Bethe ansatz of the open spin-s XXZ chain with nondiagonal boundary terms

Rajan Murgan

Department of Physics, Gustavus Adolphus College
St. Peter, MN 56082 USA

Abstract

We consider the open spin-s XXZ quantum spin chain with nondiagonal boundary terms. By exploiting certain functional relations at roots of unity, we propose the Bethe ansatz solution for the transfer matrix eigenvalues for cases where atmost two of the boundary parameters are set to be arbitrary and the bulk anisotropy parameter has values $\eta = \frac{i\pi}{3}, \frac{2i\pi}{5}, \ldots$. We present numerical evidence to demonstrate completeness of the Bethe ansatz solutions derived for $s = 1/2$ and $s = 1$.

\footnote{e-mail: rmurgan@gustavus.edu}
1 Introduction

There have been significant focus of effort in solving integrable quantum spin chains for many years. In particular, integrable quantum spin chains with boundaries (integrable open quantum spin chains) have attracted much interest over the years. As a result, models such as the open XXX and XXZ quantum spin chains have been subjected to intensive studies due to their growing applications in various fields of physics, e.g. statistical mechanics, string theory and condensed matter physics. Despite numerous success in the past [1]-[6] (also refer to [7]-[14] and references therein, for other related work on the subject.), there still remain unsolved problems in this area. Bethe ansatz (in its conventional form) for the most general case of the open XXZ quantum spin chain (even for the spin-1/2 case) with arbitrary nondiagonal boundary terms and generic bulk anisotropy parameter is yet to be found. In [14], Galleas found an interesting solution analogous to Bethe ansatz equations for the spin-1/2 case. This solution, written in terms of certain functional relations are expressed in terms of roots of the transfer matrix. Much progress have been made on the topic up to this point. In a series of publication, Bethe ansatz solutions have been derived for open spin-1/2 XXZ quantum spin chain where the boundary parameters obey certain constraint. Readers are refered to [15]-[19] for related work on the subject. Apart from this constraint, two sets of Bethe ansatz equations are needed there to obtain all $2^N$ eigenvalues, where $N$ is the number of sites. A special case of the above solution was generalized to open XXZ quantum spin chain with alternating spins by Doikou [20] using the functional relation approach, proposed by Nepomechie in [16] to solve the spin-1/2 case (which indeed the method used in this paper). In [21], related work was carried out using the method in [15]. Recently in [22], Frappat et al. further generalized the spin-1/2 XXZ Bethe ansatz solution (for boundary parameters obeying the constraint) to the spin-s case by utilizing an approach based on $Q$-operator and $T$-$Q$ equation, which was developed earlier for the spin-1/2 XXZ chain in [18] and subsequently applied to the spin-1/2 XYZ chain in [23]. As in the spin-1/2 case, two sets of Bethe ansatz equations are also needed there to produce all $(2s + 1)^N$ eigenvalues, where again $N$ represents the number of sites.

In this paper, we present Bethe ansatz solutions for open spin-s XXZ quantum spin chain without such a constraint among the boundary parameters. We follow similar approach as given in [16, 17, 24] that was used to solve the $s = 1/2$ case. It is based on fusion [4, 25, 26], the truncation of the fusion hierarchy at roots of unity [27] and the Bazhanov-Reshetikhin [28] solution of the RSOS models. As in [24], there are almost two arbitrary boundary parameters. The rest of the parameters are fixed to some values. The approach we use, which is based on functional relations obeyed by transfer matrix at roots of unity [16] yields Bethe ansatz solution which gives completely all the $(2s + 1)^N$ eigenvalues. One limitation
of the solution is that it is valid only at roots of unity, namely when the bulk anisotropy parameter has values $\eta = \frac{i\pi}{p+1}$. In this paper, we consider only even values of $p$. Lack of single set of Bethe ansatz equations that yield complete eigenvalues for the model considered here, namely where the boundary parameters are arbitrary (even at most two) has motivated us to study this problem. Moreover, we note that the relation of $s = 1$ case to the supersymmetric sine-Gordon (SSG) model [29] (here the boundary version [30, 31]), has also been part of our motivation for considering the problem.

The outline of the paper is as follows: In Sec. 2, we review the construction of the so-called fused $R$ [25, 32, 33, 34] and $K^\mp$ [4, 26] matrices from the corresponding spin-1/2 matrices. For some original work on spin-1/2 $K^\mp$ matrices, refer to [35, 36]. Construction of commuting transfer matrices from these fused matrices (using Sklyanin’s work [3], which in turn relies on Cherednik’s previous results [38]), together with some of their properties are reviewed. Fusion hierarchy and functional relations obeyed by transfer matrices are also reviewed. In Sec. 3, we present the Bethe ansatz solutions for cases with at most two arbitrary boundary parameters at roots of unity, e.g. $\eta = \frac{i\pi}{3}, \frac{i\pi}{5}, \ldots$, by exploiting the reviewed functional relations obeyed by the transfer matrices. Further, we present numerical results in Sec. 4 to illustrate the completeness of our solution, using $s = 1/2$ and $s = 1$ as examples, where the Bethe roots and energy eigenvalues derived from the Bethe ansatz equations (for some values of $p$ and $N$) are given. We remark that these energy eigenvalues coincide with the ones obtained from direct diagonalization of the Hamiltonians. Finally, we conclude the paper with discussion of the results and potential future works in Sec. 5.

2 Transfer matrices, fusion hierarchy and functional relations at roots of unity

In this section, in order to make the paper relatively self-contained, we review some crucial concepts on the construction of commuting transfer matrices for $N$-site open spin-$s$ XXZ quantum spin chain. Materials reviewed here on fused $R$, $K^\mp$ and higher spin transfer matrices are borrowed from [22], as presented there. As constructed in [3], the commuting transfer matrix for $s = 1/2$, which we denote (following notations adopted in [22]) by $t^{(\frac{1}{2}, \frac{1}{2})}(u)$, whose auxiliary space as well as each of its $N$ quantum spaces are two-dimensional, one can similarly construct a transfer matrix $t^{(j, s)}(u)$ whose auxiliary space is spin-$j$ ($(2j + 1)$-dimensional) and each of its $N$ quantum spaces are spin-$s$ ($(2s + 1)$-dimensional), for any $j, s \in \{\frac{1}{2}, 1, \frac{3}{2}, \ldots\}$ using the so-called fused $R$ [25, 32, 33, 34] and $K^\mp$ [4, 26] matrices. As for the spin-1/2 case, these $R$ and $K^\mp$ matrices serve as building blocks in the construction of the commuting transfer matrices for higher spins. We list them below along with some of their properties.
The fused-$R$ matrices can be constructed as given below,

\[ R_{\{a\}\{b\}}^{(j,s)}(u) = P_{\{a\}}^+ P_{\{b\}}^+ \prod_{k=1}^{2j} \prod_{l=1}^{2s} R_{a_k b_l}^{(1 \frac{1}{2})}(u + (k + l - j - s - 1)\eta) P_{\{a\}}^+ P_{\{b\}}^+, \tag{2.1} \]

where \(\{a\} = \{a_1, \ldots, a_{2j}\}, \{b\} = \{b_1, \ldots, b_{2s}\}\), and \(P_{\{a\}}^+\) is the symmetric projector given by

\[ P_{\{a\}}^+ = \frac{1}{(2j)!} \prod_{k=1}^{2j} \left( \sum_{l=1}^{k} \mathcal{P}_{a_l, a_k} \right), \tag{2.2} \]

\(\mathcal{P}\) is the permutation operator, with \(\mathcal{P}_{a_k, a_k} \equiv 1\); Similar definition also holds for \(P_{\{b\}}^+\).

\(R^{(1 \frac{1}{2})}(u)\) is given by

\[ R^{(1 \frac{1}{2})}(u) = \begin{pmatrix} \text{sh}(u + \eta) & 0 & 0 & 0 \\ 0 & \text{sh} u & \text{sh} \eta & 0 \\ 0 & \text{sh} \eta & \text{sh} u & 0 \\ 0 & 0 & 0 & \text{sh}(u + \eta) \end{pmatrix}, \tag{2.3} \]

where \(\eta\) is the bulk anisotropy parameter. Note that the fundamental \(R\) matrix satisfies the following unitarity relation

\[ R^{(1 \frac{1}{2})}(u) R^{(1 \frac{1}{2})}(-u) = -\xi(u) 1, \quad \xi(u) = \text{sh}(u + \eta) \text{sh}(u - \eta). \tag{2.4} \]

The \(R\) matrices in the product (2.1) are ordered in the order of increasing \(k\) and \(l\). The fused \(R\) matrices satisfy the Yang-Baxter equations \[37\]

\[ R_{\{a\}\{b\}}^{(j,k)}(u - v) R_{\{a\}\{c\}}^{(j,s)}(u) R_{\{b\}\{c\}}^{(k,s)}(v) = R_{\{b\}\{c\}}^{(k,s)}(v) R_{\{a\}\{c\}}^{(j,s)}(u) R_{\{a\}\{b\}}^{(j,k)}(u - v). \tag{2.5} \]

Having defined fused-$R$ matrices, one can analogously construct fused $K^{-}$ matrices \[4, 26\]

\[ K_{\{a\}}^{- (j)}(u) = P_{\{a\}}^+ \prod_{k=1}^{2j} \left\{ \prod_{l=1}^{k-1} R_{a_l a_k}^{(1 \frac{1}{2})} (2u + (k + l - 2j - 1)\eta) \right\} \times K_{a_k}^{- (j)}(u + (k - j - \frac{1}{2})\eta) \right\} P_{\{a\}}^+, \tag{2.6} \]

where \(K_{a_k}^{- (j)}(u)\) is the \(2 \times 2\) matrix whose components are given by \[35, 36\]

\[ K_{11}^{-}(u) = 2 \left( \text{sh} \alpha_+ \text{ch} \beta_- \text{ch} u + \text{ch} \alpha_- \text{sh} \beta_- \text{sh} u \right) \]

\[ K_{22}^{-}(u) = 2 \left( \text{sh} \alpha_+ \text{ch} \beta_- \text{ch} u - \text{ch} \alpha_- \text{sh} \beta_- \text{sh} u \right) \]

\[ K_{12}^{-}(u) = e^{\theta_-} \text{sh} 2u, \quad K_{21}^{-}(u) = e^{-\theta_-} \text{sh} 2u, \tag{2.7} \]
where $\alpha_-, \beta_-, \theta_-$ are the boundary parameters. The products of braces $\{ \ldots \}$ in (2.6) are ordered in the order of increasing $k$. The fused $K^-$ matrices satisfy the boundary Yang-Baxter equations \[ R_{\{a\},\{b\}}^{(j,s)}(u-v) K_{\{a\}}^{-(j)}(u) R_{\{a\},\{b\}}^{(j,s)}(u+v) K_{\{a\}}^{-(j)}(v) = K_{\{b\}}^{-(j)}(v) R_{\{a\},\{b\}}^{(j,s)}(u+v) K_{\{a\}}^{-(j)}(u) R_{\{a\},\{b\}}^{(j,s)}(u-v). \tag{2.8} \]
The fused $K^+$ matrices are given by
\[ K_{\{a\}}^{+(j)}(u) = \frac{1}{f^{(j)}(u)} K_{\{a\}}^{-(j)}(-u-\eta) \big|_{(\alpha_-\beta_-\theta_-)\to(-\alpha_+\beta_+\theta_+)} ^{(\alpha_-\beta_-\theta_-)}, \tag{2.9} \]
where the normalization factor is,
\[ f^{(j)}(u) = \prod_{l=1}^{2j-1} \prod_{k=1}^{l} [-\xi(2u + (l + k + 1 - 2j)\eta)]. \tag{2.10} \]

