BLOWUP SUBALGEBRAS OF THE SKLYANIN ALGEBRA

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Abstract. We describe some interesting graded rings which are generated by degree-3 elements inside the Sklyanin algebra $S$, and prove that they have many good properties. Geometrically, these rings $R$ correspond to blowups of the Sklyanin $\mathbb{P}^2$ at 7 or fewer points. We show that the rings $R$ are exactly those degree-3-generated subrings of $S$ which are maximal orders in the quotient ring of the 3-Veronese of $S$.

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1. Introduction

One of the major objectives in the theory of noncommutative projective geometry is the classification of noncommutative surfaces. The goal of this paper is to study, from a ring-theoretic standpoint, some surfaces which are birational to the generic noncommutative projective plane, the Sklyanin $\mathbb{P}^2$. We first describe our main results, and then explain in more detail how they are motivated by the classification project. In

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particular, we discuss at the end of the introduction how our results relate to Van den Bergh’s blowing up construction in [VdB].

We first review a few standard definitions. Fix an algebraically closed field $k$. In this paper, we are primarily interested in $\mathbb{N}$-graded associative $k$-algebras $A = \bigoplus_{n=0}^{\infty} A_n$ which are connected ($A_0 = k$) and finitely graded (finitely generated as a $k$-algebra), as well as domains. The Gelfand-Kirillov (GK) dimension of $A$ can be defined, in this graded setting, by $\text{GK}(A) = 1 + \limsup_{n \to \infty} \log_n (\dim_k A_n)$. We assume that $A$ is a connected finitely graded (cfg) domain with integer GK-dimension for the rest of this introduction. Any such $A$ has a graded quotient ring $Q_{\text{gr}}(A)$ formed by inverting the Ore set of all nonzero homogeneous elements, as well as the usual Goldie quotient ring $Q(A)$ formed by inverting all nonzero elements. Domains $A$, $A'$ with $Q = Q(A) = Q(A')$ are equivalent orders if there are nonzero elements $p_1, p_2, q_1, q_2 \in Q$ such that $p_1Aq_2 \subseteq A'$ and $q_1A'q_2 \subseteq A$; a maximal order is a subring $A$ of $Q$ maximal under inclusion among orders in an equivalence class.

A construction that plays a fundamental role in our results is that of a twisted homogeneous coordinate ring. Such a ring $B(X, \mathcal{L}, \sigma)$ is built out of the data of a projective $k$-scheme $X$, an invertible sheaf $\mathcal{L}$ on $X$, and an automorphism $\sigma : X \to X$. Putting $\mathcal{L}_0 = \mathcal{O}_X$ and $\mathcal{L}_n = \mathcal{L} \otimes \sigma^*(\mathcal{L}) \otimes \cdots \otimes (\sigma^{n-1})^*(\mathcal{L})$ for $n \geq 1$, then we define $B = B(X, \mathcal{L}, \sigma) = \bigoplus_{n=0}^{\infty} \mathcal{H}^0(X, \mathcal{L}_n)$, with a multiplication defined on graded pieces by $f \cdot g = f \cdot (\sigma^m)^*(g)$ for $f \in B_m, g \in B_n$. Here, $(\sigma^m)^* : \mathcal{H}^0(X, \mathcal{L}_n) \to \mathcal{H}^0(X, \mathcal{L}_m^m)$ is the map on global sections induced by pullback by $\sigma^m$, and $\cdot$ indicates the multiplication map $\mathcal{H}^0(X, \mathcal{L}_m) \otimes \mathcal{H}^0(X, \mathcal{L}_m^m) \to \mathcal{H}^0(X, \mathcal{L}_{m+n})$. We only consider this construction under an additional condition that $\mathcal{L}$ is $\sigma$-ample, which ensures that $B$ is noetherian. See Section 2 for the details.

Noncommutative projective geometry might be said to have begun with a project of Artin and Schelter to classify certain graded algebras of dimension 3 which are analogs of commutative polynomial rings (naturally, these are called AS-regular algebras today). We omit the technical definition of AS-regular, which is not needed for this paper. The classification, which was completed with Tate and Van den Bergh in [ATV1], [ATV2], in fact necessitated the development of the concept of a twisted homogeneous coordinate ring. We concentrate here on those regular algebras with three degree-1 generators and three quadratic relations, which correspond geometrically to noncommutative $\mathbb{P}^2$’s. The classification of these includes the linear regular algebras, which have the form $B(\mathbb{P}^2, \mathcal{O}(1), \sigma)$, but the more interesting examples are the elliptic ones. Such an elliptic regular algebra $A$ always has a normal element $g \in A_3$ such that $A/Ag \cong B(E, \mathcal{L}, \sigma)$ for some degree 3 divisor $E \subseteq \mathbb{P}^2$, and with $\mathcal{L} = \mathcal{O}(1)|_E$. In fact, we restrict our attention here to what is in many ways the most interesting regular algebra, the generic Sklyanin algebra $S$ (see Section 4 for a definition by presentation). In this case, $g$ is central, $E$ is a nonsingular elliptic curve, and the automorphism $\sigma : E \to E$ has infinite order.

The main objects of study in this paper will be certain subalgebras of $S$ generated in degree 3. It is convenient to work instead in the 3-Veronese $T = S^{(3)} = \bigoplus_{n \geq 0} S_{3n}$, and construct degree-1-generated
subalgebras of $T$. Now $g \in T_1$ and $T/Tg \cong B(E,L_3,\sigma^3)$. Let $\pi$ indicate the image of $x \in S$ under the quotient map $S \to S/Sg$. Given any effective Weil divisor $D$ on $E$ with $0 \leq \text{deg} D \leq 7$, we let $V(D) = \{ x \in T_1 | \pi \in H^0(E,I_D \otimes L_3) \}$, where $I_D = \mathcal{O}_E(-D)$. In other words, $V(D)$ consists exactly of those global sections of $L_3$ which vanish along the divisor $D$. We define $R(D) = k(V(D)) \subseteq T$. One goal of the paper is to prove that the rings $R(D)$ have many nice properties (with the notable exception of finite global dimension). We summarize these in the following theorem. The definitions of the properties involved are reviewed in Section 2.

**Theorem 1.1.** Let $R = R(D) \subseteq T$ as above, where $D$ is an effective divisor on $E$ of degree $d$, with $0 \leq d \leq 7$. 

1. (Theorems 5.2, 5.4) $R/Rg \cong B(E,N,\sigma^3)$, where $N = I_D \otimes L_3$ is a $\sigma^3$-ample invertible sheaf on $E$ of degree $9 - d$. The Hilbert series of $R$ is $h_R(t) = \frac{t^2 + (7 - d)t + 1}{(1 - t)^3}$ and $Q_{gr}(R) = Q_{gr}(T)$. $R$ has infinite global dimension.

2. (Theorems 6.3, 6.7) $R$ is strongly noetherian, satisfies the Artin-Zhang $\chi$ conditions, is Auslander-Gorenstein and Cohen-Macaulay. The noncommutative projective scheme $\text{proj}-R$ has cohomological dimension 2. $R$ is a maximal order.

3. (Theorem 10.4) There is a homogeneous ideal $I$ of $R$ with $\text{GK} R/I \leq 1$ which is essentially minimal among such ideals in the following sense: given a homogeneous ideal $J$ of $R$ with $\text{GK} R/J \leq 1$, then $J \supseteq I \geq n$ for some $n \geq 0$.

We make a comment on the significance of part (3) of the theorem, which is actually the part on which we expend the most effort in the paper. The generic Sklyanin algebra $S$ and its Veronese ring $T$ are known to have no homogeneous factor rings of GK-dimension 1. The rings $R(D)$ sometimes do, so the point of part (3) above is that such factor rings can be strongly controlled. This fact is needed in the proof of our main result, which can be found in Section 10. It classifies degree 1-generated orders in $T$, as follows.

**Theorem 1.2.** Let $V \subseteq T_1$ and let $A = k(V) \subseteq T$. Assume that $Q_{gr}(A) = Q_{gr}(T)$.

1. There is a unique effective divisor $D$ on $E$ with $0 \leq \text{deg} D \leq 7$ such that $A \subseteq R(D)$ with $A$ and $R(D)$ equivalent orders. In particular, the rings $R(D)$ are exactly the degree-1 generated subalgebras of $T$ which are maximal orders in $Q_{gr}(T)$.

2. If $A$ is noetherian, then in part (1) $A \subseteq R(D)$ is a finite ring extension. If $g \in A$, then $A$ is indeed noetherian.

When $g \notin A$ in the preceding theorem, then it can happen that $A$ is not noetherian and $A \subseteq R(D)$ is not a finite ring extension; see Section 11. Also, similar methods can be used to analyze the subalgebras of $S$ generated in degree 1, which was in fact the original project we attempted. Since $\dim_k S_1 = 3$, the only interesting degree-1-generated subalgebras are $A = k(V)$, where $V \subseteq S_1$ with $\dim_k V = 2$. We show in Theorem 12.2 below that for such an $A$, either the 3-Veronese ring $A^{(3)}$ is equal to the ring $R(D)$ for a divisor $D$ of degree 3, or else $A$ equals $S$ in all large degrees.
Next, we explain how our results relate to the project to classify noncommutative projective surfaces, for which we need to review a few more definitions. See [SV] for a survey of the field. If \( A \) is a cfg domain with \( \text{GK}(A) = d + 1 \), then informally we think of \( A \) as corresponding to a \( d \)-dimensional noncommutative projective variety. Let \( \text{Gr}-A \) be the category of \( \mathbb{Z} \)-graded right \( A \)-modules \( M \), and let \( \text{Tors}-A \) be the full subcategory of graded modules \( M \) such that for every \( m \in M \), \( mA_{\geq n} = 0 \) for some \( n \). The quotient category \( \text{Qgr}-A = \text{Gr}-A / \text{Tors}-A \) is called the quasi-scheme associated to \( A \). The graded quotient ring \( \text{Qgr}(A) \) is isomorphic to a skew-Laurent ring \( D[t, t^{-1}; \sigma] \), for some division ring \( D \) and automorphism \( \sigma : D \to D \). We call \( D \) the (skew) field of functions of \( A \). In general, the ring-theoretic version of the problem of classification of noncommutative surfaces has two parts. First, what are the \( (D, \sigma) \) which occur in the graded quotient rings of cfg domains \( A \) of dimension 3? Second, for a given \( Q = D[t, t^{-1}; \sigma] \) where \( D \) has transcendence degree 2, can one classify in some way the algebras \( A \) with \( \text{Qgr}(A) = Q \)?

We say that \( A \) is birationally commutative if its skew field of functions \( D \) is actually a field. The second question above has a quite satisfactory answer in the birationally commutative case. If \( B = B(X, \mathcal{L}, \sigma) \) is a twisted homogeneous coordinate ring with \( X \) an integral surface and \( \mathcal{L} \) \( \sigma \)-ample, then \( Q = \text{Qgr}(B) \cong k(X)[t, t^{-1}; \tilde{\sigma}] \), where \( \tilde{\sigma} \) is the automorphism on the field of rational functions \( k(X) \) induced by pullback by \( \sigma \). Noetherian cfg algebras \( A \) with \( \text{Qgr}(A) = \text{Qgr}(B) \) have now been classified [RS], [Si]. In large degrees, such \( A \) are either twisted homogeneous coordinate rings over varieties \( Y \) birational to \( X \), or one of a few kinds of closely related subrings of them (naive blowup rings, ADC rings, or idealizer rings in either of these.) Moreover, geometrically, the quasi-scheme \( \text{Qgr}-B \) of a twisted homogeneous coordinate ring \( B(Y, \mathcal{M}, \tau) \) is equivalent to \( \text{Qcoh}Y \), the category of quasi-coherent sheaves on \( Y \). The quasi-schemes of the other kinds of examples \( A \) behave as further blowups of the categories \( \text{Qcoh}Y \) in some generalized noncommutative sense (“naive blowups”).

The main theorem of this paper, Theorem 1.2, is an attempt to begin to classify some algebras which are not birationally commutative. It is natural to start by considering rings \( A \) with \( \text{Qgr}(A) = \text{Qgr}(S) \) for the generic Sklyanin algebra \( S \), since this is one of the most well-studied cfg algebras of dimension 3. Even in the commutative classification of surfaces, one concentrates first on nonsingular surfaces, so it is reasonable to focus our attention here on maximal orders: being a maximal order is the noncommutative analog of being integrally closed.

The intuition from the commutative case leads one to expect that the quasi-schemes of the algebras in such a classification should be related to \( \text{Qgr}-S \) via some kind of blowup procedure. In fact, the quasi-schemes \( \text{Qgr}-R(D) \) are the same as certain iterated noncommutative blowups of the Sklyanin \( \mathbb{P}^2 \), in the sense of Van den Bergh’s theory [VdB]. Starting with the Sklyanin algebra \( S \) and a divisor \( D \) of degree at most 8 on \( E \), Van den Bergh’s construction yields a blowup of the Sklyanin \( \mathbb{P}^2 \) along the divisor \( D \) (see [VdB, Section 11]); this is a quasi-scheme \( \text{Qgr}-A(D) \) for some ring \( A(D) \). Van den Bergh proves that \( \text{Qgr}-A(D) \) has many of the same formal properties as a commutative blowup, when \( D \) is in general position [VdB, Theorem 11.1.3].
Van den Bergh’s work is primarily category-theoretic, essentially defining a blowup by constructing a Rees ring over a quasi-scheme, working in an appropriate category of functors. Because the details are rather intricate, it is not obvious that the ring $A(D)$ is isomorphic to our ring $R(D)$. Only recently, after most of the work in this paper was completed, did we fully verify that this is indeed the case. We postpone the demonstration of this to a future paper, in which the connections between our results and Van den Bergh’s can be more fully explored. In the meantime, we thought would be most helpful to the reader to keep our presentation here entirely independent of [VdB]. We should remark, however, that once one knows that the rings appearing in [VdB] are the same, some of the most basic properties of the rings $R(D)$, in particular the Hilbert series (Theorem 1.1(1)), follow in a quite different way from Van den Bergh’s methods.

To close, we discuss some questions for future study.

**Question 1.3.** Can one classify all cfg maximal orders $A$ inside the generic Sklyanin algebra $S$ with $Q = Q_{gr}(A) = Q_{gr}(S^{(n)})$ for some $n$? More generally, can we classify all cfg maximal orders in $Q_{gr}(S^{(n)})$?

So far, we have classified only those subalgebras $A$ of $S$ generated in degree 1 or 3. Based on the connection with [VdB], there should also be subalgebras of $S$ corresponding to blowups at 8 points, which have presumably fallen outside the scope of our results in this paper because they will not be generated in any single degree. It seems possible that there are no other subalgebras of $S$ which are maximal orders, except those that have a Veronese ring in common with one of these blowups at 8 or fewer points. To answer the more general second question, one needs to find first the “minimal models” for this birational class. For example, presumably the regular algebra with 2 generators and 2 cubic relations of Sklyanin type (the “Sklyanin $\mathbb{P}^1 \times \mathbb{P}^1$”) is also contained in this quotient ring, and so one also has it and its subalgebras as examples.

**Question 1.4.** If one replaces the generic Sklyanin algebra $S$ by other non-PI AS-regular algebras of dimension 3, can one prove theorems analogous to the ones in this paper, or ultimately classify all cfg algebras which are maximal orders with the same quotient ring?

**Question 1.5.** What is a presentation of the ring $R(D)$ by generators and relations?

Answering this final question would be useful in order to better understand the deformation theory of these blowup algebras.

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2. Background and twisted homogeneous coordinate rings

In this section, we review some definitions and basic background material, including some facts about twisted homogeneous coordinate rings $B(X, \mathcal{L}, \sigma)$. Throughout this paper, $k$ will stand for an algebraically
closed field. All algebras \( A \) of interest will be cfg \( k \)-algebras, as defined in the introduction. We make no assumptions on the cardinality of \( k \), but if the reader is happy to assume that \( k \) is uncountable, a few later arguments can be streamlined slightly. In general, the hypotheses and notation in force in each particular section will be announced near the beginning of that section.

For completeness, we review most of the definitions we will use in the paper in this section. Some of these we will need only incidentally later, however, so the expert reader might wish to skip this section and refer back to it when necessary. We begin by reviewing the theory of noncommutative projective schemes, which was developed in [AZ1]. For convenience, assume that \( A \) is a noetherian cfg \( k \)-algebra in the following definitions. Let \( \text{Gr}-A \) be the category of \( \mathbb{Z} \)-graded right \( A \)-modules. The general convention is to use a lowercase name to indicate the subcategory of noetherian objects in a category. For example, \( \text{gr}-A \subseteq \text{Gr}-A \) is the full subcategory of finitely generated \( \mathbb{Z} \)-graded \( A \)-modules. Let \( \text{Tors}-A \subseteq \text{Gr}-A \) be the subcategory of modules \( M \) with the property that for every \( m \in M \), \( mA_{\geq n} = 0 \) for some \( n \geq 1 \). As in the introduction, the quotient category \( \text{Qgr}-A = \text{Gr}-A/\text{Tors}-A \) is called the quasi-scheme associated to \( A \). Let \( \pi : \text{Gr}-A \to \text{Qgr}-A \) be the canonical quotient functor. A review of the formalities of quotient categories can be found in [AZ1].

In the case at hand, one may think of \( \text{qgr}-A \) as the category of tails of modules; in other words, two modules \( M, N \in \text{gr}-A \) become isomorphic in \( \text{qgr}-A \) if and only if \( M_{\geq n} \cong N_{\geq n} \) for some \( n \). The noncommutative projective scheme associated to \( A \) is the pair \( \text{proj}-A = (\text{qgr}-A, \pi(A_A)) \); it remembers the distinguished object \( \pi(A) \), which plays the role of the structure sheaf. For \( M \in \text{gr}-A \) and \( n \in \mathbb{Z} \), \( M[n] \in \text{gr}-A \) is the same module as \( M \) but with the grading shifted so that \( M[n]_i = M_{i+n} \) for all \( i \in \mathbb{Z} \).

Let \( k = k_A \) be the right module \( A/A_{\geq 1} \). The noetherian algebra \( A \) satisfies \( \chi_i \) (on the right) if \( \dim_k \text{Ext}^j_A(k, M) < \infty \) for all \( M \in \text{gr}-A \) and all \( 1 \leq j \leq i \). A satisfies \( \chi \) (on the right) if it satisfies \( \chi_i \) for all \( i \geq 1 \). Of course, one defines the left \( \chi \)-conditions analogously. The cohomology groups of an object \( M \in \text{qgr}-A \) are defined by \( H^i(M) = \text{Ext}^i_{\text{qgr}-A}(\pi(A), M) \) for all \( i \geq 0 \). The cohomological dimension of \( \text{proj}-A \) is \( \text{cd}(\text{proj}-A) = \min \{ i | H^i(M) \neq 0 \text{ for some } M \in \text{qgr}-A \} \).

Next, we review definitions related to Hilbert series and growth of modules. Let \( A \) be any cfg \( k \)-algebra. Any \( M \in \text{gr}-A \) has a Hilbert function \( f_M(n) : \mathbb{Z} \to \mathbb{N} \), defined by \( f_M(n) = \dim_k M_n \); the Hilbert series of \( M \) is the corresponding formal Laurent power series \( \text{h}_M(t) = \sum_{n \in \mathbb{Z}} (\dim_k M_n) t^n \). We sometimes use the following partial order on Hilbert series: \( \sum a_n t^n \leq \sum b_n t^n \) if \( a_n \leq b_n \) for all \( n \). In this graded setting, we can define the GK-dimension of a nonzero module \( M \in \text{gr}-A \) as \( \text{GK}(M) = [\limsup_{n \to \infty} \log_n f_M(n)] + 1 \). The module \( M \in \text{gr}-A \) has a Hilbert polynomial if there is a polynomial \( q(x) \in \mathbb{Q}[x] \) such that \( q(n) = f_M(n) \) for all \( n \gg 0 \); in this case, writing \( f(x) = a_m x^m + \cdots + a_0 \) with \( a_m \neq 0 \), then \( \text{GK} M = m + 1 \) and \( m!a_m \) is an integer, the multiplicity of \( M \). For any \( d \geq 1 \), we have the \( d \)-th Veronese ring of \( A \), \( A^{(d)} = \bigoplus_{n \geq 0} A_{nd} \), which is graded by the index \( n \), unless stated otherwise. A module \( M \in \text{gr}-A \) has a Hilbert quasi-polynomial of period \( d \) for some \( d \geq 1 \), if \( \bigoplus_{n \geq 0} M_{i+nd} \) has a Hilbert polynomial as an \( A^{(d)} \)-module for each \( 0 \leq i \leq d-1 \). We say that \( A \) is generated in degree 1 if it is generated as a \( k \)-algebra by \( A_1 \). Given a cfg \( k \)-algebra \( A \) which
is generated in degree 1, we say that $M \in \text{gr}-A$ is a point module if $M$ is cyclic and $h_M(t) = 1/(1-t)$. A module $M \in \text{gr}-A$ with $\text{GK}(M) = d$ is d-critical if $\text{GK}(M/N) < d$ for all nonzero submodules $N \subseteq M$.

The $k$-algebra $A$ is strongly noetherian if $A \otimes_k C$ is noetherian for all commutative noetherian $k$-algebras $C$. As usual, a ring extension $A \subseteq B$ is finite if $B_A$ and $A_B$ are finitely generated $A$-modules. We will often use below the graded Nakayama lemma, which states that $M$ is a generating subset if and only if $\dim_k M/MA_{\geq 1} < \infty$ (and, in fact, $X \subseteq M$ is a generating subset if and only if $X$ spans $M/MA_{\geq 1}$ as a $k$-vector space.)

The last definitions we review are some important homological properties which involve ungraded modules. For a right $A$-module $M$, define $j(M_A) = \inf\{i|\text{Ext}^i(M, A) \neq 0\} \in \mathbb{N} \cup \{\infty\}$, which is called the grade of $M$. The grade of a left module is defined analogously. A right module $M$ satisfies the Auslander condition if for all $i \geq 0$ and all left submodules $N \subseteq \text{Ext}^i(M, A)$, one has $j(MN) \geq i$; the definition of Auslander for a left module is symmetric. The ring $A$ is Auslander-Gorenstein if the modules $A_A$ and $AA$ have the same finite injective dimension $d$, and every finitely generated left and right $A$-module satisfies the Auslander condition. Suppose in addition that $\text{GK}(A)$ is an integer. Then $A$ is Cohen-Macaulay if $\text{GK}(M) + j(M) = \text{GK}(A)$ for all finitely generated left and right modules $M$.

