AN INDEX THEORY FOR PATHS THAT ARE SOLUTIONS OF A CLASS OF STRONGLY INDEFINITE VARIATIONAL PROBLEMS.

PAOLO PICCIONE AND DANIEL V. TAUSK

ABSTRACT. We generalize the Morse index theorem of [12, 13] and we apply the new result to obtain lower estimates on the number of geodesics joining two fixed non conjugate points in certain classes of semi-Riemannian manifolds. More specifically, we consider semi-Riemannian manifolds \((M, g)\) admitting a smooth distribution spanned by commuting Killing vector fields and containing a maximal negative distribution for \(g\). In particular we obtain Morse relations for stationary semi-Riemannian manifolds (see [7]) and for the Gödel-type manifolds (see [3]).

1. INTRODUCTION

The standard Morse theory for variational problems on Hilbert manifolds is applicable under the assumptions that the functional involved satisfies good compactness properties (like the Palais–Smale condition), that it has only nondegenerate critical points, and that each critical point has finite Morse index. However, many variational problems that arise naturally in several contexts lead to the study of functionals that do not satisfy such assumptions. The main example that we will keep in mind as the prototype of the theory developed in this paper, is the geodesic action functional on a semi-Riemannian manifold \((M, g)\), i.e., a manifold endowed with a non positive definite nondegenerate metric tensor \(g\). It is well known that the semi-Riemannian geodesic action functional, defined in the Hilbert manifold \(\Omega_{pq}(M)\) of all curves of class \(H^1\) joining the points \(p, q \in M\), does not satisfy the Palais–Smale condition, it is unbounded both from above and from below, and it is strongly indefinite, i.e., all its critical points have infinite Morse index. Recall that the second variation of the geodesic action functional at a given geodesic \(\gamma\) is given by the so-called index form \(I_{\gamma}\), which is a symmetric bounded bilinear form in the space of all variational vector fields along \(\gamma\) vanishing at the endpoints and whose kernel consists of Jacobi fields along \(\gamma\) vanishing at the endpoints.

In order to establish the existence of at least one solution to this kind of variational problems by means of variational methods, a certain strategy has been suggested in some recent works (see for instance [3, 6, 7, 10] and the references therein). The basic idea in these theories is that, when the metric tensor \(g\) admits a suitable number of symmetries (i.e., Killing vector fields), then the variational problem can be reduced to the study of a functional which is bounded from below and it satisfies the Palais–Smale condition by “factoring out” the negative contribution in the directions of the Killing fields. For instance (see [6]), given a Lorentzian manifold \((M, g)\) (i.e., the index of \(g\) is 1) admitting a timelike Killing
vector field \( Y \), then the mentioned reduction of the variational problem consists in restricting the geodesic action functional to the set of curves \( \gamma \in \Omega_{pq}(M) \) that are geodesics in the direction of \( Y \), i.e., that satisfy the conservation law \( g(\gamma', Y) \equiv C_\gamma \) (constant). Similarly, for semi-Riemannian metrics \( g \) of arbitrary index, the reduction is done under the assumption that \( g \) admits a family \( \langle Y_i \rangle_{i=1}^r \) of commuting Killing vector fields that generate a distribution which contains a maximal negative distribution for \( g \). In Ref. [7] it is studied the case of stationary semi-Riemannian manifolds, i.e., those manifolds that admit a family of commuting Killing vector fields \( Y_i \) that span a maximal negative distribution for \( g \), while in [3] it is studied the case of Lorentzian space-times of Gödel type, which are Lorentzian manifolds admitting a pair of Killing fields, whose causal character is not necessarily constant, and that span a two-dimensional distribution where the restriction of \( g \) has index equal to 1.

It is not hard to prove that the above mentioned restricted variational problems lead to functionals whose second variation at each critical point \( \gamma \) (typically, a restriction of the index form \( I_\gamma \)) is represented by a self-adjoint operator which is a compact perturbation of a positive isomorphism, hence it has finite index. However, determining the value of such index, as well as the problem of relating this number to the geometrical properties of the corresponding geodesic, has turned out to be quite a complicated task. Observe indeed that the classical Morse index theorem fails to make sense in this situation\(^1\), nor its classical proof can be adapted to obtain an alternative statement. In Reference [5] the authors have proven that, in the case of a stationary Lorentzian manifold, the index of the restricted functional at each geodesic equals the Maslov index of the geodesic, which is a homological invariant defined as an intersection number in the Grassmannian of all Lagrangian subspaces of a symplectic space. The notion of Maslov index was first introduced by the Russian school (see [1] and the references therein) and successively extended by a large number of authors in several contexts, mainly in connection with solutions (periodic or not) of Hamiltonian systems. In the context of semi-Riemannian geodesics, the notion of Maslov index was introduced by Helfer in [8]; under generic circumstances, the Maslov index of a semi-Riemannian geodesic is given by a sort of algebraic count of the conjugate points (see [8, 11]). Using this result, in [5] the authors proved the Morse relations for geodesics of any causal character in a stationary Lorentzian manifold, that give a lower estimate of the number of geodesics joining two non conjugate points with a given Maslov index in terms of a Betti number of the loop space of the underlying manifold. The index theorem of [5] was generalized in references [13, 15]; given a geodesic \( \gamma \) in a semi-Riemannian manifold \((M, g)\) with metric \( g \) of arbitrary index, and given a maximal negative distribution \( D_t \subset T_{\gamma(t)} M \), one defines spaces \( K_D \) and \( S_D \) of variational vector fields along \( \gamma \), where \( S_D \) consists of vector fields along \( \gamma \) taking values in \( D \) and \( K_D \) consists of variational vector fields corresponding to variations of \( \gamma \) by geodesics in the directions of \( D \). Then, the index \( n_-(I_\gamma|_{K_D}) \) of the restriction of \( I_\gamma \) to \( K_D \) and the the co-index \( n_+(I_\gamma|_{S_D}) = n_-(-I_\gamma|_{S_D}) \) are finite, and their difference equals the Maslov index of \( \gamma \). In order to use this result to develop an infinite dimensional Morse theory for semi-Riemannian geodesics using the reduction argument mentioned above, one needs to restrict to the case that the distribution is spanned by commuting Killing vector fields (see Subsection 4.2). Namely, in this case the space \( K_D \) is the tangent space of a Hilbert submanifold of \( \Omega_{pq}(M) \); moreover, one has \( n_+(I_\gamma|_{S_D}) = 0 \).

\(^1\)It is well known that conjugate points along a semi-Riemannian geodesic do not in general form a finite set (see [8, 14]), and the differential operator corresponding to the Jacobi equation along the geodesic is in general not self-adjoint.
The main goal of this paper is to push beyond the stationarity assumption the results of Morse theory for semi-Riemannian geodesics, or, more generally, for solutions of unidimensional strongly indefinite variational problems that admit a sufficiently large number of symmetries. More precisely, we will consider the geodesic variational problem in a semi-Riemannian manifold \((M, g)\), with \(n_-(g) = k\), that admits linearly independent commuting Killing vector fields \((\mathcal{Y}_i)_{i=1}^r\), with \(r \geq k\), that span a distribution \(\mathcal{D}\) that contains a maximal negative distribution \(\Delta\). Observe that the distribution \(\Delta\) itself need not be spanned by Killing vector fields, i.e., we consider the case that the causal character of each one of the \(\mathcal{Y}_i\)’s need not be constant on \(M\). Such generalization enlarges significantly the class of variational problems to which the theory can be applied; as an example, in this paper we will consider the case of geodesics in semi-Riemannian manifolds of Gödel type (Subsection 4.3).

Let us give a short description of the results of the paper, as follows. Our main result concerning the index theory is stated and proved in the context of symplectic differential systems. These are linear homogeneous systems of ODE’s in \(\mathbb{R}^n \oplus \mathbb{R}^{n*}\) whose coefficient matrix is a smooth curve in the symplectic Lie algebra \(\text{sp}(2n, \mathbb{R})\). Symplectic differential systems arise naturally in connection with solutions of Hamiltonian problems in symplectic manifolds as linearizations of the Hamilton equations, using a symplectic referential along the solution (see [12, Section 3]). For instance, in the geodesic case one obtains a symplectic system (more specifically, a Morse–Sturm system) by considering the Jacobi equation along the given geodesic and using a parallel trivialization of the tangent bundle of the semi-Riemannian manifold along the geodesic. Associated to a symplectic system with Lagrangian initial data there is a notion of Maslov index and of index form; in the case of symplectic systems arising from solutions of hyper-regular Hamiltonians (for instance, geodesics), the index form coincides with the second variation of the Lagrangian action functional. The choice of a distribution \(\mathcal{D}\) determines a reduced symplectic system; we prove that when \(\mathcal{D}\) contains a maximal negative distribution then the Morse index of the restriction of the index form to the space \(\mathcal{K}_\mathcal{D}\) equals the difference between the Maslov index of the original symplectic system and the Maslov index of the reduced symplectic system, plus a suitable correction term determined by the initial condition (Theorem 3.2).

The paper is organized as follows. In Section 2 we recall from [12] the general formalism of symplectic differential systems and the index theorem of [13]. In Section 3 we state and prove our generalized index theorem; finally, in Section 4 we discuss the applications to semi-Riemannian geodesics.

## 2. Symplectic Differential Systems

We start by introducing the basic notation and terminology that will be used throughout the paper. Given real vector spaces \(V, W\) we denote by \(\text{Lin}(V, W)\) the space of linear maps from \(V\) to \(W\) and by \(\text{Bil}(V, W)\) the space of bilinear forms \(B: V \times W \to \mathbb{R}\); by \(\text{Bil}_{sym}(V)\) we denote the subspace of \(\text{Bil}(V, V)\) consisting of symmetric bilinear forms. For \(T \in \text{Lin}(V, W)\) we denote by \(T^* \in \text{Lin}(W^*, V^*)\) the transpose of \(T\), where \(V^*, W^*\) denote respectively the dual spaces of \(V\) and \(W\). The index of a symmetric bilinear form \(B \in \text{Bil}_{sym}(V)\), denoted by \(n_-(B)\), is defined as the supremum of the dimensions of the subspaces of \(V\) on which \(B\) is negative definite:

\[
n_-(B) = \sup \left\{ \dim(W): B|_W \text{ is negative definite} \right\} \in \mathbb{N} \cup \{+\infty\};
\]

the coindex of \(B\) is defined by \(n_+(B) = n_-(-B)\) and the signature of \(B\) is defined as the difference \(\text{sgn}(B) = n_+(B) - n_-(B)\), provided that one of the numbers \(n_+(B), n_-(B)\)
is finite. We also define the degeneracy of a symmetric bilinear form $B$ as the dimension of its kernel, i.e., $\text{deg}(B) = \dim(\ker(B))$.

We always implicitly identify the spaces $\text{Bil}(V,W)$ and $\text{Lin}(V,W^*)$ by the natural isomorphism $B(v,w) = B(v)(w)$. With such identification, if $V$ is finite dimensional, then a bilinear form $B \in \text{Bil}(V,V) \cong \text{Lin}(V,V^*)$ is symmetric iff $B$ equals its own transpose $B^* \in \text{Lin}(V^*,V^*) \cong \text{Bil}(V,V)$; moreover, if $B \in \text{Bil}_{sym}(V)$ is nondegenerate (i.e., $\ker(B) = \{0\}$) then $B^{-1} \in \text{Lin}(V^*,V) \cong \text{Lin}(V^*,V^*) \cong \text{Bil}(V^*,V^*)$ is the nondegenerate symmetric bilinear form on $V^*$ which equals the push-forward of $B \in \text{Bil}_{sym}(V)$ by the isomorphism $B : V \rightarrow V^*$.

