Aspects of the Equivalence Between the $f^\mu$ and $c^{\nu\mu}$ Terms in Lorentz-Violating Quantum Field Theory

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Abstract

It is known that in Lorentz-violating effective field theory, there is a classical equivalence between certain coefficients ($c$ and $f$), in spite of the fact that the operators the two types of coefficients describe appear to have opposite behaviors under CPT. This paper is a continuation of previous work extending this equivalence to the quantum level: generalizing the explicit spinorial point transformations that interconvert the $c$ and $f$ terms; demonstrating that the transformations do not give rise to any additional anomaly terms as the quantum level; and giving explicit prescriptions for modifying the $C$, $P$, and $T$ operators in the $f$ theory, so that they correspond to the correct interchanges of physical particle states.

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1 Introduction

The standard model is one of the basic pillars of modern physics. It is one of the most thoroughly tested theories ever devised and has been excellent in explaining the small-scale structure of the universe. Nevertheless, extending the standard model has been an interesting avenue of research for a long time. Searches for evidence of whatever exists beyond the standard model could eventually lead to the discovery of new specific phenomena, new elementary particles, and perhaps wholly new kinds physics.

One intriguing possibility for new physics beyond the standard model is that the basic symmetries of the theory might change. In particular, the standard model’s (and general relativity’s) Lorentz and CPT invariances might be merely low-energy approximations, not true in a more basic theory. The low-energy effective field theory containing all the new operators which incorporate local, Lorentz-violating modifications to the physics of known fields is called the standard model extension (SME) [1, 2]. The renormalizable subsector obtained from the SME by removing all the parameters with mass dimension greater than four has been termed the minimal SME. It has been pointed out by Greenberg [3] that CPT violation implies Lorentz violation in a local, stable theory—although Lorentz violation does not always imply CPT violation. Hence the action for the minimal SME contains a large number of Lorentz-violating terms, each of which may or may not be CPT violating. A rule of thumb for these new structures is that an operator with an even number of Lorentz indices is CPT even, and one with an odd number of indices is CPT odd. However, a curious case arises [4] that the apparently CPT-violating theory containing a one-index $f^\mu$ term is equivalent to a CPT-preserving theory containing only a (two-index) $c^{\nu\mu}$ term, by means of a point transformation of spinor fields. The relation established between the $c$ and $f$ terms describes an effective $c$ that is of even order in $f$, which implies that physical effects at first-order in $f$ term are not observable. Moreover, in interacting theories, these kinds of equivalences can be taken even further; for example, in the quantum electrodynamics sector of the SME, an electron $c$ term is completely equivalent to a certain kind of photon $k_F$ term.

The renormalization of the minimal SME, particularly to one-loop order, has been worked on extensively [5, 6, 7, 8, 9, 10, 11, 12, 13]. However, even this perturbative renormalization still has not been carried through in all the sectors of the minimal SME. An example would be that the one-loop renormalization of a Lorentz-violating gauge theory coupled to charged scalar fields remains incomplete, and hence the renormalization group (RG) scalings of several Lorentz-violating couplings in the $SU(2)_L$ weak gauge sector remain undetermined. Another interesting renormalization issue occurs in the minimal SME with a Chern-Simons-like term. This term not only breaks Lorentz invariance but also gauge invariance of the Lagrange density (although the action is gauge invariant), and it was quite controversial as to whether there could ever be a purely radiatively-generated Chern-Simons term, since it was found that calculations using different regulators provided different expressions for the radiative correction. (See, for example, the discussions
in Refs. [14, 15].) A similarly curious case was pointed out in Ref. [16], that the RG scalings of $c$ and $f$ appeared to be dissimilar, even though the theory with $c$ was known to be equivalent to the $f$ theory via an appropriate transformation. Fixing the renormalization at second order in $f$ revealed ambiguities in the $\beta$-functions for $c$ and $f$. However, the RG evolution of the physically observable quantity $c^{\mu\nu} - \frac{1}{2} f^\mu f^\nu$ was found to be free of such ambiguities, up to second order in $f$.

This paper is a continuation of our previous work on the relation between the SME $c$ and $f$ terms. We shall present some results that are relevant to the general structure of the corresponding $c$ and $f$ theories, with particular emphasis on how those results are related to the renormalization issues that have previously been discussed. We observed that a Lagrangian with $f$-type Lorentz violation can be converted into a Lagrangian with a $c$ instead by means of a point transformation of spinors. In section 2 we generalize the transformation discussed in Ref. [4], which showed how to convert a pure-$f$ theory into a pure-$c$ theory. The generalized transformation eliminates $f$ (at all orders) from a theory that already starts with both $f$ and $c$ terms.

Another closely related question that arises is the validity of various expressions in the context of a continuation of the one-loop renormalization to higher orders. Some of the formulas that we utilized in our $O(f^2)$ analysis of the renormalization problem may be modified at higher orders in the Lorentz violation. Since our analysis used both the Gordon decomposition identity and the closure relations for Dirac spinors, which do not generally hold without modification in the SME, in calculations beyond second order in $f$ we would need to use the generalized forms of these identities. The generalized expressions will give rise to extra higher-order terms that we would need to be careful about accounting for. The ways in which the identities in question are modified is discussed in section 3.

In section 4 we turn briefly to the question of whether the classical transformation that exchanges $f$ for $c$ might become anomalous in a quantum-mechanical context. Because of operator ordering ambiguities, classical canonical transformations may give rise to additional terms in the effective potential when a theory is rewritten in new coordinates. The question naturally arises whether the clean classical equivalence between $c$ and $f$ will also be befouled by an extra potential term at $O(h)$.

