Generalization of the discrete and continuous Shannon entropy by positive definite functions related to the constant of motion of the non-linear Lotka-Volterra system

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Abstract. In cybernetics, the Shannon formula is a positive function that measures the entropy of discrete probability distributions. The conversion of this formula to continuous probability distributions gives the infinity and negativity catastrophes, similar to the problem of renormalisation in physics. This paper suggests a solution to these two open problems, in starting from the derivation of positive definite functions from the Shannon entropy. These definite positive functions are related to the constant of motion of the non-linear Lotka-Volterra system. Then these positive definite functions are generalized for exchange of information with both discrete and continuous probability distributions. These functions generalized the Kullback-Leibler and Wiener information gain. A practical example is presented which shows a remarkable result with a scale invariance of the information gain. Then in this paper, we recall the properties of the differential equations of the non-linear Volterra predator-prey system and of the autocatalytic system of Lotka. As this paper deals with the conversion of discrete information to continuous information, we have developed the presentation of the discrete Lotka-Volterra equations. The numerical simulation of the hyperincursive discrete Lotka-Volterra equations shows an orbital stability. Moreover it is demonstrated that the hyperincursive discrete Lotka-Volterra equations are separable into two alternating discrete incursive Lotka-Volterra equations, similar to the hyperincursive discrete harmonic oscillator.

1. Introduction

The purpose of this paper deals with the proposition of the resolution of the infinity and negativity catastrophes for the continuous conversion of the discrete Shannon entropy in cybernetics, similar to the problem of renormalization in physics.

This paper will give a solution to these infinity and negativity catastrophes, in starting from the derivation of positive definite functions from the Shannon entropy. These definite positive functions are related to the constant of motion of the non-linear Lotka-Volterra system. Then these positive definite functions are generalized for exchange of information with both discrete and continuous probability distributions. These functions generalized the Kullback-Leibler and Wiener information
gain. A practical example is presented which shows a remarkable result with a scale invariance of the information gain.

Then in this paper, we recall the properties of the differential equations of the non-linear Volterra predator-prey system and of the autocatalytic system of Lotka. As this paper deals with the conversion of discrete information to continuous information, we have developed the presentation of the discrete Lotka-Volterra equations. It is shown that the hyperincursive discrete Lotka-Volterra equations show an orbital stability. Moreover it is demonstrated that the hyperincursive discrete Lotka-Volterra equations are separable into two alternating discrete incursive Lotka-Volterra equations.

The paper is organised as follows.

Section 2 deals with the definition of the discrete Shannon entropy in cybernetics. The Shannon formula is a positive function that measures the entropy of discrete probability distributions.

Then section 3 gives the derivation of positive definite functions from the Shannon entropy.

Section 4 presents the Dubois definite positive functions, $D_0(p,q), D_0(q,p), D_1(p,q)$, for exchange of discrete information. The three discrete positive definite functions generalized the Kullback-Leibler divergence. Section 5 defines the Kullback-Leibler relative entropy, information divergence or information gain.

In section 6, we present the infinity and negativity catastrophes in the Shannon continuous entropy. Section 7 defines the Norbert Wiener gain in information opposite to the continuous Shannon entropy.

Then section 8 presents the transformation of the Dubois discrete positive functions to the continuous functions. $D_0(f,g), D_0(g,f), D_1(f,g)$, that generalize the Wiener concept of continuous quantity of information as the opposite of the Shannon continuous entropy. In section 9, a practical example shows that the Dubois continuous functions give a positive quantity to the information gain with remarkable scale invariance.

Then section 10 gives the links between the harmonic oscillator, the Volterra predator-prey system and the Lotka autocatalytic system.

Finally, section 11 deals with the second order hyperincursive discrete Lotka-Volterra equations which bifurcate to 4 incursive discrete equations, similarly to the hyperincursive harmonic oscillator. Two numerical simulations of the hyperincursive discrete Lotka-Volterra equations are finally presented.

2. Definition of the discrete Shannon entropy

In cybernetics, the observer of any system can know the content in information or value of the message represented by the set of probabilities $\{p_i\}$ at any instant from the formula of Shannon-Weaver [1] in 1949 measuring the uncertainty by the entropy, $H$:

$$H = -\sum_{i=1}^{S} p_i \log p_i \geq 0,$$

$$\sum_{i=1}^{S} p_i = 1,$$

$$H \leq H_{\text{max}} = -\sum_{i=1}^{S} \frac{1}{S} \log \frac{1}{S} = \log S$$

where $p_i = p(X_i), i = 1, \ldots, S$, is the set of $X$ probabilities, sum to 1.
This entropy, $H$, is positive or null, with a maximum value for the uniform distribution, $p_i = 1/S$. Shannon and Weaver used the logarithm function in base, $b = 2$, in relation with the content of information in bit unit.

For example, the answer {yes, no} to a question, in binary base $b = 2$, shows an uncertainty, $H_B$, of 1 bit (binary digit), the unit of information:

$$H_2 = -\frac{1}{2} \log_2 \frac{1}{2} - \frac{1}{2} \log_2 \frac{1}{2} = 1 \text{ bit}$$

So, the yes or no answer to a question gives a gain of information of 1 bit.

