Magnetization current and anomalous Hall effect for massive Dirac electrons

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Existing investigations of the anomalous Hall effect (i.e. a current flowing transverse to the electric field in the absence of an external magnetic field) are concerned with the transport current. However, for many applications one needs to know the total current, including its pure magnetization part. In this paper, we employ the two-dimensional massive Dirac equation to find the exact universal total current flowing along a potential step of arbitrary shape. For a spatially slowly varying potential we find the current density $j(x)$ and the energy distribution of the current density $j'_c(x)$. The latter turns out to be unexpectedly nonuniform, behaving like a $\delta$-function at the border of the classically accessible area at energy $\varepsilon$. To demonstrate explicitly the difference between the magnetization and transport currents we consider the transverse shift of an electron ray in an electric field.

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1. Introduction. Currents flowing in topological insulators (TIs) are usually associated with the gapless edge or surface modes\cite{2,3}. Other types of currents present in these materials, which now are accessible experimentally\cite{4,5}, stem from the anomalous Hall effect (AHE)\cite{6,7}. The standard approach to describe the AHE is to employ the equation of motion for a wave packet in a crystal\cite{8,9}

$$\dot{r} = \partial \varepsilon(p)/\partial p - v_B,$$

with the Berry velocity $v_B = (e/\hbar)E \times \Omega(p)$ normal to the local electric field and the Berry curvature $\Omega(p) = i\hbar^2 \nabla_p \times \langle u_p | \nabla_p u_p \rangle$ accounting for the change of the multicomponent wave function upon moving in the Brillouin zone. What is often forgotten is that the total microscopic current has two components named transport and magnetization currents\cite{10,11}. The Berry velocity is responsible only for the transport current of electrons. Which current will be measured depends on the particular experiment. In the case of nonequilibrium electron ray injection, Eq. (1) is applicable and the transport current is observed. In this paper, however, we consider the total anomalous Hall current density in equilibrium, whose distribution can not be described by Eq. (1), but which is responsible for electro-magnetic phenomena, like the Faraday effect\cite{12,13} or topological magneto-electric effects\cite{14,15}.

Specifically, we consider the two-dimensional massive Dirac Hamiltonian, a paradigmatic model exhibiting the AHE found in various material systems of current interest. In time-reversal-symmetric systems\cite{3,4,16,17} there are several Dirac cones and the anomalous bulk current is of the valley-Hall type\cite{18}. A single massive Dirac cone is realized on the surface of a three-dimensional TI covered by a ferromagnetic insulator film\cite{19}. There, the AHE corresponds to the charge Hall current, giving rise to the fascinating physics of axion electrodynamics\cite{19}.

Our main findings concern the AHE total current density in smooth electrostatic potentials. At the end of the paper we present the calculation of the side jump of an electron ray, showing that the total (transport+magnetization) semiclassical current not only exceeds the transport AHE current predicted by Eq. (1) but even has an opposite sign. What is even more interesting, we show that the dominant contribution to the total current comes from the classical reflection (stopping) points for electrons hitting the potential step (see Eq. (15) below) and can not be described by a semiclassical formula like Eq. (11).

We start with the calculation of the total equilibrium current flowing along a potential step $U(x)$, Fig. 1. Provided that the potential is constant away from the step, this calculation is exact and valid for any shape of $U(x)$ and any value of the Fermi energy $E_F$.

![FIG. 1: Total anomalous Hall current $J_y$ flowing along the potential step $U(x)$ as a function of Fermi energy $E_F$ for a two-dimensional massive Dirac Hamiltonian with gap $2\Delta$. The labels $v$, $c$ denote valence- and conduction bands, respectively. The energy distribution of the current density $j'_c(x)$ is also shown schematically for several values of $\varepsilon$.](image-url)
velocity, \( v = 0 \)). Stimulated by the exact solution in a uniform electric field, we argue that \( \mathcal{J}(\mathbf{r}) \) has the form of a \( \delta \)-function existing only along these lines of stopping points, Eq. (13). The "quantum width" of this \( \delta \)-function scales like \( \Delta r \sim (\hbar^2/|\nabla U|)^{1/3} \), being nonperturbative in both Planck’s constant and the electric field.

2. Microscopic calculation of the equilibrium current. Consider the Hamiltonian

\[
H = v_F(\sigma_x p_x + \sigma_y p_y) + \Delta \sigma_z + U(x) \ . \tag{2}
\]

Here, the Pauli matrices act on the sublattice, valley, or spin degree of freedom depending on the material under consideration. Instead of an uniform electric field, we consider a continuous step-like potential \( U(x) \), with \( U_R = U(x \to \infty) \) and \( U_L = U(x \to -\infty) \). An arbitrarily large electric field \( eE = -\nabla U \) exists only in a certain region around \( x \approx 0 \). At large \(|x|\), the potential is constant and there is a gap in the spectrum \(|\varepsilon - U_{R,L}| < \Delta \), shifted up on the right and down on the left side of the potential step, see Fig. 1. For definiteness, we choose \( U_R - U_L < 2\Delta \). We rely here neither on semiclassical nor on weak or constant electric field approximations.

Let \( \Psi_j(x,y) = e^{i p_y y / \hbar} \psi_j(x) \) be a complete set of eigenfunctions of the Hamiltonian Eq. (2) with conserved momentum \( p_y \equiv p_y \). Our goal is to find the transverse current density defined via the velocity operator \( \mathbf{v} = i[H, \mathbf{r}] / \hbar \) for the Dirac Hamiltonian with \( e = -|e| \)

\[
j_y(x) = ev_F(\sigma_y) = ev_F \sum_{i,\varepsilon_i < \varepsilon_F} \psi_i^\dagger(x) \sigma_y \psi_i(x) \ . \tag{3}
\]

Our approach to find \( j_y(x) \) is motivated by the calculation of the out of plane current induced polarization in Rashba wires in Refs. [14][21]. Consider the spin-density for a particular stationary state \( \psi_j^\dagger(x) \sigma_z \psi_j(x) \),

\[
0 \equiv \frac{d}{dt} \psi_j^\dagger \sigma_z \psi_j = i \hbar \psi_j^\dagger [H, \sigma_z] \psi_j - v_F \partial_x (\psi_j^\dagger) \psi_j \ , \tag{4}
\]

from which we readily find

\[
\Delta \psi_j^\dagger \sigma_z \psi_j - v_F p_y \psi_j^\dagger \sigma_z \psi_j = -\frac{\hbar v_F}{2} \partial_x (\psi_j^\dagger) \psi_j \ . \tag{5}
\]

This relation is enough to find the current for electrons with \( p_y = 0 \), which has a pure AHE form along the potential step. The general case with \( p_y \neq 0 \) requires more effort.