A central object for our theory is the symplectic space $\mathbb{R}^n \oplus \mathbb{R}^{n*}$ endowed with the canonical symplectic form

$$\omega((v_1,\alpha_1),(v_2,\alpha_2)) = \alpha_2(v_1) - \alpha_1(v_2);$$

we denote by $\text{Sp}(2n,\mathbb{R})$ the symplectic group of $(\mathbb{R}^n \oplus \mathbb{R}^{n*},\omega)$, which is the closed subgroup of the general linear group $GL(2n,\mathbb{R})$ consisting of those isomorphisms of $\mathbb{R}^n \oplus \mathbb{R}^{n*}$ that preserve $\omega$. The Lie algebra $\text{sp}(2n,\mathbb{R})$ of $\text{Sp}(2n,\mathbb{R})$ consists of the linear endomorphisms $X$ of $\mathbb{R}^n \oplus \mathbb{R}^{n*}$ that are represented by matrices of the form:

$$(2.1) \quad X = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix},$$

with $B, C$ symmetric. We think of $A$ as a linear endomorphism of $\mathbb{R}^n$ and of $B, C$ as linear maps $B : \mathbb{R}^{n*} \rightarrow \mathbb{R}^n, C : \mathbb{R}^n \rightarrow \mathbb{R}^{n*}$; we also think of $B$ as a symmetric bilinear form on $\mathbb{R}^{n*}$ and $C$ as a symmetric bilinear form on $\mathbb{R}^n$. We can now give the following:

**Definition 2.1.** Let $X : [a, b] \rightarrow \text{sp}(2n, \mathbb{R})$ be a smooth curve and define $A, B$ and $C$ as in (2.1). Assume that $B(t)$ is nondegenerate for all $t \in [a, b]$. The symplectic differential system in $\mathbb{R}^n$ with coefficient matrix $X$ is the following linear homogeneous first order system of ODE's:

$$(2.2) \quad \frac{d}{dt} \begin{pmatrix} v(t) \\ \alpha(t) \end{pmatrix} = X(t) \begin{pmatrix} v(t) \\ \alpha(t) \end{pmatrix}.$$

We identify the symplectic differential system (2.2) with its coefficient matrix $X$ so that we will in general refer to the symplectic differential system $X$; the blocks $A, B, C$ will be called the coefficients of $X$. The nondegeneracy of $B(t)$ implies that the index of $B(t)$ is independent of $t \in [a, b]$; we call $B$ the fundamental coefficient of the symplectic differential system $X$ and the integer $n_- \{B(t)\}$ the index of $X$.

**Example 2.2.** A special class of symplectic differential systems are the so called Morse–Sturm systems; in our notation, Morse–Sturm systems are symplectic differential systems with $A \equiv 0$. They can be written in the form of a second order linear differential equation:

$$(2.3) \quad g^{-1}(g'v)' = Rv,$$

where $g : [a, b] \rightarrow \text{Bil}_{sym}(\mathbb{R}^n), R : [a, b] \rightarrow \text{Lin}(\mathbb{R}^n)$ are smooth curves, $g(t)$ is nondegenerate and $g(t)R(t)$ is symmetric for all $t \in [a, b]$. Equation (2.3) is identified with the symplectic differential system with coefficients $A = 0, B = g^{-1}$ and $C = gR$. Observe that when $g$ is constant, (2.3) becomes:

$$(2.4) \quad v'' = Rv.$$

Let $X$ be a symplectic differential system. Given a map $v : [a, b] \rightarrow \mathbb{R}^n$ there exists at most one map $\alpha : [a, b] \rightarrow \mathbb{R}^{n*}$ for which $(v, \alpha)$ can be a solution of $X$; such map will be
denoted by $\alpha_v$ and it is given by:

$$\alpha_v = B^{-1}(v' - Av).$$

We will usually say that a map $v : [a, b] \to \mathbb{R}^n$ is a solution of $X$ meaning that $(v, \alpha_v)$ is a solution of $X$. For later use, we give here the following formula:

$$\alpha_{fv} = f\alpha_v + f'B^{-1}v,$$

valid for all absolutely continuous maps $v : [a, b] \to \mathbb{R}^n$, $f : [a, b] \to \mathbb{R}$.

Recall that a Lagrangian subspace of $\mathbb{R}^n \oplus \mathbb{R}^n^*$ is an $n$-dimensional subspace on which the symplectic form $\omega$ vanishes. We will henceforth denote by $L_0$ the Lagrangian subspace:

$$L_0 = \{0\} \oplus \mathbb{R}^n^* \subset \mathbb{R}^n \oplus \mathbb{R}^n^*.$$  

**Definition 2.3.** Given a Lagrangian subspace $\ell_0 \subset \mathbb{R}^n \oplus \mathbb{R}^n^*$ we consider the following initial condition for the system (2.2):

$$\ell_0 = \{(v, \alpha) : v \in P, \alpha|_P + S(v) = 0 \in P^*\};$$

in terms of the pair $(P, S)$ the initial condition (2.7) can be rewritten as:

$$v(a) \in P, \alpha(a)|_P + S(v(a)) = 0 \in P^*.$$  

The initial condition (2.7) (or (2.9)) is called nondegenerate if $B(a)^{-1} \in \text{Bil}_{\text{sym}}(\mathbb{R}^n)$ is nondegenerate on $P$. This is always the case, for instance, if $\ell_0 = L_0$; in this case $P = \{0\}$ and (2.9) reduces to $v(a) = 0$.

We denote by $V$ the $n$-dimensional space of all solutions $v$ of $(X, \ell_0)$; for $t \in [a, b]$ we set:

$$V[t] = \{v(t) : v \in V\} \subset \mathbb{R}^n.$$  

For each $t \in [a, b]$, we have an isomorphism $\Phi(t)$ of $\mathbb{R}^n \oplus \mathbb{R}^n^*$ defined by the relation $\Phi(t)(v(a), \alpha(a)) = (v(t), \alpha(t))$ for every solution $(v, \alpha)$ of $X$; $\Phi$ satisfies the matrix differential equation

$$\Phi'(t) = X(t)\Phi(t),$$

with initial condition $\Phi(a) = \text{Id}$. It follows that $\Phi$ is a smooth curve in the symplectic group $\text{Sp}(2n, \mathbb{R})$. We call $\Phi$ the fundamental matrix of the system $X$. For $t \in [a, b]$, we set:

$$\ell(t) = \Phi(t)(\ell_0) = \{(v(t), \alpha_v(t)) : v \in \mathbb{V}\};$$

obviously $\ell(t)$ is a Lagrangian subspace of $\mathbb{R}^n \oplus \mathbb{R}^n^*$.

**Definition 2.4.** A focal instant for the symplectic differential system with initial data $(X, \ell_0)$ is an instant $t \in [a, b]$ such that there exists a non zero $(X, \ell_0)$-solution $v$ with $v(t) = 0$; the dimension of the space of all $(X, \ell_0)$-solutions vanishing at $t$ is called the multiplicity of the focal instant $t$ and is denoted by $\text{mul}(t)$. The signature of a focal instant $t \in [a, b]$, denoted by $\text{sgn}(t)$, is defined as the signature of the restriction of $B(t)^{-1}$ to the $B(t)^{-1}$-orthogonal complement of $\mathbb{V}[t]$; the focal instant is called nondegenerate when
such restriction is nondegenerate. In the special case where $\ell_0 = L_0$ then the focal instants of $(X, \ell_0)$ are also called conjugate instants of $X$.

An instant $t \in [a, b]$ is focal if $\mathcal{V}[t]$ is a proper subspace of $\mathbb{R}^n$, in which case $\text{null}(t)$ is the codimension of $\mathcal{V}[t]$. It is well known (see for instance [12, Theorem 2.3.3]) that nondegenerate focal instants are isolated.

Formula (2.11) defines a smooth curve in the Lagrangian Grassmannian $\Lambda$ of the symplectic space $\mathbb{R}^n \oplus \mathbb{R}^n^*$ which is the embedded submanifold of the Grassmannian of all $n$-dimensional subspaces of $\mathbb{R}^n \oplus \mathbb{R}^n^*$ consisting of all Lagrangian subspaces. We denote by $\Lambda_{\geq 1}(L_0) \subset \Lambda$ the Maslov cycle of $L_0$ which is the set of all Lagrangians $L \in \Lambda$ with $L \cap L_0 \neq \{0\}$. Each continuous curve $l$ in $\Lambda$ with endpoints outside $\Lambda_{\geq 1}(L_0)$ defines a relative singular homology class with integer coefficients in $H_1(\Lambda, \Lambda \setminus \Lambda_{\geq 1}(L_0)) \cong \mathbb{Z}$ and the corresponding integer number $\mu_{L_0}(l)$ is called the Maslov index of the curve $l$. If $\ell_0$ defines a nondegenerate initial condition for $X$ then there are no $(X, \ell_0)$-focal instants near $t = a$ and we can give the following:

**Definition 2.5.** Let $(X, \ell_0)$ be a symplectic differential system with initial data; assume that $t = b$ is not focal and that $\ell_0$ defines a nondegenerate initial condition for $X$. The Maslov index of $(X, \ell_0)$ is defined by:

$$i_{\text{Maslov}}(X, \ell_0) = \mu_{L_0}(\ell|_{[a + \varepsilon, b]}),$$

where $\varepsilon > 0$ is chosen in such a way that there are no focal instants in $[a, a + \varepsilon]$. When $\ell_0 = L_0$ we call $i_{\text{Maslov}}(X, \ell_0)$ the Maslov index of $X$ and we write just $i_{\text{Maslov}}(X)$.

For more details concerning the geometry of the Lagrangian Grassmannian $\Lambda$ of a symplectic space and the Maslov index of curves in $\Lambda$ we refer to [4, 11, 16]; for more details concerning the notion of Maslov index for symplectic differential systems with initial data we refer to [12, Subsection 2.3]. The Maslov index of a pair $(X, \ell_0)$ is stable by uniformly small perturbations of $X$. Generically, it gives an algebraic count of the focal instants of $(X, \ell_0)$:

**Proposition 2.6.** If all the focal instants of $(X, \ell_0)$ are nondegenerate, $t = b$ is not focal and $\ell_0$ defines a nondegenerate initial condition for $X$ then:

$$i_{\text{Maslov}}(X, \ell_0) = \sum_{t \in [a, b]} \text{sgn}(t).$$

**Proof.** See [12, Theorem 2.3.3].

The nondegeneracy assumption on the focal instants of $(X, \ell_0)$ is indeed generic (see [12, Proposition 2.4.1]); in the case of degenerate focal instants, the Maslov index can also be explicitly computed in terms of suitable coordinate charts on the Lagrangian Grassmannian (see [11, Proposition 4.3.1]).

