The symmetric square of a curve
and the Petri map

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Abstract

Let $M_g$ be the course moduli space of complex projective nonsingular curves of genus $g$. We prove that when the Brill-Noether number $\rho(g, 1, n)$ is non-negative the Petri locus $P_{g,n} \subset M_g$ has a divisorial component whose closure has a non-empty intersection with $\Delta_0$. In order to prove the result we show that the scheme $G^1_n(\Gamma)$ that parametrizes degree $n$ pencils on a curve $\Gamma$ is isomorphic to a component of the Hilbert scheme parametrizing certain curves on the symmetric square $\Gamma^2$ of $\Gamma$ and we study the properties of such a family of curves.

1 Introduction

This paper is devoted to some aspects of the Petri condition for linear pencils on a complex algebraic curve. Let $C$ be a nonsingular irreducible projective curve of genus $g \geq 2$ defined over $\mathbb{C}$, and let $(L, V)$ be a pair consisting of an invertible sheaf $L$ on $C$ and of an $(r+1)$-dimensional vector subspace $V \subset H^0(L)$, $r \geq 0$. $(L, V)$ is called a linear series of dimension $r$ and degree $n$, or a $g^r_n$. If $V = H^0(L)$ then the $g^r_n$ is said to be complete. A linear pencil is a $g^1_n$.

If $(L, V)$ is a $g^r_n$ then the Petri map for $(L, V)$ is the natural multiplication map

$$\mu_0(L, V) : V \otimes H^0(\omega_C L^{-1}) \longrightarrow H^0(\omega_C)$$

The Petri map for $L$ is

$$\mu_0(L) : H^0(L) \otimes H^0(\omega_C L^{-1}) \longrightarrow H^0(\omega_C)$$

Recall that $C$ is called a Petri curve if the Petri map $\mu_0(L)$ is injective for every invertible sheaf $L$ on $C$. We will say that $C$ is Petri with respect to $g^r_n$’s if the Petri map $\mu_0(L, V)$ is injective for every $g^r_n$ $(L, V)$ on $C$. We say that $C$ is Petri with respect to pencils if it is Petri w.r. to $g^1_n$’s for every $n$.

By the Gieseker-Petri theorem we know that in $M_g$, the course moduli space of nonsingular projective curves of genus $g$, the locus of

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curves which are not Petri is a proper closed subset \( P_g \), called the Petri locus. This locus decomposes as

\[
P_g = \bigcup_{r,n} P_{g,r}^n
\]

where we denote by \( P_{g,r}^n \subset \mathcal{M}_g \) the locus of curves which are not Petri w.r. to \( g_r^n \)'s. The structure of \( P_g \) and of \( P_{g,r}^n \) is not known in general: they might a priori have several components and not be equidimensional.

If the Brill-Noether number

\[
\rho(g, r, n) := g - (r + 1)(g - n + r)
\]

is nonnegative then it is conjectured that \( P_{g,r}^n \) has pure codimension one if it is non-empty. This is known to be true in some special cases, namely when \( \rho(g, r, n) = 0 \), and for \( r = 1 \) and \( n = g - 1 \) [20]. In [8] we proved that if \( \rho(g, r + 1, n) < 0 \) then \( P_{g,r}^n \) has pure codimension one outside \( P_{g,r}^{n+1} \).

When \( g \leq 13 \) then \( P_g \) has pure codimension one [4, 5, 15].

Denote by \( \mathcal{M}_g \) the moduli space of stable curves, and let

\[
\overline{\mathcal{M}_g} \setminus \mathcal{M}_g = \Delta_0 \cup \cdots \cup \Delta_{|g^2|}
\]

be its boundary, in standard notation. In [8] G. Farkas has proved the existence of at least one divisorial component of \( P_{g,1}^n \), in case \( \rho(g, 1, n) \geq 0 \) and \( n \leq g - 1 \), using the theory of limit linear series. He found a divisorial component which has a nonempty intersection with \( \Delta_1 \). Another proof has been given in [6], by degeneration to a stable curve with \( g \) elliptic tails. The method of [8] has been extended in [9] to arbitrary \( r \).

Our main result is:

**Theorem 1.1** Assume that \( g, n \geq 3 \) are such that \( \rho(g, 1, n) \geq 0 \). Then \( P_{g,1}^n \) has a divisorial component whose closure has a non-empty intersection with \( \Delta_0 \).

We obtain this result by induction on \( g \), via a preliminary study of independent interest of certain pencils on curves of genus \( g - 1 \). Namely, we consider a nonsingular curve \( \Gamma \) of genus \( \gamma = g - 1 \) which is Petri w.r. to pencils, and, for a given \( x + y \in \Gamma_2 \), the second symmetric power of \( \Gamma \), we consider those \( g_1^n \)'s \((L, V)\) such that \( x + y \) is contained in one of their divisors: we call them neutral linear pencils w.r. to \( x + y \). They form a divisor \( N_{g,1}^n(x+y) \) of \( G_1^n(\Gamma) \) which is closely related to the set of generalized pencils on the nodal curve \( X \) of arithmetic genus \( g \) obtained from \( \Gamma \) after identifying \( x = y \) (see Definition 7.3). We undertake a detailed study of the divisors \( N_{g,1}^n(x+y) \), of their singular locus, and of their relation with generalized pencils on \( X \). We prove that the only generalized \( g_1^n \)'s on \( X \) which are non-Petri are those corresponding to singular elements of \( N_{g,1}^n(x+y) \) for which neither \( x \) nor \( y \) are base points. Then we show that those \( x + y \in \Gamma_2 \) for which \( N_{g,1}^n(x+y) \) contains such singularities are contained in a curve \( R_n \subset \Gamma_2 \), and that in such a case the singularities of that sort are finitely many (Proposition 6.2). In particular, by choosing \( x + y \) outside all such curves for the possible (finite) set of values of \( n \), the curve \( X \) will be Petri w.r. to generalized pencils. From this result we
obtain an inductive proof of the Gieseker-Petri theorem for pencils. The proof of Proposition 6.2 uses Proposition 2.1 asserting that in \( \Gamma \) every invertible sheaf with \( h^0(L) \geq 2 \) satisfies \( H^1(L^2(-x)) = 0 \) for all \( x \in \Gamma \). This statement has far-reaching consequences and has been the starting point of this work.

A key role here is played by the curves \( s_{L,V} \subset \Gamma_2 \) associated to \( g_1^n \)'s (\( L, V \)), consisting of all \( x + y \in \Gamma_2 \) which are contained in some divisor of \( (L, V) \). The family of all such curves is studied in detail: in particular we show that the scheme \( G_1^n(\Gamma) \) of \( g_1^n \)'s, which parametrizes the curves \( s_{L,V} \), is isomorphic to a component of the Hilbert scheme of \( \Gamma_2 \) (Theorem 5.6).

A closer look at the above mentioned curve \( R_n \subset \Gamma_2 \) shows that it does not contain the diagonal. For this part we use the main result of [7], which implies that \( \Delta_1 \subset \mathcal{M}_g \) is not contained in the closure of \( P_g \). We also need to prove that \( R_n \) is non-empty. Here we apply a theorem of Steffen [19] to the characteristic map of the family of \( s_{L,V} \)'s (Proposition 6.1).

With the aid of all the previous results we prove Theorem 1.1 as follows. We fix a non-Petri nodal curve \( X \) as above and we consider a modular family of stable genus \( g \) curves \( f : X \longrightarrow B \) containing \( X \) among its fibres. A straightforward construction produces an irreducible variety \( \tilde{P} \) of dimension \( \geq 3g - 4 \) parametrizing pairs \( (C, (L, V)) \) such that \( (L, V) \) is non-Petri and a morphism \( \tilde{P} \longrightarrow B \). Since this map is finite above \( X \) it follows that its image is a divisorial component \( P \) of \( P_{g,n}^1 \). This component contains \( X \) and therefore it intersects \( \Delta_0 \).

The structure of the paper is as follows. In §2 we discuss the properties of the Petri map for pencils. In §3 we recall some cohomological properties of line bundles on \( \Gamma_2 \) and we introduce and study the curves \( s_{L,V} \). The schemes of neutral linear series are introduced in general in §4 and they are studied in detail in §5 in the special case of pencils. In §6 we introduce and study the curves \( R_n \). Then in §7 we interpret in terms of nodal curves what has been proved before. The final §8 is devoted to the proof of Theorem 1.1.

## 2 The Petri map for pencils

Let \( \Gamma \) be a projective nonsingular irreducible curve of genus \( \gamma \geq 3 \), \( L \) a line bundle on \( \Gamma \) and \( V \subset H^0(L) \) a subspace. If \( D \) is an effective divisor on \( \Gamma \) we define

\[
V(-D) := V \cap H^0(L(-D))
\]

and

\[
V(D) := j(V)
\]

where \( j : H^0(L) \longrightarrow H^0(L(D)) \) is the inclusion.

