INTEGRABILITY OF $q$-OSCILLATOR LATTICE MODEL

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Abstract. A simple formulation of an exactly integrable $q$-oscillator model on two dimensional lattice (in 2+1 dimensional space-time) is given. Its interpretation in the terms of 2d quantum inverse scattering method and nested Bethe Ansatz equations is discussed.

Let $Z$ be the partition function of the ice-type model [1] on the rectangular lattice with the size $N \times M$ and periodical boundary conditions. Let the lattice be completely inhomogeneous, the “weights” of $j$th vertex ($j = 1, \ldots, NM$) are parameterized by

\[ \lambda, \mu, \nu \text{ (} \nu^2 \overset{\text{def}}{=} -q^{-1}\lambda\mu \text{)}, \]

\[ q^h, x_j, y_j. \]

Here $\lambda, \mu, \nu$ are numeric parameters common for whole lattice. The key point is that we will regard the inhomogeneous “weights” $q^h, x_j, y_j$ as generators of local $q$-oscillator algebra $H$.

\[ H: \ xy = 1 - q^{2+h}, \ yx = 1 - q^h, \ xq^h = q^{h+1}x, \ yq^h = q^{-h-1}y. \]

The vertex index $j$ in Fig.1 stands for the $j$th component of the tensor power $H^\otimes_{NM}$. In this paper we will imply the Fock space representation of $q$-oscillator (Spectrum($h$) = 0, 1, 2, ...). In the limit $q \to 1$ the model becomes the completely inhomogeneous free-fermion six-vertex model [2].

Partition function $Z$ is a polynomial of two parameters $\lambda$ and $\mu$ (recall, $\nu^2 = -q^{-1}\lambda\mu$), its operator-valued coefficients belong to $H^\otimes_{NM}$. The main result of this letter is the commutativity of $Z$,

\[ Z(\lambda, \mu) Z(\lambda', \mu') = Z(\lambda', \mu') Z(\lambda, \mu) \forall \lambda, \mu, \lambda', \mu'. \]

Therefore, $Z(\lambda, \mu)$ is the layer-to-layer transfer matrix of the quantum mechanical model in wholly discrete 2 + 1 dimensional space-time. This is the simplified formulation of auxiliary $q$-oscillator lattice [3].

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Turn to the proof of Eq. (2). It is convenient to combine the weights of Fig. 1 into a six-vertex-type matrix $L_{\alpha\beta}$ acting in the product of two dimensional vector spaces $V_{\alpha} \otimes V_{\beta}$, $V \equiv \mathbb{C}^2$:

$$(3) \quad L_{\alpha\beta}(H_j; \lambda, \mu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda q^{h_j} & 0 & \nu y_j \\ 0 & \nu x_j & \mu q^{h_j} & 0 \\ 0 & 0 & 0 & \nu^2 \end{pmatrix}, \quad \nu^2 = -q^{-1} \lambda \mu .$$

Let the lines of the lattice are labeled by the indices $\alpha_n$ and $\beta_m$, $1 \leq n \leq N$ and $1 \leq m \leq M$ as it is shown in Fig. 2. The vertices of the lattice are labeled by $j = (n, m)$.

![Figure 2. Labeling of the lines of the q-oscillator lattice (a layer-to-layer transfer matrix).](image)

The “partition function” $Z$ may be written as

$$(4) \quad Z(\lambda, \mu) = \text{Trace}_{V_{\alpha} \otimes V_{\beta}} \left( L_{\alpha\beta}(\lambda, \mu) \right), \quad L_{\alpha\beta}(\lambda, \mu) = \prod_{n} \prod_{m} L_{\alpha_n\beta_m}(H_{n,m}; \lambda, \mu) ,$$

where $\prod_{n} f_n = f_N f_{N-1} \cdots f_2 f_1$, $\prod_{m} f_m = f_M f_{M-1} \cdots f_2 f_1$, $V_\alpha = \bigotimes_{n=1}^{N} V_{\alpha_n}$ and $V_\beta = \bigotimes_{m=1}^{M} V_{\beta_m}$.

