Application of the Bammann inelasticity internal state variable constitutive model to geological materials

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SUMMARY
We describe how the Bammann internal state variable (ISV) constitutive approach, which has proven highly successful in modelling deformation processes in metals, can be applied with great benefit to silicate rocks and other geological materials in modelling their deformation dynamics. In its essence, ISV theory provides a constitutive framework to account for changing history states that arise from inelastic dissipative microstructural evolution of a polycrystalline solid. In this paper, we restrict our attention to a Bammann ISV elastic-viscoplastic model with temperature and strain rate dependence and use isotropic hardening and anisotropic hardening as our two ISVs. We show the Bammann model captures the inelastic behaviour of olivine aggregates (with and without water), lherzolite (with and without water), Carrara marble and rock salt using some experimental data found in the literature. These examples illustrate that when more experimental stress–strain data are gathered on other rock materials, much more realistic numerical simulation of rock behaviour becomes feasible. Though not available in the literature, we outline a set of experiments to obtain unique Bammann ISV model constants.

Key words: Transient deformation; Plasticity, diffusion, and creep; Creep and deformation; Mechanics, theory, and modelling; Rheology: mantle.

INTRODUCTION
The rheology of silicate materials plays a central role in the internal dynamics and thermal histories of planetary bodies. To gain accurate insight into the larger scale dynamical processes, we must ultimately understand the processes that govern the mechanical behaviour of rocks at the microscale. Considerable experimental effort has been undertaken over the last several decades to characterize the microstructural and mechanical behaviour of rocks as described in reviews, for example, by Nicolas & Poirier (1976), Carter (1976), Kirby (1983), Kirby & Kronenberg (1987) and Karato & Toriumi (1989).

The most commonly used model for representing the mechanical response of rocks under varying conditions has been a power-law equation that depends on an exponent, coefficient and activation energy (cf. Goetze 1978; Chopra & Paterson 1981, 1984; Frost & Ashby 1982; Kirby 1983; Karato & Toriumi 1989). Such power-law models, however, do not possess the ability to include time-dependent history effects that arise from temperature, strain rate, pressure and deformation path. As an example of an effort to overcome some of these limitations, Covey-Crump (1994) employed the Hart internal state variable (ISV) model (Hart 1970, 1976) to study the mechanical properties of Carrara Marble, and Lerner & Kohlstedt (1981) applied the Hart ISV model to halide. Aubertin et al. (1991a,b) have also used an ISV model to analyse rock salt and other alkali halides.

Another shortcoming of the power-law approach is the inability for one set of equations to model different deformation mechanisms. A different set of flow law equations and constants are needed to model power-law creep (high temperature and low stress) and exponential creep (low temperature and high stress) for the same material. The Bammann inelasticity model is a unified creep-plasticity model that can accurately model the power-law creep, exponential creep and plasticity regimes for a polycrystalline material. One other important distinction between the Bammann inelasticity model and the power-law approach is that the power-law approach is purely empirical/phenomenological while the Bammann inelasticity model is physically motivated from dislocation mechanics.

To begin, we outline the basics of ISV theory and describe the specific features of the Bammann inelasticity model. In the first section we review the fundamentals of kinematics of large deformations. In the next section, we discuss continuum field theory. The following section details the constitutive equations where we offer a brief summary of different modelling frameworks. We then present the Bammann inelasticity ISV theory (Bammann 1988, 1990; Bammann et al. 1993, 1995). We then describe a set of
The particle velocity and acceleration are given as the time derivatives (sometimes referred to as Lagrangian coordinates) and spatial or Eulerian coordinates, $X\partial X/\partial t$, respectively. The deformation gradient, $F$, can then be expressed in indicial form as

$$F_{iJ} = x_{i,J} = \delta\tilde{x}_i(X,t)/\partial X_j.$$  \hfill (2)

As shown in Fig. 1, the deformation gradient maps an infinitesimal line segment from the reference (Lagrangian) configuration to the current (Eulerian) configuration or State 0 to State 2, respectively.

The formulation of the kinematics development follows closely that of Lee & Liu (1967). The kinematics of motion include elastic straining and inelastic flow as is illustrated by the multiplicative decomposition of the deformation gradient shown in Fig. 1. The deformation gradient, $F$, is decomposed into the deviatoric inelastic, or plastic, $F^p$, and elastic parts, $F^e$, given by

$$F = F^e F^p.$$  \hfill (3)

The deviatoric inelastic deformation gradient maps the deformation from State 0 to State 1. The elastic deformation gradient maps the deformation from State 1 to State 2.

We note that the term inelasticity is used, because it accounts for strains due to plasticity and creep (or stress relaxation) mechanisms, which can be important in silicates. At the microscale, inelastic deformation can develop from dislocation glide, dislocation climb, diffusive flow of atoms, grain boundary sliding, mechanical twinning and lattice collapse.

Eq. (3) assumes that the motion of the body is described by a smooth displacement function. This precludes the initiation of discrete failure surfaces but still allows a continuum description of damage (cf. Bammann & Aifantis 1989; Horstemeyer et al., 2000), but damage and fracture are not included in this writing. The elastic deformation gradient, $F^e$, represents lattice displacements from equilibrium. The inelastic deformation gradient, $F^p$, represents a continuous distribution of dislocations and defects whose volume preserving motion produces permanent shape changes. This can arise from creep at low strain rates or from plasticity at higher strain rates. Although dislocations are by nature discontinuities, a continuum representation is possible to simulate their effects (cf. Bilby et al. 1955; Kroner 1964).

