PERTURBATIVE SIGMA MODELS, ELLIPTIC COHOMOLOGY AND THE WITTEN GENUS

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Abstract. We construct a geometric model for elliptic cohomology with complex coefficients and provide a cocycle representative for the Witten class in this language. Our motivation stems from the conjectural connection between 2-dimensional field theories and elliptic cohomology originally due to G. Segal and E. Witten. The specifics of our constructions are informed by the work of S. Stolz and P. Teichner on super Euclidean field theories and K. Costello’s construction of the Witten genus using perturbative quantization. As a warm-up, we prove analogous results for supersymmetric quantum mechanics and K-theory.

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1. Introduction

In this paper we give a geometric model for elliptic cohomology with complex coefficients and construct a cocycle representative for the Witten genus in this language. Our motivation stems from the conjectural connection between sigma models and elliptic cohomology originally due to E. Witten [Wit87] and G. Segal [Seg88]. The specifics of our constructions are informed by the work of S. Stolz and P. Teichner on 2|1-dimensional super Euclidean field theories [ST11] and K. Costello’s construction of the Witten genus using methods of perturbative quantization [Cos10, Cos11]. Indeed, our investigation began as an attempt to develop a bridge between these two languages that relate particular 2-dimensional quantum field theories to elliptic cohomology. As a warm-up case, we will prove analogous results for 1|1-dimensional super Euclidean field theories, (twisted, Real) K-theory with complex coefficients, and the ˆA-class, showing that perturbative quantization gives rise to the familiar local index from the Atiyah-Singer theorem.

There are two main geometric constructions in the paper we wish to highlight. The first is the stack of classical vacua for the 2|1-dimensional sigma model with target a Riemannian manifold X, denoted Φ2|1(X). Roughly, these are the maps from 2|1-dimensional tori to X for which the classical action of the sigma model is zero, and (very roughly) functions on Φ2|1(X) give cocycles for TMF(X) ⊗ C, where TMF denotes the cohomology theory of topological modular forms constructed by P. Goerss, M. Hopkins and H. Miller. The second main construction is a family of kinetic operators parametrized by Φ2|1(X), denoted Δ2|1_X, that encode the linearized classical action of the sigma model on the normal bundle to the

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inclusion of classical vacua into the space of all maps of 2|1-dimensional tori to $X$. Roughly, the fiberwise determinant of the family $\Delta_X^{2|1}$ gives a function on $\Phi^{2|1}_0(X)$ that is a cocycle representative of the Witten class of $X$.

As one might expect, there are a few subtleties to confront in order to sharpen these amorphous ideas into theorems. In truth, there appear to be several ways to pass from functions on classical vacua to classes in $\text{TMF}(X) \otimes \mathbb{C}$, or from kinetic operators to the Witten class of $X$. Before charging ahead we wish to point out both the primary obstacles and the choices we’ve made to resolve them. In the case of classical vacua, some extra bells and whistles are required to enforce holomorphic dependence on the moduli stack of tori, and hence make contact with modular forms. Our choice in this case reflects the hope that supersymmetric field theories in the style of Stolz and Teichner give rise to integral classes in $\text{TMF}$, and we prove that a super geometric property of a function on $\Phi^{2|1}_0(X)$ that we call supersymmetric implies this holomorphicity. We use the same idea in the 1|1-dimensional case to construct cocycles for $K(X) \otimes \mathbb{C}$; in this dimension a function being supersymmetric implies it is invariant under the dilation action on the moduli of super circles, which is a version of supersymmetric cancellation that arises in the super traces of Dirac operators.

As for the construction of the Witten genus, the main choice involved is a regularization procedure that computes the determinant of the operator $\Delta^{2|1}_X$. We focus on two closely related options: the $\zeta$-determinant of $\Delta^{2|1}_X$ and the Quillen determinant line of $\Delta^{2|1}_X$. The former is marginally more straightforward to set up, whereas the latter makes contact with the Bismut-Freed theory of anomalies in connection with rational string structures on $X$ and the modularity of the Witten genus. Both constructions of the determinant also work in the 1|1-dimensional case, and reduce to (fairly) well-known constructions of the $\hat{A}$-genus.

1.1. Statement of results. Let $T^{2|1}_\Lambda = \mathbb{R}^{2|1}/\Lambda$ be the super torus\footnote{As usual in super geometry, one should consider families of tori. For simplicity we will suppress this family parameter in the current discussion.} associated to a lattice $\Lambda \subset \mathbb{R}^{2|1}$. Let $X$ be a smooth manifold, and $\Phi^{2|1}(X)$ denote the space (or better, stack) of fields consisting of pairs $(\Lambda, \phi)$ for $\phi: T^{2|1}_\Lambda \to X$ a smooth map. Our primary object of study is a substack, denoted $\Phi^{2|1}_{\text{susy}}(X) \subset \Phi^{2|1}(X)$, of classical vacua. Motivated by a discussion of the classical action of the sigma model (see subsection 3.1) we define $\Phi^{2|1}_{\text{susy}}(X)$ as maps whose source tori arise from lattices $\Lambda \subset \mathbb{R}^2 \subset \mathbb{R}^{2|1}$ and where the map $\phi$ factors through the projection from $T^{2|1}_\Lambda$ to $\mathbb{R}^{0|1}$ induced by the projection $\mathbb{R}^{2|1} \to \mathbb{R}^{0|1}$.

The stack $\Phi^{2|1}_{\text{susy}}(pt)$ is essentially the component of the stack of spin elliptic curves with nonbounding spin structure. As such, there is a line bundle $\omega^{1/2}$ over $\Phi^{2|1}_{\text{susy}}(pt)$ with fiber at $T^{2|1}_\Lambda$ the holomorphic sections of the square root of the canonical line of the underlying reduced torus $|T^{2|1}_\Lambda|$, where the spin structure specifies the square root. Note $\omega^{k/2} := (\omega^{1/2})\otimes^k$ tensor-squares to the line whose holomorphic sections are weight $k$ weak modular forms. Naturality of classical vacua permits us to pull back $\omega^{1/2}$ to a line bundle also denoted $\omega^{1/2}$ over $\Phi^{2|1}_{\text{susy}}(X)$.

Cocycles in our model for $\text{TMF}(X) \otimes \mathbb{C}$ are defined as supersymmetric sections of tensor powers of $\omega^{1/2}$ over $\Phi^{2|1}_{\text{susy}}(X)$, denoted $\Gamma_{\text{susy}}(\Phi^{2|1}_{\text{susy}}(X); \omega^{k/2})$. Being supersymmetric means that a section extends to one over the moduli of tori corresponding to all super lattices $\Lambda \subset \mathbb{R}^{2|1}$ where again $T^{2|1}_\Lambda$ is equipped with a map $\phi$ to $X$ that factors through $\mathbb{R}^{0|1}$. We denote the stack of such pairs $(\Lambda, \phi)$ by $\Phi^{2|1}(X)$, since such maps have nilpotent classical action. We will show that the assignment $X \mapsto \Gamma_{\text{susy}}(\Phi^{2|1}_{\text{susy}}(X); \omega^{k/2})$ defines a sheaf on the big site of manifolds manifold with values in graded algebras.

Theorem 1.1. There is a natural isomorphism of graded algebras $$\text{TMF}^*(X) \otimes \mathbb{C} \cong \Gamma_{\text{susy}}(\Phi^{2|1}_{\text{susy}}(X); \omega^{*}/2)/\text{concordance}.$$
Furthermore, sheaf-level data endows the right hand side with Mayer-Vietoris sequences such that the above is an isomorphism of multiplicative cohomology theories.

Remark 1.2. The cohomology theory $\text{TMF} \otimes \mathbb{C}$ is an ordinary cohomology theory with values in the graded ring of weak modular forms, denoted $\text{MF}^\bullet$. To justify our seemingly more complicated description we give three reasons why we believe the sheaves $\Gamma_{\text{susy}}(\Phi_0^{2|1}(-); \omega^{*'/2})$ are related to $\text{TMF}$.

1. Methods from perturbative quantization construct wrong-way maps for the cohomology theory defined by these sheaves, which we identify with the wrong-way maps coming from the string orientation of $\text{TMF}$ tensored with $\mathbb{C}$.

2. As we shall explain in subsection 1.4, these sheaves receive a map from $2$-dimensional Euclidean field theories, which Stolz and Teichner have identified as a candidate geometric model for (integral) $\text{TMF}$. We conjecture that this map gives a geometric description of the generalized Chern character for $\text{TMF}$.

3. One can use our geometric model to build equivariant theories by considering a sheaf-level data endows the right hand side with Mayer-Vietoris sequences.

A primary goal of this paper is to explain (1) in detail by constructing the Witten class as a *supersymmetric function* on $\Phi_0^{2|1}(X)$, i.e., an element

$$\text{Wit}(X) \in C^\infty_{\text{susy}}(\Phi_0^{2|1}(X)) := \Gamma_{\text{susy}}(\Phi_0^{2|1}(X); \omega^0),$$

that represents the Witten class under the isomorphism in Theorem 1.1.

First we review some facts about the Witten genus. It is a Hirzebruch genus valued in $\mathbb{Q}(q)$ determined by the characteristic series

$$(1) \quad (e^{z^2/2} - e^{-z^2/2}) \prod_{n \geq 1} \frac{1 - q^n e^{z^2/2} (1 - q^n e^{-z^2/2})}{(1 - q^n)^2} = \exp \left( \sum_{k \geq 1} \frac{E_{2k}(q)}{2k(2\pi i)^{2k}} z^{2k} \right),$$

where $E_{2k}$ denotes the (holomorphic) $2k$th Eisenstein series. The equality above was proved by D. Zagier [Zag86], and is crucial to understanding the modular properties of the Witten genus conjectured by Witten [Wit87] in his original construction. The $2k$th Eisenstein series is a modular form of weight $2k$ for $k \geq 2$; we emphasize that $E_2$ is not modular. There is a closely related genus that uses the modular (but not holomorphic) 2nd Eisenstein series, $E_2^0(q, \bar{q})$, whose characteristic series is

$$(2) \quad \exp \left( \frac{E_2^0(q, \bar{q})}{2(2\pi i)^2} z^2 + \sum_{k \geq 2} \frac{E_{2k}(q)}{2k(2\pi i)^{2k}} z^{2k} \right).$$

We call the associated genus the *nonholomorphic Witten genus*. For our purposes, we view the genus associated to $\text{mf}$ as taking values in the graded ring of weak Maass forms, denoted $\text{MF}^\bullet$, which consists of smooth functions on based lattices that transform as modular forms, but need not be nonholomorphic.

**Definition 1.3.** A rational string structure for a Riemannian manifold $X$ is a 3-form $H \in \Omega^3(X)$ such that $dH = p_1(X)$ is the first Pontryagin form associated to the Levi-Civita connection. We call the pair $(X, H)$ a rational string manifold.

For rational string manifolds, the characteristic series

$$(3) \quad \sum_{k \geq 2} \frac{E_{2k}(q)}{2k(2\pi i)^{2k}} z^{2k}$$

defines a Hirzebruch genus with values in the graded ring of weak modular forms, denoted $\text{MF}^\bullet$. The characteristic series $\text{mf}$, $\text{mf}$ and $\text{mf}$ define classes in cohomology, called
the holomorphic Witten class, the modular Witten class and the Witten class, respectively denoted [Wit(X)] ∈ H^*(X)[[q]], [Wit^+(X)] ∈ H^*(X; C^∞(L)^{SL_2(Z)}), and [Wit_H(X)] ∈ ⊕_{i=0}^{2} H^*(X; MF^i), respectively. Although the Witten class does not depend on a choice of rational string structure H (only its existence) our construction will depend on such a choice, so we include it in the notation. The pairing between the fundamental class of X and these Witten classes gives various flavors of the Witten genus of X.

Returning to our discussion of the 2|1-dimensional sigma model, for any Riemannian manifold X we shall define a vector bundle \mathcal{N} \Phi^{2|1}_0(X) over \Phi^{2|1}_0(X) that is the normal bundle to the inclusion \Phi^{2|1}_0(X) \subset \Phi^{2|1}(X). We will show that for a section \nu of the normal bundle, the linearization of the classical action takes the form

\[ S_{\text{lin}}(\nu) = \langle \nu, \Delta^{2|1}_X \nu \rangle, \]

which defines the kinetic operator \Delta^{2|1}_X. Given this description, integration of the exponentiated action over the fibers of the normal bundle amounts to taking a suitably normalized determinant (respectively, Pfaffian) of the operator \Delta^{2|1}_X restricted to even (respectively, odd) sections,

\[ \int_{\mathcal{N} \Phi^{2|1}_0(X)} \exp(-S_{\text{lin}}(\nu))d\nu = \frac{\text{pf}((\Delta^{2|1}_X)|_{\text{odd}})}{\det((\Delta^{2|1}_X)|_{\text{even}})^{1/2}} \quad \text{(formally)}. \]

We will refer to this ratio as the super determinant of \Delta^{2|1}_X, and its definition will be made precise in various contexts below.

Our first construction of the Witten class is via the \zeta-super determinant of the family of operators \Delta^{2|1}_X,

\[ \text{sdet}_\zeta(\Delta^{2|1}_X) := \frac{\text{pf}_\zeta((\Delta^{2|1}_X)|_{\text{odd}})}{\det_\zeta((\Delta^{2|1}_X)|_{\text{even}})^{1/2}}. \]

It turns out to be convenient to take this super determinant relative to the super determinant of an operator denoted \Delta^{2|1}_n, whose definition uses the trivial bundle of dimension \dim(X) in place of the tangent bundle of X. Along the way to proving Theorem 1.1 we will show that \( C^\infty(\Phi^{2|1}_0(X)) \subset \Omega^*_\text{cl}(X; \text{MF}^+), \) so a smooth function on \( \Phi^{2|1}_0(X) \) determines a cocycle in cohomology with coefficients in weak Maass forms.

**Theorem 1.4.** The relative \zeta-super determinant of the family of operators \( \Delta^{2|1}_X \) gives a function

\[ \frac{\text{sdet}_\zeta(\Delta^{2|1}_X)}{\text{sdet}_\zeta(\Delta^{2|1}_n)} =: \text{Wit}^+(X) \in C^\infty(\Phi^{2|1}_0(X)) \]

that represents the modular Witten class of X as a cocycle for cohomology with coefficients in \text{MF}^+. A choice of rational string structure on X determines a unique concordance \( W \in C^\infty(\Phi^{2|1}_0(X \times \mathbb{R})) \)

\[ i_0^*W = \text{Wit}^+(X), \quad \text{and} \quad i_1^*W =: \text{Wit}_H(X) \in C^\infty_\text{susy}(\Phi^{2|1}_0(X)) \subset C^\infty(\Phi^{2|1}_0(X)) \]

from \( \text{Wit}^+(X) \) to a supersymmetric function representing the (modular and holomorphic) Witten class, where \( i_0, i_1 : X \hookrightarrow X \times \mathbb{R} \) are the inclusions at 0 and 1, respectively.

**Remark 1.5.** The above \zeta-determinants are a bit subtle, owing to the eigenvalues of \( \Delta^{2|1}_X \) not being real. J. Quine, S. Heydari and R. Song have developed the tools we require and computed several relevant examples [QHS93].

Loosely speaking, the failure of \( \text{Wit}^+(X) \) to be holomorphic in the above construction is an example of an anomaly in the language of physics. To make contact with the mathematical theory of anomalies developed by Bismut and Freed [BF86, Fre87], we require a somewhat more sophisticated perspective on the determinant of \( \Delta^{2|1}_X \). In the footsteps of Quillen [Qui85], determinants of families of operators are not functions, but rather sections
of metrized line bundles. As we shall explain, a precise formulation of the anomaly associated to the 2|1-dimensional sigma model arises from the geometric nontriviality of the super determinant line bundle of $\Delta^{2|1}_X$.

Our situation is slightly different than the one considered by Quillen and Bismut-Freed, owing to the fact that $\Delta^{2|1}_X$ is a super family of operators. Defining a Quillen determinant line for super families of Fredholm operators requires a modicum of care: the usual method employing a cover with components $U^{(X)}$ defined as not having $\lambda$ in the spectrum of the family of operators does not make sense in super families. However, provided the operators are sufficiently nice (e.g., that certain $\zeta$-determinants exist) there is still a metrized Quillen determinant line that we shall describe presently.

Suppose we are given a family of operators, denoted $\mathcal{D}$, over a supermanifold $M$ equipped with a map to its reduced manifold, $p: M \to |M|$. We may restrict the family $\mathcal{D}$ to the reduced manifold $|M| \subset M$, obtaining a family of operators denoted $|\mathcal{D}|$. Supposing the family $|\mathcal{D}|$ is sufficiently nice, Quillen’s construction gives a metrized line bundle $\text{Det}(|\mathcal{D}|)$ over $|M|$ with determinant section $\det(|\mathcal{D}|)$. We may then pull this data back along $p$, thereby obtaining a metrized line bundle with a preferred section over $M$, i.e., we define

$$\text{Det}(\mathcal{D}) := p^*\text{Det}(|\mathcal{D}|), \quad \det(\mathcal{D}) := p^*\det(|\mathcal{D}|).$$

Since all line bundles over $M$ are isomorphic to a line bundle pulled back along $p$, natural-ity of the determinant line under restriction to its reduced submanifold dictates that any line bundle deserving the title $\text{Det}(\mathcal{D})$ on $M$ be isomorphic (topologically) to $p^*\text{Det}(|\mathcal{D}|)$. However, the metric ought to depend on how the operator $\mathcal{D}$ varies as we move around in the odd directions on $M$; the standard formula makes sense in super families,

$$\|\det(\mathcal{D})\|^2 = \det_\zeta(\mathcal{D}^\vee\mathcal{D}^*),$$

where the right hand side is the $\zeta$-determinant. It is a simple task to construct a metric with the property: we just rescale the pulled back metric on $\text{Det}(\mathcal{D})$ by the appropriate function on $M$ so that (5) becomes the definition of the metric on $\text{Det}(\mathcal{D})$.

If $\text{Det}(|\mathcal{D}|)$ has a geometric square root, an analogous discussion applies to a construction of $\text{Det}(\mathcal{D})^{1/2}$ over the super family, where we construct a metric with

$$\|\det(\mathcal{D})^{1/2}\|^2 = \det_\zeta(\mathcal{D}^\vee\mathcal{D}^*)^{1/2},$$

and similarly when $\mathcal{D}$ acts on odd sections we may define a Pfaffian line, denoted $\text{Pf}(\mathcal{D})$, with section $\text{pf}(\mathcal{D})$.

Returning to our example of interest, the families $|\Delta^{2|1}_X|$ and $|\Delta^{2|1}_m|$ give rise to a relative super determinant line (i.e., suitable tensor product of Pfaffian and determinant lines) that is trivialized by its super determinant section. The notion of a supersymmetric function on $\Phi^{2|1}_0(X)$ singles out a preferred type trivialization of $\text{sDet}(\Delta^{2|1}_X)$. In the following, $\widetilde{\Phi}^{2|1}_0(X)$ is an SL$_2(\mathbb{Z})$ covering of $\Phi^{2|1}_0(X)$ that is defined in analogy with the space of based lattices as an SL$_2(\mathbb{Z})$ covering of the moduli stack of tori.

**Definition 1.6.** Let $\mathcal{L}$ be a metrized line bundle on $\Phi^{2|1}_0(X)$ (respectively, $\widetilde{\Phi}^{2|1}_0(X)$), and let $\sigma$ be preferred nonvanishing section of $\mathcal{L}$. A supersymmetric trivialization of $(\mathcal{L}, \sigma)$ is a unit norm section $f \cdot \sigma$ for $f$ a supersymmetric function on $\Phi^{2|1}_0(X)$ (respectively, $\widetilde{\Phi}^{2|1}_0(X)$).

**Lemma 1.7.** If a supersymmetric trivialization exists, then it is unique up to a global phase.

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To avoid potential confusion, we reserve the Pfaffian terminology and notation, e.g., $\text{Pf}(\mathcal{D})$, for line bundles associated to operators acting on purely odd (i.e., fermionic) sections.
Theorem 1.8. The holomorphic Witten class \([\text{Wit}(X)]\) has a representative, \(\text{Wit}(X) \in C^\infty_{\text{susy}}(\Phi_0^{2|1}(X))\), and the section

\[
1_W := \frac{\text{sdet}(\Delta_X^{2|1}) \otimes \text{sdet}(\Delta_n^{2|1})^{-1}}{\text{Wit}(X)} \in \Gamma(\Phi_0^{2|1}(X); s\text{Det}(\Delta_X^{2|1}) \otimes s\text{Det}(\Delta_n^{2|1})^{-1})
\]

defines a supersymmetric trivialization of the relative super determinant line.

The trivialization determined by \(1_W\) fails to descend from \(\Phi_0^{2|1}(X)\) to \(\Phi_0^{2|1}(X)\) precisely because the nonmodular Witten class is not a function on \(\Phi_0^{2|1}(X)\), but rather a section of a line bundle we call the anomaly line bundle, denoted \(A\). We therefore have

\[
1_W \in \Gamma(\Phi_0^{2|1}(X); s\text{Det}(\Delta_X^{2|1}) \otimes s\text{Det}(\Delta_n^{2|1})^{-1} \otimes A^{-1}).
\]

Remark 1.9. The anomaly line bundle is a 0-categorical piece of the Chern-Simons obstruction 2-gerbe for a string structure on \(X\); the cocycle for the anomaly line is determined by the exponentiated first Pontryagin form, which is closely related to the differential cocycle associated to the Chern-Simons 2-gerbe.

Definition 1.10. A cancellation of the perturbative anomaly consists of two pieces of data: (1) a line bundle \(\mathcal{L}\) over \(\Phi_0^{2|1}(X \times \mathbb{R})\), and (2) a nonvanishing section \(\sigma\) of \(\mathcal{L}\). The pair \((\mathcal{L}, \sigma)\) is required to satisfy \(i_1^* \mathcal{L} \cong A\), \(i_0^* \mathcal{L} \cong \mathbb{C}\), and \(i_0^* \sigma = 1 \in \mathbb{C}\). In this case, we define \(1_A := i_1^* \sigma\).

Theorem 1.11. The perturbative anomaly can be cancelled if and only if \(X\) admits a rational string structure. A choice of rational string structure uniquely determines a cancellation of the perturbative anomaly for which there is an equality of trivializing sections,

\[
\frac{\text{sdet}(\Delta_X^{2|1}) \otimes \text{sdet}(\Delta_n^{2|1})^{-1}}{\text{Wit}_H(X)} = 1_W \otimes 1_A(H) \in \Gamma(\Phi_0^{2|1}(X); s\text{Det}(\Delta_X^{2|1}) \otimes s\text{Det}(\Delta_n^{2|1})^{-1})
\]

where \(1_A(H)\) denotes the section of \(A\) associated to the rational string structure \(H\), we have identified \(A^{-1} \otimes A \cong \mathbb{C}\), and \(\text{Wit}_H(X)\) is a cocycle representative of the (holomorphic and modular) Witten class of \(X\).

In the next two subsections we give brief accounts of the philosophy behind our constructions of \(\text{TMF} \otimes \mathbb{C}\) and the Witten class before describing in more detail the relation to previous work of Stolz-Teichner and Costello.

1.2. Classical vacua and Mayer-Vietoris sequences. G. Segal proposed a geometric connection between 2-dimensional field theories and elliptic cohomology [Seg88]. The fundamental objects of study are field theories over a manifold \(X\), i.e., functors with source a category whose morphisms are bordisms equipped with a smooth map to \(X\), and target a suitable category of vector spaces (to be described in greater detail in subsection 1.4). It was observed by Stolz and Teichner [ST04] that for such a model to have Mayer-Vietoris sequences the field theories are required to be fully local in the higher-categorical sense. The essential idea is that a map of a circle—viewed as an object of a 2-dimensional bordism category—into a smooth manifold \(X = U \cup V\) will generally not map to one of \(U\) or \(V\). Mayer-Vietoris sequences would require field theories over \(U\) and \(V\) to determine a field theory over \(X\), which effectively forces one to associate data to a chopped up circle, i.e., to intervals. In short, we need to require field theories over \(X\) to satisfy a higher sheaf condition; in a modern language, 2-dimensional field theories need to form a 2-stack on \(X\).

However, there is another road to locality: we can restrict maps from bordisms to \(X\) to be constant, so that such bordisms over \(X = U \cup V\) automatically lie in either \(U\) or \(V\). When the bordisms are closed and connected, a modification of this idea incorporating the relevant super geometry leads to our stack of classical vacua, \(\Phi_0^{2|1}(X)\). In this way, one can view the appearance of \(\text{TMF} \otimes \mathbb{C}\) in our construction as extracting a sheaf (of sets) on the site of manifolds from the presheaf consisting of functions on the space of fields in the 2|1-dimensional sigma model. In more detail, the line bundles \(\omega^{8/2}\) over \(\Phi^{2|1}(X)\) define
presheaves $X \mapsto \Gamma(\Phi^{2|1}(X); \omega^{*}/2)$ that fail to be sheaves, unless $X$ is zero dimensional. However, the functor $X \mapsto \Gamma(\Phi^{2|1}(X); \omega^{*}/2)$ satisfies the sheaf condition, and Theorem 1.1 identifies the appropriate sections (namely, the supersymmetric ones) that give rise to the cohomology theory $\text{TMF} \otimes \mathbb{C}$.

1.3. Perturbative quantization and the Witten class. To explain the geometry underlying our construction of the Witten class we draw an analogy to the following finite-dimensional situation: given a volume form on a manifold $M$ and an inclusion $i : N \hookrightarrow M$, restricting the volume form to $N$ requires an orientation of its normal bundle and a rapidly decaying function on the fibers. At least philosophically, quantization of the 2|1-dimensional sigma model with target $X$ amounts to constructing a volume form on the fibers of the map $\Phi(X) \rightarrow \Phi(\text{pt})$. Perturbative quantization seeks to restrict this volume form to the substack of vacua, $\Phi^{2|1}_{0}(X)$, by using the linearized exponentiated classical action as a rapidly decreasing function in the normal direction. The integral turns out to be purely quadratic, so can be computed via regularized super determinants. We explain how the result defines a (relative) volume form on $\Phi^{2|1}_{0}(X)$ with total volume the Witten genus of $X$ in subsection 5.7.

1.4. Relation to the Stolz-Teichner program. In this subsection we explain how Theorem 1.1 constructs a generalized Chern character for 2|1-dimensional Euclidean field theories over a manifold $X$ and Theorem 1.11 constructs an index map taking values in $\text{TMF}(\text{pt}) \otimes \mathbb{C}$ when $(X, H)$ a rational string manifold. First we review twisted 2|1-dimensional super Euclidean field theories as defined by Stolz and Teichner; see [ST11] for details.