The last major issue discussed in this paper, in section 5, has to do with charge conjugation, parity, and time reversal ($C$, $P$, and $T$) properties. We already know that $c$ is separately $C$ and $PT$ invariant, which makes it $CPT$ even. However, looking at the usual transformation properties for Dirac bilinears, it appears that $f$ must be $PT$ even and $C$ odd, which obviously would make it $CPT$ odd. If both the actions (either with $c$ or with $f$) represent the same quantum theory, then there must be some modified $C$, $P$, and $T$ operators which will make the $f$ term overall $CPT$ even. We shall devise a method with which we may convert our usual discrete spacetime operators that work for the $c$ term into new operators that make $f$ indeed $CPT$ even.
2 Relations Between Equivalent $c^{\mu\nu}$ and $f^\mu$ Terms

The transformation of a Lagrange density

$$\mathcal{L}_f = \bar{\psi} [i(\gamma^\mu + if^\mu\gamma_5)\partial_\mu - M] \psi$$

with solely an $f$ term into one

$$\mathcal{L}_c = \bar{\psi}' [i(\gamma^\mu + c^\mu\gamma_\nu)\partial_\mu - M] \psi'$$

with only a $c$ term has previously been laid out [4]. The necessary transformations are

$$\psi' = e^{\frac{f}{2} f^\mu \gamma_\nu \gamma_5 G(-f^2)} \psi,$$

[where $G(\xi) = \frac{1}{4\xi} \tan^{-1} \sqrt{\xi}$ is an analytic function of its argument $\xi = -f^2$], and the corresponding $\psi' = (\psi')^\dagger \gamma_0$. This converts $\mathcal{L}_f$ into the $\mathcal{L}_c$ form, with an effective $c$ term which is given by

$$c^{\mu\nu}_{\text{eff}} = \frac{f^{\mu} f^{\nu}}{f^2} \left( \sqrt{1 - f^2} - 1 \right).$$

However, this may be further generalized. (Generalizations to higher-dimensional operators, which resemble the $c$ and $f$ terms but with additional spacetime derivatives, have already been studied [17].) If we have a Lagrangian with both $c$ and $f$ terms, a relevant question is whether the theory can be converted into one which only has an effective $c$-type term or into one with only an effective $f$-type term. The answer is yes if the initial $c^{\mu\nu}$ is $c^{\mu\nu} = \alpha \nu^\mu \nu^\nu$, provided that our $f$ is also in the same direction as the vector $v$, which is the case in many theories, including single-field bumblebee models of spontaneous Lorentz violation [18]. A bumblebee models features a dynamical four-vector field which obtains a vacuum expectation value $v$ through spontaneous symmetry breaking. So while taking there to be only a single Lorentz-violating spacetime direction is a significant limitation on the space of possible Lorentz-violating theories we shall consider, it is not an unnatural or unduly severe limitation.

On the other hand, starting with the most general combination of $f$ and $c$ terms in the Lagrange density,

$$\mathcal{L} = \bar{\psi} [i(\gamma^\mu + c^{\mu\nu}\gamma_\nu + if^\mu\gamma_5)\partial_\mu - M] \psi,$$

it is clearly not possible to transfer all the Lorentz violation into the axial vector $f^\mu$ parameter. The two-index tensor $c^{\mu\nu}$ contains nine physically observable parameters. [These are not precisely identical with the traceless, symmetry part $c^{(\mu\nu)} = c^{\mu\nu} + c^{\nu\mu}$ of the tensor, when considered beyond leading order. In fact, the nine physical coefficients are the encapsulated by the traceless part of the manifestly symmetric tensor $c^{(\mu\nu)} + c^{\alpha\nu} c_\alpha^{\mu}$.] The four-component $f^\mu$ cannot possibly parameterize the full texture of the nine-dimensional space of $c^{\mu\nu}$ parameters.
Yet if all Lorentz violation in a model is described by a single underlying vectorial quantity, then the $c$ and $f$ theories are, as noted above, completely equivalent. One way to see how this works is via the following iterative approach. After the transformation of the Dirac spinor $\psi$ into $\psi'$ using relation (3) we get a Lagrange density with two kinds of Lorentz-violating terms: a term of the form $ic^{\mu\nu}f_\nu\gamma_5$, which is a $f$-type term and is of order $O(v^3)$; and an effective $c$-type term that looks like $c^{\mu\nu} - \frac{1}{2}f_\nu f_\mu$ to $O(v^3)$. If we repeat the transformation, with the new effective $f$-type term (the $ic^{\mu\nu}f_\nu\gamma_5$), this $O(v^3)$ term can be absorbed into a $O(v^6)$ $c$-type contribution, although we are again left with a $f$-type term, $i(c^{\mu\nu} - \frac{1}{2}f_\nu f_\mu)c_{\nu\alpha}f_\alpha\gamma_5$, up to the order $O(v^5)$. This process can be continued indefinitely, and we can get rid of the $f$ term totally, working order by order; the sequence of transformed Lagrangians will obviously converge if $v$ is a sufficiently small parameter (as we expect a Lorentz-violating vacuum expectation to be).

