Generalised Freud’s equation and level densities with polynomial potential

Akshat Boobna
The Creative School, E-791, C.R. Park, New Delhi-110017

Saugata Ghosh
The Creative School, E-791, C.R. Park, New Delhi-110017
(Dated: May 5, 2014)

We study orthogonal polynomials with weight \( \exp[-NV(x)] \), where \( V(x) = \sum_{k=1}^{d} a_{2k} x^{2k}/2k \) is a polynomial of order \( 2d \). We derive the generalised Freud’s equations for \( d = 3, 4 \) and \( 5 \) and using this obtain \( R_\mu = h_\mu/h_{\mu-1} \), where \( h_\mu \) is the normalization constant for the corresponding orthogonal polynomials. Moments of the density functions, expressed in terms of \( R_\mu \), are obtained using Freud’s equation and using this, explicit results of level densities as \( N \to \infty \) are derived.

PACS numbers: 02.30.Gp, 05.45.Mt

1. INTRODUCTION

Universality in random matrix theory [9, 11–13] has led people to study orthogonal [1, 2, 5] and skew-orthogonal polynomials [3] in great details. However, in the process, the non-universal level densities are neglected inspite of the possibility of its direct application in various physical systems. In this context, we study level densities of a class of non-Gaussian random matrix ensembles and thereby develop the theory of orthogonal polynomials.

Orthogonal polynomials are defined as

\[
\int_{\mathbb{R}} P_n(x) P_m(x) w(x) dx = h_n \delta_{nm}, \quad n, m \in \mathbb{N}.
\]  

(1.1)

We study orthogonal polynomials with weight function \( w(x) = \exp(-NV(x)) \), where

\[
V(x) = \sum_{k=1}^{d} a_{2k} x^{2k}/(2k), \quad a_{2d} > 0.
\]  

(1.2)

Here, we make a numerical analysis of orthogonal polynomials corresponding to \( d = 3, 4 \) and \( 5 \). We derive the corresponding Freud’s equation and calculate \( R_\mu = h_\mu/h_{\mu-1} \). We observe interesting patterns in the behavior of \( R_\mu \).

Once we have an understanding of \( R_\mu \), we use these results to obtain level densities of non-Gaussian ensembles of random matrices. We know that variation of the first \( n \)-eigenvalues of a random matrix can be studied by the \( n \)-point correlation function, \( R_\beta^{(n)}(x_1, ..., x_n) \) which is defined by

\[
R_\beta^{(n)}(x_1, ..., x_n) = \frac{N!}{(N-n)!} \int_{\mathbb{R}^{N-n}} dx_{n+1}...dx_N P_\beta,N(x_1, ..., x_N),
\]  

(1.3)

where \( \beta = 1, 2, 4 \) correspond to ensembles of random matrices invariant under orthogonal, unitary and symplectic transformations. This allows us to find the probability density of the \( n \) eigenvalues at \( x_1, ..., x_n \), irrespective of the eigenvalues at \( x_{n+1}...x_N \). \( R_1^{(2)}(x) \) is the level density, which, for \( \beta = 2 \) can be written as [19, 22]

\[
R_1^{(2)}(x) = \sum_{\mu=0}^{N-1} (h_\mu)^{-1} [P_\mu(x)]^2 e^{-NV(x)}.
\]  

(1.4)

To calculate \( R_1^{(2)}(x) \) as \( N \to \infty \), the standard method is to use the Christoffel Darboux formula and the asymptotic results of orthogonal polynomials. The latter is not always available for general polynomial potential inspite of some
serious contributions from several authors \[2, 15–18\] on the asymptotics of orthogonal polynomials with \(V(x) = x^{2d}\) using the Riemann Hilbert technique. In this paper, we use the method of resolvent to obtain the level densities as \(N \rightarrow \infty\). This needs an understanding of moments \(M_k\) defined as

\[
M_k = \int_{\mathbb{R}} x^k R^{(2)}_1(x)dx, \quad k \in \mathbb{N}.
\]  

This is derived using the values of \(R_\mu\) using generalised Freud’s equation, which we derive independently. Using this, we obtain the corresponding level densities. This gives us a good understanding of the origin of multiple band formation in the level densities in polynomial potential.

