Estimation of Multivariate Location and Covariance using the $S$-Hellinger Distance

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Abstract

This paper describes a generalization of the Hellinger distance which we call the $S$-Hellinger distance; this general family connects the Hellinger distance smoothly with the $L_2$-divergence by a tuning parameter $\alpha$ and is indeed a subfamily of the $S$-Divergence family of Ghosh et al. (2013 a, b). We use this general divergence in the context of estimating the location and covariances under (continuous) multivariate models and show that the proposed minimum $S$-Hellinger distance estimator is affine equivariant, asymptotically consistent and have high breakdown point under suitable conditions. We also illustrate its performance through an extensive simulation study which show that the proposed estimators give more robust estimator than the minimum Hellinger distance estimator for the location and correlation parameters under different types of contaminations with the contamination proportion being as high as 20%.

Keywords: $S$-Hellinger Distance, Robustness, Multivariate Location and Covariance, Breakdown Point

1. Introduction

Minimum distance estimation methods are popular in the context of robust parametric estimation; we estimate the parameter of interest by minimizing the discrepancy between the observed sample data and the assumed parametric model based on the distance under consideration. The rigorous treatment of robust density based minimum distance estimation originates Beran (1977). He considered the famous Hellinger distance with univariate parametric models and demonstrated the robustness of the corresponding minimum distance estimator, along with its full asymptotic efficiency. This work gives a remarkable breakthrough in robust estimation by exhibiting that the apparent contradiction between efficiency and robustness may be reconciled. Subsequently the literature has grown substantially and other authors extended this to several statistical divergence measures. A statistical divergences is a distance like measure which relaxes the assumptions of symmetry and triangular inequality from its definition. Several class of divergences are also seen to produce highly robust estimators with full or substantially high

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$^1$This is part of the Ph.D. research work of the first author which is ongoing at the Indian Statistical Institute

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Preprint submitted to Elsevier

March 26, 2014
asymptotic efficiency. Vajda (1989), Pardo (2006) and Basu et al. (2011) provides useful details of development and theory of such robust minimum divergence estimations. The work of Beran (1977) was limited to the case of univariate densities. Tamura and Boos (1986) extended his work on minimum Hellinger distance estimation to the case of multivariate location and covariance estimation under the restricted class of elliptically symmetric models. In the subsequent literature, there has been many attempts to generalize the Hellinger distance suitably to extract some more desirable properties along with greater robustness; see e.g. Simpson (1987) and Basu, Basu and Chaudhuri (1997). However, most, if not all, such generalization are in the case of univariate models. Recently, Ghosh et al. (2013a, b) developed a general family of divergence measures, named the $S$-divergence family, connecting the famous Cressie-Read family of power divergence (Cressie and Read, 1998) to the $L_2$-divergence smoothly in terms a parameter $\alpha$; they have argued that as the parameter $\alpha$ increases so does the robustness of the corresponding minimum $S$-divergence estimator. Here, we will consider one particular subfamily of $S$-divergences that connects the Hellinger distance (a particular member of Cressie-Read family) to the $L_2$-divergence smoothly through the parameter $\alpha$. Indeed, only this particular subclass of the $S$-divergence corresponds to a distance metric and hence satisfies some interesting properties. We will explore these properties to generate a class of robust estimators of multivariate location and covariances. We will term this particular subfamily as the “$S$-Hellinger Distance” family and derive the breakdown point of the corresponding minimum $S$-Hellinger distance estimators. Thus, our work in this paper will generalize the work of Tamura and Boos (1986) from the Hellinger distance to the case of the general family of $S$-Hellinger distances.

The rest of the paper is organized as follows: We will start with the description of the $S$-Hellinger distance (SHD) family and the corresponding minimum SHD estimators in Section 2. In Section 3 we will describe the properties of the minimum SHD estimator of multivariate location and covariance under the particular case of elliptically symmetric distributions, including its equivariance, consistency and breakdown point. The performance of the proposed estimators will be illustrated through a numerical study in Section 4. We end the paper with a small discussion in Section 5.