We can also get the exact transformation in a form similar to (3) directly. Assuming that there exists such a transformation which can get rid of the $f$-type term in the Lagrangian (as guaranteed by the preceding iterative argument), it should take the same general form as (3),

$$\psi' = A\psi = e^{\frac{i}{2}f^{\nu\gamma_5}f_\nu\gamma_5}e^{\frac{\theta}{\sqrt{f^2}}}\psi,$$  

albeit with an as yet unknown parameter $\theta$. The crucial matrix $A$ may be expanded directly as

$$A = \cosh \frac{\theta}{2} + i \frac{f^\nu}{\sqrt{f^2}} \sinh \frac{\theta}{2} \gamma_5.$$  

(7)

Since (6) is equivalent to a change in the Dirac matrices instead of the fermion field, the transformation of the matrix $\gamma^\mu$ under it is

$$(\gamma^\mu)' = e^{\frac{i}{2}f^{\nu}\gamma_5}e^{\frac{\theta}{\sqrt{f^2}}\gamma^\mu}e^{-\frac{i}{2}f^{\nu}\gamma_5}e^{-\frac{\theta}{\sqrt{f^2}}\gamma^\mu} = \gamma^\mu - \sinh \frac{\theta}{2} \frac{if^\mu}{\sqrt{f^2}} \gamma_5 + (\cosh \theta - 1) \frac{f^\mu f^\nu}{f^2} \gamma_5.$$  

(8)

This gives the transformation of the $c^{\mu\nu}\gamma_\nu = \alpha f^\nu f^\mu\gamma_\nu$ term in the Dirac Lagrangian. (We now set $v^\mu = f^\mu$ for simplicity of notation, since $v$ is no longer needed as a power counting parameter.) The transformation of the $if^{\mu}\gamma_5$ term in (8) in may be computed similarly. After the transformation we require that there be no $f^\mu$ term remaining in the transformed $\mathcal{L}'$; using this condition we may compute the angle $\theta$ and find

$$\tanh \theta = \sqrt{\frac{f^2}{1 + \alpha f^2}},$$  

(9)

observing that $\alpha = 0$ reproduces the transformation (3). With this $\theta$, we can then evaluate the final effective $c$-type term,

$$c^{\nu\mu}_{\text{eff}} = -\frac{f^\nu f^\mu}{f^2} \left[ 1 - \sqrt{(1 + \alpha f^2)^2 - f^2} \right].$$  

(10)
Naturally, this correctly reproduces the special case (4) if $\alpha = 0$. It is also clearly correct in the opposite limit of no initial $f$ term (so $f \to 0$, which corresponds to $\alpha \to \infty$), yielding the finite $c_{\text{eff}}^{\mu\nu} = c^{\mu\nu} = \alpha f^\nu f^\mu$.

### 3 Spinor and Propagator Identities

In our paper [16], we converted two $f$ vertices in an explicit momentum-space propagator to a single effective $c$ vertex in the propagator; the effective $c^{\mu\nu}$ term was given by

$$c_{\text{eff}}^{\mu\nu} = -\frac{1}{2} f^\mu f^\nu.$$  

This again reaffirms that the relation (11) is indeed correct, at least at the lowest order. We used several approximations while deriving this result which did not affect the results up to $O(f^2)$. However, when computing third-order radiative corrections, it may be necessary to include the neglected terms. Therefore, let us consider more carefully the effects of two $f$ vertex insertions into the fermion propagator.

The contribution with these insertions is equal to

$$S_{ff} = S(p)(-f^\mu \gamma_5 p_\mu)S(-f^\nu \gamma_5 p_\nu)S(p),$$

where $S(p) = \frac{i}{p-m}$ is the unmodified propagator. Upon straightforward rearrangement of the Dirac matrices, (12) gives

$$S_{ff} = S(p) \frac{1}{(p^2 - m^2)} f^\mu f^\nu p_\mu p_\nu.$$  

The trick to simplifying this further runs as follows. Use the relation $\not{p} + m = \sum u(p) \bar{u}(p)$ in the numerator of the remaining $S(p)$. Then use the Gordon decomposition to change one $p_\alpha$ to $m\gamma_\alpha$ by sandwiching it between the $u(p)$ and $\bar{u}(p)$. Then finally the mass term can be changed to one-half of $\not{p}$. After these steps, the remaining expression looks like a propagator with a $c$-type vertex insertion. However, upon closer inspection, it turns out that each of these step may potentially have additional corrections at higher orders in $f$; hence a more careful analysis is required if we are to make any statements about higher order corrections. We will therefore look at the generalizations of two key identities that are frequently used in Dirac algebra.

The modified relation (the momentum-space Dirac equation) satisfied by the spinors with the SME $c$ and $f$ terms is

$$(\not{p} + m + c^{\rho\sigma} \gamma_\rho p_\sigma + if^{\mu\nu} \gamma_5 p_\mu) u(p) = 0.$$  

The modified Gordon identity for these spinors is consequently

$$\bar{u}(p') \gamma^\mu u(p) = \bar{u}(p') \left[ \frac{(p + p')^\mu}{2m} + i(\sigma^{\mu\nu} - ic^{\rho\sigma} \gamma_\rho \gamma^\mu + f^{\nu} \gamma_5 \gamma^\mu) \frac{(p' - p)^\nu}{2m} + \frac{c^{\mu\nu} p_\nu}{m} \right] u(p).$$  

(15)
This means that when \( p' = p \) and if we consider a theory with only the \( f \) term, we can still exchange \( \gamma^\mu \) for \( mp^\mu \) as we did our previous calculation of \( S_{ff} \); that trick still works and is fine.

