ON TOTALLY REAL SUBMANIFOLDS
Ognian T. Kassabov
University of Sofia
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Many authors have investigated totally real submanifolds of Kählerian manifolds. In this paper we generalize some results in this direction obtained by B.-Y. Chen, K. Ogiue, M. Kon and K. Yano in [1] [4] [5] [6]. In particular, we study semiparallel totally real submanifolds. The notion of semiparallel submanifolds was introduced by J. Deprez in [2] [3] as extrinsic analogue for semisymmetric Riemannian spaces.

1. Preliminaries.

Let \( \tilde{M} \) be a \( 2m \)-dimensional Kählerian manifold with Riemannian metric \( g \), complex structure \( J \) and covariant differentiation \( \tilde{\nabla} \). An \( n \)-dimensional submanifold \( M \) of \( \tilde{M} \) is said to be a totally real submanifold of \( \tilde{M} \), if for each point \( p \in M \) the inclusion \( JT_pM \subset T_pM^\perp \) holds. Then \( n \leq m \). For \( X, Y \in \mathfrak{X}(M) \) we write the Gauss formula:

\[
\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y),
\]

where \( \nabla \) is the covariant differentiation on \( M \) and \( \sigma \) is a normal-bundle-valued symmetric tensor field on \( M \), called the second fundamental form of \( M \). The mean curvature vector \( H \) of \( M \) is defined by \( H = (1/n) \text{tr} \sigma \). If \( H = 0 \), \( M \) is said to be a minimal submanifold of \( \tilde{M} \). In particular, if \( \sigma = 0 \), \( M \) is called a totally geodesic submanifold of \( \tilde{M} \). For \( \xi \in \mathfrak{X}(M)^\perp \) the Weingarten formula is given by

\[
\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi,
\]

where \( -A_\xi X \) (resp. \( D_X \xi \)) denotes the tangential (resp. the normal) component of \( \tilde{\nabla}_X \xi \). It is well known that \( g(\sigma(X,Y), \xi) = g(A_\xi X, Y) \) and \( D \) is the covariant differentiation in the normal bundle. If \( D_X \xi = 0 \) for each \( X \in \mathfrak{X}(M) \), the normal vector field \( \xi \) is said to be parallel. For a normal vector field \( \xi \), let \( J\xi = P\xi + f\xi \), where \( P\xi \) (resp. \( f\xi \)) denotes the tangential (resp. the normal) component of \( J\xi \). Then \( f \) is an endomorphism of the normal bundle and \( f^3 + f = 0 \). So if \( f \) does not vanishes, it defines an \( f \)-structure in the normal bundle. If \( Df = 0 \), i.e. \( D_X f\xi - fD_X \xi = 0 \) for all \( X \in \mathfrak{X}(M) \), \( \xi \in \mathfrak{X}(M)^\perp \), the \( f \)-structure in the normal bundle is said to be parallel. In this case it is not difficult to find that
(1) \[ \sigma(X, Y) = JA_{JX}Y = JA_{JY}X , \]
(2) \[ D_XJY = J\nabla_XY , \]
(3) \[ A_\xi = 0 , \]
for \( X, Y \in \mathfrak{X}(M) \), \( \xi \perp \mathfrak{X}(M) \oplus J\mathfrak{X}(M) \), see [6, p. 46]. We note, that if \( n = m \), \( f \) vanishes and (1) and (2) hold good. Because of (1), the equation of Gauss can be written in the form

\[ \{ \tilde{R}(x, y)z \}^t = R(x, y)z - [A_{Jx}, A_{Jy}]z , \]

for \( x, y, z \in T_pM \), \( p \in M \) where \( \tilde{R} \) (resp. \( R \)) is the curvature tensor for \( \tilde{M} \) (resp. \( M \)) and \( \{ \tilde{R}(x, y)z \}^t \) denotes the tangential component of \( \tilde{R}(x, y)z \). In particular, if \( \tilde{M} \) is a complex space form \( \tilde{M}(\mu) \), i.e. a Kählerian manifold of constant holomorphic sectional curvature \( \mu \), we find

\[ \frac{\mu}{4} x \wedge y = R(x, y)z - [A_{Jx}, A_{Jy}]z . \]

If \( [A_\xi, A_\eta] = 0 \) for all \( \xi, \eta \in \mathfrak{X}(M)^\perp \), \( M \) is said to have commutative second fundamental forms. Then from (3) and (4) we obtain

**Lemma 1.** [6, p. 57] Let \( M \) be a totally real submanifold of a complex space form \( \tilde{M}(\mu) \). If the \( f \)-structure in the normal bundle is parallel, then \( M \) is of constant curvature \( \mu/4 \) if and only if \( M \) has commutative second fundamental forms.

