On limitations of learning algorithms in competitive environments

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Abstract
We discuss conceptual limitations of generic learning algorithms pursuing adversarial goals in competitive environments, and prove that they are subject to limitations that are analogous to the constraints on knowledge imposed by the famous theorems of Gödel and Turing. These limitations are shown to be related to intransitivity, which is commonly present in competitive environments.

Keywords:
learning algorithms, competitions, incompleteness theorems

1. Introduction
The idea that computers might be able to do some intellectual work has existed for a long time and, as demonstrated by recent AI developments, is not merely science fiction. AI has become successful in a range of applications, including recognising images and driving cars. Playing human games such as chess and Go has long been considered to be a major benchmark of human capabilities. Computer programs have become robust chess players and, since the late 1990s, have been able to beat even the best human chess champions; though, for a long time, computers were unable to beat expert Go players — the game of Go has proven to be especially difficult for computers.

While most of the best early AI game-playing agents were specialised to play a particular game (which is less interesting from the general AI perspective), more recent game-playing agents often involve general machine learning capabilities, and sometimes evolutionary algorithms [1, 2]. Since 2005, general game playing was developed to reflect the ability of AI to play generic games with arbitrary given rules [3]. Conceptually, the game model associated with the general game playing resembles the schematisation of algorithms introduced by Turing machines. In 2016, a new program called AlphaGo finally won a victory over a human Go champion, only to be beaten by its subsequent versions (AlphaGo Zero and AlphaZero). AlphaZero proceeded to beat the best computers and humans in chess, shogi and Go, including all its predecessors from the Alpha family [4]. Core to AlphaZero’s success is its use of a deep neural network, trained through reinforcement learning, as a powerful heuristic to guide a tree search algorithm (specifically Monte Carlo Tree Search).

The recent successes of AI in playing games are good reason to consider the limitations of learning algorithms and, in a broader sense, the limitations of AI. In the context of a particular competition (or “game”), a natural question to ask is whether a comprehensive winner AI might exist — one that, given sufficient resources, will always achieve the best possible outcome. Following the typical modus operandi of reinforcement learning, such as implemented in AlphaZero, we distinguish “learning algorithms” that require very large computational resources, and “playing programs” that have relatively modest computational needs — learning algorithms and playing

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1While relative strength of these programs has not been tested after Stockfish upgrades (AlphaZero is not publicly available), the open-source chess engine Stockfish, which incorporates elements of machine learning since version 12 of 2020, has the highest historical ELO ratings at present [5].
programs perform different roles and operate under different conditions. In this context, we focus on conceptual limitations of general learning algorithms (rather than the actual game dynamics), which are rooted in the availability of information about one’s opponents.

2. Examples of transitive and intransitive games

The outcomes that can or cannot be achieved in a particular game depend on the type of the game and the scope of information that is known about the strategies used by the opponent. It appears that, for the purposes of our analysis, numerous properties of various games can be reduced into two types of games. In this section, we illustrate these two types by basic examples shown in Figure 1. These examples are simple symmetric adversarial games, where two opponents can use the same strategies against each other.

The “dice game” shown in Figure 1(a) is fully transitive — its strategies form a transitive sequence \( 1 \prec 2 \prec 3 \prec 4 \prec 5 \prec 6 \), where “\( \prec \)” indicates a winning strategy and \( a \prec b \prec c \) always implies \( a \prec c \) when the binary relation “\( \prec \)” is transitive. It is obvious that there exists a best possible strategy, \( 6 \), which wins or at least ensures a draw. The rock-paper-scissors game shown in Figure 1(b) is strongly intransitive since “rock”, “paper” and “scissors” form an intransitive triplet \( R \prec P \prec S \prec R \). Obviously, there is no simple winning strategy in this game. The outcome of this game, however, strongly depends on whether a player is aware of the strategy used by their opponent: if player 1 knows the strategy of player 2, the first player can win easily.