The paper is organized as follows: In section 2, we study the \(d = 3\) case and observe the behaviour of \(R_\mu\) for different values of \(a_k\). Section 3 and 4 deal with \(d = 4\) and \(d = 5\) results. This is followed by our concluding remarks.

2. \(d = 3\) CASE

A. Freud’s equation

Orthogonal monic polynomials with even weight satisfy a recursion relation \[1\]

\[
 XP_\mu = P_{\mu+1} + R_\mu P_{\mu-1}, \quad \mu \in \mathbb{N},
\]  

where \(R_\mu = h_\mu/h_{\mu-1}\), for \(\mu \geq 1\) and \(R_0 = 0\).

A major development in the study of quartic weight (\(d = 2\) in eq.\((1.2)\)) polynomials \[25–27\] was the following recursive equation in \(R_\mu\) due to \[6\].

\[
\mu + 1 = NR_{\mu+1}[a_4(R_{\mu+2} + R_{\mu+1} + R_\mu) + a_2].
\]  

Now, we derive a similar Freud’s equation for sextic potential, i.e. \(d = 3\) in eq.\((1.2)\). We use the identity

\[
\int dx[P_{\mu+1}(x)P_\mu(x)e^{-NV(x)}]' = 0.
\]  

Using \(P_\mu(x) = x^\mu + \ldots\) and the orthonormality condition \[1.1\], we get

\[
\int [e^{-NV(x)}][P_{\mu+1}'(x)P_\mu(x) + P_{\mu+1}(x)P_\mu'(x)]dx + \int N[a_6x^5 + a_4x^3 + a_2x]P_{\mu+1}(x)P_\mu(x)e^{-NV(x)}]dx = 0
\]

This gives us

\[
(\mu+1)h_\mu = \int Na_6x^5P_{\mu+1}(x)P_\mu(x)e^{-NV(x)}dx + \int Na_4x^3P_{\mu+1}(x)P_\mu(x)e^{-NV(x)}dx + \int Na_2xP_{\mu+1}(x)P_\mu(x)e^{-NV(x)}dx.
\]

Using \[2.1\], we obtain

\[
\mu + 1 = NR_{\mu+1}a_6(R_{\mu+2}(R_\mu + R_{\mu+1} + R_{\mu+2} + R_{\mu+3}) + R_{\mu+1}(R_\mu + R_{\mu+1} + R_{\mu+2}) + R_\mu(R_{\mu-1} + R_\mu + R_{\mu+1})) + a_4(R_{\mu-1} + R_\mu + R_{\mu+1} + a_2).
\]  

Here we note that the corresponding Freud’s equation is cubic in nature thereby giving rise to oscillatory behavior.

B. The \(R_\mu\) plot

For the \(d = 2\) case, two main features were observed in the \(R_\mu\) plot from the original Freud’s equation: A two band structure formed by an oscillation between two values, converging to a single band.

In the sextic case, the two band and single band structures reappear, however a new, more chaotic structure is also seen, appearing either between one band and one band or one band and two band structures. Henceforth, it is termed as a “transient structure”.

For the single band structure, all the terms in the Freud’s equation are equal to each other. Thus, we obtain a $\lambda$ dependent cubic equation (where $\lambda = \mu/N$), solving which we obtain one real solution

$$D_{\pm} = \frac{27}{2} \left( (2a_4^3 - 10a_6a_4 - 100a_6^2\lambda) \pm \sqrt{(2a_4^3 - 10a_6a_4 - 100a_6^2\lambda)^2 - 4(a_6^2 - \frac{10a_6a_2}{3})^3} \right)$$

(2.5)

$$R_{\mu} = -\frac{a_4}{10a_6} - \frac{\sqrt{D_+}}{30a_6} - \frac{\sqrt{D_-}}{30a_6}$$

(2.6)

which gives the value of $R_{\mu}$ for $\lambda$ values where single band exists.

2. Two band structure

Solving the Freud’s equation assuming that two bands are formed (as seen), i.e.

$A_0 = R_0 = R_2 = R_4 = ...$

$A_1 = R_1 = R_3 = R_5 = ...$

for $N \gg \mu$, we get

$$A_0 + A_1 = -\frac{a_4 + \sqrt{a_4^2 - 4a_2a_6}}{2a_6}$$

(2.7)

It has been numerically verified that the bottom band ($A_1$) tends to 0, and we find that $A_1 \propto 1/N$ and $A_1 \propto 1/(a_4)^2$.

