On the Derived Subgroups of Some Finite Groups

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Abstract: Problem statement: In this study we focus on the derived subgroup of nonabelian 3-generator groups of order \( p^3q \), where \( p \) and \( q \) are distinct primes and \( p < q \). Our main objective is to compute the derived subgroup for these groups up to isomorphism.

Approach: In a group \( G \), the derived subgroup \( G' = [G, G] \) is generated by the set of commutators of \( G \), \( K(G) = \{[x, y] | x, y \in G \} \) and introduced by Dedekind. The relations of the group are used to compute the derived subgroup.

Results: The results show that the derived subgroup of nonabelian 3-generator groups of order \( p^3q \) is a cyclic group, \( Q_8 \) or \( A_4 \).

Conclusion/Recommendations: The problem can be considered to compute the derived subgroup of these groups without the use of the relations.

Key words: Derived subgroup, sylow theorems, finitely generated group

INTRODUCTION

Miller (1898) introduced the derived subgroup \( G' \) of a group \( G \) as the subgroup generated by \( K(G) = \{[x, y] | x, y \in G \} \), the set of commutators of \( G \). According to Miller, commutators \([x, y]\) were introduced by Dedekind a few years earlier. Commutators can act as a tool in all of group theory. For example, commutators can be used to compute Schur multiplier, Schur multiplier of a pair and nonabelian torsion squares of groups.

Basic definitions and theorems: Includes some definitions and results on the derived subgroups of nonabelian groups.

Definition 1: Hungerford (1997) let \( G \) be a group and \( X \) a subset of \( G \). Let \( \{H_i | i \in I\} \) be the family of all subgroups of \( G \) which contains \( X \). Then \( \cap H_i \) is called the subgroup of \( G \) generated by the set \( X \) and is denoted by \( < X > \).

Theorem 2: Hungerford (1997) let \( G \) be a group and \( X \) a non empty subset of \( G \). Then the subgroup \( < X > \) generated by \( X \) consists of all finite product finite product \( a_1 a_2 ... a_n \) (\( a_i \in X, n_i \in \mathbb{Z} \)). In particular for every \( a \in G \), \( < a > = \{a^n | n \in \mathbb{Z} \} \).

Definition 3: Hungerford (1997) let \( G \) be a group. The subgroup of \( G \) generated by the set \( \{x^iy^jxy | x, y \in G \} \) is called the derived subgroup of \( G \) and denoted by \( G' \).

Let \( G \) be a group and let \( G^{(1)} \) be \( G' \). Then for \( i \geq 1 \), define \( G^{(i)} = G^{(i-1)}' \). The notation \( G^{(i)} \) is called the \( i \)-th derived subgroup of \( G \). This gives a sequence of subgroups of \( G \), each normal in preceding one: \( G > G^{(1)} > G^{(2)} > ... \). Actually each \( G^{(i)} \) is a normal subgroup of \( G \).

Burnside (1911) classified all finite groups of order \( p^2q \) and Western (1898) obtained the classification of groups of order \( p^3q \), where \( p \) and \( q \) are distinct primes.

The classification of all nonabelian 2-generator groups of order \( p^3q \) is given in the following theorem.

Theorem 4: Western (1898) Let \( G \) be a nonabelian 2-generator group of order \( p^3q \), where \( p \) and \( q \) are distinct primes and \( p < q \). Then \( G \) is exactly one group of the following types Eq. 1-6:

\[
\begin{align*}
G &= < A, Q | A^4 = Q^8 = 1, \\
A^{-1}QA &= Q^{-1}; q \equiv 1 \pmod{2} 
\end{align*}
\]
where, \( a \) is any primitive root of \( a^4 \equiv 1 \pmod{q} \), \( q \equiv 1 \pmod{4} \):

\[
G = < A, Q | A^4 = Q^8 = 1, A^{-1}QA = Q^* > \tag{2}
\]

\[
G = < A, B, Q | A^4 = B^2 = Q^* = 1, \quad AB = BA, A^{-1}QA = Q^* > \tag{10}
\]

where, \( a \) is any primitive root of \( a^8 \equiv 1 \pmod{q} \), \( q \equiv 1 \pmod{8} \):

\[
G = < A, Q | A^8 = Q^* = 1, A^{-1}QA = Q^* > \tag{3}
\]

\[
G = < A, B, Q | A^8 = B^2 = Q^* = 1, \quad BAB = A^{-1}, AQ = QA, BQB = Q^{-1} > \tag{11}
\]

where, \( a \) is any primitive root of \( a^p \equiv 1 \pmod{q} \), \( q \equiv 1 \pmod{p} \):

\[
G = < A, Q | A^p = Q^* = 1, A^{-1}QA = Q^* > \tag{4}
\]

\[
G = < A, B, Q | A^p = B^2 = Q^* = 1, \quad B^2 = A^2, A^{-1}B = A^{-3}AQ = QA \quad B^{-1}QB = Q^{-1} > \tag{12}
\]

where, \( a \) is any primitive root of:

\[
a^4 \equiv 1 \pmod{q} \quad \text{and} \quad q \equiv 1 \pmod{4} \tag{14}
\]

Theorem 5: Rashid et al. (2010) Let \( G \) be a nonabelian 2-generator group of order \( p^3 q \), where \( p \) and \( q \) are distinct primes and \( p < q \). Then, \( G \cong C_q \), finite cyclic group of order \( q \).

