Flow Sampling: Accurate and Load-balanced Sampling Policies

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Abstract—Software-defined networking simplifies network monitoring by means of per-flow sampling, wherein the controller keeps track of the active flows in the network and samples the switches on each flow path to collect the flow statistics. A tradeoff in this process is between the controller’s sampling preference and the balancing of loads among switches. On the one hand, the controller may prefer to sample some of the switches on the flow path because they yield more accurate flow statistics. On the other hand, it is desirable to sample the switches uniformly so that their resource consumptions and lifespan are balanced. Focusing on the application of traffic matrix estimation, this paper formulates the per-flow sampling problem as a Markov decision process and devises policies that can achieve good tradeoffs between sampling accuracy and load balancing. Three classes of policies are investigated: the optimal policy, the state-independent policies, and the index policies, including the Whittle index and a second-order index policies. The second-order index policy is the most desired policy among all: 1) in terms of performance, it is on an equal footing with the Whittle index policy, and outperforms the state-independent policies by much; 2) in terms of complexity, it is much simpler than the optimal policy, and is comparable to state-independent policies and the Whittle index policy; 3) in terms of realizability, it requires no prior information on the network dynamics, hence is much easier to implement in practice.

Index Terms—Flow sampling, traffic matrix, load balancing, MDP, index policy.

I. INTRODUCTION

As a key part of the network management, network monitoring [1] is of critical importance to the optimal operation of the network such as capacity planning, traffic engineering, anomaly detection, and resilience analysis. In traditional IP networks, monitoring network information has proven to be very costly due to the need to deploy monitoring infrastructures [2]. With the advent of software-defined networking (SDN), however, efficient and flexible network monitoring is possible [3]–[5].

The essence of SDN (e.g., OpenFlow [4]) is to disassociate the data plane (forwarding process of network packets) from the control plane (monitoring and routing). In particular, the control plane is managed by one or more logically centralized controllers that have a global view of the network. SDN greatly simplifies the network monitoring process because all the switches are equipped with programmable interfaces, whereby the controller can query/sample the switches for statistics of each flow passing through them. In a nutshell, the controller monitors the network by simple means of per-flow sampling [6]–[9].

As an example, we elaborate on OpenTM [6], a traffic matrix (TM) estimation system built upon OpenFlow, to show how per-flow sampling works. A TM [10] is a traffic map of the network that reflects the volumes of traffic flowing between all the origin-destination (OD) pairs. In most cases where the controller only manages a local network, the controller takes its edge routers (or edge links) as sources and destinations to construct a local TM [11]. To estimate the TM, OpenTM keeps track of the statistics of all active flows in the SDN. For each active flow, the controller [6] 1) gets the routing information and determines the flow path; 2) periodically samples flow statistics such as flow byte and packet count from one of the switches on the flow path; 3) constructs the local TM by adding up statistics of flows from the same source to the same destination.

Consider sampling a single flow path. A decision to be made by the controller is which switch to sample in each sampling epoch. There are two criteria to be considered when making this decision.

Sampling preference of the controller – The controller may have a preference to sample some of the switches on the flow path. In OpenTM, for example, different switches on a flow path can observe different traffic volumes for the flow due to the packet loss. OpenTM aims to capture the amount of traffic that arrives at the destination. Thus, the controller prefers to sample the last switch of the path [6] because it is closest to the destination, and the traffic volumes sampled from it are considered as the most accurate.

The sampling preference of the controller also exists in other applications. OpenNetMon [7] and SLAM [9] are OpenFlow controller modules developed to measure per-flow latency, packet loss, and throughput. To these ends, the controller prefers to sample the first and the last switch because the difference between the statistics collected from them gives the most accurate measurements. In FlexMonitor [12], on the other hand, the controller prefers to sample the switches that yield the minimal communication cost with the controller.

Load balancing among switches – A switch consumes extra resources (e.g., CPU, memory, energy, and bandwidth) to execute the sampling tasks. A fair sampling policy should be able to distribute the sampling tasks evenly to the switches so that the resource consumptions and lifespan of different switches on the flow path are balanced [8], [13].

There is a clear tradeoff between the two criteria. As far as the sampling preference is concerned, the controller prefers to sample some of the switches more frequently as they yield
more accurate flow statistics. On the other hand, in terms of load balancing, it is preferred to sample the switches uniformly so that they carry equal average loads. An outstanding issue is how to devise a judicious sampling policy that makes the best tradeoff between these two criteria.

To fill this gap, this paper focuses on the application of per-flow sampling for TM estimation to investigate different sampling policies that balance the controller’s sampling preference (i.e., more accurate statistics) and load balancing among switches. In particular, we model the per-flow sampling problem as a discrete Markov decision process (MDP) \([14]\) with the state being a measurement of load balance among switches. The sampling policy of the controller is a mapping from state to an action (i.e., a chosen switch), and different actions yield different sampling accuracies. In successive time slots, the controller follows its sampling policy and makes a sequence of independent decisions to sample one of the switches on the flow path. The quality of an action at a state is reflected by a cost associated with this state-action pair. This cost function is designed to take both sampling accuracy and load balancing among switches into account. The optimal sampling policy is then defined as the policy that minimizes the average cost on an infinite time horizon.

Three classes of sampling policies are explored in this paper as solutions to the MDP: the optimal policy, the state-independent policies, and the index policies.

The optimal policy – The optimal policy is derived by solving the MDP using stochastic dynamic programming (DP) \([14]\). Although optimal, the relative value iteration algorithm for stochastic DP is computationally intensive: its complexity grows exponentially with the increase of the number of switches on the flow path. This limits the scalability of stochastic DP when the sampling problem involves a large number of switches.

State-independent policy – As the name suggests, state-independent policies make the sampling decision without considering the current states of the switches. We analyze two state-independent policies implemented in OpenTM \([6]\): a uniform sampling policy and a non-uniform sampling policy. For each flow path, the uniform policy instructs the controller to sample the switches uniformly at random. The non-uniform policy, on the other hand, indexes the switches on the flow path so that switches with larger indexes are closer to the destination. In each decision epoch, the non-uniform policy randomly generates two integers and instructs the controller to sample the switch indexed by the larger integer. We further generalize these two state-independent policies to a largest-order-statistic policy and a weighted-probability policy that have better performance. In particular, the weighted-probability policy is the optimal stationary state-independent policy.

Overall, state-independent policies have very low complexity, hence are easy to implement in practice. Their performance, however, is suboptimal in general.

Index policies – To devise low-complexity policies with good performance, we consider a class of index policies, in particular, the Whittle index policy \([15]\), to solve the MDP. The Whittle index refers to an index policy proposed by Whittle to solve restless multi-armed bandit (RMAB) problems \([16]\). RMAB is a sequential decision problem where, at each time, one or more choices must be made among all available Markovian arms/jobs. The Whittle index associates each arm with an index, and chooses the arm with the largest index at each decision epoch \([15]\). By so doing, the original high-dimensional decision problem is decoupled to multiple one-dimensional problems of computing the individual indexes of the jobs/arms, hence the computational complexity of the Whittle index grows linearly in the number of arms. Thanks to its low complexity and excellent performance, the framework of RMAB and the Whittle index solution has been widely used to solve the problem of route planning for unmanned military aircraft \([17]\), opportunistic communication channel usage \([18]\), \([19]\), and sensor management \([20]\), to name a few.

This paper formulates our MDP as an RMAB problem and devises a Whittle index policy to solve the MDP. The Whittle index is derived in closed form. Simulation results show that 1) the Whittle index policy performs as well as the optimal policy derived from stochastic DP when the number of switches on the flow path is small; 2) the Whittle index policy outperforms all the state-independent policies. Compared with the uniform policy and the largest order statistic policy, the Whittle index policy reduces the average cost by 66.4%. Compared with the weighted probability policy, the Whittle index policy reduces the average cost by 33.4%.

