Static elastic deformations in general relativity

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Abstract
We present a new approach to the theory of static deformations of elastic test bodies in general relativity based on a generalization of the concept of frame of reference which we identify with the concept of quo-harmonic congruence. We argue on the basis of this new approach that weak gravitational plane waves do not couple to elastic bodies and therefore the latter, whatever their shape, are not suitable antennas to detect them.

1 Introduction
The theory of the behavior of resonant elastic test bodies in the field of weak gravitational waves is based usually on the following approach: first of all one uses a Fermi frame of reference with origin the center of mass of the bar to derive a field of forces acting on the bar, or equivalently one invokes the equation of geodesic deviation between the geodesic of the center of mass and that of any other element of the bar; secondly one claims that since the force fields derived using either one of the preceding methods is very small one can use classical elasticity theory to describe the coupling between the wave and the bar. We claim that the first idea on which this approach is based is wrong, and the second needs to be understood in a more general context. Our opinion is based on the following points:
i) The identification of a Fermi congruence with a frame of reference is unacceptable except in the case where the Fermi congruence is homogeneous, i.e. it is such that any of the world-lines of the congruence can be considered to be the origin of the congruence, in which case the Fermi congruence is a Born rigid congruence. This is not the case with the congruence that is used to derive the field of forces in a resonant bar. Moreover the justification of both, the Fermi congruences and the equation of geodesic deviation, depends crucially on the implicit postulate according to which whatever the circumstances being considered the quantity:

$$\int \sqrt{g_{\alpha\beta} dx^\alpha dx^\beta}$$

(1)
calculated along any space-like geodesic, between two events, one of them lying on a world-line to this geodesic, can be identified to a physical length. This length-postulate, say, seems justified as being perfectly symmetric to the time-postulate according to which the integral (1) calculated between two events on any time-like world-line can be identified with a time interval measured with an appropriate clock. Besides their a priori esthetic value both the time and length postulates have been verified experimentally to some accuracy.

The time-postulate has been satisfactorily tested directly, it gives an explanation of the dependence of the mean-life of elementary particles on its velocity with respect a galilean frame of reference and it is daily at work, so to speak, in navigation systems. Since the length-postulate is equivalent to assuming the isotropy of the speed of light and its independence of the event where it is measured, every experiment of the Michelson-Morley type tests it to some accuracy for small values of the integral (1). The best accuracy claimed up to date corresponds to an anisotropy of less or of the order of $10^{-13}$ on the surface of the earth at the location where the latest experiment was performed. 

So, both postulates are formally symmetric and both can be checked experimentally to a high degree of accuracy. But there is an important difference. The time postulate does not bring about any theoretical difficulties

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1See [18]. The value of $10^{-15}$ mentioned in this paper is the result of some corrections that should not be made if one accepts that the observed effect could have a local origin due, say, to the earth rotation, instead of deciding that no effect could be real unless it is of cosmic origin.
and instead to accept the length-postulate as a law satisfied with infinite precision leads to the conclusion that the only frames of reference in any given space-time are the Born congruences. Since the latter are rather scarce, even in special relativity, this point of view leaves special and general relativity without a satisfactory theory of frames of reference. We prefer to accept the necessity of having such a theory and to examine the implications of this other point of view.

ii) On the other hand the classical theory of elasticity is based on some specific laws, like for instance Hooke’s law, but above that it is based on some general concepts common to every branch of classical mechanics like for instance the concept of frames of reference which are the motions of idealized objects called rigid bodies. Let us just remind for example that the concept of deformation does not make sense until one has defined an idealized standard, i.e. a frame of reference which by assumption, to a desired approximation, has not been deformed. Therefore to invoke the classical theory of elasticity in a problem of general relativity, whatever small is the gravitational field which is being considered, should require at least that we know what are the congruences which at the weak field limit coincide with the rigid motions of classical mechanics, and this supposes of course that a theory of the frames of reference has been developed in general relativity. We must realize also that a gravitational wave, whatever its field strength, does not have a limiting Newtonian equivalent and therefore to deal with gravitational waves one has to be consistent also with those other branches of physics that one uses to describe them.

