Force-free electrodynamics and foliations in an arbitrary spacetime

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Abstract

In this paper, we formulate the relationship between force-free electrodynamics and foliations. The background metric is considered predetermined and electrically neutral, but otherwise arbitrary. As it turns out, solutions to force-free electrodynamics are intimately connected to the existence of foliations of spacetime with prescribed properties. We prove a local existence and uniqueness theorem concerning foliations when the field is non-null. In the null case, we show that there will always be a class of solutions with exactly two degrees of freedom for every admissible foliation. We are also able to also prove a singularity theorem for when non-null solutions approach the null limit. All new results are clarified with examples. For a variety in the discussion, we conclude with a solution to Maxwell’s equations in FRW cosmology.

Keywords: force-free electrodynamics, general relativity, foliations

1. Introduction

Force-free electrodynamics (FFE) has become the central framework for describing the magnetospheres of active black holes. Although the governing equations of force-free electrodynamics were developed as early 1977 ([1, 2]), it did not receive much attention till the late 90s as a subject of systematic study ([3, 4]). More recently, the force-free magnetosphere in a Kerr background has played a prominent role in the study of black hole astrophysics. Properties of the force-free magnetosphere and its abilities to extract energy and angular momentum was a general feature of a numerical study of the subject ([5–8]). From a theoretical point of view, analytical solutions to a Kerr force-free magnetosphere slowly emerged as well ([9, 10], and [11]). A recent paper by Gralla and Jacobson [12] captures the current status of the theory of FFE.

This paper will focus on the connection between FFE and foliations of spacetime. In 2002, Komissarlov used the $3 + 1$ formalism to show that, in the magnetically dominated case, the

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equations of FFE are indeed hyperbolic, and that the problem is well-posed [13]. It is important to realize that these features can also depend on the formalism used. For example, the work in [14] suggested that FFE is ill-posed in a formalism relying on Euler potentials. A covariant approach to the initial value problem, in the magnetically dominated case, was also presented in [15] using a formalism developed by Geroch [16]. The electrically dominated and the null case in FFE have never been a part of systematic study thus far.

In this paper, we present a novel technique using an adapted chart formalism. In this formalism, it is possible to write down a deterministic evolution equation and an explicit formula for the solution in both the electrically and magnetically dominated case. The resulting solution is unique. We are also able to show that in the null case there will be a unique class of solutions. As we will show below, FFE is deterministic (in the generalized sense of a class of solutions) regardless of sign and value of $F^2$. Here $F$ is the electromagnetic field tensor. This should not be taken to mean that FFE is well-posed in the electrically dominated case, but rather that when (and if) such solutions exist, they have a well-defined governing equation. Additionally, we will prove a local existence and uniqueness theorem for FFE. In [17], the authors argue that the existence of a stationary, axis-symmetric, magnetically dominated, force-free electromagnetic field in a Kerr background is entirely dependent on the existence of a foliation of Kerr spacetime with certain well-prescribed properties. In the following sections, we will prove the same as a general result. In other words, we will elevate two-dimensional foliations with certain well-prescribed properties as a fundamental object for FFE in an arbitrary spacetime. These prescriptions are very natural and expected, and cannot be relaxed. Geometry (or gravity if you prefer) alone determines the existence of a force-free electromagnetic field. We do not fix the background, nor will we have any restrictions of stationarity and axis-symmetry. The foliations we refer to are integral submanifolds of an involutive distribution that is the kernel of the two-form $F$. It will also become clear that the notion of an initial value problem for FFE can be replaced by the study of FFE admissible spacetimes and foliations.

Curiously, while our formalism is able to treat the general evolution equations in a cohesive manner, it is not the case that $F^2$ can change its sign smoothly—i.e., we first describe a null field, and then separately an electrically or magnetically dominated field. In general, a smooth transition is not permitted. To be clear, this is not a failure of our formalism, but that additional restrictions are required to allow this transition, if indeed such a transition is possible. In particular, we will prove that in the case of foliations of a spacetime generated by commuting Killing vector fields, non-null solutions do not have a well defined null limit. The solution necessarily becomes unbounded in this case.

We begin with a recapitulation of the basic equations and properties of FFE, following which we will recast the equations in a coordinate system that is adapted to the foliations. It is in this adapted chart that we will prove our existence and uniqueness theorems. To illustrate the computational ability of our formalism, we will re-derive the previously obtained solutions in [9–11] using the new formalism. Spacetimes containing commuting Killing vector fields are then treated as a special case, because, as we will show they necessarily give rise to a special FFE solution under very mild restrictions. As an example, we will conclude with a pair of vacuum solutions in a Kerr background. All solutions presented in this paper are exact.

In this article, as usual, comma denotes partial differentiation, i.e.,

\[ M_r = \frac{\partial M}{\partial x^r}. \]

Also, in $\sqrt{-g}$, $g$ denotes the determinant of the metric, i.e., $\det g$. Otherwise $g$ is simply the metric, and $\epsilon_{\mu\nu\alpha\beta}$ is the Levi-Civita tensor which is completely antisymmetric in all indices such
that $\epsilon_{1234} = \sqrt{-g}$ in a positively oriented chart. Notice the unusual 1–4 labeling of indices. This will be made clear along the way.

2. Basic properties of the force-free electromagnetic field

A spacetime, for our purposes, is a four-dimensional smooth manifold $\mathcal{M}$ endowed with a metric $g$ of Lorentz signature, specifically $(-1, 1, 1, 1)$. In this work, we are concerned with the evolution of the electromagnetic field generated by a (possibly) non-trivial current density. For this reason, since we would like to account for all the electromagnetic interactions in our formalism, the background metric is taken to be fixed, and electrically neutral. We do not place any further restrictions on $g$ for our main theorem. Maxwell’s equations in the tensor form are usually written as

$$\nabla_\nu F^\mu = J^\mu, \quad (1)$$

and

$$\nabla_{[\mu} F_{\nu\lambda]} = 0. \quad (2)$$

Here, $F$ is the Maxwell field tensor and $J$ is the current density. Also, $[\mu\nu\lambda]$ denotes anti-symmetrization of the included indices in the usual manner.

Following Gralla and Jacobson ([12]), we will mainly use the formalism of exterior calculus to describe the electromagnetic interaction. Please see appendix A.1 for details. In this case, $F$ is to be viewed as a closed two-form, i.e.,

$$dF = 0, \quad (3)$$

which satisfies

$$\ast d \ast F = J. \quad (4)$$

Here $\ast$ is the Hodge-Star operator and $d$ is the exterior derivatives on forms. From the above equation and equation (1), it is clear that we use $J$ to describe both the current ‘vector’ density or the associated one-form. Recall that a vector field in geometry is also referred to as a contravariant vector field by physicists, and a one-form or a dual vector field is the usual covariant vector field. The distinction is made either by context or explicitly by denoting its index. For example, $J^\mu$ is a contravariant vector filed, while $J_\mu \equiv g_{\mu\nu} J^\nu$ is the associated covariant vector field. The same distinction is to be understood for all tensorial objects.

