Integrable Euler top and nonholonomic Chaplygin ball

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Abstract

We discuss the Poisson structures, Lax matrices, $r$-matrices, bi-hamiltonian structures, the variables of separation and other attributes of the modern theory of dynamical systems in application to the integrable Euler top and to the nonholonomic Chaplygin ball.

1 Introduction

The main aim of this paper is to prove that the integrable Euler top and the nonholonomic Chaplygin ball are very similar dynamical systems like birds of a feather flock together. Thus, on example of these twins, we want to show how all the machinery developed for integrable systems can be carried to the theory of solvable nonholonomic systems.

The integrable Euler case of rigid body motion with the fixed center of mass (the Euler top) is relatively simple in the sense that its equations of motion do not linearize on Abelian surfaces, but on the elliptic curves. Of course, this does not make the Euler top entirely trivial [16]. A classical description of the Euler top can be found in any textbook on classical mechanics, see, for instance, [1, 2, 7].

The nonholonomic Chaplygin ball [11] is that of a dynamically balanced 3-dimensional ball that rolls on a horizontal table without slipping or sliding. ‘Dynamically balanced’ means that the geometric center coincides with the center of mass but the mass distribution is not assumed to be homogeneous. Because of the roughness of the table this ball cannot slip, but it can turn about the vertical axis without violating the constraints. There is a large body of literature dedicated to the Chaplygin ball, including the study of its generalizations. See [6, 8, 12, 17, 21, 19, 25, 26, 31]. Of course, this list, as well as the bibliography of the present paper, is by far incomplete.

Section 2 starts by collecting some definitions and facts about Euler top and Chaplygin ball. In Section 3 we will consider Poisson structures associated with the Chaplygin ball as deformations of the similar standard Poisson structures for the Euler top. Section 4 contains our main results about separability at zero level of the cyclic integral of motion. In bi-hamiltonian geometry separability is an invariant geometric property of the distribution defined by mutually commuting independent integrals of motion. In fact, there is neither Hamilton-Jacobi equation, nor time which describes only partial parametrization of geometric objects. We want to show how those standard bi-Hamiltonian geometric methods may be directly applied to the nonholonomic Chaplygin ball.

In the second part of the paper we will discuss various deformations of the well-known integrable Hamiltonian systems which may be treated as generalizations of the Chaplygin ball. In Section 5 we will briefly consider deformations of integrable systems on cotangent bundles of the Riemannian manifolds and underline that the main problem is the change of time which transfers a purely mathematical construction to the sensible physical model. For all these systems the deformations of initial Poisson bracket is trivial. Finally in Section 6 we will give some examples of similar deformations on Lie algebra $e(3)$, when the deformation of the initial Poisson bracket is nontrivial.
2 Equations of motion

Let $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ and $M = (M_1, M_2, M_3)$ be the two vectors of coordinates and momenta, respectively. We postulate that they satisfy to the following differential equations

$$\dot{M} = M \times \omega, \quad \dot{\gamma} = \gamma \times \omega.$$  \hspace{1cm} (1)

For any vector function $\omega$ on the dynamical variables $x = \gamma, M$ these equations in $M = \mathbb{R}^3 \times \mathbb{R}^3$ have the following integrals of motion

$$H_1 = (\gamma, \gamma), \quad H_2 = (\gamma, M), \quad H_3 = (M, M).$$  \hspace{1cm} (2)

Six differential equations can be solved in quadratures if we know four integrals and the Jacobi multiplier \[18\]. So, we want to add some additional integral to the known integrals $H_1, H_2$ and $H_3$ and calculate the desired multiplier.

If we assume the existence of the following additional integral of motion

$$H_4 = (M, \omega)$$  \hspace{1cm} (3)

one gets

$$\frac{dH_4}{dt} = (M \times \omega, \omega) + (M, \dot{\omega}) = (M, \dot{\omega}) = 0,$$

it means that the derivative $\dot{\omega}$ has to be perpendicular to $M$. Below we stint ourselves by integrals \[4\] with

$$\omega = A_x M.$$  \hspace{1cm} (4)

In generic $A_x$ is a matrix depending on variables $x = (\gamma, M)$, which has to satisfy to the equation

$$(M, \dot{\omega}) = (A_x^\top M, M \times A_x M) + (M, \dot{A}_x M) = 0.$$  \hspace{1cm} (5)

This equation can be replaced by the particular system of very simple equations

$$A_x^\top = A_x, \quad \text{and} \quad (M, \dot{A}_x M) = 0,$$

which has constant solution associated with the Euler top

$$A_x = A, \quad A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, \quad a_k \in \mathbb{R},$$  \hspace{1cm} (6)

and solution associated with the Chaplygin ball

$$A_x = A_d, \quad \text{where} \quad A_d = A + d g(\gamma) A \gamma \otimes \gamma A,$$

$$\dot{A}_d = g(\gamma)^2 A (\gamma \otimes \beta + \beta \otimes \gamma) A, \quad \beta = (\gamma - d(\gamma, \gamma)A \gamma) \times AM.$$  \hspace{1cm} (7)

Here $A$ is given by \[6\] and function $g(\gamma)$ is equal to

$$g(\gamma) = \frac{1}{1 - d(\gamma, A \gamma)},$$  \hspace{1cm} (9)

so that $A_d$ goes to $A$ at $d \to 0$.

In the Euler-Poisson case, $M$ is the vector of the kinetic momentum, $\omega$ is the angular velocity and $\gamma$ is the unit Poisson vector \[2\] \[7\]. All these vectors are expressed in the so-called body frame, its axes coincide with the principal inertia axes so that the corresponding tensor of inertia reads as

$$J = A^{-1} = \begin{pmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{pmatrix}, \quad J_k \leq J_i + J_j.$$  \hspace{1cm} (10)
In the Chaplygin case $M$ is the vector of angular momentum of the ball with respect to the contact point, $\omega$ is the angular velocity vector of the rolling ball and $\gamma$ is the unit normal vector to the plane at the contact point. As above all these vectors are expressed in the body frame firmly attached to the ball. If the mass, the inertia tensor and radius of the rolling ball are denoted by $m$, $J$ and $a$, then matrix $A$ in (11) is equal to

$$
A = \begin{pmatrix}
\frac{1}{J_1 + d} & 0 & 0 \\
0 & \frac{1}{J_2 + d} & 0 \\
0 & 0 & \frac{1}{J_3 + d}
\end{pmatrix}, \quad d = ma^2.
$$

At $d \to 0$ equations of motion (11) and integrals of motion $H_k$ for the Chaplygin ball coincide with equations and integrals for the Euler top. Of course, it is purely mathematical and non-physical limit because $J_i \to 0$ as $d \to 0$. So, from mathematical point of view, we can say that Chaplygin ball is a deformation of the Euler top with respect to parameter $d$.

In the Euler case matrix $A$ and integral of motion $H_4$ are different from the matrix $A$ and integral of motion $H_4$ in the Chaplygin case. Nevertheless, for the brevity, we will use common notations $A$ and $H_4$ in both cases where it will not cause any confusion.

Thus, we obtain the fourth integral of motion $H_4$ for the six equations (11) at $\omega = AM$ and $\omega = A_dM$. We proceed by showing that these dynamical systems are solvable in quadratures in framework of the Euler-Jacobi last multiplier theory [18]. By definition, the Jacobi multiplier $\mu(x)$ of (11) is a function on dynamical variables $x = \gamma, M$, which has to satisfy to the equation

$$
\sum_{i=1}^{6} \frac{\partial}{\partial x_i} \mu(x) \dot{x}_i = 0, \quad \Rightarrow \quad \dot{\mu}(x) + \mu(x) \sum_{j=1}^{3} \left( \frac{\partial}{\partial \gamma_j} (\gamma \times \omega)_j + \frac{\partial}{\partial M_j} (M \times \omega)_j \right) = 0.
$$

For the solution $A$ (6) this equation is trivial

$$
\dot{\mu}(x) = 0 \quad \text{and} \quad \mu(x) = \mu \equiv c, \quad c \in \mathbb{R},
$$

but for $A_d$ one gets

$$
2g(\gamma) \dot{\mu}(x) - \mu(x) g(\gamma) = 0, \quad \text{and} \quad \mu(x) = \mu_d \equiv c \sqrt{g(\gamma)}.
$$

According to [18] the Jacobi’s multiplier is some nontrivial function in the case of constrained systems only. The integrability conditions of the nonholonomic systems formulated by Kozlov [26] include the preservation of measure related with the Jacobi multiplier.

There are many other solutions of the system (5), see review [8]. For instance, solution

$$
A_f = f(\gamma) \gamma \otimes \gamma,
$$

depending on arbitrary function $f(\gamma)$ is associated with multiplier $\mu(x) = 1$. Solutions associated with nontrivial Jacobi multiplier are given by linear in variables $\gamma$ matrices

$$
A_{abc} = A + B(\gamma \otimes c + c \otimes \gamma)B^\top,
$$

which satisfy (15), if we impose various restrictions on the numerical entries of matrices $A$, $B$ and vector $c$.

To sum up, we can easily get a lot of additional integrals $H_4$ (11) and the corresponding Jacobi multipliers of the equations (11) and, therefore, we can solve these differential equations in quadratures without any notion of the Hamilton structure, integrability by Liouville, the Poisson structure, the Lax matrices, classical $r$-matrices etc.

However, this additional and in some sense redundant information can be useful in various applications, such as the perturbation theory, the quantization theory and so on. Below we reconstruct this information starting with only integrals.
3 The Poisson brackets.

In this section our aim is to calculate the Poisson brackets for the given models without any assumptions on underlying Hamiltonian or conformally Hamiltonian structures of the equations of motion \[ [6, 8] \]. We will calculate the desired Poisson brackets only assuming that the foliation \( H_i = \alpha_i \) is a direct sum of symplectic and lagrangian foliations.

