A GAUGE-INVARIENT SUBTRACTION TECHNIQUE FOR NON-INCLUSIVE OBSERVABLES IN QCD

F. HAUTMANN
Department of Physics, Pennsylvania State University, University Park, PA 16802

Using the electromagnetic form factor of a quark as a working example, we describe a subtraction technique to treat infrared sensitive regions in non-inclusive processes.

Inclusive hard-scattering processes, characterized by a single large mass scale, are investigated in QCD by using asymptotic freedom and factorization theorems. But the application of QCD to the study of multiparticle final states, involving several mass scales, is much subtler. The main practical tool is provided by Monte Carlo event generators, modeling parton shower and hadronization. In these event generators the theory does not yet go systematically beyond the leading logarithms. To incorporate next-to-leading order QCD corrections in parton showers, extensions of the factorization theorems are necessary, for which new more precise methods are needed.

An important step in this program is to show how to decompose Feynman graphs into terms associated with particular regions in loop momentum space. In the case of a Monte Carlo event generator simulating the exclusive structure of the hadronic final states, the observables being computed are not infrared and collinear safe. It is important to develop techniques such that even for such observables the integrands to be associated with the ultraviolet region are integrable functions — and can in particular be integrated numerically through a Monte Carlo.

Let us consider as an example the virtual corrections to the electromagnetic form factor of a quark (Fig. 1). The theory for this process is well known. See for instance the review in Ref. 7. To simplify the calculations while retaining all the ingredients that are essential for our discussion, let us work in a massive abelian theory with scalar quarks. We denote the quark mass by $m$ and the gauge boson mass by $m_g$. We work in a center-of-mass frame in which the

---

*a*Talk at the International Symposium on Multiparticle Dynamics ISMD2000, Tihany, Hungary, 9-15 October 2000.
incoming quark momenta $p_A$ and $p_B$ are in the $+z$ and $-z$ directions, with $2p_A^+p_B^- = Q^2$. We consider the amplitude

$$\Gamma = i g^2 \int \frac{d^4k}{(2\pi)^4} \times \frac{(2p_A - k) \cdot (2p_B + k)}{(k^2 - m_g^2 + i\varepsilon) (p_A - k)^2 - m^2 + i\varepsilon} (p_B + k)^2 - m^2 + i\varepsilon} - UV,$$

where UV indicates the $\overline{\text{MS}}$ counterterm for the ultraviolet divergence.

Standard power counting arguments determine the regions of momentum space contributing to the leading power behavior of $\Gamma$: 1) the soft region, where all components of $k^\mu$ are much smaller than $Q$: $k^\mu \sim \lambda Q$, with $\lambda$ small; 2) the $p_A$-collinear region: $k^+ \sim Q, k_\perp \sim \lambda Q, k^- \sim \lambda^2 Q$, with $0 \leq k^+ \leq p_A^+$; 3) the $p_B$-collinear region: $k^- \sim Q, k_\perp \sim \lambda Q, k^+ \sim \lambda^2 Q$, with $-p_B^- \leq k^- \leq 0$; 4) the hard region, where all components of $k^\mu$ are of order $Q$.

Figure 1: (a) One-loop graph for the electromagnetic form factor; (b) geometry of the infrared sensitive regions in momentum space.

Ref.\cite{ref2} constructs a decomposition of $\Gamma$ into a sum of terms, one for each of these regions (Fig. 2),

$$\Gamma = S + A + B + H + \text{nonleading power} ,$$

(2)
satisfying the following requirements:

i) The splitting between the terms is to be defined gauge-invariantly: we demand that the terms be obtained from matrix elements of gauge-invariant operators.

ii) In particular, the necessary cut-offs on rapidity integrations should be gauge-invariant. As we will see, this involves the use of Wilson lines along non-lightlike directions.

iii) The evolution equations with respect to these cut-offs should be simple, in the sense that there should be no power-law remainder terms. All the power-law corrections are associated with the initial construction of the terms \( S, A, B, H \) in Eq. (2).

iv) The integrand associated with the hard region should be an integrable function even when the physical observable being computed is not infrared safe in perturbation theory.

