SATURATED ACTIONS BY FINITE DIMENSIONAL HOPF ∗-ALGEBRAS ON C*-ALGEBRAS

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Abstract. If a finite group action $\alpha$ on a unital C*-algebra $M$ is saturated, the canonical conditional expectation $E : M \to M^\alpha$ onto the fixed point algebra is known to be of index finite type with $\text{Index}(E) = |G|$ in the sense of Watatani. More generally if a finite dimensional Hopf ∗-algebra $A$ acts on $M$ and the action is saturated, the same is true with $\text{Index}(E) = \dim(A)$.

In this paper we prove that the converse is true. Especially in case $M$ is a commutative C*-algebra $C(X)$ and $\alpha$ is a finite group action, we give an equivalent condition in order that the expectation $E : C(X) \to C(X)^\alpha$ is of index finite type, from which we obtain that $\alpha$ is saturated if and only if $G$ acts freely on $X$. Actions by compact groups are also considered to show that the gauge action $\gamma$ on a graph C*-algebra $C^*(E)$ associated with a locally finite directed graph $E$ is saturated.

1. Introduction

It is known [17] that if $\alpha$ is an action by a compact group $G$ on a C*-algebra $M$, the fixed point algebra $M^\alpha$ is isomorphic to a hereditary subalgebra $e(M \times_\alpha G)e$ of the crossed product $M \times_\alpha G$ for a projection $e$ in the multiplier algebra of $M \times_\alpha G$. If $e(M \times_\alpha G)e$ is full in $M \times_\alpha G$ (that is, $e(M \times_\alpha G)e$ generates $M \times_\alpha G$ as a closed two-sided ideal), the action $\alpha$ is said to be saturated (the notion of saturated action was introduced by Rieffel [14, Chap.7]). Every action $\alpha$ with a simple crossed product $M \times_\alpha G$ is obviously saturated.

On the other hand, an action of a finite dimensional Hopf ∗-algebra $A$ on a unital C*-algebra $M$ is considered in [18] and it is shown that if the action is saturated, the canonical conditional expectation $E : M \to M^A$ onto the fixed point algebra $M^A$ is of index finite type in the sense of Watatani [19] and $\text{Index}(E) = (\dim A)/1$.

The main purpose of the present paper is to prove that the converse is also true. We see from our result that for an action $\alpha$ by a finite group $G$, $\alpha$ is saturated if and only if the canonical expectation $E : M \to M^\alpha$ is of index-finite type with index $\text{Index}(E) = |G|$.

Besides, we consider actions by compact groups to study the saturation property of a gauge action $\gamma$ on a C*-algebra $C^*(E)$ associated with a locally finite directed graph $E$ with no sinks or sources. This paper is organized as follows.

In section 2, we review the C*-basic construction from [19] and finite dimensional Hopf ∗-algebras from [18] setting up our notations. Then we prove in section 3 that if $A$ is a finite dimensional Hopf ∗-algebra acting on a unital C*-algebra $M$ such that $E : M \to M^A$ is of index finite type with $\text{Index}(E) = (\dim A)/1$, then the action is saturated (Theorem 3.3).

In section 4, we deal with the crossed product $M \times_\alpha G$ by a finite group in detail and give other equivalent conditions in order that $\alpha$ be saturated. From the

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conditions one easily see that an action with the Rokhlin property \[7\] is always saturated. Also we shall show that if \(M\) has the cancellation, an action with the tracial Rokhlin property \[12\] on \(M\) is saturated.

Note that even for an action \(\alpha\) by the finite group \(\mathbb{Z}_2\), the expectation \(E : M \to M^\alpha\) may not be of index finite type in general \[19\], Example 2.8.4. For a commutative \(C^*\)-algebra \(C(X)\) and a finite group action \(\alpha\), we give a necessary and sufficient condition that \(E : C(X) \to C(X)^\alpha\) is of index finite type (Theorem 4.10) and provide a formula for \(\text{Index}(E)\). Then as a corollary we obtain that \(\alpha\) is saturated if and only if \(G\) acts freely on \(X\).

In section 5, we consider a compact group action \(\alpha\) and investigate the ideal \(\mathcal{J}_\alpha\) of \(M \times_\alpha G\) generated by the hereditary subalgebra \(e(M \times_\alpha G)e\). Then we apply the result on \(\mathcal{J}_\alpha\) to the gauge action on a graph \(C^*\)-algebra in section 6. As a generalization of the Cuntz-Krieger algebras \[5\], the class of graph \(C^*\)-algebras \(C^*(E)\) associated with directed graphs \(E\) has been studied in various directions by considerably many authors (for example see the bibliography in the book \[13\] by Raeburn). In \[9\], Kumjian and Pask show among others that if \(\gamma\) is the gauge action on \(C^*(E)\), then \(C^*(E)^\gamma\) is stably isomorphic to the crossed product \(C^*(E) \times_\gamma \mathbb{T}\), which was done by hiring the notions of skew product of graphs and groupoid \(C^*\)-algebras. In Theorem 6.3 we shall directly show that the gauge action is actually saturated (this implies that \(C^*(E)^\gamma\) and \(C^*(E) \times_\gamma \mathbb{T}\) are stably isomorphic).

2. Preliminaries

Watatani’s index theory for \(C^*\)-algebras. In \[19\], Watatani developed the index theory for \(C^*\)-algebras, and here we briefly review the basic construction \(C^*(B, e_A)\). Let \(B\) be a \(C^*\)-algebra and \(A\) its \(C^*\)-subalgebra containing the unit of \(B\). Let \(E : B \to A\) be a faithful conditional expectation. If there exist finitely many elements \(\{v_i\}_{i=1}^n\) in \(B\) satisfying the following

\[
b = \sum_i E(bv_i)v_i^* = \sum_i v_iE(v_i^*b), \quad \text{for every } b \in B,
\]

\(E\) is said to be of \textit{index-finite type} and \(\{v_i, v_i^*\}_{i=1}^n\) is called a \textit{quasi-basis} for \(E\). The positive element \(\sum_i v_i v_i^*\) is the \textit{index} of \(E\), \(\text{Index}(E)\), which is known to be an element in the center of \(B\) and does not depend on the choice of quasi-bases for \(E\) (\[19\], Proposition 1.2.8). Let \(B\) be the completion of the pre-Hilbert module \(\mathcal{B}_0 = \{\eta(b) \mid b \in B\}\) over \(A\) with an \(A\)-valued inner product

\[
\langle \eta(x), \eta(y) \rangle = E(x^*y), \quad \eta(x), \eta(y) \in \mathcal{B}_0.
\]

Let \(\mathcal{L}_A(B)\) be the \(C^*\)-algebra of all (right) \(A\)-module homomorphisms on \(B\) with adjoints. For \(T \in \mathcal{L}_A(B)\), the norm \(\|T\| = \sup\{\|Tx\| : \|x\| = 1\}\) is always bounded. Each \(b \in B\) is regarded as an operator \(L_b\) in \(\mathcal{L}_A(B)\) defined by \(L_b(\eta(x)) = \eta(bx)\) for \(\eta(x) \in \mathcal{B}_0\). By \(e_A : B \to B\) we denote the projection in \(\mathcal{L}_A(B)\) such that \(e_A(\eta(x)) = \eta(E(x))\), \(\eta(x) \in \mathcal{B}_0\). Then the \(C^*\)-basic construction \(C^*(B, e_A)\) is the \(C^*\)-subalgebra of \(\mathcal{L}_A(B)\) in which the linear span of elements \(L_b e_A L_b^*\) (\(b, b' \in B\)) is dense.