Now we move on to a review of twisted homogeneous coordinate rings. Recall that in the introduction, we defined the twisted homogeneous coordinate ring $B(X, \mathcal{L}, \sigma) = \bigoplus_{n \geq 0} H^0(X, \mathcal{L}_n)$ for a given projective $k$-scheme $X$, invertible sheaf $\mathcal{L}$, and automorphism $\sigma : X \to X$. We now give a few more details about this construction; for more information, see [AV] and [Ke2]. For any coherent sheaf $\mathcal{F}$ on $X$ we adopt the notation $\mathcal{F}^\sigma$ for the pullback $\sigma^*\mathcal{F}$, so in this notation we have $\mathcal{L}_n = \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \ldots \mathcal{L}^{\sigma^{n-1}}$ for $n \geq 1$. The sheaf $\mathcal{L}$ is called $\sigma$-ample if for any coherent sheaf $\mathcal{F}$ on $X$, $H^i(X, \mathcal{F} \otimes \mathcal{L}_n) = 0$ for all $i \geq 1$ and $n \gg 0$. We always assume that $\mathcal{L}$ is $\sigma$-ample when constructing a twisted homogeneous coordinate ring.

Many of the definitions discussed earlier play out nicely when $A = B = B(X, \mathcal{L}, \sigma)$ is a twisted homogeneous coordinate ring, where $\mathcal{L}$ is $\sigma$-ample. If $X$ is integral (as it will be in all of the applications in this paper), the graded quotient ring of $B$ may be explicitly described as $Q_{\text{gr}}(B) \cong k[X][t, t^{-1}; \sigma]$, where $\sigma : k(X) \to k(X)$ is the map induced by pullback of rational functions by $\sigma : X \to X$. In this case, as mentioned already in the introduction, the quasi-scheme $Q_{\text{gr}}-B$ is equivalent as a category to $Q_{\text{coh}}X$, the category of quasi-coherent sheaves on $X$ [AV Theorem 1.3]. Restricting to noetherian objects for convenience, the equivalence is given explicitly by the functor $G : \text{coh} X \to Q_{\text{gr}}-B$ with formula

$$G(\mathcal{F}) = \pi(\bigoplus_{n \geq 0} H^0(X, \mathcal{F} \otimes \mathcal{L}_n)).$$

For later reference, we record here some other well-known properties of twisted homogeneous coordinate rings which follow quickly from the literature.

**Lemma 2.2.** Let $X$ be a projective scheme, $\sigma : X \to X$ an automorphism, and $\mathcal{L}$ a $\sigma$-ample invertible sheaf on $X$. Let $B = B(X, \mathcal{L}, \sigma)$. 


(1) $B$ is strongly noetherian.

(2) $B$ satisfies $\chi$ on the left and right, and $\text{cd} (\text{proj-} B) = \dim X$.

(3) If $X$ is an integral nonsingular curve, then every module $M \in \text{gr} B$ has a Hilbert polynomial of the form $\dim_k M_n = an + b$ for $n \gg 0$, some $a, b \in \mathbb{Z}$ with $a \geq 0$.

(4) If $X = E$ is a nonsingular elliptic curve, then $B$ is Auslander-Gorenstein of dimension 2 and Cohen-Macaulay.

Proof. (1) This is [ASZ, Proposition 4.13].

(2) These results are implicit in Artin and Zhang’s original paper [AZ1] but are not stated in this way. The fact that $B$ satisfies $\chi$ follows from [AZ1, Theorem 4.5, Corollary 7.5, and p. 262] and the fact that $L$ is $\sigma$-ample. For a working out of the details, see Keeler’s thesis [Ke1, Proposition 2.5.3, Proposition 2.5.8].

Since the category equivalence (2.1) takes $O_X$ to $\pi(B)$, it also shows that $\text{cd}(\text{proj-} B)$ is the same as the cohomological dimension of the scheme $X$, i.e. $\max \{i | H^i(X, F) \neq 0 \text{ for some } F \in \text{coh } X \}$, which it is well-known is equal to $\dim X$ for a projective scheme $X$.

(3) This follows quickly from (2.1) and the Riemann-Roch theorem.

(4) This is [Lev, Theorem 6.6], except that Levasseur assumes the restriction $\deg L \geq 3$, so that $L$ is very ample. But this restriction is only used in the proof of a different part of the theorem showing that every graded $B$-module has a Hilbert polynomial (and the restriction is also unnecessary to prove that, as we saw in (3)).

3. Twisted homogeneous coordinate rings over an elliptic curve

Theorem 1.1 from the introduction shows that the rings $R$ of main interest in this paper will all have a central element $g$ such that $R/Rg \cong B(E, \mathcal{L}, \sigma)$, where $E$ is a nonsingular elliptic curve and $\sigma$ has infinite order. Thus these special twisted homogeneous coordinate rings will play an important role below, and in this section we prove some results that are particular to this special case.

For the rest of this section we restrict to the following setup. Let $E$ be a nonsingular elliptic curve with fixed basepoint $p_0$ for the group structure. Let $\sigma : E \to E$ be an automorphism given by translation $x \mapsto x + r$ in the group structure, for some point $r$ of infinite order in the group. It is standard that any automorphism of infinite order on an elliptic curve is such a translation, so we are really working with any infinite order automorphism of $E$. The notation of this paragraph is fixed for the rest of this section.

We will freely use basic properties of nonsingular curves and divisors on them which can be found in [Ha, Chapter IV], for example the Riemann-Roch theorem. Another basic result we use frequently is Abel’s theorem: two Weil divisors $D, D'$ on $E$ are linearly equivalent ($D \sim D'$) if and only if $\deg D = \deg D'$ and $D$ and $D'$ give the same result when summed in the group structure of $E$. Note that the symbol $+$ is used both for the group structure of $E$ and for addition in the group $\text{Div } E$ of Weil divisors of $E$; the meaning will be clear from context.
The property of $\sigma$-ampleness is easy to understand on curves; in particular, an invertible sheaf $L$ on $E$ is $\sigma$-ample if and only if $L$ is ample, in other words if and only if $L$ has positive degree [Ke2 Theorem 1.3]. Our first goal is to show that $B(E, L, \sigma)$ is generated in degree 1 as long as $L$ does not have very small degree. This is similar to the results in [ATVI Section 7]; in fact, some parts of the next lemma also follow from [ATVI Proposition 7.17, Lemma 7.21], but we give a full proof for the reader’s convenience.

Lemma 3.1.  

(1) Let $L$ and $M$ be invertible sheaves on $E$ with $\deg L \geq 2$ and $\deg M \geq 2$. Consider the natural map $\phi : H^0(E, L) \otimes H^0(E, M) \to H^0(E, L \otimes M)$. The map $\phi$ is surjective unless $\deg L = \deg M = 2$ and $L \cong M$, in which case $\dim_k \text{im} \phi = 3$.

(2) Let $L$ be an invertible sheaf on $E$ with $\deg L \geq 2$. Then $B = B(E, L, \sigma)$ is generated in degree 1.

Proof. If $L \cong M$ with $\deg L = 2$, then the natural map is clearly not surjective. For, in that case we might as well assume that $M = L$, and $\dim_k H^0(E, L) = 2$ and $\dim_k H^0(E, L \otimes 2) = 4$ by Riemann-Roch. Then writing $H^0(E, L) = kx + ky$, we have $\phi(x \otimes y) = \phi(y \otimes x)$, and so $\dim_k \text{im} \phi \leq 3$. It is easy to check that in fact we have equality here, as asserted.

If $L \cong M$ where $\deg L \geq 3$, again we may assume that $M = L$. Then the surjectivity of $\phi$ is a special case of the standard result that any invertible sheaf of degree at least $2g + 1$ on a curve of genus $g$ is normally generated. See, for example, [Lz1 Definition 1.8.50 and Theorem 1.8.53].

Finally, assume that $M \not\cong L$, where by switching the sheaves if necessary we can assume that $\deg L \leq \deg M$. In this case, we show a slightly stronger result. It is well-known that any globally generated invertible sheaf $L$ on $E$ is generated by two sections. We claim that choosing any two sections $x, y \in H^0(E, L)$ which generate the sheaf $L$, then putting $W = kx + ky$ we even have $\phi(W \otimes H^0(E, M)) = H^0(E, L \otimes M)$. Considering the exact sequence

$$0 \to K \to W \otimes O_E \to L \to 0,$$

it is clear that $K$ is also invertible, and in fact it follows from a consideration of determinants ([Ha Exercise II.5.16(d)]) that $K \cong L^{-1}$. Tensoring with $M$ and taking the cohomology long exact sequence, we see that to prove the claim it suffices to prove that $H^1(E, L^{-1} \otimes M) = 0$. But since $N = L^{-1} \otimes M$ has $N \not\cong O_E$ and $\deg N \geq 0$, it is standard that $H^1(E, N) = 0$ [Ha Section IV.1].

(2) It suffices to show that the map

$$H^0(E, L) \otimes H^0(E, L^n) \to H^0(E, L_{n+1})$$

is surjective for all $n \geq 1$. This follows immediately from part (1), unless $\deg L = 2$, $n = 1$, and $L^n \cong L$. But since $\sigma$ is translation by a point of infinite order on $E$, $\sigma$ does not fix the linear equivalence class of any nonzero effective divisor; thus $L^n \not\cong L$. \qed

In fact, it is not hard to show that $B(E, L, \sigma)$ is generated as an algebra by $B_1$ and $B_2$ if $\deg L = 1$, but we will not need this result.
Next, we study subalgebras of $B(E,\mathcal{L},\sigma)$. The structure of these is strongly constrained by the results of Artin and Stafford on cfg domains of GK-dimension 2. Recall that (following [RRZ]) a cfg algebra $A$ is called projectively simple if every nonzero homogeneous ideal $I$ of $A$ satisfies $\dim_k A/I < \infty$. Using the Artin-Stafford theory, we now prove the following.

**Lemma 3.2.** Consider $B = B(E,\mathcal{L},\sigma)$ where $\deg \mathcal{L} \geq 1$. Let $A$ be a cfg subalgebra of $B$ with $\dim_k A_n \geq 2$ for some $n \geq 1$.

1. $\mathrm{GK} A = 2$, $A$ is noetherian, and $A$ is projectively simple.
2. Suppose that $A_n \neq 0$ for all $n \gg 0$. Then there is another nonsingular elliptic curve $F$ with automorphism $\tau$, such that $A$ is equal in all large degrees to a ring of the form
   \begin{equation}
   \bigoplus_{n \geq 0} H^0(F,\mathcal{I} \otimes \mathcal{M} \otimes \mathcal{M}^\tau \otimes \mathcal{M}^{\tau^{-1}}),(n-1),
   \end{equation}
   where $\mathcal{M}$ is an invertible sheaf on $F$ of positive degree and $\mathcal{I}$ is an ideal sheaf. Moreover, there is a finite surjective morphism $\theta : E \to F$ such that $\tau \theta = \theta \sigma$, and $\tau$ is again translation by a point of infinite order if we take basepoint $\theta(p_0)$ for the group structure of $F$.
3. Suppose that $A$ is generated in degree 1. Let $\mathcal{N}$ be the sheaf generated on $E$ by the sections in $A_1 \subseteq H^0(E,\mathcal{L})$. Then $\mathcal{I} = \mathcal{O}_E$ in (3.3), and so $A$ is equal in large degree to $A' = B(F,\mathcal{M},\tau)$. We also have $\deg_F \mathcal{M} \geq 2$; $\mathcal{N} = \theta^* \mathcal{M}$; and $\deg \mathcal{N} = (\deg \theta)(\deg \mathcal{M})$. Finally, $A \subseteq B(E,\mathcal{N},\sigma)$ is a finite ring extension.

**Proof.** (1) It is standard that since $k$ is algebraically closed, a cfg domain $A$ with $\mathrm{GK} A \leq 1$ is just a subalgebra of $k[t]$ where $t$ has positive degree. (To see this, note that $\mathrm{GK} A \leq 1$ implies that $\dim_k A_n$ is uniformly bounded, and so $Q_{gr}(A) \cong D[t,t^{-1};\rho]$ where $D$ is a finite dimensional division algebra over $k$.)

The hypothesis on $A$ thus implies that $\mathrm{GK} A = 2$, using Bergman’s gap theorem.

Note that since $A$ is a domain, $S = \{n | A_n \neq 0\}$ must be a sub-semigroup of $\mathbb{N}$, so it is easy to see that $S$ contains all large multiples of $d = \gcd(S)$. In particular, replacing $A$ by its Veronese ring $A^{(d)}$ just amounts to a regrading, and the hypotheses are preserved (since $B^{(d)} \cong B(E,\mathcal{L}_d,\sigma^d)$). Clearly the desired conclusions that $A$ is noetherian and projectively simple are unaffected by this change in grading. Thus we assume from now on that $A_n \neq 0$ for all $n \gg 0$, and we prove that $A$ is noetherian and projectively simple in part (2).

(2) Since $A$ is a domain of finite GK-dimension, its graded quotient ring exists and has the form
   
   $Q_{gr}(A) = L[t,t^{-1};\sigma|_L] \subseteq Q_{gr}(B) \cong k(E)[t,t^{-1};\sigma],$

where we can take the same $t$ of degree 1 in both quotient rings since $A_n \neq 0$ for all $n \gg 0$. Here, $\sigma : k(E) \to k(E)$ is the automorphism on rational functions induced by $\sigma : E \to E$, and necessarily $L$ is a $\sigma$-fixed subfield of $k(E)$. Since $\mathrm{GK} A = 2$, $\mathrm{tr} \deg L/k = 1$ [AS Theorem 0.1], and so $L \subseteq k(E)$ is a finite degree extension. Since $\sigma : k(E) \to k(E)$ has infinite order, $\sigma|_L$ has infinite order.
Now the Artin-Stafford theory shows that $A$ determines a projective curve $F$ with $k(F) = L$, and an automorphism $\tau : F \to F$ inducing $\sigma|_L$ on rational functions [AS Proposition 3.3]. Since $E$ is nonsingular, the inclusion of fields $k(F) \subseteq k(E)$ induces a surjective finite morphism of curves $\theta : E \to F$, such that $\tau \theta = \sigma \theta$. If $F$ has a singular locus $S$, then $S$ is $\tau$-invariant, and so $\theta^{-1}(S)$ is a $\sigma$-invariant set of points in $E$. Since $\sigma$ is a translation by a point of infinite order, it has no nonempty $\sigma$-invariant finite subsets, and so $\theta^{-1}(S)$ and hence also $S$ is empty. Thus $F$ is also nonsingular. By Hurwitz’s Theorem [Ha Corollary IV.2.4], $F$ has genus 1 or 0. If $F$ has genus 0, then $F \cong \mathbb{P}^1$ and $\tau$ necessarily has a fixed point $p$. Then $\theta^{-1}(p)$ is a finite non-empty $\sigma$-invariant subset of $E$, again a contradiction. Thus $F$ has genus 1, in other words $F$ is also a nonsingular elliptic curve. Recalling that $p_0$ is the fixed basepoint for the group structure on $E$, choosing the basepoint $\theta(p_0)$ for the group structure on $F$, then $\theta$ is a homomorphism of groups [Ha Lemma 4.9]. Since $\sigma$ is the translation $x \mapsto x + r$, then $\tau : F \to F$ is the translation $y \mapsto y + \theta(r)$, where clearly $\theta(r)$ is also a point of infinite order in the group of $F$.

Now [AS Proposition 6.4] shows that since $\tau$ has no fixed points, $A$ must be isomorphic in large degree to the ring in (3.3) for some invertible sheaf $\mathcal{M}$ of positive degree and some ideal sheaf $\mathcal{I}$ on $F$. It also follows that $A$ is noetherian [AS Theorem 5.6], and projectively simple [AS Theorem 5.11(2)].

(3) Now assume that $A$ is generated in degree 1, where $\mathcal{N} \subseteq \mathcal{L}$ is the invertible sheaf generated on $E$ by the sections in $A_1$. Part (2) applies to $A$, and so $A$ has the form given in (3.3) in large degree, necessarily with $\mathcal{I} = \mathcal{O}_F$ by [AS Theorem 4.7]. In other words, $A$ is isomorphic in large degree to the twisted homogeneous coordinate ring $A' = B(F, \mathcal{M}, \tau)$. Obviously if $\dim_k A_1 = 1$ then $A \cong k[t]$, contradicting the hypothesis, so $\dim_k A_1 \geq 2$. Then $\deg \mathcal{M} \geq 2$. Clearly also the sections in $A_1 \subseteq H^0(F, \mathcal{M})$ must generate $\mathcal{M}$ on $F$, or else $A$ cannot equal $B(F, \mathcal{M}, \tau)$ in large degree. We must then have $\theta^*(\mathcal{M}) = \mathcal{N}$, and $\deg \mathcal{N} = (\deg \theta)(\deg \mathcal{M})$, where $\deg \theta$ is the number of points in every fiber of $\theta$. To check that $A = B(F, \mathcal{M}, \tau) \subseteq B(E, \mathcal{N}, \sigma)$ is a finite extension of rings, we prove that $A B$ is finitely generated and leave the similar verification that $B A$ is finitely generated to the reader. It is enough, by the graded Nakayama lemma, to show that $\dim_k B/A_{\geq 1} B < \infty$. The right ideal $A_{\geq 1} B$ of $B$ is, by the equivalence of categories (2.1), equal in large degree to $\bigoplus_{n \geq 0} H^0(E, \mathcal{J} \otimes \mathcal{N}_n)$ for some ideal sheaf $\mathcal{J}$ on $E$. Since $A_n$ already generates $\mathcal{N}_n$ on $E$ for each $n$, necessarily $\mathcal{J} = \mathcal{O}_E$. This shows that $\dim_k B/A_{\geq 1} B < \infty$, as required.

Consider $B(E, \mathcal{N}, \sigma)$ where $\deg \mathcal{N} \geq 1$, so $\mathcal{N}$ is $\sigma$-ample. Given a module $N \in \text{gr-}B$ with GK $N \leq 1$, it will be useful later to associate a Weil divisor to $N$ which tracks the composition factors of the corresponding coherent sheaf on $E$, in the following way.

**Definition 3.4.** For any $N \in \text{gr-}B(E, \mathcal{N}, \sigma)$ with GK $N \leq 1$, then $\pi(N) \in \text{qgr-}B$ corresponds under the equivalence of categories (2.1) to a torsion sheaf $\mathcal{F} \in \text{coh} E$. Let $C(N) = \sum_{p \in E} [\text{length}_{\mathcal{O}_E, p} \mathcal{F}_p] \cdot p \in \text{Div} E$.

We want to make a few comments about point modules for $B = B(E, \mathcal{N}, \sigma)$. We only want to discuss point modules for rings which are generated in degree 1, so because of Lemma 3.1 we assume that $\deg \mathcal{N} \geq 2$. 

\[\text{Page 11}\]
In particular, in this case $\mathcal{N}$ is generated by its global sections \cite[Corollary IV.3.2]{Ha}, and it easily follows that for each $q \in E$, $P = P(q) = \bigoplus_{n \geq 0} \mathcal{H}^n(E, k(q) \otimes \mathcal{N}_n)$ is a point module for $B$, where here $k(q)$ is the skyscraper sheaf at $q$.

In the last result of this section, we will study how the divisors defined in Definition \ref{def:divisors} are related for modules over two different twisted homogeneous coordinate rings. Let $\mathcal{N} \subseteq \mathcal{N}'$ be invertible sheaves on $E$, with $2 \leq \deg \mathcal{N} = \deg \mathcal{N}' - 1$. Thus $\mathcal{N} = \mathcal{I}_p \otimes \mathcal{N}'$ for some unique point $p \in E$, where $\mathcal{I}_p = \mathcal{O}_E(-p)$ is the ideal sheaf of the point $p$. Then we have $B = B(E, \mathcal{N}, \sigma) \subseteq B' = B(E, \mathcal{N}', \sigma)$. Given $q \in E$, we have the corresponding $B'$-point module $P(q)_{B'} = \bigoplus_{n \geq 0} \mathcal{H}^n(E, k(q) \otimes \mathcal{N}'_n)$ and the corresponding $B$-point module $Q(q)_B = \bigoplus_{n \geq 0} \mathcal{H}^n(E, k(q) \otimes \mathcal{N}_n)$.

**Lemma 3.5.** Keep the notation of the previous paragraph.

1. $P(q)_B \in \text{gr-}B$ and $\pi(P(q)_B) = \pi(Q(q))$ in $\text{gr-}B$.

2. For any $N \in \text{gr-}B'$ with $\text{GK} N < 1$, then $N_B \in \text{gr-}B$ and $C(N_{B'}) = C(N_B)$ in Definition \ref{def:divisors}.

**Proof.** (1) Consider $P = P(q)_{B'} = \bigoplus_{n \geq 0} \mathcal{H}^n(E, (\mathcal{O}_E/\mathcal{I}_q) \otimes \mathcal{N}'_n)$. Given $n \geq 0$, we calculate $\{x \in B'_1 | P_n x = 0\} = \mathcal{H}^n(E, \mathcal{I}_{\tau^n(q)} \otimes \mathcal{N}')$, and so

$$\{x \in B'_1 | P_n x = 0\} = \mathcal{H}^n(E, \mathcal{I}_{\tau^n(q)} \otimes \mathcal{N'}) \cap B_1 = \begin{cases} \mathcal{H}^n(E, \mathcal{I}_{\tau^n(q)} \otimes \mathcal{N}) \subseteq B_1 & \text{if } p \neq \tau^n(q) \\ B_1 & \text{if } p = \tau^n(q) \end{cases}$$

In particular, $P_n B_1 = P_{n+1}$ if and only if $p \neq \tau^n(q)$. Since all orbits of $\tau$ are infinite, there is at most one $n$ such that $p = \tau^n(q)$. Thus for $n$ large enough, say $n \geq n_0$, we see that $(P_{\geq n})_B$ is a shifted point module for $B$. In particular, $P_B$ is finitely generated. A similar calculation to the one above also easily shows for $n \geq n_0$ that $Q(q) = \bigoplus_{n \geq 0} \mathcal{H}^n(E, k(q) \otimes \mathcal{N}_n)$ satisfies $Q_{\geq n} \cong (P_{\geq n})_B$, since $r.\text{ann}_B Q_n = r.\text{ann}_B P_n = \bigoplus_{m \geq 0} \mathcal{H}^n(E, \mathcal{I}_{\tau^n(q)} \otimes \mathcal{N}_m)$. It follows that $\pi(Q_B) = \pi(P_B)$ in $\text{gr-}B$.

