Excursion Processes Associated with Elliptic Combinatorics

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Abstract

Researching elliptic analogues for equalities and formulas is a new trend in enumerative combinatorics which has followed the previous trend of studying $q$-analogues. Recently Schlosser proposed a lattice path model in the square lattice with a family of totally elliptic weight-functions including several complex parameters and discussed an elliptic extension of the binomial theorem. In the present paper, we introduce a family of discrete-time excursion processes on $\mathbb{Z}$ starting from the origin and returning to the origin in a given time duration $2T$ associated with Schlosser’s elliptic combinatorics. The processes are inhomogeneous both in space and time and hence expected to provide new models in non-equilibrium statistical mechanics. By numerical calculation we show that the maximum likelihood trajectories on the spatio-temporal plane of the elliptic excursion processes and of their reduced trigonometric versions are not straight lines in general but are nontrivially curved depending on parameters. We analyze asymptotic probability laws in the long-term limit $T \to \infty$ for a simplified trigonometric version of excursion process. Emergence of nontrivial curves of trajectories in a large scale of space and time from the elementary elliptic weight-functions exhibits a new aspect of elliptic combinatorics.

Keywords Elliptic analogues · Lattice path models · Elliptic combinatorics · Excursion processes · Inhomogeneity in space and time · Asymptotic probability laws

1 Introduction

Recently elliptic extensions of special functions and combinatorial identities have been extensively studied [21, 19, 6, 5, 15]. In the present paper, we will report one of the trials

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to import such a trend in enumerative combinatorics and integrable systems \[3, 4\] into the study of stochastic processes describing non-equilibrium statistical mechanics models \[7\]. In the previous papers \[8, 9, 10\], one of the present authors studied the elliptic extensions of interacting particle systems in continuous space-time called the elliptic Dyson models, using the elliptic determinantal identities given by Rosengren and Schlosser \[14\]. Here we consider more fundamental models defined on a discrete spatio-temporal plane; excursion processes of a single particle associated with elliptic combinatorics \[16, 17\].

As mentioned in Section 5.11 in \[13\], the currently spreading “elliptic disease” has followed the previous “\(q\)-disease” (see also Footnote 20 in \[13\]). When we are infected by the “\(q\)-disease”, we replace every positive integer \(n\) by the \(q\)-number

\[
[n]_q \equiv 1 + q + q^2 + \cdots + q^{n-1} = \frac{1 - q^n}{1 - q},
\]

and shifted factorial \((a)_k = a(a+1)\cdots(a+k-1)\) by \(q\)-shifted factorials

\[
(\alpha; q)_k = (1 - \alpha)(1 - \alpha q)\cdots(1 - \alpha q^{k-1}) \quad \text{with} \quad \alpha = q^a. \tag{1.1}
\]

By these replacements in a classical identity we will hopefully obtain a new identity, which is regarded as the \(q\)-analogue of the original identity \[12, 5, 13, 2\]. Now, if we are infected by the “elliptic disease”, we would replace every term in the form \(1 - \alpha q^\ell\) in (1.1) by its elliptic analogue

\[
\theta(\alpha q^\ell; p) = \prod_{j=0}^{\infty} (1 - p^j \alpha q^\ell)(1 - p^{j+1}/(\alpha q^\ell)), \quad \ell = 0, 1, 2, \ldots, \tag{1.2}
\]

where \(p \in \mathbb{C}\) is a fixed complex number with \(|p| < 1\). By definition, if \(p = 0\), \(\theta(\alpha q^\ell; p)\) reduces to \(1 - \alpha q^\ell\).

In \[16\] Schlosser introduced a lattice path model in \(\mathbb{Z}^2\) consisting of positive directed unit vertical and horizontal steps with an elliptic weight function. Here we express his model as a lattice path model on a spatio-temporal plane, since we would like to discuss stochastic processes in this paper. Instead of the function \(\theta\) given by (1.2), we use the \textit{Jacobi theta function} \(\vartheta_1(v; \tau)\) which is obtained from \(\theta\) by multiplying a proper factor as

\[
\vartheta_1(v; \tau) = i e^{\pi i (\tau/4 - v)} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}) \theta(e^{2\pi i v}; e^{2\pi i \tau}), \tag{1.3}
\]

\(i = \sqrt{-1}, v, \tau \in \mathbb{C}\) with \(\Im \tau > 0\). It is holomorphic for \(|v| < \infty\) and has an expansion formula

\[
\vartheta_1(v; \tau) = 2 \sum_{n=1}^{\infty} (-1)^{n-1} e^{\pi i (n-1/2)^2} \sin\{(2n-1)\pi v\}.
\]

From this expression, we can see that \(\vartheta_1(v; \tau) \in \mathbb{R}\), if \(v \in \mathbb{R}\) and \(\tau \in i\mathbb{R}\) with \(\Im \tau > 0\), and

\[
\vartheta_1(v; \tau) \sim 2 e^{\pi i \tau/4} \sin(\pi v), \quad \text{when} \ \Im \tau \to +\infty. \tag{1.4}
\]
(Note that the present functions \( \vartheta_1(v; \tau) \) is denoted by \( \vartheta_1(\pi v, e^{\pi i \tau}) \) in [22].) Schlosser considered an ensemble of lattice paths on a spatio-temporal plane

\[
\Lambda = \{(t, x) : t \in \mathbb{N}_0, x \in \mathbb{Z}, t + x \in 2\mathbb{Z}\},
\]

(1.5)

where \( \mathbb{N} = \{1, 2, \ldots\} \) and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). The elementary steps consist of a rightward step \((t - 1, x - 1) \rightarrow (t, x)\) with weight \( q(t, x) \) and a leftward step \((t - 1, x + 1) \rightarrow (t, x)\) with weight 1, where

\[
q(t, x) = q(t, x; r, \alpha, \beta, \kappa)
\]

\[
= \vartheta_1\left(\frac{2\alpha + 3t - x}{2\pi r}; i\kappa\right) \vartheta_1\left(\frac{2\beta + t + x - 1}{2\pi r}; i\kappa\right) \vartheta_1\left(\frac{2\beta + t + x - 2}{2\pi r}; i\kappa\right) \vartheta_1\left(\frac{2\alpha - \beta - x + 2}{2\pi r}; i\kappa\right),
\]

(1.6)

which depends on four parameters, \( \alpha, \beta \in \mathbb{C}, r, \kappa > 0 \). For two spatio-temporal points \((s, x), (t, y) \in \Lambda\), if \( s \leq t \) and \( |y - x| \leq t - s \), a lattice path \( \varpi \) is defined as a sequence of spatio-temporal points \((u, z_u) \in \Lambda, u = s, s + 1, \ldots, t\) such that \( z_u - z_{u-1} \in \{-1, 1\} \), \( u = s + 1, s + 2, \ldots, t \) with \( z_s = x \) and \( z_t = y \). To each lattice path the following weight is assigned,

\[
q(\varpi) = \prod_{u=s+1}^{t} 1(z_u - z_{u-1} = 1)q(u, z_u) + 1(z_u - z_{u-1} = -1) = \prod_{u=s+1, \ldots, t, z_u-z_{u-1}=1} q(u, z_u),
\]

where \( 1(\omega) \) is an indicator function of a condition \( \omega \); \( 1(\omega) = 1 \) if \( \omega \) is satisfied and \( 1(\omega) = 0 \) otherwise. Let \( \Pi((s, x) \rightarrow (t, y)) \) be the collection of all lattice paths starting from \((s, x)\) and arriving at \((t, y)\) on \( \Lambda \). Then the transition weight from \((s, x)\) to \((t, y)\) is defined by the sum

\[
Q(s, x; t, y) = \sum_{\varpi \in \Pi((s, x) \rightarrow (t, y))} q(\varpi).
\]

(1.7)

If \( \Pi((s, x) \rightarrow (t, y)) = \emptyset \), then we put \( Q(s, x; t, y) = 0 \). Schlosser made a special choice of weight for an elementary rightward step as (1.6) so that the transition weight (1.7) can be
factorized as

\[ Q(s, x; t, y) = Q(s, x; t, y; r, \alpha, \beta, \kappa) \]

\[ = \prod_{u=1}^{(t-s)/(y-x)/2} \vartheta_1 \left( \frac{u}{\pi \tau}; i \kappa \right) \prod_{u=1}^{(t-s)-(y-x)/2} \vartheta_1 \left( \frac{u}{\pi \tau}; i \kappa \right) \times \prod_{u=(s+x)/2+1}^{(t+y)/2} \vartheta_1 \left( \frac{2+u}{\pi \tau}; i \kappa \right) \times \prod_{u=(s-x)/2+1}^{(t-y)/2} \vartheta_1 \left( \frac{\beta+1+(s-x)}{\pi \tau}; i \kappa \right) \times \prod_{u=-(s+x)/2+1}^{-(s+x)/2} \vartheta_1 \left( \frac{\alpha-\beta+u}{\pi \tau}; i \kappa \right). \]  

(1.8)

This fact can be proved by showing that (1.8) solves the recursion relation

\[ Q(s, x; t, y) = Q(s, x; t-1, y-1)q(t, y) + Q(s, x; t-1, y+1), \quad s \leq t, |y-x| \leq t-s, \]

with (1.6), and it was verified using the addition formula of theta functions (see Example 5 on page 451 of [22]),

\[ \vartheta_1(x + y; \tau) \vartheta_1(x - y; \tau) \vartheta_1(u + v; \tau) \vartheta_1(u - v; \tau) \]

\[ - \vartheta_1(x + v; \tau) \vartheta_1(x - v; \tau) \vartheta_1(y + u; \tau) \vartheta_1(-y + u; \tau) \]

\[ = \vartheta_1(y + v; \tau) \vartheta_1(y - v; \tau) \vartheta_1(x + u; \tau) \vartheta_1(x - u; \tau). \]

(1.9)

See [11] for more details of (1.9). Concerning the elementary weight (1.6) and the transition weight (1.7), we note the following three points.

(i) As a complex function of \( t, x, \alpha, \beta \), the elementary weight (1.6) is totally elliptic [19] in the sense that all free parameters \( t, x, \alpha, \) and \( \beta \) viewed as complex variables have equal periods of doubly periodicity.

(ii) By definition of lattice path models, the transition weight (1.7) satisfies the Chapman-Kolmogorov equation

\[ Q(s, x; t, y) = \sum_{z \in \mathbb{Z}} Q(s, x; u, z)Q(u, z; t, y) \]  

(1.10)

for \( 0 < s < u < t, x, y \in \mathbb{Z} \). Schlosser [10] proved that if we use the expression (1.8), a variant of Frenkel and Turaev’s \( V_1 \) summation formula of elliptic hypergeometric functions [4] is derived from (1.11), which is the elliptic extension of Jackson’s very-well-poised balanced \( _8 \phi_7 \) summation formula for \( q \)-hypergeometric functions [5].
(iii) For (1.4), if we take the limit \( \kappa \to \infty \), the elliptic weight-functions (1.6) and (1.8) become trigonometric weight-functions. Then if we put \( \alpha = i\hat{\alpha}, \beta = i\hat{\beta} \) with \( \hat{\alpha}, \hat{\beta} \in \mathbb{R} \), and take the limit \( \hat{\beta} \to \infty \) and then \( \hat{\alpha} \to -\infty \) (or \( \hat{\alpha} \to \infty, \hat{\beta} \to \infty \) in this order), we will have the following,

\[
q(t, x) \to q^{(t-x)/2},
\]

\[
Q(s, x; t, y) \to \left\{ (t-s) + (y-x) \right\}/2 \right\},  
\]

\[
q^{\{(t-s)+(y-x)/(s-x)/4  \}} q^{(t-x)/2},
\]

(1.11)

with \( q = e^{2i/r} \), where the \( q \)-binomial coefficient is defined by

\[
\left[ \begin{array}{c}
\frac{n}{k}
\end{array} \right]_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}.
\]

Moreover, if we take the further limit \( r \to \infty \), then \( q \to 1 \), \( q^{(t-s)/2} \to 1 \) and (1.11) is reduced to the usual binomial coefficient,

\[
Q(s, x; t, y) \to \left\{ (t-s) + (y-x) \right\}/2 \right\},
\]

(1.12)

The above three points verify the fact that Schlosser’s elliptic lattice-path model provides a standard elliptic extension of the binomial theorem and thus it will give a basis to build a theory of elliptic combinatorics [16, 17, 18].