Stolz and Teichner have constructed a super Euclidean bordism category over a smooth manifold $X$, denoted $2|1-\text{EBord}(X)$. The objects consist of closed 1|1-dimensional supermanifolds with a map to $X$, and the morphisms are compact 2|1-dimensional super manifolds with super Euclidean structure and a map to $X$. To account for the symmetries super Euclidean manifolds possess, this bordism category is defined as being internal to stacks on the site of supermanifolds, i.e., the collections of objects and morphisms each form stacks. As usual, disjoint union of bordisms gives this category a symmetric monoidal structure. They further define a symmetric monoidal functor $\mathcal{L}$, called a twist, from this bordism category to the category $\text{Alg}$; this target has a stack of objects consisting of topological algebras and algebra automorphisms and a stack of morphisms consisting of bimodules and bimodule automorphisms. They consider natural transformations $F$

$$\begin{array}{ccc}
2|1-\text{EBord}(X) & \downarrow F & \text{Alg}, \\
\mathcal{T}^{\otimes k} & \rightarrow &
\end{array}$$

denoting the category of such natural transformations by $2|1-\text{EFT}^{k}(X)$ for $k \in \mathbb{Z}$. They conjecture the existence of a certain higher-categorical refinement of this definition such that there is a natural ring isomorphism

$$\text{TMF}^{k}(X) \cong 2|1-\text{EFT}^{k}(X)/\text{concordance} \quad \text{(conjectural)}.$$ 

The (generalized) Chern character is a natural map $\text{TMF}^{*}(X) \rightarrow \text{TMF}^{*}(X) \otimes \mathbb{C}$, and in light of the above conjecture one might hope for a natural map from $2|1-\text{EFT}^{k}(X)$ to a geometric model for $\text{TMF}^{*}(X) \otimes \mathbb{C}$. Theorem 1.1 provides such a map, as we explain below.

Restriction of a degree $k$ twisted field theory to the subcategory of $2|1-\text{EBord}_{c}(X) \subset 2|1-\text{EBord}(X)$ of closed, connected tori equipped with a map to $X$ picks out a section of a line bundle $L_{T}$ over $2|1-\text{EBord}_{c}(X)$, where $T$ determines $L_{T}$. There is a forgetful map $u : 2|1-\text{EBord}_{c}(X) \rightarrow \Phi^{2|1}(X)$, and our line bundle $\omega^{1/2}$ is defined such that $u^{*}\omega^{1/2} \cong L_{T}$. Furthermore, Stolz and Teichner have shown that $u$ induces an isomorphism on sections

\[\text{TMF}^{k}(X) \cong 2|1-\text{EFT}^{k}(X)/\text{concordance} \quad \text{(conjectural)).}\]
(Theorem 1.15 in [ST11]) which we use to define a map denoted $\mathcal{CH}$:

$$2|1\text{-EFT}^k(X) \to \Gamma(2|1\text{-EBord}_c(X); \mathcal{L}^k_f) \xrightarrow{n^*} \Gamma(\Phi^{2|1}(X); \omega^{k/2}) \xrightarrow{\gamma} \Gamma_{\text{susy}}(\Phi^{2|1}_0(X); \omega^{k/2})$$

A section of $\omega^{k/2}$ over $\Phi^{2|1}_0(X)$ that extends to a section over $\Phi^{2|1}(X)$ trivially implies that the section is supersymmetric, whence the target of $\mathcal{CH}$ consists of supersymmetric sections.

Taking concordance classes is natural, so Theorem 1.1 shows the above composition gives a map

$$\mathcal{CH} : 2|1\text{-EFT}^k(X)/\text{concordance} \to \text{TMF}^k(X) \otimes \mathbb{C}.$$  

The map $[\mathcal{CH}]$ is independent of the choice of higher-categorical refinement under investigation by Stolz and Teichner, so gives a link between 2|1-Euclidean field theories and elliptic cohomology independent of these subtleties.

**Remark 1.12.** Stolz and Teichner have further suggested that 2|1-EFTs give a geometric refinement of TMF. In fact, we can refine the map (6) to

$$2|1\text{-EFT}^k(X) \to \bigoplus_{i+j=k} \Omega^i(X; \text{TMF}^j(pt) \otimes \mathbb{C}),$$

where the target consists of differential forms with values in $\text{TMF}^*(pt) \otimes \mathbb{C}$, the ring of weak modular forms. This provides one datum that would present 2|1-EFTs (or a quotient thereof) as a model for differential TMF in the sense of Hopkins and Singer [HS05].

Our construction of the Witten genus as a cocycle in $C^{\infty}_{\text{susy}}(\Phi^{2|1}_0(X))$ can be viewed as a piece of a conjectured Riemann-Roch or (local) index theorem for TMF. We recall that the usual local index theorem can be summarized by the commutative diagram

$$\begin{array}{ccc}
K^{k+n}(X) & \xrightarrow{\text{ch}} & K^{k+n}(X) \otimes \mathbb{C} \\
p_i^{\text{top}}, \pi_i^{\text{an}} & \downarrow & \pi_i(- \cup [\hat{A}(X)]), \int_X (- \wedge \hat{A}(X)) \\
K^k(pt) & \xrightarrow{\text{ch}} & K^k(pt) \otimes \mathbb{C}
\end{array}$$

where $\hat{A}(X)$ denotes the A-roof form associated to a choice of metric and Levi-Civita connection on $X$, and for each downward arrow there are two maps that are equal, namely the topological pushforward and the analytic pushforward. We recall that $K(X) \otimes \mathbb{C} \cong H_{\text{dR}}(X)$ where $H_{\text{dR}}$ denotes 2-periodic de Rham cohomology.

Stolz and Teichner have described an analogous picture for TMF. They conjecture that quantization of 2|1-EFTs, denoted $\pi_i^{\text{an}}$, gives an analytical index that is equal to the topological index, denoted $\pi_i^{\text{top}}$ defined by the Ando, Hopkins, Strickland and Rezk [AHS01], evaluating $\pi_i^{\text{an}}$ on concordance classes is conjectured to give a map equal to $\pi_i^{\text{top}}$ evaluated on cohomology classes. This situation is summarized by the left face of the cube in Figure 1.

As in the case for $K$-theory, the conjectured local index theorem involves the generalized Chern character. Theorem 1.11 can be interpreted as computing the expected index coming from $\pi_i^{\text{an}}$ as an integral that is local in $X$. This corresponds to a topological pushforward in $\text{TMF} \otimes \mathbb{C}$. Hence, our construction of the Witten class can be viewed as providing the right face of the cube in Figure 1 which is the local part of the conjectured index theorem.

1.5. **Relation to Costello’s construction of the Witten genus.** K. Costello constructs the Witten genus using a quantum field theory based on a (formal, derived) stack of maps from an elliptic curve to a complex manifold $X$ that are nearly constant in the language of Gelfand-Kazhdan formal geometry. Costello then interprets the Witten class as a projective volume form on this derived stack. Below, we consider maps from rigid super conformal tori to a smooth manifold $X$ that are nearly constant in the sense that their image is in a tubular neighborhood of the constant maps, and one can interpret our construction of the Witten class as determining a (relative) volume form. Morally, the way in which we geometrically encode the deformation theory of nearly constant maps in this paper
Figure 1. The above diagram summarizes a conjectured local index theorem for TMF using supersymmetric field theories. The solid arrows have been constructed, whereas the dotted arrows remain conjectural. The front face describes a local analytical index whereas the back face describes a local topological index so that commutativity of the cube would give a TMF-analog of the local Atiyah-Singer index theorem. The arrows from the front face to the back face arise from taking concordance classes.

is Koszul dual to Cosello’s: he constructs a sheaf of $L_\infty$ algebras over the odd tangent bundle, whereas here we construct an (infinite dimensional) vector bundle of deformations over the odd tangent bundle. One might hope for a precise relationship coming from the correspondence between differential graded Lie algebras and formal moduli problems as developed by Deligne, Drinfeld and Feigin and thereafter by Kontsevich-Soibelman, Lurie, Manetti and others.

In addition to these structural similarities, our construction shares many computational features with Costello’s. The functional integral in our construction can be identified with the determinant of the Hessian of the action functional of the supersymmetric sigma model, so that the Witten class is the 1-loop contribution to the partition function (e.g., see Etinghof [Eti02] Theorem 3.5). Costello shows that quantizing his theory while preserving certain symmetries only depends on a 1-loop calculation. Unfortunately, the precise form of Costello’s action functional is quite different from ours, though one might hope for a relationship in a large-volume limit of our theory.

An important difference between the approaches is that Costello requires the target manifold be Kähler, whereas our construction applies to all oriented smooth manifolds. There seems to be an underlying conceptual reason for this, coming from the geometry of 2-dimensional sigma models (for example, see [Wit07]): the 2|1-dimensional sigma model with target a Kähler manifold $X$ has extra symmetry, owing to the fact that the single supersymmetry breaks into two under the holomorphic and anti-holomorphic decomposition of $TX$. This allows one to perform a half-twist of the theory which produces a square zero odd symmetry, leading to constructions in derived geometry (in particular, AKSZ models) utilized by Costello. Physical arguments lead one to expect a special subalgebra of the

\footnote{There is an unfortunate collision of terminology: this notion of twist has nothing to do with the Stolz-Teichner twists described in the previous subsection.}
algebra of observables—called chiral differential operators by V. Gorbovnov, F. Malikov and V. Schechtman [GMS00]—that are conformally invariant. Indeed, Costello expects the factorization algebra produced by his quantization procedure to be equivalent to these chiral differential operators. In the case of a general target manifold we expect the partition function to be conformally invariant, but there need not be a nontrivial subalgebra of conformally invariant observables. Hence, the factorization algebra of observables ought to be more complicated when the target manifold is not Kähler.

1.6. Notation and terminology. Throughout, $X$ will denote an oriented, closed smooth manifold, and $\text{Mfld}$ will denote the category of smooth manifolds and smooth maps. For a smooth manifold $X$, $C^\infty(X) = C^\infty(X, \mathbb{C})$ will denote the algebra of complex-valued function on $X$.

We take a $k|\ell$-dimensional supermanifold to be a locally ringed space whose structure sheaf is locally isomorphic to $C^\infty(U) \otimes_{\mathbb{C}} \Lambda^\bullet(\mathbb{C}^\ell)$ as a super algebra over $\mathbb{C}$ for $U \subset \mathbb{R}^k$ an open submanifold. These are sometimes called cs-manifolds (e.g., [DM99]). We denote the category of supermanifolds and maps of supermanifolds by $\text{SMfld}$. For any supermanifold $N$, there is a reduced manifold we denote by $|N|$ and a canonical map $|N| \to N$ induced by the map of superalgebras $C^\infty(N) \to C^\infty(|N|)$ where $I$ denotes the ideal of nilpotent elements in the structure sheaf of $N$. We will use notation like $z, \bar{z}$ or $f, \bar{f}$ for elements of $C^\infty(N)$ that are complex conjugates in their image under the quotient $C^\infty(N) \to C^\infty(|N|)$. By M. Batchelor’s Theorem [Bat79], any supermanifold $N$ is isomorphic to $|N|, \Gamma(\Lambda^\bullet E^*)$ for $E \to |N|$ a complex vector bundle over a smooth manifold $|N|$. We denote such a supermanifold by $\pi E$. When doing geometry on supermanifolds we will use the Deligne-Morgan sign conventions [DEF+99]; in particular, differential forms on supermanifolds have a bigrading, and signs in our computations must be added accordingly.

The super manifold $\mathbb{R}^{n|m}$ is the locally ringed space with structure sheaf $C^\infty(\mathbb{R}^n) \otimes_{\mathbb{C}} \Lambda^\bullet(\mathbb{C}^m)$.

A vector bundle over a supermanifold is a finitely generated projective module over the structure sheaf. There is an important subtlety illustrated by the following example: elements of the free rank one module over $C^\infty(S)$ differ from sections of the projection $S \times \mathbb{C} \to S$. Roughly, this comes from the fact that $\mathbb{C}$, viewed as a supermanifold, behaves like $\mathbb{R}^2$ with its sheaf of complex-valued functions. However, there is a map between these two reasonable notions of section of a line bundle: given $S \to \mathbb{C}$, we can pull back the function $z \in C^\infty(\mathbb{C})$ to one on $S$, and identify this function with a section of the trivial line on $S$. Similarly, isomorphisms of the trivial line bundle over $S$ are elements of $C^\infty(S)^\times$, and these differ from maps $S \to \mathbb{C}^\times$. Again there is a map gotten by pulling back $z \in C^\infty(\mathbb{C}^\times)$, but it does not induce a bijection. In fact, the assignment $S \mapsto C^\infty(S)^\times$ is not even a representable supermanifold! When defining line bundles we will use the above maps without comment.

Generalized manifolds and generalized supermanifolds are functors $\text{Mfld}^{op} \to \text{Set}$ and $\text{SMfld}^{op} \to \text{Set}$ respectively, i.e., presheaves on manifolds and supermanifolds, respectively. We will use the notation $\text{Mfld}(M, N)$ to denote the generalized manifold $S \mapsto \text{Mfld}(S \times M, N)$, and similarly for supermanifolds. Generalized objects are representable when they can be expressed as $S \mapsto \text{Mfld}(S, M)$ for a manifold $M$, and will be denoted by $\underline{M}$ when we wish to emphasize their standing as a generalized manifold; we use similar terminology and notation for supermanifolds. For a (generalized) supermanifold $M$, we will use the notation $\mathcal{M}(S)$ to denote the set of maps $\underline{S} \mapsto \mathcal{M}$.

We will make frequent use of the isomorphism of (representable) supermanifolds $\pi TX \cong \text{SMfld}(\mathbb{R}^{0|1}, X)$. This allows us to identify $C^\infty(\text{SMfld}(\mathbb{R}^{0|1}, X)) \cong C^\infty(\pi TX) \cong \Omega^\bullet(X)$, where the target denotes differential forms on $X$. The evaluation map $\mathbb{R}^{0|1} \times \text{SMfld}(\mathbb{R}^{0|1}, X) \to \text{SMfld}(\mathbb{R}^{0|1}, X)$ coming from the action of $\mathbb{R}^{0|1}$ on itself by translation has as infinitesimal generator the de Rham $d$, viewed as an odd vector field on $\text{SMfld}(\mathbb{R}^{0|1}, X)$, i.e., an odd derivation on $C^\infty(\text{SMfld}(\mathbb{R}^{0|1}, X)) \cong \Omega^\bullet(X)$.

We view smooth stacks as objects in the bicategory associated to the double category of (generalized) super Lie groupoids and bibundles, and denote this bicategory by $\text{SmSt}$. In
particular, this allows us to identify super Lie groupoids with the stacks they present. An introduction to this perspective can be found in Section 2 of C. Blohmann’s article [Blo08].

In brief, a stack (up to isomorphism) is a Morita equivalence class of a Lie groupoid. A surjective map of stacks is a morphism of stacks that on $S$-points induces an essentially surjective, full morphism of groupoids. We will often abuse the terminology, identifying a groupoid with the stack it presents.

Following the standard convention in geometry, our symmetry groups will always act on the left unless otherwise noted. An important consequence is that for a mapping space $M = SMfd(Y,X)$ a diffeomorphism $f: Y \to Y$ acts on $x \in M$ by $x \mapsto x \circ f^{-1}$; a diffeomorphism $g: X \to X$ will act by $x \mapsto g \circ x$.

A quotient stack is a stack that admits a presentation by a quotient groupoid, $M//G$, for $G$ acting on $M$. Equivariant vector bundles on the $G$-supermanifold $M$ form a category equivalent to the category of vector bundles on the stack presented by $M//G$. In particular, a representation $\rho: G \to \text{End}(V)$ defines a vector bundle $V_\rho$ on the stack presented by $M//G$. When $V = \mathbb{C}$, sections of $V_\rho$ and $\pi V_\rho$ are

$$\Gamma(M//G, V_\rho) \cong \{ f \in C^\infty(M) \mid \mu^*(f) = p_1^*(f) \cdot p_2^*(\rho) \in C^\infty(M \times G) \}$$

where $p_1 : G \times M \to G$, $p_2 : G \times M \to M$ are the projection maps, and $\mu : G \times M \to M$ is the action map. In particular, for the trivial representation sections are precisely the $G$-invariant functions on $M$.

Any supermanifold $\mathfrak{M}$ with a left action of a super group $G$ defines a model geometry. Let an $(\mathfrak{M}, G)$-supermanifold consist of open submanifolds of $\mathfrak{M}$ glued along (restrictions of) the action of $G$ on $\mathfrak{M}$. Isometries of an $(\mathfrak{M}, G)$-supermanifold consist of diffeomorphisms that restrict locally to an action of $G$ on $\mathfrak{M}$. For an $S$-family of model supermanifolds $F \to S$, denote by $\text{Iso}_S(F)$ the group of isometries over $S$ associated to the model geometry $(\mathfrak{M}, G)$, i.e., isomorphisms $F \to F$ over $S$ that are isometries in the fibers. This defines a stack of $(\mathfrak{M}, G)$-supermanifolds on the site of supermanifolds. See [HST10] Section 6.3 for details.

The second Eisenstein series, denoted $E_2$, will refer to the holomorphic version,

$$E_2(\tau) = 2\zeta(2) + \sum_{n \in \mathbb{Z} - \{0\}} \sum_{m \in \mathbb{Z}} \frac{1}{(n + n\tau)^2}.$$ 

It is not a modular form; although invariant under translations $\tau \mapsto \tau + 1$ we have $E_2(-1/\tau) = \tau^2 E_2(\tau) - 2\pi i \tau$. Let $E_2^*$ denote the modular (but nonholomorphic) Eisenstein series,

$$E_2^*(\tau, \tau) := \lim_{\epsilon \to 0^+} \left( \sum_{(m,n) \in \mathbb{Z}^2} \frac{1}{(m\tau + n)^2|m\tau + n|^\epsilon} \right)$$

where $\mathbb{Z}^2$ denotes pairs $(m,n) \in \mathbb{Z}^2$, not both zero. We have the relationship between these Eisenstein series,

$$E_2^*(\tau, \tau) = E_2(\tau) - \frac{\pi}{\text{im}(\tau)}.$$ 

We will also have use for the Eisenstein series and Dedekind $\eta$-function as functions on the space of based lattices, $L \cong \mathfrak{h} \times \mathbb{C}^\times$. We define these is the usual way, by extending the above functions on $(\tau, 1) \in \mathfrak{h} \times \mathbb{C}^\times$ to the whole of $L$ via the appropriate dilation property determined by the weight of the function.

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2. Warm-up 1: classical vacua for the 1|1-sigma model and $K \otimes \mathbb{C}$

In this section we prove an analogous result to Theorem 1.1 for K-theory. We define a stack of classical vacua for the 1|1-dimensional sigma model, denoted $\Phi^{1|1}_0(X)$, that consists of super Euclidean circles equipped with a map to $X$ that factors through an odd line. We construct a sequence of line bundles, $\kappa^l$, over $\Phi^{1|1}_0(X)$ and consider the space of supersymmetric sections, $\Gamma_{\text{susy}}(\Phi^{1|1}_0(X); \kappa^l)$. These form a graded ring with multiplication coming from the tensor product of sections.

**Proposition 2.1.** There is a natural isomorphism of cohomology theories

$$K^*(X) \otimes \mathbb{C} \cong \Gamma_{\text{susy}}(\Phi^{1|1}_0(X); \kappa^*)/\text{concordance}$$

where the left hand side denotes the K-theory of $X$ tensored with $\mathbb{C}$.

Since K-theory with complex coefficients is simply $\mathbb{Z}/2$-graded de Rham cohomology, there are many (easy) super geometric ways of cooking up a graded ring with the above property. However, our construction and its relation to the Stolz-Teichner model for K-theory in terms of 1|1-dimensional field theories—particularly from the perspective afforded by F. Han’s thesis Han08—permits several enhancements.

1. Drawing on the theses of F. Han and F. Dumitrescu, the super holonomy of a vector bundle with connection over $X$ along super loops determines a function on $\Phi^{1|1}_0(X)$. Using Proposition 2.1, we identify the resulting class in $K^0(X) \otimes \mathbb{C}$ with the Chern character of the vector bundle in subsection 2.0.

2. The isomorphism in Proposition 2.1 can be refined to a $\mathbb{Z}/2$-equivariant one for a geometrically-defined action on $\Phi^{1|1}_0(X)$ gotten from time-reversal. In subsection 2.7 we identify concordance classes of $\mathbb{Z}/2$-invariant supersymmetric sections of $\kappa^*$ with $\text{KO}^*(X) \otimes \mathbb{C}$.

3. A gerbes with connection on $X$, denoted $\tau$, modifies the line bundle to one denoted $\kappa^{*+\tau}$. In subsection 2.8 we identify concordance classes of supersymmetric sections of $\kappa^{*+\tau}$ with $\tau$-twisted K-theory with complex coefficients.

4. In Section 3 we construct a family of operators over $\Phi^{1|1}_0(X)$ whose $\zeta$-determinant is the $A$-class of $X$ as a function on $\Phi^{1|1}_0(X)$.

5. For a finite group $G$ acting on a manifold $X$, the geometry of $\Phi^{1|1}_0(X//G)$ determines an equivariant refinement of the isomorphism in Proposition 2.1 and constructs the target of the delocalized Chern character of the $G$-equivariant K-theory of $X$ [BE11].

2.1. Motivation from the 1|1-sigma model. To motivate the geometric choices and terminology that follows we begin with a brief review of the 1|1-sigma model. A mathematical introduction can be found in D. Freed’s notes [Fre99], Chapter 2, though we have Wick-rotated the theory described by Freed; our conventions for this rotation, its affect on the Euclidean group, and its affect on the classical action functional are taken from P. Deligne and D. Freed’s article, “Classical Field Theory” in [DEF+99].

The generalized supermanifold of fields for the 1|1-dimensional sigma model has as $S$-points maps $\phi: S \times \mathbb{R}^{1|1} \to X$ from families of superpaths to a Riemannian manifold $X$. For our purposes we require these maps to satisfy a periodicity condition for a $\mathbb{Z}$-action on $S \times \mathbb{R}^{1|1}$ (to be described in detail in the next section) so that the result is a family of super circles mapping to $X$. We use the component field description of these maps, i.e., we identify $\phi$ with a pair $(x, \psi)$ for $x: S \times \mathbb{R} \to X$ a (super) family of (ordinary) paths and $\psi \in \Gamma(S \times \mathbb{R}; x^*\pi TX)$ a section of the pulled back odd tangent bundle. The classical action

\[S_{\text{class}}(\phi) = \int_{S \times \mathbb{R}} \Phi^{1|1}_0(\pi_{1|1}) \]
at this \( S \)-point is

\[
S(\phi) = S(x, \psi) := \int_0^r \langle \dot{x}, \dot{x} \rangle - i \langle \psi, \nabla_x \psi \rangle \, dt
\]

where \( dt \) is the usual volume form on \( \mathbb{R} \), and \( \langle -, - \rangle \) uses the Riemannian metric on \( X \) and the usual Hermitian pairing on complex-valued functions on the circle. In the above, \([0, r]\) is a fundamental domain for the \( S \)-family of super circles in question, where \( r \in \mathbb{R}_{>0}(S) \) measures the circumference of the family of circles over \( S \). The integral is fiberwise over \( S \), so returns a function on \( S \) for each \( S \)-point of the space of fields. By the usual functor of points yoga, \( S \) determines a smooth function on this generalized supermanifold of fields.

We will use the notation \( \int_{S^{1|1} \times S/S} \) to denote such fiberwise integrals below.

The above data of a space of fields and a classical action defines a classical field theory. From this we extract two geometrically important ideas.

1. A symmetry of the action \( S \) is an \( S \)-family of diffeomorphisms of \( \mathbb{R}^{1|1} \) that preserves the values \( S(\phi) \). Hence, \( S \) is automatically a function on a smooth stack of fields that encodes these symmetries.

2. A classical vacuum is an absolute minimum of the function \( S \) for any metric on \( X \). The first term in the action is minimized for \((S\text{-families of})\) constant paths \( x \), and the second term is minimized when the section \( \psi \) is constant, so the stack of classical vacua is the stack associated to the prestack consisting of \( S \)-families of Euclidean super circles that factorize as \( S \times \mathbb{R}^{1|1} \to S \times \mathbb{R}^{0|1} \to X \). We will give a precise definition of the stack \( \Phi_0^{1|1}(X) \) below that is motivated by these geometric considerations.

### 2.2. Super Euclidean circles.

Let \( \mathbb{R}^{1|1} \) denote the super group with multiplication

\[
(t, \theta) \cdot (t', \theta') = (t + t' + i \theta \theta', \theta + \theta'), \quad (t, \theta), (t', \theta') \in \mathbb{R}^{1|1}(S).
\]

The Lie algebra of left-invariant vector fields on this super Lie group is free on a single generator, \( D := \partial_t - i \theta \partial_\theta \) that satisfies \( D^2 = \frac{1}{2}[D, D] = -i \partial_\theta \). Define an action of \( \text{Spin}(1) \cong \mathbb{Z}/2 = \{ \pm 1 \} \) on \( \mathbb{R}^{1|1} \) by the reflection \( (t, \theta) \mapsto (t, \pm \theta) \), for \((t, \theta) \in \mathbb{R}^{1|1}(S)\). This defines the group \( \mathbb{R}^{1|1} \rtimes \mathbb{Z}/2 \). The left action of \( \mathbb{R}^{1|1} \rtimes \mathbb{Z}/2 \) on \( \mathbb{R}^{1|1} \) defines a super Euclidean model space in the sense of model geometries reviewed in subsection \textbf{1.6} where the isometries of \( \mathbb{R}^{1|1} \) comprise the supergroup \( \mathbb{R}^{1|1} \rtimes \mathbb{Z}/2 \).

\textbf{Remark 2.2.} Up to the factor of \( i \) defining the multiplication, this is the super group employed by Hohnhold, Stolz and Teichner in \textbf{HST10} in their construction of the K-theory spectrum via field theories. Although their group is isomorphic to ours as a super Lie group, the real structures are different. This minor change shows up in a few places, e.g., the Wick-rotated version of the standard Lagrangian,

\[
\int_{S \times \mathbb{R}^{1|1}} \langle D^2 \phi, D\phi \rangle d\theta dt = \int_{S \times \mathbb{R}^{1|1}/S} \left( \frac{-i}{dt} (x + \theta \psi), (\partial_t - i \theta \partial_\theta)(x + \theta \psi) \right) d\theta dt
\]

where the second equality takes a relative Berezinian integral over the fibers \( S \times \mathbb{R}^{1|1} \to S \times S^1 \). Another difference from the Hohnhold-Stolz-Teichner approach emerges in our discussion of conjugation actions in relation to real and unitary field theories in Section \textbf{2.7}.

