A Series Solution of Non-local Hyperbolic Equation

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Abstract

In this paper, a powerful, easy to use analytic tool for nonlinear problems in general, the Homotopy analysis method for approximating solution of the non-local hyperbolic equation is implemented with approximate conditions. The effectiveness of HAM is verified through illustrative examples.

Keywords: Homotopy analysis method; hyperbolic equation; non-local.

1 Introduction

In most differential equations, especially nonlinear problems, it is impossible to obtain an exact solution, so one must resort to various approximation methods of solution. It is well known that perturbation methods are strongly dependent upon small/large physical parameters, and therefore are valid in principle only for weakly nonlinear problems. On the other hand, all traditional non-perturbation methods, such as Adomian decomposition method, can not ensure the convergence of solution series. Therefore, in fact, these methods are valid only for weakly nonlinear problems, too. The Homotopy analysis method, introduced first by Liao [11, 12, 13], is a powerful analytical tool for nonlinear problems. This method, unlike all perturbation and traditional non-perturbation methods, is valid no matter whether a nonlinear problem contains small/large physical parameters or not. HAM has been applied successfully to many nonlinear problems in engineering and science. This method contains an auxiliary parameter \( \alpha \), which provide us, with a simple way, to adjust and control the convergence region of solution series for any values of \( x \) and \( t \). Recently, Abbasbandy, et al. gave the mathematical properties of \( \alpha \)-curves in HAM

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In this paper, we apply HAM to find the approximate analytical solution of the non-local hyperbolic equation. Illustrative examples show that the numerical results can be obtained by using a few iterations.

We consider the hyperbolic equation

\[ \frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} = G(x,t), \quad 0 < x < l, \quad 0 < t \leq T, \]  

(1.1)

with initial conditions

\[ u(x,0) = k(x), \quad u_t(x,0) = s(x), \quad 0 \leq x \leq l, \]  

(1.2)

and Dirichlet boundary condition

\[ u(0,t) = p(t), \quad 0 < t \leq T, \]  

(1.3)

also, subject to the non-local condition

\[ \int_{0}^{1} u(x,t)dx = q(t), \quad 0 < t \leq T, \]  

(1.4)

where \( G, k, s, p \) and \( q \) are known functions. Further we assume

\[ k(x) \in C[0,l] \cap C^2[0,l], \quad s(x) \in C[0,l] \cap C^1[0,l], \]  

(1.5)

and \( G \) is sufficiently smooth to produce a smooth classical solution \( u \).

It is worth pointing out that \( r(x) \) and \( s(x) \) satisfy the following compatibility conditions:

\[ k(0) = p(0), \quad \int_{0}^{1} k(x)dx = q(0), \]

\[ s(0) = p'(0), \quad \int_{0}^{1} s(x)dx = q'(0). \]

Non-local boundary value problems have certainly been one of the fastest growing areas in various application fields, because physical phenomena are modelled by those problems. During the last thirty years a large member of research papers devoted entirely or in part to the study of existence and uniqueness, and the numerical techniques for the solution of non-local boundary value problems. Nonlocal boundary value problems can be classified into two types; boundary value problems with nonlocal initial conditions, and boundary value problems with nonlocal boundary conditions \[5, 6\]. In this paper we consider the nonlocal boundary value problems, of the second group, for the system of hyperbolic equations. The need for studying boundary-value problems of parabolic-hyperbolic type was emphasized by Gelfand\[10\] in 1959. Later Zolina \[17\] considered several of these problems and gave some physical interpretations. The author of \[16\] has given irrigation models of wide applications of these problems. Nonlocal boundary-value problems can be solved by implicit or explicit schemes and a brief overview of recent developments can be found \[7\]. The existence and uniqueness of the solution of the problem (1.1)-(1.4) are discussed in \[4\]. Dehghan \[8\] presented several finite difference schemes for the numerical solution of problem (1.1)-(1.4). Also in \[9\] the variational method was applied for solving this problem.
2 The Homotopy Analysis Method (HAM)