The velocity gradient associated with the deformation gradient, $L = FF^{-1}$, from eq. (2) can be decomposed into

$$L = L^e + L^p,$$  \hfill (4)

where $D = \frac{1}{2}(L + L^T)$ is the symmetric part and $W = \frac{1}{2}(L - L^T)$ is the antisymmetric part of the velocity gradient tensor with analogous formulae holding for the elastic and inelastic components of the velocity gradients. The elastic rate of deformation relates to the total volumetric rate of deformation by the additive decomposition of the deformation rates similar to eq. (4),

$$D^e = D - D^p.$$  \hfill (5)

Hence, the elastic velocity gradient can be determined from the velocity gradient (Bammann 1988).

**Continuum balance equations**

In describing the material response of polycrystalline solids, one must consider the following variables related to the continuum field equations: the Cauchy stress tensor, $\sigma$, the extrinsic body force per unit mass $b$, the internal energy per unit mass $U$, the extrinsic heat supply per unit mass and time $Z$, the entropy per unit mass $\eta$, the absolute temperature $T$, the density $\rho$ and the heat flux $q$. The balance laws can be written as differential equations relating these various variables. The balance of linear momentum is given as

$$\nabla \cdot \sigma + \rho \dot{b} = \rho \dot{\gamma}.$$  \hfill (6)

The balance of angular momentum is represented by the symmetry of the Cauchy stress tensor, $(\sigma_{ij} = \sigma_{ji})$. The balance of energy is given by

$$\rho Z + \sigma : \nabla \dot{x} - \nabla \cdot q = \rho \dot{U}.$$  \hfill (7)

In addition, these equations require that the entropy production inequality be satisfied. This is given by

$$\rho \dot{\eta} - \left( \frac{\rho Q}{T} \right) + \nabla : \left( \frac{q}{\rho} \right) \geq 0.$$  \hfill (8)
The Helmholtz free energy per unit mass is often introduced, ψ = U − ηT, to rewrite the entropy inequality as

\[ −\rho \left( \psi + \eta T \right) + \sigma : \nabla \dot{x} - \left( \frac{1}{T} \right) q \cdot \nabla T \geq 0. \]  

Eq. (9) in which \( \dot{x} \) is temperature \( \dot{\sigma} \) and \( \nabla T \) should be the set of independent state variables, where \( \dot{\sigma} \) and \( T \) are elastic strain and inelastic strain, respectively. However, physical evidence dictates that inelastic strain \( \dot{e} \) and its derivative \( \eta \) is the second deviatoric invariant of stress, \( \sigma \) where \( \nabla T \) is temperature and \( \nabla Q \) is the gas constant, \( \psi \) is the Helmholtz free energy in this manner, thermodynamic conjugate variables are determined by creep tests and describe microstructural behaviour in a somewhat heuristic manner. Although not usually shown in tensorial form, we express the power-law creep rate as a second rank tensor for consistency with the kinematic development,

\[ D^n = A\sigma^n \exp \left( \frac{-Q}{RT} \right) \varepsilon, \]  

where \( A \) is the power-law coefficient, \( n \) is the power-law exponent, \( \sigma \) is the second deviatoric invariant of stress, \( \varepsilon \) is the unit stress direction tensor, \( Q \) is the activation energy, \( R \) is the gas constant, \( T \) is temperature and \( D^n \) is the inelastic rate of deformation (creep rate). Garofalo (1963) developed a form of the inelastic rate of deformation to capture a broader range of inelastic behaviour (power law and power-law breakdown regimes) by using a hyperbolic sine,

\[ D^n = \frac{A \sinh \sigma R}{\sigma} \exp \left( \frac{-Q}{RT} \right) \varepsilon, \]  

where \( B \) is a drag stress constant. For the exponential creep regime others (Kohlstedt & Goetze 1974; Tsenn & Carter 1987) have proposed for the high stress, low temperature applications

\[ D^n = A \sigma^\alpha \exp \left( \frac{-Q}{RT} \right) \varepsilon, \]  

where \( Q \) is a function of the stress.

The power-law, Garofalo and exponential creep models inherently assume an isotropic material response. Therefore, no texture or kinematic (anisotropic) hardening arising from directional dislocation substructures is considered, although anisotropy is clearly apparent in mantle rocks under finite deformations at small and large size scales (cf. Kirby & Kronenberg 1987; Egydio-Silva & Mainprice 1999). Another shortcoming of previous models is that they include no elasticity. Fundamentally, elastic–plastic couplings can dictate the mechanical stress state and are important especially when non-monotonic loading sequences arise. Covey-Crump (1994) insightfully argues that the physical interpretation of the material constants for the power-law and Garofalo equations is lacking and therefore can lead to unreliable extrapolations outside of those cases tested. Finally and most importantly, these models do not include history effects from temperature, pressure or deformation, even through mechanical properties of all real materials are functions of their history.

### Internal state variable formulation

Thermodynamically based constitutive equations that are used to capture history effects are generally cast into two classes. In the first class using hereditary integrals, the present state of the material is described by the present values and past history of observable variables. The second class is based on the concept that the present state of the material depends only on the present values of observable variables and a set of ISVs. The second approach is more appropriate to solve a wide range of boundary value problems, and it is this form that we discuss in this paper.

