An Image Enhancement Method Based on Partial Differential Equations to Improve Dark Channel Theory

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Abstract. Image applications appears in all aspects of life, such as medical imaging, electronic monitoring, face recognition, aerial photography remote sensing, fingerprint analysis, robot vision, automatic driving, intelligent car and so on in the civil field, as well as tanks, armored vehicles, artillery, missiles, bat submunitions in the military field. However, in the process of image acquisition or storage and transportation, the image quality will decline due to environmental conditions, equipment conditions or transmission compression, and so on. In order to accurately analyze the image, we need to carry out enhancement processing. In this paper, the current research status of image enhancement methods is analyzed. The traditional dark channel prior algorithm has some problems, such as edge or texture blur and so on. The processing effect is not ideal. Considering the limitations of the traditional dark channel prior algorithm, this paper proposes a improved algorithms based on the dark channel prior by combining partial differential equation. The main work includes the following aspects. First, the partial differential equation algorithm is combined with the dark channel prior method, and the improved algorithm is applied to enhance the image with fog. The contrast experiment of enhancement effect and running time is designed to verify the image enhancement effect of the improved algorithm. Second, the improved algorithms proposed in this paper are compared with traditional algorithms. After making the subjective and objective evaluation, it is found that the improved method in this paper has better image processing effect, stronger fog and noise removal effect, better edge retention effect, more information acquisition, and higher computational efficiency.

Keywords: Image Enhancement, Edge Preserving, Dark Channel Prior, Partial Differential Equation.

1. Introduction

1.1. Purpose and significance of the study
In life, images are present in all aspects of people's lives, and the application of image information can be seen in all corners of our lives, such as medical imaging, electronic surveillance, facial recognition, fingerprint security authentication, remote sensing aerial photography, robot vision and intelligent car, automatic driving, and other aspects.

In order to enhance the credibility of electronic image information, it is necessary to use some methods to make its image quality increase. With the increase of hazy weather in China in recent years, this has become a hot topic of national discussion. In this paper, we will conduct a series of studies on
the defogging process in image enhancement, and design algorithms to achieve the defogging enhancement of images.

1.2. Status of research on image enhancement methods

At present, scholars have applied two types of methods, globalized image enhancement methods and localized image enhancement methods.

1.2.1. Status of research on globalized image enhancement methods. The first category is the globalized image enhancement method [1], the core principle of this method is to make the enhancement of the image by adjusting the grayscale value of the degraded image. Currently, the following six enhancement methods are mainly used.

(1) Retinex algorithm. The fundamental idea of this algorithm is the model of color constancy in the theory of color constancy, reducing the influence of the spectral component of external illumination on the model, that is, there is no correlation with the illumination, but only through the light reflection nature of the impact of change.

(2) Global histogram equalization algorithm. The core principle of the global histogram equalization algorithm lies in its ability to turn the probability distribution of gray values of degraded image pixels with fog into a uniform distribution, thus expanding the dynamic range of gray values of pixels and widening the gray areas that contain a large amount of image information [2], by which the purpose of image contrast enhancement is achieved.

(3) Homomorphic filtering algorithm. The core concept of this algorithm is based on the illumination reflection model, which divides the pixel values of the image into two, the low-frequency and high-frequency regions of the irradiation component [3].

(4) Wavelet transform algorithm. We can decompose the image and transform the image to be enhanced into components with multiple scales and resolutions, after the transformation of components, we can target the different image degradation areas to be processed, and enlarge or reduce the components related to their degradation causes. After the processing is completed, the components are then inverted to obtain the output image with enhanced image quality [4].

(5) Curvilinear wave transform algorithm. The curved wave transform [5] algorithm is a method that Donono came out in 1999, and in a sense it is also a multi-scale image enhancement method. The principle of this method is to threshold the transformed image coefficients so that the edges of the image are enhanced, and in this method we can also achieve a relatively satisfactory image enhancement denoising effect.

(6) Algorithm based on the atmospheric modulation transfer function [6]. The basic principle of this algorithm is to improve the algorithm by using the atmospheric modulation transfer function as a model, so as to achieve the purpose of enhancing the image of foggy sky degradation.

1.2.2. Status of research on localized image enhancement methods. The core principle of localized image enhancement is to transform or transfer the function for the different characteristics of each local area of the image, and in this way, we can achieve the purpose of image enhancement by targeting the local enhancement of the image. The current research status of localized image enhancement mainly has the following two types of methods.

(1) Local histogram equalization algorithm [7]. The core principle of this algorithm is to segment the image, thus converting the problem of processing the original image into a problem of processing different small parts of the image, and we can perform histogram equalization for each small local part separately, and then recover the combination of the processed image to the original image size after completing the corresponding algorithm processing.

