SOME RESULTS ON LOCAL COHOMOLOGY MODULES

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Abstract. Let $R$ be a commutative Noetherian ring, $a$ an ideal of $R$, and let $M$ be a finitely generated $R$-module. For a non-negative integer $t$, we prove that $H^t_a(M)$ is $a$-cofinite whenever $H^t_a(M)$ is Artinian and $H^i_a(M)$ is $a$-cofinite for all $i < t$. This result, in particular, characterizes the $a$-cofiniteness property of local cohomology modules of certain regular local rings. Also, we show that for a local ring $(R, m)$, $f - \text{depth}(a, M)$ is the least integer $i$ such that $H^i_a(M) \cong H^i_m(M)$. This result in conjunction with the first one, yields some interesting consequences. Finally, we extend the non-vanishing Grothendieck’s Theorem to $a$-cofinite modules.

1. Introduction

Throughout this paper, we assume that $R$ is a commutative Noetherian ring, $a$ an ideal of $R$, and that $M$ is an $R$-module. Let $t$ be a non-negative integer. Grothendieck [4] introduced the local cohomology modules $H^t_a(M)$ of $M$ with respect to $a$. He proved their basic properties. For example, for a finitely generated module $M$, he proved that $H^t_m(M)$ is Artinian for all $t$, whenever $R$ is local with maximal ideal $m$. In particular, it is shown that $\text{Hom}_R(R/m, H^t_m(M))$ is finitely generated. Later Grothendieck asked in [5] whether a similar statement is valid if $m$ is replaced by an arbitrary ideal. Hartshorne gave a counterexample in [6], where he also defined that an $R$-module $M$ (not necessarily finitely generated) is $a$-cofinite, if $\text{Supp}_R(M) \subseteq V(a)$ and $\text{Ext}^t_R(R/a, M)$ is a finitely generated $R$-module for all $t$. He also asked when the local cohomology modules are $a$-cofinite. In this regard, the best known result is that when either $a$ is principal or $R$ is local and $\text{dim } R/a = 1$, then the modules $H^t_a(M)$ are $a$-cofinite. These results are proved in [8] and [3], respectively. Melkersson [15] characterized those Artinian modules which are $a$-cofinite. For a survey of recent developments on cofiniteness properties of local cohomology, see Melkersson’s interesting article [16]. One of the aim of this note is to show that,
for a finitely generated module $M$, the module $H^i_a(M)$ is $a$-cofinite whenever the modules $H^i_a(M)$ are $a$-cofinite for all $i < t$ and $H^t_a(M)$ is Artinian. This result, in particular, characterizes the $a$-cofiniteness property of local cohomology modules of certain regular local rings (see Remark 2.3(ii)). Next, we assume that $R$ is local with maximal ideal $m$. We prove that $f - \text{depth}(a, M)$, which was introduced in [14], is the least integer $i$ such that $H^i_a(M) \not\cong H^i_m(M)$. This result together with our first mentioned result, in turn yields some interesting consequences. Finally, we extend the non-vanishing Grothendieck’s Theorem for $a$-cofinite $R$-modules.

2. The results

The following theorem describes the behaviour of the cofiniteness and Artinian property on local cohomology modules.

**Theorem 2.1.** Let $M$ be finitely generated such that $H^i_a(M)$ is Artinian and that $H^i_a(M)$ is $a$-cofinite for all $i < t$. Then $H^i_a(M)$ is $a$-cofinite.

**Proof.** In view of [16, Proposition 4.1], it is enough to prove that $\text{Hom}_R(R/a, H^j_a(M))$ is of finite length. To prove this, by [18, Theorem 11.38], we consider the Grothendieck spectral sequence

$$E_2^{i,j} = \text{Ext}^i_R(R/a, H^j_a(M)) \Rightarrow \text{Ext}^{i+j}(R/a, M).$$

Since $E_2^{0,t} \cong E_\infty^{0,t}$ for $r$ sufficiently large, $E_\infty^{0,t}$ is isomorphic to a subquotient of $\text{Ext}^i_R(R/a, M)$ and, furthermore, $\ker d_{r-1}^{0,t} \cong E_\infty^{0,t}$ for all $r \geq 3$, where $\ker d_{r-1}^{0,t} = \ker(E_r^{0,t} \longrightarrow E_{r-1}^{r-1,t-r+2})$, we can deduce that $\ker d_{r-1}^{0,t}$ is finitely generated for $r$ sufficiently large. Next, for all $r \geq 3$, we have the exact sequence

$$0 \longrightarrow \ker d_{r-1}^{0,t} \longrightarrow E_r^{0,t} \longrightarrow E_{r-1}^{r-1,t-r+2}.$$

