TWO $q$-ANALOGUES OF EULER’S FORMULA $\zeta(2) = \pi^2/6$

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Abstract. It is well known that $\zeta(2) = \pi^2/6$ as discovered by Euler. In this paper we present the following two $q$-analogues of this celebrated formula:

$$\sum_{k=0}^{\infty} \frac{q^k(1 + q^{2k+1})}{(1 - q^{2k+1})^2} = \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^4}{(1 - q^{2n-1})^4}$$

and

$$\sum_{k=0}^{\infty} \frac{q^{2k-|(-1)^k k/2|}}{(1 - q^{2k+1})^2} = \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^2(1 - q^{4n})^2}{(1 - q^{2n-1})^2(1 - q^{4n-2})^2},$$

where $q$ is any complex number with $|q| < 1$. We also give a $q$-analogue of the identity $\zeta(4) = \pi^4/90$, and pose a problem on $q$-analogues of Euler’s formula for $\zeta(2m)$ ($m = 3, 4, \ldots$).

1. Introduction

For $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, the $q$-analogue of $n$ is defined as

$$[n]_q := \frac{1 - q^n}{1 - q} = \sum_{0 \leq k < n} q^k \in \mathbb{Z}[q].$$

Note that $\lim_{q \to 1} [n]_q = n$. For $|q| < 1$, the $q$-Gamma function introduced by F. H. Jackson [J] in 1905 is given by

$$\Gamma_q(x) := (1 - q)^{1-x} \prod_{n=1}^{\infty} \frac{1 - q^n}{1 - q^{n+x-1}}.$$ 

In view of the basic properties of the $q$-Gamma function (cf. [AAR, pp. 493–496]), we have

$$\lim_{q \to 1} (1 - q^2) \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^2}{(1 - q^{2n-1})^2} = \lim_{q \to 1} \Gamma_q \left( \frac{1}{2} \right)^2 = \Gamma \left( \frac{1}{2} \right)^2 = \pi,$$

which is essentially equivalent to Wallis’ formula

$$\prod_{n=1}^{\infty} \frac{4n^2}{4n^2 - 1} = \frac{\pi}{2}$$

since

$$\prod_{n=1}^{\infty} \frac{(1 - q^{2n})^2}{(1 - q^{2n-1})(1 - q^{2n+1})} = (1 - q) \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^2}{(1 - q^{2n-1})^2}$$

for $|q| < 1$. 

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In light of (1.1),
\[ \lim_{q \to 1} (1 - q) \prod_{n=1}^{\infty} \frac{(1 - q^{4n})^2}{(1 - q^{4n-2})^2} = \lim_{q \to 1} \frac{1 - q}{1 - q^2} \times \pi = \frac{\pi}{4} \]
and hence we may view Ramanujan’s formula
\[ \sum_{k=0}^{\infty} \frac{(-q)^k}{1 - q^{2k+1}} = \prod_{n=1}^{\infty} \frac{(1 - q^{4n})^2}{(1 - q^{4n-2})^2} \quad (|q| < 1) \]
(equivalent to Example (iv) in B. C. Berndt \[B91, \text{p. 139}\]) as a \(q\)-analogue of Leibniz’s classical identity
\[ \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}. \]

Recently, V.J.W. Guo and J.-C. Liu \[GL\] gave some \(q\)-analogues of two Ramanujan-type series for \(1/\pi\).

Let \(C\) be the field of complex numbers. The \textit{Riemann zeta function} is given by
\[ \zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{for } s \in \mathbb{C} \text{ with } \Re(s) > 1. \]

In 1734 L. Euler obtained the elegant formula
\[ \zeta(2) = \frac{\pi^2}{6}. \]

In 2011 Kh. Hessami Pilehrood and T. Hessami Pilehrood \[HP\] gave an interesting \(q\)-analogue of the known identity \(3 \sum_{n=1}^{\infty} 1/(n^2(\binom{2n}{n})) = \zeta(2)\), which states that
\[ \sum_{n=1}^{\infty} q^n (1 + 2q^n) = \sum_{n=1}^{\infty} \frac{q^n}{[n]^2_q} \quad \text{for } |q| < 1, \]
where
\[ \begin{bmatrix} 2n \\ n \end{bmatrix}_q = \frac{\prod_{k=1}^{n}[k]_q}{\prod_{j=1}^{n}[j]^2_q} \]
is the \(q\)-analogue of the central binomial coefficient \(\binom{2n}{n}\).