Finite dimensional Hopf \(*\)-algebras. As in \[18\], a finite dimensional Hopf \(*\)-algebra is a finite matrix pseudogroup of \[20\]. We review from \[18\] the definition and some basic properties of a finite dimensional Hopf \(*\)-algebra which we need in the following section.
Definition 2.1. (\[18\] Proposition 2.1) A finite dimensional $C^*$-algebra is called a finite dimensional Hopf $*$-algebra if there exist three linear maps, 

$$\Delta : A \to A \otimes A, \quad \epsilon : A \to \mathbb{C}, \quad S : A \to A$$

which satisfy the following properties

(i) $\Delta$ (comultiplication) and $\epsilon$ (counit) are $*$-homomorphisms, and $S$ (antipode) is a $*$-preserving antilinear antimultiplicative involution,

(ii) $\Delta(1) = 1 \otimes 1, \quad \epsilon(1) = 1, \quad S(1) = 1$,

(iii) $(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta$,

(iv) $(\epsilon \otimes \text{id})\Delta = \Delta(\epsilon \otimes \text{id})$,

(v) $m(S \otimes \text{id})(\Delta(a)) = \epsilon(a)1 = m(\text{id} \otimes S)(\Delta(a))$ for $a \in A$, where $m : A \otimes A \to A$ is the multiplication.

Proposition 2.2. (\[18\], \[20\]) Let $A$ be a finite dimensional Hopf $*$-algebra. Then the following properties hold.

(i) For $a \in A$, with the notation $\Delta(a) = \sum_i a_i^L \otimes a_i^R$, we have

$$\sum_i \epsilon(a_i^L)a_i^R = a = \sum_i \epsilon(a_i^R)a_i^L,$$

$$\sum_i a_i^L S(a_i^R) = \epsilon(a)1 = \sum_i S(a_i^L)a_i^R,$$

$$\sum_i a_i^R S(a_i^L) = \epsilon(a)1 = \sum_i S(a_i^R)a_i^L.$$

(ii) There is a unique normalized trace (called the Haar trace) $\tau$ on $A$ such that

$$\sum_i \tau(a_i^L)a_i^R = \tau(a)1 = \sum_i \tau(a_i^R)a_i^L, \quad a \in A.$$

(iii) There exists a minimal central projection $e \in A$ (called the distinguished projection) such that $ae = \epsilon(a)e$, $a \in A$. We have

$$\epsilon(a) = 1, \quad S(e) = e, \quad \text{and} \quad \tau(e) = (\dim A)^{-1}.$$

3. Actions by finite dimensional Hopf $*$-algebras

Throughout this section $A$ will be a finite dimensional Hopf $*$-algebra. An action of $A$ on a unital $C^*$-algebra $M$ is a bilinear map $\cdot : A \times M \to M$ such that for $a, b \in A, x, y \in M$, 

$$1 \cdot x = x,$$

$$a \cdot 1 = \epsilon(a)1,$$

$$ab \cdot x = a \cdot (b \cdot x),$$

$$a \cdot xy = \sum_i (a_i^L \cdot x)(a_i^R \cdot y),$$

$$(a \cdot x)^* = S(a^*) \cdot x^*.$$
Then the crossed product $M \rtimes A$ is the algebraic tensor product $M \otimes A$ as a vector space with the following multiplication and $*$-operation:

$$((x \otimes a)(y \otimes b)) := \sum_j x(a^L_j \cdot y) \otimes a^R_j b,$$

$$(x \otimes a)^* := \sum_i (a^L_i)^* \cdot x^* \otimes (a^R_i)^*.$$ 

Identifying $a \in A$ with $1 \otimes a$ and $x \in M$ with $x \otimes 1$, we see [18] that $M \rtimes A = \text{span}\{xa \mid x \in M, a \in A\}$.

For the definition of saturated action of $A$ on $M$, refer to section 4 of [18].

**Proposition 3.1.** ([18]) Let $M^A = \{x \in M \mid a \cdot x = \epsilon(a)x, \text{ for all } a \in A\}$ be the fixed point algebra for the action of $A$ on a unital C$^*$-algebra $M$.

(i) The action is saturated if and only if $M \rtimes A = \text{span}\{xey \mid x, y \in M\}$, where $e \in A$ is the distinguished projection.

(ii) The map $E : M \rightarrow M^A$, $E(x) = e \cdot x$, is a faithful conditional expectation onto the fixed point algebra such that

$$E((a \cdot x)y) = E(x(S(a) \cdot y)), \quad a \in A, \ x, y \in M.$$ 

(iii) The linear map $F : M \rtimes A \rightarrow M$, $F(xa) = \tau(a)x$, is a faithful conditional expectation onto $M$.

Recall that $M_0 := M$ is an $M^A$-valued inner product module by

$$\langle \eta(x), \eta(y) \rangle_{M^A} = E(x^*y)$$

(here we use the convention in [19] for the inner product as in section 2). Since every norm bounded $M^A$-module map on $M_0$ extends uniquely to the Hilbert $M^A$-module $M$, we may identify the $*$-algebra $\text{End}(M_0)$ (in [18]) of norm bounded right $M^A$-module endomorphisms of $M_0$ having an adjoint with the C$^*$-algebra $L_{M^A}(M)$ explained in section 2.

**Remark 3.2.** ([19] Proposition 1.3.3]) If $E : M \rightarrow M^A$ is of index-finite type, then

$$C^*(M, e_{M^A}) = \text{span}\{L_x e_{M^A} L_y \mid x, y \in M\} = L_{M^A}(M).$$

In fact, we see from the proof of [19] Proposition 2.1.5] that $C^*(M, e_{M^A})$ contains the unit of $L_{M^A}(M)$. Thus the ideal $\text{span}\{L_x e_{M^A} L_y \mid x, y \in M\}$ which is dense in $C^*(M, e_{M^A})$ must contain the unit of $L_{M^A}(M)$.

**Theorem 3.3.** Let $A$ be a finite dimensional Hopf $*$-algebra acting on a unital C$^*$-algebra $M$. Then the following are equivalent:

(i) The action is saturated.

(ii) $E : M \rightarrow M^A$ is of index finite type with $\text{Index}(E) = (\dim A)1$. 


Proof. (i)⇒ (ii) is shown in [18] Proposition 4.5.

(ii)⇒ (i). By Remark 3.2, \( C^*(M, e_M) = \text{span} \{ L_x e_M A L_y \mid x, y \in M \} \). Consider a map \( \phi : C^*(M, e_M) \to \mathbb{M} \times A \) given by

\[ \phi \left( \sum_i L_x e_M A L_y_i \right) = \sum_i x_i e y_i. \]

To see that \( \phi \) is well defined, let \( \sum_i L_x e_M A L_y_i = 0 \). Then for each \( z \in M \),

\[ (\sum_i L_x e_M A L_y_i)(\eta(z)) = \sum_i \eta(x_i E(y_i z)) = \eta(\sum_i x_i (e \cdot (y_i z))) = 0, \]

hence by the injectivity of \( \eta \) \( (10) \), \( \sum_i x_i (e \cdot (y_i z)) = 0 \) in \( M \). Since \( (a \cdot x)e = axe \) for \( a \in A, a \in M \) (see (7) of [18]), we thus have

\[ \sum_i x_i (e \cdot (y_i z))e = \sum_i (x_i e y_i)ze = 0 \]

in \( \mathbb{M} \times A \) for every \( z \in M \), which then implies that

\[ (\sum_i x_i e y_i)(ze z') = 0, \quad z, z' \in M. \]

