Flavour physics parameters from data and unitarity

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February 1, 2013

Abstract

The aim of the paper is to show that the nowadays experimental data from superallowed nuclear and neutron $\beta$ decays, and from leptonic and semileptonic decays allow the finding of the most probable numerical form of the KM matrix, as well as the determination of decay constants, $f_P$, and of various form factors $f^+(q^2)$, by using a genuine implementation of unitarity constraints. In particular this approach allows the determination of semileptonic form factors that is illustrated on the existing data of $D \to \pi l\nu$ and $D \to K l\nu$ decays.

1 Introduction

Within the Standard Model (SM) the flavour physics is encoded by the Kobayashi-Maskawa (KM) matrix, $[1]$, supposed to be unitary, matrix that describes the quark flavor mixing through four independent parameters: three mixing angles, $\theta_{ij}$, $ij = 12, 13, 23$, and one $CP$-violating phase,
By consequence the experimental determination of KM matrix entries is essential for the validation of the SM, and for detection of new physics beyond it.

However the determination of KM entries rises a few problems. The first one come from theory, namely the mixing angles are not invariant quantities, their numerical values depend on the original KM form, [1], or on the present day form, [2], which is not rephasing invariant. [3]. This shortcoming becomes harmless if one follows Jarlskog’s approach. Starting with her first papers on KM matrix, [4], she proposed the determination of the quark mixing matrix in terms of directly measurable quantities, and, in the same time, invariant quantities. In this context an invariant quantity is one whose numerical value does not depend on the KM matrix form, or of its rephasing invariance. Jarlskog provided two such invariants: the KM moduli, and the celebrated $J$ invariant, [5]. After few years it was realized that all the measurable quantities of the quark mixing matrix are expressible in terms of four independent KM matrix moduli, [6] and [7]. In these papers it was also shown that areas of all unitarity triangles are equivalent, and numerically equal to half of the $J$ invariant. As a conclusion we can state that the numerical values for all the measured invariant quantities should be the same irrespective of the physical processes where they are involved.

The second difficulty comes from the experimental side. If one uses the KM moduli set as independent parameters, these ones are not directly measured by experimenters. In the simplest case, that of leptonic decays, they measure branching ratios and provide numbers for products of the form $|U_{qq'}|f_P$, where $U_{qq'}$ is the corresponding KM matrix element, and $f_P$ is the decay constant. For meson semileptonic decays the physical observable is the differential decay rate, $d\Gamma/dq^2$, which up to known factors, is proportional
to $|U_{qq'} f_+ (q^2)|^2$, where $q$ denotes the transferred momentum between initial
and final mesons. In the last case the experimental teams usually provide
numerical values for products of the form $|U_{qq'} f_+ (0)|$, where $f_+ (0)$ is the
semileptonic decay form factor at zero-momentum transfer. It is clear that
from such measurements one cannot find two unknown parameters, say $|U_{qq'} |
and $f_P$. This can be done if and only if one can find independent constraints
on KM matrix moduli; fortunately these ones are provided by unitarity.

The main aim of the paper is to show how the unitarity property of the
KM matrix can be transformed into a powerful tool for the determination of
both matrix moduli and form factors directly from the experimental data.

The unitarity constraints are presented in Sec. II where they are im-
plemented in a $\chi^2$-form that depends only on KM matrix moduli. In Sec.
III we present the decay formulae for superallowed $0^+ \rightarrow 0^+$ nuclear and
neutron $\beta$ decays, and those for leptonic and semileptonic decays. They
depend on KM matrix moduli and specific decay parameters such as decay
constants, $f_P$, and form factors $f(q^2)$, etc, that are implemented in an other
$\chi^2$-piece. In Sec. IV we cite the experimental papers from which we took
the data that we used in our fit, and in Sec. V we present the numerical
results. The paper ends by Conclusion.

2 Unitarity constraints

The use of $|U_{ij}|$ as independent parameters raises an important problem,
namely the solving of the consistency problem between moduli and unitar-
ity property, which all amounts to obtaining the necessary and sufficient
conditions on the set of numbers $|U_{ij}|$ to represent the moduli of an ex-
act unitary matrix. After that we have to find a device for applying these
conditions to the experimental situation where the data are known modulo
uncertainties.

Both these problems have been solved, and a procedure for recovering KM matrix elements from error affected data was provided in [8]. These unitarity constraints say that the four independent parameters $s_{ij} = \sin \theta_{ij}$ and $\cos \delta$ should take physical values, i.e. $s_{ij} \in (0, 1)$ and $\cos \delta \in (-1, 1)$, when they are computed via equation set:

\begin{align*}
V_{ud}^2 &= c_{12}^2 c_{13}^2, \quad V_{us}^2 = s_{12}^2 c_{13}^2, \quad V_{ub}^2 = s_{13}^2 \\
V_{cb}^2 &= s_{23}^2 c_{13}^2, \quad V_{tb}^2 = c_{13}^2 c_{23}^2 \\
V_{cd}^2 &= s_{12}^2 s_{23}^2 + s_{13}^2 s_{23}^2 c_{12}^2 + 2 s_{12} s_{13} s_{23} c_{12} c_{23} \cos \delta, \\
V_{cs}^2 &= c_{12}^2 c_{23}^2 + s_{13}^2 c_{23}^2 c_{12}^2 - 2 s_{12} s_{13}^2 s_{23} c_{12} c_{23} \cos \delta, \\
V_{td}^2 &= s_{13}^2 c_{12}^2 c_{23}^2 + s_{12}^2 s_{23}^2 - 2 s_{12} s_{13} s_{23} c_{12} c_{23} \cos \delta, \\
V_{ts}^2 &= s_{12}^2 s_{13}^2 c_{23}^2 + c_{12}^2 s_{23}^2 + 2 s_{12} s_{13} s_{23} c_{12} c_{23} \cos \delta
\end{align*}  

The above relations have been obtained by using the standard KM matrix form, [2], where $V_{ij} = |U_{ij}|$, and $U_{ij}$ are the KM matrix entries. In the paper [8] it was also shown that if the independent parameters are the KM matrix moduli the reconstruction of a unitary matrix knowing its moduli is essentially unique. By consequence in the following the used independent parameters in our phenomenological analysis will be the $V_{ij}$ moduli. Although only four of them can be independent, the experimental data that are known only modulo uncertainties “force” us to use all the possible sets of four independent moduli, as it will be shown in the following. A simple combinatorial evaluation shows that there are 57 such sets.

The relations (1) are rephasing invariant, i.e. they have the same form after multiplication of all the KM matrix rows and columns by arbitrary phases. More important is that they contain a part of the unitarity con-
It is easily seen that all the six relations such as

\[ V_{ud}^2 + V_{us}^2 + V_{ub}^2 = 1 \]  \tag{2}

are a consequence of the relations \([1]\). The relations \([2]\) are the necessary conditions for unitarity fulfilment, but they are not sufficient, as we show in the following. In fact the relations \([2]\) tell us that if they are satisfied then there exists a physical solution for the mixing angles \(s_{ij}\). The relations \([2]\) are not sufficient because the unitary matrices are naturally embedded in a larger class of matrices: the set of double stochastic matrices, \([9]\).

A \(3 \times 3\) matrix \(m\) is said to be double stochastic if its entries satisfy the relations

\[ m_{ij} \geq 0, \quad \sum_{i=1}^{3} m_{ij} = 1, \quad \sum_{j=1}^{3} m_{ij} = 1 \]  \tag{3}

This set is a convex set, i.e. if \(M_1\) and \(M_2\) are double stochastic, then

\[ M = xM_1 + (1 - x)M_2, \quad 0 \leq x \leq 1 \]  \tag{4}

is double stochastic. The unitary matrices are a subset of this larger set if we define the \(m_{ij}\) entries by the relation \(m_{ij} = |U_{ij}|^2\), i.e. one get relations similar to relation \([2]\).

For example if we choose four independent moduli, e.a: \(V_{us} = a, V_{ub} = b, V_{cb} = c,\) and \(V_{cd} = d\), then the following matrix

\[
DS_1 = \begin{bmatrix}
1 - a^2 - b^2 & a^2 & b^2 \\
d^2 & 1 - c^2 - d^2 & c^2 \\
a^2 + b^2 - d^2 & c^2 + d^2 - a^2 & 1 - b^2 - c^2
\end{bmatrix}
\]  \tag{5}
is a double stochastic matrix since it exactly satisfies all the six relations similar to relation (2), as it is easily checked.

From any four independent moduli entering (1) one can get the mixing parameters $s_{ij}$ and $\cos \delta$, i.e. the four independent parameters entering the KM unitary matrix. For example with the above choice we find from the first six equations (1) all the three mixing parameters, $s_{ij}$, and $\delta$ as follows

\[
\begin{align*}
    s_{13} &= V_{ub} = b, \quad s_{12} = \frac{a}{\sqrt{1 - b^2}}, \quad s_{23} = \frac{c}{\sqrt{1 - b^2}} \\
    \cos \delta &= \frac{(1 - b^2)(d^2(1 - b^2) - a^2) + c^2(a^2 + b^2(a^2 + b^2 - 1))}{2abc\sqrt{1 - a^2 - b^2\sqrt{1 - b^2 - c^2}}} \quad (6)
\end{align*}
\]

and from the remaining relations three new $\cos \delta$ formulae similar to (6).

The above relation shows that $\cos \delta$ is an other invariant in the Jarlskog sense depending on four independent moduli, and the $CP$-violation phase can be measured via relations such as (6).

If we make use of the last four relations (1) we get only one solution for mixing parameters and $\cos \delta$. Thus depending on the chosen four independent moduli set the number of solutions varies between one and four. Because there are 57 such groups one get 165 different expressions for $\cos \delta$. They take the same numerical value when are computed via Eqs. (1), if and only if all the six relations similar to Eq.(2) are exactly satisfied. If the moduli matrix generated by four independent moduli is compatible with unitarity then $\cos \delta \in (-1, 1)$, and outside this interval when the corresponding matrix is not compatible.

