Stability criterion for bright solitary waves of the perturbed cubic-quintic Schrödinger equation

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The stability of the bright solitary wave solution to the perturbed cubic-quintic Schrödinger equation is considered. It is shown that in a certain region of parameter space these solutions are unstable, with the instability being manifested as a small positive eigenvalue. Furthermore, it is shown that in the complimentary region of parameter space there are no small unstable eigenvalues. The proof involves a novel calculation of the Evans function, which is of interest in its own right. As a consequence of the eigenvalue calculation, it is additionally shown that $N$-bump bright solitary waves bifurcate from the primary wave.

1 Introduction

The nonlinear cubic-quintic Schrödinger equation (CQNLS) is given by

$$iA_t = A_{xx} + |A|^2A + \alpha|A|^4A,$$

where $A$ is a complex-valued function of the variables $(x,t) \in \mathbb{R} \times \mathbb{R}^+$. When $\alpha = 0$, the equation becomes the focusing cubic nonlinear Schrödinger equation, and is used to describe the propagation of the envelope of a light pulse in an optical fiber which has a Kerr-type nonlinear refractive index. For short pulses and high input peak pulse power the refractive index cannot be described by a Kerr-type nonlinearity, as the index is then influenced by higher-order nonlinearities. In materials with high nonlinear coefficients, such as semiconductors, semiconductor-doped glasses, and organic polymers, the saturation of the nonlinear refractive-index change is no longer negligible at moderately high intensities and should be taken into account \([10]\). Equation (1.1) is the correct model to describe the propagation of the envelope of a light pulse in dispersive materials with either a saturable or higher-order refraction index \([10], [11]\).

Equation (1.1) cannot really be thought of as a small perturbation of the cubic nonlinear Schrödinger equation, as it has been shown that a physically realistic value for the parameter $\alpha$ is $|\alpha| \sim 0.1$ \([15]\). It turns out that the most physically interesting behavior occurs when the nonlinearity is saturating, so for the rest of this paper it will be assumed that $\alpha < 0$ \([1, 14], [12], [13], [26]\). An optical fiber which satisfies this condition can be constructed, for example, by doping with two appropriate materials \([1]\).

One of the more physically interesting phenomena associated with the double-doped optical fiber is the existence of bright solitary wave solutions ($|A(x)| \to 0$ as $|x| \to \infty$) in which the peak amplitude becomes a two-valued function of the pulse duration. These solutions were proven to be stable as solutions to the CQNLS \([6], [10], [13], [15]\).

Equation (1.1) describes an idealized fiber; therefore, it is natural to consider the perturbed CQNLS (PCQNLS)

$$iA_t = (1 + i\epsilon a)A_{xx} + i\epsilon bA + (1 + i\epsilon d_1)|A|^2A + (\alpha + i\epsilon d_2)|A|^4A,$$

where $0 < \epsilon \ll 1$ and the other parameters are real and of $O(1)$ \([21]\). The parameter $a$ describes spectral filtering, $b$ describes the linear gain or loss due to the fiber, and $d_1$ and $d_2$ describe the nonlinear gain or loss due to the fiber. Note that (1.2) is a well-defined PDE for $\epsilon > 0$ only if $a > 0$.

Solitary wave solutions to (1.2) are found by setting

$$A(x,t) = A(x)e^{-i\omega t},$$

where $0 < \epsilon \ll 1$ and the other parameters are real and of $O(1)$ \([21]\). The parameter $a$ describes spectral filtering, $b$ describes the linear gain or loss due to the fiber, and $d_1$ and $d_2$ describe the nonlinear gain or loss due to the fiber. Note that (1.2) is a well-defined PDE for $\epsilon > 0$ only if $a > 0$.
and then finding heteroclinic and homoclinic solutions for the ODE

\[(1 + i\epsilon a)A'' + (-\omega + i\epsilon b)A + (1 + i\epsilon d_1)|A|^2 A + (\alpha + i\epsilon d_2)|A|^4 A = 0,\]  
(1.4)

where \(\prime = d/dx\). Equation (1.4) has been extensively studied by many authors ([7], [8], [16], [17], [19], [22], [23], [25]). These papers have been concerned with finding various types of solutions, including fronts (kinks), bright solitary waves, and dark solitary waves. The methods employed have been both geometric ([7], [8], [16], [17], [19]) and analytic ([22], [23], [25]).

Bright solitary waves exist when there are solutions to (1.4) which are homoclinic to \(|A| = 0\). When \(\epsilon = 0\) and \(\omega > 0\) the wave is given by the expression

\[A^2(x) = \frac{4\omega}{1 + \sqrt{1 - \beta \cosh(2\sqrt{\omega} x)}}, \quad \beta = -\frac{16}{3}\alpha\omega.\]  
(1.5)

Since it is being assumed that \(\alpha < 0\), a restriction on \(\beta\) is that \(0 \leq \beta < 1\). An analytic expression for the wave exists even for \(\epsilon > 0\) ([23], [25], [27]); however, it will not be given here. For the purposes of this paper it is enough to know that the wave exists for all \(\epsilon \geq 0\).

It was previously stated that the bright solitary wave is a stable solution to (1.1). However, recent numerical work by Soto-Crespo et al [27] suggests that this wave becomes an unstable solution to (1.2) for \(\epsilon\) nonzero. The numerics suggest that this instability arises from the presence of a real eigenvalue for the linearized problem moving out of the origin and into the right-half of the complex plane. The primary purpose of this paper is to determine if this is actually the case, and to determine possible stability/instability mechanisms.

When discussing the stability of the bright solitary wave, one must locate the spectrum of the operator \(L\) found by linearizing (1.2) about the wave. The essential spectrum is easy to determine (Henry [14]). When \(\epsilon = 0\), it resides on the imaginary axis with \(|\text{Im} \lambda| \geq \omega\), while for \(\epsilon > 0\) it can be shown to be located in the left-half of the complex plane if \(a > 0\) and \(b < 0\) (equation (2.7)).

The location of the point spectrum is more problematic. It is known that when \(\epsilon = 0\), zero is an eigenvalue of multiplicity four, and there are no other point eigenvalues (Weinstein [29], [30]). For \(\epsilon \neq 0\), two of these eigenvalues will remain at the origin, due to the spatial and rotational invariance of the PCQNLS, while the other two will generically move and be of \(O(\epsilon)\). If either eigenvalue moves into the right-half plane, then the wave will be unstable. Unfortunately, one cannot conclude that if both eigenvalues move into the left-half plane, then the wave is stable. The reason is that it may be possible for eigenvalues to move out of the essential spectrum and into the right-half plane for \(\epsilon \neq 0\). This topic will be the focus of a future paper.

In this paper a determination is made as to the location of the \(O(\epsilon)\) eigenvalues for \(0 < \epsilon \ll 1\). In order to accomplish this task, it is necessary to perform detailed asymptotics for the Evans function, \(E(\lambda)\), at \(\lambda = 0\). The Evans function is an analytic function whose zeros correspond to eigenvalues, with the order of the zero being the order of the eigenvalue ([1], [24]). Since the null-space of \(L\) is at least two-dimensional, \(E(0) = E'(0) = 0\). Thus, when expanded about \(\lambda = 0\), the Evans function satisfies

\[E(\lambda) = E''(0)\frac{\lambda^2}{2!} + E'''(0)\frac{\lambda^3}{3!} + E^{(4)}(0)\frac{\lambda^4}{4!} + O(\lambda^5),\]  
(1.6)

with all the derivatives being real-valued. It turns out to be the case that

\[E''(0) = B_1 \epsilon^2 + O(\epsilon^3)\]
(equation (6.1)), while if \( E''(0) = 0 \), then
\[
E'''(0) = B_2 \epsilon + O(\epsilon^2)
\]
(equation (6.2)). Furthermore, due to a result of Weinstein ([29], [30]), \( E^{(4)}(0) = O(1) \) (Corollary 2.3). Thus, if one can determine the signs of \( B_1, B_2, \) and \( E^{(4)}(0) \), then the \( O(\epsilon) \) eigenvalues can be approximately located.

Equation (1.4) defines a four-dimensional ODE phase space. The quantity \( E''(0) \) is related to the manner in which the stable and unstable manifolds of \( A = 0 \) intersect in this phase space, and a calculation of this quantity is similar to the calculation which leads to the orientation index (Alexander and Jones [2], [3]). The calculation of \( E'''(0) \) is a different matter, however. Using the ideas presented in Kapitula [18], it is shown that \( E'''(0) \) has a relationship with the projection of a certain function onto the null-space of the linear operator. In other words,
\[
P f = B_3 E'''(0) A_N,
\]
where \( P \) represents the projection onto the null-space, \( f \) is a particular function which measures the manner in which the stable and unstable manifolds intersect, \( A_N \) is a certain basis function of the null-space of \( L \), and \( B_3 > 0 \) is a constant of proportionality. The ideas leading to the calculation of \( E'''(0) \) are generalized in an upcoming paper, as they are of interest in their own right.

For the statement of the main theorems, set
\[
\Lambda_i = \int_{\infty}^{\infty} A^i(x) \, dx,
\]
where \( A(x) \) is defined in (1.3), and let
\[
\Lambda_{24} = \frac{\Lambda_2}{\Lambda_4}, \quad \Lambda_{d_2} = -\frac{8\omega}{\beta}(\omega\Lambda_{24} - \frac{3}{4}).
\]
Since the wave \( A(x) \) depends on \( \omega \), so do the above constants. Asymptotic expansions for these constants are given in Appendix A. Note that
\[
-\frac{\partial \omega \Lambda_{d_2}}{\partial \omega \Lambda_{24}} > 0
\]
for \( 0 \leq \beta < 1 \) (Proposition 5.9).

**Theorem 1.1** Let \( 0 < \epsilon \ll 1 \). Let \( 0 \leq \beta < 1 \). Suppose that \( d_1 = d_1^* \), where \( d_1^* \) is given in Remark 1.3. Further suppose that \( a > 0 \), and that
\[
d_2 - \frac{1}{3} \alpha a < 0.
\]
Set
\[
b^* = -\frac{\partial \omega \Lambda_{d_2}}{\partial \omega \Lambda_{24}}(d_2 - \frac{1}{3} \alpha a) < 0.
\]
If \( 0 > b^* > b \), then there is one stable real eigenvalue and one real unstable eigenvalue, both of which are \( O(\epsilon) \). However, if \( 0 > b > b^* \), then there are two stable real eigenvalues and zero unstable eigenvalues of \( O(\epsilon) \). Furthermore, except for the double eigenvalue at zero, there are no other eigenvalues which are of \( O(\epsilon) \).
Remark 1.2 If either \( a < 0 \) or \( b > 0 \), then the wave is unstable due to the presence of essential spectrum in the right-half plane.

Remark 1.3 If \( \alpha = d_2 = 0 \), then the wave is unstable. Thus, Ginzburg-Landau perturbations of the cubic NLS will only support unstable solitary waves.

Remark 1.4 Since \( a > 0 \) and \( \alpha < 0 \), the constant \( b^* \) will be negative if and only if \( d_2 < 0 \). Thus, one can say that \( d_2 < 0 \) is a minimal stability condition.