Let us consider the manifold \( M, \dim M = n \), with coordinates \( x = (x_1, \ldots, x_n) \). The Jacobi last multiplier theorem \[ [5] \] ensures that \( n \) equations

\[
\frac{dx_i}{dt} = X_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n.
\]

(15)

are solvable in quadratures if we have \( n - 2 \) functionally independent integrals of motion \( \mathcal{H}_k \) and the Jacobi multiplier \( \mu \).

Let us suppose that \( M \) be a Poisson manifold endowed with a Poisson bivector \( P \), so that the corresponding Poisson bracket reads as

\[
\{ f, g \} = (P df, dg) = \sum_{i,j=1}^{\dim M} P_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}.
\]

In our examples, all the symplectic foliation associated with the Poisson bivector \( P \) is rather regular. Moreover, all the leaves are affine hyperplanes of codimension \( k \), which are the level sets of \( k \) globally defined independent Casimir functions \( C_j \)

\[
P dC_j = 0, \quad j = 1, \ldots, k,
\]

If the symplectic foliation associated with \( P \) is rather regular and \( n = 2m + k \), then we can determine the set of Hamiltonian systems on \( M \)

\[
\frac{dx_i}{dt_j} = \{ \mathcal{H}_j, x_i \}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, n - 2.
\]

(16)

The invariant volume form for all these Hamiltonian flows may be formally expressed as a square of symplectic form on symplectic leaves

\[
\nu = \Omega^m, \quad \text{where} \quad \Omega = P^{-1}_c,
\]

and \( P_c \) is restriction of \( P \) on symplectic leaves.

Furthermore if one of these Hamiltonian systems (16) is integrable by Liouville we can compare the two integrable systems (15) and (16) using change of time

\[
t \to t_j
\]

and try to extract useful information about the former system from the known properties of the latter Hamiltonian system.

According to the Liouville theorem, for the given Hamiltonian \( \mathcal{H}_j \) equations of motion (16) are integrable in quadratures on the symplectic leaves, if we have \( m \) functionally independent integrals of motion \( \mathcal{H}_1 \equiv \mathcal{H}_j, \mathcal{H}_2, \ldots, \mathcal{H}_m \) in the involution

\[
\{ \mathcal{H}_i, \mathcal{H}_j \} = 0, \quad i, j = 1, \ldots, m.
\]

We can identify all the integrals of the equations (15) with all the integrals of (16)

\[
(\mathcal{H}_1, \ldots, \mathcal{H}_{n-2}) \sim (\mathcal{H}_1, \ldots, \mathcal{H}_m; C_1, \ldots, C_k)
\]

only at \( m = 2 \), because \( n = 2m + k \) and \( n - 2 = m + k \).

Regularity of symplectic foliation is closely related to the existence of the Jacobi multiplier \( \mu \). One of the global invariants in Poisson geometry is a modular class. It is an obstruction
to the existence of measure in \( \mathcal{M} \) which is invariant under all hamiltonian flows \( 24, 47, 48 \).

For the manifold \( \mathcal{M} \) endowed with a Poisson bivector \( P \), its modular class is an element of the first Poisson cohomology group. In Section 3 we discuss some elements of the second Poisson cohomology group and the corresponding Poisson bivectors \( P' \) compatible with \( P \), which allows us to get variables of separation without any additional information.

In generic case we can identify only one integral \( H_1 = \mathcal{H}_k \) and consider not only two-dimensional systems, but other systems as well. In this case the integrals \( \mathcal{H}_j \) could be generators of some algebra of integrals with respect to the bracket \( \{ \ldots \} \), see theory of superintegrable or noncommutative integrable systems \( 5, 16 \).

In our case integrable by Euler-Jacobi equations of motion \( (11) \) with integrals \( H_1, \ldots, H_4 \) and multiplier \( \mu \) are integrable by Liouville after an appropriate change of time, if there is the Poisson bivector \( P \) such as

- \( [P, P] = 0 \), the Jacobi identity,
- \( Pd\mathcal{H}_i = Pd\mathcal{H}_j = 0 \), only two Casimir functions  \( \{ \mathcal{H}_i, \mathcal{H}_m \} = 0 \), the involution of the integrals.

Here \( [\,,\,] \) is the Schouten bracket, \( \mathcal{H}_1, \ldots, \mathcal{H}_4 \) are the four integrals \( 23 \), and \( (i, j, l, m) \) is the arbitrary permutation of \( (1, 2, 3, 4) \).

The first equation in \( (17) \) guaranties that \( P \) is a Poisson bivector. In the second equation we define two Casimir elements \( \mathcal{H}_i \) and \( \mathcal{H}_j \) of \( P \) and assume that \( \text{rank} P = 4 \). It is a necessary condition because by fixing its values one gets the four dimensional symplectic phase space of our dynamical system. The third equation provides that the two remaining integrals \( \mathcal{H}_l \) and \( \mathcal{H}_m \) are in involution with respect to the Poisson bracket associated with \( P \).

If we consider Chaplygin ball as a deformation of the Euler top, it is natural to fix for the both systems the same Casimir functions

\[
i = 1, \ j = 2, \quad \Rightarrow \quad \mathcal{H}_1 = C_1, \quad \mathcal{H}_2 = C_2
\]

and integrals of motion

\[
l = 3, \ m = 4, \quad \Rightarrow \quad \mathcal{H}_4 = H_1, \quad \mathcal{H}_3 = H_2.
\]

Solutions \( P \) associated with another choice of Casimir functions may be obtained from this solution by using classical \( r \)-matrix theory, see Section 3.3.

### 3.1 The linear in momenta Poisson bivectors.

The system \( (17) \) has infinitely many solutions and, therefore, we have to narrow the search space and try to get some particular solutions only. In this Section we assume that the entries of \( P \) are the linear functions in momenta \( M \).

**Proposition 1** In the hypotheses mentioned above the system of equations \( (17) \) has the following linear in momenta solutions:

**Euler case:** \[
P = \begin{pmatrix} 0 & \Gamma \\ \Gamma & M \end{pmatrix},
\]

**Chaplygin case:** \[
P_d = \frac{1}{\sqrt{g(\gamma)}} \begin{pmatrix} 0 & \Gamma \\ \Gamma & M \end{pmatrix} - d\sqrt{g(\gamma)} (M, A\gamma) \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

Here \( g(\gamma) \) is given by \( (9) \) and

\[
\Gamma = \begin{pmatrix} 0 & \gamma_3 & -\gamma_2 \\ -\gamma_3 & 0 & \gamma_1 \\ \gamma_2 & -\gamma_1 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & M_3 & -M_2 \\ -M_3 & 0 & M_1 \\ M_2 & -M_1 & 0 \end{pmatrix}.
\]
The proof consists in the substitution of the linear in momenta anzats for entries of the Poisson bivector

\[ P_{ij} = \sum_k f_{ijk}(\gamma) M_k \]

into (17) and in the solution of the resulting algebro-differential equations with respect to unknown coefficients \( f_{ijk}(\gamma) \). □

The Poisson brackets between variables \( x = \gamma, M \) look like

\[ \{M_i, M_j\} = \varepsilon_{ijk} M_k, \quad \{M_i, \gamma_j\} = \varepsilon_{ijk} \gamma_k \quad \{\gamma_i, \gamma_j\} = 0, \quad (21) \]

and

\[ \{M_i, M_j\}_d = \varepsilon_{ijk} \left( \frac{M_k}{\sqrt{g(\gamma)}} - d \sqrt{g(\gamma)} (M, A\gamma) \gamma_k \right), \]

\[ \{M_i, \gamma_j\}_d = \frac{\varepsilon_{ijk} \gamma_k}{\sqrt{g(\gamma)}}, \quad \{\gamma_i, \gamma_j\}_d = 0, \quad (22) \]

Here \( \varepsilon_{ijk} \) is a totally skew-symmetric tensor.

The first bracket \( \{., .\} \) is the well studied Lie-Poisson bracket on the Lie algebra \( e^*(3) \), whereas second bracket may be considered as its deformation with respect to parameter \( d \), which preserves regular symplectic foliation. The second Poisson bracket (22) has been obtained in [6].

Formally, in the both cases the Poisson bracket (21-22) allows us to rewrite the initial equations of motion (1) in the Hamiltonian form

\[ \frac{dx}{dt'} = \frac{dx}{\mu(x) dt} = \{H, x\}, \quad H = \frac{1}{2} h_4, \quad (23) \]

after changing the time variable including the corresponding Jacobi multiplier

\[ dt' = \mu(x) dt. \quad (24) \]

where \( \mu(x) = 1 \) or \( \mu(x) = \mu_d \), respectively. In the Chaplygin case this transformation has been introduced in the Chaplygin work [11] in order to get the solutions as the functions of the time variable.

It is easy to see that for the Chaplygin ball we can not directly identify this transformation with canonical transformations of the extended phase space, which change time and the Hamilton function simultaneously [30, 36, 37, 38, 39], see also discussion in [8, 13, 33, 26].

Of course, we can get similar Poisson bivectors for other solutions of (5) as well. For instance, if \( \omega = A_{ab} M \), where

\[ A_{ab} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a + b \gamma_3 \end{pmatrix}, \]

is a matrix from the family (14), then solution of (17) looks like

\[ P_{ab} = \frac{1}{\sqrt{a x_3 + b}} \left[ \begin{pmatrix} 0 & \Gamma \\ \Gamma & M \end{pmatrix} - \frac{a M_3}{2(a x_3 + b)} \begin{pmatrix} 0 & 0 \\ 0 & \Gamma \end{pmatrix} \right]. \]

(25)

In this case the equations of motion (11) are integrable by Euler-Jacobi theorem and by the Liouville theorem after the corresponding change of time.
3.2 Properties of the linear Poisson bivectors.

