Calabi-Yau Mirror Symmetry as a
Gauge Theory Duality

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Abstract. We show that there are two different dualities of two dimensional gauge
theories with \( N = (2, 2) \) supersymmetry. One is basically a consequence of 3d mirror
symmetry. The non-linear sigma model with Calabi-Yau target space on the Higgs
branch of the gauge theory is mapped into an equivalent non-linear sigma model on
the Coulomb branch of the dual theory, realizing a T-dual target space with torsion.
The second duality is genuine to two dimensions. In addition to swapping Higgs and
Coulomb branch it trades twisted for untwisted multiplets, implying a sign flip of the
left moving \( U(1)_{R} \) charge. Successive application of both dualities leads to geometric
mirror symmetry for the target space Calabi-Yau.

1. Introduction

Mirror symmetry of Calabi-Yau manifolds is one of the remarkable predictions of
type II string theory. The \( (2, 2) \) superconformal field theory associated with a string
propagating on a Ricci-flat Kähler manifold has a \( U(1)_{L} \times U(1)_{R} \) R-symmetry group,
and the Hodge numbers of the manifold correspond to the charges of \( (R, R) \) ground
states under the R-symmetry. There is a symmetry in the conformal field theory which
is a flip of the sign of the \( U(1)_{L} \) current, \( J_{L} \rightarrow - J_{L} \). If physically realized, this symmetry
implies existence of pairs of manifolds \( (\mathcal{M}, \mathcal{W}) \), which have “mirror” hodge diamonds,
\( H^{p,q}(\mathcal{M}) = H^{d-p,q}(\mathcal{W}) \), and give rise to exactly the same superconformal field theory.

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While this observation predicts existence of mirror pairs of Calabi-Yau $d$-folds, it is not constructive: one would like to know how to find $\mathcal{W}$ if one is given a Calabi-Yau manifold $\mathcal{M}$. The mirror construction has been proposed on purely mathematical grounds by Batyrev for Calabi-Yau manifolds which can be realized as complete intersections in toric varieties.

In [1], Strominger, Yau and Zaslow argued that, for mirror symmetry to extend to a symmetry of non-perturbative string theory, $(\mathcal{M}, \mathcal{W})$ must be $T^d$ fibrations, with fibers which are special lagrangian, and furthermore, that mirror symmetry is $T^d$-duality of the fibers. The argument of SYZ is local, and is very well understood only for smooth fibrations. To be able to fully exploit the idea, one must understand degenerations of special lagrangian tori, or more generally, limits in which the Calabi-Yau manifold itself becomes singular. Some progress on how this is supposed to work has been made in [1], [2], [3], and local mirror symmetry appears to be the simplest to understand in this context.

In an a priori unrelated development initiated by [4] ‘mirror pairs’ of gauge theories were found. In the field theory context, a duality in the sense of IR equivalence of two gauge theories is usually referred to as a ‘mirror symmetry’ if

- the duality swaps Coulomb and Higgs branch of the theories, trading FI terms for mass parameters,
- the R-symmetry of the gauge theory has the product form $G_L \times G_R$ and duality swaps the two factors.

The example of [4] studies $\mathcal{N} = 4$ SUSY theories in 3 dimensions. Recently it was shown [5] that this duality can even be generalized to an equivalence of two theories at all scales, a relation that will prove to be crucial for our applications.

The purpose of this note is to obtain geometrical mirror pairs by a ‘worldsheet’ construction. Since there seems to be little hope that one can do so directly in the non-linear sigma model (NL$\sigma$M), we study linear sigma models [6] which flow in the IR to non-linear sigma models with Calabi-Yau’s as their target spaces. String theory offers a direct physical interpretations of these models as world-volume theory of a D-string probe of the Calabi-Yau manifolds. We find gauge theory mirror duality for the linear sigma model which reduces to geometric mirror symmetry for Calabi-Yau target spaces.