An electromagnetic field is degenerate if there is a vector field $w$ such that the interior product of $w$ with $F$ vanishes. I.e.,

$$i_w F \equiv F(w, \cdot) = 0.$$ 

Force-free electrodynamics is a special case where the degenerate field satisfies

$$i_J F = 0,$$

where $J$ is the current density given by equation (1). In components, this can be written as

$$F_{\mu\nu} \nabla_\lambda F^{\lambda\nu} = 0. \quad (5)$$

Our central focus will be to establish to a meaningful initial data set/surface for the above equation and also to prove a local existence and uniqueness theorem.
We begin our formulation of the problem by recalling a few properties of degenerate fields (force-free electrodynamics). The interested reader is referred to [2, 3], and [4] for details. All the essential properties of FFE has been recast in an efficient way using modern notation in [12] and should serve as a reference guide for this current work. A force-free electromagnetic field is a simple two-form. I.e., there exists one-forms $\alpha$ and $\beta$ such that

$$F = \alpha \wedge \beta.$$ 

Consequently, the kernel of $F$ is a two-dimensional subspace of the tangent bundle consisting of all vector fields $v$ such that $\alpha(v) = 0 = \beta(v)$. We will then have that $i_v F = 0$.

Depending on the causal character of the kernel of $F$, denoted as $\ker F$, we can locally classify $F$ into three categories. If for any point $p$ in our spacetime, the kernel of $F$ at $p$ is spacelike, Lorentz, or if the metric when restricted to the kernel is degenerate, we say that $F$ is electrically dominated, magnetically dominated or null at $p$. This is equivalent to the requirement that the scalar

$$F^2(p) \equiv F_{\mu\nu} F^{\mu\nu}(p)$$

is less than zero, greater than zero or zero respectively. Recall, that the metric when restricted to $\ker F|_p$ is degenerate if there exists $l \in \ker F|_p$ such that $g(l, v) = 0$ for all $v \in \ker F|_p$. In particular, $l$ is a null vector.

From Cartan’s magic formula, we get that

$$v \in \ker F \text{ implies that } \mathcal{L}_v F = 0.$$ 

In the remainder of the section, we will rely on terminology and key results on foliations from differential geometry. Please see appendix A.2 for a refresher on the definitions/results used below. Now suppose $v, w \in \ker F$. Since

$$i_{[v,w]} F = [\mathcal{L}_v, i_w] F = 0,$$

we have that $\ker F$ is an involutive distribution, and therefore by Frobenius’ theorem, spacetime can be foliated by two-dimensional integral submanifolds of the distribution spanned by $\ker F$.

In particular, given any point $p \in \mathcal{M}$, there exists a coordinate chart $(U_p, \phi_p = (x^1, \ldots, x^4))$ centered about $p$, i.e.,

$$\phi_p(p) = (x^1(p), \ldots, x^4(p)) = (0, \ldots, 0),$$

that is adapted to the distribution. Without loss of generality, this means we can arrange for

$$\text{span} \left\{ \frac{\partial}{\partial x^3}, \frac{\partial}{\partial x^4} \right\} = \ker F|_{U_p}. \quad (6)$$

We will take equation (6) as part of the requirements for our adapted chart at $p$ for the foliation determined by $\ker F$. There is no preference here for a timelike coordinate, and so we label the adapted coordinates with indices ranging from 1–4, rather than the usual 0–3. In this chart, the electromagnetic field tensor can be written as

$$F = u \, dx^3 \wedge dx^4$$

for some component function $u(x^1, \ldots, x^4)$. Equation (3) now limits $u$ to only be a function of $x^3$ and $x^4$. We then obtain the needed final form of $F$:

$$F = u(x^3, x^4) dx^3 \wedge dx^4. \quad (7)$$
3. Equations of force-free electrodynamics in an adapted coordinate system

All our results will only be valid locally, and so we restrict all discussions to the adapted coordinate system described above. We will also have the occasion to require that the domain of the chart $U_p$ is \textit{starlike} about $p$. This means that if $q \in U_p$, then the line segment from $p$ to $q$ lies in $U_p$.

We seek an expression for $u$ in equation (7) satisfying equation (4) such that

$$ (J^a)^\mu = 0 \quad \text{for} \quad a = 3, 4. \tag{8} $$

Here, $\sharp$ is the raising operator defined by

$$ (J^a)^\mu = g^{\mu \nu} J_\nu. \tag{9} $$

This will ensure that the resulting current is force-free. Applying the Hodge-Star operator and the exterior derivative in the appropriate order to $F$ in equation (7) gives

$$ (J^a)^\mu = ((* \ d \ *)^a) \ ^\mu = \frac{1}{2} \epsilon^{\alpha \beta \lambda \nu} \partial_r (u \epsilon_{\mu \nu \alpha \beta} g^{\mu 3} g^{\nu 4}). \tag{9} $$

The requirements of equation (8) imply that

$$ \epsilon^{\alpha \beta \lambda \nu} \partial_r (u \epsilon_{\mu \nu \alpha \beta} g^{\mu 3} g^{\nu 4}) = 0 \quad \text{for} \quad a = 3, 4. \tag{10} $$

A solution for $u$ in the above equation will result in a force-free electromagnetic field or a trivial case of a vacuum solution where $J = 0$. To simplify the above expression when $a = 3$, define a quantity $M'$ by

$$ M' = g^{34} g^{34} - g^{33} g^{44}. $$

Naturally, despite its notational appearance, $M'$ is not a vector field. Then with the help of equation (21), it is easily seen that

$$ \epsilon^{\alpha \beta \lambda \nu} \epsilon_{\mu \nu \alpha \beta} g^{\mu 3} g^{\nu 4} = -2M'. $$

Also, by taking the derivative of $\sqrt{-g}$ we see that

$$ \partial_r \epsilon^{\alpha \beta \lambda \nu} = -(\partial_r \ln \sqrt{-g}) \epsilon^{\alpha \beta \lambda \nu}. $$

Finally, observing that

$$ \epsilon^{\alpha \beta \lambda \nu} \partial_r (u \epsilon_{\mu \nu \alpha \beta} g^{\mu 3} g^{\nu 4}) = \epsilon^{\alpha \beta \lambda \nu} \epsilon_{\mu \nu \alpha \beta} g^{\mu 3} g^{\nu 4} \partial_r u + u \partial_r (\epsilon^{\alpha \beta \lambda \nu} \epsilon_{\mu \nu \alpha \beta} g^{\mu 3} g^{\nu 4}) $$

$$ - u \epsilon_{\mu \nu \alpha \beta} g^{\mu 3} g^{\nu 4} \partial_r (\epsilon^{\alpha \beta \lambda \nu}) = 0, $$

when $a = 3$, equation (10) reduces to

$$ M^4 \frac{\partial}{\partial x^4} \ln |u| = - \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^4} (\sqrt{-gM'}). $$

