Research on Precise Trigonometric Leveling in Place of First Order Leveling

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ABSTRACT  Through the analysis of the principle, error sources and precision of trigonometric leveling, this paper points out the key problems about first order leveling replaced by trigonometric leveling; and for the first time puts forward that, in some given conditions, it is not only feasible but also valuable to replace first order leveling by precise trigonometric leveling, and proves it by experimentation as well. The content and conclusion of this paper have consulting significance and practicable value for our setting down relational criterion and production practice.

KEYWORDS  electronic distance measurements; trigonometric leveling; first order leveling; surveying for deformation monitoring

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Introduction

With the extensive applications of the total station instrument, the electronic distance measurements for trigonometric leveling (below as trigonometric leveling) has been getting more and more application in surveying and mapping with its advantages such as fleetness, simplicity and convenience and economy etc. And replacing third or fourth order level surveying with trigonometric leveling has been included in the criterion and used for production, obtaining evident economic effect. With the influence of instrument accuracy (mainly the measuring accuracy of slope distance and vertical angles) and outside factors such as the influence of the refraction of atmosphere, trigonometric leveling can not still replace first order level surveying in precise leveling surveying.

This paper puts forward that it is feasible to replace first order level surveying with trigonometric leveling in some projects. From analyzing the influence of the main error sources, the conditions and key problems about replacing first order level surveying with trigonometric leveling are pointed out clearly. And it is proved by experimentation and researches.

1 Analysis of the main error sources and precision of trigonometric leveling

Trigonometric leveling is, according to the slope distance and vertical angle measured from the station to the target point, to calculate the elevation difference of two points by the triangle formula. As Fig. 1 shows, the elevation difference \( h_{AB} \) between point A and point B (ignoring...
the influence of atmosphere refraction and the earth curvature) and the elevation of point B can be gained.

\[ h_{AB} = S\sin a + i - \nu = D\tan a + i - \nu \quad (1) \]

\[ H_B = H_A + h_{AB} \quad (2) \]

where \( D \) is the horizontal distance between point \( A \) and point \( B \).

1.1 Influence of atmosphere refraction and the earth curvature

1.1.1 Correction of two differences

The Eq. (1) is deduced with supposing that the earth surface is a horizontal plane and the line of sight is a straight one. However in fact, as the Fig. 2 shows, \( D \) is the horizontal distance and \( S \) is the slope distance actually measured between two ground-points \( A \) and \( B \), and the curves \( PE \) and \( AF \) are the level surfaces respectively passing the instrument \( P \) and the ground point \( A \). While measuring the vertical angle \( a \), the horizontal line \( PG \) intersects with plumb line of point \( B \) at \( G \), and \( GE \) is the height error produced by the earth curvature (named spherical aberration) marked with \( p \). Because of the influence of atmosphere refraction, the light curve \( NP \) enters the instrumental telescope from the target \( N \), but the axis of the line of sight locates the tangent \( PM \) of the curve \( PN \), and \( MN \) is the influence of the atmosphere refraction to trigonometric leveling (named atmosphere differences) marked with \( r \). Considering the correction of the spherical aberration and (atmosphere differences) (correction of two differences), the Eq. (1) becomes:

\[ h_{AB} = D\tan a + i - \nu + f \]
\[ f = p + r = (1 - k)D^2/(2R) \]
\[ p = D^2/(2R) \]
\[ r = -kD^2/(2R) \]

where \( R \) is the radius of the earth; \( k \) is the coefficient of atmosphere refraction; \( p \) and \( r \) are all direct proportional to \( D^2 \) and inverse ratio to \( R \), and \( r \) is also direct proportion to \( k \). The coefficient of atmosphere refraction \( k \) is related to temperature, air pressure and especially the air density, and its value is hard to measure. Generally, considering the atmosphere refraction curve as round arc approximately, and the radius of it is about 6-7 times of the earth radius. With the same principle as computes the value of \( p, r \) is:

\[ r = D^2/(2R') \approx (0.14-0.16)D^2/(2R) \]

Then \( k \approx 0.14-0.16 \). As Eq. (3) indicates; if the forth-elevation difference is positive (negative), the correction of two differences makes elevation difference increase (decrease); and then the back-elevation difference is negative (positive), with the correction of two differences, the elevation difference will decrease (increase). To average the forth and back-elevation differences can counteract the influence of ball difference completely.