To see that we make use of the numerical example $V_{us} = 2257/10^4$, $V_{ub} = 359/10^5$, $V_{cd} = 2256/10^4$, and $V_{cb} = 415/10^4$. By using the $DS_1$ matrix,
one get the $|U|$ moduli as

$$
|U| = \begin{bmatrix}
\sqrt{9490466219} & \frac{2257}{10^4} & \frac{359}{10^5} \\
\frac{141}{625} & \frac{3\sqrt{10982647}}{10^4} & \frac{83}{2\times10^4} \\
\sqrt{580181} & \sqrt{10982} & \sqrt{982648619} \\
\end{bmatrix}
$$

(7)

By using the formula from the second row of (6) one gets $\cos \delta \approx 0.64088$, showing that the above moduli matrix, (7), comes from an exact unitary matrix.

If we modify the previous numerical $V_{us}$ value by adding to it the small quantity $3 \times 10^{-4}$ the mixing parameters are still physical, only $s_{12}$ is modified by a very small quantity, and respectively three square root entries from the first two columns of (7), necessary for the exact fulfilment of all the six relations similar to (2). In this case one gets $\cos \delta \approx -1.42427$, which shows that the new moduli matrix, $|U|$, is not compatible with unitarity, even if all of its entries are positive and satisfy exactly all the six relations (2). If one computes the $J$ invariant one finds in the above two cases

$$
J^2 = 6.317 \times 10^{-10}, \text{ and } J^2 = -1.106 \times 10^{-9}
$$

(8)

Thus the physical conditions for unitarity compatibility are $\cos \delta \in (-1, 1)$, and $J^2 > 0$, respectively, and from a theoretical point of view they are equivalent.

There are even bad cases; for example if instead of $a$ and $b$ we choose the parameters $V_{ud} = f$, and $V_{ts} = g$ the corresponding double stochastic matrix has the form

$$
DS_2 = \begin{bmatrix}
f^2 & c^2 + d^2 - g^2 & 1 - c^2 - d^2 - f^2 + g^2 \\
d^2 & 1 - c^2 - d^2 & c^2 \\
1 - d^2 - f^2 & g^2 & d^2 + f^2 - g^2 \\
\end{bmatrix}
$$
With the same numerical values for $V_{cd}$, and $V_{cb}$, and $V_{ad} = f = \frac{97419}{10^{10}}$, and $V_{ts} = g = \frac{405}{10^4}$ we find
\[
V_{ub}^2 = 1 - c^2 - d^2 - f^2 + g^2 = -\frac{72761}{10^{10}}
\] (9)
and all the other $DS_2$ entries are positive. Thus $V_{ub}$ takes an imaginary value, and $\cos \delta$ and $J$ take complex values.

Similar results can be easily obtained; another example is that generated by the moduli, $V_{cd} = d, V_{cs} = h, V_{td} = k, V_{ts} = g$, whose corresponding double stochastic matrix has the form
\[
DS_3 = \begin{bmatrix}
1 - d^2 - k^2 & 1 - g^2 - h^2 & -1 + d^2 + g^2 + h^2 + k^2 \\
d^2 & h^2 & 1 - d^2 - h^2 \\
k^2 & g^2 & 1 - g^2 - k^2
\end{bmatrix}
\]
If we choose $V_{cd} = 2252/10^4, V_{cs} = 97345/10^5, V_{td} = 862/10^5, V_{ts} = 403/10^4$ one gets $\cos \delta \approx 0.3685$ which means that the corresponding $|U|$ is compatible with unitarity. If we subtract from $V_{cs}$ value the tiny quantity $5 \times 10^{-6}$ one gets $\cos \delta \approx -5.0234$, and if we subtract $9.3 \times 10^{-6}$ we get $\cos \delta \approx -1.641 i$.

For numerical computations the use of $\cos \delta$ formulae, like (6), seems to be more efficient because of their great sensitivity to small moduli variation, as the above numerical computations show.

The real physical cases are those where the central value moduli matrices, directly determined from data, or from a fit do not exactly satisfy all the six relations (2), but only approximately; for example for a good fit the difference from unity could be of the order, $10^{-5} - 10^{-7}$, i.e. rather small from a phenomenological point of view. In these cases the different formulæ for $\cos \delta$ provide different values, physical and unphysical, even if the mixing parameters take physical values as in the previous examples. Thus the
physical reality obliges us to implement the unitarity constraints

$$\cos \delta_i \approx \cos \delta_j, \ i \neq j, \ \text{all} \ \cos \delta_i \in (-1, 1)$$

(10)

into a $\chi^2$-fitting device, and our choice is

$$\chi_1^2 = \sum_{j=u,c,t} \left( \sum_{i=d,s,b} V_{ji}^2 - 1 \right)^2 + \sum_{j=d,s,b} \left( \sum_{i=u,c,t} V_{ij}^2 - 1 \right)^2$$

$$+ \sum_{i<j} (\cos \delta^{(i)} - \cos \delta^{(j)})^2, \ -1 \leq \cos \delta^{(i)} \leq 1$$

(11)

that enforces all the unitarity constraints.

