TOP–DESIGNS IN THE CATEGORY OF FORT SPACES

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Abstract. In infinite topological Fort space $X$, for nonempty subsets $C, D$ of $X$ in the following text we answer to this question “Is there any $\lambda$ and Top–design $C - (X, D, \lambda)$ of type $i$?” for $i = 1, 2, 3, 4$. We prove there exist $\lambda$ and $C - (X, D, \lambda)$, Top–design of type 2 (resp. type 4) if and only if $C$ can be embedded into $D$.

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1. Introduction

Suppose $S$ is a finite set with $n \geq 2$ elements (so $S$ is an $n$–set) and $A$ is a collection of $k$–subsets of $S$ such that each $t$–subset of $S$ occurs exactly in $\lambda$ elements of $A$, then $A$ is favorit and well studied traditional $t - (n, k, \lambda)$ combinatorial design ($t < n$ and $\lambda \geq 1$) (see [1, 3]). However these finite traditional designs has been generalized in “infinite designs” in [3], also generalized designs have been introduced for the first time in [2] as a generalization of combinatorial designs in different mathematical categories like category of well–ordered sets, topological spaces, etc.. We use term Top-design when our reference category is the category of topological spaces.

Using the same notations as in [2], in topological space $X$ for nonempty subsets $C, D$ of $X$, nonzero cardinal number $\lambda$ and collection $A$ of subsets of $X$ using statements (where by $S \approx T$ we mean $S$ and $T$ are homeomorphic spaces):

I. $\forall B \in A \ (B \approx D)$
II. $\forall B \in A \ (B \approx D \land B \approx X \ \setminus \ D)$
III. $\forall E \subseteq X \ (E \approx C \Rightarrow \text{card}(\{B \in A : E \subseteq B\}) = \lambda)$
IV. $\forall E \subseteq X \ ((E \approx C \land X \ \setminus \ E \approx X \ \setminus \ C) \Rightarrow \text{card}(\{B \in A : E \subseteq B\}) = \lambda)$

we say $A$ is a :

- $C - (X, D, \lambda)$ Top-design of type 1, if (II) and (III)
- $C - (X, D, \lambda)$ Top-design of type 2, if (I) and (III)
- $C - (X, D, \lambda)$ Top-design of type 3, if (II) and (IV)
- $C - (X, D, \lambda)$ Top-design of type 4, if (I) and (IV)

Let’s mention that if $b \in X$, equip $X$ with topology $\{U \subseteq X : b \notin U \lor (X \ \setminus \ U \text{ is finite})\}$, then we say $X$ is a Fort space with particular point $b$ [4, Counterexample 24]. One may find counterexamples regarding $C - (\{\frac{n}{n} : n \geq 1\} \cup \{0\}, D, \lambda)$ Top-designs in [2], note to the fact that $\{\frac{1}{n} : n \geq 1\} \cup \{0\}$ (with induced topology of $\mathbb{R}$) is an infinite countable Fort space, leads us to study other types of infinite Fort spaces in the approach of Top-designs.

Note 1.1. Two Fort spaces are homeomorphic if and only if they are in one–to–one correspondence. Moreover in Fort space $X$ with particular point $b$ infinite subset $Y$
of $X$ as subspace topology has Fort topology if and only if $b \in Y$ (all finite subsets of $X$ are finite discrete spaces and carry Fort topology structure).

**Convention 1.2.** In the following text suppose $X$ is an infinite Fort space with the particular point $b$.

2. Results in Top-designs on $X$

In this section we study the existence of $C - (X, D, \lambda)$ for different $Cs$ and $Ds$.

**Lemma 2.1.** For $U, V \subseteq X$ with $U \approx V$ and $X \setminus U \approx X \setminus V$ we have:

1. $b \in U$ if and only if $b \in V$ (i.e., $U \cap \{b\} = V \cap \{b\}$),
2. for infinite $U$ with $\text{card}(U) < \text{card}(X)$ and $H \subseteq X$ we have $U \approx H$ and $X \setminus U \approx X \setminus H$ if and only if $\text{card}(U) = \text{card}(H)$ and $U \cap \{b\} = H \cap \{b\}$.