Particularly, \( (\sum_i x_i e y_i)(\sum_i x_i e y_i)^* = 0 \), so that \( \sum_i x_i e y_i = 0 \) (in \( \mathbb{M} \times A \)). Thus \( \phi \) is well defined. It is tedious to show that \( \phi \) is a *-homomorphism such that the range of \( \phi(C^*(M, e_M)) = \mathbb{M} e M \) is an ideal of \( \mathbb{M} \times A \); if \( x, y, z \in M \) and \( a \in A \), then

\[ (za)(xy) = (z(a \cdot x))ey \in \mathbb{M} e M. \]

Hence it suffices to show that \( \phi(1) = 1 \). If \( \{(u_i, u_i^*)\}_{i=1}^n \) is a quasi-basis for \( E \), then

\[ \sum_i L_{u_i} e_M A L_{u_i}^* \eta(z) = \sum_i \eta(u_i E(u_i^* z)) = \eta(z), \quad z \in M, \]

which means that \( \sum_i L_{u_i} e_M A L_{u_i}^* = 1 \in \mathcal{L} M(A) \). Therefore by Proposition 2.2(iii) and Proposition 3.1(iii)

\[ F(\phi(1)) = F(\sum_{i=1}^n u_i e u_i^*) = \sum_i \tau(e') u_i u_i^* = \frac{1}{\text{dim} A} \sum_i u_i u_i^* = 1. \]

Since \( \phi \) is a *-homomorphism, \( \phi(1) \) is a projection in \( \mathbb{M} \times A \) such that \( F(1 - \phi(1)) = 0 \). But \( F \) is faithful, and \( \phi(1) = 1 \) follows. \( \square \)

4. Actions by finite groups

Throughout this section \( G \) will denote a finite group. As is well known the group \( C^* \)-algebra \( C^*(G) \) generated by the unitaries \( \{ \lambda_g \mid g \in G \} \) is a finite dimensional Hopf *-algebra with

\[ \Delta(\lambda_g) = \lambda_g \otimes \lambda_g, \quad \epsilon(\lambda_g) = 1, \quad S(\lambda_g) = \lambda_{g^{-1}} \text{ for } \lambda_g \in C^*(G). \]

The Haar trace \( \tau \) is given by \( \tau(\lambda_g) = \delta_{1,g} \), where \( \delta \) is the identity of \( G \), and the distinguished projection is \( e = \frac{1}{|G|} \sum_{g \in G} \lambda_g. \)

Let \( \alpha \) be an action of \( G \) on a unital \( C^* \)-algebra \( M \). Then it is easy to see that \( \lambda_g \cdot x := \alpha_g(x) \) for \( g \in G, x \in M \),
defines an action of $C^*(G)$ on $M$. Furthermore $M \rtimes C^*(G)$ is nothing but the usual crossed product $M \rtimes \alpha G = \text{span}\{x\lambda_g \mid x \in M, \ g \in G\}$, and the expectations $E : M \to M^\alpha(= M^{C^*(G)})$, $F : M \rtimes \alpha G \to M$ of Proposition 4.1 are given by

$$E(x) = \frac{1}{|G|} \sum_g \alpha_g(x) \quad \text{and} \quad F(\sum_g x_g \lambda_g) = x_e \ (x, x_g \in M, \ g \in G).$$

Note that for each $(i) = (i)$, we will see in Proposition 5.4 that $\sum \lambda_g$ defines an action of $\mathbb{A}$ because $\epsilon$ is a projection in $\mathbb{A}$:

$$E \eta \lambda_g = \| F(\sum_h x_h \lambda_h) \lambda_{g^{-1}} \| \leq \| \sum_h x_h \lambda_h \| \leq \| \sum_h x_h \lambda_h \|.$$ 

If $J_\alpha$ denotes the closed ideal of $M \rtimes \alpha G$ generated by the distinguished projection $e$, then Proposition 3.3(i) says that $\alpha$ is saturated if and only if $J_\alpha = M \times \alpha G$. We will see in Proposition 5.3 that

$$J_\alpha = \text{span}\{ \sum_g x\alpha_g(y) \lambda_g \mid x, y \in M \} = \text{span}\{ \sum_g x\alpha_g(x^*) \lambda_g \mid x \in M \}.$$ 

The $*$-homomorphism $\varphi : C^*(M, e_{M^\alpha}) \to M \rtimes \alpha G$ we discussed in the proof of Theorem 3.3 can be rewritten as follows.

$$\varphi(L_x e_{M^\alpha} L_y) = \frac{1}{|G|} \sum_g x\alpha_g(y) \lambda_g, \ x, y \in M$$

because $\varphi(L_x e_{M^\alpha} L_y) = xey$ and $e = \frac{1}{|G|} \sum_g \lambda_g$. If $\{(u_i, u_i^*)\}$ is a quasi-basis for $E$, we see from $\sum_i L_{u_i} e_{M^\alpha} L_{u_i^*} = 1$ and (4) that

$$\varphi(1) = \sum_i u_i e_{u_i^*} = \frac{1}{|G|} \sum_g \left( \sum_i u_i \alpha_g(u_i^*) \lambda_g \right)$$

is a projection in $M \rtimes \alpha G$. Recall that $\varphi(1) = 1$ holds if $\alpha$ is saturated.

**Theorem 4.1.** Let $M$ be a unital $C^*$-algebra and $\alpha$ be an action of a finite group $G$ on $M$. Then the following are equivalent:

(i) $\alpha$ is saturated, that is, $J_\alpha = M \times \alpha G$.

(ii) $E : M \to M^\alpha$ is of index finite type with $\text{Index}(E) = |G|$.

(iii) $E : M \to M^\alpha$ is of index finite type with $\text{Index}(E) = |G|$ and

$$\sum_i u_i \alpha_g(u_i^*) = 0, \ g \neq e$$

for a quasi-basis $\{(u_i, u_i^*)\}$ for $E$.

(iv) There exist $\{b^g_j \in M \mid g \in G, \ 1 \leq j \leq m\}$ for some $m \geq 1$ such that

(a) $\alpha_g(b^g_j) = b^g_j$, for $j = 1, \ldots, m$ and $g, h \in G$.

(b) $\sum_j b^g_j(b^g_j)^* = \delta_{gh}$.

(v) For every $\varepsilon > 0$, there exist $\{b^g_j \in M \mid g \in G, \ 1 \leq j \leq m\}$ for some $m \geq 1$ such that

(a) $\sum_j \| \alpha_g(b^g_j) - b^g_j \| < \varepsilon$,

(b) $\| \sum_j b^g_j(b^g_j)^* - \delta_{gh} \| < \varepsilon$.

**Proof.** (i) $\iff$ (ii) follows from Theorem 3.3.

(i) $\implies$ (iii). If $\{(u_i, u_i^*)\}$ is a quasi-basis for $E$, we have from (5) that $\sum_i u_i \alpha_g(u_i^*) = 0$ for $g \neq e$ since $\varphi(1) = 1$.

