Quantum Aitchison geometry

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Abstract

Multiplying a likelihood function with a positive number makes no difference in Bayesian statistical inference, therefore after normalization the likelihood function in many cases can be considered as probability distribution. This idea led Aitchison to define a vector space structure on the probability simplex in 1986. Pawlowsky-Glahn and Egozcue gave a statistically relevant scalar product on this space in 2001, endowing the probability simplex with a Hilbert space structure. In this paper we present the noncommutative counterpart of this geometry. We introduce a real Hilbert space structure on the quantum mechanical finite dimensional state space. We show that the scalar product in quantum setting respects the tensor product structure and can be expressed in terms of modular operators and Hamilton operators. Using the quantum analogue of the log-ratio transformation it turns out that all the newly introduced operations emerge naturally in the language of Gibbs states. We show an orthonormal basis in the state space and study the introduced geometry on the space of qubits in details.

1 Introduction

The very first step towards the Euclidean structure on the quantum mechanical state space was done by Aitchison in 1986 when he noticed that the operation of Bayes’s formula to change a prior probability assessment $\pi$ into a posterior one $p$ through a likelihood function $\rho$, can be viewed as an operation on the probability simplex $[1, 2, 4]$. Multiplying a likelihood function with a positive constant results the same updated probability distribution, therefore in many cases with an appropriate multiplicative factor one can transform the likelihood to a probability distribution $\rho$. Such ideas led to the definition of an abstract operation $\oplus$ for the updating process $p = \rho \oplus \pi$. The operation $\oplus$ is called perturbation, because in applications the likelihood function has just a slight effect on the prior distribution.

To be more concrete let us denote the interior of the probability simplex by $S_n$ ($n \in N \setminus \{0, 1\}$), that is

$S_n = \left\{ (p_1, \ldots, p_n) \in ]0, 1[^n \mid \sum_{i=1}^n p_i = 1 \right\}$.

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The perturbation is defined as an $S_n \times S_n \to S_n$ operation, if $p, q \in S_n$ then
\[ p \oplus q = \frac{1}{n} \sum_{i=1}^{n} p_i q_i \times (p_1 q_1, \ldots, p_n q_n) \tag{1} \]
and scalar multiplication is a $\mathbb{R} \times S_n \to S_n$ map, for $\lambda \in \mathbb{R}$ and $p \in S_n$
\[ \lambda \circ p = \frac{1}{n} \sum_{i=1}^{n} p_i^\lambda \times (p_1^\lambda, \ldots, p_n^\lambda). \tag{2} \]

It is easy to check that $(S_n, \oplus, \circ)$ is a vector space, where the zero vector is the uniform distribution. This linear structure of the simplex was studied in details [3, 7]. Note that intervals in this Aitchison geometry $(t \circ p) \oplus ((1 - t) \circ q)$ ($t \in [0, 1]$) play crucial role in hypothesis testing and have been studied in detail for decades in connection with relative entropy and Chernoff distance [8, 9, 19]. In statistic these intervals are often referred to as Hellinger arcs.

In 2001, Pawlowsky-Glahn and Egozcue using the notion of metric center and metric variation endowed this space with an inner product
\[ \langle \cdot, \cdot \rangle : S_n \times S_n \to \mathbb{R} \quad (p, q) \mapsto \frac{1}{2n} \sum_{i,j=1}^{n} \log \left( \frac{p_i}{p_j} \right) \log \left( \frac{q_i}{q_j} \right) \tag{3} \]
which turned out to be a simple tool to prove essential properties of statistical inference [10, 17]. An orthonormal basis was constructed in the Hilbert space $(S_n, \langle \cdot, \cdot \rangle)$ which has particular importance in estimation theory [12]. The norm $\| \cdot \|_\theta$ induced by the scalar product is called information evidence $I_e$, since it fulfills the natural requirements that from different prior distributions $\pi_1, \pi_2$ the same measurement inducing likelihood $\rho$, the information gain in posterior distributions $p_i = \rho \oplus \pi_i$ ($i = 1, 2$) should be the same $I_e(p_1 \oplus \pi_1) = I_e(p_2 \oplus \pi_2) = I_e(\rho)$ and the distance between the prior and posterior distributions should be the same $I_e(p_1 \ominus \pi_2) = I_e(\pi_1 \ominus \pi_2)$ [11]. The formal definition of $p \ominus \pi$ is $p \oplus ((-1) \circ \pi)$.

