PROPERTIES OF QCD VACUUM FROM LATTICE

A. DI GIACOMO
Pisa University, 2 Piazza Torricelli, Pisa, 56100 Pisa, ITALY

Advances in the understanding of the basic properties of QCD vacuum will be reported. Three main subjects will be touched:
1) Condensation of monopoles and confinement.
2) Topology, or instanton physics.
3) Gauge invariant field strength correlators, and their behaviour across the deconfining phase transition.

1 Introduction

I will report on some progress recently achieved mainly by numerical simulations on the lattice, on three subjects. 1) Condensation of monopoles and colour confinement (sect.2). 2) Topology (instanton physics) (sect.3). 3) Gauge invariant field strength correlators at short distances at $T = 0$ and at the deconfining transition (sect.4).

The conclusions of this review will be that:

1) Solid evidence exists that dual superconductivity is the mechanism of colour confinement.

2) The Witten-Veneziano formula for the mass of the $\eta'$ is definitely correct. This statement has been made possible by the construction of an improved (almost perfect) operator for the topological charge density. By use of the same technical improvement a detailed study of the topological susceptibility across the deconfining transition has been made, showing a sharp drop of it at $T_{c}$.

Studies with dynamical fermions have been started, especially with the aim of determining, in full QCD, $\chi$, its derivative $\chi'$ with respect to momentum transfer, and the matrix element of topological charge on proton states, which would allow a measurement from first principles of the so called “spin content of the proton”. All these results in full QCD have been slowed down by the discovery that the usual algorithm for Montecarlo simulation, the so called hybrid montecarlo, is very slow in changing the topological charge $Q$ of a configuration: the same $Q$ stays for few hundred updatings. This puts severe limitations on the validity of usual montecarlo studies in full QCD. A typical statistics is
indeed a few hundred configurations. Only for observables which are insensitive to topology can such a statistics be sufficient to insure proper thermalization.

3) Previous studies of the field strength correlators in the vacuum have been extended to much shorter distances ($\sim 1$ fm). The results is relevant to test stochastic models of confinement.

All these results have been made possible by the use of QUADRIX computers.

2 Condensation of monopoles and confinement.

An appealing mechanism for colour confinement in QCD is dual superconductivity of the vacuum. The chromoelectric field acting between heavy $q \bar{q}$ pairs is constrained into Abrikosov flux tubes, generating a potential energy proportional to the distance.

We test this mechanism from first principles by numerical simulation on the lattice.

The basic idea of the approach is that, to detect dual superconductivity of QCD vacuum, the vacuum expectation value of an operator with non trivial magnetic charge can be used as a probe, or as disorder parameter. Indeed a non zero value of such $v.e.v.$ signals condensation of monopoles and spontaneous breaking of the $U(1)$ symmetry related to magnetic charge conservation. The relevant magnetic $U(1)$’s are identified by the well known procedure of abelian projection. They correspond to residual $U(1)$ invariance after diagonalization of any operator belonging to the adjoint representation of the gauge group. Each operator in the adjoint representation defines an abelian projection, and hence a species of monopoles.

We have constructed a creation operator for monopole, and we use its $v.e.v.$ as a disorder parameter.

It is an open question if all abelian projections are equivalent, and identify dual superconductors. We have tested the abelian projection defined by the diagonalization of the Polyakov line, which is the local operator defined as the parallel transport on the closed path along the time axis, starting from a point and coming back to it through the periodic boundary conditions used to define finite temperature. The $v.e.v.$ of the trace of such operator is the usual order parameter for confinement. We find definite evidence for dual superconductivity in the $U(1)$ defined by such abelian projection.

The creation operator for an $U(1)$ monopole is defined as follows:

$$\mu(\vec{y}, t) = \exp \left[ i \int d^3x E(\vec{x}, t) \frac{1}{e} \vec{b}(\vec{x} - \vec{y}) \right]$$  \hspace{1cm} (1)
$\vec{E}$ is the electric field operator, and

$$\vec{b}(\vec{r}) = \frac{m}{2} \frac{\vec{r} \wedge \vec{n}}{r(r - \vec{r} \cdot \vec{n})}$$

is the vector potential describing the field of a monopole of magnetic charge $m$ in units $1/2e$. The prescription for the integral is that the Dirac string must be removed. $\mu$ is the analog of the translation operator for a particle:

$$e^{ipa} |x\rangle = |x + a\rangle$$

$\vec{A}$ is the conjugate momentum to the field $\vec{A}$. In the Schrödinger representation

$$\mu(\vec{y}, t)|\vec{A}(\vec{x}, t)\rangle = |\vec{A}(\vec{x}, t) + \frac{1}{e}\vec{b}(\vec{x} - \vec{y})\rangle$$

(2)

$\mu$ adds a monopole to the field configuration. Moreover, if $Q_M = \int d^3x \vec{\nabla} \cdot (\vec{\nabla} \wedge \vec{A})$ is the magnetic charge

$$[Q_M, \mu(\vec{y}, t)] = m\mu(\vec{y}, t)$$

(3)

i.e. $\mu$ carries magnetic charge $m$.

