FUNCTION SPACES NOT CONTAINING $\ell_1$

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Abstract. For $\Omega$ bounded and open subset of $\mathbb{R}^d$ and $X$ a reflexive Banach space with 1-symmetric basis, the function space $JF_X(\Omega)$ is defined. This class of spaces includes the classical James function space. Every member of this class is separable and has non-separable dual. We provide a proof of topological nature that $JF_X(\Omega)$ does not contain an isomorphic copy of $\ell_1$. We also investigate the structure of these spaces and their duals.

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Introduction.

The study of separable Banach spaces not containing $\ell_1$ and having a non-separable dual was initialized in the middle of 70’s with the fundamental papers of R.C. James [J] and J.Lindenstrauss-C.Stegall [L-S], where the first examples of such spaces were provided. James example is the widely known James Tree space $JT$, which is a sequence space. The space $JT$ is also investigated in [L-S] where additionally a function space sharing similar properties is defined. The later is called the James function space $JF$. The space $JT$ and its variations have been studied extensively e.g. [A-I], [A], [B-H-O], [Ha]. The most impressive member of this class of spaces has been provided by W.T.

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Gowers [G]. Mixing the JT structure with H.I constructions, Gowers was able to present a separable Banach space not containing $\ell_1$ such that every subspace has non-separable dual. Thus his space does not contain $\ell_1$, $c_0$ or a reflexive subspace. Examples lying between JT and H.I spaces are also contained in [A-T]. Non separable versions of JT are contained in [A], [F]. In the present paper we deal with function spaces related to James function space, on which the norm is defined as follows:

The space $JF$ is the completion of $L^1((0,1))$ endowed with the following norm

$$
\|f\|_{JF} = \sup \left\{ \left( \sum_{j=1}^{m} (\int_{I_j} f \, d\mu)^2 \right)^{1/2} : \{I_j\}_{j=1}^{m} \text{ interval partition of } [0,1] \right\}.
$$

Our goal is to define and study norms extending the above norm. Thus we consider the following class of spaces.

Let $\Omega$ be a bounded open subset of $\mathbb{R}^{d_0}$. We denote by $\mathcal{P}(\Omega)$ the set of all open parallelepipeds contained in $\Omega$ (i.e each $T \in \mathcal{P}(\Omega)$ has the form $T = \prod_{d=1}^{d_0} (\alpha_d, \beta_d)$, $\alpha_d < \beta_d$ for every $d \leq d_0$). For $(X, \| \cdot \|_X)$ a reflexive Banach space with $1-$symmetric basis $(e_n)_n$ the space $JF_X(\Omega)$ is defined as the completion of $L^1(\Omega)$ under the norm,

$$
\|f\|_{JF_X(\Omega)} = \sup \left\{ \left( \sum_{j=1}^{m} (\int_{T_j} f \, d\mu) e_j \|X\| : \{T_j\}_{j=1}^{m} \subset \mathcal{P}(\Omega), T_i \cap T_j = \emptyset \right) \right\}. \quad (0.1)
$$

Clearly for $\Omega = (0, 1)$ and $X = \ell_2$, $JF_X(\Omega)$ is the James function space $JF$. Throughout the paper, by $JF_X$ we shall denote the space $JF_X((0,1))$.

It is an easy observation that for $f \in L^1(\Omega)$, $\|f\|_{JF_X(\Omega)} \leq \|f\|_{L^1}$, hence $JF_X(\Omega)$ is separable. It turns out that its dual $JF^*_X(\Omega)$ is non-separable. The main result of the paper is the following :

**Theorem A.** For $\Omega$ and $X$ as before the space $JF_X(\Omega)$ does not contain an isomorphic copy of $\ell_1$.

The proof of this result is of topological nature and it is different from Lindenstrauss-Stegall’s proof for $JF$, [L-S]. Thus our argument leads also to a new proof of this result in the case of $JF$. S.V. Kisliakov, [K], has also provided another proof of the fact that $\ell_1$ does not embed in $JF$. His elegant argument is also of topological nature, and it uses the representation of $JF$ as a subspace of the space of functions of bounded 2-variation.

For $\Omega$ subset of $\mathbb{R}^{d_0}$, $d_0 > 1$, our arguments use properties of the parallelepipeds and it is not clear to us, if the non embedding of $\ell_1$ holds for norms which are defined by families of convex sets different than $\mathcal{P}(\Omega)$.

It is worth noting that the structure of $JF_X(\Omega)$ depends on the geometry of the set $\Omega$. For example if $\Omega$ is a finite union of parallelepipeds, then on the positive cone of $L^1(\Omega)$ the $\| \cdot \|_1$ and $\| \cdot \|_{JF_X(\Omega)}$ are equivalent. This property is no longer true if $\Omega$ is...
the Euclidean ball of $\mathbb{R}^{d_0}$, $d_0 > 1$. However as we show in the fourth section, $JF_X$ is isomorphic to a complemented subspace of $JF_X(\Omega)$, for any $\Omega$ bounded open subset of $\mathbb{R}^{d_0}$. Also for $d_0 > 1$, $\Omega$ bounded open subset of $\mathbb{R}^{d_0}$, $JF_X(\Omega)$ is not isomorphic to a subspace of $JF_X$.

Let’s now pass to describe how this paper is organized.

The first section is devoted to the proof of Theorem A mentioned above. We also show that the formal identity $I : L^1(\Omega) \rightarrow JF_X(\Omega)$ is a Dunford-Pettis strictly singular operator.

In the second section we show that the Haar system is a Schauder basis of $JF_X$ and that $X$ is isomorphic to a complemented subspace of $JF_X$.

In the third section we study the quotients of $JF_X^*(\Omega)$. We prove two results which show an essential difference between the structures of $JF_X(\Omega)$, when $\Omega$ is either $(0, 1)$ or a bounded open subset of $\mathbb{R}^{d_0}$ with $d_0 > 1$. In particular we show the following:

**Proposition B.** Let $\Delta$ be a countable dense subset of $(0, 1)$ and let $Y$ be the closed subspace of $JF_X$ generated by the set $\{\chi_I : I$ is an interval with endpoints in $\Delta\}$. Then

$$JF_X^*/Y = X_R^*.$$  

Here $X_R$ denotes the space endowed with the norm induced on $c_0(\mathbb{R})$ by the space $X$. Clearly $X_R^*$ is a reflexive space. In particular, in the $JF$ case, we obtain that

$$JF^*/Y \equiv \ell_2(\mathbb{R}).$$

It is well known that $JT$ shares a similar property. Next we show the following.

**Proposition C.** Let $d_0 > 1$, and $\Omega$ be a bounded open subset of $\mathbb{R}^{d_0}$. Then for every separable subspace $Y$ of $JF_X^*(\Omega)$, the quotient $JF_X^*(\Omega)/Y$ is not reflexive.

Propositions B and C yield that for $\Omega$ as in Proposition C, $JF_X^*(\Omega)$ is not a quotient of $JF_X$. Hence $JF_X(\Omega)$ does not embed in $JF_X$.

In section 4, we prove that $JF_X((0, 1)^{d_1})$ is isomorphic to a complemented subspace of $JF_X(\Omega)$, where $\Omega$ denotes a bounded open subset of $\mathbb{R}^{d_0}$ and $d_1 < d_0$.

The fifth section is devoted to the isomorphic embedding of $c_0$ in $JF$. It is stated in [L-S] that there exists a subsequence of Rademacher functions equivalent to the usual basis of $c_0$. This is a peculiar property which has as consequence that $JF$ is not embedded into $JT$. The later holds since $JT$ is $\ell_2$ saturated, [J], [A-I]. In this section we characterize those reflexive Banach spaces $X$ with $1-$symmetric basis such that the Rademacher functions in $JF_X$ contain a subsequence equivalent to $c_0$ basis. It turns out that these spaces must satisfy a property, defined as Convex Combination Property (CCP). CCP trivially holds on $\ell_p$ spaces, $1 < p < \infty$, but not in all spaces with $1-$symmetric basis. For
example the Lorentz space $d(w, p)$, where $w = (\frac{1}{n})_n$ and $1 < p < \infty$, fails this property. Concerning the CCP we prove the following.

**Theorem D.** The following are equivalent:

1. The space $X$ satisfies CCP.
2. The normalized sequence $(\frac{r_n}{\|r_n\|})_{n\in\mathbb{N}}$ in $JF_X$ of Rademacher functions contains a subsequence equivalent to the usual basis of $c_0$.

The last section contains the study of alternative descriptions of $JF_X$ and $JF_X^{**}$. Namely we introduce the space of functions of $X$-bounded variation, which is defined as follows.

$$V_X = \{f : [0, 1] \to \mathbb{R} : f(0) = 0, \|f\|_{V_X} < \infty\}$$

where

$$\|f\|_{V_X} = \sup \{\|\sum_{i=1}^{n-1} (f(t_{i+1}) - f(t_i))e_i\|_X : \mathcal{P} = \{t_i\}_{i=1}^n \text{ partition of } [0, 1]\}. $$

We also consider the closed subspace $V^0_X = \{f \in V_X : \lim_{\delta(\mathcal{P})\to 0} \alpha_X(f, \mathcal{P}) = 0\}$ of $V_X$.

**Theorem E.** The following hold:

1. $JF_X$ is isometric to $V^0_X$.
2. $JF^{**}_X$ is isometric to $V_X$.
3. On the bounded subsets of $V^0_X$ the weak topology coincides with the topology of pointwise convergence in $C[0, 1]$.

The above representations of $JF_X$ as $V^0_X$ and its second dual as $V_X$ have certain advantages. For example $JF_X$, as a completion of $L^1(0, 1)$ is not contained in the set of measurable functions, while $V^0_X$ is contained in the continuous functions $C[0, 1]$. Further $V_X$ is a set of Baire-1 functions. The use of $V^0_X$, $V_X$ appears very useful in the study of the subspaces of $V^0_X$. Indeed we first investigate the properties of $f \in C[0, 1] \cap (V_X \setminus V^0_X)$ and prove the following.

**Proposition F.** A function $f \in C[0, 1] \cap (V_X \setminus V^0_X)$ iff $f$ is a difference of bounded semicontinuous functions.

For the proof of this result we make use of some recent results from descriptive set theory, [K-L], [Ro]. As a consequence of the above result we obtain the following theorem.

**Theorem G.** A subspace $Y$ of $V^0_X$ contains $c_0$ iff $C[0, 1] \cap (\overline{V^{**}} \setminus Y) \neq \emptyset$.

Furthermore for $X = \ell_p$, $1 < p < \infty$, we get

**Theorem H.** A non reflexive subspace $Y$ of $V^0_p$ either contains $c_0$ or $\ell_p$. 


Finally we prove the following.

**Theorem I.** A closed subspace \( Y \) of \( V^0_X \) has the point of continuity property, \((PCP)\) iff \((\overline{Y}^* \setminus Y) \cap C[0,1] = \emptyset\).

Theorems G and H concern the isomorphic structure of the spaces \( JF_X \). This is not completely clarified even in the case of \( JF \). A detailed study of \( JF \) has been provided by S. Buechler’s Ph.D. Thesis [B], where the following results are included. For all \( 2 \leq p < \infty \) and \( \varepsilon > 0 \) the space \( \ell_p \) is \((1 + \varepsilon)-\)isomorphic to a subspace of \( JF \), and also every normalized weakly null sequence has an unconditional subsequence. These two results indicate the richness and the regularity of \( JF \). Moreover it is shown that for \( 1 < p < 2 \), \( \ell_p \) is not isomorphic to a subspace of \( JF \).

In the last part of the paper we present some open problems related to our investigation.

1. **The non embedding of \( \ell_1 \) into \( JF_X(\Omega) \).**

This section contains the proof that \( \ell_1 \) does not embed into \( JF_X(\Omega) \). We start with some preliminary result concerning the structure of these spaces.

We recall, from the introduction, that \( JF_X(\Omega) \) is defined for each \( \Omega \) bounded open subset of \( \mathbb{R}^{d_0} \), and \( X \) reflexive space with 1–symmetric basis, and it is the completion of \( L^1(\Omega) \) under the norm described in the introduction, see (0.1). A direct application of the triangle inequality yields that for \( f \in L^1(\Omega) \),

\[
\| f \|_{JF_X(\Omega)} \leq \| f \|_1
\]

where \( \| \cdot \|_1 \) denotes the \( L^1(\Omega) \) norm. Hence \( JF_X(\Omega) \) is a separable Banach space. Further we recall that \( \mathcal{P}(\Omega) \) denotes the set of all open parallelepipeds contained in \( \Omega \). Every \( T \in \mathcal{P}(\Omega) \) with \( \mu_{d_0}(T) > 0 \), \( \mu_d \) throughout this paper, denotes Lebesgue measure in \( \mathbb{R}^d \), defines a bounded linear functional on \( JF_X(\Omega) \) under the rule

\[
L^1(\Omega) \ni f \mapsto T^*(f) = \int_T f \, d\mu
\]

We easily see that \( \| T^* \|_{JF_X^*(\Omega)} = 1 \). Also it is easy to see that for \( T_1, T_2 \in \mathcal{P}(\Omega) \) with \( T_1 \neq T_2 \), \( \| T_1^* - T_2^* \| \geq 1 \), hence the space \( JF_X^*(\Omega) \) is non-separable.

**Remark.** The functional \( T^* \) defined by a parallelepiped \( T \) is the same if \( T \) is considered either open or closed. Hence we shall not distinguish the cases if \( T \) is open or closed.

From the definition of the norm, it is easy to see that, if \( \Omega_1 = \prod_{i=1}^{d_0} (\alpha_i, \beta_i) \) and \( \Omega_2 = \prod_{i=1}^{d_0} (\gamma_i, \delta_i) \) then \( JF_X(\Omega_1) \) is 1–isometric to \( JF_X(\Omega_2) \). Also if \( \Omega \) is open bounded subset of \( \mathbb{R}^{d_0} \) and \( T \) an open parallelepiped contained in \( \Omega \), then \( JF_X(T) \) is 1–complemented subspace of \( JF_X(\Omega) \).
Notation. Throughout the paper we denote by $X$ a reflexive Banach space with 1-symmetric basis $(e_i)_i$, and by $\| \cdot \|_X$, $\| \cdot \|_{X^*}$ the norm in $X$ and $X^*$ respectively. The space $X$ satisfies the property $\lim_{n \to \infty} \| \sum_{i=1}^n e_i \| = \infty$ (c.f. [L-T]).

The following subset of $JF_X^*(\Omega)$ plays a key role in the proof of the main result of this section:

$$S = \left\{ \sum_{n=1}^k a_n T_n : \{T_n\}_{n=1}^k \text{ pairwise disjoint elements of } \mathcal{P}(\Omega) \text{ and } \| \sum_{n=1}^k a_n e_i^* \|_{X^*} \leq 1 \right\}.$$  

Observe that the definition of the norm of $JF_X(\Omega)$ yields that for $\phi \in S$, $\| \phi \|_{JF_X(\Omega)} \leq 1$.

Lemma 1.1. The set $S$ norms isometrically the space $JF_X(\Omega)$.

Proof. Indeed, since $S$ is a subset of $BJF_X(\Omega)$ we obtain that for $f \in L^1(\Omega)$,

$$\sup \{ \langle \phi, f \rangle : \phi \in S \} \leq \| f \|_{JF_X(\Omega)}.$$

For the converse given $\varepsilon > 0$, $f \in L^1(\Omega)$ choose $\{T_i\}_{i=1}^n$ disjoint elements of $\mathcal{P}(\Omega)$ such that

$$\| f \|_{JF_X(\Omega)} - \varepsilon \leq \| \sum_{i=1}^n \left( \int_{T_i} f \, d\mu \right) e_i \|_X.$$

Next choose $\{\alpha_i\}_{i=1}^n$ such that $\| \sum_{i=1}^n \alpha_i e_i^* \|_{X^*} = 1$ and

$$\sum_{i=1}^n \alpha_i \int_{T_i} f \, d\mu = \| \sum_{i=1}^n \left( \int_{T_i} f \, d\mu \right) e_i \|_{X^*}.$$

Clearly setting $\phi = \sum_{i=1}^n \alpha_i T_i^*$ we obtain that $\phi \in S$ and

$$\| f \|_{JF_X(\Omega)} - \varepsilon \leq \langle \phi, f \rangle.$$

\square

As a consequence we obtain the following

Lemma 1.2. The set $S$ contains the extreme points of $BJF_X(\Omega)$. Hence $BJF_X(\Omega) = \overline{co}^{w^*}(S)$.

Proof. Assume on the contrary, that there exists an extreme point $x^* \in BJF_X^*$ with $x^* \notin S$. Since the $w^*$—slices of $x^*$ define a neighborhood basis for the $w^*$-topology, there exists a slice $S(x^*, f, t)$ disjoint from $S$. We may assume further that $f$ is linear combination of characteristic functions and $\| f \| = 1$. From the above we get that

1. For some $\varepsilon > 0$, $x^*(f) > \sup \{ w^*(f) : w^* \in S \} + \varepsilon$.
2. There exists $\{T_j\}_{j=1}^n \subset \mathcal{P}(\Omega)$ such that

$$1 = \| f \|_{JF_X(\Omega)} \leq \| \sum_{j=1}^n \left( \int_{T_j} f \, d\mu \right) e_j \|_X + \frac{\varepsilon}{2}.$$
Let \( x^* = \sum_{j=1}^{\infty} \beta_j e_j^* \in B_X^* \) such that \( x^*(\sum_{j=1}^{\infty} f_{T_j} \, \text{d}\mu) \leq \left\| \sum_{j=1}^{\infty} f_{T_j} \, \text{d}\mu \right\| \). We set \( w^* = \sum_{j=1}^{\infty} b_j T_j^* \) and observe that \( w^* \in S \) and also

\[
 w^*(f) = \left\| \sum_{j=1}^{n} \left( \int_{T_j} f \, \text{d}\mu \right) e_j^* \right\|. 
\]

Summing up all the above we obtain

\[
 1 - \frac{\epsilon}{T} \leq \langle f, w^* \rangle < x^*(f) - \epsilon \leq 1 - \epsilon ,
\]

a contradiction, and the proof is complete. \( \square \)

**Lemma 1.3.** Let \((T_n)_{n \in \mathbb{N}}\) be a subset of \(\mathcal{P}(\Omega)\) and suppose that \(w^* - \lim_{n \to \infty} T_n^* = s^*\). Then either \(s^*\) is equal to the characteristic function of a parallelepiped \(T\) with \(T \in \mathcal{P}(\Omega)\) and \(T_n \to T\) pointwise or \(s^* = 0\) and \(\mu_{d_h}(T_n) \to 0\).

**Proof.** Choose any subsequence \((T_{nk})_k\) such that \(T_{nk} \to S\) pointwise. If \(T^0 \neq \emptyset\) then \(S^0 \in \mathcal{P}(\Omega)\), and \(\lim_{k \to \infty} \mu(T_{nk} \Delta S) = 0\). Set \(S^0 = T\). It follows that \(\int_{T_{nk}} \psi \to \int_T \psi, \forall \psi \in L^1(\Omega)\), and hence \(\int \psi s^* = \int \psi\) for all \(\psi \in L^1(\Omega)\), which yields that \(s^* = T^*\). If \(S^0 = \emptyset\) then it is easy to see that \(\mu_{d_h}(T_n) \to 0\) and \(T^*_n \to 0\). \( \square \)

As a consequence we obtain the following corollary.

**Corollary 1.4.** Let \(n_0 \in \mathbb{N}\) be fixed, \(s_{n_0}^* = \sum_{i=1}^{n_0} a_{i,n} T_{i,n}^*\) with \(\left\| \sum_{i=1}^{n_0} a_{i,n} e_i^* \right\|_{X^*} \leq 1\). Assume that \(w^* - \lim_{n \to \infty} s_n^* = s^*\). Then there exist \((a_i)_{i=1}^n\) in \(\mathbb{R}\) and disjoint parallelepipeds \(\{T_i\}_{i=1}^{n_0}\) such that \(\left\| \sum_{i=1}^{n_0} a_i e_i^* \right\|_{X^*} \leq 1\) and \(s^* = \sum_{i=1}^{n_0} a_i T_i^*\).

We define

\[
 V = \{ \sum_{n} \alpha_n e_n^* \in B_X^* : \{\alpha_n\}_n \text{ is in decreasing order} \} .
\]

It is easy to see that \(V\) is a \(w^*\)-compact subset of \(B_X^*\).

