Some applications of uncertainty relations in quantum information

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We discuss some applications of various versions of uncertainty relations for both discrete and continuous variables in the context of quantum information theory. The Heisenberg uncertainty relation enables demonstration of the EPR paradox. Entropic uncertainty relations are used to reveal quantum steering for non-Gaussian continuous variable states. Entropic uncertainty relations for discrete variables are studied in the context of quantum memory where fine-graining yields the optimum lower bound of uncertainty. The fine-grained uncertainty relation is used to obtain connections between uncertainty and the nonlocality of retrieval games for bipartite and tripartite systems. The Robertson-Schrödinger uncertainty relation is applied for distinguishing pure and mixed states of discrete variables.

Keywords: uncertainty; purity; entanglement; nonlocality.

1. Introduction

The uncertainty principle is a central feature of quantum mechanics, prohibiting certain properties of quantum systems from being simultaneously well-defined. The Heisenberg uncertainty relation\(^1\) lower bounds the product of uncertainties, i.e., the spread measured by standard deviation, of measurement outcomes for two non-commuting observables\(^2\). An improved form of the uncertainty relation was proposed by Robertson\(^3\) and Schrödinger\(^4\), incorporating both commutators and anti-commutators of more general observables. Motivated by various physical considerations, several other versions of the uncertainty principle have since been suggested. Notable among them are reformulations that take into account the inevitable noise and disturbance associated with measurements\(^5\).

Efforts for eliminating the state-dependence of the lower bound of uncertainty have lead to the formulation of various entropic versions of the uncertainty
Entropic uncertainty relations have been tightened due to different effects, such as the presence of correlations\textsuperscript{10,11,12,13,14}. A fine-grained version of the uncertainty relation arises as a result of distinguishing the uncertainty of obtaining specific combinations of outcomes for different measurements\textsuperscript{15}. An optimal lower bound of entropic uncertainty in the presence of any type of correlations may be determined by fine-graining\textsuperscript{16}. For a recent review of uncertainty relations, see Ref.\textsuperscript{17}.

The subject of quantum information science that has seen rapid progress in recent years, was inspired originally to a great extent by the pioneering work of Einstein, Podolsky and Rosen (EPR)\textsuperscript{18}. The word ‘entanglement was first coined by Schrodinger to describe the property of spatially separated but correlated particles whose paradoxical features were highlighted by EPR. The first testable formulation of the EPR paradox was proposed\textsuperscript{19} using the position-momentum uncertainty relation, in terms of an inequality involving products of inferred variances of incompatible observables. This lead to the experimental realization\textsuperscript{20} of the EPR paradox for the case of two spatially separated and correlated light modes. A modern formulation of the EPR-Schrodinger concept of quantum steering based on violations of steering inequalities\textsuperscript{21}, akin to the Bell-type local-realist inequalities\textsuperscript{22,23}, is derived again using uncertainty relations in their entropic version. Entropic steering relations are indispensable for demonstrating steering in certain continuous variable systems where correlations are not manifest up to second order (variances of observables), as shown recently for several non-Gaussian states\textsuperscript{24}.

Several other important applications of uncertainty relations in the realm of quantum information processing have been uncovered in recent years. The uncertainty principle has been used for discrimination between separable and entangled quantum states in the realm of continuous variable systems\textsuperscript{25}. The utility of the Robertson-Schrodinger uncertainty relation\textsuperscript{3,4} has also been exploited in this context\textsuperscript{26,27}. Moreover, the Robertson-Schrodinger uncertainty relation\textsuperscript{3,4} has recently been employed in the domain of discrete variables to distinguish between pure and mixed states of single as well as bipartite qubit and qutrit systems\textsuperscript{28}. The fine-grained uncertainty relation can be used to determine the nonlocality of the underlying physical system\textsuperscript{15}, as has been demonstrated for the case of bipartite\textsuperscript{15} and tripartite\textsuperscript{29} systems, as well as in the arena of biased nonlocal games\textsuperscript{30}. The uncertainty principle plays a crucial role in the domain of quantum cryptography since security of quantum key distribution protocols relies basically on quantum uncertainty\textsuperscript{31}. Specifically, the amount of key extractable per state has been linked to the lower limit of entropic uncertainty\textsuperscript{10,32}.

Uncertainty relation in their different versions have many important applications in quantum information theory. In the present article, we review some aspects of a few of these applications, limited mainly by the areas in which the present authors have worked upon. The plan of this article is as follows. In the next Section we discuss the Robertson-Schrodinger uncertainty relation and briefly sketch how it
could be used for distinguishing pure states from mixed states of discrete variables. In Section III we focus on the topic of quantum steering where steering using the Heisenberg uncertainty relation as well as entropic steering relations are discussed in the context of continuous variables. The connection between uncertainty and nonlocality of quantum games is presented in Section IV as an application of the fine-grained steering relation. Section V contains a brief review of entropic uncertainty relations in the presence of quantum memory. Certain concluding remarks are made in Section VI.

2. Determining purity of states using the Robertson-Schrodinger uncertainty relation

In experimental protocols for information processing, the interaction with the environment inevitably affects the purity of a quantum system. A relevant issue for an experimenter is to ascertain whether a prepared pure state has remained isolated from environmental interaction. It becomes important to test whether a given quantum state is pure, in order to use it effectively as a resource for quantum information processing. The purity of a given state is also related to the entanglement of a larger multipartite system of which it may be a part. The mixedness of states can be quantified by their linear entropy, which is a nonlinear functional of the quantum state. The linear entropy can be extracted from the given state by tomography which usually is expensive in terms of resources and measurements involved.

In this section we discuss how the Robertson-Schrodinger (RS) uncertainty relation may be used to determine the mixedness of quantum states of discrete variables. For the case of continuous variable systems there exist certain pure states for which the uncertainty as quantified by the Robertson-Schrodinger uncertainty relation is minimized. The connection of purity with observable quantities of the relevant states have been found. It has been shown recently that the RS uncertainty relation can be used to distinguish between pure and mixed states of finite dimensional systems.

The RS uncertainty relation could be used as a witness of mixedness in the following way. For any pair of observables $A, B$ and for any quantum state represented by the density operator $\rho$, the RS uncertainty relation can be written as:

$$Q(A, B, \rho) \geq 0$$

where

$$Q(A, B, \rho) = (\Delta A)^2 (\Delta B)^2 - \frac{|\langle [A, B]\rangle|^2}{2}$$

$$-|\langle \{A, B\}\rangle - \langle A\rangle \langle B\rangle|^2$$

with $(\Delta A)^2$ and $(\Delta B)^2$ representing the variances of the observables, $A$ and $B$, respectively, given by $(\Delta A)^2 = (\langle A^2 \rangle) - (\langle A \rangle)^2$, $(\Delta B)^2 = (\langle B^2 \rangle) - (\langle B \rangle)^2$, and the
square (curly) brackets representing the standard commutators (anti-commutators) of the corresponding operators. The quantity $Q(A, B, \rho)$ involves the measurable quantities, i.e., the expectation values and variances of the relevant observables in the state $\rho$. States of a $d$-level quantum system are in one to one correspondence with Hermitian, positive semi-definite, unit trace operators acting on a $d$-dimensional Hilbert space. The defining properties of these density operators $\rho$ are (i) $\rho^\dagger = \rho$, (ii) $\rho \geq 0$, (iii) $\text{tr}[\rho] = 1$. Pure states correspond to the further condition $\rho^2 = \rho$ which is equivalent to the scalar condition $\text{tr}[\rho^2] = 1$. Hence, complement of the trace condition can be taken as a measure of mixedness given by the linear entropy defined for a $d$-level system as

$$S_l(\rho) = \left( \frac{d}{d-1} \right) (1 - \text{tr}(\rho^2))$$

We now describe how the quantity $Q(A, B, \rho)$ can act as an experimentally realizable measure of mixedness of a system.

Let us here discuss the case of two-level systems. The density operator for qubit systems can be expressed in terms of the Pauli matrices. The state of a single qubit can be written as $\rho(\vec{n}) = (I + \vec{n} \cdot \vec{\sigma})/2$, $\vec{n} \in \mathbb{R}^3$. Positivity of this Hermitian unit trace matrix demands $|\vec{n}|^2 \leq 1$. It follows that single qubit states are in one to one correspondence with the points on or inside the closed unit ball centred at the origin of $\mathbb{R}^3$. Points on the boundary correspond to pure states. For a pair of suitably chosen spin observables, the RS relation is satisfied as an equality for the states extremal, i.e., the pure states, and as an inequality for points other than extremals, i.e., for the mixed states. The linear entropy of the state $\rho$ can be written as $S_l(\rho) = (1 - \vec{n}^2)$. If we choose spin observables along two different directions, i.e., $A = \hat{r} \cdot \vec{\sigma}$ and $B = \hat{t} \cdot \vec{\sigma}$, then $Q$ becomes

$$Q(A, B, \rho) = (1 - (\Sigma r_i t_i)^2) S_l(\rho)$$

It thus follows that for $\hat{r}, \hat{t} = 0$, $Q$ coincides with the linear entropy. For orthogonal spin measurements, the uncertainty quantified by the RS relation, $Q$ and the linear entropy $S_l$ are exactly same for single qubit systems. Thus, it turns out that $Q = 0$ is both a necessary and sufficient condition for any single qubit system to be pure when the pair of observables are qubit spins along two different directions.

