$$

So the codimension of $X_k$ as a subset of $\Mat_{n \times m}(\mathbb{C})$ is

$$
\text{codim} X_k = \text{codim} \hat{R}_k + \text{codim}_{\hat{R}_k} X_k \geq (m-k)(n-k) + k + 1 - e =: f(k).
$$
The function \( f(k) \) is decreasing on the interval \( 0 \leq k \leq \min(m, n - 1) \). Therefore:

\[
\text{codim } X = \min_{0 \leq k \leq \min(m, n-1)} \text{codim } X_k \geq \min_{0 \leq k \leq \min(m, n-1)} f(k) = f(\min(m, n - 1)) = m + 1 - e,
\]

as claimed. This proves Theorem 5.4 modulo Theorem B.1. \( \square \)

The proof of Theorem B.1 will be given in § B.6 after we explain the necessary tools in §§ B.4-B.5.

B.4. Schubert calculus. Here we will outline some facts about the intersection of Schubert varieties. The readable expositions \([Bl, Va]\) contain more information.

A (complete) flag in \( \mathbb{C}^n \) is a sequence of subspaces \( F_0 \subset F_1 \subset \cdots \subset F_n \) with \( \dim F_j = j \). We denote \( F_\bullet = \{F_i\} \).

Given \( V \in G_k(\mathbb{C}^n) \), its rank table (with respect to the flag \( F_\bullet \)) is the datum \( \dim(V \cap F_j), j = 0, \ldots, n \). The jumping numbers are the indexes \( j \in \{1, \ldots, n\} \) such that \( \dim(V \cap F_j) - \dim(V \cap F_{j-1}) \) is positive (and thus equal to 1). Of course, if one knows the jumping numbers, one know the rank table and vice-versa. Let us define a third way to encode this information: Consider a rectangle of height \( m \) and width \( n - m \), divided in \( 1 \times 1 \) squares. We form a path of square edges: Start in the northeast corner of the rectangle. In the \( j \)th step (\( 1 \leq j \leq n \)), if \( j \) is a jumping number then we move one unit in the south direction, otherwise we move one unit in the west direction. Since there are exactly \( k \) jumping numbers, the path ends at the southwest corner of the rectangle. The Young diagram of \( V \) with respect to the flag \( F_\bullet \) is the set of squares in the rectangle that lie northwest of the path. We denote a Young diagram by \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \), where \( \lambda_i \) is the number of squares in the \( i \)th row (from north to south). Its area \( \lambda_1 + \cdots + \lambda_k \) is denoted by \( |\lambda| \).

Example B.3. Here is a possible rank table with \( k = 5, n = 12 \); the jumping numbers are underlined:

| \( j \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( \dim(W \cap F_j) \) | 0 | 0 | 0 | 1 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |

The associated path in the rectangle is:

\[
\begin{array}{cccccccccccc}
\text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} \\
\text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} \\
\text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} \\
\text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} \\
\text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} \\
\text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} \\
\text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} \\
\text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} \\
\text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} \\
\end{array}
\]

and so the Young diagram is

\[
\lambda = \begin{array}{cccc}
\text{X} & \text{X} & \text{X} & \text{X} \\
\text{X} & \text{X} & \text{X} & \text{X} \\
\text{X} & \text{X} & \text{X} & \text{X} \\
\end{array} = (5, 3, 2, 2, 1).
\]

In general, we have:

- \( \lambda = (\lambda_1, \ldots, \lambda_k) \) is a possible Young diagram if and only if \( n - k \geq \lambda_1 \geq \cdots \geq \lambda_k \geq 0 \).
- If \( j_1 < \cdots < j_k \) are the jumping numbers then \( \lambda_i = n - k - j_i + i \).

The set of \( V \in G_k(\mathbb{C}^n) \) that have a given Young diagram \( \lambda \) is called a Schubert cell, denoted by \( \Omega(\lambda) \) or \( \Omega(\lambda, F_\bullet) \). Each Schubert cell is a topological disk of real codimension \( 2|\lambda| \). The Schubert cells (for a fixed flag) give a CW decomposition of the space \( G_k(\mathbb{C}^n) \). The closure of \( \Omega(\lambda) \) (in either classical or Zariski topologies) is the set of \( V \in G_k(\mathbb{C}^n) \) such that \( \dim(V \cap F_j) \geq i \) for each \( i = 1, \ldots, n \) (where \( j_1 < \cdots < j_k \) are the jumping numbers associated to \( \lambda \)). These sets are closed irreducible varieties, called Schubert varieties. (See e.g. [Fu] §9.4.)
Example B.4. If $E \subseteq \mathbb{C}^n$ is a subspace with $\dim E = e \leq k$ then the set $S_e(E)$ defined by (B.3) is a Schubert variety $\bar{\Omega}(\lambda, F_\bullet)$, where $F_\bullet$ is any flag with $E_\bullet = E$ and

$$\lambda = \left( n-k,\ldots,n-k,0,\ldots,0 \right)$$

Let $A^*(k,n)$ denote the set of formal linear combinations with integer coefficients of Young diagrams in the $k \times (n-k)$ rectangle. This is by definition an abelian group.

**Proposition B.5.** There is a second binary operation called the cup product and denoted by the symbol $\sim$ that makes $A^*(k,n)$ a commutative ring, and is characterized by the following properties:

If $\lambda$ and $\mu$ are Young diagrams with respective areas $r$ and $s$ then their cup product is of the form:

$$\lambda \sim \mu = \nu_1 + \cdots + \nu_N,$$

where $\nu_1, \ldots, \nu_N$ are Young diagrams with area $r+s$ (possibly with repetitions, possibly $N=0$). Moreover, there are flags $F_\bullet, G_\bullet, H_\bullet$ such that the manifolds $\bar{\Omega}(\lambda, F_\bullet)$ and $\bar{\Omega}(\mu, G_\bullet)$ are transverse and their intersection is $\bigcup \bar{\Omega}(\nu_i, H_\bullet)$.