The notion of internal state was introduced into thermodynamics by Onsager (1931a,b) and was applied to continuum mechanics by Eckart (1940, 1948). The ISV formulation is a means to capture the effects of a representative volume element and not all of the complex phenomena at the local level; hence, an ISV will macroscopically average in some fashion the details of the microscopic arrangement. In essence, the complete microstructure arrangement is unnecessary as long as the macroscale ISV representation is complete (cf. Kroner 1964). As a result, the ISV must be based on physically observed behaviour and constrained by the laws of thermodynamics (Coleman & Gurtin 1967). From the viewpoint of rational thermodynamics, the ISVs provide the additional information necessary for a rational description of the thermodynamic state of the material. From the viewpoint of thermodynamics of irreversible processes (Rice 1971), the ISVs provide the information required to describe neighbouring constrained equilibrium states.

To describe our ISV formulation we first decompose the Helmholtz free energy \( \psi \) into the free energy associated with thermoelastic strain, \( \psi^T \), and the free energy associated with inelasticity, \( \psi^e \), according to

\[ \psi = \psi^T (E^*, T, \sigma, \dot{\varepsilon}, \varepsilon, \rho) + \psi^e (T, V_1, V_2, \ldots, V_n), \]  

where the thermoelastic Green strain \( E^* \) and the absolute temperature, \( T \), are termed observable state variables; \( V_i \) are the ISVs representing the effects of microstructural rearrangement at lower length scales and \( n \) is the total number of ISVs. The ISVs are sometimes referred to as generalized displacements. Each of the ISVs can be represented by any order of tensor. Consequently, its thermodynamic conjugate must be the identical order. By defining the Helmholtz free energy in this manner, thermodynamic conjugate forces arise (Chaboche 1972; Germain et al. 1983; Krajcinovic 1983) with respect to their generalized displacements in the Bammann model as

\[ \dot{\sigma} = \rho \frac{\partial \psi}{\partial E^*}, \quad \eta = -\rho \frac{\partial \psi}{\partial T}, \quad \dot{b} = -\rho \frac{\partial \psi}{\partial \sigma}, \quad \kappa = -\rho \frac{\partial \psi}{\partial R}, \quad \sigma = 0, \]  

where \( \dot{\sigma} \) is the second Piola–Kirchhoff stress defined with respect to the stress-free configuration, \( \eta \) is the specific entropy, \( \dot{b} \) is the backstress tensor corresponding to kinematic hardening \( \sigma \), \( \kappa \) is the scalar isotropic stress corresponding to isotropic hardening \( R \). The density \( \rho \) corresponds to the stress-free configuration in which these relations are written. For clarity, we might add that the relationship between the thermodynamic driving force and generalized
displacement is analogous to the well-known Newtonian force and displacement relationship characteristic of mechanical systems.

The rate equations are generally written as objective rates \((\dot{\sigma}, \dot{\alpha})\) with indifference to the continuum frame of reference assuming a Jaumann rate in which the continuum spin is assumed to equal the elastic spin \((W = W^e)\). The ISVs are functions of the observable variables (temperature, stress state and rate of deformation). In general, the rate equations of generalized displacements, or thermodynamics fluxes, describing the rate of change may be written as independent equations for each ISV or as derivatives of a suitably chosen potential function arising from the hypothesis of generalized normality (Rice 1971). An advantage of assuming generalized normality, although somewhat restrictive, is unconditional satisfaction of the Kelvin inequality of the second law of thermodynamics (non-negative intrinsic dissipation), that is,

\[
\sigma : D^m - b : \dot{\alpha} - \kappa \cdot \dot{R} \geq 0. \tag{15}
\]

Although generalized normality is not an appropriate assumption for surface rocks, it is appropriate for mantle rocks since dilatational effects on non-normality assumptions are negligible because of high confining pressures. The laboratory experimental data examined within this paper considers only the deviatoric response of a material (Poirier 1985). Clearly, compressibility effects arise in the presence of extreme pressures, and the constitutive framework described in this paper can capture some of these effects; however, further research is needed to evaluate the pressure dependence on the inelastic flow stress.

The selection of the ISVs may, in principle, be somewhat arbitrary, but the kinematic hardening and isotropic hardening are physically motivated and strongly influence the history of the material model. The Bammann ISV model accounts for deviatoric inelastic deformation resulting from the presence of dislocations in crystallographic material.

The following set of equations can be used within finite volume or finite element codes. We express each set of equations in terms of the observable state variables and ISVs. The pertinent observable state variables are the following:

\[
\dot{\sigma} = \dot{\sigma} - W^e \sigma + W^e \dot{\sigma} = \lambda \text{tr} (D^e) \mathbf{I} + 2\mu D^e. \tag{16}
\]

\[
D^e = D - D^m. \tag{17}
\]
\( D^\text{in} = f(T) \sinh \left[ \frac{\left| \sigma' - \sigma \right| - R - Y(T)}{V(T)} \right] \frac{\left| \sigma' - \sigma \right|}{\left| \sigma' - \sigma \right|}, \)  
\( (18) \)

where the elastic Lame constants are denoted by \( \lambda \) and \( \mu \) (Bammann 1990). We express the deviatoric stress \( \sigma' \) in indicial notation as
\[ \sigma'_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{kk}. \] 
\( (19) \)

The elastic rate of deformation \( D^e \) results when the total deformation \( D \), which is determined from the boundary conditions, is subtracted from the flow rule as shown in eq. (17).

