A boosted Kerr black hole solution and the structure of a general astrophysical black hole

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Abstract
A solution of Einstein’s vacuum field equations that describes a boosted Kerr black hole relative to an asymptotic Lorentz frame at the future null infinity is derived. The solution has three parameters (mass, rotation and boost) and corresponds to the most general configuration that an astrophysical black hole must have; it reduces to the Kerr solution when the boost parameter is zero. In this solution the ergosphere is north-south asymmetric, with dominant lobes in the direction opposite to the boost. However the event horizon, the Cauchy horizon and the ring singularity – which are the core of the black hole structure – do not alter, being independent of the boost parameter. Possible consequences for astrophysical processes connected with Penrose processes in the asymmetric ergosphere are discussed.

1 Introduction and derivation of the solution

One of the most successful applications of General Relativity, the mathematical theory of black holes, was developed based on the two exact black hole solutions described by the Schwarzschild metric (obtained in 1915)[1] and the Kerr metric (obtained in 1963)[2]. The Kerr metric describes a rotating black hole (with two parameters, mass and angular momentum) and has the Schwarzschild black hole as its static limit configuration. In particular the Kerr solution was of fundamental importance to the understanding of astrophysical processes involved in objects with tremendous output of energy as
quasars, pulsars and active galactic nuclei (AGNs). Recent direct observations of the gravitational wave emission from a binary black hole merger with a mass ratio of the order of 0.8, indicate that the resulting remnant black hole is a Kerr black hole which must be boosted along a particular direction, with respect to the asymptotic Lorentz frame at null infinity where such emissions have been detected. In this way we have to add to the Kerr black hole description an additional parameter – the boost parameter – connected to its motion with respect to the observation frame. The boost of the remnant black hole is actually due to the presence of a nonzero net gravitational wave momentum flux for the nonequal mass case of black hole collisions. This must also be the case of astrophysical black holes in which a boost can be inherited by the collapse of astrophysical objects with large bumps and other deformities. As we will see the presence of this further parameter does not violate theorems on the uniqueness of the structure of the Kerr black hole solution. In the present paper we use geometric units $G = c = 1$.

In the past literature several papers have dealt with the search of a boosted Kerr black hole solution, the main objective of which was the use in $1+3$ Numerical Relativity to obtain initial data for a system of spinning black holes. In [4] initial data for a boosted Kerr black hole were constructed in the axially symmetric case with view to a possible long-term numerical evolution of a single boosted black hole or a system of black holes, and apparent horizons are found numerically. In [5, 6, 7] the authors apply a Lorentz boost on the cartesian Kerr-Schild coordinates of a Kerr black hole and carry out a $1+3$ decomposition, resulting in a slicing of the Kerr-Schild spacetime from which black hole initial data can be propagated and apparent horizons are located. In [8, 9] Lorentz boosts on Kerr-Schild coordinates of a Kerr geometry were also used in evaluating a distributional energy-momentum with support in the singular region of the metric as the basis for investigating ultra relativistic limit geometries. These approaches are however distinct from the result in our paper where we obtain an exact stationary analytic solution of a boosted Kerr black hole relative to a Lorentz frame at future null infinity. This exact stationary analytic solution should be expected to correspond to the final configuration of the collision and merger of a system of spinning black holes, therefore its importance as a complementary test for the accuracy of the full numerical evolution of this system up to the final remnant black hole.

The metric of a stationary boosted Kerr black hole is obtained here by using an integration procedure analogous to that of Kerr in his original deriv-
tion of Kerr geometry\cite{2}. We make use of the simple and elegant apparatus presented in Sthephani et al.\cite{10} (sections 29.1 and 29.5) for obtaining twisting Petrov D vacuum solutions of Einstein’s equations. We start with the metric expressed as

$$ds^2 = 2\omega^1 \omega^2 - 2\omega^3 \omega^4$$

where the 1-forms \(\omega^a\) are given by

$$\omega^1 = \omega^2 = -d\xi/\bar{\rho}P, \quad \omega^3 = du + Ld\xi + \bar{L}d\bar{\xi}, \quad \omega^4 = dr + Wd\xi + \bar{W}d\bar{\xi} + H\omega^3,$$

in Robinson-Trautman-type coordinates \((u, r, \xi, \bar{\xi})\)\cite{13}, where a bar denotes complex conjugation. We also assume that the metric functions are independent of the time coordinate \(u\), namely, \(\partial/\partial u\) is a Killing vector of the geometry. \(P\) is a real function. Einstein’s vacuum equations lead then to\cite{10}

