CANNON-THURSTON MAPS FOR COXETER GROUPS
INCLUDING AFFINE SPECIAL SUBGROUPS

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Abstract. For a Coxeter group $W$ we have an associating bi-linear form $B$ on a real vector space. We assume that $B$ has the signature $(n - 1,1)$. In this case we have the Cannon-Thurston map for $W$, that is, a $W$-equivariant continuous surjection from the Gromov boundary of $W$ to the limit set of $W$. We focus on the case where Coxeter groups contain affine special subgroups.

1. Introduction

Let $(X,d_X)$ and $(Y,d_Y)$ be metric spaces equipped with an action of a countable group $G$ respectively. A map $f : X \to Y$ is called $G$-equivariant if $f$ satisfies

$$g \circ f(x) = f \circ g(x)$$

holds for all $x \in X$ and for all $g \in G$.

A continuous equivariant map between the boundary at infinity of a discrete group and its limit set is called a Cannon-Thurston map. Mitra ([18]) considered Cannon-Thurston maps for Gromov hyperbolic groups. Let $H$ be a hyperbolic subgroup of a hyperbolic group $G$ in the sense of Gromov. He asked whether the inclusion map always extends continuously to the equivariant map between the Gromov compactifications $\hat{H}$ and $\hat{G}$. Here word metrics on $H$ and $G$ defining their compactifications may differ. For this question he positively answered in the case when $H$ is an infinite normal subgroup of a hyperbolic group $G$. He also proved that the existence of the Cannon-Thurston map when $G$ is a hyperbolic group acting cocompactly on a simplicial tree $T$ such that all vertex and edge stabilizers are hyperbolic, and $H$ is the stabilizer of a vertex or edge of $T$ provided every inclusion of an edge stabilizer in a vertex stabilizer is a quasi isometric embedding ([19]). On the other hand, Baker and Riley constructed a negative example for Mitra’s question. In fact they proved that there exists a free subgroup of rank 3 in a hyperbolic group such that the Cannon-Thurston map is not well-defined ([1]). Adding to this Matsuda and Oguni showed that a similar phenomenon occurs for every non-elementary relatively hyperbolic group ([15]).

Inspired by the above results we shall consider the problem which asks whether the Cannon-Thurston map for the Coxeter groups exists. In [17] the author confirmed that the existence of Cannon-Thurston maps for Coxeter groups whose associating bilinear form has the signature $(n - 1,1)$. However the result is restricted to the case where Coxeter groups have no affine Coxeter special subgroups of rank 2010 Mathematics Subject Classification. Primary 20F55; Secondary 51F15.

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more than 3. Here a subgroup \((W', S')\) of a Coxeter group \((W, S)\) is special if \(S' \subset S\). The author verified that there exists the Cannon-Thurston map between the Gromov boundary of such a Coxeter group to the limit set. In this paper we consider the case where Coxeter groups including affine Coxeter special subgroups.

The main theorem is stated as follows.

**Theorem 1.1.** Let \(W\) be a rank \(n\) Coxeter groups whose associating bi-linear form \(B\) has the signature \((n-1, 1)\). Let \(\partial_G W\) be the Gromov boundary of \(W\) and let \(\Lambda(W)\) be the limit set of \(W\). There exists a \(W\)-equivariant, continuous surjection \(F: \partial_G W \to \Lambda(W)\).

We remark that the Gromov boundary is ordinary defined on a hyperbolic metric space. We extend the definition to arbitrary metric space by taking transitive closure due to Buckley and Kokkendorff ([4]).

**Corollary 1.2.** Let \((W, S)\) be a Coxeter system of rank \(n\) whose associated bi-linear form has the signature \((n-1, 1)\). For a special subgroup \(W'\) whose associated bi-linear form has the signature \((n-1, 1)\), if the normalized action (see §2) of \(W'\) is cocompact, then the limit set \(\Lambda(W')\) of \(W'\) is canonically embedded into the limit set of \(\Lambda(W)\).

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2. The Coxeter systems and \(B\)-reflections

2.1. **The Coxeter systems.** A group \(W\) is a Coxeter group of rank \(n\) with the generating set \(S\) if \(W\) is generated by the set \(S = \{s_1, \ldots, s_n\}\) subject only to the relations \((s_i s_j)^{m_{ij}} = 1\), where \(m_{ij} \in \mathbb{Z}_{\geq 1} \cup \{\infty\}\) for \(1 \leq i < j \leq n\) and \(m_{ii} = 1\) for \(1 \leq i \leq n\), i.e.,

\[
W = \langle s_1, \ldots, s_n \mid (s_i s_j)^{m_{ij}} \text{ for } i,j = 1, \ldots, n \rangle.
\]

A pair \((W, S)\) is said to be a Coxeter system. We refer the reader to [2, 7, 13] for the introduction to Coxeter groups.

For a Coxeter system \((W, S)\) of rank \(n\), let \(V\) be an \(\mathbb{R}\) vector space with its orthonormal basis \(\Delta = \{\alpha_s \mid s \in S\}\) with respect to the Euclidean inner product. The set \(\Delta\) is called a simple system and its elements are simple roots of \(W\). Note that by identifying \(V\) with \(\mathbb{R}^n\), we treat \(V\) as a Euclidean space. We define a symmetric bilinear form on \(V\) by setting

\[
B(\alpha_i, \alpha_j) = \begin{cases} 
-\cos \left( \frac{\pi}{m_{ij}} \right) & \text{if } m_{ij} < \infty, \\
-1 & \text{if } m_{ij} = \infty
\end{cases}
\]

for \(1 \leq i \leq j \leq n\), where \(\alpha_{s_i} = \alpha_i\). This definition of the bi-linear form is introduced in [10] to consider the based root system.

Given \(\alpha \in V\) such that \(B(\alpha, \alpha) \neq 0\), \(s_\alpha\) denotes the map \(s_\alpha : V \to V\) by

\[
s_\alpha(v) = v - 2 \frac{B(\alpha, v)}{B(\alpha, \alpha)} \alpha \quad \text{for any } v \in V,
\]
which is said to be a $B$-reflection. Since $B(\alpha, \alpha) = 1$ for $\alpha \in \Delta$, we notice that $B$-reflections preserve $B$: for any $v, u \in V$ and any $B$-reflection $s_\alpha$, we have $B(s_\alpha(v), s_\alpha(u)) = B(v, u)$ if $\alpha \in \Delta$. Thus $W$ acts on $V$ as orthonormal transformations with respect to $B$.

**Assumption 2.1.** In this paper, we always assume the following.

- The bilinear form $B$ has the signature $(n - 1, 1)$. We call such a group a Coxeter group of type $(n - 1, 1)$.
- The Coxeter matrix $B$ is not block-diagonal up to permutation of the basis. In that case, the matrix $B$ is said to be irreducible.

We only need to work on the case that $B$ is irreducible. If the matrix $B$ is reducible, then we can divide $\Delta$ into $l$ subsets $\Delta = \bigsqcup_{i=1}^{l} \Delta_i$ so that each corresponding matrix $B_i = \{B(\alpha, \beta)\}_{\alpha, \beta \in \Delta_i}$ is irreducible. Then for any distinct $i, j$, if $\alpha \in \Delta_i$ and $\beta \in \Delta_j$, $s_\alpha$ and $s_\beta$ commute. In this case we see that $W$ is a direct product

$$W = W_1 \times W_2 \times \cdots \times W_l,$$

where $W_i$ is the Coxeter group corresponding to $\Delta_i$. From this, the action of $W$ can be regarded as direct product of the actions of each $W_i$ then we see that $E = \bigsqcup_{i=1}^{l} E_i$, where $E_i$ is the set of accumulation points of roots $W_i \cdot \Delta_i$ (see Proposition 2.14 in [10]). Moreover if $B$ has the signature $(n - 1, 1)$, there exists a unique $B_k$ which has signature $(m_k - 1, 1)$ and others are positive definite. Since if the Coxeter matrix is positive definite then corresponding Coxeter group $W'$ is finite, the limit set $\Lambda(W') = \emptyset$. This ensures that $\Lambda(W)$ is distributed on $\text{conv}(\hat{\Delta}_k)$, where $\text{conv}(\hat{\Delta}_k)$ is the convex hull of $\hat{\Delta}_k$ (see §2.2 for the definition of $\hat{\cap}$). Thus $\Lambda(W)$ can be identified with $\Lambda(W_k)$. Accordingly, if there exists the Cannon-Thurston map for $W_k$ then we also have the Cannon-Thurston map for the whole group $W$. This follows from the fact that the direct product $G_1 \times G_2$ of a finite generated infinite group $G_1$ and a finite group $G_2$ has the same Gromov boundary as the Gromov boundary of $G_1$.