The independent variables for the inelastic rate of deformation are given in eq. (18) as the stress, temperature and ISVs. Eq. (18) is similar to power-law and Garofalo equations for creep except that the ISVs are now included. The deviatoric inelastic flow rule \( D^\text{in} \) is in a unified creep-plasticity form. The flow rule is a function of the temperature, the kinematic hardening ISV \( \alpha \), the isotropic hardening ISV \( R \) and the functions \( f(T) \), \( V(T) \) and \( Y(T) \), which are related to yielding with Arrhenius-type temperature dependence. The function \( Y(T) \) is the rate-independent yield stress. The function \( f(T) \) determines when the rate-dependence affects initial yielding. The function \( V(T) \) determines the magnitude of rate-dependence on yielding. These functions are determined from simple isothermal compression tests with different strain rates and temperatures (Bammann et al. 1995),
\[ V(T) = C_1 \exp \left( \frac{-C_2}{T} \right), \]
\[ Y(T) = \frac{1}{2} C_3 \exp \left( \frac{C_4}{T} \right) (1 + \tanh \left[ C_{19} \left( C_{20} - T \right) \right]), \]
\[ f(T) = C_5 \exp \left( \frac{-C_6}{T} \right). \] 
\( (20) \)

**Hardening-recovery ISV equations**

The hardening-recovery equations (Armstrong & Frederick 1966) can be thought of as the responses from thermodynamic forces arising from kinematic constraints of dislocation nucleation, motion and trapping. As such, kinematic and isotropic hardening equations are introduced in the Bammann model and have shown to cover a broad range of crystallographic materials in terms of hardening and recovery related to dislocation glide and climb. The kinematic hardening ISV \( \alpha \) reflects the effect of dislocation populations that induce anisotropic work hardening, and the isotropic hardening ISV

![Figure 3](https://example.com/figure3.png)

**Figure 3.** Comparison of model to experimental data (Chopra & Paterson 1981, 1984) for dry polycrystalline olivine.

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$R$ reflects the effect of the mean global dislocation density. Note that the inelastic rate of deformation $D^\text{in}$ in eq. (20) is directly influenced by these dislocation ISV equations. Because $D^\text{in}$ has an anisotropic component and an isotropic component, various deformation path histories can be captured such as the Bauschinger effect. As such, the hardening eqs (21) and (22) are cast in a hardening-recovery format that includes dynamic and static recovery (Bammann et al. 1993),

$$\dot{\varepsilon} = h(T)D^{\text{in}} - \left[ \frac{\sqrt{2}}{3} \varepsilon_{ij}(T) \| D^{\text{in}} \| + r_i(T) \right] \| \alpha \| \alpha,$$

(21)

### Table 1. Bammann material constants for wet and dry polycrystalline olivine aggregates, lherzolite, Carrara Marble and rock salt. Constants $C_6$–$C_{12}$ are omitted because they are zero.

| Material constants | Wet olivine aggregate | Dry olivine aggregate | Wet lherzolite | Dry lherzolite | Carrara marble | Rock salt |
|--------------------|-----------------------|-----------------------|---------------|---------------|----------------|----------|
| $C_1$ (MPa)        | 22                    | 72.1                  | 0.00001       | 0.00001       | 3.16           | 15.6     |
| $C_2$ (K)          | 0                     | 0                     | 0             | 0             | 0              | 0        |
| $C_3$ (MPa)        | 200                   | 50                    | 0.00001       | 0.00001       | 153            | 1.21     |
| $C_4$ (K)          | 1970                  | 1000                  | 0             | 0             | 0              | 0        |
| $C_5$ (s$^{-1}$)   | 0.00001               | 0.00001               | 0.00001       | 0.00001       | 0.00001        | 0.00001  |
| $C_{13}$ (MPa$^{-1}$) | 0                    | 0.083                 | 0             | 0             | 0.0525         | 129      |
| $C_{14}$ (K)       | 0                     | 0                     | 0             | 0             | 323            | 873      |
| $C_{15}$ (MPa)     | 120,000               | 19,400                | 103,000       | 103,000       | 4100           | 5420     |
| $C_{16}$ (K)       | 86.9                  | 0                     | 65.8          | 65.8          | 3.84           | 10.4     |
| $C_{17}$ (sMPa$^{-1}$) | 0.00000236             | 0.000001              | 4.15          | 0.484         | 0.000184       | 0        |
| $C_{18}$ (K)       | 0                     | 16,300                | 16,300        | 5870          | 0              | 0        |
| $C_{19}$ (1 K$^{-1}$) | 0.00383               | 0.0386                | 0             | 0             | 0              | 0        |
| $C_{20}$ (K)       | 1170                  | 1510                  | 0             | 0             | 0              | 0        |

Figure 4. Comparison of model results to experimental data (Carter & Ave Lallemant 1970) for wet lherzolite.
\[ \dot{R} = \sqrt{\frac{7}{3}} H(T) ||D^a|| - \left[ \sqrt{\frac{7}{3}} R_d(T) ||D^a|| + R_s(T) \right] R^2. \]  

(22)

