Explosion and non-explosion for the continuous-time frog model

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Abstract

We consider the continuous-time frog model on $\mathbb{Z}$. At time $t = 0$, there are $\eta(x)$ particles at $x \in \mathbb{Z}$, each of which is represented by a random variable. In particular, $(\eta(x))_{x \in \mathbb{Z}}$ is a collection of independent random variables with a common distribution $\mu$, $\mu(\mathbb{Z}_+) = 1$, $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$, $\mathbb{N} = \{1, 2, 3, \ldots\}$. The particles at the origin are active, all other ones being assumed as dormant, or sleeping. Active particles perform a simple symmetric continuous-time random walk in $\mathbb{Z}$ (that is, a random walk with $\exp(1)$-distributed jump times and jumps $-1$ and $1$, each with probability $1/2$), independently of all other particles. Sleeping particles stay still until the first arrival of an active particle to their location; upon arrival they become active and start their own simple random walks. Different sets of conditions are given ensuring explosion, respectively non-explosion, of the continuous-time frog model. Our results show in particular that if $\mu$ is the distribution of $e^{Y \ln Y}$ with a non-negative random variable $Y$ satisfying $EY < \infty$, then a.s. no explosion occurs. On the other hand, if $a \in (0, 1)$ and $\mu$ is the distribution of $e^X$, where $P\{X \geq t\} = t^{-a}$, $t \geq 1$, then explosion occurs a.s. The proof relies on a certain type of comparison to a percolation model which we call totally asymmetric discrete inhomogeneous Boolean percolation.

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1 Introduction

Denote by $A_t$ the set of sites visited by active particles by the time $t$. In this paper we investigate the various conditions on $\mu$ ensuring that the system explodes, respectively does not explode, in finite time. We exclude a trivial case and assume throughout that $\mu(0) < 1$. If $\eta(0) = 0$, then we add one active particle at the origin at time $t = 0$.

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Definition 1.1. We say that the system explodes (in finite time) if there exists \( t \in (0, \infty) \) such that \( A_t \) is infinite.

Our aim is to analyse and give conditions for explosion and non-explosion of the continuous-time frog model. An equivalent definition of explosion on \( \mathbb{Z} \) is the following: there are no sleeping particles left in a finite time. This equivalence is not entirely trivial and may not be true for the frog model on a general graph; in the one dimensional case it follows from the arguments on Page 11 at the beginning of Section 5.

The behaviour of the frog dynamics can be distinguished as follows:

- linear spread - superlinear spread but no explosion - explosion in finite time

We will review these concepts in more detail in Section 4.

The question of explosion is a classical question in the theory of branching processes [Har63, Chapter 5, Section 9] and is an important consideration in a general construction of an interacting particle system [EW03]. An explosion is a phenomenon known to take place in first-passage percolation models if a node can have sufficiently many neighbors [vdHK17]. A different type of explosion is considered in [CD16], where conditions for accumulation of an unbounded number of particles within a compact set are given for a branching random walk with non-negative displacements. In [PP94] necessary and sufficient conditions for explosion are given for first passage percolation with unit exponential weights on a spherically symmetric rooted tree. For a tree in which every vertex in generation \( n \) has \( f(n+1) \) children the probability of an explosion is shown to be 1 if and only if \( \sum_{n=1}^{\infty} \frac{1}{f(n)} < \infty \); the probability is 0 otherwise. More general (non-exponential) weights are also briefly discussed in [PP94]. Under broad assumptions, conditions for explosion of first passage percolation on spherically symmetric trees with arbitrary weight distribution are obtained in [ADGO17]. As one might expect, a lot depends on the interplay between \( f \) and the behavior of the weight distribution function near zero.

An explosion can occur for certain classes of stochastic differential equations. It is sometimes referred to as a blow-up. Conditions for explosion and non-explosion constitute a part of the classical theory [IW89, Chapter VI, Section 4]. A drift condition ensuring explosion for a multidimensional equation is given in [CK14]. Various terms may cause explosion in a stochastic differential equation with jumps [BY16].

The frog model was introduced in [AMP02] where a shape theorem for the model was proven for the frog model in discrete time with \( \mu = \delta_1 \), i.e., at \( t = 0 \) there is one frog at each site. The asymptotic properties of the spread have been studied for the frog model on various graphs: on the integer lattice [AMPR01], trees [HJJ19b], Cayley graphs [CD21], as well as multitype model on the integer lattice [DHL19]. A shape theorem in every dimension and finer results in dimension one for the continuous-time model have been obtained in [RS04, CQR07, CQR09]. A possibility of explosion for the continuous-time frog model with a general (not necessarily exponential) distribution of the time between jumps was demonstrated in [BDK21].

The results of this paper can be framed in terms of the cover time, that is, the time when every site of a graph is visited by an active particle. Explosion means that the cover time of \( \mathbb{Z} \) is finite for the continuous-time frog model; if no explosion occurs a.s., then the cover time is
infinite. For the discrete-time model the asymptotics of cover time have been studied on various finite graphs, in particular trees \cite{Her18, HJJ19a} and tori and sequences of expander graphs \cite{BFHM20}.

In this paper we give sufficient conditions for explosion and non-explosion of the continuous-time frog model. In the proofs we rely on a comparison with a certain kind of auxiliary percolation model. Using a similar proof technique, in \cite{BK20} the linear and superlinear spread of the continuous-time frog model was studied. Further description of the technique can be found in Section 4. The results of this paper demonstrate flexibility and versatility of the technique. We expect it to be applicable in various other settings when addressing the questions such as spread rate or explosion for stochastic growth models.

The paper is organized as follows. In Section 2 we formulate and discuss the main results. In Section 3 an auxiliary percolation model is introduced. In Section 4 further discussion and the main ideas of the proof are collected. Sections 5 and 6 contain the proofs of non-explosion and explosion, respectively.

2 Main results and discussion

In this section, we give sufficient conditions on the initial distribution $\mu$ of sleeping particles which lead to explosion or non-explosion. Let $A: \mathbb{N} \to (0, \infty)$ be a non-decreasing function which we interpret as a varying speed for the continuous-time frog model. We remark here that the word ‘speed’ is used loosely in this paper. We mostly have in mind an average speed over a certain interval, that is, the ratio of the distance covered to the time elapsed since the start of movement of one or several particles.

For $i, j \in \mathbb{N}$, set $\mathcal{A}(i) := \sum_{z=1}^{i} \frac{1}{A(z)}$, $\mathcal{A}(i,i+j) := \mathcal{A}(i+j) - \mathcal{A}(i) = \sum_{z=i}^{i+j} \frac{1}{A(z)}$, and $\mathcal{A}(0) = 0$.

Furthermore, let $a_0 = 0$ and for $i \in \mathbb{N}$ set $a_i := \frac{\mu}{(\mathcal{A}(i))}$.

Theorem 2.1. (i) Assume that

\[ \sum_{z=1}^{\infty} \frac{1}{A(z)} = \infty \tag{1} \]

and

\[ \sum_{i=0}^{\infty} \mu([a_i, \infty)) < \infty. \tag{2} \]

Then almost surely no explosion occurs.

(ii) Assume that

\[ \sum_{z=1}^{\infty} \frac{1}{A(z)} < \infty \tag{3} \]

and there exists $\rho > 1$ such that

\[ \sum_{m=1}^{\infty} \prod_{i=1}^{m} \mu([0, A(m)^{\rho^i}]) < \infty. \tag{4} \]

Then an explosion occurs almost surely.
Remark 2.2. If $A$ is bounded and (2) holds, then by [BK20, Theorem 1.2 (i)] not only a.s. no explosion occurs, but we know even that the spread is a.s. linear. On the other hand, condition (4) resembles the conditions in [BK20, Theorem 1.2 (ii)].

Condition (2) is shown in Section 5 to imply that in a certain sense ‘many’ sites $z \in \mathbb{N}$ are traveled over at speed below $A(z)$. Together with (1) this is then shown to imply the absence of an explosion a.s. On the other hand, (4) is used in Section 6 to obtain that in some sense ‘most’ sites $z \in \mathbb{N}$ are traveled over at speed exceeding $A(z)$. This together with (3) is then shown to imply a.s. explosion. A deeper discussion of the proof ideas can be found in Section 4. Note that since $\mu$ is concentrated on $\mathbb{Z}_+$, the function $\rho \mapsto \sum_{m=1}^{\infty} \prod_{i=1}^{m} \mu([0, A(m)^i])$ is non-decreasing, therefore condition (4) is stronger than

$$\sum_{m=1}^{\infty} \prod_{i=1}^{m} \mu([0, A(m)^i]) < \infty.$$ 

For non-explosion we need to control the tails of the initial condition (2) so that there are not too many dormant frogs at the beginning. On the other hand, in the condition for explosion we require of the initial distribution to be sufficiently heavy (4). Taking $A(x) = \frac{1}{\ln(x+1) - \ln x}$ in Theorem 2.1, (i), we get

**Corollary 2.3.** Assume that

$$\sum_{i=1}^{\infty} \mu\left(\left[\frac{i!}{(\ln(i+1))^i}, \infty\right]\right) < \infty,$$

or equivalently

$$\sum_{i=2}^{\infty} \mu\left(\left[\frac{i!}{(\ln i)^i}, \infty\right]\right) < \infty.$$

Then a.s. no explosion occurs.

Applying the inequality $1 - a \leq e^{-a}$, $a \geq 0$, to the left hand side of (4), we get

**Corollary 2.4.** Assume that (3) holds and for some $\rho > 1$

$$\sum_{m=1}^{\infty} \exp\left\{-\sum_{i=1}^{m} \mu(A(m)^{\rho i}, \infty)\right\} < \infty. \tag{5}$$

Then an explosion occurs almost surely.

Remark 2.5. In this paper we focus on the one-dimensional process. A coupling argument ([RS04, Lemma 4.1] or [BK20, Proposition 2.3]) implies that whenever explosion occurs on $\mathbb{Z}$ with probability one it also occurs on $\mathbb{Z}^d$, $d \in \mathbb{N}$, with probability one. We expect furthermore that explosion on $\mathbb{Z}$ implies explosion on regular trees by a similar coupling argument. Whether explosion on $\mathbb{Z}$ implies explosion for every connected graph with finite degrees containing a copy of the integer line as a subgraph is not clear.
Remark 2.6. Denote by $\mathcal{F}_n$ the $\sigma$-algebra generated by the walks of the particles started from within the set $\{-n, -n+1, \ldots, n\}$, that is

$$\mathcal{F}_n = \sigma\{S_{t(x,j)}^i, \eta(x), t \geq 0, -n \leq x \leq n, 1 \leq j \leq \eta(x)\}.$$  

(6)

The event \{explosion occurs\} is in $\sigma\bigcup_{n \in \mathbb{N}} \mathcal{F}_n$ and yet independent of $\mathcal{F}_n$ for every $n \in \mathbb{N}$. Therefore by the 0-1 law, $\mathbb{P}\{\text{explosion occurs}\} \in \{0, 1\}$ for any distribution $\mu$.

We close the discussion of our result by providing explicit examples of $\mu$ leading to explosion or non-explosion.