2. The $S$-Hellinger Distance (SHD) Family

We start with the definition of the $S$-Hellinger distance (SHD) family. These distances are defined, between two generic probability density function $f$ and $g$ with respect to a common measure and in terms of a tuning parameter $\alpha$ as follows:

$$SHD_{\alpha}(g, f) = \frac{2}{1 + \alpha} \left[ \int f^{1+\alpha} - 2 \int f^{\frac{1+\alpha}{2}} g^{\frac{1+\alpha}{2}} + \int g^{1+\alpha} \right]$$

$$= \frac{2}{1 + \alpha} \int \left( g^{\frac{1+\alpha}{2}} - f^{\frac{1+\alpha}{2}} \right)^2 = \frac{2}{1 + \alpha} \| g^{\frac{1+\alpha}{2}} - f^{\frac{1+\alpha}{2}} \|_2^2,$$

(1)

where, $\| \cdot \|_2$ represents the using the $L_2$-norm. It is indeed a particular subfamily of the two parameter ($\lambda$ and $\alpha$) $S$-divergence (Ghosh et al., 2013a, b) with $\lambda = -\frac{1}{2}$. Although $SHD_{\alpha}(g, f)$ in (3) is not itself a distance, it corresponds to the distances $\left[ \frac{1+\alpha}{2} SHD_{\alpha}(g, f) \right]^\frac{2}{\alpha}$. Note that, for $\alpha \leq 0$, the measure reduces to the well-known (twice, squared) Hellinger distance. For this reason we rename this particular subfamily of the $S$-Divergence as the $S$-Hellinger Distance with parameter $\alpha$ and denote it by $SHD_{\alpha}(\cdot, \cdot)$ as above. Further, at $\alpha = 1$ the $S$-Hellinger distance
measure coincides with the (squared) $L_2$-distance. Thus, the parameter $\alpha$ in the definition of the $S$-Hellinger distances provides a smooth bridge between the Hellinger distance to the $L_2$-distance. Thus, in this respect, it is also quite similar to the density power divergence family (Basu et al., 1998) which connects the Kullback-Leibler divergence with $L_2$-distance smoothly through its tuning parameter. We will see that, as in the case of density power divergence, the tuning parameter $\alpha$ in the $S$-Hellinger distance family also plays a crucial role in the robustness of the corresponding minimum distance estimators.

We will use this particular subfamily of $S$-divergence for doing multivariate location and covariance estimation because it is easily amenable to this unlike other members of $S$-divergence. This properties mainly come from the fact that it corresponds to a density metric and becomes useful in proving several properties of the corresponding minimum SHD estimation.

**Lemma 2.1.** Let $f_1, f_2, f_3$ be three densities from continuous distributions. Then for any $0 \leq s \leq 1$, we have

$$S HD_\alpha((1-s)f_1 + sf_2, f_3) \leq (1-s)S HD_\alpha(f_1, f_3) + sS HD_\alpha(f_2, f_3).$$

**proof:** The lemma follows from the fact that $(z^{1+\alpha} - 1)^2$ is a convex function for $z > 0$ and $0 \leq \alpha \leq 1$ and we can rewrite the SHD in terms of $\delta = g/f > -1$ (so that $z = \delta + 1$) as

$$S HD_\alpha(g, f) = \frac{2}{1+\alpha} \int f^{1+\alpha}((\delta + 1)^{1+\alpha} - 1)^2.$$

□

**Lemma 2.2.** Let $f_1, f_2, f_3$ be three densities from continuous distributions. Then we have

$$S HD_\alpha(f_1, f_2) \leq 2 [S HD_\alpha(f_1, f_3) + S HD_\alpha(f_2, f_3)]$$

**proof:** This follows from the triangle inequality of the $L_2$-distances and the inequality $2ab \leq a^2 + b^2$ by considering the form of the SHD as given in (1).

□

3. The Minimum SHD Estimator (MSHDE) of Multivariate Location and Covariance

Now let us consider the usual (multivariate) set-up of parametric estimation; we have $n$ independent and identically distributed observations $X_1, X_2, \cdots, X_n$ from a (multivariate) distribution $G$. We will assume that the true distribution $G$ has a density $g$ with respect to a suitable dominating measure $\nu$. We will model the true density by a parametric (multivariate) model $F = \{f_\theta : \theta \in \Theta_0 \subseteq \mathbb{R}^p\}$. Our main interest here is to estimate the unknown model parameter $\theta$ based on the observed data. We will define the measure of discrepancy between the sample data and the model family by the $S$-Hellinger distance measure between the model density $f_\theta$ and a nonparametric density estimator $\hat{g}_n$ obtained by a suitable kernel smoothing based on the observed sample. Thus the Minimum SHD Estimator (MSHDE) is given by

$$\hat{\theta}_n = t(\hat{g}_n) = \arg \min_{\theta \in \Theta} \frac{1}{1+\alpha} \|\frac{f_\theta^{1+\alpha} - f_\hat{g}_n^{1+\alpha}}{f_\hat{g}_n^{1+\alpha}}\|_2^2$$

$$= \arg \min_{\theta \in \Theta} \|\frac{f_\theta^{1+\alpha} - f_\hat{g}_n^{1+\alpha}}{f_\hat{g}_n^{1+\alpha}}\|_2^2.$$ (2)
where $\hat{g}_n$ is the kernel density estimator based on the observed sample $X_1, X_2, \cdots, X_n$, given by

$$\hat{g}_n(x) = \frac{1}{nh_n^p} \sum_{i=1}^n w\left( \frac{x - X_i}{h_n} \right),$$  \hspace{1cm} (3)

with $w(\cdot)$ being a $p-$variate density function and $\{h_n\}$ being a suitably chosen sequence of kernel bandwidths.