For two-qubit systems the states considered may be taken to be polarized along a specific known direction, say, the $z$-axis forming the Schmidt decomposition basis. In order to enable $Q(A, B, \rho)$ to be a mixedness measure, $A$ and $B$ are chosen for the two-qubit case to be of the form $A = (\hat{m} \cdot \vec{\sigma}) \otimes (\hat{n} \cdot \vec{\sigma})$, and $B = (\hat{p} \cdot \vec{\sigma}) \otimes (\hat{q} \cdot \vec{\sigma})$, respectively, where $\hat{m}, \hat{n}, \hat{p}, \hat{q}$ are unit vectors. For enabling $Z(A, B, \rho)$ to be used for determining the purity of the given two qubit state, the appropriate choice of observables $A$ and $B$ is found to be that of lying on the two dimensional $x - y$ plane (i.e., $\hat{m}, \hat{n}, \hat{p}, \hat{q}$ are all taken to be on the $x - y$ plane), normal to the $z$-axis pertaining to the relevant Schmidt decomposition basis. Then, $Q(A, B, \rho) = 0$ necessarily holds good for pure two-qubit states whose individual spin orientations are all along...
a given direction (say, the \( z \)-axis) normal to which lies the plane on which the observables \( A \) and \( B \) are defined. On the other hand, \( Q(A,B,\rho) > 0 \) holds good for most settings of \( A \) and \( B \) for two qubit isotropic states, for the Werner class of states given by \( \rho_{w} = ((1 - p)/4)I + p\rho_{s} \) (\( \rho_{s} \) is the two-qubit singlet state), as well for other types of one parameter two-qubit states which comprise of pure states whose individual spin orientations are all along the same given direction normal to the plane on which the observables \( A \) and \( B \) are defined.

The RS uncertainty relation has been shown to determine the purity of qutrit systems, as well\(^{28}\). Three-level systems are of fundamental relevance in laser physics, and have generated much recent interest from the perspective of information processing\(^{36}\). It has been shown using examples of single and bipartite class of qutrit states that the RS uncertainty relation can be satisfied as an equality for pure states while it remains an inequality for mixed states by the choice of suitable observables. An observational scheme which can detect mixedness of qutrit systems unambiguously, requires less resources compared to tomography, and is implementable through the measurement of Hermitian witness-like operators\(^{28}\). It may be relevant to note here though that the set of pure states is not convex, and hence, such witness-like operators do not arise from any geometrical separability criterion inherent to the theory of entanglement witnesses\(^{37}\), that has been applied more recently to the cases of teleportation witnesses\(^{38}\), as well as for witnesses of absolutely separable states\(^{39}\).

The operational determination of purity using the RS relation requires a few additional steps. A scheme for using the uncertainty relation to determine whether a given state is pure or mixed, provided the prior knowledge of the basis is available, has been outlined in Ref.\(^{28}\). The limitation of instrumental precision could make the observed value of \( Q \) for pure states to be a small number in stead of exactly zero. In order to take into account the experimental inaccuracy, a parameter \( \varepsilon \) is introduced in the analysis. For a single-qubit system, by choosing the measurement settings for \( A \) and \( B \) as qubit spins along \( z \) and \( x \) directions, respectively, the measured value of the uncertainty obtained as \( Q \geq \varepsilon \) leads to the conclusion that the given state is mixed. This prescription of determining mixedness holds for all single-qubit states \( \rho(\vec{n}) = \frac{(I + \vec{n} \cdot \vec{\sigma})}{2} \), except those lying in the narrow range \( 1 \geq n \geq \sqrt{1 - 2\varepsilon/3} \), as determined by putting \( Q < \varepsilon \).

To summarize, the RS uncertainty relation is able to distinguish between pure and mixed states for a broad category of two- and three-level systems. For single party systems, the scheme works for all qubits and up to three-parameter family of qutrit states\(^{40}\). For bipartite systems, the scheme has been shown to work for the mixture of two arbitrary pure states, the isotropic class, and the Werner class of states, as well. The determination of mixedness using GUR may require in certain cases a considerably lesser number of measurements compared to tomography. In the case of single qutrit states, full tomography involves the estimation of eight parameters, while through the prescription detailed in Ref.\(^{28}\) sometimes four mea-
measurements may suffice for detecting purity of a single qutrit state. A maximum of eight measurements suffices to distinguish between pure and mixed states of single qutrit up to three-parameter families. The difference in the number of required measurements is substantially enhanced for composite states. For two qubits, the RS relation requires up to five measurements compared to fifteen required by tomography. For the case of two-qutrits the measurement of at most eight expectation values suffices.

3. Quantum steering

The Einstein-Podolsky-Rosen (EPR) paradox\textsuperscript{18} has not only inspired a huge body of subsequent debate, but has played a pivotal role in the unfolding of several rich features of quantum mechanics relevant for information processing. Considering a position-momentum correlated state of two particles, and assuming the notions of spatial separability, locality, and reality to hold true at the level of quantum particles, EPR argued that that the quantum mechanical description of the state of a particle is not complete. The EPR paradox arises from the correlations between two non-commuting observables of a sub-system with those of the other sub-system, for instance, the correlations between the measurement outcomes of positions and momenta for two separated particles, i.e., $<x, p_y> \neq 0$, with $<x> = 0 = <p_y>$ individually. Due to the presence of correlations, the measurement of the position of, say, the first particle leads one to infer the correlated value of the position for the second particle (say, $x_{inf}$). Now, if the momentum of the second particle is measured giving the outcome, say $p$, the value of the product of uncertainties $(\Delta x_{inf})^2(\Delta p_{inf})^2$ may turn out to be lesser than that allowed by the uncertainty principle, viz. $(\Delta x)^2(\Delta p)^2 \geq 1$, thus leading to the paradox.

Following the work of EPR, Schrodinger\textsuperscript{41} observed that correlations between spatially separated particles entailed the possibility of steering of the state on one side merely by the choice of the measurement basis on the other side, without in any way having direct access to the affected particle. The word 'entanglement' was first coined by Schrodinger to describe the property of such spatially separated but correlated particles. Consider a bipartite entangled state which may be expressed in two different ways, as

$$|\Psi\rangle = \sum_{n=1}^{\infty} c_n |\psi_n\rangle |u_n\rangle = \sum_{n=1}^{\infty} d_n |\phi_n\rangle |v_n\rangle$$

where $\{|u_n\rangle\}$ and $\{|v_n\rangle\}$ are two orthonormal bases for one of the parties (say, Alice). If Alice chose to measure in the $\{|u_n\rangle\}$ ($\{|v_n\rangle\}$) basis, she projects Bob’s system into one of the states $|\psi_n\rangle$ ($|\phi_n\rangle$). Note that though there is no physical interaction between Alice and Bob, the ensemble of $|\psi_n\rangle$s is in general different from the ensemble of $|\phi_n\rangle$s. This ability of Alice to affect Bob’s state due to her choice of the measurement basis was dubbed as “steering” by Schrodinger\textsuperscript{41}.

A testable formulation of the EPR paradox was proposed many years later by
Reid\textsuperscript{19} for continuous variable systems using the position-momentum uncertainty relation. An inequality involving products of inferred variances of incompatible observables was derived in the context of continuous variables, as follows. Consider the quadrature phase components of two correlated and spatially separated light fields. The quadrature amplitudes associated with the fields $E_{\gamma} = C[\hat{\gamma}e^{-i\omega_\gamma t} + \hat{\gamma}^\dagger e^{i\omega_\gamma t}]$ (where, $\gamma \in \{a, b\}$, are the bosonic operators for two different modes, $\omega_\gamma$ is the frequency, and $C$ is a constant incorporating spatial factors taken to be equal for each mode) are given by

\begin{equation}
\hat{X}_\theta = \frac{\hat{a}e^{-i\theta} + \hat{a}^\dagger e^{i\theta}}{\sqrt{2}}, \quad \hat{Y}_\phi = \frac{\hat{b}e^{-i\phi} + \hat{b}^\dagger e^{i\phi}}{\sqrt{2}},
\end{equation}

where,

\begin{align*}
\hat{a} &= \frac{\hat{X} + i\hat{P}_x}{\sqrt{2}}, & \hat{a}^\dagger &= \frac{\hat{X} - i\hat{P}_x}{\sqrt{2}}, \\
\hat{b} &= \frac{\hat{Y} + i\hat{P}_y}{\sqrt{2}}, & \hat{b}^\dagger &= \frac{\hat{Y} - i\hat{P}_y}{\sqrt{2}},
\end{align*}

and the commutation relations of the bosonic operators are given by $[\hat{a}, \hat{a}^\dagger] = 1 = [\hat{b}, \hat{b}^\dagger]$. The correlations between the quadrature amplitudes $\hat{X}_\theta$ and $\hat{Y}_\phi$ are defined by the correlation coefficient, $C_{\theta, \phi}$ as\textsuperscript{19,20}

\begin{equation}
C_{\theta, \phi} = \frac{\langle \hat{X}_\theta \hat{Y}_\phi \rangle}{\sqrt{\langle \hat{X}_\theta^2 \rangle \langle \hat{Y}_\phi^2 \rangle}},
\end{equation}

where $\langle \hat{X}_\theta \rangle = 0 = \langle \hat{Y}_\phi \rangle$. The correlation is perfect for some values of $\theta$ and $\phi$, if $|C_{\theta, \phi}| = 1$, and vanishes for uncorrelated variables.