Therefore, since $E_{r-1}^{r-1,t-r+2}$ is a subquotient of $E_2^{r-1,t-r+2}$, our hypothesis give us that $E_{r-1}^{0,t}$ is finitely generated for $r$ sufficiently large. continuing in this fashion, we see that $E_2^{0,t}$ is finitely generated; and hence it is of finite length. □

The following corollary is immediate.

**Corollary 2.2.** Let $M$ be finitely generated. Suppose that the local cohomology module $H^i_a(M)$ is $a$-cofinite for all $i < t$ and that it is Artinian for all $i \geq t$. Then $H^i_a(M)$ is $a$-cofinite for all $i$. 


Remarks 2.3. (i) There is an example in [7, Example 3.4] which shows that $H^t_a(R)$ is not $a$-cofinite for $t = \text{grade}(a)$. However, by the above Theorem, $H^t_a(R)$ is $a$-cofinite, whenever it is Artinian.

(ii) Let $(R, \mathfrak{m})$ be a regular local ring of characteristic $p(>0)$ and of dimension $n$. Suppose that $R/\mathfrak{a}$ is a generalized Cohen-Macaulay local ring of dimension $d(>0)$. Then, by [20, Corollary 1.7] and Theorem 2.1, the local cohomology modules $H^i_a(R)$ are $a$-cofinite if and only if $H^{n-d}_a(R)$ is $a$-cofinite.

Let $R$ be a local ring with maximal ideal $\mathfrak{m}$ and let $M$ be a finitely generated. Following [9], a sequence $x_1, \ldots, x_n$ of elements of $R$ is said to be an $M$-filter regular sequence if, for all $p \in \text{Supp}(M) \setminus \{\mathfrak{m}\}$, the sequence $x_1/1, \ldots, x_n/1$ of elements of $R_p$ is a poor $M_p$-regular sequence. For an ideal $\mathfrak{a}$ of $R$, the $f -$ depth of $\mathfrak{a}$ on $M$ is defined as the length of any maximal $M$-filter regular sequence in $\mathfrak{a}$, denoted by $f -$ depth$(\mathfrak{a}, M)$. Here, when a maximal $M$-filter regular sequence in $\mathfrak{a}$ does not exist, we understand that the length is $\infty$. For some basic applications of these sequences see [2].

Lemma 2.4. Let $(R, \mathfrak{m})$ be a local ring and suppose that $M$ is finitely generated. Then $f -$ depth$(\mathfrak{a}, M) = \min\{i \in \mathbb{N}_0 : \text{Supp}_R H^i_a(M) \not\subseteq \{\mathfrak{m}\}\}$.

Proof. Let $x_1, \ldots, x_n$ be a maximal $M$-filter regular sequence in $\mathfrak{a}$. If there exists $p \in \text{Supp}_R(H^i_a(M)) \setminus \{\mathfrak{m}\}$ for some $0 \leq i \leq n-1$, then $x_1/1, \ldots, x_n/1$ is an $M_p$-regular sequence contained in $\mathfrak{a}R_p$. Hence $H^i_a(M)_p = 0$, which is a contradiction. It therefore follows that

$$f -$ depth$(\mathfrak{a}, M) \leq \min\{i \in \mathbb{N}_0 : \text{Supp}_R H^i_a(M) \not\subseteq \{\mathfrak{m}\}\}.$$ 

Next, by assumption on $x_1, \ldots, x_n$, there exists $p \in \text{Ass}_R(M/(x_1, \ldots, x_n)M) \setminus \{\mathfrak{m}\}$ with $\mathfrak{a} \subseteq p$. Now $p \in \text{Ass}_R(\text{Hom}_R(R/\mathfrak{a}, M/(x_1, \ldots, x_n)M))$; and hence $p \in \text{Ass}_R(\text{Ext}^n_R(R/\mathfrak{a}, M)) \setminus \{\mathfrak{m}\}$. Therefore, by [11, Proposition 1.1], $p \in \text{Supp}(H^n_a(M)) \setminus \{\mathfrak{m}\}$, and this completes the proof. □