In this study, we focus on the derived subgroups of nonabelian 3-generator groups of order \( p^3 q \) where \( p \) and \( q \) are distinct primes and \( p < q \).

The classification of all nonabelian 3-generator groups of order \( p^3 q \) is given in the following theorem.

Theorem 6: Western (1898) Let \( G \) be a nonabelian 3-generator group of order \( p^3 q \), where \( p \) and \( q \) are distinct primes and \( p < q \). Then \( G \) is exactly one group of the following types Eq. 7-21:

\[
G = < A, B, Q | A^4 = B^2 = Q^8 = 1, \quad BAB = A^{-1}, AQ = QA, BQB = Q^{-1} > \tag{7}
\]

\[
G = < A, B, Q | A^4 = B^2 = Q^8 = 1, \quad AB = BA, A^{-1}QA = Q^* > \tag{15}
\]

\[
G = < A, B, Q | A^4 = B^2 = Q^8 = 1, \quad B^{-1}AB = A^{-1}, AQ = QA, BQB = AB > \tag{16}
\]

where, \( a \) is any primitive root of:

\[
a^4 = 1 (\pmod{q}) \quad \text{and} \quad q = 1 (\pmod{4}) \tag{18}
\]

\[
G = < A, B, Q | A^4 = B^2 = Q^8 = 1, \quad AB = BA, A^{-1}QA = Q^*, BQ = QB > \tag{19}
\]

where, \( a \) is any primitive root of:

\[
a^4 = 1 (\pmod{q}) \quad \text{and} \quad q = 1 (\pmod{4}) \tag{18}
\]
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\[(\mod q), q = 1(\mod p) \text{ and } b = a, a', \ldots, a^{p-1}\]  
(20)

\[G = < A, B, Q | A^2 = B^p = Q^q = 1,\]
\[A B = B A, A^{-1} Q A = Q, B Q = Q B >\]

where, \(a\) is any primitive root of \(a^{p^2} = 1\):

\[(\mod q) \text{ and } q = 1(\mod p^2)\]  
(21)

**Main Result:**

*Theorem 7:* Let \(G\) be a nonabelian 3-generator group of order \(p^3 q\), where \(p\) and \(q\) are distinct primes and \(p < q\). Then \(G \cong C_2, C_4, C_{2q}, C_{p^2}, C_{pq}, Q_8\) or \(A_4\), where \(Q_8, A_4\) are quaternion and alternating groups, respectively.

**Proof:** By Theorem 6, \(G\) has 15 types. If \(G\) is a group of type 6.1, then \(G\) has three generators \(A, B\) and \(Q\) and relations \(AB = A^{-1}, AQ = QA\) and \(BQ = QB\). For this group we can obtain the following relations:

- \(A^j Q^i = Q^i A^j\) for all \(i, j \in \mathbb{Z}\)
- \(B^j Q^i = Q^i B^j\) for all \(i, j \in \mathbb{Z}\)
- \(AB = BA, A^2 B = BA^2, A^3 B = BA\)
- \([A, B] = A^2, [A^2, B] = 1\)

Then by mentioned relations for all \(x, y \in G, [x, y] = 1\) or \(A^2\). Therefore, \(G = \{1, A^2\}\), that is, \(G \cong C_2\).

The proof of the second type is similar to the first type.

To compute the derived subgroup for a group of type 6.3, by relations \(AB = BA, AQ = QA, BQB = Q^{-1}\) and \([Q^i, B] = Q^{-2k}\), we can obtain that \(G \cong C_q\).

The proof of types 6.4, 6.8, 6.12, 6.13 and 6.15 is similar to that type of 6.3.

For type 6.5, \(G \cong D_{4q}\), then \(G' \cong C_{2q}\).

Let \(G\) be a group of type 6.6, then by relation \(A^4 AQ = Q^i\) it is clear that \(|G| \geq pq\) and relation, \(BAB = A^{-1}\) shows that \(1, A^2 \in G'\). Thus \(|G'| = 2q\) and \(G' \cong BQ\), that is, \(G' \cong C_{2q}\).

For proving 6.7, we can use the method that we used in type 6.6.

For a group of type 6.9, \(G \cong SL(2, 3)\), where \(SL(2, 3) = < a, b, c | a^3 = b^3 = c^2 = abc >\). So \(G \cong Q_8\).

To compute \(G'\) for a group of type 6.10, by the number of generators and relations it is an immediate consequence that \(G \cong S_4\). Therefore, \(G' \cong A_4\).

Let \(G\) be a group of type 6.11, then relations \(A^{p^2} = B^p = Q^q = 1, B^{-1} AB = A^{p+1}, AQ = QA, BQ = QB\) show that \(G\) is isomorphic to \(C_p\).

Finally, for a group of type 6.14, the relations \(A^{p^2} = B^p = Q^q = 1, B^{-1} AB = A^{p+1}, AQ = QA, B^i Q = Q^i\) show that \(|G'| = pq\) and by computing the commutators, \(G\) is a cyclic group of order \(pq\).

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