LEIBNIZ’S INFINITESIMALS: THEIR FICTIONALITY, THEIR MODERN IMPLEMENTATIONS, AND THEIR FOES FROM BERKELEY TO RUSSELL AND BEYOND

MIKHAIL G. KATZ AND DAVID SHERRY

Abstract. Many historians of the calculus deny significant continuity between infinitesimal calculus of the 17th century and 20th century developments such as Robinson’s theory. Robinson’s hyperreals, while providing a consistent theory of infinitesimals, require the resources of modern logic; thus many commentators are comfortable denying a historical continuity. A notable exception is Robinson himself, whose identification with the Leibnizian tradition inspired Lakatos, Laugwitz, and others to consider the history of the infinitesimal in a more favorable light. Inspite of his Leibnizian sympathies, Robinson regards Berkeley’s criticisms of the infinitesimal calculus as aptly demonstrating the inconsistency of reasoning with historical infinitesimal magnitudes. We argue that Robinson, among others, overestimates the force of Berkeley’s criticisms, by underestimating the mathematical and philosophical resources available to Leibniz. Leibniz’s infinitesimals are fictions, not logical fictions, as Ishiguro proposed, but rather pure fictions, like imaginaries, which are not eliminable by some syncategorematic paraphrase. We argue that Leibniz’s defense of infinitesimals is more firmly grounded than Berkeley’s criticism thereof. We show, moreover, that Leibniz’s system for differential calculus was free of logical fallacies. Our argument strengthens the conception of modern infinitesimals as a development of Leibniz’s strategy of relating inassignable to assignable quantities by means of his transcendental law of homogeneity.

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2000 Mathematics Subject Classification. Primary 26E35; Secondary 01A85, 03A05.

Key words and phrases. Berkeley; continuum; infinitesimal; law of continuity; law of homogeneity; Leibniz; Robinson; Stevin; The Analyst.
**1. Introduction**

Many historians of the calculus deny any significant continuity between infinitesimal analysis of the 17th century and non-standard analysis of the 20th century, e.g., the work of Robinson. While Robinson’s non-standard analysis constitutes a consistent theory of infinitesimals, it requires the resources of modern logic\(^1\) thus many commentators are comfortable denying a historical continuity:

> [T]here is . . . no evidence that Leibniz anticipated the techniques (much less the theoretical underpinnings) of modern non-standard analysis (Earman 1975, \([32]\ p.\ 250])

The relevance of current accounts of the infinitesimal to issues in the seventeenth and eighteenth centuries is rather minimal (Jesseph 1993, \([57]\ p.\ 131])

Robinson himself takes a markedly different position. Owing to his formalist attitude toward mathematics, Robinson sees Leibniz as a kindred soul:

Leibniz’s approach is akin to Hilbert’s original formalism, for Leibniz, like Hilbert, regarded infinitary entities as ideal, or fictitious, additions to concrete mathematics. (Robinson 1967, \([127]\ p.\ 40]).

The Hilbert connection is similarly reiterated by Jesseph in a recent text\(^2\) Like Leibniz, Robinson denies that infinitary entities are real, yet he promotes the development of mathematics by means of infinitary concepts \([128]\ p.\ 45], [126] p. 282]. Leibniz’s was a remarkably modern insight that mathematical expressions need not have a referent, empirical or otherwise, in order to be meaningful. The fictional nature of infinitesimals was stressed by Leibniz in 1706 in the following terms:

> “Axiom. No reasoning about things whereof we have no idea. Therefore no reasoning about Infinitesimals.”

G. Berkeley, *Philosophical commentaries* (no. 354).

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\(^1\)It is often claimed that the hyperreals require the resources of model theory. See Appendix \([A]\) for a more nuanced view.

\(^2\)See main text at footnote \([7]\)
Figure 1. Leibniz’s law of continuity (LC) takes one from assignable to inassignable quantities, while his transcendental law of homogeneity (TLH) returns one to assignable quantities.

Philosophically speaking, I no more admit magnitudes infinitely small than infinitely great . . . I take both for mental fictions, as more convenient ways of speaking, and adapted to calculation, just like imaginary roots are in algebra. (Leibniz to Des Bosses, 11 March 1706; in Gerhardt [46, II, p. 305])

We shall argue that Leibniz’s system for the calculus was free of contradiction, and incorporated versatile heuristic principles such as the law of continuity and the transcendental law of homogeneity (see Figure 1) which were, in the fullness of time, amenable to mathematical implementation as general principles governing the manipulation of infinitesimal and infinitely large quantities. And we shall be particularly concerned to undermine the view that Berkeley’s objections to the infinitesimal calculus were so decisive that an entirely different approach to infinitesimals was required.

Jesseph suggests an explanation for the irrelevance of modern infinitesimals to issues in the seventeenth and eighteenth centuries:

The mathematicians of the seventeenth and eighteenth centuries who spoke of taking “infinitely small” quantities in the course of solving problems often left the central concept unanalyzed and largely bereft of theoretical justification (Jesseph [57, p. 131] in 1993).

According to Jesseph’s reading, the earlier approach to infinitesimals was a conceptual dead-end, and a consistent theory of infinitesimals required a fresh start. The force of this claim depends, of course, on how one understands conceptual analysis and, especially, theoretical justification. We argue that Leibniz’s defense of the infinitesimal calculus - both philosophical and mathematical - guided his successors toward an infinitesimal analysis that is rigorous by today’s standards. In order to make this case we shall show that Berkeley’s allegations of inconsistency in the calculus stem from philosophical presuppositions which are neither necessary nor desirable from Leibniz’s perspective.
2. PRELIMINARY DEVELOPMENTS

A distinction between indivisibles and infinitesimals is useful in discussing Leibniz, his intellectual successors, and Berkeley.

The term *infinitesimal* was employed by Leibniz in 1673 (see [102, series 7, vol. 4, no. 27]). Some scholars have claimed that Leibniz was the first to coin the term (e.g., Probst 2008, [124, p. 103]). However, Leibniz himself, in a letter to Wallis dated 30 March 1699, attributes the term to Mercator:

> for the calculus it is useful to imagine infinitely small quantities, or, as Nicolaus Mercator called them, infinitesimals (Leibniz [94, p. 63]).

Commentators use the term “infinitesimal” to refer to a variety of conceptions of the infinitely small, but the variety is not always acknowledged. Boyer, a mathematician and well-known historian of the calculus writes,

> In the seventeenth century, however, the infinitesimal and kinematic methods of Archimedes were made the basis of the differential and the fluxionary forms of the calculus (Boyer [20, p. 59]).

This observation is not quite correct. Archimedes’ kinematic method is arguably the forerunner of Newton’s fluxional calculus, but his infinitesimal methods are less arguably the forerunner of Leibniz’s differential calculus. Archimedes’ infinitesimal method employs *indivisibles*. For example, in his heuristic proof that the area of a parabolic segment is $4/3$ the area of the inscribed triangle with the same base and vertex, he imagines both figures to consist of perpendiculars of various heights erected on the base (ibid., 49-50). The perpendiculars are indivisibles in the sense that they are limits of division and so one dimension less than the area. *Qua* areas, they are not divisible, even if, *qua* lines they are divisible. In the same sense, the indivisibles of which a line consists are points, and the indivisibles of which a solid consists are planes. We will discuss the term “consist of” shortly.

Leibniz’s infinitesimals are not indivisibles, for they have the same dimension as the figures that consist of them. Thus, he treats curves as composed of infinitesimal lines rather than indivisible points. Likewise, the infinitesimal parts of a plane figure are parallelograms. The strategy of treating infinitesimals as dimensionally homogeneous with the objects they compose seems to have originated with Roberval or Torricelli, Cavalieri’s student, and to have been explicitly arithmetized by Wallis (Beeley 2008, [9, p. 36ff]).
Infinitesimals in this sense occur already in Democritus, who, Edwards surmises, imagined a cone to consist of infinitesimal triangular pyramids in order to deduce that its volume is \( 1/3 \) base times height (Edwards 1979, [33, p. 9]). Democritus probably used indivisibles too, in arguing that pyramids with equal heights and bases of equal (but not necessarily congruent) area have the same volume. In that case Democritus treated a pyramid as consisting of plane sections, parallel to its base.

Plutarch reports that Democritus raised a puzzle in connection with treating a cone as consisting of circular sections:

If a cone is cut by surfaces parallel to the base, then how are the sections equal or unequal? If they were unequal then the cone would have the shape of a staircase; but if they were equal, then all sections will be equal, and the cone will look like a cylinder, made up of equal circles; but this is entirely nonsensical (Plutarch quoted in Edwards 1979, [33, p. 8-9]).

This puzzle need not arise for infinitesimals of the same dimension, with an infinitesimal viewed as a frustum of a cone rather than a plane section. Zeno raised a similar but more general puzzle in connection with treating any continuous magnitude as though it consists of infinitely many indivisibles. His metrical paradox proposes a dilemma: If the indivisibles have no magnitude, then a figure which consists of them has no magnitude; but if the indivisibles have some (finite) magnitude, then a figure which consists of them will be infinite. Zeno’s paradox is, of course, a puzzle for the idea that a finite magnitude consists of indivisibles. There is a further puzzle for the idea that a magnitude consists of indivisibles. If a magnitude consists of indivisibles, then we ought to be able to add or concatenate them in order to produce or increase a magnitude. But indivisibles are not next to one another; as limits or boundaries, any pair of indivisibles is separated by what they limit. Thus, the concepts of addition or concatenation seem not to apply to indivisibles.

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3Edwards also surmises that Democritus saw that a triangular pyramid could be completed to form a triangular prism with the same base and height by adding two more prisms, each with the same base and height (ibid.).

4Kepler also used both infinitesimals of the same dimension, treating a circle, e.g., as consisting of infinitesimal triangles, and indivisibles, treating an ellipse, e.g., as consisting of its radii (Boyer 1959, [21, p. 108-9]).

5Difficulties of this sort are what lead thinkers to conceive of continuous magnitudes kinematically.
The paradox may not apply to infinitesimals in Leibniz’s sense, however. For, having neither zero nor finite magnitude, infinitely many of them may be just what is needed to produce a finite magnitude. And in any case, the addition or concatenation of infinitesimals (of the same dimension) is no more difficult to conceive of than adding or concatenating finite magnitudes. This is especially important, because it allows one to represent infinitesimals by means of numbers and so apply arithmetic operations to them. This is the fundamental difference between the infinitary methods of Archimedes (and later Cavalieri) and the infinitary methods of Leibniz and his followers.

The distinction of indivisible from infinitesimal in Leibniz’s sense is not a difficult one. Boyer distinguishes proofs by infinitesimal elements from proofs by indivisibles in various places (e.g., p. 109). But a failure to keep the distinction before one’s mind is a source of misleading claims about the 17th century calculus. In what follows, we shall say that a magnitude consists of infinitesimals just in case the infinitesimals and the original magnitude have the same dimension. Otherwise, we shall use the term indivisible.

3. A PAIR OF LEIBNIZIAN METHODOLOGIES

The existence of separate methodologies in Leibniz was already apparent to de Morgan, who quipped in 1852 that

It is also to be noticed that Leibnitz and the Bernoullis demand the method of exhaustions, or something equivalent, whenever an objection is raised to infinitesimals. They do not face a human enemy with small shot; they only use it to kill game (de Morgan 1852, [30, p. 324]).

In his seminal study of Leibniz’s methodology, H. Bos described a pair of distinct approaches to justifying the calculus:

Leibniz considered two different approaches to the foundations of the calculus; one connected with the classical methods of proof by “exhaustion”, the other in connection with a law of continuity (Bos 1974, [18, section 4.2, p. 55]).

The first approach relies on an Archimedean “exhaustion” methodology. We will therefore refer to it as the A-methodology. The second methodology relies more directly on infinitesimals. We will refer to it as the B-methodology, in an allusion to Johann Bernoulli, who, having
learned an infinitesimal methodology from Leibniz, never wavered from it.

Leibniz’s fictionalist attitude toward infinitesimals is no longer controversial today as it was in the eyes of his closest disciples such as Bernoulli, l’Hôpital, and Varignon. The fictional nature of Leibniz’s infinitesimals is clarified by G. Ferraro in the following terms:

According to Leibniz, imaginary numbers, infinite numbers, infinitesimals, the powers whose exponents were not “ordinary” numbers and other mathematical notions are not mere inventions; they are auxiliary and ideal quantities that [...] serve to shorten the path of thought (Ferraro [39, p. 35]; cf. Leibniz [99, p. 92-93]).

On Ferraro’s view, Leibniz’s infinitesimals enjoy an ideal ontological status similar to that of the complex numbers, surd (irrational) exponents, and other ideal quantities.

Leibniz’s 1702 letter to Varignon includes an important enclosure, recently analyzed by Jesseph [59]. Here Leibniz outlines a geometrical argument involving quantities $c$ and $e$ described as “not absolutely nothing”, and goes on to comment that $c$ and $e$

are treated as infinitesimals, exactly as are the elements which our differential calculus recognizes in the ordinates of curves for momentary increments and decrements (Leibniz [100, p. 104-105]).

Jesseph argues that Leibniz proposed a pair of methodologies, the first represented by his *Quadratura Arithmetica* of 1675, and the second summarized in the 1702 enclosure. Jesseph comments that

it seems that the infinitesimal here is introduced as something like a Hilbertian “ideal element” that arises when we consider limit cases and seek what Leibniz termed “the universality which enables [the calculus] to include

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6 G. Schubring attributes the first systematic use of infinitesimals as a foundational concept, to Johann Bernoulli (see [135, p. 170, 173, 187]). To note the fact of such systematic use by Bernoulli is not to say that Bernoulli’s foundation is adequate, or that that it could distinguish between manipulations with infinitesimals that produce only true results and those manipulations that can yield false results. One such infinitesimal distinction between two types of convergence was provided by Cauchy in 1853 (see [27]), thereby resolving an ambiguity inherent in his 1821 “sum theorem” (see Bråting [22]; Katz & Katz [65]; Borovik & Katz [17]; Blaszczyszyn et al. [15]).

7Jesseph’s evocation of Hilbert connects well with a viewpoint expressed by Robinson (see main text at footnote 2).
all cases, even that where the given lines disappear” (Jesseph 2011, p. 21).

Jesseph echoes Bos in emphasizing the importance of Leibniz’s law of continuity, described as “not a mathematical principle, but rather a general methodological rule with applications in mathematics, physics, metaphysics, and other sciences” (Jesseph, ibid.).

Recent Leibniz scholarship branches out into two distinct readings of Leibnizian infinitesimals. Bos, Ferraro, Horváth, Jesseph, and Laugwitz recognize the presence of a pair of methodologies, namely the A- and B-methodologies mentioned above. On the other hand, Arthur, Levey, and others adopt a syncategorematic reading which recognizes only the A-methodology, as analyzed in Subsection 5.4. The latter reading, in our view, is due to an incorrect analysis of Leibniz’s fictionalism.

4. Cum Prodiisset

Leibniz’s text Cum Prodiisset [98] (translated by Child [28]) dates from around 1701 according to modern scholars. The text is of crucial importance in understanding Leibniz’s foundational stance. We will analyze it in detail in this section.

4.1. Critique of Nieuwentijt. Leibniz begins by criticizing Nieuwentijt, who defended a conception of infinitesimal according to which the product of two infinitesimals is always zero. Mancosu’s discussion of Nieuwentijt in [111, chapter 6] is the only one to date to provide a contextual understanding of Nieuwentijt’s thought. Leibniz describes Nieuwentijt as being driven to fall back on assumptions that are admitted by no one; such as that something different is obtained by multiplying 2 by $m$ and by multiplying $m$ by 2; that the latter was impossible in any case in which the former was possible; also that the square or cube of a quantity is not a quantity or Zero (Leibniz translated by Child [28, p. 146]).

Leibniz rejects nilsquare and nilcube infinitesimals [8] which are altogether incompatible with his approach to differential calculus, as we will see in Subsection 4.6.