Theorem 2.5. (see [9, Theorem 3.10] and [14, Theorem 3.1]) Let $(R, \mathfrak{m})$ be a local ring and suppose that $M$ is finitely generated. Then $f -$ depth$(\mathfrak{a}, M) = \min\{i \in \mathbb{N}_0 : H^i_a(M) \not\cong H^i_m(M)\}$. 
Proof. If $\text{Supp}_R(M/aM) \subseteq \{m\}$, then $\sqrt{a + \text{Ann}(M)} = m$; and hence $H^i_a(M) \cong H^i_m(M)$ for all $i \geq 0$. Therefore $\min\{i \in \mathbb{N}_0 : H^i_a(M) \not\cong H^i_m(M)\} = \infty = f - \text{depth}(a, M)$; and the result follows. So, we may assume that $\text{Supp}_R(M/aM) \not\subseteq \{m\}$. Let $t = f - \text{depth}(a, M)$ and let $x_1, \ldots, x_t$ be an $M$-filter regular sequence in $a$. Then, by [19, Lemma 1.19], $H^i_a(M) \cong H^i_{(x_1, \ldots, x_t)}(M) \cong H^i_m(M)$, for all $i < t$. On the other hand, by Lemma 2.4, the $R$-module $H^t_m(M)$ is not isomorphic with $H^t_m(M)$. It therefore follows, by [9, Theorem 3.10].

Remarks 2.6. Let $M$ be finitely generated. Then
(i) in view of Theorem 2.1 and Theorem 2.5, it is clear that if $(R, m)$ is a local ring, then $H^i_a(M)$ is $a$-cofinite for all $i$ less than $f - \text{depth}(a, M)$;
(ii) it follows immediately from [9, Theorem 3.10] and Theorem 2.5 that if $(R, m)$ is local and $H^i_a(M)$ is Artinian for all $i < t$, then $H^i_a(M) \cong H^i_m(M)$ for all $i < t$.

The following lemma is needed in the proof of the next theorem. Note that if we replace $a$ by the zero ideal in the lemma, then the Grothendieck’s Theorem [4, p.88] immediately follows.

Lemma 2.7. Let $M$ be $a$-cofinite. Then for every maximal ideal $m$ of $R$ and for all $t$, $H^t_m(M)$ is Artinian.

Proof. Since $H^t_m(M)$ is an $a$-torsion module, by [13, Theorem 1.3], it is enough to prove $0 : H^t_m(M)$ a is Artinian. Let $\Phi(-)$ denote the composite functor $\text{Hom}_R(R/a, H^0_m(-))$. We get a spectral sequence arising from the composite functor as:

$$E^{i,j}_2 = \text{Ext}^i_R(R/a, H^j_m(M)) \Longrightarrow (R^{i+j}\Phi)(M).$$

Now, we use induction on $j$ (with $0 \leq j \leq t$) to show that $E^{0,j}_2$ is Artinian. Let $0 \leq j < t$ and suppose that the result has been proved for smaller values of $j$. (Note that the case $j = 0$ was proved in [15, Corollary 1.8].) We can apply [15, Theorem 1.9] and use a similar argument as in the proof of Theorem 2.1, to see that $\ker d^{0,j+1}_{r-1}$ is Artinian for $r$ sufficiently large. On the other hand, by induction, $E^{r-1,j-r+3}_{r-1}$ is
Artinian. It now follows that $E_{2}^{0,j+1}$ is Artinian. This complete the inductive step. In particular $E_{2}^{0,t}$ is Artinian. □

In the next result, we will use the concept of attached prime ideals. For more details in this subject the reader is referred to [10] or the appendix to §6 in [12].

**Theorem 2.8.** Let $(R, \mathfrak{m})$ be a local ring and let $M$ be a module of dimension $d$. If $H_{\mathfrak{m}}^{d}(M)$ is an Artinian module, then if $p$ is any of its attached prime ideals, one has $\dim R/p \geq d$.