The functions \( r_s(T) \) and \( R_s(T) \) are scalar in nature and describe the diffusion-controlled static or thermal recovery, while \( r_d(T) \) and \( R_d(T) \) are scalar functions describing dynamic recovery. Hence, the two main types of recovery that are exhibited by populations of dislocations within rocks (cf. Kirby 1983) are captured in the ISVs. The anisotropic hardening modulus is \( h(T) \), and the isotropic hardening modulus is \( H(T) \). The Bammann et al. (1995) parameters are given by

\[ r_d(T) = C_7 \exp\left(\frac{-C_8}{T}\right), \]  

(23)

\[ h(T) = C_9 - C_{10} T, \]  

(24)

\[ r_s(T) = C_{11} \exp\left(\frac{-C_{12}}{T}\right), \]  

(25)

\[ R_d(T) = C_{13} \exp\left(\frac{-C_{14}}{T}\right), \]  

(26)

\[ R_s(T) = C_{17} \exp\left(\frac{-C_{18}}{T}\right). \]  

(28)

Experiments

Experiments can be performed to uniquely determine the parameters for the Bammann model. To separate the kinematic and isotropic variables, non-monotonic testing is required to extract history effects from the evolving microstructure. Furthermore, non-monotonic and jump temperature and strain rate tests are needed to separate the effects of dynamic (strain-induced) and static (time-induced) recovery as functions of temperature. Microstructural features such as grain size, amount of water, amount of porosity/microcracking, chemistry, texture and melt fraction must be quantified on virgin polycrystalline material.

To determine the temperature and strain rate-dependent functions for yield, (eq. 20), the material should be tested at least for three different temperatures and three different strain rates under uniaxial compression. Hence, nine tests are desired. This would determine constants \( C_1 \)–\( C_6 \) and \( C_{19} \)–\( C_{20} \). If finite deformations are achieved,

\[ H(T) = C_{15} - C_{16} T, \]  

(27)

\[ R_s(T) = C_{17} \exp\left(\frac{-C_{18}}{T}\right). \]  

(28)

Figure 5. Comparison of model results to experimental data (Zimmerman & Kohlstedt 2004) for dry lherzolite.

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these nine tests can be used to determine the overall hardening and recovery constants at different strain rates and temperatures. In the absence of non-monotonic test data, one would have to assume no kinematic hardening and hence constants $C_7$–$C_{12}$ would be zero unless the Bauschinger effect was known by some other means. With no kinematic hardening assumed, the isotropic hardening moduli and dynamic recovery constants ($C_{13}$–$C_{16}$) can be determined. If monotonic and non-monotonic tests are conducted, then $C_7$–$C_{10}$ (kinematic hardening) and $C_{13}$–$C_{16}$ (isotropic hardening) can be uniquely quantified.

To determine the static recovery constants for the kinematic, $C_{10}$–$C_{12}$, and isotropic, $C_{17}$–$C_{18}$, variables, either creep or stress relaxation tests must be conducted under both monotonic and non-monotonic loading conditions for three different temperatures and three different creep rates.

Validation of the model constants can be performed by conducting jump-type tests. Once the model constants have been determined, then the model should be able to predict complex history tests, such as jump rate tests or jump temperature tests that include variations of stress states and stress state sequences.

RESULTS

Because the test data in the literature are not currently available to uniquely determine the Bammann model constants, non-unique constants can be determined using literature data for various rocks. In this section, we show the stress–strain correlations between the model and experimental data from wet and dry polycrystalline olivine, wet and dry polycrystalline lherzolite, polycrystalline Carrara Marble and rock salt. The temperature-dependent elastic moduli that were used here were obtained from the Simmons & Wang (1971). The Appendix describes the steps taken to first calibrate (determine the model constants) the model and to validate the model.

For olivine, the range of temperatures analysed ranged from $1273 \, K < T < 1573 \, K$ for wet and dry Anita Bay Dunite (Chopra & Paterson 1981, 1984). Creep data from the Chopra & Paterson (1981, 1984) to 3000 $K$ were used to determine the static recovery constants. Finally, strain rate jump tests from Chopra & Paterson (1984) and Karato et al. (1986) for wet and dry conditions were then used to validate the resulting model constants.

Experimental data for high confining pressure tests for initially isotropic wet and dry polycrystalline olivine were used to determine...
the Bammann ISV model constants. Below about 400 km olivine changes to other crystallo-
graphic phases, our constant values were appropriate only for olivine. Experimental stress–strain data for a
range of temperatures from 1273 K < \( T < \) 1573 K were analysed
to determine the yield functions, hardening moduli and dynamic
recovery functions for wet and dry Anita Bay and Aheim Bay Dunite
(Chopra & Paterson 1981, 1984) as shown in Figs 2 and 3. Dry Anita
Bay and Aheim Bay Dunite gave the same results, so Fig. 3 shows
the Aheim Bay experimental curves. Creep data from Chopra &
Paterson (1981, 1984) up to 1573 K were used to determine the
static recovery constants.

