A Pieri-type formula and a factorization formula for sums of $K$-$k$-Schur functions

Motoki Takigiku

November 9, 2018

Abstract

We give a Pieri-type formula for the sum of $K$-$k$-Schur functions $\sum_{\mu \leq \lambda} g_{\mu}^{(k)}$ over a principal order ideal of the poset of $k$-bounded partitions under the strong Bruhat order, which sum we denote by $\overline{g}_{\lambda}^{(k)}$. As an application of this, we also give a $k$-rectangle factorization formula $\overline{g}_{\mu \cup \lambda}^{(k)} = \overline{g}_{\mu}^{(k)} \overline{g}_{\lambda}^{(k)}$ where $R_t = (t^{k+1-t})$, analogous to that of $k$-Schur functions $s_{\mu \cup \lambda}^{(k)} = s_{\mu}^{(k)} s_{\lambda}^{(k)}$.

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1 Introduction

Let \( k \) be a positive integer. \( K \)-\( k \)-Schur functions \( g_{\lambda}^{(k)} \) are inhomogeneous symmetric functions parametrized by \( k \)-bounded partitions \( \lambda \), namely by the weakly decreasing strictly positive integer sequences \( \lambda = (\lambda_1, \ldots, \lambda_l) \), \( l \in \mathbb{Z}_{\geq 0} \), whose terms are all bounded by \( k \). They are \( K \)-theoretic analogues of another family of symmetric functions called \( k \)-\( \lambda \)-Schur functions \( s_{\lambda}^{(k)} \), which are homogeneous and also parametrized by \( k \)-bounded partitions. The set of \( k \)-bounded partitions is denoted by \( \mathcal{P}_k \).

In this paper we give a Pieri-type formula for a certain sum of \( K \)-\( k \)-Schur functions (Theorem 1.2 and 1.3) and a factorization formula (Theorem 1.4) involving the \( k \)-rectangle partitions \( R_t \) defined later, mainly using combinatorial properties of the strong (Bruhat) and weak orderings on the affine symmetric groups.

Historically, \( k \)-Schur functions were first introduced by Lascoux, Lapointe and Morse [LLM03], and subsequent studies led to several (conjecturally equivalent) characterizations of \( s_{\lambda}^{(k)} \): Lapointe and Morse [LM07] gave the Pieri-type formula, and Lam [Lam08] proved that \( k \)-Schur functions correspond to the Schubert basis of homology of the affine Grassmannian. Moreover, Lam and Shimozono [LS12] showed that \( k \)-Schur functions play a central role in the explicit description of the Peterson isomorphism.

These developments have analogues in \( K \)-theory. Lam, Schilling and Shimozono [LSS10] characterized the \( K \)-theoretical \( k \)-\( \lambda \)-Schur functions as the Schubert basis of the \( K \)-homology of the affine Grassmannian, and Morse [Mor12] investigated them from a combinatorial viewpoint, giving various properties including Pieri-type formulas using affine set-valued strips (the form using cyclically decreasing words are also given in [LSS10]). In this paper we start from this combinatorial characterization (see Definition 2.19).

Among the \( k \)-bounded partitions, those of the form \((t^{k+1-t}) = (t, \ldots, t)\) for \( 1 \leq t \leq k \), called \( k \)-rectangle and denoted by \( R_t \), play a special role. A notable property is the \( k \)-rectangle factorization for \( k \)-\( \lambda \)-Schur functions [LM07 Theorem 40]: if a \( k \)-bounded partition has the form \( R_t \cup \lambda \), where the symbol \( \cup \) denotes the operation of concatenating the two sequences and reordering the terms in the weakly decreasing order, then the corresponding \( k \)-\( \lambda \)-Schur function factorizes as follows:

\[
s_{R_t \cup \lambda}^{(k)} = s_{R_t}^{(k)} s_{\lambda}^{(k)}. \tag{1}
\]

It is natural to consider \( K \)-theoretic version of this formula. For several reasons below, in this regard it seems to make more sense to consider the sum of \( K \)-\( k \)-Schur functions \( \sum_{\mu \leq \lambda} g_{\mu}^{(k)} \) rather than \( K \)-\( k \)-Schur function \( g_{\lambda}^{(k)} \) (here \( \leq \) denotes the strong order, also known as the Bruhat order, which is transferred from that of the affine symmetric group \( \tilde{S}_{k+1} \) through the bijection \( \mathcal{P}_k \simeq \tilde{S}_{k+1}/S_{k+1} \). See Section 2.1.1 and 2.2.3 for the detail):

- Connection to \( K \)-Peterson isomorphism.
The (original) Peterson isomorphism, first presented by Peterson in his lectures at MIT and then published by Lam and Shimozono [LS10], states that the homology of the affine Grassmannian is isomorphic to the quantum cohomology of the flag variety after appropriate localization. As its K-theoretic version, an isomorphism between the K-homology of the affine Grassmannian and the quantum K-theory of the flag manifold, up to appropriate localization, is conjectured and called K-Peterson isomorphism:

- In their attempt in [LLMS] to verify the coincidence of the Schubert structure constants in the K-homology of the affine Grassmannian and the quantum K-theory of the flag manifold on torus-equivariant settings, Lam, Li, Mihalcea and Shimozono proved a special case of Theorem 1.4 for \( SL_2 \) (i.e. the case \( k = 1 \)) with explicit calculations, in the context of geometry:

\[
\mathcal{O}_x \mathcal{O}_{t_{-\alpha^\vee}} = \mathcal{O}_{xt_{-\alpha^\vee}},
\]

where \( x \) is any affine Grassmannian element in the affine Weyl group, \( \mathcal{O}_x \) is the Schubert class of structure sheaves on the affine Grassmannian and \( t_{-\alpha^\vee} \) is the translation by the negative of the simple coroot of \( SL_2 \). (See also Remark 2.14.)

- In [IMT], Ikeda, Iwao and Maeno gave an explicit ring isomorphism, after appropriate localization, between the K-homology of the affine Grassmannian and the presentation of the quantum Grothendieck polynomials, which is conjectured to be the quantum Schubert classes. These presentations notably involve the dual stable Grothendieck polynomials \( g_{\lambda}^{(k)} \) and their sum \( \sum_{\mu \subset R_t} g_{\mu} \) corresponding to the \( k \)-rectangles \( R_t \). Note that \( \mu \subset R_t \iff \mu \leq R_t \), and that it is conjectured that \( g_{\lambda}^{(k)} = g_{\lambda} \) for \( \lambda \subset R_t \).

- Natural appearances of \( \sum_{\mu \leq \lambda} g_{\mu}^{(k)} \) in \( k \)-rectangle factorization formulas of \( g_{\lambda}^{(k)} \).

It is suggested in [LS10] Remark 7.4 that the \( K \)-\( k \)-Schur functions should also possess similar properties to [1], including the divisibility of \( g_{R_t \cup \lambda}^{(k)} \) by \( g_{R_t}^{(k)} \), for which the author’s preceding work [Taka, Takb] gives an affirmative answer.

Let us review the results of [Taka, Takb]. It is proved that \( g_{R_t}^{(k)} \) divides \( g_{R_t \cup \lambda}^{(k)} \) in the ring \( \Lambda^{(k)} = \mathbb{Z}[h_1, \ldots, h_k] \), of which the \( K \)-\( k \)-Schur functions \( \{g_{\mu}^{(k)} \}_{\mu \in \mathcal{P}_k} \) form a basis. However, unlike [1], the quotient \( g_{R_t \cup \lambda}^{(k)}/g_{R_t}^{(k)} \) is not a single term \( g_{\lambda}^{(k)} \) but in general a linear combination of \( K \)-\( k \)-Schur functions with leading term \( g_{\lambda}^{(k)} \); for any \( \lambda \in \mathcal{P}_k \),

\[
g_{R_t \cup \lambda}^{(k)} = g_{R_t}^{(k)} \left( g_{\lambda}^{(k)} + \sum_{\mu} a_{\lambda\mu} g_{\mu}^{(k)} \right),
\]

summing over \( \mu \in \mathcal{P}_k \) such that \( |\mu| < |\lambda| \), with some coefficients \( a_{\lambda\mu} \) depending on \( R_t \). A special yet important case is the factorization of multiple \( k \)-rectangles: for \( 1 \leq t \leq k \) and \( a > 1 \),

\[
g_{R_t^a}^{(k)} = g_{R_t}^{(k)} \left( \sum_{\mu \subset R_t} g_{\mu}^{(k)} \right)^{a-1},
\]

where \( R_t^a = R_t \cup \cdots \cup R_t \) (\( a \) times). Note that \( \mu \subset R_t \iff \mu \leq R_t \). Furthermore, it is conjectured that the set of \( \mu \) appearing in (3) forms an interval under the strong order: namely, for any \( \lambda \in \mathcal{P}_k \) and \( 1 \leq t \leq k \), we expect there to exist \( \nu \in \mathcal{P}_k \) such that

\[
g_{R_t \cup \lambda}^{(k)} = g_{R_t}^{(k)} \sum_{\nu \leq \mu \leq \lambda} g_{\mu}^{(k)}.
\]

These observations hint at merit of Definition 1.1 below.
1.1 Main results

Let \( \leq_s, \leq_L, \) and \( \leq_R \) be the strong, left weak, and right weak order on \( \tilde{S}_{k+1} \) (see Section 2.2.1 for the detail).

From the observation above, we consider and denote by \( \bar{g}^{(k)}_\lambda \) the sum of \( K-k \)-Schur functions over the order ideal generated by \( \lambda \) under the strong order \( \leq_s \):

**Definition 1.1.** For any \( \lambda \in \mathcal{P}_k \), write \( \bar{g}^{(k)}_\lambda = \sum_{\mu \leq \lambda} g^{(k)}_\mu \).

Our first main theorem is a Pieri-type formula for \( \bar{g}^{(k)}_\lambda \). We start with the Pieri rule for \( g^{(k)}_\lambda \) \([LSS10,Mor12]\): for \( \lambda \in \mathcal{P}_k \) and \( 1 \leq r \leq k \),

\[
g^{(k)}_\lambda h_r = \sum_{(\lambda, \mu)} (-1)^{|\lambda|+r-|\mu|} g^{(k)}_\mu,
\]

summed over affine set-valued strips \((\mu/\lambda, A)\) of size \( r \) (See Definition 2.19 for more details). In terms of \( \bar{g}^{(k)}_\lambda \), this rule becomes relatively simple:

**Theorem 1.2.** Let \( \lambda \in \mathcal{P}_k \) and \( 1 \leq r \leq k \), and define \( \bar{h}_r = h_0 + h_1 + \cdots + h_r \). Then

\[
\bar{g}^{(k)}_\lambda \bar{h}_r = \sum_{\mu} g^{(k)}_\mu,
\]

summed over \( \mu \in \mathcal{P}_k \) such that \( \mu \leq \kappa \) for some \( \kappa \in \mathcal{P}_k \) such that \( \kappa/\lambda \) is a weak strip of size \( r \).

To express its right-hand side as a linear combination of \( \{g^{(k)}_\mu\}_\mu \), we recall that a weak strip over \( \lambda \) corresponds to a proper subset of \( I = \{0, 1, \ldots, k\} \): for \( \kappa \in \mathcal{P}_k \), \( \kappa/\lambda \) is a weak strip if and only if there exists \( A \subseteq I \) such that \( \kappa = d_A \lambda \geq_L \lambda \), where \( d_A \) is the cyclically decreasing permutation corresponding to \( A \) (see Section 2.2.2, 2.2.3, and 2.2.4 for the detail).

**Theorem 1.3.** With the setting in Theorem 1.2, we let \( d_{A_1} \lambda, d_{A_2} \lambda, \ldots \) be the list of weak strips of size \( r \) over \( \lambda \). Then

\[
\bar{g}^{(k)}_\lambda \bar{h}_r = \sum_{m \geq 1} (-1)^{m-1} \sum_{a_1 < \ldots < a_m} \bar{g}^{(k)}_{d_{A_1} \lambda \cap \cdots \cap d_{A_m} \lambda},
\]

\[
\left( = \sum_a \bar{g}^{(k)}_{d_{A_1} \lambda} - \sum_{a < b} \bar{g}^{(k)}_{d_{A_1} \lambda \cap d_{A_2} \lambda} + \sum_{a < b < c} \bar{g}^{(k)}_{d_{A_1} \lambda \cap d_{A_2} \lambda \cap d_{A_3} \lambda} - \cdots \right)
\]

(Moreover \( d_{A_n \lambda \cap \cdots \cap \lambda} = (d_{A_n} \lambda) \wedge (d_{A_{n-1}} \lambda) \wedge \ldots \), where \( \wedge \) denotes the meet in the poset \( \mathcal{P}_k \) with the strong order. See also Proposition 1.5.)

Our second main theorem is the \( k \)-rectangle factorization formula for \( \bar{g}^{(k)}_\lambda \), which holds in the same form as that for \( k \)-Schur functions \([1]\):

**Theorem 1.4.** For any \( \lambda \in \mathcal{P}_k \) and \( 1 \leq t \leq k \), we have

\[
\bar{g}^{(k)}_{R_t \lambda} = g^{(k)}_{R_t} g^{(k)}_{\lambda}.
\]

To deduce Theorem 1.4 from Theorem 1.3 is easy and discussed in Section 6. The proof of Theorem 1.2 and 1.3 on the other hand, is the technical heart of this paper and requires auxiliary work on the strong and weak orderings on the set of affine permutations as well as the structure of the set of weak strips, which are discussed in Section 3 and 4.

This paper is organized as follows.

In Section 2 we review notations and facts on combinatorial backgrounds. In Section 2.1 we treat arbitrary Coxeter groups and its strong and weak orderings. It also contains quick reviews on the generalized quotients \([BW88]\) and the Demazure products. Section 2.2 contains notations specific to the affine symmetric groups
and a review on its Young-diagrammatic treatment. In Section 2.3 we briefly review the Pieri-type formulas for $k$-Schur and $K$-$k$-Schur functions.

Section 3 contains technical lemmas on the strong and weak orders on arbitrary Coxeter groups. In Section 3.1 the lattice property of the weak order is reviewed. Although it is known that the quotient of an affine Weyl group by its corresponding finite Weyl group forms a lattice under the weak order [Wau99], we include another proof for the type affine $A$ using the $k$-Schur functions. Section 3.2 contains basic properties of the Demazure and anti-Demazure actions. In Section 3.3 we show the existence of $\min_{\leq} \{ z \in W \mid x \leq z \geq_L y \}$ and $\max_{\leq} \{ z \in W \mid x \geq_L z \leq_R y \}$, analogous to the join and meet. In Section 3.4 we consider an “interval-flipping” map $\Phi_z: [e, z]_L \rightarrow [e, z]_R; x \mapsto zx^{-1}$ and show that $\Phi_z$ is anti-isomorphic under the strong order and sends strong-meets (if exist) to strong-joins. In Section 3.5 we show the Chain Property of lower weak intervals, analogous to the Chain Property of the generalized quotients.

In Section 4, we focus on the affine symmetric groups and give results on the structure of the set of weak strips, which includes:

**Proposition 1.5** (⊂ Proposition 4.2). For any $\lambda \in \mathcal{P}_k$ and $A, B \subseteq I$ with $d_A\lambda/\lambda$ and $d_B\lambda/\lambda$ are weak strips,

1. $d_{A\cap B}\lambda/\lambda$ and $d_{A\cup B}\lambda/\lambda$ are weak strips.
2. $d_{A\cap B}\lambda = d_A\lambda \land d_B\lambda$ under the strong order.

**Proposition 1.6** (⊂ Proposition 4.12). For any $\lambda \in \mathcal{P}_k$, there exists $i_\lambda \in I (= \{0, 1, \ldots, k\})$ such that $i_\lambda \notin A$ for any weak strip $d_A\lambda/\lambda$.

Section 5 and 6 are devoted to proving the Pieri-type formula for $\tilde{g}^{(k)}_\lambda$ (Theorem 1.2 and 1.3) and the $k$-rectangle factorization formula for $\tilde{g}^{(k)}_\lambda$ (Theorem 1.4), respectively.