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[8] See further in footnotes [58] and [66]
4.2. Law of Continuity, with examples. *Cum prodiisset* erects infinitesimal calculus upon the foundation of Leibniz’s law of continuity (LC). Because it takes a variety of forms (cf. Jorgensen 2009, p. 224-229), the law of continuity is perhaps best understood as a *family resemblance concept* following Wittgenstein, i.e., a cluster of related concepts. Here Leibniz formulated LC in the following terms:

Proposito quocunque transitu continuo in aliquem terminum desinente, liceat raciocinationem communem instituere, qua ultimus terminus comprehendatur (Leibniz [98, p. 40]).

The passage can be translated as follows:

In any supposed continuous transition, ending in any terminus, it is permissible to institute a general reasoning, in which the final terminus may also be included.

The expression “final terminus” refers to the terminus mentioned earlier which is the “ending” of the said transition. We have deliberately avoided using the term *limit* in our translation. Translating *terminus* as *limit* misleadingly suggests the modern technical meaning of *limit* as a real-valued operation applied to sequences or functions. For perhaps the same reason, Bos comments that

the fundamental concepts of the Leibnizian infinitesimal calculus can best be understood as extrapolations to the

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*Boyter claims that Leibniz used this formulation of LC in “a letter to [Pierre] Bayle in 1687” (Boyter [21, p. 217]). Boyter’s claim contains two errors. First, the work in question is not a letter to Bayle but rather the Letter of Mr. Leibniz on a general principle useful in explaining the laws of nature, etc. (Leibniz 1687, [89]). Second, while this letter does deal with Leibniz’ continuity principle, it does not contain the formulation *In any supposed continuous transition, ending in any terminus, etc.*; instead, it postulates that an infinitesimal change of input should result in an infinitesimal change in the output (this principle was popularized by Cauchy in 1821 as the definition of continuity [26, p. 34]). Boyter’s erroneous claims have been reproduced by numerous authors, including M. Kline [75, p. 385].

*This is consistent with Child’s translation: “In any supposed transition, ending in any terminus, it is permissible to institute a general reasoning, in which the final terminus may also be included” [28, p. 147]. We have reinstated the adjective *continuous* modifying *transition* (deleted by Child possibly in an attempt to downplay a perceived logical circularity of defining LC in terms of continuity itself). Jorgensen [60, p. 228] cites Child’s translation and claims in footnote 21 that “this passage says nothing about continuity”.

*One scholar who was so misled was Boyer (see further in footnote [23]).*
actually infinite of concepts of the calculus of finite sequences. I use the term “extrapolation” here to preclude any idea of taking a limit (Bos [18, p. 13]).

Subsequent remarks in *Cum Prodiisset* make it clear that *terminus* encompasses inassignable quantities, for the following reasons:

- If one assumes that *terminus* is assignable, then there is no justification for applying LC to inassignables, as Leibniz does.\(^\text{12}\)
- Reading *terminus* as strictly assignable undermines the identification of LC as expressed in *Cum Prodiisset*, and LC as expressed in a February 1702 letter to Verignon, assumed by many Leibniz scholars (see Subsection \(^\text{4.3}\)).
- A finitistic reading of LC in *Cum Prodiisset* departs from the interpretation as given in Knobloch, Laugwitz, Robinson, and others.
- If all entities both in the “transition” itself and the *terminus* are finite, then LC becomes a tautology, inapplicable to the three examples Leibniz wishes to apply it to (see below), raising the question why Leibniz would have stated it at all.
- Leibniz follows Kepler in exploiting LC (see Kline [75, p. 385]). Kepler’s famous dictum (originating with Cusanus) concerning the circle being viewed as an infinitangular polygon clearly involves infinitary entities.
- In a letter to Wallis [93], Leibniz relied on LC so as to defend his use of the characteristic triangle (i.e., the relation \(ds^2 = dx^2 + dy^2\) along a curve), similarly involving infinitary entities.

Leibniz gives several examples of the application of his Law of Continuity. We will focus on the following three examples.

1. In the context of a discussion of parallel lines, he writes: when the straight line BP ultimately becomes parallel to the straight line VA, even then it converges toward it or makes an angle with it, only that the angle is then infinitely small [28, p. 148].

2. Invoking the idea that the term equality may refer to equality up to an infinitesimal error, Leibniz writes:

\[^{12}\text{Bos goes on specifically to criticize the Bourbaki’s "limite" wording "(Leibniz) se tient très près du calcul des différences, dont son calcul différentiel se déduit par un passage à la limite" (Bourbaki [19, p. 208]).}\]

\[^{13}\text{Specifically, Leibniz treats in detail an inassignable quantity he refers to as status transitus (see Subsection \(^\text{4.4}\)).}\]
when one straight line is equal to another, it is said to be unequal to it, but that the difference is infinitely small \[28\, p.\, 148\].

(3) Finally, a conception of a parabola expressed by means of an ellipse with an infinitely removed focal point is articulated in the following terms:

a parabola is the ultimate form of an ellipse, in which the second focus is at an infinite distance from the given focus nearest to the given vertex \[28\, p.\, 148\].

4.3. Souverain principe. In a 2 Feb. 1702 letter to Varignon, Leibniz formulated the law of continuity as follows:

[...\text{et il se trouve que les règles du fini réussissent dans l’infini comme s’il y avait des atomes (c’est à dire des éléments assignables de la nature) quoiqu’il n’y en ait point la matière étant actuellement sousdivisée sans fin; et que vice versa les règles de l’infini réussissent dans le fini, comme s’il y’avait des infiniment petits métaphysiques, quoiqu’on n’ en n’ ait point besoin; et que la division de la matière ne parvienne jamais à des parcelles infiniment petites: c’est parce que tout se gouverne par raison, et qu’autrement il n’aurait point de science ni règle, ce qui ne serait point conforme avec la nature du souverain principe (Leibniz [99, p. 93-94]).}]

Knobloch [77, p. 67], Robinson [126, p. 262], Laugwitz [80, p. 145], and other scholars identify this passage as an alternative formulation of the law of continuity, which can be summarized as follows: \textit{the rules of the finite succeed in the infinite, and conversely.} \[16\]

4.4. Status transitus. We resume our analysis of the law of continuity as formulated in \textit{Cum Prodiisset}. Leibniz introduces his next observation by the clause “of course it is really true that”, and notes that “straight lines which are parallel never meet” \[28\, p.\, 148\]; that “things which are absolutely equal have a difference which is absolutely nothing” \[28\, p.\, 148\]; and that “a parabola is not an ellipse at

\[14\text{Equality up to an infinitesimal is a state of transition from inequality to equality (this anticipates the law of homogeneity dealt with in Subsection 5.3).}\
\[15\text{Robinson’s attribution in [126, p. 262] contains a misprint: “(Leibniz [1701])” should be read as “(Leibniz [1702])”}\.
\[16\text{Laugwitz pointed out that this law “contains an a priori assumption: our mathematical universe of discourse contains both finite objects and infinite ones” (Laugwitz [80, p. 145]). As we have already discussed, identifying distinct A- and B-methodologies in Leibniz does not require realist commitments.}\]
all” [28, p. 149]. How does one, then, account for the examples of Subsection 4.2? Leibniz provides an explanation in terms of a state of transition (status transitus in the original Latin [98, p. 42]):

a state of transition may be imagined, or one of evanescence, in which indeed there has not yet arisen exact equality . . . or parallelism, but in which it is passing into such a state, that the difference is less than any assignable quantity; also that in this state there will still remain some difference, . . . some angle, but in each case one that is infinitely small; and the distance of the point of intersection, or the variable focus, from the fixed focus will be infinitely great, and the parabola may be included under the heading of an ellipse [28, p. 149].

A state of transition in which “there has not yet arisen exact equality” refers to example (2) in Subsection 4.2 “parallelism” refers to example (1); including parabola under the heading of ellipse is example (3).

Thus, status transitus is subsumed under terminus, passing into an assignable entity, but is as yet inassignable. Translating terminus as limit amounts to translating it as an assignable entity, the antonym of the meaning intended by Leibniz.

The observation that Leibniz’s status transitus is an inassignable quantity is confirmed by Leibniz’s conceding that its metaphysical status is “open to question”:

whether such a state of instantaneous transition from inequality to equality, . . . from convergence [i.e., lines meeting—the authors] to parallelism, or anything of the sort, can be sustained in a rigorous or metaphysical sense, or whether infinite extensions successively greater and greater, or infinitely small ones successively less and less, are legitimate considerations, is a matter that I own to be possibly open to question [28, p. 149].

17 A syncategorematic expression has no referential function. Thus, the phrase ‘the present king of France is bald’ is a syncategorematic expression, in that it doesn’t refer to any concrete individual. A syncategorematic expression serves to reveal logical relations among those parts of the sentence which are referential. In Arthur and Levey’s interpretation, the infinitesimal “serves to reveal logical relations” by tacitly encoding a quantifier applied to ordinary real values. But Leibniz clearly does not have real values in mind when he exploits the term status transitus. His status transitus is something between real values of the variable, on the one hand, and its limiting real value, on the other. Leibniz’s observation that the metaphysical (i.e., ontological) status of infinitesimals is “open to question” should
Yet Leibniz asserts that infinitesimals may be utilized independently of metaphysical controversies:

but for him who would discuss these matters, it is not necessary to fall back upon metaphysical controversies, such as the composition of the continuum, or to make geometrical matters depend thereon [28, p. 149-150].

To summarize, Leibniz holds that the inassignable status of \textit{status transitus} is no obstacle to its effective use in geometry. The point is reiterated in the next paragraph:

If any one wishes to understand these [i.e. the infinitely great or the infinitely small—the authors] as the ultimate things, or as truly infinite, it can be done, and that too without falling back upon a controversy about the reality of extensions, or of infinite continuums in general, or of the infinitely small, ay, even though he think that such things are utterly impossible; it will be sufficient simply to make use of them as a tool that has advantages for the purpose of the calculation, just as the algebraists retain imaginary roots with great profit [28, p. 150].

Leibniz has just asserted the possibility of the \textit{mathematical} infinite: “it can be done”, without \textit{ontological} commitments as to the reality of infinite and infinitesimal objects.

4.5. \textbf{Mathematical implementation of status transitus.} We will illustrate Leibniz’s concept of \textit{status transitus} by implementing it mathematically in the three examples mentioned by Leibniz (see Subsection 4.2). Example (2) can be illustrated in terms of a finite positive quantity Leibniz denotes $(d)x$.

(Bos [18, p. 57] replaced this by $dx$). The assignable quantity $(d)x$ passes via infinitesimal $dx$ on its way to absolute 0. Then the infinitesimal $dx$ is the \textit{terminus}, or the \textit{status transitus}. Zero is merely the \textit{shadow} of the infinitesimal. This particular \textit{status transitus} is the foundation rock of the Leibnizian definition of the differential quotient.

\footnotesize

apparently have put to rest any suspicions as to their alleged syncategorematic nature. After all, if an infinitesimal is merely meant as shorthand for talking about relations among sets of real values, what is the point of the lingering doubts expressed by Leibniz as to the ontological legitimacy of infinitesimals? Certainly the absence of a concrete individual counterpart of the bald king is a closed and shut question, rather than being “open to question”. See further in footnotes 38 and 66.

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Example (1) of parallel lines can be elaborated as follows. Let’s follow Leibniz in building the line through the point $(0, 1)$ parallel to the $x$-axis in the plane. Line $L_H$ with $y$-intercept 1 and $x$-intercept $H$ is given by $y = 1 - \frac{x}{H}$.

Now let $H$ be infinite. The resulting line $L_H$ has negative infinitesimal slope, meets the $x$-axis at an infinite point, and forms an infinitesimal angle with the $x$-axis at the point where they meet. We will denote by $st(x)$ the assignable (i.e., real) shadow of a finite $x$. Then every finite point $(x, y) \in L_H$ satisfies

\[
st(x, y) = (st(x), st(y)) = (st(x), st(1 - \frac{x}{H})) = (st(x), 1).
\]

Hence the finite portion of $L_H$ is infinitely close to the line $y = 1$. The line $y = 1$ is parallel to the $x$-axis, and is merely the shadow of the inassignable $L_H$. Thus, the parallel line is constructed by varying the oblique line depending on a parameter. Such variation comprises the status transitus $L_H$ defined by an infinite value of $H$.

To implement example (3), let’s follow Leibniz in deforming an ellipse, via a status transitus, into a parabola. The ellipse with vertex (apex) at $(0, -1)$ and with foci at the origin and at $(0, H)$ is given by

\[
\sqrt{x^2 + y^2} + \sqrt{x^2 + (y - H)^2} = H + 2
\]

We square (4.1) to obtain

\[
x^2 + y^2 + x^2 + (H - y)^2 + 2\sqrt{(x^2 + y^2)(x^2 + (H - y)^2)} = H^2 + 4H + 4
\]

We move the radical to one side

\[
2\sqrt{(x^2 + y^2)(x^2 + (H - y)^2)} = H^2 + 4H + 4 - (x^2 + y^2 + x^2 + (H - y)^2)
\]

and square again, to obtain after cancellation

\[
(y + 2 + \frac{2}{H})^2 - (x^2 + y^2) \left(1 + \frac{4}{H} + \frac{4}{H^2}\right) = 0.
\]

The calculation (4.1) through (4.4) depends on the following habits of general reasoning (to echo Child’s translation) with assignable quantities, which are generalized to apply to inassignable quantities (such as the terminus/status transitus) in accordance with the law of continuity:

- squaring undoes a radical;
- the binomial formula;

\[\text{The notation “st” parallels that for the standard part function in the context of the hyperreals (see Appendix A).}\]
• terms in an equation can be transferred to the other side; etc.

*General reasoning* of this type is familiar in the realm of ordinary finite real numbers, but why does it remain valid when applied to the realm of infinite or infinitesimal numbers? The validity of transferring such *general reasoning* originally *instituted* in the finite realm, to the realm of the infinite is postulated by Leibniz’s law of continuity.

We therefore apply Leibniz’s law of continuity to equation (4.4) for an infinite \( H \). The resulting entity is still an ellipse of sorts, to the extent that it satisfies all of the equations (4.1) through (4.4). However, this entity is no longer finite. It represents a Leibnizian *status transitus* between ellipse and parabola. This *status transitus* has foci at the origin and at an infinitely distant point \((0, H)\). Assuming \( x \) and \( y \) are finite, we set \( x_0 = \text{st}(x) \) and \( y_0 = \text{st}(y) \), to obtain a real shadow of this entity:

\[
\text{st} \left( \left( y + 2 + \frac{2}{H} \right)^2 - \left( x + y^2 \right) \left( 1 + \frac{4}{H} + \frac{4}{H^2} \right) \right) = \\
= \left( y_0 + 2 + \text{st} \left( \frac{2}{H} \right) \right)^2 - \left( x_0 + y_0^2 \right) \left( 1 + \text{st} \left( \frac{4}{H} + \frac{4}{H^2} \right) \right) \\
= \left( y_0 + 2 \right)^2 - \left( x_0 + y_0^2 \right) \\
= 0.
\]

Simplifying, we obtain

\[
y_0 = \frac{x_0^2}{4} - 1.
\]  

(4.5)

Thus, the finite portion of the *status transitus* (4.4) is infinitely close to its *shadow* (4.5), namely the real parabola \( y = \frac{x^2}{4} - 1 \) (in Leibniz’s terminology as translated by Child, “it is really true” that this parabola has no focus at infinity—see Subsection 4.4). This is the kind of payoff Leibniz is seeking with his law of continuity.

4.6. **Assignable versus unassignable.** In this section, we will retain the term “unassignable” from Child’s translation [28] (*inassignabiles* in the original Latin, see [98, p. 46]). After introducing finite quantities \((d)x, (d)y, (d)z\), Leibniz notes that

the unassignables \(dx\) and \(dy\) may be substituted for them by a method of supposition even in the case when they are evanescent [28, p. 153].

