Abstract

In this paper I study oriented metric ribbon graphs. I show that it’s possible to decompose these graph in some canonical way by performing surgery along appropriate multi-curves. This result provide a recursion scheme for the volumes of the moduli space of 4–valent metric ribbon graphs which can be interpreted as an oriented version of the topological recursion. I give applications to counting dessins d’enfants in a particular case.

Keywords— Metric ribbon graphs, 4–valent ribbon graphs, geometric recursion, bipartite maps, measurable foliations, dessin d’enfants.

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1 Intro:

Ribbon graphs have been studied in many places in the literature they are also called maps or fatgraphs in some places. They can be seen as cell decomposition of the surface or combinatorial model of surfaces with boundaries if we remove an open disc in each face. A metric ribbon graph is then a ribbon graph with a positive weight on each edge [12],[3]. They provide a cell decomposition of the moduli space and were use in [10] in order to prove the Witten conjecture. Ribbon graph also appear naturally in Feymann diagram expansion of correlators in models of matrix integrals[6]. In an other setting they can be used to compute volumes of spaces of abelian and quadratic differentials [5]. Then they are object of interest in this field because volume he moduli space of metric ribbon graphs can be used to understand the topology of quadratic and abelian differentials. And compute statistics for the distribution of cylinders in periodic surfaces.

Volumes of moduli space of metric ribbon graph: In this paper I investigate in some sense the geometry of metric ribbon graphs and I will use ideas which are close to the ones of [4]. I will study curves on ribbon graphs in order to decompose them in simple pieces. Results are know in the case of three valents ribbon graphs, an analogous of the Mirzakhani Mac Shan formula can be use to prove a topological recursion scheme for the volumes [3], nevertheless outside this case very few things are know on these volumes. I will mainly focus for applications to the case of 4−valents graphs with a coherent orientation of the edges of the graph. These graphs are related to orientable foliations with poles, abelian differentials with simple poles and also bipartite maps. Each stratum of the moduli space of metric ribbon graphs have a piece-wise linear structure with a lattice of integral points it allow to define a Lebesgue measure normalised by this lattice. And I will consider the volumes of the level set of surfaces with fixed boundary lengths. In the case of oriented ribbon graphs these lengths are not independents. This dependence correspond to a residues condition, each boundary come with a sign and the sum of the lengths is zero if they are weighted by these signs. It remain that I will split the boundaries in two sets, the positives and negatives and the volumes that I consider are functions of two set of variables ($L^+, L^-$)

In general these volumes are not well defined for all values of $L^+, L^-$ which satisfy the residue condition. But in the case of four valent graphs I will give the following result.

Theorem 1. The volumes $Z_{g,n^+,n^-,n^-}(L^+,L^-)$ are continuous for all $L^+, L^-$ and are homogeneous piece-wise polynomials of degrees $4g-3+n^++n^-$

The piece wise polynomiality is annoying for direct uses of these volumes.

Surgeries on the graphs In order to perform surgeries on our graphs I introduce the notion of admissible curves on a ribbon graph, this is a particular subset of homotopy classes of simple curves on which it’s possible to define the twist flow and perform surgeries. Admissible curves are the one that do not split a vertex and are related to quadratic differentials with double poles. On an oriented ribbon graph admissible curves admit an orientation and this define a structure of directed stable graph. In other word there is a sign on each boundary of the surface obtained after cutting along the multi-curve and the sign are opposites on two boundaries glued by a curve. By studying admissible curves I proved that it’ possible to find a multi-curve that separate the vertex from the rest of the surface and this choice is indeed canonical. There is a two form on each stratum of metric ribbon graphs which come from the intersection pairing in cohomology. When there is vertices of even degrees this form can degenerate and then the space have only a Poisson structure. The curve that are in the kernel of this form enjoy many interesting properties and are the ones which are used to decompose the surface.

Theorem 2. Let R an oriented metric ribbon graph with at least two vertices. For each vertex $v$ of R there exist a unique admissible multi-curve $\Gamma_v^+$ such that

- The stable graph $G_0^+$ of $\Gamma_v^+$ contain a component $c_0$ which separate $v$ from the rest of the surface.
- All the curves in $\Gamma_v$ are boundaries of $c_0$.
- $c_0$ is glued along it’s negative boundaries.

From this theorem, in the orientable case, it’s possible to find a decomposition of the graph such that each component contain a unique vertex. The main property of these stable graph is the fact that they are acyclic, this
is the reason the rigidity of the theorem. We can find numerous decomposition’s of the surface in minimal surfaces but really few of them correspond to acyclic stable graphs. Acyclic multi-curves are very rigid in some sense. This theorem can also be used to remove vertices of even degree in an unorientable ribbon graph but we do not give the detail here.

Recursion for the volumes: By using this theorem and techniques which where introduced by M.Mirzakhani \( [11] \) I can perform surgeries over the moduli space of orientable ribbon graphs and this give a recursion for the volumes \( Z_{g,n^+,n^-}(L^+|L^-) \). The form of the recursion is in some sense similar to recursion of \( [3] \) and \( [11] \) but I need to take care to the sign of the boundaries. The recursion correspond to removing a pair of pant’s which is glued along negatives boundaries positive boundaries. In this case the gluings that I consider are the following (see figure 1)

1. Removing a pant that contain two positive boundaries \((i,j)\)
2. I remove a pant that contain a positive boundary \(i\) and a negative boundary \(j\)
3. Removing a pant with one positive boundary \(i\) which is connected to the surface by two negative boundaries and do not separate the surface
4. Removing a pant with one positive boundary \(i\) which is connected to the surface by two negative boundaries and separates in the surface in two components.

Then by applying theorem \( [2] \) I obtain the following theorem \( [\cdot]_+ \) means the positive part. This theorem allow to compute the volumes inductively by a recursion scheme without using enumeration of the ribbon graphs and it’s an efficient way for low value of \( g \) and \( n\pm\) but understand the structure of the volumes could help to make this recursion more effective.

**Theorem 3.** For all value of the boundaries lengths the volumes satisfy the recursion

\[
(2g - 2 + n^+ + n^-)Z_{g,n^+,n^-}(L^+|L^-) = \sum_i \sum_j [L_i^+ - L_j^-] Z_{g,n^+,n^-}(\sum_{i \neq j} (|L_i^+ - L_j^-|) +, L_{(i,j)}^+,|L_{(i,j)}^-|)
\]

\[
+ \frac{1}{2} \sum_i \int_0^{L_i^+} Z_{g-1,n^+,n^-}(x,L_i^+ - x,|L_{(i)}^-|) x(L_i^+ - x) \ dx
\]

\[
+ \frac{1}{2} \sum_{i,j} \sum_{\{i,j\} = \{1,\ldots,n\}} x_1 x_2 Z_{g_1,n_1^+,1,n_1^-} (x_1, L_{i_1}^+|L_{i_1}^-) Z_{g_2,n_2^+,1,n_2^-} (x_2, L_{i_2}^+|L_{i_2}^-)
\]

where we use the notation

\[
x_i = \sum_{L_i^- \in I^-} L_i^- - \sum_{L_i^+ \in I^+} L_i^+,
\]

and the sets \( I^\pm \) correspond to \( \{1,\ldots,n^\pm\} \) minus the positives boundary \( i \).

**Remark 1.** Going to the Laplace transform lead to a recursion in terms of partial derivatives and things seems to going well. But the fact that our functions are supported on some affine submanifold make computations different than in the usual case.

Special case, cut and join equation and Grotendieck Dessins d’enfants: There is one case where the volumes and the recurrence are particularly nice, it correspond to the surfaces with only one negative boundary. In this case I can write the volumes as functions \( F_{g,n^+}(L) \) of only the positives boundaries. The only gluing that are allowed are the one of type (1) and (3), moreover the fact that there is only one negative boundary don’t provide enough ”space” for discontinuities.
Figure 1: The different gluing of the recursion

**Theorem 4.** The functions $F_{g,n}$ are homogeneous polynomials of degree $4g - 4 + n$ and satisfy the following recursion

$$(2g - 1 + n)F_{g,n}(L) = \frac{1}{2} \sum_{i \neq j} (L_i + L_j) F_{g,n-1}(L_i + L_j, L_{(i,j)^e})$$

$$+ \frac{1}{2} \sum_i \int_0^{L_i} F_{g-1,n+1}(x, L_i - x, L_{(i)^e}) x(L_i - x) \, dx$$

This recursion leads to a recursion for the coefficients of the polynomials. They are symmetric and then they can be indexed by partitions $\mu = (\mu^{(0)}, \mu^{(1)}, \ldots)$, I can form a generating function $\phi$ given by

$$\phi(s, t) = \sum_{\mu} s^{\frac{1}{2} \sum \mu^{(i)} \mu^{(i)}} \prod_i \frac{\mu^{(i)}}{\mu^{(i)}!} c(\mu).$$

Where $c(\mu)$ are the coefficient’s of the polynomials, then rewriting the recursion in term of this series leads to the following equation which is a cut and join equation.

**Corollary 1.** The series $\phi(s, t)$ satisfy the following equation

$$\frac{\partial \phi}{\partial s} = \sum_{i,j} (i+j) t_i t_j \frac{\partial \phi}{\partial t_{i+j}} + \sum_{i+j} (i+1)(j+1) t_{i+j-1} \frac{\partial^2 \phi}{\partial t_i \partial t_j} + t_0^2$$

This equation is not surprising because there is a bijection between orientable 4-valent graph and Dessins d’enfants with a maximal ramification over a point simple ramification over another and arbitrary ramifications over the third one. And then the last result give the cut and join for these Hurwitz numbers.

**Perspectives :**

- In a future work I plan to investigate the recursion in a more general context without the restriction on the order of the vertices. The recursion in this case but is explicit only for vertices of low order.
- In a future work I will expand the chapter on “dessins d’enfants” which is a special case of a more general result. I will give the relation between this recursion and the usual topological recursion in this case.
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2 Topology of surfaces:

In this first section I give some classical definition and result about surfaces and curves on it the main reference is [fathi2012thurston]. I will introduce directed surfaces and directed multi-curves which are important in the rest of this text. I will also introduce foliations and quadratic/abelian differentials with poles and study their elementary properties. I will also give quickly the relation between multi-arcs and foliations with poles.

2.1 Surfaces and decoration

In this paper I will consider oriented topological surfaces of genus $g$ with $n$ boundaries. All the surfaces that I consider will be stable i.e with a strictly negative Euler characteristic. Sometime I will also consider surfaces with labelled boundaries. If $M$ is the surface with boundaries I can consider three surfaces:

- The surface $M^{top}$ obtained by gluing a disc on each boundary
- The surface $M^*$ obtained by gluing a pointed disc
- $M^{db}$ the scotty double which is the surface obtained by doubling $M$ along it’s boundary.

By abuse of notations I generally denote $\partial M$ the set $\pi_0(\partial M)$. For a surface with boundaries the group $\text{Mod}(M)$ is the mapping class group of isotopy classes of homeomorphism which act trivially on the boundaries $\partial M$. I do not add marked points on the boundaries then I don’t allow twist along boundary curves.

Directed surfaces: I introduce directed surface which are surface with boundaries and an orientation of the boundaries $\epsilon$. By orientation of the boundaries I means the data of a map

$$\epsilon : \partial M \rightarrow \{\pm 1\}$$

which is non constant, I will use the notation $M^\epsilon$ for a couple $(M, \epsilon)$. The positive boundaries $\partial^+ M^\epsilon$ are in some sense the entries and the negative boundaries $\partial^- M^\epsilon$ are the exit of our surface, I will denote $n^\pm$ the respective cardinality of these sets, which is non zero by assumption.

For directed labelled surfaces I assume that each sets $\partial^\pm M^\epsilon$ is labelled from $1$ to $n^\pm$, and then a directed labelled surface is characterised by the triple $(g, n^+, n^-)$.

In what is next I will have positive weights on the boundaries which correspond to the lengths of the boundaries for a metric ribbon graph, the ”flow” through the boundary for measured foliation or the absolute value of the residue for a quadratic differential. As there is entries and exits in the surface I assume than the sum of the entry is equal to the sum of the exits. I will then consider the convex cone

$$\Lambda_{M^\epsilon} = \{ L \in (\mathbb{R}_+)^{\partial M} | \sum_{\beta} \epsilon(\beta) L_{\beta} = 0 \}. \quad (1)$$

When the boundaries are labelled I will denote the corresponding set $\Lambda_{n^+, n^-}$.

Remark 2. There is the natural orientation of the boundaries according to the orientation of the surfaces, and $\epsilon$ is an orientation of the boundary curves by assuming than the negative boundaries are oriented in the opposite direction compared to the standard orientation.

Decorations: We will briefly use some decoration on the surfaces which are partition $\mu = (1^{\mu(1)}, 2^{\mu(2)}, \ldots)$ such that

$$|\mu| - 2#\mu = 4g - 4 + 2n$$

where I use the notations

$$|\mu| = \sum_i i\mu(i) \quad #\mu = \sum_i \mu(i)$$
2.2 Simple curves and stable graphs:

Curves and multi-curves: Let $M$ is a topological surface an essential simple curves is an isotopy class of simple closed curves which are non contractibles and do not retracts on a boundary [fathi2012thurston]. A primitive multi-curve is a family of disjoint non isotopic simple closed curves. We can generalise this notion by considering integral multi-curves. An integral multi-curve can be represented formally by a sum
\[ \Gamma = \sum_{\gamma} m_{\gamma} \gamma, \]
where the sum run over the set of essential curves, the weights are strictly positive integers and two curves in the support of the sum are non intersecting.

Surgery along multi-curves and stable graphs: If $\Gamma$ is a multi-curve on $M$ along $\Gamma$. This create obtain a new topological surface $M_\Gamma$ with boundaries, which is well defined modulo isotopies. This procedure is encoded by a stable graphs. Such graph are defined by the following data’s

- A set of vertices $X_\partial G$
- A family of topological stable surfaces $(G(c))_{c \in X_\partial G}$ with boundaries
- If $X_G = \sqcup \pi_0(\partial G(c))$ then there in an involution $j : X_G \to X_G$.

These data’s define a topological surface $M_G$ by gluing the boundaries of $\sqcup_i G(c)$ which are identified by the involution. The fixed points of the involution are the boundaries of $M_G$. Let $X_1G$ the set of orbits of order two under the involution, each elements of $X_1G$ a curve on $M_G$ and the union of these curves define a primitive multi-curve on $M_G$. Reciprocally each multi-curve is associated to a stable graph $G$ characterised by $X_\partial G = \pi_0(M_\Gamma)$ and $G(c)$ is the connected component $M_\Gamma(c)$.

Two stable $G$ graph are isomorphic if there is an homeomorphism $M_{G_1} \to M_{G_2}$ that preserve the two decompositions. The set $st(M)$ of stable graphs marked by $M$ is the then equivalence class $\phi : M \to M_G$ modulo isomorphism $\alpha : M_{G_1} \to M_{G_2}$ such that $\phi_2^{-1} \circ \alpha \circ \phi_1$ is an element of $\text{Mod}(M)$.

Each stable graph come with a group of automorphisms $\text{Aut}(G)$ which are morphisms of graph $\phi : X_G \to X_G$ which fix each legs commute with $j$ and such that for each $c \in X_\partial G$ the surfaces $G(c)$ and $G(\phi(c))$ are homeomorphic.

Directed multi-curves: In the category of directed surfaces it’s natural to introduce directed multi-curves. Let $(M, \epsilon)$ a directed surface an orientation $\epsilon_\Gamma$ on a multi-curve $\Gamma$ is an orientation on the boundary of $M_\Gamma$ such that

- Each component $(M_\Gamma(c), \epsilon_\Gamma(c))$ of $M_\Gamma$ is directed
- if $\beta$ is not a fixed point of $j$ then then $\epsilon_\Gamma(j(\beta)) = -\epsilon_\Gamma(\beta)$,
- if $\beta \in \partial M$ then $\epsilon_\Gamma(\beta) = \epsilon(\beta)$.

Remark 3. Using the remark 4 a directed multi-curve is equivalent to an orientation of each curves by taking the orientation of the positive boundary.

Remark 4. We will see later all the directed multi-curves are not relevant for us and there is possible degenerations.

Directed graphs and convex cone: To a directed multi-curve we can associate a directed stable graph $G^\circ$. Directed stable graphs are defined in a similar way than directed multi-curves. Edges and leaves of such graph are oriented from $-\to +$. These directions represent flux through the directed surfaces. What I show later is that such graph can be used to decompose oriented foliations and ribbon graph in a canonical way.