Example 2.7. Assume that for some $a \in (0,1)$ for large $b > 0$

$$\mu((b, \infty)) \geq (\ln b)^{-a}. \tag{7}$$

Then by taking $A(n) = n^a$ with $\alpha > 1$ we find for $m \in \mathbb{N}$

$$\sum_{i=1}^{m} \mu((A(m)^{a_i}, \infty)) = \sum_{i=1}^{m} \mu((m^{a_i}, \infty)) \geq \frac{1}{(\alpha \ln m)^a} \sum_{i=1}^{m} \frac{1}{i^a} \geq \frac{1}{(\alpha \ln m)^a} \frac{cm^{1-a}}{1-a}$$

for some $c > 0$. Hence

$$\sum_{m=1}^{\infty} \exp \left\{ - \sum_{i=1}^{m} \mu((A(m)^{a_i}, \infty)) \right\} \leq \sum_{m=1}^{\infty} \exp \left\{ - \frac{cm^{1-a}}{(1-a)(\alpha \ln m)^a} \right\} < \infty.$$

By Corollary 2.4 an explosion occurs a.s. It follows that if $e^X \sim \mu$ ($\mu$ is the distribution of $e^X$), where $X$ is a Pareto distribution with

$$\mathbb{P}\{X \geq t\} \leq \frac{1}{t^a}, \quad t \geq 1$$

for $a \in (0,1)$, then explosion occurs a.s. Note that condition (7) is much weaker than the explosion condition in [BDK21, Remark 2.5], which was given by

$$\mu([2^{ln^2 + 1} n^{8 n^2 + 1}, \infty)) \geq \frac{1}{n - 1} \ \text{for} \ n \geq 2.$$

Example 2.8. Assume that $e^{Y \ln Y} \sim \mu$, where a non-negative random variable $Y$ has a finite expectation. Take $A(x) = \frac{1}{\ln(x+1) - \ln x}$. We write for large $i$

$$a_i = \frac{i!}{(\ln(i+1))^i} \geq e^{0.95i \ln i}$$

and

$$\mu([a_i, \infty)) = \mathbb{P}\{e^{Y \ln Y} \geq a_i\} \leq \mathbb{P}\{e^{Y \ln Y} \geq e^{0.95i \ln i}\} \leq \mathbb{P}\{Y \geq 0.9i\}.$$ 

Hence (2) holds, and Theorem 2.1, (i), implies that no explosion occurs a.s.

Remark 2.9. Let us place two preceding examples in the context of [BK20]: we know that if $\mathbb{E}Y < \infty$ for a non-negative random variable $Y$ and $e^Y \sim \mu$, then the spread is linear in time [BK20, Theorem 1.2, (i)]. On the other hand, [BK20, Theorem 1.2, (iii)] implies that the spread is superlinear if $\mathbb{E}Y = \infty$ and $e^{Y \ln^2 Y} \sim \mu$.
3 Totally asymmetric discrete inhomogeneous Boolean percolation (TADIBP)

In this section we introduce the percolation process which is used to analyze the explosion of the frog model. We introduce a general TADIBP model which is used in Section 4 to define a percolation process corresponding to the frog model. Let \( \{ \psi_z \}_{z \in \mathbb{Z}} \) be a collection of independent \( \mathbb{Z}_+ \)-valued random variables with distributions \( p_k^Z = \mathbb{P} \{ \psi_z = k \} \), \( z \in \mathbb{Z} \). We consider a germ-grain model with germs at the sites of \( \mathbb{Z} \) and grains of the form \( [x, x + \psi_x] \). The distribution of a \( \mathbb{Z}_+ \)-valued random variable \( \psi_x \) depends on the location \( x \), hence the model is inhomogeneous in space. Germ-grain models are well known and typically treated in homogeneous settings [CSKM13, Section 6.5]. The spatially homogeneous version of the model we present below was introduced by Lamperti [Lam70] and was later considered in [KW06] and [Zer18]. A continuous-space version of the model is treated in [Bez21]. We follow the interpretation introduced in [Lam70]: At each site \( x \), there is a fountain that wets integer sites in the interval \( (x, x + \psi_x] \).

We say that \( x, y \in \mathbb{Z} \), \( x \leq y \), are directly connected (denoted by \( x \xrightarrow{Z} y \)) if there exists \( z \leq x, z \in \mathbb{Z} \), such that \( z + \psi_z \geq y \). We say that \( x \) and \( y \) are connected (denoted by \( x \xrightarrow{Z} y \)) if they are directly connected, or if there exists \( z_1 \leq \ldots \leq z_n \in \mathbb{Z} \), \( z_1 \leq x, z_n \leq y \), such that \( x \in [z_1, z_1 + \psi_{z_1}], y \in [z_n, z_n + \psi_{z_n}] \), and \( z_{j+1} \in [z_j, z_j + \psi_{z_j}] \) for \( j = 1, 2, \ldots, n-1 \), or, equivalently,

\[
x \xrightarrow{Z} z_2 \xrightarrow{Z} \cdots \xrightarrow{Z} z_n \xrightarrow{Z} y.
\]

For a subset \( Q \subset \mathbb{Z} \), \( x \xrightarrow{Q} y \) and \( x \xrightarrow{Q} y \) are defined in the same way with an additional requirement that \( x, y, z, z_1, \ldots, z_n \in Q \) (in this paper we only consider \( Q = \mathbb{Z} \) and \( Q = \mathbb{Z}_+ \)). We say that \( x \in \mathbb{Z} \) is wet if the interval \( [x - 1, x] \) is contained in \( [y, y + \psi_y] \) for some \( y \in \mathbb{Z} \). In other words, \( x \in \mathbb{Z} \) is wet if for some \( y \in \mathbb{Z} \), \( y < x \) and \( y + \psi_y \geq x \). The sites that are not wet are said to be dry. Note that \( x \) is wet if and only if \( x - 1 \) and \( x \) are connected. We call the resulting random structure totally asymmetric discrete inhomogeneous Boolean percolation (TADIBP). When considering TADIBP on \( \mathbb{Z}_+ \), we also talk about ‘wet’ sites, with the understanding that both \( x \) and \( y \) are required to be from \( \mathbb{Z}_+ \). Also, we consider the origin to be wet for TADIBP on \( \mathbb{Z}_+ \).

**Definition 3.1.** For \( m \in \mathbb{Z}_+ \), denote by \( Y_m \) the difference between the rightmost site directly connected to \( m \) and \( m \), i.e.

\[
Y_m = \max \left\{ l : m \xrightarrow{Z} l \right\} - m.
\]

By definition, \( m \xrightarrow{Z} m \) and hence, \( Y_m \geq 0 \). Also, by construction, \( Y_0 = \psi_0 \) and for \( m \in \mathbb{N} \),

\[
Y_m = \psi_m \lor (\psi_{m-1} - 1) \lor \cdots \lor (\psi_1 - m + 1) \lor (\psi_0 - m).
\]

We say that \( x \) is connected to infinity, denoted by \( x \xrightarrow{Z} \infty \), if \( x \xrightarrow{Z} y \) for every \( y > x \). Note that for \( x \in \mathbb{Z}_+ \), \( x \xrightarrow{Z} \infty \) if and only if \( Y_m > 0 \) for all \( m \geq x \).

**Definition 3.2.** We say that a system \( \{ \psi_x \} \) of random variables of the TADIBP percolates if there exists \( x_0 \in \mathbb{Z}_+ \) such that \( x_0 \xrightarrow{Z} \infty \).
The following lemma is [BK20, Lemma 3.8] with a typo corrected. For completeness we also give the proof.

**Lemma 3.3.** Let \( x \in \mathbb{N} \). A.s. on \( \{ x \xrightarrow{Z} \infty \} \), every site \( y > x \) is wet, and there exists a (random) sequence \( x_0 < x_1 < x_2 < \ldots, x_i \in \mathbb{Z}_+ \), such that \( x_0 \leq x < x_1 \) and for every \( i \in \mathbb{Z}_+ \)
\[
x_{i+1} \leq x_i + \psi_{x_i} < x_{i+2}.
\]
In particular, every \( z \geq x \) belongs to no more than two intervals of the type \([x_i, x_i + \psi_{x_i}], i \in \mathbb{Z}_+\).

**Proof.** By definition of \( \xrightarrow{Z} \), every site \( y > x \) is wet a.s. on \( \{ x \xrightarrow{Z} \infty \} \). Define the elements of the sequence \( \{x_i\}_{i \in \mathbb{Z}_+} \) consecutively by setting
\[
x_0 = \max \{ y \in [0, x_0] \cap \mathbb{N} : y + \psi_y = \max \{ z + \psi_z : z = x_0, x_0 - 1, \ldots, 0 \} \}
\]
and letting for \( i \in \mathbb{Z}_+ \)
\[
x_{i+1} = \max \{ y \in [x_i + 1, x_i + \psi_{x_i}] \cap \mathbb{N} : y + \psi_y = \max \{ z + \psi_z : z = x_i + 1, \ldots, x_i + \psi_{x_i} \} \}. \quad (9)
\]
In other words, \( x_{i+1} \in [x_i + 1, x_i + \psi_{x_i}] \) is characterized by two properties:

(i) for every \( z \in [x_i + 1, x_i + \psi_{x_i}] \cap \mathbb{N} \),
\[
x_{i+1} + \psi_{x_{i+1}} \geq z + \psi_z,
\]
(ii) and for every \( z' \in [x_{i+1} + 1, x_{i+1} + \psi_{x_{i+1}}] \cap \mathbb{N} \),
\[
x_{i+1} + \psi_{x_{i+1}} > z' + \psi_{z'}
\]
(here \([a, b] = \emptyset\) if \( a > b \)). By construction, \( x_{i+1} \leq x_i + \psi_{x_i} \), so the left inequality in (8) holds. A.s. on \( \{ x \xrightarrow{Z} \infty \} \), \( x_{i+1} + \psi_{x_{i+1}} > x_i + \psi_{x_i} \), because otherwise \( x_i + \psi_{x_i} + 1 \) would not be wet. Hence a.s. on \( \{ x \xrightarrow{Z} \infty \} \) also \( x_{i+2} + \psi_{x_{i+2}} > x_{i+1} + \psi_{x_{i+1}} \). Therefore the inequality \( x_{i+2} \leq x_i + \psi_{x_i} \) is impossible a.s. on \( \{ x \xrightarrow{Z} \infty \} \) because it would contradict to (i) with \( z = x_{i+2} \). \qed

4 Notation, preliminaries, and further discussion

For each \( x \in \mathbb{Z} \) and \( j \in \mathbb{N} \), we denote by \( (S^{(x,j)}_t)_{t \geq 0} \) a simple symmetric continuous-time random walk starting at \( S^{(x,j)}_0 = 0 \). We assume that the collection
\[
\{ S^{(x,j)}_t, x \in \mathbb{Z}, j \in \mathbb{N} \}
\]
is i.i.d. For \( m, n \in \mathbb{N} \), denote \( \overline{m,n} = [m, n] \cap \mathbb{Z} \). For \( t \geq 0, x \in \mathbb{Z} \), and \( j \in \mathbb{N} \), the number \( x + S^{(x,j)}_t \) is the position of \( j \)-th particle started at location \( x \), \( t \) units of time after the sleeping particles at \( x \) were activated. Let \( (S_t, t \geq 0) \) be a simple continuous-time random walk on \( \mathbb{Z} \) and \( \tau_k \) be the \( k \)-th jump of \( (S_t, t \geq 0) \), \( \tau_0 = 0 \).
For two series \( \sum_n a_n \) and \( \sum_n b_n \) with non-negative elements we write \( \sum_n a_n \approx \sum_n b_n \) if they have the same convergence properties, that is, they either both converge or both diverge. We write \( \sum_n a_n \preceq \sum_n b_n \) if \( \sum_n b_n \) diverges, or if both \( \sum_n a_n \) and \( \sum_n b_n \) converge. This is true for example if \( a_n \leq b_n \) for large \( n \in \mathbb{N} \) (but not necessarily for all \( n \in \mathbb{N} \)). We say that two events \( A \) and \( B \) are equal a.s., or coincide a.s., if \( 1_A = 1_B \) holds a.s. Multiplication takes precedence over taking maximum and minimum: for \( a, b, c \in \mathbb{R} \), \( ab \vee c = (ab) \vee c \), \( ab \wedge c = (ab) \wedge c \).

