INSTANTS OF SMALL AMPLITUDE OF BROWNIAN MOTION AND APPLICATION TO THE KUBILIUS MODEL

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Abstract

Let \( W(t), t \geq 0, \) be the standard Brownian motion. We study the size of the time intervals \( I \) which are admissible for the long range of slow increase, namely given a real \( z > 0, \sup_{t \in I} |W(t)|/\sqrt{t} \leq z. \) We obtain optimal results in terms of class test functions and, by means of the quantitative Borel–Cantelli lemma, a finer frequency result concerning their occurrences. Some of these results extend to sums of independent random variables by means of Sakhanenko’s invariance principle. By also using this device to transfer the results to the Kubilius model, we derive applications to the prime number divisor function. We obtain refinements of some results recently proved by Ford and Tenenbaum in [4].

1. Introduction and main results

Let \( W(t), t \geq 0, \) be the standard Brownian motion. Let \( z \) be some positive real. The study of the number of occurrences of the time intervals \( I \) for which

\[
\sup_{t \in I} \frac{|W(t)|}{\sqrt{t}} \leq z
\]

is the first motivation of this work. In a second step, we will derive applications for the Kubilius model in number theory. More precisely, let \( f: [1, \infty) \rightarrow \mathbb{R}^+ \) be here and throughout a non-decreasing function such that \( f(t) \uparrow \infty \) with \( t \) and

\[
f(t) = o_{\rho}(t^\rho).
\]

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We will consider intervals of type $I = [N, Nf(N)]$. We essentially examine the case $N = e^k$, $k = 1, 2, \ldots$. The study made can be extended with no difficulty to more general geometrically increasing sequences, but this aspect will be not developed.

Put

\[
A_k(f, z) = \left\{ \sup_{e^k \leq t \leq e^k f(e^k)} \frac{|W(t)|}{\sqrt{t}} < z \right\}, \quad k = 1, 2, \ldots
\]  

(1.1)

Let $U(t) = W(e^t)e^{-t/2}, t \in \mathbb{R}$ be the Ornstein–Uhlenbeck process. It will be more convenient to work with $U$ instead of $W$. Observe that

\[
A_k(f, z) = \left\{ \sup_{k \leq s \leq k + \log f(e^k)} |U(s)| \leq z \right\}.
\]

And so, as $U$ is stationary,

\[
\mathbb{P}\{A_k(f, z)\} = \mathbb{P}\left\{ \sup_{0 \leq s \leq \log f(e^k)} |U(s)| \leq z \right\}.
\]

We say that $f \in U_z$ whenever $\mathbb{P}\{\limsup_{k \to \infty} A_k(f, z)\} = 0$, and that $f \in V_z$ if $\mathbb{P}\{\limsup_{k \to \infty} A_k(f, z)\} = 1$. By the 0–1 law (since $U$ is strongly mixing), the latter probabilities can only be 0 or 1.

Notice that if $f \in U_z$, then with probability one

\[
J(f) := \liminf_{k \to \infty} \sup_{k \leq s \leq k + \log f(e^k)} |U(s)| > z,
\]

whereas $J(f) \leq z$, almost surely if $f \in V_z$. In the latter case, it makes sense to estimate the size of the counting function

\[
N_n(f, z) = \sum_{k=1}^{n} \chi_{A_k(f, z)} \quad n = 1, 2, \ldots.
\]

Naturally this has to be done with respect to the corresponding means $\nu_n(f, z) := \mathbb{E}N_n(f, z)$.

We shall first characterize the classes $U_z$ and $V_z$ by means of a simple convergence criterion, and complete our characterization by including a frequency result concerning the class $V_z$.

**Theorem 1.1.** There exists $\lambda(z) > 0$ with $\lambda(z) \sim \frac{z^2}{4e}$ as $z \to 0$, such that if $\Sigma(f) = \sum_k f(e^k)^{-\lambda(z)}$, then

\[
f \in U_z \quad (\text{resp. } \in V_z) \quad \iff \quad \Sigma(f) < \infty \quad (\text{resp. } = \infty).
\]

Further for any $a > 3/2$,

\[
N_n(f, z) \overset{\text{as}}{=} \nu_n(f, z) + O(\nu_n^{1/2}(f, z) \log^a \nu_n(f, z)).
\]
And there are positive constants $K_1(z), K_2(z)$ depending on $z$ only, such that for all $n$

$$K_1(z) \leq \frac{\nu_n(f, z)}{\sum_{k=1}^{\infty} f(e^k)^{-\lambda(z)}} \leq K_2(z).$$

The critical value $\lambda(z)$ is the smallest eigenvalue in the Sturm–Liouville equation (2.1). See Section 2.

The class of functions $f_c(t) = \log^c t$, $c > 0$, is of special interest in view of applications to the Kubilius model. We deduce from Theorem 1.1:

**Corollary 1.2.** If $c > 1/\lambda(z)$, then $f_c \in U_z$ whereas $f_c \in V_z$ if $0 < c \leq 1/\lambda(z)$. Further, for any $0 < c \leq 1/\lambda(z)$ and $a > 3/2$,

$$N_n(f_c, z)^{1/2} \nu_n(f_c, z) + O(\nu_n^{1/2}(f_c, z) \log^a \nu_n(f_c, z)),$$

and for all $n$

$$K_1(z) \leq \frac{\nu_n(f_c, z)}{\sum_{k=1}^{\infty} k^{-c\lambda(z)}} \leq K_2(z).$$

Accordingly, if

$$I(f) := \lim inf_{k \to \infty} \sup_{e^k \leq t \leq e^{k+1}} \frac{|W(t)|}{\sqrt{t}},$$

then $\mathbb{P}\{I(f_c) \leq z\} = 1$ if and only if $0 < c \leq 1/\lambda(z)$. This is clear in view of (1.1). Noticing that $I(f) \leq I(g)$ whenever $f(N) \leq g(N)$ for all $N$ large, we therefore also deduce

**Corollary 1.3.** We have $\mathbb{P}\{I(f_c) \leq z\} = 1$ if and only if $0 < c \leq 1/\lambda(z)$, and $\mathbb{P}\{I(f) = \infty\} = 1$ if $f(t) \gg_c f_c(t)$ for all $c$.

**Remark 1.4.** This slightly improves upon Theorem 3 in [4], where it was shown that $\mathbb{P}\{I(f) < \infty\} = 1$ if $f(N) = (\log N)^b$ for some $b > 0$, whereas $\mathbb{P}\{I(f) = \infty\} = 1$, if $f(N) = (\log N)^{b(N)}$ with $b(N) \to \infty$ with $N$.