This geometrical structure of the simplex was extended to different statistical models recently. For probability density functions on bounded intervals of the real numbers [10], for $\sigma$-finite measures on a measurable space [20, 21]. In this paper a similar Hilbert space structure is presented on the quantum mechanical state space inspired by the classical Aitchison geometry.

### 2 Hilbert space structure on the quantum mechanical state space

#### 2.1 Basic notations and lemmas

We work in finite dimensional framework, that is every Hilbert space will be assumed to be finite dimensional over the complex field. Let us fix some notation. The letters $n, m$ denote integers greater than or equal to two. The symbol $M_n$ stands for the algebra of $n \times n$ complex matrices, $M_{n,sa}$ denotes the set of self-adjoint elements in $M_n$ and $M_n^+$ denotes the set of positive definite matrices in $M_{n,sa}$. The trace one elements of $M_n^+$ form the interior of the $n$-level quantum mechanical state space which is denoted by $\mathcal{M}_n$, that is $\mathcal{M}_n = \{ D \in M_n^+ | \text{Tr} \ D = 1 \}$. The linear structure of self-adjoint traceless matrices will play role in computations therefore we introduce the abbreviation $M_{n,sa}^0$ for the set of those matrices. The symbol $I_n$ is used for the $n \times n$ identity matrix and to shorten formulae for every
matrix $A \in M_n^+$ the abbreviation $\tilde{A} = \log(A)$ is used. The matrix units $E_{ij}$ will be used, $E_{ij}$ is the matrix with every component zero, except the component $ij$, which is one and the size of the matrix units will be clear from the context.

To a matrix $X \in M_n$, one can associate the left and right multiplication operators $L_X, R_X : M_n \to M_n$ that act like

$$A \mapsto L_X(A) =XA$$

$$A \mapsto R_X(A) =AX.$$  

For states $D_1, D_2 \in M_n$ the relative modular operator is defined as $\Delta_{D_1/D_2} = L_{D_1}R_{D_2}^{-1}$, that is for every $A \in M_n$ one has $\Delta_{D_1/D_2}(A) = D_1AD_2^{-1}$. In the case $D_1 = D_2$ the short notation $\Delta_D$ is used for $\Delta_{D/D}$.

The space $M_n$ is endowed with Hilbert-Schmidt inner product

$$(A, B)_n = \text{Tr } A^*B \quad \forall A, B \in M_n.$$  

For every matrix $X \in M_{n,sa}$ the multiplication operators $L_X, R_X$ and in the invertible case $\Delta_X$ too are self-adjoint, moreover, for every $D \in M_n$ the operators $L_D, R_D$ and $\Delta_D$ are positive definite.

**Lemma 1.** For matrices $A \in M_n^+$ and $B \in M_n^+$ one has for their logarithm

$$A \otimes B = \tilde{A} \otimes I_m + I_n \otimes \tilde{B} \quad \text{and} \quad A \otimes A^{-1} = \tilde{A} \otimes I_n - I_n \otimes \tilde{A}.$$  

**Proof.** The tensor product $A \otimes B$ identified as an $M_m \to M_n$ linear map acting like $X \mapsto AXB^*$. Using the commutativity of the operators $L_A$ and $R_B$

$$A \otimes B = \bar{L}_A \bar{R}_B = \bar{L}_A + \bar{R}_B = \tilde{A} \otimes I_m + I_n \otimes \tilde{B}$$

follows.  

**2.2 Elementary operations**

Now we are in the position to present the quantum analogue of the perturbation $\oplus$, multiplication $\odot$ and scalar product $\langle \cdot, \cdot \rangle_\circ$ given by Equations (1,2,3).

**Definition 1.** On the set of positive definite matrices $M_n^+$ define the following operations for every $A, B \in M_n^+$ and $\lambda \in \mathbb{R}$.

$$A \oplus B := \frac{e^{\tilde{A} + \tilde{B}}}{\text{Tr } e^{\tilde{A} + \tilde{B}}}$$

$$\lambda \odot A := \frac{e^{\lambda \tilde{A}}}{\text{Tr } e^{\lambda \tilde{A}}}$$

$$\langle A, B \rangle_\circ := \frac{1}{n} \text{Tr } (\tilde{A}B) - \frac{1}{n^2} (\text{Tr } \tilde{A})(\text{Tr } \tilde{B}).$$  

First note about these operations that $A \oplus B, \lambda \odot A \in \mathcal{M}_n$ if $A, B \in M_n^+$ and $\lambda \in \mathbb{R}$, moreover, they are defined on the rays of positive definite operators in the following sense.