(iii) $\implies$ (ii). Obvious.
(i) \implies (iv). Suppose \( J_\alpha = M \times_\alpha G \). By (3) there exist \( m \in \mathbb{N} \) and \( b_j \in M \), \( 1 \leq j \leq m \), such that
\[
\sum_g \left( \sum_j b_j \alpha_g(b_j^*) \right) \lambda_g = 1.
\]
Thus
\[
\sum_j b_j b_j^* = 1 \quad \text{and} \quad \sum_j b_j \alpha_g(b_j^*) = 0 \quad \text{for} \ g \neq \iota.
\]
Set \( b_j^g := \alpha_g(b_j) \). Then
\[
\alpha_g(b_j^g) = \alpha_g(\alpha_h(b_j)) = \alpha_{gh}(b_j) = b_{gh}^j,
\]
\[
\sum_j b_j^g(b_j^*)^g = \sum_j \alpha_g(b_j) \alpha_h(b_j^*) = \alpha_g \left( \sum_j b_j \alpha_{-1h}(b_j^*) \right) = \delta_{gh} \text{ by (2)}. \tag{7}
\]
(iv) \implies (v). Obvious.
(v) \implies (i). Let \( \varepsilon > 0 \) and let \( \{b_j^g \in M \mid g \in G, \ 1 \leq j \leq m\} \) satisfy (a) and (b) of (v). Note that (b) implies \( ||b_j^g|| < 1 + \varepsilon \) for \( g \in G, \ 1 \leq j \leq m \). Indeed from \( || \sum_j b_j^g(b_j^*)^g - 1 || \leq || \sum_j b_j^g(b_j^*)^g - 1 || < \varepsilon \), we have \( ||b_j^g||^2 \leq || \sum_j b_j^g(b_j^*)^g || < 1 + \varepsilon \). Then
\[
|| \sum_{h,j} (\sum_g b_h^g \alpha_g((b_h^g)^*) \lambda_g) - |G| ||
= || \sum_g (\sum_{h,j} b_h^g \alpha_g((b_h^g)^*) \lambda_g) - |G| ||
= || \sum_{h,j} \sum_g b_h^g \alpha_g((b_h^g)^*) \lambda_g - |G| ||
\leq \sum_{h,j} || \sum_g b_h^g \alpha_g((b_h^g)^*) \lambda_g - |G| || + \sum_{g \neq \iota} \sum_{h,j} || \sum_j b_j^g \alpha_g((b_j^g)^*) ||
\leq \sum_g \sum_{h,j} || \sum_j b_j^g \alpha_g((b_j^g)^*) - 1 || + \sum_{g \neq \iota} \sum_j \sum_{h,j} || \sum_j b_j^g \alpha_g((b_j^g)^*) ||
< \varepsilon |G| + \sum_{g \neq \iota} \sum_{j} \sum_{h} || \sum_j b_j^g \alpha_g((b_j^g)^*) - (b_j^g)^* || + \sum_{g \neq \iota} \sum_{j} \sum_{h} || b_j^g \alpha_g((b_j^g)^*) ||
< \varepsilon \left( |G| + |G|^2 \max_{g,j} ||b_j^g|| + |G|^2 \right)
< \varepsilon \left( |G| + |G|^2 (1 + \varepsilon) + |G|^2 \right).
\]
Since \( \sum_{h,j} \sum_g b_h^g \alpha_g((b_h^g)^*) \lambda_g \in J_\alpha \) and \( \varepsilon \) can be chosen to be arbitrarily small, we conclude that \( J_\alpha = M \times_\alpha G \). \( \Box \)

**Example 4.2.** Let \( w = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \) be a unitary with \( w^n = 1 \) and define an automorphism \( \alpha \) on \( M_2(\mathbb{C}) \) by \( \alpha(a) = wav^*, \ a \in M_2(\mathbb{C}) \). We will show that \( \alpha \) is saturated if and only if \( z_2 = -z_1 \). For this, recall from (5) that \( \alpha \) is saturated if and only if there exist \( x_j \in M_2(\mathbb{C}) \), \( 1 \leq i \leq m \), satisfying
\[
\sum_{k=0}^{n-1} \sum_{j=1}^{m} x_j \alpha^k(x_j^*) \lambda_k = 1_{M_2(\mathbb{C})} \tag{8}
\]
Hence, particularly for \( k = 0, 1 \), we have
\[
\sum_j x_j x_j^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \sum_j x_j \alpha(x_j^*) = \sum_j x_j w x_j^* w^* = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

With \( x_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \) and \( z_i = e^{i \theta_i}, i = 1, 2 \), this means
\[
\sum_j \left( \begin{array}{cc}
|a_j|^2 + |b_j|^2 & a_j \bar{c}_j + b_j \bar{d}_j \\
|c_j|^2 + |d_j|^2 & a_j c_j + b_j d_j
\end{array} \right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]
\[
\sum_j \left( \begin{array}{cc}
|a_j|^2 + e^{i(\theta_2-\theta_1)} |b_j|^2 & e^{i(\theta_1-\theta_2)} a_j \bar{c}_j + b_j \bar{d}_j \\
|c_j|^2 + e^{i(\theta_2-\theta_1)} |d_j|^2 & e^{i(\theta_1-\theta_2)} a_j c_j + |d_j|^2
\end{array} \right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \quad (9)
\]

Therefore, by comparing (1,1) entries of each matrices, it follows that if \( \alpha \) is saturated, then there exist positive real numbers \( a = \sum_j |a_j|^2 > 0, b = \sum_j |b_j|^2 > 0 \) such that
\[
a + b = 1 \quad \text{and} \quad a + e^{i(\theta_2-\theta_1)} b = 0. \quad (10)
\]

Note that \( b \neq 0 \) since \( b = 0 \) implies \( a = 0 \) from \( \sum_j (|a_j|^2 + e^{i(\theta_2-\theta_1)} |b_j|^2) = 0 \) in (9). There are three possible cases for \( \theta_1, \theta_2 \) as follows.

(i) If \( \theta_2 - \theta_1 \equiv 0 (\mod 2\pi) \), that is, \( \alpha \) is trivial, then (10) is not possible.

(ii) If \( \theta_2 - \theta_1 \equiv \pi (\mod 2\pi) \), then
\[
x_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad x_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}
\]
satisfy (9) with \( m = 2 \). Thus \( \alpha \) is saturated.

(iii) If \( \theta_2 - \theta_1 \neq 0, \pi (\mod 2\pi) \), then (10) is not possible for any \( a, b > 0 \). Hence \( \alpha \) is not saturated.

Remark 4.3. Let \( \alpha, \beta \in \text{Aut}(M) \) satisfy \( \alpha^n = \beta^n = \text{id}_M \) for some \( n \geq 1 \). If there is a unitary \( u \in M \) such that \( \beta = \text{Ad}(u) \circ \alpha \), then \( \alpha \) and \( \beta \) are said to be exterior equivalent, and if this is the case the crossed products are isomorphic, \( M \times_\alpha G \cong M \times_\beta G \), [13] p.45. Example 4.2 says that the property of being saturated may not be preserved under exterior equivalence. Also the case (iii) of Example 4.2 above with \( w = \text{diag}(\lambda, \lambda), \lambda = e^{\frac{2\pi i}{3}} \) (hence \( \theta_1 - \theta_2 = \frac{2\pi }{3} \neq \frac{\pi }{2} \equiv \pi (\mod 2\pi) \)), shows that \( \text{Index}(E) < |G| \) is possible even when \( E \) is of index-finite type. In fact, if \( u_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( u_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), then \( \{(u_i, u_i^*)\}_{i=1}^2 \) forms a quasi-basis for \( E \), but \( \text{Index}(E) = 2 < |Z_3| \).

Remark 4.4. Recall that the Rokhlin property and the tracial Rokhlin property (weaker than the Rokhlin property) are defined as follows and considered intensively in [11] and [12], respectively:

(a) [7] \( \alpha \) is said to have the \textbf{Rokhlin property} if for every finite set \( F \subset M \), every \( \varepsilon > 0 \), there are mutually orthogonal projections \( \{e_g \, | \, g \in G\} \) in \( M \) such that

(i) \( \|\alpha_g(e_h) - e_{gh}\| < \varepsilon \) for \( g, h \in G \).