In exactly the same way, when $a = 4$, equation (10) reduces to

$$ N^3 \frac{\partial}{\partial x^3} \ln |u| = - \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^4} (\sqrt{-gN'}). $$
where
\[ N^r = g^{33} g^{44} - g^{34} g^{43}. \]
Although \( M^r \) is not a four-vector we will still write
\[ \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^r} (\sqrt{-g} M^r) \]
as
\[ \nabla_r M^r. \]
This strictly for notational convenience. Similar remarks apply to \( N^r \) as well. With this simplification, and setting
\[ M \equiv M^4 = -N^3, \]
we can now write the equations of force-free electrodynamics described in equation (10) as
\[ M \frac{\partial}{\partial x^4} \ln |u| = -\nabla_r M^r \]
and
\[ M \frac{\partial}{\partial x^3} \ln |u| = \nabla_r N^r. \]
We have used the fact that \( u \) is only a function of \( x^3 \) and \( x^4 \) in deriving the above equations. Equations (12) and (13) will serve as our basic equations of FFE for the remainder of the paper.

4. Initial data, local existence and uniqueness theorem

As mentioned earlier, in this section, we will prove two local existence and uniqueness theorem; one for the null case, and the other for when the field is electrically or magnetically dominated. We will then supply examples for both cases. Both examples are previously obtained FFE solutions recast in the adapted chart formalism.

4.1. The null force-free field

When \( M = 0 \) there is a non-trivial solution to the equation
\[ \begin{pmatrix} g^{33} & g^{34} \\ g^{43} & g^{44} \end{pmatrix} \begin{pmatrix} \chi_3 \\ \chi_4 \end{pmatrix} = 0. \]
I.e., there exists a one-form
\[ \chi = \chi_3 \, dx^3 + \chi_4 \, dx^4 \]
such that the vector field \( \chi^i \in \ker F \), meaning
\[ \chi^i = \chi^1 \frac{\partial}{\partial x^1} + \chi^2 \frac{\partial}{\partial x^2}. \]
On the other hand, we can always write $F$ in the form
\begin{equation}
F = \bar{u} (\psi \wedge \chi),
\end{equation}
where $g(\psi, \chi) = 0$ and $\bar{u}$ is a new component function. Then $\iota_\chi F = 0$ implies that
\[ \chi(\chi^\flat) = 0 = \psi(\chi). \]
In this case, $\chi$ is a null vector and $F^2 = 2\bar{u}^2\psi^2\chi^2 = 0$. Consequently, $F$ is a null force-free field. Moreover, since
\[ g(\chi^\flat, W) = \chi(W) = 0 \]
for every $W \in \ker F$, we have that the metric, when restricted to the kernel of $F$, is degenerate as previously discussed.

We know that null force-free solutions exist, for example see [9], (and [10] for its generalizations). So, from physical grounds, it is only expected that there should be well-defined evolution equations in this case. That is almost true. It turns out that if null solutions exist, they come as a class of solutions with an arbitrary function of 2 variables.

**Theorem 1.** Let $\mathcal{F}$ be a two-dimensional foliation of a spacetime with metric $g$. Let \((U_p, \phi_p = (x^1, \ldots, x^4))\) be an adapted chart about any arbitrary point $p$. Suppose $M = 0$ (as defined in equation (11)) in $U_p$. Then $F$ given by equation (7) for any smooth function $u(x^3, x^4)$ is a unique class of force-free solution satisfying equation (6) in $U_p$ if and only if
\[ \nabla_r M' = 0 = \nabla_r N'. \]

**Proof 1.** Since $M = 0$, this is an immediate consequence of equations (12) and (13). $\square$

The above theorem would not have been easy to prove in an arbitrary chart. In the adapted chart, the equations of FFE take on a very simple form. The result however is not limiting in any way. The choice of coordinates is a necessary freedom in general relativity; it is no different from the $3 + 1$ formalism of gravity or even electrodynamics. Notice that we are not solving the initial value problem. The existence of null force-free solutions has simply been identified with certain foliations of spacetime in a definite way. Additionally, suppose a global null force-free solution exists on $\mathcal{M}$. Then clearly a global foliation of $\mathcal{M}$ exists such that locally there are adapted charts wherein $M = 0$, and $M'$ and $N'$ are divergence-free.

To illustrate the previous theorem, we will now describe the foliations generated by a null electromagnetic field denoted by $F_{\text{Null}}$ in Kerr spacetime. This solution was first obtained as an exact solution to the Blandford–Znajek equations. For details regarding its original derivation see [9]. The current density vector, in this case, is proportional to the in-falling principal null geodesic of the Kerr geometry, which in the Boyer–Lindquist coordinates takes the form:
\[ n = \left(\frac{r^2 + a^2}{\Delta}\right) \partial_t - \partial_r + \frac{a}{\Delta} \partial_\phi. \]

We will explicitly construct an adapted coordinate system and show that equations equations (12) and (13) are not violated. The kernel of $F_{\text{Null}}$ is given by (9)
\[ \ker F_{\text{Null}} = \text{span} \{ \Delta n, a \sin^2(\theta) \partial_t + \partial_\phi \}. \]

As per the theorem, the above kernel fixes the entire class of solution. For computational ease we have picked $\Delta n$ as a basis vector for the foliation. Set
\[ X_1 = \Delta n \]
and
\[ X_2 = a \sin^2(\theta) \partial_t + \partial_\phi. \]

Fix \( Q(t, r) \) by the expression
\[ -2 \left( t + r + \frac{r^2 + a^2}{r_+ - r_-} \ln |r - r_+| - \frac{r^2 + a^2}{r_+ - r_-} \ln |r - r_-| \right). \]

The defining properties of \( Q \) are
\[ Q_t = -2 \quad \text{and} \quad Q_r = -2 \left( \frac{r^2 + a^2}{\Delta} \right). \] (15)

Further, define vector fields
\[ X_3 = Q(t, r) \partial_t - \tan(\theta) \partial_\theta \]
and
\[ X_4 = \partial_\phi. \]

It is easily verified that \([X_i, X_j] = 0 \) for \( i, j = 1, \ldots, 4 \). Therefore, there exists an adapted coordinate system \((x^1, \ldots, x^4)\) such that
\[ \frac{\partial}{\partial x^i} = X_i. \]

Since there is no real need, we do not calculate the transformation functions for \( \{x^i\} \) in terms of \((t, r, \theta, \phi)\) explicitly. It is however important to note that by construction, equation (6) is automatically satisfied. The coordinate one-forms transforms as
\[ \begin{pmatrix} dx^1 \\ dx^2 \\ dx^3 \\ dx^4 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{\rho^2} & 0 & 0 \\ \frac{1}{\rho^2 \sin^2(\theta)} & \frac{Q}{\rho^2 \sin^2(\theta)} & 0 & 0 \\ -\frac{1}{\rho^2} & -\rho^2 & 0 & 0 \\ \frac{-\rho^2}{\rho^2 \sin^2(\theta)} & \frac{-Q}{\rho^2 \sin^2(\theta)} & 0 & 1 \end{pmatrix} \begin{pmatrix} dt \\ dr \\ d\theta \\ d\phi \end{pmatrix}. \] (16)

