On topologizable and non-topologizable groups

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Abstract

A group $G$ is called hereditarily non-topologizable if, for every $H \leq G$, no quotient of $H$ admits a non-discrete Hausdorff topology. We construct first examples of infinite hereditarily non-topologizable groups. This allows us to prove that $c$-compactness does not imply compactness for topological groups. We also answer several other open questions about $c$-compact groups asked by Dikranjan and Uspenskij. On the other hand, we suggest a method of constructing topologizable groups based on generic properties in the space of marked $k$-generated groups. As an application, we show that there exist no non-discrete quasi-cyclic groups of finite exponent; this answers a question of Morris and Obraztsov.

1 Introduction

Throughout this paper, we always assume topological groups and spaces to be Hausdorff. A well-known theorem of Kuratowski and Mrowka states that a topological space $X$ is compact if and only if, for any topological space $Y$, the projection $\pi_Y : X \times Y \to Y$ is closed. Motivated by this theorem, Dikranjan and Uspenskij [7] call a topological group $X$ categorically compact (or $c$-compact for brevity) if, for every topological group $Y$, the image of every closed subgroup of $X \times Y$ under the projection $\pi_Y : X \times Y \to Y$ is closed in $Y$.

Obviously every compact group is $c$-compact, while the converse was open until now even for discrete groups. More precisely, the following questions were asked in [7, Question 1.2 and Question 5.2] (see also [28, Problem 31 (i) and Question 34]).

Problem 1.1.

(a) Is every $c$-compact group compact?

(b) Is every discrete $c$-compact group finite (finitely generated, of finite exponent, countable)?

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These questions have received considerable attention in the recent years. A complete survey of recent results can be found in the book [14], which is devoted to these problems. Until now, only results in the affirmative direction were known. For example, the answer to (a) is known to be positive for solvable groups, connected locally compact groups [7], and maximally almost periodic groups [16]. Note also that every discrete c-compact group is necessarily a torsion group by [7, Theorem 5.3].

Our first goal is to show that the answer to all parts of Problem 1.1 is, in fact, negative. Our approach is based on a sufficient condition for c-compactness suggested in [7], which leads to the notion of a hereditarily non-topologizable group introduced by Lukács [15].

Recall that an abstract group is called topologizable if it admits a non-discrete Hausdorff group topology, and non-topologizable otherwise. In 1946, A. A. Markov [18] asked whether there exist non-topologizable infinite groups and the problem remained open until late 70’s. In [29], Shelah constructed first (uncountable) examples using the Continuum Hypothesis. Later Hesse [11] showed that the use of the CH in Shelah’s proof can be avoided. The affirmative answer to the Markov’s question for countable groups was obtained by the second author in [23] (see also [24, Theorem 31.5]); the proof uses the group constructed by Adjan in [1] and is essentially elementary modulo the main theorem of [1]. Since then many other examples of non-topologizable groups have been found (see, for example, [13]).

A group \( G \) is called hereditarily non-topologizable if for every \( H \leq G \) and every \( N \triangleleft H \), the quotient group \( H/N \) is non-topologizable. It is easy to prove that every hereditarily non-topologizable group is c-compact with respect to the discrete topology (see [7, Corollary 5.4]); moreover, a countable group is hereditarily non-topologizable if and only if it is c-compact with respect to the discrete topology [7, Theorem 5.5].

Using techniques developed in [21] we prove the following result (see Theorem 2.5), which completely solves Problem 1.1.

**Theorem 1.2.** There exist hereditarily non-topologizable (and hence c-compact with respect to the discrete topology) groups \( G, H, I, \) and \( J \) such that:

(a) \( G \) is infinite, finitely generated, and of bounded exponent;

(b) \( H \) is finitely generated and of unbounded exponent;

(c) \( I \) is countable, but not finitely generated;

(d) \( J \) is uncountable.

On the other hand, it is worth noting that neither of the groups constructed in [13, 23, 29] is c-compact (see Remark 2.6).

The finitely generated groups \( G \) and \( H \) from Theorem 1.2 are the so-called Tarski Monsters, i.e., infinite simple groups with all proper subgroups finite cyclic. First examples of such groups were constructed by the second author in [24]. Clearly every non-topologizable Tarski Monster is hereditarily non-topologizable. This raises the natural question of whether a Tarski Monster can be topologized. The standard way of defining a non-discrete topology using chains of subgroups obviously fails for Tarski Monsters. Moreover, most groups which are known to
be topologizable, such as infinite residually finite groups, infinite locally finite groups [2], or groups containing infinite normal solvable subgroups [11], are located on the opposite side of the group-theoretic universe.

The first (and the only known) examples of topologizable Tarski Monsters were constructed by Morris and Obraztsov in [19] using methods from [21]. An essential feature of the Morris–Obraztsov construction is that their groups have unbounded exponent and for finite exponent their method of defining a non-discrete topology seems to fail. This motivated the following.

Question 1.3. [19, Question 3] Does there exist a topologizable quasi-finite group of finite exponent?

Recall that a group is quasi-finite if all its proper subgroups are finite and is of finite exponent $n$ if $g^n = 1$ for some positive integer $n$ and every $g \in G$. In this paper we answer the Morris–Obraztsov’s question affirmatively.

Theorem 1.4. For every sufficiently large odd $n \in \mathbb{N}$ there exists a topologizable Tarski Monster of exponent $n$.

Our proof of Theorem 1.4 utilizes the notion of a generic property in a topological space. Recall that a subset $S$ of a topological space $X$ is called a $G_\delta$ set if $S$ is an intersection of a countable collection of open sets. Further one says that a generic element of $X$ has a certain property $P$ (or $P$ is generic in $X$) if $P$ holds for every $x$ from some dense $G_\delta$ subset of $X$. The Baire Category Theorem implies that in a complete metric space the intersection of any countable collection of dense $G_\delta$ sets is again dense $G_\delta$. Thus we can combine generic properties: if every property from a countable collection $\{P_1, P_2, \ldots\}$ is generic, then so is the whole collection (i.e., the conjunction of $P_1, P_2, \ldots$). In many situations this approach is useful for proving the existence of elements of $X$ simultaneously satisfying $P_1, P_2, \ldots$.

To implement this idea we consider the space of marked $k$-generated groups, $\mathcal{G}_k$, which is a compact totally disconnected metric space consisting of all $k$-generated groups with fixed generating sets. For the precise definition we refer to Section 3. The study of generic properties in subspaces of $\mathcal{G}_k$ was initiated by Champetier in [3]. The following observation is crucial for our proof of Theorem 1.4.

Proposition 1.5. For every $k \in \mathbb{N}$, the following subsets of $\mathcal{G}_k$ are $G_\delta$:

(a) the set of all topologizable groups;

(b) the set of all Tarski Monsters of any fixed finite exponent.

Using methods from the book [21], for every sufficiently large odd $n \in \mathbb{N}$ we construct a compact nonempty subset $\mathcal{T} \subseteq \mathcal{G}_2$ consisting of groups of exponent $n$ such that $\mathcal{T}$ contains a dense subset of topologizable groups and a dense subset of Tarski Monsters. Then by Proposition 1.5 the properties of being topologizable and being a Tarski Monster are generic in $\mathcal{T}$. Hence topologizable Tarski Monsters (of exponent $n$) are generic in $\mathcal{T}$. In particular, they exist.

All Tarski Monsters discussed above, as well as many other groups with “exotic” properties, are limits of hyperbolic groups. It is not difficult to see that every infinite hyperbolic
group is topologizable, but a much weaker condition also makes a group topologizable; namely, in the last section we observe that being topologizable is a generic property among limits of “hyperbolic-like” groups. More precisely, we consider the class of acylindrically hyperbolic groups introduced in [25]. This class contains all non-elementary hyperbolic groups, non-elementary relatively hyperbolic groups with proper peripheral subgroups (e.g., all non-trivial free products other than $\mathbb{Z}_2 \ast \mathbb{Z}_2$), mapping class groups of surfaces of genus $> 1$, $Out(F_n)$ for $n \geq 2$, and many other interesting examples. For the definition and more details we refer to [25] and references therein.