As a consequence of correlations in the measurement outcomes, the quadrature amplitude $\hat{X}_\theta$ can be inferred by measuring the corresponding amplitude $\hat{Y}_\phi$. In realistic situations the correlations are not perfect because of the interaction with the environment as well as finite detector efficiency. Hence, the estimated amplitudes $\hat{X}_{\theta 1}$ and $\hat{X}_{\theta 2}$ with the help of $\hat{Y}_{\phi 1}$ and $\hat{Y}_{\phi 2}$, respectively, are subject to inference errors, and given by\textsuperscript{19}

\begin{align*}
\hat{X}_{\theta 1}^e &= g_1 \hat{Y}_{\phi 1}, & \hat{X}_{\theta 2}^e &= g_2 \hat{Y}_{\phi 2},
\end{align*}

where $g_1$ and $g_2$ are scaling parameters. Now, one may choose $g_1$, $g_2$, $\phi_1$, and $\phi_2$ in such a way that $\hat{X}_{\theta 1}$ and $\hat{X}_{\theta 2}$ are inferred with the highest possible accuracy. The errors given by the deviation of the estimated amplitudes from the true amplitudes $\hat{X}_{\theta 1}$ and $\hat{X}_{\theta 2}$ are captured by $\langle (\hat{X}_{\theta 1} - \hat{X}_{\theta 1}^e)^2 \rangle$ and $\langle (\hat{X}_{\theta 2} - \hat{X}_{\theta 2}^e)^2 \rangle$, respectively. The average errors of the inferences are given by

\begin{align*}
(\Delta_{\text{inf}} \hat{X}_{\theta 1})^2 &= \langle (\hat{X}_{\theta 1} - \hat{X}_{\theta 1}^e)^2 \rangle = \langle (\hat{X}_{\theta 1} - g_1 \hat{Y}_{\phi 1})^2 \rangle, \\
(\Delta_{\text{inf}} \hat{X}_{\theta 2})^2 &= \langle (\hat{X}_{\theta 2} - \hat{X}_{\theta 2}^e)^2 \rangle = \langle (\hat{X}_{\theta 2} - g_2 \hat{Y}_{\phi 2})^2 \rangle.
\end{align*}
Authors’ Names

The values of the scaling parameters $g_1$ and $g_2$ are chosen such that \( \frac{\partial (\Delta_{\text{inf}} \hat{X}_{\theta_1})^2}{\partial g_1} = 0 = \frac{\partial (\Delta_{\text{inf}} \hat{X}_{\theta_2})^2}{\partial g_2} \), from which it follows that

\[
g_1 = \frac{\langle \hat{X}_{\theta_1} \hat{Y}_{\phi_1} \rangle}{\langle \hat{Y}_{\phi_1}^2 \rangle}, \quad g_2 = \frac{\langle \hat{X}_{\theta_2} \hat{Y}_{\phi_2} \rangle}{\langle \hat{Y}_{\phi_2}^2 \rangle}. \tag{11}
\]

The values of $\phi_1$ ($\phi_2$) are obtained by maximizing $C_{\theta_1,\phi_1}$ ($C_{\theta_2,\phi_2}$). Now, due to the commutation relations $[\hat{X}, \hat{P}_X] = i; \quad [\hat{Y}, \hat{P}_Y] = i$, it is required that the product of the variances of the above inferences $(\Delta_{\text{inf}} \hat{X}_{\theta_1})^2(\Delta_{\text{inf}} \hat{X}_{\theta_2})^2 \geq 1/4$. Hence, the EPR paradox occurs if the correlations in the field quadratures lead to the condition

\[
EPR \equiv (\Delta_{\text{inf}} \hat{X}_{\theta_1})^2(\Delta_{\text{inf}} \hat{X}_{\theta_2})^2 < \frac{1}{4}. \tag{12}
\]

Experimental realization of the EPR paradox was first carried out by Ou et al.\(^{20}\) using two spatially separated and correlated light modes. Similar demonstrations of the EPR paradox using quadrature amplitudes of other radiation fields were performed later\(^{42}\). Subsequent works have shown that the Reid inequality is effective in demonstrating the EPR paradox for systems in which correlations appear at the level of variances, though there exist several pure entangled states which do not display steering through the Reid criterion. Moreover, in systems with correlations manifesting in higher than the second moment, the Reid formulation generally fails to show occurrence of the EPR paradox, even though Bell nonlocality may be exhibited\(^{43,44}\).

On the other hand, a modern formulation of quantum steering in terms of an information theoretic task was proposed by the work of Wiseman et al.\(^{21,45}\). They considered a bipartite situation in which one of two parties (Alice) prepares a quantum state and sends one of the particles to Bob. The procedure is repeated as many times as required. Bob’s particle is assumed to possess a definite state, even if it is unknown to him (local hidden state). No such assumption is made for Alice, and hence, this formulation of steering is an asymmetric task. Alice and Bob make measurements on their respective particles, and communicate classically. Alice’s task is to convince Bob that the state they share is entangled. If correlations between Bob’s measurement results and Alice’s declared results can be explained by a local hidden state (LHS) model for Bob, he is not convinced. This is because Alice could have drawn a pure state at random from some ensemble and sent it to Bob, and then chosen her result based on her knowledge of this LHS. Conversely, if the correlations cannot be so explained, then the state must be entangled. Alice will be successful in her task of steering if she can create genuinely different ensembles for Bob by steering Bob’s state.

Using similar formulations for entanglement as well as Bell nonlocality, a clear distinction between these three types of correlations is possible using joint probability distributions, with entanglement being the weakest, steering the intermediate, and Bell violation the strongest of the three. Bell nonlocal states constitute a strict subset of steerable states which, in turn, are a strict subset of entangled states. For
the case of pure entangled states of two qubits the three classes overlap. An experimental demonstration of these differences has been performed for mixed entangled states of two qubits\textsuperscript{46}. For the case of continuous variables, Walborn et al.\textsuperscript{43} have proposed another steering condition which is derived using the the entropic uncertainty relation (EUR)\textsuperscript{6}. EUR for the position and momentum distribution of a quantum system is given by

\[ h_Q(X) + h_Q(P) \geq \ln \pi e. \]  

Walborn et al.\textsuperscript{43} considered a joint probability distribution of two parties corresponding to a non-steerable state for which there exists a local hidden state (LHS) description, given by

\[ P(r_A, r_B) = \sum_\lambda P(\lambda)P(r_A|\lambda)P_Q(r_B|\lambda), \]  

where, \( r_A \) and \( r_B \) are the outcomes of measurements \( R_A \) and \( R_B \) respectively; \( \lambda \) are hidden variables that specify an ensemble of states; \( P \) are general probability distributions; and \( P_Q \) are probability distributions corresponding to the quantum state specified by \( \lambda \). Now, using a rule for conditional probabilities

\[ P(a,b|c) = P(b|c)P(a|b) \]  

which holds when \( \{b\} \in \{c\} \), i.e., there exists a local hidden state of Bob predetermined by Alice, it follows that the conditional probability \( P(r_B|r_A) \) is given by

\[ P(r_B|r_A) = \sum_\lambda P(r_B,\lambda|r_A) \]  

with \( P(r_B,\lambda|r_A) = P(\lambda|r_A)P_Q(r_B|\lambda) \). Note that \textsuperscript{14} and \textsuperscript{15} are similar conditions for non-steerability. Next, considering the relative entropy (defined for two distributions \( p(X) \) and \( q(X) \) as \( H(p(X)||q(X)) = \sum_x p_x \ln(p_x/q_x) \) ) between the probability distributions \( P(r_B,\lambda|r_A) \) and \( P(\lambda|r_A)P(r_B|r_A) \), it follows from the positivity of relative entropy that

\[ \sum_\lambda \int dr_B P(r_B,\lambda|r_A) \ln \frac{P(r_B,\lambda|r_A)}{P(\lambda|r_A)P(r_B|r_A)} \geq 0 \]  

Using the non-steering condition \textsuperscript{15}, the definition of the conditional entropy \( h(X|Y) = -\sum_{x,y} p(x,y) \ln p(x|y) \), and averaging over all measurement outcomes \( r_A \), it follows that the conditional entropy \( h(R_B|R_A) \) satisfies

\[ h(R_B|R_A) \geq \sum_\lambda P(\lambda)h_Q(R_B|\lambda) \]  

Considering a pair of variables \( S_A, S_B \) conjugate to \( R_A, R_B \), a similar bound on the conditional entropy may be written as

\[ h(S_B|S_A) \geq \sum_\lambda P(\lambda)h_Q(S_B|\lambda) \]
Authors’ Names

For the LHS model for Bob, note that the entropic uncertainty relation \(\text{13}\) holds for each state marked by \(\lambda\). Averaging over all hidden variables, it follows that

\[
\sum_{\lambda} \mathcal{P}(\lambda) \left( h_Q(R_B|\lambda) + h_Q(S_B|\lambda) \right) \geq \ln \pi e \tag{19}
\]

Now, using the bounds \(\text{17}\) and \(\text{18}\) in the relation \(\text{19}\) one gets the entropic steering inequality given by

\[
h(R_B|R_A) + h(S_B|S_A) \geq \ln \pi e. \tag{20}
\]

Entropic functions by definition incorporate correlations up to all orders, and the Reid criterion follows as a limiting case of the entropic steering relation \(\text{43}\).