As an auxiliary tool we consider the following construction of a TADIBP. Recall that \( \{ S_t^{(x,j)} \} \) are the random walks assigned to individual particles in the frog model with initial configuration \( \eta(x) \), and let \( A : \mathbb{N} \to (0, \infty) \) be a non-decreasing function. We define the random variable \( S^{(x,j)}_t \) as:

\[
S^{(x,j)}_t = \max \left\{ k \in \mathbb{Z}_+ : \exists t > 0, j \in \eta(x) \text{ such that } t \leq \sum_{z=x+1}^{x+S_t^{(x,j)}} \frac{1}{A(z)} \text{ and } S_t^{(x,j)} \geq k \right\} \vee 0. \tag{10}
\]

(here as usual \( \max \emptyset = -\infty \)).

We consider TADIBP with \( \psi_x = \ell_x^{(A)} \). Heuristically, sites \( x \) which are wet in the TADIBP model are traversed by frogs at speed no less than \( A(x) \). Therefore, if (3) holds and (almost) all sites of the TADIBP are wet, it means that frogs traverse the space \( \mathbb{Z} \) at high speed, leading to explosion of the system. Conversely, (1) and many dry sites imply that the frog model travels at low speed, leading to non-explooding expansion.

Since \( A \) is non-decreasing, we have

\[
P\{ \ell_x^{(A)} \geq k \} \geq P\{ \ell_{x+1}^{(A)} \geq k \}, \quad x \in \mathbb{N}, k \in \mathbb{Z}_+.
\]

**Remark 4.1.** The random variable \( \ell_x^{(A)} \) can be seen as the maximal distance travelled to the right by a particle starting from \( x \) at a speed exceeding the given (varying) speed \( A \).

The following elementary lemma is used throughout the paper. In particular, it can be applied to the Poisson distribution.

**Lemma 4.2.** Assume that for a sequence of positive numbers \( \{ \alpha_j \}_{j \in \mathbb{N}} \) there exist \( r \in (0, 1) \) and \( n \in \mathbb{N} \) such that either for all \( i \geq n \)

\[
\frac{\alpha_{i+1}}{\alpha_i} \leq r \tag{11}
\]

or for all \( i \in \mathbb{N} \)

\[
\alpha_{n+1} \leq r^n \alpha_n. \tag{12}
\]

Then there exists \( C_{n,r} > 1 \) such that for \( m \in \mathbb{N} \)

\[
\sum_{i=m}^{\infty} \alpha_i \leq C_{n,r} \alpha_m. \tag{13}
\]

The constant \( C_{n,r} \) can be chosen to depend only on \( r \) and \( n \).

**Proof.** Note that (11) implies (12), and by (12)

\[
\sum_{i=n}^{\infty} \alpha_i \leq \sum_{i=n}^{\infty} r^n \alpha_n = \frac{\alpha_n}{1 - r}.
\]
Therefore $\sup_{m \in \mathbb{N}} \sum_{i=m}^{\infty} \alpha_i < \infty$, that is, (13) holds for some $C > 0$. \hfill \Box

In the article \cite{BK20} the authors described conditions for the distinction between linear and superlinear spread. To this end, for a fixed $B > 0$ they used the family $(\psi_x)_x$ given by

$$\psi_x = \max \left\{ y \geq x : \exists j, \exists t : S_t^{(x,j)} \geq B, S_t^{(x,j)} \geq y - x \right\}.$$  

The expression on the right hand side coincides with our definition of $\ell_x^{(A)}$ in (10) with $A(x) \equiv B$.

These random variables give rise to totally asymmetric discrete (homogeneous) Boolean percolation as described in Section 3. The proofs in \cite{BK20} rely on the following statements.

- If percolation occurs for every constant $B > 0$, then the spread is superlinear.
- If a positive fraction of sites is dry for some $B > 0$, then the spread is linear.

The conditions implying that percolation occurs, or that it does not occur, are then given in terms of the distribution of the initial number of particles $\mu$.

In this paper, the goal is to describe conditions separating the non-explosion and explosion as opposed to the linear and superlinear spread. The idea is to modify the family $(\psi_x)_x$ so that $\psi_x = \ell_x^{(A)}$ with an increasing function $A$ as defined in (10). Since $A$ is the “speed” at which the process propagates, the following statements should hold.

- If percolation occurs for some $A$ with
  $$\sum_{x \in \mathbb{N}} \frac{1}{A(x)} < \infty,$$
  then the process explodes.
- If percolation does not occur for some $A$ with
  $$\sum_{x \in \mathbb{N}} \frac{1}{A(x)} = \infty,$$
  then the process does not explode.

\textbf{Remark 4.3.} Note the similarity to ODE: Given

$$\dot{x} = f(x), \ x(0) = 1,$$

where $f$ is a non-negative continuous increasing function, we have explosion in finite time if

$$\int_1^{\infty} \frac{dy}{f(y)} < \infty.$$  

The proof of explosion (Theorem 2.1 (ii)) closely follows the scheme we have just outlined. In Section 6, we first show that percolation with $\psi_x = \ell_x^{(A)}$ implies explosion, and then proceed to establish that percolation occurs a.s. under assumptions in Theorem 2.1 (ii). In contrast, when considering non-explosion we do not directly rely on percolation not occurring, because a possible long range dependence makes it difficult to deduce non-explosion from non-percolation. Instead, we show that the assumptions in Theorem 2.1 (i) imply $\inf_{x \in \mathbb{N}} \mathbb{P}\{x \text{ is dry} \} > 0$, and the latter is then shown to be incompatible with explosion.
5 Proof of non-explosion

In this section, we prove the first part of Theorem 2.1. Here at the beginning of the section we lay out the roadmap of the proof. We first show that the explosions in two directions, $+\infty$ and $-\infty$, are equivalent a.s. This is formulated precisely in (42). Because of that, it suffices to rule out the possibility of the explosion in direction $+\infty$ to prove non-explosion of the system. To this end, we show next that under conditions (1) and (2) the particles left to the origin cannot contribute to an explosion in this direction (this is formulated precisely in (27)) and can thus be removed. After that totally asymmetric discrete inhomogeneous Boolean percolation enters the picture. It is introduced in Section 3 with $\ell_x^{(A)}$ defined in (10). In Proposition 5.9 the probability that a site is dry is shown to be separated from zero:

$$\inf_{m \in \mathbb{N}} \mathbb{P}\{m \text{ is dry}\} > 0.$$ 

The final stretch of the proof of Theorem 2.1, (i), starts on Page 18. There the dry sites are shown to be ‘slow’ in a certain sense, and that the inequality $\inf_{m \in \mathbb{N}} \mathbb{P}\{m \text{ is dry}\} > 0$ is incompatible with explosion.

Recall that $a_i = \frac{i}{(A(i))^{i}}$ and that in this section we work under the following assumption on $A$ and $\mu$.

**Condition 5.1.** It holds that

$$\sum_{z=1}^{\infty} \frac{1}{A(z)} = \infty \quad \text{and} \quad \sum_{i=0}^{\infty} \mu([a_i, \infty)) < \infty.$$  \hfill (14)

**Lemma 5.2.** The series $\sum_{i=1}^{\infty} \frac{1}{A(i)} \wedge \frac{1}{i}$ is divergent.

**Proof.** By the Cauchy condensation test

$$\sum_{i=1}^{\infty} \frac{1}{A(i)} \wedge \frac{1}{i} \sim \sum_{n=1}^{\infty} \frac{2^n}{A(2^n)}.$$ 

We have

$$\sum_{i=1}^{\infty} \frac{1}{A(i)} \wedge \frac{1}{i} = \sum_{n=0}^{\infty} \sum_{i=2^n}^{2^{n+1}-1} \frac{1}{A(i)} \wedge \frac{1}{i} \geq \sum_{n=0}^{\infty} 2^n \left[ \frac{1}{A(2^{n+1})} \wedge \frac{1}{2^{n+1}} \right] = \frac{1}{2} \sum_{n=1}^{\infty} 2^n \left[ \frac{1}{A(2^n)} \wedge \frac{1}{2^n} \right] := S.$$ 

If the set $Q := \{ n \in \mathbb{N} : \frac{1}{A(2^n)} \geq \frac{1}{2^n} \}$ is infinite, then $S \geq \frac{1}{2} \sum_{n \in Q} 1 = \infty$. If $Q$ is finite, then

$$S \sim \sum_{n=1}^{\infty} 2^n \frac{1}{A(2^n)} \sim \sum_{i=1}^{\infty} \frac{1}{A(i)} = \infty.$$ 

}\hfill \Box
Without loss of generality, we can replace $A(i)$ with $A(i) \lor i$: indeed, the series $\sum_{i=1}^{\infty} \frac{1}{A(i) \lor i}$ is divergent by Lemma 5.2, and (2) holds too since $\mathcal{A}$ decreases if we make $A$ greater. Thus, we assume henceforth that $A(i) \geq i, i \in \mathbb{N}$.

Define
\[
\sigma^r_\infty = \inf\{t \geq 0 : \sup A_t = \infty\} = \inf\{t \geq 0 : \text{no sleeping particles left on } [0, \infty)\}
\]
and
\[
\sigma^l_\infty = \inf\{t \geq 0 : \inf A_t = -\infty\} = \inf\{t \geq 0 : \text{no sleeping particles left on } (-\infty, 0]\}.
\]

In this paper we do not investigate the question under which $c$ ondition $\sigma^r_\infty = \sigma^l_\infty$. However, we note here that both events $\{\sigma^r_\infty < \infty\}$ and $\{\sigma^l_\infty < \infty\}$ are tail events with respect to the sequence of $\sigma$-algebras $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$ defined in (6). Hence
\[
\mathbb{P}\{\sigma^r_\infty < \infty\} \in \{0, 1\} \quad \text{and} \quad \mathbb{P}\{\sigma^l_\infty < \infty\} \in \{0, 1\}.
\]

By symmetry it follows that $\mathbb{P}\{\sigma^r_\infty < \infty\} = \mathbb{P}\{\sigma^l_\infty < \infty\}$ and hence the events $\{\sigma^r_\infty < \infty\}$ and $\{\sigma^l_\infty < \infty\}$ coincide a.s., that is, the equality
\[
\mathbb{1}\{\sigma^r_\infty < \infty\} = \mathbb{1}\{\sigma^l_\infty < \infty\}
\]
holds a.s. Therefore to prove non-explosion it is enough to show that $\mathbb{P}\{\sigma^r_\infty < \infty\} = 0$, and in the rest of the section we concentrate only on $\sigma^r_\infty$. As an aside not needed in this proof we point out that (15) justifies the discussion in the introduction on Page 2 about an equivalent definition of explosion in dimension one.

Note that $\sigma^r_\infty < \infty$ if and only if (a.s.) there exists a sequence of pairs $\{(x_n, t_n)\}_{n \in \mathbb{Z}^+}$, where $x_n \in \mathbb{Z}$, $x_0 = 0$, $0 = t_0 < t_1 < t_2 < ...$, satisfying
- the sleeping particles at $x_n$ are activated at $t_n$ by an active particle started from $x_{n-1}$, $n \in \mathbb{N}$, and
- $\lim_{n \to \infty} t_n := t_\infty < \infty$.

