Two Contrastive Boson-Pair Coherent States in Deformed Boson Scheme

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Abstract

Two types of boson-pair coherent states, which were proposed independently by the present authors, are investigated. Main conclusion is as follows: Although the two states are superposed contrastively in terms of the boson-pairs, the expectation values of the boson-pair operators for these states are the same as one another with high accuracy.
The boson coherent state has played a central role in the studies of many-body physics, and for various problems, many ideas for the modification have been proposed. The squeezed boson coherent state is one of examples and the present authors have studied it extensively. Further, recently, from the viewpoint of the $q$-deformed boson scheme, the modifications were reinvestigated by the present authors. An example is the multiboson coherent state. Under the idea by Penson and Solomon, the present authors investigated the following multiboson coherent state:

$$|c_m(+)\rangle = \left(\sqrt{\Gamma_m(+)}\right)^{-1} \sum_{n=0}^{m-1} \sum_{r=0}^{m-1} (\sqrt{n!} \sqrt{r!})^{-1} \gamma_n^r \delta_n^r \left(\sqrt{C_m(n, r)}\right)^{r+1} |mn + r\rangle ,$$

$$C_m(n, r) = n! [(mn + r)!]^{-1} .$$

Here, $\gamma_+$ and $\delta_+$ denote complex parameters and $m$, $n$, and $r$ are integers which are shown as follows:

$$m = 2, 3, 4, \cdots , \quad n = 0, 1, 2, \cdots , \quad r = 0, 1, 2, \cdots , (m-1) .$$

The quantity $\Gamma_m(\cdot)$ denotes the normalization factor given by

$$\Gamma_m(\cdot) = \sum_{r=0}^{m-1} (r!)^{-1} (|\delta_+|^2)^r \left(\sum_{n=0}^{m-1} (n!)^{-1} C_m(n, r)^{-1} (|\gamma_+|^2)^n\right) .$$

The states $|mn + r\rangle$ compose an orthonormal set:

$$|mn + r\rangle = \left(\sqrt{(mn + r)!}\right)^{-1} (\bar{c}^*)^{mn+r} |0\rangle ,$$

$$\bar{c}|0\rangle = 0 .$$

Here, $(\bar{c}, \bar{c}^*)$ denotes boson operator. The condition (3) tells us that the set $\{|mn + r\rangle\}$ is complete. Hereafter, the space spanned by the states (5) is called $\bar{B}$. We can prove that $\Gamma_m(\cdot)$ is convergent for any $m$ in the region

$$|\gamma_+|^2 < \infty . \quad (m = 2, 3, 4, \cdots)$$

This means that the state $|c_m(+)\rangle$ is normalizable for $m = 2, 3, 4, \cdots$. It may be clear that the state $|mn + r\rangle$ consists of the multiboson $(\bar{c}^*)^m$. In this sense, we call the state $|c_m(+)\rangle$ the multiboson coherent state.

On the other hand, formally, we can set up the following state:

$$|c_m(-)\rangle = \left(\sqrt{\Gamma_m(-)}\right)^{-1} \sum_{n=0}^{m-1} \sum_{r=0}^{m-1} (\sqrt{n!} \sqrt{r!})^{-1} \gamma_n^r \delta_n^r \left(\sqrt{C_m(n, r)}\right)^{r+1} |mn + r\rangle .$$

The normalization factor $\Gamma_m(\cdot)$ is written down as

$$\Gamma_m(\cdot) = \sum_{r=0}^{m-1} (r!)^{-1} (|\delta_-|^2)^r \left(\sum_{n=0}^{m-1} (n!)^{-1} C_m(n, r)^{-1} (|\gamma_-|^2)^n\right) .$$
However, $\Gamma_m(-)$ is convergent only for $m = 2$ in the region

$$|\gamma_-|^2 < 1/4. \quad (m = 2) \quad (10)$$

For the cases $m = 3, 4, 5, \ldots$, $\Gamma_m(-)$ is divergent in any region of $|\gamma_-|^2$ and in these cases, the state (8) loses its meaning. The states $|c_{m=2}(\pm)\rangle$ consist of the boson-pair $(\tilde{c}^*)^2$ and, then, we call them the boson-pair coherent states. We can see that the form of the superposition of $|c_2(\pm)\rangle$ is contrastive to that of $|c_2(-)\rangle$. The state $|c_2(-)\rangle$ was already investigated by the present authors with an interesting conclusion: With the aid of the state (8) with $m = 2$, we are able to obtain the classical counterpart of the boson-pair, which is treated as the set $(\tilde{T}_+, \tilde{T}_-, \tilde{T}_0)$ defined by

$$\tilde{T}_+ = (1/2)\tilde{c}^2, \quad \tilde{T}_- = (1/2)\tilde{c}^2, \quad \tilde{T}_0 = (1/2)\tilde{c}^*\tilde{c} + 1/4. \quad (11)$$

