Entropy and universality of Cardy-Verlinde formula in dark energy universe

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Abstract

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We study the entropy of a FRW universe filled with dark energy (cosmological constant, quintessence or phantom). For general or time-dependent equation of state $p = wp$ the entropy is expressed in terms of energy, Casimir energy, and $w$. The correspondent expression reminds one about 2d CFT entropy only for conformal matter. At the same time, the cosmological Cardy-Verlinde formula relating three typical FRW universe entropies remains to be universal for any type of matter. The same conclusions hold in modified gravity which represents gravitational alternative for dark energy and which contains terms growing at low curvature. It is interesting that BHs in modified gravity are more entropic than in Einstein gravity. Finally, some hydrodynamical examples testing new shear viscosity bound, which is expected to be the consequence of the holographic entropy bound, are presented for the early universe in the plasma era and for the Kasner metric. It seems that the Kasner metric provides a counterexample to the new shear viscosity bound.

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1 Introduction

There is growing evidence from high redshift surveys of supernovae and from WMAP data analysis that the current universe experiences a phase of cosmic speed-up. The accepted explanation for this behavior is the dominance of some dark energy contributing up to 70 percent of the critical energy density. Nevertheless, it remains unclear what this dark energy is: cosmological constant, quintessence, phantom, effective gravitational contribution or something else. In the absence of a completely consistent dark energy model a good strategy would be to explore the general properties of FRW universe with dark energy described as matter with a general (negative or time-dependent) equation of state. Surprisingly, quite a lot of information about the present and the future of such a universe may be obtained.

In particular, a number of issues related to entropy and energy of the universe and their bounds may be understood. For instance, it seems clear that the FRW equations are not so simple as they look as they may encode some quantum field theory structure via the holographic principle. In a very interesting work [1], a deep relation between the FRW equations, conformal field theory entropy, and holography was established. First, this work proposed a holographical bound on the subextensive entropy associated with Casimir energy. Second, it showed that the FRW universe entropy may be presented as a kind of Cardy entropy in conformal field theory [2]. The corresponding expression is called the Cardy-Verlinde(CV) formula. Moreover, one more relation- the universal cosmological CV formula - may be obtained by rewriting the FRW equations in terms of three holographic entropies (or energies). There is currently much activity in the study of various aspects of the CV formula (see [4] and references therein): its holographic origin, the relation to the brane-world approach, and the description via AdS duals within the AdS/CFT set-up. It is also remarkable that the CV formula should be generalized in the case of a general (constant) equation of state [5], while the cosmological CV formula remains valid.

The purpose of the present work is to discuss the entropy, Cardy-Verlinde-like formulas, related consequences of holographic entropy bound for (mainly) FRW universe filled with dark energy where the effective equation of state is negative or even time-dependent. In a similar fashion, these questions are studied for modified gravity which represents a gravitational alternative for dark energy. It is expected that a better understanding of this topic may
shed some light on questions about the origin of holographic relations in the early universe as well as in the current accelerating universe, and on the origin of dark energy itself.

The paper is organized as follows. In the next section we discuss the thermodynamic system which corresponds to FRW universe with a general equation of state which can be negative (cosmological constant, phantoms or quintessence) or time-dependent. The explicit expression for entropy of such FRW universe is found and is presented as a CV formula (in terms of energy and Casimir energy). It is remarkable that for a general equation of state, such a formula does not have simple form reminding about 2d CFT entropy. Another form of (cosmological) CV formula (which is expected to have the holographic origin and which relates three different typical entropies of FRW universe) is found to be universal, like 2d CFT entropy. The entropy bounds (including Bekenstein bound) for dark energy universe and their dependence from critical radius are briefly mentioned.

Section three is devoted to the study of the same questions for modified gravity which contains terms growing with the decrease of the curvature. Such a theory describes the current accelerating universe and represents the gravitational alternative for dark energy. It is shown that the cosmological CV formula is universal, since it remains the same in both frames (Jordan or Einstein) used to describe such a gravity. In section four the black hole thermodynamics for modified gravity is briefly discussed. It is shown that for SAdS black holes the entropy is related to the area, with a numerical coefficient different from the Einstein gravity case. The relation of such an entropy to the CV formula is briefly mentioned. Section five is related more to hydrodynamics and the early universe. Namely, it was recently suggested some universal lower bound on the relation between shear viscosity and entropy density. It is expected that such bound directly follows from Bekenstein entropy bound. As shear viscosity is typical for anisotropic universe we test the bound for hydrodynamics or Kasner universe. It seems that anisotropic universe may give some counterexample for the bound. Finally, summary and outlook are given in the last section.
2 Thermodynamics of dark energy universe: energy and entropy

Let us start from the simple thermodynamic system with the free energy $F = F(V,T)$, where $V$ is volume of the system and $T$ is temperature. The pressure $p$, the energy density $\rho$, and the entropy $S$ are given by

$$p = -\frac{\partial F}{\partial V}, \quad \rho = \frac{1}{V} \left( F - T \frac{\partial F}{\partial T} \right), \quad S = -\frac{\partial F}{\partial T}. \tag{1}$$

The first law of the thermodynamics holds automatically:

$$TdS = dE + pdV. \tag{2}$$

Here the total energy $E$ is given by $E = \rho V$. The Boltzmann constant $k_B$ is chosen to be unity ($k_B = 1$). The free energy may be chosen in the following form:

$$F = -f_0 T^\alpha V^\beta, \tag{3}$$

with some constants $f_0$, $\alpha$, and $\beta$. As a result

$$p = \beta f_0 T^\alpha V^{\beta-1}, \quad \rho = (\alpha - 1) f_0 T^\alpha V^{\beta-1}, \quad S = \alpha f_0 T^{\alpha-1} V^\beta. \tag{4}$$

Defining a parameter $w$ by $p = w\rho$ (equation of state), we obtain

$$w = \frac{\beta}{\alpha - 1}. \tag{5}$$

The case of interest is the negative equation of state, which is typical for the current, dark energy, universe. The free energy can be rewritten as

$$F = -f_0 T \left( T^{\frac{1}{w}} V \right)^\beta, \tag{6}$$

which tells that the general free energy of the matter with $w$ has the following form

$$F_w(T,V) = T \hat{F} \left( T^{\frac{1}{w}} V \right). \tag{7}$$

Here $\hat{F}(x)$ is a function depending on the matter.
For $\alpha = 4$ and $\beta = 1$, the classical radiation in 4-dimensional spacetime is restored:

\[ p = f_0 T^4, \quad \rho = 3 f_0 T^4. \]  

(8)

In order to obtain ideal gas, the free energy should look as

\[ F = f_0 \left( T^\alpha V^\beta - T \right). \]  

(9)

It is interesting that the last term does not contribute to $p$ $(\rho)$ but does contribute to the entropy $S$. In the limit that $\alpha \to 1$ and $\beta \to 0$ with finite $c_1 = f_0 (\alpha - 1)$ and $c_2 = \beta f_0$, we obtain

\[ F = T \ln \left( T^{c_1} V^{c_2} \right), \quad p = c_2 T V, \quad \rho = c_1 T V. \]  

(10)

Then $c_2$ can be identified with the number $N$ of the molecules in the gas $c_2 = N$ and $c_1 = \frac{3}{2} N$ for the monoatomic molecule. One can also obtain dust by choosing $\beta = 0$:

\[ p = 0, \quad \rho = (\alpha - 1) f_0 T^\alpha V^{-1}. \]  

(11)

We may consider the case that the entropy is constant $S = S_0$ which is typical for adiabatically expanding universe where first law of thermodynamics holds. From (4) it follows

\[ T = (\alpha f_0)^{- \frac{1}{\alpha - 1}} S_0^{\frac{1}{\alpha - 1}} V^{-w}. \]  

(12)

Here $w$ is given in (5).

Let us apply the above considerations to the $(n + 1)$-dimensional FRW metric of the form:

\[ ds^2 = g_{\mu \nu} dx^\mu dx^\nu = -d\tau^2 + a^2(\tau) \gamma_{ij} dx^i dx^j, \]  

(13)

where the $n$-dimensional metric $\gamma_{ij}$ is parametrized by $k = -1, 0, 1$. In the following, mainly the $k = 1$ case is considered. Since $V = a^n \int d^n x \sqrt{\gamma}$, the temperature of the universe is

\[ T \propto a^{-nw}. \]  

(14)

By combining (4) and (12), the total energy $E = \rho V$ is given by

\[ E = (\alpha - 1) a^{-\frac{\alpha}{\alpha - 1}} f_0^{-\frac{n}{\alpha - 1}} S_0^{\frac{n}{\alpha - 1}} V^{-w} \propto a^{-nw}. \]  

(15)
The $a$-dependence in (14) and (15) reproduces the corresponding results in [5].

Rescaling the entropy and the volume as $S_0 \to \lambda S_0$ and $V \to \lambda V$, from the expression (15), we obtain

$$E \to \lambda^{\frac{1}{n+1}+1-w} E .$$

(16)

If the energy is extensive, $E \to \lambda E$. For the extensive part of the energy it follows

$$\alpha = 1 + \frac{1}{w} , \quad \beta = 1 .$$

(17)

In order to obtain the expression of $\beta$ in (17), Eq.(5) should be used.