Given a symplectic differential system with initial data $(X, \ell_0)$ we consider the Hilbert space:

$$\mathcal{H} = \{v \in H^1([a, b], \mathbb{R}^n) : v(a) \in P, \ v(b) = 0\},$$

where $H^1([a, b], \mathbb{R}^n)$ denotes the Sobolev space of absolutely continuous $\mathbb{R}^n$-valued maps on $[a, b]$ with square integrable derivatives. On $\mathcal{H}$ we define a bounded bilinear form $I$ given by:

$$I(v, w) = \int_a^b B(\alpha_v, \alpha_w) + C(v, w) \, dt - S(v(a), w(a)).$$

(2.12)
We call \( I \) the index form associated to the pair \((X, \ell_0)\). The index form of \((X, L_0)\) will be simply referred to as the index form of \(X\); the term \(S(v(a), w(a))\) in (2.12) does not appear in the index form of \(X\) and the boundary conditions defining \(\mathcal{H}\) become \(v(a) = v(b) = 0\).

We recall from [12, Subsection 2.10] the notion of isomorphism in the class of symplectic differential systems:

**Definition 2.7.** Let \(X, \tilde{X}\) be symplectic differential systems. An isomorphism \(\phi : X \rightarrow \tilde{X}\) is a smooth curve \(\phi : [a, b] \rightarrow \text{Sp}(2n, \mathbb{R})\) with \(\phi(t)(L_0) = L_0\) for all \(t\) and such that one of the following equivalent conditions are satisfied:

(a) \(\tilde{X}(t) = \phi'(t)\phi(t)^{-1} + \phi(t)X(t)\phi(t)^{-1}\) for all \(t \in [a, b]\);

(b) \(\tilde{\Phi}(t) = \phi(t)\Phi(t)\phi(a)^{-1}\) for all \(t\), where \(\Phi, \tilde{\Phi}\) denote respectively the fundamental matrices of \(X\) and \(\tilde{X}\);

(c) \((v, \alpha)\) is a solution of \(X\) iff \((\tilde{v}, \tilde{\alpha}) = \phi(v, \alpha)\) is a solution of \(\tilde{X}\).

Given Lagrangians \(\ell_0, \tilde{\ell}_0 \subset \mathbb{R}^n \oplus \mathbb{R}^{n*}\), we say that \(\phi : (X, \ell_0) \rightarrow (\tilde{X}, \tilde{\ell}_0)\) is an isomorphism if \(\phi : X \rightarrow \tilde{X}\) is an isomorphism and \(\phi(\ell_0)(\ell_0) = \tilde{\ell}_0\).

Elements \(\phi \in \text{Sp}(2n, \mathbb{R})\) that preserve \(L_0\) are written in block matrix form as:

\[
(2.13) \quad \phi = \begin{pmatrix} Z & 0 \\ Z^{-1}W & Z^{-1} \end{pmatrix},
\]

with \(Z \in \text{Lin}(\mathbb{R}^n, \mathbb{R}^n)\) invertible and \(W \in \text{Bil}_{\text{sym}}(\mathbb{R}^n)\).

**Proposition 2.8.** Let \(\phi : (X, \ell_0) \rightarrow (\tilde{X}, \tilde{\ell}_0)\) be an isomorphism and define \(Z, W\) as in (2.13). Then:

- \((X, \ell_0)\) and \((\tilde{X}, \tilde{\ell}_0)\) have the same focal instants with same multiplicity and signature;
- a focal instant is nondegenerate for \((X, \ell_0)\) iff it is nondegenerate for \((\tilde{X}, \tilde{\ell}_0)\);
- \(\ell_0\) defines a nondegenerate initial condition for \(X\) iff \(\tilde{\ell}_0\) defines a nondegenerate initial condition for \(\tilde{X}\);
- if \(t = b\) is not focal and \(\ell_0\) defines a nondegenerate initial condition for \(X\) then \(i_{\text{Maslov}}(X, \ell_0) = i_{\text{Maslov}}(\tilde{X}, \tilde{\ell}_0)\);
- the map \(\mathcal{H} \ni v \mapsto Zv \in \mathcal{H}\) is a continuous isomorphism and it carries the index form \(I : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}\) of \((X, \ell_0)\) to the index form \(\tilde{I} : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}\) of \((\tilde{X}, \tilde{\ell}_0)\).

**Proof.** See [12, Subsection 2.10].

**Definition 2.9.** For each \(t \in [a, b]\), let \(D_t\) be an \(r\)-dimensional subspace of \(\mathbb{R}^n\), where \(r = 0, \ldots, n\) is fixed. We call \(D\) a smooth family of subspaces in \(\mathbb{R}^n\) over the interval \([a, b]\) if there exist smooth maps \(Y_i : [a, b] \rightarrow \mathbb{R}^n, i = 1, \ldots, r\), such that \(Y_i(t)\) is a basis for \(D_t\) for every \(t \in [a, b]\); the integer \(r\) is called the rank of the family \(D\) and the family of maps \(Y_i\) is called a frame for \(D\). A smooth family of subspaces \(D\) is called nondegenerate (resp., negative) for a symplectic differential system \(X\) if the symmetric bilinear form \(B(t)^{-1} \in \text{Bil}_{\text{sym}}(\mathbb{R}^n)\) is nondegenerate (resp., negative definite) on \(D_t\) for all \(t \in [a, b]\). If \(D\) is negative for \(X\) and the rank of \(D\) equals the index of \(X\) we say that \(D\) is maximal negative for \(X\). If \(D\) is nondegenerate for \(X\) then the index of \(D\) with respect to \(X\) is defined to be the index of the restriction of \(B(t)^{-1}\) to \(D_t\) (which does not depend on \(t \in [a, b]\)).

Let \(D\) be a fixed smooth family of subspaces of rank \(r\) in \(\mathbb{R}^n\) over the interval \([a, b]\). A map \(Y : [a, b] \rightarrow \mathbb{R}^n\) is called a section of \(D\) if \(Y(t) \in D_t\) for all \(t \in [a, b]\); we set:

\[
(2.14) \quad \mathcal{S}_D = \{ v \in \mathcal{H} : v \text{ is a section of } D \text{ and } v(a) = 0 \}. 
\]
An absolutely continuous map \( v : [a, b] \to \mathbb{R}^n \) is called a solution of \( X \) along \( D \) if for every absolutely continuous section \( Y : [a, b] \to \mathbb{R}^n \) of \( D \) we have that \( t \mapsto \alpha_v(t)Y(t) \) is absolutely continuous and that:

\[
(\alpha_v(Y))' = B(\alpha_v, \alpha_Y) + C(v, Y);
\]

we set:

\[
\mathcal{K}_D = \{ v \in \mathcal{H} : v \text{ is a solution of } X \text{ along } D \}.
\]

When \( v' \) is also absolutely continuous then the left hand side of (2.15) can be expanded and one obtains that \( v \) is a solution of \( X \) along \( D \) iff \( \alpha_v' - C v + A^* \alpha_v \) vanishes on \( D \); in particular, every solution \( v \) of \( X \) is a solution along \( D \). If \( (Y_i)_{i=1}^r \) is a frame for \( D \) then \( v \) is a solution of \( X \) along \( D \) iff \( \alpha_v(Y_i) \) is absolutely continuous and (2.15) is satisfied with \( Y = Y_i \) for all \( i = 1, \ldots, r \).

The spaces \( \mathcal{K}_D \) and \( \mathcal{S}_D \) are orthogonal with respect to the index form:

**Lemma 2.10.** Consider a triple \((X, \ell_0, D)\) where \((X, \ell_0)\) is a symplectic differential system with initial data and \( D \) is a smooth family of subspaces in \( \mathbb{R}^n \) over the interval \([a, b] \). Then \( I(v, w) = 0 \) for all \( v \in \mathcal{K}_D \) and all \( w \in \mathcal{S}_D \).

**Proof.** Choose a frame \((Y_i)_{i=1}^r \) for \( D \) and write \( w = \sum_{i=1}^r f_i Y_i \); the conclusion follows from an easy computation using (2.5), (2.12) and (2.15). \( \square \)

**Definition 2.11.** Given a nondegenerate smooth family of subspaces \( D \) for a symplectic differential system \( X \) and a frame \((Y_i)_{i=1}^r \) for \( D \) we define the reduced symplectic differential system corresponding to \( X \), \( D \) and \((Y_i)_{i=1}^r \) to be the following symplectic differential system in \( \mathbb{R}^r \):

\[
\begin{aligned}
&f' = -(\mathcal{B}^{-1} \circ \mathcal{A}) f + \mathcal{B}^{-1} \varphi, \\
&\varphi' = (\mathcal{C} - \mathcal{A}^* \circ \mathcal{B}^{-1} \circ \mathcal{A}) f + (\mathcal{A}^* \circ \mathcal{B}^{-1}) \varphi,
\end{aligned}
\]

where \( \mathcal{A}(t), \mathcal{B}(t), \mathcal{C}(t) \in \text{Lin}(\mathbb{R}^r, \mathbb{R}^{r*}) \) are the linear operators represented by the following matrices:

\[
\begin{aligned}
\mathcal{B}_{ij}(t) &= B(t)^{-1}(Y_i(t), Y_j(t)), & \mathcal{A}_{ij}(t) &= \alpha_{Y_j}(t)Y_i(t) \\
\mathcal{C}_{ij}(t) &= B(t)(\alpha_{Y_j}(t), \alpha_{Y_j}(t)) + C(t)(Y_i(t), Y_j(t)).
\end{aligned}
\]

The fact that \( D \) is nondegenerate for \( X \) implies that \( \mathcal{B}(t) \) is indeed nondegenerate, so that \( \mathcal{B}(t)^{-1} \) in (2.17) makes sense. Moreover, the index of the reduced system (2.17) equals the index of \( D \) with respect to \( X \). We will denote by \( X_{\text{red}} \) the coefficient matrix of (2.17) and by \( A_{\text{red}}, B_{\text{red}}, C_{\text{red}} \) the coefficients of \( X_{\text{red}} \).

The introduction of the reduced symplectic system is motivated by the following:

**Lemma 2.12.** If \( D \) is a nondegenerate smooth family of subspaces for \( X \) and \((Y_i)_{i=1}^r \) is a frame for \( D \) then \( v = \sum_{i=1}^r f_i Y_i \) is a solution of \( X \) along \( D \) iff \( f = (f_i)_{i=1}^r \) is a solution of the reduced symplectic system \( X_{\text{red}} \) corresponding to \( D \) and \((Y_i)_{i=1}^r \). In particular, \( \mathcal{K}_D \cap \mathcal{S}_D = \{0\} \) iff \( t = b \) is not conjugate for \( X_{\text{red}} \).

**Proof.** It is a straightforward computation using (2.5). \( \square \)

The index form of the reduced symplectic system can be identified with a restriction of the index form of the original system:
Lemma 2.13. If $I_{\text{red}} : \mathcal{H}_{\text{red}} \times \mathcal{H}_{\text{red}} \to \mathbb{R}$ denotes the index form of $X_{\text{red}}$ then the continuous isomorphism

\begin{equation}
\lambda : \mathcal{H}_{\text{red}} \ni f = (f_i)_{i=1}^r \longmapsto \sum_{i=1}^r f_i Y_i \in \mathcal{S}_{\mathcal{D}}
\end{equation}

carries $I_{\text{red}}$ to $I|_{\mathcal{S}_{\mathcal{D}}}$, i.e., $I(\lambda(f), \lambda(g)) = I_{\text{red}}(f, g)$ for all $f, g \in \mathcal{H}_{\text{red}}$. 

