A REMARK ON OBSERVABILITY OF THE WAVE EQUATION WITH MOVING BOUNDARY

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Abstract. We deal with the wave equation with assigned moving boundary \(0 < x < a(t)\) upon which Dirichlet or mixed boundary conditions are specified, here \(a(t)\) is assumed to move slower than the light and periodically. Moreover \(a\) is continuous, piecewise linear with two independent parameters. Our major concern will be an observation problem which is based measuring, at each \(t > 0\) of the transverse velocity at \(a(t)\). The key to the results is the use of a reduction theorem \[8\].

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1. Introduction and main results

We consider the following problems:

\[(1.1)\]
\[
\begin{cases}
    u_{tt} - u_{xx} = 0 & \text{for } 0 < x < a(t), \ t > 0, \\
    u(x, 0) = \phi(x), u_t(x, 0) = \psi(x), & 0 < x < a(0),
\end{cases}
\]

\( (\phi, \psi) \in H^1((0, a(0))) \times L^2((0, a(0))), \) with Dirichlet boundary conditions

\[(1.2)\]

\( u(0, t) = 0 \) and \( u(a(t), t) = 0, \ t > 0, \)

or with mixed boundary conditions for which \[(1.2)\] is replaced by

\[(1.3)\]

\( u(0, t) = 0 \) and \( u_x(a(t), t) = 0, \ t > 0, \)

the subscripts denote partial differentiations, here \(a\) is a strictly positive real function which is continuous, periodic, piecewise linear.

Our major concern will be to find the associated curves \(a(t)\) for which,

\[(1.4)\]

\[\int_0^T |u_x(a(t), t)|^2 \, dt \geq C^* \left( \|\phi\|^2_{H^1_0((0, a(0)))} + \|\psi\|^2_{L^2((0, a(0)))} \right), \]

for \[(1.1), (1.2)\] and

\[(1.5)\]

\[\int_0^T |u_t(a(t), t)|^2 \, dt \geq C^* \left( \|\phi\|^2_{H^1_0((0, a(0)))} + \|\psi\|^2_{L^2((0, a(0)))} \right), \]

2010 Mathematics Subject Classification. 35L05, 34K35, 93B07, 95B05.

Key words and phrases. Strings with moving ends, observability, rotation number.
for $1 \leq j \leq 3$ are valid, where $H^1_t(0, a(0)) = \{ f \in H^1(0, a(0)) \text{ such that } f(0) = 0 \}$.

Note that if $a$ is a constant, the observability inequality

$$
(1.6) \int_0^T |u_x(a, t)|^2 \, dt \geq C^* \left( \|\phi\|_{H^1_t(0, a)}^2 + \|\psi\|_{L^2(0, a)}^2 \right)
$$

holds if $T \geq 2a$ for the Dirichlet problem. Also,

$$
(1.7) \int_0^T |u_t(a, t)|^2 \, dt \geq C^* \left( \|\phi\|_{H^1_t(0, a)}^2 + \|\psi\|_{L^2(0, a)}^2 \right),
$$

holds for the mixed problem.

In [2] the author consider the system

$$
\begin{align*}
\varphi_{tt} - \varphi_{xx} &= 0 \quad \text{for} \quad 0 < x < 1, t > 0, \\
\varphi(0, t) &= \varphi(1, t) = 0, t > 0, \\
\varphi(x, 0) &= \phi(x), \varphi_t(x, 0) = \psi(x), \quad 0 < x < 1.
\end{align*}
$$

For a suitable class of curves $a(t)$, which are $a : [0, T] \to (0, L)$ in the class $C^1([0, T])$

piecewise, i.e. there exists a partition of $[0, T], 0 = t_0 < t_1 < \cdots < t_n = T,$ such that $a \in C^1([t_i, t_{i+1}])$ for all $i = 0, \cdots, n - 1$. Assume also that this partition can be chosen in such a way that $1 - |a'(t)|$ does not change the sign in $t \in [t_i, t_{i+1}]$, for all $i = 0, 1, \cdots, n - 1$. Also he makes the following hypothesis:

1. There exists constants $c_1, c_2 > 0$ and a finite number of open subintervals $I_j \subset [0, T]$ with $j = 0, \cdots, J$ such that, for each subinterval $I_j$, $a \in C^1(I_j)$ and it satisfies the following two conditions:

   • $c_1 \leq |a'(t)| \leq c_2$ for all $t \in I_j$,
   • $1 - |a'(t)|$ does not change the sign in $t \in I_j$.