(ii) \( \|e_g x - xe_g\| < \varepsilon \) for \( g \in G \) and all \( x \in F \).

(iii) \( \sum_{g \in G} e_g = 1 \).
Proof. Suppose there is a unitary \( g \in M \). Then

\[
\alpha(g) = \sum_{g \in G} e_g \quad \text{for each } g 
\]

exist a family of projections \( \{ e_g \}_{g \in G} \) in \( M \) such that

(i) \( \| \alpha(g) e_h - e_{gh} \| < \varepsilon \) for \( g, h \in G \).

(ii) \( \| e_g x - xe_g \| < \varepsilon \) for \( g \in G \) and all \( x \in F \).

(iii) With \( e = \sum_{g \in G} e_g \), the projection \( 1 - e \) is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of \( M \) generated by \( x \).

The following proposition is actually observed in [12, Lemma 1.13], and we put a proof for reader’s convenience.

**Proposition 4.5.** Let \( M \) be a unital \( C^* \)-algebra and \( \alpha \) be an action of a discrete group \( G \) on \( M \). Suppose that for every \( \varepsilon > 0 \) and every finite subset \( F \subset M \), there exist a family of projections \( \{ e_g \}_{g \in G} \) such that

1. \( \| \alpha(g) e_h - e_{gh} \| < \varepsilon \).
2. \( \| e_g x - xe_g \| < \varepsilon \) for each \( x \in F \).

Then \( \alpha \) is an outer action.

**Proof.** Suppose there is a unitary \( u \in M \) such that \( \alpha(g)(x) = uxu^* \) for every \( x \in M \) \((g \neq i)\). Put \( F = \{ u \} \) and \( 0 < \varepsilon < 1/4 \). Then there exist mutually orthogonal projections \( \{ e_g \}_{g \in G} \) such that \( \| \alpha(g) e_h - e_{gh} \| < \varepsilon \) for every finite subset \( F \subset M \). Thus \( \| e_g u - ue_g u^* \| < \varepsilon \) for each \( x \in F \). Then \( \| \alpha(g) e_i - ue_g u^* \| = 0 \). But

\[
\| \alpha(g) e_i - ue_i u^* \| = \| \alpha(g) e_i - e_g + e_g - e_i - e_i u^* \|
\geq \| e_g - e_i \| - \| \alpha(g) e_i - e_g \| - \| e_i - e_i u \|
\geq 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2}
\]

which is a contradiction.

**Remark 4.6.** If \( M \rtimes \alpha G \) is simple, \( \alpha \) is obviously saturated, and this is the case if \( G \) is a finite group, \( M \) is \( \alpha \)-simple, and \( \mathbb{T}(\alpha_g) \neq \{ 1 \} \) for all \( g \neq i \) (8 Theorem 3.1). In particular, \( M \) is saturated if \( M \) is simple and \( \alpha \) is outer.

But for a nonsimple \( M \), this may not hold. In fact, if \( \alpha \) is an outer action of \( \mathbb{Z}_n \) on \( M \) and \( u \) is a unitary in \( M \) with \( u^n = 1 \) such that the action \( Ad(u) \) on \( M \) is not saturated (as in Example 4.2), then the action \( \alpha \oplus Ad(u) \) on \( M \oplus M \) is outer but not saturated.

Now we show that if \( \alpha \) satisfies the Rokhlin property (or satisfies the tracial Rokhlin property and \( M \) has cancellation) then \( \alpha \) is saturated. For this we first review the cancellation property of \( C^* \)-algebras. For projections \( p, q \) in a \( C^* \)-algebra, we write \( p \perp q \) if \( pq = 0 \), and \( p \sim q \) if they are Murray-von Neumann equivalent.

**Definition 4.7.** A unital \( C^* \)-algebra \( M \) has the cancellation if, whenever \( p, q, r \) are projections in \( M_n(M) \) for some \( n \), with \( p \perp r, q \perp r, \) and \( (p + r) \sim (q + r) \), then \( p \sim q \).
Proposition 4.9. Let $\alpha$ be an action of a finite group $G$ on a unital $C^*$-algebra $M$. Then $\alpha$ is saturated if one of the following holds.

(i) $\alpha$ has the Rokhlin property.

(ii) $\alpha$ has the tracial Rokhlin property and $M$ has the cancellation.

Proof. (i) For an $\epsilon > 0$, there exist mutually orthogonal projections $\{e_g\}_g$ such that $\sum_g e_g = 1$ and $\|\alpha_g(e_h) - e_{gh}\| < \epsilon$. Then, with $m = 1$, the elements $b^h := e_g$ satisfy (v) of Theorem 4.1.

(ii) Now suppose $\alpha$ has the tracial Rokhlin property and $M$ has the cancellation. We shall show that $J_\alpha$ contains the unit of $M \times_\alpha G$. Let $0 < \epsilon < 1$. For each $g \in G$, choose mutually orthogonal projections $\{e^g_h\}_{h \in G}$ such that

$$\|\alpha_k(e^g_h) - e^g_{kh}\| < \frac{\epsilon}{2|G|^2},$$

and put $e^g := \sum_{h \in G} e^g_h$. If $e^g = 1$, for some $g$, then $b^h := e^g_h$ ($h \in G$) will satisfy (v) of Theorem 4.1 as in (i). If $e^g \neq 1$ for every $g \in G$, then by the tracial Rokhlin property of $\alpha$ there exist mutually orthogonal projections $\{f^g_h\}_{h \in G}$ in $M$ such that

$$\|\alpha_k(f^g_h) - f^g_{kh}\| < \frac{\epsilon}{2|G|^2}$$

and

$$(1 - \sum_{h \in G} f^g_h) \sim (e^g)' < e^g$$

for a subprojection $(e^g)'$ of $e^g$. Put $f^g := \sum_{h} f^g_h$. Then since $M$ has cancellation, it follows that $f^g \sim (1 - (e^g))' > (1 - e^g)$. Let $v_g \in M$ be a partial isometry satisfying

$$v_g^* v_g = f^g, \quad v_g v_g^* = 1 - (e^g)' ,$$

and set

$$x_g := \frac{1}{|G|} \sum_{k,h} \left( (e^k_h \alpha_g(e^k_h) + (1 - e^k)v_k f^k_h \alpha_g(f^k_h \alpha_{g^{-1}}(v^*_k))) \right), \quad g \in G.$$

Now we show that the element $x := \sum_g x_g \lambda_g \in J_\alpha$ satisfies $\|x - 1\| < \epsilon$. In fact, for $g \neq 1$,

$$\|x_g\| \leq \frac{1}{|G|} \sum_{k,h} \| (e^k_h \alpha_g(e^k_h) + (1 - e^k)v_k f^k_h \alpha_g(f^k_h \alpha_{g^{-1}}(v^*_k))) \|

\leq \frac{1}{|G|} \sum_{k,h} (\|e^k_h \alpha_g(e^k_h)\| + \|f^k_h \alpha_g(f^k_h)\|)

\leq \frac{1}{|G|} \sum_{k,h} (\|e^k_h \alpha_g(e^k_h) - e_{gh}\| + \|e^k_h e_{gh}\| + \|f^k_h \alpha_g(f^k_h - f^k_{gh})\| + \|f^k_h f^k_{gh}\|)