**Acknowledgement**

The author would like to express his gratitude to T. Ikeda for suggesting this topic to the author, many fruitful discussions and communicating to him the idea of considering the Schubert class of structure sheaves, related to the work [LLMS]. He is grateful to I. Terada for many valuable discussions and comments. He also wishes to thank Hiroshi Naruse and Mark Shimozono for helpful comments. This work was supported by the Program for Leading Graduate Schools, MEXT, Japan.

## 2 Preliminaries

In this section we review some requisite combinatorial backgrounds.

### 2.1 Coxeter groups

For basic definitions for the Coxeter groups we refer the reader to [BB05] or [Hum90].

#### 2.1.1 Strong and weak orderings

Let $(W, S)$ be a Coxeter group and $T = \{ws^{-1} \mid w \in W \}$ its set of reflections. The left weak order (or simply left order) $\leq_L$, right weak order (or right order) $\leq_R$, and strong order (or Bruhat order) $\leq$ on $W$ are generated by the covering relations:

- $u \leq_L v \iff l(v) = l(u) + 1$, $v = su$ for some $s \in S$,
- $u \leq_R v \iff l(v) = l(u) + 1$, $v = us$ for some $s \in S$,
- $u \leq v \iff l(v) = l(u) + 1$, $v = tu$ for some $t \in T$.

Note that the definition of the strong order looks different from but coincides with the classical one.
It is a few immediate observations that, for \( u, v \in W \),

\[
\begin{align*}
  u \leq_L v & \iff l(vu^{-1}) + l(u) = l(v), \tag{4} \\
  u \leq_R v & \iff l(u) + l(u^{-1}v) = l(v), \tag{5} \\
  u \leq_R uv & \iff l(u) + l(v) = l(uv) \iff v \leq_L uv. \tag{6}
\end{align*}
\]

We often use these equivalences without any mention. Using this translation from the weak order to length conditions, we can easily prove the following lemma:

**Lemma 2.1.** For \( x, y, z \in W \), we have

1. \( z \leq_L yz \leq_L xyz \iff y \leq_L xy \) and \( z \leq_L xyz \).
2. \( z \geq_L yz \geq_L xyz \iff y \leq_L xy \) and \( z \geq_L xyz \).

We often use the following notation taken from \([BW88]\): for \( w \in W \) we let \( \langle w \rangle \) denote any reduced expression for \( w \), and \( \langle u \rangle \langle v \rangle \) the concatenation of reduced expressions for \( u \) and \( v \). Hence, saying that \( \langle u \rangle \langle v \rangle \) is reduced means \( l(u) + l(v) = l(uv) \).

For \( u, v \in W \) with \( u \leq_L v \) the set \( \{ w \in W \mid u \leq_L w \leq_L v \} \) is called a **left interval** and denoted by \( \langle u, v \rangle_L \).

We define right interval \( [u, v]_R \) and strong (or Bruhat) interval \( [u, v] \) similarly. We shall use the notation \( [u, \infty)_L \) to denote the set \( \{ w \in W \mid u \leq_L w \} \), and define \( [u, \infty)_R \) and \( [u, \infty] \) similarly.

In this paper we heavily use some well-known results on the strong and weak orderings on Coxeter groups described below. See, for example, \([BB05]\) for details. Let \( v, w \in W \).

**Strong Exchange Property.** Suppose \( w = s_1s_2 \ldots s_k \ (s_i \in S) \) and \( t \in T \). If \( l(tw) < l(w) \), then \( tw = s_1 \ldots s_{i-1}s_{i'} \ldots s_k \) for some \( i' \in [k] \). Furthermore, if \( s_1s_2 \ldots s_k \) is a reduced expression then \( i' \) is uniquely determined.

**Subword Property.** Let \( w = s_1s_2 \ldots s_k \) be a reduced expression. Then \( v \leq w \) if and only if there exists a reduced expression \( v = s_i_1s_i_2 \ldots s_i_l \) with \( 1 \leq i_1 < i_2 < \cdots < i_l \leq k \).

**Chain Property.** If \( v \leq w \), then there exists a chain \( v = x_0 < x_1 < \cdots < x_k = w \).

**Lifting Property** (also known as Z-property). Let \( s \in S \). If \( sv > v \) and \( sw > w \), then the following are equivalent: (1) \( v \leq w \), (2) \( v \leq sw \), and (3) \( sw \leq v \).

### 2.1.2 Generalized quotients

For \( V \subset W \), let \( W/V = \{ w \in W \mid l(wv) = l(w) + l(v) \} \). The subsets of the form \( W/V \) are called **right generalized quotients** \([BW88]\). Similarly the set of the form \( V/W = \{ w \in W \mid l(wv) = l(v)+l(w) \} \) for all \( v \in V \) is called left generalized quotients. Note that, when \( V = W_J \), the parabolic subgroup corresponding to \( J \subset I \), the generalized quotient \( W/W_J \) is just the parabolic quotient \( W^J \).

It is shown in \([BW88\) Lemma 2.2] that if \( a, b, v \in W \) satisfy \( l(av) = l(a) + l(v) \) and \( l(bv) = l(b) + l(v) \), then \( av < bv \iff a < b \). An immediate consequence is

\[
W/\{ v \} \simeq [v, \infty)_L; w \mapsto vw \tag{7}
\]

under both the strong and left weak order.

**Chain Property for generalized quotients** ([BW88 Corollary 3.5]). If \( v, w \in W/V \) and \( v < w \), then there exists a chain \( v = x_0 < x_1 < \cdots < x_k = w \) with \( x_i \in W/V \) for all \( i \).

### 2.1.3 0-Hecke algebra and Demazure product

The **0-Hecke algebra** \( H \) associated to \( (W, S) \) is the associative algebra generated by \( \{ v_s \mid s \in S \} \) subject to the quadratic relations \( v_s^2 = -v_s \) and the braid relations of \( (W, S) \), that is, \( v_s v_t v_s \ldots = v_t v_s v_t \ldots \) for \( s, t \in S \).

---

1With the definition of \( \leq \) we employed here, this says nothing.
Proof. It follows easily from the definition of Lemma 2.2.}

Let $w \in W$ we can define without ambiguity $v_w \in H$ to be $v_{s_1} \ldots v_{s_n}$ where $s_1 \ldots s_n$ is any reduced expression for $w$. Furthermore, the elements $\{v_w | w \in W\}$ form a basis of $H$. The Demazure product (or Hecke product) $*$ on $W$ describes the multiplication of basis elements in $H$: $x * y$ is such that $v_x v_y = \pm v_{x*y}$. Some properties on the Demazure product can be found on [KM03, BM15].

We explicitly prepare the notation to denote the left multiplication in the Demazure product: for $s \in S$, we define the Demazure action $\phi_s : W \longrightarrow W$ by

$$\phi_s(x) = s * x = \begin{cases} x & \text{if } x > sx \\ sx & \text{if } x < sx \end{cases}.$$ 

Similarly we define the anti-Demazure action $\psi_s : W \longrightarrow W$ by

$$\psi_s(x) = \begin{cases} sx & \text{if } x > sx \\ x & \text{if } x < sx \end{cases}.$$ 

These maps $\{\phi_s\}_s$ and $\{\psi_s\}_s$ satisfy the quadratic relations $\phi_s^2 = \phi_s$, $\psi_s^2 = \psi_s$ and the braid relations of $(W,S)$: a direct proof (found on [Ste07, Proposition 2.1]) of this (for $\psi$) is that both $\psi_s \psi_t \psi_s \ldots$ and $\psi_t \psi_s \psi_t \ldots$ (m terms for each), where $sts \ldots = tst \ldots$ (m terms for each), send $x \in W$ to the shortest (resp. longest, when we consider $\phi$) element of the parabolic coset $W_{(s,t)}x$. Therefore we can define without ambiguity $\phi_x, \psi_x : W \longrightarrow W$ for $x \in W$ by $\phi_x = \phi_{s_1} \ldots \phi_{s_n}$ and $\psi_x = \psi_{s_1} \ldots \psi_{s_n}$ where $x = s_1 \ldots s_n$ is any reduced expression. Similarly we define right Demazure and anti-Demazure actions $\phi_x^R, \psi_x^R : W \longrightarrow W$ for $s \in S$ by $\phi_x^R(x) = \phi_x(x^{-1})^{-1}$ and $\psi_x^R(x) = \psi_x(x^{-1})^{-1}$, that is, $\phi_x^R(x) = xs$ if $x < xs$, etc. We also define $\phi_x^R$ and $\psi_x^R$ to be $\phi_{s_n}^R \ldots \phi_{s_1}^R$ and $\psi_{s_n}^R \ldots \psi_{s_1}^R$ (be careful for the order of composition) where $x = s_1 \ldots s_n$ is any reduced expression. Note that $\phi_x(y) = x * y = \phi_y^R(x)$. When $S$ is indexed with a set $I$, i.e. $S = \{s_i | i \in I\}$, we often write $\phi_i = \phi_{s_i}$ and $\psi_i = \psi_{s_i}$.

The following lemma is essentially given in [BW88 Theorem 4.2], and explicitly in [BM15 Proposition 3.1(e)]:

**Lemma 2.2.** Let $x, y, z \in W$ with $x * y = z$, that is, $\phi_x(y) = z = \phi_y^R(x)$. Let $x' = zy^{-1}$ and $y' = x^{-1}z$, that is, $z = xy' = x'y$. Then we have

1. $x, x' \leq_R z$
2. $y, y' \leq_L z$,
3. $l(z) = l(x) + l(y') = l(x') + l(y)$.
4. $x' \leq x$.
5. $y' \leq y$.

**Proof.** It follows easily from the definition of * and the Subword Property.

The proof of the following lemma is easy and similar to that of Lemma 2.2.

**Lemma 2.3.** Let $x, y, z \in W$ with $\psi_x(y) = z$. Let $x' = zy^{-1}$, that is, $z = x'y$. Then we have

1. $x' \leq x$.
2. $z \leq_L y$.
3. $x'^{-1} \leq_R y$.

We see more properties on $\phi_x, \psi_x$ in Section 3.2.
2.2 Affine symmetric groups

In this section we briefly review the connection between affine permutations, bounded partitions and core partitions. We refer the reader to [LLM+14] Chapter 2 and [Den12] for the details.

Hereafter we fix a positive integer $k$.

2.2.1 Affine symmetric group

Let $I = \mathbb{Z}_{k+1} = \{0, \ldots, k\}$. Let $[p, q] = \{p, p + 1, \ldots, q - 1, q\} \subseteq I$ for $p 
eq q - 1$. For example, $[4, 2] = \{4, 5, 0, 1, 2\}$ where $k = 5$. A subset $A \subset I$ is called connected if $A = [p, q]$ for some $p, q$. A connected component of $A \subset I$ is a maximal connected subset of $A$.

The affine symmetric group $\hat{S}_{k+1}$ is a group generated by the generators $\{s_i \mid i \in I\}$ subject to the relations $s_i^2 = 1, s_is_{i+1}s_i = s_{i+1}s_is_i$, $s_is_j = s_js_i$ for $i - j \neq 0, \pm 1$, with all indices considered mod $(k+1)$.

We often write $s_{ij\cdots}$ instead of $s_i s_j \cdots$. The parabolic quotient $\hat{S}_{k+1}/S_{k+1}$, where $S_{k+1}$ is the symmetric group $\langle s_1, \ldots, s_k \rangle$ as a subgroup of $\hat{S}_{k+1}$, is denoted by $\hat{S}^o_{k+1}$ and its elements are called affine Grassmannian elements.

For $x \in \hat{S}_{k+1}$, the set of right descents $D_R(x) = \{i \in I \mid x > xs_i\}$ ($\subseteq I$). The set of left descents $D_L(x)$ is defined similarly. For $i \in I$, an element $w \in \hat{S}_{k+1}$ is called $i$-dominant if $D_R(w) \subset \{i\}$. Note that an affine permutation is 0-dominant if and only if it is affine Grassmannian.

2.2.2 Cyclically decreasing elements

A word $a = a_1 a_2 \ldots a_m$ with letters from $I$ is called cyclically decreasing (resp. cyclically increasing) if $a_1, a_2, \ldots, a_m$ are distinct and any $j \in I$ does not precede $j + 1$ (resp. $j - 1$) in $a$. For $A \subset I$, the cyclically decreasing element $d_A$ is defined to be $s_{i_1} s_{i_2} \ldots s_{i_m}$ where $A = \{i_1, i_2, \ldots, i_m\}$ and the word $i_1 i_2 \ldots i_m$ is cyclically decreasing. The cyclically increasing element $u_A = s_{i_m} s_{i_{m-1}} \ldots s_{i_1}$ is defined similarly. Note that these definitions are independent of the choice of the word.

Example 2.4. Let $k = 5$ and $A = \{0, 1, 3, 5\} \subseteq \mathbb{Z}_6$. The possible cyclically decreasing words for $A$ are 1053, 1035, 1305 and 3105, and hence $d_A = s_1 s_0 s_3 s_5 = s_1 s_0 s_3 s_5 = s_3 s_1 s_0 s_5$.

2.2.3 Connection to bounded partitions and core partitions

In this section we review the bijection between the set of $k$-bounded partitions, $k + 1$-core partitions and affine Grassmannian elements in $\hat{S}_{k+1}$. For the details see [LLM+14] Chapter 2 and references given there.

A partition $\lambda$ is called $k$-bounded if $\lambda_1 \leq k$. Let $\mathcal{P}_k$ be the set of all $k$-bounded partitions. An $r$-core (or simply a core if no confusion can arise) is a partition none of whose cells have a hook length equal to $r$. We denote by $\mathcal{C}_r$ the set of all $r$-core partitions.

Now we recall the bijection

$$P_k \simeq \mathcal{C}_{k+1} \simeq \hat{S}^o_{k+1}. \tag{8}$$

The map $p : \mathcal{C}_{k+1} \rightarrow \mathcal{P}_k; \kappa \mapsto \lambda$ is defined by $\lambda_i = \#\{j \mid (i, j) \in \kappa, \text{hook}_{i,j}(\kappa) \leq k\}$. In fact $p$ is bijective and the inverse map $p^{-1} : \mathcal{P}_k \rightarrow \mathcal{C}_{k+1}$ is algorithmically described as a "sliding cells" procedure.

The map $s : \hat{S}^o_{k+1} \rightarrow \mathcal{C}_{k+1}$ is constructed via an action of $\hat{S}_{k+1}$ on $\mathcal{C}_{k+1}$; for $\kappa \in \mathcal{C}_{k+1}$ and $i \in I$, we define $s_i \kappa$ to be $\kappa$ with all its addable (resp. removable) corners with residue $i$ added (resp. removed), where the residue of a cell $(i, j)$ is $j - i \mod k + 1$. In fact this gives a well-defined $\hat{S}_{k+1}$-action on $\mathcal{C}_{k+1}$, which induces the bijection $s : \hat{S}^o_{k+1} \rightarrow \mathcal{C}_{k+1} ; w \mapsto w \cdot \emptyset$.

The map $P_k \rightarrow \hat{S}^o_{k+1} ; \lambda \mapsto w_\lambda$ is given by $w_\lambda = s_{i_1} s_{i_2} \ldots s_{i_l}$, where $(i_1, i_2, \ldots, i_l)$ is the sequence obtained by reading the residues of the cells in $\lambda$, from the shortest row to the largest, and within each row from right to left. See [LM05] Corollary 48 for the proof.

Thus for $\lambda \in \mathcal{P}_k$, the $k$-transpose of $\lambda$ is $p(\epsilon(\lambda)^t)$ and denoted by $\lambda^{ts}$.

Example 2.5. Let $k = 3$ and $\lambda = (3, 2, 1) \in \mathcal{P}_3$. The corresponding 4-core partition and affine permutation are $\epsilon(\lambda) = (5, 2, 1) \in \mathcal{C}_4$ and $w_\lambda = s_{203210} \in \hat{S}^o_4$. (See Figure 1)
2.2.4 Weak strips

**Definition 2.6.** For \( v, w \in \hat{S}^o_{k+1} \), we call \( v/w \) is a weak strip (or affine strip) of size \( r \) if \( v = d_A w \geq_L w \) for some \( A \subseteq I \) with \( |A| = r \). We also say \( v \) is a weak strip of size \( r \) over \( w \).

