Quantum symmetric spaces

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Abstract

Let $G$ be a semisimple Lie group, $\mathfrak{g}$ its Lie algebra. For any symmetric space $M$ over $G$ we construct a new (deformed) multiplication in the space $A$ of smooth functions on $M$. This multiplication is invariant under the action of the Drinfeld–Jimbo quantum group $\mathcal{U}_h\mathfrak{g}$ and is commutative with respect to an involutive operator $\tilde{S} : A \otimes A \to A \otimes A$. Such a multiplication is unique.

Let $M$ be a kählerian symmetric space with the canonical Poisson structure. Then we construct a $\mathcal{U}_h\mathfrak{g}$-invariant multiplication in $A$ which depends on two parameters and is a quantization of that structure.

1 Introduction

Let $G$ be a semisimple Lie group, $\mathfrak{g}$ its Lie algebra, and $r \in \wedge^2 \mathfrak{g}$ the Drinfeld–Jimbo classical $R$-matrix (see Section 2). Suppose $H$ is a closed subgroup of $G$ and $M = G/H$. Then the action of $G$ on $M$ defines a mapping $\rho : \mathfrak{g} \to \text{Vect}(M)$. So, the element $(\rho \otimes \rho)(r)$ induces a bivector field on $M$ which determines a bracket (biderivation) $\{\cdot, \cdot\}$ on the algebra $C^\infty(M)$ of smooth functions on $M$. In some cases this will be satisfy the Jacobi identity and thus define a Poisson bracket which we will call an $R$-matrix Poisson bracket. It is easy to see that the bracket may be degenerate at some points of $M$. The natural question arises whether that bracket can be quantized.

The first case when $\{\cdot, \cdot\}$ is a Poisson bracket is when the Lie algebra of $H$ contains a maximal nilpotent subalgebra. In \cite{DGM} it is proven that
in this case there exists a quantization of \{\cdot, \cdot\}, i.e., there is an associative multiplication \(\mu_h\) in \(C^\infty(M)\) of the form

\[\mu_h = m + h\{\cdot, \cdot\} + \sum_{i=2}^{\infty} h^i \mu_i(\cdot, \cdot) = m + h\{\cdot, \cdot\} + o(h),\]

where \(m\) is the usual multiplication in \(C^\infty(M)\) and \(\mu_i(\cdot, \cdot)\) are bidifferential operators. Moreover, this multiplication will be invariant under action of the Drinfeld–Jimbo quantum group \(U_{h\mathfrak{g}}\). This means that \(\mu_h\) satisfies the condition

\[x\mu_h(a, b) = \mu_h(\check{\Delta}_h(x))(a \otimes b),\]

where \(a, b \in C^\infty(M), x \in U_{h\mathfrak{g}},\) and \(\check{\Delta}_h\) is the comultiplication in \(U_{h\mathfrak{g}}\) (here we use the presentation of \(U_{h\mathfrak{g}}\) with multiplication as in \(U_{\mathfrak{g}}[[h]]\), see Section 3). In \([DG1]\) it is shown that in such a way one can obtain the \(U_{h\mathfrak{g}}\)-invariant quantization of the algebra of holomorphic sections of line bundles over the flag manifold of \(G\).

In the present paper we consider the case when \(M\) is a symmetric space. Our first result is that in this case \(\{\cdot, \cdot\}\) will also be a Poisson bracket and there is a \(U_{h\mathfrak{g}}\)-invariant quantization of this bracket. Moreover, such a quantization is unique up to isomorphism.

Suppose now that \(M\) is equipped with a \(G\)-invariant Poisson bracket \(\{\cdot, \cdot\}^{inv}\). Our second result is that in this case there exists a simultaneous \(U_{h\mathfrak{g}}\)-invariant quantization, \(\mu_{\nu,h}\), of both these brackets in the form

\[\mu_{\nu,h} = m + \nu\{\cdot, \cdot\}^{inv} + h\{\cdot, \cdot\} + o(\nu, h),\]

where \(o(\nu, h)\) includes all terms of total powers \(\geq 2\) in \(\nu, h\) with bidifferential operators as coefficients. This is the case, for example, when \(M\) is a kählerian symmetric space. Then \(\{\cdot, \cdot\}^{inv}\) coincides with the Kirillov bracket which is dual to the Kähler form on \(M\). This bracket is nondegenerate, and Melotte \([Mc]\) has proved and one can prove that there exists a deformation quantization of the Kirillov bracket, \(\mu_\nu\), that is invariant under \(G\) and \(U_{\mathfrak{g}}\). The existence of such a quantization can be also proven using the methods of the present paper. Thus, one may consider the multiplication \(\mu_{\nu,h}\) as such a quantization of the Kirillov bracket which is invariant under the action of the quantum group \(U_{h\mathfrak{g}}\).

Note that the Kirillov bracket is also generated by \(r\) in the following way. Let \(\{\cdot, \cdot\}'\) be a bracket on \(C^\infty(G)\) generated by the left-invariant extension
of \( r \) as a bivector field on \( G \). Using the projection \( G \to G/H = M \) we can consider \( C^\infty(M) \) as a subalgebra of \( C^\infty(G) \). One can check that \( C^\infty(M) \) is invariant under \( \{\cdot,\cdot\}' \) if \( H \) is a Levi subgroup. For such \( H \) the difference \( \{\cdot,\cdot\} - \{\cdot,\cdot\}' \) gives a Poisson bracket on \( M \), the so-called Sklyanin–Drinfeld Poisson bracket. The quantization of this Poisson bracket is given in [DG2].

In case \( M \) is a symmetric space the bracket \( \{\cdot,\cdot\}' \) will be a Poisson one itself and coincides with the Kirillov bracket \( \{\cdot,\cdot\}_{\text{inv}} \) (see [DG2]). In [GP] there is given a classification of all orbits in the coadjoint representation of \( G \) on which \( r \) induces the Poisson bracket.