Remark 1.5 The wave exists if \( d_1 = d_1^* \), where

\[
d_1^* = \frac{1}{4} a - \Lambda_{24} b - \Lambda_{d_2}(d_2 - \frac{1}{3} \alpha a) + O(\epsilon)
\]

(Corollary 5.7).

The constant \( d_1^* \) given in the above remark depends on \( \omega \), i.e., \( d_1^* = d_1^*(a, b, d_2, \omega) \). Let \( \omega^* \) be a fixed parameter value, so that for \( \omega \neq \omega^* \) the wave does not exist without varying \( d_1^* \). As a consequence of Corollary 5.10 and the work of Kapitula and Maier-Paape [20] one has the following theorem concerning the existence of \( N \)-pulse solutions to the PCQNLS.

**Theorem 1.6** Let the assumptions of Theorem 1.1 be satisfied with \( d_1^* = d_1^*(a, b, d_2, \omega^*) \). For each \( N \geq 2 \) there exists a bi-infinite sequence \( \{\omega_N^k\} \), with

\[
\lim_{|k| \to \infty} \omega_N^k = \omega^*,
\]

such that when \( \omega = \omega_N^k \) there is an \( N \)-pulse solution to (1.4). If \( b < b^* \), then the \( N \)-pulse is unstable, and there exist at least \( N \) unstable eigenvalues.

Remark 1.7 The interested reader should consult [20] for a more complete description of the dynamics associated with (1.4).

The paper is organized in the following manner. In Section 2 the Evans function is constructed and its asymptotic behavior is determined. Sections 3 and 4 are devoted to deriving expressions for the various derivatives of the Evans function at \( \lambda = 0 \). In Section 5 the calculations are performed. Section 6 completes the argument leading to the two main theorems of this paper.

## 2 Construction of the Evans function

After the transformation \( A \to A e^{-i\omega t} \), the PCQNLS can be written in travelling wave coordinates \( (z = x - ct) \) as

\[
iA_t = (1 + i\alpha)A_{zz} + icA_z + (-\omega + ieb)A + (1 + ied_1)|A|^2A + (\alpha + ied_2)|A|^4A,
\]

(2.1)
where \( A \) is a complex-valued function of the variables \((z, t) \in \mathbb{R} \times \mathbb{R}^+\). Upon setting \( A = A_1 + iA_2 \), and denoting \( \mathbf{A} = (A_1, A_2) \), (2.1) becomes the system

\[
J \mathbf{A}_t = (I_2 + \epsilon a J) \mathbf{A}_{zz} + cJ \mathbf{A}_z + (I_2 + \epsilon b J) |\mathbf{A}|^2 \mathbf{A} + (\alpha I_2 + \epsilon d_2 J) |\mathbf{A}|^4 \mathbf{A},
\]

where \( I_2 \) is the \( 2 \times 2 \) identity matrix and \( J \) is the skew-symmetric matrix

\[
J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

The above system can be rewritten as

\[
J \mathbf{A}_t = B \mathbf{A}_{zz} + cJ \mathbf{A}_z + F(\mathbf{A}, \omega, \epsilon),
\]

where

\[
B = I_2 + \epsilon a J
\]

and

\[
F(\mathbf{A}, \omega, \epsilon) = (\omega I_2 + \epsilon b J) \mathbf{A} + (I_2 + \epsilon d_1 J) |\mathbf{A}|^2 \mathbf{A} + (\alpha I_2 + \epsilon d_2 J) |\mathbf{A}|^4 \mathbf{A}.
\]

Let \( \tilde{\mathbf{A}} \) represent the bright solitary wave solution to (2.3) which is known to exist when \( c = 0 \) ([23], [25]). When \( \epsilon = 0 \), \( \tilde{\mathbf{A}} = (R_0, 0)^T \), where

\[
R_0^2(z) = \frac{4\omega}{1 + \sqrt{1 - \beta \cosh(2\sqrt{\omega} z)}}, \quad \beta = -\frac{16}{3} \alpha \omega
\]

(2.4) (Henry [14]). Linearizing about the wave, the eigenvalue equation is given by

\[
B \mathbf{A}'' + D \mathbf{F}(\tilde{\mathbf{A}}, \omega, \epsilon) \mathbf{A} = \lambda J \mathbf{A}, \quad \lambda = \frac{d}{dz},
\]

i.e., \(-J L \mathbf{A} = \lambda \mathbf{A}\), where

\[
-J L = -J (B \partial_z^2 + D \mathbf{F}(\tilde{\mathbf{A}}, \omega, \epsilon)).
\]

A routine calculation shows that the essential spectrum for the operator \(-J L\), hereafter referred to as \(\sigma_e(-J L)\), is given by

\[
\sigma_e(-J L) = \{ \lambda \in \mathbb{C} : \lambda = -\epsilon (a_\eta^2 - b) \pm (\eta^2 + \omega)i, \eta \in \mathbb{R} \}
\]

(2.7) (Henry [14]). Thus, for \( \epsilon > 0 \) the operator \(-J L\) is sectorial if \( a > 0 \), and the essential spectrum is in the left-half of the complex plane if \( b < 0 \). This observation leads to the following assumption, which is minimal if the solitary wave is to be stable.

**Assumption 2.1** The parameters \( a \) and \( b \) are such that \( a > 0 \) and \( b < 0 \).

After setting \( \mathbf{Y} = (\mathbf{A}, \mathbf{A}') \), the eigenvalue equation (2.3) can be rewritten as the first-order system

\[
\mathbf{Y}' = M(\lambda, z) \mathbf{Y},
\]

where \( M \) is the \( 4 \times 4 \) block matrix

\[
M(\lambda, z) = \begin{bmatrix} 0 & I_2 \\ B^{-1}(\lambda J - D \mathbf{F}(\tilde{\mathbf{A}}, \omega, \epsilon)) & 0 \end{bmatrix}.
\]
For \( \lambda \in \Omega = \mathbb{C} \setminus \sigma_{s}(L) \) there exist complex analytic functions \( Y_{i}^{s}(\lambda, z) \) and \( Y_{i}^{u}(\lambda, z) \), \( i = 1, 2 \), which are solutions to (2.8) and which satisfy

1. \( \lim_{z \to -\infty} |Y_{i}^{s}(\lambda, z)| = 0, \quad Y^{s}(\lambda, z) = (Y_{1}^{s} \wedge Y_{2}^{s})(\lambda, z) \neq 0 \)
2. \( \lim_{z \to +\infty} |Y_{i}^{u}(\lambda, z)| = 0, \quad Y^{u}(\lambda, z) = (Y_{1}^{u} \wedge Y_{2}^{u})(\lambda, z) \neq 0 \)

(2.9). The Evans function is given by

\[ E(\lambda) = (Y_{1}^{u} \wedge Y_{2}^{u} \wedge Y_{1}^{s} \wedge Y_{2}^{s})(\lambda, z), \]

and by Abel’s formula is independent of \( z \). The Evans function is such that for \( \lambda \in \Omega \) it is zero if and only if \( \lambda \) is an eigenvalue, with the order of the zero being the order of the eigenvalue (2.10). Due to the invariances of the PCQNLS, two solutions to (2.5) when \( \lambda = 0 \) are \( A = \tilde{A} \) and \( A = J\tilde{A} \). As such, one can set

1. \( Y_{1}^{s}(0, z) = Y_{1}^{u}(0, z) = \tilde{U}' \)
2. \( Y_{2}^{s}(0, z) = Y_{2}^{u}(0, z) = \tilde{U}_{J} \),

where

\[ \tilde{U} = (\tilde{A}, \tilde{A}')^{T}, \quad \tilde{U}_{J} = (J\tilde{A}, J\tilde{A}')^{T}. \]

(2.11) The below lemma describes the asymptotic behavior of the Evans function.

**Lemma 2.2** With \( Y_{i}^{u}(0, z) \) and \( Y_{i}^{s}(0, z) \) as described in (2.10), if \( \lambda \in \mathbb{R} \), then \( E(\lambda) < 0 \) as \( \lambda \to \infty \).

**Proof:** It can be assumed without loss of generality that \( \epsilon = 0 \), as if the result is true for \( \epsilon = 0 \), it will then be true for \( 0 \leq \epsilon \ll 1 \). Assume that \( \lambda \in \mathbb{R}^{+} \).

Let \( Y = (P, Q)^{T} \) in (2.3). Upon setting \( s = \sqrt{\lambda} z \) and \( Q = \sqrt{\lambda} \hat{Q} \) and letting \( \lambda \to \infty \), equation (2.8) becomes the autonomous system

\[
\begin{pmatrix}
P \\
\hat{Q}
\end{pmatrix}' =
\begin{bmatrix}
0 & I_{2} \\
J & 0
\end{bmatrix}
\begin{pmatrix}
P \\
\hat{Q}
\end{pmatrix},
\]

where \( ' = d/ds \). The eigenvalues of the above matrix are given by \( \gamma(\pm 1 \pm i) \), where \( \gamma = \sqrt{2}/2 \), so that there exists a two-dimensional unstable subspace and two-dimensional stable subspace, with the two-dimensional unstable subspace being \( \text{Span}\{ (\gamma, -\gamma, 1, 0)^{T}, (\gamma, \gamma, 0, 1)^{T} \} \) and the two-dimensional stable subspace being \( \text{Span}\{ (\gamma, -\gamma, -1, 0)^{T}, (-\gamma, -\gamma, 0, 1)^{T} \} \).

Let \( e_{i} \wedge e_{j} = e_{ij} \). In \( \Lambda^{2}(\mathbb{R}^{4}) \) the unstable subspace is represented by the vector

\[
Y^{u}(+\infty) = (\gamma, -\gamma, 1, 0)^{T} \wedge (\gamma, \gamma, 0, 1)^{T} = e_{12} - \gamma e_{13} + \gamma e_{14} - \gamma e_{23} - \gamma e_{24} + e_{34},
\]

while the stable subspace is represented by the vector

\[
Y^{s}(+\infty) = (\gamma, -\gamma, -1, 0)^{T} \wedge (-\gamma, -\gamma, 0, 1)^{T} = -e_{12} - \gamma e_{13} + \gamma e_{14} - \gamma e_{23} - \gamma e_{24} - e_{34}.
\]

Note that

\[
Y^{u}(+\infty) \wedge Y^{s}(+\infty) = -2. \tag{2.12}
\]
When \( \lambda = 0 \), for each fixed \( z \) both the unstable and stable subspaces are spanned by the vectors \( (R_0'(z), 0, R_0''(0), 0)^T \) and \((0, R_0(z), 0, R_0'(z))^T \). Set

\[
Y^u(0) = \lim_{z \to -\infty} e^{-2\sqrt{\omega}z}Y^u(0, z)
\]

\[
Y^s(0) = \lim_{z \to \infty} e^{2\sqrt{\omega}z}Y^s(0, z).
\]

Using the representation for \( R_0 \) it can then be seen that

\[
Y^u(0) = \lim_{z \to -\infty} e^{-2\sqrt{\omega}z}(R_0'(z), 0, R_0''(0), 0)^T \land (0, R_0(z), 0, R_0'(z))^T
\]

\[
= \mu (\sqrt{\omega}, 0, \omega, 0)^T \land (0, 1, 0, \sqrt{\omega})^T
\]

\[
= \mu(\sqrt{\omega}e_{12} + \omega e_{14} - \omega e_{23} + \omega^{3/2} e_{34}),
\]

where

\[
\mu = \frac{8}{(1 - \beta)^{1/2}} \omega, \quad \beta = -\frac{16}{3} \alpha \omega.
\]

A similar calculation shows that

\[
Y^s(0) = \lim_{z \to \infty} e^{2\sqrt{\omega}z}(R_0'(z), 0, R_0''(0), 0)^T \land (0, R_0(z), 0, R_0'(z))^T
\]

\[
= \mu(-\sqrt{\omega}e_{12} + \omega e_{14} - \omega e_{23} - \omega^{3/2} e_{34}).
\]

Note that

\[
Y^u(0) \land Y^s(0) = -4\omega^2 \mu^2.
\]

A more complete discussion of the following argument can be found in Alexander and Jones [3]. An orientation of \( \mathbb{R}^d \) is given by a nonzero element \( \eta \) of \( \Lambda^4(\mathbb{R}^4) \). Two ordered bases of \( \mathbb{R}^4 \), \( \{w_1, \ldots, w_4\} \) and \( \{u_1, \ldots, u_4\} \), have the same orientation if \( w_1 \land \cdots \land w_4 \) is a positive multiple of \( u_1 \land \cdots \land u_4 \) (3). Any two such bases are related by a matrix having positive determinant.