From the viewpoint of probability theory and statistical mechanics, however, we find some difficulties in Schlosser’s elliptic lattice path model and the elliptic binomial theorem. The choice of the weight (1.6) for the elementary rightward step is very suitable in order to apply the addition formula of theta functions (1.9) and thus the transition weight (1.7) can be perfectly factorized as (1.8). On the other hand, we cannot expect any compact formula for the sum

\[
c(s, x; t) = \sum_{y \in \mathbb{Z}} Q(s, x; t, y)
\]

for \( (s, x) \in \Lambda, s < t \in \mathbb{N} \). Since the transition weight (1.8) is inhomogeneous both in space and time contrary to its classical limit (1.11), the sum \( c(s, x; t) \) indeed depends on \( s, x, \) and \( t \). Therefore, if we simply define the transition probability as \( p(s, x; t, y) = Q(s, x; t, y)/c(s, x; t) \), which is well normalized as \( \sum_{y \in \mathbb{Z}} p(s, x; t, y) = 1 \), the Chapman-Kolmogorov equation will not hold for \( p(s, x; t, y) \). In addition to this normalization problem, we have to clarify the positivity condition to define proper transition probability from the transition weight (1.8) in order to construct a stochastic process. In the present paper, we introduce a family of discrete-time excursion processes in \( \mathbb{Z} \) associated with Schlosser’s totally elliptic weight functions (1.6) and (1.8) by solving the above mentioned problems. We consider the excursion processes starting from the origin and returning to the origin in a given time duration \( 2T, T \in \mathbb{N} \). The processes are inhomogeneous both in space and time and hence expected to provide new models in non-equilibrium statistical mechanics. By numerical calculation we show that
the maximum likelihood trajectories of the elliptic excursion processes on the spatio-temporal plane are not straight lines in general and are curved depending on parameters (see Figs. 3 and 6). We will analyze asymptotic probability laws in the long-term limit $T \to \infty$ for a reduced model which we call a simplified trigonometric excursion process. It should be noted that the elementary weight $q(t, x)$ is a simple function of $(t, x)$ as shown in Figs. 2 and 5. Emergence of nontrivial curves on the spatio-temporal plane is highly nontrivial and has not been expected in the previous study.

The paper is organized as follows. In Section 2, first we explain how to derive the spatio-temporal expressions for weights (1.6) and (1.8) from Schlosser’s formulas originally given for the lattice path models in $\mathbb{Z}^2$. The excursion processes are defined for finite time durations $2T, T \in \mathbb{N}$. In Section 2.3, we adopt a $T$-depending parameterization for $\alpha$ and $\beta$ to make the expressions of measures for trajectories be simpler, but this parameterization is not yet essential, since still it contains arbitrary parameters $\alpha_0$ and $\beta_0$. In our expressions using the Jacobi theta functions and trigonometric functions, it is obvious that if $\alpha_0, \beta_0 \in \mathbb{R}, r, \kappa > 0$, the obtained measures for trajectories are real-valued, but they are not non-negative definite; they are signed measures in general. In Section 2.4, we show the reductions to trigonometric and classical measures for trajectories from the elliptic measures by taking proper limits of parameters $r, \alpha_0, \beta_0$ and $\kappa$. We define two levels of trigonometric processes; the trigonometric excursion processes and its simplified version. Section 3 is devoted to giving sufficient conditions to make the measures for trajectories be non-negative for the elliptic excursion processes (Theorem 3.1) and for two kinds of trigonometric excursion processes (Corollaries 3.2 and 3.3). Then the probability measures $\mu_{2T}^{0,0}$ for the excursion processes $X(t), t \in \{0, 1, \ldots, 2T\}$ are well defined. In Section 4, results of numerical calculation are reported for the single-time probability distributions $\mu_{2T}^{0,0}(X(t) = t)$ for each time $t \in \{0, 1, \ldots, 2T\}$. We show that the maximum likelihood trajectories $x = x_{2T}^{\text{max}}(t), t \in \{0, 1, \ldots, 2T\}$ on the spatio-temporal plane exhibit nontrivial curves depending on the parameters and the level of reductions. We studied the processes by increasing the time duration $T$. The numerical results suggest the existence of scaled limit trajectories defined by

$$\lim_{T \to \infty} \left\{ \frac{x_{2T}^{\text{max}}(sT)}{T} : 0 \leq s \leq 2 \right\}$$

(1.13)

for our excursion processes. In Section 5, we concentrate on the simplified trigonometric excursion processes and analyze the asymptotic probability laws in the long-time limit $T \to \infty$. We prove that the large deviation principle is established for $T \to \infty$ by giving an integral representation for the rate function (Lemma 5.1), and characterize the scaled limit trajectories (1.13) (Theorem 5.2). At the time $t = T$, the central limit theorem for the fluctuation of trajectory is also proved (Corollary 5.3). The proofs of theorems are given in Appendices A and B. Concluding remarks are given in Section 6.
2 Preliminaries

2.1 Rewriting of Schlosser’s results

First we explain how to obtain the formulas (1.6) and (1.8) from the results reported in [16]. Schlosser introduced a lattice path model in \( \mathbb{Z}^2 \). The elliptic weight function on horizontal edges \((n - 1, m) \rightarrow (n, m)\) of \( \mathbb{Z}^2 \) is given by Eq. (2.2) in [16], which was expressed by the ‘modified theta function’ (1.2) including arbitrary complex parameters \( a, b, q, p \) with \( q \neq 0 \) and \( |p| < 1 \). (The weight on vertical edges is fixed to be 1.) We put

\[
a = e^{2i\alpha/r}, \quad b = e^{2i\beta/r}, \quad q = e^{2i\tau}, \quad p = e^{2\pi ir} = e^{-2\pi \kappa}
\]  

(2.1)

with \( r > 0 \) and \( \Re \tau \equiv \kappa > 0 \) and use the relation (1.3). Then the weight function on horizontal edges \((n - 1, m) \rightarrow (n, m)\) of \( \mathbb{Z}^2 \) is rewritten using the Jacobi theta functions as

\[
w(n, m) = \frac{\vartheta_1\left( \frac{a + n + 2m}{\pi r}; i\kappa \right) \vartheta_1\left( \frac{\beta + 2n}{\pi r}; i\kappa \right) \vartheta_1\left( \frac{\beta + 2n - 1}{\pi r}; i\kappa \right)}{\vartheta_1\left( \frac{a + n}{\pi r}; i\kappa \right) \vartheta_1\left( \frac{\beta + 2n + m}{\pi r}; i\kappa \right) \vartheta_1\left( \frac{\beta + 2n + m - 1}{\pi r}; i\kappa \right)} \times \frac{\vartheta_1\left( \frac{\alpha - \beta + 1 - n}{\pi r}; i\kappa \right) \vartheta_1\left( \frac{\alpha - \beta - n}{\pi r}; i\kappa \right)}{\vartheta_1\left( \frac{\alpha - \beta + 1 + m - n}{\pi r}; i\kappa \right) \vartheta_1\left( \frac{\alpha - \beta + m - n}{\pi r}; i\kappa \right)}.
\]  

(2.2)

Let \( w(\mathcal{P}(\ell, k) \rightarrow (n, m))) \) be the generating function of paths running from \((\ell, k) \in \mathbb{Z}^2\) to \((n, m) \in \mathbb{Z}^2\) in Schlosser’s lattice path model. The following is just a rewriting of Theorem 2.1 of Schlosser [16] using the Jacobi theta function (1.3) with the parameterization (2.1).

Theorem 2.1 (Schlosser [16]) The recursion relation of the generating function of the lattice paths

\[
w(\mathcal{P}(\ell, k) \rightarrow (n, m))) = w(\mathcal{P}(\ell, k) \rightarrow (n, m - 1))) + w(\mathcal{P}(\ell, k) \rightarrow (n - 1, m)))w(n, m)
\]  

(2.3)

with the weight function (2.2) is solved by

\[
w(\mathcal{P}(\ell, k) \rightarrow (n, m))) = \prod_{u=1}^{n-\ell} \vartheta_1\left( \frac{u}{\pi r}; i\kappa \right) \prod_{u=1}^{m-k} \vartheta_1\left( \frac{r}{\pi r}; i\kappa \right) \prod_{u=\ell+1}^{n} \vartheta_1\left( \frac{a + m + k + u}{\pi r}; i\kappa \right) \times \prod_{u=\ell+1}^{m} \vartheta_1\left( \frac{\alpha - \beta + u}{\pi r}; i\kappa \right) \prod_{u=2\ell+1}^{2n} \vartheta_1\left( \frac{\beta + u}{\pi r}; i\kappa \right) \times \prod_{u=-n+1}^{-\ell} \vartheta_1\left( \frac{\alpha - \beta + u}{\pi r}; i\kappa \right) \prod_{u= \ell+1}^{\beta + \ell + m} \vartheta_1\left( \frac{\beta + \ell + u}{\pi r}; i\kappa \right).
\]  

(2.4)
Proof. Insert (2.2) and (2.4) into (2.3). Note that
\[
\prod_{u=\ell+1}^{n} \vartheta_1 \left( \frac{\alpha + m - 1 + k + u}{\pi r}; i\kappa \right) = \prod_{u=\ell}^{n-1} \vartheta_1 \left( \frac{\alpha + m + k + u}{\pi r}; i\kappa \right),
\]
\[
\prod_{u=-n+1}^{-\ell} \vartheta_1 \left( \frac{\alpha - \beta + m - 1 + u}{\pi r}; i\kappa \right) = \prod_{u=-n}^{\ell-1} \vartheta_1 \left( \frac{\alpha - \beta + m + u}{\pi r}; i\kappa \right),
\]
\[
\prod_{u=\ell+1}^{m} \vartheta_1 \left( \frac{\beta + n - 1 + \ell + u}{\pi r}; i\kappa \right) = \prod_{u=\ell+1}^{m} \vartheta_1 \left( \frac{\beta + n + \ell + u}{\pi r}; i\kappa \right),
\]
and divide both sides of the equation by the common factor
\[
\prod_{u=1}^{n-\ell+m-k-1} \vartheta_1 \left( \frac{u}{\pi r}; i\kappa \right) \prod_{u=m-k+1}^{n} \vartheta_1 \left( \frac{u}{\pi r}; i\kappa \right) \prod_{u=\ell+1}^{n} \vartheta_1 \left( \frac{a+m+k+u}{\pi r}; i\kappa \right)
\times \prod_{u=-n+1}^{-\ell} \vartheta_1 \left( \frac{a-\beta+u}{\pi r}; i\kappa \right) \vartheta_1 \left( \frac{a-\beta-1+u}{\pi r}; i\kappa \right) \vartheta_1 \left( \frac{a-\beta-k+1+u}{\pi r}; i\kappa \right).
\]