(2) Suppose that $N$ is a 1-critical $B'$-module. Applying the category equivalence \eqref{eq:category_equivalence}, clearly $G(N)$ is a simple object of coh $E$ and so is equal to a skyscraper sheaf $k(q)$ for some $q$. Then $N_{\geq n} \cong P(q)_{\geq n}$ for some $n$. Now given any $M \in \text{gr-}B'$ with $\text{GK} M = 1$, using the fact that $M$ has constant Hilbert function by Lemma \ref{lem:hilbert_function}(3), it is easy to prove that $M$ has a filtration $0 = M_0 \leq M_1 \leq \cdots \leq M_m = M$, in which the factors $M_i/M_{i-1}$ are either finite-dimensional or 1-critical (and so equal in large degree to tails of point modules $P(q)_{\geq n}$). Clearly $C(N)$ just enumerates the points $q$ corresponding to the point module tails in this filtration. Since the same interpretation holds for GK-1 modules over the ring $B$, both claims are now immediate from part (1). \hfill $\square$

4. **The generic Sklyanin algebra**

In this section, we review some basic properties of the Sklyanin algebra $S$. We then prove a combinatorial proposition, which will be the key to the calculation of the Hilbert series of the rings $R(D)$ of main interest.
For any $a, b, c \in k$, the Sklyanin algebra $S = S(a, b, c)$ is the $k$-algebra with presentation

$$S(a, b, c) = k[x, y, z]/\{axy + byx + cz^2, ayz + bzy + cx^2, axz + bxz + cy^2\}.$$

The algebra $S$ is $\mathbb{N}$-graded where $x, y, z$ all have degree 1, and so $S$ is generated as a $k$-algebra by $S_1$. It is well-known that for very general choices of the parameters $a, b,$ and $c$, $S$ has the following properties: it is Artin-Schelter regular with Hilbert series $h_S(t) = 1/(1 - t)^3$; it has a unique up to scalar central element $g \in S_3$; $S/Sg \cong B(E, \mathcal{L}, \sigma)$, where $E$ is a nonsingular elliptic curve embedded as a degree 3 divisor in $\mathbb{P}^2$ and $\mathcal{L} = \mathcal{O}(1)|_E$; and $\sigma : E \rightarrow E$ has infinite order. See [ATV1, ATV2] for more details, including the definition of regular algebra. When $S$ has all of the above properties, we say that $S$ is a generic Sklyanin algebra. We indicate the image of any subset $V$ of $S$ under the quotient map $S \rightarrow S/Sg$ by $\overline{V}$. Since $\sigma$ has infinite order, fixing some base point $p_0 \in E$ for the group law, then $\sigma$ is a translation $x \mapsto x + r$, where $r$ has infinite order in the group. For the rest of the paper, $S$ stands for a generic Sklyanin algebra and the above notation is also fixed.

It is known that every point module for $S$ is annihilated by $g$, so the point modules for $S$ are the same as the point modules for $S/Sg = B = B(E, \mathcal{L}, \sigma)$ [ATV2, Proposition 7.7ii]. As we saw in the previous section, given a point $q \in E$ we have a corresponding point module $B/J$ for $B$, where $J = J(q) = \bigoplus_{n \geq 0} H^0(E, I_q \otimes \mathcal{L}_n)$. As an $S$-module, this is the point module $S/I$, where $I = I(q) = \{ x \in S | x \in J(q) \}$. Since $S$ has the same Hilbert series as a commutative polynomial ring in three variables, we have $\dim_k S_1 = 3$ and so $\dim_k I(q)_1 = 2$; we set $W(q) = I(q)_1$ and we call any such $W(q) \subseteq S_1$ a point space. We prove next some very useful formulas about products of point spaces and copies of $S_1$. Part (2) of the following lemma is also derived by Ajitabh [Aj] Lemma 2.3, Proposition 2.14(iv)], but we give the proof since it follows easily from Lemma 3.1.

**Lemma 4.1.** Let $p, q \in E$.

1. $S_1 W(\sigma(q)) = W(q)S_1$.

2. We have $\dim_k W(p)W(q) = 4$ if $q \neq \sigma^{-2}(p)$. On the other hand, $\dim_k W(p)W(\sigma^{-2}(p)) = 3$.

**Proof:** (1) Since $\overline{S_2}$ and $S_2$ may be identified, we just note that in $\overline{S}$, the product $\overline{S_1 W(\sigma(q))}$ amounts to the image of the multiplication map

$$H^0(E, \mathcal{L}) \otimes H^0(E, (\mathcal{I}_{\sigma(q)} \otimes \mathcal{L})^\sigma) \rightarrow H^0(E, \mathcal{L}_2),$$

which is $H^0(E, \mathcal{I}_q \otimes \mathcal{L}_2)$ by Lemma 3.1(1). In other words, the image is the space of all sections of $\mathcal{L}_2$ vanishing along the point $q$. The space $\overline{W(q)S_1}$ is equal to the same thing by a similar calculation.

(2) Again we identify $S_2$ and $\overline{S_2}$, and we are looking for the image of the multiplication map

$$\theta : H^0(E, \mathcal{I}_p \otimes \mathcal{L}) \otimes H^0(E, (\mathcal{I}_q \otimes \mathcal{L})^\sigma) \rightarrow H^0(E, \mathcal{L}_2).$$

As we saw in Lemma 3.1(1), $\text{im}(\theta)$ has dimension 3 if $\mathcal{I}_p \otimes \mathcal{L} \cong \mathcal{I}_{\sigma^{-1}(q)} \otimes \mathcal{L}^\sigma$; otherwise, we have $\text{im}(\theta) = H^0(E, \mathcal{I}_p \otimes \mathcal{I}_{\sigma^{-1}(q)} \otimes \mathcal{L}_2)$, and so $\dim_k \text{im}(\theta) = 4$. Letting $\mathcal{L} = \mathcal{O}_E(D)$ for some divisor $D$, we see that the
case \( \dim_k \text{im}(\theta) = 3 \) occurs if and only if \( D - p \sim \sigma^{-1}(D) - \sigma^{-1}(q) \). Since \( \sigma \) is a translation \( x \to x + r \), using Abel’s theorem this condition is equivalent to \( p = q + 2r \) in the group law, or \( \sigma^2(q) = p \). \( \square \)

Part (2) of the preceding lemma is also related to the following explicit free resolution of a point module.

**Lemma 4.2.** Let \( W = W(q) \subseteq S_1 \) be a point space with \( W = kw + kx \). By Lemma 4.1(2), there are elements \( y, z \in S_1 \) such that \( wy + xz = 0 \), and with \( ky + kz = W(\sigma^{-2}(q)) \). Consider the complex

\[
0 \to S[-2] \to S[-1] \oplus S[-1] \xrightarrow{(w \ x)} S \to S/(wS + xS) \to 0,
\]

where elements of the free modules are thought of as column vectors and the maps are left multiplication by the indicated matrices. Then this is an exact sequence of right \( S \)-modules, \( W(q)S = wS + xS = I(q) \), and this sequence is a free resolution of \( P(q) = S/I(q) \). In particular, \( wS \cap xS = wyS = xzS \).

**Proof.** This is a special case of [ATV2 Proposition 6.7(ii)]. \( \square \)

Using the lemmas above, we now prove an important combinatorial formula. In fact, it includes as a special case the calculation of the Hilbert function of the ring \( R(D) \) from the introduction, in case \( D = p \) is a single point. As usual, we make the convention on binomial coefficients that \( \binom{n}{k} = 0 \) if \( n < k \).

**Proposition 4.3.** Let \( W = W(p) \subseteq S_1 \) be any point space. Then for all \( m \geq 0 \) and \( n \geq 0 \) we have

\[
\dim_k S_m(WS_2)^n = \dim S_{m+3n} - \binom{n + 1}{2}.
\]

**Proof.** Let \( V(m,n) = S_m(WS_2)^n \), and put \( h(m,n) = \dim_k V(m,n) \) and \( j(m,n) = \dim_k S_{m+3n} - \binom{n + 1}{2} \). Our goal is to prove that \( h(m,n) = j(m,n) \) for all \( m, n \geq 0 \). Note that since \( \dim_k S_{m+3n} = \binom{m+3n+2}{2} \) is known, it is a triviality to check that \( j \) satisfies the following recurrence relations:

\[
\begin{align*}
(4.4) \quad j(m,n) & = j(m,n - 1) + 3m + 8n, \\
(4.5) \quad j(m,n) & = 2j(m+2,n-1) - j(m+4,n-2),
\end{align*}
\]

for all \( m \geq 0, n \geq 2 \).

Now we check the formula for small \( n \). First, \( h(m,0) = j(m,0) \) is obvious for any \( m \geq 0 \). Next, suppose that \( n = 1 \); then \( V(m,1) = S_m W(p) S_2 = W(\sigma^{-m}(p)) S_{m+2} \) by Lemma 4.1(1). Since \( S/W(\sigma^{-m}(p))S \) is a right point module for \( S \) by Lemma 4.2 we must have \( h(m,1) = \dim_k W(\sigma^{-m}(p)) S_{m+2} = \dim_k S_{m+3} - 1 = j(m,1) \), as required.

Since \( S/WS \) is a point module and \( g \) annihilates all point modules, we must have \( g \in WS_2 \). Thus for \( n \geq 1 \), the kernel of the quotient map \( V(m,n) \to \overline{V(m,n)} \) contains \( gV(m,n-1) \). In this case we have \( \dim_k \overline{V(m,n)} = 3m + 8n \), using Proposition 3.1 and the Riemann-Roch theorem. Thus we also get

\[
h(m,n) \geq h(m,n - 1) + 3m + 8n,
\]
for any $n \geq 1$. Using (144) and induction on $n$, it follows that $h(m,n) \geq j(m,n)$ for all $m,n \geq 0$.

We finish the proof by another induction on $n$. Let $n \geq 2$ and suppose that $h(m,k) = j(m,k)$ for all $m \geq 0$ and $k < n$. We have

$$V(m,n) = S_m(W(p)S_2)^n = W(\sigma^{-m}(p))S_{m+2}(WS_2)^{n-1} = wU + xU,$$

where $W(\sigma^{-m}(p)) = kw + kx$ and $U = V(m + 2, n - 1)$. By Lemma 4.2 we have $wy + xz = 0$, where $ky + kz = W(\sigma^{-m-2}(p))$, and in fact $wS \cap xS = wyS = xzS$. Then

$$wU \cap xU = \{wys|s \in S, yS \in U, zS \in U\} = \{wys|s \in S, W(\sigma^{-m-2}(p))s \subseteq U\} \supseteq wyV(m + 4, n - 2),$$

where the last inclusion holds since by Lemma 4.1(1),

$$W(\sigma^{-m-2}(p))S_{m+4}(WS_2)^{n-2} = S_{m+2}WS_2(WS_2)^{n-2} = V(m + 2, n - 1) = U.$$

Thus we have

$$h(m,n) = \dim_k(wU + xU) = 2 \dim_k U - \dim_k(wU \cap xU) \leq 2h(m + 2, n - 1) - h(m + 4, n - 2)$$

$$= 2j(m + 2, n - 1) - j(m + 4, n - 2) = j(m,n),$$

using the induction hypothesis and (145). Since we have already proved that $h(m,n) \geq j(m,n)$, the result follows. 

The last result of this section contains some other formulas about products of point spaces, which we give here because the proofs are also easy consequences of Lemma 4.2. However, we remark that these formulas are used only much later, in Sections 11 and 12, and are needed not in the proofs of the main theorems of the paper.

Lemma 4.6. (1) $\dim_k W(p)W(q)S_1 = 8$ if $q \neq \sigma^{-2}(p)$, while $\dim_k W(p)W(\sigma^2(p))S_1 = 7$. In particular, $g \in W(p)W(q)S_1$ if and only if $q \neq \sigma^{-2}(p)$.

(2) $g \in W(p)^3$ for any point $p$.

(3) Let $0 \neq f \in S_1$. Then $g \in S_1fS_1$.

Proof. All three parts are proved the same way. Given $X \subseteq S_2$, consider $W(p)X$ for some point space $W(p) = kw + kx$. Then $W(p)X = wX + xX$ and so $\dim_k W(p)X = 2 \dim_k X - \dim_k wX \cap xX$. But if $y$ and $z$ are the basis of $W(\sigma^{-2}(p))$ such that $wy + xz = 0$, as in Lemma 4.2 then

$$wX \cap xX = \{wyr|yr, zr \in X\} = wy\{r|W(\sigma^{-2}(p))r \in X\}.$$

Thus $\dim_k wX \cap xX = \dim_k Y$, where $Y = \{r \in S_1|W(\sigma^{-2}(p))r \in X\}$. In the following, since $S$ equals $\overline{S}$ in degree up to 2, we identify these low degree elements of the two rings. We also use Lemma 3.1(1) many times below without further comment, to calculate the dimension of products in $\overline{S}$. 

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1. Take $X = W(q)S_1$ above. Then $\dim_k X = 5$. If $q = \sigma^{-2}(p)$, then $Y = S_1$, and so $\dim_k W(p)X = 10 - 3 = 7$. If instead $q \neq \sigma^{-2}(p)$, then $Y = W(\sigma(q))$, and so $\dim_k W(p)X = 10 - 2 = 8$. In any case we have $\dim_k \overline{W(p)W(q)S_1} = 7$. So $g \in W(p)W(q)S_1$ if and only if $q \neq \sigma^{-2}(p)$.

2. Take $X = W(p)^3$ above. Then $\dim_k X = 4$ and $\dim_k \overline{W(p)^3} = 6$. For a section in $S_1 = H^0(E, L)$ to be in $Y$, it must vanish at both $p$ and $\sigma^{-1}(p)$, so $\dim_k Y = 1$. Thus $\dim_k W(p)^3 = 8 - 1 = 7$, and so $g \in W(p)^3$.

3. We have that $f \in H^0(E, L)$ vanishes along some divisor $D$. Letting $p$ be any point such that $D$ does not contain $\sigma^{-2}(p)$, then we claim that we even have $g \in W(p)fS_1$. We let $X = fS_1$ above, so $\dim_k X = 3$.

Hence, $\dim_k Y = 10 - e$ by the Riemann-Roch theorem, and so $\dim_k S_1 = 10 - e$ by the Riemann-Roch theorem, and so $\dim_k S_1 = 10 - e$ by the Riemann-Roch theorem, and so $\dim_k V = 10 - e$. Let $R = R(D) = k(V)$ be the $k$-subalgebra of $T$ generated by $V$. The notation of this paragraph is used throughout the rest of the paper, except in Sections 6-8, where we study a slightly more general class of rings.

The first step in our analysis of $R(D)$ is to identify the ring $\overline{R} = R/R \cap Tg \subseteq \overline{T}$. Since $R$ is generated in degree 1, $\overline{R}$ is the subalgebra of $B(E, M, \tau)$ generated in degree 1 by $H^0(E, N)$. By Lemma 4.4.2, since $\deg N \geq 2$, $\overline{R}$ is simply the subring $B(E, N, \tau)$ of $\overline{T}$. In particular, using Riemann-Roch, the Hilbert series of $\overline{R}$ is

\begin{equation}
\begin{aligned}
\overline{h}(t) &= 1 + \sum_{n \geq 1} (9 - e)nt^n = \frac{t^2 + (7 - e)t + 1}{(1 - t)^2}.
\end{aligned}
\end{equation}

Using the help of Proposition 4.3 from the last section, we now calculate the Hilbert series of $R$.

**Theorem 5.2.** Fix an effective divisor $D$ on $E$ with $0 \leq e = \deg D \leq 7$, and let $R = R(D)$. Then

1. $h_R(t) = \frac{t^2 + (7 - e)t + 1}{(1 - t)^3}$.
2. $\overline{R} = R \cap Tg$.

**Proof.** In fact, the statements in parts (1) and (2) of the theorem are equivalent. Since $R \cap Tg \supseteq Rg$ in any case and $R$ is a domain, we definitely have $\overline{h}(t) \leq (1 - t)h_R(t)$. Part (2) holds if and only if this is an equality of Hilbert series, which is if and only if (1) holds, by (5.1). We thus prove both parts of the theorem in tandem.
We first extend $k$ to an uncountable algebraically closed field if necessary. Since the Hilbert series of $R$ remains unchanged, this loses no generality.

If $\deg D = 0$ there is nothing to prove, since (2) is a tautology in this case. Now let $\deg D = 1$, say $D = p$. Since $S/W(p)S$ has the Hilbert series of a point module, by Lemma 4.3 it is easy to see that $R_1 = W(p)S_2 \subseteq T_1$. Applying Proposition 4.3 with $m = 0$, we have

$$h_R(t) = \frac{t^2 + 7t + 1}{(1-t)^3} - \frac{t}{(1-t)^3} = \frac{t^2 + 6t + 1}{(1-t)^3},$$

and so (1) (and hence also (2)) holds in this case.

Next, suppose that the theorem holds for $R' = R(D')$, for some effective divisor $D'$ with $0 \leq \deg D' < 7$, and let $D = D' + p$. We now prove that the theorem holds for $R = R(D)$, assuming the further restriction that $p$ does not lie on the $\tau$-orbit of any point occurring in $D'$. Let $R^\# = R' \cap R(p)$. Obviously $R^\# \supseteq R$; we will show that in fact we have equality. First we find $\overline{R^\#}$. We know that $\overline{R'} = B(E, I_{D'} \otimes \mathcal{M}, \tau)$ and $\overline{R(p)} = B(E, I_p \otimes \mathcal{M}, \tau)$. Thus in degree $n \geq 1$, we have

$$\overline{R(p)}_n = H^0(E, I_C \otimes \mathcal{M}_n),$$

where $C = \sum_{j=0}^{n-1} \tau^{-j}(p)$, and similarly,

$$\overline{R'}_n = H^0(E, I_{C'} \otimes \mathcal{M}_n),$$

where $C' = \sum_{j=0}^{n-1} \tau^{-j}(D')$. But the assumption on $p$ implies that the divisors $C$ and $C'$ have disjoint support, so it follows that we have

$$\overline{R(p)}_n \cap \overline{R'}_n = H^0(E, I_{C+C'} \otimes \mathcal{M}_n) = \overline{R}_n,$$

since $C + C' = \sum_{j=0}^{n-1} \tau^{-j}(D)$. But then we see that we have $\overline{R^\#} \subseteq \overline{R(p)} \cap \overline{R'} = \overline{R} \subseteq \overline{R^\#}$, and thus $\overline{R} = \overline{R^\#}$.

Now note that since $R(p) \cap Tg = R(p)g$ and $R' \cap Tg = R'g$, we have

$$R^\# \cap Tg = (R(p) \cap R') \cap Tg \subseteq (R(p)g \cap R'g) = (R(p) \cap R')g = R^\# g \subseteq R^\# \cap Tg.$$ 

Thus all inclusions here are actually equalities, and $R^\# \cap Tg = R^\# g$. Since $R \cap Tg \supseteq Rg$ and $\overline{R} = \overline{R^\#}$, it follows that $h_R(t) \geq h_{R^\#}(t)/(1-t) = h_{R^\# g}(t)/(1-t) = h_{R^\#}(t)$; since $R \subseteq R^\#$, this forces $R = R^\#$. Then $R \cap Tg = Rg$, so (2) holds for $R = R(D)$.

Finally, assume again that the theorem holds for some $D'$, where $0 \leq \deg D' < 7$, and consider $D = D' + p$ where $p \in E$ now varies. We show that $R(D)$ has the correct Hilbert series for all $p$; the theorem will then follow by induction on $\deg D$. We have a regular map $\overline{\phi} : E \to \mathbb{P}(H^0(E, I_{D'} \otimes \mathcal{M}^*))$, namely the map to projective space determined by a basis of sections for the sheaf $I_{D'} \otimes \mathcal{M}$ on $E$, which can be described explicitly as $p \mapsto H^0(E, I_{D'+p} \otimes \mathcal{M}) = \overline{V(D)}$. Thus the map $\phi : E \to \mathbb{P}(V(D'^*))$ given by $p \mapsto V(D)$ is also regular. Fixing some $n \geq 1$, we define a function $f : \mathbb{P}(V(D'^*)) \to \mathbb{N}$ by the rule $Y \mapsto \dim_k Y^n$, where $Y$ is a codimension-1 subspace of $V(D')$ and $Y^n \subseteq T_n$ is the product inside $T$. A standard argument now gives that $f$ is lower semi-continuous; in other words, $\{Y \in \mathbb{P}(V(D'^*)) | \dim_k Y^n \leq m\}$ is Zariski closed for each
m \in \mathbb{N}. Thus f \circ \phi : E \to \mathbb{N}, where f \circ \phi(p) = \dim_k V(D)^n = \dim_k R(D)_n$, is also lower semi-continuous. We showed in the previous paragraph that $R(D)$ has the correct Hilbert series for all points $p \in E$ outside of a union of finitely many $\tau$-orbits; on the other hand, $E$ has uncountably many points since $k$ is uncountable. It thus follows, from the lower-semicontinuity of $f \circ \phi$ for each $n$, that $h_{R(D)}(t) \leq \frac{t^2 + (7 - e)t + 1}{(1 - t)^2}$ for all $p \in E$, where $e = \deg D$. The reverse inequality was essentially shown in the first paragraph of the proof, so we are done.

For later use, we need the additional Hilbert series calculation given in the next lemma. The proof uses exactly the same technique as the proof of the previous result.