Using the above results, one can construct the transfer matrix $t^{(j,s)}(u)$,
\[ t^{(j,s)}(u) = \text{tr}_{\{a\}} K_{\{a\}}^{+(j)}(u) T_{\{a\}}^{(j,s)}(u) K_{\{a\}}^{-(j)}(u) T_{\{a\}}^{+(j,s)}(u), \tag{2.11} \]
where the monodromy matrices are given by products of $N$ fused $R$ matrices,
\[ T_{\{a\}}^{(j,s)}(u) = R_{\{a\},\{b[N]\}}^{(j,s)}(u) \ldots R_{\{a\},\{b[1]\}}^{(j,s)}(u), \]
\[ T_{\{a\}}^{+(j,s)}(u) = R_{\{a\},\{b[1]\}}^{(j,s)}(u) \ldots R_{\{a\},\{b[N]\}}^{(j,s)}(u). \tag{2.12} \]

These transfer matrices commute for different values of spectral parameter for any $j, j' \in \{\frac{1}{2}, 1, \frac{3}{2}, \ldots\}$ and any $s \in \{\frac{1}{2}, 1, \frac{3}{2}, \ldots\}$,
\[ [t^{(j,s)}(u), t^{(j',s)}(u')] = 0. \tag{2.13} \]

Furthermore, they also obey the fusion hierarchy \[ t^{(j-\frac{1}{s}, s)}(u-j\eta) t^{(\frac{j-1}{s}, s)}(u) = t^{(j,s)}(u-(j-\frac{1}{2})\eta) + \delta^{(s)}(u-\eta) t^{(j-1,s)}(u-(j+\frac{1}{2})\eta), \tag{2.14} \]
j = 1, $\frac{3}{2}$, ..., where $t^{(0,s)} = 1$, and $\delta^{(s)}(u)$ is given by
\[ \delta^{(s)}(u) = \delta^{(s)}_{0}(u) \delta^{(s)}_{1}(u) \tag{2.15} \]
where
\[ \delta^{(s)}_{0}(u) = \prod_{k=0}^{2s-1} \xi(u + (s - k + \frac{1}{2})\eta) ^{2N} \frac{\text{sh}(2u) \text{sh}(2u + 4\eta)}{\text{sh}(2u + \eta) \text{sh}(2u + 3\eta)} \]
\[ \delta^{(s)}_{1}(u) = 2^{4} \text{sh}(u + \alpha_- + \eta) \text{sh}(u - \alpha_- + \eta) \text{ch}(u + \beta_- + \eta) \text{ch}(u - \beta_- + \eta) \times \text{sh}(u + \alpha_+ + \eta) \text{sh}(u - \alpha_+ + \eta) \text{ch}(u + \beta_+ + \eta) \text{ch}(u - \beta_+ + \eta). \tag{2.16} \]

\[ ^{1}\text{See the appendix in [22] for more details on the fusion hierarchy.} \]
Note that the $\delta^{(s)}(u)$ in [22] differs to the one given here merely by a shift in $\eta$.

Next, we list few important properties of the rescaled “fundamental” transfer matrix $\tilde{t}^{(\frac{1}{2}, s)}(u)$ (defined below), which are useful in determining its eigenvalues. Following the definition of $\tilde{t}^{(\frac{1}{2}, s)}(u)$ as in [22], we have

$$\tilde{t}^{(\frac{1}{2}, s)}(u) = \frac{1}{g^{(\frac{1}{2}, s)}(u)2^{N}t^{(\frac{1}{2}, s)}(u)}, \quad (2.17)$$

where

$$g^{(\frac{1}{2}, s)}(u) = \prod_{k=1}^{2s-1} \text{sh}(u + (s - k + \frac{1}{2})\eta) \quad (2.18)$$

This transfer matrix has following useful properties:

$$\tilde{t}^{(\frac{1}{2}, s)}(u + i\pi) = \tilde{t}^{(\frac{1}{2}, s)}(u) \quad (i\pi - periodicity) \quad (2.19)$$

$$\tilde{t}^{(\frac{1}{2}, s)}(-u - \eta) = \tilde{t}^{(\frac{1}{2}, s)}(u) \quad (crossing) \quad (2.20)$$

$$\tilde{t}^{(\frac{1}{2}, s)}(0) = -2^3 \text{sh}^{2N}((s + \frac{1}{2})\eta) \text{ch} \eta \text{sh} \alpha_- \text{ch} \beta_- \text{sh} \alpha_+ \text{ch} \beta_+ \mathbb{I} \quad (initial \ condition) \quad (2.21)$$

$$\tilde{t}^{(\frac{1}{2}, s)}(u) \bigg|_{\eta=0} = 2^3 \text{sh}^{2N} u \left[ - \text{sh} \alpha_- \text{ch} \beta_- \text{sh} \alpha_+ \text{ch} \beta_+ \text{ch}^2 u + \text{ch} \alpha_- \text{sh} \beta_- \text{ch} \alpha_+ \text{sh} \beta_+ \text{sh}^2 u - \text{ch}(\theta_- - \theta_+) \text{sh}^2 u \text{ch}^2 u \right] \mathbb{I} \quad (semi-classical \ limit) \quad (2.22)$$

$$\tilde{t}^{(\frac{1}{2}, s)}(u) \sim -\frac{1}{2^{2N+1}} e^{(2N+4)u+(N+2)\eta} \text{ch}(\theta_- - \theta_+) \mathbb{I} \quad \text{for} \ u \to +\infty \quad (asymptotic \ behavior) \quad (2.23)$$

where $\mathbb{I}$ is the identity matrix.

Due to the commutativity property (2.13), the corresponding simultaneous eigenvectors are independent of the spectral parameter. Hence, (2.19) - (2.23) hold for the corresponding eigenvalues as well. In addition to the above mentioned properties, for bulk anisotropy values $\eta = \frac{\pi p+1}{p+1}$, with $p = 1, 2, \ldots$, the “fundamental” transfer matrix, $t^{(\frac{1}{2}, s)}(u)$ (and hence each of the corresponding eigenvalues, $\Lambda^{(\frac{1}{2}, s)}(u)$) obeys functional relations of order $p + 1$ [16]

$$t^{(\frac{1}{2}, s)}(u)t^{(\frac{1}{2}, s)}(u + \eta) \ldots t^{(\frac{1}{2}, s)}(u + p\eta)$$
\[ - \delta^{(s)}(u - \eta)t^{(1/2, s)}(u + \eta)t^{(1/2, s)}(u + 2\eta) \ldots t^{(1/2, s)}(u + (p - 1)\eta) \\
- \delta^{(s)}(u)t^{(1/2, s)}(u + 2\eta)t^{(1/2, s)}(u + 3\eta) \ldots t^{(1/2, s)}(u + p\eta) \\
- \delta^{(s)}(u + \eta)t^{(1/2, s)}(u)t^{(1/2, s)}(u + 3\eta)t^{(1/2, s)}(u + 4\eta) \ldots t^{(1/2, s)}(u + p\eta) \\
- \delta^{(s)}(u + 2\eta)t^{(1/2, s)}(u + \eta)t^{(1/2, s)}(u + 4\eta) \ldots t^{(1/2, s)}(u + p\eta) - \ldots \\
- \delta^{(s)}(u + (p - 1)\eta)t^{(1/2, s)}(u + \eta)\ldots t^{(1/2, s)}(u + (p - 2)\eta) \\
+ \ldots = f(u). \quad (2.24) \]

For example, for \( p = 2 \) and \( p = 4 \), the functional relations are
\[ \begin{split} 
    t^{(1/2, s)}(u)t^{(1/2, s)}(u + \eta)t^{(1/2, s)}(u + 2\eta) &- \delta^{(s)}(u - \eta)t^{(1/2, s)}(u + \eta) - \delta^{(s)}(u)t^{(1/2, s)}(u + 2\eta) \\
    -\delta^{(s)}(u + \eta)t^{(1/2, s)}(u) = f(u). 
\end{split} \quad (2.25) \]

and
\[ \begin{split} 
    t^{(1/2, s)}(u)t^{(1/2, s)}(u + \eta)t^{(1/2, s)}(u + 2\eta)t^{(1/2, s)}(u + 3\eta)t^{(1/2, s)}(u + 4\eta) \\
    +\delta^{(s)}(u + \eta)\delta^{(s)}(u - 2\eta)t^{(1/2, s)}(u) + \delta^{(s)}(u)\delta^{(s)}(u + 2\eta)t^{(1/2, s)}(u + 4\eta) \\
    +\delta^{(s)}(u + \eta)\delta^{(s)}(u + 2\eta)t^{(1/2, s)}(u + 4\eta) + \delta^{(s)}(u + 3\eta)t^{(1/2, s)}(u + 4\eta) \\
    +\delta^{(s)}(u + \eta)t^{(1/2, s)}(u + 2\eta)t^{(1/2, s)}(u + 3\eta)t^{(1/2, s)}(u + 4\eta) \\
    +\delta^{(s)}(u + 2\eta)t^{(1/2, s)}(u + \eta) - \delta^{(s)}(u + 2\eta)t^{(1/2, s)}(u + \eta)t^{(1/2, s)}(u + 3\eta)t^{(1/2, s)}(u + 4\eta) \\
    -\delta^{(s)}(u - 2\eta)t^{(1/2, s)}(u)t^{(1/2, s)}(u + 3\eta)t^{(1/2, s)}(u + 2\eta) \\
    -\delta^{(s)}(u - \eta)t^{(1/2, s)}(u + \eta)t^{(1/2, s)}(u + 2\eta)t^{(1/2, s)}(u + 3\eta) = f(u). 
\end{split} \quad (2.26) \]

respectively. The scalar function \( f(u) \) (which can be expressed as \( f(u) = f_0(u)f_1(u) \)) is given in terms of the boundary parameters \( \alpha_\pm, \beta_\pm, \theta_\pm \) (for even \( p \)) by
\[ \begin{aligned} 
    f_0(u) &= \begin{cases} 
    (-1)^{N+1}2^{-4pN}1^{s}N 4s^N N ((p + 1)u), & \\
    \frac{1}{2} = \frac{3}{2}, \frac{5}{2}, \ldots \\
    (-1)^{N+1}2^{-4pN}1^{s}N 4s^N N ((p + 1)u), & \\
    s = 1, 2, 3, \ldots
    \end{cases} 
\end{aligned} \quad (2.27) \]