The logarithm in base $b$, $\log_b x = \ln x / \ln b$, is related to the natural logarithm $\ln x$ in base $e$.

In this paper, we will use the natural logarithm given by the formula:

$$\log = \ln$$

3. Derivation of positive definite functions from the Shannon entropy
A generalization of this Shannon entropy was proposed by Daniel M Dubois [2] in view of taking into account temporal cyclic systems represented by probabilities depending on time $t$ as follows, $p_i = p_i(t)$, with the following entropy and normalization condition

$$H = H(t) = -\sum_{i=1}^{S} p_i(t) \log p_i(t),$$

$$\frac{1}{T} \int_0^T \sum_{i=1}^{S} p_i(t) dt = 1 \quad , \quad H_0 = \frac{1}{T} \int_0^T H(t) dt$$

where the content of the message of this cyclic system, $H_0$, is integrated over the cyclic time $T$. In defining a state of reference by the set of averaged probabilities, $\{p_{i0}\}$, the content of the message of this state of reference, $I_0$, is given by the expression

$$p_{i0} = \frac{1}{T} \int_0^T p_i(t) dt \quad , \quad I_0 = -\sum_{i=1}^{S} p_{i0} \log p_{i0}$$

The development of $H$ function in Taylor’s series around the reference state, $\{p_{i0}\}$, is given by $H = I_0 - \sum_{i=1}^{S} \left\{ (p_i - p_{i0}) \log p_{i0} + (p_i - p_{i0})^2 / 2p_{i0} + 0 - (p_i - p_{i0})^3 \right\}$ where the terms beyond the third degree are neglected [2].

In replacing $I_0$ by its equation, we obtain $H = -\sum_{i=1}^{S} \left\{ p_i \log p_{i0} + (p_i - p_{i0}) + (p_i - p_{i0})^2 / 2p_{i0} + 0 - (p_i - p_{i0})^3 \right\}$. The third term is a definite positive function called, $D_0^3$, given by $D_0^3 = \sum_{i=1}^{S} (p_i - p_{i0})^2 / 2p_{i0}$ that is applicable when the system is near its reference state [2].

In the general case, for any $\{p_i\}$ and $\{p_{i0}\}$, I defined a function $D_0$ by the following relation

$$H = -\sum_{i=1}^{S} \left\{ p_i \log p_{i0} + (p_i - p_{i0}) \right\} - D_0$$

so, I obtained immediately the function [2]:

$$D_0 = \sum_{i=1}^{S} \left\{ p_i \log(p_i/p_{i0}) + p_{i0} - p_i \right\} = D_0(t) = \sum_{i=1}^{S} \left\{ p_i(t) \log(p_i(t)/p_{i0}) + p_{i0} - p_i(t) \right\}$$

where $D_0 = D_0(t)$ is a positive definite function that is zero at the state of reference $p_i = p_{i0}$. The content of the message of a cyclic system, $H_0$, is given by
\[ H_0 = I_0 - \frac{1}{T} \int_0^T D_0 \, dt \]  

is the difference between the content of the message of the state of reference \( I_0 \) and the average of the function \( D_0 \) on a cycle, thus, \( D_0 \) is a dynamic characteristics of the system.

Similarly, I defined a second function \( D_0^{(2)} \) from \( I_0 \)

\[ I_0 = -\sum_{i=1}^S \{ p_{i0} \log p_i + (p_{i0} - p_i) \} - D_0^{(2)} \]  

so, a second function was also obtained as

\[ D_0^{(2)} = \sum_{i=1}^S \left[ p_{i0} \log(p_{i0}/p_i) + p_i - p_{i0} \right] = D_0^{(2)}(t) = \sum_{i=1}^S \left[ p_{i0} \log(p_{i0}/p_i(t)) + p_i(t) - p_{i0} \right] \]  

where \( D_0^{(2)} = D_0^{(2)}(t) \) is also a positive definite function that is zero at the reference state.

Let us remark that the two definite positive functions are symmetric

\[ D_0 = D_0(p_i, p_{i0}), \quad D_0^{(2)}(p_i, p_{i0}) = D_0(p_{i0}, p_i) \]  

When the distance between \( \{ p_i \} \) and \( \{ p_{i0} \} \) is small, \( D_0 \) and \( D_0^{(2)} \) are equal to the function \( D_0 \)

\[ D_0 = D_0^{(2)} = D_0^* = \sum_{i=1}^S (p_i - p_{i0})^2 / 2p_{i0} \]  

which means that the two symmetrical functions are different beyond the third order term.

Let us recall the following theorem about the positive definite function \( L \) of Lyapunov, after La Salle and Lefschetz [3]:

“A system is stable if it is possible to find a positive definite function, \( L \), for which the time derivative is negative or null.”

When the time derivative is null, the function is equal to a positive constant, \( L = K \), defining a cyclic system with a non-asymptotic stability, and the system keeps in memory all its fluctuations.

So, these two positive definite functions, \( D_0 \), and, \( D_0^{(2)} \), are called index of fluctuations and are also called index of diversity-stability by Dubois [4].