In [4], the behavior of the corresponding functionals $I_f$ for sums of independent random variables (assuming only second absolute moments) was also considered. In this direction, we will also establish the following result for sums of independent random variables.
**Theorem 1.5.** Let \( \{X_j, j \geq 1\} \) be independent centered random variables. Assume that, for some \( \alpha > 2 \),

\[
\sum_{j \geq 1} \mathbb{E}|X_j|^\alpha = \infty \quad \text{and} \quad v = \sup_{j \geq 1} \mathbb{E}|X_j|^\alpha < \infty. \tag{1.2}
\]

Let \( Z_n = X_1 + \cdots + X_n, \ z_n^2 = \mathbb{E}Z_n^2, \ J_n = \{j : n \leq z_j^2 \leq nf(n)\} \). Then there exists a Brownian motion \( W \) such that

\[
\lim \inf_{k \to \infty} \sup_{e^k \leq z_j^2 \leq e^k f(e^k)} \left| \frac{Z_j}{z_j} \right| \overset{a.s.}{=} \lim \inf_{k \to \infty} \sup_{e^k \leq z_j^2 \leq e^k f(e^k)} \left| \frac{W(z_j^2)}{z_j} \right|
\]

almost surely. In particular, if \( c < 1/\lambda(z) \), then

\[
\lim \inf_{k \to \infty} \sup_{e^k \leq z_j^2 \leq e^k f(e^k)} \left| \frac{Z_j}{z_j} \right| \leq z \quad \text{almost surely.}
\]

Notice in the iid case that assumption (1.2) simply reduces to the integrability condition \( \mathbb{E}|X_1|^\alpha < \infty \) for some \( \alpha > 2 \).

Now introduce the truncated prime divisor function \( \omega(m,t) = \#\{p \leq t : p \mid m\} \). Here and throughout we reserve the letter \( p \) to denote some arbitrary prime number. Put

\[
\rho(m,t) := \frac{\omega(m,t) - \log \log t}{\sqrt{\log \log t}}.
\]

The local variations of \( \rho(m,t) \) were recently investigated by Ford and Tenenbaum, who obtained in [4], after a careful study of the size of intervals of slow growth for general sums of independent random variables, quite elaborated asymptotic estimates, on the basis of the approximation formula (2.3). The results concern the functional

\[
\max_{N \leq \log \log t \leq N f(N)} \rho(m,t). \tag{1.3}
\]

Let \( f(m), g(m) \) be increasing and tending to infinity with \( m \). It is notably proved ([4, Theorem 5]) that if \( g(m) \leq (\log \log m)^{1/10} \) and \( f(N) = (\log N)^{\xi(N)} \) where \( \xi(N) \to \infty \) sufficiently slowly so that \( f(N) \leq N \), then

\[
\min_{\frac{g(m)}{N f(N)} \leq \log \log m} \max_{N \leq \log \log t \leq N f(N)} \rho(m,t) \to \infty
\]

along a set of integers \( m \) of natural density 1.

Further, if \( f(N) = (\log N)^c \) and \( g^2(m)(\log g^2(m))^c \leq \log \log m \) for \( m \) large, then on a set of integers \( m \) of density 1, we have

\[
\min_{g(m) \leq N \leq g^2(m)} \max_{N \leq \log \log t \leq N f(N)} \rho(m,t) \leq 30 \sqrt{1 + e}.
\]
This provides information on the size of intervals which are admissible for the long range of slow increase in terms of the natural density on the integers. For instance \( g(m) = \sqrt{\log \log m}/(\log \log \log m)^x \) is suitable. The principle followed in the proofs consists in modifying the proofs of the preliminary results on the size of intervals of slow growth for sums of independent random variables for the particular sequence \( \{T_n, n \geq 1\} \) (Section 2) and next applying the approximation formula (2.3).

Here we will proceed slightly differently. As we have optimal results on instants of small amplitude of Brownian motion, we directly compare the functionals (1.3) with analogous functionals of Brownian motion by means of Sakhanenko’s invariance principle (Lemma 2.4). This is done in Theorem 1.6 below. This allows to transfer our previous results to the truncated prime divisor function, not fully naturally, but sufficiently much to get new quite sharp results. More precisely, let \( 0 < M_1(x) < M_2(x), M_2(x) \uparrow \infty \) with \( x \). The previous results, as well as Theorem 1.1, Corollary 1.2, suggest to study the behaviour, for \( x \) large, of the averages

\[
\frac{1}{x} \# \left\{ m \leq x : \inf_{M_1(x) < N \leq M_2(x)} \sup_{N \leq s_j^2 \leq f(N)} \frac{|\omega(m, j) - s_j^2|}{s_j} \leq z \right\}. \tag{1.4}
\]

Here we set
\[
s_j^2 := \sum_{p \leq j} \frac{1}{p} = \frac{1}{p^2} = \log \log j + O(1), \tag{1.5}
\]
and the last relation comes from Mertens’ estimate. For technical reasons (scale invariance properties of \( W \) and Kubilius model, see next section), it turns up that it is more convenient to replace the “\( \log \log j \)” term appeared before by \( s_j^2 \). The resulting modifications are thus neglectable in the statements. We show that the asymptotic order of the averages (1.4) can be quantified by using the Ornstein–Uhlenbeck process. More precisely, let

\[
I_N = I_N(f) = \{ j : N \leq s_j^2 \leq Nf(N) \}.
\]

Let also \( N \) denote some increasing sequence of positive reals tending to infinity. The theorem below allows to reduce the study of the averages (1.4) to the one of similar questions for the Ornstein–Uhlenbeck process. Other formulations may be easily extrapolated from the proof.

**Theorem 1.6.** Assume that \( M_2(x) = O_x(x^x) \). Let \( 0 < z'' < z < z' \). As \( x \) tends to infinity,

\[
\mathbb{P} \left\{ \inf_{M_1(x) < N \leq M_2(x)} \sup_{j \in I_N} \frac{|W(s_j^2)|}{s_j} \leq z'' \right\} + o(1)
\]
\[
\leq \frac{1}{x} \# \left\{ m \leq x : \inf_{M_1(x) < N \leq M_2(x)} \sup_{j \in I_N} \frac{|\omega(m, j) - s_j^2|}{s_j} \leq z \right\}
\]
\[
\leq \mathbb{P} \left\{ \inf_{M_1(x) < N \leq M_2(x)} \sup_{j \in I_N} \frac{|W(s_j^2)|}{s_j} \leq z' \right\} + o(1).
\]
By combining with Corollary 1.2, we deduce for instance

**Corollary 1.7.** Assume that $M_2(x) = O(x^{c'})$ and $\log M_1(x) = o(\log M_2(x))$. Let $c < 1/\lambda(z)$. Then

$$\lim_{x \to \infty} \frac{1}{x} \# \left\{ 1 \leq m \leq x : \inf_{\lambda(x) \leq k \leq \log M_2(x), \epsilon k \leq s_j^2 \leq \epsilon k} \frac{\mid \omega(m, j) - s_j^2 \mid}{s_j} \leq \varepsilon \right\} = 1.$$  