$$\rho^{-1} = -(r + i\Sigma), \quad W = i \partial_\xi \Sigma,$$

$$H = \lambda/2 - \frac{mr}{r^2 + \Sigma^2},$$

$$\lambda = 2P^2 \text{ Re } (\partial_\xi \partial_\bar{\xi} \ln P),$$

$$\lambda\Sigma + P^2 \text{ Re } (\partial_\xi \partial_\bar{\xi} \Sigma) = 0,$$

$$2i\Sigma = P^2(\partial_\xi L - \partial_\bar{\xi} \bar{L}),$$

where \(m\) is a real constant and \(\lambda = \pm 1\) is the curvature of the 2-dim surface \(d\xi d\bar{\xi}/P^2\). Here we adopt \(\lambda = 1\). The \(r\)-dependence is isolated in \(\rho\) and \(H\) so that the remaining functions to be determined – \(P\), \(\Sigma\) and \(L\) – are functions of \((\xi, \bar{\xi})\) only. We should mention that in equations (3) and (4) above we have further removed an integration constant \(r_0\) by a coordinate transformation that changes the origin of the affine parameter \(r\), setting thus effectively \(r_0 = 0\). (cf. Section 29.1.4 of \cite{10}). The integration of the field equations reduces then to the integration of (5), (6) and (7). For our purposes
here we will substitute the variables \((\xi, \bar{\xi})\) by \((\theta, \phi)\) via the stereographic transformation

\[
\xi = \cot(\theta/2) \, e^{i\phi}.
\]

From (5) we integrate the real function \(P(\xi, \bar{\xi})\) by assuming that \(P\) has the form

\[
P = \frac{K(\theta, \phi)}{\sqrt{2 \sin^2(\theta/2)}}.
\]

Eq. (5) results in

\[
1 = KK_{\theta\theta} + KK_{\theta} \cot \theta - K^2_{\theta} + K^2 + \frac{(KK_{\phi\phi} - K^2_{\phi})}{\sin^2 \theta}.
\]

A general solution of (9) is given by

\[
K(\theta, \phi) = a + b \hat{x} \cdot \mathbf{n}, \quad a^2 - b^2 = 1,
\]

where \(\mathbf{x} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)\) is the unit vector along an arbitrary direction \(\mathbf{x}\) and \(\mathbf{n} = (n_1, n_2, n_3)\) is a constant unit vector satisfying \(n_1^2 + n_2^2 + n_3^2 = 1\). The solution (10) depends on three independent parameters and defines a transformation of the generalized Bondi-Metzner-Sachs (BMS) group discussed by Sachs, characterizing the general form of Lorentz boosts contained in the homogeneous Lorentz transformations of the BMS group at null infinity. In the original Kerr solution \(K(\theta, \phi) = 1\).

Here our interest will be restricted to Lorentz boosts along the \(z\)-axis only,

\[
K(\theta) = a + b \cos \theta , \quad a^2 - b^2 = 1,
\]

that will correspond to a boosted axisymmetric solution. In the realm of black hole solutions, (11) can be interpreted as corresponding to a Lorentz boost of the black hole along the \(z\)-axis. The boost parameter \(\gamma\) parametrizes \(a\) and \(b\) as \((a = \cosh \gamma, b = \sinh \gamma)\), and is associated with the velocity \(v = \tanh \gamma\) of the black hole relative to a Lorentz frame at future null infinity.

Assuming \(\Sigma = \Sigma(\theta)\) and using (11), eq. (6) in the variables \((\theta, \phi)\) reduces to

\[
\Sigma_{\theta\theta} + \cot \theta \, \Sigma_{\theta} + 2 \frac{\Sigma}{K^2(\theta)} = 0,
\]
yielding the regular solution

\[ \Sigma(\theta) = \omega \frac{(b + a \cos \theta)}{(a + b \cos \theta)}, \]

where \( \omega \) is an arbitrary constant, to be identified with the rotation parameter of the solution. Using (13) we now integrate Eq. (7). We accordingly adopt

\[ L = i\mathcal{L}(\theta)e^{-i\phi}, \]

resulting in

\[ \mathcal{L}_\theta - \mathcal{L}/\sin \theta + (1 - \cos \theta) \frac{\Sigma(\theta)}{K^2(\theta)} = 0. \]