Any lattice \( \mathbb{Z} \subset \mathbb{R}^{1|1} \) defines a pointed super Euclidean circle via the quotient, \( \mathbb{S}^{1|1} \cong \mathbb{R}^{1|1}/\mathbb{Z} \), where our convention is to quotient by the right action so that the resulting object has isometries coming from the left action of \( \mathbb{R}^{1|1} \). More precisely, we consider an \( S \)-family of generators of a \( \mathbb{Z} \)-action, \( R \in \mathbb{R}^{1|1}_>(S) \) where \( \mathbb{R}^{1|1}_> \) denotes the super manifold obtained by restricting the structure sheaf of \( \mathbb{R}^{1|1} \) on \( \mathbb{R} \) to the positive reals, \( \mathbb{R}^{1|1}_> \). Choosing coordinates on \( \mathbb{R}^{1|1}_> \), we have \( R = (r, \rho) \) for \( r \in \mathbb{R}_{>0}(S) \) and \( \rho \in \mathbb{R}^{0|1}(S) \). We have an action map

\[
\mu_R: S \times \mathbb{Z} \times \mathbb{R}^{1|1}_> \to S \times \mathbb{R}^{1|1}_>, \quad (s, n, t, \theta) \mapsto (s, n r + t + i n \rho \theta, n \rho + \theta), \quad n \in \mathbb{Z}, (s, t, \theta) \in S \times \mathbb{R}^{1|1}.
\]
The quotient is an $S$-family of pointed super Euclidean circles denoted $S \times_R \mathbb{R}^{1|1}$.

We will require an explicit description of the super Euclidean isometries with source $S \times_R \mathbb{R}^{1|1}$. Isometries are locally the left action of $\mathbb{R}^{1|1} \times \mathbb{Z}/2$ on $\mathbb{R}^{1|1}$, and lifting the action to the universal cover we observe that isometries are determined (possibly non-uniquely) by an $S$-point of $\mathbb{R}^{1|1} \times \mathbb{Z}/2$. To compute the map, choose elements $(u, \nu, \pm 1) \in (\mathbb{R}^{1|1} \times \mathbb{Z}/2)(S)$ and $(r, \rho) \in R^{1|1}(S)$. We obtain a super Euclidean isometry $S \times_R \mathbb{R}^{1|1} \to S \times_R' \mathbb{R}^{1|1}$ with

$$R = (r, \rho), \quad R' = (r \pm 2i\nu \rho, \pm \rho),$$

where $R'$ is deduced from the conjugation action of $(u, \nu) \in \mathbb{R}^{1|1}(S)$ on $(r, \rho) \in \mathbb{R}^{1|1}(S)$.

2.3. Stacks of fields and classical vacua.

Definition 2.3. The stack of fields, denoted $\Phi^{1|1}(X)$, is the stack associated to the prestack that has objects over $S$ consisting of pairs $(R, \phi)$ where $R \in \mathbb{R}_{>0}^{1|1}$ determines a family of super circles $S \times_R \mathbb{R}^{1|1}$ and $\phi: S \times_R \mathbb{R}^{1|1} \to X$ is a map. Morphisms between these objects over $S$ consist of commuting triangles

$$\begin{array}{ccc}
S \times_R \mathbb{R}^{1|1} & \xrightarrow{\sim} & S \times_R' \mathbb{R}^{1|1} \\
\phi & \circlearrowright & \phi' \\
\emptyset & \downarrow & \emptyset
\end{array}$$

where the horizontal arrow is an isomorphism of $S$-families of super Euclidean 11-manifolds.

Definition 2.4. The stack of classical vacua, denoted $\Phi^{1|1}_0(X)$, is the full substack of $\Phi^{1|1}(X)$ generated by pairs $(R, \phi)$ where $R \in \mathbb{R}_{>0}(S) \subset \mathbb{R}_{>0}^{1|1}(S)$ determines a family of super circles $S \times_R \mathbb{R}^{1|1}$ and $\phi: S \times_R \mathbb{R}^{1|1} \to X$ is a map that factors through $S \times \mathbb{R}^{0|1}$ along the map induced by the projection $\mathbb{R}^{1|1} \to \mathbb{R}^{0|1}$.

Our definition of supersymmetric sections of line bundles over $\Phi^{1|1}_0(X)$ requires the definition of a stack $\Phi^{1|1}_c(X)$ for which there is an inclusion $\Phi^{1|1}_0(X) \subset \Phi^{1|1}_c(X)$. Roughly, $\Phi^{1|1}_c(X)$ consists of super circles whose map to $X$ has nilpotent classical action. Concretely, the stack $\Phi^{1|1}_c(X)$ consists of (arbitrary) supercircles with maps to $X$ that factor through $\mathbb{R}^{0|1}$. To this end, we shall define a morphism $\text{proj}_R: S \times_R \mathbb{R}^{1|1} \to S \times_R \mathbb{R}^{0|1}$ for any $R = (r, \rho) \in \mathbb{R}_{>0}^{1|1}$. We observe that the projection $\mathbb{R}^{1|1} \to \mathbb{R}^{0|1}$ is not invariant under a lattice action $R = (r, \rho)$ for $\rho \neq 0$, so will not define such a map $\text{proj}_R$. However, we claim the map

$$\text{proj}_R: \mathbb{R}^{1|1} \times S \to \mathbb{R}^{0|1} \times S, \quad \text{proj}_R(t, \theta, s) = (\theta - {\frac{t}{r}} \rho, s)$$

is invariant under the right action of $(r, \rho) \cdot \mathbb{Z}$ on the family $\mathbb{R}^{1|1} \times S$. Denoting the action of the generator by $\mu_R$, we compute

$$\text{(proj} \circ \mu_R)(t, \theta, s) = \text{proj}(t + r + i\theta \rho, \theta + \rho, s) = \theta + \rho - {\frac{t + r + i\theta \rho}{r}} = \theta - {\frac{t}{r}} \rho.$$}

This shows the map $\text{proj}_R$ is invariant as claimed, so defines a map out of $S \times_R \mathbb{R}^{1|1}$, which in an abuse of notation we also call $\text{proj}_R: S \times_R \mathbb{R}^{1|1} \to S \times_R \mathbb{R}^{0|1}$.

Definition 2.5. The stack $\Phi^{1|1}_c(X)$ is the full substack of $\Phi^{1|1}(X)$ generated by pairs $(R, \phi)$ where $R \in \mathbb{R}_{>0}(S)$ determines a family of super circles $S \times_R \mathbb{R}^{1|1}$ and $\phi: S \times_R \mathbb{R}^{1|1} \to X$ is a map that factors through the map $\text{proj}_R$ defined above.

We observe that for $\rho = 0$, $\text{proj}_R$ is determined by the standard projection $\mathbb{R}^{1|1} \to \mathbb{R}^{0|1}$, which gives an inclusion of stacks $\Phi^{1|1}_0(X) \subset \Phi^{1|1}_c(X)$. By functoriality, this morphism lies over the inclusion $\Phi^{1|1}_0(\text{pt}) \subset \Phi^{1|1}_c(\text{pt})$. 

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**Remark 2.6.** In this remark we justify our restrictive choice of lattices in the definition of classical vacua. A map $S \times_R \mathbb{R}^{1|1} \to X$ is determined by a map $S \times \mathbb{R}^{1|1} \to X$ invariant under the $R \cdot Z$ action; we first write the morphism $S \times \mathbb{R}^{1|1} \to X$ as

$$
\phi = \hat{x}(t) + \theta \psi(t),
$$

for $t$ a coordinate on $\mathbb{R}$, $\hat{x}(t)$ a path in $X$ and $\psi(t)$ an odd tangent vector along the path. Unravelling invariance under $R \cdot Z$, we express the above as

$$
\phi = x(t) - \frac{\rho}{r} t \psi(t) + \theta \psi(t), \quad \hat{x}(t) = x(t) - \frac{\rho}{r} t \psi(t)
$$

(12)

where $x(t)$ corresponds to a smooth map $S \times S^1 \to X$ and $\psi(t)$ is an odd derivation with respect to $x(t)$. An $S$-point $(R, \phi)$ of $\Phi_0^{1|1}(X)$ must satisfy the factorization criteria, and such maps take the form

$$
\phi = x_0 - \frac{\rho}{r} t \psi_0 + \theta \psi_0,
$$

(13)

for $x_0$ a constant loop and $\psi_0$ an odd derivation with respect to $x_0$. As a consequence of the nontrivial $t$-dependence of $\phi$ when $\rho \neq 0$, the classical action of this map is *nilpotent* but generally non-zero. Our definition of $\Phi_0^{1|1}(X)$ was motivated to consist of action zero maps, which is why we have excluded families of super circles defined by $R = (r, \rho)$ with $\rho \neq 0$. Another justification from physics is that classical vacua should be static (i.e., time independent) solutions (e.g., see Freed [Fre99], pages 71-72) and the above shows that circles of super length $(R, \rho)$ with $\rho \neq 0$ vary nontrivially with time.

We note in passing that there an additional piece of this moduli space of action zero maps that has $\psi_0 = 0$ and $\rho \neq 0$, but it doesn’t seem to contain any topologically interesting information: concordance classes of functions on this piece of the moduli space of action zero maps is the trivial group.

A different reason for our restriction on the moduli of circles in defining $\Phi_0^{1|1}(X)$ comes from considering the value of 1|1-EFTs on super circles; we obtain a function on the moduli of super circles as the super trace of an operator defining the 1|1-dimensional field theory. Taylor expanding this function in the variables $(r, \rho)$, the even part comes from the super trace of an *even* operator, and the odd part comes from the super trace of an *odd* operator. But super traces of odd operators are necessarily zero. Hence, the nontrivial information about the original field theory is contained in the restriction to the moduli stack of circles with $\rho = 0$.

### 2.4. Groupoid presentations.

In the definition of $\Phi_1^{1|1}(X)$, the required factorization of the map $\phi$ allows us to identify it with a morphism $\phi_0: S \times \mathbb{R}^{0|1} \to X$, where $\phi = \phi_0 \circ \proj_R$. We use this to construct a groupoid presentation of $\Phi_1^{1|1}(X)$ whose objects over $S$ are $(R, \phi_0) \in (\mathbb{R}^{1|1}_R \times \text{SMfd}(\mathbb{R}^{0|1}, X))(S) \cong (\mathbb{R}^{1|1}_R \times \pi TX)(S)$. The morphisms in this groupoid presentation are computed as the induced action of super Euclidean isometries on $S \times \mathbb{R}^{0|1}$, i.e., the arrow that makes the diagram commute

$$
\begin{array}{ccc}
\mathbb{R}^{1|1}_R \times S & \xrightarrow{id \times \proj_R} & \mathbb{R}^{0|1} \times S \\
\downarrow & & \downarrow \\
\mathbb{R}^{1|1}_R \times S & \xrightarrow{\proj_\nu} & \mathbb{R}^{0|1} \times S,
\end{array}
$$

(14)

where the downward arrow on the left is a lift of an isometry between $S$-families of super circles and is determined by an element $(u, \nu, \theta, \pm 1) \in (\mathbb{R}^{1|1} \times \mathbb{Z}/2)(S)$, following the discussion at the end of subsection 2.2. For a coordinate $\theta$ on $\mathbb{R}^{0|1}$ and $s$ a coordinate on $S$, the assignment

$$
(\theta, s) \mapsto \left( \pm 1 \left( \theta + \nu - \rho \frac{u + i \nu}{r} \right), s \right), \quad (u, \nu, \pm 1, \theta, s) \in \text{Euc}(\mathbb{R}^{1|1}) \times \mathbb{R}^{0|1} \times S
$$

defines the unique dotted arrow satisfying this property.
Now we consider the action of $\text{Euc}_S(S \times \mathbb{R}^{1|1})$ on $S$-families of pairs $(R, \phi_0)$ where $\phi_0 : S \times \mathbb{R}^{0|1} \to X$. Choosing a coordinate $\theta$ on $\mathbb{R}^{0|1}$, we recall that $S$-points of $\text{SMfld}(\mathbb{R}^{0|1}, X) \cong \pi TX$ have the form

$$\phi_0 = x + \theta \psi, \quad x : C^\infty(X) \to C^\infty(S), \quad \psi \in \text{Der}_x(C^\infty(X), C^\infty(S))$$

where $x$ is an algebra homomorphism and $\psi$ is an odd derivation with respect to $x$. We compute the action of $\text{Euc}(\mathbb{R}^{1|1})(S)$ on these $S$-points as

$$(r, \rho, x + \theta \psi) \mapsto \left( r + 2i\nu \rho, \pm \rho, x \pm \left( \theta - \nu + \rho \frac{u + i\theta \nu}{r} \right) \psi \right),$$

where we follow our convention for group actions on mapping spaces; see Section 1.6. This shows that the stack $\Phi_{1}^{1|1}(X)$ admits the following quotient groupoid description.

**Proposition 2.7.** There is a surjective map of stacks

$$\left( \mathbb{R}_{>0} \times \pi TX \right)/\left(\mathbb{R}^{1|1} \times \mathbb{Z}/2\right) \to \Phi_{0}^{1|1}(X).$$

Restricting to $R \in \mathbb{R}_{>0}(S) \subset \mathbb{R}_{>0}^{1|1}(S)$ we obtain the following.

**Proposition 2.8.** There is an equivalence of stacks

$$\left( \mathbb{R}_{>0} \times \pi TX \right)/\left(\mathbb{R}^{1|1} \times \mathbb{Z}/2\right) \cong \Phi_{1}^{0|1}(X).$$

**Remark 2.9.** In spite of the notation, the above isn’t quite a quotient stack: the super group $\mathbb{T}^{1|1} \cong \mathbb{R}^{1|1}/R \cdot \mathbb{Z}$ depends on $R \in \mathbb{R}_{>0}$, although these groups are all isomorphic. We adopt the quotient stack notation above as a compact way of denoting this slightly more complicated groupoid.

### 2.5. Line bundles over vacua and the proof of Proposition 2.1

Define an odd line bundle $\kappa_\rho$ over $\mathbb{R}_{>0}^{1|1}/(\mathbb{R}^{1|1} \times \mathbb{Z}/2)$ via the homomorphism

$$\rho : \mathbb{R}^{1|1} \times \mathbb{Z}/2 \overset{\rho_1}{\to} \mathbb{Z}/2 \subset \mathbb{C}^\times \cong \text{Aut}(\mathbb{C}^{0|1}),$$

where $p$ is the projection and in the above $\text{Aut}(\mathbb{C}^{0|1})$ denotes the automorphisms of $\mathbb{C}^{0|1}$ as an odd vector space (not a supermanifold). As we remarked in subsection 1.6 the above defines gluing data natural in $S$ via the map $\text{SMfld}(S, \mathbb{C}^\times) \to C^\infty(S)^\times$. From Proposition 2.7, we have a map $\mathbb{R}_{>0}^{1|1}/(\mathbb{R}^{1|1} \times \mathbb{Z}/2) \to \Phi_{1}^{0|1}(pt)$. Since the super group of translations is in the kernel of $\rho$, the above defines a line bundle $\kappa$ over $\Phi_{1}^{0|1}(pt)$ that pulls back to $\kappa_\rho$ along the map in Proposition 2.7. Consider the commuting square,

$$\begin{array}{ccc}
\Phi_{0}^{1|1}(X) & \to & \Phi_{0}^{1|1}(pt) \\
\downarrow & & \downarrow \\
\Phi_{1}^{1|1}(X) & \to & \Phi_{1}^{1|1}(pt)
\end{array}$$

where the downward arrows are inclusions and the horizontal arrows are induced by the canonical map $X \to pt$. We can pullback the line bundle $\kappa$ over $\Phi_{1}^{0|1}(pt)$ along these maps, obtaining line bundles over $\Phi_{0}^{1|1}(X)$ and $\Phi_{1}^{1|1}(X)$ that we also denote by $\kappa$.

**Definition 2.10.** A section $\sigma \in \Gamma(\Phi_{0}^{1|1}(X); \kappa)$ is supersymmetric if it is in the image of the restriction map $i^* : \Gamma(\Phi_{0}^{1|1}(X); \kappa) \to \Gamma(\Phi_{1}^{1|1}(X); \kappa)$ induced by the inclusion $i : \Phi_{0}^{1|1}(X) \hookrightarrow \Phi_{1}^{1|1}(X)$. We denote the vector space of supersymmetric sections by $\Gamma_{\text{susy}}(\Phi_{0}^{1|1}(X); \kappa) := \text{image}(i^*)$.

---

As a diagram in the bicategory of stacks this is really 2-commutative, but using our super Lie groupoid presentations one can understand this as a strictly commuting diagram where the arrows are super Lie groupoid homomorphisms.
Proof of Proposition 2.1. We recall the natural isomorphism of graded \( \mathbb{C} \)-algebras,

\[
K^\bullet(X) \otimes \mathbb{C} \cong \left\{ \begin{array}{ll}
\text{H}_{\text{dR}}^\text{ev}(X) & \bullet = \text{even} \\
\text{H}_{\text{dR}}^\text{odd}(X) & \bullet = \text{odd}
\end{array} \right.
\]

with de Rham cohomology. Hence it suffices to show that concordance classes of sections \( \Gamma_{\text{dR}}(\Phi_{1}^{\infty}(X);\kappa') \) are isomorphic to 2-periodic de Rham cohomology of \( X \).

Sections of \( \kappa \) over \( \Phi_{1}^{\infty}(X) \) are in bijection with sections of the line bundle \( \kappa_{R} \) over \( (\mathbb{R}_{\geq 0} \times \pi TX)/(\mathbb{R}_{\geq 0} \times \mathbb{Z}/2) \) induced by the homomorphism \( \rho \). Therefore, we shall compute sections \( \Gamma(\Phi_{1}^{\infty}(X),\kappa') \) using Equation (3) and the groupoid presentation of the previous subsection. For \( (u,\nu) \in \mathbb{R}_{\geq 0}^{1}(S) \) and \( (r,\rho,\nu,\psi) \in (\mathbb{R}_{\geq 0}^{1} \times \pi TX)(S) \), we have the action

\[
(r,\rho,\nu,\psi) \mapsto (r + 2i\nu \rho,\rho,\nu + \rho \psi/r,\psi + i\nu \rho \psi/r).
\]

Actions by \( \mathbb{R}_{\geq 0}^{1} \) have a unique odd generator coming from the fact that the Lie algebra of \( \mathbb{R}_{\geq 0}^{1} \) is freely generated on a single odd element. For our purposes, we need only observe that an odd operator \( Q \) generates a \( \mathbb{R}_{\geq 0}^{1} \)-action via the formula \( \exp(iuQ^2 + \nu Q) \); we verify

\[
\exp(iuQ^2 + \nu Q) \exp(iu'Q^2 + \nu' Q) = \exp(iuQ^2)(1 + \nu Q) \exp(iu'Q^2)(1 + \nu' Q) = \exp(i(u + u')Q^2)(1 + (\nu + \nu') Q + \nu Q \nu' Q) = \exp(i(u + u')Q^2)(1 + (\nu + \nu') Q + i \cdot \nu Q \nu' Q) = \exp(i(u + u' + i\nu \nu')Q^2 + (\nu + \nu') Q).
\]

In such case we call \( Q \) an infinitesimal generator of the \( \mathbb{R}_{\geq 0}^{1} \)-action.

Lemma 2.11. The action (15) on \( C^\infty(\mathbb{R}_{\geq 0}^{1} \times \pi TX) \cong (C^\infty(\mathbb{R}_{\geq 0})[\rho]) \otimes \Omega^*(X) \) is determined by the infinitesimal generator

\[
Q := 2\rho \frac{d}{dr} \otimes \text{id} - \text{id} \otimes d + i\rho \frac{r}{r} \cdot \text{deg}
\]

where \( d \) is the de Rham d and \( \text{deg} \) is the degree endomorphism on differential forms.

Proof of Lemma. The action (15) is equivalent to a morphism of algebras \( C^\infty(\mathbb{R}_{\geq 0}^{1} \times \pi TX) \to C^\infty(\mathbb{R}_{\geq 0}^{1} \times \mathbb{R}_{\geq 0}^{1} \times \pi TX) \), so it suffices to verify the lemma for functions of the form

\[
G = (R_0(r) + \rho R_1(r)) \otimes f, \quad H = (R_0(r) + \rho R_1(r)) \otimes df,
\]

\( R_i(r) \in C^\infty(\mathbb{R}_{\geq 0}) \), \( f \in C^\infty(X) \supset C^\infty(\pi TX) \), since any element of \( C^\infty(\mathbb{R}_{\geq 0}^{1} \times \pi TX) \) can be written as a product of such functions. The image of \( G \) and \( H \) in \( C^\infty(\mathbb{R}_{\geq 0}^{1} \times \mathbb{R}_{\geq 0}^{1} \times \pi TX) \) under the action is

\[
G \mapsto (R_0(r + 2i\nu \rho) + \rho R_1(r + 2i\nu \rho)) \otimes (f - \nu df + \frac{\rho}{r} u df) = R_0 \otimes f + \rho R_1 \otimes f + u \rho R_0 \otimes df + 2i \nu \rho \frac{dR_0}{dr} \otimes f - \nu R_0 \otimes df - \nu \rho R_1 \otimes df
\]

\[
H \mapsto (R_0(r + 2i\nu \rho) + \rho R_1(r + 2i\nu \rho)) \otimes (df + i\nu \rho df) = R_0 \otimes df + \rho R_1 \otimes df + i R_0 \nu \frac{\rho}{r} \otimes df + 2i \nu \rho \frac{dR_0}{dr} \otimes df
\]

for \( (u,\nu) \) coordinates on \( \mathbb{R}_{\geq 0}^{1} \), where for simplicity of notation \( R_i := R_i(r) \), and the equalities follow from Taylor expansion. To derive the above formulas from the \( S \)-point description of the action, we identify the functions \( f, df \in C^\infty(\pi TX) \) with natural transformations that at an \( S \)-point give a map of sets \( \pi TX(S) \to C^\infty(S) \) defined by \( (x,\psi) \mapsto x(f) \) and \( (x,\psi) \mapsto \psi(f) \) for \( f \) and \( df \) respectively, where we view the \( S \)-point \( (x,\psi) \) as maps \( x,\psi : C^\infty(X) \to C^\infty(S) \).

From the definition of the operator \( Q \) we have

\[
\exp(iuQ^2 + \nu Q) = \exp(iuQ^2)(1 + \nu Q) = (1 + i \cdot uQ^2)(1 + \nu Q) = 1 + iuQ^2 + \nu Q.
\]
which allows us to compute the action on $G$ as

$$(1 + iuQ^2 + \nu Q)G = R_0 \otimes f + \rho R_1 \otimes f + iuQ(2i\rho \frac{dR_0}{dr} \otimes f - R_0 \otimes df + \rho R_1 \otimes df) + \nu(2i\rho \frac{dR_0}{dr} \otimes f - R_0 \otimes df + \rho R_1 \otimes df)$$

$$= R_0 \otimes f + \rho R_1 \otimes f + uR_0 \frac{\rho}{r} \otimes df + 2i\rho \frac{dR_0}{dr} \otimes f - \nu R_0 \otimes df + \nu \rho R_1 \otimes df$$

where in the first equality we have used that $\deg(f) = 0$. Similarly, the action on $H$ is

$$$(1 + iuQ^2 + \nu Q)H = R_0 \otimes df + \rho R_1 \otimes df - iuQ(2i\rho \frac{dR_0}{dr} \otimes df + i\rho \frac{R_0}{r} \otimes df) + \nu(2i\rho \frac{dR_0}{dr} \otimes df + \nu i\rho \frac{R_0}{r} \otimes df + 2i\rho \frac{dR_0}{dr} \otimes df)$$

$$= R_0 \otimes df + \rho R_1 \otimes df + \nu i\rho \frac{R_0}{r} \otimes df + 2i\rho \frac{dR_0}{dr} \otimes df.$$ 

Hence the action on $G$ and $H$ determined by $Q$ agrees with the original $\mathbb{R}^{1|1}$-action, so that $Q$ is indeed an infinitesimal generator. \hfill \Box

Returning to the proof of Proposition \ref{prop:susy-1EFT} we need to compute the image of sections under restriction to $\Phi_{0}^{1|1}(X)$, so without loss of generality we consider functions $g \in C^\infty(\mathbb{R}_{>0}^{1|1} \times \pi TX)$ whose Taylor expansion in $\rho$ has the form

$$g(r, \rho, x, \psi) = \sum g_k(r) \otimes \alpha_k,$$

i.e., the coefficients of $\rho \in C^\infty(\mathbb{R}_{>0}^{1|1})$ vanish. For such a function to be an element of $\Gamma_{\text{susy}}(\Phi_{0}^{1|1}(X); \kappa^*)$, we require the appropriate equivariance property with respect to the $\mathbb{Z}/2$-action; for this, each $g_k$ must be of even degree (respectively, of odd degree) as a differential form for $l$ even (respectively, $l$ odd). Next we need to compute the action of the infinitesimal generator $Q$ of the $\mathbb{R}^{1|1}$-action on these functions. Then we have

$$Qg = \sum \left(2i \cdot \rho \frac{dg_k}{dr} \otimes \alpha_k - g_k \otimes d\alpha_k + i \cdot \deg(\alpha_k)g_k\rho/r \otimes \alpha_k\right),$$

where $\alpha_k \in \Omega^k(X) \subset C^\infty(\pi TX)$ and $g_k \in C^\infty(\mathbb{R}_{>0})$. We first observe that invariant functions have $d\alpha_k = 0$ for all $k$ (e.g., by restricting to the locus $\rho = 0$). Furthermore,

$$\frac{dg_k}{dr} = \frac{\deg(\alpha_k)g_k}{2r},$$

so $g_k(r) = c \cdot r^{\deg(\alpha_k)/2}$ for some constant $c$. Concordance classes of closed differential forms are precisely de Rham cohomology classes (see Appendix \ref{app:cochain}). We choose an isomorphism between concordance classes of sections of $\Gamma_{\text{susy}}(\Phi_{0}^{1|1}(X); \kappa^*)$ and the de Rham cohomology ring that on cocycles sends a homogeneous element $r^{k/2} \otimes \alpha$ to $\alpha/(2\pi)^{k/2}$ where $\deg(\alpha) = k$. One can view this as a normalizing factor for the volumes of the universal family of super-circles; another reason for this choice comes from analyzing a Chern character for 1|1-EFTs constructed by F. Han \cite{Han08} that we will explain in subsection 2.6.