The following free energy for general equation of state may be considered:

$$F = -f_0 T^{1+\frac{1}{w}} V \left(1 + f_1 T^{-\frac{2}{n}} V^{-\frac{2}{n}}\right) .$$

(18)

If there is no the second term, the first term gives the extensive energy. Note that $p = w \rho$ even if the second term is included. As a result, the energy and entropy of the thermal universe follow

$$E = \frac{f_0}{w} T^{1+\frac{1}{w}} V \left(1 + \left(1 - \frac{2}{n}\right) f_1 T^{-\frac{2}{n}} V^{-\frac{2}{n}}\right) ,$$

$$S = f_0 T^{\frac{1}{w}} V \left(1 + \frac{1}{w}\right) + \left(1 + \frac{1}{w} - \frac{2}{nw}\right) f_1 T^{-\frac{2}{n}} V^{-\frac{2}{n}} .$$

(19)

As clear from (1) and (3), the entropy becomes negative (unphysical case) if $f_0$ or $\alpha$ are chosen to be negative. If the terms containing $f_1$ can be neglected, as clear from the Eqs. (19), the entropy $S$ becomes negative if

1. $f_0 < 0$ and $w < -1$ : in this case, the energy $E$ is positive.
2. $f_0 > 0$ and $0 > w > -1$ : in this case, $E$ also becomes negative.

We should also note the energy (if we neglect the terms containing $f_1$) is positive (negative) if $f_0$ is positive (negative). The case that the entropy is negative would be unphysical and should be excluded. Then the case that $w < -1$ and the energy $E$ is positive, and the case that $0 > w > -1$ and the energy $E$ is negative, should be excluded.
The sub-extensive part of the energy $E_C$, which is called the Casimir energy, is given by

$$E_C = n (E + pV - TS) = -nV^2 \frac{\partial}{\partial V} \left( \frac{F}{V} \right) = -2f_0 f_1 T^{1+\frac{\hat{w}}{2}} - \frac{2}{n} V^{1-\frac{\hat{w}}{2}} . \quad (20)$$

The extensive part of the energy $E_E$ has the following form:

$$E_E = E - \frac{1}{2} E_C = \frac{f_0}{w} T^{1+\frac{\hat{w}}{2}} \left( 1 + \left( 1 - \frac{2}{n} + \frac{1}{w} \right) f_1 T^{-\frac{\hat{w}}{2}} V^{-\frac{\hat{w}}{2}} \right) . \quad (21)$$

From the last expression in (19), we obtain

$$T \sim S^w \left( 1 + \left( 1 + \frac{1}{w} \right)^{-1} \left( 1 - \frac{2}{nw} + \frac{1}{w} \right) f_1 T^{-\frac{\hat{w}}{2}} V^{-\frac{\hat{w}}{2}} \right) , \quad (22)$$

and

$$E_E \sim S^{w+1} V^{-w} + \mathcal{O} \left( f_1^2 \right) , \quad E_C \sim S^{w+1-\frac{\hat{w}}{2}} V^{-w} + \mathcal{O} \left( f_1^2 \right) , \quad (23)$$

which reproduce the behaviors in [5]. When the size of the universe is large, the second terms in $S$ (19) and in $E_E$ (21) are sub-dominant and we obtain

$$S \sim f_0 T^{\frac{\hat{w}}{2}} V \left( 1 + \frac{1}{w} \right) , \quad E_E \sim \frac{f_0}{w} T^{1+\frac{\hat{w}}{2}} V . \quad (24)$$

Then combining (20) and (24), for the FRW metric (13) with $k = 1$, one gets

$$S \sim f_0 \left( 1 + \frac{1}{w} \right) \left( -\frac{2f_0^2 f_1}{n} \right)^{-\frac{n}{2(w+1)n-1}} V^{\frac{nw}{(w+1)n-1}} \left[ a^{nw} \sqrt{E_E E_C} \right]^{\frac{n}{(w+1)n-1}}$$

$$A \equiv f_0 \left( 1 + \frac{1}{w} \right) \left( -\frac{4f_0^2 f_1}{n} \right)^{-\frac{n}{2(w+1)n-1}} V^{\frac{nw}{(w+1)n-1}} . \quad (25)$$

Here $V_0 = \int d^n x \sqrt{\gamma}$. Eq.(25) reproduces Eq.(20) in [5] if we identify

$$A = \left( \frac{2\pi}{\sqrt{\alpha \beta}} \right)^{\frac{n}{(w+1)n-1}} . \quad (26)$$
This expression represents one of the forms of Cardy-Verlinde formula [1] for general equation of state.

As there are astrophysical indications that dark energy currently dominates at the thermal universe our main interest is related with the case where \( w \) can be negative. One usually denotes the matter as quintessence if \(-\frac{1}{3} > w > -1\) and as phantom [6] if \( w < -1 \). When \( w = -1 \), the situation corresponds to the cosmological constant. First we should note that entropy \( S \) (25) becomes singular at \( w = -1 + \frac{1}{n} \), which occurs since the product \( ECEE \) becomes independent of the temperature. If the entropy \( S \) is conserved, Eq.(25) indicates that the product \( ECEE \) increases if the size of the universe \( a \) increases when \( w \) is negative. The entropy may be conserved but we may consider the variation of the entropy as a change of the initial condition.

When \( 0 > w > -1 + \frac{1}{n} \), if we keep \( ECEE \) to be constant, Eq.(25) shows that \( S \) decreases if \( a \) increases. When \( w < -1 + \frac{1}{n} \), \( S \) increases if \( a \) increases but \( S \) decreases if \( ECEE \) increases. As is seen from (19), the specific heat \( \frac{dE}{dT} \) with fixed volume (\( V \) is a constant) becomes negative, when \( 0 > w > -1 \). For the phantom (\( w < -1 \)), the specific heat is positive and for the cosmological constant, the specific heat vanishes.

For the current realistic universe the case that there are many kinds of matter (with dark energy dominance ) is typical. In such a case the free energy may be written as sum over various contributions

\[
F = -\sum_i f_{i0} T^{1 + \frac{1}{w_i}} V \left( 1 + f_{i1} T^{-\frac{2}{nw_i}} V^{-\frac{2}{n}} \right).
\]

Then one gets

\[
E = \sum_i \frac{f_{i0}}{w_i} T^{1 + \frac{1}{w_i}} V \left( 1 + \left( 1 - \frac{2}{n} \right) f_{i1} T^{-\frac{2}{nw_i}} V^{-\frac{2}{n}} \right),
\]

\[
S = \sum_i f_{i0} T^{-\frac{1}{w_i}} V \left( \left( 1 + \frac{1}{w_i} \right) + \left( 1 + 2 w_i - \frac{2}{nw_i} \right) f_{i1} T^{-\frac{2}{nw_i}} V^{-\frac{2}{n}} \right),
\]

\[
E_C = -2 \sum_i f_{i0} f_{i1} T^{1 + \frac{1}{w_i} - \frac{2}{nw_i}} V^{1 - \frac{2}{n}},
\]

\[
E_E = \sum_i \frac{f_{i0}}{w_i} T^{1 + \frac{1}{w_i}} V \left( 1 + \left( 1 - \frac{2}{n} + w_i \right) f_{i1} T^{-\frac{2}{nw_i}} V^{-\frac{2}{n}} \right).
\]
Thus, in case that there are several types of matter, we cannot obtain a simple relation (25). Nevertheless, an inequality follows

\[ S \geq S_i \sim A_i \left[ a^{n w_i} \sqrt{(2 E_i - E_{C_i}) E_{C_i}} \right]^{(w_i+1)/n-1} . \]  

Here

\[ A_i \equiv f_{0i} \left( 1 + \frac{1}{w_i} \right) \left( -\frac{4 f_{0i}^2 f_{1i}}{n} \right)^{-2((w_i+1)/n-1)} V_0^{\frac{w_i}{(w_i+1)/n-1}} . \]  

As \( S = \sum_i S_i \) and \( S_i \geq 0 \), the inequality (29) holds for arbitrary \( i \). With the entropy \( S \) (28), at high temperature the matter with small and positive \( w_i \) dominates. We now denote the quantities related with the matter with smallest but positive \( w_i \) by the index “min”. At high temperature, instead of (25), one gets

\[ S \sim A_{\text{min}} \left[ a^{n w_{\text{min}}} \sqrt{(2 E_{\text{min}} - E_{C_{\text{min}}}) E_{C_{\text{min}}}} \right]^{(w_{\text{min}}+1)/n-1} . \]  

On the other hand, at low temperature as in current universe, if all the \( w_i \)'s are positive, the matter with large \( w_i \) dominates. We now denote the quantities related with the matter for largest \( w_i \) by the index “max”. Then at low temperature

\[ S \sim A_{\text{max}} \left[ a^{n w_{\text{max}}} \sqrt{(2 E_{\text{max}} - E_{C_{\text{max}}}) E_{C_{\text{max}}}} \right]^{(w_{\text{max}}+1)/n-1} . \]  

If there is a dark energy (say, phantom) with negative \( w \), such a matter dominates at low temperature

\[ S \sim A_p \left[ a^{n w_p} \sqrt{(2 E_p - E_{pC}) E_{pC}} \right]^{(w_p+1)/n-1} . \]  

Here we have denoted the quantities related with the phantom matter by the index “\( p \)”. Note that for negative equation of state the above universe entropy formula does not remind one about the well-known Cardy formula in CFT. Since the entropy is given by

\[ S \sim f_{p0} T^{\frac{1}{2p}} V \left( 1 + \frac{1}{w_p} \right) + \left( 1 + \frac{1}{w_p} - \frac{2}{nw_p} \right) f_{p1} \left( T^{\frac{4}{2p}} V \right)^{-\frac{2}{n}} \]  

(34)
for conserved entropy, \( T \frac{1}{w_p} V \) is a constant:

\[
T \frac{1}{w_p} V = C .
\]

Then the energy \( E \) can be rewritten as

\[
E \sim f_{p0} CT \left( 1 + \left( 1 - \frac{2}{n} \right) f_{p1} C^{-\frac{2}{n}} \right),
\]

\[
= f_{p0} C^{w_p + 1} V^{-w_p} \left( 1 + \left( 1 - \frac{2}{n} \right) f_{p1} C^{-\frac{2}{n}} \right),
\]

\[
= f_{p0} C^{w_p + 1} V_0^{-w_p} a^{-nw_{wp}} \left( 1 + \left( 1 - \frac{2}{n} \right) f_{p1} C^{-\frac{2}{n}} \right).
\]

Thus, the energy is linear with the temperature. In the last line, we have considered the FRW metric (13). Generally in the FRW metric, if we have the relation \( p = w \rho \), we find \( \rho \propto a^{-n(1+w)} \) (energy conservation) and \( E = \rho V \propto a^{-nw} \), which is consistent with (36). If there is only dark matter with \( w < 0 \) in the universe, the relation (35) is valid even at high temperature. When the universe expands and the radius grows, the temperature grows too and also the energy \( E \) and the energy density \( \rho \) behave as \( E \sim a^{-nw_{wp}} \) and \( \rho \sim a^{-n(w_{wp}+1)} \). As a result the density becomes large and might generate some future singularities (like Big Rip).