Figure 1: Examples of simple symmetric games: a) fully transitive: dice game, b) strongly intransitive: rock-paper-scissors game. The arrows point from losing to winning strategies.

Although the dice game is not significantly affected by information about the opponent (strategy \( 6 \) is always the best in Figure 1(a)), knowing more about the opponent might still be practically useful. Indeed, a player needs to examine \( 6^2 = 36 \) possible outcomes but, if the strategy of the opponent is known, can reduce this number to 6. Even if the exact opposing strategy is not known, the experience gained by playing against opponents is still useful: we may be able to focus our exploration on strong strategies (e.g. \( 4 \), \( 5 \) and \( 6 \) in the dice game). If the game is transitive but complicated, this can reduce the time and effort needed for strategy optimisation. If the game is intransitive, the advantage given by information about the opponent becomes crucial for success. Presence of intransitivity in competitive environments is a common source of systemic and evolutionary complexity [9], and this tends to increase the importance of information about one’s competitors. In general, more realistic games can involve various mixtures of transitive and intransitive rules. In this case, strategies that are a priori deemed to be transitively superior can be unexpectedly beaten by seemingly weak but intransitively effective alternatives.

As noted above, games can be played by computer programs at the levels that routinely exceed human abilities. Intransitivity is common in competitions among programs. According to CCRL website [5], the result matrix of the 12 leading chess engines is intransitive. For example, the
While the chess rules allow, at least in principle, for transitive dominance of the best algorithms, predicting winners can be even more difficult when game rules are explicitly intransitive. The idea that difficulties of predicting winners in intransitive competitions is related to algorithmic insolvability of the halting problem was first suggested by Pudlak [10] [7]. The present work proves that, in the context of learning algorithms, insolvability of the halting problem is indeed related to intransitivity and this imposes conceptual limitations on the extent of learning that can be mastered by algorithms in competitive environments.

Considering learning algorithms playing a game against each other, we note that the character of the game and information available about the opposing algorithm are major factors affecting the outcome of the game. These factors are taken into account in more accurate definitions of competitions between algorithms, which are introduced further in the paper.

3. Learning algorithms and playing programs

Let AI be represented by a learning algorithm $L$, which is run on computers that are practically unconstrained in terms of resources and speed, to produce $L \rightarrow p$ a playing program $p$, which is then run in real time with limited resources. Due to these limitations, one can assume that there is a large but finite number of possible programs. The learning algorithm is understood in a general way. For example, algorithm $L$ can simply print a certain program $p$, or $L$ may perform extensive calculations to optimise its output. At this stage we consider only algorithms that produce output and halt.

Adversarial competition between $p'_i$ and $p''_j$ (which are selected from a finite set of allowed programs $p'_1, \ldots, p'_n$, and $p''_1, \ldots, p''_m$) is evaluated for a particular game by computable algorithm $C$. The simplest way to encode a program $p_i$ is by its number $i$, although $p_i$ can also be a program in any computer language that can be executed by $C$. The rules of the game are reflected by $C$, and deemed to be known to the learning algorithms, i.e. $L' = L'[C] \rightarrow p'_i$ and $L'' = L''[C] \rightarrow p''_j$. For our purposes, it is sufficient to consider zero-sum win/draw/loss two-player games, i.e. in which $C$ has the form

$$C(p'_i, p''_j) = \begin{cases} +1, & p'_i \succ p''_j, \text{ (win for player 1)} \\ 0, & p'_i \sim p''_j, \text{ (draw)} \\ -1, & p'_i \prec p''_j, \text{ (loss for player 1)} \end{cases}$$

As long as the resource allocations are sufficient, any finite game of this category can be represented by such an algorithmic function $C(\_, \_)$; thus we can consider $C(\_, \_)$ to itself be a definition of a game. The specific form of the game rules — with perfect or imperfect information, with simultaneous or sequential moves, etc. — does not affect our interpretation. As schematically illustrated in Figure 1 and accurately defined below, we only need to distinguish two types of games (although these types are, of course, affected by the game rules: for example, imperfect information tends to stimulate intransitivity — see ref. [8]). The algorithm $C$ allows only limited time (or a limited number of steps) for execution of the programs. For example, if a program fails to make a move within given time limits, it might immediately be judged as the loser; the time limits, however, do not apply to the learning stage. Hence, each program is allowed to execute only a finite number of steps (determined by the time limit) and the game has only a finite number of states; resource limitations imply that the size of the programs must be finite (and, perhaps, limited by the game rules); therefore, the numbers of possible programs $n'$ and $n''$ are large but finite.

