Solving Linear Programming Problem in $\mathbb{Z}_n$ by Graphical Method

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Abstract. The aim of this paper is to find a new type of linear programming problems depending on the set $\mathbb{Z} / n$ by using the graphical method. We gave new definitions that is characterized by the set $\mathbb{Z} / n$. We found that in this set many basic concepts in classic linear programming changed.

Keywords: Partially and Totally Ordered Relation, Linear Programming, Graphical Method.

1. Introduction

A Linear Programming (LP) problem is one of the mathematical programming problems, a restricted optimization problem, it is one of the most useful and extensively used techniques of operational research [1], the aim is to find a set of values for continuous or distinct variables $x_i$, $i = 1, 2, ..., q$ which maximizes or minimizes a linear objective function $Z$, which satisfy a set of linear constraints (linear equations and/or inequalities) [2]. The development of exact optimization methods for LP optimization problems during the last 60 years has been very successful [3]. In the other hand algebra and number theory, have found extensive applications in theoretical computer science to divisibility tests and to block ciphers in cryptography and security, coding theory, and much more. The Chinese remainder theorem (CRT) is one of the oldest theorems in mathematics, it was used to calculate calendars as early as the first century AD [4]. However in [5] Alhusiny had suggested two methods to solve the equation of the form $ax \equiv b \pmod{n}$, and the roots of the equation of the form $ax^n \equiv b \pmod{n}$. In this paper, we tried to give a new concept of LP based on $\mathbb{Z}_n$, the new field of optimization depends on the graphical concepts in $\mathbb{Z}_n$ which showed a new style of LP.

2. Some Basic Concepts:

Def.1: Let $\mathbb{Z}_n$ be any class on $n$. Any subset $R$ of $\mathbb{Z}_n \times \mathbb{Z}_n$ ($R \subseteq \mathbb{Z}_n \times \mathbb{Z}_n$) is called a relation from $\mathbb{Z}_n$ to $\mathbb{Z}_n$.

In our work, $\bar{a}$ is an element of $\mathbb{Z}_n$, i.e. $\bar{a} \in \mathbb{Z}_n$.

Def.2: Let $\bar{a}, \bar{b} \in \mathbb{Z}_n$, if $a \leq b$ then $\bar{a} \leq \bar{b}$ where $a, b \in \mathbb{Z}$.

Lemma: Let $a, b \in \mathbb{Z}$, $a \leq b$ if and only if $\bar{a} \leq \bar{b}$ where $\bar{a}, \bar{b} \in \mathbb{Z}_n$.
Proof: $\Rightarrow$) By Def.2, if $a \leq b$ then $\bar{a} \leq \bar{b}$ where $a, b \in \mathbb{Z}$ and $\bar{a}, \bar{b} \in \mathbb{Z}_n$.
$\Leftarrow$) Let $\bar{a} \leq \bar{b}$ to prove $a \leq b$

since $\bar{a}, \bar{b} \in \mathbb{Z}_n$, then $0 \leq a \leq n - 1$ and $0 \leq b \leq n - 1$
$\Rightarrow a \leq b$

Let $R$ be a relation on $\mathbb{Z}_n$ s.t. $R = \{(\bar{a}, \bar{b}) \in \mathbb{Z}_n \times \mathbb{Z}_n : \bar{a} \leq \bar{b}\}$.

3. Partially Ordered Relation:

Def.3 The relation $R$ on a set $\mathbb{Z}_n$ be Partially Ordered Relation (POR) if it is reflexive, anti-symmetric and transitive relations. The pair $(\mathbb{Z}_n, R)$ is called Partially Ordered Set (POS).
In LP the constructional constraints must be equations and/or inequalities.

**Lemma 2:** The pair \((\mathbb{Z}/g_1, \leq)\) is a POS.

**Proof:** To show \(\leq\) is reflexive, anti-symmetric and transitive relations.

Let \(\bar{a}, \bar{b}, \bar{c} \in \mathbb{Z}/g_1\),

To prove \(\leq\) is reflexive: since \(\bar{a} \leq \bar{a} \Rightarrow (\bar{a}, \bar{a}) \in R\).