\leq \frac{1}{|G|} \sum_{k,h} \left( \frac{\epsilon}{2|G|^2} + \frac{\epsilon}{2|G|^2} \right) = \frac{\epsilon}{|G|}.
and
\[ x_\iota = \frac{1}{|G|} \left( \sum_k e^k + \sum_k (1 - e^k) v_k f^k v_k^* \right) \]
\[ = \frac{1}{|G|} \left( \sum_k e^k + \sum_k (1 - e^k)(1 - (e^k)'(1 - e^k)) \right) \]
\[ = 1. \]

For the rest of this section we consider a finite group action on a commutative
\( C^* \)-algebra \( C(X) \). If \( G \) acts on a compact Hausdorff space \( X \), it induces an action,
say \( \alpha \), on \( C(X) \) by
\[ \alpha_g(f)(x) = f(g^{-1}x), \quad f \in C(X). \]
For each \( x \in X \), let \( G_x = \{ g \in G : gx = x \} \) be the isotropy group of \( x \) and for a
subgroup \( H \) of \( G \) (\( H < G \)), put
\[ X_H = \{ x \in X : G_x = H \}. \]
It is readily seen that \( X_H \) and \( X_{H'} \) are disjoint if \( H \neq H' \), and \( X \) is partitioned as
\[ X = \bigcup_{H < G} X_H. \]

**Theorem 4.10.** Let \( X \) be a compact Hausdorff space and \( G \) a finite group acting
on \( X \). If \( \alpha \) is the induced action of \( G \) on \( C(X) \), the following are equivalent:
(i) \( E : C(X) \to C(X)^\alpha \) is of index finite type.
(ii) \( X_H \) is closed for each \( H < G \).
Moreover, if this is case the index of \( E \) is \( \text{Index}(E) = \sum_{H < G} \frac{|G|}{|H|} \chi_{X_H} \), where \( \chi_{X_H} \)
is the characteristic function on \( X_H \).

**Proof.** (i) \( \implies \) (ii). If \( E \) is of index-finite type and \( \{ (u_i, u_i^*) \}_{i=1}^k \) is a quasi-basis for
\( E \), then
\[ \sum_i u_i E(u_i^* f) = f, \]
that is,
\[ \frac{1}{|G|} \sum_i u_i(x) \left( \sum_{g \in G} u_i^*(g^{-1}x) f(g^{-1}x) \right) = f(x), \quad (11) \]
for \( f \in C(X) \) and \( x \in X \). For each \( x \in X \), choose a continuous function \( f_x \in C(X) \)
satisfying \( f_x|_{Gx \setminus \{ x \}} = 0 \) and \( f_x(x) = 1 \). Then (11) with \( f_x \) in place of \( f \) gives
\[ \frac{1}{|G|} \sum_i u_i(x) \left( \sum_{g \in G} u_i^*(g^{-1}x) f_x(g^{-1}x) \right) = f_x(x), \quad (12) \]
and so we have
\[ \frac{|G_x|}{|G|} \sum_i u_i(x) u_i^*(x) = 1. \quad (13) \]
To show that each $X_H$ is closed, let $\{x_n \in X_H : n = 1, 2, \ldots\}$ be a sequence of elements in $X_H$ with limit $x \in X_H$. Then (11) gives

$$f_x(x_n) = \frac{1}{|G|} \sum_i u_i(x_n) \left( \sum_{g \in G} u_i^*(g^{-1}x_n)f_x(g^{-1}x_n) \right)$$

$$= \frac{1}{|G|} \sum_i u_i(x_n) \left( \sum_{g \in H} u_i^*(g^{-1}x_n)f_x(g^{-1}x_n) + \sum_{g \not\in H} u_i^*(g^{-1}x_n)f_x(g^{-1}x_n) \right)$$

$$= \frac{1}{|G|} \sum_i u_i(x_n) \left( |H|u_i^*(x_n)f_x(x_n) + \sum_{g \not\in H} u_i^*(g^{-1}x_n)f_x(g^{-1}x_n) \right).$$

Taking the limit as $n \to \infty$, we have

$$f_x(x) = \frac{1}{|G|} \sum_i u_i(x) \left( |H|u_i^*(x)f_x(x) + \sum_{g \not\in H} u_i^*(g^{-1}x)f_x(g^{-1}x) \right)$$

$$= \frac{1}{|G|} \sum_i u_i(x) \left( |H|u_i^*(x)f_x(x) + |H' \setminus H|u_i^*(x)f_x(x) \right)$$

$$= \frac{|H| + |H' \setminus H|}{|G|} \sum_i u_i(x)u_i^*(x)f_x(x).$$

Therefore, comparing with (13), we obtain

$$|H'| = |H| + |H' \setminus H|$$

since $G_x = H'$ and $f_x(x) = 1$. Hence

$$H \subset H'. $$

On the other hand, since $G_{x_n} = H$, again by (13), \( \frac{|H|}{|G|} \sum_i u_i(x_n)u_i^*(x_n) = 1 \) with the limit \( \frac{|H|}{|G|} \sum_i u_i(x)u_i^*(x) = 1 \) as $n \to \infty$. But also \( \frac{|H'|}{|G|} \sum_i u_i(x)u_i^*(x) = 1 \) by (13), and thus $|H| = |H'|$. Consequently we have

$$H = H'$$

because $H \subset H'$. This shows that $X_H$ is closed.

(ii) $\implies$ (i). Assume that $X_H$ is closed for every subgroup $H$ of $G$. Then $X_H$ is open since there are only finitely many such subsets. Let $\mathcal{U}_H = \{U_{H,i_H} : i_H = 1, 2, \ldots, n_H\}$ be an open covering of $X_H$ such that

$$x \in U_{H,i_H} \implies g^{-1}x \not\in U_{H,i_H} \text{ or } g^{-1}x \not\in X_H \text{ whenever } g^{-1}x \neq x.$$ 

Let $\{v_{H,i_H}\}$ be a partition of unity subordinate to $\mathcal{U}_H$. We understand that the domain of $v_{H,i_H}$ is $X$ by assigning 0 to $x \not\in X_H$. Let $u_{H,i_H} = \sqrt{v_{H,i_H}}$.

We claim that

$$\left\{ \left( \frac{G}{H} u_{H,i_H}, \sqrt{\frac{|G|}{|H|} u_{H,i_H}^*} \right) : H < G, i_H = 1, 2, \ldots, n_H \right\}$$

(14)
is a quasi-basis for $E$. For $f \in C(X)$ and $x \in X$, let $F < G$ and $1 \leq j \leq n_F$ be such that $x \in X_F$ and $x \in U_{F,j}$. Then

$$\sum_{H < G} \sum_{i_H = 1}^{n_H} \left( \frac{|G|}{|H|} u_{H,i_H} E \left( \frac{|G|}{|H|} u_{H,i_H}^* f \right) \right)(x)$$

$$= \frac{1}{|G|} \sum_{H < G} \sum_{i_H = 1}^{n_H} \left( \sqrt{\frac{|G|}{|H|}} u_{H,i_H}(x) \sum_{g \in G} \sqrt{\frac{|G|}{|H|}} u_{H,i_H}^*(g^{-1}x)f(g^{-1}x) \right)$$

$$= \sum_{i_F = 1}^{n_F} \frac{1}{|F|} u_{F,i_F}(x) \left( \sum_{g \in G} u_{F,i_F}^*(g^{-1}x)f(g^{-1}x) \right)$$

$$= \frac{1}{|F|} \sum_{i_F = 1}^{n_F} u_{F,i_F}(x) \left( \sum_{g \in G} u_{F,i_F}^*(g^{-1}x)f(g^{-1}x) \right)$$

$$= \frac{1}{|F|} \sum_{i_F = 1}^{n_F} u_{F,i_F}(x)|F|u_{F,i_F}^*(x)f(x)$$

$$= \sum_{i_F = 1}^{n_F} v_{F,i_F}(x)f(x)$$

$$= f(x),$$

as claimed.