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## 2 \( R \)-matrix Poisson brackets on symmetric spaces

Let \( \mathfrak{g} \) be a simple Lie algebra over the field of complex numbers \( \mathbb{C} \). Fix a Cartan decomposition of \( \mathfrak{g} \) with corresponding root system \( \Omega \) and choice of positive roots, \( \Omega^+ \). We consider the Drinfeld–Jimbo classical \( R \)-matrix

\[
r = \sum_{\alpha \in \Omega^+} X_\alpha \wedge X_{-\alpha} \in \wedge^2 \mathfrak{g},
\]

where \( X_\alpha \) are the elements from the Cartan–Chevalley basis of \( \mathfrak{g} \) corresponding to \( \Omega \), and \( \Omega^+ \) denotes the set of positive roots. We shall use the notation \( r = r_1 \otimes r_2 \) as a shorthand for \( \sum_i r_{1i} \otimes r_{2i} \) in denoting this \( R \)-matrix. The same convention of suppressing the summation sign and the index of summation will be used throughout the paper.

This \( r \) satisfies the so-called modified classical Yang-Baxter equation which means that the Schouten bracket of \( r \) with itself is equal to an invariant element \( \varphi \in \wedge^3 \mathfrak{g} \):

\[
[r, r]_{\text{sch}} = [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = \varphi. \tag{1}
\]

Here we use the usual notation: \( r^{12} = r_1 \otimes r_2 \otimes 1 \), \( r^{13} = r_1 \otimes 1 \otimes r_2 \), and so on. Note that any invariant element in \( \wedge^3 \mathfrak{g} \) is dual up to a multiple to the three-form \( (x, [y, z]) \) on \( \mathfrak{g} \), where \( (\cdot, \cdot) \) denotes the Killing form. Therefore, \( \varphi \) will be also invariant under all automorphisms of the Lie algebra \( \mathfrak{g} \).
The $R$-matrix $r$ obviously satisfies the following conditions: a) it is invariant under the Cartan subalgebra $c$, and b) $\theta r = -r$ where $\theta$ is the Cartan involution of $\mathfrak{g}$, $\theta X_\alpha = -X_{-\alpha}$, $\theta|_c = -1$. These conditions determine $r$ uniquely up to a multiple (see [SS] §11.4).

In case $\mathfrak{g}$ is a semisimple Lie algebra with a Cartan decomposition, let $r \in \wedge^2 \mathfrak{g}$ satisfy the equation (1) for some invariant $\varphi \in \wedge^3 \mathfrak{g}$ and the previous conditions a) and b). Then $r$ will be a linear combination of the Drinfeld–Jimbo $R$-matrices on the simple components of $\mathfrak{g}$. We will also call such $r$ the Drinfeld–Jimbo $R$-matrix.

Let $\mathfrak{g}_R$ be a real form of a semisimple (complex) Lie algebra $\mathfrak{g}$, and $G$ a connected Lie group with $\mathfrak{g}_R$ as its Lie algebra. Suppose $\sigma$ is an involutive automorphism of $G$, and $H$ is a subgroup of $G$ such that $G^\sigma_0 \subset H \subset G^\sigma$, where $G^\sigma$ is the set of fixed points of $\sigma$ and $G^\sigma_0$ is the identity component of $G^\sigma$. The automorphism $\sigma$ induces an automorphism of the both Lie algebras $\mathfrak{g}_R$ and $\mathfrak{g}$ which we will also denote by $\sigma$. Thus, the space of left cosets $M = G/H$ is a symmetric space (see [He]). We denote by $o$ the image of unity by the natural projection $G \to M$. The mapping $\tau : M \to M, gH \mapsto \sigma(g)H$, is well defined and has $o$ as an isolated fixed point, therefore, the differential $\dot{\tau} : T_o \to T_o$ of $\tau$ at the point $o$ multiplies the vectors of the tangent space $T_o$ by $(-1)$.

The action of $G$ on $M$ defines the mapping of $\mathfrak{g}_R$ into the Lie algebra of real vector fields on $M$, $\rho : \mathfrak{g}_R \to \text{Vect}_R(M)$, that extends to a mapping $\rho : \mathfrak{g} \to \text{Vect}(M)$ of the complexification of $\mathfrak{g}_R$ into the Lie algebra of complex vector fields $\text{Vect}(M)$ on $M$.

The mapping $\rho$ induces on $M$ a skew-symmetric bivector field in the following way. The element $\rho(r_1) \otimes \rho(r_2) \in \wedge^2 \text{Vect}(M)$ (tensor product over $\mathbb{C}$ not $C^\infty(M)$) generates a bracket on the algebra $C^\infty(M)$ of smooth complex-valued functions on $M$, $\{f, g\} = \rho(r_1)f \rho(r_2)g$, where $f, g \in C^\infty(M)$ and $\rho(r_1)f$ is the derivative of $f$ along the vector field $\rho(r_1)$. It is obvious that this defines a skew-symmetric biderivation, therefore it is defined by a bivector field, i.e., a section of $\wedge^2$ of the tangent bundle, which we denote by $\rho(r)$.

From now on we will suppose that the invariant element $\varphi \in \wedge^3 \mathfrak{g}$ is invariant under $\sigma$ as well. In case $\mathfrak{g}$ is a simple Lie algebra this will be satisfied automatically.

**Proposition 2.1** The bracket $\{\cdot, \cdot\}$ is a Poisson bracket on $M$. 
Since $\rho(\varphi)$ is a $G$-invariant three-vector field on $M$, therefore it is defined by its value at the point $o$, $\rho(\varphi)_o$. Since $\varphi$ is $\sigma$-invariant, $\rho(\varphi)$ has to be $\tau$-invariant, which implies that $\dot{\tau}\rho(\varphi)_o = \rho(\varphi)_o$. But the operator $\dot{\tau}$ acts on $T_o$ by multiplying by $(-1)$, so that $\dot{\tau}\rho(\varphi)_o = -\rho(\varphi)_o$. Therefore, $\rho(\varphi) = 0$. This means that the Schouten bracket $[\rho(r), \rho(r)]$ is equal to zero, which is equivalent to the bracket $\{\cdot, \cdot\}$ satisfying the Jacobi identity.

