An Accelerating Universe from Dark Matter Interactions with Negative Pressure

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As an explanation for the acceleration of the universe, we propose dark matter with self-interactions characterized by a negative pressure; there is no vacuum energy whatsoever in this Cardassian model. These self-interactions may arise due to a long-range “fifth force” which grows with the distance between particles. We use the ordinary Friedmann equation, $H^2 = \frac{8\pi G \rho}{3}$, and take the energy density to be the sum of two terms: $\rho = \rho_M + \rho_K$. Here $\rho_M \equiv m n_M$ is the ordinary mass density and $\rho_K \equiv \rho_K(\rho_M)$ is a new interaction term which depends only on the matter density. For example, in the original version of the Cardassian model, $\rho_K = b \rho_M^n$ with $n < 2/3$; other examples are studied as well. We use the ordinary four-dimensional Einstein’s equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ and assume a perfect fluid form for the energy momentum tensor. Given this ansatz, we can compute the accompanying pressure, and use the conservation of the energy-momentum tensor to obtain modified forms of the Euler’s equation, Poisson equation, and continuity equation. With this fully relativistic description, one will then be able to compute growth of density perturbations, effects on the Cosmic Background Radiation, and other effects with an eye to observational tests of the model.
I. INTRODUCTION

Observations of Type IA Supernovae [1,2], as well as concordance with other observations, including the microwave background [3] and galaxy power spectra [4], indicate that the universe is flat and accelerating. Many authors have explored possible explanations for the acceleration: a cosmological constant, a decaying vacuum energy [5,6], quintessence [7–9], and gravitational leakage into extra dimensions [10].

Freese and Lewis [11] proposed Cardassian expansion as an explanation for acceleration which invokes no vacuum energy whatsoever. In this model the universe is flat and accelerating, and yet consists only of matter and radiation. They proposed additional terms in the Friedmann equation, which becomes \( H^2 = g(\rho_M) \), where \( \rho_M \) contains only matter and radiation (no vacuum), \( H = \dot{a}/a \) is the Hubble constant (as a function of time), \( G = 1/m^2_{\text{pl}} \) is Newton’s universal gravitation constant, and \( a \) is the scale factor of the universe. The function \( g(\rho_M) \) returns to the usual \( 8\pi\rho_M/(3m^2_{\text{pl}}) \) during the early history of the universe, but takes a different form that drives an accelerated expansion after a redshift \( z \sim 1 \). Such modifications to the Friedmann equation may arise, e.g., as a consequence of our observable universe living as a 3-dimensional brane in a higher dimensional universe [12].

In this paper we propose an alternative origin for the same behavior. This origin is a “fifth force” in ordinary four spacetime dimensions. We use the standard Friedmann equation

\[
H^2 = \frac{8\pi G \rho}{3},
\]

but allow the dark matter to have self-interactions that contribute a negative pressure: these self-interactions drive accelerated expansion, and may arise due to a long-range “fifth force.”

We speculate on a form of the force between particles that may be responsible for such an interaction: a confining force.

We take the energy density \( \rho \) on the right hand side of Eq.(1) to be the sum of two terms: the ordinary contributions from matter and radiation plus a new interaction term. During the matter dominated era, we take
\[ \rho = \rho_M + \rho_K, \]  

where \( \rho_M = mn_M \) (mass \( m \) times number density \( n_M \) of matter) is the ordinary mass density and \( \rho_K \) is a new interaction term which is a function only of the mass density, \( \rho_K \equiv \rho_K(\rho_M) \). We take the mass density to scale as usual with the redshift,

\[ \rho_M = \rho_{M,0}(a/a_0)^{-3} \]  

where subscript 0 refers to today.