Section 2 is a short review of the definition of quo-harmonic congruences that we also call meta-rigid motions to emphasize our choice of interpreting them as frames of reference. Quo-harmonic congruences are a generalization of Born congruences and have already been discussed in two preceding papers [23], [24]. Section 3 contains the presentation of our point of view on the basic principles of the relativistic theory of elasticity. The ingredients in our theory, some common to other approaches and some new, are the following:

- A definition of the generating vector field of a deformation.
- The strain tensor field defining a deformation understood as relative to

\[2\] In [22] we discussed the implications concerning the speed of light using a theory of frames of reference which is an ancestor of that summarized in section 2
a given frame of reference.

- The energy-momentum tensor
- The Hooke’s law, and
- The Beltrami-Michell integrability conditions.

We shall consider only static deformations. These are defined as those deformations for which the motion of the deformed body can be described by the same congruence as that of the body before deformation. In section 4 we work out explicitly, as an example which illustrates the role of each of the ingredients above, the special relativistic generalization of a bar being accelerated with constant acceleration. Section 5 contains an analysis of the problem of the detection of weak plane gravitational waves. Is this analysis which has led us to claim that such waves do not couple to elastic bodies as we stated in the abstract.

2 Frames of reference

Frames of reference must be a class of congruences with two main properties, besides some more specific ones:

i) its local definition must be intrinsic and independent of any particular space-time. On the contrary global conditions compatible with the local definition may be dependent on the space-time being considered.

ii) any 3-dimensional sub-bundle of world-lines of a frame of reference must characterize the whole congruence, and satisfy the same local intrinsic condition.

Among acceptable congruences to be used in General relativity as frames of reference are Born motions, [1], when they exist but their number is notoriously small, whatever the space-time being considered, including Minkowski space-time, [3], [4].

Fermi congruences, either in their original form, [5], [10], or including rotation of the axis on the world-line which generates the congruence, [8], [17], are often used as a substitute to Born congruences. This is in our opinion a serious mistake. Fermi congruences obviously satisfy condition i) above, but they only satisfy condition ii), and are therefore acceptable as frames
of reference, when they are homogeneous, i.e., when any of their world-lines could be considered as a generator of the same congruence. This implies (See for instance [23]) that a Fermi congruence is in fact a Born congruence, and we are again led to the difficulties stated in the preceding paragraph.

The problem of generalizing the class of Born congruences has already been considered in many papers. Born himself, [4], proposed a second definition and since then many other generalizations have been proposed (See for example more references in [20]). The definition of a frame of reference to be used below has been derived from tentatives proposed in several preceding papers [21], [22], [23], [24].

Let us consider a time-like congruence \( \mathcal{R} \) and let \( u^\alpha \) be its tangent unit vector field. We shall use the following definitions and notations:

\[
\hat{g}_{\alpha\beta} \equiv g_{\alpha\beta} + u_\alpha u_\beta
\]

which is the projector associated to \( u^\alpha \),

\[
\Lambda^\alpha \equiv -u^\rho \nabla_\rho u^\alpha
\]

which is, up to the sign, the intrinsic acceleration field, and we shall call occasionally the Newtonian Field. And

\[
\Sigma_{\alpha\beta} \equiv c(\hat{\nabla}_\alpha u_\beta + \hat{\nabla}_\beta u_\alpha)
\]

where, \( T_{\beta_1\beta_2\ldots\beta_s} \) being any tensor,

\[
\hat{\nabla}_\alpha T_{\beta_1\beta_2\ldots\beta_s} \equiv \hat{g}_{\alpha\lambda} \hat{g}_{\beta_1\beta_2} \cdots \hat{g}_{\beta_s} \nabla_\lambda T_{\mu_1\mu_2\ldots\mu_s},
\]

which is called the rate of ‘deformation’ field\[^3\].