Then we obtain the equation
\[
\vartheta_1 \left( \frac{n-\ell+m-k}{\pi r}; i\kappa \right) \vartheta_1 \left( \frac{\beta+n+\ell+m}{\pi r}; i\kappa \right) \vartheta_1 \left( \frac{a-\beta+\ell}{\pi r}; i\kappa \right)
\times \vartheta_1 \left( \frac{a+m+k+\ell}{\pi r}; i\kappa \right) \vartheta_1 \left( \frac{a-\beta-m+n}{\pi r}; i\kappa \right) \vartheta_1 \left( \frac{a-\beta+m-n}{\pi r}; i\kappa \right)
\times \vartheta_1 \left( \frac{a-n}{\pi r}; i\kappa \right) \vartheta_1 \left( \frac{a-\beta+2m+n}{\pi r}; i\kappa \right) \vartheta_1 \left( \frac{a-\beta+2m-n}{\pi r}; i\kappa \right) \vartheta_1 \left( \frac{a-\beta+m-n}{\pi r}; i\kappa \right).
\]

If we multiply both sides by
\[
\vartheta_1 \left( \frac{n-\ell}{\pi r}; i\kappa \right) \vartheta_1 \left( \frac{m-k}{\pi r}; i\kappa \right) \vartheta_1 \left( \frac{\alpha + m + k + n}{\pi r}; i\kappa \right)
\times \vartheta_1 \left( \frac{\beta + 2n + m}{\pi r}; i\kappa \right) \vartheta_1 \left( \frac{\alpha - \beta + m - n}{\pi r}; i\kappa \right),
\]

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then we obtain the equation

\[
\vartheta_1 \left( \frac{n - \ell + m - k}{\pi r}; i\kappa \right) \vartheta_1 \left( \frac{\alpha + m + k + n}{\pi r}; i\kappa \right) \\
\times \vartheta_1 \left( \frac{\beta + n + \ell + m}{\pi r}; i\kappa \right) \vartheta_1 \left( \frac{\alpha - \beta + m - n}{\pi r}; i\kappa \right) \\
= \vartheta_1 \left( \frac{m - k}{\pi r}; i\kappa \right) \vartheta_1 \left( \frac{\alpha + m + k + \ell}{\pi r}; i\kappa \right) \\
\times \vartheta_1 \left( \frac{\beta + 2n + m}{\pi r}; i\kappa \right) \vartheta_1 \left( \frac{\alpha - \beta + m - \ell}{\pi r}; i\kappa \right) \\
+ \vartheta_1 \left( \frac{n - \ell}{\pi r}; i\kappa \right) \vartheta_1 \left( \frac{\alpha + n + 2m}{\pi r}; i\kappa \right) \\
\times \vartheta_1 \left( \frac{\beta + n + \ell + k}{\pi r}; i\kappa \right) \vartheta_1 \left( \frac{\alpha - \beta + k - n}{\pi r}; i\kappa \right).
\]

This is a special case of the addition formula of theta functions (1.9) with

\[x = \frac{1}{2\pi r}(\alpha + 2m + \ell), \quad y = \frac{1}{2\pi r}(\alpha + 2m + 2n - \ell),\]

\[u = \frac{1}{2\pi r}(2\beta - \alpha + 2n + \ell), \quad v = -\frac{1}{2\pi r}(\alpha + \ell + 2k),\]

and \(\tau = i\kappa\), since \(\vartheta_1(-v; \tau) = -\vartheta_1(v; \tau)\). Thus the proof is completed.

The weight (1.6) for the elementary rightward step \((t - 1, x - 1) \to (t, x)\) and the transition weight (1.8) from \((s, x)\) to \((t, y)\) in the spatio-temporal plane (1.5) are obtained from the above as

\[q(t, x) = w((t + x)/2, (t - x)/2),\]

\[Q(s, x; t, y) = w(P(((s + x)/2, (s - x)/2) \to ((t + y)/2, (t - y)/2))).\]

### 2.2 Excursion processes

Assume that

\[T \in \mathbb{N}.\] (2.5)

Consider an excursion process on \(\mathbb{Z}\) which starts from the origin \(0\) at time \(t = 0\) and returns to the origin \(0\) at time \(t = 2T\); \(X(t), t \in \{0, 1, \ldots, 2T\}\). Denote the measure for trajectories of this excursion process by \(\nu_{2T}\). For an arbitrary integer \(M \in \mathbb{N}\) and for an arbitrary strictly increasing series of times \(\{t_m \in \mathbb{N}_0 : m = 0, 1, \ldots, M\}\) such that

\[0 = t_0 < t_1 < t_2 < \cdots < t_M \leq 2T,\] (2.6)

if

\[x^{(m)} \in \mathbb{Z}, \quad t_m + x^{(m)} \in 2\mathbb{Z}, \quad \text{for } m = 1, 2, \ldots, M,\] (2.7)
then the multi-time joint measure of \( X(t), t \in \{0, 1, \ldots, 2T\} \) is given by

\[
P_{2T}^0(X(t_m) = x^{(m)}, m \in \{1, 2, \ldots, M\}) = \prod_{m=1}^{M} Q(t_{m-1}, x^{(m-1)}; t_m, x^{(m)}) \frac{Q(t_M, x^{(M)}; 2T, 0)}{Q(0, 0; 2T, 0)}.
\] (2.8)

If the condition (2.7) is not satisfied, then we assume

\[
P_{2T}^0(X(t_m) = x^{(m)}, m \in \{1, 2, \ldots, M\}) = 0.
\] (2.9)

By the Chapman-Kolmogorov equation (1.10) for \( Q(s, x; t, y) \),

\[
\sum_{x^{(m')} \in \mathbb{Z}} \sum_{x^{(m)} \in \mathbb{Z}} P_{2T}^0(X(t_m) = x^{(m)}, m \in \{1, 2, \ldots, M\}) = P_{2T}^0(X(t_1) = x^{(1)})
\]

\[
= Q(0, 0; t_1, x^{(1)}; 2T, 0) \frac{Q(t_1, x^{(1)}; 2T, 0)}{Q(0, 0; 2T, 0)}.
\]

Again by the Chapman-Kolmogorov equation (1.10) for \( Q(s, x; t, y) \), we have

\[
\sum_{x^{(1)} \in \mathbb{Z}} P_{2T}^0(X(t_1) = x^{(1)}) = 1.
\] (2.10)

That is, this measure \( P_{2T}^0 \) is well normalized. When \( \alpha, \beta \in \mathbb{R}, \ r > 0, \kappa > 0 \), the measure \( P_{2T}^0(\cdot) = P_{2T}^0(\cdot; r, \alpha, \beta, \kappa) \) is real-valued, but it is signed in general.

Excursion processes are also called bridges, pinned processes, or tied-down processes. See Part I. Section IV.4 in [1] for a Brownian bridge.

### 2.3 Parameterization for excursion processes

An explicit expression for the single-time measure is obtained by applying (1.8) to the formula

\[
P_{2T}^0(X(t) = x) = Q(0, 0; t, x) \frac{Q(t, x; 2T, 0)}{Q(0, 0; 2T, 0)}, \quad t \in \{0, 1, \ldots, 2T\}.
\] (2.11)

We found that the following parameterization makes the expression much simpler,

\[
\alpha = \alpha_0 - \frac{1}{2}(3T + 1), \quad \beta = \beta_0 - \frac{1}{2}(3T + 1),
\] (2.12)
where \( \alpha_0 \) and \( \beta_0 \) are still arbitrary. Let

\[
\Lambda_{2T} = \{(t, x) \in \Lambda : 0 \leq t \leq T, -t \leq x \leq t\}
\]

\[
\cup \{(t, x) \in \Lambda : T + 1 \leq t \leq 2T, t - 2T \leq x \leq -t + 2T\}.
\]

The obtained expression is the following,

\[
\mathbb{P}_{2T}^0(X(t) = x) = \mathbb{P}_{2T}^0(X(t) = x; r, \alpha_0, \beta_0, \kappa)
\]

\[
= \begin{cases} 
0, & \text{if } (t, x) \in \Lambda_{2T}, \\
\prod_{n=1}^{(t+x)/2} \vartheta_1 \left( \frac{n \pi}{\pi \tau}, i\kappa \right) \prod_{n=1}^{(t-x)/2} \vartheta_1 \left( \frac{n \pi}{\pi \tau}, i\kappa \right) \prod_{n=1}^{(t+2T-x)/2} \vartheta_1 \left( \frac{n \pi}{\pi \tau}, i\kappa \right) \prod_{n=1}^{(t-x+2T)/2} \vartheta_1 \left( \frac{n \pi}{\pi \tau}, i\kappa \right) & \text{if } (t, x) \notin \Lambda_{2T},
\end{cases}
\]

where

\[
c_1(t; T) = \frac{\prod_{n=1}^t \vartheta_1 \left( \frac{n \pi}{\pi \tau}, i\kappa \right) \prod_{n=1}^{T-t} \vartheta_1 \left( \frac{n \pi}{\pi \tau}, i\kappa \right) \prod_{n=1}^T \vartheta_1 \left( \frac{n \pi}{\pi \tau}, i\kappa \right)}{\vartheta_1 \left( \frac{\alpha_0 - \beta_0}{\pi \tau}, i\kappa \right) \prod_{n=1}^{T-t} \vartheta_1 \left( \frac{n \pi}{\pi \tau}, i\kappa \right) \prod_{n=1}^{T-t} \vartheta_1 \left( \frac{n \pi}{\pi \tau}, i\kappa \right) \prod_{n=1}^{T-t} \vartheta_1 \left( \frac{n \pi}{\pi \tau}, i\kappa \right)} \times \prod_{n=1}^T \left\{ \vartheta_1 \left( \frac{n \pi}{\pi \tau}, i\kappa \right) \right\}^2 \vartheta_1 \left( \frac{\alpha_0 - \beta_0}{\pi \tau}, i\kappa \right) \vartheta_1 \left( \frac{\alpha_0 - \beta_0}{\pi \tau}, i\kappa \right) \vartheta_1 \left( \frac{\alpha_0 - \beta_0}{\pi \tau}, i\kappa \right).
\]

From now on, we assume the parameterization (2.12) with \( \alpha_0, \beta_0 \in \mathbb{R}, r, \kappa > 0 \) for the signed measure \( \mathbb{P}_{2T}^0(\cdot) = \mathbb{P}_{2T}^0(\cdot; r, \alpha_0, \beta_0, \kappa) \).