**Lemma 5.3.** Let $D'$ be an effective divisor with $0 \leq \deg D' \leq 6$, let $p \in E$ and let $D = D' + p$. Let $e = \deg D = \deg D' + 1$. Let $R = R(D)$ and $R' = R(D')$. Consider the right $R$-module $M = k + R'_1 R$. Then

$$h_M(t) = h_R(t) + \frac{t}{(1 - t)^2} = \frac{(8 - e)t + 1}{(1 - t)^3}.$$  

**Proof.** An easy calculation, using Lemma 3.1(1), shows that $M/(\langle Tg \cap M \rangle = \overline{M} = k + R'_1 R$ has Hilbert series $(8 - e)t + 1$ $(1 - t)^2$. Since multiplication by $g$ is an injective homomorphism of $M$, we have

$$h_M(t) = \frac{h_{M/Mg}(t)}{(1 - t)} = \frac{h_{\overline{M}}(t)}{(1 - t)} = \frac{(8 - e)t + 1}{(1 - t)^3}.$$  

Now suppose that $p$ does not lie on the $\tau$-orbit of any point in the support of $D'$. Then we showed in the course of the proof of Theorem 5.2 that $R' \cap R(p) = R$. Since all Hilbert series of the terms in this equation are now known by Theorem 5.2, it easily follows that $R' + R(p) = T$. This certainly implies that $R' + (k + T_1 R(p)) = T$. But the Hilbert series of $k + T_1 R(p)$ follows from Proposition 4.3 so this determines immediately the Hilbert series of $N = R' \cap (k + T_1 R(p))$, which a calculation shows to be $h_N(t) = \frac{(8 - e)t + 1}{(1 - t)^3}$. Since $M \subseteq N$, the estimate of the first paragraph forces $M = N$, and thus the Hilbert series of $M$ is as stated.

Now just as in the proof of the previous theorem, we can reduce to the case where $k$ is uncountable. A completely analogous argument as that of the last paragraph of the previous proof also shows that for any $n \geq 1$, the $k$-dimension of $M_n = R'_1 R_{n-1}$ is a lower semi-continuous function of $p \in E$. The estimate $h_M(t) \geq \frac{(8 - e)t + 1}{(1 - t)^3}$ holds for all $p$, and was shown above to be an equality for all but countably many $p$; this forces $h_M(t) = \frac{(8 - e)t + 1}{(1 - t)^3}$ for all $p$, as required.

We close this section with a few more properties of $R(D)$ that follow easily at this point. Together with Theorem 5.2, the following result finishes the proof of Theorem 1.1(1).

**Theorem 5.4.** Let $R = R(D)$ for some effective divisor $D$ with $0 \leq \deg D \leq 7$.

1. $R$ has infinite global dimension.
2. $Q_{gr}(R) = Q_{gr}(T)$.
(3) $R^\text{op} \cong R(\bar{D})$ for some divisor $\bar{D}$ with $\deg \bar{D} = \deg D$.

Proof. (1) A standard argument (for example, see Proposition 2.9), using the minimal graded free resolution of $R = R/R_{\geq 1}$, shows that if $R$ has finite global dimension, then $h_\infty(t) = \frac{1}{p(t)}$ for some polynomial $p(t) \in \mathbb{Z}[t]$. But it is easy to prove that the Hilbert series of $R$, as calculated in Theorem 6.2, cannot be written in this form.

(2) If $D = 0$ there is nothing to prove, so assume $\deg D > 0$ and write $D = D' + p$ for some divisor $D'$. We will show that $Q_{gr}(R) = Q_{gr}(R')$, where $R = R(D)$ and $R' = R(D')$. Then the result follows by induction on $\deg D$. By Lemma 5.3, $\dim_k R'_1R_1 - \dim_k R_2 = 2$. We may write $R'_1 = R_1 + kz$ for some element $z \in R'_1$, and so $R'_1R_1 = zR_1 + R_2$. Since $\dim_k R_1 \geq 3$ (given that $\deg D \leq 7$), we must have $zR_1 \cap R_2 \neq 0$. Then $z \in Q_{gr}(R)$, so $R' \subseteq Q_{gr}(R)$ and thus $Q_{gr}(R) = Q_{gr}(R')$.

(3) Consider the ring $B = B(E, \mathcal{M}, \tau)$ and the set of all possible subspaces of the form $H^0(E, J_D \otimes \mathcal{M}) \subseteq B_1$ for divisors with $0 \leq \deg D \leq 7$. We claim that this set can also be characterized as the set of all subspaces $W \subseteq B_1$ such that the ring $k(W)$ satisfies $Q_{gr}(k(W)) = Q_{gr}(B)$ and $\dim_k W^n = n(\dim_k W)$ for all $n \geq 1$. In fact, the claim follows easily from Lemma 5.2(3). As a consequence, it is easy to see that any graded anti-automorphism $\phi: B \to B$ must permute this set of subspaces.

Now, it is well known that $S^\text{op} \cong S$; for example, the map induced by $x \mapsto y, y \mapsto x, z \mapsto z$ gives an anti-automorphism $\phi: S \to S$, as one easily sees from the presentation. Passing to the 3-Veroneses, this induces an anti-automorphism $\phi: T \to T$. Since the central element $g$ is unique up to scalar, $\phi$ must descend to an anti-automorphism $\psi: T/Tg \to T/Tg$, where $T/Tg \cong B = B(E, \mathcal{M}, \tau)$. By the first paragraph, $\psi$ must permute the set of subspaces of the form $V(D) \subseteq T_1$ for divisors $0 \leq \deg D \leq 7$. So for given $D$, $\phi$ restricts to an anti-isomorphism from $R(D)$ to $R(\bar{D})$ for some other divisor $\bar{D}$, which clearly must have the same degree. $\square$

6. Rings with a twisted homogeneous coordinate ring factor

In this section, we prove that the rings $R(D)$ have many additional nice properties; in particular, we prove Theorem 3.1(2). Since we now know that $R(D)/R(D)g \cong B(E, \mathcal{N}, \tau)$, the main technique is to observe that many good properties pass from the factor ring $B$ to the ring $R(D)$. This technique is well-known and holds much more generally. Because it may be useful for future reference, and is not really any more difficult, in this section and the next two we work not just with the rings $R(D)$, but with the more general class of rings satisfying the following hypothesis.

**Hypothesis 6.1.** Let $R$ be a cfg- $k$-algebra which is generated in degree 1. Suppose that $R$ is a domain with a homogeneous central element $g$ of degree $d > 0$ such that $R/Rg \cong B(E, \mathcal{N}, \tau)$, where $E$ is a nonsingular elliptic curve with fixed basepoint $p_0$, $\tau: E \to E$ is a translation automorphism $x \mapsto x + s$ for a point $s$ of infinite order, and $\deg \mathcal{N} \geq 2$. We continue to use the notation $\overline{Y}$ for the image of a subset $Y$ of $R$ under the map $R \to R/Rg$. 
Hypothesis 6.1 is assumed throughout the rest of this section. We remark that for some of the results of this and the next few sections, it is sufficient that \( g \) be a normal element \( (Rg = gR) \); in particular, this is true of Theorems 6.3 and 6.7 below.

The following definition is standard in this setting.

**Definition 6.2.** Given any \( \text{cfg} \) algebra \( A \) with a central element \( g \) and \( M \in \text{gr}-A \), we define its \( g \)-torsion submodule \( Z(M) = \{m \in M | mg^n = 0 \text{ for some } n > 0\} \). \( M \) is \( g \)-torsion if \( Z(M) = M \) and \( g \)-torsionfree if \( Z(M) = 0 \).

This torsion theory has the usual formal properties. For example, it is easy to verify that as long as \( A \) is noetherian, then for any \( M \in \text{gr}-A \), \( Z(M)g^n = 0 \) for some \( n \geq 1 \), and \( M/Z(M) \) is \( g \)-torsionfree.

In the next theorem, we give a number of properties of rings satisfying Hypothesis 6.1 which follow quite immediately from the properties of twisted homogeneous coordinate rings studied earlier, together with results from the literature about when good properties “pass up” from a factor ring modulo a central (or normal) element.

**Theorem 6.3.** Let \( R \) satisfy Hypothesis 6.1.

1. \( R \) is strongly noetherian.
2. Every \( M \in \text{gr}-R \) has a Hilbert quasi-polynomial of period \( d \).
3. \( R \) is Auslander-Gorenstein and Cohen-Macaulay.
4. \( R \) satisfies \( \chi \) on the left and right, and \( \text{proj}\-R \) has cohomological dimension 2.

**Proof.** (1) Since \( B(E, N, \tau) \) is strongly noetherian by Lemma 2.2(1), \( R \) is strongly noetherian by [ASZ Proposition 4.9(1)].

(2) For any module \( M \in \text{gr}-R \), consider its \( g \)-torsion submodule \( Z(M) \). Since \( Z(M)g^n = 0 \) for some \( n \), \( Z(M) \) has a filtration with finitely many factor modules \( Z(M)g^i/Z(M)g^{i+1} \), each of which is a graded \( B \)-module; thus \( Z(M) \) has a Hilbert polynomial, by Lemma 2.2(3). Since \( M' = M/Z(M) \) is \( g \)-torsionfree, \( H_{M'}(t) = \frac{H_{M'/g}(1-t^d)}{1-t^d} \). Here, \( M'/g \) again is a \( B \)-module and so has a Hilbert polynomial. The form of the Hilbert series of \( M' \) then implies that \( M' \) has a Hilbert quasi-polynomial of period \( d \). Thus \( M \) does also.

(3) The ring \( B(E, N, \tau) \) is Auslander-Gorenstein of dimension 2 and Cohen-Macaulay, by Lemma 2.2(4). Then \( R \) is Auslander-Gorenstein of dimension 3 and Cohen-Macaulay, by [Lev Theorem 5.10].

(4) We know that \( B \) satisfies \( \chi \) and \( \text{cd}(\text{proj}-B) = 1 \), by Lemma 2.2(2), so \( R \) satisfies \( \chi \) by [AZ1 Theorem 8.8], and \( \text{cd}(\text{proj}-R) \leq 2 \) by [AZ1 Theorem 8.8]. We need to show equality in the last statement. The higher cohomology in \( \text{proj}-R \) can be calculated as a limit of \( \text{Ext} \) groups:

\[
\bigoplus_{m \in \mathbb{Z}} H^i(\pi(M)[m]) = \lim_{n \to \infty} \text{Ext}^{i+1}_R(R/R_{\geq n}, M),
\]

for any \( i \geq 1 \) and \( M \in \text{gr}-R \) [AZ1 Proposition 7.2]. Here, the direct limit is induced by the obvious factor maps \( R/R_{\geq (n+1)} \to R/R_{\geq n} \). Now since \( R \) is Cohen-Macaulay by part (3), for any \( N \) with \( \text{GK} N = 0 \) we
have \( \text{Ext}_R^i(N,R) = 0 \) if \( i < 3 \), but \( \text{Ext}_R^3(N,R) \neq 0 \). It follows from the long exact sequence in Ext that the maps in the direct limit \( \lim_{n \to \infty} \text{Ext}^3(R/R_{\geq n},R) \) are injective, and thus \( \bigoplus_{m \in \mathbb{Z}} H^2(\pi(R)[m]) \neq 0 \). So \( \text{cd}(\text{proj-} R) = 2 \) exactly. \( \square \)

Next, we study homogeneous ideals of rings \( R \) satisfying Hypothesis \text{[6.1]}. Here are a few basic properties of such ideals.

**Lemma 6.4.** Let \( R \) satisfy Hypothesis \text{[6.1]}.

1. If \( J \subseteq R \) is a nonzero homogeneous ideal, then \( J = Ig^n \) for some \( n \geq 0 \) and some ideal \( I \) with \( \text{GK} R/I \leq 1 \).
2. If \( I \) is a homogeneous ideal of \( R \) with \( \text{GK} R/I \leq 1 \), then there is a unique homogeneous ideal \( I' \supseteq I \) such that \( \dim_k I'/I < \infty \) and \( R/I' \) is \( g \)-torsionfree.
3. If \( M \in \text{gr}-R \) is \( g \)-torsionfree with \( \text{GK} M \leq 1 \), then \( I = r. \text{ann}_R(M) \) is an ideal such that \( \text{GK} R/I = 1 \) and \( R/I \) is \( g \)-torsionfree.

**Proof.** (1) Since \( J \neq 0 \), there is a maximal \( n \geq 0 \) such that \( J \subseteq Rg^n \). Then \( J = Ig^n \) for some homogeneous ideal \( I \). Since \( I \nsubseteq Rg \) by choice of \( n \), \( \overline{T} \neq 0 \) in \( R/Rg = B(E,N,\tau) \), which is a projectively simple ring by Lemma \text{[5.2]}(1). Thus \( \overline{R} = R/I \) satisfies \( \dim_k \overline{R}/\overline{Rg} < \infty \), from which it follows that \( \text{GK} \overline{R} \leq 1 \).

(2) Let \( I \) be such an ideal. Since \( \text{GK} R/Rg = 2 \), \( I \nsubseteq Rg \) and so the same argument as in part (1) shows that \( \dim_k R/(I + Rg) < \infty \). But then since \( (I + Rg^i)/(I + Rg^{i+1}) \) is a finitely generated \( R/(I + Rg)-\)module for each \( i \), \( \dim_k R/(I + Rg^n) < \infty \) for each \( n \). Write \( I'/I = Z((R/I)_R) \); clearly \( I' \) is a homogeneous ideal. Then \( I'/I \) is annihilated by some \( g^n \), and thus is a finitely generated \( R/(I + Rg^n)-\)module. So \( \dim_k I'/I < \infty \), as claimed.

(3) Since \( M \) has a Hilbert quasi-polynomial of period \( d \), by Theorem \text{[6.3]}(2), \( \text{GK} M \leq 1 \) implies that \( M \) has eventually periodic Hilbert function of period \( d \). In particular, since \( M \) is \( g \)-torsionfree, \( Mg \) is equal in large degree to \( M \), so there is \( n_1 \) such that \( (Mg)_{\geq n_1} = M_{\geq n_1} \). Now since \( g \) is central, we conclude that \( M_n = (M_{n-d})g \) and \( M_{n-d} \) have the same right annihilator in \( R \), for all \( n \geq n_1 \). Thus \( I = r. \text{ann}_R M = r. \text{ann}_R \bigoplus_{n=n_0}^{n_1-1} M_n \), where \( n_0 \) is an integer such that \( M_n = 0 \) for \( n < n_0 \). But since \( \bigoplus_{n=n_0}^{n_1-1} M_n \) is a finite-dimensional subspace of the module \( M \) of \( \text{GK} \)-dimension \( \leq 1 \), this forces \( \text{GK} R/I \leq 1 \) as needed. It is easy to check that \( R/(r. \text{ann}_R M) \) is automatically \( g \)-torsionfree, because \( M \) is.

We see from part (1) of the preceding result that we will understand all of the ideals of \( R \) if we understand the ideals \( I \) such that \( \text{GK} R/I = 1 \). It is convenient to make the following definitions.

**Definition 6.5.** Let \( A \) be a \( \text{cfg} \) \( k \)-algebra. If \( I \) is a homogeneous ideal of \( A \) such that \( \text{GKA}/I \leq 1 \), then we call \( I \) *special* and we call \( A/I \) a *special factor ring*. A special ideal \( I \) of \( A \) is said to be *minimal in tails* if \( \pi(I) \) is (uniquely) minimal among the set of objects \( \{ \pi(J) \in \text{qgr}-A | J \text{ is a special ideal of } A \} \). In other words, this means that every other special ideal \( J \) of \( A \) satisfies \( J \supseteq I \) for some \( n \geq 0 \). If \( A \) has a special ideal which is minimal in tails, then (speaking loosely) we say that \( A \) has a *minimal special ideal*. 

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The reason for the terminology “special” is that the rings $S$ and $T$ have no special factor rings except those of finite $k$-dimension, as we recall in Lemma 10.3 below. The rings $R(D)$ sometimes do have $GK$-1 factor rings (see Example 11.1), but they are rather rare, exactly in the sense that $R(D)$ always has a minimal special ideal. Proving this is the goal of much of the next several sections.

**Remark 6.6.** Suppose that $R$ satisfies Hypothesis 6.1 and that $R$ has a minimal special ideal. Using the results of Lemma 6.3 we can say more about this situation. Since clearly any two ideals which are both minimal in tails agree in large degree, using Lemma 6.4(2) there is a unique ideal $I$ which is minimal in tails with the additional property that $R/I$ is $g$-torsionfree. In other words, $R$ has a unique largest homogeneous $g$-torsionfree special factor ring $R/I$. Also, by Lemma 6.4(3), it is then easy to see that this same ideal $I$ is also the unique largest ideal which annihilates all $GK$-1 $g$-torsionfree right $R$-modules. Because of Theorem 5.3(3), $I$ must also be the unique largest ideal annihilating all $GK$-1 $g$-torsionfree left $R$-modules.

We now show that rings satisfying the main hypothesis of this section are maximal orders, as a quick consequence of the Cohen-Macaulay property. That the maximal order property is closely connected to such homological properties more generally is well-known; for example, see [St]. Together with Theorem 6.3 this completes the proof of Theorem 1.1(2).

**Theorem 6.7.**
Let $R$ satisfy Hypothesis 6.1. Then $R$ is a maximal order in its Goldie quotient ring.

**Proof.** For a homogeneous ideal $J$ of $R$, we define $O_r(J) = \text{Hom}_R(R,J) = \{x \in Q_{gr}(R)|Jx \subseteq J\}$ and $O_t(J) = \text{Hom}_R(J,J) = \{x \in Q_{gr}(R)|xJ \subseteq J\}$. To show that $R$ is a maximal order, it suffices to show that for all homogeneous ideals $J$ of $R$, we have $O_r(J) = O_t(J) = R$ [Ro, Lemma 9.1]. We prove that $O_t(J) = R$; the proof that $O_r(J) = R$ is symmetric, and is left to the reader.

Any homogeneous ideal $J$ of $R$ has the form $J = J'g^n$ where $J'$ is a special ideal, by Lemma 6.3(1). Then

$$O_t(J) = \{x \in Q_{gr}(R)|xJ'g^n \subseteq J'g^n\} = \{x \in Q_{gr}(R)|xJ' \subseteq J'\} = O_t(J').$$

So it will be enough to consider special homogeneous ideals $J$, and show that $\text{Hom}_R(J,J) = R$. Since $\text{Hom}_R(J,R) \subseteq \text{Hom}_R(J,J)$, it is enough to show that $\text{Hom}_R(J,R) = R$. But we have the following portion of the long exact sequence in $\text{Ext}$:

$$\ldots \rightarrow R = \text{Hom}_R(R,R) \rightarrow \text{Hom}_R(J,R) \rightarrow \text{Ext}_R^1(R/J,R) \rightarrow \ldots$$

Since $GK(R/J) \leq 1$, we have $\text{Ext}_R^1(R/J,R) = 0$ by the Cohen-Macaulay property of $R$, which holds by Theorem 6.3(3). Thus $\text{Hom}_R(J,R) = R$ and the result follows.

7. Subrings

In order to prove our main classification result, Theorem 1.2 we need to get some understanding of when a $cfg$ subring $A$ of a ring $R(D)$ is an equivalent order to $R(D)$. Thus, in this section we study when this
happens for subrings $A$ which are close to $R(D)$, in the sense that $A \subseteq R(D)$ is a finite ring extension. This is a natural assumption, because it is what we will actually know when we apply these results later in the proof of Theorem \[1.2\].

As in the previous section, we actually work with a slightly more general setup. For the rest of this section, we fix the following notation.

**Hypothesis 7.1.** Let $R$ be a ring satisfying Hypothesis \[6.1\] and let $A \subseteq R$ be a cfg subalgebra such that $Q_{gr}(A) = Q_{gr}(R)$ and $A \subseteq R$ is a finite ring extension. Let $C = A(g)$ be the subring generated by $A$ and the central element $g \in R_d$, so that $A \subseteq C \subseteq R$.

We first make a convenient definition concerning ring extensions. Let $U \subseteq V$ be an inclusion of cfg $k$-algebras which are domains with $Q_{gr}(U) = Q_{gr}(V)$. We say that $U$ and $V$ have a common ideal if there is a homogeneous ideal $0 \neq I$ of $V$ such $I \subseteq U$. Clearly if $U$ and $V$ have a common ideal, then they are equivalent orders. Here are some other simple facts about this concept.

**Lemma 7.2.** Let $U \subseteq V \subseteq W$ be cfg $k$-algebras which are domains with $Q_{gr}(U) = Q_{gr}(V) = Q_{gr}(W)$.

1. If $U \subseteq V$ is a finite ring extension, then $U$ and $V$ have a common ideal.
2. If $U$ and $V$ have a common ideal and $V$ and $W$ have a common ideal, then $U$ and $W$ also have a common ideal.

*Proof.* (1) This argument may be found in the proof of \[AS\] Theorem 4.1], but we repeat it here for convenience. Since $U$ and $V$ have the same graded quotient ring, every element of $(V/U)_U$ has nonzero annihilator in $U$. Moreover, $V(U/V)_U$ is a bimodule which is finitely generated on both sides. It follows that $I = r.\text{ann}_U(V/U)$ is a nonzero homogeneous right ideal of $U$ which also satisfies $VI \subseteq I$. A symmetric argument gives that $J = l.\text{ann}_U(V/U)$ is a nonzero homogeneous left ideal of $U$ such that $JV \subseteq J$. Thus $0 \neq IJ$ is a homogeneous ideal of $V$ which is contained in $U$.

(2) Let $0 \neq I$ be a common ideal for $U$ and $V$ and $0 \neq J$ a common ideal for $V$ and $W$. Then $0 \neq IJ$ is a common ideal for $U$ and $W$. \[\square\]

As mentioned earlier, one of our main concerns is to understand when the ring $A$ in Hypothesis \[7.1\] is an equivalent order to $R$. We will prove this below only under a further assumption that $R$ has a minimal special ideal. The case where $g \in A$, in other words the case of the ring $C$, is much more straightforward, and so we study it separately first.

**Lemma 7.3.** Assume Hypothesis \[7.1\].

1. $C \subseteq R$ is a finite extension and $C$ is a noetherian ring.
2. Every special factor ring of $C$ has eventually periodic Hilbert function of period $d$. Also, if $R$ has a minimal special ideal, then so does $C$. 

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Lemma 3.2(1). Also, \( k \dim C \subseteq R \) implies that \( \dim C \subseteq R \). Proof. (1) The graded Nakayama lemma shows that \( R_C \) is a finitely generated module if and only if \( \dim_k R/R_C \geq 1 < \infty \). Since \( g \in C \geq 1 \), this is also equivalent, by Nakayama again, to \( R = R/rg \) being a finitely generated right \( C \)-module, which is true by hypothesis. A similar argument holds on the left, so \( C \subseteq R \) is a finite extension.