and
\[ \begin{aligned} 
    f_1(u) &= (-1)^N2^{-3-2p} \left( \\
    \begin{array}{c} 
    \sin ((p + 1)\alpha_-) \sin ((p + 1)\beta_-) \sin ((p + 1)\alpha_+) \sin ((p + 1)\beta_+) \sin^2 ((p + 1)u) \\
    - \cos ((p + 1)\alpha_-) \cos ((p + 1)\beta_-) \sin ((p + 1)\alpha_+) \sin ((p + 1)\beta_+) \sin^2 ((p + 1)u) \\
    - (-1)^N \sin ((p + 1)(\theta_- - \theta_+)) \sin^2 ((p + 1)u) \sin^2 ((p + 1)u) \\
    \end{array} \right). \end{aligned} \quad (2.28) \]
for $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ and
\[
\begin{align*}
f_1(u) &= (-1)^{N+1}2^{3-2p} \left( \sh ((p+1)\alpha_-) \ch ((p+1)\beta_-) \sh ((p+1)\alpha_+) \ch ((p+1)\beta_+) \ch^2 ((p+1)u) \\
&\quad - \ch ((p+1)\alpha_-) \sh ((p+1)\beta_-) \ch ((p+1)\alpha_+) \sh ((p+1)\beta_+) \sh^2 ((p+1)u) \\
&\quad - \ch ((p+1)(\theta_- - \theta_+)) \sh^2 ((p+1)u) \ch^2 ((p+1)u) \right). 
\end{align*}
\]
(2.29)

for $s = 1, 2, 3, \ldots$, Note that $f(u)$ satisfies
\[
f(u + \eta) = f(u), \quad f(-u) = f(u).
\]
(2.30)

and
\[
f_0(u)^2 = \prod_{j=0}^{p} \delta_0^{(s)}(u + j\eta).
\]
(2.31)

where $\delta_0^{(s)}(u)$ is given by (2.16).

3 Bethe ansatz

In this section, we give main results of this paper. We derive Bethe ansatz equations for various cases where atmost two of the boundary parameters $\{\alpha_-, \alpha_+, \beta_-, \beta_+\}$ are arbitrary by adopting the steps given in [24]. By considering atmost two boundary parameters, we find certain factors in the calculation become perfect squares. This facilitate the computations that follow. More on this is explained below.

3.1 $\alpha_+,$ $\alpha_-$ arbitrary

Here, we take both $\alpha_-$ and $\alpha_+$ to be arbitrary while setting $\beta_\pm = \eta, \theta_- = \theta_+ = \theta,$ where $\theta$ is arbitrary. In order to obtain Bethe ansatz equations for the transfer matrix eigenvalues $\Lambda^{(\frac{1}{2}, s)}(u)$, we shall recast the functional relations (2.24) as the condition that the determinant of a certain matrix vanishes (following [28]). We find that the functional relations (2.24) for the transfer matrix eigenvalues can be written as
\[
\det \mathcal{M} = 0,
\]
(3.1)
where \( \mathcal{M} \) is given by the \((p + 1) \times (p + 1)\) matrix

\[
\mathcal{M} = \begin{pmatrix}
\Lambda^{(\frac{1}{2}, s)}(u) & -h(u) & 0 & \ldots & 0 & -h(-u + p\eta) \\
-h(-u) & \Lambda^{(\frac{1}{2}, s)}(u + p\eta) & -h(u + p\eta) & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-h(u + p^2\eta) & 0 & 0 & \ldots & -h(-u - p(p - 1)\eta) & \Lambda^{(\frac{1}{2}, s)}(u + p^2\eta)
\end{pmatrix}
\] (3.2)

(whose successive rows are obtained by simultaneously shifting \( u \to u + p\eta \) and cyclically permuting the columns to the right) provided that there exists a function \( h(u) \) with the following properties

\[
h(u + 2i\pi) = h(u + 2(p + 1)\eta) = h(u), \quad (3.3)
\]

\[
h(u + (p + 2)\eta) \ h(-u - (p + 2)\eta) = \delta^{(s)}(u), \quad (3.4)
\]

\[
\prod_{j=0}^{p} h(u + 2j\eta) + \prod_{j=0}^{p} h(-u - 2j\eta) = f(u). \quad (3.5)
\]

From (3.3)-(3.5), we see that the problem of finding \( h(u) \) then reduces to solving the following quadratic equation in \( z(u) \),

\[
z(u)^2 - z(u)f(u) + \prod_{j=0}^{p} \delta^{(s)}(u + (2j - 1)\eta) = 0, \quad (3.6)
\]

where

\[
z(u) = \prod_{j=0}^{p} h(u + 2j\eta). \quad (3.7)
\]

For the cases considered here and in subsequent sections, the discriminants of the corresponding quadratic equations are perfect squares, and the factorizations such as (3.7) can be readily carried out. However, when all boundary parameters are arbitrary, the discriminant is no longer a perfect square; and factoring the result becomes a formidable challenge.

Solving the quadratic equation (3.6) for \( z(u) \), making use of the explicit expressions (2.16) and (2.27)-(2.29) for \( \delta^{(s)}(u) \) and \( f(u) \), respectively, we obtain the following for \( h(u) \),

\[
h(u) = h_0(u)h_1(u), \quad (3.8)
\]

with

\[
h_0(u) = (-1)^{2sN}4 \left[ \prod_{k=0}^{2s-1} \text{sh}(u + (s - k + \frac{1}{2})\eta) \right]^{2N} \frac{\text{sh}(2u + 2\eta)}{\text{sh}(2u + \eta)} \quad (3.9)
\]
and

\[ h_1(u) = \begin{cases} \cosh^2(u-\eta) \sinh(u-\alpha_-) \sinh(u + (1)^N \alpha_+) \frac{\cosh\left(\frac{1}{2}(u+\alpha_-+\eta)\right)}{\cosh\left(\frac{1}{2}(u-\alpha_-+\eta)\right)} \frac{\cosh\left(\frac{1}{2}(u+(1)^N \alpha_-+\eta)\right)}{\cosh\left(\frac{1}{2}(u+\alpha_-+\eta)\right)}, \\
\quad s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \\
\cosh^2(u-\eta) \sinh(u-\alpha_-) \sinh(u + \alpha_+) \frac{\cosh\left(\frac{1}{2}(u+\alpha_-+\eta)\right)}{\cosh\left(\frac{1}{2}(u-\alpha_-+\eta)\right)} \frac{\cosh\left(\frac{1}{2}(u+(1)^N \alpha_-+\eta)\right)}{\cosh\left(\frac{1}{2}(u+\alpha_-+\eta)\right)}, \\
\quad s = 1, 2, 3, \ldots \end{cases} \]

(3.10)

Further, the structure of the matrix \( M \) (3.2) suggests that its null eigenvector has the form \( (Q(u), Q(u+p\eta), \ldots, Q(u+p^2\eta)) \), where \( Q(u) \) has the periodicity property

\[ Q(u + 2i\pi) = Q(u). \]

(3.11)

It suggests that the transfer matrix eigenvalues are given by

\[ \Lambda^{\frac{1}{2},s}(u) = h(u) \frac{Q(u+p\eta)}{Q(u)} + \tilde{h}(-u + p\eta) \frac{Q(u-p\eta)}{Q(u)}, \]

(3.12)

which is of the Baxter’s \( TQ \) relation form. Noting that the functions \( h(u) \) and \( \tilde{h}(-u+p\eta) \) (see (3.8)-(3.10)) have the factor \( g^{\frac{1}{2},s}(u) \) in common (since \( g^{\frac{1}{2},s}(u) = g^{\frac{1}{2},s}(-u+p\eta) \)), we can rewrite (3.12) in terms of the eigenvalues of \( \tilde{t}^{\frac{1}{2},s}(u) \) (see (2.17)) as

\[ \tilde{\Lambda}^{\frac{1}{2},s}(u) = \tilde{h}(u) \frac{Q(u+p\eta)}{Q(u)} + \tilde{h}(-u + p\eta) \frac{Q(u-p\eta)}{Q(u)}, \]

(3.13)

where

\[ \tilde{h}(u) = \tilde{h}_0(u) h_1(u) \]

(3.14)

with

\[ \tilde{h}_0(u) = (-1)^{2sN} 4 \sinh(2^N(u + (s + \frac{1}{2})\eta)) \frac{\sinh(2u+2\eta)}{\sinh(2u+\eta)} \]

(3.15)

and

\[ Q(u) = \prod_{j=1}^{M} \sinh\left(\frac{1}{2}(u-u_j)\right) \sinh\left(\frac{1}{2}(u+u_j-p\eta)\right), \]

(3.16)

with the periodicity (3.11) as well as the crossing property

\[ Q(-u + p\eta) = Q(u). \]

(3.17)

where

\[ M = 2sN + 2p + 1, \]

(3.18)
which is confirmed numerically for small values of $N$ and $p$. We stress here that the $h(u)$ given above is not the only solution. It is obtained largely by trial and error, verifying numerically for small values of $N$ that the eigenvalues can indeed be expressed as (3.13) with $Q(u)$'s of the form given by (3.16). We also remark that (3.18) is consistent with the asymptotic behavior (2.23). Making use of the analyticity of $\tilde{\Lambda}(^{1/2}s)(u)$, we have the following for the Bethe ansatz equations,

$$\frac{\tilde{h}(u_j)}{h(-u_j + p\eta)} = -\frac{Q(u_j - p\eta)}{Q(u_j + p\eta)}, \quad j = 1, \ldots, M. \tag{3.19}$$

### 3.2 $\beta_+, \beta_-$ arbitrary

In the following, we set $\beta_-$ and $\beta_+$ arbitrary while setting $\alpha_+ = \eta$, $\theta_+ = \theta_+ = \theta$. As before, we write the functional relations (2.24) for the transfer matrix eigenvalues in the form of (3.1), where for this case, the matrix $M$ is given by

$$M = \begin{pmatrix}
\Lambda(^{1/2}s)(u) & -h(u) & 0 & \ldots & 0 & -h(-u - \eta) \\
-h(-u - (p + 1)\eta) & \Lambda(^{1/2}s)(u + p\eta) & -h(u + p\eta) & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-h(u + p^2\eta) & 0 & 0 & \ldots & -h(-u - (p^2 + 1)\eta) & \Lambda(^{1/2}s)(u + p^2\eta)
\end{pmatrix},$$

(3.20)

if $h(u)$ satisfies

$$h(u + 2i\pi) = h(u + 2(p + 1)\eta) = h(u), \tag{3.21}$$

$$h(u + (p + 2)\eta) h(-u - \eta) = \delta(s)(u), \tag{3.22}$$

$$\prod_{j=0}^p h(u + 2j\eta) + \prod_{j=0}^p h(-u - (2j + 1)\eta) = f(u). \tag{3.23}$$