3 Decay Formalism

In this section we show how the used decay formalism for the description of available experimental data allows us to define a second piece of the $\chi^2$-function by taking into account as much as possible the experimental information.

For example the $V_{ud}$ parameter enters the description of superallowed $0^+ \rightarrow 0^+ \beta$ nuclear decay, and neutron $\beta$ decay. Superallowed $0^+ \rightarrow 0^+ \beta$ decay between $T=1$ analog states depends uniquely on the vector part of the weak interaction and, according to the conserved vector current hypothesis, its experimental $ft$ value is related to the vector coupling constant, which is a fundamental constant and by consequence has the same value for all such transitions, see [10], [11], [12] and [13]. This means that the following relation should hold

$$ft = \frac{K}{2|G_V|^2 |M_F|^2} = \text{const}$$

(12)

where $K/(\hbar c)^6 = 2\pi^3 \hbar \ln2/(m_e c^2)^5$, $G_V$ is the vector coupling constant for
semi-leptonic weak interactions, and $M_F$ is the Fermi matrix element which in this case is equal to $\sqrt{2}$. The $f_t$ value that characterizes any $\beta$ transition depends on the total transition energy $Q_{EC}$, the half-life, $t_{1/2}$, of the parent state, and the branching ratio for the particular studied transition, [11]. However the above relation is only approximately satisfied by a restricted data set, and for this set one defines a "corrected" value $\mathcal{F}t \equiv f_t(1 + \delta'_R)(1 + \delta_{NS} - \delta_C)$, where $\delta'_R$ and $\delta_{NS}$ comprise the transition-dependent part of the radiative correction, while $\delta_C$ depends on details of the nuclear structure. In the above formula one takes $|G_V^2| = g_V^2 V_{ud}^2$, with $g_V = 1$, and (12) is written in the new form

$$\mathcal{F}t = \frac{K}{2V_{ud}^2(1 + \Delta^V_R)}$$

(13)

where $\Delta^V_R$ is the transition-independent part of the radiative corrections whose last estimation given in [11] is

$$\Delta^V_R = (2.361 \pm 0.038)\%$$

(14)

Similarly for neutron $\beta$ decay we make use of the formula

$$V_{ud}^2(1 + 3\lambda^2) = \frac{4908.7(1.9)s}{\tau_n}$$

(15)

see [14], where $\tau_n$ is the neutron mean life and $\lambda = g_A/g_V$. In our approach $V_{ud}$, $\Delta^V_R$, and $\lambda$ are free parameters to be found from fit.

In SM the purely leptonic decay of a $P$ meson, $P \rightarrow l\bar{\nu}_l$, proceeds via annihilation of the quark pair to a charged lepton and neutrino through exchange of a virtual $W$ boson, and the branching fraction, up to radiative
corrections, has the form

\[ B(P \to l\bar{\nu}_l) = \frac{G_F^2 M_P m_l^2}{8\pi \hbar} \left(1 - \frac{m_l^2}{M_P^2}\right)^2 f_P^2 V_{qq'}^2 \tau_P \]  

(16)

where \( G_F \) is the Fermi constant, \( M_P \) and \( m_l \) are the \( P \) meson and \( l \) lepton masses, respectively, \( f_P \) is the decay constant, \( V_{qq'} \) is the modulus of the corresponding KM matrix element, and \( \tau_P \) is \( P \) lifetime.

The next simple decays involving \( V_{ij} \) moduli are the semileptonic decays of heavy pseudoscalar mesons, \( H \), into lighter ones, \( P \), whose physical observable is the differential decay rate, written as

\[ \frac{d\Gamma(H \to P l\nu_l)}{dq^2} = \frac{G_F^2 V_{qq'}^2}{192\pi^3 M_H^3} \lambda^{3/2}(q^2)|f_+(q^2)|^2 \]  

(17)

where \( q = p_H - p_P \) is the transferred momentum, and \( f_+(q^2) \) is the global form factor which is a combination of the two form factors generated by the vector part of the weak current. When the leptons are electrons, or muons whose masses are low compared to the mass difference, \( m_H - m_P \), \( \lambda(q^2) \) is the usual triangle function

\[ \lambda(q^2) = (M_H^2 + M_P^2 - q^2)^2 - 4M_H^2 M_P^2 \]  

(18)

For the decay \( \bar{B} \to Dl\nu \) the experimenters make use of an other variable, namely \( w = (M_B^2 + M_D^2 - q^2)/(2M_B M_D) \). When the \( \tau \) lepton is involved the above formulae are a little bit more complicated, see [15].

Usually, till now, the experimenters provided numerical values for products of the form \( V_{qq'} f_+(0) \), and a few for \( V_{qq'} |f_+(q^2)| \).

The second \( \chi^2 \)-component which takes into account the experimental
data has the form
\[ \chi^2_2 = \sum_i \left( \frac{d_i - \tilde{d}_i}{\sigma_i} \right)^2 \]  
(19)

where \( d_i \) are the physical parameters one wants to be found from fit, \( \tilde{d}_i \) are the numerical values that describe the corresponding experimental data, while \( \sigma_i \) is the uncertainty associated to \( \tilde{d}_i \). For semileptonic decays \( d_i \) could be of the form \( d_i = |f_+(q_i^2)|V_{kl} \). In the following our fitting \( \chi^2 \)-function will be

\[ \chi^2 = \chi^2_1 + \chi^2_2 \]  
(20)

As it is easily seen the \( V_{ij} \) moduli enter naturally in all the formulae that describe the leptonic and semileptonic decays being, in our opinion, a strong argument for their use as fit parameters.