If $G^\circ$ is an oriented stable graph then I consider convex cone
\[ \Lambda_{G^\circ} = \{ L \in \prod_c \Lambda_{G^\circ(c)} \mid L(\beta) = L(j(\beta)) \}. \]  
\[ (2) \]
If $\mathcal{G}^\circ$ is marked by $M^\circ$ there is a map

$$L_{\partial} : \Lambda_{\mathcal{G}^\circ} \longrightarrow \Lambda_{M^\circ}$$

and the image is a convex cone. For each $L$ in the image I will consider $\Lambda_{\mathcal{G}^\circ}(L)$ the level set which is an affine submanifold. The space $\Lambda_{\mathcal{G}^\circ}$ is the intersection of $\mathbb{R}^X_{\mathcal{G}^\circ}$ with the subspace $T_{\mathcal{G}^\circ}$ defined by the exact sequence

$$0 \longrightarrow T_{\mathcal{G}^\circ} \longrightarrow \mathbb{R}^X_{\mathcal{G}^\circ} \longrightarrow \mathbb{R}^X_{1\mathcal{G}^\circ} \longrightarrow 0.$$  

The dimension of $T_{\mathcal{G}^\circ}$ is always equal to $\# X_1\mathcal{G} + \# \partial\mathcal{G} - \# X_0\mathcal{G} + 1$ but it can happen that the dimension of $\Lambda_{\mathcal{G}^\circ}$ is smaller than expected. This occurs when constraints induced by the orientation are incompatibles and force curve to be equal to zeros. This is due to the topology of the directed graph and such graph have no physical meaning I this paper. I say that a graph is not degenerate if the dimension of $\Lambda_{\mathcal{G}^\circ}$ is $\# X_1\mathcal{G} + \# \partial\mathcal{G} - \# X_0\mathcal{G} + 1$.

The spaces $\Lambda_{\mathcal{G}^\circ}, \Lambda_{\mathcal{G}^\circ}(L)$ carries natural affine measures normalised by the set of integral points, we will denote them $d\sigma_{\mathcal{G}^\circ}$ and $d\sigma_{\mathcal{G}^\circ}(L)$. The measure $d\sigma_{\mathcal{G}^\circ}(L)$ is the conditional measure of $d\sigma_{\mathcal{G}^\circ}$ with respect to $d\sigma_{M^\circ}$.

**Topology of the graph :** If $\mathcal{G}^\circ$ is a directed graph and $\gamma$ is an edges in $X_1\mathcal{G}^\circ$ then there is a projection

$$l_{\gamma} : \Lambda_{\mathcal{G}^\circ} \longrightarrow \mathbb{R}_+.$$  

According to the topology of the graph several things can happen

- The length function is equal to zero on $\Lambda_{\mathcal{G}^\circ}$ and then the graph is degenerate
- The length is non zero but it is constant in $\Lambda_{\mathcal{G}^\circ}(L)$ and then its a function of $L$
- The length is non zero but it is bounded in $\Lambda_{\mathcal{G}^\circ}(L)$
- The length is non zero and unbounded in $\Lambda_{\mathcal{G}^\circ}(L)$ and then its a function of $L$

An absolute directed cycle in a directed graph is closed cycle which contain only edges oriented positively. A relative directed cycle is a directed path that goes from a positive to a negative boundary, absolute cycles and relative cycles define elements of $\Lambda_{\mathcal{G}^\circ}$, and we can say that it’s primitive if it can’t be written as a sum of two cycles. We have the following proposition.

**Proposition 1.** The extremal points of $\Lambda_{\mathcal{G}^\circ}$ are the primitive cycles.

**Proof.** I don’t give details here but if $u$ is an element of the cone and let $\gamma$ an edge in the support of $u$ i.e $u_\gamma > 0$. The fact that the sum of the entries and the exists is vanishing at each vertex of the graph imply that we can find an cycle $c$ that pass through this edge and with support contained in the one of $u$. This is a consequence of an exploration process in the graph. Moreover I can choose $c$ such that there is an edge with $u_\gamma' = c_\gamma' > 0$ and then the support of $u - c$ is strictly contained in the one of $u$. By induction of the cardinal of the support it give the claim.

From this proposition I can derive the following corollary.

**Corollary 2.** The following characterisation is true

- An edge $\gamma$ is degenerated iff $\gamma$ is not contained in any cycle in $\mathcal{G}^\circ$.
- The edge $\gamma$ is bounded iff $\gamma$ is not contained in any absolute cycle of $\mathcal{G}^\circ$.
- The edge $\gamma$ is constant iff $\gamma$ separate the graph in two connected components

**Acyclic graph and constant one :** I will use a lot a particular kind of directed stable graph which are the ones which are acyclic. This means that the graph does not contain any absolute cycle. In this case the orientation of the edges define a partial order relation on the vertices and this fact characterise acyclic graphs. From the result of proposition (2)we see that all the acyclic graphs are non degenerate.

It will be convenient for use to label the component of an acyclic graph and a natural way to do it is to assume that the enumeration is compatible with the order relation.

It’s possible to say that a graph is bounded if all the edges are bounded. From proposition 2.
Corollary 3. The graph $G^\circ$ is bounded iff it’s acyclic.

For an acyclic graph it make sense to compute the integral of a continuous function over the set $\Lambda_{G^\circ}(L)$.

An other particular kind of directed graph are the one such that all the edges are constant. According to proposition 2 this means that all the edges separate the graph and then.

Corollary 4. Constant graphs are directed trees stable.

For such graph $G^\circ$ the length of an edge factor through $L$ and is the restriction of a linear function

$$l_{G^\circ,\gamma} : \Lambda_{M^\circ} \rightarrow \mathbb{R}$$

A constant edge separate the graph in two connected components and if $I_1, I_2$ are the boundaries in these components we have

$$l_{G^\circ,\gamma} = \left| \sum_{\beta \in I_i} \epsilon(\beta) l_{\beta} \right|,$$

which is linear on the cell.

2.3 Foliations and differentials with poles :

Definition : For $M$ a surface with boundaries measurable foliations on $M$ can be define in several possible ways. Some references on the subject are for instance [fathi2012thurston]. Let $MF(M)$ as the space of foliations on $M^\bullet$ such that the one form that locally define the foliation is given (locally) by the real part of a quadratic differential with a simple poles at the puncture. I assume that the quadratic differential have no simple poles in this case. These objects are considered up to isotopies and whitehead moves. It’s possible to find a circle around each puncture of $M^\bullet$ such that the leaves of the foliation are all transverse or all tangent to the circle and cut the surface along these circles. This induce a foliation on $M$ which is ”transverse” to the boundaries in some sense. When the foliation is tangent to the circle there is no canonical choice of circles.

In what follow I will be mainly interested into orientable foliations. A foliation is locally given by a closed one form but this choice is not canonical there is a sign ambiguity. A foliation is orientable if it have a trivial monodromy
and in this case it can be represented globally by a closed one form.

The residue of an element \( \lambda \) in \( \mathcal{MF}(M) \) is the absolute value of the residue of a one form that locally represent \( \lambda \) near the poles. In the case of orientable surface there is no sign ambiguities and then residues are reals. Stoke theorem then imply than the sum of the residues is equal to zeros. When none of the residues is vanishing the signs define an orientation of the boundaries of \( M \). So it’s possible to consider the subspace \( \mathcal{MF}(M^\circ) \) of oriented foliation on a directed surface \( M^\circ \).

**Multi-arcs and foliations :** On a surface with boundaries we can consider arcs that rely two boundaries. Here I assume than these arcs are simple and non trivial in the sense that the surface obtained but cutting \( M \) along the arc does not contain component’s homeomorphic to a disc. A weighted multi-arc is the a sum

\[
\sum_a m_a a
\]

of arcs which are pairwise non intersecting. I will denote \( \mathcal{MA}_{k}(M) \) the space of multi-arcs.

A foliation define in a natural way a weighted multi-arc by looking the leaf which intersect the boundaries. This procedure is the exploration of the surface from the boundaries and it give a map

\[
\mathcal{MF}(M) \rightarrow \mathcal{MA}_{k}(M) \cap \{0\}.
\]

To construct this map I take a circle that surround each poles of the foliation with non vanishing residues. By choosing the circles close enough I can assume than the foliation intersect transversely these circles. Each circle define a contractible neighbourhood \( U_y \) of a pole \( y \) and if a leaf enter in such neighbourhood it cannot escape. Then the intersection of the singular leaf of the foliation and the circles is finite a finite set \( X_0 \lambda \) and each circle \( C_y \) is divided in a finite number of intervals. We denote \( X \lambda \) the set of intervals. If \( x \in C_y \) is a point in one of these intervals, it’s possible to consider the half leaf starting to \( x \) in the direction opposite to \( U_y \). By assumption this leaf does not hit any singularities. By an adaptation of Thurston recurrence lemma \([fathi2012thurston]\) it’s then possible to show that such leaf must intersect an other circle \( C_y \) at a point \( T(x) \). The map \( T \) is well defined on the union of the intervals and induce a map

\[
s_1 : X\lambda \rightarrow X\lambda
\]

such that \( T \) map \( I \) to \( s_1(I) \). The map \( s_1 \) is an involution. A leaf of the foliation that join \( I \) and \( s_1(I) \) define an arc on the surface. The intersection product of these arcs with the boundary curves is non trivial then the arcs are non trivial. By surgeries it’s also possible to see that two arc are necessarily non homotopic. And then the foliation define a multi-arcs Moreover the transverse measure on \( \lambda \) induce a measure on each intervals and the total mass give a weight \( m_I(\lambda) \) on each interval \( I \). The map \( T \) preserve these measures and then \( m_I(\lambda) \) define weight on the arc \( a \) and this give the desired map.

A multi-arc define a partial foliation of the surface which can be extended by the Thurston enlargement procedure \([fathi2012thurston]\). So the map is surjective but it’s not injective in general. In some singular case the foliation can have interior leaves which can’t be detected by an exploration from the boundaries.

We the foliation is orientable the trajectories can be oriented in a way such that they goes from the positive boundaries to the negative boundaries. A directed multi-arc on \( M^\circ \) is then a multi-arc such that all the arcs join a positive and a negative boundary. Then in this case the restriction induce a map

\[
\mathcal{MF}(M^\circ) \rightarrow \mathcal{MA}_{\rightarrow}(M^\circ)
\]

I assume that the residues are non vanishing.

**Foliation with vanishing residues :** A particular subset \( \mathcal{MF}_0(M) \) of foliations with poles are the one with zero residues. \( \mathcal{MF}_0(M) \) contain a trivial element which is the Jenkin Strebel foliation it is periodic and all the non singular trajectories retract to a pole. This foliation will be denoted \( 0 \), it’s non trivial that it’s unique up to Whitehead moves. A non trivial foliation contains leaves which are not homotopic to boundary curves. It’s then possible to pinch the boundaries and obtain a foliation on the punctured surface \( M^* \), the foliation have a marked singularity on each puncture of \( M^* \) which is a conical singularity.
Proposition 2. The contraction define a bijection

\[ MF_0(M) \setminus \{0\} \longrightarrow MF(M^*) \]

The bijection is characterised by the fact that the two foliation have the same intersection number on the essential simple closed curves in \( M^* \).

Stratum of abelian differential: If \( M^\circ \) is a directed surface I will consider the Teichmüller space of abelian differential with simple poles \( \mathcal{HT}(M^\circ) \) and such that the sign of the real part of the residue correspond to the sign of the boundaries of \( M^\circ \). I will also consider the subspace \( \mathcal{HT}_0(M^\circ) \) of abelian differentials with real residues.

For a non directed surface I will consider the space \( QT(M) \) of quadratic differentials with double poles on \( M^* \) and \( QT_0(M) \) the subspace of surfaces with real residues.

If \( \mu \) is a decoration on \( M \) (paragraph 2.1). Then we will denote \( QT(M, \mu) \) the stratum of quadratic differentials with \( \mu(i) \) zeros of order \( i-2 \) and in a similar way we denote \( \mathcal{HT}(M^\circ, \mu) \) the abelian differentials with \( \mu(i) \) zeros of order \( \frac{i-2}{2} \).

Pair of transverse foliations: Two measurable foliations \( \lambda_1, \lambda_2 \) on a surface without boundaries are transverse iff for all essential simple curves they satisfy

\[ \iota(\lambda_1, \gamma) + \iota(\lambda_2, \gamma) > 0. \]

Quadratic differentials define pair of transverse foliations and the two notions are in fact equivalent due to the Hubard-Masur theorem [9].

Transversality have a straightforward generalisation for foliations with poles but I replace essential by non contractible which means that we include the boundary curves. As before a quadratic differentials with double poles define a pair of transverse foliations with poles. I do not give statement of the converse in general but a doubling argument allow to prove the following result by using the Hubard Masur theorem

Proposition 3. The subspace of pair of transverse foliations in \( MF(M) \times MF_0(M) \) is identified with the space \( QT_0(M) \) of quadratic differential with real residues.
3 Metric ribbon graphs and their moduli spaces:

In this section I give classical definitions on ribbon graph and the relation with multi-arcs. I will define orientable ribbon graph and study some of their properties. Metric ribbon graph are the most important object of this text and I will give some classical definition and the relation with weighted multi arcs and foliation with poles. A simple construction that I call zippered rectangle construction allow to construct foliation from metric ribbon graphs and this construction will be used in the next section. Finally I will define the Teichmüller and moduli space of metric ribbon graphs.

3.1 Ribbon graphs

Combinatorial ribbon graph Following M.Kontsevich [10] a combinatorial ribbon graph $R$ is defined by the following way. Let $XR$ a set of oriented edges or half edges and let

- $s_1 : XR \rightarrow XR$ an involution without fixed point,
- $s_0 : XR \rightarrow XR$ a cyclic order which define the vertices of the graph.

The boundary permutation is defined as $s_2 = s_1 \circ s_0^{-1}$ and these data's satisfy $s_0 s_1 s_2 = id$. I denote $X_0 R$ the set of vertices, $X_1 R$ the unoriented edges and $X_2 R$ are the faces or boundaries. We have

$$X_i R = XR / \langle s_i \rangle.$$ 

For all $e \in XR$ I denote $[e]_i \in X_i R$ the projection and $\# [e]_i$, the cardinal of the orbit of $e$ under $s_i$.

For each ribbon graph I consider $\mu_R$ the decoration of $M$ that count the number of vertices of $R$ with a given degree.

$$\mu_R (i) = \text{number of vertices of degree } i$$

Orientation of a ribbon graph Let $R$ a ribbon graph, an orientation on $R$ is defined as a map

$$\epsilon : XR \rightarrow \{\pm 1\},$$

such that

$$\epsilon \circ s_2 = \epsilon \quad \epsilon \circ s_1 = -\epsilon.$$ 

I will say that $R$ is orientable if it admits an orientation and oriented if some orientation is fixed. I can use the notation $R^\epsilon = (R, \epsilon)$ for an oriented ribbon graph.

The orientations satisfy the following trivial properties

- if $R$ is connected there is at most two orientations because the group generated by $s_1, s_2$ act transitively on the set of oriented edges,
- if $R$ is orientable then $R$ have only vertex of even degree because $\epsilon \circ s_0 = -\epsilon$.
- An orientation of the graph is a map from the set of half edges which is constant on the boundaries because we have $\epsilon \circ s_2 = \epsilon$. Then it induce a map

$$\epsilon : X_2 R \rightarrow \{\pm 1\}.$$ 

And then it define a partition of the boundaries $X_2 R = X_2^+ R \sqcup X_2^- R$

- For each ribbon graph there is a natural double cover $\tilde{R}$ which is oriented and is ramified over the vertices of odd degree.

Automorphisms: The data’s $(XR, s_0, s_1)$ characterise the ribbon graph, two triples are equivalents iff there is a bijection between the sets of oriented edges that preserve the data’s. An automorphism of the graph is a bijection $XR \rightarrow XR$ that preserve these data’s and fix each boundary. We denote $\text{Aut}(R)$ the group of automorphisms, as it’s noticed in [12] it can happen than an automorphism act trivially on the set of edges $X_1 R$ but not on the half edge. Nevertheless as we consider orientable ribbon graph such automorphism necessarily reverse the orientation of the edges and then permute the boundaries. In this case the action of $\text{Aut}(R)$ on the half edges is necessarily free.
Figure 3: An oriented ribbon graph

**Zippered rectangles** In this section I give simple construction called in the rest of the text zippered rectangles. I restrict to the case of orientable ribbon graph is this text but this construction is more general by taking the double cover. A ribbon graph is naturally associated to a surface with boundaries or a surface with a cellular decomposition by gluing rectangles.

Let $R$ a combinatorial ribbon graph and consider for all $e \in X^+ R$ a rectangle

$$R_e = [0, 1] \times [-1, 1].$$

It’s possible to glue these rectangles by using

$$\{1\} \times [0, 1] \subset R_e \rightarrow \{0\} \times [0, 1] \subset R_{e^2}, \quad \{0\} \times [-1, 0] \subset R_e \rightarrow \{1\} \times [-1, 0] \subset R_{e^1 e^2 e}$$

There is conical singularities at the points $(0, 0), (1, 0) \in R_e$ but they are removable.

I then obtain a surface with boundaries $M_R$ with an embedded graph on it’s given by the union of $[0, 1] \times \{0\}$.

The surface $M_R$ retract on the graph $R$ and there is an identification

$$X_2 R = \pi_0(\partial M_R),$$

then the orientation on $R$ induce an orientation of the boundaries of $M_R$.

**Remark 5.** We can also construct the surface $M_R^{top}$ obtained by capping the boundaries of $M_R$ the graph $R$ induce a cell decomposition of the surface which is a map. The orientability is then equivalent to the fact that this map is bipartite, in the sense that face are labelled by $\pm 1$ and two adjacent faces have opposite signs.