A general solution for (15) is given by

\[ \mathcal{L}(\theta) = \left\{ \frac{1 - \cos \theta}{\sin \theta} \right\} \left\{ C_1 - \frac{\omega}{2b^2} \left( \frac{a^2 + 2ab \cos \theta + b^2}{(a + b \cos \theta)^2} \right) \right\}, \]

where \( C_1 \) is an arbitrary constant. The apparent singular behaviour of the above solution for a zero boost \((b^2 = 0)\) can be eliminated either by fixing \( C_1 = \omega/2b^2 \) or by a coordinate transformation on the final metric. In both cases it finally results

\[ \mathcal{L}(\theta) = -\omega \frac{(1 - \cos \theta)}{2(a + b \cos \theta)^2} \sin \theta = -2\omega \frac{\cos \theta/2 \sin^2 \theta/2}{(a + b \cos \theta)^2}. \]

Also the equation defining \( W \) yields

\[ W = i\omega \frac{\sin \theta}{(a + b \cos \theta)^2} \left( \sin^2 \theta/2 \right) e^{-i\phi}, \]

implying that

\[ W d\xi + \bar{W} d\bar{\xi} = -\frac{\omega}{(a + b \cos \theta)^2} \sin^2 \theta \, d\phi. \]

Analogously

\[ L = -i \frac{2\omega}{(a + b \cos \theta)^2} \cos \theta/2 \sin^3 \theta/2 \, e^{-i\phi} \]
results in
\[ Ld\xi + \bar{L}d\bar{\xi} = \frac{\omega}{(a + b\cos\theta)^2} \sin^2 \theta \ d\phi . \] (21)

We also obtain
\[ \frac{1}{\rho} = -(r + i\Sigma) = -(r + i\omega \frac{b + a\cos\theta}{a + b\cos\theta}) . \] (22)

The metric (1) results finally
\[ ds^2 = \frac{r^2 + \Sigma^2(\theta)}{(a + b\cos\theta)^2} (d\theta^2 + \sin^2 \theta d\phi^2) - \\
- \left( du + \frac{\omega \sin^2 \theta}{(a + b\cos\theta)^2} d\phi \right) \left( dr - \frac{\omega \sin^2 \theta}{(a + b\cos\theta)^2} d\phi \right) \\
- \left( du + \frac{\omega \sin^2 \theta}{(a + b\cos\theta)^2} d\phi \right)^2 \frac{1}{r^2 + \Sigma^2(\theta)} \left( r^2 - 2mr + \Sigma^2(\theta) \right) , \] (23)

where \( \Sigma(\theta) = \omega (b + a\cos\theta)/(a + b\cos\theta) \). This metric describes a boosted Kerr black hole, the boost being along its axis of rotation, with respect to an asymptotic Lorentz frame at future null infinity. For \( b = 0 \) the metric (23) is the Kerr metric in retarded Robinson-Trautman coordinates. For \( \omega = 0 \) it represents a boosted Schwarzschild black hole along the \( z \)-axis[11].

2 Properties of the solution: the ergosphere and horizons

A direct examination of (23) shows that \( \partial/\partial u \) and \( \partial/\partial \phi \) are Killing vectors of the geometry. The boosted Kerr geometry also presents an ergosphere, defined by the limit surface for static observers, namely, the locus where the Killing vector \( \partial/\partial u \) becomes null[14]. In the coordinate system of (23) the equation for the static limit surface \( g_{au} = 0 \) results in
\[ r^2 - 2mr + \Sigma^2(\theta) = 0 , \] (24)

namely,
\[ r_{stat}(\theta) = m + \sqrt{m^2 - \omega^2 \left( \frac{b + a\cos\theta}{a + b\cos\theta} \right)^2} . \] (25)
The horizons of the boosted Kerr metric are the surfaces where
\[ g^{rr} = \frac{r^2 - 2mr + \omega^2 (\sin^2 \theta + (b + a \cos \theta)^2)/K^2(\theta)}{r^2 + \Sigma^2(\theta)} = 0, \] (26)
resulting, after some algebra,
\[ r_{\pm} = m \pm \sqrt{m^2 - \omega^2}. \] (27)

We see that the event horizon \( r_+ \) and the Cauchy horizon \( r_- \) do not alter by the effect of the boost. This should be expected since both horizons are lightlike surfaces and therefore Lorentz invariant, contrary to the case of the static limit surface.

