Angular correlation in $\gamma - \gamma$ coincidences: a quantitative study

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Abstract

A classical experiment that is nowadays ordinarily performed in Nuclear Physics laboratory classes is the measurement of the angular distribution of maximally correlated annihilation gamma rays radiated in coincidence, like those emitted from a $^{22}$Na source. For the first time we present an analytic expression for such angular distribution, which can be easily tested and confronted with the laboratory measurements.
1 Introduction

The study of angular correlations in gamma rays radiated in coincidence is a standard classroom experiment in undergraduate Physics courses. The counts of coincidences are given in the literature in terms of the geometric efficiency of the detectors [1], but no expression is given for such efficiency. The goal of this article is to provide an analytical expression for such efficiency in terms of variables that can be determined in the laboratory, so that a more complete comparison between the predicted and the measured counts can be made.

The article is organized as follows. In section 2 we review the main concepts associated to the experiment. We distinguish between uncorrelated and correlated pairs of gamma rays, we compute the geometrical efficiency associated to a detector and, based on that efficiency, we determine the rate of coincidences for pairs of uncorrelated gamma rays. In section 3 we compute the rate of coincidences for pairs of fully correlated gamma rays, like those emitted from a $^{22}$Na source. Finally, in section 4 we discuss further possible applications of our results.

2 Setup and description of the experiment

Consider the detection of an emission of specific gamma rays by some radioactive nucleus. The counting rate, in s$^{-1}$, of the counter associated to the detector is given by

\[ N = A \varepsilon \frac{\Omega}{4\pi}. \]  

(2.1)

Here $A$ is the number of gamma rays per second emitted by the isotropic radioactive source: it is given as the product of the source activity by the fraction of its decays that result in the gamma rays we are considering. $\varepsilon$ is the intrinsic efficiency of the detector for the corresponding gamma ray energy, and $\Omega$ is the solid angle subtended at the source by the face of the detector. One often calls the fraction $\frac{\Omega}{4\pi}$ the geometrical efficiency of the detector [2].

Indeed, we assume that the face of the detector is a circle of finite radius $r$, at a distance $d$ of the isotropic source it is facing. The fraction of events impinging on the detector over the events emitted by the source corresponds to the area of the spherical cap limited by the face of the detector (and inserted on a sphere having the source at its center) divided by the area of the whole sphere. The radius of that sphere is $R = \sqrt{d^2 + r^2}$, and its area is of course $4\pi R^2$. The area of the spherical cap is given by an elementary surface integral as $2\pi R^2 \left(1 - \frac{d}{R}\right)$ or, defining the angle $\beta = \arctan \frac{r}{d}$, $2\pi R^2 \left(1 - \cos \beta\right)$. Dividing by the area of the sphere $4\pi R^2$, we get for the geometrical efficiency in (2.1) [3]

\[ \frac{\Omega}{4\pi} = \frac{1}{2} \left(1 - \cos \beta\right). \]  

(2.2)

The value of $\beta$ can be varied by choosing the distance $d$ between the detector and the source (of course, given the restrictions in each laboratory). Typical values of $\beta$ are small. Smaller values of $\beta$ mean a larger angular resolution but a smaller geometrical efficiency.
Consider now the almost simultaneous emission of two gamma rays, each one being detected by its own detector. Depending on if their emissions are independent events or not, the respective photons are said to be *uncorrelated* or, otherwise, *correlated*. If their detections occur simultaneously, the photons are said to be *intemporal coincidence*. The relative probability that a photon will be emitted at an angle $\theta$ with respect to a previously emitted photon is called angular distribution function and denoted $W(\theta)$.

In general, when two gamma rays are emitted in succession from an atomic nucleus, their directions are correlated due to the physics of the emission process. In particular, when an excited nuclear state decays to the ground state through one or more intermediate states, the spin of the nucleus affects the angular distribution of the photons emitted during each transition. In these cases, the angular distribution function $W(\theta)$ can depend both on the spin of the states involved in the transitions and on the multipole order of the emitted radiation [4]. In the limit case when the two photons are uncorrelated, the angular distribution function is uniform and isotropic: $W(\theta) = \frac{1}{2\pi}$.

The other limit case occurs in the simple process when an isotope undergoes $\beta^+$ decay, after which the resulting positron is captured by an electron, and they both annihilate to produce a pair of 511 keV gamma rays. Because of conservation of momentum, the two photons must be emitted in exactly opposite directions, with a relative angle $\theta = \pi$. Therefore, in this case the two photons are *totally correlated* and the corresponding angular distribution function is simply given by

$$W(\theta) = \delta(\theta - \pi); \quad (2.3)$$

this is what happens with sodium-22 ($^{22}$Na).