It remains to prove that \( C \) is noetherian. We claim that in fact the following fact holds: given a finitely generated \( \mathbb{Z} \)-graded right \( C \)-module \( M \), if \( (M/Mg)_C \) is noetherian then \( M_C \) is noetherian. First note that since \( M \) is finitely generated, \( M \) is right bounded (\( M_n = 0 \) for \( n \ll 0 \)). Using this, it is easy to prove that if \( M \) has a non-finitely generated submodule, it has a homogeneous non-finitely generated submodule. Since \( g \) is obviously still a central (and hence normal) element of \( C \), the rest of the proof of the claim follows by exactly the same argument as [Atiyah-MacDonald, Lemma 8.2], which proves the case where \( M = C \).

Apply the claim to \( M = R \); this shows that \( R_C \) is noetherian as long as \( (R/rg)_C \) is noetherian. In fact, \( R = R/rg \) is a \( C = C/(C \cap rg) \) module, and so it is enough to prove that \( R_C \) is noetherian. This follows since \( C \subseteq R \) is a finite ring extension, and \( C \) is noetherian by Lemma 3.2(1). Thus \( R_C \) is noetherian, and a symmetric argument shows that \( CR \) is noetherian. Of course, then, \( C \) is a noetherian ring.

(2) Suppose that \( I \) is an arbitrary special ideal of \( C \). Since \( C \subseteq R \) is a finite extension by part (1), Lemma 7.2(1) implies that \( C \) and \( R \) have a common nonzero homogeneous ideal, say \( 0 \neq H \). We can assume that \( H \subseteq I \), by replacing \( H \) by \( HIH \) if necessary. By Lemma 6.4(1), \( H = Kg^n \) for some special ideal \( K \) of \( R \) and \( n \geq 0 \). By Lemma 6.4(2), \( K \) has a finite dimensional extension \( L \) such that \( R/L \) is \( g \)-torsionfree; by increasing \( n \) if necessary, we can take \( H = Lg^n \) as our common ideal.

Now given any \( i \geq 1 \), notice that the right \( C \)-module \( M = ((C \cap Lg^{i-1}) + I)/(C \cap Lg^i + I) \) is killed by \( (C \cap rg) + I \), so \( M \) is a module over \( C/(C \cap rg) + I) = \overline{C}/\overline{T} \). The ring \( \overline{C} \) is projectively simple, by Lemma 5.2(1). Also, \( \overline{T} \neq 0 \) since \( GKC/I = 1 \), so necessarily \( \dim_k \overline{C}/\overline{T} < \infty \). We know that \( C \) is noetherian by part (1), so \( M \in gr-C \) and thus \( \dim_k M < \infty \). Since \( (C \cap Lg^n) + I = I \), we conclude by induction on \( i \) that \( \dim_k((C \cap L) + I)/I < \infty \), and so \( I \supseteq (C \cap L)_{\geq m} \) for some \( m \).

To prove that \( C/I \) has eventually \( d \)-periodic Hilbert function, we might as well now replace \( I \) by \( I+(C \cap L) \), so we can assume that \( C/I \) is a factor of \( C/(C \cap L) \). Now \( N = C/(C \cap L) \) is \( g \)-torsionfree and has bounded Hilbert function, since \( R/L \) has these properties. An easy argument shows then that \( \dim_k N/Ng < \infty \). Then \( N' = C/I \) is a surjective image of \( N \) and so also \( \dim_k N'/Ng < \infty \). Then another easy argument shows that this forces \( N' \) to have eventually \( d \)-periodic Hilbert function, as needed.

Finally, if \( R \) has a minimal special ideal, then as in Remark 6.6 there is a unique ideal \( L' \) of \( R \) which is minimal in tails and such that \( R/L' \) is \( g \)-torsionfree. Then necessarily \( L' \subseteq L \), and so we can replace \( L \) in the above argument by \( L' \). Then any special ideal \( I \) of \( C \) has \( I \supseteq (L' \cap C)_{\geq m} \) for some \( m \), and this implies that \( L' \cap C \) is an ideal of \( C \) which is minimal in tails.

Next, we handle the more complicated case of subrings of \( R \) that need not contain \( g \). We now need to impose the existence of a minimal special ideal to gain more traction.
Theorem 7.4. Assume Hypothesis [7.7] and assume in addition that $R$ has a minimal special ideal.

1. $A$ and $R$ have a common ideal; in particular, $R$ and $A$ are equivalent orders.

2. If $A$ is noetherian, then $A \subset R$ is a finite extension.

Proof. (1) We have $C = A(g) = A + Ag + Ag^2 + \cdots$. Note for each $n \geq 0$ that $CA_{\geq n} = A_{\geq n} + A_{\geq n}g + A_{\geq n}g^2 + \cdots = A_{\geq n}C$. In particular, $CA_{\geq n}$ is an ideal of $C$. Moreover, the Hilbert series of $C/CA_{\geq n}$ is at most as large as $\sum_{i \geq 0} p(t)t^id$, where $p(t) = h_{A/CA_{\geq n}}(t)$ is a polynomial since $\dim_k A/CA_{\geq n} < \infty$. So $GK(C/CA_{\geq n}) \leq 1$, and $CA_{\geq n}$ is a special ideal of $C$. Now $C$ has a minimal special ideal, and all special factor rings of $C$ have eventually $d$-periodic Hilbert function, by Lemma [7.3(2)]. It is easy then to see that any descending chain of special ideals of $C$ must eventually stabilize in tails. Thus $\dim_k CA_{\geq n_0}/CA_{\geq \infty} < \infty$ for all $n \geq n_0$, some $n_0$.

We claim that $CA_{\geq n_0}$ is a finitely generated right $A$-module. By the graded Nakayama Lemma, to show this it suffices to show that $\dim_k CA_{\geq n_0}/CA_{\geq n_0}A_{\geq 1} < \infty$. Though we don’t know that $A$ is noetherian, we do have that $A$ is a finitely generated $k$-algebra by hypothesis. If $e$ is the maximum degree of its generators, then clearly $A_{\geq (e+n_0)} \subseteq A_{n_0}A_{\geq 1}$. So it is enough to show that $\dim_k CA_{\geq n_0}/CA_{\geq (e+n_0)} < \infty$, which we already showed above; the claim is proved. A symmetric proof shows that $CA_{\geq n_0}$ is finitely generated on the left over $A$. In conclusion, setting $\tilde{A} = A + CA_{\geq n_0}$, then $A \subseteq \tilde{A}$ is a finite ring extension.

We have constructed a series of ring extensions as follows, where all rings have the same graded quotient ring:

$$A \subseteq \tilde{A} = A + CA_{\geq n_0} \subseteq C \subseteq R.$$ (7.5)

Here, both of the outer extensions $A \subseteq \tilde{A}$ and $C \subseteq R$ are finite ring extensions (using Lemma [7.3(1)]), and thus have a common ideal, by Lemma [7.2(1)]. Also, the extension $A + CA_{\geq n_0} \subseteq C$ certainly has a common ideal, namely $CA_{\geq n_0}$. By Lemma [7.2(2)], the extension $A \subseteq R$ has a common ideal, and $A$ and $R$ are equivalent orders.

(2) Suppose that $A$ is noetherian, and consider the argument in part (1) above and the chain of ring extensions (7.5). Since $\tilde{A} = A + CA_{\geq n_0}$ is finite over $A$ on both sides, it is also a noetherian ring in this case. Choosing any nonzero homogeneous element $x \in A_{\geq n_0}$, we see that $xC \subseteq A$ and so $C_A$ embeds in $A$; thus $C_A$ is Noetherian. Similarly, $AC$ is noetherian. Thus $A \subseteq C$ and $C \subseteq R$ are finite ring extensions, and so $A \subseteq R$ is also a finite extension. In fact, it is easy to see that $\dim_k C/\tilde{A} < \infty$ in this case. \qed

If in Hypothesis [7.4] we have that $\mathfrak{m} \subset \mathfrak{m}$ is not just a finite extension but an equality in large degrees, then one can expect to say more about how close $A$ is to $R$. The following result studies this. There is a strong result for the ring $C$, and a rather weaker one for a general $A$. We note that this result is not needed in the proof of Theorem [1.2], though it is used in Section [12] below in the classification of subrings of $S$ generated in degree 1. We also do not know if the extra hypotheses in part (1) of the following proposition are necessary.

Proposition 7.6. Assume the setup in Hypothesis [7.7] and suppose that $\dim_k \mathfrak{m}/\mathfrak{m} < \infty$. 


(1) If in addition $A \subseteq R$ is a finite extension and $(A \cap Rg)/(A \cap Rg^2) \neq 0$, then $A$ contains a special ideal of $R$.

(2) $C$ contains a special ideal of $R$.

Proof. (1) Since $A \subseteq R$ is a finite extension with $Q_{gr}(A) = Q_{gr}(R)$, $A$ and $R$ have a common ideal by Lemma 7.2(1). This ideal must have the form $Jg^n$ for some special ideal $J$ of $R$, by Lemma 6.4(1). $J$ has a finite-dimensional extension $J'$ such that $R/J'$ is $g$-torsionfree (Lemma 6.4(2)), so possibly after increasing $n$ we can replace $J$ by $J'$ and assume that $R/J$ is $g$-torsionfree.

We have the following exact sequence:

$$0 \to (J \cap Rg)/Jg \to J/Jg \to R/Rg \to R/(J + Rg) \to 0.$$ 

Here, $(J \cap Rg)/Jg \cong (M/J)[-d]$, where $M = \{x \in R | xg \in J\}$. Then $M/J$ is contained in the $g$-torsion submodule $Z(R/J)$ of $R/J$. By choice of $J$, $M = J$ and so $(Rg \cap J)/Jg = 0$. Thus $J/Jg$ injects into $R/Rg$, and this is even an inclusion of $\overline{R} - \overline{R}$-bimodules. For the purposes of this proof, following the definition for rings, we call a bimodule $M$ projectively simple if every sub-bimodule $0 \neq N$ of $M$ satisfies dim$_k M/N < \infty$.

Since $R/Rg$ is a projectively simple $\overline{R} - \overline{R}$-bimodule, by Lemma 3.2(1), clearly $J/Jg$ is also a projectively simple $\overline{R} - \overline{R}$-bimodule. Since multiplication by $g$ induces a bimodule isomorphism $Jg^{-1}/Jg^i \to Jg^i/Jg^{i+1}$ for each $i$, $Jg^i/Jg^{i+1}$ is also a projectively simple $\overline{R} - \overline{R}$-bimodule. Since dim$_k \overline{R}/\overline{A} < \infty$, it is then easy to see that $Jg^i/Jg^{i+1}$ is also a projectively simple $\overline{A} - \overline{A}$-bimodule for each $i$.

By hypothesis, we can choose $x \in A \cap Rg \setminus Rg^2$. Clearly we can also choose $y \in A \cap J \setminus Rg$, since we have GK $A/(A \cap J) \leq 1$, while GK $A/A \cap Rg = GK \overline{A} = 2$. Then $x'y \in A \cap Jg^i \setminus Rg^{i+1}$, and thus $(A \cap Jg^i)/(A \cap Jg^{i+1}) \neq 0$. Since $(A \cap Jg^i)/(A \cap Jg^{i+1})$ is a $\overline{A} - \overline{A}$ sub-bimodule of $Jg^i/Jg^{i+1}$, we conclude that $(A \cap Jg^i)/(A \cap Jg^{i+1})$ equals $Jg^i/Jg^{i+1}$ in large degree.

Now recall that $Jg^n \subseteq A$. So there exists an $i \geq 0$ minimal with the property that $(Jg^i)_{\geq m} \subseteq A$ for some $m \geq 0$. Suppose $i > 0$. Since $(A \cap Jg^{i-1})/(A \cap Jg^i)$ is equal in large degree to $Jg^{i-1}/Jg^i$, and $A \cap Jg^i \supseteq (Jg^i)_{\geq m}$ for some $m$, we conclude that $A \cap Jg^{i-1}$ is equal in large degree to $Jg^{i-1}$. Then $(Jg^{i-1})_{\geq m'} \subseteq A$ for some $m'$, contradicting the choice of $i$. So $i = 0$, and $A$ contains a special ideal of $R$ of the form $J_{\geq m}$.

(2) We have that $C \subseteq R$ is a finite extension by Lemma 7.3(1), and clearly $(C \cap Rg)/(C \cap Rg^2) \neq 0$ since $g \in C$. Taking $A = C$ in part (1), the result is immediate. \hfill \Box

8. Divisors of modules

The concept of a divisor associated to a GK-2 module is a basic tool in the study of the module category of the Sklyanin algebra $S$, and AS-regular algebras more generally. It was first studied by Ajitabh in [Aj1], and then also by Van den Bergh, De Naeghel, and others, for example see [Aj2], [AjV], [DeN]. We will see that this concept is also very important for the study of the rings $R(D)$. In this section, we recall some of the basic definitions and properties of divisors of modules. At the end of the section, we apply the theory we have
We will associate divisors only to admissible modules. For a dmissible module \( M/Mg \) this is equivalent to the requirement that \( M \) is a \( \text{GK} \)-2 domain and \( M/Mg \) is normal, and that \( \text{GK} M \otimes Rg \) is torsionfree, in fact \( \text{Tor}^1(M, R/Rg) = 0 \) for all \( i \geq 2 \). We now define the divisor associated to an admissible module; this is a variation on the original definition of Ajitabh, following Van den Bergh [VdB, p. 66], which has the advantage of exactness as in Lemma 8.3(4) below.

**Definition 8.2.** Given \( M \in \text{gr}-R \) which is admissible, let \( N = M/Mg \) and \( N' = \text{Tor}^1_1(M, R/Rg) \). We define \( \text{div} M = C(N) − C(N') \in \text{Div} E \), in the notation of Definition 3.4.

For the preceding definition to make sense, we need \( \text{GK} \text{Tor}^1_1(M, R/Rg) \leq 1 \) when \( M \) is admissible. This follows from (8.1) and part (1) of the next lemma, which also collects several other easy properties of the divisor.

**Lemma 8.3.** Let \( M \in \text{gr}-R \).

1. \( M \) is admissible if and only if \( \text{GK} M \leq 2 \) and \( \text{GK} Z(M) \leq 1 \).
2. If \( M \) is admissible and \( g \)-torsionfree, then \( \text{div} M = C(M/Mg) \); in particular, \( \text{div} M \) is effective.
3. For admissible \( M \) and \( i \in \mathbb{Z} \), \( \text{div} M[i] = \sigma^i(\text{div} M) \).
4. The map \( M \mapsto \text{div} M \) is additive on short exact sequences of admissible modules in \( \text{gr}-R \).

**Proof.** (1) Applying \( - \otimes R/Rg \) to the exact sequence \( 0 \to Z(M) \to M \to M'/Z(M) \to 0 \), one obtains an exact sequence

\[
\cdots \to \text{Tor}^1(M', R/Rg) \to Z(M)/Z(M)g \to M/Mg \to M'/M'g \to 0.
\]

Since \( M' \) is \( g \)-torsionfree, in fact \( \text{Tor}^1(M', R/Rg) = 0 \). Now if \( \text{GK} M \leq 2 \) and \( \text{GK} Z(M) \leq 1 \), then \( \text{GK} Z(M)/Z(M)g \leq 1 \). Then since \( \text{GK} M' \leq 2 \) and \( M' \) is \( g \)-torsionfree, \( \text{GK} M'/M'g \leq 1 \) by a simple Hilbert series argument. So \( \text{GK} M/Mg \leq 1 \) from the exact sequence. Conversely, if \( \text{GK} M/Mg \leq 1 \), then \( \text{GK} Z(M)/Z(M)g \leq 1 \) from the exact sequence. Since \( Z(M) \) has a finite filtration \( Z(M) \supseteq Z(M)g \supseteq \cdots \supseteq Z(M)g^n = 0 \) for some \( n \), and \( Z(M)g^i/Z(M)g^{i+1} \) is a (shifted) surjective image of \( Z(M)g^{i-1}/Z(M)g^i \) for each \( i \), we see that \( \text{GK} Z(M) \leq 1 \). Also, \( \text{GK} M/Mg \leq 1 \) implies that \( \text{GK} M \leq 2 \), by Hilbert series again.

(2) This is immediate from the definition of \( \text{div} M \) and (8.1).
(3) It follows from a definition chase that for \( N \in \text{gr}-B \) with \( \text{GK} N \leq 1 \), we have \( C(N[i]) = \tau^i(C(N)) \), which implies the result.

(4) Apply \( - \otimes R/Rg \) to a given exact sequence \( 0 \to M \to N \to P \to 0 \) of admissible modules. Then the desired result follows immediately from the corresponding long exact sequence in Tor (recall that \( \text{Tor}_2^R(P, R/Rg) = 0 \)), together with the fact that \( C(\cdot) \) is additive on short exact sequences of GK-1 modules in gr-\( B \).

The divisor \( \text{div} M \) is an important tool in understanding \( g \)-torsionfree modules \( M \in \text{gr}-R \) with \( \text{GK} M = 2 \). Namely, to get a picture of such modules \( M \), one first tries to understand which linear equivalence classes of divisors occur as \( \text{div} M \). Such linear equivalence classes have been completely characterized when \( R = S \) is the generic Sklyanin algebra, by work of De Naeghel [DeN]. The complete answer is quite intricate, but in the sequel we only need the weak consequence in the following lemma, which already follows from Ajitabh’s first paper [Aj1].

**Definition 8.4.** Let \( R \) satisfy Hypothesis 6.1. Then we define

\[
\text{div}_2(R) = \{ \text{div} M | M \in \text{gr}-R \text{ is } g\text{-torsionfree and } \text{GK}(M) = 2 \} \subseteq \text{Div} E.
\]

**Lemma 8.5.** Let \( S \) be a generic Sklyanin algebra. Then the image of \( \text{div}_2(S) \) in \( \text{Pic} E \) is countable.

**Proof.** Every \( g \)-torsionfree \( M \in \text{gr}-S \) with \( \text{GK} M = 2 \) has an extension \( M' \supseteq M \) with \( \dim_k M'/M < \infty \), where \( M' \) has projective dimension 1 (see the discussion following [Aj1 Lemma 2.5]). Then by [Aj1] Lemma 2.5], the linear equivalence class of \( \text{div} M = \text{div} M' \) is uniquely determined by the sequence of graded Betti numbers appearing in the minimal projective resolution of \( M' \), and there are clearly countably many such sequences of Betti numbers. (Technically, Ajitabh uses a different definition of the divisor \( \text{div} M \) in this paper, which it is well-known is equivalent to ours for the modules \( M \) in question [AjV p. 1636].) \( \square \)

While it would be interesting to study the divisors of modules over \( R(D) \) more carefully, our goal in this paper is more modest: we just want to prove the analog of Lemma 8.5 for the rings \( R(D) \), which we will do in the next section. It simplifies arguments considerably to work modulo the divisors of GK-1 modules. This is accomplished by the next definition, following an idea of Van den Bergh [VdB p. 111].

**Definition 8.6.** Let \( H_\tau \) be the subgroup of \( \text{Div} E \) generated by \( \{ p - \tau^i(p) | p \in E, i \in \mathbb{Z} \} \). Let \( H_\ell \) be the subgroup of \( \text{Div} E \) of divisors linearly equivalent to 0, so that \( \text{Pic} E = \text{Div} E/H_\ell \) as usual. We define \( \overline{\text{Div}} E = \text{Div} E/H_\tau \), and \( \overline{\text{Pic}} E = \text{Div} E/(H_\ell + H_\tau) \). For \( M \in \text{gr}-R \), let \( \overline{\text{div}} M \) be the image in \( \overline{\text{Div}} E \) of \( \text{div} M \). Let \( \overline{\text{div}}(R) = \{ \overline{\text{div}} M | M \in \text{gr}-R \text{ is admissible} \} \subseteq \overline{\text{Div}} E \).

**Lemma 8.7.**

1. If \( M \in \text{gr}-R \) with \( \text{GK} M \leq 1 \), then \( \overline{\text{div}} M = 0 \). In particular, if \( M \in \text{gr}-R \) is admissible then \( \overline{\text{div}}(M) = \overline{\text{div}}(M/Z(M)) \).

2. The image of \( \overline{\text{div}}_2(R) \) in \( \overline{\text{Pic}} E \) is countable if and only if the image of \( \overline{\text{div}}(R) \) in \( \overline{\text{Pic}} E \) is countable.
Lemma 8.8. Keep the notation of the preceding paragraph. Given any admissible $M \in \text{gr-R}$ with $\text{GK} M \leq 1$, then $\text{div} M = 0$, it suffices to prove this when $M$ is either $g$-torsion or $g$-torsionfree. If $M$ is $g$-torsionfree, then $\text{GK} M/Mg = 0$, and so $\text{div} M = 0$. If $M$ is $g$-torsion, then it has a filtration by finitely many $B$-modules of GK-dimension at most 1. If $N \in \text{gr-B}$, then $N' = \text{Tor}^R_1(N, R/Rg) \cong N[-d]$, and so $\text{div} N = C - \tau^{-d}(C)$ for some effective divisor $C$; clearly then $\text{div}(N) = 0$ by definition. The second statement follows, using Lemma 8.3(1).

(2) Let $\theta : \text{Pic} E \to \text{Pic} \overline{E}$ be the quotient map. For any point $p$ and $i \in \mathbb{Z}$, we have $p - \tau^i(p) \sim -is$, where $s$ is the point such that $\tau$ is the translation $\tau(x) = x + s$ in the group structure. Thus $H_\ell + H_\tau = H_\ell + Zs$. It follows that $\theta$ is a countable-to-1 map. Now let $P$ be the image of $\text{div}_2(R)$ in $\text{Pic} E$. Since by part (1) only $g$-torsionfree modules of $\text{GK}$-2 matter in determining which divisors occur in $\text{div}(R)$, it is easy to see that $\theta(P) \cup \{0\}$ is equal to the image of $\text{div}(R)$ in $\text{Pic} \overline{E}$. The result follows, since $P$ is countable if and only if $\theta(P)$ is countable.