**Proof.** 1) First suppose $U$ is infinite, so $V$ is infinite too. Since $b$ is the unique limit point of any infinite subset of $X$, $U$ contains a limit point if and only if $b \in U$ on the other hand $U$ contains a limit point if and only if $V$ contains a limit point which means $b \in V$ in its turn.

Now suppose $U$ is finite, thus $X \setminus U$ is infinite and using a similar method described above, we have $b \in X \setminus U$ if and only if $b \in X \setminus V$ which completes the proof.

2) Suppose $\text{card}(U) = \text{card}(H)$ and $U \cap \{b\} = H \cap \{b\}$, then $\text{card}(U \setminus \{b\}) = \text{card}(H \setminus \{b\})$, thus there exists bijection $f : U \setminus \{b\} \rightarrow H \setminus \{b\}$. If $b \notin U$, then $b \notin H$ and $f : U \setminus \{b\} = U \rightarrow H \setminus \{b\}$ is a homeomorphism (of discrete spaces) too. If $b \in U$, then $b \in H$ too and $\tilde{f} : U \rightarrow H$ with $\tilde{f}(U \setminus \{b\}) = f$ and $\tilde{f}(b) = b$ is a homeomorphism of infinite Fort spaces ($U$ and $H$ with particular point $b$). So $U \approx H$.

On the other hand if $\text{card}(U) = \text{card}(H) < \text{card}(X)$, then $\text{card}(X \setminus U) = \text{card}(X \setminus H) = \text{card}(X)$. Also if $U \cap \{b\} = H \cap \{b\}$, then $(X \setminus U) \cap \{b\} = (X \setminus H) \cap \{b\}$.

So if $\text{card}(U) = \text{card}(H) < \text{card}(X)$ and $U \cap \{b\} = H \cap \{b\}$, then $\text{card}(X \setminus U) = \text{card}(X \setminus H)$ and $(X \setminus U) \cap \{b\} = (X \setminus H) \cap \{b\}$ which shows $X \setminus U \approx X \setminus H$ by the above argument.

Use item (1) to complete the proof of (2). □

Note that if there exists a $C - (X, D, \lambda)$, Top-design of type $i$, then there exists $U \approx D$ with $C \subseteq U$, so $\text{card}(C) \leq \text{card}(U) = \text{card}(D)$. Therefore $\text{card}(C) = \text{min}(\text{card}(D), \text{card}(C)) \leq \text{card}(X)$.

**Theorem 2.2.** Regarding 1st type of Top-designs for nonempty subsets $C, D$ of $X$ we have:

a. suppose $b \notin C \cup D$:
   a1. if $C$ is finite, then there is not any $C - (X, D, \lambda)$, Top-design of type 1,
   a2. if $C$ is infinite and $\text{card}(C) < \text{card}(X)$, then there exist $\lambda$ and a $C - (X, D, \lambda)$ Top-design of type 1,
   a3. if $\text{card}(C) = \text{card}(D) = \text{card}(X)$, then there exist $\lambda$ and a $C - (X, D, \lambda)$ Top-design of type 1 if and only if $D = X \setminus \{b\}$,

b. if $b \in C \setminus D$, then there is not any $C - (X, D, \lambda)$, Top-design of type 1,

c. suppose $b \in D$:
   c1. for finite $C$ there exist $\lambda$ and a $C - (X, D, \lambda)$ Top-design of type 1 if and only if $\text{card}(C) + 2 \leq \text{card}(D)$,
c2. if $C$ is infinite and $\text{card}(C) = \min(\text{card}(D), \text{card}(C)) < \text{card}(X)$, then there exist $\lambda$ and a $C - (X, D, \lambda)$ Top–design of type 1,

c3. if $\text{card}(C) = \text{card}(D) = \text{card}(X)$, then there exist $\lambda$ and a $C - (X, D, \lambda)$ Top–design of type 1 if and only if $D = X$.

