Measures of Noncompactness in $(\tilde{N}_{\Delta}^q)$ Summable Difference Sequence Spaces

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Abstract. In the given paper we first introduce $\tilde{N}_{\Delta}^q$ summable difference sequence spaces and prove some properties of these spaces. We then obtain the necessary and sufficient conditions for infinite matrices $A$ to map these sequence spaces on the spaces $c_0, c$ and $\ell_\infty$, the Hausdorff measure of noncompactness is then used to obtain the necessary and sufficient conditions for the compactness of the linear operators defined on these spaces.

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1 Introduction and Preliminaries

We write $w$ for the set of all complex sequences $x = (x_k)_{k=0}^\infty$ and $\phi, c_0, c$ and $\ell_\infty$ for the sets of all finite, convergent sequences and sequences convergent to zero, and bounded respectively. The sequence $e$ is given by $e = (1, 1, 1, \ldots)$ and $e^{(n)}$ is the sequence with 1 as only nonzero term at the $n$th place for each $n \in \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. Further by $cs$ and $\ell_1$ we denote the spaces of all sequences whose series is convergent and absolutely convergent respectively.

The $\beta$–dual of a subset $X$ of $w$ is defined by

$$X^{\beta} = \{a \in w : ax = (a_kx_k) \in cs \text{ for all } x = (x_k) \in X\}$$

If $A$ is an infinite matrix with complex entries $a_{nk} \ n, k \in \mathbb{N}$, we write $A_n$ for the sequence in the $n$th row of $A$, $A_n = (a_{nk})_{k=0}^\infty \ n \in \mathbb{N}$. The

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A−transform of any $x = (x_k) \in w$ is given by $Ax = (A_n(x))_{k=0}^{\infty}$, where

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk}x_k \quad n \in \mathbb{N}$$

the series on right must converge for each $n \in \mathbb{N}$.

If $X$ and $Y$ are subsets of $w$, we denote by $(X, Y)$, the class of all infinite matrices that map $X$ into $Y$. So $A \in (X, Y)$ if and only if $A_n \in X^\beta$, $n = 0, 1, 2, \ldots$ and $Ax \in Y$ for all $x \in X$. The matrix domain of an infinite matrix $A$ in $X$ is defined by

$$X_A = \{x \in w : Ax \in X\}$$

If $X$ and $Y$ are Banach Spaces, then by $\mathcal{B}(X, Y)$ we denote the set of all bounded (continuous) linear operators $L : X \rightarrow Y$, which is itself a Banach space with operator norm $\|L\| = \sup_x \{\|L(x)\|_Y : \|x\| = 1\}$ for all $L \in \mathcal{B}(X, Y)$. The linear operator $L : X \rightarrow Y$ is said to be compact if the domain of $L$ is all of $X$ and every bounded sequence $(x_n) \in X$, the sequence $(L(x_n))$ has a sub-sequence which converges in $Y$. The operator $L \in \mathcal{B}(X, Y)$ is said to be of finite rank if $\dim R(L) < \infty$, where $R(L)$ denotes the range space of $L$. A finite rank operator is clearly compact.

The concept of difference sequence spaces was first introduced by Kizmaz [1] and later several authors studied new sequence spaces defined by using difference operators like Mursaleen and Nouman [2] and many more.

In the past, several authors studied matrix transformations on sequence spaces that are the matrix domains of the difference operator, or of the matrices of the classical methods of summability in spaces such as $\ell_p$, $c_0$, $c, \ell_\infty$, or others. For instance, some matrix domains of the difference operator were studied in ([1], [3]), of the Riesz matrices in [4], and so on. In this paper, we first define a new summable difference sequence space as the matrix domains $X_T$ of arbitrary triangles $\tilde{N}_q$ and $\Delta^-$ and obtain it basis, $\beta$ dual of the new sequence spaces. We then find out the necessary and sufficient condition for the exists of matrix transformations and finally obtain the results related to the compactness of the linear operators on these new sequence spaces.
2 \( \tilde{N}_\Delta^q \) Summable Difference Sequence Spaces

Define the difference operator as

\[
\Delta^- x_k = x_{k-1} - x_k , k = 0, 1, 2, \ldots \quad \text{where} \quad x_{-1} = 0
\]

(1)