EPR steering for Gaussian as well as non-Gaussian states has been studied in the literature \(\text{43, 24, 47}\). Non-Gaussian states may be generated by the process of photon subtraction and addition \(\text{48}\), and these states generally have higher degree of entanglement than the Gaussian states. We conclude this section by discussing the example of steering by one such non-Gaussian state, viz., the eigenstate of the two-dimensional harmonic oscillator. The energy eigenfunctions of the two-dimensional harmonic oscillator may be expressed in terms of Hermite-Gaussian (HG) functions given by \(\text{48}\)

\[
u_{nm}(x, y) = \sqrt{\frac{2}{\pi}} \left( \frac{1}{2^{n+m} w^{2n+m}} \right)^{1/2} \times H_n \left( \frac{\sqrt{2} x}{w} \right) H_m \left( \frac{\sqrt{2} y}{w} \right) e^{-\left(\frac{x^2 + y^2}{w^2}\right)} ,
\]

\[
\int |\nu_{nm}(x, y)|^2 dx dy = 1 \tag{21}
\]

Entangled states may be constructed from superpositions of HG wave functions

\[
\Phi_{nm}(\rho, \theta) = \sum_{k=0}^{n+m} \nu_{n+m-k, k}(x, y) \frac{\rho^k}{k!} (\sqrt{-1})^k \times \sqrt{\frac{k!(n+m-k)!}{n!m!2^{n+m}}} \tag{22}
\]

\[
f_k^{(n,m)} = \frac{\partial^k}{dt^k} ((1-t)^n (1+t)^m)|_{t=0}, \tag{23}
\]

where \(\Phi_{nm}(\rho, \theta)\), the Laguerre-Gaussian (LG) functions are given by \(\text{48}\)

\[
\Phi_{nm}(\rho, \theta) = e^{i(n-m)\theta} e^{-\rho^2/w^2} (-1)^{\min(n,m)} \left( \frac{\rho \sqrt{2}}{w} \right)^{\left| n-m \right|} \times \sqrt{\frac{2}{\pi n! m! w^2}} L_{\min(n,m)}^{\left| n-m \right|} \left( \frac{2\rho^2}{w^2} \right) \min(n, m)!
\]

\[
\times \sqrt{\frac{2}{\pi n! m! w^2}} L_{\min(n,m)}^{\left| n-m \right|} \left( \frac{2\rho^2}{w^2} \right) \min(n, m)!
\]
with \( \int |\Phi_{nm}(\rho, \theta)|^2 d\rho d\theta = 1 \), where \( w \) is the beam waist, and \( L^l_p(x) \) is the generalized Laguerre polynomial. The superposition (22) is like a Schmidt decomposition thereby signifying the entanglement of the LG wave functions.

In terms of dimensionless quadratures \( \{X, P_X\} \) and \( \{Y, P_Y\} \), given by \( x(y) \rightarrow \frac{w}{\sqrt{2}} X(Y) \), and \( p_x(p_y) \rightarrow \frac{\sqrt{2} w}{\hbar} P_X(P_Y) \), the canonical commutation relations are \( [\hat{X}, \hat{P}_X] = i; [\hat{Y}, \hat{P}_Y] = i \), and the operator \( \hat{P}_X \) and \( \hat{P}_Y \) are given by \( \hat{P}_X = -i \frac{\partial}{\partial X} \) and \( \hat{P}_Y = -i \frac{\partial}{\partial Y} \), respectively. The Wigner function corresponding to the LG wave function in terms of the scaled variables is given by

\[
W_{nm}(X, P_X; Y, P_Y) = \frac{(-1)^{n+m}}{(\pi)^2} L_n[4(Q_0 + Q_2)] \exp(-4Q_0)
\]

(25)

where \( Q_0 = \frac{1}{4} [X^2 + Y^2 + P_X^2 + P_Y^2] \), and \( Q_2 = \frac{X P_Y - Y P_X}{2} \). It was shown in Ref.\(^{24}\) that the Reid criterion is unable to reveal steering for the LG wave function. The entropic steering inequality in this case may be written in terms of the conjugate pairs of dimensionless quadratures, (20) given by

\[
h(X|Y) + h(P_X|Y) \geq \ln \pi e,
\]

(26)

where \( X, Y, P_X \) and \( P_Y \) are the outcomes of measurements \( \mathcal{X}, \mathcal{Y}, \mathcal{P}_X \) and \( \mathcal{P}_Y \) respectively. For \( n = 0 \) and \( m = 0 \), the LG wave function factorizes into a product state with the corresponding Wigner function given by

\[
W_{00}(X, P_X; Y, P_Y) = \frac{e^{-X^2 - Y^2 - P_X^2 - P_Y^2}}{\pi^2}.
\]

(27)

In this case the relevant entropies turn out to be \( h(\mathcal{X}|\mathcal{P}_Y) = h(\mathcal{P}_X|\mathcal{Y}) = \ln \pi e \) and \( h(Y) = h(P_Y) = \frac{1}{4} \ln \pi e \), and hence, the entropic steering inequality becomes saturated\(^{24}\), i.e.,

\[
h(X|Y) + h(P_X|Y) = \ln \pi e.
\]

(28)

For \( n = 1 \) and \( m = 0 \), the Wigner function has the form

\[
W_{10}(X, P_X; Y, P_Y) = e^{-X^2 - Y^2 - P_X^2 - P_Y^2} \times \frac{(P_X - Y)^2 + (P_Y + X)^2 - 1}{\pi^2}
\]

(29)

and the relevant entropies are given by \( h(\mathcal{X}|\mathcal{P}_Y) = h(\mathcal{P}_X|\mathcal{Y}) \approx 2.41509 \), and \( h(Y) = h(P_Y) \approx 1.38774 \). Hence, the entropic steering relation in this case becomes

\[
h(X|Y) + h(P_X|Y) \approx 2.05471 < \ln \pi e
\]

(30)

Steering is thus demonstrated here. For higher values of angular momentum, the violation of the inequality becomes stronger for higher values of \( n \), as shown in Ref.\(^{24}\).
It may be noted that the Laguerre-Gaussian functions are physically realizable field configurations with interesting topological and coherence properties, and are considered to be potentially useful for several information processing applications. Steering has been demonstrated using the entropic steering relation for other classes of non-Gaussian states such as photon subtracted squeezed vacuum states and N00N states in Ref. where it has been proposed that it may be easier to detect entanglement in some such states using steering compared to the manifestation of Bell violation. Note also that further generalizations of entropic steering inequalities to the case of symmetric steering, loss-tolerant steering, as well as to the case of steering with quantum memories have also been proposed recently.

4. Fine-graining and its connection with nonlocality

Uncertainty relations impose restrictions on the knowledge about the properties of a system described by its state of a system. The Heisenberg uncertainty relation prohibits the certain prediction of the measurement outcomes of two non-commuting observables. For example, when one predicts certainly the spin orientation of a qubit along the $z$-axis, the knowledge of spin orientation of that qubit along the $x$-axis is completely uncertain, as the probability of getting spin up and down are equal. With the motivation of distinguishing the uncertainty inherent in obtaining any combination of outcomes for different measurements, Oppenheim and Wehner proposed a fine-grained form of the uncertainty relation. Such fine-graining is aimed at capturing the plurality of simultaneous possible outcomes of a set of measurements. Considering bipartite systems they formulated a fine-grained uncertainty relation for a special class of nonlocal retrieval games for which there exist only one winning answer for one of the two parties. The upper bound of the uncertainty relation which is also the maximum winning probability of the retrieval game was shown to specify the degree of nonlocality of the underlying physical theory. In particular, such an upper bound is applicable to discriminate between the degree of nonlocality pertaining to classical theory, quantum theory, and no-signalling theory with maximum nonlocality for bipartite systems. Similar formulations of fine-graining in the context of nonlocal games have been later used to distinguish the nonlocality of tripartite systems, as well as in the context of biased bipartite and tripartite games.

