The work in this paper concerns the study of different approximations for one-dimensional one-phase Stefan-like problems with a space-dependent latent heat. It is considered two different problems, which differ from each other in their boundary condition imposed at the fixed face: Dirichlet and Robin conditions. The approximate solutions are obtained by applying the heat balance integral method (HBIM), the modified HBIM and the refined integral method (RIM). Taking advantage of the exact analytical solutions, we compare and test the accuracy of the approximate solutions. The analysis is carried out using the dimensionless generalised Stefan number (Ste) and Biot number (Bi). It is also studied the case when Bi goes to infinity in the problem with a convective condition, recovering the approximate solutions when a temperature condition is imposed at the fixed face. Some numerical simulations are provided in order to assert which of the approximate integral methods turns out to be optimal. Moreover, we pose an approximate technique based on minimising the least-squares error, obtaining also approximate solutions for the classical Stefan problem.

Key words: Stefan problem, variable latent heat, heat balance integral method, refined heat balance integral method, exact solutions

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1 Introduction

Stefan problems model heat transfer processes that involve a change of phase. They constitute a broad field of study since they appear in a great number of mathematical and industrial significance problems [1, 6, 10, 13]. A large bibliography on the subject is given in [25] and a review on analytical solutions in [26].

The Stefan problem with a space-dependent latent heat can be found in several physical processes. In [23], it was developed a mathematical model for the shoreline movement in a sedimentary basin using an analogy with the one-phase melting Stefan problem with a variable latent heat. Besides, in [31], it was introduced a two-phase Stefan problem with a general type of space-dependent latent heat from the background of the artificial ground-freezing technique.
The assumption of variable latent heat not only becomes meaningful in the study of the shoreline movement or in the soil freezing techniques but also in the nanoparticle melting [18] and in the one-dimensional consolidation with threshold gradient [29]. More references dealing with non-constant latent heat can be found in [3, 4, 7, 9, 14, 17, 21, 27, 30, 32, 33].

In this paper, we are going to consider two different Stefan-like problems (P) and \( (P_h) \) with space-dependent latent heat imposing different conditions at the fixed boundary. The first problem to consider can be stated as follows:

**Problem (P).** Find the location of the free boundary \( x = s(t) \) and the temperature \( T = T(x, t) \) at the liquid region \( 0 < x < s(t) \) such that

\[
\frac{\partial T}{\partial t} = a^2 \frac{\partial^2 T}{\partial x^2}, \quad 0 < x < s(t), \quad t > 0, \tag{1.1a}
\]

\[
T(0, t) = \theta_\infty t^{\alpha/2}, \quad t > 0, \tag{1.1b}
\]

\[
T(s(t), t) = 0, \quad t > 0, \tag{1.1c}
\]

\[
k \frac{\partial T}{\partial x}(s(t), t) = -\gamma s(t)^{\alpha/2} \dot{s}(t), \quad t > 0, \tag{1.1d}
\]

\[
s(0) = 0, \tag{1.1e}
\]

Equation (1.1a) is the heat conduction equation in the liquid region where \( a^2 = \frac{k}{\rho c} \) is the diffusion coefficient being \( k \) the thermal conductivity, \( \rho \) the density mass and \( c \) the specific heat capacity. At \( x = 0 \), a Dirichlet condition (1.1b) is imposed. It must be noticed that the temperature at the fixed boundary is time-dependent and it is characterised by a parameter \( \theta_\infty > 0 \). In addition, condition (1.1c) represents the fact that the phase change temperature is assumed to be 0 without loss of generality, condition (1.1d) is the corresponding Stefan condition and (1.1e) is the initial position of the free boundary.

The remarkable feature of the problem is related to the condition at the interface given by the Stefan condition (1.1d), where the latent heat by unit of volume is space-dependent defined by a power function of the position \( \frac{k}{\rho} x^{\alpha}(t) \) with \( \gamma \) a given positive constant and \( \alpha \) an arbitrarily non-negative real value.