Proceeding in a similar way to the previous case and setting $h(u) = h_0(u)h_1(u)$ we find

$$h_0(u) = (-1)^{2sN}4 \left[ \prod_{k=0}^{2s-1} \frac{\text{sh}(u + (s - k + \frac{1}{2})\eta)}{\text{sh}(2u + \eta)} \right]^{2N} \frac{\text{sh}(2u + 2\eta)}{\text{sh}(2u + \eta)} \tag{3.24}$$

and

$$h_1(u) = \begin{cases}
\text{sh}(u - \eta)\text{sh}(u + \eta)(\text{ch} u - i \text{sh} \beta_-)(\text{ch} u + (-1)^N i \text{sh} \beta_+), \\
s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \\
\text{sh}(u - \eta)\text{sh}(u + \eta)(\text{ch} u + i \text{sh} \beta_-)(\text{ch} u - i \text{sh} \beta_+) \}
\end{cases}, \tag{3.25}$$
The transfer matrix eigenvalues are now given by

\[ \Lambda^{(\frac{1}{2},s)}(u) = h(u) \frac{Q(u + pm)}{Q(u)} + h(-u - \eta) \frac{Q(u - pm)}{Q(u)}, \]  

(3.26)

As before, due to the common factor \( g^{(\frac{1}{2},s)}(u)^{2N} \) (see 2.18), and using the crossing symmetry \( g^{(\frac{1}{2},s)}(u) = \pm g^{(\frac{1}{2},s)}(-u - \eta) \), we conclude that the eigenvalues of \( \tilde{\Lambda}^{(\frac{1}{2},s)}(u) \) are given by

\[ \tilde{\Lambda}^{(\frac{1}{2},s)}(u) = \tilde{h}(u) \frac{Q(u + pm)}{Q(u)} + \tilde{h}(-u - \eta) \frac{Q(u - pm)}{Q(u)}, \]  

(3.27)

where

\[ \tilde{h}(u) = \tilde{h}_0(u) h_1(u) \]  

(3.28)

and

\[ \tilde{h}_0(u) = (-1)^{2sN} 4 \sh^{2N}(u + (s + \frac{1}{2})\eta) \frac{\sh(2u + 2\eta)}{\sh(2u + \eta)} \]  

(3.29)

The ansatz for \( Q(u) \) is given by

\[ Q(u) = \prod_{j=1}^{M} \sh \left( \frac{1}{2}(u - u_j) \right) \sh \left( \frac{1}{2}(u + u_j + \eta) \right), \]  

(3.30)

which satisfies \( Q(u + 2i\pi) = Q(u) \) and \( Q(-u - \eta) = Q(u) \); and

\[ M = 2sN + p. \]  

(3.31)

Moreover, the Bethe ansatz equations for the zeros \( u_j \) take the form

\[ \frac{\tilde{h}(u_j)}{h(-u_j - \eta)} = -\frac{Q(u_j - pm)}{Q(u_j + pm)}, \quad j = 1, \ldots, M. \]  

(3.32)

where we find the number of Bethe roots (3.31) is consistent with the asymptotic behaviour (2.23).

### 3.3 One arbitrary \( \beta \) and one arbitrary \( \alpha \)

Finally, we consider combinations where the arbitrary parameters consist of one of the \( \beta \)'s and any one of the \( \alpha \)’s. To keep the expressions general, we drop the subscripts \( \pm \) from the boundary parameters, \( \alpha_{\pm}, \beta_{\pm} \). The remaining boundary parameters are fixed, e.g., \( \beta_+, \alpha_- \) arbitrary, \( \beta_- = \eta, \alpha_+ = \frac{i\pi}{2} \) or other similar combinations. Also, as in previous cases, we let \( \theta_- = \theta_+ = \theta \). The matrix \( \mathcal{M} \) is identical in form as in (3.2).
We once again find \( h(u) = h_0(u)h_1(u) \), with the same \( h_0(u) \) as for the earlier cases. For \( h_1(u) \), we take the following,

\[
h_1(u) = \begin{cases} 
\text{ch} u \text{ch}(u - \eta)(\text{sh} u + (-1)^N \text{i ch} \beta) \text{sh}(u - \alpha) \frac{\text{ch}(\frac{1}{2}(u+\alpha+\eta))}{\text{ch}(\frac{1}{2}(u-\alpha-\eta))}, & s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \\
\text{ch} u \text{ch}(u - \eta)(\text{sh} u + \text{i ch} \beta) \text{sh}(u - \alpha) \frac{\text{ch}(\frac{1}{2}(u+\alpha+\eta))}{\text{ch}(\frac{1}{2}(u-\alpha-\eta))}, & s = 1, 2, 3, \ldots
\end{cases}
\] (3.33)

The above \( h(u) \) satisfies (3.3)-(3.5). The eigenvalues of the transfer matrix and Bethe ansatz equations are given by (3.12), (3.13), (3.16) and (3.19), with

\[
M = 2Ns + p
\] (3.34)

which again is consistent with (2.23). We note that for \( s = 1/2 \), our solutions for all the above cases coincide with the corresponding solutions found in [24].

4 Energy eigenvalues and Bethe roots

In this section, we illustrate the completeness of the Bethe ansatz solutions derived in Sec. 3. We provide numerical evidence for cases \( s = 1/2 \) and \( s = 1 \), namely the complete energy levels together with the Bethe roots used in the computation (see Tables 1 and 2).

4.1 \( s = 1/2 \) case

The Hamiltonian for the open spin-1/2 XXZ quantum spin chain is given by [35, 36]

\[
\mathcal{H} = \frac{1}{2} \sum_{n=1}^{N-1} (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \text{ch} \eta \sigma_n^z \sigma_{n+1}^z) \\
+ \frac{1}{2} \text{sh} \eta \left[ \text{coth} \alpha_- \text{tanh} \beta_- \sigma_n^z + \text{csch} \alpha_- \text{sech} \beta_- (\text{ch} \theta_- \sigma_1^x + \text{i sh} \theta_- \sigma_1^y) \\
- \text{coth} \alpha_+ \text{tanh} \beta_+ \sigma_N^z + \text{csch} \alpha_+ \text{sech} \beta_+ (\text{ch} \theta_+ \sigma_N^x + \text{i sh} \theta_+ \sigma_N^y) \right],
\] (4.1)

where \( \sigma^x, \sigma^y, \sigma^z \) are the standard Pauli matrices, \( \eta \) is the bulk anisotropy parameter, \( \alpha_\pm, \beta_\pm, \theta_\pm \) are arbitrary boundary parameters, and \( N \) is the number of spins.

We compute the energy eigenvalues of (4.1) (from Bethe ansatz) for a particular case derived in Sec. 3. For the purpose of illustration, it is sufficient to consider the case where
the two arbitrary boundary parameters are $\alpha_-$ and $\beta_-$. The steps here can be repeated for any other desired combinations of boundary parameters. The Hamiltonian (4.1) is related to the first derivative of the transfer matrix, $\mathcal{H} = c_1^{(\frac{\eta}{2})} \frac{d}{du} \tilde{t}^{(\frac{\eta}{2}), \frac{\eta}{2}}(u) \bigg|_{u=0} + c_2^{(\frac{\eta}{2})} \mathbb{I}$, (4.2)
where
\[
c_1^{(\frac{\eta}{2})} = \frac{1}{16 \sh \alpha_- \ch \beta_- \sh \alpha_+ \ch \beta_+ \sh^{2N-1} \eta \ch \eta},
\]
\[
c_2^{(\frac{\eta}{2})} = -\frac{\sh^2 \eta + N \ch^2 \eta}{2 \ch \eta},
\]
and $\mathbb{I}$ is the identity matrix. Moreover, (4.2) implies that the energy eigenvalues are given by
\[
E = c_1^{(\frac{\eta}{2})} \frac{d}{du} \tilde{\Lambda}^{(\frac{\eta}{2}), \frac{\eta}{2}}(u) \bigg|_{u=0} + c_2^{(\frac{\eta}{2})},
\]
(4.4)
Hence, using the results (3.13)-(3.16) and (3.33) one arrives at the following result for the energy eigenvalues in terms of Bethe roots \{u_j\},
\[
E = \frac{1}{2} \sh \eta \ch \frac{\eta}{2} \sum_{j=1}^{M} \frac{1}{\sh(\frac{\eta}{2} u_j) \ch(\frac{\eta}{2} (u_j + \eta))} + \frac{1}{2} N \ch \eta - \frac{1}{2} \ch \frac{2 \eta}{\ch \eta}
\]
\[
- \frac{1}{2} \sh(\coth \alpha_- + i \sech \beta_- - \tanh(\frac{\alpha_- + \eta}{2})).
\]
(4.5)
where $M = N + p$ (see (3.34)).

In Table 1, we tabulate the energy eigenvalues computed using (4.5) for $N = 4$ together with the Bethe roots (These roots are obtained using a method developed by McCoy and his collaborators [39] which is also explained in [17]). This numerical result illustrates the completeness of Bethe ansatz equations derived in Sec. 3. We have verified that the energies given in Table 1 coincide with those obtained from direct diagonalization of (4.1).

4.2 $s = 1$ case

In this section, we repeat the analysis for $s = 1$. We shall consider the case investigated in Sec. 3.2, namely the case with arbitrary $\beta_-, \beta_+$. The integrable Hamiltonian for the open spin-1 XXZ quantum spin chain is given by (adopting notations used in [22])
\[
\mathcal{H} = \sum_{n=1}^{N-1} H_{n,n+1} + H_b.
\]
(4.6)

\[\text{Note that for } s = 1/2, \tilde{t}^{(\frac{\eta}{2}), \frac{\eta}{2}}(u) = \tilde{t}^{(\frac{\eta}{2}), \frac{\eta}{2}}(u)\]

\[\text{The function } \tilde{h}(u) \text{ used here coincides with the one found in [24].} \]
$H_{n,n+1}$ represents the bulk terms. Explicitly, these terms are given by [40],

\[
H_{n,n+1} = \sigma_n - (\sigma_n)^2 + 2\text{sh}^2 \eta \left[\sigma_n^z + (S_n^z)^2 + (S_{n+1}^z)^2 - (\sigma_n^z)^2\right] \\
\quad - 4\text{sh}^2\left(\frac{\eta}{2}\right) \left(\sigma_n^x \sigma_n^y + \sigma_n^y \sigma_n^x\right),
\]

(4.7)

where

\[
\sigma_n = \vec{S}_n \cdot \vec{S}_{n+1}, \quad \sigma_n^x = S_n^x S_{n+1}^x + S_n^y S_{n+1}^y, \quad \sigma_n^z = S_n^z S_{n+1}^z,
\]

(4.8)

and $\vec{S}$ are the $su(2)$ spin-1 generators. $H_b$ represents the boundary terms which have the following form (see e.g., [22, 41])

\[
H_b = a_1(S_1^z)^2 + a_2S_1^z + a_3(S_1^z)^2 + a_4(S_1^z)^2 + a_5S_1^z S_1^z + a_6S_1^z S_1^- \\
\quad + a_7S_1^z S_1^+ + a_8S_1^- S_1^z + (a_j \leftrightarrow b_j \text{ and } 1 \leftrightarrow N),
\]