---

\[\text{When the *general reasoning* being transferred to the infinite realm is generalized to encompass arbitrary elementary properties (i.e. first order properties), one obtains the Loś-Robinson transfer principle (see Appendix A).}\]
Leibniz proceeds to derive his multiplicative law in the case $ay = xv$. Simplifying the differential quotient, Leibniz obtains

$$\frac{ady}{dx} = \frac{x dv}{dx} + v + dv. \quad (4.6)$$

At this point Leibniz proposes to transfer “the matter, as we may, to straight lines that never become evanescent”, obtaining

$$\frac{a (d)y}{dx} = \frac{x (d)v}{dx} + v + dv. \quad (4.7)$$

The advantage of (4.7) over (4.6) is that the expressions $\frac{(d)y}{dx}$ and $\frac{(d)v}{dx}$ are assignable (real). Leibniz points out that “$dv$ is superfluous”. The reason given is that “it alone can become evanescent”. The law of homogeneity (see Subsection 5.3) is not mentioned explicitly in $Cum$ $Prodissset$; therefore the rationale for this step is so far unsatisfactory. Discarding the $dv$ term, one obtains the expected product formula in this case. Note that thinking of the left hand side of (4.7) as the assignable shadow of the right hand side is consistent with Leibniz’s example (2) (see Subsection 4.2).

A final item worth noting is the division by second differentials occurring on page 157:

$$\frac{ddy}{ddx} = \frac{x dv}{a ddx} \frac{v}{a} + 2 \frac{dx dv}{a ddx} + \frac{2dv}{a} + 2 \frac{dx dv}{a ddx} + \frac{dv}{a}. \quad (4.8)$$

The final formula on page 158 in Leibniz’s text (in Child’s translation) is the assignable version of (4.8):

$$\frac{ddv}{ddx} = \frac{x dy}{a ddx} + v + 2 \frac{dy dv}{a ddx}. \quad (4.9)$$

Formula (4.9) similarly involves division by second order differentials. Division by second order unassignable differentials is incompatible with the nilsquare approach.

---

20 Child incorrectly transcribes formula (4.7) from Gerhardt, replacing the equality sign in Gerhardt by a plus sign. Note that Leibniz himself used the sign $\equiv$ (see McClenon [116, p. 371]).

21 Child’s transcription of formula (4.8) contains numerous errors: the numerator of the fraction $\frac{x}{a}$ is missing; the expression $\frac{dy dv}{ddx}$ appears with a $y$ in place of $v$ in the numerator; the expression $\frac{2dx dv}{ddx}$ appears with a $ddx$ in place of $dv$ in the numerator.

22 See further in footnotes 38 and 66.
5. LAWS OF CONTINUITY AND HOMOGENEITY

As discussed in Section 4, in his 1701 text Leibniz views parallel lines through the lens of a terminus, or status transitus, of intersecting lines forming an infinitesimal angle.

5.1. From secant to tangent. A related technique, involved in the determination of the tangent line from an equation for a secant line, is found in Leibniz’s 1684 text Nova Methodus:

We have only to keep in mind that to find a tangent means to draw a line that connects two points of the curve at an infinitely small distance, or the continued side of a polygon with an infinite number of angles, which for us takes the place of the curve (Leibniz [88]; translation from Struik [145, p. 276]).

Leibniz’s final clause here indicates that he viewed a curve as an infinite-sided polygon with infinitesimal sides. In the terminology of Subsection 4.4, the polygon is the status transitus, while the circle itself is merely its assignable shadow.

To elaborate on Leibniz’s construction of the tangent, if we have a formula for a secant line through a pair of variable points \(A, A'\) whose distance \(|AA'|\) tends to zero, the formula remains valid for the terminus, or status transitus, when \(|AA'|\) is infinitesimal (a similar calculation was already detailed in Subsection 4.5). Note that the limits of \(A\) and \(A'\), in the modern mathematical sense, are necessarily the same point, making it impossible to build the tangent line. In other words, the limit of \(|AA'|\) is 0, and a distance of 0 doesn’t correspond to a line, but simply to a single point, and is disconnected from the geometrical notion of tangent. To understand Leibniz’s construction of the tangent line in Nova Methodus, we must steer clear of limits in the modern sense, and rely instead on terminus or status transitus as elaborated in Cum Prodiisset.23

23S. L’Huilier (1750–1840) understood Leibniz’s law of continuity similarly: “if a variable quantity at all stages enjoys a certain property, its limit will enjoy the same property” [103, p. 167]. L’Huilier, writing a century before Weierstrass, is using the term limit in its generic sense close to terminus/status transitus. Blinded by the modern limit doctrine, Boyer comments as follows: “The falsity of this doctrine is immediately apparent from the fact that irrational numbers may easily be defined as the limit of sequences of rational numbers, or from the observation that the properties of a polygon inscribed in a circle are not those of the limiting figure—the circle” [21, p. 256]. But Boyer’s “limiting figure” is an anachronistic imposition, of a post-triumvirate variety, upon both L’Huilier and Leibniz. What Leibniz had in mind was a status terminus whose shadow is the circle (see also footnote 11).
5.2. The arithmetic of infinities in *De Quadratura Arithmetica*. Leibniz’s “masterwork on the calculus”, *De Quadratura Arithmetica*, was written near the end of his stay in Paris (ca. 1676) (Arthur 2001, p. 393, note 5); French translation in Leibniz (86). This text makes it clear that Leibniz introduced a distinction between equality on the nose, on the one hand, and approximate equality, or infinite closeness, on the other. Knobloch puts it as follows:

While up to then two quantities were called equal if their difference was zero, Leibniz called two quantities equal if their difference can be made arbitrarily, that is infinitely small [77, p. 63].

The rule governing infinitesimal calculation that Knobloch represents as Leibniz’s rule 2.2, states:

\[ x, y \text{ finite, } x = (y + \text{ infinitely small}) \iff x - y \approx 0 \]

(not assignable difference) (Knobloch [77, p. 67]).

Here Knobloch represents the relation of being infinitely close by a pair of wavy lines. Concerning Leibniz’s rules for the arithmetic of the infinite, Knobloch comments as follows:

In his treatise Leibniz used a dozen rules which constitute his arithmetic of the infinite. He just applied them without demonstrating them, only relying on the law of continuity: The rules of the finite remains valid in the domain of the infinite (ibid.).

Knobloch is alluding to Leibniz’s formulation of the law of continuity in a 2 feb. 1702 letter to Varignon (99) (see Subsection 4.3). The last of Leibniz’s rules is represented by Knobloch as rule 12:

\[ 12: x \text{ divided by } y = \frac{x + \text{ infinitely small}_1}{y + \text{ infinitely small}_2} \]

(Knobloch [77, p. 68]).

In other words, rule 12 authorizes a replacement of the right-hand side, “\((x + \text{ infinitely small}_1) \text{ divided by } (y + \text{ infinitely small}_2)\)” , by the left-hand side, “\(x \text{ divided by } y\)”, in infinitesimal calculations. As we shall see, Rule 12 is crucial to Leibniz’s conception of the differential quotient, \(dy/dx\).

Thus, to find \(dy/dx\) when \(y = x^2\) one starts with infinitesimal \(\Delta x\) and forms the infinitesimal quotient \(\Delta y/\Delta x\). One then simplifies the infinitesimal quotient, relying on Leibniz’s *law of continuity* (see Subsection 4.2) to justify each simplification (algebraic manipulations valid for ordinary numbers, are similarly valid for infinitesimals), so as to obtain the familiar quantity \(2x + \Delta x\). Next, to produce the expected answer, \(2x\), for the “differential quotient” (today called the derivative), one applies
Leibniz’s Rule 12 so as to discard the infinitesimal part $\Delta x$. Rule 12 amounts to an application of the transcendental law of homogeneity (see Subsection 5.3).

5.3. Transcendental law of homogeneity. Leibniz introduces a law called the transcendental law of homogeneity (TLH), governing equations involving differentials (as well as higher-order differentials). Bos summarizes the law in the following terms:

A quantity which is infinitely small with respect to another quantity can be neglected if compared with that quantity. Thus all terms in an equation except those of the highest order of infinity, or the lowest order of infinite smallness, can be discarded. For instance,

$$a + dx = a$$

$$dx + ddy = dx$$

etc. The resulting equations satisfy this [...] requirement of homogeneity (Bos [18] p. 33 paraphrasing Leibniz [101, p. 381-382]).

The title of Leibniz’s text is *Symbolismus memorabilis calculi algebraici et infinitesimalis in comparatione potentiarum et differentiarum, et de lege homogeneorum transcendentali*. The inclusion of the transcendental law of homogeneity (*lege homogeneorum transcendentali*) in the title of the text attests to the importance Leibniz attached to TLH.

Leibniz’s TLH has the effect of eliminating higher-order terms. The TLH is also mentioned in Leibniz’s *Nova Methodus* [88] (GM V, 224) of 1684 (cf. Bos [18] p. 33).

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24 Our analysis of Berkeley’s criticism of the proof of the product rule for differentiation appears in Section 7.

25 Kline opines that “In response to criticism of his ideas, Leibniz made various, unsatisfactory replies” [73] p. 384, and proceeds to quote a passage from a letter to Wallis from 30 March 1699 (Kline reports an incorrect year 1690): “It is useful to consider quantities infinitely small such that when their ratio is sought, they may not be considered zero but which are rejected as often as they occur with quantities incomparably greater [...] Thus if we have $x + dx$, $dx$ is rejected [...] Similarly we cannot have $xdx$ and $dx dx$ standing together. Hence if we are to differentiate $xy$ we write $(x + dx)(y + dy) = xy + xdy + ydx + dx dy$. But here $dx dy$ is to be rejected as incomparably less than $xdy + ydx$” (Leibniz [94] p. 63). This summary of the law of homogeneity is dismissed as “unsatisfactory” by Kline. In fairness it must be added that Kline wrote two years before the appearance of the seminal study by Bos [18].
5.4. Syncategorematics. At variance with Bos’ and Jessep’s reading, which accords the Leibnizian, fictional, infinitesimal the status of a separate approach as distinct from an Archimedean approach, syncategorematically inclined scholars maintain that the Leibnizian infinitesimal is merely shorthand for exhaustion à la Archimedes. This approach originates with the second, 1990 edition of Ishiguro’s book [56]. This is a kind of fictionalism, which Ishiguro describes as logical fictionalism, and we think of as reductive fictionalism: Propositions that refer apparently to fictions may be reduced to propositions that refer only to standard mathematical entities. Levey summarizes the approach as follows:

by April of 1676, with his early masterwork on the calculus, *De Quadratura Arithmetica*, nearly complete, Leibniz has abandoned any ontology of actual infinitesimals and adopted the syncategorematic view of both the infinite and the infinitely small as a philosophy of mathematics and, correspondingly, he has arrived at the official view of infinitesimals as fictions in his calculus (Levey [83, p. 107]).

According to Levey, Leibniz’s fictionalism “may be styled Archimedean” [83, p. 133]. Thus, the syncategorematic reading seeks to reduce the B-methodology to the A-methodology. More precisely, there is no separate B-methodology, syncategorematically speaking. Levey argues his claim by citing Leibniz’s comments in the month of April, 1676, even though Leibniz spent the next forty years publishing mathematics that employed infinitesimal techniques.

Not content with syncategorematizing infinitesimals right out of Leibniz’s thought, Levey pursues an even more radical thesis concerning their alleged sudden disappearance:

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Levey elaborates his position as follows: “The syncategorematic analysis of the infinitely small is […] fashioned around the order of quantifiers so that only finite quantities figure as values for the variables. Thus,

1. the difference \(|a - b|\) is infinitesimal

2. does not assert that there is an infinitely small positive value which measures the difference between \(a\) and \(b\). Instead it reports,

3. For every finite positive value \(\varepsilon\), the difference \(|a - b|\) is less than \(\varepsilon\).

Elaborating this sort of analysis carefully allows one to express the now-usual epsilon-delta style definitions, etc.” (Levey [83, p. 109-110]). To summarize: no B-methodology, syncategorematically speaking. What support does Levey provide for his nominalistic interpretation? Leibniz’s comments in April, 1676.
[...] within a short few weeks, it’s all over for the infinitely small [...] Good-bye to all the wonderful limit entities: good-bye parabolic ellipse with one focus at infinity, [...] (Levey [83, p. 114-115]).

But did Leibniz indeed bid “good-bye” to such “wonderful entities”? In point of fact, Leibniz does refer to just such a “parabolic ellipse with one focus at infinity” in 1701, in his text Cum Prodiisset [98, p. 46-47], a quarter of a century after an exaggerated report of its demise by Levey. The relevant passage was cited at the end of Subsection 4.2.

An even more explicit statement of Leibniz’s position, in terms of the status transfusus, was cited in Subsection 4.4, ending with the words the parabola may be included under the heading of an ellipse (Child, [28, page 149]).

The idea that “parabola may be included under the heading of an ellipse” is one manifestation of Leibniz’s law of continuity, “a general methodological rule” as Jesseph puts it. The reason Leibniz gives so much detail here (see the full passage cited in Subsection 4.4) is readily understood if we assume he is developing the strategy implicit in a B-methodology. Namely, to the extent that he is asserting a non-trivial philosophical or heuristic principle, he seeks to present a plausible justification concerning the reliance on fictional objects such as infinitesimals and parabolas with foci at infinity, and their properties. On the other hand, if one assumes syncategorematically that infinitesimals and foci at infinity are merely shorthand for relations among finite objects expressed by means of a series of quantifiers, why does Leibniz bother to formulate, and appeal to, the law of continuity? If the syncategorematic interpretation is correct, then the law of continuity can only be asserting a tautology: a sequence of standard entities consists of standard entities arranged in a sequence.

Knobloch [77] and Arthur [7] claim that Leibniz’s Theorem 6 in [85] (referred to as Leibniz’s Proposition 6 in [7]) was a major step toward the development of Riemann sums and a syncategorematic account of infinitesimals. Leibniz described the result as ‘most thorny’ (spinosis-sima). However, Jesseph [59] points out that Leibniz’s argument here involves the construction of auxiliary curves, and the latter “requires that we have a tangent construction that will apply to the original curve” [59]. The full title of Leibniz’s work is Arithmetical quadrature of the circle, ellipse and hyperbola, suggesting that it is not intended as a general method of quadrature. In order to achieve such generality, the transmutation theorem must be invoked, and that requires a fully general method of tangent construction. Jesseph notes, however, that
although Leibniz’s investigations in 1675-76 could show how conic sections and other well-behaved curves could be handled without recourse to infinitesimals, he himself understood that there were limitations to what could be achieved with these methods. Indeed, I suspect that he set aside the *Arithmetical Quadrature* without publishing it because he had turned his attention to more powerful methods that he would introduce in the 1680s in what he called “our new calculus of differences and sums, which involves the consideration of the infinite,” and “extends beyond what the imagination can attain” (GM 5: 307) (Jesseph [59]).

The syncategorematic interpretation is at odds with the historical studies by Bos [18] and Horváth [55]. One of the most salient points is the following. Leibniz defended his intuitions of an arbitrary-order infinitesimal against Nieuwentijt’s intuition of a nilsquare infinitesimal, by passing to the reciprocals and arguing in support of arbitrary-order infinite quantities. Whatever the merits of such an argument, the salient point here is that Leibniz did *not* adopt what would have been a more straightforward and powerful defense for someone viewing infinitesimals as merely logical fictions, eliminable by a suitable paraphrase. Had that been the case, Leibniz would have pointed out that an infinitesimal is merely a syncategorematic expression indicative of a logical analysis involving only finite quantities, in the exhaustive spirit of the A-methodology. Unless Levey is prepared to declare nilsquare infinitesimals similarly syncategorematic even in the eyes of Nieuwentijt, Leibniz’s response to the latter furnishes evidence in favor for his commitment to pursuing a separate B-methodology (see additional details in Section 12).

The adjective *syncategorematic* refers to non-denoting phrases like ‘the’ and ‘a’, whose logical role was analysed first by medieval scholastics. But its application to Leibniz’s infinitesimals actually amounts to a claim that Leibniz anticipated Weierstrass’s $\epsilon, \delta$ techniques [27].

Interpreting Leibniz as if he had read Weierstrass already would appear to fall into the category of feedback-style ahistory criticized by Grattan-Guinness [51, p. 176]. Anticipations of Weierstrass and Cantor are merely reflections of a philosophical disposition in favor of a sparse ontology prevalent since the launching of the great experiment of eliminating infinitesimals from analysis, favoring a tendentious rewriting of its history. Exactly this prompts Mancosu to observe that

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[27] See footnote [26] for Weierstrassian epsilonic details in Levey.
the literature on infinity is replete with such ‘Whig’ history. Praise and blame are passed depending on whether or not an author might have anticipated Cantor and naturally this leads to a completely anachronistic reading of many of the medieval and later contributions (Mancosu [113, p. 626]).