**Example B.6.** Working in $A^*(2,4)$, let us compute the products of the Young diagrams $\lambda = $ and $\mu = $. Fix a flag $F_\bullet$. Then $\bar{\Omega}(\lambda, F_\bullet)$ is the set of $W \in G_2(\mathbb{C}^4)$ that contain $F_1$, and $\bar{\Omega}(\mu, F_\bullet)$ is the set of $W \in G_2(\mathbb{C}^4)$ that are contained in $F_3$. Take another flag $G_\bullet$, which is in general position with respect to $F_\bullet$, that is $F_1 \cap G_{4-i} = \{0\}$. Then:

- The set $\bar{\Omega}(\lambda, F_\bullet) \cap \bar{\Omega}(\mu, F_\bullet)$ contains a single element, namely $F_1 \oplus G_1$, and thus equals $\bar{\Omega}(2,2, H_\bullet) = \{H_2\}$ for an appropriate flag $H_\bullet$. This shows that $\lambda \sim \mu = $.
- The space $F_3 \cap G_3$ is 2-dimensional and thus is the single element of $\bar{\Omega}(\mu, F_\bullet) \cap \bar{\Omega}(\lambda, G_\bullet)$. So $\mu \sim \lambda = $.
- The set $\bar{\Omega}(\lambda, F_\bullet) \cap \bar{\Omega}(\mu, G_\bullet)$ is empty, thus $\lambda \sim \mu = 0$.

However, if we work in $A^*(4,8)$ then it can be shown that:

$$\begin{align*}
\lambda \sim \lambda &= \begin{array}{c}
\lambda \sim \lambda = \begin{array}{c}
\lambda \sim \lambda = \begin{array}{c}
\lambda \sim \lambda = \begin{array}{c}
\lambda \sim \lambda = \begin{array}{c}
\lambda \sim \lambda = \begin{array}{c}
\lambda \sim \lambda = \begin{array}{c}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{align*}$$

If we drop the terms that do not fit in a $2 \times 2$ rectangle, we reobtain the results for $G_2(\mathbb{C}^4)$.

The general computation of the product $\lambda \sim \mu$ is not simple and can be done in various ways. For our purposes, however, it will be sufficient to know when the product is zero or not. The answer is provided by the following simple lemma:

**Lemma B.7** ([Fu], p. 148–149). Let $\lambda$ and $\mu$ be Young diagrams in the $k \times (n-k)$ rectangle. The following two conditions are equivalent:

1. $\lambda \sim \mu \neq 0$.
2. If one draws inside the $k \times (n-k)$ rectangle the Young diagrams of $\lambda$ and $\mu$, being the latter rotated by $180^\circ$ and put in the southeast corner, then the two figures do not overlap (see Fig. 3). Equivalently, $\lambda_i + \mu_{k+1-i} \leq n-k$ for every $i = 1, \ldots, n$.

**B.5. Intersection of subvarieties of the Grassmannian.** Next we explain how the Schubert calculus sketched above can be used to obtain information about intersection of general subvarieties of the Grassmannian, by means of cohomology and Poincaré duality. See [Fu, Appendix B] and [Hu] for further details.
Any topological space $X$ has singular homology groups $H_i X$ and cohomology groups $H^i X$ (here taken always with integer coefficients). With the cup product $H^i X \times H^j X \to H^{i+j} X$, the cohomology $H^* X = \bigoplus H^i X$ has a ring structure.

If $X$ is a real compact oriented manifold of dimension $d$ then the homology group $H_d X$ is canonically isomorphic to $\mathbb{Z}$, with a generator $[X]$ called the fundamental class of $X$. In addition, there is Poincaré duality isomorphism $H^i X \to H_{d-i} X$, which is given by $\alpha \mapsto \alpha \smile [X]$ (taking the cap product with the fundamental class). Let us denote by $\omega \mapsto \omega^*$ the inverse isomorphism.

Next suppose $Y$ and $Z$ are compact oriented submanifolds of $X$, of codimensions $i$ and $j$ respectively. Also suppose that $Y$ and $Z$ have transverse intersection $Y \cap Z$, which therefore is either empty or a compact submanifold of codimension $i+j$, which is oriented in a canonical way. The images of the fundamental classes of $Y$, $Z$, and $Y \cap Z$ under the inclusions into $X$ define homology classes that we denote (with a slight abuse of notation) by $[Y] \in H_{d-i} X$, $[Z] \in H_{d-j} X$, $[Y \cap Z] \in H_{d-(i+j)} X$. Then their Poincaré duals $[Y]^* \in H^i X$, $[Z]^* \in H^j X$, and $[Y \cap Z]^* \in H^{i+j} X$ are related by:

$$[Y]^* \smile [Z]^* = [Y \cap Z]^*.$$  
That is, cup product is Poincaré dual to intersection.

Now consider the case where $X$ is a projective nonsingular (i.e., smooth) complex variety, and $Y$ and $Z$ are irreducible subvarieties of $X$. Obviously, the fundamental class $[X]$ makes sense, because $X$ is a compact manifold with a canonical orientation induced from the complex structure. A deeper fact (see [Fu, Appendix B]) is that fundamental classes $[Y]$ and $[Z]$ can also be canonically associated to the (possibly singular) subvarieties $Y$ and $Z$, and the Poincaré duality between cup product and intersection works in this situation. More precisely, suppose that $Y$ and $Z$ are transverse in the algebraic sense: $Y \cap Z$ is a union of subvarieties $W_1, \ldots, W_\ell$ whose codimensions are the sum of the codimensions of $Y$ and $Z$, and for each $i = 1, \ldots, \ell$, the tangent spaces $T_w Y$ and $T_w Z$ are transverse for all $w$ in a Zariski-open subset of $W_i$. Then each $W_i$ has its canonical fundamental class, and the following duality formula holds:

$$[Y]^* \smile [Z]^* = [W_1]^* + \cdots + [W_\ell]^*.$$  

In our application of this machinery, $X$ will be the Grassmannian $G_k(\mathbb{C}^n)$. In this case:

- The fundamental classes of the Schubert varieties $[\Omega(\lambda, F_ullet)]$ do not depend on the flag $F_ullet$.
- Let $\sigma_\lambda$ denote the Poincaré dual of $[\Omega(\lambda, F_ullet)]$. Then $H^{2r} G_k(\mathbb{C}^n)$ is a free abelian group and the elements $\sigma_\lambda$ with $|\lambda| = r$ form a set of generators. (The cohomology groups of odd codimension are zero.)
- The cup product on cohomology agrees with the “cup” product of Young diagrams explained in the previous section.
B.6. End of the proof. We are now able to prove Theorem B.1.

Proof of Theorem B.1. Let $1 \leq e \leq k < n$. Let $E \subset \mathbb{C}^n$ be a subspace of dimension $e$, and consider the set $S_k(E)$ defined by (B.5). Recall from Example B.4 that this is the Schubert variety for the Young diagram $\lambda$ given by (B.6).