Using the language of cocycle theories reviewed in Appendix \ref{app:cochain} to endow the sequence of sheaves $\Gamma_{\text{susy}}(\Phi_{0}^{1|1}(-); \kappa^*)$ with the structure of a cohomology theory, we need to specify a suspension class and desuspension map. The rapidly decaying section

$$\sigma := r^{1/2}e^{-y^2}dy \in \Gamma_{\text{susy,cvs}}(\Phi_{0}^{1|1}(\mathbb{R}); \kappa^1)$$

for $y$ a coordinate on $\mathbb{R}$ defines a suspension class for desuspension maps

$$\Gamma_{\text{susy,cvs}}(\Phi_{0}^{1|1}(X \times \mathbb{R}); \kappa^{l+1}) \xrightarrow{f(-)r^{-1/2}} \Gamma_{\text{susy}}(\Phi_{0}^{1|1}(X); \kappa^l).$$
where $\int$ is integration of differential forms along the fibers of the projection $X \times \mathbb{R} \to X$. The image of $\sigma$ for $X = \text{pt}$ is $1 \in \Gamma_{\text{susy}}(\Phi_{0}^{1|1}(\text{pt}); \kappa^{0})$, and by standard arguments this induces a suspension isomorphism on concordance classes. Hence, $(\Gamma_{\text{susy}}(\Phi_{0}^{1|1}(-); \kappa^{*}), \sigma, \int)$ defines a cohomology theory isomorphic to $K^{*} \otimes \mathbb{C}$.

**Remark 2.12.** The function $r^{k/2}$ is related to the kernel of the Dirac operator on the circle of radius $r$: there is a line bundle over the moduli of (spin) Euclidean circles whose fiber at a given point is the kernel of Dirac operator, and the nonvanishing section, $\sqrt{dr}/r$, has norm squared equal to $r$. Hence, the quantity $\sqrt{dr}/\sqrt{r}$ is a unit norm section which trivializes the line bundle. Under this choice of trivialization, multiplication by $r^{k/2}$ essentially allows us to identify a function on the moduli of circles with a section of the $k^{\text{th}}$ tensor power of this line bundle.

2.6. The Chern character for 1|1-EFTs and classical vacua. Drawing on the work of F. Han [Han08] and F. Dumitrescu [Dum06], in this subsection we explain how vector bundles with connection give functions on the stack of classical vacua. This will explain our choice of isomorphism $\mathbb{H}_{\text{dR}}^{*}(X) \cong \Gamma_{\text{susy}}(\Phi_{0}^{1|1}(X); \kappa^{*})/\sim$ above, and will allow us to identify an index map for 1|1-EFTs over $X$ that agrees with the usual index of twisted Dirac operators in Section 4.7.

Let 1|1-EFT$(X)$ denote the category of 1|1-dimensional (untwisted) field theories, defined similarly to the discussion in Section 1.4 coming from the bicategories 1|1-Euclidean bordisms 1|1-EBord$(X)$ and algebras, both viewed as categories fibered over supermanifolds. Let $\text{Vect}^{V}(X)$ denote the groupoid of vector bundles with connection on $X$ and connection preserving bundle automorphisms. We claim that the following diagram commutes

\[
\begin{array}{ccc}
\text{Vect}^{V}(X) & \xrightarrow{F} & 1|1\text{-EFT}(X) \\
\downarrow & & \uparrow \text{CH} \\
K^{0}(\text{X}) & \xrightarrow{\text{ch}} & \Gamma_{\text{susy}}(\Phi_{0}^{1|1}(X); \kappa^{*})
\end{array}
\]

where $\text{ch}$ is the usual Chern character, $\text{CH}$ evaluates a 1|1-Euclidean field theory on classical vacua, conc denotes taking concordance classes of sections and applying our chosen isomorphism to de Rham cohomology, and $F$ is the functor constructed by F. Dumitrescu [Dum06] in his thesis that produces a super Euclidean field theory from a vector bundle with connection using super parallel transport. It remains to show that for our choice of conc, the above commutes.

**Remark 2.13.** The composition of $\text{CH}$ and conc is very close to being the Chern character map constructed in F. Han’s thesis [Han08]. Ignoring some subtle issues in the Euclidean geometry, the salient difference is that his map evaluates on a fixed family of super Euclidean circles (of radius $1/2\pi$) whereas ours evaluates along a nontrivial stack of super circles before choosing an isomorphism with de Rham cohomology. One upshot of our more complicated approach is that the characterization of supersymmetric functions on classical vacua shows that the value of a 1|1-dimensional field theory on a super circle is independent of the circumference. This gives an alternate proof of the usual supersymmetric cancelation property for the super trace of a Dirac operator in the case of a 1|1-EFT over the point defined by a Dirac operator (see [HST10] for a precise description such examples of 1|1-EFTs).

We will now verify the commutativity of the diagram above. In the case that a 1|1-EFT comes from a vector bundle, the resulting function on the stack $\Phi_{0}^{1|1}(X)$ reads off the holonomy of the vector bundle along super loops whose bosonic part is constant; the result is an even function on the moduli stack of classical vacua, and a calculation reveals it is $\text{Tr}(\exp(-irF))$ for $F$ the curvature 2-form of the vector bundle and $r$ the usual coordinate on $\mathbb{R}_{\geq 0}$. One can infer this from F. Han’s thesis [Han08]; a different treatment is given in the note by F. Dumitrescu [Dum12] where at the end of the calculation.
on page 5 one takes the path to have length $r$ rather than length 1, and one should set up the computation using the super Euclidean structure on $\mathbb{R}^{1|1}$ where the infinitesimal generator of translations is $\partial_x - i\partial_{\theta}$. Hence, for the desired commuting diagram we require $\text{Tr}(\exp(-i\theta F)) \mapsto \text{Tr}(\exp(F/(2\pi i)))$, which is in agreement with our choice of isomorphism $H_{\text{dR}}^\bullet(X) \cong \Gamma_{\text{susy}}(\Phi_0^{1|1}(X); \kappa^?)$.

### 2.7. Time reversal, conjugation and KO-theory.

On the stack $\text{Vect}^\nabla(X)$ of vector bundles with connection, there is a $\mathbb{Z}/2$-action gotten from reversing the complex structure on a vector bundle, denoted $E \mapsto \overline{E}$. This $\mathbb{Z}/2$-action descends to one on complex K-theory and has as its homotopy fixed points KO-theory. In this section we explain how to extend this action to one on $1|1$-Euclidean manifolds (and hence, 1|1-EFTs). There is a closely related action on $1|1$-Euclidean manifolds coming from time reversal. We show that the restriction of either of these actions to classical vacua gives rise to cocycles for $\text{KO} \otimes \mathbb{C}$.

The situation for time reversal is simpler, so shall begin with it. **Time reversal** is an automorphism of the $1|1$-Euclidean isometry group determined by $(t, \theta) \mapsto (-t, i\theta)$. This acts on the charts of a $1|1$-Euclidean manifold by precomposition and on gluing maps by conjugation, so determines an automorphism of the stack of $1|1$-Euclidean manifolds, i.e., over each $S$ there is an endofunctor on the groupoid of super Euclidean manifolds over $S$. This determines an action on the category of $1|1$-Euclidean bordisms and (by restriction) on the stack of classical vacua that we denote by $t$.

**Definition 2.14.** A $t$-invariant structure on a line bundle $\mathcal{L}$ over $\Phi_0^{1|1}(X)$ is an isomorphism $\tau : t^*\mathcal{L} \to \mathcal{L}$. A $t$-invariant section $s$ is one for which $t^*s = \tau^{-1}(s)$.

The conjugation action is a bit more complicated owing to the fact that (untwisted) field theories are functors from $1|1$-EBord$(X)$ to TV fibered over supermanifolds, not just manifolds: conjugation acts trivially on manifolds, but not on supermanifolds. For $E \to S$ a vector bundle over a supermanifold, reversing the complex structure on $E$ requires that we also reverse the complex structure on the structure sheaf of $S$, i.e., we take the same real sheaf of functions where the complex numbers act by precomposition with complex conjugation. We denote the image of this action by $E \to \overline{S}$. This gives an action on the fibered category of vector spaces. To promote it to an action on field theories we also require a conjugation action on $1|1$-EBord$(X)$. For without such an action the composition of a field theory with conjugation is not a fibered functor over supermanifolds, but rather a functor that lies over a non-identity endofunctor on supermanifolds. Hence, for a family of $1|1$-Euclidean manifolds $F \to S$, we need to endow $\overline{F} \to \overline{S}$ with the structure of a $1|1$-Euclidean manifold. The relevant framework was given by Hohnhold, Stolz and Teichner in [HST10] near Definition 6.18, as we shall review presently.

A real structure on a supermanifold is an isomorphism $r_S : S \to \overline{S}$ such that $\tau_S \circ r_S = \text{id}_S$. A map $f : S \to S'$ between supermanifolds with real structure is real if $r_{S'} \circ f = f \circ r_S$. A real structure on a super Lie group is a real structure on the underlying supermanifold such that the unit, inverse and multiplication maps are all real. A real structure on a model geometry is a real structure on the model space and a real structure on the super Lie group of isometries such that the action map is real.

For the super Lie group $\text{Euc}(\mathbb{R}^{1|1}) \cong \mathbb{R}^{1|1} \times \mathbb{Z}/2$, there is a real structure determined by $(t, \theta, \pm 1) \mapsto (\bar{t}, i\bar{\theta}, \pm 1)$. We observe that $\tau \circ r = \text{id}$, so it remains to check compatibility with the Lie group structure. For the unit and inversion compatibility is immediate, and for the multiplication map $m : \mathbb{R}^{1|1} \times \mathbb{R}^{1|1} \to \mathbb{R}^{1|1}$ we verify

$$r \circ m(t, \theta, t', \theta') = (\bar{t} + i\bar{\theta}', i(\bar{\theta} + \theta')) = \overline{m \circ (r \times r)(t, \theta, t', \theta')} \quad \text{coordinates on } \mathbb{R}^{1|1} \times \mathbb{R}^{1|1}.$$ 

The above conjugations define an endofunctor on line bundles over classical vacua as follows. Let a line bundle $\mathcal{L}$ over $\Phi_0^{1|1}(X)$ be given, and choose an $S$-point $(R, \phi) \in \Phi_0^{1|1}(X)(S)$. Conjugation takes the $S$-family of $1|1$-Euclidean manifolds defined by $R$ to an $\overline{S}$-family of supermanifolds. We give this $\overline{S}$-family a $1|1$-Euclidean structure using the real structure $r$.
above. Conjugation of the map $\phi$ to $X$ gives a map $\bar{\phi}$ to $\tilde{X} \cong X$, since any ordinary manifold has a canonical real structure. In total, we have obtained a $\tilde{S}$-point of $\Phi_{0}^{[1]}(X)$.

The line bundle $\mathcal{L}$ over $\Phi_{0}^{[1]}(X)$ defines a line bundle over $\tilde{S}$, and we apply the conjugation action on vector bundles over supermanifolds to obtain a line bundle denoted $r^*\mathcal{L}$ over $\tilde{S}$.

For a section $s$ of $\mathcal{L}$ defined over the universal family $S = \mathbb{R}_{>0} \times \pi TX$ (and by naturality, over any $S$-point of $\Phi_{0}^{[1]}(X)$) the above construction gives a section $r^*s$ of $r^*\mathcal{L}$.

**Definition 2.15.** A real structure on a line bundle $\mathcal{L}$ over $\Phi_{0}^{[1]}(X)$ is an isomorphism $\rho: r^*\kappa \to \kappa$. A real section $s$ is one for which $r^*s = \rho^{-1}(s)$.

**Proposition 2.16.** There are isomorphisms $\rho_{l}: r^*\kappa^{l} \to \kappa^{l}$ and $\tau_{l}: t^*\kappa^{l} \to \kappa^{l}$ unique up to sign and compatible with tensor products that give $\kappa^{l}$ the structure of real and $t$-invariant line bundles over $\Phi_{0}^{[1]}(X)$.

**Proof.** Pick a cocycle representative for $\kappa$ that pulls back from the groupoid presentation (2.8) of $\Phi_{0}^{[1]}(pt)$. Then $r^*\kappa$ has the identical cocycle since $r^*\kappa$ must be nontrivial, and there is precisely one such nontrivial cocycle in our groupoid presentation of $\Phi_{0}^{[1]}(pt)$. Therefore, an isomorphism $r^*\kappa \cong \kappa$ comes from an isomorphism $\mathbb{C} \to \mathbb{C}$. A simple computation identifies $(r^*)^2$ with the grading endomorphism of $\kappa$, and compatibility with this requires we take the isomorphism $\mathbb{C} \to \mathbb{C}$ determined by $1 \mapsto \pm i$. We choose the map $1 \mapsto +i$, which in turn gives isomorphisms $r^*\kappa^{l} \equiv \kappa^{l}$ by taking tensor products.

The situation for the $t$-action is nearly identical: $t^*\kappa$ has the same cocycle as $\kappa$ because $t$ acts trivially on the objects in our groupoid presentation of $\Phi_{0}^{[1]}(pt)$, and so acts trivially on the cocycle for $\kappa$. Geometrically, $(t^*)^2$ is the same automorphism on super circles as the one implementing the grading endomorphism of $\kappa$, so we again choose the isomorphism $\mathbb{C} \to \mathbb{C}$ determined by $1 \mapsto i$, and extend by tensor products to obtain isomorphisms $t^*\kappa^{l} \equiv \kappa^{l}$. □

**Proposition 2.17.** Concordance classes of real supersymmetric sections of $\kappa^{l}$ over $\Phi_{0}^{[1]}(X)$ are in bijection with $\text{KO}^{l}(X) \otimes \mathbb{C}$, as are concordance classes of $t$-invariant supersymmetric sections of $\kappa^{l}$.

**Proof.** Using Proposition 2.11 throughout we will implicitly identify supersymmetric sections of $\kappa^{l}$ with closed differential forms. A section $\alpha \in \Gamma_{\text{susy}}(\Phi_{0}^{[1]}(X); \kappa^{l})$ is real if

$$r^*\alpha = \rho^{-1}(\alpha) = i^{-k}\alpha.$$

The action of $r$ on $\alpha$ can be traced through the map $r: \mathbb{R}^{[1]} \to \mathbb{R}^{[1]}$ restricted to $\mathbb{R}^{[0,1]}$ and the associated action on $C^\infty(\pi TX) \cong C^\infty(\text{SMfld}(\mathbb{R}^{[0,1]}, X))$. We compute that $r^*\alpha = (i)^{-\deg(\alpha)}\alpha$. Together with the above equation, this implies that $\alpha$ can be identified with an element of $\bigoplus_{i \in \mathbb{Z}} \Omega^{i+4i}_{\text{cl}}(X)$, and conversely such an element gives rise to a real section. In the case of the $t$-action, it is easy to see the action on forms is through $(i)^{-\deg(\alpha)}$, yielding the same result. Since $\text{KO}^{l}(X) \otimes \mathbb{C} \cong \bigoplus_{i \in \mathbb{Z}} H^{i+4i}_{\text{dR}}(X)$, the proposition is proved. □

**Remark 2.18.** If we include time reversal as an isometry of super Euclidean manifolds, we obtain the unoriented $1|1$-Euclidean model geometry with isometry group $\mathbb{R}^{[1,1]} \rtimes \mathbb{Z}/4$ acting on the model space $\mathbb{R}^{[1]}$. From this we may construct the stack of unoriented classical vacua, which has the groupoid presentation $(\mathbb{R}_{>0} \times \pi TX)/(\mathbb{T}^{[1]} \rtimes \mathbb{Z}/4)$ gotten from identical arguments as in the proof of Proposition 2.7. This groupoid has a naturally defined 4-periodic line bundle from inclusion $\mathbb{Z}/4 \to \mathbb{C} \cong \text{End}(\mathbb{C}^{[0,1]})$ as roots of unity, and supersymmetric sections are cocycles for $\text{KO}(X) \otimes \mathbb{C}$. This is in keeping with the construction of spectrum KO by Hohnhold, Stolz and Teichner using unoriented super Euclidean field theories [HST10]. Proposition 2.17 points toward a possible construction of the spectrum K as a $\mathbb{Z}/2$-equivariant spectrum using oriented $1|1$-Euclidean field theories together with either the homotopy action of time-reversal or conjugation.
2.8. Gerbes and twisted K-theory. Bundle gerbes with connection provide a geometric model for twists of K-theory. For a gerbe with curvature 3-form $H$, twisted K-theory with complex coefficients can be computed by the de Rham complex with the twisted differential $d + H$. To do this in a manner which can be made functorial, for a cover $\{U_i\}$ of $X$, a class in twisted cohomology consists of forms $\beta_i$ on $U_i$ such that $\beta_j = \exp(F_{ij})\beta_i$ on $U_i \cap U_j$, where $F_{ij}$ denotes the curvature of the line bundle with connection defined on $U_i \cap U_j$ in the cocycle data for a gerbe with connection on $X$; e.g., see Atiyah and Segal’s account in [AS05], particularly Section 6. In this subsection we show that a gerbe with connection on $X$ determines a line bundle over classical vacua whose sections are classes in twisted K-theory with complex coefficients.

Let cocycle data for a bundle gerbe with connection on $X$ be given, i.e., suppose we have a cover $\{U_i\}$ of $X$ together with line bundles with connection $(L_{ij}, \nabla_{ij})$ for each overlap $U_i \cap U_j$ satisfying a cocycle condition. These line bundles determine super parallel transport functors for super paths in $U_i \cap U_j$, and restriction to $\Phi_0^{1|1}(X)$ gives a function on $\mathbb{R}_{\geq 0} \times \pi T(U_i \cap U_j)$ coming from the holonomy around these super paths; when restricted to $\mathbb{R}_{> 0} \times \pi T(U_i \cap U_j)$ this is precisely the function $\exp(iF_{ij})$ for $F_{ij}$ the curvature of the connection $\nabla_{ij}$. These nonvanishing functions on overlaps determine a line bundle on $\mathbb{R}_{\geq 0} \times \pi TX$, where the cocycle condition can be verified immediately via the assumed cocycle condition for the gerbe with connection. We can restrict attention to sections invariant under the $\mathbb{R}^{1|1} \times \mathbb{Z}/2$-action on $\mathbb{R}_{> 0} \times \pi TX$, which defines a line bundle on $\Phi_0^{1|1}(X)$ we denote by $\tau$. We define

$$
\Gamma_{\text{susy}}(\Phi_0^{1|1}(X); \kappa^{l + \tau}) := \text{image} \left( \Gamma(\Phi_0^{1|1}(X); \kappa^l \otimes \tau) \to \Gamma(\Phi_0^{1|1}(X); \kappa^l \otimes \tau) \right)
$$

The vector space $\Gamma_{\text{susy}}(\Phi_0^{1|1}(X); \kappa^{l + \tau})$ is spanned by functions $\beta_i \in \Gamma_{\text{susy}}(\Phi_0^{1|1}(U_i); \kappa^l)$ on each $U_i$ such that $\beta_j = \exp(iF/\tau)\beta_i$. Hence, we’ve shown that a gerbe with connection leads to a natural isomorphism of algebras,

$$
K^{* + \tau}(X) \otimes \mathbb{C} \cong \Gamma_{\text{susy}}(\Phi_0^{1|1}(X); \kappa^{l + \tau})/\text{concordance}
$$

where the left hand side denotes K-theory twisted by the gerbe, taken with complex coefficients, and the right hand side denotes sections of the line bundle over classical vacua constructed out of $\tau$.

3. Classical vacua of the 2|1-sigma model and TMF $\otimes \mathbb{C}$

Just as in the 1|1-dimensional case, 2|1-dimensional super tori arise as a quotient of a model space by a lattice. Our construction of the stack of 2|1-dimensional classical vacua, denoted $\Phi_0^{2|1}(X)$, is a hybrid between the classical story for the stack of elliptic curves and the construction of $\Phi_0^{1|1}(X)$ in the previous section. We define a line bundle $\omega^{1/2}$ over $\Phi_0^{2|1}(X)$ related to the Pfaffian line bundle over the moduli stack of spin elliptic curves. With these constructions in place, proving Theorem 1.1 amounts to a computation.

3.1. Motivation from the 2|1-sigma model. We give a brief overview of the 2|1-sigma model as motivation for the definitions that follow.

The generalized supermanifold of fields for the 2|1-sigma model has as $S$-points maps $\phi: S \times \mathbb{R}^{2|1} \to X$ satisfying certain periodicity conditions with respect to a $\mathbb{Z} \times \mathbb{Z}$-action that will be made precise in the next subsection. We will describe such maps in terms of their component fields

$$
x: S \times \mathbb{R}^2 \to X, \quad \psi \in \Gamma(S \times \mathbb{R}^2, x^* \pi TX), \quad \phi = x + \theta \psi.
$$

The classical action is

$$
S(\phi) = S(x, \psi) = \frac{1}{2} \int_{S \times \mathbb{R}^{2|1}/S} \left( \langle \partial_{\bar{z}} x, \partial_z x \rangle + i \langle \nabla_{\partial_z} \psi, \psi \rangle \right) dzd\bar{z},
$$
where $\partial_\z$ is a vector field on $\mathbb{R}^{2|1}$ we will define below and the integral is fiberwise over fundamental domains for the given $S$-family of supertori. The norm is taken with respect to the usual pairing on complex-valued functions on tori and the Riemannian metric on the target; in fact this only depends on a conformal structure on the source. The second term in the action is dilation-invariant, as we will verify in Remark 3.1 below.

As in the 1|1-dimensional case, we extract two geometrically important ideas from the classical 2|1-sigma model.

1. The symmetries of the classical action $\mathcal{S}$ come from super Euclidean isometries of $\mathbb{R}^{2|1}$ and the dilations specified above, and so $\mathcal{S}$ is naturally a function on a stack of fields we will define below.

2. The classical vacua for the 2|1-sigma model consist of constant maps $x$ and covariantly constant sections $\psi$, so we find that $S$-points of classical vacua can be identified with maps $S \times \mathbb{R}^{2|1} \rightarrow S \times \mathbb{R}^{0|1} \rightarrow X$. These maps have zero action, which motivates our definition of $\Phi_{S|0}^{2|1}(X)$.

### 3.2. Super Euclidean tori and modular forms.

The relevant 2|1-dimensional model space arises from the super group $\mathbb{R}^{2|1}$ with multiplication

$$(z, \bar{z}, \theta) \cdot (z', \bar{z}', \theta') = (z + z', \bar{z} + \bar{z}' + i\theta \theta', \theta + \theta'), \quad (z, \bar{z}, \theta), \ (z', \bar{z}', \theta') \in \mathbb{R}^{2|1}(S)$$

and there is an action of $\mathbb{C}^\times$ on this supergroup by

$$(q, \bar{q}) \cdot (z, \bar{z}, \theta) = (q^2 z, \bar{q}^2 \bar{z}, q \bar{q} \theta), \quad (q, \bar{q}) \in \mathbb{C}^\times(S), \ (z, \bar{z}, \theta) \in \mathbb{R}^{2|1}(S).$$

From this we form the supergroup $\mathbb{R}^{2|1} \times \mathbb{C}^\times$ which acts on $\mathbb{R}^{2|1}$ and defines a rigid super conformal model space (this is the conformal Euclidean model space in the terminology of Stolz-Teichner [ST11]). This gives a stack of 2|1-dimensional rigid conformal manifolds on the site of supermanifolds, whose value on $S$ is the groupoid of fiber bundles of rigid super conformal manifolds and fiberwise isometries.

**Remark 3.1.** To verify that the above is indeed the symmetry group of the classical action functional, we compute

$$\int_{S \times T^{2|1}/S} \langle D^2 \phi, D\phi \rangle d\theta d\bar{\theta} d\bar{z} d\bar{\bar{z}} = \int_{S \times T^{2|1}/S} (-i\partial_\z(x + \theta \psi), (\partial_\theta - i\partial_\bar{\theta})(x + \theta \psi))$$

$$= \frac{1}{2} \int_{S \times T^{2|1}/S} (\partial_\z x, \partial_\bar{\z} x) + i(\nabla \partial_\z \psi, \psi) d\bar{z} d\bar{\bar{z}} = \mathcal{S}(\phi),$$

where $D = \partial_\theta - i\partial_\bar{\theta}$ is a right-invariant vector field on $\mathbb{R}^{2|1}$, and the second equality is a Berezinian integral along the fibers $S \times T^{2|1} \rightarrow S \times T^2$. Since $S$ is constructed out of right-invariant vector fields, it is invariant under the left action of $\mathbb{R}^{2|1}$. To see invariance under the dilation action of $\mathbb{C}^\times$ we observe that $(q, \bar{q}) \in \mathbb{C}^\times$ acts by $d\theta \mapsto \bar{q}^{-1} d\theta$; $dz \mapsto q^2 dz$ and $d\bar{z} \mapsto \bar{q}^2 d\bar{z}$; and finally, $D \mapsto \bar{q}^{-1} D$. Using the conjugate-linearity of the pairing, the result follows.

An (oriented, based) lattice $\Lambda \subset \mathbb{R}^{2|1}$ gives a super torus, $\mathbb{R}^{2|1}/\Lambda$. The data of such a $\Lambda$ is over $S$ is a pair $\tilde{\ell} = (\ell, \bar{\ell}, \sigma)$, $\tilde{\bar{\ell}} = (\ell', \bar{\ell'}, \sigma') \in \mathbb{R}^{2|1}(S)$ satisfying

1. $\tilde{\ell} \cdot \tilde{\bar{\ell}} = \tilde{\ell}' \cdot \tilde{\bar{\ell}}'$ where $\cdot$ denotes multiplication in $\mathbb{R}^{2|1}(S)$ defined above;
2. for any map $\phi$: pt $\rightarrow S$, the complex numbers $\phi^* \ell$ and $\phi^* \ell'$ are linearly independent over $\mathbb{R}$ and $\phi^* \ell / \phi^* \ell' \in \mathfrak{h}$ where $\mathfrak{h} \subset \mathbb{C}$ is the upper half plane.

We denote the generalized supermanifold with these $S$-points by $sL$, the generalized supermanifold of based lattices. The first condition is equivalent to $\sigma \sigma' = 0$, from which one can deduce that $sL$ is not representable. However, it is affine in the sense that maps $S \rightarrow sL$ can be identified with algebra maps $C^\infty(L)[\sigma, \sigma']/(\sigma \sigma') \rightarrow C^\infty(S)$. An $S$-point $\Lambda = (\ell, \bar{\ell})$ of $sL$ allows us to form the $S$-family of super rigid conformal tori as the quotient $(S \times \mathbb{R}^{2|1})/\Lambda =: S \times_A \mathbb{R}^{2|1}$, where $A$ acts on $\mathbb{R}^{2|1}$ by translations.
Let \( L \subset sL \) denote the supermanifold of (ordinary) lattices, \( \Lambda \subset \mathbb{R}^2 \subset \mathbb{R}^{2|1} \). Following our description of functions on \( sL \) in the previous paragraph, \( L \) is the reduced manifold of \( sL \). There is a diffeomorphism \( L \cong \mathfrak{h} \times \mathbb{C}^\times \) where \( \mathfrak{h} \) is the upper half plane.