Proof. Recall that the domain $\mathcal{H}_{\text{red}}$ of the index form of $X_{\text{red}}$ is given by:

$$\mathcal{H}_{\text{red}} = \{ f \in H^1([a, b], \mathbb{R}^r) : f(a) = f(b) = 0 \},$$

so that $\lambda$ is indeed a continuous isomorphism. The conclusion follows by a straightforward computation using (2.5). \hfill \Box

Remark 2.14. The reduced symplectic system $X_{\text{red}}$ depends on the choice of the frame for $\mathcal{D}$ and not only on $\mathcal{D}$; however, different choices of a frame for $\mathcal{D}$ produce isomorphic reduced systems. In particular, when discussing notions that are invariant by isomorphisms (like Maslov index, focal instants), we do not need to specify a frame for $\mathcal{D}$.

We recall the following result announced in [13] and proven in [15]:

Theorem 2.15. Consider a triple $(X, \ell_0, \mathcal{D})$, where $(X, \ell_0)$ is a symplectic differential system with initial data such that $\ell_0$ defines a nondegenerate initial condition for $X$ and $\mathcal{D}$ is a maximal negative smooth family of subspaces for $X$. If $t = b$ is not $(X, \ell_0)$-focal then:

$$i_{\text{Maslov}}(X, \ell_0) = n_-(I|_{\mathcal{K}_\mathcal{D}}) - n_+(I|_{\mathcal{S}_\mathcal{D}}) = n_-(B(\alpha)^{-1}|_{\mathcal{D}}).$$

\hfill \Box

The following lemma\footnote{In [12, Corollary 2.6.10] the result of Lemma 2.16 is proven for a maximal negative family $\mathcal{D}$; here we adapt the proof to the case where $\mathcal{D}$ is only nondegenerate.} is an addendum to the result of Theorem 2.15:

Lemma 2.16. Let $(X, \ell_0, \mathcal{D})$ be a triple where $(X, \ell_0)$ is a symplectic differential system with initial data and $\mathcal{D}$ is a nondegenerate smooth family of subspaces for $X$. If $t = b$ is not a conjugate instant for the reduced symplectic system corresponding to $(X, \ell_0)$ and $\mathcal{D}$ then we have a direct sum decomposition $\mathcal{H} = \mathcal{K}_\mathcal{D} \oplus \mathcal{S}_\mathcal{D}$.

Proof. If $(Y_i)_{i=1}^r$ denotes a frame for $\mathcal{D}$ then $\mathcal{K}_\mathcal{D}$ is the kernel of the bounded linear operator $F : \mathcal{H} \to L^2([a, b], \mathbb{R}^r)/\text{Const}$ defined by

$$F(v)(t) = \alpha_v(t) Y_i(t) - \int_a^t B(\alpha_v, \alpha_{Y_i}) + C(v, Y_i) \, ds \quad \text{mod Const},$$

for all $v \in \mathcal{H}$, $t \in [a, b]$, $i = 1, \ldots, r$, where $L^2([a, b], \mathbb{R}^r)$ denotes the Hilbert space of $\mathbb{R}^r$-valued square integrable maps on $[a, b]$ and $\text{Const}$ denotes the $r$-dimensional subspace of $L^2([a, b], \mathbb{R}^r)$ consisting of constant maps. If $\lambda$ denotes the isomorphism defined in (2.19) then a straightforward computation using (2.5) shows that $F \circ \lambda$ is given by:

\begin{equation}
(F \circ \lambda)(f)(t) = \mathcal{B}(t) f'(t) + \mathcal{A}(t) f(t) - \int_a^t \mathcal{A}^* f' + \mathcal{C} f \, ds \quad \text{mod Const},
\end{equation}

for all $f = (f_i)_{i=1}^r \in \mathcal{H}_{\text{red}}$, $t \in [a, b]$. The kernel of $F \circ \lambda$ are the solutions $f$ of the reduced symplectic system with $f(a) = f(b) = 0$; it follows that $F \circ \lambda$ is injective. We will now show that $F \circ \lambda$ is a Fredholm operator of index zero; this will imply that $F \circ \lambda$ (and
therefore $F|_{S_{\nu}}$ is an isomorphism and that will conclude the proof. To prove that $F \circ \lambda$ is a Fredholm operator of index zero, observe first that by the additivity of the Fredholm index by composition of operators and by the invertibility of $B$, the map

$$
\mathcal{H}_{\text{red}} \ni f \mapsto Bf' \mod \text{Const} \in L^2([a, b], \mathbb{R}^r)/(\text{Const})
$$

is a Fredholm operator of index zero. By (2.20) and the compact inclusion $H^1 \hookrightarrow C^0$, the operator $F \circ \lambda$ is a compact perturbation of (2.21). This concludes the proof. □

**Remark 2.17.** The kernel of the index form $I : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is the space of solutions $v : [a, b] \to \mathbb{R}^n$ of $(X, t_0)$ with $v(b) = 0$ (see [12, Subsection 2.5]). It follows from Lemmas 2.10 and 2.16 that if $t = b$ is not conjugate for the reduced symplectic system then the kernel of $I|_{X_{\text{red}}}$ coincides with the kernel of $I$ in $\mathcal{H}$.

**Remark 2.18.** In some situations it is useful to consider the symplectic differential system $X_{\text{red}}$ which is isomorphic to $X_{\text{red}}$ and whose coefficients $A_{\text{red}}, B_{\text{red}}, C_{\text{red}}$ are given by:

$$
\begin{align*}
\tilde{A}_{\text{red}}(t) &= -B(t)^{-1} \circ A_{\text{ant}}(t), \\
\tilde{B}_{\text{red}} &= B(t)^{-1}, \\
\tilde{C}_{\text{red}}(t) &= C(t) - A_{\text{sym}}(t) + A_{\text{ant}}(t) \circ B(t)^{-1} \circ A_{\text{ant}}(t),
\end{align*}
$$

for all $t \in [a, b]$, where $A_{\text{sym}}, A_{\text{ant}}$ denote respectively the symmetric and anti-symmetric components of $A$:

$$
\begin{align*}
A_{\text{sym}}(t) &= \frac{A(t) + A(t)^*}{2}, \\
A_{\text{ant}}(t) &= \frac{A(t) - A(t)^*}{2}.
\end{align*}
$$

An explicit isomorphism from $X_{\text{red}}$ to $\tilde{X}_{\text{red}}$ is given by (2.13) with:

$$
Z(t) = \text{Id}, \quad W(t) = -A_{\text{sym}}(t), \quad t \in [a, b].
$$

**Remark 2.19.** If a nondegenerate smooth family of subspaces $\mathcal{D}$ for a symplectic differential system $X$ admits a frame $(Y_i)_{i=1}^r$ consisting of solutions of $X$ satisfying the symmetry condition

$$
\alpha_{Y_i}(Y_j) = \alpha_{Y_j}(Y_i), \quad i, j = 1, \ldots, r,
$$

then the coefficients of the reduced symplectic system $\tilde{X}_{\text{red}}$ defined in Remark 2.18 are $\tilde{A}_{\text{red}} = 0, \tilde{B}_{\text{red}} = B^{-1}, \tilde{C}_{\text{red}} = 0$. The system $\tilde{X}_{\text{red}}$ becomes the differential equation

$$
B f' = \text{constant}.
$$

An instant $t \in [a, b]$ is conjugate for $\tilde{X}_{\text{red}}$ iff the integral:

$$
B^j(t) = \int_a^t B(s)^{-1} \, ds
$$

is a degenerate (symmetric) bilinear form in $\mathbb{R}^r$, in which case the multiplicity of $t$ equals the degeneracy of $B^j(t)$. If $t = b$ is not conjugate for $\tilde{X}_{\text{red}}$ then, by Proposition 2.6, the Maslov index of $\tilde{X}_{\text{red}}$ is given by:

$$
i_{\text{Maslov}}(\tilde{X}_{\text{red}}) = \sum_{t \in [a, b]} \text{sgn}\left( B(t)|_{\text{im}(B^j(t))} \right),
$$

provided that $B(t)$ is nondegenerate on the image of $B^j(t)$ for those $t \in [a, b]$ such that $B^j(t)$ is degenerate. In (2.26) we have denoted by $\perp$ the orthogonal complement with respect to $B(t)$. 
3. The Index Theorem

In this section we consider the following setup:

- $(X, \ell_0)$ is a symplectic differential system with initial data on $\mathbb{R}^n$ over the interval $[a, b]$;
- $\mathcal{D}$ and $\Delta$ are nondegenerate smooth families of subspaces for $X$ with $\Delta_t \subset \mathcal{D}_t$ for all $t$;
- $(Y_i)_{i=1}^r$ is a frame for $\mathcal{D}$ such that $(Y_i)_{i=1}^r$ is a frame for $\Delta$;
- $X_{\text{red}}$ is the reduced symplectic system corresponding to $\mathcal{D}$ and $(Y_i)_{i=1}^r$;
- $I : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is the index form of $(X, \ell_0)$ and $\mathcal{K}_{\mathcal{D}}, \mathcal{S}_{\mathcal{D}}$ (resp., $\mathcal{K}_\Delta, \mathcal{S}_\Delta$) are the subspaces of $\mathcal{H}$ defined in analogy with (2.16) and (2.14) for $(X, \ell_0)$ and $\mathcal{D}$ (resp., $\Delta$);
- $I_{\text{red}} : \mathcal{H}_{\text{red}} \times \mathcal{H}_{\text{red}} \to \mathbb{R}$ is the index form of $X_{\text{red}}$;
- $\lambda : \mathcal{H}_{\text{red}} \to \mathbb{S}_{\mathcal{D}}$ is the continuous isomorphism defined in (2.19);
- $\Delta_{\text{red}}$ is the (constant) smooth family of subspaces $\Delta_{\text{red}} \equiv \mathbb{R}^k \oplus \{0\} \subset \mathbb{R}^r$ in $\mathbb{R}^r$ over the interval $[a, b]$;
- $\mathcal{K}_{\Delta_{\text{red}}}$ and $\mathcal{S}_{\Delta_{\text{red}}}$ are the subspaces of $\mathcal{H}_{\text{red}}$ defined in analogy with (2.16) and (2.14) for the symplectic differential system with initial data $(X_{\text{red}}, \{0\} \oplus \mathbb{R}^r)$ relatively to the smooth family of subspaces $\Delta_{\text{red}}$;

The following facts are immediate:

1. $\mathcal{K}_{\mathcal{D}} \subset \mathcal{K}_\Delta$ and $\mathcal{S}_\Delta \subset \mathcal{S}_{\mathcal{D}}$;
2. $\lambda(\mathcal{S}_{\Delta_{\text{red}}}) = \mathcal{S}_\Delta$;
3. $\Delta_{\text{red}}$ is a nondegenerate family of subspaces for $X_{\text{red}}$.

We prove the following preparatory lemma:

**Lemma 3.1.** An absolutely continuous map $f : [a, b] \to \mathbb{R}^r$ is a solution of $X_{\text{red}}$ along $\Delta_{\text{red}}$ iff $v = \sum_{i=1}^r f(Y_i)$ is a solution of $X$ along $\Delta$. In particular, $\lambda(\mathcal{K}_{\Delta_{\text{red}}}) = \mathcal{K}_\Delta \cap \mathcal{S}_{\mathcal{D}}$.