5.5. Punctiform and non-punctiform continua. Critics have objected to historical continuity between Leibniz and Robinson on the grounds that Leibniz’s continuum was non-punctiform, while Robinson’s is punctiform.

Leibniz does not appear to have thought of the continuum as being made up of points. Rather, points merely mark locations on the continuum. The view of the continuum as made up of points (therefore “punctiform continuum”) as later pursued by Cantor, is tied up with a set-theoretic foundation prevalent in modern mathematics.

On the other hand, Leibniz’s mathematics can arguably be imbedded into modern mathematics as was done by Robinson. The ingredients introduced, such as the axiom of infinity and the law of excluded middle, do not appear to contradict what is found in Leibniz and, on the contrary, provide a basis for a mathematical implementation of key insights already found in Leibniz, such as the law of continuity (which became the transfer principle) and the transcendental law of homogeneity (whose special case became the standard part function). The latter seems to have been strangely ignored by scholars, in spite of the detailed discussion in (Bos 1974, [18]).

Thus, the punctiform nature of the modern approach, be it standard or non-standard, may be irrelevant to interpreting Leibniz and his calculus. Certainly traditional historians acknowledge historical continuity between the calculus of Newton and Leibniz, on the one hand, and the calculus of today, on the other, in spite of the punctiform underpinnings of the latter. This is because the punctiform nature of the modern approach only comes to the fore in phenomena, not in calculus, but in real analysis as it emerged in the 19th century. The same remark applies to Robinson’s approach.

6. “Marvellous sharpness of Discernment”

“It is curious to observe, what subtilty and skill this great Genius employs to struggle with an insuperable Difficulty;

28Thus, Boyer writes: “The traditional view […] ascribes the invention of the calculus to […] Newton and […] Leibniz” [21, p. 187].

29Berkeley’s spelling.
and through what Labyrinths he endeavours to escape the Doctrine of Infinitesimals”. — G. Berkeley, *The Analyst*

The title of this section is taken from Berkeley [13, Section XVII]. In an analysis of George Berkeley’s criticism as expressed in *The Analyst* [13], Sherry [138] identifies what are actually two distinct criticisms that are frequently conflated in the literature. Sherry describes them as

1. the metaphysical criticism, and
2. the logical criticism

(see [138, p. 457]).

6.1. **Metaphysical criticism.** Sherry describes Berkeley’s *metaphysical criticism* as targeting a purportedly contradictory nature of the fluxions and evanescent increments. Berkeley similarly targets infinitesimals:

> The foreign Mathematicians are supposed by some, even of our own, to proceed in a manner, less accurate perhaps and geometrical, yet more intelligible. Instead of flowing Quantities and their Fluxions, they consider the variable finite Quantities, as increasing or diminishing by the continual Addition or Subduction of infinitely small Quantities [13, Section V].

Berkeley’s criticism emanates from his philosophical commitment to a theory of perception [32] anchored in the 18th century empiricist dogma.

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30Sherry’s dichotomy was picked up by Jesseph [58, p. 124].

31Three additional significant aspects of Berkeley’s criticism could be mentioned: (3) a belief in naive indivisibles (this a century after Cavalieri), i.e., a rejection of infinite divisibility that was already commonly accepted by mathematicians as early as Wallis and others; (4) empiricism, i.e., a belief that a theoretical entity is only meaningful insofar as it has an empirical counterpart, or referent; this belief ties in with Berkeley’s theory of perception which he identifies with a theory of knowledge; (5) Berkeley’s belief that Newton’s attempt to escape a reliance on infinitesimals is futile (see the epigraph to this Section [6]). The latter belief is contrary to a consensus of modern scholars. Thus, Pourciau [123] argues that Newton possessed a clear kinetic conception of limit similar to Cauchy’s, and cites Newton’s lucid statement to the effect that “Those ultimate ratios . . . are not actually ratios of ultimate quantities, but limits . . . which they can approach so closely that their difference is less than any given quantity . . . ” See Newton [120, p. 39] and [121, p. 442]. The same point, and the same passage from Newton, appeared a century earlier in Russell [132, item 316, p. 338-339].

32See Sections 8 and 9 for more detailed comments on Berkeley’s philosophy in relation to infinitesimals.
that meaningful expressions refer to particular perceptions. Berkeley’s metaphysical criticism is anchored in his empiricist epistemology, which does not allow infinite divisibility. This is illustrated by Berkeley’s Question 5, which appears in a long list of questions at the end of The Analyst:

Qu. 5. Whether it doth not suffice, that every assignable number of Parts may be contained in some assignable Magnitude? And whether it be not unnecessary, as well as absurd, to suppose that finite Extension is infinitely divisible?

This has the unwanted, even absurd consequence that nearly all of traditional geometry must be abandoned. It is for this reason that we claim that Berkeley’s criticism of the calculus stands on shakier grounds than Leibniz’s defense.

A number of years later, the metaphysical criticism of infinitesimals will be expressed most forcefully by Karl Marx. Marx referred to the theory of Leibniz and Newton as “the mystical differential calculus”. Echoes of class struggle reverberate through rage and cry of opposition as Marx notes that Leibniz and Newton

believed in the mysterious character of the newly discovered calculus, that yielded true (and moreover, particularly in the geometrical application, astonishing) results by a positively false mathematical procedure. They were thus selfmystified, valued the new discovery all the higher, enraged the crowd of old orthodox mathematicians all the more, and thus called forth the cry of opposition, that even in the lay world has an echo and is necessary in order to pave the way for something new (Marx, p. 168], cited in Kennedy [73, p. 307]).

Struik [144] p. 187-189] concurred, and Carchedi queried: “Which view of social reality is hidden behind and informs Marx’s method of differentiation? Marx differentiates with the eyes of the social scientist, of the dialectician” [25, p. 424]. Yet one can’t help wondering whether a, dialectical, elimination of infinitesimals as a class, with its attendant trading of a simple algorithmic technique for the manipulation of multiple quantifiers (and quantifier alternations) genuinely serves the interest of either the “lay world” or the proletariat.

Note here the similarity between the empiricist dogma and the syncategorical interpretation of infinitesimals. Both assert that meaningfulness consists ultimately in referring to some favored type of entity.

Berkeley uses extension in the sense of what we would call today a continuum.
6.2. **Logical criticism.** Meanwhile, Berkeley’s *logical criticism* targets a purported *shift in hypothesis*, when a proof is guilty of a *fallacia suppositionis*, that is, of gaining certain points by means of one supposition, but subsequently attaining the final goal by retaining the points just won, in combination with additional points obtained by replacing the original supposition by its contradictory (Sherry [138, p. 257]).

Thus, Berkeley’s “ghosts of departed quantities” criticism of the definition of the “differential quotient” amounted to the following query: how can a quantity \((dx)\) possess a “ghost” \((dx \neq 0)\), and at the same time be “departed” \((dx = 0)\)? Alternatively, how can the infinitesimal analyst have his cake \((dx \neq 0)\) and eat it, too \((dx = 0)\)? Or, as Berkeley colorfully put it,

> I shall now . . . observe as to the method of getting rid of such Quantities, that it is done without the least Ceremony (Berkeley [13, Section XVIII]).

The *exposition* of Berkeley’s critique of the calculus by Jesseph [57] is described as “definitive” by Sherry [139, p. 127], who finds, however, shortcomings in Jesseph’s *evaluation* of that critique. Sherry notes Berkeley’s double standard in his attitude toward arithmetic and geometry. Thus, Berkeley accepts the practice of arithmetic on the purely pragmatic grounds of its utility, reflecting an instrumentalist position. For Berkeley, arithmetic lacks empirical content, i.e., numerals do not denote particular perceptions; the same length can be 3 (feet) or 36 (inches).

Meanwhile, in the case of geometry, including the infinitesimal calculus, Berkeley adopts a non-instrumentalist approach which insists upon a subject matter. Thus, Berkeley criticizes infinitesimals for possessing no referent, unlike the classical geometry of Euclid, which does possess a referent, in Berkeley’s view. Sherry notes that Jesseph

> doesn’t really address the question why Berkeley was reluctant to explicate the calculus by the tools of his philosophy of arithmetic and its supporting semantical doctrine that meaningfulness lies in the use to which terms can be put [139, p. 127].

Both Berkeley’s metaphysical criticism and his logical criticism stem from philosophical blunders, rather than inherent defects in infinitesimal calculus. On the one hand, Berkeley is committed to the absurd rejection of traditional geometry, and on the other, he abandons his
empiricist dogma as soon as it proves inconvenient for his purposes. Berkeley’s clericalist agenda may have affected such a selective application of his empiricist dogma, raising issues of intellectual integrity already alluded to by de Morgan in his characteristically caustic style:

> Dishonesty must never be insinuated of Berkeley. But the Analyst was intentionally a publication involving the principle of Dr. Whateley’s argument against the existence of Buonaparte; and Berkeley was strictly to take what he found. The Analyst is a tract which could not have been written except by a person who knew how to answer it. But it is singular that Berkeley, though he makes his fictitious character nearly as clear as afterwards did Whateley, has generally been treated as a real opponent of fluxions (de Morgan [30, p. 329]).

In Section 7, we will consider a Berkeleyian response to the law of homogeneity and the product rule.

7. Berkeley’s critique of the product rule: “The acme of lucidity”?

> “Berkeley attacked the logic of the method of fluxions or infinitesimal calculus, holding that the infinitesimal […] was self-contradictory. His two ways of bringing this out are the acme of lucidity; one concerns the fluxion of a power, the other that of a product.” J. Wisdom [154]

In this section, we analyze Berkeley’s critique of the product rule.

7.1. Berkeley’s critique. Berkeley developed a detailed criticism of the proof of the product rule for differentiation. Berkeley refers to

> Leibnitz and his followers in their calculus differentialis making no manner of scruple, first to suppose, and secondly to reject Quantities infinitely small (Berkeley [13, Section XVIII]) [emphasis in the original].

Berkeley illustrates the unscrupulous rejection, in the context of a proof of the product rule, as follows:

> in the calculus differentialis the main Point is to obtain the difference of such Product. Now the Rule for this is got by rejecting the Product or Rectangle of the Differences. And in general it is supposed, that no Quantity is bigger or lesser for the Addition or Subduction of its

35 The reference is to Richard Whately (1787–1863).
Infinitesimal: and that consequently no error can arise from such rejection of Infinitesimals [13, Section XVIII].

Berkeley continues:

XIX. And yet it should seem that, whatever errors are admitted in the Premises, proportional errors ought to be apprehended in the Conclusion, be they finite or infinitesimal: and that therefore the ἀκριβεία of Geometry requires nothing should be neglected or rejected.

Here Berkeley is objecting to the last step in the calculation

\[ d(uv) = (u + du)(v + dv) - uv = udv + vdu + du dv = udv + vdu. \] (7.1)

Is Berkeley’s objection valid? The last step in calculation (7.1), namely

\[ udv + vdu + du dv = udv + vdu \]

is an application of Leibniz’s transcendental law of homogeneity [37] (see Bos [18, p. 33]) already mentioned in Subsection 5.2. The law justifies dropping the \( du dv \) term on the grounds that, given an equation whose two sides contain differentials of different orders, one is authorized to discard the higher-order ones.

In his 1701 text Cum Prodiisset [98, p. 46-47], Leibniz presents an alternative justification of the product rule (see Bos [18, p. 58]). Here he divides by \( dx \) and argues with differential quotients rather than differentials [38]. Adjusting Leibniz’s notation (see Subsection 4.6) to fit with the calculation (7.1), we obtain an equivalent calculation

\[
\frac{d(uv)}{dx} = \frac{(u + du)(v + dv) - uv}{dx} = \frac{udv + vdu + du dv}{dx} = \frac{udv + vdu}{dx} + \frac{du dv}{dx} = \frac{udv + vdu}{dx}.
\]

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36 Accuracy

37 Leibniz had two laws of homogeneity, one for dimension and the other for the order of infinitesimalness. Bos states that they ‘disappeared from later developments’ [18, p. 35], referring to Euler and Lagrange.

38 Leibniz freely inverts his infinitesimals, making it difficult to interpret his infinitesimals in terms of modern nilsquare ones, as Arthur attempts to do in [7] (see also footnote 66).
Under suitable conditions the term $\frac{du \, dv}{dx}$ is infinitesimal, and therefore the last step

$$\frac{udv + vdu}{dx} + \frac{du \, dv}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

(7.2)

is legitimized as an instance of the transcendental law of homogeneity (see Subsection 5.3), which authorizes one to discard the higher-order term in an expression containing infinitesimals of different orders.

7.2. Rebuttal of Berkeley’s logical criticism. Berkeley’s logical criticism of the calculus is that the evanescent increment is first assumed to be non-zero to set up an algebraic expression, and then treated as zero in discarding the terms that contained that increment when the increment is said to vanish.

It is open to Leibniz to rebut Berkeley’s logical criticism by noting that the evanescent increment is not “treated as zero”, but, rather, merely discarded through an application of the transcendental law of homogeneity.

Jesseph quoted approvingly a passage from Cajori who characterized Berkeley’s arguments as “so many bombs thrown into the mathematical camp”. On the contrary, Berkeley’s criticisms reveal more about Berkeley’s own mathematical and philosophical limitations than about the shortcomings of the mathematicians he attempted to criticize. Jesseph discusses a defense of Newton by James Jurin (1684–1750) in the following terms:

On Jurin’s analysis, there is no inconsistency in dividing by an increment $o$ to simplify a ratio and then dismissing any remaining $o$-terms as “vanished”.

Jesseph mentions that Jurin defended Newton “even to the point of insisting that a ratio of evanescent increments could subsist even as the quantifies forming the ratio vanish”, but misses the essential point that, while it is incorrect to say that “there is no inconsistency in dividing by an increment $o$ to simplify a ratio and then dismissing any remaining $o$-terms as ‘vanished’”, if one deletes the last two words from Jurin’s phrase, this does become correct:

there is no inconsistency in dividing by an increment $o$ to simplify a ratio and then dismissing any remaining $o$-terms.

This is not to say that Leibniz’s system for differential calculus satisfied modern standards of rigor. Rather, we are rejecting the claim by Berkeley and triumvirate historians to the effect that Leibniz’s system contained logical fallacies.
Leibniz could have his cake and eat it, too—but not at Berkeley’s empiricist table, as we discuss in the next section.

8. “A THORN IN THEIR SIDES”

“Hobbes […] sought to mount an empirical ladder in providing a physicalist explanation of geometry, and had the ladder pulled out from under him by [J.] Wallis. . . . Berkeley, in his criticism of Newton, was not so easily routed.” E. Strong 142 p. 92-93

Strong apparently felt that Berkeley’s empiricist ladder was sturdier than Hobbes’. Was it?

Berkeley’s dual criticism is essentially unanswerable from the viewpoint of his empiricist logic. He would have similarly rejected modern theories of Archimedean continua stemming from the work of Cantor, Dedekind, and Weierstrass, because they involve infinite aggregates, as well as rejecting infinitesimal-enriched continua, since both involve infinitary constructions. Meanwhile, Berkeley’s empiricism outlaws all infinite objects, as is evident, for instance, from the following item:

Qu. 21. Whether the supposed infinite Divisibility of finite Extension hath not been a Snare to Mathematicians, and a Thorn in their Sides? [13, Question 21].

Once Berkeley’s empiricist epistemology is rejected, so are the obstacles to responding to his pair of criticisms. A response to the metaphysical criticism lies in a presentation of a more stratified (hierarchical) structure, with an A-continuum englobed inside a B-continuum (see Figure 2 and Section 9).

8.1. Felix Klein on infinitesimal calculus. In 1908, Felix Klein described a rivalry of such continua in the following terms. Having outlined the developments in real analysis associated with Weierstrass and his followers, Klein pointed out that

The scientific mathematics of today is built upon the series of developments which we have been outlining. But an essentially different conception of infinitesimal calculus has been running parallel with this [conception] through the centuries [74] p. 214.

Such a different conception, according to Klein,

40See footnote 34.
41Weierstrass’s nominalistic reconstruction (as C. S. Peirce called it, and as Burgess might have) was analyzed in [64].
harks back to old metaphysical speculations concerning the structure of the continuum according to which this was made up of [...] infinitely small parts (ibid.).