The set $(\tilde{T}_+, \tilde{T}_-, \tilde{T}_0)$ obeys the $su(1,1)$-algebra. Further, by the requantization, we get the Holstein-Primakoff type boson representation for the set $(\tilde{T}_+, \tilde{T}_-, \tilde{T}_0)$. The above-mentioned scheme presents us a problem: To investigate that the state (1) for $m = 2$ can produce the above-mentioned feature. The above task is the purpose of this paper.

By adopting the basic idea of the MYT boson mapping, we investigate the structure of the states (1) and (8) for $m = 2$ from the viewpoint of the deformed boson scheme. In order to get the images of $|c_2(\pm)\rangle$, we must prepare a new space, which is called, hereafter, $\hat{B}$. Following the remarks mentioned in the final part of Ref., we construct the space $\hat{B}$ in terms of the boson $(\tilde{c}, \tilde{c}^*)$ and the operator $(\tilde{d}, \tilde{d}^*)$. For the boson $(\tilde{c}, \tilde{c}^*)$, we have orthonormal set in the form

$$|n\rangle = (\sqrt{n}!)^{-1}(\tilde{c}^*)^n|0\rangle, \quad (n = 0, 1, 2, \cdots) \quad (12)$$
$$\tilde{c}|0\rangle = 0. \quad (13)$$

Let the operator $(\tilde{d}, \tilde{d}^*)$ obey the following commutation relation:

$$[\tilde{d}, \tilde{d}^*] = 1 - (m/(m - 1)!)(\tilde{d}^*)^{m-1}(\tilde{d})^{m-1}. \quad (14)$$

For the operator $(\tilde{d}, \tilde{d}^*)$, we presuppose the existence of the state $|0\rangle$ obeying

$$\tilde{d}|0\rangle = 0. \quad (15)$$

Then, with use of the relations (12) and (15), we are able to obtain the following set:

$$|r(m)\rangle = \begin{cases} (\sqrt{r}!)^{-1}(\tilde{d}^*)^r|0\rangle, & (r = 0, 1, 2, \cdots, m - 1) \\ 0 & (r = m, m + 1, \cdots) \end{cases} \quad (16)$$
The operation of $\hat{d}^*$ on the state $|r = m - 1(m)\rangle$ automatically leads us to

$$\hat{d}^*|r = m - 1(m)\rangle = 0. \quad (17)$$

The relation (17) gives us the form in the lower equation of (16). We can also prove the relation

$$\tilde{N}_d|r(m)\rangle = r|r(m)\rangle, \quad (r = 0, 1, 2, \cdots, m - 1), \quad \tilde{N}_d = \hat{d}^* \hat{d}. \quad (18)$$

Therefore, the set $\{|r(m)\rangle; r = 0, 1, 2, \cdots, m - 1\}$ is orthonormal and complete. Then, Dirac’s sense, we can set up the following relation for $(\hat{d}, \hat{d}^*)$ :

$$(\hat{d})^m = 0, \quad (\hat{d}^*)^m = 0. \quad (19)$$

Combining the states (12) with the states (16), we define the orthonormal set $\{|n, r(m)\rangle\}$ :

$$|n, r(m)\rangle = |n\rangle \otimes |r(m)\rangle. \quad (20)$$

The set consisting of the state (20) composes the space $\hat{B}$. Of course, $\{|n, r(m)\rangle\}$ is complete.

Hereafter, we treat the case $m = 2$ concretely.

The normalization factors $\Gamma_2(\pm)$ are calculated as follows :

$$\Gamma_2(+) = \cosh |\gamma_+| + |\delta_+|^2 \cdot |\gamma_+|^{-1} \sinh |\gamma_+|, \quad (21a)$$

$$\Gamma_2(-) = \left(\sqrt{1 - 4|\gamma_-|^2}\right)^{-1} + |\delta_-|^2 \cdot \left(\sqrt{1 - 4|\gamma_-|^2}\right)^{-3}. \quad (21b)$$

The images of $|c_2(\pm)\rangle$ in the space $\hat{B}$, which we denote $|c_2(\pm)\rangle$, are expressed in the form

$$|c_2(\pm)\rangle = U|c_2(\pm)\rangle. \quad (22)$$

Here, $U$ is given as

$$U = \sum_{n=0}^{\infty} \sum_{r=0}^{1} |n, r(2)\rangle \langle 2n + r|. \quad (23)$$