As an example the system with dust and quintessence or phantom, where \( w \) is negative, may be considered. If we assume that there is no internal structure in the dust, the energy of the dust does not depend on the temperature and the free energy, corresponding to (11), becomes a constant: \( F = E_{D0} \). Then the total free energy can be assumed to be given by

\[
F = E_{D0} - f_{p0} T^{1 + \frac{1}{w_p}} \left( 1 + f_{p1} T^{-\frac{2}{nw_{wp}}} V^{-\frac{2}{n}} \right). \tag{37}
\]

Thus, one obtains

\[
E = E_{D0} + f_{p0} T^{1 + \frac{1}{w_p}} \left( 1 + \left( 1 - \frac{2}{n} \right) f_{p1} \left( T^{-\frac{1}{w_p}} V \right)^{-\frac{2}{n}} \right),
\]

\[
S = f_{p0} T^{\frac{1}{w_p}} \left( \left( 1 + \frac{1}{w_p} \right) + \left( 1 + \frac{1}{w_p} - \frac{2}{nw_{wp}} \right) f_{p1} \left( T^{\frac{1}{w_p}} V \right)^{-\frac{2}{n}} \right). \tag{38}
\]
Note that dust does not contribute to the entropy. The energy of the dust is not extensive nor sub-extensive. The extensive and sub-extensive (Casimir) parts of the energy of the phantom or quintessence are given by

\[
E_{pC} = -2f_{p0} f_{p1} T^{1+\frac{2}{n_{wp}}} a^{-\frac{2}{n_{wp}}} V^{1+\frac{2}{n}},
\]
\[
E_{pE} = \frac{f_{p0}}{w_p} T^{1+\frac{2}{w_p}} V \left(1 + \left(1 - \frac{2}{n} + w_p\right) f_{p1} T^{-\frac{2}{w_p}} V^{-\frac{2}{n}}\right).
\]

If we assume the entropy \( S \) is conserved, from the expression of \( S \) (38), we find \( T^\frac{1}{w_p} V \) is a constant:

\[
T^\frac{1}{w_p} V = C.
\]

Then the energy \( E \) (38) can be rewritten as

\[
E = E_{D0} + \frac{f_{p0}}{w_p} C T \left(1 + \left(1 - \frac{2}{n}\right) f_{p1} C^{-\frac{2}{n}}\right),
\]
\[
= E_{D0} + \frac{f_{p0}}{w_p} C^{w_p+1} V^{-w_p} \left(1 + \left(1 - \frac{2}{n}\right) f_{p1} C^{-\frac{2}{n}}\right),
\]
\[
= E_{D0} + \frac{f_{p0}}{w_p} C^{w_p+1} V_0^{-w_p} a^{-nw_p} \left(1 + \left(1 - \frac{2}{n}\right) f_{p1} C^{-\frac{2}{n}}\right).
\]

Then energy is again linear in the temperature. In the last line, we have considered the FRW metric (13). Generally in the FRW metric, if we have the relation \( p = w \rho \), we find \( \rho \propto a^{n(1+w)} \) and \( E = \rho V \propto a^{-nw} \), which is consistent with the last expression for the phantom or quintessence in (41).

Taking into account the recent cosmological considerations of variations of fundamental constants, one may start from the case that \( w_p \) depends on the time \( t \). Of course, this may be negative (or sign-changing) function. The energy conservation condition looks like

\[
0 = \dot{\rho}_p + \frac{\dot{a}}{a} (\rho_p + p_p),
\]
by assuming \( \rho_p = w_p(t)p_p \). The following expression may be found

\[
\rho_p = a^{-n(1+w_p(t))} e^{\int t \dot{w}_p(t) \ln a(t) dt}.
\]

The energy in such a universe is

\[
E_p = \rho_p V = a^{-nw_p(t)} e^{\int t \dot{w}_p(t) \ln a(t) dt} V_0.
\]

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If the spacetime expansion is adiabatic and thermodynamical quantities can be defined, Eqs. (1) are valid. Thus, if we define a free energy as in the phantom part of Eq.(37), we can obtain the entropy and energy as in (38) and the extensive and sub-extensive parts of the energy as in (39). Then if we define a variable \( \xi \) by

\[
T = V^{-w_p(t)} \xi ,
\]  

(45)
extracting the phantom part \( E_p \) from the expression of \( E \) in (38), we obtain

\[
E_p = \frac{f_p0}{w_p(t)} a^{-nw_p(t)} V_0^{-w_p(t)} \xi^{1 + \frac{1}{w_p(t)}} \left( 1 + \left( 1 - \frac{2}{n} \right) f_p1 \xi^{-\frac{2}{nw_p(t)}} \right) .
\]

(46)

By comparing (44) with (46), one finds

\[
\xi = \left( \frac{w_p(t)}{f_p0} \right)^{\frac{w_p(t)}{w_p(t)+1}} V_0^{w_p(t)} e^{-\frac{w_p(t)}{f_p0}} \int t \dot{w}_p(t') \ln a(t')dt' 
\times \left\{ 1 - f_p1 \frac{1 - \frac{2}{n} \xi}{1 + \frac{1}{w_p(t)}} \left( w_p(t) \right)^{-\frac{2}{nw_p(t)+1}} V_0^{-\frac{2}{n} e^{-\frac{2}{w_p(t)+1}} \int t \dot{w}_p(t') \ln a(t')dt'} \right\} 
+ O \left( f_p^2 \right) .
\]

(47)

From Eqs.(38) and (39), the expressions of the entropy \( S_p \), the extensive part of the energy \( E_{pE} \) and the Casimir energy \( E_{pC} \) may be evaluated:

\[
S_p = f_p0 \frac{1}{w_p(t)} \left( \left( 1 + \frac{1}{w_p(t)} \right) + \left( 1 + \frac{1}{w_p(t)} - \frac{2}{nw_p(t)} \right) f_p1 \xi^{-\frac{2}{nw_p(t)}} \right) ,
\]

\[
E_{pE} = f_p0 \frac{a^{-nw_p(t)} V_0^{-w_p(t)} \xi^{1 + \frac{1}{w_p(t)}} \left( 1 + \left( 1 - \frac{2}{n} + w_p(t) \right) f_p1 \xi^{-\frac{2}{nw_p(t)}} \right) ,
\]

\[
E_{pC} = -2f_p0 f_p1 a^{-nw_p(t)} V_0^{-w_p(t)} \xi^{1 + \frac{1}{w_p(t)}} - \frac{2}{nw_p(t)} .
\]

(48)

As \( w_p(t) \) and \( \xi \) are time-dependent, the entropy is not constant and not conserved. Nevertheless, from (48) the Cardy-Verlinde [1] like formula ( a la Youm [5]) (33) is still valid:

\[
S_p \sim A_p \left[ a^{nw_p} \sqrt{(2E_p - E_{pC}) E_{pC}} \right]^{\frac{n}{nw_p+1}n-1} .
\]

(49)
We should note, however, since

\[ A_p = f_{p0} \left( 1 + \frac{1}{w_p(t)} \right) \left( -\frac{4f^2_{p0}f_{p1}}{n} \right)^{-\frac{n}{2(\omega_p(t)+1)\omega(n-1)}} V_0^{\frac{\omega n}{\omega_p(t)+1}\omega(n-1)} \]  

(50)

and \( w_p(t) \) depend on time, \( A_p \) is not a constant but a function of the time \( t \). Thus, the entropy of the expanding universe with (negative) time-dependent equation of state is found.

Now, the FRW equations for the universe filled with matter with pressure \( p \) and energy density \( \rho \) are given by

\[
H^2 = \frac{16\pi G}{n(n-1)} \rho - \frac{k}{a^2}, \quad \dot{H} = -\frac{8\pi G}{n-1} (\rho + p) + \frac{k}{a^2}.
\]  

(51)

As in [1], if we define the Hubble entropy \( S_H \), the Bekenstein-Hawking energy \( E_{BH} \), and the Hawking temperature \( T_H \) by

\[
S_H \equiv \frac{(n-1)HV}{4G}, \quad E_{BH} \equiv \frac{n(n-1)V}{8\pi Ga^2}, \quad T_H \equiv -\frac{\dot{H}}{2\pi H},
\]  

(52)

the FRW equations can be rewritten in universal form as

\[
S_H = \frac{2\pi a}{n} \sqrt{E_{BH}(2E - kE_{BH})}, \\
kE_{BH} = n (E + pV - T_H S_H),
\]  

(53)

Furthermore with the Bekenstein entropy \( S_B \) and the Bekenstein-Hawking entropy \( S_{BH} \) as

\[
S_B \equiv \frac{2\pi a}{n} E, \quad S_{BH} \equiv \frac{(n-1)V}{4Ga},
\]  

(54)

we obtain well-known relation between entropies

\[
S_H^2 = 2S_B S_{BH} - kS_{BH}^2.
\]  

(55)

In case of \( k = 1 \), Eq.(55) can be rewritten as

\[
S_H^2 + (S_B - S_{BH})^2 = S_B^2.
\]  

(56)

Then we find \( S_H \leq S_B \). For the system with limited self-gravity, there occurs the Bekenstein bound [3]:

\[
S \leq S_B.
\]  

(57)
This bound is useful for the case that the system has relatively low energy or small volume. Then Bekenstein entropy $S_B$ scales as $S_B \to \lambda^{1+\frac{1}{n}} S_B$ under the scale transformation $V \to \lambda V$ and $E \to \lambda E$ [1].