Irrespective of the specific nature of the game, the competition can be fully specified by a finite $n' \times n''$ table $C_{ij} = C(p'_i, p''_j)$, which is computable for the learning algorithms. It is easy to see that each program $p'_i$ can be interpreted as strategy number $i$ out of the set of $n'$ possible game strategies, while $p''_j$ is just the strategy number $j$ out of the set of $n''$ strategies. Indeed the table $C_{ij}$ is simply a normal-form representation of the game. A Turing machine can easily
compute $C(p_i', p_j')$, for example, by selecting $C_{ij}$ from the game table. In the special case of a symmetric game, the two sets of strategies are the same $n' = n'' = n$, $p_i' = p_i'' = p_i$, while the payoff matrix must be antisymmetric: $C_{ij} = -C_{ji}$. A symmetric game, obviously, requires that $C_{ii} = C(p_i, p_i) = 0$, but in asymmetric games $C(p_i, p_i) \neq 0$ is possible (assuming that strategy $p_i$ is available to the first and to the second player). For example, the outcome $C(p_i, p_i) = 1$ might be common in a game with a first-mover advantage (though the order of players in $C(\cdot, \cdot)$ is down to convention).

Since each program $p_i$ can be interpreted as a game strategy, the outcomes of competition between these pure strategies can be defined in terms of the classical game theory for zero-sum games with pure strategies. This game may or may not have a Nash equilibrium in pure strategies. For our purposes, it is sufficient to consider two alternatives: A) a game with transitive domination, which must have a pure strategy Nash equilibrium and B) a strongly intransitive game, which cannot. In the case with transitive domination, there is a strategy $p_i'$ that is transitive superior over all opposing strategies: $p_i' \succ p_j'$ for all $j$ (without loss of generality, we assume that player 1 has this strategy). Note that in a symmetric game, strict transitive domination $p_i' \succ p_j'$ is not possible since $p_i' = p_d \sim p_d = p_j'$. The game is strongly intransitive if each strategy $p_i'$ has at least one strategy $p_j'' = S' (p_j')$ (dependent on $p_j'$) such that $p_i' \prec p_j''$, and each strategy $p_j''$ has at least one strategy $p_i'' = S'' (p_i')$ (dependent on $p_i'$) such that $p_i'' \succ p_j''$. The functions $S'(\cdot)$ and $S''(\cdot)$ are “best response” functions, which pick out (potentially out of several winning options) a player’s best answer to a fixed strategy from their opponent. In general $S'(\cdot)$ and $S''(\cdot)$ are different functions, but one can choose $S'(\cdot) = S''(\cdot)$ in a symmetric game.

It is obvious that a learning algorithm can guarantee victory as player 1 (or at least a draw if the game is symmetric) in the transitive case (A), as long as it finds the transitive superior strategy $L' \rightarrow p_i'$, but any learning algorithm can be defeated in the intransitive case (B): $L' \rightarrow p_i'$ is defeated by $L'' \rightarrow S''(p_i')$, which is in turn defeated by $L' \rightarrow S'(S''(p_i'))$, which is defeated by $L'' \rightarrow S''(S'(S''(p_i')))$. Of course, if one were to allow random mixing of strategies, an algorithm supplemented by a (quantum) random generator might be able to eke out some statistical edge by playing a mixed-strategy Nash equilibrium, but this is far from winning every game that in principle can be won.