**Lemma 1.5.** Let \(s_n^* \in S\) and \(w^* - \lim_{n \to \infty} s_n^* = s^*\). Then \(s^*\) is of the form \(\sum_{i=1}^{\infty} a_i T_i^*\), with \(\left\| \sum_{i=1}^{\infty} a_i e_i^* \right\| \leq 1\) and \(\{T_i\}_i \subset \mathcal{P}(\Omega)\) pairwise disjoint. Moreover

\[
 \left\| \sum_{i} a_i T_i^* \right\| \leq \left\| \sum_{i} a_i e_i^* \right\|_{X^*} , \tag{1.1}
\]

and hence \(S^0 = S^w\).

**Proof.** Let \(s_n^* = \sum_{i=1}^{k_n} a_{i,n} T_{i,n}^*\), where \(\{T_{i,n}\}_{i=1}^{k_n}\) are pairwise disjoint and \(\left\| \sum_{i=1}^{k_n} a_{i,n} e_i^* \right\| \leq 1\). From the previous corollary we may assume that \(\lim \ k_n = \infty\), and also that for each \(n \in \mathbb{N}\), \(|a_{i,n}| \geq |a_{i+1,n}|\) for \(i = 1, 2, \ldots, k_n - 1\). Hence the sequence \((\sum_{i=1}^{k_n} a_{i,n} e_i^*)_n \subset V\). Since \(V\) is \(w^*\)-compact, by passing to a subsequence if it is necessary we assume that there exists a \(x^* = \sum_{i=1}^{\infty} a_i e_i^* \in V\) such that \(w^* - \lim_{n \to \infty} \sum_{i=1}^{k_n} a_{i,n} e_i^* = x^*\). We may also assume that there exists a sequence \(\{T_i\}_i\) of disjoint parallelepipeds in \(\mathcal{P}(\Omega)\) such that
\[ \lim_{n \to \infty} \mu_{d_0} (T_{i,n} \triangle T_i) = 0, \text{ for every } i = 1, 2, \ldots \] We set \( u^* = \sum_{i=1}^{\infty} \alpha_i T_i^* \). Since \( x^* \in B_X \), we have that \( u^* \in B_{JF_X^*(\Omega)} \).

**Claim.** \( w^* - \lim_{n \to \infty} s_n^* = u^* \).

**Proof of the Claim.** Let \( \varepsilon > 0 \) and \( f = \chi_T, T \) a parallelepiped in \( \mathcal{P}(\Omega) \). Since for every \( n \in \mathbb{N} \) the sequence \( \{a_{i,n}\}_i \) is decreasing and the space \( X \) is reflexive, we obtain that there exists \( i_0 \in \mathbb{N} \) such that \( |a_{i,n}| < \varepsilon \) for every \( i \geq i_0 \) and every \( n \in \mathbb{N} \). We may also assume that \( \| \sum_{i>i_0} a_i e_i^* \| < \varepsilon \). Let \( n_0 \in \mathbb{N} \) such that for every \( n \geq n_0 \), \( k_n \geq i_0 \), \( |a_{i,n} - a_i| \leq \frac{\varepsilon}{i_0} \), and \( \mu_{d_0} (T_{i,n} \triangle T_i) \leq \frac{\varepsilon}{i_0} \| f \| \) for every \( i = 1, 2, \ldots, i_0 \). Then for every \( n \geq n_0 \) we have that,

\[
\left| \sum_{i=1}^{i_0} a_{i,n} T_{i,n}^* (\chi_T) - \sum_{i=1}^{\infty} a_i T_i^* (\chi_T) \right|
\leq \sum_{i=1}^{i_0} a_{i,n} T_{i,n}^* (\chi_T) - \sum_{i=1}^{i_0} a_i T_i^* (\chi_T) + \sum_{i=i_0+1}^{k_n} a_{i,n} T_{i,n}^* (\chi_T) + \sum_{i=i_0+1}^{\infty} a_i T_i^* (\chi_T)
\leq \varepsilon \| \chi_T \| + \varepsilon \| \chi_T \| + \varepsilon \| \chi_T \| + \varepsilon \| \chi_T \| \leq 4\varepsilon \| \chi_T \| .
\]

Since the linear span of the characteristic of parallelepipeds in \( \Omega \) is a dense subset of \( JF_X(\Omega) \), we have the result. Inequality (1.1) follows directly from the definition of the norming set \( S \) and immediately implies that \( \bar{S}^w = \bar{S}^w^* \).

**Corollary 1.6.** Let \((f_n)_n \) be a bounded sequence in \( JF_X(\Omega) \). Then \((f_n)_n \) is weakly Cauchy iff for every \( T \in \mathcal{P}(\Omega) \) the numerical sequence \( T^*(f_n)_n \) is convergent.

**Proof.** Assume that \((T^*(f_n))_n \) is convergent for all \( T \in \mathcal{P}(\Omega) \). Then the same remains valid for every \( x^* \) in the norm closure of the linear span of the set \( \{T^* : T \in \mathcal{P}(\Omega)\} \).

The later subspace of \( JF_X^*(\Omega) \) contains \( \bar{S}^w \) which from, Lemma [1.5] coincides with \( \bar{S}^w^* \). Lemma [1.2] yields that \( \bar{S}^w^* \) contains the extreme points of the ball of \( JF_X^*(\Omega) \), and the result is obtained from Rainwater theorem [R]. The other direction is obvious.

We pass now to prove that \( \ell_1 \not\hookrightarrow JF_X(\Omega) \) for every \( \Omega \) open and bounded subset of \( \mathbb{R}^{d_0} \), and every reflexive Banach space \( X \) with 1-symmetric basis. Lemmas [1.7][1.8] will be the main ingredients for the proof. For a given sequence \((f_n)_n \), using these lemmas we choose constructively a \( w - \text{Cauchy} \) subsequence. As we have mentioned in the introduction, it is not clear to us, if we can have corresponding results for norms defined by convex subsets different from parallelepipeds.

**Notation.** Let \( T = \Pi_{i=1}^{d_0} (\alpha_d, \beta_d) \) be an element of \( \mathcal{P}(\Omega) \). In the sequel for \( d \leq d_0 \) we denote by \( m_d(T) \) the number \( \beta_d - \alpha_d \).
Let \((f_n)_n\) be a normalized sequence in \(JFX(\Omega) \cap L_1(\Omega)\). We shall show that \((f_n)_n\) contains a \(w\)–Cauchy subsequence. We start with the following lemma.

**Lemma 1.7.** For every \(\varepsilon > 0\), there exists \(n(\varepsilon) \in \mathbb{N}\) such that for every \(1 \leq d \leq d_0\), \(L \in [\mathbb{N}]\) and \(\delta > 0\) there exist \(L' \in [L]\) and \(n(\varepsilon)\) disjoint parallelepipeds \(T_{\varepsilon,d,1}, \ldots, T_{\varepsilon,d,n(\varepsilon)}\) with \(m_d(T_{\varepsilon,d,i}) < \delta\) for all \(i \leq n(\varepsilon)\), satisfying the following property:

For each \(d \leq d_0\) and every parallelepiped \(T \in \mathcal{P}(\Omega)\) which is disjoint from \(\{T_{\varepsilon,d,i}\}_{i=1}^{n(\varepsilon)}\), with the property \(m_d(T) < \delta\) we have that

\[
\limsup_{n \in L'} \int_T f_n \, d\mu \leq \varepsilon.
\]

In the above lemma, when \(d_0 = 1\) we require the measure of the interval be arbitrarily small. In higher dimensions, for our consideration, it is not sufficient that \(\mu(T)\) be arbitrarily small, but we require that \(m_d(T)\) be arbitrarily small for each of the sides of the parallelepiped. This is due mainly to the geometry of parallelepipeds of higher dimensions.

**Proof.** On the contrary, suppose that the conclusion does not hold. Then there exists \(\varepsilon_0 > 0\) such that for every \(n \in \mathbb{N}\) there exist \(d_n \leq d_0\), \(L(n) \in [\mathbb{N}]\) and \(\delta_n > 0\) such that for every \(L' \in [L(n)]\) and disjoint parallelepipeds \(T_1, T_2, \ldots, T_n\), with \(m_d(T_j) < \delta_n\), \(j \leq n\), there exists a parallelepiped \(T\) disjoint from \(T_j\), \(j \leq n\), with \(m_d(T) < \delta_n\) and

\[
\limsup_{n \in L'} \int_T f_n \, d\mu > \varepsilon_0.
\]

Let \(m \in \mathbb{N}\) be such that \(\varepsilon_0 \| \sum_{i=1}^{m} e_i \| > 1\). We set \(d = d_m\), \(\delta = \delta_m\) and \(L = L(m)\). Next we inductively choose disjoint parallelepipeds \(\{S_j\}_{j=1}^m\) with \(m_d(S_j) < \delta\) and \(L_1 \supset L_2 \supset \ldots \supset L_m\) of \(L\), such that for every \(1 \leq j \leq m\) and \(n \in L_j\), \(\|f_{S_j} f_n\| \geq \varepsilon_0\). The choice goes as follows:

We choose \(S_1 \in \mathcal{P}(\Omega)\) with \(m_d(S_1) < \delta\). From our assumption \(\limsup_{L} \|f_{S_1} f_n\| > \varepsilon_0\), hence there exists \(L_1 \subseteq L\) such that \(\|f_{S_1} f_n\| \geq \varepsilon_0\). This completes the choice of \(S_1\) and \(L_1\). Assume that \(S_1, \ldots, S_{j-1}, L_1 \supset \ldots \supset L_{j-1}\), \(j \leq m\), have been chosen satisfying the inductive assumption. Since \(j - 1 < m\) and \(L_{j-1} \subset L\), there exists \(S_j \in \mathcal{P}(\Omega)\) disjoint from \(S_1, \ldots, S_{j-1}\) with \(m_d(S_j) < \delta\) and \(\limsup_{L_{j-1}} \|f_{S_j} f_n\| > \varepsilon_0\). We choose \(L_j \subset L_{j-1}\) such that \(\|f_{S_j} f_n\| \geq \varepsilon_0\) for all \(n \in L_j\) and this completes the inductive construction.

Take any \(n \in L_m\). Then \(n \in L_j\) for all \(1 \leq j \leq m\) and hence, \(\|f_{L_j} f_n \mu\| \geq \varepsilon_0\). Hence

\[
\|f_n\| \geq \| \sum_{j=1}^{m} \left( \int_{S_j} f_n \, d\mu \right) e_j \| \geq \varepsilon_0 \| \sum_{j=1}^{m} e_j \| > 1,
\]

which contradicts our assumption that \(\|f_n\| = 1\).

**Selection of a \(w\)–Cauchy subsequence.**
To obtain the desired \( w - \text{Cauchy} \) subsequence of \((f_n)_n\) we shall apply repeatedly Lemma 1.7 in the following manner:

Fix \( \varepsilon = \frac{1}{m} \), and \( 1 \leq d \leq d_0 \). Then Lemma 1.7 yields that there exists \( n(k) \) such that for every \( L \in [N] \) and every \( \delta = \frac{1}{m} \), a finite sequence \( \{T_{k,d,1}, \ldots, T_{k,d,n(\varepsilon)}\} \) and \( L_m^d(\varepsilon) \subset L \) are defined so that the conclusion of Lemma 1.7 is fulfilled.

Therefore for a fixed \( k \) we inductively define sequences \( \{L_m^d(k)\}_m, \{T_{k,d,1}^d, \ldots, T_{k,d,n(k)}^d\}_m \) such that

1. \( \{L_m^d(k)\}_m \) is a decreasing sequence of subsets of \( N \).
2. For each \( m \in N \) the pair \( \{T_{k,d,1}^d, \ldots, T_{k,d,n(k)}^d\} \) and \( L_m^d(k) \) satisfies the conclusion of Lemma 1.7 for \( \delta = \frac{1}{m} \) i.e.

\[
\limsup_{n \in L_m^d(k)} \int_T |f_n d\mu| \leq \frac{1}{m},
\]

(1.2)

Notice that for a fixed \( k \) and \( d \leq d_0 \), \( \lim_{m \to \infty} m_d(T_{k,d,i}^d) = 0 \) for every \( i \leq n(k) \). Hence passing to a subsequence we may assume that there exist \( n(k) \) faces \( H_{k,d,1}, \ldots, H_{k,d,n(k)} \), each one of the form

\[
\prod_{i=1}^{d_0} [\alpha_i, \beta_i] \ 	ext{with} \ \alpha_d = \beta_d,
\]

and moreover

\[
\lim_{m \to \infty} dist_H(H_{k,d,i}, T_{k,d,i}^d) = 0 \ \text{for all} \ i \leq n(k),
\]

(1.3)

where \( \text{dist}_H \) denotes the Hausdorff distance. We set \( A_{k,d} \) be the set of all coordinates of the extremes points of the faces \( H_{k,d,1}, \ldots, H_{k,d,n(k)} \). Clearly \( A_{k,d} \) is a finite set.

Let \( L_m^{d}(k) \) denote any diagonal set of the decreasing sequence \( \{L_m^d(k)\}_m \). Applying the above procedure inductively we obtain

\[
L_m^{d_0}(k) \supset \ldots \supset L_m^{d}(k) = L^{d_0}(k),
\]

(1.4)

\[
\{T_{k,d,1}^d, \ldots, T_{k,d,n(k)}^d\}_{m \in N}, \ 1 \leq d \leq d_0,
\]

\[
\{A_{k,d}\}, \ 1 \leq d \leq d_0, \ A_{k,d} \text{ is finite subset of } \mathbb{R},
\]

such that for every \( d \leq d_0, \ L^{d}(k), \ \{T_{k,d,1}^d, \ldots, T_{k,d,n(k)}^d\}_{m \in N} \) satisfies (1.2), and \( A_{k,d} \) is defined as above.

Proceeding now by induction for \( k = 1, 2, \ldots \) we choose \( \{L^{d_0}(k)\}_k \) a decreasing sequence of infinite sets, \( \{\{T_{k,d,1}^d, \ldots, T_{k,d,n(k)}^d\}_{m \in N, d \leq d_0}\}_k \) and \( \{\{A_{k,d}\}_{d=1}^{d_0}\}_k \) such that for any \( k \in N \), the corresponding families satisfies (1.4). Set

\[
F = \bigcup_{k \in N} \bigcup_{d=1}^{d_0} A_{k,d} \cup Q.
\]
Clearly \( F \) is a countable set and hence the set \( M \) of the parallelepipeds in \( \mathcal{P}(\Omega) \) with vertices in \( F^{d_0} \) is also countable. Therefore there exists a diagonal subset \( L \) of \( \{ L_\infty(k) \}_k \) such that

\[
\lim_{n \in L} \int_T f_n \, d\mu \quad \text{exists for every } T \in M .
\]

Our intention is to show that \((f_n)_{n \in L} \) is \( w-Cauchy \). This follows from the next lemma.

**Lemma 1.8.** For any \( T \) in \( \mathcal{P}(\Omega) \)

\[
\lim_{n \in L} \int_T f_n \, d\mu \quad \text{exists} .
\]

**Proof.** Let \( k \in \mathbb{N} \). It is enough to show that

\[
\limsup_{n \in L} \int_T f_n \, d\mu - \liminf_{n \in L} \int_T f_n \, d\mu < \frac{4d_0}{k} .
\]

Let \( T = \{ (x_1, x_2, \ldots, x_{d_0}) : \alpha_d < x_d < \beta_d, \forall d \leq d_0 \} \) and for \( d \leq d_0 \),

\[
\Pi^1_d = \{ (x_1, x_2, \ldots, x_{n_0}) \in T : x_d = \alpha_d \text{ and } \alpha_j < x_j < \beta_j, \text{ for } j \neq d \} ,
\]

\[
\Pi^2_d = \{ (x_1, x_2, \ldots, x_{n_0}) \in T : x_d = \beta_d \text{ and } \alpha_j < x_j < \beta_j, \text{ for } j \neq d \} .
\]

We set

\[
I_1 = \{ 1 \leq d \leq d_0 : \{ a_d, b_d \} \not\subset F \} .
\]

We assume that \( I_1 \) is not empty, otherwise \( T \) belongs in \( \mathcal{M} \) and therefore \( \lim_{n \in L} \int_T f_n \, d\mu \) exists. For every \( d \leq d_0 \) with \( d \in I_1 \), \#\{\( a_d, b_d \)\} \cap \( F \geq 1 \). For simplicity we assume that for every \( d \in I_1 \), \#\{\( a_d, b_d \)\} \cap \( F = 2 \). The proof of the general case follows similar arguments.

Hence, we assume that for every \( d \in I_1 \),

\[
\{ \alpha_d, \beta_d \} \cap \{ pr_d(H_{k,d,i}) , i \leq n(k) \} = \emptyset ,
\]

where \( pr_d \) denotes the \( d \)-projection in \( \mathbb{R}^{d_0} \).

For every \( d \in I_1 \), we set

\[
\delta_d^1 = \min \{ \text{dist}(a_d, \text{pr}_d(H_{k,d,i})) : i \leq n(k) \} > 0 ,
\]

and

\[
\delta_d^2 = \min \{ \text{dist}(b_d, \text{pr}_d(H_{k,d,i})) : i \leq n(k) \} > 0 .
\]

Set \( \delta_d = \min \{ \delta_d^1, \delta_d^2 \} \), and \( \delta_0 = \frac{1}{4} \min \{ \delta_d : d \in I_1 \} > 0 \). We choose \( m \in \mathbb{N} \) such that

\[
\frac{1}{m} < \frac{\delta_0}{10} \quad \text{and} \quad \text{dist}_H(H_{k,d,i}, T_{k,d,i}) < \frac{\delta_0}{10} , \text{ for every } i \leq n(k) \text{ and every } d \in I_1 .
\]

For each \( d \in I_1 \) we choose \( p_d, q_d \in F \) such that

\[
0 < p_d - \alpha_d < \frac{1}{m} \quad \text{and} \quad 0 < \beta_d - q_d < \frac{1}{m} .
\]

For \( d \in I_1 \), we consider the following parallelepipeds

\[
S^1_d = \{ (x_1, \ldots, x_{d_0}) \in T : \alpha_d < x_d < p_d \text{ and } \alpha_j < x_j < \beta_j \text{ for } j \neq d \} .
\]
\[ S_d^i = \{(x_1, \ldots, x_d_0) \in T : q_d < x_d < \beta_d \text{ and } \alpha_j < x_j < \beta_j \text{ for } j \neq d \} . \]

We observe that \( m_d(S_d^i) < \frac{1}{m} \), for \( i = 1, 2 \) and \( d \in I_1 \). Furthermore \( S_d^i \) is disjoint from the elements of the set \( \{T_{k,d,i}^n \mid i \leq n(k)\} \). Clearly the above two properties are also satisfied by any parallelepiped \( R \) which is contained in \( S_d^i \). The properties of \( \{T_{k,d,i}^n\}_{k=1}^n \), yield that for every parallelepiped \( R \) contained in \( S_d^i \) we have that
\[
\limsup_{n \in L} \int_R f_n d\mu - \liminf_{n \in L} \int_R f_n d\mu \leq \frac{2}{k}.
\] (1.5)

Let \( K \) be the parallelepiped
\[ K = \{(x_1, \ldots, x_n_0) \in T : p_d \leq x_d \leq q_d, \text{ for } d \in I_1, \text{ otherwise } \alpha_d < x_d < \beta_d \} . \]

Clearly the parallelepiped \( K \) has vertices in \( F_0^d \) and hence
\[
\limsup_{n \in L} \int_K f_n d\mu - \liminf_{n \in L} \int_K f_n d\mu = 0.
\] (1.6)

For every \( d \in I_1 \) let,
\[ T_d^1 = \{(x_1, \ldots, x_d_0) \in T : x_j \in pr_d(K) \text{ for every } j < d, \alpha_d < x_d < p_d \text{ and } \alpha_j < x_j < \beta_j \text{ for } j > d \} \]

and
\[ T_d^2 = \{(x_1, \ldots, x_d_0) \in T : x_j \in pr_d(K) \text{ for every } j < d, q_d < x_d < \beta_d \text{ and } \alpha_j < x_j < \beta_j \text{ for } j > d \} . \]

The following properties are easily established.

