Spontaneous breaking of spatial symmetries in collective neutrino oscillations

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(Dated: December 23, 2014)

A dense neutrino medium can experience collective oscillations or self-induced flavor transformation through nonlinear neutrino-neutrino refraction. To make the problem of collective neutrino oscillations more tractable, all previous studies on this subject have assumed some spatial symmetry or symmetries in the neutrino medium (e.g., translation symmetries in the early universe and spherical symmetry in core-collapse supernovae). We point out that the collective oscillation modes studied in such models are very special. Using a simple toy model we show that spatial symmetries can be broken spontaneously in collective neutrino oscillations. We also show that the spatial-symmetry-breaking (SSB) modes of neutrino oscillations can exist for both neutrino mass hierarchies and even in the regimes where collective neutrino oscillations were previously thought to be suppressed. This finding calls for study of collective neutrino oscillations in multi-dimensional models.

PACS numbers: 14.60.Pq, 97.60.Bw

Introduction. Hot and dense astrophysical environments such as core-collapse supernovae are of great importance to our understanding of the origin of elements. Through

$$\nu_e + n \rightleftharpoons p + e^-, \quad \bar{\nu}_e + p \rightleftharpoons n + e^+$$  \hspace{1cm} (1)

and other processes neutrinos play crucial roles in these environments by, e.g., extracting energy from and depositing energy into matter and changing the n-to-p ratio.

It has been firmly established by various experiments that neutrinos can oscillate among different flavors or weak interaction states during propagation (see, e.g., [1] for a review). Because neutrinos of different flavors are emitted with different luminosities and energy spectra in a core-collapse supernova, it is natural to wonder if neutrino oscillations can have some effects on supernova dynamics and nucleosynthesis. However, a quick examination of the matter density profile of the supernova envelope (e.g., in [2]) shows that, if neutrinos change their flavors only through the standard Mikheyev-Smirnov-Wolfenstein (MSW) mechanism [3 4], the temperature and density of the ordinary matter and/or the fluxes of neutrinos are too low for neutrino oscillations to have a significant impact on supernova dynamics or nucleosynthesis (however, see [5]). Nonetheless, neutrino oscillations will alter the energy spectra of supernova neutrinos (e.g., [6 7]). An accurate interpretation of the neutrino signals from a future nearby core-collapse supernova will improve our knowledge of the physical conditions inside the nascent neutron star at the center of the supernova. It may also facilitate the determination of whether the neutrino masses have a normal hierarchy (NH) or an inverted one (IH), i.e. whether $|\nu_3|$ is the most massive of the three neutrino mass eigenstates $|\nu_i\rangle \, (i = 1, 2, 3)$ or not.

It has long been suspected that the presence of the dense neutrino medium can affect neutrino oscillations and other物理 phenomena in a core-collapse supernova through the neutrino-neutrino refraction or neutrino self-coupling potential $\mathcal{V}_{\nu\nu}$ [8 12]. The self-consistent calculations of neutrino oscillations with neutrino self-coupling in a one-dimensional, spherically symmetric model, dubbed as the “neutrino bulb model”, revealed some surprising and interesting results [13 14]. For example, it was shown that collective neutrino oscillations, which were first discovered in numerical calculations in the context of the early universe [15 15], can result in split spectra in which neutrinos (and antineutrinos) of different flavors swap energy spectra in certain energy ranges. Through a simple toy model of a dense gas of mono-energetic neutrinos and antineutrinos, it is intuitively understood that the neutrino media in the neutrino bulb model (and also in a strictly homogeneous and isotropic environment) has a flavor instability in the IH scenario [17 19]. This instability is analogous to that of an inverted pendulum. Since then many more interesting aspects of collective neutrino oscillations have been discovered (e.g., [20 32]; see [33] for a recent but incomplete review).

Recently it was shown that, because of the current-current nature of the weak interaction, a class of multi-azimuthal-angle (MAA) flavor instabilities can exist in the NH scenario [34 34]. The MAA flavor instabilities will trigger collective neutrino oscillations which break the axial symmetry around any radial direction in the neutrino bulb model spontaneously.