Given a subset $S \subseteq G_k$, we denote by $\overline{S}$ its closure in $G_k$.

**Theorem 1.6.** Let $S \subseteq G_k$ be a subset consisting of (marked $k$-generated) acylindrically hyperbolic groups. Then being topologizable is a generic property in the set $S$.

The proof of Theorem 1.6 is accomplished by proving that every acylindrically hyperbolic group is topologizable (see Lemma 5.1). Then Proposition 1.5 yields the claim. Finally we sketch a possible application of Theorem 1.6 to constructing non-discrete groups with all nontrivial elements conjugate (for details and motivation see Section 5).

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## 2 Hereditarily non-topologizable groups

Recall that a subset $V$ of a group $G$ is called *elementary algebraic* if there exist $a_1, \ldots, a_k \in G$ and $\varepsilon_1, \ldots, \varepsilon_k \in \mathbb{Z}$ such that $V$ is the set of all solutions of the equation $a_0 x^{\varepsilon_1} a_1 x^{\varepsilon_2} \cdots a_k x^{\varepsilon_k} = 1$ in $G$.

**Lemma 2.1** (A.A. Markov [18]). A countable group $G$ is non-topologizable if and only if $G \setminus \{1\}$ is a finite union of elementary algebraic sets.

We say that a group $G$ is given by a presentation over a free product $G_0 \ast G_1 \ast \ldots$ if $G$ is presented in the form

$$G = (G_0 \ast G_1 \ast \ldots) / \langle \langle R_1, R_2, \ldots \rangle \rangle,$$

where $\langle \langle R_1, R_2, \ldots \rangle \rangle$ denotes the minimal normal subgroup of the free product $G_0 \ast G_1 \ast \ldots$ containing $R_1, R_2, \ldots$. We are interested in presentations satisfying *condition R* introduced in [21]. The exact definition of this condition is rather technical and will not be used in our paper. For our purpose it suffices to know that R is a condition on the additional relators $R_1, R_2, \ldots$, which allows one to apply the machinery from the book [21]. One of these applications is the following theorem.

**Obraztsov’s Theorem.** [20] (see also [21] Theorem 35.1). There exists $N \in \mathbb{N}$ such that for any odd $n_0 \geq N$ and any countable (finite or infinite) family of nontrivial countable groups $G_0, G_1, \ldots$ without elements of order 2, there is an infinite simple group $O(G_0, G_1, \ldots)$ such that the following conditions hold.
(a) \(O(G_0, G_1, \ldots)\) contains all \(G_i\) as distinct maximal subgroups.

(b) Any two distinct maximal subgroups of \(O(G_0, G_1, \ldots)\) intersect trivially.

(c) Any proper subgroup of \(O(G_0, G_1, \ldots)\) is either cyclic of order dividing \(n_0\) or conjugate to a subgroup of some \(G_i\).

Moreover, if our collection \(G_0, G_1, \ldots\) contains at least two groups, we also have the following.

(d) \(O(G_0, G_1, \ldots)\) is generated by any pair of non-trivial elements \(x, y\) satisfying \(x \in G_i\) and \(y \notin G_i\) for some \(i\).

(e) \(O(G_0, G_1, \ldots)\) has a presentation satisfying condition \(R\) over the free product \(G_0 \ast G_1 \ast \ldots\).

The proof of the next lemma essentially uses the machinery developed in the book [21].

Even brief definitions of all notions used below (condition \(R\), reduced diagram, numerical parameters, etc.) would take many pages, so we choose not to explain them here and simply refer to [21].

**Lemma 2.2.** If a group \(U\) has a presentation with condition \(R\) over a free product \(G_0 \ast G_1 \ast \ldots\), where \(G_i\) are countable nontrivial groups without elements of order 2, then for any elements \(g_0 \in G_0 \setminus \{1\}\) and \(u \in U \setminus G_0\), the element \(g_0 u^a\) is not conjugate to any element of any group \(G_i\).

Henceforth, \(g^h\) means \(h^{-1}gh\) if \(g\) and \(h\) are elements of a group.

**Proof.** Suppose that \(g_0 u^a\) is conjugate to an element \(h \in G_i\). Let \(X \in G_0 \ast G_1 \ast \ldots\) be a word (in the alphabet \(G_0 \cup G_1 \cup \ldots\)) representing \(u\).

It follows that there is a diagram of conjugacy \(\Sigma\) of the word \(g_0 X^{-1} g_0 X\) and the letter \(h\). That is, \(\Delta\) is an annular van Kampen diagram over the presentation of \(U\) with labels of the boundary components equal \(g_0 X^{-1} g_0 X\) and \(h\) (read in the appropriate direction). One can identify the subpaths of the boundary of \(\Sigma\) labeled by \(X\) and \(X^{-1}\) and obtain a diagram \(\Delta_0\) on a sphere with 3 holes. Its boundary components \(p_1, p_2\) and \(q\) are labeled (e.g., in clockwise manner) by the letters \(g_0, g_0,\) and \(h\), respectively. A simple path \(x\) labeled by \(X\) connects the origins of the paths \(p_1\) and \(p_2\).

If \(\Delta_0\) is not a reduced diagram, then one can make reductions described in Section 13 of [21] and obtain a reduced diagram \(\Delta\) with the same boundary labels. Moreover, there is a simple path in \(\Delta\) connecting the origins of \(p_1\) and \(p_2\), whose label is equal to \(X\) in \(U\).

We obtain a reduced diagram \(\Delta\) on a sphere with 3 holes, but the length of every its boundary component is equal to 1 (in the metric of diagrams over free products). It follows that the rank of \(\Delta\) is 0. Indeed, Lemmas 33.3 – 34.2 [21] extend the theory of presentations with the condition \(R\) to the presentations over free products; and so if \(r(\Delta) > 0\), then the length of one of the boundary components of \(\Delta\) must be \(> \epsilon n > 1\) by Theorem 22.2. Here \(\epsilon\) and \(n\) are some parameters from [21] satisfying \(n \gg \frac{1}{\epsilon} \gg 1\); their exact values are not essential for us. (For more details about parameters see Section 4.)
Thus the word $X$ is equal in $U$ to a word $X'$ such that the element $g_0g_0^{X'}$ is conjugate to an element of $G_i$ in the free product $G_0 * G_1 * \ldots$. But this can happen only if $X'$ represents an element of $G_0$. Hence $u \in G_0$. This contradiction completes the proof. \hfill \Box

**Theorem 2.3.** For any sufficiently large integer $n_0$ and any countable nonempty family of nontrivial countable groups $G_1, G_2, \ldots$ without elements of order 2, there exists an infinite simple group $O'(G_1, G_2, \ldots)$ such that the following conditions hold:

(a) $O'(G_1, G_2, \ldots)$ contains all $G_i$ as distinct maximal subgroups.

(b) Any two distinct maximal subgroups of $O'(G_1, G_2, \ldots)$ intersect trivially.

(c) Any proper subgroup of $O'(G_1, G_2, \ldots)$ is either cyclic of order at most $n_0$ or conjugate to a subgroup of some $G_i$.

(d) Any pair of non-trivial elements $x, y$ satisfying $x \in G_i$ and $y \notin G_i$ for some $i$ generates $O'(G_1, G_2, \ldots)$.

(e) There exists an equation with one unknown having precisely one non-solution in $O'(G_1, G_2, \ldots)$. In particular, $O'(G_1, G_2, \ldots)$ is non-topologizable.