**Definition 2.7.** For \( v, w \in \hat{S}^o_{k+1} \) and \( A \subseteq I \), we call \( (v/w, A) \) an affine set-valued strip of size \( r \) if \( v = d_A * w (= \phi_{d_A}(w)) \) and \( |A| = r \). We also say \( (v, A) \) is a affine set-valued strip of size \( r \) over \( w \).

Note that if \( (v/w, A) \) is an affine set-valued strip of size \( r \) then \( v/w \) is an affine strip of size \( \leq r \).

**Remark 2.8.** Identifying \( \lambda, \alpha_\lambda \) and \( w_\lambda \) through the bijection \( P_k \simeq C_{k+1} \simeq \hat{S}^o_{k+1} \), we often say \( \mu/\gamma \) (resp. \( \kappa/\gamma \)) is a weak strip for \( \lambda, \mu \in P_k \) (resp. \( \kappa, \gamma \in C_{k+1} \)), etc.

**Remark 2.9.** Regarding \( v, w \in \hat{S}^o_{k+1} \) as bounded (or core) partitions as above, we see these notions are variants of the horizontal strip. For example, \( w_\lambda/w_\mu \) is a weak strip if and only if the corresponding cores \( \alpha_\lambda/\alpha_\mu \) form a horizontal strip and \( w_\mu \geq_L w_\lambda \), and the term “affine set-valued” originates in affine set-valued tableaux. See, for example, [LLM14] and [Mor12] for more details.

**Example 2.10.** Let \( k = 3 \) and \( \lambda = (3, 2, 1) \in P_3 \), and thus \( w_\lambda = s_{203210} \) and \( \alpha_\lambda = (5, 2, 1) \). Figure 2 lists all \( v \) such that \( v/w_\lambda \) is a weak strip (the corresponding core partitions are displayed).

### 2.2.5 \( k \)-code

The content of this section is mostly cited from [Den12].

A \( k \)-code is a function \( \alpha : I \rightarrow \mathbb{Z}_{\geq 0} \) such that there exists at least one \( i \in I \) with \( \alpha(i) = 0 \). We often write \( \alpha_i = \alpha(i) \). The diagram of a \( k \)-code \( \alpha \) is the Ferrers diagram on a cylinder with \( k + 1 \) columns indexed by \( I \), where the \( i \)-th column contains \( \alpha_i \) boxes. A \( k \)-code \( \alpha \) may be identified with its filling, which is the diagram of \( \alpha \) with all its boxes marked with their residues, that is, \( i - j (\in I) \) for one in the \( i \)-th column and \( j \)-th row. A flattening of the diagram of a \( k \)-code \( \alpha \) is what is obtained by cutting out a column with no boxes (that is, column \( j \) with \( \alpha_j = 0 \)). A reading word of \( \alpha \) is obtained by reading the rows of the flattening of \( \alpha \) from right to left, beginning with the last row. Note that, though a \( k \)-code may have multiple columns with no boxes, the affine permutation given by the reading word of \( \alpha \) is independent of the choice of a flattening. Indeed, for a \( k \)-code \( \alpha \) with \( m \) rows, letting \( A_i \) be the set of the residues of the boxes in the \( i \)-th row in the diagram of \( \alpha \), we have that \( d_{A_m} \ldots d_{A_2} d_{A_1} \) is the affine permutation corresponding to \( \alpha \). In fact this correspondence is bijective (Theorem 2.11); an algorithm to obtain a \( k \)-code from an affine permutation is explained below.

**Maximizing moves.**

For a cyclically decreasing decomposition \( w = d_{A_m} \ldots d_{A_2} d_{A_1} \), there corresponds a “skew \( k \)-code diagram”, that is, a set of boxes in the cylinder with \( k + 1 \) columns indexed by \( I \) for which \( A_i \) is the set of the residues of the boxes in the \( i \)-th row. To justify it to the bottom, we consider the following “two-row move”: pick any consecutive two rows \( A_i \) and \( A_{i+1} \), and let \( i, j \in I \) with \( j \neq i - 1 \). Then,

1. if \( i - 1 \notin A_{i+1}, [i, j] \subset A_{i+1}, [i+1, j] \subset A_i \), and \( i, j + 1 \notin A_i \), then we replace \( A_i \) and \( A_{i+1} \) with \( A_i \cup \{i\} \) and \( A_{i+1} \setminus \{j\} \), reflecting the equation \((s_j s_{j-1} \ldots s_i)(s_j \ldots s_{i+1}) = (s_{j-1} \ldots s_i)(s_j \ldots s_{i+1} s_i)\).
Figure 2: The weak strips over $w_\lambda$ where $\lambda = (3, 2, 1)$. Left weak covers are represented as solid lines, and strong covers are solid or dotted lines. A solid edge between $v$ and $w$ is labelled with $s_i$ if $v = s_iw$. 
Then \( \text{RD}(w) \) (Figure 3) where \( k = 3 \) and \( w = s_2s_{30}s_{431} \).

(2) if \( i - 1 \not\in A_{a+1}, [i,j] \subset A_{a+1}, [i,j] \subset A_a, \) and \( j + 1 \not\in A_a, \) then we conclude this decomposition does not give a reduced expression, reflecting the fact that \((s_i s_{j-1} \ldots s_1)(s_j \ldots s_{i+1}s_i)\) is not a reduced expression.

\[
\begin{array}{cccc}
 & i & \cdots & j-1 \\
i & i+1 & \cdots & j \\
\end{array}
\]

: not reduced

Note that these moves look simpler when \( i = j \):

(1) \( \sim \)

(2) : not reduced

It is shown in [Den12] Section 3] that, for any decomposition \( w = d_{A_m} \ldots d_{A_1} \) that gives a reduced expression, we can apply a finite series of moves of type (1) to justify its diagram to the bottom and obtain a \( k \)-code, which is in fact uniquely determined from \( w \) and denoted by \( \text{RD}(w) \), and gives the maximal decreasing decomposition \( w = d_{B_n} \ldots d_{B_1} \), that is, the vector \((|B_1|, \ldots, |B_n|)\) is maximal in the lexicographical order among such decompositions for \( w \). Furthermore, this procedure bijectively maps affine permutations to \( k \)-codes:

**Theorem 2.11 (Den12 Theorem 38)**. The map \( w \mapsto \text{RD}(w) \) gives a bijection between \( \tilde{S}_{k+1} \) and the set of \( k \)-codes.

**Example 2.12**. Let \( k = 3 \) and \( w = s_2s_{30}s_{431} \) (this expression gives the maximal decreasing decomposition). Then \( \text{RD}(w) = (0,2,0,1,3) \). (See Figure 3)

Note that this construction also works if maximal decreasing decomposition is replaced with maximal increasing decompositions, that is, the unique decomposition \( w = u_{B_n} \ldots u_{B_1} \) into cyclically increasing elements with the vector \((|B_1|, \ldots, |B_n|)\) being maximal in the lexicographical order, by modifying the notion of the filling of a \( k \)-code so that the box in the \( i \)-th column and \( j \)-th row is marked with \( j-i \) instead of \( i-j \). The resulting \( k \)-code is denoted by \( \text{RI}(w) \). The map \( w \mapsto \text{RI}(w) \) also gives a bijection between \( \tilde{S}_{k+1} \) and the set of \( k \)-codes.

It is proved [Den12 Corollary 39] that, \( w \in \tilde{S}_{k+1} \) is \( i \)-dominant if and only if the flattening of the corresponding \( k \)-code \( \text{RD}(w) \) forms a \( k \)-bounded partition with residue \( i \) in its lower left box, that is, \( \text{RD}(w)_i \geq \text{RD}(w)_{i+1} \geq \cdots \geq \text{RD}(w)_{i-2} \geq \text{RD}(w)_{i-1} = 0 \). When \( i = 0 \), this mapping from 0-dominant permutations to \( k \)-bounded partitions coincides with the one described earlier in Section 2.2.5. Moreover, it is proved [Den12 Proposition 51] that, for \( w \in W^o \) the two corresponding \( k \)-codes \( \text{RD}(w) \) and \( \text{RI}(w) \), regarded as \( k \)-bounded partitions, are transformed into each other by taking \( k \)-transpose: \( \text{sh}(\text{RI}(w)) = (\text{sh}(\text{RD}(w)))^{\omega_k} \) where \( \text{sh}(\alpha) \in \mathcal{P}_k \) is defined by \( \text{sh}(\alpha)_j = |\{i | \alpha_i \geq j\}| \).

It is also proved in [Den12 Proposition 56] that if \( x \leq_k y \) then \( \text{RD}(x) \subset \text{RD}(y) \) and \( \text{RI}(x) \subset \text{RI}(y) \).

**Example 2.13**. Let \( k = 3 \) and \( w = s_6s_1s_{32}s_{63}s_{210} = s_1s_0s_3s_{12}s_{61}s_{30} \) (these presentations give the maximal decreasing and increasing decompositions). Then \( \text{RD}(w) = (5,3,1,0) \) and \( \text{RI}(w) = (6,3,0,0) \), and thus \( \text{sh}(\text{RD}(w)) = (3,2,2,1,1) = (2,2,2,1,1)^{\omega_3} = \text{sh}(\text{RI}(w))^{\omega_3} \). (See Figure 4)
Lemma 2.15. \( \leq 0 \)

For any \( \lambda \) dense \( d \) of the fundamental coweight, modulo left multiplication by the length zero elements.

Proof. Let \( \lambda \) be a fundamental coweight, modulo left multiplication by the length zero elements. In fact \( wR_i \) is congruent, in the extended affine Weyl group, to the translation \( t_{-\mathbf{v}'} \) by the negative of a fundamental coweight, modulo left multiplication by the length zero elements.

The next lemma describes the mapping \( \lambda \mapsto R_t \cup \lambda \) in terms of affine permutations. For \( A \subseteq I \) and \( 0 \leq t \leq k \), we write \( A + t = \{ a + t \mid a \in A \} (\subseteq I) \).

Lemma 2.16. Let \( 1 \leq t \leq k \). Define a group isomorphism

\[
\tilde{f}_t: \tilde{S}_{k+1} \to \tilde{S}_{k+1} \mid s_i \mapsto s_{i+t} \quad \text{for } i \in I.
\]

For any \( \lambda \in P_k \), we have

\[
w_{R_t \cup \lambda} = \tilde{f}_t(w_\lambda)w_{R_t}.
\]

Proof. Let \( d_{A_1} \ldots d_{A_t} \) and \( d_{B_{k+1-t}} \ldots d_{B_k} \) be the maximal decreasing decompositions of \( w_\lambda \) and \( w_{R_t} \). Then \( d_{A_{m+t}} \ldots d_{A_{k+1-t}} \) is the maximal decomposition of \( f_t(w_\lambda) \). Stacking the \( k \)-code diagram of \( f_t(w_\lambda) \) on that of \( w_{R_t} \), we obtain the diagram (not necessarily justified to the bottom) corresponding to the (not necessarily maximal) decreasing decomposition \( f_t(w_\lambda)w_{R_t} = d_{A_{m+t}} \ldots d_{A_{k+1-t}} \) (See Figure 5).

The next lemma explains the correspondence between weak strips over \( \lambda \) and weak strips over \( R_t \cup \lambda \).

Lemma 2.17. Let \( \lambda \in P_k \).

1. For \( A \subseteq I \), if \( \lambda A/\lambda \) is a weak strip then \( R_t \cup (d_A \lambda) = d_{A+t}(R_t \cup \lambda) \).
2. \( R_t \cup (d_A \lambda), R_t \cup (d_A \lambda) \), ... is the list of all weak strips over \( R_t \cup \lambda \) (of size \( r \)).
3. \( d_{A_{1+t}}(R_t \cup \lambda), d_{A_{2+t}}(R_t \cup \lambda), \ldots \) is the list of all weak \( r \)-strips over \( R_t \cup \lambda \) (of size \( r \)).

Proof. (2) is [LM94, Theorem 20]. (3) follows from (1) and (2).

1. It suffices to show the case \( |A| = 1 \), that is, \( R_t \cup (s_i \lambda) = s_{i+t}(R_t \cup \lambda) \). This is essentially shown in the process of proving [LM94, Theorem 20] by seeing correspondence between addable corners of \( c(\lambda) \) with residue \( i \) and addable corners of \( c(R_t \cup \lambda) \) with residue \( i + t \). Yet we here give another proof: by Lemma 2.15 it follows \( w_{R_t \cup (s_i \lambda)} = f_t(w_{s_i \lambda})w_{R_t} = f_t(s_i w_\lambda)w_{R_t} = s_i + t f_t(w_\lambda)w_{R_t} = s_i + t w_{R_t \cup \lambda} \).

2.3 Symmetric functions

For basic definitions for symmetric functions, see for instance [Mac95, Chapter I].
Figure 5: Justifying process with maximizing moves, where $k = 5$, $t = 2$, $R_2 = (2^4)$, and $\lambda = (4, 3, 3, 1)$.

2.3.1 Symmetric functions

Let $\Lambda = \mathbb{Z}[h_1, h_2, \ldots]$ be the ring of symmetric functions, generated by the complete symmetric functions $h_r = \sum_{i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} \ldots x_{i_r}$. For a partition $\lambda$ we set $h_\lambda = h_{\lambda_1} h_{\lambda_2} \ldots h_{\lambda_l(\lambda)}$. The set $\{h_\lambda\}_{\lambda \in \mathcal{P}}$ forms a $\mathbb{Z}$-basis of $\Lambda$.

2.3.2 Schur functions

The Schur functions $\{s_\lambda\}_{\lambda \in \mathcal{P}}$ are the family of symmetric functions satisfying the Pieri rule:

$$h_r s_\lambda = \sum_{\mu / \lambda, \text{horizontal strip of size } r} s_\mu.$$

2.3.3 $k$-Schur functions

We recall a characterization of $k$-Schur functions given in [LM07], since it is a model for and has a relationship with $K$-$k$-Schur functions.

**Definition 2.17** ($k$-Schur function via $k$-Pieri rule). $k$-Schur functions $\{s_w^{(k)}\}_{w \in \tilde{S}_{k+1}^c}$ are the family of symmetric functions such that

$$s_c^{(k)} = 1,$$

$$h_r s_w^{(k)} = \sum_v s_v^{(k)} \quad \text{for } 1 \leq r \leq k \text{ and } w \in \tilde{S}_{k+1}^c,$$

summed over $v \in \tilde{S}_{k+1}^c$ such that $v/w$ is a weak strip of size $r$.

It is known that $\{s_w^{(k)}\}_{w \in \tilde{S}_{k+1}^c}$ forms a basis of $\Lambda^{(k)} = \mathbb{Z}[h_1, \ldots, h_k] \subset \Lambda$, and $s_w^{(k)}$ is homogeneous of degree $l(w)$. We regard $s_\lambda^{(k)}$ as $s_w^{(k)}$ for $\lambda \in \mathcal{P}_k$. It is proved in [LM07 Theorem 40] that

**Proposition 2.18** ($k$-rectangle property). For $1 \leq t \leq k$ and $\lambda \in \mathcal{P}_k$, we have $s_{R_t \cup \lambda}^{(k)} = s_{R_t}^{(k)} s_\lambda^{(k)} (= s_{R_t} s_\lambda^{(k)})$. 

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2.3.4 K-k-Schur functions

In this paper we employ the following characterization with the Pieri rule ([LSS10 Corollary 7.6], [Mor12 Corollary 50]) of the K-k-Schur function as its definition.

**Definition 2.19** (K-k-Schur function via K-k-Pieri rule). K-k-Schur functions \( \{g_w^{(k)}\}_{w \in \tilde{S}_{k+1}^n} \) are the family of symmetric functions such that \( g_{\emptyset}^{(k)} = 1 \) and

\[
 h_r \cdot g_w^{(k)} = \sum_{(A,v)} (-1)^{r+l(w)-l(v)} g_v^{(k)},
\]

for \( w \in \tilde{S}_{k+1}^n \) and \( 1 \leq r \leq k \), summed over \( v \in \tilde{S}_{k+1}^n \) and \( A \subseteq I \) such that \( (v/w, A) \) is an affine set-valued strip of size \( r \).