**Lemma 2.** For every $A, B \in M_n^+$ and $c \in \mathbb{R}^+$ we have the following scale invariance.

$$A \oplus B = (cA) \oplus B = A \oplus (cB)$$

$$\lambda \odot A = \lambda \odot (cA)$$

$$\langle A, B \rangle_\circ = \langle cA, B \rangle_\circ = \langle A, cB \rangle_\circ$$
Proof. Elementary computation.

Because of scale invariance one can factorize by rays, which means that is enough to consider only the set of trace one matrices i.e. the set of quantum states. It turns out that the interior of the state space is a Euclidean space with these operations.

**Theorem 1.** The state space $\mathcal{M}_n$ is a real Hilbert space with addition $\oplus$, null vector $\frac{1}{\sqrt{n}}I_n$, multiplication $\odot$ and inner product $\langle \cdot, \cdot \rangle_\circ$.

**Proof.** Not every requirement of Hilbert spaces will be proven, just two of them to give insight to such calculations. Lemma 2 gives the idea to define temporarily the relation $a \simeq b$ if there is a $c \in \mathbb{R}^+$ such that $a = cb$.

First let us prove the distributivity for $A, B \in \mathcal{M}_n$ and $\lambda \in \mathbb{R}$.

\[
\lambda \odot (A \oplus B) \simeq \lambda \odot e^{\tilde{A} + \tilde{B}} \simeq e^{\lambda \tilde{A} + \lambda \tilde{B}} \simeq e^{\lambda \tilde{A}} \oplus e^{\lambda \tilde{B}} \simeq (\lambda \odot A) \oplus (\lambda \odot B)
\]

Now let us prove the additivity of the scalar product for $A, B, C \in \mathcal{M}_n$.

\[
\langle A \oplus B, C \rangle_\circ = \frac{1}{n} \text{Tr} ((\tilde{A} + \tilde{B})\tilde{C}) - \frac{1}{n^2}(\text{Tr} (\tilde{A} + \tilde{B}))(\text{Tr} \tilde{C}) = \langle A, C \rangle_\circ + \langle B, C \rangle_\circ.
\]

Because of the group property of the addition, every state $A$ has an additive inverse, which will be denoted by $\ominus A$. It can be expressed by multiplication as $\ominus A = (-1) \odot A$.

As in the classical case the intervals $(t \odot A) \oplus ((1 - t) \odot B)$ ($t \in [0,1]$) has special interests in quantum hypothesis testing [6, 14, 19]. They can be considered as one generalization of Hellinger arc.

The introduced operations are unitary invariant, which means that they preserve symmetries in quantum mechanics.

**Lemma 3.** For every state $A, B \in \mathcal{M}_n$, $\lambda \in \mathbb{R}$ and unitary matrix $U \in \mathcal{M}_n$ we have the identities

\[
(UAU^*) \oplus (UBU^*) = U(A \oplus B)U^*
\]

\[
\lambda \odot (UAU^*) = U(\lambda \odot A)U^*
\]

\[
\langle UAU^*, UBU^* \rangle_\circ = \langle A, B \rangle_\circ.
\]

**Proof.** Simple application of the formula $\tilde{UAU^*} = U\tilde{A}U^*$.

The tensor product plays a key role in quantum information theory [15, 19] since it describes composite systems. The following theorem shows that the introduced scalar product acts on composite systems as they were direct summand of subspaces.

**Theorem 2.** For states $A_1, A_2 \in \mathcal{M}_n$ and $B_1, B_2 \in \mathcal{M}_m$ the scalar product of their tensor product can be expressed as

\[
\langle A_1 \otimes B_1, A_2 \otimes B_2 \rangle_\circ = \langle A_1, A_2 \rangle_\circ + \langle B_1, B_2 \rangle_\circ.
\]

**Proof.** Elementary computation using Lemma 1.

An immediate consequence of the previous theorem is the Pythagorean theorem for composite quantum systems.