The fine-grained uncertainty relation (or rather, a set of relations) as proposed by Oppenheim and Wehner is given by

$$ P(\sigma, x) := \sum_{t=1}^{n} p(t)p(x(t)|t)_\sigma \leq \zeta_x(\mathcal{T}, \mathcal{D}) \quad (31) $$

where $P(\sigma, x)$ is the probability of possible outcomes written as a string $x = \{x^{(1)}, ..., x^{(n)}\}$ corresponding to a set of measurements $\{t\} (\in \mathcal{T})$ chosen with probabilities $\{p(t)\} (\in \mathcal{D}$, the probability distribution of choosing measurements), $p(x(t)|t)_\sigma$ is the probability of obtaining outcome $x^{(t)}$ by performing measurement
labeled ‘t’ on the state of a general physical system \( \sigma \), \( n(=|T|) \) is the total number of different measurement settings, and \( \zeta_x(T, D) \) is given by

\[
\zeta_x(T, D) = \max_x \sum_{t=1}^n p(t)p(x^{(t)}|t)_\sigma
\]

where the maximum is taken over all possible states allowed on a particular system. The uncertainty of measurement outcome occurs for the value of \( \zeta_x(T, D) < 1 \). The value of \( \zeta_x(T, D) \) is bound by the particular physical theory. The no-signaling theory with maximum nonlocality gives the upper bound \( \zeta_x(T, D) = 1 \). For the case of the single qubit in quantum theory, the form of the fine-grained uncertainty relation is given by

\[
P(T, \sigma_A) = \sum_{t=1}^n p(t)p(a = x^{(t)}|t)_{\sigma_A} \leq \zeta_x(T, D)
\]

where \( p(a = x^{(t)}|t)_{\sigma_A} = Tr[A^a_t A^a_A] \) with \( A^a_t \) being the measurement operator corresponding to measurement setting ‘t’ giving outcome ‘a’, and \( \zeta_x(T, D) = \max_{\sigma_A} P(T, \sigma_A) \). Here the maximum is taken over all possible single qubit states. The value of \( \zeta_x(T, D) \) that occurs for the spin measurements along the z-axis and along the x-axis with equal probability ( \( p(t) = 1/2 \)) on the eigenstates of \( (\sigma_x + \sigma_z)/\sqrt{2} \) and \( (\sigma_x - \sigma_z)/\sqrt{2} \), is \( (1 + \frac{1}{2\sqrt{2}}) \).

The connection between fine-graining and nonlocality was observed by Oppenheim and Wehner\(^{15} \) for the case of bipartite systems. They provided specific examples of nonlocal retrieval games (for which there exist only one winning answer for one of the two parties) for the purpose of discriminating different types of theories by the upper bound of \( \zeta \) (the degree of nonlocality). According to these games, Alice and Bob receive questions ‘s’ and ‘t’ respectively, with some probability distribution \( p(s, t) \) (for simplicity, \( p(s, t) = p(s)p(t) \)); and their answer ‘a’ or ‘b’ will be winning answers determined by the set of rules, i.e., for every setting ‘s’ and the corresponding outcome ‘a’ of Alice, there is a string \( x_{s,a} = (x_{s,a}^{(1)}, ..., x_{s,a}^{(n)}) \) of length \( n = |T| \) that determines the correct answer \( b = x_{s,a}^{(1)} \) for the question ‘t’ for Bob. In the particular game considered, Alice and Bob share a state \( \rho_{AB} \) which is emitted and distributed by a source. Alice and Bob are spatially separated enough so that no signal can travel while experimenting. Alice performs either of her measurements \( A_0 \) and \( A_1 \) and Bob, either of \( B_0 \) and \( B_1 \) at a time. These measurements having the outcomes +1 and −1, can be chosen by Alice and Bob without depending on the choice made by the other. The CHSH inequality\(^{23} \)

\[
\frac{1}{4}[E(A_0 B_0) + E(A_0 B_1) + E(A_1 B_0) - E(A_1 B_1)] \leq \frac{1}{2}
\]

holds for any local hidden variable model and can be violated when measurements are done on quantum particles prepared in entangled states. Here \( E(A_i B_j) \) are the averages of the product of measurement outcomes of Alice and Bob with \( i, j = 0, 1 \).

In the context of the above game, Alice and Bob receive respective binary questions \( s, t \in \{0, 1\} \) (i.e., representing two different measurement settings on each
and they win the game if their respective outcomes (binary) $a, b \in \{0, 1\}$ satisfy the condition $a \oplus b = s.t$. At the starting of the game, Alice and Bob discuss their strategy (i.e., choice of shared bipartite state and also measurement). They are not allowed to communicate with each other once the game has started. The probability of winning the game for a physical theory described by bipartite state ($\sigma_{AB}$) is given by

$$P_{\text{game}}(S, T, \sigma_{AB}) = \sum_{s,t} p(s,t) \sum_{a} p(a, b = x^t_{s,a} | s, t)_{\sigma_{AB}}$$

where the form of $p(a, b = x^t_{s,a} | s, t)_{\sigma_{AB}}$ in terms of the measurements on the bipartite state $\sigma_{AB}$ is given by

$$p(a, b = x^t_{s,a} | s, t)_{\sigma_{AB}} = \sum_{b} V(a, b | s, t) \langle (A^a_s \otimes B^b_t) \rangle_{\sigma_{AB}}$$

where $A^a_s = \frac{(I+(-1)^a A)}{2}$ is a measurement of the observable $A_s$ corresponding to setting 's' giving outcome 'a' at Alice’s side; $B^b_t = \frac{(I+(-1)^b B)}{2}$ is a measurement of the observable $B_t$ corresponding to setting ‘t’ giving outcome ‘b’ at Bob’s side, and $V(a, b | s, t)$ is the winning condition given by

$$V(a, b | s, t) = 1 \quad \text{iff} \quad a \oplus b = s.t$$

$$= 0 \quad \text{otherwise}$$

Using Eqs. (35), (36), (37) and taking $p(s,t) = p(s)p(t) = 1/4$, the expression of $P_{\text{game}}(S, T, \sigma_{AB})$ for the bipartite state $\sigma_{AB}$ is obtained to be

$$P_{\text{game}}(S, T, \sigma_{AB}) = \frac{1}{2} \left( 1 + \frac{\langle B_{\text{CHSH}} \rangle_{\sigma_{AB}}}{4} \right)$$

where

$$B_{\text{CHSH}} = A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1$$

corresponds to the Bell-CHSH operator$^{22,23}$. To characterize the allowed distribution under the theory, we need to know the maximum winning probability, maximized over all possible strategies for Alice and Bob. The maximum winning probability is given by

$$P_{\text{max}} = \max_{S, T, \sigma_{AB}} P_{\text{game}}(S, T, \sigma_{AB})$$

The value of $P_{\text{max}}(S, T, \sigma_{AB})$ allowed by classical physics is $\frac{3}{4}$ (as classically, the Bell-CHSH inequality is bounded by 2), by quantum mechanics is $\left( \frac{1}{2} + \frac{1}{2\sqrt{2}} \right)$ (due to the maximum violation of Bell inequality, $\langle B_{\text{CHSH}} \rangle = 2\sqrt{2}$), and by no-signaling theories with maximum Bell violation $\langle B_{\text{CHSH}} \rangle = 4$, that occurs for the PR-box$^{56}$ is 1. The connection of Eq. (37) with the no-signalling constraint for the general case of a bipartite system was elaborated by Barrett et al.$^{57}$

The above description refers to the scenario when the two parties have no bias towards choosing a particular measurement. Nonlocality in the context of biased
games has been discussed in Ref.\textsuperscript{30} using the fine-grained uncertainty relation. In the particular game chosen\textsuperscript{58} the biased game, the intention of Alice is to choose $A_0$ with probability $p(0 \leq p \leq 1)$ and $A_1$ with probability $(1 - p)$. Bob intends to choose $B_0$ and $B_1$ with probabilities $q(0 \leq q \leq 1)$ and $(1 - q)$, respectively. The measurements and their outcomes are coded into binary variables pertaining to an input-output process. Alice and Bob have binary input variables $s$ and $t$, respectively, and output variables $a$ and $b$, respectively. Input $s$ takes the values 0 and 1 when Alice measures $A_0$ and $A_1$, respectively. Output $a$ takes the values 0 and 1 when Alice gets the measurement outcomes +1 and −1, respectively. The identifications are similar for Bob’s variables $t$ and $b$. Now, the rule of the game is that Alice and Bob’s particles win (as a team) if their inputs and outputs satisfy

$$a \oplus b = s.t$$

(41)

where $\oplus$ denotes addition modulo 2. Input questions $s$ and $t$ have the probability distribution $p(s, t)$ (for simplicity, $p(s, t) = p(s)p(t)$ where $p(s = 0) = p, p(s = 1) = (1 - p), p(t = 0) = q$ and $p(t = 1) = (1 - q)$). The fine-grained uncertainty relation is now invoked. The expression of $P_{\text{game}}$ is given by

$$P_{\text{game}}(S, T, \rho_{AB}) = \frac{1}{2}[1 + \langle \text{CHSH}(p, q) \rangle_{\rho_{AB}}]$$

(42)

with $\text{CHSH}(p, q) = [pqA_0 \otimes B_0 + p(1 - q)A_0 \otimes B_1 + (1 - p)qA_1 \otimes B_0 - (1 - p)(1 - q)A_1 \otimes B_1]$ being the form of CHSH-function after introducing bias.