(2) Enhancement algorithm based on local variance. The basic idea and theory of this algorithm is to use different processing methods to enhance different areas of the image to be processed according to the characteristics of the area.
2. Dark channel prior model and partial differential equation algorithm

This chapter introduces the common method of image defogging processing in image enhancement, which is also the main research object of this paper - the dark channel a priori model. Since the current research work on dark channel a priori image enhancement technology is based on the physical model of atmospheric scattering, this chapter investigates the algorithm principle of the dark channel a priori model for image enhancement processing on the premise of studying and analyzing the atmospheric scattering phenomenon and the physical model of atmospheric scattering. In this paper, an improved algorithm is proposed to address the limitations of the dark channel a priori model, and its principles are basically introduced in this chapter.

2.1. Atmospheric scattering model

According to the literature [8], atmospheric scattering has two important effects on light propagation, one is that the presence of the atmosphere causes a certain degree of attenuation of the reflected light from the target in the air, and the other is the scattering effect of some invisible small volume particles in the air on the reflected light from the scene. When the light propagates from one medium to another, with the increase in propagation distance, the light will have different degrees of attenuation, the amount of attenuation can be expressed as the following formula.

\[ dE(x, \lambda) = -\beta(\lambda)E(x, \lambda)dx \]  

(1)

\[ \beta(\lambda) \] represents the magnitude of the scattering coefficient, whose value characterizes the magnitude of the scattering effect of light by the medium through which the light travels. The result of the attenuation of light after the scattering effect of particles in the atmosphere can be written in the following equation.

\[ E(d, \lambda) = \frac{L_\infty(\infty, \lambda)\rho \exp(-\beta(\lambda)d)}{d^2} \]  

(2)

In the above equation, \( L_\infty(\infty, \lambda) \) is the intensity of light propagating from an infinite distance; \( \rho \) refers to the reflection of light by the target or other objects in the air, in addition to the invisible mixture of gas molecules, there are many visible but very small. The volume of tiny particles is

\[ dV = d\omega \cdot x^2 \cdot dx \]  

(3)

Then the intensity of the scattered light of the propagating beam caused by it is

\[ dI(x, \lambda) = dV \cdot m \cdot \beta(\lambda) = d\omega \cdot x^2 \cdot dx \cdot \beta(\lambda) \]  

(4)

Where \( m \) is a constant, and \( m \) is different for different light sources. By equation (2.4), we can know that the light intensity after scattering of ambient light is

\[ dE(x, \lambda) = \frac{dI(x, \lambda)\exp(-\beta(\lambda)x)}{x^2} \]  

(5)

Integrating (4) and equation (5) together and \( L_\infty(\infty, \lambda) = m \) Integrate on the interval \( x \in [0, d] \) to get

\[ L(d, \lambda) = L_\infty(\infty, \lambda)\left[1 - \exp(-\beta(\lambda)d)\right] \]  

(6)
In the formula, the scattering coefficient $\beta$ is a function of wavelength $\lambda$, and the relationship between $\beta$ and $\lambda$ is as follows

$$\beta(\lambda) \propto \frac{1}{\lambda^\gamma}$$  \hspace{1cm} (7)

In the formula, $0 \leq \gamma \leq 4$, its value is determined by the size of the scattering particles.

In summary, when applying this recovery model in RGB space.

$$E(d) = \frac{L_b(x, \lambda) \rho \exp(-\beta d)}{d^2} \tilde{D} + L_b(x, \lambda)(1 - \exp(-\beta d))\tilde{A}$$ \hspace{1cm} (8)

In the formula, $\tilde{D}$ denotes the unit vector of a specific pixel point in a certain direction in the fogged image; $\tilde{A}$ denotes the unit vector in a certain direction in the atmospheric light model, and the final mathematical expression is

$$I(x) = J(x) \cdot t(x) + A(1 - t(x))$$ \hspace{1cm} (9)

Where is the $A$ intensity of a fixed light source in the atmosphere; $I(x)$ is the input image, i.e., the original blurred image containing fog; $J(x)$ is the output image, i.e., the image after enhanced defogging, $t(x)$ is the atmospheric transfer function, which represents the attenuation of light after passing through the atmosphere.

