Hankel operators induced by radial Bekollé–Bonami weights on Bergman spaces

José Ángel Peláez¹ · Antti Perälä² · Jouni Rättyä³

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Abstract

We study big Hankel operators $H^\nu_\omega f : A^p_\omega \to L^q_\nu$ generated by radial Bekollé–Bonami weights $\nu$, when $1 < p \leq q < \infty$. Here the radial weight $\omega$ is assumed to satisfy a two-sided doubling condition, and $A^p_\omega$ denotes the corresponding weighted Bergman space. A characterization for simultaneous boundedness of $H^\nu_\omega$ and $H^q_\nu$ is provided in terms of a general weighted mean oscillation. Compared to the case of standard weights that was recently obtained by Pau et al. (Indiana Univ Math J 65(5):1639–1673, 2016), the respective spaces depend on the weights $\omega$ and $\nu$ in an essentially stronger sense. This makes our analysis deviate from the blueprint of this more classical setting. As a consequence of our main result, we also study the case of anti-analytic symbols.

Keywords Hankel operator · Bekollé–Bonami weight · Bergman space · Bergman projection · doubling weight

Mathematics Subject Classification Primary 47B35; Secondary 32A36

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¹ Departamento de Análisis Matemático, Universidad de Málaga, Campus de Teatinos, 29071 Málaga, Spain
² Department of Mathematical Sciences, Chalmers University of Technology and the University of Gothenburg, 412 96 Gothenburg, Sweden
³ Department of Physics and Mathematics, University of Eastern Finland, P.O.Box 111, 80101 Joensuu, Finland
1 Introduction and main results

Let $\mathcal{H}(\mathbb{D})$ denote the space of analytic functions in the unit disc $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$. A function $\omega : \mathbb{D} \to [0, \infty)$, integrable over the unit disc $\mathbb{D}$, is called a weight. It is radial if $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$. For $0 < p < \infty$ and a weight $\omega$, the Lebesgue space $L^p_\omega$ consists of (equivalence classes of) complex-valued measurable functions $f$ in $\mathbb{D}$ such that

$$\|f\|_{L^p_\omega} = \left( \int_{\mathbb{D}} |f(z)|^p \omega(z) \, dA(z) \right)^{\frac{1}{p}} < \infty,$$

where $dA(z) = dx \, dy / \pi$ denotes the normalized Lebesgue area measure on $\mathbb{D}$. The weighted Bergman space $A^p_\omega$ is the space of analytic functions in $L^p_\omega$. As usual, $A^0_\omega$ denotes the weighted Bergman space induced by the standard radial weight $(\alpha + 1)(1 - |z|^2)^\alpha$. If $\nu$ is a radial weight then $A^p_\nu$ is a closed subspace of $L^p_\nu$, and the orthogonal projection from $L^p_\nu$ to $A^p_\nu$ is given by

$$P_\nu(f)(z) = \int_{\mathbb{D}} f(\xi) B^\nu_\xi(\xi) \omega(\xi) \, dA(\xi), \quad z \in \mathbb{D},$$

where $B^\nu_\xi$ are the reproducing kernels of $A^p_\nu$, $f(z) = \langle f, B^\nu_\xi \rangle_{A^p_\nu}$ for all $z \in \mathbb{D}$ and $f \in A^p_\nu$.

The study of the boundedness of weighted Bergman projections on $L^p$-spaces is a compelling topic that has attracted a considerable amount of attention during the last decades. A well known result due to Bekollé and Bonami [4,5] describes the weights $\omega$ such that the Bergman projection $P_\eta$, induced by the standard weight $(\eta + 1)(1 - |z|^2)^\eta$, is bounded on $L^p_\omega$ for $1 < q < \infty$. We denote this class of weights by $B_q(\eta)$, and write $B_q = \cup_{\eta \geq 1} B_q(\eta)$ for short. In the case of a standard weight, the Bergman reproducing kernels are given by the neat formula $(1 - \overline{z}\xi)^{-1}$. However, for a general radial weight $\nu$ the Bergman reproducing kernels $B^\nu_\xi$ may have zeros [18] and such explicit formulas for the kernels do not necessarily exist. This is one of the main obstacles in dealing with $P_\nu$ [9,16]. Nonetheless, we shall prove in Proposition 6 below that if $\nu \in B_q$ is radial, then $P_\nu : L^q_\nu \to L^q_\nu$ is bounded for each $1 < q < \infty$. The proof of this relies on accurate estimates for the integral means of $B^\nu_\xi$, recently obtained in [16, Theorem 1], and the result itself plays an important role in the proof of the main discovery of this paper.

All the above makes the class of radial weights in $B_q$ an appropriate framework for the study of the big Hankel operator

$$H^\nu_\beta(f)(z) = (I - P_\nu)(fg)(z), \quad f \in L^q_\nu, \quad z \in \mathbb{D},$$

on weighted Bergman spaces. For an analytic function $f$, the Hankel operator $H^\beta_\beta$, induced by a standard projection, has been widely studied on Bergman spaces since the pioneering work of Axler [3], which was later extended in [1]. In the case of a rapidly decreasing weight $\nu$ and $f \in \mathcal{H}(\mathbb{D})$, Galanopoulos and Pau [10] did an extensive research on $H^\nu_\nu$ on $A^2_\nu$; see [2] for further results. For general symbols, Zhu [21] was the first to build up a bridge between Hankel operators and the mean oscillation of the symbols in the Bergman metric, and this idea has been further developed in several contexts [6–8,22]; see [23] and the references therein for further information on the theory of Hankel operators. More recently, Pau et al. [12] described the complex valued symbols $f$ such that the Hankel operators $H^\beta_\nu$ and $H^\nu_\beta$ are simultaneously bounded from $A^0_\nu$ to $L^q_\nu$, provided $1 < p \leq q < \infty$. Our primary aim is to extend this last-mentioned result to the context of radial $B_q$-weights. To do this, some definitions are needed. For a radial weight $\omega$, we assume throughout the paper that $\hat{\omega}(z) = \int_{|z|}^1 \omega(s) \, ds > 0$ for all $z \in \mathbb{D}$, for otherwise the Bergman space $A^0_\omega$ would contain
all analytic functions in $D$. A radial weight $\omega$ belongs to the class $\mathcal{D}$ if there exists a constant $C = C(\omega) > 1$ such that $\hat{\omega}(r) \leq C \frac{1}{r^r}$ for all $0 \leq r < 1$. Moreover, if there exist $K = K(\omega) > 1$ and $C = C(\omega) > 1$ such that

$$\hat{\omega}(r) \geq C \frac{1 - r}{K}, \quad 0 \leq r < 1,$$  \quad (1.1)$$

then $\omega \in \mathcal{D}$. We write $D = \mathcal{D} \cap \mathcal{D}$ for short. For basic properties of these classes of weights and more, see [13,14] and the references therein. Let $\beta(z, \zeta)$ denote the hyperbolic distance between $z, \zeta \in \mathbb{D}$, $\Delta(z, r)$ the hyperbolic disc of center $z$ and radius $r > 0$, and $S(z)$ the Carleson square associated to $z$. For $0 < p, q < \infty$ and radial weights $\omega, \nu$, define

$$\gamma(z) = \gamma_{\omega, \nu, p, q}(z) = \frac{\tilde{\nu}(z) \frac{1}{2} (1 - |z|)^{\frac{1}{2}}}{\tilde{\omega}(z) \frac{1}{2} (1 - |z|)^{\frac{1}{2}}}, \quad z \in \mathbb{D}. \quad (1.2)$$

Further, for $f \in L^1_{\nu, \text{loc}}$, write $\hat{f}_{r, \nu}(z) = \frac{\int_{\Delta(z, r)} f(\zeta) \nu(\zeta) dA(\zeta)}{\nu(\Delta(z, r))}$ and

$$\text{MO}_{\nu, q, r}(f)(z) = \left( \frac{1}{\nu(\Delta(z, r))} \int_{\Delta(z, r)} |f(\zeta) - \hat{f}_{r, \nu}(z)|^q \nu(\zeta) dA(\zeta) \right)^{\frac{1}{q}}$$

for all $z \in \mathbb{D}$. It is worth noticing that for prefixed $r > 0$, the quantity $\nu(\Delta(z, r))$ may equal to zero for some $z$ arbitrarily close to the boundary if $\nu \in \mathcal{D}$. However, if $\nu \in D$, then there exists $r_0 = r_0(\nu) > 0$ such that $\nu(\Delta(z, r)) \geq \nu(S(z)) > 0$ for all $z \in \mathbb{D}$ if $r \geq r_0$. The space $\text{BMO}(\Delta)_{\omega, \nu, p, q, r}$ consists of $f \in L^q_{\nu, \text{loc}}$ such that

$$\|f\|_{\text{BMO}(\Delta)_{\omega, \nu, p, q, r}} = \sup_{z \in \mathbb{D}} \left( \text{MO}_{\nu, q, r}(f)(z) \gamma(z) \right) < \infty.$$ 

We will show that if $\nu \in D$, then $\text{BMO}(\Delta)_{\omega, \nu, p, q, r}$ does not depend on $r$ for $r \geq r_0$. In this case, we simply write $\text{BMO}(\Delta)_{\omega, \nu, p, q}$. The main result of this study reads as follows and it will be proved in Sect. 5.

**Theorem 1** Let $1 < p \leq q < \infty$, $\omega \in D$, $\nu \in B_q$ a radial weight and $f \in L^q_v$. Then $H^v_f$, $H^u_f: A^p_\omega \to L^q_v$ are bounded if and only if $f \in \text{BMO}(\Delta)_{\omega, v, p, q}$. 

The approach employed in the proof of this result follows the guideline of [12, Thorem 4.1], however a good number of steps cannot be adapted straightforwardly and need substantial modifications. In Sect. 2 we prove some results concerning the classes of weights involved in this work and the boundedness of the Bergman projection $P_\nu$, while in Sect. 3 we introduce and study two spaces of functions on $\mathbb{D}$. One of them is denoted as $\text{BA}(\Delta)_{\omega, \nu, p, q}$, and although its initial definition depends on $r$, it can be described in terms of an appropriate Berezin transform or simply observing that $f \in \text{BA}(\Delta)_{\omega, v, p, q}$ if and only the multiplication operator $M_f(g) = fg$ is bounded from $A^p_\omega$ to $L^q_v$ [15]. The second one, denoted by $\text{BO}(\Delta)_{\omega, v, p, q}$, consists of continuous functions on $\mathbb{D}$ such that the oscillation in the Bergman metric is bounded in terms of the auxiliary function $\gamma$ given in (1.2). We show that $f \in \text{BO}(\Delta)_{\omega, v, p, q}$ if and only if

$$|f(z) - f(\zeta)| \lesssim \|f\|_{\text{BO}(\Delta)_{\omega, v, p, q}} (1 + \beta(z, \zeta)) \Gamma(z, \zeta), \quad z, \zeta \in \mathbb{D},$$
where
\[
\Gamma_\zeta(z, \zeta) = \left( \frac{1-|z|^2}{\max\{1-|z|^2, 1-|\zeta|^2\}} \right) \frac{1}{p} \frac{1}{q} \frac{1}{z+1} \hat{\omega} \left( 1 - \frac{2|1-|\zeta|^2|}{\max\{1-|z|^2, 1-|\zeta|^2\}} \right) \frac{1}{p} \min \left\{ \frac{\hat{\omega}(z)}{(1-|z|^2)^\frac{1}{p}}, \frac{\hat{\omega}(\zeta)}{(1-|\zeta|^2)^\frac{1}{q}} \right\},
\]
for an appropriate (small) constant \( \tau = \tau(\omega, \nu) > 0 \). If \( \omega \) and \( \nu \) are standard weights, then \( \Gamma_\zeta \) does not coincide with the function playing the corresponding role in [12, Lemma 3.2]; in the latter case the function is simpler in many aspects and does not depend on the additional parameter \( \tau \). Then, we show that
\[
\text{BMO}(\Delta)_{\omega, \nu, p, q} = \text{BA}(\Delta)_{\omega, \nu, p, q} + \text{BO}(\Delta)_{\omega, \nu, p, q}.
\]  
(1.3)

In order to prove this decomposition, due to the complex nature of \( \Gamma_\zeta(z, \zeta) \), we are forced to split \( \mathbb{D} \) into several regions depending on \( z \), establish sharp estimates for \( \Gamma_\zeta(z, \zeta) \) in each region and then apply properties of weights in \( \mathcal{D} \). The identity (1.3) together with a description of the boundedness of the integral operator
\[
T_{b,c} f(z) = \int_{\mathbb{D}} f(\zeta) \left( \frac{(1-|z|^2)^b}{(1-z\zeta)^c} \right) dA(\zeta)
\]
and its maximal counterpart from \( A^p_\omega \) to \( L^q_\nu \), see Sect. 4 below, are key tools to prove that each \( f \in \text{BMO}(\Delta)_{\omega, \nu, p, q} \) induces a bounded Hankel operator from \( A^p_\omega \) to \( L^q_\nu \). Theorem 1 will be proved in Sect. 5.

Finally, in Sect. 6, as a byproduct of Theorem 1, we describe the analytic symbols such that \( H^\nu_T : A^p_\omega \to L^q_\nu \) is bounded. The space \( \mathcal{B}_d \) consists of \( f \in \mathcal{H}(\mathbb{D}) \) such that
\[
\|f\|_{\mathcal{B}_d} = \sup_{z \in \mathbb{D}} |f'(z)| (1 - |z|) \gamma(z) + |f(0)| < \infty,
\]
where \( \gamma \) is given by (1.2).

**Theorem 2** Let \( 1 < p \leq q < \infty \), \( \omega \in \mathcal{D} \), \( \nu \in B_q \) a radial weight and \( f \in A^1_\nu \). Then \( H^\nu_T : A^p_\omega \to L^q_\nu \) is bounded if and only if \( f \in \mathcal{B}_d \).

Throughout the paper \( \frac{1}{p} + \frac{1}{p'} = 1 \) for \( 1 < p < \infty \). Further, the letter \( C = C(\cdot) \) will denote an absolute constant whose value depends on the parameters indicated in the parenthesis, and may change from one occurrence to another. We will use the notation \( a \lesssim b \) if there exists a constant \( C = C(\cdot) > 0 \) such that \( a \leq Cb \), and \( a \gtrsim b \) is understood in an analogous manner. In particular, if \( a \lesssim b \) and \( a \gtrsim b \), then we will write \( a \asymp b \).