2.4 Reduction to trigonometric and classical measures

By the form (1.6) and the asymptotic property (1.4) of \( \vartheta_1 \),

\[
\tilde{q}(t, x; r, \alpha_0, \beta_0) \equiv \lim_{\kappa \to \infty} q(t, x; r, \alpha_0, \beta_0, \kappa)
\]

is well defined (see Section 3.2). Let \( \mathbb{P}_{2T}^0(\cdot; r, \alpha_0, \beta_0) \) be the signed measure obtained from \( \mathbb{P}_{2T}^0(\cdot; r, \alpha_0, \beta_0, \kappa) \) by taking the limit \( \kappa \to \infty \). For example, the single-time measure (2.14).
gives
\[ \bar{\mathbb{P}}_{2T}^{0,0}(X(t) = x) = \bar{\mathbb{P}}_{2T}^{0,0}(X(t) = x; r, \alpha_0, \beta_0) \]
\[
\begin{cases}
\hat{c}_1(t; T) \sin \left( \frac{\alpha_0 - \beta_0 - x}{2r} \right) \\
\cdot \prod_{n=1}^{(t+x)/2} \sin \left( \frac{2\alpha_0 - (2n + 3T - 2t - 1)}{2r} \right) \\
\cdot \prod_{n=1}^{(t-x)/2} \sin \left( \frac{2\beta_0 - (2n + 3T - 2t - 1)}{2r} \right)
\end{cases}
\]
\[
\hat{c}_1(t; T) = \prod_{\alpha=1}^{T} \sin \left( \frac{\alpha}{r} \right) \prod_{n=1}^{2T} \sin \left( \frac{\alpha_0 - \beta_0 - n}{2r} \right)
\]
\[
\cdot \prod_{n=1}^{T} \sin \left( \frac{\alpha_0 - \beta_0 - n}{2r} \right) \prod_{n=1}^{T} \sin \left( \frac{\alpha_0 - \beta_0 + n}{2r} \right)
\]
\[
\text{if } (t, x) \in \Lambda_{2T},
\]
\[
\text{otherwise},
\]
\[
\hat{c}_1(t; T) = \frac{\prod_{n=1}^{t} \sin \left( \frac{\alpha}{r} \right) \prod_{n=1}^{2T-t} \sin \left( \frac{\alpha}{r} \right) \prod_{n=1}^{T} \sin \left( \frac{\alpha}{r} \right)}{\prod_{n=1}^{2T} \sin \left( \frac{\alpha}{r} \right) \prod_{n=1}^{T} \sin \left( \frac{\alpha}{r} \right) \prod_{n=1}^{T} \sin \left( \frac{\alpha}{r} \right)}
\]
\[
\text{if } (t, x) \in \Lambda_{2T},
\]
\[
\text{otherwise},
\]

Now we consider the further limit. We see that
\[
\tilde{q}(t, x; r, \alpha_0) = \lim_{\beta_0 \to \infty} \tilde{q}(t, x; r, \alpha_0, i\beta_0) = \frac{\sin \left[ \{2\alpha_0 - (3T + 1) + (3t - x)\} / 2r \right]}{\sin \left[ \{2\alpha_0 - (3T + 1) + (t + x)\} / 2r \right]}.
\]

We write the corresponding signed measure as \( \bar{\mathbb{P}}_{2T}^{0,0} \) with (2.16) gives
\[
\bar{\mathbb{P}}_{2T}^{0,0}(X(t) = x) = \bar{\mathbb{P}}_{2T}^{0,0}(X(t) = x; r, \alpha_0)
\]
\[
\begin{cases}
\hat{c}_1(t; T) \prod_{n=1}^{(t+x)/2} \sin \left( \frac{2\alpha_0 - (2n + 3T - 2t - 1)}{2r} \right) \\
\cdot \prod_{n=1}^{(t-x)/2} \sin \left( \frac{2\beta_0 - (2n + 3T - 2t - 1)}{2r} \right)
\end{cases}
\]
\[
\hat{c}_1(t; T) = \prod_{\alpha=1}^{T} \sin \left( \frac{\alpha}{r} \right) \prod_{n=1}^{2T-t} \sin \left( \frac{\alpha}{r} \right) \prod_{n=1}^{T} \sin \left( \frac{\alpha}{r} \right)
\]
\[
\cdot \prod_{n=1}^{t} \sin \left( \frac{\alpha}{r} \right) \prod_{n=1}^{T} \sin \left( \frac{\alpha}{r} \right) \prod_{n=1}^{T} \sin \left( \frac{\alpha}{r} \right)
\]
\[
\text{if } (t, x) \in \Lambda_{2T},
\]
\[
\text{otherwise},
\]

We can show that if we put \( \alpha_0 = i\tilde{\alpha}_0 \) in (2.16) and (2.17) and take the limit \( \tilde{\alpha}_0 \to \infty \), then we obtain the result (2.18) and (2.19) with the following replacement,
\[
\alpha_0 \to \beta_0, \quad x \to -x.
\]
Figure 1: The collection $\Lambda_{2T}^*$ of points $(t, x)$ such that any trajectory of the excursion process from 0 to 0 with time duration $2T$ can contain the rightward step $(t - 1, x - 1) \to (t, x)$.

For the trigonometric measures $\mathbb{P}^{0,0}_{2T}$ and $\tilde{\mathbb{P}}^{0,0}_{2T}$, if we take the further limit $r \to \infty$, we will obtain a measure $\mathbb{P}^{0,0}_{2T,cl}$ such that

$$
\mathbb{P}^{0,0}_{2T,cl}(X(t) = x) = \begin{cases} 
\frac{t}{(2T - t)} \left( \frac{2T - t}{2} \right) \left( \frac{(2T - t) + x}{2} \right), & \text{if } (t, x) \in \Lambda_{2T}, \\
0, & \text{otherwise.}
\end{cases}
$$

This gives the single-time probability measure for the excursion process of the classical random walk (that is, the simple and symmetric random walk) on $\mathbb{Z}^2$ starting from 0 and returning to 0 with time duration $2T$.

3 Positivity Conditions for Measures

3.1 Elliptic excursion processes

With the parameterization (2.12), the elementary weight-function $q(t, x)$ given by (1.6) becomes the following,

$$
q(t, x) = q(t, x; r, \alpha_0, \beta_0, \kappa) = \prod_{j=1}^{5} \frac{\vartheta_1(\zeta_j/\pi; i\kappa)}{\vartheta_1(\eta_j/\pi; i\kappa)},
$$

(3.1)
where

\[ \begin{align*}
\zeta_1 &= \frac{\alpha_0}{r} - \frac{3T + 1}{2r} + \frac{3t - x}{2r}, \\
\zeta_2 &= \frac{\beta_0}{r} - \frac{3T + 1}{2r} + \frac{t + x}{2r}, \\
\zeta_3 &= \zeta_2 - \frac{1}{r}, \\
\zeta_4 &= \frac{\alpha_0 - \beta_0}{r} - \frac{t + x}{2r}, \\
\zeta_5 &= \zeta_4 + \frac{1}{r},
\end{align*} \]

(3.2)

and

\[ \begin{align*}
\eta_1 &= \frac{\alpha_0}{r} - \frac{3T + 1}{2r} + \frac{t + x}{2r}, \\
\eta_2 &= \frac{\beta_0}{r} - \frac{3T + 1}{2r} + \frac{3t + x}{2r}, \\
\eta_3 &= \eta_2 - \frac{1}{r}, \\
\eta_4 &= \frac{\alpha_0 - \beta_0}{r} - \frac{x}{2r}, \\
\eta_5 &= \eta_4 + \frac{1}{r}.
\end{align*} \]

(3.3)

On the spatio-temporal plane \( \Lambda \), the collection of all points \((t, x)\) such that any trajectory of the present excursion process from 0 to 0 with time duration \(2T\) can contain the rightward step \((t - 1, x - 1) \to (t, x)\) weighted with (3.1) is given by

\[ \Lambda_{2T}^* = \{(t, x) \in \Lambda : 1 \leq t \leq T, -t + 2 \leq x \leq t \} \]

\[ \cup \{(t, x) \in \Lambda : T + 1 \leq t \leq 2T, t - 2T \leq x \leq -t + 2T \}. \]

(3.4)

See Fig. 1. If

\[ q(t, x) \geq 0, \quad \forall (t, x) \in \Lambda_{2T}^*. \]

(3.5)

then the transition weight (1.8) is non-negative, and hence the multi-time joint measure (2.8) is non-negative definite. In this case \( \mathbb{P}_{0, 0}^{2T} \) gives a probability measure and the excursion process \( \{X(t)\}_{t \in \{0, 1, \ldots, 2T\}} \) is well-defined.

**Theorem 3.1** Put

\[ \lambda = \frac{3T - 1}{2\pi r}. \]

(3.6)

The following are sufficient conditions so that \( \mathbb{P}_{2T}^{0, 0} \) is non-negative and gives a probability measure for the excursion process \( X(t), t \in \{0, 1, \ldots, 2T\} \),

\[ 0 \leq \lambda < \frac{1}{2}, \]

(3.7)

\[ \pi \lambda < \frac{\alpha_0}{r} < \pi (1 - \lambda), \]

(3.8)

\[ \pi \lambda < \frac{-\beta_0}{r} < \pi (1 - \lambda), \]

(3.9)

\[ \frac{\alpha_0 - \beta_0}{r} < \pi \left\{ 1 - \frac{2(T + 1)}{3T - 1} \lambda \right\}. \]

(3.10)
Proof. In the region $\Lambda^*_2T, T \in \mathbb{N}$, given by (3.11), the following inequalities hold,

$$1 \leq t \leq T, \quad x \geq -t + 2, \quad x \leq t,$$

or

$$T + 1 \leq t \leq 2T, \quad x \geq t - 2T, \quad x \leq -t + 2T. \quad (3.12)$$

When the inequalities (3.11) hold,

$$-t \leq -x \leq t - 2 \quad \Rightarrow \quad 2t \leq 3t - x \leq 4t - 2,$$

$$2 \leq t + x \leq 2t \quad \Rightarrow \quad 2 \leq t + x \leq 2T. \quad (3.13)$$

By the first inequality of (3.11), $2 \leq 2t \leq 2T$. Combining this with (3.13), we have

$$4 \leq 3t + x \leq 4T. \quad (3.14)$$

Similarly, when the inequalities (3.12) hold, we have

$$6 \leq 3t - x \leq 6T, \quad 2 \leq t + x \leq 2T, \quad 2T + 4 \leq 3t + x \leq 6T. \quad (3.15)$$

Therefore, the following inequalities hold in $\Lambda^*_2T, T \in \mathbb{N}$,

$$2 \leq 3t - x \leq 6T, \quad 2 \leq t + x \leq 2T, \quad 4 \leq 3t + x \leq 6T. \quad (3.16)$$

By the first and the second inequalities in (3.16), we have

$$\frac{\alpha_0}{r} - \frac{3T - 1}{2r} \leq \zeta_1 \leq \frac{\alpha_0}{r} + \frac{3T - 1}{2r}, \quad \frac{\alpha_0}{r} - \frac{3T - 1}{2r} \leq \eta_1 \leq \frac{\alpha_0}{r} + \frac{3T - 1}{2r} - \frac{2T}{r}. \quad (3.17)$$

Hence, if

$$0 < \frac{\alpha_0}{r} - \frac{3T - 1}{2r} \quad \text{and} \quad \frac{\alpha_0}{r} + \frac{3T - 1}{2r} < \pi$$

$$\iff \frac{3T - 1}{2r} < \frac{\alpha_0}{r} < \pi - \frac{3T - 1}{2r}, \quad (3.17)$$

then $0 < \zeta_1 < \pi, 0 < \eta_1 < \pi$, and

$$0 < \frac{\vartheta_1(\zeta_1/\pi; i\kappa)}{\vartheta_1(\eta_1/\pi; i\kappa)} < \infty. \quad (3.18)$$

Similarly, by the second and the third inequalities in (3.16), we have

$$\frac{\beta_0}{r} - \frac{3T - 1}{2r} + \frac{1}{r} \leq \zeta_2 \leq \frac{\beta_0}{r} + \frac{3T - 1}{2r} - \frac{T}{r}, \quad \frac{\beta_0}{r} - \frac{3T - 1}{2r} \leq \zeta_3 \leq \frac{\beta_0}{r} + \frac{3T - 1}{2r} - \frac{T - 1}{r}, \quad (3.19)$$

$$\frac{\beta_0}{r} - \frac{3T - 1}{2r} + \frac{1}{r} \leq \eta_2 \leq \frac{\beta_0}{r} + \frac{3T - 1}{2r}, \quad \frac{\beta_0}{r} - \frac{3T - 1}{2r} \leq \eta_3 \leq \frac{\beta_0}{r} + \frac{3T - 1}{2r} - \frac{1}{r}. \quad (3.20)$$
Hence, if
\[-\pi < -\frac{\beta_0}{r} - \frac{3T - 1}{2r} \quad \text{and} \quad -\frac{\beta_0}{r} - \frac{3T - 1}{2r} < 0\]
\[\iff \quad \frac{3T - 1}{2r} < \frac{\beta_0}{r} < \pi - \frac{3T - 1}{2r},\] (3.19)
then \(-\pi < \zeta_j < 0, -\pi < \eta_j < 0\) for \(j = 2, 3\) and
\[0 < \frac{\vartheta_1(\zeta_4/\pi; i\kappa)}{\vartheta_1(\zeta_5/\pi; i\kappa)} < \infty.\] (3.20)