2.2. Dark channel prior model

By conducting probabilistic statistics on a large number of fog-free images, KaiMing He et al \cite{9} found that the processing of the original image among clear images that do not contain fog can be done by using a delineation block. The image is divided and the result is overlapped with the RGB space, and after the mathematical statistics, it is found that each small block has at least one channel with relatively dark pixel points. The eigenvalues of these relatively darker pixel points are infinitely close to zero, in such a way that the dark primary color a priori statistical law. On this basis, the blurred photos taken in foggy days can also be applied to this theory by classifying the original photos according to the statistical fog content of the photos, and restoring the photos according to the color of each part, thus achieving the purpose of defogging and clearing the photos. The dark primary colors can be expressed in the following form.

$$J^{\text{dark}}(x) = \min_{y \in \Omega(x)} \left( \min_{c \in \{r, g, b\}} J^c(y) \right) \approx 0$$ \hspace{1cm} (10)

In the formula, $J^c$ denotes each channel of the color image $J$. According to the literature \cite{22}, the relationship between the components of the dark primary color and $t(x)$ can be expressed in the following form.

$$\tilde{t}(x) = 1 - \omega \min_{y \in \Omega(x)} \left( \min_{c} \frac{J^c(y)}{A^c} \right)$$ \hspace{1cm} (11)
In the formula, \( \omega \) is a positive number not exceeding 1.

The transmittance map \( t(x) \) can be roughly estimated by Equation (11).

In He's algorithm [9], the pixel points of the image are ordered in terms of brightness magnitude, and the maximum value of the unprocessed points of \( I(x) \) the fogged image corresponding to the first 0.1% of the pixel points is used as the value of atmospheric light intensity \( A \). After processing, the expression of the defogged image based on the dark primary color model changes to the form shown in equation (12).

\[
J(x) = \frac{I(x) - A}{\max(t(x), t_0)} + A
\]  

(12)

In the formula, \( t_0 \) is a fixed constant, which can avoid the divisor being zero.

2.2.1. Estimation of atmospheric light values. According to the literature [9], regardless of the method used to calculate the atmospheric light intensity \( A \), there are more or less shortcomings. This situation is bound to be corrected, and a method of estimating the propagation of light is mentioned in the literature [10] to correct the above problem. This is done by sorting the dark channels in the fogged image according to the number of dark channels counted, taking the average of the top 30 and using it as an estimate of the light, \( A = (A_r + A_g + A_b) / 3 \). Atmospheric light is whiter than other light, so when using the average estimate, the number can be adjusted for different light applications, as long as it meets the following conditions.

\[
\max(|A_r - \bar{A}|, |A_g - \bar{A}|, |A_b - \bar{A}|) \leq B_{\text{max}}
\]  

(13)

In this case, the value \( B_{\text{max}} \) of 0.01 can be determined to improve the color fidelity of the recovered image more precisely.

2.2.2. Atmospheric scattering map in precise transmittance space. In this paper, the method described in the literature [11] is adopted to solve the minimum value of the color channel using the following equation.

\[
w(x) = \min_{d \in \{r, g, b\}} \left[ I^d(x) \right]
\]  

(14)

The guided filtering method can be used to \( w(x) \) properly remove the details of the image, and the result of the removal is \( V'(x) \). The guided filtering process has the following two advantages: first, it can effectively reduce the chance of halo effect at the edges, and second, it can ensure that there is no error due to the inverse gradient at the edges of the image. After the above processing, the calculation formula is

\[
V(x) = \max\left[ \min(p \times V'(x), w(x)), 0 \right]
\]  

(15)

The transmittance obtained \( t(x) \) from \( V(x) = A(1-t(x)) \) is

\[
t(x) = 1 - 0.95 \times V / A
\]  

(16)
From equations (15) and (16), it is clear that the color distortion, such as oversaturation, will increase when \( p \) it is relatively large \( V(x) \), but at this time \( t(x) \) it can be taken to a relatively small value, which can lead to distortion after processing.

2.2.3. Contrast enhancement and brightness adjustment. Although there are various ways to enhance the image traditionally, increasing the contrast is relatively easy, but after processing, there is also a high probability that it will lead to brighter light parts and darker dark parts. In this case, after increasing the contrast, we can do targeted enhancement of the dark parts for the relatively dark parts of the image to ensure the best possible restoration of the dark parts. The method of brightness adjustment is proposed in the literature [12] as \( y = f(x) \), such that \( f(255) = 255 \), which yields

\[
y = \frac{255}{\ln 256} \times \frac{\ln(x + 1)}{\ln 2} \left( 2 + 8 \left( \frac{x}{255} \right)^{\ln b_{\ln 0.5}} \right)
\]

2.3. Theoretical Foundations of Partial Differential Equation Image Enhancement

2.3.1. Theory of partial differential equations. For partial differential equations, there are various ways to classify them, but the more mainstream way is to classify them according to their equation forms. In general, equations containing partial derivatives of unknown functions can be called partial differential equations, and their general forms are as follows.