**Definition 3.2.** Weak modular forms of weight \( n/2 \) are holomorphic functions \( h \) on \( L \) that are \( \text{SL}_2(\mathbb{Z}) \)-invariant and have the property that \( h(q \cdot \Lambda) = q^{-n/2}h(\Lambda) \) for \( q \in \mathbb{C}^\times \). Taking products of holomorphic functions gives a graded ring, denoted \( \text{MF} \) whose degree \( n \) piece, denoted \( \text{MF}^n \), are the weight \( n/2 \) weak modular forms.

**Remark 3.3.** The above is equivalent to the more common definition of modular forms of weight \( n/2 \) as holomorphic functions \( h \) on the upper half plane \( \mathfrak{h} \) with the property \( h\left( \frac{\sigma + h}{c\tau + d} \right) = \epsilon(\tau) h(\tau) \) for \( \tau \in \mathfrak{h} \), \( \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right] \in \text{SL}_2(\mathbb{Z}) \) and \( \epsilon(\tau) \) a holomorphic square root of \( c\tau + d \). However, the first definition is better-suited for our computations below.

### 3.3 Stacks of fields and classical vacua

**Definition 3.4.** The stack of fields, denoted \( \Phi^{2|1}(X) \), is the stack associated to the prestack that has objects over \( S \) consisting of pairs \( (\Lambda, \phi) \) where \( \Lambda \in sL(S) \) determines a family of super tori \( S \times_{\Lambda} \mathbb{R}^{2|1} \) and \( \phi : S \times_{\Lambda} \mathbb{R}^{2|1} \rightarrow X \) is a map. Morphisms between these objects over \( S \) consist of commuting triangles

\[
S \times_{\Lambda} \mathbb{R}^{2|1} \xrightarrow{\phi} S \times_{\Lambda'} \mathbb{R}^{2|1} \\
\phi' \downarrow \quad \quad \quad \quad \quad \downarrow \phi
\]

where the horizontal arrow is an isomorphism of \( S \)-families of super rigid conformal 2|1-manifolds.

**Definition 3.5.** The stack of classical vacua, denoted \( \Phi^0_0^{2|1}(X) \), is the full substack of \( \Phi^{2|1}(X) \) generated by pairs \( (\Lambda, \phi) \) where \( \Lambda \in \text{L}(S) \subset sL(S) \) determines a family of super tori \( S \times_{\Lambda} \mathbb{R}^{2|1} \) and \( \phi : S \times_{\Lambda} \mathbb{R}^{2|1} \rightarrow X \) is a map that factors through the map induced by the projection \( \mathbb{R}^{2|1} \rightarrow \mathbb{R}^{0|1} \).

Analogous arguments as in Remark 2.6 apply to justify our restriction on lattices in the definition of the stack of classical vacua.

The description of a supersymmetric section of a line bundle over \( \Phi^{2|1}_0(X) \) runs in parallel to the 1|1-dimensional case, and so requires us to define a stack \( \Phi^{2|1}(X) \subset \Phi(X) \) that has \( \Phi^{2|1}_0(X) \subset \Phi^{2|1}(X) \). In particular, we require a morphism \( \text{proj}_{\Lambda} : S \times_{\Lambda} \mathbb{R}^{2|1} \rightarrow S \times \mathbb{R}^{0|1} \) for any \( \Lambda = (\ell, \bar{\ell}) \in sL(S) \). For \( (\ell, \bar{\ell}, \sigma, \ell', \bar{\ell}', \sigma') \in sL(S) \) define a map

\[
\text{proj}_{\Lambda} : \mathbb{R}^{2|1} \times S \rightarrow \mathbb{R}^{0|1}, \quad \text{proj}_{\Lambda}(z, \bar{z}, \theta, s) := \theta - \sigma \frac{\bar{z} \ell' - z \ell'}{\ell' - \ell} - \sigma' \frac{z \bar{\ell} - \bar{z} \ell}{\ell' - \ell}.
\]

This map is invariant under translations in the lattice,

\[
\text{proj}_{\Lambda}(z + \ell, \bar{z} + \bar{\ell} + \theta \sigma, \theta + \sigma, s) = \theta + \sigma - \sigma \frac{(z + \bar{\ell}) \ell' - (z + \ell) \bar{\ell}}{\ell' - \ell} = \text{proj}_{\Lambda}(z, \bar{z}, \theta, s) + \sigma - \sigma
\]

\[
\text{proj}_{\Lambda}(z + \ell', \bar{z} + \bar{\ell} + \theta \sigma', \theta + \sigma', s) = \theta + \sigma' - \sigma' \frac{(z + \bar{\ell}) \ell' - (z + \ell) \bar{\ell}}{\ell' - \ell} = \text{proj}_{\Lambda}(z, \bar{z}, \theta, s) + \sigma' - \sigma',
\]

where we use that \( \sigma^2 = \sigma'^2 = \sigma \sigma' = 0 \). Therefore \( \text{proj}_{\Lambda} \) descends to a map we also denote by \( \text{proj}_{\Lambda} : S \times_{\Lambda} \mathbb{R}^{2|1} \rightarrow S \times \mathbb{R}^{0|1} \).
Definition 3.6. The stack $\Phi^{21}_0(X)$ is the full substack of $\Phi^{21}_-(X)$ generated by pairs $(\Lambda, \phi)$ where $\Lambda \in sL(S)$ determines a family of super circles $S \times A \mathbb{R}^{21}$ and $\phi: S \times A \mathbb{R}^{21} \to X$ is a map that factors through the map $\text{proj}_A$ defined above.

For $\Lambda \in L(S) \subset sL(S)$, we observe that $\text{proj}_A$ is induced by the ordinary projection $S \times \mathbb{R}^{21} \to S \times \mathbb{R}^{01}$, so there is an inclusion of stacks $\Phi^{21}_0(X) \subset \Phi^{21}_-(X)$. We now turn our attention to finding groupoid presentations of these stacks.

3.4. Groupoid presentations. We will require an explicit description of the super rigid conformal isometries with source $S \times A \mathbb{R}^{21}$. There are two sorts of isometries to consider: those arising from the $\text{SL}_2(\mathbb{Z})$-action that changes the choice of basis for the lattice, and those arising from the action of the rigid conformal group on $\mathbb{R}^{21} \times S$. Starting with the latter, by lifting to the universal cover such isometries are determined by the left action of an element of $\text{RConf}(\mathbb{R}^{21})(S)$ on $\mathbb{R}^{21}$. By an identical computation as in the 1-dimensional case, an element of $F \in \text{RConf}(\mathbb{R}^{21})(S)$ represents the identity isometry if and only if $(\ell, \ell')$ have $\sigma = \sigma' = 0$ and $F \in \mathbb{R} \times \mathbb{Z} \subset \mathbb{C}^2 \subset \mathbb{R}^{21}$.

We will now derive an explicit formula for the target of the rigid conformal isometry $S \times A \mathbb{R}^{21} \to S \times A \mathbb{T}^{21}$ for $\Lambda, \Lambda' \in sL(S)$ determined by an $S$-point of $(\mathbb{R}^{21} \times \mathbb{C}^\times) \times \text{SL}_2(\mathbb{Z})$. The action of $\mathbb{R}^{21}$ changes the basepoint, so acts on lattices through conjugation. The $\mathbb{C}^\times$ action is through dilation, and the $\text{SL}_2(\mathbb{Z})$-action changes the lattice in the usual manner, so we have

$$
\Lambda' := \left( \begin{array}{cccc} a & b \\ c & d \end{array} \right), u, v, q, q', (\ell, \ell', \ell, \ell', \sigma, \sigma') \mapsto \left( \begin{array}{c} q^2(a\ell + b\ell'), q^2(a\ell + 2i\nu\sigma) + b(\ell + 2i\nu\sigma'), \\ q^2(c\ell + d\ell'), q^2(c\ell + 2i\nu\sigma) + d(\ell + 2i\nu\sigma') \end{array} \right)
$$

for

$$(\ell, \ell', \ell, \ell', \sigma, \sigma') \in sL(S), (u, v, q) \in \mathbb{R}^{21}(S), (q, q') \in \mathbb{C}^\times(S), \left[ \begin{array}{cccc} a & b \\ c & d \end{array} \right] \in \text{SL}_2(\mathbb{Z})(S).$$

Given $\phi: S \times A \mathbb{R}^{21}$, let $\phi_0: S \times \mathbb{R}^{01} \to X$ by the map such that $\phi = \phi_0 \circ \text{proj}_A$. We define an action of $\text{SL}_2(\mathbb{Z}) \times \text{RConf}(\mathbb{R}^{21})$ on $S \times \mathbb{R}^{01}$ by finding an arrow that makes the diagram commute

$$
\begin{array}{ccc}
S \times \mathbb{R}^{21} & \xrightarrow{\text{id} \times \text{proj}_A} & S \times \mathbb{R}^{01} \\
\cong \downarrow & & \downarrow \\
S \times \mathbb{R}^{21} & \xrightarrow{\text{proj}_A} & S \times \mathbb{R}^{01}.
\end{array}
$$

where the vertical left arrow is the lift of a rigid conformal isometry to $S \times \mathbb{R}^{21}$ determined by an $S$-point of $\text{RConf}(\mathbb{R}^{21})$ that, together with an $S$-point of $\text{SL}_2(\mathbb{Z})$, determines $\Lambda'$. First, we observe that the maps $\text{proj}_A$ are invariant under the $\text{SL}_2(\mathbb{Z})$-action, i.e., for $A \in \text{SL}_2(\mathbb{Z})$, we have $\text{proj}_A = \text{proj}_{A \cdot A}$. To see this we compute

$$\Lambda' = (a\ell + b\ell', a\ell + b\ell', a\sigma + b\sigma', c\ell + d\ell', c\ell + d\ell', c\sigma + d\sigma')$$

and observe that the image of $\ell \ell' - \ell \ell$ is

$$(a\ell + b\ell')(c\ell + d\ell') - (c\ell + d\ell')(a\ell + b\ell') = \ell \ell' - \ell \ell$$

where we use that $ad - bc = 1$, and then we compute

$$\text{proj}_A(z, \bar{z}, \theta, s) = \theta - (a\sigma + b\sigma')\frac{z(c\ell + d\ell') - z(c\ell + d\ell')}{\ell \ell' - \ell \ell} - (c\sigma + d\sigma')\frac{z(a\ell + b\ell') - z(a\ell + b\ell')}{\ell \ell' - \ell \ell}$$

$$= \theta - \sigma\frac{\ell \ell' - \ell \ell}{\ell \ell' - \ell \ell} - \sigma\frac{\ell \ell' - \ell \ell}{\ell \ell' - \ell \ell} = \text{proj}_A(z, \bar{z}, \theta, s),$$

where we again use that $ad - bc = 1$. 

25
Turning attention to a candidate action of $R\text{Conf}(\mathbb{R}^{2|1})$ on $S \times \mathbb{R}^{0|1}$ in diagram (19) we claim the map

$$\theta \mapsto \bar{\theta} \left( \theta + \nu - \frac{(\sigma \ell' - \sigma' \ell')(\bar{u} + i\theta \nu) - (\sigma \ell' - \sigma' \ell')u}{\ell' - \ell \ell} \right)$$  

(20)

is the unique such map making the diagram (19) commute, where $(u, \bar{u}, \nu, q) \in (\mathbb{R}^{2|1} \rtimes \mathbb{C}^*) (S) \cong R\text{Conf}(\mathbb{R}^{2|1})(S)$. This follows from a direct computation: we apply $\text{proj}_\Lambda$ and $\text{proj}_{\Lambda'}$ to the source and target of the action (18) respectively (with the trivial $SL_2(\mathbb{Z})$-action); we have

$$(\ell, \bar{\ell}, \sigma, \ell', \bar{\ell}', \sigma') \mapsto (q^2 \ell, q^2(\ell + 2i\sigma \nu), \bar{q} \nu, q^2 \ell', q^2(\ell + 2i\sigma' \nu), \bar{q} \nu')$$

and

$$(z, \bar{z}, \theta) \mapsto (z + u, \bar{z} + \bar{u} + i\theta \nu, \theta + \nu),$$

so

$$\text{proj}_{\Lambda'}(q^2(z + u), q^2(z + \bar{u} + i\theta \nu), \bar{q}(\theta + \nu)) = \bar{q} \left( \theta + \nu - \frac{\bar{z} + \bar{u} + i\theta \nu)\ell + (z + u)\bar{\ell}}{\ell' - \ell \ell} \right)$$

\[-\bar{\sigma}'(z + u)\bar{\ell} - (\bar{z} + \bar{u} + i\theta \nu)\bar{\ell}' \right)$$

where we have used that $\sigma^2 = \sigma'^2 = \sigma' = 0$. We compare the above to $\text{proj}_{\Lambda}(z, \bar{z}, \theta)$, and find the difference to be exactly (20).

The previously computed $SL_2(\mathbb{Z}) \times R\text{Conf}(\mathbb{R}^{2|1})$-action on $sL$ together with Equation (20) defines a $SL_2(\mathbb{Z}) \times R\text{Conf}(\mathbb{R}^{2|1})$-action on $sL \times \pi TX \cong sL \times SMfld(\mathbb{R}^{0|1}, X)$ (where we will follow conventions set forth in Section 1.6 regarding left actions on mapping spaces), yielding the following.

**Proposition 3.7.** There is a surjective map of stacks,

$$(sL \times \pi TX)/((SL_2(\mathbb{Z}) \times (\mathbb{R}^{2|1} \rtimes \mathbb{C}^*)) \to \Phi^2_0(X).$$

Restricting to the locus where $\sigma = \sigma' = 0$, we have the following.

**Proposition 3.8.** There is an equivalence of stacks,

$$(L \times \pi TX)/((SL_2(\mathbb{Z}) \times (\mathbb{T}^{2|1} \rtimes \mathbb{C}^*)) \cong \Phi^2_0(X).$$

**Remark 3.9.** The above is not quite a quotient stack in the usual sense: the group $\mathbb{T}^{2|1} \cong \mathbb{R}^{2|1}/\Lambda$ depends on $\Lambda \in L$, so perhaps one should view this as a groupoid whose morphisms are a bundle of groups (whose fibers are isomorphic) over $L \times \pi TX$. Moreover, notice that $\mathbb{T}^2 \subset \mathbb{T}^{2|1}$ acts trivially for all $\Lambda$. For simplicity, we will stick with the quotient groupoid notation.

We will have use for the following $SL_2(\mathbb{Z})$-covering of $\Phi^2_0(X)$ and $\Phi^2_0(X)$.

**Definition 3.10.** Define $\Phi^2_0(X)$ (respectively, $\Phi^2_0(X)$) as the stack whose objects are the same as $\Phi^2_0(X)$ (respectively, $\Phi^2_0(X)$), and whose morphisms are commuting triangles (17) where we require the rigid conformal isometries to come from elements of $(\mathbb{R}^{2|1} \rtimes \mathbb{C}^*)(S)$, i.e., we exclude $SL_2(\mathbb{Z})$-alterations of the lattice.

**Corollary 3.11.** The stack $\Phi^2_0(X)$ is presented by $(L \times \pi TX)/((\mathbb{T}^{2|1} \rtimes \mathbb{C}^*))$, and the stack $\Phi^2_0(X)$ admits a surjective map from $(sL \times \pi TX)/((\mathbb{R}^{2|1} \rtimes \mathbb{C}^*))$ that is compatible with the surjection in Proposition 3.7.
3.5. Line bundles over stacks of fields and the proof of Theorem [1.1] Define an odd line bundle $\omega^{1/2}_\rho$ over $\Phi_\epsilon^2(\text{pt}) \cong sL/(\text{SL}_2(\mathbb{Z}) \times (\mathbb{R}^{2|1} \rtimes \mathbb{C}^\times))$ via the homomorphism
\[
\rho: \text{SL}_2(\mathbb{Z}) \times (\mathbb{R}^{2|1} \rtimes \mathbb{C}^\times) \to \mathbb{C}^\times \cong \text{Aut}(\mathbb{C}^{0|1})
\]
given by the projection composed with the map $z \mapsto z^{-1}, z \in \mathbb{C}^\times$. Since the super group of translations is in the kernel of $\rho$, we obtain a line bundle $\omega^{1/2}$ over $\Phi_\epsilon^2(\text{pt})$ that pulls back to $\omega^{1/2}_\rho$ along the surjective map of Proposition [3.7]. Furthermore, we obtain isomorphisms on spaces of sections,
\[
\Gamma\left(sL// (\text{SL}_2(\mathbb{Z}) \times (\mathbb{R}^{2|1} \rtimes \mathbb{C}^\times)) \right) ; \omega^{k/2} \cong \Gamma\left(\Phi_\epsilon^2(X) ; \omega^{k/2}\right).
\]
We have the following commutative square
\[
\begin{array}{ccc}
\Phi_0^{2|1}(X) & \to & \Phi_0(\text{pt}) \\
\downarrow & & \downarrow \\
\Phi_\epsilon^{2|1}(X) & \to & \Phi_\epsilon(\text{pt})
\end{array}
\]
where the vertical arrows are the usual inclusions, and the horizontal arrows are induced by the map $X \to \text{pt}$. We pull back the line bundle $\omega^{1/2}$ over $\Phi_\epsilon^{2|1}(X)$ along these maps.

Definition 3.12. A section $\sigma \in \Gamma(\Phi_0^{2|1}(X) ; \omega^{1/2})$ is supersymmetric if it is in the image of the restriction map $i^*: \Gamma(\Phi_0^{2|1}(X) ; \omega^{1/2}) \to \Gamma(\Phi_\epsilon^{2|1}(X) ; \omega^{1/2})$ induced by the inclusion $i: \Phi_0^{2|1}(X) \hookrightarrow \Phi_\epsilon^{2|1}(X)$. We denote the vector space of supersymmetric sections by $\Gamma_{\text{susy}}(\Phi_0^{2|1}(X) ; \omega^{1/2}) := \text{image}(i^*)$. Define supersymmetric sections over $\Phi_\epsilon^{2|1}(X)$ analogously.

Proof of Theorem [1.1] Since TMF $\otimes \mathbb{C}$ is a rational cohomology theory, it is represented by a wedge of Eilenberg-MacLane spectra and we have an isomorphism
\[
\text{TMF}^k(X) \otimes \mathbb{C} \cong \bigoplus_{i+j=k} \text{H}^i(X ; \text{MF}^j).
\]
with ordinary cohomology valued in weak modular forms. Since $\text{MF}^*$ is a graded $\mathbb{C}$-algebra, this ordinary cohomology theory can be modeled by de Rham cohomology with values in weak modular forms. It remains to show that concordance classes of sections $\Gamma_{\text{susy}}(\Phi_0^{2|1}(X) ; \omega^{1/2})$ can be identified with such de Rham cohomology classes.

We begin by computing the action of $\mathbb{R}^{2|1}$ at an $S$-point $(\phi, \Lambda) \in \Phi_\epsilon^{2|1}(X)(S)$ using (20); the image of an $S$-point $(\ell, \bar{\ell}, \sigma, \ell', \bar{\ell}', \sigma', x, \psi) \in (sL \times \pi TX)(S)$ under this action is
\[
\left(\ell, \bar{\ell} + 2i\nu\sigma, \sigma, \bar{\ell}', \bar{\ell}', 2i\nu\sigma', \sigma', x - \left(\frac{\nu - \sigma\ell' - \sigma'\bar{\ell}}{\ell\ell' - \ell\bar{\ell}} \bar{u} - \frac{\sigma\ell' - \sigma'\bar{\ell}}{\ell\ell' - \ell\bar{\ell}} u\right) \psi, \left(1 - i\frac{\sigma\ell' - \sigma'\bar{\ell}}{\ell\ell' - \ell\bar{\ell}} \nu\right) \psi\right).
\]

Lemma 3.13. The above $\mathbb{R}^{2|1}$-action on $g \in C^\infty(sL) \otimes \Omega^*(X)$ can be expressed as
\[
g \mapsto \exp(uR + iuQ_0^2 + \nu Q_0)g,
\]
for infinitesimal generators
\[
R := \frac{\sigma\ell' - \sigma'\bar{\ell}}{\ell\ell' - \ell\bar{\ell}} \otimes d, \quad Q = 2i\nu \sigma \partial_\ell \otimes \text{id} + 2i\nu' \sigma' \partial_{\bar{\ell}} \otimes \text{id} - \text{id} \otimes d + i\frac{\sigma\ell' - \sigma'\bar{\ell}}{\ell\ell' - \ell\bar{\ell}} \otimes \text{deg},
\]
where $d$ is the de Rham $d$ and $\text{deg}$ is the degree endomorphism on differential forms.

Proof of Lemma. The action is through algebra maps $C^\infty(sL \times \pi TX) \to C^\infty(\mathbb{R}^{2|1} \times sL \times \pi TX)$, so it suffices to check the lemma on functions
\[
G = (l_0 + \sigma l_1 + \sigma' l_2) \otimes f, \quad H = (l_0 + \sigma l_1 + \sigma' l_2) \otimes df
\]
for \( l_0, l_1, l_2 \in C^\infty(L) \) and \( f \in C^\infty X \subset C^\infty(\pi TX) \). The image of these functions under the action determined by the \( S \)-point formula displayed before the statement of the lemma is

\[
(u, \bar{u}, \nu) \cdot G = (l_0 + \sigma l_1 + \sigma' l_2) \otimes f - \nu(l_0 - \sigma l_1 - \sigma' l_2) \otimes df + 2i \nu(\sigma \partial_{\ell} + \sigma' \partial_{\ell'}) l_0 \otimes df + \left( \frac{\sigma \ell' - \sigma' \ell}{\ell' - \ell} + \frac{u \sigma \ell' - \sigma' \ell}{\ell' - \ell} \right) l_0 \otimes df.
\]

\[
(u, \bar{u}, \nu) \cdot H = (l_0 + \sigma l_1 + \sigma' l_2) \otimes df + 2i \nu(\sigma \partial_{\ell} + \sigma' \partial_{\ell'}) l_0 \otimes df + i \nu \frac{\sigma \ell' - \sigma' \ell}{\ell' - \ell} l_0 \otimes df.
\]

The formulas follow from Taylor expanding functions on \( sL \), together with the identification of the functions \( f, df \in C^\infty(\pi TX) \) with natural transformations that at an \( S \)-point give a map of sets \( \pi TX(S) \rightarrow C^\infty(S) \) defined by \( (x, \psi) \mapsto x(f) \) and \( (x, \psi) \mapsto \psi(f) \) for \( f \) and \( df \) respectively, where we view the \( S \)-point \( (x, \psi) \) as maps \( x, \psi: C^\infty(X) \rightarrow C^\infty(S) \).

Next we calculate for the operators \( Q \) and \( R \),

\[
\exp(uR + i\bar{u}Q^2 + \nu Q) = \exp(uR)(1 + i\bar{u}Q^2)(1 + \nu Q) = (1 + uR)(1 + i\bar{u}Q^2 + \nu Q).
\]

Applying this to the function \( G \) we find

\[
(1 + uR)(1 + i\bar{u}Q^2 + \nu Q)G = (1 + uR) \left( (l_0 + \sigma l_1 + \sigma' l_2) \otimes f + i\bar{u}Q(2i(\sigma \partial_{\ell} + \sigma' \partial_{\ell'}) l_0 \otimes f - (l_0 - \sigma l_1 - \sigma' l_2) \otimes df \right)
\]

\[
+ (1 + uR) \left( (l_0 + \sigma l_1 + \sigma' l_2) \otimes f - i\bar{u}(2i(\sigma \partial_{\ell} + \sigma' \partial_{\ell'}) l_0 \otimes f - (l_0 - \sigma l_1 - \sigma' l_2) \otimes df \right)
\]

\[
= (1 + uR) \left( (l_0 + \sigma l_1 + \sigma' l_2) \otimes f + \frac{\sigma \ell' - \sigma' \ell}{\ell' - \ell} l_0 \otimes df + u \frac{\sigma \ell' - \sigma' \ell}{\ell' - \ell} l_0 \otimes df \right)
\]

and similarly for the function \( H \),

\[
(1 + uR)(1 + i\bar{u}Q^2 + \nu Q)H = (1 + uR) \left( (l_0 + \sigma l_1 + \sigma' l_2) \otimes df + \bar{u}Q(2i(\sigma \partial_{\ell} + \sigma' \partial_{\ell'}) l_0 \otimes df + i \frac{\sigma \ell' - \sigma' \ell}{\ell' - \ell} \otimes df \right)
\]

\[
+ (1 + uR) \left( (l_0 + \sigma l_1 + \sigma' l_2) \otimes df + i \frac{\sigma \ell' - \sigma' \ell}{\ell' - \ell} \otimes df \right)
\]

\[
= (1 + uR) \left( (l_0 + \sigma l_1 + \sigma' l_2) \otimes df + \frac{\sigma \ell' - \sigma' \ell}{\ell' - \ell} l_0 \otimes df + \nu(2i(\sigma \partial_{\ell} + \sigma' \partial_{\ell'}) l_0 \otimes df + i \frac{\sigma \ell' - \sigma' \ell}{\ell' - \ell} \otimes df \right)
\]

Comparing with the formulas for \( (u, \bar{u}, \nu) \cdot G \) and \( (u, \bar{u}, \nu) \cdot H \), we see that the action determined by the infinitesimal generators agrees with the original \( \mathbb{R}^{2\dim} \)-action.

Returning to the proof of Theorem 1.1, we observe that the restriction homomorphism on functions \( i^*: C^\infty(sL \times \pi TX) \rightarrow C^\infty(L \times \pi TX) \) is precisely the quotient by the ideal generated by \( \sigma \) and \( \sigma' \). So to compute function on \( \Phi^0_{2\dim}(X) \) in the image of the restriction
map it suffices to consider those functions on $\Phi^2_{\ell}(X)$ with Taylor expansions

$$f(\ell, \ell', x, \psi) = \sum f_k(\ell, \ell', \ell') \otimes \alpha_k$$

for $f_k \in C^\infty(L)$ and $\alpha_k \in \Omega^*(X)$, i.e., the coefficients of $\sigma$ and $\sigma'$ vanish.

In the groupoid description of $\Phi^2_{\ell}(X)$, $\text{SL}_2(\mathbb{Z})$ acts trivially on $\pi TX$ (and hence on differential forms) so invariance under this action means that the $f_k$ are $\text{SL}_2(\mathbb{Z})$-invariant. Invariance under $R$ implies that the $\alpha_k$ are closed. It remains to understand invariance under $Q$ and dilation equivariance.