If the air condition of forth-measurement is almost same as that of back-measurement, the value of \( k \) varies little, so the atmosphere difference can counteract mostly. This is the reason that we adopt bilateral observation in precise trigonometric leveling.

![Fig. 2 Indication about correction of atmosphere refraction and the earth curvature](image)

1.1.2 A method to measure the value of \( k \)

The coefficient of atmosphere refraction \( k \) is a variable value because of the influence of the atmosphere density, and the influence of atmosphere refraction can’t be eliminated thoroughly by correction and averaging the forth and back-elevation differences. In some specific conditions, we can measure the local value of \( k \) with the following method. The idea of this method is that with the correction of this two kinds of differences, the trigonometric leveling difference should be equal to the precise level elevation. Then we can get the calculation formula of \( k \) from Eq. (3).

\[ k = (-2R)r/(D^2) = (-2R)(h_L - D\tan a - i + \nu - p)/D^2 \quad (4) \]

where \( h_L \) is the precise level elevation difference.
Observing the vertical angle \( \alpha \) and horizontal distance \( D \) and measuring the height of instrument and target at two points whose precise level elevations are known, the value of \( k \) for the station in this direction can be calculated. If observing in different period of time (for example, one time every hour), we can get a series of discrete values, and can get a curve of \( k \) values varied with time. It can be used as the values of \( k \) in that region.

1.2 Error source and accuracy analysis

Making derivations of the first formula and \( f \) of Eq. (3), we can get the following formulas according to the error propagation law,

\[
\begin{align*}
m_{\alpha}^2 &= \tan^2 \alpha \cdot m_{\alpha}^2 + \frac{D^2}{\cos^2 \alpha} \cdot \rho^2 + m_i^2 + m_{\gamma}^2 + m_{\delta}^2 \\
m_{\beta}^2 &= (1 + k^2) \frac{D^2}{R^2} m_{\beta}^2 + \frac{D^2}{4R} m_{\delta}^2 \\
m_{\gamma}^2 &= \frac{D^2}{\cos^2 \alpha} \cdot \rho^2 m_{\gamma}^2 + \left( \tan^2 \alpha + \frac{(1 + k^2)}{R^2} D^2 \right) \cdot m_{\delta}^2 + \frac{D^2}{4R} m_{\gamma}^2 + m_{\delta}^2 + m_i^2
\end{align*}
\]

Combining the above two formulas, we can get:

\[
m_{\alpha}^2 = \frac{-D^2}{\cos^2 \alpha} \cdot \rho^2 m_{\gamma}^2 + \left( \tan^2 \alpha + \frac{(1 + k^2)}{R^2} D^2 \right) \cdot m_{\delta}^2 + \frac{D^2}{4R} m_{\gamma}^2 + m_{\delta}^2 + m_i^2
\]

From Eq. (7), it is clear that the main factors affecting the accuracy of the trigonometric leveling are as follows (\( \rho = 206 265 \), \( R = 6 370 \) km).

1) Error of \( \alpha \). Adopting the electronic total station instrument TCA2003 for full-automatic surveying, and measuring 9 cycles with face left and face right and then averaging the all values, \( \delta \) can amount to \( \pm 0.5'' \) to \( \pm 0.7'' \). Because the influence of \( \delta \) is proportional to square of line-length, \( D \) is not proper to be long. The conclusion, in this paper, is that it's proper when \( D \) is not over 600 m. Supposing \( \delta = \pm 0.7'' \), while \( D = 100 \) m, \( \alpha = 30^\circ \), the first part of the right side of Eq. (7) equals 0.20 mm², and while \( D = 600 \) m, \( \alpha = 30^\circ \), the first part of the right side of Eq. (7) equals 7.37 mm².

2) Influence of \( \delta \) and \( \nu \). If all triangulation points have forced centering observation stand, measuring the heights of instrument and target at four directions with the examined bar steel rules, the accuracy of \( \nu \) and \( \nu \) can amount to 0.3 mm. It requests that the stand face is level and the difference of two directions is less than 0.3 mm, and then the average of the four direction values is the final value.