Eq.(53) has a form similar to the second equation in (25) with $w = \frac{1}{n}$ and this equation is called the cosmological Cardy-Verlinde formula. The second equation in (52) has a form similar to (20) and $E_{BH}$ may correspond to the Casimir energy $E_C$. In [1], the following cosmological bound has been proposed:

$$E_C \leq E_{BH}.$$  \hspace{1cm} (58)

As seen from the definition of $E_{BH}$ in (52), we find $E_{BH} \sim a^{n-2}$. If we consider phantom or quintessence as a matter field, as seen from the last expression in (41), the behavior of the Casimir energy is given by $E_C \sim a^{-nw}$. Then if

$$w < -1 + \frac{2}{n}$$ \hspace{1cm} (59)

and $E_C$ is positive, there is a critical radius $a_c$ where $E_C = E_{BH}$ and if the radius $a$ of the universe is larger than the critical radius: $a > a_c$, the bound in (58) is violated. Formally $a_c$ is given by

$$a_c = \left[ -\frac{16\pi G f_{p0} f_{p1} V_0^{-w_p-1} C^{1-\frac{2}{n}}}{n(n-1)} \right]^{\frac{1}{nw_p+n-2}},$$ \hspace{1cm} (60)

with the parameters $f_{p0}$, $f_{p1}$, and $C$, which may be determined by some initial conditions. If we consider 4-dimensional spacetime ($n = 3$), Eq.(59) gives $w < -\frac{1}{3}$, then for the quintessence ($-1 < w < -\frac{1}{3}$), the cosmological constant ($w = -1$), and the phantom ($w < -1$), there is always a critical radius $a_c$ and the bound (58) is violated if $a > a_c$.

Similarly, one can discuss the entropy bounds for dark energy universe as in [5] even if $w_p$ depends on time. Although the entropy is not conserved, the expression of the entropy $S_p$ (49) still holds. The quantity $(2E_p - E_p) E_{pC}$ inside the square root of (49) has a maximum $E_p^2$ when $E_{pC} = E_p$. Then

$$S \leq A_p [a^{nw_p} E_p]^{\frac{n}{(wp+1)n-1}}$$ \hspace{1cm} for $w_p > -1 + \frac{1}{n}$,

$$S \geq A_p [a^{nw_p} E_p]^{\frac{n}{(wp+1)n-1}}$$ \hspace{1cm} for $w_p < -1 + \frac{1}{n}$. \hspace{1cm} (61)
As $w_p$ depends on time, at some time, we may have $w_p > -1 + \frac{1}{n}$ and at another time, $w_p < -1 + \frac{1}{n}$. If we define the Bekenstein entropy $S_{pB}$ for the dark energy as in (54): $S_{pB} \equiv \frac{2\pi a}{n} E_p$, we find, even if $w_p$ depends on time, the relation as in [5]:

$$
S \leq S_0 \left[ a^{nwp-1} S_B \right]^{\frac{n}{(wp+1)n-1}} \quad \text{for} \quad w_p > -1 + \frac{1}{n},
$$

$$
S \geq S_0 \left[ a^{nwp-1} S_B \right]^{\frac{n}{(wp+1)n-1}} \quad \text{for} \quad w_p < -1 + \frac{1}{n}.
$$

(62)

Here $S_0$ is given by

$$
S_0 = A_p \left( \frac{n}{2\pi} \right)^{\frac{n}{(wp+1)n-1}}.
$$

(63)

However, as $w_p$ and $A_p$ depend on time, $S_0$ also depends on time. If $w < -1 < -1 + \frac{1}{n}$, the entropy can be negative (unphysical case) even if the energy is positive. If $-1 < w < \frac{1}{n}$, the entropy becomes negative only when the energy is negative.

## 3 Entropy and energy in modified gravity

In [7, 8], a gravitational alternative was suggested for the dark energy modifying the standard Einstein action at low curvature by $1/R$ term. Such modified gravity may produce the current cosmic speed-up [7] and may be naturally generated by string/M-theory [9]. It represents some kind of higher derivative and non-local gravity, and as such it may contain some instabilities [11]. Nevertheless, with some mild modifications at high curvature region the theory is shown to be stable [12] which is also supported by quantum field theory [12]. Modified gravity was studied in Palatini form [10], and it seems that it may be viable also in such a version. Classically, its action may be mapped to an equivalent scalar-tensor theory. We discuss below the entropy, the energy and CV formula for accelerated universe in modified gravity which provides the gravitational dark energy.

Let us start from the rather general 4-dimensional action:

$$
\dot{S} = \frac{1}{\kappa^2} \int d^4 x \sqrt{-g} f(R),
$$

(64)
where $\kappa^2 = 16\pi G$, $R$ is the scalar curvature, and $f(R)$ is some arbitrary function. By using the conformal transformation

$$g_{\mu\nu} \rightarrow e^\sigma g_{\mu\nu},$$

with

$$\sigma = -\ln f'(R),$$

etc., the action (64) is rewritten as

$$\hat{S}_E = \frac{1}{\kappa^2} \int d^4 x \sqrt{-g} \left( R - \frac{3}{2} g^{\rho\sigma} \partial_\rho \sigma \partial_\sigma \sigma - V(\sigma) \right).$$

Here

$$V(\sigma) = e^\sigma g \left( e^{-\sigma} \right) - e^{2\sigma} f \left( g \left( e^{-\sigma} \right) \right) = \frac{A}{f'(A)} - \frac{f(A)}{f'(A)^2}.$$ 

Here $g(B)$ is given by solving the equation $B = f'(A)$ with respect to $A$: $A = g(B)$ and $A$ in (68) is given by $A \equiv -e^{2\sigma}$. This is the standard form of the scalar-tensor theories where the scalar field is fictitious [12].

We now consider the FRW cosmology in modified gravity. FRW metric in the physical (Jordan) frame is given by:

$$ds^2 = -dt^2 + \hat{a}(t)^2 \sum_{i,j=1}^{3} \gamma_{ij} dx^i dx^j.$$ 

The FRW equation in the Einstein frame has the following form:

$$3H_E^2 + \frac{3k}{2\hat{a}_E^2} = \frac{\kappa^2}{2} \left( \rho(\sigma E) + \rho_{(m)} \right).$$

Here $\rho_{(m)}$ is the energy density of the matter but for simplicity, we neglect the matter. We also concentrate on the $k = 0$ case but the obtained results are correct even for $k \neq 0$ case if the radius of the universe is large enough. The Hubble constant $H_E$ in the Einstein frame is defined by

$$H_E \equiv \frac{\dot{\hat{a}}_E}{\hat{a}_E},$$

with the scale factor $\hat{a}_E$ in the Einstein frame:

$$\hat{a}_E = e^{-\frac{\sigma}{2}} \hat{a}.$$
The contribution from the $\sigma$ field to the energy-momentum tensor $\rho_{(\sigma E)}$ is given by

$$\rho_{(\sigma E)} \equiv \frac{1}{\kappa^2} \left( \frac{3}{2} \dot{\sigma}^2 + V(\sigma) \right). \quad (73)$$

In the Einstein frame, the equation of motion for $\sigma$ has the following form:

$$0 = 3 (\ddot{\sigma} + 3H_E\dot{\sigma}) + V'(\sigma). \quad (74)$$

Assuming that when the curvature is small the action is given by

$$S = \frac{1}{\kappa^2} \int d^4x \sqrt{-g} \left( R - \frac{\tilde{a}}{\tilde{R}} \right), \quad (75)$$

the potential is given by

$$V(\sigma) \sim \frac{2}{\sqrt{a}} e^{\frac{2}{3}\sigma}. \quad (76)$$

Since $\sigma = -\ln f'(R) \sim -\ln \frac{\tilde{a}}{\tilde{R}^2}$, $\sigma$ is negative and large. Then the solution of equations (70) and (74) is given by

$$\hat{a}_E = a_{E0} \left( \frac{t_E}{t_0} \right)^{\frac{4}{3}}, \quad \sigma = -\frac{4}{3} \ln \frac{t_E}{t_0}, \quad \frac{t_0^2}{\sqrt{a}} = 4. \quad (77)$$

Here $t_E$ is the time coordinate in the Euclidean frame, which is related to the time coordinate $t$ in the (physical) Jordan frame by $e^{\frac{2}{3}\sigma} dt_E = dt$. As a result

$$3t_E^{\frac{1}{3}} = t, \quad (78)$$

and in the physical (Jordan) frame the power law inflation occurs

$$\dot{a} = e^{\frac{2}{3}\sigma} \dot{a}_E \propto t_E^{\frac{2}{3}} \propto t^2, \quad (79)$$

In general, if $p = w\rho$, the scale factor $a$ behaves as

$$\dot{a} \sim t^{\frac{2}{3(1+w)}}. \quad (80)$$

Then as we can see from (79), in the Jordan frame we find $w = -\frac{2}{3}$ and from (79), in the Einstein frame, $w = -2$. In fact, in the Einstein frame one has

$$\rho_{(\sigma E)} = \frac{1}{\kappa^2} \left( \frac{3}{2} \dot{\sigma}^2 + V(\sigma) \right) \sim \frac{32}{3\kappa^2 t_E^2},$$

$$p_{(\sigma E)} = \frac{1}{\kappa^2} \left( \frac{3}{2} \dot{\sigma}^2 - V(\sigma) \right) \sim -\frac{16}{3\kappa^2 t_E^2}. \quad (81)$$
Although the Jordan frame is physical, as the separation to the gravity and the matter is more easy in the Einstein frame, we work in the Einstein frame for a while. FRW equation (70) can be rewritten in the form of the cosmological CV formula with \( n = 3 \) as

\[
S_E^H = \frac{2\pi a}{3} \sqrt{E_{BH}^E (2E^E - kE_{BH}^E)} .
\]  