Using in particular a system of coordinates adapted to \( \mathcal{R} \), i.e., such that \( u^i = 0 \) the metric of the space-time can be conveniently described in the following form:

\[
ds^2 = -(\dot{\vartheta}^0)^2 + d\hat{s}^2
\]

with

\[^3\text{We write the word deformation in quotation marks here to avoid a confusion with the concept of deformation to be defined in the next section.}\]
\[ \vartheta^0 = \xi(dx^0 - \varphi_i dx^i), \quad d\hat{s}^2 = \hat{g}_{ij}(t, x^k)dx^i dx^j, \quad x^0 = ct \] (7)

where the following notations have been used

\[ \xi = \sqrt{-g_{00}} \quad \varphi_i = \xi^{-2}g_{0i}, \] (8)

and

\[ \hat{g}_{ij} = g_{ij} + \xi^2 \varphi_i \varphi_j \] (9)

which we shall call the Fermat quo-tensor[4] and it is the object whose components are the space components of the projector (4). The quo-tensors whose components are the space components of \( \Lambda^\alpha \) and \( \Sigma_{\alpha\beta} \) are

\[ \Lambda_i = -c^2 (\hat{\partial}_i \ln \xi + \partial_0 \varphi_i) \] (10)

where

\[ \hat{\partial}_i \cdot \equiv \partial_i \cdot + \varphi_i \partial_0. \] (11)

and

\[ \Sigma_{ij} = c\hat{\partial}_0 \hat{g}_{ij} \] (12)

where

\[ \hat{\partial}_0 \equiv \xi^{-1} \partial_0 \] (13)

We shall also need to consider the following symbols, first used in [4] and in [7]:

\[ \hat{\Gamma}^k_{ij} = \frac{1}{2}\hat{g}^{is}(\hat{\partial}_j \hat{g}_{ks} + \hat{\partial}_k \hat{g}_{js} - \hat{\partial}_s \hat{g}_{jk}), \] (14)

We shall say that a function \( f(x^\alpha) \) is a quo-harmonic function in the quotient space \( V_3 = V_4/R \), where \( R \) here is the equivalence relation defined by the congruence \( \mathcal{R} \), if it is a solution of the following equation

\[ \hat{\Delta}f - \hat{\partial}_0^2 f = 0 \] (15)

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4The prefix quo- indicates that it is a tensor orthogonal to the vector \( u^\alpha \) written in a system of coordinates adapted to this vector.
where

\[ \hat{\triangle} \equiv \hat{g}^{ij}(\hat{\partial}_i \hat{\partial}_j - \hat{\Gamma}^k_{ij} \hat{\partial}_k) \quad (16) \]

This quo-harmonicity concept is an intrinsic concept associated to a congruence. In fact, using a general system of coordinates, eq. (15) can be written in a manifestly covariant way as follows

\[ \Box f + (\Lambda^\alpha + \Sigma u^\alpha) \partial_\alpha f = 0 \quad (17) \]

where:

\[ \Box \equiv g^{\alpha\beta}(\partial_\alpha \partial_\beta - \Gamma^\rho_{\alpha\beta} \partial_\rho) \quad (18) \]

is the intrinsic d’Alembertian of the space-time and

\[ \Sigma = \hat{g}^{\alpha\beta} \Sigma_{\alpha\beta} \quad (19) \]

We shall say that a congruence \( \mathcal{R} \) is quo-harmonic if there exist three independent functions of the space coordinates only, \( f^a(x^i) \), which are quo-harmonic, i.e. such that:

\[ \hat{\triangle} f^a = 0 \quad (20) \]

Born congruences are quo-harmonic. This follows from the fact that for Born congruences the Fermat tensor \( \hat{g}^{ij}(x^k) \) does not depend on time and therefore the operator \( \hat{\triangle} \) above becomes the usual 3-dimensional Laplacian in a Riemannian manifold. Eqs. (20) admit then in the neighborhood of any point a system of three independent solutions (See for example [12]) and therefore quo-harmonic congruences are an acceptable generalization of Born congruences.