The region between the surfaces \( r_{\text{stat}}(\theta) \) and \( r_+ \) is the ergosphere, where the Penrose process\([15, 16]\) takes place. The ergosphere is deformed by the boost as a consequence of the corresponding deformation of \( r_{\text{stat}}(\theta) \), as illustrated in Figure 1 for a boosted Kerr black hole with mass \( m = 200 \) and rotation parameter \( \omega = 195 \), in geometrical units. The presence of the boost makes the ergosphere north-south asymmetric with dominant lobes in the direction opposite to the boost (\( v = \tanh \gamma \)). The deformation increases as the boost increases. However we can see that the event horizon \( r_+ \) is not altered by the boost. The direction of the boost is along the positive \( z \)-axis (\( \gamma > 0 \)).

3 The singularity

From the Kretschmann curvature invariant of the spacetime we have that the metric (23) is truly singular at
\[ r^2 + \Sigma^2(\theta) = 0. \] (28)
The nature of the singularity can be analyzed by transforming (23) into the Kerr-Schild form via the transformation
\[ x + iy = (r - i\omega)e^{i\phi} \sin \theta/K(\theta), \quad z = r \frac{b + a \cos \theta}{a + b \cos \theta}, \quad u = t - r, \] (29)
where \( r \) is defined by
\[ r^4 - (R^2 - \omega^2)r^2 - \omega^2 z^2 = 0, \quad R^2 = x^2 + y^2 + z^2, \] (30)
Figure 1: The structure of the boosted Kerr black hole for mass $m = 200$ and rotation parameter $\omega = 195$. The presence of the boost turns the ergosphere north-south asymmetric as shown by the dominant lobes for $\gamma = 1.9$ (dashed blue curve) and $\gamma = 3.6$ (continuous blue curve). The event horizon $r_+$ is not altered by the boost (dashed red curve). The direction of the boost is along the positive $z$-axis ($\gamma > 0$).
so that in this coordinate system the singularity \((r = 0, \cos \theta = -b/a)\) corresponds to the circle

\[
x^2 + y^2 = \omega^2 a^2 \sin^2(\theta_s), \quad \theta_s = \arccos(-b/a)
\]

located at the plane \(z = 0\). A careful manipulation of \((31)\) results in \(x^2 + y^2 = \omega^2\), showing that the singularity is not altered by the boost, analogously to the event and Cauchy horizons. In this way the core of the black hole structure – the event horizon, the Cauchy horizon and the singularity – does not alter, being independent of the boost parameter. This establishes the invariance of the black hole structure under Lorentz boosts of the Bondi-Metzner-Sachs group at future null infinity\[12, 11\].

4 Final comments and conclusions

The metric \((23)\) derived here, solution of Einstein’s vacuum equations, describes the most general configuration that an astrophysical black hole must have. The solution has three parameters, mass \(m\), rotation \(\omega\) and boost \(\gamma\), which are necessary for the description of black holes in nature. In fact the processes involving the formation of an astrophysical black hole, like the merger of binary inspirals of unequal mass rotating black holes, imply that the resulting remnant black hole must be a boosted Kerr black hole with respect to an asymptotic Lorentz frame at null infinity. This is the case of the recent detection, by the LIGO Scientific Collaboration\[17\], of the gravitational waves emitted by a binary black hole merger\[3\] with a mass ratio of the order of 0.8 – indicating that the resulting remnant black hole of this collision must be a Kerr black hole boosted along a particular direction, with respect to the asymptotic Lorentz frame at null infinity where such emission has been detected. This must also be the case of astrophysical black holes in which a boost can be inherited by the collapse of astrophysical objects with large bumps and other deformities.

The presence of this further parameter does not violate previous theorems on the uniqueness of the Kerr black hole solution. The core of the black hole structure of the boosted Kerr solution, namely, the event horizon, the Cauchy horizon and the ring singularity remain invariant under the introduction of a boost in the solution.

On the other hand the static limit surface (which defines the ergosphere) and the ergosphere itself are affected by the boost. The presence of the boost
turns the ergosphere asymmetric, with dominant lobes in the direction opposite to the direction of the boost. This asymmetry increases as the boost parameter increases, as shown in Figure 1. This asymmetry can have possible consequences for astrophysical mechanisms connected with the Penrose processes in the asymmetric ergosphere. Among these processes are the magnetohydrodynamic Blandford-Znajek mechanisms\cite{18, 19} by which rotational energy can be extracted of Kerr black holes in the form of electromagnetic and kinetic energy. We argue that the application of such mechanisms as engines of relativistic jets from quasars, pulsars and AGNs should be properly treated and modified in the neighborhood of a boosted Kerr black hole (that describes a general astrophysical black hole) taking into account the large north-south asymmetry of the ergosphere.

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