AN ESTIMATION FOR THE ESSENTIAL NORM OF COMPOSITION OPERATORS ACTING ON BLOCH-TYPE SPACES

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Abstract. Let $\mu$ be any weight function defined on the unit disk $\mathbb{D}$ and let $\phi$ be an analytic self-map of $\mathbb{D}$. In the present paper we show that the essential norm of composition operator $C_\phi$ mapping from the $\alpha$-Bloch space, with $\alpha > 0$, to $\mu$-Bloch space $B^\mu$ is comparable to

$$\limsup_{|a|\to 1^-} \|\sigma_a \circ \phi\|_{B^\mu},$$

where, for $a \in \mathbb{D}$, $\sigma_a$ is a certain special function in $\alpha$-Bloch space.

Keywords: Bloch spaces, Composition operators.

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1. Introduction

Let $\mu$ be denotes what we call a weight on the unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$; that is, $\mu$ is a bounded, continuous and strictly positive function defined on $\mathbb{D}$, and let $H(\mathbb{D})$ be the space of all holomorphic functions on $\mathbb{D}$, which is equipped with the topology of uniform convergence on compact subsets of $\mathbb{D}$. The $\mu$-Bloch space $B^\mu(\mathbb{D})$, which we denote more briefly by $B^\mu$, consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_\mu := \sup_{z \in \mathbb{D}} \mu(z) |f'(z)| < \infty.$$

$\mu$-Bloch spaces are called weighted Bloch spaces. For weights $\mu$ on $\mathbb{D}$, a Banach space structure on $B^\mu$ arises if it is given the norm

$$\|f\|_{B^\mu} := |f(0)| + \|f\|_\mu.$$

These Banach spaces provide a natural setting in which one can study properties of various operators. For instance, Attele in [1] proved that if $\mu_1(z) := w(z) \log \frac{2}{w(z)}$, where $w(z) := 1 - |z|^2$ and $z \in \mathbb{D}$, then the Hankel operator $H_f$ induced by a function $f$ in the Bergman space $A^2(\mathbb{D})$ (see [4, Ch. 2]) is bounded if and only if $f \in B^{\mu_1}$, thus giving one reason, and not the only reason, why log-Bloch-type spaces are of interest. When $\mu(z) = v_\alpha(z) := (1 - |z|^2)^\alpha$ with $\alpha > 0$ fixed, then we get back the $\alpha$-Bloch space which is denoted as $B^\alpha$ and when $\alpha = 1$ we obtain the Bloch space $B$.

A holomorphic function $\phi$ from the unit disk $\mathbb{D}$ into itself induces a linear operator $C_\phi$, defined by $C_\phi(f) = f \circ \phi$, where $f \in H(\mathbb{D})$. $C_\phi$ is called the
composition operator with symbol φ. Composition operators continue to be widely studied on many subspaces of $H(D)$ and particularly in Bloch-type spaces.

The study of the properties of composition operators on Bloch-type spaces began with the celebrated work of Madigan and Matheson in [8], where they characterized the continuity and compactness of composition operators acting on the Bloch space $B$. Many extensions of the Madigan and Matheson’s results have appeared (see for instance [12] and a lot of references therein). In particular, Xiao in [17] has extended the results by Madigan and Matheson in [8] to composition operators $C_\phi$ acting between $\alpha$-Bloch spaces. Recently, many authors have found new criteria for the continuity and compactness of composition operators acting on Bloch-type spaces in terms of the $n$-th power of the symbol $\phi$ and the norm of the $n$-th power of the identity function on $D$. The first result of this kind appears in 2009 and it is due to Wulan, Zheng, and Zhu ([16]), in turn, their result was extended to $\alpha$-Bloch spaces by Zhao in [19]. Another criterion for the continuity and compactness of composition operators on Bloch space is due to Tjani in [14] (see also [15] or more recently [16]), she showed the following result:

**Theorem 1.1** ([14]). The composition operator $C_\phi$ is compact on $B$ if and only if $\phi \in B$ and

$$\lim_{|a| \to 1^-} \| \varphi_a \circ \phi \|_B = 0,$$

where $\varphi_a$ is a Möbius transformation from the unit disk onto itself; that is, $\varphi_a(z) = (a - z) / (1 - \overline{a}z)$, with $z \in D$.

This last result has been recently extended to $\alpha$-Bloch spaces by Malavé and Ramos-Fernández in [9].