The Hamiltonian Eq. (2) depends on two parameters \( \Delta \) and \( p_y \), i.e. \( H = H(\Delta, p_y) \). The two eigenfunctions \( \psi(\Delta, \pm p_y) \) of \( H \) (still depending on the coordinate \( x \)) are related to the eigenfunctions of the same Hamiltonian with vanishing \( y \)-momentum and enlarged mass

\[
\psi(\Delta, \pm p_y) = e^{\pm i \theta \sigma_z} \psi(\sqrt{\Delta^2 + v_F^2 p_y^2}, 0) \ , \quad \tan \theta = \frac{v_F p_y}{\Delta} \ . \tag{6}
\]

Calculating the expectation values of Eq. (2), we find

\[
\Delta \psi_j^\dagger \sigma_z \psi_j + v_F p_y \psi_j^\dagger \sigma_y \psi_j = \Delta \psi_j^\dagger \sigma_z \psi_j - v_F p_y \psi_j^\dagger \sigma_y \psi_j \ , \tag{7}
\]

where \( \psi_+ = \psi(\Delta, p_y) \) and \( \psi_- = \psi(\Delta, -p_y) \). Here, we used that due to Eq. (3) the values of \( \psi_j^\dagger \sigma_z \psi_j \) do not depend on the sign of \( p_y \).

To find the total current we only need to know the sum \( \psi_j^\dagger \sigma_y \psi_j \psi_+ + \psi_j^\dagger \sigma_y \psi_- \). Combining Eq. (7) and Eq. (6), we can eliminate the terms \( \psi_j^\dagger \sigma_z \psi_j \) and find

\[
\psi_j^\dagger \sigma_y \psi_+ + \psi_j^\dagger \sigma_y \psi_- = -\frac{\Delta h v_F}{2(\Delta^2 + v_F^2 p_y^2)} \left[ \psi_j^\dagger \psi_j + \psi_j^\dagger \psi_j \right] \ . \tag{8}
\]

The r.h.s. is effectively a derivative of the electron density. Substituting Eq. (8) into Eq. (3) and integrating across the potential step leads to the total AHE current

\[
J_y = J_R - J_L \ , \quad J_R = -\frac{e \Delta h v_F}{2} \sum_{i R(L)} \psi_i^\dagger_{R(L)} \psi_i_{R(L)} \ . \tag{9}
\]

Here, the summations over all occupied states \( i_R \) and \( i_L \) are performed at some points \( x_R, x_L \) to the right and to the left of the step, where the potential is constant. Interference terms between the left- and right-moving waves present in \( \psi_j^\dagger \psi_j \) average out if these points are far away from the step leading to an exact AHE current, universal for an arbitrary shape of the potential (compare to Fig 2 below).

In the case of a slowly varying potential \( U(x) \) considered in the next section, Eq. (9) allows us to find the part of the current flowing in a strip \( x_L < x < x_R \) even for points \( x_{R(L)} \) inside the step region.

We introduce the valence and conduction band contributions \( J_{R(L)} = J_{R(L)}^c + J_{R(L)}^v \) in Eq. (9). Then

\[
J_{R(L)}^v = \frac{e}{2\hbar} (U_{R(L)} + \Delta - \varepsilon_F) \ , \tag{10}
\]

for \( \varepsilon_F > \Delta + U_{R(L)} \) and \( J_{R(L)}^v = 0 \) otherwise [21]. Integration over the momentum direction in Eq. (9) is done as \( \int_0^{2\pi} d\phi / (a^2 + \cos^2 \phi) = 2\pi / a \sqrt{a^2 + 1} \). The current Eq. (10) is proportional to the difference between the Fermi energy and the conduction band bottom. To find the contribution to the current from the valence band electrons, we perform the summation in Eq. (9) over all occupied states in that band with energies higher than some large negative energy \( \varepsilon_{min} \). This gives

\[
J_{R(L)}^v = \left\{ \begin{array}{ll}
\frac{e}{2\hbar} (\varepsilon_{min} + \Delta - U_{R(L)}) & , \varepsilon_F > U_{R(L)} - \Delta \\
\frac{e}{2\hbar} (\varepsilon_{min} - \varepsilon_F) & , \varepsilon_F < U_{R(L)} - \Delta \end{array} \right\} \ . \tag{11}
\]

The dependance on \( \varepsilon_{min} \) disappears in the current \( J_y \) defined in Eq. (9). For electrons with energies smaller than \( \varepsilon_{min} \), the density \( \psi_j^\dagger \psi_j \) is constant and the current
density, being a derivative of the particle density, Eq. 8 vanishes. With the help of Eqs. 10, 11 we find the total current as a piecewise linear function of \( \varepsilon_F \) (see Fig. 1)

\[
J_y = \begin{cases} 
0 & \varepsilon_F > U_R + \Delta \\
\frac{\partial}{\partial x} (\varepsilon_F - U_R - \Delta) & U_R > \varepsilon_F - \Delta > U_L \\
\frac{\partial}{\partial x} (U_L - U_R) & U_L + \Delta > \varepsilon_F > U_R - \Delta \\
\frac{\partial}{\partial x} (U_L - \Delta - \varepsilon_F) & U_R > \varepsilon_F + \Delta > U_L \\
0 & U_L - \Delta > \varepsilon_F 
\end{cases}
\] (12)

valid for any shape of the potential step \( U(x) \) (we don’t even require monotonous \( U(x) \)). Generalization of Eq. 12 for \( U_R - U_L > 2\Delta \) is presented in the supplementary [21].

Interestingly, the currents in either valence or conduction band coincide (by absolute value) with the half of a single-Landau level quantum Hall drift current [22].

Rather surprisingly, for energy regions, \( \varepsilon_F > U_R + \Delta \) and \( \varepsilon_F < U_L - \Delta \), the current Eq. 12 vanishes abruptly, \( J_y \equiv 0 \). A source of the AHE current is attributed to the Berry anomalous velocity Eq. 1 with \( \Omega_z(p) = \frac{1}{2} \hbar^2 \Delta v_F^2 / (\Delta^2 + p^2 v_F^2)^{3/2} \) [10]. In the case of a monotonous \( U(x) \), the last term in Eq. 1 has always the same sign, meaning there is no current cancellation upon summation over electronic states. A vanishing \( J_y \) in Eq. 12 is possible only because of another, not captured by Eq. 1, contribution to the microscopic current density, called magnetization current [10, 12], arising because of an in-homogeneity of electron wave-packet rotation.