1. For \( d_1, d_2 \in I_1, i, j \in \{1, 2\} \) such that \( T_{d_1}^i \neq T_{d_2}^j \), we have that \( \mu_{d_0}(T_{d_1}^i \cap T_{d_2}^j) = 0 \).
2. \( T = \bigcup_{d \in I_1} (T_d^1 \cup T_d^2) \cup K \).
3. \( T_d^i \) is contained in \( S_d^i \) for every \( d \in I_1, i = 1, 2 \).

Then (3) and (1.5) yield
\[
\limsup_{n \in L} \int_{T_d^1} f_n d\mu - \liminf_{n \in L} \int_{T_d^1} f_n d\mu \leq \frac{2}{k}.
\] (1.7)

Finally (1), (2), (1.6) and (1.7) yield
\[
\limsup_{n \in L} \int_T f_n d\mu - \liminf_{n \in L} \int_T f_n d\mu \leq \frac{4d_0}{k}.
\]

The proof is complete.

**Theorem 1.9.** Let \( \Omega \) be an open bounded subset of \( \mathbb{R}^{d_0} \) and \( X \) be a reflexive Banach space with symmetric basis. Then \( \ell_1 \) does not embed into \( JF_X(\Omega) \).
Proof. It is enough to show that every normalized sequence \((f_n)_n\) in \(JF_X(\Omega)\) has a \(w\)-Cauchy subsequence. A perturbation argument yields that \((f_n)_n\) could be assumed to belong to \(L^1(\mu)\). Lemmas\[1.7\] and \[1.8\] yield that it contains a \(w\)-Cauchy subsequence. □

In [Pe] it has been shown that the identity \(I : L^1([0,1]) \hookrightarrow JF_X\) is a strongly regular operator. This result is naturally extended to the corresponding \(I : L^1(\Omega) \hookrightarrow JF_X(\Omega)\). The proof of it uses martingale techniques. A rather simple argument yields that \(I\) is Dunford-Pettis operator (i.e maps weakly compact sets to norm compact) a property weaker than the strong regularity. For sake of completeness we include a proof of it. The proof uses the following

**Lemma 1.10.** [Pe] Let \(X\) be a Banach space with 1-symmetric basis, not containing \(\ell_1\). Then for each \(\varepsilon > 0\) there exists \(\delta > 0\) such that for every \(\{\alpha_i\}_{i=1}^k\) with \(\sum_{i=1}^k |\alpha_i| \leq 1\) and \(\max\{|\alpha_i| : i \leq k\} \leq \delta\) we have

\[
\| \sum_{i=1}^k \alpha_i e_i \| < \varepsilon.
\]

**Proposition 1.11.** Let \((f_n)_n\) be a weakly null sequence in \(L^1(\Omega)\). Then \(\|f_n\|_{JF_X(\Omega)} \to 0\).

Proof. Assume on the contrary that there exists a normalized weakly null sequence \((f_n)_n\) in \(L^1(\Omega)\) so that for all \(n \in \mathbb{N}\)

\[
\|f_n\|_{JF_X(\Omega)} > \varepsilon > 0.
\]

For each \(f_n\) choose a family \(\{T_{n,j}\}_{j=1}^{k_n} \in \mathcal{P}(\Omega)\) such that

\[
\| \sum_{j=1}^{k_n} \left( \int_{T_{n,j}} f_n d\mu \right) e_j \|_X > \frac{\varepsilon}{2}. \tag{1.8}
\]

Further we observe that

\[
\sum_{j=1}^{k_n} \left| \int_{T_{n,j}} f_n d\mu \right| \leq 1. \tag{1.9}
\]

Hence Lemma\[1.10\] yields that there exists \(\delta > 0\) such that for each \(n \in \mathbb{N}\) there exists \(j_n\) satisfying

\[
\left| \int_{T_{n,j_n}} f_n d\mu \right| > \delta.
\]

Passing, if it is necessary, to a subsequence we may assume that \(T_{n,j_n} \to T\).

The uniform integrability of \((f_n)_n\) guarantees that \(\lim_{n \to \infty} \left| \int_T f_n d\mu \right| > \frac{\varepsilon}{2}\), which contradicts the weak convergence of \((f_n)_n\) to zero. □

**Corollary 1.12.** The identity \(I : L^1(\Omega) \hookrightarrow JF_X(\Omega)\) is Dunford-Pettis and strictly singular operator.
Proof. Proposition [1.11] yields immediately that the identity is a Dunford - Pettis operator. The strict singularity follows from a well known property of $L_1(\Omega)$, namely every closed subspace $Z$ of it either is reflexive or contains $\ell_1$. Since the identity is D-P operator we obtain that it is not isomorphism on any reflexive subspace of $L_1(\Omega)$, while Theorem [1.9] yield that the identity is not isomorphism on any subspace containing $\ell_1$. Hence it is strictly singular. □

2. The Haar system in $JF_X$.

In this section we give a simple proof that the Haar system $(h_n)_n$ is a basis for $JF_X$. We also prove that $X$ is isomorphic to a complemented subspace of $JF_X$. Our approach is based on elementary properties of symmetric sequences.

Notation. For every $Q = \{I_j\}_{j=1}^m$ partition of $[0,1]$, we set 

$$\tau(Q, f) = \| \sum_{j=1}^m (\int_{I_j} f) e_j \|_X.$$ 

Proposition 2.1. Let $f \in JF_X$ and $P_0 = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[0,1]$ such that $f|_{(t_{i-1}, t_i)} = \alpha_i$ for every $i = 1, 2, \ldots, n$. Then for every partition $Q = \{I_j\}_{j=1}^n$ of $[0,1]$ into disjoint intervals, there exists a partition $P \subseteq P_0$ such that $\tau(Q, f) \leq \tau(P, f)$.

This result is very useful for computing norms of functions in $JF_X$ and studying the structure of the space. One immediate consequence is that the Haar system is a Schauder basis for $JF_X$.

To give an idea of the proof consider a partition $P_0 = \{0 = t_0 < t_1 < t_2 < t_3 = 1\}$ of $[0,1]$ into three subintervals, $T_i = [t_{i-1}, t_i]$, $i = 1, 2, 3$. Assume that the function $f$ on $[0,1]$ takes value $\alpha_i$ on $T_i$, $i \leq 3$, and $\alpha_1, \alpha_3 \geq 0$ and $\alpha_2 \leq 0$. One can check that 

$$\|f\|_{JF} = \max \{ \int_0^1 f, ||(\int_{T_1} f) e_1 + (\int_{T_2} f) e_2 + (\int_{T_3} f) e_3 \|_X \} .$$

The proof for the general case uses finite induction to replace any partition $Q$ of $[0,1]$ by a partition $Q' \subset P_0$, so that $\tau(f, Q) \leq \tau(f, Q').$

This proposition will give us as corollary the result of this section. In the proof we shall use the following that restates a result from [Pe].

Lemma 2.2. Let $X$ be a Banach space with a $1-$symmetric basis $(e_i)_{i \in \mathbb{N}}$ and $x = \sum_{n=1}^k \alpha_n e_n$ a linear combinations with $\alpha_n \geq 0$. Then for every $\{\sigma_j\}_{j=1}^\ell$ disjoint partition of $\{1, \ldots, n\}$ we have that 

$$\|x\|_X \leq \| \sum_{j=1}^\ell \left( \sum_{n \in \sigma_j} \alpha_n \right) e_j \|_X .$$
Lemma 2.3. Let \( X \) be a Banach space with 1-symmetric basis \( (e_i)_i \). Let \( x = \sum_{i=1}^{j-1} \beta_i e_i + (\beta_j + \lambda_j \alpha) e_j + \lambda_{j+1} \alpha e_{j+1} + \sum_{i=j+2}^{r} \beta_i e_i \), where \( \lambda_j, \lambda_{j+1} \geq 0 \) and \( \lambda_j + \lambda_{j+1} \leq 1 \). Then

\[
\|x\| \leq \begin{cases} 
\|\sum_{i=1}^{j-1} \beta_i e_i + (\beta_j + (\lambda_j + \lambda_{j+1}) \alpha e_j + \sum_{i=j+2}^{r} \beta_i e_i\| & \text{if } \alpha \cdot \beta_j > 0 \\
\|\sum_{i=1}^{j-1} \beta_i e_i + \beta_j e_j + (\lambda_j + \lambda_{j+1}) \alpha e_{j+1} + \sum_{i=j+2}^{r} \beta_i e_i\| & \text{if } \alpha \cdot \beta_j < 0 
\end{cases}
\]

Proof. Indeed, if \( \beta_j \cdot \alpha > 0 \) using the Lemma 2.2 for the block \( (\beta_j + \lambda_j \alpha) e_j + \lambda_{j+1} \alpha e_{j+1} \), we have the inequality. If \( \beta_j \cdot \alpha < 0 \), we may assume that \( \alpha > 0 \) since the basis is 1-unconditional. Then \( |\beta_j + \lambda_j \alpha| \leq \max\{|\beta_j|, |\lambda_j \alpha|\} \). Substituting the coefficient of \( e_j \) by the coefficient \( \max\{|\beta_j|, |\lambda_j \alpha|\} \) and adding the term \( \min\{|\beta_j|, |\lambda_j \alpha|\} e_m \) and reordering if necessary, we have that the norm will increase, due to the symmetric property of the basis. Using Lemma 2.2 we have the result. \( \square \)

Proof of Proposition 2.4 We use finite induction in order to replace the partition \( Q = \{I_j\} \) by a partition \( P = \{S_i\} \) with endpoints in \( P_0 \) such that \( \tau(Q, f) \leq \tau(P, f) \). In each step of the induction we shall replace some of the intervals \( I_j \) by appropriate intervals \( S_j \).

For every partition \( P = \{R_j\}_j \) we may assume that \( \max R_{j-1} = \min R_j \), for every \( j \), otherwise we add the interval \( (\min R_{j-1}, \max R_j) \) in the partition, and we have that \( \tau(P, f) \) increases, due the symmetric property of the basis of \( X \).

For every \( i = 1, 2, \ldots, n \) we set \( T_i = [t_{i-1}, t_i] \). For every \( i = 1, 2, \ldots, n \), let \( A_i = \{1 \leq j \leq m : I_j \subseteq T_i\} \). We also set \( B = \{1 \leq j \leq m : I_j \cap T_i \neq \emptyset \text{ for at least two } i's, 1 \leq i \leq n\} \). Let \( A_i = \{j_{i1}, j_{i1}+1, \ldots, j_{ii}+r_i\} \neq \emptyset \). From Lemma 2.2 we have that

\[
\tau(Q, f) \leq \|\sum_{j \in B} (\int_{I_j} f) e_j + \sum_{i:A_i \neq \emptyset} \sum_{j \in A_i} (\alpha_i \mu(\cup_{j \in A_i} I_j)) e_{j_{i1}}\|.
\]

For every \( i \) such that \( A_i \neq \emptyset \) we replace \( \cup_{j \in A_i} I_j \) by the maximal interval contained in \( T_i \) and it is disjoint with \( I_j, j \in B \). We have that \( \tau(f, Q) \) increases, due the symmetric property of the basis. In the sequel, from the above observation, we shall assume that if two intervals, in the inductive construction, intersect an interval \( T_i \), at least one of them intersects another interval \( T_j \) as well.

Assume that we have replaced the partition \( \{I_i\}_i \) by a partition \( \{S_1, \ldots, S_{l-1}, I_l, I_{l+1}, \ldots, I_m\} \) which increases \( \tau(Q, f) \), such that the intervals \( S_i, i \leq l-1 \), have endpoints in \( P_0 \) and also that we have replaced \( I_l \) by an interval such that the initial point belongs to \( P_0 \). Let \( t_{ji} \in P_0 \) be such that \( t_{ji-1} < \max I_i < t_{ji} \). We distinguish two cases.

Case 1. \( I_l \subset T_{ji} = [t_{ji-1}, t_{ji}] \).
We have assumed that $I_{l+1}$ is not contained in $T_{j_i}$. From the hypothesis for $f$ we have that

$$
\int_{I_{l+1}} f = \int_{I_{l+1} \cap T_{j_i}} f + \int_{I_{l+1} \setminus T_{j_i}} f = \alpha_{j_i} \mu(I_{l+1} \cap T_{j_i}) + \int_{I_{l+1} \setminus T_{j_i}} f .
$$

We have two subcases:

Subcase 1a. $(\int_{I_{l+1} \setminus T_{j_i}} f) \cdot \alpha_{j_i} < 0$.

Then, from Lemma [2.3] for the block $(\int_{I_l} f) e_l + (\int_{I_{l+1}} f) e_{l+1}$, and the inductive hypothesis, replacing $I_l$ by $S_l = T_{j_i} \supset I_l \cup (I_{l+1} \cap T_{j_i})$ we have that

$$
\tau(Q, f) \leq \left\| \sum_{j=1}^{l-1} \left( \int_{S_j} f \right) e_j + \left( \int_{S_l} f \right) e_l + (\int_{I_{l+1} \setminus T_{j_i}} f) e_{l+1} + \sum_{j \geq l+1} \left( \int_{I_j} f \right) e_j \right\|_X .
$$

We also replace the interval $I_{l+1}$ by the interval $S_{l+1} = I_{l+1} \setminus T_{j_i}$ which has initial point in $\mathcal{P}_0$.

Subcase 1b. $(\int_{I_{l+1}} f) \cdot \alpha_{j_i} > 0$.

In this case, using Lemma [2.3] we can replace the interval $I_l$ by the interval $S_l = [\min T_{j_i}, \max I_{l+1}]$, which has initial point in $\mathcal{P}_0$ and we delete the interval $I_{l+1}$ and therefore the coefficient of $e_{l+1}$. The norm increases, since

$$
\left| \alpha_{j_i} \mu(I_l) + \alpha_{j_i} \mu(I_{l+1} \cap T_{j_i}) + \int_{I_{l+1} \setminus T_{j_i}} f \right| \leq \left| \alpha_{j_i} \mu(T_{j_i}) + \int_{I_{l+1} \setminus T_{j_i}} f \right| = \left| \int_{S_l} f \right| .
$$

For the interval $[\min T_{j_i}, \max I_{l+1}]$, it could be the case that $\max I_{l+1} \not\in \mathcal{P}_0$. If such a case occurs, then in case 2 we show how to replace it.

Case 2. $I_l \not\subseteq T_{j_i} = [t_{j_i-1}, t_{j_i}]$.

Then the interval $I_l$ intersects an interval $T_i$ for some $i < j_i$, since $\max I_l < t_{j_i}$. From the hypothesis for $f$ we have that

$$
\int_{I_l} f = \int_{I_l \setminus T_{j_i}} f + \alpha_{j_i} \mu(I_l \cap T_{j_i}) .
$$

We have two subcases:

Subcase 2a. $(\int_{I_l \setminus T_{j_i}} f) \cdot \alpha_{j_i} > 0$.

By our assumptions we have that $I_{l+1} \cap \not= \emptyset$. Using Lemma [2.2] for the block $(\int_{I_l} f) e_l + (\int_{I_{l+1}} f) e_{l+1}$ we may assume that $I_{l+1} \not\subseteq T_{j_i}$. From the hypothesis for $f$ we have

$$
\int_{I_{l+1}} f = \alpha_{j_i} \mu(I_{l+1} \cap T_{j_i}) + \int_{I_{l+1} \setminus T_{j_i}} f .
$$

If $\alpha_{j_i} \cdot (\int_{I_{l+1} \setminus T_{j_i}} f) > 0$, applying Lemma [2.2] for the block $(\int_{I_l} f) e_l + (\int_{I_{l+1}} f) e_{l+1}$ we get that,

$$
\tau(Q, f) \leq \left\| \sum_{j=1}^{l-1} \left( \int_{S_j} f \right) e_j + \left( \int_{I_l \cup T_{j_i} \cup I_{l+1}} f \right) e_l + \sum_{j \geq l+2} \left( \int_{I_j} f \right) e_j \right\|_X .
$$
We have replaced the intervals \( I_i, I_{i+1} \) by the interval \( S_i = I_i \cup T_{j_i} \cup I_{i+1} \) which has initial point in \( P_0 \).

If \( (\int_{I_i \setminus T_{j_i}} f) \cdot \alpha_{j_i} < 0 \) using the symmetric property of the basis and Lemma 2.3 replacing \( I_i \) by \( S_i = [\min I_i, \max T_{j_i}] \), we get that

\[
\tau(Q, f) \leq \| \sum_{j=1}^{l-1} (\int_{S_j} f) e_j + (\int_{S_i} f) e_i + (\int_{I_{i+1} \setminus T_{j_i}} f) e_{i+1} + \sum_{j>l+1} (\int_{I_j} f) e_j \| \cdot X.
\]

We also replace the interval \( I_{i+1} \) by the interval \( S_{i+1} = I_{i+1} \setminus T_{j_i} \) which has initial point in \( P_0 \).

**Subcase 2b.** \( (\int_{I_i \setminus T_{j_i}} f) \cdot \alpha_{j_i} < 0 \).

In this case we replace \( I_i \) by \( S_i = I_i \setminus T_{j_i} \), which has endpoints in \( P_0 \), and we add the term \( (\int_{I_i \setminus T_{j_i}} f) e_{i+1} \), transferring the sum \( \sum_{j \geq l+1} (\int f) e_j \). The interval \( I_i \setminus T_{j_i} \) has initial point in \( P_0 \), and we follow the arguments of Case 1 for the interval \( I_i \setminus T_{j_i} \).

Following the above arguments for the intervals which we get in the above cases, with initial point in \( P_0 \), we have the result. \( \square \)

Let us recall the definition of the Haar system \((h_n)_n\). We set \( h_1 = \chi_{[0,1]} \) and

\[
h_{2^k+i} = \chi_{[\frac{2^k+i}{2^k+1}, \frac{2^k+i+1}{2^k+1}]} - \chi_{[\frac{2^k+i+1}{2^k+1}, \frac{2^k+i+2}{2^k+1}]} \quad \text{for every} \quad 1 \leq i \leq 2^k, \, k = 0, 1, \ldots
\]

**Corollary 2.4.** The Haar system is a Schauder basis for \( JF_X \).

**Proof.** It is enough to show that

\[
\| \sum_{i=1}^{n+1} a_i h_i \|_{JF_X} \leq \| \sum_{i=1}^{n+1} a_i h_i \|_{JF_X}.
\]

Set \( P = (I_j)_{j=1}^m \) the partition corresponding to the simple function \( f = \sum_{i=1}^{n+1} a_i h_i \), by Proposition 2.1 Standard properties of the Haar system yield that \( \text{supph}_{n+1} \) is contained in some \( I_j \). Choose a subset \( Q \) of \( P \) such that \( \tau(f, Q) = \| f \|_{JF_X} \). Clearly \( \tau(f, Q) = \tau(g, Q) \), where \( g = \sum_{i=1}^{n+1} a_i h_i \), and this completes the proof. \( \square \)

**Remark.** S.Bellenot [Be], has provided a proof that the Haar system is a basis for \( JF \). His proof is based on the notion a neighborly basis, introduced by R.C.James.

**Lemma 2.5.** Let \( A_n \) be a sequence of successive intervals such that \( \mu(A_{2n-1}) = \mu(A_{2n}) \) for every \( n \in \mathbb{N} \). We set \( y_n = \chi_{A_{2n-1}} - \chi_{A_{2n}} \) for every \( n \in \mathbb{N} \). Then we have that

\[
\| \sum_{n} \alpha_n e_n \|_X \leq \| \sum_{n} \alpha_n \frac{y_n}{\mu(A_{2n-1})} \|_{JF_X} \leq 2 \| \sum_{n} \alpha_n e_n \|_X.
\]

Hence \( \left( \frac{y_n}{\mu(A_{2n-1})} \right)_n \) is 2-equivalent to the unit vector basis \((e_n)_n\) of \( X \).