The Whittle index policy has satisfactory average-cost performance and low computation complexity. Yet, as the optimal policy does, it relies on perfect knowledge of the network dynamics for “planning”. This prior knowledge, however, may not be available to the controller in practice. In view of this, this paper further puts forth a second-order index policy inspired by the form of the Whittle index. The second-order index policy is the most desired policy among all as it requires no prior knowledge of the network dynamics while having all the advantages of the Whittle index. Simulation results show that the performance gap between the second-order index policy and the Whittle index is negligible.

II. Problem Formulation

Consider an SDN where the controller monitors all active flows in the network to estimate the traffic matrix. In particular, the controller monitors each flow independently in a time-slotted manner, and the monitoring slot boundaries for different flows can be misaligned. Without loss of generality, we focus on a single flow path \(AB\) from origin A to Destination B. As shown in Fig. 1, \(AB\) travels through \(M\) switches indexed by \(\{i : i = 1, 2, \ldots, M\}\), and switches with larger indexes being closer to the destination. At the beginning of time slots \(\{t : t = 0, 1, 2, \ldots\}\), the controller has to decide which of the \(M\) switches to sample to collect real-time flow statistics.

Due to packet loss, the most accurate statistics on the traffic volume from an origin to a destination can be obtained from the \(M\)-th switch because it is closest to the destination \(B\) \([6]\). Sampling the last switch in successive slots gives the controller the most accurate statistics but imposes a substantial load on the last switch at the same time. If we deterministically sample
the last switch on each flow, all the ingress/egress switches at the edge of the network will be heavily burdened with the querying loads. In this light, a judicious sampling policy should be devised to balance the tradeoff between sampling accuracy and fair loading among switches.

Definition 1 (Accuracy). Suppose the controller samples the $i$-th switch in slot $t$. The accuracy of the collected statistics is $\varphi_i = \sigma^{M-t-i}$, where $\sigma \in (0,1]$ is a constant. Larger $\varphi_i$ is desired in each sampling operation.

To measure the querying load imposed on each switch, we let the controller maintain $M$ counters, each of which is associated with a switch.

Definition 2 (Counters). The $i$-th counter $n_i$, which associates the $i$-th switch, records the number of slots since the last slot the $i$-th switch was sampled. Over time, $n_i$ evolves in the following way:

$$n_{i}^{t+1} = \begin{cases} 0, & \text{if the } i\text{-th switch is sampled in slot } t; \\ n_{i}^{t} + 1, & \text{otherwise}, \end{cases}$$

where we use superscript to denote time, and subscript to denote the index of the switch. The counters are updated at the end of a time slot.

We emphasize that the evolution of the counters in (1) can be triggered by not only the sampling of the controller on path $AB$, but also the sampling operation on any other flow path which intersects with $AB$. Take Fig. 1 for example. There are $M = 3$ switches on path $AB$, and there is another flow path $AB'\bar{B}$ that intersects with $AB$ at the second switch (i.e., the second switch is a crosspoint). Suppose the counter array of the three switches on $AB$ are updated to $\{2, 3, 1\}$ at the end of slot $t-1$, and the controller decides to sample the third switch of $AB$ in slot $t$.

a) If the controller also samples path $AB'$ at the crosspoint, the counter array associated with $AB$ evolves to $\{3, 0, 0\}$ because both the second and third switches are sampled by the controller in slot $t$.

b) Otherwise, if the controller does not sample path $AB'$ at the crosspoint, the counter array evolves to $\{3, 4, 0\}$.

In a nutshell, the counter of a switch will be reset to 0 as long as it is sampled during slot $t$, whether it is sampled by flow path $AB$ or by any other flow paths.

Consider the $M$ switches on path $AB$. We model the event that a switch is sampled by other flow paths (other than $AB$) as a random variable. Specifically, define the event $H_i^t$: the $i$-th switch is sampled by flows other than $AB$ in slot $t$. We assume $H_i^t, \forall i$ follows independent Bernoulli distribution with parameter $p_i$, and is time-invariant (constant over time). That is, the $i$-th switch is sampled by flows other than $AB$ with probability $p_i$ in a time slot. We can rewrite the evolution of $n_i^t$ in (1) as follows:

a) If the $i$-th switch is sampled by $AB$ in slot $t$.

$$n_{i}^{t+1} = 0, \text{ w. p. } 1;$$

b) If the $i$-th switch is not sampled by $AB$ in slot $t$.

$$n_{i}^{t+1} = \begin{cases} 0, & \text{w. p. } p_i; \\ n_{i}^{t} + 1, & \text{w. p. } 1 - p_i. \end{cases}$$

Remark. An alternative way to define the counters $n_i^t$ is the number of times that the $i$-th switch has been sampled up until slot $t$. That is, counter $n_i^t$ is increased by 1 if the $i$-th switch is sampled in slot $t$, and frozen otherwise (a setup akin to the Gittins index [21]). However, given this definition, the MDP associated with our flow sampling problem is very tricky to handle because $n_i^t$ grows indefinitely over time. In particular, the states of the MDP do not communicate. The definition in (1) circumvents this issue and renders the problem solvable.

The definition of counters is analogous to the definition of Age of Information (AoI) [22] with a different physical meaning. AoI measures the freshness of the collected information, whereas the value of a counter measures the querying loads imposed on the corresponding switch. That is, the $M$ switches carry the same information (information about the same flow) with different accuracies, and we aim to achieve a balance between accuracy and switch load.

Our objective is to discover a sampling policy $\mu^*$ that makes the best tradeoff between sampling accuracy and load balancing among switches. To this end, we introduce the definitions of immediate cost and average cost below to measure the quality of a policy.

Definition 3 (Immediate cost and average cost). Let the state of the controller in slot $t$ be the values of the $M$ counters, i.e., $s^t = \{n_i^t : i = 1, 2, ..., M\}$. The immediate cost of being in state $s^t$ is defined as

$$C(s^t) = \sum_{i=1}^{M} \varphi_i n_i^t.$$ (4)

A given policy $\mu$ instructs the controller to traverse through a series of states. The average cost incurred by this policy over the infinite-time horizon is defined as

$$J_{\mu} = \lim_{T \to \infty} \mathbb{E}_\mu \left[ \frac{1}{T} \sum_{t=0}^{T-1} C(s^t) \right].$$ (5)

The optimal policy, denoted by $\mu^*$, is the policy that minimizes the average cost over the infinite-time horizon, giving.

$$\mu^* = \arg \min_{\mu} J_{\mu}.$$ (6)

A lower bound to the average cost $J_{\mu}$ is given in Theorem 1.
Theorem 1 (Lower bound of the average cost). A lower bound to $J_\mu$ is

$$L_B(J_\mu) = \frac{1}{2} \sum_{i=1}^M \varphi_i \left( \frac{1}{(1-p_i)\alpha_i^* + p_i} - 1 \right),$$

where $\{\alpha_i^* : i = 1, 2, ..., M\}$ is a distribution given by

$$\alpha_i^* = \left( \sqrt{\frac{\varphi_i}{1-p_i}} - \frac{p_i}{1-p_i} \right)^+,$$

function $(x)^+ = x$ if $x \geq 0$, and $(x)^+ = 0$ if $x < 0$ \(23\); $\varphi$ is chosen such that

$$\sum_{i=1}^M \left( \sqrt{\frac{\varphi_i}{1-p_i}} - \frac{p_i}{1-p_i} \right)^+ = 1.$$

Proof. See Appendix A.