**Theorem 3.** For states $A \in \mathcal{M}_n$ and $B \in \mathcal{M}_m$ we have

\[
\|A \otimes B\|_\circ^2 = \|A\|_\circ^2 + \|B\|_\circ^2.
\]
The relative entropy is a key concept in quantum information theory [15, 19], which can be defined via modular operators for states $D_1, D_2 \in \mathcal{M}_n$ as
\[
S(D_1, D_2) = -\left\langle D_1^{1/2}, \Delta_{D_2/D_1} D_1^{1/2} \right\rangle.
\]
This formula is nothing else but Araki’s definition of the relative entropy in a general von Neumann algebra [5]. Later Petz used the concept of modular operators to define quasi entropies [18] and in 2002 Hiai and Petz presented a unified approach to quantum relative entropies, monotone metrics and quasi entropies via modular operators [13]. Note that the logarithm of the modular operator occurs in the formula. It turns out that the scalar product $\langle \cdot, \cdot \rangle_\circ$ can be expressed in terms of the logarithm of modular operators.

**Theorem 4.** For states $A, B \in \mathcal{M}_n$ the scalar product can be written as
\[
\langle A, B \rangle_\circ = \frac{1}{2n^2} \left\langle \Delta A, \Delta B \right\rangle.
\]
**Proof.** The theorem is equivalent to the equality
\[
\langle A, B \rangle_\circ = \frac{1}{2n^2} \text{Tr} \left( \left( A \otimes A^{-1} \right) \left( B \otimes B^{-1} \right) \right),
\]
which can proved using Lemma 1.

It is worth noting the remarkable similarities of the Equations (3) and (4).

### 2.3 Aitchison geometry on the space of qubits

As an application we present the Aitchison geometry on the space of qubits $\mathcal{M}_2$ via explicit formulae. The state space $\mathcal{M}_2$ can be identified with the interior of the unit ball in the three dimensional Euclidean space with the map
\[
\{(x, y, z) \in \mathbb{R}^3 \mid \| (x, y, z) \| < 1 \} \to \mathcal{M}_2 \quad (x, y, z) \mapsto \frac{1}{2} \begin{pmatrix} 1 + z & x + iy \\ x - iy & 1 - z \end{pmatrix}.
\]
This unit ball is called Bloch ball and its elements are referred to as quantum states. The origin of the Bloch ball is the origin in the Aitchison geometry. The Hilbert space operations on the space $\mathcal{M}_2$ is summarized in the following lemma.

**Lemma 4.** Assume that $D_1$ and $D_2$ are elements in the Bloch ball, such that their Euclidean distance from the origin is $R, r$ and the angle between them is $\vartheta$. We have for their scalar product
\[
\langle D_1, D_2 \rangle_\circ = \text{artanh}(r) \text{artanh}(R) \cos(\vartheta),
\]
the length of the vectors are
\[
\| D_1 \|_\circ = \text{artanh}(R), \quad \| D_2 \|_\circ = \text{artanh}(r),
\]
so the angle between them in the Aitchison geometry is $\vartheta$ too. The distance square between them is
\[
\| D_1 \otimes D_2 \|_\circ^2 = \text{artanh}^2(R) + \text{artanh}^2(r) - 2 \cos(\vartheta) \text{artanh}(R) \text{artanh}(r).
\]
The additive inverse is
\[
\ominus D_1 = I_2 - D_1,
\]
that is the reflection to the origin in the Bloch ball. The multiplication is a dilatation, for $\lambda \in \mathbb{R}^+$ the Euclidean distance the state $\lambda \otimes D_1$ from the origin in the Bloch ball is $\tanh(\lambda \text{artanh}(R))$. 

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Proof. The unitary invariance of the metric implies that the norm of a state in the Bloch ball depends just on the distance from the origin. To simplify the calculations we can assume that the first state lays on the positive part of the axis $z$ and the second state has no $y$ component. Using the parameterization for states

$$D_1(R) = \frac{1}{2} \begin{pmatrix} 1 + R & 0 \\ 0 & 1 - R \end{pmatrix}, \quad D_2(r, \vartheta) = \frac{1}{2} \begin{pmatrix} 1 + r \cos \vartheta & r \sin \vartheta \\ r \sin \vartheta & 1 - r \cos \vartheta \end{pmatrix},$$

calculations gives us Equations (5,6,7,8).