For example, consider the original Cardassian model proposed in [11]:

\[ H^2 = \frac{8\pi}{3m^2_{\text{pl}}} \rho_M + B\rho_M^n \quad \text{with} \quad n < 2/3, \]  

or, equivalently,

\[ H^2 = \frac{8\pi G}{3} \rho_M \left[ 1 + \left( \frac{\rho_{\text{Card}}}{\rho_M} \right)^{1-n} \right], \]  

where \( \rho_{\text{Card}} \) is the matter density at which the two terms are equal. The second term in square brackets is negligible initially (when \( \rho_M \gg \rho_{\text{Card}} \)), and only comes to dominate at redshift \( z \approx 1 \) (once \( \rho_M \sim \rho_{\text{Card}} \)). Once it dominates, it causes the universe to accelerate: the scale factor grows as \( a \sim t^{2/3n} \) with \( n < 2/3 \) so that \( \ddot{a} > 0 \).

There are two possible interpretations of Eq.(5). Previously, [11] interpreted it as a modified Friedmann equation which arose from the physics of extra dimensions [12]. Here, and in the longer paper [13], we treat this equation as an ordinary Friedmann equation in which the second term describes self-interactions of the dark matter particles! with

\[ \rho_K = \frac{3m^2_{\text{pl}}}{8\pi} B\rho_M^n. \]  

In a ‘generalized Cardassian model’ other functions \( g(\rho_M) \) of the matter density on the right hand side of the Friedmann equation can also drive an accelerated expansion in

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1This interpretation may have a four-dimensional origin or serve as an effective description of higher dimensional physics.
the recent past of the universe without affecting its early history [14,13]. Several of these
alternative functions will be discussed below [see Eqs.(23) and (26)].

In the model we discuss here, we use the ordinary four-dimensional Einstein’s equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}.$$  \hspace{1cm} (7)

We take the energy momentum tensor to be made only of matter and radiation, with the
perfect fluid form

$$T^{\mu\nu} = pg^{\mu\nu} + (p + \rho)u^\mu u^\nu,$$  \hspace{1cm} (8)

where $\rho$ is the total energy density given in Eq.(2), $p = p_M + p_K$ is the accompanying
pressure, and $u^\mu$ is the fluid four-velocity. The Cardassian contribution to the pressure can
be computed in our class of models and turns out to be a negative pressure, $p_K < 0$. This
negative pressure is responsible for the universe’s acceleration. We can find models with any
negative equation of state $w_K = p_K/\rho_K < 0$, both constant and time-dependent\(^2\), including
$w_K < -1$; we also find models in which the pressure depends inversely on the energy density,
$p \sim -1/\rho^r$.

We use the conservation of the energy-momentum tensor to obtain modified forms of
the Euler’s equation, Poisson equation, and continuity equation. With this fully relativis-
tic description, one can then compute growth of density perturbations, effects on Cosmic
Background Radiation and cluster abundance, and other effects with an eye to observational
tests of the model. An initial study of density perturbations, performed in the Newtonian
limit, can be found in our longer paper [13].

The Cardassian model also has the attractive feature that matter alone is sufficient to
provide a flat geometry. The geometry of Eq.(1) is flat, as required by observations of the

\(^2\)We remark that in our class of models, $w_K < -1$ is not connected to a violation of the weak
energy condition. Since $\rho_K$ and $p_K$ are not independent of $\rho_M$, the weak energy condition is a
condition on the total energy-momentum tensor: $(p_M + p_K)/(\rho_M + \rho_K) \geq -1.$

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microwave background [3]. The numerical value of the critical mass density for which the universe is flat can be modified; we take the value of the critical mass density to be 0.3 of the usual value. Hence the matter density can have exactly this new critical value and satisfy all the observational constraints, e.g., from the baryon cluster fraction and the galaxy power spectrum. Throughout this paper, we will assume that there is no vacuum energy at all. We do not solve the cosmological constant problem; we simply set it to zero.

We speculate on the possible origin of an interaction energy with negative pressure in Sect. II, present a general fluid formulation in Sect. III, and then give specific examples in Sect. IV. We conclude in Section V.