To prove \(\leq\) is anti-symmetric: i.e. to prove \((\bar{a}, \bar{b}) \in R \land (\bar{b}, \bar{a}) \in R \Rightarrow \bar{a} = \bar{b}\)

\(\Rightarrow\) Let \(\bar{a} \leq \bar{b} \land \bar{b} \leq \bar{a}\), by Lemma 1 \(a \leq b \land b \leq a\)

\(\Rightarrow a = b\) and by properties of equivalence classes

We get, \(a = b\).

\(\Rightarrow\) \(\leq\) is anti-symmetric

To prove \(\leq\) is transitive: i.e. To prove \((\bar{a}, \bar{b}) \in R \land (\bar{b}, \bar{c}) \in R \Rightarrow (\bar{a}, \bar{c}) \in R\)

Let \(\bar{a} \leq \bar{b} \land \bar{b} \leq \bar{c}\) it means \(a \leq b \land b \leq c\).

And so, \(a \leq c\).

\(\Rightarrow\) \(\bar{a} \leq \bar{c} \Rightarrow (\bar{a}, \bar{c}) \in R \Rightarrow R\) is transitive.

That means \((\mathbb{Z}/g_1, \leq)\) is a POS.

\(\blacksquare\)

**Def. 4:** Let \(R\) be a POR on a set \(\mathbb{Z}_n\) and let \(\bar{a}, \bar{b} \in \mathbb{Z}_n\). Then \(\bar{a}, \bar{b}\) are called comparable with respect to \(R\) if \((\bar{a}, \bar{b}) \in R\) or \((\bar{b}, \bar{a}) \in R\).

4. Totally Ordered Relation

**Def. 5:** A relation \(\leq\) on a set \(\mathbb{Z}_n\) is called *Totally Ordered Relation (TOR)* if:

1. \(R\) is POR.
2. \(\bar{a}, \bar{b}\) are comparable \(\forall \bar{a}, \bar{b} \in \mathbb{Z}_n\).

The pair \((\mathbb{Z}_n, R)\) is called *Totally Ordered Set (TOS)*.

Therefore \(\leq\) is TOR and the pair \((\mathbb{Z}_n, \leq)\) is a TOS.

5. Equation of the Line:

**Def. 6:** The slope of a line in the plane \(\mathbb{Z}_n^2 = \mathbb{Z}_n \times \mathbb{Z}_n\) passing throw the points \((\bar{x}_1, \bar{y}_1), (\bar{x}_2, \bar{y}_2)\) is \(\bar{s} = \frac{y_2-y_1}{x_2-x_1}\), where \([s]: \mathbb{Z}_n^2 \to \mathbb{Z}_n\) is the step function.

**Def. 7:** The general equation of the line \(L\) in the plane \(\mathbb{Z}_n^2\) passing throw the point \((\bar{x}_0, \bar{y}_0)\) with slope \(\bar{s}\) is \(\bar{y} - \bar{y}_0 = \bar{s}(\bar{x} - \bar{x}_0)\)

6. Linear Programming (LP):

6.1 The Structure of a Linear Program Model [1]:

The general formula for LP problem is:

\[
\text{Max} (\text{or Min}) Z = c_1x_1 + c_2x_2 + \cdots + c_qx_q
\]

\(s.t.
\]

\[
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1q}x_q(\leq, =, \geq) b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2q}x_q(\leq, =, \geq) b_2 \\
\vdots \\
a_{p1}x_1 + a_{p2}x_2 + \cdots + a_{pq}x_q(\leq, =, \geq) b_p
\]

\(x_i \geq 0\)

The model has the following elements:
An objective function of the decision variables $x_i$. Decision variables are affected by the cost coefficients $c_j$.