## 2 Auxiliary results

For a radial weight \( \omega, K > 1 \) and \( 0 \leq r < 1 \), let \( \rho_n^r = \rho_n^r(\omega, K) \) be defined by \( \hat{\omega}(\rho_n^r) = \hat{\omega}(r) K^{-n} \) for all \( n \in \mathbb{N} \cup \{0\} \). Write \( \rho_n = \rho_n^0 \) for short. For \( x \geq 1 \), write \( \omega_x = \int_0^1 r^x \omega(r) \, dr \). Denote
\[
\omega^*(z) = \int_{|z|}^1 \log \frac{s}{|z|} \omega(s) s \, ds, \quad z \in \mathbb{D} \setminus \{0\}.
\]

Throughout the proofs we will repeatedly use several basic properties of weights in the classes \( \mathcal{D} \) and \( \mathcal{D} \). For a proof of the first lemma, see [13, Lemma 2.1]; the second one can be proved by similar arguments.
Lemma A Let $\omega$ be a radial weight. Then the following statements are equivalent:

(i) $\omega \in \hat{D}$;

(ii) There exist $C = C(\omega) > 0$ and $\beta = \beta(\omega) > 0$ such that

\[ \hat{\omega}(r) \leq C \left( \frac{1-r}{1-t} \right)^\beta \hat{\omega}(t), \quad 0 \leq r \leq t < 1; \]

(iii) There exist $C = C(\omega) > 0$ and $\gamma = \gamma(\omega) > 0$ such that

\[ \int_0^t \left( \frac{1-t}{1-s} \right)^\gamma \omega(s) \, ds \leq C \hat{\omega}(t), \quad 0 \leq t < 1; \]

(iv) There exists $\lambda = \lambda(\omega) \geq 0$ such that

\[ \int_{\mathbb{D}} \frac{dA(z)}{|1-\bar{z}\zeta|^{\lambda+1}} \leq C(1-|\zeta|)\lambda, \quad \zeta \in \mathbb{D}; \]

(v) There exist $K = K(\omega) > 1$ and $C = C(\omega, K) > 1$ such that $1 - \rho_n^r(\omega, K) \geq C(1 - \rho_{n+1}^r(\omega, K))$ for some (equivalently for all) $0 \leq r < 1$ and for all $n \in \mathbb{N} \cup \{0\}$.

Lemma B Let $\omega$ be a radial weight. Then $\omega \in \hat{D}$ if and only if there exist $C = C(\omega) > 0$ and $\alpha = \alpha(\omega) > 0$ such that

\[ \hat{\omega}(t) \leq C \left( \frac{1-t}{1-r} \right)^\alpha \hat{\omega}(r), \quad 0 \leq r \leq t < 1. \]

Two more results on weights of more general nature than Lemmas A and B are also needed.

Lemma 3 Let $\omega$ be a radial weight. Then the following statements are equivalent:

(i) $\omega \in \hat{D}$;

(ii) For some (equivalently for each) $\nu \in D$ there exists a constant $C = C(\omega, \nu) > 0$ such that

\[ \int_r^1 \frac{\omega(t) \tilde{\nu}(t)}{\hat{\omega}(t)} \, dt \leq C \tilde{\nu}(r), \quad 0 \leq r < 1; \]

(iii) For some (equivalently for each) $\nu \in D$ there exists a constant $C = C(\omega, \nu) > 0$ such that

\[ \int_0^r \frac{\omega(t)}{\hat{\omega}(t) \tilde{\nu}(t)} \, dt \leq \frac{C}{\tilde{\nu}(r)}, \quad 0 \leq r < 1. \]

**Proof** Let first $\omega \in \hat{D}$ and $0 \leq r < 1$, and consider $\rho_n^r(\omega, K)$ for all $n \in \mathbb{N} \cup \{0\}$. Then Lemma B, applied to $\nu \in D \subset \hat{D}$, and Lemma A(v), applied to $\omega$, imply

\[ \int_r^1 \frac{\omega(t) \tilde{\nu}(t)}{\hat{\omega}(t)} \, dt = \sum_{n=0}^{\infty} \int_{\rho_n^r}^{\rho_{n+1}^r} \frac{\omega(t) \tilde{\nu}(t)}{\hat{\omega}(t)} \, dt \leq \sum_{n=0}^{\infty} \tilde{\nu}(\rho_n^r) \int_{\rho_n^r}^{\rho_{n+1}^r} \frac{\omega(t)}{\hat{\omega}(t)} \, dt \]

\[ \leq \tilde{\nu}(r) \log K \sum_{n=0}^{\infty} \frac{1}{(C^\beta)^n} = \tilde{\nu}(r) \log K \frac{C^\beta}{C^\beta - 1}, \quad 0 \leq r < 1, \]
for a suitably fixed $K = K(\omega) > 1$, and thus (ii) is satisfied. Conversely, (ii) implies
\[ C\hat{\nu}(r) \geq \int_r^1 \frac{\omega(t)\hat{v}(t)}{\hat{\omega}(t)} \, dt \geq \int_r^{1+r/2} \frac{\omega(t)\hat{v}(t)}{\hat{\omega}(t)} \, dt \geq \hat{\nu}\left(\frac{1+r/2}{2}\right) \log \frac{\hat{\omega}(r)}{\hat{\omega}\left(\frac{1+r/2}{2}\right)} , \quad 0 \leq r < 1 ,
\] and since $v \in D \subset \hat{D}$ by the hypothesis, we deduce $\hat{\omega}(r) \lesssim \hat{\omega}\left(\frac{1+r/2}{2}\right)$ for all $0 \leq r < 1$. Thus $\omega \in \hat{D}$.

Let $\omega \in \hat{D}$ and $0 \leq r < 1$, and consider $\rho_n = \rho_n(\omega, K)$ for all $n \in \mathbb{N} \cup \{0\}$. Fix $k = k(\omega, K) \in \mathbb{N} \cup \{0\}$ such that $\rho_k \leq r < \rho_{k+1}$. Then
\[ \int_0^r \frac{\omega(t)}{\hat{\omega}(t)\hat{v}(t)} \, dt = \sum_{n=0}^{k-1} \int_{\rho_n}^{\rho_{n+1}} \frac{\omega(t)}{\hat{\omega}(t)\hat{v}(t)} \, dt + \int_{\rho_k}^r \frac{\omega(t)}{\hat{\omega}(t)\hat{v}(t)} \, dt , \quad 0 \leq r < 1 ,
\] where, by Lemma B, applied to $v \in D \subset \hat{D}$, and Lemma A(v), applied to $\omega$,
\[ \sum_{n=0}^{k-1} \int_{\rho_n}^{\rho_{n+1}} \frac{\omega(t)}{\hat{\omega}(t)\hat{v}(t)} \, dt \leq \sum_{n=0}^{k-1} \frac{1}{\hat{v}(\rho_{n+1})} \int_{\rho_n}^{\rho_{n+1}} \frac{\omega(t)}{\hat{\omega}(t)} \, dt \leq \log K \frac{(1-\rho_k)^{\alpha}}{\hat{\nu}(r)} \sum_{n=0}^{k-1} \frac{1}{(C^\alpha)^n} \leq \log K \frac{C^\alpha}{\hat{v}(r)} , \quad k \in \mathbb{N} ,
\] for some $\alpha = \alpha(v) > 0$ and for a suitably fixed $K = K(\omega) > 1$, and similarly,
\[ \int_{\rho_k}^r \frac{\omega(t)}{\hat{\omega}(t)\hat{v}(t)} \, dt \leq \frac{1}{\hat{v}(r)} \log \frac{\hat{\omega}(\rho_k)}{\hat{\omega}(r)} \leq \frac{\log K}{\hat{v}(r)} , \quad k \in \mathbb{N} \cup \{0\} .
\]

The statement (iii) follows from these estimates.

Conversely, by replacing $r$ by $\frac{1+r}{2}$ in (iii) we obtain
\[ \frac{C}{\hat{v}\left(\frac{1+r}{2}\right)} \geq \int_0^{1+(r/2)^2} \frac{\omega(t)}{\hat{\omega}(t)\hat{v}(t)} \, dt \geq \int_r^{1+(r/2)^2} \frac{\omega(t)}{\hat{\omega}(t)\hat{v}(t)} \, dt \geq \frac{1}{\hat{v}(r)} \log \frac{\hat{\omega}(r)}{\hat{\omega}\left(\frac{1+r}{2}\right)} , \quad 0 \leq r < 1 ,
\] and since $v \in D \subset \hat{D}$ by the hypothesis, we deduce $\hat{\omega}(r) \lesssim \hat{\omega}\left(\frac{1+r}{2}\right)$ for all $0 \leq r < 1$. Thus $\omega \in \hat{D}$.

\[ \square \]

**Lemma 4** Let $\omega, v \in D$, and denote $\sigma = \sigma_{\omega, v} = \omega \hat{v}/\hat{\omega}$. Then $\hat{\sigma} \asymp \hat{v}$ on $[0, 1)$, and hence $\sigma \in D$.

**Proof** Lemma 3(ii) implies $\hat{\sigma} \lesssim \hat{v}$ on $[0, 1)$. The argument used to prove (i) $\Rightarrow$ (ii) in the said lemma shows that $\hat{\sigma} \gtrsim \hat{v}$ on $[0, 1)$, provided $\omega \in \hat{D}$ and $v \in D$. Thus $\hat{\sigma} \asymp \hat{v}$, and $\sigma \in D$ by Lemmas A(ii) and B.

\[ \square \]

The next lemma says that in many instances concerning $A^p$-norms we may replace $\omega$ by $\tilde{\omega} = \hat{\omega}/(1 - |\cdot|)$ if $\omega \in D$. This result has the flavor of radial Carleson measures and indeed can be established by appealing to the characterization of Carleson measures for the Bergman space $A^p_\omega$ induced by $\omega \in \hat{D}$ given in [15]. That approach requires showing that the involved

\[ \square \]
integration by parts shows that (2.1) is equivalent to

\[ \int_{\mathcal{D}} |f(z)|^p (1 - |z|^\kappa) \omega(z) dA(z) \asymp \int_{\mathcal{D}} |f(z)|^p (1 - |z|)^{\kappa-1} \tilde{\omega}(z) dA(z), \quad f \in \mathcal{H}(\mathbb{D}). \]

(2.1)

**Proof** The function \((1 - |\cdot|^\kappa)^{-1} \tilde{\omega} \) is a weight for each \(\kappa > -\alpha\) by Lemma B. Therefore an integration by parts shows that (2.1) is equivalent to

\[ \int_0^1 \frac{d}{dr} M^p_\rho(r, f) \left( \int_r^1 (1 - t)^\kappa \omega(t) dt \right) dr \asymp \int_0^1 \frac{d}{dr} M^p_\rho(r, f) \left( \int_r^1 (1 - t)^{\kappa-1+\beta(\rho)} dt \right) dr. \]

Another integration by parts reveals that both integrals from \(r\) to 1 above are bounded by a constant times \(\tilde{\omega}(r)(1 - r)^\kappa\). But Lemma A(ii) implies

\[ \int_r^1 (1 - t)^{\kappa-1+\beta(\rho)} dt \gtrsim \frac{\tilde{\omega}(r)}{(1 - r)^{\beta(\rho)}} \int_r^1 (1 - t)^{\kappa-1+\beta(\rho)} dt \asymp \tilde{\omega}(r)(1 - r)^\kappa, \quad 0 \leq r < 1, \]

and

\[ \int_r^1 (1 - t)^\kappa \omega(t) dt \gtrsim \frac{\tilde{\omega}(r)}{(1 - r)^{\beta(\rho)}} \int_r^1 \omega(t)(1 - t)^{\kappa+\beta(\rho)} dt \asymp \tilde{\omega}(r)(1 - r)^\kappa, \quad 0 \leq r < 1, \]

by Lemma 4. The assertion follows. \(\square\)

The last auxiliary results shows that each radial weight in the Bekollé–Bonami class \(B_q\) belongs to \(\mathcal{D}\), and for each \(v \in \mathcal{D}\) the maximal Bergman projection

\[ P_v^+(f)(z) = \int_{\mathcal{D}} f(\zeta)|B_v^+(\zeta)|\nu(\zeta) dA(\zeta), \quad z \in \mathbb{D}, \]

is bounded on \(L^q_v\). It is worth noticing that obviously \(\mathcal{D} \not\subset \cup_{1 < q < \infty} B_q\) because \(v \in \mathcal{D}\) may vanish on a set of positive measure.

**Proposition 6** Let \(1 < q < \infty\) and \(v \in B_q\) a radial weight. Then \(v \in \mathcal{D}\). Moreover, \(P_v^+: L^q_v \to L^q_v\) is bounded for all \(v \in \mathcal{D}\).

**Proof** If \(v \in B_q\), then by [5] there exists \(\beta > -1\) such that

\[ \left( \int_{S(a)} v(z) dA(z) \right)^{\frac{1}{q'}} \left( \int_{S(a)} \left( \frac{1 - |z|^\beta}{v(z)} \right)^{\frac{q'}{q}} (1 - |z|^\beta) dA(z) \right)^{\frac{1}{q'}} \lesssim (1 - |a|)^{(2+\beta)}, \quad a \in \mathbb{D}. \]

Since \(v\) is radial, this condition easily implies \(v \in \mathcal{D}\).