The functions \( Y^u_\lambda(\lambda, z) \) and \( Y^s_\lambda(\lambda, z) \) can be used to get a basis for \( \mathbb{R}^4 \) for each \( \lambda \) and \( z \). In particular, in a manner similar to (2.13) proper scalings of

\[
\lim_{z \to \infty} Y^u_\lambda(\lambda, z), \quad \lim_{z \to -\infty} Y^s_\lambda(\lambda, z)
\]

determine a basis for each \( \lambda \). Since

\[
[Y^u(0) \land Y^s(0)][Y^u(+\infty) \land Y^s(+\infty)] > 0,
\]

the matrix taking the basis at \( \lambda = 0 \) to that at \( \lambda = +\infty \) has positive determinant, so that the two bases have the same orientation. The sign of \( E(\lambda) \) for large positive \( \lambda \) is then determined by equation (2.12), from which the conclusion of the lemma follows.

**Corollary 2.3** When \( \epsilon = 0 \), the Evans function satisfies \( E(0) = E'(0) = E''(0) = E'''(0) = 0 \), with \( E^{(4)}(0) < 0 \). Furthermore, \( E(\lambda) < 0 \) for \( \lambda > 0 \).

**Proof:** The fact that the first three derivatives of the Evans function at \( \lambda = 0 \) are zero, with the fourth derivative being nonzero, is a direct consequence of the work of Weinstein (29, 30). Furthermore, it is known that when \( \epsilon = 0 \) the bright solitary wave is stable, so that there exist no positive eigenvalues (3, 13); hence, the Evans function is nonzero for \( \lambda > 0 \). The fact that the fourth derivative is negative then follows from Lemma 2.2. ■
3 Calculation of derivatives

For \( \epsilon > 0 \) it will be generically true that \( E(0) = E'(0) = 0 \) with \( E''(0) \neq 0 \). Since \( E^{(4)}(0) < 0 \), by calculating \( E''(0) \) one will be able to determine the location of the zeros of \( E(\lambda) \) which are \( O(\epsilon) \), and hence the location of the small eigenvalues. When \( E''(0) = 0 \), an eigenvalue will be passing through the origin. A determination of \( E'''(0) \) will enable one to decide whether the eigenvalue is passing into the right-half or left-half of the complex plane.

This section is devoted to determining these quantities, and relating them to properties of the wave.

Time independent solutions to (2.3) satisfy the ODE

\[
BA'' + cJA' + F(A, \omega, \epsilon) = 0, \quad ' = \frac{d}{dz},
\]

which can be written as the first-order system

\[
U' = G(U, c, \omega, \epsilon),
\]

where \( U = (U_1, U_2) \in \mathbb{R}^4 \) and

\[
G(U, c, \omega, \epsilon) = \begin{pmatrix} U_2 \\ B^{-1}(-F(U_1, \omega, \epsilon) - cJU_2) \end{pmatrix}.
\]

The bright solitary wave corresponds to a solution homoclinic to \( U = 0 \), and is realized as the nontrivial intersection of the two-dimensional unstable manifold, \( W^u(z, c, \omega, \epsilon) \), with the two-dimensional stable manifold, \( W^s(z, c, \omega, \epsilon) \). Due to the rotational symmetry associated with the PCQNLS, there exists no distinguished trajectory in \( W^u(z, c, \omega, \epsilon) \cap W^s(z, c, \omega, \epsilon) \). However, this rotational symmetry allows one to choose a trajectory so that \( \tilde{A}_2(0) = 0 \), which uniquely defines a trajectory in the two-dimensional manifold. Set \( \tilde{U} = (\tilde{A}, \tilde{A}') \), so that \( \tilde{U} \subset W^u(z, c, \omega, \epsilon) \cap W^s(z, c, \omega, \epsilon) \) is a distinguished solution.

Before continuing, the following proposition is needed. It follows immediately upon examination of (2.3).

**Proposition 3.1** The Frechet derivative of the nonlinearity \( F \) satisfies

\[
DF_\omega(A, \omega, \epsilon) = -A.
\]

Since \( G \) depends smoothly on the parameters, so do the manifolds. The bright solitary wave is manifested as the nontrivial intersection of \( W^u(z, c, \omega, \epsilon) \) and \( W^s(z, c, \omega, \epsilon) \). Differentiating (3.2) with respect to the parameters \( c \) and \( \omega \) and evaluating over the wave \( \tilde{U} \) yields the systems

\[
(\partial_c W^r)' = DG_U(\tilde{A}, 0, \omega, \epsilon) \partial_c W^r + (0, -B^{-1}J\tilde{A}')^T \quad \text{(3.4)}
\]

and

\[
(\partial_\omega W^r)' = DG_U(\tilde{A}, 0, \omega, \epsilon) \partial_\omega W^r + (0, B^{-1}\tilde{A})^T. \quad \text{(3.5)}
\]

In these equations \( r \in \{u, s\} \), the result of Proposition 3.1 is implicitly used, and

\[
DG_U(\tilde{A}, 0, \omega, \epsilon) = \begin{bmatrix} 0 & I_2 \\ -B^{-1}DF_A(\tilde{A}, \omega, \epsilon) & 0 \end{bmatrix}.
\]
Note that a consequence of these equations is that $\partial_c(W^u - W^s)$ and $\partial_w(W^u - W^s)$ are solutions to the linear system

$$\delta U' = DG_U(\tilde{A}, 0, \omega, \epsilon) \delta U.$$  \hfill (3.6)

If one sets $\tilde{U}_J = (J\tilde{A}, J\tilde{A}')$, then the following proposition is realized.

**Proposition 3.2** Four solutions to (3.6) are given by $\tilde{U}', \tilde{U}_J, \partial_c(W^u - W^s), \partial_w(W^u - W^s)$; furthermore, if $D_2 = [\partial_c(W^u - W^s) \wedge \partial_w(W^u - W^s) \wedge \tilde{U}' \wedge \tilde{U}_J](z, 0, \omega, \epsilon)$ is nonzero, then the solutions are linearly independent.

**Proof:** It has already been seen that these four functions are solutions to the linear system. When $D_2 \neq 0$, the linear independence of the solutions follows from the fact that $D_2$ is the Wronskian. \hfill \Box

**Remark 3.3** By Abel’s formula, $D_2$ is independent of $z$.

Set

$$\delta U_1 = \partial_c(W^u - W^s), \delta U_2 = \partial_w(W^u - W^s), \delta U_3 = \tilde{U}', \delta U_4 = \tilde{U}_J,$$  \hfill (3.7)

so that

$$D_2 = (\delta U_1 \wedge \delta U_2 \wedge \delta U_3 \wedge \delta U_4)(z, 0, \omega, \epsilon).$$

Assuming that $D_2 \neq 0$, the functions $\delta U_1$ and $\delta U_2$ grow exponentially fast in the supremum norm as $|z| \to \infty$, while the functions $\delta U_3$ and $\delta U_4$ decay exponentially fast.

Let $H : R \to R^3$ be a uniformly bounded measurable function. Suppose that the solution to

$$\delta U' = DG_U(\tilde{A}, 0, \omega, \epsilon) \delta U + H$$  \hfill (3.8)

is desired, and further suppose that one wishes the solution to be bounded for either $z \to -\infty$ or $z \to \infty$. Denoting the solution by $\delta U^\pm$, with $|\delta U^\pm(z)| \leq M < \infty$ as $z \to \pm\infty$, by following the discussion in Kapitula [18] it can be seen that

$$\delta U^\pm = \frac{1}{D_2}(c_1^\pm(H)\delta U_1 + c_2^\pm(H)\delta U_2 + c_3(H)\delta U_3 + c_4(H)\delta U_4),$$  \hfill (3.9)

where

$$c_1^\pm(H) = \int_{\pm\infty}^z |H\delta U_2 \delta U_3 \delta U_4|(s) \, ds, \quad c_2^\pm(H) = \int_{\pm\infty}^z |\delta U_1 H \delta U_3 \delta U_4|(s) \, ds,$$

$$c_3(H) = \int_0^z |\delta U_1 \delta U_2 H \delta U_4|(s) \, ds, \quad c_4(H) = \int_0^z |\delta U_1 \delta U_2 \delta U_3 H|(s) \, ds.$$  \hfill (3.10)

The following lemma can now be proved.

**Lemma 3.4** Set

$$H_1 = (0, -B^{-1}J\tilde{A})^T, \quad H_2 = (0, B^{-1}\tilde{A})^T.$$

Then

$$D_2 = \int_{-\infty}^{\infty} |H_1 \delta U_2 \delta U_3 \delta U_4|(s) \, ds = \int_{-\infty}^{\infty} |\delta U_1 H_2 \delta U_3 \delta U_4|(s) \, ds,$$

while

$$0 = \int_{-\infty}^{\infty} |H_2 \delta U_2 \delta U_3 \delta U_4|(s) \, ds = \int_{-\infty}^{\infty} |\delta U_1 H_1 \delta U_3 \delta U_4|(s) \, ds.$$
Proof: Using equations (3.4) and (3.5) along with equation (3.9), one can see that
\[
\partial_t W^u = \frac{1}{D_2} (c_1^-(H_1) \delta U_1 + c_2^-(H_1) \delta U_2 + c_3(H_1) \delta U_3 + c_4(H_1) \delta U_4)
\]
\[
\partial_t W^s = \frac{1}{D_2} (c_1^+(H_1) \delta U_1 + c_2^+(H_1) \delta U_2 + c_3(H_1) \delta U_3 + c_4(H_1) \delta U_4),
\]
and
\[
\partial_\omega W^u = \frac{1}{D_2} (c_1^-(H_2) \delta U_1 + c_2^-(H_2) \delta U_2 + c_3(H_2) \delta U_3 + c_4(H_2) \delta U_4)
\]
\[
\partial_\omega W^u = \frac{1}{D_2} (c_1^+(H_2) \delta U_1 + c_2^+(H_2) \delta U_2 + c_3(H_2) \delta U_3 + c_4(H_2) \delta U_4).
\]
Subtracting and using the definition of \( \delta U_1 \) and \( \delta U_2 \) then yields
\[
\delta U_1 = \frac{1}{D_2} [(c_1^-(H_1) - c_1^+(H_1)) \delta U_1 + (c_2^-(H_1) - c_2^+(H_1)) \delta U_2]
\]
\[
\delta U_2 = \frac{1}{D_2} [(c_1^-(H_2) - c_1^+(H_2)) \delta U_1 + (c_2^-(H_2) - c_2^+(H_2)) \delta U_2].
\]
Using the definitions of the \( c_i^\pm \)'s and the fact that \( \delta U_i \) are linearly independent functions yields the final result. □

Define
\[
e_i^* = -\delta U_1 \wedge \delta U_3 \wedge \delta U_4, \quad e_2^* = -\delta U_2 \wedge \delta U_3 \wedge \delta U_4.
\] (3.11)
The functions \( e_i^* \in \mathcal{A}^3(\mathbb{R}^4) \) for \( i = 1, 2 \); furthermore, since \( \lambda = 0 \) is an isolated eigenvalue, both of these functions satisfy an estimate of the type
\[
|e_i^*(z)| \leq C e^{-\mu |z|}, \quad i = 1, 2
\] (3.12)
for some positive constants \( C \) and \( \mu \) (Kapitula [18]). For a given bounded continuous function \( F : \mathbb{R} \to \mathbb{R}^2 \), define
\[
< e_i^*, F > = \int_{-\infty}^{\infty} (H \wedge e_i^*)(s) \, ds, \quad i = 1, 2,
\]
where \( H = (0, F)^T \). With the above discussion in mind, one can rewrite Lemma 3.4 in the following manner.