(4.9)

where $S^z = S^x + iS^y$. The coefficients \(\{a_i\}\) of the boundary terms at site 1 are functions of the boundary parameters ($\alpha_-, \beta_-, \theta_- \text{ and } \eta$). They are given by,

\[
\begin{align*}
    a_1 &= \frac{1}{4}a_0 \left(\text{ch} 2\alpha_- - \text{ch} 2\beta_- + \text{ch} \eta \right) \text{sh} 2\eta \text{sh} \eta, \\
    a_2 &= \frac{1}{4}a_0 \text{sh} 2\alpha_- \text{sh} 2\beta_- \text{sh} 2\eta, \\
    a_3 &= -\frac{1}{8}a_0 e^{2\theta_-} \text{sh} 2\eta \text{sh} \eta, \\
    a_4 &= -\frac{1}{8}a_0 e^{-2\theta_-} \text{sh} 2\eta \text{sh} \eta, \\
    a_5 &= a_0 e^{\theta_-} \left(\text{ch} \beta_- \text{sh} \alpha_- \text{ch} \frac{\eta}{2} + \text{ch} \alpha_- \text{sh} \beta_- \text{sh} \frac{\eta}{2}\right) \text{sh} \eta \text{ch}^2 \eta, \\
    a_6 &= a_0 e^{-\theta_-} \left(\text{ch} \beta_- \text{sh} \alpha_- \text{ch} \frac{\eta}{2} + \text{ch} \alpha_- \text{sh} \beta_- \text{sh} \frac{\eta}{2}\right) \text{sh} \eta \text{ch}^2 \eta, \\
    a_7 &= -a_0 e^{\theta_-} \left(\text{ch} \beta_- \text{sh} \alpha_- \text{ch} \frac{\eta}{2} - \text{ch} \alpha_- \text{sh} \beta_- \text{sh} \frac{\eta}{2}\right) \text{sh} \eta \text{ch}^3 \eta, \\
    a_8 &= -a_0 e^{-\theta_-} \left(\text{ch} \beta_- \text{sh} \alpha_- \text{ch} \frac{\eta}{2} - \text{ch} \alpha_- \text{sh} \beta_- \text{sh} \frac{\eta}{2}\right) \text{sh} \eta \text{ch}^3 \eta,
\end{align*}
\]

(4.10)

where

\[
a_0 = \left[\text{sh}(\alpha_- - \frac{\eta}{2}) \text{sh}(\alpha_- + \frac{\eta}{2}) \text{ch}(\beta_- - \frac{\eta}{2}) \text{ch}(\beta_- + \frac{\eta}{2})\right]^{-1}.
\]

(4.11)

Similarly, the coefficients \(\{b_i\}\) of the boundary terms at site $N$ which are functions of the boundary parameters ($\alpha_+, \beta_+, \theta_+ \text{ and } \eta$), are given by the following correspondence,

\[
b_i = a_i \bigg|_{\alpha_- \to \alpha_+, \beta_- \to -\beta_+, \theta_- \to -\theta_+}.
\]

(4.12)
To derive the energy formula similar to (4.5) for $s = 1$ case, we once again begin by expressing the spin-1 Hamiltonian in terms of the first derivative of spin-1 transfer matrix, namely $t^{(1,1)}(u)$. One can construct $t^{(1,1)}(u)$ from $t^{(1,1)}(u)$ by using the fusion hierarchy formula (2.13),

$$t^{(1,1)}(u) = t^{(\frac{1}{2},1)}(u - \frac{\eta}{2}) t^{(\frac{1}{2},1)}(u + \frac{\eta}{2}) - \delta^{(1)}(u - \frac{\eta}{2})$$

(4.13)

where $\delta^{(1)}(u)$ is given by (2.15)-(2.16) with $s = 1$. Following [22], we work with rescaled transfer matrix given by

$$\tilde{t}^{(1,1)}(u) = \frac{\text{sh}(2u) \text{sh}(2u + 2\eta)}{[\text{sh} u \text{sh}(u + \eta)]^{2N}} t^{(1,1)}(u),$$

(4.14)

where $t^{(1,1)}(u)$ is the transfer matrix constructed from “gauge”-transformed $R^{(1,1)}(u)$ and $K^{±(1)}(u)$ matrices. Note: The rescaled transfer matrix does not vanish at $u = 0$. The Hamiltonian $\mathcal{H}$ (4.16), according to [3], is related to the first derivative of $\tilde{t}^{(1,1)}(u)$,

$$\mathcal{H} = c_1^{(1)} \frac{d}{du} \tilde{t}^{(1,1)}(u) \bigg|_{u=0} + c_2^{(1)},$$

(4.15)

which in turn implies that the energy eigenvalues in terms of transfer matrix eigenvalues $\tilde{\Lambda}^{(1,1)}(u)$, are given by

$$E = c_1^{(1)} \frac{d}{du} \tilde{\Lambda}^{(1,1)}(u) \bigg|_{u=0} + c_2^{(1)},$$

(4.16)

where

$$c_1^{(1)} = \text{ch} \eta \left\{ 16 \text{sh} 2\eta \text{sh} \eta \text{sh} \eta \frac{2N}{3} \text{sh} \alpha_- \frac{\eta}{2} \text{sh} \alpha_+ \frac{\eta}{2} \text{ch} \alpha_- \frac{\eta}{2} \text{ch} \alpha_+ \frac{\eta}{2} \right\}^{-1}.$$

(4.17)

and

$$c_2^{(1)} = \frac{-a_0}{4} b \text{ch} \eta - (N - 1)(4 + \text{ch} 2\eta) + 2N \text{ch}^2 \eta$$

$$- \frac{\text{sh} \eta}{2d} \left\{ -2 \text{ch} 2\alpha_+ \left( \text{ch} \eta(3 + 7 \text{ch} 2\eta + \text{ch} 4\eta) + \text{ch} 2\beta_+(4 + 5 \text{ch} 2\eta + 2 \text{ch} 4\eta) \right) \right. \right.$$

$$+ \left. \left. 2 \text{ch} \eta \left( \text{ch} 2\beta_+(3 + 7 \text{ch} 2\eta + \text{ch} 4\eta) + \text{ch} \eta(5 + 3 \text{ch} 2\eta + 3 \text{ch} 4\eta) \right) \right\} \right.$$

$$- \frac{\text{sh} 2\eta}{2d} \left\{ \text{ch} 2\beta_+(2 + 4 \text{ch} \eta \text{ch} 3\eta) + \text{ch} \eta(5 \text{ch} 2\eta + \text{ch} 4\eta) - 2 \text{ch} 2\alpha_+ \left( 1 + \text{ch} 2\eta \right. \right.$$

$$+ \left. \text{ch} 2\beta_+(\text{ch} \eta + 2 \text{ch} 3\eta) + \text{ch} 4\eta \right) \right\}.$$

(4.18)

---

4 One reason for such a transformation is to bring these matrices to a more symmetric form. For a detailed discussion on this, refer to Sec. 4 of [22].
where
\[ b = 2\left(-\ch 2\beta - \ch^3\eta + \ch 2\alpha_-(1 + \ch 2\beta_+ \ch \eta)\right) \quad (4.19) \]
and
\[ d = -4 \sh 3\eta \sh(\alpha_+ + \frac{\eta}{2}) \sh(\alpha_+ - \frac{\eta}{2}) \ch(\beta_+ + \frac{\eta}{2}) \ch(\beta_+ - \frac{\eta}{2}) \quad (4.20) \]
Furthermore, using the fact that \( \Lambda^{(1,1)}(u) = \Lambda^{(1,1)}(u) \) and \( \Lambda^{(3,2)} \), \( \Lambda^{(4,1)} \), we obtain the energy in terms of Bethe roots \( \{u_j\} \),
\[ E = -\frac{1}{2} \sh \eta \sh 2\eta \sum_{j=1}^{M} \frac{1}{\ch(\frac{1}{2}(u_j + \frac{3\eta}{2})) \ch(\frac{1}{2}(u_j - \frac{\eta}{2}))} + \frac{1}{2} \sh 2\eta \left( A'(0) + B'(0) \right) \]
+ \( c_1^{(1)} C'(0) + c_2^{(1)} \).
\[ (4.21) \]
where
\[ A(u) = \tilde{h}(u + \frac{\eta}{2}) \tilde{h}(u - \frac{\eta}{2}) \]
\[ B(u) = -\tilde{h}(u + \frac{\eta}{2}) \tilde{h}(-u - \frac{\eta}{2}) \]
\[ C(u) = -\left( \frac{\ch(u + \frac{\eta}{2}) - i \sh \beta_-}{\ch(u + \frac{\eta}{2}) + i \sh \beta_-} \right) \left( \frac{\ch(u + \frac{\eta}{2}) + i \sh \beta_+}{\ch(u + \frac{\eta}{2}) - i \sh \beta_+} \right) B(u) \]
\[ \tilde{h}(u) = 4 \sh 2N(u + \frac{3\eta}{2}) \sh(2u + 2\eta) \sh(u + \eta) \sh(u - \eta) \]
\[ \times \left( \ch u + i \sh \beta_- \right) \left( \ch u - i \sh \beta_+ \right) \quad (4.22) \]
Also, \( M = 2N + p \) (see (3.31)).

We tabulate the energies computed using (4.21) for \( N = 3 \) with the Bethe roots (which are obtained using similar method as for the \( s = 1/2 \) case above) in Table 2. These numerical results once again illustrate the completeness of Bethe ansatz equations derived in Sec. 3. We have verified that the energies given in Table 2 coincide with those obtained from direct diagonalization of (4.6). One can proceed to repeat the analysis for higher spin values, namely \( s > 1 \). However, due to tedious computations, we avoid from pursuing it here.

5 Discussion

We have determined Bethe ansatz solutions of the open spin-\( s \) XXZ quantum spin chain for cases with nondiagonal boundary terms \( (3.13)-(3.19) \) and \( (3.27)-(3.32) \), by following the
method used earlier in [16, 24] to solve the spin-1/2 case. This method relies on functional relations (2.24) that the “fundamental” transfer matrices, $t^{(1/2)}(u)$ obey at roots of unity. However, these solutions hold only for $\eta = \frac{i\pi}{3}, \frac{i\pi}{5}, \ldots$. Unlike Bethe ansatz solutions found in earlier works on the open spin-$s$ XXZ chain with nondiagonal boundary terms, we emphasize that Bethe ansatz solutions found here hold for arbitrary values of boundary parameters (atmost two). We have checked these solutions for chains of length up to $N = 4$, and have verified that indeed they give the complete set of $(2s + 1)^N$ eigenvalues. Moreover, we also presented numerical evidence for the completeness of the Bethe ansatz solutions found (using $s = 1/2$ and $s = 1$ as examples) in Tables 1 and 2. Perhaps the completeness of the Bethe ansatz equations for spin-$s$ can readily be concluded from completeness of the corresponding Bethe ansatz equations for spin-1/2 case and the fusion hierarchy (2.14) which is used in the construction of higher spin-$s$ transfer matrices.

There remain many problems worth investigating. As mentioned in the Introduction, due to the relation of $s = 1$ case to supersymmetric sine-Gordon (SSG) model, one can carry out similar analysis as in [31], but now for spin-1 chain with nondiagonal boundary terms. One could also try to extend the solutions presented here to cases with multiple $Q(u)$’s as in [42]. Also, to our knowledge, conventional form of Bethe ansatz solution for the open XXZ quantum spin chain, where all six boundary parameters are arbitrary with generic values of bulk anisotropy parameter $\eta$, has not been found. We remark that through a series of important work on the spectrum of XXZ spin chain based on representation theory of the $q$-Onsager algebra [11], Baseilhac and Koizumi argue that obtaining such a conventional Bethe ansatz solution for the most general case is unlikely. It would be interesting to explore their results further and compare their approach with the Bethe ansatz approach.