III. THE OPTIMAL POLICY

The problem of discovering the optimal flow sampling policy in \(9\) can be described as a discrete MDP. Specifically, at the beginning of a slot $t$, the controller observes a state of the counter array $s^t = \{n_i^t : i = 1, 2, ..., M\}$. Given this observation, the controller chooses an action $a^t$ (i.e., which switch to sample) following its sampling policy $\mu$, and executes $a^t$ in slot $t$. The action produces two results: 1) an immediate cost $C(s^t)$ is incurred, and 2) the system evolves to a new state $s^{t+1}$ in the next slot as per the transition probability defined below.

$$P(s^{t+1} | s^t, a^t = j) = \prod_{i=1,2,\ldots,M,i\neq j} \frac{p_i}{1-p_i} \mathbb{I}_{n_i^{t+1} = 0} + (1-p_i) \mathbb{I}_{n_i^{t+1} = n_i^t + 1},$$

where $\mathbb{I}$ is an indicator function, and

$$s^t = (n_1^t, n_2^t, \ldots, n_{j-1}^t, n_j^t, n_{j+1}^t, \ldots, n_M^t),$$

$$s^{t+1} = (n_1^{t+1}, n_2^{t+1}, \ldots, n_{j-1}^{t+1}, n_j^{t+1}, n_{j+1}^{t+1}, \ldots, n_M^{t+1}).$$

Eq. \(7\) defines the probability that the controller evolves from $s^t$ to $s^{t+1}$ if action $a^t = j$ is executed in slot $t$. Specifically, 1) the $j$-th counter $n_j^{t+1}$ is reset to 0 deterministically; 2) the $i$-th counter $n_i^{t+1}$, $i \neq j$ is reset to 0 with probability $p_i$, and evolves to $n_i^t + 1$ with probability $1 - p_i$. The evolutions of all counters are independent. Thus, $P(s^{t+1} | s^t, a^t = j)$ is a product of $M - 1$ terms, each of which is $p_i$ or $1-p_i$, depending on the value of $n_i^{t+1}$, $i \neq j$.

The same decision problem is faced by the controller in all the subsequent slots, but with different observations and corresponding actions.

An example of the state transitions is given in Fig. 2 wherein $M = 4$. As can be seen, the system starts with state $s^0 = (0, 0, 0, 0)$. In the beginning of slot $t = 0$, the controller takes action $a^0 = 4$, and no event $H_2^0$ happens during slot 0. Thus, the state transits to $s^1 = (1, 1, 1, 0)$ at the end of slot 0 because only the fourth switch is sampled. In slot 1, the controller takes action $a^1 = 3$, and there is an event $H_2^1$, meaning that the second switch is a crosspoint and is sampled by another flow during slot 1. Thus, the state transits to $s^2 = (2, 0, 0, 1)$ at the end of slot 1 because both the second and third switches are sampled. In each slot, an immediate cost is incurred as the penalty of being in state $s^t$, as defined in \(4\).

As in \(6\), the controller aims to discover the optimal policy $\mu^*$ that minimizes the average cost on an infinite time horizon. For this average-cost MDP, the optimal solution can be computed via a relative value iteration process. Specifically, the Bellman equation for the average-cost optimality criterion is given by \(8\) below \(13\). We only consider stationary policies, thus the index $t$ is removed in the rest of this section.

$$g^e \Phi + h^* = T(h^*)$$

where $g^e$ is the gain of MDP, i.e., the average cost incurred per time step when the system is in equilibrium; $e$ is an all-ones column vector; $h^*$ is a vector with each element being the relative value function of a state. The relative value function of a state $s$ (also named the cost-to-go function), is the difference between the total cost incurred by a system that starts with state $s$ and the total cost incurred by a system that starts with a steady-state state over an infinite time horizon, i.e., the extra cost incurred by the transient behavior of being in state $s$. The operator $T$ is a Bellman operator given by

$$T(h)[s] = \min_{a} \left\{ C(s) + \sum_{s'} P(s'|s, a) h[s'] \right\},$$

where $h[s]$ is an element in the vector $h$ that corresponds to state $s$, $P(s'|s, a)$ is defined in \(7\).

The solution $(g^e, h^*)$ to \(8\) can be computed by a relative value iteration process given in Algorithm 1 the convergence of which is guaranteed as the Bellman operator is a span contraction \(14\). Based on the computed $(g^e, h^*)$, the optimal policy $\mu^*$ can be extracted from $h^*$ by acting greedy (i.e., choose the action that gives the minimal future cost), giving,

$$\mu^*(s) = \arg \min_{a} \left\{ C(s) + \sum_{s'} P(s'|s, a) h^*[s'] \right\},$$

\(9\)
Relative value iteration gives us the optimal solution to (6), but it also presents several problems.

1) To compute (9), the number of states of the MDP must be finite so that \( \sum_{s'} P(s'|s, a) = 1 \). However, the state size in our problem is infinite, because the value of a counter can be any non-negative integers, i.e., \( n_i \in \mathbb{N}_0 \). As a result, we must set an upper limit, \( U \), for each counter to enable the computation of (9) in each iteration (i.e., a counter value larger than \( U \) is set to \( U \)). In order not to affect the optimality of the relative value iteration, \( U \) must be set large enough so that \( P(n_i > U) \) is negligible for a set of policies in the neighborhood of the optimal policy.

2) Given the upper limit \( U \) for each counter, the state size is now \( |S| = U^M \), and the decision space is \( |S| \times |A| \times |S| = MU^{2M} \). The computational complexity of relative value iteration grows exponentially with the increase of \( M \). This largely limits the scalability of the relative value iteration, and makes the optimal policy prohibitively expensive to compute for large \( M \).

3) Relative value iteration solves the MDP by optimal planning. A prerequisite is that the controller must have perfect knowledge \( p_i \), a parameter determined by the local volatility of each switch, to compute the transition probability \( P(s'|s, a) \) in (9). This prior information, however, may not be available to the controller in practice.

In summary, a realistic and decent sampling policy should have 1) good performance in terms of minimizing the average cost, 2) low computational complexity, 3) very little reliance on the prior information of network dynamics. The above optimal policy does not satisfy the second and third requirements. In this context, we have to consider other low-complexity solutions that scale well with the number of switches and do not rely on knowledge \( p_i \).

IV. STATE-INDEPENDENT POLICIES

A sampling policy can be state-dependent or state-independent. State-dependent policies, e.g., the optimal policy \( \mu^* \) in (6), make the sampling decision based on the current states of the counters. State-independent policies, on the other hand, make the sampling decision regardless of the counter states. Compared with the optimal policy in (6), state-independent policies are suboptimal in general, but they have low complexity, hence are very easy to implement in practice. Moreover, some of the state-independent policies do not rely on prior information of the network parameters. This section focuses on two state-independent policies implemented in (6), i.e., the uniform policy and the non-uniform policy, and their generalizations. Their performance is analyzed in terms of the average cost over the infinite-time horizon.

A simple sampling strategy is uniform sampling. In each slot, a uniform policy samples one of the \( M \) switches on path \( \overline{AB} \) uniformly at random. The performance of uniform sampling is characterized in Proposition 2.

**Proposition 2** (performance of the uniform policy). The average cost of the uniform policy over the infinite-time horizon is given by

\[
J_{\text{uniform}} = \sum_{i=1}^{M} \frac{\varphi_i(M-1)(1-p_i)}{M - (M-1)(1-p_i)}. \tag{10}
\]

**Proof.** See Appendix B □

Consider a homogeneous network wherein \( p_1 = p_2 = \ldots = p_M = p \). Eq. (10) can be simplified to

\[
J_{\text{uniform}} = \frac{(1 - \sigma^M)(M-1)(1-p)}{(1-\sigma)[M - (M-1)(1-p)]}. \tag{11}
\]

Eq. (11) suggests that \( J_{\text{uniform}} \) monotonically increases with \( M \). Let \( M \to \infty \).

\[
\lim_{M \to \infty} J_{\text{uniform}} = \frac{1-p}{(1-\sigma)p}. \tag{12}
\]

The uniform policy samples each switch with the same probability \( 1/M \) in each slot. Obviously, this policy is suboptimal when we have a sampling preference over different switches (i.e., when different switches have different \( \varphi_i \) and \( p_i \)). To tackle this problem, (6) further proposed a non-uniform sampling policy: in each slot, the controller randomly generates two random integers between 1 and \( M \) with replacement, and then samples the switch indexed by the larger integer. By so doing, the switches closer to the destination are more likely to be sampled. This matches the system model in (6) because the controller is more inclined to sample the switches closer to the destination as they yield more accurate statistics. This non-uniform sampling policy can be generalized as follows.

**Definition 4** (largest-order-statistic policy). In each slot, the largest-order-statistic policy randomly generates \( G \) integers between 1 and \( M \) with replacement, and samples the switch indexed by the largest integer (i.e., the largest order statistic of the uniform distribution).