### 2.2. $B$-reflections and the normalized actions of $W$.

Recall that a matrix $A$ is non-negative if each entry of $A$ is non-negative.

**Lemma 2.2.** Let $o$ be an eigenvector for the negative eigenvalue of $B$. Then all coordinates of $o$ have the same sign.

**Proof.** This follows from Perron-Frobenius theorem for irreducible non-negative matrices. Let $I$ be the identity matrix of rank $n$. Then $-B + I$ is irreducible and non-negative. Note that since $-B + I$ and $B$ are symmetric, all eigenvalues are real. By Perron-Frobenius theorem, we have a positive eigenvalue $\lambda'$ of $-B + I$ such that $\lambda'$ is the maximum of eigenvalues of $-B + I$ and each entry of corresponding eigenvector $u$ is positive. On the other hand, for each eigenvalue $a$ of $B$ there exists an eigenvalue $b$ of $-B + I$ such that $a = 1 - b$. Let $\lambda$ be the negative eigenvalue of $B$. Then an easy calculation gives $\lambda = 1 - \lambda'$. Therefore $Ru = Ro$. \hfill $\square$

We fix $o \in V$ be the eigenvector corresponding to the negative eigenvalue of $B$ whose euclidean norm equals to 1 and all coordinates are positive. Hence if we write $o$ in a linear combination $o = \sum_{i=1}^{n} a_i o_i$ of $\Delta$ then $a_i > 0$. Given $v \in V$, we define $|v|$ by $\sum_{i=1}^{n} a_i v_i$ if $v = \sum_{i=1}^{n} v_i o_i$. Note that a function $| \cdot |_1 : V \rightarrow \mathbb{R}$ is actually a norm in the set of vectors having nonnegative coefficients. Let $V_i = \{v \in V : |v|_1 = i\}$, where $i = 0, 1$. For $v \in V \setminus V_0$, we write $\hat{v}$ for the “normalized”
vector $\frac{w}{|w|} \in V_1$. We also call $o$ the normalized eigenvector (corresponding to the negative eigenvalue of $B$). Also for a subset $A \subset V \setminus V_0$, we write $\hat{A} = \{ \hat{a} \mid a \in A \}$. We notice that $B(x, \alpha) = [\alpha]_1 B(x, \hat{\alpha})$ hence the sign of $B(x, \alpha)$ equals to the sign of $B(x, \hat{\alpha})$ for any $x \in V$ and $\alpha \in \Delta$.

We denote by $q(v) = B(v, v)$ for $v \in V$. Let $Q = \{ v \in V \mid q(v) = 0 \}$, $Q_- = \{ v \in V \mid q(v) < 0 \}$ then we have

$$\hat{Q} = V_1 \cap Q, \quad \hat{Q}_- = V_1 \cap Q_-. $$

We see that $\hat{Q}$ is an ellipsoid since $B$ has the signature $(n-1, 1)$.

**Remark 2.3.** We have

$$W(V_0) \cap Q = \{ 0 \},$$

where $0$ is the origin of $\mathbb{R}^n$. To see this we only need to verify that $V_0 \cap Q = \{ 0 \}$ since $Q$ is invariant under $B$-reflections. We notice that $V_0 = \{ v \in V \mid B(v, o) = 0 \}$. For $i = 1, \ldots, n-1$, let $p_i$ be an eigenvector of $B$ corresponding to a positive eigenvalue $\lambda_i$. For any $v \in V_0$, we can express $v$ in a linear combination $v = \sum_{i=1}^{n-1} \lambda_i v_i^2 \| p_i \|^2 \geq 0$ where $\| * \|$ denotes the euclidean norm. Since $\lambda_i > 0$ for $i = 1, \ldots, n-1$, we have $B(v, v) = 0$ if and only if $v = 0$.

For $v \in V \setminus V_0$, we define $\hat{v} = \frac{v}{|v|}$. Then the normalized action of $w \in W$ is given by

$$w \cdot v := \hat{w}(v), \quad v \in V \setminus W(V_0).$$

The normalized action preserves the region

$$D := \hat{Q}_- = V_1 \cap Q_-.$$

This is an open set in $V_1$ with the subspace topology and $\partial D = \hat{Q}$. Then by Remark 2.3, the normalized action is a continuous action on $D$. The region $D$ may protrude from the convex hull conv($\hat{\Delta}$) of $\hat{\Delta}$. In that case letting $R := D \setminus \text{conv}(\hat{\Delta})$, we can restrict the normalized action on

$$D' := D \setminus \bigcup_{w \in W} w \cdot R,$$

since $\bigcup_{w \in W} w \cdot R$ is invariant under the normalized action of $W$.

**Notation 2.4.** Our main purpose in this paper is to study the limit set of $W$ for the normalized action. Therefore we work in $V_1$ with the subspace topology unless otherwise indicated. For a subset $A \subset V_1$, int($A$), $\overline{A}$ and conv($A$) denote the interior, the closure and the convex hull of $A$ each other with respect to the subspace topology on $V_1$.

### 2.3. The word metric.

Let $G$ be a finitely generated group and fix a generating set $S$. Then all elements in $G$ can be represented by a product of elements in $S$. We say such a representation to be a word and let $\langle S \rangle$ be the set of all words generated by $S$. For a word $w \in \langle S \rangle$ we define the word length $\ell_S(w)$ as the number of generators $s \in S$ in $w$. We denote the minimal word length of words representing $g \in G$ by $|g|_S$. An expression of $g$ realizing $|g|_S$ is called the reduced expression or the geodesic word. Using the word length, we can define so-called the word metric with respect to $S$ on $G$, i.e, for $g, h \in G$, their distance is $|g^{-1}h|_S$. In this paper for a Coxeter system $(W, S)$ we always work on the generating set $S$. For this reason
we omit the subscript and denote the word length (resp. the minimal word length) for \( S \) simply by \( \ell \) (resp. \(|\ast|\)).

The word metric on a group \( G \) with respect to a generating set \( S \) can be regarded as a metric on the Cayley graph of \( G \) with respect to \( S \).

3. The Hilbert Metric

For four vectors \( a, b, c, d \in V \) with \( c - d, b - a \neq 0 \), we define the cross ratio \([a, b, c, d]\) with respect to \( B \) by

\[
[a, b, c, d] := \frac{\|y - a\| \|x - b\|}{\|y - b\| \|x - a\|},
\]

where \( \|\ast\| \) denotes the Euclidean norm. Using this we obtain a distance \( d \) on \( D \) as follows. For any \( x, y \in D \), take \( a, b \in \partial D \) so that the points \( a, x, y, b \) lie on the segment connecting \( a, b \) in this order. Then \( y - b, x - a \neq 0 \). We define

\[
d(x, y) := \log[a, x, y, b],
\]

and call this the Hilbert metric on \( D \).

We collect some geometric properties of \((D, d)\). Since \( D \) is an ellipsoid, it is well-known that \((D, d)\) is a proper (i.e. every closed ball is compact) and complete (uniquely) geodesic space (for example see [17]). More precisely geodesics in this space are Euclidean lines. Furthermore we see that \((D, d)\) is isometric to the hyperbolic space. To prove this we need following two lemmas.

Lemma 3.1. Any linear transformation \( L : V \to V \) preserves the cross ratio \([a, x, y, b]\) for any \( \{a, x, y, b\} \) which are collinear in this order (namely \( x, y \) are on the segment connecting \( a \) and \( b \)) and \( a, b \in \partial D \), \( x, y \in D \).

Proof. For any linear transformation \( L \) and any metric space \((X, d_X)\) with the Hilbert metric in \( \mathbb{R}^n \) we can define the Hilbert metric \( d_{L(X)} \) in \( L(X) \) since \( L(X) \) is bounded and convex. Then we have \( \|y - b\| \leq \|x - b\|, \|x - a\| \leq \|y - a\| \) with respect to the Euclidean norm \( \|\ast\| \). From the collinearity, each pair \( \{y - b, x - b\} \) and \( \{x - a, y - a\} \) have same direction respectively. So there exist constants \( k, l \geq 1 \) such that \( x - b = k(y - b) \) and \( y - a = l(x - a) \). Since any linear transformation sends lines to lines, we have

\[
[L(a), L(x), L(y), L(b)] = \frac{\|L(y - a)\| \|L(x - b)\|}{\|L(y - b)\| \|L(x - a)\|} = l\frac{\|L(x - a)\| \|L(y - b)\|}{\|L(y - b)\| \|L(x - a)\|} = lk = l\frac{\|x - a\| \|y - b\|}{\|y - b\| \|x - a\|} = \frac{\|y - a\| \|x - b\|}{\|y - b\| \|x - a\|} = [a, x, y, b].
\]

\[\Box\]

Lemma 3.2. Let \( a_1, a_2, a_3, a_4 \) be points in \( V \) which are collinear and \( a_1 - a_4 \neq 0 \). Let \( b_1, b_2, b_3, b_4 \in V \) satisfying

- for each \( i \), \( b_i \) lies on a ray \( R_i \) connecting \( a_i \) and some point \( p \in V \),

- \( b_1, b_2, b_3, b_4 \) are non-collinear.