Let now \(1 < q < \infty\) and \(v \in \mathcal{D}\), and define \(h = \nu^{-\frac{1}{q'}}\). Then \(\int_0^1 h(s)^{q'} \nu(s) ds \asymp \tilde{\nu}(t)^{\frac{1}{q'}}\) for all \(0 \leq t < 1\). Therefore Lemma B yields

\[ \int_0^1 \int_0^1 h(s)^{q'} \nu(s) ds \tilde{\nu}(t)(1 - t) dt \asymp \int_0^1 \frac{dt}{\tilde{\nu}(t)^{\frac{1}{q'}}(1 - t)} \lesssim \frac{1}{\tilde{\nu}(r)^{\frac{1}{q'}}} = h^{q'}(r), \quad 0 \leq r < 1. \]

(2.2)
Moreover, by symmetry, (2.2) with \( q' \) in place of \( q \) is satisfied. Since \( \nu \in \hat{D} \), we may apply [16, Theorem 1] and (2.2) to deduce
\[
\int_{\mathbb{D}} |B^v_\nu(\zeta)|h^p(\zeta)v(\zeta) \ dA(\zeta) \lesssim h^p(\zeta), \quad \zeta \in \mathbb{D},
\]
and
\[
\int_{\mathbb{D}} |B^v_\nu(\zeta)|h^p(\zeta)z(\zeta) \ dA(\zeta) \lesssim h^p(\zeta), \quad \zeta \in \mathbb{D}.
\]
It follows from Schur’s test [23, Theorem 3.6] that the maximal Bergman projection \( P^+_{\nu} : L^p_\nu \to L^p_\nu \) is bounded. \( \square \)

3 Some spaces of functions

Recall that
\[
\gamma(z) = \gamma_{\omega,v,p,q}(z) = \frac{\vbar{\nu}(z)^{1/2}(1-|z|)^{1/4}}{\omega(z)^{1/2}(1-|z|)^{1/4}}, \quad z \in \mathbb{D},
\]
and \( \hat{f}_{r,v}(z) = \frac{\int_{\Delta(z,r)} f(\zeta)v(\zeta) \ dA(\zeta)}{v(\Delta(z,r))} \) for \( f \in L^1_{v,\text{loc}} \), and
\[
\text{MO}_{v,q,r}(f)(z) = \left( \frac{1}{v(\Delta(z,r))} \int_{\Delta(z,r)} |f(\zeta) - \hat{f}_{r,v}(z)|^q v(\zeta) \ dA(\zeta) \right)^{1/q},
\]
for all \( z \in \mathbb{D} \). If \( \nu \in \hat{D} \), then by the definition there exist \( K = K(\nu) > 1 \) and \( C = C(\nu) > 1 \) such that
\[
\int_{r}^{1-\frac{1-r}{K}} \nu(s) \ ds \geq (C-1)\vbar{\nu} \left( 1 - \frac{1-r}{K} \right) > 0, \quad 0 \leq r < 1.
\]
It follows that there exists \( r_v \in (0, \infty) \) such that \( v(\Delta(z,r)) > 0 \) for all \( z \in \mathbb{D} \) if \( r \geq r_v \).

The space \( \text{BMO}(\Delta) = \text{BMO}(\Delta)_{\omega,v,p,q,r} \) consists of \( f \in L^q_{v,\text{loc}} \) such that
\[
\|f\|_{\text{BMO}(\Delta)} = \sup_{z \in \mathbb{D}} \left( \text{MO}_{v,q,r}(f)(z)\gamma(z) \right) < \infty.
\]
The following lemma is easy to establish; see [12, Lemma 3.1] for a similar result.

Lemma 7 Let \( 1 \leq p, q < \infty \), \( \omega \) a radial weight, \( \nu \in \hat{D} \) and \( r_v \leq r < \infty \). Then
\[
\text{MO}_{v,q,r}(f)(z) \leq 2 \left( \frac{1}{v(\Delta(z,r))} \int_{\Delta(z,r)} |f(\zeta) - \lambda|^q v(\zeta) \ dA(\zeta) \right)^{1/q}, \quad z \in \mathbb{D}, \ \lambda \in \mathbb{C}, \ f \in L^q_v,
\]
and therefore \( f \in L^q_v \) belongs to \( \text{BMO}(\Delta) \) if and only if for each \( z \in \mathbb{D} \) there exists \( \lambda_z \in \mathbb{C} \) such that
\[
\sup_{z \in \mathbb{D}} \left( \frac{\gamma(z)^q}{v(\Delta(z,r))} \int_{\Delta(z,r)} |f(\zeta) - \lambda_z|^q v(\zeta) \ dA(\zeta) \right) < \infty.
\]
For $0 < p, q < \infty$, $0 \leq \tau < \infty$ and radial weights $\omega, \nu$, let

$$
\Gamma_{\tau}(z, \xi) = \left( \frac{|1 - \overline{\xi} \xi|^2}{\max\{1 - |\xi|^2, 1 - |\xi|^2\}} \right)^{\frac{1}{p}} \hat{\omega} \left( 1 - \frac{2|1 - \overline{\xi} \xi|^2}{\max\{1 - |\xi|^2, 1 - |\xi|^2\}} \right)^{\frac{1}{p}} \min \left\{ \frac{\hat{\nu}(\xi)}{(1 - |\xi|^2)^{\frac{1}{p}}}, \frac{\hat{\nu}(\zeta)}{(1 - |\zeta|^2)^{\frac{1}{p}}} \right\}, \quad z, \xi \in \mathbb{D}, \quad (3.2)
$$

with the understanding that $\hat{\omega}(t) = \hat{\omega}(0)$ when $t < 0$. The following lemma explains the behavior of $\Gamma_{\tau}$ near the diagonal.

**Lemma 8** Let $0 < p, q, r < \infty$, $0 \leq \tau < \infty$ and $\omega, \nu \in \mathcal{D}$. Then

$$
\Gamma_{\tau}(z, \xi) \preceq \gamma(z)^{-1} \times \gamma(\xi)^{-1}, \quad \beta(z, \xi) \leq r.
$$

**Proof** Clearly

$$
|1 - \overline{\xi} \xi| \geq 1 - |\xi| > 1 - |\xi|, \quad \beta(z, \xi) \leq r,
$$

and hence there exist $0 < m_r < 1 < M_r < \infty$ such that

$$
m_r (1 - |z|) \leq \frac{2|1 - \overline{\xi} \xi|^2}{\max\{1 - |\xi|^2, 1 - |\xi|^2\}} \leq M_r (1 - |z|), \quad \beta(z, \xi) \leq r.
$$

Since $\omega \in \mathcal{D}$ by the hypothesis, and $\hat{\omega}(t) = \hat{\omega}(0)$ for $t < 0$, Lemma A(ii) implies

$$
\hat{\omega}(z) \leq \frac{C}{m_r^p} \hat{\omega}(1 - m_r (1 - |z|)) \leq \frac{C}{m_r^p} \hat{\omega} \left( 1 - \frac{2|1 - \overline{\xi} \xi|^2}{\max\{1 - |\xi|^2, 1 - |\xi|^2\}} \right), \quad \beta(z, \xi) \leq r,
$$

and

$$
\hat{\omega} \left( 1 - \frac{2|1 - \overline{\xi} \xi|^2}{\max\{1 - |\xi|^2, 1 - |\xi|^2\}} \right) \leq C M_r^\beta \hat{\omega}(1 - M_r (1 - |z|)) \leq C M_r^\beta \hat{\omega}(z), \quad \beta(z, \xi) \leq r,
$$

for some $C = C(\omega) > 0$ and $\beta = \beta(\omega) > 0$. Further, $\hat{\nu}(z) \preceq \hat{\nu}(\xi)$ and $\hat{\omega}(z) \preceq \hat{\omega}(\xi)$ if $\beta(z, \xi) \leq r$ by Lemma A(ii). The assertion follows from these estimates.

For continuous $f : \mathbb{D} \to \mathbb{C}$ and $0 < r < \infty$, define

$$
\Omega_r f(z) = \sup \{ |f(z) - f(\xi)| : \beta(z, \xi) < r \}, \quad z \in \mathbb{D},
$$

and let $BO(\Delta) = BO(\Delta)_{\omega, \nu, p, q, r}$ denote the space of those $f$ such that

$$
\|f\|_{BO(\Delta)} = \sup_{z \in \mathbb{D}} (\Omega_r f(z) \gamma(z)) < \infty.
$$

Lemma 9 shows that the space $BO(\Delta) = BO(\Delta)_{\omega, \nu, p, q, r}$ is independent of $r$.

**Lemma 9** Let $0 < p \leq q < \infty$, $0 < r < \infty$, $\omega, \nu \in \mathcal{D}$ and $\gamma(z) = \gamma_{\omega, \nu, p, q}(z) = \frac{\hat{\nu}(z)\theta(1 - |z|^2)^{\frac{1}{q}}}{\hat{\omega}(z)\theta(1 - |z|^2)^{\frac{1}{p}}}$. Let $f : \mathbb{D} \to \mathbb{C}$ be continuous, and $0 < \tau < \min\{q\alpha(\omega)/p, \alpha(\nu)\}$, where $\alpha(\nu)$ and $\alpha(\omega)$ are those from Lemma B. Then the following statements are equivalent:

(i) $f \in BO(\Delta)$;

(ii) $|f(z) - f(\xi)| \lesssim \|f\|_{BO(\Delta)} (1 + \beta(z, \xi)) \Gamma_{\tau}(z, \xi)$ for all $z, \xi \in \mathbb{D}$. 
The election of \( \tau \) together with Lemma B shows that the functions \( \hat{\omega}/(1-r)^{\tau} \) and \( \hat{\nu}/(1-r)^{\tau} \) are essentially decreasing on \([0, 1]\). Therefore the inequalities (3.4) and \(|z_j| \leq \max(|z|, |\xi|)\) yield

\[
\|f\|_{BO(\Delta)} \|A(z, \zeta) N \|_{BO(\Delta)}(1 + \beta(z, \zeta)) A(z, \zeta), \quad \beta(z, \zeta) > r.
\]

Therefore (ii) is satisfied. \( \Box \)

For \( 0 < p, q < \infty, 0 < r < \infty \) and radial weights \( \omega, \nu, \) the space \( BA(\Delta) = BA(\Delta)_{p,q,r} \) consists of \( f \in L^q_{v, loc} \) such that

\[
\|f\|_{BA(\Delta)} = \sup_{z \in \mathbb{D}} \left( \frac{1}{\nu(A(z, r))} \int_{A(z,r)} |f(\zeta)|^q v(\zeta) dA(\zeta) \right)^{\frac{1}{q}} \gamma(z) < \infty.
\]

For \( c, \sigma \in \mathbb{R} \) and a radial weight \( v, \) the general Berezin transform of \( \varphi \in L^q_{v,(1-|z|)^{\sigma}} \) is defined by

\[
B(\varphi)(z) = B_{v,c,\sigma}(\varphi)(z) = \frac{(1-|z|^2)^{c+1}}{\nu(z)} \int_{\mathbb{D}} \varphi(\zeta) \frac{(1-|\zeta|^2)^\sigma}{|1-z\zeta|^{2+c+\sigma}} v(\zeta) dA(\zeta), \quad z \in \mathbb{D}.
\]
The next lemma shows, in particular, that the space \( BA(\Delta) = BA(\Delta)_{\omega,v,p,q,r} \) is independent of \( r \) as long as \( r \) is sufficiently large depending on \( v \in D \).

**Lemma 10** Let \( 0 < p \leq q < \infty, 0 < r < \infty \) and \( \omega, v \in D \), \( \gamma(z) = \gamma_{\omega,v,p,q}(z) = \frac{\gamma(z)^{\frac{1}{q}}}{\omega(z)^{\frac{1}{p}} (1-|z|)^{\frac{1}{p}}} \). If \( f \in L^q_v \), then the following statements are equivalent:

(i) There exists \( r_0 = r_0(v) > 0 \) such that \( f \in BA(\Delta) = BA(\Delta)_{\omega,v,p,q,r} \) for all \( r \geq r_0 \);
(ii) \( |f|^q v dA \) is a \( q \)-Carleson measure for \( A^p_\omega \);
(iii) The identity operator \( I_d : A^p_\omega \to L^q_v \) is bounded;
(iv) The multiplication operator \( M_f(g) = fg \) is bounded from \( A^p_\omega \) to \( L^q_v \);
(v) \( \sup_{z \in D} \gamma(z)^q B(|f|^q)(z) < \infty \) for all \( \sigma > 1 - \frac{q}{p} (1 + \alpha) \) and \( c > \max\{-1 - \sigma, \frac{q}{p} (1 + \beta) - 2\} \), where \( \alpha = \alpha(\omega) > 0 \) and \( \beta = \beta(\omega) > 0 \) are those of Lemmas \( A(ii) \) and \( B \).

**Proof** It is obvious that (ii), (iii) and (iv) are equivalent by the definitions. Assume (ii) is satisfied, that is,

\[
\left( \int_D |g(\zeta)|^q |f(\zeta)|^q v(\zeta) dA(\zeta) \right)^{\frac{1}{q}} \lesssim \|g\|_{A^p_\omega}, \quad g \in A^p_\omega. \tag{3.5}
\]

For \( z \in D \), let \( g_z(\zeta) = \left( \frac{1-|z|}{1-|\zeta|} \right)^{\frac{\lambda - 1}{q}} \), where \( \lambda = \lambda(\omega) > 0 \) is that of Lemma \( A(iv) \). Further, since \( v \in D \) by the hypothesis, there exists \( r_v \in (0, \infty) \) such that \( v(\Delta(z, r)) > 0 \) for all \( r \geq r_v \). For \( g = g_z \) and \( r \geq r_v \), (3.5) yields

\[
\left( \frac{1}{v(\Delta(z, r))} \int_{\Delta(z, r)} |f(\zeta)|^q v(\zeta) dA(\zeta) \right)^{\frac{1}{q}} \lesssim \frac{\|g_z\|_{A^p_\omega}}{v(\Delta(z, r))} \lesssim \frac{(\omega(\zeta)(1-|z|))^{\frac{1}{p}}}{\nu(\Delta(z, r))^{\frac{1}{q}}}, \quad z \in D.
\]

But since \( v \in D \), applications of Lemmas \( A(ii) \) and \( B \) show that

\[
v(\Delta(z, r)) \asymp \tilde{\nu}(z)(1-|z|), \quad z \in D, \tag{3.6}
\]

if \( r \) is sufficiently large. It follows that \( f \in BA(\Delta) = BA(\Delta)_{\omega,v,p,q,r} \) for all such \( r \), and thus (i) is satisfied.

Conversely, if (i) is satisfied, then by using (3.6) we deduce

\[
\left( \int_{\Delta(z, r)} |f(\zeta)|^q v(\zeta) dA(\zeta) \right)^{\frac{1}{q}} \lesssim \omega(z)^{\frac{1}{p}} (1-|z|)^{\frac{1}{p}}, \quad z \in D.
\]

Therefore \( |f|^q v dA \) is a \( q \)-Carleson measure for \( A^p_\omega \) by [17, Theorem 3].