The Poisson-Lichnerowicz cohomology of the Poisson manifold was defined in [29], and it provides a good framework to express the deformation and the quantization obstructions, see [24, 47, 48].

Let us remind some necessary facts from the Poisson geometry. The Poisson manifold $\mathcal{M}$ is a smooth (or complex manifold) endowed with the Poisson bivector $P$ fulfilling the Jacobi condition

$$[P, P] = 0$$

with respect to the Schouten bracket on the algebra of the multivector fields on $\mathcal{M}$. Other Poisson bivector $P'$ is compatible with $P$ if any of its linear combination $P + \lambda P'$ is the Poisson bivector, i.e. if

$$[P, P'] = 0.$$

Bivectors $P'$ are the 2-cocycles in the Poisson-Lichnerowicz cohomology defined by Poisson bivector $P$ on the Poisson manifold $\mathcal{M}$. They must be compared with the bivectors $P^{(X)} = \mathcal{L}_X (P) \Rightarrow [P, P^{(X)}] = 0$ (26)

which are the Lie derivative of $P$ along any vector field $X$ on $\mathcal{M}$. Bivectors $P^{(X)}$ are 2-coboundaries and 2-cocycles simultaneously. However not all cocycles are coboundaries. If $X$ is such vector field that the Jacobi condition

$$[[P^{(X)}, P^{(X)}] = 0$$

is satisfied, then $P^{(X)}$ (26) is called the trivial deformation of the Poisson bivector $P$.

Now let us go back to our physical models. The first bivector $P^{(20)}$ is the well studied Lie-Poisson bivector on the Lie algebra $e^*(3)$ of Lie group $E(3)$ of Euclidean motions of $\mathbb{R}^3$. The second bivector $P_d^{(20)}$ can be treated as its “nonholonomic” deformation related with the Chaplygin ball.

**Proposition 2** Bivector $P_d^{(20)}$ is 2-cocycle in the Poisson-Lichnerowicz cohomology defined by canonical Poisson bivector $P$ on $e^*(3)$.

The proof is a straightforward calculation of the Schouten bracket

$$[P, P_d] = 0.$$

Recall that the Schouten bracket $[[R, Q]$ of two bivectors $R$ and $Q$ is trivector and its entries in local coordinates $x$ look like

$$[R, Q]^{ijk} = - \sum_{m=1}^{dim \mathcal{M}} \left( Q^{mk} \frac{\partial R^{ij}}{\partial x^m} + R^{mk} \frac{\partial Q^{ij}}{\partial x^m} + \text{cycle}(i, j, k) \right). \quad (27)$$

□

**Proposition 3** In the generic case the Poisson bivector

$$P_d = \frac{1}{\sqrt{g(\gamma)}} \left( \begin{array}{cc} 0 & \Gamma \\ \Gamma & M \end{array} \right) - d\sqrt{g(\gamma)} (M, A \gamma) \left( \begin{array}{cc} 0 & 0 \\ 0 & \Gamma \end{array} \right) \quad (28)$$

is a nontrivial deformation of the standard Lie-Poisson bivector $P$, which is a sum of two Lie derivatives of $P$

$$P_d = \mathcal{L}_Y (P) + \frac{d\sqrt{g(\gamma)} (\gamma, M)}{2} \mathcal{L}_Z (P). \quad (29)$$

Here entries of the vector fields $Y = \sum Y^i \partial_j$ and $Z = \sum Z^i \partial_j$ are given by

$$Y^i = Z^i = 0, \quad Y^{i+3} = - \frac{M_j}{\sqrt{g(\gamma)}}, \quad Z^{i+3} = \left( \left( \text{tr} A \cdot \text{Id} - A \right) \gamma \right)_i, \quad i = 1, 2, 3. \quad (30)$$
In order to prove first part of this proposition we have to try to solve the following equation
\[ P_d = \mathcal{L}_X(P), \] (31)
with respect to unknown \( X \). Recall that in local coordinates the Lie derivative of a bivector \( P \) along a vector field \( X \) reads
\[
(\mathcal{L}_X(P))^{ij} = \sum_{k=1}^{\dim M} \left(X^k \frac{\partial P^{ij}}{\partial x^k} - P^{kj} \frac{\partial X^i}{\partial x^k} - P^{ik} \frac{\partial X^j}{\partial x^k}\right).
\]

Using the modern software for symbolic calculations it is easy to prove that entries of (31) form the inconsistent system of differential equations. It means that (31) is infeasible equation and, therefore, cocycle \( P_d \) is no coboundary. The second part of the proposition is verified by direct calculations. □

It is easy to see, that if \((\gamma, M) = 0\) then \( P_d = \mathcal{L}_Y(P) \) is a trivial deformation with all the pleasant mathematical and physical consequences, see [29, 47] and [8, 11] respectively.

In the finite-dimensional case local Poisson geometry begins with the splitting theorem, which says that in the neighborhood of any point in the Poisson manifold \( M \), there are coordinates \((q_1, \ldots, q_m, p_1, \ldots, p_m, C_1, \ldots, C_k)\) such as
\[
P = \sum_{i=1}^{m} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \frac{1}{2} \sum_{i,j=1}^{k} \phi_{ij}(C) \frac{\partial}{\partial C_i} \wedge \frac{\partial}{\partial C_j} \quad \text{and} \quad \phi_{ij}(0) = 0.
\]
So, if the compatible bivectors \( P \) and \( P' \) have a common set of Casimirs \( C_1, \ldots, C_k \), we can identify the Darboux coordinates \((q, p)\) of \( P \) with the Darboux coordinates \((q', p')\) of \( P' \) and obtain the local map \( \phi : M \to M \), which pulls back \( P' \) to \( P \). In Section 4.3 we prove that in our case this local map \( P_d \to P \) can be extended to the global one at \( C_2 = 0 \).

If we come back to the general theory, the second Poisson-Lichnerowicz cohomology group \( H^2_P \) on \( M \) is precisely the set of bivectors \( P' \) solving \([P, P'] = 0\) modulo the solutions of the form \( P(X) = \mathcal{L}_X(P) \). We can interpret \( H^2_P \) as the space of infinitesimal deformations of the Poisson structure modulo trivial deformations. We should keep in mind that cohomology reflects the topology of the leaf space and the variation in the symplectic structure as one passes from one leaf to another [29, 24, 47, 48].

### 3.3 The \( r \)-matrices

It is known that equations (1) can be rewritten in the Lax form
\[
\frac{dL}{dt} = [L, \Omega], \quad L = M + \frac{\Gamma}{\lambda}, \quad \lambda \in \mathbb{R}.
\] (32)

if we identify \((\mathbb{R}^3, \times)\) and \((so(3), [\cdot, \cdot])\) by using a well known isomorphism
\[
z = (z_1, z_2, z_3) \to Z = \begin{pmatrix} 0 & z_3 & -z_2 \\ -z_3 & 0 & z_1 \\ z_2 & -z_1 & 0 \end{pmatrix},
\] (33)
where \( \times \) is a cross product in \( \mathbb{R}^3 \) and \([\cdot, \cdot]\) is a matrix commutator in \( so(3) \). Another possibility is to use \( 4 \times 4 \) antisymmetric matrices
\[
M = \begin{pmatrix} 0 & M_3 & -M_2 & 0 \\ -M_3 & 0 & M_1 & 0 \\ M_2 & -M_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & \omega_3 & -\omega_2 & 0 \\ -\omega_3 & 0 & \omega_1 & 0 \\ \omega_2 & -\omega_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]
and symmetric matrix

\[
\mathbf{\Gamma} = \begin{pmatrix}
0 & 0 & 0 & \gamma_1 \\
0 & 0 & 0 & \gamma_2 \\
0 & 0 & 0 & \gamma_3 \\
\gamma_1 & \gamma_2 & \gamma_3 & 0
\end{pmatrix}.
\]

In the both cases bilinear Lax matrix \( \mathbf{L}(\lambda) \) belongs to a huge family of Lax matrices described in the book [34], see also [35]. Below we will consider only \( 3 \times 3 \) Lax matrices.

The Lax equation implies that the spectral invariants of the Lax matrix \( \mathbf{L}(\lambda) \) are conserved quantities under the Hamiltonian evolution, but their involutivity and functionally independence must be checked case by case. In a noteworthy paper [3], Babelon and Viallet showed that if all the spectral invariants of the \( m \times m \) matrix \( \mathbf{L}(\lambda) \) are in involution with respect to some Poisson bracket \( \{ \ldots \} \) on a given phase space, then there is a matrix \( r_{12}(\lambda, \mu) \) of order \( m^2 \times m^2 \) such that the Poisson brackets between the entries of \( L \) are represented in the commutator form

\[
\{ \mathbf{L}(\lambda), \mathbf{L}(\mu) \}_k = [r_{12}(\lambda, \mu), \mathbf{L}] - [r_{21}(\lambda, \mu), \mathbf{L}(\mu)].
\]

(34)

Here \( \mathbf{L}(\lambda) = \mathbf{L}(\lambda) \otimes \mathbf{I} \), \( \mathbf{L}(\mu) = \mathbf{I} \otimes \mathbf{L}(\mu) \) and \( r_{12}(\lambda, \mu) \) is a classical \( r \)-matrix and

\[
r_{21}(\lambda, \mu) = \mathbf{P} r_{12}(\mu, \lambda),
\]

where \( \mathbf{P} \) is a permutation operator: \( \forall x \otimes y = y \otimes x, \forall x, y \in \mathbb{C}^m \) [34].