It is known that \( \{g_w^{(k)}\}_{w \in \tilde{S}_{k+1}^n} \) forms a basis of \( \Lambda^{(k)} \). Besides, though \( g_w^{(k)} \) is an inhomogeneous symmetric function in general, the degree of \( g_w^{(k)} \) is \( l(w) \) and its homogeneous part of highest degree is equal to \( s_w^{(k)} \). In this paper, for \( f = \sum_w c_w g_w^{(k)} \in \Lambda^{(k)} \) we write \( [g_v^{(k)}](f) = c_v \).

3 Properties on the strong and weak orderings on Coxeter groups

In this section we let \((W, S)\) be an arbitrary Coxeter group.

Recall that for a poset \((P, \leq)\) and a subset \( A \subset P \), if the set \( \{z \in P \mid z \leq y \text{ for any } y \in A\} \) has the maximum element \( z_0 \) then \( z_0 \) is called the meet of \( A \) and denoted by \( \wedge A \), and if \( \{z \in P \mid z \geq y \text{ for any } y \in A\} \) has the minimum element then it is called the join of \( A \) and denoted by \( \vee A \). When \( A = \{x, y\} \), its meet and join are simply called the meet and join of \( x \) and \( y \), and denoted by \( x \wedge y \) and \( x \vee y \). A poset for which any nonempty subset has the meet is called a complete meet-semilattice. A poset for which any two elements have the meet and join is called a lattice. A subset of a complete meet-semilattice has the join if it has a common upper bound, since the join is the meet of all its common upper bounds then.

In this paper we denote the meet of \( x, y \in W \) under the strong (resp. left, right) order by \( x \wedge y \) (resp. \( x \wedge_L y \), \( x \wedge_R y \)) and call it the strong meet (resp. left meet, right meet) of \( \{x, y\} \). We define \( x \vee y \), \( x \vee_L y \) and \( x \vee_R y \) similarly.

3.1 Lattice property of the weak order

It is known that the weak order on any Coxeter group or its parabolic quotient forms complete meet semilattice (see, for example, [BB05 Theorem 3.2.1]). The join of two elements in them, however, does not always exist, but it is known that the quotient of an affine Weyl group by its corresponding finite Weyl group forms a lattice under the weak order [Wau99]. We here include another proof for the type affine A case for the sake of completeness.

**Lemma 3.1.** For any \( v, w \in \tilde{S}_{k+1}^n \), their join \( v \vee_L w \) under the left weak order exists.

**Proof.** Since \( \tilde{S}_{k+1}^n \) is a meet complete semilattice, it remains to show the existence of a common upper bound of \( v \) and \( w \) under the left order. Let \( s_v^{(k)} \) and \( s_w^{(k)} \) denote the k-Schur functions corresponding to \( v \) and \( w \). In the expansion of their product in the k-Schur function basis \( s_v^{(k)} s_w^{(k)} = \sum_u c_u s_u^{(k)} \), every \( u \) appearing in the right-hand side satisfies \( w \leq_L u \) because \( s_u^{(k)} \) can be written as a polynomial in \( h_1, \ldots, h_k \) and by the Pieri rule \( h_1 s_u^{(k)} \) is in general a linear combination of \( s_y^{(k)} \) with \( y \geq_L x \). By the same reason we have \( v \leq_L u \).

We proved the following corollary in the proof of the lemma above:

**Corollary 3.2.** For any \( v, w \in \tilde{S}_{k+1}^n \), every \( u \) appearing with a nonzero coefficient in the right-hand side of \( s_v^{(k)} s_w^{(k)} = \sum_u c_u s_u^{(k)} \) satisfies \( u \geq_L v \vee_L w \).
With the $K$-$k$-Pieri rule instead of the $k$-Pieri in hand, the same holds for the $K$-$k$-Schur functions:

**Corollary 3.3.** For any $v, w \in \tilde{S}_{k+1}$, every $u$ appearing with a nonzero coefficient in the right-hand side of $g^{(k)}_w g^{(k)}_u = \sum_a d^{(k)}_{vu} \delta^{(k)}_u$ satisfies $u \geq_L v \lor_L w$.

### 3.2 Properties of Demazure and anti-Demazure actions

**Lemma 3.4.** Let $x \in W$ and $\phi_x, \psi_x$ be the Demazure and anti-Demazure actions defined in Section 3.2.6

1. $\phi_x(w) \geq_L w$ and $\psi_x(w) \leq_L w$ for any $w \in W$.
2. $\phi_x$ and $\psi_x$ are order-preserving under $\leq$. Namely, if $v \leq w$ then $\phi_x(v) \leq \phi_x(w)$ and $\psi_x(v) \leq \psi_x(w)$.
3. For any $y \in W$, the map $(x \mapsto \phi_x(y))$ is order-preserving and the map $(x \mapsto \psi_x(y))$ is order-reversing under $\leq$.
4. $\phi_x \psi_{x^{-1}}(y) \geq y$ and $\psi_{x^{-1}} \phi_x(y) \leq y$ for any $y \in W$.
5. $\phi_x$ preserves strong meets and $\psi_x$ preserves strong joins. Namely, for $v, w \in W$,
   - (a) if $v \land w$ exists then $\phi_x(v) \land \phi_x(w)$ exists and equals to $\phi_x(v \land w)$.
   - (b) if $v \lor w$ exists then $\psi_x(v) \lor \psi_x(w)$ exists and equals to $\psi_x(v \lor w)$.

**Remark 3.5.** This lemma also works for $\phi_x^R$ and $\psi_x^R$ instead of $\phi_x$ and $\psi_x$.

**Remark 3.6.** For the statements on $\phi_x$, (1) of this lemma is done in [BM15 Proposition 3.1(d)]; (2) and (3) in [BM15 Proposition 3.1(c)].

**Proof.** (1) is clear from the definition of $\phi_x$ and $\psi_x$. (2) is from the Lifting Property. (3) is clear from (1) and the Subword Property. (4) The case $x = s \in S$ is clear from the definition of $\phi_s, \psi_s$, and the general case follows from this and (2).

(5) (a) It suffices to prove it when $x = s \in S$. Write simply $\phi = \phi_s$ and $\psi = \psi_s$. Assume $v \land w$ exists. We have $\phi(v \land w) \leq \phi(v), \phi(w)$ by (2). To show that $\phi(v \land w)$ is the meet of $\phi(v)$ and $\phi(w)$, take Arbitrary $u$ with $u \leq \phi(v), \phi(w)$. Then $\psi(u) \leq \psi(v), \psi(w)$ from the Lifting Property, and hence $\psi(u) \leq v \land w$, which implies $\psi(u) \leq v \land w$. Applying $\phi$, we have $\phi(u) = \phi(\psi(u)) \leq \phi(v \land w)$, and hence $u \leq \phi(v \land w)$. (b) is essentially the same as (a).

**Remark 3.7.** The map $\phi_x$ (resp. $\psi_x$) does not preserve strong joins (resp. meets) in general. For example, letting $W = S_4$, we have $s_{212} \land s_{223} = s_2$ but $\psi_2(s_{122}) \land \psi_2(s_{223}) = s_{12} \land s_{32} = s_2 = \psi_2(s_2)$, where we write $s_{ij...}$ instead of $s_i s_j ...$. Mapping everything above via $x \mapsto x w_0$ where $w_0$ is the longest element of $W$, we obtain a counterexample for $\phi_x$ preserving joins.

**Corollary 3.8.** Let $u, v, x, y \in W$ with $\langle u \rangle(x)$ and $\langle v \rangle(y)$ are reduced and $ux = vy$ (or namely, $u \leq_L ux = vy \geq_L v$). Then $u \geq v \iff x \leq y$.

**Proof.** By Lemma 3.3 we have $u \geq v \iff u^{-1} \geq v^{-1} \iff (x =) \psi_{u^{-1}}(ux) \leq \psi_{v^{-1}}(vy) (= y)$. The other direction is similar.

### 3.3 Half-strong, half-weak meets and joins

Analogous to the meets and joins under the weak order, we show the existence of the minimum element (under $\leq$) of the set

$$\{ z \in W \mid x \leq z \geq_L y \},$$

and the maximum of

$$\{ z \in W \mid x \geq_L z \leq y \}.$$
Proposition 3.10. Let \( x, y \in W \).

1. The set \( \{ u \in W \mid x \leq \phi_u(y) \} \) has the minimum element \( \psi_{y^{-1}}(x) \) under the strong order.

2. The set \( \{ u \in W \mid \psi_{y^{-1}}(x) \leq y \} \) has the minimum element \( \psi_{y^{-1}}(x) \) under the strong order.

Proof. (1) We prove it by induction on \( l(y) \). The base case \( l(y) = 0 \) being clear, we assume \( l(y) > 0 \). Take \( s \in S \) such that \( y > ys \). Let \( x' = \psi_{y}(x) = \min(x, xS) \) and \( y' = ys \). Since \( y = y' * s \), for any \( u \) we see \( u*y = u*y'*s \), whence by the Lifting Property \( x \leq u*y \iff x' \leq u*y' \). Hence \( D(x, y) = D(x', y') \) where \( D(x, y) = \{ u \in W \mid x \leq \phi_u(y) = u*y \} \). By the induction hypothesis it follows that \( D(x, y) \) has the minimum element \( \psi_{y^{-1}}(x') \), which equals to \( \psi_{y^{-1}}(x) \).

(2) Let \( E(x, y) = \{ u \in W \mid \psi_{y^{-1}}(x) \leq y \} \). It suffices to show \( D(x, y) = E(x, y) \). By Lemma 3.4, we have \( x \leq \phi_u(y) \implies \psi_{y^{-1}}(x) \leq \psi_{y^{-1}}(y) \leq y \) and \( \psi_{y^{-1}}(x) \leq y \implies x \leq \phi_u \psi_{y^{-1}}(x) \leq \phi_u(x) \). □

Proposition 3.11. Let \( x, y \in W \).

1. The set \( \{ z \in W \mid x \leq z \geq_L y \} \) has the minimum element \( \psi_{y^{-1}}(x)y \) under the strong order.

2. The set \( \{ z \in W \mid x \geq_L z \leq y \} \) has the maximum element \( \psi_{y^{-1}}(x)^{-1} x \) under the strong order.

Proof. (1) By (7), we have \( D(x, y) \supset \{ u \mid x \leq uy \geq_L y \} \simeq \{ z \mid x \leq z \geq_L y \}; u \mapsto uy \), where the isomorphism is under \( \leq \). The minimum element \( u \) of \( D(x, y) \) satisfies \( u*y = uy \) i.e. \( uy \geq y \) and since otherwise \( u*y^{-1}(y) \) is a smaller element of \( D(x, y) \). Hence by Proposition 3.10, we have \( \psi_{y^{-1}}(x)y = min_{\leq} \{ z \mid x \leq z \geq_L y \} \).

(2) By Corollary 3.8, we have \( E(x, y) \supset \{ u \mid x \geq_L u^{-1}x \leq y \} \simeq \{ z \mid x \geq_L z \leq y \}; u \mapsto u^{-1}x \), where the anti-isomorphism is under \( \leq \). For a similar reason to (1) we have \( max_{\leq} \{ z \mid x \geq_L z \leq y \} = (min_{\leq} E(x, y))^{-1} x = (\psi_{y^{-1}}(x))^{-1} x \).

From the proposition above, we define

\[
x_{S \lor \\hat{L} \ y} = y_{S \lor \\hat{L} \ x} := \min_{\leq} \{ z \in W \mid x \leq z \geq_L y \} = \psi_{y^{-1}}(x)y,
\]

\[
x_{\hat{L} \land S \ y} = y_{\hat{L} \land S \ x} := \max_{\leq} \{ z \in W \mid x \geq_L z \leq y \} = (\psi_{y^{-1}}(x))^{-1} x.
\]

We define \( x_{S \lor \hat{R} \ y} \) and \( x_{\hat{L} \land \hat{R} \ y} \) similarly.

3.4 Flipping lower weak intervals

For any \( z \in W \), define the map

\[
\Phi_z : [e, z]_L \to [e, z]_R; x \mapsto zx^{-1}
\]

and its inverse

\[
\Psi_z : [e, z]_R \to [e, z]_L; y \mapsto y^{-1}z.
\]

Proposition 3.12 below demonstrates that these maps behave well along with the strong order on \( W \) and its meet/join operations.

Proposition 3.12. Let \( z \in W \).

1. \( \Phi_z \) and \( \Psi_z \) are anti-isomorphisms under the strong order.

2. \( l(\Phi_z(x)) = l(z) - l(x) \) for any \( x \in [e, z]_L \) and \( l(\Psi_z(y)) = l(z) - l(y) \) for any \( y \in [e, z]_R \).
We only prove (2). Let us consider the following claim we have
\[ \Phi_{z}(x \wedge y) = \Phi_{z}(x) \vee \Phi_{z}(y). \]

**Proof.** (1) is done in Corollary 3.13 and (2) is obvious.

(3) \( \Phi_{z} \) and \( \Psi_{z} \) send strong meets to strong joins. Namely,

(a) for \( x, y \in [e, z]_{L} \) such that \( x \wedge y \) exists and \( x \wedge y \in [e, z]_{L} \), we have \( \Phi_{z}(x \wedge y) = \Phi_{z}(x) \vee \Phi_{z}(y) \).

(b) for \( x, y \in [e, z]_{R} \) such that \( x \wedge y \) exists and \( x \wedge y \in [e, z]_{R} \), we have \( \Psi_{z}(x \wedge y) = \Psi_{z}(x) \vee \Psi_{z}(y) \).

(Note that the meets and joins are not taken in \( [e, z]_{L} \) or \( [e, z]_{R} \) but in \( W \).)

**Proof.** Let \( x, y, x \wedge y \in [e, z]_{L} \). From (1) it follows that \( \Phi_{z}(x \wedge y) \geq \Phi_{z}(x), \Phi_{z}(y) \). To show the minimality of \( \Phi_{z}(x \wedge y) \), let us take arbitrary \( w \in W \) such that \( w \geq \Phi_{z}(x), \Phi_{z}(y) \). From Proposition 3.11 we can let \( w' = z_{R} \wedge z_{S} w \). Since \( \Phi_{z}(x), \Phi_{z}(y) \in [e, z]_{R} \wedge [e, w] \), we have \( \Phi_{z}(x), \Phi_{z}(y) \leq w' \). Since \( w' \leq z_{R} \), applying \( \Psi_{z} \) (\( = \Phi_{z}^{-1} \)), we have \( x, y \geq \Psi_{z}(w') \). Hence \( x \wedge y \geq \Psi_{z}(w') \). Applying \( \Phi_{z} \), we have \( \Phi_{z}(x \wedge y) \leq w' \), and hence \( \Phi_{z}(x \wedge y) \leq w \). Therefore \( \Phi_{z}(x \wedge y) \) is the join of \( \{ \Phi_{z}(x), \Phi_{z}(y) \} \).

**Remark 3.13.** It seems to be true that \( \Phi_{z} \) and \( \Psi_{z} \) send strong joins to strong meets. Its proof would require that there be the strong-minimum element of \( \{ z \mid x \leq z \leq y \} \) and the strong-maximum of \( \{ z \mid x \leq L z \leq y \} \) for any \( x, y \in W \), analogous to Proposition 3.11.

### 3.5 Chain Property for lower weak intervals

In this section we prove the Chain Property for the lower weak intervals \([e, u]_{L}\) and \([e, u]_{R}\) for arbitrary Coxeter group \( W \) and its element \( u \in W \). This is similar to that for the generalized quotients, in that \([e, u]_{L} = \{ x \mid x \leq u \} \) whereas \( W/\{ u \} = \{ x \mid x \geq u \} \). Besides it is shown in [BWSS Corollary 4.5] that the class of right generalized quotients and lower left intervals coincide for finite \( W \). When \( W \) is infinite, however, these do not, as we give a counterexample below. Beforehand we recall [BWSS Theorem 4.10]: for any Coxeter group \( W \), the left generalized quotients and the right generalized quotients are in bijection by \( U \mapsto W/U \) and \( V \mapsto V/W \), and a subset \( U \subset W \) is a right generalized quotient if and only if \( U = W/(U \setminus W) \).