Using brane constructions T-dual to the D1 brane probe on $\mathcal{M}$ one can easily construct gauge theories which flow to mirror manifolds. We will show that two different dualities of $d = 2, \mathcal{N} = (2, 2)$ supersymmetric gauge theories arise: One has a realization in string theory as ‘$S$-duality†’ of ‘interval’ brane setups. This duality is a consequence of mirror symmetry of $d = 3, \mathcal{N} = 2$ field theories where both the Coulomb branch and the Higgs branch are described by non-compact Calabi-Yau manifolds. This duality in $d = 3$ maps a Calabi Yau manifold to an identical one, so while it is a non-trivial field theory

† where by $S$-duality in type IIA setups we mean a flip of the 2 and the 10 direction.
statement, this does not provide a linear $\sigma$ model construction of the mirror Calabi-Yau manifold. There exists another $d = 2, \mathcal{N} = (2, 2)$ field theory duality obtained as S-duality of the diamond brane constructions of [3]. This duality maps a theory whose Coulomb branch is dual to Calabi-Yau manifold $\mathcal{M}$ in the “boring” way described above, to a theory whose Higgs branch is the mirror manifold $\mathcal{W}$. The composition of these two dualities, therefore, flows to Calabi-Yau mirror symmetry. While we consider in this note a very particular family on non-compact Calabi-Yau manifolds, the generalization to arbitrary affine toric varieties is possible.

The organization of the note is as follows: In the next section we discuss different possible dualities in two dimensions as obtained via brane constructions for the case of the conifold. Section three generalizes this discussion to sigma models built from branes dual to more general non-compact CY manifolds and to the non-abelian case, describing $N$ D-brane probes on the singular CY. In the last section we present a detailed study of the moduli space and argue that the composition of two dualities is Calabi-Yau mirror symmetry.

2. Mirror duality in 2d gauge theories

2.1. From three to two dimensions

The original $D = 3, \mathcal{N} = 4$ of [4] upon compactification implies also a duality relation in $\mathcal{N} = (4, 4)$ theories in 2 dimensions, as noted in [7, 8]. The recent results of [5] are needed to make this precise. The nature of this duality with 8 supercharges will teach us, how we should understand the $\mathcal{N} = (2, 2)$ examples. Since in 2d the concept of a moduli space is ill defined, equivalence of the IR physics does not require the moduli spaces and metrics to match point by point, but only that the NL $\sigma$Ms on the moduli space§ are equivalent, as we will see in several examples.

Start with the 3d theory compactified on a circle. This is the setup analyzed in [11]. It is governed by two length scales, $g_{YM}^2$, the 3d Yang-Mills coupling, and $R_2$, the compactification radius. To flow to the deep IR is equivalent to sending both length scales to zero. However physics still might depend on the dimensionless ratio

$$\gamma = g_{YM}^2 R_2.$$  

As shown in [11], while the Higgs branch metric is protected, the Coulomb branch indeed does depend on $\gamma$. For $\gamma \gg 1$ we first have to flow into the deep IR in 3d and then compactify, resulting in a 2d NL$\sigma$M on the 3d quantum corrected Coulomb branch. The resulting target space is best described in terms of the dual photon in 3d, a scalar of radius $\gamma$. For ‘the mirror of the quiver’ (U(1) with $N_f$ electrons) it turns out to be an ALF space with radius $\gamma$. For small $\gamma$ we should first compactify, express the theory in terms of the Wilson line, a scalar of radius $\frac{1}{\gamma}$, and obtain as a result a tube metric with § or in the non-compact CY examples we are considering the two disjoint CFTs of Coulomb and Higgs branch [9, 10]
torsion, corresponding to the metric of an NS5 brane on a transverse circle of radius $\frac{1}{7}$ [11, 12]. Indeed these two NL$\sigma$Ms are believed to be equivalent [13] and exchanging the dual photon for the Wilson line amounts to the T-duality of NS5 branes and ALF space in terms of the IR NL$\sigma$M $\parallel$.

In order to obtain linear $\sigma$-model description of this scenario, one has to use the all scale mirror symmetry of [5]. They show that $g^2_{YM}$ maps to a Fermi type coupling in the mirror theory, or more precisely: one couples the gauge field via a BF coupling to a twisted gauge field, the gauge coupling of the twisted gauge field being baptised Fermi coupling. For the case of the quiver theory with Fermi coupling, one obtains the same ALF space, this time on the Higgs branch. In the same spirit we will present two different dualities for $\mathcal{N} = (2, 2)$ theories.