In our case for the Euler top and for the Chaplygin ball we have one Lax matrix \( \mathbf{L}(\lambda) \) and two different Poisson brackets. First bracket \( \{ \ldots \} \) [21] associated with the bivector \( \mathbf{P} \) [20] yields the standard \( r \)-matrix

\[
r_{12}(\lambda, \mu) = \frac{\mu}{\mu - \lambda} \sum_{i=1}^{3} \mathbf{S}_i \otimes \mathbf{S}_i.
\]

(35)

Here \( \mathbf{S}_i \) form a basis in the space of \( 3 \times 3 \) antisymmetric matrices

\[
\mathbf{S}_1 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & 0
\end{pmatrix}, \quad \mathbf{S}_2 = \begin{pmatrix}
0 & 0 & -1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix}, \quad \mathbf{S}_3 = \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

For the nonholonomic bracket \( \{ \ldots \}_d \) [22] associated with the bivector \( \mathbf{P}_d \) [20] the \( r \)-matrix will be a more complicated dynamical \( r \)-matrix.

**Proposition 4** The Lax matrix for nonholonomic Chaplygin ball [34] satisfies the linear \( r \)-matrix algebra [34] with the following \( r \)-matrix

\[
r_{12}(\lambda, \mu) = \frac{\mu}{\mu - \lambda} \left( \frac{1}{\sqrt{g(\gamma)}} - d\lambda \sqrt{g(\gamma)} (\mathbf{M}, \mathbf{A} \gamma) \right) \sum_{i=1}^{3} \mathbf{S}_i \otimes \mathbf{S}_i.
\]

(36)

The proof is straightforward verification of [34] for the given Lax matrix.. \( \Box \) Usually, the bilinear Lax matrices [32] are not very useful in integration of equations of motion. However, they are very effective in various geometric applications, for instance see one of the latest applications in discrete differential geometry [27].

In bi-Hamiltonian geometry we can use bilinear Lax matrices in order to get solutions of (17) associated with another choice of the Casimir elements. Namely, if \( r_{12} \) is classical \( r \)-matrix and \( \varphi \) is intertwining operator, then \( r_{12} \circ \varphi \) is also a classical \( r \)-matrix [34]. For a given matrix \( r_{12} \) the \( r \)-matrices \( r_{12} \circ \varphi \) form a linear Lie pencil, which generates a family of compatible Lie-Poisson brackets. For instance, if we take trivial intertwining operators from [34]

\[
\varphi_1 = \mu^{-1}, \quad \text{and} \quad \varphi_2 = \mu^{-2}
\]
and \( r \)-matrix \[33\], then one gets the following well-known Poisson bivectors for the Euler top
\[
P_1 = \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad P_2 = \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix},
\]
see \[32\]. Construction of other brackets on \( e^*(3) \) associated with classical \( r \)-matrices \[35,36\] and more sophisticated relations between various classical \( r \)-matrices \[42\] will be discussed in forthcoming publication.

4 Separation of variables at \((\gamma, M) = 0\).

Now we address the problem of separation of variables within the theoretical scheme of bi-Hamiltonian geometry \[15, 44, 45\]. According to \[11\] we can start with the case \((\gamma, M) = 0\) and then reduce the generic case to this particular one.

In geometry, instead of an additive separation of variables in the partial differential equation called the Hamilton-Jacobi equation, we have some invariant geometric property of the Lagrangian distribution defined by \( m \) independent functions \( H_1, \ldots, H_m \) \[15, 44, 45\].

Namely, an \( m \)-tuple \( H_1, \ldots, H_m \) of functionally independent functions defines a separable foliation on symplectic leaves of \( M \), if there are variables of separation \((q_1, \ldots, q_m, p_1, \ldots, p_m)\) and \( m \) separated relations of the form
\[
\Phi_i = 0, \quad \text{with} \quad \det \left[ \frac{\partial \Phi_i}{\partial H_j} \right] \neq 0. \quad (37)
\]
It simple means that the common level surfaces of \( H_1, \ldots, H_m \) form rather regular foliation of symplectic leaves and every leaf of this lagrangian foliation may be represented as a direct product of one-dimensional geometric objects defined by separated relations \[37\].

Now let us remind how to get variables of separation in framework of the bi-Hamiltonian geometry. The bi-Hamiltonian manifold \( M \) is a smooth (or complex) manifold endowed with two compatible Poisson bivectors \( P \) and \( P' \). Dynamical systems on \( M \) with the integrals of motion in involution with respect to the both brackets
\[
\{H_i, H_j\} = \{H_i, H_j\}' = 0, \quad i, j = 1, \ldots, m, \quad (38)
\]
are called bi-integrable systems \[44, 45\]. The bi-involutivity of the integrals of motion \[48\] is equivalent to the existence of the control matrix \( F \) defined by
\[
P' dH_i = P \sum_{j=1}^{m} F_{ij} dH_j, \quad i = 1, \ldots, m. \quad (39)
\]
The eigenvalues \((q_1, \ldots, q_m)\) of \( F \) are the coordinates of separation, whereas the suitable normalized left eigenvectors of \( F \) form the generalized St"ackel matrix \( S \)
\[
F = S^{-1} \text{diag} (q_1, \ldots, q_m) S
\]
which defines the separation relations
\[
\Phi_i = \sum_{j=1}^{n} S_{ij} (q_i, p_i) H_j + U_i (q_i, p_i) = 0, \quad i = 1, \ldots, m. \quad (40)
\]
Here the entries of St"ackel matrix \( S_{ij} \) and the St"ackel potentials \( U_i \) depend only on one pair \((q_i, p_i)\) of the canonical variables of separation, Casimir functions \( C_j \) and, in generic case, on the integrals of motion \[15, 44, 45\].

In our case the St"ackel matrix and the St"ackel potentials depend only on variables of separation, and it allows us to calculate the canonical transformation from the initial variables \( \gamma, M \) to the variables of separation explicitly.
4.1 Darboux-Nijenhuis coordinates

In order to get variables of separation according to the general usage of bi-hamiltonian geometry firstly we have to calculate the bi-hamiltonian structure for the given systems with integrals of motion \( H_1, H_2 \) on manifold \( M \) with the canonical Poisson bivector \( P \) and its deformation \( P_d \).

**Proposition 5** Let us introduce two vector fields \( X = \sum X^j \partial_j \) and \( X_d = \sum X_d^j \partial_j \), with the following entries:

\[
X^i = 0, \quad X_d^i = 0, \quad X^{i+3} = \left[ \gamma \times A(\gamma \times M) \right]_i, \quad i = 1, 2, 3.
\]

The Poisson bivectors

\[
P' = \mathcal{L}_X P \quad \text{and} \quad P'_d = \mathcal{L}_{X_d} P_d
\]

are compatible with the bivectors \( P \) and \( P_d \) respectively. Bivectors (42) have common symplectic leaves

\[
P' dC_1 = P'_d dC_1 = 0, \quad P' dC_2 = P'_d dC_2 = 0,
\]

whereas integrals of motion \( H_1, H_2 \) are in the bi-involution

\[
\{ H_1, H_2 \} = \{ H_1, H_2 \}' = 0, \quad \{ H_1, H_2 \}_d = \{ H_1, H_2 \}'_d = 0
\]

with respect to the corresponding Poisson brackets at \( (\gamma, M) = 0 \) only.

The proof is a straightforward verification of the corresponding Schouten and Poisson brackets in local coordinates. □ Thus, we proved that the Euler top and the nonholonomic Chaplygin ball are bi-integrable systems at \( (\gamma, M) = 0 \). At second step we have to calculate the corresponding control matrices \( F \) and \( F_d \) defined by (39). For the Euler top we have

\[
F = \begin{pmatrix} 0 & (A^\gamma \gamma, \gamma) \\ -\langle \gamma, \gamma \rangle & \langle (\text{tr}A \cdot \text{Id} - A) \gamma, \gamma \rangle \end{pmatrix}
\]

and similar to the Chaplygin ball

\[
F_d = \begin{pmatrix} 0 & (A_d^\gamma \gamma, \gamma) \\ -\langle \gamma, \gamma \rangle & \langle (\text{tr}A_d \cdot \text{Id} - A_d) \gamma, \gamma \rangle \end{pmatrix}
\]

Here \( A^\gamma = (\det A) A^{-1} \) is adjoint or cofactor matrix.

The Darboux-Nijenhuis coordinates associated with the bivectors (42) and control matrices (45) are the roots of their characteristic polynomials

\[
\tau(\lambda) = \lambda^2 - \left( \langle \text{tr}A \cdot \text{Id} - A \rangle \gamma, \gamma \right) \lambda + \langle \gamma, \gamma \rangle \langle A^\gamma \gamma, \gamma \rangle = 0,
\]

\[
\tau_d(\lambda) = \lambda^2 - \left( \langle \text{tr}A_d \cdot \text{Id} - A_d \rangle \gamma, \gamma \right) \lambda + \langle \gamma, \gamma \rangle \langle A_d^\gamma \gamma, \gamma \rangle = 0.
\]

By definition (15), the Darboux-Nijenhuis variables are canonical with respect to the symplectic form \( \Omega \) associated with first bivector \( P \) and put the recursion operator \( N = P' P^{-1} \) in diagonal form on symplectic leaves of \( M \).

Below we prove that these Darboux-Nijenhuis coordinates (47) are the variables of separation for the bi-lagrangian foliation defined by integrals \( H_1, H_2 \) on symplectic leaves of \( e^t(3) \) fixed by \( C_1 = 1 \) and \( C_2 = 0 \).