Assume that \( (L, V) \) is a base-point free pencil on \( \Gamma \), i.e. a linear pencil such that \( V \) generates \( L \). The Koszul complex of \( V \) gives the exact sequence:

\[
0 \longrightarrow \Lambda^2 V \otimes L^{-2} \longrightarrow V \otimes L^{-1} \longrightarrow \mathcal{O}_\Gamma \longrightarrow 0
\]  

(1)

If we tensor it by \( LM \) where \( M \) is any line bundle, we obtain:

\[
0 \longrightarrow \Lambda^2 V \otimes L^{-1}M \longrightarrow V \otimes M \longrightarrow LM \longrightarrow 0
\]
and taking global sections we obtain the exact sequence:

\[ 0 \longrightarrow \bigwedge^2 V \otimes H^0(L^{-1}M) \longrightarrow V \otimes H^0(M) \longrightarrow H^0(LM) \tag{2} \]

where \( m \) is the multiplication map. Therefore we have a canonical identification:

\[ \ker(m) = \bigwedge^2 V \otimes H^0(L^{-1}M) \tag{3} \]

If we take \( M = \omega_{\Gamma}L^{-1} \) then (2) becomes

\[ 0 \longrightarrow \bigwedge^2 V \otimes H^0(\omega_{\Gamma}L^{-2}) \longrightarrow V \otimes H^0(\omega_{\Gamma}L^{-1}) \longrightarrow H^0(\omega_{\Gamma}) \tag{4} \]

where \( \mu_0 = \mu_0(L,V) \). Therefore we have a canonical identification:

\[ \ker(\mu_0(L,V)) = \bigwedge^2 V \otimes H^0(\omega_{\Gamma}L^{-2}) \]

After tensoring (1) by \( L^2 \) we have an exact sequence:

\[ 0 \longrightarrow \bigwedge^2 V \otimes \mathcal{O}_\Gamma \longrightarrow V \otimes L \longrightarrow L^2 \longrightarrow 0 \tag{5} \]

which induces:

\[ \bigwedge^2 V \otimes H^1(\mathcal{O}_\Gamma) \longrightarrow V \otimes H^1(L) \longrightarrow H^1(L^2) \longrightarrow 0 \tag{6} \]

and dualizing:

\[ 0 \longrightarrow H^0(\omega_{\Gamma}L^{-2}) \longrightarrow V^\vee \otimes H^0(\omega_{\Gamma}L^{-1}) \longrightarrow \bigwedge^2 V^\vee \otimes H^0(\omega_{\Gamma}) \]

If we now tensor by \( \bigwedge^2 V \) and recall that \( V = \bigwedge^2 V \otimes V^\vee \) we obtain the sequence (6) again. This implies that \( \mu_0(L,V) \) and \( \rho \) are the same map. Therefore we have an isomorphism

\[ \ker(\mu_0(L,V)) \cong \ker(\rho) = H^1(L^2)^\vee \tag{7} \]

Suppose now that \( (L,V) \) is any linear pencil on \( \Gamma \) and let \( B \) be its fixed divisor. Then, applying the above analysis to the fixed point free linear pencil \( (L(-B),V(-B)) \) we obtain that, for any line bundle \( M \),

\[ \ker[V \otimes H^0(M)] \cong \ker[H^0(LM(-B))] \]

In particular we obtain:

\[ \ker[\mu_0(L,V)] \cong \ker[V(-B) \otimes H^0(\omega_{\Gamma}L^{-1})] \cong H^0(\omega_{\Gamma}L^{-2}(B)) \]

We have the following proposition:

**Proposition 2.1** If \( \Gamma \) is Petri with respect to pencils then:

(i) \( H^1(L^2) = 0 \) for every \( L \in \text{Pic}(\Gamma) \) such that \( h^0(L) \geq 2. \)
(ii) \( H^1(L^2(-x)) = 0 \) for every \( L \in \text{Pic}(\Gamma) \) such that \( h^0(L) \geq 2 \) and for every \( x \in \Gamma \).

**Proof.** We only need to prove (ii). Let \( L \in \text{Pic}(\Gamma) \) be such that \( h^0(L) \geq 2 \) and assume that \( H^1(L^2(-x)) \neq 0 \) for some \( x \in \Gamma \). Consider the exact sequence:

\[
0 \rightarrow L^2(-x) \rightarrow L^2 \rightarrow L^2 \otimes \mathcal{O}_x \rightarrow 0
\]

If \( x \) is not base-point of \( L \) then \( h^1(L^2) > 0 \) and this contradicts (i). If \( x \) is base-point of \( L \) then it is also a base point of \( L \). Therefore \( L = M(x) \) with \( h^0(M) = h^0(L) \geq 2 \). We have \( L^2(-x) = M^2(x) \) so that:

\[
0 < h^1(L^2(-x)) = h^1(M^2(x)) \leq h^1(M^2)
\]

again contradicting (i).

\[\square\]

A special case of part (ii) of the proposition is in [18], Lemma 4.2.

Some of these remarks can be extended to linear series of higher dimension, as follows. If \( \dim(V) = r + 1 \geq 3 \) and \( V \) generates \( L \), from the Koszul complex we obtain the exact sequence:

\[
0 \rightarrow \bigwedge^{r+1} V \otimes L^{-r-1} \rightarrow \bigwedge^r V \otimes L^{-r} \rightarrow \varphi_V^* \Omega_p^{r-1} \rightarrow 0
\]

where \( \varphi_V : \Gamma \rightarrow P(V^\vee) = \mathbb{P} \) is the morphism defined by \( V \). After tensoring by \( L^{r+1} \) and taking cohomology we obtain the exact sequence:

\[
\bigwedge^{r+1} V \otimes H^1(\mathcal{O}_x) \rightarrow \bigwedge^r V \otimes H^1(L) \rightarrow H^1(\varphi_V^* \Omega_p^{r-1}(r + 1)) \rightarrow 0
\]

Again, the map \( \rho^V \) can be identified with \( \mu_0(L, V)^{\vee} \), after having identified

\[
V^\vee = \bigwedge^{r+1} V \otimes \bigwedge^r V
\]

and we find the well-known isomorphism:

\[
\ker(\mu_0(L, V)) \cong H^1(\Gamma, \varphi_V^* \Omega_p^{r-1}(r + 1))^{\vee}
\]

### 3 The second symmetric power

Let \( \Gamma_2 \) be the second symmetric power of \( \Gamma \). Denote by

\[
\sigma : \Gamma \times \Gamma \rightarrow \Gamma_2
\]

the natural degree two quotient morphism and by

\[
\Gamma \xleftarrow{\pi_1} \Gamma \times \Gamma \xrightarrow{\pi_2} \Gamma
\]

the projections. Given \( L, M \) invertible sheaves on \( \Gamma \) we will let

\[
L \boxtimes M := \pi_1^* L \otimes \pi_2^* M
\]
Let $D \subset \Gamma \times \Gamma$ be the diagonal and

$$\delta = \sigma(D) = \{2x : x \in \Gamma\} \subset \Gamma^2$$

We have

$$\sigma^*(\delta) = 2D$$

Moreover, the invertible sheaf $\mathcal{O}_{\Gamma^2}(\delta)$, which we will shortly denote by $\delta$, is divisible by two. Precisely, by general properties of double covers, there is an invertible sheaf $\delta/2$ on $\Gamma^2$ satisfying:

$$\sigma^* \mathcal{O}_{\Gamma \times \Gamma} = \mathcal{O}_{\Gamma^2} \oplus \mathcal{O}_{\Gamma^2}(-\delta/2)$$

and

$$\sigma^*(\delta/2) = D$$

(9)

In particular we have:

$$(\delta/2 \cdot \delta/2) = 1 - \gamma, \quad \mathcal{O}_{\delta}(-\delta/2) = \omega_{\delta}$$

(10)

The numerical class of the irreducible curve in $\Gamma^2$:

$$x + \Gamma = \{x + y : y \in \Gamma\} \subset \Gamma^2$$

is independent of $x \in \Gamma$ and will be denoted by $X$. If $L = \mathcal{O}(\sum n_i x_i)$ is an invertible sheaf on $\Gamma$ we will denote by $LX$ the invertible sheaf $\mathcal{O}_{\Gamma^2}(\sum n_i (x_i + \Gamma))$ on $\Gamma^2$. If $\deg(L) = n$ then

$$LX \equiv_{num} nX$$

It is immediate to check that

$$\sigma^*(LX) = L \boxtimes L$$

(11)

and

$$X \cdot \delta/2 = 1 = X^2$$

(12)

Lemma 3.1

$$\omega_{\Gamma^2} = \omega_{\Gamma}X - \delta/2$$

Proof. It is a special case of a well-known formula of Mattuck [16]. Let’s give a direct proof. Since we have

$$\omega_{\Gamma \times \Gamma} = \omega_{\Gamma} \boxtimes \omega_{\Gamma}$$

by the Hurwitz formula for the double cover $\sigma$ (2, (19) p. 41) we obtain:

$$\omega_{\Gamma} \boxtimes \omega_{\Gamma}(-D) = \sigma^* \omega_{\Gamma^2}$$

On the other hand, by (9) and (11) we have:

$$\sigma^* \left[\omega_{\Gamma}X - \frac{\delta}{2}\right] = \omega_{\Gamma} \boxtimes \omega_{\Gamma}(-D)$$

In other words:

$$\sigma^* \omega_{\Gamma^2} = \sigma^* \left[\omega_{\Gamma}X - \frac{\delta}{2}\right]$$

Since $\sigma^* : \text{Pic}(\Gamma^2) \to \text{Pic}(\Gamma \times \Gamma)$ is injective it follows that:

$$\omega_{\Gamma^2} = \omega_{\Gamma}X - \frac{\delta}{2}$$

\[\square\]
Remark 3.2 Comparing (10) with the adjunction formula we obtain:

\[ \mathcal{O}_\delta(-\delta/2) = \mathcal{O}_\delta(\omega_\Gamma X + \delta/2) \]

and therefore \( \mathcal{O}_\delta(\omega_\Gamma X + \delta) = \mathcal{O}_\delta \). This implies that the effective divisor \( \omega_\Gamma X + \delta \) on \( \Gamma_2 \) is not ample.