The time derivatives of these positive definite functions are given by the following expressions [2]:

\[ \frac{dD_0}{dt} = \sum_{i=1}^S \frac{dp_i}{dt} \log(p_i/p_{i0}) \], \quad \frac{dD_0^{(2)}}{dt} = \sum_{i=1}^S \frac{dp_i}{dt} \left( 1 - \frac{p_{i0}}{p_i} \right) \], \quad \frac{dD_0^*}{dt} = \sum_{i=1}^S \frac{dp_i}{dt} \left( \frac{p_i}{p_{i0}} - 1 \right) \]

and three invariants, \( K_1, K_2 \) and \( K^* \), can be deduced in annulling these time derivatives.

After some mathematical developments, with \( S = 2 \), from the first INVARIANT, \( D_0 = K_1 \), two differential equations were deduced:

\[ \frac{dp_1}{dt} = -F \log \frac{p_2}{p_{20}} \]

\[ \frac{dp_2}{dt} = +F \log \frac{p_1}{p_{10}} \]

where \( F \) is any function of \( p_i, t \).

Also, from the second INVARIANT, \( D_0^{(2)} = K_2 \), two other differential equations were deduced:
\[
\begin{align*}
\frac{dp_1}{dt} &= -Fp_1(p_2 - p_{20}) \\
\frac{dp_2}{dt} &= +Fp_2(p_1 - p_{10})
\end{align*}
\]

where \(F\) is any function of \(p_i, t\).

The remarkable fact is that this system of equations, for \(F = 1\), is the well-known Volterra model [5] of prey, \(p_1(t)\), and predator, \(p_2(t)\), and also the well-known Lotka model [6] of an autocatalytic chemical kinetics.

From the third INVARIANT, \(D^*_0 = K^*\), for the case \(S = 2\), two differential equations are deduced

\[
\begin{align*}
\frac{dp_1}{dt} &= +F(p_2 - p_{20})/p_{20} \\
\frac{dp_2}{dt} &= -F(p_1 - p_{10})/p_{10}
\end{align*}
\]

where \(F\) is any function of \(p_i, t\), that is the model of the harmonic oscillator.

4. **The Dubois definite positive functions for exchange of discrete information**

From the functions, \(D_0, D_0^{(2)}\), we can define two generalized discrete definite positive functions, where the reference state \(\{p_{i0}\}\) is replaced by a general discrete probability distribution \(\{q_i\}\).

So, the first definite positive function, with the temporal normalization, is defined by Dubois [7] as follows

\[
D_0(p, q) = \sum_{i=1}^{S}[p_i \log(p_i/q_i) + q_i - p_i], \quad \frac{1}{T} \int_{0}^{T} \sum_{i=1}^{S} p_i dt = 1, \quad \frac{1}{T} \int_{0}^{T} \sum_{i=1}^{S} q_i dt = 1 \tag{11}
\]

where \(p_i = p(X_i(t))\) and \(q_i = q(X_i(t))\) are two discrete probability distributions depending on continuous time, \(t\), defining both discrete and continuous probability distributions.

In inverting \(p_i\) and \(q_i\), the second definite positive function is defined by Dubois [7]

\[
D_0(q, p) = \sum_{i=1}^{S}[q_i \log(q_i/p_i) + p_i - q_i] \tag{12}
\]

As Shown by Dubois [7], in adding these two functions, divided by 2, a symmetrical averaged function is obtained as follows

\[
D_1(p, q) = \frac{1}{2}(D_0(p, q) + D_0(q, p)) = D_1(q, p) = \frac{1}{2} \sum_{i=1}^{S}(p_i - q_i) \log(p_i/q_i) \tag{13}
\]

These three discrete definite positive functions are equal to zero for \(p_i = q_i, i = 1, \ldots, S\).

These discrete probabilities depend on continuous time.

The practical result of this generalization is the fact that it can formalize the bi-directional exchange of information in a transceiver both ways, or between transmitter and receiver systems.

The first important result of this paper deals with the fact that these temporal discrete functions \(D_0(p, q), D_0(q, p), D_1(p, q)\) are a generalization of the Kullback-Leibler [8] divergence, also called relative entropy, information divergence or information gain.

5. **The Kullback-Leibler relative entropy, information divergence or information gain**

The Kullback-Leibler [8] divergence is a measure of the difference between two normalized discrete probability distributions
\[ D(P\|Q) = \sum_{i=1}^{S} p_i \ln \frac{p_i}{q_i}, \quad \sum_{i=1}^{S} p_i = 1, \quad \sum_{i=1}^{S} q_i = 1 \]  

if \( q_i = 0 \) implies \( p_i = 0 \). It is a measure of information lost when \( Q \) represents a model of \( P \), so \( P \) represents the actual distribution of data.

6. The infinity and negativity catastrophes in the Shannon continuous entropy

There are two well-known problems when the discrete Shannon entropy is converted into continuous Shannon entropy.