It is easy to see that at \( (\gamma, M) = 0 \) the passage from the Euler top to the nonholonomic Chaplygin ball consists of the replacement of the constant matrix \( A \) on dynamical one \( A_d \).
in the equations of motion (1), in the Hamiltonian $H_1 = (M, AM)$ and equations (31, 35, 37) only. Similar to geometric quantization theory, simplicity of this deformation is a sequence of the equation (29)

$$P_d = L_Y (P),$$

properties of the Lie derivative $L$ and of the vector fields $Y$ (30) and $X_d$ (41).

### 4.2 Elliptic coordinates

At $C_2 = 0$ and $C_1 = 1$ we can identify the corresponding symplectic leaf of $\mathcal{M} = e^* (3)$ with the cotangent bundle $T^* \mathbb{S}$ of the unit two-dimensional Poisson sphere $\mathbb{S}$.

If we put $C_1 = (\gamma, \gamma) = 1$, then, dividing characteristic polynomials (47) on $\det (A - \lambda I_d)$ we get the standard definitions of elliptic coordinates $u, v$ on the sphere and their nonholonomic deformations $u$, $v:

$$e(\lambda) = \frac{\gamma_1^2}{\lambda - a_1} + \frac{\gamma_2^2}{\lambda - a_2} + \frac{\gamma_3^2}{\lambda - a_3} = \frac{(\lambda - u)(\lambda - v)}{(\lambda - a_1)(\lambda - a_2)(\lambda - a_3)},$$

(48)

and

$$e_d(\lambda) = g(\gamma) \left( \frac{\gamma_1^2 (1 - da_1)}{\lambda - a_1} + \frac{\gamma_2^2 (1 - da_2)}{\lambda - a_2} + \frac{\gamma_3^2 (1 - da_3)}{\lambda - a_3} \right)$$

(49)

$$= \frac{(\lambda - u)(\lambda - v)}{(\lambda - a_1)(\lambda - a_2)(\lambda - a_3)},$$

respectively. Function $g(\gamma)$ in coordinates $u, v$ reads as

$$g(\gamma) = \frac{(1 - du)(1 - dv)}{(1 - da_1)(1 - da_2)(1 - da_3)}.$$  

(50)

Equation (48) is a standard definition of the elliptic coordinates $u, v$ on the unit sphere, whereas equation (49) determines the nonholonomic elliptic coordinates $u, v$. In [9], the coordinates of the form (49) are used for the integration of the sphere-sphere problem under the name quasi-spheroconical coordinates, see discussion in [46].

We have to point out that our aim is the calculation of the variables of separation without any additional assumptions. Thus, we have to calculate the conjugated momenta $p_u, p_v$ and $p_u, p_v$ in framework of bi-hamiltonian geometry. It is easy to prove that in our case we have identical Stäckel matrices

$$S = \begin{pmatrix} 1 & 1 \\ -u & -v \end{pmatrix} \quad \text{and} \quad S_d = \begin{pmatrix} 1 & 1 \\ -u & -v \end{pmatrix},$$

and identical Stäckel potentials

$$U_1^{(1)} = u H_2 - H_1, \quad U_2^{(1)} = v H_2 - H_1,$$

$$U_1^{(3)} = u H_2 - H_1, \quad U_2^{(3)} = v H_2 - H_1,$$

(51)

where

$$H_1 = H_4 = (M, \omega), \quad H_2 = H_3 = (M, M).$$

According to [44, 45], the notion of the Stäckel potentials allows us to find unknown conjugated momenta using the Poisson brackets only.

For instance, the following recurrence chain of the Poisson brackets

$$\phi_1 = \{ u, U_1^{(1)} \}_1, \quad \phi_2 = \{ u, \phi_1 \}_1, \ldots, \quad \phi_i = \{ u, \phi_{i-1} \}_1$$

(52)
breaks down on the third step $\phi_3 = 0$. It means that $U_1^{(1)}(u, p_u)$ is the second order polynomial in momentum $p_u$ and, therefore, we can define this unknown momentum in the following way

$$p_u = \frac{\phi_1}{\phi_2} = \sum_{i,j,m} \varepsilon_{ijm} \frac{\gamma_j \gamma_m (a_j - a_m)(a_i - u)}{2(u-v)(a_j - a_i)(a_m - a_i)} M_i$$

up to the canonical transformations $p_u \to p_u + f(u)$. As above, $\varepsilon_{ijk}$ is a totally skew-symmetric tensor.

Similar calculation with $U_2^{(1)}(v, p_v)$ yields to the definition of the second momentum $p_v$. In nonholonomic case we can perform completely identical calculations too. The results obtained so far can be summarized in the following statement.

**Proposition 6** The initial coordinates $x = \gamma, M$ are expressed via elliptic coordinates $u, v$ and $p_u, p_v$

$$\gamma_i = \sqrt{\frac{(u - a_i)(v - a_i)}{(a_j - a_i)(a_m - a_i)}}, \quad i \neq j \neq m,$$

$$M_i = \frac{2\varepsilon_{ijm} \gamma_i \gamma_m (a_j - a_m)}{u - v} \left( (a_i - u)p_u - (a_i - v)p_v \right),$$

In terms of the nonholonomic elliptic coordinates $u, v$ and $p_u, p_v$ the same variables look like

$$\gamma_i = \sqrt{\frac{(1 - da_j)(1 - da_m)}{(1 - du)(1 - dv)}}, \quad \sqrt{\frac{(u - a_i)(v - a_i)}{(a_j - a_i)(a_m - a_i)}} \times i \neq j \neq m,$$

$$M_i = \frac{2\varepsilon_{ijk} \gamma_i \gamma_k (a_j - a_k) \sqrt{g(\gamma)}}{u - v} \left( (a_i - u)(1 - du)p_u - (a_i - v)(1 - dv)p_v \right).$$

where $g(\gamma)$ is given by (57).

It is simple combination of definitions of the Casimir functions $C_{1,2}$, coordinates (48-49), momentum (53) and other momenta. □

### 4.3 The reduction of the Poisson brackets

The two sets of variables of separation $u, v, p_u, p_v$ and $u, v, p_u, p_v$ are the Darboux variables with respect to the brackets $\{., .\}$ (21) and $\{., .\}_d$ (22) on the symplectic leaf $C_1 = 1$ and $C_2 = 0$. Of course, we can identify these variables and get the diffeomorphism $\phi : \mathcal{M} \to \mathcal{M}$, which pulls back the nonholonomic bracket $\{., .\}_d$ to the standard Lie-Poisson bracket $\{., .\}$ on the Lie algebra $e^*(3)$ at $C_2 = 0$.

**Proposition 7** At $(\gamma, M) = 0$ the Poisson bracket $\{., .\}_d$ (22) between the variables $\gamma, M$ coincides with the Lie-Poisson bracket $\{., .\}$ (21) between the variables

$$\hat{\gamma}_j = \sqrt{g(\gamma)} \left( 1 - d(\gamma, \gamma) a_j \right) \gamma_j, \quad j = 1, 2, 3,$$

$$\hat{M}_j = \frac{1}{\prod_{i \neq j} \left( 1 - d(\gamma, \gamma) a_i \right)} \left( \frac{M_j}{\sqrt{g(\gamma)}} + d\sqrt{g(\gamma)} (M, \mathbf{A}_\gamma) \gamma_j \right).$$

This mapping identifies variables $u, v$ with the usual elliptic coordinates $u, v$ on the sphere, which were used by Chaplygin [17].

13
It is a direct sequence of the previous Proposition.

So, at \((\gamma, M) = 0\) we can map the nonholonomic Poisson bracket to the standard Poisson bracket on the cotangent bundle of the sphere. It means that any integrable system on the sphere has an integrable counterpart with respect to the nonholonomic bracket and vice versa. The list of the known integrable systems on the sphere can be found in [4, 7, 49].

In the next section we prove that we can not identify the Euler top and the nonholonomic Chaplygin ball using this mapping because they have different separated relations even at \((\gamma, M) = 0\).

4.4 Separation relations

Substituting variables \(\gamma, M\) into the St"ackel potentials (51), we obtain a pair of separation relations (40) for the Euler top and the Chaplygin ball. These separated equations define some algebraic curves and we can say that the equations of motion (1) are linearized on the symmetrized product of these curves.

Proposition 8 In holonomic case at \(\omega = AM\) the variables of separation lie on two copies of the hyperelliptic curve of genus one

\[
C^{(1)}: \quad 4(a_1 - x)(a_2 - x)(a_3 - x) y^2 - (xH_2 - H_1) = 0, \tag{57}
\]

where \(x = u, v\) and \(y = p_u, p_v\).

In nonholonomic case at \(\omega = A_dM\) the variables of separation lie on two copies of the following hyperelliptic curve of genus two

\[
C^{(3)}: \quad 4(1 - dx)(a_1 - x)(a_2 - x)(a_3 - x) y^2 - (xH_2 - H_1) = 0, \tag{58}
\]

where \(x = u, v\) and \(y = p_u, p_v\).

Initial variables as functions on variables of separation are given by (54) and (55). It allows us to express integrals of motion \(H_1, H_2\) in terms of variables of separation. Substituting the resulting formulae for \(H_1, H_2\) into the separated relation we prove this proposition.

In fact, we obtain the variables of separation and the separated equations geometrically, i.e. without the equations of motion, the time variable and the underlying Hamiltonian or conformally Hamiltonian structures. We only suppose that the foliation defined by the integrals \(H_{1,2}\) on symplectic leaves of the corresponding Poisson brackets is bi-lagrangian foliation.