Lemma 3.3 Let \( L \in \text{Pic}(\Gamma) \). Then

\[
\begin{align*}
H^0(\Gamma, LX) & = S^2H^0(\Gamma, L) \\
H^1(\Gamma, LX) & \cong H^0(\Gamma, L) \otimes H^1(\Gamma, L) \\
H^2(\Gamma, LX) & = \bigwedge^2 H^1(\Gamma, L)
\end{align*}
\]

and

\[
\begin{align*}
H^0(\Gamma, LX - \delta/2) & = \bigwedge^2 H^0(\Gamma, L) \\
H^1(\Gamma, LX - \delta/2) & \cong H^0(\Gamma, L) \otimes H^1(\Gamma, L) \\
H^2(\Gamma, LX - \delta/2) & = S^2H^1(\Gamma, L)
\end{align*}
\]

Proof. The involution \( \iota : \Gamma \times \Gamma \to \Gamma \times \Gamma \) acts on

\[
H^0(\Gamma \times \Gamma, L \boxtimes L) = H^0(\Gamma, L) \otimes H^0(\Gamma, L) = S^2H^0(\Gamma, L) \oplus \bigwedge^2 H^0(\Gamma, L)
\]

by interchanging the factors in the tensor product. On the other hand, by the projection formula we have

\[
H^0(\Gamma \times \Gamma, L \boxtimes L) = H^0(\Gamma \times \Gamma, \sigma^*(LX)) = H^0(\Gamma_2, LX) \oplus H^0(\Gamma_2, LX - \delta/2)
\]

and the two summands are respectively the \( \iota \)-invariant and \( \iota \)-antinvariant part. This proves the equalities for \( H^0 \)’s. The equalities for \( H^2 \)’s are proved similarly using Serre duality and the expression of \( \omega_{\Gamma_2} \).

The isomorphisms for \( H^1 \)’s follow from the Kunneth formula and direct computation.

Lemma 3.4 There are natural identifications:

(i) \( H^1(\Gamma_2, \mathcal{O}_{\Gamma_2}) = H^1(\Gamma, \mathcal{O}_\Gamma) \)

(ii) \( H^2(\Gamma_2, \mathcal{O}_{\Gamma_2}) = \bigwedge^2 H^1(\Gamma, \mathcal{O}_\Gamma) = H^0(\Gamma_2, \omega_{\Gamma_2})^{\vee} \)

(iii) \( H^1(\Gamma_2, \omega_{\Gamma_2}) = H^0(\Gamma, \omega_\Gamma) \)

Proof. It is a special case of Lemma 3.3.
Remark 3.5  After making the natural identification $\delta = \Gamma$, from Lemma 3.3 and from (10) and (12) it follows that the restriction map:

$$H^0(\Gamma_2, LX) \to H^0(\delta, LX \otimes O_{\delta})$$

is identified with the natural restriction:

$$S^2 H^0(\Gamma, L) \to H^0(\Gamma, L^2)$$

and therefore

$$H^0(\Gamma_2, LX - \delta) = \ker \left[ S^2 H^0(\Gamma, L) \to H^0(\Gamma, L^2) \right]$$

Similarly the map:

$$H^0(\Gamma_2, LX - \delta/2) \to H^0(\delta, (LX - \delta/2) \otimes O_{\delta})$$

is identified with the Wahl map:

$$w_L : \Lambda^2 H^0(\Gamma, L) \to H^0(\Gamma, \omega_{\Gamma} \otimes L^2)$$

and $H^0(\Gamma_2, LX - 3\delta/2) = \ker(w_L)$.

Consider the universal divisor

$$\Delta := \{(x, x + y) : x, y \in \Gamma \} \subset \Gamma \times \Gamma_2$$

and the diagram:

$$\begin{array}{ccc}
\Delta & \xrightarrow{\epsilon} & \Gamma \times \Gamma_2 \\
\downarrow p & & \downarrow q \\
\Gamma & & \Gamma_2
\end{array}$$

We have an isomorphism:

$$\epsilon : \Gamma \times \Gamma \to \Delta$$

and we can identify the map $\sigma$ with the composition

$$\Gamma \times \Gamma \xrightarrow{\epsilon} \Delta \xrightarrow{q} \Gamma_2$$

Given a line bundle $L$ on $\Gamma$ we can consider the locally free sheaf of rank two on $\Gamma_2$:

$$E_L := q_* (p^* L|_\Delta)$$

which is called the secant bundle of $L$. Since

$$p \cdot \epsilon = \pi_1 : \Gamma \times \Gamma \to \Gamma$$

we may as well write:

$$E_L = \sigma_* (\pi_1^* L)$$
Let \((L, V)\) be a \(g^1\) on \(\Gamma\). On \(\Gamma_2\) consider the composition

\[
\phi : V \otimes \mathcal{O}_{\Gamma_2} \xrightarrow{\cdot q^*p^*L} E_L
\]

where the second homomorphism is obtained by pushing down the restriction homomorphism on \(\Gamma \times \Gamma_2\):

\[
p^*L \xrightarrow{p^*\mathcal{O}_{\Gamma_2}} q^*p^*L_{|\Delta}
\]

\(\phi\) is a homomorphism of locally free sheaves of rank two. We define an effective divisor \(s_{L,V}\) on \(\Gamma_2\) by:

\[
s_{L,V} := D_1(\phi) = D_0(\bigwedge^2 \phi)
\]

where

\[
\bigwedge^2 \phi : \bigwedge^2 V \otimes \mathcal{O}_{\Gamma_2} \rightarrow \bigwedge^2 E_L
\]

Set-theoretically we have:

\[
s_{L,V} = \{x + y \in \Gamma_2 : \dim[V(-x - y)] \geq 1\}
\]

**Lemma 3.6** Let \((L, V)\) be a \(g^1\) on \(\Gamma\).

(i) We have

\[
\mathcal{O}(s_{L,V}) = \bigwedge^2 V \otimes \bigwedge^2 E_L \cong L^X - \delta/2
\]

(ii) \(H^0(\Gamma_2, \mathcal{O}_{\Gamma_2}(s_{L,V})) = \bigwedge^2 V \otimes H^0(\Gamma, L)\)

\(H^1(\Gamma_2, \mathcal{O}_{\Gamma_2}(s_{L,V})) = \bigwedge^2 V \otimes H^0(\Gamma, L) \otimes H^1(\Gamma, L)\)

\(H^2(\Gamma_2, \mathcal{O}_{\Gamma_2}(s_{L,V})) = \bigwedge^2 V \otimes S^2 H^1(\Gamma, L)\)

(iii) The arithmetic genus \(g(s_{L,V})\) of \(s_{L,V}\) satisfies:

\[
2g(s_{L,V}) - 2 = (n - 2)(n + 2\gamma - 3) - 2
\]

and

\[
s_{L,V} \cdot s_{L,V} = (n - 1)^2 - \gamma
\]

**Proof.** (i) See [10], Ex. 14.4.17, p. 263. (ii) follows from Lemma 3.3. (iii) is left to the reader. \(\square\)

Let \((L, V)\) be a \(g^1\) on \(\Gamma\), and let:

\[
t_{L,V} := \{(x, y) : V(-x - y) \neq 0\} \subset \Gamma \times \Gamma
\]

Consider the morphisms:

\[
\begin{array}{cc}
\pi_1 & t_{L,V} \\
\Gamma & s_{L,V}
\end{array}
\]

(13)
where $\pi_1$ is induced by the first projection. Assume that $(L, V)$ is base-point free. Then the above diagram can be completed as:

$$
\begin{array}{c}
\pi_1 \\
\downarrow \\
\sigma \\
\downarrow \\
\varphi_{L,V} \\
\downarrow f \\
\Gamma \\
\end{array}
\xrightarrow{t_{L,V}}
\begin{array}{c}
\downarrow \\
s_{L,V} \\
\end{array}
\xrightarrow{\varphi_{L,V}}
\begin{array}{c}
P(V^*) \\
\end{array}
$$

where $f$ is the obvious morphism. The degrees of the morphisms are:

$$
\deg(\pi_1) = n - 1, \quad \deg(\varphi_{L,V}) = n, \quad \deg(\sigma) = 2, \quad \deg(f) = \binom{n}{2}
$$

Lemma 3.7 Let $(L, V)$ be a $g^1_n$ on $\Gamma$. Then:

(i) $t_{L,V}$ and $s_{L,V}$ are both connected.

(ii) If $(L, V)$ is base-point free and with simple ramification then $t_{L,V}$ and $s_{L,V}$ are both irreducible and nonsingular.

(iii) Assume that $\Gamma$ is very general and that $n \leq \gamma - 1$. If $s_{L,V}$ is reducible then the fixed divisor $B$ of $(L, V)$ is positive and $s_{L,V} = BX \cup s_{L(-B), V(-B)}$.

Proof. (i) Let’s prove that $s_{L,V}$ is connected. We may assume that $n \geq 2$. Since $s_{L,V}$ is effective we have $H^0(\mathcal{O}_{\Gamma_2}(-s_{L,V})) = 0$. Therefore from the exact sequence on $\Gamma_2$:

$$
0 \longrightarrow \mathcal{O}(-s_{L,V}) \longrightarrow \mathcal{O}_{\Gamma_2} \longrightarrow \mathcal{O}_{s_{L,V}} \longrightarrow 0
$$

we see that it suffices to prove that $H^1(\mathcal{O}_{\Gamma_2}(-s_{L,V})) = 0$. Consider the exact sequence:

$$
0 \longrightarrow \mathcal{O}(-LX - \frac{4}{3}) \longrightarrow \mathcal{O}(-s_{L,V}) \longrightarrow \mathcal{O}_B(-s_{L,V}) \longrightarrow 0
$$

We have $H^1(\mathcal{O}(-LX - \frac{4}{3}) = H^0(-L) \otimes H^1(-L) = 0$ (Lemma 3.3). Thus it suffices to prove that $\partial : H^1(\mathcal{O}_B(-s_{L,V})) \longrightarrow H^2(\mathcal{O}(-LX - \frac{4}{3})$ is injective, or equivalently that $\partial^\gamma$ is surjective. But $\mathcal{O}_B(-s_{L,V}) = \mathcal{O}_\Gamma(-\omega_{\Gamma} - 2L)$ and $H^2(\mathcal{O}(-LX - \frac{4}{3}) = S^2H^1(\Gamma, -L)$ (Lemma 3.3). Therefore

$$
\partial^\gamma : S^2H^0(\omega_{\Gamma} + L) \longrightarrow H^0(2(\omega_{\Gamma} + L))
$$

This map is the natural multiplication, and it is surjective because $\deg(\omega_{\Gamma} + L) \geq 2g + 1$ ([17]). This concludes the proof of the connectedness of $s_{L,V}$. Since $\sigma$ is ramified over the points $2x \in s_{L,V}$, where $x \in \Gamma$ is a ramification point of $\varphi_{L,V}$, we deduce that $t_{L,V}$ is connected as well.
(ii) Assume that \((L, V)\) is base-point free. Let \(\text{Sing}(t_{L,V}) \subset t_{L,V}\) and \(\text{Sing}(s_{L,V}) \subset s_{L,V}\) be the singular loci. Since \(\Gamma\) and \(P(V^*)\) are nonsingular we have:

\[
\text{Sing}(s_{L,V}) \subset \text{Ram}(f) = \{x + y \in s_{L,V} : \text{either } x \text{ or } y \text{ is a ram. pt of } (L, V)\}
\]

\[
\text{Sing}(t_{L,V}) \subset \text{Ram}((\pi)) = \{(x, y) : V(-x - 2y) \neq 0\}
\]

Assume now that \((L, V)\) has only simple ramification and that \((x, y) \in \text{Ram}(\pi_1)\). Then \(x \neq y\) because otherwise \(V(-3x) \neq 0\). Moreover \((y, x) \notin \text{Ram}(\pi_1)\) because otherwise \(V(-2x - 2y) \neq 0\). Therefore \((y, x) \notin \text{Sing}(t_{L,V})\).