With the largest-order-statistic policy, the \( M \) switches are sampled by path \( \overline{AB} \) in a non-uniform manner. Proposition 3 gives the performance of such a scheme.

**Proposition 3** (performance of the largest-order-statistic policy). The average cost of the largest-order-statistic policy over the infinite-time horizon is given by

\[
J_{\text{order}} = \sum_{i=1}^{M} \frac{\varphi_i(1 - q_i)(1-p_i)}{1 - (1 - q_i)(1-p_i)}. \tag{12}
\]

where

\[
q_i = \frac{\varphi_i^G - (i-1)^G}{M^G}. \tag{13}
\]

Let \( M \to \infty \), \( M \gg G \), \( p_1 = p_2 = \ldots = p_M = p \). \( J_{\text{order}} \) converges to the same value as the uniform policy,

\[
\lim_{M \to \infty} J_{\text{order}} = \frac{1-p}{(1-\sigma)p}. \tag{14}
\]

**Proof.** See Appendix C □

The largest order statistic policy is better than the uniform policy in that the sampling distribution takes the different \( \varphi_i \) and \( p_i \) of different switches into account. A natural question is
Theorem 1. Let us consider a decoupled problem of sampling only one switch. To simplify the notations, we remove the subscript \( i \) for all the definitions in Section \( III \) since there is only one switch. The state of the switch is then \( s = \{ n : n = \mathbb{N}_0 \} \), and the action space is \( a = \{ 0, 1 \} \) where 0 and 1 correspond to “rest” and “sample”, respectively. The state transition probability is given by
\[
\begin{align*}
P( s^{t+1} = 0 \mid s^t = n, a^t = 1) &= 1, \\
P( s^{t+1} = 0 \mid s^t = n, a^t = 0) &= p, \\
P( s^{t+1} = n + 1 \mid s^t = n, a^t = 0) &= 1 - p,
\end{align*}
\]

The immediate cost incurred by being in state \( s^t \) and executing \( a^t \) is
\[
\begin{align*}
C( s^t = n, a^t = 1) &= c + \varphi n, \\
C( s^t = n, a^t = 0) &= \varphi n,
\end{align*}
\]
where \( \varphi \) is the accuracy associated with this switch, and \( c \geq 0 \) is a fixed sampling cost (defined later).

The optimal policy \( \overline{p}^* \) for the decoupled problem is defined as
\[
\overline{p}^* = \arg \min_{\overline{p}} \lim_{T \to \infty} E \left[ \frac{1}{T} \sum_{t=0}^{T-1} C(s^t, a^t) \right].
\]

Compared with the original \( M \)-switch sampling problem, the decoupled problem introduces a fixed sampling cost \( c \). Without this fixed sampling cost, the controller would keep sampling the switch to minimize \( \overline{p}^* \). To avoid this, we artificially introduce a fixed cost \( c \) for each sampling operation. As per Whittle’s argument, we aim to find the sampling cost \( c^* \) for which it is equally optimal to “sample” and “rest” (i.e., the expected costs incurred by “sample” and “rest” are the same). In doing so, \( c^* \), i.e., the Whittle index, acts as a measurement of how much the controller is willing to pay to sample this switch.

In the original \( M \)-switch sampling problem, we could compute the corresponding Whittle index for individual switches in each decision epoch, and sample the switch with the largest Whittle index.

B. Solving the Decoupled Problem

The decoupled problem is also a controlled MDP. Given a sampling cost \( c \), the optimal solution to the decoupled problem can be modified from (8) as
\[
g^* + h^*[n] = \min \{ c + \varphi n + h^*[0], \varphi n + ph^*[0] + (1-p)h^*[n+1] \},
\]
where the two terms inside the minimization operation correspond to the costs incurred by the actions “sample” and “rest”, respectively. Without loss of generality, we choose state \( n = 0 \) as the reference state and set \( h^*[0] = 0 \). Thus,
\[
h^*[n] = \varphi n + \min \{ c, (1-p)h^*[n+1] \} - g^*.
\]

Eq. (17) defines the relative value function of each state \( n \) under the optimal policy for a given sampling cost \( c \).

Proposition 5 (solution to the decoupled problem). The optimal policy \( \overline{p}^* \) to the decoupled problem is a threshold policy.
For a given sampling cost $c$, there exists an integer threshold $\Gamma(c)$ such that 1) if a state $n < \Gamma(c)$, the optimal policy is to “rest”, and 2) if a state $n \geq \Gamma(c)$, the optimal policy is to “sample”.

Proof. We first assume the optimal policy to the decoupled problem has a threshold structure, and derive the relationship between the threshold and the gain of the MDP. Then, we verify that the relationship satisfies the Bellman equation in [17], hence the threshold policy is the optimal policy to the decoupled problem.

Let us assume there exists a threshold $\Gamma$ such that: if a state $n < \Gamma$, the optimal policy at this state is “rest”, and 2) if a state $n \geq \Gamma$, the optimal policy at this state is “sample”.

From [17], we have
\[c \geq (1-p)h^*[n+1], \quad \forall n < \Gamma,\] (18)
\[h^*[n] = \varphi n + (1-p)h^*[n+1] - g^*, \quad \forall n < \Gamma,\] (19)
\[h^*[0] = 0.\] (20)

For $n \geq \Gamma$, we have
\[c \leq (1-p)h^*[n+1], \quad \forall n \geq \Gamma,\] (21)
\[h^*[n] = c + \varphi n - g^*, \quad \forall n \geq \Gamma,\] (22)

Next, we show that $h^*[n]$ is a monotonically increasing function of $n$ if the optimal policy is a threshold policy.

1) For $n \geq \Gamma$, [22] indicates that $h^*[n]$ monotonically increases with the increase of $n$.

2) At the threshold,
\[h^*[\Gamma] - h^*[\Gamma-1] = h^*[\Gamma] - \varphi(\Gamma-1) - (1-p)h^*[\Gamma] + g^*\]
\[= p(c + \varphi \Gamma - g^*) - \varphi(\Gamma-1) + g^*\]
\[= (1-p)g^* + pc - \varphi(\Gamma-1) = (1-p)g^* + pc - \varphi(\Gamma - 1 - p).\] (23)

Let $n = \Gamma - 1$ in (18),
\[c \geq (1-p)h^*[\Gamma] = (1-p)(c + \varphi \Gamma - g^*)\]
\[(1-p)g^* + pc \geq \varphi \Gamma(1-p)\] (24)

Substituting (24) into (23) gives us
\[h^*[\Gamma] - h^*[\Gamma-1] > 0.\]

3) For $n < \Gamma$, it follows from (19) that
\[h^*[n] - h^*[n-1] = \varphi + (1-p)(h^*[n+1] - h^*[n]).\]

Since $h^*[\Gamma] - h^*[\Gamma-1] > 0$, we have
\[h^*[n] - h^*[n-1] > 0, \quad \forall n < \Gamma.\]

Overall, $h^*[n]$ is a monotonically increasing function of $n$.

Next, we show that the threshold policy satisfies the Bellman equation in [17], and hence is the optimal policy to the decoupled problem.

Consider any state $n$. If the optimal action at state $n$ is “sample”, we must have $n \geq \Gamma$ for the threshold policy. Thus, $h^*[n+1] \geq h^*[\Gamma+1]$ because $h^*[n]$ is a monotonically increasing function of $n$.

\[h^*[n+1] \geq h^*[\Gamma+1] \because h^*[n] \text{ is a monotonically increasing function of } n.\]

Then,
\[(1-p)h^*[n+1] \geq (1-p)h^*[\Gamma+1] \geq c.\]

This is consistent with [17] if the optimal action at state $n$ is “sample”.

On the other hand, if the optimal action at state $n$ is “rest”, we must have $n \leq \Gamma - 1$ for the threshold policy, and $h^*[n+1] \leq h^*[\Gamma]$. Thus,
\[(1-p)h^*[n+1] \leq (1-p)h^*[\Gamma] \leq c.\]

This is consistent with [17] if the optimal action at state $n$ is “rest”.