However, in order to get the solutions of the separated equations \(x(t)\) and \(y(t)\) we have to explicitly introduce a time variable \(t\). Solving separated equations with respect to \(H_{1,2}\) one gets the Hamilton functions for the Euler top

\[
H_1 = \frac{4v(a_1 - u)(a_2 - u)(a_3 - u)}{u - v} p_u^2 + \frac{4v(a_1 - v)(a_2 - v)(a_3 - v)}{v - u} p_v^2, \tag{59}
\]

and for the Chaplygin ball

\[
H_1 = \frac{4v(1 - du)(a_1 - u)(a_2 - u)}{u - v} p_u^2 + \frac{4u(1 - dv)(a_1 - v)(a_2 - v)}{v - u} p_v^2. \tag{60}
\]

By definition the variables of separation are canonical variables and, therefore, we have

\[
\{H_1, x_1\} = \frac{4x_2 \sqrt{P(x_1)}}{x_1 - x_2} \quad \{H_1, x_2\} = \frac{4x_1 \sqrt{P(x_2)}}{x_2 - x_1}, \tag{61}
\]

\[
\{H_1, x_1\}_d = \frac{4x_2 \sqrt{P_d(x_1)}}{x_1 - x_2} \quad \{H_1, x_2\}_d = \frac{4x_1 \sqrt{P_d(x_2)}}{x_2 - x_1}.
\]

14
Here variables $x_{1,2}$ are coordinates of separation $u,v$ or $u,v$, respectively. Polynomials $P(x)$ and $P_d(x)$ are the polynomials of degree 4 and 5 in $x$ variable

$$P(x) = \ (a_1 - x)(a_2 - x)(a_3 - x)(xH_2 - H_1),$$

$$P_d(x) = \ (1 - dx)(a_1 - x)(a_2 - x)(a_3 - x)(xH_2 - H_1).$$

On the other hand, according to (23), the brackets (61) are equal to

$$\{H_1, x_{1,2}\} = 2 \mu \frac{dx_{1,2}}{d\tau}$$

and

$$\{H_1, x_{1,2}\}_d = 2 \mu_d \frac{dx_{1,2}}{d\tau},$$

where

$$\mu = 1, \quad \text{and} \quad \mu_d = \sqrt{g(\gamma)} = \sqrt{\frac{(1 - du)(1 - dv)}{(1 - da_1)(1 - da_2)(1 - da_3)}}$$

are Jacobi multipliers (12-13).

For the Chaplygin ball, in order to get the solutions $x_{1,2}(t)$ of the equations of motion, we have to consider the Jacobi inversion problem for the equations

$$\beta_1 - 2 \int \mu_d \, dt = \int \frac{dx_1}{\sqrt{P_d(x_1)}} + \int \frac{dx_2}{\sqrt{P_d(x_2)}},$$

$$\beta_2 = \int \frac{x_1 dx_1}{\sqrt{P_d(x_1)}} + \int \frac{x_2 dx_2}{\sqrt{P_d(x_2)}},$$

where $\beta_{1,2}$ are the constants of integration. The change of time variable (24) reduces these equations to the standard Abel-Jacobi equations [11, 26].

It is easy to prove that the right hand side in $\beta_2$ (62) coincides with an additional Euler-Jacobi quadrature emerged in the Jacobi last multiplier theory. Of course, for the Chaplygin ball this quadrature can be obtained without any change of time variable.

5 Generalizations of the nonholonomic Chaplygin ball at $C_2 = 0$.

Equations of motion, Poisson brackets, Lax matrices and classical $r$-matrices for the Chaplygin ball are deformation of the same objects for the Euler top by parameter $d$. Moreover, at $C_1 = 1$ and $C_2 = 0$ we know how to deform the corresponding variables of separation and the separated relations. Because at $C_1 = 1$ and $C_2 = 0$ our phase space is equivalent to the cotangent bundle $T^*S$ of the unit two-dimensional Poisson sphere, we can obtain similar deformations of other integrable systems on cotangent bundles of the Riemannian manifolds.

5.1 The $2 \times 2$ Lax matrices.

In variables of separation we deal with the uniform Stäckel systems [39] and, therefore, we can get $2 \times 2$ Lax matrices associated with the Abel-Jacobi equations [62] in a standard way, see [14, 28, 37, 38, 41] as well as the relevant references therein.

According to [37, 38, 41], let us introduce the following functions on the canonical variables of separation and spectral parameter $\lambda$

$$h(\lambda) = -\frac{1}{8} \left\{ H_2, e(\lambda) \right\}, \quad h_d(\lambda) = -\frac{1}{8(1 - d\lambda)} \left\{ H_2, e_d(\lambda) \right\}_d,$$
and
\[
\begin{align*}
    f(\lambda) &= \frac{1}{4} \left( \left\{ H_2, h(\lambda) \right\} - e(\lambda)H_2 \right), \\
    f_d(\lambda) &= \frac{1}{4(1-d\lambda)} \left( \left\{ H_2, h_d(\lambda) \right\}_d - \left( 1 + \frac{\text{tr}A - 2(u + v)}{1 - 2d\lambda} \right) e_d(\lambda)H_2 \\
    &\quad - \frac{e_d(\lambda)H_1}{1 - d\lambda} \right).
\end{align*}
\]

Here \( e(\lambda) \) and \( e_d(\lambda) \) are given by (48-49).

For the brevity below we will use the following denotation, index \( x \) is a white space for the Euler top and its Hamiltonian generalizations, whereas \( x = d \) for the Chaplygin ball and the corresponding generalizations.

**Proposition 9** At \((\gamma, M) = 0\) the Lax matrices
\[
\begin{align*}
    \mathcal{L}_x &= \begin{pmatrix} h_x & e_x \\ f_x & -h_x \end{pmatrix}, \\
    \mathcal{A}_x &= \frac{1}{\mu_x e_x} \begin{pmatrix} -e'_x & 0 \\ 2h'_x & e'_x \end{pmatrix},
\end{align*}
\]

satisfy to the Lax equation
\[
\frac{d}{dt} \mathcal{L}_x(\lambda) = \frac{\mu_x}{2} \left\{ H_1, \mathcal{L}_x \right\}_x = \left[ \mathcal{L}_x(\lambda), \mathcal{A}_x(\lambda) \right].
\]  

Here \( z' = \{ z, H_1 \}_x \) is a time derivative up to Jacobi multiplier (12-13).

The proof is a straightforward verification of the equations (64) for the given Lax matrices. □

As usual, substituting \( \lambda = x \) into the determinants of the Lax matrices
\[
\det \mathcal{L}_x(\lambda) = -h^2_x(\lambda) - e_x(\lambda)f_x(\lambda),
\]

which are equal to
\[
\begin{align*}
    \det \mathcal{L}(\lambda) &= -\frac{\lambda H_2 - H_4}{4(a_1 - \lambda)(a_2 - \lambda)(a_1 - \lambda)}, \\
    \det \mathcal{L}_d(\lambda) &= -\frac{\lambda H_2 - H_4}{4(1-d\lambda)(a_1 - \lambda)(a_2 - \lambda)(a_1 - \lambda)},
\end{align*}
\]

one gets separated relations (67) and (68) because \( e_x(x) = 0 \) and \( b_x(x) = y \).

In [11] Chaplygin reduces the generic case at \((\gamma, M) \neq 0\) to the particular case at \((\gamma, M) = 0\). By applying the inverse map to the Lax matrices (62) one gets the Lax matrices for the generic case of the nonholonomic Chaplygin ball. These matrices and the corresponding \( r \)-matrix algebra will be studied in a forthcoming separate publication.

Matrices \( \mathcal{L}_x(\lambda) \) are associated with the uniform Stäckel systems and, therefore, they satisfy to the linear \( r \)-matrix algebra (64) with the well-studied dynamical \( r \)-matrices [14, 28, 37, 38, 41].

In contrast with the previous 3 \( \times \) 3 Lax matrices (62) it allows us to obtain some well studied generalizations of these 2 \( \times \) 2 matrices in the next paragraphs.

### 5.2 Chaplygin ball and separable potentials.

We are going to demonstrate that the Chaplygin ball at \((\gamma, M) = 0\) is still integrable in the force fields associated with a huge family of the so-called separable potentials [4, 14, 49].

It is well known of how to get various generalizations of the separable systems using the deformations of their separated equations [18]. For instance, let us consider the following deformations of the separation relations (67) and (68)
\[
4(a_1 - x)(a_2 - x)(a_3 - x)y^2 - (xH_2 - H_1) + V(x) = 0
\]

16
$$4(1 - dx)(a_1 - x)(a_2 - x)(a_3 - x) y^2 - (xH_2 - H_1) + V(x) = 0,$$

where the potential $V$ is some function on $x$. Usually, potential $V$ is a linear combination of the trivial separable potentials $V_m = \alpha_m x^m$, where $m$ is a positive or negative integer \[4, 49\].

In order to get the same deformations in the initial variables $\gamma, M$ we can use the generating function \[49\]
$$\Phi(\lambda) = \frac{\phi(\lambda)}{e_x(\lambda)},$$

or the determinant of the corresponding deformations the Lax matrix $L_x(\lambda)$ \[63\]
$$f_x \to f_x + \left[\frac{\phi(\lambda)}{e_x(\lambda)}\right]_{MN}.$$ 

Here $\phi(\lambda)$ is a parametric function on spectral parameter and $[\xi(\lambda)]_{MN}$ is a linear combination of the Laurent projections of $\xi(\lambda)$ by $\lambda$ \[14, 37, 38\].