Introducing the above results to Eq. (7), and then we have \( m_{\alpha}^2 = 7.37 + 0.75 + 1.28 + 0.3 + 0.3 \). According to the principle of equal-effect and neglecting to account, the influence of \( \delta \), \( \nu \), and \( \nu \) can be neglected to account.

According to the above analysis, error of \( \alpha \) is the biggest influence on \( m_{\alpha} \), and it is the main error source of trigonometric leveling, and the influence of error of \( k \) on \( m_{\alpha} \) is the second one, and the influence of error of \( D \) on \( m_{\alpha} \) is very small. While \( D > 600 \) m, the influence of \( k \) error will increase sharply, so \( D \) is not proper to be too large. Furthermore, the value of \( k \) should be measured accurately and meanwhile we should correct the elevation differences.

Supposing \( k = 0.16 \), \( m_{\alpha} = \pm 0.4 \) mm, \( D = 600 \) m, the second part of the right side of Eq. (7) equals 0.75 mm².

3) Error of \( k \). The coefficient of atmosphere refraction \( k \), decided primarily by temperature grads and atmosphere density, is commonly considered that it varies greatly in morning and evening but is relatively stable at noon and most stable in cloudy days and the nighttime. Measuring and computing \( k \) with the method put forward in Section 1.1.2, and then averaging the forth and back elevation differences after correcting the two differences, can weaken the influence of \( k \) greatly. Experiments indicate that the mid-error of \( k \) is roughly \( \pm 0.03 \pm 0.05 \). Supposing \( m_{\alpha} = \pm 0.04 \), while \( D = 600 \) m, the third part of the right side of Eq. (7) equals 0.75 mm².

4) Influence of \( i \) and \( v \). If all triangulation points have forced centering observation stand, measuring the heights of instrument and target at four directions with the examined bar steel rules, the accuracy of \( i \) and \( v \) can amount to 0.3 mm. It requests that the stand face is level and the difference of two directions is less than 0.3 mm, and then the average of the four direction values is the final value.

Introducing the above results to Eq. (7), and then we have \( m_{\alpha}^2 = 7.37 + 0.75 + 1.28 + 0.3 + 0.3 \). According to the principle of equal-effect and neglecting to account, the influence of \( m \) and \( m \) can be neglected to account.

According to the above analysis, error of \( \alpha \) is the biggest influence on \( m_{\alpha} \), and it is the main error source of trigonometric leveling, and the influence of error of \( k \) on \( m_{\alpha} \) is the second one, and the influence of error of \( D \) on \( m_{\alpha} \) is very small. While \( D > 600 \) m, the influence of \( k \) error will increase sharply, so \( D \) is not proper to be too large. Furthermore, the value of \( k \) should be measured accurately and meanwhile we should correct the elevation differences.

Supposing \( m_{\alpha} = \pm 0.4 \) mm, \( m_{\alpha} = \pm 1.5 \) mm, \( k = 0.16 \), \( m_{\alpha} = \pm 0.04 \), \( R = 6 370 \) km, neglecting \( m \), and \( m \), taking \( \pm 3 \) mm as limit error for the MSE of trigonometric leveling, choosing different \( \alpha \) and \( D \) for Eq. (7), then we will get Table 1.
and Table 2 as follows.

| Table 1 | Fix D, change a, and calculate m_Ab |
|---------|-----------------------------------|
| D/m     | 100  200  300  400  500  600 |
| a/(°)   | 30  30  30  30  30  30 |
| m_Ab/(±mm) | 0.98  1.26  1.63  2.07  2.55  3.07 |

| Table 2 | Fix a, change D, and calculate m_Ab |
|---------|-----------------------------------|
| D/m     | 600  600  600  600  600  600 |
| a/(°)   | 5  10  15  20  25  30 |
| m_Ab/(±mm) | 2.35  2.40  2.49  2.63  2.81  3.07 |

As the Table 1 and Table 2 show, while fix the a, m_Ab will increase quickly with D's increase; When fix the D, m_Ab will increase slowly with a's increase. And it is easy to conclude that D is not proper over 600 m, and a is not proper over 30°.