3. Transient structure

When the transient structure is divided modulo 3 into three bands ($b_0$, $b_1$ and $b_2$ in fig. 1), it is seen that each of the three separate bands continuously oscillate, and converge to a common value.
The sum mod 3 for the duration of the transient structure oscillates above the value

\[ b_0 + b_1 + b_2 \approx \frac{-a_4 + \sqrt{a_4^2 - 4a_2a_6}}{2a_6}, \]  

(2.8)

where \( b_0, b_1 \) and \( b_2 \) correspond to three consecutive values from each of the bands.

### C. Critical \( a_4 \)'s

On analyzing the roots of the sextic potential, we obtain the critical value of \( a_4 \), denoted here by \( a_{4c} \), where the structure of the \( R_\mu \) plot changes. We find the points at which potential plot touches 0, and obtain

\[ a_{4c} = -\sqrt{\frac{48}{9a_2a_6}} \]  

(2.9)

In the \( R_\mu \) plot, when (i) \( a_4 < a_{4c} \), we observe a two band structure, followed by a transient structure, which converges to single band. (ii) \( a_4 > a_{4c} \), we see a one band structure, followed by a transient structure, which converges to single band.

We now analyze the behavior of \( R_\mu \) as \( a_4 \) approaches \( a_{4c} \) (fig. 2). It is observed the transient structure resolves into three distinct bands near \( a_{4c} \). Exactly at the critical value, two of these bands coincide to form an upper band, and the third band forms a lower band, creating a pseudo two band structure.

**FIG. 2:** \( R_\mu \) plot as \( a_4 \) approaches \( a_{4c} \) for \( a_6=1, a_2=1, N=500 \)
D. Level Density

In this subsection, we derive the level density using the method of resolvent \[3, 10\] as \(N \to \infty\). The result is expressed in terms of moments \(M_k \underbrace{[1.5]}\) which are derived using results from the Freud’s equation. We then compare the result with the level density for \(N = 30\) using \(1.4\).

FIG. 3: Level density plots for \(d = 3, a_6 = 1, a_4 = -3, a_2 = 1\) \(N = 30\). For the theoretical plot, calculated moments are \(M_2 = 62.536, M_4 = 164.770\).

Here, we recall \[3\] that we use the scaling \(V(x) \to V(x)/2\) to obtain the corresponding results.

\[
\left[ \pi R^{(2)}_1(x) \right]^2 = N \int_{-\infty}^{\infty} \frac{V'(z) - V'(x)}{z - x} R^{(2)}_1(x) \, dx - N^2 \left[ \frac{V'(x)}{2} \right]^2
\]

\[
= N \left[ a_6(x^4N + x^2M_2 + M_4) + a_4(x^2N + M_2) + a_2N \right] - N^2 x^2 \left( \frac{a_6x^4 + a_4x^2 + a_2}{2} \right)^2.
\]

Finally,

\[
\left[ \pi R^{(2)}_1(x) \right] \frac{N}{x} = \left( \frac{a_6M_4 + a_4M_2}{N} + a_2 \right) + x^2 \left[ \left( \frac{a_6x^2}{N} + a_4 \right) - \frac{1}{4} \left( \frac{a_6x^4 + a_4x^2 + a_2}{2} \right)^2 \right]. \tag{2.10}
\]

Now that we have derived \(R^{(2)}_1(x)\) in terms of \(M_2\) and \(M_4\), we would be interested to calculate them using Freud’s equation. We use

\[
M_k = \sum_{\mu} \int \frac{x^k}{h_\mu} P^\mu(x) P^\nu(x) w(x) \, dx
\]

\[
= \sum_{\mu} \int \frac{(x^k P^\mu(x)) P^\nu(x)}{h_\mu} w(x) \, dx
\]

\[
= \sum_{\mu} \sum_{\nu} \int \frac{C_\nu P^\nu(x) P^\mu(x)}{h_\mu} w(x) \, dx
\]

\[
= \sum_{\mu} C_\mu, \tag{2.11}
\]