For the purposes of illustration of the HAM, first we write problem in standard form. So we transform problem (1.1)-(1.4) to an equivalent problem, where not involved unusual boundary condition (1.4). As done in [9] we introduce a new unknown function

\[ w(x, t) = u(x, t) - h(x, t), \quad 0 < x < l, \quad 0 < t \leq T, \]  

(2.6)

where

\[ h(x, t) = (1 - \frac{2x}{l})p(t) + (\frac{2x}{l^2})q(t). \]  

(2.7)

Then (1.1)-(1.4) are converted to the following problem:

\[ \frac{\partial^2 w(x, t)}{\partial t^2} - \frac{\partial^2 w(x, t)}{\partial x^2} = G(x, t), \quad 0 < x < l, \quad 0 < t \leq T, \]  

(2.8)

with initial conditions

\[ w(x, 0) = E(x), \quad w_t(x, 0) = \sigma(x), \quad 0 \leq x \leq l, \]  

(2.9)

and Dirichlet boundary condition

\[ w(0, t) = 0, \quad 0 < t \leq T, \]  

(2.10)

subject to the non-local condition

\[ \int_0^l w(x, t)dx = 0, \quad 0 < t \leq T, \]  

(2.11)

where

\[ G(x, t) = G(x, t) - (1 - \frac{2x}{l})p''(t) - (\frac{2x}{l^2})q''(t), \]  

\[ E(x, t) = k(x, t) - (1 - \frac{2x}{l})p(0) - (\frac{2x}{l^2})q(0), \]  

and

\[ \sigma(x, t) = s(x, t) - (1 - \frac{2x}{l})p'(0) - (\frac{2x}{l^2})q'(0). \]  

(2.12)

Note that the Dirichlet and integral conditions are now homogeneous. To illustrate the procedure HAM, we consider the following system:

\[ \mathcal{N}[w(x, t)] = 0, \]  

(2.13)

where \( \mathcal{N} \) is a nonlinear operator, \( x \) and \( t \) denote independent variables, \( w(x, t) \) is an unknown function. By means of generalizing the traditional homotopy analysis method constructed Zero-order deformation equation

\[ (1 - q)\mathcal{L}[\phi(x; t; q) - w_0(x, t)] = qh\mathcal{N}[\phi(x; t; q)], \]  

(2.14)

where \( q \in [0, 1] \) is the embedding parameter, \( h \neq 0 \) is an auxiliary parameter, \( \mathcal{L} \) is an auxiliary linear operator, \( w_0(x, t) \) is an initial guess of \( w(x, t) \) and \( \phi(x; t; q) \) is an unknown
function. It is important that one has great freedom to choose auxiliary parameter $h$ and $L$ in homotopy analysis method. Obviously, when $q = 0$ and $q = 1$, both
\begin{equation}
\phi(x, t; 0) = w_0(x, t) \quad \text{and} \quad \phi(x, t; 1) = w(x, t),
\end{equation}
hold. Thus as $q$ increases from 0 to 1, the solution $\phi(x, t; q)$ varies from the initial guess $w_0(x, t)$ to the solution $w(x, t)$. Expanding the function $\phi(x, t; q)$ in a Taylor series with respect to $q$, one has
\begin{equation}
\phi(x, t; q) = w_0(x, t) + \sum_{m=1}^{+\infty} w_m(x, t) q^m,
\end{equation}
where
\begin{equation}
w_m(x, t) = \frac{1}{m!} \frac{\partial^m \phi(x, t; q)}{\partial q^m}
|_{q=0}, \quad m = 0, 1, \ldots
\end{equation}
If the auxiliary linear operator, initial guess and the auxiliary parameter $h$ are so properly chosen, then the series equation (2.16) converges at $q = 1$, one has
\begin{equation}
\phi(x, t; 1) = w_0(x, t) + \sum_{m=1}^{+\infty} w_m(x, t).
\end{equation}
According to (2.17), the governing equation can be deduced from the Zero-order deformation equations (2.14). Define the vector
\begin{equation}
\vec{w}_m(x, t) = \{w_0(x, t), w_1(x, t), \ldots, w_m(x, t)\}.
\end{equation}
Differentiating (2.14) $m$ times with respect to the embedding parameter $q$, and then setting $q = 0$ and finally dividing them by $m!$, we have the so-called $m^{th}$ order deformation equation
\begin{equation}
\mathcal{L}[w_m(x, t) - \chi_m w_{m-1}(x, t)] = h \mathcal{R}_m(\vec{w}_{m-1}),
\end{equation}
where
\begin{equation}
\mathcal{R}_m(\vec{w}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(x, t; q)]}{\partial q^{m-1}} |_{q=0},
\end{equation}
and
\begin{equation}
\chi_m = \begin{cases}
0, & m \leq 1, \\
1, & m > 1.
\end{cases}
\end{equation}