2 Conditions and key problems of replacing the first order level surveying with trigonometric leveling

2.1 Conditions

For replacing the first order level surveying with trigonometric leveling following conditions should be met. Having constructed control network and deformation monitoring network with extra-high accuracy request, measuring with the distance and angle surveying method and the network are composed of tough graphs of triangles or ground-quadrangle; the network-points have forced centering observation stand; the distance between two neighboring points is less than 600 m, and the vertical angle of it is less than 30°; having points that distributed equally and the height of those points can be measured easily by first geometrical level surveying as known-points (occupy 1/3 or so ) for rigorous adjustment for trigonometric leveling network. By and large, there are a number of networks with above conditions.

2.2 Key problems

It is best to adopt high accuracy electronic total station instruments such as TCA2003 for full-automatic measuring; and the network should be surveyed when the atmosphere is stable, and the best time is cloudy day or nighttime. As the full-automatic surveying is high-intellectualized, it needs less surveying time and can collect a great large number of data, the influence of atmosphere is easy to find out and be eliminated. Choose some representative sides spanning river or valley and the influence of atmosphere vertical refraction is great, and whose elevation difference can be easily gotten. And calculate the value of k and its variety from trigonometric leveling and the known elevation difference, draw the values curve of k, on these grounds we can correct the forth and back elevation differences and average them. As handling the data, there is need of some known-elevation points as known points and adjusts them rigorously as trigonometric leveling network.

3 Experiment and analysis

The plane and vertical network for safety monitoring of a dam (as Fig. 3 shows) needs period surveying, and the accuracy must reach the first order request that the plane and height accuracies of weakest point are all ±1.5 mm. And the net is ordinarily measured with first order side-angle net and first geometrical level. Whole net has total 20 points, at all of the points forced centering observation stand has established. The longest side of the net is 610 m, and the shortest side is 140 m, the rough average is 350 m; and the biggest vertical angle is 30°, the minimum is 2.5°, rough average is 15°.

![Fig. 3 Trigonometric leveling surveying net](image-url)
In order to prove whether the trigonometric leveling can replace first order level surveying, besides using TCA2003 to conduct side-angle surveying and first order level surveying according to the first order net request, we also measure the $k$ value with the method above mentioned, and measure the heights of stand, instrument and target accurately, and correct the elevation differences of 52 sides and average the forth and back differences as observed values of elevation differences, and then take 6 distributing evenly and easy to be measured with first order level surveying points from all 20 points as known-elevation points for adjustment, lastly we compare the triangulation elevations of 14 points with the first order level elevations after adjusting, and the statistical results is shown in Table 3, where $h_T$ is the triangulation elevation; $h_L$ is the first level elevation. According to the MSE of first order level point $m_{h_L} (± 1.5 \text{ mm})$, the limited error of the two heights is $\delta_m = ± 2\sqrt{2}m_{h_L} = ± 4.24 \text{ mm}$ (where $m_{h_L}$ is the MSE of first order level height). As Table 3 shows, the differences between triangulation elevation and first order level elevation of 14 points are all less than 2.5 mm, which are less than limited error great. It explains that it is feasible to replace first order level surveying with trigonometric leveling.

| $h_T-h_L$/mm | $±1.0$ | $±1.0±2.0$ | $±2.5$ |
|--------------|--------|------------|--------|
| Count of points | 2  | 10 | 2 |  

What demanded to point out is that all 14 points are mostly located in hill, it makes first order level surveying very difficult and dangerous. It costs 15 days to survey the level elevations of these points. One among those 14 points is roughly 1 km in level route far away from its neighbor point, and the elevation difference of them achieves 160 m, and the stages of level surveying are more than 120 and it costs 3 days to do forth and back surveying for this point. But for all 20 points trigonometric leveling costs just 6 days, and the work is very easy. The trigonometric leveling is producted full-automatically along with side-angle surveying, which improves work efficiency greatly, and saves the manpower and material resources.

### 4 Conclusions

Aiming at the demand of some projects, by analyzing the error source and accuracy of trigonometric leveling, this paper puts forward a method that ascertain the value of $k$, and expatiates the key technical problems about replacing first order level surveying with the precise trigonometric leveling; and should point out that in some particular conditions, it is not only feasible but also high economical to replace first order level surveying with the precise trigonometric leveling, it has been proved in practice. The content and conclusion of this paper have great consulting significance and practicable value for our setting down relational criterion and production practice.

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