- A set of $p$ constraints, in which a linear combination of the variables affected by coefficients $a_{ij}$ has to be less or equal than its right hand side value $b_i$.
- The bounds of the decision variables. In this case, all decision variables have to be non-negative.

### 6.2 Linear Programming (LP) in $\mathbb{Z}_n$:

In the beginning, we write the mathematical formulation of the LP in $\mathbb{Z}_n$.

$$
\max \text{ (or min) } Z_{z_n} = \tilde{c}_1 \cdot \bar{x}_1 + \tilde{c}_2 \cdot \bar{x}_2 + \cdots + \tilde{c}_q \cdot \bar{x}_q \\
\text{s.t. } \\
\bar{a}_{11} \cdot \bar{x}_1 + \bar{a}_{12} \cdot \bar{x}_2 + \cdots + \bar{a}_{1q} \cdot \bar{x}_q (\leq, =, \geq) \bar{b}_1 \\
\bar{a}_{21} \cdot \bar{x}_1 + \bar{a}_{22} \cdot \bar{x}_2 + \cdots + \bar{a}_{2q} \cdot \bar{x}_q (\leq, =, \geq) \bar{b}_2 \\
\vdots \\
\bar{a}_{p1} \cdot \bar{x}_1 + \bar{a}_{p2} \cdot \bar{x}_2 + \cdots + \bar{a}_{pq} \cdot \bar{x}_q (\leq, =, \geq) \bar{b}_p \\
\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_q, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_q, \bar{b}_1, \bar{b}_2, \ldots, \bar{b}_q, \bar{a}_{11}, \bar{a}_{12}, \ldots, \bar{a}_{1q}, \bar{a}_{21}, \bar{a}_{22}, \ldots, \bar{a}_{2q}, \ldots, \bar{a}_{pq}, \in \mathbb{Z}_n
$$

We can write the mathematical formulation of the LP in $\mathbb{Z}_n$ by other form.

$$
\max \text{ (or min) } Z_{z_n} = \tilde{c}^T \bar{x} \\
\text{s.t. } \\
\tilde{A} \bar{x} (\leq, =, \geq) \bar{b} \\
\bar{x}_i \in \mathbb{Z}_n, \ i \in I,
$$

where $\tilde{c} \in \mathbb{Z}_n^q$, $\bar{b} \in \mathbb{Z}_n^p$, $\tilde{A} \in \mathbb{Z}_n^{p \times q}$ and $I \subseteq \{1, \ldots, q\}$ are the index sets of the $\mathbb{Z}^+$.

The set

$$
\mathcal{F} = \{ \bar{x} \in \mathbb{Z}_n^q | \tilde{A} \bar{x} (\leq, =, \geq) \bar{b}, \bar{x}_i \in \mathbb{Z}_n, \forall \ i \in I \}
$$

$\mathcal{F}$ is the feasible region of the problem.

The values $\bar{x} \in \mathcal{F}$ which satisfy the objective function if there exists be optimal solution ($\max$($\text{or min}$)).

Note: We will examine the solutions resulting from the boundary points of the intersection of lines that constitute the area of the feasible region and therefore we will claim that our solutions are Efficient Solutions (ESs).

Examples: There are several kinds of $\mathbb{Z}_n$ depends on the number $n$ such as prime, non-prime, and the non-prime are into several kinds, so we will provide more than one example depending on the number $n$.

First, we will compare between LP in $\mathbb{Z}$ and $\mathbb{Z}_n$ for the straight lines, feasible region and multiple solution.