**Proof.** For the left inequality we consider the partition \((A_{2n-1})_n\). Then

\[
\| \sum_{n} \alpha_n \frac{y_n}{\mu(A_{2n-1})} \|_{JF_X} \geq \| \sum_{k} (\int_{A_{2k-1}} \sum_{n} \alpha_n \frac{y_n}{\mu(A_{2n-1})} e_k) \|_X \geq \| \sum_{n} \alpha_n e_n \|_X.
\]

For the right inequality, let \((I_j)_j\) be any partition of \([0,1]\). The function \( \sum_{n} \alpha_n \frac{y_n}{\mu(A_{2n-1})} \) satisfies the assumptions of Proposition 2.1, so we may assume that each of the intervals...
\( I_j \) is a finite union of successive \( A_n \). It is easy to see that for each interval \( I_j \), we have the following estimates

\[
\left| \int_{I_j} \sum_n \alpha_n \frac{y_n}{\mu(A_{2n-1})} \right| \leq \begin{cases} 
0 & \text{if } I_j = [A_{2k-1}, A_{2m}] , k \leq m \\
|\alpha_m| & \text{if } I_j = [A_{2k-1}, A_{2m-1}] , k \leq m \\
|\alpha_k| & \text{if } I_j = [A_{2k}, A_{2m}] , k \leq m \\
| - \alpha_k + \alpha_m | & \text{if } I_j = [A_{2k}, A_{2m-1}] , k < m 
\end{cases}
\]

Since the intervals are successive, we have that each \( \alpha_n \) appears at most two times, and therefore

\[
\| \sum_n \alpha_n \frac{y_n}{\mu(A_{2n-1})} \|_{JFX} \leq 2 \| \sum_n \alpha_n e_n \|_X .\tag{2.1}
\]

\( \square \)

**Notation.** In the sequel we denote by \( \langle A \rangle \) the linear subspace generated by a subset \( A \) of a normed space \( Y \).

**Theorem 2.6.** \( X \) is isomorphic to a complemented subspace of \( JFX \).

**Proof.** Let \( (y_n) \) be the sequence defined in the previous lemma. We prove that the space generated by this sequence is a complemented subspace of \( JFX \). Consider the map \( P : JFX \mapsto \langle \{ y_n/\mu(A_{2n-1}) : n \in \mathbb{N} \} \rangle \) defined by the rule

\[
f \mapsto \sum_n A^*_{2n-1}(f) \frac{y_n}{\mu(A_{2n-1})} .
\]

From the definition of the map \( T \) we have that \( A^*_{2n-1}(\frac{y_k}{\mu(A_{2k-1})}) = \delta_{n,k} \), and from inequality (2.1) we have that

\[
\| \sum_n A^*_{2n-1}(f) \frac{y_n}{\mu(A_{2n-1})} \|_{JFX} \leq 2 \| \sum_n A^*_{2n-1}(f)e_n \|_X
\]

\[
= 2 \| \sum_n (\int_{A_{2n-1}} f)e_n \|_X \leq 2\|f\|_{JFX} .
\]

It follows that \( P \) is a projection with \( \|P\| \leq 2 \). \( \square \)

**Remark.** It is clear that we can choose subsequences of the Haar system which fulfill the assumptions of Lemma and therefore are weakly null. However Haar system \( (h_n) \) is not a weakly null sequence. We describe subsequences of the Haar system which does not converge weakly to 0.

Consider \( \beta_1 \in \mathbb{N} \) and \( k_1 \in \mathbb{N} \). For \( n \geq 2 \) set \( \beta_n = 8\beta_{n-1} + 2 \), and set

\[
x_n = \chi_{\left[ \frac{\beta_n}{2^{k_1}a^{n-1}}, \frac{\beta_n+1}{2^{k_1}a^{n-1}} \right]} - \chi_{\left[ \frac{\beta_n+1}{2^{k_1}a^{n-1}}, \frac{\beta_n+2}{2^{k_1}a^{n-1}} \right]} \text{ for } n \geq 1.
\]

The following are easily established.
We denote by \( JF \). It is obvious that \( X \). Let \( \beta \). Theorem 3.2. Let \( (0 \leq h \leq 1) \). Let \( \beta \) is the initial point of a Haar function, then the sequence \((x_n)_n \) is a subsequence of the Haar system, and \( (\frac{x_n}{\|x_n\|_{JF}})_n \) does not converges weakly to 0 in \( JF_X \). On the other hand if \((h_n)_n \in M \) is a subsequence of the Haar system such that \( \max \supp h_n = \alpha \) for all \( n \in M \), it is not hard to see that the there exists a subsequence \((h_n)_n \in L \) of \((h_n)_n \in M \) such that \( (\frac{h_n}{\|h_n\|})_n \in L \) is weakly null, and in particular is equivalent to the unit vector basis of \( X \).

3. Quotients of \( JF_X(\Omega) \).

This section is devoted to the study of quotients \( JF_X(\Omega)/Y \), where \( Y \) is a separable subspace of \( JF_X(\Omega) \).

**Definition 3.1.** Let \( X \) be a Banach space with a symmetric basis, and \( \Gamma \) an infinite set. We denote by \( X_\Gamma \) the completion of \( c_{00}(\Gamma) \) under the norm
\[
\| \sum_{i=1}^{n} \alpha_i e_{\gamma_i} \| = \| \sum_{i=1}^{n} \alpha_i e_i \|_X, \text{ where } \gamma_i \neq \gamma_j \text{ for } i \neq j.
\]
It is obvious that \( X_\Gamma \) is reflexive iff \( X \) is reflexive.

We prove the following

**Theorem 3.2.** Let \( \Delta \) be any countable dense subset of \( [0,1] \), and \( \{I_i\}_{i=1}^{\infty} \) be an enumeration of the subintervals of \( (0,1) \) with endpoints in \( \Delta \). We set \( Y = \langle \{I_j^* : j = 1,2,\ldots\} \rangle \). The quotient space \( JF_X(\Omega)/Y \) is isomorphic to \( X_\Gamma^* \), where the set \( \Gamma \) has the cardinality of the continuum.

Before passing to the proof of the theorem we make some preliminary observations. We denote by \( \widehat{x}^* \), the equivalence class of the functional \( x^* \in JF_X \). Since \( \ell_1 \) does not embed into \( JF_X \), we have that \( B_{JF_X} = \overline{c_0} (\text{Ext } B_{JF_X}) \). Hence Lemmas 1.2, 1.5 yields that \( JF_X = \langle \{I^* : I = (\alpha, \beta) \subseteq (0,1)\} \rangle \). Therefore \( \langle \{I^*: I = (\alpha, \beta) \subseteq (0,1)\} \rangle \) is dense in \( JF_X(\Omega)/Y \).

**Lemma 3.3.** The \( \langle \{I^*: I = (\alpha, \delta) : \alpha \notin \Delta, \delta \in \Delta\} \rangle \) is dense in \( JF_X(\Omega)/Y \).
Proof. For simplicity we assume that \( \{0,1\} \subset \Delta \). First we observe that for \( I = (\alpha, \beta) \), \( \hat{I}^* \neq 0 \) iff \( \{\alpha, \beta\} \not\subset \Delta \) and also since \( \Delta \) is a dense subset of \( (0,1) \) we obtain for any \( I = (\alpha, \beta) \), \( I^* = I_1 + I_2 \), where \( I_1 = (\alpha, \delta) \), \( I_2 = (\delta, \beta) \) with some \( \delta \in \Delta \). To complete the proof we observe that for \( I_1 = (\delta, \beta) \), \( I_2 = (\beta, \delta') \), \( \delta, \delta' \in \Delta \) we have that \( -I_1^* \in \hat{I}_2^* \). \( \square \\

Proof of Theorem 3.3. Let \( \alpha_1 < \alpha_2 < \ldots < \alpha_n \) be such that \( \alpha_i \not\subset \Delta \) for \( i = 1,2,\ldots,n \) and \( \hat{S}^*_{i} = \hat{\chi}^*_{(\alpha_i,\delta_i)} \), \( \hat{S}^*_n = \hat{\chi}^*_{(\alpha_n,\delta_n)} \), be elements of \( JF_X^*/Y \) with \( \delta_i \in \Delta \), \( i \leq n \).

We shall prove that

\[
\frac{1}{2} \| \sum_{j=1}^{n} \lambda_j e_j^* \|_{X^*} \leq \| \sum_{j=1}^{n} \lambda_j \hat{S}^*_j \|_{JF_X^*/Y} \leq \| \sum_{j=1}^{n} \lambda_j e_j^* \|_{X^*},
\]

for every finite sequence \( (\lambda_j) \) of reals. This together with Lemma 3.2 implies that \( JF_X^*/Y \) is isomorphic to \( X^*(\Gamma) \), where \( \Gamma = \{ \alpha \in (0,1) \setminus \Delta \} \). Let’s observe that we may assume that \( \alpha_1 < \delta_1 < \alpha_2 < \delta_2 < \ldots \).

Let \( \{I_j^* : I_j = (d_1,d_2) : d_1, d_2 \in \Delta \} \) be the set of intervals generating the subspace \( Y \). We set \( Y_m = \{I_{i,m}^* \}_{i=1}^{m} \). Since \( \cup_{m=1}^{\infty} Y_m \) is dense in \( Y \) we obtain that

\[
dist(\sum_{j=1}^{n} \lambda_j S_j^*, Y) = \lim_{m \to \infty} dist(\sum_{j=1}^{n} \lambda_j S_j^*, Y_m).
\]

For \( m \in \mathbb{N} \) and \( j = 1, \ldots, n \), there exists \( q_j^m \in (0,1) \) such that the interval \( (\alpha_j - q_j^m, \alpha_j + q_j^m) \) has the following properties:

1. For every \( i \leq m \), either \( (\alpha_j - q_j^m, \alpha_j + q_j^m) \cap I_i = \emptyset \) or \( (\alpha_j - q_j^m, \alpha_j + q_j^m) \cap I_i = \emptyset \).

2. \( (\alpha_j, \alpha_j + q_j^m) \subset (\alpha_j, \delta_j) \).

Set \( Q_j,m = (\alpha_j - q_j^m, \alpha_j) \) and \( Q_j,m = (\alpha_j, \alpha_j + q_j^m) \) and choose \( \{\beta_j\}_{j=1}^{n} \) such that

\[
\| \sum_{j=1}^{n} \beta_j e_j \|_X = 1, \quad \| \sum_{j=1}^{n} \lambda_j e_j^* \|_{X^*} = \sum_{j=1}^{n} \lambda_j \beta_j.
\]  

(3.1)

Set finally

\[
f = \sum_{j=1}^{n} \beta_j \frac{\chi_{Q_j,m} - \chi_{Q_j,m}^*}{\mu(Q_j,m)}.
\]

Lemma 2.5 yields that

\[
\|f\|_{JF_X} \leq 2
\]  

(3.2)

Also, for every \( 1 \leq i \leq m \) property (1) yields that \( \langle I_i^*, f \rangle = 0 \). Hence (3.1), (3.2) implies

\[
dist(\sum_{j=1}^{n} \lambda_j S_j^*, Y_m) \geq \frac{1}{2} \sum_{j=1}^{n} \lambda_j S_j^*, f \geq \frac{1}{2} \sum_{j=1}^{n} \lambda_j \beta_j = \frac{1}{2} \sum_{j=1}^{n} \lambda_j e_j^* \|_{X^*}.
\]

This proves the left inequality, namely

\[
\frac{1}{2} \| \sum_{j=1}^{n} \lambda_j e_j^* \|_{X^*} \leq \| \sum_{j=1}^{n} \lambda_j \hat{S}^*_j \|_{JF_X^*/Y}.
\]  

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The right inequality follows immediately from the disjointness of \( \{S_j\}_{j=1}^\infty \), since
\[
\left\| \sum_{j=1}^n \lambda_j \hat{S}_j \right\|_{JF^*_X/Y} \leq \left\| \sum_{j=1}^n \lambda_j S_j^* \right\|_{JF^*_X} \leq \left\| \sum_{j=1}^n \lambda_j e_j^* \right\|_{X^*}.
\]
The proof is complete. \( \square \)

**Remark.** If \( Y \) denotes the subspace of \( JF^*_X \) generated by the biorthogonal functionals of the Haar system, the previous theorem yields that \( JF^*_X/Y \) is isomorphic to \( X^*_r \). In the particular case of James function space \( JF \), the corresponding quotient is isomorphic to \( \ell_2(\Gamma) \). This result is the analogue of the corresponding result for the James tree space \( JT \) \([LS]\). Next we will see that these results are no longer valid for the class of the spaces \( JF^*_X(\Omega) \), \( \Omega \) open bounded subset of \( \mathbb{R}^{d_0} \), \( d_0 > 1 \).

**Proposition 3.4.** Let \( d_0 > 1 \) and \( \Omega = (0,1)^{d_0} \). Let \( (T_j)_{j \in J} \) be a family in \( \mathcal{P}(\Omega) \) such that there exists \( 1 \leq d \leq d_0 \) with
\[
(0,1) \notin \{pr_d(x) : x \text{ is a vertex of } T_j \text{ for some } j \in J\}.
\]
Then if \( Y = \langle T_j^* : j \in J \rangle \), the quotient \( JF^*_X(\Omega)/Y \) is not reflexive.

**Proof.** Assume that for \( d = 1 \) there exists \( \alpha \in (0,1) \setminus \{pr_1(x) : x \text{ is a vertex of } T_j \text{ for some } j \in J\} \). We choose \( r_n \in (0,1) \) strictly decreasing to zero and set \( S_n = (\alpha,1) \times (0,r_n)^{d_0-1} \).

**Claim.** \( \{\hat{S}_n^*\}_n \) has no weakly converging subsequences.

**Proof of the Claim.** For \( F \) a finite subset of \( J \) (i.e \( F \in \mathcal{P}_{<\omega}(J) \)) we set \( Y_F = \langle T_j^* : j \in F \rangle \). Clearly for every \( x^* \in JF^*_X(\Omega) \),
\[
\|x^*\| = \text{dist}(x^*, Y) = \inf \{\text{dist}(x^*, Y_F) : F \in \mathcal{P}_{<\omega}(J)\}.
\]
To show that \( \{\hat{S}_n^*\}_n \) does not have \( w \)–convergent sequence it is enough to prove that for \( \varepsilon = \frac{1}{\|e_1 + e_2\|_{X}} \) and every \( \{R_k^*\}_{k \in \mathbb{N}} \) convex block subsequence of \( \{S_n^*\}_n \) there exists \( k_1 < k_2 \) such that
\[
\|\hat{R}_{k_1}^* - \hat{R}_{k_2}^*\| \geq \varepsilon.
\]
To see this, we consider \( G_1, G_2 \) finite subsets of \( \mathbb{N} \) with \( \max G_1 = \ell < q = \min G_2 \) and \( R_1 = \sum_{n \in G_1} \alpha_n S_n^* \), \( R_2 = \sum_{n \in G_2} \alpha_n S_n^* \). Observe that
\[
\sum_{n \in G_1} \alpha_n \chi_{S_n}(w) = 1 \text{ for all } w \in (\alpha,1) \times (r_q,r_\ell)^{d_0-1}.
\]
Next we consider any \( F \in \mathcal{P}_{<\omega}(J) \) and we show that
\[
\text{dist}(R_1^* - R_2^*, Y_F) \geq \frac{1}{\|e_1 + e_2\|_{X}},
\]
which immediately yields the claim.
Indeed, set
\[ \delta = \min \left\{ \{ \alpha - \text{pr}_1(x) : x \text{ is a vertex of } T_j, j \in F \} \cup \{ \alpha, 1 - \alpha \} \right\} > 0. \]
Moreover we choose \( r'_q < r'_\ell \) such that \( r_q < r'_q < r'_\ell < r_\ell \) and
\[ (r'_q, r'_\ell) \cap \{ \text{pr}_d(x) : x \text{ is a vertex of } T_j, j \in F \} = \emptyset \] for all \( d = 2, 3, \ldots, d_0 \).
We set
\[ Q_1 = (\alpha - \frac{\delta}{2}, \alpha) \times (r'_q, r'_\ell)^{d_0-1} \quad \text{and} \quad Q_2 = (\alpha, \alpha + \frac{\delta}{2}) \times (r'_q, r'_\ell)^{d_0-1}. \]
Observe the following.
(1) For \( j \in F \) either \( Q_1 \cup Q_2 \subset T_j \) or \((Q_1 \cup Q_2) \cap T_j = \emptyset \).
(2) \( \sum_{n \in G_1} \alpha_n \chi_{S_1}(w) = 0, \forall w \in Q_1 \) and \( \sum_{n \in G_1} \alpha_n \chi_{S_2}(w) = 1, \forall w \in Q_2 \).
(3) \( \sum_{n \in G_2} \alpha_n \chi_{S_1}(w) = 0, \forall w \in Q_1 \cup Q_2 \).
Consider now the element of \( JF_X(\Omega) \) defined by
\[ f = \frac{\chi_{Q_2} - \chi_{Q_1}}{\|Q_2\|}. \]
An easy computation yields that \( \|f\| = \|e_1 + e_2\|_X \). Properties (1), (2) and (3), stated above, imply that
\[ \text{dist}(R_1^*, R_2^*, Y_F) \geq \frac{1}{\|e_1 + e_2\|_X} = \frac{1}{\|e_1 + e_2\|_X} \cdot \frac{1}{\|e_1 + e_2\|_X} \]
This completes the proof of the claim, and the proof of the proposition. \( \square \)
Let’s pass now to some consequences of the above proposition.

**Proposition 3.5.** Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^{d_0}, d_0 > 1 \), and \( \{T_j\} \subset P(\Omega) \) such that for some \( 1 \leq d \leq d_0 \)
\[ \text{pr}_d(\Omega) \not\subset \{ \text{pr}_d(x) : x \text{ is a vertex of } T_j, j \in J \}. \]
If \( Y = \langle T_j^+ : j \in J \rangle \), then \( JF_X^*(\Omega)/Y \) is not reflexive.

**Proof.** Since \( \text{pr}_d(\Omega) \) is open, there exists \( S = \prod_{i=1}^{d_0} (\alpha_i, \beta_i) \subset \Omega \) such that
\[ (\alpha_d, \beta_d) \not\subset \{ \text{pr}_d(x) : x \text{ is a vertex of } T_j, j \in J \}. \]
The result is obtained with the same arguments as in the previous proposition. \( \square \)

**Theorem 3.6.** Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^{d_0}, d_0 > 1 \). Then for every separable subspace \( Y \) of \( JF_X^*(\Omega) \) the quotient \( JF_X^*(\Omega)/Y \) is not reflexive.
Proof. Since $\ell_1$ does not embed into $JF_X(\Omega)$, by Lemmas 1.2, 1.5 and Haydon’s theorem [H], we obtain that $JF^*_X(\Omega) = \langle T^* : T \in \mathcal{P}(\Omega) \rangle$. Hence for any $Y$ separable subspace of $JF^*_X(\Omega)$ there exists a sequence $\{T_n^*\}_n$ such that $Y \hookrightarrow Z = \langle T_n^* : n \in \mathbb{N} \rangle$. Clearly $\{T_n\}_n$ satisfies the assumption of Proposition 3.5, hence $JF^*_X(\Omega)/Z$ is not reflexive. This implies that $JF^*_X(\Omega)/Y$ is also not reflexive. □

Corollary 3.7. Let $\Omega$ be a bounded open subset of $\mathbb{R}^{d_0}$, $d_0 > 1$. Then the space $JF^*_X(\Omega)$ is not isomorphic to a quotient of $JF^*_X$. In particular $JF^*_X(\Omega)$ is not isomorphic to a subspace of $JF^*_X$.

Proof. On the contrary, assume that there exists a subspace $Z$ of $JF^*_X$ such that $JF^*_X(\Omega)$ is isomorphic to the quotient space $JF^*_X/Z$. Let $\Delta$ be a countable dense subset of $(0, 1)$ and $Y = \langle \{I^* : I \text{ has endpoints in } \Delta \} \rangle$. It is well known that $JF^*_X/W$ is isometric to $JF^*_X/Y/W/Y$ and also isometric to $JF^*_X/Z/W/Z$. Moreover the quotient space $W/Z$ is separable. Hence we have that

$$JF^*_X/W \approx JF^*_X/Z/W/Z \approx JF^*_X(\Omega)/W/Z,$$

and therefore by Theorem 3.6 $JF^*_X/W$ is not reflexive. On the other hand, by Theorem 3.2 we have that $JF^*_X/Y$ is reflexive, and therefore

$$JF^*_X/Y \approx JF^*_X/Y/W/Y \approx X^*_1/W/Y.$$

From (3.3) and (3.4) we derive a contradiction.

