SUPERMEMBRANES AND SUPERMATRIX MODELS

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We briefly review recent developments in the theory of supermembranes and supermatrix models. In a second part we discuss their interaction with background fields. In particular, we present the full background field coupling for the bosonic case. This is a short summary of the talk at the workshop. A more extended version will appear elsewhere.

1 Supermembranes and matrix models

It has been known for some time that certain supersymmetric quantum-mechanical models characterized by the presence of zero-potential valleys and Gauss-type constraints play an important role in the quantization of fundamental supermembranes. In the light-cone formulation with a flat target space, the supermembrane theory exhibits a residual invariance under area-preserving diffeomorphisms of the membrane surface. This infinite-dimensional group can be truncated in many cases to a finite-dimensional group, e.g. to U(N), such as to lead to a quantum-mechanical model based on a finite number of degrees of freedom. The relevant Hamiltonian equals

\[ H = g^{-1} \text{Tr} \left[ \frac{1}{2} \mathbf{P}^2 - \frac{1}{4} [X^a, X^b]^2 + \frac{1}{2} g \theta^T \gamma_a [X^a, \theta] \right]. \] (1)

Here, \( \mathbf{X} \), \( \mathbf{P} \) and \( \theta \) take values in the Lie algebra of the gauge group and are represented by matrices that span this Lie algebra. Furthermore, these momenta and coordinates transform as vectors and spinors under the 'transverse'
SO(9) rotation group. The Hamiltonian (1) can also be interpreted as the zero-volume limit of 10-dimensional supersymmetric Yang-Mills theory with a corresponding gauge group. As explained above, for the supermembrane this gauge group consists of area-preserving diffeomorphisms. The coupling constant $g$ is then equal to the light-cone momentum of the membrane, denoted by $P_0^+ = (P_-)_0$.

Physical states must be invariant under the gauge group. Classical zero-energy configurations require all the commutators $[X^a, X^b]$ to vanish. Dividing out the gauge group implies that zero-energy configurations are parametrized by $R^9/S_N$. For the membrane, zero-energy configurations correspond to zero-area stringlike configurations of arbitrary length $\mathcal{R}$.

At the quantum level the models described by (1) have a continuous energy spectrum $\mathcal{E}$. Whether or not a normalizable ground state exists at the beginning of the continuum is a subtle issue $\mathcal{E}$. Such a ground state must be annihilated by the supercharges. For the supermembrane, it is expected to comprise the physical states of 11-dimensional supergravity.

More recently it was shown that the very same quantum-mechanical matrix models based on $U(N)$ describe the short-distance dynamics of $N$ D0-branes $\mathcal{E}$. Further interest in these models has been triggered by a conjecture according to which the degrees of freedom captured in M-theory, are in fact described by the $U(N)$ supersymmetric matrix and related models in the $N \rightarrow \infty$ limit $\mathcal{E}$. M-theory is defined as the strong-coupling limit of type-IIA string theory and is supposed to contain all the relevant degrees of freedom of the known string theories, both at the perturbative and the nonperturbative level $\mathcal{E}$. In this description the various string-string dualities play a central role. At large distances M-theory is described by 11-dimensional supergravity $\mathcal{E}$.

So it turns out that M-theory, supermembranes and super-matrix models are intricately related. A direct relation between supermembranes and type-IIA theory was emphasized in particular by Townsend $\mathcal{E}$, based on the relation between $d = 10$ extremal black holes in 10-dimensional supergravity and the Kaluza-Klein states of 11-dimensional supergravity. From the string point of view these states carry Ramond-Ramond charges, just as the D0-branes $\mathcal{E}$. Strings can arise from membranes by a so-called double-dimensional reduction $\mathcal{E}$. Supermembranes were, for example, also employed to provide evidence for the duality of M-theory on $\mathcal{R}^{10} \times S^1/Z_2$ and 10-dimensional $E_8 \times E_8$ heterotic strings $\mathcal{E}$.

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*b* The supermembrane can live in $D = 11, 7, 5$ or 4 space-time dimensions, so that the transverse rotations are $SO(D - 2)$. Here we restrict our attention exclusively to $D = 11$.

*c* For discussions on the existence of massless states, see refs. 1, 7–9. According to ref. 9 such states do indeed exist in $D = 11$. 