Modern supernova hydrodynamic simulations show that large convection overturns and standing accretion shock instabilities are common in supernova explosions (e.g., [37 39]). But to make the problem of collective oscillations of supernova neutrinos more tractable, all the studies on this subject have assumed the neutrino bulb model, including [10] which investigates the impact of supernova deleptonization asymmetry on collective neutrino oscillations. The spherical symmetry of the neu-
trino bulb model implies that the neutrinos which are emitted from different source points but otherwise have the same initial conditions have identical flavor evolution history. Consequently, the same set of neutrinos keep on interacting with each other at all radii in the neutrino bulb model which seems rather artificial.

In this letter we want to point out that a new class of spatial-symmetry-breaking (SSB) flavor instabilities can exist in the neutrino medium for both the NH and IH scenarios. Because of the SSB instabilities, collective neutrino oscillations can break the spatial symmetry or symmetries in the equations of motion (such as the translation symmetries in a homogeneous neutrino gas and the spherical symmetry in the neutrino bulb model) spontaneously.

A two-dimensional toy model. We consider an infinite neutrino plane which lies at $z = 0$. It constantly emits mono-energetic neutrinos and antineutrinos of energy $E$ in only two directions which are denoted by unit vectors

$$
\hat{v}_\zeta = [u_\zeta, 0, v_z] \quad (\zeta = L, R),
$$

where $0 < v_z < 1$ and $u_R = -u_L = \sqrt{1 - v_z^2}$. We assume that neutrinos experience refraction only and that the space at $z > 0$ is filled with uniform and dense matter. We also assume that the physical conditions are translationally symmetric along the $y$ direction and periodic along the $x$ direction. Clearly, our toy model is stationary and two-dimensional because it does not depend on time or the $y$ dimension.

In this letter we consider the mixing of two active neutrino flavors, $\nu_\mu$ and $\nu_\tau$, the latter of which is a linear superposition of $\nu_\mu$ and $\nu_\tau$. We describe neutrinos by flavor (density) matrices $\rho_\zeta(x, z)$ with trace 1, where the diagonal elements are the probabilities for the neutrinos to be in different flavors, and where $x$ and $z$ are the space coordinates of the neutrinos. The flavor matrices $\tilde{\rho}$ for antineutrinos are defined in a similar way.

We adopt the natural units with $\hbar = c = 1$. In our model it is convenient to express all lengths in terms of the inverse of the vacuum neutrino oscillation frequency

$$
\omega = \frac{|\Delta m^2|}{2E} \approx (1.6 \text{ km})^{-1} \left( \frac{|\Delta m^2|}{\Delta m_{atm}^2} \right) \left( \frac{10 \text{ MeV}}{E} \right),
$$

where $\Delta m^2$ is the mass-squared difference between two neutrino mass states, and $\Delta m_{atm}^2 \approx 2.4 \times 10^{-3} \text{ eV}^2$ is the measured atmospheric mass-squared difference. The strength of the neutrino self-interaction is given by

$$
\mu = \sqrt{2}(1 - \hat{v}_L \cdot \hat{v}_R)G_F j_\nu,
$$

where $G_F \approx (293 \text{ GeV})^{-2}$ is the Fermi coupling constant, and $j_\nu$ is the total number flux of the neutrinos of all flavors (but excluding antineutrinos) in either $L$ or $R$ direction. We assume that the number fluxes of both neutrinos and antineutrinos are independent of $x$ and $\zeta$.

The neutrino self-interaction Hamiltonian for neutrinos at point $(x, z)$ is

$$
H_{\nu\nu,\zeta}(x, z) = \mu [\rho_\zeta(x, z) - \alpha \tilde{\rho}_\zeta(x, z)],
$$

where $\zeta = R, L$ are the opposites of $\zeta$, and $\alpha$ is the ratio of the total number flux of antineutrinos to that of neutrinos.