The second problem \((P_h)\) arises by imposing a convective (Robin) condition at the fixed face \( x = 0 \) instead of a Dirichlet one. In mathematical terms, we can define \((P_h)\) as follows:

**Problem \((P_h)\).** Find the location of the free boundary \( x = s_h(t) \) and the temperature \( T_h = T_h(x, t) \) at the liquid region \( 0 < x < s_h(t) \) such that equations (1.1a) and (1.1c)–(1.1e) are satisfied, together with the Robin condition

\[
k \frac{\partial T}{\partial x}(0, t) = \frac{h}{\sqrt{t}} \left[ T(0, t) - \theta_\infty t^{\alpha/2} \right], \quad t > 0, \tag{1.1b*}
\]

Condition (1.1b*) states that the incoming heat flux at the fixed face is proportional to the difference between the material temperature and the ambient temperature. Here, \( \theta_\infty t^{\alpha/2} \) characterises the bulk temperature at a large distance from the fixed face \( x = 0 \) and \( h \) represents the heat transfer at the fixed face. We will work under the assumption that \( h > 0 \) and \( 0 < T_h(0, t) < \theta_\infty t^{\alpha/2} \) in order to guarantee the melting process.

The exact solution to problem \((P)\) was given in [32] for integer non-negative values of \( \alpha \) and was generalised in [33] by taking \( \alpha \) as a real non-negative constant. Besides, the exact solution of the problem \((P_h)\) was provided in [3].
It is known that due to the non-linear nature of the Stefan problem, exact solutions are limited to a few cases and therefore it is necessary to solve them either numerically or approximately.

The idea in this paper is to take advantage of the exact solutions available in the literature testing the accuracy of different approximate integral methods.

The heat balance integral method (HBIM), introduced by Goodman [8], is an approximate technique which is usually employed for solving the location of the free boundary in phase-change problems. It consists in the transformation of the heat equation into an ordinary differential equation in time, assuming a quadratic profile in space for the temperature. For those profiles, several variants have been introduced in [28] and [20]. In addition, in [11, 12, 15, 16] this method has been applied defining new accurate temperature profiles. Moreover, for the case $\alpha = 0$, the explicit solution to the problem ($P_h$) for the two-phase process was given in [24] and this was useful to obtain the accuracy of different HBIMs to problem ($P_h$) in [2].

The paper will be structured as follows: in Section 2 we will give a brief introduction about the approximate methods to be implemented. Then, in Section 3, we will recall the exact solution to problem (P) that considers a Dirichlet condition at the fixed face and we will get some different approximate solutions that will be tested with the exact one. In Section 4, we will present the exact solution to the problem with a Robin condition at the fixed face, i.e. problem ($P_h$). We are going to implement the different approximate methods and we will test their accuracy. In all cases, we are going to provide numerical examples and comparisons. In addition, we will show that the approximate solutions to problem ($P_h$) converge to the approximate solutions to problem (P) when the heat transfer coefficient $h$ goes to infinity. Finally, in Section 5, we will implement an approximate method that consists in minimising the least-squares error as in [19]. For the case $\alpha = 0$, we obtain different approximations for the problems (P) and ($P_h$) by using the least-squares approximate method.

## 2 Heat balance integral methods

The classical HBIM, described for first time in [8], was designed to approximate problems involving phase changes. This method consists in changing the heat equation (1.1a) by an ordinary differential equation in time that arises by assuming a suitable temperature profile consistent with the boundary conditions, integrating (1.1a) with respect to the spacial variable in an appropriate interval, and replacing the Stefan condition (1.1d) by a new equation obtained from the phase-change temperature (1.1c).

Therefore, if we derive condition (1.1c) with respect to time and take into account the heat equation (1.1a), we get

$$\frac{\partial T}{\partial x}(s(t), t)\dot{s}(t) + a^2 \frac{\partial^2 T}{\partial x^2}(s(t), t) = 0. \quad (2.1)$$

Clearing $\dot{s}$ and replacing it in the Stefan condition (1.1d) it gives

$$\frac{k}{\gamma s'(t)} \left[ \frac{\partial T}{\partial x}(s(t), t) \right]^2 = a^2 \frac{\partial^2 T}{\partial x^2}(s(t), t). \quad (1.1d^*)$$

This last condition is going to substitute the Stefan condition in the approximated problem obtained from the classical HBIM.
On the other hand, using equation (1.1a) and the condition (1.1c), we have

\[
\frac{d}{dt} \int_0^{s(t)} T(x, t)dx = \int_0^{s(t)} \frac{\partial T}{\partial t}(x, t)dx + T(s(t), t)\dot{s}(t) = \int_0^{s(t)} a^2 \frac{\partial^2 T}{\partial x^2}(x, t)dx = a^2 \left[ \frac{\partial T}{\partial x}(s(t), t) - \frac{\partial T}{\partial x}(0, t) \right].
\]

Then, by applying the Stefan condition (1.1d) it results that

\[
\frac{d}{dt} \int_0^{s(t)} T(x, t)dx = -a^2 \left[ \frac{\gamma}{k} s(t)\dot{s}(t) + \frac{\partial T}{\partial x}(0, t) \right]. \tag{1.1a^*}
\]

The classical HBIM suggests to solve an approximate problem (P) through a new problem that arises from replacing the heat equation (1.1a) by (1.1a*) and the Stefan condition (1.1d) by (1.1d*) keeping the rest of the conditions of (P) the same. In short, the method consists in solving the problem governed by (1.1a*), (1.1b), (1.1c), (1.1d*) and (1.1e). A priori, this method will work better than the classical one due to the fact that it changes less conditions from the exact problem.