   We assume, without loss of generality, that there exists $j_1$ with $-1 \leq j_1 \leq J$ such that $a(t)$ is decreasing in $I_j$ for $0 \leq j \leq j_1$, and $a(t)$ is increasing in $I_j$ for $j_1 < j \leq J$.

   The case $j_1 = -1$ corresponds to that where $a(t)$ is increasing in all the subintervals $I_j$. Analogously, $j_1 = J$ corresponds to the case where $a(t)$ is decreasing in all the subintervals $I_j$.

2. For each $j = 0, \cdots, J$, let $U_j$ be the subintervals defined as follows:

   $$
   U_j = \begin{cases} 
   \{ s - a(s) \text{ with } s \in I_j \} & \text{if } j \leq j_1 \\
   \{ s + a(s) \text{ with } s \in I_j \} & \text{if } j > j_1.
   \end{cases}
   $$

   Then, there exists an interval $W_1$ with length $(W_1) > 2L$ such that

   $$
   W_1 \subset \bigcup_{j=0}^J U_j.
   $$

3. For each $j = 0, \cdots, J$, let $V_j$ be the subintervals defined as follows:

   $$
   V_j = \begin{cases} 
   \{ s + a(s) \text{ with } s \in I_j \} & \text{if } j \leq j_1 \\
   \{ s - a(s) \text{ with } s \in I_j \} & \text{if } j > j_1.
   \end{cases}
   $$

   He gets the following observability estimate:

   $$
   (1.8) \int_0^T \left| \frac{d}{dt} [\varphi(a(t), t)] \right|^2 \, dt \geq C^* \left( \|\phi\|_{H^1_t(0, 1)}^2 + \|\psi\|_{L^2(0, 1)}^2 \right),
   $$

where $T$ is given by an optical geometric condition requiring that any ray, starting anywhere in the domain and with any initial direction, must meet the dissipation...
The function \( \lambda \).

Assumption 1.3. \( D \) belongs to \( \mathcal{H} \).

There exist \( \alpha \).

Assumption 1.2. \( H \).

Before stating our main results, let us specify some hypotheses on \( F \) functions \( a \) curves \( \phi \).

Here we introduce a new approach that provides (1.4) and (1.5) for another class of \( H \). We give an example where assumptions 1.2 and 1.3 are guaranteed as in [3].

Remark 1.4. We make less assumptions and get on the occasion a larger class of \( \rho \). Rigorous studies pointing out the use of rotation numbers has led to fruitful contributions, one of which is an elegant and important result (see [5, section II] for more details):

Let \( \text{Lip}(\mathbb{R}) \) be the space of Lipschitz continuous functions on \( \mathbb{R} \). We shall denote the Lipschitz constant of a function \( F \) by

\[
L(F) := \sup_{x,y \in \mathbb{R}, x \neq y} \frac{|F(x) - F(y)|}{|x - y|}.
\]

Furthermore, denote by \( D_p \) the set of functions continuous and strictly increasing of the form \( x + g(x) \), where \( g(x) \) is a periodic continuous function.

**Proposition 1.1.** Let \( a \) be a periodic function. Then

\[
F := (I + a) \circ (I - a)^{-1}
\]

belongs to \( D_p \). Moreover, the rotation number \( \rho(F) \) defined by

\[
\rho(F) = \lim_{n \to \infty} \frac{F^n(x) - x}{n}
\]

exists, and the limit is equal for all \( x \in \mathbb{R} \).

Assume that \( a(t) \) is a periodic function, \( a(t) > 0 \), \( a \in \text{Lip}(\mathbb{R}) \) such that \( L(a) \in (0, 1) \). Assume also that \( |a'(t)| < 1 \) for all \( t \in \mathbb{R} \) and \( \rho(F) \in \mathbb{R} \setminus \mathbb{Q} \) such that there exists a function \( H \in D_p \).

\[
H^{-1} \circ F \circ H(\xi) = \xi + \rho(F).
\]

Before stating our main results, let us specify some hypotheses on \( H \).