Congruences in general, and quo-harmonic congruences in particular, can be classified according to their order of genericity, a concept that was defined in [24] and indicates how general is the tensor \( \Sigma_{ij} \). Let us remind here only that the quo-harmonic congruences with smallest genericity are the Born congruences, which have a genericity 2, that the congruences with genericity 3 are the shear-free ones, i.e. those satisfying:
\[
\dot{\partial}_0 \dot{g}^{ij} + \frac{1}{3} \Sigma \dot{g}^{ij} = 0 \quad (21)
\]

and that those with genericity 6 are those for which there exist a function \(c_1\) such that:

\[
\ddot{\partial}_0^2 \dot{g}^{ij} + c_1 \partial_0 \dot{g}^{ij} = 0 \quad (22)
\]

The current orthodoxy claims that general covariance allows to say that any congruence can be considered to be an acceptable frame of reference, a point of view that leaves this concept without any meaning. That this is not the answer to the problem of defining the concept of frame of reference can be argued as follows: In 2-dimensional space-times the number of degrees of freedom of Born congruences is 1 like for rigid motions in classical mechanics with the same dimensions. Therefore in two dimensions one can have both general covariance and a correct theory of frames of reference. In 3-dimensional space-times it can be conjectured on the basis of preliminary results, [23], that the set of congruences, call it \(B_2\), that are Born or shear-free congruences have 3 degrees of freedom, again as much as rigid motions in classical mechanics with the same dimensions. The simple fact that this conjecture might be true is again an indication that there is no conceptual contradiction between general covariance and the possibility of having a more restrictive theory of frames of reference. Our point of view is that that this problem will be solved also for 4-dimensional space-times in one way or another. Quo-harmonic congruences, which generalize Born congruences in 4 dimensions, and include all shear-free congruences in 3 dimensions, [23], are for the time being our best educated guess.

3 Basic concepts of relativistic elasticity theory

Many different approaches to a theory of elasticity in general relativity have been proposed in the literature (See for instance [11], [13], [14], [16], and [19] which contains a review of many proposals). Our approach here differs from all other we know of by insisting that to make sense of concepts such as the concept of deformation of a body one needs to have a well defined concept
of reference frame allowing to compare the initial state of the body with the result of acting on it, either with universal forces like the gravitational ones or either with more specific ones. A definition of this concept has been proposed in the preceding section and it is used in this one in an essential way in conjunction with common ingredients used to describe the behavior of macroscopic bodies like the conservation of the energy-momentum tensor or a relativistic generalization of Hooke’s law.

Let us consider a system of coordinates adapted to a frame of reference $\mathcal{R}$ and let us consider a second congruence $\mathcal{B}$ whose parametric equations, $x^0$ being the parameter, in this system of coordinates are

$$y^\alpha = x^\alpha + \zeta^\alpha(x^\rho)$$  \hspace{1cm} (23)

If $\zeta^\alpha$ is a small vector so that the products of components can be neglected we shall say that the congruence $\mathcal{B}$ is a deformation of the frame of reference $\mathcal{R}$ and that $\zeta^\alpha$ is the generator of this deformation. A short and explicit calculation proves that if we note $w^\alpha$ the unit tangent vector to the congruence $\mathcal{B}$ and use the following notation

$$w^\alpha = u^\alpha + \delta u^\alpha$$  \hspace{1cm} (24)

one has in this system of adapted coordinates:

$$\delta u^0 = -\xi^{-1} \varphi_i \partial_i \zeta^i, \quad \delta u^i = \xi^{-1} \partial_i \zeta^i,$$  \hspace{1cm} (25)

a result which can be written in the manifestly covariant form:

$$\delta u^\alpha = \hat{g}^\alpha_\lambda \mathcal{L}(\zeta) u^\lambda$$  \hspace{1cm} (26)

where $\mathcal{L}(\cdot)$ is the Lie derivative operator. It follows from this result that the generator of a deformation is defined up to the transformation

$$\zeta^\alpha \rightarrow \zeta^\alpha + k u^\alpha$$  \hspace{1cm} (27)

where $k$ is any function. This property can thus be used to assume, without any loss of generality, that $u^\alpha$ and $\zeta^\alpha$ are orthogonal:

$$u_\alpha \zeta^\alpha = 0$$  \hspace{1cm} (28)