In [28], a modified integral balance method is presented. It postulates to change only the heat equation keeping the same the rest of conditions, even the Stefan condition. It means that it consists in solving an approximate problem given by (1.1a*), (1.1b), (1.1c), (1.1d) and (1.1e).

On the other hand, from the heat equation (1.1a), and the condition (1.1c) we have

\[
\int_0^x \int_0^{s(t)} \frac{\partial T}{\partial t}(z, t)dzdx = \int_0^x \int_0^{s(t)} a^2 \frac{\partial^2 T}{\partial z^2}(z, t)dz dx = a^2 \int_0^{s(t)} \frac{\partial T}{\partial x}(x, t)dx - \frac{\partial T}{\partial x}(0, t) dx,
\]

that is to say

\[
\int_0^x \int_0^{s(t)} \frac{\partial T}{\partial t}(z, t)dzdx = -a^2 \left[ T(0, t) + \frac{\partial T}{\partial x}(0, t)s(t) \right]. \tag{1.1a^†}
\]

The refined integral method (RIM) introduced in [20] suggests to solve an approximate problem given by (1.1a†), (1.1b), (1.1c), (1.1d) and (1.1e). That is to say, to replace the heat equation (1.1a) by (1.1a†).

In all cases, to solve the above approximated problems, it is necessary to adopt a suitable profile for the temperature. Throughout this paper, we will assume a quadratic profile in space

\[
\bar{T}(x, t) = \frac{r^2}{\theta^2} \left[ \hat{A} \left( 1 - \frac{x}{s(t)} \right) + \hat{B} \left( 1 - \frac{x}{\bar{s}(t)} \right)^2 \right]. \tag{2.2}
\]
where \( \tilde{T} \) and \( \tilde{s} \) will be approximations of \( T \) and \( s \), respectively. We can notice that in the chosen profile a power function of time arises in order to be compatible with the boundary conditions imposed in the exact problem.

It is worth to mention that for the approximations to the problem \((P_h)\), it will be enough to consider the same approximate problems stated for \((P)\), changing only the boundary condition \((1.1b)\) by \((1.1b^*)\).

3 One-phase Stefan problem with Dirichlet condition

3.1 Exact solution

Before introducing the different approaching methods for problem \((P)\), we present the exact solution, which was given in [32] and [33] for the cases when \( \alpha \in \mathbb{N}_0 \) and \( \alpha \in \mathbb{R}^+ \setminus \mathbb{N}_0 \), respectively.

Let us define the following non-dimensional parameter:

\[
\text{Ste} = \frac{k\theta_\infty}{\gamma \alpha^{a+2}}
\]

which is called generalised Stefan number \((\text{Ste})\). We use the word ‘generalised’ since in case that the latent heat \( l \) is constant, i.e. \( \alpha = 0 \), we can recover the usual formula for the Ste, which assuming a zero phase-change temperature is given by \( \text{Ste} = \frac{c\theta_\infty}{l} \). Notice that if we take \( \alpha = 0 \) then the Dirichlet condition at the fixed face is given by \( \theta_\infty \) and from the Stefan condition \((1.1d)\) the latent heat becomes \( l = \gamma / \rho \).

Then, if we combine the results found in [32] and [33], we can rewrite the solution of the problem \((P)\) (as it was done in the appendix of [5]), obtaining for each \( \alpha \in \mathbb{R}^+ \) that

\[
T(x, t) = t^{\alpha/2} \left[ A M \left( -\frac{\alpha}{2}, 1, -\eta^2 \right) + B \eta M \left( -\frac{\alpha}{2} + \frac{1}{2}, 3, -\eta^2 \right) \right],
\]

\[
s(t) = 2a \nu \sqrt{t},
\]

where \( \eta = \frac{x}{2a \sqrt{t}} \) is the similarity variable,

\[
A = \theta_\infty, \quad B = \frac{-\theta_\infty M \left( -\frac{\alpha}{2}, 1, -\nu^2 \right)}{\nu M \left( -\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\nu^2 \right)},
\]