It is known [4], the commutativity of layer-to-layer transfer matrices follows from a tetrahedron equation. In particular, the commutativity (2) follows from

$$(5) \quad L_{\alpha'\beta'}(H; \lambda', \mu') L_{\alpha\beta}(H; \lambda, \mu) = M_{\alpha\alpha'}(H_0; \frac{\mu}{\mu'}) M_{\beta\beta'}(H_0; \frac{\lambda}{\lambda'}) L_{\alpha'\beta'}(H; \lambda', \mu'),$$

where matrix elements of $M(H_0; \xi)$ belong to an additional copy $H_0$ of q-oscillator:

$$(6) \quad M(H_0; \xi) = \begin{pmatrix} \xi^{h_0} & 0 & 0 & 0 \\ 0 & \lambda_0(-\xi)^{h_0} & \nu_0 \xi^{-1/2+h_0} y_0 & 0 \\ 0 & \nu_0 \xi^{1/2+h_0} x_0 & \mu_0(-\xi)^{h_0} & 0 \\ 0 & 0 & 0 & \nu_0^2 \xi^{h_0} \end{pmatrix}, \quad \nu_0^2 = q^{-1} \lambda_0 \nu_0 .$$

Equation (5) may be verified directly (in auxiliary spaces $V_\alpha \otimes \cdots \otimes V_\mu$, it is just $16 \times 16$ matrix equation), the commutativity (2) may derived by the repeated use of (5) for the forms (4).
The layer-to-layer transfer matrix $Z(\lambda, \mu)$ may be interpreted in the terms of the two-dimensional quantum inverse scattering method and quantum groups ([3] and e.g. [5, 6]). Transfer matrix \( \mathcal{L}_\beta \) is the 2d transfer matrix $Z(\lambda, \mu) = \text{Trace}_{V_{\beta}} \bigcap_n \mathcal{L}_\beta^{(n)}(\lambda, \mu)$ for the length-$N$ chain of the Lax operators

\[
\mathcal{L}_\beta^{(n)}(\lambda, \mu) = \text{Trace}_{V_{\alpha_n}} \bigcap_m \mathcal{L}_{\alpha_n, \beta_m}(\mathcal{H}_{n, m}; \lambda, \mu),
\]

Due to the six-vertex structure of (9), the Lax operator (7) has the block-diagonal form (in the combinatorial formulation of Fig.1 it means the conservation of the number of bold edges on the left and right of $\alpha_n^{th}$ column of Fig.2):

\[
\mathcal{L}_\beta(\lambda, \mu) = \bigoplus_{m=0}^M (-q^{-1}\mu)^m L_{\omega_m}(\lambda)
\]

where $L_{\omega_m}$ is the Lax operator for $m^{th}$ fundamental representation $\pi_{\omega_m}$ of $\mathcal{U}_q(\hat{\mathfrak{sl}}_M)$ in the auxiliary space\(^1\). In the quantum space, the Lax operator (7) acts in $F^{\otimes M}$, where $F$ is a representation space of $q$-oscillator $\mathcal{H}$. The Lax operator (7) has the central element

\[
q^{J_n} = q^{h_{n, 1} + h_{n, 2} + \cdots + h_{n, M}}.
\]

For the Fock space representation of $q$-oscillators, $F^{\otimes M} = \bigoplus_{J=0}^\infty \pi_{j\omega_1}$ is the direct sum of rank-$J$ symmetrical tensor representations of $\mathcal{U}_q(\hat{\mathfrak{sl}}_M)$. The $R$-matrix for the Lax operators (7) follows from (5):

\[
R_{\beta, \beta'} \left( \frac{\lambda}{\chi} \right) = \text{Trace}_{\mathcal{H}_0} \bigcap_m M_{\beta_m, \beta'_m} \left( \mathcal{H}_0; \frac{\lambda}{\chi} \right).
\]

Matrix elements of $M$ (10) depend on two extra parameters $\lambda_0, \mu_0$. They produce the decomposition

\[
R_{\omega_m, \omega_{m'}} \left( \frac{\lambda}{\chi} \right) = \bigoplus_{m, m'=0}^\infty \lambda_0^m \mu_0^{m'} R_{\omega_m, \omega_{m'}} \left( \frac{\lambda}{\chi} \right)
\]

where $R_{\omega_m, \omega_{m'}}$ is the $R$-matrix for $\pi_{\omega_m} \otimes \pi_{\omega_{m'}}$ fundamental representations of $\mathcal{U}_q(\hat{\mathfrak{sl}}_M)$.