The \( \Delta^- = (\delta_{nk})_{n,k=0}^\infty \) is a triangular matrix written as

\[
\delta_{nk} = \begin{cases} 
-1 & k = n \\
1 & k = n - 1 \\
0 & k > n
\end{cases}
\]

The inverse of this matrix is \( S = (s_{nk}) \) given as

\[
s_{nk} = \begin{cases} 
-1 & 0 \leq k \leq n \\
0 & k > n
\end{cases}
\]

Let \( (q_k)_{k=0}^\infty \) be positive sequences and \( (Q_n)_{n=0}^\infty \) be the sequence defined as \( Q_n = \sum_{i=0}^n q_i \). The \((\tilde{N}, q)\) transform of the sequence \((x_k)_{k=0}^\infty\) is defined as

\[
t_n = \frac{1}{Q_n} \sum_{i=0}^n q_i x_i
\]

The matrix \( \tilde{N}_q \) for this transformation can be written as

\[
(\tilde{N}_q)_{nk} = \begin{cases} 
q_k & 0 \leq k \leq n \\
Q_n & k > n
\end{cases}
\]

The inverse of this matrix is [5]

\[
(\tilde{N}_q)^{-1} = \begin{cases} 
(-1)^{n-k} \frac{Q_k}{q_n} & n-1 \leq k \leq n \\
0 & 0 \leq k \leq n-2, k > n
\end{cases}
\]

We define the \( \tilde{N}_\Delta^q \) summable difference sequence spaces as

\[
(\tilde{N}_\Delta^-)_0 = (c_0, \Delta^-)_{\tilde{N}_q} = \left\{ x \in w : \tilde{N}_q \Delta^- x = \left( \frac{1}{Q_n} \sum_{k=0}^n q_k \Delta^- x_k \right)_{n=0}^\infty \in c_0 \right\}
\]

\[
(\tilde{N}_\Delta^-) = (c, \Delta^-)_{\tilde{N}_q} = \left\{ x \in w : \tilde{N}_q \Delta^- x = \left( \frac{1}{Q_n} \sum_{k=0}^n q_k \Delta^- x_k \right)_{n=0}^\infty \in c \right\}
\]

\[
(\tilde{N}_\Delta^-)_\infty = (\ell_\infty, \Delta^-)_{\tilde{N}_q} = \left\{ x \in w : \tilde{N}_q \Delta^- x = \left( \frac{1}{Q_n} \sum_{k=0}^n q_k \Delta^- x_k \right)_{n=0}^\infty \in \ell_\infty \right\}
\]
For any sequence \( x = (x_k)_{k=0}^{\infty} \), define \( \tau = \tau(x) \) as the sequence with \( n \)th term given by

\[
\tau_n = (N^q_{\Delta^{-}})_{n}(x) = \frac{1}{Q_{n}} \sum_{k=0}^{n} q_k \Delta^{-} x_k \quad (n = 0, 1, 2, \ldots) \quad (2)
\]

For any two sequences \( x \) and \( y \), the product \( xy = (x_k y_k)_{k=0}^{\infty} \).

### 2.1 Basis for the new sequence spaces

**Proposition 2.1.** [[6], 1.4.8, p.9]

Every triangle \( T \) has a unique inverse \( S = (s_{nk})_{n,k=0}^{\infty} \) which is also a triangle, and \( x = T(S(x)) = S(T(x)) \) for all \( x \in w \).

**Proposition 2.2.** [[7], Theorem 2.3] If \( (b^{(n)})_{n=0}^{\infty} \) is a basis of the linear metric space \( (X,d) \), then \( (S(b^{(n)})_{n=0}^{\infty} \) is a basis of \( Z = X_T \) with metric \( d_T \) defined by \( d_T(z,\bar{z}) = d(T(z),T(\bar{z})) \) for all \( z,\bar{z} \in Z \). Where \( S \) is the inverse of the matrix \( T \).