A priori it may be that infinitely many elements of $\{x_n\}_{n \in \mathbb{Z}^+}$ are negative. Over the next few pages we show that under Condition 5.1 this is impossible (as formulated in Corollary 5.6 and (27)). Recall that $a_i = \frac{\mu((x^r(i)), i \in \mathbb{N}$ and $a_0 = 0$, and set and $b_i = \mu((a_{i-1}, a_i]), b_1 = \mu([0, a_1])$. The next two lemmas have an auxiliary character and are used later to bound certain series.

**Lemma 5.3.** There exists $C_a > 1$ such that
\[
\sum_{i=j}^{\infty} \frac{1}{a_i} \leq \frac{C_a}{a_j} \quad j \in \mathbb{N}.
\]

**Proof.** Recall that we have assumed $A(i) \geq i$ for $i \in \mathbb{N}$, which we can do due to Lemma 5.2. Let $\varepsilon \in (0, 0.1)$. For large $n \in \mathbb{N}$
\[
\sum_{j=1}^{n} A^{-1}(j) \leq \frac{\varepsilon}{n}
\]
and hence
\[ \left[ \frac{\mathcal{A}(n+1)}{\mathcal{A}(n)} \right]^n = \left[ 1 + \frac{A^{-1}(n+1)}{\sum_{j=1}^{n} A^{-1}(j)} \right]^n \leq \left[ 1 + \frac{\varepsilon}{n} \right]^n \leq e^\varepsilon. \] (17)

Consequently for large \( n \in \mathbb{N} \)
\[ \frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(\mathcal{A}(n+1))^{n+1}} : \frac{n!}{(\mathcal{A}(n))^{n}} = \frac{n+1}{\mathcal{A}(n+1)} \left[ \frac{\mathcal{A}(n)}{\mathcal{A}(n+1)} \right]^n \geq \frac{n+1}{e^\varepsilon \mathcal{A}(n+1)}. \]

It remains to note that \( \mathcal{A}(n) \leq \sum_{j=1}^{n} \frac{1}{j} \leq 2 + \ln n, n \in \mathbb{N}, \) and hence \( \frac{n+1}{\mathcal{A}(n+1)} \xrightarrow{n \to \infty} \infty. \)

**Lemma 5.4.** Let \( \{\alpha_i\}_{i \in \mathbb{N}} \) be an increasing sequence of natural numbers satisfying for some \( c_\alpha > 0 \)
\[ \sum_{i=j}^{\infty} \frac{1}{\alpha_i} \leq \frac{c_\alpha}{\alpha_j}, \quad j \in \mathbb{N}, \]
and let \( \beta_i = \mu((\alpha_{i-1}, \alpha_i]), \beta_1 = \mu([0, \alpha_1]). \) Then
\[ \sum_{i=1}^{\infty} \frac{1}{\alpha_i} \sum_{k,k \geq 0, k \leq \alpha_i} \mu(k)k \leq c_\alpha, \] (18)
and
\[ \sum_{i=1}^{\infty} \frac{1}{\alpha_i} \sum_{k,k \geq 0, k \leq \alpha_i} \mu(k)k \leq c_\alpha \frac{\alpha_1}{\alpha_2} + c_\alpha (1 - \beta_1). \] (19)

**Proof.** Set \( \alpha_0 = 0. \) We have
\[ \sum_{i=1}^{\infty} \frac{1}{\alpha_i} \sum_{k,k \geq 0, k \leq \alpha_i} \mu(k)k = \sum_{i=1}^{\infty} \frac{1}{\alpha_i} \sum_{j=1}^{\alpha_j} \sum_{k=k=\alpha_{j-1}+1}^{\infty} \mu(k)k \]
\[ \leq \sum_{i=1}^{\infty} \frac{1}{\alpha_i} \sum_{j=1}^{\alpha_j} \beta_j \alpha_j \leq \sum_{j=1}^{\infty} \beta_j \alpha_j \sum_{i=1}^{\infty} \frac{1}{\alpha_i} \leq \sum_{j=1}^{\infty} \beta_j \alpha_j \frac{c_\alpha}{\alpha_j} = c_\alpha \sum_{j=1}^{\infty} \beta_j \leq c_\alpha. \]

Similarly
\[ \sum_{i=2}^{\infty} \frac{1}{\alpha_i} \sum_{k,k \geq 0, k \leq \alpha_i} \mu(k)k = \sum_{i=2}^{\infty} \frac{1}{\alpha_i} \sum_{j=1}^{\alpha_j} \sum_{k=k=\alpha_{j-1}+1}^{\infty} \mu(k)k \]
\[ \leq \sum_{i=2}^{\infty} \frac{1}{\alpha_i} \sum_{j=1}^{\alpha_j} \beta_j \alpha_j \leq \sum_{j=1}^{\infty} \beta_j \alpha_j \sum_{i=1}^{\infty} \frac{1}{\alpha_i} \leq \sum_{j=1}^{\infty} \beta_j \alpha_j \frac{c_\alpha}{\alpha_j \alpha_{j+1}} = c_\alpha \beta_1 \frac{\alpha_1}{\alpha_2} + c_\alpha (1 - \beta_1), \]
which gives (19).

Let \( (N_t^{(x,j)}, t \geq 0) \) be a Poisson process obtained from \( (S_t^{(x,j)}, t \geq 0) \) by making all jumps be \( +1: \) for \( q > 0 \) the number \( N_q^{(x,j)} \) can be seen as the number of jumps of \( (S_t^{(x,j)}, t \geq 0) \) before the time \( q. \) Clearly, a.s. \( S_t^{(x,j)} \leq N_t^{(x,j)} \) for all \( x \in \mathbb{Z}, j \in \mathbb{N}, t \geq 0. \) Also, let \( (N_t, t \geq 0) \) be the Poisson process with the same jumps as \( (S_t, t \geq 0). \) The next lemma is key in establishing that under Condition 5.1 particles left to the origin cannot contribute to explosion.
Lemma 5.5. There exists an increasing sequence \( \{d_q\}_{q \in \mathbb{N}} \) satisfying
\[
\sum_{q \in \mathbb{N}} \mathbb{P}\{ \max\{N_q^{(x,j)} + i : x < 0, 1 \leq j \leq \eta(x)\} \geq d_q\} < \infty.
\]

Note that the time \( q \) takes discrete values here.

**Proof.** By Lemma 4.2 for \( t \geq 0 \) there exists \( C_t > 0 \) such that for all \( m, i \in \mathbb{N} \)
\[
\mathbb{P}\{N_t \geq m + i\} \leq C_t e^{-t} (m+i)! < C_t e^{-t} \frac{e^{m+i} e^{m+i}}{(m+i)^{m+i}} = C_t e^{-t} \left( \frac{te}{m+i} \right)^{m+i}.
\]

By Condition 5.1 for \( n \in \mathbb{N} \) there exists \( \kappa_n \in \mathbb{N} \) such that
\[
\sum_{i=1}^{\infty} \mu[a_{i+\kappa_n}, \infty) = \sum_{i=\kappa_n+1}^{\infty} \mu[a_i, \infty) < \frac{1}{2^n}.
\]

For \( q, i \in \mathbb{N} \) set \( c_{q,i} := C_q e^{-\eta \left( \frac{q}{d_{q,i}^{-1}} \right)^{d_{q,i}}} \), where \( C_q \) is the constant in (20), and choose the sequence \( d_1, d_2, \ldots, d_q \geq 2^q \), in such a way that \( c_{q+1,i} < \frac{1}{2^n} c_{q,i}, c_{1,i} \leq \frac{1}{a_i} \),
\[
a_{i+\kappa_n} < c_{q,i}^{-1}, \quad i, q \in \mathbb{N},
\]

\[
c_{q,1} \sum_{k: k \geq 0, k < c_{q,1}^{-1}} \mu(k, k) \leq \frac{1}{2^n}, \quad \text{and} \quad \mu([c_{q,1}^{-1}, \infty)) \leq \frac{1}{2^n}.
\]

The sequence \( d_1, d_2, \ldots \) can be constructed successively: given \( d_1, \ldots, d_n, d_{n+1} \) can be chosen large enough to satisfy all the conditions. It is important for (22) that by Condition 5.1 the asymptotic growth rate of \( j \mapsto a_j \) is actually lower than that of \( j \mapsto \left( \frac{d+j}{c} \right)^j \) for constants \( c, d > 0 \): that is, for every \( c, d > 0 \) for large \( j \)
\[
a_j < \left( \frac{d+j}{c} \right)^j < \left( \frac{d+j}{c} \right)^j.
\]

For (23) it is important that
\[
\lim_{Q \to \infty} \frac{1}{Q} \sum_{k: k \geq 0, k < Q} \mu(k, k) \leq \lim_{Q \to \infty} \sum_{k: k \geq 0} \mu(k, k) \left[ \frac{k}{Q} \wedge 1 \right] = 0.
\]

We have for \( q \in \mathbb{N} \) by (20)
\[
\mathbb{P}\{ \max\{N_q^{(i,j)} : i < 0, 1 \leq j \leq \eta(i)\} \geq d_q\} \leq \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \mu(k, k) \left[k \mathbb{P}\{N_q \geq d_q + i\} \wedge 1\right] \\
\leq \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \mu(k, k) \left[k c_{q,i} \wedge 1\right] \\
= \sum_{i=1}^{\infty} \mu(c_{q,i}^{-1}, \infty) + \sum_{i=1}^{\infty} c_{q,i} \sum_{k: k \geq 0, k < c_{q,i}^{-1}} \mu(k, k).
\]
Taking the sum over \( q \) in (24) we get
\[
\sum_{q \in \mathbb{N}} \mathbb{P}\{ \max\{ N_q^{(i,j)} + i : i < 0, 1 \leq j \leq \eta(x) \} \geq d_q \}
\leq \sum_{q=1}^{\infty} \sum_{i=1}^{\infty} \mu(c_{q,i}^{-1}, \infty) + \sum_{q=1}^{\infty} \sum_{i=1}^{\infty} \sum_{k:k \geq 0, k < c_{q,i}^{-1}} \mu(k)k. \quad (25)
\]

Our conditions on \( d_q \) and \( c_{q,i} \) now imply that both sums on the right hand side of (25) are finite. The first is finite since by (21) and (22)
\[
\sum_{q=1}^{\infty} \sum_{i=1}^{\infty} \mu(c_{q,i}^{-1}, \infty) \leq \sum_{q=1}^{\infty} \sum_{i=1}^{\infty} \mu(a_{\alpha_q+i}, \infty) \leq \sum_{q=1}^{\infty} \frac{1}{2^q}.
\]

To show that the second sum on the right hand side in (25) is finite, we split the sum into two and apply (23) and Lemma 5.4 to the internal sums with \( \alpha_i = \lfloor c_{q,i}^{-1} \rfloor \). In notation of Lemma 5.4 we can take \( c_{\alpha} = 2 \) for every \( q \in \mathbb{N} \), and use that \( \frac{d_q}{2^q} \leq d_q^{-1} \leq 2^{-q} \) and \( \beta_1 \geq 1 - 2^{-q} \) by the second inequality in (23). We have
\[
\sum_{q=1}^{\infty} \sum_{i=1}^{\infty} c_{q,i} \sum_{k:k \geq 0, k < c_{q,i}^{-1}} \mu(k)k \leq \sum_{q=1}^{\infty} \sum_{i=1}^{\infty} c_{q,i} \sum_{k:k \geq 0, k < c_{q,i}^{-1}} \mu(k)k \leq \sum_{q=1}^{\infty} \frac{1}{2^q} + \sum_{q=1}^{\infty} \left( \frac{1}{2^q} + \frac{1}{2^{2q}} \right) < \infty.
\]