By the second inequality in (3.16) and the inequality \(-T \leq x \leq T\) satisfied in \(\Lambda_{2T}^*, T \in \mathbb{N}\), we have
\[\frac{\alpha_0 - \beta_0}{r} - \frac{T}{r} \leq \zeta_4 \leq \frac{\alpha_0 - \beta_0}{r} - \frac{1}{r},\]
\[\frac{\alpha_0 - \beta_0}{r} - \frac{T}{r} + \frac{1}{r} \leq \zeta_5 \leq \frac{\alpha_0 - \beta_0}{r},\]
\[\frac{\alpha_0 - \beta_0}{r} - \frac{T}{r} \leq \eta_4 \leq \frac{\alpha_0 - \beta_0}{r} + \frac{T}{r},\]
\[\frac{\alpha_0 - \beta_0}{r} - \frac{T}{r} + \frac{1}{r} \leq \eta_5 \leq \frac{\alpha_0 - \beta_0}{r} + \frac{T + 1}{r}.\]

Hence, if
\[0 < \frac{\alpha_0 - \beta_0}{r} - \frac{T}{r} \quad \text{and} \quad \frac{\alpha_0 - \beta_0}{r} + \frac{T + 1}{r} < \pi\]
\[\iff \quad \frac{T}{r} < \frac{\alpha_0 - \beta_0}{r} < \pi - \frac{T + 1}{r},\] (3.21)
then \(0 < \zeta_j < \pi, 0 < \eta_j < \pi\) for \(j = 4, 5\) and
\[0 < \frac{\vartheta_1(\zeta_4/\pi; i\kappa)}{\vartheta_1(\zeta_5/\pi; i\kappa)} < \infty.\] (3.22)

Using \(\lambda\) defined by (3.6), the conditions (3.17) and (3.19) are expressed as (3.18)-(3.19). If we combine (3.17) and (3.19), we obtain
\[\frac{3T - 1}{r} < \frac{\alpha_0 - \beta_0}{r} < 2\pi - \frac{3T - 1}{r}.\]

For \(T \in \mathbb{N}, T/r < (3T - 1)/r\). We can also verify that if \(\lambda < 1/2\), then
\[2\pi - \frac{3T - 1}{r} > \pi - \frac{T + 1}{r} = \pi \left\{1 - \frac{2(T + 1)}{3T - 1}\lambda\right\}.
\]
Therefore, if (3.10) is satisfied in addition to (3.17)-(3.19), (3.21) holds. Thus (3.18), (3.20), and (3.22) imply that the elementary weight-function \(q(t, x)\) given by (3.1) satisfies (3.5). The proof is complete.
We call \( \{X(t)\}_{t \in \{0, 1, \ldots, 2T\}} \) with the parameters satisfying the conditions (3.7)-(3.10) the \textit{elliptic excursion process} with time duration \( 2T \).

As implied by the above proof, \( \alpha_0 \) and \( \beta_0 \) can be interchanged in (3.8)-(3.10) to obtain another set of sufficient conditions for non-negative \( P_{2T}^{0,0} \), which results in reflection of excursion trajectories as \( x \leftrightarrow -x \).

### 3.2 Trigonometric excursion processes

By (1.4), we see that
\[
\hat{q}(t, x) = \hat{q}(t, x; r, \alpha_0, \beta_0) \equiv \lim_{\kappa \to \infty} q(t, x; r, \alpha_0, \beta_0, i\kappa) = \prod_{j=1}^{5} \frac{\sin \zeta_j}{\sin \eta_j}, \tag{3.23}
\]
with (3.2) and (3.3). If \( 0 \leq \zeta \leq \pi \), then \( 0 \leq \vartheta_1(\zeta/\pi; i\kappa) < \infty \) for \( 0 < \kappa < \infty \) and \( 0 \leq \sin \zeta \leq 1 \), while if \( -\pi \leq \zeta \leq 0 \), then \( 0 \leq \vartheta_1(\zeta/\pi; i\kappa) < \infty \) for \( 0 < \kappa < \infty \) and \( 0 \leq \sin \zeta \leq 1 \). Therefore, the proof of Theorem 3.1 implies the following corollary.

\textbf{Corollary 3.2} If the conditions (3.7)-(3.10) are satisfied, then \( \hat{P}_{2T}^{0,0} \) gives a probability measure for the excursion process \( X(t), t \in \{0, 1, \ldots, 2T\} \).

We call \( \{X(t)\}_{t \in \{0, 1, \ldots, 2T\}}, \hat{P}_{2T}^{0,0} \) with the parameters satisfying the conditions (3.7)-(3.10) the \textit{trigonometric excursion process} with time duration \( 2T \).

### 3.3 Simplified trigonometric excursion processes

For (3.23), we see that
\[
\tilde{q}(t, x) = \tilde{q}(t, x; r, \alpha_0) \equiv \lim_{\hat{\beta}_0 \to \infty} \hat{q}(t, x; t, \alpha_0, i\hat{\beta}_0) = \frac{\sin \zeta_1}{\sin \eta_1}, \tag{3.24}
\]
where \( \zeta_1 \) and \( \eta_1 \) are given by (3.2) and (3.3).

\textbf{Corollary 3.3} Assume that the conditions (3.7) and (3.8) are satisfied. Then \( \tilde{P}_{2T}^{0,0} \) gives a probability measure for the excursion process \( X(t), t \in \{0, 1, \ldots, 2T\} \).

We call \( \{X(t)\}_{t \in \{0, 1, \ldots, 2T\}}, \tilde{P}_{2T}^{0,0} \) with the parameters satisfying the conditions (3.7) and (3.8) the \textit{simplified trigonometric excursion process} with time duration \( 2T \).

### 4 Numerical Study of Trajectories

As shown at the end of Section 2.4, if we take the limit \( r \to \infty \) with a fixed finite \( T < \infty \), the present process is reduced to the excursion process of a classical random walk, whose
trajectory is distributed according to (2.20). Here we consider the case such that the long-term limit $T \to \infty$ is taken as well as $r \to \infty$. Using Stirling’s formula, $n! \sim \sqrt{2\pi n} n^ne^{-n}$ in $n \to \infty$, it is easy to verify that

$$
\lim_{T \to \infty} \frac{\sqrt{T}}{2} e^{0.02T,cl}(X(Ts)) = \sqrt{T} \xi
$$

$$
= \frac{1}{\sqrt{\pi s(2-s)}} e^{-\xi^2/(s(2-s))}, \quad s \in (0, 2), \quad \xi \in \mathbb{R}.
$$

(4.1)

This is nothing but the probability density at time $s$ of the Brownian bridge (see, for instance, Part I. Section IV.4 in [1]) starting from 0 and returning to 0 with time duration 2.

In this section, we report the numerical study of the long-term behavior in $T \to \infty$ of the present elliptic and trigonometric excursion processes. In order to realize any non-classical behavior in $T \to \infty$, the parameter $\lambda$ introduced as (3.6) should be non-zero. It implies that we have to take the double limit $r \to \infty$, $T \to \infty$ in which $r$ should be proportional to $T$ at least asymptotically in this limit. In this section, we assume a proportional relation

$$
\pi r = \sigma T,
$$

(4.2)

with a fixed factor $\sigma$. Since

$$
\lambda < \frac{3T}{2\pi r} = \frac{3}{2\sigma},
$$

(4.3)

the condition (3.7) of Theorem 3.1 is satisfied, if

$$
\sigma \geq 3.
$$

(4.4)

Moreover, by (4.3) and the fact that $1/3 < (T + 1)/(3T - 1) \leq 1$ for $T \in \mathbb{N}$, we can verify that if

$$
\frac{3}{2\sigma} \pi \leq \frac{\alpha_0}{r} \leq \left(1 - \frac{3}{2\sigma}\right) \pi,
$$

(4.5)

$$
\frac{3}{2\sigma} \pi \leq -\frac{\beta_0}{r} \leq \left(1 - \frac{3}{2\sigma}\right) \pi,
$$

(4.6)

$$
\frac{\alpha_0 - \beta_0}{r} \leq \left(1 - \frac{3}{\sigma}\right) \pi,
$$

(4.7)

then the conditions (3.8)-(3.10) of Theorem 3.1 are satisfied.

4.1 Simplified trigonometric excursion processes

The conditions of Corollary 3.3 are satisfied, if (4.4) and (4.5) are valid. As mentioned above, the parameter $\lambda$ will represent the degree of deviation from the classical processes; the $\lambda \to 0$ limit corresponds to the classical cases. Here we will concentrate on the case in which $\lambda$ is
maximized, that is, the parameter $\sigma$ takes its minimal value $\sigma = 3$. Then the condition (4.5) determines the value of $\alpha_0/r$ uniquely as follows,

$$\sigma = 3, \quad \frac{\alpha_0}{r} = \frac{\pi}{2}. \quad (4.8)$$

Figure 2 shows the elementary weight $\bar{q}(t, x)$ as a function of $(t, x)$ in $\Lambda^*_2 T$ for $T = 100$. The dependence on $(t, x)$ is very simple. In this special parameterization, (2.18) and (2.19) are written as follows,

$$\bar{c}_2(t; T) = \left\{ \begin{array}{ll}
\prod_{n=1}^{(t+x)/2} \cos \left( \frac{(2n+3T-2t-1)\pi}{6T} \right) 
\prod_{n=1}^{(t-x)/2} \sin \left( \frac{n\pi}{3T} \right) 
\prod_{n=1}^{(2T-t)-x/2} \cos \left( \frac{(2n+2T-T-1)\pi}{6T} \right) 
\prod_{n=1}^{(2T-t)+x/2} \sin \left( \frac{n\pi}{3T} \right) 
\prod_{n=1}^{2T} \sin \left( \frac{n\pi}{3T} \right) \cos \left( \frac{(2n-T-1)\pi}{6T} \right), & \text{if } (t, x) \in \Lambda_{2T}, \\
0, & \text{otherwise},
\end{array} \right. \quad (4.9)$$

where

$$\bar{c}_2(t; T) = \frac{\prod_{n=1}^{t} \sin \left( \frac{n\pi}{3T} \right) \prod_{n=1}^{2T-t} \sin \left( \frac{n\pi}{3T} \right) \prod_{n=1}^{T} \sin^2 \left( \frac{n\pi}{3T} \right) \prod_{n=1}^{2T} \cos \left( \frac{(2n-T-1)\pi}{6T} \right)}{\prod_{n=1}^{2T} \sin \left( \frac{n\pi}{3T} \right) \cos \left( \frac{(2n-T-1)\pi}{6T} \right)}. \quad (4.10)$$

By definition of excursion processes

$$\bar{P}_{2T}^0(X(t) = x) = \bar{P}_{2T}^0(X(2T) = x) = \delta_{x0}.$$
Figure 3: Time-evolution of the probability distribution $\tilde{P}_{2T}^{0,0}(X(t) = x)$ of the simplified trigonometric excursion process with $T = 100$. The maximum likelihood trajectory is reverse S-shaped.