**Assumption 1.2.** There exist \( \lambda_1 > 0 \) and \( \lambda_2 > 0 \) such that

\[
\lambda_1 \leq H'(t) \leq \lambda_2, \quad t \in \mathbb{R}.
\]

**Assumption 1.3.** The function \( b(t) := \frac{H'(a(t) + t) - H'(a(t) + t)}{H'(a(t) + t) + H'(a(t) + t)} \) satisfies

\[
c_1 \leq b(t) \leq c_2, \quad c_1, c_2 > 0, \quad \text{for all } t \in \mathbb{R}.
\]

**Remark 1.4.** We make less assumptions and get on the occasion a larger class of functions \( a(t) \) in connexion with the work of Castro [4].

We give an example where assumptions 1.2 and 1.3 are guaranteed as in [3]. Let \( a \) be continuous and periodic on \( \mathbb{R} \), \( a > 0 \), be such that

\[
a(t) := \begin{cases}
\alpha t + \frac{\alpha(1-\alpha)(1+\beta)}{2(\alpha-\beta)} & \text{if } \frac{\alpha(1+\beta)}{2(\alpha-\beta)} \leq t \leq \frac{\alpha(1+\beta)-2\beta}{2(\alpha-\beta)}, \\
\beta t - \beta + \frac{\alpha(1-\alpha)(1+\beta)}{2(\alpha-\beta)} & \text{if } \frac{\alpha(1+\beta)-2\beta}{2(\alpha-\beta)} \leq t \leq \frac{\alpha(1+\beta)-3\beta}{2(\alpha-\beta)}.
\end{cases}
\]
with $\alpha, \beta \in (-1, 1)$. Let $l_1 := \frac{1 + a}{1 + \beta}$, $l_2 := \frac{1 + \beta}{1 + a}$. The definition of $a$ is chosen such that $F$ is directly given on $[0,1]$ and we extend $F$ through the formula: $F(x + 1) = F(x) + 1$ for any $x \in \mathbb{R}$. The function $F$ is defined by:

$$F(x) := (I + a) \circ (I - a)^{-1}(x) = \begin{cases} l_1 x + F_0 & \text{if } 0 \leq x \leq x_0 \\ l_2 x + F_0 + 1 - l_2 & \text{if } x_0 < x < 1, \end{cases}$$

with $F_0 := \frac{l_2(l_1-1)}{l_2 - l_1}$, $x_0 := \frac{l_1}{l_2 - l_1}$. Also the rotation number is given by the expression:

$$\rho(F) = \frac{\ln l_1}{\ln \left(\frac{l_2}{l_1}\right)},$$

and the function $H$ given by (1.10) is done by

$$H(x) = h_0 \ln(|x + h_1|) + h_1,$$

where $h_0 = \frac{1}{\ln\left(\frac{l_2}{l_1}\right)}$, $h_1 = \frac{l_2 - l_1}{l_2 - l_1}$ and $h_2 = -\ln(|h_1|)$, and satisfies the following inequalities: if $l_1 > l_2$,

$$\frac{1}{\ln\left(\frac{l_2}{l_1}\right)} \frac{l_1 - l_2}{l_1} \leq H'(x) \leq \frac{1}{\ln\left(\frac{l_2}{l_1}\right)} \frac{l_1 - l_2}{l_2},$$

and if $l_1 < l_2$,

$$\frac{1}{\ln\left(\frac{l_2}{l_1}\right)} \frac{l_1 - l_2}{l_2} \leq H'(x) \leq \frac{1}{\ln\left(\frac{l_2}{l_1}\right)} \frac{l_1 - l_2}{l_1}.$$  

The function $b$ which is 1-periodic is defined on $[0,1)$ by

$$b(t) = -\frac{a(t)}{t + \frac{l_2 - l_1}{l_2 - l_1}}.$$

Assuming that $l_1 < l_2$, this function satisfies for all $t \in \mathbb{R}$

$$\frac{a_{\min}(l_2 - l_1)}{l_2} \leq \frac{a(t)(l_2 - l_1)}{l_2} \leq b(t) \leq \frac{a(t)(l_2 - l_1)}{l_1} \leq \frac{a_{\max}(l_2 - l_1)}{l_1}.$$  

On the existence of solutions to the Dirichlet or the mixed problem, we refer the reader to [3]. We have the following proposition:

**Proposition 1.5.** If $a \in Lip(\mathbb{R})$, $L(a) \in [0,1)$, $a > 0$ and

$$(\varphi_0, \varphi_1) \in H^1_0((0,a(0))) \times L^2((0,a(0)))$, or in $H^1_0((0,a(0))) \times L^2((0,a(0)))$, denote by $Q := (0,a(t)) \times \mathbb{R}^+$ and $Q_\tau := (0,a(t)) \times (0,\tau), \tau \in \mathbb{R}^+$. There exists a unique weak solution $u$ of either the Dirichlet or the mixed problem satisfying the initial conditions $u(x,0) = \phi(x), u_t(x,0) = \psi(x)$, $0 < x < a(0)$. Moreover there exists $f \in H^1_{loc}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that

$$u(x, t) = f(t + x) - f(t - x) \quad a.e. \text{ in } Q,$$

and $u \in L^\infty(Q) \cap H^1(Q_\tau)$.

Our main results are stated as follows:

**Theorem 1.6** (Neumann observability). Under the assumption (1.2) there exist $T, C^* > 0$ such that for all $u$ solution of the system (1.1) with the Dirichlet boundary condition (1.3) and initial data $(\phi, \psi) \in H^1_0((0,a(0))) \times L^2((0,a(0)))$, we have

$$\int_0^T \|u_x(a(t), t)\|^2 \, dt \geq C^* \left(\|\phi\|^2_{H^1_0((0,a(0)))} + \|\psi\|^2_{L^2((0,a(0)))}\right).$$

\(^{1}u \in H^1(Q_\tau)\) is called a weak solution of either the Dirichlet or the mixed problem if $u_{tt} - u_{xx} = 0$ in $\mathcal{D}'(Q)$ and the boundary conditions are satisfied.
Lemma 2.4. The next lemma will be very useful for the proof of our main results.

\( K \)

Proposition 2.3. \( \xi \) and \( R \)

Corollary 1.8. Based on the observability estimate mentioned above, we get:

Proposition 2.2.

The following propositions can essentially be found in \[7\], we reproduce them here for the reader’s convenience and for (\( x,t \)) we give further comments on the quasi periodic case.

The paper is organized as follows: In section 2 we prove our main results and in the last section we give further comments on the quasi periodic case.

2. PROOF OF THE MAIN RESULTS

We shall construct a transformation of the time-dependent domain \([0, a(t)] \times \mathbb{R}\) onto \([0, \rho(F)/2] \times \mathbb{R}\) that preserves the D’Alembertian form of the wave equations. This preserving property will reveal very important. Using \( H \) given by \((1.11)\), we define a domain transformation \( \Phi: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) as follows:

\[
\begin{align*}
\xi &= (H(x + t) - H(-x + t))/2, \\
\tau &= (H(x + t) + H(-x + t))/2,
\end{align*}
\]

for \((x,t) \in \mathbb{R}^2\).

Remark 2.1. The following propositions can essentially be found in \[7\] (see also the references therein), we reproduce them here for the reader’s convenience and because our presentation is synthetic.

Proposition 2.2. The transformation \( \Phi \) is a bijection of \([0, a(t)] \times \mathbb{R}\) to \([0, \rho(F)/2] \times \mathbb{R}\) and \( \Phi \) maps the boundaries \( x = 0 \) and \( x = a(t) \) onto the boundaries \( \xi = 0 \) and \( \xi = \rho(F)/2 \) (resp).

Proposition 2.3. Let \( u(x,t) \) satisfying \((\partial_x^2 - \partial_t^2)u(x,t) = 0 \) and \( V(\xi, \tau) \) defined by \( u(\Phi^{-1}(\xi, \tau)) \). Then the following identity holds

\[
(\partial_x^2 - \partial_t^2)u(x,t) = K(\xi, \tau)(\partial_x^2 - \partial_t^2)V(\xi, \tau)
\]

where \( K(\xi, \tau) \) is defined by

\[
4H' \circ H^{-1}(\xi + \tau)H' \circ H^{-1}(-\xi + \tau) \circ H^{-1}(\xi + \tau).
\]

The next lemma will be very useful for the proof of our main results.