The lattice version of $\mu$ is

$$\mu = \exp - \langle \beta S_M \rangle$$

(4)

with

$$S_M = \sum_{\vec{n}} \Pi_{0i}(\vec{n}, n_0) \left( e^{ib_i} - 1 \right)$$

(5)

$\frac{1}{e} \text{Im} \Pi_{0i}$ is the lattice version of the electric field. The disorder parameter is

$$\langle \mu \rangle = \frac{\int dU \exp [-\beta(S_W + S_M)]}{\int dU \exp (-\beta S_W)}$$

(6)

Eq. (6) corresponds to Eq. (3) at the lowest order in $\vec{b}$, but it is a compactified version of it. By use of the definition Eq. (4) the gauge arbitrariness in the choice of $\vec{b}$ to describe the monopole field is reabsorbed in the Haar measure of the Feynman integral. To avoid problems with fluctuations we do not measure $\langle \mu \rangle$ directly, but $\rho = \frac{d}{dx} \ln \mu$ and reconstruct $\langle \mu \rangle$ as

$$\langle \mu \rangle = \exp \int_0^\beta \rho(x) \, dx$$

(7)
$\langle \mu \rangle$ determined in this way converges to a nonzero constant as $V \to \infty$ for $T < T_c$ and goes to zero as $(T_c - T)^\delta$ at the deconfining temperature. The typical form of $\rho$ is shown in fig. 1 for $SU(3)$ on a $12^3 \times 4$ lattice. The sharp negative peak at $T_c$ signals the drop of $\langle \mu \rangle$ at $T_c$. $\langle \mu \rangle$ is zero for $T > T_c$ only in the limit $V \to \infty$, as is typical of disorder parameters.

![Fig.1 SU(3) gauge theory. Lattice $12^3 \times 4$. $\rho = \frac{d}{d\beta} \ln \langle \mu \rangle$](image)

We have repeated the construction for the $XY$ model in 3-d, both as a test of the method and as an amusing application of it. The model describes superfluid $^4$He. It can be viewed as the euclidean version of a free massless particle in $2 + 1$ dimension.

The field variable is an angle $\theta(x)$. On the lattice:

$$S = \beta \sum_{\mu, i} [1 - \cos(\Delta_i \theta)] \simeq \frac{\beta}{2} (\Delta_i \theta)^2$$  \hspace{1cm} (8)

The theory admits soliton configurations which have the topology of vortices

$$\tilde{\theta}(\vec{x} - \vec{y}) = \arctg \frac{(\vec{x} - \vec{y})_2}{(\vec{x} - \vec{y})_1}$$  \hspace{1cm} (9)

is an example. Vortices condense in the ground state in the superfluid phase: they play a similar role as monopoles in QCD. As for QCD we can define a creation operator for a vortex

$$\mu(\vec{y}, t) = \exp \left[ i \int \partial_\theta(\vec{x}, t) \tilde{\theta}(\vec{x} - \vec{y}) d^2x \right]$$  \hspace{1cm} (10)
The operator which counts vortices, $V$ can be written as

$$V = \frac{1}{2\pi} \int d^2 x \left( \tilde{\nabla} \wedge \tilde{A} \right) = \tilde{\nabla} \theta$$  \hspace{1cm} \text{(11)}$$

and

$$[V, \mu] = \mu$$  \hspace{1cm} \text{(12)}$$

On the lattice by the same procedure of compactification used for monopoles

$$\mu(\vec{y}, x_0) = \exp \left\{ -\beta \left[ \sum_{\vec{n} \neq 0} \cos \left[ \Delta_0 \theta(\vec{n}) + \bar{\theta}(\vec{n} - \vec{y}) \right] - \cos \left[ \Delta_0 \theta(\vec{n}) \right] \right] \right\}$$  \hspace{1cm} \text{(13)}$$

As in the case of monopoles we define $\langle \mu \rangle$ in terms of the correlator of a vortex antivortex by cluster property

$$\lim_{|x| \to \infty} \langle \mu(x) \mu(0) \rangle \simeq A e^{-\alpha |x|} + \langle \mu \rangle^2$$  \hspace{1cm} \text{(14)}$$

and instead of $\langle \mu \rangle$ we measure

$$\rho = \frac{d}{d\beta} \ln \langle \mu \rangle$$  \hspace{1cm} \text{(15)}$$

The behaviour of $\rho$ is shown in Fig. 2.