where \(C_\mu\) are coefficients which can be expressed in terms of \(R_\mu\). A few typical examples are

\[
M_2 = \sum_{\mu=0}^{N-1} (R_{\mu+1} + R_\mu), \tag{2.12}
\]
\[ M_4 = \sum_{\mu=0}^{N} \left( R_\mu^2 + R_{\mu+1}^2 + 2R_{\mu}R_{\mu+1} + R_{\mu+1}R_{\mu+2} + R_{\mu}R_{\mu-1} \right), \]  
(2.13)

and

\[ M_6 = \sum_{\mu=0}^{N} \left( R_{\mu-2}R_{\mu-1}R_{\mu} + R_{\mu-1}^2R_{\mu} + 2R_{\mu-1}R_{\mu}^2 + R_{\mu}^3 + 2R_{\mu-1}R_{\mu}R_{\mu+1} ight) \]
\[ + 3R_{\mu}^2R_{\mu+1} + 3R_{\mu}R_{\mu+1}^2 + R_{\mu+1}^3 + 2R_{\mu}R_{\mu+1}R_{\mu+2} + 2R_{\mu+1}R_{\mu+2} \]
\[ + R_{\mu+1}^2R_{\mu+2} + R_{\mu+1}R_{\mu+2}R_{\mu+3} \). \]  
(2.14)

The expression for \( M_6 \) and higher moments are extremely cumbersome but can be easily calculated using the aforementioned algorithm.

3. \( d = 4 \) CASE

A. Freud’s equation

![Graph](image)

FIG. 4: \( d = 4 \) case plot for \( a_8 = 1, a_6 = -5, a_4 = 6, a_2 = -1, N = 200 \)

As in the \( d = 3 \) case, we start with the identity

\[
\int dx [P_{\mu+1}(x)P_{\mu}(x)e^{-NV(x)}]' = 0, \]

(3.1)
where $V(x)$ is defined as in eq. (1.2), but with $d = 4$. Using the recursion relation for orthogonal polynomials, we get

\[
\mu + 1 = NR_{\mu + 1}[a_8(R_{\mu + 2}R_{\mu + 3})^{\mu + 4} + R_{\mu + 2}^{\mu + 3} + R_{\mu + 1}^{\mu + 2} + R_{\mu + 1}^{\mu + 3} + R_{\mu + 1}^{\mu + 4}] \\
+ a_8(R_{\mu + 2}R_{\mu + 1} + R_{\mu + 1}R_{\mu + 2} + R_{\mu + 1}^2 + R_{\mu + 1}^3 + R_{\mu + 1}^4) \\
+ a_6(R_{\mu + 2} + R_{\mu + 1}^2 + R_{\mu + 1}^3 + R_{\mu + 1}^4) \\
+ a_4 \sum_{i=\mu-1}^{\mu+1} R_i + a_2 \]

(3.2)

Here we note that due to the non-linear nature of the Freud’s equation, we observe oscillations in the solution for $R_{\mu}$. These oscillations can be seen when the plot is divided modulo 4 into 4 residual bands ($b_0, b_1, b_2$ and $b_3$ in fig. 4).

**B. Level density**

Using the obtained moments and the formulation for finding level density (sec. 2D), we derive the function for $R_{1}(x)$ for the $d = 4$ case (3.3). The plot of $R_{1}(x)$ obtained using this function is compared with the $R_{1}(x)$ calculated from (1.4) in fig. 5.

We note that irregularities in the form of small oscillations around the expected value are seen near the peaks. This is because the numerically calculated level density is for a finite value of $N$. These oscillations gradually disappear as the value of $N$ increases.

\[
\left[\frac{\pi R_{1}^{(2)}(x)}{N}\right]^2 = \left[\frac{a_8M_6 + a_6M_4 + a_4M_2}{N} + a_2\right] \\
+ x^2 \left(\frac{a_8x^4 + a_6x^2 + a_4 + a_8M_2 + a_6M_2}{N} - \frac{(a_8x^6 + a_6x^4 + a_4x^2 + a_2)^2}{4}\right). 
\]  

(3.3)