To compute \( \{M^i\} \) and \( \{N^i\} \) in the adapted frame we need to first calculate the components of the inverse metric. This is easily done by noting that
\[ g^{ij} = g(dx^i, dx^j). \]

So we apply the above expressions in equation (16) to equation (22), and for example observe that
\[ g^{11} = g(dx^1, dx^1) = \frac{1}{\rho^2 \Delta}. \]

Proceeding in a similar manner and using the definitions of \( \{M^i\} \) and \( \{N^i\} \) we get that the only non-vanishing components of \( M^i \) and \( N^i \) are
\[ M^1 = -\frac{\cos^2(\theta)}{\sin^2(\theta)} \frac{1}{a \rho^2 \Delta} \]
and
\[ N^1 = \frac{\cos^2(\theta)}{\sin^3(\theta)} \frac{Q}{a^2 \rho^2 \Delta}. \]

In particular, as expected, here \( M = 0 \), and our previous theorem applies. Once the components of the metric are found, its determinant is easily computed as well, and we get that
\[ \sqrt{-g} = a \rho^2 \sin^3(\theta) \Delta \tan(\theta). \]

Then \( M^1 \sqrt{-g} = -\cos(\theta) \), and clearly
\[ \frac{\partial}{\partial \chi^1} (M^1 \sqrt{-g}) = X_1(M^1 \sqrt{-g}) = 0. \]

Therefore \( M^1 \) is divergence free (recall that the term divergence is used only due to the similarity in expression). Since
\[ N^1 \sqrt{-g} = \frac{Q \cos(\theta)}{a \sin^3(\theta)}, \]
we get that
\[ \sqrt{-g} \nabla_r N^1 = X_1(N^1 \sqrt{-g}) \]
\[ = \frac{\cos(\theta)}{a \sin^3(\theta)} \left[ (r^2 + a^2) Q_r - \Delta Q_r \right] = 0 \]
as required. The above equality follows from equation (15). Since \( \{M^1\} \) and \( \{N^1\} \) are divergence free, we have met all the requirements of the theorem. Therefore, equations (7) and (16) imply that
\[ F_{\text{Null}} = u(\theta, x^4) \, d\theta \wedge \left[ dt + \frac{\rho^2}{\Delta} d\rho - a \sin^2(\theta) d\varphi \right] \]
satisfies the force-free equations for any smooth, but otherwise arbitrary \( u(\theta, x^4) \). Here we have rewritten \( u \) as a function of \( \theta \) and \( x^4 \). This is an acceptable trade considering the transformation given by equation (16). We have also made the substitution
\[ \frac{u}{a \tan(\theta) \sin^3(\theta)} \rightarrow u. \]

Since we have not paused to write out the explicit coordinate transformation for the adapted chart, the above solution can seem abstract and not very recognizable. To alleviate this confusion, note that \( u(\theta, x^4) \) is not to be a function of \( x^4 \) and \( x^5 \). In particular, this means that
\[ X_1(u) = 0 = X_2(u). \]

When written in the infalling Kerr–Schild coordinates
\[ F_{\text{Null}} = -u(\theta, x^4) \, d\theta \wedge [d\bar{t} - a \sin^2(\theta) d\bar{\varphi}] = -u(\theta, x^4) \, d\theta \wedge n^5, \]
where \( n^5 \) is the one-form defined by
\[ (n^5)_\mu = g_{\mu\nu} n^\nu, \]
and \( u(\theta, x^4) = u(\bar{t}, \bar{r}, \theta, \bar{\varphi}) \) is subject to the constraints
\[ X_1(u) = -\partial_{\bar{t}} u = 0, \]
\[ X_2(u) = -\partial_{\bar{r}} u = 0, \]
\[ X_3(u) = 0, \]
\[ X_4(u) = 0. \]
and
\[ X_2(u) = a \sin^2 \theta \ u_3 + u_{\varphi} = 0. \]
In fact, the solution \( F\) \(_{\text{Null}}\) as presented above is the \( t \) and \( \varphi \) dependent generalization of the original derivation in [9]. This generalization was first noted in [10]. In our development here it is clear why we must necessarily have this generalization. In [12], the authors constructed a further generalization. It should also be clear why we do not get that generalization here: particular class of solutions depend entirely on the chosen null foliation!

### 4.2. \( F^2 \neq 0 \)

In this subsection we will require that \( M \neq 0 \) in \( U_p \). Further, if
\[
\left( \frac{\nabla_r M'}{M} \right)_{\alpha^3} + \left( \frac{\nabla_r N'}{M} \right)_{\alpha^4} = 0,
\]
we say that \( M' \) and \( N' \) satisfies the smoothness condition. From equations (12) and (13) it is clear that smoothness condition will ensure that
\[ u_{\alpha^3, \alpha^4} = u_{\alpha^4, \alpha^3}. \]
Also, if
\[
\left( \frac{\nabla_r M'}{M} \right)^{\alpha_2} = \left( \frac{\nabla_r N'}{M} \right)^{\alpha_2} = 0 \quad \text{for} \ a = 1, 2,
\]
we say that \( M' \) and \( N' \) satisfies the closed-ness condition; when this happens, from equation (7) we get that \( dF = 0 \).

**Theorem 2.** Let \( \mathcal{F} \) be a two-dimensional foliation of a spacetime with metric \( g \). Let \( (U_p, \phi_p = (x^1, \ldots, x^4)) \) be an adapted, starlike chart centered about any point \( p \). Let \( \mathcal{F}_p \) be the maximal submanifold containing \( p \) in \( \mathcal{F} \), and let \( S \) be the connected slice in \( \mathcal{F}_p \cap U_p \) containing \( p \). Let \( M' \) and \( N' \) be as defined above such that \( M \neq 0 \). Then, there exists a unique (up to an integration constant), non-null, smooth, force-free electromagnetic field \( F \) on \( U_p \) such that \( \ker F \) are given by the integral submanifolds of the foliation if and only if \( M' \) and \( N' \) satisfies the smoothness and closed-ness conditions.

**Proof 2.** As mentioned previously, the adapted chart is such that equation (6) is satisfied. As we have shown above, in \( U_p \), the force-free electromagnetic field \( F \) can be written as a simple two-form given by equation (7). Since \( u \) is only a function of \( x^3 \) and \( x^4 \), clearly the closedness condition must be satisfied, and prescribing \( F|_S \) is simply a matter of picking \( u|_S = u_0 \) for some constant \( u_0 \). From the paragraph above the statement of the theorem, the smoothness condition on \( u \) is unavoidable for the class of solutions we are looking for. Therefore, the ‘only if’ portion of the theorem is proved.