The relation (22) presents the following expression for $|c_2(\pm)\rangle$ :

$$|c_2(\pm)\rangle = \left(\sqrt{\Gamma_2(\pm)}\right)^{-1} \exp \left(2^{\pm 1} \gamma_+ \hat{c}^\dagger \left(\sqrt{\tilde{N}_c + \tilde{N}_d + 1/2}\right)^{\mp 1}\right) \exp \left(\delta_\pm \hat{d}^*\right) |0\rangle. \quad (24)$$

Here, $|0\rangle$ denotes the vacuum for $\hat{c}$ and $\hat{d}$. Since $(\hat{d}^*)^2 = 0$, which comes from the relation (19), we should note the form

$$\exp \left(\delta_\pm \hat{d}^*\right) |0\rangle = |0\rangle + \delta_\pm \hat{d}^*|0\rangle. \quad (25)$$
For the states (24), we define the following operators:

\[
\hat{\gamma}_\pm = 2^{\pm 1} \left( \sqrt{N_c + \hat{N}_d + 1/2} \right)^{\pm 1} \hat{c},
\]
\[
\hat{\delta}_\pm = \left( \sqrt{(N_c + \hat{N}_d + 1/2) \cdot (\hat{N}_d + 1/2)^{-1}} \right)^{\pm 1} \hat{d}.
\]  

(26)  
(27)

The operators \(\hat{\gamma}_\pm\) and \(\hat{\delta}_\pm\) satisfy the commutation relations

\[
[\hat{\gamma}_\pm, \hat{\gamma}_\mp^*] = 1, \quad [\hat{\delta}_\pm, \hat{\delta}_\mp^*] = 0, \quad [\hat{\gamma}_\pm, \hat{\delta}_\pm] = 0.
\]  

(28)

The operation of \(\hat{\gamma}_\pm\) and \(\hat{\delta}_\pm\) on the states \(|c_2(\pm)\rangle\) gives us

\[
\hat{\gamma}_\pm |c_2(\pm)\rangle = \gamma_\pm |c_2(\pm)\rangle, \quad (\hat{\gamma}_\mp^* |c_2(\pm)\rangle = 0),
\]
\[
\hat{\delta}_\pm |c_2(\pm)\rangle = \delta_\pm \hat{P}_1 |c_2(\pm)\rangle. \quad (\hat{P}_1 = |0\rangle\langle 0|)
\]  

(29a)  
(29b)

The relation (29) tells us that the states \(|c_2(\pm)\rangle\) are regarded as coherent states for the operators \(\hat{\gamma}_\pm\) and \(\hat{\delta}_\pm\), respectively.

It may be interesting to introduce new parameters \((c, c^*)\) and \((d, d^*)\) in our present system, which obey the canonicity condition

\[
\langle c_2(\pm)|\partial_c|c_2(\pm)\rangle = c^*/2, \quad \langle c_2(\pm)|\partial_d|c_2(\pm)\rangle = d^*/2.
\]  

(30)

As was shown in the TDHF theory in canonical form, \((c, c^*)\) and \((d, d^*)\) obeying the condition (30) can be regarded as canonical variables in the boson type. The condition (30) gives us

\[
c = \gamma_\pm \sqrt{(\partial \Gamma_2(\pm)/\partial |\gamma_\pm|^2) \cdot \Gamma_2(\pm)^{-1}}, \quad d = \delta_\pm \sqrt{(\partial \Gamma_2(\pm)/\partial |\delta_\pm|^2) \cdot \Gamma_2(\pm)^{-1}}.
\]  

(31)

Since \(\Gamma_2(\pm)\) are functions of \(|\gamma_\pm|^2\) and \(|\delta_\pm|^2\), we can express \(\gamma_\pm\) and \(\delta_\pm\) in terms of \((c, c^*)\) and \((d, d^*)\). As was done in Ref.5, the expression for \((\gamma_-, \delta_-)\) is easily obtained in the form

\[
\gamma_- = (1/2)c(\sqrt{|c|^2 + |d|^2 + 1/2})^{-1},
\]
\[
\delta_- = d\sqrt{(1/2 + |d|^2)(1 - |d|^2)^{-1}(\sqrt{|c|^2 + |d|^2 + 1/2})^{-1}}.
\]  

(32a)  
(32b)