In conclusion, the optimal policy for the decoupled problem is a threshold policy.

Given the threshold structure of the optimal policy, the decoupled problem is essentially a unichain with a single recurrent class. All the states $n \geq \Gamma$ are transient states. The circles in the figure are states, while the rectangles are actions.

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\[h^*[n+1] \geq h^*[\Gamma+1] \because h^*[n] \text{ is a monotonically increasing function of } n.\]

This is consistent with [17] if the optimal action at state $n$ is “sample.”
As a result, the decoupled problem for each switch is indexable, hence the original M-switch sampling problem is indexable.

Given the indexability condition established in Lemma 6, the Whittle index policy is captured by Theorem 7 below:

**Theorem 7 (Whittle index policy).** At the beginning of a slot $t$, the controller computes a Whittle index $c^*(n_i)$ separately for each switch as a function of its current state $n_i$, and then samples the switch with the greatest index. The whittle index is given by

$$c^*(n_i) = \frac{\phi_i(1-p_i)}{p_i^2} \left[ (1-p_i)^{n_i+2} + (n_i+2)p_i - 1 \right].$$  \hfill (27)

**Proof.** The Whittle index at state $n$ is the sampling cost $c^*$ for which “sample” and “rest” make no difference. As suggested by the Bellman equation in (17), we have

$$c^* = (1-p)h^*[n+1],$$  \hfill (28)

That is, if $c < c^*$, the optimal policy at the state $n$ is to sample, and if $c > c^*$, the optimal policy at the state $n$ is to rest. When the sampling cost is exactly $c^*$, it is equally optimal to “sample” and “rest”, and state $n$ is the threshold.

Substituting (28) into (22) gives us

$$pc^* = (1-p)\varphi(n+1) - (1-p)g^*,$$  \hfill (29)

Next, we deduce $g^*$ as a function of $c^*$.

Let $n = 0, 1, 2, ..., n-1$ in (19), we have

$$h^*[1] = g^* \frac{1}{1-p},$$

$$h^*[2] = g^* \left( \frac{1}{1-p} + \frac{1}{(1-p)^2} \right) - \varphi \left( \frac{1}{1-p} \right),$$

$$h^*[3] = g^* \left( \frac{1}{1-p} + \frac{1}{(1-p)^2} + \frac{1}{(1-p)^3} \right) - \varphi \left( \frac{2}{1-p} + \frac{1}{(1-p)^2} \right),$$

$$\vdots$$

$$h^*[n] = g^* \left( \frac{1}{1-p} + \frac{1}{(1-p)^2} + \cdots + \frac{1}{(1-p)^n} \right) - \varphi \left( \frac{n-1}{1-p} + \frac{n-2}{(1-p)^2} + \cdots + \frac{1}{(1-p)^n} \right)$$

$$= g^* \frac{(1-p)^{-n}-1}{p} - \varphi \frac{(1-p)^{1-n} - (1-p) - np}{p^n}.\tag{30}$$

On the other hand, since $n$ is the threshold, we have (31) from (22).

$$h^*[n] = c^* + \varphi n - g^*.$$

Equating (30) and (31) gives us

$$g^* = \frac{p(1-p)^n}{1-(1-p)^n}c^* + \varphi n - \frac{(1-p)^{n+1} - (1-p) + np}{p[1-(1-p)^{n+1}]} \varphi\tag{32}$$

Substituting (32) into (29), we finally have

$$c^* = \frac{\varphi(1-p)}{p^2} \left[ (1-p)^{n+2} + (n+2)p - 1 \right].$$

This is the cost that the controller is willing to pay to sample a switch when it is in state $n$. At a decision epoch, the controller computes a $c^*(n_i)$ for each switch based on its current state and samples the one with the greatest $c^*(n_i)$. ■

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**D. The second-order index policy**

In Fig. 4, the optimal policy, the state-independent policies, and the Whittle index policy are evaluated in a three-dimensional coordinate system. The positive direction of the $x$-axis means the policy requires higher computational complexity, the positive direction of the $y$-axis means the policy yields larger average cost (poorer performance), and the positive direction of the $z$-axis means the policy requires prior-information $p_i$, a parameter determined by the local volatility of each switch.

As shown, the Whittle index policy is preferred to the optimal policy and the state-independent policies thanks to its low complexity and decent average-cost performance. Yet, the execution of the Whittle index policy hinges on the accurate estimation of $p_i$, as the optimal policy does. Since the accurate estimates of $p_i$ may not be available to the controller in practice, we put forth a second-order index policy in the following that does not rely on prior information $p_i$, while inheriting all the advantages of the Whittle index.

**Definition 5 (second-order index policy).** At the beginning of a slot $t$, the controller computes a second-order index $I(n_i)$ separately for each switch as a function of its current state $n_i$, and then samples the switch with the greatest index. For the $i$-th switch, the second-order index is given by

$$I(n_i) = \lim_{p_i \to 0} c^*(n_i) = \frac{\varphi_i}{2} (n_i + 1)(n_i + 2).\tag{33}$$

It is plausible that the second-order index policy performs well when $p_i$ is small, because the second-order index is inferred from the Whittle index by assuming a switch undergoes very light traffic with $p_i \to 0$. However, one may ask, does this second-order index perform well when some of the switches undergo moderate or heavy traffic with relatively large $p_i$? We answer this question affirmatively by the simulation results in section VI, where it is shown that the second-order index policy performs well for both small and large $p_i$.

Overall, the second-order index policy is the most desired policy among other policies. As shown in Fig. 4, it has low computation complexity, no reliance on the prior-information $p_i$, and comparable average-cost performance to the Whittle index policy.
Remark. When $p_i$ of the $i$-th switch is large, an alternative to the second-order index in (33) is a first-order index

$$I(n_i) = \lim_{p_i \to 1} \frac{c^*(n_i)}{1 - p_i} = \varphi_i(n_i + 1). \quad (34)$$

This gives us the following heuristic index policy.

**Heuristic index policy** – Assume the controller has a rough idea of whether $p_i$ is larger or smaller than a threshold probability $\overline{p}$ for each switch. At a decision epoch, the controller takes the second-order index in (33) as the heuristic index for switches whose $p_i < \overline{p}$; and the first-order index in (34) as the heuristic index for switches whose $p_i \geq \overline{p}$. Then, the controller samples the switch with the largest heuristic index.

This heuristic index policy is evaluated at the end of section VII. It is shown that the heuristic index policy only yields minor gains over the second-order policy. Yet, it requires the controller to know a certain amount of prior knowledge $p_i$, and the threshold probability $\overline{p}$ must be chosen very carefully. Overall, the second-order index is good enough to ensure a minor gap to the Whittle index policy.

VI. NUMERICAL AND SIMULATION RESULTS

A. The optimal policy and the lower bound

As stated in Section III, the computational complexity of relative value iteration is prohibitively high. This makes the optimal policy in (6) very expensive to obtain, especially when the number of switches $M$ is large. In view of this, we first consider a simple case where there are only three switches to evaluate the performance gap between the optimal policy and the lower bound given in Theorem I.

Fig. 5 presents the average costs achieved by the optimal policy, the uniform policy, and the Whittle index policy benchmarked against the lower bound on a flow path with $M = 3$ switches. In this figure, we fix $\sigma = 0.8$, i.e., the accuracies of statistics collected from the three switches are 0.64, 0.8, and 1, respectively. The probability that a switch is sampled by flows other than $\overline{AB}$ is set to $p_1 = p_2 = p_3 = p$, and we increase $p$ from 0.025 to 0.2. To execute relative value iteration and compute the optimal policy, we set the upper limit $U$ of each counter to 10 (i.e., a counter no longer grows when it reaches 10). The size of the state space is then $|S| = U^M = 1000$, and the decision space is $|S| \times |A| \times |S| = M|U|^{2M} = 3 \times 10^6$.