In the expression (4.9) it is obvious to see the following symmetry,

$$\tilde{P}_{2T}^{0,0}(X(t) = x) = \tilde{P}_{2T}^{0,0}(X(2T - t) = -x), \quad 0 \leq t \leq 2T, \quad x \in \mathbb{Z}. \quad (4.11)$$

Hence, at $t = T$ the distribution should be symmetric,

$$\tilde{P}_{2T}^{0,0}(X(T) = x) = \tilde{P}_{2T}^{0,0}(X(T) = -x), \quad x \in \mathbb{Z}.$$ 

Figure 3 shows the time-dependence of $\tilde{P}_{2T}^{0,0}(X(t) = x)$ given by (4.9) and (4.10) for $T = 100$. This figure clarifies that at each time $t \in \{0, 1, \ldots, 2T\}$ there is a single peak at a point which we denote by $\tilde{x}_{2T}^{\text{max}}(t)$, and

$$\tilde{x}_{2T}^{\text{max}}(t) < 0 \quad \text{for} \ 1 \leq t \leq T - 1,$$

$$\tilde{x}_{2T}^{\text{max}}(t) = 0 \quad \text{at} \ t = T,$$

$$\tilde{x}_{2T}^{\text{max}}(t) > 0 \quad \text{for} \ T + 1 \leq t \leq 2T - 1.$$

The line $\{\tilde{x}_{2T}^{\text{max}}(t) : 0 \leq t \leq 2T\}$ expresses the maximum likelihood trajectory of the simplified trigonometric excursion process. As shown by Fig. 3 it is reverse S-shaped.

Figure 4 shows $\{\tilde{x}_{2T}^{\text{max}}(sT)/T : 0 \leq s \leq 2\}$ for $T = 100, 150, 200, 250$. We find accumulation of points with different values of $T$ into a curve, which is called data-collapse in the scaling plots studied in statistical mechanics. It suggests that the values of $\tilde{x}_{2T}^{\text{max}}(t)$ are proportional to $T$ in $T \to \infty$ and thus the following scaled limit trajectory exists,

$$\{\tilde{v}(s) : s \in [0, 2]\} = \lim_{T \to \infty} \left\{ \frac{\tilde{x}_{2T}^{\text{max}}(sT)}{T} : 0 \leq s \leq 2 \right\},$$

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Figure 4: Numerical values of $\tilde{x}^{\text{max}}_T(t)/T$, $t \in \{0,1,\ldots,2T\}$ are plotted for various time durations, $T = 100$ (circles $\bigcirc$), 150 (crosses $+$), 200 (triangles $\triangle$), 250 (squares $\blacksquare$). The observed data-collapse suggests the existence of a limit curve $\tilde{v}(t), t \in [0, 2]$ for the simplified trigonometric excursion process in the long-term limit $T \to \infty$.

and it exhibits a nontrivial curve. Due to (4.11), it has the following symmetry,

$$\tilde{v}(t) = -\tilde{v}(2 - t), \quad t \in [0, 2],$$

which proves $\tilde{v}(1) = 0$.

4.2 Trigonometric excursion processes

For the trigonometric excursion process ($\{X(t)\}_{t \in \{0,1,\ldots,2T\}}$, $\tilde{x}^{0,0}_T$), we have to choose the parameters satisfying the conditions (4.5)-(4.7). Since the combination of (4.5) and (4.6) gives

$$\frac{3}{\sigma} \pi \leq \left(1 - \frac{3}{\sigma}\right) \pi \iff \sigma \geq 6.$$

Here we choose the smallest value $\sigma = 6$ to make $\lambda$ be the largest. In this case (4.5)-(4.7) determine the other parameters uniquely as

$$\sigma = 6, \quad \frac{\alpha_0}{r} = -\frac{\beta_0}{r} = \frac{\pi}{4}.$$

(4.12)

Figure 5 shows the elementary weight $\tilde{q}(t, x)$ as a function of $(t, x)$ in $\Lambda^*_T$ for $T = 100$. The dependence on $(t, x)$ is very simple. In this special parameterization, (2.16) and (2.17) are
Figure 5: The elementary weight $\hat{q}(t, x)$ is shown as a function of $(t, x)$ in $\Lambda_{2T}$ for $T = 100$. The dependence on $(t, x)$ is very simple.

written as follows,

$$
\widehat{P}_{2T}^{0,0}(X(t) = x) = \widehat{P}_{2T}^{0,0}(X(t) = x; r, \alpha_0, \beta_0)_{\pi r = 6T, \alpha_0/r = \pi/4, \beta_0/r = -\pi/4},
$$

where

$$
\hat{c}_2(t; T) \cos \left(\frac{\pi}{6T}\right) \times \prod_{n=1}^{(t+x)/2} \sin \left(\frac{n\pi}{6T}\right) \prod_{n=1}^{(t-x)/2} \sin \left(\frac{n\pi}{12T}\right) = \left\{ \begin{array}{ll}
\prod_{n=1}^{(2T-t)-x} \sin \left(\frac{n\pi}{6T}\right) \prod_{n=1}^{(2T-t)+x} \sin \left(\frac{n\pi}{12T}\right) & \text{if } (t, x) \in \Lambda_{2T}, \\
0, & \text{otherwise},
\end{array} \right. \tag{4.13}
$$

and

$$
\hat{c}_2(t; T) = 2^T \prod_{n=1}^{T} \sin \left(\frac{n\pi}{6T}\right) \prod_{n=1}^{2T-t} \sin \left(\frac{n\pi}{6T}\right) \prod_{n=1}^{T} \cos \left(\frac{(2nT-1)\pi}{6T}\right). \tag{4.14}
$$

By definition of excursion process

$$
\widehat{P}_{2T}^{0,0}(X(0) = x) = \widehat{P}_{2T}^{0,0}(X(2T) = x) = \delta_{x0}.
$$

In the above expression, it is obvious to see the following symmetry,

$$
\widehat{P}_{2T}^{0,0}(X(t) = x) = \widehat{P}_{2T}^{0,0}(X(2T - t) = x), \quad 0 \leq t \leq 2T, \quad x \in \mathbb{Z}. \tag{4.15}
$$
Figure 6: Time-evolution of the probability distribution $\hat{P}_{2T}^0(X(t) = x)$ of the trigonometric excursion process with $T = 100$. The maximum likelihood trajectory is $C$-shaped.

Figure 6 shows the time-dependence of $\hat{P}_{2T}^0(X(t) = x)$ given by (4.13) and (4.14) for $T = 100$. This figure suggests that at each time $t \in \{0, 1, \ldots, 2T\}$ there is a single peak at $x = \hat{x}_{2T}^{\text{max}}(t)$, and

$$\hat{x}_{2T}^{\text{max}}(t) < 0 \quad \text{for } 1 \leq t \leq 2T - 1.$$

The line $\{\hat{x}_{2T}^{\text{max}}(t) : 0 \leq t \leq 2T\}$ expresses the maximum likelihood trajectory of the trigonometric excursion process. As shown by Fig. 6, it is $C$-shaped.

Figure 7 shows $\{\hat{x}_{2T}^{\text{max}}(sT)/T : 0 \leq s \leq 2\}$ for $T = 100, 150, 200, 250$. The data-collapse is observed and it suggests that the values of $\hat{x}_{2T}^{\text{max}}(t)$ are proportional to $T$ in $T \to \infty$ and thus the following scaled limit trajectory exists,

$$\{\hat{v}(t) : t \in [0, 2]\} = \lim_{T \to \infty} \left\{ \frac{\hat{x}_{2T}^{\text{max}}(sT)}{T} : 0 \leq s \leq 2 \right\}.$$

This limit trajectory exhibits a nontrivial curve. Due to (4.15), it has the following symmetry,

$$\hat{v}(t) = \hat{v}(2 - t), \quad t \in [0, 2].$$

### 4.3 Elliptic excursion processes

For the elliptic excursion process $\{X(t)\}_{t \in \{0, 1, \ldots, 2T\}, \mathbb{P}_{2T}^0}$, here we consider the same parameterization (4.12) as in the previous trigonometric process. The behavior of the elementary weight $q(t, x)$ on $\Lambda_{2T}^*$ is similar to $\tilde{q}(t, x)$ shown by Fig. 5 with slight modification depending on $\kappa$. It is a plain function of $(t, x)$ in $\Lambda_{2T}^*$.
metric excursion process in the long-term limit $T \to \infty$. The observed data-collapse suggests the existence of a limit curve $\hat{v}(t), t \in [0, 2]$ for the trigonometric excursion process in the long-term limit $T \to \infty$.

are written as follows,

$$
P_{2T}^{0,0}(X(t) = x) = P_{2T}^{0,0}(X(t) = x; r, \alpha_0, \beta_0, \kappa) \bigg|_{\pi r = 6T, \alpha_0/r = \pi/4, \beta_0/r = -\pi/4}
$$

$$
= \left\{ \begin{array}{ll}
c_2(t; T) \frac{1}{\varphi_1(\frac{i}{12T}; i \kappa)} & 
\prod_{n=1}^{(t+x)/2} \frac{\vartheta_1(\frac{1}{4} - \frac{2n+3T-2t-1}{12T}; i \kappa)}{\vartheta_1(\frac{n}{6T}; i \kappa)}
\prod_{n=1}^{(t-x)/2} \frac{\vartheta_1(\frac{1}{4} + \frac{2n+3T-2t-1}{12T}; i \kappa)}{\vartheta_1(\frac{n}{6T}; i \kappa)} \\
\prod_{n=1}^{(2T-t-\varepsilon)/2} \frac{\vartheta_1(\frac{1}{4} + \frac{2n-2t+1}{12T}; i \kappa)}{\vartheta_1(\frac{n}{6T}; i \kappa)} & 
\prod_{n=1}^{(2T-t+\varepsilon)/2} \frac{\vartheta_1(\frac{1}{4} - \frac{2n+3T-2t-1}{12T}; i \kappa)}{\vartheta_1(\frac{n}{6T}; i \kappa)}
\end{array} \right\},
$$

where

$$
\vartheta_2(v; \tau) = \vartheta_1(v + 1/2; \tau).
$$

When $\kappa > 0$ is large, the behavior of the elliptic excursion process $\{X(t)\}_{t \in \{0, 1, ..., 2T\}}, P_{2T}^{0,0}$ is very similar to that of the trigonometric excursion process $\{X(t)\}_{t \in \{0, 1, ..., 2T\}}, \hat{\mathbb{P}}_{2T}^{0,0}$. Let $x_{2T}^{\text{max}}(t; \kappa)$ denote the maximum likelihood trajectory of the elliptic excursion process with parameter $\kappa > 0$. Figure 4 shows $\{x_{2T}^{\text{max}}(sT; \kappa)/T : 0 \leq s \leq 2\}$ with $\kappa = 0.5$ and $\kappa = 10$.
1. We scale the spatio-temporal coordinate \((t, x) \in \Lambda_{2T}\) in a unit of \(T\) as
\[
t = sT, \quad x = vT.
\]

Then we expect that there is a non-negative function \(I(s, v)\) such that the following asymptotics holds,
\[
\tilde{\mathbb{P}}^{0,0}_{2T}(X(sT) = vT) \simeq e^{-TI(s,v)} \quad \text{as } T \to \infty,
\] (5.1)

for \(T = 100, 150, 200, 250\). The data-collapse suggests that the values of \(x_{2T}^{\text{max}}(t; \kappa)\) are proportional to \(T\) in \(T \to \infty\) and thus the scaled limit trajectory
\[
\{v(s; \kappa) : s \in [0, 2]\} = \lim_{T \to \infty} \left\{ \frac{x_{2T}^{\text{max}}(sT; \kappa)}{T} : 0 \leq s \leq 2 \right\},
\]
exists for each value of \(\kappa > 0\). The deviation of this limit trajectory \(v(t, \kappa), t \in [0, 2]\) from the straight line is enhanced as \(\kappa \to 0^+\).

5 Asymptotic Analysis of Simplified Trigonometric Excursion Processes

In this section, we report our trials to analyze asymptotics of the probability law \(\tilde{\mathbb{P}}^{0,0}_{2T}\) of the simplified trigonometric excursion processes in the long-term limit \(T \to \infty\).