To understand better the spatial distribution of the current Eq. 9 together with the role of interference oscillations between the incoming and outgoing waves, neglected in Eqs. 10, 12, we describe below a numerically exact current density in a constant electric field.

3. Uniform electric field. Let the potential in Eq. 2 be \( U(x) = Fx - \Delta \). In this paper we consider Dirac Hamiltonians with a gap large enough to neglect tunneling between the bands. The large insulating gap means that in order to investigate the development of the AHE current along the border of an electron Fermi liquid it is enough to take only the conduction band electrons with energies slightly above the gap into account, i.e. the non-relativistic limit \( |p_y v_F|, |Fx| \ll \Delta \) with the spinor wave-function [21]

\[
\psi^T = e^{ip_y y} \phi(x), 0 \), \( \phi(x) = \Delta i ((x-x_c)/x_0) \) .
\] (13)

Here, \( \Delta i(x) \) is the Airy function, \( x_0 = (\hbar^2 v_F^2 / (\Delta F))^{1/3} \) and \( x_c = \varepsilon / F - p_y^2 v_F^2 / (2\Delta F) \) and the energy of an electron in the conduction band is \( |\varepsilon| \ll \Delta \).

The current due to a single state Eq. 13 is considered in the supplementary material. In Fig. 2 we show the total density of the conduction band current, \( j_y(x) \), for \( \varepsilon_F = 0 \) obtained after summation over the energy and \( p_y \)-momentum [21]. Besides the interference oscillations, the figure agrees well with the step function form

\[
j_y(x) = \frac{e}{2\hbar} F \theta(-x) .
\] (14)

The width of the \( \delta \)-function, nonperturbative both in \( \hbar \) and in the electric field, is \( r_0 = (\hbar^2 v_F^2 / (\Delta \sqrt{|\varepsilon|}))^{1/3} \). Eq. 15 is valid if \( \sqrt{|\partial^2 U / \partial x^2|} r_0 \ll \sqrt{|\Delta U|} \). Absence of inter-band tunneling requires \( r_0 |\nabla U| \ll \Delta \).

The \( \delta \)-function Eq. 15 is peaked at the border of the area accessible classically at the energy \( \varepsilon \). This defines the line of stopping points, where both components of the momentum of an electron reaching such a point vanish.

![FIG. 2: Anomalous Hall current density (conduction band, \( e = -|e| \) for \( U(x) = Fx - \Delta \) and \( \varepsilon_F = 0 \). Distances are measured in units of \( x_0 = (\hbar^2 v_F^2 / (\Delta F))^{1/3} \). The vertical axis has arbitrary units. The thin solid line is the current density \( j_y(x) \). The step-like current density is formed at distances \( \sim x_0 \) from the edge. The dashed line shows the energy distribution of the current density \( j_y^n(x) \) and the thick solid line is \( j_y^m(x) \) smoothed with the function \( \exp(-|x-x'|^2/x_0^2) \).](image-url)
Inside this area \( j^t(r) = 0 \). We describe the nature of the cancellations leading to the disappearance of \( j^t(r) \) in the supplementary.

For the valence band Eq. 14 changes sign. For a smooth potential, the total current density \( j^t_y(x) + j^{\bar{v}}_y(x) \) vanishes everywhere where the Fermi energy stays locally inside the conduction or the valence band. For example, the nondissipative AHE current vanishes inside an electronic puddle in the conduction band, but exists due to the valence band current outside the puddle.

4. Ray dynamics and magnetization currents. The eigenfunction of the Hamiltonian Eq. 2 in the semiclassical limit may be written in the form (we use \( \hbar = v_F = 1 \) in this section)

\[
\psi(x) = \sqrt{1/v_F}e^{i[p(x)dx']} [1 + \beta(x)\sigma_y]\phi .
\] (16)

Here, \( \phi^T(x) = \frac{1}{\sqrt{2}}(\sqrt{1 + \xi}, \sqrt{1 - \xi}) \), \( \xi = \Delta/\sqrt{\Delta^2 + p^2} \) and \( p(x) = \sqrt{(x - U(x))^2 - \Delta^2} \). For illustrative purposes we only consider the case of an incident ray parallel to the potential gradient. The classical velocity of the ray is \( v_c = p/\sqrt{\Delta^2 + p^2} = p/(-\Delta) \). (Arbitrary incident angles may be considered with the help of Eq. 6)

The coefficient \( \beta \) in Eq. 15, responsible for the expectation value of the anomalous velocity, may be found by acting with \( \psi(x) \) on the Hamiltonian Eq. 2 or may be extracted in the linear approximation in \( U' = dU/dx \) from the exact Eq. 8. Doing so, we find

\[
\langle v_y \rangle = \frac{\psi^\dagger\sigma_y\psi}{\psi^\dagger\psi} = 2\beta = \frac{-U'\Delta}{2p^2\sqrt{\Delta^2 + p^2}} .
\] (17)

This velocity is bigger, and even much bigger if \( p(x) \ll \Delta \), than the AHE velocity deduced from Eq. 1. However, Eq. 17 does not provide the true information about the transverse transport, since the solution \( \psi(x) \) extends indefinitely along the y-axis. Even more, as we show below, the term \( \sim \beta \) in the wave function Eq. 16 simply does not contribute to the electrons’ trajectory.

To find the actual bending of the trajectory, we need to consider a ray of electrons

\[
\Psi(x,y) = \int f(p_y)\psi_{p_y}(x,y)dp_y ,
\] (18)

where \( \psi_{p_y}(x,y) \) are the solutions of the Dirac equation Eq. 2 having all the same energy \( \varepsilon \) and the narrow function \( f(p_y) \) is peaked at \( p_y = 0 \).

The relation between the eigenfunctions of \( H \) in Eq. 2 with different \( p_y \) was found in Eq. 8, which, for small \( p_y \), gives \( \psi_{p_y}(x,y) = e^{ip_yy}(1 + \sigma_x p_y/(2\Delta))\psi(x) \). Substituting this into Eq. 13 we obtain

\[
\Psi = \left(1 + \frac{\sigma_x}{2\Delta} \frac{\partial}{\partial y}\right) g(y)\psi(x) ,
\] (19)

where \( g(y) = \int dk e^{iky} f(k) \) is a smooth envelope function in \( y \) direction of a ray propagating mostly along the \( x \)-axis. It is convenient to choose \( g(y) \) almost flat within some region \( dy \gg 1/p(x) \) which smoothly goes to zero outside the region, see Fig. 3.