By definition the strain tensor of the congruence $\mathcal{B}$ with respect to the frame of reference $\mathcal{R}$ is the tensor:
\[ \epsilon_{\alpha\beta} = \hat{g}_\alpha^\lambda \hat{g}_\beta^\mu \mathcal{L}(\zeta) \hat{g}_{\lambda\mu} \text{ or } \epsilon_{\alpha\beta} = \hat{\nabla}_\alpha \zeta_\beta + \hat{\nabla}_\beta \zeta_\alpha \] (29)

Notice that \( \epsilon_{\alpha\beta} \) defines \( \zeta_\alpha \) only up to a transformation

\[ \zeta_\alpha \rightarrow \zeta_\alpha + s_\alpha \] (30)

where \( s_\alpha \) is the generator of a symmetry of the Fermat quotient, i.e., a solution of the following equations:

\[ \hat{\nabla}_\alpha \zeta_\beta + \hat{\nabla}_\beta \zeta_\alpha = 0 \] (31)

We shall say that a deformation is static if

\[ \delta u^\alpha = 0 \] (32)

Notice that this does imply neither that the generator vector is zero nor that the strain tensor is zero. Static deformations are the simplest and they are the only ones that we shall consider in the following two sections.

An elastic test body is described by its energy-momentum tensor:

\[ T^\alpha_\beta = \rho u^\alpha u_\beta - c^{-2} \pi^\alpha_\beta, \quad \pi^\alpha_\beta u_\beta = 0 \] (33)

where \( \rho \) is its density, \( u^\alpha \) is its unit four-velocity field and where \( \pi^\alpha_\beta \) is the tensor describing the stresses which are present as a result of the volume forces acting on the body and those acting on its surface. The body is described also by its shape, and this supposes a geometry of space to which to refer it. If no other forces than gravitational ones act on the body then its energy-momentum tensor must be conserved:

\[ \nabla_\alpha T^\alpha_\beta = 0, \] (34)

otherwise the r-h-s would be equal to the volume applied forces. On the surface of the body we must have

\[ \pi_{\alpha\beta} n^\alpha = f_\beta \] (35)

where \( n^\alpha \) is the normal to the surface and \( f_\beta \) are the applied forces on the surface.

Projecting eqs. (34) along \( u^\alpha \) one gets
\[ u^\alpha \partial_\alpha \rho + \frac{1}{2} \rho \Sigma = \frac{1}{2c^2} \pi^{\alpha \beta} \Sigma_{\alpha \beta} \]  

(36)

where \( \Sigma_{\alpha \beta} \) and \( \Sigma \) have been defined in (4) and (19).

On the other hand projecting eqs. (37) on the tangent hyperplane orthogonal to \( u^\alpha \) one gets

\[ \hat{D}_\alpha \pi^\alpha_{\beta} = -\rho \Lambda_\beta \]  

(37)

where

\[ \hat{D}_\alpha \equiv \hat{\nabla}_\alpha - c^{-2} \Lambda_\alpha \]  

(38)

with \( \Lambda_\alpha \) and \( \hat{\nabla}_\alpha \) already defined in (3) and (5).

Let us consider all space-time metrics for which systems of coordinates exist such that the behavior of its components when \( c \to \infty \) is:

\[ g_{00} = -1 + \frac{2U(t, x^k)}{c^2}, \quad g_{0i} = \frac{U_i(t, x^k)}{c^3}, \quad g_{ij} = \delta_{ij} \]  

(39)

Since this behavior is preserved by the infinite-dimensional classic group of rigid motions:

\[ t' = t, \quad x'^i = R^i_j(t)(x^j + S^j(t)) \]  

(40)

where the matrix \( R \) is a rotation matrix, we shall call the behavior above the classical limit, if it exists, of a space-time metric. At this limit the Fermat tensor is, neglecting terms of order greater than \( c^{-2} \),

\[ \hat{g}_{ij} = \delta_{ij} \]  

(41)

and at the same approximation we have

\[ \Lambda_i = \partial_i U - \frac{1}{c^2} \partial_i U_i, \quad \Sigma_{ij} = 0 \]  

(42)

Equations (36) and (37) become then

\footnote{The metric components \( g_{ij} \) can be considered as defining the space-time metric of an exact ‘extended Newtonian theory’ invariant under the group of rigid motions \( R \). This theory was presented in [20].}
\[ \partial_t \rho = 0, \quad \partial_i \pi^i_j = -\rho \Lambda_i, \]  

the latter being the equilibrium equations of the body.