To prove the existence and uniqueness of a solution, on \( U_p \), define a one-form \( \omega \) by
\[
\omega = \frac{\nabla_r N'}{M} \ dx^3 - \frac{\nabla_r M'}{M} \ dx^4.
\]
The smoothness condition gives that \( d\omega = 0 \). The Poincare’ lemma then implies that on \( U_p \), there exists a function \( \tilde{u} \) such that
\[ \omega = d\tilde{u}. \]
The Poincare’ lemma also gives a formula for the construction of the potential function as well, and it is given by

\[
\tilde{u} = \int_0^1 \left[ \frac{\nabla r N^r}{M} (tx^3, tx^4) x^3 - \frac{\nabla r M^r}{M} (tx^3, tx^4) x^4 \right] dt.
\]

In the above integral, the integrands are evaluated along \((tx^3, tx^4)\) for \(t \in [0, 1]\). Since \(U_p\) is starlike the integrands are well defined. Then

\[
\partial_3 \tilde{u} = \int_0^1 \left( \frac{\nabla r N^r}{M} \right)_3 (tx^3, tx^4) x^3 t \, dt + \int_0^1 \frac{\nabla r N^r}{M} (tx^3, tx^4) \, dt \\
- \int_0^1 \left( \frac{\nabla r M^r}{M} \right)_3 (tx^3, tx^4) x^4 t \, dt \\
= \int_0^1 \left( \frac{\nabla r N^r}{M} \right)_3 (tx^3, tx^4) x^3 t \, dt + \int_0^1 \frac{\nabla r N^r}{M} (tx^3, tx^4) \, dt \\
+ \int_0^1 \left( \frac{\nabla r N^r}{M} (tx^3, tx^4) \right)_4 x^4 t \, dt.
\]

The last term in the right-hand side above was modified using the smoothness condition. Therefore

\[
\partial_3 \tilde{u} = \int_0^1 \frac{d}{dt} \left( \frac{\nabla r N^r}{M} \right) dt = \frac{\nabla r N^r}{M}.
\]

After a similar calculation of \(\partial_4 \tilde{u}\), we see that \(d(\ln u) = d\tilde{u}\). This can be solved to give

\[
u = u_0 \exp \tilde{u}, \tag{17}
\]

which satisfies all the requirements of the theorem. Now suppose \(u_1\) and \(u_2\) are two solutions to force free equations on \(U_p\) that agree on \(S\), then \(d(u_1 - u_2) = d\tilde{u}_1 - d\tilde{u}_2 = 0\). I.e., \(u_1 = u_2 + c\), where \(c\) is a constant which must vanish since the two solutions agree on \(S\). \(\square\)

Notice what the theorem enables us to do: in the non-null case, albeit locally, the existence of force-free solutions is directly dependent on the existence of foliations where the components of the metric tensor satisfy a prescribed set of properties. Additionally, suppose a global non-null force-free solution exists on \(\mathcal{M}\). Then clearly a global foliation of \(\mathcal{M}\) exists such that locally there are adapted charts wherein in the smoothness and the closed-ness condition hold. The search for force-free solutions can now be a topic of study for geometers as well.

In a recent paper \([11]\), we found an exact solution to the force-free magnetosphere of a Kerr black hole. The solution was not globally well behaved. Nonetheless, it will be instructive to reconstruct the solution using a coordinate system that is adapted to the foliation generated by the kernel of \(F\).

In the present formalism, since the foliations take the primary role, let us begin by describing the Kernel of \(F\). The solution is magnetically dominated, and this implies that \(M \neq 0\), and the above formalism applies. We will denote this solution as \(F_{Mag}\). Define vector fields in the Boyer–Lindquist coordinates of the Kerr geometry by

\[
X_1 = (r^2 + a^2) \partial_t + a \partial_\varphi,
\]
and
\[ X_2 = a \sin^2(\theta) \partial_t + L \sin(\theta) \partial_\theta + \partial_\varphi, \]
where \( L = L(r) \) is of the form
\[ L = \frac{f}{\sqrt{C^2 - f^2}}, \]
where \( f \) is an arbitrary function of \( r \), and \( C \) is an integration constant. Then ([11])
\[ \ker F_{Mag} = \text{span}\{X_1, X_2\}. \]

Since our goal is to illustrate the method by which we construct a foliation adapted coordinate system, for computational ease, we set \( L \) (and hence \( f \)) as a constant. This is tantamount to the vacuum case since in the \( F_{Mag} \) solution the current vector \( J \) is given by (as shown in ([11]))
\[ J_{Mag} = \frac{f_r}{\rho^2 \sin^2(\theta)} X_2. \]

When \( L, f \) is a constant, define vector fields
\[ X_3 = P(t, r) \partial_t + (r^2 + a^2) \partial_r \]
where
\[ P(t, r) = \frac{2ra}{L} \cos(\theta) + 2rt, \]
and for consistency and efficiency of notation, set
\[ X_4 = \partial_\varphi. \]

It is easy to verify that \([X_i, X_j] = 0\) for \( i, j = 1, \ldots, 4 \). I.e., just as in the previous example, there exists an adapted coordinate system \((x^1, \ldots, x^4)\) such that
\[ \frac{\partial}{\partial x^i} = X_i. \] (18)

Here the bases one-forms are given by
\[
\begin{pmatrix}
\frac{dx^1}{dx^3} \\
\frac{dx^2}{dx^3} \\
\frac{dx^3}{dx^3} \\
\frac{dx^4}{dx^3}
\end{pmatrix} = \begin{pmatrix}
1 \\
0 \\
0 \\
-1
\end{pmatrix} \begin{pmatrix}
1 & -P & -a \sin(\theta) & 0 \\
\frac{-P}{r^2 + a^2} & L & \frac{L}{r^2 + a^2} & 0 \\
0 & \frac{aP}{r^2 + a^2} & -\rho^2 & 0 \\
-\frac{a}{r^2 + a^2} & \frac{L}{\sin(\theta)} & \frac{L}{\sin(\theta)} & r^2 + a^2
\end{pmatrix} \begin{pmatrix}
dr \\
d\theta \\
d\varphi \\
d\varphi
\end{pmatrix}.
\]

In this chart,
\[ \sqrt{-g} = \rho^2 \sin^2(\theta)(r^2 + a^2)^2 L. \]

We already know that \( F_{Mag} \) must take the form
\[ F_{Mag} = u(x^3, x^4) \ dx^3 \wedge dx^4. \]
\[
\frac{dr}{r^2 + a^2} - \frac{\rho^2}{(r^2 + a^2)L} \sin(\theta) d\theta + dl \phi = u \left[ \frac{-a}{r^2 + a^2} + \left( \frac{\rho^2}{(r^2 + a^2)L} \sin(\theta) d\theta + dl \phi \right) \right].
\]