Studying angular correlations can be very useful for the analysis of nuclear decay schemes and the assignment of spin and parity to excited nuclear states. Assuming one of the detectors is fixed and the other one can be moved along a circumference having the source at its center, the rate of coincident counts (i.e. the counts of photons in temporal coincidence) can be measured for different values of the position of the moving detector, which can in principle be identified with the angle $\theta$. Up to an overall normalization, this rate can be identified with the angular distribution function, at least if the correlation between the pair of gamma rays being considered is not very high.

Let $\epsilon_1, \epsilon_2$ be the intrinsic efficiencies and $\Omega_1, \Omega_2$ be the solid angles subtended by the faces of the two detectors associated to the coincidence counts. According to (2.1), their counting rates are respectively given by $N_1 = A\epsilon_1 \frac{\Omega_1}{4\pi}$, $N_2 = A\epsilon_2 \frac{\Omega_2}{4\pi}$.

If the pair of gamma rays being considered is uncorrelated, or if its correlation is low, the rate of “true” coincidences is given by [1, 3]

$$C_U = A\epsilon_1 \epsilon_2 \frac{\Omega_1 \Omega_2}{4\pi}. \quad (2.4)$$

The solid angles $\Omega_1, \Omega_2$ can be obtained from (2.2), assuming for each detector geometric configurations with angles $\beta_1, \beta_2$. Like equation (2.1), (2.4) contains an overall normalization, which we mean “true” coincidences, coming from events originating in the same decay. They should be distinguished from “accidental” coincidences, due to the accidental combination of two separate events from independent decays that occur closely spaced in time.
normalization factor, $A$, and probabilistic factors, the intrinsic and geometrical efficiencies. This formula evidences the fact that, since the emissions of the gamma rays are independent events, their joint probability is the product of the probabilities of each separate event. It does not have any dependence on the angle $\theta$, as it should.

For correlated photons, there is a strong angular dependence on the rate of coincidence counts, as we mentioned. In general, the rate of coincidences includes a factor depending on the correlation of the pair of photons \[3\]; such factor is quite difficult to determine in principle. But in the limit when the gamma rays are fully correlated, like when the angular distribution function is given by \[2.3\], the rate of coincidences can be worked out. That is the main goal of this article.

3 Angular distribution of coincidence counts for maximally correlated gamma rays

For the remainder of the article, we will assume the existence of two detectors. For simplicity, we will assume that the two detectors are identical (that is what typically happens in practice, although the following discussion can be generalized to detectors having different geometries). Yet, the detectors are distinguishable: one of them is fixed, and the other one can be moved along a circumference. In the center of that circumference, there is a source of $^{22}$Na, emitting pairs of gamma rays in opposite directions, with an angular distribution function given by \[2.3\]. Each of the two detectors is placed facing the $^{22}$Na source (fig. 1).

If the detectors were point-like, in order to detect a coincidence they would have to be also facing each other, in a straight line. The angular dependence of the the rate of coincidences would be similar to \[2.3\] and, up to a normalization, it could be identified with the angular distribution function, as we mentioned. But the detectors have a finite size, expressed on the finite solid angle $\Omega$ in \[2.1\]. Because of that finite size, it is possible to detect coincident photons as long as the two detectors are placed so that they can be hit by them, even if the detectors are not facing each other. To illustrate that, let’s make a small geometric digression.

![Figure 1: The fixed and the rotating detectors and definitions of $\alpha, \beta, r, d$.](image)

As we saw, each of the detectors has a face, a circle of radius $r$ at a distance $d$ to
the source. Associated to each detector there is a spherical cap, limited by its face, and corresponding to a solid angle $\Omega$. Consider now the reflection of the spherical cap of the moving detector across the plane that passes by the source and is perpendicular to the fixed detector. If the moving detector is in its original position, facing the fixed detector, opposite to it, the reflected spherical cap of the moving detector will coincide with the original spherical cap of the fixed detector. The counts in each of the detectors should be given by (2.1), and they should match the number of coincidences, since each count would correspond to a coincidence. But if one rotates the moving detector by some angle $\alpha$ around a circumference of radius $d$ (so that its distance to the source remains constant), the positions of the reflected and the original spherical caps no longer coincide. If the angle $\alpha$ of displacement of the moving detector is smaller than a critical value, the intersection of the two spherical caps is not empty. In this configuration, for each detector there are photons hitting it that do not reach the other detector. From (2.1), the probability that a photon emitted from the source hits the detector is given by the geometrical efficiency $\frac{\Omega}{4\pi}$. The probability that each of the two fully correlated photons hits a detector, therefore forming a coincidence, is therefore, by the same reasoning, given by

$$C_C = A\epsilon_1\epsilon_2 \frac{\Delta\Omega}{4\pi},$$

(3.5)

$\Delta\Omega$ being the solid angle corresponding to the intersection of the original (fixed) and the reflected (rotating) spherical caps. $\frac{\Delta\Omega}{4\pi}$ is the geometrical efficiency corresponding to the counting of coincidences. This is what we wish to compute, in terms of geometrical variables which can be determined in the laboratory.