The second very useful quantity is the spin-summed outer product \( \sum u(p)\bar{u}(p) \) in the theory with just a \( f \) term. To calculate this, we can evaluate the expression in the fermion rest frame and then boost the expression (taking care to boost the Lorentz-violating backgrounds as well). The result is

\[
\sum u(p)\bar{u}(p) = \slashed{p} + m + if^\mu\gamma_5 p_\mu.
\]  

This relation can be used, along with the modified Gordon decomposition, to simplify calculations at higher orders in \( f \).

However, we need to be careful when we include a \( c \) term in the starting Lagrangian along with an equivalent \( f \), as we can see that we get an additional effective \( c \) term which is second order in \( f \), and according to (10), the effective \( c \)-type terms are not simply additive. Calculations may be carried through order by order in \( v \), keeping \( c \) and \( f \) contributions of equivalent orders. Working in this fashion, all contributions at odd orders in \( v \) should cancel in physical observables. Alternatively, this problem can be circumvented if we convert the starting theory into an equivalent theory which only contains the \( c \) or \( f \) term and work entirely in that theory, transforming back to the original only that the very end of calculation.

### 4 Quantum Considerations

Another issue is whether the relation (10) is exact in quantum theory. Relations that hold at the classical level are sometimes no longer valid after the quantization of a theory. The problem typically comes form the fact that several quantum theories may have the same classical limit, due to operator ordering ambiguities. For example, it has been pointed out [19, 20] that in the path integral formalism, when we perform a change of coordinates from Cartesian to polar, we get additional (effective) potential energy terms, which compensate for the fact that the new canonical coordinates are not linear in the previous coordinates. The rotation symmetry (corresponding to the conserved angular momentum) gives rise to a zero-energy mode, and although we can get rid of this zero mode, we obtain an additional contribution of \( \frac{L^2}{2mr^2} \) to the effective potential—the classical centrifugal term. Another important example would be the case of kink solitons in 1+1 dimensions [21]. The kinks, which are topological solutions of the classical equations of motion for a scalar field, have a translation symmetry. A consequence of this zero-mode symmetry is that perturbative computations diverge while computing the higher order corrections, as the corrections have \( \omega_0 \) (the zero-mode frequency) in the denominator [22]. We can get rid of this problem by performing a point transformation and computing the path integral in suitable coordinates, so that we get rid of the troublesome zero-mode
coordinate; but again the transformation gives rise to additional new terms in the effective potential.

If the transformation that takes a $f$ theory to a $c$ theory is associated with an additional effective potential term at $O(\hbar)$, then it could give further corrections to the theory’s $c_{\text{eff}}$. This would mean that the renormalization process at higher orders might behave differently from what we have already computed at the lowest nontrivial orders.

The exact form of the additional potential accompanying a point transformation is (including the explicit loop-counting parameter $\hbar$) \[23\],

\[
\Delta V' (\hat{Q}) = \frac{1}{4} \hbar \left( \frac{1}{2} g^{ij}(\hat{Q})_{,ij} - 2g(\hat{Q})^{-\frac{3}{2}} \left\{ g^{ij}(\hat{Q}) \left[ g^{\frac{3}{2}} g(\hat{Q})^{-\frac{1}{2}} \right]_{,i} \right\}_j \right),
\]

(17)

where $Q$ is the new coordinate variable; its canonically conjugate momentum would be $P$. The new metric $g$ for the revised coordinate space is given by

\[
g_{ij} = \Sigma_a F^a(Q)_{,i} F^a(Q)_{,j};
\]

(18)

with the commas in (17) and (18) indicating differentiation, and $F$ giving the relationship between the old canonically conjugate variables $q$ are $p$ and the new $Q$ and $P$, according to

\[
q^a = F^a(Q) \quad p^a = F^a_{,i} (Q) g^{ij} P_j.
\]

(19)

The Dirac Lagrangian can be cast in the Hamiltonian form with momenta and coordinates $\psi^\dagger$ and $\psi$ corresponding to $p$ and $q$, respectively (at least in flat Minkowski spacetime). The relation (17) is formulated for a bosonic system. For the fermionic fields, results are analogous \[24\], with the quantum correction to a canonical point transformation originating in two ways: either due to the nonlinearity of the transformation, or due to an anomalous Jacobian for the transformation. We shall discuss each of these in turn.

If we perform the change of canonical coordinates as suggested by the relation (3) then the metric $g$ appearing in (18) will be a constant matrix, dependent only on the $\gamma$-matrices and $f$—not on position or field strength $\psi$. It is then immediately apparent that the additional potential term given analogously to (17) is identically zero. Another, perhaps easier, way to see this is by observing the fact that the additional potential terms arise only when there is a nonlinear change in the dynamical variables, which does not happen with (3) or (6). Hence it turns out that the formula (10) must be stable even with the inclusion of quantum corrections.

The equivalence between $f$ and $c$ is obviously most interesting if the equivalence persists even in the presence of interactions, and this has guided our choice of methods for demonstrating the equivalence. In this connection, we should note that there is no chiral anomaly associated with the transformation $A$, even if a gauge interaction is added to the action. The gauge invariant regulation of the fermion measure leads to a nontrivial Jacobian $J$ exactly if

\[
-i \log J = \lim_{\Lambda \to \infty} \int d^4 x \langle x | \text{tr} \left[ (\log A) e^{(i\partial)^2/\Lambda^2} \right] | x \rangle
\]

(20)
is nonzero [25]. However, since log A is proportional to $\gamma_\nu \gamma_5$, the expansion of the exponential regulator always produces traces of odd numbers of basic $\gamma$-matrices, which uniformly vanish.