One can also consider the case of playing multiple games, when a program $p_i' (i = 1, \ldots, n')$ competes against $p_j' (j = 1, \ldots, n'')$ in the games $C_{ij}^{(1)}, \ldots, C_{ij}^{(m)}$ of $m$ games with $n' \times n''$ possible payoffs. The outcome of the series (win, draw or loss) is given by an $n' \times n''$ overall payoff matrix $C_{ij}$ that depends on the outcomes of each game $C_{ij}^{(1)}, \ldots, C_{ij}^{(m)}$ as defined by the conditions of the series. Note that the number $m_n$ of different games $\mathbb{1}$ with $n' \times n''$ payoffs is finite and cannot exceed $m_n \leq 3^{n' \times n''}$, therefore $C_{ij}$ must be one of these $m_n$ games. Although playing multiple games can make the structure of the effective payoff matrix $C_{ij}$ more complicated and therefore increase practical difficulties experienced by the competitors, this does not impose any new principal constraints on our analysis.

### 4. Learning algorithms in open-source competitions

There is, however, a more interesting case, in which the learning algorithms can receive information about the strategies played by their opponents. The case when the first algorithm $L'$ knows the opposing program $p_j'$ (that is, $L'[p_j'] \rightarrow p_i'(p_j')$ but $L'' \rightarrow p_i''$ does not depend on $p_i'$) gives a very significant advantage to $L'$ that, obviously, can be translated into optimal strategies $p_i'(p_j')$, even for intransitive games. We are interested in the more complicated case in which the learning algorithms are placed in symmetric or comparable conditions. Under these conditions, the two opposing learning algorithms $L'$ and $L''$ receive each other’s source code and then train two opposing programs $p_i'$ and $p_j'$ that compete in real time. This implies that $L'[L''] \rightarrow p_i'(L'')$; that is, $L'$ has $L''$ as an input to produce $p_i'$ depending on $L''$, and $L''[L'] \rightarrow p_j'(L')$, i.e. $L''$ has $L'$ as an input to produce $p_j'$ depending on $L'$. This information exchange is fully symmetric, although the game, which is determined by $C(p_i', p_j')$, may be symmetric or asymmetric. Note that it is generally unknown whether a particular algorithm can halt and produce an output for a particular input.
These learning algorithms that compete in open-source competitions can be formally specified using a universal Turing machine, $U(M)[D_0] \rightarrow D_1$. Here, the universal Turing machine $U$ has header $M$, is applied to input $D_0$ and produces output $D_1$. The learning algorithms associated with open-source competitions are given by $U(L')[C, L''] = p'_L$ and $U(L''[C, L'] \rightarrow p''_L$, where the computable algorithmic function $C = C(p'_L, p''_L)$ (or the corresponding game payoff table $C_{ij}$) is emphasised to be available to both competing Turing machines. The two algorithms implemented by the universal Turing machine $U$ represent two adversarial goals and cannot be reduced to a single algorithm. Therefore it is possible to consider learning algorithms that are expected to play any game with number of payoffs not exceeding $n' \times n''$.

An algorithm $L'$ can win over algorithm $L''$ by either producing a winning program $L'[L''] \rightarrow p'_L(L'') \triangleright p''_L(L')$, or by producing an output $L'[L''] \rightarrow p'_L(L'')$ and demonstrating that the competing algorithm does not halt $L''[L'] \rightarrow \emptyset''$ (at this point our analysis assumes existence of an agreed axiomatic system that determines correctness of demonstrations). Therefore, at least some of the algorithms that fail to halt (e.g. a computer program with an infinite loop) can be defeated. Note that the outcome of the competition may remain undecided (e.g. when both algorithms run indefinitely).