II. ORIGIN OF INTERACTION ENERGY WITH NEGATIVE PRESSURE

Here we speculate on a possible origin for an interaction energy with a negative pressure. Dark matter particles may be subject to a new interparticle force which is long-range and confining. This force may be of gravitational origin or maybe a fifth force. It is the confinement property that gives rise to a negative pressure (see [13]). That a confining force can give rise to an effective negative pressure is well-known in particle physics, where the MIT bag model is just such an effective description of quark confinement.

To be somewhat quantitative, let us consider a simple example of a power law interparticle potential

\[
U_{ij} = Ar_{ij}^\alpha, \tag{9}
\]

where \(\alpha > 0\), \(r_{ij}\) is the distance between particles and \(A\) is a normalization constant. The corresponding force is a power law, \(F(r) \propto r^{\alpha-1}\). The total new interaction energy of a system of \(N\) particles occupying a volume of radius \(R\) will be \(U_{\text{new}} \simeq AN^2R^\alpha\), to within a numerical factor of order 1 dependent on the geometry. The total gravitational potential energy of the same system is, also within a factor of order unity, \(U_{\text{grav}} \simeq \frac{GM^2}{R}\), where \(M\) is the total mass of the system. To play a cosmological role at the present time, the new
energy must be of the same order of the gravitational energy when $R \simeq R_H$, the current size of the horizon. Imposing that $U_{\text{new}} \simeq U_{\text{grav}}$ at $R \simeq R_H$ gives us the normalization $A = \frac{Gm^2}{R_H}$, where $m = M/N$ is the mass of a single particle. The magnitude of the new force on galactic scales is negligible compared to the gravitational force. This Newtonian formulation must be modified at large distances because of the finite speed of light and issues of causality. Notice that the Cardassian index $n$ in the $\rho^n$ model is connected to the exponent $\alpha$ in the confining force law through $\alpha = 3(1 - n)$.

III. BASIC EQUATIONS

A. Perfect Fluid

As discussed in the Introduction, we use the ordinary four-dimensional Einstein’s equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ where the energy-momentum tensor is made only of matter and radiation and has the perfect fluid form, $T^{\mu\nu} = p g^{\mu\nu} + (p + \rho) u^\mu u^\nu$. Here the total energy density $\rho$ for matter includes not only the mass density $\rho_M$ (mass times number density) but also any interactions as in Eq.(2).

B. Conservation laws and Evaluation of Pressure

The Bianchi identities guarantee the conservation of energy and momentum,

$$T^{\mu\nu ; \nu} = 0. \quad (10)$$

In a comoving frame, energy-momentum conservation gives the (fully relativistic) energy conservation and Euler equations [15,16]

$$\dot{\rho} = -u^\mu ; \mu (\rho + p),$$

$$\dot{u}_\mu = -\frac{h^{\nu}_\mu p_{,\nu}}{\rho + p}. \quad (12)$$
where the dot denotes a derivative with respect to comoving time and the tensor $h_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu$ projects onto comoving hypersurfaces. We impose in addition that mass (or equivalently particle number) is conserved,

$$\left(\rho_{M} u^\mu\right)_{,\mu} = 0. \quad (13)$$

This will give us the usual dependence of the mass density on the scale factor of the universe, Eq.(3).

From Eqs.(1) and (11), or equivalently from adiabaticity, we can find the pressure due to the new interactions,

$$p_K = \rho_{M} \left( \frac{\partial \rho_K}{\partial \rho_{M}} \right)_s - \rho_K. \quad (14)$$

C. Newtonian limit

Now we obtain the basic equations in the Newtonian limit. In Minkowski space, we write $u^\alpha = \gamma(1, \vec{v})$ and the metric $\eta^{\alpha\beta} = \text{diag}(-1, 1, 1, 1)$, where $\gamma = 1/\sqrt{1 - v^2}$ and $\vec{v}$ is the 3-dimensional fluid velocity. Then, $T_{\mu\nu} = \text{diag}(-\rho, p, p, p)$. We consider a weak static field produced by a nonrelativistic mass density. Poisson’s equation becomes

$$\nabla^2 \phi = 4\pi G (\rho + 3p). \quad (15)$$

From $T^{\alpha\beta}_{;\beta} = 0$ we find the continuity and Euler’s equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot [(\rho + p)\vec{v}] = 0, \quad (16)$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{\nabla p}{\rho + p} - \nabla \phi. \quad (17)$$

Notice that we do not assume $p \ll \rho$ in the right hand side of Eqs. (15-17).