As can be seen from Fig. 5,

1) The Whittle index policy performs as well as the optimal policy for small $M$ (the two curves coincident with each other). However, the optimality of the Whittle index is unknown in the case of large $M$ due to the unavailability of the optimal policy.

2) The performance gap between the optimal policy and the lower bound is minor when $p$ is small, but gets larger as $p$ increases. This is not surprising because to derive the lower bound, we have assumed in Theorem I that the variance of the inter-sampling time of each switch is negligible relative to the mean of the inter-sampling time. Thus, the lower bound is supposed to be tighter in the case of larger $M$ and smaller $p$.

Assuming a large number of switches, the following parts evaluate the performance of state-independent policies proposed in (6) and our second-order index policy. Keeping in mind that the Whittle index policy can be suboptimal, and the lower bound may not be tight, we will take them as the benchmarks.

B. State-independent policies

This subsection evaluates the average costs achieved by different state-independent policies and their generalizations, i.e., the uniform sampling policy, the largest-order-statistic policy, and the weighted-probability policy.

The numerical and simulation results of the above three state-independent sampling policies are presented in Fig. 6, where we fix $\sigma = 0.8$, and $p_1 = p_2 = \cdots = p_M = p = 0.1$. The analytical results match with the simulation results very well.

As can be seen from Fig. 6,

1) Uniform sampling gives the worst performance. The average cost, as predicted in (11), increases monotonically...
with the increase of $M$. As $M$ goes to infinity, the average cost converges to $\frac{1-\pi}{(1-\pi)p} = 45$.

2) The performance of the largest-order-statistic policy depends on the value of $G$, i.e., the number of random integers generated each time. For a fixed $G \ll M$, (33) indicates that the average cost converges to the same value $\frac{1-\pi}{(1-\pi)p} = 45$ as the uniform policy.

3) The weighted probability policy outperforms both the uniform policy and the largest-order-statistic policy. This outcome is expected because we have optimized the sampling probability over all switches to devise the weighted probability policy. As indicated in (14), the performance of the weighted probability policy is twice of the lower bound. With the increase of $M$, the average cost converges to around 22.64.

4) The Whittle index policy outperforms all three state-independent policies. Compared with the uniform policy and the largest-order-statistic policy, the Whittle index policy reduces the average cost by 66.4% when $M$ goes to infinity. Compared with the weighted-probability policy, the Whittle index policy reduces the average cost by 33.4% when $M$ goes to infinity.

C. The Second-order index policy

The Whittle index policy outperforms the state-independent policies by much, but it requires accurate estimates of $p_i$ to compute the indexes. An alternative to the Whittle index is the second-order index given in (33), the computation of which does not require any prior information $\pi_i$. This subsection verified the performance of the second-order index policy benchmarked against the Whittle index policy. We consider an asymmetric network where switches undergo two kinds of sampling-request traffic: 1) all the odd-indexed switches undergo light traffic with small $p_i = \pi_0$; and 2) all the even-indexed switches undergo moderate/heavy traffic with relatively large $p_i = \pi_1$. In the simulation, we fix $\pi_0$ to 0.01, and vary $\pi_1$. For the Whittle index policy, $\pi_0$ and $\pi_1$ are assumed to be known to the controller such that the Whittle index can be computed. For the second-order index policy, the controller computes the second-order index directly from (33).

Fig. 7 presents the average costs achieved by the second-order index and the Whittle index policies in the considered network, wherein $M = 40$. As shown, for different $\pi_1$, the performance gaps between the two policies are very small. The second-order index policy is a good substitute for the Whittle index policy given the same low-complexity property and comparable average-cost performance. Better yet, the second-order index policy requires no prior-information of $p_i$.

Finally, we evaluate the heuristic index policy in the same network. As per the heuristic index policy, the controller has to compute a heuristic index for each switch in a decision epoch. To this end, we first set the threshold probability $\bar{\pi} = 0.3$. That is, the heuristic index of the $i$-th switch is the second-order index given in (33) if $p_i < 0.3$, and is the first-order index given in (34) if $p_i \geq 0.3$. The controller then samples the switch with the largest heuristic index.

The performance of the heuristic index policy is plotted in Fig. 7. As shown, when $\pi_1 < 0.3$, the performance of the heuristic index policy is the same as that of the second-order index, because all $p_i$ in the network are smaller than the threshold probability 0.3. On the other hand, when $\pi_1 \geq 0.3$, the indexes of all even-indexed switches are the first-order indexes rather than the second-order indexes. The heuristic index policy is slightly better than the second-order index policy. However, the downsides are that the controller has to know a certain amount of information about $p_1$, and the threshold probability $\bar{\pi}$ must be chosen very carefully (an ill-chosen $\bar{\pi}$ easily leads to substantial performance degradations).

VII. CONCLUSION

In software-defined networking (SDN), the controller samples each active flow to gather network information for traffic engineering and management. A good sampling policy should sample the switches to meet the controller’s sampling preference and balance the query loads on the switches. In addition, a practical sampling policy should be computation-friendly, and has little reliance on prior knowledge of the network dynamics since they may be unavailable in practice. The policies that meet these requirements, to our knowledge, are lacking in the literature.

To fill this research gap, this paper investigated per-flow sampling for traffic matrix (TM) estimation, and studied the performance of different policies with the above criteria. Our main contributions are as follows:

1) We formulated the per-flow sampling problem as a Markov decision process (MDP). The optimal policy to this MDP is defined as the policy that makes the best tradeoffs between sampling accuracy and load balancing among switches. We solved the MDP by a relative value iteration algorithm and derived the optimal policy.

2) We analyzed two state-independent policies previously proposed by others and generalized them to a largest-order-statistic policy and a weighted probability policy. The weighted probability policy was shown to be the optimal stationary state-independent policy. The performance of these policies was derived and validated by simulation results.
3) We transformed the MDP into a restless multi-armed bandit (RMAB) problem that admits a Whittle index policy. The Closed-form Whittle index was derived. The Whittle index policy is near-optimal and has better performance than the previously proposed state-independent policies and their generalizations. The Whittle index policy, however, requires prior knowledge of the network dynamics.

4) Inspired by the Whittle index policy, we put forth a second-order index policy. This policy meets all the expectations we have for a practical policy: it is easy to compute, achieves very good tradeoffs between sampling accuracy and load balancing, and does not require any prior knowledge of the network dynamics.

**APPENDIX A**

A LOWER BOUND TO THE AVERAGE COST

This appendix proves Theorem 1

**Proof.** Let us focus on the $i$-th counter $n_i^t$, and study how it evolves. As per (2) and (3), $n_i^t$ is reset to 0 once the $i$-th switch is sampled. In between two sampling slots of the $i$-th switch, $n_i^t$ increases from 0 to $d - 1$ if the inter-sampling time is $d$ slots.

Following the sample-path analysis [19, 24], we consider one sampling trajectory of the controller, and assume the $i$-th switch is sampled at slot $t_i,1, t_i,2, \ldots, t_i,K_i$. As $K_1, K_2, \ldots, K_M \to \infty$, $t_{1,K_1} = t_{2,K_2} = \ldots = t_{M,K_M} = T \to \infty$. The sample mean and sample variance of the inter-sampling time $d_{i,k} = t_{i,k} - t_{i,k-1}$ $(k = 1, 2, \ldots, K_i, t_{i,0} = 0)$ are defined as

$$E[d_i] = \frac{1}{K_i} \sum_{k=1}^{K_i} d_{i,k},$$

$$V[d_i] = \frac{1}{K_i} \sum_{k=1}^{K_i} (d_{i,k} - E[d_i])^2 = \frac{1}{K_i} \sum_{k=1}^{K_i} d_{i,k}^2 - E^2[d_i].$$