It follows that \(\sigma(x, y) = \sigma(y, x) \notin \text{Sing}(s_{L,V})\) because \(\sigma\) is etale over \(x + y\). But then also \((x, y) \notin \text{Sing}(t_{L,V})\). The conclusion is that \(t_{L,V}\) and \(s_{L,V}\) are both nonsingular. Since they are connected they are irreducible as well.

(iii) If \(\Gamma\) is very general, from [ACGH] Lemma p. 359, it follows that \(\text{NS}(\Gamma_2) = \langle x, \delta/2 \rangle\), so that in particular the numerical class of every effective curve is a combination of \(x\) and \(\delta/2\) with integer coefficients. Let us assume that \(s_{(L,V)} = A \cup B\), where \(A\) and \(B\) are effective curves on \(\Gamma_2\).

We write \(A = a_1x + a_2\delta/2\), \(B = b_1x + b_2\delta/2\) for their numerical classes.

We have \(a_2 + b_2 = -1\) and we will assume that \(a_2 \geq b_2\).

Suppose that \(a_2 \geq 1\), so that \(b_2 \leq -2\). Since \(\delta\) is irreducible and is not contained in any component of \(s_{(L,V)}\) we have

\[\delta \cdot A \geq 0\]

so that

\[a_1 \geq a_2(\gamma - 1) \geq n.\]

It then follows that \(b_1 = n - a_1 \leq 0\). But then \(x \cdot B = b_1 + b_2 \leq -2\), and \(x\) is the class of an ample divisor.

It then follows that \(a_2 \leq 0\), so that in fact \(a_2 = 0\) because \(a_2 + b_2 = -1\) and \(a_2 \geq b_2\). Then \(A = a_1x\) numerically and in particular \((L, V)\) has a base locus of degree \(a_1\). If \(B\) is reducible we repeat the argument until we get down to a base point free pencil.

\[\Box\]

Remarks 3.8

(a) If \((L, V)\) does not have simple ramification then \(t_{L,V}\) and \(s_{L,V}\) can be singular. Assume for example that the pencil contains a divisor of the form \(2x + 2y + z_1 + \cdots + z_{n-4}\), with \(x \neq y\). Then \(V(-x - 2y) \neq 0 \neq V(-2x - y)\). This implies that \((x, y) \in t_{L,V}\) is in \(\text{Ram}(\pi_1) \cap \text{Ram}(\pi_2)\), where \(\pi_1, \pi_2 : \Gamma \times \Gamma \to \Gamma\) are the projections. Therefore \((x, y) \in \text{Sing}(t_{L,V})\), and \(\sigma(x, y) \in \text{Sing}(s_{L,V})\).

(b) Part (iii) of Lemma [5,7] will not be needed in the rest of the paper.

(c) The curves \(t_{L,V}\) are also considered in the recent preprint [12], where they are called trace curves.

4 The schemes of neutral linear series

We shall adopt the standard notation \(G^r_n(\Gamma)\) to denote the scheme of \(g^r_n\)'s on the curve \(\Gamma\). For its definition and main properties we refer to [11].
Definition 4.1 \( \text{Let } (L, V) \text{ be a } g^n_r \text{ on } \Gamma \text{ and } x + y \in \Gamma_2. \) We say that \((L, V)\) is neutral w.r. to \(x + y\) if \(\dim(V(-x - y)) \geq r\).

Consider the following diagram where all the arrows are projections:

\[
\begin{array}{c}
\Gamma \times \Gamma_2 \xrightarrow{\pi_{12}} \Gamma \times \Gamma_2 \times G^r_n \xrightarrow{\pi_{13}} \Gamma \times G^r_n \\
\downarrow \quad \downarrow \quad \downarrow \\
\Gamma_2 \xrightarrow{\pi_2} \Gamma_2 \times G^r_n \xrightarrow{\pi_2} G^r_n
\end{array}
\]

and where \(G^r_n = G^r_n(\Gamma)\). Let \(P\) be the pullback on \(\Gamma \times G^r_n\) of a Poincaré line bundle on \(\Gamma \times \text{Pic}^n(\Gamma)\) and let \(E^r_n \subset u_\ast P\) be the tautological locally free subsheaf of rank \(r + 1\) on \(G^r_n\).

We consider on \(\Gamma \times \Gamma_2 \times G^r_n\) the natural restriction homomorphism:

\[
\pi_2^\ast P \rightarrow \pi_{13}^\ast P \otimes \pi_{12}^\ast \mathcal{O}_\Delta
\]

Then we push it down to \(\Gamma_2 \times G^r_n\) and we compose with the inclusion

\[
\pi_2^\ast E^r_n \subset \pi_2^\ast u_\ast P = \pi_{23}^\ast \pi_{13}^\ast P
\]

We obtain a homomorphism of locally free sheaves on \(\Gamma_2 \times G^r_n\):

\[
\Phi : \pi_2^\ast E^r_n \rightarrow \pi_{23}^\ast [\pi_{13}^\ast P \otimes \pi_{12}^\ast \mathcal{O}_\Delta]
\]

of ranks \(r + 1\) and two respectively. Consider the closed subscheme:

\[
N^r_n(\Gamma) := D_1(\Phi) \subset \Gamma_2 \times G^r_n
\]

and the projections:

\[
N^r_n(\Gamma) \xleftarrow{p_1} \Gamma_2 \times G^r_n \xrightarrow{p_2} G^r_n
\]

For each \(x + y \in \Gamma_2\) we have

\[
p_1^{-1}(x + y) = D_1(\Phi(x + y)) \subset G^r_n
\]

where

\[
\Phi(x + y) : E^r_n \xrightarrow{u_\ast P} P \otimes \mathcal{O}_{x + y}
\]

We define

\[
N^r_n(x + y) := p_1^{-1}(x + y)
\]

Set-theoretically:

\[
N^r_n(x + y) = \{(L, V) : \dim(V(-x - y)) \geq r\}
\]

Therefore we call \(N^r_n(x + y)\) the scheme of neutral linear \(g^n_r\)'s with respect to \(x + y\). If \(r = 1\) then for every \((L, V) \in G^n_1\) we have an isomorphism:

\[
p_2^{-1}(L, V) \cong s_{L, V}
\]

In particular we see that \(N^1_n(\Gamma) \neq \Gamma_2 \times G^1_n\) and therefore it is a proper divisor in \(\Gamma_2 \times G^1_n\).
The case of pencils

In this section we assume that $\Gamma$ is Petri w.r. to pencils and that

$$\gamma \geq \rho(\gamma, 1,n) := 2n - 2 - \gamma \geq 1 \quad (17)$$

Lemma 5.1 Under the assumptions above $G^1_n = G^1_n(\Gamma) \neq \emptyset$ is nonsingular connected of dimension $\rho(\gamma, 1, n)$, and consists generically of complete and base point free $g^1_n$’s.

Proof. For the first assertion see [1], Prop. 4.1 p. 187. If a component of $G^1_n$ consists generically of $g^1_n$’s with base points, then $G^1_{n-1}$ has a component of dimension $\geq \rho(\gamma, 1, n) = \rho(\gamma, 1, n - 1) + 2$, contradicting the hypothesis that $\Gamma$ is Petri w.r. to pencils. A similar argument holds in case a component of $G^1_n$ consists of incomplete pencils. \qed

We have a natural inclusion:

$$\Gamma_2 \times G^1_{n-2} \rightarrow \Gamma_2 \times G^1_n$$

$$(x + y, (A, U)) \rightarrow (x + y, (A + x + y), U(x + y))$$

Lemma 5.2

$$\iota(\Gamma_2 \times G^1_{n-2}) = D_0(\Phi) \subset \text{Sing}(N^1_n(\Gamma))$$

Proof. $\iota(\Gamma_2 \times G^1_{n-2})$ is supported on the points $(x + y, (L, V)) \in \Gamma_2 \times G^1_n$ such that $x + y$ is contained in the fixed divisor of $(L, V)$, and these are exactly the points where $\Phi$ vanishes. The inclusion is obvious. \qed

Note that $\Gamma_2 \times G^1_{n-2} = \emptyset$ if $\rho(\gamma, 1, n) \leq 3$, and it is nonsingular of pure dimension

$$\dim(\Gamma_2 \times G^1_{n-2}) = \rho(\gamma, 1, n - 2) + 2 = \rho(\gamma, 1, n) - 2$$

if $\rho(\gamma, 1, n) \geq 4$.