### 4 Experimental Data

In our analysis we used the data on superallowed \( 0^+ \rightarrow 0^+ \) nuclear \( \beta \) decays published in the papers [10]-[13], and data on the neutron lifetime from four papers: [16], [17], [18], [19], for \( V_{ud} \) determination. We also used four values for the \( \beta \)-asymmetry parameter \( A_0 \), from papers [20], [21], [22], [23], and one for the electron-antineutrino correlation coefficient \( a_0 \), [24], for the \( \lambda \) determination.

\( V_{us} \) modulus is involved in the kaon and pion leptonic and semileptonic decays, but also in the ratio \( V_{us}/V_{ud} \), that appears in the Marciano relation,
that we write as

\[
V_{us}^2 f_{\pi K}^2 (1 + C_r) = \frac{\mathcal{B}(K \to \mu \bar{\nu}_\mu (\gamma)) \tau_\pi m_\pi (1 - \frac{m_\mu^2}{m_\pi^2})^2}{\mathcal{B}(\pi \to \mu \bar{\nu}_\mu (\gamma)) \tau_K m_K (1 - \frac{m_\mu^2}{m_K^2})^2}
\]  

(21)

where \(C_r\) is a radiative correction stemming from both \(\pi\) and \(K\) hadronic structures. In fact relations of the form (21) can be written for all the leptonic and semileptonic decays, which leads to a cancellation of the experimental errors appearing on the right side.

The last numerical values for the product \(f_{\pi K}^{0}(0)V_{us}\) are given by KLOE collaboration, [26], and by FlaviaNet Working Group, [27], and in fit we made use of all the (little) different \(f_{\pi}^{0}(0)V_{us}\) values corresponding to the five channels. We also used results from [28], [29], [30], [31], [32], [33], [34], [35], [36], [37], [38], [39] that give only a “mean value” for the above product.

\(V_{ub}\) is the most poorly determined modulus although there is much experimental information coming from decays \(B \to \pi l\nu\), see [41], [42], [43], [44], [46], [47], [48]. In these papers the experimenters have been confronted with the known difficulty, getting two distinct parameters, \(V_{ub}\) and \(|f_{B\pi}^{0}(q^2)|\), from their product measured from experimental data. Thus they used the form factor lattice computations to obtain \(V_{ub}\) values depending on \(q^2\), see [41], [42], and [43]. For fit we found three measurements for \(f_{B}V_{ub}\), [44], [46], and three phenomenological determinations involving \(f_{B\pi}^{0}(0)V_{ub}\), [47], [48], [49].

\(V_{cd}\) and \(V_{cs}\) moduli enter the leptonic decays \(D \to l\nu\), [50], [51], [52], and \(D_{s}^{+} \to l\nu\), [53], [54], [55], [56], [57], respectively, as well as the semileptonic decays \(D \to \pi l\nu\), and \(D \to Kl\nu\), [59], [60], [61]. In the last three papers one find \(V_{cq}f_{\pi}^{+}(0)\) values, for \(q = d\), and \(q = s\), and for the first time numerical
values for the product
\[ |f_+(q^2)|V_{eq} = \sqrt{\frac{d\Gamma}{dq^2} \frac{24\pi^3 \rho_K^{K,\pi}}{G_F^2}} \]  
that allow the form factor extraction directly from data. The above semileptonic decays also allow the measurement of the ratio \( V_{cd} f^{D\pi^+}(0)/V_{cs} f^{DK^+}(0) \), see [60], [61], [62], [63], [64], [65], and give an independent determination of the ratio \( f^{DK^+}(0)/f^{D\pi^+}(0) \), which in fit was considered a new independent parameter.

Finally from the semileptonic decays \( \bar{B} \to Dl\nu \) and \( \bar{B} \to D^*l\nu \), [66], [67], [68], [69], [70], [71], [72], [73], [74], [75], [76], [77], one find the \( V_{cb} \), \( G(1) \) and \( F(1) \) parameters.

5 Numerical results

Data from the above cited papers were used to define \( \chi_2^2 \), the second component of full \( \chi^2 \), which has a parabolic form in \( V_{ij}^2 \). The first component \( \chi_1^2 \), [11], that contain all unitarity constraints, has a parabolic part, and one that is highly nonlinear in all \( V_{ij} \). Thus we had to test the stability of the expected physical values against the strong non-linearity implied by unitarity. Eventually the chosen method was to modify all the measured central values in the same sense, plus and minus, respectively, proportional to their corresponding uncertainties.

An important assumption included in our approach was: the numerical \( V_{ij} \) moduli values must be the same irrespective of the physical processes where they are implied. Accordingly the other parameters, such as decay constants, \( f_P \), form factors, \( f_+(0) \), \( \lambda \), etc., that parametrize each given experiment, have been considered as independent parameters to be obtained.
from fit, by applying the usual technique to obtain their mean values and uncertainties.