Corollary 3.5 The constant \( D_2 \) is given by
\[
D_2 = < e_1^*, B^{-1} \tilde{A} > = < e_2^*, B^{-1} J \tilde{A} >.
\]
Furthermore,
\[
0 = < e_1^*, B^{-1} J \tilde{A} > = < e_2^*, B^{-1} \tilde{A} >.
\]

It is now possible to relate the Evans function to the structural stability of the wave. The proof of the first part of the below lemma is an alternate to that found in Alexander and Jones [3], and may be of interest in its own right.

Lemma 3.6 The Evans function satisfies
\[
E''(0) = 2D_2,
\]
where \( D_2 \) is defined in Proposition 3.3. Alternatively,
\[
E''(0) = 2 < e_1^*, B^{-1} \tilde{A} > = 2 < e_2^*, B^{-1} J \tilde{A} >.
\]
Proof: In this proof, the dependence of functions on the variable $z$ will be suppressed. Upon differentiating $E'(\lambda)$ and evaluating at $\lambda = 0$, one sees that

$$E''(0) = 2(\partial_\nu (Y_u - Y_1) \wedge \partial_\lambda (Y_u - Y_2) \wedge Y^s \wedge Y^s)(0).$$

In the above calculation the fact that $Y_1^s(0) = Y_2^s(0)$ was implicitly used. Since $Y_1^s(0) = \tilde{U}$ and $Y_2^s(0) = \tilde{U}_0$, all that is left to do is show the equivalence with the first two entries making up $E''(0)$.

Differentiating (2.8) with respect to $\lambda$ and evaluating at $\lambda = 0$ one sees that

$$(\partial_\lambda Y^s_1)' = M(0, z) \partial_\lambda Y^s_1 - (0, -B^{-1}J\tilde{A}')^T$$

and

$$(\partial_\lambda Y^s_2)' = M(0, z) \partial_\lambda Y^s_2 - (0, B^{-1}\tilde{A})^T,$$

where $\epsilon \in \{u, s\}$. In the above equation, the fact that $J^2 = -I_2$ is implicitly used. Since $M(0) = DG_U(\tilde{A}, 0, \omega, \epsilon)$, by following the proof of Lemma (3.4) and using the definitions of $H_i$ presented therein it can be shown that

$$\partial_\lambda (Y^s_1 - Y^s_1) = -\frac{1}{D_2}[(c_1^-(H_1) - c_1^+(H_1))\delta U_1 + (c_2^-(H_1) - c_2^+(H_1))\delta U_2]$$

$$\partial_\lambda (Y^s_2 - Y^s_2) = -\frac{1}{D_2}[(c_1^-(H_2) - c_1^+(H_2))\delta U_1 + (c_2^-(H_2) - c_2^+(H_2))\delta U_2].$$

By the result of Lemma (3.4) it is then seen that actually

$$\partial_\lambda (Y^s_1 - Y^s_1) = -\delta U_1, \quad \partial_\lambda (Y^s_2 - Y^s_2) = -\delta U_2.$$

Upon substituting the above into the expression for $E''(0)$ the first part of the lemma is proved.

The second part of the lemma follows immediately from Corollary (3.3). □

Remark 3.7 Note that a consequence of the above argument is that

$$\partial_\lambda Y^r_1 = -\partial_\nu W^r, \quad \partial_\lambda Y^r_2 = -\partial_\omega W^r$$

for $\epsilon \in \{u, s\}$.

Remark 3.8 Using the expansion of the Evans function given in (1.4) one can then write

$$E(\lambda) = \beta_1 B^(-1) \tilde{A} > \lambda^2 + E'''(0) \frac{\lambda^3}{3!} + E^{(4)}(0) \frac{\lambda^4}{4!} + O(\lambda^5)$$

$$= \beta_2 B^(-1) J\tilde{A}' > \lambda^2 + E'''(0) \frac{\lambda^3}{3!} + E^{(4)}(0) \frac{\lambda^4}{4!} + O(\lambda^5).$$

The above discussion is predicated on the assumption that $D_2 \neq 0$, which is equivalent to the manifolds $W^u$ and $W^s$ intersecting transversely. There are instances in which this intersection will not be transverse, in which case $D_2 = 0$. In this circumstance $E'''(0) = 0$ (Lemma (3.4)), so that an eigenvalue is passing through the origin. In order to determine the direction in which the eigenvalue is moving through the origin, it would be helpful to know $E''(0)$. 

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Suppose that $D_2 = 0$ due to the fact that $\delta U_2 = 0$, i.e., because $\partial_\omega W^u = \partial_\omega W^s$, while $e_1^* \neq 0$. Note that this implies that $|\partial_\omega W^u| \to 0$ exponentially fast as $|z| \to \infty$. Let $\delta \tilde{U}_2$ be any solution to (3.8) such that

$$D_3 = (\delta \tilde{U}_2 \wedge e_1^*)(z, 0, \omega, \epsilon)$$

(3.13)
is nonzero. The above discussion concerning the construction of solutions to (3.8) can be recreated by substituting $\delta U_2$ with $\delta \tilde{U}_2$ and $D_2$ with $D_3$. Let $\pi : \mathbb{R}^4 \to \mathbb{R}^2$ be the projection operator onto the first two components. The following lemma can now be proven.

**Lemma 3.9** Suppose that $E''(0) = 0$ with $e_1^* \neq 0$. Then

$$E'''(0) = 6 < e_1^*, B^{-1} J\pi(\partial_\omega W^u) > .$$

**Proof:** In this proof the dependence of solutions on $z$ will be suppressed. Since $E''(0) = 0$ and $e_1^* \neq 0$ implies that $\delta U_2 = 0$, by the proof of Lemma 3.6 it is necessarily true that $\partial_\lambda(Y_2^u - Y_2^s)(0) = 0$. Using this fact, a tedious calculation then shows that

$$E'''(0) = 3(\partial_\lambda(Y_1^u - Y_1^s) \wedge \partial_\lambda(Y_2^u - Y_2^s) \wedge Y_1^s \wedge Y_2^s)(0).$$

Define the projection matrix

$$Q = \begin{bmatrix} 0 & 0 \\ B^{-1}J & 0 \end{bmatrix},$$

and set $H_3 = Q\partial_\omega W^u = (0, B^{-1} J\pi(\partial_\omega W^u))^T$. Note that $Q = M_\lambda(0, z)$. By Remark 3.7

$$\partial_\lambda Y_2^s = \partial_\lambda Y_2^u = -\partial_\omega W^u.$$  

Using this, differentiating (2.8) twice with respect to $\lambda$, and evaluating at $\lambda = 0$ gives

$$(\partial_\lambda^2 Y_2^r)' = M(0, z)\partial_\lambda^2 Y_2^r - 2Q\partial_\omega W^u,$$

where $r \in \{u, s\}$. Using the definition of $H_3$, the solutions to this ODE are

$$\partial_\lambda^2 Y_2^u(0) = -\frac{2}{D_3}(c_1^-(H_3)\delta U_1 + c_2^- (H_3) \delta \tilde{U}_2 + c_3(H_3)\delta U_3 + c_4(H_3)\delta U_4)$$

$$\partial_\lambda^2 Y_2^s(0) = -\frac{2}{D_3}(c_1^+(H_3)\delta U_1 + c_2^+ (H_3) \delta \tilde{U}_2 + c_3(H_3)\delta U_3 + c_4(H_3)\delta U_4),$$

where the above functions $c_i$ are such that in their definitions $\delta U_2$ has been replace with $\delta \tilde{U}_2$. Upon subtracting one gets that

$$\partial_\lambda^2 (Y_2^u - Y_2^s)(0) = -\frac{1}{D_3}[(c_1^-(H_3) - c_1^+(H_3))\delta U_1 + (c_2^- (H_3) - c_2^+ (H_3))\delta \tilde{U}_2].$$

(3.14)

By the previous lemma it is known that

$$\partial_\lambda(Y_1^u - Y_1^s)(0) = -\delta U_1.$$  

After substituting the above expressions into that for $E'''(0)$ and using the definition of $D_3$ one gets that

$$E'''(0) = 6(c_2^- (H_3) - c_2^+ (H_3)).$$

Evaluating this expression gives the final step in the proof. □

**Remark 3.10** Using the expansion of the Evans function given in (2.8), if $E''(0) = 0$, then

$$E(\lambda) = \langle e_1^*, B^{-1} J\pi(\partial_\omega W^u) \rangle > \lambda^3 + E^{(4)}(0) \frac{\lambda^4}{4!} + O(\lambda^5).$$
4 Alternative expressions for the derivatives

Now that expressions are known for the various derivatives of the Evans function, it is desirable to reduce them to computable quantities. This section is devoted to that task.

It will be convenient to write everything in polar coordinates, i.e.,

$$A = (r \cos \theta, r \sin \theta).$$

In polar coordinates, \( U = T(r, \theta, s, \phi) \), where

$$T(r, \theta, s, \phi) = \begin{bmatrix} r \cos \theta \\ r \sin \theta \\ rs \cos \theta - r \phi \sin \theta \\ rs \sin \theta + r \phi \cos \theta \end{bmatrix},$$

with \( s = r'/r \) and \( \phi = \theta' \). A routine calculation shows that

$$DT(r, \theta, s, \phi) = \begin{bmatrix} \cos \theta & -r \sin \theta & 0 & 0 \\ \sin \theta & r \cos \theta & 0 & 0 \\ s \cos \theta - \phi \sin \theta & -r(s \sin \theta + \phi \cos \theta) & r \cos \theta & -r \sin \theta \\ s \sin \theta + \phi \cos \theta & r(s \cos \theta - \phi \sin \theta) & r \sin \theta & r \cos \theta \end{bmatrix},$$

with

$$|DT(r, \theta, s, \phi)| = r^3,$$

so that the transformation is nonsingular except at the origin.