The maximum probability $P_{\text{game}}$ of winning the biased game was obtained\textsuperscript{30} by maximizing the function $\langle \text{CHSH}(p, q) \rangle$ for different theories. Such maximization was earlier performed in the literature for the unbiased scenario\textsuperscript{59} and subsequently, for the biased case as well\textsuperscript{58}, in the latter case by considering two halves of the ranges of the parameters $p$ and $q$. First, for the case of $p, q \geq 1/2$, the classical maximum is obtained using an extremal strategy where the values of all the observables are +1 giving the maximum value of the biased CHSH-function to be $1 - 2(1 - p)(1 - q)$. With this classical maximum, the winning probability is given by

$$P_{\text{game}}(S, T, \rho_{AB})_{\text{classical maximum}} = 1 - (1 - p)(1 - q)$$

(43)

This reduces to the value $\frac{3}{4}$ for the unbiased case when $p = q = \frac{1}{2}$. For the quantum strategy, the parameter space is divided in two regions of $[p, q]$ with the first region corresponding to $1 \geq p \geq (2q)^{-1} \geq \frac{1}{2}$. Here $\langle \text{CHSH}(p, q) \rangle \leq 1 - 2(1 - p)(1 - q)$ leads to

$$P_{\text{game}}(S, T, \rho_{AB})_{\text{region 1}} = 1 - (1 - p)(1 - q)$$

(44)

showing that the upper bound is the same as achieved by classical theory. Thus, quantum correlations (entanglement) offers no advantage over classical correlations in performing the specified task in this region. However, in the other region $1 \geq (2q)^{-1} > p \geq \frac{1}{2}$ one gets the value $\langle \text{CHSH}(p, q) \rangle \leq \sqrt{2}\sqrt{q^2 + (1 - q)^2}\sqrt{p^2 + (1 - p)^2}$ that is greater than the classical bound. So, the
biasing parameters in this region enable discrimination among classical and quantum correlations. The upper bound of the fine-grained uncertainty relation is in this case given by,

$$P_{\text{game}}(S, T, \rho_{AB}) \mid_{\text{quantum maximum}} = \frac{1}{2}[1 + \sqrt{2} \sqrt{q^2 + (1 - q)^2} \sqrt{p^2 + (1 - p)^2}]$$ (45)

The extent of non-locality that can be captured by the fine-grained uncertainty relation is regulated by the bias parameters.

The fine-grained uncertainty relation has been applied to study the nonlocality of tripartite systems, as well\textsuperscript{29}. In this case a nonlocal retrieval game similar to CHSH-game for bipartite systems is considered, as follows. Three parties, Alice, Bob and Charlie receive respective binary questions ‘s’, ‘t’, and ‘u’ ∈ {0, 1} (corresponding to their two different measurement settings at each side), and they win the game if their respective outcomes (binary) ‘a’, ‘b’, and ‘c’ ∈ {0, 1} satisfy certain rules. Three kinds of no-signaling boxes, known as full-correlation boxes have been considered, for which all one-party and two-party correlation in the system vanish\textsuperscript{60}. The game is won if their answers satisfy the set of rules, either

$$a \oplus b \oplus c = s.t \oplus t.u \oplus u.s$$ (46)

or

$$a \oplus b \oplus c = s.t \oplus s.u$$ (47)

or else

$$a \oplus b \oplus c = s.t.u$$ (48)

All the above boxes violate the Mermin inequality\textsuperscript{61}, whereas the Svetlichny inequality\textsuperscript{62} is violated only by the box given by Eq. (46) (known as the Svetlichny box). The winning probability of the game under a physical theory described by a shared tripartite state $\sigma_{ABC}$ (among Alice, Bob and Charlie) is given by

$$P_{\text{game}}(S, T, U, \sigma_{ABC}) = \sum_{s,t,u} p(s,t,u) \sum_{a,b} p(a,b,c = x_{s,t,a,b}^{(u)}|s,t,u)_{\sigma_{ABC}}$$ (49)

where $p(s,t,u)$ is the probability of choosing the measurement settings ‘s’ by Alice, ‘t’ by Bob and ‘u’ by Charlie, and $p(a,b,c|s,t,u)_{\sigma_{ABC}}$ the joint probability of getting outcomes ‘a’, ‘b’ and ‘c’ for corresponding settings ‘s’, ‘t’ and ‘u’ given by

$$p(a,b,c = x_{s,t,a,b}^{(u)}|s,t,u)_{\sigma_{ABC}} = \sum_c V(a,b,c|s,t,u) \langle A^a_s \otimes B^b_t \otimes C^c_u \rangle_{\sigma_{ABC}}$$ (50)

where $A^a_s$, $B^b_t$ and $C^c_u$ are the measurements corresponding to setting ‘s’ and outcome ‘a’ at Alice’s side, setting ‘t’ and outcome ‘b’ at Bob’s side, and setting ‘u’
and outcome ‘c’ at Charlie’s side, respectively; and $V(a, b, c | s, t, u)$ (the winning condition) is non zero (= 1) only when the outcomes of Alice, Bob and Charlie are correlated by either of Eqs. (46), (47) or (48), and is zero otherwise. The maximum winning probability over all possible strategies (i.e., the choice of the shared tripartite state and measurement settings by the three parties) for any theory is given by

$$P_{\text{game}}^{\text{max}} = \max_{S, T, U, \sigma_{ABC}} P_{\text{game}}(S, T, U, \sigma_{ABC})$$

which is a signature of the allowed probability distribution under that theory.

The cases corresponding to classical, quantum and no-signalling theories with super-quantum correlations for the above different full-correlation boxes (rules of the nonlocal game) have been studied in Ref. 29. For the case of the winning condition given by Eq. (46), the expression of $P_{\text{game}}(S, T, U, \sigma_{ABC})$ for the shared tripartite state $\sigma_{ABC}$ is given by

$$P_{\text{game}}(S, T, U, \sigma_{ABC}) = \frac{1}{2} \left[ 1 + \frac{\langle S_1 \rangle_{\sigma_{ABC}}}{8} \right]$$

where

$$S_1 = A_0 \otimes B_0 \otimes C_0 + A_0 \otimes B_0 \otimes C_1 + A_0 \otimes B_1 \otimes C_0 + A_1 \otimes B_0 \otimes C_0 + A_0 \otimes B_1 \otimes C_1 - A_1 \otimes B_0 \otimes C_1 - A_0 \otimes B_1 \otimes C_1 - A_1 \otimes B_0 \otimes C_1 - A_1 \otimes B_1 \otimes C_1$$

The value of $P_{\text{game}}^{\text{max}}$ allowed in classical physics is $3/4$ which follows from the Svetlichny inequality. For the case of quantum physics, the maximum violation of the Svetlichny inequality is $4\sqrt{2}$ which occurs for the GHZ-state. The value of $P_{\text{game}}^{\text{max}}$ allowed in quantum physics is $\left( \frac{1}{2} + \frac{1}{\sqrt{2}} \right)$. For the case of the no-signalling theory, the algebraic maximum of the Svetlichny inequality is 8, and the value of $P_{\text{game}}^{\text{max}}$ in this case is 1, corresponding to a correlation with maximum nonlocality.

It was found in Ref. 29 that none of the other two full correlation Mermin boxes (47) and (48) are able to distinguish classical theory from quantum theory in terms of their degree of nonlocality. The fine-grained uncertainty relation determines the degree of nonlocality as manifested by the Svetlichny inequality for tripartite systems corresponding to the winning condition given by (46), in the same way as it determines the nonlocality of bipartite systems manifested by Bell-CHSH inequality. One is able to differentiate the various classes of theories (i.e., classical physics, quantum physics and no-signaling theories with maximum nonlocality or superquantum correlations) by the value of $P_{\text{game}}^{\text{max}}$ for tripartite systems. A biased tripartite system had also been explored. However, it was observed using a bipartition model that there is a zone specified by the biasing parameters where even the Svetlichny inequality cannot perform the discrimination between various physical systems based on their degree of nonlocality.
5. Quantum memory

In quantum information theory, an uncertainty relation in terms of entropy is regarded to be more useful than that in terms of standard deviation. The uncertainty relating to the outcomes of observables is reformulated in terms of Shannon entropy instead of standard deviation. Entropic uncertainty relations for two observables in the context of discrete variables was introduced by Deutsch\(^7\). An improved version was conjectured by Kraus\(^8\), given by

\[ H(R) + H(S) \geq \log_2 \frac{1}{c} \]

and later proved by Maassen and Uffink\(^9\). Here \( H(i) \) denotes the Shannon entropy of the probability distribution of the measurement outcomes of observable \( i \) \( (i \in \{ R, S \}) \) and \( \frac{1}{c} \) quantifies the complementarity of the observable. For non-degenerate observables,\( c = \max_{i,j} c_{i,j} = \max_{i,j} |\langle a_i | b_j \rangle|^2 \), where \( |a_i\rangle \) and \( |b_j\rangle \) are eigenvectors of \( R \) and \( S \), respectively.