The average cost $J_\mu$ in (5) can then be manipulated as follows:

$$J_\mu = \lim_{T \to \infty} E_\mu \left\{ \sum_{i=1}^{M} \varphi_i \left( \frac{1}{T} \sum_{t=0}^{T-1} n_i^t \right) \right\}$$

$$= \lim_{T,K_i \to \infty} E_\mu \left\{ \frac{1}{T} \sum_{i=1}^{M} \varphi_i \sum_{k=1}^{K_i} d_{i,k-1} \sum_{n=0}^{n_i^t} n \right\}$$

$$= \lim_{T,K_i \to \infty} E_\mu \left\{ \frac{1}{2T} \sum_{i=1}^{M} \varphi_i \sum_{k=1}^{K_i} (d_{i,k}^2 - d_{i,k}) \right\}$$

$$= \lim_{T,K_i \to \infty} \frac{1}{2T} \sum_{i=1}^{M} \varphi_i K_i \left( V[d_i] + E^2[d_i] - E[d_i] \right).$$

Since $E[d_i] = T/K_i$ as $T, K_i \to \infty$, we have

$$J_\mu = \lim_{T,K_i \to \infty} \frac{1}{2T} \sum_{i=1}^{M} \varphi_i K_i \left( V[d_i] + \frac{T^2}{K_i^2} - \frac{T}{K_i} \right)$$

$$= \lim_{T,K_i \to \infty} \frac{1}{2T} \sum_{i=1}^{M} \varphi_i \left( V[d_i] + \frac{T}{K_i} - 1 \right)$$

Equation (36) gives us a lower bound on $J_\mu$:

$$L_B(J_\mu) = \min_{T,K_i \to \infty} \frac{1}{2T} \sum_{i=1}^{M} \varphi_i \left( \frac{T}{K_i} - 1 \right).$$

This bound can be further refined as follows [25].

The behavior of the $i$-th switch in one sampling trajectory can be understood from Fig. 8. As shown, there are overall $T$ slots, among which the $i$-th switch is sampled by flow $A瓘$ for $i \in \mathcal{A}$ times for some $\alpha_i > 0$. For the other $\alpha_i T$ slots, the $i$-th switch can be sampled by flow paths other than $A瓘$, or simply rest. Overall, the number of slots that the $i$-th switch is sampled is given by

$$K_i = \alpha_i T + p_i (1 - \alpha_i) T.$$

Also, we have

$$\sum_{i=1}^{M} \alpha_i T = T,$$

because flow path $A瓘$ samples a switch in every slot. Substituting (38) into (37), we have

$$L_B(J_\mu) = \min \frac{1}{2T} \sum_{i=1}^{M} \varphi_i \left( \frac{1}{(1 - p_i) \alpha_i + p_i} - 1 \right).$$

Eq. (39) and (40) give us the following linear program:

$$L_B(J_\mu) = \min \frac{1}{2T} \sum_{i=1}^{M} \varphi_i \left( \frac{1}{(1 - p_i) \alpha_i + p_i} - 1 \right)$$

s.t. $\sum_{i=1}^{M} \alpha_i = 1$, $\alpha_i \geq 0$. (41)

Form the Lagrangian as

$$f(\alpha_i, \lambda_0, \lambda_i) = \frac{1}{2T} \sum_{i=1}^{M} \varphi_i \left( \frac{1}{(1 - p_i) \alpha_i + p_i} - 1 \right) + \lambda_0 \left( \sum_{i=1}^{M} \alpha_i - 1 \right) - \lambda_i \alpha_i.$$ (42)

The optimal solutions $\{\alpha_i^* : i = 1, 2, \ldots, M\}$ to (41) must satisfy the Karush-Kuhn-Tucker (KKT) conditions as follows:

$$\frac{\partial f(\alpha_i, \lambda_0, \lambda_i)}{\partial \alpha_i} \bigg|_{\alpha_i^*} = \frac{\varphi_i (1 - p_i)}{2(1 - p_i) \alpha_i^* + p_i} - \lambda_0 - \lambda_i = 0,$$
\[ \frac{\partial f(\alpha_i, \lambda_0, \lambda_i)}{\partial \lambda_0} \bigg|_{\alpha_i^*} = \sum_{i=1}^{M} \alpha_i^* - 1 = 0,\]

\[ \lambda_i \alpha_i^* = 0, \]

\[ \lambda_0 \geq 0, \]

\[ \lambda_i \geq 0. \]

A valid solution to (41) is given by

\[ \alpha_i^* = \left( v \sqrt{\frac{\phi_i}{1-p_i} - \frac{p_i}{1-p_i}} \right)^+, \tag{43} \]

where \( (x)^+ = x \) if \( x \geq 0 \), and \( (x)^+ = 0 \) if \( x < 0 \); \( v \) is chosen such that

\[ \sum_{i=1}^{M} \left( v \sqrt{\frac{\phi_i}{1-p_i} - \frac{p_i}{1-p_i}} \right)^+ = 1. \]

It is easy to verify that (43) satisfies the KKT condition by setting

\[ \lambda_0 = \frac{1}{2v^2} \geq 0, \]

and

\[ \lambda_i = \frac{1}{2} \left( \frac{1}{v^2} - \frac{\phi_i(1-p_i)}{p_i} \right) \geq 0. \]

The lower bound is thus given by

\[ L_B(J_\mu) = \frac{1}{2} \sum_{i=1}^{M} \phi_i \left( \frac{1}{(1-p_i)\alpha_i^* + p_i} - 1 \right), \]

\[ \text{APPENDIX B} \]

\[ \text{PERFORMANCE OF THE UNIFORM SAMPLING POLICY} \]

\[ \text{Proof.} \] Under the uniform sampling policy, we can rewrite (5) as

\[ J_{\text{uniform}} = \lim_{M \to \infty} \mathbb{E} \left[ \frac{1}{T} \sum_{t=0}^{T-1} \sum_{i=1}^{M} \phi_i n_i^t \right] \]

\[ = \lim_{M \to \infty} \mathbb{E} \left[ \sum_{i=1}^{M} \phi_i \left( \frac{1}{T} \sum_{t=0}^{T-1} n_i^t \right) \right] \]

\[ = \sum_{i=1}^{M} \phi_i \mathbb{E} [n_i], \tag{44} \]

where the last equality holds because the uniform policy is stationary, i.e., the probability that a switch is sampled does not change over time. Given (44), our target is to derive the average value of each counter.

For the \( i \)-th switch, the probability that it is sampled by flow paths other than \( A_i B \) is \( p_i \), and the probability that it is sampled by path \( A_i B \), denoted by \( q_i \), is \( 1/M \). Thus, in any slot, the probability that the \( i \)-th switch is not sampled by the controller is

\[ a_i = (1-q_i)(1-p_i), \tag{45} \]

and \( 1-a_i \) is the probability that the \( i \)-th switch is sampled in a slot.

As shown in Fig. (9), the state transitions of the \( i \)-th switch form a Markov Chain [28], the equilibrium distribution of which is given by

\[ \pi_n = a_i^n(1-a_i), n = 0, 1, 2, \ldots \]

Thus,

\[ \mathbb{E}[n_i] = \sum_{n=0}^{\infty} n \pi_n = \frac{a_i}{1-a_i}, \tag{46} \]

and

\[ J_{\text{uniform}} = \sum_{i=1}^{M} \phi_i \mathbb{E}[n_i] = \sum_{i=1}^{M} \phi_i \frac{a_i}{1-a_i} \]

\[ = \sum_{i=1}^{M} \phi_i (M-1)(1-p_i). \tag{47} \]

Let \( M \to \infty \), \( p_1 = p_2 = \cdots = p_M = p \), we have \( a_i \to 1-p \). Eq. (47) can be refined as

\[ \lim_{M \to \infty} J_{\text{uniform}} = \frac{1-p}{(1-\sigma)p}. \]

\[ \text{APPENDIX C} \]

\[ \text{PERFORMANCE OF THE LARGEST-ORDER-STATISTIC POLICY} \]

\[ \text{Proof.} \] The largest-order-statistic policy is stationary. Thus, we can follow the proof of Proposition 2 to derive its performance. In particular, the state transitions of the \( i \)-th switch are given by the same Markov chain in Fig. (9) with

\[ q_i = \frac{i^G-(i-1)^G}{M^G}, \]

\[ a_i = (1-q_i)(1-p_i). \]

Thus, we have

\[ \mathbb{E}[n_i] = \frac{a_i}{1-a_i} = \frac{(1-q_i)(1-p_i)}{1-(1-q_i)(1-p_i)}, \]

and

\[ J_{\text{order}} = \sum_{i=1}^{M} \phi_i \mathbb{E}[n_i] \]

\[ = \sum_{i=1}^{M} \phi_i (1-q_i)(1-p_i). \tag{48} \]

Let \( M \to \infty \), and \( p_1 = p_2 = \cdots = p_M = p \), we have \( q_i \to 0 \), and

\[ \lim_{M \to \infty} J_{\text{order}} = \frac{1-p}{(1-\sigma)p}. \]
APPENDIX D

PERFORMANCE OF WEIGHTED-PROBABILITY POLICY

Proof. A weighted-probability policy samples the $M$ switches in each slot following the same distribution $\{w_i: i = 1, 2, \ldots, M\}$. To derive the minimal average cost achieved by the weighted-probability policy, we have to find the optimal distribution $\{w_i^*: i = 1, 2, \ldots, M\}$.