By integrating only over \( \Delta(z, r) \) in (v) and using (3.6) we obtain (i) from (v). To complete the proof of the lemma, it remains to show the converse implication. To do this, pick up a sequence \( \{a_j\} \) and \( 0 < r < \infty \) in accordance with [23, Lemma 4.7], and observe that \( \omega \) is essentially constant in each hyperbolically bounded region by Lemma \( A(ii) \). Then by using (3.6), the hypothesis (i), the election of \( c \) and \( \sigma \), and finally Lemmas \( A(ii) \) and \( B \), we deduce
The following statements are equivalent:

\[ \beta(\omega), \beta(\nu), \gamma(\nu) > \]

where \( \nu \) is well defined and continuous. Since \( \nu \) follows from the case (ii) of the next theorem.

and thus (v) is satisfied.

With these preparations we are ready to show that \( \text{BMO}(\Delta) = \text{BA}(\Delta) + \text{BO}(\Delta) \). This follows from the case (ii) of the next theorem.

**Theorem 11** Let \( 1 \leq p \leq q < \infty, \omega, v \in \mathcal{D}, \gamma(z) = \gamma_{\omega,v,p,q}(z) = \frac{\nabla(z)\frac{1}{q}(1-|z|)^{\frac{1}{q}}}{\omega(z)^{\frac{1}{p}}(1-|z|)^{\frac{1}{p}}} \) and \( f \in L_v^q \). Further, let \( r \geq r_v, \sigma > 0 \) and

\[ c > \frac{2q}{p} (\beta(\omega) + 1) + \sigma + \max \{2\beta(v), \gamma(v)\}, \]

where \( \beta(\omega), \beta(v), \gamma(v) > 0 \) are associated to \( v \) and \( \omega \) via Lemma A(ii), (iii). Then the following statements are equivalent:

(i) There exists \( r_0 = r_0(v) \geq r_v \) such that \( f \in \text{BMO}(\Delta) = \text{BMO}(\Delta)_{\omega,v,p,q,r} \) for all \( r \geq r_0 \);

(ii) \( f = f_1 + f_2 \), where \( f_1 \in \text{BA}(\Delta) \) and \( f_2 = \hat{f}_{r,v} \in \text{BO}(\Delta) \);

(iii) \( \sup_{z \in \mathbb{D}} (B(|f - \hat{f}_{r,v}(z)|^q)\gamma(z)^q) < \infty \);

(iv) For each \( z \in \mathbb{D} \) there exists \( \lambda_z \in \mathbb{C} \) such that \( \sup_{z \in \mathbb{D}} (B(|f - \lambda_z|^q)\gamma(z)^q) < \infty \).

**Proof** Obviously, (iii) implies (iv). Next assume (iv). The relation (3.6) shows that there exists \( r_0 = r_0(v) > 0 \) such that

\[ \frac{1}{v(\Delta(z,r))} \int_{\Delta(z,r)} |f(\zeta) - \lambda_z|^q v(\zeta) \, dA(\zeta) \]

\[ \leq \frac{(1 - |z|)^{q+1}}{\nabla(z)} \int_{\mathbb{D}} |f(\zeta) - \lambda_z|^q \frac{(1 - |\zeta|^2)^\sigma}{|1 - z\bar{\zeta}|^{2+c+\sigma}} v(\zeta) \, dA(\zeta), \quad z \in \mathbb{D}, \quad r_0 \leq r < \infty, \]

which together with Lemma 7 shows that (i) is satisfied.

Assume now (i), and let \( f_2 = \hat{f}_{r,v} \). Since \( f \in L_v^q, q \geq 1 \) and \( r \geq r_v \), the function \( f_2 \) is well defined and continuous. Since \( \omega, v \in \mathcal{D} \) by the hypothesis, one may use Lemmas A(ii)
and B together with the argument in [12, 1651–1652] with minor modifications to show that \( f_2 = \hat{f}_{r,v} \in \text{BO}(\Delta) \) and \( f_1 = f - \hat{f}_{r,v} \in \text{BA}(\Delta) \). Thus (ii) is satisfied.

To complete the proof it suffices to show that (ii) implies (iii), so assume \( f = f_1 + f_2 \), where \( f_1 \in \text{BA}(\Delta) \) and \( f_2 = \hat{f}_{r,v} \in \text{BO}(\Delta) \). Since \( \hat{f}_{r,v} = \hat{f}_{1r,v} + \hat{f}_{2r,v} \), it suffices to prove the condition in (iii) for \( f_1 \) and \( f_2 \) separately. First observe that by Lemma A(iii) the constant function 1 satisfies

\[
B(1)(z) \lesssim \frac{(1 - |z|)^{c+1}}{\nu(z)} \left( \int_0^{|z|} \frac{v(t)}{(1 - t)^{1+c}} dt + \frac{1}{(1 - |z|)^{1+c+\sigma}} \int_{|z|}^1 (1 - t)\nu(t) dt \right) \lesssim 1, \quad z \in \mathbb{D},
\]

because \( c > \max\{|\gamma(v)|, \sigma\} - 1 \) by the hypothesis. This together with Hölder’s inequality and Lemma 10 yields

\[
B \left( |f_1 - \hat{f}_{1r,v}(z)|^q \right) \gamma(z)^q \lesssim \left( B(|f_1|^q)(z) + |\hat{f}_{1r,v}(z)|^q \right) \gamma(z)^q \\
\leq \left( B(|f_1|^q)(z) + |\hat{f}_{1r,v}(z)|^q \right) \gamma(z)^q \lesssim 1, \quad z \in \mathbb{D},
\]

and thus (iii) for \( f_1 \in \text{BA}(\Delta) \) is satisfied.

To deal with \( f_2 \in \text{BO}(\Delta) \), pick up \( \tau \) satisfying the hypothesis of Lemma 9. Then

\[
|f_2(\zeta) - \hat{f}_{2r,v}(z)| = \left| \frac{1}{v(\Delta(z), r)} \int_{\Delta(z,r)} (f_2(\zeta) - f_2(u))v(u) dA(u) \right| \\
\leq \frac{1}{v(\Delta(z), r)} \int_{\Delta(z,r)} |f_2(\zeta) - f_2(u)| v(u) dA(u) \\
\lesssim \frac{1}{v(\Delta(z), r)} \int_{\Delta(z,r)} (1 + \beta(\zeta, u))\Gamma_\tau(\zeta, u) v(u) dA(u) \\
\lesssim (1 + \beta(z, \zeta))\Gamma_\tau(z, \zeta), \quad z, \zeta \in \mathbb{D},
\]

because \( \Gamma_\tau(\zeta, u) \asymp \Gamma_\tau(z, \zeta) \) for all \( u \in \Delta(z, r) \) by Lemma A(ii); see the proof of Lemma 8 for similar estimates. Hence it suffices to show that

\[
\frac{(1 - |z|)^{c+1}\gamma(z)^q}{v(z)} \int_\mathbb{D} \left| (1 + \beta(z, \zeta))\Gamma_\tau(z, \zeta) \right|^q \frac{(1 - |\zeta|^2)^\sigma}{|1 - z\zeta|^2 (1 + \beta(z, \zeta))} v(\zeta) dA(\zeta) \lesssim 1, \quad z \in \mathbb{D},
\]

(3.7)

to obtain (iii) for \( f_2 \in \text{BO}(\Delta) \). The proof of (3.7) is involved and will be divided into four separate cases. Before dealing with each case, we observe that since \( \beta(z, \zeta) \) grows logarithmically, we may pick up \( 0 < \delta < \min \left\{ \sigma, \frac{\sigma}{\tilde{\beta}(\omega)} + \beta(\nu) + \frac{\sigma}{\pi} \right\} \) and a constant \( C = C(\delta) > 0 \) such that

\[
1 + \beta(z, \zeta) \leq C \left| (1 - |\varphi_2(\zeta)|)^{-\frac{\delta}{\sigma}} \right| = C \left( \frac{|1 - z\zeta|^2}{(1 - |z|)(1 - |\zeta|)} \right)^{\frac{\delta}{\sigma}}, \quad z, \zeta \in \mathbb{D}.
\]

(3.8)

Case I

If

\[
\zeta \in D_1(z) = \left\{ w \in \mathbb{D} : 1 - \frac{2|1 - z\overline{w}|^2}{1 - |z|^2} \leq 0 \right\},
\]

\[ Springer\]
then $1 - |z| \lesssim |1 - z\tilde{w}|^2$ and

$$\Gamma(z, \zeta)^q \leq \left( \frac{|1 - z\tilde{w}|^2}{\max(1 - |z|, 1 - |\zeta|)} \right)^{\frac{q}{p} - \tau - 1} \mathcal{D}(0)^{\frac{q}{p} - \tau - 1} \left( \frac{1 - |z|}{1 - |\zeta|} \right)^{\tau} \chi_{D(0, |z|)}(\zeta)$$

$$+ \left( \frac{|1 - z\tilde{w}|^2}{1 - |z|} \right)^{\frac{q}{p} - \tau - 1} \left( \frac{1 - |\zeta|}{1 - |z|} \right)^{\tau} \chi_{D(0, |\zeta|)}(\zeta), \quad z \in \mathbb{D}, \quad \zeta \in D_1(z),$$

because of how $\tau$ is chosen in Lemma 9. Therefore (3.8) together with Lemmas A(ii) and 3(ii) yields

$$\frac{(1 - |z|)^{c + 1} \gamma(z)^q}{\nu(z)} \int_{D_1(z)} |(1 + \beta(z, \zeta)) \Gamma(z, \zeta)^q \frac{(1 - |\zeta|^2)^\sigma}{|1 - z\tilde{w}|^{2 + c + \sigma}} \nu(\zeta) \, dA(\zeta)$$

$$\lesssim \frac{(1 - |z|)^{c + 2 + 2\tau - \delta - \frac{q}{p}} \gamma(z)^q}{\nu(z)^2} \int_{D_1(z) \cap D(0, |z|)} \frac{(1 - |\zeta|^2)^{\sigma - \delta}}{|1 - z\tilde{w}|^{4 + c + \sigma - 2(\frac{q}{p} + \delta - \tau)}} \nu(\zeta) \, dA(\zeta)$$

$$+ \frac{(1 - |z|)^{c + 2 + \tau - \delta - \frac{q}{p}} \gamma(z)^q}{\nu(z)} \int_{D_1(z) \cap D(0, |z|)} \frac{(1 - |\zeta|^2)^{\sigma - \delta + \tau}}{\nu(\zeta)^2} \nu(\zeta) \, dA(\zeta)$$

$$\lesssim \frac{(1 - |z|)^{\frac{q}{p} + \tau - \frac{q}{p}} \gamma(z)^q}{\nu(z)^2} \int_0^{|z|} (1 - s)^{\sigma - \delta} \nu(s) \, ds$$

$$+ \frac{(1 - |z|)^{\frac{q}{p} + 1 + \tau - \delta - \frac{q}{p}}}{\nu(z)} \int_0^{|z|} (1 - s)^{\sigma - \delta + \tau} \nu(s) \, ds$$

$$\lesssim \frac{(1 - |z|)^{\frac{q}{p} + 1 - \beta + \gamma - \beta(v) - \beta(\omega)}}{\nu(z)^{\frac{q}{p}}} \lesssim 1, \quad z \in \mathbb{D},$$

where the last estimate is an immediate consequence of the choices of $c$ and $\delta$.

**Case 2** If

$$\zeta \in D_2(z) = \left\{ w \in \mathbb{D} : 1 - \frac{2|1 - zw|^2}{1 - |z|^2} \geq |z| \geq |w| \right\},$$

then $|1 - z\tilde{w}| \gg 1 - |z|^2 \leq 1 - |\zeta|^2$, which together the fact that $\frac{\tilde{w}(\zeta)}{(1 - r)^\tau}$ and $\frac{\tilde{w}(w)}{(1 - r)^\tau}$ are essentially decreasing on $[0, 1)$ gives

$$\Gamma(z, \zeta)^q \lesssim \gamma(z)^{-q}, \quad z \in \mathbb{D}, \quad \zeta \in D_2(z).$$