Fig. 2 shows the comparison of the model results to the ex-
perimental data for wet polycrystalline dunite from Aheim Bay
(900 \( \mu \)m grain size) and Anita Bay (100 \( \mu \)m grain size). Due to the
lack of non-monotonic test data for all materials considered in this
study constants \( C_7–C_{12} \) are set to zero, which implies anisotropic
hardening is not included. Anisotropic hardening can be added,
of course, when the appropriate experimental measurements are
performed. The data that are available reveal strong temperature de-
pendence. The experimental stress–strain data available in Chopra
& Paterson (1981, 1984) does not show different strain rates so
only the temperature dependence is considered for comparison pur-
poses. Table 1 shows the proposed material constants for wet and
dry olivine. Constants \( C_2, C_6, C_{12}, C_{18} \) and \( C_{20} \) are assumed to be
zero for this parameter determination.

Fig. 3 shows the comparison of the model results to the ex-
perimental data for dry monolithic polycrystalline dunite. From
the Chopra & Paterson (1981, 1984) data, two different tempera-
tures and two different strain rates were used for the compar-
ison. Chopra & Paterson (1981, 1984) noted that no grain size
effect was evident in material without water, so the Aheim Bay
and Anita Bay Dunite yielded essentially the same stress–strain
behaviour.

Fig. 4 shows a comparison of the ISV model to three different
temperatures and three different strain rates for wet lherzolite data
(Carter & Ave Lallemant 1970). Lherzolite is a fine-grained nearly
equiaxed granular rock comprising 60–70 per cent olivine, 20–
30 per cent enstatite, 5–10 per cent diopside and less than
1 per cent spinel. The temperatures ranged between 1123 K and
1333 K and the strain rates ranged between \( 7.8 \times 10^{-6} \) s\(^{-1} \) and
\( 7.8 \times 10^{-4} \) s\(^{-1} \). Although the experimental data in Fig. 4 go from
nearly perfect plasticity at the higher temperatures to almost bilinear
hardening in the lower temperatures, note that the model is robust
enough to capture such behaviour. Table 1 displays the constants
used for wet lherzolite. Fig. 5 displays the comparison of the ISV model to a temperature of 1473 K and a strain rate of $3.7 \times 10^{-5} \text{s}^{-1}$ for dry lherzolite (Zimmerman & Kohlstedt 2004). The constants for dry lherzolite are also listed in Table 1. Note the only material constant different between wet and dry lherzolite is $C_{17}$, which is the dynamic recovery constant.

Fig. 6 shows a comparison of the ISV model to three different temperatures and two different strain rates for Carrara Marble (Covey-Crump 1994). Again, the Bammann model captures the stress–strain behaviour fairly well for these ranges of temperatures and strain rates. The constants for Carrara Marble are listed in Table 1.

Fig. 7 shows the correlation of the model results to the experimental data for polycrystalline rock salt (Aubertin et al. 1991b). Rock salt (halite) was included among the materials we considered to show other non-silicate geomaterials can be treated successfully using the Bammann ISV model. Only two strain rates and temperatures were included in the comparison, and clearly more data would be desirable to capture a larger range of temperatures and strain rates. The constants for rock salt are listed in Table 1. Data do exist for single crystal halide from the work done by Lerner et al. (1979), Lerner & Kohlstedt (1981) and Stone et al. (2004); however, this was not included in this study due to the anisotropy seen in single crystals.

**DISCUSSION**

The previous section documents the Bammann ISV model’s ability to reproduce the experimental stress–strain data for high-temperature geomaterials. One way to validate the constants determined in the previous section is to run experimental jump-type tests and compare the model with the experiment. The jump tests give a better indication of the model’s ability to capture the material’s response, because the constants are not determined from them. The power-law model will also be plotted with the experimental data and Bammann ISV model. Both strain rate and temperature jump tests can be performed for comparison.

For the dry dunite, strain rate jump tests were included in the experiments performed by Chopra & Paterson (1984). Fig. 8 shows a direct comparison of the Bammann Paterson (1984) experiments of the Bammann ISV Model and the power-law model with the Chopra & Paterson (1984) experiments. Chopra & Paterson (1984) determined the power-law parameters, $A$, $Q$ and $n$, based on their experiments for dry olivine, which are shown in

![Comparison of Bammann ISV model and power-law model to experimental data (Chopra & Paterson 1984) for dry polycrystalline olivine for a varying strain rate history at 1573 K.](https://example.com/figure8.png)
Table 2. Power-law material parameters for dry olivine and lherzolite.

| Material constants | Dry olivine | Wet lherzolite |
|--------------------|-------------|----------------|
| $A$ (s$^{-1}$ MPa$^{-n}$) | 22840       | 803.8          |
| $Q$ (kJ mol$^{-1}$)  | 535         | 334.1          |
| $n$                | 3.6         | 2.3            |

Table 2. The power-law model does not consider the elastic portion of the stress–strain curve, so the stress is constant for each of the given strain rates. A given strain rate simply yields the unique steady state stress. The non-steady state or transient portion of the stress–strain behaviour is completely neglected. With good constants the power-law model can represent the steady state portion of the stress–strain curve for various strain rates and temperatures. Fig. 8 shows that the constants for dry olivine do an adequate job of capturing the steady state stresses for the given strain rates. However, the Bammann ISV model constants for dry olivine do a better job of capturing the steady state stress while accurately modelling the transient portion of the stress–strain behaviour.