A finite size scaling analysis, described in ref.\textsuperscript{14} gives, for the critical index of the correlation length

$$\nu = 0.669 \pm 0.065 \hspace{1cm} (0.670(7))$$  \hspace{1cm} \text{(16)}$$

for the critical temperature

$$\beta_c = 0.4538 \pm 0.0003 \hspace{1cm} (0.45419(2))$$  \hspace{1cm} \text{(17)}$$

and for the critical index of $\langle \mu \rangle$

$$\delta = 0.740 \pm 0.029$$  \hspace{1cm} \text{(18)}$$

The first two quantities agree with the determinations by other methods, which are shown in parentheses, showing that the method is correct and effective.
3 Topology (instantons)

Almost all of the Gell-Mann current algebra, which was originally abstracted from massless free quark model, has been preserved with the advent of QCD, except for the singlet axial current, \( j_5^\mu = \sum_f \bar{\psi}_f \gamma^\mu \gamma^5 \psi_f \), which was conserved in the quark model, but is anomalous in QCD.

\[ \partial_\mu j_5^\mu = 2N_f Q \]  

(19)

\( Q(x) = \frac{g^2}{16\pi^2} \varepsilon^{\mu\nu\rho\sigma} G^a_{\mu\nu} G^a_{\rho\sigma} \) is the density of topological charge. It is the second Chern number, is known as topological charge and takes on integer values on smooth configurations. The non conservation of \( j_5^\mu \) in QCD gives a handle to solve the so called \( U(1) \) problem: if \( j_5^\mu \) were conserved and the corresponding symmetry were a Wigner symmetry, parity doublets should exist, if it were realized à la Goldstone then one should have \( m_{\eta'} \leq \sqrt{3} m_\pi \) and neither of these possibilities is realized in nature.

However in an expansion in \( 1/N_c \) \( Q(x) \) is non leading, being \( \propto g^2 = \lambda/N_c \) (\( \lambda = g^2 N_c \)). In a philosophy in which the leading order in \( 1/N_c \) describes the essentials of hadron physics, \( U(1) \) can be considered a symmetry. The anomaly acts there as a perturbation and displaces the \( \eta' \) mass from zero, which would correspond to Goldstone symmetry, to its true value. The
quantitative relation is}

\[
\frac{2N_f}{f_\pi^2} \chi = m_n^2 + m_{\pi}^2 - 2m_K^2 \quad \text{or} \quad \chi = (180 \text{ MeV})^4
\]  

(20)

\(\chi\) is the topological susceptibility of the vacuum

\[
\chi = \int d^4x \langle 0|T(Q(x)Q(0))|0\rangle
\]  

(21)

at the leading order, i.e., in the absence of fermions.

The prediction Eq. 20 can be tested on lattice and involves the computation of \(\chi\) in the quenched approximation. The correct way to compute \(\chi\) is to define a regularized version of \(Q\), \(Q_L\), and compute the lattice topological susceptibility

\[
\chi_L = \sum_n \langle Q_L(n)Q_L(0) \rangle
\]  

(22)

Like in any regularization scheme the regularized operator mixes, in the limit in which the cutoff is removed, with all the operators having the same quantum numbers and smaller or equal dimension in mass.

21

By use of this prescription one gets

\[
Q_L = Z(\beta)Q
\]

\[
\chi_L = Z^2(\beta)\chi a^4 + B(\beta)G_2 a^4 + P(\beta)
\]  

(24)

In Eq. 24 \(G_2 = \frac{G_\mu^a G_\nu^a}{8\pi} \langle G_{\mu\nu}^a G_{\rho\sigma}^a \rangle\) is the gluon condensate, and \(P(\beta)\) describes the mixing to the identity operator, which is usually called perturbative tail. Both the last two terms in Eq. 24 come from the singularity in the definition (22) of \(\chi\) as \(x \to 0\). \(Z(\beta), B(\beta), P(\beta)\) depend on the choice of \(Q_L\), which is arbitrary for terms of \(O(a^6)\) or higher. For the most simple choice of \(Q_L\), in terms of the plaquette \(\Pi^{\mu\nu}\)

\[
Q_L = -\frac{1}{32\pi^2} \varepsilon_{\mu\nu\rho\sigma} \text{Tr} \{\Pi^{\mu\nu}(n)\Pi^{\rho\sigma}(n)\}
\]  

(25)

\(Z = 0.18\) and the correction to continuum in Eq. 24, namely the sum of the last two terms in Eq. 24 is much bigger than the first term.