(82)

by defining

\[
S_E^H \equiv \frac{H_E V_E}{2G}, \quad E^E \equiv \rho_{(\sigma E)} V_E, \quad E_{BH}^E \equiv \frac{3V_E}{4\pi G \hat{a}_E^2}, \quad V_E \equiv \hat{a}_E^3 \int d^3 \sqrt{-\gamma},
\]

(83)

and \( \kappa^2 = 16\pi G \). The second FRW equation can be given by considering the derivative of the (first) FRW equation (70) with respective Einstein time \( t_E \) and can be rewritten as

\[
kE_{BH}^E = 3 \left( E^E + p_{(\sigma E)} V_E - T_H^E \dot{S}_H^E \right) .
\]

(84)

Here

\[
T_H^E \equiv -\frac{1}{2\pi H_E} \frac{dH_E}{dt_E}.
\]

(85)

and we find

\[
p_{(\sigma E)} = -\frac{1}{3H_E} \frac{d\rho_{(\sigma E)}}{dt_E} - \rho_{(\sigma E)} .
\]

(86)

In the physical Jordan frame, since \( \dot{a} = e^{\sigma} \dot{a}_E \) and \( e^{\sigma} dt_E = dt \), the Hubble parameter is

\[
H \equiv \frac{1}{\dot{a}} \frac{d\dot{a}}{dt} = \frac{1}{\dot{a}_E} \frac{d\dot{a}_E}{dt_E} \frac{dt_E}{dt} + \frac{1}{2} \frac{d\sigma}{dt} = H_E e^{-\frac{\sigma}{2}} + \frac{\dot{\sigma}}{2}.
\]

(87)

Then in the Jordan frame, the FRW equation can be rewritten as

\[
3H^2 + \frac{3k}{\dot{a}_E^2} = \frac{\kappa^2}{2} \rho_{(\sigma)} , \quad \rho_{(\sigma)} \equiv \rho_{(\sigma E)} e^{-\sigma} + H \dot{\sigma} - \frac{\dot{\sigma}^2}{4} .
\]

(88)

Defining

\[
S_H \equiv \frac{HV}{2G}, \quad E \equiv \rho_{(\sigma)} V, \quad E_{BH} \equiv \frac{3V}{4\pi G \hat{a}^2}, \quad V \equiv \hat{a}^3 \int d^3 \sqrt{-\gamma}.
\]

(89)
we obtain the cosmological Cardy-Verlinde formula:
\[
S_H = \frac{2\pi a}{3} \sqrt{E_{BH} (2E - kE_{BH})}. \tag{90}
\]

By differentiating the FRW equation (88) with respect to \(t\), one gets the second FRW equation:
\[
\frac{dH}{dt} - \frac{k}{a^2} = \frac{\kappa^2}{2} \left( \rho(\sigma) + p(\sigma) \right), \quad p(\sigma) \equiv -\frac{1}{3H} \frac{d\rho(\sigma)}{dt} - \rho(\sigma). \tag{91}
\]

With the definition of the temperature \(T_H\) by
\[
T_H \equiv -\frac{1}{2\pi H} \frac{dH}{dt}, \tag{92}
\]

it follows
\[
kE_{BH} = 3 \left( E + p(\sigma)V - T_H S_H \right). \tag{93}
\]

For the case of \(k = 0\), by substituting (77), (78), and (79) into the expressions of \(\rho(\sigma)\) in (88) and \(p(\sigma)\) in (91), we find
\[
\rho(\sigma) = \frac{\rho(\sigma)0}{t^2}, \quad \rho(\sigma)0 \equiv \frac{22(27)^{\frac{2}{3}}}{3\kappa^2 t_0^3} - 12, \quad p(\sigma) = -\frac{2}{3} \frac{\rho(\sigma)0}{t^2}. \tag{94}
\]

Eventually, it follows \(w = -\frac{2}{3}\) in the Jordan frame.

At the low temperature, as the field with lowest (negative) \(w\) dominates, we may have an equation similar to (33) with \(n = 3:\)
\[
S \sim A_\sigma \left[ a^{3w_\sigma} \sqrt{(2E_\sigma - E_{\sigma C}) E_{\sigma C}} \right]^{\frac{3}{3w_\sigma + 2}}, \tag{95}
\]
\[
A_\sigma \equiv f_{\sigma 0} \left( 1 + \frac{1}{w_\sigma} \right) \left( -\frac{4f_{\sigma 0} f_{\sigma 1}}{3} \right)^{-\frac{3}{2(3w_\sigma + 2)}} V_0^{\frac{3w_\sigma}{3w_\sigma + 2}}. \tag{95}
\]

Since \(w_\sigma = -\frac{2}{3}\), the exponents in (95) diverges. Then in order that the entropy is finite, the condition appears
\[
\left( -\frac{4f_{\sigma 0} f_{\sigma 1}}{3} \right)^{-\frac{3}{2}} V_0^{\frac{3w_\sigma}{3w_\sigma + 2}} \left[ a^{3w_\sigma} \sqrt{(2E_\sigma - E_{\sigma C}) E_{\sigma C}} \right]^3 = 1. \tag{96}
\]
We should also note that the solution (77) or (79) is for \( k = 0 \) case. Then the Casimir force should vanish. In order to find the Casimir force, we need to consider the \( k \neq 0 \) case. As the expansion over \( k \) corresponds to the expansion with respect to the inverse of the radius of the universe, we may consider the perturbation with respect to \( k \) in order to obtain the Casimir energy.

We should also note that, as discussed after (19), since now \( w \) is greater than \(-1\) but negative, the entropy \( S \) could be negative only if the energy is negative.

4 Black hole thermodynamics

We now consider the black hole solution in the modified gravity, whose action is given by (75). As it will be shown, its thermodynamical properties are also related to the CV formula. If we assume \( R_{\mu\nu} \propto g_{\mu\nu} \), the equation of motion is given by

\[
0 = \left(1 + \frac{\tilde{a}}{R^2}\right) R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \left( R - \frac{\tilde{a}}{R} \right),
\]

(97)

Then

\[
R = \pm \sqrt{3\tilde{a}} , \quad R_{\mu\nu} = \pm \frac{\sqrt{3\tilde{a}}}{4} g_{\mu\nu} .
\]

(98)

A large class of solutions of (98) is given by the family of metrics

\[
ds^2 = -e^{2\rho} dt^2 + e^{-2\rho} dr^2 + \sum_{i,j=1,2} g_{ij}^{(2)} dx^i dx^j ,
\]

\[
e^{2\rho} = \frac{1}{r} \left( -\mu + k^{(2)} r - \frac{\Lambda r^3}{3} \right) , \quad \Lambda = \mp \frac{\sqrt{3\tilde{a}}}{4} .
\]

(99)

embracing de Sitter (dS) and anti-de Sitter (AdS) black holes with any horizon topology. Here \( k^{(2)} \) is the Ricci curvature of the transverse manifold, as given by the Ricci tensor \( R_{ij}^{(2)} \) of the metric \( g_{ij}^{(2)} \), i.e. \( R_{ij}^{(2)} = k^{(2)} g_{ij}^{(2)} \). If \( \Lambda < 0 \) (\( \Lambda > 0 \)), the spacetime is asymptotically anti-deSitter (deSitter). In both cases the curvature radius will be defined by \( L^2 = 3/|\Lambda| = 12/\sqrt{3\tilde{a}} \).
We shall mainly study the SAdS metric although our results apply equally well to any horizon topology. The thermodynamical free energy can be obtained according to a quantum gravity tree-level formula involving the Euclidean action $I_E$

$$F(\beta) = \beta^{-1} I_E = \frac{\kappa}{2\pi} I_E$$

where $\kappa$ is the surface gravity of the black hole. To pursue this program one has to regularize the volume divergences. In anti-de Sitter gravity one can achieve this, essentially, by two well known methods. One is the counterterm method inspired by the Maldacena duality with conformal field theories, the other a background subtraction chosen to correspond to the vacuum of the CFT. This uniquely identifies it as anti-de Sitter space itself, with no matter inside. The unregularized Euclidean action will be

$$I_E = -\frac{1}{16\pi G} \int d^4x \left( R - \frac{\tilde{a}}{R} \right) |g|^{1/2} - \frac{1}{8\pi G} \oint K |h|^{1/2} d^3x$$

The Euclidean SAdS solution is given by (99) taking $k^{(2)} = 1$ and the metric $g_{ij}^{(2)}$ to be that of a round two-sphere

$$ds^2 = \left( 1 - \frac{\mu}{r} + \frac{r^2}{L^2} \right) d\tau^2 + \left( 1 - \frac{\mu}{r} + \frac{r^2}{L^2} \right)^{-1} dr^2 + r^2 d\omega_2^2$$

where $d\omega_2^2$ is the line element of a 2-sphere with unit radius and volume $\omega_2 = 4\pi$. Moreover $\tau \simeq \tau + \beta$ is periodically identified up to $\beta$ and the curvature radius is $L^2 = 12/\sqrt{3\tilde{a}}$. This is a solution of (100) with $R = -\sqrt{3\tilde{a}}$. Therefore it represents a spherically symmetric black hole immersed in anti-de Sitter space.