Using entanglement between the state of the observed system and another quantum system (memory), Berta et al.\(^10\) have shown that the lower bound of entropic uncertainty (given by Eq.\((55)\)) can be improved in the presence of quantum correlations. The entropic uncertainty relation in the presence of quantum memory is given by\(^10\)

\[ S(R|B) + S(S|B) \geq \log_2 \frac{1}{c} + S(A|B) \]

where \( S(R|B) \) (\( S(S|B) \)) is the conditional von Neumann entropy of the state given by \( \sum_j (|\psi_j\rangle \langle \psi_j | \otimes I) \rho_{AB} (|\psi_j\rangle \langle \psi_j | \otimes I) \), with \( |\psi_j\rangle \) being the eigenstate of observable \( R(S) \), and \( S(R|B) \) (\( S(S|B) \)) quantifies the uncertainty corresponding to the measurement \( R(S) \) on the system “A” given information stored in the system “B” (i.e., quantum memory). \( S(A|B) \) quantifies the amount of entanglement between the quantum system possessed by Alice and the quantum memory possessed by Bob. For example, the sum of uncertainties of two measurement outcomes \( (H(R) + H(S)) \) for measurement of two observables \( (R, S) \) on the quantum system (“A”, possessed by Alice) can be reduced to 0 (i.e., there is no uncertainty) if that system is maximally entangled with another system, called quantum memory (“B”, possessed by Bob). Here, Bob is able to reduce his uncertainty about Alice’s measurement outcome with the help of communication from Alice regarding the choice of her measurement performed, but not its outcome.

Recently, Coles and Piani\(^14\) have made the lower bound of entropic uncertainty in the presence of quantum memory tighter. Their modified form of the entropic uncertainty relation is given by

\[ S(R_A|B) + S(S_A|B) \geq \text{c'}(\rho_A) + S(A|B) \]

where \( \text{c'}(\rho_A) = \max \{ \text{c'}(\rho_A, R_A, S_A), \text{c'}(\rho_A, S_A, R_A) \} \). \( \text{c'}(\rho_A, R_A, S_A) \) and
where uncertainty for the measurement of the observable \( S \) is measured by \( R \) where \( \Pi \) with \( I \) is defined by
\[ \rho_{\text{AB}} = \sum_j p_j^{rA(\rho_{\text{AB}})} \Pi_{j}^{R_{\text{A}}(\rho_{\text{AB}})} \otimes \rho_{\text{Bij}}^{R_{\text{A}}(\rho_{\text{AB}})}, \]

where \( \Pi_{j}^{R_{\text{A}}(\rho_{\text{AB}})} \)'s are the orthogonal projectors on the eigenstate \( |\psi_j\rangle_{R_{\text{A}}(\rho_{\text{AB}})} \) of observable \( R_{\text{A}}(\rho_{\text{AB}}) \), \( p_j^{R_{\text{A}}(\rho_{\text{AB}})} = \text{Tr}[(|\psi_j\rangle_{R_{\text{A}}(\rho_{\text{AB}})} \langle \psi_j| \otimes I)\rho_{\text{AB}}(|\psi_j\rangle_{R_{\text{A}}(\rho_{\text{AB}})} \langle \psi_j| \otimes I)] \), \( \rho_{\text{Bij}}^{R_{\text{A}}(\rho_{\text{AB}})} = \text{Tr}_{\text{A}}[(|\psi_j\rangle_{R_{\text{A}}(\rho_{\text{AB}})} \langle \psi_j| \otimes I)\rho_{\text{AB}}(|\psi_j\rangle_{R(\rho_{\text{AB}})} \langle \psi_j| \otimes I)]/p_j^{R_{\text{A}}(\rho_{\text{AB}})} \) and \( \rho_{\text{AB}} \) is the state of joint system ‘\( \text{A} \)’ and ‘\( \text{B} \)’.

In another work, Pati et al.\(^{11}\) have extended the concept of memory to include more general quantum correlations beyond entanglement. This leads to the improvement of the lower bound given by
\[ S(R_{\text{A}}|B) + S(S_{\text{A}}|B) \geq c'(\rho_{\text{A}}) + S(A|B) + \max\{0, D_{\text{A}}(\rho_{\text{AB}}) - C_{A}^{M}(\rho_{\text{AB}})\}, \]

where the quantum discord \( D_{\text{A}}(\rho_{\text{AB}}) \) is given by\(^{65}\)
\[ D_{\text{A}}(\rho_{\text{AB}}) = \mathcal{I}(\rho_{\text{AB}}) - C_{A}^{M}(\rho_{\text{AB}}), \]

with \( \mathcal{I}(\rho_{\text{AB}}) = S(\rho_{\text{A}}) + S(\rho_{\text{B}}) - S(\rho_{\text{AB}}) \) being the mutual information of the state \( \rho_{\text{AB}} \) which contains the total correlation present in the state \( \rho_{\text{AB}} \) shared between the system \( \text{A} \) and the system \( \text{B} \), and the classical information \( C_{A}^{M}(\rho_{\text{AB}}) \) for the shared state \( \rho_{\text{AB}} \) (when Alice measures on her system) is given by
\[ C_{A}^{M}(\rho_{\text{AB}}) = \max_{\Pi_{\text{A}}} \{ S(\rho_{\text{B}}) - \sum_{j=0}^{R_{\text{A}}(\rho_{\text{AB}})} p_j^{R_{\text{A}}(\rho_{\text{AB}})} S(\rho_{\text{Bij}}^{R_{\text{A}}(\rho_{\text{AB}})}) \}. \]

Experiments have demonstrated the effectiveness of reducing quantum uncertainty using quantum memory, for the case of pure\(^{66}\) as well as mixed\(^{67}\) entangled states. For the purpose of experimental verification of inequality \( \mathcal{I}(\rho_{\text{AB}}) \geq S(\rho_{\text{A}}) + S(\rho_{\text{B}}) - S(\rho_{\text{AB}}) \), the entropic uncertainty is recast in the form of the sum of the Shannon entropies \( H(p_d^{R(\rho_{\text{S}})}) + H(p_d^{S(\rho_{\text{S}})}) \) when Alice and Bob measure the same observables \( R(\rho_{\text{S}}) \) on their respective systems and get different outcomes whose probabilities are denoted by \( p_d^{R(\rho_{\text{S}})} \) and \( H(p_d^{R(\rho_{\text{S}})}) = -p_d^{R(\rho_{\text{S}})} \log_2 2 p_d^{R(\rho_{\text{S}})} - (1 - p_d^{R(\rho_{\text{S}})}) \log_2 (1 - p_d^{R(\rho_{\text{S}})}) \). Making use of Fano's
inequality\textsuperscript{68}, it follows that $\mathcal{H}(p_d^R) + \mathcal{H}(p_d^S) \geq S(R|B) + S(S|B)$ which using Eq. (56) gives

$$\mathcal{H}(p_d^R) + \mathcal{H}(p_d^S) \geq \log_2 \frac{1}{c} + S(A|B) \quad (63)$$

The right hand side of the inequality (63) can be determined from the knowledge of the state and the measurement settings. The entropic uncertainty relation has been used for verifying the security of key distribution protocols\textsuperscript{69}. Devetak and Winter\textsuperscript{70} derived that the amount of key $K$ that Alice and Bob are able to extract per state should always exceed the quantity $S(R|E) - S(R|B)$, where the quantum state $\rho_{ABE}$ is shared between Alice, Bob and the evesdropper Eve ($E$). Extending this idea by incorporating the effect of shared quantum correlation between Alice and Bob, Berta et al.\textsuperscript{11} reformulated their relation (56), in the form of $S(R|E) + S(R|B) \geq \log_2 \frac{1}{c}$ enabling them to derive a new lower bound on the key extraction rate, given by $K \geq \log_2 \frac{1}{c} - S(R|B) - S(S|B)$.