In polar coordinates, let the manifolds be denoted by \( W^u_p \) and \( W^s_p \). In these coordinates it is a routine calculation to show that

$$\begin{align*}
\delta U_1 &= DT \partial_r (W^u_p - W^s_p), \\
\delta U_2 &= DT \partial_\omega (W^u_p - W^s_p), \\
\delta U_3 &= DT \partial_z W^u_p, \\
\delta U_4 &= DT \partial_\theta W^u_p.
\end{align*}$$

(4.2)

When \( s = 0 \) the manifolds can be parameterized as

$$W^u_p = (r^u(\theta, \beta), \theta, 0, \phi^u(\theta, \beta))^T, \quad W^s_p = (r^s(\theta, \beta), \theta, 0, \phi^s(\theta, \beta))^T,$$

(4.3)

where \( \beta = (c, \omega, \epsilon) \). Now, let the underlying wave be denoted by

$$\tilde{A}(z) = (R(z) \cos \Theta(z), R(z) \sin \Theta(z)).$$

Due to the fact that the wave is even it can be assumed that

$$R'(0) = \Theta'(0) = 0,$$

(4.4)

while the rotational symmetry of the PCQNLS allows one to set

$$\Theta(0) = 0.$$

(4.5)

Under these assumptions, when \( z = 0 \), i.e., when \( S = R'/R = 0 \),

$$\begin{align*}
\partial_r (W^u_p - W^s_p) &= \partial_r ((r^u - r^s), 0, 0, (\phi^u - \phi^s))^T, \\
\partial_\omega (W^u_p - W^s_p) &= \partial_\omega ((r^u - r^s), 0, 0, (\phi^u - \phi^s))^T, \\
\partial_z W^u_p &= (0, 0, S'(0), \Phi'(0))^T, \\
\partial_\theta W^u_p &= (0, 1, 0, 0)^T.
\end{align*}$$

(4.6)

Combining the above with (4.2) allows one to prove the following lemma.
The statement of Lemma 3.6 then gives the result.

Since (9), (28). Thus, in order to finish the calculation of $\epsilon^*(0, 0, \omega, \epsilon)$, we have

$$|DT| = \langle DT \rangle = 2 \langle DT \rangle.$$

By (4.6) and the calculation for $|DT|$, it can be seen that

$$D_2 = -R^2(0)S'(0) \begin{vmatrix} \partial_c(r^u - r^w) & \partial_\omega(r^u - r^w) \\ \partial_c(\phi^u - \phi^w) & \partial_\omega(\phi^u - \phi^w) \end{vmatrix}.$$

The steady-state equations in polar coordinates are given by equation (5.3). As a consequence of Proposition 5.1,

$$r^u(0, c, \omega, \epsilon) = r^w(0, -c, \omega, \epsilon), \quad \phi^u(0, c, \omega, \epsilon) = -\phi^w(0, -c, \omega, \epsilon).$$

Thus, it can be concluded that

$$\partial_\omega(r^u - r^w)(0, 0, \omega, \epsilon) = \partial_\omega(\phi^u - \phi^w)(0, 0, \omega, \epsilon) = 0,$$

while

$$\partial_c(r^u - r^w)(0, 0, \omega, \epsilon) = 2\partial_c\phi^u(0, 0, \omega, \epsilon) = 2\partial_\omega\phi^u(0, 0, \omega, \epsilon).$$

Since $S'(0) = R''(0)/R(0)$, this then yields that

$$D_2 = -4R^2(0)R''(0) \partial_c\phi^u(0, 0, \omega, \epsilon) \partial_\omega\phi^u(0, 0, \omega, \epsilon).$$

The statement of Lemma 3.6 then gives the result. ■

Now that a computable expression for $E''(0)$ is known, it would be beneficial to have an expression for $E''(0)$. Observation of Lemma 5.3 yields that one must first better understand

$$e_1^* = -\delta U_1 \wedge \delta U_3 \wedge \delta U_4 \in \Lambda^3(R^4).$$

By (4.2) the above quantity can be rewritten as

$$e_1^* = -(DT \partial_c(W_p^u - W_p^w)) \wedge (DT \partial_\omega W_p^u) \wedge (DT \partial_\omega W_p^u) = -DT(3) \langle \partial_c(W_p^u - W_p^w) \wedge \partial_\omega W_p^u \wedge \partial_\omega W_p^u \rangle,$$

where $DT(3)$ is the $4 \times 4$ matrix induced by $DT$ which maps $\Lambda^3(R^4)$ to itself. The matrix $DT(3)$ is formed by taking all the $3 \times 3$ minors of $DT$, and is given by

$$DT(3) = r^2 \begin{pmatrix} \cos \theta & \sin \theta & -(s \cos \theta + \phi \sin \theta) & -(s \sin \theta - \phi \cos \theta) \\ -\sin \theta & \cos \theta & s \sin \theta - \phi \cos \theta & -(s \cos \theta + \phi \sin \theta) \\ 0 & 0 & \cos \theta & \sin \theta \\ 0 & 0 & -r \sin \theta & r \cos \theta \end{pmatrix}$$

(9), (28). Thus, in order to finish the calculation of $e_1^*$, all that is left to determine is

$$(e_1^*)_p = \partial_c(W_p^u - W_p^w) \wedge \partial_\omega W_p^u \wedge \partial_\omega W_p^u.$$
Set
\[
\begin{align*}
\xi_1 &= \partial_z W^u = (R', \Theta', S', \Phi')^T \\
\xi_2^- &= \partial_c W^u_p, \quad \xi_2^+ = \partial_c W^s_p \\
\xi_3 &= \partial_b W^u_p = (0, 1, 0, 0).
\end{align*}
\]  
(4.8)

Let the vectors \(e_i, i = 1, \ldots, 4\), be the unit vectors in \(\mathbb{R}^4\), and define
\[
e_{ijk} = e_i \wedge e_j \wedge e_k.
\]

The collection of vectors \(\{e_{123}, e_{124}, e_{134}, e_{234}\}\) form a basis for \(\Lambda^3(\mathbb{R}^4)\), so that \((e_1^*)_p\) can be written in terms of these vectors. Now define
\[
P_{ij}^\pm = \begin{pmatrix}
(\xi_1)_i \\
(\xi_2)_i \\
(\xi_3)_i
\end{pmatrix},
\]
and set \(\tilde{P}_{ij} = P_{ij}^- - P_{ij}^+\). Using (4.8), a routine calculation then shows that
\[
(e_1^*)_p = \tilde{P}_{13}e_{123} + \tilde{P}_{14}e_{124} - \tilde{P}_{34}e_{234}.
\]

A consequence of the above discussion is the following lemma.

**Lemma 4.2** Let \(M > 0\) be given, and suppose that \(\tilde{P}_{ij} = O(\epsilon)\) for \(|z| \leq M\). Then
\[
e_1^* = \begin{cases}
-R^2(\tilde{P}_{13}e_{123} + (\tilde{P}_{14} + S\tilde{P}_{34})e_{124} - R\tilde{P}_{34}e_{234}) + O(\epsilon^2), & |z| \leq M \\
O(e^{-|z|})\epsilon, & |z| \geq M,
\end{cases}
\]
where \(\mu > 0\).

**Proof:** Let \(\eta_1 > 0\) be such that \(R^2(z) = O(e^{-\eta_1|z|})\), i.e., \(\eta_1 = 2\sqrt{\omega} + O(\epsilon)\). Since \(\Phi = O(\epsilon)\) implies that \(\Theta = O(\epsilon)z\), one can easily see that for \(|z| \leq M\)
\[
DT^{(3)} = R^2 \begin{bmatrix}
1 & 0 & -S & 0 \\
0 & 1 & 0 & -S \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & R
\end{bmatrix} + O(\epsilon),
\]
while \(\|DT^{(3)}\| = O(e^{-\eta_1|z|})\) for \(|z| \geq M\). Therefore, given the assumption on the functions \(\tilde{P}_{ij}\), for \(|z| \leq M\),
\[
e_1^* = DT^{(3)}(e_1^*)_p \\
= \tilde{P}_{13}e_{123} + (\tilde{P}_{14} + S\tilde{P}_{34})e_{124} - R\tilde{P}_{34}e_{234} + O(\epsilon^2).
\]

Since \(\tilde{P}_{ij} = O(\epsilon)\) for \(|z| \leq M\), it is necessarily true that for \(|z| \geq M\), \(\tilde{P}_{ij} = O(e^{\eta_2|z|})\epsilon\), which then implies that \((e_1^*)_p = O(e^{\eta_2|z|})\epsilon\). Thus, for \(|z| \geq M\) one sees that
\[
|e_1^*| \leq \|DT^{(3)}\| |(e_1^*)_p| \\
= O(e^{-\eta_1|z|})O(e^{\eta_2|z|})\epsilon \\
= O(e^{(\eta_2-\eta_1)|z|})\epsilon.
\]

Setting \(\mu = \eta_1 - \eta_2\), the fact that \(e_1^*\) approaches zero exponentially fast (equation (3.12)) guarantees that \(\mu > 0\).
Since $B = I_2 + \epsilon aJ$, a simple calculation shows that $B^{-1}J = J + O(\epsilon)$. Thus, using the fact that $\Theta = O(\epsilon)$ for $|z| \leq M$, it is not difficult to see that when $\delta U_2 = 0$,

$$B^{-1}J\pi(\partial_\omega W^u) = \begin{cases} (0, \partial_\omega R_0)^T + O(\epsilon), & |z| \leq M \\ O(e^{-\eta_3|z|}), & |z| \geq M, \end{cases}$$

(4.10)

where $\eta_3 > 0$.

**Lemma 4.3** Let $M > 0$ be given, and suppose that $\tilde{P}_{ij} = O(\epsilon)$ for $|z| \leq M$. When $E''(0) = 0$, the third derivative of the Evans function satisfies

$$E'''(0) = 6 \int_{-\infty}^{\infty} R_0^2(s) \partial_\omega R_0(s) \tilde{P}_{13}(s) ds + O(e^{-\eta_4M})\epsilon + O(\epsilon^2),$$

where $\eta_4 > 0$.

**Proof:** The integrand associated with $E'''(0)$ is given by $H_3 \wedge e_1^*$, where

$$H_3 = (0, B^{-1}J\pi(\partial_\omega W^u))^T.$$  

Using (4.10) and Lemma 4.2, one then sees that for $|z| \leq M$

$$H_3 \wedge e_1^* = -R_0^2 \partial_\omega R_0 \tilde{P}_{13} e_{4123} + O(\epsilon^2)$$

$$= R_0^2 \partial_\omega R_0 \tilde{P}_{13} + O(\epsilon^2),$$

while for $|z| \geq M$

$$H_3 \wedge e_1^* = O(e^{-\eta_4M|z|})\epsilon.$$

In the above calculation, the fact that $e_{4123} = e_4 \wedge e_{123} = -1$ is used. 