In this paper, we will consider only the minimum SHD estimator of multivariate location and covariance under suitable assumptions and describe some important properties of it. We will restrict ourselves only to parametric families having elliptically symmetric distribution with density function given by

$$f_\theta(x) \propto |\Sigma|^{-1/2} \psi((x - \mu)^T \Sigma^{-1} (x - \mu)),$$  \hspace{1cm} (4)

where $\theta = \{\mu, \Sigma\}$ and the parameter space is $\Theta = \mathbb{R}^p \times \mathcal{S}$, with $\mathcal{S}$ being the set of all $p \times p$ positive definite matrices.

### 3.1. Equivariance

Any parametric estimator of the (multivariate) location and covariance can potentially be used to infer about the orientation and shape of data-points in a multi-dimensional space. Since there is no one universal way of measuring data, a minimal requirement for these estimators is that they should be independent of the coordinate system. So for the estimation of multivariate location and covariance, we only look at the estimators that are affine equivariant and affine covariant respectively for location and covariance estimation (see Tamura and Boos, 1986, for relevant definitions). We can derive some sufficient conditions, under which the Minimum SHD estimator of multivariate location and covariance satisfies the requirement of equivariance.

**Lemma 3.1.** Suppose that

1. the model family is chosen to be elliptical having density of the form (4) and
2. the kernel density estimator satisfies the condition $\hat{g}_n(x) = \hat{g}_n(A^{-1}(x - b))/|A|$.

Then the Minimum SHD estimator of Multivariate Location and Covariance are affine equivariant and affine covariant respectively.

The proof is straightforward and is omitted. In a practical situation it is necessary to choose a kernel density estimator satisfying condition (2). We can construct a density estimate of the radial type satisfying (2), whenever we have an initial affine covariant estimate $\hat{\Sigma}_0$ of covariance based on the observed data and it will be of the form

$$\hat{g}_n(x) = \frac{1}{nh_n^p |\Sigma_0|^{1/2}} \sum_{i=1}^n w\left( \frac{||x - X_i||}{\hat{h}_n^{1/2}} \right),$$  \hspace{1cm} (5)

where $||x||^2_\Sigma = x^T \Sigma^{-1} x$. Then the MSHDE of the location and covariance using this kernel density estimate will be both affine equivariant and affine covariant within the multivariate elliptical family (4).
3.2. Consistency

Suppose the observed sample data $X_1, X_2, \ldots, X_n$ come from the true common density $g$. Then the minimum SHD estimator can be shown to be consistent under suitable conditions on the Kernel density estimator $\hat{g}_n$ and the model family $f_\theta, \theta \in \Theta$.

**Theorem 3.2.** Suppose that, with probability one, the minimum SHD Estimator $t(\hat{g}_n)$ exists for all sufficiently large $n$ and as $n \to \infty$,

$$\|\hat{g}_n^{1/2} - f_\theta^{1/2}\|_2 \to 0,$$

and

$$\|f_\theta^{1/2} - \hat{g}_n^{1/2}\|_2 \to 0 \Rightarrow \theta_n \to \theta,$$

for any sequence $\{\theta_n : \theta_n \in \Theta\}$. Then we have, with probability one,

$$t(\hat{g}_n) \to \theta \quad \text{as} \quad n \to \infty.$$  \hspace{1cm} (8)

**Proof:** Denote $t_n = t(\hat{g}_n)$ so that the triangle inequality gives

$$\|f_\theta^{1/2} - \hat{g}_n^{1/2}\|_2 \leq \|f_\theta^{1/2} - \hat{g}_n^{1/2}\|_2 + \|\hat{g}_n^{1/2} - f_\theta^{1/2}\|_2$$

$$\leq 2\|\hat{g}_n^{1/2} - f_\theta^{1/2}\|_2 \to 0,$$

by (6). Then, by (7), $t_n = t(\hat{g}_n) \to \theta$ as $n \to \infty$ with probability one. □

Note that the first condition, namely (6), needed for the consistency of the minimum SHD estimator simply says that the kernel density estimator $\hat{g}_n$ converges to the true density $g$ in the metric corresponding to the SHD. This is a very intuitive assumption for the consistency and other asymptotic properties of the any minimum distance estimator. However, in this case, it can be proved using the uniform convergence or by $L_1$ convergence of the kernel estimator under some mild assumption on the corresponding bandwidth sequence.