**FIG. 5:** Level density plots for $d = 4$, $a_8 = 1$, $a_6 = -5$, $a_4 = 6$, $a_2 = -1$, $N = 20$. For the theoretical plot, calculated moments are $M_2 = 43.475$, $M_4 = 134.555$, $M_6 = 438.400$. 
4. \( d = 5 \) CASE

A. Freud’s equation

As in the \( d = 3 \) case, we start with the identity

\[
\int dx [P_{\mu + 1}(x)P_{\mu}(x)e^{-NV(x)}]' = 0,
\]

(4.1)

where \( V(x) \) is defined as in eq. (1.2), but with \( d = 5 \). From here, one can understand that finding the Freud’s equation is algorithmic in nature and we leave it to the reader to derive it explicitly. Here, we will show the generic plot of the \( R_\mu \) function.

![FIG. 6: \( d = 5 \) case plot modulo 5 for \( a_{10} = 10, a_8 = -80, a_6 = 210, a_4 = -200, a_2 = 48, N = 50 \)]

B. Level density

Having derived the moments \( M_2, M_4, M_6 \) and \( M_8 \) using the general formulation given in sec. 2.1, we use the derivation provided to obtain the function for \( R^{(2)}_1(x) \) for the \( d = 5 \) case (4.2). The plot of \( R^{(2)}_1(x) \) obtained using this function is compared with the \( R^{(2)}_1(x) \) calculated from (1.3) in fig. 7.

Once again, we note that small oscillations around the expected value are seen. This is because we are calculating \( R^{(2)}_1(x) \) for a finite value of \( N \), and these become smooth as \( N \to \infty \)

\[
\left[ \frac{\pi R^{(2)}_1(x)}{N} \right]^2 = \frac{a_{10}M_8 + a_8M_6 + a_6M_4 + a_4M_2 + a_2}{N} + x^2 \left( a_{10}x^6 + a_8x^4 + a_6x^2 + a_4 + \frac{a_{10}x^4M_2 + a_8x^2M_2 + a_6M_2 + a_4x^2M_4 + a_10M_6 + a_8M_4}{N} \right)
\]

(4.2)

\[- \frac{x^2}{4} (a_{10}x^8 + a_8x^6 + a_6x^4 + a_4x^2 + a_2)^2.\]
5. CONCLUSION

In this paper, we obtain the Freud’s equation for polynomials with weight function $\exp[-NV(x)]$, where $V(x) = \sum_{k=1}^{d} a_{2k}x^{2k}/2k$ is a polynomial of order $2d$. We derive the generalised Freud’s equations for $d = 3$, 4 and 5. We observe limit cycle behavior of $R_\mu$. We use these results and the method of resolvent to obtain the level densities of the corresponding random matrix models. However, this involves an explicit calculation of the higher moments which we calculate numerically and insert in the analytic results of the level density. It would be nice to obtain explicit results of these moments as was done for the quartic case. But we have failed in this investigation due to the complex nature of the Freud’s equation.

Here, we might recall that for $d = 2$, the Freud’s equation is quadratic while for higher $d$, it becomes cubic ($d = 3$), quartic ($d = 4$) and so on. This results in oscillations in the $R_\mu$ function and hence studying the limit cycle behavior becomes increasingly complicated. Further investigation is needed to study these non-linear Freud’s equations, specially in the context of integrability and hence the existence of Lax pairs. We wish to come back to these questions in a later publication.

[1] Szego G., 1939, Orthogonal Polynomials, Amer. Math. Soc. Colloq. Pub., vol. 23, Amer. Math. Soc., New York, NY

[2] Deift, P., 2000, Orthogonal polynomials and random matrices: A Riemann-Hilbert approach, Courant Lec. Notes No. 3. Amer. Math. Society, Providence, Rhode Island.