Now consider a (nonempty) subvariety $Y \subset G_k(\mathbb{C}^n)$ that is disjoint from $S_k(E)$. We want to give a lower bound for the codimension $c$ of $Y$. We can of course assume that $Y$ is irreducible.

Let $[Y]^*$ be the dual of fundamental class of $Y$. This is a nonzero element of $H^{2e}G_k(\mathbb{C}^n)$. It can be expressed as $\sum n_i \sigma_{\mu_i}$, where $\mu_i$ are Young diagrams with area $|\mu_i| = c$, and $n_i$ are nonzero integers. In fact we have $n_i > 0$, because of the canonical orientations induced by complex structure.

Since the intersection between $S_k(E)$ and $Y$ is empty (and in particular transverse), Poincaré duality gives $[S_k(E)]^* \sim [Y]^* = 0$. Therefore we have $\sigma_\lambda \sim \sigma_{\mu_i} = 0$ for each $i$.

By Lemma B.7, if we draw the Young diagram of $\mu_i$ rotated by $180^\circ$ and put in the southeast corner of the $k \times (n-k)$ rectangle, then it overlaps the Young diagram $\lambda$ pictured in (B.6). This is only possible if $c \geq k - e + 1$; indeed the Young diagram $\mu$ with least area such that $\lambda \sim \mu \neq 0$ is

$$\mu = \begin{pmatrix} 1, \ldots, 1, 0, \ldots, 0 \\ k-e+1 \text{ times} \quad e-1 \text{ times} \end{pmatrix},$$

for which the overlapping picture becomes:

![Diagram](https://via.placeholder.com/150)

This concludes the proof of Theorem B.1. \qed

As explained in §B.3.2, Theorem 5.4 follows.

APPENDIX C. STRATIFICATIONS AND TRANSVERSALITY

C.1. Stratifications. This appendix contains fundamental for the understanding of Section 6. We recall a few notions about stratifications and transversality, and prove Proposition 6.1. We refer the reader to [GWPL, Ma] for more details and proofs.

Let $X$ be a smooth (i.e., $C^\infty$) manifold. A smooth stratification of a closed subset $\Sigma \subset X$ is a filtration by closed subsets

$$\Sigma = \Sigma_n \supset \Sigma_{n-1} \supset \cdots \supset \Sigma_0$$

such that for each $i$, the set $\Gamma_i = \Sigma_i \setminus \Sigma_{i-1}$ (where $\Sigma_{-1} := \emptyset$) is a smooth submanifold of $X$ without boundary and the dimension of $\Gamma_i$ decreases strictly with increasing $i$. Each connected component of $\Gamma_i$ is called a stratum. The codimension in $X$ of a stratification is the codimension of the stratum of largest dimension. A stratification of a set $\Sigma$ is not unique, but this codimension in $X$ does not depend on the choice of the stratification.

Actually, apart for discrete subsets $\Sigma \subset X$, if there is one smooth stratification, then there are infinitely many others. However, the subsets we deal with are endowed with certain canonical stratifications:
Theorem C.1 (Existence of canonical stratifications). Any algebraic set \( \Sigma \subset \mathbb{C}^N \) admits a canonical smooth stratification whose strata are complex submanifolds of \( \mathbb{C}^N \). Any closed semialgebraic set \( \Sigma \subset \mathbb{R}^N \) admits a canonical smooth stratification whose strata are semialgebraic submanifolds of \( \mathbb{R}^N \).

In the case of an irreducible algebraic set \( \Sigma \subset \mathbb{C}^n \), the canonical stratification can be obtained as follows: The connected components of the set of regular (i.e., non-singular) points form the higher-dimensional strata; then one decomposes the set of singular points of \( \Sigma \) into irreducible components and proceeds by induction.

In any case, those canonical stratifications are uniquely characterized by a certain minimality property. In particular, the canonical stratifications are equivariant under polynomial automorphisms of the ambient space.

Another important property of the canonical stratifications is that they satisfy the so-called Whitney conditions\( p_a q \) and \( p_b q \):

For any sequence of points \( x_n \) in a stratum \( \Gamma \) of dimension \( i \) converging to a point \( y \) in a stratum \( \Delta \) of dimension \( < i \), if the sequence of tangent spaces \( T_{x_n} \Gamma \) converges to an \( i \)-space \( E \subset T_y X \), then we have

(a) \( E \) contains \( T_y \Delta \),

(b) in a local chart, if a sequence \( y_n \in \Delta \) converges to \( y \) and if the lines \( x_n y_n \) converge to a line \( L \subset T_y \Delta \), then \( L \subset E \).

A smooth stratification that satisfies the Whitney conditions is called a Whitney stratification. Let us write down some properties.

Proposition C.2 (Basic properties of Whitney stratifications). Let \( X, Y \) be smooth manifolds. Let

\[
(C.1) \quad \Sigma_n \supset \cdots \supset \Sigma_0
\]

be a filtration of a set \( \Sigma \subset X \). Then:

1. Being a Whitney stratification is a local property of a filtration: So if \( (C.1) \) is a Whitney stratification then \( \Sigma_n \cap U \supset \cdots \supset \Sigma_0 \cap U \) is a Whitney stratification, and conversely if each point in \( \Sigma \) has an open neighborhood \( U \subset X \) such that \( \Sigma_n \cap U \supset \cdots \supset \Sigma_0 \cap U \) is a Whitney stratification then \( (C.1) \) is a Whitney stratification.

2. If \( (C.1) \) is a Whitney stratification of codimension \( m \) in \( X \), then \( \Sigma_n \times Y \supset \cdots \supset \Sigma_0 \times Y \) is a Whitney stratification of codimension \( m \) in \( X \times Y \).

3. If \( (C.1) \) is a Whitney stratification and \( f: X \to Y \) is a smooth diffeomorphism then \( f(\Sigma_n) \supset \cdots \supset f(\Sigma_0) \) is a Whitney stratification in \( Y \).

Let us now discuss how stratifications behave with respect to transversality. Let \( f: X \to Y \) be a \( C^1 \) map. Let \( \Sigma = \Sigma_d \supset \cdots \supset \Sigma_0 \) be a stratification of a closed subset \( \Sigma \) of \( Y \). One says that \( f \) is transverse to that stratification (in symbols, \( f \pitchfork \Sigma \)) if it is transverse to each of its strata. Transversality to a general stratification is not an open condition. However, we obtain openness if the stratification is Whitney.