Recall that an action $G$ on $X$ is free if $gx \neq x$ for $g \in G$, $g \neq 1$, and $x \in X$.

**Corollary 4.11.** Let $X$ be a compact Hausdorff space and $G$ a finite group acting on $X$. If $\alpha$ is the induced action on $C(X)$, the following are equivalent.

(i) $G$ acts freely on $X$.

(ii) $E : C(X) \to C(X)^\alpha$ is of index-finite type with $\text{Index}(E) = |G|$.

(iii) $\alpha$ is saturated.

**Proof.** (i) $\implies$ (ii) is proved in [19, Proposition 2.8.1].

To show (ii) $\implies$ (i), let $E$ be of index-finite type with $\text{Index}(E) = |G|$. Then from Theorem 4.10, we have

$$|G| = \text{Index}(E) = \sum_{H < G} \frac{|G|}{|H|} \chi_{X_H},$$

which implies that $H = \{e\}$ is the only subgroup of $G$ such that $X_H \neq \emptyset$. Hence $X = X_{\{e\}}$, that is, $G$ acts freely on $X$. (ii) $\iff$ (iii) comes from Theorem 4.1. □

## 5. Saturated actions by compact groups

**Notation** 5.1. Let $M$ be a $C^*$-algebra and $\alpha$ be an action of a compact group $G$ on $M$. For $x, y \in M$, define continuous functions $f_{x,y}, f_{x,1}, f_{1,y} \in C(G, M)$ from $G$ to
$M$ as follows:

\[ f_{x,y}(t) = x \alpha_t(y), \]

\[ f_{x,1}(t) = x, \quad f_{1,y}(t) = \alpha_t(y) \quad \text{for } t \in G. \]

Then it is easily checked that $f_{x,y} = f_{x,1} \ast f_{1,y}$ and $f_{x,y}^* = f_{y^*,x^*}$.

Recall that $C(G, M)$ is a dense $*$-subalgebra of $M \times_\alpha G$ with the multiplication and involution defined by

\[ f \ast g(t) = \int_G f(s) \alpha_s(g(s^{-1}t))ds, \]

\[ f^*(t) = \alpha_t(f(t^{-1})^*), \]

where $dg$ is the normalized Haar measure ([13, 7.7], [6, 8.3.1]). Hence if $G$ is a finite group, $f_{x,y}$ can be written as

\[ f_{x,y} = \frac{1}{|G|} \sum_g x \alpha_g(y) \lambda_g. \]

If $	ilde{M}$ denotes the smallest unitization of $M$ (so $	ilde{M} = M$ if $M$ is unital), the function $e : G \to \tilde{M}$, $e(s) = 1$, for every $s \in G$ is a projection of the multiplier algebra of $M \times_\alpha G$ ([17]).

**Proposition 5.2.** ([17]) Let $\alpha$ be an action of a compact group $G$ on a $C^*$-algebra $M$. Then identifying $x \in M^\alpha$ and the constant function in $C(G, M)$ with the value $x$ everywhere we see that

\[ x \mapsto f_{x,1} : M^\alpha \to e(M \times_\alpha G)e \]

is an isomorphism of $M^\alpha$ onto the hereditary subalgebra $e(M \times_\alpha G)e$ of the crossed product $M \times_\alpha G$.

The notion of saturated action is introduced by Rieffel for a compact group action on a $C^*$-algebra, and we adopt the following equivalent condition as the definition.

**Definition 5.3.** (Rieffel, see [14, 7.1.9 Lemma]) Let $M$ be a $C^*$-algebra and $\alpha$ be an action of compact group $G$ on $M$. $\alpha$ is said to be saturated if the linear span of $\{f_{a,b} \mid a, b \in M\}$ is dense in $M \times_\alpha G$ (see Notation 5.1). We denote

\[ J_\alpha = \overline{\text{span}} \{f_{a,b} \mid a, b \in M\}. \]

**Proposition 5.4.** Let $\alpha$ be an action of a compact group $G$ on a $C^*$-algebra $M$. Then $J_\alpha$ is the ideal of $M \times_\alpha G$ generated by the hereditary subalgebra $e(M \times_\alpha G)e$. Moreover $J_\alpha = \overline{\text{span}} \{f_{a,a^*} \in C(G, M) \mid a \in M\}$.

**Proof.** We first show that $J_\alpha$ is an ideal of $M \times_\alpha G$. Let $x \in C(G, M)$ and $a, b \in M$. Then $x \ast f_{a,b} \in J_\alpha$. Indeed,

\[ (x \ast f_{a,b})(t) = \int x(s) \alpha_s(f_{a,b}(s^{-1}t))ds \]

\[ = \int x(s) \alpha_s(a) \alpha_t(b)ds \]

\[ = (\int x(s) \alpha_s(a)ds) \alpha_t(b), \]
hence \( x * f_{a,b} = f_{c,b} \in \mathcal{J}_\alpha \), where \( c = \int x(s)\alpha_s(a)\,ds \in M \). Also \( f_{a,b}^* = f_{b^*,a^*} \) implies that \( \mathcal{J}_\alpha = \mathcal{J}_{\alpha}^* \) is an ideal of \( \mathcal{M} \times G \).

Let \( \mathcal{J} := (\mathcal{M} \times G) e(\mathcal{M} \times G) \) be the closed ideal generated by \( e(\mathcal{M} \times G) e \). Now we show that \( \mathcal{J}_\alpha \subset \mathcal{J} \). From 

\[
(f_{a,b} * e)(t) = \int f_{a,b}(s)\alpha_s(e(s^{-1}t))\,ds = \int a\alpha_s(b)\,ds = a \int \alpha_s(b)\,ds,
\]

we have \( f_{a,b} * e = f_{aE(b),1} \), where \( E(b) = \int \alpha_s(b)\,ds \in M^\alpha \). Hence for \( a, b, c, \) and \( d \) in \( M \), we have

\[
f_{a,b} * e * f_{c,d} = (f_{a,b} * e) * (f_{c,d}^{} * e)^* = f_{aE(b),1}^{} * (f_{dE(c^{}),1}^{} * e)^*
\]

\[
= f_{aE(b),1}^{} * f_{1,E(c^{}),d}^{}
= f_{aE(b),E(c^{}),d}^{}
\]

which means that \( f_{x,y} \in \mathcal{J} \) for any \( a, d \in M \) and \( x, y \in M^\alpha \). Since \( M^\alpha \) contains an approximate identity for \( M \), it follows that \( f_{a,b} \in \mathcal{J} \) for \( a, b \in A \).

For the converse inclusion \( \mathcal{J} \subset \mathcal{J}_\alpha \), note that if \( x \in C(G, M) \), then \( (x * e)(t) = \int x(s)\,ds \) for \( t \in G \). With notations \( x' = \int x(s)\,ds \) and \( x'' := \int \alpha_s(x(s^{-1}))\,ds \in M \), we see that

\[
(x * e * y)(t) = \int (x * e)(s)\alpha_s(y(s^{-1}t))\,ds
\]

\[
= x' \int \alpha_s(y(s^{-1}t))\,ds
\]

\[
= x' \alpha_t(\int \alpha_s(y(s^{-1}))\,ds)
\]

\[
= x' \alpha_t(y'')
\]

\[
= f_{x',y''}(t)
\]

belongs to \( \mathcal{J}_\alpha \) for \( x, y \in C(G, M) \).