\[
F \left( x_1, \ldots, x_n, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial^m u}{\partial x_1^m \partial x_2^m \ldots \partial x_n^m} \right) = 0
\]

Where, \( x = (x_1, x_2, \ldots, x_n) \) is the independent variable of the equation, \( u(x) = (x_1, x_2, \ldots, x_n) \) is the vector of unknowns in the equation, and the \( m = m_1 + m_2 + \ldots + m_n \) total order of the partial differential equation is the sum of the orders of the variables. The partial differential equations can be classified into different classes of equations with reference to the number of unknown functions, the order of the equation and the power of the function and its derivatives. In addition, there is a special classification of partial differential equations more commonly used in practical problems, namely, binary second-order equations, whose basic form is shown below.

\[
a \frac{\partial^2 u}{\partial x^2} + b \frac{\partial^2 u}{\partial x \partial y} + c \frac{\partial^2 u}{\partial y^2} + d_1 \frac{\partial u}{\partial x} + d_2 \frac{\partial u}{\partial y} + eu = f(x, y)
\]

The above equation is characterized by
The criteria of the model used for two-dimensional image processing correspond to the dimensionality of the variables of the binary second-order partial differential equation, and in fact, both fluctuation and thermal diffusion equations have been introduced in the image denoising and recovery problems.

### 2.3.2. Partial differential equation model in image enhancement domain

There are more applications of partial differential equations, and depending on some specific physical problems, a single partial differential equation can also be chosen. The definition of a planar gray scale diagram is as follows.

\[
\mathcal{D} : \mathbb{R}^n \rightarrow \mathbb{R} \quad (x \rightarrow u(x))
\]

In the formula, \( \Omega \) is the domain of the image, and \( n \in \mathbb{N}^+ \) is the dimension of the space, \( n = 2, x = (x, y) \). The derivative of image \( u \) with respect to \( a \) is \( u_a = \frac{\partial u}{\partial a} \). Use \( \nabla u \) to the derivative of variable \( x, y \) : \( \nabla u = \left( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) = (u_x, u_y) \). The modulus of the image gradient is \( |\nabla u| = \sqrt{u_x^2 + u_y^2} \).

The pattern of change of the image with time \( t \) is shown in the following equation

\[
\frac{\partial u}{\partial t} = \lambda_1 \cdot u_{\omega_0} + \lambda_2 \cdot u_{vv} \tag{22}
\]

Where, \( u : \mathbb{R}^2 \rightarrow \mathbb{R} \) denotes the grayscale image, and \( u(x, y) \) denotes its grayscale value at the location \( (x, y) \in \mathbb{R}^2 \). \( \omega \) denotes the tangential direction of the contour lines with the same gray value in the image, and \( v \) is used to represent the gradient direction. \( u_{\omega_0} \) of the image, and denotes \( u \) the differential form in the coordinate system along the tangential direction, \( u_{vv} \) denotes the differential form in the normal direction, and \( \lambda_1, \lambda_2 \) denotes the diffusion multiplier of the image along the tangential direction and the normal direction, respectively. The \( u(x, y, t) \) solution of the differential equation is the result of the filtering operation of the image at the \( t \) scale.

### 3. Combined dark channel a priori and partial differential equations for the improved algorithm solution

As the gradient change of fogged images is relatively flat, the dense fog can easily mask the image information, and the process of enhancing and defogging using the traditional dark channel a priori algorithm will lead to incomplete defogging or remove the image information of the region together, resulting in poor edge retention and low brightness of the image enhancement processing. In this chapter, we transform the image information from the spatial domain to the gradient domain for processing, take the gradient on both sides of the atmospheric physical scattering model, establish a new image processing model, solve the block transmittance and point transmittance by using the dark channel theory, get the high and low frequency information by wavelet transform, then combine the low frequency part of the block transmittance with the high frequency part of the point transmittance...
to obtain the new fused transmittance by using the inverse wavelet transform. Finally, the finite difference method is used to solve this model to realize the image enhancement in the gradient domain, and to achieve the effect of edge preservation and brightness enhancement.

