VANDERMONDE DETERMINANTAL IDEALS

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Abstract. We show that the ideal generated by maximal minors (i.e., \(k + 1\)-minors) of a \((k+1) \times n\) Vandermonde matrix is radical and Cohen-Macaulay. Note that this ideal is generated by all Specht polynomials with shape \((n-k, 1, \ldots, 1)\).

1. Introduction

Let \(n, k\) be integers with \(n > k \geq 1\). Consider the polynomial ring \(R = K[x_1, \ldots, x_n]\) over a field \(K\), and the following non-square Vandermonde matrix

\[
M_{n,k} := \begin{pmatrix}
1 & 1 & \cdots & 1 \\
x_1 & x_2 & \cdots & x_n \\
x_1^2 & x_2^2 & \cdots & x_n^2 \\
\vdots & \vdots & \ddots & \vdots \\
x_1^k & x_2^k & \cdots & x_n^k
\end{pmatrix}.
\]

Let \(I_{n,k}^{\text{Vd}}\) denote the ideal of \(R\) generated by all maximal minors (i.e., \(k + 1\) minors) of \(M_{n,k}\).

The purpose of this paper is to prove the following.

Theorem 1.1. \(R/I_{n,k}^{\text{Vd}}\) is a reduced Cohen-Macaulay ring with \(\dim R/I_{n,k}^{\text{Vd}} = k\) and \(\deg R/I_{n,k}^{\text{Vd}} = S(n, k)\), where \(S(n, k)\) stands for the Stirling number of the second kind.

The present paper can be seen as the precursor of our ongoing project [6] on Specht ideals. For a partition \(\lambda\) of \(n\), we can consider the ideal

\[
I_{\lambda}^{\text{Sp}} = (\Delta_T \mid T \text{ is a Young tableau of shape } \lambda)
\]

of \(R\), where \(\Delta_T \in R\) denotes the Specht polynomial corresponding to \(T\) (see [2]). Then we have \(I_{n,k}^{\text{Vd}} = I_{(n-k,1,\ldots,1)}^{\text{Sp}}\). Note that the \(K\)-vector subspace \(\langle \Delta_T \mid T \text{ is a Young tableau of shape } \lambda \rangle\) of \(R\) is the Specht module associated with \(\lambda\) as an \(S_n\)-module. The Specht modules are often constructed in different manner (e.g., using Young tabloids), and play crucial role in the representation theory of symmetric groups (see, for example [5]). General Specht ideals are much more delicate than the Vandermont case \(I_{n,k}^{\text{Vd}}\). For example, \(R/I_{\lambda}^{\text{Sp}}\) is not even pure dimensional. for many \(\lambda\), and the Cohen-Macaulayness of \(R/I_{\lambda}^{\text{Sp}}\) may depend on \(\text{char}(K)\) for some fixed \(\lambda\). In [6], we will use the representation theory of symmetric groups.

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It is noteworthy that Fröberg and Shapiro \[1\] also studied some variants of \(R/I_{n,k}^{\text{vd}}\) in a different context.

## 2. Results and proofs

Extending the base field, we may assume that \(K\) is algebraically closed. Theoretically, this assumption is not necessary in the following argument, but it makes the expositions more readable.

For an ideal \(I \subset R\), set \(V(I) := \{p \mid p \in \text{Spec } R, p \supset I\}\) as usual. For \(a = (a_1, \ldots, a_n) \in K^n\), let \(m_a\) denote the maximal ideal \((x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n)\) of \(R\). By abuse of notation, we just write \(a \in V(I)\) to mean \(m_a \in V(I)\). Clearly, \(a \in V(I)\) if and only if \(f(a) = 0\) for all \(f \in I\).

### Proposition 2.1

We have \(\dim R/I_{n,k}^{\text{vd}} = k\) and

\[
\text{deg } \left( R/\sqrt{I_{n,k}^{\text{vd}}} \right) = S(n, k),
\]

where \(S(n, k)\) stands for the Stirling number of the second kind, that is, the number of ways to partition the set \(\{1, 2, \ldots, n\}\) into \(k\) non-empty subsets.