The first problem deals with the appearance of infinity in converting the discrete probabilities to a continuous probability distribution,

\[ f(x) = \frac{1}{(b-a)^2}, \quad x \in [a, b] \]

which shows that the continuous entropy \( h \) is negative for the interval smaller than unity,

\[ h(u) = -\int_a^b u(x) \log u(x) dx = \log(b-a) < 0 \]

for \( (b-a) < 1 \)

Let us take a second case of a linear continuous distribution, \( y(x) \), in the interval, \( a \) and \( b \), \( f(x) = y(x) = 2(b-x)/(b-a)^2 \), in \( x \in [a, b] \) that shows that the continuous entropy, \( h(y) \), \( h(y) = -\int_a^b y(x) \log y(x) dx = \frac{1}{2} \left[ 1 - \log \left( \frac{4}{(b-a)^2} \right) \right] < 0 \) for

\[ \frac{4}{(b-a)^2} > e \]

is negative for \( (b-a) < 2/\sqrt{e} \cong 3 \).

7. The Norbert Wiener gain in information opposite to the continuous Shannon entropy

Norbert Wiener (2014, page 144), defined the quantity of information, \( Q_2 \) in base 2,

\[ Q_2(f) = -h_2 = \int_{-\infty}^{+\infty} f(x) \log_2 f(x) dx, \quad \int_{-\infty}^{+\infty} f(x) dx = 1 \]
He defined *a posteriori* information, given by the uniform density, 
\[ f_1(x) = u_1(x) = 1/(b - a), \]
in \[ x \in [a, b], \quad 0 \leq a < b \leq 1 \]
and *a priori* information, given by a uniform density, 
\[ f_2(x) = u_2(x) = 1/(1 - 0), \]
in \[ x \in [0, 1] \]
Then Wiener defined a gain in information, \( Q_2(u_1, u_2) \), as the measure of the difference between *a posteriori* information and *a priori* information, as follows
\[
Q_2(u_1, u_2) = Q_2(u_1) - Q_2(u_2) = \log \left( \frac{1}{b - a} \right)
\]  
(19)

**8. Transformation of the Dubois discrete functions to continuous positive definite functions**

The functions given by the equations (11,12,13) \[ D_0(p, q), D_0(q, p), D_1(p, q) \], defined by Dubois [7] can be easily transformed to continuous functions.

Indeed, in replacing the discrete distributions, \( p, q \) by the continuous distributions, \( f(x, t)dx, g(x, t)dx \), in the interval, \( x \in [a, b] \), and the summation by an integral, with normalization conditions, continuous information is defined by definite positive function.

The continuous functions \( D_0(p, q), D_0(q, p) \), are defined by Dubois [7]
\[
D_0(f, g) = \int_{x_0}^{x} \left[ f \log(f/g) + g - f \right] dx, \\
D_0(g, f) = \int_{x_0}^{x} \left[ g \log(g/f) + f - g \right] dx
\]
(20a,b)

with the normalization conditions
\[
\frac{1}{T} \int_{0}^{T} \int_{x_0}^{x} f(x, t) dx dt = 1, \\
\frac{1}{T} \int_{0}^{T} \int_{x_0}^{x} g(x, t) dx dt = 1
\]
(21a,b)

and the continuous function, \( D_1(p, q) \), is defined by Dubois [7]
\[
D_1(f, g) = D_1(g, f) = \frac{1}{2} \int_{x_0}^{x} [(f - g) \log(f/g)] dx
\]
(22)

where there is no infinity in the logarithm, because, \( \log[f(x, t)dx/g(x, t)dx] = \log(f/g) \).

These three functions are null for \( f = g \).

Let us remark that it is also possible to define the time-dependent function, \( f = f(x, t) \), with a function, \( g = g(x) \), that does not depend on time, or both functions, \( f = f(x) \), and, \( g = g(x) \), that do not depend on time.

These Dubois continuous functions, \( D_0(f, g), D_0(g, f), D_1(f, g) \), are a generalisation of the Wiener [9] concept of continuous quantity of information as the opposite of the Shannon continuous entropy.

**9. The Dubois continuous functions give a positive information gain with scale invariance**

Finally, let us apply these continuous functions, \( D_0(f, g), D_0(g, f), D_1(f, g) \), to the linear
\[
f(x) = y(x) = 2(b - x)/(b - a)^2
\]
and uniform distributions,
\[
g(x) = u(x) = 1/(b - a),
\]
in the interval, \( x \in [a, b] \). So, we obtain
\[ D_0(y, u) = \int_a^b y \log(y/u) \, dx = \log 2 - \frac{1}{2}, \quad D_0(u, y) = \int_a^b u \log(u/y) \, dx = 1 - \log 2 \]

\[ D_1(y, u) = D_1(u, y) = \frac{1}{2 \pi} \int_a^b [f - g] \log(f/g) \, dx = \frac{1}{2} [D_0(y, u) + D_0(u, y)] = \frac{1}{4} \]

(23)

(24)

that define positive quantities of continuous information independent of the interval \([a, b]\), with a scale invariance [7], what is a totally remarkable result, in comparison with the entropy given by the equations (16) and (17).

Before giving the discrete equation of the Lotka-Volterra system, let us give some properties of the harmonic oscillator, the Volterra predator-prey system and the Lotka autocatalytic reaction system.