Moreover, the fact that the effects of (6) do not produce any additional potential terms can also be seen in yet another way—via an analogy to a slightly simpler transformation that is known not to be associated with complications of this sort. In the path integral formalism, $\Delta V'$ arises out of the nontrivial transformation of the path integral measure. When moving from integrals over Grassmann variables to path integrals over the Grassmann-valued fields, it is standard to change the measure from $D\psi^* D\psi$ to $D\bar{\psi} D\psi$, often without even mentioning the change, even in pedagogical treatments. The measures are, of course, completely equivalent, but there is a unitary transformation that effects the change, $\bar{\psi}_a = (\gamma_0)_{ab} \psi_b$. For the reasons discussed in the previous two paragraphs, this transformation to new dynamical variables does not introduce any additional potentials; the transformation simply reshuffles the components of one of the Dirac fields, in the same fashion at all spacetime locations. This is exactly what (6) represents as well—a position- and momentum-independent similarity transformation of the components of the Dirac spinor fields. Just as $D\psi^* D\psi \rightarrow D\bar{\psi} D\psi$ is an uncomplicated transformation of the measure, so is $D\psi D\psi \rightarrow D\bar{\psi}' D\psi'$.

5 Discrete Symmetries

The last major topic we shall address is the discrete symmetries of the $f$ theory. This is an issue that has the potential to cause a fair amount of confusion, since—to naive appearances—a theory with only a $c$-type term must be even under $C$ and $PT$, while a theory with just a $f$ has different behavior. That might make it difficult to see how theories with $c$ and $f$ could be equivalent. Indeed, a Lagrangian with just $c^{00}$ or $c^{jk}$ terms manifestly corresponds to a $C$-even, $P$-even, $T$-even theory. However, it appears that $f^0$ is actually separately odd under $C$, $P$, and $T$, which presents a puzzle.

The solution to the puzzle is that the $C$, $P$, and $T$ operators have to be modified if the theory contains an $f$ term. The usual forms of these operators are derived from the fact that they leave the Lagrangians for certain theories (such as the free Dirac theory, or its extension to quantum electrodynamics) invariant. However, while the way the symmetries act on coordinates is fixed by the fact that certain coordinates must be inverted or preserved, the way the operators act in the four-dimensional Dirac spinor space may be adjusted and is not so strictly fixed. It is familiar, for example, that the phases associated with the discrete operators cannot be uniquely determined, or that each scalar field in a theory may be assigned an even or odd intrinsic parity. In the $f$ theory, the discrete operators are modified in a much more profound way, but they are still valid representations of the $CPT$ algebra; and as long as a representation exists that leaves the action of the theory invariant, the physical observables of the theory will be likewise
invariant.

Once this is recognized, it is not too challenging to determine the forms of the modified operators, since it is already known how to transform the $f$ theory into a theory with a $c_{\text{eff}}$. For simplicity, we shall consider a theory with only a $f$, but the generalization to a theory with both types of coefficients is straightforward—simply using the transformation (3) instead of (3) in what follows.

5.1 Parity and Reflections

There are other minimal SME terms besides $f^0$ that share the property of being odd under the standard $C$, $P$, and $T$ operations—spatial $a^j$ and $e^j$ terms, for instance. However, there are additional, less-discussed discrete symmetries that distinguish these operators. The parity operator is defined as inverting all three spatial coordinates, taking $\vec{x} \rightarrow -\vec{x}$. However, this $P$ may be broken down into the product of three separate spatial reflections, $P = R_1 R_2 R_3$, where $R_j$ inverts just one of the coordinates, taking $x_j \rightarrow -x_j$ and leaving the two orthogonal coordinates unaffected. In Lorentz-invariant theories, little distinction is made between the behavior of a quantities under $P$ and $R_j$, because $P$ can also be written as $R_j$ followed by a $\pi$ rotation around the $x_j$-axis. However, in the absence of rotation symmetry, the behavior of observables under $P$ and $R_j$ need not be the same.

While particular spacelike components $a^j$ and $e^j$ are odd under $R_j$, which inverts the axis along which those coefficients point, they are even under reflections along the two perpendicular axes. However, the behavior of $f^0$ different; using the standard reflection operator $R_j$ (which acts on the Dirac spinor $\psi$ by the matrix $S_{R_j} = i\gamma^j\gamma_5$), the $f^0$ term appears to be odd under each of the $R_j$ separately (or, indeed under any reflection whatsoever, including ones along oblique directions). This reflection behavior is unique in the minimal SME; it is shared by no other term.

Considering the theory with a $f$ term, the Lagrange density contains the Dirac matrix operator $\gamma^\mu + if^\mu\gamma_5$ sandwiched between the fermion fields. Converting this theory into the theory with the $c_{\text{eff}}$ term is equivalent to using the conjugation operation

$$\gamma^\mu + if^\mu\gamma_5 \rightarrow A(\gamma^\mu + if^\mu\gamma_5)A^{-1} \equiv e^{\frac{1}{2}i\nu^\mu\gamma_5\frac{\theta}{\sqrt{f^2}}}(\gamma^\mu + if^\mu\gamma_5)e^{-\frac{1}{2}i\nu^\mu\gamma_5\frac{\theta}{\sqrt{f^2}}} = \gamma^\mu + c_{\text{eff}}^{\mu\nu}\gamma_\nu \quad (21)$$