**Proof.** For \(a = (a_1, \ldots, a_n) \in K^n\), \(a \in V(I_{n,k}^{\text{vd}})\) if and only if

\[
\text{rank}(M_{n,k}(a)) \leq k,
\]

where \(M_{n,k}(a)\) is the matrix given by putting \(x_i = a_i\) for each \(i\) in \(M_{n,k}\). The latter condition is equivalent to that \(#\{a_1, \ldots, a_n\} \leq k\). This is also equivalent to that there is a partition \(\Pi = \{F_1, \ldots, F_k\}\) of the set \([n] := \{1, 2, \ldots, n\}\) such that \(a_i = a_j\) for all \(i, j \in F_l\) \((l = 1, 2, \ldots, k)\). For the above partition \(\Pi\), let \(P_\Pi\) denote the prime ideal

\[
(x_i - x_j \mid i, j \in F_l \text{ for } l = 1, \ldots, k)
\]

of \(R\). Since

\[
\sqrt{I_{n,k}^{\text{vd}}} = \bigcap_{\Pi: \text{partition of } [n]} P_\Pi,
\]

\(\dim R/P_\Pi = k\) for all \(\Pi\), and \(\text{deg } R/P_\Pi = 1\), we are done. \(\square\)

Applying elementary column operations to \(M_{n,k}\), we get the following matrix

\[
M'_{n,k} := \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
x_1 & x_2 - x_1 & x_3 - x_1 & \cdots & x_n - x_1 \\
x_1^2 & x_2^2 - x_1^2 & x_3^2 - x_1^2 & \cdots & x_n^2 - x_1^2 \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
x_1^k & x_2^k - x_1^k & x_3^k - x_1^k & \cdots & x_n^k - x_1^k
\end{pmatrix}
\]

Consider its \(k \times (n-1)\) submatrix

\[
N_{n,k} := \begin{pmatrix}
x_2^2 - x_1^2 & x_3 - x_1 & \cdots & x_n - x_1 \\
x_2^3 - x_1^3 & x_3^2 - x_1^2 & \cdots & x_n^2 - x_1^2 \\
\vdots & \vdots & \cdots & \vdots \\
x_2^{k-1} - x_1^{k-1} & x_3^k - x_1 & \cdots & x_n^{k-1} - x_1^{k-1}
\end{pmatrix}
\]
Clearly, $I_{n,k}^V$ is generated by all maximal minors (i.e., $k$-minors) of $N_{n,k}$.

**Theorem 2.2.** $R/I_{n,k}^V$ is Cohen-Macaulay. Moreover, its minimal graded free resolution is given by the Eagon-Northcott complex (see, for example [4]) associated to the matrix $N_{n,k}$.

**Proof.** Since $ht(I_{n,k}^V) = \dim R - \dim R/I_{n,k}^V = n - k = (n - 1) - k + 1$, $I_{n,k}^V$ is a standard determinantal ideal in the sense of [4]. Hence the assertion is immediate from well-known properties of this notion (c.f. §1.2 of [4]). \qed

When we construct the Eagon-Northcott resolution of $I_{n,k}^V$, we use a symmetric power $\text{Sym}_i V$ of a $k$-dimensional vector space $V$ with a basis $e_1, \ldots, e_k$ such that $\deg e_i = i$ for each $i$. Set

$$p_{i,j}^m := \# \{(a_1, \ldots, a_m) \in \mathbb{N}^m \mid a_1 + a_2 + \cdots + a_m = i, a_1 + 2a_2 + \cdots + ma_m = j\}.$$

For simplicity, set $p_{0,j}^m := \delta_{0,j}$. The following facts are easy to see.

- $p_{i,j}^m \neq 0$ if and only if $i \leq j \leq im$,
- $\sum_j p_{i,j}^m = \binom{m+i-1}{i}$.