**Example 3.14.** Let \( W = \tilde{S}_{k+1} = \langle s_{0}, s_{1}, \ldots, s_{k} \rangle \). Let \( w_{0} \) be the longest element of \( S_{k+1} = \langle s_{1}, \ldots, s_{k} \rangle \). From the following claim we have \( s_{0}w_{0} \in \tilde{S}_{k+1}/(\tilde{S}_{k+1} \setminus \tilde{S}_{k+1}) \), and thereby \( S_{k+1} = [e, w_{0}]_{L} \) is not a right generalized quotient of \( \tilde{S}_{k+1} \).

**Claim.** For any \( z \in \tilde{S}_{k+1}, \) the product \( \langle w_{0} \rangle(z) \) is reduced if and only if \( \langle s_{0}w_{0} \rangle(z) \) is reduced.

**Proof of Claim.** The “if” direction is clear. Toward the “only if” direction, assume \( \langle w_{0} \rangle(z) \) is reduced, that is, \( \langle z^{-1} \rangle \langle w_{0} \rangle(z) \) is reduced. Since \( z^{-1}w_{0} \geq L w_{0} \), we have \( RD(z^{-1}w_{0}) \supset RD(w_{0}) \). Hence, since the first row of \( RD(w_{0}) \) is \( \{ 1, \ldots, k \} \) and the rows of a \( k \)-code are proper subsets of \( \{ 0, 1, \ldots, k \} \), the first row of \( RD(z^{-1}w_{0}) \) is also \( \{ 1, \ldots, k \} \). Thus, inserting \( s_{0} \) into \( RD(z^{-1}w_{0}) \) from the bottom and justifying it to the bottom with maximizing moves, we successfully obtain \( RD(z^{-1}w_{0}s_{0}) \), the \( i \)-th column of which is

- the \( k \)-th column of \( RD(z^{-1}w_{0}) \) with an \( s_{0} \) added, when \( i = 0 \),
- the \( i \)-th column of \( RD(z^{-1}w_{0}) \) when \( i = 1, \ldots, k-1 \),
- empty when \( i = k \).

(See Figure 4) In particular \( \langle z^{-1}w_{0} \rangle(z) \) is reduced. Combining this with that \( \langle z^{-1} \rangle \langle w_{0} \rangle(z) \) is reduced, we have \( \langle z^{-1} \rangle \langle w_{0} \rangle(z) \) is reduced, and hence so is \( \langle s_{0} \rangle \langle w_{0} \rangle(z) \), as desired.

The proof of the following proposition is parallel to that of [BWSS Theorem 3.4]. Beforehand we recall that, for \( x, y \in W \) with \( x \geq y \) and any fixed reduced expression \( x = s_{1} \ldots s_{m} \), there exists \( 1 \leq j_{1} < j_{2} < \cdots < j_{l} \leq m \) such that \( x = y^{(0)} > y^{(1)} > \cdots > y^{(l)} = y \) where

\[ y^{(a)} = s_{1} \ldots \hat{s}_{j_{1}} \ldots \hat{s}_{j_{a}} \ldots s_{m}. \]

See, for example, [BWSS Section 3] or [BWS2] for the detail.
Properties on the weak strips

Theorem 3.16. For any cyclically decreasing element corresponding to $A$ the weak order has the Chain Property.

Proof. Suppose to the contrary that there exists $a$ such that $y^{(a)}u \not\leq_L u$. Since $y^{(l)}u = yu \leq_L u$, we can take such $a$ that $y^{(a)}u \not\leq_L u$ and $y^{(a+1)}u \leq_L u$.

Since $xu \leq_L u$, we have $s_{j_a+1} \ldots s_m u \leq_L u$. Hence there exists $p < j_a$ such that

$$s_p z u \not\leq_L u \quad \text{and} \quad z u \leq_L u,$$

where we put

$$z = s_{p+1} \ldots \hat{s}_{j_a} \ldots s_{j_{a+1}} \ldots s_m,$$

where there may be more indices omitted between $s_{p+1}$ and $\hat{s}_{j_a}$, according to the omissions in $y^{(a)} = s_1 \ldots \hat{s}_{j_1} \ldots s_{j_a} \ldots s_m$. Since $y^{(a+1)}u \leq_L u$, we have

$$s_p \hat{z} u \leq_L u \quad \text{and} \quad \hat{z} u \leq_L u,$$

where we put

$$\hat{z} = s_{p+1} \ldots \hat{s}_{j_a} \ldots s_{j_{a+1}} \ldots s_m.$$

We have $zu < s_p z u$ by (9) and $\hat{z} u > s_p \hat{z} u$ by (10). Besides, since $y^{(a)} > y^{(a+1)}$ it follows $z > \hat{z}$, and thereby $zu < \hat{z} u$. Hence we have $s_p z u = \hat{z} u$ by the Lifting Property and length arguments. Therefore $s_p z = \hat{z} < z$, which contradicts the fact that $s_p z$ is a consecutive subword of a reduced expression for $y^{(a)}$.

As a corollary, we have the Chain Property for weak lower intervals:

Theorem 3.16. For any $u \in W$, the principal order ideal $[e,u]_L$ (resp. $[e,u]_R$) under the left (resp. right) weak order has the Chain Property.

Proof. The statement for left lower intervals follows from Proposition 3.15 and that $\{ x \in W \mid xu \leq_L u \} = [e,u^{-1}]_L$ for $u \in W$, which follows from $xu \leq_L u \iff x^{-1} \leq_R u \iff x \leq_L u^{-1}$. The statement for right intervals is proved parallely.

4 Properties on the weak strips

Hereafter, throughout this paper, we restrict our attention to $\tilde{S}_{k+1}$ rather than general Coxeter groups and let $W = \tilde{S}_{k+1}$ and $W^0 = \tilde{S}^0_{k+1}$. In Section 2.2 we put $I = \{0,1,\ldots,k\}$ and let $d_A$ denote the cyclically decreasing element corresponding to $A \subseteq I$.

In this section we prove some properties on weak strips. First we define for any $u \in W$,

$$Z_{u,+} = \{ v \in W \mid v = d_A u \geq_L u \text{ for } \exists A \subseteq I \},$$

$$Z'_{u,+} = \{ A \subseteq I \mid d_A u \geq_L u \} = \{ A \subseteq I \mid d_A u \in Z_{u,+} \},$$

$$Z_{u,-} = \{ v \in W \mid v = d_A^{-1} u \leq_L u \text{ for } \exists A \subseteq I \},$$

$$Z'_{u,-} = \{ A \subseteq I \mid d_A^{-1} u \leq_L u \} = \{ A \subseteq I \mid d_A^{-1} u \in Z_{u,-} \}.$$

Figure 6: Inserting $s_0$ into $RD(z^{-1}w_0)$ and justify it to obtain a k-code for $z^{-1}w_0s_0$
It is an immediate observation from the Subword Property that

- The map \((Z'_{u,+}, \subset) \to (Z_{u,+}, \leq)\); \(A \mapsto d_A u\) is an isomorphism of posets.
- The map \((Z'_{u,-}, \subset) \to (Z_{u,-}, \leq)\); \(A \mapsto d_A^{-1} u\) is an anti-isomorphism of posets.

Since if \(u \in W^o\) and \(v \leq_L u\) then \(v \in W^o\), for \(u \in W^o\) we have

\[ Z_{u,-} = \{v \mid v/u\text{ is a weak strip}\}. \]

On the other hand, the set \(Z_{u,+}\) does not coincide with the set of \(v\) such that \(v/u\) is a weak strip. More precisely, for \(u \in W^o\) we have by definition

\[ v/u \text{ is a weak strip } \iff v \in Z_{u,+} \text{ and } v \in W^o. \]

Recalling that \(v \in W^o \iff vw_{0}^J \geq_L w_{0}^J\) where \(J = \{1, \ldots, k\}\) and \(w_{0}^J\) is the longest element of \(W_J = S_{k+1}\), by Lemma 2.1 we have

\[ v/u \text{ is a weak strip } \iff vw_{0}^J \in Z_{uw_{0}^J,+} \iff v = d_A u \text{ with } A \in Z'_{uw_{0}^J,+}. \]

In other words, defining

\[ Z_{u,+}^o = \{v \mid v/u\text{ is a weak strip}\}, \]
\[ Z_{u,+}^J = \{A \subseteq I \mid d_A u/u\text{ is a weak strip}\} = \{A \subseteq I \mid d_A u \in Z_{u,+}^o\}, \]

we have

\[ Z_{u,+}^o \simeq Z_{uw_{0}^J,+} : v \mapsto vw_{0}^J, \]
\[ Z_{u,+}^J = Z_{uw_{0}^J,+}. \]

**Example 4.1.** Figure 7 illustrates the same example as Example 2.10.

From the example above, we would expect these properties:

1. \(Z'_{u,\pm}\) is closed under intersection and union.
2. \(Z'_{u,\pm}\) has the maximum element.
3. \(Z_{u,\pm}\) and \(Z'_{u,\pm}\) have the Chain Property. (See Section 4.3 for the detail)

(1), (2), (3) are proved in Section 4.1, 4.2, 4.3, respectively.

### 4.1 Intersection and union

In this section we prove the following proposition as the compilation of Lemma 4.5, 4.9 and 4.10

**Proposition 4.2.** For \(u \in W\), we have

1. \(A, B \in Z'_{u,\pm}\) and \(A \cup B \neq I \implies A \cup B \in Z'_{u,\pm}\).
2. \(A, B \in Z'_{u,\pm}\) \(\implies A \cap B \in Z'_{u,\pm}\).
3. \(A, B \in Z'_{u,+} \implies d_{A \cap B} u = (d_A u) \land (d_B u).\)
Figure 7: The posets $Z_{u,+}^\circ (\simeq Z_{w_{0d},+}^\circ)$ and $Z_{u,-}^\circ = (Z_{w_{0d},-}^\circ)$ for $u = w_\lambda$ where $k = 3$ and $\lambda = (3, 2, 1) \in \mathcal{P}_3$ (and $w_{0d}$ is the longest element of $S_4$). Left weak covers are represented as solid lines, and strong covers are solid or dotted lines. A solid edge between $v$ and $w$ is labelled with $i$ if $v = s_iw$.

(4) $A, B \in Z_{u,-}^\circ \implies d_{A \cap B}^{-1}u = (d_A^{-1}u) \cup (d_B^{-1}u)$.

In this section we say $A, B \subseteq I$ are strongly disjoint if for any $i \in A$ and $j \in B$ it holds that $i - j \neq 0, \pm 1$, and $x, y \in W$ are strongly commutative if any Coxeter generator $s_i$ appearing in a reduced expression of $x$ and any $s_j$ appearing in that of $y$ satisfy $i - j \neq 0, \pm 1$. The next lemma is straightforward.

Lemma 4.3. Let $A, B \subseteq I$ and $x, y \in W$.

(1) If $A, B$ are strongly disjoint, then $d_A, d_B$ are strongly commutative.

(2) For the decomposition $A = A_1 \sqcup \cdots \sqcup A_m$ into connected components, $A_1, \ldots, A_m$ are pairwise strongly disjoint and $d_{A_1}, \ldots, d_{A_m}$ are pairwise strongly commutative.

(3) For $x' \leq x$ and $y' \leq y$, if $x, y$ are strongly commutative then so are $x', y'$.

(4) If $x, y$ are strongly commutative, then $x, y$ are commutative and $l(xy) = l(x) + l(y)$.

Lemma 4.4. Let $x, y, z \in W$ with $x, y$ are strongly commutative. Then

(1) $z \leq_L xyz \iff z \leq_L xz, yz$.

(2) $z \geq_L xyz \iff z \geq_L xz, yz$.

Proof. (1) The “only if” direction immediately follows by the definition of the weak order and commutativity of $x, y$. We prove the “if” direction by induction on $l(x) + l(y)$. It is clear when $l(x) = 0$ or $l(y) = 0$. In particular the case $l(x) + l(y) \leq 1$ is done and we may assume $l(x) + l(y) \geq 2$ and $l(x), l(y) > 0$.

Step A: the case $l(x) + l(y) = 2$, i.e. $l(x) = l(y) = 1$.

We can write $x = s_i$ and $y = s_j$ with $s_i \neq s_j$, $s_is_j = s_js_i$ from the strong commutativity. We have $s_is_jz \geq_L z$ by the assumption. Hence $z \in W/W_{\{i,j\}}$, where $W_{\{i,j\}} = \langle s_i, s_j \rangle = \{e, s_i, s_j, s_is_j\}$. Therefore $s_is_jz \geq_L z$.

Step B: the case $l(x) + l(y) > 2$.

From the commutativity of $x, y$ we may assume $l(y) \geq l(x)$; in particular $l(y) > 1$. Take a reduced expression of $y = s_{i_1} \cdots s_{i_t}$ and put $y' = s_{i_1} \cdots s_{i_{t-1}}$, $z' = s_{i_t}z$. Since $z \leq_L yz$ and $s_i \leq_L y$, we have $z \leq_L z'$. Now we can obtain $z' \leq_L xy'z'$, which implies $z \leq_L z' \leq_L xy'z' = xyz$ as desired, by applying the induction hypothesis for $(x, y, z) := (x, y', z')$, having its assumption satisfied as follows:
\[ x, y' \text{ are strongly commutative.} \]

Proof. From Lemma 4.3 (3).

\[ z' \leq z \] \[ y' z'. \]

Proof. Since \( z \leq y z \) and \( s_i \leq y \), by Lemma 2.1 (1) we have \( z' = s_i z \leq y z = y' z'. \)

\[ z' \leq z \] \[ x z'. \]

Proof. Since \( l(x) + l(y) > l(x) + l(s_i) \), we can obtain \( z \leq x z' \) by applying the induction hypothesis for \((x, y, z) := (x, s_i, z)\), having that its assumption described below is clearly satisfied:

- \( x \) and \( s_i \) are strongly commutative.
- \( z \leq x z \).
- \( z \leq s_i z \).

Besides \( s_i \leq x s_i \), hence we have \( z' \leq x z' \) by Lemma 2.1 (1).

(2) is proved similarly to (1).

Lemma 4.5. Let \( w \in W \) and \( A, B \subseteq I \) with \( w \leq d_Aw, dBw \).

(1) \( w \leq d_{A \cap B}w \).

(2) The element \( d_{A \cap B}w \) is the strong meet of \( d_Aw \) and \( dBw \).

Remark 4.6. The same statement with all \( d_X \) replaced with \( u_X \) is proved similarly.

Remark 4.7. It does not generally hold that if \( w \leq x w, yw \) and \( x \wedge y \) exists then \( w \leq (x \wedge y)w \); a counterexample is \( W = S_4, x = s_{21}, y = s_{23}, w = s_2 \).

Proof. (1) Within this proof we call \( x \in W \) satisfies (*) if \( w \leq x w \).

Decomposing \( A, B \) into connected components \( A = A_1 \sqcup \cdots \sqcup A_m \) and \( B = B_1 \sqcup \cdots \sqcup B_n \), we have \( A \cap B = \bigcup_{i,j} (A_j \cap B_j) \). Each nonempty \( A_j \cap B_j \) has at most two connected components, each component \( C \) of which satisfies \( d_{A_j} = xd_C \) for some \( x \in W \) or \( d_{B_j} = yd_C \) for some \( y \in W \) as easily seen. Having that both \( d_A (\geq d_{A_j}) \) and \( d_B (\geq d_{B_j}) \) satisfy (*) and that lower bounds in \( \leq \) inherit (*), we see each \( d_C \) satisfies (*). Besides \( (A_j \cap B_j) \cap (A_j' \cap B_j') = (A_j \cap A_j') \cap (B_j \cap B_j') \) is empty unless \( (i, j) = (i', j') \), we thus have \( A \cap B \) decomposes as \( A \cap B = C_1 \sqcup \cdots \sqcup C_l \) into connected components, where each \( d_{C_j} \) satisfies (*). Now it follows from Lemma 4.4 (1) that \( d_{A \cap B} = d_{C_1} \ldots d_{C_l} \) satisfies (*), as desired.