For example, if $V = \alpha x^2$ one gets the integrable system
$$H_2 = (M, M) + \alpha(\gamma, A\gamma), \quad H_1 = (M, A_1 M) - \frac{\alpha}{a_1 a_2 a_3}(\gamma, A^{-1}\gamma), \quad (65)$$

which can be identified with the Neumann system on the sphere, and its nonholonomic counterpart

$$H_2 = (M, M) + \alpha g(\gamma) \left(\gamma, A\gamma - d(A\gamma, A\gamma)\right), \quad (66)$$

$$H_1 = (M, A_3 M) - \alpha g(\gamma) \left(\gamma, A^{-1}\gamma\right) a_1 a_2 a_3 - da_1 a_2 a_3.$$ 

Another nonholonomic analog of the Neumann system with the polynomial in $\gamma$ potential has been proposed by Kozlov \[26\] at

$$V = -\alpha x^2 + (a_1 + a_2 + a_3)x + \frac{\alpha(a_1 - x)(a_2 - x)(a_3 - x)}{1 - dx}, \quad (67)$$

see integrals of motion in \[85\]. At $(\gamma, M) = 0$ this system is separable in the Chaplygin coordinates \[17\].

If $V = \beta x^3$ we obtain a forth order polynomial potential on the sphere

$$H_1 = (M, A_1 M) + \beta \frac{(\gamma, A^{-1}\gamma)}{a_1 a_2 a_3} \left(\gamma, A\gamma - trA\right), \quad (68)$$

and its nonholonomic analog

$$H_1 = (M, A_3 M) + \beta g(\gamma) \left(\frac{(\gamma, A^{-1}\gamma)}{a_1 a_2 a_3} - da_1 a_2 a_3\right) \times$$

$$\times \left[g(\gamma) \left(\gamma, A\gamma - d(A\gamma, A\gamma) - trA\right)\right]. \quad (69)$$

Similarly we can get other well-known integrable systems on the sphere \[4, 49\], such as Braden and Rosochatius systems, and their nonholonomic counterparts separable in the nonholonomic elliptic coordinates.

5.3 Deformations of natural Hamiltonian systems on Riemannian spaces of constant curvature.

At $(\gamma, M) = 0$ the Euler top is a dynamical system describing free motion on the two-dimensional sphere, which may be identified with a particular case of the Gaudin magnet \[28\].
It is well-known how to describe similar $N$-dimensional integrable systems on any Riemannian space of constant curvature and then how to add separable potentials to these systems, see [14, 22, 28, 37, 38, 40, 41] and references within. There are many different tools to investigations of such systems [4, 7, 34]. Below we will use only one of them based on separation of variables method.

The key ingredient of this construction is $2 \times 2$ Lax matrix associated with the $sl(2)$ Gaudin magnet [14, 28, 40, 41], which is completely defined by the rational function $e(\lambda)$ on a given Riemannian manifold. The list of all admissible functions may be found in [22].

Let us start with elliptic coordinates $(q, p)$ on the sphere $S_N$ in $N+1$-dimensional Euclidean space $\mathbb{R}_N$. In this case deformation of the free motion similar to Chaplygin ball consists of three steps:

- we have to change the separation relations from

$$\prod_{j=1}^{N+1} (a_j - q_i) p_i^2 - (q_i^{N-1} H_N + \cdots + q_i H_2 + H_1) = 0, \quad i = 1, \ldots, N,$$

(70)

and to

$$(1 - dq_i) \prod_{j=1}^{N+1} (a_j - q_i) p_i^2 - (q_i^{N-1} H_N + \cdots + q_i H_2 + H_1) = 0,

(71)

compare with (57-58);

- we have to change the definition of $q_j$ in term of cartesian coordinates in $\mathbb{R}_N$ from

$$e(\lambda) = \sum_{k=1}^{N+1} \gamma_k^2 \prod_{i=1}^{N} (\lambda - q_i) \prod_{j=1}^{N+1} (\lambda - a_j),$$

(72)

that implies $\sum_{i=1}^{N+1} \gamma_i^2 = 1$, to

$$e_d(\lambda) = g(\gamma) \sum_{k=1}^{N+1} \gamma_k^2 (1 - da_k) \prod_{i=1}^{N} (\lambda - q_i) \prod_{j=1}^{N+1} (\lambda - a_j),$$

(73)

where $g(\gamma)$ is defined by residue of $e_d(\lambda)$ at infinity;

- we have to change the time variable in order to attach some nonholonomic physical meaning to the proposed pure mathematical integrals $H_1, \ldots, H_N$ and the corresponding equations of motion, see [19, 21].

Remind that rewriting separated relations (70) or (71) in the Stäckel form (40)

$$\Phi_i = p_i^2 + \sum_{j=1}^{m} S_{ij}(q_i) H_j = 0, \quad i = 1, \ldots, m,$$

we can easily determine integrals of motion in a standard way [37, 38]

$$H_k = \sum_{i=1}^{m} C_{ki} p_i^2,$$

where $C = S^{-1}$ is the inverse matrix to the Stäckel matrix $S$, which is a standard transpose Brill-Noether matrix with entries divided by $\prod_{j=1}^{N+1} (a_j - q_i)$ or $(1 - dq_i) \prod_{j=1}^{N+1} (a_j - q_i)$. The same integrals may be rewritten in the following form [41]

$$H_k = \text{res}_{\lambda = \infty} \lambda^{N-k} e^{-1}(\lambda).$$
Of course, instead of one parametric deformation we can consider multi-parameter deformations replacing terms \((1 - dq_j)\) and \((1 - da_j)\) on \(\prod(1 - d_mq_j)\) and \(\prod(1 - d_m a_k)\) in (71) and (73), respectively.

On the other hand instead of (72) we can start with any other coordinate system and the corresponding separated equations on the Riemannian spaces of constant curvature [28, 41]. For instance, we can take elliptic coordinates

\[
e(\lambda) = 1 + \sum_{k=1}^{N} \frac{\gamma_k^2}{\lambda - e_k} = \prod_{j=1}^{N} (\lambda - q_j) \prod_{i=1}^{N} (\lambda - e_i).
\]

or parabolic coordinates

\[
e(\lambda) = \lambda - 2\gamma_N - \sum_{k=1}^{N-1} \frac{\gamma_k^2}{\lambda - e_k} = \prod_{j=1}^{N} (\lambda - q_j) \prod_{i=1}^{N-1} (\lambda - e_i)
\]

in \(N\)-dimensional Euclidean space.

In order to consider systems with potential we can add Stäckel potential \(U(q_i)\) to \(i\)-th separated equation (70) and (71).

For all these deformations we easy calculate \(2 \times 2\) Lax matrices in terms of variables of separation, because all these deformations are uniform Stäckel systems associated with various hyperelliptic curves [28, 37, 38, 40, 41].

So, there are not mathematical problems in the construction of such “nonholonomic” dynamical systems associated with any orthogonal coordinate system and their potential generalizations. The main problem is definition of a suitable change of time variable, which may be associated with an interesting physical model.

6 Generalizations of the nonholonomic Chaplygin ball at \((\gamma, M) \neq 0\).

In the previous section we consider various deformations of our dynamical systems at \((\gamma, M) = 0\) using the variables of separation method for the Hamiltonian systems and the suitable time reparametrization. We proceed by discussing some possible deformations of the Chaplygin ball in generic case.

As above index \(x\) is a white space for generalizations of the Euler top and \(x = d\) for generalizations of the Chaplygin ball.

Let us consider the deformation of the equations (1)

\[
\dot{M} = M \times \omega + \gamma \times b, \quad \dot{\gamma} = \gamma \times \omega,
\]

where \(\omega = A_x M\) and vector \(b\) is an arbitrary function on \(\gamma\) and \(M\). We want to discuss only an existence of integrals of motion in involution with respect to the Poisson bracket \(\{\ldots\}_d\), which is a necessary condition for the Liouville integrability of the corresponding Hamiltonian systems. The invariant measures and the corresponding time transformations are considered in review [8].

It is clear that \(\mathcal{H}_{1,2}\) (2) remain the integrals of equations (70) and upon the same basis we can identify these integrals with Casimir functions (15) on our Poisson manifold. It allows us to look for two additional integrals \(\mathcal{H}_3\) and \(\mathcal{H}_4\) in involution with respect to the same Poisson brackets \(\{\ldots\}\) (21) and \(\{\ldots\}_d\) (22)

\[
\{\mathcal{H}_3, \mathcal{H}_4\} = 0, \quad \{\mathcal{H}_3, \mathcal{H}_4\}_d = 0,
\]

where \(\{\ldots\}_d\) is the “nonholonomic” deformation of canonical bracket \(\{\ldots\}\) on \(e^*(3)\). For the first Poisson bracket all possible integrable deformations are well known [2, 7, 34]. So, we can try to get “nonholonomic” deformations of the Lagrange and Kowalevski tops, or of the Kirchhoff, Clebsch and Steklov-Lyapunov systems.
If the Hamilton function reads as
\[ 2H = \mathcal{H}_4 = (M, \omega) + 2V(\gamma), \quad \omega = A_x M, \]
then in holonomic case equations (76) are identified with the Euler-Poisson equations [2, 7]
\[ \dot{M} = M \times \frac{\partial H}{\partial M} + \gamma \times \frac{\partial H}{\partial \gamma}, \quad \dot{\gamma} = \gamma \times \frac{\partial H}{\partial M}, \quad H = \frac{1}{2} \mathcal{H}_4, \quad (77) \]
whereas in nonholonomic case first equation has to be replaced to
\[ \dot{M} = M \times \frac{\partial H}{\partial M} + \gamma \times \frac{\partial V}{\partial \gamma}, \quad (78) \]
according to the procedure of elimination of the undetermined Lagrange multipliers, see review [8] and references within.

6.1 Linear integrals of motion.
Let us briefly consider the Lagrange top [2, 7, 34] and its nonholonomic twin [10, 20]. Recall that Lagrange top is a special case of rotation of a rigid body around a fixed point in a homogeneous gravitational field, characterized by the following conditions: the rigid body is rotationally symmetric, i.e. two of its three principal moments of inertia coincide, and the fixed point lies on the axis of rotational symmetry. In much the same way second system is the Chaplygin ball with the rotationally symmetric mass distribution in the homogeneous gravitational field.