By the above lemma a.s. only finitely many events \{\max\{ N_t^{(x,j)} : x < 0, 1 \leq j \leq \eta(x) \} \geq d_t \}, \( t \in \mathbb{N} \), occur. In particular, we have

**Corollary 5.6.** *A.s. for all \( t > 0 \)
\[
\sup_{x<0, 1 \leq j \leq \eta(x)} (S_t^{(x,j)} + x) < \infty. \quad (26)
\]

This means that a.s. the particles at \(-1, -2, \ldots\) do not contribute to the explosion toward \(+\infty\). More precisely, let us modify our process by removing all the sleeping particles left to the origin at the beginning and then proceeding as usual. For this modified process let \( \theta_n, n \in \mathbb{N} \), be the moment when site \( n \) is visited by an active particle for the first time, and let \( \theta_\infty = \lim_{n \to \infty} \theta_n \). Then clearly a.s. \( \sigma_\infty^x \leq \theta_\infty \), however in view of Corollary 5.6, a.s.
\[
1{\{ \sigma_\infty^x = \infty \}} = 1{\{ \theta_\infty = \infty \}}. \quad (27)
\]

The equality (27) represents a major stepping stone in the proof of Theorem 2.1, (i). It allows us to remove at time \( t = 0 \) all sleeping particles left to the origin. From here on out we only consider the modified process with particles left to the origin removed; equivalently, we set \( \eta(y) = 0 \) for \( y < 0 \).
Recall that $\ell_{i}^{(A)}$ was defined in (10). In the remaining part of the proof of Theorem 2.1, (i), we consider totally asymmetric discrete inhomogeneous Boolean percolation from Section 3 on $\mathbb{Z}_{+}$ with $\psi_{i} = \ell_{i}^{(A)}$, $i \in \mathbb{Z}_{+}$. The probability that $m \in \mathbb{N}$ is dry is given by

$$
P\{m \text{ is dry} \} = P\{Y_{m} = 0 \} = \prod_{i=0}^{m-1} P\{\ell_{i}^{(A)} \leq m - i \} = \prod_{i=0}^{m-1} \left(1 - P\{\ell_{i}^{(A)} > m - i \} \right).
$$

(28)

In the next few lemmas we work towards establishing that $\inf_{m \in \mathbb{N}} P\{m \text{ is dry} \} > 0$; this is achieved in Proposition 5.9. The next two lemmas, Lemma 5.7 and Lemma 5.8, are auxiliary tools in the proof of Proposition 5.9.

**Lemma 5.7.** There exists $C > 0$ such that for $i \in \mathbb{Z}_{+}$, $j \in \mathbb{N}$

$$
P\left\{ \exists t > 0 : t \leq \sum_{z=i+1}^{i+S_{t}} \frac{1}{A(z)} \text{ and } S_{t} \geq j \right\} \leq C \exp\{-\mathcal{A}(i,i+j)\} \frac{(\mathcal{A}(i,i+j))^{j}}{j!}.
$$

**Proof.** Recall that $(S_{t}, t \geq 0)$ is a simple continuous-time random walk on $\mathbb{Z}$, $\tau_{k}$ is the $k$-th jump of $(S_{t}, t \geq 0)$, $\tau_{0} = 0$, and $(N_{t}, t \geq 0)$ is the Poisson process with jumps at $\tau_{1}, \tau_{2}, ...$. Note that

$$
P\left\{ \exists t > 0 : t \leq \sum_{z=i+1}^{i+S_{t}} \frac{1}{A(z)} \text{ and } S_{t} \geq j \right\} = P\{\exists t > 0 : t \leq \mathcal{A}(i+S_{t}) - \mathcal{A}(i) \text{ and } S_{t} \geq j \}
$$

$$
\leq P\{\exists t > 0 : t \leq \mathcal{A}(i+N_{t}) - \mathcal{A}(i) \text{ and } N_{t} \geq j \}.
$$

Now

$$
P\{\exists t > 0 : t \leq \mathcal{A}(i+N_{t}) - \mathcal{A}(i) \text{ and } N_{t} \geq j \} = P\{\exists n \in \mathbb{N}, n \geq j : \tau_{n} \leq \mathcal{A}(i,i+n) \}
$$

$$
= \sum_{n=j}^{\infty} P\{N_{\mathcal{A}(i,i+n)} \geq n \}
$$

$$
= \sum_{n=j}^{\infty} \sum_{k=n}^{\infty} e^{-\mathcal{A}(i,i+n)} \frac{\left(\mathcal{A}(i,i+n)\right)^{k}}{k!} = \sum_{k=j}^{\infty} \frac{1}{k!} \sum_{n=j}^{k} e^{-\mathcal{A}(i,i+n)} \left(\mathcal{A}(i,i+n)\right)^{k}
$$

$$
\leq e^{-\mathcal{A}(i,i+j)} \sum_{k=j}^{\infty} \frac{1}{k!} \sum_{n=j}^{k} \left(\mathcal{A}(i,i+k)\right)^{k}
$$

$$
= e^{-\mathcal{A}(i,i+j)} \sum_{k=j}^{\infty} \frac{(k-j+1)}{k!} \left(\mathcal{A}(i,i+k)\right)^{k}.
$$

(29)

Note that $\mathcal{A}(i,i+k) \leq \mathcal{A}(k) \leq \ln k + 2$ and similarly to (17)

$$
\frac{\left(\mathcal{A}(i,i+k+1)\right)^{k+1}}{\left(\mathcal{A}(i,i+k)\right)^{k}} \leq \left[1 + \frac{1}{k}\right]^{k} \leq e.
$$

Therefore the sequence $\{s_{k}\}_{k \in \mathbb{N}}$ satisfies the conditions of Lemma 4.2 uniformly in $i$ and $j$:

$$
\frac{s_{k}}{s_{k+1}} = \frac{k+1}{\mathcal{A}(i,i+k+1)} \frac{k-j+1}{k-j+2} \left(\frac{\left(\mathcal{A}(i,i+k+1)\right)^{k}}{\left(\mathcal{A}(i,i+k)\right)^{k}}\right) \geq \frac{k+1}{2e(\ln k + 2)}.
$$
We see that the the convergence \( \frac{s_k}{s_{k+1}} \xrightarrow{k \to \infty} \infty \) takes place uniformly in \( i \) and \( j \). Consequently there exists \( C > 0 \) such that for all \( i, j \in \mathbb{N} \)

\[
\sum_{k=j}^{\infty} s_k \leq Cs_j.
\]

Therefore by (29)

\[
\mathbb{P} \{ \exists t > 0 : t \leq \mathcal{A}(i + N_t) - \mathcal{A}(i) \text{ and } N_t \geq j \} \leq Ce^{-\mathcal{A}(i,i+j)}(\mathcal{A}(i,i+j))t^j.
\]

\[\square\]

**Lemma 5.8.** Let \( 0 < \alpha_1 \leq \alpha_2 \leq \ldots \) be a sequence of positive numbers such that \( \alpha_n \xrightarrow{n \to \infty} \infty \), \( \limsup_{n \to \infty} \frac{\alpha_{n+1}}{\alpha_n} > 1 \), and for some \( C \alpha > 1 \)

\[
\sum_{i=m}^{\infty} \frac{1}{\alpha_i} \leq \frac{C\alpha}{\alpha_m}.
\]

Then for \( C \geq 1 \)

\[
\sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \mu(k) (1 \wedge kC\alpha_i^{-1}) \simeq \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \mu(k) (1 \wedge k\alpha_i^{-1}).
\]

**Proof.** We have

\[
\sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \mu(k) (1 \wedge kC\alpha_i^{-1}) \leq \sum_{i=1}^{\infty} \sum_{k=kC\alpha_i}^{\infty} \mu(k) + \sum_{i=1}^{\infty} \sum_{k<kC\alpha_i} \mu(k)kC\alpha_i^{-1}
\]

\[
= \sum_{i=1}^{\infty} \mu([C\alpha_i, \infty)) + C \sum_{i=1}^{\infty} \sum_{k<kC\alpha_i} \mu(k)k\alpha_i^{-1}.
\]

(30)

Since \( C \geq 1 \)

\[
\sum_{i=1}^{\infty} \mu([C\alpha_i, \infty)) \lesssim \sum_{i=1}^{\infty} \mu([\alpha_i, \infty)).
\]

(31)

The second sum in (30) is always finite. Indeed, set \( \alpha_0 = 0 \) and \( \beta_i = \mu((C\alpha_{i-1}, C\alpha_i]) \), then

\[
\sum_{i=1}^{\infty} \sum_{k<kC\alpha_i} \mu(k)k\alpha_i^{-1} = \sum_{i=1}^{\infty} \sum_{j=1}^{i} \sum_{k=kC\alpha_{j-1}}^{kC\alpha_j} \mu(k)k\alpha_i^{-1} \leq \sum_{i=1}^{\infty} \sum_{j=1}^{i} \beta_jC\alpha_j\alpha_i^{-1}
\]

\[
= C \sum_{j=1}^{\infty} \beta_jC\alpha_j \sum_{i=j}^{\infty} \alpha_i^{-1} \leq CC\sum_{j=1}^{\infty} \beta_jC\alpha_j \alpha_i^{-1} \leq CC\alpha.
\]

Thus (31) yields

\[
\sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \mu(k) (1 \wedge kC\alpha_i^{-1}) \lesssim \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \mu(k) (1 \wedge k\alpha_i^{-1}).
\]

Since

\[
\sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \mu(k) (1 \wedge kC\alpha_i^{-1}) \geq \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \mu(k) (1 \wedge k\alpha_i^{-1}),
\]

the statement of the lemma follows. \[\square\]
Proposition 5.9. We have
\[ \sup_{m \in \mathbb{N}} \sum_{i=0}^{m-1} \mathbb{P}\{\ell_i^A > m - i\} < \infty \] \tag{32} 
and
\[ \inf_{m \in \mathbb{N}} \mathbb{P}\{m \text{ is dry}\} > 0. \] \tag{33}

Proof. The inequality \( 1 - x \geq e^{-x}, \ x \in (0,1) \), implies
\[ \mathbb{P}\{m \text{ is dry}\} = \prod_{i=0}^{m-1} \left(1 - \mathbb{P}\{\ell_i^A > m - i\}\right) \geq \exp\left(-g_m \sum_{i=0}^{m-1} \mathbb{P}\{\ell_i^A > m - i\}\right), \] \tag{34}
where
\[ g_m = \frac{1 - \max_{0 \leq i \leq m-1} \mathbb{P}\{\ell_i^A > m - i\}}{\mathbb{P}\{\ell_i^A > m - i\}}. \]

Note that \( g_m \xrightarrow{m \to \infty} 1 \) since \( \max_{0 \leq i \leq m-1} \mathbb{P}\{\ell_i^A > m - i\} = \max_{0 \leq i \leq m-1} \mathbb{P}\{\ell_{m-i} > i\} \xrightarrow{m \to \infty} 0 \) which holds due to \( A(x) \xrightarrow{x \to \infty} \infty \).