For the vector space $V$ discussed above, the dimension of the degree $j$ part of $\text{Sym}_i V$ is $p_{i,j}^k$.

**Corollary 2.3.** For $i \geq 1$, we have

$$\beta_{i,j}(R/I_{n,k}^V) = p_{i-1,j-\frac{k(k+1)}{2}}^k \times \binom{n-1}{k+i-1}.$$  

**Proof.** Since the minimal free resolution of $R/I_{n,k}^V$ is given by the Eagon-Northcott complex, we have

$$\beta_{i,j}(R/I_{n,k}^V) = \left(\dim_K [(\text{Sym}_{i-1} V) \otimes_K \bigwedge^k V]_j\right) \times (\dim_K \bigwedge^{k+i-1} W)$$

$$= (\dim_K [\text{Sym}_{i-1} V]_{j-\frac{k(k+1)}{2}}) \times (\dim_K \bigwedge^{k+i-1} W)$$

$$= p_{i-1,j-\frac{k(k+1)}{2}}^k \times \binom{n-1}{k+i-1},$$

where $V$ is the $K$-vector space considered above, and $W$ is a $K$-vector space of dimension $n - 1$. \qed

**Example 2.4.** Since $p_{i,j}^2 = 0$ or 1 for all $i, j$, we have $\beta_{i,j}(R/I_{n,2}^V) = 0$ or $\binom{n-1}{i+1}$ for all $i \geq 1$. For example, the Betti table of $R/I_{6,2}^V$ is the following.

| total: | 1 | 10 | 20 | 15 | 4 |
|--------|---|----|----|----|---|
| 0:     | 1 | .   | .  | .  | . |
| 1:     | .  | .   | .  | .  | . |
| 2:     | .  | 10  | 10 | 5  | 1 |
| 3:     | .  | .   | 10 | 5  | 1 |
| 4:     | .  | .   | .  | 5  | 1 |
| 5:     | .  | .   | .  | .  | 1 |
The following are the Betti tables of $R/I_{Vd,6}^{6,3}$ and $R/I_{Vd,7}^{7,3}$, respectively.

\[
\begin{array}{cccc}
\text{total:} & 1 & 10 & 15 & 6 \\
0: & 1 & . & . & . \\
1: & . & . & . & . \\
2: & . & . & . & . \\
3: & . & . & . & . \\
4: & . & . & . & . \\
5: & . & 10 & 5 & 1 \\
6: & . & . & 5 & 1 \\
7: & . & . & 5 & 2 \\
8: & . & . & . & 1 \\
9: & . & . & . & 1 \\
\end{array}
\]

\[
\begin{array}{cccc}
\text{total:} & 1 & 20 & 45 & 36 & 10 \\
0: & 1 & . & . & . & . \\
1: & . & . & . & . & . \\
2: & . & . & . & . & . \\
3: & . & . & . & . & . \\
4: & . & . & . & . & . \\
5: & . & 20 & 15 & 6 & 1 \\
6: & . & . & 15 & 6 & 1 \\
7: & . & . & 15 & 12 & 2 \\
8: & . & . & 6 & 2 & . \\
9: & . & . & 6 & 2 & . \\
10: & . & . & . & . & 1 \\
11: & . & . & . & . & 1 \\
\end{array}
\]

**Theorem 2.5.** We have

\[
\deg R/I_{Vd,n,k}^{n} = S(n, k).
\]

**Proof.** Since $I_{Vd,n,1}^{n}$ is an ideal generated by linear forms, we have $\deg R/I_{Vd,n,1}^{n} = 1 = S(n, 1)$ for all $n \geq 2$. Since $I_{Vd,n,n-1}^{n}$ is a principal ideal generated by a polynomial of degree $\binom{n}{2}$, we have $\deg R/I_{Vd,n,n-1}^{n} = \binom{n}{2} = S(n, n - 1)$. It is well-known that the Stirling numbers of the second kind satisfy the recurrence relation

\[
S(n, k) = S(n - 1, k - 1) + kS(n - 1, k).
\]

So it suffices to show that $\deg R/I_{Vd,n,k}^{n}$ also satisfies the corresponding relation

\[
\deg R/I_{Vd,n,k}^{n} = \deg R'/I_{Vd,n-1,k-1}^{n-1} + k(\deg R'/I_{Vd,n-1,k}^{n-1})
\]

for $n - 1 > k$, where $R'$ is the polynomial ring $K[x_1, \ldots, x_{n-1}]$.