(2) By the Subword Property we have \( d_{A \cap B} = d_A \wedge dB \). From the assumption and (1), we have \( \phi_w^R(d_X) = d_X w \) for \( X = A, B, A \cap B \). Hence by Lemma 3.3 (5) we have \( d_{A \cap B}w = d_Aw \wedge dBw \).

Corollary 4.8. Let \( \lambda \in P_k \), and \( k^{(1)}, k^{(2)} \) be weak strips over \( \lambda \). Write \( k^{(i)} = d_{A_i} \lambda \) for each \( i \) with \( A_i \subseteq I \). Then \( d_{A_i \cap A_j} \lambda \) is a weak strip over \( \lambda \) and is the meet of \( k^{(i)}, k^{(2)} \) in the poset \( P_k \) with the strong order.

Proof. Let \( w_\lambda \in W^w \) be the affine Grassmannian permutation corresponding to \( \lambda \), and \( w_0 \lambda \) the longest element of \( S_{k+1} \). By Lemma 2.1, the condition \( d_A \lambda / \lambda \) is a weak strip is equivalent to \( d_{Aw_\lambda} \lambda \geq L \) \( w_\lambda w_0 \). From this and Lemma 4.3 (1) we see \( d_{A \cap A_2} \lambda / \lambda \) is a weak strip. From Lemma 4.3 (2) we have \( d_{A \cap A_2} \lambda = (d_{A_1} \lambda) \wedge (d_{A_2} \lambda) \) in \( W \). Since \( W^w \subset W \) is a subposet, this is also the meet in \( W^w \simeq P_k \).

Lemma 4.9. Let \( w \in W \) and \( A, B \subseteq I \) with \( d_A^{-1}w, d_B^{-1}w \leq L w \).

(1) \( d^{-1}_{A \cap B}w \leq L w \).

(2) The element \( d^{-1}_{A \cap B}w \) is the strong join of \( d^{-1}_{A}w \) and \( d^{-1}_{B}w \).

Proof. (1) is proved parallely to Lemma 4.5 (1), making use of Lemma 4.4 (2) instead of Lemma 4.4 (1).

(2) We have \( d_A^{-1}w, d_B^{-1}w, d^{-1}_{A \cap B}w \in \{ w, L \} \) by (1). The map \( \Phi_w \) in Lemma 3.12 sends \( d_A^{-1}w, d_B^{-1}w, d_{A \cap B}^{-1}w \) to \( d_A \lambda, dB, d_{A \cap B} \lambda \) respectively. Since \( d_{A \cap B} = d_A \wedge dB \), sending them back via \( \Psi_w \), we have \( d_A^{-1}w \vee (d_B^{-1}w) \).

\[ d^{-1}_{A \cap B}w = d_A^{-1}w \vee (d_B^{-1}w) \] by Lemma 3.12 (3).
Lemma 4.10. Let $u \in W$ and $A, B \subsetneq I$ with $A \cup B \neq I$.

(1) If $d_A u, d_B u \geq_L u$, then $d_{A \cup B} u \geq_L u$.

(2) If $d^-_A u, d^-_B u \leq_L u$, then $d^-_{A \cup B} u \leq_L u$.

Proof. We only give a proof of (1) since that of (2) is quite similar.

Assume $d_A u, d_B u \geq_L u$. Take the decomposition $A = A_1 \cup \cdots \cup A_n$ and $B = B_1 \cup \cdots \cup B_m$ into connected components. Since $d_{A_i} \leq d_A$, we have $d_{A_i} u \geq_L u$ for any $i$, and similarly $d_{B_j} u \geq_L u$ for any $j$. Since $A \cup B = (\cdots (A \cup B_1) \cup \cdots ) \cup B_n$, we only need to prove it when $B$ is connected. Assume $B$ is connected. It is also easy to see, from Lemma 4.4 and Lemma 4.4(1), that it suffices to prove it when $A, B$ and $A \cup B$ are connected. We therefore assume $A, B$ and $A \cup B$ are connected. The case $A \subset B$ or $B \subset A$ being clear, we assume $A \not\subset B \not\subset A$; namely we let $A = [i, j]$ and $B = [p, q]$ with $p \leq i \leq q + 1 \leq j + 1$ without loss of generality, where we employ an ordering $r + 1 < \cdots < k < 0 < \cdots < r - 1$ of $I \setminus \{r\}$ with an arbitrarily fixed element $r \in I \setminus (A \cup B)$. Since $d_B = s_q \ldots s_p \geq_L s_{i-1} \ldots s_p = d_{B \setminus A}$ and $d_B u \geq_L u$, we have $d_{B\setminus A} u \geq_L u$. Hence we may replace $B$ by $B \setminus A = [p, i - 1]$.

Let $B' = B \setminus \{i - 1\}$ and $u' = d_{B'} u$. Since $d_{B'} \leq d_B$ and $d_B u \geq_L u$, it follows that $u' \geq_L u$. Since $s_{i-1} u' = d_B u \geq_L u$ and $d_{B'} u = d_{d_B u} \geq_L u$, the latter of which is from Lemma 4.4(1), it easily follows that $s_{i-1} u' \geq_L u'$ and $d_{d_B u} \geq_L u'$ from Lemma 4.4.

Toward a contradiction, suppose $d_{A \cup B} u \not\leq_L u$. Then we have $d_{A \cup B} u \geq_L u$ since $d_{A \cup B} u = d_{d_B u}$ by $u \leq u'$. Since $s_{i-1} u' \geq_L u'$, there exists $a \in [i, j]$ such that $x_{s_{i-1} u'} \geq_L u'$ and $s_a x_{s_{i-1} u'} \not\geq_L u'$, which implies $s_a x_{s_{i-1} u'} < x_{s_{i-1} u'}$, where we write $x = s_{i-1} s_{a-2} \ldots s_{i+1} s_i$. On the other hand, since $d_B u \geq_L u'$ we have $s_a x u' \geq_L u'$ and $x u' \geq_L u'$. Besides we have $x_u' \geq_L x u'$ from the Subword Property. Hence the Lifting Property implies that $x u' \leq s_a x_{s_{i-1} u'}$, which is actually an equality since both sides have the same length. Therefore we have $(s_{a-1} s_{a-2} \ldots s_{i+1} s_i) = s_a x_{s_{i-1} u'} = \emptyset$, which is absurd.

Remark 4.11. Unlike the “cap” case, it does not always hold that $d_{A \cup B} u = (d_A u) \lor (d_B u)$ in (1), or $d^-_{A \cup B} u = (d^-_A u) \land (d^-_B u)$ in (2). A counterexample for (1) is given by $W = S_3$, $u = e$, $A = \{1\}$ and $B = \{2\}$.

4.2 Non-appearing indices

Proposition 4.12. (1) For any $w \in W$, there exists $i^-_w \in I$ such that $i^-_w \not\in A$ for any $A \subsetneq I$ with $d^-_A u \leq_L u$.

(2) For any $w \in W^\circ$, there exists $i^+_w \in I$ such that $i^+_w \not\in A$ for any $A \subsetneq I$ with $d_A u \geq_L u$ and $d_A u \in W^\circ$.

Proof. (1) For any $A \subsetneq I$, we have

$$d^-_A w \leq_L w \iff d_A w \leq_R w$$

$$\iff u_A \leq_L w^{-1}$$

$$\iff RI(u_B) \subset RI(w^{-1}),$$

and the last condition is equivalent to $A$ being included by the first row of $RI(w^{-1})$. Hence we can take $i^-_w$ from the complement of the first row of $RI(w^{-1})$.

(2) By Lemma 3.1 we may take $z := \bigvee \{d_B w \mid A \subsetneq I \text{ s.t. } d_B w \geq_L w, d_B w \in W^\circ\}$, the left join of all weak strips over $w$. Take any $A \subsetneq I$ such that $d_A w \geq_L w$ and $d_A w \in W^\circ$. Since $w, d_A w \leq_L z$, we have $zw^{-1} \geq_R z(d_B w)^{-1} = zw^{-1}u_B$, which is equivalent to $wz^{-1} \geq_L d_A wz^{-1}$. Hence, similarly to the proof of (1) we have $A$ is a subset of the first row of $RD((wz^{-1})^{-1}) = RD(zw^{-1})$, which is a proper subset of $I$ and independent of $A$, and therefore we can take $i^+_w$ from its complement.

Remark 4.13. The index $i^+_w$ in (2) above is in fact uniquely determined as follows: a bounded partition $\lambda \in P_k$, corresponding to a 0-dominant affine permutation $w_\lambda \in W^\circ$, has the unique weak strip of size $k$, namely $(k) \cup \lambda$. Since the corresponding core $c((k) \cup \lambda)$ has $k$ more boxes in the first row than $c(\lambda)$ does, the only possibility for $i^+_w$ is what is determined by the following equivalent descriptions:

- The residue of the rightmost box in the first row of $c(\lambda)$.  

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Proof. By Proposition 4.2 (1) and Proposition 4.12.

Remark example, let \( c(\lambda) = (5, 2, 1) \). The dotted shape on the left figure represents \( c((k) \cup \lambda) \), and the solid one does \( c(\lambda) \). In this case \( w_{(k)\cup\lambda} = s_3s_2s_1w_\lambda = d_{(1; 2; 3)}w_\lambda \) and therefore \( i^+_{w_\lambda} = 0 \).

- The negative of the residue written in the leftmost box in the last row of \( \text{RI}(w_\lambda) = \lambda^{\omega_k} \).
- \( m - 1 \), where \( w_\lambda = u_A \ldots u_A \) is the maximal increasing decomposition for \( w_\lambda \). (Note that \( A_m = \{ i, i + 1, \ldots, m - 2, m - 1 \} \) for some \( i \).

Remark 4.14. We cannot drop the assumption on 0-dominantness of \( d_Aw \) in (2) of the proposition. For example, let \( k = 3 \) and \( w = s_3s_0 \). Then \( w = u_{(3, 0)} \) is the maximal increasing decomposition and hence \( i^+_w \) should be 0, but \( d_{(10)}w = s_6s_3s_0 \geq_L w \).

Corollary 4.15. Let \( u \in W \).

1. \( Z'_{u,+} \) has the maximum element under \( < \). Hence, \( Z_{u,+} \) has the maximum element under \( \leq \).
2. \( Z'_{u,-} \) has the maximum element under \( < \). Hence, \( Z_{u,-} \) has the minimum element under \( \leq \).

Proof. By Proposition 4.2 (1) and Proposition 4.12.

4.3 Chain Property

Proposition 4.16. The sets \( Z_{u,+} \) and \( Z_{u,-} \) have the Chain Property. Namely, for any \( x, y \in Z_{u,\pm} \) such that \( x \leq y \), there exists a sequence \( x = \exists z(0) < \exists z(1) \leq \ldots < \exists z(i) = y \) such that \( z(i) \in Z_{u,\pm} \) for any \( i \).

Proof. First we note a few immediate observations:

- For a poset \( P \) and a subposet \( Q \subset P \), if \( A \subset P \) is an order ideal then \( A \cap Q \) is an order ideal of \( Q \).
- If a subset \( X \) of a Coxeter group \( W \) has the Chain Property and \( Y \subset X \) is an order ideal, then \( Y \) also has the Chain Property.

Let \( D = \{ d_A | A \subset I \} \). Since \( D \subset W \) is an order ideal, the set \( \{ d_A | d_A \leq_R u \} = D \cap [e, u]_R \) is an order ideal of \( [e, u]_R \) and hence has the Chain Property since so does \( [e, u]_R \) as proved in Theorem 3.18. Hence \( Z_{u,-} \) also has the Chain Property since it is the image of \( \{ d_A | d_A \leq_R u \} \) under the the anti-isomorphism \( \Psi_u: [e, u]_R \rightarrow [e, u]_L; x \mapsto x^{-1}u \).

Similarly, the set \( \{ d_A | d_Au \geq_L u \} = D \cap (W/\{u\}) \) has the Chain Property since it is an order ideal of \( W/\{u\} \), which has the Chain Property [BWSS Corollary 3.5]. Hence, since \( Z_{u,+} \) is the image of \( \{ d_A | d_Au \geq_L u \} \) under the isomorphism \( \cdot(u): W/\{u\} \rightarrow [u, \infty]_L \), we conclude that \( Z_{u,+} \) has the Chain Property.

From the isomorphism \( (Z_{u,+}, \leq) \simeq (Z'_{u,+}, \subset) \) and the anti-isomorphism \( (Z_{u,-}, \leq) \simeq (Z'_{u,-}, \subset) \), we have the Chain Property for \( Z'_{u,\pm} \):

Corollary 4.17. The sets \( Z'_{u,+} \) and \( Z'_{u,-} \) have the Chain Property. Namely, for any \( A, B \in Z_{u,\pm} \) with \( A \subset B \), there exists a sequence \( A = \exists C(0) \subset \exists C(1) \subset \ldots \subset \exists C(i) = B \) such that \( C(i) \in Z'_{u,\pm} \) for any \( i \).
5 Proof of the Pieri rule for $\gamma^{(k)}$

This section is devoted for the proof of Theorem 1.2 and 1.3.

5.1 Outline

Let $w = w_\lambda \in W^\circ$ be the affine Grassmannian element corresponding to $\lambda$. Recall the Pieri rule for $g^{(k)}_\lambda$ (Definition 2.19):

$$g^{(k)}_v h_i = \sum_{A \subseteq I, |A| = i} (-1)^{i-\ell(d_A \ast v) - \ell(v)} g^{(k)}_{d_A \ast v}.$$  

Summing this up over $v \in W^\circ \cap [e, w]$ and $i \in \{0, 1, \ldots, r\}$, we have

$$\gamma^{(k)}_w h_r = \sum_{v \leq w} \sum_{A \subseteq I, |A| \leq r} (-1)^{|A| - \ell(d_A \ast v) - \ell(v)} g^{(k)}_{d_A \ast v},$$

and its coefficient of $g^{(k)}_u$ (for $u \in W^\circ$) is

$$[g^{(k)}_u \gamma^{(k)}_w h_r] = \sum_{v \leq w} \sum_{A \subseteq I, |A| \leq r, u = d_A \ast v} (-1)^{|A| - \ell(u) - \ell(v)}. \hspace{1cm} (11)$$

We shall illustrate, in the example below, that if the summation above is not empty then there is a “matching” on the set of appearing $(A, v)$’s with an unmatched element, and the corresponding summands cancel accordingly, and consequently the value of the summation is equal to 1.

Example 5.1. Let $k = 3$ and $u = s_{310} = w_\lambda \in S_7^\circ$, where $\lambda = (2, 1) \in P_3$. Table I lists the pairs $(v, A)$ such that $d_A \ast v = u$, organized according to the size of $A$. Apparently there are the same number of pairs $(v, A)$ with $|A| = r'$ and $(-1)^{|A| - \ell(u) - \ell(v)} = +1$, and those with $|A| = r'$ and $(-1)^{|A| - \ell(u) - \ell(v)} = -1$, for each fixed $r' > 0$. Furthermore, introducing the condition $v \leq w$ for $w = s_{210}$, say, we see that the set of the pairs $(v, A)$ with $d_A \ast v = u$ and $v \leq w$ is $\{(s_{10}, \{3\}), (s_{10}, \{1, 3\}), (s_{10}, \{1, 3\})\}$. and that the number of such pairs $(v, A)$ with $|A| = r'$ and $(-1)^{|A| - \ell(u) - \ell(v)} = +1$ and those with $|A| = r'$ and $(-1)^{|A| - \ell(u) - \ell(v)} = -1$ coincide whenever $r' \neq 1$, and differ by 1 when $r' = 1$.

| $|A|$ | $(v, A)$ | $(-1)^{|A| - \ell(u) - \ell(v)}$ |
|-----|----------|-------------------------------|
| 0   | $(s_{310}, \emptyset)$ | +1                           |
| 1   | $(s_{30}, \{1\})$ | +1                           |
|     | $(s_{310}, \{1\})$ | -1                           |
|     | $(s_{10}, \{3\})$ | +1                           |
|     | $(s_{310}, \{3\})$ | -1                           |
| 2   | $(s_{10}, \{1, 3\})$ | +1                           |
|     | $(s_{10}, \{1, 3\})$ | -1                           |
|     | $(s_{30}, \{1, 3\})$ | -1                           |
|     | $(s_{310}, \{1, 3\})$ | +1                           |

According to the observation above, we let

$$X_{A,u} = \{v \in W \mid d_A \ast v = u\} = \{v \in W \mid \phi_{d_A}(v) = u\},$$

$$Y_{A,u} = X_{A,u} \cap [e, w].$$
Figure 9: The poset of 4-cores (and corresponding elements in $\tilde{S}_4^{\circ}$) up to those of size 4. The left weak covers are represented by solid lines, and the strong covers are dotted or solid lines. A solid edge labelled with $i$ corresponds to the left multiplication by $s_i$.

for $u \in W^\circ$ and $A \subsetneq I$. Note that, for any $v \in X_{A,u}$, Lemma 3.3(1) implies $v \leq_L u$, and hence it follows from $u \in W^\circ$ that $v \in W^\circ$. Hence

$$[g_u^{(k)}(\tilde{g}_w^{(k)})h_r] = \sum_{|A| \leq r} \sum_{v \in Y_{A,u}} (-1)^{|A|-(l(u)-l(v))}.$$  \hfill (12)

The flow of the proof is as follows:

Step 1. Every element of $X_{A,u}$ has the form $d_B^{-1}u$ with $B \subset A$, and thereby $X_{A,u}$ is anti-isomorphic to a subposet of $[\varnothing, A]$, denoted later by $X'_{A,u}$, by $d_B^{-1}u \mapsto B$.