Proposition 10 If \( \omega = A_x M \),
\[ a_1 = a_2, \quad \text{and} \quad b = (0, 0, b_3), \]
then the integrals of the equations (77)-(78)
\[ \mathcal{H}_3 = (M, M) + 2a_1^{-1}(b, \gamma) \quad \text{and} \quad \mathcal{H}_4 = (M, \omega) + 2(b, \gamma), \quad (79) \]
are in the involution with respect to the Poisson brackets \{,\} [21] and \{,\}_d [22], respectively.

The proof is a straightforward calculation of the Poisson brackets between integrals of motion.
In holonomic case the linear in momenta integral
\[ K = (b, M) = M_3, \quad \{ \mathcal{H}_k, K \} = 0, \quad k = 1, 2, 3, \]
can be obtained from the quadratic integrals (79) in a standard way
\[ \sqrt{\mathcal{H}_4 - a_1 \mathcal{H}_3} = \sqrt{a_3 - a_1 M_3} = \sqrt{a_3 - a_1 K}. \quad (80) \]
In nonholonomic case the linear integral looks like
\[ K = \sqrt{\mathcal{g}(\gamma)} \left( M_3 + \frac{a_1 x_3(\gamma, M)}{1 - da_1(\gamma, \gamma)} \right), \quad \{ \mathcal{H}_k, K \}_d = 0 \quad k = 1, 2, 3. \quad (81) \]
It can be represented via quadratic integrals (79) and the Casimir functions according to the relation
\[ \mathcal{H}_4 - a_1 \mathcal{H}_3 + (a_1 - a_3)(1 - da_1(\gamma, \gamma)) K^2 - \frac{a_1^2}{1 - da_1(\gamma, \gamma)} (\gamma, M)^2 = 0. \]
In both cases the equations of motion (70) are equivalent to the Lax equation
\[ \frac{dL}{dt} = [L, \Omega + \lambda B], \quad L = M + \frac{\Gamma}{\lambda}, \quad \lambda \in \mathbb{R}. \]
It is obvious, that the Lax matrix $L$ satisfies the linear $r$-matrix brackets (34) with the same $r$-matrices (35) and (36).

In holonomic case by $\omega = A_1 M$ we have $b \times (M - \omega) = 0$ at $a_1 = a_2 = 1$ and $b_1 = b_2 = 0$. It allows us to get another well-known Lax representation for the Lagrange top [2, 34]:

$$\frac{dL}{dt} = [L, \Omega + \lambda B], \quad L = \lambda B + M + \frac{\Gamma}{\lambda},$$

where

In nonholonomic case $b \times (M - \omega) \neq 0$ and we have no such Lax matrix at all. Of course, it is a superficial argument because the main point is that the nonholonomic system is related with the genus three algebraic curve instead of the elliptic curve for the Lagrange top.

Namely, using the Euler angles and their conjugated momenta, for the Lagrange top we can easily prove that the pair of canonical variables

$$u = \gamma_3 = \cos(\theta), \quad p_u = \frac{\gamma_2 M_1 - \gamma_1 M_2}{\gamma_1^2 + \gamma_2^2} = -\frac{p_\theta}{\sin(\theta)}, \quad \{u, p_u\} = 1,$$

lies on the elliptic curve defined by equation

$$a_1 p_u^2 + \frac{2ub + \beta - a_3 \alpha^2}{u^2 - 1} + \frac{a_1 (\alpha u + \ell)^2}{(u^2 - 1)^2} = 0,$$

where we fix the values of the integrals of motion

$$H_1 = (\gamma_1, \gamma) = 1, \quad H_3 = (\gamma, M) = \ell, \quad K = \alpha, \quad H_4 = \beta.$$ (83)

For the nonholonomic system canonical variables

$$u = u, \quad \hat{p}_u = \frac{p_u}{\sqrt{g}}, \quad \{u, \hat{p}_u\} = 1,$$

satisfy to the following separated equation

$$a_1 \hat{p}_u^2 + \frac{2ub + \beta - a_3 \rho \alpha^2}{u^2 - 1} g + \frac{a_1 (\alpha u + \ell)^2}{(1 + da_1(u^2 - 1)(u^2 - 1)^2) = 0.}$$ (84)

Here we fix the values of the integrals of motion as in (83) and

$$g = \frac{d}{1 - da_1 + d(a_1 - a_3)u^2}, \quad \rho = \frac{(1 - da_1)^2}{1 + da_1(u^2 - 1)},$$

where

$$\frac{d}{1 - da_1 + d(a_1 - a_3)u^2}, \quad \rho = \frac{(1 - da_1)^2}{1 + da_1(u^2 - 1)}, \quad g = \frac{d}{1 - da_1 + d(a_1 - a_3)u^2}, \quad \rho = \frac{(1 - da_1)^2}{1 + da_1(u^2 - 1)},$$

If $\ell = (\gamma, M) = 0$, one gets the elliptic curve, but in generic case rewriting the equation (84) in polynomial form we obtain the algebraic curve of genus three.

Only at $(\gamma, M) = 0$ we can get the solutions in the terms of elliptic functions and, therefore, only in this particular case we can try to reconstruct the Lax matrix associated with the elliptic curve.

Three different bi-Hamiltonian structures for the Lagrange top have been obtained in [43]. These structures are related with different variables of separation and, therefore, different quadratures. If we get similar dynamical Poisson bivectors for its nonholonomic counterpart, one gets various quadratures, which could be associated with the distinct Lax matrices and underlying $r$-matrix algebras.

### 6.2 Second order integrals of motion

For the Kirchhoff problem, the integrable cases by Kirchhoff, Clebsch, and Steklov-Lyapunov are known. In this section we begin with the Clebsch case.
Proposition 11 If $\omega = A_x M$ then the integrals of the equations (77)-(78) \[
H_3 = (M, M) - (A, \gamma, \gamma) \quad \text{and} \quad H_4 = (M, \omega) + (A, \gamma, \gamma) \quad (85)
\]
are in the involution with respect to the Poisson brackets $\{., .\}$ (21) and $\{., .\}_d$ (22), respectively. As above proof is a straightforward calculation of the Poisson brackets between integrals of motion.

In holonomic case we have the well-studied Clebsch problem. In nonholonomic case this deformation of the Chaplygin system has been proposed by Kozlov [26] in framework of the Euler-Jacobi integration procedure, i.e. without notion of the Poisson bracket.

There are some different Lax matrices for the Clebsch model [7, 34]. For example, 
\[
L = \lambda A + M + \gamma \times \gamma \lambda.
\]
We can not directly generalize this matrix to the nonholonomic case, because $\dot{A} = 0$ in contrast with $\dot{A} = 0$ above. We suppose that the nonholonomic Kozlov system is related with the algebraic curve of higher genus and, therefore, the corresponding Lax matrices will be more complicated deformations of the known Lax matrices for the Clebsch problem.

At $(\gamma, M) = 0$ the Clebsch system becomes the so-called Neumann system on the sphere, which is separable in the elliptic coordinates $u, v$ (48) [7]. Its nonholonomic counterpart is the separable system in Chaplygin coordinates $u, v$ (49) [17] and, therefore, we can get $2 \times 2$ Lax matrices $L(\lambda)$ (63) for this nonholonomic system as well.

The Clebsch case is equivalent to the Brun case of integrability in the Euler-Poisson equations [7] and, moreover, it is trajectory isomorphic to the Kowalevski gyrostat [23]. We can hope to get a nonholonomic analog of the Kowalevski top by using similar isomorphism.

Now let us briefly discuss the integrable Steklov-Lyapunov case in the Kirchhoff equation and the corresponding integrals of motion \[
H_3 = (M, M) - 2(M, A\gamma) + (\gamma, C^2\gamma),
\]
\[
H_4 = (M, \omega) + 2(M, A\gamma) + (A\gamma, C^2\gamma), \quad (86)
\]
where $C = \text{diag}(a_2 - a_3, a_3 - a_1, a_1 - a_2)$. These integrals are in the involution with respect to the first bracket $\{H_3, H_4\} = 0$.

If we replace $\omega = AM$ on $\omega = A_d M$ then $\{H_3, H_4\}_d = 0$ only if two parameters $a_i$ coincide with each other. So, for the nonholonomic bracket $\{., .\}_d$ we have to propose some more complicated deformations of the integrals of motion (86).

It is known that the Steklov-Lyapunov system is equivalent to the integrable system on the sphere with forth order potential (69) [69, 10]. We suppose that a similar transformation of the system (69) separable in nonholonomic elliptic coordinates allows us to get a nonholonomic counterpart of the Steklov-Lyapunov system.

7 Conclusion

We consider two very similar dynamical systems, which evolve on coadjoint orbits of Lie algebra $e(3)$ and their non-trivial symplectic deformations.

Close ties between the integrable Euler top and the nonholonomic Chaplygin ball allow us to get Lax matrices, $r$-matrices and bi-hamiltonian structure for this nonholonomic system. Moreover, in framework of the Jacobi method of separation of variables we describe a huge family of separable potentials, which can be added to nonholonomic Hamiltonian and briefly discuss how to get the $N$-dimensional nonholonomic systems on the Riemannian spaces of constant curvature.

In [11] Chaplygin transforms the generic case of the rolling ball to the particular case of horizontal angular momentum $(\gamma, M) = 0$. It allows us to solve the equations of motion using
the same variables of separation \( u, v \) [19], which will be the non-canonical variables with respect to initial Poisson bracket \( \{\cdot, \cdot\}_d \) [22] after this map. We will discuss this Chaplygin map in framework of the Poisson geometry in separate publication, as well as the corresponding \( 2 \times 2 \) Lax matrices and the underlying \( r \)-matrix algebra.

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