From now on, we assume that $n - 1 > k$. Note that the matrices

\[
N_{n-1,k-1} := \begin{pmatrix}
x_2 - x_1 & x_3 - x_1 & \cdots & x_{n-1} - x_1 \\
x_2^2 - x_1^2 & x_3^2 - x_1^2 & \cdots & x_{n-1}^2 - x_1^2 \\
\vdots & \vdots & \ddots & \vdots \\
x_2^{k-1} - x_1^{k-1} & x_3^{k-1} - x_1^{k-1} & \cdots & x_{n-1}^{k-1} - x_1^{k-1}
\end{pmatrix}
\]
and
\[
N_{n-1,k} := \begin{pmatrix}
x_2 - x_1 & x_3 - x_1 & \cdots & x_{n-1} - x_1 \\
x_2^2 - x_1^2 & x_3^2 - x_1^2 & \cdots & x_{n-1}^2 - x_1^2 \\
\vdots & \vdots & \ddots & \vdots \\
x_k^k - x_1^k & x_k^k - x_1^k & \cdots & x_k^k - x_1^k
\end{pmatrix}
\]
can be regarded as submatrices of $N_{n,k}$. Let $J_1$ and $J_2$ be the ideals (of $R$) generated by all maximal minors of $N_{n-1,k-1}$ and $N_{n-1,k}$, respectively. By [3, Lemma 2.3 (2)], we have
\[\deg R/I_{n,k}^{Vd} = \deg R/J_1 + k(\deg R/J_2)\]
On the other hand, we have $R/J_1 \cong (R/I_{n-1,k-1}^{Vd})[x_n]$, and hence $\deg R/J_1 = \deg R/I_{n-1,k-1}^{Vd}$. Similarly, $\deg R/J_2 = \deg R/I_{n-1,k}^{Vd}$. Now (2.1) is clear. \qed

Remark 2.6. In the first version of this paper, the key formula (2.1) was shown by a direct computation from Corollary 2.3. More precisely, the equations
\[\beta_{1,j}(R/I_{n,k}^{Vd}) = \beta_{1,j-k}(R/I_{n-1,k-1}^{Vd}) + \beta_{1,j}(R/I_{n-1,k}^{Vd})\]
and
\[\beta_{i,j}(R/I_{n,k}^{Vd}) = \beta_{i,j-k}(R/I_{n-1,k-1}^{Vd}) + \beta_{i,j}(R/I_{n-1,k}^{Vd}) + \beta_{i-1,j-k}(R/I_{n-1,k}^{Vd})\]
hold for $i \geq 2$. One can check this in Example 2.4. Anyway, we see that these equations imply (2.1).

Now we know that (2.1) is a direct consequence of [3, Lemma 2.3 (2)]. It is noteworthy that this lemma is a result of Gorenstein liaison theory.

Corollary 2.7. $R/I_{n,k}^{Vd}$ is reduced.

Proof. Since $A := R/I_{n,k}^{Vd}$ is Cohen–Macaulay, any non-zero ideal $I \subset A$ satisfies $\dim I = \dim A$ as an $A$-module. Hence if $A$ is not reduced, then $\deg A > \deg A/\sqrt{(0)}$. However, it contradicts the fact that
\[\deg (R/I_{n,k}^{Vd}) = S(n,k) = \deg \left(R/\sqrt{I_{n,k}^{Vd}}\right)\]
\[\square\]

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