...
four vectors $b_1, b_2, b_3, b_4$ are collinear and $b_1 - b_4 \neq 0$.

Then we have

$$[a_1, a_2, a_3, a_4] = [b_1, b_2, b_3, b_4].$$

**Proof.** From the assumption, all eight points are located on the two dimensional subspace $\text{Span}(a_1 - p, a_4 - p)$ spanned by $a_1 - p$ and $a_4 - p$ in $V$.

Let $\ell_0$ be a line in $\text{Span}(a_1 - p, a_4 - p)$ through $a_1$ and $a_4$. Consider two lines $\ell_2$ and $\ell_3$ in $\text{Span}(a_1 - p, a_4 - p)$ parallel to $\ell_0$ with $b_2 \in \ell_2$ and $b_3 \in \ell_3$. Let $B_i \in R_i \cap \ell_2$ and $B'_i \in R_i \cap \ell_3$ for $i = 1, 2, 3, 4$. Then we have $b_2 = B_2$ and $b_3 = B'_3$, and there is a positive constant $k$ such that $B'_i - p = k(B_i - p)$ for $i = 1, 2, 3, 4$. Since two triangles with vertices $\{b_4, b_2, B_4\}$ and $\{b_4, b_3, B'_4\}$ are similar,

$$\frac{\|b_2 - b_4\|}{\|b_3 - b_4\|} = \frac{\|B_2 - B_4\|}{\|B'_3 - B'_4\|}$$

By the similar reason, we also have

$$\frac{\|b_2 - b_1\|}{\|b_3 - b_1\|} = \frac{\|B_2 - B_1\|}{\|B'_3 - B'_4\|}$$

In addition since $\ell_0$ and $\ell_2$ are parallel, there exists a constant $m$ so that $B_i - B_j = m(a_i - a_j)$, for all $i, j \in \{1, 2, 3, 4\}$. Therefore, we obtain

$$[b_1, b_2, b_3, b_4] = \frac{\|b_3 - b_1\|\|b_2 - b_4\|}{\|b_3 - b_4\|\|b_2 - b_1\|} = \frac{\|B'_3 - B'_1\|\|B_2 - B_4\|}{\|B'_3 - B'_1\|\|B_2 - B_1\|}$$

$$= \frac{\|k(B_3 - B_1)\|\|B_2 - B_4\|}{\|k(B_3 - B_1)\|\|B_2 - B_1\|} = \frac{\|B_3 - B_1\|\|B_2 - B_4\|}{\|B_3 - B_1\|\|B_2 - B_1\|}$$

$$= \frac{\|a_3 - a_1\|\|a_2 - a_4\|}{\|a_3 - a_4\|\|a_2 - a_1\|} = [a_1, a_2, a_3, a_4],$$

which implies what we wanted. \(\square\)

**Lemma 3.3.** The metric space $(D, d)$ is isometric to $\mathbb{H}^{n-1}$.

**Proof.** Let $A$ be a diagonal matrix $(1, \ldots, 1, -1)$. We consider the region

$$\tilde{D} = \{v \in V \mid A(v, v) < 0, v_n = 1\}$$

where $A(,)$ is the bilinear form defined by $A$ and $v_n$ is the $n$-th coordinate of $v$. The region $\tilde{D}$ can be regard as a hyperbolic space of the projective model which is equipped with the Hilbert metric $d_A$ defined in the same way as $d$.

Since $B$ is a symmetric bi-linear form, we can diagonalize the Gram matrix $(B(\lambda_i, \alpha_j))_{i,j}$ by an orthogonal matrix $L$. Let $\{\lambda_1, \ldots, \lambda_{n-1}, -\lambda_n\} (\lambda_i > 0)$ be the eigenvalues of $B$ and let $L'$ be a diagonal matrix $(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})$. Then we have $^t(LL')B(LL') = A$ where $^tM$ denotes the transpose of a matrix $M$. We set a linear transformation $g(v) = L'^{-1}L^{-1}v$ for $v \in V$. Then we notice that $g(D) \subset \{v \in V \mid 0 < v_n, A(v, v) < 0\}$. We define a projection $p : g(D) \to \tilde{D}$ by $v \mapsto v/v_n$. Obviously $p \circ g$ is a bijection. Take any $x, y \in D$ and $a, b \in \partial D$ such that $d_D(x, y) = [a, x, y, b]$. Then by Lemma 3.1 we have $[a, x, y, b] = [g(a), g(x), g(y), g(b)]$. Adding to this by Lemma 3.2 we have $[g(a), g(x), g(y), g(b)] = [p \circ g(a), p \circ g(x), p \circ g(y), p \circ g(b)]$. Thus $p \circ g$ gives an isometry from $(D, d)$ to $(\tilde{D}, d_A)$.

\(\square\)
By this lemma we see that the space $D$ is a proper (complete) CAT(0) space (see [3, PART II.]). Thus its CAT(0) boundary $\partial D$ (see [3, II.8.1]) is homeomorphic to the sphere $S^{n-2}$, hence $\partial D \simeq \partial D$ where $\partial D$ is the Euclidean boundary of $D$.

4. Isometric action on $(D, d)$

4.1. $W$ acts isometrically. We define two open sets (with respect to subspace topology of $V_1$)

$$K := \{ v \in D \mid \forall \alpha \in \Delta, B(\alpha, v) < 0 \} \quad \text{and} \quad K' := K \cap \text{int}(\text{conv}(\hat{\Delta})).$$

For $\alpha \in \Delta$ we set $P_\alpha = \{ v \in V_1 \mid \alpha\text{-th coordinate of } v \text{ is } 0 \}$ and $H_\alpha = \{ v \in V_1 \mid B(v, \alpha) = 0 \}$. We define

$$\mathcal{P} = \{ v \in V_1 \mid \forall \alpha \in \Delta, B(\alpha, v) < 0 \} \quad \text{and} \quad \mathcal{P}' = \mathcal{P} \cap \text{int}(\text{conv}(\hat{\Delta})).$$

Since $\mathcal{P}$ (resp. $\mathcal{P}'$) is bounded by finitely many $n-1$ dimensional subspaces $\{H_\alpha \mid \alpha \in \Delta\}$ (resp. $\{H_\alpha \mid \alpha \in \Delta\}$ and $\{P_\alpha \mid \alpha \in \Delta\}$), actually $\mathcal{P}$ (resp. $\mathcal{P}'$) is a polyhedron. In general, $\mathcal{P}$ is not a simplex.

**Remark 4.1 ([17]).** We have the following:

- $K = \mathcal{P} \cap D$ and $K' = K \cap \text{int}(\text{conv}(\hat{\Delta})) = \mathcal{P}' \cap D$.
- $K'$ (hence $K$) is not empty.

**Definition 4.2.** We assume that a group $G$ acts on a metric space $X$ isometrically. We denote the action $G \acts X$ by $g.x$ for $g \in G$ and $x \in X$.

- An open set $A \subset X$ is a fundamental region if $\overline{G.A} = X$ and $g.A \cap A = \emptyset$ for any $g \in G$ where $\overline{G.A}$ is topological closure of $G.A$.
- An open set $A \subset X$ is the Dirichlet region at $o \in A$ if $A$ equals to the set
  $$\{ x \in D \mid d(o, x) < d(o, w \cdot x) \text{ for } w \in W \setminus \{id\} \}.$$  
  - The action $G \acts X$ is discrete or properly discontinuous if for any compact set $K$ the set
    $$\{ g \in G \mid g(K) \cap K \neq \emptyset \}$$
    is finite.

In [17], following propositions are proved ([17, Lemma 4.11]).

**Proposition 4.3.** Let $W$ be a Coxeter group of type $(n-1,1)$. The normalized action of $W$ on $D$ is isometric for the Hilbert metric $d$ and properly discontinuous.

**Lemma 4.4.** For any $x \in K$, $K$ is the Dirichlet region at $x$ and a fundamental region.

We say a rank $n$ Coxeter system $(W, S)$ is affine if its associating bi-linear form $B$ with respect to $S$ has the signature $(n-1, 0)$. Fixing a generating set $S$ we simply say a Coxeter group $W$ is affine if $(W, S)$ is affine. Affine Coxeter groups have infinite order and whose set of accumulation points of normalized roots is a singleton ([10, Corollary 2.15]). We notice that if the rank $> 2$ then there is no simple roots $\alpha, \beta \in \Delta$ with $B(\alpha, \beta) \leq -1$ for any affine Coxeter group. In fact if $B(\alpha, \beta) \leq -1$ then the reflection subgroup generated by $s_\alpha, s_\beta$ has infinite order hence $E \cap \text{conv}(\{\alpha, \beta\}) \neq \emptyset$. The set $\text{conv}(\{\alpha, \beta\})$ is an edge of $\text{conv}(\hat{\Delta})$. On the other hand, it is well-known that any Coxeter element of an infinite Coxeter group, that is an element given by multiplying all simple $B$-reflections, has infinite order. Then an accumulation
point given by multiplying a Coxeter element infinitely many times should lie on \( \text{int}(\text{conv}(\hat{\Delta})) \) provided that \( B \) is irreducible. This is a contradiction.