It is obvious that \( (c_0,\Delta^{-})_{N_q} = (c_0)_{N_q} \Delta^{-} \), So the basis for new spaces are given by \( (N_q \cdot \Delta^{-})^{-1} \left( e^{(n)} \right) = (\Delta^{-})^{-1} \left( N_q \right)^{-1} \left( e^{(n)} \right) \) we have

**Theorem 2.3.** Let \( \lambda_k = (N^q_{\Delta^{-}} x)_k \) for all \( k \in \mathbb{N} \). Define the sequence \( s^{(k)} = \{ s^{(k)}_n \}_{n \in \mathbb{N}} \) of the elements of \( (c_0,\Delta^{-})_{N_q} \) as

\[
s^{(k)}_n = \begin{cases} \sum_{j=1}^{k} Q_j \left( \frac{1}{q_{j+1}} - \frac{1}{q_j} \right) & 0 \leq k < n \\ -\frac{Q_k}{q_k} & k = n \\ \sum_{j=1}^{k-1} Q_j \left( \frac{1}{q_{j+1}} - \frac{1}{q_j} \right) + \frac{Q_k}{q_k} & k > n \end{cases}
\]

for every fixed \( k \in \mathbb{N} \). Then

i) The sequence \( \{ s^{(k)} \}_{k \in \mathbb{N}} \) is a basis for the space \( (c_0,\Delta^{-})_{N_q} \) and any \( x \in (c_0,\Delta^{-})_{N_q} \) can be uniquely represented in the form

\[
x = \sum_k \lambda_k s^{(k)}
\]
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ii) The set \( \{e, s^{(k)}\} \) is a basis for the spaces \((c, \Delta^-)\tilde{N}_q\) and any \( x \in (c, \Delta^-)\tilde{N}_q \) has a unique representation in the form

\[
x = ls_n^{(-1)} + \sum_k (\lambda_k - l)s^{(k)}
\]

where for all \( k \in \mathbb{N} \), \( l = \lim_{k \to \infty} ((\tilde{N}_\Delta^-) x)_k \).

**Proof.** Since \((X, \Delta^-)\tilde{N}_q = (X)\tilde{N}_q, \Delta^- \) for \( X = c_0, c, \ell_\infty \). Now \( e = (e^{(k)})_k \) is the standard basis for \( c \) and by Now \( \tilde{N}_q \) is a triangle and \( \Delta^- \) is triangle so \( \tilde{N}_q, \Delta^- \) is also a triangle and

\[
(\tilde{N}_q \cdot \Delta^-)^{-1} = (\Delta^-)^{-1} \cdot (\tilde{N}_q)^{-1} = \begin{cases} 
Q_k \left( \frac{1}{q_{k+1}} - \frac{1}{q_k} \right) & 0 \leq k < n \\
\frac{1}{q_k} & k = n \\
0 & k > n
\end{cases}
\]

Hence \( \{s^{(k)}\}_{k \in \mathbb{N}} \) is a basis for the space \((c_0, \Delta^-)\tilde{N}_q\) and the results i) and ii) are obvious to follow. □

**Note:** We consider the standard basis to find the general results related to our sequence spaces.

**Theorem 2.4.** The sequence spaces \((\tilde{N}_\Delta^-)_0, (\tilde{N}_\Delta^-)1 \) and \((\tilde{N}_\Delta^-)\ell_\infty \) are BK-spaces with norm \( \|x\|_{\tilde{N}_\Delta^-} \) given by

\[
\|x\|_{\tilde{N}_\Delta^-} = \sup_{n} \left| \frac{1}{Q_n} \sum_{k=0}^{n} q_k \Delta^- x_k \right|
\]

If \( Q_n \to \infty (n \to \infty) \), then \((\tilde{N}_\Delta^-)_0\) has AK, and every sequence \( x = (x_k)_{k=0}^\infty \in (\tilde{N}_\Delta^-)_0 \) has unique representation

\[
x = le + \sum_k (\lambda_k - l)e^{(k)} \tag{3}
\]

where \( l \in \mathbb{C} \) is such that \( x - le \in (\tilde{N}_\Delta^-)_0 \)

**Proof.** Since \((X, \Delta^-)\tilde{N}_q = (X)\tilde{N}_q, \Delta^- \) for all \( X = c_0, c, \ell_\infty \) and the spaces \( c_0, c, \ell_\infty \) are BK spaces with respect to natural norm [8], p.217-218] and the matrix \( \tilde{N}_q \cdot \Delta^- \) is a triangle so by Theorem 4.3.12, [6], gives \((\tilde{N}_\Delta^-)_0, (\tilde{N}_\Delta^-)_1 \) and \((\tilde{N}_\Delta^-)\ell_\infty \) are BK spaces The space \((\tilde{N}_\Delta^-)_0\) has AK and the unique representation of elements of \((\tilde{N}_\Delta^-)_0 \) are simply followed from Theorem 2 of [9] and [10]. □
2.2 \( \beta \) dual of the new spaces