For minimal submanifolds we have

**Lemma 2.** [6, p. 57] Let \( M \) be a totally real minimal submanifold with commutative second fundamental forms of a Kählerian manifold \( \tilde{M} \). If the \( f \)-structure in the normal bundle is parallel, then \( M \) is totally geodesic.

We note that because of (1)

\[ M \text{ is minimal if and only if } \sum_{i=1}^n A_{Je_i}e_i = 0 \]

for any orthonormal basis \( \{e_1, \ldots, e_n\} \) of a tangent space of \( M \).

Let \( R^\perp \) be the curvature tensor of the normal connection, i.e.
\[
R^\perp(X,Y)\xi = D_X D_Y \xi - D_Y D_X \xi - D_{[X,Y]} \xi,
\]
for \(X, Y \in \mathfrak{X}(M), \xi \in \mathfrak{X}(M)\perp\). Using (2) we find that
\[
(6) \quad R^\perp(X,Y)JZ = JR(X,Y)Z.
\]

Let \(\nabla\) denote the covariant differentiation with respect to the connection of van der Waerden-Bortolotti. For example
\[
(\nabla_X \sigma)(Y, Z) = D_X \sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z).
\]
If \(\nabla \sigma = 0\), then \(M\) is said to have parallel second fundamental form or to be a parallel submanifold. More generally \(M\) is said to be a semiparallel submanifold if \(\nabla\) is constant, if
\[
(\nabla(X,Y)\sigma)(Z, U) = (\nabla_X(\nabla_Y \sigma))(Z, U) - (\nabla_Y(\nabla_X \sigma))(Z, U),
\]
or equivalently
\[
(\nabla(X,Y)\sigma)(Z, U) = R^\perp(X,Y)\sigma(Z, U) - \sigma(R(X,Y)Z, U) - \sigma(Z, R(X,Y)U).
\]
In Euclidean spaces submanifolds of this kind are considered by J. Deprez [2] [3]. Using (1) and (6) we derive

**Lemma 3.** Let \(M\) be a totally real submanifold with parallel \(f\)-structure in the normal bundle of a Kählerian manifold \(\tilde{M}\). Then \(M\) is semiparallel if and only if
\[
R(x,y)A_{Jz}u = A_{Jz}R(x,y)u + A_{Ju}R(x,y)z
\]
for all \(x, y, z, u \in T_pM, p \in M\).

On the other hand, as a generalization of the submanifolds with parallel mean curvature vector, the submanifolds with semiparallel mean curvature vector are defined by \(R^\perp(X,Y)H = 0\). We note that the class of submanifolds with semiparallel mean curvature vector includes also the semiparallel submanifolds.

**2. Submanifolds of constant curvature.**

**Proposition 1.** Let \(M\) be an \(n\)-dimensional \((n > 1)\) totally real submanifold of constant curvature \(c\), with parallel \(f\)-structure in the normal bundle of a Kählerian
manifold $\tilde{M}$. If the mean curvature vector $H$ of $M$ is semiparallel, then $M$ is minimal or flat.

**Proof.** First we note that $H \in J\mathfrak{X}(M)$, because of (1). Now using (6) we obtain:

$$0 = R^\perp(X,Y)H = -JR(X,Y)JH,$$

i.e.:

$$R(X,Y)JH = 0.$$  

Since $M$ is of constant curvature $c$, this implies that

$$(7) \quad c\{g(y, JH)x - g(x, JH)y\} = 0$$

for all $x, y \in T_pM$. Let $c \neq 0$. Putting in (7) $y = (JH)_p, x \perp y, x \neq 0$ we obtain $H_p = 0$. Hence $M$ is minimal.

Now we prove the main result in this section.

**Theorem 1.** Let $M$ be an $n$-dimensional ($n > 1$) totally real semiparallel submanifold of constant curvature $c$ with parallel $f$-structure in the normal bundle of a Kählerian manifold $\tilde{M}$. Then $M$ is flat, i.e. $c = 0$ or $M$ is a totally geodesic submanifold of $\tilde{M}$.