We compute

$$Qf = \sum_k \left( 2i(\sigma\partial_{\ell} f + \sigma'\partial_{\ell'} f) \otimes \alpha_k - f_k \otimes d\alpha_k + \deg(\alpha_k) f_k \frac{\sigma\ell' - \sigma'\ell}{\ell' - \ell'} \otimes \alpha_k \right)$$

and so we have

$$\partial_{\ell} f_k = \frac{\deg(\alpha_k)}{2} \frac{\ell'}{\ell' - \ell'} f_k, \quad \partial_{\ell'} f_k = -\frac{\deg(\alpha_k)}{2} \frac{\ell}{\ell' - \ell'} f_k,$$

from which we deduce that the $f_k$ have the form $(\ell' - \ell')^{\deg(\alpha_k)/2} F_k$ for $F_k \in C^\infty(sL)$ holomorphic, i.e., $\partial_{\ell} F_k = \partial_{\ell'} F_k = 0$. The dilation action on a $k$-form is through $q^{-k}$ (where we are using our convention for group actions on mapping spaces, see Section 1.6), so we have

$$(q, q) \cdot ((\ell' - \ell')^{k/2} F \otimes \alpha) = (q^2 q^2 \ell' - \ell')^{k/2} (q, q) \cdot F \otimes q^{-k} \alpha = q^k (\ell' - \ell')^{k/2} (q, q) \cdot F \otimes \alpha,$$

so for $(\ell' - \ell')^{k/2} F \otimes \alpha$ to be a section of $E^j$, we require that $(q, q) \cdot F = q^{i-k} F$, and so—together with holomorphicity and $\text{SL}_2(\mathbb{Z})$-invariance—$F$ is a weak modular form of weight $(j-k)/2$.

From this we see that supersymmetric sections of $\omega^{k/2}$ are generated by sums of functions $(\ell' - \ell')^{\deg(\alpha)/2} F \otimes \alpha$ for $F \in M^{j} \otimes \alpha \in \Omega^{2j}(X)$ such that $i + j = k$. We choose the identification of such functions with differential forms valued in modular forms that takes a homogeneous element $(\ell' - \ell')^{\deg(\alpha)/2} F \otimes \alpha$ to $F \otimes (2\pi)^{\deg(\alpha)/2}$. Concordance classes therefore yield a ring isomorphic to the de Rham cohomology ring required by the discussion at the beginning of this proof. This concludes the proof of the isomorphism of rings claimed in the statement of the theorem.

To verify that we obtain an isomorphism of cohomology theories, we require sheaf level data on $\Gamma_{\text{susy}}(\Phi^2_{0}(X); \omega^{k/2})$ that endows concordance classes with the structure of a cohomology theory. We accomplish this through the language of cocycles theories developed by Stolz and Teichner, reviewed in Appendix A. The upshot of this approach is that we need only provide a sheaf-level suspension isomorphism for $\Gamma_{\text{susy}}(\Phi^2_{0}(X); \omega^{k/2})$ in order to identify concordance classes with a cohomology theory.

We claim that the desired suspension isomorphism is implemented by any rapidly decreasing 1-form $s$ on $\mathbb{R}$ with total integral 1. Such a 1-form can be promoted to a section $(\ell' - \ell')^{-1/2} \otimes s \in \Gamma_{\text{susy},cvs}(\mathbb{R}; \omega^{1/2})$. If $p_1 : X \times \mathbb{R} \to X$, $p_2 : X \times \mathbb{R} \to \mathbb{R}$ are the projections, we have the map

$$\times s : \Gamma_{\text{susy}}(\Phi^2_{0}(X); \omega^{k+1/2}) \to \Gamma_{\text{susy},cvs}(\Phi^2_{0}(X \times \mathbb{R}); \omega^{k/2})$$

$$\alpha \mapsto (p_1^* \alpha) \otimes (p_2^* s).$$

There is a map

$$\int : \Gamma_{\text{susy},cvs}(\Phi^2_{0}(X \times \mathbb{R}); \omega^{k+1/2}) \to \Gamma_{\text{susy}}(\Phi^2_{0}(X); \omega^{k/2})(X),$$

that first multiplies by $(\ell' - \ell')^{-1/2}$ and then integrates compactly supported differential forms over the fibers of the projection $X \times \mathbb{R} \to X$. By the usual arguments (together with the computation of concordance classes above) $\times s$ and $\int$ are inverses at the
level of concordance classes. Then, by Theorem A.3, we can identify the functor, $X \mapsto \Gamma_{\text{susy}}(\Phi_0^{1|1}(X); \omega^{*}/2)/\text{concordance with the cohomology theory } X \mapsto \text{TMF}^*(X) \otimes \mathbb{C}$. □

Remark 3.14. The factors of $(\hat{\ell}' - \hat{\ell})^{1/2}$ are related to the kernel of the Dirac operator on the torus determined by the corresponding lattice. In the standard description of elliptic curves in terms of a parameter $\tau \in \mathfrak{h}$ in the upper half plane, the function $(\hat{\ell}' - \hat{\ell})^{1/2}$ can be identified with $\sqrt{\im(\tau)}$. In this description, the line bundle over the moduli stack of spin tori coming from the kernel of the Dirac operator has a trivializing section $\sqrt{\det(\tau)}$ with norm squared $\im(\tau)$, e.g., see the discussion in [Pre87] immediately following Proposition 4.5.

Hence, the factor $(\hat{\ell}' - \hat{\ell})^{1/2}$ essentially identifies a function on the moduli space of tori with a section of the $k$th tensor power of the line bundle coming from the kernel of the Dirac operator.

4. Warm-up 2: the perturbative $1|1$-sigma model and the $\hat{A}$-genus

Define the $\hat{A}$-class of a Riemannian manifold $X$ to be the polynomial in Pontryagin forms defined by the usual characteristic series $\frac{x/2}{\sinh(x/2)}$, and denote this closed, even differential form by $\hat{A}(X) \in \Omega^2_{\text{cl}}(X)$. By the proof of Proposition 2.1, we can view this element as a supersymmetric section on the $0$th tensor power of $\kappa$ on classical vacua, i.e., a supersymmetric function, which we also denote by $\hat{A}(X) \in C^\infty_{\text{susy}}(\Phi_0^{1|1}(X))$. The goal of the present section is to identify the function $\hat{A}(X)$ with the relative determinant of a family of operators, denoted $\Delta^{1|1}_X$, parametrized by $\Phi_0^{1|1}(X)$.

The operators $\Delta^{1|1}_X$ are called kinetic operators, and they encode the linearization of the classical action of the sigma model in the following manner

$$ S_{\text{lin}}(\nu) := \int_{S^1} \langle \nabla_D \nu, \nabla_D \nu \rangle = \int_{S^1} \langle \nu, \Delta^{1|1}_X \nu \rangle dt $$

where $\nu$ is a section of the normal bundle $N\Phi_0^{1|1}(X)$ to the inclusion $\Phi_0^{1|1}(X) \subset \Phi^{1|1}(X)$. We will compute the $\zeta$-super determinant of $\Delta^{1|1}_X$ relative to an operator $\Delta^{1|1}_n$; roughly this has the effect that all constructions involving $TX$ will be taken relative to a trivial bundle $\mathbb{R}^n$ with $n = \dim(X)$. We will show that this relative $\zeta$-super determinant is a representative of the $\hat{A}$-class of $X$.

**Proposition 4.1.** The relative $\zeta$-super determinant of the family of operators $\Delta^{1|1}_X$ gives a supersymmetric function

$$ \frac{s\text{det}_\zeta(\Delta^{1|1}_X)}{s\text{det}_\zeta(\Delta^{1|1}_n)} = : \hat{A}(X) \in C^\infty_{\text{susy}}(\Phi_0^{1|1}(X)) $$

that represents the $\hat{A}$-class of $X$, i.e., $[\hat{A}(X)] \in H^*_\text{dR}(X)$.

For future convenience, we will also prove a variation on the above that brings Bismut-Freed-Quillen determinant lines into the fray. Let $\text{sDet}(\Delta^{1|1}_X) \otimes \text{sDet}(\Delta^{1|1}_n)^{-1}$ denote the relative super determinant line of $\Delta^{1|1}_X$, and $s\text{det}(\Delta^{1|1}_X) \otimes s\text{det}(\Delta^{1|1}_n)^{-1} \in \Gamma(\text{sDet}(\Delta^{1|1}_X) \otimes \text{sDet}(\Delta^{1|1}_n)^{-1})$ its relative determinant section.

**Proposition 4.2.** The section

$$ \frac{s\text{det}(\Delta^{1|1}_X) \otimes s\text{det}(\Delta^{1|1}_n)^{-1}}{\hat{A}(X)} \in \Gamma(\text{Det}(\Delta^{1|1}_X) \otimes \text{Det}(\Delta^{1|1}_n)^{-1}) $$

defines a geometric trivialization of the relative super determinant line as a real line bundle.

At the end of the section we will explain how the above constructions of $\hat{A}(X)$ define a local index map for $1|1$-Euclidean field theories that is closely related to the local index in the Atiyah-Singer theorem.
The essence of the above pair of propositions has occurred in various guises elsewhere: most notably, we learned the basic pieces of the ζ-determinant computation in our Proposition 4.1 from E. Witten’s article in [DEF+99], particularly pages 476-485. Our approach is also very similar in spirit to that of R. Grady and O. Gwilliam in their construction of the A-class [GG12]; there are some language barriers between the two approaches, but roughly their version of topological quantum mechanics arises from the large-volume limit of our linearized classical action functional.

4.1. The tangent space of fields restricted to the stack of classical vacua. The tangent space to a smooth mapping space \( \mathsf{Mfld}(N, M) \) can be described as the pullback of the tangent bundle of \( M \) along the evaluation map

\[
ev: N \times \mathsf{Mfld}(N, M) \to M,
\]

viewed as a vector bundle over \( \mathsf{Mfld}(N, M) \). By restricting attention to an S-point of a map \( f \in \mathsf{Mfld}(N, M)(S) \), we compute the fiber

\[
T_{f} \mathsf{Mfld}(N, M) \cong \Gamma(S \times N, f^{*}TM),
\]

where on the right hand side \( f \) denotes the image under the adjunction \( \mathsf{Mfld}(S, \mathsf{Mfld}(N, M)) \cong \mathsf{Mfld}(S \times N, M) \). In this subsection we apply this construction to the stack of classical vacua.

We begin by describing this tangent stack in terms of the groupoid \( \mathbb{R}_{>0} \times \mathsf{SMfld}(\mathbb{R}^{0}[1], X)/(S^{1}[1] \times \mathbb{Z}/2) \). Let \( \mathbb{R}_{>0} \times \mathbb{R}^{[1]} \) denote the relevant universal family of super circles, i.e., the family with \( S = \mathbb{R}_{>0} \) and \( R \in \mathbb{R}_{>0}(\mathbb{R}_{>0}) \) the identity map. The corresponding universal bundle for \( \Phi^{[1]}(X) \) is equipped with an evaluation map

\[
(\mathbb{R}_{>0} \times \mathbb{R}^{[1]}) \times \mathsf{SMfld}(\mathbb{R}^{0}[1], X) \xrightarrow{\text{proj} \times \text{id}} \mathbb{R}_{>0} \times \mathbb{R}^{0}[1] \times \mathsf{SMfld}(\mathbb{R}^{0}[1], X) \xrightarrow{\text{id} \times ev} \mathbb{R}_{>0} \times X.
\]

The evaluation map is invariant under the action of the Euclidean group. Hence, the above determines a morphism of super Lie groupoids and therefore a morphism of stacks. In an abuse of notation, we denote this morphism by \( ev \).

There is a morphism of stacks

\[
p: \mathbb{R}_{>0} \times \mathbb{R}^{[1]} \times \mathsf{SMfld}(\mathbb{R}^{0}[1], X)/(\mathbb{T}^{[1]} \times \mathbb{Z}/2) \to \Phi^{[1]}_{0}(X)
\]

induced by the projection

\[
\mathbb{R}_{>0} \times \mathbb{R}^{[1]} \times \mathsf{SMfld}(\mathbb{R}^{0}[1], X) \to \mathbb{R}_{>0} \times \mathsf{SMfld}(\mathbb{R}^{0}[1], X).
\]

We consider the pullback bundle \( ev^{*}T(\mathbb{R}_{>0} \times X) \cong ev^{*}(T\mathbb{R}_{>0} \oplus TX) \) and take sections along the projection \( p \), which we denote by \( p_{s} \). Intuitively, a section of \( p_{s} ev^{*}TX \) is an infinitesimal deformation of a map \( \phi: (\mathbb{R}_{>0} \times \mathbb{R}^{[1]})/\mathbb{Z} \to X \), and a section of \( ev^{*}T\mathbb{R}_{>0} \) is an infinitesimal deformation of the lattice parameter \( R = (r, 0) \) defining the family of curves. In the following we will only need to consider deformations of the first type, so we define a tangent bundle

\[
\mathcal{T}(\Phi^{[1]}_{0}(X)) := p_{s} ev^{*}TX,
\]

where \( p_{s} \) denotes taking sections along the fiber of the projection \( p \) above. For various normalizations that follow, we will also require the bundle

\[
\tilde{Z}_{n}(\Phi^{[1]}_{0}(X)) := p_{s} ev^{*}\mathbb{R}^{n},
\]

where \( \mathbb{R}^{n} \to X \) is the trivial bundle.

We observe that an S-point of \( \mathcal{T}(\Phi^{[1]}_{0}(X)) \) is a pair \( (R, \phi) \in \Phi^{[1]}_{0}(X)(S) \) together with a section \( v \in \Gamma(S \times R \mathbb{R}^{[1]}, \phi^{*}TX) \). In fact, this S-point description gives an equivalent definition of \( \mathcal{T}\Phi^{[1]}_{0}(X) \), which comes equipped with an obvious inclusion into a similarly defined stack \( \mathcal{T}\Phi^{[1]}_{0}(X) \) consisting of S-points \( (R, \phi) \in \Phi^{[1]}_{0}(X) \) and sections \( v \in \Gamma(S \times R \mathbb{R}^{[1]}, \phi^{*}TX) \). This inclusion is precisely the one coming from the restriction of the tangent bundle of \( \Phi^{[1]}(X) \) to the substack \( \Phi^{[1]}_{0}(X) \). There is a similar S-point description for \( \tilde{Z}_{n}(\Phi^{[1]}_{0}(X)) \), replacing \( TX \) by \( \mathbb{R}^{n} \).
The Riemannian metric and Levi-Civita connection on $TX$ pull back along $\text{ev}$ to give a metric and connection that we will utilize below. Similarly, the standard metric and trivial connection on $\mathbb{R}^n$ can be pulled back along $\text{ev}$.

4.2. An exponential map on sections of $T\Phi^{11}_0(X)$. We now will describe an exponential map $T\Phi^{11}_0(X) \to \Phi^{11}(X)$; this is essentially a reproduction of E. Witten’s discussion on page 482 of [DEF+99] and serves to motivate our definition of $\mathcal{N}\Phi^{11}_0(X)$ in the next subsection.

Given a section $\nu \in \Gamma(S \times_R \mathbb{R}^{11}, \phi^*TX)$ along a map $\phi$ that factors through the projection to $S \times \mathbb{R}^{01}$, we will use the Levi-Civita connection on $TX$ and the exponential map of the Riemannian manifold $X$ to define a map $S \times_R \mathbb{R}^{11} \to X$. We identify $\nu$ with a map $S \times \mathbb{R}^{11} \to TX$ satisfying a periodicity condition. Consequently, for any compact subset of $S$ the image of $\nu$ is a compact subset of $TX$. We consider the composition

$$[0, \delta] \times S \times \mathbb{R}^{11} \to TX \xrightarrow{\exp} X$$

for exp the exponential map with respect to the Riemannian metric; there exists a fixed $\delta$ for which this exponential map is defined provided that $S$ is compact, or if one insists on noncompact families we may choose a smooth strictly positive function on $S$ we also denote by $\delta$, and we consider the bundle of compact intervals $[0, \delta] \times S \subset \mathbb{R} \times S$ over $S$. We denote the family of maps parametrized by $[0, \delta] \times S$ defined by the composition above by $\phi + \delta \nu$.

4.3. The normal bundle to the stack of classical vacua. To define the normal bundle to the inclusion, we need to identify the orthogonal complement to sections $\nu$ such that $\phi + \delta \nu \in \Phi^{11}_0(X) \subset \Phi^{11}(X)$, i.e., sections whose image under the exponential map remains in the substack $\Phi^{11}_0(X)$. Such sections are precisely constant ones, meaning those in the kernel of $(\phi^*\nabla)_{\partial_t}$. The metric on $TX$ and the standard volume form on $\mathbb{R}$ allow us to take the orthogonal complement of these sections, and we observe that this subspace is invariant under the action of super Euclidean isometries because $\partial_t$ is even (so fixed by the $\mathbb{Z}/2$-action) and $[\partial_t, D] = 0$ where $D = \partial_\theta - i\theta \partial_t$ is the odd generator of the super Lie algebra of translations. Hence, this gives a well-defined vector bundle over the stack $\Phi^{11}_0(X)$.

Definition 4.3. Define $\mathcal{N}\Phi^{11}_0(X) \subset T\Phi^{11}_0(X)$ as having S-points sections in the orthogonal complement of the constant sections.

Similarly for the bundle $\tilde{Z}_n(\Phi^{11}_0(X))$, we make the following definition.

Definition 4.4. Define $Z_n(\Phi^{11}_0(X)) \subset \tilde{Z}_n(\Phi^{11}_0(X))$ as having $S$-points sections in the orthogonal complement of the constant sections.

4.4. The linearized classical action functional. In this subsection we define a functional on sections of $\mathcal{N}\Phi^{11}_0(X)$ that will be used to construct the $\hat{A}$-genus. Let

$$S_{\text{lin}}(\phi, \nu) :=\int_{S \times S^{11}} \langle (\phi^*\nabla)_D \nu, (\phi^*\nabla)_D \nu \rangle \, dt d\theta$$

where $D = \partial_\theta - i\theta \partial_t$ is the left-invariant vector field on the family $S \times S^{11}$, and the integral is the Berezinian integral over the fibers of the projection $S \times_R \mathbb{R}^{11} \to S$. Since we have constructed the above out of right-invariant vector fields on $\mathbb{R}^{11}$, the functional is automatically invariant under the left-action of the Euclidean group, which is the relevant action to construct an invariant function on $\Phi^{11}_0(X)$ when viewed as a subspace of a mapping space.

Notation 4.5. We will make repeated use of the identification between differential forms on $X$ and functions on $\pi TX \cong \text{SMfld}(\mathbb{R}^{01}, X)$. We will often want to emphasize the role of a form $\omega \in \Omega^*(X)$ as a natural transformation from maps $\phi : S \times \mathbb{R}^{01} \to X$ to functions on $S$, where we denote the Taylor expansion of $\phi$ by $x_0 + \theta \psi_0$. In this case we will write $\omega(\psi_0, \psi_0, \ldots, \psi_0)$. Another way of describing $\omega(\psi_0, \psi_0, \ldots, \psi_0) \in C^\infty(\pi TX)$ is the
pullback of $\omega$ along the projection $\pi TX \to X$ postcomposed with $k$-fold iterated contraction with the odd vector field on $\pi TX$ determined by the de Rham operator, where $\deg(\omega) = k$. Since $\pi TX \cong \text{SMfld}(\mathbb{R}^{0|1}, X)$ is the universal case, this function pulls back to any $S$.

We will require an identity coming from geometry of vector bundles over supermanifolds.

**Lemma 4.6.** Let $\phi_0: S \times \mathbb{R}^{0|1} \to X$ be a map with Taylor expansion $\phi_0 = x_0 + \theta \psi_0$ and $i_0: S \to S \times \mathbb{R}^{0|1}$ the inclusion at $\theta = 0$. The $C^\infty(S)[\theta]$-linear map

$$v \mapsto i_0^*v + \theta i_0^*((\phi_0^*\nabla_{\partial_0})v), \quad \Gamma(S \times \mathbb{R}^{0|1}, \phi_0^*E) \to \Gamma(S, x_0^*E)[\theta] \cong \Gamma(S \times \mathbb{R}^{0|1}, \phi_0^*x_0^*E)$$

gives an isomorphism of vector bundles over $S \times \mathbb{R}^{0|1}$. The image of $\phi^*\nabla_{\partial_0}$ along this inverse of this isomorphism is the operator $\partial_0 + \theta F(\psi_0, \psi_0)$, where $F(\psi_0, \psi_0)$ denotes the curvature of the connection $\nabla$ viewed as an $\text{End}(x_0^*E)$-valued function on $S$.

**Proof.** By the Yoneda lemma, the universal case is $S = \pi TX = \text{SMfld}(\mathbb{R}^{0|1}, X)$ and the universal map to $X$ is $\phi_0 = \text{ev}: \mathbb{R}^{0|1} \times \text{SMfld}(\mathbb{R}^{0|1}, X) \to X$, the evaluation map. Identifying functions on $\pi TX$ with differential forms on $X$, $\text{ev}^*: C^\infty(X) \to C^\infty(\mathbb{R}^{0|1} \times \pi TX)$ is given by $f \mapsto f + \theta df$ where $d$ is the de Rham operator. To avoid notational confusion below with the trivial connection, we denote the evaluation map by $f \mapsto f + \theta L_d f$. The notation comes from identifying the de Rham operator on $X$ with the Lie derivative with respect to the odd vector field on $\pi TX$ that when viewed as a derivation on functions is the de Rham operator. We emphasize that $L_d$ is an odd derivation of cohomological degree zero.

To compute $\phi_0^*\nabla_{\partial_0}$ on $\Gamma(S \times \mathbb{R}^{0|1}, \phi_0^*E)$ we use a modified version of a computation of F. Han; see Theorem 4.3 of [Han08]. Locally on $X$ we write $\nabla = d + A$ for $d$ the de Rham operator twisted by a trivialization of the bundle $E$ and $A$ an $\text{End}(E)$-valued 1-form, so $\text{ev}^*\nabla = d + \text{ev}^* A$. Locally we may diagonalize the endomorphism part of $A$, so it suffices to compute $\text{ev}^* a$ for $a$ a 1-form. In turn, it suffices to consider the case $a = B d\psi C$ for $B$ and $C$ functions on $X$:

$$(\text{ev}^* B) d(\text{ev}^* C) = (B + \theta L_d B) d(C + \theta L_d C)$$

$$= (B + \theta L_d B) dC + (B + \theta L_d B) d\theta(L_d C) + (B + \theta L_d B) d\theta d(L_d C).$$

We observe that $B L_d C$ is precisely the 1-form $a$ viewed as a function on $\pi TX$, and $L_d B L_d C$ is the 2-form $da$ viewed as a function on $\pi TX$; respectively these are denoted by $a(\psi_0)$ and $da(\psi_0, \psi_0)$. We have $\phi_0^*\nabla_{\partial_0} = i_{\partial_0}(\phi_0^*\nabla)$, where $i$ denotes contraction with a vector field. This picks out the part of $\text{ev}^* A$ involving $d\theta$, so we obtain

$$(\phi_0^*\nabla_{\partial_0}) = \partial_0 + A(\psi_0) + \theta dA(\psi_0, \psi_0).$$

Next we turn attention to the claimed isomorphism of vector bundles. One can deduce that the asserted map is a bijection abstractly, but we will require an explicit inverse that is locally given by the formula

$$v \mapsto v - \theta A(\psi_0) v, \quad \Gamma(S, x_0^*E) \to \Gamma(S \times \mathbb{R}^{0|1}, \phi_0^*E),$$

using the same notation as above. One can verify directly that the image of $v$ is in the kernel of $i_0^* \circ (\phi_0^*\nabla_{\partial_0})$ so that the map above is injective with left inverse the map in the statement of the lemma. Extending $C^\infty(\mathbb{R}^{0|1})$-linearly, we observe that the requisite compositions are identities so that this map is the claimed inverse.

Now we compute the image of the $\phi_0^*\nabla_{\partial_0}$ along the isomorphism described above, which we denote by $h$, as

$$h(\phi_0^*\nabla_{\partial_0}(h^{-1}(v))) = h(\phi_0^*\nabla_{\partial_0}(v - \theta A(\psi_0)v))$$

$$= h(\partial_0 v - \theta A(\psi_0)\partial_0 v - A(\psi_0)v + A(\psi_0)v)$$

$$+ \theta A(\psi_0) A(\psi_0) v + \theta dA(\psi_0, \psi_0)v)$$

$$= h(\partial_0 v + \theta F(\psi_0, \psi_0)v - \theta A(\psi_0)\partial_0 v),$$

$$= \partial_0 v + \theta F(\psi_0, \psi_0)v$$

yielding the desired operator in the universal case, and hence for arbitrary $S$-families. □
Notation 4.7. In the case at hand in this section, \( \nabla \) will be the Levi-Civita connection on \( X \), and we denote the image of \( \phi_0^* \nabla \) as in Lemma 4.6 by \( \partial_\theta - \theta R \). We use the same notation for the pullback of the operator along the projection \( \mathbb{S} \times R \mathbb{R}^{1|1} \to R \mathbb{R}^{0|1} \), so that we obtain a formula for \( \phi^* \nabla \) for \( \phi : \mathbb{S} \times R \mathbb{R}^{1|1} \to X \) a map factoring through the projection. The isomorphism in Lemma 4.6. is a type of Taylor expansion on sections, where we observe that \( \phi_0^* \theta_0^* E = \phi_0^* \theta_0^* \) evaluates at \( \theta = 0 \). For a section \( \nu \in \Gamma(\mathbb{S} \times R \mathbb{R}^{1|1}, \phi^* TX) \), we denote this expansion (i.e., the image under the isomorphism specified by Lemma 4.6) by \( a + \theta \eta \). For simplicity, we will sometimes write \( \nu = a + \theta \eta \).