Euler’s celebrated formula (1.2) plays very important roles in modern mathematics. Though \(\sum_{n=1}^{\infty} q^n / [n]^2_q \) (with \(|q| < 1\)) is a natural \(q\)-analogue of \(\zeta(2)\), it seems hopeless to use it to give a \(q\)-analogue of (1.2). As nobody has given a \(q\)-analogue of Euler’s formula (1.2) before, we aim to present two \(q\)-analogues of (1.2) in this paper.

Our main result is as follows.

**Theorem 1.1.** For any \(q \in \mathbb{C}\) with \(|q| < 1\), we have
\[ \sum_{k=0}^{\infty} \frac{q^k(1 + q^{2k+1})}{(1 - q^{2k+1})^2} = \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^4}{(1 - q^{2n-1})^4}, \]
Therefore (1.3) and (1.4) indeed give

\[
\sum_{k=0}^{\infty} \frac{q^{2k-\lfloor (-1)^k k/2 \rfloor}}{(1-q^{2k+1})^2} = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^2(1-q^{4n})^2}{(1-q^{2n-1})^2(1-q^{4n-2})^2}.
\]

Clearly,

\[
\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{3}{4} \zeta(2)
\]

and hence (1.2) has the equivalent form

\[
(1.5) \quad \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.
\]

Now we explain why (1.5) follows from (1.3) or (1.4). In view of (1.1),

\[
\lim_{q \to 1} (1-q^2)^2 \prod_{n=1}^{\infty} \frac{(1-q^{2n})^4}{(1-q^{2n-1})^4} = \pi^2,
\]

\[
\lim_{q \to 1} (1-q^2)(1-q^4) \prod_{n=1}^{\infty} \frac{(1-q^{2n})^2(1-q^{4n})^2}{(1-q^{2n-1})^2(1-q^{4n-2})^2} = \pi^2.
\]

Thus

\[
(1.6) \quad \lim_{q \to 1} (1-q^2)^2 \prod_{n=1}^{\infty} \frac{(1-q^{2n})^4}{(1-q^{2n-1})^4} = \frac{\pi^2}{4}
\]

and

\[
\lim_{q \to 1} (1-q^2) \prod_{n=1}^{\infty} \frac{(1-q^{2n})^2(1-q^{4n})^2}{(1-q^{2n-1})^2(1-q^{4n-2})^2} = \frac{\pi^2}{8}.
\]

On the other hand,

\[
\lim_{q \to 1} (1-q^2) \sum_{k=0}^{\infty} \frac{q^k(1+q^{2k+1})}{(1-q^{2k+1})^2} = \lim_{q \to 1} \sum_{k=0}^{\infty} \frac{q^k(1+q^{2k+1})}{[2k+1]^2} = \sum_{k=0}^{\infty} \frac{2}{(2k+1)^2},
\]

\[
\lim_{q \to 1} (1-q^2) \sum_{k=0}^{\infty} \frac{q^{2k-\lfloor (-1)^k k/2 \rfloor}}{(1-q^{2k+1})^2} = \lim_{q \to 1} \sum_{k=0}^{\infty} \frac{q^{2k-\lfloor (-1)^k k/2 \rfloor}}{[2k+1]^2} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}.
\]

Therefore (1.3) and (1.4) indeed give \( q \)-analogues of (1.2).

We also deduce a \( q \)-analogue of the known formula \( \zeta(4) = \pi^4/90 \), which has the equivalent form

\[
(1.7) \quad \sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} = \frac{\pi^4}{96}.
\]

**Theorem 1.2.** For any \( q \in \mathbb{C} \) with \( |q| < 1 \), we have

\[
(1.8) \quad \sum_{k=0}^{\infty} \frac{q^{2k}(1 + 4q^{2k+1} + q^{4k+2})}{(1-q^{2k+1})^4} = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^8}{(1-q^{2n-1})^8}.
\]
As is clearly seen, the left-hand side of (1.8) times \((1 - q)^4\) tends to 
\[6 \sum_{k=0}^{\infty} \frac{1}{(2k + 1)^4}\] 
as \(q \to 1\). In view of (1.6), the right-hand side of (1.8) times \((1 - q)^4\) has the limit \(\pi^4/16\) as \(q \to 1\). So (1.8) implies (1.7) and hence it gives a \(q\)-analogue of the formula \(\zeta(4) = \pi^4/90\).

We will show Theorems 1.1 and 1.2 in the next section.