The stability tests provided sets of different moduli matrices that have been used to obtain a mean value matrix and its corresponding error matrix. The mean and uncertainty matrices have been computed by embedding the unitary matrices into the double stochastic matrix set, see papers [8] and [81].

The central values and uncertainties of data used in fit are those published in the above cited papers, and we combined the statistical and systematic uncertainties in quadrature when the experimenters provided both of them. The numerical values obtained from fit for the decay constants, $f_\pi, f_K, f_B, f_D, f_{Ds}$, the semileptonic form factors $f_+(0), \Delta V^i_R, \lambda$, as well as for the ratios $f_K/f_\pi$ and, $f^{DK}_+/f^{D\pi}_+(0)$ are given in Table 1. All of them are in the expected range although many have big uncertainties.

Our approach allows a “fine structure analysis” of all experiments measuring one definite quantity, such as $\Delta V^i_R$, or $f^{K\pi}_+(0)$, etc. For example KLOE collaboration data, [26], and FlaviaNet Working Group data, [27], on $f^{K\pi}_+(0)V_{us}$ lead to

$$f^{K\pi}_+(0)_{KLOE} = 950.38 \pm 5.56 \text{ and } f^{K\pi}_+(0)_{Flavia} = 955.06 \pm 4.31$$

(23)

respectively, whose central values are a little bit different, but compatible between them at 1σ, and all together provide

$$f^{K\pi}_+(0)_{KLOE+Flavia} = 952.72 \pm 5.30$$

(24)

$f^{K\pi}_+(0)$ from Table 1 has a precision of 1%, and the above value obtained from ten measurements has a precision of 0.56%.
Table 1. Numerical values for decay constants $f_P$ and form factors $f_+(0)$ in MeV units, and $\Delta V_R$ and $\lambda$

| Parameter | Central Value | Uncertainty |
|-----------|---------------|-------------|
| $f_\pi$   | 131.131       | 1.522       |
| $f_K$     | 154.97        | 2.17        |
| $f_K/f_\pi$ | 1.1818     | 0.0042      |
| $f_B$     | 222.8         | 25.0        |
| $f_D$     | 207.6         | 9.8         |
| $f_{D_s}$ | 271.0         | 18.0        |
| $f_{K\pi}^+(0)$ | 955.34 | 9.27        |
| $f_{B\pi}^+(0)$ | 214.9      | 13.4        |
| $f_{D\pi}^+(0)$ | 653.2      | 19.1        |
| $f_{DK}^+(0)$ | 751.8      | 10.4        |
| $f_{DK}^+(0)/f_{D\pi}^+(0)$ | 1.171 | 0.049        |
| $\mathcal{F}(1)$ | 957.5       | 57.7        |
| $\mathcal{G}(1)$ | 1,125.3     | 40.7        |
| $\Delta V_R \%$ | 2.373       | 0.096       |
| $\lambda$ | -1.2686       | 0.0057      |
| $C_r$     | 0.002         | 0.0001      |

$^1$ Data on $V_{cd}f_{D\pi}^+/V_{cs}f_{DK}^+$ from papers $^{60-65}$

In this approach one could obtain information about lepton universality in leptonic decays. Because the corresponding $f_M$ decay constant should be same for both decays $M \to \mu \nu$ and $M \to \tau \nu$, a big difference between them could show a possible violation. As an example we chose $M = D_{s}^+$ meson since lattice computations provided a number for this decay constant,
\[ f_{D^+} = 241(3), \text{ see paper } [78], \text{ with a very small error. Our results are} \]

\[ f_{D^+_s \rightarrow \mu \nu} = 265.0 \pm 14.0, \quad f_{D^+_s \rightarrow \tau \nu} = 276.0 \pm 20.0 \]  

(25)

which are consistent with lepton universality. The above two numbers together with that from Table 1 completely disagree with that provided by lattice computations, being far away from theoretical prediction at 8\(\sigma\), 13\(\sigma\), and 10\(\sigma\), respectively, where, \(\sigma = 3\), is the lattice uncertainty. The experimental spreading is, 246.0 \(\leq\) \(f_{D^+_s}\) \(\leq\) 311.0, and the minimal and maximal values correspond to the branching ratios obtained for \(B(D_s^+ \rightarrow \mu^+\nu_\mu) = (5.15 \pm 0.63 \pm 0.20 \pm 1.29) \times 10^{-3}\), see [58], and to the branching ratio \(B(D_s^+ \rightarrow \tau\nu) = (8.0 \pm 1.3 \pm 0.6)\%\), given by Eq.(6) in paper [57], respectively. The new result obtained in [79] does not improve the situation.

On the other hand if one computes the difference between each one of the above three values and that provided by lattice computation, divided by the corresponding experimental error \(\sigma\) obtained from the fit one finds the same value, 1.17, which shows again the consistency of experimental data. In our opinion the lattice number is much underestimated. In contradiction their value for \(f_K/f_\pi = 1.189(7)\) is not far from the fit value \(f_K/f_\pi = 1.1818(42)\).