Lemma 2.4. Denote by

\[
E_u(t) = \frac{1}{2} \int_0^{a(t)} \left[ |u_t(x,t)|^2 + |u_x(x,t)|^2 \right] dx
\]

the energy of the field \( u \), and

\[
E_V(\tau) = \int_0^{\rho(F)/2} \left( |V_\xi(\xi, \tau)|^2 + |V_\tau(\xi, \tau)|^2 \right) d\xi,
\]

the energy of the field \( V \). There are two positive constants \( C_1 \) and \( C_2 \) such that

\[
C_1 E_V(\tau) \leq E_u(t) \leq C_2 E_V(\tau).
\]

(2.21)
Proof. We calculate,
\[ \partial_t u = \partial_{\xi} V \partial_{\xi} \xi + \partial_{\tau} V \partial_{\tau} \]
and so,
\[ E_u(t) = \frac{1}{2} \int_0^t \left[ |u_t(x,t)|^2 + |u_x(x,t)|^2 \right] dx = \]
\[ \frac{1}{2} \int_0^t \left\{ |V_{\xi} \xi_t + V_{\tau} \tau_t|^2 + |V_{\xi} \xi_x + V_{\tau} \tau_x|^2 \right\} dx. \]

Make use of:
\[ \xi_x = (\partial_{\xi} \xi) = (\partial_{\tau} \tau) = \tau_x = [H'(x + t) + H'(-x + t)]/2, \]
\[ \xi_t = (\partial_{\xi} \xi) = (\partial_{\tau} \tau) = \tau_x = [H'(x + t) - H'(-x + t)]/2. \]

Hence,
\[ \xi^2 + \xi_t^2 = \tau^2 + \tau_x^2 = \frac{1}{2} \left[ |H'(x + t)|^2 + |H'(-x + t)|^2 \right], \]
\[ \xi \tau \tau = \xi_x \tau_x = \frac{1}{4} \left[ (H'(x + t))^2 - (H'(-x + t))^2 \right]. \]

Back to the energy,
\[ E_u(t) = \int_0^t \frac{1}{2} \left[ |V_{\xi} \xi|^2 + |V_{\tau} \tau|^2 \right] \left[ |H'(x + t)|^2 + |H'(-x + t)|^2 \right] dx \]
\[ + \int_0^t \frac{1}{2} |V_{\xi} V_{\tau}| \left[ |H'(x + t)|^2 - |H'(-x + t)|^2 \right] dx. \]

Also, differentiating \( x = (H^{-1}(\xi + \tau) - H^{-1}(-\xi + \tau))/2, \) we obtain
\[ dx = 1/2((H^{-1})'(\xi + \tau) + (H^{-1})'(\xi - \tau))d\xi. \]

The inequality \( |V_{\xi} V_{\tau}| \leq 1/2 \left( V_x^2 + V_{\tau}^2 \right) \) yields to:
\[ (2.22) \quad C_1 E_V(\tau) \leq E_u(t) \leq C_2 E_V(\tau). \]

for positive constants \( C_1, C_2. \)

\( \Box \)

Remark 2.5. Applying the transformation \( \Phi, \) the system (1.1)-(1.2) becomes:
\[ \left\{ \begin{array}{l}
\partial^2_t V - \partial^2_{\xi} V = 0, \quad \text{for} \quad 0 < \xi < \rho(F)/2, \quad \tau \in \mathbb{R}, \\
V(0, \tau) = 0, \quad V(\rho(F)/2, \tau) = 0, \quad \tau \in \mathbb{R}, \\
V(\xi, 0) = \psi_1(\xi), \quad V_x(\xi, 0) = \psi_1(\xi), \quad \xi \in (0, \rho(F)/2).
\end{array} \right. \]  

(2.23)

We need the following Lemma.

Lemma 2.6. If \( T > \rho(F), \) there exists \( C(T) > 0 \) such that for all \((\phi_1, \psi_1) \in H^1_0(0, \rho(F)/2) \times L^2(0, \rho(F)/2)\) we have
\[ C(T) \int_0^T |V_{\xi}(\rho(F)/2, \tau)|^2 d\tau \geq \|\phi_1\|^2_{H^1_0(0, \rho(F)/2)} + \|\psi_1\|^2_{L^2(0, \rho(F)/2)}. \]

Proof of Theorem 1.6. We consider (1.1)-(1.2) and state:
\[ \partial_x u = \partial_{\xi} V \partial_{\xi} \xi + \partial_{\tau} V \partial_{\tau} \]

Next we have:
\[ \partial_x u(a(t), t) = \partial_{\xi} V(\rho(F)/2, \tau) \partial_{\xi} \xi(a(t), t) + \partial_{\tau} V(\rho(F)/2, \tau) \partial_{\tau} \tau(a(t), t) \]