With the additional constraint of particle number conservation, we have the ordinary continuity equation for matter,

$$\frac{\partial \rho_{M}}{\partial t} + \nabla \cdot (\rho_{M} \vec{v}) = 0. \quad (18)$$

Hence our basic nonrelativistic equations are Eqs.(15 - 18).
IV. THREE EXAMPLES OF CARDASSIAN MODELS

A. Original $\rho^n$ Cardassian Model

In the original Cardassian model of Ref. [11],

$$\rho_K = b \rho^n_M = \rho_M \left( \frac{\rho_{\text{Card}}}{\rho_M} \right)^{1-n}, \quad (19)$$

with $n < 2/3$ as in Eqs.(4) and (5). The pressure associated with this model in the fluid approach follows from Eq. (14) as

$$p_K = -(1-n)\rho_K. \quad (20)$$

This model therefore has a constant negative $w_K = p_K/\rho_K = -(1-n)$. Any observational test of $\rho^n$ Cardassian that depends only on $a(t)$ thus has similar behavior to quintessence; however, tests that rely, e.g. on modified Poisson’s equations will have different observational consequences.

One can now obtain the basic equations in the Newtonian limit for the $\rho^n$ Cardassian model by substituting $\rho_K$ and $p_K$ of Eqs.(19) and (20) into Eqs.(15 - 18). In particular, the modified Poisson’s equation becomes

$$\nabla^2 \phi = 4\pi G \left[ \rho_M - (2-3n) \left( \frac{\rho_{\text{Card}}}{\rho_M} \right)^{1-n} \rho_M \right], \quad (21)$$

The modified Euler’s equation of Eq.(17) gives rise to a new force

$$\left. \frac{d\vec{r}}{dt} \right|_{\text{new}} = -\nabla p_K \rho_M = +n(1-n) \left( \frac{\rho_{\text{Card}}}{\rho_M} \right)^{1-n} \nabla \rho_M. \quad (22)$$

On galactic scales, this force destroys flat rotation curves (velocities tend to increase as one goes out to large radii in an unacceptable way). This problem for rotation curves persists in a relativistic generalization of this argument. The fluid $\rho^n$ Cardassian case must therefore be thought of as an effective model that applies only on cosmological scales. The examples in the following two subsections present alternatives to address the issues on galactic scales.
B. Polytropic Cardassian

Another class of models has

$$\rho = \rho_{\text{internal}} + \rho_{\text{Card}} \left[ 1 + \left( \frac{\rho_M}{\rho_{\text{Card}}} \right)^q \right]^{\frac{1}{q-1}}$$

with $q \neq 0$. This model can be used on all scales (see below), but it does not quite fit the criteria of “Cardassian” since it does have a vacuum term. At late times in the future of the universe, when $\rho_M \ll \rho_{\text{Card}}$, this model becomes cosmological constant dominated with $\Lambda = \rho_{\text{Card}}$. Eq.(23) is very similar to a model that was derived earlier [10] due to gravitational leakage into extra dimensions.

Here, the pressure is

$$p = p_M - \rho_{\text{Card}} \left[ 1 + \left( \frac{\rho_M}{\rho_{\text{Card}}} \right)^q \right]^{\frac{1}{q-1}}. \quad (24)$$

When the ordinary pressure $p_M$ can be neglected, this model obeys a polytropic equation of state

$$p = -\rho_{\text{Card}} \left( \frac{\rho_{\text{Card}}}{\rho} \right)^{q-1}, \quad (25)$$

with negative pressure and negative polytropic index $N = -1/q$. For $q > 1$, the pressure varies inversely with the energy density (to the power $q - 1$).