Further, the second condition, namely (7), needed for the consistency of the minimum SHD estimator is a model specific condition. It is expected to hold for many elliptically symmetric densities. We have checked this to hold for the most common multivariate normal model.

3.3. Influence Function

To define the influence function of the minimum SHD estimator, we need to redefine it in terms of the statistical functionals. The minimum $S$-Hellinger distance functional $\theta^\alpha = T_\alpha(G)$ at the distribution $G$ is given by

$$T_\alpha(G) = \arg\min_{\theta \in \Theta} \|g^{1/2\alpha} - f_\theta^{1/2\alpha}\|_2,$$  \hspace{1cm} (10)

provided such a minimum exists. We will also denote its value $\theta^\alpha$ at the true distribution $G$ as the best fitting value of the parameter. Then using corresponding estimating equation, it is easy to derive the influence function of the minimum SHD functional; but we can also use the corresponding expression of the minimum $S$-divergence functional derived in Ghosh et al. (2013a) and substitute $\lambda = -\frac{1}{2}$ there. Thus, the influence function of minimum SHD functional $T_\alpha$ at the distribution $G$ with degenerate contamination at the point $y$ turns out to be

$$IF(y; T_\alpha, G) = J^{-1}(y)\left[ u_{\theta^\alpha}(y)f_\theta^{1/2\alpha}(y)g^{1/2\alpha}(y) - \xi \right],$$  \hspace{1cm} (11)
where  \( \xi = \xi(\theta^0) \), \( J = J(\theta^0) \) with  \( \xi(\theta) = \int u_0 f_0^{1+\alpha} g^{1+\alpha} \) and \( J(\theta) = \int u_0^2 f_0^{1+\alpha} + \int (\frac{\xi}{1+\alpha} - u_0^2)(g^{1+\alpha} - f_0^{1+\alpha}) \). But \( i_0(x) = -\nabla[u_0(x)] \). However, if the true density  \( g \) belongs to the model family with \( g = f_0 \), the influence function simplifies to

\[
IF(y; \theta, F_\theta) = \left[ \int u_0 u_0^T f_0^{1+\alpha} \right]^{-1} \left\{ u_0(y)^T f_0^\alpha(y) - \int u_0 f_0^{1+\alpha} \right\}.
\]  

Note that the influence function of the minimum SHD functional at the model is exactly the same as that of the density power divergence and all other subfamilies of the \( S \)-divergence measure with any fixed \( L \). This is expected from the work of Ghosh et al. (2013a) on the properties of the minimum \( S \)-divergence estimators. Further, for most of the parametric models, this influence function is unbounded only at \( \alpha = 0 \) which corresponds to the case of minimum Hellinger distance estimator; but all the other minimum SHD estimators have bounded influence function at the model.

As a particular example, we will consider the case of most common elliptically symmetric density of \( p \)-variate normal with mean vector \( \mu \) and covariance matrix \( \Sigma \). Then, the influence function of the MSHDE of \( \theta = (\mu, \Sigma) \) can be written in terms of that of the individual parameters, namely \( T_0^\mu(F_\theta) \) and \( T_0^\Sigma(F_\theta) \) of \( \mu \) and \( \Sigma \), as

\[
IF(y; \theta, F_\theta) = (IF(y; T_0^\mu, F_\theta), IF(y; T_0^\Sigma, F_\theta)).
\]

Also, it is then easy to derive that the influence function of the MSHDE of location \( \mu \) is given by

\[
IF(y; T_0^\mu, F_\theta) = (1 + \alpha)\tilde{\xi}^\alpha (y - \mu)e^{-\tilde{\xi}(y-\mu)^T\Sigma^{-1}(y-\mu)},
\]

and that of the covariance \( \Sigma \) is given by

\[
IF(y; T_0^\Sigma, F_\theta) = \xi(1 + \alpha)\tilde{\xi}^\alpha \left\{ (y - \mu)(y - \mu)^T\Sigma^{-1} - p \right\} e^{-\tilde{\xi}(y-\mu)^T\Sigma^{-1}(y-\mu)} - (1 - p(1 + \alpha)) \Sigma,
\]

with  \( \xi = 2(1 + \alpha)(3 - 2p(1 + \alpha) + p^2(1 + \alpha)^2)^{-1} \).