3.1. Modeling of partial differential equations for image enhancement

We first establish a connection between the image taken without fog and the image taken with fog. We first obtain the atmospheric physical scattering model (2.10) for the image with fog, and then the gradient of the expression for both sides of the model equation simultaneously yields

\[ \nabla I = T \nabla E + E \nabla T - E_\infty \nabla T \]

And

\[ \nabla I = T \nabla E + (E - E_\infty) \nabla T \]  \hspace{1cm} (23)

The left and right sides of Equation (23) show that in order to make the fogged image recover better to a fog-free image, \( E \) needs to be chosen so that Equation (23) holds approximately. In order to minimize the value of the error on both sides of Equation (23), we set up the problem of finding the minimal value of the generalized function shown in the following equation.

\[ J = \arg \min_{I} \int \int_{\Omega} \| \nabla I - (T \nabla E + (E - E_\infty) \nabla T) \|^{2} d\Omega \] \hspace{1cm} (24)

Thus the problem of recovering the image with fog is transformed into the problem of finding the minimal value of the generalized function. The integrated function in Equation (24) can be further written as

\[ F = \big[ I_x - (TE_x + (E - E_\infty) T_x) \big]^2 + \big[ I_y - (TE_y + (E - E_\infty) T_y) \big]^2 \] \hspace{1cm} (25)

The next equation (25) is given by the product function and equation (24) to find the extreme value of the general function, easy to obtain the function \( I(x, y), E(x, y) \) need to satisfy the following relationship.

\[ \begin{cases} 
F_E - \frac{\partial}{\partial x} F_{E_x} - \frac{\partial}{\partial y} F_{E_y} = 0 \\
F_I - \frac{\partial}{\partial x} F_{I_x} - \frac{\partial}{\partial y} F_{I_y} = 0
\end{cases} \] \hspace{1cm} (26)

We can then transform solving Eq. (3.3), into the problem of solving the system of equations obtained from Eq. (26) consisting of the functions \( I(x, y), E(x, y) \), which in turn leads to

\[ \frac{\partial}{\partial x} \left( I_x - (TE_x + (E - E_\infty) T_x) \right) + \frac{\partial}{\partial y} \left( I_y - (TE_y + (E - E_\infty) T_y) \right) = 0 \]

And
\[
\text{div}(T\nabla E) + \text{div}((E - E_\infty)\nabla T) - \Delta I = 0
\]  

Eq. (27) is actually the Euler equation corresponding to Eqs. (24) and (26). We further solve it to solve the enhanced defogged image with the boundary condition taken as

\[
E|_{\Omega} = \left[ I - E_\infty (1 - T) \right]/T|_{\Omega}
\]

3.2. Solution of partial differential equation model for image enhancement

3.2.1. Solving for transmittance and atmospheric light intensity. We assume that the intensity of the atmospheric light \(A_\infty\) is known, and we will use the method in the literature [9] to estimate the block and point transmittance; then, we use the wavelet transform technique to decompose the low-frequency and high-frequency information of the image, and then fuse the decomposed information to obtain the fused transmittance for the image processing here [14].

(1) Estimation of block transmittance

The transmittance estimation equation proposed by He [9] is

\[
t_1(x,y) = 1 - \min_{(x',y') \in \Omega (x,y)} \min_{c \in \{R,G,B\}} \left( \frac{I^c(x',y')}{A^c_\infty} \right)
\]

Where \(I^c\) denotes the first \(c (c \in \{R,G,B\})\) color channel of the image \(I\), \(A^c_\infty\) denotes the atmospheric light intensity of that channel, and \(\Omega(x, y)\) denotes the local area of the image.

Here, we introduce another constant \(\omega\), the transmittance estimation formula used in Eq. (3.8) is rewritten as

\[
t_1(x,y) = 1 - \omega \min_{(x',y') \in \Omega (x,y)} \min_{c \in \{R,G,B\}} \left( \frac{I^c(x',y')}{A^c_\infty} \right)
\]

In this case, we choose different \(\omega\) values to determine the degree of defogging of the final image. Through a large number of experimental verification, this paper takes \(\omega\) as 0.85.

(2) Estimation of point transmittance

For the estimation of the point transmittance, we still refer to the method of the literature [9]. From the dark primary color a priori theory, first taking the minimum of both sides of Eq. (31) yields

\[
\min_{c \in \{R,G,B\}} \left( I^c(x,y) \right) = t(x,y) \min_{c \in \{R,G,B\}} \left( A^c(x,y) \right) + A^c_\infty (1 - t(x,y))
\]

Then the dark primary color channel value of the clear image approximates to zero

\[
A^c_{\text{dark}}(x,y) = \min_{c \in \{R,G,B\}} \left( A^c(x,y) \right) = 0
\]

Substituting Eq. (32) into Eq. (31) yields the expression for the transmittance \(t_2(x,y)\) as

\[
t_2(x,y) = 1 - \min_{c \in \{R,G,B\}} \left( \frac{I^c(x,y)}{A^c_\infty} \right)
\]
In the same way as for the estimation part of the block transmittance, also in order to retain a part of the fog information on the final resulting image, a constant $\omega$ is introduced in Eq. (33)

$$t_2(x, y) = 1 - \omega \min_{c \in [R, G, B]} \left( \frac{I^c(x, y)}{A_c} \right)$$

We still use the point transmittance calculation method proposed in the previous subsection to obtain the transmittance image with fog, and combine it with the atmospheric light intensity to finally recover the image without fog.