The Bammann ISV model also has the added benefit of remembering its deformational history through its ISVs. For dry olivine, only the isotropic hardening state variable is being used as a history tracker. Physically speaking, isotropic hardening in the Bammann ISV is used to represent statistically the number of stored dislocations that cause a yield surface to expand in a uniform manner. When the stress reaches steady state at a constant strain rate and temperature, the statistical production (hardening) and destruction (recovery) of stored dislocations balance out and the isotropic hardening variable achieves a steady state. The value of the isotropic hardening variable is remembered when the strain rate changes. For the case shown in Fig. 8, the isotropic hardening variable eventually reaches a lower steady state value based on the lower strain rate, and the change in isotropic hardening decreases the stress.

Carter & Ave Lallemant (1970) performed both strain rate and temperature jump tests that can also be used to validate the model. Fig. 9 shows the comparison of the Bammann ISV model and the power-law model to a strain rate jump test from Carter & Ave Lallemant (1970) at a constant temperature of 1333 K. Wet lherzolite is a different test of a model’s robustness, because much of the experimental data done by Carter & Ave Lallemant (1970) do not reach steady state especially at lower temperatures, $T < 1123$ K, and the higher tested strain rates, rate $> 7.8 \times 10^{-5}$ s$^{-1}$. Carter & Ave Lallemant (1970) use the steady state values of stress for...
Figure 10. Comparison of Bammann ISV model and power-law model to experimental data (Carter & Ave Lallemant 1970) for wet lherzolite for a varying temperature history at a constant strain rate of $7.8 \times 10^{-5} \, \text{s}^{-1}$.

Fig. 10 compares the Bammann ISV model and the power-law model to the experimental temperature jump test by Carter & Ave Lallemant (1970) at a constant strain rate of $7.8 \times 10^{-5} \, \text{s}^{-1}$. The same power-law constants for wet lherzolite are used as shown in Table 2. The strain rate used for the temperature jump test is right on the borderline in terms of the usefulness of the power-law constants. The lower temperatures, 1123 K and 1223 K, are extremely overestimated as shown in Fig. 10. Because the values for the power-law stresses are so high, 5.1 GPa and 1.43 GPa at 1123 K and 1223 K, respectively, we do not show them in the plot because they are so far off the chart. The steady state values for stress are closer for the 1333 K and 1443 K case but still are overestimated. Again the Bammann ISV model constants prove to be far superior for this temperature jump test.

Figs 8 and 9 show that for the given temperatures and strain rates the Bammann ISV model can accurately capture the transient effects that the power law could never capture. The Bammann ISV model even does a good job of extrapolating past the fitted range as shown by the 1443 K temperature in Fig. 10. The 1443 K temperature stress–strain behaviour was not originally fit as observed in Fig. 4. Care must be taken to not use too much extrapolation beyond the experimental data, because different physical mechanisms could be involved, which the model may not capture with a given set of constants. The larger the range of temperatures and strain rates for which the Bammann model constants are fit, the greater is range over which the model can usefully be applied.

CONCLUSIONS

The Bammann inelasticity ISV elastic-viscoplastic model appears to offer important advantages for treating the time-dependent phenomenological behaviour of upper-mantle silicates and non-silicates. This model includes temperature dependence, strain...
rate dependence, isotropic hardening and anisotropic hardening effects for polycrystalline rocks. It also captures history dependence arising from strain rate and/or temperature changes, a feature that power-law models cannot achieve. We obtain the required model constants for several rocks from experimental data available in the current literature. The constants reported in this paper include those for wet olivine, dry olivine, wet ilherzolite, dry ilherzolite, marble and rock salt. Finally, although the material constants determined in this initial survey were non-unique relative to the experimental data currently available, we present an experimental matrix by which unique constants can be obtained.

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REFERENCES

Armstrong, P.J. & Frederick, C.O., 1966. A mathematical representation of the multiaxial bauschinger effect, CEGB Report RD/B/N, pp. 731.

Aubertin, M., Gill, D.E. & Ladanyi, B., 1991a. An internal variable model for the creep of rocksalt, Rock Mech. Rock Eng., 24, 81–97.

Aubertin, M., Gill, D.E. & Ladanyi, B., 1991b. A unified viscoplastic model for the inelastic flow of alkali halides, Mech. Mater., 11, 63–82.

Bammann, D.J., 1988. Modelling the large strain-high temperature response of metals, in Modelling and Control of Casting and Welding Processes IV, pp. 329–338, eds Giami, A.F. & Abbaschian, G.J., TMS Publications, Warrendale, PA.

Bammann, D.J., 1990. Modelling temperature and strain rate dependent large deformations of metals, Appl. Mech. Res., 1, 312–318.

Bammann, D.J. & Afantis, E.C., 1989. A damage model for ductile metals, Nucl. Eng. Des., 116, 355–362.

Bammann, D.J., Chiesa, M.L., Horstemeyer, M.F. & Weingarten, L.I., 1993. Failure in ductile materials using finite element methods, in Structural Crushworthiness and Failure, eds Jones, N. & Weirzbicki, T., Elsevier Applied Science, London.

Bammann, D.J., Chiesa, M.L. & Johnson, G.C., 1995. Modelling large deformation anisotropy in sheet metal forming, in Simulation of Materials Processing: Theory, Methods, and Applications, eds Shen & Dawson, Balkema, Rotterdam.

Baumgardner, J.R., 1985. Three-dimensional treatment of convective flow in the Earth’s mantle, J. Stat. Phys., 39, 501–511.

Bilby, B.A., Bullough R., & Smith, E., 1955. Continuous distributions variables, J. Chem. Phys., 25(6), 1219–1244.