Finally an orthonormal basis $D_1, D_2, D_3$ is presented in $\mathcal{M}_2$ with respect to the scalar product $\langle \cdot, \cdot \rangle_0$.

$$D_1 = \frac{1}{2} \begin{pmatrix} 1 & \tanh 1 \\ \tanh 1 & 1 \end{pmatrix}, \quad D_2 = \frac{1}{2} \begin{pmatrix} 1 & i \tanh 1 \\ -i \tanh 1 & 1 \end{pmatrix}, \quad D_3 = \frac{1}{e + e^{-1}} \begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix}.$$  

3 Connection to Gibbs states

Definition 2. In quantum mechanical setting an operator $H \in \mathcal{M}_{n,sa}$ is called Hamiltonian and for a parameter $\beta \in \mathbb{R}^+$ the state

$$D = \frac{e^{-\beta H}}{Tr e^{-\beta H}}$$

is called Gibbs state of $H$ at inverse temperature $\beta$, or shortly Gibbs state.

Note that the Hamilton operators $H$ and $H + cI_n$ determine the same Gibbs state, so one can assume that the Hamiltonian is traceless. Using the definition of Gibbs states one can assign a state to every Hamiltonian. The counterpart of this mechanism in classical probability theory is called softmax function. Now we define the inverse of this map.

Definition 3. The centered log-ratio transformation is defined as

$$\text{clr} : \mathcal{M}_n \to \mathcal{M}^{0}_{n,sa}, \quad D \mapsto \tilde{D} - \frac{1}{n} I_n \text{Tr} \tilde{D}.$$  

Below we show that the centered log-ratio transformation is scale invariant, linear, isometric and preserve tensor product in some sense.

Theorem 5. For states $A, B \in \mathcal{M}_n$, $C \in \mathcal{M}_m$, scalars $\lambda \in \mathbb{R}$, $c \in \mathbb{R}^+$ and unitary operator $U \in \mathcal{M}_n$ we have

$$\text{clr}(c A) = c \text{clr} A,$$

$$\text{clr}(\lambda \odot A) = \lambda \text{clr} A,$$

$$\text{clr}(A \oplus B) = \text{clr} A + \text{clr} B,$$

$$\langle A, B \rangle_0 = \frac{1}{n} \langle \text{clr} A, \text{clr} B \rangle_n,$$

$$\text{clr}(A \odot C) = \text{clr} A \odot I_m + I_n \odot \text{clr} C,$$

$$\text{clr}(U A U^*) = U (\text{clr} A) U^*.$$  

Proof. Simple application of Lemmas 1, 2 and 3.

This theorem gives an interpretation of the vector operations in the state space. The sum of Gibbs states $D_1 \odot D_2$ generated by Hamiltonians $H_1$ and $H_2$ at the same inverse temperature $\beta$ is just the Gibbs state corresponding to the Hamiltonian $H_1 + H_2$. The state $\lambda \odot D_1$ ($\lambda \in \mathbb{R}$) can be
interpreted as a Gibbs state generated by either $\lambda H_1$ at $\beta$ or $H_1$ at $\lambda \beta$. Finally, $\langle D_1, D_2 \rangle$ is just the Hilbert-Schmidt inner product of the Hamiltonians $(H_1, H_2)$ up to a normalizing factor.

The above mentioned properties of the clr function indicate a simple proof for Theorem 2. For states $A_1, A_2 \in M_n$ and $B_1, B_2 \in M_m$ the calculations below prove the theorem.

\[
(A_1 \otimes B_1, A_2 \otimes B_2)_o = \frac{1}{nm} \langle \text{clr}(A_1 \otimes B_1), \text{clr}(A_2 \otimes B_2) \rangle_n
\]

\[
= \frac{1}{nm} \langle \text{clr}(1) \otimes I_m + I_n \otimes \text{clr}B_1, \text{clr}A_2 \otimes I_m + I_n \otimes \text{clr}B_2 \rangle_n
\]

\[
= \frac{1}{nm} \left( \text{Tr} \left( (\text{clr}A_1 \text{clr}A_2) \otimes I_m \right) + \text{Tr} \left( \text{clr}A_1 \otimes \text{clr}B_2 \right) + \text{Tr} \left( \text{clr}A_2 \otimes \text{clr}B_1 \right) + \text{Tr} \left( I_n \otimes (\text{clr}B_1 \text{clr}B_2) \right) \right)
\]

\[
= \frac{1}{n} \text{Tr} \left( \text{clr}A_1 \text{clr}A_2 \right) + \frac{1}{nm} \text{Tr} \left( \text{clr}A_1 \right) \text{Tr} \left( \text{clr}B_2 \right) + \frac{1}{nm} \text{Tr} \left( \text{clr}A_2 \right) \text{Tr} \left( \text{clr}B_1 \right) + \frac{1}{m} \text{Tr} \left( B_1 \text{clr}B_2 \right)
\]

\[
= \frac{1}{n} \langle \text{clr}(A_1), \text{clr}(A_2) \rangle_n + \frac{1}{m} \langle \text{clr}(B_1), \text{clr}(B_2) \rangle_n
\]

\[
= \langle A_1, A_2 \rangle_o + \langle B_1, B_2 \rangle_o
\]