Next, we study how the divisors of modules over $R$ are related to divisors of modules over a Veronese ring. Suppose that $m \geq 1$ divides the degree $d$ of the central element $g$ of $R$. Then the $m$th Veronese ring $R' = R^{(m)} = \bigoplus_{n \geq 0} R_{nm}$ still satisfies the basic setup of this section: setting $d' = d/m$, we have $g \in R'_{d'}$ is central in $R'$ and $R'/R'g \cong B^{(m)} \cong B(E, N_m, \tau_m)$. Given a module $M \in \text{gr-R}$, we can define a module $M^{(m)} = \bigoplus_{n=0}^\infty M_{nm} \in \text{gr-R}'$; the rule $M \mapsto M^{(m)}$ defines a functor $F : \text{gr-R} \to \text{gr-R}'$.

Lemma 8.8. Keep the notation of the preceding paragraph. Given any admissible $M \in \text{gr-R}$, then $\text{div}_R M = \text{div}_{R'} M^{(m)}$. Moreover, $\text{div}(R) = \text{div}(R')$.

Proof. First, suppose that $N \in \text{gr-B}$ is isomorphic in large degree to $\bigoplus_{n=0}^\infty H^0(E, \mathcal{F} \otimes N_n)$ for some torsion sheaf $\mathcal{F}$. Then $N^{(m)}$ is isomorphic as a $B' = R'/R'g = B(E, N_m, \tau_m)$-module to $\bigoplus_{n=0}^\infty H^0(E, \mathcal{F} \otimes N_{mn})$ in large degree. Thus in the notation of Definition 3.4, $C_B(N) = C_{B'}(N^{(m)})$.

Given $M \in \text{gr-R}$, we have $M^{(m)}/M^{(m)}g = (M/Mg)^{(m)}$. Since all $B$-modules and $B'$-modules have a Hilbert polynomial of the form $f(n) = an + b$ for $n \gg 0$, by Lemma 2.2(3), it is easy to check then that $\text{GK} R M/Mg = \text{GK} R' M^{(m)}/M^{(m)}g$; thus $M$ is admissible if and only if $F(M) = M^{(m)}$ is. Now given an admissible $M \in \text{gr-R}$, setting $P = \{m \in M | mg = 0\}$, we also have

$$\text{Tor}^R_1(M^{(m)}, R'/R'g) \cong P^{(m)}[-d/m] \cong P[-d]^{(m)} \cong \text{Tor}_1^R(M, R/Rg)^{(m)}.$$ 

Thus $\text{div}_R M = \text{div}_{R'} M^{(m)}$ follows from the previous paragraph.

Finally, because $R$ is generated in degree 1, the functor $F$ descends to an equivalence of categories $\overline{F} : q\text{gr-R} \to q\text{gr-R}'$ [AZ1 Proposition 5.10(3)]. In particular, any admissible $M' \in \text{gr-R}'$ is isomorphic in large degree to $F(M)$ for some $M \in \text{gr-R}$, which must also then be admissible, as we have seen. The conclusion $\overline{\text{div}}(R) = \overline{\text{div}}(R')$ follows.  

\[ \square \]
Recall that a module $M \in \text{gr-}R$ is called a line module if $M$ is cyclic, generated in degree 0, and $h_M(t) = 1/(1-t)^2$. In the last result of this section, we completely describe the divisors of such a line module and its submodules, under the further condition that the central element $g$ is in degree 1. In this case, by Theorem 6.3(2), every module $N \in \text{gr-}R$ has a Hilbert polynomial, and so a well-defined multiplicity.

**Lemma 8.9.** Let $R$ satisfy Hypothesis 6.1 and in addition assume that $g \in R_1 \ (d = 1)$. Let $M \in \text{gr-}R$ be a line module for $R$.

1. $M$ is $g$-torsionfree, admissible, and $\text{div} \ M = p$ is a single point.

2. If $0 \neq N \subseteq M$ is a graded submodule, then there is a graded submodule $\hat{N}$ with $N \subseteq \hat{N} \subseteq M$ such that $\dim_k \hat{N}/N < \infty$ and $\hat{N} \cong L[-m]$ for some line module $L$. In particular, $M$ is $\text{GK-2}$-critical. Moreover, $m$ is the multiplicity of $M/N$ and $\text{div} \ L = \tau^{m-n}(p)$, where $n$ is the multiplicity of $Z(M/N)$.

**Proof.** (1) Suppose first that $\text{GK} \ Z(M) = 2$. Then $M$ must contain a subfactor isomorphic to a shift of $B$. However, since $\dim_k B_n = (\deg N)n$ for $n \gg 0$ by Riemann-Roch, any shift of $B_B$ is a module with multiplicity $\deg N \geq 2$, whereas $M$ has multiplicity 1 by the definition of a line module. This contradiction shows that $\text{GK} \ Z(M) \leq 1$ and thus $M$ is admissible by Lemma 8.3(1).

Now $M' = M/\text{Z}(M)$ is a $g$-torsionfree module with $\text{GK} \ M' = 2$. Note that $\text{GK} M'/M'g = 1$. Since $R$ is generated in degree 1, the smallest possible Hilbert series of a $\text{GK}-1$ module which is cyclic and generated in degree 0 is the Hilbert series of a point module, $1/(1-t)$. Thus $H_{M'/M'g}(t) \geq 1/(1-t)$. Since $M'$ is $g$-torsionfree, $H_{M'}(t) \geq 1/(1-t)^2$, forcing $M' = M$ and $Z(M) = 0$. This also shows that $M/Mg \in \text{gr-}B$ actually is a point module, and $\text{div} \ M = p$ is a single point as claimed.

(2) Suppose that $N$ is a nonzero homogeneous submodule of $M$. Let $n \geq 0$ be maximal such that $N \subseteq M^{g^n}$. Then $N^{g^{-n}} \subseteq M$. Let $N^{g^{-n}} \subseteq K \subseteq M$ be chosen so that $K/N^{g^{-n}}$ is the largest finite-dimensional submodule of $M/N^{g^{-n}}$.

By choice of $n$, $K \nsubseteq M^{g^n}$. Since $M$ is $g$-torsionfree, $\text{Tor}^R_1(M, R/Rg) = 0$ and so we have an exact sequence

$$0 \rightarrow \text{Tor}^R_1(M/K, R/Rg) \rightarrow K/Kg \rightarrow M/Mg \rightarrow M/(K + Mg) \rightarrow 0.$$  

By assumption, $\theta \neq 0$. As we saw in the proof of part (1), $M/Mg$ is a point module. Since $R$ is generated in degree 1, necessarily $\text{im} \ \theta = (M/Mg)_{\geq m'}$ for some $m' \geq 0$. In particular, this forces $\text{GK} K/Kg = 1$, and since $K$ is $g$-torsionfree, $\text{GK} K = 2$. Since $M$ has Hilbert function $\dim_k M_n = n + 1$, there is no choice but for $K$ to have a Hilbert polynomial of the form $\dim_k K_n = n + b$ for $n \gg 0$, for some $b \leq 1$. Then $\dim_k (K/Kg)_n = 1$ for $n \gg 0$. Since $\text{im} \ \theta$ is also equal to a point module in large degree, this forces $\text{ker} \ \theta = \text{Tor}^R_1(M/K, R/Rg)$ to be finite-dimensional over $k$. On the other hand, $\text{Tor}^R_1(M/K, R/Rg)$ is (a shift of) the largest submodule of $M/K$ killed by $g$; by choice of $K$, $\text{ker} \ \theta = \text{Tor}^R_1(M/K, R/Rg) = 0$.

Thus $K/Kg \cong (M/Mg)_{\geq m'}$ is a tail of a point module, generated in degree $m'$. Since $K$ is $g$-torsionfree, $K$ then has the Hilbert function of a line module shifted by $-m$; since $K$ is also generated in degree $m'$
by the graded Nakayama lemma, we must have \( K \cong L[-m'] \) for some line module \( L \). Setting \( \widetilde{N} = Kg^n \), we have \( \dim_k \widetilde{N}/N < \infty \) and \( \widetilde{N} \cong L[-m' - n] \). In particular, it certainly follows from Hilbert series that \( \text{GK} M/N \leq 1 \), and so \( M \) is \( \text{GK} \)-2-critical.

Since we proved that \( \text{Tor}^R_1(M/K, R/Rg) = 0 \), we know that \( M/K \) is \( g \)-torsionfree with \( \text{GKM}/K \leq 1 \), whereas clearly \( K/N \) is \( g \)-torsion by construction. It follows that \( \text{div} M = \text{div} K \), and so \( \text{div} K = p \) and thus \( \text{div} L = \tau m'(p) \), by Lemma 8.3(3). The number \( n \) is clearly the multiplicity of \( Ng^{-n}/N \), so also the multiplicity of \( K/N = Z(M/N) \), and \( m' \) is the multiplicity of \( M/K \). Thus \( m = m' + n \) is the multiplicity of \( M/N \).  

\[ \Box \]

### 9. The Exceptional Line Module for \( R(D) \)

Beginning in this section, we refocus attention on the specific rings \( R(D) \) of interest. For the rest of the paper, we use the notation introduced in Sections 4 and 5. Fix an effective divisor \( D' \) on \( E \) with \( 0 \leq \text{deg} D' \leq 6 \), and let \( D = D' + p \in \text{Div} E \) for some point \( p \in E \). Throughout this section, let \( R = R(D) \) and \( R' = R(D') \). By Theorem 5.2, \( \overline{R} = R/Rg = B(E, N', \tau) \) and \( \overline{R'} = R'/R'g = B(E, N'', \tau) \), where \( N = I_D \otimes M \) and \( N' = I_{D'} \otimes M \). As usual, we write \( \overline{Y} \) for the image of any subset \( Y \subseteq S \) under the homomorphism \( S \to S/Sg \).

Our goal is to study the ring extension \( R \subseteq R' \), and show that intuitively the quasi-scheme \( \text{Qgr}-R \) behaves as a noncommutative blowup of \( \text{Qgr}-R' \) at the point \( p \). In particular, we identify a line module in \( \text{gr}-R \) which behaves as an exceptional object for this blowup. We also study the relationship between divisors of modules over the two rings \( R \) and \( R' \). This will also allow us to show, by induction on \( \text{deg} D \), that \( \text{div}_2(R) \) is contained in countably many linear equivalence classes of divisors.

We now construct the exceptional line module for the extension \( R \subseteq R' \). It is closely related to the structure of \( (R'/R)_R \).

**Lemma 9.1.** Fix \( R \subseteq R' \) as above.

1. \( J = \{ y \in R | (R'_1)y \in R \} \) is a right ideal of \( R \) such that \( L_R = R/J \) is a line module. Moreover, \( \text{div} L = \tau(p) \).
2. \( (R'/R)_R \cong \bigoplus_{i=0}^{\infty} L[-i-1] \).

**Proof.** (1) Fix throughout the proof some \( z \in R'_1 \) such \( R'_1 = R_1 + kz \). Clearly \( J = \{ x \in R | zx \in R \} \). Let \( M = k + R'_1 R = R + R'_1 R \). Then we have

\[
M/R = zR + R/R \cong zR/(zR \cap R) \cong R/J[-1].
\]

Also, \( h_{M/R}(t) = \frac{t}{(1-t)^2} \) by Lemma 5.3. Thus \( R/J \) has the Hilbert series of a line module, and is obviously generated in degree 0, so it is a line module.

To compute the divisor of \( L \), recall that \( L/Rg = R/(Rg + J) = \overline{R}/\overline{J} \) must be a point module. The equation \( \overline{R}/J \subseteq \overline{R} \) forces \( \overline{J} \subseteq \bigoplus_{n \geq 0} H^0(E, I_{\tau(p)} \otimes N_n) \), by considering vanishing conditions. This forces the
point module \( P(\tau(p)) = \bigoplus_{n \geq 0} H^0(E,k(\tau(p)) \otimes N_n) \) to be a factor of \( R/J \), so we must have \( R/J \cong P(\tau(p)) \) and \( \text{div} \ L = \tau(p) \).

(2) Let \( z \) be as in part (1). For each \( n \geq 0 \), we claim that we can choose an element \( w_n \in R_n \) so that \( w_n z \not\in R'_n R_1 \). Clearly we can take \( w_0 = 1 \). If we cannot choose such a \( w_n \) for some \( n \geq 1 \), then \( R_n z \subseteq R'_n R_1 \), and this forces \( R_n R'_1 \subseteq R'_n R_1 \), and thus \( R_n R'_1 \subseteq R'_n R_1 \). But thinking of these as subspaces of \( R'_n R_1 = H^0(E,N'_n) \) determined by certain vanishing conditions (using Lemma 3.1), clearly this is impossible: \( R_n R'_1 \) is the subspace of sections vanishing along the divisor \( p + \tau^{-1}(p) + \cdots + \tau^{-n+1}(p) \), while \( R'_n R_1 \) is the subspace of sections vanishing along \( \tau^{-n}(p) \). This proves the claim, so we fix such elements \( w_n \).

We claim that the \( R \)-submodules \( M^{(n)} = (w_n z R + R)/R \) of \( R'/R \) for \( n \geq 0 \) are independent and span \( R'/R \). First we show spanning. We claim that for each \( n \geq 1 \), we have \( N^{(n)} = M^{(0)} + M^{(1)} + \cdots + M^{(n-1)} = (R + R'_{\leq n})R/R \), from which spanning will immediately follow. The base case \( n = 1 \) is by definition. Now assume the equation holds for some \( n \), and consider \( N^{(n+1)} = (R + R'_{\leq n} + w_n z R)/R \). We have \( \dim_k R'_{n+1} - \dim_k R_n R_1 = 1 \); to see this, note that \( \dim_k R'_1 - \dim_k R'_n R_1 = 1 \) using Lemma 3.1 and \( R'_{n+1} \cap T g = R'_n g \subseteq R'_n R_1 \). Moreover, \( w_n z \in R'_{n+1} \setminus R'_n R_1 \) by choice of \( w_n \); thus \( R'_{n+1} = R'_n R_1 + kw_n z \), and it follows that \( N^{(n+1)} = (R + R'_{\leq (n+1)} R)/R \).

Next, note that \( M^{(n)} \) is cyclic and generated in degree \( n + 1 \). Then \( M^{(n)} \cong (R/K^{(n)})[-n - 1] \), where \( K^{(n)} = \{ y \in R | w_n z y \in R \} \). Clearly we have \( J \subseteq K^{(n)} \), so \( M^{(n)} \) is isomorphic to a factor module of \( L[-n - 1] \). However, Theorem 5.2 implies that \( h_{R'/R}(t) = \frac{t}{(1-t)^3} = \sum_{n \geq 0} \frac{t^{n+1}}{(1-t)^2} \), which is clearly the same as the Hilbert series of \( \bigoplus_{n=0}^\infty L[-n - 1] \). Since \( \sum_{n \geq 0} M^{(n)} = R'/R \), as we showed in the previous paragraph, the only possible conclusion is that \( K^{(n)} = J \) for all \( n \), and that the submodules \( M^{(n)} \) are also independent. Thus \( R'/R \cong \bigoplus_{n=0}^\infty M^{(n)} \cong \bigoplus_{n=0}^\infty L[-n - 1] \).

**Definition 9.2.** We call the line module \( L \) appearing in the preceding theorem the *exceptional line module* (for the extension \( R = R(D) \subseteq R(D') = R' \)). A full subcategory of an abelian category is called a *Serre subcategory* if its set of objects is closed under subobjects, factor objects, and extensions. Let \( \tilde{C} \) be the smallest Serre subcategory of \( \text{Gr}-R \) containing \( L \) and closed under shifts and direct limits, and let \( C = \pi(\tilde{C}) \subseteq \text{Qgr}-R \) be the corresponding full subcategory of \( \text{Qgr}-R \). We call \( C \subseteq \text{Qgr}-R \) the *exceptional category* (again, for a fixed choice of extension \( R \subseteq R' \)).

We do not spend much time studying the geometric properties of the quasi-schemes \( \text{Qgr}-R(D) \) in this paper; a lot of information about these quasi-schemes will flow from [VdB], once the connections with that paper are fully established. Because it is easy, however, we do offer one theorem which further justifies the idea that \( \text{Qgr}-R \) should be thought of as a blowup of \( \text{Qgr}-R' \). The following result shows that the difference between \( \text{Qgr}-R \) and \( \text{Qgr}-R' \) is determined entirely by the exceptional category.
Theorem 9.3. Let \( \overline{C} \) and \( C \) be as in Definition 9.2. Let \( \overline{D} \) be the subcategory of \( \text{Gr-}R' \) consisting of all objects \( N_{R'} \) such that \( N_R \in \overline{C} \), and let \( D = \pi(\overline{D}) \subseteq Qgr-R' \). Then there is an equivalence of quotient categories \( Qgr-R/\overline{C} \simeq Qgr-R'/D \).

Proof. Define functors \( F : \text{Gr-}R \rightarrow \text{Gr-}R' \) and \( G : \text{Gr-}R' \rightarrow \text{Gr-}R \) by \( F(M) = M \otimes_R R' \) and \( G(N_{R'}) = N_R \). We claim that \( F \) descends to a functor \( \overline{F} : Qgr-R/\overline{C} \rightarrow Qgr-R'/\overline{D} \). For this, we first note that if \( M \in \text{Gr-}R \), then we have an exact sequence in \( \text{Gr-}R \) as follows:

\[
0 \rightarrow \text{Tor}^1_R(M, R') \rightarrow \text{Tor}^1_R(M, R'/R) \rightarrow M \rightarrow M \otimes_R R' \rightarrow M \otimes_R (R'/R) \rightarrow 0.
\]

By Lemma 9.1(2), clearly \( (R'/R) \in \overline{C} \). It easily follows that \( \text{Tor}^1_R(M, R'/R) \) and \( M \otimes_R (R'/R) \) are also in \( \overline{C} \). Thus if \( M \in \overline{C} \), then \( (M \otimes_R R')^R \) is in \( \overline{C} \) also, so \( F(M) \in \overline{D} \). We also see from the exact sequence that \( \text{Tor}^1_R(M, R') \in \overline{D} \) for any \( M \in \text{Gr-}R \). Thus the induced functor \( \tilde{F} : \text{Gr-}R \rightarrow \text{Gr-}R'/\overline{D} \) is an exact functor such that \( \tilde{F}(\overline{C}) = 0 \). By the universal property of the quotient category, \( \tilde{F} \) induces a functor \( \overline{F} : Qgr-R/\overline{C} \rightarrow Qgr-R'/\overline{D} \), as claimed. Since \( G \) is already itself an exact functor, a similar but easier argument shows that \( G \) descends to a functor \( \overline{G} : Qgr-R'/\overline{D} \rightarrow Qgr-R/\overline{C} \). In addition, the same exact sequence (9.4) above easily yields that \( \overline{G} \overline{F} \) is naturally isomorphic to the identity functor.

Now for \( N \in \text{Gr-}R' \), there is a natural map \( \phi_N : FG(N) \rightarrow N \) given by the multiplication map \( \phi : N \otimes_R R' \rightarrow N \). We claim that this descends to give a natural isomorphism between \( \overline{F} \overline{G} \) and the identity functor. Since \( \phi \) is surjective, it suffices to show that \( \ker \phi \in \overline{D} \), or equivalently that \( (\ker \phi)^R \in \overline{C} \). Suppose that \( 0 \neq s = \sum (n_i \otimes r_i) \in \ker \phi \) is a homogenous element, where \( n_i \in N \) and \( r_i \in R' \) are homogeneous. Define for each \( n \geq 1 \) the right \( R \)-ideal \( J^{(n)} = \{ y \in R | (R'\leq_n)y \subseteq R \} \). We can choose some \( n \geq 1 \) for which \( r_i J^{(n)} \subseteq R \) for all \( i \), and then clearly \( sJ^{(n)} = 0 \). Thus \( (\ker \phi)^R \) is a direct limit of factors of the modules \( R/J^{(n)} \); it suffices to show that \( R/J^{(n)} \in \overline{C} \) for all \( n \geq 1 \). However, letting \( \{v_1, \ldots, v_m\} \) be a \( k \)-basis of \( R'_{\leq n} \), we have that \( J^{(n)} = \bigcap_{i=1}^m r \cdot \text{ann}_R(v_i + R) \), where \( v_i + R \in R' \) for each \( i \). Since \( R' \) it easily follows that each \( R/r \cdot \text{ann}_R(v_i + R) \), and hence also \( R/J^{(n)} \), is in \( \overline{C} \) as needed.

Intuitively, modding out the exceptional category \( C \) from \( Qgr-R \) corresponds to removing the exceptional line. In accord with the intuition that \( Qgr-R \) should be thought of as a blowup of \( Qgr-R' \) at the point \( p \), one might hope that it would be enough to mod out from \( Qgr-R' \) a subcategory generated by the simple object corresponding to the point \( p \). However, in general it seems that some \( R' \)-modules of \( GK-2 \) are entangled and must get included in \( \overline{D} \). See Proposition 11.2(4) below for an example.

In the next result, we compare the divisors of admissible modules over the two rings \( R \) and \( R' \). Not surprisingly, the difference between \( \overline{\text{div}}(R) \) and \( \overline{\text{div}}(R') \) is generated entirely by the divisor of the exceptional line module.

**Proposition 9.5.** Let \( R = R(D) \subseteq R' = R(D') \) as above.

1. Suppose that \( M \in \text{gr-}R' \) is admissible. Then \( M_R \in \text{gr-}R, M_R \) is admissible, and \( \text{div} M_R = \text{div} M_{R'} \).
(2) If \( M \in \text{gr-}R \) is in the exceptional category \( \bar{C} \) of Definition \( 9.2 \), then \( \text{div}M = np \in \text{Div}E \) for some \( n \geq 0 \).

(3) \( \text{div}(R) \subseteq \{ G + np \mid G \in \text{div}(R') , n \in \mathbb{Z} \} \subseteq \text{Div}E \).

**Proof.** (1) Recall that \( R'/R'g = B' = B(E, N', \tau) \) and \( R/Rg = B = B(E, N, \tau) \). For any \( N \in \text{gr-}B' \) with \( \text{GK} N \leq 1 \), we have \( N_B \in \text{gr-}B \), and \( C(N_B') = C(N_B) \), by Lemma \( 3.3 \).