### 3.4. Breakdown Point

Now, we will consider the breakdown point of the minimum \( S \)-Hellinger distance functional in the case of the estimation of multivariate location and covariance as a measure of its robustness. We consider the set-up with \( X \) representing the original data set of a given size \( n \) and \( Y \) be a contaminating data set of size \( m \) \((m \leq n)\). The estimator \( \hat{\theta}_n \) will be said to break down if, through proper choice of the elements of the data set \( Y \), the difference \( \hat{\theta}_n(X|Y) - \hat{\theta}_n(X) \) can be made arbitrarily large. If \( m^* \) is the smallest number of the contaminating values for which the estimator breaks down, then the breakdown point of the corresponding estimator at \( X \) is  \( \frac{m^*}{n} \). In the case of multivariate location and covariance, the breakdown point of the joint estimation of location and covariance has been defined by Donoho (1982) through the following measure of discrepancy

\[
B(\theta_1, \theta_2) = tr(\Sigma_1 \Sigma_2^{-1} + \Sigma_1^{-1} \Sigma_2) + \|\mu_1 - \mu_2\|^2
\]

between parameter values \( \theta_1 = (\mu_1, \Sigma_1) \) and \( \theta_2 = (\mu_2, \Sigma_2) \), where \( tr \) represents the trace of a matrix and \( ||.|| \) represents the Euclidean norm. The joint estimate of multivariate location and covariance will break down when the supremum of the discrepancy, as given in the above equation, between the pure data estimate at \( X \) and the contaminated data estimate at \( X|Y \), is infinite.
We will derive the breakdown point of the minimum SHD estimators under specific assumptions on the true and model densities. Let $\mathcal{G}$ denote a subclass of all probability densities satisfying
\[ SHD_\alpha(g, f) \leq \kappa(\alpha), \quad \forall g, f \in \mathcal{G}. \quad (15) \]
Then, we will assume that the model family $\mathcal{F}$ and true density $g$ both belong to the family $\mathcal{G}$. Also let $\hat{g}_n(c_n) = \hat{g}_n(c_n, x)$ be the kernel density having bandwidth $c_n$ and $f_\theta(c_n)$ denotes the model density nearest to the above kernel estimator in terms of the $S$-Hellinger distance. We assume that such a model density exists. Define
\[ a_{n,m} = SHD_\alpha(\hat{g}_n(c_{n+m}), f_\theta(c_n)), \quad (16) \]
and $v^* = \lim \inf_{\theta_1, \theta_2} SHD_\alpha(f_{\theta_1}, f_{\theta_2})$ where, the limit is taken as $B(\theta_1, \theta_2) \to \infty$. \quad (17)

**Theorem 3.3.** Assume that the model family $\mathcal{F}$ and true density $g$ both belong to $\mathcal{G}$. Then, the Minimum SHD Estimator $\theta_\alpha(c_n)$ has the breakdown point $\varepsilon^*(\theta_\alpha(c_n))$ satisfying
\[ \varepsilon^*(\theta_\alpha(c_n)) \geq \frac{\sqrt{\mu - a_{n,m}}}{\kappa(\alpha) - a_{n,m}}. \quad (18) \]

**Proof:** Note that, using the definition of $f_\theta(c_n)$ and Lemma 2.2, we get
\[ 2SHD_\alpha(\hat{g}_{n,m}(c_{n+m}), f_\theta(c_n)) \geq 2SHD_\alpha(\hat{g}_{n,m}(c_{n+m}), f_{\hat{\theta}_{n,m}}(c_{n+m})) \geq SHD_\alpha(f_{\hat{\theta}_{n,m}}(c_{n+m}), f_\theta(c_n)) - 2SHD_\alpha(\hat{g}_{n,m}(c_{n+m}), f_\theta(c_n)). \]
Then by definition, the occurrence of breakdown will imply
\[ SHD_\alpha(f_{\hat{\theta}_{n,m}}(c_{n+m}), f_\theta(c_n)) \geq v^*, \quad \Rightarrow \quad SHD_\alpha(\hat{g}_{n,m}(c_{n+m}), f_\theta(c_n)) \geq \frac{v^*}{4}. \]
Now, let us denote the empirical distributions of $X, Y$ and $X \cup Y$ by $G_n, G_m$ and $G_{n+m}$ respectively so that we will have
\[ G_{n+m} = \frac{n}{n+m}G_n + \frac{m}{n+m}G_m. \]
Then applying the above to the kernel defining equation (5), we will get
\[ \hat{g}_{n,m}(c_{n+m}) = \frac{n}{n+m}\hat{g}_n(c_{n+m}) + \frac{m}{n+m}\hat{g}_m(c_{n+m}). \]
Now applying the Lemma 2.1,
\[ SHD_\alpha(\hat{g}_{n,m}(c_{n+m}), f_\theta(c_n)) \leq \frac{n}{n+m}SHD_\alpha(\hat{g}_n(c_{n+m}), f_\theta(c_n)) + \frac{m}{n+m}SHD_\alpha(\hat{g}_m(c_{n+m}), f_\theta(c_n)). \]
Combining the above equations and using the assumptions of the Theorem and (15), we get
\[ \frac{v^*}{4} \leq SHD_\alpha(\hat{g}_{n,m}(c_{n+m}), f_\theta(c_n)) \leq \left[ 1 - \frac{m}{n+m} \right] SHD_\alpha(\hat{g}_n(c_{n+m}), f_\theta(c_n)) + \left[ \frac{m}{n+m} \right] \kappa(\alpha) \]
\[ = \left[ 1 - \frac{m}{n+m} \right] a_{n,m} + \left[ \frac{m}{n+m} \right] \kappa(\alpha). \]
Collecting the coefficients of $\frac{m}{n+m}$ that gives the proportion of data contamination resulting in breakdown, we get the desired as result.