To determine the governing equations for \(u\) we need to compute quantities \(\{M_r\}\), and \(\{N_r\}\) as defined in the previous section. To impose equation (12), we begin by calculating the components of \(\{M_r\}\):

\[
M^1 = \frac{(L^2 + 1)}{L^2} \frac{a\Delta}{\rho^2(r^2 + a^2)^3},
\]

\[
M^2 = \frac{1}{L^2 \rho^2 \sin^2(\theta) (r^2 + a^2)^3},
\]

and

\[
M = M^4 = \frac{(L^2 + 1)}{L^2} \frac{\Delta}{\sin^2(\theta)(r^2 + a^2)^3}.
\]

Then from equation (18) we get that

\[
\nabla_r M = \frac{1}{\sqrt{-g}} \left[ X_1(M^1 \sqrt{-g}) + X_2(M^2 \sqrt{-g}) + X_4(M^4 \sqrt{-g}) \right] = 0.
\]

From equation (12) and definition of \(X_4\) we must then have that \(u, \phi = 0\). To impose equation (13), we perform a similar calculation using the components of \(\{N_r\}\). Here,

\[
N^1 = \frac{(L^2 + 1)}{L^2} \frac{P\Delta}{\rho^2(r^2 + a^2)^3 \sin^2(\theta)},
\]

\[
N^2 = \frac{aP\Delta}{\rho^2(r^2 + a^2)^3 L^2 \rho^2 \sin^2(\theta)},
\]

and of course \(N^3 = -M\). Just as above, a careful calculation of \(\nabla_r N = \) yields that

\[
\frac{1}{M} \nabla_r N = -\frac{2}{\Delta} \left[ (r^2 + a^2)(r - M) - 2r\Delta \right].
\]

Since \(u, \phi = 0\), and

\[
\frac{\partial u}{\partial x^1} = 0 = X_1(u),
\]

we have that \(u_t = 0\). Therefore,

\[
\frac{\partial u}{\partial x^3} = X_3(u) = (r^2 + a^2)u_r.
\]

Equation (13) then requires that

\[
\frac{r^2 + a^2}{u} \frac{\partial u}{\partial x} = -\frac{2}{\Delta} \left[ (r^2 + a^2)(r - M) - 2r\Delta \right].
\]

This is easily integrated to give that

\[
u = u_0 \frac{(r^2 + a^2)^2}{\Delta}.
\]
where $u_0$ is an integration constant. Inserting the above expression into equation (19), and as shown in [11], we get that

$$F_{\text{Mag}} = \frac{u_0}{A^2} dr \wedge \left[ a \ dt + \frac{r^2}{L \sin(\theta)} d\theta - (r^2 + a^2) d\phi \right]$$

is a vacuum solution in Kerr geometry. Incidentally, since $F_{\text{Mag}}$ is a vacuum solution, its Hodge-Star dual, denoted by $\tilde{F}_{\text{Mag}}$ is also a vacuum solution, and is given by

$$\tilde{F}_{\text{Mag}} = \frac{u_0}{\sin(\theta)} d\theta \wedge \left[ a \sin^2(\theta) d\phi - dt \right] + \frac{dt}{L} \wedge d\phi.$$

### 4.3. Foliation by commuting Killing vector fields

In the previous theorem, one type of stumbling block arises when $\det g$ and the components $\{M^a\}$ and $\{N^a\}$ are functions of $x^1$ and $x^2$. This is because the only variable $u$ that we have is not dependent on the first two coordinates of the adapted chart. But in the event, we are spared of the dependence on coordinates $x^1$ and $x^2$; in the following theorem we will show that the smoothness condition is automatically satisfied. Moreover, in this case, we get an explicit expression of the electromagnetic field.

**Theorem 3.** Suppose, in an adapted coordinate chart

- $M \neq 0$,
- $\partial_a \det g = 0 = \partial_a M$ for $a = 1, 2$,
- $\partial_1 M^1 + \partial_2 M^2 = 0$, and
- $\partial_1 N^1 + \partial_2 N^2 = 0$.

Then there exists a smooth, force-free electromagnetic field $F$ on $U_p$ such that $\ker F$ are given by the integral submanifolds of the foliation. As before, the solution is unique whenever $F|_S$ is prescribed and is given by

$$F = \frac{q}{M \sqrt{-g}} dx^3 \wedge dx^4,$$

for some constant $q$.

**Proof 3.** The conditions of the theorem imply that

$$\nabla_a M' = \partial_a M + M \partial_4 \sqrt{-g}$$

and

$$\nabla_a N' = -\partial_a M - M \partial_3 \sqrt{-g}.$$  

Since we have already required the non-null condition, it is now a trivial matter to check the gauge and smoothness conditions. Finally, note that

$$d \ln u = -d \ln(M \sqrt{-g}).$$

This is easily integrated to give the expression stated in the theorem. □

This theorem clarifies an important point. Note that in general, one cannot expect a non-full force-free field to smoothly become null. In the limit that $M \to 0$, $F$ becomes undefined. So,
in reality, the force-free condition must break down. If there are indeed cases where a smooth limit occurs, other restricting conditions have to be met to allow this smooth transition. This is why our general formalism treats the null case separately. As is clear from equation (20), we are able to formulate a singularity theorem of FFE in the following way.

**Corollary 1.** When the requirements of theorem 3 holds, in the limiting case, the solution to FFE becomes singular as the electromagnetic field becomes null.

The conditions for the theorem above are not easily met in general, and there is no direct way to recognize them from the start save in the case when spacetime admits 2 commuting Killing vector fields. But, when this happens, there is an easy expression for at least 1 force-free solution when \( M \neq 0 \). The following corollary is an immediate consequence of theorem 3 above.

**Corollary 2.** Let \((\mathcal{M}, g)\) admit two commuting Killing vector fields \( X_1 \) and \( X_2 \). Then in the adapted coordinate system, where \( \partial_a = X_a \) for \( a = 1, 2 \), if \( M \neq 0 \), there exists a non-null force-free electromagnetic solution given by equation (20).

### 4.4. Vacuum solutions

In the adapted coordinate system, we already have that \( (J^*)_a = 0 \) for \( a = 3, 4 \). In the event

\[
g^{\mu 3} g^{\nu 4} = 0
\]

whenever \( \mu, \nu \neq 3, 4 \), in equation (9), \( \alpha, \beta = 1, 2 \). In this case, we have that \( J = 0 \), and our formalism reduces to the case of vacuum solutions. A simple example in Kerr spacetime will illustrate the, albeit limited, power of equation (20).