Proof. Let $\mathcal{W} = \{E \subseteq X : E \approx D \wedge X \setminus E \approx X \setminus D\}$. By item (1) in Lemma 2.1, it’s evident that $b \in D$ if and only if $b \in \bigcup \mathcal{W}$ (resp. $b \in \bigcap \mathcal{W}$).

a1) Choose $k \in C$, if $\mathcal{A}$ is a $C - (X, D, \lambda)$, Top–design of type 1, then $\mathcal{A} \subseteq \mathcal{W}$ and $\mathcal{A}$ is a $(C \setminus \{k\}) \cup \{b\} - (X, D, \lambda)$, Top–design of type 1 too, which is a contradiction since $b \notin \bigcup \mathcal{W}$.

a2) We have the following sub–cases:

• $\text{card}(C) \leq \text{card}(D) < \text{card}(X)$. In this case by item (2) in Lemma 2.1 we have $\mathcal{W} = \{E \subseteq X \setminus \{b\} : \text{card}(E) = \text{card}(D)\}$. Using $\text{card}(X \setminus \{b\}) = \text{card}(X \setminus (C \cup \{b\}))$ for $\mathcal{W}' = \{E \subseteq X \setminus (C \cup \{b\}) : \text{card}(E) = \text{card}(D)\}$ we have $\text{card}(\mathcal{W}) = \text{card}(\mathcal{W}')$. It’s evident that $\mathcal{W}' \rightarrow \mathcal{W}$ is one–to–one, so $\text{card}(\mathcal{W}') \leq \text{card}((F \in \mathcal{W} : C \subseteq F)) \leq \text{card}(\mathcal{W})$. Thus $\text{card}((F \in \mathcal{W} : C \subseteq F)) = \text{card}(\mathcal{W})$. Since $C$ is infinite and $b \notin C$, for all subset $E$ of $X$ with $C \approx E$ we have $\text{card}(C) = \text{card}(E)$ and $b \notin E$, so by a similar method described for $C$ we have $\text{card}((F \in \mathcal{W} : E \subseteq F)) = \text{card}(\mathcal{W})$. Hence $\mathcal{W}$ is a $C - (X, D, \text{card}(\mathcal{W}))$ Top–design of type 1.

• $\text{card}(C) < \text{card}(D) = \text{card}(X)$. In this case by Lemma 2.1, $\mathcal{W} = \{E \subseteq X \setminus \{b\} : \text{card}(E) = \text{card}(D) \wedge \text{card}(X \setminus E) = \text{card}(X \setminus D)\}$. Since $\text{card}(C) < \text{card}(D)$ and $C, D$ carry discrete topologies thus $C$ can be embedded in $D$ and without any loss of generality we may suppose $C \subseteq D$. By infiniteness of $D$, at least one of the sets $D \setminus C$ or $C$ is infinite and

$$\text{card}(C) < \text{card}(X) = \text{card}(D) = \text{card}(C) + \text{card}(D \setminus C) = \max(\text{card}(C), \text{card}(D \setminus C))$$

so we have $\text{max}(\text{card}(C), \text{card}(D \setminus C)) = \text{card}(D \setminus C) = \text{card}(X)$. Since $2\text{card}(D \setminus C) = \text{card}(D \setminus C)$, we may choose $H \subseteq D \setminus C$ with

$$\text{card}(H) = \text{card}(D \setminus C) \setminus H = \text{card}(D \setminus (C \cup H)) = \text{card}(D \setminus C) = \text{card}(X)$$.

Let $K = \{F \subseteq D \setminus (H \cup C) : \text{card}(F \cup \{b\}) = \text{card}(X \setminus D)\}$, and consider the following claim:

Claim. For $F \in K$ we have $C \subseteq X \setminus (F \cup \{b\}) \in \mathcal{W}$. Suppose $F \in K$, so $F \subseteq D \setminus (H \cup C)$ so $H \subseteq X \setminus (F \cup \{b\}) \subseteq X$ thus $\text{card}(X \setminus (F \cup \{b\})) = \text{card}(X) = \text{card}(D)$ and $\text{card}(X \setminus D) = \text{card}(F \cup \{b\}) = \text{card}(X \setminus (X \setminus (F \cup \{b\})))$, therefore $X \setminus (F \cup \{b\}) \in \mathcal{W}$. Also $F \subseteq D \setminus (H \cup C)$ and $b \notin C$ show $C \subseteq X \setminus (F \cup \{b\})$.