**Proof.** From the right exactness of $H_{\mathfrak{m}}^{d}(-)$ on modules of dimension $\leq d$, we get $H_{\mathfrak{m}}^{d}(M/pM) \cong H_{\mathfrak{m}}^{d}(M)/pH_{\mathfrak{m}}^{d}(M)$, which is $\neq 0$, since $p$ is an attached prime ideal of $H_{\mathfrak{m}}^{d}(M)$. But $M/pM$ is a module over $R/p$. Therefore $\dim R/p \geq d$. □

In the following theorem, which establishes the non-vanishing Grothendieck Theorem for $a$-cofinite modules.

**Theorem 2.9.** Let $(R, \mathfrak{m})$ be a local ring and let $M$ be a non-zero $a$-cofinite $R$-module of dimension $n$. Then $H_{\mathfrak{m}}^{n}(M) \neq 0$.

**Proof.** Firstly note that, in view of the hypotheses, $0 :_{M} a$ is a finitely generated $R$-module of dimension $n$. Now, we prove the theorem by induction on $n(\geq 0)$. If $n = 0$, then $0 :_{M} a$ is Artinian; and hence, by [13, Theorem 1.3], $M$ is Artinian. Therefore $H_{\mathfrak{m}}^{0}(M) = M \neq 0$.

Suppose, inductively, that $n \geq 1$ and the result has been proved for $n - 1$. We may assume that $M$ is $\mathfrak{m}$-torsion free. Also, by [15, Corollary 1.4], we may assume that $\text{Ass}(M)$ is a finite set. Then, there exists a non-zero divisor $x \in \mathfrak{m}$ on $M$. Suppose the contrary that $H_{\mathfrak{m}}^{n}(M) = 0$. Then, for any such $x$, we can consider the exact sequence $0 \rightarrow M \xrightarrow{x} M \rightarrow M/\mathfrak{m}M \rightarrow 0$ to see that $H_{\mathfrak{m}}^{n-1}(M)/xH_{\mathfrak{m}}^{n-1}(M) \cong H_{\mathfrak{m}}^{n-1}(M/\mathfrak{m}M)$,

$n - 1 = \dim(0 :_{M} a)/x(0 :_{M} a) \leq \dim(0 :_{M/\mathfrak{m}M} a) = \dim M/\mathfrak{m}M \leq n - 1$, and that, by [15, Remark(a)], $M/\mathfrak{m}M$ is $a$-cofinite. Therefore, by induction hypothesis, $H_{\mathfrak{m}}^{n-1}(M)/xH_{\mathfrak{m}}^{n-1}(M) \neq 0$. Note that, by Lemma 2.7, $H_{\mathfrak{m}}^{n-1}(M)$ is Artinian. If $\mathfrak{m} \notin \text{Att} H_{\mathfrak{m}}^{n-1}(M)$, then, for any $y \in \mathfrak{m} \setminus \bigcup_{p \in \text{Att} H_{\mathfrak{m}}^{n-1}(M)} p \bigcup_{q \in \text{Ass}(M)} q$,
we have $H^{n-1}_m(M) = yH^{n-1}_m(M)$, which is a contradiction. Thus $m \in \text{Att } H^{n-1}_m(M)$.

Let $\text{Att } H^{n-1}_m(M) = \{p_1, \ldots, p_t, m\}$ and let $z \in m \setminus \bigcup_{i=1}^t p_i \bigcup_{q \in \text{Ass}(M)} q$. Then, by the above argument, we have $H^{n-1}_m(M)/zH^{n-1}_m(M) \cong H^{n-1}_m(M/zM)$. Hence, by [17, Proposition 5.2], $\text{Att } H^{n-1}_m(M/zM) = \text{Supp}(R/(zR)) \cap \text{Att } H^{n-1}_m(M) = \{m\}$. Therefore, by [1, Corollary 7.2.12], $H^{n-1}_m(M/zM)$ has finite length. If we show that $H^{n-1}_m(M/zM) = 0$, then we achieved at the required contradiction. To this end, first let $n = 1$. Then we have the exact sequence

$$0 \to H^0_m(M) \stackrel{\cdot z}{\longrightarrow} H^0_m(M) \to H^0_m(M/zM) \to H^1_m(M).$$

By our hypothesis $H^0_m(M) = 0 = H^1_m(M)$; and so $H^0_m(M/zM) = 0$. Now, we assume that $n > 1$. Then, Theorem 2.8 implies that attached prime ideals of $H^{n-1}_m(M/zM)$ is empty; and so $H^{n-1}_m(M/zM) = 0$. □

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