3. Estimation of fused transmittance

To combine the characteristics of both transmittance maps, Z. Wang et al [14] proposed a method to form a new fused transmittance after processing them. The basic idea is to eliminate the high-frequency information part of the block transmittance estimation method so that the whole is close to the real image and free of halo, and to select the high-frequency part of the point transmittance estimation method so as to obtain a better detail part without overall distortion.

4. Estimation of atmospheric light intensity

We use the method in the literature [14] to estimate the intensity of atmospheric light, i.e., we operate on each channel of the initially obtained image, from which we select the channel grayscale maximum to be the atmospheric light intensity value for this calculation. Since the method operates on the image with pixel point values in all three color signal channels within the value range from zero to one, the values $E$ in the three channels are approximated as (1,1,1).

3.2.2. Numerical solution of the model. In the following, the model equation (6) and equation (7) we have obtained are discretized directly with respect to the time and space variables using the finite difference method. The terms in equation (6) can be expressed as

$$\text{div}(TVE) = \left[ \frac{\partial}{\partial x} \cdot \frac{\partial}{\partial y} \left( T \frac{\partial E}{\partial x}, T \frac{\partial E}{\partial y} \right) \right] = T_s \frac{\partial E}{\partial x} + T_y \frac{\partial E}{\partial y} + TAE$$

$$\text{div}(ETV) = \left[ \frac{\partial}{\partial x} \cdot \frac{\partial}{\partial y} \left( E \frac{\partial T}{\partial x}, E \frac{\partial T}{\partial y} \right) \right] = T_s \frac{\partial E}{\partial x} + T_y \frac{\partial E}{\partial y} + EAT$$

$$\text{div}(E \nabla T) = E_s \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \left( \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y} \right) \right) = E_s \Delta T$$

Where $\Delta$ is the Laplace operator, so that equation (6) is

$$2T_s \frac{\partial E}{\partial x} + 2T_y \frac{\partial E}{\partial y} + EAT + TAE = \Delta I + E_s \Delta T$$

$E_{i,j}$ is the pixel point value of the de-fogged image at the point $(i, j)$, due to the pixel point arrangement is to satisfy the requirement of the equipartition principle in the finite difference method, the value of the spatial step in both directions of the 2D image can be taken to be 1. The discretization formula (38) is approximated by the first-order central difference quotient $\frac{\partial E}{\partial x}, \frac{\partial E}{\partial y}$
\[ \frac{\partial E}{\partial x}_{i,j} \approx \frac{1}{2} \left( E_{i+1,j} - E_{i-1,j} \right) \]

\[ \frac{\partial E}{\partial y}_{i,j} \approx \frac{1}{2} \left( E_{i,j+1} - E_{i,j-1} \right) \] (39)

Laplace operator \( \Delta E \) is approximated using a five-point difference format

\[ \Delta E_{i,j} \approx E_{i+1,j} + E_{i-1,j} + E_{i,j+1} + E_{i,j-1} - 4E_{i,j} \] (40)

Then the discrete form of equation (40) is

\[
\left\{
\begin{aligned}
(\Delta T_{i,j} - 4T_{i,j})E_{i,j} + \left(2T_{x_{i,j}} + T_{i,j}\right)E_{i+1,j} + \\
(-2T_{x_{i,j}} + T_{i,j})E_{i-1,j} + \left(2T_{x_{i,j}} + T_{i,j}\right)E_{i,j+1} + \\
(-2T_{y_{i,j}} + T_{i,j})E_{i,j-1} = E_x \Delta T_{i,j} + \Delta I_{i,j}
\end{aligned}
\right.
\]

\[ i = 2, 3 \cdots, M - 1; j = 2, 3 \cdots, N - 1 \]

\[ E_{i,j} = \frac{I_{i,j} - E_x}{T_{i,j}} + E_x, i = 1, M; j = 1, N \] (41)