Another unexpected result concerns the \(\Delta R^V\) constancy, usually assumed in all the four papers [10]-[13], assumption that is not confirmed by our analysis. However our result \(\Delta R^V = 2.373 \pm 0.096\) obtained from data [10] is in good concordance with the value given by relation (14), see Table 2, but our uncertainty is 2.5\(\sigma\) higher than the theoretical one. In this case the \(\Delta R^V\) spreading that results from paper [10] is, 2.2027 \(\leq\) \(\Delta R^V\) \(\leq\) 2.472, which corresponds to 7.1\(\sigma\) where \(\sigma = 0.038\%\) is the theoretical uncertainty. The extremal nuclei are \(^{22}\text{Mg}\) and \(^{54}\text{Co}\), respectively. The spreading obtained
from Savard et al data, [13], corresponds to 10.8σ, and the extremal nuclei are 74Rb and 34Cl. However the difference between the central results from [10] and [13] is 2σ, which suggests that there is still room for numerical improvements. The mean values and the corresponding uncertainties for all data from papers [10]-[13] are given in Table 2.

Table 2. ∆\(V_R\) central values and uncertainties provided by fit when using data from Refs. [10]-[13]

| Ref. | [10] | [11] | [12] | [13] | [11] |
|------|------|------|------|------|------|
| ∆\(V_R\)% | 2.373(96) | 2.399(112) | 3.361(196) | 2.294(136) | 2.361(38)\(^1\) |

\(^1\) The “constant” ∆\(V_R\) and its uncertainty, from [11]

The precision of \(f_P\) and \(f_{+}(0)\) determinations in Table 1 varies from 1% for \(f_P^{Kπ}(0)\) to 11% for \(f_B\). More about variability of the above parameters could be learnt from KM moduli matrix. Our fit result for KM central moduli values is

\[
V_c = \begin{bmatrix}
0.974022 & 0.226415 & 0.004251 \\
0.226253 & 0.973323 & 0.038108 \\
0.009569 & 0.037131 & 0.999265
\end{bmatrix}
\]  

(26)

\(V_c\) and its associated uncertainty matrix, \(\sigma_{V_c}\), have been obtained by using the convexity property, (4), and \(\sigma_{V_c}\) has the form

\[
\sigma_{V_c} = \begin{bmatrix}
1.1 \times 10^{-6} & 1.9 \times 10^{-5} & 2.1 \times 10^{-5} \\
3.6 \times 10^{-5} & 2.7 \times 10^{-4} & 3.1 \times 10^{-4} \\
3.5 \times 10^{-5} & 2.9 \times 10^{-4} & 3.9 \times 10^{-4}
\end{bmatrix}
\]  

(27)

The last matrix has been obtained with the help of stability tests. One such matrix is (28) that was obtained when all the central measured values
have been modified with plus one tenth from the corresponding uncertainty. Although such a modification is highly improbable from an experimental point of view, it brings to light the variation direction for all parameters entering the fit.

\[
V_+ = \begin{pmatrix}
0.974021 & 0.226332 & 0.007484 \\
0.226114 & 0.973083 & 0.044512 \\
0.012432 & 0.043391 & 0.998891 \\
\end{pmatrix}
\] (28)

For example the above three matrices show that \( V_{ud} \) is precisely determined with four digits, while \( V_{us}, V_{cd}, \) and \( V_{cs} \) only with three digits. \( V_+ \) matrix also shows the high \( V_{ub} \) volatility, such that future data could lead to higher values for it than that given by \( V_c \) matrix. In fact matrices \( V_+ \) and \( V_c \) show that \( f_B \in (123.5, 222.8) \).

Although our value for \( V_{ub} = (4.25 \pm 0.02) \times 10^{-3} \) is higher than that from PDG fit, [2], there are experimental determinations which are compatible with it, one example being Ref. [48], whose value is \( V_{ub} = (4.1 \pm 0.2_{st} \pm 0.2_{syst} \pm 0.2_{FF}) \times 10^{-3} \).

The new data from \( D \) leptonic, [50]-[52], and semileptonic decays, [59]-[61], combined with the new data from \( \bar{B} \rightarrow Dl\nu \) and \( \bar{B} \rightarrow D^*l\nu \) [66]-[70], changed the KM moduli values, in particular those from the last row and column, see [2], p. 150.

Our approach allows the form factor determination. The paper [61] provided for the first time results on \( V_{cq}|f_+(q^2)|, q = d, s, \) and in Table 3 are given our determinations.

A big step forward will be the measurement of \( q^2 \) dependence for products of the form \( |f_+(q^2)U_{qb}| \), where \( q = u, c, \) for \( \bar{B} \rightarrow \pi l\nu, \bar{B} \rightarrow Dl\nu \) and
$B \to D^* l \nu$ decays, similar to that done for $D$ semileptonic decays.