**Proof.** Let us take a sufficiently large odd number $n_0$ as in Obraztsov’s Theorem and a finite cyclic group $G_0 = \langle g \rangle$ of odd order coprime to $n_0$ (of order three, for instance). Let $O'(G_1, G_2, \ldots) = O(G_0, G_1, G_2, \ldots)$.

Parts (a)-(d) of the theorem follow immediately from the corresponding parts of Obraztsov’s theorem. Thus we only need to prove (e). To construct an equation with precisely one non-solution in $O'(G_1, G_2, \ldots)$, we take any element $v \in O'(G_1, G_2, \ldots) \setminus G_0$ and consider the equation

$$[(g^x)^{n_0}, (g^y)^{v^x})^n_0] = 1. \quad (1)$$

Let us first show that the identity element is not a solution to this equation. Indeed assume that it is. Note that the equality $[g^{2n_0}, g^{2n_0}] = 1$ implies $[g, g^v] = 1$, because the order of $g$ is coprime to $2n_0$. Thus the group $\langle g, g^v \rangle$ is abelian. In particular, $\langle g, g^v \rangle$ is a proper subgroup of $O'(G_1, G_2, \ldots)$. Since $G_0 = \langle g \rangle$ is abelian in $O'(G_1, G_2, \ldots)$ by Obraztsov’s theorem, we have $\langle g, g^v \rangle = \langle g \rangle$. This in turn implies that $\langle g, v \rangle$ is metabelian. Again $\langle g, v \rangle$ is a proper subgroup of $O'(G_1, G_2, \ldots)$ (for example, because the latter group is simple) and hence $\langle g, v \rangle = G_0$ by maximality of $G_0$. However this contradicts our choice of $v$.

We note that if an element $x$ does not belong to $G_0$, then by Lemma 2.2 and part (e) of Obraztsov’s theorem the element $g^x$ is not conjugate to an element of any $G_i$. Therefore, $g^x$ generates a cyclic subgroup of order dividing $n_0$ by Obraztsov’s theorem. Hence for every $x \notin G_0$, the first argument of the commutator in (1) is 1. For the same reason, the second argument of the commutator is 1 whenever $x \notin (G_0)^v$. Observe that $G_0 \cap (G_0)^v = \{1\}$, since both subgroups are maximal and distinct by Obraztsov’s theorem. Thus all nonidentity elements of $O'(G_1, G_2, \ldots)$ are solutions to (1). In particular, $O'(G_1, G_2, \ldots)$ is non-topologizable by Lemma 2.1. \hfill \Box
Remark 2.4. It is known that

- any group embeds into a non-topologizable group [30];
- there exists a (non-topologizable) torsion-free group of any infinite cardinality such that some equation has exactly one non-solution in this group [13];
- there exists an infinite (non-topologizable) group naturally isomorphic to its automorphism group such that some equation has exactly one non-solution in this group [31].

Theorem 2.5. There exist infinite hereditarily non-topologizable simple torsion groups \( G, H, I, \) and \( J \) such that:

(a) \( G \) is 2-generated, quasi-cyclic, and of bounded exponent;
(b) \( H \) is 2-generated, quasi-cyclic, and of unbounded exponent;
(c) \( I \) is not finitely generated, countable, and of bounded exponent;
(d) \( J \) is uncountable and of bounded exponent.

Proof. Let \( n_0 \) be a sufficiently large odd integer as in Theorem 2.3. Let

\[
G = O'(\mathbb{Z}_{n_0}), \quad H = O'(\mathbb{Z}_{n_0}, \mathbb{Z}_{n_0+2}, \mathbb{Z}_{n_0+4}, \ldots), \quad I = \bigcup_{i=0}^{\infty} G^i,
\]

where \( G^0 = G_0 = \langle g \rangle \) is a finite cyclic group of odd order coprime to \( n_0 \), and \( G^{i+1} = O(G^i, \mathbb{Z}_{n_0}) \) for \( i > 0 \).

The uncountable group \( J \) (of the first uncountable cardinality) is constructed similarly to \( I \) but using the transfinite induction (up to the first uncountable ordinal).

The groups \( G \) and \( H \) are quasi-cyclic and simple by Theorem 2.3. The groups \( I \) and \( J \) are simple, because they are unions of increasing chains of simple subgroups.

All four groups are non-topologizable. For \( G \) and \( H \), this follows directly from Theorem 2.3. The groups \( I \) and \( J \) are non-topologizable, since the set of non-solutions of the equation \((gg^x)^{n_0} = 1\) is nonempty and finite (it is contained in \( G_0 \)) by Lemma 2.2.

Let us show that the groups \( I \) and \( J \) contain no proper infinite subgroups except for subgroups conjugate to \( G^i \). Indeed let \( P \leq I \) be a proper infinite subgroup. If \( P \leq G^k \) for some \( k \), part (c) of Theorem 2.3 implies (by induction) that \( P \) is conjugate to \( G^i \) for some \( i \leq k \). Thus it suffices to rule out the case when for every \( k \in \mathbb{N} \), \( P \) contains an element that does not belong to \( G^k \). Fix any \( k_0 \in \mathbb{N} \). There exists \( k_1 \geq k_0 \) such that \( P \) contains a non-trivial element \( x \in G^{k_1} \). By our assumption there also exists \( k_2 > k_1 \) such that \( P \) contains an element \( y \in G^{k_2} \setminus G^{k_2-1} \). Now part (d) of Theorem 2.3 implies that the subgroup \( \langle x, y \rangle \) coincides with \( G^{k_2} \). Since \( G^{k_0} \leq G^{k_1} \leq G^{k_2} \), we obtain that \( G^{k_0} \leq P \). As this holds true for any \( k_0 \in \mathbb{N} \), we have \( P = I \), which contradicts properness of \( P \). This completes the proof for \( I \). For the group \( J \), the proof is analogous but one has to use transfinite induction instead of the standard one.

Since \( G^i \) is simple and non-topologizable for every \( i \), it follows that \( I \) and \( J \) are hereditary non-topologizable.
Remark 2.6. Note that neither of the groups constructed in [13, 23, 29] is c-compact. Indeed it is immediate from the definition that c-compactness is preserved by taking closed subgroups (i.e., any subgroups in the discrete case). Recall that a discrete countable group is c-compact if and only if it is hereditarily non-topologizable. Since (discrete) groups from [29] and [13] contain infinite cyclic subgroups, they are not c-compact.

The countable non-topologizable group constructed in [23] is also not hereditarily non-topologizable, since it has the free Burnside group $B(m, n)$ with $m \geq 2$ generators and of large odd exponent $n$ as a quotient. The latter group admits a non-discrete topology defined by a nested chain of normal subgroup. This can be extracted from [21, Theorem 39.3]; for $m = 2$ this also follows from Corollary 4.8 applied to $J = \emptyset$. Alternatively one can argue as follows. By [21, Theorem 39.1] the group $B(m, n)$ contains $B(\infty, n)$.

3 The space of marked groups and $G_\delta$ sets

Let $F_k$ be the free group of rank $k$ with basis $X = \{x_1, \ldots, x_k\}$ and let $G_k$ denote the set of all normal subgroups of $F_k$. Given $M, N \triangleleft F_k$, let

$$d(M, N) = \begin{cases} \max \left\{ \frac{1}{|w|} \mid w \in N \triangle M \right\}, & \text{if } M \neq N \\ 0, & \text{if } M = N, \end{cases}$$

where $| \cdot |$ denotes the word length with respect to the generating set $X$. It is easy to see that $(G_k, d)$ is a compact Hausdorff totally disconnected (ultra)metric space [10].