Since \( \partial_{\xi} \xi = [H'(x + t) + H'(-x + t)]/2 \) and \( \partial_{\tau} \tau = [H'(x + t) - H'(-x + t)]/2, \)

it follows that:
\[ |\partial_x u(a(t), t)|^2 = |\partial_{\xi} V(\rho(F)/2, \tau) \partial_{\xi} \xi(a(t), t) + \partial_{\tau} V(\rho(F)/2, \tau) \partial_{\tau} \tau(a(t), t)|^2 \]
\[ = \frac{1}{4} \left( |\partial_{\xi} V(\rho(F)/2, \tau)|^2 [H'(x + t) + H'(-x + t)] \right)^2. \]
Make use of Young inequalities, (1.18) is a consequence of the inequalities (1.15) and (1.14), Lemma 2.4 and Lemma 2.6.

Proof of Corollary 1.8. Let us consider

\[
\begin{cases}
\partial^2_t V - \partial^2_\xi V = 0, & \text{for } 0 < \xi < \rho(F)/2, \tau \in \mathbb{R}, \\
V(0, \tau) = 0, \quad V(\rho(F)/2, \tau) = 0, & \tau \in \mathbb{R}, \\
V(\xi, 0) = V_0(\xi), \quad V_\tau(\xi, 0) = V_1(\xi), & \xi \in (0, \rho(F)/2).
\end{cases}
\]

The system (2.24) is exactly observable at time $\rho(F)$ that is: there exists $C > 0$ such that for all $\tau \geq \rho(F)$, we have

\[
C(T) \int_0^T |V_\xi(\rho(F)/2, \tau)|^2 d\tau \geq \|\phi_1\|^2_{L^2_t(0, \rho(F)/2)} + \|\psi_1\|^2_{L^2_t(0, \rho(F)/2)},
\]

and so the following problem

\[
\begin{cases}
\partial^2_t \tilde{V} - \partial^2_\xi \tilde{V} = 0, & 0 < \xi < \rho(F)/2, \tau \in \mathbb{R}, \\
\tilde{V}(0, \tau) = 0, \quad \tilde{V}(\rho(F)/2, \tau) = \varphi(\tau), & \tau \in \mathbb{R}, \\
\tilde{V}(\xi, 0) = \tilde{V}_0(\xi), \quad \tilde{V}_\tau(\xi, 0) = \tilde{V}_1(\xi), & \xi \in (0, \rho(F)/2).
\end{cases}
\]

is exactly controllable at $\rho(F)$ that is for all $(\tilde{V}_0, \tilde{V}_1) \in L^2(0, \rho(F)/2) \times H^{-1}(0, \rho(F)/2)$, there exists $g \in L^2(0, \rho(F))$ such that $\tilde{V}(\xi, \tau) = 0$ for all $\tau \geq \rho(F)$.

Moreover, $g := V_\xi(\rho(F)/2, \tau)\chi(0, \rho(F))(\tau)$.

So the following transformed system

\[
\begin{cases}
\partial^2_t u - \partial^2_\xi u = 0, & 0 < x < a(t), \quad t \in \mathbb{R}, \\
u(0, t) = 0, \quad u(a(t), t) = f(t), & t \in \mathbb{R}, \\
u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in (0, a(0))
\end{cases}
\]

is exactly controllable with a time of control $T := |e^{\varphi(F)-h_0} - h_1|$ and a control $f(t)$ is given by $f(t) = g \left( \frac{H(a(t)+t)+H(-a(t)+t)}{2} \right)$.

\[\square\]

3. Further comments: The quasi periodic case

One can try to generalize the previous results to the case when $a$ is no longer periodic but has some sort of quasiperiodicity\(^2\).

The problem is much more complicated, since there is no rotation number. However, in [7] the author uses a weaker notion of upper (resp. lower) rotation number of $F$ at every point $x$ as follows:

\[
\overline{\rho}(F) = \limsup_{n \to +\infty} \frac{F_n(x) - x}{n}
\]

(resp. $\underline{\rho}(F) = \liminf_{n \to +\infty} \frac{F_n(x) - x}{n}$).

As a consequence, it is shown that under the same Diophantine condition [11], [10] satisfied by $\overline{\rho}(F)$ (resp. $\underline{\rho}(F)$), the rotation number of $F$ exists and coincides with the lower (resp. upper) rotation number.