**Embedded ribbon graphs**: If $M^g$ is a surface of type $(g, n^+, n^-)$ and $R$ an oriented ribbon graph, an embedding of $(R, \epsilon)$ is an isotopy class of homeomorphism $M_R \rightarrow M$ which preserve the orientation on the boundaries. We denote Rib$(M^g)$ the set of embedded ribbon graph. The mapping class group act on the set of embedded ribbon graphs and the quotient is the space rib$(M)$ of combinatorial ribbon graph with the same topology than $M^g$ and marked boundaries. An embedded ribbon graph is generic if it contain only vertices of degree four and I will denote Rib$^*(M^g)$, rib$^*(M^g)$ the set of generic ribbon graph.

**Ribbon graph and filling multi arcs**: Let $R$ an embedded ribbon graph on $M$ with no vertices of degree one or two, then each edge $e \in X_1 R$ belong to two boundaries and there is a unique arc $e^*$ that rely these two boundaries and intersect only this edge. The union of all the $e^*$ form a multi-arc $A_R$ (figure [3.1]). When there is vertices of
order one or two $A_R$ is still a multi arcs on the surface obtained by removing these points. I make this choice to be consistent with the non triviality of the arcs.

All the multi-arcs does not define a ribbon graph. In fact it’s possible to cut the surface along multi-arc and the result is a family of surface $M_A$ with boundaries and corner at the boundaries. A multi-arc is filling iff $\iota(A, \gamma) > 0$ for all $\gamma$ non contractible which is equivalent to the fact that all the components of $M_A$ are topological polygons with $2i$ sides ($i \geq 3$). For a filling multi-arc I denote $\mu_A$ the partition such that $\mu_A(i)$ is the number of face with $2i$ sides.

The multi-arcs $A_R$ is always filling, the vertices of degree $i$ in $R$ correspond to faces with $2i$ sides in $A_R$. Reciprocally a multi-arc also define an embedded ribbon graph and then there is a bijection between embedded ribbon graph and filling multi-arcs that preserve the profile $\mu_R = \mu_{A_R}$

A foliation is filling if it’s intersection pairing with any non contractible simple closed curve is non zeros. In other word it’s filling if up to Whitehead moves it’s equivalent to a foliation without saddle connection. We can define similar notion for multi-arcs.

**Proposition 4.** The set of filling foliations is identified with the set of filling weighted multi-arcs

Later I will give an explicit construction of the inverse of

$$\lambda \longrightarrow A_\lambda$$

**3.2 Metric ribbon graph, weighted arcs and measured foliations :**

**Metric ribbon graph :** Let $M^\circ$, an oriented metric ribbon graph marked by $M^\circ$ is a pair $S = (R, m)$ where

- $R \in \text{Rib}(M)$ is an embedded oriented ribbon graph
- $m$ is a metric on $R$, which is a map $m : X_1R \longrightarrow \mathbb{R}_{>0}$. it assign a positive length to each edge of the graph.

I denote $\text{Met}(R) = \mathbb{R}_{>0}^{X_1R}$ the set of metrics on $R$ and I will use the notation $T_R = \mathbb{R}_{X_1R}$ for the tangent space OF $\text{Met}(R)$. $R^\circ$ is oriented it give a canonical identification

$$T_R = H^1(M, X_0R, \mathbb{R}).$$

When the graph is four-valent the following formula is valid formula

$$\mathcal{d}_R := \dim \text{Met}(R) = 4g - 4 + 2n$$
**Weighted multi-arcs** Metric ribbon graphs can be seen in a different way. In paragraph (3.1) I show that we can associate to each ribbon graph \( R \) a filling multi-arc \( A_R \). In this way we can obtain a bijection between filling multi-arcs and embedded ribbon graphs. In the same way if \( S = (R, m) \) is a metric ribbon graph it’s possible to associate the weighted multi-arc

\[
A_S = \sum_{e \in X_1 R} m_e(S)e^*,
\]

and then metric ribbon graphs are weighted filling multi-arcs. The two definition can have advantages in different situations.

**Lengths of the boundaries and orientability** For all \( \beta \in \pi_0(\partial M) \) and \( S = (R, m) \) a metric ribbon graph on \( M \) it’s possible to define the length on \( \beta \) as

\[
l_\beta(S) = \sum_{e \in X_1 R, [e]_2 = \beta} m_{[\cdot]_1}(S).
\]

this define a linear function

\[
l_\beta : \text{Met}(R) \rightarrow \mathbb{R}_{>0}
\]

In the language of multi arcs the lengths of the boundaries is just the pairing between the arc and a curve homotopic to the boundary.

The fact that the dual of an oriented ribbon graph is bipartite imply the following condition on the boundary lengths

\[
\sum_\beta \epsilon(\beta)l_\beta = 0.
\]

for all metric on \( R \). Then the image of the application \( L_0 \) lie into the affine subspace \( \Lambda_{M^\ast} \) (equation [1]).

The derivative \( dl_\beta \) of \( l_\beta \) is an element in \( T_R^\ast \) and I will denote \( H_R \) the subspace generated by the linear form \( dl_\beta \) and \( K_R \) the annihilator of \( H_R \) i.e the subspace of \( T_R \) defined by

\[
0 \rightarrow K_R \rightarrow T_R \rightarrow H_R^\ast \rightarrow 0.
\]

The following proposition characterise the orientability and despite it simplicity it’s very useful in practice.

**Proposition 5.** Let \( R \) a ribbon graph not necessarily orientable with \( n \) boundaries, then the dimension of \( H_R \) is

- \( n \) if \( R \) is not orientable
- \( n - 1 \) if \( R \) is orientable and the only relation is given by the orientation

\[
\sum_\beta \epsilon(\beta)dl_\beta = 0.
\]

And then in the case of four-valent ribbon graph we have

\[
d_R = \dim K_R = 4g - 3 + n
\]

**Proof.** Let \( \epsilon : X_2 R \rightarrow \mathbb{R} \) such that

\[
\sum_x \epsilon_x d\mu_{comb} = 0.
\]

We can consider a leaf \( \epsilon : X R \rightarrow \mathbb{R} \) we have \( \epsilon(s_2 e) = \epsilon(e) \) and

\[
\sum_{[e]_1} (\epsilon(e) \epsilon(s_1 e)) dm_{[\cdot]_1} = 0.
\]

And then \( \epsilon(e) = -\epsilon(s_1 e) \), up to a multiplicative constant the only relation are given by the orientations. If the graph is connected there is at most one orientation up to a sign, then we see that the tangent map is either surjective if the graph is not orientable or the image is of codimension one if it’s orientable.
Metric ribbon graphs and orientable foliations: A metric ribbon graph define in a natural way a foliation with poles which is the real part of a quadratic differential. If $S = (R, m)$ is a metric ribbon graph then the Jenkin Strebel differential $q_S(0)$ is defined on $R^*_e$ by $m_e^2(dx)^2$. These forms can be glued and define a quadratic differential with double poles and real residues. The real part of this quadratic differential define a foliation $\lambda_S$ which is locally given by $m_e|dx|$. This construction induce a map

$$\text{Met}(R) \longrightarrow \mathcal{MF}(M)$$

and using the part [5.1] it give a map

$$\mathcal{MA}_0^g(M) \rightarrow \mathcal{MF}(M)$$

this map is the opposite of the map constructed in section [2.3]. When the graph is oriented then the foliation $\lambda_S$ is naturally oriented and it’s the real part of an abelian differential $\alpha$. This map is the opposite of the map constructed in section [2.3]. When the graph is oriented then the foliation $\lambda_S$ is naturally oriented and it’s the real part of an abelian differential $\alpha_S(0)$. Then the following proposition is true

Proposition 6. The set of filling foliations $\mathcal{MF}^0(M)$ is identified with the set of filling weighted multi-arcs $\mathcal{MA}_0^g(M)$. The set of filling foliations $\mathcal{MF}^0(M^\circ)$ is identified with $\mathcal{MA}_0^g(M^\circ)$

3.3 Moduli space and volumes:

Construction of the moduli space: The Teichmüller space $T^{\text{comb}}(M^\circ)$ of oriented metric ribbon graph on $M^\circ$ is the space of all embedded oriented metric ribbon graph in $M^\circ$. Which is the union of the disjoint cells

$$T^{\text{comb}}(M^\circ) = \bigcup_{R^0 \in \text{Rib}(M^\circ)} \text{Met}(R^0),$$

A ribbon graph $R$ can degenerate to an other $R'$ by contracting a set of edges which does not contain loop and this induce a map

$$\text{Met}(R') \rightarrow \text{Met}(R)$$

and define a structure of linear cell complex on $T^{\text{comb}}(M^\circ)$. The complex $T^{\text{comb}}(M^\circ)$ is naturally embedded into the larger complex $T^{\text{comb}}(M)$ of all metric ribbon graphs by forgetting the orientation. The degeneration of ribbon graph preserve the orientation and then $T^{\text{comb}}(M^\circ)$ is a closed subcomplex of codimension $2g - 2 + n$. The space $T^{\text{comb}}(M)$ can be used to find a cell decomposition of the decorated Teichmüller space $T_{g,n} \times (\mathbb{R}_+)^n$ [10]. The top cells of $T^{\text{comb}}(M^\circ)$ is the space of four valent ribbon graphs which will be denoted $T^{\text{comb},\circ}(M^\circ)$.

There is a natural action of the mapping class group on $T^{\text{comb}}(M^\circ)$ and the moduli space is the quotient $\mathcal{M}^{\text{comb}}(M^\circ)$. The space $\mathcal{M}^{\text{comb},\circ}(M^\circ)$ is then the disjoint union of the quotients $\text{Aut}(R^0) \backslash \text{Met}(R^0)$ where $R^0$ run over the elements in $\text{rib}(M^\circ)$.

The lengths of boundaries are defined on the Teichmüller space and induce a map

$$L_0 : \mathcal{M}^{\text{comb}}(M^\circ) \rightarrow \Lambda_{M^\circ}$$

and I will consider the level set $\mathcal{M}^{\text{comb}}(M^\circ, L) \subset \mathcal{M}^{\text{comb}}(M^\circ)$ which is a cell complex and locally an affine submanifold of codimension $n - 1$.

Combinatorial Teichmüller and foliations: From the result of the last section there is injective map

$$T^{\text{comb}}(M) \longrightarrow \mathcal{MA}_0^g(M) \longrightarrow \mathcal{MF}(M)$$

the image of the first is the set of filling multi-arcs and then

$$T^{\text{comb}}(M) = \mathcal{MA}_0^g(M)$$

the space $\mathcal{MA}_0^g(M)$ is a cell complex which is the closure of $T^{\text{comb}}(M)$. In a similar way

$$T^{\text{comb}}(M^\circ) \rightarrow \mathcal{MA}_0^g(M^\circ) \rightarrow \mathcal{MF}(M^\circ)$$

and

$$T^{\text{comb}}(M^\circ) = \mathcal{MA}_0^g(M^\circ).$$

In both cases the spaces $T^{\text{comb},\circ}(M), T^{\text{comb},\circ}(M^\circ)$ are identified with the subset $\mathcal{MF}^{\text{comb},\circ}(M), \mathcal{MF}^{\text{comb},\circ}(M^\circ)$ of foliations with no saddle connection.
Measures on the Teichmüller space of metric ribbon graphs: The Teichmüller space of metric ribbon graphs possesses a set of integral points \( T_{\text{comb}}^\circ \subseteq T_{\text{comb}} \) of metric ribbon graphs such that the lengths of each edge is a positive integer. In each cell \( I \) have

\[
\text{Met} \cap_{\mathbb{Z}} (M^\circ) = \text{Met} \cap_{\mathbb{Z}} (M^\circ) \cap_{X_I} \mathbb{R}_M.
\]

The measure \( d\nu_{M^\circ} \) is the affine measure normalised by the set of integral points. It’s defined on each to cells of \( T_{\text{comb}}^\circ \) by

\[
d\nu_{M^\circ} = \left| \bigwedge_{e \in X_I} dm_e \right|.
\]

By definition the set \( T_{\text{comb}, \ast}^\circ \) is negligible for this measure. For now there is no orientation on the cells and then \( d\nu_{M^\circ} \) is not defined by a volume form.

The measure \( d\nu_{M^\circ} \) is the counting measure for \( T_{\text{comb}}^\circ \), for all open set \( U \subseteq T_{\text{comb}}^\circ \) with a negligible boundary we have

\[
\nu_{M^\circ}(U) = \lim_{t \to +\infty} \frac{\# t \cdot U \cap T_{\text{comb}}(M^\circ)}{t^{X_I} M}.
\]

For each \( L \) there is a natural affine measure \( d\nu_{M^\circ}(L) \) on \( T_{\text{comb}}^\circ(L) \). For all \( S = (R, m) \in T_{\text{comb}, \ast}(M^\circ, L) \) the exponential map for the affine structure

\[
\exp_S : U \subseteq K_R \longrightarrow \text{Met}(R)
\]

In our case the exponential is just \((R, m) \rightarrow (R, m + u)\). The measure \( d\nu_{M^\circ}(L) \) on \( \exp_S(U) \) is the Lebesgue measure normalised by the lattice of integral points \( K_R(\mathbb{Z}) \) in \( K_R \). If we choose an other base point for the exponential then the change of coordinates are translation and preserve the volume form so the measure \( \nu_{M^\circ}(L) \) is well defined.

**Volumes of the moduli space:** For each \( M^\circ \) the moduli space \( M_{\text{comb}}^\circ \) is also equipped by the affine measure \( \nu_{M^\circ} \). The action of the mapping class group act linearly and preserve the set of integral points and then the measure. This space have an infinite volume, the measure \( dZ_{M^\circ} \) on \( \Lambda_{M^\circ} \) is defined as the pushforward of \( d\nu_{M^\circ} \) under the map \( L_\partial \)

\[
dZ_{M^\circ} = L_\partial \ast d\nu_{M^\circ}.
\]

This measure is characterised by the relation

\[
\int_{\Lambda_{M^\circ}} f(L) \, dZ_{M^\circ} = \int_{M_{\text{comb}}(M^\circ)} f(L(S)) \, d\nu_{M^\circ}.
\]

For each \( L \in \Lambda_{M^\circ} \) we can also consider the level set \( M_{\text{comb}, \ast}(M^\circ, L) \) is equipped by it’s affine measure \( d\nu_{M^\circ}(L) \). Again the action of the mapping class group is linear and preserve the lattice of integral points

\[
g : K_R(\mathbb{Z}) \longrightarrow K_{g \cdot R}(\mathbb{Z}).
\]

We denote \( Z_{M^\circ}(L) \) the volume of this affine submanifold

\[
Z_{M^\circ}(L) = \int_{M_{\text{comb}}(M^\circ, L)} d\nu_{M^\circ}(L)
\]

The volume are naturally related to the measure \( dZ_{M^\circ} \) the measure can be decomposed as

\[
d\nu_{M^\circ} = d\nu_{M^\circ}(L) \times d\sigma_{M^\circ}
\]

and then we have the following lemma

**Lemma 1.** The volumes \( dZ_{M^\circ} \) are absolutely continuous with respect to \( d\sigma_{M^\circ} \) and on \( \Lambda_{M^\circ} \) we have the relation

\[
\frac{dZ_{M^\circ}}{d\sigma_{M^\circ}} = Z_{M^\circ}(L)
\]

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Figure 5: Curve on a metric ribbon graph

4 Curves on orientable ribbon graphs:

In this section I study curves on ribbon graph, on a metric ribbon graph a curve have a unique combinatorial representation and it’s possible to define it’s combinatorial length. I will explain how to perform surgery along a curve on a ribbon graph. I will also introduce admissible curves and give several definitions. These curves will be used in the rest of the text and are the one which seems relevant when the graph have vertices of degree higher than three. I will give coordinate for admissible curves and foliations which use zippered rectangle construction.

4.1 Curves on ribbon graph:

Combinatorial representation:

Let $\gamma$ a curve on $M$ and $R$ an embedded ribbon graph. As the surface retract on the graph we can see that $\gamma$ define a "curve" on $R$ (figure 5). We can associate to an oriented curve a combinatorial model which is a sequence of oriented edges $e_0, ..., e_r$, such that $e_0 = e_r$ and $[e_i, e_{i+1}] = [s_1 e_i]_0 \forall i$. Such curves is defined modulo the action of $\mathbb{Z}$ by shifting the sequence, but there is still several representations of an isotopy class of curves. The minimal representation is the one which minimise the number of edges in the sequence up to homotopy. To construct this minimum it’s possible to process by induction or use the universal cover $R^\infty$ of the ribbon graph $R$ on $M$. The graph $R^\infty$ is an infinite tree. A representation of the curve lift to an infinite path $(e^\infty_i)$ on $R^\infty$. As the graph is a tree this path is represented by a unique minimal path that traverse each edge at most one time. The projection of this path is homotopic to the original path and minimise the combinatorial length.

Using this construction we have a bijection between homotopy class of curves on surfaces and minimal unoriented combinatorial curves on the ribbon graph.

Length of a curve on a metric ribbon graph:

Let $S = (R, m)$ a metric ribbon graph then for all curve $\gamma$ we can define it’s length by the following way. It suffice to set an orientation and a minimal representation $(e_0, ..., e_{r+1})$ then the length is defined in a straightforward way by

$$l_\gamma(S) = \sum_{i=0}^{r} m_{[e_i]}(S)$$

The combinatorial length is then minimum of the lengths of the combinatorial paths homotopic to $\gamma$, computed with the metric. If $A_S$ is the mult-arc associated to $S$ then the length of a curve $\gamma$ is also given by using the intersection pairing.