For values of $\alpha$ larger (modulo $2\pi$, of course) than a critical value, the intersection of the two caps becomes empty. That critical value of $\alpha$ can be easily determined from fig. 2; it is given by

$$\alpha_{\text{crit}} = 2\beta.$$  

(3.6)

$\Delta\Omega$ (and the number of coincidences) reach a maximum when $\alpha \equiv 0$ and a minimum (0) when $\alpha \equiv \alpha_{\text{crit}}$, always modulo $2\pi$.

Despite the area in which we are interested being embedded into a sphere, the problem of calculating it does not have spherical symmetry. Indeed, such symmetry is broken by the existence of an axis around which one of the detectors rotates. Therefore spherical coordinates are not the most adequate for this case. But one can use normal cartesian coordinates. We take the plane of rotation as the $x-y$ plane, with the source located at the origin. The rotation takes place therefore around the $z$ axis. Initially the two detectors are facing each other. One then rotates the moving detector by an angle $\alpha$ and so does the spherical cap it defines. For convenience, we place the center of the face of the fixed detector at the position $(d\sin\left(\frac{\alpha}{2}\right), -d\cos\left(\frac{\alpha}{2}\right), 0)$, while the center of the one of the moving detector is at $(d\sin\left(\frac{\alpha}{2}\right), d\cos\left(\frac{\alpha}{2}\right), 0)$. The center of the reflected face of the moving detector is at $(-d\sin\left(\frac{\alpha}{2}\right), -d\cos\left(\frac{\alpha}{2}\right), 0)$. In fig. 2 we can see the projection in the $x-y$ plane of the intersection of the original fixed and of the reflected moving spherical caps. We want to compute the area of such intersection.

It is straightforward to figure out that points $P, Q$ in fig. 2 have coordinates

$$P = \left(-R\sin\left(\beta - \frac{\alpha}{2}\right), -R\cos\left(\beta - \frac{\alpha}{2}\right), 0\right),$$

and
Figure 2: The fixed and the rotating spherical caps and their intersection. The original (fixed) cone is drawn in full line, while the reflected (rotating) cone is dashed.

\[ Q = \left( R \sin \left( \beta + \frac{\alpha}{2} \right), -R \cos \left( \beta + \frac{\alpha}{2} \right), 0 \right), \]

from which we get the equation of the plane, perpendicular to the \( x-y \) plane, which passes through these two points (a straight line in fig. 2):

\[ y = \tan \left( \frac{\alpha}{2} \right) \left( x + R \sin \left( \beta - \frac{\alpha}{2} \right) \right) - R \cos \left( \beta - \frac{\alpha}{2} \right). \]

The surface whose area we wish to compute can be split into four equal parts. The first one can be seen as the surface plot of \( Z(x, y) = \sqrt{R^2 - x^2 - y^2} \) (positive \( z \) coordinate, above the \( x-y \) plane), with \( x-y \) range defined in the third quadrant \((x < 0, y < 0)\) as

\[ -\sqrt{R^2 - x^2} \leq y \leq \tan \left( \frac{\alpha}{2} \right) \left( x + R \sin \left( \beta - \frac{\alpha}{2} \right) \right) - R \cos \left( \beta - \frac{\alpha}{2} \right), \]

\[ -R \sin \left( \beta - \frac{\alpha}{2} \right) \leq x \leq 0. \]

The second part is given as the surface plot of the same function, but with \( x-y \) range defined in the fourth quadrant \((x > 0, y < 0)\). This range is analogous to the one of the first part, but reflected around the \( y \) axis. The third and fourth parts have the same \( x-y \) ranges of the first and second part, respectively, but they are the surface plots of \( Z(x, y) = -\sqrt{R^2 - x^2 - y^2} \) (negative \( z \) coordinate, above the \( x-y \) plane). They are the reflections of the first and second parts around the \( x-y \) plane. Clearly the areas of the four parts are equal, and we can take for the total area four times the area of the first part, given by the following surface integral:

\[ 4 \int_{-R \sin \left( \beta - \frac{\alpha}{2} \right)}^{0} \int_{-\sqrt{R^2 - x^2}}^{\tan \left( \frac{\alpha}{2} \right) \left( x + R \sin \left( \beta - \frac{\alpha}{2} \right) \right) - R \cos \left( \beta - \frac{\alpha}{2} \right)} ddy dx \tag{3.7} \]
Because we are only interested in the solid angle, we can simply take \( R = 1 \) in the previous formula, obtaining:

\[
\Delta \Omega = 4 \int_{-\sin(\beta - \frac{\alpha}{2})}^{\tan(\frac{\alpha}{2})} \left( x + \sin(\beta - \frac{\alpha}{2}) \right) \cos(\beta - \frac{\alpha}{2}) \frac{1}{\sqrt{1 - x^2 - y^2}} dy \ dx \quad (3.8)
\]