In order to find the \( \beta \) dual we need the results of [11] which are

**Lemma 2.5.** \( A \in (c_0 : l_1) \) if and only if

\[
\sup_{K \in F} \left| \sum_{k \in K} a_{nk} \right| < \infty
\]

**Lemma 2.6.** \( A \in (c_0 : c) \) if and only if

\[
\sup_n \sum |a_{nk}| < \infty,
\]

\[
\lim_{n \to \infty} a_{nk} - \alpha_k = 0.
\]

**Lemma 2.7.** \( A \in (c_0 : \ell_\infty) \) if and only if

\[
\sup_n \sum |a_{nk}| < \infty,
\]

**Theorem 2.8.** Let \((q_k)_{k=0}^\infty\) be positive sequences, \(Q_n = \sum_{i=0}^n q_i\) and \(a = (a_k) \in w\) we define a matrix \(C = (c_{nk})_{n,k=0}^\infty\) as

\[
c_{nk} = \begin{cases} 
Q_k \left( \frac{1}{q_{k+1}} - \frac{1}{q_k} \right) \sum_{j=k+1}^n a_j & \text{if } 0 \leq k < n \\
\frac{Q_k a_k}{q_k} & \text{if } k = n \\
0 & \text{if } k > n
\end{cases}
\]

and consider the sets

\[
c_1 = \left\{ a \in w : \sup_n \sum_k |c_{nk}| < \infty \right\} ; c_2 = \left\{ a \in w : \lim_{n \to \infty} c_{nk} \text{ exists for each } k \in \mathbb{N} \right\}
\]

\[
c_3 = \left\{ a \in w : \lim_{n \to \infty} \sum_k |c_{nk}| = \sum_k \lim_{n \to \infty} c_{nk} \right\} ; c_4 = \left\{ a \in w : \lim_{n \to \infty} \sum_k c_{nk} \text{ exists} \right\}
\]

Then \( [\tilde{N}_{\Delta -} q]_0^\beta = c_1 \cap c_2 \), \( [\tilde{N}_{\Delta -} q]_\infty^\beta = c_1 \cap c_2 \cap c_4 \) and \( [\tilde{N}_{\Delta -} q]_{\infty}^\beta = c_2 \cap c_3 \).
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**Proof.** We prove the result for \( \left[ (\tilde{N}_{\Delta_{-}}^{q})_{0} \right]^{\beta} \) for the other two same procedure can be followed. Let \( x \in \left( \tilde{N}_{\Delta_{-}}^{q} \right)_{0} \) then there exists a \( y \) such that \( y = \tilde{N}_{\Delta_{-}}^{q} x \).

Hence

\[
\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n} a_k \left( \tilde{N}_{\Delta_{-}}^{q} \right)^{-1} y_k = \sum_{k=0}^{n} a_k \left[ \sum_{j=0}^{k-1} Q_j \left( \frac{1}{q_{j+1}} - \frac{1}{q_j} \right) y_j - \frac{Q_k}{q_k} y_k \right] = \sum_{k=0}^{n} \left[ Q_{k-1} \left( \frac{1}{q_k} - \frac{1}{q_{k-1}} \right) \sum_{j=k+1}^{n} a_j - \frac{Q_k a_k}{q_k} \right] y_k = (Cy)_n
\]

So \( ax = (a_n x_n) \in cs \) whenever \( x \in \left( \tilde{N}_{\Delta_{-}}^{q} \right)_{0} \) if and only if \( Cy \in cs \) whenever \( y \in c_0 \).

Using Lemma 2.6 we get \( \left[ (\tilde{N}_{\Delta_{-}}^{q})_{0} \right]^{\beta} = c_1 \cap c_2 \) In the same way we can show the other two results as well. \( \square \)

By Theorem 7.2.9, \([6]\) we know that if \( X \) is a BK-space and \( a \in w \) then

\[
\|a\|^{*} = \sup \left\{ \sum_{k=0}^{\infty} a_k x_k : \|x\| = 1 \right\}
\]

provided the term on the right side exists and is finite, which is the case whenever \( a \in X^{\beta} \).