10. The harmonic oscillator, Volterra predator-prey system and Lotka autocatalytic system

Let us recall that the harmonic oscillator with the oscillating mass, \(m\), and the spring constant, \(k\), is represented by the ordinary differential equations:

\[ \frac{dx(t)}{dt} = v(t), \quad \frac{dv(t)}{dt} = -\omega^2 x(t) \]

(25a,b)

where \(x(t)\) is the position and \(v(t)\) the velocity as functions of the time \(t\), and the pulsation \(\omega\) is related to \(k\) and \(m\) by

\[ \omega^2 = k/m \]

(26)

The solution, with the initial conditions \(x(0)\) and \(v(0)\), is given by

\[ x(t) = x(0) \cos(\omega t) + \left[ \frac{v(0)}{\omega} \right] \sin(\omega t) \]

\[ v(t) = -\omega x(0) \sin(\omega t) + v(0) \cos(\omega t) \]

(27a,b)

In the phase space, given by \((x(t), v(t))\), the solutions are given by closed curves (orbital stability).

The period of oscillations \(T\) is given by

\[ T = 2\pi/\omega \]

(28)

The energy \(e(t)\) of the harmonic oscillator is constant \(e_0\) and is given by

\[ e(t) = kx^2(t)/2 + mv^2(t)/2 = kx^2(0)/2 + mv^2(0)/2 = e(0) = e_0 \]

(29)

Now, let us recall the properties of the non-linear Lotka-Volterra system. The Lotka-Volterra system may be given by the two differential equations given by Dubois [2]:

\[ \frac{dP_1}{dt} = +P_1 P_{20} - P_1 P_2, \quad \frac{dP_2}{dt} = -P_{10} P_2 + P_1 P_2 \]

(30a,b)

Where the set \(\{P_i(t)\}, i = 1,2\), are two populations dependent of time, \(t\), and the set \(\{P_{10}\}, i = 1,2\), are two constant parameters.

The non-trivial stationary state, for

\[ \frac{dP_i(t)}{dt} = 0 \]

(31)

is given by

\[ P_i = P_{i0}, \quad i = 1,2. \]

(31a)

Firstly, in making the change of variables:

\[ P_i = N_i/\left(\alpha_1 + \alpha_2\right), \quad i = 1,2 \]

(32a)
\( P_{10} = \frac{\alpha_2}{(\alpha_1 + \alpha_2)} \), \hspace{1cm} (32b) \\
and
\( P_{20} = \frac{\alpha_1}{(\alpha_1 + \alpha_2)} \), \hspace{1cm} (32c) \\
and
\[ t = \frac{\tau}{(\alpha_1 + \alpha_2)} \] \hspace{1cm} (32d)

the original predator-prey Volterra equations given by Volterra [5] in 1931 are obtained as

\[ \frac{dN_1}{d\tau} = +\alpha_1 N_1 - \lambda_1 N_1 N_2, \quad \frac{dN_2}{d\tau} = -\alpha_2 N_2 + \lambda_1 N_1 N_2 \] \hspace{1cm} (33a,b)

where \( N_1(\tau) \) and \( N_2(\tau) \) are the temporal populations of prey and predator, respectively, depending on the time \( \tau \), and \( \alpha_1 \) is the constant growth rate of the prey and \( \alpha_2 \) the constant death rate of the predator, and \( \lambda_1 \) is the constant predation rate.

Secondly, in making the change of variables:

\[ \frac{P_1}{\lambda_1} = \frac{\alpha_1}{(\alpha_1 + \alpha_2)} \hspace{1cm} (34a) \]
\[ \frac{P_2}{\lambda_2} = \frac{\alpha_2}{(\alpha_1 + \alpha_2)} \hspace{1cm} (34b) \]
\[ P_{10} = \frac{k_1}{(k_1 A + k_3)} \hspace{1cm} (34c) \]
\[ P_{20} = \frac{k_2}{(k_1 A + k_3)} \hspace{1cm} (34d) \]
\[ t = \frac{\tau}{(k_1 A + k_3)} \] \hspace{1cm} (34e)

the original autocatalytic Lotka equations given by Lotka [6] in 1956 are obtained as

\[ \frac{dX}{d\tau} = k_1 A X - k_2 XY \]
\[ \frac{dY}{d\tau} = -k_3 Y + k_2 XY \] \hspace{1cm} (35a,b)

where \( X(\tau) \) and \( Y(\tau) \) are the temporal concentrations of the two chemical products, respectively, depending on the time \( \tau \), and the \( k_1, k_2, \) and \( k_3 \), are the constants of reaction of the chemical kinetics of the following autocatalytic reaction:

\[ A + X \overset{k_1}{\rightarrow} 2 X, \quad X + Y \overset{k_2}{\rightarrow} 2 Y, \quad Y \overset{k_3}{\rightarrow} B \] \hspace{1cm} (36a,b,c)

where \( A \) and \( B \) are constant concentrations of initial and final chemical products, respectively.