**Remark 4.4** A similar calculation leads to the conclusion that

$$E'''(0) \approx 2 \int_{-\infty}^{\infty} R_0(s)(\tilde{P}_{14}(s) + S_0(s)\tilde{P}_{34}(s)) ds.$$  

5 Asymptotics

Now that an expression for $E'''(0)$ has been derived, in order to determine the location of the eigenvalues near zero the expressions $\partial_r r^u$ and $\partial_\omega \phi^u$ must be calculated. In addition, in order to calculate $E'''(0)$, one must determine $\tilde{P}_{ij}$ and show that the quantities are $O(\epsilon)$ for $|z| \leq M$.

Set

$$A(z) = (r(z) \cos \theta(z), r(z) \sin \theta(z)).$$

Let the known underlying solitary wave be denoted by $(R, \Theta, S, \Phi)$. Note that

$$S = R'/R = \frac{d}{dz} \ln R.$$  

(5.1)

Recall the analytic expression for the wave given in (2.4) when $\epsilon = 0$, i.e.,

$$R_0^2(z) = \frac{4\omega}{1 + \sqrt{1 - \beta \cosh(2\sqrt{\omega} z)}}, \quad \beta = -\frac{16}{3} \alpha \omega$$

(5.2)
This wave will henceforth be denoted by \((R_0, 0, S_0, 0)^T\). By defining
\[
\Delta = 1 + \epsilon^2 a^2,
\]
the steady-state ODE is
\[
\begin{align*}
\dot{r} &= rs \\
\dot{\theta} &= \phi \\
\Delta \dot{s} &= -\Delta s^2 + \Delta \phi^2 - c(\epsilon a s - \phi) - (-\omega + \epsilon^2 ab) - (1 + \epsilon^2 a d_1) r^2 - (\alpha + \epsilon^2 a d_2) r^4 \\
\Delta \dot{\phi} &= -2\Delta s \phi - c(s + \epsilon a \phi) - \epsilon[(b + a \omega) + (d_1 - a) r^2 + (d_2 - a a \alpha) r^4].
\end{align*}
\] (5.3)

It should be noted that the equation for \(\theta\) is superfluous, and hence is usually ignored; however, it is included here for completeness. After dropping the \(O(\epsilon^2)\) terms the variational equations are given by
\[
\begin{align*}
\delta \dot{r}' &= S \delta r + R \delta s \\
\delta \theta' &= \delta \phi \\
\delta \dot{s}' &= -2R(1 + 2\alpha R^2) \delta r - 2S \delta s + 2\Phi \delta \phi + \delta \omega - (\epsilon a S - \Phi) \delta c - 2\epsilon [(b + a \omega) + (d_1 - a) R^2 + (d_2 - a a \alpha) R^4] \delta \epsilon \\
\delta \dot{\phi}' &= -2\epsilon \Phi[(d_1 - a) + 2(d_2 - a a \alpha) R^2] \delta r - 2\Phi \delta s - 2S \delta \phi - \epsilon a \delta \omega - (\epsilon a \Phi + S) \delta c - [(b + a \omega) + (d_1 - a) R^2 + (d_2 - a a \alpha) R^4] \delta \epsilon \\
\delta \omega' &= 0 \\
\delta \epsilon' &= 0 \\
\delta c' &= 0.
\end{align*}
\] (5.4)

An observation yields the following proposition.

**Proposition 5.1** Equation (5.3) is invariant under
\[(z, c, \omega, r, \theta, s, \phi) \rightarrow (-z, -c, \omega, r, \theta, -s, -\phi).\]

An expression for \(\partial \omega \phi(u)(0)\) will first be determined. Let
\[
\phi_\epsilon(z) = \delta \phi(\partial_t W_\epsilon^u(z, c, \omega)).
\] (5.5)

Since
\[
\delta c(\partial_t W_\epsilon^u(z, c, \omega)) = \delta \omega(\partial_t W_\epsilon^u(z, c, \omega)) = 0,
\]
by using (5.4) it can be seen that when \(\epsilon = 0\)
\[
\phi_\epsilon' = -2S_0 \phi_\epsilon - [(b + a \omega) + (d_1 - a) R_0^2 + (d_2 - a a \alpha) R_0^4].
\] (5.6)

By definition \(\phi_\epsilon\) is uniformly bounded as \(z \to -\infty\), so that upon using (5.1) the solution to (5.6) can be written as
\[
R_0^u(0) \phi_\epsilon(z) = -[(b + a \omega) \int_{-\infty}^{z} R_0^u(s) \, ds + (d_1 - a) \int_{-\infty}^{z} R_0^t(s) \, ds + (d_2 - a a \alpha) \int_{-\infty}^{z} R_0^s(s) \, ds].
\] (5.7)
By definition the function $\phi_\epsilon$ describes, up to $O(\epsilon)$, the location of the $\phi$-component of $W^u_p(z, c, \omega)$, so that
\[
\phi^u(0) = \epsilon \phi_\epsilon(0) + O(\epsilon^2). \tag{5.8}
\]
Thus, when performing calculations on $\phi^u(0)$, for $\epsilon > 0$ small enough it is sufficient to perform them on $\phi_\epsilon(0)$. Given (5.7) and the fact that an exact expression exists for $R^2_0(z)$, this then implies that rather detailed information can be gathered regarding the variation of $\phi^u(0)$ with respect to $\omega$ for $\epsilon$ sufficiently small. Set
\[
\Lambda_m = \int_{-\infty}^{\infty} R^m_0(s) \, ds, \quad \Lambda'_2 = \int_{-\infty}^{\infty} (R'_0(s))^2 \, ds. \tag{5.9}
\]

**Lemma 5.2** The function $\phi^u(0)$ is given by
\[
\phi^u(0) = \epsilon \phi_\epsilon(0) + O(\epsilon^2),
\]
where
\[
2R^2_0(0)\phi_\epsilon(0) = \Lambda'_2 a - \Lambda_2 b - \Lambda_4 d_1 - \Lambda_6 d_2.
\]

**Proof:** Since $R_0$ is an even function,
\[
\int_{-\infty}^{0} R^m_0(s) \, ds = \frac{1}{2} \int_{-\infty}^{\infty} R^m_0(s) \, ds
\]
for any positive integer $m$. Thus, when (5.7) is evaluated at $z = 0$,
\[
2R^2_0(0)\phi_\epsilon(0) = (-\omega \Lambda_2 + \Lambda_4 + \alpha \Lambda_6)a - \Lambda_2 b - \Lambda_4 d_1 - \Lambda_6 d_2.
\]
The function $R_0$ satisfies
\[
R''_0 - \omega R_0 + R^3_0 + \alpha R^5_0 = 0.
\]
Upon multiplying the above equation by $R_0$ and integrating by parts one sees that
\[
\Lambda'_2 = -\omega \Lambda_2 + \Lambda_4 + \alpha \Lambda_6,
\]
from which the conclusion of the lemma follows. \qed

**Remark 5.3** The expressions $\Lambda_2$ and $\Lambda_4$ are evaluated in Appendix A.

It is known that the wave exists for all $\epsilon > 0$, with the perturbation being regular ([23], [25]). A necessary condition for the existence of the bright solitary wave is that $\phi_\epsilon(z)$ remains uniformly bounded as $z \to \infty$. Since $R_0(z) \to 0$ as $z \to \infty$, this then yields the next lemma.

**Lemma 5.4** A necessary condition for the existence of the bright solitary wave is that
\[
\Lambda'_2 a - \Lambda_2 b - \Lambda_4 d_1 - \Lambda_6 d_2 = 0.
\]

**Proof:** Evaluating (5.7) at $z = \infty$ and requiring that the right-hand side be zero at the limit yields
\[
0 = (-\omega \Lambda_2 + \Lambda_4 + \alpha \Lambda_6)a - \Lambda_2 b - \Lambda_4 d_1 - \Lambda_6 d_2
\]
\[
= \Lambda'_2 a - \Lambda_2 b - \Lambda_4 d_1 - \Lambda_6 d_2. \quad \blacksquare
\]
Remark 5.5 An examination of Lemmas 5.2 and 5.4 shows that the necessary condition for the existence of the wave implies that 

$$\phi_r(0) = 0.$$ 

The expression present in the above lemma can clearly be solved for $$d_1$$ in terms of the other parameters. Before doing so, however, it will be desirable to simplify the above expression. As the following proposition illustrates, there is a simple relationship between the above quantities.

Proposition 5.6 The relations

1. $$\Lambda_6 = \frac{3}{2\alpha}(\omega\Lambda_2 - \frac{3}{4}\Lambda_4)$$
2. $$\Lambda'_2 = \frac{1}{2}\omega\Lambda_2 - \frac{1}{8}\Lambda_4$$

hold true.

Proof: As mentioned in the proof of Lemma 5.2, the function $$R_0$$ satisfies the ODE

$$R_0'' - \omega R_0 + R_0^3 + \alpha R_0^5 = 0.$$ 

Multiplying by $$R_0$$ and integrating by parts yields that

$$-\Lambda'_2 - \omega\Lambda_2 + \Lambda_4 + \alpha\Lambda_6 = 0,$$

while multiplying by $$R'_0$$ and integrating yields

$$\Lambda'_2 - \omega\Lambda_2 + \frac{1}{2}\Lambda_4 + \frac{1}{3}\alpha\Lambda_6 = 0.$$ 

Upon subtracting the above two equations one sees that

$$\Lambda'_2 - \frac{1}{4}\Lambda_4 - \frac{1}{3}\alpha\Lambda_6 = 0.$$ 

The conclusion of the first part of the proposition is now clear.

The proof for the second part follows in a similar manner. Simply add the two equations to get the relation

$$\omega\Lambda_2 - \frac{3}{4}\Lambda_4 - \frac{2}{3}\alpha\Lambda_6 = 0,$$

from which one immediately gets the second part of the proposition. ■

Note that for $$\beta = -16\alpha\omega/3$$ the relation for $$\Lambda_6$$ can be rewritten as

$$\Lambda_6 = -\frac{8\omega}{\beta}(\omega\Lambda_2 - \frac{3}{4}\Lambda_4).$$

With this observation, define

$$\Lambda_{24} = \frac{\Lambda_2}{\Lambda_4}, \quad \Lambda_{d_2} = -\frac{8\omega}{\beta}(\omega\Lambda_{24} - \frac{3}{4}).$$

(5.10)

Note that as a consequence of Proposition 5.6, $$\Lambda_{d_2} = \Lambda_6/\Lambda_4.$$
Corollary 5.7 In order for the wave to exist the parameter $d_1$ must equal $d_1^*$, where up to $O(\epsilon)$

$$d_1^* = \frac{1}{4}a - \Lambda_{24}b - \Lambda_{d_2}(d_2 - \frac{1}{3}\alpha a).$$

**Proof:** By Lemma 5.4, in order for the wave to exist it must be true that

$$d_1 = \frac{\Lambda_{d_2}a - \Lambda_{d_2}b - \Lambda_{d_2}d_2}{\Lambda_{d_2}}.$$  

The conclusion of the corollary follows after one uses the relationships described in Proposition 5.6.  

Now that an expression for the function $\phi^u(0)$ is known (Lemma 5.2), it is possible to understand its behavior when the parameter $\omega$ is varied. A consequence of Lemma 5.2 is that it is sufficient to understand the manner in which $\phi(0)$ varies. Since $\phi(0) = 0$ when $d = d_1^*$, a simple application of the implicit function theorem yields that

$$\partial_\omega \phi(0) + \partial_{d_1} \phi(0) \partial_\omega d_1^* = 0. \quad (5.11)$$

The quantities $\partial_{d_1} \phi(0)$ and $\partial_\omega d_1^*$ are accessible, so that the term $\partial_\omega \phi(0)$ can be calculated.