**Proof.** Since $M$ is of constant curvature $c$, $R(x, y) = cx \wedge y$ holds good. Hence Lemma 3 implies:

$$(8) \quad c\{g(y, A_{Jz}u)x - g(x, A_{Jz}u)y\} = c\{g(y, z)A_{Ju}x - g(x, z)A_{Ju}y$$

$$+ g(y, u)A_{Jz}x - g(x, u)A_{Jz}y\}$$

for all $x, y, z, u \in T_pM, p \in M$. Let $c \neq 0$ and $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $T_pM$. We put $x = u = e_i$ in (8) and we add for $i = 1, \ldots, n$ using (5) and Proposition 1. The result is

$$(n + 1)cA_{Jz}y = 0.$$  

Hence the assertion follows, because of (3).

Using Lemma 1 we obtain

**Corollary.** Let $M$ be an $n$-dimensional ($n > 1$) totally real semiparallel submanifold of constant curvature $c$, with parallel $f$-structure in the normal bundle
of a complex space form \( \widetilde{M}(\mu) \). Then \( M \) is flat, i.e. \( c = 0 \) or \( M \) is a totally geodesic submanifold of \( \widetilde{M}(\mu) \), i.e. \( c = \mu/4 \).

For parallel minimal submanifolds this Corollary is proved in [1], see also [6, p.61].

3. Minimal submanifolds and the sign of the scalar curvature.

Let \( M \) be an \( n \)-dimensional totally real minimal semiparallel submanifold with parallel \( f \)-structure in the normal bundle of a complex space form \( \widetilde{M}(\mu) \). According to (4) and Lemma 3:

\[
\frac{\mu}{4} \{ g(y, A_{Jz}u)x - g(x, A_{Jz}u)y \} + [A_{Jx}, A_{Jy}]A_{Jz}u
\]

\[
= \frac{\mu}{4} \{ g(y, z)A_{Ju}x - g(x, z)A_{Ju}y \\
+ g(y, u)A_{Jz}x - g(x, u)A_{Jz}y \} \\
+ A_{Ju}[A_{Jx}, A_{Jy}]z + A_{Jz}[A_{Jx}, A_{Jy}]u
\]

holds good and hence we derive the relation:

\[
\frac{\mu}{4} \{ g(y, A_{Jz}A_{Ju}v)g(x, w) - g(x, A_{Jz}A_{Ju}v)g(y, w) \} \\
+ g([A_{Jx}, A_{Jy}]A_{Jz}A_{Ju}v, w)
\]

(9)

\[
= \frac{\mu}{4} \{ g(y, z)g(A_{Jx}A_{Ju}v, w) - g(x, z)g(A_{Jy}A_{Ju}v, w) \\
+ g(y, A_{Jv}v)g(A_{Jz}x, w) - g(x, A_{Ju}v)g(A_{Jz}y, w) \} \\
+ g([A_{Jx}, A_{Jy}]z, A_{Ju}A_{Jv}v) + g(A_{Jz}[A_{Jx}, A_{Jy}]A_{Ju}v, w)
\]

for all \( x, y, z, u, v, w \in T_pM, p \in M \). Let \( \{e_1, \ldots, e_n\} \) be an orthonormal basis of \( T_pM \). In (9) we put \( x = w = e_i, y = u = e_j, z = v = e_k \) and we add for \( i, j, k = 1, \ldots, n \); this gives:

\[
\frac{\mu}{4}(n+1) \sum_{i=1}^{n} \text{tr} A_{Je_i}^2 - 2 \sum_{i,j=1}^{n} \text{tr} A_{Jei}^2 A_{Jej}^2 - \sum_{i,j=1}^{n} \text{tr} A_{Je_i}^2 A_{Je_j}^2 A_{Je_i} A_{Je_j}
\]

(10)

\[
+ 2 \sum_{i,j=1}^{n} \text{tr} A_{Je_i} A_{Je_j} A_{Je_i} A_{Je_j} = 0.
\]

On the other hand it is not difficult to find that
\[ \sum_{i,j=1}^{n} \text{tr} A_{Je_i}^2 A_{Je_j}^2 = \sum_{i,j=1}^{n} \text{tr} A_{Je_i} A_{Je_j}^2 A_{Je_i}. \]

Applying this to (10) we obtain:

\[ \frac{\mu}{4} (n + 1) \sum_{i=1}^{n} \text{tr} A_{Je_i}^2 + \sum_{i,j=1}^{n} \left( \text{tr} (A_{Je_i} A_{Je_j} - A_{Je_j} A_{Je_i})^2 - \text{tr} A_{Je_i}^2 A_{Je_j}^2 \right) = 0. \]