After performing the integrations, the final result is given by

\[
\Delta \Omega = 4 \left( \arccot \left( \frac{\sqrt{2} \sin \frac{\alpha}{2}}{\sqrt{\cos \alpha - \cos 2\beta}} \right) - \arccot \left( \frac{\sqrt{2} \cos \beta \sin \frac{\alpha}{2}}{\sqrt{\cos \alpha - \cos 2\beta}} \right) \cos \beta \right), \quad (3.9)
\]

with \( \arccot(x) = \frac{\pi}{2} - \arctan(x) \) being the inverse cotangent function.

We can check that the result \((3.9)\) for \( \Delta \Omega \) has some properties that one should expect. It is a periodic function, with period \( 2\pi \), for the values of \( \alpha \) where it is defined. On a neighborhood of \( \alpha = 0 \), it is defined (and positive) only for \( \alpha \leq 2\beta \). It vanishes for \( \alpha = 2\beta \), the critical value \( \alpha_{\text{crit}} \) from \((3.6)\). It is not defined as a real function in the intervals \( 2\beta < \alpha < 2\pi - 2\beta \) and \( -2\pi + 2\beta < \alpha < -2\beta \), but that does not have any physical or geometrical meaning: for those values of \( \alpha \), the rate of coincidences should be 0. For \( \alpha = 0 \) (no rotation of the moving detector), \((3.9)\) reduces to \((2.2)\), as it should. In the limit \( \beta \to 0 \), when the detectors become point-like, \((3.9)\) reduces to \((2.3)\) with \( \alpha = \pi - \theta \); in this limit, the angular resolution of the detectors is the highest, and the rate of coincidences matches the angular distribution function for the given source, as we saw.

In the book [1] there is a detailed presentation of the experiment of measuring the angular correlations in \( \gamma - \gamma \) coincidences, focusing on the \( ^{22}\text{Na} \) source, including a complete description of the experimental apparatus. One can also find there estimates of the rate of accidental coincidences and of coincidences due to other uncorrelated gamma rays emitted by the same source (the 1.277 MeV gamma ray from the decay of \( ^{22}\text{Ne} \)), showing that they are both neglectable when compared to the rate of coincidences we have been considering. In the same reference, and also in [5], one can find tables and plots of experimental results of the rate of coincidences as a function of the angle. In future experiments, these results can be fitted to the expression \((3.9)\) we have derived.

4 Further applications and conclusions

In this article, we have derived expression \((3.9)\) which, together with \((2.4)\), gives us the rate of coincidences for pairs of fully correlated gamma rays like those emitted by a \( ^{22}\text{Na} \) source. Although the final result \((3.9)\) seems simple, the calculation associated to it is quite nontrivial and was not yet available in the literature.

Previously, one could just measure the rates of counts at different angles, in order to determine the angular distribution of maximally correlated annihilation gamma rays radiated in coincidence and check that there was a peak at the central angle. Having obtained \((3.9)\), one can actually fit the theoretical rate of coincidences \((2.4)\) to experimental data, in order to determine the source activity and the intrinsic efficiencies of the detectors, and compare the results with others obtained from different methods.
A very complete discussion on the determination of the intrinsic efficiency can be found in \[6\].

Throughout this article we always considered a point-like source; a case we did not consider was that of a source with finite size. This study is presented in \[3\] for the counting rate of a single detector: the result is given in terms of Bessel functions. Extending such study to the rate of coincidences we have studied could be the topic of a future project. But, in general, considering a point-like source is a good approximation, and allows for many relevant measurements to be made.

A totally different application of eq. (3.9) could be in the field of Gamma-Ray Astronomy. Compton telescopes use pixel detectors in an efficient way: each unit acts both as a scattering and as detection elements. In theory, in a Compton process there are two ways of obtaining the Compton angle \( \vartheta \): the “theoretical” value \( \vartheta_e \) obtained from the Compton equation, and the direct measurement \( \vartheta_m \). \( \vartheta_e \) is calculated by the measured energy deposits of the recoil electron and the scattered photon according to the Compton scattering kinematics; \( \vartheta_m \) is calculated from measured interaction positions and the direction of the incident gamma ray. In theory, \( \vartheta_e \) and \( \vartheta_m \) should coincide; in practice, their difference \( \vartheta_e - \vartheta_m \) is called the angular resolution measure (ARM) \[7\]. The angular resolution of the telescope is determined by the FWHM of its ARM distribution, which has a well-shaped peak. That distribution can be modeled using eq. (3.9).

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