Proposition C.3 (Transversality is open). Let \( X, Y \) be \( C^\infty \) manifolds without boundary. Let \( \Sigma = \Sigma_d \supset \cdots \supset \Sigma_0 \) be a Whitney stratification of a closed subset of \( Y \). Then the set \( \mathcal{O} = \{ f \in C^1(X,Y); f \pitchfork \Sigma \} \) is open in \( C^1(X,Y) \) (with respect to the strong topology).

Actually, only Whitney condition \( (a) \) is necessary here (use the \( (1) \Rightarrow (3) \) implication of Trotman’s theorem \( [Tt] \)).
C.2. Jets and jet transversality. We recall the basic notions on jets and state
the transversality theorems we will need; see [Hi] for details.

Let $X$, $Y$ be smooth manifolds without boundary. If $1 \leq r < \infty$, an $r$-jet from
$X$ to $Y$ is an equivalence class of pairs $(x, f)$, where $x \in X$, $f$ is a $C^r$ map from
a neighborhood of $x$ to $Y$, and where $(x, f)$ is equivalent to $(x', f')$ if $x = x'$ and $f$
and $f'$ have same derivatives at $x$ up to order $r$. We denote by $J^r(X, Y)$ the space
of r-jets from $X$ to $Y$. It is a smooth manifold.

For all $1 \leq s \leq \infty$, we denote by $C^s(X, Y)$ the space of $C^s$-maps from $X$ to $Y$,
endowed with the strong topology.

Given $1 \leq r < s \leq \infty$ and a map $g \in C^s(X, Y)$, the $r$-jet extension is the map
$j^r g : X \to J^r(X, Y)$ that sends $x$ to the equivalence class $j^r g(x)$ of $(x, g)$. Then the mapping

$$j^r : C^s(X, Y) \to C^{s-r}(X, J^r(X, Y))$$

is continuous.

**Theorem C.4** (Jet transversality). Let $1 \leq r < s \leq \infty$. Let $X$ and $Y$ be $C^\infty$ mani-

folds without boundary. Let $W \subset J^r(X, Y)$ be a $C^\infty$ submanifold without boundary.
Then the $C^s$-maps $g : X \to Y$ for which the $r$-jet extension $j^r g$ is transverse to $W

form a residual subset of $C^s(X, Y)$.

We finally prove the proposition stated in §B.

**Proof of Proposition B.2.** By Proposition C.3, the set $\{ F : X \to j^1(X, Y); F \not\equiv \Sigma \}$

is open in $C^1(X, J^1(X, Y))$. Hence the set $O := \{ f : X \to Y; j^1 f \not\equiv \Sigma \}$ is open in $C^2(X, Y)$.

Fix $r \geq 2$. Given a Whitney stratification $\Sigma_0 \supset \cdots \supset \Sigma_i$ of $\Sigma$, let $Z_i = \Sigma_i \setminus \Sigma_{i-1}$

be the corresponding decomposition into smooth submanifolds. By the jet transversality theorem (Theorem C.4), each set $R_i = \{ f \in C^r(X, Y); j^1 f \not\equiv Z_i \}$ is residual. Thus $O \cap C^r(X, Y) = \bigcap_i R_i$ is $C^r$-dense. This concludes the proof. $\square$

**APPENDIX D. PROOF OF THE RESULT IN THE HOLOMORPHIC SETTING**

**Proof of Theorem 1.4.** Let $U \subset \mathbb{C}^m$ be an open subset. We may identify the set of
1-jets from $U$ to $GL(d, \mathbb{C})$ with

$$U \times GL(d, \mathbb{C}) \times gl(d, \mathbb{C})^m.$$ 

As we did in Section B and using $GL(d, \mathbb{C})$ instead of Theorem 1.4, we obtain
that the set of poor 1-jets from $U$ to $GL(d, \mathbb{C})$ is the algebraic subset $U \times P_m^{(\mathbb{C})}$ of
the space of 1-jets. Hence it admits a stratification

$$U \times P_m^{(\mathbb{C})} = U \times \Sigma_n \supset \cdots \supset U \times \Sigma_0.$$ 

Write $U \times P_m^{(\mathbb{C})}$ as the disjoint union $\bigsqcup_{0 \leq i \leq n} X_i$ where each $X_i$ is a smooth sub-

manifold of dimension $i$ in the jet space $J^1(U, GL(d, \mathbb{C}))$, and $X_n$ has codimension $m$.

Fix now a map $A \in \mathcal{H}(U, GL(d, \mathbb{C}))$. For all $v = (a, b_1, \ldots b_m) \in \mathbb{C}^{m+1}$ and $u = (u_1, \ldots, u_m) \in \mathbb{C}^m$, write

$$P_v(u) = a + \sum_{i=1}^m b_i u_i.$$ 

For all $v = (v_{i,j})_{1 \leq i, j \leq d} \in (\mathbb{C}^m)^d$, write $P_v = [P_{v_{i,j}}]_{1 \leq i, j \leq d}$ and define the map

$$\Phi_v = A + P_v.$$ 

One can write the 1-jet extension $j^1 A$ at the point $u \in U$ as

$$j^1 A(u) = [u, A(u), B_1, \ldots, B_m] \in U \times GL(d, \mathbb{C}) \times [Mat_{d \times d}(\mathbb{C})]^m.$$
The same way, if we put \( v_{i,j} = (a_{i,j}, b_{1,i,j}, \ldots, b_{m,i,j}) \), we have
\[
j^iP_v(u) = [u, P_v(u), (b_{1,i,j})_{1 \leq i, j \leq d}, \ldots, (b_{m,i,j})_{1 \leq i, j \leq d}].
\]
Define the map \( F : v \mapsto F_v = j^i\Phi_v \). The evaluation map of \( F \) is:
\[
F^{ev} : \left\{(\mathbb{C}^{m+1})^d \times U \to U \times \text{Mat}_{d \times d}(\mathbb{C}) \times [\text{Mat}_{d \times d}(\mathbb{C})]^m, (v, u) \mapsto F_v(u)\right\}
\]
Hence,
\[
F^{ev}(v, u) = j^1(A + P_v) = \left[u, (A + P_v)(u), (b_{1,i,j})_{1 \leq i, j \leq d}, \ldots, (b_{m,i,j})_{1 \leq i, j \leq d}\right]
\]
Claim D.1. For all \( u \), the map \( F^{ev} \) restricts to a submersion from the \((\cdot, u)\)-fiber to the \([u, \cdot]\)-fiber.