Acknowledgments

I would like to thank R. I. Nepomechie for useful suggestions. I also thank P. Baseilhac for crucial correspondence.

References

[1] M. Gaudin, “Boundary Energy of a Bose Gas in One Dimension,” Phys. Rev. A4, 386 (1971);
M. Gaudin, La fonction d’onde de Bethe (Masson, 1983).
[2] F.C. Alcaraz, M.N. Barber, M.T. Batchelor, R.J. Baxter and G.R.W. Quispel, “Surface exponents of the quantum XXZ, Ashkin-Teller and Potts models,” J. Phys. A20, 6397 (1987).

[3] E.K. Sklyanin, “Boundary conditions for integrable quantum systems,” J. Phys. A21, 2375 (1988).

[4] L. Mezincescu, R.I. Nepomechie and V. Rittenberg, “Bethe Ansatz solution of the Fateev-Zamolodchikov quantum spin chain with boundary terms,” Phys. Lett. A147, 70 (1990).

[5] E.C. Fireman, A. Lima-Santos and W. Utiel, “Bethe Ansatz solution for quantum spin-1 chains with boundary terms,” Nucl. Phys. B626, 435 (2002) [nlin/0110048].

[6] W. Galleas and M.J. Martins, “Solution of the SU(N) Vertex Model with Non-Diagonal Open Boundaries,” Phys. Lett. A335, 167 (2005) [nlin.SI/0407027]; C.S. Melo, G.A.P. Ribeiro and M.J. Martins, “Bethe ansatz for the XXX-S chain with non-diagonal open boundaries,” Nucl. Phys. B711, 565 (2005) [nlin.SI/0411038].

[7] A. Doikou and A. Babichenko, “Principal chiral model scattering and the alternating quantum spin chain,” Phys. Lett. B515, 220 (2001); A. Doikou, “The XXX spin s quantum chain and the alternating $s^1$, $s^2$ chain with boundaries,” Nucl. Phys. B634, 591 (2002); A. Doikou and P.P. Martin, “On quantum group symmetry and Bethe ansatz for the asymmetric twin spin chain with integrable boundary,” J. Stat. Mech. P06004 (2006) [hep-th/0503019]; A. Doikou, “The Open XXZ and associated models at q root of unity,” J. Stat. Mech. P09010 (2006) [hep-th/0603112].

[8] J. de Gier and P. Pyatov, “Bethe Ansatz for the Temperley-Lieb loop model with open boundaries,” J. Stat. Mech. P03002 (2004) [hep-th/0312235]; A. Nichols, V. Rittenberg and J. de Gier, “One-boundary Temperley-Lieb algebras in the XXZ and loop models,” J. Stat. Mech. P03003 (2005) [cond-mat/0411512]; J. de Gier, A. Nichols, P. Pyatov and V. Rittenberg, “Magic in the spectra of the XXZ quantum chain with boundaries at $\Delta = 0$ and $\Delta = -1/2$,” Nucl. Phys. B729, 387 (2005) [hep-th/0505062]; J. de Gier and F.H.L. Essler, “Bethe Ansatz Solution of the Asymmetric Exclusion Process with Open Boundaries,” Phys. Rev. Lett. 95, 240601 (2005) [cond-mat/0508707]; J. de Gier and F.H.L. Essler, “Exact spectral gaps of the asymmetric exclusion process with open boundaries,” J. Stat. Mech. P12011 (2006) [cond-mat/0609645].
[9] D. Arnaudon, J. Avan, N. Crampé, A. Doikou, L. Frappat and E. Ragoucy, “General boundary conditions for the sl(N) and sl(M|N) open spin chains,” J. Stat. Mech. P08005, (2004) [math-ph/0406021].

[10] W.-L. Yang, Y.-Z. Zhang and M. Gould, “Exact solution of the XXZ Gaudin model with generic open boundaries,” Nucl. Phys. B698, 503 (2004) [hep-th/0411048]; W.-L. Yang and Y.-Z. Zhang, “Exact solution of the $A_{n-1}^{(1)}$ trigonometric vertex model with non-diagonal open boundaries,” JHEP 01, 021 (2005) [hep-th/0411190]; W.-L. Yang, Y.-Z. Zhang and R. Sasaki, “$A_{n-1}$ Gaudin model with open boundaries,” Nucl. Phys. B729, 594 (2005) [hep-th/0507148].

[11] P. Baseilhac and K. Koizumi, “A deformed analogue of Onsager’s symmetry in the XXZ open spin chain,” J. Stat. Mech. P10005 (2005) [hep-th/0507053]; P. Baseilhac, “The $q$-deformed analogue of the Onsager algebra: beyond the Bethe ansatz approach,” Nucl. Phys. B754, 309 (2006) [math-ph/0604036]; P. Baseilhac and K. Koizumi, “Exact spectrum of the XXZ open spin chain from the $q$-Onsager algebra representation theory,” J. Stat. Mech. P09006 (2007) [hep-th/0703106]; P. Baseilhac, “New results in the XXZ open spin chain,” [0712.0452].

[12] A. Nichols, “The Temperley-Lieb algebra and its generalizations in the Potts and XXZ models,” J. Stat. Mech. P01003 (2006) [hep-th/0509069]; A. Nichols, “Structure of the two-boundary XXZ model with non-diagonal boundary terms,” J. Stat. Mech. L02004 (2006) [hep-th/0512273].

[13] Z. Bajnok, “Equivalences between spin models induced by defects,” J. Stat. Mech. P06010 (2006) [hep-th/0601107].

[14] W. Galleas, “Functional relations from the Yang-Baxter algebra: Eigenvalues of the XXZ model with non-diagonal twisted and open boundary conditions,” Nucl. Phys. B790, 524 (2008) [0708.0009]

[15] J. Cao, H.-Q. Lin, K.-J. Shi and Y. Wang, “Exact solutions and elementary excitations in the XXZ spin chain with unparallel boundary fields,” cond-mat/0212163; J. Cao, H.-Q. Lin, K.-J. Shi and Y. Wang, “Exact solution of XXZ spin chain with unparallel boundary fields,” Nucl. Phys. B663, 487 (2003).

[16] R.I. Nepomechie, Nucl. Phys. B622, 615 (2002); Addendum, Nucl. Phys. B631, 519 (2002) [hep-th/0110116]; R.I. Nepomechie, “Functional relations and Bethe Ansatz for the XXZ chain,” J. Stat.
[17] R.I. Nepomechie and F. Ravanini, “Completeness of the Bethe Ansatz solution of the open XXZ chain with nondiagonal boundary terms,” *J. Phys. A37*, 433 (2004) [hep-th/0304092].

[18] W.-L. Yang, R.I. Nepomechie and Y.-Z. Zhang, “Q-operator and T-Q relation from the fusion hierarchy,” *Phys. Lett. B633*, 664 (2006) [hep-th/0511134].

[19] W.-L. Yang and Y.-Z. Zhang, “On the second reference state and complete eigenstates of the open XXZ chain,” *JHEP 04*, 044 (2007) [hep-th/0703222].

[20] A. Doikou, “Fused integrable lattice models with quantum impurities and open boundaries,” *Nucl. Phys. B668*, 447 (2003) [hep-th/0303205].

[21] A. Doikou, “A note on the boundary spin s XXZ chain,” *Phys. Lett. A366*, 556 (2007) [hep-th/0612268].

[22] L. Frappat, R.I. Nepomechie and E. Ragoucy, “Complete Bethe ansatz solution of the open spin-s XXZ chain with general integrable boundary terms,” *J. Stat. Mech. P09008* (2007) [math-ph/0707.0653].

[23] W.-L. Yang and Y.-Z. Zhang, “T-Q relation and exact solution for the XYZ chain with general nondiagonal boundary terms,” *Nucl. Phys. B744*, 312 (2006) [hep-th/0512154].

[24] R. Murgan and R.I. Nepomechie, “Bethe Ansatz derived from the functional relations of the open XXZ chain for new special cases,” *J. Stat. Mech. P05007* (2005); Addendum, *J. Stat. Mech. P11004* (2005) [hep-th/0504124].

[25] P.P. Kulish and E.K. Sklyanin, “Quantum spectral transform method, recent developments,” Lecture Notes in Physics, Vol. 151, (Springer, 1982) 61; P.P. Kulish, N.Yu. Reshetikhin and E.K. Sklyanin, “Yang-Baxter equation and representation theory. I,” *Lett. Math. Phys.* 5 (1981) 393; P.P. Kulish and N.Yu. Reshetikhin, “Quantum linear problem for the sine-Gordon equation and higher representation,” *J. Sov. Math.* 23 (1983) 2435; A.N. Kirillov and N.Yu. Reshetikhin, “Exact solution of the Heisenberg XXZ model of spin s,” *J. Sov. Math.* 35 (1986) 2627; “Exact solution of the integrable XXZ Heisenberg model with arbitrary spin. I. The ground state and the excitation spectrum,” *J. Phys. A20* (1987) 1565;
[26] L. Mezincescu and R.I. Nepomechie, “Fusion procedure for open chains,” J. Phys. A25, 2533 (1992); Y.-K. Zhou, “Row transfer matrix functional relations for Baxter’s eight-vertex and six-vertex models with open boundaries via more general reflection matrices,” Nucl. Phys. B458, 504 (1996) [hep-th/9510095].

[27] V.V. Bazhanov, S.L. Lukyanov and A.B. Zamolodchikov, “Integrable structure of conformal field theory, quantum KdV theory and thermodynamic Bethe ansatz,” Commun. Math. Phys. 177 (1996) 381 [hep-th/9412229]; “Integrable structure of conformal field theory III. The Yang-Baxter relation,” Commun. Math. Phys. 200 (1999) 297 [hep-th/9805008]; A. Kuniba, K. Sakai and J. Suzuki, “Continued fraction TBA and functional relations in XXZ model at root of unity,” Nucl. Phys. B525[FS] (1998) 597 [math/9803056].

[28] V.V. Bazhanov and N.Yu. Reshetikhin, “Critical RSOS Models And Conformal Field Theory,” Int. J. Mod. Phys. A4, 115 (1989).

[29] P. Di Vecchia and S. Ferrara, “Classical solutions in two-dimensional supersymmetric field theories,” Nucl. Phys. B130, 93 (1977); J. Hruby, “On the supersymmetric sine-Gordon model and a two-dimensional ‘bag’,” Nucl. Phys. B131, 275 (1977); S. Ferrara, L. Girardello and S. Sciuto, “An infinite set of conservation laws of the supersymmetric sine-Gordon theory,” Phys. Lett. B76, 303 (1978); R. Shankar and E. Witten, “The S matrix of the supersymmetric nonlinear sigma model,” Phys. Rev. D17, 2134 (1978); C. Ahn, D. Bernard and A. LeClair, “Fractional supersymmetries in perturbed coset CFTs and integrable soliton theory,” Nucl. Phys. B346, 409 (1990); C. Ahn, “Complete S matrices of supersymmetric sine-Gordon theory and perturbed superconformal minimal model,” Nucl. Phys. B354, 57 (1991).