Note that one can naturally identify $G_k$ with the set of all marked $k$-generated groups, i.e., pairs $(G, (x_1, \ldots, x_k))$, where $G$ is a group and $(x_1, \ldots, x_k)$ is a generating $k$-tuple of $G$. (By abuse of notation, we keep the same notation for the generators $x_1, \ldots, x_k$ of $F_k$ and their images in $G$.) For this reason the space $G_k$ with the metric defined above is called the space of marked groups with $k$ generators. For brevity, we simply call elements of $G_k$ groups instead of marked $k$-generated groups.

Let $L_k$ be the first order language that contains the standard group operations $\cdot, -1$, the constant symbol 1, and constant symbols $x_1, \ldots, x_k$. Every element $(G, (x_1, \ldots, x_k)) \in G_k$ can be naturally thought of as an $L_k$-structure.

The following lemma is obvious.

Lemma 3.1. Let $w$ be a word in the alphabet $X \cup X^{-1}$. Then for every $k \in \mathbb{N}$, the set of groups in $G_k$ satisfying $w = 1$ (or $w \neq 1$) is clopen.

Proof. If $w = 1$ (or $w \neq 1$) in a group $(G, (x_1, \ldots, x_k)) \in G_k$ and $w$ has length $r$, then $w = 1$ (respectively, $w \neq 1$) in every other group $(H, (x_1, \ldots, x_k)) \in G_k$ such that $d(G, H) < 1/r$. Thus the set of groups satisfying $w = 1$ (respectively, $w \neq 1$) is open and the claim of the lemma follows. \qed

Recall that a sentence in a first order language is called an $\forall \exists$-sentence if it has the form

$$\forall a_1 \ldots \forall a_m \exists b_1 \ldots \exists b_n \Phi(a_1, \ldots, a_m, b_1, \ldots, b_n),$$

(2)
where $\Phi(a_1, \ldots, a_m, b_1, \ldots, b_n)$ is a quantifier-free formula. If such a sentence only contains existential (respectively, universal) quantifiers, it is called existential (respectively, universal). We say that a subset $S \subseteq G_k$ is \(\forall \exists\)-definable if there exists an \(\forall \exists\)-sentence $\Sigma$ in $L_k$ such that
\[
S = \{ P \in G_k \mid P \models \Sigma \},
\]
i.e., $S$ is exactly the set of all elements of $G_k$ satisfying $\Sigma$. Similarly we define existentially definable and universally definable subsets.

Observe that if $(G, (x_1, \ldots, x_k)) \in G_k$, then we know that $x_1, \ldots, x_k$ generate $G$. This allows us to use the following quantifier elimination procedure. Let $R(u)$ be a (not necessarily first order) property of marked $k$-generated groups which depends on some parameter $u$ interpreted as a group element. Enumerate all words \(\{w_1, w_2, \ldots\}\) in the alphabet $X \cup X^{-1}$. Then we obviously have
\[
\{ P \in G_k \mid P \models \forall u R(u) \} = \bigcap_{i=1}^{\infty} \{ P \in G_k \mid P \models R(w_i) \} \quad (3)
\]
and
\[
\{ P \in G_k \mid P \models \exists u R(u) \} = \bigcup_{i=1}^{\infty} \{ P \in G_k \mid P \models R(w_i) \}. \quad (4)
\]

The first part of the following lemma is well-known although we were unable to find an exact reference.

**Proposition 3.2.**

(a) [Folklore] Every existentially defined subset of $G_k$ is open.

(b) Every \(\forall \exists\)-definable subset of $G_k$ is a $G_\delta$ set.

**Proof.** Every existential sentence is equivalent to a sentence
\[
\exists b_1 \cdots \exists b_n \ \Phi_1(b_1, \ldots, b_n) \lor \cdots \lor \Phi_q(b_1, \ldots, b_n), \quad (5)
\]
such that each $\Phi_i$ is a system of equations and inequations of the form $w = 1$ (respectively, $w \neq 1$), where $w$ is a word in the alphabet \(\{x_1^{\pm 1}, \ldots, x_k^{\pm 1}\} \cup \{b_1^{\pm 1}, \ldots, b_n^{\pm 1}\}\). Thus the first claim follows from Lemma 3.1 and the quantifier elimination \(\ref{quantifier_elimination}\) applied to all quantifiers in \((5)\). To prove (b) we have to eliminate all universal quantifiers in \((2)\) according to \((3)\) and apply (a). \(\square\)

It would be interesting to find other sufficient conditions in the spirit of \[17\] and \[5\] for a (not necessarily first order) sentence to define a $G_\delta$ subset of $G_k$. In particular, we ask the following.

**Question 3.3.** Which second order sentences define $G_\delta$ subsets of $G_k$?

The next proposition provides some particular non-trivial examples of $G_\delta$ subsets of $G_k$, which are relevant to our paper. Recall that by a *Tarski Monster* we mean a finitely generated infinite simple group with all proper subgroups finite cyclic.
Proposition 3.4. For every $k \in \mathbb{N}$, the following subsets of $G_k$ are $G_\delta$:

(a) The set of all topologizable groups.
(b) The set of all infinite groups.
(c) The set of all groups satisfying a given identity.
(d) The set of all simple groups.
(e) The set of Tarski Monsters of any fixed finite exponent.
(f) The set of groups with all non-trivial elements conjugate.

Proof. Let $E = \{E_1, E_2, \ldots\}$ denote the set of all finite collections of equations over the free group $F_k$ with one unknown. Given an element $(G, (x_1, \ldots, x_k)) \in G_k$ (i.e., an epimorphism $\varepsilon : F_k \to G$), we can think of each $E_n$ as a collection of equations over $G$ by projecting all coefficients to $G$ via $\varepsilon$. Consider the following condition:

$C_n$: Some nontrivial element of $G$ satisfies neither of the equations from $E_n$ or $1$ satisfies at least one of the equations from $E_n$.

Clearly every $C_n$ can be expressed by an existential formula in $L_k$. Hence the set $C_n$ of elements of $G_k$ satisfying $C_n$ is open for every $n$ by Proposition 3.2. By Lemma 2.1 the set of all topologizable groups in $G_k$ coincides with $\bigcap_{n \in \mathbb{N}} C_n$ and hence it is a $G_\delta$ set by definition.

To prove (b) we first observe that the set of all groups of order $\leq m$ in $G_k$ is finite, hence the set $I_m$ of groups having more than $m$ elements is open. Consequently, the set of all infinite groups is a $G_\delta$ set being the intersection of all $I_m$.

Part (c) follows from part (b) of Proposition 3.2 and the obvious fact that the subset of $G_k$ consisting of groups satisfying a given identity can be defined by a universal sentence.

Let us prove (d). Fix some word $w$ in $X \cup X^{-1}$ and enumerate all words $\{u_1, u_2, \ldots\}$ in the normal closure of $w$ in $F_k$. Observe that the property

$D_w$: The normal subgroup of $G$ generated by $w$ is trivial or coincides with $G$

can be expressed by the (infinite) disjunction of formulas

$$x_1 = u_{i_1} \land \ldots \land x_k = u_{i_k}$$

for $\{i_1, \ldots, i_k\} \in \mathbb{N}^k$ and $w = 1$. Hence the set $D_w$ of elements of $G_k$ satisfying $D_w$ is the union of open subsets of $G_k$ by Lemma 3.1. Consequently, $D_w$ is open. It is easy to check that a group $(G, (x_1, \ldots, x_k)) \in G_k$ is simple if and only if it belongs to $\bigcap_{w \in F_k} D_w$. Thus we obtain (d).