Now, just as Theorem 8.1 in [6, p. 69], we can prove the following

**Theorem 2.** Let \( M \) be a totally real minimal semiparallel submanifold with parallel \( f \)-structure in the normal bundle of a complex space form \( \tilde{M}(\mu) \). If the square of the length of the second fundamental form is constant (or equivalently, if \( M \) has constant scalar curvature \( \tau \)), then \( M \) is totally geodesic or \( \tau \geq 0 \). Moreover, if \( \tau = 0 \), then \( M \) is flat.

4. **Commutative second fundamental forms.**

To begin this section we note, that if \( M \) is a totally real submanifold with commutative second fundamental forms and parallel \( f \)-structure in the normal bundle of a complex space form \( \tilde{M}(\mu) \), then the equation of Gauss (4) reduces to

\[ R(x, y) = \frac{\mu}{4} x \wedge y. \]

**Proposition 2.** Let \( M \) be an \( n \)-dimensional \((n > 1)\) totally real semiparallel submanifold with commutative second fundamental forms of a complex space form \( \tilde{M}(\mu) \). If the \( f \)-structure in the normal bundle is parallel, then \( M \) is totally geodesic or flat.

**Proof.** According to (11), \( M \) is of constant curvature \( \mu/4 \). Now the assertion follows from Theorem 1.

More generally, from (11), Lemma 2 and Proposition 1 we derive

**Proposition 3.** Let \( M \) be an \( n \)-dimensional \((n > 1)\) totally real submanifold with commutative second fundamental forms and parallel \( f \)-structure in the normal bundle of a complex space form \( \tilde{M}(\mu) \). If the mean curvature vector \( H \) of \( M \) is semiparallel, then \( M \) is totally geodesic or \( \mu = 0 \).

Propositions 2 and 3 generalize some results in [6, p. 62].
If \( n = 2 \) and \( M \) has parallel mean curvature vector, we can weaken the assumptions of Proposition 3. Namely, we have

**Proposition 4.** Let \( M \) be a totally real surface with commutative second fundamental forms and parallel \( f \)-structure in the normal bundle of a Kähler manifold \( \tilde{M} \). If the mean curvature vector of \( M \) is parallel, then \( M \) is flat or totally geodesic.

This follows from Lemma 2 and the following

**Proposition 5.** Let \( M \) be a totally real surface with parallel \( f \)-structure in the normal bundle of a Kähler manifold \( \tilde{M} \). If the mean curvature vector \( H \) of \( M \) is parallel, then \( M \) is flat or minimal.

**Proof.** Since \( H \) is parallel, it has constant length. Let \( H \neq 0 \), i.e. \( M \) is not minimal. As in Proposition 1 we have \( R(X, Y)JH = 0 \). Hence \( R(X, JH, JH, X) = 0 \), and since \( H \) does not vanish, then \( M \) is flat.

**Theorem 3.** Let \( M \) be an \( n \)-dimensional (\( n > 1 \)) complete totally real submanifold with parallel mean curvature vector and commutative second fundamental forms of a \( 2m \)-dimensional simply connected complete complex space form \( \tilde{M}(\mu) \). If the \( f \)-structure in the normal bundle is parallel and \( M \) is not totally geodesic, then \( M \) is a pythagorean product of the form

\[ S^1(r_1) \times \ldots \times S^1(r_p) \times \mathbb{R}^{n-p} \]

in a \( \mathbb{C}^n \) in \( \mathbb{C}^m \), where \( 1 \leq p \leq n \).

**Proof.** According to Proposition 3, \( \mu = 0 \). Then \( \tilde{M}(\mu) \) is (isometric to) \( \mathbb{C}^m \) and the assertion follows from Theorem 7.1 in [6, p. 65].

For parallel submanifolds, Theorem 3 is proved in [5], (see also [6, p. 66]).

**Corollary.** Under the same assumptions as in Theorem 3, if \( M \) is compact, it is pythagorean product of the form

\[ S^1(r_1) \times \ldots \times S^1(r_n) \]

in a \( \mathbb{C}^n \) in \( \mathbb{C}^m \).

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University of Sofia
Faculty of Mathematics and Mechanics
5, Anton Ivanov Street
1126 Sofia
BULGARIA