Step 2. The poset $X'_{A,u} \subset [\varnothing, A]$ has the minimum element $B$ and is a boolean poset; $X'_{A,u} = [B, A]$.

Step 3. The subset $Y_{A,u}$ of $X_{A,u}$ being an order ideal, its image $Y'_{A,u}$ under $X_{A,u} \simeq X'_{A,u}$ is an order filter of $X'_{A,u}$. Moreover $Y'_{A,u}$ is closed under intersection, reflecting join-closedness of $Y_{A,u}$. Hence $Y'_{A,u}$ is also a boolean lattice. Therefore, the value of the summation over $v \in Y_{A,u}$ in (12) is 0 unless $|Y_{A,u}| = 1$ since its summands cancel out, and 1 if $|Y_{A,u}| = 1$.

Step 4. If $u \leq d_B w$ for some $B \subsetneq I$ with $|B| = r$ and $d_B w \geq_L w$, then there uniquely exists $A$ such that $|Y_{A,u}| = 1$, and hence the value of the right-hand side in (12) is 1. If there does not exist such $B$, then neither does such $A$, and hence (12) is 0.

Remark 5.2. The set $X_{A,u}$ is a fiber of the Demazure action $\phi_d^A$. In Step 2 (Corollary 5.11) this fiber is shown to be a boolean poset. Meanwhile, for the longest element $w_J$ of a finite parabolic subgroup $W_J$, any fiber of its Demazure action $\phi_w^J$ is a parabolic coset $W_J x$, whence isomorphic to $W_J$. More generally it might be interesting to find fibers of the Demazure action $\phi_w$ of an arbitrary element $w$.

5.2 Proof of Theorem 1.2 and 1.3

We fix $u \in W^\circ$. 

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Figure 10: Relation between $Z_{u,-}, Z'_{u,-}, X_{A,u}, X'_{A,u}, Y_{A,u}, Y'_{A,u}$. 

$B \mapsto d_B^{-1}u$

$\{\emptyset, I\} \supset Z'_{u,-} \sim_{\text{anti}} Z_{u,-} \subset [e, u]_L$

$\{\emptyset, A\} \supset X'_{A,u} \sim_{\text{anti}} X_{A,u} \subset [e, u]_L \cap [d_A^{-1}u, u]$ 

$Y'_{A,u} \sim_{\text{anti}} Y_{A,u} = X_{A,u} \cap [e, w]$

5.2.1 Step 1

We fix $A \subset I$. Since $Y_{A,u} \subset X_{A,u}$, the summation over $v$ in (12) is 0 when $X_{A,u} = \emptyset$. We hence assume $X_{A,u} \neq \emptyset$, since otherwise such $A$ does not contribute to the value of the right-hand side of (12). Take arbitrary $v \in X_{A,u}$. From Lemma 2.2 and the definition of $X_{A,u}$ we have

(1) $v, d_A^{-1}u \leq_L u$,

(2) $d_A^{-1}u \leq v$.

From Proposition 3.12 (1) and (1) above, (2) is equivalent to

(3) $uv^{-1} \leq d_A$.

The Subword Property and (3) imply $uv^{-1} = d_B$, or equivalently $v = d_B^{-1}u$, for some $B \subset A$. We have $A, B \in Z_{u,-}'$ from (1).

We let

$$X'_{A,u} = \{ B \subset I \mid d_B^{-1}u \in X_{A,u} \},$$

$$Y'_{A,u} = \{ B \subset I \mid d_B^{-1}u \in Y_{A,u} \}.$$

The argument above is restated as follows (see also Figure 10):

**Lemma 5.3.**

(1) $X_{A,u} \neq \emptyset \implies A \in Z_{u,-}'$.

(2) $X_{A,u} \subset [d_A^{-1}u, u]$.

(3) $(X'_{A,u}, \subset)$ and $(X_{A,u}, \leq)$ are anti-isomorphic by $B \mapsto d_B^{-1}u$.

(4) $X'_{A,u} \subset [\emptyset, A]$.

(5) $X'_{A,u} \subset Z_{u,-}'$.

**Proof.** It remains to show that the mapping $B \mapsto d_B^{-1}u$ in (3) is order-reversing, which follows from Proposition 3.12 (1) and the Subword Property.

5.2.2 Step 2 and 3

Let us start with an example to describe the situation.

**Example 5.4.** Let $k = 5$, $\lambda = (5, 3, 2, 1)$, $\mu = (5, 2, 2, 2)$, $u = w_\lambda$ and $w = w_\mu$ (see Figure 11). When $A = \{5, 0, 1\}$, for example, $X_{A,u} = Y_{A,u} = \{s_1u, s_{01}u, s_{11}u, s_{01}u\}$ and $X'_{A,u} = Y'_{A,u} = \{1\}, \{5, 0, 1\}$. Similarly, when $A = \{3, 5, 1\}$ we see $X'_{A,u} = \{\emptyset, \{3, 5, 1\}\}$ and $Y'_{A,u} = \{1\}, \{3, 5, 1\}$. 

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Lemma 5.5. $X_A$ and $Y_A$ are convex under the strong order. Namely, if $v \leq v' \leq v''$ and $v, v'' \in X_A$ (resp. $Y_A$) then $v' \in X_A$ (resp. $Y_A$).

Proof. It follows from Lemma 5.4(2).

Remark 5.6. It is not a very immediate consequence of Lemma 5.5 that $X_{A,u}'$ and $Y_{A,u}'$ are convex in the boolean poset $[\varnothing, I]$, yet it is shown to be true in Corollary 5.11.

In this section we write $\{i_1, \ldots, i_m\} <$ to denote the set $\{i_1, \ldots, i_m\}$ for which the condition that $(i_1, \ldots, i_m)$ is cyclically increasing is imposed.

Lemma 5.7. (1) $B, C \in X_{A,u}' \implies B \cap C \in X_{A,u}'$.

(2) $B, C \in Y_{A,u}' \implies B \cap C \in Y_{A,u}'$.

Proof of Claim 1. We only give a proof of the statement for $B$ since that for $C$ is the same.

(1) When $i_1 \notin B$, we see $d_{B}^{-1}u = d_{B}^{-1}u = s_{i_1}d_{B}^{-1}u < d_{B}^{-1}u$, and hence both (a) and (b) is clear.

(2) When $i_1 \notin B$, we claim that $s_{i_1}d_{B}^{-1}u < d_{B}^{-1}u$; suppose, on the contrary, $s_{i_1}d_{B}^{-1}u > d_{B}^{-1}u$. Then we have $s_{i_1}d_{B}^{-1}u \not\leq L u$ since $l(s_{i_1}d_{B}^{-1}u) > l(u) - l(s_{i_1}d_{B}^{-1}u)$. On the other hand, $u = \phi_{dA}(d_{B}^{-1}u) = \phi_{dA}(s_{i_1}d_{B}^{-1}u)$ since $s_{i_1}d_{B}^{-1}u > d_{B}^{-1}u$, and therefore $s_{i_1}d_{B}^{-1}u \leq L u$ by Lemma 2.2 which is in contradiction. Therefore $s_{i_1}d_{B}^{-1}u < d_{B}^{-1}u$. Now (a) is clear since $d_{B}^{-1}u = d_{B}^{-1}u$, and (b) follows from $d_{B}^{-1}u = s_{i_1}d_{B}^{-1}u$.

Claim 1 is proved.

Claim 2. $\phi_{i_1}(d_{B}^{-1}C)u = d_{B}^{-1}C'u$.

Proof of Claim 2. By Claim 1(b) and Proposition 1.2(2), we have $B'' \cap C'' \in Z_{u,-}'$, that is, $u \geq L d_{B'' \cap C''}u$. Since $B'' \cap C'' = (B' \cap C') \cup \{i_1\}$, we have $d_{B'' \cap C''}u = s_{i_1}d_{B'' \cap C''} > d_{B'' \cap C''}u$, and hence $d_{B'' \cap C''}u = s_{i_1}d_{B'' \cap C''}u < d_{B'' \cap C''}u$.

In this example we follow the cyclic ordering $3 < 4 < 5 < 1$ on $I \setminus \{2\}$, as we see $i_1 = 2$, i.e. every element of $Z_{u,-}'$ is a subset of $I \setminus \{2\}$.
Claim 3. \( B' \cap C' = X'_{A', u} \)

Proof of Claim 3. By Claim 1(a) and that \( B \in X'_{A, u} \), we have \( u = \phi_{d_A} (d_B^{-1} u) = \phi_{d_{A'}} \phi_i (d_B^{-1} u) = \phi_{d_{A'}} (d_B^{-1} u) \), and hence \( B' \in X'_{A', u} \). Similarly \( C' \in X'_{A', u} \). Hence \( B' \cap C' = X'_{A', u} \) by the induction hypothesis. Claim 3 is proved.

Now we have

\[
\phi_{d_A} (d_B^{-1} u) = \phi_{d_{A'}} \phi_i (d_B^{-1} u) = \phi_{d_{A'}} (d_B^{-1} u) = u.
\] (by Claim 2)

(2) follows from (1) and the definition of join and \( Y_{A, u} \).

Lemma 5.8. Let \( A, A' \in Z'_{u, -} \) with \( A' \subset A \) and \( |A \setminus A'| = 1 \). Then \( A' \in X'_{A, u} \).

Proof. Let \( A = \{i_1, \ldots, i_m\} \) and \( A' = \{i_1, \ldots, \hat{i}_k, \ldots, i_m\} \).

Since \( u \geq L d_A^{-1} u = s_{i_1} \ldots \hat{s}_k \ldots s_{i_m} u \),

- \( \phi_{i_j} (s_{i_1} \ldots s_{i_{k-1}} s_{i_{k+1}} \ldots s_{i_m} u) = s_{i_{j+1}} \ldots s_{i_{k-1}} s_{i_{k+1}} \ldots s_{i_m} u \) for \( 1 \leq j < k \),
- \( \phi_{i_j} (s_{i_1} \ldots s_{i_m} u) = s_{i_{j+1}} \ldots s_{i_m} u \) for \( k < j \leq m \).

Since \( u \geq L d_A^{-1} u = s_{i_1} \ldots s_{i_m} u \),

- \( \phi_{i_k} (s_{i_{k+1}} \ldots s_{i_m} u) = s_{i_{k+1}} \ldots s_{i_m} u \).

Hence

\[
\phi_{d_A} (d_A^{-1} u) = \phi_{i_m} \ldots \phi_{i_{k+1}} \phi_{i_k} \phi_{i_{k-1}} \ldots \phi_{i_1} (s_{i_1} \ldots s_{i_{k-1}} s_{i_{k+1}} \ldots s_{i_m} u) = \phi_{i_m} \ldots \phi_{i_{k+1}} \phi_{i_k} (s_{i_{k+1}} \ldots s_{i_m} u) = \phi_{i_m} \phi_{i_{k+1}} (s_{i_{k+1}} \ldots s_{i_m} u) = u.
\]
Lemma 5.9. Let $A = \{i_1, \ldots, i_m\} \subset Z'_{u,-}$ and $B \subset X'_{A,u}$. By Lemma 5.3(4) we can write $B = \{i_1, \ldots, \widehat{i_{k}}, \ldots, i_{j}, \ldots, i_m\}$ for some $1 \leq k \leq j \leq m$. Let $A(a) = \{i_{j+1}, i_{j+2}, \ldots, i_{m-1}, i_m\}$ and $B(a) = B \cap A(a) = \{i_{j+1}, \ldots, \widehat{i_{j+k}}, \ldots, i_{j}, \ldots, i_m\}$ for each $a \in \{1, \ldots, l\}$. Then, for each $1 \leq a \leq l$,

\[
s_{i_j}d^{-1}_{B(a)}u < d^{-1}_{B(a)}u.
\]

Proof. We carry out induction on $l = |A \setminus B|$, with trivial base case $l = 0$. Assume $l > 0$. From Lemma 5.3(5), we have $u \geq_{L} d_{B}^{-1}u = s_{i_1} \ldots s_{i_{j-1}}d_{B(i)}^{-1}u$, and hence $d_{B(i)}^{-1}u \geq_{L} s_{i_1} \ldots s_{i_{j-1}}d_{B(i)}^{-1}u$ by Lemma 2.1. Hence

\[
u = \phi_{d_A}(d_{B}^{-1}u) = \phi_{d_A(i)} \phi_{i_{j-1}} \phi_{i_{j-2}} \ldots \phi_{i_1}(s_{i_1} \ldots s_{i_{j-1}}d_{B(i)}^{-1}u) = \phi_{d_A(i)} \phi_{i_{j-1}}(d_{B(i)}^{-1}u).
\]

We now claim $s_{i_j}d_{B(i)}^{-1}u < d_{B(i)}^{-1}u$; suppose to the contrary that $s_{i_j}d_{B(i)}^{-1}u > d_{B(i)}^{-1}u$. Then we have $s_{i_j}d_{B(i)}^{-1}u \not\subset_{L} u$ since $l(s_{i_j}d_{B(i)}^{-1}u) > l(u) = l(s_{i_j}d_{B(i)}^{-1}u)$. On the other hand, $s_{i_j}d_{B(i)}^{-1}u > d_{B(i)}^{-1}u$ implies $\phi_{i_{j-1}}(d_{B(i)}^{-1}u) = s_{i_j}d_{B(i)}^{-1}u$, which implies $\phi_{d_A(i)}(s_{i_j}d_{B(i)}^{-1}u) = u$ by (13), which implies $s_{i_j}d_{B(i)}^{-1}u \subset_{L} u$ by Lemma 2.2 which is in contradiction.

Therefore $s_{i_j}d_{B(i)}^{-1}u < d_{B(i)}^{-1}u$, that is, $\phi_{i_{j}}(d_{B(i)}^{-1}u) = d_{B(i)}^{-1}u$, and hence $\phi_{d_A(i)}(d_{B(i)}^{-1}u) = u$ by (13). Hence, since $|A(1) \setminus B(1)| = |A \setminus B| - 1$, we obtain $s_{i_{j-a}}d_{B(a)}^{-1}u < d_{B(a)}^{-1}u$ for $a = 2, \ldots, l$ by the induction hypothesis applied for $(A, B) := (A(1), B(1))$.

Lemma 5.10. Let $A, B \subset Z'_{u,-}$ with $B \subset A$. The following are equivalent:

1. $B \in X'_{A,u}$. 
2. $B \cup \{i\} \subset Z'_{u,-}$ for any $i \in A \setminus B$.
3. $B \cup \{i\} \subset X'_{A,u}$ for any $i \in A \setminus B$.
4. $A \setminus \{i\} \subset Z'_{u,-}$ for any $i \in A \setminus B$.
5. $A \setminus \{i\} \subset X'_{A,u}$ for any $i \in A \setminus B$. 