The essential norm of a continuous linear operator $T$ between normed linear spaces $X$ and $Y$ is its distance from the compact operators; that is, $\|T\|_{e}^{X \to Y} = \inf \{ \|T - K\|^{X \to Y} : K : X \to Y$ is compact $\}$, where $\| \cdot \|^{X \to Y}$ denotes the operator norm. Notice that $\|T\|_{e}^{X \to Y} = 0$ if and only if $T$ is compact, so that estimates on $\|T\|_{e}^{X \to Y}$ lead to conditions for $T$ to be compact. The essential norm of a composition operator on $B$ was calculated by A. Montes-Rodríguez in [10]. He obtained similar results for essential norms of weighted composition operators between weighted Banach spaces of analytic functions in [11]. Other results in this direction appear in the paper by Contreras and Hernández-Díaz in [3]; in particular, formulas for the essential norm of weighted composition operators on the $\alpha$-Bloch spaces of $B_\alpha$ were obtained (see also the paper of MacCluer and Zhao [7]). Recently, many extensions of the above results have appeared in the literature; for instance, the reader is referred to the paper of Yang and Zhou [18] and several references therein. Zhao in [19] gave a formula for the essential norm of $C_\phi : B^\alpha \to B^\beta$ in terms of an expression involving norms of powers of $\phi$. 
More precisely, he showed that
\[ \| C_\phi \|_{e}^{B^\alpha \to B^\beta} = \left( \frac{e}{2\alpha} \right)^{\alpha} \limsup_{j \to \infty} j^{\alpha - 1} \| \phi^j \|_{B^\beta}. \]

It follows from the discussion at the beginning of this paragraph that \( C_\phi : B^\alpha \to B^\beta \) is compact if and only if
\[ \lim_{j \to \infty} j^{\alpha - 1} \| \phi^j \|_{B^\beta} = 0. \]

The Zhao’s results in [19] have been extended recently to the weighted Bloch spaces by Castillo, Clahane, Farías and Ramos-Fernández in [2]. Also, Hyvärinen, Kemppainen, Lindström, Rautio and Saukko in [6] obtained necessary and sufficient conditions for boundedness and an expression characterizing the essential norm of a weighted composition operator between general weighted Bloch spaces \( B^\mu \), under the technical requirements that \( \mu \) is radial, and that it is non-increasing and tends to zero toward the boundary of \( \mathbb{D} \).

The goal of the present paper is to give an estimate of the essential norm of composition \( C_\phi \) mapping from \( B^\alpha \) to \( B^\mu \) which implies Theorem 1.1 and the result given by Malavé and Ramos-Fernández in [9]. More precisely, in the next section we will show the following result.

**Theorem 1.2.** Let \( \phi \) be an analytic self-map of the unit disk \( \mathbb{D} \). Then for the essential norm of the composition operator \( C_\phi : B^\alpha \to B^\mu \) we have

\[ \| C_\phi \|_{e}^{B^\alpha \to B^\mu} \sim \limsup_{|a| \to 1^-} \| \sigma_a \circ \phi \|_{B^\mu}. \]

The relation (1) means that there is a positive constant \( M_\alpha \), depending only on \( \alpha \), such that
\[ \frac{1}{M_\alpha} \| C_\phi \|_{e}^{B^\alpha \to B^\mu} \leq \limsup_{|a| \to 1^-} \| \sigma_a \circ \phi \|_{B^\mu} \leq M_\alpha \| C_\phi \|_{e}^{B^\alpha \to B^\mu} \]

and the functions \( \sigma_a \) with \( a \in \mathbb{D} \) will be defined at the begin of the next section.

2. **Proof of Theorem 1.2**

The key to our results lies in considering the following family of functions. For \( a \in \mathbb{D} \) fixed, we define
\[ \sigma_a(z) = (1 - |a|) \left( (1 - \overline{a}z)^{-\alpha} - 1 \right), \quad (z \in \mathbb{D}). \]

Clearly, for each \( a \in \mathbb{D} \), the function \( \sigma_a \) has bounded derivative and for this reason we have that \( \sigma_a \in B^\alpha \). In fact, it is easy to see that
\[ \sup_{a \in \mathbb{D}} \| \sigma_a \|_{B^\alpha} \leq \alpha 2^\alpha. \]
Furthermore, it is clear that if \( \frac{1}{2} < |a| < 1 \), then
\[
(2) \quad \left| \sigma_a^\alpha(a) \right| \geq \frac{\alpha}{4(1 - |a|^2)}.
\]
Also, we can see that \( \sigma_a \) goes to zero uniformly on compact subsets of \( D \) as \( |a| \to 1^- \). Also, we will need the following lemma which is well known and is consequence of a more general result due to Tjani in [13]:

**Lemma 2.1.** The composition operator \( C_\phi : \mathcal{B}^\alpha \to \mathcal{B}^\mu \) is compact if and only if given a bounded sequence \( \{f_n\} \) in \( \mathcal{B}^\alpha \) such that \( f_n \to 0 \) uniformly on compact subsets of \( D \), then \( \|C_\phi(f_n)\|_{\mathcal{B}^\mu} \to 0 \) as \( n \to \infty \).

Now we can show Theorem 1.2.

**Proof of Theorem 1.2.**

We set
\[
L = \limsup_{|a| \to 1^-} \|\sigma_a \circ \phi\|_{\mathcal{B}^\mu}.
\]
Let \( K : \mathcal{B}^\alpha \to \mathcal{B}^\mu \) be any compact operator, \( a \in D \) fixed and define
\[
f_a(z) = \frac{1}{\alpha^2} \sigma_a(z), \quad (z \in D).
\]
Then \( f_a \) goes to zero uniformly on compact subsets of \( D \) as \( |a| \to 1^- \), \( \|f_a\|_{\mathcal{B}^\alpha} \leq 1 \) for all \( a \in D \) and
\[
\|C_\phi - K\|_{\mathcal{B}^\alpha \to \mathcal{B}^\mu} \geq \|(C_\phi - K) f_a\|_{\mathcal{B}^\mu} \geq \frac{1}{\alpha^2} \|\sigma_a \circ \phi\|_{\mathcal{B}^\mu} - \|K f_a\|_{\mathcal{B}^\mu}.
\]
Hence, taking \( \limsup_{|a| \to 1^-} \) and using Lemma 2.1, we obtain
\[
(3) \quad \|C_\phi\|_{\mathcal{B}^\alpha \to \mathcal{B}^\mu} \geq \frac{1}{\alpha^2} \limsup_{|a| \to 1^-} \|\sigma_a \circ \phi\|_{\mathcal{B}^\mu}.
\]

Now, we go to show that there exists a constant \( M_\alpha > 0 \), depending only on \( \alpha \), such that
\[
\|C_\phi\|_{\mathcal{B}^\alpha \to \mathcal{B}^\mu} \leq M_\alpha \limsup_{|a| \to 1^-} \|\sigma_a \circ \phi\|_{\mathcal{B}^\mu}.
\]
Bearing this in mind, we define, for \( r \in [0, 1] \), the linear dilation operator \( K_r : H(D) \to H(D) \) by \( K_r f = f_r \), where \( f_r \), for each \( f \in H(D) \), is given by \( f_r(z) = f(rz) \). It is clear that if \( f \in H(D) \) then \( r f_r \to f \) uniformly on compact subsets of \( D \) as \( r \to 1^- \). Also, the following statements hold:

1. For \( r \in [0, 1] \), the operator \( K_r \) is compact on \( \mathcal{B}^\alpha \),
2. for each \( r \in [0, 1] \)
\[
\|K_r\|_{\mathcal{B}^\alpha \to \mathcal{B}^\alpha} \leq 1.
\]
Hence, if we consider a sequence \( \{r_n\} \subset (0, 1) \) such that \( r_n \to 1 \) as \( n \to \infty \) and define \( K_n = K_{r_n} \), then for all \( n \in \mathbb{N} \), the operator \( C_\phi K_n \) is a compact from \( \mathcal{B}^\alpha \) into \( \mathcal{B}^\mu \) and by definition of the essential norm we have
\[
\|C_\phi\|_{\mathcal{B}^\alpha \to \mathcal{B}^\mu} \leq \limsup_{n \to \infty} \|C_\phi - C_\phi K_n\|_{\mathcal{B}^\alpha \to \mathcal{B}^\mu}.
\]
Thus, we have to show that
\[ \limsup_{n \to \infty} \| C_\phi - C_{\phi^2}K_n \|_{B^\alpha \to B^\mu} \leq M_\alpha L. \]
To see this, consider any \( f \in B^\alpha \) such that \( \| f \|_{B^\alpha} = 1 \), then since
\[ \| (C_\phi - C_{\phi^2}K_n) f \|_{B^\mu} = \| f(\phi(0)) - f(r_n\phi(0)) \| + \| (f - f_{r_n}) \circ \phi \| \mu \]
and \( |f(\phi(0)) - f(r_n\phi(0))| \to 0 \) as \( n \to \infty \), it is enough to show that
\[ \limsup_{n \to \infty} \| (f - f_{r_n}) \circ \phi \| \mu \leq M_\alpha L. \]
Furthermore, since \( r_n(f')r_n \to f' \) uniformly on compact subsets of \( D \) as \( n \to \infty \), we have
\[ \limsup_{n \to \infty} \sup_{|\phi(z)| \leq r_N} \mu(z) \left| (f - f_{r_n})'(\phi(z)) \right| |\phi'(z)| = 0, \]
where \( N \in \mathbb{N} \) is large enough such that \( r_n \geq \frac{1}{2} \) for all \( n \geq N \). Hence we only have to show that
\[ S := \limsup_{n \to \infty} \sup_{|\phi(z)| > r_N} \mu(z) \left| (f - f_{r_n})'(\phi(z)) \right| |\phi'(z)| \leq M_\alpha L. \]
Indeed, we write \( S \leq \limsup_{n \to \infty} (S_1 + S_2) \), where
\[ S_1 = \sup_{|\phi(z)| > r_N} \mu(z) \left| f'(\phi(z)) \right| |\phi'(z)| \text{ and } S_2 = \sup_{|\phi(z)| > r_N} \mu(z)r_n \left| f'(r_n\phi(z)) \right| |\phi'(z)|. \]
Then we have
\[
S_1 = \sup_{|\phi(z)| > r_N} \mu(z) \left| f'(\phi(z)) \right| |\phi'(z)| \frac{v_\alpha (\phi(z))}{v_\alpha (\phi(z))} \frac{\sigma_\phi' (\phi(z))}{\sigma_\phi' (\phi(z))} \\
\leq \frac{4}{\alpha} \| f \|_{B^\alpha} \sup_{|\phi(z)| > r_N} \mu(z) \left| \sigma_\phi' (\phi(z)) \right| |\phi'(z)| \\
\leq \frac{4}{\alpha} \sup_{|\phi(z)| > r_N} \mu(z) |\sigma_\phi' (\phi(z))| |\phi'(z)| \leq \frac{4}{\alpha} \sup_{|\alpha| > r_N} \| \sigma_\phi \circ \phi \|_{B^\mu},
\]
where we have used the relation (2) in the first inequality and the fact that \( \| f \|_{B^\alpha} \leq 1 \) in the second one. Taking limit as \( N \to \infty \) we obtain
\[ \limsup_{n \to \infty} S_1 \leq \frac{4}{\alpha} L. \]
In similar way, we have
\[
S_2 \leq \frac{4}{\alpha} \| f \|_{B^\alpha} \sup_{|\phi(z)| > r_N} \mu(z) \left| \sigma_\phi' (\phi(z)) \right| |\phi'(z)| \frac{r_n v_\alpha (\phi(z))}{v_\alpha (r_n\phi(z))} \\
\leq \frac{4}{\alpha} \sup_{|\alpha| > r_N} \| \sigma_\phi \circ \phi \|_{B^\mu},
\]
since \( rv_\alpha(z) < v_\alpha(rz) \) for all \( r \in (0, 1) \) and all \( z \in \mathbb{D} \). Therefore

\[
\|C_\phi\|_{B_\alpha \to B_\mu} \leq \frac{8}{\alpha} \limsup_{|a| \to 1^-} \|\sigma_a \circ \phi\|_{B_\mu}.
\]

This completes the proof of the theorem. \(\blacksquare\)

As an immediate consequence of Theorem 1.2, we have the following corollary which generalize a result obtained recently by Malavé and Ramos-Fernández in [9] and extend a result due to Tjani in [14]. A similar result was found by Giménez, Malavé and Ramos-Fernández in [5], but for the composition operator \( C_\phi : B \to B_\mu \), where the weight \( \mu \) can be extended to non vanishing, complex valued holomorphic function that satisfy a reasonable geometric condition on the Euclidean disk \( D(1, 1) \).

**Corollary 2.2.** The composition operator \( C_\phi \) is compact from \( B_\alpha \) into \( B_\mu \) if and only if \( \phi \in B_\mu \) and

\[
\lim_{|a| \to 1^-} \|\sigma_a \circ \phi\|_\mu = 0.
\]

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