Klein appears to imply that those in the Leibnizian tradition are committed to the reality of infinitesimals. While this is not true for Leibniz, at least, Klein’s comments do indicate the seriousness with which some leading mathematicians of the post-triumvirate era viewed the tradition of a B-continuum — inspite of the “official” line as to its alleged banishment by Cantor, Dedekind, and Weierstrass.

H. Poincaré expressed himself similarly in his essay *Science and hypothesis*. Having discussed what is recognizably an Archimedean continuum, Poincaré proceeds to ask the following question: “Is the creative power of the mind exhausted by the creation of the mathematical continuum?” and concludes: “No: the works of Du Bois-Reymond demonstrate it in a striking way”. Poincaré then details his position as follows:

We know that mathematicians distinguish between infinitesimals of different orders and that those of the second order are infinitesimal, not only in an absolute way, but also in relation to those of the first order. It is not difficult to imagine infinitesimals of fractional or even of irrational order, and thus we find again that scale of the mathematical continuum... (Poincaré [122, p. 50]).

8.2. Did George slay the infinitesimal? Not all commentary on the history of the calculus has been as perceptive as Klein’s and Poincaré’s. In fact, commentators have generally regarded Berkeley’s criticisms as
Figure 3. George’s attempted slaying of the infinitesimal, following E. T. Bell and P. Uccello

a crucial step toward banning infinitesimals from mathematics once and for all; see e.g., (Cajori 1917, [24, p. 151-154]) and (Boyer 1949, [20]). In Subsection 5.2 we saw that the means of answering Berkeley’s pair of criticisms were already available to Leibniz himself. A historiographic failure to dissect Berkeley’s criticism into its two component parts has led to it being overrated by both historians and mathematicians, for it prevented them from appreciating the rebuttals available to Leibniz. In the course of the 19th century, infinitesimals came to be thought of as something of an intellectual embarrassment. Such a view ultimately found expression in even the popular (pseudo)historical narratives of E. T. Bell. Bell waxed poetic about infinitesimals having been

- slain [10, p. 246],
- scalped [10, p. 247], and
- disposed of [10, p. 290]

by the cleric of Cloyne (see Figure 3). ‘Scalps’ of departed quantities continue to litter the closets of historical studies of infinitesimal calculus, as we illustrate in Section 11.

9. Varieties of Continua

Section 8 distinguished Berkeley’s logical criticism from his metaphysical criticism. In 1966 Robinson wrote:

44Dismissing Bell’s martial flourishes as merely verbal excesses would be missing the point. Bell has certainly been criticized for other fictional excesses of his purportedly historical writing (thus, Rothman writes: “[E. T.] Bell’s account [of Galois’s life], by far the most famous, is also the most fictitious” 134, p. 103); however, his confident choice of martial imagery here cannot but reflect Bell’s perception of a majority view among professional mathematicians. Bell is convinced that Berkeley refuted infinitesimals only because triumvirate historians and mathematicians told him so.
The vigorous attack directed by Berkeley against the foundations of the Calculus in the forms then proposed is, in the first place, a brilliant exposure of their logical inconsistencies. But in criticizing infinitesimals of all kinds, English or continental, Berkeley also quotes with approval a passage in which Locke rejects the actual infinite ... It is in fact not surprising that a philosopher in whose system perception plays the central role, should have been unwilling to accept infinitary entities [126, p. 280-281].

Implicit in this passage is Robinson’s awareness of the two facets of Berkeley’s criticism. Unfortunately, he had no access to Leibniz’s manuscript De Quadratura Arithmetica first published in 1993 (see Subsection 5.2). We therefore cannot agree with his description of Berkeley’s Analyst as “a brilliant exposure” of “logical inconsistencies”. Robinson’s praise pays lip service to the received views on the history of the calculus. As we showed in Subsection 5.2 though, Leibniz possessed the conceptual tools to formulate a logically unassailable theory of the calculus, in particular, his use of Rule 12 as a special case of the transcendental law of homogeneity.

Over the centuries, historians, mathematicians, and philosophers have envisioned (at least) two distinct theories of the continuum, as discussed in Section 8:

- an A-continuum (for Archimedes), and
- a B-continuum (for Johann Bernoulli followed by Leibniz).

The former is a “thin” continuum, exemplified by what are called today the real numbers; the latter is a “thick” continuum incorporating infinitesimals.

One possible way of explaining the relation of the two continua is the following. All the values in the A-continuum are (theoretically) possible results of measurement. The B-continuum has values, like \( x + dx \), which could never be the result of measurement.

The contents of the A-continuum would correspond, in Leibniz’s terms, to all and only assignable magnitudes. Meanwhile, the contents of the B-continuum would contain, in Leibniz’s terms, inassignable magnitudes, as well. The law of homogeneity as used in (7.2) allows

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[45] See footnote 4
[46] Simon Stevin’s decimals are at the foundation of the common number system; see footnote 62 below for additional details.
[47] In this context, it may be interesting to note that a close relationship exists between Cantor’s construction of the usual A-continuum in terms of equivalence
one to pass from a relation between an inassignable and an assignable quantity, to an equality between assignable ones. Note that an equality may be a relation between two assignable magnitudes, or possibly two inassignable ones.

In the world of L. Carnot and A. Cauchy, the assignable/inassignable distinction translates into a dichotomy of “constant” quantities versus “variable” quantities, an infinitesimal being viewed as generated by a null sequenc[13] (see K. Bråting \[22\]). Cauchy, like most contemporary authors, did not typically refer to infinitesimals as numbers because a wide (but not universal) consensus since at least Stevin had been to use number to refer to the result of counting or measuring. The term quantity, as in quantité variable, in Carnot and Cauchy refers to a more general spectrum of possibilities, typically elements of an ordered system. Thus, complex numbers are never referred to as quantities by Cauchy, but rather as expressions. In Cauchy’s presentation of the material, the starting point are variable quantities, i.e., sequences. To Cauchy, a sequence “becomes” an infinitesimal if it tends to zero (cf. Borovik & Katz \[17\]). More generally, a sequence defines a variable quantity, that decomposes as a sum of a constant quantity (i.e. Stevin number) plus a null sequence (that becomes an infinitesimal). For example, the “variable quantity” defined by the sequence

\[(3.1, 3.14, 3.141, 3.1415, \ldots)\]

will decompose as the sum of the real number $\pi$ and a negative infinitesimal. It can easily be shown that a similar decomposition (for finite elements) necessarily holds in any ordered field properly containing the reals.

10. A CONTINUITY BETWEEN LEIBNIZIAN AND MODERN INFINITESIMALS?

Are modern theories of infinitesimals, legitimate heirs to Leibniz’s theory? Robinson wrote in 1966:

classes of Cauchy sequences, on the one hand, and one of the more straightforward constructions of the B-continuum, on the other; see Appendix \[A\].

Schubring \[135, p. 454\] notes that both Cauchy and Carnot approached infinitesimals dynamically, in terms of sequences (sometimes referred to as “variables”, understood as a succession of values) which tend to zero. A. Youschkevitch quotes Carnot to the effect that an infinitesimal is a variable quantity all of whose values are determinate and finite (see Gillispie \[47, p. 242\]). To note the fact of the identical definition of infinitesimals found in Carnot in Cauchy is not to imply total agreement; thus, Cauchy rejected Carnot’s definition of the differential.
It is shown in this book that Leibniz’s ideas can be fully vindicated and that they lead to a novel and fruitful approach to classical Analysis and to many other branches of mathematics [126, p. 2].

Such claims on Leibniz’s heritage have encountered stiff resistance. A claim of a historical continuity between Leibniz’s and Robinson’s infinitesimals is a formidable challenge to the received triumvirate scholarship. There are two ways of deflecting the challenge posed by modern infinitesimals.

One of the objections is that Robinson’s theory represents a radical upheaval of the foundations of classical analysis, and the other, than it is a trivial reformulation thereof. While these objections are obviously at odds with each other, they bear scrutiny. We will summarize and analyze these objections below.

(1) There is no historical continuity at all. Historical infinitesimals were unsound as analyzed by Berkeley, and became obsolete in 1870. Robinson’s infinitesimals are similar to the historical infinitesimals in name only. In reality Robinson’s invention relies on sophisticated model theory and an upheaval of the foundations of mathematics unimaginable to the pioneers of infinitesimal calculus.

(2) Robinson’s approach is merely a re-packaging of the old Weierstrassian ideas. If you unwind Robinson’s definitions, you find at bottom the same ideas that revolutionized mathematics in the 1870s, namely the Cantorian set-theoretic revolution that radically transformed our understanding of the continuum, from which the modern punctiform conception has emerged.

While on the face of it, objections (1) and (2) are mutually contradictory, there is a grain of truth in both, even though both are off target, as we argue below.

Analysis of criticism (1): Robinson chose to present his results in the form of maximal generality, exploiting powerful compactness theorems (see Malcev [110]) to prove the most general existence results for non-standard extensions of $\mathbb{R}$. The price one pays for generality is that the intuitive source of the notion of an infinitesimal, namely the null sequence, is well-hidden and buried deep inside the axiom system, through the introduction of a new symbol $\epsilon$ and the system of inequalities

$$\epsilon < \frac{1}{n}, \quad n \in \mathbb{N}.$$
On the other hand, the intuitive notion of a null sequence is closer to the surface in the ultrapower construction (see Appendix A). The latter is less general but has the advantage of offering a lucid cognitive and formal link to historical infinitesimals, generated as they were by variable quantities or null sequences.

Analysis of criticism (2): Robinson places himself squarely in the tradition of classical logic which emerged in the work of Frege, Peano, and Hilbert at about the same time Cantor, Dedekind, and Weierstrass banished infinitesimals by reconstructing analysis on the basis of Stevin numbers. Thus, Robinson’s foundational apparatus is tame compared to programs proposed by Brouwer and much later Lawvere [82], in the framework of intuitionistic logic.

The crucial point remains that Robinson’s use of infinitesimals is in no way influenced by the fact that they can be defined by means acceptable to the triumvirate. Rather, his use of infinitesimals is governed by Leibniz’s law of continuity (see Subsection 4.2); that is, the inferences that Robinson draws by means of infinitesimals are those that the transfer principle—a precise formulation of the law of continuity—licenses.

By introducing the hyperreal field, an extension of \( \mathbb{R} \), Robinson gains the problem solving power and convenience of an extended number system. Describing such an extension as a trivial modification of Weierstrassian analysis makes no more sense than claiming that the latter is a trivial modification of the Greek idea of number, rooted exclusively in \( 2, 3, 4, \ldots \).

Fields medalist T. Tao summed up the advantage of the hyperreal framework by noting that it allows one to rigorously manipulate things such as “the set of all small numbers”, or to rigorously say things like “\( \eta \) is smaller than anything that involves \( \eta_0 \)”, while greatly reducing epsilon management issues by automatically concealing many of the quantifiers in one’s argument (Tao [149, p. 55]).

\[49\] Not every hyperreal field can be obtained this way, though a more general construction called limit ultrapower can be used to construct a maximal class hyperreal field (see Ehrlich [35] and Borovik, Jin, and Katz [16]).

\[50\] From a strict set-theoretic viewpoint, each of the successive number systems \( \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \) can be reduced to the previous one by the familiar set-theoretic constructions. Yet it is generally recognized that each successive enlargement, when additional entities come to be viewed as individuals (or atomic entities), constitutes a conceptual advantage over the previous one, with a gain in problem solving power.
The connection to Weierstrass has to run through each of the, distinctly un-Weierstrassian, developments listed below. Namely, Robinson’s theory is anchored in a series of developments each of which was thought breathtaking in its time:

1. (Zermelo 1904, [156]) isolated the axiom of choice which until then was a hidden hypothesis in proofs;
2. (Tarski 1930, [150]) proved the existence of ultrafilters using the axiom of choice;
3. (Skolem 1934, [140]) constructed non-standard models of Peano arithmetic;
4. (Hewitt 1948, [54]) constructed hyper-real fields using a form of the ultrapower construction relying on ultrafilters;
5. (Łoś 1955, [107]) proved his theorem the consequence of which is the transfer principle for hyper-real fields.

11. Commentators from Russell onward

“There is no such thing as an infinitesimal stretch; if there were, it would not be an element of the continuum; the Calculus does not require it, and to suppose its existence leads to contradictions.”

B. Russell, *The principles of mathematics*, p. 345.

11.1. Russell’s non-sequiturs. Russell’s *The principles of mathematics* [132] dates from 1903. Russell opens his discussion of infinitesimals by citing the Archimedean property:

If $P$, $Q$ be any two numbers, or any two measurable magnitudes, they are said to be finite with respect to each other when, if $P$ be the lesser, there exists a finite integer $n$ such that $nP$ is greater than $Q$ [132, p. 332, start of paragraph 310].

This is the first time the word-root of “measure, measurable” appears in Russell’s chapter XL. It will play a crucial role in Russell’s argument. Its meaning needs to be established carefully. The term “measurable” can be used in at least the following three senses:

1. as a measure-theoretic term, e.g., in the expression *Lebesgue-measurable set* (this is the most commonly used sense in contemporary mathematics);
2. as a number-theoretic or analytic term, as in “measurable quantity”, meaning a quantity that can be multiplied by other quantities (such as the integer $n$ in Russell’s comment cited above);
3. as an empirical term, signifying “accessible perceptually to the senses or to physical measuring devices”. 
The intended meaning here is clearly the number-theoretic meaning (2). Russell assumes that $P$ violates the Archimedean property, and notes that

if it were possible for $Q$ to be ... finite, $P$ would be ... infinitesimal--a case, however, which we shall see reason to regard as impossible [132, p. 332, end of paragraph 310].

Russell is treading on dangerous ground here, following Cantor’s ill-fated attempt to prove infinitesimals to be inconsistent. On page 333, Russell for the first time uses the term “transfinite” in place of “infinite”:

But it must not be supposed that the ratio of the divisibilities of two wholes, of which one at least is transfinite, can be measured by the ratio of the cardinal numbers of their simple parts [132, p. 333, paragraph 311].

The implication is that the terms “infinite” and “transfinite” have the same meaning. In other words, Russell is assuming that any infinite entity will be a Cantorian infinite entity. He thus imposes the strait-jacket of Cantor’s theory of cardinality upon a discussion that should be independent of such theories.

Following a discussion of some mathematical uses of the term “infinitesimal” that Russell finds unobjectionable, Russell makes the following declaration:

What makes these various infinitesimals somewhat unimportant, from a mathematical standpoint, is, that measurement essentially depends on the axiom of Archimedes, and cannot, in general, be extended by means of transfinite numbers [132, p. 333, paragraph 311].

This statement comes at the end of a long paragraph dealing with purely mathematical matters. What strikes the reader of Russell’s text is the extra-mathematical nature of Russell’s declaration concerning the nature of measurement. Clearly Russell is now using the root “measurement” in the empirical sense (3). It is the empirical sense that lends the phrase its plausibility. However, as a logical link in Russell’s argument, what is required here is the number-theoretic meaning (2), as discussed above.

The notion of infinitesimal in Russell’s time was a heuristic concept that has not been defined yet. The entire enterprise by Cantor and Russell (to prove the non-well-foundedness of a heuristic concept that has not been defined yet) retroactively strikes one as ill-conceived.
The conflation of meanings (2) and (3) is not a novelty found in Russell. Rather, it can be traced back at least to Berkeley, to whom no expression is meaningful unless it possesses an empirical counterpart.

Russell’s error is therefore two-fold: first, the identification of infinity with Cantorian transfinitude, which is understandable to a certain extent since Cantor’s theory was the only theory of infinity considered reliable by mathematicians at the time; and Russell’s conflation of separate meanings of “measurable” following Berkeley, and disregarding several centuries of intervening philosophy, a feat somewhat less pardonable.

11.2. Earman and the st-function. A few months after Robinson’s death J. Earman published a text claiming to refute any meaningful connection between Leibniz’s and Robinson’s infinitesimals. Did he succeed?

Earman opens his text by introducing a distinction between two types of Leibnizian infinitesimals, denoted, respectively, infinitesimal$_1$ and infinitesimal$_2$.