Let us assume that we have an elastic body which is isotropic and homogeneous, then we shall say that this body has been statically deformed with respect to some given frame of reference \( R \) if the 4-velocity of each element of the body is tangent to \( R \) and if a static deformation of the frame of reference can be found such that the stress tensor of the body can be related to the strain tensor according to the relativistic generalization of the Hooke’s law

\[ \epsilon_{\alpha\beta} = \frac{1}{3(3\lambda + 2\mu)} \pi \hat{g}_{\alpha\beta} + \frac{1}{2\mu}(\pi_{\alpha\beta} - \frac{1}{3} \hat{g}_{\alpha\beta} \pi), \quad \pi = \hat{g}^{\alpha\beta} \pi_{\alpha\beta} \]  

where \( \lambda \) and \( \mu \) are the Lame’s parameters of the body. Of course not every congruence which is a frame of reference can be the motion of a body which has been statically deformed. In the next section we shall consider a case for which this is the case because the congruence is a Killing congruence, and in section 5 we shall consider a case for which this is also the case but only at a desired approximation.

Since the strain tensor \( \epsilon_{\alpha\beta} \) has a well defined structure one can derive from eqs. (29) necessary integrability conditions that this tensor, or the displacement vector \( \zeta_\alpha \) will have to satisfy. Because of Hooke’s law the stress tensor will have to satisfy also some supplementary equations besides eqs. (44). We are not going to detail here these supplementary equations but we shall remind what they are at the classical limit because we shall refer to them in the next section.

From the definition (29) we have at this limit

\[ \epsilon_{ij} = \partial_i \zeta_j + \partial_j \zeta_i \]  

whose integrability conditions are

\[ \partial_{ik} \epsilon_{jl} + \partial_{jl} \epsilon_{ik} - \partial_{il} \epsilon_{jk} - \partial_{jk} \epsilon_{il} = 0 \]  

But since the l-h-s of these equations has the geometrical structure of a linearized Riemann tensor in three dimensions these equations are equivalent to its contraction with \( \delta^{il} \), i.e.:

\[ \Delta \epsilon_{ik} + \partial_{ik} \epsilon^s_s - \partial_i(\partial_s \epsilon^s_k) - \partial_k(\partial_s \epsilon^s_i) = 0 \]  

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and using Hooke’s law and the equilibrium equations (43) one obtains:

$$\Delta \pi_{ik} + \frac{2(\mu - \lambda)}{2\mu + 3\lambda} \partial_{ik} \pi = \rho (\partial_i \Lambda_k + \partial_k \Lambda_i + \frac{\lambda}{2\mu - \lambda} \partial_s \Lambda^s \delta_{ik})$$ (48)

These equations are called the Beltrami-Michell equations (See for instance [13]). Their importance comes from the fact, known as the fundamental theorem of the theory of elasticity theory in classical mechanics, that joined to the equilibrium equations (43) and the conditions on the surface (35) they determine a unique solution for the stress tensor and, through Hooke’s law, a unique solution for the strain tensor. The displacement vector is then uniquely defined up to a rigid motion.

4 Constant acceleration in Minkowski space-time

Let us consider an homogeneous and isotropic test body. From the elasticity point of view this body will be characterized by its density and its Lamé’s parameters. And from the geometrical point of view it will be characterized by its shape, a concept that supposes a geometry of the space on which it is embedded. This geometry, from our point of view, is that of the quotient manifold $V_3$ and the geometric ingredients defined in (8) and (9).