Therefore (3.8) and Lemma A(iii) yield

$$\frac{(1 - |z|)^{c + 1 + \delta} \gamma(z)^q}{\nu(z)} \int_{D_2(z)} |(1 + \beta(z, \zeta)) \Gamma(z, \zeta)^q \frac{(1 - |\zeta|^2)^\sigma}{|1 - z\tilde{w}|^{2 + c + \sigma}} \nu(\zeta) \, dA(\zeta)$$

$$\lesssim \frac{(1 - |z|)^{c + 1 - \delta}}{\nu(z)} \int_{D_2(z)} \frac{(1 - |\zeta|^2)^{\sigma - \delta}}{|1 - z\tilde{w}|^{2 + c + \sigma - 2\delta}} \nu(\zeta) \, dA(\zeta)$$

$$\lesssim \frac{(1 - |z|)^{c + 1 - \delta}}{\nu(z)} \int_0^{|z|} \nu(r) \frac{1}{(1 - r)^{c + 1 - \delta}} \, dr \lesssim 1, \quad z \in \mathbb{D}.$$
Case 3 If
\[ \zeta \in D_3(z) = \left\{ w \in \mathbb{D} : \min \left\{ 1 - \frac{2|1-z\overline{w}|^2}{1-|z|^2}, |w| \right\} \geq |z| \right\}, \]
then \(|1-z\overline{\zeta}| \geq 1-|z|^2 \), which together the fact that \( \frac{\tilde{\omega}(r)}{(1-r)^{\sigma}} \) and \( \frac{\tilde{\omega}(r)}{(1-r)^{\frac{\delta}{p}}} \) are essentially decreasing on \([0, 1]\), implies
\[ \Gamma_r(\zeta, \zeta)^q \lesssim \frac{\tilde{\omega}(z)^q}{\tilde{v}(z)} \frac{(1-|z|)^{q-1}}{|1-z\overline{\zeta}|^2 + \sigma}, \quad z \in \mathbb{D}, \quad \zeta \in D_3(z). \]
Therefore (3.8) and Lemma 3(ii) imply
\[
\frac{(1-|z|)^{c+1}\gamma(z)^q}{\tilde{v}(z)} \int_{D_3(z)} |(1 + \beta(z, \zeta)) \Gamma_r(z, \zeta)|^q \frac{(1-|\zeta|^2)\sigma}{|1-z\overline{\zeta}|^{2+c+\sigma}} v(\zeta) dA(\zeta)
\lesssim (1-|z|)^{c+1-\delta} \int_{D_3(z)} \frac{(1-|\zeta|^2)^{\sigma-\delta}}{|1-z\overline{\zeta}|^{2+c+\sigma-2\delta} \tilde{v}(z)} v(\zeta) dA(\zeta)
\lesssim (1-|z|)^{\delta-\sigma} \int_{|z|}^{1} \frac{(1-s)^{\sigma-\delta} v(s)}{\tilde{v}(s)} ds \lesssim 1, \quad z \in \mathbb{D}.
\]
Case 4 If
\[ \zeta \in D_4(z) = \left\{ w \in \mathbb{D} : 1 - \frac{2|1-z\overline{w}|^2}{1-|z|^2} < |z| \right\}, \]
then Lemma A(ii) gives
\[ \tilde{\omega} \left( 1 - \frac{2|1-z\overline{\zeta}|^2}{1-|\zeta|^2} \right) \lesssim \left( \frac{|1-z\overline{\zeta}|}{1-|\zeta|} \right)^{2\beta(\omega)} \tilde{\omega}(z), \quad z \in \mathbb{D}, \quad \zeta \in D_4(z), \]
and hence
\[ \Gamma_r(\zeta, \zeta)^q \lesssim \left( \frac{|1-z\overline{\zeta}|}{1-|\zeta|} \right)^{2\beta(\omega)} \tilde{\omega}(z)^q \left( \frac{|1-z\overline{\zeta}|^2}{1-|\zeta|^2} \right)^{\frac{q}{p}-\tau-1} \frac{(1-|z|)^{\tau}}{\tilde{v}(z)} \chi_{D(0,|z|)}(\zeta)
\]
\[ + \left( \frac{|1-z\overline{\zeta}|^2}{1-|\zeta|^2} \right)^{\frac{q}{p}-\tau-1} \frac{(1-|\zeta|^2)^{\tau}}{\tilde{v}(\zeta)} \chi_{\mathbb{D}\setminus D(0,|z|)}(\zeta), \quad z \in \mathbb{D}, \quad \zeta \in D_4(z). \]
Therefore (3.8) and Lemmas A(iii) and 3(ii) yield
\[
\frac{(1-|z|)^{c+1}\gamma(z)^q}{\tilde{v}(z)} \int_{D_4(z)} |(1 + \beta(z, \zeta)) \Gamma_r(z, \zeta)|^q \frac{(1-|\zeta|^2)^{\sigma}}{|1-z\overline{\zeta}|^{2+c+\sigma}} v(\zeta) dA(\zeta)
\lesssim (1-|z|)^{c+2-2\delta-\frac{q}{p} \beta(\omega) + \tau} \int_{D_4(z)} \frac{(1-|\zeta|^2)^{\sigma-\delta+\tau+1}}{|1-z\overline{\zeta}|^{4+c+\sigma-2\delta-2\beta(\omega) \frac{q}{p} - \frac{2q}{p} + 2\tau} \tilde{v}(z)} v(\zeta) dA(\zeta)
\]
\[ + (1-|z|)^{c+2-2\delta-\frac{q}{p} \beta(\omega) + \tau+1} \int_{D_4(z)\setminus D(0,|z|)} \frac{(1-|\zeta|^2)^{\sigma-\delta+\tau+1}}{|1-z\overline{\zeta}|^{4+c+\sigma-2\delta-2\beta(\omega) \frac{q}{p} - \frac{2q}{p} + 2\tau} \tilde{v}(z)} v(\zeta) dA(\zeta)
\]
\[ \lesssim (1-|z|)^{c+2-2\delta-\frac{q}{p} \beta(\omega) + \tau+1} \int_{|z|}^{1} \frac{v(r)}{(1-r)^{2+c-\delta-2\beta(\omega) \frac{q}{p} - \frac{2q}{p} + \tau}} dr \lesssim 1, \quad z \in \mathbb{D}. \]
Since $\mathbb{D} = \bigcup_{j=1}^{4} D_j(z)$ for each $z \in \mathbb{D}$, by combining the four cases we obtain (3.7). Thus (ii) implies (iii), and the proof is complete.

4 Boundedness of integral operators

In order to deal with the boundedness of Hankel operators, we need a technical result concerning certain integral operators. For $f \in L^1_b$ and $b, c \in \mathbb{R}$, define

$$T_{b,c}(f)(z) = \int_{\mathbb{D}} f(\zeta) \frac{(1 - |\zeta|^2)^b}{(1 - z^2)^c} \, dA(\zeta), \quad z \in \mathbb{D},$$

and

$$S_{b,c}(f)(z) = \int_{\mathbb{D}} f(\zeta) \frac{(1 - |\zeta|^2)^b}{|1 - z^2|^c} \, dA(\zeta), \quad z \in \mathbb{D}.$$

In the analytic case the operator $T_{b,c}$ can be interpreted as a fractional differentiation or integration depending on the parameters $b$ and $c$ [20]. The boundedness of these operator between $L^p$ spaces induced by standard weights has been characterized in [19].

Lemma A(ii) shows that for $\eta \in \hat{\mathbb{D}}$ there exists a constant $c_0 = c_0(\sigma) > 1$ such that hypotheses (i) and (ii) of the next lemma are satisfied for all $c \geq c_0$.

**Lemma 12** Let $1 < p \leq q < \infty, b > -1, c > 1$ and $\sigma, \eta \in \mathbb{D}$ such that

(i) $\int_{r}^{1} \frac{(1 - t)^{c-2}}{\hat{\eta}(t)^{\frac{1}{q}}} \, dt \lesssim \frac{(1 - r)^{c-1}}{\hat{\eta}(r)^{\frac{1}{q}}}, \quad 0 \leq r < 1$;

(ii) $\int_{0}^{r} \frac{\eta(t)}{(1 - t)^{\frac{c}{q}} - \hat{\eta}(t)^{\frac{1}{q}}} \, dt \lesssim \frac{\hat{\eta}(r)^{\frac{1}{q}}}{(1 - r)^{\frac{c}{q}-1}}, \quad 0 \leq r < 1$.

Then the following statements are equivalent:

1. $S_{b,c} : A^p_b \rightarrow L^q_b$ is bounded;
2. $T_{b,c} : A^p_b \rightarrow L^q_b$ is bounded;
3. $\sup_{0 < r < 1} (1 - r)^{2b-c+\frac{1}{q} - \frac{1}{p}} \frac{\hat{\eta}(r)^{\frac{1}{q}}}{\hat{\sigma}(r)^{\frac{1}{q}}} < \infty$.

**Proof** Obviously (1) implies (2). Assume now (2), and for each $\zeta \in \mathbb{D}$ and $N \in \mathbb{N}$, define $f_{\zeta,N} \in H^\infty$ by $f_{\zeta,N}(z) = \frac{z^N}{\sigma(S(\zeta))^{\frac{1}{p}}} \left(\frac{1 - |\zeta|^2}{1 - \zeta z}\right)^{2b+N}$ for all $z \in \mathbb{D}$. By differentiating the reproducing formula of $A^2_b$ we obtain

$$g^{(N)}(z) = M_1 \int_{\mathbb{D}} \frac{\hat{u}^N g(u)(1 - |u|^2)^b}{(1 - \hat{u}z)^{2b+N}} \, dA(u), \quad z \in \mathbb{D}, \quad N \in \mathbb{N}, \quad g \in A^2_b, \quad (4.1)$$

where $M_1 = M_1(N, b) > 0$ is a constant. Therefore

$$T_{b,c}(f_{\zeta,N})(z) = \frac{(1 - |\zeta|^2)^{2b+N}}{\sigma(S(\zeta))^{\frac{1}{p}}} \int_{\mathbb{D}} \frac{u^N (1 - |u|^2)^b}{(1 - u\zeta)^{2b+N}(1 - \hat{u}z)^c} \, dA(u)$$

$$= \frac{(1 - |\zeta|^2)^{2b+N}}{\sigma(S(\zeta))^{\frac{1}{p}}} \int_{\mathbb{D}} \frac{\hat{u}^N (1 - |u|^2)^b}{(1 - \zeta \hat{u})^{2b+N}(1 - \zeta u)^c} \, dA(u)$$

$$= M_2 \frac{(1 - |\zeta|^2)^{2b+N}}{\sigma(S(\zeta))^{\frac{1}{p}}} \frac{z^N}{(1 - \zeta z)^{c+N}},$$
where $M_2 = M_2(b, c, N) > 0$. Fix $N > \max \left\{ \frac{\lambda_\eta+1}{q} - c, \frac{\lambda_\sigma+1}{p} - b - 2 \right\}$. Then Lemma A(iv) gives $\|f_{\xi,N}\|_{L_q^p} = 1$ and

$$\int_{\mathbb{D}} \frac{\eta(z)}{|1 - \xi z|^{(c+N)q}} dA(z) \asymp \frac{\eta(S(\xi))}{(1 - |\xi|)^{(c+N)q}}, \quad \xi \in \mathbb{D}. $$

Therefore (2) yields

$$\infty > \|f_{\xi,N}\|_{L_q^p}^q \gtrsim \|T_{b,c}(f_{\xi,N})\|_{L_q^p}^q \asymp \left( \frac{(1 - |\xi|^2)^{2+b+N}}{\sigma(S(\xi))^{1/p}} \right)^q \int_{\mathbb{D}} \frac{\eta(z)}{|1 - \xi z|^{(c+N)q}} dA(z)$$

$$\asymp (1 - |\xi|^2)^q(2+b-c) \frac{\eta(S(\xi))}{\sigma(S(\xi))^{1/p}}, \quad \xi \in \mathbb{D},$$

thus (3) holds.

Assume (3) holds and let $h(\xi) = \tilde{\sigma}(\xi)^{1/p'} (1 - |\xi|^2)^{b-1/2}(1 - \frac{1}{p'})$ for all $\xi \in \mathbb{D}$. Then Hölder’s inequality yields

$$|S_{b,c}f(z)| \leq \left( \int_{\mathbb{D}} |f(\xi)|^p h(\xi)^p \frac{dA(\xi)}{|1 - z\xi|^c} \right)^{1/p} \left( \int_{\mathbb{D}} \frac{(1 - |\xi|^2)^b}{h(\xi)} \frac{dA(\xi)}{|1 - z\xi|^c} \right)^{1/p'}$$

$$= I_1(z)^{1/p} \cdot I_2(z)^{1/p'},$$

where

$$I_2(z) = \int_{\mathbb{D}} \frac{(1 - |\xi|^2)^b}{|1 - z\xi|^c} dA(\xi) \asymp \int_0^1 \frac{(1 - r)^{b-\frac{1}{p} + \frac{1}{q} + 1} \tilde{\sigma}(r)^{1/p'}}{(1 - r|z|)^{c-1}} dr$$

$$= \int_0^{|z|} \frac{(1 - r)^{b-\frac{1}{p} + \frac{1}{q} + 1}}{\tilde{\sigma}(r)^{1/p'}} (1 - r|z|)^{c-1} dr + \int_{|z|}^1 \frac{(1 - r)^{b-\frac{1}{p} + \frac{1}{q} + 1}}{\tilde{\sigma}(r)^{1/p'}} (1 - r|z|)^{c-1} dr = J_{|z|} + J_{|z|},$$

Lemma B together with the assumption (3) yields

$$J_{|z|} \lesssim \int_0^{|z|} \frac{dr}{\tilde{\eta}(r)^{1/p}} \lesssim \frac{1}{\tilde{\eta}(z)^{1/p}}, \quad z \in \mathbb{D},$$

since $\eta \in \mathcal{D} \subset \tilde{\mathcal{D}}$ by the hypothesis. In a similar fashion, (3) together with the hypothesis (i) gives

$$J_{|z|} \lesssim \frac{1}{(1 - |z|)^{c-1}} \int_{|z|}^1 \frac{dr}{\tilde{\eta}(r)^{1/p}} \lesssim \frac{1}{(1 - |z|)^{c-1}} \int_{|z|}^1 \frac{dr}{\tilde{\eta}(z)^{1/p}} \lesssim \frac{1}{\tilde{\eta}(z)^{1/p}}, \quad z \in \mathbb{D},$$

and hence $I_2(z) \lesssim \tilde{\eta}(z)^{-1/p}$ for all $z \in \mathbb{D}$. This estimate and Minkowski’s integral inequality (Fubini’s theorem in the case $q = p$) now yield

$$\|S_{b,c}(f)\|_{L_q^p} \lesssim \left( \int_{\mathbb{D}} \left( \int_{\mathbb{D}} |f(\xi)|^p h(\xi)^p \frac{dA(\xi)}{|1 - z\xi|^c} \right)^{1/p} \frac{\eta(z)}{\tilde{\eta}(z)^{1/p'}} \frac{dA(z)}{|1 - z\xi|^c} \right)^{1/p}$$

$$\leq \int_{\mathbb{D}} |f(\xi)|^p \tilde{\sigma}(\xi) I_3(\xi) \frac{dA(\xi)}{|1 - z\xi|^c},$$
Lemma B. Then Lemmas 9 and 13 yield

\[ I_3(\xi) = \frac{h(\xi)}{\sigma(\xi)} \left( \int_{\mathbb{D}} \frac{\eta(z)dA(z)}{|1 - z\xi|^{c/\gamma q - 1}} \right)^{\frac{1}{q}} \approx \frac{h(\xi)}{\sigma(\xi)} \left( \int_{0}^{1} \frac{\eta(r)}{(1 - r|\xi|^{c/\gamma q - 1})} dr \right)^{\frac{1}{q}}. \]

Since

\[ \int_{0}^{1} \frac{\eta(r)}{(1 - r|\xi|^{c/\gamma q - 1})^{1/p'}} dr \leq \int_{0}^{1} \frac{\eta(r)}{(1 - r|\xi|^{c/\gamma q - 1})^{1/p}} dr \leq \frac{\eta(\xi)^{1/p}}{(1 - |\xi|^{c/\gamma q - 1})}, \quad \xi \in \mathbb{D}, \]

by the hypothesis (ii), and

\[ \int_{1}^{1} \frac{\eta(r)}{(1 - r|\xi|^{c/\gamma q - 1})^{1/p}} dr \leq \frac{1}{(1 - |\xi|^{c/\gamma q - 1})} \int_{1}^{1} \frac{\eta(r)}{\eta(\xi)^{1/p}} dr \leq \frac{\eta(\xi)^{1/p}}{(1 - |\xi|^{c/\gamma q - 1})}, \quad \xi \in \mathbb{D}, \]

we deduce

\[ I_3(\xi) \lesssim (1 - |\xi|)^{2b-c+1/\gamma q - 1} \frac{\eta(\xi)^{1/p}}{\eta(\xi)^{1/p}} \lesssim 1, \quad \xi \in \mathbb{D}, \]

by the assumption (3). It follows that \( \|S_{b,c}(f)\|_{L^q_{\omega}} \lesssim \|f\|_{A_{\sigma}^q} \). This finishes the proof because \( \|f\|_{A_{\sigma}^q} \approx \|f\|_{A_{\sigma}^q} \) for all \( f \in \mathcal{H}(\mathbb{D}) \) by Lemma 5 provided \( \sigma \in \mathcal{D} \).

\[ \square \]

5 Proof of Theorem 1

In order to prove the sufficiency part of Theorem 1 we shall use the next result which follows from the argument used in the proof of [12, Lemma 4.5].