It has been recently realized that a further improvement in the lower bound of entropic uncertainty is possible using fine graining. A new form of the uncertainty relation in the presence of quantum memory was derived\textsuperscript{16}, in which the lower bound of entropic uncertainty corresponding to the measurement of two observables is determined by fine-graining of the possible measurement outcomes. The fine-grained uncertainty relation\textsuperscript{15}, as discussed in the previous section, is here considered in the context of a quantum game played by Alice and Bob who share a two-qubit state $\rho_{AB}$ which is prepared by Alice. Bob’s qubit which he receives from Alice, represents the quantum memory. Bob’s uncertainty of the outcome of Alice’s measurement of one of two incompatible observables (say, $R$ and $S$), is reduced with the help of fine-graining, when Alice helps Bob by communicating her measurement choice of a suitable spin observable on her system. In this game Alice and Bob are driven by the requirement of minimizing the value of the quantity $\mathcal{H}(p_d^R) + \mathcal{H}(p_d^S)$ which forms the left hand side of the entropic uncertainty relation (63). The minimization is over all incompatible measurement settings such that $R \neq S$, i.e.,

$$\mathcal{H}(p_d^R) + \mathcal{H}(p_d^S) \geq \min_{R \neq S \neq R} [\mathcal{H}(p_d^R) + \mathcal{H}(p_d^S)] \quad (64)$$

To find the minimum value, the choice of the variable $R$ was fixed without the loss of generality to be $\sigma_z$ (spin measurement along the $z$-direction), and then the minimization was performed over the other variable $S$. The uncertainty defined by the entropy $\mathcal{H}(p_d^S)$ is minimum when the certainty of the required outcome is maximum, corresponding to an infimum value for the probability $p_d^S$. In order to obtain the infimum value of $p_d^S$, the fine-grained uncertainty relation was used in a form relevant to the game considered where the infimum value of the winning probability (corresponding to minimum uncertainty) is given by

$$p_d^{S_{\inf}} = \inf_{S(\neq \sigma_z)} \sum_{a,b} V(a,b) Tr[(A^a_2 \otimes B^b_2).\rho_{AB}], \quad (65)$$
with the winning condition \( V(a, b) \) given by

\[
V(a, b) = \begin{cases} 
1 & \text{iff } a \oplus b = 1 \\
0 & \text{otherwise.} 
\end{cases}
\] (66)

with \( A^a_S \) being a projector for observable \( S \) with outcome ‘a’, given by \( S^\alpha = \frac{I + (-1)^\alpha \vec{n}_S \cdot \vec{\sigma}}{2} \) (and similarly for \( B^b_S \)), where \( \vec{n}_S(\equiv \{ \sin(\theta_S) \cos(\phi_S), \sin(\theta_S) \sin(\phi_S), \cos(\theta_S) \}) \); \( \vec{\sigma} = \{ \sigma_x, \sigma_y, \sigma_z \} \) are the Pauli matrices; \( \alpha \) takes the value either 0 (for spin up projector) or 1 (for spin down projector). The above winning condition proposed in Ref.16 is different from the winning conditions used in Refs.15,29,30 for the purpose of capturing the nonlocality of quantum systems. Here the fine-grained uncertainty relation is to make it directly applicable to the experimental situation of quantum memory66,67.

The form of the entropic uncertainty relation obtained by fine-graining is given by\(^{16}\) to be

\[
\mathcal{H}(p^{R}_{\sigma_z}) + \mathcal{H}(p^{S}_{\sigma_z}) \geq \mathcal{H}(p^{S}_{\sigma_z}) + \mathcal{H}(p^{S}_{\sigma_{inf}}) 
\] (67)

The value of \( p^{S}_{\sigma_{inf}} \) has been calculated for various quantum states such as the Werner state, Bell-diagonal state and a state with maximally mixed marginals\(^{16}\). The above uncertainty relation (67) is able to account for the experimental results obtained for the case of maximally entangled states\(^{66}\) and mixed Bell-diagonal states\(^{67}\). Moreover, the limit set by (67) prohibits the attainment of the lower bound of entropic uncertainty\(^{10}\) as defined by the right hand side of equation (56) for the class of two-qubit states with maximally mixed marginals.

The uncertainty relation (67) is independent of the choice of measurement settings as it optimizes the reduction of uncertainty quantified by the conditional Shannon entropy over all possible observables. Given a bipartite state possessing quantum correlations, inequality (67) provides the fundamental limit to which uncertainty in the measurement outcomes of any two incompatible variables can be reduced. Since the uncertainty principle in its entropic form could be used for verifying the security of key distribution protocols, there exist ramifications of Eq. (67) on the key extraction rate in quantum key generation. It is possible to obtain a tighter lower bound on the key rate\(^{16}\) given by \( K \geq \log_2 \frac{1}{2} - \mathcal{H}(p^{R}_{\sigma_z}) + \mathcal{H}(p^{S}_{\sigma_{inf}}) \) when the two parties involved in the protocol retain data whenever they make the same choice of measurement on their respective sides. The relation (67) is the optimized lower bound of entropic uncertainty, which represents the ultimate limit to which uncertainty of outcomes of two non-commuting observables can be reduced by performing any set of measurements in the presence of quantum memory.

6. Conclusions

In this article we have discussed various applications of different versions of uncertainty relations. Much of the review presented here deals with various formulations of entropic uncertainty relations\(^{6,8,9,10,11,14,16}\) in different situations for the case of
both discrete and continuous variables. However, we have also briefly discussed the
Heisenberg uncertainty relation\textsuperscript{1} and its Robertson-Schrodinger variant\textsuperscript{3,4} in the
context of two specific applications, namely, demonstration of EPR-steering\textsuperscript{18,19},
and determination of the purity of states\textsuperscript{28}, respectively. We conclude with a section-
wise summary of the main results discussed in this article, and a few possible future
directions of study.

We have discussed in Section II how the Robertson-Schrodinger uncertainty
relation may be connected to the property of purity and mixedness of single and bi-
partite qubit systems\textsuperscript{28}. The uncertainty corresponding to the measurement of suit-
able observables vanishes for pure states, and is positive definite for mixed states.
Using this feature a scheme was proposed to distinguish pure and mixed states
belonging to the classes of all single-qutrit states up to three parameters, as well as
several classes of two-qutrit states, when prior knowledge of the basis is available\textsuperscript{28}.
A possible implementation of the proposed witnesses for detecting mixedness here
could be through techniques involving measurement of two-photon polarization-
entangled modes for qutrits\textsuperscript{71}. Since the class of all pure states is not convex, the
witnesses proposed for detecting mixedness do not arise from the separability cri-
teron that holds for the widely studied entanglement witnesses\textsuperscript{37}, as well as the
recently proposed teleportation witnesses\textsuperscript{38}, and witnesses for absolutely separable
states\textsuperscript{39}. However, a similar prescription of distinction of categories of quan-
tum states based on the measurement of expectation values of Hermitian operators is
followed.

In Section III a discussion of EPR steering\textsuperscript{18,19,41} is presented in the context of
continuous variable entangled states. Though entangled states form a strict subset
of steerable states\textsuperscript{21,45}, several entangled pure states fail to reveal steering through
the Reid criterion\textsuperscript{19} for wide ranges of parameters. Using the entropic uncertainty
relation for continuous variables\textsuperscript{6}, an entropic steering condition can be derived\textsuperscript{43}.
Examples of various non-Gaussian states for which entropic steering can be demon-
strated, such as, the two-dimensional harmonic oscillator states, the photon sub-
tracted squeezed vacuum state, and the N00N state have been studied\textsuperscript{24}. Steering
with such states may be demonstrated by computing the relevant conditional en-
tropies using the Wigner function whose non-Gaussian nature plays an important
role. These examples reiterate the fact that though Bell violation guarantees steer-
ability, the two types of quantum correlations are distinct from each other. More-
over, the presence of quantum correlations in certain class of states may be more
easily detected through the violation of the entropic steering inequality compared
to the violation of the Bell inequality\textsuperscript{24}. This could be useful for detecting and ma-
nipulating correlations in non-Gaussian states for practical purposes in information
processing and quantum metrology.

The relation between uncertainty and nonlocality is discussed in Section IV. The
connection between the degree of nonlocality of the underlying physical theory and
the fine-grained uncertainty relation has been proposed\textsuperscript{15}, as expressed in terms of
the maximum winning probability of certain nonlocal games. A generalization of
this connection to the case of tripartite systems has been formulated\textsuperscript{29}. The fine-grained uncertainty relation determines the degree of nonlocality as manifested by the Svetlichny inequality\textsuperscript{62} for tripartite systems in the same way as it determines the nonlocality of bipartite systems manifested by Bell-CHSH inequality\textsuperscript{22,23}. With the help of the fine-grained uncertainty relation, one is able to differentiate the various classes of theories (i.e., classical physics, quantum physics and no-signaling theories with maximum nonlocality or superquantum correlations) by the value of the maximum winning probability of the relevant retrieval game. The fine-grained uncertainty relation\textsuperscript{15} has been further employed\textsuperscript{30} to distinguish between classical, quantum and super-quantum correlations based on their strength of nonlocality, in the context of biased games\textsuperscript{58} involving two or three parties. Discrimination among the underlying theories with different degrees of nonlocality is in this case possible for a specific range of the biasing parameters where quantum correlations offer the advantage of winning the particular nonlocal game over classical correlations. Analytical generalizations to multiparty nonlocal games may further be feasible using such an approach\textsuperscript{30}.

Section V deals with the issue of entropic uncertainty relations for discrete variables in the presence of quantum memory\textsuperscript{10}. The optimized lower bound of entropic uncertainty in the presence of quantum memory has been derived\textsuperscript{16} with the help of the fine-grained uncertainty principle\textsuperscript{15}. Since entropy (or uncertainty) is directly related to probability, the analysis of fine-graining involves the minimization (or maximization) of probability in order to minimize uncertainty. In measurements and communication involving two parties, the lower bound of entropic uncertainty cannot fall below the bound derived using fine-graining, as is illustrated with several examples of pure and mixed states of discrete variables\textsuperscript{16}. After fine-graining the entropic uncertainty relation furnishes a fundamental limitation on the precision of outcomes for measurement of two incompatible observables in the presence of quantum memory. Implications on the key rate for secure key generation is also discussed. Further work along this direction may be able to shed light on the information theoretic resources entailed in the process of fine-graining.

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