When operated with the weighted-probability policy, the state transitions of a single switch are the shown in Fig. 9 where $a_i$ is the probability that the i-th switch is not sampled in a slot, giving

$$a_i = (1 - w_i)(1 - p_i).$$

Similar to (46) and (47), we can compute the steady-state distribution of all states in equilibrium, and write the average cost achieved by the weighted-probability policy with distribution $\{w_i: i = 1, 2, \ldots, M\}$ as

$$J_{\text{weighted}} = \sum_{i=1}^{M} \sum_{i=1}^{M} \frac{a_i}{1 - a_i} = \sum_{i=1}^{M} \frac{\varphi_i(1 - w_i)(1 - p_i)}{1 - (1 - w_i)(1 - p_i)}. \quad (49)$$

and the minimal average cost

$$J_{\text{weighted}}^* = \min_{\left\{w_i\right\}} J_{\text{weighted}} = \min_{\left\{w_i\right\}} \sum_{i=1}^{M} \varphi_i(1 - w_i)(1 - p_i)$$

s.t. $\sum_{i=1}^{M} w_i = 1, \quad w_i \geq 0. \quad (50)$

The optimal solution to this linear program is the same as (41), i.e., the optimal distribution is

$$w_i^* = \left(\frac{v}{\sqrt{\varphi_i} - p_i} \right) / \frac{1}{1 - p_i}, \quad (51)$$

and $v$ is chosen such that

$$\sum_{i=1}^{M} \left(\frac{v}{\sqrt{\varphi_i}} - \frac{p_i}{1 - p_i} \right) = 1.$$

The minimal average cost achieved by the weighted-probability policy is thus twice the lower bound in Theorem 1 giving

$$J_{\text{weighted}}^* = \sum_{i=1}^{M} \varphi_i \left(\frac{1}{1 - p_i}w_i^* + p_i - 1 \right).$$

REFERENCES

[1] P.-W. Tsai, C.-W. Tsai, C.-W. Hsu, and C.-S. Yang, “Network monitoring in software-defined networking: a review,” IEEE Syst. J., vol. 12, no. 4, pp. 3958–3969, 2018.

[2] A. Pras, J. Schonwalder, M. Burgess, O. Fester, G. M. Perez, R. Studler, and B. Stiller, “Key research challenges in network management,” IEEE Commun. Mag., vol. 45, no. 10, pp. 104–110, 2007.

[3] H. Kim and N. Feamster, “Improving network management with software defined networking,” IEEE Commun. Mag., vol. 51, no. 2, pp. 114–119, 2013.

[4] D. Kreutz, F. M. Ramos, P. E. Verissimo, C. E. Rothenberg, S. Azodolmolky, and S. Uhlig, “Software-defined networking: a comprehensive survey,” Proc. IEEE, vol. 103, no. 1, pp. 14–76, 2014.

[5] Y. Shao, A. Rezace, S. C. Liew, and V. Chan, “Significant sampling for shortest path routing: a deep reinforcement learning solution,” IEEE J. Sel. Areas Commun., 2020.

[6] A. Tootoonchian, M. Ghobadi, and Y. Ganjali, “OpenTM: traffic matrix estimator for openflow networks,” in Int. Conf. Passive Active Netw. Meas. (PAM). Springer, 2010, pp. 201–210.

[7] N. L. Van Adrichem, C. Doerr, and F. A. Kuipers, “OpenNetMon: network monitoring in openflow software-defined networks,” in IEEE Netw. Operat. Manag. Symp. (NOMS). IEEE, 2014, pp. 1–8.

[8] Y. Yu, C. Qian, and G. Jiang, “Distributed and collaborative traffic monitoring in software-defined networks,” in ACM SIGCOMM Workshop HotSDN, 2014, pp. 85–90.

[9] C. Yu, C. Lumezanu, A. Sharma, Q. Xu, G. Jiang, and H. V. Madhyastha, “Software-defined latency monitoring in data center networks,” in Int. Conf. Passive Active Netw. Meas. (PAM). Springer, 2015, pp. 360–372.

[10] A. Soule, A. Lakhiha, N. Taft, K. Papagiannaki, K. Salamatian, A. Nucci, M. Crovella, and C. Diet, “Traffic matrices: balancing measurements, inference and modeling,” in ACM SIGMETRICS, 2005, pp. 362–373.

[11] P. Tate, M. Roughan, H. Haddadi, and O. Bonaventure, “Internet traffic matrices: a primer,” Recent Adv. Netw., vol. 1, pp. 1–56, 2013.

[12] B. Wang and J. Su, “Flexmonitor: a flexible monitoring framework in SDN,” Symmetry, vol. 10, no. 12, p. 713, 2018.

[13] M. Mosshref, M. Yu, and R. Govindan, “Resource/accuracy tradeoffs in software-defined measurement,” in ACM SIGCOMM workshop HotSDN, 2014, pp. 73–78.

[14] M. L. Puterman, Markov decision processes: discrete stochastic dynamic programming. John Wiley & Sons, 2014.

[15] P. Whittle, “Restless bands: activity allocation in a changing world,” J. Appl. Prob., pp. 287–298, 1988.

[16] J. Gittins, K. Glazebrook, and R. Weber, Multi-armed bandit allocation indices. John Wiley & Sons, 2011.

[17] J. Le Ny, M. Dahleh, and E. Feron, “Multi-UAV dynamic routing with partial observations using restless bandit allocation indices,” in Amer. Control Conf. IEEE, 2008, pp. 4220–4225.

[18] Y. Sun, E. Uysal-Biyikoglu, R. Singh, and E. Modiano, “Scheduling policies for minimizing age of information in broadcast wireless networks,” IEEE/ACM Trans. Netw., vol. 26, no. 6, pp. 2637–2650, 2018.

[19] R. Singh, X. Guo, and P. R. Kumar, “Index policies for optimal mean-variance trade-off of inter-delivery times in real-time sensor networks,” in IEEE Conf. Comput. Commun. (INFOCOM). IEEE, 2015, pp. 505–512.

[20] R. Weber et al., “On the Gittins index for multiarmed bandits,” The Ann. Appl. Prob., vol. 2, no. 4, pp. 1024–1033, 1992.

[21] Y. Sun, E. Uysal-Biyikoglu, R. D. Yates, C. E. Koksal, and N. B. Shroff, “Update or wait: how to keep your data fresh,” IEEE Trans. Inf. Theory, vol. 63, no. 11, pp. 7492–7508, 2017.

[22] D. P. Palomar and J. R. Fonollosa, “Practical algorithms for a family of waterfilling solutions,” IEEE Trans. Signal Process., vol. 53, no. 2, pp. 686–695, 2005.

[23] F. Baccelli and W. A. Massey, “A sample path analysis of the M/M/1 queue,” J. Appl. Prob., vol. 26, no. 2, pp. 418–422, 1989.

[24] Y.-W. Shao and S. C. Liew, “Flexible subcarrier allocation for interleaved frequency division multiple access,” Wireless Networks, vol. 11, no. 10, pp. 1361–1372, 2005.