The background metric will be (101) with $\mu = 0$, i.e. anti-de Sitter space at finite temperature $T = \kappa/2\pi$. This has zero gravitational entropy, since there is no horizon. The action (100) for the metric (101) is easily seen to be

$$I_E = \frac{\sqrt{\tilde{a}} \beta}{3\sqrt{12G}} (R_m^3 - r_+^3) + \text{“boundary terms”}$$

where $R_m$ is an upper bound for the radial integration and $r_+$ is the radius of the horizon. The action of the background is

$$I_{EB} = \frac{\sqrt{\tilde{a}} \beta}{3\sqrt{12G}} R_0^3 + \text{“background boundary terms”}$$
where again $R_0$ is a radial cutoff. Now a meaningful comparison of the black hole free energy with the vacuum free energy (empty AdS space) requires that the vacuum metric on the surface $r = R_0$ be asymptotically coincident with the actual metric on the surface $r = R_m$. This matching condition ensures that the boundary temperatures in the black hole and the background, be equal. A simple check gives the matching condition that, asymptotically

$$R_0 = R_m - \frac{\mu L^2}{6R_m^2}$$

Using this into (103) and subtracting the result from (102), gives the regularized action

$$\Delta I_E = \frac{\sqrt{\tilde{a}} \beta}{6 \sqrt{12G}} (\mu L^2 - 2r_+^3)$$

(104)

We note that the mass parameter $\mu$ and $r_+$ are functions of $\beta$ through the defining relations

$$\mu = r_+ + \frac{r_+^3}{L^2}$$

(105)

$$\beta = \frac{4\pi L^2 r_+}{\mu L^2 + 2r_+^3}$$

(106)

Hence the entropy could be computed by the familiar thermodynamical relation

$$S = \beta \partial_\beta \Delta I_E - \Delta I_E$$

Instead we may use an easier way. We note that both $R - \tilde{a}/R$ as well as $R - 2\Lambda$ are proportional to $\sqrt{\tilde{a}}$, so $I_E$ must be proportional to the action as computed in Einstein gravity. Denoting this as $I_{AdS}$, a simple computation gives

$$I_E = \frac{4}{3} I_{AdS}$$

(107)

We know that the entropy in Einstein gravity is $A/4G$, so we immediately conclude that in $1/R$ gravity the entropy must be

$$S = \frac{4}{3} \frac{A}{4G} = \frac{A}{3G}$$

(108)

One finds that the boundary terms do not contribute to the final result.
So black holes in modified gravity are a little bit more entropic than expected. We may confirm this result by using the Noether charge method. In this case the formula is[13]

\[ S = 4\pi \int_{S^2} \frac{\partial \mathcal{L}}{\partial R} d^2x \]  

(109)

where \( \mathcal{L} = \mathcal{L}(R) \) is the Lagrangian density and the integral is over the horizon at \( r = r_+ \). In our case \( \mathcal{L} = \sqrt{g}(R - \tilde{a}/R)/16\pi G \), so

\[ S = \frac{A}{4G} \left( 1 + \frac{\tilde{a}}{R^2} \right) = \frac{4}{3} \frac{A}{4G} \]

as a simple computation will confirm using \( R^2 = 3\tilde{a} \). These calculations can be done in any spacetime dimensions, say \( d \). Then (108) generalizes to

\[ S = \frac{2d}{d + 2} \frac{A}{4G} \]  

(110)

Note that for the black hole with the size of FRW universe, the entropy is defined by the Bekenstein-Hawking entropy \( S_{BH} \) (54). Then the above result (108) and (110) indicates that \( S_{BH} \) should also be modified by the factor \( \frac{2d}{d+2} \) if compared with the FRW universe in Einstein gravity.

The higher entropy of black hole in 1/R-gravity means that they are more massive than in Einstein theory, since by the first law \( dM = TdS \). The precise prediction should just be that \( M \) is larger by the factor \( z = 2d/(d+2) \).

An asymptotically \( SAdS_d \) black hole in General Relativity has an excitation energy over the AdS vacuum which can be computed by canonical methods, by means of the formula

\[ M = \frac{1}{8\pi G} \int N(\Theta - \Theta_0)\sqrt{\sigma}d^{d-2}x \]  

(111)

Here we integrate over a \((d-2)\)-dimensional sphere at infinity, contained in a Cauchy surface of equal time, the lapse function \( N = \sqrt{-g_{tt}} \), times the trace of the second fundamental form of the sphere as embedded in the Cauchy surface, after a regularizing subtraction from empty AdS space. For the metric (101) one finds

\[ M = \frac{(d-2)\omega_{d-2}}{16\pi G} \mu \]  

(112)
This can be expressed as a function of the black hole radius by using the condition $N(r_+) = 0$, which is

$$\mu = r_+^{d-3} + \frac{r_+^{d-1}}{L^2}. \quad (113)$$

In theories with an AdS dual, this relation can be interpreted as the energy of a CFT living on the boundary of AdS spacetime, and leads to a CV formula for AdS black holes. In higher derivatives gravity, and this is just our case, things may be not so straightforward. For a theory whose Lagrangian $L = L(R)$ is a function of the scalar curvature, the above mass can be related to a Noether charge[14] which is proportional to $\partial L/\partial R$, as in the entropy derivation given above. More than this, it is this Noether charge that enters the formulation of the first law for stationary black holes in diffeomorphism covariant theories of gravity[13, 14]. The result is the mass formula (111), except that the integrand gets multiplied with $16\pi G \partial L/\partial R$ evaluated on the background solution, where $L = (R - \tilde{a}/R)/16\pi G$ is the actual Lagrangian. This gives all masses an extra coefficient

$$1 + \frac{\tilde{a}}{R^2} = \frac{4}{3}$$

It is therefore clear that the Cardy-Verlinde formula for AdS black holes[15, 16], being the square root of a quadratic function of all the relevant energies, will give the entropy the $4/3$ coefficient too, in accord with our calculations.

## 5 Hydrodynamical examples testing the holographic entropy bound

The suggestion of Kovtun et al. [17] that there may exist in cosmology a universal lower bound on $\eta/s$ - $\eta$ being the shear viscosity and $s$ the entropy content per unit volume - is interesting, since it may be of fundamental importance. These authors are concerned with the infrared properties of theories whose gravity duals contain a black brane with a nonvanishing Hawking temperature, the point being that the infrared behavior is governed by hydrodynamical laws. If we for definiteness consider a stack of $N$ non-extremal
D3 branes in type IIB supergravity, the metric near the horizon is given by

\[
ds^2 = \frac{r^2}{R^2}[-f(r)dt^2 + dx^2 + dy^2 + dz^2] + \frac{R^2}{r^2f(r)}dr^2 + R^2dΩ^2_5,
\]

(114)

where \( R \propto N^{1/4} \) is a constant, and \( f(r) = 1 - r_0^4/r^4 \) with \( r_0 \) being the horizon. The Hawking temperature of this metric is \( T = r_0/\pi R^2 \), and \( \eta \) and \( s \) are given by

\[
\eta = \frac{1}{8}\pi N^2 T^3, \quad s = \frac{1}{2}\pi^2 N^2 T^3.
\]

(115)

Thus, in dimensional notation

\[
\frac{\eta}{s} = \frac{\hbar}{4\pi k_B} = 6.08 \times 10^{-13} \text{ K s}.
\]

(116)

The conjecture of Kovtun et al. (see also [18]) is that the value in Eq. (116) is a lower bound for \( \eta/s \). Since this bound does not involve the speed of light, the authors even conjecture that this bound exists for all systems, including non-relativistic ones.

The idea has recently been further elaborated in [19], arguing that the bound follows from the generalized covariant entropy bound. From Eq.(55), there is the Bekenstein (and also the holographic) entropy bound, which is used to prove the new bound to shear viscosity.

The purpose of this section is to elucidate this holographic idea by considering some examples explicitly. We will choose examples from general physics. Our scope is thus wider than in the previous sections; our aim is to investigate the generality of the entropy bound. We will consider three examples, the first taken from ordinary hydrodynamics, the second from the theory of the universe in the beginning of its plasma era, and finally the third taken from the very early universe under conditions corresponding to the Kasner metric. The third example is presumably the one of main interest; the shear viscosity concept is after all a concept that relates to a physical situation that is anisotropic. Moreover, we will discuss the validity of the Cardy-Verlinde entropy formula in the case of viscous cosmology, thus elaborating on the previous treatment on this topic in [20].

The central inequality that we intend to analyze, is thus

\[
\frac{\eta/s}{\hbar/4\pi k_B} > 1.
\]

(117)
Example 1. Hydrodynamics: Small Reynolds number flow. The following setup taken from ordinary hydrodynamics involves both the shear viscosity $\eta$ and the entropy density $s$: Assume that a solid sphere with radius $R$ and with high thermal conductivity $\lambda$ is immersed in a uniform flow passing it at small Reynolds numbers. We take the origin in the center of the sphere, and use spherical coordinates with the polar axis in the direction of the undisturbed velocity $u$ of the stream. The equation of thermal conduction is

$$\nabla^2 T = -\frac{\eta}{2\lambda} (v_{i,k} + v_{k,i})^2,$$

where $v$ is the fluid velocity for $r \geq R$. Inserting Stokes’ formula (applicable at low Reynolds numbers) for $v$, the solution for the temperature distribution $T(r)$ can be written as \[\text{\cite{21}}\]

$$T(r) - T_0 = \frac{9u^2\eta}{4\lambda} \left\{ \left( \frac{3R^2}{4r^2} - \frac{5R^3}{3r^3} + \frac{R^4}{r^4} - \frac{1}{12} \frac{R^6}{r^6} \right) \cos^2 \theta ight.$$ \[+ \frac{2R}{3r} - \frac{3R^2}{4r^2} + \frac{5R^3}{9r^3} - \frac{1}{6} \frac{R^4}{r^4} - \frac{1}{36} \frac{R^6}{r^6} \right\},$$

where $T_0$ is the constant reference temperature at infinity. The boundary conditions are $T = T_1 = \text{const}$ and $\int (\partial T/\partial r)r^2 \sin \theta d\theta = 0$ for $r = R$. From Eq. (119) it is seen that $\Delta T \equiv T_1 - T_0 = 5u^2\eta/8\lambda$.