Let us find a bound for the sum in the exponent of (34). Conditioning on the number of particles on the site \( i \) we get
\[ \mathbb{P}\{\ell_i^A > m - i\} \leq \sum_{k=0}^{\infty} \mu(k) \left(1 \land k \mathbb{P}\left\{\exists t > 0 : t \leq \sum_{z=i+1}^{i+S_i} \frac{1}{A(z)} \text{ and } S_t \geq m - i\right\}\right). \] \tag{35}

By (35) and Lemma 5.7 for some \( C \geq 1 \),
\[ \sum_{i=0}^{m-1} \mathbb{P}\{\ell_i^A > m - i\} \leq \sum_{i=0}^{m-1} \sum_{k=0}^{\infty} \mu(k) \left(1 \land k C \exp\left\{-\mathcal{A}(i,m)\right\} \right) \left(\frac{(\mathcal{A}(i,m))^{m-i}}{(m-i)!}\right) \]
\[ \leq \sum_{i=1}^{m-1} \sum_{k=0}^{\infty} \mu(k) \left(1 \land k C \exp\left\{-\mathcal{A}(m-i,m)\right\} \right) \left(\frac{(\mathcal{A}(m-i,m))^{i}}{i!}\right) \]
\[ \leq \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \mu(k) \left(1 \land k C \exp\left\{-\mathcal{A}(m-i,m)\right\} \right) \left(\frac{(\mathcal{A}(m-i,m))^{i}}{i!}\right). \] \tag{36}

Recall that \( a_i = \frac{i}{(\mathcal{A}(i))^{i}} \). Since \( A \) is non-decreasing for \( i \in \mathbb{Z}_+ \) and \( m \in \mathbb{N} \), \( m > i \), we have
\[ a_i^{-1} = \frac{(\mathcal{A}(i))^{i}}{i!} \geq \exp\left\{-\mathcal{A}(m-i,m)\right\} \left(\frac{(\mathcal{A}(m-i,m))^{i}}{i!}\right). \] \tag{37}

Hence by (36) and Lemma 5.8
\[ \sum_{i=0}^{m-1} \mathbb{P}\{\ell_i^A > m - i\} \leq \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \mu(k) \left(1 \land k C a_i^{-1}\right) \]
\[
\sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \mu(k) \left( 1 \wedge k a_i^{-1} \right) \\
= \sum_{i=1}^{\infty} \sum_{k:k \geq a_i} \mu(k) \left( 1 \wedge k a_i^{-1} \right) + \sum_{i=1}^{\infty} \sum_{k:k < a_i} \mu(k) \left( 1 \wedge k a_i^{-1} \right) \\
= \sum_{i=1}^{\infty} \mu(\{a_i, \infty\}) + \sum_{i=1}^{\infty} \sum_{k:k < a_i} \mu(k) k a_i^{-1} \\
= S_1 + S_2.
\]

By (14)
\[
S_1 \leq \sum_{i=0}^{\infty} \mu(\{a_i, \infty\}) < \infty. \tag{38}
\]
To bound \(S_2\) recall that \(a_0 = 0\) and \(b_i = \mu([a_i-1, a_i])\). By Lemma 5.3 for some \(C_a > 1\) we have
\[
\sum_{i=1}^{\infty} \frac{1}{a_i} \leq \frac{C_a}{a_j}, \quad j \in \mathbb{N},
\]
and hence
\[
S_2 = \sum_{i=1}^{\infty} \sum_{j=1}^{i} \sum_{k:a_j-1 \leq k < a_j} \mu(k) \frac{1}{a_i} \\
\leq \sum_{i=0}^{\infty} \sum_{j=1}^{i} b_j a_j \frac{1}{a_i} \leq \sum_{j=1}^{\infty} b_j a_j \sum_{i=j}^{\infty} \frac{1}{a_i} \leq \sum_{j=1}^{\infty} b_j a_j \frac{C_a}{a_j} = C_a \sum_{j=1}^{\infty} b_j \leq C_a.
\]
Thus (32) is proven. Since \(g_m \xrightarrow{m \to \infty} 1\), (33) follows from (32) and (34).

**Proof of Theorem 2.1, (i).** Recall that \(A_t\) is the set of sites visited by active particles by the time \(t\),
\[
\theta_n = \min\{t \geq 0 : n \in A_t\}, \quad n \in \mathbb{N},
\]
\[
\theta_\infty = \lim_{n \to \infty} \theta_n,
\]
and all sleeping particles left to the origin are removed at the beginning. The event \(\{\theta_\infty < \infty\}\) is a tail event with respect to the \(\sigma\)-algebra \(\sigma\{S_t(x, j), t \geq 0, 0 \leq x \leq n, 1 \leq j \leq \eta(x)\}\). Hence
\[
\mathbb{P}\{\theta_\infty < \infty\} \in \{0, 1\}. \tag{39}
\]
We have a.s.
\[
\{\text{explosion occurs}\} = \{\theta_\infty < \infty\}.
\]
Let us now point out that for every site \(x \in \mathbb{N}\) there is a finite sequence \((y_j, k_j, s_j)_{j \in \{0, 1, \ldots, m\}}\) such that \(0 = y_0 < y_1 < \ldots < y_m = x\), and sleeping particles at \(y_j\) are activated at time \(s_j\) by the particle \((y_{j-1}, k_{j-1})\) started at \(y_{j-1}, j = 1, \ldots, m\). Furthermore, for every \(x \in \mathbb{N}\) such a sequence is a.s. uniquely defined.

Consider a random sequence of particles \((x_j, k_j)_{j \in \mathbb{I}}\) such that the site \(x_{j+1} \in \mathbb{Z}_+\) is activated by the particle \((x_j, k_j), 1 \leq k_j \leq \eta(x_j)\), and \(x_0 = 0\). Denote also by \(t_j\) the (random) time when the site \(x_j\) was activated. The index set \(\mathbb{I}\) is either \(\mathbb{Z}_+\) or \{0, 1, \ldots, m\} for some \(m \in \mathbb{N}\). Note that
we have $0 = x_0 < x_1 < x_2 < ...$ because all particles left of the origin are removed, and because we know exactly the order in which the sites of $\mathbb{Z}_+$ are getting visited by active particles. We call the interval $(x_n, x_{n+1}]$ fast if
\[
    t_{n+1} - t_n \leq \sum_{j=x_{n+1}}^{x_{n+1}} \frac{1}{A(j)}; \tag{40}
\]
otherwise we call the interval $(x_n, x_{n+1}]$ slow. Note that while it is not necessarily true that every wet site belongs to a fast interval, it is true that every dry site belongs to a slow interval. Indeed, take $y \in \mathbb{N}$. Consider the event \{y is dry\}. We are going to show that a.s. on this event $y$ belongs to a slow interval $(x_n, x_{n+1}]$ for some $n \in \mathbb{N}$. By definition a.s. on this event
\[
z + \ell_z^{(A)} < y, \quad \forall z < y.
\]
Therefore by definition of $\ell_z^{(A)}$ a.s. on \{y is dry\}
\[
\forall z < y \forall k \geq y - z \forall t \geq 0 \forall j \in 1, \eta(z) : S_t^{(z,j)} \geq k \Rightarrow t > \sum_{m=x_{n+1}}^{z+S_t^{(z,j)}} \frac{1}{A(m)}. \tag{41}
\]
Now for $n \in \mathbb{Z}_+$ consider the event \{y $\in$ $(x_n, x_{n+1}]$ and y is dry\}. Taking $z = x_n$ and $k = x_{n+1} - x_n$ in (41) we find that a.s. on \{y $\in$ $(x_n, x_{n+1}]$ and y is dry\}
\[
\forall t \geq 0 \forall j \in 1, \eta(x_n) : S_t^{(x_n,j)} \geq x_{n+1} - x_n \Rightarrow t > \sum_{m=x_{n+1}}^{x_{n+1}+S_t^{(x_n,j)}} \frac{1}{A(m)},
\]
hence a.s. on \{y $\in$ $(x_n, x_{n+1}]$ and y is dry\}
\[
\forall t \geq 0 \forall j \in 1, \eta(x_n) : t \leq \sum_{m=x_{n+1}}^{x_{n+1}} \frac{1}{A(m)} \Rightarrow S_t^{(x_n,j)} < x_{n+1} - x_n. \tag{42}
\]
By construction the site $x_{n+1}$ is activated at time $t_{n+1}$ by a particle started at $x_n$, and the site $x_n$ was activated at time $t_n$. Therefore a.s.
\[
t_{n+1} - t_n = \inf \left\{ t > 0 | \exists j \in 1, \eta(x_n) : S_t^{(x_n,j)} = x_{n+1} - x_n \right\}
\]
From (42) and the right-continuity of the random walk trajectories it follows that a.s. on \{y $\in$ $(x_n, x_{n+1}]$ and y is dry\}
\[
t_{n+1} - t_n > \sum_{m=x_{n+1}}^{x_{n+1}} \frac{1}{A(z)},
\]
that is, that the $(x_n, x_{n+1}]$ is slow. Taking a union over $n$ we see that indeed a.s. every dry sight belongs to a slow interval.

Since particles at the site $x_{n+1}$ are activated by a particle started at $x_n$ we have $\theta_{x_n} = t_n$. Next we bound $\theta_{x_n}$ from below by imagining that fast intervals are traveled over instantaneously, whereas slow intervals take the time equal to the expression on the right hand side of (40) to traverse. By definition of a slow interval we have for $m \in \mathbb{N}$ a.s.
\[ \theta_{x_m} = t_m = (t_m - t_{m-1}) + (t_{m-1} - t_{m-2}) + \ldots + (t_1 - t_0) \geq \sum_{j=1}^{x_m} \frac{1}{A(j)} \mathbb{1}\{j \text{ belongs to a slow interval}\}. \]

Since a.s. every dry sight belongs to a slow interval we also have a.s.

\[ \theta_{x_m} \geq \sum_{j=1}^{x_m} \frac{1}{A(j)} \mathbb{1}\{j \text{ is dry}\}. \]

and for \( n < m \)

\[ \theta_{x_m} - \theta_{x_n} \geq \sum_{j=x_{n+1}}^{x_m} \frac{1}{A(j)} \mathbb{1}\{j \text{ is dry}\}. \quad (43) \]

By Proposition 5.9 for some \( c \in (0, 1] \)

\[ \mathbb{P}(D_n) \geq c, \quad n \in \mathbb{N}, \quad (44) \]

where \( D_n = \{ \text{n is dry}\} \). For \( n \in \mathbb{N} \) let \( r_n \) be the minimal element of \( \{x_j\}_{j \in \mathbb{Z}^+} \) to the right of \( n \):

\[ r_n = \min \{x : x \in \{x_j\}_{j \in \mathbb{Z}^+}, x \geq n\}. \]

Assume that

\[ \mathbb{P}\{\theta_\infty < \infty\} = 1. \quad (45) \]

Since a.s. \( \theta_m \to \theta_\infty, \ m \to \infty \), there exists \( N \in \mathbb{N} \) such that the event \( B := \{\theta_\infty \leq \theta_N + 1\} \) satisfies \( P(B) \geq 1 - \frac{c}{3} \). Note that a.s. on \( B \)

\[ \theta_\infty \leq \theta_n + 1 \leq \theta_{r_n} + 1, \quad n \geq N. \quad (46) \]

We have

\[ \mathbb{P}(D_i \cap B) \geq \mathbb{P}(D_i) + \mathbb{P}(B) - 1 \geq \frac{2}{3} c. \]

There exists \( N' \in \mathbb{N}, \ N' \geq N, \) such that

\[ \mathbb{P}\{r_N \leq N' - 1\} \geq 1 - \frac{c}{3}. \]

We have then \( \mathbb{P}(D_i \cap B \cap \{r_N \leq N'\}) \geq \frac{c}{3} \), and hence by (43)

\[
\begin{align*}
\mathbb{E}[\theta_\infty - \theta_N] \mathbb{1}[B] &\geq \mathbb{E}[\theta_\infty - \theta_N] \mathbb{1}[B] \mathbb{1}\{r_N \leq N'\} \\
&\geq \mathbb{E}\left[ \mathbb{1}[r_N \leq N'] \sum_{i=N'}^{\infty} \mathbb{1}\{j \text{ is dry}\} \frac{1}{A(j)} \right] \\
&= \sum_{i=N'}^{\infty} \frac{1}{A(j)} \mathbb{E}[\mathbb{1}[B] \mathbb{1}\{r_N \leq N'\} \mathbb{1}\{j \text{ is dry}\}] \\
&= \sum_{i=N'}^{\infty} \frac{1}{A(j)} \mathbb{P}(B \cap \{r_N \leq N'\} \cap D_j) \geq \sum_{i=N'}^{\infty} \frac{c}{3A(i)} = \infty,
\end{align*}
\]

but this contradicts (46) since by (46) it should hold that \( \mathbb{E}(\theta_\infty - \theta_N) \mathbb{1}[B] \leq 1 \). Thus (45) cannot hold, and we have by (39)

\[ \mathbb{P}\{\theta_\infty < \infty\} = 0, \quad (47) \]

that is, the probability of explosion is zero. \( \square \)
6 Proof of explosion

This section is devoted to the proof of explosion of the frog model. First we relate the associated TADIBP to the explosion of the frog model. Next we state conditions for percolation, and this gives the desired result.