Now we are ready to express the linearized classical action in terms of familiar geometric quantities. Since the map \( x_0 \) is independent of \( t \), we observe that \( \phi^* \nabla_{D^2} = \phi^* \nabla_{-i \partial_\theta} = -id/dt \) on the sections \( a \) and \( \eta \) for the chosen Taylor decomposition of \( \nu \) (which, by construction, is compatible with the above formula for the covariant derivative with respect to \( \partial_\theta \)). With these identities in place, we compute

\[
\mathcal{S}_{\text{lin}}(\phi, \nu) = \int_{\mathbb{S} \times \mathbb{S}^{1|1}/\mathbb{S}} (-i \dot{a} - i \theta \dot{\eta}, i \theta \dot{a} + \eta - \theta Ra) d\theta dt
\]

\[
= \int_{\mathbb{S} \times \mathbb{S}^{1}/\mathbb{S}} ((\dot{a}, \dot{a}) + i(\dot{a}, -Ra) + i(\dot{\eta}, \eta)) dt
\]

\[
= \int_{\mathbb{S} \times \mathbb{S}^{1}/\mathbb{S}} (-\bar{\dot{a}}, a) + i(\bar{\dot{a}}, a) + i(\dot{\eta}, \eta)) dt
\]

where \( da/dt = \dot{a}, \ d\eta/dt = \dot{\eta} \), the Berezinian integral is determined by \( \int \theta d\theta = 1 \), and in the last line we have integrated by parts in two places. Hence, we can express the linearized classical action in terms of a pair of operators, \( (\Delta_X^{1|1})_{\text{bos}} \) and \( (\Delta_X^{1|1})_{\text{fer}} \) with

\[
\mathcal{S}_{\text{lin}}(\phi, \nu) = \int_{\mathbb{S} \times \mathbb{S}^{1}/\mathbb{S}} (\langle (\Delta_X^{1|1})_{\text{bos}} a, a \rangle + \langle (\Delta_X^{1|1})_{\text{fer}} \eta, \eta \rangle) dt,
\]

\[
(\Delta_X^{1|1})_{\text{bos}} := -\text{Id}_{TX} \otimes \frac{d^2}{dt^2} + iR \otimes \frac{d}{dt}, \ (\Delta_X^{1|1})_{\text{fer}} := -i \cdot \text{Id}_{TX} \otimes d/dt.
\]

Replacing the Levi-Civita connection in formula 22 by the trivial connection on \( \mathbb{R}^n \) gives rise to a functional on sections of \( Z_n(\Phi_0^{1|1}(X)) \),

\[
\int_{\mathbb{S}^{1}/\mathbb{S}} \langle (\Delta_n^{1|1})_{\text{bos}} a, a \rangle + \langle (\Delta_n^{1|1})_{\text{fer}} \eta, \eta \rangle dt, \quad (\Delta_n^{1|1})_{\text{bos}} := -\text{Id}_n \otimes \frac{d^2}{dt^2}, \ (\Delta_n^{1|1})_{\text{fer}} := -i \cdot \text{Id}_n \otimes d/dt.
\]

Remark 4.8. In this remark we will make explicit the induced pairing on \( \pi TX \) coming from the metric on \( X \). Following our conventions, functions are complex valued and bundles are by definition modules over functions, so all bundles are taken over \( \mathbb{C} \), i.e., \( TX \) denotes the complexification of the real tangent bundle. The bundle \( \pi TX \) is the parity reversed projective module over the \( \mathbb{C} \)-algebra \( C^\infty(X) \) coming from \( TX \). Explicitly, the module associated to \( \pi TX \) is \( \Gamma(TX) \otimes C^{0|1} \) where \( C^{0|1} \) is the odd super line (as a super vector space). The metric pairing is then the composition

\[
(\Gamma(TX) \otimes C^{0|1}) \otimes (\Gamma(TX) \otimes C^{0|1}) \cong (\Gamma(TX) \otimes \Gamma(TX)) \otimes (C^{0|1} \otimes C^{0|1})
\]

\[
\cong \Gamma(TX) \otimes \Gamma(TX) \overset{\partial_\theta}{\to} \Gamma(TX)
\]

where the first isomorphism is the braiding isomorphism for the tensor product of super vector spaces, the second isomorphism uses the isomorphism of super vector spaces \( C^{0|1} \otimes C^{0|1} \cong \mathbb{C} \), and the last map is the conjugate linear extension of the Riemannian metric \( g \) on \( X \) to the complexified tangent bundle.

4.5. The \( \hat{A} \)-class as a relative \( \zeta \)-determinant.

Proof of Proposition 4.7. One can compute the relevant \( \zeta \)-determinants directly (see our proof of Proposition 4.2), but for the sake of variety we will reduce the computation to one
very similar to those that arise from Feynman diagrams. This will result in the $\hat{A}$-class as the genus associated to the characteristic series on the right hand side below:

$$x/2 \sinh(x/2) = \exp \left( \sum_{k=1}^{\infty} \frac{x^{2k}}{2k(2\pi i)^{2k}} 2\zeta(2k) \right).$$

Pointwise, the $\zeta$-determinants of the operators $\text{Id}_{TX} \otimes d^{2}/dt^{2}$ and $\text{Id}_{TX} \otimes d/dt$ only depend on $TX$ as a metric space of dimension $n$. We recall that for $A$ a determinant class Fredholm operator, there is an equality $\det_{\zeta}(AB) = \det_{Fr}(A)\det_{\zeta}(B)$ where $\det_{Fr}(A)$ denotes the Fredholm determinant. Using this feature of $\zeta$-determinants, we obtain

$$\frac{s\det_{\zeta}(\Delta^{11}_{TX})}{s\det_{\zeta}(\Delta^{11}_{n})} = \det_{Fr} \left( \text{Id}_{TX} \otimes \text{id} - i\mathcal{R} \otimes \left( \frac{d}{dt} \right)^{-1} \right)^{-1/2}$$

where we will show below that the right hand side is indeed a determinant class operator.

We use functional calculus to compute

$$\left( \det_{Fr} \left( \text{id} - \mathcal{R} \otimes \left( \frac{d}{dt} \right)^{-1} \right) \right)^{-1/2} = \exp \left( -\frac{1}{2} \text{Tr} \left( \log \left( 1 - i\mathcal{R} \otimes \left( \frac{d}{dt} \right)^{-1} \right) \right) \right) = \exp \left( \sum_{k=1}^{\infty} \frac{\text{Tr} \left( i\mathcal{R} \otimes \left( \frac{d}{dt} \right)^{-1} \right)^{k}}{2k} \right)$$

The result will now follow from a pair of computations, using the fact that

$$\text{Tr} \left( (i\mathcal{R} \otimes (d/dt)^{-1})^{k} \right) = i^{k} \text{Tr} (\mathcal{R}^{k}) \cdot \text{Tr}((d/dt)^{-k}).$$

where the two traces on the right hand side are over sections of the pullback of $TX$ and the orthogonal complement of the constant functions in $C^{\infty}(S^{1})$, respectively. To compute the trace of $(d/dt)^{-k}$ we choose the basis $\{\exp(2\pi in/r)\}_{n \in \mathbb{Z}}$ with $k \neq 0$ for the orthogonal complement of constant functions in $C^{\infty}(S^{1})$ and obtain the absolutely convergent series

$$\text{Tr}((d/dt)^{-2k}) = \sum_{n \neq 0} \frac{1}{(2\pi in)^{2k}} = \frac{2\pi^{2k}}{(2\pi i)^{2k}} \sum_{n \in \mathbb{N}} \frac{1}{n^{2k}} = \frac{2\pi^{2k}\zeta(2k)}{(2\pi i)^{2k}}$$

where $\zeta$ is the Riemann zeta function. Note that the trace of any odd power of $d/dt$ vanishes by spectral symmetry.

For $F$ the curvature 2-form of a real vector bundle $V$ with connection, the differential-form valued Pontryagin character of $V$ is defined by

$$\text{Tr}(F^{2k}) = (2k)! (2\pi i)^{2k} \text{ch}_{2k}(V \otimes \mathbb{C}) = (2k)! (2\pi i)^{2k} \text{ph}_{k}(V),$$

where $\text{ch}_{2k}$ denotes the $4k^{th}$ component of the Chern character, and $\text{ph}_{k}$ denotes the $4k^{th}$ component of the Pontryagin character. In our cochain model, following considerations in subsection 2.6 we have

$$(ir)^{2k} (1/2) \text{Tr} (\mathcal{R})^{2k} = (2k)! \text{ph}_{k}(TX),$$

where (in an abuse of notation) the right hand side above denotes the $k^{th}$ component of the Pontryagin character as a function on $\Phi^{11}_{0}(X)$. We observe that the left hand side is a supersymmetric function on $\Phi^{11}_{0}(X)$. Putting this together we get

$$\frac{s\det_{\zeta}(\Delta^{11}_{TX})}{s\det_{\zeta}(\Delta^{11}_{n})} = \exp \left( \sum_{k=1}^{\infty} \frac{(2k)! \text{ph}_{k}(TX) \cdot 2\zeta(2k)}{2k \cdot (2\pi i)^{2k}} \right)$$

which we identify as the $\hat{A}$-class on $X$ as a supersymmetric function on $\Phi^{11}_{0}(X)$. □

Remark 4.9. We will have some use below for the square root of the $\zeta$-determinant of $d^{2}/dt^{2} \otimes \text{Id}_{TX}$: as a function on $\mathbb{R}_{>0} \times \pi TX$ it is $r^{n}$, which follows from Example 2 of [QHS93].
4.6. The $\hat{A}$-class as a trivialization of a Quillen determinant line. The operator $\Delta_X^{1|1}$ can also be viewed as a perturbation of the Laplacian on $S^1$ acting on even sections and of the Dirac operator on $S^1$ acting on odd sections, where the perturbation is parametrized by the supermanifold $\mathbb{R}_{>0} \times \pi TX$. In this section we will explain how the $\hat{A}$-class arises when trivializing a Bismut-Freed-Quillen (super) determinant line over a super family.

To begin, we define the relevant determinant and Pfaffian lines in question. The restriction of our family of operators to the reduced manifold is the family $d^2/dt^2 \otimes \text{Id}_{TX}$ over $\mathbb{R}_{>0} \times X \subset \mathbb{R}_{>0} \times \pi TX$, that over $(r,x) \in \mathbb{R}_{>0} \times X$ is the operator $d^2/dt^2$ acting on functions on the circle of radius $r$ twisted by the tangent space $T_x X$. So on the reduced manifold we define line bundles

$$s\text{Det}(|\Delta_X^{1|1}|) := \text{Pf}(d/dt \otimes \text{Id}_{TX}) \otimes \text{Det}(d/dt \otimes \text{Id}_{TX})^{-1},$$
$$s\text{Det}(|\Delta_n^{1|1}|) := \text{Pf}(d/dt \otimes \text{Id}_{\mathbb{R}^{2n}}) \otimes \text{Det}(d/dt \otimes \text{Id}_{\mathbb{R}^{2n}})^{-1}.$$ 

Notice that we have taken the square root of the Laplacian to define the determinant line on the bosonic piece, in accordance with equation $\langle 1 \rangle$. Because the operators in question are invertible, these lines bundles have nonvanishing super determinant sections,

$$s\text{det}(|\Delta_X^{1|1}|) := \frac{\text{Pf}(d/dt \otimes \text{Id}_{TX})}{\text{det}(d/dt \otimes \text{Id}_{TX})}, \quad s\text{det}(|\Delta_n^{1|1}|) := \frac{\text{Pf}(d/dt \otimes \text{Id}_{\mathbb{R}^{2n}})}{\text{det}(d/dt \otimes \text{Id}_{\mathbb{R}^{2n}})}.$$ 

We define the super determinant line over the super family as the pullback along the projection $p: \mathbb{R}_{>0} \times \pi TX \to \mathbb{R}_{>0} \times X$ of the above lines:

$$s\text{Det}(|\Delta_X^{1|1}|) := p^*s\text{Det}(|\Delta_X^{1|1}|), \quad s\text{Det}(|\Delta_n^{1|1}|) := p^*s\text{Det}(|\Delta_n^{1|1}|)$$

and the super determinant sections are similarly defined by pullback. The metric on these super determinant lines is a rescaling of the pullback metrics,

$$g_{s\text{Det}(|\Delta_X^{1|1}|)} := \frac{\text{det}_c((d^2/dt^2 \otimes \text{Id}_{TX})(d^2/dt^2 \otimes \text{Id}_{TX})^\ast)^{1/2}}{\text{det}_c((d^2/dt^2 \otimes \text{Id}_{TX} + id/dt\mathcal{R})(d^2/dt^2 \otimes \text{Id}_{TX} + id/dt\mathcal{R})^\ast)^{1/2}},$$
$$g_{p^*s\text{Det}(|\Delta_X^{1|1}|)} := p^*g_{s\text{Det}(|\Delta_X^{1|1}|)}.$$ 

We observe that the norm of the super determinant section $s\text{det}(|\Delta_X^{1|1}|)$ over $\mathbb{R}_{>0} \times \pi TX$ is the inverse square root of the $\zeta$-determinant of the bosonic part of $\Delta_X^{1|1}$ multiplied by the $\zeta$-pfaflan of the fermionic part of $\Delta_X^{1|1}$. Thus, our definition leads to a simple generalization of the Bismut-Freed formula for the norm of the super determinant section over a super family.

**Proof of Proposition 4.2.** In order to trivialize the relative determinant line, it suffices to specify a unit norm real section. The section $s\text{det}(|\Delta_X^{1|1}|)$ is real and nonvanishing. Its norm squared can be calculated directly through the definition of the metric on $s\text{det}(|\Delta_X^{1|1}|)$ as

$$\|s\text{det}(\Delta_X^{1|1})\|^2 = \frac{\|\text{det}_c((d^2/dt^2 \otimes \text{Id}_{TX})(d^2/dt^2 \otimes \text{Id}_{TX})^\ast)^{1/2}}{\|\text{det}_c((d^2/dt^2 \otimes \text{Id}_{TX} + id/dt\mathcal{R})(d^2/dt^2 \otimes \text{Id}_{TX} + id/dt\mathcal{R})^\ast)^{1/2}}.$$ 

The norm squared of the relative super determinant is

$$\frac{\|s\text{det}(\Delta_X^{1|1})\|^2}{\|s\text{det}(\Delta_n^{1|1})\|^2} = \frac{\|\text{det}_c((d^2/dt^2 \otimes \text{Id}_{\mathbb{R}^{2n}})(d^2/dt^2 \otimes \text{Id}_{\mathbb{R}^{2n}})\ast)^{1/2}}{\|\text{det}_c((d^2/dt^2 \otimes \text{Id}_{TX} + id/dt\mathcal{R} + id/dt\mathcal{R})^\ast)^{1/2}}.$$ 

$$= \frac{\|\text{det}_c((d^2/dt^2 \otimes \text{Id}_{TX}) + id/dt\mathcal{R})\ast\text{det}_\mathcal{F}(1 + id/dt\mathcal{R})\ast\text{det}_\mathcal{F}(1 - id/dt\mathcal{R})\|^2}{\|\text{det}_\mathcal{F}(1 + id/dt\mathcal{R})\ast\text{det}_\mathcal{F}(1 - id/dt\mathcal{R})\|^2}.$$ 

$$= \hat{A}(X) \cdot \hat{A}(X)$$

where the last line follows from the computation in the proof of Proposition 4.1. Finally, we observe that $\hat{A}(X)$ is a real-valued function on $\mathbb{R}_{>0} \times \pi TX$, and so the proposed section is indeed real and of unit norm, as required. 

□
4.7. A local index map for 1|1-EFTs. When $X$ is oriented, the supermanifold $\pi TX$ has a canonical volume form coming from integration of differential forms on $X$. Any nonvanishing function on $\pi TX$ can be used to modify the volume form. More generally, given a function on $\mathbb{R}_{>0} \times \pi TX$, we may define a relative volume form for the projection, $\mathbb{R}_{>0} \times \pi TX \rightarrow \mathbb{R}_{>0}$.

It follows from Remark 4.9 that the $\zeta$-super determinant of $\Delta^{1|1}_X$ (not the relative determinant) is $r^{-n/2}A(X)$. The relative volume form determined by this function gives a map between supersymmetric sections,

$$\Gamma_{\text{susy}}(\Phi^{1|1}_0(X); \kappa^k) \rightarrow \Gamma_{\text{susy}}(\Phi^{1|1}_0(pt); \kappa^{k-n})$$

where $\int_X$ denotes integration of differential forms over $X$ and $\dim(X) = n$. As such, sdet$_{\zeta}(\Delta^{1|1}_X)$ determines a sort of supersymmetric volume form along the fibers of $\Phi^{1|1}_0(X) \rightarrow \Phi^{0|1}_0(pt)$. This integration can be identified with the wrong-way map coming from the spin orientation of KO tensored with $\mathbb{C}$. Furthermore, the total volume of $\Phi^{1|1}_0(X)$ with respect to this choice of volume for is the $A$-genus of $X$.

We obtain an index map for 1|1-EFTs over an oriented manifold $X$ by precomposing with the restriction map that evaluates a field theory on the super circles over $X$ defining the stack of classical vacua:

$$1|1-\text{EFT}^k(X) \rightarrow \Gamma_{\text{susy}}(\Phi^{1|1}_0(X); \kappa^k) \rightarrow \Gamma_{\text{susy}}(\Phi^{0|1}_0(pt); \kappa^{k-n}) = \begin{cases} \mathbb{C} & k - n \text{ even} \\ 0 & k - n \text{ odd} \end{cases}$$

For field theories over $X$ defined in terms of vector bundles with connection, we recounted in subsection 2.6 that the first map gives $\text{Tr}(\exp(-irF)) \in C^{\infty}_{\text{susy}}(\Phi^{1|1}_0(X))$ for $F$ the curvature of the connection. Tracing through the various factors of $2\pi$ and $r$, we see that its image under the above composition is the usual index map for the Atiyah-Singer theorem, namely we obtain the $A$-genus of $X$ twisted by the Chern character of the bundle.

5. The perturbative 2|1-sigma model and the Witten genus

Define the Witten class of a Riemannian manifold $X$ to be the polynomial in Pontryagin forms defined by the characteristic series $[1]$, and denote this closed, even differential form with coefficients in holomorphic functions on based lattices with Wit($X$). If the first Pontryagin class of $X$ vanishes, Wit($X$) determines a class $[\text{Wit}(X)] \in \text{TMF}^0(X) \otimes \mathbb{C}$.

The proof of Theorem 1.1 allows us to view Wit($X$) as a function on the $\text{SL}_2(\mathbb{Z})$-cover $\tilde{\Phi}^{2|1}_0(X)$ of $\Phi^{0|1}_0(X)$, but since the 2nd Eisenstein series is not a modular form (i.e., is not $\text{SL}_2(\mathbb{Z})$-invariant) this function does not descend to a function on $\Phi^{2|1}_0(X)$ unless the first Pontryagin form happens to vanish. In general the Witten class is a section of a line bundle over $\tilde{\Phi}^{2|1}_0(X)$; we will construct a trivialization of this line determined by a choice of rational string structure on $X$.

The first goal of this section is to prove Theorem 1.4 by constructing the (possibly nonmodular) Witten class as a relative the $\zeta$-determinant of a family of operators, denoted $\Delta^{2|1}_X$, parametrized by $\tilde{\Phi}^{2|1}_0(X)$. This is a straightforward generalization of the 1|1-dimensional case. The second goal is to prove Theorem 1.11 by constructing the relative determinant line bundle of $\Delta^{2|1}_X$, and explain how a trivialization can be built from the Witten class together with a rational string structure.

5.1. The tangent space of fields restricted to the stack of classical vacua. We begin by describing the tangent stack to classical vacua in terms of the groupoid $L \times \text{SMfld}(\mathbb{R}^{0|1}, X)/(\mathbb{T}^{2|1} \rtimes \mathbb{C}^\times)$. The universal bundle of super tori over this stack is equipped with an evaluation map

$$L \times_A \mathbb{R}^{2|1} \times \text{SMfld}(\mathbb{R}^{0|1}, X) \rightarrow L \times \mathbb{R}^{0|1} \times \text{SMfld}(\mathbb{R}^{0|1}, X) \stackrel{\text{proj} \times \text{id}_{\text{ev}}}{\longrightarrow} L \times X.$$
Hence, the above evaluation map determines a morphism of super Lie groupoids and hence a morphism of stacks. In an abuse of notation, we denote this morphism by ev. There is a morphism of stacks

\[ p: L \times_A \mathbb{R}^{2|1} \times SMfld(\mathbb{R}^{0|1}, X)/(\mathbb{T}^{1|1} \times \mathbb{Z}/2) \to \Phi^{2|1}_0(X) \]

coming from a morphism of super Lie groupoids induced by the projection,

\[ L \times_A \mathbb{R}^{2|1} \times SMfld(\mathbb{R}^{0|1}, X) \to L \times SMfld(\mathbb{R}^{0|1}, X). \]

We consider the pull back bundle \( ev^* TX \) coming from a morphism of super Lie groupoids induced by the projection, \( \mathbb{R}^{2|1} \to \mathbb{R}^{0|1} \). Geometrically, a section of \( ev^* TX \) is an infinitesimal deformation of a map \( \phi: (L \times \mathbb{R}^{2|1})/\mathbb{Z} \to X \), and a section of \( ev^* TL \) is an infinitesimal deformation of the lattice parameter \( L \) defining the family of curves. In the following we will only need to consider deformations of the first type, so we define

\[ T(\Phi^{2|1}_0(X)) := p_* ev^* TX, \]

where \( p_* \) denotes taking sections along the fiber of the projection. We also define

\[ \mathcal{Z}_n(\Phi^{2|1}_0(X)) := p_* ev^* \mathbb{R}^n, \]

which will be helpful for normalizing determinants.

**Remark 5.1.** The notation \( \mathcal{Z}_n \) is intended to remind one of the normalizing constants, typically denoted \( Z \), appearing in path integral computations.

**5.2. The normal bundle to the stack of classical vacua.** In an identical manner to the 1|1-dimensional case, for pair \((\phi, \nu) \in T\Phi^{2|1}_0(X)(S)\) the composition

\[ [0, \delta] \times S \times \mathbb{R}^{2|1} \xrightarrow{\phi} TX \xrightarrow{\exp} X \]

with the Riemannian exponential map on \( X \) yields an exponential map on sections of \( T\Phi^{2|1}_0(X) \). As before, we denote the \([0, \delta] \times S\)-family of maps determined by the composition above by \( \phi + \delta \nu \). Using the metric on \( TX \) and the volume form on \( \mathbb{R}^2 \), we take the orthogonal complement of the sections that are in the kernel of both \((\phi^* \nabla)_{\partial_x}\) and \((\phi^* \nabla)_{\partial_z}\). Since these operators commute with the super Lie algebra of \( \mathbb{R}^{2|1} \) and the kernel is preserved by the dilation action of \( \mathbb{C}^\times \) on \( \mathbb{R}^{2|1} \), the orthogonal complement defines a vector bundle over the stack \( \Phi^{2|1}_0(X) \).

**Definition 5.2.** Define \( N\Phi^{2|1}_0(X) \subset T\Phi^{2|1}_0(X) \) as having \( S \)-points sections in the orthogonal complement of the constant sections.

**Definition 5.3.** Define \( Z_n(\Phi^{2|1}_0(X)) \subset \mathcal{Z}_n(\Phi^{2|1}_0(X)) \) as having \( S \)-points sections in the orthogonal complement of the constant sections.

**5.3. The linearized classical action.** Let

\[ S_{\text{lin}}(\phi, \nu) := \int_{S \times \mathbb{R}^{2|1}/S} \langle (\phi^* \nabla)_{Dz} \nu, (\phi^* \nabla)_{D\bar{z}} \nu \rangle d\theta d\bar{z}. \]

where \( D = \partial_\theta - i \partial_{\bar{z}} \) is the usual right-invariant vector field on the family \( S \times \mathbb{R}^{2|1} \). Since we have defined the above using right-invariant vector fields, \( S_{\text{lin}} \) is automatically left-invariant under the action of translations, \( \mathbb{T}^{2|1} \). Using the same argument as in Remark 2.2, \( S_{\text{lin}} \) is also invariant under the \( \mathbb{C}^\times \)-action on \( \mathbb{R}^{2|1} \), and so invariant under the action of the full rigid conformal group.

To understand the action functional in terms of more familiar geometric quantities, we proceed as in the 1|1-dimensional case using Lemma 4.6 and that \((\phi^* \nabla)_{\partial_z} = \partial_z\) since the map \( x_0 \) is independent of \( \bar{z} \). We also apply the Taylor expansion \( \nu = a + \theta \eta \) that uses the
Levi-Civita connection on $X$. From this we compute

$$S_{\text{lin}}(\phi, \nu) = \int_{S \times T^2 \times S} \langle -i\partial \zeta (a + \theta \eta), -\theta i\partial \zeta a - \theta R a + \eta \rangle d\theta d\zeta$$

$$\quad = \int_{S \times T^2 \times S} \left( \langle \partial \zeta a, \partial \zeta a \rangle + \langle -i \partial \zeta a, -R a \rangle + i \langle \partial \zeta a, \eta \rangle \right) d\zeta d\zeta$$

$$\quad = \int_{S \times T^2 \times S} \left( -\left( a, \frac{\partial^2}{\partial \zeta^2} a \right) + \left( a, iR \frac{\partial}{\partial \zeta} a \right) + i \left( \eta, \frac{\partial}{\partial \zeta} \eta \right) \right) d\zeta d\zeta$$

Hence, we can express the linearized classical action in terms of a pair of operators, $(\Delta^{21}_X)_{\text{bos}}$ and $(\Delta^{21}_X)_{\text{fer}}$ with

$$S_{\text{lin}}(\phi, \nu) = \int_{T^2 \times S} \langle (\Delta^{21}_n)_{\text{bos}} a, a \rangle + \langle (\Delta^{21}_n)_{\text{fer}} \eta, \eta \rangle dt,$$

$$(\Delta^{21}_n)_{\text{bos}} := -\text{Id}_{TX} \otimes \frac{\partial^2}{\partial \zeta^2} + R \otimes \frac{\partial}{\partial \zeta}, \quad (\Delta^{21}_n)_{\text{fer}} = \text{Id}_{TX} \otimes i \frac{\partial}{\partial \zeta}.$$

Replacing the Levi-Civita connection in formula (24) by the trivial connection on $\mathbb{R}^n$ and repeating the above discussion defines a functional on sections of $Z_n(\Phi_{0}^{21}(X))$ given by the expression

$$\int_{T^2 \times S} \langle (\Delta^{21}_n)_{\text{bos}} a, a \rangle + i \langle (\Delta^{21}_n)_{\text{fer}} \eta, \eta \rangle dt,$$

$$\quad = -\text{Id}_n \otimes \frac{\partial^2}{\partial \zeta^2}, \quad (\Delta^{21}_n)_{\text{fer}} = \text{Id}_n \otimes i \frac{\partial}{\partial \zeta}.$$

5.4. The modular Witten class as a $\zeta$-determinant.

**Proof of Theorem 1.4** We need to compute

$$\frac{\text{sdet}_\zeta(\Delta^{21}_X)}{\text{sdet}_\zeta(\Delta^{21}_n)} = \frac{\text{pf}_\zeta((\Delta^{21}_X)_{\text{fer}})\text{det}_\zeta((\Delta^{21}_n)_{\text{bos}})^{1/2}}{\text{det}_\zeta((\Delta^{21}_X)_{\text{bos}})^{1/2}\text{pf}_\zeta((\Delta^{21}_n)_{\text{fer}}))}$$

The pfaffians can be computed following Example 5 in [QHS93]. However, since the $\zeta$-functions associated to these pfaffians are equal, their overall contribution cancels. The computation of the $\zeta$-determinant of the Laplacian can be found in a variety of places; e.g., Example 9 in [QHS93], and we get

$$\text{det}_\zeta(\partial \zeta \otimes \text{Id}_{\mathbb{R}^n}) = (\bar{\ell}e - \ell \bar{e}) \eta(\ell, \ell')|^{4n}$$

where $\eta(\ell, \ell')$ denotes the Dedekind $\eta$-function.