**Theorem 2.9.** For \( \left[ (\tilde{N}_{\Delta_{-}}^{q})_{0} \right]^{\beta} \), \( \left[ (\tilde{N}_{\Delta_{-}}^{q})_{1} \right]^{\beta} \) and \( \left[ (\tilde{N}_{\Delta_{-}}^{q})_{\infty} \right]^{\beta} \) the norm \( \|a\|^{*} \) is defined as

\[
\|a\|^{*} = \sup_{n} \left[ \sum_{k=0}^{n-1} Q_k \left( \frac{1}{q_{k+1}} - \frac{1}{q_k} \right) \sum_{j=k+1}^{n} a_j + \frac{Q_n a_n}{q_n} \right]
\]
Proof. If $x^{[n]}$ denotes the $n$th section of the sequence $x \in (\bar{N}^q_\Delta)_0$ then using (2) we have

$$\tau_k^{[n]} = \tau_k(x^{[n]}) = \frac{1}{Q_k} \sum_{j=0}^k q_j \Delta^- x_j^{[n]}$$

Let $a \in \left( (\bar{N}^q_\Delta)_0 \right)^\beta$, then for any non-negative integer $n$ define the sequence $d^{[n]}$ as

$$d^{[n]}_k = \begin{cases} Q_k \left( \frac{1}{q_{k+1}} - \frac{1}{q_k} \right) \sum_{j=k+1}^n a_j & 0 \leq k < n \\ \frac{Q_k a_k}{q_k} & k = n \\ 0 & k > n \end{cases}$$

Let $\|a\|_\Pi = \sup_n \|d^{[n]}\|_1 = \sup_n \left( \sum_{k=0}^\infty |d^{[n]}_{k+1}| \right)$ where $\Pi = \left( (\bar{N}^q_\Delta) \right)^\beta$. The inequality $\|a\|_\Pi \leq \|a\|^*$ is obvious.

Also

$$\left| \sum_{k=0}^\infty a_k x_k^{[n]} \right| = \left| \sum_{k=0}^{n-1} a_k \left( \sum_{j=0}^k \frac{1}{q_j} (Q_j \tau_j^{[n]} - Q_{j-1} \tau_{j-1}^{[n]}) \right) \right|$$

$$\leq \left| \sum_{k=0}^{n-1} Q_k \left( \frac{1}{q_{k+1}} - \frac{1}{q_k} \right) \left( \sum_{j=k+1}^n a_j \right) \tau_k^{[n]} \right| + \left| \frac{a_n Q_n}{q_n} \right| \tau_n^{[n]}$$

$$\leq \sup_k |\tau_k^{[n]}| \cdot \left( Q_k \left( \frac{1}{q_{k+1}} - \frac{1}{q_k} \right) \sum_{j=k+1}^n a_j + \left| \frac{a_n Q_n}{q_n} \right| \right)$$

$$= \|x^{[n]}\|_{\bar{N}^q_\Delta} \|d^{[n]}\|_1$$

$$= \|a\|_\Pi \|x^{[n]}\|_{\bar{N}^q_\Delta}$$

Hence $\|a\|^* \leq \|a\|_\Pi$

From the above inequalities we get the required conclusion. \[\square\]

Following are some well known results
Proposition 2.10. (cf. [12], Theorem 7) Let $X$ and $Y$ be BK spaces, then $(X, Y) \subset \mathcal{B}(X, Y)$ that is every matrix $A$ from $X$ into $Y$ defines an element $L_A$ of $\mathcal{B}(X, Y)$ where

$$L_A(x) = A(x) \quad \forall \ x \in X$$

Also $A \in (X, \ell_\infty)$ if and only if

$$\|A\|^* = \sup_n \|A_n\|^* = \|L_A\| < \infty$$

If $(b^{(k)})_{k=0}^{\infty}$ is a basis of $X, Y$ and $Y_1$ are FK spaces with $Y_1$ a closed subspace of $Y$, then $A \in (X, Y_1)$ if and only if $A \in (X, Y)$ and $A(b^{(k)}) \in Y_1$ for all $k = 0, 1, 2, \ldots$.