Aitchison et al. gave an orthonormal basis on the probability simplex in 2002 [12]. Here we present the noncommutative analogue of their basis.

**Theorem 6.** For a given $n$ define $a = \sqrt{\frac{2}{n}}$ and the following matrices.

\[
1 \leq k < l \leq n : \quad A_{kl} = \frac{1}{n - 2 + 2 \cosh a} (I_n + (\cosh a - 1)(E_{kk} + E_{ll}) + (\sinh a)(E_{kl} + E_{lk}))
\]

\[
1 \leq k < l \leq n : \quad B_{kl} = \frac{1}{n - 2 + 2 \cosh a} (I_n + (\cosh a - 1)(E_{kk} + E_{ll}) + (\sinh a)(E_{kl} - E_{lk}))
\]

\[
1 \leq k \leq n - 2 : \quad C_k = \frac{1}{(k + 1) e^a + e^{-(k+1)a} + k - 2} \text{Diag} \left( \frac{e^a, \ldots, e^a, e^{-(k+1)a}, 1, \ldots, 1, e^a}{n-k-2} \right)
\]

where $\alpha = \sqrt{\frac{n}{k^2 + 3k + 2}}$

\[
k = n - 1 : \quad C_{n-1} = \frac{1}{n - a + 2 \cosh a} (I_n + (e^a - 1)E_{11} + (e^{-a} - 1)E_{nn})
\]

The set of matrices $\{A_{kl}, B_{kl}\}_{1 \leq k < l \leq n} \cup \{C_k\}_{1 \leq k \leq n-1}$ form an orthonormal basis in $M_n$ with respect to the scalar product $\langle \cdot, \cdot \rangle_o$.

**Proof.** The matrices $A_{kl}, B_{kl}$ and $C_k$ are the Gibbs states generated by Hamiltonians

\[
1 \leq k < l \leq n : \quad A_{kl} = a(E_{kl} + E_{lk})
\]

\[
1 \leq k < l \leq n : \quad B_{kl} = ia(E_{kl} - E_{lk})
\]

\[
1 \leq k \leq n - 2 : \quad C_k = \sqrt{\frac{n}{k^2 + 3k + 2}} \text{Diag} \left( \frac{1, \ldots, 1, -k + 1, 0, \ldots, 0, 1}{n-k-2} \right)
\]

\[
k = n - 1 : \quad C_{n-1} = a(E_{11} - E_{nn}).
\]
Simple computation shows that these matrices form an orthonormal basis in $M^0_{n,n_0}$ with respect to the normalized Hilbert-Schmidt scalar product $\frac{1}{n} \langle \cdot, \cdot \rangle_n$. According to Theorem 5 the corresponding Gibbs states form an orthonormal basis with respect to $\langle \cdot, \cdot \rangle_\circ$ in $M_n$.

4 Conclusions

In this paper the Aitchison geometry was generalized from classical statistical models to the quantum mechanical finite dimensional state space, that is a Hilbert-space structure on the state space was presented. The intervals in this space was already used in quantum hypothesis testing, as in the classical case. The introduced scalar product, inspired by the classical one, surprisingly preserve the tensor product and can be expressed in terms of modular operators and Hamilton operators. The Hilbert space structure of the state space is turned out to be unitary invariant. The analogue of the log-ratio transformation was given in this quantum setting which helped give an orthonormal basis in the state space. The above mentioned notions were studied in detail on the space of qubits.

As in the classical case, where the Aitchison geometry first was presented on the simplex and later extended to more and more complicated statistical models, in quantum setting, the presented geometry is just the first step in this direction. In the future the generalization to infinite dimensional state spaces case could be the counterpart of the classical continuous case. The generalization to the Radon-Nykodim derivative of states in a von Neumann algebra could be considered as a noncommutative version of the existing extension to Radon-Nykodim derivative of measures.

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