5.1 Large deviation principle and scaled limit trajectory

We scale the spatio-temporal coordinate \((t, x) \in \Lambda_{2T}\) in a unit of \(T\) as
\[
t = sT, \quad x = vT.
\]

Then we expect that there is a non-negative function \(I(s, v)\) such that the following asymptotics holds,
\[
\tilde{\mathbb{P}}^{0,0}_{2T}(X(sT) = vT) \simeq e^{-TI(s,v)} \quad \text{as } T \to \infty,
\] (5.1)
for
\[
\mathcal{X}_2 = \{(s, v) \in \mathbb{R}^2 : 0 \leq s \leq 1, -s \leq v \leq s\} \cup \{(s, v) \in \mathbb{R}^2 : 1 < s \leq 2, -(2 - s) \leq v \leq 2 - x\}.
\]

When the asymptotic of the form (5.1) is valid, we say that the large deviation principle is established, and the function \(I\) is called the rate function (see, for instance, [20]). If so in the present system, the scaled limit trajectory, which was denoted by \(v = \tilde{v}(s), s \in [0, 2]\) and numerically studied in the previous section, will be determined as zeros of \(I\),

\[I(s, \tilde{v}(s)) = 0, \quad s \in [0, 2], \quad (5.2)\]

since (5.1) with (5.2) implies

\[
\lim_{T \to \infty} \mathbb{P}_{2T}^0(X(sT) = \tilde{v}(s)T) = 1, \quad s \in [0, 2].
\]

This is the law of large numbers in the present problem. As shown by (4.9), \(\mathbb{P}_{2T}^0(X(sT) = vT)\) consists of several products of trigonometric functions. For example, it contains the product

\[
B_{2T}^{(1)}(s, v) = \prod_{n=1}^{\infty} \cos \left( \frac{2n + (3 - 2s)T - 1 \pi}{6T} \right).
\]

If we take the logarithm of this, we find that

\[
\frac{\pi}{3T} \log B_{2T}^{(1)}(s, v) = \frac{\pi}{3T} \sum_{n=1}^{\infty} \log \cos \left( \frac{2n + (3 - 2s)T - 1 \pi}{6T} \right)
\]

\[
\to \int_{(3-2s)\pi/6}^{(3-s+v)\pi/6} \log \cos u \, du \quad \text{in} \ T \to \infty.
\]

The following Fourier expansion formulas are useful to evaluate the integrals of logarithms of trigonometric functions,

\[
\int dx \log \sin x = -x \log 2 - \frac{1}{2} \sum_{n=1}^{\infty} \frac{\sin(2nx)}{n^2}, \quad 0 \leq x < \pi, \quad (5.4)
\]

\[
\int dx \log \cos x = -x \log 2 + \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sin(2nx)}{n^2}, \quad |x| < \frac{\pi}{2}. \quad (5.5)
\]

We can derive the following integral representation for the rate function. The proof is given in Appendix A.
Lemma 5.1 For the simplified trigonometric excursion process \( \{X(t)\}_{t \in \{0,1,\ldots,2T\}, \overline{\mu}^{0,0}_{2T}} \) with parameters \((4,8)\), the large deviation principle \((5.1)\) is established with the rate function \(I\) expressed by

\[
I(s,v) = 3 \int_{0}^{(1-s-v)/6} \log \frac{\sin(\pi(v+1/6))}{\sin(\pi(-v+1/6))} dv + 6 \int_{0}^{(1-s+v)/6} \log \frac{\sin(\pi(v+1/6))}{\sin(\pi(-v+1/6))} dv + 6 \int_{0}^{(1-s)/3} \log \frac{\cos(\pi(v+1/6))}{\cos(\pi(-v+1/6))} dv \quad \text{for } (x,v) \in \Lambda_2.
\]

By differentiating the equation \((5.2)\) by \(s\), we obtain \(dI(s,\tilde{v}(s))/ds = 0\) and then

\[
\frac{d\tilde{v}(s)}{ds} = -\frac{\partial_s I(s,\tilde{v}(s))}{\partial I(s,\tilde{v}(s))},
\]

where

\[
\partial_s I(s,v) = \frac{\partial I(s,v)}{\partial s}, \quad \partial_v I(s,v) = \frac{\partial I(s,v)}{\partial v}.
\]

Lemma 5.1 gives the following differential equation,

\[
\frac{d\tilde{v}(s)}{ds} = -\log \frac{\sin(\pi(s + \tilde{v}(s))/6)\sin^2(\pi(s - \tilde{v}(s))/6)\cos^4(\pi(2s - 1)/6)}{\sin(\pi(2 - s - \tilde{v}(s))/6)\sin^2(\pi(2 - s + \tilde{v}(s))/6)\cos^4(\pi(3 - 2s)/6)}.
\]

We can prove the following.

Theorem 5.2 Consider the simplified trigonometric excursion process with parameters \((4,8)\). The scaled limit trajectory \(v = \tilde{v}(s), s \in [0,2]\) passes the origin at \(s = 1\), that is, \(\tilde{v}(1) = 0\), and in the time region in the vicinity of \(s = 1\), it behaves as

\[
\tilde{v}(s) = \frac{1}{3} (s - 1) - \frac{2^4\pi^2}{3^6} (s - 1)^3 + O((s - 1)^5).
\]

Proof. Assume that \(|s - 1| \ll 1\) and \(|v| \ll 1\) in \((5.6)\). For \(|v| \ll 1\),

\[
\sin(\pi(v+1/6)) = \frac{1}{2} \left( 1 + \sqrt{3} \pi v - \frac{\pi^2}{2} v^2 - \frac{\sqrt{3} \pi^3}{6} v^3 + \frac{\pi^4}{24} v^4 + \frac{\sqrt{3} \pi^5}{120} v^5 \right) + O(v^6);
\]

\[
\cos(\pi(v+1/6)) = \frac{\sqrt{3}}{2} \left( 1 - \frac{\sqrt{3} \pi}{3} v - \frac{\pi^2}{2} v^2 + \frac{\sqrt{3} \pi^3}{18} v^3 + \frac{\pi^4}{24} v^4 - \frac{\sqrt{3} \pi^5}{360} v^5 \right) + O(v^6).
\]
Then
\[
\int_0^{(1-s\pm v)/6} \log \frac{\sin(\pi(\varphi + 1/6))}{\sin(\pi(-\varphi + 1/6))} d\varphi
\]
\[
= \int_0^{(1-s\pm v)/6} \log \sin(\pi(\varphi + 1/6)) d\varphi - \int_{-(1-s\pm v)/6}^{0} \log \sin(\pi(\varphi + 1/6)) d\varphi
\]
\[
= \frac{\sqrt{3\pi}}{2^2 \times 3^2} (1 - s \pm v)^2 + \frac{2\sqrt{3\pi^3}}{2^4 \times 3^5} (1 - s \pm v)^4
\]
\[
+ \frac{11\sqrt{3\pi^5}}{2^4 \times 3^8 \times 5} (1 - s \pm v)^6 + O(|1 - s \pm v|^8),
\]
and
\[
\int_0^{(1-s)/3} \log \frac{\cos(\pi(\varphi + 1/6))}{\cos(\pi(-\varphi + 1/6))} d\varphi
\]
\[
= \int_0^{(1-s)/3} \log \cos(\pi(\varphi + 1/6)) d\varphi - \int_{-(1-s)/3}^{0} \log \cos(\pi(\varphi + 1/6)) d\varphi
\]
\[
= -\frac{\sqrt{3\pi}}{3^2} (1 - s)^2 - \frac{2\sqrt{3\pi^3}}{3^7} (1 - s)^4 - \frac{2^2 \sqrt{3\pi^5}}{3^9 \times 5} (1 - s)^6 + O(|1 - s|^8).
\]
Together these give
\[
I(s, v) = \frac{\sqrt{3\pi}}{2^2 \times 3^2} (s - 1 - 3v)^2
\]
\[
- \frac{\sqrt{3\pi^3}}{2^4 \times 3^6} \left\{ 5(s - 1)^4 + 36v(s - 1)^3 - 162v^2(s - 1)^2 + 36v^3(s - 1) - 27v^4 \right\}
\]
\[
- \frac{\sqrt{3\pi^5}}{2^4 \times 3^8 \times 5} \left\{ 29(s - 1)^6 + 198v(s - 1)^5 - 1485v^2(s - 1)^4
\]
\[
+ 660v^3(s - 1)^3 - 1485v^4(s - 1)^2 + 198v^5(s - 1) - 99v^6 \right\}
\]
\[
+ O(|s - 1|^8, |v|^8).
\]
(5.8)
The first line of (5.8) vanishes when \(v = (s - 1)/3\). So we can assume that
\[
v = \frac{1}{3} (s - 1) + c(s - 1)^3 + O((s - 1)^5)
\]
with an unknown coefficient \(c\). In this assumption (5.8) is written as
\[
I(s, v) = \frac{\sqrt{3\pi}}{2^2} \left( c + \frac{2^4 \pi^2}{3^6} \right)^2 (s - 1)^6 + O((s - 1)^8),
\]
and hence it is verified that \(I(s, \bar{v}(s)) = O((s - 1)^8)\) with (5.7). The proof is complete.
Figure 9: Numerical values of $\tilde{x}^{\max}(t)/T, t \in \{0, 1, \ldots, 2T\}$ are plotted versus $s = t/T$ for $T = 500$ by dots. In the vicinity of $s = 1$, the dots are well approximated by the cubic curve (5.9) starting from $(0, -v_0)$ and ending at $(2, v_0)$ with $v_0 = 1/3 - 2^4\pi^2/3^6 = 0.116 \cdots$.

In Fig. 9, the numerical values of $\tilde{x}^{\max}(t)/T, t \in \{0, 1, \ldots, 2T\}$ with $T = 500$ are plotted versus $s = t/T$ by dots, which will well simulate the scaled limit trajectory $v = \tilde{v}(s), s \in [0, 2]$, since $T = 500$ is large enough. The cubic curve

$$v = \frac{1}{3}(s - 1) - \frac{2^4\pi^2}{3^6}(s - 1)^3, \quad s \in [0, 2];$$

(5.9)

is also shown, which starts from $(0, -v_0)$ and ends at $(2, v_0)$ with $v_0 = 1/3 - 2^4\pi^2/3^6 = 0.116 \cdots$. Coincidence of this cubic curve with the dots is excellent in the vicinity of $s = 1$.

5.2 Central limit theorem at time $t = T$

In Theorem 5.2, $\tilde{v}(1) = 0$ was proved. As shown by (5.3), this means that the trajectory with time duration $2T$ passes through the origin 0 at time $t = T$ with probability one, when $T \to \infty$.

Now we study the fluctuation of the trajectory around 0 at time $t = T$. We can prove the following central limit theorem.

**Proposition 5.3** For the simplified trigonometric excursion process $\{X(t)\}_{t \in \{0, 1, \ldots, 2T\}}$, $\tilde{P}^{0,0}_{2T}$ with parameters (4.8),

$$\lim_{T \to \infty} \sqrt{T} \tilde{P}^{0,0}_{2T}(X(T) = \sqrt{T} \xi) = f(\xi)$$

(5.10)

with

$$f(\xi) = \frac{3^{1/4}}{2} \exp\left(-\frac{\sqrt{3}\pi}{4} \xi^2\right), \quad \xi \in \mathbb{R}.$$  

(5.11)

That is, at time $t = T$, the fluctuation is proportional to $\sqrt{T}$ in the limit $T \to \infty$, and its coefficient is normally distributed with variance $\sigma^2 = 2/(\sqrt{3}\pi)$.  

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Figure 10 shows the numerical plots of \((\sqrt{T}/2)\tilde{\mathbb{P}}^{0,0}\sqrt{T}\xi\) with parameters (4.8) for \(T = 100\) and the function \(f(\xi)\) given by (5.11). We found that \(T = 100\) is large enough to see a good coincidence.