**Proof.** The map $f$ is a solution of $X_{\text{red}}$ along $\Delta_{\text{red}}$ iff $(\mathcal{B}f')_i$ is absolutely continuous for $i = 1, \ldots, k$ and

$$[\left(I - \mathcal{A} f' \mathcal{C} f \right)]_i = (\mathcal{C} f + \mathcal{A}^* f')_i, \quad i = 1, \ldots, k.$$

Using (2.5) one easily checks that this is also the condition for $v$ to be a solution of $X$ along $\Delta$. \hfill $\Box$

We can now prove the main theorem of the section.

**Theorem 3.2** (generalized index theorem). Consider a triple $(X, \ell_0, \mathcal{D})$ where $(X, \ell_0)$ is a symplectic differential system with initial data in $\mathbb{R}^n$ over the interval $[a, b]$ such that $\ell_0$ defines a nondegenerate initial condition for $X$ and $\mathcal{D}$ is a smooth family of subspaces in $\mathbb{R}^n$ over $[a, b]$ whose index with respect to $X$ equals the index of $X$. Denote by $X_{\text{red}}$ the reduced symplectic system corresponding to $X$ and $\mathcal{D}$ (see Remark 2.14). If $t = b$ is neither a focal instant for $(X, \ell_0)$ nor a conjugate instant for $X_{\text{red}}$ then:

$$n_-(I|_{\mathcal{K}_{\mathcal{D}}}) = i_{\text{Maslov}}(X, \ell_0) - i_{\text{Maslov}}(X_{\text{red}}) + n_- (B(a)^{-1}|_P).$$

**Proof.** Since the index of $\mathcal{D}$ with respect to $X$ equals the index of $X$ then $\mathcal{D}$ is nondegenerate for $X$ and there exists a maximal negative family $\Delta$ for $X$ with $\Delta \subset \mathcal{D}$. We may choose a frame $(Y_i)_{i=1}^r$ of $\mathcal{D}$ such that $(Y_i)_{i=1}^r$ is a frame for $\Delta$, so that we are in the setup specified at the beginning of the section. Obviously $\Delta_{\text{red}}$ is a maximal negative
family for $X_{\text{red}}$, so that we may apply Theorem 2.15 to the triple $(X, \ell_0, \Delta)$ and to the triple $(X_{\text{red}}, \{0\} \oplus \mathbb{R}^n, \Delta_{\text{red}})$ obtaining:

\[
\begin{align*}
\text{i}_{\text{Maslov}}(X, \ell_0) &= n_-(I|_{\mathcal{K}_{\Delta}}) + n_+(I|_{\mathcal{S}_\Delta}) - n_-(B(a)^{-1}|_{P}), \\
\text{i}_{\text{Maslov}}(X_{\text{red}}) &= n_-(I_{\text{red}}|_{\mathcal{K}_{\Delta_{\text{red}}}}) + n_+(I_{\text{red}}|_{\mathcal{S}_{\Delta_{\text{red}}}}).
\end{align*}
\]

From Lemma 2.13 we get:

\[
\begin{align*}
n_-(I_{\text{red}}|_{\mathcal{K}_{\Delta_{\text{red}}}}) &= n_-(I_{\lambda|_{\mathcal{K}_{\Delta_{\text{red}}}}}), \\
n_+(I_{\text{red}}|_{\mathcal{S}_{\Delta_{\text{red}}}}) &= n_+(I_{\lambda|_{\mathcal{S}_{\Delta_{\text{red}}}}});
\end{align*}
\]

Using item (2) on page 11, we have:

\[
\begin{align*}
n_+(I|_{\lambda|_{\mathcal{S}_{\Delta_{\text{red}}}}}) &= n_+(I|_{\mathcal{S}_\Delta}).
\end{align*}
\]

Lemmas 2.10, 2.16 and item (1) on page 11 imply that we have an $I$-orthogonal direct sum decomposition $\mathcal{K}_\Delta = \mathcal{K}_\Delta + (\mathcal{S}_\Delta \cap \mathcal{K}_\Delta)$, where $\mathcal{S}_\Delta \cap \mathcal{K}_\Delta = \lambda(I_{\mathcal{K}_{\Delta_{\text{red}}}})$ by Lemma 3.1. Hence:

\[
\begin{align*}
n_-(I|_{\mathcal{K}_{\Delta}}) &= n_-(I|_{\mathcal{K}_\Delta}) + n_-(I_{\lambda|_{\mathcal{K}_{\Delta_{\text{red}}}}}).
\end{align*}
\]

The conclusion now follows from equalities (3.2)—(3.6). □

4. GEODESICS IN SEMI-RIEMANNIAN MANIFOLDS

Let $(M, g)$ be an $n$-dimensional semi-Riemannian manifold, where $g$ is a (nondegenerate) metric tensor of index $k$. We denote by $\nabla$ the Levi-Civita connection of $g$ and by $\mathcal{R}(X, Y) = \nabla_X \nabla_Y Y - \nabla_Y \nabla_X X - \nabla_{[X, Y]}$ the curvature tensor of $\nabla$. Given a geodesic $\gamma : [a, b] \to M$ then the Jacobi equation along $\gamma$

\[
\begin{align*}
v'' = \mathcal{R}(\gamma', v) \gamma'
\end{align*}
\]

produces a Morse–Sturm system of the form (2.4) in $\mathbb{R}^n$ by means of a parallel trivialization of the tangent bundle of $M$ along $\gamma$ (the bilinear form $g \in \text{Bil}(\mathbb{R}^n, \mathbb{R}^n)$ corresponds to the metric tensor $g$ and the endomorphism $R(t) \in \text{Lin}(\mathbb{R}^n, \mathbb{R}^n)$ corresponds to $\mathcal{R}(\gamma(t), \cdot) \gamma'(t)$). In (4.1) the prime denotes covariant derivative along $\gamma$; this notation will be used whenever a curve $\gamma$ is fixed by the context.

Different parallel trivializations of $TM$ along $\gamma$ produce Morse–Sturm systems that are isomorphic as symplectic differential systems\(^3\).

Let $\mathcal{P} \subset M$ be a smooth submanifold with $\gamma(a) \in \mathcal{P}$ and $\gamma'(a) \in T_{\gamma(a)}\mathcal{P}^\perp$; a $\mathcal{P}$-Jacobi field along $\gamma$ is a Jacobi field $v$ satisfying the initial conditions:

\[
\begin{align*}
v(a) \in T_{\gamma(a)}\mathcal{P}, \quad g(v'(a), \cdot)|_{T_{\gamma(a)}\mathcal{P}} + \mathbb{I}_{\gamma'(a)}(v(a), \cdot) = 0 \in T_{\gamma(a)}\mathcal{P}^*,
\end{align*}
\]

where $\mathbb{I}_{\gamma'(a)} \in \text{Bil}_{\text{sym}}(T_{\gamma(a)}\mathcal{P})$ denotes the second fundamental form of $\mathcal{P}$ in the normal direction $\gamma'(a)$. If $\mathcal{P} \subset \mathbb{R}^n$, $S \in \text{Bil}_{\text{sym}}(\mathcal{P})$ correspond to $T_{\gamma(a)}\mathcal{P}$ and $\mathbb{I}_{\gamma'(a)}$ by means of the chosen parallel trivialization of $TM$ along $\gamma$ then $\mathcal{P}$-Jacobi fields correspond to $(X, \ell_0)$-solutions, where $X$ is the Morse–Sturm system (2.4) and $\ell_0 \subset \mathbb{R}^n \oplus \mathbb{R}^n^*$ is the Lagrangian (2.8). Also, the index form of the pair $(X, \ell_0)$ corresponds to the second variation of the geodesic action functional

\[
\begin{align*}
E(z) = \frac{1}{2} \int_a^b g(z', z') \, dt
\end{align*}
\]

\(^3\)More generally, non parallel trivializations of $TM$ along $\gamma$ yield symplectic differential systems from the Jacobi equation along $\gamma$. All symplectic differential systems obtained in this way are isomorphic (see [12, Section 3]).
at the critical point $\gamma$. The domain of $E$ is the Hilbert manifold $\Omega_{\gamma}(M)$ consisting of $H^1$ curves $z : [a, b] \to M$ with $z(a) \in \mathcal{P}$, $z(b) = q$, where $q = \gamma(b)$. Recall that the critical points of $E$ in $\Omega_{\gamma}(M)$ are the geodesics starting orthogonally at $\mathcal{P}$ and ending at $q$.

The Lagrangian $\ell_0$ defines a nondegenerate initial condition for $X$ iff the submanifold $\mathcal{P}$ is nondegenerate at $\gamma(a)$, i.e., if $\mathfrak{g}$ is nondegenerate on $T_{\gamma(a)}\mathcal{P}$. Focal instants for $(X, \ell_0)$ correspond to $\mathcal{P}$-focal points along $\gamma$. The case where the initial submanifold $\mathcal{P}$ is a single point corresponds to the case where $\ell_0 = L_0$ (recall (2.6)); in this case, the initial condition defined by $\ell_0$ is always nondegenerate.

When $\mathcal{P}$ is nondegenerate at $\gamma(a)$ and $\gamma(b)$ is not $\mathcal{P}$-focal along $\gamma$ then we can define the Maslov index $\Maslov(\gamma, \mathcal{P})$ of the geodesic $\gamma$ with respect to the initial submanifold $\mathcal{P}$ to be the Maslov index of the pair $(X, \ell_0)$; by Proposition 2.8, the Maslov index of $\gamma$ with respect to $\mathcal{P}$ does not depend on the parallel trivialization used to produce the pair $(X, \ell_0)$. When $\mathcal{P}$ is a single point we call $\Maslov(\gamma, \mathcal{P})$ the Maslov index of $\gamma$ and we write simply $\Maslov(\gamma)$.

If $(\mathcal{Y}_i)_{i=1}^r$ are smooth vector fields along $\gamma$ such that $(\mathcal{Y}_i(t))_{i=1}^r$ is the basis of a nondegenerate subspace of $T_{\gamma(t)}M$ for all $t \in [a, b]$ then the parallel trivialization along $\gamma$ produce maps $Y_i : [a, b] \to \mathbb{R}^n$ which form a frame for a nondegenerate smooth family of subspaces for $X$. The operators $A, B, C \in \text{Lin}(\mathbb{R}^r, \mathbb{R}^r)$ which appear in the corresponding reduced symplectic system (2.17) are given by:

$$
B_{ij} = \mathfrak{g}(\mathcal{Y}_i, \mathcal{Y}_j), \quad A_{ij} = \mathfrak{g}(\mathcal{Y}_i', \mathcal{Y}_j),
$$

$$
C_{ij} = \mathfrak{g}(\mathcal{Y}_i', \mathcal{Y}_j') + \mathfrak{g}(\mathcal{R}(\gamma'), \mathcal{Y}_i)\mathcal{Y}_j'.
$$

**Definition 4.1.** Let $\gamma : [a, b] \to M$ be a geodesic and $(\mathcal{Y}_i)_{i=1}^r$ smooth vector fields along $\gamma$ such that $(\mathcal{Y}_i(t))_{i=1}^r$ is the basis of a nondegenerate subspace of $T_{\gamma(t)}M$ for all $t \in [a, b]$. Consider the symplectic differential system $X_{\text{red}}$ defined in (2.17) with $A, B, C$ defined in (4.4). If $t = b$ is not conjugate for $X_{\text{red}}$ then the reduced Maslov index of the geodesic $\gamma$ (with respect to the fields $\mathcal{Y}_i$) is defined by:

$$
\Maslov_{\text{red}}(\gamma) = \Maslov(X_{\text{red}}).
$$

In this geometrical context, the Index Theorem 3.2 gives a generalized Morse index theorem for semi-Riemannian geodesics. Observe that the term $n\cdot (B(a)^{-1})_{ij}$ appearing in equality (3.1) is the index of the metric $\mathfrak{g}$ in the tangent space $T_{\gamma(a)}\mathcal{P}$ of the initial submanifold.