We will call the bracket $\{\cdot, \cdot\}$ an $R$-matrix Poisson bracket. Note that this bracket is not $g$-invariant and may be degenerate in some points of $M$.

Suppose now that there is on $M$ a $g$-invariant Poisson bracket $\{\cdot, \cdot\}_{inv}$. The case will be if the Poisson structure on $M$ is dual to a $G$ invariant symplectic form, as in the case of a kählerian symmetric space. For example, if $M$ is a hermitian symmetric space the kählerian form is the imaginary part of the hermitian form on $M$.

**Proposition 2.2** The $R$-matrix and any invariant Poisson brackets are compatible, i.e. for any $a, b \in C$ the bracket $a\{\cdot, \cdot\} + b\{\cdot, \cdot\}_{inv}$ is a Poisson one.

**Proof** The straightforward computation following from the fact that $\{\cdot, \cdot\}$ is expressed in terms of vector fields coming from $g$ and $\{\cdot, \cdot\}_{inv}$ is $G$ invariant. (see [DGM]).

**3 Three monoidal categories**

We recall that a monoidal category is a triple $(\mathcal{C}, \otimes, \phi)$ where $\mathcal{C}$ is a category equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called a tensor product functor, and a functorial isomorphism $\phi_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ called associativity constraint, which satisfies the pentagon identity (omitting subscripts on $\phi$), i.e. the diagram

\[
\begin{array}{ccc}
((X \otimes Y) \otimes Z) \otimes U & \xrightarrow{\phi} & (X \otimes Y) \otimes (Z \otimes U) \\
\phi \otimes \text{id} \downarrow & & \downarrow \phi \\
(X \otimes (Y \otimes Z)) \otimes U & \xrightarrow{\phi} & X \otimes ((Y \otimes Z) \otimes U)
\end{array}
\]

\[
\overset{id \otimes \phi}{\uparrow}
\]

is commutative.
If \((\tilde{C}, \otimes, \tilde{\phi})\) is another monoidal category, then a morphism from \(C\) to \(\tilde{C}\) is given by a pair \((\alpha, \beta)\) where \(\alpha : C \to \tilde{C}\) is a functor and \(\beta : \alpha(X \otimes Y) \to \alpha(X) \otimes \alpha(Y)\) is a functorial isomorphism such that the diagram

\[
\begin{array}{ccc}
\alpha((X \otimes Y) \otimes Z) & \xrightarrow{\beta} & \alpha(X \otimes Y) \otimes \alpha(Z) \\
\alpha(\phi) & \downarrow & \downarrow \phi \\
\alpha(X \otimes (Y \otimes Z)) & \xrightarrow{\beta} & \alpha(X) \otimes \alpha(Y) \otimes \alpha(Z)
\end{array}
\]

is commutative.

The morphism \((\alpha, \beta)\) of monoidal categories allow us to transfer additional structures given on objects of \(C\) to objects from \(\tilde{C}\). For example, let \(X \in \text{Ob}(C)\). A morphism will be called \(C\)-associative (or \(\phi\)-associative) if we have the following equality of morphisms of \((X \otimes X) \otimes X \to X\)

\[
\mu(\mu \otimes \mu) = \mu(\mu \otimes \mu)\phi.
\]

Then, for \(\alpha(X) \in \text{Ob}(\tilde{C})\) the naturally defined morphism \(\alpha(\mu) : \alpha(X) \otimes \alpha(X) \to \alpha(X)\) will be \(\tilde{C}\)-associative (\(\tilde{\phi}\)-associative).

Let \(A\) be a commutative algebra with unit, \(B\) a unitary \(A\)-algebra. The category of representations of \(B\) in \(A\)-modules, i.e. the category of \(B\)-modules, will be a monoidal category if the algebra \(B\) is equipped with additional structures \([D]\). Suppose we have an algebra morphism, \(\Delta : B \to B \otimes_A B\), which is called a comultiplication, and \(\Phi \in B \otimes^3\) is an invertible element such that \(\Delta\) and \(\Phi\) satisfy the conditions

\[
(id \otimes \Delta)(\Phi) = (1 \otimes \Phi) \cdot (id \otimes \Delta)(\Phi) \cdot (\Phi \otimes 1), \quad (id \otimes \Delta)(\Phi) = (1 \otimes \Phi) \cdot (\Phi \otimes \Delta)(\Phi), \quad b \in B,
\]

(3)

(4)

We define a tensor product functor which we will denote \(\otimes_C\) for \(C\) the category of \(B\) modules or simply \(\otimes\) when there can be no confusion in the following way: given \(B\)-modules \(M, N M \otimes_C N = M \otimes_A N\) as an \(A\)-module with the action of \(B\) defined as \(b(m \otimes n) = b_1 m \otimes b_2 n\) where \(b_1 \otimes b_2 = \Delta(b)\).

The element \(\Phi\) gives an associativity constraint \(\Phi : (M \otimes N) \otimes P \to M \otimes (N \otimes P), (m \otimes n) \otimes p \mapsto \Phi_1 m \otimes (\Phi_2 n \otimes \Phi_3 p)\), where \(\Phi_1 \otimes \Phi_2 \otimes \Phi_3 = \Phi\).

By virtue of (3) \(\Phi\) induces an isomorphism of \(B\)-modules, and by virtue of (4) the pentagonal identity (1) holds. We call the triple \((B, \Delta, \Phi)\) a Drinfeld...
algebra. Thus, the category $\mathcal{C}$ of $B$-modules for $B$ a Drinfeld algebra becomes a monoidal category. When it becomes necessary to be more explicit we shall denote $\mathcal{C}(B, \Delta, \Phi)$.