Proof. We want to prove that
\[
v \mapsto \left[\left(u, (A + P_v)(u), (b_{1,i,j})_{1 \leq i, j \leq d}, \ldots, (b_{m,i,j})_{1 \leq i, j \leq d}\right)\right]
\]
is a submersion, or equivalently that
\[
v \mapsto \left[\left(P_v(u), (b_{1,i,j})_{1 \leq i, j \leq d}, \ldots, (b_{m,i,j})_{1 \leq i, j \leq d}\right)\right]
\]
is a submersion. Noting that \( v = (a_{i,j}, b_{k,i,j})_{1 \leq i, j \leq d} \), this comes easily from the fact that \((a_{i,j}) \mapsto P_v(u)\) is a submersion, for any fixed set of coefficients \((b_{k,i,j})_{1 \leq i, j \leq d} \).

That claim immediately implies that \( F^{ev} \) is a submersion. In particular it is transverse to each \( X_i \). By the parametric transversality theorem (see [Hi, p. 79]), there is a residual subset of parameters \( v \) in \((\mathbb{C}^{m+1})^d\) such that \( F_v = j^i\Phi_v \) is transverse to \( X_i \), for all \( i \).

When \( v \) goes to 0, \( \Phi_v \) tends to \( A \) in \( H(U, GL(d, \mathbb{C})) \). This shows the denseness in \( H(U, GL(d, \mathbb{C})) \) of the maps \( \hat{A} \) such that \( j^1\hat{A} \) is transverse to \( X_i \), for all \( i \). Take such a map \( \hat{A} \); for all \( i \), the image of \( j^1\hat{A} \) does not intersect \( X_0 \sqcup \cdots \sqcup X_{n-1} \) and intersects \( X_n \) (which has codimension \( m \)) only in a discrete subset.

Fix \( K' \subset U \) a compact set that contains \( K \) in its interior. The image \( j^1\hat{A} \) restricted to \( K' \) can only intersect \( X_n \) in a finite set \( \Gamma \); indeed, any accumulation point of that intersection set would have to be in \( X_0 \sqcup \cdots \sqcup X_{n-1} \), since \( X_0 \sqcup \cdots \sqcup X_{n-1} \) is closed, and this would contradict the fact that \( j^1\hat{A} \) does not intersect \( X_0 \sqcup \cdots \sqcup X_{n-1} \).

By the choice of our topology, a small perturbation \( \hat{A} \) of \( \hat{A} \) is \( C^0 \) close to \( \hat{A} \) by restriction to \( K' \). By Cauchy’s formula, the map \( \hat{A} \) is \( C^2 \) close to \( \hat{A} \) over the set \( K \). Hence, the (compact) image of \( j^1\hat{A} \) restricted to \( K \) is still far from \( X_0 \sqcup \cdots \sqcup X_{n-1} \), and intersects \( X_n \) transversally in some \( \epsilon \)-neighborhood of \( \Gamma \) inside \( X_n \). Thus it also has to intersect \( X_n \) only on a finite set.

So we have found an open and dense subset of holomorphic maps whose 1-jets above \( K \) intersect the set of \( N \)-poor jets only on a finite number of points. As a consequence, for such maps, there are only finitely many constant singular inputs in \( K^N \) for the system 1.4. This concludes the proof of Theorem 1.6.

**APPENDIX E. SINGULAR CONSTANT INPUTS OF GENERIC TYPE**

In this appendix we prove Theorem 1.2 and the other assertions made at the end of § 1.2. We also discuss other control-theoretic properties of generic semilinear systems that are related to universal regularity.
E.1. The poor data of generic type. Recall from §A.1 the definition of an unconstrained matrix. Let $(e_1, \ldots, e_d)$ denote the canonical basis of $\mathbb{C}^d$.

Lemma E.1. Suppose that the datum $A = (A, B_1, \ldots, B_m) \in \text{GL}(d, \mathbb{C}) \times \mathfrak{gl}(d, \mathbb{C})^m$ has the following properties:

1. $A$ is an unconstrained diagonal matrix;
2. there are indices $i_0, j_0 \in \{1, \ldots, d\}$ with $i_0 \neq j_0$ such that for each $k \in \{1, \ldots, m\}$, the $(i_0, j_0)$ entry of the matrix $B_k$ vanishes;
3. the off-diagonal vanishing entry position $(i_0, j_0)$ above is unique.

Then:
1. There is a single direction $[v] \in \mathbb{C}P^{d-1}$ such that $\Lambda(A) \cdot v \neq \mathbb{C}^d$, namely $[e_{j_0}]$.
2. The space $\Lambda(A) \cdot e_{j_0}$ has codimension 1; in fact, it equals $\text{span}\{e_i; i \neq i_0\}$.

Proof of Lemma E.1. By arguments as in the proof of Lemma A.1, we see that

$$\Lambda(A) \ni \{(y_{ij}) \in \mathfrak{gl}(d, \mathbb{C}); y_{11} = \cdots = y_{dd}, y_{i_0 j_0} = 0\}.$$ The conclusions follow easily. □

Recall from §2.3 that a set $\mathcal{Z} \subset [\text{Mat}_{d \times d}(\mathbb{K})]^{1+m}$ is called saturated if $(A, B_1, \ldots, B_m) \in \mathcal{Z}$ implies that:

- for all $P \in \text{GL}(d, \mathbb{K})$ we have $(P^{-1}AP, P^{-1}B_1P, \ldots, P^{-1}B_mP) \in \mathcal{Z}$;
- for all $Q = (q_{ij}) \in \text{GL}(m, \mathbb{K})$, letting $B'_i = \sum_j q_{ij}B_j$, we have $(A, B'_1, \ldots, B'_m) \in \mathcal{Z}$.

Remark E.2.
1. A subset $[\text{Mat}_{d \times d}(\mathbb{K})]^{1+m}$ is saturated if and only if it is invariant under a certain action of the group $\text{GL}(d, \mathbb{K}) \times \text{GL}(m, \mathbb{K})$.
2. The real part of a complex saturated set is saturated (in the real sense).