3 Illustrative examples

Example 3.1. Consider (1.1)-(1.4) with $l = 1$, $T = 0.5$ and [8, 15]
\begin{align*}
G(x, t) &= 0, \quad 0 < x < 1, \quad 0 < t < 0.5, \\
k(x, t) &= 0, \quad s(x) = \pi \cos(\pi x), \quad 0 < x < 1, \\
p(t) &= \sin(\pi t), \quad q(t) = 0, \quad 0 < t < 0.5.
\end{align*}
The exact solution is $u(x, t) = \cos(\pi x) \sin(\pi t)$. For this problem we obtain:
\begin{align*}
\overline{G}(x, t) &= \pi^2(1 - 2x) \sin(\pi t),
\end{align*}
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\[ k(x, t) = 0, \]
\[ s(x, t) = \pi \cos(\pi x) - (1 - 2x)\pi. \]

We select the following initial guess
\[ w_0(x, t) = t\pi \cos(\pi x) - t(1 - 2x)\pi, \]
and the auxiliary linear operator is
\[ L[\phi(x, t; q)] = \frac{\partial^2 \phi(x, t; q)}{\partial t^2}, \tag{3.23} \]
with the property
\[ L[c_1 + tc_2] = 0, \tag{3.24} \]
where \( c_1 \) and \( c_2 \) are integral constants. Also the nonlinear operator is
\[ N[\phi(x, t; q)] = \frac{\partial^2 \phi(x, t; q)}{\partial t^2} - \frac{\partial^2 \phi(x, t; q)}{\partial x^2} - \pi^2(1 - 2x) \sin(\pi t). \tag{3.25} \]

Then \( m \)th order deformation equation is
\[ L[w_m(x, t) - \chi_m w_{m-1}(x, t)] = hR_m(\vec{w}_{m-1}), \tag{3.26} \]
with the initial condition
\[ w_m(x, 0) = 0, \tag{3.27} \]
and
\[ R_m(\vec{w}_{m-1}) = \frac{\partial^2 w_{m-1}(x, t; q)}{\partial t^2} - \frac{\partial^2 w_{m-1}(x, t; q)}{\partial x^2} \tag{3.28} \]
\[ -(1 - \chi_m)(\pi^2(1 - 2x) \sin(\pi t)). \tag{3.29} \]

Using symbolic computation systems such as Maple or Mathematica, we recursively obtain
\[ w_1(x, t) = -h\pi t + 2h\pi tx + \frac{1}{6}h\pi^3 t^3 \cos(\pi x) + h\sin(\pi t) - 2hx \sin(\pi t), \]
\[ w_2(x, t) = -h\pi t - h^2\pi t + 2h\pi t x + 2h^2\pi t^2 x + \frac{1}{6}h\pi^3 t^3 \cos(\pi x) + \frac{1}{6}h^2\pi^3 t^3 \cos(\pi x) \]
\[ + \frac{1}{120}h^2 \pi^5 t^5 \cos(\pi x) + h\sin(\pi t) + h^2 \sin(\pi t) - 2hx \sin(\pi t) - 2h^2 x \sin(\pi t), \]