The definition of the “partition function” $Z$ was initially $N \leftrightarrow M$ invariant. In particular, the alternative to (7) product $\text{Trace}_{V_{\beta}} \bigcap_n \mathcal{L}_{\alpha_n, \beta}(\mathcal{H}_n; \lambda, \mu)$ is the Lax operator for $\mathcal{U}_q(\hat{\mathfrak{sl}}_N)$ with the spectral parameter $\mu$, while $M$ becomes the length of $\mathcal{U}_q(\hat{\mathfrak{sl}}_N)$ chain. Central elements of $\mathcal{U}_q(\hat{\mathfrak{sl}}_N)$ $L$-operators are (cf. (11))

\[
q^{K_m} = q^{h_{1, m} + h_{2, m} + \cdots + h_{N, m}}.
\]

Since both sets of occupation numbers $J_n$ and $K_m$ are integrals of motion, their eigenvalues define a sub-sector of the $q$-oscillator model. For example, if $M = 2$ and the lattice is interpreted as the $\mathcal{U}_q(\hat{\mathfrak{sl}}_2)$ chain with the length $N$, the choice $J_n = 1$ gives us the six-vertex model (in general, $\pi_{j\omega_1}$ is spin $J/2$ representation of $\mathcal{U}_q(\mathfrak{sl}_2)$), whereas $K_1$ and $K_2$ stand for the numbers of spins up and spins down.

We would like to conclude the letter by the announcement of the universal form of the nested Bethe Ansatz equation for $Z$. The explicit polynomial decomposition of $Z$ is

\[
Z(\lambda, \mu) = \sum_{n=0}^N \sum_{m=0}^M \lambda^m \mu^n z_{n, m} = \sum_{m=0}^\infty \mu^m T^M_{\omega_m}(\lambda^M) = \sum_{n=0}^N \lambda^N T^N_{\omega_n}(\mu^N) .
\]

\(^1\)It is implied, $\omega_m$ are the dominant weights of $\mathcal{U}_q(\hat{\mathfrak{sl}}_M)$, $\pi_{\omega_0}$ and $\pi_{\omega_M}$ stand for scalar representations.
Let \( u, v \) be an additional auxiliary Weyl pair, \( uv = q^2 vu \), serving the following notations:

\[
\langle Q | u \rangle = Q(u), \quad \langle Q | u | u \rangle = uQ(u), \quad \langle Q | v \rangle = vQ(q^2u) .
\]

Let now (cf. (13), where the last but one expression is related to \( \mathcal{U}_q(\hat{sl}_M) \) Bethe Ansatz)

\[
J(u, v) = \sum_{n=0}^{N} \sum_{m=0}^{M} (-q)^{-nm} u^n v^m z_{m,n} \equiv \sum_{m=0}^{M} v^m T^{(sl_M)}_{\omega_m} \left( (-q)^m u \right).
\]

Then the nested Bethe Ansatz equation for \( \mathcal{U}_q(\hat{sl}_M) \) chain (notations (14) are taken into account) is

\[
\langle Q | J(u, v) | u \rangle \equiv \sum_{m=0}^{M} v^m Q(q^{2m}u) T^{(sl_M)}_{\omega_m} \left( (-q)^m u \right) = 0 .
\]

Extra condition \( Q(0) = 1 \) is an \( M \) th power equation for \( v \) entering the definition (14), its \( m \) th solution \( v = v_m \) corresponds to an order \( K_m \) polynomial \( Q = Q_m(u) \). The set of \( [v_m, Q_m(u)]_{m=1..M} \) is a complete set of independent solutions of \( M \) th order linear equation (15). In the same way and with the same results one may consider other realizations of Weyl algebra, for instance equation \( \langle v | J(u, v) | \bar{Q} \rangle = 0 \) for polynomials \( \langle v | \bar{Q} \rangle = \bar{Q}(v) \) and the last expression in (15) gives the dual \( \mathcal{U}_q(\hat{sl}_N) \) Bethe Ansatz.

In this letter we considered rectangular lattice with homogeneous \( \lambda, \mu \). The results of this paper may be generalized to the case of a lattice of any shape with inhomogeneous set of \( \lambda_j, \mu_j \). This, as well as the \( 3d \)-invariant derivation of (16), is the subject of forthcoming papers.

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\(^2\)Objects \( \langle u | \bar{Q} \rangle \) and \( \langle Q | v \rangle \) are polynomials for anti-Fock space representation \( \text{Spectrum}(\hbar) = -1, -2, -3, \ldots \).