To compute the final $\zeta$-determinant, we choose a basis of functions on the torus

$$F_{n,m}(\bar{z}, \bar{z}) := \exp \left( \frac{2\pi i}{\ell \ell' - \bar{\ell} \bar{e}} (-z(n \ell + m \bar{e}) + \bar{z}(n \ell + m \bar{e})) \right), \quad (m, n) \in \mathbb{Z} \times \mathbb{Z},$$

and form the $\zeta$-function

$$\zeta(s) := \sum_{(m,n) \in \mathbb{Z}^2} \text{Tr} \left( \text{Id}_{TX} \otimes \frac{4\pi^2}{(\ell \ell' - \bar{e} \bar{e})^2} |m \ell + n \bar{e}|^2 + R \otimes \frac{2\pi}{(\ell \ell' - \bar{e} \bar{e})} (m \ell + n \bar{e})^s \right).$$

We consider the binomial expansion,

$$\zeta(s) = \sum_{(m,n) \in \mathbb{Z}^2} \text{Tr} \left( \left( \text{Id}_{TX} + \frac{(\ell \ell' - \bar{e} \bar{e})}{2\pi} R \otimes (m \ell + n \bar{e})^{-1} \right)^s \left( \frac{4\pi^2}{(\ell \ell' - \bar{e} \bar{e})^2} |m \ell + n \bar{e}|^2 \right)^s \right)$$

$$= \sum_{(m,n) \in \mathbb{Z}^2} \sum_{k=0}^{\text{finite}} \text{Tr} \left( R^k \otimes \frac{s(s-1) \cdots (s-k+1)}{k!(2\pi)^k} (m \ell + n \bar{e})^k (\ell \ell' - \bar{e} \bar{e})^k \left( \frac{2\pi |m \ell + n \bar{e}|}{(\ell \ell' - \bar{e} \bar{e})} \right)^{2s} \right)$$

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where the sum is finite because $\mathcal{R}$ is nilpotent. Focusing on the part of the sum starting from $k = 3$, we differentiate under the sum to obtain the following contribution to $\zeta'(0)$:

$$
\sum_{(n,m) \in \mathbb{Z}^2} \sum_{k=3}^{\infty} \frac{\text{Tr} \left( \mathcal{R}^k \frac{(-1)^{k-1}}{k} \left( \ell' - \ell \right)^k (m\ell + n\ell')^{-k} \right)}{(2\pi)^k} = - \sum_{k=2}^{\infty} \frac{\text{Tr}(\mathcal{R}^{2k})(\ell' - \ell)^{2k} E_{2k}(\ell, \ell')}{2k(2\pi)^{2k}}
$$

where we have used that odd powers of $\mathcal{R}$ have trace zero.

It remains to compute the contribution from the terms with $k = 0$ and $k = 2$. For $k = 0$, we obtain the same $\zeta$-function in the calculation of $\text{det}_x(\partial_x \partial_x \otimes \text{Id}_{\mathbb{R}^n})$, so this cancels in the final relative determinant computation. For $k = 2$, the derivative at $s = 0$ is essentially the definition of the modular 2nd Eisenstein series, and we obtain

$$
\lim_{s \to 0} -\frac{d}{ds} \text{Tr} \left( \mathcal{R}^2 \sum_{(m,n) \in \mathbb{Z}^2} \frac{s(s-1)}{2} \left( \frac{2\pi|m\ell + n\ell'|}{(\ell' - \ell \ell)} \right)^{2s-2} \right) = - \frac{\text{Tr}(\mathcal{R}^2)(\ell' - \ell)^2 E_2(\ell, \ell')}{2(2\pi)^2}.
$$

The standard differential-form valued Chern character (and Pontryagin character) is Equation (23). The initial homomorphism $\mathbb{C} \to \text{MF}^\bullet$ of algebras induces a natural transformation $H^\bullet_{\text{dir}}(X) \to H^\bullet_{\text{dir}}(X; \text{MF}^0)$ that is injective for each $X$; this allows us to consider the Pontryagin character of $X$ as a class in $\otimes_{i+j=\bullet} H^i_{\text{dir}}(X; \text{MF}^3) \simeq \text{TMF}^\bullet(X) \otimes \mathbb{C}$, which in an abuse of notation we also denote by $\text{ph}_k(TX)$. In our cochain model we have

$$
(i(\ell' - \ell \ell))^{2k}(1/2) \text{Tr}(\mathcal{R}^2) = (2k)! \text{ph}_k(TX) \in \Gamma_{\text{susy}}(\Phi^{2|1}_0(X); \omega^{\bullet/2}),
$$

which we will use below to identify traces of the curvature operator with graded components of the Pontryagin character.

Putting together the calculations above, we have

$$
\frac{\text{sdet}_x(\Delta_{2|1}^X)}{\text{sdet}_x(\Delta_{n|1}^X)} = \exp \left( \frac{E_2(\ell, \ell, \ell', \ell')}{(2\pi)^2} \text{ph}_1(TX) + \sum_{k=2}^{\infty} \frac{(2k)! E_{2k}(\ell, \ell')}{2k(2\pi)^{2k}} \text{ph}_k(TX) \right),
$$

which is the modular Witten class as a smooth function on $\Phi^{2|1}_0(X)$.

Viewing a rational string structure as a function $H \in C^\infty(\pi TX)$, we observe that $L_{\partial_b} H = p_1$ as functions on $\pi TX$, where $L_{\partial_b}$ is the Lie derivative along the canonical odd vector field on $\pi TX$. Hence, the function

$$
\frac{\text{sdet}_x(\Delta_{2|1}^X)}{\text{sdet}_x(\Delta_{n|1}^X)} \cdot \exp \left( \frac{E_2(\ell, \ell, \ell', \ell')}{(2\pi)^2} (\ell' - \ell \ell)^2 L_{\partial_b}(tH) \right) \in C^\infty(\Phi^{2|1}_0(X \times \mathbb{R})),
$$

at $t = 0$ is the original relative super determinant, and at $t = 1$ we obtain the modular and holomorphic Witten class of $X$.

\[ \square \]

5.5. The holomorphic Witten class as a trivialization of a determinant line. We now proceed to define the super determinant line of $\Delta_{2|1}^X$. When restricted to the reduced manifold, the operator $\Delta_{2|1}^X$ is the Laplacian on $T^2, -\partial^2/\partial z \partial z$, acting on even sections, and the operator $i \partial_{\bar{z}}$ acting on odd sections. So, mimicking the $1|1$-dimensional case, on the reduced manifold we define line bundles

$$
s\text{Det}(\Delta_{2|1}^X) := \text{Pf}(i\partial_{\bar{z}} \otimes \text{Id}_{\pi TX}) \otimes \text{Det}(i\partial_{\bar{z}} \otimes \text{Id}_{\pi TX})^{-1},
$$

$$
s\text{Det}(\Delta_{n|1}^X) := \text{Pf}(i\partial_{\bar{z}} \otimes \text{Id}_{\mathbb{R}^n}) \otimes \text{Det}(i\partial_{\bar{z}} \otimes \text{Id}_{\mathbb{R}^n})^{-1}.
$$

Because the operators in question are invertible, these lines bundles have nonvanishing super determinant sections,

$$
s\text{det}(\Delta_{2|1}^X) := \frac{\text{pf}(i\partial_{\bar{z}} \otimes \text{Id}_{\pi TX})}{\text{det}(i\partial_{\bar{z}} \otimes \text{Id}_{\pi TX})},
$$

$$
s\text{det}(\Delta_{n|1}^X) := \frac{\text{pf}(i\partial_{\bar{z}} \otimes \text{Id}_{\mathbb{R}^n})}{\text{det}(i\partial_{\bar{z}} \otimes \text{Id}_{\mathbb{R}^n})}.
$$

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We define the super determinant line over the super family as the pullback along the projection \( p : L \times \pi TX \to L \times X \) of the above lines:

\[
\text{sDet}(\Delta^2_{X}) := p^* \text{sDet}(|\Delta^2_{X}|), \quad \text{sDet}(\Delta^2_{n}) := p^* \text{sDet}(|\Delta^2_{n}|)
\]

and the super determinant sections are similarly defined by pullback. The metric on these super determinant lines is a rescaling of the pullback metrics,

\[
g_{\text{sDet}(\Delta^2_{X})} := \frac{\text{det}_S((i\partial_{\bar{z}} \otimes \text{Id}_{\pi TX})(i\partial_{\bar{z}} \otimes \text{Id}_{\pi TX})^*)^{1/2}}{\text{det}_S((-i\partial_{\bar{z}} \otimes \text{Id}_{\pi TX} + i\partial_z \otimes \text{Id}_{\pi TX})^2)^{1/2}} p^* g_{\text{sDet}(\Delta^2_{X})},
\]

\[
g_{\text{sDet}(\Delta^2_{n})} := p^* g_{\text{sDet}(\Delta^2_{n})},
\]

**Proof of Lemma 1.4** Any pair of unit norm sections must differ by a \( U(1) \)-valued function on \( L \times \pi TX \). Since the claim is a local one, we can write such a function as \( \exp(2\pi i f) \) for \( f \) a real valued function. Assuming \( \sigma \exp(g) \) and \( \sigma \exp(f) \) are supersymmetric trivializations, we will show that \( f - g \) is constant. Without loss of generality, we may take \( g = 0 \) and then it suffices to show that \( f \) is constant.

By assumption \( f \) is a supersymmetric function on \( \Phi^2_{01}(X) \). This requires that \( f \) be a sum of holomorphic functions tensored with closed forms (up to some factors coming from the volume of the tori). The only real-valued holomorphic functions are constant, so \( f \) must be a weak modular form of weight zero tensored with a closed 0-form. Hence, \( f \) is constant, and so (since \( X \) is connected) \( \sigma \) and \( \exp(2\pi i f)\sigma \) differ by a global phase. \( \square \)

**Proof of Theorem 1.4** By definition, a supersymmetric trivialization will consist of a unit norm section that is a supersymmetric function multiplied by the given (relative) super determinant section. The norm squared of the super determinant section of \( \Delta^2_{X} \) can be calculated directly through the definition of the metric on \( \text{sDet}(\Delta^2_{X}) \) as

\[
||\text{sDet}(\Delta^2_{X})||^2 = \frac{||\text{det}_S((i\partial_{\bar{z}} \otimes \text{Id}_{\pi TX})(i\partial_{\bar{z}} \otimes \text{Id}_{\pi TX})^*)^{1/2}}{||\text{det}_S((-i\partial_{\bar{z}} \otimes \text{Id}_{\pi TX} + i\partial_z \otimes \text{Id}_{\pi TX})^2)^{1/2}} .
\]

The norm squared of the relative super determinant is

\[
\frac{||\text{sDet}(\Delta^2_{X})||^2}{||\text{sDet}(\Delta^2_{n})||^2} = \frac{||\text{det}_S((i\partial_{\bar{z}} \otimes \text{Id}_{\pi TX})(i\partial_{\bar{z}} \otimes \text{Id}_{\pi TX})^*)^{1/2}}{||\text{det}_S((-i\partial_{\bar{z}} \otimes \text{Id}_{\pi TX} + i\partial_z \otimes \text{Id}_{\pi TX})^2)^{1/2}} .
\]

Calculation of the numerator on the right hand side follows from the Kronecker limit formula (see also example 9 of [QHS93]) and is

\[
\text{det}_S((i\partial_{\bar{z}} \otimes \text{Id}_{\pi TX})(i\partial_{\bar{z}} \otimes \text{Id}_{\pi TX})^*) = (\ell \ell' - \ell' \ell)^{4n} |\eta(\ell, \ell')|^{8n}
\]

where \( \eta(\ell, \ell') \) denotes the Dedekind \( \eta \)-function. To compute the denominator, we form the \( \zeta \)-function associated to the operator and largely use the same tricks as in the previous subsection. We have the \( \zeta \)-function

\[
\zeta(s) := \sum_{n,m} \text{Tr} \left( \frac{4\pi^2 |n\ell + m\ell'|}{(\ell \ell' - \ell' \ell)^2} + \frac{2\pi(n\ell + m\ell')}{(\ell \ell' - \ell' \ell)^2} \right)^s = \sum_{n,m} \text{Tr} \left( \frac{\mathcal{R}(\ell \ell' - \ell' \ell)}{2\pi(n\ell + m\ell')} \right)^s(\ell \ell' - \ell' \ell)^{4s}
\]

\[
= \sum_{n,m} \text{Tr} \left( \sum_k \frac{s(s-1) \cdots (s-k+1)}{k!} \left( \frac{\mathcal{R}(\ell \ell' - \ell' \ell)}{2\pi(n\ell + m\ell')} \right)^k \right).
\]

where as usual the sums are over \( m, n \) not both zero. Applying the product rule to evaluate the derivative at \( s = 0 \) yields two pieces: the \( \zeta \)-determinant of the previous subsection, and
its complex conjugate. Given the formula \( E_2^2 + E_2^2 = E_2 + \overline{E_2} \) relating the 2nd Eisenstein series, we have
\[
\exp(-C(0)) = \text{Wit}^+(X)\overline{\text{Wit}^+(X)}(\ell\ell' - \overline{\ell}\overline{\ell'})^{4n}|\eta(\ell, \ell')|^{8n} = \text{Wit}(X)\overline{\text{Wit}(X)}(\ell\ell' - \overline{\ell}\overline{\ell'})^{4n}|\eta(\ell, \ell')|^{8n},
\]
and so the norm squared of the relative super determinant section is
\[
\frac{\|\text{sdet}(\Delta_2^{(1)})\|^2}{\|\text{sdet}(\Delta_2^{(1)})\|^2} = \text{Wit}(X)\overline{\text{Wit}(X)}.
\]
Hence, the asserted section is unit norm. Furthermore, since \( \text{Wit}(X) \) is a supersymmetric function on \( \Phi_0^{2|1}(X) \), the asserted trivialization is a supersymmetric one. \( \square \)

5.6. **Trivializing the anomaly and the modular Witten genus.** The determinant line and its metric is equivariant for the \( \text{SL}_2(\mathbb{Z}) \)-action on tori, and hence descends along the quotient map \( \Phi_0^{2|1}(X) \rightarrow \Phi_0^{2|1}(X) \). However, \( 1_W \) does not yield a trivialization because \( \text{Wit}(X) \) is not a function on the quotient; instead, it is a section of a line bundle we denote by \( \mathcal{A} \). The goal of this section is to trivialize \( \mathcal{A} \) with a section \( 1_\mathcal{A} \) so that
\[
\frac{\text{sdet}(\Delta_2^{(1)}) \otimes \text{sdet}(\Delta_2^{(1)})^{-1} \otimes 1_\mathcal{A}}{\text{Wit}(X)} \in \Gamma(\text{sDet}(\Delta_2^{(1)}) \otimes \text{sDet}(\Delta_2^{(1)})^{-1} \otimes \mathcal{A} \otimes \mathcal{A}^{-1})
\]
is a trivialization of the relative super determinant line.
First we characterize \( \mathcal{A} \) in terms of an equivariant cocycle, i.e., a map \( \alpha : \text{SL}_2(\mathbb{Z}) \rightarrow C^\infty(L \times \pi TX)^\times \) satisfying the cocycle condition
\[
\alpha_{AB}(x) = \alpha_A(Bx) \cdot \alpha_B(x), \quad A, B \in \text{SL}_2(\mathbb{Z}), \ x \in L \times \pi TX.
\]
As such, it suffices to describe \( \alpha \) in terms of its values on generators of \( \text{SL}_2(\mathbb{Z}) \). Let
\[
T = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.
\]
Define \( \alpha_T := 1 \), the constant function, and \( \alpha_S = \exp(p_1/\ell\ell') \) where we identify \( p_1 \) with a function on \( \pi TX \). This defines the line bundle \( \mathcal{A} \) over \( L \times \pi TX \). We observe that
\[
\exp(E_2 \cdot \ell\ell' - \overline{\ell}\overline{\ell'})^2 \cdot p_1/(2\pi i)^2
\]
is a nonvanishing section of \( \mathcal{A} \) (as can be checked from the \( \text{SL}_2(\mathbb{Z}) \) transformation properties of the holomorphic 2nd Eisenstein series), and hence any other section of \( \mathcal{A} \) differs from this one by a complex-valued function on \( \Phi_0^{2|1}(X) \).

**Proof of Theorem 5.11.** Let \( H \) be a rational string structure on \( X \), viewed as a function on \( \pi TX \). We choose the line bundle \( \mathcal{L} \) to be given by the cocycle that on generators \( S \) and \( T \) of \( \text{SL}_2(\mathbb{Z}) \) is trivial on \( T \) and on \( S \) is \( \exp(d(tH)/\ell\ell') \) for \( t \) a coordinate on \( \mathbb{R} \). As required, this is the trivial cocycle at \( t = 0 \) and is the cocycle for \( \mathcal{A} \) at \( t = 1 \). Conversely, the existence of a line bundle \( \mathcal{L} \) satisfying the required conditions is equivalent to a concordance of closed differential forms the restricts to \( 0 \) and \( p_1 \) at the boundaries. Such a concordance is determined by a form \( H \) with \( dH = p_1 \); see Appendix A.
We take the section \( \sigma = \exp(E_2 \cdot \ell\ell' - \overline{\ell}\overline{\ell'}^2 \cdot d(tH)/(2\pi i)^2) \), viewed as a function on \( L \times \pi TX \). The associated section \( 1_\mathcal{A}(H) \) of \( \mathcal{A} \) is
\[
1_\mathcal{A}(H) = \exp \left( \frac{E_2(\ell, \ell')}{(2\pi i)^2} (\ell\ell' - \overline{\ell}\overline{\ell'})^2 d(tH) \right) \in \Gamma(\Phi_0^{2|1}(X \times \mathbb{R}); \mathcal{L})
\]
from which the asserted equality of sections follows immediately. \( \square \)
5.7. An index map for 2|1-EFTs. Just as in the 1|1-dimensional case, there is a canonical volume form on the fibers of $\Phi_{0}^{2|1}(X) \to \Phi_{0}^{2|1}(pt)$, and the Witten class can be used to modify the associated integration map. The rescaling, $(\bar{\ell}^{\ell} - \bar{\ell}^{\ell})^{-n/2}\text{Wit}_{H}(X)$, gives rise to a map denoted $\pi_{1}$ on supersymmetric sections,

$$\Gamma_{\text{susy}}(\Phi_{0}^{2|1}(X); \omega^{k/2}) \frac{(\bar{\ell}^{\ell} - \bar{\ell}^{\ell})^{-n/2}\text{Wit}_{H}(X)}{\mathbb{C}^{\infty}(L \times \pi T X)} \xrightarrow{f_{3}} \Gamma_{\text{susy}}(\Phi_{0}^{2|1}(pt); \omega^{k-n/2})$$

so that $(\bar{\ell}^{\ell} - \bar{\ell}^{\ell})^{-n/2}\text{Wit}_{H}(X)$ determines a sort of supersymmetric volume form on $\Phi_{0}^{2|1}(X)$. The map $\pi_{1}$ can be identified with the wrong-way map coming from the string orientation of TMF tensored with $\mathbb{C}$. Furthermore, the total relative volume of $\Phi_{0}^{2|1}(X)$ with respect to this choice of volume for is the Witten genus of $X$. Notice that this type of volume form fails to exist when $X$ is not rationally string.

We obtain an index map for 2|1-EFTs over a rational string manifold $X$ by precomposing with the restriction map that evaluates a field theory on the super tori over $X$ defining the stack of classical vacua:

$$\text{2|1-EFT}^{*}(X) \xrightarrow{\text{res}} \Gamma_{\text{susy}}(\Phi_{0}^{2|1}(X); \omega^{*}) \xrightarrow{\pi_{1}} \Gamma_{\text{susy}}(\Phi_{0}^{2|1}(pt); \omega^{*}) \cong \text{MF}^{*}.$$ 

Our computation in dimension 1|1 related the index of twisted Dirac operators to an analogous index for 1|1-EFTs. As such, we view the above as a candidate geometric construction of a local analytical index for TMF.

APPENDIX A. CONCORDANCE AND COCYCLE THEORIES

Let $\mathcal{F}: \text{Mfd}^{\text{pt}} \to \text{Set}$, be a sheaf of pointed sets.

**Definition A.1.** Two sections $s_{+}, s_{-} \in \mathcal{F}(X)$ are concordant if there exists a section $s \in \mathcal{F}(X \times \mathbb{R})$ such that $i_{+}^{*} s = \pi_{1} s_{+}$ and $i_{-}^{*} s = \pi_{1} s_{-}$ where

$$i_{+}: X \times (1, \infty) \hookrightarrow X \times \mathbb{R}, \quad \pi_{+}: X \times (1, \infty) \to X$$

$$i_{-}: X \times (-\infty, -1) \hookrightarrow X \times \mathbb{R}, \quad \pi_{-}: X \times (-\infty, -1) \to X$$

are the usual inclusion and projection maps. Concordance defines an equivalence relation; we denote the set of sections up to concordance by $\mathcal{F}[X]$ and the concordance class of a section $s$ by $[s]$.

Let $\mathcal{F}_{\text{cs}}(X)$ denote the compactly supported sections of $\mathcal{F}$ evaluated on $X$, and $\mathcal{F}_{\text{cvs}}(X \times \mathbb{R})$ denote the sections with compact vertical support in the $\mathbb{R}$ direction. We use the same notation for rapidly decaying sections, e.g., $e^{-x^{2}} \in C^{\infty}_{\text{cs}}(\mathbb{R}) \subset C^{\infty}(\mathbb{R})$.

**Definition A.2.** A cocycle theory is a sequence of sheaves $\mathcal{F}^{k}$ of pointed sets, for $k \in \mathbb{Z}$ and natural structure maps

$$\sigma: \mathcal{F}^{k}(X) \to \mathcal{F}^{k+1}(X \times \mathbb{R}), \quad \int: \mathcal{F}^{k+1}_{\text{cvs}}(X \times \mathbb{R}) \to \mathcal{F}(X),$$

with the property that $\sigma$ and $\int$ are inverses to one another on concordance classes. A multiplicative cocycle theory has in addition natural maps

$$\times: \mathcal{F}^{k}(X) \wedge \mathcal{F}^{l}(Y) \to \mathcal{F}^{k+l}(X \times Y),$$

that are associative and are compatible with $\int$,

$$\int(\alpha \times \beta) = \int(\alpha) \times \beta, \quad \int(\beta \times \alpha) = \beta \times \int(\alpha),$$

for $\alpha \in \mathcal{F}^{k+1}_{\text{cvs}}(X \times \mathbb{R})$ and $\beta \in \mathcal{F}^{l}(Y)$.

**Theorem A.3** (Stolz-Teichner). Let $(\mathcal{F}^{*}, \times, \sigma)$ be a multiplicative cocycle theory. Then the assignment $X \mapsto \mathcal{F}^{*}[X]$ is a multiplicative cohomology theory.
A sketch of a proof. Our sketch makes use of a result of I. Madsen and M. Weiss [MW07] (see their Appendix A.1): any sheaf $\mathcal{F}$ has a naturally defined classifying space $|\mathcal{F}|$ with the property that

$$\mathcal{F}[X] \cong [X, |\mathcal{F}|],$$

where $[-,-]$ denotes homotopy classes of maps. The morphisms of sheaves $\sigma: \mathcal{F}(-) \to \mathcal{F}_{cvs}(- \times \mathbb{R})$ give homotopy equivalences $|\mathcal{F}[k]| \to |\mathcal{F}|^{k+1}$ of pointed spaces, so that the sequence of spaces has the structure of a spectrum, and homotopy classes of maps into it form a cohomology theory.

We observe that a cup product $\cup: \mathcal{F}^k(X) \times \mathcal{F}^l(X) \to \mathcal{F}^{k+l}(X)$ gives rise to a cross product $\times: \mathcal{F}^k(X) \times \mathcal{F}^l(Y) \to \mathcal{F}^{k+l}(X \times Y)$ in the usual manner: we define

$$\omega \times \eta := p_1^*\omega \cup p_2^*\eta, \quad \omega \in \mathcal{F}^k(X), \quad \eta \in \mathcal{F}^l(Y),$$

where $p_1: X \times Y \to X$, $p_2: X \times Y \to Y$.

**Proposition A.4.** Two closed $k$-forms $\alpha_0, \alpha_1 \in \Omega^k(M)$ are cohomologous if and only if they are concordant. Furthermore, concordances between closed $k$-forms are in bijection with $(k-1)$-forms.

**Proof.** Suppose $\alpha_0, \alpha_1 \in \Omega^k(M)$ are cohomologous, and let $\alpha$ be an $(k-1)$ form such that $d\alpha = \alpha_0 - \alpha_1$. Define

$$\tilde{\alpha} = \alpha_0 - d(t\alpha) \in \Omega^k(X \times \mathbb{R})$$

Then it is clear that $i^*_\tau \tilde{\alpha} = \beta_j$ and $d\tilde{\alpha} = 0$, so that $\tilde{\alpha}$ is indeed a concordance from $\alpha_0$ to $\alpha_1$. Furthermore, this gives a map from $(n-1)$-forms to concordances.

Now suppose that $\tilde{\alpha}$ exists. Define a linear operator

$$Q: \Omega^k(X \times \mathbb{R}) \to \Omega^{k-1}(X), \quad Q\alpha = \int_0^1 i^*_\tau (i_\partial \alpha) \, d\tau$$

where $t$ is the coordinate for the $\mathbb{R}$ component of $X \times \mathbb{R}$, $\partial_t$ is the vector field on $\mathbb{R}$ arising from this coordinate, and $i$ is the usual contraction operator. We claim

$$dQ + Qd = i_1^* - i_0^*$$

so that

$$d(Q\tilde{\alpha}) = (i_1^* - i_0^* - Qd)\tilde{\alpha} = \alpha_1 - \alpha_0,$$

since $\tilde{\alpha}$ is closed. To verify this we compute

$$dQ\alpha + Qd\alpha = \int_0^1 i^*_\tau (i_\partial \alpha) \, d\tau + \int_0^1 i^* \alpha (d(\partial_\tau + i_\partial \alpha) \, d\tau$$

$$= \int_0^1 i^*_\tau ((d\partial_\tau + i_\partial \alpha) \, d\tau$$

$$= \int_0^1 i^*_\tau (sL_\partial \alpha) \, d\tau$$

$$= \int_0^1 \frac{d}{d\tau} (i^*_\tau \alpha) \, d\tau = (i_1^* - i_0^*)\alpha.$$
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