Here \( \partial_0 \) and \( \partial_\varphi \) are Killing vector fields, and so we set \( \{x^1 = t, x^2 = \varphi, x^3 = r, x^4 = \theta\} \) for our adapted coordinate system. I.e., we are simply using the Boyer–Lindquist coordinate system. In this case

\[
M = -\frac{\Delta}{\rho^2},
\]

and so from equation (20)

\[
F_{KV} = q \frac{\rho^2}{\Delta \sin(\theta)} \, dr \wedge d\theta
\]

is easily verified to be a vacuum solution. Here, \( q \) is the integration constant, and KV stands for Kerr vacuum. The above expression for \( F_{KV} \) is exactly the same in appearance in the horizon penetrating Kerr–Schild coordinate system as well. As expected, \( F_{KV} \) is undefined at the event horizon, given by \( \Delta = 0 \), even though the metric is not singular in the Kerr–Schild coordinate system. In our case the solution is necessarily singular due to our singularity theorem (corollary 1). The metric, when restricted to the kernel of \( F_{KV} \), becomes degenerate and hence the solution approaches the null limit. The null Killing vector

\[
(r^2 + a^2) \, \partial_t + a \, \partial_\varphi
\]

has a vanishing inner product with every tangent vector of the kernel.

Notice that \( F_{KV} \) is one of the terms in \( F_{Mag} \). But, when treated as separate solutions, their individual kernels are entirely different distributions. As mentioned before, since \( F_{KV} \) is a
vacuum solution, its Hodge-Star dual given by

\[ \tilde{F}_{KV} = q \, dt \wedge d\varphi \]

is also a vacuum solution in Kerr geometry.

5. Discussion and conclusion

We have made great progress in understanding the connection between FFE and foliations. The initial data surfaces take the form of two-dimensional submanifolds whose tangent space agrees with the kernel of the degenerate electromagnetic field. One is not free to pick the submanifolds. Indeed, the entire difficulty in the theory of FFE is in finding the appropriate foliation of spacetime with suitable submanifolds. Once we find the foliation, a solution is guaranteed.

When \( F^2 \neq 0 \), the solution is unique modulo an integration constant (which is the only choice in local initial data). In the null case, we get a class of solutions depending on 2 different parameters (coordinates \( x^3 \) and \( x^4 \)) of the theory.

Further, we are able to explain why the general theory of FFE without any further restrictions separates into the null case and the non-null case. This is because smooth transitions are not generally allowed. In a certain class of solutions, one that is generated by a pair of commuting Killing vector fields, solutions necessarily become undefined as one approaches the null limit.

It is important to mention that our formalism includes the Blandford–Znajek mechanism. In addition to the worked-out examples, any new solution to the Blandford–Znajek mechanism must satisfy equations (12) and (13). This is not surprising since the equations governing \( u \) are derived from the fully covariant force-free equations of electrodynamics in curved spacetime. Unlike the Blandford–Znajek mechanics however, this paper does not place any further restrictions like stationarity or axis-symmetry, nor do we fix the background metric. Also, since the exteriors of black holes are covered by a single coordinate chart, new numerical recipes can be constructed to search for foliations with the required properties. Once the foliations are found, one can retroactively integrate to find closed-form solutions given by equation (7) in the null case or equation (17) otherwise.

This work expects a follow up in several directions. The connection between FFE and foliations has been tackled by breaking covariance. I.e., we have heavily relied on the adapted coordinate system. The recasting of the theory in completely geometric terms should now be tractable.

In the magnetically dominated case, the work by Komissarov [18], and Carrasco and Reula [15] suggests that indeed such foliations exists. The hyperbolic nature of the governing equations guarantees the existence of solutions and hence foliations as prescribed in this paper. It will be insightful to understand how the initial value problem in the magnetically dominated case implies the existence of foliations satisfying the smoothness and closed-ness conditions used in theorem 2.

This paper also develops a well-defined evolution equation for the electrically dominated case. This suggests that the equations of FFE in the electrically dominated case leads to unique solutions, and is a topic for further study.

Finally, the issue of a smooth transitions between solutions describing null and non-null fields have to be studied in greater detail. It is not clear whether the singularity theorem we have proved can be extended to a larger class of solutions, and is certainly a topic of future research.
Appendix A

A.1. Exterior calculus

In this section, we simply list the relevant formulae of exterior calculus we have used throughout the paper. Let a differential form \( \omega \) be given by

\[
\omega = \omega_{i_1 \ldots i_k} \, dx^{i_1} \otimes \ldots \otimes dx^{i_k},
\]

where the component functions \( \omega_{i_1 \ldots i_k} \) are completely antisymmetric, then

\[
\omega = \frac{1}{k!} \omega_{i_1 \ldots i_k} \, dx^{i_1} \wedge \ldots \wedge dx^{i_k}.
\]

The exterior derivative of \( \omega \) is defined by the expression

\[
d\omega = \frac{1}{k!} \omega_{i_1 \ldots i_k} \, dx^i \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}.
\]

In an \( n \) dimensional manifold, the Hodge-Star operator * takes a \( k \) form to an \( n-k \) form. It is defined by the formula

\[
*\omega = \frac{1}{k!(n-k)!} (\ast \omega)_{i_1 \ldots i_{n-k}} \, dx^{i_1} \wedge \ldots \wedge dx^{i_{n-k}},
\]

where

\[
(\ast \omega)_{i_1 \ldots i_{n-k}} = \epsilon_{j_1 \ldots j_k i_{n-k}} \omega^{j_1 \ldots j_k}.
\]

The components of \( \omega \) are raised as usual by the inverse metric tensor. I.e.,

\[
\omega^{i_1 \ldots k} = g^{i_1 j_1} \ldots g^{i_k j_k} \omega_{j_1 \ldots j_k}.
\]

The Poincare’ lemma tells us that all closed forms are locally exact: let \( U \) be an open, starlike set about a point \( p \) in the manifold. Let \( \omega \) be a \( k \) form on \( U \) such that \( d\omega = 0 \). Then there is a \( k-1 \) form \( \alpha \) on \( U \) such that

\[
\omega = d\alpha.
\]

Contractions of the Levi-Civita tensor can be taken using the formula

\[
\epsilon^{a_1 a_2 \ldots a_{j+1} \ldots a_n} \epsilon_{a_1 a_2 \ldots a_j b_{j+1} \ldots b_n} = (-1)^s (n-j)! \, \delta_{b_{j+1}}^{a_{j+1}} \ldots \delta_{b_n}^{a_n}.
\]

(21)

Here \( s \) is the index of the metric. In the case of general relativity, for our choice of signature, \( s = 1 \). The square brackets indicate that the indices in between have to be summed in an antisymmetric fashion.

Cartan’s magic formula for differential forms is given by

\[
\mathcal{L}_v F = di_v F + i_v dF.
\]

Here \( \mathcal{L}_v F \) is the Lie derivative of \( F \) with respect to \( v \).
A.2. Useful results from differential geometry

Details of proofs of all the results in this section can be found in [19].