4.2. Three cases. We recall that \( \text{conv}(\hat{\Delta}) \) is a simplex and \( \partial D \cup D \) is an ellipsoid. Following three distinct situations can happen due to the bilinear form \( B \);

(i) the region \( D \cup \partial D \) is included in \( \text{int}(\text{conv}(\hat{\Delta})) \);
(ii) there exist some faces of \( \text{conv}(\hat{\Delta}) \) which are tangent to the boundary \( \partial D \);
(iii) \( D \cup \partial D \not\subset \text{int}(\text{conv}(\hat{\Delta})) \) and no face of \( \text{conv}(\hat{\Delta}) \) is tangent to \( \partial D \).

In this paper we concentrate in the case (ii). For the other cases, the existence of the Cannon-Thurston maps are argued in [17]. We rephrase these cases as follows.

**Proposition 4.5.** For each case, we have the followings:

(a) The case (i) \( \iff \overline{K'} = \overline{K} \subset D \),
\( \iff \) every Coxeter subgroup of \( W \) of rank \( n-1 \) generated by a subset of \( S \) is finite:

(b) The case (ii) \( \iff \overline{K} \) or \( \overline{K'} \) has some vertices in \( \partial D \),
\( \iff \) \( W \) includes at least one affine special subgroup:

(c) The case (iii) \( \iff \) all the vertices of \( \overline{K} \) are not always in \( \partial D \) and at least one of them is not in \( D \),
\( \iff \) every special subgroup of \( W \) of rank \( n' < n \) is of type \((n'-1,1)\) or \((n',0)\).

From Proposition 4.5 we deduce that the fundamental region \( K \) (resp. \( K' \)) is bounded if the case (i) (resp. the case (ii)) occurs. If \( K' \) is not compact, then \( \partial D \) must be tangent to some faces of \( \text{conv}(\hat{\Delta}) \). In this case \( K' \) has some cusps at points of tangency of \( \partial D \). This happens if and only if (ii). Because of this we call each cases as follows: The normalized action of \( W \) on \( D \) is

- cocompact if the case (i) happens;
- with cusps if the case (ii) happens;
- convex cocompact if the case (iii) happens.

In the case (ii) the rank of cusp \( v \) is the minimal rank of the affine Coxeter subgroup generated by a subset of \( S \) which fixes \( v \).

For a metric space, its isometries are classified into three types by the translation length. The translation length of an isometry \( \gamma \) of a metric space \((X,d)\) is the value \( \text{trans}(\gamma) := \inf\{d(x,\gamma(x)) \mid x \in X\} \).

**Definition 4.6.** For a metric space \( X \) an isometry \( \gamma \) of \( X \) is called

1. elliptic if \( \text{trans}(\gamma) = 0 \) and attains in \( X \),
2. hyperbolic if \( \text{trans}(\gamma) \) attains a strictly positive minimum,
3. parabolic if \( \text{trans}(\gamma) \) does not attain its minimum.

**Remark 4.7.** Let \( X \) be a CAT(0) space and \( \gamma \) be an isometry on \( X \). It is clear that \( \gamma \) is elliptic if and only if there exists at least one fixed point of \( \gamma \) in \( X \). In the case where \( X \) is proper and the group \( \langle \gamma \rangle \) acts on \( X \) discretely, if \( \gamma \) is elliptic then \( \langle \gamma \rangle \) is finite. For the hyperbolic isometry, we have a fixed geodesic line in \( X \) ([3, Theorem 6.8(1)])]. This means that if \( \gamma \) is hyperbolic then there are at least two fixed points by \( \gamma \) on \( \partial X \) where \( \partial X \) is the CAT(0) boundary of \( X \). Accordingly if \( \gamma \) is of infinite order and has only one fixed point in \( \partial X \) then it is parabolic.

As noted before we focus on the case (ii) in this paper. Hence we assume that \( W \) includes at least one affine Coxeter subgroup \( W' \) with a generating set \( S' \subset S \).
Lemma 4.8. If \( w \in W \) has an eigenvector \( \xi \) on \( \hat{Q} \) corresponding to the eigenvalue 1 as a product of \( B \)-reflections. Then for any eigenvector of \( w \) corresponding to the eigenvalue 1 on \( \hat{Q} \) lies on the set \( \mathbb{R} \xi \).

Proof. We assume that \( w \) has another eigenvector \( \eta \) included in \( \hat{Q} \) which is not in \( \mathbb{R} \xi \). Then by [12, Lemma A.3] (since this lemma holds for any Coxeter groups) the number of eigenvectors with norm 1 precisely equals to two and corresponding eigenvalues \( \lambda \) and \( \lambda' \) satisfy an equation \( \lambda \lambda' = 1 \). By the assumption of our claim, we have \( \lambda = 1 = \lambda' \).

Consider the normalized action of \( W \) on \( D \). Notice that we have \( w \cdot \hat{\xi} = \hat{\xi} \) and \( w \cdot \hat{\eta} = \hat{\eta} \). The region \( D \) is a proper CAT(0) and Gromov hyperbolic space. Then by Remark 4.7, \( w \) must be a hyperbolic element and hence it does not have a fixed point in \( D \). However a point \( t \xi + (1 - t)\hat{\eta} \) for \( t \in [0, 1] \) is fixed by \( w \) and lies in \( D \) which is a contradiction.

Proposition 4.9. Any infinite order element \( w \) of an arbitrary affine Coxeter subgroup \( W' \) of \( W \) with a generating set \( S' \subset S \) is a parabolic isometry of \( (D, d) \).

Proof. Let \( \Delta' \) be a subset of \( \Delta \) corresponding to \( S' \) and let \( B' \) be a submatrix corresponding to \( \Delta' \). We remark that the \( B \)-reflection on \( \text{conv}(\hat{\Delta}') \) coincides with the \( B' \)-reflection. Because \( D \) is proper metric space, by Remark 4.7, \( w \) is either hyperbolic or parabolic.

Now by [10, Corollary 2.15] and [17, Theorem 1.2] the limit set of \( W' \) is a point \( \xi \). Moreover it equals to \( \hat{Q}' = \{ v \in V'_1 \mid B'(v, v) = 0 \} \). Then \( \xi \) should be a fixed point in \( \partial D \) by \( w \) since \( B' \)-reflections preserve the bilinear form \( B' \). Since \( W \) action by \( B \)-reflections are linear we see that \( \xi \) is an eigenvector of \( w \). On the other hand, by the definition of \( \hat{Q}', \xi \) is also an eigenvector of \( B' \) corresponding to the eigenvalue 0. This shows that \( \xi \) corresponds to the eigenvalue 1 of \( w \). Lemma 4.8 says that such an element in \( W \) fixes only one point in \( \partial D \). By Remark 4.7, we see that \( w \) is parabolic.

5. Horoballs and higher rank cusps

It is known that a tangent point \( p \in \text{conv}(\hat{\Delta}') \cap \partial D \) in the Case (ii) for some \( \Delta' \subset \Delta \) can be expressed as the intersection of \( \{ H_\alpha \mid \alpha \in \Delta' \} \). It is easy to see that the converse is also true namely for \( \Delta' \subset \Delta \) if \( \{ H_\alpha \mid \alpha \in \Delta' \} \) intersect each other at \( x \in \partial D \) then \( x \) is a point on \( \text{conv}(\hat{\Delta}') \). In fact if there exists an intersection point \( p \) of \( \{ H_\alpha \mid \alpha \in \Delta' \} \) on \( \partial D \) then the bilinear form \( B' \) given by restricting \( B \) to \( \Delta' \) has the eigenvalue 0 and \( p \) is its eigenvector. Now let \( w = \prod_{\alpha \in \Delta' \setminus S_\alpha} \). Then we have \( w(p) = p \) hence \( w \) has an eigenvalue 1. By Proposition 4.9, \( p \) is the unique fixed point of the normalized action of \( w \). On the other hand \( W' \) generated by \( \{ s_\alpha \mid \alpha \in \Delta' \} \subset S \) is affine since \( B' \) has the eigenvalue 0. Accordingly there must be the unique fixed point \( p' \) of the normalized action of \( w \) on \( \text{conv}(\hat{\Delta}') \). Together with the argument above we have \( p = p' \). We define a set \( PF \) of such points:

\[
PF := \{ p \in \partial D \mid \exists \Delta' \subset \Delta \text{ s.t. } p = (\cap_{\alpha \in \Delta} H_\alpha) \cap (\cap_{\delta \in \Delta \setminus \Delta'} P_\delta) \}.
\]

Then we notice that \( PF \) is the set of vertices of \( K' \) which are on \( \partial D \) by Proposition 4.5 (b).