Surprisingly, Eq. 15 is all what we need in order to find the transverse displacement of the ray. To see that, we rewrite Eq. 15 as

\[
\Psi = g \left(y + \frac{1}{2\Delta}\right) \psi_+(x) + g \left(y - \frac{1}{2\Delta}\right) \psi_-(x) ,
\] (20)

where we use the decomposition of \( \psi(x) \) into the \( \sigma_x \) eigenvectors, \( \psi = \psi_+ + \psi_- \) and \( \sigma_x \psi = \psi_+ - \psi_- \). The trajectory \( y(x) \) is now found as (see Fig. 3)

\[
y(x) = \frac{1}{2\Delta} \frac{-\psi^\dagger(x)\sigma_x\psi(x)}{\psi^\dagger(x)\psi(x)} = \frac{-p(x)}{2\sqrt{\Delta^2 + p(x)^2}} .
\] (21)

The shift of the trajectory \( y(x) \) exists already for the wave function Eq. 10 in zeroth order in \( U' \) when the term \( \sim \beta \) is omitted. The transverse transport velocity is now

\[
v_{ytr} = v_x \frac{dy}{dx} = \frac{U'\Delta}{2\sqrt{\Delta^2 + p(x)^2}}^{3/2} ,
\] (22)

in accordance with Eq. 11.

Both the distribution of the local velocity Eq. 17 and the shift of the ray Eqs. 21, 22 are shown in Fig. 3. Corresponding total and transport currents may be shown to satisfy the relation \( j = j_{tr} + \nabla \times M(r) \) 10, 21, where

![FIG. 3: The electron ray with width \( \delta y \) traversing the region of a smooth potential step \( U(x) \) with energy \( \varepsilon > \Delta + U \). Blue lines are the borders of the ray with approximately constant electron density inside. Red lines show the velocity field \( \langle \mathbf{v} \rangle = \langle \sigma \rangle \). There is a side jump of the ray during crossing the electric field area, \( dU/dx \neq 0 \), consistent with the anomalous velocity \( v_B = \Theta \times \nabla U/\hbar \) with \( \Theta \) being the Berry curvature. However, the value of the transverse velocity inside the ray always exceeds this value, \( |v_y| > v_B \). Also shown is the ray envelope function \( g(y) \). According to Eqs. 20, 21 the ray is always shifted downwards compared to \( g(y) \). The magnitude of the shift is bigger for larger velocity \( v_x \) or, equivalently, for larger \( \varepsilon - U(x) \). Circles inside the ray show how the uniform electron’s magnetization results in a current. The density of the circles (magnetization), rotating counter-clockwise as required by the sign of the transverse current, increases \( \sim |\Psi|^2 \) to the right on the figure.](image)
∇ × M(r) is the magnetization current and M(r) is the density of the magnetic moment (this we illustrate in Fig. 3 by circles with a coordinate dependent density).

The transport (22) and the total (17) anomalous Hall currents flow in opposite directions, as indicated in Fig. 3. One should remember, however, that the figure shows the semiclassical current Eq. (17) due to a single injected electron ray. In the case of the equilibrium AHE there will be many such rays, including the ones reflected by the potential. The total equilibrium transverse current – flowing in the direction suggested by Eq. (1) – originates from the narrow reflection region, Eq. (15), not captured by Eqs. (20) (21). Interestingly, the anomalous shift of the trajectory Eqs. (20) (21) turned out to have an exact upper bound, y < 1/(2Δ).

Conclusions. In this paper, we calculated the total local AHE current for electrons described by the massive Dirac Hamiltonian. The exact results turned out to be strongly universal in the case where the potential U(x) depends only on one coordinate. What is even more surprising, the current which we found for an arbitrarily smooth potential U(r) turns out to be much stronger (and its energy/coordinate dependence much sharper) than the usually considered Berry curvature-induced currents. For example, the equilibrium anomalous Hall currents exist if the Fermi energy lies inside the insulating gap, but disappear abruptly if ε_F is shifted into the conduction or valence band. The width of the transition is governed by the weakness of the electric field, the smoothness of the potential or the size of the sample. It will be interesting to see this sharp Fermi energy dependence in measurements of the quantum Kerr and Faraday effects (13), which recently became available for the surface states in three-dimensional topological insulators (14).

Another promising direction for further research lies in the understanding of the relations between the non-dissipative equilibrium bulk AHE currents considered here and the protected edge currents found in the topological insulators (23).

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APPENDIX A: ANOMALOUS HALL CURRENT FOR THE CASE $U_R - U_L > 2\Delta$

Here, we consider the generalization of the formula Eq. (12) to the case of a large potential step $U(x)$ with a magnitude that exceeds the value of the gap in the electron spectrum, $U_R - U_L > 2\Delta$.

As in the main text, we assume that the gap $\Delta$ is large enough and $U(x)$ varies sufficiently slowly such that tunnelling between the bands may be neglected. That means that Eqs. (11, 12) are still valid, i.e. (Note that the electron’s charge is negative, $e = -|e|$.)

$$J_R^v = \begin{cases} \frac{e}{2h}(U_R + \Delta - \varepsilon_F) , & \varepsilon_F > U_R + \Delta \\ 0 , & \varepsilon_F < U_R + \Delta \end{cases}$$

$$J_L^v = \begin{cases} \frac{e}{2h}(U_L + \Delta - \varepsilon_F) , & \varepsilon_F > U_L + \Delta \\ 0 , & \varepsilon_F < U_L + \Delta \end{cases}$$

and

$$J_R^v = \begin{cases} \frac{e\Delta}{2h} (\varepsilon_{\text{min}} + \Delta - U_R) , & \varepsilon_F > U_R - \Delta \\ \frac{e\Delta}{2h} (\varepsilon_{\text{min}} - \varepsilon_F) , & \varepsilon_F < U_R - \Delta \end{cases}$$

$$J_L^v = \begin{cases} \frac{e\Delta}{2h} (\varepsilon_{\text{min}} + \Delta - U_L) , & \varepsilon_F > U_L - \Delta \\ \frac{e\Delta}{2h} (\varepsilon_{\text{min}} - \varepsilon_F) , & \varepsilon_F < U_L - \Delta \end{cases}$$

As we explained in the main text, $\varepsilon_{\text{min}}$ is the lower bound of the integral(sum) over energies in Eq. (9) which cancels out from the current $J_y$. Electrons with energies below $\varepsilon_{\text{min}}$ do not contribute to the current since for them the derivative of the density $\partial_x \psi^\dagger \psi$ vanishes in Eq. (3). We assume that there is no contribution to AHE from the bottom (ultraviolet cutoff) of the valence band. The formal proof of the latter will be given elsewhere.