Lemma 5.3 Let $(L, V) \in N^1_n(x + y)$ for some $x + y \in \Gamma_2$, and consider the multiplication map:

$$M : V \otimes H^0(\omega_L^{-2}(x + y)) \rightarrow H^0(\omega_L(x + y))$$

Let $B$ be the fixed divisor of $(L, V)$. Then

(a) $\ker(M) \cong H^0(\omega_L^{-2}(x + y + B)) \cong H^1(L^2(-x - y - B))^\vee$

(b) $\dim[\ker(M)] \leq 1$.

(c) If $x < B$ then $M$ is injective.

Proof. (a) is a direct consequence of [5] and (b) is left to the reader. (c) From Proposition 2.11i it follows that $H^1(L^2(-2B)) = 0$, and then also $H^1(L^2(-2B - y)) = 0$, by Prop. 2.11i. But, since $x < B$, we have a surjection

$$H^1(L^2(-2B - y)) \twoheadrightarrow H^1(L^2(-x - y - B)) \cong \ker(M)^\vee$$

and therefore $\ker(M) = 0$. \qed
Proposition 5.4 Let \((x + y, (L, V)) \in N^1_n(\Gamma)\). Then \((L, V)\) is a nonsingular point of \(N^1_n(x + y)\) if and only if \((x + y, (L, V)) \notin \iota(\Gamma_2 \times G^1_{n - 2})\) and \(M\) is injective.

Proof. For every \(x + y \in \Gamma_2\) the points of \(N^1_n(x + y) \cap \iota(\Gamma_2 \times G^1_{n - 2})\) are singular for \(N^1_n(x + y)\), by Lemma 5.2. Therefore we can assume that \((x + y, (L, V)) \in N^1_n(\Gamma) \setminus \iota(\Gamma_2 \times G^1_{n - 2})\). We need to describe the tangent space \(T_{(L,V)}N^1_n(x + y) \subset T_{(L,V)}G^1_n\). Note that

\[\rho - 1 \leq \dim \left[T_{(L,V)}N^1_n(x + y)\right] \leq \rho\]  

(18)

where \(\rho := \rho(\gamma, 1, n)\) and the first equality holds if and only if \(N^1_n(x + y)\) is nonsingular of dimension \(\rho - 1\) at \((L, V)\). Denote by

\[\pi : G^1_n \longrightarrow \text{Pic}^n(\Gamma)\]

the Abel-Jacobi map. Consider the well-known exact sequence ([1], p. 187):

\[0 \longrightarrow \text{Hom}(V, H^0(L)/V) \longrightarrow T_{(L,V)}N^1_n(x + y) \longrightarrow T_{(L,V)}G^1_n \stackrel{d\pi}{\longrightarrow} H^1(\Omega_V) \longrightarrow V^\vee \otimes H^1(L)\]  

(19)

From this sequence and from [18] we see that \((L, V)\) is a nonsingular point of \(N^1_n(x + y)\) if and only if either \(\text{Hom}(V, H^0(L)/V) \not\subset T_{(L,V)}N^1_n(x + y)\) or \(d\pi(T_{(L,V)}N^1_n(x + y)) \neq d\pi(T_{(L,V)}G^1_n)\).

Assume first that the complete linear series \((L, H^0(L))\) is not neutral w.r. to \(x + y\); then in particular \(V \neq H^0(L)\) and \(\text{Hom}(V, H^0(L)/V) \not\subset T_{(L,V)}N^1_n(x + y)\). It follows that \(N^1_n(x + y)\) is nonsingular at \((L, V)\). On the other hand we have \(H^0(\omega_L^{-1}(x + y)) \cong H^0(\omega_L^{-1})\) and from the commutative diagram:

\[\begin{array}{ccc}
V \otimes H^0(\omega_L^{-1}) & \longrightarrow & H^0(\omega_L) \\
\downarrow & & \downarrow \\
V \otimes H^0(\omega_L^{-1}(x + y)) & \longrightarrow & H^0(\omega_L(x + y))
\end{array}\]

we see that \(M\) is injective. Therefore the Proposition is true in this case.

Assume now that the complete series \((L, H^0(L))\) is neutral w.r. to \(x + y\). In this case

\[\text{Hom}(V, H^0(L)/V) \subset T_{(L,V)}N^1_n(x + y)\]

and therefore \((L, V)\) is a nonsingular point of \(N^1_n(x + y)\) if and only if

\[d\pi(T_{(L,V)}N^1_n(x + y)) \neq d\pi(T_{(L,V)}G^1_n) = \ker(\mu^\vee)\]

We have \(\dim[V(-x - y)] = 1\) because \((x + y, (L, V)) \notin D_0(\Phi)\). Let \(\mu = \mu_0(L(-x,y), V(-x-y))\), so that

\[\mu^\vee : H^1(\Omega_V) \longrightarrow V(-x-y)^\vee \otimes H^1(L(-x-y))\]
and \( \text{ker}(\mu^\vee) \) consists of the tangent directions along which \( L(-x - y) \) deforms carrying along the section of \( V(-x - y) \).

Then
\[
\text{ker}(\mu_0^\vee) \cap \text{ker}(\mu^\vee)
\]

Consider the following commutative diagram:

\[
\begin{array}{ccc}
V^\vee \otimes H^1(L) & \xrightarrow{\nu} & P \\
\mu_0^\vee & \downarrow & \downarrow \\
H^1(\mathcal{O}_L) & \xrightarrow{\nu} & V(-x - y)^\vee \otimes H^1(L) \\
\mu^\vee & \downarrow & \downarrow \\
V(-x - y)^\vee \otimes H^1(L(-x - y)) & \xrightarrow{\nu} & \end{array}
\]

where
\[
P := [V^\vee \otimes H^1(L)] \times_{V(-x - y)^\vee \otimes H^1(L)} [V(-x - y)^\vee \otimes H^1(L(-x - y))]
\]

and where \( \nu \) is the map induced by the universal property of \( P \). By construction
\[
\text{ker}(\nu) = \text{ker}(\mu_0^\vee) \cap \text{ker}(\mu^\vee)
\]

Observing that \( h^1(L(-x - y)) = h^1(L) + 1 \), we deduce that \( \dim(P) = 2h^1(L) + 1 \). Therefore, recalling (20) and (21), we see that \((L, V)\) is a nonsingular point of \( N^1_n(x + y) \) if and only if \( \nu \) is surjective.

We have another commutative diagram:

\[
\begin{array}{ccc}
V^\vee \otimes H^1(L(-x - y)) & \xrightarrow{n} & P \\
\mu^\vee & \downarrow & \downarrow \\
V(-x - y)^\vee \otimes H^1(L(-x - y)) & \xrightarrow{\nu} & \end{array}
\]

where \( n \) is induced by the two surjective maps \( e : H^1(L(-x - y)) \to H^1(L) \) and \( p : V^\vee \to V(-x - y)^\vee \). It follows from linear algebra that \( n \) is surjective and that \( \text{ker}(n) = \text{ker}(e) \cap \text{ker}(p) \) is one-dimensional. Therefore there is induced the following commutative and exact diagram:

\[
\begin{array}{ccc}
H^1(\mathcal{O}_L(-x - y)) & \xrightarrow{M^\vee} & V^\vee \otimes H^1(L(-x - y)) \\
\xrightarrow{\nu} & \xrightarrow{n} & \xrightarrow{\text{coker}(M^\vee)} \\
H^1(\mathcal{O}_L) & \xrightarrow{\nu} & P \\
\xrightarrow{n} & \xrightarrow{\text{coker}(\nu)} & \end{array}
\]

where the vertical map on the right is surjective. Therefore, if \( M^\vee \) is surjective then \( \nu \) is surjective. Assume that \( M^\vee \) is not surjective. Then
neither $x$ nor $y$ is a base points of $(L, V)$ (Lemma 5.3(c)) and $\text{Im}(M) \not\subset H^0(\omega)$: it follows that $\ker(M^\alpha) \cap \ker(\epsilon) = 0$, and this implies that $(0) \neq \text{coker}(M^\alpha) \cong \text{coker}(\nu)$. In conclusion $M^\alpha$ is surjective if and only if $\nu$ is surjective if and only if $(L, V)$ is a nonsingular point of $N_1^1(x+y)$, and this proves the Proposition. \[\square\]

**Corollary 5.5** Assume that $(x+y, (L, V)) \in N_1^1(G) \setminus \nu(G \times G_{n-2})$. Let $B$ be the fixed divisor of $(L, V)$ and $b := \deg(B)$. The following conditions are equivalent:

1. $h^1(L^2(-x - y - B)) = 1$.
2. There exists a unique effective divisor $z_1 + \cdots + z_{2n+b} \in \omega \cap (x+y+B)$.
3. $(L, V)$ is a singular point of $N_1^1(x+y)$.

In particular, if $x < B$ then $(L, V)$ is a nonsingular point of $N_1^1(x+y)$.

**Proof.** The equivalence of (i) and (ii) is obvious. By Proposition 5.4 condition (iii) is satisfied if and only if $M$ is not injective, and this is equivalent to $\dim(\ker(M)) = h^1(L^2(-x - y - B)) = 1$, by Lemma 5.3. The last assertion is Lemma 5.3(c). \[\square\]

Let’s consider the diagram (10) in the case $r = 1$:

\[
\begin{array}{ccc}
N_1^1(\Gamma)^r & \rightarrow & \Gamma_2 \times G_1^1 \\
p_1 \downarrow & & \downarrow p_2 \\
\Gamma_2 & \rightarrow & G_1^1
\end{array}
\]

The projection $p_2$ defines a flat family of curves in $\Gamma_2$, namely the family of all curves $s_{L,V}$. Thus, by functoriality, we have a morphism:

\[\zeta : G_1^1 \rightarrow \text{Hilb}^{\Gamma_2}\]

**Theorem 5.6** The morphism $\zeta$ is an isomorphism of $G_1^1$ onto an irreducible component of $\text{Hilb}^{\Gamma_2}$.