(with $\theta = \tanh^{-1}\sqrt{f^2}$) on the kinetic operator in spinor space. [Note that the $A$ that appears on left in $A(\gamma^\mu + if^\mu\gamma_5)A^{-1}$ actually arises—due to the presence of $\bar{\psi} = \psi^\dagger\gamma^0$ on the left-most end of the operator product in the action—as the inverse of $A^{-1} = \gamma^0(A^{-1})^\dagger\gamma^0 = A^{-1}$.] Under this conjugation, the $if^\mu\gamma_5$ mixes with the $\gamma^\mu$ to produce the linear combination $\gamma^\mu + c_{\text{eff}}^{\mu\nu}\gamma_\nu$ of $\gamma$-matrices. To understand the physical behavior of the $f$ theory under a spacetime transformation, we should look at how the full kinetic sector of the theory behaves—considering the $C$, $P$, and $T$ properties of $\gamma^\mu + if^\mu\gamma_5$ as a single block, rather than looking at the properties of $f$ or $c$ terms on their own. With this in mind, we consider a discrete symmetry (involution) operator $X'$ that acts on $\psi'$, where $\psi'$ is the
fermion field in a theory transformed to have solely a $c$-type term in Lagrangian. The action of conjugation by $X'$ on the field operator is represented by a linear transformation with matrix $S'_X$, according to $X'\psi'X' = S'_X\psi'$.

Acting with $S'_X$ on $\psi'$ is equivalent to conjugating $\gamma^\mu + c^{\mu\nu}\gamma^\nu$ by $S'_X$, and if $X$ is a symmetry of the action, then the conjugation operation must have an eigenvalue $(-1)^X$,

$$(-1)^X(\gamma^\mu + c^{\mu\nu}\gamma^\nu) = S'_X(\gamma^\mu + c^{\mu\nu}\gamma^\nu)S'_X.$$

To find $S_X$, the matrix representation of how $X$ acts in the untransformed theory with $f$, we may simply insert factors of $A$ and $A^{-1}$ into (22) and use (21) on each side,

$$(-1)^X A^{-1}(\gamma^\mu + c^{\mu\nu}\gamma^\nu)A = A^{-1} S'_X(\gamma^\mu + c^{\mu\nu}\gamma^\nu)S'_X A \quad (23)$$

$$(-1)^X (\gamma^\mu + if^\mu\gamma_5) = A^{-1} S'_X A(\gamma^\mu + if^\mu\gamma_5)A^{-1} S'_X A, \quad (24)$$

then read off the expression $S_X = A^{-1} S'_X A$.

In the $\psi'$ theory, $\gamma^0$ acts as the parity transformation matrix $S'_P$. It is straightforward to find the equivalent operation of $P$ in the $\psi$ theory that contains $f$ but no $c$,

$$P\psi P = S_P \psi(t, -\vec{x}) = e^{-i f^\nu\gamma^\nu\gamma^5\sqrt{f^2}} e^{i f^\nu\gamma^\nu\gamma^5\sqrt{f^2}} \psi(t, -\vec{x}) \quad (25)$$

$$= \left[ \gamma^0 + i\frac{f^0}{\sqrt{1 - f^2}} \gamma_5 + \frac{1 - \sqrt{1 - f^2}}{f^2 \sqrt{1 - f^2}} f^\nu \gamma^\nu \right] \psi(t, -\vec{x}), \quad (26)$$

much like in (8). In spite of the presence of a $f^2$ factor in the denominator of the last bracketed term of $S_P$ in (26), the whole expression is regular as $f^2 \to 0$, since the numerator contains $1 - \sqrt{1 - f^2}$, which is approximately $\frac{1}{2} f^2$ in that limit.

Note that $S_P$ in (26) is not Lorentz invariant, but instead it has, just like $S'_P = \gamma^0$, exactly one free time index in each term. Moreover, if $f^0 = 0$, then the $S_P$ matrix is not modified, which is correct, since theories in which either $f$ is purely spacelike or $c$ has only space-space (that is, $c^{ij}$) components, there is no negative sign under the action of $S'_P$ on $\psi$. For a less trivial example, let us consider the next-simplest case, in which $f^\mu = (f^0, 0, 0, 0)$ is purely timelike. Then the new parity matrix is

$$S_P = \frac{1}{\sqrt{1 - (f^0)^2}} \gamma^0 + i\frac{f^0}{\sqrt{1 - (f^0)^2}} \gamma_5. \quad (27)$$

This is directly proportional to the matrix that multiplies the time derivative $\partial_0$ in the Dirac Lagrange density $L_f$, and it also anticommutes with all the spatial $\gamma^j$ matrices. Thus one can easily see that the $\gamma^\mu \partial_\mu + if^0\gamma_5 \partial_0$ term in the Lagrange density is even under the action of the conjugation by this modified parity matrix. This makes sense because in this theory $c_{\text{eff}}$ only has a $c^{00}$ component, and the term $c^{00}\gamma_5 \partial_0$ is also even under parity.
The same method may be applied to find the correct reflection operators in the $f$ theory. For example, for inversion of the $x_1$-direction, the new $x_1$-axis reflection operator $S'_{R_1}$ is, following the above line of argument,

$$S'_{R_1} = A^{-1}S'_{R_1}A = A^{-1}i\gamma^1\gamma_5A$$

$$= \frac{i}{\sqrt{1-f^2}}\gamma^1\gamma_5 + \frac{f^1}{\sqrt{1-f^2}}I + \frac{f^\nu}{\sqrt{1-f^2}}\gamma^1\gamma_\nu + \frac{i(1-\sqrt{1-f^2})f^1f^\nu}{f^2\sqrt{1-f^2}}\gamma_\nu\gamma_5$$

(29)

(where $I$ is the identity matrix in $4 \times 4$ spinor space). The results for other spatial reflections are obviously analogous.