Let $(B, \Delta, \Phi)$ be a Drinfeld algebra and $F \in B^\otimes 2$ an invertible element. Put

$$\tilde{\Delta}(b) = F\Delta(b)F^{-1}, \ b \in B,$$

and

$$\tilde{\Phi} = (1 \otimes F) \cdot (id \otimes \Delta)(F) \cdot \Phi \cdot (\Delta \otimes id)(F^{-1}) \cdot (F \otimes 1)^{-1}. \ (6)$$

Then $\tilde{\Delta}$ and $\tilde{\Phi}$ satisfy (3) and (4), therefore the triple $(B, \tilde{\Delta}, \tilde{\Phi})$ also becomes a Drinfeld algebra which generates the corresponding monoidal category $\tilde{\mathcal{C}}(B, \tilde{\Delta}, \tilde{\Phi})$. Note that the categories $\mathcal{C}$ and $\tilde{\mathcal{C}}$ consist of the same objects as $B$-modules, and the tensor products of two objects are isomorphic as $A$-modules. The categories $\mathcal{C}$ and $\tilde{\mathcal{C}}$ will be equivalent. The equivalence $\mathcal{C} \rightarrow \tilde{\mathcal{C}}$ is given by the pair $(\alpha, \beta) = (\text{Id}, F)$, where $\text{Id}: \mathcal{C} \rightarrow \tilde{\mathcal{C}}$ is the identity functor of the categories (considered without the monoidal structures, but only as categories of $B$-modules), and $F: M \otimes_A N \rightarrow M \otimes_{\tilde{\mathcal{C}}} N$ is defined by $m \otimes n \mapsto F_1 m \otimes F_2 n$ where $F_1 \otimes F_2 = F$. By virtue of (5) $F$ gives an isomorphism of $B$-modules, and (6) implies the commutativity of diagram (2).

Assume $M$ is a $B$-module with a multiplication $\mu : M \otimes_A M \rightarrow M$ which is a homomorphism of $A$-modules. We say that $\mu$ is invariant with respect to $B$ and $\Delta$ if it is a morphism in the monoidal category $\mathcal{C}(B, \Delta, \Phi)$. This means that

$$b\mu(x \otimes y) = \mu \Delta(b)(x \otimes y) \ \text{for} \ b \in B, \ x, y \in M. \ (7)$$

When $\mu$ is $\mathcal{C}$-associative, $\mathcal{C} = \mathcal{C}(B, \Delta, \Phi)$, then we shall also say that $\mu$ is a $\Phi$-associative multiplication, i.e. we have the equality

$$\mu(\mu \otimes id)(x \otimes y \otimes z) = \mu(id \otimes \mu)\Phi(x \otimes y \otimes z) \ \text{for} \ x, y, z \in M. \ (8)$$

Since the pair $(\text{Id}, F)$ realizes an equivalence of the categories, the multiplication $\tilde{\mu} = \mu F^{-1} : M \otimes_A M \rightarrow M$ will be $\tilde{\Phi}$-associative and invariant in the category $\tilde{\mathcal{C}}$.

Now we return to the situation of Section 2. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ with a fixed Cartan decomposition and an involution $\sigma$. Let $U_\mathfrak{g}$ be the universal enveloping algebra with the usual comultiplication
\[ \Delta : U_g \to U_g \otimes U_g \] generated as a morphism of algebras by the equations 
\[ \Delta(x) = 1 \otimes x + x \otimes 1 \] for \( x \in g \) and extended multiplicatively.

We will deal with the category \( \text{Rep}(U_g[[h]]) \). Objects of this category are representations of \( U_g[[h]] \) in \( \mathbb{C}[[h]] \)-modules of the form \( E[[h]] \) for some vector space \( E \). We denote here by \( E[[h]] \) the set of formal power series in an indeterminate \( h \) with coefficients in \( E \). By tensor product of two \( \mathbb{C}[[h]] \)-modules we mean the completed tensor product in \( h \)-adic topology, i.e. for two vector spaces \( E_1 \) and \( E_2 \) we have \( E_1[[h]] \otimes E_2[[h]] = (E_1 \otimes_{\mathbb{C}} E_2)[[h]] \).

As usual, morphisms in this category are morphisms of \( \mathbb{C}[[h]] \)-modules that commute with the action of \( U_g[[h]] \). A representation of \( U_g[[h]] \) on \( E[[h]] \) can be given by a power series \( R_h = R_0 + h R_1 + \cdots + h^n R_n + \cdots \in \text{End}(E)[[h]] \)
where \( R_0 \) is a \( \mathbb{C} \) representation of \( U_g \) in \( E \) and \( R_i \in \text{Hom}_\mathbb{C}(U(g), \text{End}(E)) \).

Hence, \( R_h \) may be considered as a deformation of \( R_0 \). By misuse of language, we will say that \( R_h \) is a representation of \( U_g \) in the space \( E[[h]] \). The functor \( \otimes_{\mathbb{C}} \mathbb{C}[[h]] \) sending a representation of \( U_g \) to a representation of \( U_g[[h]] \) defines an equivalence of categories between the category \( \text{Rep}(U_g) \) of representations of \( U_g \) and the category \( \text{Rep}(U_g[[h]]) \) so we will shorten notation denote the latter by \( R(U_g) \) as well.

Since the comultiplication \( \Delta \) on \( U_g \) gives rise to a comultiplication on \( U_g[[h]] \) and is coassociative, the triple \( (U_g[[h]], \Delta, 1 \otimes 1 \otimes 1 = 1) \) becomes a Drinfeld algebra and the category \( \text{Rep}(U_g) \) turns into a monoidal category \( \text{Rep}(U_g, \Delta, 1) \) with the identity associativity constraint. This is the classical way to introduce a monoidal structure in the category \( \text{Rep}(U_g) \). Another possibility arises from the theory of quantum groups due to Drinfeld. In the following proposition we suppose that the element \( \varphi = [r, r]_{sch} \) is invariant under the involution \( \sigma \).