**Proof.** $\zeta$ is injective: in fact $(L, V)$ can be reconstructed from $s_{L,V}$ as the pencil generated by the two divisors $(a + \Gamma) \cap s_{L,V}$ and $(b + \Gamma) \cap s_{L,V}$ for distinct general $a, b \in \Gamma$. Therefore, since $G_1^1$ is nonsingular, it will suffice to prove that $d_{(L,V)}\zeta$ is injective for all $(L, V)$ and that it is an isomorphism in case $(L, V)$ is complete.

Recalling Lemmas 3.3 and 3.6 we see that the cohomology sequence associated to the exact sequence

\[
0 \rightarrow \mathcal{O}_{\Gamma_2} \rightarrow \mathcal{O}_{\Gamma_2}(s_{L,V}) \rightarrow N_{s_{L,V}} \rightarrow 0
\]

is:

\[
0 \rightarrow \bigwedge^2 V^\vee \otimes \bigwedge^2 H^0(V, L) \otimes H^0(\mathcal{O}_{\Gamma_2}, s_{L,V}) \rightarrow H^0(s_{L,V}, N_{s_{L,V}}) \rightarrow H^1(\mathcal{O}_{\Gamma_2}) \rightarrow \bigwedge^2 V^\vee \otimes H^0(\Gamma, L) \otimes H^1(\Gamma, L)
\]

(24)
Comparing with the exact sequence \[19\] we deduce a commutative diagram:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \text{Hom}(V, H^0(L)/V) & \longrightarrow & T_{(L,V)}G^1_n & \longrightarrow & H^1(\mathcal{O}_V) \\
& & \downarrow & & \downarrow \partial_{(L,V)\zeta} & \longrightarrow & \mu_0(L,V)^{\vee} \\
0 & \longrightarrow & \Lambda^2 H^0(G, L) & \longrightarrow & H^0(s_{L,V}, \mathcal{N}_{s,L,V}) & \longrightarrow & H^1(\mathcal{O}_V) \\
& & \downarrow & & \downarrow & \longrightarrow & \Lambda^2 V^\vee \otimes H^0(\Gamma, L) \otimes H^1(\Gamma, L)
\end{array}
\]

where the left vertical arrow is the differential at \(V \subset H^0(L)\) of the Plücker embedding of \(\text{Grass}(2, H^0(L))\). Clearly \(d_{(L,V)\zeta}\) is injective and it is an isomorphism when \(V = H^0(L)\). \(\Box\)

\section{The singularities of \(p_1\)}

We keep the same assumptions as in \(\Box\) namely that \(\Gamma\) is Petri w.r. to pencils and that \(\Box\) holds. Let

\[\mathcal{I} := \mathcal{O}(-N^1_n(\Gamma)) \subset \mathcal{O}_{\Gamma_2 \times G^1_n}\]

be the ideal sheaf of \(N^1_n(\Gamma)\) inside \(\Gamma_2 \times G^1_n\). Consider the natural homomorphism of locally free sheaves on \(N^1_n(\Gamma)\):

\[\partial : \mathcal{I}/\mathcal{I}^2 \longrightarrow \pi_2^* \Omega^1_{G^1_n} = \Omega^1_{\Gamma_2 \times G^1_n/\mathcal{I}^2N^1_n}\]

We will denote by \(D_0(\partial)\) the vanishing scheme of \(\partial\).

**Proposition 6.1** Assume that \(\Gamma\) is Petri w.r. to pencils. Then

(i) \(\text{Supp}(D_0(\partial))\) coincides with the relative singular locus of \(p_1 : N^1_n(\Gamma) \longrightarrow \Gamma_2\).

In particular \(\mathcal{I}(\Gamma_2 \times G^1_{n-2}) \subset \text{Supp}(D_0(\partial))\).

(ii) Let \(U \subset G^1_n\) be the open set of \(g^1_n\)’s which are base point free and complete. Then \(D_0(\partial) \cap p_2^{-1}(U) \neq \emptyset\) and every irreducible component of \(F := D_0(\partial) \cap p_2^{-1}(U)\) has dimension \(\geq 1\).

**Proof.** (i) Since we have

\[\Omega^1_{\Gamma_2 \times G^1_n/\mathcal{I}^2} = \pi_2^* \Omega^1_{G^1_n}\]

the homomorphism \(\partial\) fits into the exact sequence:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{I}/\mathcal{I}^2 & \longrightarrow & p_2^* \Omega^1_{G^1_n} & \longrightarrow & \Omega^1_{N^1_n/\mathcal{I}^2} & \longrightarrow & 0
\end{array}
\]

Dualizing we obtain the exact sequence:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \text{Hom}(\Omega^1_{N^1_n/\mathcal{I}^2}, \mathcal{O}_{N^1_n}) & \longrightarrow & p_2^* T_{G^1_n} & \longrightarrow & \partial^\vee N^1_n & \longrightarrow & T_{p_1} & \longrightarrow & 0
\end{array}
\]

where \(T_{p_1}\) is the first relative cotangent sheaf of \(p_1\) and \(N^1_n = N^1_n(\Gamma)/\mathcal{I}^2 \times G^1_n\). Since \(D_0(\partial) = D_0(\partial^\vee)\), we see that \(\text{Supp}(D_0(\partial)) = \text{Supp}(T_{p_1})\), and this
is precisely the relative singular locus of \( p_1 \). By Lemma 5.2 for each \( x + y \in \Gamma_2 \) we have

\[
\iota(\Gamma_2 \times G_{n-2}) \cap N^1_t(x + y) \subset \text{Sing}(N^1_t(x + y))
\]

and therefore \( \iota(\Gamma_2 \times G_{n-2}) \subset \text{Supp}(D_0(\partial)) \).

(ii) Note that \( U \) is dense in \( G_2 \) (Lemma 5.1). Since domain and codomain of \( \partial \) have ranks 1 and \( \rho \) respectively, it follows that every irreducible component of \( F \) has codimension \( \leq \rho \) in \( N^2_\omega(\Gamma) \), i.e. it has dimension \( \geq 1 \). Assume that \( \Gamma \) is a general curve of genus \( g \), then a general \( (L, V) \in U \) has simple ramification (by a straightforward count of parameters), and therefore \( s_{L,V} \) is nonsingular and irreducible, by Lemma 5.7(ii). If we restrict \( \partial^\vee \) to \( s_{L,V} = p_2^{-1}(L, V) \), we obtain:

\[
T_{(L,V)}G_2^1 \otimes \mathcal{O}_{s_{L,V}} \longrightarrow N_{s_{L,V}/\Gamma_2}
\]

Since \( N_{s_{L,V}/\Gamma_2} \) has positive degree \((n - 1)^2 - \gamma \) it follows that the vector bundle \( p_2^*G^1_2 \otimes N_{N^2_\omega} \) is \( p_2 \)-relatively ample. We now apply Theorem 0.3 of [19] to deduce that \( F \neq \emptyset \). This proves the non-emptiness statement in case \( \Gamma \) is general. But the same property is preserved under specialization, and therefore it is also true only assuming that \( \Gamma \) is Petri w.r. to pencils.

**Proposition 6.2** In the same hypothesis as before let \( R_n = p_1(F) \subset \Gamma_2 \). Then:

(i) \( F \) is purely one-dimensional, \( R_n \subset \Gamma_2 \) is a curve and the projection \( F \longrightarrow R_n \) is finite.

(ii) For all \( (x + y, (L, V)) \in F \), \( x \) is not a base point of the pencil \( (L, V) \).

(iii) If \( \Gamma \) is general then the curve \( R_n \) does not contain the diagonal \( \delta \subset \Gamma_2 \).

**Proof.** (i) It will suffice to prove the following:

A) The restriction to \( D_0(\partial) \cap p_2^{-1}(U) \) of the projection \( p_1 : N^2_\omega(\Gamma) \to \Gamma_2 \) has no positive dimensional fibres.

B) \( p_1(D_0(\partial) \cap p_2^{-1}(U)) \) does not contain the general point of the curve \( x + \Gamma \) for all \( x \in \Gamma \).

Let’s prove (A). Assume that for some \( x + y \in \Gamma_2 \) there is an irreducible curve \( T \subset p_2^{-1}(x + y) \cap [D_0(\partial) \cap p_2^{-1}(U)] \subset N^2_\omega(x + y) \cap F \). Then, for each \( t \in T \) the corresponding \( (L_t, V_t) \) is base-point free and complete and, by Proposition 6.1, is a singular point of \( N^2_\omega(x + y) \). By the criterion (ii) of Corollary 5.3, there is a unique effective divisor \( D_t = z_1(t) + \cdots + z_{2n-2n}(t) \) in \( |\omega_t L_t^{-2}(x + y)| \). Since the divisors \( D_t \) are not constant w.r. to \( t \in T \), for some \( t_0 \in T \) we have \( x \in \text{Supp}(D_{t_0}) \). This implies that \( h^1(L_{\omega t_0}(-y)) > 0 \), contradicting Proposition 2.1(ii). This concludes the proof of (A).

Let’s prove (B). Assume that, given \( x \), for a general \( y \in \Gamma \) there is

\[
(L_y, V_y \in p_1^{-1}(x + y) \cap [D_0(\partial) \cap p_2^{-1}(U)]
\]

Then, by the criterion (ii) of Corollary 5.3, for each such \( y \) there is a unique effective divisor \( D_y \in |\omega_t L_y^{-2}(x + y)| \). Arguing as before we deduce that \( x \in \text{Supp}(D_y) \) for some \( y \) and we contradict Proposition 2.1(ii) again.
(ii) Clearly, if \((x + y, (L, V)) \in D_0(\partial) \cap p_2^{-1}(U)\) there is nothing to prove; moreover \(h^1(L^2(-x - y)) = 1\), by Corollary 5.5. Consider now \((x + y, (L, V)) \in F\) arbitrary, and let \(B\) be the fixed divisor of \((L, V)\). By upper-semicontinuity we have \(h^1(L^2(-x - y)) > 0\), and therefore \(h^1(L^2(-x - y - B)) > 0\) as well. If \(x < B\) then \(h^1(L^2(-x - y - B)) = 0\), By Lemma 5.3(c), and this is a contradiction.