2.2. Mirror symmetry from the interval

One way to ‘derive’ field theory duality is to embed the field theory into string theory and then field theory duality is a consequence of string duality. A construction of this sort was implemented in [14] for the $\mathcal{N} = 4$ theory in d=3 via brane configurations. One uses an interval construction with the 3 basic ingredients: NS5 along 012345, D5 along 012789 and D3 along 0126. The two R-symmetries are $SU(2)_{345}$ and $SU(2)_{789}$. D3 brane segments between NS5 branes give rise to vector multiplets, with the 3 scalars in the 3 of $SU(2)_{345}$. D3 brane segments between D5 branes are hypermultiplets with the four scalars transforming as 2 doublets of $SU(2)_{789}$.

Under S-duality the D5 branes turn into NS5 branes and vice versa while D3 branes stay invariant. One obtains the same kind of setup but with D5 and NS5 branes interchanged. S-duality of type IIB string theory is mirror symmetry in the gauge theory $\parallel$.

Now let us move on to the 2d theories. The brane realization of this duality is via an interval theory in IIA with NS and NS’ branes and D2 branes along 016 [15]. The IIA analog of S-duality, the 2-10 flip, takes this into D4 and D4’ branes. The following parameters define the interval brane setup and the gauge theory:

- The separation of NS and NS’ brane along 7 is the FI term. It receives a complex partner, the 10 separation which maps to the 2d theta angle.
- The separation of the D4 branes along 2 and 3 gives twisted masses to the flavors.

Mirror symmetry maps the FI term to the twisted masses. A twisted mass sits in a background vector multiplet and has to be contrasted with the standard mass from the $\parallel$This picture is obvious from the string theory perspective. Studying a D2 D6 system on a circle, going to the IR first lifts us to an M2 on an ALF space which becomes a fundamental string on the ALF, while going to 2d first makes us T-dualize to D1 D5, leaving us with the $\sigma$ model of a string probing a 5-brane background. $\|$ To be precise, the S-dual theory will really contain a gauge theory of twisted hypers coupled to twisted vectors. If in addition one performs a rotation taking 345 into 789 space, the two R-symmetries are swapped and the theory is written in terms of vectors and hypers.
superpotential which sits in a background chiral multiplet. Like the real mass in $\mathcal{N} = 2$ theories in $d = 3$, it arises from terms like

$$\int d^4 \theta Q^\dagger e^{V_B} Q.$$ 

where $V_B$ is a background vector multiplet.

**An Example:** As an example let us discuss the interval realization of the small resolution of the conifold. As shown in [16, 17] by performing $T_6$ T-duality on a D-string probe of the conifold we get an interval realization of the conifold gauge theory in terms of an elliptic IIA setup with D2 branes stretched on a circle with one NS and one NS’ brane. In this IIA setup the separation of the NS branes in 67 is the small resolution, while turning on the diamond mode would be the deformation of the conifold.

The gauge group on the worldvolume of the D-string on the conifold is [18] a $U(1) \times U(1)$ gauge group with 2 bifundamental flavors $A_1, A_2, B_1$ and $B_2$. We can factor out the decoupled center of mass motion, the diagonal $U(1)$, which does not have any charged matter and hence is free. We are left with an interacting $U(1)$ with 2 flavors. The scalar in the decoupled vector multiplet is the position of the D1 brane in the 23 space transverse to the conifold. While the Coulomb branch describes separation into fractional branes, the Higgs branch describes motion on the internal space and reproduces the conifold geometry. The complexified blowup mode for resolving the conifold is the FI term and the $\theta$ angle.

After 2-10 flip, the dual brane setup is again an elliptic model, this time with one D4 and one D4’ brane. The gauge theory is a single $\mathcal{N} = (8,8)$ $U(1)$ from the D2 brane with 2 additional $\mathcal{N} = (2,2)$ matter flavors from the D4 and D4’ brane. That is we have

- 3 ‘adjoints’, that is singlet fields $X, Y$ and $Z$ and
- matter fields $Q, \tilde{Q}, T$ and $\tilde{T}$ with charges $+1,-1,+1,-1$.
- They couple via a superpotential

$$W = QX\tilde{Q} + TXT\tilde{T}.$$ 

- The singlet $Z$ is decoupled and corresponds to the center of mass motion.