The constant of motion, \( K_2 \), of the Lotka-Volterra equations (30a,b) is given by Dubois [2]

\[ \frac{dK_2}{d\tau} = \left[ P_{10} \log(P_{10}/P_1(\tau)) - P_{10} + P_1(\tau) \right] + \left[ P_{20} \log(P_{20}/P_2(\tau)) - P_{20} + P_2(\tau) \right] = K_2 \geq 0 \] \hspace{1cm} (37a)

that is a definite positive function, null at the stationary state, and its time derivative is null:

\[ \frac{dK_2}{d\tau} = 0 \] \hspace{1cm} (37b)

Following Lyapunov [3], a system is stable if it is possible to find a positive definite function, \( L \), for which the time derivative is negative or null.

When the time derivative is null, the function is equal to a positive constant, \( L = K \), defining a cyclic system with a non-asymptotic stability, what is called an orbital stability.

When the values of the variables \( P_i(\tau) \) are near to the stationary state \( P_{i0} \),
\[ q_i(t) = P_i(t) - P_{i0}, \quad (38) \]

the Taylor development of the constant of motion (37a) becomes \[2\]
\[ D^*_0(t) = (P_i(t) - P_{i0})^2/2P_{i0} + (P_2(t) - P_{20})^2/2P_{20} \quad (39) \]

and in making the change in variables,
\[ q_i(t) = P_i(t) - P_{i0} \quad (40) \]

the Lotka-Volterra equations (30a,b) become
\[ dq_1(t)/dt = -P_{i0}q_2(t) - q_1(t)q_2(t) \]
\[ dq_2(t)/dt = +q_1(t)P_{20} + q_1(t)q_2(t) \quad (41a,b) \]

so, the linearization of the Lotka-Volterra system, in neglecting the non-linear terms, \(q_1, q_2\), gives
\[ dq_1(t)/dt = -P_{i0}q_2(t) \]
\[ dq_2(t)/dt = +q_1(t)P_{20} \quad (42a,b) \]

So, in defining the position and velocity variables by
\[ q_2(t) = kx(t), \]
\[ q_1(t) = v(t) \quad (43a,b) \]

and the mass and string constant by
\[ P_{i0} = 1/m, \]
\[ P_{20} = k \quad (44a,b) \]

the harmonic oscillator (25a,b) is obtained as follows
\[ dv(t)/dt = -(k/m)x(t) \]
\[ dx(t)/dt = v(t) \quad (45a,b) \]

and the constant of motion (39) becomes the constant energy of the harmonic oscillator (29)
\[ D^*_0(t) = (q_1(t))^2/2P_{i0} + (q_2(t))^2/2P_{20} = + m v^2/2 + k x^2/2 = T + V = e(t) = e_0 \quad (46) \]

Now let us consider the discrete Lotka-Volterra equations.

11. The time-symmetric second order hyperincursive discrete Lotka-Volterra equations

The general form of the Lotka-Volterra system is given by
\[ dX(t)/dt = F(X(t), Y(t)), \]
\[ dY(t)/dt = G(X(t), Y(t)) \quad (47a,b) \]

As usually made in computer science, let us now introduce the discrete time \(t_k\), as follows
\[ t_k = t_0 + k\Delta t, k = 0,1,2,... \quad (48a) \]

where \(k\) is the integer time increment and the discrete variables are then noted as
\[ X(t_k) = X(k), \]
\[ Y(t_k) = Y(k) \quad (48b,c) \]

The time-symmetric second order hyperincursive discrete equations of the differential equations (47a,b) are given by Dubois [10] as follows
\[ X(k + 1) = X(k - 1) + 2H F(X(k - 1), Y(k)), \quad (49a) \]
\[ Y(k + 1) = Y(k - 1) - 2H G(X(k), Y(k - 1)), \quad (49b) \]

where

\[ H = \Delta t \quad (49c) \]

So, the hyperincursive algorithm of the Lotka-Volterra differential equations 3.1ab is given by

\[
P_1(k + 1) = P_1(k - 1) + 2h P_1(k - 1)[P_{20} - P_2(k)],\]

\[
P_2(k + 1) = P_2(k - 1) - 2h P_2(k - 1)[P_{10} - P_1(k)],
\]

or

\[
P_1(k + 1) = P_1(k - 1)[1 + 2h[P_{20} - P_2(k)]],
\]

\[
P_2(k + 1) = P_2(k - 1)[1 - 2h[P_{10} - P_1(k)]], \quad (50a,b)
\]

with the four boundary conditions for \( P_1(k), P_2(k) \), at the discrete times \( k = -1 \) and \( k = 0 \),

\[
P_1(-1), \quad P_2(-1), \quad \text{and} \quad P_1(0), P_2(0). \quad (51a,b,c,d)
\]

Tables 1 and 2 give two numerical simulations of the hyperincursive discrete equations (50a,b).