Remark 5.1. By [2, Proposition 4.2.5], for \( w \in W \) and \( s_\alpha \in S \) if \( \ell(sw) > \ell(w) \) then all the coordinates of \( w^{-1}(\alpha) \) are non-negative.
Remark 5.2. For $p \in PF$ so that \( \{p\} = (\cap_{\alpha \in \Delta} H_\alpha) \cap (\cap_{\beta \in \Delta \setminus \Delta'} P_\beta) \), the affine subgroup $W'$ generated by \( \{s_\alpha \mid \alpha \in \Delta'\} \) fixes $p$. Conversely if a Coxeter subgroup fixes $p$ then it is a subgroup of $W'$. In fact for $w \in W \setminus W'$ there exists at least one $\beta \in \Delta \setminus \Delta'$ such that the $\beta$-th coordinate of $w \cdot p$ is not 0. This can be seen by the induction for the word length. Recall that $H_\alpha$ cannot intersect with $\text{conv}(\Delta \setminus \{\beta\})$ for any $\alpha \in \Delta$. For $\beta \in \Delta \setminus \Delta'$, then the $\beta$-th coordinate of $s_\beta(p)$ is $-2B(p, \beta) > 0$ since $p \in K$ and $p \notin H_\beta$ for any $\beta \in \Delta \setminus \Delta'$ hence $B(p, \beta) < 0$. We assume that the claim holds for all $w' \in W$ with the length below $k$. For $w \in W$ with $\ell(w) = k + 1$ and $w = s_\beta w'$ ($\ell(w') = k$), we have $w(p) = w'(p) - 2B(w'(p), \beta)$. Then by the inductive assumption there exists at least one $\gamma \in \Delta \setminus \Delta'$ such that the $\gamma$-th coordinate of $w'(p)$ is $\neq 0$. If $\gamma \neq \beta$ then we obtain the claim. For the other case, i.e., $\gamma = \beta$ and the $\beta$-th coordinate of $w'(p)$ is not zero, since all the coordinates of $w'^{-1}(\beta)$ are non-zero (see, Remark 5.1) and $p \in K$ we have $B(w'(p), \beta) = B(p, w'^{-1}(\beta)) \leq 0$. This shows our claim.

We notice that the argument above only needs the assumption $p \notin H_\beta$ for any $\beta \in \Delta \setminus \Delta'$. Since $H_\beta$ cannot intersect with $\text{conv}(\Delta \setminus \{\beta\})$, we have the same result for any Coxeter subsystem $(W', S')$ such that $S' \subset S$, namely, for any $w \in W \setminus W'$ and any $x \in \Lambda(W) \cap \text{conv}(\Delta')$ there exists at least one $\beta \in \Delta \setminus \Delta'$ such that the $\beta$-th coordinate of $w \cdot x$ is not 0.

Definition 5.3. Let $(X, d)$ be a CAT(0) space. Fix a point $o \in X$ and take $k \in \mathbb{R}$. For $x \in \partial X$, we take a geodesic $c$ from $x$ to $\xi$. A horoball at $\xi$ with $k$ (based at $o$) is a set

$$O_{\xi, k} = \left\{ x \in X \mid \lim_{t \to \infty} d(c(t), x) - t < k \right\}.$$ 

The boundary of a horoball $\partial O_{\xi, k}$ is called a horosphere, that is,

$$\partial O_{\xi, k} = \left\{ x \in X \mid \lim_{t \to \infty} d(c(t), x) - t = k \right\}.$$

The function $b_\xi(x) := \lim_{t \to \infty} d(c(t), x) - t$ defining the horoball is said to be a Busemann function associated with $c$. It is known that Busemann functions are well defined, convex and 1-Lipschitz. We remark that $O_{\xi, k} \subset O_{\xi, k'}$ for $k < k'$ and $O_{p, k}$ tends to $p$ for $k \to -\infty$. In this paper, we always take the normalized eigenvector for the negative eigenvalue of $B$ as the base point $o$.

Lemma 5.4. There exists $k \in \mathbb{R}$ such that for any $p, p' \in PF$ and $w \in W$, if $O_{p, k} \neq w \cdot O_{p', k}$ then $O_{p, k} \cap w \cdot O_{p', k} = \emptyset$.

Proof. For any $k$ and $x \notin O_{p, k}$ we have $k \leq \lim_{t \to \infty} d(c(t), x) - t$ where $c$ is the geodesic from $o$ to $p$. If $x \in K \setminus O_{p, k}$ then since $K$ equals to the Dirichlet region at $x$ we have

$$k \leq d(c(t), x) - t \leq d(c(t), w \cdot x) - t$$

for all $w \in W$ and $t \in \mathbb{R}_{\geq 0}$. Hence we have $w \cdot x \notin O_{p, k}$.

Take a closed ball $B(o, r)$ centered at $o$ and the radius $r$ satisfying the condition $r > \max\{d(o, [\alpha, \beta]) \mid \alpha, \beta \in \Delta\}, \{d(o, P_\alpha) \mid \alpha \in \Delta\}$ where $[\alpha, \beta]$ is the segment connecting $\alpha$ and $\beta$. This maximum always exists since $\Delta$ is finite. By the definition of $r$, each component of $K \setminus B(o, r)$ includes just one vertex of $K$. Since Busemann functions are continuous and $B(o, r)$ is compact, we have $k < 0$ such that $O_{p, k} \cap B(o, r) = \emptyset$ for all $p \in PF$. 

If there exists \( w \in W \) such that \( O_{p,k} \neq w \cdot O_{p',k} \) and \( O_{p,k} \cap w \cdot O_{p',k} \neq \emptyset \) then there must be \( x \in K \cap O_{p,k} \cap w' w \cdot O_{p',k} \) for some \( w' \in W \) by the above argument. Let \( \xi \) be the Euclidean segment from \( p \) to \( x \) and let \( \eta \) be the Euclidean segment from \( w' w \cdot p' \) to \( x \). By the definition of \( k > 0 \), \( \xi \) does not intersect with any face of \( \partial K \) which does not contain \( p \). \( \eta \) also does not intersect with any face of \( w' w \cdot \partial K' \) which does not contain \( w' w \cdot p' \). Thus we have \( x \in K' \cap w' w \cdot K' \). This contradicts to the discreteness of the normalized action. \( \square \)

Fix a constant \( k \) which is smaller than the constant in the claim of Lemma 5.4. Let \( o \in D \) be the eigenvector corresponding to the negative eigenvalue of \( B \) as a basepoint. Then \( o \in K \) by [17, Lemma 5]. For each \( p \in PF \), we take a horoball at \( p \) with \( k \) (based at \( o \)) and denote it by \( O_p \). By Proposition 4.5 we have an affine special subgroup corresponding to each \( p \in PF \) uniquely. If \( W' \subset W \) is an affine subgroup corresponding to \( p \in PF \) then \( w \cdot O_p = O_{w \cdot p} = O_p \) for any \( w \in W' \) since \( p \) is fixed by \( W' \). We set \( O := \{ O_p \}_{p \in PF} \).

We remove the orbits of \( O \) from \( D \) and denote it by \( D'' \):
\[
D'' = D' \setminus \text{conv}(\tilde{\Delta}).
\]
Note that \( D'' \) is closed in \( D \) because \( O \) and \( R = D \setminus \text{conv}(\tilde{\Delta}) \) are open. The following is obvious.

**Lemma 5.5.** The set \( D'' \) is invariant under the normalized action of \( W \).

We define \( K'' := K \cap D'' \). Then we can assume that \( o \in K'' \) by taking sufficiently small \( k \). Recall that \( O \) contains all horoballs at the vertices of \( \tilde{K} \) which lie on \( \partial D \). This indicates that \( \tilde{K}'' \) is bounded closed set hence compact since \( D \) is proper.

Since \( K \) is a fundamental region of the normalized action, Lemma 5.5 says that \( K'' \) is a fundamental region of the normalized action on \( D'' \). Define a metric \( d' \) on \( D'' \) by letting \( d'(x,y) \) be the minimum length of a path in \( D'' \) connecting \( x \) and \( y \).

Now we assume that \( k \) is small enough so that the geodesic arc between \( o \) and \( s \cdot o \) is in \( D'' \) for each \( s \in S \).