The structural role infinitesimals$_1$, $i = 1, 2$, play in Earman’s text is transparent. Namely, Earman was confronted with overwhelming evidence that Leibniz thought of infinitesimals as “ideal” entities. Earman was seeking to refute such a notion. Therefore he introduced the distinction between allegedly two types of infinitesimals: infinitesimal$_1$, $i = 1, 2$. Such a distinction allowed him to claim that Leibniz was describing only infinitesimal$_1$ as “ideal,” ($j = 2, 3$), but not infinitesimal$_2$.

What is the meaning of Earman’s dichotomy? He says infinitesimal$_1$ is “intrinsically small”, whereas infinitesimal$_2$ “is incomparably small with respect to ordinary quantities.” Is Earman’s distinction coherent? If we do have an absolute/intrinsic scale of “ordinary quantities” as Earman puts it, then we also have an absolute/intrinsic scale of “smallness”, and therefore infinitesimal$_2$ is englobed in infinitesimal$_1$.

Furthermore, whenever Leibniz deals with an allegedly “intrinsic” infinitesimal $e$, he does not hesitate to consider $e^2$ and higher powers, showing that his intrinsically small infinitesimal$_1$ is englobed in infinitesimal$_2$. None of the recent Leibniz scholarship seems to have picked up Earman’s infinitesimal$_{i=1,2}$ dichotomy.

On page 239, lines 7-8 Earman claims that Leibniz is referring to infinitesimal$_1$ in a quote he provides from Leibniz, cited out of both

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Note that the second sense ties in well with Louis Narens’ approach to measurement where he transforms it into a relative notion: certain entities are measurable compared to others. This allows him to do measurement theory in non-Archimedean contexts (see [118]).
Loemker [106] and Gerhardt [45]. However, the quote he provides contains no indication why Leibniz would think these are allegedly infinitesimals\textsubscript{1} rather than infinitesimals\textsubscript{2}, and Earman himself does not provide any argument to buttress his claim.

On page 239, lines 15-19 Earman makes a similarly unsubstantiated and ahistorical claim that Leibniz was not referring to infinitesimals in an “ordinal sense”, but rather to “transfinite cardinal numbers”. Such an alleged deduction from Leibniz is a non-sequitur. Meanwhile, Earman’s ahistorical claim plays a key role in his argument. Thus, on page 240, lines 1-3 he argues that a “reciprocal of an infinite cardinal” infinitesimal\textsubscript{1} is “not well defined”, and buttresses his argument in footnote 5 by a reference to Fraenkel’s Abstract Set Theory [41] from 1966. Now such operations on infinite cardinals are indeed not well defined, but the assumption that Leibnizian infinitesimals are built from cardinals rather than from, say, ordinals or other material, is neither Leibniz’s nor Fraenkel’s, but rather Earman’s. In point of fact, one can indeed build a consistent theory of infinitesimals starting with infinite ordinals. Such a theory is called the surreal. P. Ehrlich [35] recently constructed an isomorphism between maximal surreals and maximal hyperreals.

But Earman’s most serious error occurs on page 249, where he discusses second order infinitesimals in Leibniz, such as $(dx)^2$. He declares:

> We can always arrange that our non-standard model has second order infinitesimals (Earman [32, p. 249]).

Earman goes on to interpret such second-order infinitesimals in a non-standard model, in his own novel way. Namely, he seeks to interpret them as elements of the secondary non-standard extension, which he denotes by

$$\mathbb{R}^{**},$$

of the hyperreal extension $\mathbb{R}^*$ of $\mathbb{R}$, so that one has $\mathbb{R} \subset \mathbb{R}^* \subset \mathbb{R}^{**}$. What he means by $\mathbb{R}^{**}$ is $(\mathbb{R}^*)^*$. This can be obtained, for example, by applying the compactness theorem [110]. Earman’s conclusion is that

unfortunately, these second-order infinitesimals do not have the critical property that Leibniz assigned them; namely, if $\epsilon$ is a first order infinitesimal, then $(\epsilon)^2$ is second order (ibid.).

Earman is claiming that hyperreal infinitesimals do not have the property that the square of a first-order infinitesimal is a second-order infinitesimal.

Now it is true that a square of an infinitesimal in $\mathbb{R}^*$ will never yield an element of $\mathbb{R}^{**}$ which is incomparably smaller than every element
of its subfield $\mathbb{R}^* \subset \mathbb{R}^{**}$. But Earman’s dubbing $\mathbb{R}^{**}$ as being “second order” is artificial and unnecessary. The field $\mathbb{R}^{**}$ can indeed be described as a “secondary” non-standard extension of $\mathbb{R}$, but this notion of secondary has nothing to do with second-order infinitesimals. On the other hand, the ordinary square of an infinitesimal in Robinson’s hyperreals $\mathbb{R}^*$ will indeed be “second order” in Leibniz’s sense. Earman’s “second order” criticism of the hyperreals is merely a play on words and an instance of a strawman fallacy.

Earman grudgingly concedes that

It is true that at points non-standard analysis gives the appearance of following Leibniz’s strategy of neglecting infinitesimal terms, but the appearance is only a superficial one [32, p. 250].

What is superficial about such an appearance? Earman elaborates:

In the non-standard definition of derivative, it is true, for example that the $dx$ terms on the right hand side ... is [sic] in a sense ‘dropped’ in obtaining the answer ... But the ‘dropping’ comes from taking the standard part of the quantities ... [32, p. 250].

Earman feels that “dropping” the $dx$ terms by Leibniz is dissimilar from “dropping” them by means of the standard part function. Is it dissimilar? Discarding (‘dropping’) the remaining terms by means of the standard part function is indeed a modern mathematical implementation of the transcendental law of homogeneity (see Subsection 5.3), following Leibniz’s strategy. Earman declares that

Leibniz’s basic strategy of neglecting infinitesimal terms in comparison with finite ones is not followed in non-standard analysis [32, p. 250].

Earman claims that Leibniz’s strategy is not followed in Robinson’s theory. Or perhaps it is? Indeed, such “neglecting” was the object of Berkeley’s logical criticism analyzed in Subsection 5.2 and Section 4, where we saw that the binary relation of equality up to infinitesimal (via the law of homogeneity) was familiar to Leibniz. A similar strategy is indeed followed in non-standard analysis so as to produce a logically

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53To elaborate, Earman is simply mistaken to think that higher-order infinitesimals require higher order hyperreals. Thus, given a ‘first-order’ infinitesimal $dx \in \mathbb{R}^*$, a second-order infinitesimal in Leibniz’s sense would correspond to the square $dx^2 \in \mathbb{R}^*$, so that we don’t need to consider $\mathbb{R}^{**}$ at all.
consistent definition. Earman’s remarks amount to criticizing Robinson for providing an answer to Berkeley’s logical criticism. Earman’s remarks are therefore in error. He further announces that

there is . . . no evidence that Leibniz anticipated the techniques . . . of modern non-standard analysis [32, p. 250].

Here Earman contradicts his own discussion on page 245 of Leibniz’s law of continuity, in considering the question of “which of the statements true of ordinary quantities are also true of infinitesimal and infinite quantities, and vice versa?”:

Leibniz clearly perceived one form of this question, and in answer he says that “the rules of the finite are found to succeed in the infinite . . . and conversely, the rules of the infinite apply to the finite” (Earman [32, p. 245]).

Here Earman is citing Leibniz’s law of continuity, an antecedent of the transfer principle (see Subsections 4.2, 4.3, and 5.2). In this sense it can be said that Leibniz anticipated a technique of Robinson’s theory.

On page 251, in footnote 16, Earman cites Bos’ article from 1974, and writes: “The reader is urged to consult this excellent article”. Let us therefore turn to Bos.

11.3. Bos’ appraisal. A detailed 1974 scholarly study of Leibniz by Bos contains a brief appendix dedicated to non-standard issues [18, Appendix 2]. Bos notes that

to every given function \( f : \mathbb{R} \to \mathbb{R} \), is assigned a unique extension \( f^* : \mathbb{R}^* \to \mathbb{R}^* \), which preserves certain properties of \( f \). The field \( \mathbb{R}^* \) provides the framework for the development of the differential and integral calculus by means of infinitely small and infinitely large numbers (Bos [18, p. 81]).

To spell out Bos’ suggestion, we choose an infinitesimal increment \( \Delta x \) and calculate the corresponding increment \( \Delta y = f^*(x + \Delta x) - f^*(x) \). Bos [18, p. 82] goes on to reproduce a version of Robinson’s definition of the derivative:

\[
f'(x) := \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.
\]  (11.1)
Here the little circle to the upper left of the parenthetical expression on the right-hand-side of (11.1) stands for the standard part (of the expression included between the parentheses) Bos proceeds to express the following sentiment:

I do not think that the appraisal of a mathematical theory, such as Leibniz’s calculus, should be influenced by the fact that two and three quarter centuries later the theory is “vindicated” in the sense that it is shown that the theory can be incorporated in a theory which is acceptable by present-day mathematical standards (Bos [18, p. 82]).

Bos appears to feel that the appraisal of a mathematical theory, routinely described as logically inconsistent, should not be influenced by a demonstration that it can in fact be implemented by means of a consistent mathematical theory. Or perhaps it should be so influenced, as Bos himself pointed out earlier in his article? In fact, Bos acknowledged seventy pages earlier that Robinson’s hyperreals do indeed provide a preliminary explanation of why the calculus could develop on the insecure foundation of the acceptance of infinitely small and infinitely large quantities (Bos [18, p. 13]).

Is that not an instance of an appraisal being influenced by later vindication? Furthermore, the clarification of Leibniz’s heuristic law of continuity (see Subsections 4.2 and 4.3): “rules in the finite domain transfer to the infinite domain”, in Leibniz’s terminology, in terms of a precise transfer principle provided by Łos’s theorem is surely an instance of a remarkable vindication. Such a vindication certainly influences our appreciation of the historical theory. Thus, Laugwitz points out that Robinson’s infinitesimals can be seen as a revival of Leibniz’s fictions, without explicit mention of their well-foundedness. The latter is [...] made precise in a principle of transfer: the rules are founded on the rules of “finite totalities” [80, p. 152].

The most serious error of Bos’ examination of Robinson’s theory is that Bos misunderstands the usage of the term “transfer” by Robinson. Let us analyze the relevant passage from Robinson:

\[54\] See Appendix A Subsection A.3 Robinson did not work with the “st” notation, explained in Sections 8, 9, and Appendix A

\[55\] We reproduce Bos’ passage in first person.
Leibniz did say, in one of the passages quoted above, that what succeeds for the finite numbers succeeds also for the infinite numbers and vice versa, and this is remarkably close to our *transfer of statements* from \( \mathbb{R} \) to \( \mathbb{R}^* \) and in the opposite direction. (Robinson [126, p. 266]).

Bos expresses his disagreement in the following terms:

I cannot agree with [Robinson] that this is “remarkably close to our transfer of statements from \( \mathbb{R} \) to \( \mathbb{R}^* \) and in the opposite direction”, and in the context of this passage Robinson himself shows that Leibniz did not, and could not have, provided such a proof (Bos [18, p. 83]).

This may appear to be a disagreement among scholars as to a proper interpretation of historical theories. But pay close attention to the terminology Bos chooses to employ. Bos quotes Robinson’s phrase about “transfer”, but rejects a suggestion that Leibniz could have provided “proof”. Apparently he understood Robinson’s phrase as a spurious claim of a Leibnizian source for the detailed work of systematically “transferring” (i.e., proving) each of the \( \mathbb{R} \)-statements in an \( \mathbb{R}^* \)-context. None of such detailed work is to be found in Leibniz, to be sure. Has Bos then refuted Robinson? Certainly not.

Indeed, Robinson’s reference to “transfer” is shorthand for “transfer principle” (terminology introduced by later authors), which makes all real statements that may have been used by Leibniz *automatically* true in an \( \mathbb{R}^* \)-context. Robinson was *not* claiming that Leibniz was busy proving that real statements also apply to infinitesimals, as it were anticipating such work in non-standard analysis. What Robinson did claim is that the heuristic law of continuity as expressed by Leibniz (see Subsection 4.2), found expression in a precise metamathematical principle in Robinson’s theory, a point apparently missed by Bos. The point was not lost on Urquhart, who commented that some of Bos’ criticisms of Robinson involve absurd and impossible demands - for example, his first criticism (Bos, 1974, p. 83) is that Robinson proves the existence of his infinitesimals, whereas Leibniz does not! (Urquhart 2006, [152]).

Bos proceeds to make an apt criticism of Leibniz’s theory of infinitesimals by noting its inability to handle \( \sqrt{h} \) for infinitesimal \( h \) ([18, p. 83, last line]), due to the fact that Leibniz’s infinitesimals come only in
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Bos notes that Euler was aware of this problem [18, p. 84-86]. We may add that Cauchy developed a theory of infinitesimals of arbitrary (positive) real orders in 1829 (see Laugwitz [78, p. 272]), anticipating later theories of non-Archimedean continua due to Stolz and du Bois-Reymond in terms of rates of growth of functions, which in turn anticipated Skolem’s work on non-standard models of arithmetic [140] (see the historical discussion in Robinson [126, p. 278]). Some 28 years after Bos, Serfati will take up the subject of Robinson and the transfer principle again.

**11.4. Serfati’s creationist epistemology.** In 2002, M. Serfati challenged Robinson and the transfer principle in [136, p. 317-318]. Serfati recognizes that Robinson proved such a transfer principle between different mathematical domains, which Serfati describes as a significant accomplishment in metamathematics. Serfati proceeds to add, however, that such an accomplishment does not concern the creation of those mathematical domains in the first place. Serfati therefore describes Robinson’s concern as having an “objectif juridique”. Meanwhile, Serfati’s own “principe de prolongement” is described as dealing with the creation of mathematical objects, and therefore is touted as being concerned with the epistemology of (mathematical) creation, as opposed to what he describes as “du juridique”, i.e., Robinson’s contribution.

Serfati feels that the novelty in this particular area is the creation of the mysterious entities such as infinitesimals and infinite numbers. Once they are created, we can worry about “juridical” issues of comparing them with the usual entities, issues of secondary importance according to Serfati.

The essential point Serfati misses is that non-Archimedean mathematical objects themselves were already old hat by the time Robinson came on the scene. A long line of work on non-Archimedean systems (Stolz, du Bois-Reymond, Levi-Civita, Hilbert, Skolem, Hewitt, Schmieden–Laugwitz, to name only a few) testifies to Serfati’s misconception. The real novelty resides in the transfer principle allowing one productively to apply such systems in mathematics.

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56Leibniz did consider expressions like $\frac{d^{1/2}x}{2}$ and even gave an explanation of this expression; see his letter to L’Hospital from 30 September 1695 (Leibniz, [92]).

57It is interesting to note a criterion of success of a theory of infinitesimals as proposed by Adolf Abraham Fraenkel and, before him, by Felix Klein. In 1908, Klein formulated a criterion of what it would take for a theory of infinitesimals to be successful. Namely, one must be able to prove a mean value theorem for arbitrary intervals, including infinitesimal ones [74, p. 219]. In 1928, A. Fraenkel [40, pp. 116-117] formulated a similar requirement in terms of the mean value theorem. Such
On the last line on page 317, Serfati mentions Leibniz’s principle of continuity, implying that it fits better with his “creationist epistemology” than with Robinson’s “juridical metamathematics”. In fact, the opposite is true: Leibniz’s law of continuity (see Subsection 4.2) was precisely a principle relating two domains: the finite and the infinite. Infinitesimals were widely used before Leibniz. The novelty of Leibniz’s law of continuity is not the introduction of infinitesimals, but rather the formulation of a resilient heuristic principle by Leibniz in an attempt to relate the finite to the infinite. Serfati’s presentation of his epistemological creationist principle as being on a higher level than Robinson’s transfer principle remains unsupported. At any rate, what Serfati’s article does reveal is that Serfati is perfectly aware of the connection between Leibniz’s law of continuity, on the one hand, and the transfer principle for the hyperreals, on the other.