The Bernoulli numbers \(B_0, B_1, \ldots\) are given by 
\[B_0 = 1\] 
and 
\[\sum_{k=0}^{n} \binom{n+1}{k} B_k = 0 \quad (n = 1, 2, 3, \ldots).\]

Euler proved (cf. [IR, pp. 231–232]) that for each \(m = 1, 2, 3, \ldots\) we have
\[(1.9) \quad \zeta(2m) = (-1)^{m-1} \frac{2^{2m-1} \pi^{2m}}{(2m)!} B_{2m}.\]

To seek for \(q\)-analogues of (1.9) is our novel idea in this paper. We don’t know whether one can find a \(q\)-analogue of (1.9) similar to (1.3), (1.4) and (1.8) for \(m = 3, 4, 5, \ldots\), and this problem might stimulate further research.

2. Proofs of Theorems 1.1 and 1.2

Recall that the triangular numbers are the integers
\[T_n := \frac{n(n+1)}{2} \quad (n = 0, 1, 2, \ldots).\]

As usual, we set
\[(2.1) \quad \psi(q) := \sum_{n=0}^{\infty} q^{T_n} \quad \text{for } |q| < 1.\]

**Lemma 2.1.** For \(|q| < 1\) we have
\[(2.2) \quad \psi(q) = \prod_{n=1}^{\infty} \frac{1 - q^{2n}}{1 - q^{2n-1}}.\]

**Remark 2.2.** This is a well-known result due to Gauss (cf. Berndt [B06, (1.3.14), p. 11]).

**Lemma 2.3.** Let \(n \in \mathbb{N}\) and
\[t_4(n) := |\{(w, x, y, z) \in \mathbb{N}^4 : T_w + T_x + T_y + T_z = n\}|.\]

Then
\[(2.3) \quad t_4(n) = \sigma(2n + 1),\]

where \(\sigma(m)\) denotes the sum of all positive divisors of a positive integer \(m\).
Remark 2.4. This is also a known result, see [B06 (3.6.6.), p.72]. In contrast with (2.3), for any positive integer \( n \) Jacobi showed that

\[
r_4(n) := |\{(w, x, y, z) \in \mathbb{Z}^4 : w^2 + x^2 + y^2 + z^2 = n\}| = 8 \sum_{4 \mid d \mid n} d
\]

(cf. [B06 (3.3.1), p. 59]).

Lemma 2.5. For \(|q| < 1\) we have

\[
\sum_{k=0}^{\infty} q^k(1 + q^{2k+1}) \frac{1}{(1 - q^{2k+1})^2} = \sum_{n=0}^{\infty} \sigma(2n + 1)q^n.
\]

Proof. For each \( k \in \mathbb{N} \), clearly

\[
\frac{q^k(1 + q^{2k+1})}{(1 - q^{2k+1})^2} = 2q^k(1 - q^{2k+1})^{-2} - q^k(1 - q^{2k+1})^{-1}
\]

\[
= 2q^k \sum_{j=0}^{\infty} \binom{-2}{j}(-q^{2k+1})^j - q^k \sum_{j=0}^{\infty} q^{(2k+1)j}
\]

\[
= \sum_{j=0}^{\infty} (2j + 1)q^{(2j+1)(k+1/2)-1/2}.
\]

Thus

\[
\sum_{k=0}^{\infty} q^k(1 + q^{2k+1}) \frac{1}{(1 - q^{2k+1})^2} = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (2j + 1)q^{((2j+1)(2k+1)-1)/2}
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{d \mid 2n+1} d \right) q^{(2n+1-1)/2} = \sum_{n=0}^{\infty} \sigma(2n + 1)q^n.
\]

This concludes the proof. \( \square \)

Lemma 2.6. For each \( n \in \mathbb{N} \), we have

\[
|\{(u, v, x, y) \in \mathbb{N}^4 : T_u+T_v+2T_x+2T_y = n\}| = \sum_{d \mid 4n+3} \frac{d - (-1)^{(d-1)/2}}{4}.
\]

Remark 2.7. This is a known result due to K. S. Williams [W].