Table 3. $|f_+(q^2)|$ form factors from $D \to \pi l \nu$ and $D \to K l \nu$ decays

| Bin | $f_+^\pi(q^2) (\pi^0 e^+ \nu_e)$ | $f_+^\pi(q^2) (\pi^- e^+ \nu_e)$ | $f_+^K(q^2) (K^- e^+ \nu_e)$ | $f_+^K(q^2) (\bar{K}^0 e^+ \nu_e)$ |
|-----|--------------------------------|--------------------------------|--------------------------------|--------------------------------|
| 1   | 0.6895 ± 0.0576 | 0.7072 ± 0.0355 | 0.7798 ± 0.0136 | 0.7808 ± 0.0208 |
| 2   | 0.7514 ± 0.0709 | 0.7735 ± 0.0400 | 0.8281 ± 0.0146 | 0.8096 ± 0.0239 |
| 3   | 0.8442 ± 0.0841 | 0.7956 ± 0.0488 | 0.8435 ± 0.0156 | 0.8466 ± 0.0259 |
| 4   | 0.8928 ± 0.0974 | 0.9812 ± 0.0532 | 0.9113 ± 0.0198 | 0.8856 ± 0.0289 |
| 5   | 1.1005 ± 0.115  | 1.017 ± 0.065  | 0.9945 ± 0.0229 | 0.9175 ± 0.0331 |
| 6   | 1.3083 ± 0.151  | 1.101 ± 0.084  | -                  | 2.024 ± 0.301  |
| 7   | 1.5779 ± 0.221  | 1.635 ± 0.111  | -                  |                          |
| 8   | -                | 1.759 ± 0.2358 | -                  |                          |
| 9   | -                | 2.024 ± 0.301  | -                  |                          |

The simplest case is that of $|f_+(q^2)U_{ub}|$ because there are measured data
which in principle could be transformed in values for the product $|f_+(q^2)U_{ub}|$.

Such measurements will allow a more precise determination of the moduli $V_{ub}$ and $V_{cb}$, and, by consequence, they will provide better values for all KM matrix moduli from the the first two rows, and, perhaps, the first hints for new physics beyond SM, if any.

The $V_c$ matrix, (28), provides numerical values for $\delta$ and the angles of the standard unitarity triangle, as follows

$$\delta = (89.96 \pm 0.36)^\circ, \quad \alpha = (64.59 \pm 0.27)^\circ,$$

$$\gamma = (89.98 \pm 0.06)^\circ, \quad \beta = (25.49 \pm 0.28)^\circ$$

(30)

The $\delta$ value could be interpreted as a maximal violation of $CP$ symmetry because $\sin \delta \approx 1$. A surprising change is the shape of the standard unitarity triangle that is now a right triangle, since $\gamma \approx 90^\circ$. The Jarlskog invariant is $J = (3.567 \pm 0.007) \times 10^{-5}$.

Similar results are obtained from $V_+$ matrix even if $V_{ub}$ is almost twice bigger than that from $V_c$; they are

$$\delta = (90.0 \pm 0.2)^\circ, \quad \alpha = (54.1 \pm 0.1)^\circ,$$

$$\gamma = (90.0 \pm 0.4)^\circ, \quad \beta = (36.0 \pm 0.2)^\circ$$

(31)

The above results show that $\delta$ and $\gamma$ angles are independent of $V_{ub}$ variation, while $\alpha$ and $\beta$ are moderately dependent. Thus experimental results on the product $|f_+(q^2)U_{ub}|$ could lead to a better determination of all the angles of the standard unitarity triangle.

The small angles uncertainties show that $V_c$ and $V_+$ matrices are highly compatible with unitarity constraints and all the 165 cosine $\delta$ formulae provide very close each other values for all of them.

A comparison with lattice computations from [80] shows that $f_+^{D\pi}(0)$ and
$f^{DK}_+(0)$, as well as their ratio are in good agreement with the experimental values obtained by us, and our uncertainties are smaller.

6 Conclusion

In this paper we presented a phenomenological tool that allows determination of the KM moduli, semileptonic form factors and decay constants directly from experimental data. It is based on a new implementation of unitarity constraints that makes use of KM matrix moduli as fit parameters. These constraints are strong enough and give a consistent picture of nowadays flavour physics, and until now provide no signals for physics beyond the SM.

A feature of our tool is that all the measurable parameters are each other strongly correlated, a little modification of one of them propagates to all the other parameters, property which is a consequence of unitarity constraints.

The new data from the $D$ leptonic and semileptonic decays, and those from $B \to D(D^*)\ell\nu$ led to a significant change of KM moduli and of standard unitarity triangle shape. A crucial step forward would be the measurement of $B \to \pi\ell\nu$ form factors in bins of $0.5$ GeV$^2$, similar to that done in [61] for $D \to \pi\ell\nu$ and $D \to K\ell\nu$, that will allow a better $V_{ub}$ determination, and a stabilization of moduli values entering the first two rows.

As a final conclusion one can say that in contradistinction to experiments from other physics branches, those from semileptonic flavour physics have to be considered as a single experiment because only using all of them, they could provide numbers for all the physical quantities: form factors, decay constants, $V_{ij}$ moduli, etc.

Acknowledgements. We acknowledge a partial support from ANCS Contract No 15EU/06.04.2009.
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