Therefore

$$\eta : K \rightarrow \{B \in \mathcal{W} : C \subseteq B\}$$

is well–defined and clearly one–to–one.

Thus $\text{card}(K) \leq \text{card}((\{B \in \mathcal{W} : C \subseteq B\}) \leq \text{card}(\mathcal{W})$, however using $\text{card}(D \setminus (H \cup C)) = \text{card}(X \setminus \{b\})$ we have:

$$\text{card}(\mathcal{W}) \leq \text{card}((\{E \subseteq X \setminus \{b\} : \text{card}(E) = \text{card}(X \setminus (D \cup \{b\}))\})$$

$$\text{card}(\mathcal{W}) \leq \text{card}((\{F \subseteq D \setminus (H \cup C) : \text{card}(F) = \text{card}(X \setminus (D \cup \{b\}))\})$$

$$\text{card}(K)$$
which leads to \( \text{card}(K) = \text{card}(\{ B \in \mathbb{W} : C \subseteq B \}) = \text{card}(\mathbb{W}) \).

For \( E \subseteq X \) with \( E \approx C \) (so \( b \notin E \), we have \( D' = (D \setminus C) \cup E \in \mathbb{W} \) and \( \mathbb{W} = \{ E \subseteq X \setminus \{ b \} : \text{card}(E) = \text{card}(D') \wedge \text{card}(E \setminus X) = \text{card}(X \setminus D') \} \). Using a similar method described above, we have \( \text{card}(\{ B \in \mathbb{W} : E \subseteq B \}) = \text{card}(\mathbb{W}) \), thus \( \mathbb{W} \) is a \( C - (X, D, \text{card}(\mathbb{W})) \) Top–design of type 1.

\( a_3 \) In this case if \( \mathcal{A} = C - (X, D, \lambda) \) Top–design of type 1, then there exists \( \mathbb{B} \in \mathcal{A} \) with \( X \setminus \{ b \} \subseteq B \) (since \( C \approx X \setminus \{ b \} \)), which leads to \( D = X \setminus \{ b \} \), and \( \mathbb{W} = \{ X \setminus \{ b \} \} \) is a \( C - (X, D, 1) \) Top–design of type 1.

\( b \) Use the fact that if \( b \notin D \), then for all \( B \subseteq X \) with \( D \approx B \) and \( X \setminus D \approx X \setminus B \) we have \( b \notin B \), and in particular \( C \not\subseteq B \).

\( c_3 \) First suppose \( \mathcal{A} = C - (X, D, \lambda) \) Top–design of type 1 and \( D \) is finite, then for all subsets \( H \) of \( X \) with \( \text{card}(H) = \text{card}(C) \), there exists \( B \in \mathcal{A} \) with \( H \subseteq B \), however we may assume \( b \notin H \), using \( b \in B \) we have \( \text{card}(H) \leq \text{card}(B \setminus \{ b \}) = \text{card}(D) - 1 \). Hence \( \text{card}(C) + 1 \leq \text{card}(D) \). If \( \text{card}(C) + 1 = \text{card}(D) \) then any subset of \( X \setminus \{ b \} \) with \( \text{card}(C) \) elements occurs in just one element of \( \mathcal{A} \) and \( \mathcal{A} = \{ S \cup \{ b \} : S \subseteq X \setminus \{ b \} \wedge \text{card}(S) = \text{card}(C) \} \) now choose a subset \( J \) of \( X \setminus \{ b \} \) with \( \text{card}(C) - 1 \) elements, then infinite elements of \( \mathcal{A} \) contain \( J \cup \{ b \} \approx C \) which is in contradiction with \( \lambda = 1 \), so \( \text{card}(C) + 1 < \text{card}(D) \) and \( \text{card}(C) + 2 \leq \text{card}(D) \).