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An obvious question is whether fundamental supermembranes themselves can provide the degrees of freedom of M-theory. This question is often answered negatively in view of a suspected quantum-mechanical inconsistency of the supermembrane theory. Here we note that not too much is known about the actual consistency or inconsistency of the supermembrane theory, which, incidentally, is linked directly to the large-$N$ behaviour of the U($N$) matrix models. A possibly worrisome feature is also, that the fundamental supermembrane does not seem to leave much room for its dual (‘magnetic’) partner, the fivebrane. Irrespective on how one judges these potential shortcomings, supermembranes give rise to an independent perspective on many aspects of M-theory. Furthermore they have many features in common with matrix models.

From a more technical point of view, the supermembrane provides us with a well-defined tool for tackling various questions. For instance the spherical supermembrane captures the nature of the $N \to \infty$ “decompactification” limit for the matrix models as well as the emergence of the 11-dimensional super-Poincaré invariance. One can also study open membranes and topologically nontrivial membranes with or without winding around compact directions. As it turns out, it is not always guaranteed that one can obtain a matrix-model prescription in these situations. For instance, the approach followed for compactification of matrix models, which is strongly guided by the application to D-branes and T-duality, does not make immediate contact with winding supermembranes, albeit that there are certain similarities. At present the significance of this situation is not completely clear to us.

A rather prominent question concerns matrix models in curved backgrounds with a target-space tensor field. In the second part of this talk we demonstrate how such couplings can be obtained from supermembrane theory formulated in a curved superspace background.

### 2 Supermembranes in curved backgrounds

Let us consider a generic superspace with coordinates $Z^M = (X^\mu, \theta)$. World indices are $M = (\mu, \alpha)$ and corresponding tangent-space indices are denoted by $A = (r, a)$. The supermembrane requires super-vielbeine $E^A_M$ and a 3-rank tensor gauge field $B_{MNP}$. The action for a supermembrane is written in terms of embedding coordinates $Z^M(\zeta)$, which are functions of the three world-volume coordinates $\zeta^i$. It takes the following form,

$$S = \int d^3 \zeta \left[ -\sqrt{-g(Z(\zeta))} - \frac{1}{6} \epsilon^{ijk} \Pi_i^A \Pi_j^B \Pi_k^C B_{CBA}(Z(\zeta)) \right], \quad (2)$$
where $\Pi_i^A = \partial Z^M / \partial \zeta^i E_M^A$ and the induced metric equals $g_{ij} = \Pi_i^x \Pi_j^x \eta_{rs}$, with $\eta_{rs}$ the constant Lorentz-invariant metric.

Flat superspace is characterized by

$$
E_\mu^r = \delta_\mu^r, \quad E_\mu^a = 0, \\
E_\alpha^a = \delta_\alpha^a, \quad E_\alpha^r = - (\bar{\theta} \Gamma^r)_\alpha, \\
B_{\mu\nu\alpha} = (\bar{\theta} \Gamma_{\mu\nu})_\alpha, \quad B_{\mu\alpha\beta} = (\bar{\theta} \Gamma_{\mu\nu})_{(\alpha} (\bar{\theta} \Gamma^\nu)_{\beta)}, \\
B_{\alpha\beta\gamma} = (\bar{\theta} \Gamma_{\mu\nu})_{(\alpha} (\bar{\theta} \Gamma^\nu)_{\beta}) (\bar{\theta} \Gamma^\mu)_{\gamma}, \quad B_{\mu\nu\rho} = 0.
$$

(3)

These quantities receive corrections in the presence of background fields. Here we consider a background induced by a nontrivial target-space metric, a target-space tensor field and a target-space gravitino field, corresponding to the fields of (on-shell) 11-dimensional supergravity. This background can in principle be incorporated into superspace by a procedure known as ‘gauge completion’ \cite{27}. For 11-dimensional supergravity, the first steps of this procedure have been carried out long ago \cite{28}, but unfortunately only to first order in fermionic coordinates $\theta$. Meanwhile we have also determined the second-order contributions, which will be reported elsewhere.