**Lemma 13** Let \( 1 < q < \infty \) and \( \nu, \omega \) weights such that \( P_\omega : L^q_\nu \rightarrow L^q_\nu \) is bounded. Then

\[ \|H_\nu^q(g)\|_{L^q_\nu} \leq (1 + \|P_\omega\|_{L^q_\nu \rightarrow L^q_\nu}) \|H_\nu^q(g)\|_{L^q_\nu} \quad \text{for } f \in L^q_\nu, \quad g \in H^\infty. \]

**Proposition 14** Let \( 1 < p \leq q < \infty \), \( \nu \in B_q \) a radial weight and \( \omega \in \mathcal{D} \). If \( f \in BO(\Delta) \), then \( H_\nu^q : A_{\omega}^q \rightarrow L^q_\nu \) is bounded.

**Proof** By [5] there exists a constant \( s_0 = s_0(\nu) > -1 \) such that \( P_s : L^q_\nu \rightarrow L^q_\nu \) is bounded for each \( s > s_0 \). Let \( 0 < \tau < \min\{q\alpha(\omega)/p, \alpha(\nu)\} \), where \( \alpha(\nu) \) and \( \alpha(\omega) \) are those from Lemma B. Then Lemmas 9 and 13 yield

\[ \|H_\nu^q(g)\|_{L^q_\nu} \lesssim \|H_\nu^q(g)\|_{L^q_\nu} \leq \int_{\mathbb{D}} \left( \int_{\mathbb{D}} \frac{|f(z) - f(\zeta)| |g(\zeta)|}{|1 - \xi|^{2+\theta}} dA(\zeta) \right)^q v(z) dA(z) \]

\[ \leq \int_{\mathbb{D}} \left( \int_{\mathbb{D}} |g(\zeta)| (\frac{\beta(\nu, \zeta) + 1) \Gamma_\tau(\nu, \zeta)}{|1 - \xi|^{2+\theta}} dA(\zeta) \right)^q v(z) dA(z), \quad g \in H^\infty. \]

Let \( s > \max\{s_0, 2(\beta(\nu) + \beta(\nu) + 2\alpha(\nu))\}, \delta < \min\{\frac{\tau}{q}, \frac{\alpha(\nu)}{q}\} \) and \( K > 1 \) to be fixed later. Then applying (3.8), we get

\[ \|H_\nu^q(g)\|_{L^q_\nu} \lesssim \sum_{j=1}^{5} \int_{\Omega_j(\nu)} \left( \int_{\Omega_j(\nu)} |g(\zeta)| \frac{\Gamma_\tau(\nu, \zeta) dA(\zeta)}{|1 - \xi|^{2+\theta - 2\delta(1 - |\xi|^2)^{\delta - s}}} \right)^q v(z) dA(z) \]

\[ = \sum_{j=1}^{5} I_j(g), \quad (5.1) \]

\[ \square \]
where

\[
\begin{align*}
\Omega_1(z) &= \left\{ \zeta \in \mathbb{D} : \frac{1}{|1 - \overline{z}\zeta|^2} \leq \frac{2}{\max\{1 - |z|^2, 1 - |\zeta|^2\}} \right\} \cap D(0, |z|), \\
\Omega_2(z) &= \left\{ \zeta \in \mathbb{D} : \frac{1}{|1 - \overline{z}\zeta|^2} \leq \frac{2}{\max\{1 - |z|^2, 1 - |\zeta|^2\}} \right\} \cap (\mathbb{D} \setminus D(0, |z|)), \\
\Omega_3(z, K) &= \left\{ \zeta \in \mathbb{D} : \frac{1 - |\zeta|}{K} \geq \frac{2|1 - \overline{z}\zeta|^2}{\max\{1 - |z|^2, 1 - |\zeta|^2\}} \right\}, \\
\Omega_4(z, K) &= \left\{ \zeta \in \mathbb{D} : \frac{1 - |\zeta|}{K} < \frac{2|1 - \overline{z}\zeta|^2}{\max\{1 - |z|^2, 1 - |\zeta|^2\}} < 1 \right\} \cap D(0, |z|), \\
\Omega_5(z, K) &= \left\{ \zeta \in \mathbb{D} : \frac{1 - |\zeta|}{K} < \frac{2|1 - \overline{z}\zeta|^2}{\max\{1 - |z|^2, 1 - |\zeta|^2\}} < 1 \right\} \cap (\mathbb{D} \setminus D(0, |z|)).
\end{align*}
\]

The quantities \(I_j(g), j = 1, \ldots, 5\), will be estimated separately.

**Case \(I_1(g)\)** By using the definition of \(\Omega_1(z)\), and the fact that \(\frac{\check{\nu}(x)}{(1 - x)^r}\) is essentially decreasing on \([0, 1)\) we deduce

\[
\Gamma_r(z, \zeta) \lesssim \left( \frac{|1 - \overline{x}\zeta|^2}{\max\{1 - |x|^2, 1 - |\zeta|^2\}} \right)^{\frac{1}{p} - \frac{1}{q}} \lesssim \left( \frac{|1 - \overline{z}\zeta|^2}{1 - |\zeta|^2} \right)^{\frac{1}{p} - \frac{1}{q}} \left( \frac{(1 - |z|)^{\frac{r}{q} - \delta}}{\check{\nu}(z)} \right)^{\frac{1}{q}}, \quad z \in \mathbb{D}, \ \zeta \in \Omega_1(z).
\]

Then the estimate

\[
M_1(r, f) \leq M_p(r, f) \lesssim \|f\|_{A^p_\omega} - \frac{1}{p}, \quad 0 \leq r < 1, \quad f \in \mathcal{H}(\mathbb{D}), \quad (5.2)
\]

and Lemma 3(ii) yield

\[
\begin{align*}
I_1(g) &\lesssim \int_\mathbb{D} \left( \int_{\Omega_1(z)} |g(\zeta)| \frac{(1 - |\zeta|)^{s - \delta} - 1}{|1 - \overline{z}\zeta|^2} \frac{1}{p} + \frac{1}{q} dA(\zeta) \right)^q \frac{\nu(z)(1 - |z|)^{r - \delta q}}{\check{\nu}(z)} dA(z) \\
&\lesssim \|g\|_{A^p_\omega}^q \left( \int_0^1 (1 - t)^{\frac{s - 1}{p}} dA(t) \right)^q \lesssim \|g\|_{A^p_\omega}^q \left( \int_0^1 (1 - t)^{\frac{s - 1}{p} - \frac{\beta(\omega)}{p}} dA(t) \right)^q \lesssim \|g\|_{A^p_\omega}^q, \quad g \in H^\infty.
\end{align*}
\]

**Case \(I_2(g)\)** The definition of \(\Omega_2(z)\) and the fact that \(\frac{\check{\nu}(x)}{(1 - x)^r}\) is essentially decreasing imply

\[
\Gamma_r(z, \zeta) \lesssim \left( \frac{|1 - \overline{x}\zeta|^2}{\max\{1 - |x|^2, 1 - |\zeta|^2\}} \right)^{\frac{1}{p} - \frac{1}{q}} \lesssim \left( \frac{|1 - \overline{z}\zeta|^2}{1 - |\zeta|^2} \right)^{\frac{1}{p} - \frac{1}{q}} \left( \frac{(1 - |z|)^{r}}{\check{\nu}(z)} \right)^{\frac{1}{q}}, \quad z \in \mathbb{D}, \ \zeta \in \Omega_2(z).
\]
Therefore (5.2) and Lemmas A and B yield

\[
I_2(g) \lesssim \int_D \left( \int_{\Omega_2(z)} |g(\xi)| \frac{(1 - |\xi|)^{s-\delta-\frac{1}{p} + \frac{1+\xi}{q}}}{\widehat{v}(\xi)^{\frac{1}{q}}} dA(\xi) \right)^q (1 - |z|)^{-\delta q} v(z) dA(z)
\]

\[
\lesssim \left( \int_D |g(\xi)| \frac{(1 - |\xi|)^{\frac{s-\delta-\frac{1}{p} + \frac{1+\xi}{q}}}{\widehat{v}(\xi)^{\frac{1}{q}}} dA(\xi) \right)^q \int_0^1 (1 - r)^{-\delta q} v(r) dr
\]

\[
\lesssim \|g\|_{A^p_q}^q \left( \int_0^1 (1 - r)^{\frac{s-\delta-\frac{1}{p} + \frac{1+\xi}{q}}{\min \left\{ \frac{\widehat{v}(t)}{(1 - |\xi|)^t}, \frac{\widehat{\nu}(t)}{(1 - |\xi|)^t} \right\}} dA(\xi) \right)^q \lesssim \|g\|_{A^p_q}^q, \quad g \in H^\infty.
\]

**Case I_3(g)** To deal with \(I_3(g)\), note first that now \(2K(1 - \bar{z}^2) \leq (1 - |\xi|) \max\{1 - |z|^2, 1 - |\xi|^2\} \leq 2(\max\{|1 - |z|, 1 - |\xi|\})^2\) for all \(\xi \in \Omega_3(z, K)\). Hence \(\xi \in \Delta(z, R)\) for some \(R = R(K) \in (0, \infty)\) if \(K \geq 1\) is sufficiently large. Fix such a \(K\), and note that then \(\widehat{v}(\xi) \asymp \widehat{v}(z)\) for all \(\xi \in \Omega(z, K)\) by Lemma A(ii). By using this and the fact that \(\frac{\widehat{v}(x)}{(1-x)^{\frac{1}{\tau}}}\) is essentially decreasing on \([0, 1]\), we deduce

\[
\Gamma_{\tau}(z, \xi) \lesssim \left( \frac{\max\{1 - |z|^2, 1 - |\xi|^2\}}{(1 - |\xi|)^{\frac{1}{\tau}} \min \left\{ \frac{\widehat{v}(z)}{(1 - |\xi|)^t}, \frac{\widehat{\nu}(\xi)}{(1 - |\xi|)^t} \right\}} \right)^{-\frac{1}{\tau}}
\]

\[
\lesssim \left( \frac{1 - |\xi|}{\widehat{v}(\xi)^{\frac{1}{\tau}}} \right)^{-\frac{1}{\tau}}, \quad z \in D, \quad \xi \in \Omega_3(z, K),
\]

and it follows that

\[
I_3(g) \lesssim \int_D \left( \int_{\Delta(z, R)} \left( |g(\xi)| \frac{1 - |\xi|^2}{\widehat{v}(\xi)^{\frac{1}{\tau}}} \right)^{\frac{1+\xi}{q}} dA(\xi) \right)^q \frac{v(z)(1 - |z|)^{\frac{1}{\tau} - q\delta}}{\widehat{v}(z)} dA(z)
\]

\[
\lesssim \int_D \left( \int_{\Delta(z, R)} \left( |g(\xi)| \frac{1 - |\xi|^2}{\widehat{v}(\xi)^{\frac{1}{\tau}}} \right)^{\frac{1+\xi}{q}} dA(\xi) \right)^q \frac{v(z)(1 - |z|)^{\frac{1}{\tau} - \frac{\rho(\tau)}{q} - \frac{\rho(\tau) - \delta}{q}}}{\widehat{\nu}(z)} dA(z)
\]

\[
\leq \int_D \left( \int_{\Delta(z, R)} \left( |g(\xi)| \frac{1 - |\xi|^2}{\widehat{v}(\xi)^{\frac{1}{\tau}}} \right)^{\frac{1+\xi}{q}} dA(\xi) \right)^q \frac{v(z)(1 - |z|)}{\widehat{v}(z)} dA(z)
\]

\[
= \left\| S_{s, s+2} \left( 1 - \frac{1}{p} + \frac{1}{q} \right) \left( |g| \frac{1}{\widehat{v}(\xi)^{\frac{1}{q}}} \right) \right\|_{L^q}^q = \left\| S_{b,c} \left( |g| \frac{1}{\widehat{v}(\xi)^{\frac{1}{q}}} \right) \right\|_{L^q}^q, \quad g \in H^\infty,
\]

where \(\eta(z) = \frac{v(z)(1 - |z|)}{\widehat{v}(z)}\) for all \(z \in D\). To apply Lemma 12 with \(\sigma \equiv 1\), we must check that its hypotheses are satisfied. To do this, first observe that \(\eta \in D\) and \(\bar{\eta}(r) \asymp (1 - r)\) for all
0 \leq r < 1 \text{ by Lemma 4. Hence}
\int_0^1 \frac{(1-t)^{c-2}}{\eta(t)^{\frac{1}{q}}} dt \asymp \int_r^1 \frac{(1-t)^{s-\frac{2}{p}+\frac{1}{q}}}{\eta(t)^{\frac{1}{q}}} dt \asymp (1-r)^{1+s-\frac{2}{p}+\frac{1}{q}} \asymp \frac{(1-r)^{c-1}}{\eta(r)^{\frac{1}{q}}}, \quad 0 \leq r < 1,
and, by Lemma 3(iii),
\int_0^r \frac{\eta(t)}{(1-t)^{\frac{c}{p}-1}\eta(t)^{\frac{1}{p}}} dt \asymp \int_0^r \frac{\nu(t)}{\eta(t)(1-t)^{\frac{c}{p}}(s+2(1-\frac{1}{p}+\frac{1}{q}))} dt \asymp \frac{1}{(1-r)^{\frac{c}{p}-1}}, \quad 0 \leq r < 1,
so the hypotheses of Lemma 12 are satisfied. Moreover,
(1-r)^{2+b-c+\frac{1}{q}-\frac{1}{p}} \frac{\eta(r)^{\frac{1}{q}}}{\sigma(r)^{\frac{1}{p}}} \asymp 1, \quad 0 \leq r < 1,
and consequently (5.3) and Lemmas 12 and 5 yield \( I_3(g) \lesssim \|g\|_{A^q_0}^q \lesssim \|g\|_{A^q_0}^q \) for all \( g \in H^\infty \).