In order to complete the proof, we have the following cases:

**Case 1.** \( X \) is uncountable and \( D \) is infinite. In this case choose infinite countable subset \( I \) of \( D \setminus \{ b \} \). By the proof of \((a_2)\) for

\[
\mathbb{W}_{-b} = \{ E \subseteq X : E \approx D \setminus \{ b \} \wedge X \setminus E \approx X \setminus (D \setminus \{ b \}) \}
\]

is a \( I - (X, D \setminus \{ b \}, \text{card}(\mathbb{W}_{-b})) \) Top–design of first type. We show \( \mathbb{W} \) is a \( C - (X, D, \text{card}(\mathbb{W})) \) Top–design of first type. Consider \( H \subseteq X \) with \( H \approx C \). There exists \( J \subseteq X \setminus \{ b \} \) with \( H \setminus \{ b \} \subseteq J \) and \( J \approx I \) so

\[
\text{card}(\mathbb{W}_{-b}) \geq \text{card}(\{ B \in \mathbb{W}_{-b} : H \setminus \{ b \} \subseteq B \}) \geq \text{card}(\{ B \in \mathbb{W}_{-b} : J \subseteq B \}) = \text{card}(\mathbb{W}_{-b})
\]

therefore \( \text{card}(\{ B \in \mathbb{W}_{-b} : H \setminus \{ b \} \subseteq B \}) = \text{card}(\mathbb{W}_{-b}) \). Considering bijection \( \eta : \mathbb{W}_{-b} \to \mathbb{W} \), and \( b \in \bigcap \mathbb{W} \) we have \( \text{card}(\{ B \in \mathbb{W} : H \setminus \{ b \} \subseteq B \}) = \text{card}(\{ B \in \mathbb{W} : H \setminus \{ b \} \subseteq B \}) \).

**Case 2.** \( X, D \) and \( X \setminus D \) are infinite countable. In this case we may suppose \( X \setminus \{ b \} = \{ p_n : n \geq 1 \} \) and \( D = \{ p_{2n} : n \geq 1 \} \cup \{ b \} \) with distinct \( p_n \)'s. Let \( \mathcal{A} = \{ X \setminus \{ p_{2k+1} : k \geq s \} : s \geq 1 \} \), then \( \mathcal{A} = C - (X, D, \mathbb{N}) \) Top–design of type 1.

**Case 3.** \( X \) and \( D \) are infinite countable and \( X \setminus D \neq \emptyset \) is finite. In this case \( \mathbb{W} \) is infinite countable and \( C - (X, D, \mathbb{N}) \) Top–design of type 1.

**Case 4.** \( X = D \) is infinite countable. In this case \( \mathbb{W} = \{ X \} \) is a \( C - (X, D, 1) \) Top–design of type 1.

**Case 5.** \( D \) is finite and \( \text{card}(C) + 2 \leq \text{card}(D) \). In this case \( \text{card}(\mathbb{W}) = \text{card}(X) \) (since for infinite set \( X \) we have \( \text{card}(X) = \text{card}(P_{\text{fin}}(X)) \), where \( P_{\text{fin}}(X) \) is the collection of all finite subsets of \( X \)) and \( \mathbb{W} \) is a \( C - (X, D, \text{card}(X)) \) Top–design of type 1.

\( c_2 \) In this case by the proof of \((a_2)\), \( \mathbb{W} \) is a \( C \setminus \{ b \} - (X, D \setminus \{ b \}, \text{card}(\mathbb{W})) \) Top–design of type 1, using \( b \in \bigcap \mathbb{W} \), shows that \( \mathbb{W} \) is a \( C - (X, D, \text{card}(\mathbb{W})) \) Top–design of type 1 too.

\( c_3 \) Use a similar method described in the proof of \((a_3)\).