**Remark 1.8.** Let $d > 1$. There is no loss when restricting to $x^{1/d} \leq m \leq x$ in the above ratios. But $M_1(x) < e^k \leq M_2(x)$ imply $M_1(m) < e^k \leq M_2(m^d)$. This allows to deduce from Corollary 1.7 a result similar to those in [4] previously described, namely for any $d > 1$,

$$\lim_{x \to \infty} \frac{1}{x} \# \left\{ 1 \leq m \leq x : \inf_{\lambda(x) \leq k \leq \log M_2(m^d), \epsilon k \leq s_j^2 \leq \epsilon k} \frac{\mid \omega(m, j) - s_j^2 \mid}{s_j} \leq \varepsilon \right\} = 1,$$

Taking for instance $M_2(x) = \log x$, $M_1(x) = \log^{c'}(x)$, $\varepsilon(x) \to 0$, we get

$$\lim_{x \to \infty} \frac{1}{x} \# \left\{ 1 \leq m \leq x : \inf_{\epsilon k \leq \log m \leq \epsilon k, \epsilon k \leq \epsilon k} \frac{\mid \omega(m, j) - s_j^2 \mid}{s_j} \leq \varepsilon \right\} = 1.$$

The relation $c < 1/\lambda(z)$ asymptotically becomes $\pi \sqrt{e}/2 \leq z$, $z \to 0$.

**Remark 1.9.** If instead of the condition $\log M_1(x) = o(\log M_2(x))$, we have the weaker assumption $\log M_1(x) = \rho \log M_2(x), 0 < \rho < 1$, then by operating similarly and using the 0–1 law, we would also get for $\rho$ sufficiently small

$$\lim_{x \to \infty} \frac{1}{x} \# \left\{ 1 \leq m \leq x : \inf_{M_2(m) \epsilon k \leq \log M_2(m), \epsilon k \leq \epsilon k} \frac{\mid \omega(m, j) - s_j^2 \mid}{s_j} \leq \varepsilon \right\} = 1.$$  

However, we have no idea about a suitable precise value of $\rho$.

We will further establish a delicate frequency result for the truncated divisor function, which is in the spirit of Theorem 1.1.

**Theorem 1.10.** Let $0 \leq M_1(x) < M_2(x)$, $M_1(x) \uparrow \infty$ be such that $M(x) = O(x^c)$. For any $c' > c > 0$ and $c < 1/\lambda(z)$, there exists a constant $\kappa > 0$ be such that

$$\lim_{x \to \infty} \frac{1}{x} \# \left\{ m \leq x : \inf_{M_1(x) \leq n \leq M_2(x)} \frac{\# \{ k \leq n : \sup_{\epsilon k \leq \epsilon k} \frac{\mid \omega(m, j) - s_j^2 \mid}{s_j} \leq \varepsilon \}}{n^{1-c\lambda(z)}} \geq \kappa \right\} = 1.$$
2. Auxiliary results

We first list the needed probabilistic results. Next we briefly describe the Kubilius model and extract from the fundamental inequality a useful lemma. The underlying small deviation problem, namely the study for small $z$ of

$$\Pr \left\{ \sup_{0 \leq s \leq t} |U(s)| < z \right\}$$

is intimately linked to the Sturm–Liouville equation

$$\psi''(x) - x\psi'(x) = -\lambda \psi(x), \quad \psi(-z) = \psi(z) = 0. \quad (2.1)$$

Let $\lambda_1 \leq \lambda_2 \leq \cdots$ and $\psi_1(x), \psi_2(x), \cdots$ respectively denote the eigenvalues and normed eigenfunctions of equation (2.1). Here $\lambda_i, \psi_i$ depend on $z$ and it is known that $\psi_1, \psi_2, \ldots$ form an orthonormal sequence with respect to the weight function $e^{-x^2/2}$. According to Newell’s result (see [7], see also (3.16) in [1])

$$\Pr \left\{ \sup_{0 \leq s \leq t} |U(s)| < z \right\} = \frac{1}{(2\pi)^{1/2}} \sum_{k=1}^{\infty} e^{-\lambda_k t} \left( \int_{-z}^{z} \psi_k(x)e^{-x^2/2}dx \right)^2.

Let $\lambda(z)$ denote the smallest eigenvalue ($\lambda(z) = \lambda_1$). Then $\lambda(z) > 0$ is a strictly decreasing continuous function of $z$ on $(0, \infty)$. Further

$$\lambda(z) \sim \frac{\pi^2}{4z^2} \quad \text{as } z \to 0.$$

See Lemma 3.1 in [1], see also Lemma 2.2 for the following result.

**Lemma 2.1.** (Csáki’s estimate) For $z > 0$, $t > 0$ we have

$$\frac{e^{-\lambda(z)t}}{(2\pi)^{1/2}} \left( \int_{-z}^{z} \psi_1(x)e^{-x^2/2}dx \right)^2 \leq \Pr \left\{ \sup_{0 \leq s \leq t} |U(s)| < z \right\} \leq \frac{e^{-\lambda(z)t}}{1 - e^{-t}}.

It follows that for $z > 0$, there exist positive constants $K_1(z), K_2(z)$ such that for all $t \geq 1$

$$K_1(z)e^{-\lambda(z)t} \leq \Pr \left\{ \sup_{0 \leq s \leq t} |U(s)| < z \right\} \leq K_2(z)e^{-\lambda(z)t}. \quad (2.2)$$

Now let $\mathcal{E}_s^t$ denote the vector space generated by $U(u)$, $s \leq u \leq t$, and introduce the maximal correlation coefficient

$$\rho(\tau) = \sup_{\xi \in \mathcal{E}_s^t, \eta \in \mathcal{E}_s^u} \frac{|\mathbb{E}(\xi - \mathbb{E}\xi)(\eta - \mathbb{E}\eta)|}{\sqrt{\mathbb{E}(\xi - \mathbb{E}\xi)^2}\mathbb{E}(\eta - \mathbb{E}\eta)^2}}^{1/2}.$$

By stationarity, this one does not depend on $t$. Stationary Gaussian processes such that $\rho(\tau) \to 0$ as $\tau \to \infty$ are called completely regular. The spectral density of $U$ has the form $|\Gamma(\lambda)|^{-2}$ with $\Gamma(\lambda) = 1 + i\lambda$, which is obviously an entire function. Moreover, we also have $\log |\Gamma(\lambda)| \in L^1(\mathbb{R})$. Further $\Gamma$ has $i$ as unique imaginary zero. As $\Im(\frac{1}{1+\lambda}) = \frac{1}{1+\lambda^2}$, it follows that $\sup_{\lambda \in \mathbb{R}} |\Im(\frac{1}{1+\lambda})| < \infty$. Our next lemma is therefore just a direct consequence of [5, Section VI.6, Theorem 6].
Lemma 2.2. The process $U$ is completely regular, and further
\[ \rho(\tau) = O(e^{-(1-\varepsilon)\tau}). \]

This result, which is due to Kolmogorov and Rozanov ([6], see Theorem 1 and remarks at the end of p. 207), will be crucial in the proof of Theorem 1.1.