and \( \nu \) is the unique positive solution to the following equation:

\[
\text{Ste} \frac{\nu}{2^{\alpha+1}} f(\nu) = \nu^{\alpha+1}, \quad \nu > 0,
\]

where is defined by

\[
f(\nu) = \frac{1}{\nu M \left( \frac{\alpha}{2} + 1, \frac{3}{2}, \nu^2 \right)}
\]

and \( M(a, b, z) \) is the Kummer function defined by

\[
M(a, b, z) = \sum_{s=0}^{\infty} \frac{(a)_s}{(b)_s s!} z^s, \quad \text{(b cannot be a non-positive integer)}
\]
First of all we shall notice that if $M$ is defined by
\[ M = \frac{\text{Ste}}{2^a + 1}, \]
where the constants $A, B$ are defined as a function of $v_1$ by
\[ A = -\frac{2\left[32^a v_1^{a+2} + \text{Ste}((-3 + (1 + \alpha)v_1^2))\right]}{\text{Ste}(3 + (1 + \alpha)v_1^2)}, \]
\[ B = \frac{3\left[2^{a+1}v_1^{a+2} + \text{Ste}(-1 + (1 + \alpha)v_1^2)\right]}{\text{Ste}(3 + (1 + \alpha)v_1^2)}, \]
and the coefficient $v_1$ is a solution to the following equation:
\[ z^{2a+4}(-3)2^{2a+1}(\alpha - 2) + z^{2a+2}(-9)2^{2a+1} + z^{4+\alpha}(-3)2^a(\alpha - 3)(\alpha + 1)\text{Ste} \]
\[ + z^{\alpha+2}(-3)2^{a+1}(\alpha + 7)\text{Ste} + z^{a9}2^a\text{Ste} + z^{42}(\alpha + 1)^2\text{Ste}^2 \]
\[ + z^2(-12)(\alpha + 1)\text{Ste}^2 + 18\text{Ste}^2 = 0, \quad z > 0. \]

Proof First of all we shall notice that if $T_1$ adopts the profile (3.9), it is clear evident that the condition (1.1c) is automatically verified. From the imposed Dirichlet condition at the fixed boundary (1.1b) we get
\[ A_1 + B_1 = 1. \]
Approximate solutions to one-phase Stefan-like problems

\[ \frac{\partial^2 T_1}{\partial x^2}(x, t) = t^{\alpha/2} \theta \frac{2B_1}{s_1^2(t)}. \]

Therefore, from condition (1.1d*), we claim

\[ \frac{k}{\gamma s_1^2(t)} t^{\alpha/2} \theta \frac{A_1^2}{s_1^2(t)} = a^2 t^{\alpha/2} \theta \frac{2B_1}{s_1^2(t)}. \]

Then, it follows that

\[ s_1(t) = \left( \frac{A_1^2 k \theta}{2B_1 \gamma a^2} \right)^{1/\alpha} \sqrt{t}. \]

Defining \( v_1 \) such that \( v_1 = \frac{1}{2a} \left( \frac{A_1^2 k \theta}{2B_1 \gamma a^2} \right)^{1/\alpha} \), we deduce that

\[ s_1(t) = 2a v_1 \sqrt{t}, \]

where \( v_1, A_1 \) and \( B_1 \) are related as

\[ A_1^2 = \frac{2^{\alpha+1} v_1^{\alpha}}{\text{Ste}} B_1. \]

Condition (1.1a*) and

\[ \frac{d}{dt} \int_0^{s_1(t)} T_1(x, t) \, dx = \frac{d}{dt} \int_0^{s_1(t)} t^{\alpha/2} \theta \left[ A_1 \left( 1 - \frac{x}{s_1(t)} \right) + B_1 \left( 1 - \frac{x}{s_1(t)} \right)^2 \right] \, dx \]

\[ = \theta \left( \frac{A_1}{2} + \frac{B_1}{3} \right) \left( 2^{-\alpha/2} s_1(t) + \theta^{\alpha/2} s_1(t) \right) \]

give

\[ \theta \left( \frac{4}{2} + \frac{B_1}{3} \right) \left( \frac{v_1^{\alpha/2-1}}{2} s_1(t) + \theta^{\alpha/2} s_1(t) \right) = -a^2 \left[ \frac{v_1}{2} s_1^2(t) s_1(t) + t^{\alpha/2} \theta \frac{(A_1 + 2B_1)}{s_1(t)} \right]. \]

According to (3.15), it results that

\[ A_1 \left( (\alpha + 1) v_1^{2} - 1 \right) + B_1 \left( \frac{2}{3} (\alpha + 1) v_1^{2} - 2 \right) = \frac{-2^{\alpha+1} v_1^{\alpha+2}}{\text{Ste}}. \]

Thus, we have obtained three equations (3.14), (3.16) and (3.18) for the unknown coefficients \( A_1, B_1 \) and \( v_1 \).