Two years ago, a new text by Serfati appeared as a book chapter [137]. Serfati reproduces an interesting Leibniz quote on page 14 where Leibniz speaks explicitly of equality as encompassing “equivalence up to an infinitely small”, the word “equivalence” originating in Leibniz. Serfati proceeds to mention non-standard analysis in the following terms:

contemporary mathematics has been able through non-standard analysis to give a meaning (in a certain sense and at the cost of the well-known complications) [to] the Leibnizian infinitely small [137, p. 15]

Serfati does not mention the fact that non-standard analysis gives meaning to the law of continuity; he only mentions the fact that it gives meaning to infinitesimals.

On page 26 Serfati discusses the disagreement between Cauchy and Poncelet. Poncelet exploited an extension of the principle of continuity, known in the 18th and 19th centuries as the principle of the “generality of algebra”, to study properties of algebraic curves, and Cauchy was critical of this attitude, consistent with his opposition to the “generality of algebra” in his Cours d’Analyse and elsewhere (curiously, Serfati does not mention the term “generality of algebra”). Serfati proceeds to point out that “Cauchy himself had failed in his work due to an intuitive application—this time, false—of the schema” (of the principle of continuity). Here Serfati is referring to Cauchy’s sum theorem of 1821, which he mentioned as an example (without using the name) a few pages earlier. Serfati is assuming that Cauchy made an error in 1821.
and ignores the essential ambiguity of the 1821 result (see Bråting [22], Cutland et al. [29]). Serfati is more explicit about Cauchy’s “error” on page 28 and in footnote 33 (whose text appears on page 30). Serfati goes on to write:

Nowadays, therefore, 300 years after Leibniz, the “principle of continuity” belongs to an interiorized set of methodological rules. Like the principle of symmetry (considered as normative) or that of generalisation–extension (considered as a standard procedure of construction of algebraic or topological objects), they constitute part of the daily mathematical practice.

Serfati concludes:

Yet, they are never made explicit as such. Sometimes there is talk about this or that proof by continuity, but no manual discusses the principle of such proofs [137, p. 28].

Now Keisler’s instructor’s manual [71] (companion to his Elementary Calculus [72]) does discuss both Leibniz’s law of continuity, the transfer principle, and the relation between them (see Subsection 4.2 above for some examples). Thus, the significance of Leibniz’s heuristic principle, both philosophically and mathematically, was and is clearly realized by the practitioners of non-standard analysis.

Claiming that Leibniz’s system for differential calculus was free of logical fallacies may appear similar to claiming the possibility of squaring the circle\(^{58}\) to a historian, but only if the latter internalized a triumviratist spin on the history of mathematics as an ineluctable march, away from logically fallacious infinitesimals, and toward the yawning heights of Weierstrassian epsilontics.

11.5. Cauchy and Moigno as seen by Schubring. To illustrate the significance of the distinction between Berkeley’s pair of criticisms, consider Cauchy’s proof of the intermediate value theorem [26, Note III, p. 460-462]. Fowler [43] discussed Cauchy’s proof in the context of the decimal representation, and made the connection between Cauchy and Stevin.\(^{59}\) Cauchy constructs an increasing sequence \(a_n\) and a decreasing

\(^{58}\)Such was indeed the tenor of a recent referee report, see [http://u.cs.biu.ac.il/~katzmik/straw2.html](http://u.cs.biu.ac.il/~katzmik/straw2.html)

\(^{59}\)Fowler notes that “Stevin described an algorithm for finding the decimal expansion of the root of any polynomial, the same algorithm we find later in Cauchy’s proof of the intermediate value theorem” [43, p. 733]. The matter is discussed in detail in Blaszczyk et al. [15]. See also footnote [62].
sequence $b_n$ of successive approximations, $a_n$ and $b_n$ becoming successively closer than any positive distance. At this stage, the desired point is considered to have been exhibited, by Cauchy. A triumvirate scholar would object that Cauchy has not, and could not have, proved the existence of the limit. But imagine that the polytechnicien Auguste Comte had asked M. le Professeur Cauchy the following question:

Consider a decimal rank $k > 0$. What is happening to the $k$-th decimal digit $a_n^k$ of $a_n$, and $b_n^k$ of $b_n$?

Cauchy would have either sent Comte to the library to read Simon Stevin (1548-1620), or else provided a brief argument to show that for $n$ sufficiently large, the $k$-th digit stabilizes, noting that special care needs to be taken in the case when $a_n$ is developing a tail of 9s and $b_n$ is developing a tail of 0s. Clearly the arguments appearing in Cauchy’s book are sufficient to identify the Stevin decimal expression of the limit. From the modern viewpoint, the only item missing is the remark that a Stevin decimal is a number, by definition (modulo the identification of the pair of tails).

Schubring reports on a purportedly successful effort by the cleric Moigno, a student of Cauchy’s, “to pick apart” (see [135, p. 445]) infinitesimal methodologies. Here Moigno puzzles over how an infinitesimal magnitude can possibly be less than its own half:

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60 Cauchy’s notation for the two sequences is $x_0, x_1, x_2, \ldots$ and $X, X', X'', \ldots$ [26, p. 462].

61 Comte’s notes of Cauchy’s lectures have been preserved (see [135, p. 437]).

62 Fearnley-Sander writes that “the modern concept of real number […] was essentially achieved by Simon Stevin, around 1600, and was thoroughly assimilated into mathematics in the following two centuries” [38, p. 809]. Fearnley-Sander’s sentiment is echoed by van der Waerden [153, p. 69]. Stevin had anticipated Cauchy’s proof of the intermediate value theorem, and produced a fine-tuned version of the iteration, where each step of the iteration produces an additional digit of the decimal expansion of the solution. The algorithm is discussed in more detail in [141, §10, p. 475-476]. Stevin subdivides the interval into ten equal parts, resulting in a gain of a new decimal digit of the solution at every iteration of the algorithm. Who needs the “existence” of the real numbers when Stevin constructs an explicit decimal representation of the solution? See also footnote 59.

63 The pioneers of infinitesimal calculus were aware of the non-uniqueness of decimal representation (at least) as early as 1770, see Euler [37, p. 170].

64 Once the real numbers have been defined, considerable technical difficulties remain in the definition of the multiplication and other algebraic operations. They were overcome by Dedekind (see Fowler [43]).
these magnitudes, [assumed to be] smaller than any given magnitude, still have substance and are divisible; however, their existence is a chimera, since, necessarily greater than their half, their quarter, etc., they are not actually less than any given magnitude (as quoted by Schubring [135, p. 456]).

Moigno is clearly focusing on Berkeley’s metaphysical criticism, not the logical criticism. As far as the metaphysical criticism is concerned, the solution was readily available in Cauchy’s work, and in fact already in the work of Leibniz. The solution is in terms of the distinction between variable magnitude and constant magnitude; namely, Cauchy’s stratified hierarchy of constant quantities (A-continuum) englobed inside an enriched B-continuum of variable quantities including infinitesimals generated by null sequences (see Section 8).

Schubring described Moigno as

the first writer to pick apart the traditional claim in favor of their purported simplicité [135, p. 445].

However, Moigno did not take apart the simplicity of infinitesimals, either purported or real. Rather, Moigno was confused, as many a modern triumvirate scholar. Similarly, in his 2007 anthology [53], S. Hawking reproduces Cauchy’s infinitesimal definition of continuity on page 639; but claims on the same page 639, in a comic non-sequitur, that Cauchy “was particularly concerned to banish infinitesimals”. In the same vein, historian J. Gray lists continuity among concepts Cauchy allegedly defined

using careful, if not altogether unambiguous, limiting arguments [52, p. 62] [emphasis added—authors],

whereas in reality limits appear in Cauchy’s definition only in the sense of the endpoints of the domain of definition.

11.6. Bishop and Connes. Analyses of the critiques of Robinson’s infinitesimals by E. Bishop and A. Connes appear respectively in [67] and [68].

65 On occasion, Cauchy uses inequalities, rather than equations involving infinitesimals, as, for instance, in Theorems I and II in section 3 of chapter 2 of the Cours d’analyse (see Grabiner [50]). However, the thrust of his foundational approach, pace Grabiner, is to use infinitesimals (generated by null sequences) as inputs to functions, i.e., as individuals/atomic entities; to define continuity in terms of infinitesimals; and to apply infinitesimals to a range of problems, including an infinitesimal definition of the “Dirac” delta function (see Freudenthal [42, p. 136] and Laugwitz [81]).
12. Horváth's Analysis

In a 1986 text in *Studia Leibnitiana*, M. Horváth [55], takes a critical view of a certain trend in contemporary Leibniz scholarship. We summarize some of Horváth’s main points below.

12.1. Leibniz and Nieuwentijt. Bernard Nieuwentijt possessed dramatically different intuitions about infinitesimals as compared to Leibniz’s. Nieuwentijt favored nilsquare infinitesimals: if $dx$ is infinitesimal, then one should have $dx^2 = 0$. Meanwhile, Leibniz was attached to infinitesimals $dx^n \neq 0$ of arbitrary order. With hindsight, we know that both were right: Leibniz’s intuitions are implemented in Robinson’s theory, whereas Nieuwentijt’s, in Lawvere’s Smooth Infinitesimal Analysis (see J. Bell [11, 12]).

What interests us here is Leibniz’s response to Nieuwentijt’s criticism, discussed in [55, p. 63]. Nieuwentijt insists that $dx^2 = 0$. Leibniz responded in letters to both L’Hopital [90, p. 288] and to Huygens [91, p. 207] dating from 1695, twenty years after the *Quadratura Arithmetica* (similar comments in *Cum Prodiisset* were discussed in Subsection 4.1).

Leibniz responds that arbitrary orders of infinitesimals are necessary so as to accommodate arbitrary orders of infinity (by inversion). Note that Nieuwentijt’s criticism could have been answered more simply, by affirming that $dx$ is not an actual infinitesimal but merely a manner of speaking, representing a shorthand for exhaustion à la Archimedes. Certainly Nieuwentijt did not hold that ordinary Stevin numbers [66] can be nilsquare. The fact that Leibniz does not do so demonstrates that he and Nieuwentijt were of one mind as to the non-Archimedean nature of $dx$, and only disagreed about “higher-order” matters.

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66 It is therefore puzzling to find R. Arthur [7] insisting, in J. Bell’s name, on similarities between Leibniz’s approach and that of Smooth Infinitesimal Analysis (SIA). Arthur analyzes Bell’s notion of an (intuitionistic) infinitesimal $x$ as satisfying the relation $\neg\neg x = 0$. Bell describes such an $x$ as “indistinguishable from 0”. In more detail, Bell’s infinitesimal $x$ satisfies $\neg\neg(\neg(\neg(x = 0)))$. In classical logic this would imply $x = 0$, but not in intuitionistic logic. This can certainly sound like a “fictional” entity to a classically-trained audience, but no intuitionist has been known to have embraced fictionalism about anything in mathematics (certainly not E. Bishop—see [67]; indeed, many intuitionists insist that all mathematical expressions refer to constructible objects), and at any rate such “fictionality” certainly has nothing to do with Leibniz’s. Certainly Arthur’s claim that SIA infinitesimals are variable quantities unlike Robinson’s is incorrect. Arthur proceeds to describe Bell’s infinitesimals as “fictional”, and bases his analogy with Leibniz on the latter term, which procedure strikes us as an unconvincing pun (see also footnote 38).
12.2. Leibniz’s pair of dual methodologies for justifying the calculus. Leibniz pursued two different tracks for answering the critics of his calculus: an Archimedean track and a track exploiting infinitesimals with the law of continuity. The latter track appears in texts posterior to his *Quadratura Arithmetica*, e.g., texts dating from 1680 and 1701 (see below). Horváth writes:

The first approach is based on the fact that Leibniz regards his infinitesimal calculus as an abbreviated language for proofs given by the Greek method of exhaustion. In connection with this method Leibniz often refers to Archimedes by name (Horváth [55, p. 65-66]).

We refer to this approach briefly as the A-methodology. Horváth proceeds to describe the alternative methodology in the following terms:

the second approach lies in the fact that, in Leibniz’s mind, the rules of his calculus can be proved by his so-called principle of continuity [55, p. 66].

The approach based on the law of continuity involves infinitesimals (see Subsection 12.2 and Section 5). We refer to it briefly as the B-methodology.

Leibniz explains the A-methodology in his letter to Pinson as follows:

In our calculations there is no need to conceive the infinite in a rigorous way. For instead of the infinite or the infinitely small, one takes quantities as large, or as small, as necessary in order that the error be smaller than the given error, so that one differs from Archimedes’ style only in the expressions, which are more direct in our method and conform more to the art of invention (Leibniz [96] cited in Horváth [55, p. 66]).

Leibniz clarifies his fictionalist position concerning the B-methodology as involving “ideal concepts” in a 1702 letter to Varignon in the following terms:

even if someone refuses to admit infinite and infinitely small lines in a rigorous metaphysical sense and as real things, he can still use them with confidence as ideal concepts which shorten the reasoning (Leibniz [99, p. 92], cited in Horváth [55, p. 66]).

Leibniz presented the dual pair of methodologies in an article dating from the 1700s (see Subsection 12.5).

12.3. Incomparable quantities and the Archimedean property. Leibniz uses the term “incomparable quantities” for quantities that
are supposed to implement the reduction of the calculus to the A-methodology in a passage that also conveys the embattled situation the defenders of the calculus found themselves in, in the face of their “opponents”:

These incomparable quantities are not at all fixed or determined but can be taken to be as small as we wish in our geometrical reasoning and so have the effect of the infinitely small in the rigorous sense. If any opponent tries to contradict this proposition, it follows from our calculus that the error will be less than any possible assignable error, since it is in our power to take that incomparably small quantity small enough for that purpose, inasmuch as we can always take a quantity as small as we wish (Leibniz [99, p. 92] cited in Horváth [55, p. 66]).

On the other hand, elsewhere Leibniz defines such “incomparable quantities” in terms of the violation of what today is called the Archimedean property. Thus, Leibniz writes in a letter to l’Hôpital:

I call incomparable quantities of which the one can not become larger than the other if multiplied by any finite number. This conception is in accordance with the fifth definition of the fifth book of Euclid (Leibniz [91, p. 288], cited in Horváth [55, p. 63]).

Here Leibniz employs the term “incomparable quantity” in the sense of a non-Archimedean quantity, in the context of a B-continuum.

12.4. From Leibniz’s *Elementa* to his *Nova Methodus*. Leibniz’s manuscript *Elementa calculi novi* [87] dates from 1680. The text *Elementa* is a preliminary draft of his famous paper *Nova Methodus* [88] dating from 1684.

The text had undergone a significant transformation between 1680 and 1684. Thus, in *Elementa*, Leibniz describes quantities he denotes \(1D_2C, 2C_3D, \ldots\) as “*incrementa momentanea*”, i.e., infinitely small. By 1684, however, a change occurs:

Leibniz does not use differentials but only differences in the sense of fixed, small, finite quantities. Leibniz [presumably] does not use the term “infinitely small” in his article [1684] in order to avoid controversies which

\(^{67}\text{Leibniz is apparently referring to the *fourth* definition.}\)
most likely would have arisen in connection with this notion (Horváth [55, p. 62]).

Thus, in 1675 and 1684, Leibniz emphasizes the A-methodology, but five years after *Quadratura Arithmetica* and four years before *Nova Methodus*, he relies upon the B-methodology.

12.5. **Justification du calcul des infinitesimales.** Leibniz penned an additional defense of his calculus in the early 1700s. The 1700 version is entitled *Defense du calcul des differences* [95]. The 1701 version is entitled *Justification du Calcul des infinitesimales...* [97], and is the published version of the article. In his *Defense*,

> Leibniz appeals to the method of Archimedes in the first part of the draft, while the allusion to the law of continuity can be found in the second part of his draft [55, footnote 27, p. 69].

The order is reversed in his *Justification*. Thus, both the A-methodology and the B-methodology are present at this late stage, and it is the latter that gets right-of-way, a quarter century after *Quadratura Arithmetica*.

13. A Leibniz–Robinson route out of the labyrinth?

We have dissected Berkeley’s critique into its component parts following Sherry, and have revealed the implausibility of some of the assumptions underlying that critique. We have discussed both the critique’s ill-informed nature, and Berkeley’s contradictory attitude when writing about a different field of mathematics, such as arithmetic.