Recall also the classical form of the Borel–Cantelli quantitative lemma ([8, Theorem 3] or [12, Theorem 8.3.1]).

Lemma 2.3. Let \{\(A_k, k \geq 1\)\} be a sequence of events satisfying
\[ P(A_k \cap A_\ell) \leq P(A_k)P(A_\ell) + \gamma_{\ell-k}P(A_k) \] (\(\forall \ell \geq k \geq 1\)),

where \(\gamma_i \geq 0\) and \(\sum_{i=0}^{\infty} \gamma_i < \infty\). Let \(\psi_n = \sum_{k=1}^{n} P(A_k)\) and assume that \(\psi_n \to \infty\) with \(n\). Then for every \(a > 3/2\),
\[ \sum_{k=1}^{n} \chi_{A_k} \xrightarrow{a.s.} \psi_n + O(a^{1/2}(\log \psi_n)^a). \]

We finally need a suitable invariance principle for sums of independent random variables. This one is due to Sakhanenko (see [10, Theorem 1]). We give its most appropriate formulation for our purpose. Let \{\(\xi_j, j \geq 1\)\} be independent centered random variables with absolute second moments. Let \(t_k = \sum_{j=1}^{k} E\xi_j^2, S_k = \sum_{j=1}^{k} \xi_j\) and let \{\(r_k, k \geq 1\)\} be some non-decreasing sequence of positive reals. Let \(\alpha \geq 2, y > 0\). Put successively
\[ \Delta_n = \sup_{k \leq n} |S_k - W(t_k)|, \]
\[ \Delta = \sup_{n \geq 1} \frac{\Delta_n}{r_n}, \]
\[ \overline{\xi} = \sup_{j \geq 1} \frac{|\xi_j|}{r_j}, \]
\[ L_\alpha(y) = \sum_{j \geq 1} E\min \left\{ \frac{|\xi_j|^\alpha}{y^{\alpha}r_j^\alpha}, \frac{|\xi_j|^2}{y^2r_j^2} \right\}. \]

Lemma 2.4. There exists an absolute constant \(C\) such that for any fixed \(\alpha\) there exists a Brownian motion \(W\) such that for all \(x > 0\)
\[ P\{\Delta \geq Cx\alpha x\} \leq L_\alpha(x). \]

Now we pass to the Kubilius model. Recall that \(p\) denotes some arbitrary prime number. Let \{\(Y_p, p \geq 1\)\} be a sequence of independent binomial random
variables such that $\mathbb{P}\{Y_p = 1\} = 1/p$ and $\mathbb{P}\{Y_p = 0\} = 1 - 1/p$. We can view $Y_p$ as modelling whether or not an integer taken at random is divisible by $p$. Let

$$T_n = \sum_{p \leq n} Y_p, \quad S_n = T_n - ET_n.$$ 

Then $\mathbb{E}S_n^2 = s_n^2 = \log \log n + O(1)$ by (1.5). The sequence $\{T_n, n \geq 1\}$ is known to asymptotically behave as the truncated prime divisor function

$$\omega(m, t) = \#\{p \leq t : p|m\},$$

at least when $t$ is not too close to $m$. More precisely, let

$$\omega_r(m) = (\omega(m, 1), \ldots, \omega(m, r)),$$

where $r$ is some integer with $2 \leq r \leq x$, and put $u = \frac{\log x}{\log r}$. Then, given an arbitrary $c < 1$, we have uniformly in $x, r$ and $Q \subset \mathbb{Z}^r$,

$$\frac{\#\{m \leq x : \omega_r(m) \in Q\}}{x} = \mathbb{P}\{(T_1, \ldots, T_r) \in Q\} + O(x^{-c} + e^{-u \log u}). \quad (2.3)$$

See Lemmas 3.2, 3.5 in [3, Chapter 3]. See also [11, Theorem 1] for a more precise result involving the Dickman function.

**Remark 2.5.** There are natural restrictions in the application of this estimate to asymptotic studies, due to the error term $e^{-u \log u}$. To make it small, it requires if $r = r(x)$ that $r(x) = O(x^{\epsilon})$ for all $\epsilon > 0$. This amounts to truncate the prime divisor function $\omega(m)$ at level $O(x^{\epsilon})$, which is satisfactory as long as $m \ll x$. However, these integers have a neglectable contribution on the size of the left term of (2.3). Therefore the model is mostly adapted to the analysis of the distribution of the small divisors of an integer. See [3, p. 122], see also [11, Introduction] for a complete and precise analysis of this point.

Estimate (2.3) can be for instance used to estimate the number of integers having no prime divisors in prescribed sets. Let $I = [p, q]$, $q \leq x$; as

$$\#\{m \leq x : p|m \Rightarrow p \notin I\} = \#\{m \leq x : \omega(m, p) = \cdots = \omega(m, q)\},$$

it follows that

$$\frac{1}{x} \#\{m \leq x : p|m \Rightarrow p \notin I\} = \prod_{p \in I} \left(1 - \frac{1}{p}\right) + O(x^{-c} + e^{-u \log u}).$$

Choosing $I = [2, y]$, next $I = [y, x]$ allows to recover a known formula on the smallest or largest prime divisors of $m$. 
Clearly, the approximation formula (2.3) can be used to transfer properties from \((T_k)\) to \(\omega\). Let indeed \(f\) be such that \(f(N) = o_p(N^\gamma)\). Recall that \(I_N = \{j : N \leq s_j^2 \leq Nf(N)\}\) and let \(\mathcal{N}\) be some fixed increasing sequence of reals. Moreover, let \(M_i : \mathbb{N} \to \mathbb{R}^+\) be non-decreasing with \(\lim_{x \to \infty} M_i(x) = \infty, i = 1, 2,\) and such that

\[
1 \leq M_1(x) < M_2(x), \quad M_2(x) = O_e(x^\gamma).
\]

Let \(r = r(x) \sim M_2(x)f(M_2(x)), r\) an integer. Then

\[
u = \nu(x) = \log x \log r(x) \sim \log x \log M_2(x) \to \infty
\]

with \(x\). Put

\[
Q_x = \bigcup_{M_i(x) < N \leq M_2(x)} \left\{ (\nu_1, \ldots, \nu_r) \in \mathbb{Z}^r : \sup_{j \in I_N} \left| \omega(m, j) - \frac{\nu_j - s_j^2}{s_j} \right| \leq z \right\}.
\]

By applying (2.3) with \(Q = Q_x\), we get the following useful comparison relation:

**Lemma 2.6.** For any \(z > 0\), as \(x\) tends to infinity,

\[
\frac{1}{x} \# \left\{ m \leq x : \inf_{N \leq s \leq M_2(x)} \sup_{j \in I_N} \left| \omega(m, j) - \log \log j \right| \leq z \right\} = P \left\{ \inf_{N \leq s \leq M_2(x)} \sup_{j \in I_N} \left| T_j - \log \log j \right| \leq z \right\} + o(1).
\]

**3. Proof of Theorem 1.1**

By stationarity and by using (2.2),

\[
\frac{K_1(z)}{f(e^k \lambda(z))} \leq P(A_k(f, z)) = P \left\{ \sup_{0 \leq s \leq \log f(e^k)} |U(s)| < z \right\} \leq \frac{K_2(z)}{f(e^k \lambda(z))}.
\]

By summing up,

\[
K_1(z) \sum_{k=1}^n f(e^k)^{-\lambda(z)} \leq \nu_0(f, z) \leq K_2(z) \sum_{k=1}^n f(e^k)^{-\lambda(z)}.
\]

If the series \(\Sigma(f) = \sum_k f(e^k)^{-\lambda(z)}\) converges, by the first Borel–Cantelli lemma

\[
P \left\{ \sup_{k \leq s \leq k + \log f(e^k)} |U(s)| > z, \ k \text{ eventually} \right\} = 1.
\]
Hence \( f \in \mathcal{U}_2 \). Now consider the case \( \Sigma(f) = \infty \). We shall prove that \( f \in \mathcal{V}_2 \). Let \( 0 < c_1 < 1/\lambda(z) < c_2 \) and put

\[
f_1(t) = \log^{c_1} t, \quad f_2(t) = \log^{c_2} t.
\]

We may assume \( f_1 \leq f \leq f_2 \). This is a standard device. Indeed, as \( f_2 \in \mathcal{U}_2 \), we have the implication: \((f_1 \lor f) \land f_2 \in \mathcal{V}_2 \Rightarrow (f_1 \lor f) \in \mathcal{V}_2 \Rightarrow f \in \mathcal{V}_2 \). So it suffices to prove that \((f_1 \lor f) \land f_2 \in \mathcal{V}_2 \). We now use the simplified notation \( A_k(f, z) = A_k \)

and notice that \( K'(z)k^{-c_2\lambda(z)} \leq \mathbb{P}(A_k) \leq K''(z)k^{-c_1\lambda(z)} \). By Lemma 2.2, for every \( \ell > k \),

\[
\frac{|\mathbb{P}(A_k \cap A_\ell) - \mathbb{P}(A_k)\mathbb{P}(A_\ell)|}{\sqrt{\mathbb{P}(A_k)(1 - \mathbb{P}(A_k))\mathbb{P}(A_\ell)(1 - \mathbb{P}(A_\ell))}} \leq C_1 e^{-C_2(\ell - k)},
\]

\( C_1, C_2 \) being absolute constants. Hence

\[
\mathbb{P}(A_k \cap A_\ell) - \mathbb{P}(A_k)\mathbb{P}(A_\ell) \leq C_1 e^{-C_2(\ell - k)}\sqrt{\mathbb{P}(A_k)\mathbb{P}(A_\ell)}
\]

\[
\leq C_1 e^{-C_2(\ell - k)}\mathbb{P}(A_\ell)(\frac{\mathbb{P}(A_k)}{\mathbb{P}(A_\ell)})^{1/2}
\]

\[
\leq C(z)\mathbb{P}(A_\ell)e^{-C_2(\ell - k)}\left(\frac{c_2}{k^{c_1}}\right)^{\lambda(z)/2}.
\]

But for some absolute constant \( C_3 < C_2 \) and \( C_4 > 0 \) depending on \( z \), we have

\[
e^{-C_2(\ell - k)}\left(\frac{c_2}{k^{c_1}}\right)^{\lambda(z)/2} \leq C_4 e^{-C_3(\ell - k)}.
\]

Indeed, let \( \ell = (H + 1)k, H \geq 0 \). This amounts to showing that

\[
(H + 1)c_2\lambda(z)/2k^{(c_2 - c_1)\lambda(z)/2} \leq C_4 e^{(C_2 - C_3)Hk}.
\]

We use the following inequality. Let \( \delta, \beta, \varepsilon \) be positive reals with \( \delta \geq \beta \). Then there exists \( C \) depending on \( \delta, \beta, \varepsilon \) only such that \( H^\delta k^\beta \leq C e^{Hk} \) for all non-negative reals \( H, k \) with \( k \geq 1 \). Indeed, if \( 0 \leq H \leq 1 \), then \( H^\delta k^\beta \leq (Hk)^\beta \leq C e^{Hk} \). And if \( H > 1 \), \( H^\delta k^\beta \leq (Hk)^\beta \leq C e^{Hk} \).

Applying this with \( \delta = c_2\lambda(z)/2, \beta = (c_2 - c_1)\lambda(z)/2 \) yields

\[
H^{c_2\lambda(z)/2k^{(c_2 - c_1)\lambda(z)/2}} \leq C e^{Hk},
\]

which implies our claim. Therefore

\[
|\mathbb{P}(A_k \cap A_\ell) - \mathbb{P}(A_k)\mathbb{P}(A_\ell)| \leq C_4 e^{-C_3(k - \ell)}\mathbb{P}(A_\ell).
\]

Lemma 2.3 thus applies, and we deduce (for every \( a > 3/2 \))

\[
\sum_{k=1}^{n} \chi A_k \overset{a.s.}{=} \nu_n(f, z) + O_a(\nu_n(f, z)^{1/2} \log^{a} \nu_n(f, z)).
\]
In particular
\[ P\left\{ \sup_{k \leq s \leq k + \log f(e^k)} |U(s)| \geq z, \; k \text{ infinitely often} \right\} = 1. \]

Hence also \( f \in V_z \).

Corollary 1.2 follows easily. Indeed, let \( 0 < c \leq 1/\lambda(z) \). By Theorem 1.1, 
\( N_n(f_c, z) \uparrow \infty \) almost surely. And so \( P\{J(f_c) \leq z\} = 1 \). Now if \( c > 1/\lambda(z) \), in view of estimate (2.2) the series \( \sum_{k=1}^{\infty} P\{A_k(f_c, z)\} \) converges. And by the first Borel–Cantelli lemma \( P\{J(f_c) > z\} = 1 \). Corollary 1.3 is just a reformulation of Corollary 1.2 using the change of variable \( s = e^t \).