Now for \( M \in \text{gr-}R' \) admissible, let \( N = M/Mg \in \text{gr-}B' \), where \( \text{GK} N \leq 1 \). Then \( N_B \in \text{gr-}B \), and it follows from the graded Nakayama lemma that \( M_R \in \text{gr-}R \). Clearly \( M_R \) is admissible. Moreover,

\[
\text{Tor}^R_1(M, R'/R'g)_R \cong \{ m \in M \mid mg = 0 \}[-1] \cong \text{Tor}^R_1(M_R, R/Rg).
\]

Thus \( \text{div} M_R = \text{div} M_{R'} \) follows from the definition of the divisor.

(2) We have that \( M \) is filtered by subfactors of shifts of the exceptional line module \( L \), so since \( \text{div} \) is additive on exact sequences, it suffices to prove the result when \( M \) is a subfactor of a shift of \( L \). Since \( L \) is \( \text{GK} \)-2-critical by Lemma \( 8.9(2) \), using Lemma \( 8.7(1) \) it suffices to assume \( M \) is a nonzero submodule of a shift of \( L \). But then \( \text{div} M = \tau^i(p) \) for some \( i \in \mathbb{Z} \) by Lemma \( 8.9 \) and Lemma \( 9.1(1) \), so \( \text{div} M = p \).

(3) Let \( M \in \text{gr-}R \) be admissible. Consider again the exact sequence of right \( R \)-modules

\[
\cdots \rightarrow \text{Tor}^1_R(M, R'/R) \rightarrow M \rightarrow M' \rightarrow M \otimes_R (R'/R) \rightarrow 0,
\]

where \( M' = M \otimes_R R' \). As we already observed in the proof of Theorem \( 9.3 \), the terms \( \text{Tor}^1_R(M, R'/R) \) and \( M \otimes_R (R'/R) \) must be in the exceptional category \( \bar{C} \), and thus \( \ker \theta \in \bar{C} \) and \( \text{im} \theta \in \bar{C} \). Note that \( M' \in \text{gr-}R' \), and that \( M'/M'g = (M/Mg) \otimes_R R' = (M/Mg) \otimes_B B' \). We have \( \text{GK}_B M/Mg \leq 1 \), so that \( M/Mg \) is a Goldie-torsion module over the domain \( B \). It follows that \((M/Mg) \otimes_B B' \) is in the exceptional category \( \bar{C} \), and thus \( \text{div}(M') \) must be in the exceptional category \( \bar{C} \). Now, by part (1), \( M' \) is admissible, \( \text{div}(M') = \text{div} M'_R \), and \( \text{div}(M) \subseteq \{ G + np \mid G \in \text{div}(R'), n \in \mathbb{Z} \} \) follows immediately from the exact sequence and part (2). \( \blacksquare \)

**Corollary 9.6.** Let \( R = R(D) \) for any effective \( D \) with \( 0 \leq \deg D \leq 7 \). Then the image of \( \text{div}_2(R) \) in \( \text{Pic} E \) is countable.

**Proof.** Let \( S \) be the generic Sklyanin algebra. By Lemma \( 8.5 \) we know that the image of \( \text{div}_2(S) \) in \( \text{Pic} E \) is countable. Thus the image of \( \text{div}(S) \) in \( \text{Pic} E \) is countable, by Lemma \( 8.7 \). Letting \( T = S^{(3)} \) as usual, then since \( \text{div}(S) = \text{div}(T) \) by Lemma \( 8.8 \), the image of \( \text{div}(T) \) in \( \text{Pic} E \) is countable. It now follows, by Proposition \( 9.5(3) \) and induction on \( \deg D \), that the image of \( \text{div}(R(D)) \) in \( \text{Pic} E \) is countable. Thus the image of \( \text{div}_2(R) \) in \( \text{Pic} E \) is countable, by Lemma \( 8.7 \) again. \( \blacksquare \)

As an application of the previous results, we can now prove the following interesting result about the divisors of the line modules of the ring \( R(D) \).

**Theorem 9.7.** Let \( R = R(D) \) where \( 0 \leq \deg D \leq 7 \). Then the set \( \{ \text{div} L \mid L \) is a line module for \( R \} \) is finite.
Proof. First, we reduce to the case of an uncountable base field. Let \( k \subseteq \ell \) be a field extension where \( \ell \) is uncountable and algebraically closed. If \( M \in \text{gr}-R \) is a line module, then \( M \otimes_k \ell \) is a line module over \( R_\ell = R \otimes_k \ell \), which is just \( R(D) \) constructed over the bigger field \( \ell \). Also, identifying the points of \( E \) with a subset of \( E \times \text{Spec}_k \text{Spec} \ell \), \( \text{div}_M = \text{div}(M \otimes_k \ell)_{R_\ell} \). Thus we see that it is enough to prove the result over the base field \( \ell \). Using this, we assume for the rest of the proof that \( k \) is uncountable.

Now since \( R \) is strongly noetherian, by Theorem 5.3(1), the work of Artin and Zhang shows that there is a projective scheme \( X \), the line scheme, which parameterizes the line modules for \( R \). We recall its construction: for any fixed degree \( m \), one considers the set of all graded subspaces \( J_0 \oplus J_1 \oplus \cdots \oplus J_m \subseteq R_0 \oplus R_1 \oplus \cdots \oplus R_m \) such that \( \dim_k(R/J)_n = n + 1 \) for all \( 0 \leq n \leq m \), and \( J_iR_j \subseteq J_{i+j} \) for all \( i, j \) with \( i + j \leq m \). This is a closed subscheme of a product of projective Grassmannians, and so is a projective scheme \( X_m \). Now Corollary E4.5 shows that there is some finite \( m \) such that any such \( \bigoplus_{n=0}^m J_n \) generates a right ideal \( J \) of \( R \) such that \( R/J \) is a line module; then for this \( m \), \( X = X_m \) is the line scheme.

Recall that the divisor of a line module over \( R \) is a single point, by Lemma 8.9. We claim that given any point \( q \in E \), the set of line modules \( M \) such that \( \text{div} \ M = q \) forms a closed subset of \( X \), which we write as \( X_q \). Given a line module \( M = R/J \), the condition \( \text{div} \ M = q \) is equivalent to \( R/J + Rq \cong P(q) \), where \( P(q) \in \text{gr}-B \) is the point module associated to the point \( q \). But there is a unique right ideal \( I \supseteq Rq \) of \( R \) such that \( R/I \cong P(q) \), so it is also equivalent to demand that \( \bigoplus_{n=0}^m J_n \subseteq \bigoplus_{n=0}^m I_n \). This is clearly a further closed condition in the construction above, proving the claim.

Now Corollary 9.6 shows that the image of \( \text{div}_2(R) \) in \( \text{Pic} E \) is countable. Since every line module \( M \) is GK-2 and \( g \)-torsionfree (Lemma 8.9(1)), \( \Omega = \{ \text{div} M \mid M \text{ is a line module for } R \} \subseteq \text{div}_2(R) \). But no two single point divisors on \( E \) are linearly equivalent, so \( \Omega \) is a countable set of points of \( E \). But then \( X = \bigcup_{q \in \Omega} X_q \) expresses \( X \) as a countable disjoint union of closed subvarieties. Since \( k \) is uncountable, this is possible only if the set \( \Omega \) is finite. \( \square \)

10. The main classification theorem

The first two lemmas of this section begin to make clear how the special ideals of the ring \( R(D) \) are closely connected to the line modules of \( R(D) \). We will then use the fact that the divisors of line modules of \( R(D) \) are very restricted to show, in Theorem 10.4 below, that \( R(D) \) has a minimal special ideal, completing the proof of Theorem 1.1. Our main classification result, Theorem 1.2, will then quickly follow from our previous results in Section 7.

For the next two lemmas, we maintain the setup of the previous section. So let \( D' \) be an effective divisor on \( E \) with \( 0 \leq \deg D' \leq 6 \), and let \( D = D' + p \) for some point \( p \in E \). Let \( R = R(D) \) and \( R' = R(D') \), and let \( L_R \) be the exceptional line module for the extension \( R \subseteq R' \), as in Definition 9.2.

Lemma 10.1. There is a unique smallest nonzero submodule \( N \subseteq L \) such that \( L/N \) is \( g \)-torsionfree.
Proof. Recall that \( \text{div } L = \tau(p) \) by Lemma \([9.4](1)\). Suppose that \( L/N \) is \( g \)-torsionfree for some submodule \( 0 \neq N \subseteq L \). By Lemma \([8.9](2)\), \( N \cong L'[-m] \) is a shift of a line module \( L' \) with \( \text{div } L' = \tau^{m+1}(p) \). By Theorem \([9.7]\) there can be only finitely many values of \( m \) for which \( \tau^{m+1}(p) \) is the divisor of a line module over \( R \). Thus there is an upper bound on the possible value of \( m \), which is also the multiplicity of the \( \text{GK} \)-1 module \( L/N \).

Thus we may pick a submodule \( 0 \neq N \subseteq L \) such that \( L/N \) is \( g \)-torsionfree and the multiplicity \( m \) of \( L/N \) is maximal among all such choices. If \( 0 \neq N' \subseteq L \) is another submodule with \( L/N' \) \( g \)-torsionfree, then \( L/(N \cap N') \) is also \( g \)-torsionfree. \( L/(N \cap N') \) must then also have multiplicity \( m \), so \( N \cap N' \) is equal to \( N \) in large degree. By \( g \)-torsionfreeness, \( N \cap N' = N \) and so \( N \subseteq N' \). \( \square \)

**Lemma 10.2.** There is a special ideal \( I \) of \( R \) with the following property: given any (possibly infinite) direct sum of shifts of \( L \), say \( M = \bigoplus L[a_i] \), and \( N \in \text{gr-}R \) such that \( N \) is isomorphic to a subfactor of \( M \), \( \text{GK } N \leq 1 \), and \( N \) is \( g \)-torsionfree, then \( I \subseteq r. \text{ann}_R N \).

**Proof.** Let \( N \) be as in the statement. Since \( N \) is finitely generated, we can choose submodules \( N'' \subseteq N' \subseteq M \) with \( N', N'' \in \text{gr-}R \) such that \( N \cong N'/N'' \). Then \( N' \) is contained in finitely many terms in the direct sum comprising \( M \), so we may replace \( M \) if necessary by a finite sum \( M = \bigoplus_{i=1}^m L[a_i] \) for some shifts \( a_i \in \mathbb{Z} \). We adjust the choices of \( N' \) and \( N'' \) further. If \( N' \cap L[a_j] = 0 \) for some \( j \), then \( N' \) embeds in \( M = \bigoplus_{i \neq j} L[a_i] \), so we might as well replace \( M \) by a direct sum of fewer terms. Thus we can assume that \( N' \cap L[a_i] \neq 0 \) for all \( 1 \leq i \leq m \). If \( N'' \cap L[a_i] = 0 \) for some \( i \), then the nonzero submodule \( N' \cap L[a_i] \) of \( L[a_i] \) embeds in \( N \). But \( L[a_i] \) is \( \text{GK} \)-2-critical and so \( \text{GK } N' \cap L[a_i] = 2 \), contradicting \( \text{GK } N = 1 \); thus \( N'' \cap L[a_i] \neq 0 \) for all \( i \). Now for each \( i \), consider the \( g \)-torsion submodule \( Z(L[a_i]/(N'' \cap L[a_i])) = P_i/(N'' \cap L[a_i]), \) where \( P_i \subseteq L[a_i], \) and let \( P = \bigoplus P_i \subseteq M. \) Since \( N \) is \( g \)-torsionfree, \( N \cong (N' + P)/(N'' + P), \) so \( N \) is isomorphic to a subfactor of \( M/P. \) But now \( L[a_i]/P_i \) is a \( g \)-torsionfree proper factor module of \( L[a_i]. \) Let \( \tilde{L} \) be the unique minimal nonzero submodule of \( L \) such that \( L/\tilde{L} \) is \( g \)-torsionfree, as constructed in Lemma \([10.1]\), then \( \tilde{L}[a_i] \subseteq P_i. \) Thus \( N \) is a subfactor of \( \bigoplus L[a_i]/\tilde{L}[a_i]. \) By Lemma \([6.4](3)\), \( I = r. \text{ann}_R L/\tilde{L} \) is a special ideal, and clearly \( I \) kills \( N \). \( \square \)

We are now about ready to show, in the following theorem, that \( R(D) \) always has a minimal special ideal. First, we recall the following fact, which is essentially proved in \([ATV2]\).

**Lemma 10.3.** The generic Sklyanin algebra \( S \) and its Veronese ring \( T = S^{(3)} \) have no homogeneous factor rings of \( \text{GK} \)-dimension 1.

**Proof.** Looking at the localization \( \Lambda = T[g^{-1}] = S[g^{-1}]^{(3)} \), the degree-0 piece \( \Lambda_0 \) is a simple ring \([ATV2]\) Corollary 7.9). Then if \( T \) has a homogeneous factor ring \( T/I \) of \( \text{GK} \) 1, it has such a factor ring \( T/I' \) which is also \( g \)-torsionfree, by Lemma \([6.4](2)\); in this case, \( (T/I')[g^{-1}]_0 \) is a proper nonzero factor ring of \( \Lambda_0 \), a contradiction. The result for \( S \) follows the same way. \( \square \)
Theorem 10.4. (Theorem 1.1(3)) Let \( R = R(D) \) for any effective \( D \) with \( 0 \leq \deg D \leq 7 \). Then \( R \) has a minimal special ideal.

Proof. (1) The proof is by induction on \( \deg D \). For the base case, \( D = 0 \) and \( R(D) = T \), so the result is trivial by Lemma 10.3 (\( T \) itself is minimal in tails.)

Now assume that \( \deg D \geq 1 \), let \( D = D' + p \) for some \( p \), and suppose that we have proven the theorem for \( R' = R(D') \). Consider a special ideal \( J \) of \( R \). We have a chain of special \( R \)-ideals \( J \subseteq R'J \cap R \subseteq R'JR' \cap R \). It is easy to see that \( M = (R'JR' \cap R)/(R'J \cap R) \) is isomorphic, as a right \( R \)-module, to a subfactor of some direct sum of copies of shifts of \( (R'/R)_R \). Also, by Lemma 5.4(2), each \( (R'/R)_R \) is isomorphic to a direct sum of copies of shifts of the exceptional line module \( L \) for \( R \subseteq R' \), as defined in Definition 11.2. Since \( R/(R'J \cap R) \) is a special factor ring, \( \dim_k Z(M) < \infty \) by Lemma 6.4(2). Thus if \( M' = M/Z(M) \), then \( M' \) is a \( g \)-torsion-free, GK-dimension 1 subfactor of a direct sum of shifts of \( L \). If \( I \) is the special ideal constructed in Lemma 10.2, we conclude that \( M'I = 0 \). Thus \( M' \geq m_1 = 0 \) for some \( m_1 \).

Similarly, \( N = (R'J \cap R)/J \) is isomorphic as a left \( R \)-module to a subfactor of some direct sum of copies of shifts of \( R/(R'J/R) \). Because of Theorem 5.4(3), all right-sided results in this paper have left-sided analogs. A left-sided version of the argument in the last paragraph then constructs a special ideal \( H \) of \( R \) satisfying \( H \geq m_2 N = 0 \) for some \( m_2 \).

Now \( R'JR' \) is clearly a special ideal of \( R' \) (otherwise by Lemma 6.4(1), \( R'JR' \subseteq R'g \) and so \( J \subseteq R'g \cap R = Rg \), a contradiction). By the induction hypothesis, \( R' \) has a special ideal \( K \) which is minimal in tails, and so \( K \geq m_3 \subseteq R'JR' \) for some \( m_3 \). Then \( H \geq m_3 (K \geq m_3 \cap R)/J \geq m_1 \subseteq J \). This implies that \( [H(K \cap R)/I] \geq m_4 \subseteq J \), for some \( m_4 \). Since \( G = H(K \cap R)I \) is a special ideal of \( R \) which is independent of \( J \), we see that \( G \) is an ideal which is minimal in tails for \( R \).

At this point, we have done all of the hard work necessary to prove the main classification theorem of the paper, Theorem 1.2, and we now put all of the pieces together.

Proof of Theorem 1.2. Recall the setup of the theorem: we have \( V \subseteq T_1 \) and \( A = k(V) \subseteq T \), and we assume that \( Q_{gr}(A) = Q_{gr}(T) \). As usual, let an overline indicate the image of subsets under the homomorphism \( T \rightarrow T/Tg \equiv B(E, M, \tau) \). Now \( \overline{V} \) generates some subsheaf \( \overline{N} \) of \( \overline{M} \) on \( E \). Then \( \overline{N} = \overline{I}_D \otimes \overline{M} \) for some divisor \( D \) on \( E \), and \( A \subseteq R(D) \). We claim that \( 0 \leq \deg D \leq 7 \), or equivalently that \( \deg \overline{N} \geq 2 \). If not, then \( H^0(E, \overline{N}) \) \( \leq 1 \) by Riemann-Roch, and thus either \( V = kg \) or \( V = kx + kg \) for some \( x \in T_1 \). In either case, since \( g \) is central, clearly \( A \) is commutative, and so \( Q_{gr}(A) = Q_{gr}(T) \) cannot possibly hold; this contradiction proves the claim.

Now Lemma 9.2(3) makes clear that \( \overline{A} \) is equal in large degree to some \( B(F, \overline{N'}, \tau') \subseteq B(E, \overline{N}, \tau) \), where \( F \) may be a different elliptic curve, but in any case \( \overline{A} \subseteq \overline{R(D)} = B(E, \overline{N}, \tau) \) is a finite ring extension. Thus Hypothesis 7.4 holds, and the results of Section 7 apply. Since \( R(D) \) is now known to have a minimal special ideal by Theorem 10.4, Theorem 7.4(1) shows that \( A \) and \( R(D) \) are equivalent orders. Since the rings \( R(D) \) are maximal orders by Theorem 6.4, it is now clear that the rings \( R(D) \) are the only subrings of \( T \) generated
in degree 1 which have the same graded quotient ring as $T$ and which are maximal orders. Clearly a given $A$ is an equivalent order only to the smallest $R(D)$ containing it, so the $D$ with this property is unique.

Theorem 7.4(2) also implies that $A \subseteq R(D)$ is a finite ring extension if $A$ is noetherian; in particular, this does hold if $g \in A_1$, by Lemma 7.3(1). \hfill \Box

11. Examples

Some of the complicated (and interesting) behavior of the rings $R(D)$ happens when $D$ contains multiple points on the same $\tau = \sigma^3$-orbit. In this section, we work out some of the features of one of the simplest such cases. This will provide explicit examples of a number of phenomena studied in previous sections. In this section and the next, we again need the notation for point spaces $W(p) \subseteq S_1$, as in Section 4.

The following is the example that will occupy us for most of this section.

Example 11.1. Fix any $p \in E$, let $D = p + \sigma^{-3}(p) \in \text{Div } E$, and set $R = R(D)$. It will be interesting to compare the two different ways in which $R(D)$ can be thought of as an iterated blowup of $T$, so we need some more notation. Set $R' = R(p)$ and $R'' = R(\sigma^{-3}(p))$. The first sequence of blowups is $R = R(D) \subseteq R' = R(p) \subseteq T$. Let $J = \{x \in R|R'_1x \subseteq R\}$, so that $L = R/J$ is the exceptional line module for the blowup $R \subseteq R'$ as in Definition 9.2; similarly, let $J' = \{x \in R'|T_1x \subseteq R'\}$, so that $L' = R'/J'$ is the exceptional line module of $R' \subseteq T$. The other sequence of blowups is $R = R(D) \subseteq R'' = R(\sigma^{-3}(p)) \subseteq R$. Let $J^\circ = \{x \in R|R''_1x \subseteq R\}$, so that $L^\circ = R/J^\circ$ is the exceptional line module for $R \subseteq R''$; and let $J'' = \{x \in R''|T_1x \subseteq R''\}$, so that $L'' = R''/J''$ is the exceptional line module of $R'' \subseteq T$.

Proposition 11.2. Example 11.1 has the following properties.

1. Let $V = W(p)W(\sigma^{-2}(p))S_1$, thought of as a subspace of $R_1$. Then $I = VR$ is a special ideal of $R$ such that $R/I$ is a $g$-torsionfree point module.

2. The subring $A = k(V)$ of $R$ is equal to $k + I$, and $A$ is neither right nor left noetherian.

3. $J \subseteq I$, and $(I/J)_R$ is isomorphic to the shifted line module $R/J^\circ[-1]$.

4. $L'_R \cong L''_R$. In particular, $L'_{R'} \in \text{gr } R'$ is a GK-2 module such that $L'_R \in \tilde{C}$ and $L'_{R'} \in \tilde{D}$, in the notation of Theorem 9.3.

Before proving the proposition, we make some remarks on the significance of these properties. Part (1) gives an explicit example of a special ideal in a ring $R(D)$. Part (2) shows that degree-1 generated subrings $A$ of $T$ which do not contain the central element $g$ can indeed be non-noetherian; also, $A \subseteq R$ need not be a finite ring extension. Part (3) gives an explicit example where an exceptional line module (in this case $L \cong R/J$) contains another shifted line module (in this case $L^\circ[-1]$, a shift of the exceptional module for the other blowup sequence), where the factor is GK-1 $g$-torsionfree. This is exactly the phenomenon that Lemma 10.1 is designed to control. Finally, part (4) shows, as was claimed earlier, that GK-2 modules are sometimes among those that need to be quotiented out from $\text{Qgr } R'$ in the equivalence of categories $\text{Qgr } R'/C \sim \text{Qgr } R'/D$ in Theorem 9.3.
The Hilbert series of $J''$ is easily determined, since the Hilbert series of $R''$ is known and $R''/J''$ is a line module, to be $h_{J''}(t) = \frac{t^2 + 7t}{(1-t)^3}$. In particular, we see from this that $V = J''$, since $\dim_k J'' = 7$ also. We claim that $J'' = VR''$, so that $J''$ is generated in degree 1 as a right $R''$-ideal. Since $R''/J''$ is $g$-torsionfree, by Lemma [8.9(1)], we have $J'' \cap R'' = J''g$. Then $h_{VR''}(t) = \frac{t^2 + 7t}{(1-t)^2}$. Also $VR'' \subseteq J''$, and the Hilbert series of $VR''$ is easily calculated, using Lemma [3.1] to be the same as that of $J''$, so that $VR'' = J''$. Then we have $h_{VR''}(t) \geq h_{VR''}(t)/(1-t) = h_{VR''}(1-t) = h_{J''}(t)$, forcing $VR'' = J''$, which proves the claim.