This section is meant to be a simple application of the relativistic theory of static deformations. Therefore to keep things as simple as possible we shall assume that the space-time is Minkowski’s and that the body is a cylindrical body at rest initially with respect to a galilean frame of reference. We shall assume also that the basis of the cylinder is pushed with a uniform force along a fixed direction. We assume finally that after some transient elastic oscillations, that we shall not consider here, the motion of the body is the Killing congruence $u^\alpha$ obtained constructing the irrotational Fermi congruence with origin any of the world-lines of the basis on which acts the constant force.

Using a system of coordinates adapted to $u^\alpha$ such that

$$u^0 = (1 + c^{-2}a_k x^k)^{-1}, \quad u^i = 0$$ (49)

the line-element of Minkowski’s space-time is (See for instance [8])
\[ ds^2 = -[1 + c^{-2}a_ix^i]^2c^2dt^2 + \delta_{ij}dx^i dx^j \]  

(50)

where \( a_i \) is the intrinsic acceleration of the world-line which has been chosen as origin of the Fermi construction. The corresponding Fermat tensor is:

\[ \hat{g}_{ij} = \delta_{ij} \]  

(51)

i.e., the space manifold \( \mathcal{V}_3 \) associated with the frame of reference \( \mathcal{R} \) is euclidean like in classical mechanics. The components of \( \Lambda_i \) and \( \Sigma_{ij} \) are:

\[ \Lambda_i = -a_i(1 + c^{-2}a_k x^k)^{-1}, \quad \Sigma_{ij} = 0 \]  

(52)

Note that each element of the cylinder describes a world line with constant intrinsic acceleration but this acceleration will depend on the distance to the basis of the cylinder. Equations (36) become in this example

\[ \partial_t \rho = 0 \]  

(53)

and equations (37) become

\[ \partial_i \pi_j^i + c^{-2}(1 + c^{-2}a_k x^k)^{-1}a_i \pi_j^i = (1 + c^{-2}a_k x^k)^{-1}\rho a_j \]  

(54)

As we see, at the classical limit, these are the equilibrium equations

\[ \partial_i \pi_j^i = \rho a_j \]  

(55)

of a body which moves with uniform constant acceleration.

Let us assume now that the basis of the cylinder is parallel to the \( x - y \) plane and it is being pushed in the positive \( z \) direction. Taking into account the symmetries of the problem and the fact that \( a_1 = a_2 = 0 \) it is easy to find a particular solution to these equations satisfying the boundary conditions (35), namely:

\[ \pi_3^3 = -(1 + c^{-2}a_z)^{-1}\rho a(l - z), \quad a = a_3 \]  

(56)

\( l \) being the height of the cylinder and the other components of the stress tensor being zero. At the classical limit we have:

\[ \pi_3^3 = -\rho a(l - z) \]  

(57)
which is the solution that leads to the correct solution for the generator vector $\zeta_i$. But in Special relativity (56) it is not. In fact from (29) and (31) it follows that the strain tensor is again the same expression (13) as in classical mechanics; therefore the integrability conditions for these equations are the Beltrami-Michell equations (48) that we derived in the preceding section and a direct substitution of expression (56), the other components of the stress tensor being zero, would prove that this tensor does not satisfy these eqs. (48).

To solve the problem in this case we could look for a solution in terms of power series of $1/c^2$

$$\pi_{ij} = \pi_{ij}^{(0)} + 1/c^2 \pi_{ij}^{(1)} + \cdots$$

starting with the classical solution (55). At each order of approximation one has to solve then a classical problem of elasticity with different volume forces.