Turning on the FI term and the $\theta$ angle in the original theory is a motion of the NS brane along the 7 and 10 direction respectively. It maps into a 23 motion for the D4 brane, giving a twisted mass to $Q$ and $\tilde{Q}$.

This analysis can also be performed by going to the T-dual picture of D1 branes probing D5 branes intersecting in codimension 2, that is over 4 common directions. Aspects of this setup and its T-dual cousins in various dimensions have already been studied by numerous authors, e.g. for the D3 D7 D7’ system in [19] or for the D0 D4 D4’ in [20]. The resulting gauge theory agrees with what we have found by applying the standard interval rules.
2.3. Twisted mirror symmetry from diamonds

A second T-dual configuration for D1 brane probes of singular CY manifolds is D3 branes ending on a curve of NS branes, called diamonds in [3]. These setups are the $T_{48}$-duals of D1 brane probes of the $C_{kl}$ spaces. Indeed it was this relation that allowed us to derive the diamond matter content to begin with [3]. In order to use the diamond construction to see mirror symmetry, we use S-duality of string theory, as in the original work of [14]. Let us first consider the parameters defining a diamond and how they map under S-duality:

- the complex parameter defining the NS brane diamond contains the FI term which is paired up with the 2d $\theta$ angle,
- the S-dual D5-brane diamond is defined by a complex parameter which is derived from a superpotential mass term.

FI term and theta angle are contained in a background twisted chiral multiplet. Under the duality this twisted chiral multiplet is mapped to a background chiral multiplet containing the mass term. Since ordinary mirror symmetry mapped FI terms to mass terms in twisted chiral multiplet, the map of operators under the two versions of duality will be different.

An Example: Let us start once more with the simplest example, the D1 string on the blowup of the conifold. That is, we consider a single diamond, one NS and one NS' brane, on a torus. After S-duality this elliptic model with NS5 and NS5' brane turns into an elliptic model with D5 and D5' brane. Since we have only D-branes in this dual picture, the matter content can be analyzed by perturbative string techniques. To shortcut, we perform $T_{48}$ duality to the D1 D5 D5' system as in the interval setup. For the special example of the conifold the two possible mirrors do not differ in the gauge and matter content, only in the parameter map. This will not be the case in the more general examples considered below.

As analyzed above, the corresponding dual gauge theory is a U(1) gauge group with 3 neutral fields $X$, $Y$ and $Z$ and two flavors $Q$, $\tilde{Q}$, $T$ and $\tilde{T}$ with charges $+1$, $-1$, $+1$, $-1$ respectively. The superpotential in the singular case is $W = QX\tilde{Q} + TY\tilde{T}$.

By S-duality, as in the NS NS' setup, turning on the D5-brane diamonds corresponds turning on vevs for the d=4 hypermultiplets from the D5 D5' strings. Under $\mathcal{N} = (2,2)$ these hypermultiplets decompose into background chiral multiplets and hence appear as parameters in the superpotential. If we call those chiral multiplets $h$ and $\tilde{h}$ the corresponding superpotential contributions are [20] $Qh\tilde{T} + \tilde{Q}hT$, so that all in all the full superpotential reads

$$W = QX\tilde{Q} + TY\tilde{T} + Qh\tilde{T} + \tilde{Q}hT.$$
3. More mirror pairs

3.1. Other singular CY spaces

According to the analysis of [16, 3], D1 brane probes on the blowup of spaces of the form

\[ G_{kl} : \ xy = u^k v^l \]

are \( T_6 \) dual to an interval setup with \( k \) NS and \( l \) NS’ branes. The gauge group is a \( U(1)^{k+l-1} \) with bifundamental matter. It is straight forward to construct interval mirrors via the 2-10 flip in terms of a \( U(1) \) with 2 singlets and \( k+l \) flavors. The \( k+l-1 \) complexified FI terms map into the \( k + l - 1 \) independent twisted mass terms (one twisted mass can be absorbed by redefining the origin of the Coulomb branch).