**Corollary 3.4.** Let $g$ be the true density. If $SHD_{\alpha}(g, f_{\theta_n}(c_n)) \to a$ and $\hat{g}_{0+m}(c_{n+m}) \to g(x)$, for each $x$, we have $a_{m,n} \to a$ almost surely. Then, by the theorem,

$$\liminf_{n \to \infty} \varepsilon^*(\theta_n(c_n)) \geq \frac{\varepsilon - a}{\kappa(\alpha) - a},$$

(19)

Further, if the true distribution belongs to the model family, i.e., $g = f_{\theta}$ for some $\theta \in \Theta$, we get $a = 0$ and then

$$\liminf_{n \to \infty} \varepsilon^*(\theta_n(c_n)) \geq \frac{\nu^*}{4\kappa(\alpha)}.$$  

(20)

**Remark:** Whenever $\nu^* = \kappa(\alpha)$ and the true distribution belongs to the model family, the corollary yields

$$\liminf_{n \to \infty} \varepsilon^*(\theta_n(c_n)) \geq \frac{1}{4}.$$  

That is, breakdown cannot occur for the minimum SHD estimator in this case for $\varepsilon < \frac{1}{4}$.

**4. Numerical Illustration**

We will illustrate the performance of the proposed minimum $S$-Hellinger distance estimator through an extensive simulation study. We generate data $(X_{1,1}, X_{2,1}), \ldots, (X_{1,n}, X_{2,n})$ from the bivariate normal distribution with mean vector $(0, 0)^T$ and covariance matrix $\begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$. We will model it by the bivariate normal density with mean parameter $(\mu_1, \mu_2)^T$ and covariance matrix $\begin{pmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho \\ \sigma_1\sigma_2\rho & \sigma_2^2 \end{pmatrix}$, where $\rho$ represents the correlation between the two components. To obtain the MSHDE of the parameters $\theta = (\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)^T$ we need to minimize the SHD measure between the model density and the kernel density estimate $\hat{g}_n$ of the true density. To achieve equivariance of the estimators, we consider the estimator $\hat{g}_n$ satisfying the condition (5) with $p = 2$ and $w(\cdot)$ being the Gaussian kernel. We will follow the normal reference rule (Scott, 2001) to select the bandwidth $h_n = (h_{n,1}, h_{n,2})$ given by $h_{n,i} = 1.06\sigma_i n^{1/5}$, for all $i = 1, 2$, with $\sigma_i$ being the standard deviation of $X_i$, which we will replace by its robust estimate

$$\sigma_i = \frac{\text{Median}_{k} |X_{ik} - \text{Median}_j X_{jk}|}{0.6745}.$$  

Further, using the form of the bivariate normal density $f_{\theta}(\cdot)$, the SHD measure between the model and the kernel estimator $\hat{g}_n$ can be expressed as

$$SHD_{\alpha}(\hat{g}_n, f_{\theta}) = \frac{2}{1 + \alpha} \left[ \frac{1}{2 \pi \sigma_1^2 \sigma_2^2 (1 - \rho^2)} - 2 \int f_{\theta}^{\frac{1}{2}} \hat{g}_n^{\frac{1}{2}} \right]$$

$$+ \text{(constant independent of } \theta \text{)}.$$  

(21)
Thus, we will obtain the MSHDE of $\theta$ by minimizing the first two terms within the square bracket in the above expression with respect to $\theta \in \mathbb{R} \times \mathbb{R} \times [0, \infty) \times [0, \infty) \times [-1, 1]$. We will use the sample size $n = 50$ and use 1000 replications of simulated samples to compute the empirical bias and MSE of the MSHDEs under the present set-up.