Lemma E.3. There exists a saturated algebraically closed set $S_m^{(C)} \subset \text{GL}(d, \mathbb{C}) \times [\text{Mat}_{d \times d}(\mathbb{C})]^m$ of codimension at least $m + 1$ such that for all $(A, B_1, \ldots, B_m) \in S_m^{(C)} \cap S_m^{(C)}$, the following properties hold:

1. $A$ is unconstrained;
2. if $P \in \text{GL}(d, \mathbb{C})$ is such that $P^{-1}AP$ is a diagonal matrix then there are indices $i_0, j_0 \in \{1, \ldots, d\}$ with $i_0 \neq j_0$ such that for each $k \in \{1, \ldots, m\}$, the $(i_0, j_0)$ entry of the matrix $P^{-1}B_kP$ vanishes;
3. for each choice of $P$ above, the off-diagonal vanishing entry position $(i_0, j_0)$ is unique.

In order to prove the lemma, we begin by checking algebraicity of the constraints:

Lemma E.4. The set $K \subset \text{GL}(d, \mathbb{C})$ of constrained matrices is an algebraically closed subset of codimension 1.

Proof. Multiply all constraints, obtaining a polynomial in the variables $\lambda_1, \ldots, \lambda_d$. This polynomial is symmetric, and therefore (see e.g. [La Thm. IV.6.1]) can be written as a polynomial function of the elementary symmetric polynomials in the variables $\lambda_1, \ldots, \lambda_d$. Now substitute each elementary symmetric polynomial in this expression by the corresponding coefficient of the characteristic polynomial.
of the matrix $A$. This gives a polynomial function on the entries of the matrix $A$ that vanishes if and only if $A$ is constrained. It is obvious that the corresponding algebraic set $K$ has codimension 1.

Now we check algebraicity of double vanishing:

**Lemma E.5.** There exists a saturated algebraically closed subset $D$ of $\text{GL}(d, \mathbb{C}) \times [\text{Mat}_{d \times d}(\mathbb{C})]^m$ such that if $(A, B_1, \ldots, B_m) \in D$ and $A$ has simple spectrum then property $\mathcal{H}$ from Lemma E.3 is satisfied, but property $\mathcal{S}$ is not.

**Proof.** First, consider the subset $X \subset [\text{Mat}_{d \times d}(\mathbb{C})]^{1+m} \times (\mathbb{C}P^{d-1})^2$ formed by tuples $(A, B_1, \ldots, B_m, [v], [w])$ such that

$$[Av] = [v], \quad [A^*w] = [w], \quad w^*v = 0, \quad w^*B_kv = 0 \text{ for each } k = 1, \ldots, m,$$

where $v$ and $w$ are regarded as column-vectors and the star denotes transposition. The set $X$ is obviously algebraic; thus, by Proposition 5.1, so is its projection $Y$ on $[\text{Mat}_{d \times d}(\mathbb{C})]^{1+m}$.

Let $A$ be a matrix with simple spectrum. Then $(A, B_1, \ldots, B_m)$ belongs to $Y$ if and only if property $\mathcal{H}$ from Lemma E.3 is satisfied. In particular, the fiber of $Y$ over $A$ is a union of affine subspaces of $[\text{Mat}_{d \times d}(\mathbb{C})]^m$. Intersections of those affine spaces correspond to points where the uniqueness property $\mathcal{S}$ is not satisfied. These points of intersection are singular points of $Y$. Conversely, it is clear that the variety $Y$ is smooth at the points on the fiber over $A$ where property $\mathcal{S}$ is satisfied.

So let $Z$ be the (algebraically closed) set of singular points of $Y$. It is straightforward to see that the set $Y$ is saturated. Recalling Remark E.2 (part 1) and the fact that a group acting on a variety preserves singular points, we see that the set $Z$ is saturated as well.

We define $D$ as the set $Z$ minus the tuples $(A, B_1, \ldots, B_m)$ with $\det A = 0$. Then $D$ has all the required properties. \hfill $\square$

Now we combine the facts above with Scholium 5.9 to prove Lemma E.3.

**Proof of Lemma E.3.** For simplicity of writing we will omit the $m$ subscripts and the $(\mathbb{C})$ superscripts.

Let $\pi: P \to \text{GL}(d, \mathbb{C})$ be the projection on the first matrix. Define

$$S = \pi^{-1}(K) \cup (D \cap P),$$

where $K$ and $D$ come respectively from Lemmas E.4 and E.5. Then $S$ is a saturated algebraically closed subset of $P$. If $A = (A, B_1, \ldots, B_m) \in P \smallsetminus S$ then:

- $A \notin K$, which is property $\mathcal{H}$;
- since $A \in P$, it follows from Lemma A.1 that $A$ is conspicuously poor, and so property $\mathcal{S}$ holds;
- since $A \notin D$, property $\mathcal{S}$ also holds.

To complete the proof of the lemma, we need to show that $\text{codim} S \geq m + 1$. We will use the following inclusion:

$$(E.1) \quad S \subseteq \mathcal{F} \cup \left( \pi^{-1}(K) \cap \mathcal{F} \right) \cup \left( (\mathcal{D} \cap P) \cap \pi^{-1}(K) \right),$$

where $\mathcal{F}$ comes from Scholium 5.3. Recall that $\mathcal{F}$ equals $\pi^{-1}(C_j)$, where $C_j$ is given by (5.39), and it has codimension at least $m + 1$.

We apply Lemma 5.2 and Remark 5.3 to the set $\mathcal{F}' \subset Y' \times [\text{gl}(d, \mathbb{C})]^m$, where $Y' = \text{GL}(d, \mathbb{C}) \smallsetminus C_j - 1$. Since $K$ has codimension at least 1 in $Y'$, and the fibers of $\mathcal{F}'$ all have codimension at least $m$, we conclude that that codim $\mathcal{F}' \geq m + 1$.

Next, we want to apply Lemma 5.2 and Remark 5.3 to the set $\mathcal{F}'' \subset Y'' \times [\text{gl}(d, \mathbb{C})]^m$, where $Y'' = \text{GL}(d, \mathbb{C}) \smallsetminus K$. For each $A \in Y''$, it follows from Lemma E.3
that the fiber of $F''$ over $A$ (which is the same as the fiber of $D$ over $A$) has codimension $2m$ in $[\text{gl}(d, \mathbb{C})]^m$, corresponding to the $2m$ different matrix entries that must vanish. We conclude that $\text{codim } F'' \geq 2m$.

We have seen that each of the three sets on the right-hand side of (E.1) has codimension at least $m + 1$. So the same is true for $S$, as we wanted to prove. □

E.2. Proof of the addendum to the Main Theorem 1.1.

Proof of Theorem 1.3. Consider the set $S_m^{(C)}$ given by Lemma E.3 and let $S_m^{(R)}$ be its real part. This is an algebraically closed saturated subset of $GL(d, \mathbb{R}) \times \text{[gl}(d, \mathbb{R})]^m$ which, by Proposition 5.1, has codimension at least $m + 1$.