An interesting corollary of the nonlinearity of the relationship between $f$ and $c_{\text{eff}}$ is that the individual components of $f$ do not possess well-defined parities. The parity transformation just of the Dirac matrices in the $c$-type fermion kinetic term is

$$P'(\gamma^\mu + c^\mu_\nu\gamma_\nu)P' = (-1)^\mu\gamma^\mu + (-1)^\nu c^{\nu\mu}\gamma_\nu,$$

(30)

where $(-1)^\mu$ is $-1$ if $\mu$ is any spacelike index $j$. Including the derivative term

$$P'(\gamma^\mu + c^\mu_\nu)P'\partial_\mu P' = \gamma^\mu + (-1)^\nu(-1)^\mu c^{\nu\mu}\gamma_\nu.$$  

(31)

Different components of $c$ evidently have different behavior under $P'$; in particular, while $c^{00}$ and $c^{jk}$ terms are even, the mixed time-space term $c^{0j}$ is $P'$ odd. (The individual $c^{jk}$ terms can similarly be further classified by their behavior under individual reflections.) This means that a $f$ theory with either a purely timelike or purely spacelike $f$ is even under $P$. However, if there are both a nonzero $f^0$ and nonzero $f^j$, then the theory includes physical parity-violating effects. So the transformation of a term in the $f$ theory into an equivalent $c$-type term does not give nice $P$ (or, in a similar fashion, $T$) eigenvalues in general. However, since every single $c$ term in $L_c$ is symmetric under $P'T'$ and $C'$ separately, we expect that every $f$ term should likewise be even under the modified action of $PT$, $C$, and thus $CPT$ operators.

5.2 Charge Conjugation

The purely spatial transformations $P$ and $R_j$ are the simplest of the the discrete symmetries to work with. Like purely spatial rotations, they are represented by unitary matrices that act on the spinor structure of $\psi$. The remaining operations, $C$ and $T$, are potentially trickier, since there are complex conjugations involved. For example, because $C$ exchanges particle and antiparticle states, $C\psi C = S_C\psi^*$. (Note that another convention for the definition of $S_C$ also exists, differing from ours by a factor of $\gamma^0$.)

The requirement on the $S_C'$ matrix (which implements charge conjugation in the standard Dirac theory, or the theory with only $c$-type Lorentz violation) is that it should satisfy $S_C'\gamma^\mu = -(\gamma^\mu)^*S_C'$. This ensures that $C'$ interchanges fermion and antifermion
creation (and, separately, annihilation) operators. The analogous condition in the $f$-type theory is that

$$S_C(\gamma^\mu + if^\mu \gamma_5) = -(\gamma^\mu + if^\mu \gamma_5)^* S_C. \tag{32}$$

In both the Dirac and Weyl representations of the Dirac algebra (which differ only by the interchange of $\gamma^0$ and $-\gamma_5$), $\gamma^0$, $\gamma^1$, and $\gamma^3$ are real, so $S_C'$ is proportional to the imaginary $\gamma^2$: $S_C' = -i\gamma^2$ with a standard choice of phase. [The alternate convention for $S_C$ mentioned above is based on the fact that $C'\psi' C' = -i(\bar{\psi}' \gamma^0 \gamma^2)^T$ and $C'\bar{\psi}' C' = (-i\gamma^0 \gamma^2 \psi')^T$ in the Dirac and Weyl representations.] To find a matrix $S_C$ that satisfies (32), we can actually just transform $S_C'$ in much the same way as $S_P'$—just taking account of the extra complex conjugation in $C\psi = S_C\psi^*$. Doing this, we have

$$C\psi C = S_C\psi^* = A^T S_C' A^* \psi^*. \tag{33}$$

[Note that while $A$ can expressed in the representation-independent form (7), the matrices $A^*$ and $A^T$ cannot, because of the way they depend on which of the representation matrices are imaginary.] As an example, we consider again the theory in which $f$ is purely timelike, so that (7) gives

$$A = \cosh \frac{\theta}{2} + i \frac{f^0}{|f^0|} \sinh \frac{\theta}{2} \gamma^0 \gamma_5. \tag{34}$$

Applying this $A$, along with (9) with $\alpha = 0$, the expression for the charge conjugation matrix becomes

$$S_C = -\frac{i}{\sqrt{1 - (f^0)^2}} \gamma^2 + \frac{f^0}{\sqrt{1 - (f^0)^2}} \gamma^0 \gamma^2 \gamma_5, \tag{35}$$

which can be checked to satisfy (32).

In the less-used Majorana representation, all four Dirac $\gamma^\mu$ matrices are purely imaginary, obeying $\gamma^\mu_T = -(-1)^\mu \gamma^\mu$ and $\gamma^5_T = -\gamma_5_T$. The usual charge conjugation matrix in this representation is $S_C' = -iI$. In fact, this is not actually that puzzling, since if all four $\gamma^\mu$ are imaginary, then $S_C' \gamma^\mu = -(-\gamma^\mu)^* S_C'$ just requires that $S_C'$ commute with them all; and by Schur’s Lemma, only a multiple of the identity has this property. The simplicity of $S_C'$ makes the evaluation of $S_C$ also quite simple (even for arbitrary $f$),

$$S_C = A^T(-i)A^* = -i(A^*)^2 = -\frac{i}{\sqrt{1 - f^2}} I - \frac{f^\nu}{\sqrt{1 - f^2}} \gamma^\nu \gamma_5. \tag{36}$$