We will first consider the performance of the proposed MSHDEs under pure data with no contamination. Table 1 presents the empirical bias and MSE of the MSHDEs under data generated from the true density with parameter value $\theta = (0, 0, 1, 0.5)$. It is clear from the table that there is a loss of efficiency of the MSHDEs in terms of their MSE as the parameter $\alpha$ increases form 0 to 1, but this loss is not very significant under pure data. However the variance parameters do exhibit a decline in performance for large $\alpha$ compared to $\alpha = 0$, although the mean square errors are not necessarily monotone over $\alpha$.

Table 1: Empirical Bias and MSE of the MSHDE of parameter $\theta = (\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$ under no contamination for $n = 50$

|       | $\alpha = 0$ | $\alpha = 0.05$ | $\alpha = 0.1$ | $\alpha = 0.25$ | $\alpha = 0.4$ | $\alpha = 0.5$ | $\alpha = 0.7$ | $\alpha = 0.8$ | $\alpha = 1$ |
|-------|--------------|-----------------|----------------|-----------------|----------------|----------------|----------------|----------------|-------------|
| $\mu_1$ | 0.056        | 0.055           | 0.054          | 0.049           | 0.042          | 0.037          | 0.028          | 0.024          | 0.018       |
| $\mu_2$ | 0.010        | 0.010           | 0.009          | 0.009           | 0.008          | 0.007          | 0.005          | 0.004          | 0.002       |
| $\sigma_1$ | $-0.046$    | 0.010           | 0.067          | 0.209           | 0.279          | 0.294          | 0.289          | 0.280          | 0.258       |
| $\sigma_2$ | 0.021        | 0.080           | 0.140          | 0.285           | 0.349          | 0.358          | 0.341          | 0.325          | 0.293       |
| $\rho$   | $-0.142$     | $-0.141$        | $-0.140$       | $-0.135$        | $-0.129$       | $-0.125$       | $-0.120$       | $-0.118$       | $-0.117$    |

Now to illustrate the expected robustness of the MSHDEs, we will repeat the simulation study presented above but contaminate a certain percentage of the data with observations from different densities representing different contamination scenarios. We consider three different contamination proportions — light contamination of 5%, moderate contamination of 10% and heavy contamination of 20% for each scenario. For brevity we will report only a few interesting common cases below. First consider the contamination only in the direction of the mean parameter where we will consider (a) equal contamination in both the means from a bivariate normal density with parameter $\theta = (5, 5, 1, 1, 0.5)$ and (b) high contamination in only one mean from a bivariate normal density with parameter $\theta = (10, 0, 1, 1, 0.5)$. The empirical values of the absolute bias and the MSE of the MSHDEs of $\mu_1, \mu_2$ and $\sigma_1$ are shown in the Figure 1. It is clear from the figure that whether the contamination is in one mean or both means, its effect is similar on estimators of $\mu_1$ and $\mu_2$ (although the scales of change are somewhat different). The bias (in magnitude) and MSE both decrease drastically as the tuning parameter $\alpha$ increases for the heavy contamination case. For moderate and light contamination the estimators are competitive or better for large values of $\alpha$. However the bias and the MSE of the scale parameter $\sigma_1$ is sometimes marginally better when $\alpha = 0$ compared to large values of $\alpha$. The results for the parameter $\sigma_2$ is similar to that of $\sigma_1$. The estimator of $\rho$ on the other hand does not appear to be affected significantly due to the insertion of contamination in means. Hence the graphs for $\sigma_2$ and $\rho$ are not presented here.

In Figure 2 we provide the absolute bias and MSE of the MSHDE of $\mu_1, \sigma_1$ and $\rho$ for two different contamination scenarios — (c) contamination in both mean and correlation parameter by bivariate normal with $\theta = (5, -5, 1, 1, -1)$, (d) contamination in variance and correlation
parameter by bivariate normal with \( \theta = (0, 0, 10, 1, -1) \). The bias and the MSE of MSHDE of all the three parameters indicate that the estimators corresponding to large values of \( \alpha \) are generally competitive with \( \alpha = 0 \) case, but appear to be significantly more stable under heavy contamination. The estimator of \( \mu_2 \) and \( \sigma_2 \) behave similarly to those of \( \mu_1 \) and \( \sigma_1 \) respectively.