Each edge $e$ of the graph is associated to an arc $e^*$ that rely the two boundaries $[e], [s_1 e]_2$ then it’s possible to define the Thurston intersection pairing of an element of a curve with $e^*$ and we set

$$y_e(\gamma) = \iota(\gamma, e).$$

With these notations the length of $\gamma$ is given by the following formula

$$l_\gamma(S) = \iota(A_S, \gamma) = \sum_{e \in X^1_R} m_e(S)y_e(\lambda).$$
The proof is direct because the minimal combinatorial representation of the path minimise the number of edges in the path and then the number of intersection with $A_S$. An other way to set that is to look at the universal cover of our graph the number of edge crossed by the minimal path is exactly the intersection number with $A_S$.

**Surgery along a curve:** Let $R$ a ribbon graph on $M$ and $\Gamma$ a curve. We can define the ribbon graph $R_{\Gamma}$ obtained by cutting $R$ along $\Gamma$ (see figure 6). It’s not easy to define it by using ribbon graphs but it’s straightforward for multi-arcs.

If $A$ is a multi-arc and $\Gamma$ a multi-curve then up to isotopies I can assume than they are in minimal position, which means that they minimise their number of intersection. Then it possible to cut the surface along the curve and the result is a family of arcs on each connected components of $M_{\Gamma}$. There is possibly pairs of homotopic arcs but I identify and it define a multi-arc $A_{\Gamma}$ on $M_{\Gamma}$. If $A$ is filling then using the paragraph 3.1 $A_{\Gamma}$ is filling to.

If $A_R$ is the multi-arc associated to $R$, the multi-arc $(A_R)_{\Gamma}$ is filling and then it’s associated to a ribbon graph $R_{\Gamma}$ on $M_{\Gamma}$ such that $A_{R_{\Gamma}} = (A_R)_{\Gamma}$. The multi arc $A_{R_{\Gamma}}$ is characterized by the following condition

$$\iota(A_{\Gamma}, \gamma) = \iota(A, \gamma)$$

for all $\gamma$ with $\iota(\gamma, \Gamma) = 0$.

If $R$ have a metric $m$ it induce weights on $A_R$, then it’s possible to obtain weights on $(A_R)_{\Gamma}$ by summing the weights of homotopic arcs in $A \cap M_{\Gamma}$. This induce a linear map

$$\text{cut}_{\Gamma} : \text{Met}(R) \rightarrow \text{Met}(R_{\Gamma}).$$

and this map is continuous at the intersection of two cells. I will also use the notation $S_{\Gamma}$ for $\text{cut}_{\Gamma}(S)$ to be consistent with the other notations. The metric ribbon graph $S_{\Gamma}$ is characterised by

$$l_{\gamma}(S_{\Gamma}) = l_{\gamma}(S)$$

for all $\gamma \in S(M_{\Gamma}, \partial M_{\Gamma})$

**Multi-curves on an oriented surface** In the case of oriented ribbon graph it’s natural to consider orientable multi-curves. A curve is orientable if the orientation $\epsilon$ is constant along it’s minimal representation and in this case It is possible to orient the curve such that $\epsilon = 1$ when we travel along the curves (see figure 7). The following lemma assert that the orientability is stable under surgeries along orientable curves and conversely if we glue oriented graph by identifying positive and negative boundaries the result is still oriented.

**Lemma 2.** Let $\Gamma$ a multi-curve with stable graph $G$

1. If $R^\epsilon$ is an oriented ribbon graph such that $\Gamma$ is orientable then the graph $R_{\Gamma}$ posses a natural orientation $\epsilon_{\Gamma, R}$ which induce an orientation on $G$.

2. If $R$ is a metric ribbon graph such that $R_{\Gamma}$ is oriented and this orientation induce an orientation on $G$. Then the surface $R$ is also oriented and this orientation is compatible with the orientation on $\Gamma$.
4.2 Admissible curves:

**Definition:** When the graph $R$ is not generic it may happen that a curve split a vertex into several vertices of lower degrees (see figure 8). For each graph there is a partition $\mu_R = (\mu_R(i))_i$ such that $\mu_R(i)$ is a number of vertices of degree $i$. For all multi-curve there is a partition $\mu_{R_{\Gamma}(c)}$ for each component of $R_{\Gamma}$ obtained after cutting along the curve. The curve $\Gamma$ is then naturally decorated by $R$. Let $\mu_{R,\Gamma}$ the decoration on $M$ obtained by summing the decorations of each connected components of $M_{\Gamma}$. Then we have $\#\mu_{R,\Gamma} \geq \#\mu$ and the curve does not split any singularity iff the two decorations coincide. Then we put the following definition.

**Definition 1.** A multi-curve $\Gamma$ is admissible on $R$ iff it does not split any vertices of the graph. Which is equivalent to $\mu_{R,\Gamma} = \mu$ or even $\#\mu_{R,\Gamma} = \#\mu_R$.

Admissible curves on a metric ribbon graphs are intimately related to quadratic differential with poles and prescribed singularities. For a quadratic differential with poles it’s possible to consider the decoration $\mu_q$ which count the zeros of order $i-2$. Then I give the following generalisation of admissibility for foliations.

**Definition 2.** A foliation $\lambda \in \mathcal{MF}(M)$ is admissible on $S$ if there is $q_S(\lambda)$ a quadratic differential with double poles such that

$$\Re(q_S(\lambda)) = S \quad \Im(q_S(S)) = \lambda.$$

moreover I assume that it satisfy $\mu_q = \mu_S$.

This definition coincide with the first one for multi-curves.

**Lemma 3.** The integral foliations are the admissible multi-curves

$$\mathcal{MS}_Z(R^\circ) = \mathcal{MF}_0(R^\circ) \cap \mathcal{MS}_Z(M).$$

**Proof.** Let $\Gamma = \sum_{\gamma} m_{\gamma, \gamma} \in \mathcal{MS}_Z(M)$ and $S$ then the quadratic differential can be obtained in the following way. I consider the surface $S_{\Gamma}$, on this surface there is a Jenkin-Strebel $q_{S_{\Gamma}}(0)$ differential on each of its connected component. The trajectories of these differentials are periodic and surround a pole. It’s possible to glue an horizontal cylinders of height $m_{\gamma}$ to the two boundaries $\gamma^1, \gamma^2$ associated to $\gamma \in \Gamma$. The result is Jenkin-Strebel differential $q$ such that

$$\Re(q) = S \quad \Im(q) = \Gamma.$$

by uniqueness (proposition 3) $q = q_{\Gamma}(S)$ and by construction the profile of the quadratic differential is $\mu_{S,\Gamma}$. Then the curve is admissible iff $q_{\Gamma}(S)$ have $\mu = \mu_{S,\Gamma} = \mu_q$. 

\[\square\]
Figure 8: A non admissible curve

Combinatorial representation of an admissible curve: I present a way to describe admissible curves by their combinatorial representation. Let $R$ a ribbon graph, I introduce the two permutations

$$s_2^+ = s_2, \quad s_2^- = s_1 s_2^{-1} s_1.$$

A admissible curve cannot turn around a vertex in some sense they can only turn around boundaries.

**Lemma 4.** A curve is admissible on $R$ iff it’s simple and admit a representation of the form $e_0, ..., e_r$ with

$$e_{i+1} = s_2^+ e_i.$$

and $\epsilon(e_0) = 1$

And such representation is minimal and then necessarily unique up to a cyclic permutation and reversing the order. Then an admissible curve can be encoded by a starting edge and a finite word $\{\pm 1\}$.

Admissible multi-curves on an oriented ribbon graph: If $R^o$ is oriented we have the following lemma which give the relation between admissible curve and orientable curves (see figure 9).

**Lemma 5.** Let $R^o$ an oriented ribbon graph then all the admissible curves are orientable.

- all the admissible curves are orientable,
- Admissible curves and orientable curves coincide iff the graph have only vertices of degree two or four.

**Proof.** This is a consequence of the representation (lemma 4) because the two permutations $s^\pm$ preserve the orientation.

**Proof.** To prove the second part we need proposition 7. If we have a vertex $v$ of degree $2k > 4$ it’s possible to add an edge $e_0$ which split this vertex in two vertex of degree $2k_i \geq 4$. Let $R'$ this new ribbon graph, it’s oriented and their is a map $K_R \rightarrow K_{R'}$

By computing the dimension $\dim K_{R'} = \dim K_R + 1$ and from proposition 7 any curve in $K_{R'}(\mathbb{Z}) \setminus K_R(\mathbb{Z})$ will be orientable and split the vertex $v$ on $R$. If there is only vertices of degree 4 it’s easy to see than an orientable curve is admissible.

**Remark 6.** In the orientable case the fact that the curve are canonically oriented have the following consequence. An admissible curve turn around the positive poles in the direct direction and it turn around the negatives poles in the indirect direction. Then an admissible curve orbit around a boundary during a time before leave it to an other boundary according to figure 9. But admissibility is less restrictive than zigzag 7.

A corollary of the this result is the following

**Corollary 5.** A foliation $\lambda$ is orientable on $S^o$ iff there is an abelian differential $\alpha$ such that

$$\Re \alpha = S^o, \quad \Im \alpha = \lambda$$

and $\alpha$ have $\mu_S(i)$ zeros of order $\frac{i-2}{2}$.

For the converse result we need proposition 7.
4.3 Zippered rectangles and coordinates for admissible foliations:

Coordinates for $\mathcal{MF}(R)$ and $\mathcal{MF}_0(R)$: For all $R$ I denote $Q(R)$ the quadratic differential marked by $M$ such that $\Re(q) \in \text{Met}(R)$ and $\mu_q = \mu_R$ is other word it correspond to the space of admissible foliations on a surface in $\text{Met}(R)$. I also consider $Q_0(R)$ the subspace of differentials with real residues. we also consider $\mathcal{H}_0(R) = \mathcal{H}(R) \cap \mathcal{H}_0T(M)$ the abelian differentials with real residues.

Proposition 7. For all $S$ the spaces $\mathcal{MF}(S), \mathcal{MF}_0(S)$ depend only of the ribbon graph and there is homeomorphism which preserve the integral structure

$$Q(R) \rightarrow \text{Met}(R) \times T_R, \quad Q_0(R) \rightarrow \text{Met}(R) \times K_R.$$ 

In particular there is an homeomorphism

$$\mathcal{MF}(R) \simeq T_R, \quad \mathcal{MF}_0(R) \simeq K_R.$$ 

Assume than $R^c$ is oriented on $M^c$, I consider in a similar way the space of abelian differentials $\mathcal{H}(R^c), \mathcal{H}(R^c)$. As a corollary I obtain the following fact

Corollary 6. All the admissible foliation on $R$ are orientable

$$\mathcal{MF}(R) \subset \mathcal{MF}(M^c)$$

and the quadratic differential $q_S(x)$ is the square of an abelian differential $\alpha_S(x)$ and under the identification

$$K_R = H^1(M_R^{\text{top}}, X_0R, \mathbb{R}), \quad T_R = H^1(M_R, X_0R, \mathbb{R})$$

the map correspond to the period coordinates.

An other important corollary is the following

Corollary 7. The set of integral multi-curves $\mathcal{MF}_\mathbb{Z}(R)$ is identified with the lattice of integral points $K_R(\mathbb{Z})$ under the period coordinates $x$

Remark 7. In the case of unorientable ribbon graph it’s possible to construct a canonical double cover $\tilde{R}$ of $R$ which orientable. The tangent space is naturally identified with the space

$$H^1(M_R, X_0\tilde{R}, \mathbb{R})^-$$

of one form anti-invariant under the galois involution. In this case the map $x$ correspond to the period map with value in this space.

I now prove proposition 7.
Proof. I first construct the map which is the period map of the imaginary foliation along the edge of the embedded graph $R$

$$\mathcal{MF}(S) \longrightarrow T_R,$$

we use the zippered rectangle construction which is a decomposition of foliations with poles. For all $q$ and all $e \in XR$, there is a maximal embedded infinite rectangle $R^*_e \rightarrow M^*$ such that $\Re q$ is locally given by $|dx|$ on $R^*_e$. Moreover I assume than the orientation of $[0, 1]$ correspond with the orientation on $E$. $q$ have no singularities on the interior of $R^*_e$ but the maximality imply that there is at least one singularity on each boundaries. As $q$ have the same profile than $R^*_e$, $\mu_q = \mu_{R^*_e}$ then $q$ have no vertical saddle connections there and then there is only one singularity on each boundary of $R^*_e$. It’s possible to choose a square root’s $\alpha$ of $q$ on $R^*_e$ such that $\Re q = dx$. After this choice I denote $x^e_-$ the singularity on the left boundary and $x^e_+$ the one on the right. Let $I_e \subset R^*_e$ the horizontal maximal open interval oriented according $e$ such that the left extremity is $x^e_-$. I have an isomorphism $I_e \times \mathbb{R} \rightarrow R^*_e$ 

$$(x, y) \longrightarrow v_y(x).$$

If $(x, y)$ are complex coordinates then the pull back of $\alpha$ under this map is equal to $dz$. In the local coordinates given by $\phi_e$ I can define

$$x^e_+ = m_e(\alpha) + ix^e_e(\alpha),$$

which is the relative period of $\Re q$ along the edge $e$. There is no sign ambiguities because I assume than the real part is positive and then $x_{s1e} = x_e$. This define an element of the tangent space $T_R$.

By construction the data $(S, x)$ are enough to recover $q$. It suffice to glue the rectangles $R^*_e$ of length $m_e$ by performing a shear of parameter $x_e$ on the right boundary. There is no constraint on the parameter $(m, x)$ to perform the construction. I obtain in this way a riemann surface marqued by $M_R$ and the one form $(dz)^2$ on each rectangle induce a quadratic differential $q_S(x)$. The two construction are the inverse of each other then I can conclude that there is a bijection

$$\mathcal{MF}(S) \longrightarrow T_R.$$ 

The imaginary part of the quadratic differential $\alpha_S(x)$ define a foliation $\lambda_R(x)$ which is does not depend of the metric on the edges, so the space $\mathcal{MF}(S)$ depend only of $R$.

To conclude

$$\text{Res}_\beta \lambda = \sum_{e \in X_e} y_e(\beta)x_e(\lambda).$$

The RHS is $dl_\beta$ evaluated at $\sum_e x_e(\lambda)\partial_{m_e} \in T_R$. Then the elements of $\mathcal{MF}(S)$ correspond exactly to the vectors in $K_R$.

Irreducible ribbon graphs: Irreducible ribbon graph are generalised pair of pants in some sense, they are interesting class of ribbon graph but I wont use them so much here.

Definition 3. A ribbon graph is irreducible iff $\mathcal{MS}_\mathbb{Z}(R) = 0$
They are irreducible because we can reduce their topology by admissible surgeries. In some sense they are minimal objects in the category of surfaces with decoration. From the results of the last section I can derive the following fact. A ribbon graph is irreducible if it satisfy

$$K_R = \{0\}.$$

For such graph the length of each edge is an explicit function of the length of the boundaries. The irreducible ribbon graph can be classified, we have the following proposition which is given by computing the dimension of $K_R$.

**Proposition 8.** A ribbon graph is irreducible iff it’s of genus zeros and it satisfy one of the two following conditions

- it have only two vertices of odd degree,
- or it’s orientable and have only one vertex.

**Dual coordinates on $\mathcal{MF}(R, \partial R)$ and $\mathcal{MF}_0(R)$** For each $R$ I define $\mathcal{MF}(M, \partial M)$ as the space of formal sum

$$X = \lambda + \sum \beta h_{\beta \beta}$$

where $h \in \mathbb{R}^{\partial M}$. An element of this space is a foliation in $\mathcal{MF}_0(R)$ marked by a choice of a periodic and possibly singular trajectory around each pole. Let $R$ an embedded ribbon graph I define the subspace $\mathcal{MF}(R, \partial R)$ of admissible foliations on $R$. There is a natural inclusion of $\mathcal{MF}_0(R)$ in $\mathcal{MF}(R, \partial R)$ and a forget full map

$$\mathcal{MF}(R, \partial R) \rightarrow \mathcal{MF}_0(R)$$

The "coordinates" $(y_e)$ of equation [4] are well defined for the elements of $\mathcal{MF}(R, \partial R)$ and more generally $\mathcal{MF}(M, \partial M)$. But in general the map

$$y : \mathcal{MF}(R, \partial R) \rightarrow \mathbb{R}^{X_1 R}$$

is neither injective or surjective. I define other parameters in the following way. For each $e \in X R$ let $\gamma_e$ the unoriented arc that join $[e]_2$ and $[e]_0$ and I denote

$$z_e = \iota(\lambda, \gamma_e)$$

which is the distance between the singularity $[e]_2$ and the boundary curve $[e]_0$. They satisfy the relation (see figure 10)

$$x_e(\lambda) = z_e(\lambda) - z_{s2e}(\lambda) \quad y_e(\lambda) = z_{t0}^{-1} e(\lambda) + z_e(\lambda)$$

and the $(z_e)$ satisfy also the constraints

$$z_{s21} e(\lambda) + z_e(\lambda) = z_{s2e}(\lambda) + z_{s1} e(\lambda) \quad (4)$$

Let $W_R^+$ the set of $(z_e) \in \mathbb{R}^{X_R}$ that satisfy the last relations. And $W_R$ the same space but with coefficients in $\mathbb{R}$. Then we have the following fact

**Lemma 6.** There is a train track $\tau_R$ such that $W_R^+$ is the set of weights on $\tau_R$.

I construct the train track in the following way (see figure 4.3). For each oriented edge I associate a vertex $v_e$. The edges of $\tau_R$ are of two types

- there is an edge $s_e$ for all $e \in X_1 R$ that join the two vertices labelled by the two orientations of $e$,
• there is an edge $s'_e$ for all $e \in XR$ that join $v_{[e]}$, and $v_{[s_e]}$.