\( M, N \) denotes the image size, \( T_{x_{i,j}}, T_{y_{i,j}} \) is the central difference quotient at \((i, j)\) of \( T_x, \Delta T_{i,j} \), \( \Delta I_{i,j} \) is the five-point difference format at \((i, j)\) of \( \Delta T, \Delta I \), respectively, we can obtain the following equation

\[
\left\{
\begin{aligned}
(\Delta T_{i,j} - 4T_{i,j})E_{i,j}^{k+1} + \left(2T_{x_{i,j}} + T_{i,j}\right)E_{i+1,j}^{k+1} + \\
(-2T_{x_{i,j}} + T_{i,j})E_{i-1,j}^{k+1} + \left(2T_{x_{i,j}} + T_{i,j}\right)E_{i,j+1}^{k+1} + \\
(-2T_{y_{i,j}} + T_{i,j})E_{i,j-1}^{k+1} = E_x \Delta T_{i,j} + \Delta I_{i,j}
\end{aligned}
\right.
\]

\[ i = 2, 3 \cdots, M - 1; j = 2, 3 \cdots, N - 1 \]

\[ E_{i,j}^{k+1} = \frac{I_{i,j} - E_x}{T_{i,j}} + E_x, i = 1, M; j = 1, N \]

\[ k = 0, 1, 2 \cdots, K \] (43)

Using the Gauss-Seidel (G-S) iterative method to solve the discrete system of equations, the final iterative equation is obtained as

\[ (\Delta T_{i,j} - 4T_{i,j})E_{i,j}^{k+1} + \left(2T_{x_{i,j}} + T_{i,j}\right)E_{i+1,j}^{k+1} + \]

\[ (2T_{y_{i,j}} + T_{i,j})E_{i,j+1}^{k+1} + \left(-2T_{y_{i,j}} + T_{i,j}\right)E_{i,j-1}^{k+1} = E_x \Delta T_{i,j} + \Delta I_{i,j} \]

\[ i = 2, 3 \cdots, M - 1; j = 2, 3 \cdots, N - 1 \]

\[ E_{i,j}^{k+1} = \frac{I_{i,j} - E_x}{T_{i,j}} + E_x, i = 1, M; j = 1, N \]

\[ k = 0, 1, 2 \cdots, K \]

Where \( K \) is the number of iterations.
3.3. Experimental results and analysis

In this section, we introduce an outdoor scene database (named O-HAZE), which consists of real haze images, haze images are captured under real haze generated by a professional haze meter, and O-HAZE contains 45 different outdoor scenes that depict the same visual content recorded under the same lighting parameters. In Figure 2 we can see 12 images, which are the foggy outdoor environment images for the experiments conducted in this subsection, selected from the O-HAZE dataset. The images in Figure 3 are the processing results of the traditional dark channel algorithm for the fogged images in Figure 2. Figure 4 shows the processing results of the new algorithm after improving the dark channel a priori algorithm combined with partial differential equations in this section, Figure 5 shows the processing results of the Single Scale Retinex algorithm (hereinafter referred to as the S algorithm), Figure 6 shows the processing results of Multi Scale Retinex algorithm (hereinafter referred to as M algorithm), Figure 7 shows the processing results of Anisotropic Diffusion algorithm (hereinafter referred to as A algorithm), Figure 8 shows the processing results of Gradient Algorithm (hereinafter referred to as G algorithm), Figure 9 shows the results of Fast Dual Minimization algorithm (hereinafter referred to as F algorithm).

![Figure 2](image1.png)

**Figure 2** Foggy outdoor environment image

![Figure 3](image2.png)

**Figure 3** Processing results of outdoor environment images in foggy days under the dark channel algorithm

![Figure 4](image3.png)

**Figure 4** Processing results of outdoor environment images in foggy days with the improved algorithm in this section

![Figure 5](image4.png)

**Figure 5** Processing results of outdoor environment images in foggy days under S-algorithm
From the above experiments, it can be seen that, in terms of the overall enhancement effect, the improved model designed in this section is very effective for processing haze images, with significant results for both fog and noise removal in the images, and a brighter and clearer overall image.

From the figure can be seen in Figure 2 image due to the thick fog, visibility is relatively low, visual perception does not add, we need to observe objects in the fog is hidden, more difficult to observe. After the dark channel a priori algorithm for image enhancement and defogging, the image effect does become clearer, but the overall image tone is darker, and the distinction between the observation object and the surrounding background and environment is not obvious, and observation is still more difficult. After the S algorithm processing, the image noise interference was removed, but a large part of the texture details in the image were blurred out as a result. After the M algorithm processing, the image edge and texture maintenance effect is relatively general. After the A algorithm, the image showed a certain degree of overexposure, and the image contrast showed no obvious signs of improvement. After the G algorithm, the overexposure phenomenon is more serious and the analysis value is lower. After the F algorithm processing, the overall effect of the image is better, but there is a slight distortion situation. After processing by the improved algorithm combined with partial differential equation in this section, the image texture information is retained significantly better, the edges and contours of the observed objects are clearer, and the brightness of the image is also greatly

Figure 6 Processing results of outdoor environment images in foggy days under M algorithm

Figure 7 Processing results of outdoor environment images in foggy days under A algorithm

Figure 8 Processing results of outdoor environment images in foggy days under G algorithm

Figure 9 Processing results of outdoor environment images in foggy days under F algorithm
improved, and we can clearly identify the objects to be observed from the image, and the overall results show that the algorithm enhanced defogging effect in this chapter is more prominent.