Therefore, combining all the findings and results it would appear that the proposed MSHDEs with larger values of \( \alpha \) give us highly robust estimators of the mean and the correlation parameters compared to those based on usual Hellinger distance (\( \alpha = 0 \)) for any contamination scenario even if the contamination proportion is as high as 20%. The corresponding estimators of the variance parameters appear to be competitive with the Hellinger distance based methods under data contamination. We believe that this, together with the theoretical properties, provides sufficient indication that the method based on the SHD has enough potential as a useful robust estimator for multivariate location and covariance and warrant further exploration.

5. Discussion

Robust estimation of multivariate location and scatter is a difficult problem as it is often not very easy to maintain the degree of stability of an estimator with increasing dimension. We propose a class of estimators in this connection that has attractive breakdown properties irrespective of the data dimension. In this respect the proposed methods expand the scope of application beyond the Hellinger distance. It appears that several members of this family may have better stability properties compared to the ordinary Hellinger distance. We trust that the proposed methods will turn out to be very useful practical tools for the data analyst and the applied statistician.

References

[1] Basu, A., I. R. Harris, N. L. Hjort, and M. C. Jones (1998). Robust and efficient estimation by minimizing a density power divergence. Biometrika 85, 549-559.
[2] Basu, A., S. Basu, and G. Chaudhuri (1997). Robust minimum divergence procedures for the count data models. Sankhya B, 59, 11–27.
[3] Basu, A., Shioya, H., and Park, C. (2011). Statistical Inference: The Minimum Distance Approach. Chapman & Hall/CRC, Boca Raton, FL.
[4] Beran, R. J. (1977). Minimum Hellinger distance estimates for parametric models. Ann. Statist., 5, 445–463.
[5] Donoho, D. L. (1982). Breakdown Properties of Multivariate Location Estimators, unpublished qualifying paper, Harvard University, Statistics Dept.
[6] Ghosh, A., Haris, I., Maji, A., Basu, A., Pardo, L. (2013a). A Generalized Divergence for Statistical Inference. Technical Report, Bayesian and Interdisciplinary Research Unit, Indian Statistical Institute, Kolkata.
[7] Ghosh, A., Maji, A., Basu, A. (2013b). Robust Inference based on Divergences in Reliability Systems. Applied Reliability Engineering and Risk Analysis: Probabilistic Models and Statistical Inference. Eds, Inference, Ilia Frenkel, Alex, Karagrigoriou, Anatoly Lisnianski and Andre Kleyner, Dedicated to the Centennial of the birth of Boris Gnedenko. John Wiley & Sons, Ltd. 290–307
[8] Pardo, L. (2006). Statistical Inference based on Divergences. CRC/Chapman-Hall.
[9] Simpson, D. G. (1987). Minimum Hellinger distance estimation for the analysis of count data. Journal of the American Statistical Association 82, 802-807.
[10] Tamura, R. N. and D. D. Boos (1986). Minimum Hellinger distance estimation for multivariate location and covariance. Journal of the American Statistical Association 81, 223-229.
[11] Vajda, I. (1989). Theory of Statistical Inference and Information. Dordrecht: Kluwer Academic.
Figure 1: Plot of empirical bias and MSE of the MSHDEs of means $\mu_1$, $\mu_2$ and standard error $\sigma_1$ under contamination in the mean parameter for $n = 50$ [marker $\ast$ : contamination from bivariate normal with parameter $\theta = (5, 5, 1, 1, 0.5)$, marker $+$ : contamination from bivariate normal with parameter $\theta = (10, 0, 1, 1, 0.5)$; solid line : 5% contamination, dashed-dotted line : 10% contamination, Dotted line : 20% contamination]

*Plot for the case with representation "Dotted line - marker $\ast$" is shown in multiple of $10^{-1}$ of original value to manage scaling of the figure*
Figure 2: Plot of empirical bias and MSE of the MSHDEs of mean $\mu_1$, standard errors $\sigma_1$, and correlation $\rho$ under contamination in multiple parameters for $n = 50$ [marker $*$: contamination from bivariate normal with parameter $\theta = (5, -5, 1, 1, -1)$, marker $+$: contamination from bivariate normal with parameter $\theta = (0, 0, 10, 1, -1)$; solid line: 5% contamination, dashed-dotted line: 10% contamination, Dotted line: 20% contamination]

*Plot for the case with representation “Dotted line - marker *” is shown in multiple of $10^{-1}$ of original value for the parameters $\mu_1$ and $\rho$ to manage scaling of the figure*