Of course, we used the relation (21b). In the case of \((\gamma_+, \delta_+)\), the relations (21a) and (31) give us

\[
|\gamma_+|^2 = (1 - |d|^2)^{-1} \cdot (2|c|^2 + |d|^2(1 - h(|\gamma_+|))) \cdot h(|\gamma_+|),
\]
\[
|\delta_+|^2 = |d|^2(1 - |d|^2)^{-1} \cdot h(|\gamma_+|),
\]
\[
h(|\gamma_+|) = |\gamma_+| \coth |\gamma_+|.
\]  

(33a)  
(33b)  
(34)
As is clear from the form (33), it may be impossible to have simple expression for \(|\gamma_+|^2, |\delta_+|^2\) in terms of \(|c|^2, |d|^2\). Therefore, we try to give approximate expressions in the two extreme regions: \(|\gamma_+| \to 0\) and \(|\gamma_+| \to \infty\). First, we note that the following expressions can be derived:

\[
h(|\gamma_+|) = 1 + (1/3)|\gamma_+|^2 - (1/45)|\gamma_+|^4 + (2/945)|\gamma_+|^6 - \cdots, \quad \text{for} \quad |\gamma_+| < \pi , \quad (35)
\]

\[
h(|\gamma_+|) = |\gamma_+|(1 + 2e^{-2|\gamma_+|} + 2e^{-4|\gamma_+|} + 2e^{-6|\gamma_+|} + \cdots), \quad \text{for} \quad |\gamma_+| \geq 0 . \quad (36)
\]

The forms (35) and (36) are useful for the regions \(|\gamma_+|^2 \to 0\) and \(|\gamma_+|^2 \to \infty\), respectively. By using the relation (35), the form (33a) is reduced to the following for the region \(|\gamma_+| \to 0\):

\[
|\gamma_+|^2 = 2|c|^2 + (2/3)(|c|^2 + |d|^2)|\gamma_+|^2 - (2/45)(|c|^2 + 2|d|^2)|\gamma_+|^4 + \cdots . \quad (37)
\]

The relation (37) leads us to the approximate expressions for \((\gamma_+, \delta_+)\) in the region \(|\gamma_+| \to 0\):

\[
\gamma_+ = 2c\sqrt{(1/3)(|c|^2 + |d|^2) + 1/2} , \quad \delta_+ = d\left(\sqrt{1 - |d|^2}\right)^{-1}\sqrt{(2/3)|c|^2 + 1} . \quad (38)
\]

The relation (36) gives us the following form which is reduced from the relation (33) for the region \(|\gamma_+| \to \infty\):

\[
|\gamma_+|^2 = (2|c|^2 + |d|^2)^2 + 4\left((2|c|^2 + |d|^2)^2 - 2|d|^2(2|c|^2 + |d|^2)|\gamma_+|\right)e^{-2|\gamma_+|}
\]

\[
+ 8\left((2|c|^2 + |d|^2)^2 - 4|d|^2(2|c|^2 + |d|^2)|\gamma_+| + 2|d|^4|\gamma_+|^2\right)e^{-4|\gamma_+|} + \cdots . \quad (39)
\]

Then, we have the approximate expressions for \((\gamma_+, \delta_+)\) in the region \(|\gamma_+| \to \infty\):

\[
\gamma_+ = 2c\sqrt{|c|^2 + |d|^2 + |d|^4/4|c|^2} , \quad \delta_+ = d\left(\sqrt{1 - |d|^2}\right)^{-1}\sqrt{2|c|^2 + |d|^2} . \quad (40)
\]

Since \(|c_2(+)\) is different from \(|c_2(-)\), it is meaningless to compare the results (38) and (40) with (32) directly.

The comparison of both results can be done through the expectation values of the images of \(\tilde{T}_{\pm,0}\), defined in the relation (31). The operators \(\hat{T}_{\pm,0}\) defined by \(U\hat{T}_{\pm,0}U^\dagger\) are given by

\[
\hat{T}_+ = \hat{c}^*\sqrt{\hat{N}_c + \hat{N}_d + 1/2} , \quad \hat{T}_- = \sqrt{\hat{N}_c + \hat{N}_d + 1/2} \hat{c} , \quad \hat{T}_0 = \hat{N}_c + \hat{N}_d/2 + 1/4 . \quad (41)
\]

The expectation value of \(\hat{T}_+\) for \(|c_2(-)\) is shown in Ref.5:

\[
\langle c_2(-)|\hat{T}_+|c_2(-)\rangle = c^*\sqrt{|c|^2 + |d|^2 + 1/2} . \quad (42)
\]
In the case of $|c_2(+)\rangle$, we note the relation $\hat{T}_+ = \hat{\gamma}_+^*/2$. Then, in the region $|\gamma_+| \to 0$, i.e., $|c| \to 0$, the first equation in (38) gives us

$$\langle c_2(+) | \hat{T}_+ | c_2(+) \rangle = c^* \sqrt{(1/3)(|c|^2 + |d|^2) + 1/2} \ .$$  \hspace{1cm} (43a)