The second part follows from a duality argument. □

4. The embedding of $JF^*_X$ into $JF^*_X(\Omega)$.

In this section we prove the following:

Theorem 4.1. Let $1 \leq d_0 \leq d_1$ and $\Omega$ be a bounded open subset of $\mathbb{R}^{d_1}$. Then $JF^*_X((0, 1)^{d_0})$ is isometric to a complemented subspace of $JF^*_X(\Omega)$.

Since $JF^*_X((0, 1)^{d_1})$ is isometric to $1 - \text{complemented}$ subspace of $JF^*_X(\Omega)$, it is enough to prove the result for $\Omega = (0, 1)^{d_1}$.

We set $\mathcal{D}$ be the dense subspace of $L^1((0, 1)^{d_0})$ consisting of the functions of the form

$$x = \sum_{j=1}^m \alpha_j \chi_{R_j},$$

where $\{R_j\}_j$ are disjoint parallelepipeds in $\mathcal{P}((0, 1)^{d_0})$.

To each $x = \sum_{j=1}^m \alpha_j \chi_{R_j} \in \mathcal{D}$ we correspond the vector $\tilde{x} = \sum_{j=1}^m \alpha_j \chi_{R_j \times (0, 1)^{d_1-d_0}}$ of $JF^*_X(\Omega)$. Denote by

$$U : \mathcal{D} \rightarrow JF^*_X(\Omega)$$

the above assignment, which is a linear operator.
Lemma 4.2. Let $x \in \mathcal{D}$ and $x^* \in JF_X^*(\Omega)$ such that $x^* = \sum_{i=1}^n b_i T_i^*$, $\{T_i\}_{i=1}^n \subset \mathcal{P}(\Omega)$ disjoint, and $\|\sum_{i=1}^n b_i \epsilon_i\|_{X^*} \leq 1$. Then

(1) $\|x\|_{JF_X((0,1)^d_0)} \leq \|\tilde{x}\|_{JF_X((0,1)^d_0)}$.

(2) $x^*(\tilde{x}) \leq \|x\|_{JF_X((0,1)^d_0)}$.

Proof. (1). This follows easily. Indeed for every $\{S_i\}_{i=1}^k \subset \mathcal{P}((0,1)^d_0)$ disjoint, we consider the disjoint family $\{S_i \times (0,1)^{d_i-d_0}\}_{i=1}^k$ of $(0,1)^{d_i}$, and notice that

$$\|\sum_i \left( \int_{S_i} \sum_j^m \alpha_{j} \chi_{R_j} \right) \epsilon_i \| = \| \sum_i \left( \int_{S_i} \sum_j^m \alpha_{j} \chi_{R_j} \right) \epsilon_i \|.$$  

Taking the supremum in both sides we obtain (1).

(2) To see the second inequality, assume that $x = \sum_{j=1}^m \alpha_j \chi_{R_j}$, $x^* = \sum_{i=1}^n b_i T_i^*$ are given. Denote by $\pi_1 : \mathbb{R}^{d_1} \mapsto \mathbb{R}^{d_0}, \pi_2 : \mathbb{R}^{d_1} \mapsto \mathbb{R}^{d_1-d_0}$ the natural projections of $\mathbb{R}^{d_1}$ onto the two orthogonal subspaces $\mathbb{R}^{d_0}, \mathbb{R}^{d_1-d_0}$.

Assume additionally, that the families $\{R_j\}_{j=1}^m, \{T_i\}_{i=1}^n$ satisfy the following property

For every $j = 1, \ldots, m$, $i = 1, \ldots, n$ either $R_j \subset \pi_1(T_i)$ or $R_j \cap \pi_1(T_i) = \emptyset$. (4.1)

If (4.1) fails, then we rewrite $x$ as $\sum_{\ell=1}^k \alpha_{\ell} \chi_{R_\ell}$ so that the families $\{R_\ell\}_\ell, \{T_i\}_i$ satisfying (4.1).

Next we choose family $\{Q_\ell\}_{\ell=1}^k \subset \mathcal{P}((0,1)^{d_1-d_0})$ such that

(1) $\{Q_\ell\}_{\ell=1}^k$ are pairwise disjoint parallelepipeds.

(2) For every $\ell = 1, \ldots, k$, $i = 1, \ldots, n$ either $Q_\ell \subset \pi_2(T_i)$ or $Q_\ell \cap \pi_2(T_i) = \emptyset$.

(3) If $B_i = \{\ell : Q_\ell \subset \pi_2(T_i)\}$, then $T_i = \bigcup_{\ell \in B_i} (\pi_1(T_i) \times Q_\ell)$ almost everywhere.

We also set

$$A_j = \{i : R_j \subset \pi_1(T_i)\} \quad \text{for } j = 1, \ldots, m.$$  

The above properties yield that

$$x^*(\tilde{x}) = \sum_{i=1}^n b_i T_i^* = \sum_{j=1}^m \alpha_j \mu_{d_0}(R_j) \sum_{i \in A_j} b_i \sum_{\ell \in B_i} \mu_{d_1-d_0}(Q_\ell) = \sum_{\ell=1}^k \mu_{d_1-d_0}(Q_\ell) \sum_{i \in B_i} b_i \sum_{j \in A_j} \alpha_j \mu_{d_0}(R_j) = \sum_{\ell=1}^k \mu_{d_1-d_0}(Q_\ell) \sum_{i \in B_i} b_i (\pi_1(T_i))^*(\sum_{j \in A_j} \alpha_j \chi_{R_j}) = \sum_{\ell=1}^k \mu_{d_1-d_0}(Q_\ell) \sum_{i \in B_i} b_i (\pi_1(T_i))^*(x).$$
Recall that $\mu_{d_0}$, $\mu_{d_1-d_0}$ denotes the Lebesgue measure on $\mathbb{R}^{d_0}$, $\mathbb{R}^{d_1-d_0}$ respectively. To see that $\sum_{i} b_i T_i^*(\tilde{x}) \leq \|x\|_{JF_X((0,1)^{d_0})}$ we notice that property (1) yields that

$$\sum_{\ell} \mu_{d_1-d_0}(Q_\ell) \leq 1.$$  \hfill (4.2)

For fixed $\ell$, if $1 \leq i_1 \neq i_2 \leq n$ are such that $\ell \in B_{i_1}$, $\ell \in B_{i_2}$, it holds that $\pi_1(T_{i_1}) \cap \pi_1(T_{i_2}) = \emptyset$. Therefore setting

$$y_{i}^\ell = \sum_{i: \ell \in B_{i}} b_i \pi_1(T_{i})^* \text{ for } \ell = 1, \ldots, k,$$

we conclude that $y_k^* \in B_{JF_X}$.

Finally set

$$y^* = \sum_{\ell=1}^{k} \mu_{d_0 - 1}(Q_\ell) y_{i}^\ell.$$

Observe that (4.2) yields that $y^*$ is a subconvex combination of $\{y_{i}^\ell\}_{\ell=1}^{k}$, hence $y^* \in B_{JF_X((0,1)^{d_0})}$ and $y^*(x) = x^*(\tilde{x})$. This completes the proof of the lemma. \hfill $\square$

**Proof of Theorem 4.1.** From Lemma 4.2 it follows that $U : \mathcal{D} \rightarrow JF_X(\Omega)$ is an isometry, which is extended to an isometry of $JF_X((0,1)^{d_0})$ into $JF_X(\Omega)$. It remains to show that $U(JF_X((0,1)^{d_0}))$ is a 1-complemented subspace. Indeed, we set

$$Y = (\mathbb{R} \times (0,1)^{d_1-d_0} : R \in \mathcal{P}((0,1)^{d_0})) \subset JF_X(\Omega),$$

and $Q : JF_X(\Omega) \rightarrow JF_X(\Omega)/Y_\perp$. Since $U$ is an isometry, we obtain that $By 1-norms$ the subspace $U(JF_X((0,1)^{d_0}))$, hence $Q \circ U$ is also an isometry. To see that is onto, we observe that for every $T \in \mathcal{P}(\Omega)$ there exists $0 \leq \lambda \leq 1$, such that $\chi_T - \lambda \chi_{\pi_1(T) \times (0,1)^{d_1-d_0}} \in Y_\perp$. This completes the proof of the theorem. \hfill $\square$

5. **SUBSEQUENCES OF RADEMACHER FUNCTIONS EQUIVALENT TO $c_0$ BASIS.**

Before stating the next definition we introduce some notation. Let $(n_k)_k$ be an increasing sequence of $\mathbb{N}$ and $(\sigma_k)_k$ a sequence of successive subsets of $\mathbb{N}$ with $\# \sigma_k = 2^{n_k}$. We denote by $\lambda_k^{\perp} = \|\sum_{n \in \sigma_k} e_n\|_X$, and we set $u_k = \lambda_k \sum_{n \in \sigma_k} e_n$, which clearly satisfies $\|u_k\|_X = 1$.

**Definition 5.1.** Let $X$ be a reflexive Banach space with 1-symmetric basis $(e_n)_n$. The space $X$ satisfies the Convex Combination Property (CCP) if there exist a strictly increasing sequence $(n_k)_k$ and $C > 0$ such that the following is fulfilled:

For $(\sigma_k)_k$, $(u_k)_k$ as above with $\# \sigma_k = 2^{n_k}$, every $(I_k)_k$ with $I_k \subset \sigma_k$ and $\sum_k \frac{\# I_k}{\# \sigma_k} \leq 1$ we have that

$$\|\sum_{k=1}^{\infty} \lambda_k \sum_{n \in I_k} e_n\|_X \leq C.$$  \hfill \hfill (5.1)

Our goal is to prove the following:

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Theorem 5.2. The following are equivalent:

(1) The space $X$ satisfies CCP.

(2) The normalized sequence $(\frac{r_n}{\|r_n\|})_{n \in \mathbb{N}}$ in $JF_X$ of Rademacher functions contains a subsequence equivalent to the usual basis of $c_0$.

Remark. It is an easy exercise that $\ell_p$, $1 < p < \infty$, have CCP. Lorentz space $d(w_1, p)$, $1 < p < \infty$, $w_1 = (\frac{1}{n})_n$ fails CCP. Indeed, choose a rapidly increasing sequence $(n_k)_k$ of integers, $(\sigma_k)_k$ subsets of $\mathbb{N}$ with $\#\sigma_k = 2^n_k$ and $I_k \subset \sigma_k$ with $\#I_k = \frac{w_1}{2^n_k}$. From the definition of the norm of the space $d(w_1, p)$, using that $\ln(n) \approx \sum_{i=1}^{n} \frac{1}{i}$, we easily see that

$$\|\sum_{n \in \sigma_k} e_n\|_{d(w_1, p)} \approx (\ln(2^n_k))^{1/p},$$

and

$$\left(\sum_{k} \frac{\ln(2^n_k - k) - \ln(2^n_{k-1} - k + 2)}{\ln(2^n_k)}\right)^{1/p} \to \infty.$$

We do not know if $JF_{d(w_1, p)}$ contains $c_0$.

The next proposition yields that CCP implies a formally stronger property.

Proposition 5.3. Let $X$ have the CCP and $(n_k)_k$, $(\sigma_k)_k$, $(u_k)_k$, $(\lambda_k)_k$, $C > 0$ as before. Then for every sequence $(\alpha_n)_{n \in \sigma}$, $\sigma = \cup_k \sigma_k$ such that

$$0 \leq \alpha_n \leq 1 \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{\sum_{n \in \sigma_k} \alpha_n}{\#\sigma_k} \leq 1,$$  \hspace{1cm} (5.1)

we have that

$$\|\sum_{k=1}^{\infty} \sum_{n \in \sigma_k} \lambda_k \alpha_n e_n\|_X \leq C + 1.$$  \hspace{1cm} (5.2)

Proof. It is enough to show that for every $k_0 \in \mathbb{N}$ and every $(\alpha_n)_{n \in \cup_{k=1}^{k_0} \sigma_k}$ satisfying (5.1) we have that

$$\|\sum_{k=1}^{k_0} \sum_{n \in \sigma_k} \lambda_k \alpha_n e_n\|_X \leq C + 1.$$  

Fix $k_0$ and set

$$K = \left\{ \sum_{k=1}^{k_0} \sum_{n \in \sigma_k} \lambda_k \alpha_n e_n : (\alpha_n)_{n \in \cup_{k=1}^{k_0} \sigma_k} \text{ satisfies (5.1)} \right\}.$$

Then $K$ is a closed convex bounded subset of a finite dimensional subspace of $X$, hence it is convex and compact. It is easy to see that

If $(\alpha_n)_{n \in \cup_{k=1}^{k_0} \sigma_k}$ satisfies (5.1) and there exists $n_1, n_2 \in \cup_{k=1}^{k_0} \sigma_k$ with $n_1 \neq n_2$ and $0 < \alpha_{n_1}, \alpha_{n_2} < 1$ then $\sum_{k=1}^{k_0} \sum_{n \in \sigma_k} \lambda_k \alpha_n e_n$ is not an extreme point of $K$. 

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Hence every $x \in Ex(K)$ is of the form $x = \sum_{k=1}^{\ell_0} \sum_{n \in I_k} \lambda_k e_n + \alpha_n \lambda_k e_n$, and from CCP we obtain that $\|x\|_X \leq C + 1$. This yields that for every $x \in K$, $\|x\|_X \leq C + 1$. □

Next we summarize some simple properties of Rademacher functions $(r_n)_n$.

**Lemma 5.4.** The following hold

1. For every interval $I \subset [0,1]$, $n \in \mathbb{N}$, $| \int_I r_n d\mu | \leq \frac{1}{2^n}$.
2. If $\lambda_k^{-1} = \| \sum_{i=1}^{2^n} e_i \|_X$ then $\| r_n \|_{JF_X} = \lambda_n^{-1} 2^{-n}$.
3. Denote $\tilde{r}_n = \frac{r_n}{\| r_n \|_{JF_X}} = \lambda_n 2^n r_n$. Then $\int_{I} \tilde{r}_n = \pm \lambda_n$.
4. The reflexivity of $X$ implies that $\lambda_n^{-1} \to 0$, $\lambda_n^{-1} 2^{-n} \to 0$.
5. For each interval $I$, $\int_I \tilde{r}_n | \leq \lambda_n^{-1} 2^{-n}$, hence $(\tilde{r}_n)_n$ is weakly null sequence in $JF_X$.

**Proof.** (1) is well known, (2) follows from Proposition 2.1. (3) It is easy. (4) It is well known, $[L^\infty, \| \cdot \|_{JF_X}]$. (5) follows from (1) and (4). □

**Notation.** (a) In the Section 2 for $f \in L^1(\mu)$ and $Q = \{I_j\}_{j=1}^n$ partition of $(0,1)$, the quantity $\tau(Q,f)$ was defined. This is extended to each $f \in JF_X$ as follows:

$$\tau(Q,f) = \| \sum_{j=1}^{n} I_j^* (f) e_j \|_X .$$

(b). We recall that for $Q = \{I_j\}_{j=1}^n$ partition of $(0,1)$, the width of $Q$ is defined as $\delta(Q) = \max\{ \mu(I_j) : j = 1, \ldots, n \}$.

**Lemma 5.5.** Let $f \in JF_X(\Omega)$. Then

$$\lim_{\varepsilon \to 0} \sup \{ \tau(Q,f) : \delta(Q) \leq \varepsilon \} = 0 .$$

**Proof.** Notice that if $X$ has a symmetric basis $(e_n)_n$ and it does not contain $\ell_1$, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $(\alpha_n)_n$, $0 \leq \alpha_n \leq 1$, $\max\{ \alpha_n : n \in \mathbb{N} \} \leq \delta$ and $\sum_n \alpha_n \leq 1$ we have that $\| \sum_{n=1}^{\infty} \alpha_n e_n \|_X < \varepsilon$, Lemma 1.10. This property together with the uniform integrability of $f \in L^1(\mu)$ yields the conclusion for $f \in L^1(\mu)$. The general case follows easily from the density of $L^1(\mu)$ in $JF_X$. □

**Proposition 5.6.** Assume that $(\tilde{r}_n)_{n \in M}$ is a subsequence of the normalized Rademacher functions which endowed with $\| \cdot \|_{JF_X}$ is equivalent to the unit vector basis of $c_0$. Then $X$ satisfies the CCP.

**Proof.** Choose $C > 0$ such that for any finite subset $F$ of $M$, $\| \sum_{n \in F} \tilde{r}_n \|_{JF_X} \leq C$. Next choose inductively $n_1 < \ldots < n_k < \ldots$, such that $\{n_k\} \subset M$ and

$$\sup \left\{ \tau(\tilde{r}_{n_k},Q) : \delta(Q) \leq \frac{1}{2^{k+1}} \right\} \leq \frac{1}{2^k} .$$

(5.3)

We claim that $\{n_k\}_k, C+1$ satisfy CCP in $X$. 27
Indeed given $A_1 \subset \sigma_1, \ldots, A_k \subset \sigma_k$ such that $\sum_{j=1}^{k} \frac{\#A_j}{\#\sigma_j} \leq 1$, setting $d_j = \#L_j$, we choose a partition $\{I_1, \ldots, I_d, \ldots, I_{d_k}\}$ of $(0, 1)$ as follows:

Let $A_1 = [0, \frac{\#A_1}{\#\sigma_1}]$ and $A_j = [\sum_{i<j} \frac{\#A_i}{\#\sigma_i}, \sum_{i\leq j} \frac{\#A_i}{\#\sigma_i}]$ for $j = 2, \ldots, k$. For every $j \leq k$ we consider $d_j$ intervals $\{I_i\}_{i=1}^{d_j}$ which form a partition of $A_j$, and each $I_i$ is of the form $I_i = [\frac{\#A_j}{\#\sigma_j}, \frac{\#\sigma_j}{\#\sigma_j}]$, for an appropriate $\ell$. Observe that $\mu(\bigcup_{i=1}^{d_j} I_i) = \frac{\#A_j}{\#\sigma_j}$. Further for every $1 \leq j \leq k, i = 1, \ldots, d_j$

\[ |\int_{I_i} \hat{r}_{n_j}| = \lambda_j = \| \sum_{n \in \sigma_j} e_n \|_X^{-1}. \]

Therefore

\[ \| \sum_{j=1}^{k} \sum_{n \in \Lambda_j} \lambda_j e_n \|_X = \| \sum_{j=1}^{k} \sum_{i=1}^{d_j} \left( \int_{I_i} \hat{r}_{n_j} d\mu \right) e_n \|_X. \] (5.4)

Further

\[ C \geq \| \sum_{j=1}^{k} \hat{r}_{n_j} \|_{JF_X} \geq \| \sum_{j=1}^{k} \sum_{i=1}^{d_j} \left( \int_{I_i} \hat{r}_{n_j} d\mu \right) e_n \|_X \]

\[ = \| \sum_{j=1}^{k} \sum_{i=1}^{d_j} \left( \int_{I_i} \hat{r}_{n_j} d\mu \right) e_n \|_X \]

\[ \geq \| \sum_{j=1}^{k} \sum_{i=1}^{d_j} \left( \int_{I_i} \hat{r}_{n_j} d\mu \right) e_n \|_X - \sum_{j=1}^{k} \tau(\hat{r}_{n_j}, Q_j), \] (5.5)

where $Q_j = \bigcup_{i>0} \{I_i\}_{i=1}^{d_j}$. Condition (5.3) yields that

\[ \sum_{j=1}^{k} \tau(\hat{r}_{n_j}, Q_j) \leq 1. \]

Setting together (5.4), (5.5) we obtain the desired result.

**Proposition 5.7.** Assume that $X$ satisfies CCP. Then there exists a subsequence of Rademacher functions equivalent, in $JF_X$ norm, to the unit vector basis of $c_0$.

**Proof.** Let $(n_k)_k, C > 0$ witness the presence of CCP in $X$. We inductively choose a subsequence $(n_{k_\ell})$ of $(n_k)_k$ and a decreasing sequence $(\varepsilon_\ell)_\ell \subset (0, 1)$ satisfying the following properties:

1. For each $\ell \in \mathbb{N}$,

   \[ \sup \left\{ \tau(\hat{r}_{n_{k_\ell}}, Q) : \delta(Q) \leq \varepsilon_\ell \right\} < \frac{1}{4^\ell}. \]

2. For each $\ell > 1$,

   \[ \sup \{ \tau(\hat{r}_{n_{k_\ell}}, Q) : \min \{\mu(I) : I \in Q\} \geq \varepsilon_{\ell-1} \} < \frac{1}{4^\ell}. \]
The inductive choice proceeds as follows. We set \( k_1 = 1 \) and from Lemma 5.5 there exists \( \varepsilon_1 > 0 \) such that for \( \ell = 1 \) the inductive assumption (1) is fulfilled. Observe that for every partition \( Q \) of \((0,1)\) such that \( \min\{\mu(Q) : I \in Q\} \geq \varepsilon_1 \) satisfies \( \#Q < \frac{1}{\varepsilon_1} \). Hence Lemma 5.4(1) yields that there exists \( k_2 \) such that the inductive assumption (2) is fulfilled. The general inductive step follows the same argument.