Proposition 2.11. (cf. [13], Proposition 3.4) Let $T$ be a triangle

(i) If $X$ and $Y$ are subsets of $w$, then $A \in (X, Y_T)$ if and only if $B = TA \in (X, Y)$.

(ii) If $X$ and $Y$ are BK spaces and $A \in (X, Y_T)$, then

$$\|L_A\| = \|L_B\|$$

Using Proposition 2.10 and Theorem 2.9 we can easily conclude that following:

Corollary 2.12. Let $(q_k)_{k=0}^{\infty}$ be a positive sequence, $Q_n = \sum_{k=0}^{n} q_k$ and $\Delta^-$ be the difference operator as defined in (1), then

i) $A \in \left( (N_{\Delta^-}^q)_{\infty}, \ell_\infty \right)$ if and only if

$$\sup_{m,n} \left[ \sum_{k=0}^{m-1} Q_k \left( \frac{1}{q_{k+1}} - \frac{1}{q_k} \right) \sum_{j=k+1}^{m} a_{nj} + \left| \frac{Q_m a_{nm}}{q_m} \right| \right] < \infty \quad (4)$$

and

$$\frac{A_n Q_n}{q} \in c_0 \ \forall \ n = 0, 1, \ldots \quad (5)$$
\[ A \in \left( \left( N_{\Delta}^q \right)_0, \ell_\infty \right) \text{ if and only if condition (4) holds and} \]
\[
\frac{A_n Q}{q} \leq c \quad \forall \ n = 0, 1, 2, \ldots
\]  
\hspace{1cm} (6)

\[ A \in \left( \left( N_{\Delta}^q \right)_0, \ell_\infty \right) \text{ if and only if condition (4) holds.} \]

\[ A \in \left( \left( N_{\Delta}^q \right)_0, c_0 \right) \text{ if and only if condition (4) holds and} \]
\[
\lim_{n \to \infty} a_{nk} = 0 \quad \text{for all } k = 0, 1, 2, \ldots
\]  
\hspace{1cm} (7)

\[ A \in \left( \left( N_{\Delta}^q \right)_0, c_0 \right) \text{ if and only if conditions (4), (5) and (7) holds and} \]
\[
\lim_{n \to \infty} \sum_{k=0}^{\infty} a_{nk} = 0 \quad \text{for all } k = 0, 1, 2, \ldots
\]  
\hspace{1cm} (9)

Again Using Proposition 2.11 and Theorem 2.4 we have the following corollary:

\section{Hausdorff Measure of Noncompactness}

From Mursaleen et. al. [14] we let \( S \) and \( M \) be the subsets of a metric space \((X, d)\) and \( \epsilon > 0 \). Then \( S \) is called an \( \epsilon \)-net of \( M \) in \( X \) if for every \( x \in M \) there exists \( s \in S \) such that \( d(x, s) < \epsilon \). Further, if the set \( S \) is finite, then the \( \epsilon \)-net \( S \) of \( M \) is called finite \( \epsilon \)-net of \( M \). A subset of a metric space is said to be \textit{totally bounded} if it has a finite \( \epsilon \)-net for
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every \(\epsilon > 0\).

If \(\mathcal{M}_X\) denotes the collection of all bounded subsets of metric space \((X, d)\). If \(Q \in \mathcal{M}_X\) then the Hausdorff Measure of Noncompactness of the set \(Q\) is defined by

\[
\chi(Q) = \inf \{ \epsilon > 0 : Q\text{ has a finite }\epsilon\text{-net in }X \}
\]

The function \(\chi : \mathcal{M}_X \to [0, \infty)\) is called Hausdorff Measure of Noncompactness \([15]\).

The basic properties of Hausdorff Measure of Noncompactness can be found in \([5, 16, 15]\). Some of those properties are:

If \(Q, Q_1\) and \(Q_2\) are bounded subsets of a metric space \((X, d)\), then

\[
\chi(Q) = 0 \iff Q\text{ is totally bounded set},
\]

\[
\chi(Q) = \chi(\bar{Q}),
\]

\[
Q_1 \subseteq Q_2 \Rightarrow \chi(Q_1) \leq \chi(Q_2),
\]

\[
\chi(Q_1 \cup Q_2) = \max \{ \chi(Q_1), \chi(Q_2) \},
\]

\[
\chi(Q_1 \cap Q_2) = \min \{ \chi(Q_1), \chi(Q_2) \}. 
\]