Consider the set $\tilde{\Gamma}$ of 1-jets $J \in J^1(\mathcal{U}, GL(d, \mathbb{C}))$ that have a local expression $(u, A(u), B_1, \ldots, B_m)$ with $(A(u), B_1, \ldots, B_m) \in S_m^{(R)}$. This does not depend on the choice of the local coordinates, because $S_m^{(R)}$ is saturated. By the same arguments as in the proof of Theorem 1.1, the set $\tilde{\Gamma}$ admits a Whitney stratification. Its codimension is at least $m + 1$. Applying Proposition 6.1, we obtain a $C^\infty$-dense set $\tilde{\mathcal{O}} \subset C^\infty(\mathcal{U}, GL(d, \mathbb{C}))$ formed by maps $A$ that are transverse to the stratification.

Let $\mathcal{O}$ be the set provided by Theorem 1.1 and consider a map $A \in \mathcal{O} \cap \tilde{\mathcal{O}}$. Then whenever a jet $j^1 A(u)$ is poor, it does not belong to $\tilde{\Gamma}$. Recalling Lemma E.3, we see that the local expression of $j^1 A(u)$ satisfies (after a change of basis) the hypotheses of Lemma E.3. Therefore parts 1 and 2 of the theorem follow respectively from conclusions 1 and 2 of the lemma. □

Remark E.6. The proof of Theorem 1.2 also gives more information about the 1-jets that appear generically for singular constant inputs $(u, \ldots, u)$: any associated matrix datum is conspicuously poor and the matrix $A(u)$ is unconstrained.

Remark E.7. Properties 1 and 2 in Theorem 1.2 are in fact dual to each other. If $A$ is the datum representing the 1-jet of $A(u)$ at a point, and $\Lambda = \Lambda(A)$, then property 1 means that there is an unique direction $[v] \in \mathbb{R}P^{d-1}$ such that $\Lambda \cdot v \neq \mathbb{C}^d$. Then property 2 means that there is an unique direction $[w] \in \mathbb{R}P^{d-1}$ such that $\Lambda^* : w \neq \mathbb{C}^d$, where $\Lambda$ is the set of the transposes of the matrices in $\Lambda$. This fact can be proved easily using the dual characterization of Lemma 3.11.

E.3. Local persistence of singular inputs. Let $A \in C^\infty(\mathcal{U}, GL(d, \mathbb{R}))$, $r \geq 1$. We will work upon Lemma 2.3 in order to obtain a more practical way to detect that the 1-jet of $A$ at a point corresponds to a conspicuously poor datum (which as mentioned in Remark E.6 is the only type of poor data that appear generically). For example, in the $m = 1$, $d = 2$ case, we will see that conspicuous poorness means that the angular velocity of one of the eigendirections vanishes (see Remark E.8 below).

Suppose that $u_0 \in \mathcal{U}$ is such that the matrix $A(u_0)$ is diagonalizable over $\mathbb{R}$ and with simple eigenvalues only. By Proposition 2.8, there is a neighborhood $\mathcal{U}_0$ of $u_0$ and $C^r$-maps $\lambda_1, \ldots, \lambda_d : \mathcal{U}_0 \to \mathbb{C}$ such that for all $u \in \mathcal{U}_0$, the complex numbers $\lambda_i(u)$ are all distinct, and form the spectrum of $A(u)$; moreover there exist a $C^r$ map $P : \mathcal{U}_0 \to GL(d, \mathbb{R})$ such that for all $u \in \mathcal{U}_0$, $A(u) = P(u) \Delta(u) P^{-1}(u)$, where $\Delta(u) = \text{Diag}(\lambda_1(u), \ldots, \lambda_d(u))$.

(E.2) $A(u) = P(u) \Delta(u) P^{-1}(u)$, where $\Delta(u) = \text{Diag}(\lambda_1(u), \ldots, \lambda_d(u))$.

For simplicity, let us consider first case where $\mathcal{U}$ is an interval in $\mathbb{R}$ (in particular $m = 1$). Then the normalized derivative of $A$ at a point $u$ can be identified with $N(u) := A'(u) A^{-1}(u)$. Consider the expression of $N(u)$ in the basis that diagonalizes $A(u)$, that is, $B(u) := P^{-1}(u) N(u) P(u)$. Since $\frac{d}{du} P^{-1}(u) = -P^{-1}(u) P'(u) P^{-1}(u)$, we compute that

$$B(u) = \Delta(u) \Delta^{-1}(u) + Q(u) - \Delta(u) Q(u) \Delta^{-1}(u),$$
where
\[ Q(u) := P^{-1}(u) P'(u). \]
So the off-diagonal entries of the matrices \( B(u) \) and \( Q(u) \) are related by
\[ b_{ij}(u) = (1 - \lambda_i(u)/\lambda_j(u)) q_{ij}(u) \quad (i \neq j). \]
In view of Lemma 2.7, we conclude the following: if for some \( p \cdot A \) satisfies conditions (E.5) and (E.6) actually exist; moreover we can always find \( U \) such that \( \tilde{\Phi} \).

**Remark E.8.** Let us give a geometrical interpretation of condition (E.3). The columns of \( P \) form a basis \( (v_1, \ldots, v_d) \) of eigenvectors of \( A \), and the rows of \( P^{-1} \) form a basis \( (f_1, \ldots, f_d) \) of eigenfunctionals of \( A \) (in the sense that \( f_i \circ A = \lambda_i f_i \)); these two bases are related by \( f_i(v_j) = \delta_{ij} \). So \( q_{ij} = f_i \left( \frac{\partial u}{\partial v_j} \right) \) is the component of the velocity of \( v_j \) in the direction of \( v_i \). For example, for \( d = 2 \), condition (E.3) means that one of the eigendirections of \( A \) has zero angular speed at instant \( u = u_* \).

It is trivial to adapt the previous calculations to the higher dimensional case and then conclude the following:

**Proposition E.9.** Let \((u_1, \ldots, u_m)\) be coordinates in a chart domain \( U_0 \subset \mathcal{U} \) where expression (E.2) holds. Consider matrices
\[ Q_k(u) := P^{-1}(u) \frac{\partial P}{\partial u_k}(u). \]
If for some \( u_* \in U_0 \) there is an off-diagonal entry position \((i, j)\) such that
\[ (E.4) \quad \text{for each } k = 1, \ldots, m, \text{the } (i, j)-\text{entry of the matrix } Q_k(u_*) \text{ vanishes} \]
then the 1-jet \( j^1 A(u_*) \) is poor, that is, the constant input \((u_*, \ldots, u_*)\) is singular.