**Lemma 5.8** When $d_1 = d_1^*$,

$$\partial_\omega \phi(0) = -\frac{\Lambda_4}{2R_0^2(0)} \left( \partial_\omega \Lambda_{24}b + \partial_\omega \Lambda_{d_2}(d_2 - \frac{1}{3}\alpha a) \right).$$

**Proof:** First, an examination of Lemma 5.2 shows that

$$\partial_{d_1} \phi(0) = -\frac{\Lambda_4}{2R_0^2(0)}.$$  

Upon differentiating the expression for $d_1^*$ given in Corollary 5.7 and using (5.11), one then arrives at the conclusion of the lemma.  

It is important to understand how $\partial_\omega \phi(0)$ varies with the parameters. Before making a definitive statement, the following proposition is needed.

**Proposition 5.9** Set

$$\beta = -\frac{16}{3}\alpha \omega.$$  

When $0 \leq \beta < 1$, $\partial_\omega \Lambda_{24} < 0$, $\partial_\omega \Lambda_{d_2} > 0$, so that

$$\frac{\partial_\omega \Lambda_{d_2}}{\partial_\omega \Lambda_{24}} < 0.$$  

**Proof:** Using the definition of $\Lambda_{24}$ and Corollary A.2 one sees that

$$\partial_\omega \Lambda_{24} = \frac{(\Lambda_4 - 4\omega \Lambda_2)\partial_\omega \Lambda_2}{\Lambda_4^2}.$$  

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Upon using the Taylor expansions given in Corollary A.3 one sees that
\[ \Lambda_4 - 4\omega \Lambda_2 = -16\omega^{3/2} \sum_{n=0}^{\infty} \frac{\beta^n}{(2n+1)(2n+3)}, \]
which is clearly negative. Since Corollary A.2 states that \( \partial_\omega \Lambda_2 > 0 \) for \( 0 \leq \beta < 1 \), it is now clear that \( \partial_\omega \Lambda_4 < 0 \).

Since \( \partial_\omega \beta = \beta/\omega \),
\[ \partial_\omega \Lambda_{d_2} = -\frac{8\omega}{\beta} \partial_\omega (\omega \Lambda_{24}). \]
Using the Taylor series expansions given in Corollary A.3, after some tedious manipulations one can see that
\[ \partial_\omega (\omega \Lambda_{24}) = C \sum_{n=0}^{\infty} (a_n - b_n)\beta^n, \]
where
\[ C = \left( 4\omega \left( \sum_{n=0}^{\infty} \frac{1}{2n + 3} \beta^n \right)^2 \right)^{-1} > 0 \]
and
\[ a_n = \sum_{j=0}^{n} \frac{j}{2j + 1} \frac{1}{2(n - j) + 3}, \quad b_n = \sum_{j=0}^{n} \frac{j}{2j + 3} \frac{1}{2(n - j) + 1}. \]

The claim regarding \( \partial_\omega \Lambda_{d_2} \) will be proven as soon as it can be shown that \( a_n - b_n < 0 \).

Upon combining terms,
\[ a_n - b_n = 4 \sum_{j=0}^{n} \frac{j}{f(j, n)} \frac{n - 2j}{f(j, n)}, \]
where
\[ f(j, n) = (2j + 1)(2(n - j) + 1)(2j + 3)(2(n - j) + 3). \]

By the integral test,
\[ a_n - b_n \leq 4 \int_{0}^{n} xg(x, n) \, dx, \]
where
\[ g(x, n) = \frac{n - 2x}{f(x, n)}. \]

Set \( y = x - n/2 \). Then
\[ g(y, n) = -2 \frac{y}{f(y, n)}, \]
with
\[ f(y, n) = \frac{1}{2} (4y^2 - (n + 1)^2)(4y^2 - (n + 3)^2), \]
so that \( g(y, n) \) is odd in \( y \) with \( yg(y, n) < 0 \). Therefore,
\[ \int_{0}^{n} xg(x, n) \, dx = \int_{-n/2}^{n/2} (y + \frac{n}{2})g(y, n) \, dy \]
\[ = \int_{-n/2}^{n/2} yg(y, n) \, dy \]
\[ < 0, \]
so that $a_n - b_n < 0$. □

Combining the above results yields the following corollary, which concerns the variation of $\phi^u(0)$ with $\omega$.

**Corollary 5.10** Suppose that $d = d_1^*$. Set

$$b^* = -\frac{\partial^2 \Lambda_{d_2}}{\partial^2 \Lambda_{24}} (d_2 - \frac{1}{3} \alpha a).$$

For $\epsilon > 0$ sufficiently small, if $b > b^*$, then $\partial_\omega \phi^u(0) > 0$; otherwise, $\partial_\omega \phi^u(0) < 0$. Furthermore, for $0 \leq \beta < 1$

$$\frac{\partial_\omega \Lambda_{d_2}}{\partial_\omega \Lambda_{24}} < 0.$$

**Proof:** Since $\Lambda_4/R_0^2(0) > 0$ and $\partial_\omega \Lambda_{24} < 0$, the result follows immediately from Lemma 5.8 and Proposition 5.9. □

Now that $\partial_\omega \phi^u(0)$ is known, the quantities $\partial_\omega \tau^u(0)$ and $\tilde{P}_{ij}$ must be calculated. This can be accomplished simultaneously. As in (4.8), set

$$\xi_1 = \partial_x W_p^u, \quad \xi_2 = \partial_c W_p^u, \quad \xi_2^+ = \partial_c W_p^s.$$

Letting $P_{x_ix_j}$ denote $\delta x_i \land \delta x_j$, as in (4.9) set

$$P_{rs}^+ = P_{rs}(\xi_1, \xi_2), \quad P_{r\phi}^+ = P_{r\phi}(\xi_1, \xi_2), \quad P_{s\phi}^+ = P_{s\phi}(\xi_1, \xi_2).$$

Note that the computation of $E'''(0)$ requires that $P_{rs}^+ - P_{rs}^-$ be known (Lemma 4.3). Before continuing, a preliminary lemma is needed.

**Lemma 5.11** Set

$$\phi_{c}^\pm = \delta \phi(\xi_2^\pm).$$

When $\epsilon = 0$, $\phi_{c}^\pm = -1/2$.

**Proof:** It is easy to see from the variational equation (5.4) that when $\epsilon = 0$

$$(\phi_{c}^\pm)' = -2S_0 \phi_{c}^\pm - S_0. \quad (5.12)$$

This equation is easily solved, and one then finds that

$$R_0^2(z) \phi_{c}^\pm(z) = -\int_{\pm \infty}^z R_0^2(s) S_0(s) \, ds$$

$$= -\frac{1}{2} \int_{\pm \infty}^z \partial_s (R_0^2(s)) \, ds,$$

which yields the conclusion. □

Armed with the above lemma, a statement regarding $P_{r\phi}^\pm$ and $P_{s\phi}^\pm$ can now be made.

**Lemma 5.12** When $\epsilon = 0$,

$$P_{r\phi}^\pm = -\frac{1}{2} R_0^d, \quad P_{s\phi}^\pm = -\frac{1}{2} S_0^d.$$
Proof: Since \( \phi' = 0 \) when \( \epsilon = 0 \), a simple observation yields that

\[
\begin{align*}
P^\pm_{r\phi} &= R'_{0\phi} c, \\
P^\pm_{s\phi} &= S'_{0\phi} c.
\end{align*}
\]

The conclusion now follows from the above lemma.

**Corollary 5.13** For \( |z| \leq M \),

\[
|P^+_{r\phi} - P^-_{r\phi}| = O(\epsilon), \quad |P^+_{s\phi} - P^-_{s\phi}| = O(\epsilon).
\]

**Remark 5.14** By definition, \( \tilde{P}_{14} = P^+_{r\phi} - P^-_{r\phi} \) and \( \tilde{P}_{34} = P^+_{s\phi} - P^-_{s\phi} \) in Lemma 4.2.

It is now desirable to compute \( P^\pm_{rs} \). First, note that

\[
P_{r\epsilon}(\xi_1, \xi_2^\pm) = P_{r\omega}(\xi_1, \xi_2^\pm) = 0,
\]

and that

\[
P_{r\epsilon}(\xi_1, \xi_2^\pm) = RS.
\]

Since

\[
P'_{rs} = -SP_{rs} + 2\Phi P^\pm_{r\phi} + P_{r\omega} - (\epsilon a S - \Phi) P_{rc} - 2\epsilon a [(b + a \omega) + (d_1 - a)R^2 + (d_2 - a \alpha)R^4] P_{r\epsilon},
\]

upon substitution of the above relations one sees that

\[
(P^{\pm}_{rs})' = -SP^{\pm}_{rs} + 2\Phi P^{\pm}_{r\phi} - (\epsilon a S - \Phi)RS.
\]

The solution to this equation is given by

\[
R(z)P^\pm_{rs}(z) = -\epsilon a \int_{\pm \infty}^{z} R^2(s)S^2(s) ds + \int_{\pm \infty}^{z} R(s) \Phi(s)(R(s)S(s) + 2P^\pm_{r\phi}(s)) ds.
\]  

(5.15)

Using Lemma 5.12 and the fact that \( R' = RS \) yields that for bounded \( z \),

\[
R(z)S(z) + 2P^\pm_{r\phi}(z) = O(\epsilon),
\]

which implies, since \( \Phi = O(\epsilon) \), that the second integral is \( O(\epsilon^2) \). The above argument gives the following lemma.

**Lemma 5.15** Let \( M > 0 \) be given. Then

\[
R_0(z)P^-_{rs}(z) = -\epsilon a \int_{-\infty}^{z} (R'_0)^2(s) ds + O(\epsilon^2), \quad z \in (-\infty, M]
\]

\[
R_0(z)P^+_{rs}(z) = \epsilon a \int_{z}^{\infty} (R'_0)^2(s) ds + O(\epsilon^2), \quad z \in [-M, \infty).
\]

Proof: The conclusion follows immediately from (5.15), taking asymptotic expansions for \( R \) and \( S \), and using the fact that \( R' = RS \).

From this lemma one can derive the following three corollaries.
Corollary 5.16 Let $M > 0$ be given. Then for $|z| \leq M$,

$$R_0(z) \tilde{P}_{rs}(z) = -\Lambda_2 a \epsilon + O(\epsilon^2),$$

where $\tilde{P}_{rs} = P^-_{rs} - P^+_{rs}$.

Corollary 5.17 The quantity $\partial_c r^u(0)$ has the asymptotic expansion

$$\partial_c r^u(0) = N a \epsilon + O(\epsilon^2),$$

where $N < 0$ is given by

$$N = \frac{\Lambda_2'}{2R_0''(0)}.$$

Proof: Given the result of Lemma 5.15, the conclusion follows immediately from the facts that

$$P^-_{rs}(0) = -S'(0) \partial_c r^u(0),$$

and that $S'(0) = R''(0)/R(0)$. ■

Recall the expression given for $E'''(0)$ in Lemma 4.3. Given the results of Corollary 5.13 and Lemma 5.15, a definitive statement can now be made about this quantity.