**Claim.** The sequence \( (\tilde{r}_{nk\ell})_\ell \) is equivalent to the unit vector basis of \( c_0 \).

For this, we show that the inductive assumptions (1) and (2) together with CCP in \( X \) yield that for every \( d \in \mathbb{N} \),

\[
\| \sum_{\ell=1}^{d} \tilde{r}_{nk\ell} \|_{JF_X} \leq 3(C + 1) + 1. \tag{5.6}
\]

Since every subsequence of \( (\tilde{r}_{nk\ell})_\ell \) satisfies (1) and (2), we obtain that every subsequence of \( (\tilde{r}_{nk\ell})_\ell \) also satisfies (5.6), and this will end the proof.

To see (5.6), we consider \( Q = \{I_j\}_{j=1}^{q} \) arbitrary partition of \((0,1)\) and show that

\[
\tau(\sum_{\ell=1}^{d} \tilde{r}_{nk\ell}, Q) \leq 3(C + 1) + 1. \tag{5.7}
\]

Hence

\[
\tau(\sum_{\ell=1}^{d} \tilde{r}_{nk\ell}, Q) \leq \tau(\sum_{\ell=1}^{d} (\tilde{r}_{nk\ell}|_{Q_\ell}), Q) + \sum_{\ell=1}^{d} \tau(\tilde{r}_{nk\ell}, \cup_{\ell' \neq \ell} Q_{\ell'})
\]

\[
\leq \tau(\sum_{\ell=1}^{d} (\tilde{r}_{nk\ell}|_{Q_\ell}), Q) + 1. \tag{5.8}
\]

Here \( \tilde{r}_{nk\ell}|_{\cup Q_\ell} \) denotes the restriction of \( \tilde{r}_{nk\ell} \) on the set \( \cup\{I : I \in Q_\ell\} \). In the last step we show that

\[
\tau(\sum_{\ell=1}^{d} (\tilde{r}_{nk\ell}|_{Q_\ell}), Q) \leq 3(C + 1) .
\]

For this we split each \( Q_\ell \) into three set \( Q_\ell^1, Q_\ell^2, Q_\ell^3 \) as follows:

\[
Q_\ell^1 = \{I \subset Q_\ell : \exists 0 \leq m < 2^{nk\ell}, (m, \frac{m+1}{2^{nk\ell}}) \subset I\} ,
\]

\[
Q_\ell^2 = \{I \subset Q_\ell : \exists 0 \leq m < 2^{nk\ell}, I \subset (m, \frac{m+1}{2^{nk\ell}})\} ,
\]

\[
Q_\ell^3 = \{I \subset Q_\ell : 0 \leq m < 2^{nk\ell}, \frac{m}{2^{nk\ell}} < \min I < \frac{m+1}{2^{nk\ell}} < \max I \leq \frac{m+2}{2^{nk\ell}}\}. 
\]
Clearly $Q_1^i \cup Q_2^i \cup Q_3^i = Q_i$, and we set $Q_i^i = \bigcup_{\ell=1}^d Q_i^\ell$ for $i = 1, 2, 3$. With the aid of CCP we show that

$$\tau(\sum_{\ell=1}^d \tilde{r}_{n_k\ell} | Q_i^\ell, Q_i^i) \leq C + 1,$$

which yields the entire proof.

We prove it for $i = 2$, which is the most complicated case. The other two cases follow from similar arguments.

For this, we choose $\sigma'_1 < \ldots < \sigma'_d$ successive subsets of $\mathbb{N}$ with $\#\sigma'_\ell = \#Q_i^2\ell$. Then

$$\tau(\sum_{\ell=1}^d \tilde{r}_{n_k\ell} | Q_i^2, Q_i^i) = \| \sum_{\ell=1}^d \sum_{m \in \sigma'_\ell} \left( \int_{I_m} \tilde{r}_{n_k\ell} d\mu \right) e_m \|_X.$$

We decompose each $Q_i^2\ell$ into $Q_i^\ell, s$ where $Q_i^\ell, s = \{ I \in Q_i^2\ell : I \subset (\frac{s}{2^{n_k\ell}}, \frac{s+1}{2^{n_k\ell}}) \}$. Observe that for $I \in Q_i^\ell, s$

$$| \int_I \tilde{r}_{n_k\ell} d\mu | = \mu(I) \cdot \lambda_{k\ell} \cdot 2^{n_k\ell},$$

where $\lambda_{k\ell} = \| \sum_{i=1}^{2^{n_k\ell}} e_i \|^{-1}_X$. Hence

$$\sum_{I \in Q_i^\ell, s} | \int_I \tilde{r}_{n_k\ell} d\mu | = \mu(\bigcup_{I \in Q_i^\ell, s} I) \lambda_{k\ell} 2^{n_k\ell}.$$

Lemma 2.2 yields that

$$\left\| \sum_{\ell=1}^d \sum_{m \in \sigma'_\ell} \left( \int_{I_m} \tilde{r}_{n_k\ell} d\mu \right) e_m \right\|_X \leq \left\| \sum_{\ell=1}^d \sum_{m \in \sigma'_\ell} \mu(\bigcup_{I \in Q_i^\ell, s_m} I) \lambda_{k\ell} 2^{n_k\ell} e_m \right\|_X,$$

where $\sigma'_\ell \subset \sigma'_d$ with $\#\sigma'_\ell \leq 2^{n_k\ell}$. Here $(\ell, s)$ denotes a one to one corresponding of $\sigma'_\ell$ onto $\{ \ell, s \}_{s=0}^{2^{n_k\ell}-1}$. Also $\sum_{\ell=1}^d \sum_{m \in \sigma'_\ell} \mu(\bigcup_{I \in Q_i^\ell, s_m} I) \lambda_{k\ell} 2^{n_k\ell} e_m$ satisfies the assumptions of Proposition 5.3 hence

$$\left\| \sum_{\ell=1}^d \sum_{m \in \sigma'_\ell} \mu(\bigcup_{I \in Q_i^\ell, s_m} I) \lambda_{k\ell} 2^{n_k\ell} e_m \right\|_X \leq C + 1.$$

□

Proof of Theorem 5.2. It follows from Proposition 5.6 and 5.7 □

6. The space $V_X$ of functions of bounded $X$–variation.

In the final section we present a representation of $JF_X$ and $JF_X^{**}$ as function spaces of bounded $X$-variation, which generalizes the representation of $JF_p$ as spaces of functions of bounded $p$–variation pointed out by J.Lindendstaruss and C.Stegall [L–S], and used also by S.V.Kisliakov [K], in his proof that $\ell_1$ does not embed in $JF_p$.

Let $f : [0, 1] \to \mathbb{R}$. We adopt the following notation.
For $\mathcal{P} = \{t_i\}_{i=0}^{n-1}$ a partition of $[0, 1]$ and $X$ a reflexive Banach space with 1-symmetric basis we set

$$\alpha_X(f, \mathcal{P}) = \| \sum_{i=0}^{n-1} (f(t_{i+1}) - f(t_i))e_i \|_X,$$

and

$$\|f\|_{V_X} = \sup \{ \alpha_X(f, \mathcal{P}) : \mathcal{P} \text{ partition of } [0, 1] \} .$$

We also set

$$V_X = \{ f : f(0) = 0 \text{ and } \|f\|_{V_X} < \infty \} ,$$

and

$$V_X^0 = \{ f \in V_X : \lim_{\varepsilon \to 0} \sup \{ \alpha_X(f, \mathcal{P}) : \delta(\mathcal{P}) \leq \varepsilon \} = 0 \} .$$

It is easy to see that $V_X, V_X^0$ are Banach spaces endowed with $\| \cdot \|_{V_X}$. Further Lemma 5.4 yields that the Volterra operator $V(f)(t) = \int_0^t f(x)d\mu$ defines an isometry from $JF_X$ into $V_X^0$, which is actually onto. Let’s observe that the dual isometry maps $I^*$, where $I = [t_1, t_2]$, to $\delta_{t_2} - \delta_{t_1}$. Hence the set

$$\mathcal{S} = \left\{ \sum_{n=1}^\infty a_n I_n^* : \{I_n\}_{n=1}^\infty \text{ pairwise disjoint and } \| \sum_{n=1}^\infty a_n e_n^* \|_{X^*} \leq 1 \right\}$$

is mapped onto the set

$$\mathcal{K} = \left\{ \sum_{n=1}^\infty a_n (\delta_{d_n} - \delta_{c_n}) : \{(c_n, d_n)\}_n \text{ pairwise disjoint and } \| \sum_{n=1}^\infty a_n e_n^* \|_{X^*} \leq 1 \right\} , \quad (6.1)$$

and so the set $\mathcal{K}$ shares the properties proved for the set $\mathcal{S}$. Namely

1. $\mathcal{K}$ is $w^*$-compact and norming.
2. $B(V_X^0)_r = \overline{co}(\|K\|)$.

These two properties yield that

3. The space $V_X = (V_X^0)^{**}$.
4. A bounded sequence $(f_n)_n \subset V_X^0$ is $w-$Cauchy iff $(f_n(t))_n$ is convergent for all $t \in [0, 1]$.

The next theorem summarize the above observations.

**Theorem 6.1.** Let $X$ be a reflexive Banach space with a symmetric basis. Then

(i) $JF_X$ is isometric to $V_X^0$.
(ii) $(V_X^0)^{**} = V_X$.
(iii) On the bounded subsets of $V_X^0$ the weak topology coincides with the topology of pointwise convergence.

**Remark.** It is clear that $V_X^0$ is a subspace of $C[0, 1]$ and further the identity $I : V_X^0 \to C[0, 1]$ is a bounded operator. Next we shall see that any function $f \in C[0, 1] \cap (V_X \setminus V_X^0)$ has a remarkable property.
Definition 6.2. Let $K$ be a compact metric space. Following \[H-O-R, Ro\] we denote by $D(K)$ the set of all bounded functions on $K$ which are differences of bounded semi-continuous functions.

If $X$ is a separable Banach space and $K = (B_{X^*}, w^*)$ then as it is shown in \[H-O-R\] the classical Bessaga- Pełczynski theorem,\[B-P\], yields that there exists $x^{**} \in D(K) \cap (X^{**} \setminus X)$ iff $c_0$ embeds into $X$.

Therefore $D(K)$ provides a characterization of the embedding of $c_0$ into Banach spaces.

Our intention is to prove the following.

Theorem 6.3. If $K = (B_{(V_0^*)^*}, w^*)$ then

$$D(K) \cap V_X = C[0,1] \cap V_X.$$  

As a corollary we obtain the following characterization.

Corollary 6.4. Let $X$ be a reflexive space with $1$–symmetric basis. Then the following are equivalent

1. $c_0$ is isomorphic to a subspace of $V_0 \equiv JF_X$.
2. There exists a function $f \in C[0,1]$ such that $f \in V_X \setminus V_0^\circ$.

As we have mentioned in the introduction for the proof of the above stated theorem we shall make use of methods from descriptive set theory. We start with the following notation and definition.

Notation. (a) Let $K$ be a metric space, $f : K \to \mathbb{R}$ and $s \in K$. We set

$$\lim_{s' \to s} f(s') = \inf \{ \sup f(V) : V \text{ is a neighborhood of } s \} .$$

(b) For $f$ as above we denote by $Uf$ the upper semicontinuous envelope of $f$, which alternatively is defined as follows:

$$Uf(s) = \lim_{s' \to s} f(s') .$$

Definition 6.5. Let $K$ be a compact metric space and $f : K \to \mathbb{R}$ be a bounded function. For each countable ordinal $\xi$ the function $osc_\xi(f) : K \to \mathbb{R} \cup \{ \infty \}$ is defined inductively as follows.

For $\xi = 0$ we set $osc_\xi(f)(s) = 0 \forall s \in K$.

If $0 < \xi < \omega_1$ and $osc_\xi f$ has been defined, we first set

$$\tilde{osc}_{\xi+1} f(s) = \lim_{s' \to s} \{|f(s) - f(s')| + osc_\xi f(s')\} ,$$

and then we set

$$osc_{\xi+1} f = U \tilde{osc}_{\xi+1} f .$$
If $\xi$ is a limit ordinal and for $\zeta < \xi$, $\text{osc}_\zeta f$ has been defined then we set
\[
\tilde{\text{osc}}_\xi f(s) = \sup_{\zeta < \xi} \text{osc}_\zeta f(s),
\]
and finally
\[
\text{osc}_\xi f = U \tilde{\text{osc}}_\xi f.
\]
This completes the inductive definition.

The family $\{\text{osc}_\xi f\}_{\xi < \omega_1}$ was introduced by A.Kechris and A.Louveau, [K-L]. H. Rosenthal, [Ro], recognized the key role of this family in the study of non trivial $w$–$\text{Cauchy}$ sequences in Banach spaces. The definition presented here is due to H.Rosenthal and is a modification of the original one. Some recent results related to this family are obtained in [A-K]. The basic property of the family $\{\text{osc}_\xi f\}_{\xi < \omega_1}$ is described by the next proposition.

**Proposition 6.6.** [K-L], [Ro] Let $K$ be a compact metric space and $f : K \mapsto \mathbb{R}$ be a bounded function. The following are equivalent.
(a) The function $f$ is a difference of bounded semicontinuous functions.
(b) For each $\xi < \omega_1$, the function $\text{osc}_\xi f$ is a bounded function.

**Lemma 6.7.** Let $f \in C[0,1] \cap V_X$ and $K$ be the $w^*$–compact subset of $B(V_0^*)$, defined in (6.7). Then for all $\xi < \omega_1$ we have that
\[
\|\text{osc}_\xi f\|_\infty \leq \|f|_K\| = \|f\|_{V_X}.
\] (6.2)

Clearly (6.2) and Proposition 6.6 yields that $f \in D(K)$. Before passing to the proof, we state some abbreviations and notations.

**Notation.** In the sequel we consider the set $K$ endowed with the weak$^*$ topology. Let $s \in K$. Then $s = \sum_{i=1}^{\infty} \alpha_i(\delta_{t_i^2} - \delta_{t_i^1})$ where $\{(t_i^1, t_i^2)\}_{i=1}^{\infty}$ is a family of disjoint intervals and $\|\sum_{i=1}^{\infty} \alpha_i e_{t_i^1}^*\|_{X^*} \leq 1$. It is obvious that any permutation of $\mathbb{N}$ yields a new representation of the vector $s$. For a fixed representation we denote by
\[
 s_{|n_0} = \sum_{i=1}^{n_0} \alpha_i(\delta_{t_i^2} - \delta_{t_i^1}) \in K,
\]
\[
 s_{|n_o} = \sum_{i=n_0+1}^{\infty} \alpha_i(\delta_{t_i^2} - \delta_{t_i^1}) \in K,
\]
\[
 s_{|[n_0,n_1]} = \sum_{i=n_0+1}^{n_1} \alpha_i(\delta_{t_i^2} - \delta_{t_i^1}) \in K.
\]
Moreover, observe that if $(s_k)_k$, $s \in K$ with $s_k \rightarrow s$, then $s = \sum_{i=1}^{\infty} \alpha_i(\delta_{t_i^2} - \delta_{t_i^1})$ and for every $n_0 \in \mathbb{N}$ there exists a representation of $s_k = \sum_{i=1}^{\infty} \alpha_{i,k}(\delta_{t_i^2_{i,k}} - \delta_{t_i^1_{i,k}})$ such that for every $i \leq n_0$, $\lim_{k \rightarrow \infty} \alpha_{i,k} = \alpha_i$, $\lim_{k \rightarrow \infty} t_{i,k}^1 = t_i^1$, $\lim_{k \rightarrow \infty} t_{i,k}^2 = t_i^2$. 


We pass to the following

**Lemma 6.8.** For every \( f \in V_X \) and every \( s \in K \), \( \lim f(s_{>n}) = 0 \).

This is an immediate consequence of the property \( \lim_{n} \| s_{>n} \| = 0 \).

**Lemma 6.9.** Let \( f \in C[0,1] \cap V_X \) and \((s_k)_{k}\), \( s \in K \) such that \( s_k \rightarrow s \). Then for every \( n_0 \in \mathbb{N}, \varepsilon > 0 \) there exists \( k \in \mathbb{N} \) such that for \( 0 \leq n_1 < n_2 \leq n_0 \)
\[
|f(s_{(n_1,n_2)}) - f(s_{k|(n_1,n_2)})| < \varepsilon.
\]

This is also easy and follows from the convergence of \((s_k)_{k}\) to \( s \) and the continuity of the function \( f \). It is worth noticing that this is the only point where the continuity of the function \( f \) is used.

**Proof of Lemma 6.9.** The proof follows from the next inductive hypothesis.

For every \( s \in K, \varepsilon > 0, \delta > 0 \) and \( n_0 \in \mathbb{N} \), if \( s = \sum_{i=1}^{\infty} \alpha_i (s_{i} - s_{i+1}) \) and \( \text{osc}_{\xi} f(s) > \delta \), there exist \( s' \in K \) and \( n_1 > n_0 \) such that

(i) \( \quad \text{For } 0 \leq m_1 < m_2 \leq n_0, \quad \text{it holds } |f(s_{[m_1,m_2]}) - f(s'_{[m_1,m_2]})| < \varepsilon \).

(ii) \( \quad |f(s'_{n_1})| > \delta \).

A proof of the inductive hypothesis immediately yields a proof of the lemma. We proceed by induction.

For \( \xi = 0 \) is trivial.

Assume that for some \( \xi < \omega_1 \) the inductive hypothesis has been established. Let \( s \in K, \varepsilon > 0, \delta > 0 \) and \( n_0 \in \mathbb{N} \) such that \( \text{osc}_{\xi+1} f(s) > \delta \). From the definition of \( \text{osc}_{\xi+1} f \) and Lemma 6.9, we choose \( \hat{s} \) such that
\[
\hat{\text{osc}}_{\xi+1} f(\hat{s}) > \delta, \quad (6.3)
\]
and for \( 0 \leq m_1 < m_2 \leq n_0 \)
\[
|f(s_{[m_1,m_2]}) - f(\hat{s}_{[m_1,m_2]})| < \frac{\varepsilon}{4}. \quad (6.4)
\]

Next we choose a sequence \((s_k)_{k}\) such that

(i) \( \quad s_k \rightarrow \hat{s} \).

(ii) \( \quad \lim_k \{ |f(\hat{s}) - f(s_k)| + \text{osc}_{\xi} f(s_k) \} = \hat{\text{osc}}_{\xi+1} f(\hat{s}) \).

(iii) \( \quad \lim_k |f(\hat{s}) - f(s_k)| = \alpha \).

(iv) \( \quad \lim_k \text{osc}_{\xi} f(s_k) = \beta \).

Assume that both \( \alpha, \beta \) are positive (If at least one of them is equal to zero the proof is simpler). Set \( \delta_1 = \hat{\text{osc}}_{\xi+1} f(\hat{s}) - \delta > 0 \) and since \( \alpha + \beta = \hat{\text{osc}}_{\xi+1} f(\hat{s}) \) there exist \( 0 < c_1 < \alpha, 0 < c_2 < \beta \) such that \( c_1 + c_2 > \delta + \frac{3\delta_1}{4} \). Next choose \( n_1 \in \mathbb{N} \) such that
\[
|f(\hat{s}_{>n})| < \frac{\alpha - c_1}{16} \quad \text{for all } n > n_1. \quad (6.5)
\]
Further choose \( k \in \mathbb{N} \) such that

For \( 0 \leq m_1 < m_2 \leq n_1 \), \( |f(\hat{s}_{(m_1,m_2)}) - f(s_k|(m_1,m_2))| < \min\left\{ \frac{\varepsilon}{4}, \frac{\alpha - c_1}{8} \right\} \),  \( (6.6) \)

\[
|f(\hat{s}) - f(s_k)| > c_1 + \frac{\alpha - c_1}{2} ,
\]
\( (6.7) \)

osc\( f(s_k) > c_2 \) .  \( (6.8) \)

For this \( s_k \) we choose \( n_2 > n_1 \) such that for all \( n \geq n_2 \),

\[
|f(s_k|_{>n})| < \frac{\alpha - c_1}{8} .
\]
\( (6.9) \)

Applying the inductive hypothesis for the ordinal \( \xi \), the element \( s_k, \min\{\frac{\varepsilon}{\delta}, \frac{\alpha - c_1}{8}\}, c_2 \) and \( n_2 \) we obtain \( s'' \in \mathcal{K} \) and \( n_3 > n_2 \) such that

for \( 0 \leq m_1 < m_2 \leq n_2 \), \( |f(s_k|(m_1,m_2)) - f(s''|(m_1,m_2))| < \min\left\{ \frac{\varepsilon}{4}, \frac{\alpha - c_1}{8} \right\} \),  \( (6.10) \)

\[
f(s''|_{>n_3}) > c_2 .
\]
\( (6.11) \)

For the element \( s'' \) we have the following estimates.