Can a learning algorithm, say $L'$, defeat all opposing algorithms when information about the algorithms is exchanged? In the case of an asymmetric game with transitive domination (A), any opponent that halts and produces an output $p''_L$ is defeated by: $L'[L''] \rightarrow p'_L \triangleright p''_L$ — the winning algorithm $L'$ simply ignores its input $L''$ and selects $p'_L = p''_L$. The case of strongly intransitive competition (B) is, by contrast, quite a bit more complicated. In this case, any strategy $p''_L$ that $L'$ produces could (at least in principle) be beaten by an opponent that outputs $S''(p''_L)$: thus an analogous algorithm that ignores its input seems especially foolish.

More specifically, we consider an algorithm $L'$ to be a universal winner for a given game $C(...)$ when it can defeat every opposing algorithm $L''$. At this point we can formulate the following theorem

**Theorem 1.** Any algorithm competing in an open-source competition associated any strongly intransitive game cannot be a universal winner.

**Proof.** Indeed, let algorithm $L' = L_w$ be such a universal winning algorithm that can defeat any opposing algorithm $L''$. The winning algorithm $L_w[L'']$ must always halt for any $L''$, and must produce an output $L_w[L''] \rightarrow p'_L(L'')$ dependent on $L''$. Consider the following $L''$: it runs $L_w[L'']$ to obtain its output, $p''_L = p'_L(L'')$, and then selects $p''_L = S''(p''_L)$, which defeats $L_w$. Hence $L_w$ is defeated by at least one algorithm $L''$ and cannot be a universal winner. 

\[ H(L, D) = H(L[D]) = \begin{cases} 1, & \text{when } L[D] \text{ halts} \\ 0, & \text{when } L[D] \text{ does not halt} \end{cases} \] (2)

that uses the agreed axiomatic system to determine whether algorithm (program) $L$ halts or runs forever when applied to input data $D$. We can show that the impossibility of an algorithm implementing the universal halting function follows from our considerations.

**Theorem 2.** (Turing) A universal computable halting function $H(L, D) = H(L[D])$ does not exist — it cannot be computable for all algorithms $L$ and all data sets $D$.  

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Proof. If the halting function \( H(\ldots) \) is universally computable, one can easily construct a universal winning algorithm \( L' = L_w \):

\[
L_w[L''] : \begin{cases} 
H(L'', L_w) = 0, & \rightarrow p'_1 \\
H(L'', L_w) = 1, & \rightarrow S\left(p''_j(L_w)\right)
\end{cases}
\]

for a selected strongly intransitive game \( C(\ldots) \). That is, if \( L_w[L''] \) determines that \( L''[L_w] \) runs forever, then \( L_w \) can print step-by-step execution of \( H(\ldots) \) to demonstrate \( H(L'', L_w) = 0 \), produce any output, say \( L_w[L''] \rightarrow p'_1 \), halt and declare its victory. If \( L_w[L''] \) determines that \( L''[L_w] \) halts and \( L''[L_w] \rightarrow p''_j(L_w) \), then \( S\left(p''_j(L_w)\right) \) is selected. Since a universal winning algorithm cannot exist according to theorem 3 this also prohibits the existence of a universal halting function and proves theorem 2.

The halting problem is also related to the incompleteness theorem of G"odel, which, in the context of our consideration, points to the existence of correct but unprovable statements (e.g. the statement “\( L[D] \) runs forever”). It appears that incompleteness of formal systems is promoted by intransitivity, at least under conditions of competitive environments considered in theorems 1 and 2. In simple terms, intransitivity makes our knowledge incomplete and relativistic.

6. Failure to halt as a game strategy

In the context of the competitions considered here, the incompleteness theorems allow us to broaden the statement of theorem 1.

Theorem 3. Any algorithm competing in an open-source competition cannot be a universal winner.

Proof. Let us assume that algorithm \( L' = L_w \) is such a universal winning algorithm. Due to theorem 2 this algorithm cannot implement a universal halting function and there must exist an algorithm \( L_h \), whose halting status remains unknown when applied to some data \( D_h \) (note that \( L_h \) and \( D_h \) may or may not have some direct relevance to the current game). Hence, the algorithm \( L''[L_w] \) that ignores \( L_w \) and executes \( L_h[D_h] \) is not defeated by \( L_w[L''] \). This leads us to a contradiction.