The equations of motion for neutrinos and antineutrinos are

$$
i\nabla \tilde{v}_\zeta \rho_\zeta(x, z) = [\Omega + H_{\nu\nu,\zeta}(x, z), \rho_\zeta(x, z)],
$$

$$
i\nabla \tilde{v}_\zeta \bar{\rho}_\zeta(x, z) = [-\Omega + H_{\nu\nu,\zeta}(x, z), \bar{\rho}_\zeta(x, z)].
$$

In the above equations $\nabla \tilde{v}_\zeta$ is the gradient along the direction of $\tilde{v}_\zeta$ so that

$$
\nabla \tilde{v}_\zeta \rho_\zeta(x, z) = u_\zeta \partial_x \rho_\zeta + v_z \partial_z \rho_\zeta,
$$

and

$$
\Omega = \frac{\eta \omega}{2} \left[ \begin{array}{cc} -\cos 2\theta_{\text{eff}} & \sin 2\theta_{\text{eff}} \\ \sin 2\theta_{\text{eff}} & \cos 2\theta_{\text{eff}} \end{array} \right]
$$

is the effective Hamiltonian for the neutrino in matter, where the effects of the uniform and dense matter is included by assuming an effective neutrino mixing angle $\theta_{\text{eff}} \ll 1$ [17,18], and $\eta = \text{sgn}(\Delta m^2) = +1$ and $-1$ for NH and IH, respectively.

We define

$$
\rho_{\zeta,m}(z) = \frac{1}{L} \int_0^L e^{-imkx} \rho_\zeta(x, z) \, dx,
$$

where $m$ is an integer, $L$ is the size of the periodic box in the $x$ direction, and $k = 2\pi/L$. We also define $\tilde{\rho}_{\zeta,m}(z)$ for the antineutrino in a similar way. Using relations

$$
\rho_\zeta = \sum_m e^{ikm} \rho_{\zeta,m},
$$

$$
\nabla \tilde{v}_\zeta \rho_\zeta = \sum_m e^{ikm} [v_z \rho'_\zeta,m + i mk u_\zeta \rho_{\zeta,m}],
$$

and Eq. [6] we obtain

$$
i v_z \frac{d}{dz} \rho_{\zeta,m} = mk u_\zeta \rho_{\zeta,m} + [\Omega, \rho_{\zeta,m}] + \mu \sum_{m'} [\rho_{\zeta,m'} - \alpha \tilde{\rho}_{\zeta,m'}, \rho_{\zeta,m'}].
$$

The equation of motion for $\tilde{\rho}_{\zeta,m}$ is the same as Eq. [13] except with the sign of $\Omega$ reversed. 

Linearized flavor stability analysis. We assume that neutrinos and antineutrinos are in almost pure electron flavor at $z = 0$:

$$
\rho \approx \begin{bmatrix} 1 & \delta \rho \\ 0 & 0 \end{bmatrix}, \quad \tilde{\rho} \approx \begin{bmatrix} 1 & \delta \tilde{\rho} \\ 0 & 0 \end{bmatrix},
$$

where $\delta \rho = \rho_{\zeta,m} - \rho_{\zeta,m'}$, $\delta \tilde{\rho} = \tilde{\rho}_{\zeta,m} - \tilde{\rho}_{\zeta,m'}$, and $\zeta = R, L$. We require that $\tilde{\rho} = \tilde{\rho} = 0$ when $\rho = \rho = 0$. This gives

$$
\delta \rho = \frac{\Omega_{\text{eff}}}{\mu} \delta \tilde{\rho},
$$

where $\Omega_{\text{eff}} = \eta \omega / 2$ is the effective Hamiltonian for the neutrino in matter, and $\delta \rho_{\zeta,m}$ and $\delta \tilde{\rho}_{\zeta,m}$ are the neutrino and antineutrino flavor changes, respectively.
Taking $\theta_{\text{eff}} = 0$ and keeping only the terms up to $O(|\delta \bar{\rho}|)$ in Eq. (13) and its counterpart for the antineutrino we arrive at
\begin{equation}
\Lambda_m = v_z^{-1} \begin{bmatrix}
0 & -\eta \omega & ma & 0 \\
-\eta \omega & -\mu & 0 & ma \\
ma & 0 & 2\mu & \eta \omega \\
0 & ma & -\eta \omega + \mu & \mu \\
\end{bmatrix}
\end{equation}
with $a = (2\pi/L)\sqrt{1 - v_z^2}$ and $\mu_{\pm} = \mu(1 \pm \alpha)$.