**Proposition 3.1**

1. There is an invariant element \( \Phi_h \in U_g[[h]] \otimes U_g[[h]] \) of the form \( \Phi_h = 1 \otimes 1 \otimes 1 + h^2 \varphi + \cdots \) satisfying the following properties:
   a) it depends on \( h^2 \), i.e. \( \Phi_h = \Phi_{-h} \);
   b) it satisfies the equations (3) and (4) with the usual \( \Delta \);
   c) \( \Phi_h^{-1} = \Phi_h^{s21} \), where \( \Phi_h^{s21} = \Phi_3 \otimes \Phi_2 \otimes \Phi_1 \) for \( \Phi = \Phi_1 \otimes \Phi_2 \otimes \Phi_3 \);
   d) \( \Phi_h \) is invariant under the Cartan involution \( \theta \) and \( \sigma \);
   e) \( \Phi_h \Phi_s^h = 1 \), where \( s \) is the antipode, i.e., an antiinvolution of \( U_g \) such that \( s(x) = -x \) for \( x \in g \), and \( \Phi_s^h = (s \otimes s \otimes s)(\Phi_h) \).

2. There is an element \( F_h \in U_g[[h]] \otimes U_g[[h]] \) of the form \( F_h = 1 \otimes 1 + h r + \cdots \) satisfying the following properties:
a) it satisfies the equation \((6)\) with the usual \(\Delta\) and with \(\Phi = 1 \otimes 1 \otimes 1\);  
b) it is invariant under the Cartan subalgebra \(\mathfrak{c}\);  
c) \(F_{-h} = F_{h}^{\theta} = F_{h}^{21}\);  
d) \(F_{h}(F_{s}(F_{h}))^{21} = 1\)

Proof. Existence and properties a)–c) for \(\Phi\) are proven by Drinfeld \([Dr]\). From his proof which is purely cohomological it is seen that \(\Phi\) can be chosen invariant under all those automorphisms under which the element \(\varphi\) is invariant. This proves 1 d). Similarly 1 e) can be deduced from the cohomological construction by restricting to a suitable subcomplex, \([DS]\).

Existence and the property a) for \(F\) are also proven by Drinfeld \([Dr]\). In his proof he used the explicit existence of the Drinfeld–Jimbo quantum group \(U_{h}\). A purely cohomological construction of \(F\), not assuming the existence of the Drinfeld–Jimbo quantum group, and establishing the properties listed in 2 b)–2 d) is given in \([DS]\). \(\Box\)

So, we obtain two nontrivial Drinfeld algebras: \((U_{\mathfrak{g}}(\Delta, \Phi))\) with the usual comultiplication and \(\Phi\) from Proposition 3.1, and \((U_{\mathfrak{g}}, \tilde{\Delta}, id)\) where \(\tilde{\Delta}(x) = F_{h}\Delta(x)F_{h}^{-1}\) for \(x \in U_{\mathfrak{g}}\). The corresponding monoidal categories \(\text{Rep}(U_{\mathfrak{g}}, \Delta, \Phi)\) and \(\text{Rep}(U_{\mathfrak{g}}, \tilde{\Delta}, 1)\) are isomorphic, the isomorphism being given by the pair \((Id, F_{h})\). Note that the bialgebra \((U_{\mathfrak{g}}[[h]], \tilde{\Delta})\) is coassociative one and is isomorphic to Drinfeld-Jimbo quantum group \(U_{h}\) by Drinfeld’s uniqueness theorem, for proof see \([SS]\). So that the category \(\text{Rep}(U_{\mathfrak{g}}, \tilde{\Delta}, 1)\) with the trivial associativity constraint is called the category of representation of quantum group. Note that if we “forget” the monoidal structures all three categories are isomorphic to the category \(\text{Rep}(U_{\mathfrak{g}})\).

Remark. Corresponding to the category \(\text{Rep} = \text{Rep}(U_{\mathfrak{g}}, \Delta, \Phi)\) define a category \(\text{Rep}'\) with the reversed tensor product, \(V \otimes' W = W \otimes V\), and the associativity constraint \(\Phi'(V \otimes' W \otimes' U) = \Phi^{-1}(U \otimes (W \otimes V))\). Denote by \(S: V \otimes W \to W \otimes V\) the usual permutation, \(v \otimes w \mapsto w \otimes v\), which we will consider as a mapping \(V \otimes W \to V \otimes' W\). Then the condition 1 c) for \(\Phi\) implies that the pair \((Id, S)\) defines an equivalence of the categories \(\text{Rep}\) and \(\text{Rep}'\).

The antiinvolution \(s\) defines an antipode on the bialgebra \(U_{\mathfrak{g}}\). The existence of the antipode and property 1 e) for \(F_{h}\) makes \(\text{Rep}\) into a rigid monoidal category. The property 2 c) for \(F_{h}\) gives an equivalence of the
categories \( \text{Rep}(U_g, \Delta, \Phi) \) and \( \text{Rep}(U_g, \bar{\Delta}, 1) \) as rigid monoidal categories (see [DS] for more details).

## 4 Quantization

Let \( A \) be the sheaf of smooth functions on a smooth manifold \( M \). Let \( \text{Diff}(M) \) be the sheaf of linear differential operators on \( M \). A \( C \)-linear mapping \( \lambda : \bigotimes^n_C A \to A \) is called an \( n \)-differential cochain if there exists an element \( \hat{\lambda} \in \bigotimes^n \text{Diff}(M) \) such that \( \lambda(a_1 \otimes \ldots \otimes a_n) = \hat{\lambda}_1 a_1 \cdot \hat{\lambda}_2 a_2 \cdots \hat{\lambda}_n a_n \), where \( \hat{\lambda} = \hat{\lambda}_1 \otimes \cdots \otimes \hat{\lambda}_n \) (summation understood). It is easy to see that the element \( \hat{\lambda} \) is uniquely determined by the cochain \( \lambda \). We say that \( \lambda \) is "null on constants", if \( \lambda(a_1 \otimes \ldots \otimes a_n) = 0 \) in case at least one of \( a_i \) is a constant. Such \( \lambda \) is presented by \( \hat{\lambda} \in \bigotimes^n \text{Diff}(M)_0 \) where \( \text{Diff}(M)_0 \) denotes differential operators which are zero on constants. From now on we only consider \( n \)-differential cochains that are zero on the constants. Denote by \( H^n(A) \) the Hochschild cohomology defined by the complex of such spaces.