Then the solution obtained by HAM is as follows
\[ w(x, t) = \sum_{m=0}^{\infty} w_m(x, t). \tag{3.30} \]

It is easily seen that the \( m \)th order HAM approximation when \( h = -1 \) reads
\[ w(x, t) \approx \cos(\pi x) \sum_{k=0}^{m} (-1)^k (\pi t)^{2k+1} (2k + 1)! - (1 - 2x) \sin(\pi t), \tag{3.31} \]
which obviously converges to the exact solution \( \cos(\pi x) \sin(\pi t) - (1 - 2x) \sin(\pi t) \) as \( m \to \infty \).

Fig. 1, shows the \( h \)-curve of HAM series solution \( \frac{\partial}{\partial t} w_{15}(0, 0) \) and \( \frac{\partial^9}{\partial t^9} w_{15}(0, 0) \) to get a proper interval for convergence-controller parameter. The valid region of convergence-controller parameter is \(-1.12 < h < -0.84\). Also the error of norm 2 with HAM by 15\(^{th}\)-order and 10\(^{th}\)-order approximation, i.e.,

\[
\text{Residual Error} = \left[ \int_0^1 \int_0^1 (w_n(x, t) - w(x, t))^2 \, dt \, dx \right]^{\frac{1}{2}},
\]

(3.32)

With respect to \( h \) is plotted in Fig. 2. Absolute error for the 15\(^{th}\) order approximation by HAM for \( w(x, t) \) is plotted in Fig. 3 for \( h = -0.85 \).

![Figure 1: The \( h \)-curve of \( w_1^{(7)} \) and \( w_1^{(9)} \) for the 15\(^{th}\)-order approximation. Solid line: \( w_{15}^{(7)}(0, 0, h) \); Dashed line: \( w_{15}^{(9)}(0, 0, h) \).](image1)

![Figure 2: Error of norm 2 for the 15\(^{th}\)-order and 10\(^{th}\)-order approximation by HAM for per \( w(x, t) \) per \( h \). Solid line: \( w_{15} \); Dashed line: \( w_{10} \).](image2)
Example 3.2. Consider (1.1)-(1.4) with \( l = 1, T = 0.5 \) where [14]
\[
G(x, t) = x^2 e^t - 2e^t, \quad 0 < x < 1, \quad 0 < t < 0.5,
\]
\[
k(x, t) = x^2, \quad s(x) = x^2, \quad 0 < x < 1,
\]
\[
p(t) = 0, \quad q(t) = \frac{1}{3}e^t, \quad 0 < t < 0.5.
\]
The exact solution is \( u(x, t) = x^2 e^t \). For this problem we obtain:
\[
\overline{G}(x, t) = x^2 e^t - \frac{2}{3}xe^t - 2e^t,
\]
\[
\overline{k}(x, t) = x^2 - \frac{2}{3}x,
\]
\[
\overline{s}(x, t) = x^2 - \frac{2}{3}x.
\]
We choose the initial guess
\[
w_0(x, t) = x^2 - \frac{2}{3}x + t(x^2 - \frac{2}{3}x),
\]
and the auxiliary linear operator
\[
L[\phi(x,t;q)] = \frac{\partial^2 \phi(x,t;q)}{\partial t^2},
\]
with the property
\[
L[c_1 + tc_2] = 0,
\]
where \( c_1, c_2 \) are integral constants, and the nonlinear operator,
\[
N[\phi(x,t;q)] = \frac{\partial^2 \phi(x,t;q)}{\partial t^2} - \frac{\partial^2 \phi(x,t;q)}{\partial x^2} - (x^2 e^t - \frac{2}{3}xe^t - 2e^t).
The $m$th order deformation equation is