We must make sure that at the scales of galaxies and clusters the Cardassian pressure can be neglected compared to the ordinary pressure. At large matter densities, the Cardassian pressure is $|p_K| \approx \rho_M^{1-q} \rho_{\text{Card}}^q$. In a galaxy or cluster with velocity dispersion $\sigma$, the ordinary pressure is $p_M \approx \rho_M \sigma^2$. We want $|p_K|/p_M \approx (\rho_{\text{Card}}/\rho_M)^q/\sigma^2 \ll 1$. Taking $\sigma \approx 300 \text{ km/s}$ and assuming $p_K$ is unimportant out to $\approx 100 \text{ kpc}$ where $\rho_M \approx 10^2 \rho_{\text{Card}}$, this condition amounts to $q \gtrsim 3$.

The models that match the rotation curves have pressure that scales inversely with the energy density. The condition $q \gtrsim 3$ requires that $p \propto -\frac{1}{\rho^r}$ where $r = q - 1 \gtrsim 2$. This model is similar to the Chaplygin gas [18] which has $r \leq 1$; note that we differ from the Chaplygin gas in the value of the exponent that we find to be required.
Fits to supernova data [17] show that $q \lesssim 2$. Hence we find difficulties in making polytropic cardassian models, or generalized Chaplygin gas models, work at both cosmological and galactic scales.

C. Modified polytropic Cardassian

Let us return now to the original Cardassian proposal in which there is no vacuum energy whatsoever. A Cardassian model that can be used on all scales is

$$\rho = \rho_{\text{internal}} + \rho_M \left[ 1 + \left( \frac{\rho_{\text{Card}}}{\rho_M} \right)^{q(1-n)} \right]^{\frac{1}{q}}. \quad (26)$$

For $q = 1$ this reduces to the original $\rho^n$ Cardassian model. The pressure follows as

$$p = p_M - (1 - n)\rho_M \left[ 1 + \left( \frac{\rho_{\text{Card}}}{\rho_M} \right)^{q(1-n)} \right]^{\frac{1}{q}-1} \left( \frac{\rho_{\text{Card}}}{\rho_M} \right)^{q(1-n)}. \quad (27)$$

This model is interesting because the two parameters $n$ and $q$ are important on different scales. The parameter $n$ sets the current value of $w \simeq -(1 - n)$, and so can be chosen to fit the supernova data, while the parameter $q$ governs the suppression of the Cardassian pressure at high densities, and can therefore be chosen not to interfere with galactic rotation curves and cluster dynamics. On galactic scales, the pressure again depends inversely on the energy density, but on large scales of the universe it depends linearly. Concrete comparisons with data will be presented in [17].

V. CONCLUSIONS

An interpretation of Cardassian expansion as an interacting dark matter fluid with negative pressure is developed. The Cardassian term on the right hand side of the Friedmann equation (and of Einstein’s equations) is interpreted as an interaction term. The total energy density contains not only the matter density (mass times number density) but also interaction terms. These interaction terms give rise to an effective negative pressure which drives cosmological acceleration. These interactions may be due to interacting dark matter,
e.g. with a long-range confining force or a fifth force between particles. Alternatively, such interactions may be an effective description of higher dimensional physics.

A fully relativistic fluid model of Cardassian expansion has been developed, in which energy, momentum, and particle number are conserved, and the modified Poisson’s and Euler’s equations have been derived. In [13], a preliminary study of density fluctuations in the early universe has also been presented. There we developed a Newtonian theory of perturbations, but discovered curious gauge ambiguities in the relativistic theory that must be resolved in a future study.

One of our goals is to allow predictions of various observables that will serve as tests of the model. The Cardassian model will have unique predictions, particularly due to the modified Poisson’s equation. For example, one can now calculate the effect on the Integrated Sachs Wolfe component in the Cosmic Microwave Background, as well as the effect on cluster abundances at different redshifts. These predictions can then be tested against measurements of these quantities. Comparison with existing and upcoming supernova data is being studied in another paper [17].

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