It is known that the space \( H^n(A) \) is isomorphic to the space of the antisymmetric \( n \)-vector fields on \( M \). Suppose that a group \( G \) acts on \( M \) and there exists a \( G \)-invariant connection on \( M \). In this case Lichnerowicz proved ([Li]) for \( n \leq 3 \) that \( H^n_G(A) \) is isomorphic to the space of the \( G \)-invariant antisymmetric \( n \)-vector fields on \( M \). Here \( H_G(A) \) is the cohomology of the subcomplex of \( G \)-invariant cochains.

We will consider cochains \( \lambda_h : A[[h]] \otimes^2 \to A[[h]] \) given by power series from \( \text{Diff}(M) \otimes^2 [[h]] \) of the form \( \lambda_h = 1 + \sum h^i \lambda_{1i} \otimes \lambda_{2i} \). (By our convention, each \( \lambda_{1i} \otimes \lambda_{2i} \) is a sum over a second index \( j \).) This means that \( \lambda_h(a \otimes b) = \sum_i h^i \lambda_{1i}(a) \lambda_{2i}(b) \), where \( \lambda_0(a, b) = ab \). We will also write \( \lambda_h : A \otimes^2 \to A \). The cochain \( \mu_h : A \otimes^2 \to A \) is called equivalent to \( \lambda_h \) if there exists a differential 1-cochain \( \xi_h : A[[h]] \to A[[h]] \), \( \xi_h = 1 + \sum h^i \xi_i \) such that \( \mu_h(a \otimes b) = \xi_h^{-1} \lambda_h(\xi_h a \otimes \xi_h b) \), where inverse is computed in the sense of formal power series.

Let \( M \) be a symmetric space, as in Section 2. Consider the space \( A = C^\infty(M) \) as an object of the category \( \text{Rep}(U_g, \Delta, \Phi_h) \) where \( \Phi_h \) is from Proposition 3.1.

### Proposition 4.1
There is a multiplication \( \mu_h \) on \( A \) with the properties:
a) $\mu_h$ is $\Phi_h$-associative, i.e.

$$\mu_h(\mu_h \otimes id)(a \otimes b \otimes c) = \mu_h(id \otimes \mu_h)(a \otimes b \otimes c), \quad a, b, c \in A;$$

b) $\mu_h$ has the form

$$\mu_h(a \otimes b) = ab + \sum_{i \geq 4} h^i \mu_i(a \otimes b),$$

where $\mu_i$ are two-differential cochains, null on constants. Moreover, $\mu_h = \mu_{-h}$, i.e., $\mu_h$ depends only on $h^2$.

c) $\mu_h$ is invariant under $g$ and $\tau$;

d) $\mu_h$ is commutative, i.e.

$$\mu_h(a \otimes b) = \mu_h(b \otimes a).$$

The multiplication with such properties is unique up to equivalence.

**Proof** We use arguments from [Li], proceeding by induction. We may put $\mu_1 = \mu_2 = 0$, because the usual multiplication $m(a \otimes b) = ab$ satisfies a) modulo $h^4$. This follows because $\Phi_h$ is a series in $h^2$ and the $h^2$-term $\varphi = 0$ on $M$. Suppose we have constructed $\mu_i$ for even $i < n$, such that $\mu_h^n = \sum' \mu_i h^i$ satisfies a)–d) modulo $h^n$, where $\sum'$ denotes sum over even indices. Then,

$$\mu_h^n(\mu_h \otimes id) = \mu_h^n(id \otimes \mu_h^n)\Phi_h + h^n \eta \mod h^{n+2}, \quad (9)$$

where $\eta$ is an invariant three-cochain.

The following direct computation using the pentagon identity for $\Phi_h$ shows that $\eta$ is a Hochschild cocycle. By definition

$$d\eta = m(id \otimes \eta) - \eta(m \otimes id^\otimes2) + \eta(id \otimes m \otimes id) - \eta(id^\otimes2 \otimes m) + m(\eta \otimes id).$$

Using (9) and calculating modulo $h^{n+2}$ we can replace $m$ with $\mu_h^n$. Furthermore, the $G$-invariance of $\mu_h^n$ implies that

$$\Phi(\mu_h^n \otimes id^\otimes2) = (\mu_h^n \otimes id^\otimes2)(\Delta \otimes id^\otimes2)\Phi,$$

$$\Phi(id \otimes \mu_h^n \otimes id) = (id \otimes \mu_h^n \otimes id)(id \otimes \Delta \otimes id)\Phi,$$

$$\Phi(id^\otimes2 \otimes \mu_h^n) = (id^\otimes2 \otimes \mu_h^n)(id^\otimes2 \otimes \Delta)\Phi.$$
Therefore we have the following equations modulo \( h^{n+2} \),

\[
\begin{align*}
\mu_h^n(id \otimes \mu_h^n)(id \otimes \mu_h^n \otimes id) & - \mu_h^n(id \otimes \mu_h^n)(id \otimes id \otimes \mu_h^n) = h^nm(id \otimes \eta) \\
\mu_h^n(\mu_h \otimes id)(\mu_h \otimes id^{\otimes 2}) & - \mu_h^n(id \otimes \mu_h^n)(\mu_h \otimes id^{\otimes 2}) = h^nm(\eta \otimes id^{\otimes 2}) \\
\mu_h^n(\mu_h \otimes id)(id \otimes \mu_h^n \otimes id) & - \mu_h^n(id \otimes \mu_h^n)(id \otimes id \otimes \mu_h^n) = h^nm(\eta \otimes m \otimes id) \\
\mu_h^n(\mu_h \otimes id)(id^{\otimes 2} \otimes \mu_h^n) & - \mu_h^n(id \otimes \mu_h^n)(id^{\otimes 2} \otimes \mu_h^n) = h^n(id^{\otimes 2} \otimes \Delta)(\Phi_h) \\
\mu_h^n(\mu_h \otimes id)(\mu_h \otimes id^{\otimes 2}) & - \mu_h^n(id \otimes \mu_h^n)(\mu_h \otimes id)(\Phi_h \otimes 1) = h^nm(\eta \otimes \mu_h^n).
\end{align*}
\]