### 4.1. A variational principle for semi-Riemannian geodesics

We now consider fixed an $n$-dimensional semi-Riemannian manifold $(M, \mathfrak{g})$ with metric tensor $\mathfrak{g}$ of index $k$, a smooth submanifold $\mathcal{P} \subset M$, a point $q \in M$ and smooth vector fields $(\mathcal{Y}_i)_{i=1}^r$ on $M$ such that $(\mathcal{Y}_i(m))_{i=1}^r$ is a basis for a nondegenerate subspace of $T_mM$ for all $m \in M$. We say that an absolutely continuous curve $\gamma : [a, b] \to M$ is a geodesic along the fields $\mathcal{Y}_i$ if $\mathfrak{g}(\gamma', \mathcal{Y}_i)$ is absolutely continuous on $[a, b]$ and

$$
\mathfrak{g}(\gamma', \mathcal{Y}_i') = \mathfrak{g}(\gamma', \mathcal{Y}_i'),
$$

for $i = 1, \ldots, r$. If $\gamma$ is of class $C^2$ then $\gamma$ is a geodesic along the fields $\mathcal{Y}_i$ iff $\gamma''$ is orthogonal to the distribution spanned by the $\mathcal{Y}_i$; in particular, if $\gamma$ is a geodesic then $\gamma$ is a geodesic along the fields $\mathcal{Y}_i$. Moreover, if the vector fields $\mathcal{Y}_i$ are Killing4 then $\gamma$ is a geodesic along the fields $\mathcal{Y}_i$ iff $\mathfrak{g}(\gamma', \mathcal{Y}_i)$ is constant for all $i = 1, \ldots, r$.

---

4Recall that $\mathcal{Y}$ is a Killing vector field iff the bilinear form $\mathfrak{g}(\nabla \mathcal{Y}, \cdot)$ is skew-symmetric.
Consider the following subset of the Hilbert manifold $\Omega_{Pq}(M)$:

$$\mathcal{N}_{Pq}(M) = \{ \gamma \in \Omega_{Pq}(M) : \gamma \text{ is a geodesic along the fields } \mathcal{Y}_i \}. $$

We are interested in determining conditions that imply that $\mathcal{N}_{Pq}(M)$ is a Hilbert submanifold of $\Omega_{Pq}(M)$ and that the critical points of the restriction of the geodesic action functional $E$ to $\mathcal{N}_{Pq}(M)$ are the geodesics $\gamma : [a, b] \to M$ starting orthogonally to $P$ and ending at $q$. These conditions are given in the following:

**Theorem 4.2.** Let $\gamma \in \mathcal{N}_{Pq}(M)$ be fixed; consider the following homogeneous system of linear ODE's in $\mathbb{R}^r \oplus \mathbb{R}^{r^*}$:

$$
\begin{align*}
\dot{f} &= -(\mathfrak{B}^{-1} \circ A)f + \mathfrak{B}^{-1} \varphi, \\
(\varphi + \mathfrak{E}f)' &= (C + \mathfrak{E} - (A^* + \mathfrak{E}) \circ \mathfrak{B}^{-1} \circ A)f + ((A^* + \mathfrak{E}) \circ \mathfrak{B}^{-1})\varphi,
\end{align*}
$$

where $A, \mathfrak{B}, C \in \text{Lin}(\mathbb{R}^r, \mathbb{R}^{r^*})$ are defined in (4.4) and $\mathfrak{E}, \mathfrak{E} \in \text{Lin}(\mathbb{R}^r, \mathbb{R}^{r^*})$ are defined by:

$$
\mathfrak{E}_{ij} = g(\nabla_{\mathcal{Y}_i} \mathcal{Y}_j, \cdot), \quad \mathfrak{E}'_{ij} = g((\nabla_{\mathcal{Y}_i} \mathcal{Y}_j)', \cdot').
$$

Assume that the system (4.5) does not admit a non zero solution $(f, \varphi)$ with $f(a) = f(b) = 0$. Then:

1. $\gamma$ has a neighborhood $\mathfrak{U}$ in $\mathcal{N}_{Pq}(M)$ which is a Hilbert submanifold of $\Omega_{Pq}(M)$;
2. $\gamma$ is a critical point of $E|_{\mathfrak{U}}$ iff $\gamma$ is a geodesic starting orthogonally at $P$;
3. if $\gamma$ is a critical point of $E|_{\mathfrak{U}}$ then the degeneracy of the second variation of $E|_{\mathfrak{U}}$ at $\gamma$ is equal to the multiplicity of $\gamma(b)$ as a $P$-focal point along $\gamma$.

Assume that $\gamma$ is a critical point of $E|_{\mathfrak{U}}$ such that $\gamma(b)$ is not $P$-focal along $\gamma$ and $P$ is nondegenerate at $\gamma(a)$. If the index of $g$ restricted to the distribution spanned by $(\mathcal{Y}_i)_{i=1}^r$ equals the index of $g$ then the Morse index of $E|_{\mathfrak{U}}$ at $\gamma$ is given by:

$$
n_-(\text{d}^2(E|_{\mathfrak{U}})(\gamma)) = i_{\text{Maslov}}(\gamma) - i^\text{red}_{\text{Maslov}}(\gamma) + n_-(g|_{T_{\gamma(a)}P}).$$

Before getting into the proof of Theorem 4.2, which will take us up almost entirely the rest of the subsection, we will make a few remarks about its statement. First we observe that if $\gamma$ is a geodesic then $\mathfrak{E} = \mathfrak{E}'$, so that (4.5) becomes the reduced symplectic system (2.17); in particular, the hypothesis of the theorem is satisfied precisely when $t = b$ is not a conjugate instant for the reduced symplectic system.

Let us look now at the particular case where each $\mathcal{Y}_i$ is a Killing vector field; this obviously implies that $\mathfrak{E} = -A^*$. Another remarkable equality that holds in this case is $C = -\mathfrak{E}$; to prove it, recall that the Hessian of a vector field $\mathcal{Y}$ is the $(2,1)$-tensor field defined by $\text{Hess}(\mathcal{Y}) = \nabla \nabla \mathcal{Y}$, i.e., $\text{Hess}(\mathcal{Y})(V, W) = \nabla_V \nabla_W \mathcal{Y} - \nabla_{\nabla_V W} \mathcal{Y}$. Observe that $\text{Hess}(\mathcal{Y})(V, W) - \text{Hess}(\mathcal{Y})(W, V) = \mathcal{R}(V, W) \mathcal{Y}$; moreover, if $\mathcal{Y}$ is Killing then $g(\text{Hess}(\mathcal{Y})(V, W), Z)$ is skew-symmetric in the variables $W$ and $Z$, because $g(\nabla \mathcal{Y}, \cdot)$ is skew-symmetric. Using all these formulas we compute:

$$
\mathfrak{E}_{ij} = g((\nabla_{\mathcal{Y}_j} \mathcal{Y}_i, \cdot'), \cdot') = g(\text{Hess}(\mathcal{Y}_i)(\gamma', \cdot), \gamma') + g(\nabla_{\mathcal{Y}_i} \mathcal{Y}_i, \gamma') = g(G \gamma, \mathcal{Y}_i, \gamma') + g(\nabla_{\mathcal{Y}_i} \mathcal{Y}_i, \gamma') = -C_{ij}.
$$

We have proven that if the fields $\mathcal{Y}_i$ are Killing then the system (4.5) becomes (recall (2.23)):

$$
\mathfrak{B} \mathcal{Y}' + 2A_{\text{ant}} f = \text{constant}.
$$

Moreover, if the fields $\mathcal{Y}_i$ commute, i.e., $[\mathcal{Y}_i, \mathcal{Y}_j] = \nabla_{\mathcal{Y}_i} \mathcal{Y}_j - \nabla_{\mathcal{Y}_j} \mathcal{Y}_i = 0$ for all $i, j = 1, \ldots, r$ then $A$ is symmetric and (4.5) becomes (2.24).
Remark 4.3. In the case where the $Y_i$’s are commuting Killing vector fields then the hypothesis of Theorem 4.2 is satisfied iff the symmetric bilinear form $B^f(b)$ (recall (2.25)) is nondegenerate. Moreover, if $\gamma$ is a geodesic starting orthogonally at $P$ then, by Remark 2.19, the reduced Maslov index of $\gamma$ is given by the righthand side of (2.26) provided that $B(t)$ is nondegenerate on $\Im(B^f(t))$ for those $t \in [a, b]$ such that $B^f(t)$ is degenerate.

Proof of Theorem 4.2. We start by considering the smooth map
\begin{equation}
(4.7) \quad \mathcal{F} : \Omega_{\mathcal{P}q}(M) \rightarrow L^2([a, b], \mathbb{R}^r^+) / \text{Const}
\end{equation}
defined by:
\begin{equation*}
\mathcal{F}(\gamma)(t)_i = g(\gamma'(t), Y_i(\gamma(t))) - \int_a^t g(\gamma', Y'_i) \, ds \mod \text{Const},
\end{equation*}
for all $\gamma \in \Omega_{\mathcal{P}q}(M)$, $t \in [a, b]$ and $i = 1, \ldots, r$. In (4.7) we have denoted by $\text{Const}$ the subspace of $L^2([a, b], \mathbb{R}^r^+)$ consisting of constant maps. Obviously:
\begin{equation}
(4.8) \quad \mathcal{N}_{\mathcal{P}q}(M) = \mathcal{F}^{-1}(0).
\end{equation}
The differential of $\mathcal{F}$ is computed as:
\begin{equation}
(4.9) \quad d\mathcal{F}_\gamma(v)(t)_i = g(v'(t), Y_i(\gamma(t))) + g(\gamma'(t), \nabla_{v(t)} Y_i)
\end{equation}
\begin{equation*}
- \int_a^t g(v', Y'_i) + g(R(\gamma', v) Y_i', Y_i) + g((\nabla_v Y_i)', Y_i') \, ds \mod \text{Const},
\end{equation*}
for all $t \in [a, b]$, $i = 1, \ldots, r$ and all $v \in T_\gamma \Omega_{\mathcal{P}q}(M)$. Consider the subspace $\mathcal{S}_\gamma$ of $T_\gamma \Omega_{\mathcal{P}q}(M)$ consisting of vector fields that vanish at the endpoints and that take values in the span of the fields $Y_i$, i.e.:
\begin{equation*}
\mathcal{S}_\gamma = \left\{ \sum_{i=1}^r f_i Y_i : f_i : [a, b] \rightarrow H^1, f_i(a) = f_i(b) = 0 \right\} \subset T_\gamma \Omega_{\mathcal{P}q}(M).
\end{equation*}
The central point of the proof is showing that the restriction of $d\mathcal{F}_\gamma$ to $\mathcal{S}_\gamma$ is an isomorphism; for $v = \sum_{i=1}^r f_i Y_i \in \mathcal{S}_\gamma$, (4.9) can be rewritten as:
\begin{equation}
(4.10) \quad d\mathcal{F}_\gamma(v)(t) = B(t) f'(t) + (A(t) + \mathcal{E}(t)) f(t) - \int_a^t (A^* + \mathcal{E}) f' + (C + \mathcal{F}) f \, ds \mod \text{Const},
\end{equation}
for all $t \in [a, b]$, where $f = (f_i)_{i=1}^r : [a, b] \rightarrow \mathbb{R}^r$. The righthand side of (4.10) defines an $L^2([a, b], \mathbb{R}^r^+)$-valued Fredholm operator of index zero in the Hilbert space $H^1_0([a, b], \mathbb{R}^r)$ of $H^1$ maps $f : [a, b] \rightarrow \mathbb{R}^r$ with $f(a) = f(b) = 0$. This is proven by an argument similar to the one used in the proof of Lemma 2.16, except that here we also need the compact inclusion $W^{1,1} \hookrightarrow L^2$. Setting $\varphi = Af + B f'$ then the righthand side of (4.10) vanishes iff $f$ is a solution of (4.5) with $f(a) = f(b) = 0$; it follows that $d\mathcal{F}_\gamma|_{\mathcal{S}_\gamma}$ is injective and therefore an isomorphism onto $L^2([a, b], \mathbb{R}^r^+)$.