[30] T. Inami, S. Odake and Y.-Z. Zhang, “Supersymmetric extension of the sine-Gordon theory with integrable boundary interactions,” Phys. Lett. B359, 118 (1995) [hep-th/9506157]; R.I. Nepomechie, “The boundary supersymmetric sine-Gordon model revisited,” Phys. Lett. B509, 183 (2001) [hep-th/0103029]; Z. Bajnok, L. Palla and G. Takács, “Spectrum of boundary states in N = 1 SUSY sine-Gordon theory,” Nucl. Phys. B644, 509 (2002) [hep-th/0207099].

[31] C. Ahn, R.I. Nepomechie and J. Suzuki, “Finite size effects in the spin-1 XXZ and supersymmetric sine-Gordon models with Dirichlet boundary conditions,” Nucl. Phys. B767, 250 (2007) [hep-th/0611136].
[32] M. Karowski, “On the bound state problem in (1+1)-dimensional field theories” *Nucl. Phys.* **B153**, 244 (1979).

[33] H.M. Babujian, “Exact solution of the isotropic Heisenberg chain with arbitrary spins: thermodynamics of the model,” *Nucl. Phys.* **B215**, 317 (1983);
L.A. Takhtajan, “The picture of low-lying excitations in the isotropic Heisenberg chain of arbitrary spins,” *Phys. Lett.* **87A**, 479 (1982).

[34] K. Sogo, “Ground state and low-lying excitations in the Heisenberg XXZ chain of arbitrary spin S,” *Phys. Lett.* **A104**, 51 (1984);
H.M. Babujian and A.M. Tsvelick, “Heisenberg magnet with an arbitrary spin and anisotropic chiral field,” *Nucl. Phys.* **B265** [FS15], 24 (1986);
A.N. Kirillov and N.Yu. Reshetikhin, “Exact solution of the Heisenberg XXZ model of spin s,” *J. Sov. Math.* **35**, 2627 (1986);
A.N. Kirillov and N.Yu. Reshetikhin, “Exact solution of the integrable XXZ Heisenberg model with arbitrary spin. I. The ground state and the excitation spectrum,” *J. Phys.* **A20**, 1565 (1987).

[35] H.J. de Vega and A. González-Ruiz, “Boundary K-matrices for the six vertex and the \( n(2n-1) \) \( A_{n-1} \) vertex models,” *J. Phys.* **A26**, L519 (1993) [hep-th/9211114].

[36] S. Ghoshal and A.B. Zamolodchikov, “Boundary S-Matrix and Boundary State in Two-Dimensional Integrable Quantum Field Theory,” *Int. J. Mod. Phys.* **A9**, 3841 (1994) [hep-th/9306002].

[37] R. J. Baxter, “Partition function of the eight-vertex lattice model,” *Ann. Phys. (NY)* **70**, 193 (1972) [Ann. Phys. (NY) **281**, 187 (2000)]; “Asymptotically degenerate maximum eigenvalues of the eight-vertex model transfer matrix and interfacial tension,” *J. Stat. Phys.* **8**, 25 (1973); “Exactly Solved Models in Statistical Mechanics,” (Academic Press) (1982)

[38] I.V. Cherednik, “Factorizing particles on a half line and root systems,” *Theor. Math. Phys.* **61**, 977 (1984).

[39] S. Dasmahapatra, R. Kedem and B.M. McCoy, “Spectrum and completeness of the three state superintegrable chiral Potts model” *Nucl. Phys.* **B396**, 506 (1993) [hep-th/9204003];
K. Fabricius and B.M. McCoy, “Bethe’s equation is incomplete for the XXZ model at roots of unity” *J. Stat. Phys.* **103**, 647 (2001) [cond-mat/0009279].

22
[40] A.B. Zamolodchikov and V.A. Fateev, “Model factorized $S$ matrix and an integrable Heisenberg chain with spin 1,” *Sov. J. Nucl. Phys.* **32**, 298 (1980).

[41] T. Inami, S. Odake and Y.-Z. Zhang, “Reflection $K$ matrices of the 19 vertex model and XXZ spin 1 chain with general boundary terms,” *Nucl. Phys.* **B470**, 419 (1996) [hep-th/9601049].

[42] R. Murgan and R.I. Nepomechie, “Generalized $T-Q$ relations and the open XXZ chain,” *J. Stat. Mech.* **P08002** (2005) [hep-th/0507139]; R. Murgan, R.I. Nepomechie and C. Shi, “Exact solution of the open XXZ chain with general integrable boundary terms at roots of unity,” *J. Stat. Mech* **P08006** (2006) [hep-th/0605223].
| $E$     | Bethe roots $u_j$                                                                 |
|---------|----------------------------------------------------------------------------------|
| -3.19769 | $0.222018 + 2.91719 i, 0.0900395 + 2.91719 i, -2.6018 i,$                          |
|         | $1.01834 - 1.7952 i, 2.15279 i, 0.267003 - 1.7952 i,$                           |
|         | $1.00769 i, 1.0165 + 1.3464 i, 0.0900395 - 0.224399 i, 0.222018 - 0.224394 i$      |
| -2.42188 | $0.324807 - 3.13487 i, 0.319576 + 2.68662 i, 0.0958764 + 2.91717 i,$            |
|         | $-2.60637 , 2.15279 i, 0.356519 - 1.7952 i,$                                    |
|         | $1.09238 i, 0.0958764 - 0.224378 i, 0.324807-0.455523 i,$                       |
|         | $0.319576 + 0.00617674 i$                                                        |
| -1.87006 | $0.530712 + 2.91722 i, 0.0853747 + 2.91719 i, -2.60166 i,$                      |
|         | $0.946517 - 1.7952 i, 2.15279 i, 0.25634 - 1.7952 i,$                           |
|         | $1.00255 i, 0.943646 + 1.3464 i, 0.0853747 - 0.224399 i,$                      |
|         | $0.530712 - 0.224428 i$                                                         |
| -1.30053 | $0.0805934 + 2.91719 i, 1.4454 - 1.7952 i, -2.6016 i,$                           |
|         | $0.607877 - 1.7952 i, 0.212672 - 1.7952 i, 0.592602 + 1.3464 i,$                |
|         | $0.992082 i, 0.54 i, 1.44531 + 1.3464 i,$                                     |
|         | $0.0805934 - 0.224399 i$                                                         |
| -0.874711| $0.284541 + 2.91781 i, 0.251638 - 3.13998 i, 0.245572 + 2.69145 i,$             |
|         | $2.15279 i, 0.359151 - 1.7952 i, 1.59561 i,$                                    |
|         | $-0.983447 i, 0.251638 - 0.450414 i, 0.245572 + 0.00134169 i,$                  |
|         | $0.284541 - 0.225016 i$                                                         |
| -0.674656| $0.518223 + 2.91722 i, 0.199566 + 2.91719 i, 0.940199 - 1.7952 i,$             |
|         | $2.15279 i, 0.255178 - 1.7952 i, 1.00201 i,$                                    |
|         | $-0.988749 i, 0.937209 + 1.3464 i, 0.199566 - 0.2244 i,$                        |
|         | $0.518223 - 0.224431 i$                                                         |
| -0.203476| $0.182373 + 2.91719 i, 1.43967 - 1.7952 i, 0.604391 - 1.7952 i,$               |
|         | $0.211794 - 1.7952 i, 0.588802 + 1.3464 i, 0.991943 i, -0.988797 i,$            |
|         | $0.54 i, 1.43958 + 1.3464 i, 0.182373 - 0.224399 i$                             |

Table 1: The 16 energies and corresponding Bethe roots given by $\tilde{\Lambda}^{(1/2,1/2)}(u)$ for $N = 4, s = 1/2, p = 6, \eta = i\pi/7, \alpha_- = 0.54 i, \beta_- = 0.2, \theta_- = 0, \alpha_+ = i\pi/2, \beta_+ = \eta, \theta_+ = 0$
| $E$ (continued) | Bethe roots $u_j$ (continued) |
|---------------|-----------------------------|
| 0.343441      | 0.149446 - 3.14159 i, 0.149313 + 2.69279 i, 1.05439 - 1.7952 i, 2.15279 i, 0.273366 - 1.7952 i, 1.05294 + 1.3464 i, -2.60198 i, 0.149446 - 0.448806 i, 0.149313 |
| 0.761262      | 0.0287807 - 2.72544 i, 0.487517 - 1.7952 i, 0.290846 + 1.3464 i, 0.0287807 - 0.864949 i, -0.619831 i, 0.54 i, 0.278493 i, -0.277755 i, -0.0641203 i, 0.0641203 i |
| 0.846541      | 0.249771 - 3.14157 i, -3.01053 i, 3.01052 i, 0.186824 - 2.39256 i, 0.259537 + 2.25 i, 2.3744 i, 0.186824 - 1.19783 i, 0.54 i, 0.249771 - 0.44882 i, 0.259537 + 0.442796 i |
| 0.883622      | 0.357565 + 3.13829 i, 0.357656 + 2.69611 i, 0.985138 - 1.7952 i, 2.15279 i, 0.260208 - 1.7952 i, 1.00394 i, -0.98875 i, 0.982836 + 1.3464 i, 0.357565 - 0.445494 i, 0.357656 - 0.00331156 i |
| 1.00689       | 0.351296 + 2.91719 i, 1.42001 - 1.7952 i, 0.593281 - 1.7952 i, 0.209089 - 1.7952 i, 0.576676 + 1.3464 i, 0.991553 i, -0.988798 i, 0.54 i, 1.41991 + 1.3464 i, 0.351296 - 0.224399 i |
| 1.20648       | 0.452936 + 2.92003 i, 0.43968 - 2.9168 i, 0.439123 + 2.46302 i, -2.6019 i, 2.15279 i, 0.328707 - 1.7952 i, 1.04938 i, 0.43968 - 0.673593 i, 0.439123 + 0.229771 i, 0.452936 - 0.227233 i |
| 1.50502       | 0.625154 - 3.1108 i, 0.624995 + 2.66205 i, -2.6016 i, 0.830961 - 1.7952 i, 0.238271 - 1.7952 i, 1.69665 i, 0.825207 + 1.3464 i, 0.54 i, 0.625154 - 0.479587 i, 0.624995 + 0.0307393 i |
| 1.82374       | 0.755163 + 2.9172 i, -2.60159 i, 1.3168 - 1.7952 i, 0.550839 - 1.7952 i, 0.199615 - 1.7952 i, 0.530191 + 1.3464 i, 0.990548 i, 1.31659 + 1.3464 i, 0.54 i, 0.755163 - 0.224402 i |
| 2.16601       | 1.72415 - 1.7952 i, 0.893009 - 1.7952 i, 0.447923 - 1.7952 i, 0.176866 - 1.7952 i, 1.70344 i, 0.416589 + 1.3464 i, 0.891068 + 1.3464 i, -0.988799 i, 0.54 i, 1.72413 + 1.3464 i |