Proof of Theorem 1.1. In view of Lemmas 2.1 and 2.3

\[
\prod_{n=1}^{\infty} \frac{(1 - q^{2n})^4}{(1 - q^{2n-1})^4} = \psi(q)^4 = \sum_{n=0}^{\infty} t_4(n)q^n = \sum_{n=0}^{\infty} \sigma(2n + 1)q^n.
\]

Combining this with (2.4) we immediately obtain (1.3).
By Lemmas [2.1] and [2.6], we have

\[
\prod_{n=1}^{\infty} \frac{(1 - q^{2n})^2(1 - q^{4n})^2}{(1 - q^{2n-1})^2(1 - q^{4n-2})^2}
\]

\[
= \psi(q)^2 \psi(q^2)^2 = \sum \{ (u, v, x, y) \in \mathbb{N}^4 : T_u + T_v + 2T_x + 2T_y = n \} |q^n|
\]

\[
= \sum_{n=0}^{\infty} \sum_{d \mid 4n+3} d - (-1)^{(d-1)/2} \frac{4}{q^n}
\]

\[
= \sum_{j=0}^{\infty} \frac{(4j + 1)}{4} \sum_{k=0}^{\infty} q^{((4j+1)(4k+3)-3)/4}
\]

\[
+ \sum_{j=0}^{\infty} \frac{(4j + 3)}{4} \sum_{k=0}^{\infty} q^{((4j+3)(4k+1)-3)/4}
\]

\[
= \sum_{k=0}^{\infty} \left( q^{4k} \sum \frac{2j^3}{4} + q^{3k} \sum (j + 1) q^{j(4k+1)} \right)
\]

For \(|z| < 1\), clearly

\[
\sum_{j=0}^{\infty} (j + 1) z^j = \sum_{j=1}^{\infty} j z^{j-1} = \sum_{j=0}^{\infty} d j z^j = d \frac{1}{1 - z} = \frac{1}{(1 - z)^2}.
\]

Therefore

\[
\prod_{n=1}^{\infty} \frac{(1 - q^{2n})^2(1 - q^{4n})^2}{(1 - q^{2n-1})^2(1 - q^{4n-2})^2}
\]

\[
= \sum_{k=0}^{\infty} \left( q^{4k} \frac{q^{4k+3}}{(1 - q^{4k+3})^2} + \frac{q^{3k}}{(1 - q^{4k+1})^2} \right) = \sum_{m=0}^{\infty} q^{2m - \lfloor -1 \rfloor m/2} \frac{1}{(1 - q^{2m+1})^2}.
\]

This proves (1.4). The proof of Theorem 1.1 is now complete. \(\square\)

**Remark 2.8.** In light of (1.6) and (2.6), we have the curious formula

(2.7) \[
\lim_{q \to 1} (1 - q)^2 \sum_{n=0}^{\infty} \sigma(2n + 1) q^n = \frac{\pi^2}{4}.
\]

**Lemma 2.9.** Let \(n \in \mathbb{N}\) and

\[
t_8(n) = |\{(x_1, \ldots, x_8) \in \mathbb{N}^8 : T_{x_1} + T_{x_2} + \cdots + T_{x_8} = n\}|.
\]

Then

(2.8) \[
t_8(n) = \sum_{2 \mid d | n+1} \frac{(n + 1)^3}{d^3}.
\]

**Remark 2.10.** This is a result due to A. M. Legendre, see [B06, p. 139] or [ORW, Theorem 5].
Proof of Theorem 1.2. For $z \in \mathbb{C}$ with $|z| < 1$, we have
\[
\frac{z}{(1-z)^4} = \sum_{k=0}^{\infty} \binom{-4}{k} (-z)^k = \sum_{k=0}^{\infty} \binom{k+3}{3} z^{k+1} = \sum_{k=1}^{\infty} \frac{k(k+1)(k+2)}{6} z^k
\]
and hence
\[
\frac{z(1+4z+z^2)}{(1-z)^4} = (1+4z+z^2) \sum_{k=1}^{\infty} k(k+1)(k+2) \frac{z^k}{6}
\]
\[
= \sum_{k=1}^{\infty} (k(k+1)(k+2) + 4(k-1)k(k+1) + (k-2)(k-1)k) \frac{z^k}{6}
\]
\[
= \sum_{k=1}^{\infty} k^3 z^k.
\]
Thus
\[
\sum_{k=0}^{\infty} q^{2k+1} \frac{(1+4q^{2k+1}+q^{4k+2})}{(1-q^{2k+1})^4} = \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} m^3 q^{(2k+1)m} = \sum_{n=1}^{\infty} \left( \sum_{2|d|n} n^3 \right) q^n.
\]
Combining this with Lemma 2.9, we obtain
\[
\sum_{k=0}^{\infty} q^{2k} (1+4q^{2k+1}+q^{4k+2}) \frac{1}{(1-q^{2k+1})^4} = \sum_{n=1}^{\infty} t_n(n-1)q^{n-1} = \psi(q)^8.
\]
So, with the help of Lemma 2.1, we get the desired identity (1.8). This completes the proof.

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