We can now prove all the assertions made in the statement of the theorem. Assertion (1) follows from (4.8) and from the fact that $\gamma$ is a regular point for $\mathcal{F}$. Moreover:
\begin{equation}
(4.11) \quad T_\gamma \mathcal{B} = \text{Ker}(d\mathcal{F}_\gamma).
\end{equation}

\footnote{A Killing vector field restricts to a Jacobi field along any geodesic, so that the fields $Y_i$ indeed correspond to solutions of $X$.}
Since \( d\mathcal{F}_\gamma |_{\mathcal{S}_\gamma} \) is an isomorphism, we have:

\[
T_\gamma \mathcal{O}_{pq}(M) = T_\gamma \mathcal{O} \oplus \mathcal{S}_\gamma.
\]

Assertion (2) will follow once we establish that \( dE_\gamma \) vanishes on \( \mathcal{S}_\gamma \). To see this, recalling that \( \gamma \) is a geodesic along the fields \( \gamma_i \), we compute as follows for \( v = \sum_{i=1}^r f_i \gamma_i \in \mathcal{S}_\gamma \):

\[
dE_\gamma(v) = \int_a^b g(\gamma', v') \, dt = \sum_{i=1}^r \int_a^b [f_i g(\gamma', \gamma_i')]' \, dt = 0.
\]

Assume now that \( \gamma \) is a geodesic starting orthogonally at \( \mathcal{P} \). As in the beginning of the section, we choose a parallel trivialization of \( TM \) along \( \gamma \) and consider the Morse–Sturm system with initial data \((X, \ell_0)\) corresponding to (4.2) and to the Jacobi equation along \( \gamma \). As it was observed, the index form \( I \in \text{Bil}_{sym}(\mathcal{H}) \) of \((X, \ell_0)\) corresponds to the second variation \( d^2 E_\gamma \in \text{Bil}_{sym}(T_\gamma \mathcal{O}_{pq}(M)) \); moreover, \( \mathcal{S}_\gamma \) corresponds to the space \( \mathcal{S}_D \) in (2.14). Since \( \gamma \) is a geodesic, integration by parts in (4.9) show that \( v \in T_\gamma \mathcal{O}_{pq}(M) \) is in the kernel of \( d\mathcal{F}_\gamma \) iff \( g(\gamma', \gamma_i') \) is absolutely continuous and

\[
g(\gamma', \gamma_i') = g(\gamma', \gamma_i') + g(\mathcal{R}(\gamma', v) \gamma', \gamma_i'),
\]

for all \( i = 1, \ldots, r \). From (4.11) we conclude that the tangent space \( T_\gamma \mathcal{O} \) corresponds by the chosen parallel trivialization of \( TM \) along \( \gamma \) to the space \( \mathcal{K}_D \) in (2.16) (here \( \mathcal{D} \) is the nondegenerate family of subspaces for \( X \) which has as a frame the maps \( \gamma_i : [a, b] \to \mathbb{R}^n \) corresponding to the fields \( \gamma_i \)). Since \( \gamma \) is a geodesic, the system (4.5) coincides with the reduced symplectic system \( X_{\text{red}} \), so that \( t = b \) is not conjugate for \( X_{\text{red}} \). The remaining assertions in the statement of the theorem now follow immediately from Remark 2.17 and the generalized Index Theorem 3.2.

\[ \square \]

4.2. Geodesics in stationary semi-Riemannian manifolds. In this subsection we apply our theory to obtain Morse relations for geodesics in stationary semi-Riemannian manifolds. For simplicity, we consider the case of geodesics between two fixed points. A Ljusternik–Schnirchelman theory for this situation was developed in [7]; we remark that in this case we will need only the Index Theorem of [13] (Theorem 2.15) and not the generalized Index Theorem 3.2.

Let \((M, g)\) be an \( n \)-dimensional semi-Riemannian manifold with \( g \) a metric tensor of index \( r \). We will call \((M, g)\) stationary if it admits Killing vector fields \( (\gamma_i)_{i=1}^r \) such that \( [\gamma_i, \gamma_j] = 0 \) for all \( i, j = 1, \ldots, r \) and such that \((\gamma_i(m))_{i=1}^r \) is a basis of a subspace \( D_m \) of \( T_m M \) on which \( g \) is negative definite for all \( m \in M \).

Let \( p, q \in M \) be fixed and define \( \mathcal{N}_{pq}(M) \) and \( \mathcal{O}_{pq}(M) \) as in Subsection 4.1 with \( \mathcal{P} = \{p\} \). Since \( g \) is negative definite on \( \mathcal{D} \), the bilinear form \( \mathcal{B}(t) \in \text{Bil}_{sym}(\mathbb{R}^r) \) defined in (4.4) is always negative definite and therefore also \( \mathcal{B}(t) \) is negative definite for all \( t \in [a, b] \) (see (2.25)). It follows from Remark 4.3 that the hypothesis of Theorem 4.2 is satisfied for every curve \( \gamma \in \mathcal{N}_{pq}(M) \) and that the reduced Maslov index of any geodesic is zero. Theorem 4.2 implies the following facts:

- \( \mathcal{N}_{pq}(M) \) is a Hilbert submanifold of \( \mathcal{O}_{pq}(M) \);
- the critical points of \( E|_{\mathcal{N}_{pq}(M)} \) (see (4.3)) are precisely the geodesics on \( M \) from \( p \) to \( q \);
- if \( q \) is not conjugate to \( p \) then all the critical points of \( E|_{\mathcal{N}_{pq}(M)} \) are nondegenerate;
- if \( \gamma \) is a nondegenerate critical point of \( E|_{\mathcal{N}_{pq}(M)} \), then its Morse index is given by:

\[
n_{\neq} \left( d^2 E|_{\mathcal{N}_{pq}(M)}(\gamma) \right) = i_{\text{Maslov}}(\gamma).
\]
Definition 4.4. We say that $E$ is pseudo-coercive on $\mathcal{N}_{pq}(M)$ if given a sequence $(\gamma_n)_{n \geq 1}$ in $\mathcal{N}_{pq}(M)$ with $\sup_{n \geq 1} E(\gamma_n) < +\infty$ then $(\gamma_n)_{n \geq 1}$ admits a uniformly convergent subsequence.

Examples and sufficient conditions for $E$ to be pseudo-coercive on $\mathcal{N}_{pq}(M)$ are given in [7, Appendix B].

Let $\mathcal{D}^\perp$ denote the orthogonal complement of $\mathcal{D}$ with respect to $\mathfrak{g}$ and let $\mathfrak{g}_+$ be the Riemannian metric in $M$ such that $\mathcal{D}$ and $\mathcal{D}^\perp$ are $\mathfrak{g}_+$-orthogonal, $\mathfrak{g}_+$ equals $\mathfrak{g}$ on $\mathcal{D}^\perp$ and $\mathfrak{g}_+$ equals $-\mathfrak{g}$ on $\mathcal{D}$. We define a Riemannian metric on the Hilbert manifold $\mathcal{N}_{pq}(M)$ by:

$$\langle v, w \rangle_{H^1} = \int_a^b g_+(v', w') \, dt, \quad v, w \in T_\gamma \mathcal{N}_{pq}(M), \quad \gamma \in \mathcal{N}_{pq}(M),$$

where the prime denotes the covariant derivative along $\gamma$ with respect to the Levi-Civita connection of $\mathfrak{g}_+$.

Proposition 4.5. If $E$ is pseudo-coercive on $\mathcal{N}_{pq}(M)$ then $E|_{\mathcal{N}_{pq}(M)}$ has complete sublevels, it is bounded from below and it satisfies the Palais–Smale condition. Moreover, if the fields $\mathfrak{g}_i$ are complete then $\mathcal{N}_{pq}(M)$ has the same homotopy type of the loop space of $M$.

Proof. See [7, Proposition 3.3, Theorem 4.1, Proposition 4.3, Proposition 5.2].

Theorem 4.6 (Morse relations for geodesics in stationary semi-Riemannian manifolds).

Let $(M, g)$ be a stationary semi-Riemannian manifold and let $p$ and $q$ in $M$ be two nonconjugate points. For $i \in \mathbb{N}$, set:

$$n_i(p, q) = \text{number of geodesics } \gamma \text{ in } M \text{ from } p \text{ to } q \text{ with } i_{\text{Maslov}}(\gamma) = i.$$ 

Then, under all the assumptions of Proposition 4.5, we have the following equality of formal power series in the variable $\lambda$:

$$\sum_{i=0}^{\pm \infty} n_i(p, q) \lambda^i = \Psi_\lambda(\Omega^{(0)}(M); \mathbb{K}) + (1 + \lambda)Q(\lambda),$$

where $\mathbb{K}$ is an arbitrary field, $\Omega^{(0)}(M)$ is the loop space of $M$, $\Psi_\lambda(\Omega^{(0)}(M); \mathbb{K})$ is its Poincaré polynomial with coefficients in $\mathbb{K}$ and $Q(\lambda)$ is a formal power series in $\lambda$ with coefficients in $\mathbb{N} \cup \{+\infty\}$.

Proof. It follows from Proposition 4.5 using standard Morse theory on Hilbert manifolds (see for instance [2]).

4.3. Geodesics in Gödel-type spaces. In this subsection we will apply our theory to obtain Morse relations for geodesics in semi-Riemannian manifolds of Gödel-type; again, we will consider the case of geodesics between two fixed points. A Ljusternik-Schnirelman theory for this situation (in the Lorentzian case) was developed in [3]; to obtain the Morse relations we will need the new index Theorem 3.2 rather than the index theorem of [13].

Let $(M_0, g^0)$ be a Riemannian manifold and let $\rho : M_0 \to \text{Bil}_{\text{sym}}(\mathbb{R}^r)$ be a smooth map such that $\rho(x)$ is a nondegenerate symmetric bilinear form of index $k$ in $\mathbb{R}^r$ for all $x \in M_0$. Consider the product $M = M_0 \times \mathbb{R}^r$ endowed with the semi-Riemannian metric $\mathfrak{g}$ defined by:

$$\mathfrak{g}(x, u)(((\xi_1, \eta_1), (\xi_2, \eta_2))) = g^0_2(\xi_1, \xi_2) + \rho(x)(\eta_1, \eta_2),$$
for all \( x \in M_0, u \in \mathbb{R}^r \), \( \xi_1, \xi_2 \in T_x M_0 \) and \( \eta_1, \eta_2 \in \mathbb{R}^r \). In analogy with [3, Definition 1.1] we will call \((M, g)\) a semi-Riemannian manifold of Gödel type. In [3] it is considered the case where \( r = 2 \) and \( k = 1 \).