Corollary 5.18 Suppose that $E''(0) = 0$. Then

$$E'''(0) = -(\tilde{N} a + O(\epsilon^{-n_1 M})) \epsilon + O(\epsilon^2),$$

where $\tilde{N} > 0$ is given by

$$\tilde{N} = 3\Lambda_2' \partial_\omega \Lambda_2.$$

Proof: Substitution of the expression for $\tilde{P}_{rs}$ into the expression for $E'''(0)$ yields that

$$E'''(0) = -6c a \Lambda_2' \int_{-\infty}^{\infty} R_0(s) \partial_\omega R_0(s) ds + O(\epsilon^{-n_1 M}) \epsilon + O(\epsilon^2)$$

$$= -3\Lambda_2' \partial_\omega \Lambda_2 a + O(\epsilon^{-n_1 M}) \epsilon + O(\epsilon^2).$$

The fact that the constant $\tilde{N}$ is positive follows immediately from Corollary A.2. ■

6 Final Arguments

By Lemma 4.1, Lemma 5.8, and Corollary 5.17 it can be seen that

$$E''(0) = C_1 a (b - C_2 (d_2 - \frac{1}{3} \alpha a)) \epsilon^2 + O(\epsilon^3),$$

where

$$C_1 = 2\Lambda_2 \Lambda_2 a < 0$$

and

$$C_2 = -\frac{\partial_\omega \Lambda_2}{\partial_\omega \Lambda_2} > 0.$$
Furthermore, when $E''(0) = 0$, by Corollary 5.18

$$E'''(0) = -C_3 a \epsilon + O(\epsilon^2),$$

(6.2)

where $C_3 > 0$.

The proof of Theorem 1.1 is now essentially complete. The result follows immediately from the expansions given in equations (6.1) and (6.2), and the fact that $E^{(4)}(0) < 0$ (Corollary 2.3). The reason the eigenvalues are $O(\epsilon)$ and real follows immediately from the fact that $E''(0) = O(\epsilon^2)$, while $E'''(0) = O(\epsilon)$.

The conclusion of Theorem 1.6 follows immediately from the work of Kapitula and Maier-Paape [20]. In order to use that work to conclude the existence of multiple pulse orbits, all that is necessary is to show that $\partial_\omega \phi^u(0) \neq 0$. This condition is met as a consequence of Corollary 5.10. The fact that the multiple pulse solutions are unstable for $b < b^*$ follows immediately from the fact that the primary pulse is unstable for $b < b^*$ (Alexander and Jones [3]). The minimal number of unstable eigenvalues also follows from that work.
A Appendix: Evaluation of Constants

In the following, \( \beta = \frac{-16}{3} \alpha \omega \).

**Proposition A.1** The quantities \( \Lambda_2 \) and \( \Lambda_4 \) satisfy

1. \( \Lambda_2 = 4\sqrt{\omega} \frac{1}{\sqrt{\beta}} Tanh^{-1}(\sqrt{\beta}) \)

2. \( \Lambda_4 = -16\omega \frac{1}{\beta}(\sqrt{\omega} - \frac{1}{4}\Lambda_2) \).

**Proof:** Parts 1. and 2. follow immediately from a direct integration, and can be verified with the help of Maple V (Release 4).

**Corollary A.2** The functions \( \Lambda_2 \) and \( \Lambda_4 \) satisfy

1. \( \omega \partial_\omega \Lambda_2 = \frac{2}{1-\beta} \omega^{-1/2} \)

2. \( \omega \partial_\omega \Lambda_4 = 4\omega \partial_\omega \Lambda_2. \)

Using the fact that

\[ Tanh^{-1}(x) = \frac{1}{2} \ln \frac{1+x}{1-x}, \]

Taylor series can be generated for various quantities.

**Corollary A.3** When \( 0 \leq \beta < 1 \) one has the Taylor expansions

1. \( \Lambda_2 = 4\omega^{1/2} \sum_{n=0}^{\infty} \frac{\beta^n}{2n+1} \)

2. \( \Lambda_4 = 16\omega^{3/2} \sum_{n=0}^{\infty} \frac{\beta^n}{2n+3} \)

3. \( \Lambda_{24} = \frac{3}{4\omega}(1 - \frac{2^2}{3 \cdot 5} \beta - \frac{2^2 \cdot 3^2}{3 \cdot 5^2 \cdot 7} \beta^2 - \frac{2^2 \cdot 23}{3 \cdot 5^3 \cdot 7} \beta^3 + O(\beta^4)) \)

4. \( \Lambda_{d_2} = \frac{8}{5}(1 + \frac{3^2}{5 \cdot 7} \beta + \frac{23}{5^2 \cdot 7} \beta^2 + \frac{3 \cdot 1879}{5^3 \cdot 7^2 \cdot 11} \beta^3 + O(\beta^4)) \)

5. \( \partial_\omega \Lambda_{d_2} = \frac{8}{5}(1 + 2 \frac{3^2}{5 \cdot 7} \beta + 3 \frac{23}{5^2 \cdot 7} \beta^2 + 4 \frac{3 \cdot 1879}{5^3 \cdot 7^2 \cdot 11} \beta^3 + O(\beta^4)) \)

6. \( \partial_\omega \Lambda_{24} = -\frac{3}{4\omega^2}(1 - \frac{2^2}{3 \cdot 5} \beta - \frac{2^2 \cdot 3^2}{3 \cdot 5^2 \cdot 7} \beta^2 - 4 \frac{2^2 \cdot 23}{3 \cdot 5^3 \cdot 7} \beta^3 + O(\beta^4)) \)

7. \( \frac{\partial_\omega \Lambda_{d_2}}{\partial_\omega \Lambda_{24}} = -\frac{32}{15}\omega^2(1 + \frac{2 \cdot 11}{3 \cdot 7} \beta + \frac{73}{3^2 \cdot 7} \beta^2 + \frac{2^2 \cdot 59 \cdot 2017}{3^3 \cdot 5^2 \cdot 7^2 \cdot 11} \beta^3 + O(\beta^4)). \)
References

[1] J. Alexander, R. Gardner, and C.K.R.T. Jones. A topological invariant arising in the stability of travelling waves. *J. reine angew Math.*, 410:167–212, 1990.

[2] J. Alexander and C.K.R.T. Jones. Existence and stability of asymptotically oscillatory triple pulses. *Z. angew Math. Phys.*, 44:189–200, 1993.

[3] J. Alexander and C.K.R.T. Jones. Existence and stability of asymptotically oscillatory double pulses. *J. reine angew Math.*, 446:49–79, 1994.

[4] C. De Angelis. Self-trapped propagation in the nonlinear cubic-quintic Schrödinger equation: a variational approach. *IEEE J. Quantum Elect.*, 30(3):818–821, 1994.

[5] R. Bishop and S. Goldberg. *Tensor Analysis on Manifolds*. Dover Publications, Inc., New York, 1968.

[6] G. Boling and W. Yaping. Orbital stability of solitary waves for the nonlinear derivative Schrödinger equation. *J. Diff. Eq.*, 123(1):35–55, 1995.

[7] A. Doelman and W. Eckhaus. Periodic and quasi-periodic solutions of degenerate modulation equations. *Physica D*, 53:249–266, 1991.

[8] J. Duan and P. Holmes. Fronts, domain walls, and pulses in a generalized Ginzburg-Landau equation. *Proc. Edinburgh Math. Soc.*, 38:77–97, 1995.

[9] H. Flanders. *Differential Forms with Applications to the Physical Sciences*. Dover Publications, Inc., New York, 1989.

[10] S. Gatz and J. Herrmann. Soliton propagation in materials with saturable nonlinearity. *J. Opt. Soc. Am. B*, 8(11):2296–2302, 1991.

[11] S. Gatz and J. Herrmann. Soliton collision and soliton fusion in dispersive materials with a linear and quadratic intensity depending refraction index change. *IEEE J. Quantum Elect.*, 28(7):1732–1738, 1992.

[12] S. Gatz and J. Herrmann. Soliton propagation and soliton collision in double-doped fibers with a non-Kerr-like nonlinear refractive-index change. *Optics Lett.*, 17(7):484–486, 1992.

[13] M. Grillakis, J. Shatah, and W. Strauss. Stability theory of solitary waves in the presence of symmetry, I. *Journal of Functional Analysis*, 74:160–197, 1987.

[14] D. Henry. Geometric theory of semilinear parabolic equations. In *Lecture Notes in Mathematics 840*. Springer-Verlag, New York, 1981.

[15] J. Herrmann. Bistable bright solitons in dispersive media with a linear and quadratic intensity-dependent refraction index change. *Optics Comm.*, 87:161–165, 1992.

[16] C.K.R.T. Jones, T. Kapitula, and J. Powell. Nearly real fronts in a Ginzburg-Landau equation. *Proc. Roy. Soc. Edin.*, 116A:193–206, 1990.

[17] T. Kapitula. Bifurcating bright and dark solitary waves of the nearly nonlinear cubic-quintic Schrödinger equation. (submitted).
[18] T. Kapitula. On the stability of travelling waves in weighted $L^\infty$ spaces. *J. Diff. Eq.*, 112(1):179–215, 1994.

[19] T. Kapitula. Singular heteroclinic orbits for degenerate modulation equations. *Physica D*, 82(1&2):36–59, 1995.

[20] T. Kapitula and S. Maier-Paape. Spatial dynamics of time periodic solutions for the Ginzburg-Landau equation. *Z. angew math. Phys.*, 47(2):265–305, 1996.

[21] Y. Kodama, M. Romagnoli, and S. Wabnitz. Soliton stability and interactions in fibre lasers. *Elect. Lett.*, 28(21):1981–1983, 1992.

[22] B. Malomed and A. Nepomnyashchy. Kinks and solitons in the generalized Ginzburg-Landau equation. *Physica D*, 73:305, 1994.

[23] P. Marcq, H. Chaté, and R. Conte. Exact solutions of the one-dimensional quintic complex Ginzburg-Landau equation. *Physica D*, 56D:303–367, 1992.

[24] R. Pego and M. Weinstein. Eigenvalues, and instabilities of solitary waves. *Phil. Trans. R. Soc. Lond. A*, 340:47–94, 1992.

[25] W. Van Saarloos and P. Hohenberg. Fronts, pulses, sources, and sinks in the generalized complex Ginzburg-Landau equation. *Physica D*, 56D:303–367, 1992.

[26] A. Sombra. Bistable pulse collisions of the cubic-quintic nonlinear Schrödinger equation. *Optics Comm.*, 94:92–98, 1992.

[27] J. Soto-Crespo, N. Akhmediev, and V. Afanasjev. Stability of the pulselike solutions of the quintic complex Ginzburg-Landau equation. *J. Opt. Soc. Am. B*, 13(7):1439–1449, 1996.

[28] M. Spivak. *Differential Geometry: Volume I*. Publish or Perish, Inc., Houston, 1970.

[29] M. Weinstein. Modulational stability of ground states of nonlinear Schrödinger equations. *SIAM J. Math. Anal.*, 16(3):472–491, 1985.

[30] M. Weinstein. Lyapunov stability of ground states of nonlinear dispersive evolution equations. *Comm. Pure Appl. Math.*, 39:51–68, 1986.