The band structure in the presence of a large potential step is shown in Fig. 4 and needs to be compared to Fig. 1. The five regimes of different current behaviors $J_y(\varepsilon_F)$ in Fig. 4 represent (from lower to higher energies $\varepsilon_F$) metallic, half-metallic, insulating, half-metallic and metallic phases. Here, metallic refers to $\varepsilon_F$ lying inside the conduction or valence band, insulating means that $\varepsilon_F$ always lies inside the gap and half-metallic means that $\varepsilon_F$ is inside the gap only on one side (half plane) of the potential step. The sequence of regimes in Fig. 4 is metallic, half-metallic, metal-metal, half-metallic and metallic where the metal-metal regime corresponds to two half-plane metals separated by an impenetrable tunnel barrier. The total anomalous Hall current for the setup in Fig. 4 found from Eqs. (9, 23, 24) is

$$J_y = \begin{cases} 0 , & \varepsilon_F > U_R + \Delta \\ \frac{e\Delta}{2h} (\varepsilon_F - U_R - \Delta) , & U_R + \Delta > \varepsilon_F > U_R - \Delta \\ -\frac{e\Delta}{h} , & U_R - \Delta > \varepsilon_F > U_L + \Delta \\ \frac{e\Delta}{2h} (U_L - \Delta - \varepsilon_F) , & U_L + \Delta > \varepsilon_F > U_L - \Delta \\ 0 , & U_L - \Delta > \varepsilon_F \end{cases}$$

valid again for any shape of the potential step $U(x)$. The main difference to Eq. (12) is the smaller current in the central metal-metal regime, i.e. $J_y = -\frac{e\Delta}{h}$ instead of $J_y = \frac{e\Delta}{h}(U_L - U_R)$.

APPENDIX B: CURRENT DENSITY IN AN UNIFORM ELECTRIC FIELD

Like in the main text, we consider here a linear potential $U(x) = Fx - \Delta$ and assume the nonrelativistic limit of the conduction band electrons, i.e. $|Fx| \ll \Delta$, $|p_y v_F| \ll \Delta$. The Dirac equation Eq. (2) for the spinor wave function $\psi^T = (\phi, \chi)$ reduces in this case to a non-relativistic Schrödinger equation for the upper component, $\phi$, with the lower component being negligibly small, $|\chi| \ll |\phi|$,

$$\left( \frac{p^2}{2m} + F x \right) \phi = \varepsilon_F \phi , \quad \chi = \frac{p_x + ip_y}{2mv_F} \phi .$$

Here, the mass $m$ is related to the gap parameter via $\Delta = mv_F^2$. The Schrödinger equation for $\phi$ is solved by the Airy function as used in Eq. (13) of the main text. In this limit, the anomalous Hall current density Eqs. (9, 10) becomes proportional to the derivative of the electron
charge density

\[ j_y(x) = -e \frac{\hbar}{2m} \partial_x \rho(x) . \tag{27} \]

Also without the loss of generality we may put Fermi energy equal to zero, \( \varepsilon_F = 0 \).

Current for a single electron

We start by considering the current density Eq. (27) contributed by a single electronic state described by the wave function Eq. (13). The result of such a calculation, shown in Fig. 5 for \( p_y = \varepsilon = 0 \), demonstrates strong interference oscillations between incoming and reflected electron waves. As follows from the asymptotic form of the Airy function for large negative \( x \), the oscillation amplitude stays constant but its period decreases like \( \sim 1/|x|^{3/2} \).

In order to visualize the semiclassical part of the current density we smooth out the interference oscillations by averaging the density at each point \( x \) with the weight functions \( e^{-((x-x')/x_0)^2} \) and \( e^{-((x-x')^2/x_0)^2} \) in Fig. 5. Assuming an oscillation amplitude of unity, the semiclassical current density at large negative \( x \) is small like \( 1/(4|x|^{3/2}) \). Therefore, we have to enlarge the smoothed current density in the figure in order to make the semiclassical current density visible.

The smoothed current density shown in Fig. 5 (thick solid line) has two distinct features. The first was already mentioned. It is the positive power law tail of \( j_y(x) \) for large negative values of \( x \). (Only at this tail the current density may be explained by Eq. (17) for the special case of a weak linear potential.) The second is the large negative (\( \varepsilon = -|\epsilon| \)) bump of the current density at the reflection point \( x = 0 \). The current density, Eq. (27), is proportional to the derivative of the electron density, which in the semiclassical limit is \( \rho(x) \sim 1/\sqrt{-x} \) at large negative \( x \) and \( \rho(x) \approx 0 \) at positive \( x \). That is why both the power law tail of \( j_y(x) \) and the negative bump are inevitable and survive the smoothing procedure. The total single-electron current integrated over \( x \) vanishes in the non-relativistic approximation Eq. (13). The negative bump of the current density at the reflection point is responsible for the vanishing of the current density energy distribution \( j^c(r) \) inside the classically accessible area, see Eq. (15). (Note that \( j^c(r) \) is a sum of currents due to many electrons with the same energy \( \varepsilon \), while Fig. 5 shows only the single electron current.)

Summing up the current contributions due to many electrons

The semiclassical electron density in the conduction band is given by the integral

\[ \rho = \int_{\varepsilon_0 < \varepsilon < \varepsilon_F} \frac{d^2p}{(2\pi \hbar)^2} = \frac{2}{(2\pi \hbar)^2} \int \frac{d\varepsilon_x dp_y}{d\varepsilon_x / dp_x} , \tag{28} \]

where \( \varepsilon_0 = p_y^2/(2m) \) and the coordinate-dependence emerges through the limits of integration, \( \varepsilon = p^2/2m + Fx < \varepsilon_F \). To make a connection between this formula and the wave function Eq. (13), we notice that the smooth part of the squared Airy function, representing the semiclassical density, using the negative \( x \) asymptotics may be written as (here \( p_y = 0 \), generalization for finite \( p_y \) is obvious, see Eq. (30) below)

\[ \langle \text{Ai}^2 \left( \frac{x}{x_0} \right) \rangle = \frac{\sqrt{x_0}}{2\pi \sqrt{-x}} = \frac{\sqrt{2mFx_0}}{2\pi p_x(x)} = \frac{\sqrt{2F/\varepsilon_0}}{2\pi d\varepsilon_x / dp_x} , \tag{29} \]

where we have used \( p_x(x) = \sqrt{-2mFx} \) and \( \langle ... \rangle \) denotes the average procedure. Consequently, we may replace the semiclassical density of the conduction band electrons Eq. (28) by the exact formula

\[ \rho(x) = \frac{1}{\pi \hbar^2 \sqrt{2Fx_0/m}} \times \int_{\varepsilon_x > 0} \text{Ai}^2 \left( \frac{\varepsilon_x + p_y^2/(2m) + Fx}{Fx_0} \right) d\varepsilon_x dp_y . \tag{30} \]
The absence of electrons with energies above the Fermi energy, $\varepsilon_F = 0$, is taken care of automatically due to the exponential suppression of the Airy function for positive arguments, so there is no need to introduce an upper limit of integration over either $\varepsilon_x$ or $|p_y|$.