After the above subjective evaluation, we choose peak signal-to-noise ratio (PSNR), structural similarity (SSIM) and information entropy, respectively, for the result analysis. The higher the value of the peak signal-to-noise ratio, the better the denoising effect of the algorithm; the higher the value of the structural similarity, the closer the restored image is to the original image and the better the effect; the higher the value of the information entropy, the more effective information can be obtained from the image. Meanwhile, in order to compare the computational efficiency of each of the seven algorithms, the data of CPU running time of each algorithm were experimentally counted.

We counted the quantitative values of each objective evaluation result mentioned above and plotted them in Table 1.

| Evaluation Criteria                  | PSNR  | SSIM  | Information entropy | CPU running time(s) |
|--------------------------------------|-------|-------|---------------------|---------------------|
| Fogged images                        | None  | None  | 6.5707              | None                |
| Dark channel algorithm               | 9.4268| 0.4897| 7.1913              | 17.9008             |
| Algorithm of this paper              | 11.1987| 0.5331| 7.5636              | 16.3783             |
| S algorithm                          | 11.6832| 0.5134| 6.9831              | 19.3686             |
| M algorithm                          | 10.3684| 0.4357| 7.1067              | 18.6492             |
| A algorithm                          | 10.1395| 0.4931| 6.8394              | 17.6348             |
| G algorithm                          | 9.3691 | 0.4351| 6.4317              | 14.3165             |
| F algorithm                          | 11.4359| 0.5264| 7.5943              | 25.3194             |

By analyzing the data in the above table we can conclude that: after objective evaluation, on the whole, the improved dark channel a priori model combined with partial differential equations has more obvious advantages compared to the original algorithm. The peak signal-to-noise ratio value of the improved algorithm is relatively high, which shows that the image denoising effect is better, while the denoising effect of the S algorithm is better, but the method is not desirable because the denoising leads to incomplete image information acquisition; the value of the structural similarity value shows that the image processed by the partial differential equation improved algorithm is closer to the original image, the edge preservation effect is better, and more effective information is retained in the image, while the M and G algorithms have The value is relatively low; the improvement of the information entropy value shows that more effective information can be captured from the image, except for the algorithm used in the G algorithm, all other algorithms become significantly higher compared to the original image, among which, the algorithm in this section has the highest corresponding value with the F algorithm. Meanwhile, after improvement, the computational efficiency is relatively high, and the value of each data of F algorithm is relatively high, but the computational efficiency is the lowest among all algorithms in this experiment.

Through the above quality evaluation of the traditional dark channel a priori model with the improved algorithm model combined with partial differential equations and other five common algorithms from both subjective and objective evaluation perspectives, we can finally conclude that the improved effect has a certain degree of improvement in all aspects compared with the original algorithm and has some advantages compared with several common algorithms.

4. Summary of thesis research work
This paper firstly elaborates the research purpose and significance of the topic, and introduces the current status and research status of image enhancement methods in China and abroad, and explains the principle and basic theory of image enhancement based on atmospheric physical model and dark channel theory, and briefly analyzes it, and explains the theoretical basis of partial differential
equation method for image enhancement, which gives a more solid foundation for the later research and application. This paper innovatively combines the dark channel theory with partial differential equations, and proposes the improved algorithms for color image enhancement and defogging.

The research work in this paper is as follows.

1. The principle of atmospheric scattering model and dark channel theory are elaborated, and the relevant parameters to be adjusted for image enhancement and defogging by dark channel theory are clarified, and preliminary studies on atmospheric light value, transmittance and contrast are made.

2. An improved algorithm combining dark channel prior and partial differential equation is designed to transform the image information from spatial domain to gradient domain for processing, and a new image processing model is established, in which the block transmittance and point transmittance are solved by using dark channel theory, the high and low frequency information is obtained by wavelet transform, and then the low frequency part of the block transmittance is combined with the high frequency part of the point transmittance to obtain the new fused transmittance by using inverse wavelet transform. Finally, this model is solved by the finite difference method.

3. Designing comparison experiments, choosing the public dataset O-HAZE, and conducting experimental comparisons of the original dark channel algorithm, the algorithms improved in this paper and the five common image enhancement and defogging algorithms on the dataset, comparing the experimental results using seven methods with the actual degraded images for enhancement processing. The results are analyzed and compared, and the evaluation results verify that the algorithms designed in this paper not only have good enhancement effects in image enhancement, but also can better maintain the detailed information of the image such as edges and textures.

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