Similarly we can construct diamond mirrors for D1 brane probes of \( C_{kl} \) spaces,

\[ C_{kl} : \ xy = z^k, \ uv = z^l. \]

The gauge group for the D1 brane probe is \( U(1)^{2kl-1} \). The mirror is once more a single \( U(1) \) with 2 singlets and \( k+l \) flavors. This time \( (k+1)(l+1) - 3 \) complexified FI terms map to superpotential masses. Note that, while the D1-brane gauge theory has \( 2kl-1 \) FI terms, only \( (k+1)(l+1) - 3 \) lead to independent deformations of the moduli space. This is a consequence of the fact the D1 brane gauge theory is not the minimal linear sigma model of \( C_{kl} \), which is just a \( U(1)^{(k+1)(l+1)-3} \) (the same phenomenon arises in the case of \( \mathbb{C}^3/\Gamma \) orbifolds [21]).

3.2. Generalization to non-abelian gauge groups

Our realization in terms of brane setups gives us for free the non-abelian version of the story, the mirror dual of \( N \) D1 branes sitting on top of the conifold. Let us spell out the dual pairs once more in the simple example of the conifold. Generalization to arbitrary \( G_{kl} \) and \( C_{kl} \) spaces is straight forward. The gauge group on \( N \) D1 branes on the blowup of the conifold is [18]

\[ SU(N) \times SU(N) \times U(1) \]

where we already omitted the decoupled center of mass VM. The matter content consists of 2 bifundamental flavors \( A_{1,2}, B_{1,2} \). They couple via a superpotential

\[ W = A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1. \]

The diamond mirror of this theory is a single \( U(N) \) gauge groups with 3 adjoints \(+ X, Y \) and \( Z \) and 2 fundamental flavors \( Q, \tilde{Q}, T, \tilde{T} \) coupling via a superpotential:

\[ W = X[Y, Z] + Q X \tilde{Q} + T Y \tilde{T} + h Q \tilde{T} + \tilde{h} \tilde{Q} T \]

where \( h \) and \( \tilde{h} \) are the same background parameters determining the diamond as in the abelian case.

* And here by adjoint we really mean a \( U(N) \) adjoint, that is a \( SU(N) \) adjoint and a singlet. The singlet in \( Z \) once more corresponds to the overall center of mass motion and decouples.
4. Geometric mirror symmetry from linear sigma models

The basic conjecture is that applying both dualities successively maps the LσM for a given Calabi-Yau to the LσM on the mirror. The parameter map we have presented above implies that the dual theory is formulated in terms of twisted multiplets, realizing the required flip in the R-charge.

In order to support our conjecture, let us do the calculation for the single D1 brane probe on a $C_{kl}$ space. By construction the Higgs branch of the gauge theory we start with is the blowup $C_{kl}$ space. The twisted mirror of this theory is a U(1) gauge theory coupled to $k+l$ flavors $Q$, $\tilde{Q}$ and $T$, $\tilde{T}$ and two singlet fields $X$ and $Y$. The superpotential takes the form

$$W = \sum_{i=1}^{k} Q_i (X - a_i) \tilde{Q}^i + \sum_{a=1}^{l} T_a (Y - b_a) \tilde{T}^a + \sum_{ia} Q_i h^i a T^a + \tilde{Q}^i \tilde{h}^a T_a, \quad (1)$$

where $h$ and $\tilde{h}$ are background hypermultiplets parametrizing the diamonds and the $a_i$ and $b_a$ are the relative positions of the D5 and D5' branes in the D1 D5 D5' picture along 45 and 89 respectively; $\sum a_i = \sum b_a = 0$.

According to the conjecture we now must find the ordinary mirror of this theory, whose Higgs branch, it is claimed, will be the mirror manifold. Ordinary mirror symmetry derives from 3d mirror symmetry. In three dimensions the Higgs branch of the mirror theory is the same as the quantum corrected Coulomb branch of the original one. For the purpose of computing the mirror of the $C_{kl}$ space it suffices therefore to calculate the effective Coulomb branch of the 3d U(1) gauge theory with $k+l$ flavors and superpotential eq.(1).