A nonperturbative determination both of \(Z\) and of the additive renormalization \(M(\beta) = B(\beta)G_2 a^4 + P(\beta)\) is possible. The idea is based on the fact that changing the number of instantons by local updating procedure is much slower process than thermalizing the short range fluctuations which are responsible for renormalizations. So, starting from a zero field configuration, where
$Q = 0$ and $\chi = 0$, and heating it will only produce $M(\beta)$. Similarly putting one instanton of charge $Q$ on a lattice and heating it will leave $Q$ unchanged for a large number of sweeps, but will produce the fluctuations necessary to build up $Z$: a plateau will be reached where $Q_L = ZQ$, and $Z$ can be read on it.

In any case it is unpleasant that most of the observed signal is artifact to be removed. Recently, playing on the arbitrariness by higher terms in $a$ in the definition of $Q_L$, an improved operator has been constructed for which lattice artifacts are reduced by 2 orders of magnitude; $Z$ is now of order 0.6, instead of 0.18, and $M(\beta)$ is less than 10% of the whole signal in a wide scaling window.

Counterterms are removed by the same non-perturbative technique, but now they are a small part of $\chi_L$. In particular $P(\beta)$ is negligible in a wide range of $\beta$’s ($\sim 2\%$ on the entire signal).

![Graph](image)

**Fig. 3** Determination of $\chi$ with the naive operator, Eq. 25 (0 smear) and with the once and twice improved operators.

The results are summarized in Fig. 3 and in Fig. 4. Fig. 3 shows the determination of $\chi$ at $T = 0$: the result is $\chi = 175 \pm 5$ MeV in excellent agreement with the Witten Veneziano prediction, and in agreement with previous
determinations within their large errors. A new result is the behaviour of $\chi$ across the deconfining transition shown in Fig. 5, which is certainly of interest for the models of QCD vacuum based on instanton liquid.

**Fig. 4** $M(\beta)/a^4(\beta)$. The slight deviations from a constant allow to extract $P(\beta)$. The result is $P(\beta) < 2-3\% \chi_L$

**Fig. 5** $\chi$ across $T_c$. 

- **2-smear data**
- **1-smear data**

- **$\chi/\Lambda_L^4$, $T=0$**
- **1-smear data**
- **2-smear data**
4 Gauge invariant field correlators.

The correlators are defined as

\[ D_{\mu\nu\rho\sigma}(x) = \langle 0 | T\left( G_{\mu\nu}(x)S(x)G_{\rho\sigma}(0)S^\dagger(x) \right) | 0 \rangle \quad (26) \]

with \( G_{\mu\nu} = \sum_a T^a G^a_{\mu\nu}(x) \) (\( T^a \) generators of the gauge group) and \( S(x) \) the parallel transport from \( 0 \) to \( x \)

\[
S(x) = \exp\left( \int_0^1 A_\mu(tx)x^\mu dt \right) \quad (27)
\]

\[ A_\mu = \sum_a T^a A_\mu^a(x). \]

These correlators are of interest for stochastic models of confinement, where they play a fundamental role, being the lowest order in a cluster expansion of the correlation functions.

The correlators had been determined on lattice a few years ago\(^{11}\) in the range of distances from 0.4 to 1 fm.

Measuring \( D_{\mu\nu\rho\sigma}(x) \) on the lattice naively, taking for \( G_{\mu\nu} \) the open plaquette operator, is difficult because of lattice artefacts due to short range fluctuations.

If these fluctuations are smeared, e.g. by a local cooling procedure, a plateau will eventually be reached in the cooling process, where short range effects have been removed, but large distance physics is left unchanged. The basic idea is that in a local cooling process a distance \( d \) is reached after a number of steps \( t \) which is governed by a sort of diffusion equation

\[ t \propto d^2 \]

The minimum distance which can be explored by this technique is a few (~4) lattice spacings.