Consider the commuting Killing vector fields \( \mathcal{Y}_i = (0, \frac{\partial}{\partial \rho}) \), \( i = 1, \ldots, r \) in \( M \). An absolutely continuous curve \( \gamma = (\gamma_0, u) : [a, b] \to M \) is a geodesic along the fields \( \mathcal{Y}_i \) iff

\[
\rho(\gamma_0(t)) u'(t) = \text{constant} \in \mathbb{R}^r^* 
\]

for \( t \in [a, b] \). Let \( p = (p_0, u_0), q = (q_0, u_1) \in M \) be fixed and define \( \mathcal{N}_{pq}(M) \) and \( \Omega_{pq}(M) \) as in Subsection 4.1 with \( \mathcal{P} = \{ p \} \). For \( \gamma = (\gamma_0, u) \in \mathcal{N}_{pq}(M) \), the bilinear form \( \mathbb{B}(t) \in \text{Bil}_{\text{sym}}(\mathbb{R}^r) \) corresponding to \( \gamma \) defined in (4.4) is given by \( \mathbb{B}(t) = \rho(\gamma_0(t)) \); the bilinear form \( \mathbb{B}^f(t) \in \text{Bil}_{\text{sym}}(\mathbb{R}^r) \) defined in (2.25) is given by:

\[
\mathbb{B}^f_{\gamma_0}(t) = \int_a^t \rho(\gamma_0(s))^{-1} \, ds \in \text{Bil}_{\text{sym}}(\mathbb{R}^r^*),
\]

where we write \( \mathbb{B}^f_{\gamma_0}(t) \) rather than \( \mathbb{B}^f(t) \) to keep the dependence on \( \gamma_0 \) explicit. By Remark 4.3, the hypothesis of Theorem 4.2 is satisfied for \( \gamma \) iff \( \mathbb{B}^f_{\gamma_0}(b) \) is nondegenerate\(^6\).

Assume now that for every \( \gamma_0 \in \mathcal{N}_{pq}(M_0) \) the bilinear form \( \mathbb{B}^f_{\gamma_0}(b) \) is nondegenerate. Then a curve \( \gamma = (\gamma_0, u) \in \mathcal{N}_{pq}(M) \) is uniquely determined by \( \gamma_0 \); namely, from (4.12), we get:

\[
u(t) = u_0 + \mathbb{B}^f_{\gamma_0}(t)\mathbb{B}^{-1}_{\gamma_0}(b)(u_1 - u_0).
\]

By Theorem 4.2, \( \mathcal{N}_{pq}(M) \) is a Hilbert submanifold of \( \Omega_{pq}(M) \); moreover, we obtain a diffeomorphism \( \phi : \Omega_{pq}(M_0) \to \mathcal{N}_{pq}(M) \) given by \( \phi(\gamma_0) = (\gamma_0, u) \) with \( u \) defined in (4.14), if \( E \) denotes the geodesic action functional of \( M \) (see (4.3)) then the composite map \( E_0 = E \circ \phi : \Omega_{pq}(M_0) \to \mathbb{R} \) is given by:

\[
E_0(\gamma_0) = \frac{1}{2} \int_a^b \mathbb{g}_0(\gamma'_0, \gamma'_0) \, dt + \frac{1}{2} \mathbb{B}^{-1}_{\gamma_0}(b)(u_1 - u_0, u_1 - u_0),
\]

for all \( \gamma_0 \in \Omega_{pq}(M_0) \). Theorem 4.2 implies that the critical points of \( E_0 \) are precisely the curves \( \gamma_0 \in \Omega_{pq}(M_0) \) for which \( \gamma = \phi(\gamma_0) \) is a geodesic; moreover, \( \gamma_0 \) is a nondegenerate critical point of \( E_0 \) iff \( q \) is not conjugate to \( p \) along \( \gamma \). The index of the second variation of \( E_0 \) at a nondegenerate critical point \( \gamma_0 \) is given by:

\[
n - \left( d^2 E_0(\gamma_0) \right) = i_{\text{Maslov}}(\gamma) - i_{\text{red}}^{\text{Maslov}}(\gamma).
\]

By Remark 4.3, the reduced Maslov index \( i_{\text{red}}^{\text{Maslov}}(\gamma) \) can be generically computed by formula (2.26).

The Palais-Smale condition and the boundedness from below for the functional \( E_0 \) are satisfied under certain technical hypothesis on \( g \). In the result below we will assume that the Hilbert manifold \( \Omega_{pq}(M_0) \) is endowed with the Riemannian metric:

\[
\langle \xi_1, \xi_2 \rangle_{H^1} = \int_a^b \mathbb{g}_0(\xi'_1, \xi'_2) \, dt, \quad \xi_1, \xi_2 \in T_{\gamma_0} \Omega_{pq}(M_0), \quad \gamma_0 \in \Omega_{pq}(M_0),
\]

where the prime denotes covariant derivative along \( \gamma_0 \) in the Levi-Civita connection of \((M_0, g_0)\). Recall that if \( g_0 \) is complete then the metric \( \langle \cdot, \cdot \rangle_{H^1} \) is also complete (see [9]).

\(^6\) In the notation of [3] (where it is considered the case \( r = 2, k = 1 \)), the nondegeneracy of (4.13) is the condition "\( |\mathcal{L}(\phi)| > 0 \)."
Proposition 4.7. Assume that \((M_0, g_0)\) is a complete Riemannian manifold, that \(B_{\gamma_0}^I(b)\) is nondegenerate for all \(\gamma_0 \in \Omega_{p_0q_0}(M_0)\) and that:
\[
\sup_{x \in M_0} \|\rho(x)^{-1}\| < +\infty, \quad \sup_{\gamma_0 \in \Omega_{p_0q_0}(M_0)} \|B_{\gamma_0}^I(b)^{-1}\| < +\infty.
\]
Then the functional \(E_0 : \Omega_{p_0q_0}(M_0) \rightarrow \mathbb{R}\) is bounded from below and it satisfies the Palais-Smale condition.

Proof. This is proved in [3, Lemmas 3.5 and 3.7] in the case \(r = 2, k = 1\). The proof of the general case is analogous.

The technical hypotheses in the statement of Proposition 4.7 are satisfied under suitable boundedness assumptions on \(\rho\) (see [3, Remark 1.4] for examples).

Theorem 4.8 (Morse relations for geodesics in Gödel-type manifolds). Let \((M, g)\) be a semi-Riemannian manifold of Gödel-type. Let \(p = (p_0, u_0)\) and \(q = (q_0, u_1)\) in \(M\) be two non conjugate points; for \(i \in \mathbb{N}\), set:
\[
n_i(p, q) = \text{number of geodesics } \gamma \text{ in } M \text{ from } p \text{ to } q \text{ with } i_{\text{Maslov}}(\gamma) - i_{\text{red-Maslov}}(\gamma) = i.
\]
Then, under the assumptions of Proposition 4.7, we have the following equality of formal power series in the variable \(\lambda\):
\[
\sum_{i=0}^{+\infty} n_i(p, q) \lambda^i = Q_3(\Omega^{(0)}(M) ; \mathbb{K}) + (1 + \lambda)Q(\lambda),
\]
where \(\mathbb{K}\) is an arbitrary field, \(\Omega^{(0)}(M)\) is the loop space of \(M\), \(Q_3(\Omega^{(0)}(M) ; \mathbb{K})\) is its Poincaré polynomial with coefficients in \(\mathbb{K}\) and \(Q(\lambda)\) is a formal power series in \(\lambda\) with coefficients in \(\mathbb{N} \cup \{+\infty\}\).

Proof. It follows from Proposition 4.7 by using standard Morse theory on Hilbert manifolds (see for instance [2]) and observing that the loop space of \(M\) has the same homotopy type of \(\Omega_{p_0q_0}(M_0)\). "

REFERENCES

[1] V. I. Arnol’d, Characteristic Class Entering in Quantization Conditions, Funct. Anal. Appl. 1 (1967), 1–13.
[2] R. Bott, Lectures on Morse Theory, Old and New, Bull. AMS, 7 (2), 1982, 331–358.
[3] A. M. Candela, M. Sánchez, Geodesic Connectedness in Gödel Type Space-Times, Diff. Geom. Appl. 12 (2000), 105–120.
[4] J. J. Duistermaat, On the Morse Index in Variational Calculus, Adv. in Math. 21 (1976), 173–195.
[5] F. Giannoni, A. Masiello, P. Piccione, D. Tausk, A Generalized Index Theorem for Morse–Sturm Systems and Applications to semi-Riemannian Geometry, Asian Journal of Mathematics Vol. 5, no. 3 (2001).
[6] F. Giannoni, P. Piccione, An Intrinsic Approach to the Geodesical Connectedness of Stationary Lorentzian Manifolds, Commun. Anal. Geom. 7, n. 1 (1999), p. 157–197.
[7] F. Giannoni, P. Piccione, R. Sampalmieri, On the Geodesic Connectedness for a Class of Semi-Riemannian Manifolds, Journal of Mathematical Analysis and Applications 252 (2000), no. 1, 444–476.
[8] A. D. Helfer, Conjugate Points on Spacelike Geodesics or Pseudo-Self-Adjoint Morse-Sturm-Liouville Systems, Pacific J. Math. 164, n. 2 (1994), 321–340.
[9] W. Klingenberg, Riemannian Geometry, De Gruyter Studies in Mathematics, 1982.
[10] A. Masiello, Variational Methods in Lorentzian Geometry, Pitman Research Notes in Mathematics 309, Longman, London 1994.
[11] F. Mercuri, P. Piccione, D. Tausk, Stability of the Focal and the Geometric Index in semi-Riemannian Geometry via the Maslov Index, Technical Report RT-MAT 99-08, Mathematics Department, University of São Paulo, Brazil, 1999. (LANL math.DG/9905096)
[12] P. Piccione, D. V. Tausk, An Index Theorem for Non Periodic Solutions of Hamiltonian Systems, Proceedings of the London Mathematical Society (3) 83 (2001), 351–389.
[13] P. Piccione, D. V. Tausk, *The Maslov Index and a Generalized Morse Index Theorem for Non Positive Definite Metrics*, Comptes Rendus de l’Académie de Sciences de Paris, vol. 331, 5 (2000), 385–389.

[14] P. Piccione, D. V. Tausk, *On the Distribution of Conjugate Points along semi-Riemannian Geodesics*, preprint 2000, to appear on Communications in Analysis and Geometry. (LANL math.DG/0011038)

[15] P. Piccione, D. V. Tausk, *The Morse Index Theorem in Semi-Riemannian Geometry*, preprint 2000 (LANL math.DG/0011090), to appear in Topology.

[16] J. Robbin, D. Salamon, *The Maslov Index for Paths*, Topology 32, No. 4 (1993), 827–844.

DEPARTAMENTO DE MATEMÁTICA,
UNIVERSIDADE DE SÃO PAULO, BRAZIL

E-mail address: piccione@ime.usp.br, tausk@ime.usp.br

URL: http://www.ime.usp.br/~piccione, http://www.ime.usp.br/~tausk