[3] Ghosh S., 2009, Skew-orthogonal polynomials and Random matrix theory, CRM Monograph Series, AMS

[4] Ghosh S., Pandey A., Puri S., Saha R., Non-Gaussian random-matrix ensembles with banded spectra, Phys. Rev. E 67(2003), no.2, 025201

[5] Sheen R.-C. Plancherel-Rotach type asymptotics for orthogonal polynomials associated with $\exp(-x^6)$ J. Approx. Theory, 50 (1987), 232-293

[6] Freud G., On the Coefficients in the recursion formulae of orthogonal polynomials, Proc. Royal Irish Acad., 76A (1976), 1-6

[7] Levin A.L. and Lubinsky D.S., Orthogonal polynomials and Christoffel functions for $\exp(- | x |^\alpha)$, $\alpha \leq 1$, J. Approx. Theory, 80 (1995), 219-252

[8] Random matrix theories in quantum physics: Common concepts, T. Guhr, A. Mueller-Groeling, H. A. Weidenmuller Phys. Rep.299, 189(1998).
[9] A. W. J. Beenakker, Rev. Mod. Phys., 69, 731 (1997).

[10] Ghosh S., 2002, Thesis, Jawaharlal Nehru University, New Delhi (unpublished).

[11] Deift P., Gioev D., 2007, Universality in Random Matrix Theory for orthogonal and symplectic ensembles, Int. Math. Res. Pap. IMRP, 2, rpm004.

[12] Deift P., Gioev D., 2007, Universality at the edge of the spectrum for unitary, orthogonal and symplectic ensembles of random matrices, Comm. Pure Appl. Math. 60, 867-910.

[13] Deift P., Universality for mathematical and Physical systems, math-ph/0603038.

[14] Bleher, P.M. and Its, A.R., 1999, Semiclassical asymptotics of orthogonal polynomials, Riemann-Hilbert problems, and universality in the matrix model. Ann. of Math. (2) 150, 185–266.

[15] Deift, P.A., Kriecherbauer, T. and McLaughlin, K. T-R., New results for the asymptotics of orthogonal polynomials and related problems via the Lax-Levermore method. Announcement in Proceedings of Symposia in Applied Mathematics, 54 (1998) 87-104. Full text in J. Approx. Theory, 95 (1998), 388-475, under the title, New results on the equilibrium measure for logarithmic potentials in the presence of an external field.

[16] Deift, P.A., Kriecherbauer, T., McLaughlin, K. T-R., Venakides, S. and Zhou, X., 1999, Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory. Comm. Pure Appl. Math. 52, 1335–1425.

[17] Deift, P.A., Kriecherbauer, T., McLaughlin, K. T-R., Venakides, S. and Zhou, X., 1999, Strong asymptotics of orthogonal polynomials with respect to exponential weights. Comm. Pure Appl. Math. 52, 1491–1552.

[18] Plancherel M. and Rotach W., 1929, Sur les valeurs asymptotiques des polynomes d’Hermie $H_n(x) = (−1)^n e^{x^2/2} d^n(e^{-x^2/2})/dx^n$, Comment. Math Helv.1227-54.

[19] Dyson F. J., 1972, A Class of Matrix Ensembles, J. Math. Phys.13, 90-97.

[20] Dyson F. J. and Mehta M. L., 1963, Statistical theory of energy levels of complex systems IV, J. Math. Phys. 4,701-712.

[21] Random Matrices, Mehta M. L., 2004 (The Netherlands, Elsevier, 3rd ed.).

[22] Matrix Theory: Selected Topics and Useful Results, Mehta M. L., 1988 (Delhi 110007, India, Hindustan Publishing Corporation, 2nd ed.).

[23] Nagao T. and Wadati M., 1991, Correlation functions of random matrix ensembles related to classical orthogonal polynomials, J. Phys. Soc. Japan.60, 3298-3322.

[24] Nagao T. and Wadati M., 1992, Correlation functions of random matrix ensembles related to classical orthogonal polynomials II, J. Phys. Soc. Japan.61, 78-88.

[25] Stojanovic A., 2000, Universality in Orthogonal and Symplectic Invariant Matrix Models with Quartic Potential, Mathematical Physics, Analysis And Geometry 3 (4): 339-373.

[26] Stojanovic A., 2004, Errata: Universality in Orthogonal and Symplectic Invariant Matrix Models with Quartic Potential, Mathematical Physics Analysis And Geometry s 7 (4): 347-349 (erratum).

[27] Stojanovic A., 2000, Une approche par les polynomes orthogonaux pour des classes de matrices alatoires orthogonalement et symplectiquement invariantes: application l’universalit de la statistique locale des valeurs propres, Preprint, Boieledfeld 00-01-006 (www.physik.uni-bielefeld.de/bibos/).