**Proposition 5.6.** \( W \) acts on \( (D'', d') \) isometrically.

**Proof.** Fix \( w \in W \) and \( a, b \in D'' \) arbitrary. Let \( \sigma \) be a path in \( D'' \) connecting \( a \) and \( b \). Then for any \( \epsilon > 0 \) there exists a partition \( \{ c_0 = a, c_1, \ldots, c_n = b \} \) of \( \sigma \) such that
\[
\ell(\sigma) \leq \sum_{i=1}^{n} d(c_{i-1}, c_i) + \epsilon.
\]
Since \( W \) acts on \( (D, d) \) isometrically we have
\[
\sum_{i=1}^{n} d(c_{i-1}, c_i) = \sum_{i=1}^{n} d(w \cdot c_{i-1}, w \cdot c_i) \leq \ell(w \cdot \sigma).
\]
Hence \( \ell(\sigma) \leq \ell(w \cdot \sigma) \). This implies that \( d'(a, b) \leq d'(w \cdot a, w \cdot b) \) and the reverse is showed in the same way. Thus we have \( d'(a, b) = d'(w \cdot a, w \cdot b) \). \( \square \)

**Lemma 5.7.** Under the notations above, there exist constants \( l \) and \( l' \) so that
\[
l(\sigma) \leq d'(o, w \cdot o) \leq l' \ell(\sigma)
\]
for all \( w \in W \).

We see this by the same way as [9, Lemma in p.213].
Proof. The right hand inequality holds with \( l' = \max\{ d'(s, o) \mid s \in S \} \) by the triangle inequality.

We prove the other inequality. Let \( d(K'') \) be the diameter of \( K'' \) and let \( C = \max\{ \ell(w) \mid w \in W, d'(s, o) \leq 7d(K'') \} \). For any \( w \in W \), we divide a geodesic from \( o \) to \( w \cdot o \) in \( D'' \) into intervals of length \( 5d(K'') \). We have \( d'(s, o) / 5d(K'') \) intervals whose length is \( 5d(K'') \) and one shorter interval. This gives the estimate

\[
\ell(w) \leq C \left( 1 + \frac{d'(o, w \cdot o)}{5d(K'')} \right),
\]

and hence the lemma holds.

We need to compute how the metric \( d' \) differs from the metric \( d \). Now Lemma 3.3 ensures that horoballs in \( D \) are mapped to horoballs in \( \tilde{D} \). It is well known that the space \( (\tilde{D}, d_A) \) in Lemma 3.3 is isometric to the hyperbolic space \( (\mathbb{H}^n, d_\mathbb{H}) \) of the upper half plane model. In \( (\mathbb{H}^n, d_\mathbb{H}) \) we can compare the hyperbolic distance of two points on a horosphere and the length of a path on that horosphere (for the precise computation see [9, p.214-p.215]). For \( x, y \) on horosphere in \( (\mathbb{H}^n, d_\mathbb{H}) \) we denote \( c \) as an arc on horosphere joining \( x \) and \( y \). Then we have

the hyperbolic length of \( c \leq \exp \left( \frac{d_\mathbb{H}(x, y)}{2} \right) \),

and hence

\[
2 \left( \log d'(x, y) \right) \leq d(x, y).
\]

Consequently we obtain a constant \( C > 0 \) from Lemma 5.7, we have the following.

**Lemma 5.8.** For a Coxeter group \( W \) of type \((n - 1, 1)\), there exists a constant \( C > 0 \) so that

\[
2(\log \ell(w)) - C \leq d(o, w \cdot o)
\]

for all \( w \in W \).

6. The Gromov Boundary and the Cannon-Thurston Map

In this section we discuss the existence of the Cannon-Thurston map from the Gromov boundary \( \partial_G W \) to the limit set \( \Lambda(W) \) for a Coxeter group with higher rank cusps.

6.1. The Gromov boundaries. The Gromov boundary of a hyperbolic space is one of the most studied boundary at infinity. In this section we define it for an arbitrary metric space due to [4].

Let \((X, d, o)\) be a metric space with a base point \( o \). We denote simply \( (s|s) \) as the Gromov product with respect to the base point \( o \). A sequence \( x = \{x_i\}_i \) in \( X \) is a Gromov sequence if \( (x_i|x_j)_z \to \infty \) as \( i, j \to \infty \) for any base point \( z \in X \). Note that if \( (x_i|x_j)_z \to \infty \) \( (i, j \to \infty) \) for some \( z \in X \) then for any \( z' \in X \) we have \( (x_i|x_j)_z' \to \infty \) \( (i, j \to \infty) \).

We define a binary relation \( \sim_G \) on the set of Gromov sequences as follows. For two Gromov sequences \( x = \{x_i\}_i, y = \{y_i\}_i \), \( x \sim_G y \) if \( \liminf_{i, j \to \infty} (x_i|y_j) = \infty \). Then we say that two Gromov sequences \( x \) and \( y \) are equivalent \( x \sim y \) if there exist a finite sequence \( \{x = x_0, \ldots, x_k = y\} \) such that

\[ x_{i-1} \sim_G x_i \text{ for } i = 1, \ldots, k. \]
It is easy to see that the relation ∼ is an equivalence relation on the set of Gromov sequences. The Gromov boundary ∂G\,X is the set of all equivalence classes \([x]\) of Gromov sequences \(x\). If the space \(X\) is a finitely generated group \(G\) then the Gromov boundary of \(G\) depends on the choice of the generating set in general. In this paper we always define the Gromov boundary of a Coxeter group \(W\) using the generating set of the Coxeter system \((W, S)\). We shall use without comment the fact that every Gromov sequence is equivalent to each of its subsequences. To simplify the statement of the following definition, we denote a point \(x\) the fact that every Gromov sequence is equivalent to each of its subsequence. To simplify the statement of the following definition, we denote a point \(x\) the fact that every Gromov sequence is equivalent to each of its subsequences. To simplify the statement of the following definition, we denote a point \(x\) the fact that every Gromov sequence is equivalent to each of its subsequences. To simplify the statement of the following definition, we denote a point \(x\) the fact that every Gromov sequence is equivalent to each of its subsequences. To

We denote \(\text{inf} \{ \lim \inf_{i,j \to \infty} (x_i|y_j) \mid [x] = a, [y] = b \}, \quad \text{if } a \neq b, \quad \infty, \quad \text{if } a = b.

We set \(U(x, r) := \{ y \in \partial G\,X \mid (x|y) > r \}\) for \(x \in \partial G\,X\) and \(r > 0\) and define \(U = \{ U(x, r) \mid x \in \partial G\,X, r > 0 \}\). The Gromov boundary \(\partial G\,X\) can be regarded as a topological space with a subbasis \(U\).

If the space \(X\) is δ-hyperbolic in the sense of Gromov, then this topology is equivalent to a topology defined by the following metric. For \(\epsilon > 0\) satisfying \(\epsilon \delta \leq 1/5\), we define \(d_\epsilon\) as follows:

\[ d_\epsilon(a, b) = e^{-\epsilon |a|} \quad (a, b \in \partial G\,X). \]

Then it follows from 5.13 and 5.16 in [22] that \(d_\epsilon\) is actually a metric. In this paper, we always take \(\delta \leq 1/5\) for all \(\delta\) hyperbolic spaces \(X\) and assume that \(\partial G\,X\) is equipped with \(d_\epsilon\)-topology.

**Remark 6.1.** Since \(D\) is isometric to \(\mathbb{H}^{n-1}\), it is a proper complete Gromov hyperbolic CAT(0) space. Hence we see that \(\partial D\) is homeomorphic to \(\partial D\) ([3, H.I.H.3.7 and H.II.11.2]).

If an isometric group action \(G \curvearrowright X\) on a Gromov hyperbolic space \(X\) is properly discontinuous and cocompact then the group \(G\) is also hyperbolic in the sense of Gromov and it is called a hyperbolic group (see [22]).

**Remark 6.2.** Next we consider a geodesic (hence a Euclidean line) \(\gamma\) in \(D\) between two points on a horoball. By Lemma 4.17 two metrics \(d_H\) on \(H^n\) and \(d\) on \(D\) shows that a horoball in \(D\) is mapped to a horoball in \(H^n\). Hence we can map a horoball in \(D\) to a horoball \(O,H = \{(x_1, \ldots, x_n) \in H^n \mid x_n > c \}\) for some \(c > 0\). We denote \(o' \in H^n\) as the image of the base point \(o \in D\). Let \(x', y' \in H^n\) be the image of the end points \(x, y\) of \(\gamma\) and let \(z'\) be the image of the nearest point \(z\) of \(\gamma\) from \(o\). The geodesic \(\gamma'\) in \(H^n \setminus O,H\) for the length metric is a straight line on \(\partial H\) and it is nothing but the nearest point projection of the image of \(\gamma\) to the plane \(\{(x_1, \ldots, x_n) \in H^n \mid x_n = c \}\). Therefore the distance from \(o'\) to \(\gamma'\) is bounded above by \(2d_H(o', z')\).