The proof of (e) is similar. Fix some $n \in \mathbb{N}$. For two words $u, v$ in $X \cup X^{-1}$, consider the set $S_{u,v}$ of all $(G, (x_1, \ldots, x_k)) \in G_k$ such that the subgroup of $G$ generated by $\{u, v\}$ is contained in a cyclic subgroup of order dividing $n$. It is easy to see that $S_{u,v}$ can be defined by an existential formula in $L_k$. E.g., for $n = 2$ the following formula works:

$$\exists z \ (z^2 = 1 \land ((u = 1 \land v = 1) \lor (u = 1 \land v = z) \lor (u = z \land v = 1) \lor (u = z \land v = z))).$$

Thus \( S_{u,v} \) is open in \( G_k \).

Further let \( \{ w_1, w_2, \ldots \} \) be the set of all elements of the subgroup of \( F_k \) generated by \( u \) and \( v \). For any \((i_1, \ldots, i_k) \in \mathbb{N}^k\), denote by \( \mathcal{R}_{u,v,i_1,\ldots,i_k} \) the set of all elements of \( G_k \) satisfying

\[
x_1 = w_{i_1} \land \ldots \land x_k = w_{i_k}.
\]

By Lemma 3.1 every \( \mathcal{R}_{u,v,i_1,\ldots,i_k} \) is also open. Thus the set

\[
\mathcal{Q}_{u,v} = S_{u,v} \cup \left( \bigcup_{(i_1,\ldots,i_k) \in \mathbb{N}^k} \mathcal{R}_{u,v,i_1,\ldots,i_k} \right)
\]

is open and hence the set

\[
\mathcal{T}_0 = \bigcap_{u,v \in F_k} \mathcal{Q}_{u,v}
\]

is a \( G_\delta \) set. It is easy to see that \( \mathcal{T}_0 \) has the property:

\[
\textbf{T: For every } (G,(x_1,\ldots,x_k)) \in \mathcal{T}_0, \text{ every 2-generated subgroup of } G \text{ is either cyclic of order dividing } n \text{ or coincides with } G.
\]

Let

\[
\mathcal{T} = \mathcal{T}_0 \cap \mathcal{I} \cap \mathcal{S},
\]

where \( \mathcal{I} \) is the set of all infinite groups and \( \mathcal{S} \) is the set of all simple groups in \( G_k \). Then \( \mathcal{T} \) is a \( G_\delta \) subset of \( G_k \). We want to show that \( \mathcal{T} \) is exactly the subset of all (marked \( k \)-generated) Tarski Monsters satisfying the identity \( x^n = 1 \).

Indeed suppose \((G,(x_1,\ldots,x_k)) \in \mathcal{T}\). Let \( H \) be a proper subgroup of \( G \). According to \( \textbf{T} \) every 2-generated subgroup of \( H \) is cyclic of order dividing \( n \). This obviously implies that \( H \) itself is cyclic of order dividing \( n \). Note also that \( G \) is infinite, simple, and satisfies \( x^n = 1 \) by the definition of \( \mathcal{T} \). Conversely, it is easy to see that every Tarski Monster satisfying the identity \( x^n = 1 \) belongs to \( \mathcal{T} \).

Finally to prove (f) it suffices to note that the subset of \( G_k \) consisting of groups with 2 conjugacy classes can be defined by the \( \forall \exists \)-formula

\[
\forall x \forall y \exists t \left( x = 1 \lor y = 1 \lor t^{-1}xt = y \right).
\]

Now applying part (b) of Proposition 3.2 finishes the proof.

\[\square\]

### 4 Topologizable Tarski Monsters

Our proof of Theorem 1.4 makes use of a particular variant of the general construction described in [21, Sections 25-27]. The variant used here is similar to that from [21, Section 39.2]. Below we briefly recall it and refer the reader to [21] for details.

Given a group \( G \) generated by a set \( X \), we write “\( A \equiv B \)” for two words in the alphabet \( X \cup X^{-1} \) if they coincide as words (i.e., letter-by-letter) and “\( A = B \) in \( G \)” if \( A \) and \( B \)
represent the same elements of $G$; by abuse of notation we identify words in $X \cup X^{-1}$ and elements represented by them. As in [21], given a word $A$ in some alphabet, $|A|$ denotes its length.

The general construction in [21, Sections 25-27] uses a sequence of fixed positive small parameters

\[ \alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \iota. \]

The exact relations between the parameters are described by a system of inequalities, which can be made consistent by choosing each parameter in this sequence to be sufficiently small as compared to all previous parameters. In [21] and below, this way of ensuring consistency is referred to as the lowest parameter principle (see [21, Section 15.1]). Below we will use the following auxiliary parameters (which are assumed to be integers):

\[ h = \delta - 1, \quad d = \eta - 1, \quad n = \iota - 1. \]

We also fix a sufficiently large odd $n_0 \in \mathbb{N}$ satisfying

\[ n_0 > \max \left\{ (h + 1)n, \frac{h(d + n + 2h - 2)}{1 - \alpha} \right\}. \]  

(7)

**Remark 4.1.** Our notation in this section is borrowed from [21] and is different from the notation in the introduction: the exponent denoted by $n$ in Theorem 1.4 is denoted by $n_0$ here.

Given a subset $J \subseteq \mathbb{N}$, we construct groups $G(i, J)$ by induction on $i \in \mathbb{N}$ as follows. Let $G(0, J) = F(a_1, a_2)$ be the free group with basis \{a_1, a_2\}. Suppose now that $G(i - 1, J)$ is already constructed for some $i \geq 1$, and that for each $1 \leq j \leq i - 1$ we have already defined a set $X_j$ of words of length $j$ in \{a_\pm 1, b_\pm 1\} called *periods of rank $j*.

The set of periods of rank $i$, $X_i$, is defined to be a maximal set of words of length $i$ in the alphabet \{a_\pm 1, b_\pm 1\} such that no $A \in X_i$ is conjugate to a power of a word of length $< i$ in the group $G(i - 1, J)$, and if $A$ is conjugate to $B$ or $B^{-1}$ in $G(i - 1, J)$ for some $A, B \in X_i$ then $A \equiv B$.

The group $G(i, J)$ is obtained from $G(i - 1, J)$ by adding a set of relations $S_i$ constructed as follows. First for each period $A \in X_i$, $S_i$ contains the relation

\[ A^{n_0} = 1 \]  

(8)

called a *relation of the first type of rank $i*.

If $i \notin J$, no other relations are included in $S_i$. If $i \in J$, then for each $A \in X_i$ we fix some maximal set of words $Y_A$ such that:

(a) For any $T \in Y_A$, we have $1 \leq |T| \leq d|A|;

(b) Every double coset $\langle A \rangle g \langle A \rangle$ in $G(i - 1, J)$ contains at most one word from $Y_A$ and this word has minimal length among all words representing elements of $\langle A \rangle g \langle A \rangle$ in $G(i - 1, J)$.
If \( a_1 \notin \langle A \rangle \) in \( G(i-1) \), then for every \( T \in \mathcal{Y}_A \) such that \( T \notin \langle A \rangle a_1\langle A \rangle \) in \( G(i-1,J) \), we add the relation

\[
a_1 A^nT A^{n+2} \ldots T A^{n+2h-2} = 1
\]  
(9)

to the set \( S_i \). Further if \( a_2 \notin \langle A \rangle \cup \langle A \rangle a_1\langle A \rangle \) and \( T \notin \langle A \rangle a_2\langle A \rangle \) in \( G(i-1,J) \), then we also add the relation

\[
a_2 A^{n+1}T A^{n+3} \ldots T A^{n+2h-1} = 1
\]  
(10)

to \( S_i \). These relations are called \textit{relations of the second type of rank} \( i \).