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Figure 14: For Lemma 5.9

\begin{equation}
\sum_{v \in Y_{A,u}} (-1)^{|A|-(l(u)-l(v))} = \sum_{B \in Y'_{A,u}} (-1)^{|A|-(l(u)-l(d_B^{-1}u))}
\end{equation}

(6) \( [B,A] \subset Z'_{u,-} \).

(7) \( [B,A] \subset X'_{A,u} \).

**Proof.** (2) \( \iff \) (4) \( \iff \) (6) and (6) \( \implies \) (2) are obvious. (2) \( \implies \) (4) \( \implies \) (6) is from Lemma 4.2(1).

(1) \( \implies \) (7): We already proved (1) \( \implies \) (2) \( \iff \) (6). Hence, since \( A,B \in X'_{A,u} \) and \( [B,A] \subset Z'_{u,-} \), by Lemma 5.5 we have \( [B,A] \subset X'_{A,u} \).

(1) \( \iff \) (3) \( \iff \) (5) \( \iff \) (7): It is obvious that (7) \( \implies \) (3), (5). From Lemma 5.7(1) we have (3) \( \implies \) (1) and (5) \( \implies \) (1). Besides we already proved (1) \( \implies \) (7).

(4) \( \implies \) (5): By Lemma 5.8.

We write \( \bigcap X = \bigcap_{x \in X} x \) for a set \( X \) of sets.

**Corollary 5.11.** We have \( X'_{A,u} = \left( \bigcap X'_{A,u} \right) / A \) if \( X'_{A,u} \neq \emptyset \), and \( Y'_{A,u} = \left( \bigcap Y'_{A,u} \right) / A \) if \( Y'_{A,u} \neq \emptyset \). In particular, \( X'_{A,u} \) and \( Y'_{A,u} \) are isomorphic to boolean posets, and therefore so are \( X_{A,u} \) and \( Y_{A,u} \).

**Proof.** Assume \( Y'_{A,u} \) is nonempty. Then \( Y'_{A,u} \) has the minimum element \( C = \bigcap Y'_{A,u} \) by Lemma 5.7(2). By Lemma 5.10(1) \( \implies \) (7) we have \( [C,A] \subset X'_{A,u} \). Moreover, since \( Y_{A,u} \) is an order ideal of \( X_{A,u} \) we have \( Y'_{A,u} \) is an order filter of \( X'_{A,u} \) and therefore \( [C,A] \subset Y'_{A,u} \). The opposite inclusion \( Y'_{A,u} \subset [C,A] \) is implied by minimality of \( C \). Therefore \( Y'_{A,u} = [C,A] \).

It is proved similarly that \( X'_{A,u} = [\bigcap X'_{A,u},A] \) whenever \( X'_{A,u} \neq \emptyset \).

Therefore we have

\begin{equation}
\sum_{v \in Y_{A,u}} (-1)^{|A|-(l(u)-l(v))} = \sum_{B \in Y'_{A,u}} (-1)^{|A|-(l(u)-l(d_B^{-1}u))}
\end{equation}
5.2.3 Step 4

Next we discuss which \( A \) satisfies the condition \( |Y_{A,u}| = 1 \).

Since \( Z_{u,-} \subseteq [e,w]_L \) is an order filter, so is \( Z_{u,-} \cap [e,w] \subseteq [e,u]_L \cap [e,w] \). Hence, if \( (w_S \wedge_L u = \max([e,u]_L \cap [e,w]) \notin Z_{u,-} \), then \( Z_{u,-} \cap [e,w] = \emptyset \), and hence \( Y_{A,u} = \emptyset \) for any \( A \) since \( Y_{A,u} = X_{A,u} \cap [e,w] \subseteq Z_{u,-} \cap [e,w] \). We hence assume \( w_S \wedge_L u \in Z_{u,-} \) and write \( w_S \wedge_L u = d_{A_0}^{-1}u \) with \( A_0 \in Z_{u,-} \). Write \( Z_{u,-} = Z_{u,-} \cap [e,w] \). Note that \( w_S \wedge_L u = \max Z_{u,-} \).

Example 5.12. Recall Example 5.4. In that case \( \max(Z_{u,-} \cap [e,w]) = s_1u \) and hence \( A_0 = \{ 1 \} \). It is easily checked that \( X_{\{ 1 \},u} = \{ u, s_1u \} \) and \( Y_{\{ 1 \},u} = \{ s_1u \} \).

Lemma 5.13. \( [Y_{A,u}] = 1 \iff A = A_0 \).

Proof. ( \( \Rightarrow \) ) Clearly \( d_{A_0}^{-1}u \in Y_{A_0,u} \). On the contrary, take any \( u \in Y_{A_0,u} \). Then \( v = d_{B}^{-1}u \) for some \( B \in Y_{A_0,u}' \). Since \( Y_{A_0,u}' \subseteq X_{A_0,u} \subseteq [\emptyset, A], \) we have \( B \subseteq A_0 \). On the other hand, since \( v \in Y_{A_0,u} = X_{A_0,u} \cap [e,w] \subseteq Z_{u,-} \), we have \( v \leq \max Z_{u,-} = d_{A_0}^{-1}u, \) and hence \( B \supset A_0 \). Therefore \( B = A_0 \).

( \( \Leftarrow \) ) If \( A \notin Z_{u,-} \), then \( |Y_{A,u}| \leq |X_{A,u}| = 0 \) from Lemma 5.8(1). We hence assume \( A \in Z_{u,-} \). Then \( d_{A}^{-1}u \notin Z_{u,-} \).

If \( d_{A}^{-1}u \notin w \), then \( Y_{A,u} = \emptyset \) since \( d_{A}^{-1}u \) is the minimum element of \( X_{A,u} \) and \( Y_{A,u} = X_{A,u} \cap [e,w] \) is an order ideal of \( X_{A,u} \).

Hence we assume \( d_{A}^{-1}u \leq w \). Since \( d_{A}^{-1}u = \max Z_{u,-}^{\geq w} \), we have \( d_{A}^{-1}u \leq d_{A_0}^{-1}u, \) and hence \( A_0 \subseteq A \). Suppose \( A_0 \subseteq A \). By Corollary 4.14 there exists an \( A' \in Z_{u,-} \) such that \( A_0 \subset A' \subset A \) and \( |A \setminus A'| = 1 \). By Lemma 5.8 and that \( d_{A_0}^{-1}u \leq d_{A_0}^{-1}u \leq w \) we have \( d_{A'}^{-1}u \in Y_{A,u} \). Hence \( Y_{A,u} \supset \{ d_{A_0}^{-1}u, d_{A_0}^{-1}u \} \).

Therefore, substituting (14) and the result of Lemma 5.13 into the right-hand side of (12) and noting that \( |A_0| = l(u) - l(w_S \wedge_L u) \), we have

\[
[g_e^{(k)}][g_w^{(k)}]_{B,r} = \begin{cases} 1 & \text{if } w_S \wedge_L u \in Z_{u,-} \text{ and } l(u) - l(w_S \wedge_L u) \leq r, \\ 0 & \text{otherwise}. \end{cases}
\]

Finally, we show the following:

Lemma 5.14. The following are equivalent:

(1) \( w_S \wedge_L u \in Z_{u,-} \) and \( l(u) - l(w_S \wedge_L u) \leq r \).

(2) There exists \( A \) such that \( |A| \leq r \) and \( u \geq_L d_{A}^{-1}u \leq w \).

(3) There exists \( A \) such that \( |A| \leq r \) and \( u \leq d_{A}w \geq_L w \).

(4) There exists \( A \) such that \( |A| = r \) and \( u \leq d_{A}w \geq_L w \).

Proof. (1) \( \iff \) (2): Clear.

(3) \( \iff \) (4): (4) \( \implies \) (3) is obvious. (3) \( \implies \) (4) follows from the fact \( Z'_{u,+} \) has the Chain Property and the maximum element of size \( k \), which corresponds to the maximum element of \( Z_{u,+} \).

(2) \( \implies \) (3): Assume \( u \geq_L d_{A}^{-1}u \leq w \). Then \( u = \phi_{d_{A}}(d_{A}^{-1}u) \leq \phi_{d_{A}}(w) \) by Lemma 5.14(2). Besides, we have \( \phi_{d_{A}}(w) = d_{B}w \geq_L w \) for some \( B \subseteq A \) by Lemma 2.2 and \( |B| \leq |A| \leq r \).

(3) \( \implies \) (2): Proved similarly to (2) \( \implies \) (3), with Lemma 5.13 instead of Lemma 2.2. \( \square \)
Now we finished, from Lemma 5.14 (1) $\iff$ (4), the proof of Theorem 1.2

$$\overline{g}_{w}^{(k)}\overline{h}_{r} = \sum_{u}q_{u}^{(k)},$$

summed over $u \in W^c$ such that $u \leq d_{A}w$ for some $A \subseteq I$ with $|A| = r$ and $d_{A}w \geq_{L} w$.

Theorem 1.3 follows from Theorem 1.2 Corollary 4.8 and the Inclusion-Exclusion Principle.

6 Proof of the $k$-rectangle factorization formula

This section is devoted for the proof of Theorem 1.4.

The idea of the proof is similar to that of Proposition 2.18 we consider a linear map $\Theta : \Lambda^{(k)} \longrightarrow \Lambda^{(k)}$ extending $\gamma_{\lambda}^{(k)} \mapsto \gamma_{R_{i} \cup \lambda}^{(k)}$, having that $\{\gamma_{\lambda}^{(k)}\}_{\lambda \in P_{k}}$ forms a basis of $\Lambda^{(k)}$. It suffices to show $\Theta$ is a $\Lambda^{(k)}$-homomorphism, since it implies $\gamma_{R_{i} \cup \lambda}^{(k)} = \Theta(\gamma_{\lambda}^{(k)}) = \gamma_{\lambda}^{(k)}\Theta(1) = \gamma_{\lambda}^{(k)}\Theta(\gamma_{\lambda}^{(k)}) = \gamma_{\lambda}^{(k)}\gamma_{R_{i}}$. Since $\{\overline{h}_{i}\}_{1 \leq i \leq k}$ generate $\Lambda^{(k)}$, we only need to show

$$\Theta(\overline{h}_{i}\gamma_{\lambda}^{(k)}) = \overline{h}_{i}\Theta(\gamma_{\lambda}^{(k)}).$$

Let $d_{A_{1}}\lambda, d_{A_{2}}\lambda, \ldots$ be the list of all weak strips over $\lambda$ of size $r$. Applying Theorem 1.3 to both sides of (15), we have

$$(\text{LHS}) = \Theta \left( \sum_{a}g_{d_{A_{a}}\lambda}^{(k)} - \sum_{a<b}g_{d_{A_{a} \cap A_{b}}\lambda}^{(k)} + \ldots \right)$$

$$= \sum_{a}g_{R_{i} \cup (d_{A_{a}}\lambda)}^{(k)} - \sum_{a<b}g_{R_{i} \cup (d_{A_{a} \cap A_{b}}\lambda)}^{(k)} + \ldots, \quad (16)$$

and by Lemma 2.16 (3) we have

$$(\text{RHS}) = \overline{h}_{i}\gamma_{R_{i} \cup \lambda}^{(k)}$$

$$= \sum_{a}g_{d_{A_{a}+t}(R_{i} \cup \lambda)}^{(k)} - \sum_{a<b}g_{d_{A_{a}+t}(A_{b} \cap \lambda)}^{(k)}(R_{i} \cup \lambda) + \ldots. \quad (17)$$

Since $(A_{a} + t) \cap (A_{b} + t) \cap \cdots = (A_{a} \cap A_{b} \cap \cdots) + t$, by Lemma 2.16 (1) we have $16 = 17$.

Now Theorem 1.4 is proved.

References

[ACT] David Anderson, Linda Chen, and Hsian-Hua Tseng, *On the quantum K-ring of the flag manifold*. arXiv:1711.08414.

[BB05] Anders Björner and Francesco Brenti, *Combinatorics of Coxeter groups*, Graduate Texts in Mathematics, vol. 231, Springer, New York, 2005. MR2133266

[BM15] Anders S. Buch and Leonardo C. Mihalcea, *Curve neighborhoods of schubert varieties*, J. Differential Geom. 99 (2015), no. 2, 255–283.

[BW82] Anders Björner and Michelle Wachs, *Bruhat order of Coxeter groups and shellability*, Adv. in Math. 43 (1982), no. 1, 87–100. MR646668

[BW88] Anders Björner and Michelle L. Wachs, *Generalized quotients in Coxeter groups*, Trans. Amer. Math. Soc. 308 (1988), no. 1, 1–37. MR946427

[Den12] Tom Denton, *Canonical decompositions of affine permutations, affine codes, and split k-Schur functions*, Electron. J. Combin. 19 (2012), no. 4, Paper 19, 41. MR3001656

[Deo87] Vinay V. Deodhar, *A splitting criterion for the bruhat orderings on coxeter groups*, Communications in Algebra 15 (1987), no. 9, 1889–1894.

[Hum90] James E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, Cambridge, 1990. MR1066460
Takeshi Ikeda, Shinsuke Iwao, and Toshiaki Maeno, *Peterson isomorphism in K-theory and relativistic Toda lattice*, International Mathematics Research Notices (2018), rny051, available at https://oup.backfile/content_public/journal/imrn/pap/10.1093_imrn_rny051/2/rny051.pdf

Allen Knutson and Ezra Miller, *Subword complexes in Coxeter groups*, Advances in Mathematics 184 (2004), no. 1, 161–176.

Thomas Lam, *Schubert polynomials for the affine Grassmannian*, J. Amer. Math. Soc. 21 (2008), no. 1, 259–281.

Luc Lapointe, Alain Lascoux, and Jennifer Morse, *Tableau atoms and a new Macdonald positivity conjecture*, Duke Math. J. 116 (2003), no. 1, 103–146.

Thomas Lam, Changzheng Li, Leonardo C. Mihalcea, and Mark Shimozono, *A conjectural peterson isomorphism in K-theory*. arXiv:1705.03435.

Thomas Lam, Luc Lapointe, Jennifer Morse, Anne Schilling, Mark Shimozono, and Mike Zabrocki, *k-Schur functions and affine Schubert calculus*, Fields Institute Monographs, vol. 33, 2014.

L. Lapointe and J. Morse, *Order ideals in weak subposets of Young’s lattice and associated unimodality conjectures*, Ann. Comb. 8 (2004), no. 2, 197–219.

Luc Lapointe and Jennifer Morse, *Tableaux on k + 1-cores, reduced words for affine permutations, and k-Schur expansions*, J. Combin. Theory Ser. A 112 (2005), no. 1, 44–81.

Luc Lapointe and Jennifer Morse, *A k-tableau characterization of k-Schur functions*, Adv. Math. 213 (2007), no. 1, 183–204.

Thomas Lam and Mark Shimozono, *Quantum cohomology of G/P and homology of affine Grassmannian*, Acta Math. 204 (2010), no. 1, 49–90. MR2600433

Thomas Lam and Mark Shimozono, *From quantum Schubert polynomials to k-Schur functions via the Toda lattice*, Math. Res. Lett. 19 (2012), no. 1, 81–93.

Thomas Lam, Anne Schilling, and Mark Shimozono, *K-theory Schubert calculus of the affine Grassmannian*, Compos. Math. 146 (2010), no. 4, 811–852.

Ian G. Macdonald, *Symmetric functions and Hall polynomials*, Second, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.

Jennifer Morse, *Combinatorics of the K-theory of affine Grassmannians*, Adv. Math. 229 (2012), no. 5, 2950–2984.

Mark Shimozono, private communication. 2018.

John R. Stembridge, *A short derivation of the Möbius function for the Bruhat order*, J. Algebraic Combin. 25 (2007), no. 2, 141–148. MR2310418

Motoki Takigiku, *Factorization formulas of K-k-Schur functions I*. arXiv:1704.08643.

Motoki Takigiku, *Factorization formulas of K-k-Schur functions II*. arXiv:1704.08660.

Debra J. Waugh, *Upper bounds in affine Weyl groups under the weak order*, Order 16 (1999), no. 1, 77–87 (2000). MR1740744

Graduate School of Mathematical Sciences, the University of Tokyo, Japan

E-mail address: takigiku@ms.u-tokyo.ac.jp