It is easy to see that $V \subseteq R_1$, where $\dim_k R_1 = 8$, so clearly we have $V + kg = R_1$. Consider $V^2 = W(p)W(\sigma^{-2}(p))S_1W(p)S_1$. By moving point spaces, we see that $W(\sigma^{-2}(p))S_1W(p) = W(\sigma^{-2}(p))W(\sigma^{-1}(p))S_1$, and this contains $g$ by Lemma [4.6(1)]. Thus $V^2 \supseteq W(p)gW(\sigma^{-2}(p))S_1 = Vg$. This implies that $R_1V = (V + kg)V = V^2$, and similarly $VR_1 = V^2$. Also, then $V^2 + kg^2 = R_1V + Vg + kg^2 = R_1V + R_1g = R_1(V + kg) = R_2$. An inductive argument gives $V^n + kg^n = R_n$ and $VR_{n-1} = V^n = R_{n-1}V$ for all $n \geq 1$. Finally, note that $g^n \notin V^n$ for all $n \geq 1$, because $V^n \subseteq VR'' = J''$, and $g^n \notin J''$ for all $n \geq 1$ since $R''/J''$ is $g$-torsionfree. In conclusion, $I = VR = RV$ is an ideal of $R$ such that $\{1, g, g^2, \ldots \}$ form a $k$-basis for $R/I$. So $I$ is a special ideal of $R$, and moreover $R/I$ is a right $g$-torsionfree $R$-point module.

(2) The calculation in (1) also shows that $A = k\langle V \rangle = k + I$. Clearly $A$ is not right noetherian, since $R_A$ is inhabited, but $R_A$ embeds as a submodule of $A$ by taking any $0 \neq x \in I$ and considering $xR \subseteq I \subseteq A$. Similarly, $A$ is not left noetherian.

(3) Note that $J'' \subseteq R'$. This is because by part (1), we have $J'' = W(p)W(\sigma^{-2}(p))S_1R''$, which is easily checked to be contained in $R'$ by moving point spaces (in fact, we even have $W(p)W(\sigma^{-2}(p))S_1T_1 = R_2$.) So $J'' \subseteq R' \cap R''$. Since $R''/J''$ is a line module, $R''/J''$ is a point module, in particular each graded piece has dimension 1. By considering vanishing conditions, it is easy to see that $(R'' \cap R')_n \nsubseteq R''_n$ for $n \geq 1$, and so this forces $R'' \cap R'/J'' = k$. Since $(R'' \cap R')/J''$ is $g$-torsionfree, being an $R$-submodule of $R''/J''$, we conclude that $(R'' \cap R')/J''$ is a $g$-torsionfree point module, which must have $\{1, g, g^2, \ldots \}$ as a $k$-basis. Then the natural map $R/(J'' \cap R') \rightarrow (R' \cap R'')/J''$ is an isomorphism of $R$-modules. Clearly $I = VR \subseteq VR'' = J''$. Thus $I \subseteq J'' \cap R$ and so we have $I = J'' \cap R$ by Hilbert functions.

Now it is straightforward to check that $J \subseteq J''$, as follows. If $x \in J$, then $R_1x \subseteq R$. Choosing any $z \in R_1 \setminus R_1'$, we have $R_1 + kz = T_1$ and so $T_1x = (R_1x + kzx) \subseteq R + R_1R \subseteq R''$. Thus $x \in J''$. So $J \subseteq J'' \cap R = I$.

It is known by the proof of part (1) that $I = VR$ is generated in degree 1 as a right $R$-ideal. So $I/J$ is a cyclic module, generated in degree 1, with the Hilbert series of a line module shifted by $-1$. Examine $X = VJ^\circ$. By moving point spaces, it is easy to see that $R_1^\circ V = VR_1^\circ$. Thus $X$ satisfies $R_1^\circ X = R_1^\circ VJ^\circ$.
\( VR^n J^o \subseteq VR \subseteq R \). So \( X \subseteq J \). This implies that \( J^o \subseteq r.\text{ann}(y) \), where \( 0 \neq y \in V/J_1 \) is a generator of \( I/J \). Thus there is a surjection \( R/J^o[-1] \to I/J \). This is a surjection between \( R \)-modules with the same Hilbert function, so it is an isomorphism.

(4) Consider the \( R \)-module \( R/(J' \cap R) \). This module is certainly \( g \)-torsionfree (being an \( R \)-submodule of the line module \( R'/J' \)). Moreover, by considering vanishing conditions one easily sees that \( \overline{R/J' \cap R} \) has the Hilbert series of a point module at the very smallest, as follows: since we showed that \( J'' = VR^n \) in part (1), an analogous argument gives that \( J' = J'_1 R' \), where \( J'_1 = W(\sigma^3(p))W(\sigma(p))S_1 \); thus, for all \( n \geq 0 \) every element of \( \overline{J'_n} \) vanishes at \( \sigma^3(p) \), which is not true of every element of \( \overline{R_n} \). Then \( R/(J' \cap R) \) has the Hilbert series of a line module at the very smallest. Now we have \( J^o \subseteq J' \), by a completely analogous proof to the one showing \( J \subseteq J'' \) in part (3) (just switch the roles of \( R' \) and \( R'' \)). Since \( R/J^o \) is a line module and \( J^o \subseteq (J' \cap R) \), we conclude that \( J^o = J' \cap R \). Thus \( R/J^o = R/(J' \cap R) \to R'/J' \) is an injection of line modules and so is in fact an isomorphism.

Combined with (3), this shows that \( L'_R \) is isomorphic to a shift of a submodule of \( R/J = L \), so that \( L'_R \in \mathcal{C} \) is in the exceptional category of the blowup \( R \subseteq R' \). Then \( L'_R \in \mathcal{D} \) by definition.

Having a divisor \( D \) which contains two points on the same \( \sigma^3 \)-orbit is not the only situation in which one expects the ring \( R(D) \) to have special ideals. Such ideals also occur when the points of \( D \) are not in general position in \( \mathbb{P}^2 \), for example if \( D \) is the sum of three collinear points, as follows.

**Example 11.3.** Let \( 0 \neq f \in S_1 \), and let \( D = p + q + r \) be the hyperplane section of \( E \) where \( f \in H^0(E, \mathcal{L}) \) vanishes. Let \( R = R(D) \). Let \( V = fS_2 \), so that \( \dim_k V = 6 \). We have \( V = \overline{R} \), both being the set of sections of \( L_3 \) vanishing along \( D \). Since \( \dim_k R_1 = 7 \), we have \( g \notin V \) and \( V + kg = R_1 \). Consider \( V^2 = fS_2fS_2 \). The space \( S_1fS_1 \) contains \( g \) by Lemma 4.6.3, and so \( fS_2fS_2 \supseteq fS_1gS_1 = Vg \). It then follows, similarly as in the proof of part (1) of the preceding proposition, that \( I = RV = VR \) is a special ideal of \( R \) with \( V^n = I_n \) and \( V^n + kg^n = R_n \) for all \( n \geq 1 \). Moreover, since \( Sg \) is a completely prime ideal of \( S \), it is easy to check that \( g^n \notin fS \), so that \( g^n \notin V^n \) for all \( n \geq 0 \). Thus \( R/I \) is a factor ring which is also a \( g \)-torsionfree point module.

**12. Subrings of \( S \) generated in degree 1**

In this final section, we study the behavior of subrings of a generic Sklyanin algebra \( S \) which are generated in degree 1. Because \( \dim_k S_1 = 3 \), it is not surprising that there are rather few possibilities. Thus we can give a much more specific classification result than in the case of degree-3-generated algebras, but at the expense of some picky calculations. Still, we offer this result as a step towards a general theory of subalgebras generated in an arbitrary degree of \( S \). Throughout this section, let \( A = k(V) \), where \( 0 \neq V \subseteq S_1 \). Certainly if \( \dim_k V = 1 \), then \( A \cong k[z] \), and this case is both boring and does not have \( Q_{\gcd}(A) \neq Q_{\gcd}(S) \). So the only case worth studying is \( \dim_k V = 2 \), and we will see in Theorem 12.2 below that the classification is a dichotomy depending on whether or not \( V \) is a point space.
The major technical part of the work is contained in the next lemma. Though what \( \overline{A} \) is equal to in large degree follows easily from previous lemmas, in case \( V \) is not a point space, our method of proof in the next theorem requires a more careful analysis of \( \dim_k A_n \) for small \( n \).

**Lemma 12.1.** Let \( V \subseteq H^0(E, L) \) such that \( \dim_k V = 2 \) and \( V \) generates \( L \), and let \( A = k(V) \subseteq S \), so that \( \overline{A} = k(V) \subseteq B = B(E, L, \sigma) \).

1. \( \overline{A} \) is equal to \( B \) in large degree.
2. \( A_1 S_1 = S_2 = S_1 A_1 \).
3. \( \dim_k A_2 = 4 \) and \( \dim_k A_3 \geq 7 \).

**Proof.** Throughout the proof of this lemma, we identify \( A \leq 2 \) with \( \overline{A} \leq 2 \).

1. By Lemma [3.2](3), \( \overline{A} \) is equal in large degree to some \( B(F, M, \tau) \). Here, there is a finite morphism of nonsingular elliptic curves \( \theta : E \to F; 3 = \deg L = (\deg \theta)(\deg M) \), where \( M \) is the sheaf on \( F \) generated by \( \overline{A} \); and \( \deg M \geq 2 \). The only possibility is that \( \theta \) is the identity, and \( \overline{A} \) is equal in large degree to \( B = B(E, L, \sigma) \).

2. This was actually already shown in the course of the proof of Lemma [3.1](1). Namely, since \( L \not\cong L^* \), the argument there shows not only that the natural map

\[
\theta : H^0(E, L) \otimes H^0(E, L^*) \to H^0(E, L_2)
\]

is surjective, but that it is surjective even when restricted to \( V \otimes H^0(E, L^*) \) or \( H^0(E, L) \otimes V^* \).

3. Certainly \( \dim_k A_2 \leq 4 \). Choosing \( z \in S_1 \setminus A_1 \), part (2) shows that \( (kz + A_1)A_1 = S_2 \), and so \( zA_1 + A_2 = S_2 \). Since \( \dim_k S_2 = 6 \), this forces \( \dim_k A_2 = 4 \).

Now we prove that \( \dim_k A_3 \geq 7 \) (we haven’t tried to figure out whether the exact value of \( \dim_k A_3 \) is 7 or 8.) The idea is to show that for generic choice of a \( k \)-basis \( y, z \) for \( A_1 \), we have \( \dim_k zA_2 \cap yS_2 = 1 \).

Certainly, then, \( A_3 = yA_2 + zA_2 \) will have dimension at least \( 2(4) - 1 = 7 \).

Since \( V \) generates \( L \), a basis for the sections in \( V \) determines a morphism \( \phi : E \to \mathbb{P}(V^*) \cong \mathbb{P}^1 \). The map \( \phi \) is the composition of the embedding \( i : E \to \mathbb{P}(B_1^*) \cong \mathbb{P}^2 \) with a projection map given by projection away from the point \( x \in \mathbb{P}(B_1^*) \) determined by \( V \), where \( x \notin i(E) \). Thus the hyperplane sections of \( \phi \) are the degree 3 divisors which are the intersections of lines in \( \mathbb{P}^2 \) through \( x \) with \( E \). Let \( D \) be a Weil divisor such that \( L \cong \mathcal{O}_E(D) \), and let \( |D| \) be the complete linear system of \( D \). Let \( \mathfrak{d} \subseteq |D| \) be the (incomplete) linear system of hyperplane sections of \( \phi \). Now suppose that \( A_1 = ky + kz \) and \( zu = yv \) in \( S_3 \), with \( u \in A_2, v \in S_2 \). Then \( \mathfrak{d} \mathfrak{u} = \mathfrak{v} \mathfrak{u} \) in \( B_3 \). Now \( \mathfrak{u} \) vanishes along some divisor \( C \in \mathfrak{d} \) and \( \mathfrak{v} \) does not vanish at any of the points in \( C \), because lines through \( x \) in \( \mathbb{P}^2 \) are uniquely determined by any point in \( E \) they go through. So necessarily \( \mathfrak{u} \) vanishes along \( \sigma(C) \), in other words \( \mathfrak{w} \in H^0(E, \mathcal{I}_{\sigma(C)} \otimes L_2) \cap A_2 \). We will show that for a generic choice of \( y \) (equivalently a generic choice of \( C \in \mathfrak{d} \)), vanishing along \( \sigma(C) \) presents 3 linearly independent conditions to sections in \( A_2 \). Then \( \dim_k H^0(E, \mathcal{I}_{\sigma(C)} \otimes L_2) \cap A_2 = 1 \), and completing \( y \) to a basis \( \{ y, z \} \) we will conclude that \( \dim_k zA_2 \cap yS_2 = 1 \) as needed.
We want a more geometric interpretation for the sections in $A_2$. A basis for the sections in $V_\sigma \subseteq H^0(E, \mathcal{L}_\sigma)$ determines a map $\phi': E \to \mathbb{P}((V_\sigma)^*) \cong \mathbb{P}^1$, where $\phi' = \phi \circ \sigma$. The sections in $A_2 \subseteq H^0(E, \mathcal{L}_2)$ determine a map $\theta: E \to \mathbb{P}(A_2^*) \cong \mathbb{P}^3$, and since we know that in fact $A_2 \cong V \otimes_k V_\sigma$ by dimension count, the map $\theta$ factors through the map $\rho = \phi \times \phi': E \to \mathbb{P}(V_\sigma)^* \times \mathbb{P}((V_\sigma)^*) \cong \mathbb{P}^1 \times \mathbb{P}^1$; in other words, $\theta$ is $\rho$ followed by a Segre embedding. This allows us to interpret the linear system of hyperplane sections of $\theta$ as (1,1)-hyperplane sections of $\rho$. Moreover, we claim that $\rho$ is generically one-to-one (and so birational onto its image). For we have $\phi = \pi_1 \circ \rho$, where $\pi_1$ is the first projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. Thus if $d$ is the degree of $\rho: E \to \rho(E)$ (the number of points in a generic fiber of this map), then $d$ divides the degree of $\phi$, which is 3. Moreover, $d < 3$, because no fiber of $\phi$ can also be a fiber of $\phi'$, given that $D \not\sim \sigma^{-1}(D)$. So $d = 1$.

Now a generic $C \in \mathfrak{d}$ must consist of 3 distinct points; otherwise, every line in $\mathbb{P}^2$ through $x$ will be tangent to $E$, and this sort of thing happens for plane curves rarely, and certainly not for an embedded elliptic curve [Ha Theorem IV.3.9]. Since $\rho$ is also generically one-to-one, choosing a generic $C \in \mathfrak{d}$, then $\rho(\sigma(C))$ will consist of 3 distinct points. Moreover, those three points will present 3 linearly independent conditions to sections of $\mathcal{O}(1,1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$, unless $\rho(\sigma(C))$ lies entirely on a line in one of the two rulings. But this would happen only if $\sigma(C) \in \mathfrak{d}$ or $\sigma^2(C) \in \mathfrak{d}$, both of which are impossible. Thus for a generic choice of $C \in \mathfrak{d}$, $\sigma(C)$ presents 3 linearly independent conditions to the sections of $A_2$, as required. □

**Theorem 12.2.** Let $V \subseteq S_1$ with $\dim_k V = 2$ and let $A = k(V)$.

(1) If $V = W(p)$ is a point space, then $A^{(3)} = R(D)$, where $D = p + \sigma^{-1}(p) + \sigma^{-2}(p)$. In this case, $A \cap Sg = Ag$ and $h_A(t) = \frac{t^2 + 1}{(1-t)^2(1-t^2)}$. In particular, $A$ satisfies Hypothesis [6.4] and thus all of the theorems in Section [6].

(2) If $V$ is not a point space, then $A$ is equal to $S$ in all large degrees.

**Proof.** (1). To show that $A^{(3)} = R(D)$, it suffices to show that $V^3 = R(D) = \{x \in S_3 | xg^i \in A \}$ for each $i \geq 0$. Obviously $A \subseteq N^{(i)}$. Moreover, since $R(D) \cap Tg^i = R(D)g^i$, we have $N^{(i)}_{3n} = A_{3n}$ for all $n \geq 0$. Consider $\overline{A} \subseteq \overline{N^{(i)}} \subseteq \overline{S}$. Since $(\overline{N^{(i)}}/\overline{A})_{3n} = 0$ for all $n \geq 0$, and $\overline{A}$ is generated in degree 1, we see that $\overline{N^{(i)}}/\overline{A}$ is a direct limit of graded $\overline{A}$-modules which are finite-dimensional over $k$. On the other hand, it is clear that $\overline{Q}_g(R(D)) = Q_g(T)$, so $\overline{Q}_g(\overline{A}) = Q_g(\overline{S})$, so that $\overline{A} \subseteq \overline{N^{(i)}}$ is an essential extension of right $\overline{A}$-modules. Also, $\text{Ext}_2^2(k, \overline{A}) = 0$, since $\overline{A}$ is Cohen-Macaulay by Proposition [2.24 (4)]. This forces $\overline{N^{(i)}}/\overline{A}$ for each $i$. In other words, $N^{(i)} \subseteq A + Sg$. Now note that $N^{(1)} \subseteq A + Sg^0 = S$, and suppose that $N^{(1)} \subeq A + Sg^i$ for some $i \geq 0$. Then for $x \in N^{(1)}$, writing $x = a + sg^i$ with $a \in A$, $s \in S$, we see that $xg = ag + sg^{i+1} \in A$ and so $s \in N^{(i+1)}$. Since $N^{(i+1)} \subseteq A + Sg$, we have $x \in A + Sg^{i+1}$, so $N^{(1)} \subeq A + Sg^{i+1}$. By induction on $i$, $N^{(1)} \subeq \bigcap_{i \geq 0} A + Sg^i = A$, so $A \cap Sg = Ag$. The Hilbert series follows immediately since $h_A(t) = \frac{t^2 + 1}{(1-t)^2}$. 

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(2) By Lemma [12.1(1)], we have that \( \overline{A} \) is equal to \( S \) in large degree. We wish to apply Proposition [7.6(1)]. For this we need to check the hypotheses of that proposition, namely that \( A \subseteq S \) is a finite extension, that \( Q_{\text{gr}}(A) = Q_{\text{gr}}(S) \), and that \( (A \cap S g)/(A \cap S g^2) \neq 0 \). Suppose we prove these things, then Proposition [7.6(1)] will show that \( A \) contains a special ideal of \( S \). But \( S \) has no such ideals except those of finite-\( k \)-codimension, by Lemma [10.3]. Thus \( A \) will then equal \( S \) in large degree, as required.

The remaining needed facts follow from the other parts of Lemma [12.1]. First, the proof of Lemma [12.1(3)] shows that \( S_2 = zA_1 + A_2 = A_1 z + A_2 \) where \( z \in S_1 \setminus A_1 \). Then by the graded Nakayama lemma, \( S = A + zA = Az + A \). So \( A \subseteq S \) is a finite extension. We also have from Lemma [12.1(3)] that \( \dim_k A_2 = 4 \), as well as the rather hard-earned estimate \( \dim_k A_3 \geq 7 \). Since \( S_3 = zA_2 + A_3 = A_2 z + A_3 \), with \( \dim_k S_3 = 10 \), this implies that \( (zA \cap A)_3 \neq 0 \). Choose \( 0 \neq x \in A_2 \) such that \( zx \in A_3 \). Since \( g \in S_3 = A_2 z + A_3 \), we have \( gx \in A_2 zx + A_3 x \subseteq A_5 \). But then \( 0 \neq gx \in (A \cap S g) \setminus (A \cap S g^2) \), so that \( (A \cap S g)/(A \cap S g^2) \neq 0 \). Also, since we have seen that \( zA \cap A \neq 0 \), then \( z \in Q_{\text{gr}}(A) \). So \( S_1 \subseteq Q_{\text{gr}}(A) \), and thus \( Q_{\text{gr}}(S) = Q_{\text{gr}}(A) \). All of the hypotheses of Proposition [7.6(1)] are now verified, and we are done.

As a prelude to future work, we close with the following example of a subring of \( S \) generated in degree 2. It points to a new issue that arises in trying to understand algebras generated in that degree, and presumably all other degrees; degrees 1 and 3 were special because they divide the degree of the central element in \( S \).

**Example 12.3.** Let \( p \in E \) be given, and let \( A = k(V) \subseteq S^{(2)} \), where \( V = W(p)S_1 \subseteq S_2 \) is the set of all sections of \( S_2 = B(E, \mathcal{L}_2) \) vanishing along \( p \). Since we know from the theorems in this paper that both the degree-3 generated algebra \( k(W(p))S_2 \cong R(p) \) and the degree-1-generated algebra \( k(W(p)) \) are very nice rings, we might hope \( A \) is similarly good.

Now \( V^2 = W(p)S_1 W(p)S_1 = S_1 W(\sigma(p)) W(p)S_1 \subseteq S_1 g \) by Lemma [1.6(1)]. Then \( V^3 \supseteq VS_1 g \) and \( V^3 \supseteq S_1 g V = S_1 V g \). Since it is easy to prove that \( VS_1 + S_1 V = S_3 \), we have \( V^3 \supseteq S_3 g \). An similar inductive argument shows that \( \bigoplus_{i \geq 0} S_{(2i+1)} g \subseteq A \). We claim then that \( A \) is not left noetherian. Indeed, if \( A \) were left noetherian, then \( A' = \bigoplus_{i \geq 0} S_{2i} \) would be a finitely generated left \( A \)-module, since \( A' \cong S' g \) (as ungraded modules) and \( S' g \subseteq A \). But then \( \overline{A'} \) would be finitely generated, where \( \overline{A} = \bigoplus_{i \geq 0} H^0(E, \mathcal{L}_{2i+1}) \), and by the graded Nakayama lemma, we would have to have \( \dim_k \overline{A}/(A_{(2i+1)}) < \infty \). However, since \( \overline{A} = B(E, \mathcal{N}', \sigma^2) \) with \( \mathcal{N}' = I_p \otimes \mathcal{L}_2 \), we have \( (A_{(2i+1)}) \subseteq \bigoplus_{i \geq 0} H^0(E, I_p \otimes \mathcal{L}_{2i+1}) \) for all \( i \geq 0 \), from which it is clear that \( \dim_k \overline{A}/(A_{(2i+1)}) = \infty \).

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