5 Weak plane gravitational waves

Let us consider the line element of a weak plane gravitational wave that we shall assume for simplicity to be rectilinearly polarized. To be more precise, we consider the line element:

$$ds^2 = -c^2 dt^2 + (1 + h) dx^2 + (1 - h) dy^2 + dz^2, \quad h = h(u), \quad u = ct - z \ (59)$$

This is actually the wave-zone expression of a particular approximate solution

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} + \cdots \ (60)$$

of the linearized Einstein equations:

$$\square h_{\alpha\beta} = T_{\alpha\beta}, \quad h_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2} \eta_{\alpha\beta}, \quad h = \eta^{\alpha\beta} h_{\alpha\beta} \ (61)$$

To obtain this solution it has been assumed that the source of this gravitational field was confined in a bounded region of the space associated with a galilean frame of reference of Minkowski space-time. Namely the congruence $\mathcal{R}$ with parametric equations
\begin{equation}
x^i = \text{const.}
\end{equation}

and unit tangent vector with components \( u^0 = 1, \ u^i = 0 \). The concept of a Galilean frame of reference is a well defined concept if the space-time is Minkowski’s but not for a space-time with a metric like (60). What one needs is a definition that generalizes the concept of a frame of reference acceptable in any space-time. Our approach of section 2 to this problem allows us to interpret the congruence defined by eqs. (62) as a frame of reference with respect to which the source of the field is at rest, not because it is a galilean frame of reference of Minkowski space-time but because it is, at the approximation being considered, a quo-harmonic congruence which will reduce, in this particular case, to a galilean frame of reference if \( h \) in (59) were zero. In fact the geometrical objects associated with the congruence (62) are the following: the Fermat tensor is

\[\hat{g}_{ij} = g_{ij} = \delta_{ij} + \hat{h}_{ij}, \quad \hat{h}_{ij} = h(\delta_{1i}\delta_{1j} - \delta_{2i}\delta_{2j});\] (63)

the Newtonian field of forces is zero

\[\Lambda_i = 0\] (64)

and the non zero components of the rate of 'deformation' field are

\[\Sigma_{11} = c\partial_0 h, \quad \Sigma_{22} = -c\partial_0 h\] (65)

From (63) it follows that

\[\hat{\Delta}x^k = -\frac{1}{2}\delta^{ij}\delta^{ks}(\partial_i \hat{h}_{js} + \partial_j \hat{h}_{is} - \partial_s \hat{h}_{ij})\] (66)

which yields:

\[\hat{\Delta}x^1 = \partial_1 h = 0, \quad \hat{\Delta}x^2 = \partial_2 h = 0, \quad \hat{\Delta}x^3 = 0\] (67)

meaning that the congruence \( \mathcal{R} \) is quo-harmonic. At the desired approximation we also have

\[\hat{g}^{11} = 1 - h, \quad \hat{g}^{22} = 1 + h, \quad \hat{g}^{33} = 1\] (68)

the other components being zero. From these expressions we can see that the relations (62) are satisfied with
\[ c_1 = -\frac{\partial^2_0 h}{\partial_0 h} \] (69)

which means that the congruence has locally genericity 6.

Let us examine the problem of deciding how an elastic bar couples to a wave like (59). To do that we ask whether an elastic bar at rest or in uniform motion with respect to the source of the wave can be statically deformed with respect to the frame of reference \( R \), and if so under which forces acting on its surface. To implement the idea that the bar is at rest with respect to the source we have to accept that the motion of the bar is itself described by the congruence \( R \). Taking into account the smallness of the deviation with respect to the euclidean metric of the Fermat tensor associated with \( R \), the smallness of the stress tensor, and eqs. (64) the equilibrium equations (37) become here

\[ \partial_i \pi^i_j = 0 \] (70)

and the expression of the strain tensor (29) is the same as in the classical limit

\[ \epsilon_{ij} = \partial_i \zeta_j + \partial_j \zeta_i \] (71)

Since

\[ \pi_{ij} = 0 \] (72)

is an obvious solution of the equilibrium equations (70) which satisfies the boundary conditions (35) with zero applied forces and since the integrability conditions of eqs. (71) are also trivially satisfied it follows from Hooke’s law (44) that \( \epsilon_{ij} = 0 \) and that \( \zeta_i \) is either zero or it defines an infinitesimal fixed rigid motion. This result can be translated in words as follows: an elastic bar, if left at rest in the wave zone, or in uniform motion with respect to a source of gravitational radiation, will remain in the same state of motion and no stresses will be induced at this approximation by the wave.

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