$$L[w_m(x, t) - \chi_m w_{m-1}(x, t)] = hR_m(\vec{w}_{m-1}), \quad (3.33)$$

with the initial condition

$$w_m(x, 0) = 0,$$

where

$$R_m(\vec{w}_{m-1}) = \frac{\partial^2 w_{m-1}(x, t; q)}{\partial t^2} - \frac{\partial^2 w_{m-1}(x, t; q)}{\partial x^2}$$

$$- (1 - \chi_m)(x^2e^t - \frac{2}{3}xe^t - 2e^t). \quad (3.34)$$

Therefore, we recursively obtain

$$w_1(x, t) = -2h + 2e^t h - 2ht - h^2 - \frac{h^3}{3} - \frac{2hx}{3} + \frac{2e^t hx}{3} - \frac{2thx}{3} + hx^2 - e^t hx^2 + htx^2,$$

$$w_2(x, t) = -2h + 2e^t h - 4h^2 + 4e^t h^2 - 2ht - 4h^2 t - h^2 t^2$$

$$- \frac{ht^3}{3} - \frac{2ht^3}{3} - \frac{2hx}{3} + \frac{2e^t hx}{3} - \frac{2thx}{3} + \frac{2e^t hx}{3} - \frac{2thx}{3} + \frac{2h^2 tx}{3}$$

$$- \frac{2h^2 tx}{3} + hx^2 - e^t hx^2 + h^2 x^2 - e^t h^2 x^2 + htx^2 + h^2 tx^2,$$

$$\vdots$$

Accordingly, the $m$th order approximation solution of the HAM, $W_m(x, t)$, is in the form

$$w(x, t) \approx W_m(x, t) = \sum_{i=0}^{m} w_i(x, t) \quad (3.36)$$

Fig. 4, shows the $h$-curve of HAM series solution $\frac{\partial^4}{\partial t^4}w_{30}(0, 0)$ and $\frac{\partial^5}{\partial t^5}w_{30}(0, 0)$ to get a proper interval for convergence-controller parameter. The valid region of convergence-controller parameter is $-1.5 < h < -0.2$. Also the error of norm 2 with HAM by 25th-order and 30th-order approximation, i.e.,

$$\text{Residual Error} = \left[ \int_0^1 \int_0^1 (w_n(x, t) - w(x, t))^2 dt dx \right]^{\frac{1}{2}}, \quad (3.37)$$

with respect to $h$ is plotted in Fig. 5. Absolute error for the 30th-order approximation by HAM for $w(x, t)$ is plotted in Fig. 6 for $h = -0.68$. 
Example 3.3. Consider (1.1)-(1.4) with $l = 1, T = 0.5$:

$$G(x, t) = -(\pi^2 x^2 - 2) \cos(\pi t), \quad 0 < x < 1, \quad 0 < t < 0.5,$$
\[ k(x,t) = x^2, \quad s(x) = 0, \quad 0 < x < 1, \]
\[ p(t) = 0, \quad q(t) = \frac{1}{3} \cos(\pi t), \quad 0 < t < 0.5. \]

The exact solution for this problem is \( u(x,t) = x^2 \cos(\pi t) \). For this problem, we obtain:

\[ \mathcal{G}(x,t) = -\cos(\pi t)(\pi^2 x^2 - \frac{2}{3} x) + 2), \]
\[ \mathcal{G}(x,t) = x^2 - \frac{2}{3} x, \]
\[ s(x,t) = 0. \]