Finally the following polarization identity proves the last assertion.

\[
a\alpha_t(b) = \frac{1}{4} \sum_{k=0}^{3} i^k(b + i^ka^*)^* \alpha_t(b + i^ka^*).
\]
orthogonal projections such that
\[ s_e^* s_e = p_{r(e)} \quad \text{and} \quad p_v = \sum_{s(e)=v} s_e s_e^* \] if \( s^{-1}(v) \neq \emptyset \).

It is now well known that there exists a C*-algebra \( C^*(E) \) generated by a universal CK \( E \)-family \( \{ s_e, p_v \mid e \in E, v \in E^0 \} \), in this case we simply write \( C^*(E) = C^*(s_e, p_v) \). For the definition and basic properties of graph C*-algebras, see, for example, \( [1, 2, 10, 11, 15] \) among others. If \( \alpha = \alpha_1 \alpha_2 \cdots \alpha_{|\alpha|} (\alpha_i \in E^1) \) is a finite path, by \( s_\alpha \) we denote the partial isometry \( s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_{|\alpha|}} \) (\( s_v = s_v^* = p_v \), for \( v \in E^0 \)).

We will consider only locally finite graphs and it is helpful to note the following properties of graph C*-algebras.

**Remark 6.1.**

(i) Let \( C^*(E) = C^*(s_e, p_v) \) be the graph C*-algebra associated with a row finite graph \( E \), and let \( \alpha, \beta \in E^* \) be finite paths in \( E \). Then
\[
s_\alpha^* s_\beta = \begin{cases} s_{\mu}, & \text{if } \alpha = \beta \mu \\ s_\nu, & \text{if } \beta = \alpha \nu \\ 0, & \text{otherwise.} \end{cases}
\]

Therefore \( C^*(E) = \operatorname{span}\{s_\alpha^* s_\beta \mid \alpha, \beta \in E^*\} \).

(ii) Note that \( s_\alpha^* s_\beta = 0 \) for \( \alpha, \beta \in E^* \) with \( r(\alpha) \neq r(\beta) \).

(iii) If \( \alpha, \beta, \mu, \) and \( \nu \) in \( E^n \) are the paths of same length,
\[
(s_\alpha^* s_\beta)(s_\mu^* s_\nu) = \delta_{\beta,\mu} s_\alpha^* s_\nu.
\]

Thus for each \( n \in \mathbb{N} \) and a vertex \( v \) in a locally finite graph \( E \), we see that \( \operatorname{span}\{s_\alpha^* s_\beta \mid \alpha, \beta \in E^n \text{ and } r(\alpha) = r(\beta) = v\} \)

is a *-algebra which is isomorphic to the full matrix algebra \( M_m = (M_m(\mathbb{C})) \), where \( m = \left| \{ \alpha \in E^n \mid r(\alpha) = v \} \right| \).

Recall that the gauge action \( \gamma \) of \( \mathbb{T} \) on \( C^*(E) = C^*(s_e, p_v) \) is given by
\[
\gamma_z(s_e) = z s_e, \quad \gamma_z(p_v) = p_v, \quad z \in \mathbb{T}.
\]

\( \gamma \) is well defined by the universal property of the CK \( E \)-family \( \{ s_e, p_v \} \). Since
\[
\int_{\mathbb{T}} \gamma_z(s_\alpha^* s_\beta)dz = \int_{\mathbb{T}} z^{||\alpha|-|\beta||}(s_\alpha^* s_\beta)dz = 0, \quad |\alpha| \neq |\beta|,
\]
one sees that
\[
C^*(E)^\gamma = \operatorname{span}\{s_\alpha^* s_\beta \mid \alpha, \beta \in E^*, |\alpha| = |\beta|\}.
\]

If \( Z \) denotes the following graph:

\[
Z : \quad \cdots \quad \bullet \cdots \bullet \cdots \bullet \cdots \bullet \cdots \cdot \cdots 2 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2
\]

then \( C^*(Z) \) is isomorphic to the C*-algebra \( K \) of compact operators on an infinite dimensional separable Hilbert space, hence \( C^*(Z) \) is itself a simple AF algebra. But \( C^*(Z)^\gamma \) coincides with the commutative subalgebra \( \operatorname{span}\{s_\alpha^* s_\alpha \mid \alpha \in Z^*\} \) which is far from being simple, and thus we know that the simplicity of \( C^*(E) \) does not imply that of \( C^*(E)^\gamma \) in general.
In [3], the Cartesian product of two graphs $E$ and $F$ is defined to be the graph $E \times F = (E^0 \times F^0, E^1 \times F^1, r, s)$, where $r(e, f) = (r(e), r(f))$ and $s(e, f) = (s(e), s(f))$. Since the graph $Z \times E$ has no loops for any row-finite graph $E$, we know that $C^*(E)^\gamma$ is an AF algebra (13) by the following proposition.

**Proposition 6.2.** ([9]) Let $E$ be a row finite graph with no sources. Then the following hold:

(a) $C^*(E)^\gamma$ is stably isomorphic to $C^*(E) \times_\gamma \mathbb{T}$.

(b) $C^*(E) \times_\gamma \mathbb{T} \cong C^*(Z \times E)$.

Now we show that a gauge action is saturated. For this, note that the linear span of the continuous functions of the form

$$t \mapsto f(tx), \quad f \in C(G), \quad x \in A$$

is dense in $C(G, A)$ [13, 7.6.1]. Hence by Remark 6.1(i), one sees that

$$C^*(E) \times_\gamma \mathbb{T} = \text{span}\{z^n s_\alpha s_\beta^* | \alpha, \beta \in E^* \ n \in \mathbb{Z}\}. \quad (15)$$

**Theorem 6.3.** Let $E$ be a locally finite graph with no sinks and no sources. Then the gauge action $\gamma$ on $C^*(E)$ is saturated.

**Proof.** We show that $J_\gamma = C^*(E) \times_\gamma \mathbb{T}$. By (15) it suffices to see that

$$z^n s_\alpha s_\beta^* \in J_\gamma \quad \text{for all } \alpha, \beta \in E^*, \ n \geq 0$$

(because $z^{-n}s_\alpha s_\beta^* = (z^n s_\beta s_\alpha^*)^*$ for $n \geq 0$).

Now fix $\alpha, \beta \in E^*$ and $n \geq 0$. Put $l = n - (|\alpha| - |\beta|)$. There are two cases.

(i) $l \geq 0$: One can choose a path $\mu$ such that $|\mu| = l$ and $r(\mu) = s(\alpha)$. Then

$$z^n s_\alpha s_\beta^* = z^{|l+|\alpha|-|\beta|} s_\mu s_\alpha s_\beta^* s_\mu^* s_\mu = s_\mu^* g_z(s_\mu s_\beta s_\alpha^*) = f_{s_\mu^* s_\mu, s_\beta s_\alpha^*}(z),$$

where the function $f_{s_\mu^* s_\mu, s_\beta s_\alpha^*}$ belongs to $J_\gamma$.

(ii) $l < 0$: Choose a path $\nu$ with $|\nu| = |\beta| + n$ and $r(\nu) = r(\alpha)$. With $a = s_\alpha s_\mu^*$, $b = s_\nu s_\beta^*$, we have $f_{a, b} \in J_\gamma$ and

$$z^n s_\alpha s_\beta^* = s_\alpha s_\mu^* g_z(s_\mu s_\beta s_\alpha^*) = f_{a, b}(z).$$

$\square$

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