Finally we define

\[
G(i,J) = \langle a_1, a_2 \mid R_{i-1} \cup S_i \rangle.
\]

Note that there is some freedom in choosing periods in every rank and sets \( \mathcal{Y}_A \). We additionally require our construction to be \textit{uniform} in the following sense: if \( I \cap [1,r] = J \cap [1,r] \) for some \( r \in \mathbb{N} \), then the sets of periods and the corresponding sets \( \mathcal{Y}_A \) in \( G(i,I) \) and \( G(i,J) \) coincide for all \( 1 \leq i \leq r \). In particular, \( G(i,I) \) and \( G(i,J) \) have the same relations for all \( 1 \leq i \leq r \).

Let \( G(\infty,J) \) denote the limit group of the sequence \( G(0,J) \to G(1,J) \to \ldots \). That is,

\[
G(\infty,J) = \langle a_1, a_2 \mid \bigcup_{i=1}^{\infty} S_i \rangle.
\]

The presentations of \( G(i,J) \), \( i \in \mathbb{N} \cup \{ \infty \} \), constructed above will be called \textit{canonical}.

\textbf{Remark 4.2.} In our notation, the groups \( G(i,j) \) constructed in \cite[Section 39.2]{21} are exactly \( G(i,\{j+1,j+2,\ldots\}) \).

We will need analogues of Lemma 39.5 and Lemma 39.6 from \cite{21}. Recall that the condition \( R \) is a technical condition which allows to apply the techniques developed in \cite[Sections 25-27]{21}. For the definition, we refer to \cite[Section 25]{21}.

\textbf{Lemma 4.3.}  
(a) For every \( i \in \mathbb{N} \) and \( J \subseteq \mathbb{N} \), the presentation of the group \( G(i,J) \) constructed as above satisfies the condition \( R \).

(b) For every \( J \subseteq \mathbb{N} \), the group \( G(\infty,J) \) is infinite and torsion of exponent \( n_0 \).

(c) If \( J \) contains all but finitely many natural numbers, then every proper subgroup of \( G(\infty,J) \) is cyclic of order dividing \( n_0 \).

\textbf{Proof.} The proof of the first statement almost coincides with the proof of Lemma 27.2 in \cite{21}. The only difference is that in our construction we choose \( n_0 \) to satisfy \( (7) \), while in \cite{21} one takes \( n_0 \) such that \( n = \lfloor (h+1)^{-1}n_0 \rfloor \). However the latter equality is not essential for the proof of Lemma 27.2. What is really used there is the inequality \( (h+1)n \leq n_0 \) (see the last line of the proof), which follows from \( (7) \).

Now part (a) allows us to apply Theorems 26.1 and 26.2 from \cite{21}, which yield (b). Finally the proof of (c) repeats the proof of \cite[Lemma 39.6]{21} verbatim after replacing \( G(\infty,j) \) with \( G(\infty,J) \), and \( j \) with \( \max(\mathbb{N} \setminus J) \). The key point here is that all relations of the second type of rank \( \geq \max(\mathbb{N} \setminus J) \) are imposed in \( G(\infty,J) \). \( \Box \)
In the next lemma, we could replace “arbitrary large” with “every”. However the weaker statement is sufficient for our goals.

**Lemma 4.4.** For any $J \subseteq \mathbb{N}$, there exist periods of arbitrary large rank. That is, for every $r \in \mathbb{N}$, the set of periods $X_i$ is non-empty for some $i > r$.

**Proof.** We repeat the main argument from the proof of [21, Theorem 19.3] with obvious changes. Fix some $r \in \mathbb{N}$. By [21, Lemma 4.6] there exists a 6-aperiodic word $X$ in the alphabet $\{a_1, a_2\}$ of length at least $20r$. Assume first that $X^{n_0} = 1$ in $G(r, J)$. Arguing as in the second paragraph of the proof of [21, Theorem 19.1] (and replacing the reference to [21, Theorem 16.2] there with the reference to [21, Theorem 22.2]) we conclude that the cyclic word $X^{n_0}$ contains a subword of the form $A^{20}$ for some non-trivial $A$ of length at most $r$. Since the length of $X$ is greater than $20r$, this contradicts the assumption that $X$ is 6-aperiodic. This contradiction shows that $X^{n_0} \neq 1$ in $G(r, J)$. In particular, we have $G(\infty, J) \neq G(r, J)$ as $X^{n_0} = 1$ in $G(\infty, J)$ by part (b) of Lemma 4.3. Therefore periods of rank $> r$ exist.

□

Every group $G(i, J)$ comes with a natural generating set, namely the image of $\{a_1, a_2\}$ under the natural homomorphism $F_2 \rightarrow G(i, J)$. By abuse of notation we denote the image of $\{a_1, a_2\}$ in $G(i, J)$, $i \in \mathbb{N} \cup \{\infty\}$, by $\{a_1, a_2\}$ as well. In what follows we say that a homomorphism $\varepsilon: G(\infty, I) \rightarrow G(\infty, J)$ is natural if $\phi(a_1) = a_1$ and $\phi(a_2) = a_2$.

**Lemma 4.5.** Let $J \subseteq \mathbb{N}$ and let $I = J \cap [1, r]$ for some $r \in \mathbb{N}$. Then the following hold:

(a) There exists a natural homomorphism $\varepsilon: G(\infty, I) \rightarrow G(\infty, J)$.

(b) Ker $\varepsilon$ does not contain nontrivial elements of $G(\infty, I)$ of length $\leq r$ with respect to the generating set $\{a_1, a_2\}$.

**Proof.** We first note that claim (a) is not obvious as, in general, the set of defining relations in the canonical presentation of $G(\infty, I)$ is not a subset of the set of relations in the canonical presentation of $G(\infty, J)$. However it is possible to construct other presentations of $G(\infty, I)$ and $G(\infty, J)$ for which this is the case.

Let $\mathcal{R}_I$ and $\mathcal{R}_J$ be the sets of relations of the second type in the canonical presentations of $G(\infty, I)$ and $G(\infty, J)$, respectively. By uniformness of our construction, we have $\mathcal{R}_I \subseteq \mathcal{R}_J$. Since both $G(\infty, I)$ and $G(\infty, J)$ are torsion of exponent $n_0$ by part (b) of Lemma 4.3 and all relations of the first type have the form $X^{n_0} = 1$ for some word $X$ in the alphabet $\{a_1^\pm, a_2^\pm\}$, we can represent the groups $G(\infty, I)$ and $G(\infty, J)$ as follows:

$$G(\infty, I) = \langle a_1, a_2 \mid \mathcal{R}_I, X^{n_0} = 1 \forall X \rangle$$

and

$$G(\infty, J) = \langle a_1, a_2 \mid \mathcal{R}_J, X^{n_0} = 1 \forall X \rangle,$$

where the relations $X^{n_0} = 1$ are imposed for all words $X$ in $\{a_1^\pm, a_2^\pm\}$. Now part (a) of the lemma becomes obvious.
Further part (a) of Lemma \[1.3\] allows us to apply Lemma 23.16 from [21], which implies that every nontrivial element from $\ker \varepsilon$ has length at least $(1 - \alpha)$ times the minimal possible length of a relator of rank $> r$. It is easy to see from [8]-[10], that the length of every relator of rank $> r$ is at least $(r + 1) \min \{n_0, (2h - 1)n\} > rn$. By the lowest parameter principle we can assume that $(1 - \alpha)n > 1$. Hence every nontrivial element from $\ker \varepsilon$ has length at least $r$.