6.1 Connecting TADIBP and explosion

As stated above, our first step is to relate percolation of the TADIBP to explosion of the frog model. This approach uses the activation times of certain sites related to the TADIBP process, similar to Proposition 4.2 in [BK20].

Proposition 6.1. Assume that the TADIBP for \( \{ \ell_x^{(A)} \}_{x \in \mathbb{Z}_+} \) percolates, where \( A \) satisfies
\[
\sum_{z=1}^{\infty} \frac{1}{A(z)} < \infty.
\]

Then for any \( x_0 \in \mathbb{N} \), explosion occurs almost surely on \( \{ x_0 \xrightarrow{\mathbb{Z}_+} \infty \} \).

Proof. Consider TADIBP with \( \psi_x = \ell_x^{(A)} \). Take a percolation sequence \( \{ x_n \}_{n \in \mathbb{N}} \) given by Lemma 3.3 corresponding to the set \( \{ x_0 \xrightarrow{\mathbb{Z}_+} \infty \} \). Set \( y_n = x_n + \ell_x^{(A)} \) and denote by \( \sigma_x \) the activation time of location \( x \), i.e. the first time an active frog visits site \( x \). Note that a.s. \( \sigma_x < \infty \) for every \( x \in \mathbb{Z} \) since at \( t = 0 \) there is at least one active particle at the origin. By definition of \( \ell_x^{(A)} \) in (10) a.s. on \( \{ x_0 \xrightarrow{\mathbb{Z}_+} \infty \} \) there exists \( j_n \in \mathbb{I}, \eta(x_n) \) such that
\[
\sigma_{y_n} - \sigma_{y_{n-1}} \leq \sigma_{y_n} - \sigma_{x_n} \leq \sum_{z=x_n+1}^{x_n+S(y_n,j_n)} \frac{1}{A(z)}.
\]
Furthermore, since each point \( z \) is in at most two of the intervals \( [x_i, x_i + \ell_x^{(A)}] \), we have that
\[
\sum_{n=1}^{\infty} (\sigma_{y_n} - \sigma_{y_{n-1}}) \leq 2 \sum_{z=1}^{\infty} \frac{1}{A(z)} < \infty.
\]
Therefore \( \sigma_\infty = \lim_{n \to \infty} \sigma_n < \infty \), i.e. the total activation time “up to infinity” is finite.

6.2 Conditions for percolation

The next step is to find conditions on the tail distribution of TADIBP with \( \psi_x = \ell_x^{(A)} \) such that the system percolates. The random variables \( \psi_x = \ell_x^{(A)} \) are independent but not identically distributed. Recall that the Markov chain \( \{ Y_m \}_{m \in \mathbb{Z}_+} \) was defined in Definition 3.1. Note that \( Y_m > 0 \) for all but finitely many \( m \in \mathbb{Z}_+ \) is equivalent to percolation of the system \( \{ \psi_x \}_{x \in \mathbb{Z}_+} \).

Lemma 6.2. Assume that
\[
\sum_{m=1}^{\infty} \prod_{i=0}^{m} (1 - \mathbb{P}\{ \psi_{m-i} > i \}) < \infty.
\]

Then a.s. there exists \( x_0 \in \mathbb{Z}_+ \) connected to \( \infty \):
\[
\mathbb{P}\{ x_0 \xrightarrow{\mathbb{Z}_+} \infty \text{ for some } x_0 \in \mathbb{Z}_+ \} = 1.
\]
Proof. Since all $\psi_x$ are independent, the following identity holds by Definition 3.1:
\[
\mathbb{P}\{Y_m = 0\} = \prod_{i=0}^{m} (1 - \mathbb{P}\{\psi_{m-i} > i\}).
\]
By Borel-Cantelli, the assumption implies that
\[
\mathbb{P}\{Y_m = 0 \text{ infinitely often}\} = 0,
\]
and the system percolates.

Next, we need to find out which conditions on the initial distribution imply that
\[
\sum_{m=1}^\infty \prod_{i=1}^{m} \mu([0, A(m)^{\rho_1}]) \lesssim \sum_{m=1}^\infty \prod_{i=1}^{m} \mu([0, \tilde{A}(m)^{\rho_1}]).
\]
where we set $r^{i}_{m-i} = \mathbb{P}\{\psi_{m-i} > i\}$. To this end, we establish inhomogeneous analogues of the lemmas from [BK20]. We write $r^{i}_{m-i}(A)$ if the coefficient corresponds to $\psi^{i}_{m-i} = \ell(A)_{m-i}$. Note that $i$ and $m - i$ are interchangeable in (48).

The following lemma shows that we may assume that $A(m) > 1$ for all $m \in \mathbb{N}$.

**Lemma 6.3.** Set
\[ z_0 := \min\{z \in \mathbb{N}: A(z) > 1 \text{ and } \mu([0, A(z)]) > 0\}. \]
Furthermore, define $\tilde{A}(m) := A(m + z_0 - 1)$. Then for any $\rho > 1$,
\[
\sum_{m=1}^\infty \prod_{i=1}^{m} \mu([0, A(m)^{\rho_1}]) \simeq \sum_{m=1}^\infty \prod_{i=1}^{m} \mu([0, \tilde{A}(m)^{\rho_1}]).
\]
Note that $\tilde{A}(m) > 1$ for all $m \in \mathbb{N}$.

**Proof.** We show $\lesssim$ first, i.e., assume that
\[
\sum_{m=1}^\infty \prod_{i=1}^{m} \mu([0, A(m)^{\rho_1}]) < \infty.
\]
We have
\[
\sum_{m=1}^\infty \prod_{i=1}^{m} \mu([0, A(m)^{\rho_1}]) \geq \sum_{m=1}^\infty \prod_{i=1}^{m} \mu([0, A(m)^{\rho_1}]) \\
= \sum_{m=1}^\infty \prod_{i=1}^{m+z_0-1} \mu([0, A(m + z_0 - 1)^{\rho_1}]) \\
= \sum_{m=1}^\infty \prod_{i=1}^{m+z_0-1} \mu([0, \tilde{A}(m)^{\rho_1}]) \\
\geq \prod_{i=1}^{z_0-1} \mu([0, \tilde{A}(1)^{\rho_1}]) \sum_{m=1}^\infty \prod_{i=1}^{m} \mu([0, \tilde{A}(m)^{\rho_1 + \rho z_0 - \rho}])
\]
\[
\mu([0, \tilde{A}(1)^{2\rho_i}]) \leq \prod_{i=1}^{\infty} \mu([0, \tilde{A}(m)^{\rho_i}])
\]

For the direction “\(\preceq\)”, assume that
\[
\sum_{m=1}^{\infty} \prod_{i=1}^{m} \mu([0, \tilde{A}(m)^{\rho_i}]) < \infty.
\]

Then
\[
\sum_{m=1}^{\infty} \prod_{i=1}^{m} \mu([0, A(m)^{\rho_i}]) \leq z_0 - 1 + \sum_{m=1}^{\infty} \prod_{i=1}^{m+z_0-1} \mu([0, A(m+z_0-1)^{\rho_i}])
\]
\[
= z_0 - 1 + \sum_{m=1}^{\infty} \prod_{i=1}^{m+z_0-1} \mu([0, \tilde{A}(m)^{\rho_i}])
\]
\[
\leq z_0 - 1 + \sum_{m=1}^{\infty} \prod_{i=1}^{m} \mu([0, \tilde{A}(m)^{\rho_i}]),
\]

which proves the second direction.

In view of Lemma 6.3, we may assume from now on that \(A(m) > 1\) for all \(m \in \mathbb{N}\). To estimate \(r_{m-i}^n\), we need the following lemma, which is the inhomogeneous analogue to Lemma 4.6 of [BK20]: Recall that \((S_t)_{t \geq 0}\) is a continuous-time simple random walk on \(\mathbb{Z}\).

Lemma 6.4. Assume that \(A: \mathbb{N} \to (1, \infty)\) is non-decreasing and
\[
\lim_{z \to \infty} A(z) = \infty. \tag{49}
\]

Then for any \(n, x \in \mathbb{Z}_+\),
\[
P\{\exists t \geq 0: t \leq \sum_{z=x+1}^{x+S_t} \frac{1}{A(z)}, S_t > n\} \geq \frac{\left(1 - \frac{1}{A(x+n+1)}\right)(n+1)}{A(x+n+1)^{n+1}e^{2n+1}\sqrt{n+1}}.
\]

Proof. Recall that \(\tau_n\) is the \(n\)-th jump of \((S_t, t \geq 0)\) and thus has the Erlang distribution as the sum of \(n\) independent unit exponentials. In particular,
\[
P\{\tau_n \leq b\} \geq e^{-b}n^b b^b\frac{\sqrt{\pi}}{n!}.
\]

Since \(\frac{1}{A(z)} < 1\) for all \(z \in \mathbb{N}\),
\[
P\{\exists t \geq 0: t \leq \sum_{z=x+1}^{x+S_t} \frac{1}{A(z)}, S_t > n\} \geq P\{\exists t \geq 0: t \leq \frac{n+1}{A(x+n+1)}}, S_t = n + 1\}
\[
\geq P\{\frac{\tau_{n+1}}{n+1} \leq A(x+n+1)^{-1}\}P(S_{\tau_j} - S_{\tau_{j-1}} = 1, j \in [1, n+1])
\]
\[
\geq e^{-\frac{n+1}{A(x+n+1)}}(n+1)^{n+1}2^{-(n+1)}
\]
\[
A(x+n+1)^{n+1}(n+1)!^2
\]

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\[
\geq e^{(1 - \frac{1}{A(x+n+1)})^n(1+1)} \frac{A(x+n+1)^n e^{2n+1} \sqrt{n+1}}{n+1}
\]

For the convergence analysis below we set for \( m \in \mathbb{N} \) and \( i \in \{0, 1, \ldots, m\} \)

\[
E_2(i, m) = \frac{1}{e \sqrt{i+1}} \left( \frac{e^{1 - \frac{1}{A(m+1)}}}{2A(m+1)} \right)^{i+1}.
\]

Recall that \( r_{m-i}^i \) was defined right after (48).

**Lemma 6.5** (cf. Lemma 5.4, [BK20]). Assume that \( A(m) > 1 \) for all \( m \in \mathbb{N} \) and

\[
\lim_{z \to \infty} A(z) = \infty.
\]

Then for the function \( E_2(i, m) \) defined above we have

\[
r_{m-i}^i(A) \geq 1 - \sum_{k=0}^{\infty} \mu(k) \left( 1 - E_2(i, m) \right)^k
\]

for all \( m \in \mathbb{N} \) and \( i \in \{0, 1, \ldots, m\} \).