To elucidate our general strategy, let us just confine ourselves to the purely bosonic case and present the light-cone formulation of the membrane in a background consisting of the metric $G_{\mu\nu}$ and the tensor gauge field $C_{\mu\nu\rho}$. The Lagrangian density for the bosonic membrane follows directly from (2),

$$
\mathcal{L} = - \sqrt{-g} + \frac{1}{6} \varepsilon^{ijk} \partial_i X^\mu \partial_j X^\nu \partial_k X^\rho C_{\mu\nu\rho},
$$

(4)

where $g_{ij} = \partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu}$. For the light-cone formulation, the coordinates are decomposed in the usual fashion as $(X^+, X^-, X^a)$ with $a = 1 \ldots 9$. Furthermore we use the diffeomorphisms in the target space to bring the metric in a convenient form

$$
G_{--} = G_{\alpha-} = 0.
$$

(5)

Subsequently we identify the time coordinate of the target space with the world-volume time, by imposing the condition $X^+ = \tau$. Following the same steps as for the membrane in flat space \cite{1}, one arrives at a Hamiltonian formulation of the theory in terms of coordinates and momenta. These phase-space variables are subject to a constraint. It is a pleasant surprise that this constraint takes the same form as for the membrane theory in flat space, namely,

$$
\phi_\tau = P_a \partial_\tau X^a + P_- \partial_\tau X^- \approx 0.
$$

(6)
Of course, the definition of the momenta in terms of the coordinates and their
derivatives does involve the background fields, but at the end all explicit de-
pendence on the background cancels out.

The Hamiltonian now follows straightforwardly. As it turns out, the b ack-
ground tensor field appears in the combinations
\begin{align*}
 C_a &= -\varepsilon^{rs}\partial_r X^- \partial_s X^b C_{-ab} + \frac{1}{2}\varepsilon^{rs}\partial_r X^b \partial_s X^c C_{abc}, \\
 C_\pm &= \frac{1}{2}\varepsilon^{rs}\partial_r X^a \partial_s X^b C_{\pm ab}, \\
 C_{+-} &= \varepsilon^{rs}\partial_r X^- \partial_s X^a C_{+-a}.
\end{align*}
We can further simplify the background by imposing a gauge where
\begin{align}
 C_{+-a} &= 0 , \quad C_{-ab} = 0 , \quad G_{+-} = 1 .
\end{align}
In that case the Hamiltonian takes the form
\begin{align}
 H &= \int d^2\sigma \left\{ \frac{1}{P_-} \left[ \frac{1}{2}(P_a - C_a - P_+ G_{a+})^2 + \frac{1}{4}(\varepsilon^{rs}\partial_r X^a \partial_s X^b)^2 \right] \\
 &- \frac{1}{2} P_+ G_{++} - \frac{1}{2}\varepsilon^{rs}\partial_r X^a \partial_s X^b C_{+-ab} \right\},
\end{align}
where the coordinates \( \sigma^{1,2} \) parametrize the spacesheet of the membrane.

As we want to avoid explicit time dependence of the background fields, we assume the metric and the tensor field to be independent of \( X^+ \). If we assume, in addition, that they are independent of \( X^- \), it turns out that \( P_- \) becomes \( \tau \)-independent. This allows us to set \( P_-(\sigma) = (P_-)_0 \sqrt{w(\sigma)} \), exactly as in flat space. Here \( \sqrt{w} \) is some density on the spacesheet, whose integral is normalized to unity. With these restrictions, it is possible to write down a gauge theory of area-preserving diffeomorphisms for the membrane in the presence of background fields. Its Lagrangian density equals
\begin{align}
 w^{-1/2} \mathcal{L} &= \frac{1}{2}(D_0 X^a)^2 + D_0 X^a \left[ \frac{1}{2} C_{abc} \{ X^b , X^c \} + G_{a+} \right] \\
 &- \frac{1}{2} \{ X^a , X^b \}^2 + \frac{1}{2} G_{++} + \frac{1}{2} C_{+-ab} \{ X^a , X^b \},
\end{align}
where we used the metric \( G_{ab} \) to contract transverse indices. Furthermore we adopted the usual definitions for the Poisson bracket and the covariant derivative,
\begin{align}
 \{ A , B \} &= \frac{\varepsilon^{rs}}{\sqrt{w}} \partial_r A \partial_s B , \\
 D_0 X^a &= \partial_0 X^a - \{ \omega , X^a \} .
\end{align}
For convenience we have set \((P_-)_0 = 1\). The above Lagrangian density is manifestly invariant under area-preserving diffeomorphisms in the presence of the background fields. It is now straightforward to write it in terms of a matrix model, by truncating the mode expansion for coordinates and momenta in the standard fashion. A more explicit derivation of these results and their supersymmetric generalization will appear in a separate publication.

While preparing this report, we received two papers dealing with matrix models in certain backgrounds. We believe that the background interactions proposed in these papers are contained in our above results.

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