4. Proof of Theorem 1.6

Let \( \varepsilon, \eta \) be positive reals. Let \( \alpha \) be sufficiently large so that \( \varepsilon \alpha > 1 + \eta \). Apply Lemma 2.4 to \( S_n \) (here \( \xi_p = Y_p - EY_p \)). Choose \( r_p = (\log \log p)^{1+\eta} \) and recall that \( E|\xi_p|^\alpha \sim 1/p \) for \( p \) large. Then
\[
\sum_p E[|\xi_p|^\alpha] / r_p^\alpha \leq C \sum_p \frac{1}{p(\log \log p)^{1+\eta}} \leq C \sum_j \frac{1}{j \log j(\log \log j)^{1+\eta}} < \infty.
\]

We have used the fact that if \( p_j \) denotes the \( j \)-th prime number in the increasing order, then \( p_j \sim j \log j \). Now notice the following simple estimate valid for all positive \( y \):
\[
L_\alpha(y) \leq y^{-\alpha} \sum_{j \geq 1} E[|\xi_j|^\alpha] / r_j^\alpha.
\]

We deduce that \( L_\alpha(y) \leq C_\alpha y^{-\alpha} \). Recall that \( E S_n^2 = s_n^2 \). Therefore there exists a Brownian motion \( W \) such that if
\[
\Upsilon_\varepsilon = \sup_{n \geq 1} \frac{\sup_{j \leq n} |S_j - W(s_j^2)|}{(\log \log n)^\varepsilon},
\]
then \( E\Upsilon_\varepsilon < \infty, \beta < \alpha \). We will just use the fact that \( E\Upsilon_\varepsilon < \infty \). Let \( z' > z \). By using Lemma 2.6, we have
\[
\frac{1}{x} \#\left\{ m \leq x : \inf_{N \in N} \sup_{j \in I_N} \frac{|\omega(m,j) - s_j^2|}{s_j} \leq z \right\} = P\left\{ \inf_{N \in N} \sup_{j \in I_N} \frac{|S_j|}{s_j} \leq z \right\} + o(1) \quad (4.1)
\]
\[
\leq P\left\{ \inf_{N \in N} \sup_{j \in I_N} \frac{|W(s_j^2)|}{s_j} \leq z' \right\} + P\{A\} + o(1),
\]
where we set

\[ A = \left\{ \inf_{N \in \mathbb{N}} \sup_{j \in I_N} \frac{|W(s_j^2)|}{s_j} > z', \inf_{N \in \mathbb{N}} \sup_{j \in I_N} \frac{|S_j|}{s_j} \leq z \right\}. \]

We have

\[
\left| \inf_{N \in \mathbb{N}} \sup_{j \in I_N} \frac{|W(s_j^2)|}{s_j} - \inf_{N \in \mathbb{N}} \sup_{j \in I_N} \frac{|S_j|}{s_j} \right| \\
\leq \sup_{N \in \mathbb{N}} \sup_{j \in I_N} \frac{|S_j - W(s_j^2)|}{s_j} \leq \sup_{N \in \mathbb{N}} \sup_{j \in I_N} \frac{|S_j - W(s_j^2)|}{s_j},
\]

where \( j^* \) denotes the largest index such that \( s_j^2 \in I_N \) of \( I_N \). Thus

\[
\mathbb{P}\{A\} \leq \mathbb{P}\left\{ \sup_{N \in \mathbb{N}} \sup_{j \leq j^*} \frac{|S_j - W(s_j^2)|}{s_j} > z' - z \right\}.
\]

Let \( \varepsilon' > \varepsilon \). Since \( f(N) = o_p(N^\rho) \) by assumption and \( s_j^2 \sim \log \log j \) by (1.5), we have for all \( N \) sufficiently large, \( N \geq N(\varepsilon, \varepsilon') \) say,

\[
\sup_{j \leq j^*} |S_j - W(s_j^2)| \leq \mathcal{Y}_\varepsilon (\log \log j^*)^\varepsilon \leq C\mathcal{Y}_\varepsilon (Nf(N))^\varepsilon \leq C\mathcal{Y}_\varepsilon N^{\varepsilon'}. \]

Then

\[
\sup_{j \leq j^*} \frac{|S_j - W(s_j^2)|}{s_j} \leq CN^{-\frac{1}{2}} \sup_{j \leq j^*} |S_j - W(s_j^2)| \leq C\mathcal{Y}_\varepsilon N^{-\frac{1}{2}+\varepsilon'}. \tag{4.2}
\]

Therefore, for \( N \geq N(\varepsilon, \varepsilon') \),

\[
\sup_{N \in \mathbb{N}} \sup_{j \leq j^*} \frac{|S_j - W(s_j^2)|}{s_j} \leq C\mathcal{Y}_\varepsilon M_1(x)^{-\frac{1}{2}+\varepsilon'}.
\]

It follows that

\[
\mathbb{P}\{A\} \leq \mathbb{P}\{\mathcal{Y}_\varepsilon > C(z' - z)M_1(x)^{-\frac{1}{2}+\varepsilon'}\} \leq \frac{C}{(z' - z)M_1(x)^{-\frac{1}{2}+\varepsilon'}} \mathbb{E}\mathcal{Y}_\varepsilon.
\]

Consequently \( \mathbb{P}\{A\} = o(x) \), and we deduce from (4.1) that

\[
\frac{1}{x} \# \left\{ m \leq x : \inf_{N \in \mathbb{N}} \sup_{j \in I_N} \frac{|\omega(m, j) - s_j^2|}{s_j} \leq z \right\} \leq \mathbb{P}\left\{ \inf_{N \in \mathbb{N}} \sup_{j \in I_N} \frac{|W(s_j^2)|}{s_j} \leq z' \right\} + o(1).
\]
Now let $0 < z'' < z$. As

$$\mathbb{P}\left\{ \inf_{N \in \mathbb{N}} \sup_{j \in I_N} \frac{|W(s_j^2)|}{s_j} \leq z'', \sup_{N \in \mathbb{N}} \sup_{j \in I_N} \frac{|S_j - W(s_j^2)|}{s_j} \leq z - z'' \right\}$$

$$\leq \mathbb{P}\left\{ \inf_{N \in \mathbb{N}} \sup_{j \in I_N} \frac{|S_j|}{s_j} \leq z \right\},$$

we deduce from Lemma 2.6

$$\frac{1}{x} \# \left\{ m \leq x : \inf_{N \in \mathbb{N}} \sup_{j \in I_N} \frac{\omega(m, j) - s_j^2}{s_j} \leq z \right\}$$

$$= \mathbb{P}\left\{ \inf_{N \in \mathbb{N}} \sup_{j \in I_N} \frac{|S_j|}{s_j} \leq z \right\} + o(1)$$

$$\geq \mathbb{P}\left\{ \inf_{N \in \mathbb{N}} \sup_{j \in I_N} \frac{|W(s_j^2)|}{s_j} \leq z'' \right\} - \mathbb{P}\{ B \} + o(1),$$

where we set

$$B = \left\{ \inf_{N \in \mathbb{N}} \sup_{j \in I_N} \frac{|W(s_j^2)|}{s_j} \leq z'', \sup_{N \in \mathbb{N}} \sup_{j \in I_N} \frac{|S_j - W(s_j^2)|}{s_j} > z - z'' \right\}.$$

By operating similarly, we also get

$$\frac{1}{x} \# \left\{ m \leq x : \inf_{N \in \mathbb{N}} \sup_{j \in I_N} \frac{\omega(m, j) - s_j^2}{s_j} \leq z \right\}$$

$$\geq \mathbb{P}\left\{ \inf_{N \in \mathbb{N}} \sup_{j \in I_N} \frac{|W(s_j^2)|}{s_j} \leq z'' \right\} + o(1).$$

The proof is now complete.