The strategy we use to construct such a decomposition is similar to the R-operation techniques for renormalization. See Ref. for a related approach. We proceed from smaller to larger regions (Fig. 2). For each region, we construct a term that, added to the terms for smaller regions, gives a good leading-power approximation to the original amplitude in that region, and does not receive leading contributions from regions that are smaller or have an overlap with the region being treated.

To see how this works, let us look at the form of the result for one of the terms. The term \( S \) associated with the soft region is

\[
S = \frac{-ig^2}{(2\pi)^4} \int d^2k \left( \frac{1}{(k^2 - m_g^2 + i\varepsilon)} \right) \left[ \frac{1}{(k^- - i\varepsilon)(k^+ + i\varepsilon)} \right]
\]

\[
- \frac{u_B^-}{(k^- - i\varepsilon) (u_B^- k^+ + u_B^+ k^- + i\varepsilon)}
- \frac{u_A^+}{(u_A^+ k^- + u_A^- k^+ - i\varepsilon) (k^+ + i\varepsilon)}
- \text{UV}
\]

The first term in the square brackets is just obtained by taking the soft approximation to Eq. (1). This term still has singularities from the ultraviolet and collinear regions. The ultraviolet singularity is to be dealt with by the standard subtractive renormalization procedure. We treat the collinear singularities in a similar fashion: the next two terms in the square brackets are subtractions terms designed to cancel the collinear contributions.

To define these terms we have introduced two vectors \( u_A = (u_A^+, u_A^-, 0) \), \( u_B = (u_B^+, u_B^-, 0) \), lying along directions away from the light cone. The second
term in the square brackets subtracts the divergence from the region collinear to $p_A$, i.e., $k^-/k^+ \to 0$. The non-lightlike vector $u_B$ in this term provides a cut-off on the region of small $k^+$. Similarly, the third term subtracts the divergence from the region collinear to $p_B$, with the vector $u_A$ providing a cut-off on the region of small $k^-$. Further inspection of the contour integrations in Eq. (3) for $k^+ k^- \ll k^2$ shows that $u_A$ and $u_B$ must be spacelike, $u_A^+, u_B^- > 0$, $u_A^-, u_B^+ < 0$. Note that the collinear-to-$p_A$ subtraction term has no collinear-to-$p_B$ singularity; indeed it is power suppressed in this region. The same is true with $A$ and $B$ exchanged.

The important point is that the cut-offs thus introduced are defined gauge-invariantly: the counterterms in Eq. (3) can be obtained from matrix elements of path-ordered exponentials of the gluon field along non-lightlike lines. For a generic direction $n$, define

$$V_q(n) = \mathcal{P} \exp \left( i g \int_{-\infty}^{0} dz A(z n) \cdot n \right),$$

$$V_{\bar{q}}(n) = \mathcal{P} \exp \left( -ig \int_{-\infty}^{0} dz A(z n) \cdot n \right).$$

Consider the product of vacuum expectation values

$$\langle 0 | V_q(p_A) V_{\bar{q}}(p_B) | 0 \rangle \langle 0 | V_q(u_A) | 0 \rangle \langle 0 | V_{\bar{q}}(u_B) | 0 \rangle.$$

At one loop the first factor in the numerator gives the unsubtracted soft term in the first line of Eq. (3), while the two factors in the denominator, involving
Wilson lines along spacelike directions, give the collinear subtractions. The remaining factors in the numerator cancel factors of a complete external propagator for the Wilson line. Given the one-loop formulas, this result appears to be unique, if we simply assume that the quantity which we calculate is the product of vacuum expectation values of some Wilson line operators.

The vectors \( u_A, u_B \) introduced by the subtractions are not physical parameters. Their utility comes from the fact that evolution equations in \( y_A = (1/2) \ln |u_A^+/u_A^-|, y_B = (1/2) \ln |u_B^+/u_B^-| \) can be applied to the terms in Eq. (2) to extract effects associated with large logarithms \( \ln \). One of the advantages of the subtraction procedure described here is that the corresponding evolution equations are homogeneous. This can be contrasted, e.g., with the case of Ref. 7, where the evolution equations have power-law corrections. The simpler structure of the equations may be helpful in more complicated cases, such as the factorization needed for the inclusion of next-to-leading corrections in Monte Carlo event generators.