Lemma 3.1. Assume that $a(t)$ is an $\eta-q,p$ function, $\hat{a}(\theta)$ is real analytic and satisfy $|\hat{a}'(\theta)| < 1$ for $\eta, \theta \in \mathbb{R}^m$ and set $\beta = (\frac{2\pi}{\eta_1}, \ldots, \frac{2\pi}{\eta_m})$. Assume also that there exists $C_0 > 0$ depending on $\beta$ such that $|\langle k, \beta \rangle + \pi l/\overline{\rho}(F)| > C_0 \parallel k \parallel_{m+1}$. Then, there

\(^2\)A function $a(t), t \in \mathbb{R}$ is called quasiperiodic with basic frequencies $\omega = (\omega_1, \ldots, \omega_m) \in \mathbb{R}^m$ (brieily $2\pi/\omega-q,p$) if there exists a continuous function $\hat{g}(\theta), \theta = (\theta_1, \ldots, \theta_m) \in \mathbb{R}^m$ that is $2\pi$-periodic in each $\theta_i, i = 1, \ldots, m$ such that $a(t) = \hat{a}(\omega t)$ holds. $\hat{g}(\theta)$, is called the corresponding function and $2\pi/\omega = (\frac{2\pi}{\omega_1}, \ldots, \frac{2\pi}{\omega_m})$ the basic periods of $a$. 
exists a real analytic function \( H(\xi) = \xi + h(\xi) \), where \( h(\xi) \) is an \( \eta \)-q.p. function, such that

\[
H^{-1} \circ F \circ H(\xi) = \xi + \mathcal{P}(F).
\]

**Remark 3.2.** Thanks to Lemma \( \mathcal{L} \) Theorem 1.1 is easily extended by similar arguments.

**Remark 3.3.** Generalizations of the foregoing results may be obtained in a 3D context, assuming that solutions and given data are functions of only \( r = (x^2 + y^2 + z^2)^{1/2} \) with respect to the space variables. Let \( \Omega \) be the domain \( 0 < r < a(t) \) and consider,

\[
\begin{align*}
\tag{3.27}
& u_{tt} - u_{rr} - (2/r)u_r = 0 \quad \text{in} \quad \Omega, \ t > 0, \\
& \text{with boundary conditions} \quad u(0, t) = u(a(t), t) = 0, \ t > 0, \\
& \text{and initial conditions} \quad u(r, 0) = \phi(r), \ u_t(r, 0) = \psi(r), \ 0 < r < a(0).
\end{align*}
\]

Introducing the transformation \( u(r, t) = w(r, t)/r \) leads to the problem:

\[
\begin{align*}
\tag{3.28}
& \begin{cases}
  w_{tt} = w_{rr} & 0 < r < a(t), \ t > 0, \\
  w(0, t) = w(a(t), t) = 0, \ t > 0, \\
  w(r, 0) = r\phi(r), \ w_t(r, 0) = r\psi(r), \ 0 < r < a(0).
\end{cases}
\end{align*}
\]

4. APPENDIX

In this section, we treat the Dirichlet observability.

**Theorem 4.1** (Dirichlet observability). Under the assumptions \( \mathcal{L} \) and \( \mathcal{M} \) suppose moreover that \( l_1 < l_2 \), there exist \( T, \ C > 0 \) such that for all solution \( u \) of the system (1.1) with the mixed boundary condition (1.3) and initial data \( (\phi, \psi) \in H^2_1(0, a(0)) \times L^2(0, a(0)) \), we have

\[
\tag{4.29}
\int_0^T |u_t(a(t), t)|^2 \, dt \geq C^* \left( \|\phi\|_{H^2_1(0, a(0))}^2 + \|\psi\|_{L^2(0, a(0))}^2 \right).
\]

**Remark 4.2.** Using \( \Phi \) given by (2.20), we transform the system (1.1)–(1.3) into:

\[
\tag{4.30}
\begin{align*}
  & \partial^2_t V - \partial^2_\xi V = 0, \quad \text{for} \quad 0 < \xi < \rho(F)/2, \quad \tau \in \mathbb{R}, \\
  & V(0, \tau) = 0, \quad V(\rho(F)/2, \tau) + b(t(\tau))V(\rho(F)/2, \tau) = 0, \quad \tau \in \mathbb{R}, \\
  & V(\xi, 0) = \phi_2(\xi), \quad V(\xi, 0) = \psi_2(\xi), \quad \xi \in (0, \omega/2).
\end{align*}
\]

For the proof of Theorem 4.1 we need the following lemmas.