**Case I_4(g)** By using the definition of \( \Omega_4(z, K) \), Lemma A(ii) and the fact that \( \tilde{\sigma}(x) \frac{1}{(1-x)^r} \) is essentially decreasing on \([0, 1)\), we deduce
\begin{align*}
\Gamma_r(z, \zeta) &\lesssim \frac{\left|1-\zeta \right|^2}{\max\left|1-z|x|^2, 1-|\zeta|^2\right|} \frac{1}{\sigma} \left(1 - \frac{2|1-\zeta|^2}{\max\left|1-z|x|^2, 1-|\zeta|^2\right|}\right) \frac{1}{\eta(t)^{\frac{1}{q}} \min \left\{ \frac{\nu(t)}{1-|z|^2}, \frac{\hat{\nu}(t)}{1-|\zeta|^2} \right\}^{\frac{1}{q}}}, \quad 0 \leq r < 1.
\end{align*}
Therefore
\begin{align}
I_4(g) &\lesssim \left\| \int_{\mathbb{D}} \left( \frac{1}{|g(z)|} \tilde{\sigma}(\zeta) \frac{1}{\eta(t)^{\frac{1}{q}}} \right)^{\frac{1}{q}} \left|1-\zeta \right|^{s-\delta} \frac{2B(\omega)}{p} + \frac{1}{q} - \frac{r}{q} \frac{dA(z)}{\tilde{\nu}(z)} \right\|_{L^q_{b,c}}^{q}.
\end{align}
where \( b = s - \delta - \frac{2B(\omega)}{p} + \frac{1}{q} - \frac{r}{q} \), \( c = 2 + s - 2\delta - \frac{2B(\omega)}{p} - \frac{2}{p} + \frac{2}{q}\) and \( \eta(z) = \frac{\nu(z)}{\tilde{\nu}(z)} \) for all \( z \in \mathbb{D} \). We will appeal to Lemma 12 with \( \sigma \equiv 1 \). First observe that \( \eta \in \mathcal{D} \) and \( \tilde{\eta}(r) \asymp (1-r)^{r-\delta} \) for all \( 0 \leq r < 1 \) by Lemma 4. Hence
\begin{align*}
\int_0^1 \frac{(1-t)^{c-2}}{\eta(t)^{\frac{1}{q}}} dt &\asymp \int_r^1 \frac{(1-t)^{s-\frac{2}{p}+\frac{1}{q}}}{\eta(t)^{\frac{1}{q}}} dt \asymp (1-r)^{1+s-\frac{2}{p}+\frac{1}{q}} \asymp \frac{(1-r)^{c-1}}{\eta(r)^{\frac{1}{q}}}, \quad 0 \leq r < 1,
\end{align*}
and, by Lemma 3(iii),

\[
\int_0^r \frac{\eta(t)}{(1-t)^{\frac{c_4}{p}} - \frac{1}{\eta(t)}} dt \asymp \int_0^r \frac{v(t)}{\tilde{\nu}(t)(1-t)^{\frac{2}{p} + \frac{2}{\beta(\nu)} - 1 - \frac{\tau - \delta}{p}} dt \\
\leq \frac{1}{(1-r)^{\frac{2}{p} + \frac{2}{\beta(\omega)} - 1 - \frac{\tau - \delta}{p}}} \frac{\tilde{\eta}(r)^{\frac{1}{p}}}{\tilde{\sigma}(r)^{\frac{1}{p}}} \quad 0 \leq r < 1,
\]

so the hypotheses of Lemma 12 are satisfied. Moreover,

\[
(1 - r)^{2 + b - c + \frac{1}{q} - \frac{1}{p}} \frac{\tilde{\eta}(r)^{\frac{1}{p}}}{\tilde{\sigma}(r)^{\frac{1}{p}}} \asymp 1, \quad 0 \leq r < 1,
\]

and hence (5.4) and Lemmas 12 and 5 imply \( I_4(g) \lesssim \|g\|^q_{A^+_\Omega} \lesssim \|g\|^q_{A^+_\Omega} \) for all \( g \in H^\infty \).

Case \( I_5(g) \) By using the definition of \( \Omega_5(z, K) \), Lemma A(ii) and the fact that \( \tilde{\nu}(x) \) is essentially decreasing on \([0, 1]\), we deduce

\[
\Gamma_T(z, \xi) \lesssim \left( \frac{|1| - 2|z|}{\max\{1 - |z|^2, 1 - |\xi|^2\}} \right)^{\frac{1}{p} - \frac{1}{q}} \frac{1}{\tilde{\omega}(z)} \left( \frac{1}{1 - \max\{1 - |z|^2, 1 - |\xi|^2\}} \right)^{\frac{1}{p}}
\leq \left( \frac{1}{1 - |\xi|^2} \right)^{\frac{1}{p} - \frac{1}{q}} \frac{1}{\tilde{\omega}(z)} \left( \frac{1}{1 - |\xi|^2} \right)^{\frac{1}{p}}
\leq \left( \frac{1}{1 - |\xi|^2} \right)^{\frac{1}{p} - \frac{1}{q}} \frac{2\beta(\omega)}{\tilde{\nu}(z)^{\frac{1}{p}}}, \quad z \in \mathbb{D}, \quad \xi \in \Omega_5(z, K).
\]

Therefore Lemma A(ii) yields

\[
I_5(g) \lesssim \int_{\mathbb{D}} \int_{\Omega_5(z, K)} \left( |g(z)| |\tilde{\omega}(\xi)| \frac{1}{\tilde{\nu}(z)^{\frac{1}{p}}} \right)^q \left( \frac{1 - |z|^2}{1 - |\xi|^2} \right)^{s - \delta - \frac{2\beta(\omega)}{p} - \frac{1}{q} + \frac{\beta(\nu)}{q}} |dA(z)|
\lesssim \int_{\mathbb{D}} \int_{\Omega_5(z, K)} \left( |g(z)| |\tilde{\omega}(\xi)| \frac{1}{\tilde{\nu}(z)^{\frac{1}{p}}} \right)^q \left( \frac{1 - |z|^2}{1 - |\xi|^2} \right)^{s - \delta - \frac{2\beta(\omega)}{p} - \frac{1}{q} + \frac{\beta(\nu)}{q}} |dA(z)|
= \|S_{b,c} \left( \frac{1}{|z|^{\frac{1}{q}}} \right) g \|^q_{L^q_{\tilde{\eta}}}, \quad g \in H^\infty,
\]

where \( b = s - \delta - \frac{2\beta(\omega)}{p} + \frac{1}{q} - \frac{\beta(\nu)}{q}, c = 2 + s - 2\delta - \frac{2\beta(\omega)}{p} - \frac{2}{p} + \frac{2}{q} \) and \( \eta(z) = \frac{v(z)(1 - |z|)^{\beta(\nu) - 3q}}{v(z)^{\frac{1}{p}}} \) for all \( z \in \mathbb{D} \). Again we will appeal to Lemma 12 with \( \sigma \equiv 1 \). First observe that \( \eta \in \mathbb{D} \) and \( \tilde{\eta}(r) \leq (1 - r)^{\delta - \beta(\nu) - \delta_1} \) for all \( 0 \leq r < 1 \) by Lemma 4. Hence

\[
\int_r^1 \frac{(1 - t)^{c - 2}}{\tilde{\eta}(t)^{\frac{1}{q}}} dt \asymp \int_r^1 (1 - t)^{s - \delta + \frac{2\beta(\omega)}{p} - \frac{2}{p} + \frac{2}{q} - \frac{\beta(\nu)}{q}} dt \asymp (1 - r)^{1 + s - \delta + \frac{2\beta(\omega)}{p} - \frac{2}{p} + \frac{2}{q} - \frac{\beta(\nu)}{q}}
\leq \frac{(1 - r)^{c - 1}}{\tilde{\eta}(r)^{\frac{1}{q}}}, \quad 0 \leq r < 1,
\]
and, by Lemma 3(iii),
\[
\int_0^r \frac{\eta(t)}{(1-t)^{\frac{c_2}{p}-1}\eta(t)^{1/p'}} dt \geq \int_0^r \frac{v(t)}{v'(t)(1-t)^{\frac{c_2}{p}-1}\eta(t)^{1/p'}} dt
\]
\[
\lesssim \frac{1}{(1-r)^{\frac{c_2}{p}-1}} \frac{\hat{\eta}(r)^{\frac{1}{p}}}{\hat{\eta}(r)^{\frac{1}{p'}}} 0 \leq r < 1,
\]
so the hypotheses of Lemma 12 are satisfied. Moreover,
\[
(1-r)^{2-b-c+\frac{1}{q}-\frac{1}{p}} \frac{\hat{\eta}(r)^{\frac{1}{p}}}{\hat{\eta}(r)^{\frac{1}{p'}}} \chi, 0 \leq r < 1,
\]
and hence (5.5) together with Lemmas 5 and 12 imply \( I_5(g) \lesssim \|g\|_{A^p_{\eta}}^{q} \|g\|_{A^p_{\eta}}^{q} \) for all \( g \in H^\infty \). This finishes the proof of the proposition. \( \Box \)

In order to prove the necessity part of Theorem 1 some definitions are needed. For \( \eta > -1 \) and a radial weight \( \omega \), let \( b^\eta_{z,\omega} = B^\eta_z / \|B^\eta_z\|_{A^p_\omega} \) for \( z \in \mathbb{D} \), where \( B^\eta_z(\zeta) = (1-\bar{z}\zeta)^{(-2+\eta)} \).

For each \( f \in L^q_\nu \), define
\[
g^\eta_{z,\omega}(\zeta) = \frac{P_\nu(\overline{f} b^\eta_{z,\omega})(\zeta)}{b^\eta_{z,\omega}(\zeta)}, \quad \zeta \in \mathbb{D},
\]
and note that \( g^\eta_{z,\omega} \) is a well-defined analytic function in \( \mathbb{D} \) because the standard Bergman kernel \( b^\eta_{z,\omega} \) has no zeros. If \( \nu, \omega \) are weights, \( \eta > -1 \) and \( 0 < p, q < \infty \), let us consider the global mean oscillation
\[
\| f b^\eta_{z,\omega} - g^\eta_{z,\omega}(z)b^\eta_{z,\omega} \|_{L^q_\nu}^q, \quad z \in \mathbb{D}.
\]

**Proposition 15** Let \( 1 < p \leq q < \infty \), \( f \in L^q_\nu \), \( \omega \in \mathcal{D} \), \( \nu \in B_q \) a radial weight and \( \gamma(z) = \gamma_{\omega,\nu,p,q}(z) = \frac{\hat{\omega}(z)^{\frac{1}{q}}}{\hat{\omega}(z)^{\frac{1}{q}}} \). If \( H^\nu_{H^v}, H^v_{H^\omega} : A^p_\omega \to L^q_\nu \) are bounded, then there exists \( \eta_0 = \eta_0(\nu, \omega) > -1 \) such that
\[
\sup_{z \in \mathbb{D}} \| f b^\eta_{z,\omega} - g^\eta_{z,\omega}(z)b^\eta_{z,\omega} \|_{L^q_\nu}^q \leq \| H^\nu_f \|_{A^p_\omega \to L^q_\nu} + \| P_\eta \|_{L^q_\nu \to L^q_\nu} \left( \| H^v_{H^\omega} \|_{A^p_\omega \to L^q_\nu} + \| H^\nu_{H^v} \|_{A^p_\omega \to L^q_\nu} \right).
\]
for each \( \eta \geq \eta_0 \). Moreover, there exists \( r_0 = r_0(\nu) > 0 \) such that for each fixed \( r \geq r_0 \) and \( \eta \geq \eta_0 \),
\[
\sup_{z \in \mathbb{D}} \| f b^\eta_{z,\omega} - g^\eta_{z,\omega}(z)b^\eta_{z,\omega} \|_{L^q_\nu} \leq \sup_{z \in \mathbb{D}} \gamma(z) \left( \frac{1}{v(\Delta(z,r))} \int_{\Delta(z,r)} |f(\zeta) - \overline{g_{z,\omega}(\zeta)} b^\eta_{z,\omega} dA(\zeta) \right)^{\frac{1}{q}}.
\]