With the help of Eq. (27) we may find the current density

$$j_y(x) = \frac{eF}{2\pi m \sqrt{2F \varepsilon_x/m}} \int_{\varepsilon_y > 0} A_1^2 \left( \frac{\varepsilon_y + Fx}{Fx_0} \right) dp_y.$$  

This formula is used for calculating the current density in Fig. 2. The easiest way to find the energy distribution of the current density $j_y^c(x)$ is by differentiating Eq. (31) to obtain

$$j_y^c(0) = \frac{1}{F} \frac{\partial}{\partial x} j_y(x).$$  

This result is shown in Fig. 2.

Replacing the squared Airy function by its asymptotic behavior we find the semiclassical density of the anomalous Hall current (valid only for negative $x$)

$$j_y(x) = \frac{eF}{4\pi^2 \hbar} \int_0^{-Fx} \frac{dy}{\sqrt{\varepsilon_y(Fx - \varepsilon_y)}}.$$  

Upon energy integration we arrive at Eq. (14), $j_y(x) \propto \theta(-x)$.

**Comparison to existing results**

Our results for the anomalous Hall current density for the massive Dirac Hamiltonian (e.g. Eq. (14) of the main text) differ substantially from the calculation of dissipationless bulk currents in a recent paper Ref. [18] where only the transport contribution to the anomalous Hall current caused by the Berry velocity in Eq. (1) was taken into account. For the potential $U = Fx$ (shifted by $\Delta$ from what we used before) and the Fermi energy $\varepsilon_F = 0$ the authors of Ref. [18] have found the total (conduction plus valence band) density of anomalous current in the form

$$j_y(x) = \begin{cases} j_0, & |x| < x_0 \\ j_0 x_0/|x|, & |x| > x_0 \end{cases} \quad j_0 = \frac{|e|F}{2\hbar}.$$  

Here $x_0 = \Delta/F$ and $x = \pm x_0$ are the classical turning points for the conduction and valence band electrons at the Fermi energy. (Note that the total current $\int j_y(x) dx$ diverges logarithmically.)

Our approach for a sufficiently small $F$ would give instead of Eq. (31) the current density

$$j_y(x) = \begin{cases} j_0, & |x| < x_0 \\ 0, & |x| > x_0 \end{cases} \quad j_0 = \frac{|e|F}{2\hbar}.$$  

The current density is constant and proportional to the electric field only when the Fermi energy lies in the gap between two bands.

The results Eq. (31) and Eq. (35) are compared in Fig. 3. As we mentioned already, the reason for the difference between Eqs. (31) and (35) is that the authors of Ref. [18] calculate only the transport current caused by the Berry velocity contribution Eq. (1) to the motion of the center of the wave packet. Our derivation, starting from the calculation of the expectation value of the velocity operator automatically includes both the motion of its center and the inhomogeneous rotation of the wavepacket, creating the magnetization current. Although it was argued [10–12] that the magnetization current is irrelevant for transport phenomena, it is necessary for finding e.g. the magnetic moment of the electron gas.

Taking into account only the conduction band contribution to the current near the Fermi energy crossing with the bottom of the conduction band in Eq. (35) gives

$$j_y(x) = \frac{e}{2\hbar} F \theta(-x_0 - x),$$  

which is the same as Eq. (11) with shifted coordinate due to a different definition of the constant electric field potential. Analogously, extracting the conduction band current from Eq. (33) at $x \approx -x_0$ gives for the Fermi energy above the top of the valence band and only slightly above the bottom of the conduction band, valence band electrons give a large constant contribution to the current, while the linear in $x$ (at $|x - x_0| \ll x_0$) part of $j_y(x)$ comes from the conduction band electrons

$$j_y(x) = -\frac{e}{2\hbar} \frac{(x_0 + x)F^2}{\Delta} \theta(-x_0 - x).$$  

FIG. 6: Top - Energy bands structure. Bottom - Comparison of the density of the anomalous Hall current found in this paper (solid-blue) with the current density due to side jumps only [18] (dashed-brown) around the band crossing by the Fermi energy in a strong electric field.
which is parametrically smaller than our result. The linear increase of the current in Eq. (37) reflects the fact that the electron density in a two-dimensional non-relativistic electron gas in a constant electric field increases linearly with the coordinate. All these electrons have the same anomalous velocity Eq. (31). The current density of Eq. (30) is much more singular than Eq. (37) and leads to the $\delta$-function energy distribution of the current density $j^r(r)$, Eq. (15).

**Vanishing of $j^r(r)$**

Integration over energy in Eq. (33) gives a step-shaped current density Eq. (14). Differerentiating this result like in Eq. (32) gives a $\delta$-function energy distribution of the current density $j^r_\epsilon(x)$. Vanishing of $j^r_\epsilon(x)$ almost everywhere except the close vicinity of the line $\Delta + U(x) - \epsilon = 0$ allows us to suggest a general $\delta$-function formula for the current in arbitrary smooth potential $U(r)$, Eq. (15).

The $\delta$-function Eq. (15) peaks at the border of the area accessible classically at the energy $\epsilon$. This is the line of stopping points where both components of momentum of an electron reaching the point vanish. Inside this area $j^r(r) = 0$, which is somewhat surprising since the semiclassical current density (given by the $\sim 1/|x|^{3/2}$ tail of the smoothed density in Fig. 5) for each electron is always of the same sign. The only current of the "wrong" sign, ensuring the vanishing of the current density distribution, is the negative bump in the smoothed current density in Fig. 5 at the classical turning point. Through every point r there exist two trajectories with energy $\epsilon$ and momentum p normal to the local electric field. These are trajectories having a turning point in the sense of Fig. 5 and the "negative bump" in the current density at r.