For \( 0 \leq m_1 < m_2 < n_0 \), \( (6.4) \), \( (6.6) \) and \( (6.10) \) yield that

\[
|f(s|(m_1,m_2)) - f(s''|(m_1,m_2))| < \varepsilon .
\]

Next observe that \( (6.5) \), \( (6.6) \), \( (6.7) \) and \( (6.9) \) yield

\[
|f(s_k|(n_1,n_2))| >
\]
\[
> |f(\hat{s}) - f(s_k)| - |f(\hat{s}_{|n_1}) - f(s_k|_{|n_1})| - |f(\hat{s}|_{>n_1})| - |f(s_k|_{>n_2})| \\
> c_1 + \frac{\alpha - c_1}{2} - \frac{\alpha - c_1}{8} - \frac{\alpha - c_1}{8} - \frac{\alpha - c_1}{8} \\
= c_1 + \frac{\alpha - c_1}{8}
\]

and from \( (6.10) \) we get

\[
|f(s''|(n_1,n_2))| > c_1 + \frac{\alpha - c_1}{8} - \frac{\alpha - c_1}{8} = c_1 .
\]

Observe that

\[
s''|_{>n_3} = s''|(n_1,n_2] + s''|(n_2,n_3] + s''|_{>n_3} ,
\]

and since \( f \) is an affine function on \( \mathcal{K} \) we obtain that there exist \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{-1,1\} \) such that

\[
f(\varepsilon_1 s''|(n_1,n_2)] \geq 0, \quad f(\varepsilon_2 s''|(n_2,n_3]) \geq 0, \quad f(\varepsilon_3 s''|_{>n_3}) \geq 0 .
\]

It is easy to check that the element

\[
s' = s''|_{n_1} + \varepsilon_1 s''|(n_1,n_2] + \varepsilon_2 s''|(n_2,n_3] + \varepsilon_3 s''|_{>n_3}
\]

satisfies the conclusion of the inductive hypothesis.
If \( \xi \) is a limit ordinal and for some \( s \in \mathcal{K} \), such that \( \text{osc}_\xi f(s) > \delta \), then from the definition of \( \text{osc}_\xi f \) and Lemma 6.9 there exist \( \zeta < \xi \) and \( \hat{s} \) such that

\[
\text{osc}_\zeta f(\hat{s}) > \delta,
\]

and for \( 0 \leq m_1 < m_2 \leq n_0 \), \( |f(s_{(m_1,m_2)}) - f(\hat{s}_{(m_1,m_2)})| < \frac{\delta}{2} \).

Applying the inductive hypothesis for \( \zeta \) the element \( \hat{s}, \frac{\delta}{2}, \delta \) and \( n_0 \) we get an element \( s' \), which it is easily seen that satisfies the conclusion of the lemma. The proof is complete.

As a consequence of the previous lemma we obtain the following

**Proposition 6.10.** Let \( f \in C[0,1] \cap \mathcal{V}_X \). Then \( f \) is a difference of bounded semicontinuous functions defined on \( L = (B(\mathcal{V}_X^\infty), w^*) \).

**Proof.** If \( f \in \mathcal{V}_X^0 \) the conclusion trivially holds. Hence assume that \( f \notin \mathcal{V}_X \setminus \mathcal{V}_X^0 \). From Lemma 6.7 we obtain that \( f \in D(\mathcal{K}) \). Since \( \ell_1 \) does not embed in \( \mathcal{V}_X^0 \), Odell- Rosenthal’s theorem [O-R], yields that there exists a bounded sequence \( (f_n)_n \) in \( \mathcal{V}_X^0 \) converging \( \text{weak}^* \) to \( f \). Hence \( (f_n|_\mathcal{K})_n \subset C(\mathcal{K}) \) and converges pointwise to \( f|_\mathcal{K} \). From [H-O-R] we obtain that there exists \( (g_n)_n \) block of convex combinations of \( (f_n)_n \) such that \( (g_n|_\mathcal{K})_n \) is isomorphic to the summing basis of \( c_0 \). Since \( \mathcal{K} \) norms \( \mathcal{V}_X^0 \), we obtain that \( (g_n)_n \) remains isomorphic to the summing basis and converges \( \text{weak}^* \) to \( f \). Hence \( f \in D(L) \).

This proposition and known results yield the following.

**Proposition 6.11.** Let \( Y \) be a non reflexive subspace of \( \mathcal{V}_X^0 \). Assume that there exists \( f \in \mathcal{V}_X^\infty \) such that \( f \in C[0,1] \cap (\mathcal{V}_X \setminus \mathcal{V}_X^0) \). Then \( c_0 \) is isomorphic to a subspace of \( Y \).

Up to this point we have proved one direction of the Theorem 6.3. For the inverse part we need the following lemma.

**Lemma 6.12.** Let \( (f_n)_n \) be a normalized weakly null sequence in \( \mathcal{V}_X^0 \). Let \( \varepsilon > 0 \), \( ([\alpha_n, \beta_n])_n \) be a sequence of intervals such that \( \lim_n (\beta_n - \alpha_n) = 0 \) and \( |f_n(\alpha_n) - f_n(\beta_n)| > \varepsilon \), for all \( n \in \mathbb{N} \). Then there exists a subsequence \( (f_{n_i})_i \) of \( (f_n)_n \) which admits a lower \( X \)-estimate, i.e. there exists \( c > 0 \) such that

\[
c \| \sum \alpha_i e_i \|_X \leq \| \sum \alpha_i f_{n_i} \|_{\mathcal{V}_X^\infty} \text{ for all } (\alpha_i)_i^\infty \subset \mathbb{R}.
\]

**Proof.** Passing to a subsequence, we may assume that \( (\alpha_n)_n \), \( (\beta_n)_n \) are monotone sequences, such that \( \lim_n \alpha_n = \alpha = \lim_n \beta_n \). Using that \( (f_n)_n \) is weakly null and the fact that finite subsets of \( C(0,1) \) are equicontinuous, by a diagonal process we obtain a subsequence \( ([c_n, d_n])_{n \in L}, L \subset \mathbb{N} \), such that

(1) \( [c_n, d_n] \subset [\alpha_n, \beta_n] \) for all \( n \in L \).
(2) For \( n_1, n_2 \in L, n_1 < n_2 \), we have that \([c_{n_1}, d_{n_1}] \cap [\alpha_{n_2}, \beta_{n_2}] = \emptyset\).

(3) \(|f_n(d_n) - f_n(c_n)| > \frac{\varepsilon}{2}, \) for all \( n \in L \).

(4) For all \( n \in L, \sum_{k \in L} |f_k(d_n) - f_k(c_n)| < \frac{\varepsilon}{2^k}, \) from the equicontinuity.

(5) For all \( n \in L, \sum_{L \ni k > n} |f_k(d_n) - f_k(c_n)| < \frac{\varepsilon}{2^k}, \) since \((f_n)_n\) is weakly null.

Set \((f_n)_{n \in L} = (f_n)_{l \in \mathbb{N}}\). We prove that the subsequence \((f_n)_{l \in \mathbb{N}}\) admits a lower \(\frac{\varepsilon}{8} - X\) estimate. Indeed,

\[
\left\| \sum_{l=1}^{k} \alpha_l f_l \right\|_{V_X^0} \geq \left\| \sum_{l=1}^{k} \left( \sum_{l=1}^{k} \alpha_l f_l(d_j) - \sum_{l=1}^{k} \alpha_l f_l(c_j) \right) e_j \right\|_{X} \\
\geq \left\| \sum_{l=1}^{k} \alpha_l f_l(d_n_l) - \sum_{l=1}^{k} \alpha_l f_l(c_n_l) \right\|_{X} \geq \varepsilon \frac{\varepsilon}{8} \geq \frac{\varepsilon}{8} \left\| \sum_{l=1}^{k} \alpha_l e_l \right\|_{X}.
\]

This completes the proof of the lemma.

\[\Box\]

**Proposition 6.13.** Let \( f \in V_X \setminus C[0, 1] \). Then for every bounded sequence \((f_n)_n\) converging weak* to \( f \) there exists a subsequence \((f_{n_j})_j\) such that \((f_{n_{j+1}} - f_{n_j})_j\) has a lower \(X\) estimate.

\[\text{Proof.}\] Since \( f \) is not continuous there exists \( t \in [0, 1], (t_k)_k \) converging to \( t \) and \( \varepsilon > 0 \) such that \( |f(t) - f(t_k)| > \varepsilon \). Therefore there exist subsequences \((f_{n_j})_j\) and \((t_{k_j})_j\) such that \( |f_{n_j}(t) - f_{n_j}(t_{k_j})| > \varepsilon \). Further since each \( f_{n_j} \) is continuous we may assume that \( |f_{n_j}(t_{k_{j+1}}) - f_{n_j}(t_k)| < \frac{\varepsilon}{8} \). Clearly \((f_{n_{j+1}} - f_{n_j})_j\) satisfies the assumption of Lemma 6.12 and hence has a further subsequence with lower \(X\)–estimate.

\[\Box\]

As a corollary of Proposition 6.13 we get the following.

**Corollary 6.14.** If \( f \in V_X \setminus C[0, 1] \), then \( f \not\in D(L) \), where \( L = (B(V_X^0)^*, w^*) \).

\[\text{Proof of Theorem 6.3.}\] Follows from Proposition 6.10 and Corollary 6.14 \[\Box\]

We pass now to give a criterion for upper \(X\)–estimate.

**Lemma 6.15.** Let \((f_n)_n\) be a normalized weakly null sequence in \(V_X^0\) such that \(\|f_n\|_{\infty} \geq C\) for every \( n \in \mathbb{N}\). Then for every \( \varepsilon > 0 \) there exist \( M \in \mathbb{N}\) and \( t_1, \ldots, t_{k(\varepsilon)} \) points such that, for every \( \delta > 0 \) there exists \( n_0 \), such that

\[\|f_n|_{[0,1]} \|_{\ell_{k(\varepsilon) \delta_H(t, \delta)}} \|_{\infty} < \varepsilon \text{ for every } n > n_0, n \in M.\]

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Proof. Suppose that the conclusion does not hold. Inductively we choose \( M_1 \supset \ldots \supset M_k \), \((t^n)_n \in M_j, s_j, \delta_j > 0, j \leq k \) such that

1. \( |f_n(t^n)| > \varepsilon \) for every \( n \in M_j \).
2. \( \lim_{n \in M_j} t^n = s_j \) for every \( j \leq k \) and \( s_i \neq s_j \) for \( i \neq j \).
3. \( \|f_n|_{[0,1]}|_{\cup_{i=1}^k B(t_i, \delta_i)} \|_{\infty} > \varepsilon \) for every \( n \in M_{j+1} \).

Passing to a further subset of \( M_k \) we may assume that for every \( n \in M_k \) it holds, \(|f_n(s_j)| < \frac{\varepsilon}{2} \) for every \( j \leq k \). Then for \( n \in M_k \) sufficiently large, we have that

\[
\|f_n\|_{V_0^0} \geq \frac{\varepsilon}{2} \sum_{i=1}^{k} e_i \|X\], 
\]
a contradiction for large \( k \), since \( \sum_{i=1}^{k} e_i \|X\| \to \infty \). \( \square \)

**Lemma 6.16.** Let \((f_n)_n\) be a normalized weakly null sequence in \( V_0^0 \) such that \( \|f_n\|_{\infty} \geq C \) for every \( n \in \mathbb{N} \). Then for every \( \varepsilon > 0 \) there exists \( M \in [\mathbb{N}] \) such that \( \{t : |f_n(t)| \geq \varepsilon\} \) are pairwise disjoint.

**Proof.** Let \( \varepsilon > 0 \). Passing to a subsequence, by Lemma 6.15 we may assume that there exists \( k(\varepsilon) \) points \( t_1, \ldots , t_{k(\varepsilon)} \) such that

\[
\|f_n|_{[0,1]}|_{\cup_{i=1}^{k(\varepsilon)} B(t_i, \delta_i)} \|_{\infty} < \varepsilon \quad \text{for every} \quad n \in \mathbb{N},
\]
and also \( |f_n(t_i)| < \frac{\varepsilon}{2} \) for every \( n \in \mathbb{N} \) and every \( i = 1, \ldots , k(\varepsilon) \). Set \( f_{n_1} = f_1 \). Since \( f_{n_1} \) is continuous, we find for every \( i \leq k(\varepsilon) \) a neighborhood \( B_1(t_i, \delta_1) \subset B(t_i, \delta) \) of \( t_i \) such that \( |f_{n_1}|_{B_1(t_i, \delta_1)} < \frac{\varepsilon}{4} \). Set \( U_1 = \{t : |f_{n_1}(t)| > \varepsilon\} \). Then \( U_1 \cap (\cup_{i=1}^{k(\varepsilon)} B_1(t_i, \delta_1)) = \emptyset \).

Using Lemma 6.15 we pass to a subsequence \((f_{n_{M_2}})_n \) such that

\[
\|f_{n_{M_2}}|_{[0,1]}|_{\cup_{i=1}^{k(\varepsilon)} B_1(t_i, \delta_1)} \|_{\infty} < \varepsilon \quad \text{for every} \quad n \in M_2.
\]

Set \( f_{n_2} = f_{\min M_2} \). Since \( f_{n_2} \) is continuous, we find for every \( i \leq k(\varepsilon) \) a neighborhood \( B_2(t_i, \delta_2) \subset B_1(t_i, \delta_1) \) of \( t_i \) such that \( |f_{n_2}|_{B_2(t_i, \delta_2)} < \frac{\varepsilon}{4} \). Set \( U_2 = \{t : |f_{n_2}(t)| > \varepsilon\} \). Then \( U_2 \subset \cup_{i=1}^{k(\varepsilon)} (B_1(t_i, \delta_1) \setminus B_2(t_i, \delta_2)) \) and \( U_2 \cap U_1 = \emptyset \). Continuing in the same manner we get the desired subsequence. \( \square \)

The following corollary of the above lemmas seems to be of independent interest.

**Corollary 6.17** (Splitting lemma). Let \((f_n)_n\) be a normalized weakly null sequence in \( V_0^0 \). Then there exists a subsequence \((f_{n_{M}})_n \) of \((f_n)_n\) such that \( f_n = g_n + h_n \) for every \( n \in M \), \( \lim_{n} \|g_n\|_{\infty} = 0 \) and \( \lim_{n} \mu(\text{supp} h_n) = 0 \).

**Proof.** For the sequence \((f_n)_n\) we may assume that there exists a constant \( C > 0 \) such that \( \|f_n\|_{\infty} \geq C \) for all \( n \in \mathbb{N} \). Otherwise there exists a subsequence \((f_{n_{M}})_n \) of \((f_n)_n\) such that \( \lim_{n_{M}} \|f_{n_{M}}\|_{\infty} = 0 \), and the result follows immediately, setting \( g_n = f_n \) and \( h_n = 0 \).
Applying Lemma 6.15 inductively for \( \varepsilon = \frac{1}{2^j} \) we get a decreasing sequence \((M_j)_j\) of infinite subsets of \( \mathbb{N} \) and subsequences \((f_n)_{n \in M_j}\) and \(t_1^j, \ldots, t_{k(j)}^j\) points such that

\[
\|f_n|_{[0,1]} \setminus \bigcup_{i=1}^{k(j)} B(t_i, \delta_j) \|_\infty < \frac{1}{2^j} \text{ for every } n \in M_j,
\]

where \( \delta_j = \frac{k(j)}{2^j} \). Let \( n_j = \min M_j \). For every \( i \leq k(j) \), let \( g_i^j \) be a linear function defined in \([t_i^j - \delta_j, t_i^j + \delta_j]\) with endpoints \( f(t_i^j - \delta_j), f(t_i^j + \delta_j) \). Define \( g_j(t) = f_{n_j}(t) \) for every \( t \not\in \bigcup_{i=1}^{k(j)} B(t_i^j, \delta_j) \), while \( g_j(t) = g_i^j(t) \) if \( t \in B(t_i^j, \delta_j) \), \( i \leq k(j) \). We also set \( h_j = f_{n_j} - g_j \).

It is easy to see that \( g_j, h_j \) have the properties we claim. \( \square \)

**Remark.** The isomorphic structure of the subspaces of \( JF_X(\Omega) \) remains unclear, even in the case of James function space. We have been informed by E.Odell that in the Ph.D. Thesis of his student S.Buechler [B], is included the following result. Every normalized weakly null sequence \((x_n)_n\) and \( \varepsilon > 0 \) there exists a subsequence \((x_n)_{n \in M}\) admitting an \( \sqrt{2 + \varepsilon} \) upper \( \ell_2 \)-estimate. We present a slightly more general result.

**Definition 6.18.** Let \( X \) be a Banach space with 1-symmetric basis \((\epsilon_i)_i\). The space \( X \) has the block dominated property, if there exists \( C > 0 \) such that for every normalized block sequence \((u_i)_i\) we have that

\[
\left\| \sum_i \alpha_i u_i \right\| \leq C \left\| \sum_i \alpha_i \epsilon_i \right\|.
\]

As a consequence of Lemmas 6.15, 6.16 we get the following theorem.

**Theorem 6.19.** Let \( X \) have the block dominated property. Then every normalized weakly null sequence \((f_n)_n\) in \( V_X^0 \), has a further subsequence \((f_n)_{n \in M}\) which admits an upper \( X \)-estimate.

**Proof.** We distinguish two cases for the sequence \((f_n)_n\). The proof is almost identical in the two cases. We present the proof of the first case, and we shall indicate at the end the modification for the second case.

**Case.** 1. There exists \( C > 0 \) such that \( \|f_n\|_\infty \geq C \) for all \( n \in \mathbb{N} \).

Applying inductively Lemma 6.16 we choose a subsequence \((f_{n_j})_j\) of \((f_n)_n\), \((\delta_j)_j\), \((\varepsilon_j)_j\) sequences of real numbers such that

1. \( \tau(f_{n_j}, \mathcal{P}) < \frac{\varepsilon_j}{2^j} \) for every \( \mathcal{P} \) with \( \delta(\mathcal{P}) \leq \delta_j \).
2. \( \varepsilon_1 > \ldots > \varepsilon_k > \ldots \) and \( \varepsilon_{j+1} < \varepsilon_j \frac{1}{2^j} \) for every \( j > 1 \).
3. The sets \( U_i = \{ t : |f_{n_i}(t)| \geq \frac{\varepsilon_i}{2^i} \} \) are pairwise disjoint for every \( i \geq j \).