**Remark 4.1.** It follows from (4.2) that for all $0 < \delta < 1/2$ and $d \geq 0$

$$\mathbb{E} \sup_{N} \frac{1}{N^\delta} \sup_{j \in I_N} \frac{|S_j - W(s_j^2)|}{s_j}^d < \infty. \quad (4.3)$$

Consequently, for any increasing unbounded sequence of reals $N$,

$$\lim\inf_{k \to \infty} \sup_{j \in I_{N_k}} \frac{|S_j|}{\sqrt{\log \log j}} \overset{a.s.}{=} \lim\inf_{k \to \infty} \sup_{j \in I_{N_k}} \frac{|W(s_j^2)|}{\sqrt{\log \log j}}. \quad (4.4)$$
Let $z'' < z$. Let $f = f_c$ with $c < 1/\lambda(z'')$, $\mathcal{N} = \{e^k, k \geq 1\}$. Let also $0 < \gamma < 1$. Observe that

$$\mathbb{P}\left\{ \inf_{M_1(x) < N = e^k \leq M_2(x)} \sup_{j \in I_N} \frac{|W(s_j^2)|}{s_j} \leq z'' \right\}$$

$$\geq \mathbb{P}\left\{ \inf_{M_1(x) < N = e^k \leq M_2(x)} \sup_{e^k \leq t \leq e^k c} \frac{|W(t)|}{\sqrt{t}} \leq z'' \right\}$$

$$\geq \mathbb{P}\left\{ \sum_{\log M_1(x) < k \leq \log M_2(x)} \chi\left\{ \sup_{e^k \leq t \leq e^k c} \frac{|W(t)|}{\sqrt{t}} \leq z'' \right\} > 0 \right\}$$

$$= \mathbb{P}\left\{ \sum_{\log M_1(x) < k \leq \log M_2(x)} \chi\{A_k(z'')\} = 0 \right\}$$

$$\geq \mathbb{P}\left\{ \sum_{\log M_1(x) < k \leq \log M_2(x)} \chi\{A_k(z'')\} \geq \gamma \sum_{\log M_1(x) < k \leq \log M_2(x)} \mathbb{P}\{A_k(z'')\} \right\}.$$

Thus

$$\mathbb{P}\left\{ \inf_{M_1(x) < N = e^k \leq M_2(x)} \sup_{j \in I_N} \frac{|W(s_j^2)|}{s_j} \leq z'' \right\}$$

$$= \mathbb{P}\left\{ \frac{N_{\log M_2(x)}(f_c, z'')} - N_{\log M_1(x)}(f_c, z'')}{\nu_{\log M_2(x)}(f_c, z'') - \nu_{\log M_1(x)}(f_c, z'')} \geq \gamma \right\}. \tag{5.1}$$

By Corollary 1.2,

$$\lim_{n \to \infty} \frac{N_n(f_c, z'')}{\nu_n(f_c, z'')} = 1 \quad \text{and} \quad K_1(z'') \leq \frac{\nu_n(f_c, z'')}{n^{1-c\lambda(z'')}} \leq K_2(z'').$$

By assumption, we have $\log M_1(x) = o(\log M_2(x))$. Thus $\nu_{\log M_1(x)}(f_c, z'') = o(\nu_{\log M_2(x)}(f_c, z'')).$ It follows that

$$\lim_{n \to \infty} \frac{N_{\log M_2(x)}(f_c, z'')} - N_{\log M_1(x)}(f_c, z'')]}{\nu_{\log M_2(x)}(f_c, z'') - \nu_{\log M_1(x)}(f_c, z'')} = 1.$$

Consequently

$$\lim_{x \to \infty} \inf \mathbb{P}\left\{ \frac{N_{\log M_2(x)}(f_c, z'')} - N_{\log M_1(x)}(f_c, z'')]}{\nu_{\log M_2(x)}(f_c, z'') - \nu_{\log M_1(x)}(f_c, z'')} \geq \gamma \right\} = 1.$$

By combining this with (5.1), we get

$$\lim_{x \to \infty} \inf \mathbb{P}\left\{ \inf_{M_1(x) < N = e^k \leq M_2(x)} \sup_{j \in I_N} \frac{|W(s_j^2)|}{s_j} \leq z'' \right\} = 1.$$
In view of Theorem 1.6, this also implies
\[
\lim_{x \to \infty} \frac{1}{x^\# \{ m \leq x : \inf_{\log M_1(x) < k \leq \log M_2(x)} \sup_{e^k \leq s_j \leq e^{k+1}} |\omega(m, j) - s_j^2| \leq z \} } = 1.
\]

The proof is now complete.

6. Proof of Theorem 1.10

The sets \( A_k(c, z) \) being introduced before Theorem 1.1, we also define
\[
B_k(c, z) = \left\{ \sup_{j \in \mathbb{I}_N} \frac{|\omega(m, j) - s_j|}{s_j} \leq z \right\},
\]
\[
C_k(c, z) = \left\{ \sup_{j \in \mathbb{I}_N} \frac{|S_j|}{s_j} \leq z \right\},
\]
\[
D_k(c, z) = \left\{ \sup_{j \in \mathbb{I}_N} \frac{|W(s_j^2)|}{s_j} \leq z \right\}.
\]

Fix \( u > 0 \) and let \( \eta > 0 \). By (4.3), on a measurable set of full measure, we have for all \( k \) large enough, \( D_k(c, u) \subseteq C_k(c, u + \eta) \). Let \( 0 < c < 1 / \lambda(z) \). By Theorem 1.1,
\[
\lim_{n \to \infty} \frac{\sum_{k=1}^{n} \chi_{A_k(c, z)}}{\sum_{k=1}^{n} \mathbb{P}(A_k(c, z))} \overset{a.s.}{=} 1.
\]

Let \( 0 \leq M_2(x) \uparrow \infty \) and such that \( M_2(x) = O(z^2) \). Obviously,
\[
\lim_{x \to \infty} \inf_{M_1(x) \leq n \leq M_2(x)} \frac{\sum_{k=1}^{n} \chi_{A_k(c, z)}}{\sum_{k=1}^{n} \mathbb{P}(A_k(c, z))} \overset{a.s.}{=} 1.
\]