In the region $|\gamma_+| \to \infty$, i.e., $|c| \to \infty$, the first equation in (40) leads us to

$$\langle c_2(+) | \hat{T}_+ | c_2(+) \rangle = c^* \sqrt{|c|^2 + |d|^2 + |d|^4/4|c|^2} \ .$$  \hspace{1cm} (43b)

For $\hat{T}_0$, both forms $|c_2(\pm)\rangle$ give us the common result :

$$\langle c_2(\pm) | \hat{T}_0 | c_2(\pm) \rangle = |c|^2 + |d|^2/2 + 1/4 \ .$$  \hspace{1cm} (44)

The operator $\hat{c}^*$ defined by $U \hat{c} U^\dagger$ is also interesting. It is given by

$$\hat{c}^* = \sqrt{2}(\hat{c}^* \hat{d} + \hat{d}^* \hat{c} + \hat{d}^* \hat{d} + 1/2) \ .$$ \hspace{1cm} (45)

The expectation value for $|c_2(-)\rangle$ is calculated as

$$\langle c_2(-) | \hat{c}^* | c_2(-) \rangle = \sqrt{(1 - |d|^2)(1/2 + |d|^2)^{-1}} \left(c^* d + d^* \sqrt{|c|^2 + |d|^2 + 1/2} \right) \ .$$ \hspace{1cm} (46)

The above is shown in Ref.[5]. The expectation value for $|c_2(+)\rangle$ is given in the form

$$\langle c_2(+) | \hat{c}^* | c_2(+) \rangle = (\gamma_+^* \delta_+ + \delta_+^* h(|\gamma_+|))(|\delta_+|^2 + h(|\gamma_+|))^{-1} \ .$$ \hspace{1cm} (47)

In the region $|\gamma_+| \to 0$, i.e., $|c| \to 0$, the relation (47) is approximated as

$$\langle c_2(+) | \hat{c}^* | c_2(+) \rangle = \sqrt{(1 - |d|^2)(1/2)^{-1}} \left(c^* d + d^* \sqrt{(1 + (2/3)(|c|^2 + |d|^2)(1 + (2/3)|c|^2)^{-1}}

+ d^* \sqrt{(1/3)|c|^2 + 1/2} \right) \ .$$ \hspace{1cm} (48a)

Further, the approximate form in the region $|\gamma_+| \to \infty$, i.e., $|c| \to \infty$, is obtained as follows :

$$\langle c_2(+) | \hat{c}^* | c_2(+) \rangle = \sqrt{(1 - |d|^2)(1/2)^{-1}} \left(c^* d + d^* \sqrt{|c|^2 + |d|^2/2} \right) \ .$$ \hspace{1cm} (48b)

The expectation values of $\hat{T}_+$ and $\hat{c}^*$ shown in the relations (12) and (13), respectively, for the state $|c_2(-)\rangle$ are exactly calculated in any region of $|c|^2$ and the case of $\hat{T}_0$ shown in the form (14) is also in the same situation as the above. Further, the expectation values of $\hat{T}_{\pm,0}$ give us the Poisson brackets under the form which are completely the same as those of the commutation relations for $\hat{T}_{\pm,0}$ in Dirac’s sense. This fact was shown in Ref.[5]. From the above reason, we can see that the state $|c_2(-)\rangle$ gives us the classical counterpart for the set $(\hat{T}_+, \hat{T}_-, \hat{T}_0)$. On the other hand, we gave the approximate expressions in the two extreme regions for the state $|c_2(+)\rangle$. However, the leading terms with respect to $|c|^2$ are exact
and, the forms are almost the same as those in the case of $|c_2(-)\rangle$ in the framework of the leading term. Therefore, we can able to reproduce the same aspects as those for $|c_2(-)\rangle$ with rather high accuracy in spite of contrastive superpositions for the two boson-pair coherent states. Further, it can be expected that the approximation shown in the relation (38) in Ref.2 may be permitted. We note again that the state $|c_2(+)\rangle$ can be extended to the case $m = 3, 4, 5, \cdots$, but the state $|c_2(-)\rangle$ cannot be done. From the above reason, $|c_m(+)\rangle$ may be superior to $|c_m(-)\rangle$. This is our conclusion. It may be interesting to apply the state $|c_2(+)\rangle$ to the Lipkin model, the prototype of which was already presented in Ref.9).

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