25
| $E$  | Bethe roots $u_j$                                                                 |
|------|----------------------------------------------------------------------------------|
| -12.4557 | 0.484779 - 3.10162 i, 0.411886 + 2.48641 i, 0.106801 - 3.14045 i, 0.0868008 + 2.51253 i, 0.348418 - 1.88496 i, 0.640811 + 1.25664 i, 0.106801 - 0.629463 i, 0.0868008 + 0.0000685 i, 0.411886 + 0.0268653 i |
| -9.695  | 0.575021 + 2.83026 i, 0.0719595 - 3.14151 i, 0.0613009 + 2.51321 i, 0.80436 - 1.88496 i, 0.0365392 + 1.72227 i, 0.0365392 + 0.791004 i, 0.943625 + 1.25664 i, 0.0719595 - 0.628405 i, 0.0613009 + 0.0000685 i, 0.575021- 0.316986 i |
| -8.36086 | 0.500526 + 2.8322 i, 0.0403235 - 2.64238 i, 0.0383788 + 2.64283 i, 0.0383947 + 2.38377 i, 0.670337 - 1.88496 i, 0.0403235 - 1.12753 i, 0.873976 + 1.25664 i, 0.0383788 - 0.129554 i, 0.0383947 + 0.129508 i, 0.500526 - 0.318927 i |
| -7.49773 | 0.507933 + 2.83108 i, 0.196307 + 3.14091 i, 0.207919 + 2.51414 i, 0.723661 - 1.88496 i, 0.0062148 + 1.85904 i, 0.899811 + 1.25664 i, 0.0062148 + 0.654233 i, 0.196307 - 0.627633 i, 0.207919 - 0.00868076 i, 0.507933 - 0.317806 i |
| -7.43354 | 1.31309 -1.88496 i, 0.0437839 -3.14159 i, 0.0387234 +2.51327 i, 0.46288 -1.88496 i, 0.00846623 +1.83461 i, 0.635229 +1.25664 i, 0.00846623 +0.678666 i, 0.0437839 -0.628325 i, 0.0387234, 1.33352 +1.25664 i |
| -6.81246 | 1.28291 -1.88496 i, 0.00540715 + 2.56635 i, 0.00749169 - 2.55718 i, 0.0054066 + 2.46021 i, 0.418583 - 1.88496 i, 0.00749169 - 1.21273 i, 0.603838 + 1.25664 i, 0.0054066 + 0.530635 i, 0.00540715 - 0.53073 i, 1.30642 + 1.25664 i |
| -6.75338 | 0.534826 - 3.13136 i, 0.366787 + 2.83126 i, 0.11739 - 2.52032 i, 0.0961446 + 2.51326 i, 0.433162 + 1.84874 i, 0.11739 - 1.24959 i, 0.0961446 + 0.0000111021 i, 0.433162 + 0.664539 i, 0.366787 - 0.317989 i, 0.534826 - 0.638555 i |

Table 2: The 27 energies and corresponding Bethe roots given by $\tilde{\Lambda}^{(1)}(u)$ for $N = 3, s = 1, p = 4, \eta = i\pi/5, \alpha_\pm = \eta, \beta_\pm = 0.35, \theta_\mp = 0.54, \alpha_+ = \eta, \beta_+ = 0.76, \theta_+ = 0.54$
| $E$ (continued) | Bethe roots $u_j$ (continued) |
|-----------------|--------------------------------|
| -6.37707        | 0.450862 - 3.1307 i, 0.3548 + 2.83363 i, 0.203444 + 3.14135 i, 0.313088 + 1.93502 i, 0.10853 + 1.81328 i, 0.10853 + 0.699993 i, 0.203444 - 0.62808 i, 0.313088 + 0.578258 i, 0.450862 - 0.639212 i, 0.3548 - 0.320357 i |
| -5.9431         | 0.15167 + 3.14151 i, 1.28717 - 1.88496 i, 0.152799 + 2.51336 i, 0.431361 - 1.88496 i, 0.000660592 + 1.88095 i, 0.610477 + 1.25664 i, 0.000660592 + 0.632323 i, 0.15167 - 0.628234 i, 1.3102 + 1.25664 i, 0.152799 - 0.0000868625 i |
| -4.96948        | 0.245899 - 3.14157 i, 1.09088 - 1.88496 i, 0.0674913 - 2.51035 i, 0.056464 + 2.51327 i, 0.236615 + 1.88684 i, 0.0674913 - 1.25957 i, 1.14613 + 1.25664 i, 0.236615 + 0.626432 i, 0.245899 - 0.628337 i, 0.056464 |
| -4.75362        | 0.118717 + 3.04878 i, 0.118716 - 3.04877 i, 1.08872 - 1.88496 i, 0.120353 + 1.97697 i, 0.0725214 + 1.8268 i, 0.0725214 + 0.686472 i, 0.118716 - 0.721137 i, 0.118717 - 0.535507 i, 0.120353 + 0.536301 i, 1.14446 + 1.25664 i |
| -4.29889        | 0.611823 - 3.10726 i, 0.585252 + 2.48522 i, 0.283735 + 2.82749 i, 0.545972 - 1.88496 i, 0.00037444 + 1.88259 i, 0.773596 + 1.25664 i, 0.00037444 + 0.630686 i, 0.283735 - 0.314213 i, 0.611823 - 0.662648 i, 0.585252 + 0.0280531 i |
| -4.09589        | 0.792496 + 2.51421 i, 0.79911 - 2.5049 i, 0.0126395 + 2.51327 i, 0.0132493 - 2.51136 i, 0.0132493 - 1.25855 i, 0.0832428 + 1.25664 i, 0.79911 - 1.26501 i, 0.939738 + 1.25664 i, 0.0126395, 0.792496 - 0.000938409 i |
| -4.05775        | 0.79092 + 2.51422 i, 0.797571 - 2.50483 i, 0.0134204 + 3.14159 i, 0.0128183 + 1.8868 i, 0.0564763 + 1.25664 i, 0.797571 - 1.26508 i, 0.938917 + 1.25664 i, 0.0134204 - 0.628319 i, 0.0128183 + 0.62647 i, 0.79092 - 0.000943414 i |
| -3.93649        | 0.886933 - 3.14031 i, 0.0107824 + 2.51327 i, 0.0112295 - 2.51164 i, 0.876475 + 1.88715 i, 0.725212 - 1.88496 i, 0.0112295 - 1.25828 i, 0.074371 + 1.25664 i, 0.0107824, 0.876475 + 0.626128 i, 0.886933 - 0.629601 i |
| -3.90338        | 0.886042 - 3.1403 i, 0.011339 + 3.14159 i, 0.875518 + 1.88714 i, 0.723621 - 1.88496 i, 0.0108981 + 1.88655 i, 0.0516604 + 1.25664 i, 0.011339 - 0.628319 i, 0.0108981 + 0.626729 i, 0.875518 + 0.626131 i, 0.886042 - 0.629609 i |
| $E$ (continued) | Bethe roots $u_j$ (continued) |
|----------------|-------------------------------|
| -3.5973        | 0.336944 -3.14041i, 0.335901 +2.51212i, 1.23958 -1.88496i, |
|                | 0.391865 -1.88496i, 0.000094209 +1.88434i, 0.568642 +1.25664i, |
|                | 0.000094209 +0.628934i, 1.26813 +1.25664i, 0.336944 -0.629498i, |
|                | 0.335901 +0.0115798i          |
| -2.69459       | 0.779303 +2.82839i, 0.232072 +2.82743i, 1.05217 -1.88496i, |
|                | 0.329315 -1.88496i, 0.0000460866 +1.88465i, 0.475658 +1.25664i, |
|                | 0.0000460866 +0.628622i, 1.10962 +1.25664i, 0.232072 -0.31416i, |
|                | 0.779303 -0.315113i           |
| -2.39712       | 0.503174 +2.79201i, 0.504144 -2.78706i, 0.506179 +2.23281i, |
|                | 0.745784 -1.88496i, 0.000133753 +1.88409i, 0.944351 +1.25664i, |
|                | 0.000133753 +0.629187i, 0.504144 -0.982846i, 0.503174 -0.278731i, |
|                | 0.506179 +0.280461i           |
| -1.91101       | 0.745717 -3.13841i, 0.476924 -2.52358i, 0.476385 +2.51311i, |
|                | 0.712375 +1.88751i, 0.0000130876 +1.88425i, 0.476924 -1.24633i, |
|                | 0.000108768 +0.629027i, 0.476385 +0.000165845i, 0.712375 +0.625767i, |
|                | 0.745717 -0.631505i           |
| -1.65102       | 0.591714 +3.13021i, 0.594377 +2.52503i, 1.08954 -1.88496i, |
|                | 0.39176 -1.88496i, 0.00001303025 +1.88483i, 0.467049 +1.25664i, |
|                | 0.00001303025 +0.62844i, 1.14296 +1.25664i, 0.591714 -0.616939i, |
|                | 0.594377 -0.0117591i          |
| -1.3681        | 1.52581 -1.88496i, 0.205857 +2.82743i, 0.710251 -1.88496i, |
|                | 0.25594 -1.88496i, 1.88494 i, 0.349688 +1.25664i, |
|                | 0.813986 +1.25664i, 0.628371 i, 0.205857 -0.314159i, |
|                | 1.53319 +1.25664i            |
| -1.24743       | 0.75169 +2.86018i, 0.738919 -2.84659i, 0.731307 +2.15882i, |
|                | 0.34774 -1.88496i, 0.0000130362 +1.88481i, 0.564622 +1.25664i, |
|                | 0.0000130362 +0.62846i, 0.738919 -0.92332i, 0.731307 +0.354451i, |
|                | 0.75169 -0.346911 i           |
| -0.451049      | 0.496986 +2.82743i, 1.4849 -1.88496i, 0.679662 -1.88496i, |
|                | 0.240941 -1.88496i, 1.88494 i, 0.326913 +1.25664i, |
|                | 0.789576 +1.25664i, 0.628339 i, 1.49385 +1.25664i, |
|                | 0.496986 -0.314155 i          |
| -0.278905      | 0.901463 -3.10659i, 0.894513 +2.47987i, 0.816494 -1.88496i, |
|                | 0.258214 -1.88496i, 1.88493 i, 0.358314 +1.25664i, |
|                | 0.927674 +1.25664i, 0.628344 i, 0.901463 -0.663324 i, |
|                | 0.894513 +0.0334058 i         |
| $E$ (continued) | Bethe roots $u_j$ (continued) |
|-----------------|--------------------------------|
| 0.72331         | 1.02774 +2.82771 i, 1.29908 -1.88496 i, 0.601685 -1.88496 i, 0.212716 -1.88496 i, 1.88495 i, 0.282851 +1.25664 i, 0.716771 +1.25664 i, 0.628324 i, 1.31854 +1.25664 i, 1.02774 -0.314435 i |
| 1.69087         | 1.74631 -1.88496 i, 0.943277 -1.88496 i, 0.514308 -1.88496 i, 0.183505 -1.88496 i, 1.88495 i, 0.238533 +1.25664 i, 0.614563 +1.25664 i, 0.62832 i, 0.990458 +1.25664 i, 1.7488 +1.25664 i |