We choose the initial guess

\[ w_0(x,t) = x^2 - \frac{2}{3} x, \]

and the auxiliary linear operator

\[ \mathcal{L}[\phi(x,t;q)] = \frac{\partial^2 \phi(x,t;q)}{\partial t^2}, \tag{3.38} \]

with the property

\[ \mathcal{L}[c_1 + tc_2] = 0, \tag{3.39} \]

where \( c_1, c_2 \) are integral constants, and the nonlinear operator,

\[ \mathcal{N}[\phi(x,t;q)] = \frac{\partial^2 \phi(x,t;q)}{\partial t^2} - \frac{\partial^2 \phi(x,t;q)}{\partial x^2} + \cos(\pi t)(\pi^2 x^2 - \frac{2}{3} x) + 2). \tag{3.40} \]

The \( m \)th order deformation equation, for \( m \geq 1 \) becomes

\[ \mathcal{L}[w_m(x,t) - \chi_m w_{m-1}(x,t)] = h\mathcal{R}_m(\vec{w}_{m-1}), \tag{3.41} \]

with the initial condition

\[ w_m(x,0) = 0, \tag{3.42} \]

where

\[ \mathcal{R}_m(\vec{w}_{m-1}) = \frac{\partial^2 w_{m-1}(x,t;q)}{\partial t^2} - \frac{\partial^2 w_{m-1}(x,t;q)}{\partial x^2} \]
\[ -(1 - \chi_m)(- \cos(\pi t)(\pi^2 x^2 - \frac{2}{3} x) + 2)). \tag{3.43} \]

Therefore, we recursively obtain

\[ w_1(x,t) = \frac{2h}{\pi^2} - \frac{\pi^2}{\pi^2} + \frac{2h \cos(\pi t)}{\pi^2} + \frac{2}{3}h \pi \cos(\pi t) - \frac{\pi^2}{\pi^2} \cos(\pi t), \]
\[ w_2(x,t) = \frac{2h}{\pi^2} + \frac{4h^2}{\pi^2} - \frac{\pi^2}{\pi^2} - \frac{2h^2 x^2}{\pi^2} + \frac{2h x^2}{\pi^2} + \frac{2h^2 x^2}{\pi^2} - \frac{2h \cos(\pi t)}{\pi^2} \]
\[ - \frac{4h^2 \cos(\pi t)}{\pi^2} + \frac{2}{3}h \pi \cos(\pi t) + \frac{2}{3}h^2 x^2 \cos(\pi t) - \frac{\pi^2}{\pi^2} x^2 \cos(\pi t) - \frac{2h^2 \cos(\pi t)}{\pi^2}, \]
\[ \vdots \]
Then the solution obtained by HAM is as follows

\[ w(x, t) = w_0(x, t) + w_1(x, t) + w_2(x, t) + \ldots. \]  \hspace{1cm} (3.45)

Fig. 7, shows the h-curve of HAM series solution \( \frac{\partial^4}{\partial t^4}w_{25}(0, 0) \) and \( \frac{\partial^6}{\partial t^6}w_{25}(0, 0) \) to get a proper interval for convergence-controller parameter. The valid region of convergence-controller parameter is \(-1.6 < h < -0.4\). Also the errors of norm 2 with HAM by 25th order and 20th order approximation, i.e.,

\[
\text{Residual Error} = \left[ \int_0^1 \int_0^1 (w_n(x, t) - w(x, t))^2 \, dt \, dx \right]^{\frac{1}{2}}, \quad (3.46)
\]

with respect to \( h \) are plotted in Fig. 8. Finally, absolute error for the 25th order approximation by HAM for \( w(x, t) \) is plotted in Fig. 9 for \( h = -0.55 \).

![Figure 7: The h-curve of \( w_4^{(4)} \) and \( w_6^{(6)} \) for the 30th-order approximation. Solid line: \( w_6^{(6)}(0, 0, h) \); Dashed line: \( w_4^{(4)}(0, 0, h) \).](image1)

![Figure 8: Error of norm 2 for the 25th-order and 20th-order approximation by HAM for \( w(x, t) \) for per \( h \). Solid line: \( w_{25} \); Dashed line: \( w_{20} \).](image2)
4 Conclusions

The goal of this work has been to derive an approximation for solution of hyperbolic equation. We have achieved this goal by applying the Homotopy analysis method. The HAM contains the auxiliary parameter $h \neq 0$, which provides us with a simple way to adjust and control the convergence region of solution. So the HAM overcomes the difficulties arises in the perturbation and Adomian decomposition method and traditional non-perturbation methods.