In general the real matrix $\Lambda_m$ has four eigenvalues $\lambda_m^{(i)} (i = 1, 2, 3, 4)$. If $\lambda_m^{(i)}$ is complex, then its complex conjugate is also an eigenvalue of $\Lambda_m$. A positive $\kappa_m^{(i)} = \text{Im}(\lambda_m^{(i)})$ implies that the neutrino gas is unstable in flavor evolution and that collective oscillations can start. Because the characteristic polynomial of $\Lambda_m$ contains only even powers of $m$, the eigenvalues of $\Lambda_{-m}$ are the same as those of $\Lambda_m$. Further, the values of $\kappa_m^{(i)}$ are independent of the neutrino mass hierarchy in our toy model because $\Lambda_m \to 2(\mu_+ / v_z)I - \Lambda_{-m}$ under transformation

\[ \eta \to -\eta, \quad (D^\pm_m, S^\pm_m) \to (D^\mp_m, S^\mp_m). \]

We note that, in our model, the equations of motion have both the left-right ($L \leftrightarrow R$) symmetry and the translation symmetry along the $x$ direction. The initial conditions of the system at $z = 0$ also have these two symmetries, although approximately. The left-right symmetry will be preserved if all the anti-symmetric modes $(D^\pm_m, S^\pm_m)$ remain stable. The translation symmetry will be preserved if $\kappa_m^{(0)} = 0$ for all $m \neq 0$.

Refs. [3, 4] studied a toy model with two neutrino beams emitted from a source with exact translation symmetries. In this model only the $m = 0$ modes can exist, and the evolution of the symmetric mode $(D^0_0, S^0_0)$ and
anti-symmetric mode \((D_0^\alpha, S_0^\alpha)\) are decoupled in the linear regime [see Eq. (17)]. According to the above discussion, if the symmetric mode has an instability in certain neutrino mass hierarchy, then the anti-symmetric mode is unstable in the opposite mass hierarchy and breaks the left-right symmetry.

The highlight of our work is that the values of \(\kappa_{m \neq 0}^{(i)}\) are not necessarily 0. This implies the existence of the spatial-symmetry-breaking or SSB flavor instabilities which break the translation symmetry in the \(x\) direction spontaneously. Because the symmetric and anti-symmetric modes are coupled when \(m \neq 0\), the SSB instabilities also break the left-right symmetry. Further, because the values of \(\kappa_{m \neq 0}^{(i)}\) are independent of the sign of \(\eta\), SSB instabilities can exist for both neutrino mass hierarchies.

In Fig. 1 we plot \(\kappa_{m \neq 0}^{\text{max}}(\mu)\), the largest of all \(\kappa_{m \neq 0}^{(i)}(\mu)\), as functions of \(\mu\) and \(m\) for the cases with \(v_z = 0.5\), \(L = 20\pi/\omega\) and \(\alpha = 0.8\) and 0.5, respectively. Fig. 1 shows that, in our two-dimensional neutrino gas model, collective oscillation modes of different \(|m|\) values are unstable in different physical regimes. As \(|m|\) increases, the flavor unstable region shifts to larger \(\mu\), and its width also increases. In addition, as the asymmetry in the number fluxes of neutrinos and antineutrinos decreases, the flavor unstable regions move to larger \(\mu\), and the collective oscillation modes develop faster (because \(\kappa_{m \neq 0}^{\text{max}}\) are larger).