Further
\[
\kappa_1 \leq \frac{\sum_{k=1}^{n} \mathbb{P}(A_k(c, z))}{n^{1-c(\lambda(z))}} \leq \kappa_2,
\]
for some positive constants \( \kappa_1, \kappa_2 \). Let \( z' > z \). Since \( A_k(c, z) \subseteq D_k(c, z') \), it follows that with probability one
\[
1 \overset{a.s.}{=} \lim_{x \to \infty} \inf_{M_1(x) \leq n \leq M_2(x)} \frac{\sum_{k=1}^{n} \chi_{A_k(c, z)}}{\sum_{k=1}^{n} \mathbb{P}(A_k(c, z))} \leq \limsup_{x \to \infty} \inf_{M_1(x) \leq n \leq M_2(x)} \frac{\sum_{k=1}^{n} \chi_{C_k(c, z')}}{\sum_{k=1}^{n} \mathbb{P}(A_k(c, z))}.
\]

Let \( 0 < \varepsilon < 1 \) and put
\[
Q_x = \left\{ (\nu_1, \ldots, \nu_r) \in \mathbb{Z}^r : \inf_{M_1(x) \leq n \leq M_2(x)} \frac{1}{\sum_{k=1}^{n} \mathbb{P}(C_k(c, z'))} \sum_{k=1}^{n} \chi \left\{ \sup_{j \in \mathbb{I}_N} \frac{|\nu_j - s_j^2|}{s_j} \leq z' \right\} \leq \varepsilon \right\}.
\]
By applying (2.3) with \( Q = Q_x \), we get
\[
\frac{1}{x} \# \left\{ m \leq x : \inf_{M_1(x) \leq n \leq M(x)} \frac{1}{\sum_{k=1}^{n} P(C_k(c, z'))} \sum_{k=1}^{n} \chi \{ B_k(c, z') \} \leq \varepsilon \right\} \\
= \P \left\{ \inf_{M_1(x) \leq n \leq M(x)} \frac{1}{\sum_{k=1}^{n} P(C_k(c, z'))} \sum_{k=1}^{n} \chi \{ C_k(c, z') \} \leq \varepsilon \right\} + o(1).
\]
Thus
\[
\limsup_{x \to \infty} \frac{1}{x} \# \left\{ m \leq x : \inf_{M_1(x) \leq n \leq M(x)} \frac{1}{\sum_{k=1}^{n} P(C_k(c, z'))} \sum_{k=1}^{n} \chi \{ B_k(c, z') \} \leq \varepsilon \right\} \\
= \limsup_{x \to \infty} \P \left\{ \inf_{M_1(x) \leq n \leq M(x)} \frac{1}{\sum_{k=1}^{n} P(C_k(c, z'))} \sum_{k=1}^{n} \chi \{ C_k(c, z') \} \leq \varepsilon \right\} = 0.
\]
This being true for all \( 0 < \varepsilon < 1 \), we infer that
\[
\lim_{x \to \infty} \frac{1}{x} \# \left\{ m \leq x : \inf_{M_1(x) \leq n \leq M(x)} \frac{1}{\sum_{k=1}^{n} P(C_k(c, z'))} \sum_{k=1}^{n} \chi \{ B_k(c, z') \} \geq 1 \right\} = 1.
\]
Finally, for some \( \kappa > 0 \) depending on \( z \),
\[
\lim_{x \to \infty} \frac{1}{x} \# \left\{ m \leq x : \inf_{M_1(x) \leq n \leq M(x)} \frac{1}{n^{1-\alpha M(z)}} \sum_{k=1}^{n} \chi \{ B_k(c, z') \} \geq \kappa \right\} = 1.
\]

7. Proof of Theorem 1.5

Let \( 1/\alpha < \beta < 1/2 \). Take \( r_n = (\sum_{i=1}^{j} E |X_i|^2)^{\beta} = z_n^{2\beta} \). We notice that
\[
\sum_{j=1}^{\infty} \frac{E |X_j|^\alpha}{r_j^{\beta}} = \sum_{j=1}^{\infty} \frac{E |X_j|^\alpha}{(\sum_{i=1}^{j} E |X_i|^2)^{\alpha \beta}} \leq C \sum_{j=1}^{\infty} \frac{E |X_j|^\alpha}{(\sum_{i=1}^{j} E |X_i|^\alpha)^{\alpha \beta}} < \infty,
\]
since \( \alpha \beta > 1 \). Thus
\[
L_\alpha(y) \leq y^{-\alpha} \sum_{j=1}^{\infty} \frac{E |X_j|^\alpha}{r_j^\beta} \leq Cy^{-\alpha}.
\]
By Lemma 2.4, there exists a Brownian motion \( W \) such that if
\[
Y = \sup_n \frac{1}{r_n} \sup_{j \leq n} |Z_j - W(z_j^2)|
\]
then \( EY^{\alpha'} < \infty, \alpha' < \alpha \). Now let \( j^*_p = \max \{ j : r_j \leq 2^p \} \). As
\[
\sup_{2^p-1 < r_j \leq 2^p} \frac{|Z_j - W(z_j^2)|}{r_j} \leq 2 \sup_{j \leq j^*_p} |Z_j - W(z_j^2)|,
\]
whenever \( \{ j : 2^{p-1} < r_j \leq 2^p \} \neq \emptyset \), it follows that
\[
\sup_{r_j \geq 1} \frac{|Z_j - W(z_j^2)|}{r_j} \leq 2\Upsilon.
\]

Let \( j(N) = \max(J_N) \). Hence
\[
\frac{\left| \sup_{j \in J_N} \frac{|Z_j|}{z_j} - \sup_{j \in J_N} \left| \frac{W(z_j^2)}{z_j} \right| \right|}{\sup_{j \leq j(N)} \left| \frac{1}{z_j^{1-2\beta}} \right|} \leq \left( \sup_{j \in J_N} \frac{1}{z_j^{1-2\beta}} \right) \sup_{j \leq j(N)} \left| \frac{Z_j - W(z_j^2)}{z_j^{2\beta}} \right| \to 0,
\]
as \( N \to \infty \) almost surely, since \( \beta < 1/2 \). By specifying this for \( N = e^k \), we therefore deduce
\[
\liminf_{k \to \infty} \sup_{e^k \leq z_j^2 \leq e^k f(e^k)} \frac{|Z_j|}{z_j} = \liminf_{k \to \infty} \sup_{e^k \leq z_j^2 \leq e^k f(e^k)} \frac{|W(z_j^2)|}{s_j}
\]
almost surely. This together with Corollary 1.2 concludes the proof.

8. Concluding remarks

Clearly, the approximation formula (2.3) applies to strongly additive arithmetic functions \( f(n) = \sum_{p \mid n} f(p) \), and associated truncated functions. For additive arithmetic functions \( f(n) = \sum_{p^\nu \mid n} f(p) \), the comparison is made with the sums of independent random variables \( \xi_p \) defined by \( \mathbb{P}\{\xi_p = f(p^\nu)\} = \left( 1 - 1/p \right) p^{-\nu} \), \( \nu = 0, 1, \ldots \). See [3], [11]. Special cases will be investigated elsewhere.

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