Then $W^+_R$ is the set of positive weights on the train track $\tau_R$. The train track is non degenerate, if $W_R$ is the same space with real weights, then $\dim W_R = \dim W^+_R = \#X_1 R + 1$.

Then the following proposition is true

**Proposition 9.** The map

$$\mathcal{MF}(R, \partial R) \rightarrow W^+_R$$

$$\lambda \mapsto z(\lambda)$$

is a bijection and identify $W^+_R(\mathbb{Z})$ with $\mathcal{MS}_R(M, \partial M)$

**Remark 8.** In the orientable case I can consider for each $e \in XR$ the vector $\gamma_e$ in $H_1(M^+_R, X_0 R \cap X_2 R, \mathbb{R})$. Then the space $W_R$ is identified with the cohomology $H^1(M^+_R, X_0 R \cap X_2 R, \mathbb{R})$. if we perform the change of variable $(e(e)z_e)$.

**Corollary 8.** The one form $dx_\lambda$ is equal to zeros on $T_R$ iff the foliation $\lambda$ is trivial.

As I will show later this is not true if we restrict to $K_R$. I prove the proposition 9 if I restrict to abelian differential for simplicity

**Proof.** I use zippered rectangles as in the last section the space . For all $(S, z) \in \text{Met}(R) \times W^+_R$ let $x(z)$ the $x$-coordinates given by the last relation. From proposition 7 I can construct an abelian differential $\alpha_S(x)$ on $M^*$. As we have

$$x_e = z_{s_2}e - z_e$$

the sum along a boundary is zeros and then $x(z)$ is in $K_R$ so $3\alpha_S(x)$ is in $\mathcal{MF}_0(R)$. For each $e$ we can consider the trajectory along $[e]_2$ which pass trough the point $(0, z_e) \in R_e$ this is well defined due to the constraints $A$. The horizontal foliation and the trajectory does not depend of the choice of $S \in \text{Met}(R)$ and then we have the inverse map

$$W^+_R \rightarrow \mathcal{MF}(R, \partial R).$$

\[ \square \]

### 4.4 Deformations of metric ribbon graphs, twist and horocyclic flow:

I use curve and foliations to study deformations of metric ribbon graph. As in [3] I consider twist flow along admissible curves, I rely this flow to the horocyclic flow on the space of quadratic differential with poles, where it’s is much easier to understand it.

**Combinatorial twist :** We give a first intuitive definition of the twist flow which is the same than the definition of the twist flow along a geodesic on an hyperbolic Riemann surface. Let $\gamma \in S(M)$ then after cutting $S$ along $\gamma$ there is two new boundaries $\gamma_1, \gamma_2$ in $M$, that correspond to $\gamma$. If $S \in \text{Met}(R)$, a point $x \in \gamma$ induce two points $x^1 \in \gamma^1$. It is possible to glue the two boundaries $\gamma_1, \gamma_2$ of $S_1$ by identifying these two points and the result is $S$. For $t$ small enough it’s also possible to do translation of $x^2$ by a distance $-t$ according to the orientation of the boundary $\gamma^2$. This give a new point $x^2_t$ and then it’s possible to glue $S_t$ by identifying $x^1_t$ and $x^2_t$. The new surface is denoted $\phi^t_t(S)$ and its a well defined metric ribbon graph for $t$ small enough which does not depend of the choice of the base point $x$ and the label on the two boundaries.

The flow $\phi^t_t(S)$ is contained in $\text{Met}(R)$ for $t$ small enough iff the curve $\gamma$ is admissible. When the curve is not admissible it split a vertex and then the twist flow will split this vertex for arbitrary small times so it’s not included in $\text{Met}(R)$.

For two disjoint curves $\gamma_1, \gamma_2$ there is the relation have $(S_{\gamma_1})_{\gamma_2} = (S_{\gamma_2})_{\gamma_1} = S_{\gamma_1 \cup \gamma_2}$ and then $\phi_{\gamma_1} \circ \phi_{\gamma_2} = \phi_{\gamma_2} \circ \phi_{\gamma_1}$ and it’s possible to define for all $\Gamma \in \mathcal{MS}_R(M)$

$$\phi^t_t = \prod_{\gamma} \phi_{\gamma}^{m_{\gamma-1}},$$

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for $t$ small enough.

**Twist flow in coordinates**: Let $\gamma$ an admissible curves on $R$. From lemma [3] there is a combinatorial representation $(e_i)$ with $e_{i+1} = s^\pm e_i$. Then it’s possible to define the signed intersection number $\iota_\gamma(i)$ by

$$
\iota_\gamma(i) = \begin{cases} 
1 & \text{if } e_i = s^{-1} e_{i-1} \\
0 & \text{if } e_i = s^{-1} e_{i-1} \\
0 & \text{else}.
\end{cases}
$$

Then $\iota_\gamma(e)$ for $e \in X_1 R$ is defined as the sum of the $\iota_\gamma(i)$ over the $i$ such that $[e_i]_1 = e$. These coefficients are independent of the combinatorial representation they are well defined for integral multi-curves by linearity and satisfy

**Lemma 7.** The twist flow along $\Gamma$ is given locally by

$$m_e(\phi^t_\lambda(S)) = m_e(S) + t x_e(\lambda).$$

In fact we can see that $\iota_e(\Gamma) = x_e(\Gamma)$ and it’s a “combinatorial” formula for the period coordinates.

**Twist flow and horocyclic flow**: An other way to define the twist flow is the following. For all multi-curve $\Gamma$ it’s possible to consider the subspace $M_F \Gamma(M)$ of $M_F(M)$ of foliations transverse to $\Gamma$. As $\Gamma$ correspond to an element of $M_{F_0}(M)$ from [3] each element of $M_F \Gamma(M)$ is the real part of a unique quadratic differential $q_\gamma(\lambda)$ with imaginary foliation given by $\Gamma$. If $\gamma$ an essential curve $q_\gamma(\lambda)$ is a Jenkins-Strebel differential with only one cylinder of core curve $\gamma$ and height one. Then the intuitive notion of twist correspond to a shear along this cylinder which can be defined by using the horocyclic flow $U_t$. The horocyclic flow preserve the imaginary foliation and then it define a map

$$\phi_t : M_F \Gamma(M) \rightarrow M_F \Gamma(M).$$

As we will see later there is coordinate on $M_F \Gamma$ on wich $\phi_t$ is continuous. Moreover $T^{comb}(M) \subset M_F \gamma(M)$ is open for these coordinates then twist flow is well defined for small time on $T^{comb}(M)$. We have a straightforward generalisation for weighted multi-curves and the flow satisfy the following properties

**Lemma 8.** Twist flow and horocyclic flow coincident on weighted multi-curves.

More generally for all foliation $\lambda \in M_F(M)$ it’s possible to define the twist flow on the space of foliations transverse to $\lambda$ by using the horocyclic flow.

**Proposition 10.** The twist flow along $\lambda$ preserve $Met(R)$ for small time if $\lambda$ is in $M_F(R)$. In this case the twist flow is locally a translation generated by the locally constant vector field $\xi_\lambda$

$$\xi_\lambda = \sum_{e} x_e(\lambda) \partial_{m_e}.$$ 

Then the tangent vector of the twist flow is just given by the relative periods $x$ of the foliation along the edges of the graph. From the results of the last section we see that all the vector in $T_R$ is tangent to a unique trajectory of the twist flow.

**Proof.** For all $S \in Met(R)$ and $\lambda \in M_F(R)$ the horocyclic flow $\phi^t_\lambda(S)$ correspond to an horizontal stretch on the rectangles $[0, m_e(S)] \times \mathbb{R}$ and then we see that it’s well defined for small times. Moreover by computing the period coordinates give

$$m_e(\phi^t_\lambda(S)) = m_e(S) + t x_e(\lambda).$$

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Gluing and bundles for oriented surfaces: Let $M^o$ a directed surface and $\Gamma^o$ a directed multi-curve on it. I consider the subset $\mathcal{MF}_\Gamma(M^o)$ of orientable foliations on $M^o$ transverse to $\Gamma$ and which are represented by an abelian differential. Such differential induce a direction on the curve of $\Gamma$ and I assume that it correspond to $\Gamma^o$.

For all stable graph $G^o$ let $T^{comb}(G^o)$ the affine subset of $\prod_c T^{comb}(G^o(c))$

$$T^{comb}(G^o) = \left\{ S = (S(c)) \in \prod_c T^{comb}(G^o(c))| l_\beta(S) = l_{l(\beta)} \quad \forall \beta \in XG \right\}$$

Let $\Gamma^o$ an oriented multi-curve with stable graph $G^o$ then the following proposition i true

**Proposition 11.** $\mathcal{MF}_\Gamma(M^o)$ is stable under the twist flow along each curves in $\Gamma$ and there is a surjective map

$$cut_{\Gamma} : \mathcal{MF}_\Gamma(M^o) \longrightarrow T^{comb}(G^o).$$

Moreover this map is an affine $\mathbb{R}^\Gamma$ bundle compatible with the integral structure.

We will proove the following lemma which is the existence of local twist coordinates and give the proposition as a corollary.

**Lemma 9.** For each $S \in T^{comb}(G^o)$ there exist $V \subset T^{comb}(G^o)$ and $U \subset \mathcal{MF}_\Gamma(M^o)$ such that

- $V$ is an open neighbourhood of $S$ invariant by dilatation and $U = cut_{\Gamma}^{-1}(V)$
- for all $\gamma \in \Gamma$ there is $t_\gamma : U \longrightarrow \mathbb{R}$ such that

$$t_\gamma(\phi^\gamma_{\gamma'}) = t_\gamma + t_\delta_{\gamma,\gamma'}$$

for all $\gamma' \in \Gamma$. And we have a piece-wise linear isomorphism

$$U \longrightarrow V \times \mathbb{R}^\Gamma,$$

which induce a bijection

$$U \cap \mathcal{MS}_Z(M) \longrightarrow \mathbb{Z} \times \mathbb{Z}^\Gamma,$$

where $U_Z, V_Z$ are the integral points.

Proof. Let $S \in T^{comb,*}_\Gamma(M^o)$ and let $(R_{\Gamma}(c))$ such that $R_{\Gamma} \in \prod_c Met(R_{\Gamma}(c))$, we define $U$ as the set of foliation $\lambda \in \mathcal{MF}_\Gamma(M^o)$ such that $\lambda_{\Gamma} \in \prod_c Met(R_{\Gamma}(c))$ and we take $V$ as the set $S' \in \prod_c Met(R_{\Gamma}(c))$ with $l_\beta(S') = l_{R_{\Gamma}^{\beta}}(S')$. For each $\gamma \in \Gamma$ we have

$$\phi^\gamma_{\gamma'}(\lambda) = \mathbb{R}U_{\lambda q_\gamma(\lambda)}$$

so $cut_{\Gamma}(\phi^\gamma_{\gamma'}(\lambda)) = cut_{\Gamma}(\lambda)$ and then $U$ in invariant under the twist flow. For all $\gamma \in \Gamma$ we denote $C^{\Delta}_\gamma(\lambda)$ the cylinder in $q_\gamma(\lambda)$ associated with $\gamma$. It’s possible to fix two singularities $s^\gamma_1, s^\gamma_2$ one on each boundary of $C^{\Delta}_\gamma(\lambda)$ and an arc $a_\gamma$ contained in $C^{\Delta}_\gamma(\lambda)$ that join these two singularities. Then I can consider

$$t_\gamma(\lambda) = \frac{\mathbb{R}(q_\gamma(\lambda), a_\gamma)}{3(q_\gamma(\lambda), a_\gamma)} = \frac{\mathbb{R}(q_\gamma(\lambda), a_\gamma)}{3(q_\gamma(\lambda), a_\gamma)}$$

which does not depend of the choice of the roots of the quadratic differential. I have the relation

$$t_\gamma(\phi^\gamma_{\gamma'}(\lambda)) = t_\gamma(\lambda) + t_\delta_{\gamma,\gamma'}$$

and if $\lambda \in U_Z$ then $t_\gamma(\lambda) \in \mathbb{Z}$ and $cut_{\Gamma}(\lambda) \in V_Z$ because by definition the period are integral. By gluing it’s possible to construct the inverse map

$$g \lambda : V \times \mathbb{R}^\Gamma \longrightarrow U$$

by construction the maps $cut_{\Gamma}, t_\gamma, g \lambda$ are piece wise linear and preserve the set of integral points.
Covering of admissible curves: Let $\Gamma$ a directed curve on $M$ let $T_{\Gamma}^{\text{comb},*}(M)$ the set of generic metric ribbon graph $S$ such that $\Gamma$ is admissible and the orientation are compatible $\Gamma_S = \Gamma$. There is a natural inclusion
\[ T_{\Gamma}^{\text{comb},*}(M) \hookrightarrow \mathcal{MF}_{\Gamma}(M) \]
and the restriction of the cutting map define a map
\[ T_{\Gamma}^{\text{comb},*}(M) \rightarrow T^{\text{comb},*}(M) . \]
Let $\mathcal{MF}_{\Gamma}(M)$ the subset of foliations with no saddle connection at all. This subset correspond to the subset of foliation represented by an abelian differential with simple zeros. The foliations in $\mathcal{MF}_{\Gamma}(M)$ is necessarily represented by a four-valent ribbon graph. And then there is a bijection
\[ \mathcal{MF}_{\Gamma}(M) \rightarrow T_{\Gamma}^{\text{comb},*}(M) . \]
5 Acyclic decomposition:

In this part I give one of the main result of this text which allow to give recursion for the volumes. I will study the Poisson structure on the space of metric ribbon graph give several expression for the pairing which have been studied first in [10]. I will show that the combinatorial length is in some sense the hamiltonian of the twist flow which generalise result of [3]. I will briefly speak about irreducible ribbon graph but they not play a central role in this text. Finally I will show that I possible to separate a vertex from the rest of the surface. The proof use degeneration of the symplectic structure and I tried to use as little combinatorics as possible

5.1 Symplectic geometry on the space of metric ribbon graph:

The anti-symmetric pairing on $K_R$:

In the case of orientable ribbon graph we have identifications

$$T_R \cong H^1(M_R, X_0 R, R), \quad K_R \cong H^1(M_R^{top}, X_0 R, R)$$

given by a map $f_R$. We have an exact sequence for the relative cohomology

$$0 \rightarrow H^0(M^{top}, \mathbb{R}) \rightarrow H^0(X_0 R, \mathbb{R}) \rightarrow H^1(M^{top}, X_0 R, \mathbb{R}) \rightarrow H^1(M_R^{top}, \mathbb{R}) \rightarrow 0.$$  

The space $H^1(M_R^{top}, \mathbb{R})$ is a symplectic vector space for the intersection pairing

$$\langle \omega_1, \omega_2 \rangle = \int_{M_R^{top}} \omega_1 \wedge \omega_2.$$  

This pairing induce a two form $\Omega_R$ on $K_R$ which is degenerate in general and then there is only a Poisson structure.

$$\Omega_R(f_R(\omega_1), f_R(\omega_2)) = \int_{M_R^{top}} \omega_1 \wedge \omega_2.$$  

The long exact sequence in the relative cohomology is use full to study degeneration of the symplectic structure. The image of $H^0(X_0 R, \mathbb{R})$ in the exact sequence of relative cohomologies measure how the pairing is degenerate. We denote $\tilde{H}_R$ this subspace in $T_R$. Then the pairing $\Omega_R$ induce a non degenerate pairing on $K_R/\tilde{H}_R$.

Lemma 10. If the graph is orientable the dimension of $\tilde{H}_R$ is given by $\#X_0 R - 1$

Then we deduce that when the graph is orientable the form $\Omega_R$ is non degenerate the graph have only one vertex. And such graph will be called minimal graphs.

Definition 4. A ribbon graph is minimal if it’s orientable with only one vertex i.e iff the Poisson structure on $K_R$ is symplectic.

In figure [12] I give series of minimal graphs for low degree. In general a minimal ribbon graph is not necessarily irreducible.

The dual pairing: There is also an identification

$$T_R^* \cong H^1(M_R^{top}\setminus X_0 R, X_2 R, \mathbb{R}).$$

Let $\hat{K}_R$ the subspace

$$H^1(M_R^{top}, X_2 R, \mathbb{R})$$

using the Mayer Vietoris sequence in this case the space $\hat{K}_R$ is the kernel of the map

$$T_R^* \rightarrow \hat{H}_R$$

and then is equal to $(T_R/\hat{H}_R)^*$. The intersection pairing induce a anti-symmetric pairing $\hat{\Omega}_R$ on $\hat{K}_R$. As before we have an exact sequence for the relative cohomology

$$0 \rightarrow H^0(M_R^{top}, \mathbb{R}) \rightarrow H^0(X_2 R, \mathbb{R}) \rightarrow H^1(M_R^{top}, X_2 R, \mathbb{R}) \rightarrow H^1(M_R^{top}, \mathbb{R}) \rightarrow 0.$$  

The image of $H^0(X_2 R, \mathbb{R})$ correspond to the space $H_R$ and $\hat{\Omega}_R$ induce a symplectic structure on $\hat{K}_R/\hat{H}_R$.  