In the situation of Proposition E.9, assume additionally that the map
\[ (E.6) \quad \Phi : \begin{cases} U_0 & \to \text{Im } \Phi \subset \mathbb{K}^m \\ u & \mapsto [\text{the } (i, j)\text{-entry of } Q_k(u)]_{1 \leq k \leq m} \end{cases} \]
is a diffeomorphism.

In that case, the existence of a poor jet is persistent in the following way: If \( \tilde{A} \) is sufficiently \( C^2 \)-close to \( A \) then by Proposition 2.8 we can express \( \tilde{A}(u) = \tilde{P}(u) \Delta(u) \tilde{P}^{-1}(u) \) for \( u \) close to \( u_* \), where \( \tilde{P} \) and \( \Delta \) are \( C^2 \)-close to \( P \) and \( \Delta \) respectively, and \( \Delta \) is diagonal. The corresponding matrices \( \tilde{Q}_k = \tilde{P}^{-1} \frac{\partial \tilde{P}}{\partial u_k} \) are \( C^1 \)-close to \( Q_k \) and the map
\[ \tilde{\Phi} : u \mapsto [\text{the } (i, j)\text{-entry of } \tilde{Q}_k(u)]_{1 \leq k \leq m} \]
is \( C^1 \)-close to \( \Phi \). By (E.6) the fact that \( \Phi(u_*) = (0, \ldots, 0) \), there is \( \tilde{u} \) close to \( u_* \) such that \( \tilde{\Phi}( \tilde{u} ) = (0, \ldots, 0) \). In particular the 1-jet \( j^1 \tilde{A}(\tilde{u}) \) is poor.

Now, concerning existence: It is evident that a domain \( U_0 \) and 2-jets \( j^2 P(u_*) \) satisfying conditions (E.5) and (E.6) actually exist; moreover we can always find a map \( P : U \to \text{GL}(d, \mathbb{R}) \) with a prescribed 2-jet at a point \( u_* \). In view of the discussion above, we conclude the following:

**Proposition E.10** (Persistence of singular inputs). For any \( d \geq 1 \) and any \( d \)-dimensional smooth manifold \( \mathcal{U} \), there exists a \( C^2 \)-open nonempty subset of maps \( A \in C^2(\mathcal{U}, \text{GL}(d, \mathbb{R})) \) with the following property: there exists \( u \in \mathcal{U} \) such that the constant inputs \((u, \ldots, u)\) of any length are all singular for the system (E.4).
That is, one cannot improve Theorem 1.1 replacing “discrete set” by “empty set”.

Given any map $A$ such that (E.5) holds at some point, we can $C^1$-perturb $A$ (by $C^1$-perturbing $P$) in a way such that (E.5) now holds for a non-discrete set of points. This shows that the statement of Theorem 1.1 with “$C^2$-open” replaced by “$C^1$-open” is not true. Using the same idea and Baire’s theorem, one can also show that the conclusion of Theorem 1.1 is not true for $C^1$-generic maps $A$; actually for $C^1$-generic $A$, the points $u \in \mathcal{U}$ corresponding to singular constant controls form a perfect set.

E.4. Other control-theoretic properties. We now introduce a few control-theoretic notions related to accessibility and regularity, and discuss the validity of statements similar to Theorem 1.1 for these notions.

Consider a general control system $1.1$. Fix a time length $N$, and let $\phi_N$ denote the response map as in (1.2). We say that a trajectory determined by $(x_0; u_0, \ldots, u_{N-1})$ is:

- **locally accessible** if for every neighborhood $V$ of $(u_0, \ldots, u_{N-1})$ in $\mathcal{U}^N$, the set $\phi_N(\{x_0\} \times V)$ has nonempty interior.
- **strongly locally accessible** if for every neighborhood $V$ of $(u_0, \ldots, u_{N-1})$ in $\mathcal{U}^N$, the set $\phi_N(\{x_0\} \times V)$ contains in its interior the final state $\phi_N(x_0; u_0, \ldots, u_{N-1})$.

The following implications are immediate:

regular $\Rightarrow$ strongly locally accessible $\Rightarrow$ locally accessible.

We say that an input $(u_0, \ldots, u_{N-1})$ is **universally locally accessible** (resp. **universally strongly locally accessible**) if the trajectory determined by $(x_0; u_0, \ldots, u_{N-1})$ is locally accessible (resp. strongly locally accessible).

Now we come back to the context of projective semilinear control systems $1.4$. A (relatively weak) corollary of Theorem 1.1 is that for generic maps $A$, universal local accessibility holds at all constant inputs:

**Proposition E.11.** Let $N \in \mathbb{N}$ and $\mathcal{O} \subset C^2(\mathcal{U}, \text{GL}(d, \mathbb{R}))$ be as in Theorem 1.1. For any $A \in \mathcal{O}$, every constant input sequence of length $N$ is universally locally accessible.

**Proof.** If $A \in \mathcal{O}$ then for every constant input sequence of length $N$ we can find a regular input sequence nearby. $\square$

As we have shown in Proposition E.10, it is not possible to improve Proposition E.11 by replacing “local accessible” by “regular”. Neither it is possible to replace “local accessible” by “strongly local accessible”, as the following simple example (in dimensions $m = 1$, $d = 2$) shows:

**Example E.12.** For $u \in \mathbb{R}$, define

$$P(u) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad \Delta(u) = \text{Diag}(2, 1).$$

Let $\mathcal{U}$ be an small open interval containing 0, and define $A: \mathcal{U} \to \text{GL}(2, \mathbb{R})$ by (E.2). Let $\xi_0 \in \mathbb{R}^1$ correspond to the direction of the vector $(1, 0)$. Then for any subinterval $V$ of 0, and any $N > 0$, the set

$$\phi_N(\{\xi_0\} \times V^N) = \{A(u_{N-1}) \cdots A(u_0) \cdot \xi_0 u_i \in V\}$$

is an interval of $\mathbb{R}^1$ containing $\xi_0 = \phi_N(\xi_0; 0, \ldots, 0)$ in its boundary. Therefore the input $(0, \ldots, 0)$ is not universally strongly locally accessible. A similar situation occurs for any $C^2$-perturbation of $A$. 
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