Kohlstedt, D.L. & Goetze, C., 1974. Low-stress high-temperature creep in olivine single crystals, J. geophys. Res., 79, 2045–2051.

Krajcinovic, D., 1983. Constitutive equations for damaging materials, J. appl. Mech., 50, 355–360.

Kroner, E., 1964. Course 3: Continuum Theory of Defects, Institute für Theoretische und Angewandte Physik der Universität Stuttgart and Max Planck Institut für Metallforschung, Stuttgart, Germany.

Lee, E.H. & Liu, D.T., 1967. Finite strain elastic-plastic theory with application to plane-wave analysis, J. appl. Physics, 38, 391–408.

Lerner, I. & Kohlstedt, D.L., 1981. Effect of γ radiation on plastic flow of NaCl, J. Am. Ceram. Soc., 64, 105–108.

Lerner, I., Chiang, S.-W. & Kohlstedt, D.L., 1979. Load relaxation studies of four alkali halides, Acta Metallurgica, 27(7), 1187–1196.

Nicolas, A. & Poirier, J.P., 1976. Crystal Plasticity and Solid State Flow in Metamorphic Rocks, Wiley, New York.

Onat, E.T. & Fardisheh, F., 1972. Representation of creep of metals, ORNL Report 4783.

Onsager, L., 1931a. Reciprocal relations in irreversible processes I, Phys. Rev., 37, 405–426, doi:10.1103/PhysRev.37.405.

Onsager, L., 1931b. Reciprocal relations in irreversible processes II, Phys. Rev., 38, 2265–2279, doi:10.1103/PhysRev.38.2265.

Rice, J.R., 1971. Inelastic constitutive relations for solids: an internal-variable theory and its application to metal plasticity, J. Mech. Phys. Solids, 9, 433–455.

Simmons, G. & Wang, H. 1971. Single Crystal Elastic Constants and Calculated Aggregate Properties: A Handbook, MIT Press, Cambridge, MA.

Stone, D.S., Pookphol, T. & Cooper, R.F., 2004. Similarity and scaling in creep and load relaxation of single-crystal halite (NaCl), J. geophys. Res., 109, B12201, 1–17.

Tsehn, M. C. & Carter, N. L., 1987. Upper limits of power law creep of rocks, Tectonophysics, 136, 1–26.
**APPENDIX: STEPS IN FITTING THE BAMMANN INELASTICITY ISV MODEL**

Below is a summary of determining the Bammann inelasticity ISV model constants. Although there are many ways to determine the constants, this way has been shown to be efficient. In all of the steps, if a constant value could be zero and the model still could be fit in the optimization procedure, then that constant would stay at zero. For example, if there are no experimental data for differing temperatures, the even constants ($C_2$, $C_4$, etc.) will be zero.

**Model calibration**

Step 1: Put the stress–strain data into a true stress-true strain format and separate the highest and lowest temperature and strain rate data. In the end you will try to match all of the data but at first you will examine the temperature difference and then the strain rate difference.

**Determination of yield parameters**

Step 2: Determine the yield related parameters first. Retrieve stress–strain data with two extreme temperatures at one applied strain rate. Solve for $C_3$ and $C_4$ first. If there are data near melt, you will also need to solve for $C_{19}$ and $C_{20}$ as well otherwise, just $C_3$ and $C_4$ will get most of the data range.

Step 3: Determine the strain rate sensitivity of the yield-related parameters. Retrieve stress–strain data with two extreme strain rates at one temperature. Solve for $C_1$ and $C_5$. Retrieve the stress–strain data with the two extreme strain rates but retrieve also the two extreme temperatures now and solve for $C_2$ and $C_6$.

Step 4: With $C_1$–$C_6$ and $C_{19}$–$C_{20}$ determined, use a fitting routine to optimize these constants to the data that it was originally fit to. Some of the values might change because this multi-objective optimization will weigh everything equally, whereas the previous steps did not include weighting from the other constants because they were independently determined.

**Determination of hardening parameters**

Step 5: Now that the yield parameters are determined, we must determine the hardening parameters. First, we start with the isotropic hardening and then the kinematic (anisotropic) hardening. Retrieve the stress–strain data with two temperatures and one applied strain rate. Solve for the isotropic hardening constants first $C_{11}$–$C_{16}$. If there is not a good fit, then one can solve for the kinematic hardening constants next $C_7$–$C_{10}$. Usually, the kinematic hardening values are determined from the load–unload tests as it was originally designed to capture the Bauschinger effect. In the absence of those data, one can use the kinematic hardening to capture the short range hardening transient and isotropic hardening can be used to capture the long range transient.

Step 6: Retrieve the two extreme strain rates with the two extreme temperature regimes. Optimize $C_7$–$C_{10}$ and $C_{11}$–$C_{16}$ considering all of these constants together.

Step 7: If there are creep data available, then retrieve those data and determine the values of $C_{11}$–$C_{12}$ and $C_{17}$–$C_{18}$ for the kinematic and isotropic static recovery terms.

Step 8: Retrieve all of the stress–strain data at the different temperatures and strain rates and optimize $C_7$–$C_{16}$ considering all of these constants together.

**Model validation**

Step 9: Retrieve jump rate stress–strain data and compare the model calibration constants prediction to the experimental data.

Step 10: Retrieve jump temperature stress–strain data and compare the model calibration constants prediction to the experimental data.