| Table 1. Simulation of the hyperincursive discrete Lotka-Volterra equations |
|---|
| | \( P_{10} = 1, P_{20} = 1 \) |
| | \( N \) | \( h \) | \( k \) | \( P_1(k) \) | \( P_2(k) \) | \( D(k) \) |
|---|---|---|---|---|---|
| 12 | 0.5 | -1 | 1.0500 | 1.0866 | 0.005 |
| | | 0 | 1.0000 | 1.1000 | 0.005 |
| | | 1 | 0.9450 | 1.0866 | 0.005 |
| | | 2 | 0.9134 | 1.0395 | 0.005 |
| | | 3 | 0.9077 | 0.9925 | 0.005 |
| | | 4 | 0.9202 | 0.9435 | 0.005 |
| | | 5 | 0.9589 | 0.9133 | 0.005 |
| | | 6 | 1.0000 | 0.9048 | 0.005 |
| | | 7 | 1.0502 | 0.9133 | 0.005 |
| | | 8 | 1.0867 | 0.9502 | 0.005 |
| | | 9 | 1.1025 | 0.9925 | 0.005 |
| | | 10 | 1.0948 | 1.0476 | 0.005 |
| | | 11 | 1.0500 | 1.0866 | 0.005 |
| | | 12 | 1.0000 | 1.1000 | 0.005 |

In Tables 1 and 2, the iterations are given by \( k = -1, 0, 1, 2, 3, \ldots, N \), where \( N = 12 \) with an interval of discrete time, \( h = 0.5 \).

In these simulations, the values of the variables are near the stationary state so we can choose the constant of motion, \( D(k) \), given by equation (39)

\[
D(k) \equiv D_0(k) = (P_1(k) - P_{10})^2 / 2P_{10} + (P_2(k) - P_{20})^2 / 2P_{20} \quad (52)
\]

In Table 1, the initial conditions are given by

\[
P_1(k) = P_{10} - 0.1 \sin(2k\pi/N), k = -1 \text{ and } k = 0,
\]

\[
P_2(k) = P_{20} + 0.1 \cos(2k\pi/N), k = -1 \text{ and } k = 0, \quad (53a)
\]

where the stationary state is given by \( P_{10} = 1 \) and \( P_{20} = 1 \).
Table 2. Simulation of the hyperincursive discrete Lotka-Volterra equations

| N  | h | k | P1(k) | P2(k) | D(k) |
|----|---|---|-------|-------|------|
| 12 | 0.5 | -1 | 2.02000 | 0.51732 | 0.0004 |
|   |   | 0  | 2.00000 | 0.52000 | 0.0004 |
|   | 1  | 1.97960 | 0.51732 | 0.0004 |
|   | 2  | 1.96536 | 0.50939 | 0.0004 |
|   | 3  | 1.96101 | 0.49940 | 0.0004 |
|   | 4  | 1.96654 | 0.48953 | 0.0004 |
|   | 5  | 1.98154 | 0.48269 | 0.0004 |
|   | 6  | 2.00058 | 0.48049 | 0.0004 |
|   | 7  | 2.02019 | 0.48297 | 0.0004 |
|   | 8  | 2.03465 | 0.49020 | 0.0004 |
|   | 9  | 2.04000 | 0.49971 | 0.0004 |
|   | 10 | 2.03525 | 0.50980 | 0.0004 |
|   | 11 | 2.02000 | 0.51732 | 0.0004 |
|   | 12 | 2.00000 | 0.52000 | 0.0004 |

In Table 2, the initial conditions are given by

\[ P_1(k) = P_{10} - 0.04 \sin(2k \pi/N) \text{, } k = -1 \text{ and } k = 0, \]

\[ P_2(k) = P_{20} + 0.02 \cos(2k \pi/N) \text{, } k = -1 \text{ and } k = 0, \]

where the stationary state is given by \( P_{10} = 2 \) and \( P_{20} = 0.5 \).

In these Tables 1 and 2, the simulations are limited to one period given by \( N = 12 \) iterations, that is one orbit in the phase space, where the orbital stability of the hyperincursive discrete equations

\[ P_1(11) = P_1(-1), \quad P_2(11) = P_2(-1) \]

\[ P_1(12) = P_1(0), \quad P_2(12) = P_2(0) \]

is well verified.

The first 4 iterations of the hyperincursive discrete Lotka-Volterra equations (50a,b) are given in Table 3.

Table 3. The first 4 iterations of the hyperincursive discrete Lotka-Volterra equations (50a,b)

| Iterations | \( P_1(k+1) = \frac{P_1(k-1)[1+2h(P_{20} - P_2(k))]}{P_1(k-1)[1+2h(P_{20} - P_2(0))]} \) | \( P_2(k+1) = \frac{P_2(k-1)[1-2h(P_{10} - P_1(k))]}{P_2(k-1)[1-2h(P_{10} - P_1(0))]} \) |
|------------|---------------------------------------------------------------------------------|------------------------------------------------------------------|
| k = 0:     | \( P_1(1) = P_1(-1)[1+2h(P_{20} - P_2(0))] \)                                | \( P_2(1) = P_2(-1)[1-2h(P_{10} - P_1(0))] \) |
| k = 1:     | \( P_1(2) = P_1(0)[1+2h(P_{20} - P_2(1))] \)                                | \( P_2(2) = P_2(0)[1-2h(P_{10} - P_1(1))] \) |
| k = 2:     | \( P_1(3) = P_1(1)[1+2h(P_{20} - P_2(2))] \)                                | \( P_2(3) = P_2(1)[1-2h(P_{10} - P_1(2))] \) |
| k = 3:     | \( P_1(4) = P_1(2)[1+2h(P_{20} - P_2(3))] \)                                | \( P_2(4) = P_2(2)[1-2h(P_{10} - P_1(3))] \) |

As it is well seen in Table 3, the hyperincursive discrete Lotka-Volterra equations are separable into two alternating incursive equations, at odd and even values of the discrete time.