**Lemma 4.3.** Assume that \( l_1 < l_2 \), then there exist positive constants \( C \) and \( \omega \) such that

\[
\tag{4.31}
E_V(\tau) \leq Ce^{-\omega \tau} E_V(0).
\]

**Proof.** Define the Lyapunov function:

\[
E_1(\tau) = \frac{1}{2} \int_0^{\rho(F)} [V_\xi^2(\xi, \tau) + V_\tau^2(\xi, \tau)]d\xi + \delta \int_0^{\rho(F)} \xi V_\xi(\xi, \tau) V_\tau(\xi, \tau)d\xi.
\]

We obtain for \( \delta < \frac{1}{\rho(F)} \),

\[
\tag{4.32}
0 < (1 - \delta \rho(F))E_V(\tau) \leq E_1(\tau) \leq (1 + \delta \rho(F))E_V(\tau).
\]

We derive \( E_1 \) with respect to \( \tau \), we get

\[
E_1'(\tau) = [V_\xi V_\tau]_{\xi=0}^{\xi=\rho(F)} - \delta \int_0^{\rho(F)} [V_\xi^2(\xi, \tau) + V_\tau^2(\xi, \tau)]d\xi + \frac{\delta}{2} \xi(V_\xi^2 + V_\tau^2)_{\xi=0}^{\xi=\rho(F)}.
\]

\[
E_1'(\tau) = \left[ \frac{\delta}{2} (1 + b(t(\tau))^2 - b(t(\tau))V_\xi^2(\rho(F), \tau) - \frac{\delta}{2} \int_0^{\rho(F)} [V_\xi^2(\xi, \tau) + V_\tau^2(\xi, \tau)]d\xi.
\]

We choose \( \delta \) small enough, taking into account (1.16) and (4.32) we get

\[
E_1'(\tau) \leq -\omega E_1(\tau).
\]

The proof is complete. \( \square \)
Lemma 4.4. If $T > \rho(F)$, then there exists $C(T) > 0$ such that for all $(\phi_2, \psi_2) \in H^1_1(0, \rho(F)/2) \times L^2(0, \rho(F)/2)$ we have

\begin{equation}
C(T) \int_0^T |V_\xi(\rho(F)/2, \tau)|^2 \, d\tau \geq \|\phi_2\|_{H^1_1(0, \rho(F)/2)}^2 + \|\psi_2\|_{L^2(0, \rho(F)/2)}^2,
\end{equation}

and

\begin{equation}
C(T) \int_0^T |V_\tau(\rho(F)/2, \tau)|^2 \, d\tau \geq \|\phi_2\|_{H^1_1(0, \rho(F)/2)}^2 + \|\psi_2\|_{L^2(0, \rho(F)/2)}^2.
\end{equation}

Proof. The energy identity for the system (4.30) gives:

$E_V(T) - E_V(0) = - \int_0^T b(t(\tau))|V_\tau(\rho(F)/2, \tau)|^2 \, d\tau.$

Using (1.16) and (4.31), we obtain

$\int_0^T |V_\tau(\rho(F)/2, \tau)|^2 \, d\tau \geq C \int_0^T b(t(\tau))|V_\tau(\rho(F)/2, \tau)|^2 \, d\tau \geq C(E_V(0) - E_V(T)) \geq CE_V(0)(1 - e^{-\omega T}).$

This permit to conclude the second inequality in Lemma 4.4. For the first inequality, it suffices to use (4.33) and (1.16). $\square$

Proof of Theorem 4.1. For the proof of (4.29), we state as above:

$\partial_t u = \partial_\xi V \partial_\xi \xi + \partial_\tau V \partial_\tau \tau.$

Next we have:

$|\partial_t u(a(t), t)|^2 = \frac{1}{4} \{ \partial_\xi V(\rho(F)/2, \tau) [H'(x + t) - H'(-x + t)] + \partial_\tau V(\rho(F)/2, \tau) [H'(x + t) + H'(-x + t)] \}^2.$

Make use of Young inequalities, Lemma 2.4, 4.33, 4.34 and (1.15), we obtain the desired result. $\square$

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