This result should be compared with (4.1) which shows the convergence of the excursion process of classical random walk to the Brownian bridge. We note that the variance \(\sigma^2 = 2/(\sqrt{3}\pi) = 0.367\cdots\) of the present process is smaller than the classical value \(\sigma_{cl} = 1/2\) at time \(s = 1\).

6 Concluding Remarks

The weight functions of Schlosser’s lattice model [16] are complex functions which are totally elliptic in the sense that all variables representing the coordinates \(t, x\) and parameters \(\alpha, \beta\) have equal periods of doubly periodicity, if they are view as complex variables [19]. The totally elliptic function giving the weight for the elementary step of lattice paths (1.6) was cleverly chosen [16] so that the generating functions of lattice paths, which are defined as summations of products of elementary weight-functions, are completely factorized as (1.8).

In the present paper, we have constructed a family of one-dimensional excursion processes from Schlosser’s two-dimensional lattice path models. Here the spatio-temporal coordinates \((t, x)\) as well as parameters are real variables and the weight functions are real functions of them. In order to obtain probability measures for trajectories of stochastic processes, we have considered the excursion processes and given sufficient conditions to make the measures be non-negative definite.

Although our probability measures have lost the totally elliptic property as complex functions, the product formulas originally given for the generating functions in Schlosser’s lattice path models remain in the probability measures for the excursion processes. The
obtained processes are inhomogeneous both in space and time, and in order to describe
the behavior of trajectories, we have to precisely evaluate the ratios of products of Jacobi’s
theta functions for the elliptic excursion processes and of the trigonometric functions for the
reduced processes.

We have studied the excursion processes starting from the origin and returning to the
origin in time duration $2T, T \in \mathbb{N}$. If we considered the excursion process of the simple and
symmetric random walk, or its diffusion scaling limit realized as the Brownian bridge, the
maximum likelihood trajectories are the straight lines connecting $(0,0)$ and $(0,2T)$ on the
spatio-temporal plane. Contrary to such classical results, the maximum likelihood trajec-
tories in the present elliptic and trigonometric excursion processes exhibit nontrivial curves
connecting $(0,0)$ and $(0,2T)$ in the spatio-temporal plane. Moreover, data-collapse observed
in the scaling plots for variety of time durations suggests the existence of scaled limit trajec-
tories in the $T \to \infty$ limit. There we have concentrated on the single-time measures (2.11)
for the excursion processes. A general formula was given for any multi-time joint measure
by (2.13), which provides a well-defined probability measure for trajectories, if the conditions
(3.7)-(3.10) of Theorem 3.1 are satisfied for the parameterization (2.12). Spatio-temporal
correlations of each trajectory should be systematically studied in the future. In Section 4,
we reported about our numerical study only in the special cases with $\sigma = 3$ for the sim-
plicated trigonometric excursion process and with $\sigma = 6$ for the trigonometric and elliptic
excursion processes, since in these cases the degree of deviation from the classical processes
$\lambda$ is maximized. Numerical study of other cases with lower values of $\lambda$ is now in progress.

In Section 5, we have reported about the asymptotic analysis in the long-term limit
$T \to \infty$ only for the simplified trigonometric excursion process. There the Fourier analysis
for the integrals of logarithms of trigonometric functions are used. In order to analyze
asymptotic probability laws for the elliptic processes, we need to develop the present method
to the elliptic level. It will be an interesting future problem. Emergence of nontrivial curves
of trajectories shown by Figs. 3 and 6 from the simple elementary weight-functions (see
Figs. 2 and 5) is a new aspect of the elliptic combinatorics [16, 17, 18] and expected to lead
a way to elliptic probability theory.

In the paper by Schlosser [16], factorization formulas are given for the Karlin-McGregor
determinants for plural lattice paths with the totally elliptic weights. They are systems of
nonintersecting lattice paths weighted by complex functions. It will be a challenging future
problem to define probability measures for ensembles of nonintersecting lattice paths and
construct noncolliding particle systems which are indeed inhomogeneous both in space and
time.

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A Proof of Lemma 5.1

Let
\[ A_{2T}(\gamma) = \prod_{n=1}^{\gamma T} \sin \left( \frac{n\pi}{3T} \right), \]
\[ B_{2T}^{(1)}(s, v) = \prod_{n=1}^{(s+v)T/2} \cos \left( \frac{\{2n + (3 - 2s)T - 1\} \pi}{6T} \right), \]
\[ B_{2T}^{(2)}(s, v) = \prod_{n=1}^{\{(2-s)-v\}T/2} \cos \left( \frac{\{2n + (2s - 1)T - 1\} \pi}{6T} \right), \]
\[ B_{2T}^{(3)} = \prod_{n=1}^{T} \cos \left( \frac{(2n - T - 1) \pi}{6T} \right). \]

For fixed \( \gamma < \infty \),
\[
\frac{\pi}{3T} \log A_{2T}(\gamma) = \frac{\pi}{3T} \sum_{n=1}^{\gamma T} \log \sin \left( \frac{n\pi}{3T} \right) \rightarrow \int_{0}^{\gamma \pi/3} \log \sin u \, dy \quad \text{in} \ T \to \infty.
\]

If we use (5.4), we have the following expression for the limit,
\[
- \lim_{T \to \infty} \frac{1}{T} \log A_{2T}(\gamma) = \gamma \log 2 + \frac{3}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \left( \frac{2\gamma n\pi}{3} \right).
\]

Similarly, we obtain the following limits,
\[
- \lim_{T \to \infty} \frac{1}{T} \log B_{2T}^{(1)}(s, v)
= \frac{s + v}{2} \log 2 - \frac{3}{2\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \left[ \sin \left( \frac{(3 - s + v) n \pi}{3} \right) - \sin \left( \frac{(3 - 2s) n \pi}{3} \right) \right],
\]
\[
- \lim_{T \to \infty} \frac{1}{T} \log B_{2T}^{(2)}(s, v)
= \frac{2 - s - v}{2} \log 2 - \frac{3}{2\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \left[ \sin \left( \frac{(1 + s - v) n \pi}{3} \right) - \sin \left( \frac{(2s - 1) n \pi}{3} \right) \right],
\]
\[
- \lim_{T \to \infty} \frac{1}{T} \log B_{2T}^{(3)} = \log 2 - \frac{3}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \sin \left( \frac{n\pi}{3} \right).
\]
After some calculation using the addition formulas of trigonometric functions and the equality
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \sin \left( \frac{n\pi}{3} \right) = \frac{2}{3} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \left( \frac{n\pi}{3} \right),
\]
we arrive at the following Fourier expansion for the rate function,
\[
I(s, v) \equiv \lim_{T \to \infty} \frac{1}{T} \log \tilde{P}_{0, 0}^T(X(st) = vT)
\]
\[
= \frac{6}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \sin \left( \frac{n\pi}{3} \right) \cos \left( \frac{2(1-s)n\pi}{3} \right) + \frac{5}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \left( \frac{n\pi}{3} \right) \\
- \frac{6}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \left( \frac{n\pi}{3} \right) \cos \left( \frac{(1-s+v)n\pi}{3} \right) \\
- \frac{3}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \left( \frac{n\pi}{3} \right) \cos \left( \frac{(1-s-v)n\pi}{3} \right). 
\]
Let
\[
J(\varphi) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \left( \frac{n\pi}{3} \right) \cos (n\varphi), \quad K(\varphi) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \sin \left( \frac{n\pi}{3} \right) \cos (n\varphi).
\]
Then by (5.4) and (5.5), we have
\[
J(\varphi_1, \varphi_2) \equiv J(\varphi_1) - J(\varphi_2)
\]
\[
= -\frac{1}{2} \left[ \int_{\varphi_1}^{\varphi_2} \log \sin \left( \frac{u}{2} + \frac{\pi}{6} \right) du - \int_{-\varphi_1}^{-\varphi_2} \log \sin \left( \frac{u}{2} + \frac{\pi}{6} \right) du \right],
\]
\[
K(\varphi_1, \varphi_2) \equiv K(\varphi_1) - K(\varphi_2)
\]
\[
= \frac{1}{2} \left[ \int_{\varphi_1}^{\varphi_2} \log \cos \left( \frac{u}{2} + \frac{\pi}{6} \right) du - \int_{-\varphi_1}^{-\varphi_2} \log \cos \left( \frac{u}{2} + \frac{\pi}{6} \right) du \right].
\]
Note that the equality (A.1) is written as $K(0) = (2/3)J(0)$. Then we can show that (A.2) is rewritten as

$$I(s, v) = -\frac{3}{\pi} J\left(\frac{1-s-v}{3}, \frac{1-s}{3}\right) - \frac{5}{\pi} J\left(\frac{1-s}{3}, 0\right) + \frac{1}{\pi} J\left(\frac{1-s}{3}, \frac{1-s+v}{3}\right) - \frac{4}{\pi} J\left(\frac{1-s}{3}, 0\right) + \frac{6}{\pi} K\left(\frac{2(1-s)}{3}, 0\right)$$

$$= -\frac{3}{\pi} J\left(\frac{1-s-v}{3}, 0\right) - \frac{6}{\pi} J\left(\frac{1-s+v}{3}, 0\right) + \frac{6}{\pi} K\left(\frac{2(1-s)}{3}, 0\right).$$

(A.3)

It is easy to verify that (A.3) is equal to (5.6). The proof is hence complete. 

**B Proof of Proposition 5.3**

By a similar calculation to that given in the proof of Lemma 5.1 in Appendix A, we can show that

$$-\frac{1}{T} \log \tilde{P}_{2T}^0(X(T) = x) = \frac{27}{2\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \sin \left(\frac{n\pi}{3}\right)$$

$$- \frac{9}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \left(\frac{n\pi}{3}\right) \cos \left(\frac{xn\pi}{3T}\right) + o(T) \quad \text{in} \quad T \to \infty.$$ 

Since the equality (A.1) holds, the above is written as

$$\int_{\pi/3}^{\pi/3} \log \left(\frac{u}{2} + \frac{\pi}{6}\right) du - \int_{-\pi/3T}^{\pi/3T} \log \left(\frac{u}{2} + \frac{\pi}{6}\right) du.$$ 

(B.1)

Using (5.4), we can verify that (B.1) is equal to

$$\frac{9}{2\pi} \left[ - \int_0^{\pi/3} \log \sin \frac{u}{2} du + \frac{1}{2} \int_0^{\pi(1+x/T)/3} \log \sin \left(\frac{u}{2} + \frac{\pi}{6}\right) du + \frac{1}{2} \int_0^{\pi(1-x/T)/3} \log \sin \left(\frac{u}{2} + \frac{\pi}{6}\right) du \right]$$

$$= \frac{9}{2\pi} \left[ \int_0^{\pi/3} \log \left(\frac{u}{2} + \frac{\pi}{6}\right) du - \int_{-\pi/3T}^{\pi/3T} \log \left(\frac{u}{2} + \frac{\pi}{6}\right) du \right].$$

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Now we put $x = \sqrt{T} \xi$. Since

$$\log \sin \left(\frac{u}{2} + \frac{\pi}{6}\right) = -\log 2 + \frac{\sqrt{3}}{2} u - \frac{1}{2} u^2 + O(u^3),$$

we obtain the evaluation

$$\log \tilde{P}_{2T}^{0,0}(X(T) = \sqrt{T} \xi) = -\frac{\sqrt{3} \pi}{4} \xi^2 + O(T^{-2}) \quad \text{in } T \to \infty.$$

This implies that

$$\tilde{P}_{2T}^{0,0}(X(T) = \sqrt{T} \xi) \simeq \text{const.} \times \exp \left(-\frac{\sqrt{3} \pi}{4} \xi^2\right) \quad \text{in } T \to \infty.$$

The constant factor is determined by the normalization condition and the probability density function (6.11) is obtained. The proof is thus complete. \(\square\)

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