One may ask: What is the appropriate value to be inserted for the entropy density $s$? Taking water as an example, one might use the handbook value for $s$, resulting in $\eta/s = 2.3 \times 10^{-10}$ K s, as in Ref. \cite{19}. However, in our opinion the physically most natural value to use for $s$ in the present example is the one associated with the temperature difference $\Delta T$. This amounts to setting

$$s = \rho c_p \int_{T_0}^{T_1} \frac{dT}{T} \simeq \rho c_p \frac{\Delta T}{T_0},$$

$c_p$ being the specific heat capacity at constant pressure. We then get

$$\frac{\eta}{s} = \frac{8\nu T_0}{5u^2 Pr},$$

where $\nu = \eta/\rho$ is the kinematic viscosity and $Pr = \nu \rho c_p/\lambda$ the Prandtl number. We choose the moderate velocity $u = 1 \text{ mm/s}$ to keep the Reynolds
number small, and take $T = 300$ K. Then, with $\nu = 0.010 \text{cm}^2/\text{s}$, $Pr = 6.75 \text{[21]}$ we get

$$\eta_s = 71 \text{ K s}$$

(122)
as a typical value. The inequality (117) is obviously satisfied.

**Example 2. Plasma era in the early universe.** As the next step we consider the initial stage of the the plasma era in the early universe. This can be taken to occur at about $t = 1000$ s after the big bang, when the universe is characterized by ionized H and He in approximate equilibrium with radiation (cf., for instance, [22, 23, 24, 25]). The number densities of electrons and photons are equal, $n \simeq 10^{19} \text{ cm}^{-3}$, the temperature is $T \simeq 4 \times 10^8$ K, and the energy density is $\rho c^2 = a_r T^4$, where $a_r = \pi^2 k_B^4/(15 \hbar^3 c^3) = 7.56 \times 10^{-15} \text{ erg cm}^{-3} \text{K}^{-4}$ is the radiation constant. The pressure is $p = \rho c^2/3$. The presence of energy dissipation and viscosity coefficients in the cosmic fluid is due to the fact that the thermal equilibrium is not quite perfect. From relativistic kinetic theory one can calculate the viscosity coefficients. Let $x = m_e c^2/k_B T$ be the ratio between electron rest mass and thermal energy; when $x \gg 1$ it is convenient to use the polynomial approximations [26] (cf. also [24]) for the evaluation of the shear viscosity $\eta$ and the bulk viscosity $\zeta$:

$$\eta = \frac{5m_e^6 e^8 \zeta(3)}{9\pi^3 \hbar^3 e^4 n} x^{-4},$$

(123)

$$\zeta = \frac{\pi c^2 \hbar^3 n}{256 e^4 \zeta(3)} x^3,$$

(124)

$\zeta(3) = 1.202$ being the Riemann zeta function. At $T = 4 \times 10^8$ K one has $x = 14.8$, leading to

$$\eta = 2.8 \times 10^{14} \text{ g cm}^{-1} \text{ s}^{-1}, \quad \zeta = 7.0 \times 10^{-3} \text{ g cm}^{-1} \text{ s}^{-1}.$$ (125)

We note that both $\eta$ and $\zeta$ now contain $\hbar$, and also that $\eta$ is enormously larger than $\zeta$.

The entropy density, in view of the radiation dominance, is given by

$$s = \frac{4}{3} a_r T^3 = 6.45 \times 10^{11} \text{ erg cm}^{-3} \text{K}^{-1},$$

(126)

and so

$$\frac{\eta}{s} = 435 \text{ K s}.$$ (127)
This value is surprisingly enough of the same order of magnitude as the value given in Eq. (122). There seems to be no simple reason why this should be so; the physical conditions in the two cases are widely different.

So far, we assumed a radiation dominated FRW universe. What happens if the universe is instead filled with matter obeying the relation \( p = w \rho c^2 \), with \( w \) constant and negative? To investigate this point let us go back to Eq. (24), in which the sub-extensive parts are neglected. For the ratio \( s/\rho c^2 \), where \( s = S/V \) and \( \rho c^2 = E/V \), we obtain

\[
\frac{s}{\rho c^2} = 1 + \frac{w}{T}.
\]

This expression is seen to be independent of the prefactor \( f_0 \). Let us assume that the energy density at \( T = 4 \times 10^8 \) K is the same as before, i.e., \( \rho c^2 = a_r T^4 = 1.94 \times 10^{20} \) erg cm\(^{-3} \). Then \( s \) is found from (128), and taking the shear viscosity to be given by (124) as before, we obtain the following simple equation

\[
\frac{\eta}{s} = \frac{578}{1 + w}.
\]

We see that except in the case where \( w \) is close to \(-1\), the order of magnitude of \( \eta/s \) is roughly the same as above. It is moreover evident that the expression (129) is physically meaningful only when \( w > -1 \) (the viscosity \( \eta \) has always to be positive, for general thermodynamical reasons). We thus see that the inclusion of shear viscosity implies that it is only the case of quintessence that is of physical interest. The case of phantoms, \( w < -1 \), leads to negative entropies and is in the present context excluded.

**Example 3. The Kasner universe.** Our third example is taken from the theory of the very early universe. From ordinary hydrodynamics we know that the shear viscosity comes into play whenever there are fluid sheets sliding with respect to each other. Correspondingly, in a relativistic formulation, the most natural circumstances under which \( \eta \) is expected to be of significance are when anisotropy is brought into consideration. It becomes natural to focus attention on the anisotropic Kasner metric

\[
ds^2 = -dt^2 + t^{2p_1} dx^2 + t^{2p_2} dy^2 + t^{2p_3} dz^2,
\]

where the numbers \( p_1, p_2, p_3 \) are constants. The two numbers \( P \) and \( Q \) are
defined by
\[ P = \sum_{i=1}^{3} p_i, \quad Q = \sum_{i=1}^{3} p_i^2. \] (131)

In a vacuum Kasner space, \( P = Q = 1 \). Here, we assume that there is an isotropic fluid with energy density \( \rho \) and pressure \( p \) immersed in this space. Both \( \rho \) and \( p \), as well as the viscosity coefficients \( \eta \) and \( \zeta \), are assumed to be dependent on time but independent of position. If \( U^\mu = (U^0, U^i) \) is the fluid’s four-velocity, the energy-momentum tensor is
\[ T_{\mu\nu} = \rho U_\mu U_\nu + (p - \zeta \theta) h_{\mu\nu} - 2\eta \sigma_{\mu\nu}, \] (132)
where \( h_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu \) is the projection tensor, \( \theta = U^\mu U_\mu \) is the scalar expansion, \( \theta_{\mu\nu} = \frac{1}{2}(U_{\mu;\alpha} h_{\nu}^{\alpha} + U_{\nu;\alpha} h_{\mu}^{\alpha}) \) is the expansion tensor, and \( \sigma_{\mu\nu} = \theta_{\mu\nu} - \frac{1}{3} h_{\mu\nu} \theta \) is the shear tensor.

Consider now the Einstein equations, taking the cosmological constant \( \Lambda \) to be zero. With \( \kappa^2 = 16\pi G \) we obtain from \( R_{\mu\nu} = \frac{1}{2} \kappa^2 (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\alpha_\alpha) \) the two equations
\[ P - Q + \frac{3}{4} \kappa^2 t \zeta P = \frac{1}{4} \kappa^2 t^2 (\rho + 3p), \] (133)
\[ p_i (1 - P - \kappa^2 t \eta) + \frac{1}{4} \kappa^2 t (\zeta + \frac{4}{3} \eta) P = -\frac{1}{4} \kappa^2 t^2 (\rho - p). \] (134)

The structure of the Einstein equations leads to the time relationships
\[ \rho(t) = \rho_0 (t_0/t)^2, \quad p(t) = p_0 (t_0/t)^2, \] (135)
\[ \zeta(t) = \zeta_0 t_0/t, \quad \eta(t) = \eta_0 t_0/t, \] (136)
where \( \{\rho_0, p_0, \zeta_0, \eta_0\} \) refer to the chosen initial instant \( t = t_0 \). We can then write the equations such that they contain time-independent quantities only:
\[ P - Q + \frac{3}{4} \kappa^2 \zeta_0 t_0 P = \frac{1}{4} \kappa^2 t_0^2 (\rho_0 + 3p_0), \] (137)
\[ p_i (1 - P - \kappa^2 \eta_0 t_0) + \frac{1}{4} \kappa^2 t_0 (\zeta_0 + \frac{4}{3} \eta_0) P = -\frac{1}{4} \kappa^2 t_0^2 (\rho_0 - p_0). \] (138)

Let us consider the production of entropy. First, for the Bianchi type-I spaces the average expansion anisotropy parameter \( A \) is defined as [27]
\[ A = \frac{1}{3} \sum_{i=1}^{3} \left( 1 - \frac{H_i}{H} \right)^2, \] (139)
where $H_i = \dot{a_i}/a_i$ with $a_i = t^\mu_i$ are the directional Hubble factors and $H = \frac{1}{3} \sum H_i$ is the average Hubble factor. Accordingly, in our case

$$A = \frac{3Q}{P^2} - 1. \quad (140)$$

Next, the entropy current four-vector is $S^\mu = n\sigma U^\mu$, where $n$ is the baryon number density and $\sigma = s/n$ the nondimensional entropy per baryon. In general,

$$S_{\mu;\nu} = 2\eta T \sigma_{\mu\nu} + \frac{\zeta}{T} \theta^2, \quad (141)$$

meaning in the comoving frame of reference ($\dot{\sigma} = d\sigma/dt$)

$$\dot{\sigma} = \frac{3P^2}{nTt^2} \left( \zeta + \frac{2}{3} \eta A \right). \quad (142)$$

As we would expect, the anisotropy in general provides a significant contribution to the growth of entropy, in view of the large magnitude of $\eta$. However, let us go back to Eq. (138): this equation tells us that all the $p_i$ have to be equal in the present case. With $p_1 = p_2 = p_3 \equiv b$ we get for the isotropic Kasner space

$$b = \frac{1}{6} \left[ 1 + \frac{3}{4} \kappa^2 t_0 \zeta_0 + \sqrt{\left( 1 + \frac{3}{4} \kappa^2 t_0 \zeta_0 \right)^2 + 3 \kappa^2 t_0^2 (\rho_0 - p_0)} \right]. \quad (143)$$

It is seen that the shear viscosity is absent. Equation (142) reduces to

$$\dot{\sigma} = \frac{3P^2}{nk_BT t^2} \zeta, \quad (144)$$

when written in dimensional form.