Note that the universal winner \( L_w \) is shown to be defeated in the intransitive conditions of theorem 1 by a known, specific strategy of the opposition, \( L''[L_w] \rightarrow p''_j = S\left(p''_i\right) \); while \( L_w \) only fails to defeat \( L'' \) in theorem 3. In the latter case, \( L'' \) executes the algorithm \( L_h[D_h] \), which is only known to exist but not explicitly specified or nominated. Generally, both \( L'' \) and \( L' \) might not know whether a specific \( L_h[D_h] \) halts or not since, if the fact that \( L_h[D_h] \) does not halt is known to \( L'' \) using \( L_h \), then \( L' \) might be able to demonstrate this and win. Hence, competitors may have to use algorithms without complete knowledge of their performance. The abstract strategies used to prove theorem 3 may have more relevance to the real world than one might think. For example, the weaker side \( L'' \) may realise that it would lose to \( p'_i = p'_d \) with any strategy \( p''_j \); it can muster, and simply refuse to halt \( L''[L'] \rightarrow \mathcal{O}'' \), while hiding its intentions from the opposition so that \( L' \) cannot prove that \( H(L'', \ldots) = 0 \). Does this, perhaps, resemble the situation in the last US presidential elections?

One can see that the logic of the adversarial game converts a failure to solve the problem and halt into a new game strategy, \( \mathcal{O}'' \), assuming that \( H(L'', \ldots) \) is not computable. Even if \( p''_j \) is transitivity dominant, the strategy \( \mathcal{O}'' \) introduces some intransitivity into the game since \( p''_j > p''_i = p'_d \) (with some suitable selection of \( j \) and \( i \)), but \( p''_j \sim \mathcal{O}'' \sim p'_i \), although this intransitivity is weaker than that in strongly intransitive games. Intransitivity imposes limitations on universal winning strategies, which, as determined by theorems 1 and 2, are associated with the impossibility of universal computable halting functions.
7. Discussion and conclusion

Considering the capacity of learning algorithms to learn and adapt to succeed in competitions, we note the importance of information about one’s competitors. One-sided availability of such information makes competition highly uneven. It is not a surprise that, given both substantial computational resources and full information about a particular opponent, a good learning algorithm should be able to defeat this opponent (assuming, of course, that this is permitted by the game rules). We demonstrate, however, that if information is exchanged pari passu with the opposition, one’s ability to manage the outcomes of adversarial games is necessarily limited by algorithmic incompleteness, and no learning algorithm can become a universal winner in complex competitions, especially when the relevant competition rules are intransitive. In the context of adversarial games, we do not presume algorithmic incompleteness by invoking the incompleteness theorems, but demonstrate incompleteness in game conditions by using the potential intransitivity of competitive environments. Our consideration involves a standard, Turing-like interpretation of computer algorithms but in conditions when the algorithms are subordinated to the presence of conflicting goals pursued by the competitors.

A basic learning algorithm to evolve a chess engine towards better performance can be created without major difficulties, but achieving successful learning under intransitive arrangements appears to be much more problematic [2]. A major question related to AI is whether AI can perform well in more complex and uncertain situations, especially when significant intransitivity is present in a competitive environment. Our analysis illustrates that mere existence of intransitivity under these conditions is sufficient for demonstrating incompleteness of our knowledge. We show that no algorithm can become undefeatable and exercise full control over an intransitive competitive environment by subjugating all other agents to its own goals, unless this algorithm is advantageously benefited by asymmetric availability of information or resources. Good learning algorithms should be capable of converting knowledge about the opposing algorithms into a significant advantage. The impossibility of controlling competitive environments in intransitive conditions points to emergence of complexity, although the question of whether AI systems can reach the higher levels of complexity associated with known complex evolutionary systems (e.g. that of biological, social and technological systems, and of human intelligence and organisation) remains open.

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