In physical units the limitation comes from the requirement that the lattice size \( L a \) must be larger than the typical scale of 1 fm. At \( \beta \approx 6.1 \) 1 fm is 8 lattice spacings so that distances from 0.5 to 1 fm can be explored on a \( 16^4 \) lattice. This was the range of distances in ref.\(^{11}\). To have a distance of 3-4 lattice spacing corresponding to 0.2 fm one needs \( \beta \approx 7 \) and a lattice size \( L \geq 32 \). Fig. 6 shows the two independent form factors \( D \) and \( D_1 \) which parametrize \( D_{\mu\nu\rho\sigma} \) as a function of distance, extracted from the joint sample of data of ref.\(^{11}\) and \(^{29}\).

Across \( T_c \) the magnetic correlators are the same as at \( T = 0 \), while the electric correlators drop by an order of magnitude\(^{11}\). This is consistent with the vanishing of the string tension at \( T > T_c \).
Fig. 6 \( D_\perp = D + D_1 \) (upper curve) and \( D_\parallel = D + D_1 + x^2 \partial D_1 / \partial x^2 \) (lower curve).

References

1. G. ’t Hooft, in “High Energy Physics”, EPS International Conference, Palermo 1975, ed. A. Zichichi; G. ’t Hooft, Nucl Phys. B 190 (1981) 455.
2. S. Mandelstam, Phys. Rep. C 23, 245 (1976).
3. G. Parisi, Phys. Rev. D 11, 971 (1975).
4. L. Del Debbio, A. Di Giacomo, G. Paffuti, Phys. Lett. B 349, 513 (1995).
5. L. Del Debbio, A. Di Giacomo, G. Paffuti, P. Pieri, Phys. Lett. B 355, 255 (1995).
6. E. Witten, Nucl. Phys. B 156, 269 (1979).
7. G. Veneziano, Nucl. Phys. B 159, 213 (1979).
8. C. Christou, A. Di Giacomo, H. Panagopoulos, E. Vicari, Phys. Rev. D 53, 2619 (1996).
9. B. Alles, M. D’Elia, A. Di Giacomo, IFUP-TH 26/96; hep-lat 9605013.
10. B. Alles, G. Boyd, M. D’Elia, A. Di Giacomo, E. Vicari, in preparation.
11. A. Di Giacomo, H. Panagopoulos, Phys. Lett. B 285, 133 (1992).
12. G. ’t Hooft, Nucl. Phys. B 190, 455 (1981).
13. L.P. Kadanoff, H.Ceva, Phys. Rev. B 3, 3918 (1971).
14. Di Cecio, A. Di Giacomo, G. Paffuti, M. Trigiante, cond-mat/9603139.
15. G. ’t Hooft, Phys. Rev. Lett. 37, 8 (1976).
16. S. Weinberg, Phys. Rev. D 11, 3583 (1975).
17. G. ’t Hooft, *Nucl. Phys.* B **72**, 461 (1974).
18. G. Veneziano, *Nucl. Phys.* B **117**, 519 (1976)
19. E. Witten, *Nucl. Phys.* B **117**, 269 (1979).
20. G. Veneziano, *Nucl. Phys.* B **159**, 213 (1979).
21. M. Campostrini, A. Di Giacomo, H. Panagopoulos, *Phys. Lett.* B **277**, 491 (1992).
22. M. Campostrini, A. Di Giacomo, H. Panagopoulos, E. Vicari, *Nucl. Phys.* B **329**, 683 (1990).
23. M. Campostrini, A. Di Giacomo, Y. Günduc, M.P. Lombardo, H. Panagopoulos, R. Tripiccione, *Phys. Lett.* B **252**, 436 (1990).
24. A. Di Giacomo, E. Vicari, *Phys. Lett.* B **275**, 429 (1992).
25. B. Alles, M. Campostrini, A. Di Giacomo, Y. Günduc, E. Vicari, *Phys. Rev.* D **48**, 2284 (1993).
26. E. Shuryak, *Comments in Nucl. and Particle Physics*, 21, 235, (1994).
27. H.G. Dosh, *Phys. Lett.* B **190**, 177 (1987)
28. H.G. Dosh, Yu. A. Simonov, *Phys. Lett.* B **205**, 339 (1988).
29. A. Di Giacomo, E. Meggiolaro, H. Panagopoulos, *IFUP-TH* 14/96; hep-lat 9603017
30. A. Di Giacomo, E. Meggiolaro, H. Panagopoulos, *IFUP-TH* 14/96; hep-lat 9603018