A significant, and widely denied, aspect of the story is the existence of a direct perceptual, cognitive, and even formal connection between historical infinitesimals as they were practiced by giants like Leibniz and Cauchy, via the work of Stolz, Paul du Bois-Reymond, Veronese, and others at the turn of the century, and the emergence of hyper-real fields in the middle of the 20th century.

Berkeley’s *Philosophical Commentaries* [14] is an early work, dating from approximately 1709. Here Berkeley asserts that Euclidean geometry is full of paradoxes. He backs away from this assertion as he matures, but later in his *Principles*, his most mature work, it turns out that he won’t accept infinite divisibility and sees it on a par with infinitesimals. How is it that Berkeley has no difficulty with imaginary roots and yet balks at infinite divisibility?

Berkeley is prepared to be instrumentalist about imaginary roots, yet refuses to be instrumentalist about certain idealisations of the continuum involved in the calculus. Another puzzling aspect of his position
is the following. If he indeed does not accept the latter, why does he provide a “cancellation of errors” justification of the latter? The justification turns out to be meaningless symbol-pushing, according to a recent article by Andersen [2]; or more precisely circular symbol pushing: the fact that everything works out in the end is based on a result due to Apollonius of Perga [4, Book I, Theorem 33] on the tangents of the parabola, which is equivalent to the calculation of the derivative in the quadratic case. Thus, Berkeley’s logical criticism is further weakened by the circular logic he relied upon in his “compensation of errors” approach. Given Berkeley’s fame among historians of mathematics for allegedly spotting logical flaws in infinitesimal calculus, it is startling to spot circular logic at the root of Berkeley’s own doctrine of compensation of errors, seeing that his new, improved calculation of the derivative of $x^2$ relies upon Apollonius’ determination of the tangent to a parabola.

There is indeed an irreducible conceptual clash between Leibniz and Berkeley. One of the sides had to give way. What historians sometimes do not fully appreciate is the fact that the weaker side was Berkeley’s, not Leibniz’s. Modern mathematics would not even start without non-referential concepts that would have been ridiculed by Berkeley as meaningless due to his empiricist bias.

Edwin Hewitt [54] in 1948 constructed infinitesimal-enriched continua and introduced the term hyper-real. In 1955, J. Loś [107] proved what has come to be known as Loś’s theorem (for ultraproducts), whose consequence is the transfer principle for hyper-real fields, which embodies a mathematical implementation of Leibniz’s heuristic law of continuity.

ACKNOWLEDGEMENTS

We are grateful to H. Jerome Keisler for helpful remarks that helped improve an earlier version of the manuscript. The influence of Hilton Kramer (1928-2012) is obvious.

APPENDIX A. RIVAL CONTINUA

The historical roots of infinitesimals go back to Cauchy, Leibniz, and ultimately to Archimedes. Cauchy’s approach to infinitesimals is not a variant of the hyperreals. Rather, Cauchy’s work on the rates of growth of functions anticipates the work of late 19th century investigators such as Stolz, du Bois-Reymond, Veronese, Levi-Civita, Dehn, and others, who developed non-Archimedean number systems against virulent opposition from Cantor, Russell, and others (see Ehrlich [34].
Figure 4. Zooming in on infinitesimal $\epsilon$ (here $\text{st}(\pm \epsilon) = 0$)

and Katz and Katz [64] for details). The work on non-Archimedean systems motivated the work of T. Skolem on non-standard models of arithmetic [130], which subsequently stimulated work culminating in the hyperreals of Hewitt, Los, and Robinson.

The relation between the rival theories of the continuum distinguished by Felix Klein (see Subsection 8.1) can be summarized as follows. A Leibnizian definition of the differential quotient

$$\frac{\Delta y}{\Delta x},$$

whose logical weakness was criticized by Berkeley, was modified by A. Robinson by exploiting a map called the standard part, denoted “st”, from the finite part of a “thick” B-continuum (i.e., a Bernoullian continuum) to a “thin” A-continuum (i.e., an Archimedean continuum), as illustrated in Figures 2 and 4. The derivative is defined as $\text{st} \left( \frac{\Delta y}{\Delta x} \right)$, rather than the differential quotient $\frac{\Delta y}{\Delta x}$ itself. Robinson wrote that “this is a small price to pay for the removal of an inconsistency” (Robinson [126, p. 266]). However, the process of discarding the higher-order infinitesimals has solid roots in Leibniz’s law of homogeneity (see Subsection 5.3).

A.1. Hyperreals via maximal ideals. We summarize a 20th century implementation of an alternative to an Archimedean continuum, namely an infinitesimal-enriched continuum. Such a continuum is not to be confused with incipient notions of such a continuum found in earlier centuries. We refer to such a continuum as a B-continuum.68

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68See footnote [6]
We begin with a heuristic representation of a B- or “thick” continuum, denoted \( \mathbb{IR} \), in terms of an infinite resolution microscope (see Figure 4). One presentation of such a structure is in Robinson \([125]\). Such an infinitesimal-enriched continuum is suitable for use in calculus, analysis, and elsewhere. Robinson built upon earlier work by E. Hewitt \([54]\), J. Loś \([107]\), and others. In 1962, W. Luxemburg \([108]\) popularized a presentation of Robinson’s theory in terms of the ultrapower construction in the mainstream foundational framework of the Zermelo–Fraenkel set theory with the axiom of choice (ZFC).

The construction can be viewed as a relaxing, or refining, of Cantor’s construction of the reals. This can be motivated by a discussion of rates of convergence as follows. In Cantor’s construction, a real number \( u \) is represented by a Cauchy sequence \( \langle u_n : n \in \mathbb{N} \rangle \) of rationals. But the passage from \( \langle u_n \rangle \) to \( u \) in Cantor’s construction sacrifices too much information. We would like to retain a bit of the information about the sequence, such as its “speed of convergence”. This is what one means by “relaxing” or “refining” Cantor’s construction of the reals (cf. Giordano et al. \([48]\)). When such an additional piece of information is retained, two different sequences, say \( \langle u_n \rangle \) and \( \langle u'_n \rangle \), may both converge to \( u \), but at different speeds. The corresponding “numbers” will differ from \( u \) by distinct infinitesimals. If \( \langle u_n \rangle \) converges to \( u \) faster than \( \langle u'_n \rangle \), then the corresponding infinitesimal will be smaller. The retaining of such additional information allows one to distinguish between the equivalence class of \( \langle u_n \rangle \) and that of \( \langle u'_n \rangle \) and therefore obtain distinct hyperreals infinitely close to \( u \).

At the formal level, we proceed as follows. We construct a hyperreal field as a quotient of the collection of arbitrary sequences, where a sequence
\[
\langle u_1, u_2, u_3, \ldots \rangle
\]
converging to zero generates an infinitesimal (the kernel of the quotient homomorphism is the maximal ideal \( \mathcal{M} \) described below). Arithmetic operations are defined at the level of representing sequences; e.g., addition and multiplication are defined term-by-term. Thus, we start with the ring \( \mathbb{Q}^\mathbb{N} \) of sequences of rational numbers. Let
\[
\mathcal{C}_Q \subset \mathbb{Q}^\mathbb{N}
\]

\[69\)Note that both the term “hyper-real”, and an ultrapower construction of a hyperreal field, are due to E. Hewitt in 1948, see \([54\) p. 74]. Luxemburg \([108\ also clarified its relation to the competing construction of Schmieden and Laugwitz \([134\), also based on sequences, which used a different kind of filter.\]
denote the subring consisting of Cauchy sequences. The reals are by
definition the quotient field
\[ \mathbb{R} := \mathbb{C}_\mathbb{Q} / \mathbb{F}_{\text{null}}, \quad (A.3) \]
where \( \mathbb{F}_{\text{null}} \) is the ideal containing all null sequences (i.e., sequences
tending to zero)\(^{70}\). Note that \( \mathbb{Q} \) is imbedded in \( \mathbb{Q}^\mathbb{N} \) by constant se-
quences. An infinitesimal-enriched extension of \( \mathbb{Q} \) may be obtained by modifying \((A.3)\). Now consider the subring
\[ \mathcal{F}_{\text{ez}} \subset \mathbb{F}_{\text{null}} \]
of sequences that are “eventually zero”, i.e., vanish at all but finitely
many places. Then the quotient \( \mathbb{C}_\mathbb{Q} / \mathcal{F}_{\text{ez}} \) naturally surjects onto \( \mathbb{R} = \mathbb{C}_\mathbb{Q} / \mathbb{F}_{\text{null}} \). The elements in the kernel of the surjection
\[ \mathbb{C}_\mathbb{Q} / \mathcal{F}_{\text{ez}} \to \mathbb{R} \]
are prototypes of infinitesimals\(^{71}\). Note that the quotient \( \mathbb{C}_\mathbb{Q} / \mathcal{F}_{\text{ez}} \) is
not a field, as \( \mathcal{F}_{\text{ez}} \) is not a maximal ideal. To obtain a field, we must
replace \( \mathcal{F}_{\text{ez}} \) by a maximal ideal.

It is more convenient to describe the modified construction using the
ring \( \mathbb{R}^\mathbb{N} \) rather than \( \mathbb{C}_\mathbb{Q} \) of \((A.2)\).

We therefore redefine \( \mathcal{F}_{\text{ez}} \) to be the ring of real sequences in \( \mathbb{R}^\mathbb{N} \) that
eventually vanish, and choose a maximal proper ideal \( \mathcal{M} \) so that we
have
\[ \mathcal{F}_{\text{ez}} \subset \mathcal{M} \subset \mathbb{R}^\mathbb{N}. \quad (A.4) \]
Then the quotient
\[ \mathbb{I}\mathbb{R} := \mathbb{R}^\mathbb{N} / \mathcal{M} \quad (A.5) \]
is a hyperreal field. The foundational material needed to ensure the
existence of a maximal ideal \( \mathcal{M} \) satisfying \((A.4)\) is weaker than the axiom
of choice. This concludes the construction of a hyperreal field \( \mathbb{I}\mathbb{R} \) in
the traditional foundational framework, ZFC.

The construction is equivalent to the usual ultrapower construction
as popularized by Luxemburg\(^{72}\). Thus it is not entirely accurate to
suppose, as Jesseph does, that a consistent theory of infinitesimals

\(^{70}\)Namely, the traditional construction of the real field, usually attributed to
Cantor, views a real number as an equivalence class of Cauchy sequences of rational
numbers. Null sequences comprise the equivalence class corresponding to the real
number \( 0 \in \mathbb{R} \).

\(^{71}\)Such elements could be called “infinitesimal” to the extent that they violate
the Archimedean property suitably interpreted, but the ring they are elements of
has unsatisfactory properties.

\(^{72}\)Analyzing \( \mathcal{M} \), one discovers that its structure is controlled by a free ultrafilter
on \( \mathbb{N} \). The order relation on \( \mathbb{I}\mathbb{R} \) is defined relative to the ultrafilter. Additional
details may be found in \([15] \) Appendix A.\]
requires the resources of model theory. The resources of a rigorous undergraduate course in abstract algebra suffice.

A.2. Example. To give an example, the sequence
\[ \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \tag{A.6} \]
represents a nonzero infinitesimal, in the sense that its class \([\frac{1}{n}]\) in (A.5) is nonzero and satisfies \([\frac{1}{n}] < r\) for every positive real number \(r\).73

A.3. Construction of standard part. In the field \(\mathbb{R}\) of (A.5), consider the subring \(I \subset \mathbb{R}\) consisting of infinitesimal elements (i.e., elements \(e\) such that \(|e| < \frac{1}{n}\) for all \(n \in \mathbb{N}\)). Denote by \(I^{-1}\) the set of inverses of nonzero elements of \(I\). The complement \(\mathbb{R} \setminus I^{-1}\) consists of all the finite (sometimes called limited) hyperreals. Constant sequences provide an inclusion \(\mathbb{R} \subset \mathbb{R}\). Every element \(x \in \mathbb{R} \setminus I^{-1}\) is infinitely close to some real number \(x_0 \in \mathbb{R}\). The standard part function, denoted “st”, associates to every finite hyperreal, the unique real infinitely close to it:

\[ st : \mathbb{R} \setminus I^{-1} \rightarrow \mathbb{R}, \text{ with } x \mapsto x_0. \]

The real \(x_0\) is sometimes called the shadow of \(x\). If \(x\) happens to be the equivalence class of a Cauchy sequence \(\langle x_n : n \in \mathbb{N} \rangle\), then the shadow of \(x\) is the limit of \(\langle x_n \rangle\):

\[ st(x) = \lim_{n \to \infty} x_n. \]

As explained in Subsection 5.3, the standard part function can be seen as an implementation of Leibniz’s transcendental law of homogeneity.

A.4. The transfer principle. The transfer principle is a mathematical implementation of Leibniz’s heuristic law of continuity: “what succeeds for the finite numbers succeeds also for the infinite numbers and vice versa” (see Robinson [126, p. 266]). The transfer principle, allowing an extension of every first-order real statement to the hyperreals, is a consequence of the theorem of J. Loś in 1955 (see [107]), and can therefore be referred to as a Leibniz-Łoś transfer principle. A Hewitt-Łoś framework allows one to work in a B-continuum satisfying the transfer principle.

A helpful “semicolon” notation for presenting an extended decimal expansion of a hyperreal was described by A. H. Lightstone [105]. See also P. Roquette [130] for infinitesimal reminiscences. A discussion of infinitesimal optics is in K. Stroyan [143], J. Keisler [72], D. Tall [146],

73See footnote 72.
74Some examples were provided in Subsection 4.5 following formula (4.4).
Figure 5. Differentiating $y = f(x) = x^2$ at $x = 1$ yields

$$\frac{\Delta y}{\Delta x} = \frac{f(.9\ldots) - f(1)}{.9\ldots - 1} = \frac{(.9\ldots)^2 - 1}{.9\ldots - 1} = \frac{(.9\ldots - 1)(.9\ldots + 1)}{.9\ldots - 1} = .9\ldots + 1 \approx 2.$$ 

Here $\approx$ is the relation of being infinitely close. Hyperreals of the form $.9\ldots$ are discussed in [63].

L. Magnani & R. Dossena [109], [31], and Bair & Henry [8]. Applications of the B-continuum range from aid in teaching calculus [36, 62, 63, 147, 148] (see illustration in Figure 5) to the Boltzmann equation (see L. Arkeryd [5, 4]); modeling of timed systems in computer science (see H. Rust [133]); Brownian motion and economics (see Anderson [2]); mathematical physics (see Albeverio et al. [1]); etc. The hyperreals can be constructed out of integers (see Borovik, Jin, & Katz [16]). The traditional quotient construction using Cauchy sequences, usually attributed to Cantor, can be factored through the hyperreals (see Giordano & Katz [48]).

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Mikhail G. Katz is Professor of Mathematics at Bar Ilan University, Ramat Gan, Israel. His joint study with P. Blaszczyk and D. Sherry entitled Ten misconceptions from the history of analysis and their debunking is due to appear in Foundations of Science. A joint study with A. Borovik entitled Who gave you the Cauchy–Weierstrass tale? The dual history of rigorous calculus appeared in Foundations of Science (online first). A joint study with A. Borovik and R. Jin entitled An integer construction of infinitesimals: Toward a theory of Eudoxus hyperreals is due to appear in Notre Dame Journal of Formal Logic 53 (2012), no. 4. A joint study with Karin Katz entitled A Burgessian critique of nominalistic tendencies in contemporary mathematics and its historiography, appeared in Foundations of Science 17 (2012), no. 1, 51–89. A joint study with Karin Katz entitled Meaning in classical mathematics: is it at odds with Intuitionism? appeared in Intelectica 56 (2011), no. 2, 223-302. A joint study with Karin Katz, Stevin numbers and reality, appeared in Foundations of Science (online first). A joint study with Eric Leichtnam, Commuting and non-commuting infinitesimals, is due to appear in American Mathematical Monthly. A joint study with David Tall entitled The tension between intuitive infinitesimals and formal mathematical analysis, appeared as a chapter in a book edited by Bharath Sriraman, Crossroads in the History of Mathematics and Mathematics Education, The Montana Mathematics Enthusiasts Monographs in Mathematics Education 12, Information Age Publishing, Inc., Charlotte, NC, 2012.

David Sherry is fortunate to be professor of philosophy at Northern Arizona University in the cool pines of the Colorado Plateau.