Since the equations are congruences modulo \( h^{n+2} \) and \( h^n\Phi = h^n \mod h^{n+2} \) the equations remain valid if we multiply on the left by any expression in \( \Phi \) and leave the right side unchanged. Multiply the left side of the first equation by \( ((id \otimes \Delta \otimes id)\Phi)(\Phi \otimes 1) \), the left side of the third equation by \( \Phi \otimes 1 \), the left side of the fourth equation by \( (\Delta \otimes id \otimes id)\Phi \), leave the remaining equations unchanged, then add the five equations with alternating signs. Using the pentagon identity in \( \Phi \) and the identity \( (\mu_h \otimes id)(id \otimes id \otimes \mu_h^n) = \mu_h^n \otimes \mu_h^n = (id \otimes \mu_h^n)(\mu_h^n \otimes id \otimes id) \), we conclude that \( d\eta = 0 \).

Since \( g \) is semisimple the cochains invariant under \( g \) and \( \tau \) form a subcomplex which is a direct summand. The arguments from the proof of Proposition 2.3 show that there are no three-vector fields on \( M \) invariant under \( g \) and \( \tau \). Hence the cohomology of this subcomplex is equal to zero, i.e. \( \eta \) is a coboundary. Further, there is a \( g \) and \( \tau \) invariant connection on \( M \) (see [He] 4.A.1). The property \( \Phi_h^{-1} = \Phi_h^{21} \) and commutativity of \( \mu_h \) imply that \( \eta(a \otimes b \otimes c) = \eta(c \otimes b \otimes a) \). It follows that there is an invariant commutative two-cochain \( \mu_h \) such that \( d\mu_h = \eta \), which shows that \( \mu_h^n + h^n\mu_h \) satisfies a)–d) modulo \( h^{n+2} \). Therefore, proceeding step-by-step we can build the multiplication \( \mu_h \).

The equivalence of any two such multiplications follows from the fact that any symmetric Hochschild differential-two-cochain bounds.

Now we suppose that on the algebra \( A \) there is a \( g \) and \( \tau \) invariant multiplication \( \mu : (A \otimes A)[[\nu]] \to A[[\nu]] \) which is associative in the usual sense.
and such that $\mu_0 = m$ where $m$ is the usual multiplication on $A$. The multiplication $\mu_\nu$ exists when $M$ is a kählerian symmetric space. In this case $\mu_\nu$ can be constructed as the deformation quantization of the Poisson bracket $\{\cdot, \cdot\}_{\text{inv}}$ which is the dual to the kählerian form on $M$. Such a quantization also can be constructed using the arguments of Proposition 4.1 and has the form

$$\mu_\nu(a, b) = ab + \frac{1}{2} \nu \{a, b\}_{\text{inv}} + o(\nu).$$

Moreover, $\mu_\nu$ satisfies the property

$$\mu_\nu(a, b) = \mu_\nu(-b, a).$$

Denote by $A_\nu$ the corresponding algebra. Let $H^n(A_\nu)$ be the Hochschild cohomology of this algebra. Since $H^2(\mathfrak{g}, \tau)(A_0) = 0$ it is easy to see that $H^3(\mathfrak{g}, \tau)(A_\nu) = 0$ as well. Using the same arguments as in the proof of Proposition 4.1, we have the following

**Proposition 4.2** Let $M$ be a kählerian symmetric space, $\mu_\nu$ the quantization of the Kirillov bracket. Then there is a multiplication $\mu_{\nu, h}$ on $A = C^\infty(M)$ depending on two formal variables with the properties:

a) $\mu_{\nu, h}$ is $\Phi_h$-associative, i.e.

$$\mu_{\nu, h}(\mu_{\nu, h} \otimes \text{id})(a \otimes b \otimes c) = \mu_{\nu, h}(\text{id} \otimes \mu_{\nu, h})(a \otimes b \otimes c), \quad a, b, c \in A;$$

b) $\mu_{\nu, h}$ has the form

$$\mu_{\nu, h}(a \otimes b) = \mu_\nu(a \otimes b) + \sum_{i \geq 4} h^i \mu_{\nu, i}(a \otimes b),$$

where $\mu_{\nu, i} : (A \otimes A)[[\nu]] \to A[[\nu]]$ are 2-differential cochains null on constants. Moreover, $\mu_{\nu, h}$ depends only of $h^2$, i.e. $\mu_{\nu, h} = \mu_{\nu, -h}$, and $\mu_{\nu, h}(a, b) = \mu_{-\nu, h}(b, a)$.

c) $\mu_{\nu, h}$ is invariant under $\mathfrak{g}$ and $\tau$;

d) $\mu_{\nu, 0}$ coincides with $\mu_\nu$, and $\mu_{0, h}$ coincides with $\mu_h$ from Proposition 4.1.

The multiplication with such properties is unique up to equivalence.