Let $\mathcal{M}$ be a smooth manifold of dimension $m$. Let $d < m$, $\mathcal{D}(p)$ is a $d$-dimensional subspace of $\mathcal{T}_p(\mathcal{M})$. $\mathcal{D}$ is a smooth distribution provided $\mathcal{D}$ is locally spanned by smooth vector fields. $\mathcal{D}$ is involutive (or completely integrable) if for any smooth vector fields $X$ and $Y$ that lie in $\mathcal{D}$, $[X, Y]$ lies in $\mathcal{D}$. A submanifold $\mathcal{N}$ of $\mathcal{M}$ is an integral manifold of a distribution $\mathcal{D}$ if the tangent space of every $p \in \mathcal{N}$ is given by $\mathcal{D}(p)$. If $\mathcal{D}$ is a smooth distribution in $\mathcal{M}$ such that there is an integral manifold passing through every point of $\mathcal{M}$, then $\mathcal{D}$ is clearly involutive.

The following powerful theorem by Frobenius proves the converse of the previous statement: let $\mathcal{D}$ be a $d$-dimensional involutive distribution in $\mathcal{M}$. Let $p \in \mathcal{M}$. There exists a weakly embedded integral manifold of $\mathcal{D}$ through $p$. Additionally, there exists a cubic coordinate system $(U, \phi)$ centered at $p$ and

$$\phi = (x^1, \ldots, x^m)$$

such that $x^i = \text{constant}$ for $i = d + 1, \ldots, m$ are integral manifolds of $\mathcal{D}$.

This final result will help us identify useful, and in our case adapted coordinate functions. In an $m$-dimensional manifold, let $X_1, \ldots, X_k, k \leq m$ be smooth, point wise linearly independent, commuting fields on an open set about some $p \in \mathcal{M}$. Then there exists a coordinate a chart $\{x^i\}_{i=1}^m$ about $p$ such that for $i = 1, \ldots, k$, locally

$$X_i = \frac{\partial}{\partial x^i}.$$

Finally, let $\mathcal{F}$ be a collection of submanifolds $\{\mathcal{F}\}_\alpha$ of fixed dimension. Let $\mathcal{F}_\alpha \cap \mathcal{F}_\beta = \emptyset$ for $\alpha \neq \beta$ such that $\bigcup_\alpha \mathcal{F}_\alpha$ is the entire manifold $\mathcal{M}$, then we say that $\mathcal{F}$ is a foliation of $\mathcal{M}$, and $\mathcal{F}_\alpha$ are the leaves of the foliation.

A.3. Kerr geometry in Boyer–Lindquist/outgoing Kerr–Schild coordinates

From the analysis in the main body of the paper, it is clear that we need the contravariant form of the Kerr metric. In the Boyer–Lindquist coordinate system, $(t, r, \theta, \phi)$, this takes the following form:

$$g^{\mu\nu} = \begin{bmatrix}
-\frac{\Sigma^2}{\rho^2 \Delta} & 0 & 0 & -\frac{\alpha \Sigma}{\Delta} \\
0 & \frac{\Delta}{\rho^2} & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\frac{\alpha \Sigma}{\Delta} & 0 & 0 & \frac{\Delta - \alpha^2 \sin^2(\theta)}{\rho^2 \Delta \sin^2(\theta)}
\end{bmatrix}. \quad (22)$$

Here,

$$\rho^2 = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 - 2Mr + a^2,$$

$$\Sigma^2 = (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta, \quad z = \frac{2M}{\rho^2}.$$
and
\[ \sqrt{-g} = \rho^2 \sin \theta. \]
Here \( r = r_+ = M + \sqrt{M^2 - a^2} \) locates the outer event horizon.

The infalling Kerr–Schild coordinates are \((\bar{t}, r, \theta, \bar{\varphi})\). They are related to the Boyer–Lindquist coordinates by the following relations:
\[
d\bar{t} = dr + \frac{r^2 + a^2}{\Delta} dr, \quad d\bar{\varphi} = d\varphi + \frac{a}{\Delta} dr.
\]
In the Kerr–Schild outgoing coordinates, the metric components in the basis \(\{\bar{t}, r, \theta, \bar{\varphi}\}\) become
\[
\bar{g}_{\mu\nu} = \begin{bmatrix}
z - 1 & 1 & 0 & -za \sin^2 \theta \\
1 & 0 & 0 & -a \sin^2 \theta \\
0 & 0 & \rho^2 & 0 \\
-za \sin^2 \theta & -a \sin^2 \theta & 0 & \Sigma^2 \sin^2 \theta / \rho^2
\end{bmatrix}.
\] (23)

Appendix B. An example in FRW cosmology

To further illustrate the computational merits of the formalism described above, we include a new solution to the Maxwell field in a Friedman–Robertson–Walker spacetime (as far as the author is aware). For concreteness, we will pick the choice where the sectional curvature \(k\) is set to 1. In the hyper-spherical coordinate system the metric, in this case, takes the form
\[
g = -dt^2 + a^2(t)dr^2 + a^2(t) \sin^2 r d\Omega^2.
\]
Here \(d\Omega^2\) is the metric of a unit two-sphere. We do not place any restrictions on the matter content of this Universe and consequently \(a(t)\) is not fixed. To fix the foliation, let us pick an adapted frame given by
\[ X_1 = \partial_\theta, \quad X_2 = \partial_\varphi, \quad X_3 = \partial_t, \quad \text{and} \quad X_4 = \partial_r. \]
We shall denote the resulting solution as \(F_{\text{FRW}}\). By this choice, we are setting
\[ \ker F_{\text{FRW}} = \text{span}\{\partial_\theta, \partial_\varphi\}. \]
Clearly, \(\ker F_{\text{FRW}}\) is not degenerate. Further since it is spacelike, we will see that \(F_{\text{FRW}}\) is electrically dominated. Also, since the metric is diagonal in the adapted coordinate, as previously mentioned, if a solution exists, it is guaranteed to be a vacuum solution. Following equation (7), we shall tentatively write
\[ F_{\text{FRW}} = u(t, r) dt \wedge dr. \]
A direct calculation shows that the only non-trivial components of \(M^r\) and \(N^r\) are given by
\[ M^4 = \frac{1}{a^2} = -N^3. \]
Force-free equations (12) and (13) reduce to the tractable form given by
\[ \partial_t \ln |u| = -\partial_r \ln \sin^2 r \]
and
\[ \partial_t \ln |u| = -\partial_t \ln |a|. \]

The above two equations are easily integrated to give
\[ F_{\text{FRW}} = \frac{u_0}{a(t) \sin^2 r} \, dt \wedge dr. \]

It is easy to check that this solution satisfies Maxwell’s vacuum equations. Additionally, the dual, magnetically dominated solution in this case is given by
\[ \tilde{F}_{\text{FRW}} = *F_{\text{FRW}} = -u_0 a(t) \sin^2 \theta \, d\theta \wedge d\phi. \]

In both the cases, \( u_0 \) is the usual integration constant.

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