In what follows we think of $G(\infty, J)$ (or, more precisely, $(G(\infty, J), \{a_1, a_2\})$ as an element of $G_2$. Let $\mathcal{T}$ be the subspace of $G_2$ consisting of $G(\infty, J)$ for all $J \subseteq \mathbb{N}$. To apply the Baire Theorem to $\mathcal{T}$ we need to know that $\mathcal{T}$ is complete as a metric space. We will prove this by showing that $\mathcal{T}$ is a continuous image of the Cantor set. Recall that the Cantor set $C$ can be identified with $2^\mathbb{N}$, where the distance between any two distinct subsets $I, J \subseteq \mathbb{N}$ is defined by

$$
d(I, J) = \frac{1}{\min(I \triangle J)}.
$$

**Corollary 4.6.** The map from the Cantor set $C$ to $\mathcal{T}$ defined by $J \mapsto (G(\infty, J), (a_1, a_2))$ is Lipschitz. In particular, this map is continuous and $\mathcal{T}$ is compact.

**Proof.** Let $I, J \subseteq \mathbb{N}$. Suppose now that $d(I, J) = 1/r$ for some $r \geq 1$ in $C$. Let $K = I \cap [1, r - 1] = J \cap [1, r - 1]$. By part (b) of Lemma \[1.3\], we have $d(G(\infty, I), G(\infty, K)) \leq 1/r$ and $d(G(\infty, J), G(\infty, K)) \leq 1/r$. Since $d$ is an ultrametric, we obtain

$$
d(G(\infty, I), G(\infty, J)) \leq \max\{d(G(\infty, I), G(\infty, K)), d(G(\infty, J), G(\infty, K))\} \leq 1/r = d(I, J).
$$

Thus the map $C \to \mathcal{T}$ is $1$-Lipschitz. \qed

Our next goal is to show that $\mathcal{T}$ contains a dense subset of topologizable groups. We begin with an auxiliary result.

**Lemma 4.7.** Let $I$ be a finite subset of $\mathbb{N}$. Then for every nontrivial element $g \in G(\infty, I)$, there exists a nontrivial normal subgroup $N \triangleleft G(\infty, I)$ such that $g \notin N$.

**Proof.** Let $l$ denote the word length of the element $g$ with respect to the generating set $\{a_1, a_2\}$. By Lemma \[1.3\] there exists a period $A$ of some rank

$$
i > \max\{l, \max I\}, \quad (11)
$$

Since $G(\infty, I)$ is infinite by part (b) of Lemma \[1.3\] we can additionally assume that balls of radius $i$ in $G(\infty, I)$ contain more than $n_0^2$ elements.

Note that the double coset $\langle A \rangle a_1 \langle A \rangle$ in $G(\infty, I)$ contains at most $n_0^2$ elements as $A'^{n_0} = 1$ in $G(\infty, I)$. Therefore, by our choice of $i$, there exists a word $T$ of length $1 \leq |T| \leq i < d_i$ such that $T$ does not belong to $\langle A \rangle a_1 \langle A \rangle$ in $G(\infty, I)$. Hence $T$ does not belong to $\langle A \rangle a_1 \langle A \rangle$ in $G(i - 1, I)$. Replacing $T$ with the shortest word among all words representing elements of the double coset $\langle A \rangle T \langle A \rangle \leq G(i - 1, I)$ if necessary, we can assume that $T \in Y_\alpha$. 15
Let now $J = I \cup \{i\}$. By \eqref{11}, Lemma 4.5 applies to $I$ and $J$ with $r = i - 1 \geq l$. Let $N$ be the kernel of the natural homomorphism $G(\infty, I) \to G(\infty, J)$. Then by part (b) of Lemma 4.3 we have $g \not\in N$.

It remains to show that $N$ is nontrivial. To this end, we will show that
\[
1 \neq a_1 A^n T A^{n+2} \cdots T A^{n+2h-2} \in N
\]
for $G(\infty, I)$. Indeed $a_1 A^n T A^{n+2} \cdots T A^{n+2h-2} \in N$ by the construction of $G(\infty, J)$. Suppose that $a_1 A^n T A^{n+2} \cdots T A^{n+2h-2} = 1$ in $G(I, \infty)$. Let $\Delta$ be the corresponding reduced disk diagram over $G(I, \infty)$. Then $\Delta$ is a $B$-map by \cite[Lemma 26.5]{21} and part (a) of Lemma 4.3.

Note that $\Delta$ does not contain faces of rank $\geq i$. Indeed otherwise the centralizer $C(\infty, J)$ of Lemma 4.3 we obtain that there are no relations of the second type of rank $\geq i$ in $G(I, \infty)$. However by our choice of $n_0$ this is impossible since these faces are “too large”; more precisely, by \cite[Lemma 23.16]{21}, the perimeter of each face in $\Delta$ is at most
\[
\frac{|\partial \Delta|}{1 - \alpha} \leq \frac{hi(d + n + 2h - 2)}{1 - \alpha} < n_0 i
\]
(see \eqref{7}), while the length of every relation of the first type of rank $\geq i$ is at least $n_0 i$.

Thus $\Delta$ is a diagram over $G(i - 1, I)$. Since $G(i - 1, I) = G(i - 1, J)$, $\Delta$ is also a diagram over $G(i - 1, J)$. Hence the relation $a_1 A^n T A^{n+2} \cdots T A^{n+2h-2} = 1$ can be derived from relations of rank $< i$ in $G(\infty, J)$. This contradicts \cite[Corollary 25.1]{21}, which guarantees that the relations of the canonical presentation of $G(\infty, J)$ are independent. The contradiction shows that $a_1 A^n T A^{n+2} \cdots T A^{n+2h-2} \neq 1$ in $G(I, \infty)$ and therefore $N$ is nontrivial.

Corollary 4.8. Let $J$ be a finite subset of $\mathbb{N}$. Then $G(\infty, J)$ is topologizable.

Proof. It suffices to construct a sequence of infinite normal subgroups
\[
N_1 \triangleright N_2 \triangleright \ldots \tag{12}
\]
of $G(\infty, J)$ with trivial intersection. Then taking $\{N_i\}_{i \in \mathbb{N}}$ as the base of neighborhoods of 1, we obtain a group topology on $G(\infty, J)$ which is Hausdorff as $\bigcap_{i \in \mathbb{N}} N_i = \{1\}$ and is non-discrete as every $N_i$ is infinite.

To this end, we first note that every non-trivial normal subgroup $M \triangleleft G(\infty, J)$ is infinite. Indeed otherwise the centralizer $C_{G(\infty, J)}(M)$ has finite index in $G(\infty, J)$. By \cite[Theorem 26.5]{21} the centralizer of every element in $G(\infty, J)$ is cyclic. Since $G(\infty, J)$ is torsion by part (b) of Lemma 4.3 we obtain that $C_{G(\infty, J)}(M)$ is finite and hence so is $G(\infty, J)$. However this contradicts part (b) of Lemma 4.3.

Now we construct the desired sequence \eqref{12} by induction. Let $G(\infty, J) = \{1, g_1, g_2, \ldots\}$. By Lemma 4.7 we can find a non-trivial subgroup $N_1 \triangleleft G(\infty, J)$ that does not contain $g_1$. Suppose that $N_j$ is already constructed for some $j \geq 1$ and $\{g_1, \ldots, g_j\} \cap N_j = \emptyset$. Applying Lemma 4.7 again, we can find a non-trivial subgroup $N \triangleleft G(\infty, J)$ such that $g_{j+1} \not\in N$. Let $N_{j+1} = [N_j, N]$. Obviously $\{g_1, \ldots, g_{j+1}\} \cap N_{j+1} = \emptyset$. Note also that $N_{j+1}$ is nontrivial. Indeed otherwise $N_j \leq C_{G(\infty, J)}(N)$ and arguing as in the previous paragraph we obtain that $N_j$ is finite; however this contradicts the fact that every non-trivial normal subgroup of $G$ is infinite. This completes the inductive step. Obviously $\bigcap_{i \in \mathbb{N}} N_i = \{1\}$ and thus the lemma is proved.
Remark 4.9. If $D$ is a $G_δ$ subset of a topological space $X$ and $Y$ is a subspace of $X$, then $D \cap Y$ is a $G_δ$ subset of $Y$. This observation will be used several times below.