First let us study the classical moduli space. The D-term equations require

$$\sum_{i=1}^{k} |Q_i|^2 - |\tilde{Q}^i|^2 + \sum_{a=1}^{l} |T_a|^2 - |\tilde{T}^a|^2 = 0$$

the F-term requirements for the $Q$, $T$, $\tilde{Q}$ and $\tilde{T}$ fields are

$$N \begin{pmatrix} Q \\ T \end{pmatrix} = 0, \quad (\tilde{Q}, \tilde{T})N^T = 0$$

where $N$ is the $k+l$ by $k+l$ matrix

$$N = \begin{pmatrix} \text{diag}\{X - a_1, X - a_2, \ldots, X - a_k\} \\ \tilde{h} \end{pmatrix} \begin{pmatrix} h \end{pmatrix} \quad \text{diag}\{Y - b_1, Y - b_2, \ldots, Y - b_l\}$$

In addition the scalar potential contains the standard piece

$$2\sigma^2 \left( \sum_{i=1}^{k} |Q_i|^2 + |\tilde{Q}^i|^2 + \sum_{a=1}^{l} |T_a|^2 + |\tilde{T}^a|^2 \right)$$

from the coupling of the scalar $\sigma$ in the vector multiplet to the matter fields and the F-terms for $X$ and $Y$. The classical Coulomb branch is three complex dimensional
parametrized by $X$, $Y$ and $\sigma + i\gamma$, where $\gamma$ is the dual photon. Along this branch, $Q$, $\tilde{Q}$, $T$ and $\tilde{T}$ are zero. The Coulomb branch meets the Higgs branch along the curve $^*$

$$\det(N) = 0.$$ 

Now consider the quantum Coulomb branch. As shown in [22] the quantum Coulomb branch of a $U(1)$ theory with $N_f = k + l$ flavors has an effective description in terms of chiral fields $V_+$ and $V_-$ and a superpotential

$$W_{eff} = -N_f(V_+V_- \det(M))^{1/N_f}.$$ 

$M$ is the $k + l$ by $k + l$ meson matrix

$$M = \begin{pmatrix}
Q_i \tilde{Q}^j & Q_i \tilde{T}^b \\
T_a \tilde{Q}^j & T_a \tilde{T}^b
\end{pmatrix}.$$ 

Far out on the Coulomb branch $V_\pm$ are related to the classical variables via $V_\pm \sim e^{\pm 1/g^2 (\sigma + i\gamma)}$. Adding the tree level superpotential eq.(1) written in the compact form

$$\text{Tr } (NM)$$

to this effective superpotential, the $M$ F-term equations describing our quantum Coulomb branch read

$$N_{\beta\gamma} - (V_+V_-)^{1/N_f} \frac{H_{\beta\gamma}}{\det(M)^{1-1/N_f}} = 0 \quad (2)$$

where

$$H_{\beta\gamma} = \frac{\partial \det(M)}{\partial M^{\beta\gamma}}.$$ 

Taking the determinant in eq.(2) we find that the quantum Coulomb branch is described by a hypersurface

$$\det(N) = V_+V_-.$$ 

This is precisely the mirror manifold of $C_{kl}$ [23]. Since the origin $V_+ = V_- = X = Y = 0$ is no longer part of this branch of moduli space, we arrive at a smooth solution even so we started from the effective superpotential of [22] that is singular at the origin.

We here considered only mirror symmetry for $C_{kl}$ spaces. Since any affine toric CY can be imbedded in $C_{kl}$ for sufficiently large $k$ and $l$, mirror symmetry for all such spaces follows by deformation.

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$^*$ Note that this is the defining equation of the curve the NS5-branes wrap, the diamond [3]. It is also the defining equation of the complex structure of the local mirror manifold for the blownup $C_{kl}$, the deformed $\mathcal{G}_{kl}$, whose defining equation obtained by adding the ‘quadratic pieces’ $UV - \det(N) = 0$ which do not change the complex structure.
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