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Figure 12: List of minimal orientable graphs with a vertex of degree 4, 6, 8, the red ones are not irreducible

**Pairing on** $W_R$: Let the space $H^1(M_R, X_2 R \cup X_0 R, \mathbb{R})$

there is two canonical forget full map

$$p_R : H^1(M_R, X_2 R \cup X_0 R, \mathbb{R}) \longrightarrow H^1(M_R, X_0 R, \mathbb{R}),$$

$$p_R^* : H^1(M_R, X_2 R \cup X_0 R, \mathbb{R}) \longrightarrow H^1(M_R, X_2 R, \mathbb{R}).$$

which are surjective.

On each of the cohomology spaces we have a canonical intersection pairing $\langle ., . \rangle$ and moreover

$$\langle \alpha_1, \alpha_2 \rangle = \langle p_R(\alpha_1), p_R(\alpha_2) \rangle = \langle p_R^*(\alpha_1), p_R^*(\alpha_2) \rangle$$

If $W_R$ is the vector space defined in paragraph 4.3 then

**Lemma 11.** If $R$ is orientable the space $H^1(M_R, X_2 R \cap X_0 R \mathbb{R})$ is isomorphic to $W_R$ via the map

$$\omega \longrightarrow (z_c(\omega))_{c \in X_R}$$

where $z_c(\omega) = \langle \omega, \gamma_c \rangle$

Let $\Omega_R$ the pairing induced on $W_R$. Then the following proposition is true

**Lemma 12.** For all $z_1, z_2 \in W_R$ we have

$$\Omega_R(z_1, z_2) = \Omega_R(p_R(z_1), p_R(z_2)) = \Omega_R(p_R^*(z_1), p_R^*(z_2)) = \langle \hat{p}_R(z_2), p_R(z_1) \rangle$$

the last bracket is the one between $T_R^*$ and $T_R$.

**Hamiltonian of the horocyclic flow**  Here I give a sketch of the prove the following theorem which was also proved in [2] for the principal stratum. This result is valid of course for non orientable ribbon graph by using the two cover. In this case the Poisson structure is defined in a similar way by using the intersection product on the anti-invariant cohomology of the two cover the period coordinate are naturally with value in this space.
Theorem 5. Let \( \lambda \in \mathcal{MF}_0(R) \) then we have
\[
\Omega_R(\xi, \cdot) = dl_\lambda
\]

Proof. The proof is immediate, for all \( \lambda \in \mathcal{MF}_0(R) \) we construct in section 4.3 an element \( z(\lambda) \in W_R^+ \) and by definition of \( p_R \).
\[
p_R(z(\lambda)) = \xi \quad \hat{p}_R(z(\lambda)) = dl_\lambda
\]

From this lemma and lemma 12 I obtain
\[
\Omega_R(\xi, \xi) = \Omega_R(p_R(z(\lambda)), \xi) = \langle \hat{p}_R(z(\lambda)), \xi \rangle = dl_\lambda(\xi).
\]

Computation of the pairing in coordinates: In this section I give several expressions for the pairing. On \( W_R \) it’s given by the following formula
\[
\Pi_R = -\frac{1}{2} \sum_{e \in X_1 R} x_e \wedge y_e
\]
and then \( \Pi_R \) is the pull back of the canonical symplectic form under a map
\[
W_R \to T_R \times T_R^*
\]
In term of the \( z \) coordinate we also have the following expression
\[
\Pi_R = \frac{1}{2} \sum_{e \in X_1 R} z_e \wedge z_{s_0 e}
\]
which is in some sense the Thurston form because it have the same form than the thurston two form on the train track \( \tau_R \) (ref []).

There is a dual expression of this formula. The role of \( s_0 \) and \( s_2 \) are in some sense symmetric and we have
\[
\Pi_R = \frac{1}{2} \sum_{e \in X_1 R} z_e \wedge z_{s_2 e}
\]

It’s also possible to give expression in term of the forms \( x \) and \( y \). In M.Kontsevich introduce for each boundary \( \beta \) a two form on \( K_R \) defined in the following way. Let an edge \( e \) with \( [e]_2 = \beta \) and assume than the boundary contain \( r \) edges then
\[
\omega_\beta = \sum_{0 \leq i < j < r} x_{s_1} \wedge x_{s_2} = -\sum_{1 \leq j \leq r} z_{s_1} \wedge z_{s_2}^{-1}
\]
and then by abuse of notations
\[
\Omega_R = \Pi_R = -\frac{1}{2} \sum_{\beta} \omega_\beta.
\]
In a similar way for each vertex \( v \) we can fix an edge \( e \) with \( [e]_0 = v \) if the vertex is of degree \( r \) then I can set
\[
\hat{\omega}_v = \sum_{0 \leq i < j < r} (-1)^i+j y_{s_0} \wedge y_{s_1} = -\sum_{1 \leq j \leq r} z_{s_0} \wedge z_{s_1}^{-1}
\]
and then
\[
\Pi_R = -\frac{1}{2} \sum_{\beta} \hat{\omega}_v.
\]
Figure 13: Symplectic curve on an orientable graph

**Degeneration of the structure** From the result of the last section I show that the structure is degenerated on the space $\hat{H}_R$ which is the image of

$$H^0(X, \tilde{R}, \mathbb{R}) \rightarrow K_R.$$  

And could be also called the kernel foliation. There is also a map

$$d\lambda : MF_0(R) \rightarrow K^*.$$  

And from the result of the last section the following proposition is true

**Proposition 12.** An element $\xi$ is in $\hat{H}_R$ iff $d\lambda(\xi) = 0$ for all $\lambda \in MF_0(R)$

This result is of some interest it means that two metric ribbon graphs in $Met(R)$ are on the same leaf of the kernel foliation $\hat{H}_R$ iff they have the same geometry. In the sense that they have the same length spectrum on admissible closed curves. Note that this is not true for non admissible curves.

### 5.2 Acyclic decomposition :

In this section I construct canonical curves that lies in the kernel foliations and we will use them to decompose surfaces with vertices of even degrees.

**Statement of the result :** We say that an admissible multi-curve separate a vertex $v$ in $R$ from the rest of the surface if the component that contain $v$ in $R$ contain no other vertices.

**Theorem 6.** Let $R$ an oriented metric ribbon graph with at least two vertices. For each vertex $v$ of $R$ there exist a unique admissible multi-curve $\Gamma_v^+$ such that

- The stable graph $G_v^+$ of $\Gamma_v^+$ contain a component $c_0$ which separate $v$ from the rest of the surface.
- All the curves in $\Gamma_v$ are boundaries of $c_0$.
- $c_0$ is glued along it’s negative boundaries.

These multi-curves are intimately related to degeneration of the symplectic structure $\Omega_R$. These multi-curves satisfy several elementary properties

- The multi-curve is functorial in the sense that it’s well behaved under the action of the mapping class group.

- We have the dual result for negative boundaries $\Gamma_v^-(S)$ is defined by $\Gamma_v^+(-S)$, where $-S$ is obtained by reversing the orientation of $S$. If $\xi_v^\pm$ is the twist flow along $\Gamma_v^\pm(S)$ then we have $\xi_v^- = -\xi_v^+$

- The tangent vectors of the twist flow $\xi_v^-$ are in $\hat{H}_R$. 

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Theorem 6 is quite surprising it means that the local structure of the graph around an even vertex admit some model which correspond to minimal ribbon graph. And it’s possible to cut the graph around the vertex in a canonical way. This process allow to recover the structure on the graph inductively on the number of vertices and the recursion scheme is very similar to an oriented version of the topological recursion as we will see later. By applying this theorem several times we obtain the following corollary wich show how the orientable ribbon graph are rigid.

**Corollary 9.** Les $R$ an oriented ribbon graph with labelled vertices, then there is a unique admissible multi-curves $\Gamma$ such that

1. The components of $R_\Gamma$ are minimals,
2. The oriented stable graph $G^\circ$ associated to $\Gamma$ is acyclic,
3. The labels on the vertices is compatible with the order on the graph.

In other word in the orientable case if we enumerate the vertices we have can have a canonical decomposition of our graph into minimal pieces. This result is different in general that a decomposition into Fenchel Nielsen coordinate where it’s not assumed that the graph is acyclic and where the component are supposed to be irreducible. With this result we are not allowed to cut the surface along a curve which is an handle. Then some genus remain in the components of the decomposition. On each component of the acyclic decomposition the two form is non degenerate and on a minimal oriented surface an admissible curve is necessarily unbounded. Then the acyclic multi-curve of corollary 9 are necessarily maximal. And then I can reformulate this corollary by saying that if the order on the vertices is fixed then there is a maximal acyclic admissible multi-curve which is compatible with this order.

This theorem can be interpreted in term of foliations with poles. If the foliation is orientable and if we enumerate the singularities we can decompose it in a canonical way as a family of foliation with one singularity. The local structure is more easily understandable and the global structure is the one of a directed acyclic graph (figure 2).

In the case of the sphere the minimal ribbon graph are irreducible then we have the following corollary.

**Corollary 10.** Les $R$ an oriented ribbon graph on the sphere with labelled vertices, then there is a unique maximal admissible multi-curves $\Gamma$ such that

1. The oriented stable graph $G^\circ$ associated to $\Gamma$ is acyclic,
2. The labels on the vertices is compatible with the order on the graph.

**Proof of the theorem :** Let $R^\circ = (R, \epsilon)$ an oriented ribbon graph and let $v$ a vertex and $e$ an edge such that $[e]_0 = v$. We will construct the tangent vector associated to the twist flow along $\Gamma^+_v$, we consider

$$\xi^+_v (R) = \sum_{i=0}^{\deg_R(v)-1} (-1)^i \partial_{[v]_0 e}.$$

Then we have the following fact

**Lemma 13.** The vector $\xi^+_v (R)$ belong to $\hat{H}_R(\mathbb{Z})$ and the only relation between the $(\xi^+_v (R))_{v \in X_0 R}$ are proportional to

$$\sum_v \xi^+_v (R) = 0.$$

**Proof.** From part 4.3 an admissible foliation in $\mathcal{MF}_0(R)$ we can associate y "coordinates" and we have the formula

$$dl_{\gamma} (\xi^+_v (R)) = \sum_{i=0}^{2k-1} y_{[v]_0 e} (\gamma)(-1)^i,$$

where $e$ is an half edge such that $[e]_0 = v$ and $\epsilon(e) = 1$. By using part 4.3 we have the formula

$$y_{e} (\gamma) = z_{e} (\gamma) + z_{a_e e} (\gamma)$$

and then we see that we have $dl_{\lambda} (\xi^+_v (R)) = 0$. 

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If we consider a linear relation of the form
\[ \sum_r r_v \xi_v^+ = 0 \]
we can lift \( r_v \) to \( XR \) we obtain a map which is invariant under \( s_0 \) such that
\[ r_v - r_{s_1 v} = 0 \]
and then there is only one such relation (up to a constant) given by
\[ \sum_v \xi_v^+ = 0. \]

Remark 9. We have the exact sequence of cohomology
\[ 0 \rightarrow H^0(M_R^{top}, \mathbb{R}) \rightarrow H^0(X_0 R, \mathbb{R}) \rightarrow H^1(M_R^{top}, X_0 R, \mathbb{R}) \rightarrow H^1(M_R, \mathbb{R}) \]
the second space is the dual of \( \mathbb{Z}^{X_0 R} \) and it’s generated by the canonical basis \( \eta_v \) which satisfy \( \eta_v(v') = \delta_{v,v'} \). The first map is then the diagonal embedding
\[ x \rightarrow x \sum_v \eta_v. \]
The second map is the boundary map \( \delta \) and the we have for all oriented edges
\[ \delta \eta_v e = \delta_{v,[s_1 v]_0} - \delta_{v,[e]_0}. \]
then we see that \( \xi_v^+ \) represent the vector \( \eta_v \) in the cohomology.

Using proposition 7 we have the following corollary

Corollary 11. There is a unique admissible multicurve \( \Gamma_v^+ \) such that \( \xi_v^+ = \xi_v^+(R) \). We now give an elementary construction of the curve \( \Gamma_v^+ \). Fix an oriented edge \( e_0 \) with \( \epsilon(e_0) = 1 \) and \( [e_0]_0 = v \). Then we can turn around the boundary \([e]_2\) until we reach \( v \) again i.e we consider the smaller \( k \) such that \( e_k = [s_0 s_k e]_0 = v \) and then we leave the boundary to the one which is on his right by setting \( e_{k+1} = s_1 s_0^{-1} s_1 e_k \) after some time we will come back to \( e_0 \). If the curve do not contain all the oriented edges \( e \) with \( \epsilon(e) = 1 \) and \( [e]_0 = v \) then we restart the procedure on an other edge. And the end we obtain a minimal representation of a multicurve \( \tilde{\Gamma}_v^+ \). In other word to construct the curve we take all the positive boundaries that meet \( v \) and we perform cutting and gluing at \( v \) which are described in figure 14. This curve is in \( MSZ(M, \partial M) \), all connected component are either in \( S(M) \) or are homotopic to a negative boundary. Let \( B^+ \) the positive boundaries of \( R \) that are adjacents to \( v \) and \( B^- \) the negative boundaries of \( R \) that are adjacent to \( v \) only. From the constuction used in section 4.4 we have the following lemma

Lemma 14. The curve constructed in this way is \( \Gamma_v^+ + \sum_{\beta \in B^-} \beta \) and we have on \( T_R \)
\[ dl_{\Gamma_v^+} = \sum_{\beta \in B^+} dl_{\beta} - \sum_{\beta \in B^-} dl_{\beta} \]
where we sum over the boundaries that contain \( v \) and are oriented positively.

We know proove the first part of the theorem 6

Lemma 15. The curve \( \Gamma_v^+ \) satisfy the desired properties

Proof. Let \( R_v^+ \) the connected component of \( R_v^{top} \) that contain \( v \) and \( G_v^0 \) the stable graph associted to \( \Gamma_v^+ \) we can decompose \( \Gamma_v^+ \) into three sets of curves \( A_i \), \( i = 1...3 \). Where
\begin{itemize}
  \item \( A_1 \) are the curve that join two boundaries of \( R_v^+ \)
  \item \( A_2 \) are the curves that join a boundary of \( R_v^+ \) to an other vertex of \( G_v^+ \)
  \item \( A_3 \) are the curves that are not connected to \( R_v^+ \)
\end{itemize}
Let \( \tilde{R}_v \) the ribbon graph obtained by cutting \( R \) along the curves in \( A_2 \), \( \tilde{R}'_v \) the component that contain \( v \) and \( \tilde{R}''_v \) the union of the other components. We have the map \( \text{cut}''_v : T_R \to T_{\tilde{R}''_v} \), wich is surjective. All the boundaries in \( B^+ \) and \( B^- \) are in \( \tilde{R}_v \) and forall \( \gamma \in A_3 \), \( dl'_\gamma \) on \( T_R \) is the pull back of \( dl'_\gamma \) on \( T_{\tilde{R}''_v} \) under the map \( \text{cut}''_v \). Then we obtain

\[
\sum_{\gamma \in A_3} m_\gamma dl'_\gamma = 0
\]
on \( T_{\tilde{R}''_v} \). All the coefficient are positive and then the multicurves must be empty we have \( A_3 = 0 \).

We have the following functoriality for \( \xi^+_v(R) \)

**Lemma 16.** For all admissible curves \( \Gamma \) we have

\[
\text{cut}_v \xi^+_v(R) = \xi^+_v(R_{\Gamma}).
\]

Then we have

\[
\text{cut}_v (\xi^+_v(R)) = \xi^+_v(R^+_v)
\]
by definition of \( \Gamma^+_v \) we have \( \text{cut}_v (\xi^+_v(R)) = 0 \) and then \( \xi^+_v(R^+_v) = 0 \). Then by using the following lemma we can conclude that \( R^+_v \) is symplectic and the \( v \) is the only vertex in \( R^+_v \).

**Lemma 17.** For all \( R^v \) oriented and \( v \in X_0 R \) we have \( \xi^+_v(r^v) = 0 \) iff \( R \) is symplectic.

On \( K_{\tilde{R}''_v} \) we have the relation

\[
\sum_{\gamma \in A_1} m_\gamma dl'_\gamma = 0
\]
as \( R^+_v \) is symplectic this is also the case of \( \tilde{R}'_v \) and then we have the following lemma

**Lemma 18.** Let \( R \) any symplectic ribbon graph and \( \Gamma \) an admissible curve then the \( dl_\gamma, \gamma \in \Gamma \) are independent on \( K_R \).

With this lemma we conclude that we must have \( m_\gamma = 0 \) for all \( \gamma \in A_1 \). Finally we obtain the relation

\[
\sum_{\gamma \in A_2} m_\gamma dl'_\gamma = \sum_{\beta \in B^+} dl'_\beta - \sum_{\beta \in B^-} dl'_\beta
\]
on \( T_{R^+_v} \). If \( \epsilon_v \) is the orientation induced on \( R^+_v \) then the only relation between the boundaries are given by \( \epsilon_v \) and then we see that as \( m_\gamma \) are positives then all the boundaries in \( A_2 \) are negatives, and the boundary of \( R^+_v \) is the union of \( A_2, B^+, B^- \). Then the curve \( \Gamma^+_v \) satisfy the desired properties.