(iii) For a general \(x \in \Gamma\) the curve \(X = \Gamma \cup E\), where \(E\) is a general elliptic curve and \(\Gamma \cap E = \{x\}\), is a general element of \(\Delta_1 \subset \mathcal{M}_g\), where \(g = \gamma + 1\). If \(\delta \subset R_n\) then a pencil \((L, V) \in F \cap p_1^{-1}(2x)\) descends to a pencil on \(X\) whose Petri map is not injective. But from [7] we know that the general curve in \(\Delta_1\) is Petri, and we get a contradiction. \(\square\)

7 Nodal curves

Let \(X\) be an integral projective curve of arithmetic genus \(g \geq 4\) whose only singularity is an ordinary node \(\bar{z}\). Let

\[\nu: \Gamma \longrightarrow X\]

be the normalization, and let \(\gamma := g - 1\) be its genus. Denote by \(x + y = \nu^{-1}(\bar{z})\). We have \(\nu^\ast \omega_X = \omega_\Gamma(x + y)\).

A torsion free rank 1 sheaf \(L\) on \(X\) is either invertible or it is not invertible precisely at the point \(\bar{z}\). In this case \(L = \nu^\ast \tilde{L}\) where \(\tilde{L}\) is an invertible sheaf on \(\Gamma\). In particular the ideal sheaf \(\mathcal{I}_\bar{z}\) of \(\bar{z}\), which is torsion free but not locally free, is

\[\mathcal{I}_\bar{z} = \nu^\ast \mathcal{O}_\Gamma(-x - y)\]

Lemma 7.1 (i) There is a canonical isomorphism of \(\mathcal{O}_X\)-modules

\[\text{Hom}_X(\nu^\ast \mathcal{O}_\Gamma, \mathcal{O}_X) \cong \mathcal{I}_\bar{z}\]

associating to \(\varphi : \nu^\ast \mathcal{O}_\Gamma \longrightarrow \mathcal{O}_X\) the section of \(\mathcal{O}_X\) corresponding to the composition

\[\mathcal{O}_X \longrightarrow \nu^\ast \mathcal{O}_\Gamma \overset{\varphi}{\longrightarrow} \mathcal{O}_X\]

(ii) For every ideal sheaf \(\mathcal{N} \subset \mathcal{O}_\Gamma\) we have \(\mathcal{I}_\bar{z}(\nu^\ast \mathcal{N}) \subset \mathcal{O}_X\) (\(\mathcal{I}_\bar{z}\) is the conductor of \(\mathcal{O}_\Gamma\) in \(\mathcal{O}_X\)).

(iii) Let \(\mathcal{N} \subset \mathcal{O}_\Gamma\) be an ideal sheaf. Then there is a canonical isomorphism:

\[\nu^\ast [\text{Hom}_\Gamma(\mathcal{N}, \mathcal{O}_\Gamma)] \cong \text{Hom}_X(\mathcal{I}_\bar{z}(\nu^\ast \mathcal{N}), \mathcal{O}_X)\]

Proof. (i) is elementary and (ii) is true by definition.

(iii) is local around \(\bar{z}\). So let \(O = \mathcal{O}_{X,\bar{z}}\), \(A\) the integral closure of \(O\), \(M = \text{Hom}_O(A, O) \subset O\) the maximal ideal, and \(N = \mathcal{N}_{\bar{z}}\). We have

\[\text{Hom}_A(N, A) = \text{Hom}_A(MN, M) = \text{Hom}_A(MN, \text{Hom}_O(A, O)) = \text{Hom}_O(MN, O)\]

(because \(M\) is invertible)
The last equality is obtained by associating to \( f : MN \to \text{Hom}_O(A, O) \) the \( O \)-homomorphism \( \varphi : MN \to O \) defined by \( \varphi(a) = f(a)(1) \).

The degree of a torsion free rank 1 sheaf \( L \) on \( X \) is

\[
\text{deg}(L) = \chi(L) - \chi(\mathcal{O}_X)
\]

**Lemma 7.2** If \( L \) is not locally free, then \( L = \nu_*\tilde{L} \) for some invertible sheaf \( \tilde{L} \) on \( \Gamma \) and

\[
\text{deg}(\tilde{L}) = \text{deg}(L) - 1 \tag{25}
\]

In particular

\[
\text{deg}(\mathcal{I}_z) = -1
\]

and

\[
\text{deg}(\nu_*\mathcal{O}_\Gamma) = 1
\]

**Proof.** In fact \( \text{deg}(\tilde{L}) = \chi(\tilde{L}) - \chi(\mathcal{O}_\Gamma) \); we have \( \chi(\tilde{L}) = \chi(L) \) because \( \nu \) is finite, and

\[
\chi(\mathcal{O}_\Gamma) = 1 - \gamma = 1 - g + 1 = \chi(\mathcal{O}_X) + 1
\]

We need the following two definitions.

**Definition 7.3** A \( g^n_r \) on \( X \) is a pair \((L, V)\) where \( L \) is an invertible sheaf of degree \( n \) and \( V \subset \mathcal{H}^0(L) \) is a subspace of dimension \( r + 1 \). A generalized \( g^n_r \) on \( X \) is a pair \((L, V)\), where \( L \) is a torsion free rank 1 sheaf of degree \( n \) and \( V \subset \mathcal{H}^0(L) \) is a vector subspace of dimension \( r + 1 \). Obviously, every \( g^n_r \) is also a generalized \( g^n_r \).

**Definition 7.4** Let \((L, V)\) be a generalized \( g^n_r \) on \( X \). The natural map

\[
\mu_0(V) : V \otimes \text{Hom}(L, \omega_X) \longrightarrow \mathcal{H}^0(\omega_X)
\]

is called the Petri map of \((L, V)\). If \( L \) is invertible then \( \mu_0(V) \) is just the ordinary Petri map of \((L, V)\):

\[
\mu_0(V) : V \otimes \mathcal{H}^0(\omega_X \cdot L^{-1}) \longrightarrow \mathcal{H}^0(\omega_X)
\]

If \( \mu_0(V) \) is injective then \((L, V)\) is called a Petri (generalized) \( g^n_r \). Given \( r, n \) the curve \( X \) is called a Petri curve w.r. to generalized \( g^n_r \)'s if every generalized \( g^n_r \) on \( X \) is Petri. Similarly, given \( r \geq 0 \), we call \( X \) a Petri curve w.r. to generalized \( g^n_r \)'s if for each \( n \) every limit \( g^n_r \) on \( X \) is Petri. We call \( X \) a Petri curve if it is Petri w.r. to generalized \( g^n_r \)'s for all \( r, n \).

If \( L \) is a torsion free rank 1 sheaf of degree \( n \) on \( X \) which is not locally free, then \( L = \nu_*\tilde{L} \) where \( \tilde{L} \) is a line bundle of degree \( n - 1 \) on \( \Gamma \); we have

\[
\mathcal{H}^0(X, L) = \mathcal{H}^0(\Gamma, \tilde{L})
\]

and

\[
\text{Hom}_X(L, \omega_X) = \text{Hom}_X(\nu_*\tilde{L}, \omega_X)
= \text{Hom}_X(\mathcal{I}_z[\nu_*L(x + y)], \omega_X)
= \nu_*[\text{Hom}_V(\tilde{L}(x + y), \nu^*\omega_X)] \quad \text{(by Lemma 7.1(iii))}
= \nu_*[\text{Hom}_V(\tilde{L}, \omega_V)]
\]
so that
\[
\text{Hom}_X(L, \omega_X) = \text{Hom}_\Gamma(\tilde{L}, \omega_T) = H^0(\Gamma, \omega_T \tilde{L}^{-1})
\] 
(26)

If \( L \) is invertible on \( X \) then we have an inclusion:
\[
H^0(X, L) \subset H^0(X, \nu^*L) = H^0(\Gamma, \nu^*L)
\]
(27)

After these preliminaries we can now prove a proposition which relates the Petri maps on \( X \) with certain maps on \( \Gamma \).

**Proposition 7.5** Let \((L, V)\) be a generalized \( g^*_1 \) on \( X \). Then

(i) If \( L \) is invertible we have the commutative diagram:
\[
\begin{array}{ccc}
V \otimes H^0(\omega_X L^{-1}) & \xrightarrow{\mu_0(V)} & H^0(\omega_X) \\
\| & & \| \\
V \otimes H^0(\omega_T \nu^* L^{-1}(x + y)) & \xrightarrow{M} & H^0(\omega_T(x + y))
\end{array}
\]

where \( M \) is the natural multiplication map. If moreover \( \bar{z} \) is a base point of \((L, V)\) then we have a commutative diagram:
\[
\begin{array}{ccc}
V \otimes H^0(\omega_X L^{-1}) & \xrightarrow{\mu_0(V)} & H^0(\omega_X) \\
\| & & \| \\
V \otimes H^0(\omega_T \nu^* L^{-1}(x + y)) & \xrightarrow{M} & H^0(\omega_T(x + y))
\end{array}
\]

(ii) If \( L \) is not invertible then \( L = \nu^* \tilde{L} \) for some invertible sheaf \( \tilde{L} \) on \( \Gamma \) and we have a commutative diagram:
\[
\begin{array}{ccc}
V \otimes \text{Hom}(L, \omega_X) & \xrightarrow{\mu_0(V)} & H^0(\omega_X) \\
\| & & \| \\
V \otimes H^0(\omega_T \tilde{L}^{-1}) & \xrightarrow{H^0(\omega_T)} & H^0(\omega_T)
\end{array}
\]

Proof. (i) In the first diagram the left vertical inclusion is induced by the inclusion \([27]\) applied to \( \omega_X L^{-1} \). The commutativity is clear. The second diagram holds because \( V = V(-x - y) \) since \( \bar{z} \) pulls back to \( x + y \).