The hyperincursive discrete Lotka-Volterra equations bifurcate to the following two incursive discrete Lotka-Volterra systems.

The first incursive discrete Lotka-Volterra system is given by the two incursive equations
that are computed by sequential incursive iterations with the following two initial conditions,

\[ P_1(-1), P_2(0), \]  

where, in the first equation, the variable

\[ P_1(2k + 1), k = 0, 1, 2, \ldots, \]

is computed at odd values of time and then transmitted to the second equation, where the variable

\[ P_2(2k + 2), k = 0, 1, 2, \ldots, \]

is computed at even values of time.

The second incursive discrete Lotka-Volterra system is given by the two incursive equations

\[ P_2(2k + 1) = P_2(2k - 1) \left[ 1 - 2h[P_1 - P_2(2k)] \right], \]
\[ P_1(2k + 2) = P_1(2k) \left[ 1 + 2h[P_2 - P_1(2k + 1)] \right], \]  

(57a,b)

that are computed are by sequential incursive iterations with the following two different initial conditions,

\[ P_2(-1), P_1(0), \]  

where, in the first equation, the variable

\[ P_2(2k + 1), k = 0, 1, 2, \ldots, \]

is computed at odd values of time and then transmitted to the second equation, where the variable

\[ P_1(2k + 2), k = 0, 1, 2, \ldots, \]

that is now computed at even values of time.

These two incursive discrete Lotka-Volterra equations are completely independent of each other.

The second incursive discrete Lotka-Volterra equations are the inverse of the first incursive Lotka-Volterra equations by discrete time inversion \( T: \Delta t \rightarrow -\Delta t \).

More mathematical developments about the discrete hyperincursive and incursive discrete Lotka-Volterra system are given in Dubois [11] with various numerical simulations.

As we have shown, when the values of the variables \( P_i(t) \) are near to the stationary state \( P_i0 \), the Lotka-Volterra equations transform, by linearization, to the equations of the harmonic oscillator.

So, the hyperincursive discrete Lotka-Volterra transform, by linearization, to the hyperincursive discrete harmonic oscillator that bifurcates also to 4 incursive discrete equations.

So, all the mathematical developments made on the hyperincursive discrete harmonic oscillator can be extended to the theory of the hyperincursive discrete non-linear Lotka-Volterra equations.

Let us cite the series of papers by Adel F Antippa and Daniel M Dubois on the harmonic oscillator via the discrete path approach [12], on anticipation, orbital stability, and energy conservation in discrete harmonic oscillators [13], on the dual incursive system of the discrete harmonic oscillator [14], on the superposed hyperincursive system of the discrete harmonic oscillator [15], on the incursive discretization, system bifurcation, and energy conservation [16], on the hyperincursive discrete harmonic oscillator [17], on the synchronous discrete harmonic oscillator [18], on the discrete harmonic oscillator, a short compendium of formulas [19], on the time-symmetric discretization of the
harmonic oscillator [20], and finally, on the discrete harmonic oscillator, evolution of notation and cumulative erratum [21].

12. Conclusion

The Shannon formula is a positive function that measures the entropy of discrete probability distributions. The conversion of this formula to continuous probability distributions gives the infinity and negativity catastrophes. There is a similar problem in physics that is partially resolved by the so-called renormalization. This paper deals with a tentative solution to these two open problems.

From the Shannon entropy, we present the derivation of positive definite functions. These definite positive functions are related to the constant of motion of the non-linear Lotka-Volterra system. Then these positive definite functions are generalized for exchange of information with both discrete and continuous probability distributions.

Firstly, the three discrete positive definite functions, $D_0(p, q), D_0(g, p), D_1(p, q)$, are given by the equations (11, 12, 13) defined by Dubois [7]. These discrete functions generalized the Kullback-Leibler divergence. Secondly, these three functions are transformed to the continuous functions $D_0(f, g), D_0(g, f), D_1(f, g)$, given by the equations (20-a-b, 22) defined by Dubois [7]. These continuous functions generalized the Wiener concept of continuous quantity of information as the opposite of the Shannon continuous entropy. A practical example shows that the Dubois continuous functions give a positive quantity to the information gain with a remarkable scale invariance of the information gain.

Then we presented the links between the Volterra predator-prey system, the autocatalytic system of Lotka and the harmonic oscillator, given by differential equations. The discretization of the Lotka-Volterra equations is then proposed with the second order hyperincursive discrete Lotka-Volterra equations. We present two numerical simulations of the two hyperincursive discrete Lotka-Volterra equations, limited to one period that corresponds to one orbit in the phase space, where the orbital stability of the hyperincursive discrete equations is well verified. These 2 hyperincursive discrete Lotka-Volterra equations bifurcate to 4 incursive discrete equations that are separable into two alternating discrete incursive Lotka-Volterra equations, similarly to the hyperincursive discrete harmonic oscillator.

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