More generally we consider a geodesic with respect to \(d'\) of \(x, y \in D'\). We denote \(\xi\) as the geodesic connecting \(x, y\). Let \(\{O_i\}\) be the set of horoballs which intersects with \(\xi\). Then by [16, Theorem 8.1] we see that the geodesic with respect to \(d'\) lies in the \(R\) neighborhood of \(\xi \cup \bigcup_i O_i\) where \(R\) is a universal constant. Now we consider segments \(\xi_i = \xi \cap O_i\) and let \(x_i, y_i\) be the end points of \(\xi_i\). Let \(\xi'_i\) be the geodesic
in $D''$ connecting $x_i, y_i$ and let $\xi'$ be the path given by replacing each $\xi_i$ with $\xi'_i$. Recall that the geodesic in $\mathbb{H}^n$ is unique and the fact that the geodesic between two points on a horoball $O_B = \{(x_1, \ldots, x_n) \in \mathbb{H}^n \mid x_n > c\}$ for some $c > 0$ is an Euclidean straight line on $\partial O_B$. Because of this, we see that there exists another universal constant $R'$ such that the geodesic $\xi$ lies in the $R'$ neighborhood of $\xi'$.

Let $F : W \to D''$ be the quasi isometry defined by $F(w) = w \cdot o$ for every $w \in W$ and if $w = w's$ for some $s \in S$ then $F$ maps the edge joining the vertices $w, w' \in W$ to the geodesic $[w \cdot o, w' \cdot o]$.

**Lemma 6.3.** There exists a constant $P > 0$ satisfying the following. For any $x, y \in W \cdot o$ there exists a geodesic $\gamma$ in $W$ which $F(\gamma)$ connects $x, y$ such that is in bounded Hausdorff distance with $P$ from a geodesic connecting $x, y$ in $D''$.

**Proof.** Take the geodesic $\xi$ between $x$ and $y$ in $D$. Then $\xi$ crosses some orbits of $K$ and some horoballs. We claim that a curve $\tau$ given by replacing all segments of $\xi$ which cross horoballs with the geodesics in $D''$ connecting each end points crosses the same orbits of $K''$ as $\xi$.

We remark that $\xi$ gives a geodesic $\gamma$ in $W$ in the following way. Let $w_1 \cdot K, \ldots, w_k \cdot K$ be the orbits which are crossed by $\xi$. Then for each $i \in \{1, \ldots, k - 1\}$, there exists a $B$-reflection $s_i \in S$ such that $w_{i+1} = s_i w_i$. We see that $\gamma = s_{k-1} \cdots s_1$ is a geodesic in $W$ by the deletion condition of Coxeter system $(W, S)$ (more precisely, see the proof of [17, Proposition 4.12] or [7, Corollary 3.2.7]). We see that $F(\gamma)$ equals to the path connecting $w_1 \cdot o, \ldots, w_k \cdot o$ with geodesics.

Let $O$ be a horoball in $D$ which intersects with $\xi$ and let $O'$ be the horoball $\{(x_1, \ldots, x_n) \in \mathbb{H}^n \mid x_n > c\}$ in $\mathbb{H}^n$ isometric to $O$. We denote $x$ and $y$ as the endpoints of a segment $\xi_0$ in $\xi$ which crosses $O$, and let $x', y' \in \mathbb{H}^n$ be the corresponding points to $x, y$ each other. We notice that $x'$ and $y'$ lie on $\partial O' = \{(x_1, \ldots, x_n) \in \mathbb{H}^n \mid x_n = c\}$. Consider the geodesic $l$ in $\mathbb{H}^n \setminus O'$ connecting $x'$ and $y'$ for the length metric. Then $l$ is the Euclidean straight line connecting $x', y'$ and it crosses the same image of orbits of $K$ as image of $\xi_0$. This is because all the orbits of $K$ crossing to $O$ are the orbits of an affine special subgroup whose elements of infinite order fix the point of tangency on $O$, and $l$ is given by the orthogonal projection to $\partial O'$ of a hyperbolic geodesic with respect to the Euclidean inner product.

Thus resulting curve $\tau$ crosses the same orbits of $K''$ as $\xi$. More of this since $\tau$ lies in the orbit of $K''$ and the diameter of $K''$ is bounded, the Hausdorff distance between the paths $F(\gamma)$ and $\tau$ is bounded by the diameter of $K''$.

By the remark described before this lemma, we see that the geodesic in $D''$ between $x, y$ is on bounded distance with universal constant $R'$ from $\tau$.

Putting $P = R' + (\text{diameter of } K'')$, we have the conclusion. \qed

**Definition 6.4.** Let $(W, S)$ be a Coxeter system. For a sequence $\{w_k\}_k$ in $W$, a path in $V_1$ is a sequence path for $\{w_k\}_k$ if the path is given by connecting segments $[w_k \cdot o, w_{k+1} \cdot o]$ for all $k \in \mathbb{N}$.

In [14, Theorem 5.2], Karlsson and Noskov showed the following useful theorem.

**Theorem 6.5 (Karlsson-Noskov).** Let $\{x_n\}_n$ and $\{z_n\}_n$ be two sequences consisting of points in $\bar{D}$. Assume that $x_n \to \overline{\tau} \in \partial \bar{D}$, $z_n \to \overline{\tau} \in \partial \bar{D}$ and $[\overline{\tau}, \overline{\tau}] \not\subset \partial \bar{D}$, where $[\overline{\tau}, \overline{\tau}]$ is a segment connecting $\overline{\tau}$ and $\overline{\tau}$. Then there exists a constant $M = M(\overline{\tau}, \overline{\tau})$ such that for the Gromov product $(x_n|z_n)_y$ in Hilbert distance relative to
some fixed point \( y \in \tilde{D} \) we have
\[
\limsup_{n \to \infty} (x_n z_n)_y \leq M.
\]

This implies that if two unbounded sequences \( x_n \) and \( z_n \) in \( \tilde{D} \) are equivalent in the sense of Gromov, then these sequences converge to the same point in \( \partial \tilde{D} \).

We have the Cannon-Thurston map for a Coxeter group with higher rank cusps directly. We remind the following fact. Let \((X, d)\) be a \( \delta \)-hyperbolic space. For any \( x, y, o \in X \), let \( z \) be an arbitrary point on a geodesic connecting \( x, y \). In a \( \delta \)-hyperbolic space, by the definition, \( \delta \geq \min\{d(z, [o, x]), d(z, [o, y])\} \). Hence we have \( d(o, z) \geq (x|y)_o \). If \( z \) is the nearest point of a geodesic \([x, y]\) from \( o \), then we obtain \((x|y)_o \geq d(o, z) - \delta \) ([22, 2.33]). Thus
\[
d(o, z) \geq (x|y)_o \geq d(o, z) - \delta
\]
for such a point.

**Proposition 6.6.** Assume that \( W \) includes rank \( m > 2 \) cusps. Let \( F : W \to D'' \) be the quasi isometry defined by \( F(w) = w \cdot o \) for every \( w \in W \). Then \( F \) extends to \( \tilde{F} : \partial_G(W, S) \to \Lambda(W) \) continuously. Moreover \( \tilde{F} \) is surjective and \( W \)-equivariant.

**Proof.** In this proof we denote by \( C \) a generic constant whose value may change line to line. We show that the Gromov product of any two orbits \( w \cdot o, w' \cdot o \) of \( o \in D'' \) for the metric \( d \) is bounded below by the Gromov product of \( w \) and \( w' \) with respect to the unit id \( o \in W \) for the word metric. If this is true, then we have the well definedness of \( \tilde{F} \) by Theorem 6.5 and the continuity by the fact that \( \partial_G D \) and \( \partial D \) are homeomorphic. More of this for any limit point \( \xi \) which is not in \( W \cdot PF \) by taking the geodesic on \((D, d)\) from \( o \) to \( \xi \) we can construct a sequence path. The corresponding sequence for that sequence path is actually a geodesic in \( W \) by Lemma 6.3. If \( \xi \in W \cdot PF \) then \( w \cdot \xi \in PF \) for some \( w \in W \). In this case there exists a Coxeter subsystem \((W', S')\) of \((W, B)\) such that \( S' \subset S \) and \( W' \) fixes \( w \cdot \xi \) by Proposition 4.5. Since \( W' \) is affine, there exists at least one Gromov sequence. Then for any Gromov sequence \( \{w_i^j\}_i \) consists of elements in \( W' \), the sequence \( \{w_i^j \cdot o \}_i \) converges to \( \xi \). Thus we see that \( \tilde{F}^{-1}(\xi) \) is not empty and hence \( \tilde{F} \) is surjective. The \( W \)-equivariantness of \( \tilde{F} \) is trivial by the construction.