To conclude we need the converse statement

**Lemma 19.** If \( \Gamma \) satisfy the desired properties it’s \( \Gamma^+_v \)

**Proof.** Let \( \Gamma \) such curve and let \( c \) a connected component of it’s stable graph that does not contain \( v \) then we have the relation on \( K_R \)

\[
\sum_{v \in X_0 R^v(c)} \xi^-_v = \sum_{\gamma \in X^\Gamma(c)} \xi^-_\gamma.
\]
and then we get by summing over \( c \)

\[
\xi_\Gamma = \sum_{v \not\in c} \xi^-_v = \xi^+_v
\]
Case of non orientable surface  When consider non orientable surface with an even vertex we still have degeneration of the symplectic structure and we can also find canonical curves that separate this vertex from the rest of surface.

Theorem 7. If $R$ is any ribbon graph and $v$ a vertex of even degree there is exactly two admissible multi-curve $\Gamma_{+}^{\pm}$ such that

- $\Gamma_{+}^{\pm}$ separate $v$ from the rest of the surface
- The component $R_{+}^{\pm}$ is orientable and admit an orientation such that it’s glued along it’s negative boundaries.
- All curves in $\Gamma_{+}^{\pm}$ are boundaries of $R_{+}^{\pm}$

This theorem contain the case of theorem 6. In this case the group of automorphisms of the surface can eventually exchange the two multi-curve $\Gamma_{+}^{\pm}$. This happen for instance for the torus with one boundary.

Extracting a pair of pant on an oriented surface:  In the generic case when there is only vertices of degree 4 then the only minimal surfaces are topological pair of pants. In this case minimal and irreducible surfaces coincide and then the last theorem give particular family of canonical Fenchel-Nielsen decomposition of our graph.

Let $S^{\circ}$ a generic oriented metric ribbon graph an embedded bounded pair of pant in $S$ is an orientable curve $\Gamma$ such that

- There is a component $c_0$ of $G^{\circ}$ which is a pair of pant,
• All the curve in $\Gamma$ are in the boundaries of $c_0$.
• $c_0$ is glued along it’s negative boundaries.

The terminology is justified because the third condition imply that length of such curve is necessarily bounded by the lengths of the boundaries. This imply by 7 that there is only a finite number of bounded embedded pair of pants.

These three conditions also imply that the choice of the marked marked component $c_0$ is canonical. Moreover as the only orientable ribbon graph on a pair of pant contain a unique vertex of order four. Then a bounded pair of pant on a generic oriented metric ribbon graph separate a vertex from the rest of the surface.

**Theorem 8.** For each generic oriented metric ribbon graph $S$ and each vertex $v$ there exist a unique bounded pair of pants $\Gamma^+_v$ that separate $v$ from the rest of the surface.

A reformulation of corollary 9 give the following result

**Corollary 12.** Les $R^\circ$ a generic ribbon graph with labelled vertices, then there is a unique orientable multi-curves $\Gamma$ such that

1. $\Gamma$ is maximal i.e components of $R_\Gamma$ are pair of pant’s,
2. The oriented stable graph $G^\circ$ associated to $\Gamma$ is acyclic,
3. The labels on the vertices is compatible with the order on the graph.
6 Surgery on the volumes and recurrence relation:

In this part I give the recursion for the volumes of moduli space. For each directed surface the moduli space $\mathcal{M}^{\text{comb}}(M^\circ, L)$ posses a measure $d\nu_{M^\circ}(L)$. I will denote $Z_{M^\circ}(L)$ the volume which is function on $\Lambda_{M^\circ}$. It’s well defined because the space is the a union of a finite number of relatively compact cells. In general the boundaries are labelled and then I choose to separate the positive and negative variable and write $Z_{g,n^+,n^-}(L^+|L^-)$ the volumes.

6.1 Surgery at the level of the volumes and stable graph

In this section I give results which generalise ideas of [3] outside the generic case. I study space of combinatorial surfaces marked by an admissible curves and perform surgeries along these curves.

Covering and decomposition of the measures: Let $M^\circ$ and $G^\circ$ a directed stable graph, there is a natural bundle over $\mathcal{M}^{\text{comb}}(G^\circ)$ which correspond to all the possible gluing of surfaces in $\mathcal{M}^{\text{comb}}(G^\circ)$. If $\Gamma^\circ$ is a multi-curve that represent $G^\circ$ and let $\mathcal{M}\mathcal{F}_{\Gamma^\circ}(M^\circ)$[4.4]. This space carry an action of $\text{Stab}(\Gamma^\circ)$ and it’s possible to form the quotient

$$BM^{\text{comb}}(G^\circ) = \text{Stab}(\Gamma^\circ)\backslash \mathcal{M}\mathcal{F}_{\Gamma^\circ}(M^\circ).$$

The reasons of this choice are the following

- The space $\mathcal{M}\mathcal{F}_{\Gamma^\circ}(M^\circ)$ contain in a natural way $\mathcal{T}_{\text{comb}}(G^\circ)$.
- It’s possible to cut an element of $\mathcal{M}\mathcal{F}_{\Gamma^\circ}(M^\circ)$ along the curves in $\Gamma$ and the result is an element of $\mathcal{T}_{\text{comb}}(G^\circ)$ and this induce a map

$$\text{cut} : BM^{\text{comb}}(G^\circ) \longrightarrow \mathcal{M}^{\text{comb}}(G^\circ).$$

- Moreover the twist flow along the curves in $\Gamma$ is well defined on $\mathcal{M}\mathcal{F}_{\Gamma^\circ}(M^\circ)$ which was not the case of $\mathcal{T}_{\text{comb}}(M^\circ)$ and this induce a affine orbifold torus action on $BM^{\text{comb}}(G^\circ)$.

This space is a piece-wise linear orbifold with a natural measure normalised by it’s set of integer points and It denoted it $d\tilde{\nu}_{G^\circ}$. Using the results of 4.4 the

$$\text{cut} \times d\tilde{\nu}_{G^\circ} = \prod_{\gamma \in X_1G^\circ} l_\gamma \times d\nu_{G^\circ}$$

For any stable graph and any positive symmetric function

$$F : R_{+}^{X_1G^\circ} \longrightarrow R_+$$

it’s possible to consider the integral

$$Z_{G^\circ}(F)(L) = \int_{BM^{\text{comb}}(G^\circ, L)} F(L_{G^\circ}(S))d\tilde{\nu}_{G^\circ}(L)$$

where $L_{G^\circ}$ is the lengths of the curves of the stable graph and $L$ is a variable indexed by the boundaries.

Proposition 13. For all $G^\circ$ a stable graph the function $Z_{G^\circ}(F)(L)$ is given by

$$Z_{G^\circ}(F)(L) = \frac{1}{\# \text{Aut}(G^\circ)} \int_{L \in \Lambda_{G^\circ}(L)} F(l) \prod_{c} Z_{G^\circ}(c)(L(c), l(c)) \prod_{\gamma} l_\gamma d\sigma_{G^\circ}(L)$$

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Statistic for multi-curves covering and integration formula. If $\Gamma^o$ is an oriented multi-curve on $M^o$, I denote
\[
\mathcal{T}^\text{comb.}^o(M^o) = \{ S \in \mathcal{T}^\text{comb.}^o(M^o) | \Gamma \in \mathcal{MF}_0(S) \text{ and } \Gamma^o = \Gamma_S^o \}
\]
the space of generic oriented surfaces $S^o$ such that $\Gamma$ is orientable on $S^o$ and the orientation on it induced by $S^o$ correspond to $\Gamma^o$. Let $G^o$ the stable graph of $\Gamma^o$, there is an action of Stab($\Gamma^o$) on $\mathcal{T}^\text{comb.}^o(M^o)$ and I denote $\mathcal{M}_G^\text{comb.}^o(M^o)$ the quotient
\[
\mathcal{M}_G^\text{comb.}^o(M^o) = \text{Stab}(\Gamma^o) \backslash \mathcal{T}_G^\text{comb.}^o(M^o)
\]
There is a canonical map
\[
\pi_G^o : \mathcal{M}_G^\text{comb.}^o(M^o) \to \mathcal{M}_G^\text{comb.}^o(M^o)
\]
and the fiber over $S$ is the set of admissible curves on $S$ which are in the orbit of $\Gamma^o$ or equivalently the admissible curves with stable graph given by $G^o$. The map $\pi_G^o$ is a covering over each cells and the $\mathcal{M}_G^\text{comb.}^o(M^o)$ is equipped by the pull back of the measure on $\mathcal{M}^\text{comb.}^o(M^o)$.

As in [3] I consider the following statistic for the distribution of the length of multi-curves. We define $N_{G^o} F(S)$ as the sum of $F(L_{\Gamma}(S))$ over all the orientable curves with stable graph $G^o$.
\[
N_{G^o} F(S) = \sum_{\Gamma \in \pi_{G^o}^{-1}(S)} F(L_{\Gamma}(S)).
\]
This is by definition the push forward of $F \circ L_{\Gamma}$ under $\pi_{G^o}$. The function $N_{\Gamma} F$ is well defined on the moduli space $\mathcal{M}^\text{comb.}(M)$ because of the relation
\[
F(L_{g \cdot \Gamma}(g \cdot S)) = F(L_{\Gamma}(S))
\]
and because the map
\[
g : \mathcal{MS}(R) \to \mathcal{MS}(g \cdot R)
\]
preserve the orientation on the stable graphs. Then it satisfy the following relation which is the formula for a push forward under a covering
\[
\int_{\mathcal{M}_G^\text{comb.}^o(M^o, L)} N_{G^o} F(S) d\nu_{M^o}(L) = \int_{\mathcal{M}_G^\text{comb.}^o(M, L)} F(L_{\Gamma}(S)) d\nu_{M^o}(L).
\]
There is a canonical map
\[
\mathcal{M}_G^\text{comb.}^o(M^o) \to \text{BM}_G^\text{comb.}^o(G^o)
\]
this map is not surjective but the following lemma allow to avoid this problem.

**Lemma 21.** The subset $\mathcal{M}_G^\text{comb.}^o(M^o)$ is of full measure in $\text{BM}_G^\text{comb.}^o(G^o)$ and this is also true for $\mathcal{M}_G^\text{comb.}^o(M^o, L)$ in $\text{BM}_G^\text{comb.}^o(G^o, L)$ for all $L$.

We don’t give a detailed proof of this lemma it’s based on the fact that the complementary set is formed by foliations that contain saddle connection. In the orientable case such saddle connections can be generic. But this phenomenon cannot happen if the foliation is transverse to a multi curve. Then the only configuration of saddle connection that are possible are not generic and then the surface are in some dimension one submanifold. There is only a countable number of such submanifolds and then the problem are in a set of zero measure. But writing correctly this argument is not very interesting here and could take a lot of space.

Moreover in the case of acyclic gluing that I will use the situation is indeed much more simple because the gluing allway create multi-arcs which make the situation much more simpler in fact if the stable graph is acyclic the space $\mathcal{MF}_{\Gamma^o}(M^o)$ is included in $\mathcal{MA}_G(M^o)$, because gluing cannot create a cycle.

Using this lemma I obtain the relation
\[
Z_{G^o} F(L) = \int_{\mathcal{M}_G^\text{comb.}^o(M^o, L)} N_{G^o} F(S) d\nu_{M^o}(L)
\]
Then it give the following proposition which was first proved by M.Mirzakhani in the case of hyperbolic surfaces.

**Proposition 14.** The statistics satisfy the following integral formula
\[
\int_{\mathcal{M}_G^\text{comb.}^o(M^o, L)} N_{G^o} F(S) d\nu_{M^o}(L) = \frac{1}{\text{Aut}(\Gamma^o)} \int_{L \in \Lambda_{G^o}(L)} F(l) \prod_{\gamma} Z_{G^o}(L(c), l(c)) \prod_{\gamma} l_{\gamma} d\sigma_{M^o}
\]

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6.2 Recursion for the volumes:

In this part I use the theorem and the results of the last part to obtain recursion for the volumes $Z_{g,n^+\,n^-}(L^+\,L^-)$.

A kind of degenerated geometric recursion formula: We denote $P(M^\circ)$ the set of stable graphs on $M^\circ$ which correspond to bounded pair of pants. There is four family of such stable graph which are represented in figure I and are also given in the introduction.

1. We remove a pant that contain two positive boundaries $(i,j)$
2. We remove a pant that contain a positive boundary $i$ and a negative boundary $j$
3. We remove a pant with one positive boundary $i$ which is connected to the surface by two negative boundaries and do not separate the surface
4. We remove a pant with one positive boundary $i$ which is connected to the surface by two negative boundaries and separate the surface in two components.

Then we can rewrite the theorem in the following strange way which is a kind of degeneration of the geometric recursion formula. The advantage of this formulation is that it is straight forward to integrate it by using the results of the last section. In other word it means that the covering associated to the bounded of pant’s is of degree $2g - 2 + n^+ + n^-$

**Lemma 22.** For all $S \in M^{comb.\ast}(M^\circ, L)$ we have

$$2g - 2 + n^+ + n^- = \sum_{G^0 \in P(M^\circ)} N_{G^0}(1)(S).$$

**Proof.** This is just a reformulation of theorem by using the definition of the functions $N_{G^0}(1)(S)$. \qed

Recursion for volumes: In this section we make effective theorem in the case of surface with vertices of degree four.

**Theorem 9.** For all value of the boundary lengths the volumes satisfy the recursion

$$(2g - 2 + n)Z_{g,n^+\,n^-}(L^+\,L^-) = \sum_{i} \sum_{j} [l_i^+ - l_j^+] Z_{g,n^+\,n^- - 1}([L_i^+ - L_j^+]_i, L_j^+\,L_{ij}^+\,L_{ij}^-)$$

$$+ \frac{1}{2} \sum_{i \neq j} (l_i^+ + l_j^+) Z_{g,n^+\,n^- - 1,n^-}(L_i^+ + L_j^+, L_{ij}^+\,L_{ij}^-\,L^-)$$

$$+ \frac{1}{2} \sum_{i} \int_{0}^{L_i^+} Z_{g-1,n^+\,n^- - 1}(x, L_i^+ - x, L_{ij}^+\,L^-) x(L_i^+ - x) \, dx$$

$$+ \frac{1}{2} \sum_{i} \sum_{g_1 + g_2 = g} x_1 x_2 Z_{g_1,n_1^+\,n_1^-}(x_1, L_{i1}^{g_1}\,L_{i1}^-) Z_{g_2,n_2^+\,n_2^-}(x_2, L_{i2}^{g_2}\,L_{i2}^-)$$

where we use the notation

$$x_l = \sum_{i \in I_l^-} L_i^+ - \sum_{i \in I_l^+} L_i^-$$

For each $L \in \mathbb{R}_+^E$ and $I \subset E$ we use the notation $L_I = (L_x)_x \in \mathbb{R}_+^I$.

As corollary of this proposition I obtain the following fact

**Corollary 13.** The volumes $Z_{g,n^+\,n^-}(L^+\,L^-)$ are continuous piece-wise polynomials, homogeneous of degree $4g - 3 + n$ which are symmetric under permutations of booth sets of variables.

**Proof.** To obtain the theorem I multiply the formula of proposition by the measure $dv_{M^\circ}(L)$ and I integrate over the moduli space. By using the result of section I obtain the following formula

$$(2g - 2 + n)Z_{M^\circ} = \sum_{G^0 \in P(M^\circ)} N_{G^0}(1).$$
The covering of bounded pair of pants split into several coverings which correspond to the fourth types of gluing with all the possible choice of boundaries. Now from the result of the last section we have

\[ Z_{G^\circ}(1)(L^+|L^-) = \frac{1}{\#\text{Aut}(G^\circ)} \int_{L \in \Lambda_{G^\circ}(L)} \prod_c Z_{G^\circ(c)}(L^+(c), L^-(c)) \prod_\gamma I_{c, \sigma G^\circ}(L). \]

Then there is a unique component in our graph which is a pair of pant glued along it’s negative boundary. The volumes associated to this component are constant equal to one because there is only one oriented graph on an oriented pair of pant then this term disappear in the formula. To finish the proof it remain to compute the domain of integration in each cases. All the multi-curves used are rigid in the sense that \( \Lambda_{G^\circ}(L) \) is reduced to a point with one exception when the genus is reduced by one which correspond to type 3 of figure 1.

6.3 Some properties of the recursion and the volumes:

Graphical expansion: As in the case of the topological recursion the formula (9) admit a graphical expansion obtained by iterating the recursion. Proposition 12 give canonical maximal acyclic multi-curve which decompose the surface. In the case of four valent vertices this is a pant decomposition and then the stable graph are trivalent. It’s possible to write the volume of surface with labelled vertices as a sum over these graph Proposition 15.

The volumes satisfy

\[ (2g - 2 + n)! Z_{g,n^+,n^-}(L^+|L^-) = \sum G^\circ \frac{1}{\#\text{Aut}(G^\circ)} \int_{L \in \Lambda_{G^\circ}(L)} \prod_\gamma I_{c, \sigma G^\circ}(L) \]

where we sum over all the directed acyclic pant decomposition with an enumeration of the vertices and with \( n^+ \) positives and \( n^- \) negatives labeled leg’s.