**APPENDIX C: EXPLICIT PROOF OF THE FORMULA FOR $j^r(r)$.**

In this section we first give a proof of Eq. (15) in the nonrelativistic limit by calculating the current carried by the conduction band electrons for a smooth two-dimensional potential and a Fermi energy only slightly above the insulating gap.

Consider the Dirac Hamiltonian

$$H = v_F (\sigma_x p_x + \sigma_y p_y) + \Delta \sigma_z + U(r) - \Delta.$$  
(38)

Calculation of the anomalous Hall current (carried e.g. by the conduction band electrons) becomes especially easy to calculate when $|U(r)| \ll \Delta$.

Let the eigenfunction of $H$ in Eq. (38) have the form

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} .$$  
(39)

Substituting this into Eq. (38) in the limit $|\epsilon| \ll \Delta, |U| \ll \Delta$, one readily recovers the nonrelativistic Schrödinger equation

$$\left( \frac{p^2}{2m} + U(r) \right) \phi = \epsilon \phi , \quad \chi = \frac{-i\hbar \partial_x + h \partial_y}{2mv_F} \phi .$$  
(40)

Here, $\Delta = mv_F^2$. We refer to the electron state described by the Dirac Hamiltonian Eq. (38) with at least one component of momentum effectively comparable to $\Delta/v_F$ as relativistic and the electron described by the Eq. (40) as nonrelativistic. The two components of the current $j = ev_F(\sigma)$ are now found as (compare to Eq. (3))

$$\psi^\dagger \sigma_y \psi = -\frac{\hbar v_F}{2\Delta} \partial_x \phi^* \partial_x \phi - \frac{i\hbar v_F}{2\Delta} [\phi^* \partial_y \phi - (\partial_y \phi^*)\phi] ,$$

$$\psi^\dagger \sigma_x \psi = \frac{\hbar v_F}{2\Delta} \partial_y \phi^* \partial_y \phi - \frac{i\hbar v_F}{2\Delta} [\phi^* \partial_x \phi - (\partial_x \phi^*)\phi] .$$  
(41)

Here, the second terms on the r.h.s. of both equations have the form of the usual currents in non-relativistic quantum mechanics, whereas anomalous Hall current is given by the first terms. The electron density in the semiclassical and non-relativistic approximation is

$$\sum_{\epsilon_0, \epsilon \leq \epsilon_F} \psi^\dagger \psi \psi^\dagger \psi = \int \epsilon^2 d\epsilon = \frac{2}{3\hbar^2} \frac{m(\epsilon_F - \epsilon)}{4\pi \epsilon_F^2} \frac{\epsilon}{|\epsilon|} .$$  
(42)

Combining Eqs. (41) and (42) we obtain

$$\frac{j}{2\hbar} \theta(\epsilon_F - U(r)) \frac{\epsilon}{|\epsilon|} \phi^* \phi .$$  
(43)

where $n$ is the unit vector normal to the $(x,y)$ plane. Differentiating this with respect to $\epsilon_F$ gives $j^r(r)$ of Eq. (15).

Our derivation of Eq. (15) for a smooth two-dimensional potential $U(r)$ relies on the nonrelativistic approximation Eq. (10). The energy distribution of the current density $j^r(r)$ found from Eq. (15) vanishes everywhere except in the narrow region around the line of stopping points $\epsilon - U(r) = 0$. But the electron in a smooth potential near a stopping point is always nonrelativistic. This suggests that the range of validity of the result Eq. (15) may be larger than just the nonrelativistic limit $|U(r)| \ll \Delta$, but it may be valid for an arbitrarily large (and still smooth) two-dimensional potential $U(r)$.

Indeed, Eqs. (12) (14) of the main text (second of those is the exact analog of Eq. (13)) were found for an arbitrarily large smooth potential but only depending on one coordinate $U(r) \equiv U(x)$. That means, all what is left to do in order to prove Eqs. (13) (15) for an arbitrarily large Fermi energy is to show that the contribution to the anomalous current from the electrons with high energy has the form of a local expansion in powers of the gradients of the potential. In other words, the anomalous current at a point $r$ caused by the electrons with $\epsilon - U(r) \gg \Delta$ must be a function of the gradient, $\nabla U(r)$, found at the same point.
To find the current density for an arbitrarily strong two-dimensional potential one may take the wave function Eq. (16) and sum up the currents due to all occupied states. However, we may simply notice that the anomalous velocity Eq. (17) due to the solution Eq. (16) depends only on the local derivative of the potential \( U' \). This means that if Eqs. (15, 43) are valid for arbitrary \( \varepsilon_F \) in an arbitrarily strong uniform electric field, they should be also valid for an arbitrarily smooth potential \( U(\mathbf{r}) \) for a large Fermi energy \( \varepsilon_F \sim \Delta \).

APPENDIX D: CALCULATION OF \( \beta(x) \).

In this section, we calculate explicitly starting from the semiclassical solution of Eq. (2) the coefficient \( \beta(x) \) entering the wave function Eq. (16) of the main text. This coefficient is responsible for the emergence of a finite expectation value of the anomalous velocity \( \langle v_y \rangle \) in Eq. (17). In the main text we avoid the direct calculation of \( \beta(x) \) and use instead Eq. (8) to calculate \( \langle v_y \rangle \). Similar to the corresponding part of the main text, we use here \( h = v_F = 1 \).

Let the conduction band electron described by the wave function Eq. (18) be in a state \( \phi_+ \) with an arbitrary incident angle \( \phi \). The scattering density \( \rho \) is proportional to \( \phi_+ \phi_+^* \). The components of \( \phi_+ \) are the positive and negative energy eigenvectors in the limit of a flat \( U(x) \).

The wave function Eq. (15) is in principle exact, provided one can find the coefficients \( a(x) \) and \( b(x) \) to all orders in the small \( U' \). Substituting Eq. (15) into the Dirac equation leads to an (exact) system of linear equations

\[
-a' + i \frac{U' \Delta}{2p \sqrt{\Delta^2 + p^2}} b + 2 \Delta b = 0 ,
\]

\[
-b' - i \frac{U' \Delta}{2p \sqrt{\Delta^2 + p^2}} a + 2p b = 0 .
\]

Approximate solutions of the system of equations (47) (which we are interested in) may be found iteratively. First, in the second equation, we may neglect a small derivative \( b' \) compared to \( 2p b \), leading to

\[
b \approx i \frac{U' \Delta}{4p^2 \sqrt{\Delta^2 + p^2}} a .
\]

Substituting this into the first equation of Eq. (47) and neglecting the small second order term \( \sim U' b \) we find

\[
a \approx \sqrt{\frac{\Delta^2 + p^2}{p}} .
\]

The fact that this solution reproduces correctly the classical electron density

\[
\rho \approx |a|^2 \sim 1/v_x ,
\]

is an additional crosscheck.