Proof of Theorem 1.4. Note that the set of finite subsets of $\mathbb{N}$ is dense in the Cantor set $C$. Hence its image is dense in $T$ by Corollary 4.6. Using Lemma 4.8 we obtain that $T$ contains a dense subset of topologizable groups. Then by Proposition 3.4 and Remark 4.9 we conclude that the property of being topologizable is generic in $T$.

Further the set of all cofinite subsets of $\mathbb{N}$ is also dense in the Cantor set. Using Lemma 4.3 and arguing as in the previous paragraph, we obtain that the property of being a Tarski Monster (of exponent $n_0$) is also generic in $T$. Since $T$ is compact, we can apply the Baire Category Theorem, which implies that the property of being a topologizable Tarski Monster (of exponent $n_0$) is also generic in $T$. In particular, such groups exist.

5 Further speculations

One can produce many other examples of “exotic” topologizable groups using the fact that most limits of “hyperbolic-like” groups are topologizable. More precisely, we recall that an isometric action of a group $G$ on a metric space $S$ is called acylindrical if for every $\varepsilon > 0$ there exist $R, N > 0$ such that for every two points $x, y$ with $d(x, y) \geq R$, there are at most $N$ elements $g \in G$ satisfying $d(x, gx) \leq \varepsilon$ and $d(y, gy) \leq \varepsilon$.

A group $G$ is called acylindrically hyperbolic if it acts acylindrically and non-elementary on a (Gromov) hyperbolic space. Recall also that non-elementarity of the action can be defined in this context by requiring that $G$ is not virtually cyclic and has unbounded orbits. For details we refer to [25].

The class of acylindrically hyperbolic groups contains many examples of interest: non-virtually cyclic hyperbolic groups, non-virtually cyclic relatively hyperbolic groups with proper peripheral subgroups, all but finitely many mapping class groups of punctured closed surfaces, $Out(F_n)$ for $n \geq 2$, groups acting properly on proper $CAT(0)$ spaces and containing rank 1 elements, and so forth [6, 25].

The proof of the following lemma relies heavily on results of [6]; we will refer the reader to [6] for definitions of the auxiliary notions used in the proof. In the particular cases of hyperbolic and relatively hyperbolic groups one could alternatively use results of [22] or [27].

Lemma 5.1. Every acylindrically hyperbolic group is topologizable.

Proof. Let $G$ be an acylindrically hyperbolic group. By [25, Theorem 1.2], $G$ contains non-degenerate hyperbolically embedded subgroups (see [6] for the definition). This allows us to apply [6, Theorem 2.23], which guarantees that there exists a hyperbolically embedded subgroup $H \leq G$ such that $H \cong \mathbb{Z} \times K$, where $K$ is a finite group. In particular, $H$ contains an infinite chain of infinite normal (in $H$) subgroups $N_1 \triangleright N_2 \triangleright \ldots$ with trivial intersection. Let $M_i$
denote the normal closure of \(N_i\) in \(G\). Since \(H\) is hyperbolically embedded, the group-theoretic Dehn surgery theorem (see [6, Theorem 2.25 (c)]) implies that \(\bigcap_{i \in \mathbb{N}} M_i = \{1\}\). Now we can use the chain \(M_1 \supset M_2 \supset \ldots\) to define a Hausdorff topology on \(G\), taking \(\{M_i \mid i \in \mathbb{N}\}\) as the base of neighborhoods at 1. Since every \(N_i\) is infinite, so is \(M_i\) and hence the topology is non-discrete.

Using chains of normal subgroups \(N_1 \supset N_2 \supset \ldots\) as bases of neighborhoods is a fairly standard approach to defining a topology on a given group \(G\). It is interesting to ask whether one can topologize a “hyperbolic-like” group in an essentially different way. The question can be formalized as follows: Under what conditions does an acylindrically hyperbolic group admit a topology with respect to which it is topologically simple?

Note that acylindrically hyperbolic groups are very far from being abstractly simple [6]. Of course, \(G\) is not topologically simple in the topology defined in the proof of Lemma 5.1 as well, since every \(N_i\) is closed. However, we conjecture the following.

**Conjecture 5.2.** Suppose that an acylindrically hyperbolic group \(G\) has no non-trivial finite normal subgroups. Then \(G\) admits a topology with respect to which it is topologically simple.

Note that the absence of non-trivial finite normal subgroups is necessary as finite subgroups are always closed.

Conjecture 5.2 holds for hyperbolic groups. Indeed, Chaynikov [4] proved that every non-elementary hyperbolic group \(G\) without non-trivial finite normal subgroups admits a faithful action on \(\mathbb{N}\) which is \(k\)-transitive for every \(k \in \mathbb{N}\). This action defines a dense embedding \(G \to S(\mathbb{N})\), where \(S(\mathbb{N})\) is the group of all permutations of \(\mathbb{N}\) endowed with the topology of pointwise convergence. Let \(A_{ fin}(\mathbb{N}) = \bigcup_{n \in \mathbb{N}} A_n\) and \(S_{ fin}(\mathbb{N}) = \bigcup_{n \in \mathbb{N}} S_n\), where \(A_n\) and \(S_n\) are the groups of even permutations and all permutations of \(\{1, \ldots, n\}\), respectively, naturally embedded in \(S(\mathbb{N})\). Then \(A_{fin}(\mathbb{N})\) and \(S_{fin}(\mathbb{N})\) are the only proper non-trivial normal subgroups of \(S(\mathbb{N})\) (see [8, Theorem 8.1A]). Obviously both of them are dense and hence \(S(\mathbb{N})\) is topologically simple. Now using the fact that the image of \(G\) is dense in \(S(\mathbb{N})\) it is straightforward to verify that \(G\) is topologically simple with respect to the topology induced by the embedding. In seems plausible that the Chaynikov’s result can be generalized to groups from \(H_k\), which would imply Conjecture 5.2 in the full generality.

**Proof of Theorem 1.6.** The theorem obviously follows from Lemma 5.1 part (a) of Proposition 3.4 and Remark 4.9.

To illustrate usefulness of Theorem 1.6 we outline here the proof of the existence of a topologizable groups with 2 conjugacy classes. Details will appear in the forthcoming paper [12].

First examples of groups with 2 conjugacy classes other than \(\mathbb{Z}/2\mathbb{Z}\) were constructed by Higman, B.H. Neumann and H. Neumann in 1949; first finitely generated examples were constructed by the third author in [26]. Motivated by the recent study of groups with the Rokhlin property, (i.e., topological groups with a dense conjugacy class) Glassner and Weiss ask in [9] whether there exist topological analogues of these constructions. Specifically, they ask whether
there exists a non-discrete locally compact topological group with 2 conjugacy classes. Our approach allows to construct a non-discrete group with 2 conjugacy classes; local compactness can not be ensured by our methods although our group will be compactly generated (and even finitely generated in the abstract sense).

To construct a topologizable group with 2 conjugacy classes we first recall that groups with exactly two conjugacy classes form a $G_\delta$ subset of $G_k$ by part (f) of Proposition 3.4. Further let $k \geq 2$ and let $\mathcal{AH}_{tf}$ denote the subset of all groups from $G_k$ that are torsion free and acylindrically hyperbolic. The technique developed in [26] can be extended to acylindrically hyperbolic groups to show that $\mathcal{AH}_{tf}$ contains a dense $G_\delta$ subset of groups with 2 conjugacy classes; the proof can be found in [12, Corollary 8.10]. Combining this with Theorem 1.6 we obtain that a generic group in $\mathcal{AH}_{tf}$ is topologizable and all its non-trivial elements are conjugate.

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