The collinear terms \( A, B \) in Eq. (2) can also be given an operator definition in terms of spacelike Wilson lines. The hard term \( H \) is obtained by taking the massless approximation to \( \Gamma - A - B - S \). The result for \( H \) reads

\[
H = -\frac{g^2}{8\pi^2} \int \frac{d\mathbf{k}^2}{\mathbf{k}^2} \left\{ \ln \left( \frac{\mathbf{k}^2}{Q^2} \right) + i\pi + \frac{1 - \mathbf{k}^2/Q^2}{R} \left[ \ln \left( \frac{1 + R}{1 - R} \right) - i\pi \right] \right\} - \text{UV} \tag{6}
\]

where \( Q^2 = 2p_A^+ p_B^- \) and

\[
R = \begin{cases} 
\sqrt{1 - 4\mathbf{k}^2/Q^2} & \text{if } 4\mathbf{k}^2/Q^2 \leq 1 , \\
i \sqrt{4\mathbf{k}^2/Q^2 - 1} & \text{if } 4\mathbf{k}^2/Q^2 > 1 .
\end{cases} \tag{7}
\]

\( H \) is independent of the choice of the vectors \( u_A, u_B \). As a result of the infrared subtractions, the \( \mathbf{k}^2 \) integration in Eq. (6) is regular at small \( \mathbf{k}^2 \). The large \( \mathbf{k}^2 \) behavior is to be dealt with via an ultraviolet counterterm.

In conclusion, the subtractive procedure that we have applied gives a finite coefficient for the hard part of the form factor, while the counterterms have a simple meaning in terms of gauge-invariant operators and obey homogeneous evolution equations. Similar procedures can be defined in real emission processes for collinear and soft contributions. Calculational schemes of this kind will be needed to improve the accuracy of Monte Carlo calculations for multiparticle final states beyond the leading order.
Acknowledgments

The results presented in this talk have been obtained in collaboration with J. Collins. This research is supported in part by the US Department of Energy. I thank the organizers of ISMD2000 for their invitation and for the excellent organization of the conference.

References

1. See, for instance, R.K. Ellis, W.J. Stirling and B.R. Webber, *QCD and collider physics*, Cambridge University Press, Cambridge, 1996; *QCD and beyond*, Proceedings of the Theoretical Advanced Study Institute TASI ’95, edited by D.E. Soper (World Scientific, Singapore, 1996).
2. J.C. Collins and F. Hautmann, Phys. Lett. B 472, 129 (2000).
3. J.C. Collins, JHEP 0005:004 (2000).
4. J.C. Collins and F. Hautmann, JHEP 0103:016 (2001); F. Hautmann, hep-ph/0101006.
5. C. Friberg and T. Sjöstrand, hep-ph/9906316 in Proceedings of the DESY Workshop “Monte Carlo Generators for HERA Physics”, eds. A.T. Doyle, G. Grindhammer, G. Ingelman and H. Jung (Hamburg, 1999), p.181.
6. I.I. Balitsky and V.M. Braun, Nucl. Phys. B311, 541 (1989).
7. J.C. Collins, in *Perturbative Quantum Chromodynamics*, edited by A.H. Mueller, World Scientific 1989, p. 573.
8. S.B. Libby and G. Sterman, Phys. Rev. D 18, 3252 (1978); *ibid*. 18, 4737 (1978).
9. G.P. Korchemsky and A.V. Radyushkin, Sov. J. Nucl. Phys. 45, 910 (1987).
10. G.P. Korchemsky, Mod. Phys. Lett. A4, 1257 (1989), Phys. Lett. B 220, 629 (1989).
11. F.V. Tkachov, hep-ph/9703423 Int. J. Mod. Phys. A8, 2047 (1993).