(ii) The left vertical equality is a consequence of \([26]\) and the commutativity is clear in this case too. \[\square\]

**Corollary 7.6** If \( \Gamma \) is Petri w.r. to pencils and the map
\[
M : \nu^* V \otimes H^0(\omega_T(\nu^* L)^{-1}(x + y)) \rightarrow H^0(\omega_T(x + y))
\]
(28)
is injective for all \( g^*_1 \)'s on \( \Gamma \) of the form \( (\nu^* L, \nu^* V) \) for which \( x \) is not a base point then \( X \) is Petri w.r. to generalized \( g^*_1 \)'s.
Proof. If $\Gamma$ is Petri w.r. to pencils, then the map (28) is injective whenever $x$ is a base point of $(\nu^*L, \nu^*V)$, because $(\nu^*L, \nu^*V) \in N^1_0(x+y)$ and by Lemma (5.2(c)). Therefore the corollary is an immediate consequence of Proposition 7.4 since, by 7.5(ii), the Petri map is injective for generalized $g^1_1$'s $(L, V)$ for which $L$ is not locally free.

As a consequence of all the above we have:

**Theorem 7.7** Assume that $\Gamma$ is Petri w.r. to pencils and let $R_n \subset \Gamma_2$ be the curve introduced in Proposition 6.2. Then:

(i) If $x + y \in \Gamma_2 \setminus R_n$ then $X$ is Petri w.r. to generalized $g^1_n$'s.

(ii) If $x + y \in R_n$ then $X$ has finitely many generalized $g^1_n$'s $(L, V)$ for which the Petri map $\mu_0(V)$ is not injective, and for all of them $L$ is not locally free or $\bar{z}$ is a base point. Let $(L, V)$ be a generalized $g^1_n$ on $X$ for which $\bar{z}$ is not a base point and $L$ is invertible. If the map (28) is not injective then it follows from Propositions 5.4 and 6.1 that $(\nu^*L, \nu^*V) \in F$. Since the map $F \rightarrow p_1(F)$ is finite, we deduce that there are finitely many $g^1_n$'s on $X$ such that the map (28) is not injective. The assertion about the completeness is a consequence of the definition of $F$.

(iii) If $x + y$ is general then it is not on the finitely many curves $R_n$, $\frac{1}{2}g \leq n \leq g$, and therefore $X$ is Petri w.r. to generalized $g^1_n$'s for all such $n$'s, by (i) above, and therefore it is Petri w.r. to generalized pencils.

**8 Proof of Theorem 1.1**

Proof. Let $X$ be as in Theorem 7.7(ii), with $x + y \in R_n$ general. Then $X$ is irreducible and 1-nodal of arithmetic genus $g$, has a non-zero finite number of generalized $g^1_n$'s $(L, V)$ which are non-Petri, for all of them $L$ is invertible, they are base-point free and complete. We consider a modular family of deformations of $X$:

$$
\begin{array}{c}
X \rightarrow \mathcal{X} \leftarrow X \\
\downarrow f \\
\text{Spec}(\mathbb{C}) \rightarrow B
\end{array}
$$

In other words $B$ is nonsingular of dimension $3g - 3$ and the Kodaira-Spencer map

$$
\kappa_b : T_bB \rightarrow \text{Ext}^1(\Omega^1_{X(b)}, \mathcal{O}_{X(b)})
$$
is an isomorphism for all \( b \in B \). We will further assume that all fibres of \( f \) are in \( \mathcal{M}_g \cup \Delta_0 \) and that \( f \) has a section \( \sigma : B \to \mathcal{X} \). We have a commutative diagram:

\[
\begin{array}{ccc}
J_n(\mathcal{X}/B) \times_B \mathcal{X} & \xrightarrow{p_2} & \mathcal{X} \\
\downarrow \quad p_1 & & \downarrow f \\
J_n(\mathcal{X}/B) & \xrightarrow{q} & B
\end{array}
\]

(29)

where \( J_n(\mathcal{X}/B) \) is the relative Picard variety parametrizing invertible sheaves of degree \( n \) on the fibres of \( f \). We now consider the relative scheme \( G^1_n(\mathcal{X}/B) \) of \( g^1 \)'s on the fibres of \( f \). It can be defined as in the (absolute) case of a single curve, using the existence of a relative Poincaré sheaf \( \mathcal{P} \) on \( J_n(\mathcal{X}/B) \times_B \mathcal{X} \). We need to recall its construction. One fixes an integer \( m \geq 2g - 1 - n \) and considers the relative grassmannian

\[
\pi : G(2, p_1, \mathcal{P} \otimes p_2^* \mathcal{O}_{\mathcal{X}}(m\sigma(B))) \to J_n(\mathcal{X}/B)
\]

and the tautological locally free sheaf of rank two \( \mathcal{E} \subset \pi^* \mathcal{P} \otimes p_2^* \mathcal{O}(m\sigma(B)) \). Then the evaluation homomorphism:

\[
\mathcal{P} \otimes p_2^* \mathcal{O}_{\mathcal{X}}(m\sigma(B)) \to \mathcal{P} \otimes p_2^* \mathcal{O}_{\sigma(B)}(m\sigma(B))
\]

induces a homomorphism:

\[
\pi^* p_1_\ast [\mathcal{P} \otimes p_2^* \mathcal{O}_{\mathcal{X}}(m\sigma(B))] \to \pi^* p_1_\ast [\mathcal{P} \otimes p_2^* \mathcal{O}_{\sigma(B)}(m\sigma(B))]
\]

and \( G^1_n(\mathcal{X}/B) \) is the vanishing scheme of the composition:

\[
\mathcal{E} \xrightarrow{\pi^* p_1_\ast [\mathcal{P} \otimes p_2^* \mathcal{O}_{\mathcal{X}}(m\sigma(B))]} \pi^* p_1_\ast [\mathcal{P} \otimes p_2^* \mathcal{O}_{\sigma(B)}(m\sigma(B))]
\]

The restriction \( \mathcal{E}|_{G^1_n(\mathcal{X}/B)} \) will be denoted by \( \tilde{\mathcal{E}} \).

In a similar way one defines the relative scheme \( G^{g-n}_{2g-2-n}(\mathcal{X}/B) \) as a closed subscheme of the relative grassmannian

\[
\chi : G(g-n+1, p_1, \mathcal{P}^{-1} \otimes p_2^* \omega_{\mathcal{X}/B}(e\sigma(B))) \to J_{2g-2-n}(\mathcal{X}/B)
\]

for some \( e \geq n+1 \), and the restricted tautological sheaf \( \tilde{\mathcal{F}} \) of rank \( g-n+1 \) on \( G^{g-n}_{2g-2-n}(\mathcal{X}/B) \). We denote by

\[
\Upsilon : G^{g-n}_{2g-2-n}(\mathcal{X}/B) \to J_n(\mathcal{X}/B)
\]

the composition of \( \chi \) with the isomorphism \( J_{2g-2-n}(\mathcal{X}/B) \to J_n(\mathcal{X}/B) \) induced by residuation with respect to \( \omega_{\mathcal{X}/B} \). Now consider the following diagram:

\[
\begin{array}{ccc}
\tilde{G} & \xrightarrow{q_2} & G^{g-n}_{2g-2-n}(\mathcal{X}/B) \\
\downarrow q_1 & & \downarrow \Upsilon \\
G^1_n(\mathcal{X}/B) & \xrightarrow{\pi} & J_n(\mathcal{X}/B) & \xrightarrow{q} & B
\end{array}
\]

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where we denoted by $\tilde{G} = G_1^1(\mathcal{X}/B) \times_{\mu_0(\mathcal{X}/B)} G_{2g-2-n}^n(\mathcal{X}/B)$. Let $\omega := (q\pi q_1)^* f_*(\omega_{\mathcal{X}/B})$. It is locally free of rank $g$ on $\tilde{G}$, and we have a natural multiplication homomorphism:

$$\tilde{\mu}_0 : q_1^* \tilde{E} \otimes q_2^* \tilde{F} \longrightarrow \omega$$

Consider the degeneracy locus:

$$P_{g,n}^1(\mathcal{X}/B) := D_{2(g-n+1)-1}(\tilde{\mu}_0) \subset \tilde{G}$$

It is non-empty because its projection to $G_1^1(\mathcal{X}/B)$ contains every $g_1^n(L, V)$ on $X$ such that $\mu_0(L, V)$ is not injective. Moreover $P_{g,n}^1(\mathcal{X}/B)$ has expected codimension

$$g - 2(g - n + r) + 1 = \rho(g, 1, n) + 1$$

in $\tilde{G}$. Therefore every irreducible component $\tilde{P}$ of $P_{g,n}^1(\mathcal{X}/B)$ satisfies:

$$\dim(\tilde{P}) \geq \dim(\tilde{G}) - [\rho(g, 1, n) + 1] \geq 3g - 4$$

Consider in particular a component $\tilde{P}$ such that $q_1(\tilde{P})$ contains an $(L, V)$ on $X$ with $\mu_0(L, V)$ not injective. Let

$$P := (q\pi q_1)(\tilde{P}) \subset B$$

Then $b_0 \in P$ and the induced morphism $\tilde{P} \longrightarrow P$ is finite above $b_0$, since $q_1$ is birational above $(L, V)$ (because $(L, V)$ is complete) and $q_1(\tilde{P}) \cap G_1^1(\mathcal{X}/B)(b_0)$ is finite. Therefore $\dim(P) = \dim(\tilde{P}) \geq 3g - 4$. But $\dim(P) \leq 3g - 4$ as well because $P$ is a closed locus parametrizing curves that are non-Petri w.r.t. pencils, and therefore it does not contain all the points of $B$ parametrizing curves in $\Delta_0$, by Theorem 7.7(iii). Let $\beta : B \longrightarrow M_g \cup \Delta_0$ be the functorial morphism, which is finite by construction. Then $\beta(P)$ is an irreducible divisor containing $[X]$, but not entirely contained in $\Delta_0$ (Theorem 7.7) and $\beta(P) \cap M_g$ is a divisorial component of $P_{g,n}^1$ having the required properties. This concludes the proof of Theorem 1.1.

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