### 5.3 Time Reversal

Finally, let $T$ be the time reversal operator and $S_T$ the corresponding matrix. Attention has to be paid to the fact that time reversal is anti-unitary. The additional anti-unitarity means that when the matrix is transformed to a different basis, the $A^{-1}$ on the left (as in $S_X = A^{-1}S_X'A$) must be complex conjugated. The time reversal matrix for the Lagrangian with $c$ is $S_T' = \gamma^1 \gamma^3$ in the Dirac and Weyl representations. A useful identity
in these representation is \((A^{-1})^* = -\gamma^2 A^{-1} \gamma^2\), so it turns out that the matrix by which \(T\) operates in the \(f\) theory is

\[
T\psi T = (A^{-1})^* \gamma^1 \gamma^3 A\psi(-t, \vec{x})
\]

\[
= \left[ -\frac{i}{\sqrt{1-f^2}} \gamma^1 \gamma^3 + \frac{f^\nu}{\sqrt{1-f^2}} \gamma^1 \gamma^3 \gamma^5 \gamma^\nu + \frac{f^0}{\sqrt{1-f^2}} \gamma^2 \gamma^2 \gamma^5 \right] \psi(-t, \vec{x}).
\]

(37)

An alternative guiding principle for verifying the action of \(T\) is that the combination \(TP\) of operators must have the same eigenvalue for its conjugation action on \(\gamma^\mu + if^\mu \gamma^5\) as \(T'P'\) has on \(\gamma^\mu + c^{\mu\nu} \gamma^\nu\). The combined action of both operators is

\[
TP\psi PT = (A^{-1})^* \gamma^0 \gamma^1 \gamma^3 A\psi(-t, -\vec{x})
\]

\[
= \left( -\frac{i}{\sqrt{1-f^2}} \gamma^2 \gamma^5 + \frac{f^\nu}{\sqrt{1-f^2}} \gamma^2 \gamma^5 \gamma^\nu \right) \psi(-t, -\vec{x}),
\]

(39)

which has the desired properties. Moreover, if the action of the new operators \(TP\) on the creation and annihilation operators are

\[
TPa^s_\vec{p} PT = a^{-s}_\vec{p}
\]

(41)

\[
TPb^s_\vec{p} PT = b^{-s}_\vec{p}
\]

(42)

(where the \(-s\) indicates a reversal of all the spin projections), the action of these operators on the Dirac field in momentum space is

\[
TP\psi PT = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{E_\vec{p}}} \left[ a^{-s}_\vec{p} u^s(\vec{p})^* e^{ipx} + (b^{-s}_\vec{p})^* v^s(\vec{p})^* e^{-ipx} \right],
\]

where \(u^s(\vec{p})\) is the plane wave fermion solution of the modified Dirac equation. It can be shown with some algebra that

\[
(A^{-1})^* \gamma^0 \gamma^1 \gamma^3 A\psi^{-s}(\vec{p}) = u^s(\vec{p})^*,
\]

(44)

and a similar relation also holds for antiparticle spinors \(v^s(p)\). (This generalizes the relations that holds for the eigenspinors in the usual, Lorentz-invariant Dirac theory.) So the preceding formulas show the invariance of the \(f\) theory under physical inversion of all the spacetime coordinates.
6 Discussion

In this paper, we have demonstrated a number of useful facts and formulas describing the SME fermion sector in the presence of a $f$ term, based on the relationship between such a theory and a similar one with a $c$ term instead. One important result was that the field redefinition that interchanges the two types of Lorentz violation does not produce any extra effective potential terms in the Lagrangian, which means that the formula (10) for $c_{\text{eff}}$ is correct in even quantum theory. We previously noticed how the effective $\beta$-function for the nonlinear combination of coefficients $c^{\mu\nu} - \frac{1}{2} f^{\mu} f^{\nu}$ is free of ambiguities up to second order in $f$. Since the equivalence of $c$ and $f$ is not anomalous at the quantum level, we expect that the generalized quantity (10) will continue to exhibit a scheme-independent RG scale dependence at all orders in $f$. Explicit calculations at the higher orders should be possible, facilitated by the modified identities given in section 3.

Yet in spite of these results, there are still potentially interesting technical questions about the equivalence between the $c$ and $f$ theories remaining to be answered. We have shown how the mode expansion of the fermion field in the $f$ theory is affected by the $f$ theory’s $\text{PT}$ operators (43), but we did not explicitly show how this worked out with the $C$, $P$, or $R_j$ (reflection) operators. In fact, we cannot naively assume the transformation of the creation and annihilation operators under the action of $C$, $P$, or $T$ to be exactly the same as in a Lorentz-invariant free field theory. To see the issue that arises, suppose that we begin in a theory with only a $c$, in which we may assume that the $C'$, $P'$, and $T'$ transformations of fermion and antifermion creation and annihilation should be the same as in the standard field theory. Then we perform the spinor transformation to obtain instead an equivalent theory with a $f$ term. However, the spinor redefinitions are not straightforward symmetry transformations, because they are not generally unitary. This means that the Fock spaces need not be same in the old and new theories. In short, the creation and annihilation operators possibly will have some complicated nonunitary reshuffling of their eigenspinor coefficients, and so a direct mode expansion of the field to see the effect of the $C$ operator or of the $P$ operator alone is potentially nontrivial. Of course, nonunitary implementations of symmetry transformations are nothing new in relativistic quantum field theory; because the Lorentz group is not compact, boost transformations acting on spinors are also represented by nonunitary matrices. However, understanding the relationships between the state spaces in the SME $c$ and $f$ theories is an interesting further problem to be addressed.

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