Eq. (10) of the main text is reproduced after we notice that

\[
\beta = ib/a .
\]

APPENDIX E: MAGNETIZATION CURRENT VS. TRANSPORT AHE CURRENT.

In this section, we consider in more details the division of the total anomalous Hall current into the transport and magnetization currents discussed in the last part of the main paper. Like in the main text, we put \( h = v_F = 1 \). Also like in the main text we are going to consider the transverse shift and transverse current distribution in a wide electron ray injected parallel to the x-axis and parallel to the electric field \( (U(r) \equiv U(x)) \) described by the wave function Eq. (18). Ray injection with an arbitrary incident angle may be considered with the help of Eq. (19) of the main text. It is constructive to consider the envelope function \( g(y) \) (Eq. (19) to be flat, \( g(y) \approx 1 \), over the large region \( \delta y \) (i.e. \( p_x \delta y \gg 1 \)) with also very smooth steps towards \( g = 0 \) outside this region, cf. Fig. 8.

By calculating the longitudinal current \( j_x = e \Psi^\dagger \sigma_x \Psi \) and the density \( \rho = e \Psi^\dagger \Psi \) from Eq. (20) we find

\[
j_x = eg^2 \left( y + \frac{1}{2\Delta} \right) \psi_+^\dagger \psi_+ - eg^2 \left( y - \frac{1}{2\Delta} \right) \psi_-^\dagger \psi_- ,
\]

\[
\rho = eg^2 \left( y + \frac{1}{2\Delta} \right) \psi_+^\dagger \psi_+ + eg^2 \left( y - \frac{1}{2\Delta} \right) \psi_-^\dagger \psi_- .
\]

We remind that \( \Psi = \Psi(x,y) \) is the wave function of a ray propagating mostly in x-direction, Eq. (18), and \( \psi = \psi(x) \) is the plane-wave solution Eq. (19) with momentum parallel to the electric field \( (p_y \equiv 0) \). The components of
ψ(x) in the σx eigenvalue basis are ψ+_x = 1/2(1 + σx)ψ and ψ−_x = 1/2(1 − σx)ψ. The second equation (19) was used in the main text to calculate the side-jump of the trajectory Eq. (21).

Now we may use the fact that for our choice of the normalization of the wave function Eq. (10) ψ+_xψ−_x = 1 and ψ+_1ψ−_1 = 1/vx and write instead of Eq. (22)

\[ j_x = eg^2(y) + eg'g/(\Delta v_x) , \]
\[ \rho = eg^2(y)/v_x + eg'g/\Delta , \]

where vx = vx(x) acquires a dependance on x in the case of a smooth potential U(x),

\[ v_x = \frac{p(x)}{\sqrt{\Delta^2 + p(x)^2}} , \]
\[ p(x) = \sqrt{(\varepsilon - U(x))^2 - \Delta^2} . \]

The second equality in Eq. (59) may be rewritten as

\[ \rho = eg^2(y + vx/(2\Delta))/vx , \]

which immediately gives the trajectory (equivalent to Eq. (21) of the main text)

\[ y(x) = -\frac{v_x(x)}{2\Delta} . \]

The possible time evolution of the quantum ray Eq. (18) should proceed in agreement with the continuity equation. Since the ray is built from waves of the same energy, the charge distribution is stationary and the continuity equation reduces to the vanishing of the divergence of the current, div j = 0. Thus we write

\[ \frac{\partial j_x}{\partial x} = -e g' g' v_x / \Delta v_x^2 = -\frac{\partial j_y}{\partial y} , \]

where g' = dg/dy and v'_x = dv_x/dx. The current j_y here can not be derived directly from the wave function Eq. (19). It appears due to the correction ∼ βσ_y to the semiclassical wave-function in Eq. (10), which was omitted in Eq. (19) and consequently in Eqs. (22). Still the continuity equation Eq. (57) allows us to find this current. The total anomalous current density j_y is expected to be independent of the transverse coordinate y only inside the ray, where g(y) ≈ 1. At the borders of the ray j_y increases from zero (outside) to this constant value. This increase is described by Eq. (57). Integration over y of the second equality in Eq. (57) gives the value of the anomalous current inside the ray

\[ j_y = e g^2 v'_x / 2\Delta v_x^2 . \]

The calculation of the derivative of the longitudinal velocity v'_x here gives

\[ v'_x = \frac{dv_x}{dx} = \frac{\Delta^2}{2p^3} \frac{U'}{U} \sqrt{\Delta^2 + p^2} = \frac{-U' \Delta}{p(\Delta^2 + p^2)} , \]

leading to

\[ j_y = -e g^2 U' \Delta \frac{p}{2p^3} , \]
\[ \langle v_y \rangle = \frac{j_y}{\rho} = \frac{-U' \Delta}{2p^3} \sqrt{\Delta^2 + p^2} , \]

in agreement with Eq. (17) of the main text. Thus we see that the anomalous transverse current inside the ray (and the velocity Eq. (17)), which is not captured by Eqs. (18) (22) follows from them through the continuity equation. Eq. (60) shows the total microscopic AHE current for which the corresponding velocity ⟨v_y⟩ can not be deduced from Eq. (1).

As was written in the main text, the difference between the total microscopic and transport current densities is naturally attributed to the magnetization current [10]

\[ j = j_{tr} + \dot{\mathbf{J}}_{mag} = \nabla \times \mathbf{M}(r) , \]

with M(r) being the density of the magnetic moment. According to Ref. [10] the magnetic moment of an electron subject to the massive two-dimensional Dirac Hamiltonian is

\[ \mu(p) = \frac{e}{2\hbar(m^2v_F^2 + p^2)} = \frac{e\Delta}{2(\Delta^2 + p^2)} . \]

That gives the magnetization density

\[ M = \mu \psi^\dagger \psi = \mu/v_x = \frac{e\Delta}{2p^3} \frac{v_F}{v_x} . \]

Consequently, we find, in agreement with the general expectation [10, 12].

\[ -\frac{\partial M}{\partial x} = \frac{-eU' \Delta}{2p^3} \frac{1}{p^2} \left( \frac{1}{p^2} + \frac{1}{p(\Delta^2 + p^2)} \right) = j_y - j_{tr} . \]

With that we have explicitly demonstrated that the spatially inhomogeneous magnetization is responsible for the difference between the total current and the transport current.