Let us evaluate the expression (144). Due to the proportionality to the small bulk viscosity we can insert for $n$ and $T$ as if the cosmic fluid were ideal. Thus from conservation of particle number, $n \propto a^{-3}$, and from conservation of entropy, $a \propto T^{-1}$. As moreover $t \propto T^{-2}$, we can write Eq. (144) as

$$\dot{\sigma} = \frac{3P^2 \zeta_0}{n_0k_BT_0 t_0^3} \frac{1}{t}. \quad (145)$$
Thus $\sigma - \sigma_0 \propto \ln(t/t_0)$ is the increase in specific entropy when $t$ increases from $t_0$ to $t$. Multiplying with the particle density $n$ we obtain an expression for the corresponding increase $s - s_0$ in entropy density. Recalling the expression in Eq. (136) for $\eta$ we then derive as our main result the following expression for the sought ratio:

$$\frac{\eta}{s} = \frac{\eta_0 t_0}{s_0 t} \left[1 + \frac{3P_0^2 \zeta_0}{s_0 T_0 t_0} \left(\frac{t_0}{t}\right)^{3/2} \ln \frac{t}{t_0}\right]^{-1}. \quad (146)$$

It is of interest to evaluate this expression at $t = t_0$. Let us identify $t_0$ with the instant at which $T = 10^{12}$ K, i.e., at $t = 2 \times 10^{-4}$ s. Then $n_0 = 6 \times 10^{29}$ cm$^{-3}$, $\rho_0 = 4.5 \times 10^{34}$ erg cm$^{-3}$. This temperature is a kind of limit for standard cosmological theory. If $T > 10^{12}$ K the universe consists of many kinds of particles and antiparticles, but when $T$ has fallen below this value the large number of hadrons has disappeared, and the universe consists of leptons, antileptons, photons, and nucleons. We then have [26]

$$\eta = \frac{3\pi c h^4}{608 m_e G_F^2} x, \quad \zeta = \frac{\pi c h^4}{1776 m_e G_F^2} x^5, \quad (147)$$

which is valid when $x = m_e c^2 / k_B T$ is small. Here, the weak coupling constant is given by $G_F c / \hbar^3 = 10^{-5} m_p^{-2}$.

With $T_0 = 10^{12}$ K we get $x = 5.94 \times 10^{-3}$, and so we have at this instant

$$\eta_0 = 1.8 \times 10^{23} \text{ g cm}^{-1} \text{ s}^{-1}, \quad \zeta_0 = 6.0 \times 10^{12} \text{ g cm}^{-1} \text{ s}^{-1}. \quad (148)$$

The entropy density is calculated approximatively by assuming radiation dominance, such as before. Then, from $s = s_0 \propto T^3$ we get

$$s_0 = 1.0 \times 10^{22} \text{ erg cm}^{-3} \text{ K}^{-1}. \quad (149)$$

Thus,

$$\frac{\eta_0}{s_0} = 18 \text{ K s}. \quad (150)$$

Once again, we end up with the same order-of-magnitude result for the ratio $\eta/s$ as before, when we choose to work at the instant $t = t_0$. However, Eq. (146) tells us that $\eta/s$ diminishes with increasing $t$, and approaches zero when $t \to \infty$. It means that $\eta/s$ cannot in this case be subject to a lower
bound. The Kasner case thus provides a counterexample to the suggestion in Eq. (117). Of course, this can be considered as rather academical as current universe is not the anisotropic one.

Actually, it follows already from the thermodynamical formalism that the lower bound in Eq. (117) cannot be universal. At least this is so in a phenomenological theory, in which $\eta$ and $\zeta$ are arbitrary input parameters. Namely, from Eqs. (141) or (142) it is seen that the specific entropy rate of change involves both $\eta$ and $\zeta$. Let us imagine that $\eta$ is kept constant while $\zeta$ is changing. Therewith $\dot{\sigma}$, and accordingly $\sigma$ itself, as well as the ratio $\eta/s$, change. If this ratio were subject to a lower bound, this would correspond to the existence of a maximum value of $s$. However, we may make $\sigma$ and $s$ as large as we wish, by inserting sufficiently large value of $\zeta$ in Eq. (142). Recall in this context the way in which viscosity coefficients are introduced in fluid mechanics: they are based on the assumption that first order velocity gradients are sufficient to construct the contribution to the stress tensor due to deviations from thermal equilibrium. The theory is thus approximate already from the outset.

The discussion of Verlinde [1] about the holographic bound on the subextensive entropy associated with the Casimir energy, assumed a radiation dominated FRW universe. As shown in [20], the same entropy formula holds if the fluid possesses a constant, though small, bulk viscosity. Similarly, the generalized entropy formula [5] for the case that the state equation is $p = w\rho$ with $w$ a constant (still assuming a FRW metric), was also found to hold in the presence of the same kind of viscosity [28].

One may ask: How does the entropy formula look if the cosmic fluid possesses both a shear viscosity and a bulk viscosity? The answer is immediate, if the anisotropy is originally introduced via the Kasner metric. As shown above the Einstein equations wash out the anisotropies, and we are left with an isotropic Kasner metric whose scale factor is $t^b$, where $b$ is given by Eq. (143). The anisotropy factor $A$ vanishes, and the production of entropy is governed by the bulk viscosity $\zeta$; cf. Eq. (142).

Let us assume that $\zeta$ is constant and small, so that we can adopt the same expression for $a(t)$ as in the case of a nonviscous fluid. The argument can be given similarly to that given in [28]: Taking $n = 3$ we see that the quantity $\rho a^{3(w+1)}$ can be considered as a function of $n\sigma$. Since $E \sim \rho a^3$ and $S \sim n\sigma a^3$, it follows that $Ea^{3w}$ is independent of $V$ and is a function of $S$ only. The conventional decomposition of the total energy $E(S,V)$ into an extensive
part and a sub-extensive part, \( E(S, V) = E_E(S, V) + \frac{1}{2}E_C(S, V) \), together with the scale transformations \( E_E(\lambda S, \lambda V) = \lambda E_E(S, V) \), \( E_C(\lambda S, \lambda V) = \lambda^{1/3}E_C(S, V) \), leads to

\[
E_E = \alpha \frac{4\pi a^{3w}}{3w+1}, \quad E_C = \frac{\beta}{2\pi a^{3w}} S^{w+1/3}, \quad (151)
\]

where \( \alpha \) and \( \beta \) are constants. We thus get, when we reinstate \( a = t^b \),

\[
S = \left[ \frac{2\pi t^{3bw}}{\sqrt{\alpha \beta}} \sqrt{(2E - E_C)E_C} \right]^{\frac{3}{2w+1}}. \quad (152)
\]

This is the Kasner-induced form of our previous expression (25), when \( n = 3 \). It will be of interest to understand better the connection between CV formula and shear viscosity bound. However, this requires the non-trivial generalization of CV formula for anisotropic universe with shear viscosity.

## 6 Discussion

In summary, we studied the entropy of FRW universe filled with dark energy and its representation in the form of holographic CV formula. This investigation shows that the expression of the entropy in terms of energy and Casimir energy depends on the equation of state in a quite complicated form. It is only for a radiation dominated FRW universe the corresponding CV formula acquires the form typical for 2d CFT entropy. On the same time, for negative or time-dependent equation of state such a formula seems to have nothing to do with 2d CFT, being still related with holography. Nevertheless, there exists another, cosmological CV formula which is very useful to derive the entropy bounds and which is the same for any type of matter under consideration. It is remarkable that universality of cosmological CV formula together with the fact that it predicted by the form of FRW equations proves its holographical origin. Of course, the actual reasons for such a manifestation of the holographic principle in the modern universe remain to be obscure. (Some hints maybe drawn from brane-world approach.) Furthermore, all above conclusions remain to be true in the modified gravity which is considered as gravitational alternative for dark energy. This should not seem strange after all as modified gravity maybe re-written in the classically equivalent form as kind of scalar-tensor gravity with matter described by scalar field.

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The black hole thermodynamics in modified gravity is also considered. The black hole entropy law is slightly different (by numerical factor) from the standard case of the Einstein gravity. In the last section we analyze the recently proposed bound for ratio of shear viscosity with entropy density. This bound seems to follow from the Bekenstein entropy bound. As shear viscosity is absent in the current isotropic universe, we concentrate on the early universe at plasma era or anisotropic Kasner universe where newly proposed bound seems to be violated.

The important lesson drawn from this and other studies of the entropy of FRW universe is that holographic principle does not distinguish whether dark energy is present or not. For instance, the cosmological CV formula is the same whatever is the equation of state. This indicates that the origin of dark energy should be searched within fundamental theory, perhaps within string/M-theory.

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