Further if \(X\) is a normed space the \(\chi\) has the additional properties connected with the linear structure.

\[
\chi(Q_1 + Q_2) \leq \chi(Q_1) + \chi(Q_2)
\]

\[
\chi(\eta Q) = |\eta|\chi(Q) \quad \eta \in \mathbb{C}
\]

If \(X\) and \(Y\) are normed space, then for \(A \in \mathcal{B}(X, Y)\) the Hausdorff Measure of Noncompactness of \(A\), is denoted by \(\|A\|_\chi\) and is defined as

\[
\|A\|_\chi = \chi(AB)
\]

Where \(B = \{x \in X : \|x\| = 1\}\) is the unit ball in \(X\).

Also \(A\) is said to be compact if and only if \(\|A\|_\chi = 0\) and \(\|A\|_\chi \leq \|A\|\).

**Proposition 3.1.** ([15], Theorem 6.1.1, \(X = c_0\)) Let \(Q \in M_{c_0}\) and \(P_r : c_0 \to c_0 \ (r \in \mathbb{N}\) be the operator defined by \(P_r(x) = (x_0, x_1, \ldots, x_r, 0, 0, \ldots)\) for all \(x = (x_k) \in c_0\). Then, we have

\[
\chi(Q) = \lim_{r \to \infty} \left( \sup_{x \in Q} \|(I - P_r)(x)\| \right)
\]
where $I$ is the identity operator on $c_0$.

**Proposition 3.2.** ([15], Theorem 6.1.1) Let $X$ be a Banach space with a Schauder basis $\{e_1, e_2, \ldots\}$, and $Q \in M_X$ and $P_n : X \to X$ ($n \in \mathbb{N}$) be the projector onto the linear span of $\{e_1, e_2, \ldots, e_n\}$. Then, we have

$$
\frac{1}{a} \lim_{n \to \infty} \sup_{x \in Q} \| (I - P_n)(x) \| \leq \chi(Q) \leq \inf_n \left( \sup_{x \in Q} \| (I - P_n)(x) \| \right) \leq \lim_{n \to \infty} \sup_{x \in Q} \| (I - P_n)(x) \|
$$

where $a = \lim_{n \to \infty} \sup \| I - P_n \|$. If $X = c$ then $a = 2$. (see [15], p.22).

### 4 Compact operators on the spaces $\left( N^q_\Delta \right)_0$, $\left( N^q_\Delta \right)$ and $\left( N^q_\Delta \right)_\infty$

**Theorem 4.1.** Consider the matrix $A$ as in Corollary 2.12, and for any integers $n, s$, $n > s$ set

$$
\| A \|^{(s)} = \sup_{n > p} \sup_{m} \left( \sum_{j=0}^{m-1} Q_j \left( \frac{1}{q_{j+1}} - \frac{1}{q_j} \right) \sum_{i=j+1}^{m} a_{ni} + \left| Q_m a_{nm} q_m \right| \right) \tag{11}
$$

If $X$ be either $\left( N^q_\Delta \right)_0$ or $\left( N^q_\Delta \right)$ and $A \in (X, c_0)$. Then

$$
\| L_A \|_\chi = \lim_{s \to \infty} \| A \|^{(s)}. \tag{12}
$$

If $X$ be either $\left( N^q_\Delta \right)_0$ or $\left( N^q_\Delta \right)$ and $A \in (X, c)$. Then

$$
\frac{1}{2} \cdot \lim_{s \to \infty} \| A \|^{(s)} \leq \| L_A \|_\chi \leq \lim_{r \to \infty} \| A \|^{(s)}. \tag{13}
$$

and if $X$ be either $\left( N^q_\Delta \right)_0$, $\left( N^q_\Delta \right)_\infty$ or $\left( N^q_\Delta \right)_\infty$ and $A \in (X, \ell_\infty)$. Then

$$
0 \leq \| L_A \|_\chi \leq \lim_{s \to \infty} \| A \|^{(s)}. \tag{14}
$$

**Proof.** Let $F = \{ x \in X : \| x \| \leq 1 \}$ if $A \in (X, c_0)$ and $X$ is one of the spaces $\left( N^q_\Delta \right)_0$ or $\left( N^q_\Delta \right)$, then by Proposition 3.1

$$
\| L_A \|_\chi = \chi(AF) = \lim_{s \to \infty} \left[ \sup_{x \in F} \| (I - P_s)A x \| \right] \tag{15}
$$
Again using Proposition 2.10 and Corollary 2.12 we have

$$\|A\|^s = \sup_{x \in F} \|(I - P_s)Ax\|$$  \hspace{1cm} (16)