The subsequence \((f_{n_j})_j\) admits an \( (3C + 2\varepsilon) \)-upper \( X \)-estimate i.e.

\[
\left\| \sum_{i=1}^{m} \alpha_i f_{n_i} \right\|_2 \leq (3C + 2\varepsilon) \left\| \sum_{i=1}^{m} \alpha_i \epsilon_i \right\|_X \quad \text{for every } \{\alpha_i\}_{i=1}^{m} \subset \mathbb{R}.
\]
Indeed, let $Q = \{ t_i \}_{i=1}^r$ be a partition of $[0,1]$. Consider the partition of $Q$ into $Q_j = \{ t_i : \delta_j < t_{i+1} - t_i \leq \delta_{j-1} \}$, where $\delta_0 = 1$, $j \leq m$. Property (1) implies that

$$\| \sum_{j=1}^m \sum_{t_i \in Q_j} \left( \sum_{k \leq j} \alpha_k (f_{nk}(t_{i+1}) - f_{nk}(t_i)) \right) e_i \| X \leq \sum_{k=1}^m \tau(f_{nk}, \cup_{j \geq k} Q_j) \leq \varepsilon .$$

Also for fixed $j$ and $t_i \in Q_j$, property (3) implies that for at most three $k \geq j$, $k_1, k_2, k_3$, $|f_{nk}(t_{i+1}) - f_{nk}(t_i)| \geq \varepsilon_{j+1}$. For every $t_i$, let $k_{t_i}$ be the $k_j$, $j \leq 3$, which realize $\max \{ |\alpha_{k_j} f_{nk}(t_{i+1}) - f_{nk}(t_i)| : j \leq 3 \}$. Therefore

$$\| \sum_{j=1}^m \sum_{t_i \in Q_j} \left( \sum_{k \geq j} \alpha_k (f_{nk}(t_{i+1}) - f_{nk}(t_i)) \right) e_i \|$$

$$\leq 3 \| \sum_{k=1}^m \alpha_k \sum_{j=1}^m \sum_{t_i \in Q_j: k_{t_i} = k} (f_{nk}(t_{i+1}) - f_{nk}(t_i)) e_i \| + \sum_{j=1}^m \# Q_j \cdot \varepsilon_{j+1}$$

$$\leq 3C \| \sum_{k=1}^m \alpha_k e_k \| + \varepsilon ,$$

since $X$ has the block dominated property.

**Case. 2** There exists a subsequence $(f_n)_{n \in M}$ of $(f_n)_n$ such that $\lim_{n \in M} \| f_n \|_\infty = 0$.

In this case we proceed as in Case 1, choosing a further subsequence $(f_{n_j})_j$ of $(f_n)_{n \in M}$, $(\delta_j)_j$, $(\varepsilon_j)_j$ sequences of real numbers such satisfying (1), (2) as above, and (3) is replaced by

$$\langle 3' \rangle \text{ For every } j \in \mathbb{N}, \text{ it holds that } \| f_{n_i} \|_\infty < \frac{\varepsilon_{j+1}}{2} \text{ for every } i > j .$$

Then, following the arguments of Case 1, we easily seen that $(f_{n_j})_j$ admits an $(C + 2\varepsilon)$ upper estimate. \hfill \Box

The following two results follows from our previous work.

**Proposition 6.20.** Let $X$ have the block dominated property and $(f_n)_n$ be a normalized $w-Cauchy$ sequence in $V_X^0$ which converges weak$^*$ to $f \in V_X \setminus C[0,1]$. Then there exists a subsequence $(f_{nk})_k$ of $(f_n)_n$ such that $(f_{n_{2k+1}} - f_{n_{2k}})_k$ is equivalent to the basis of $X$.

**Proof.** By Proposition 6.13 and the discontinuity of $f$ there exist $\varepsilon > 0$, and a subsequence $(f_n)_{n \in M}$ of $(f_n)_n$ such that $(f_{2n+1} - f_{2n})_{n \in M}$ has a lower estimate, and $\| f_{2n+1} - f_{2n} \|_\infty \geq \varepsilon$ for all $n \in M$. The sequence $(f_{2n+1} - f_{2n})_{n \in M}$ satisfies the assumptions of Theorem 6.19 and therefore there exists a further subsequence $(f_{n_{2k+1}} - f_{n_{2k}})_k$ which admits an upper $X-$estimate. \hfill \Box

Proposition 6.11 and Proposition 6.20 immediately yield the following.
Theorem 6.21. Let X have the block dominated property and Y be a non reflexive subspace of $V_0^X$. Then Y contains isomorphically $c_0$ or X.

The following result answer partially the question on the structure of the subspaces of $JFX$, X having the block dominated property.

Proposition 6.22. Let X have the block dominated property, and $(x_n)_n$ be a normalized $\delta-$separated (i.e $\|x_n - x_m\| > \delta > 0$) sequence in $JFX \cap L^1$, such that $(\|x_n\|_{L^1})_n$ is bounded. Then the subspace Y generated by $(x_n)_n$ contains isomorphically X.

In the proof we shall use the following lemma, which holds for any reflexive Banach space with 1-symmetric basis.

Lemma 6.23. Let $(x_n)_n$ be a normalized weakly null sequence in $JFX \cap L^1(\mu)$. Let $\varepsilon > 0$, $([\alpha_n, \beta_n])_n$ be a sequence of intervals such that $\lim (\beta_n - \alpha_n) = 0$ and $\int_{\alpha_n}^{\beta_n} x_n \, d\mu > \varepsilon$ for all $n \in \mathbb{N}$. Then there exists a subsequence $(x_n)_l$ of $(x_n)_n$ which admits a lower X-estimate, i.e there exists $c > 0$ such that

$$c \| \sum_l \alpha_l e_i \|_X \leq \| \sum_l \alpha_l x_n \|_{JFX} \text{ for all } (\alpha_l)_l \subset \mathbb{R}.$$ 

The proof of this lemma follows immediately from Lemma 6.12, considering $JFX$ as the space $V_0^X$.

Proof of Proposition 6.22. Since $\ell_1$ does not embed into $JF$, passing to a subsequence and taking differences of successive terms, we assume that $(x_n)_n$ is normalized weakly null sequence.

Claim. There exist $\varepsilon > 0$, $M$ an infinite subset of $\mathbb{N}$ and a sequence $([\alpha_n, \beta_n])_{n \in M}$ of intervals such that

$$| \int_{\alpha_n}^{\beta_n} x_n \, d\mu | > \varepsilon \text{ for all } n \in M \text{ and also } \lim (\beta_n - \alpha_n) = 0.$$

Proof of the Claim. Since $(x_n)_n$ is bounded in $L^1$-norm from Lemma 1.10 (see also proof of Proposition 1.11), we conclude that there exist $\varepsilon > 0$ and a sequence $([\alpha_n, \beta_n])_n$ of intervals, such that

$$| \int_{\alpha_n}^{\beta_n} x_n \, d\mu | > \varepsilon \text{ for all } n \in \mathbb{N} \ . \ (6.12)$$

If $\lim (\beta_n - \alpha_n) = 0$, we have finished, otherwise let’s observe the following.

There exist $\varepsilon' > 0$, $M$ an infinite subset of $\mathbb{N}$, such that for all $n \in M$, there exists an interval $[c_n, d_n]$ such that

1. $| \int_{c_n}^{d_n} x_n \, d\mu | > \varepsilon'$ for all $n \in M$,
2. $\lim (d_n - c_n) = 0.$
Indeed, choose monotone subsequences \((\alpha_n, \beta_n)_{n \in M}\) such that \(\alpha_n \to \alpha, \beta_n \to \beta\). For simplicity assume that \([\alpha, \beta] \subseteq [\alpha_n, \beta_n]\). Since \((x_n)_n\) is weakly null, we choose \(n_0 \in M\) such that
\[
\left| \int_{\alpha}^{\beta} x_n \, d\mu \right| < \frac{\varepsilon}{4} \quad \text{for all } n \in M, n > n_0.
\]
Clearly (6.12) implies that for \(n \in M, n > n_0\),
\[
either \left| \int_{\alpha}^{\alpha_n} x_n \, d\mu \right| > \frac{\varepsilon}{4} \text{ or } \left| \int_{\beta}^{\beta_n} x_n \, d\mu \right| > \frac{\varepsilon}{4}.
\]
If the former holds, set \([c_n, d_n] = [\alpha_n, \alpha]\) otherwise set \([c_n, d_n] = [\beta, \beta_n]\) and let \(\varepsilon' = \frac{\varepsilon}{4}\).
We easily conclude that \((x_n)_{n \in M}, ([c_n, d_n])_{n \in M}, \varepsilon'\) satisfy the desired properties. The proof of the Claim is complete. \(\square\)

To complete the proof of Proposition 6.22, Lemma 6.23 yields a further subsequence \((x_{n_k})_{k \in \mathbb{N}}\) which admits a lower \(X\)-estimate. From Theorem 6.19 \((x_{n_k})_{k \in \mathbb{N}}\) has a subsequence \((x_{n_k})_{k \in M}\) with an upper \(X\)-estimate. Clearly \((x_{n_k})_{k \in M}\) is equivalent to the basis of \(X\). \(\square\)

A direct consequence of Proposition 6.22 is the following.

**Corollary 6.24.** Let \(X\) have the block dominated property. Then every subsequence \((h_{n_k})_k\) of the Haar system in \(JF_X\), generates a subspace containing \(X\).

In the last part of this section we prove the equivalence between the Point Continuity Property (PCP), and the non embedding of \(c_0\) in \(V_0^X\). We recall that a bounded subset \(W\) of a Banach space \(X\) has PCP if for every \(w\)-closed subset \(A\) of \(W\) and \(\varepsilon > 0\) there exists a relatively weakly open subset \(U\) of \(A\) with \(\text{diam}(U) < \varepsilon\). For a comprehensive study of PCP and its relation with other properties we refer to [Bo]. We start with the following.

**Lemma 6.25.** Let \(W\) be a bounded subset of \(V_0^X\). Then for every relatively weakly open \(U \subseteq W\) and every \(\varepsilon > 0\) there exists \(g \in U\) such that for every weakly open neighborhood \(V\) of \(g\) and every \(g' \in U \cap V\) we have that \(\|g - g'\|_\infty \leq \varepsilon\).

**Proof.** Assume on the contrary. There exists a bounded set \(W\) and a relatively weakly open subset \(U\) of \(W\) such that for every \(g \in U\) there exists a net \((g_i)_{i \in I} \subseteq U\) with \(g_i \xrightarrow{w} g\) and \(\|g - g_i\|_\infty \geq \varepsilon\). Since this property remains invariant under translations of \(W\) we assume that \(0 \in U\) and that for every \(g \in W\), \(\|g\|_{V_X} \leq C\).

Choose \(n_0 \in \mathbb{N}\) such that
\[
\| \sum_{i=1}^{n_0} e_i \|_{V_X} \geq \frac{4C}{\varepsilon}.
\]

**Claim.** There exists \(g \in U\) with \(\|g\|_{V_X} > 2C\).
Clearly a proof of this claim derives a contradiction and completes the proof of the lemma.

Proof of the Claim. To prove the claim, we proceed by induction on \( k = 0, \ldots, 2n_0^2 \) choosing

1. A partition \( \mathcal{P}_k \) of \([0, 1]\),
2. \( g_k \in U \),
3. \([\alpha_k, \beta_k], 0 \leq \alpha_k < \beta_k \leq 1\), such that the following are fulfilled.
   a. \( g_0 = 0, \mathcal{P}_0 = \{0, 1\}, \alpha_0 = 0, \beta_0 = 1 \).
   b. \( \mathcal{P}_0 \subset \mathcal{P}_1 \subset \ldots \subset \mathcal{P}_{2n_0^2} \).
   c. \( \{\alpha_k, \beta_k\} \subset \mathcal{P}_{k+1} \).
   d. For \( k = 1, \ldots, 2n_0^2 \), \(|(g_k - g_{k-1})(t)| > \varepsilon \) for all \( t \in [\alpha_{k+1}, \beta_{k+1}] \).
   e. \(|(g_k - g_{k+1})(s)| < \frac{s}{8\varepsilon + 1} \) for \( s \in \mathcal{P}_{k+1} \).
   f. \( \text{diam}\left(\sum_{j=0}^{k-1} |g_j - g_{j+1}([s_j^{k+1}, s_{j+1}^{k+1}])\right) < \frac{\varepsilon}{8} \), where \( \mathcal{P}_{k+1} = \{0 = s_0^{k+1} < \ldots < s_{k+1}^{k+1} = 1\} \).

Assume that for some \( 0 \leq k < 2n_0^2 \), \( \mathcal{P}_k, g_j, \{\alpha_j, \beta_j\} \) have been defined for \( 0 \leq j \leq k \).

Then we first choose \( \mathcal{P}_{k+1} \) such that \( \mathcal{P}_k \cup \{\alpha_k, \beta_k\} \subset \mathcal{P}_{k+1} \) and condition (f) is satisfied.

The latter is possible because of the continuity of the function \( \sum_{j=1}^{k-1} |g_j - g_{j+1}| \). Next we choose a net \((g_i)_{i \in I} \subset U\) such that \( g_i - g_k \xrightarrow{w} 0 \), \( \|g_i - g_k\|_{\infty} > \varepsilon \), and clearly there exists \( g_{i_0} \equiv g_{k+1}\) satisfying condition (e). Finally choose \( \alpha_{k+1} < \beta_{k+1} \) such that \(|g_k - g_{k+1}(t)| > \varepsilon \) for all \( t \in [\alpha_{k+1}, \beta_{k+1}] \). Let’s observe the following.

(i) For all \( k = 1, \ldots, 2n_0^2 \) there exists \( s_i^k \in \mathcal{P}_k \) such that \( s_i^k < \alpha_k < \beta_k < s_i^{k+1} \). This follows from conditions (d) and (e). Set \( c_k = s_i^k \) and \( d_k = s_{i+1}^k \).

(ii) Let \( D = \{(c_k, \alpha_k), (\beta_k, d_k) : k = 1, \ldots, 2n_0^2\} \). Then any \( I_1, I_2 \in D \) satisfy

either \( I_1 \subset I_2 \) or \( I_2 \subset I_1 \) or \( I_1 \cap I_2 = \emptyset \).

(iii) For all \( k = 1, \ldots, 2n_0^2 \),

\[
|g_{2n_0^2}(c_k) - g_{2n_0^2}(\alpha_k)| = \sum_{j=1}^{2n_0^2} |(g_j - g_{j-1})(c_k) - (g_j - g_{j-1})(\alpha_k)| \geq \sum_{j \neq k} |(g_k - g_{k-1})(c_k) - (g_k - g_{k-1})(\alpha_k)| - \sum_{j \neq k} |(g_j - g_{j-1})(c_k) - (g_j - g_{j-1})(\alpha_k)| \geq \frac{7\varepsilon}{8} - \frac{2\varepsilon}{8} > \frac{\varepsilon}{2},
\]

and using similar reasoning

\[
|g_{2n_0^2}(\beta_k) - g_{2n_0^2}(d_k)| > \frac{\varepsilon}{2}.
\]
Next we assert that there exists a family $\mathcal{D}' \subset \mathcal{D}$ with $\# \mathcal{D}' \geq n_0$ consisting of pairwise disjoint intervals.

If such a family $\mathcal{D}'$ exists then the choice of $n_0$ and relation (iii) yield that $\|g_{2n_0^2}\|_{V_X} \geq 2C$, which proves the claim and completes the proof of the lemma.

Hence consider $\mathcal{D}$ and assume that any $\mathcal{D}' \subset \mathcal{D}$ consisting or pairwise disjoint intervals satisfies $\# \mathcal{D}' < n_0$. Clearly $\mathcal{D}$ with the order of inclusion (i.e $I_1 \prec I_2$ iff $I_1 \supset I_2$) defines a finite tree and from our assumption each level has less than $n_0$ elements, while $\# \mathcal{D} > 2n_0^2$. Hence there exists a branch $\{I_{k_1} \supseteq I_{k_2} \supseteq \ldots \supseteq I_{k_{2n_0}}\}$ of $\mathcal{D}$ and we may assume that there exists $\{I_{k_1} \supseteq I_{k_2} \supseteq \ldots \supseteq I_{k_{2n_0}}\}$ such that $I_{k_i} = (c_{k_i}, a_{k_i})$ for all $i \leq n_0$, or $I_{k_i} = (\beta_{k_i}, d_{k_i})$ for all $i \leq n_0$. Then it is easy to see that if the first alternative holds the family $\{(\beta_{k_i}, d_{k_i}) : i \leq n_0\}$ consists of pairwise disjoint intervals. A similar conclusion holds if the second alternative occurs. This proves our assertion and the proof of the lemma is complete. □

This lemma yields the following.

**Proposition 6.26.** Let $W$ be a bounded subset of $V_X^0$ and $\delta > 0$ such that for every relatively weakly open $U \subset W$ we have that $\text{diam}(U) > \delta$. Then

$$\overline{W}^* \cap C[0,1] \cap (V_X \setminus V_X^0) \neq \emptyset.$$  

**Proof.** Applying inductively Lemma 6.25 we obtain $(g_n)_n \subset W$ such that

1. $\|g_n - g_{n+1}\|_{V_X} > \frac{\delta}{2}$.
2. $\|g_n - g_{n+1}\|_{\infty} < \frac{\delta}{2}$.

Hence $(g_n)_n$ converges uniformly to a continuous function $g$. Also $(g_n)_n$ converges weak$^*$ to the same function in the space $V_X$. Standard perturbation arguments yield that $(g_n)_n$ could be chosen such that the limit function $g \in V_X \setminus V_X^0$. This completes the proof. □

As a consequence we obtain the following.

**Theorem 6.27.** Let $Y$ be a subspace of $V_X^0$. The following are equivalent.

1. $c_0$ does not embed into $Y$.
2. The space $Y$ has the Point of Continuity Property (PCP)

In particular if $c_0$ does not embed into $V_X^0$, then $V_X^0$ has PCP.

7. **Remarks and Problems.**

We present some problems related to the spaces $JF_X(\Omega)$ and certain remarks related to these problems.
Problem 1. Let $\Omega$ be an open bounded subset of $\mathbb{R}^{d_0}$, $d_0 > 1$. For $Q(\Omega)$ a family of convex bodies contained in $\Omega$ we consider the space $JF(Q(\Omega))$ endowed with the following norm
\[
\|f\|_{JF(Q(\Omega))} = \sup \left\{ \left( \sum_{i=1}^{n} \left( \int_{V_i} f \right)^2 \right)^{1/2} : \{V_i\}_{i=1}^{n} \subset (Q(\Omega)), V_i \cap V_j = \emptyset \right\}.
\]
For what families $Q(\Omega)$ the following hold:
(i) The space $JF(Q(\Omega))$ does not contain $\ell_1$.
(ii) The space $JF^*(Q(\Omega))$ is non-separable.

Remark. Our proof for the family of the parallelepipeds $P(\Omega)$ does not yield similar results for other families. In particular the above problem is open if $Q(\Omega)$ is either the Euclidean balls or the $\ell_{d_0}^\infty$ balls contained in $\Omega$.

Problem 2. Suppose that $\Omega, \Omega'$ are open and bounded subsets of $\mathbb{R}^{d_0}$. Is it true that $JF_X(\Omega)$ is isomorphic to $JF_X(\Omega')$.

Remark. As we have mentioned at the beginning of the first section $JF_X((0, 1)^{d_0})$ is isomorphic to a complemented subspace of $JF_X(\Omega)$ for any open $\Omega \subset \mathbb{R}^{d_0}$.

Problem 3. Is it possible for $1 < d_1 < d_0$ the space $JF_X((0, 1)^{d_0})$ be isomorphic to a subspace of $JF_X((0, 1)^{d_1})$.

Corollary 3.7 yields that this is not possible if $d_1 = 1$. But the argument used for this result is not extended in higher dimensions.

Problem 4. Does $c_0$ embed into $JF_X$ for any $X$ reflexive with 1-symmetric basis.

This problem is related to our results presented in the last two sections. There are two ways to approach a positive answer to this problem. The first is the following

Question. Does there exist a property similar to CCP valid in any reflexive space $X$ with 1-symmetric basis which implies the existence of a sequence in $JF_X$ equivalent to $c_0$ basis.

The second concerns the following which summarize some of our results from section 6.

Theorem. The following are equivalent.

(1) $c_0$ embeds into $JF_X$.
(2) $C[0, 1] \cap (V_X \setminus V_0^X) \neq \emptyset$.
(3) $V_0^X$ fails PCP.
(4) The identity $I : V_0^X \to C[0, 1]$ is not semi-embedding.
Hence a second approach is to show that some of the above equivalents holds for any $JF_X$ space.

The last problem concerns the structure of $JF$.

**Problem 5.** Does every subspace of $JF$ contains either $c_0$ or $\ell_p$, $2 \leq p < \infty$.

As we have mentioned in the introduction the space $JF$ contains $\ell_p$, for $2 \leq p < \infty$ (\[B\]). From the results of section 6, follows that this problem is reduced to the case of subspaces $Y$ of $V_\infty^X$ which are reflexive and the identity $I : Y \to C[0,1]$ is a compact operator.

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