Times inversion and other symmetries: A trivial property of the recursion is the symmetry under permutation of the variable which is due to the fact that we sum symmetric function over a set which is invariant under permutation of the variable. This is different from the symmetry of topological recursion which is deeper statement.

Proposition 16. If the \( Z'_g \) are function obtained by the recursion of theorem 9 which satisfy \( Z'_g(x_1, x_2|y_1) = Z'_g(y_1|x_1, x_2) \). Then the \( Z'_g \) are symmetric under permutation of each set of the variables

An second and more interesting properties is the times inversion which correspond to the operation

\[ Z(L^+|L^-) \rightarrow Z(L^-|L^+). \]

The recursion of theorem 9 is obtained by removing positive boundary but it’s also possible to consider the recursion along negative boundaries.

Proposition 17. If the coefficients satisfy \( Z_0(x_1, x_2|y_1) = Z_0(y_1|x_1, x_2) \) then the function \( Z_g(L^+|L^-) \) also satisfy this relation

Proof. This proposition is obtained by the graph expansion off proposition 15. In fact the set of directed acyclic graph have an involution given by reversing the orientation.

String equation for the volumes: The functions \( Z_{g,n^+,n^-}(L^+|L^-) \) satisfy very simple relations when contract a variables which is an analogous of the string equation. We will investigate special case of this recursion later and derive a dilaton equation in these cases.

Proposition 18. The volumes satisfy the following relation when \( L_i^+ = 0 \)

\[ Z_{g,n^+,n^-}(0, L^+|L^-) = \left( \sum_i L_i^- \right)^\gamma \left( \sum_i L_i^+ \right) Z_{g,n^+,n^-}(L^+|L^-) \]
A physical interpretation of the recursion: Here we interpret the volumes and the recursion as an expansion of the partition function associated to a physical toy model. A surface $M^0$ can be viewed as an interaction between strings in some sense. We consider the negative boundaries as the exits of our system and the positive boundaries are the entries. An element in $\Lambda_{M^0}$ correspond to a positive weight on each string such that the total weight of the entries is equal to the total weight of the exits. These weights can be seen as the energy of the string and we have the conservation of the energy during the interaction. The function $Z_{M^0}$ define a kernel operator. To a function
\[ f : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+ \]
we can associate
\[ Z_{M^0}(f)(L^+) = \int_{\mathbb{R}_+^2} f(L^-) dZ_{M^0} \]
which is now a function on $\mathbb{R}_+^2$, we recall that $I^\pm$ is the decomposition of $\partial M$ into positives and negatives boundaries.

Then we can see a surface with one vertex as an elementary interaction between these string. A surface $M^0$ with several vertices is then a complicated interaction between our entries and $Z_{M^0}$ is the transition kernel associated to this interaction. It’s not a probability kernel because it’s not normalised.

Laplace transform of the recursion: In this section we compute the Laplace transform of the last recursion
\[ Z_{g,n^+,n^-}(\lambda^+|\lambda^-) = \int e^{-\lambda^+L^+ - \lambda^-L^-} dZ_{g,n^+,n^-}(L^+|L^-) \]
The fact that the support of the measure is contained in $\{\sum_i t_i^+ = \sum_i t_i^-\}$ implies the following symmetry for this function
\[ Z_{g,n^+,n^-}(\lambda^+ + t|\lambda^- - t) = Z_{g,n^+,n^-}(\lambda^+|\lambda^-). \]

**Theorem 10.** The Laplace transform satisfy the recursion

\[
Z_{g,n^+,n^-}(\lambda^+|\lambda^-) = -\sum_{i,j} \frac{\partial^+ Z_{g,n^+,n^-}(\lambda_i^+,\lambda_{i,j}^+|\lambda_{i,j}^-)}{\lambda_i^+ + \lambda_{i,j}^-} Z_{g,n^+,n^-}(\lambda_i^+,\lambda_{i,j}^+|\lambda^-) + \frac{1}{2} \sum_{i \neq j} \frac{1}{\lambda_i^+ - \lambda_j^+} Z_{g-n^+1,n^-}(\lambda_i^+,\lambda_{i,j}^+|\lambda^-) + \frac{1}{2} \sum_i \sum_{g_1 + g_2 = g} \partial^+ Z_{g_1,n^+_1+n^-_1}(\lambda_i^+,\lambda_{i,j}^+|\lambda^-) \partial^+ Z_{g_2,n^+_2+n^-_2}(\lambda_i^+,\lambda_{i,j}^+|\lambda^-)
\]

Where we denote $\partial^+ \xi$ the derivative with respect to $\lambda_i^+$

**Proof.** To prove this theorem we only need to compute the Laplace transform of the term in the recursion. For instance if $f$ is a continuous function and $\mathcal{L}f$ its Laplace transform we have
\[
\int_{x_1 < x_2} e^{-\lambda_1 x_1 - \lambda_2 x_2} (x_1 - x_2)f(x_1 - x_2)dx_1 dx_2 = \frac{1}{\lambda_1 + \lambda_2} \int_{x} e^{-\lambda_1 x} f(x)dx = -\frac{\partial f(\lambda_1)}{\lambda_1 + \lambda_2}
\]
and then
\[
\int e^{-\lambda^+L^+ - \lambda^-L^-} [L_i^+ - L_j^-]_+ Z_g([L_i^+ - L_j^-]_+, L_{i,j}^+|L_{i,j}^-)d\sigma = -\frac{\partial^+ Z_g(\lambda_i^+ + \lambda_{i,j}^+|\lambda_{i,j}^-)}{\lambda_i^+ + \lambda_{i,j}^-}
\]
for the other terms of the recursion I use the formulas
\[
\int_{x_1, x_2} e^{-\lambda_1 x_1 - \lambda_2 x_2} f(x_1 + x_2)dx_1 dx_2 = \frac{\partial \mathcal{L}f(\lambda_1) - \partial \mathcal{L}f(\lambda_2)}{\lambda_1 - \lambda_2} \tag{5}
\]
\[
\int_{x_1, x_2} e^{-\lambda_1 x_1 - \lambda_2 x_2} f(x_1 - x_2)dx_1 dx_2 = \partial_1 \partial_2 \mathcal{L}f(\lambda, \lambda). \tag{6}
\]
Here the function $f$ is continuous and piece-wise polynomial.
7 Case of surfaces with one negative boundary and application to counting ”Dessin d’enfants”

In this section I will study in more details the case of surfaces with only one negative boundary. As we will see the recursion in this case take a much simpler and it’s possible to rely it to cut and join equation.

7.1 Surfaces with one negative boundary:

First of all if $n^- = 1$ there is an identification

$$\Lambda_{n^+,1} \cong \mathbb{R}^{n^+},$$

and then it’s possible to drop the negative boundaries and write $Z_{g,n^+,n^-}(L^+|L^-)$ as

$$Z_{g,n^+,n^-}(L^+|L^-) = F_{g,n^+}(L^+),$$

where $F_{g,n^+}(L)$ is a function on $\mathbb{R}^{n^+}$.

As we say the recursion of theorem 9 allow preserve the ”sub-category” of surfaces with one negative boundary is stable under the recursion. Extracting a bounded pair of pant on a surface with only one negative boundary create only surfaces with one negative boundary. Moreover these gluing are necessarily non separating and can’t be of type 2. Then the recursion take the following form

**Corollary 14.** The functions $F_{g,n}$ are homogeneous polynomials of degree $4g - 4 + n$ and satisfy the following recursion

$$(2g + n - 1)F_{g,n}(L) = \frac{1}{2} \sum_{i \neq j} (L_i + L_j)F_{g,n-1}(L_i + L_j, L_{(i,j)})$$

$$+ \frac{1}{2} \sum_i \int_0^{l_i} F_{g-1,n+1}(x, L_i - x, L_{(i)}) x(L_i - x) \, dx$$

For this proposition we can deduce $F_{g,n}(L)$ from the case $F_{0,2}(L_1, L_2) = 1$.

**Proof.** To prove this proposition we remark that when we use theorem 8 to a surface with only one boundary then the cutting which are allowed are necessarily of type 1 or 3, in the other case a component of the surface must contain only positive boundaries which is impossible. Reciprocally performing gluing of type 1 and 3 preserve the subcategory of surfaces with only one negative boundary then we can deduce the formula. The form of the recursion preserve the space of polynomial which conclude the proof by using corollary 9.

String and dilaton equations:

The function $F_n$ satisfy two series of equations which are similar to string and dilaton equations. The string equation is given by rewriting the formula of proposition 18. We obtain the formula

$$F_{g,n+1}(0, L) = \left( \sum_i L_i \right) F_{g,n}(L).$$

This formula is obtained by computing volume of the space of ribbon graph with only one edge in the first boundary. By looking at the volume of ribbon graph with at most two edges it’s possible to compute the first order of $F_{g,n+1}$ at $L_1 = 0$. This give the following relation which is an analogous of dilaton equation

**Proposition 19.** The volume $F_{g,n+1}$ satisfy the relation

$$\frac{\partial F_{g,n+1}}{\partial L_1}(0, L) = (2g + n - 1)F_{g,n}(L)$$

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Case of the sphere: A particular case is the one of the sphere, rewriting the last formula we obtain the relation

\[(n - 1)F_{0,n}(L) = \sum_{i < j} (l_i + l_j)F_{0,n-1}(l_i + l_j, L_{\{i,j\}})\]

But in the case of genus zeros the surface with only one positive boundary is also preserved by the recursion along positive boundaries. If we denote \(|x| = \sum x_i\), there is the following formula

\[(n - 1)Z_0(t^*_1|L^*_1) = \frac{1}{2} \sum_{l_1^*, l_2^*} |L^*_1|L^*_2|Z_0(|L^*_1|L^*_2)Z_0(|L^*_1||L^*_2).\]

Then using time inversion it’s possible to obtain another recurrence relation for the function \(F_{0,n}(L)\)

\[(n - 1)F_{0,n}(L) = \frac{1}{2} \sum_{l_1, l_2} |L_{l_1}|L_{l_2}|F_{0,n+1}(L_{l_1})F_{0,n+1}(L_{l_2}).\]

These two recurrence relation are in fact directly related to planted numbered tree and correspond to remove a pair of leafs or the roots.

**Recursion for the coefficients string, dilaton and cut and join relations:** In this section I investigate the recursion obtained for the coefficients of the polynomial \(F_{g,n}(L)\), I can write

\[F_{g,n}(L) = \sum_{\alpha} L^\alpha c(\alpha)\]

With the condition

\[|\alpha| = 4g - 2 + n \quad \#\alpha = n\]

so it’s possible drop the indices \((g, n)\) in the notation. By expanding the last formula I derive the following relation for these coefficients.

**Corollary 15.** The coefficients \((c(\alpha))\) satisfy the following recursion

\[(2g - 1 + n)c(\alpha) = \frac{1}{2} \sum_{i \neq j} \frac{(\alpha_i + \alpha_j)!}{\alpha_i!\alpha_j!} c(\alpha_i + \alpha_j - 1, \alpha_{\{i,j\}}) + \frac{1}{2} \sum_{i, x_1 + x_2 = n, \alpha_i} \frac{(x_1 + 1)!(x_2 + 1)!}{\alpha_i!} c(x_1, x_2, \alpha_{\{i\}})\]

The coefficients \(c(\alpha)\) satisfy an important symmetry they are invariant under permutations. Then as for the intersection numbers of the tautological class over the moduli space it’s possible to see \(c\) as a function on the set of generalised partitions, we write \(c(\mu)\) where \(\mu = (\mu(0), \mu(1), \ldots)\). Then I can consider the following formal serie with infinitely many variables.

\[\phi(s, t) = \sum_{\mu} s^{\mu(0) + \#\mu} \prod_{i} \mu(i)! \mu^{(i)} c(\mu)\]

Then I obtain the following result

**Corollary 16.** The series \(\phi(s, t)\) satisfy the following equation

\[\frac{\partial \phi}{\partial s} = \frac{1}{2} \sum_{i,j} (i + j)t_{ij}\frac{\partial \phi}{\partial t_{i+j-1}} + \frac{1}{2} \sum_{i+j} (i + 1)(j + 1)t_{i+j-3}\frac{\partial^2 \phi}{\partial t_i \partial t_j} + t_0^2\]

In fact the variable \(\epsilon\) and \(t\) are not independent we have the relation

\[\phi(s, t) = \phi(1, t(s)) = \psi(t(s)) \quad \text{with} \quad t_i(s) = s^\frac{i+1}{2} t_i\]

And then we have by taking the derivative and evaluating at \(s = 1\)

\[\frac{\partial \phi}{\partial s} = \sum_{i} \frac{i + 1}{2} t_i \frac{\partial \psi}{\partial t_i}\]

Then we can obtain

\[\sum_{i} (i + 1)t_i \frac{\partial \psi}{\partial t_i} = \sum_{i,j} (i + j)t_{ij}\frac{\partial \psi}{\partial t_{i+j-1}} + \sum_{i+j} (i + 1)(j + 1)t_{i+j-3}\frac{\partial^2 \psi}{\partial t_i \partial t_j} + t_0^2\]
7.2 Dual problem and Hurwitz number:

**Lemma 23.** The oriented generic ribbon graph of type \((g, n^+, n^-, µ)\) where \(µ\) is the partition that encode the degrees of the vertices, are in bijection with the coverings of the sphere ramified over three points \((0, 1, ∞)\) such that

- There is \(µ(i)\) ramification of order \(\frac{i+2}{2}\) over 1.
- There is \(n^-\) ramifications over ∞ and \(n^+\) over 0 which are labeled.

These covering are called dessins d’enfants and where studied in many place.

**Proof.** Let \(R^g\) an oriented ribbon graph made by gluing rectangle \(R_e\) where \(e \in XR\) is oriented positively. From [] we have a canonical one form \(ω_R\) given locally by \(dz\) on each \(R_e\) if we fix a vertices \(v\) of the graph then the period map

\[ z \mapsto \int_v^z ω_R \]

is well defined on the universal cover of \(R\) and the image of the fundamental group is contained in \(Z\) then we have a well defined map is

\[ R → \mathbb{C}/Z \]

which is a cylinder. The graph is then the pre-image under of the circle \(R/Z\). The vertices of the graph are mapped to 0. The map is ramified at a vertex \(v\) of degree \(2i > 2\) with degree \(i - 1\) because an interval of the form \([0, ϵ]\) have \(i\) pre-image that contain \(v\). They correspond to the positive edges \(e\) with \([e]_0 = v\). When we pull back a differential \(w\) with a simple pole at zeros under \(φ = z^k\) we have \(Res_ω φ^e w = kResω w\). We can see \(C/Z\) as a sphere with two removed points at \(±∞\), under our assumption the positive boundaries are mapped to the pole at \(i∞\) (resp \(-i∞\). The residue of \(w_R\) correspond to the length of the boundary which is also equal to the number of edge that contain a boundary.

In other hand for all covering ramified over 0 and \(±i∞\) it’s possible obtain a ribbon graph on the surface by looking at the pre-image of the circle the orientation on the circle induce an orientation on the graph.

Then the oriented generic ribbon graph with one negative boundaries correspond to dessins d’enfants with a maximal ramification over \(-i∞\) and \(2g - 2 + n\) simple ramification over 0. Let \(h_{α, (1^{2g-1+n}), (4g-2+2n)}\) the corresponding Hurwitz number. I assume than the ramifications over the first point are labelled.

The last lemma and an explicit computation of the volumes in this case allow use to write the following formula.

**Lemma 24.** The volumes \(F_{g,n}\) are polynomials which are naturally related to Hurwitz number in the following way.

\[ F_{g,n}(x_1, ..., x_n) = \sum_α \prod_i \frac{x_1^{α(i)-1}}{(α(i) - 1)!} h_{α, (1^{2g-1+n}), (4g-2+2n)} \]

**Proof.** To prove this result I compute the volume \(F_{g}(L)\) associated to an oriented ribbon graph with a single negative boundary. This is an integral over some affine subspace in \(Met(R)\). The relation on \(Met(R)\) are given by \(L_i^+(m) = x_i\) for each \(i = 1, ..., n^+\). As the graph is orientable the dual is bipartite then an edge of the graph appear in exactly one of these equations with a weight equal to 1. In other word there is an identification

\[ Met(R^g, x) = \prod_i \{(m_e)_{[e]_2 = β^+_i} \sum_e m_e = x_i \} \]

Then the volumes are given by the volumes of simplices. Each of the factor is equipped with the affine measure and

\[ \int_{\sum_j y_j = x} dσ = \frac{x^{n-1}}{(n - 1)!} \]

Then the volumes of \(Met(R^g, x)\) is

\[ \prod_i \frac{x_1^{α_i-1}}{(α_i - 1)!} \]

where \(α_i\) is the number of edges in the boundary \(β^+_i\). By summing the contribution of all the graphs

\[ \sum_R \frac{F_R}{#Aut(R)} \]

the coefficient in front of \(\prod_i \frac{x_1^{α_i-1}}{(α_i - 1)!}\) is the number of four valent ribbon graph with \(α_i\) edge on the \(i\)-ieme positive boundary counted with automorphisms. By using proposition 23 we can conclude the proof.  \[\square\]
From this proposition and proposition [15] give recurrence relation for the generating serie of unlabeled Hurwitz number.

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