Implications for real physical systems. Albeit a simple toy model, our two-dimensional neutrino gas model bears many similarities to the models which were used to study collective neutrino oscillations. For example, all the previous works on collective oscillations of supernova neutrinos have assumed the spherically symmetric neutrino bulb model, and most of them have also assumed the axial symmetry around any radial direction from the center of the supernova. The spherical and axial symmetries in the neutrino bulb model correspond to the translation and left-right symmetries in our two-dimensional model, respectively.

In [34] it was shown that, in the neutrino bulb model, the multi-zenith-angle (MZA) and multi-azimuthal-angle (MAA) instabilities can arise in the IH and NH scenarios, respectively. Collective neutrino oscillations triggered by the MAA instability break the axial symmetry spontaneously. Because the numerical calculations before [34] all have assumed the axial symmetry, no collective neutrino oscillations were found to exist in the NH scenario. However, when the neutrinos emitted with different azimuthal angles are allowed to evolve independently, collective neutrino oscillations are indeed found to exist in the NH scenario which break the axial symmetry as expected [35].

The collective neutrino oscillation modes due to the MZA and MAA instabilities correspond to the \(m = 0\) symmetric and anti-symmetric modes, respectively, in our toy model. As we have discussed above, if one mode is unstable in the IH scenario, then the other is unstable in the NH scenario. More importantly, our results show that, if one allows neutrinos emitted from different source points to evolve independently, the spatial symmetry (i.e., the translation symmetry along the \(x\) direction in our two-dimensional toy model and the spherical symmetry in the neutrino bulb model) can be broken spontaneously by the SSB instabilities \((m \neq 0)\). Because of the usage of the models with special spatial symmetry or symmetries, the collective neutrino oscillation modes studied in literature so far all belong to the special kind which preserves the spatial symmetry.

Our results also show that, unlike the \(m = 0\) instabilities, the SSB instabilities can arise for both the NH and IH scenarios and even in the regimes where the \(m = 0\) instabilities are suppressed. For core-collapse supernovae this implies that collective neutrino oscillations may have a larger impact on nucleosynthesis of heavy elements than what was previously expected (e.g., [45]).

In our simple model it seems that the central value \(\mu\) of the flavor unstable region increases linearly with \(\sqrt{|m|}\) when \(|m|\) is sufficiently large. In a real physical system there must exists a cutoff value \(m_{\text{max}}\) so that the SSB modes with \(|m| > m_{\text{max}}\) are suppressed. After all, for too small distance scales Eq. (6) becomes invalid because it is based on the assumption of the coherent forward scattering of neutrinos by many particles in the medium. In core-collapse supernovae the SSB modes of tiny scales may be suppressed because they are unstable in the regime where the matter and/or neutrino densities are very large. A very large matter density can suppress collective neutrino oscillations in the neutrino bulb model. This is because the propagation distances are generally different for the neutrinos meeting at any given point [46]. Also in the neutrino bulb model, very large neutrino number densities can suppress collective oscillations because the differences among the neutrino self-interaction potentials \(H_{\nu\nu}\) for different neutrino beams are large [26]. However, it remains to be seen what effects these two suppression mechanisms may have on the SSB modes.

Even though the SSB modes with different \(m\) values may be decoupled in the linear regime [where Eq. (10) or a similar linearized equation of motion is valid], this is unlikely to be true when these modes grow out of the linear regime. Consequently, a large \(m_{\text{max}},\) say \(\gtrsim 1000\), portends a great challenge to follow flavor oscillations numerically in a dense neutrino gas.

Conclusions. Using a simple, two-dimensional neutrino gas model we have demonstrated that the spatial-symmetry-breaking or SSB flavor instabilities can exist for both neutrino mass hierarchies. Because of the existence of the SSB instabilities, collective neutrino oscillations can develop in the regimes where they were previously thought to be suppressed. Such collective os-
oscillation modes break the spatial symmetry or symmetries in the neutrino media spontaneously. This calls for numerical simulations of neutrino oscillations in multidimensional models which can be a very challenging task.

We would like to thank Sajad Abbar and S. Vahid Noormofidi for useful discussions. This work was supported by DOE EPSCoR grant #DE-SC0008142 at UNM.