**Proof.** We want to estimate \( r_{m-i}^i(A) = \mathbb{P}\{\ell_{m-i}^A > i\} \). To this end, note that

\[
(\ell_{m-i}^A > i)^c = \bigcup_{k=0}^{\infty} \left( \eta(m-i) = k \right) \cap \left\{ \forall t \geq 0 \ \forall j \in \{1, \ldots, k\}: t > \sum_{z=m-i+1}^{m-i+S_{t}(m-i,j)} 1 = 1 - \mathbb{P}\{\exists t \geq \tau_i: t > \sum_{z=m-i+1}^{m-i+S_{t}(m-i,j)} 1 \}ight\}
\]

where the union is disjoint. Since all random walks are independent, this in turn implies that

\[
r_{m-i}^i(A) = 1 - \mathbb{P}\{\ell_{m-i}^A \leq i\} = 1 - \sum_{k=0}^{\infty} \mu(k) \left( \mathbb{P}\{\forall t \geq \tau_i: t > \sum_{z=m-i+1}^{m-i+S_{t}(m-i,j)} 1 \}ight)^k.
\]

We may replace the condition \( t > 0 \) by \( t \geq \tau_i \) above, since it is impossible for the process \( (S_t)_t \) to be larger than \( i \) before the \( i \)-th jump (even before the \( (i+1) \)-st jump). By Lemma 6.4, we have for every \( m \) and \( i \in \{0, 1, \ldots, m\} \)

\[
\mathbb{P}\{\forall t \geq \tau_i: t > \sum_{z=m-i+1}^{m-i+S_{t}(m-i,j)} 1 \} = 1 - \mathbb{P}\{\exists t \geq \tau_i: t \leq \sum_{z=m-i+1}^{m-i+S_{t}(m-i,j)} 1 \} \leq 1 - \frac{e^{(1 - \frac{1}{A(m+1)})^n(1+1)}}{A(m+1)^n e^{2n+1} \sqrt{n+1}} = 1 - E_2(i, m).
\]

\[
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\]
Remark 6.6. Fix $m_0, i_0 \in \mathbb{N}$. We have

$$
\sum_{m=1}^{\infty} \prod_{i=0}^{m} (1 - r_{m-i}) = \sum_{m=m_0}^{m-1} \prod_{i=0}^{m} (1 - r_{m-i}) + \sum_{m=m_0}^{\infty} \prod_{i=1}^{m} (1 - r_{m-i}) \\
\leq m_0 - 1 + \sum_{m=m_0}^{\infty} \prod_{i=0}^{m} (1 - r_{m-i}).
$$

Therefore, for the convergence of (48), it suffices to consider

$$
\sum_{m=m_0}^{\infty} \prod_{i=0}^{m} (1 - r_{m-i})
$$

for some $m_0, i_0 \in \mathbb{N}$, or

$$
\sum_{m=m_0}^{\infty} \prod_{i=0}^{m} (1 - r_{m-i}),
$$

because $i$ and $m - i$ are interchangeable and hence, the two expressions above are equal.

Since we want to find conditions on the convergence of (48), by Lemma 6.5, we may use the inequality

$$
\sum_{m=m_0}^{\infty} \prod_{i=1}^{m} (1 - r_{m-i}(A)) \leq \sum_{m=m_0}^{\infty} \prod_{i=1}^{m} \sum_{k=0}^{\infty} \mu(k)(1 - E_2(i, m))^k.
$$

Let us now bound $E_2(i, m)$ from below by a quantity involving $A$. This bound will be used in the final part of the proof of explosion.

Lemma 6.7. Assume that the non-decreasing function $A: \mathbb{N} \to (1, \infty)$ fulfills

$$
\sum_{z=1}^{\infty} \frac{1}{A(z)} < \infty.
$$

Then there exists $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$ and all $0 \leq i \leq m$,

$$
E_2(i, m) \geq \left( \frac{1}{2A(m+1)} \right)^{i+2}.
$$

Proof. First of all, note the following properties of $A$:

- $A$ diverges, i.e. (49) is fulfilled.

- There exists $m_0$ such that

$$
\frac{1}{A(m+1)} \leq \frac{2}{e\sqrt{m+1}} \text{ for all } m \geq m_0. \quad (50)
$$

This can be seen as follows: assume that this is not the case. Then for each $n_0$, there exists $m \geq n_0$ such that

$$
\frac{1}{A(m+1)} > \frac{2}{e\sqrt{m+1}}. \quad (51)
$$
Since $A$ is non-decreasing, we have for all $n \leq m$,

$$
\sum_{j=1}^{m+1} \frac{1}{A(j)} \geq \sum_{j=1}^{m+1} \frac{1}{A(m+1)} > \frac{2(m+1)}{e\sqrt{m+1}} = 2\sqrt{\frac{m+1}{e}}.
$$

Choosing a sequence $\{m_k\}_{k \in \mathbb{N}}$ such that $m_k \to \infty$ and (51) holds for all $k \in \mathbb{N}$, we have

$$
\lim_{k \to \infty} \frac{m_{k+1}}{m_k} = +\infty
$$

which contradicts the assumption of the lemma. Hence (50) is true.

Let $m \geq m_0$, where $m_0 \in \mathbb{N}$ is chosen such that (50) holds. Using that $e^{1-\epsilon} \geq 1$ and taking into account that $i \leq m$, we have

$$
E_2(i, m) = \frac{1}{e\sqrt{i+1}} \left( \frac{e^{1-\frac{2}{A(m+1)\sqrt{m+1}}}}{2A(m+1)} \right)^{i+1} \geq \left( \frac{1}{2A(m+1)} \right)^{i+2}.
$$

Next, we want to translate the condition above into something more useful, i.e. dependent on the initial distribution of the frog model.

**Lemma 6.8.** We have

$$
\sum_{m=m_0}^{\infty} \prod_{i=i_0}^{\infty} \mu(k)(1 - E_2(i, m))^k \lesssim \sum_{m=m_0}^{\infty} \prod_{i=i_0}^{\infty} \mu([0, 2A(m)^\alpha]). \quad (52)
$$

**Proof.** By Lemma 6.7, we have

$$
\sum_{k=0}^{\infty} \mu(k)(1 - E_2(i, m))^k \leq \sum_{k=0}^{\infty} \mu(k) \left( 1 - \left( \frac{1}{2A(m+1)} \right)^{i+2} \right)^k
\leq \sum_{k=0}^{\infty} \mu(k) \left( 1 - \left( \frac{1}{2A(m+2)} \right)^{i+2} \right)^k. \quad (53)
$$

Since we may start the convergence analysis at arbitrary $m_0$ and $i_0$, we may shift the index $m + 2 \mapsto m$ and $i + 2 \mapsto i$ and consider

$$
\sum_{k=0}^{\infty} \mu(k) \left( 1 - \left( \frac{1}{2A(m)} \right)^{i} \right)^k.
$$

Set $\nu = \rho - 1 > 0$. We have

$$
\sum_{k=0}^{\infty} \mu(k) \left( 1 - 2A(m)^{-i} \right)^k \leq \sum_{k=0}^{\infty} \mu(k)e^{-2kA(m)^{-i}}
$$

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\[
\sum_{k=0}^{\left[2A(m)^{\mu_0}\right]} \mu(k) \varepsilon^{-2k A(m)^{-i}} + \sum_{k=[2A(m)^{\mu_1}]+1}^{\infty} \mu(k) \varepsilon^{-2k A(m)^{-i}} \leq \mu([0, 2A(m)^{\rho_1}]) + e^{-2A(m)\rho_1}.
\]

Next, let \( n_0 \in \mathbb{N} \) such that \( n_0 \geq m_0 \) as well as \( \mu([0, 2A(m)^{\rho_1}]) \geq \frac{1}{2} \), where \( m_0 \) is given as in Lemma 6.7. Then we have

\[
\frac{\prod_{i=i_0}^{m} \mu([0, 2A(m)^{\rho_1}]) + e^{-2A(m)^{\rho_1}}}{\prod_{i=i_0}^{m} \mu([0, 2A(m)^{\rho_1}])} = \prod_{i=i_0}^{m} \left( 1 + \frac{e^{-2A(m)^{\rho_1}}}{\mu([0, 2A(m)^{\rho_1}])} \right)
\]

\[
\leq \prod_{i=i_0}^{n_0-1} \left( 1 + \frac{e^{-2A(m)^{\rho_1}}}{\mu([0, 2A(m)^{\rho_1}])} \right) \cdot \prod_{i=n_0}^{m} \left( 1 + 2e^{-2A(m)^{\rho_1}} \right).
\]

The first product is finite and the second one converges, since the series

\[
\sum_{i=n_0}^{\infty} e^{-2A(n)^{\rho_1}} \leq \sum_{i=n_0}^{\infty} e^{-2A(n_0)^{\rho_1}}
\]

converges absolutely by the ratio test:

\[
e^{-2A(n_0)^{\rho_1}(i+1)} e^{-2A(n_0)^{\rho_1}} = e^{2A(n_0)^{\rho_1}(1-2A(n_0)^{\rho_1})} \leq e^{1-2A(n_0)^{\rho_1}} < 1.
\]

Therefore the fraction on the left hand side of (55) is bounded. This means that

\[
\sum_{m=m_0}^{\infty} \prod_{i=i_0}^{m} \left( \mu([0, 2A(m)^{\rho_1}]) + e^{-2A(m)^{\rho_1}} \right) = \sum_{m=m_0}^{n_0-1} \prod_{i=i_0}^{m} \left( \mu([0, 2A(m)^{\rho_1}]) + e^{-2A(m)^{\rho_1}} \right) + C_1 \sum_{m=n_0}^{\infty} \prod_{i=n_0}^{m} \left( \mu([0, 2A(m)^{\rho_1}]) + e^{-2A(m)^{\rho_1}} \right)
\]

\[
\approx \sum_{m=n_0}^{\infty} \prod_{i=n_0}^{m} \mu([0, 2A(m)^{\rho_1}]).
\]

and therefore by (53) and (54)

\[
\sum_{m=m_0}^{\infty} \sum_{i=i_0}^{\infty} \mu(k)(1 - E_2(i, m)) \leq \sum_{m=m_0}^{\infty} \prod_{i=i_0}^{m} \mu([0, 2A(m)^{\rho_1}]).
\]

\( \square \)

Finally, replacing in Lemma 6.8 the sequence \( \{A(m)\}_{m \in \mathbb{N}} \) by \( \{B(m)\}_{m \in \mathbb{N}} \) with \( B(m) = 2A(m) \) for all \( m \in \mathbb{N} \) and putting all lemmas and calculations together, we arrive at the following result:
**Theorem 6.9.** Assume that there exists a non-decreasing function $B : \mathbb{N} \to (0, \infty)$ such that
\[
\sum_{z=1}^{\infty} \frac{1}{B(z)} < \infty.
\]
Furthermore, assume there exists $\rho > 1$ such that the initial distribution of the frog model satisfies
\[
\sum_{m=1}^{\infty} m \prod_{i=1}^{m} \mu([0,B(m)\rho^{i}]) < \infty.
\]
Then the frog process explodes a.s.

**Proof.** By the above calculation and Lemma 6.5, we have that
\[
\sum_{m=1}^{\infty} m \prod_{i=0}^{m} (1 - r_{m-i}(A)) < \infty.
\]
By Lemma 6.2, the corresponding TADIBP percolates. By Proposition 6.1, the system explodes. \qed

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