\( \square \)
Lemma 2.3. For nonempty subsets $C, D$ of $X$, $C$ can be embedded into $D$ if and only if
\[ \text{“} C \text{ is finite or } b \notin C \setminus D \text{”, and “} \text{card}(C) \leq \text{card}(D) \text{”}. \]

Proof. Suppose $C$ can be embedded in $D$ and choose $E \subseteq D$ with $E \approx C$, so $\text{card}(C) = \text{card}(E) \leq \text{card}(D)$. If $C$ is infinite and $b \in C$ then any subset of $X$ homeomorphic with $C$ contains $b$, thus $b \in E(\subseteq D)$ and $b \notin C \setminus D$. \hfill \Box

Theorem 2.4. For nonempty subsets $C, D$ of $X$, there exist $\lambda$ and a $C \setminus (X, D, \lambda)$, Top–design of type 2 if and only if $C$ can be embedded into $D$.

Proof. If we can not embed $C$ into $D$ it’s evident that there is not any $C \setminus (X, D, \lambda)$, Top–design of type 2.

Conversely suppose $C$ can be embedded in $D$, so by Lemma 2.3 $\text{card}(C) \leq \text{card}(D)$ and “$C$ is finite or $b \notin C \setminus D$”. Let $L = \{E \subseteq X : E \approx D\}$. We have the following cases:

- $\text{card}(C) \leq \text{card}(D)$ and $C$ is finite. In this case $L$ is a $C \setminus (X, D, \lambda)$ Top–design of type 2 with:

\[ \lambda = \begin{cases} 1 & \text{card}(C) = \text{card}(D), \\ \text{card}(\{E \subseteq X : \text{card}(E) = \text{card}(D)\}) & \text{otherwise}. \end{cases} \]

For this aim use the fact that $\eta : \{E \subseteq X \setminus C : \text{card}(E) = \text{card}(D)\} \rightarrow \{E \subseteq X : \text{card}(E) = \text{card}(D)\}$ with $\eta(E) = E \cup C$ is bijective.

- $\text{card}(C) = \min(\text{card}(C), \text{card}(D)) < \text{card}(X)$ and $b \notin C \setminus D$. In this case by Theorem 2.2 there exists $\lambda$ and $C \setminus (X, D, \lambda)$ Top–design of type 1, so it is a $C \setminus (X, D, \lambda)$ Top–design of type 2 too.

- $\text{card}(C) = \min(\text{card}(C), \text{card}(D)) = \text{card}(X)$ and $b \notin C \setminus D$. In this case $\lambda = \{(X \setminus \{b\}) \cup (D \cap \{b\})\}$ is a $C \setminus (X, D, 1)$ Top–design of type 2. \hfill \Box

Theorem 2.5. Regarding 3rd type of Top–designs for nonempty subsets $C, D$ of $X$, there exist $\lambda$ and a $C \setminus (X, D, \lambda)$, Top–design of type 3 if and only if $b \notin C \setminus D$, $\text{card}(C \setminus \{b\}) \leq \text{card}(D \setminus \{b\})$ and $\text{card}(X \setminus (D \cup \{b\})) \leq \text{card}(X \setminus (C \cup \{b\}))$.

Proof. Let $\mathcal{W} = \{E \subseteq X : E \approx D \land X \setminus E \approx X \setminus D\}$. If $\mathcal{A}$ is a $C \setminus (X, D, \lambda)$, Top–design of type 3, then $\mathcal{A} \subseteq \mathcal{W}$ and we have the following cases:

- Case 1. $b \in C \setminus D$. In this case for all $E \in \mathcal{A}(\subseteq \mathcal{W})$, we have $b \notin E$ and $C \not\subseteq E$ thus $\mathcal{A}$ is not a $C \setminus (X, D, \lambda)$, Top–design of type 3.

- Case 2. $\text{card}(C \setminus \{b\}) > \text{card}(D \setminus \{b\})$, and “$b \in C \cap D$ or $b \notin C \cup D$”. In this case we have $\text{card}(C) > \text{card}(D)$ so we can not embed $C$ into $D$ and it’s evident that there is not any $C \setminus (X, D, \lambda)$, Top–design of type 3.