Now let us consider $A = C^\infty(M)$ as an object of the category $\text{Rep}(U_{\mathfrak{g}}, \tilde{\Delta}, 1)$ of representations of the Drinfeld–Jimbo quantum group $U_{h\mathfrak{g}}$. As we have
seen in Section 3, the multiplications $\mu_h$ and $\mu_{\nu,h}$ can be transferred to this category in the following way:

$$\tilde{\mu}_h = \mu_h F_h^{-1}$$

$$\tilde{\mu}_{\nu,h} = \mu_{\nu,h} F_h^{-1}.$$  

We may obviously assume that $F_h$ has the form

$$F_h = 1 \otimes 1 - \frac{1}{2} h \{\cdot, \cdot\} + o(h).$$

Then we have the following

**Theorem 4.3** Let $M$ be a symmetric space over a semisimple Lie group. Then the multiplications $\tilde{\mu}_h$ and $\tilde{\mu}_{\nu,h}$ (the second exists when $M$ is a kählerian symmetric space) satisfy the following properties:

a) $\tilde{\mu}_h$ and $\tilde{\mu}_{\nu,h}$ are associative;

b) $\tilde{\mu}_h$ and $\tilde{\mu}_{\nu,h}$ have the form

$$\tilde{\mu}_h(a \otimes b) = ab + \frac{1}{2} h \{a, b\} + o(h)$$

$$\tilde{\mu}_{\nu,h}(a \otimes b) = ab + \frac{1}{2} (h \{a, b\} + \nu \{a, b\}_{inv}) + o(h, \nu)$$

c) $\tilde{\mu}_h$ and $\tilde{\mu}_{\nu,h}$ are invariant under action of the Drinfeld–Jimbo quantum group $U_{h^G}$;

d) $\tilde{\mu}_{\nu,0}$ coincides with $\mu_{\nu}$, and $\tilde{\mu}_{0,h}$ coincides with $\tilde{\mu}_h$;

e) Let $\tilde{S} = F_h S F_h^{-1}$ where $S$ denotes the usual transposition, $S(a \otimes b) = b \otimes a$ for $a, b \in A$. Then $\tilde{\mu}_h$ is $\tilde{S}$-commutative:

$$\tilde{\mu}_h(a \otimes b) = \tilde{\mu}_h \tilde{S}(a \otimes b) \text{ for } a, b \in C^\infty(M).$$

For $\tilde{\mu}_{\nu,h}$ one has:

$$\tilde{\mu}_{\nu,h}(a \otimes b) = \tilde{\mu}_{-\nu,0} \tilde{S}(a \otimes b) \text{ for } a, b \in C^\infty(M).$$

The multiplications with such properties are unique up to equivalence.
Remarks.

1. The action of the real Lie group \( G \) and \( \tau \) on \( M \) induces an action on \( C^\infty(M)[[h]] \). It follows from Propositions 3.1 and 3.2 that \( \mu_h \) and \( \mu_{\nu,h} \) are invariant under \( G \) and \( \tau \). This implies that \( \bar{\mu}_h \) and \( \bar{\mu}_{\nu,h} \) will be invariant under a “quantized” action of \( G \) and \( \tau \). This new action appears by taking of tensor products of \( C^\infty(M) \). Namely, let \( g \) be either an element of \( G \) or \( g = \tau \), then for \( a, b \in C^\infty(M) \) define \( g \circ_h a = g \circ a, g \circ_h (a \otimes b) = F_h(g \otimes g) F_h^{-1}(a \otimes b) \), where \( \circ \) denotes the usual action. The multiplications \( \bar{\mu}_h \) and \( \bar{\mu}_{\nu,h} \) are invariant under this quantized action, i.e., for example,

\[
g \circ_h \bar{\mu}_h(a, b) = \bar{\mu}_{\nu,h} \circ_h (a \otimes b).
\]

2. We may consider a complex symmetric space \( M = G/H \), where \( G \) is a complex semisimple Lie group and \( H \) a complex subgroup. As above, one can construct the multiplications \( \mu_h \) and \( \bar{\mu}_h \) on the space \( C^\infty(M) \) that also will give a multiplication on the space of holomorphic functions on \( M \). The previous remark remains valid for the complex group \( G \).

In particular, the group \( G \) itself may be considered as a symmetric space, \( G = (G \times G)/D \) where \( D \) is the diagonal. The action of \( G \times G \) on \( G \) is \((g_1, g_2) \circ g = g_1 g g_2^{-1}, (g_1, g_2) \in G \times G, g \in G \). In this case \( \sigma(g_1, g_2) = (g_2, g_1) \), \( \tau(g) = g^{-1} \). In order for \( \varphi \) to be \( \sigma \)-invariant the corresponding \( R \)-matrix can be taken in the form \( \tilde{r} = (r, r) \in \wedge^2 \mathfrak{g}_1 \oplus \wedge^2 \mathfrak{g}_2 \subset \wedge^2 (\mathfrak{g}_1 \oplus \mathfrak{g}_2) \) or \( \tilde{r} = (r, -r) \), where the Lie subalgebras \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \) correspond to \( (G \times 1) \) and \( (1 \times G) \). In this example \( U_\mathfrak{g} = (U_\mathfrak{g})^{\otimes 2} \supset U_\mathfrak{g} \oplus U_\mathfrak{g} \) and in the both cases the element \( \Phi_h \) has the form \( \hat{\Phi}_h = (\Phi_h, \bar{\Phi}_h) \) with \( \Phi_h \) from Proposition 3.1. In case \( \tilde{r} \) the corresponding \( \hat{F}_h \) has the form \( \hat{F}_h = (F_h, F_h) \) with \( F_h \) from Proposition 3.1. In case \( \tilde{r} \) the corresponding \( \hat{F}_h \) has the form \( \hat{F}_h = (F_h, F_{-h}) \) with \( F_h \) from Proposition 3.1. Then, \( \rho(\hat{\Phi}_h) = id \), so that for \( \mu_h \) one can take the usual multiplication \( m \) on \( C^\infty(G) \), and \( \bar{\mu}_h(a, b) = m(F_h(a \otimes b) F_h^{-1}) \) in the case \( \tilde{r} \), and \( \bar{\mu}_h(a, b) = m(F_h(a \otimes b) F_{-h}^{-1}) \) in the case \( \tilde{r} \). Therefore, in the both cases \( C^\infty(G) \) may be considered as an algebra in the category Rep(\( (U_\mathfrak{g})^{\otimes 2}, \hat{\Delta}, 1 \)) with the multiplication \( \bar{\mu}_h \). Note that the first multiplication is a quantization of the Poisson bracket \((r - r') \) on \( G \) where \( r \) and \( r' \) denote the extensions of \( r \) as right- and left-invariant bivector fields on \( G \), whereas the second multiplication is a quantization of the Poisson bracket \((r + r') \) on \( G \).
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