**Proof** The definition of the Hankel operator along with triangle inequality gives
\[
\| f b^\eta_{z,\omega} - g^\eta_{z,\omega}(z)b^\eta_{z,\omega} \|_{L^q_\nu}^q \leq \| H^\nu_f(b^\eta_{z,\omega}) \|_{L^q_\nu}^q + \| P_\nu(f b^\eta_{z,\omega}) - g^\eta_{z,\omega}(z)b^\eta_{z,\omega} \|_{L^q_\nu}^q
\]
\[
\leq \| H^\nu_f \|_{A^p_\omega \to L^q_\nu} \| b^\eta_{z,\omega} \|_{A^p_\omega} + \| P_\nu(f b^\eta_{z,\omega}) - g^\eta_{z,\omega}(z)b^\eta_{z,\omega} \|_{L^q_\nu}^q
\]
\[
= \| H^\nu_f \|_{A^p_\omega \to L^q_\nu} + \| P_\nu(f b^\eta_{z,\omega}) - g^\eta_{z,\omega}(z)b^\eta_{z,\omega} \|_{L^q_\nu}^q.
\]
If \( g \in A^p_{\eta} \), then the reproducing formula for the standard weighted Bergman projection yields \( \overline{g(z)} b^\eta_{z,\omega} = P_\eta(\overline{g} b^\eta_{z,\omega}). \) Since \( \nu \in B_q \) is radial and \( f \in L^q_\nu \), we have \( \nu \in \mathcal{D} \) and \( P_\nu(f b^\eta_{z,\omega}) \in A^q_{\nu} \).
by Proposition 6. Therefore \( g^\eta_z \in A_1^\eta \) for all \( z \in \mathbb{D} \). Moreover, \( A_0^\eta \subset A_1^\eta \subset A_1^1 \) if \( \eta > \frac{B(v)}{q} - 1 \) by Lemma A(ii). It follows that

\[
\| P_v(f g^\eta_{z,\omega}) - g^\eta_{z,\omega}(z) b^\eta_{z,\omega} \|_{L^q_v} = \| P_v(f g^\eta_{z,\omega}) - P_\eta(g^\eta_{z,\omega}) b^\eta_{z,\omega} \|_{L^q_v} = \| P_\eta(P_v(f g^\eta_{z,\omega}) - g^\eta_{z,\omega} b^\eta_{z,\omega}) \|_{L^q_v}, \quad z \in \mathbb{D}.
\]

By [5], there exists \( \eta_1 = \eta_1(v) > \frac{B(v)}{q} - 1 \) such that \( P_\eta : L^q_v \to L^q_v \) is bounded if \( \eta \geq \eta_1 \). Therefore

\[
\| P_v(f g^\eta_{z,\omega}) - g^\eta_{z,\omega}(z) b^\eta_{z,\omega} \|_{L^q_v} \leq \| P_\eta \|_{L^q_v} \| P_v(f g^\eta_{z,\omega}) - g^\eta_{z,\omega} b^\eta_{z,\omega} \|_{L^q_v}, \quad z \in \mathbb{D}, \quad \eta \geq \eta_1.
\]

The triangle inequality yields

\[
\| P_v(f g^\eta_{z,\omega}) - g^\eta_{z,\omega}(z) b^\eta_{z,\omega} \|_{L^q_v} \leq \| f g^\eta_{z,\omega} - P_v(f g^\eta_{z,\omega}) \|_{L^q_v} + \| f g^\eta_{z,\omega} - g^\eta_{z,\omega} b^\eta_{z,\omega} \|_{L^q_v} = \| H^\eta_f (b^\eta_{z,\omega}) \|_{L^q_v} + \| \overline{f b^\eta_{z,\omega}} - g^\eta_{z,\omega} b^\eta_{z,\omega} \|_{L^q_v} \leq \| H^\eta_f \|_{A^\eta_0 \to L^q_v} \| b^\eta_{z,\omega} \|_{A^\eta_0} + \| \overline{f b^\eta_{z,\omega}} - P_v(f b^\eta_{z,\omega}) \|_{L^q_v} = \| H^\eta_f \|_{A^\eta_0 \to L^q_v} \| b^\eta_{z,\omega} \|_{A^\eta_0} + \| H^\eta_f \|_{A^\eta_0 \to L^q_v} \leq \| H^\eta_f \|_{A^\eta_0 \to L^q_v} + \| H^\eta_f \|_{A^\eta_0 \to L^q_v}.
\]

By combining the above estimates we deduce

\[
\| f b^\eta_{z,\omega} - g^\eta_{z,\omega}(z) b^\eta_{z,\omega} \|_{L^q_v} \leq \| H^\eta_f \|_{A^\eta_0 \to L^q_v} \left( \| H^\eta_f \|_{A^\eta_0 \to L^q_v} + \| H^\eta_f \|_{A^\eta_0 \to L^q_v} \right),
\]

for any \( \eta \geq \eta_1(v) \).

To see the second one, first observe that [16, Corollary 2] and Lemma A(ii) give

\[
\| b^\eta \|_{A^p_0}^p \leq \int_0^{|z|} \frac{|\omega(t)|}{(1 - t)^{p(2 + \eta)}} dt \leq \int_0^{|z|} \frac{1}{(1 - t)^{p(2 + \eta)} - \beta(\omega)} dt \leq \frac{|\omega(z)|}{(1 - |z|)^{p(2 + \eta) - 1}}, \quad |z| \to 1^{+},
\]

provided \( \eta > \frac{\beta(\omega) + 1}{p} - 2 \). Moreover, by (3.6) there exists \( r_0 = r_0(v) > 0 \) such that \( (1 - |z|) \tilde{\omega}(z) > \nu(\Delta(z, r_0)) \) for any \( r \geq r_0 \). Hence, for each \( r \geq r_0 \) we have

\[
\| f b^\eta_{z,\omega} - g^\eta_{z,\omega}(z) b^\eta_{z,\omega} \|_{L^q_v} \leq \int_{\Delta(z, r)} \frac{1}{|f(\xi) - g^\eta_{z,\omega}(\xi)|^q |b^\eta_{z,\omega}(\xi)|^q \nu(\xi) dA(\xi)} \leq \frac{1}{|\omega(z)|^q (1 - |z|)^q} \int_{\Delta(z, r)} \frac{|f(\xi) - g^\eta_{z,\omega}(\xi)|^q \nu(\xi) dA(\xi)} = \frac{1}{|\omega(z)|^q (1 - |z|)^q} \int_{\Delta(z, r)} \frac{|f(\xi) - g^\eta_{z,\omega}(\xi)|^q \nu(\xi) dA(\xi)} \leq \frac{1}{|\omega(z)|^q (1 - |z|)^q} \int_{\Delta(z, r)} \frac{|f(\xi) - g^\eta_{z,\omega}(\xi)|^q \nu(\xi) dA(\xi)} = \frac{1}{|\omega(z)|^q (1 - |z|)^q} \int_{\Delta(z, r)} \frac{|f(\xi) - g^\eta_{z,\omega}(\xi)|^q \nu(\xi) dA(\xi)}.
\]

The second claim for \( \eta_0 = \max|\eta_1, \frac{\beta(\omega) + 1}{p} - 2 \) is now proved. \( \square \)

**Proof of Theorem 1** If \( H^\nu_f, H^\nu_T : A^p_0 \to L^q_v \) are bounded, then \( f \in \text{BMO}(\Delta) \) by Proposition 15 and Theorem 11.

Conversely, let \( f \in \text{BMO}(\Delta) \). Then \( f \) can be decomposed as \( f = f_1 + f_2 \), where \( f_1 \in \text{BA}(\Delta) \) and \( f_2 \in \text{BO}(\Delta) \), by Theorem 11(ii). Proposition 14 shows that \( H^\nu_{f_2}, H^\nu_{f_2} :
$A^p_\omega \to L^q_v$ are bounded. Moreover, since $v \in B_q$ is radial, $v \in \mathcal{D}$ and $P_v : L^q_v \to L^q_v$ is bounded by Proposition 6. Therefore Lemma 10 yields

$$\|H^v_f(g)\|^q_{L^q_v} \leq \|f_1 g\|_{L^q} + \|P_v(f_1 g)\|_{L^q_v} \lesssim \|f_1 g\|_{L^q_v} \lesssim \|g\|_{A^p_\omega}, \quad g \in H^\infty.$$  

It follows that $H^v_f, H^v_{-f} : A^p_\omega \to L^q_v$ are bounded. \hfill \QED
Fix $\sigma > \max \left\{ 0, 1 - \frac{q}{p} (1 + \alpha(\omega)) + q \beta(v) \right\}$ and $c$ satisfying (6.2). Then

$$c > \max \left\{ \beta(v) - 1, -2 + \beta(v) + \frac{q}{p} (1 + \beta(\omega)) - q \alpha(v) \right\}.$$ 

Therefore, [11, Lemma 7] together with Lemmas A(ii) and B gives

$$I_1(z) \lesssim (1 - |z|)^{c+1} + f(z)^q \int_\mathbb{D} |f'(\xi)|^q \frac{(1 - |\xi|^2)^{\sigma+q-1}}{|1 - \xi|^2 + c + \sigma} \ dA(\xi)$$

$$\lesssim \|f\|_{\mathcal{B}_d^q}(1 - |z|)^{c+1} + f(z)^q \int_\mathbb{D} \gamma(\xi)^{-q} \frac{(1 - |\xi|^2)^{\sigma-1}}{|1 - \xi|^2 + c + \sigma} \ dA(\xi)$$

$$\lesssim \|f\|_{\mathcal{B}_d^q}(1 - |z|)^{c+1} + f(z)^q \int_0^1 \frac{\widehat{\omega}(s)^{\frac{q}{p}} (1 - s)^{\frac{\beta(v)}{2} + \frac{\sigma - 2}{2} + \frac{\sigma + 2}{p} \beta(\omega) - \beta(v)}}{(1 - s)^{\frac{3 - c}{p} - \frac{q}{p} \beta(\omega) + \alpha(v) - \beta(v)}} \ ds$$

and

$$I_2(z) \lesssim (1 - |z|)^{c+1} - \beta(\omega) + f(z)^q \int_\mathbb{D} |f'(\xi)|^q \frac{(1 - |\xi|^2)^{\beta(v)+\sigma-1}}{|1 - \xi|^2 + c} \ dA(\xi)$$

$$\lesssim \|f\|_{\mathcal{B}_d^q}(1 - |z|)^{c+1} - \beta(\omega) + f(z)^q \int_\mathbb{D} \gamma(\xi)^{-q} \frac{(1 - |\xi|^2)^{\beta(v)+\sigma-1}}{|1 - \xi|^2 + c + \sigma} \ dA(\xi)$$

$$\lesssim \|f\|_{\mathcal{B}_d^q}(1 - |z|)^{c+1} - \beta(\omega) + f(z)^q \int_0^1 \frac{\widehat{\omega}(s)^{\frac{q}{p}} (1 - s)^{\frac{\beta(v)+\sigma}{2} + \frac{\sigma + 2}{p} \beta(\omega) - \beta(v)}}{(1 - s)^{\frac{3 - c}{p} - \frac{q}{p} \beta(\omega) + \alpha(v) - \beta(v)}} \ ds$$

By combining these estimates we deduce $f \in \text{BMO}(\Delta)$, and thus $\mathcal{B}_d^q \subset \mathcal{H}(\mathbb{D}) \cap \text{BMO}(\Delta)$.

Assume now that $f \in \mathcal{H}(\mathbb{D}) \cap \text{BMO}(\Delta)$. Then (6.3) holds for some $\sigma > 1$ and $c$ satisfying (6.2). Therefore (3.6) implies

$$\infty > \sup \frac{(1 - |z|)^{c+1} + f(z)^q}{\widehat{\nu}(z)} \int_\mathbb{D} |f(\xi) - f(z)|^q \frac{(1 - |\xi|^2)^{\sigma-1}}{|1 - \xi|^2 + c + \sigma} \widehat{\nu}(\xi) dA(\xi)$$

$$\geq \frac{\gamma(z)^q}{(1 - |z|^2)^{2} \widehat{\nu}(z)} \int_{\Delta(z,r)} |f(\xi) - f(z)|^q \widehat{\nu}(\xi) dA(\xi)$$

$$\geq \frac{\gamma(z)^q}{|\Delta(z,r)| \Delta(z,r)} \int_{\Delta(z,r)} |f(\xi) - f(z)|^q dA(\xi), \quad z \in \mathbb{D}.$$
By arguing as in [12, 1653–1654] we deduce $\mathcal{H}(\mathbb{D}) \cap \text{BMO}(\Delta) \subset B_{dy}$. \hfill $\Box$

The space $B_{dy}$ consists of constant functions only if $\limsup_{|z| \to 1^{-}}((1-|z|)\gamma(|z|))^{-1} = 0$. Moreover, $B_{dy}$ is a subset of the disc algebra if $((1-x)\gamma(x))^{-1} \in L^1(0, 1)$, and $B_{dy}$ coincides with a Bloch-type space if $\gamma$ is decreasing.

**Proof of Theorem 2** Since $f \in A^q$, the operator $H_{f}^{q}$ is densely defined. If $H_{f}^{q} : A^p_{q(v)} \to L^q_{v}$ is bounded, choosing $g \equiv 1$ it follows that $f \in A^q_{q(v)}$, and therefore $f \in B_{dy}$ by Theorem 1 and Proposition 16.

Conversely, assume $f \in B_{dy}$. Since $v \in B_q$ is radial, Proposition 6 implies $v \in \mathcal{D}$. Therefore Lemmas A(ii) and B yield

$$
\|f\|_{A^q} \lesssim \int_{\mathbb{D}} \left( \int_{0}^{1} \left| \frac{f'}{s} \right|^q \left( \frac{z}{|z|} \right)^q ds \right)^{q} v(z) dA(z) + |f(0)|^q 
$$

for all $f \in \mathcal{H}(\mathbb{D})$. If $\frac{1+\beta(v)}{q} - \frac{1+\alpha(v)}{p} > 0$, Lemma 3(ii) gives

$$
\|f\|_{A^q} \lesssim \|f\|_{B_{dy}} \left( 1 + \int_{0}^{1} \frac{\hat{\omega}(t)^{q} (1-t)^{\frac{q}{p}-1}}{v(t)} v(t) dt \right) 
$$

If $\frac{1+\beta(v)}{q} - \frac{1+\alpha(v)}{p} = 0$, then Lemmas B and 3(ii) yield

$$
\|f\|_{A^q} \lesssim \|f\|_{B_{dy}} \left( 1 + \int_{0}^{1} \frac{\hat{\omega}(t)^{q} (1-t)^{\frac{q}{p}-1}}{v(t)} \log \frac{e}{1-t} v(t) dt \right) 
$$

Finally, if $\frac{1+\beta(v)}{q} - \frac{1+\alpha(v)}{p} < 0$, then Lemma 3(ii) gives

$$
\|f\|_{A^q} \lesssim \|f\|_{B_{dy}} \left( 1 + \int_{0}^{1} \frac{\hat{\omega}(t)^{q} (1-t)^{\frac{q}{p}-1}}{v(t)} v(t) dt \right) 
$$

Therefore $f \in A^q_{q(v)}$, and thus $B_{dy} \subset A^q_{q(v)}$. This together with Theorem 1 and Proposition 16 finishes the proof. \hfill $\Box$
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