- Case 3. $\text{card}(C \setminus \{b\}) > \text{card}(D \setminus \{b\})$, $b \in D \setminus C$. In this case for all $B \in \mathcal{A}$, $b \in B$ and $\text{card}(C) = \text{card}(C \setminus \{b\}) > \text{card}(D \setminus \{b\}) = \text{card}(B \setminus \{b\})$ so $C \not\subseteq B \setminus \{b\}$ and $\mathcal{A}$ is not a $C \setminus (X, D, \lambda)$, Top–design of type 3.

- Case 4. $\text{card}(X \setminus (D \cup \{b\})) > \text{card}(X \setminus (C \cup \{b\}))$. In this case for all $E \in \mathcal{W}$ we have $\text{card}(X \setminus (D \cup \{b\})) > \text{card}(X \setminus (C \cup \{b\}))$, thus $X \setminus (E \cup \{b\}) \not\subseteq X \setminus (C \cup \{b\})$ and $C \not\subseteq E$, so there is not any $C \setminus (X, D, \lambda)$, Top–design of type 3.

Considering the above cases $b \notin C \setminus D$, $\text{card}(C \setminus \{b\}) \leq \text{card}(D \setminus \{b\})$ and $\text{card}(X \setminus (D \cup \{b\})) \leq \text{card}(X \setminus (C \cup \{b\}))$. Conversely, suppose $b \notin C \setminus D$, $\text{card}(C \setminus \{b\}) \leq \text{card}(D \setminus \{b\})$ and $\text{card}(X \setminus (D \cup \{b\})) \leq \text{card}(X \setminus (C \cup \{b\}))$, then $\mathcal{W}$ is a $C \setminus (X, D, \lambda)$, Top–design of type 3 for
\[ \lambda = \text{card}\{E \in \mathcal{W} : C \subseteq E\} \] (note that for \(F \subseteq X\) with \(F \approx C\) and \(X \setminus F \approx X \setminus C\), the map \(\{E \in \mathcal{W} : F \subseteq E\} \to \{E \in \mathcal{W} : C \subseteq E\}\) is bijective). \(\Box\)

**Theorem 2.6.** For nonempty subsets \(C, D\) of \(X\), there exist \(\lambda\) and a \(C-(X, D, \lambda)\), Top–design of type 4 if and only if \(C\) can be embedded into \(D\).

**Proof.** If \(C\) can be embedded into \(D\), then there exist \(\lambda > 0\) and a \(C-(X, D, \lambda)\) Top–design of type 2 like \(\mathbb{A}\) by Theorem 2.4, so \(\mathbb{A}\) is a \(C-(X, D, \lambda)\) Top–design of type 4 too.

Conversely, it’s evident that if \(\mathbb{A}\) is a Top–design of type \(i\) (for \(i = 1, 2, 3, 4\)), then there exists \(E \in \mathbb{A}\) with \(C \subseteq E\), using \(E \approx D\) leads us to the fact that \(C\) can be embedded into \(D\). \(\Box\)

**Theorem 2.7.** For nonempty subsets \(C, D\) of \(X\) the following statements are equivalent:

- there is not any \(C-(X, D, \lambda)\), Top–design of type 2,
- there is not any \(C-(X, D, \lambda)\), Top–design of type 4,
- “\(C\) is infinite and \(b \in C \setminus D\)”, or “\(\text{card}(C) > \text{card}(D)\)”
- \(C\) can not be embedded into \(D\).

**Proof.** Theorems 2.4, 2.6 and Lemma 2.3. \(\Box\)

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**References**

[1] I. Anderson, *Combinatorial designs: construction methods*, Ellis Horwood Ltd., 1990.
[2] F. Ayatollah Zadeh Shirazi and M. Bagherian, Generalized-designs, *Missouri Journal of Mathematical Sciences*, 23, no. 1 (2011), 27–47.
[3] P. J. Cameron, and B. S. Webb, What is an infinite design?, *Journal of Combinatorial Designs*, 10:2 (2002), 79–91.
[4] L. A. Steen and J. A. Seebach Jr., *Counterexamples in topology*, New York-Montreal, Holt, Rinehart and Winston Inc., 1970.

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