From (15) and (16) we get

$$\|LA\|_\chi = \lim_{s \to \infty} \|A\|^{(s)}.$$

Since every sequence $x = (x_k)_{k=0}^\infty \in c$ has a unique representation

$$x = le + \sum_{k=0}^\infty (x_k - l)e^{(k)} \quad \text{where} \quad l \in \mathbb{C} \quad \text{is such that} \quad x - le \in c_0$$

We define $P_s : c \to c$ by $P_s(x) = le + \sum_{k=0}^{s}(x_k - l)e^{(k)}$, $s = 0, 1, 2, \ldots$. Then $\|I - P_s\| = 2$ and using (16) and Proposition 3.2 we get

$$\frac{1}{2} \cdot \lim_{s \to \infty} \|A\|^{(s)} \leq \|LA\|_\chi \leq \lim_{s \to \infty} \|A\|^{(s)}$$

Finally we define $P_s : \ell_\infty \to \ell_\infty$ by $P_s(x) = (x_0, x_1, \ldots, x_s, 0, 0 \ldots)$, $x = (x_k) \in \ell_\infty$.

Clearly $AF \subset P_s(AF) + (I - P_s)(AF)$

So using the properties of $\chi$ we get

$$\chi(AF) \leq \chi[P_s(AF)] + \chi[(I - P_s)(AF)]$$

$$= \chi[(I - P_s)(AF)]$$

$$\leq \sup_{x \in F} \|(I - P_s)A(x)\|$$

Hence by Proposition 2.10 and and Corollary 2.12 we get

$$0 \leq \|LA\|_\chi \leq \lim_{s \to \infty} \|A\|^{(s)}$$

□

A direct corollary of the above theorem is

**Corollary 4.2.** Consider the matrix $A$ as in Corollary 2.12, and $X = (N^q_{\Delta-})_0$ or $X = (N^q_{\Delta-})$ then if $A \in (X, c_0)$ or $A \in (X, c)$ we have

$LA$ is compact if and only if $\lim_{s \to \infty} \|A\|^{(s)} = 0$
Further, for $X = (N^q_{\Delta -})_0$, $X = (N^q_{\Delta -})$ or $X = (N^q_{\Delta -})_{\infty}$, if $A \in (X, \ell_{\infty})$ then we have

$$L_A \text{ is compact if } \lim_{s \to \infty} \|A\|^{(s)} = 0 \quad (17)$$

In (17) it is possible for $L_A$ to be compact although $\lim_{s \to \infty} \|A\|^{(s)} \neq 0$, that is the condition is only sufficient condition for $L_A$ to be compact.

For example, let the matrix $A$ be defined as $A_n = e^{(1)}n = 0, 1, 2, \ldots$ and $q^n = 3^n$, $n = 0, 1, 2, \ldots$

Then by (4) we have

$$\sup_{m,n} \left[ \sum_{k=0}^{m-1} Q_k \left( \frac{1}{q_{k+1}} - \frac{1}{q_k} \right) \sum_{j=k+1}^{m} a_{nj} + \left| \frac{Q_m a_{nm}}{q_m} \right| \right] = \sup_{n} \left( \frac{2}{3} + \frac{1}{2}(1 - 3^{-n}) \right) < 2$$

Now by Corollary 2.12 we know $A \in \left( (N^q_{\Delta -})_{\infty}, \ell_{\infty} \right)$.

But

$$\|A\|^{(s)} = \sup_{n > s} \left[ \frac{2}{3} + \frac{1}{2}(1 - 3^{-n}) \right] = \frac{7}{6} - \frac{1}{2 \cdot 3^{r+1}} \quad \forall r$$

Which gives $\|A\|^{(s)} = \frac{7}{6} \neq 0$.

Since $A(x) = x_1$ for all $x \in (N^q_{\Delta -})_{\infty}$, so $L_A$ is compact operator.

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