References

[1] S. Abbasbandy, E. Shivanian, K. Vajravelu, Mathematical properties of $h$-curve in the frame work of the homotopy analysis method, Commun Nonlinear Sci Numer Simulat. 16 (2011) 4268-4275. 
http://dx.doi.org/10.1016/j.cnsns.2011.03.031.

[2] S. Abbasbandy, Homotopy analysis method for the Kawahara equation, Nonlinear Analysis: Real World Applications 11 (2010) 307-312. 
http://dx.doi.org/10.1016/j.nonrwa.2008.11.005.

[3] S. Abbasbandy, T. Hayat, On series solution for unsteady boundary layer equations in a special third grade fluid, Commun Nonlinear Sci Numer Simulat 16 (2011) 3140-3146. 
http://dx.doi.org/10.1016/j.cnsns.2010.11.018.

[4] S.A. Berlin, Existence of solution for one-dimensional wave equations with nonlocal conditions, Electron J Diff Eqn 76 (2001) 1-8.
[5] J. Chabrowski, On nonlocal problems for parabolic equations, Negaya Math. J. 93 (1984) 109-131.

[6] Y.S. Choi, K.Y. Chan, A parabolic equation with nonlocal boundary conditions arising from electrochemistry, Nonlinear Anal. Theory Methods Appl. 18 (1992) 317-331. http://dx.doi.org/10.1016/0362-546X(92)90148-8.

[7] M. Dehghan, Efficient techniques for the second-order parabolic equation Subject to nonlocal specifications, Applied Numerical Mathematics 52 (2005) 39-62. http://dx.doi.org/10.1016/j.apnum.2004.02.002.

[8] M. Dehghan, On the solution of an initial- boundary value problem that combines Neumann and integral condition for the wave equation, Numer Methods Partial Diff Eqn 21 (2005) 24-40. http://dx.doi.org/10.1002/num.20019.

[9] M. Dehghan, Variational iteration method for solving the wave equation Subject to an integral conservation condition, Chaos, Solitons and Fractals 41 (2009) 1448-1453. http://dx.doi.org/10.1016/j.chaos.2008.06.009.

[10] I.M. Gelfand, Some questions of analysis and differential equations, UMN, Ser. 3 (1995) 3-19.

[11] S.J. Liao, Beyond Perturbation: Introduction to the Homotopy Analysis Method, Chapman and Hall/ CRC Press, Boca Raton, 2003.

[12] S.J. Liao, on the homotopy analysis method for nonlinear problems, Appl. Math. Comput. 147 (2004) 499-513. http://dx.doi.org/10.1016/S0096-3003(02)00790-7.

[13] S.J. Liao, Comparison between the homotopy analysis method and homotopy perturbation method, Appl. Math. Comput. 169 (2005) 1186-1194. http://dx.doi.org/10.1016/j.amc.2004.10.058.

[14] S. Momani, Analytic approximate solution for fractional heat-like and wave-like equations with variable coefficients using the decomposition method, Appl. Math. Comput. 165 (2005) 459-472. http://dx.doi.org/10.1016/j.amc.2004.06.025.

[15] A. Saadatmandi, M. Dehghan, Numerical solution of the one-dimensional wave equation with an integral condition, Numer Methods Partial Diff Eq 23 (2007) 282-292. http://dx.doi.org/10.1002/num.20177.

[16] L.I. Serbina, Nonlocal mathematical models of movement in watertransit system, MosCow, Nauka, 2007.

[17] L.A. Zolina, On a boundary-value problem for hyperbolic-parabolic equation, computational mathematics and mathematical physics ofUSSR 6 (1966) 63-78.