Example1: Let we have two examples in $\mathbb{Z}$ and $\mathbb{Z}_{256}$ where $n = 256$
\[
\begin{align*}
\text{max } Z &= 64x_1 + 16x_2 \\
\text{s.t. } &x_1 + 2x_2 \leq 191 \\
&x_1 + 3x_2 \leq 223 \\
&x_1, x_2 \in \mathbb{Z}^+
\end{align*}
\]

\[
\begin{align*}
\text{max } Z &= \frac{64 \cdot 256}{256} \bar{x}_1 + \frac{16 \cdot 256}{256} \bar{x}_2 \\
\text{s.t. } &\bar{x}_1 + \frac{256}{256} \bar{x}_2 \leq 191 \\
&\bar{x}_1 + \frac{256}{256} \bar{x}_2 \leq 223 \\
&\bar{x}_1, \bar{x}_2 \in \mathbb{Z}_{256}
\end{align*}
\]

In the Fig.1(a) shows intersect between two district lines in \(\mathbb{Z}\), when the Fig.1(b) shows the lines becomes parallel pieces in \(\mathbb{Z}_{256}\). In the Fig.1(c) have a single area of feasible region in \(\mathbb{Z}\), when the Fig.1(d) shows a multi-area of feasible regions in \(\mathbb{Z}_{256}\). In the Fig.1(e) shows one Efficient Solution (ES) in \(\mathbb{Z}\), when the Fig.1(f) shows more than one ESs in \(\mathbb{Z}_{256}\).

Fig.1 Comparable between LP for \(\mathbb{Z}\) and \(\mathbb{Z}_{256}\)

**Example2:** If we have the same constraints in Example1 on condition \(n = 225, 256\) and 257, respectively, with different objective functions:

\[
\begin{align*}
\text{max } Z_{\mathbb{Z}_n} &= \frac{72}{n} \cdot x_1 + \frac{24}{n} \cdot x_2 & \ldots (1) \\
\text{max } Z_{\mathbb{Z}_n} &= \frac{71}{n} \cdot \bar{x}_1 + \frac{23}{n} \cdot \bar{x}_2 & \ldots (2) \\
\text{max } Z_{\mathbb{Z}_n} &= \frac{64}{n} \cdot \bar{x}_1 + \frac{32}{n} \cdot \bar{x}_2 & \ldots (3)
\end{align*}
\]

With \(\bar{x}_1, \bar{x}_2 \in \mathbb{Z}_n\) and \(n = 225, 256\) and 257.
For the Eq.1, the Fig.2 explain number the solutions for $n = 225$, 256 and 257. In the Fig.2 (a) and (b) shows that we have multi $ES$ where $n = 225$ and 256 (non-prime and even numbers, respectively), while the Fig.2 (c) shows only one $ES$ where $n = 257$ (prime number).

Fig.2 Comparable among $LP$ in $\mathbb{Z}_n$ for Eq.1 where $n = 225$, 256 and 257.

For the Eq.2, the Fig.3 explain number the solutions for $n = 225$, 256 and 257. This figure shows that all the cases gives multi $ES$ for $n = 225$, 256 and 257 (non-prime, even and prime numbers, respectively).

Fig.3 Comparable among $LP$ in $\mathbb{Z}_n$ for Eq.2 where $n = 225$, 256 and 257.

For the Eq.3, the Fig.4 explain number the solutions for $n = 225$, 256 and 257. In the Fig.4 (a) and (c) shows that we have only one $ES$ where $n = 225$ and 257 (non-prime and prime numbers, respectively), while the Fig.2 (b) shows we have multi $ES$ where $n = 256$ (even number).
Fig. 4 Comparable among $LP$ in $\mathbb{Z}_n$ for Eq. 3 where $n = 225, 256$ and $257$

7. Conclusion:
One of the most important things observed and concluded after the comparison between $LP$ in $\mathbb{Z}$ and $\mathbb{Z}_n$ are that the straight line in $\mathbb{Z}_n$ does not remain on one straightness but turns into parallel pieces, and in $\mathbb{Z}_n$, two straight lines non-intersect or intersect at more points. We have shown that in $\mathbb{Z}_n$, the feasible region will not remain single and becomes closed in $\mathbb{Z}_n$ for $n$ of the elements while it is open to infinity in $\mathbb{Z}$. And in $\mathbb{Z}_n$, the possibility of appearing more than optimal solution is yelled even if the objective function is not parallel to one of the constraints.

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