Take \( x, y \in W \cdot o \) arbitrarily and let \( \tau \) be the geodesic on \((D, d)\) connecting \( x = w_x \cdot o \) and \( y = w_y \cdot o \). Let \( z \) be the nearest point from \( o \) to \( \tau \). Let \( \gamma \) be a geodesic in \( W \) which is constructed in the same way as the proof of Lemma 6.3. We construct a path \( \tau' \) in \( D'' \) by replacing segments of \( \tau \) which cross horoballs with the geodesics in \( D'' \) connecting each end points. We denote by \( z' \) the nearest point from \( o \) to \( \tau' \). Now we have \( d(o, z) \geq Cd(o, z') \) by Remark 6.2. Adding to this we put \( z'' = w_z \cdot o \in W \cdot o \) as the nearest point from \( z' \) to \( F(\gamma) \). Then \( d(o, z') \geq d(o, z'') - C \) since the diameter of \( K'' \) is bounded.

Furthermore by the inequality (1) we have
\[
\log \left(Cd'(p, q)^2\right) \leq d(p, q)
\]
for any \( p, q \in D'' \).
Then we have
\[
(x|y)_0 \geq d(o, z) - C \geq Cd(o, z') - C \geq Cd(o, z'') - C \\
\geq \log (Cd(o, z'')^2) \geq \log (C|w_z|^2) \\
\geq \log (C(w_x|w_{p_0}|_o)^2).
\]

Thus we have the claim. □

This proves that the existence of the Cannon-Thurston maps for the case (ii) which is described at the beginning of Section 4. For the other cases we have already done in [17] as mentioned in Section 1. Thus we obtain the conclusion. □

Remark 6.7. For a cusp \( p \in PF \) there exists corresponding affine special Coxeter subsystem \((W', S')\) of \((W, B)\) and \( W' \) fixes \( p \). If the rank of \( p \) is 2 then there exist \( \alpha, \beta \in \Delta \) such that \( B(\alpha, \beta) = -1 \) and the Coxeter subgroup \( W' \) generated by \( \{s_\alpha, s_\beta\} \) is affine. Since an affine Coxeter group has only one limit point, \( \{(s_\alpha s_\beta)^k \cdot o\}_k \) and \( \{(s_\beta s_\alpha)^k \cdot o\}_k \) converges to the same limit point. However in the Gromov boundary of \((W, S)\), \( \{(s_\alpha s_\beta)^k\}_k \) and \( \{(s_\beta s_\alpha)^k\}_k \) lie in distinct equivalence class. In fact, considering another action of \((W, S)\) defined by another bi-linear form \( B' \) such that \( B'(\alpha, \beta) < -1 \), then the limit set \( \Lambda_{B'}(W') \subset \Lambda_{B'}(W) \) consists of two points. In this case the limit points of \( \{(s_\alpha s_\beta)^k \cdot o\}_k \) and \( \{(s_\beta s_\alpha)^k \cdot o\}_k \) are distinct. On the other hand the map \( \partial_G W \rightarrow \Lambda_{B'}(W) \) is well defined hence \( F' \) can never be injective. For the case where \( p \) is not a rank 2 cusp, the author does not know whether the preimage of \( p \) is a point or not.

For other limit points, the map \( F \) is injective. Let \( \xi \in \Lambda(W) \setminus W \cdot PF \) and let \( \{x_i\}_i \) be a sequence in \( D \) converging to \( \xi \) with \( |x_i| \geq i \). Then by the properness and CAT(0)-ness of \( D \), we can construct a “good” subsequence \( \{y_i\}_i \) of \( \{x_i\}_i \), as follows. This method is due to [4]. Let \( [o, x_i] \) be the geodesic segment connecting \( o \) and \( x_i \) for each \( i \in \mathbb{N} \). For \( k < i \), let \( P_k(x_i) \) be the intersection point of \([o, x_i]\) with \( S(o, k) \) where \( S(o, k) = \{y \in D \mid d(o, y) = k\} \). Writing \( x^0_i = x_i \), we inductively define a sequence of nested subsequences. Given a subsequence \( \{x^k_{i-1}\}_{i=1}^{\infty} \) of \( \{x_i\}_i \), where \( k \in \mathbb{N} \) is the inductive index, by the compactness of \( S(o, k) \) we can take a subsequence \( \{x^k_{i\infty}\}_{i=1}^{\infty} \) of \( \{x^k_{i-1}\}_{i=1}^{\infty} \) such that \( x^k_n = x_n \) for some \( n \geq k \), and such that all the points \( P_k(x^k_i), i \in \mathbb{N} \), lie within a distance 1 of each other and converge to some point \( z_k \in S(o, k) \) as \( i \rightarrow \infty \). We write \( z_0 = o \). Then a ray given by joining geodesic segments \( [z_{k-1}, z_k], k \in \mathbb{N} \) is the geodesic \([o, \xi]\) from \( o \) to \( \xi \). Now define \( \{y_i\}_i \) as the diagonal sequence, i.e. \( y_i = x^i_i, i \in \mathbb{N} \).

For any Gromov sequence \( \{w_i\}_i \) in \( W \) such that \( \{w_i \cdot o\}_i \) converges to \( \xi \), letting \( x_i = w_i \cdot o \) for each \( i \in \mathbb{N} \) in the argument above, we have a subsequence \( \{y_j\}_j = \{w'_j \cdot o\}_j \). Then since \( \xi \in \Lambda(W) \setminus W \cdot PF \), we have a geodesic \( \{v_0, v_1, \ldots, v_n, \ldots\} \) in \( W \) such that each \( v_i \cdot o \) lie within a bounded distance from \( [o, \xi] \). We can take a sequence \( \{p_i\}_i \) in \([o, \xi]\) satisfying \( B(p_i, 1) \subset D' \) for each \( i \in \mathbb{N} \). Let \( r_i \) be the distance from \( o \) to \( p_i \). We consider the subsequence \( \{v'_i\} \) of \( \{v_i\} \) so that \( d(v'_i \cdot o, p_i) < C \) where \( C \) is the diameter of \( K' \). Then for each \( i \in \mathbb{N} \), \( P_{r_i}(u'_i \cdot o) \) lie within a distance 1 from \( p_i \). Let \( \{u'_i\}_i \) be a sequence in \( W \) so that each \( u'_i \cdot o \) is the nearest orbit of \( o \) from \( P_{r_i}(u'_i \cdot o) \). Now, the geodesic word of \( u'_1 \) is a subword of the geodesic word of \( w'_j \). Moreover there exists a constant \( C' > 0 \) such that
\[
|u'_i| - C' \leq |v'_i| \leq |u'_i| + C'.
\]
We need only finitely many orbits of \( K' \cap D'' \) to cover both \( p_i \) and \( P, (w'_j \cdot o) \) for each \( i, j \) so that the union is connected and such numbers are bounded uniformly. This means that there exists another constant \( C'' \) such that

\[
|v'_i u'_j| \leq C'',
\]

for any \( i, j \). Thus we have

\[
2(u'_i | w'_j) = |v'_i| + |w'_j| - |v'_i u'_j| \\
\geq |v'_i| + |w'_j| - (|v'_i u'_j| + |v'_i| | w'_j|) \\
\geq |v'_i| + |w'_j| - |v'_i| | w'_j| - C'' \\
= |v'_i| + |w'_j| - C'' \\
\geq 2|v'_i| - C' - C''.
\]

Since \(|w| \leq Cd(w \cdot o), |v'_i| \to \infty \) as \( i \to \infty \). This shows that \( \{v_i \} \sim \{v'_i \} \sim \{w'_i \} \sim \{w_i \} \) and hence \( \bar{F}^{-1}(\xi) \) is a singleton.

**Proof of Corollary 1.2.** Moreover since \((W, S)\) is a Coxeter system, by the deletion condition ([7, pp.35, (D)]) any expression of an element \( w \) of \( W \) includes a reduced expression of \( w \). Therefore for a Coxeter subsystem \((W', S')\) of \((W, S)\), any reduced expression of an element in \((W', S')\) is also a reduced expression as an element in \((W, S)\). This shows that the Cayley graph of \((W', S')\) is embedded into the Cayley graph of \((W, S)\) isometrically. If the normalized action of \((W', S')\) on its phase space is cocompact, this embedding extends continuously to an embedding of the Gromov boundary of \((W', S')\) into the Gromov boundary of \((W, S)\). Together with Remark 6.7, the limit set \( \Lambda(W') \) is naturally identified with \( \Lambda(W) \cap \text{conv}(\hat{\Delta}') \) via the Cannon-Thurston map of \((W, S)\). \( \square \)

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