From (3.14) and (3.18), it is obtained that \( A_1 \) and \( B_1 \) are given as a function of \( v_1 \) by (3.11) and (3.12), respectively.

Then, equation (3.16) leads to the fact that \( v_1 \) must be a positive solution to (3.13).

For the existence of solution to problem (P1), it remains to prove that the function \( w_1 = w_1(z) \), defined as the left-hand side of equation (3.13), has at least one positive root. This can be easily check by evaluating \( w_1(0) = 18 \text{Ste}^2 > 0 \) and

\[ w_1(1) = -a^2 (3 2^\alpha - 2 \text{Ste}) \text{Ste} - 2 \alpha (3 4^\alpha + 4 \text{Ste}^2) - 2 (3 4^\alpha + 3 2^{\alpha+2} \text{Ste} - 4 \text{Ste}^2) \]
From the assumption that $0 < \text{Ste} < 1$, we obtain 

$$3 \, 4^\alpha + 3 \, 2^\alpha + 2 \text{Ste} - 4 \text{Ste}^2 > 2^\alpha + 2 \text{Ste} - 4 \text{Ste}^2 = 4 \text{Ste}(3 \, 2^\alpha - \text{Ste}) > 0.$$ 

Therefore $w_1(1) < 0$. Consequently, we can assure that there exists at least one positive solution to equation (3.13) in the interval $(0, 1)$.

**Remark 3.3** The approximated free boundary $s_1$ behaves as a square root of time just like the exact one $s$, it means that $s_1(t) = 2av_1\sqrt{t}$ while $s(t) = 2av\sqrt{t}$.

**Remark 3.4** After Theorem 3.2 follows the question about uniqueness of solution. We found that there exists different values for $\alpha$ and $0 < \text{Ste} < 1$ that leads to multiple roots of equation (3.13), i.e. $w_1(z) = 0$, $z > 0$ (see Figure 1).

However our study must be reduced to find the roots of $w_1(z)$ located in the interval $(0, 1)$ in view of the proof of Theorem 3.2 but also in view of Remark 3.1. For the particular case of $\alpha = 0$ the uniqueness analysis was given in [2].

Although we could not prove it analytically, by setting different values for $\alpha$ and Ste we can see that there exists just one root of the polynomial $w_1(z)$ located in the interval $(0, 1)$. In Figure 2, we illustrate this fact setting $\alpha = 0.5, 1, 1.5, 2, 3, 5, 10$ and $\text{Ste} = 0.5$. We have just plot between $0 \leq z \leq 0.5$ in order to appreciate better this fact.

With the purpose of testing the classical integral balance method and in view of the above remark we will only compare graphically the coefficient $v_1$ that characterises the approximated free boundary problem $s_1$ with the coefficient $v$ that characterises the exact free boundary $s$. In Figure 3, we illustrate this comparisons for different values of $0 < \text{Ste} < 1$ and $\alpha$.

For the comparisons we have assumed that $0 < \text{Ste} < 1$ not only due to the hypothesis in Theorem 3.2, but also because of the fact that in general, the majority of phase change materials under a realistic temperature present an Ste that does not exceed 1 (see [22]).
Approximate solutions to one-phase Stefan-like problems

Now, we will turn to the modified integral balance method. In this case we state an approxi-
mated problem (P₂) for the problem (P) that is stated as follows: find the free boundary \( s_2 = s_2(t) \)
and the temperature \( T_2 = T_2(x, t) \) in \( 0 < x < s_2(t) \) such that equation (1.1a*) and conditions (1.1b),
(1.1c), (1.1d) and (1.1e) are satisfied.

Assuming a quadratic profile in space for \( T_2 \) we obtain the next theorem

**Theorem 3.5** The problem (P₂) has a unique solution given by

\[
T_2(x, t) = t^{\alpha/2} \theta_\infty \left[ A_2 \left( 1 - \frac{x}{s_2(t)} \right) + B_2 \left( 1 - \frac{x}{s_2(t)} \right)^2 \right],
\]

where the constants \( A_2 \) and \( B_2 \) are given by

\[
A_2 = \frac{6 \text{Ste} - 2 \text{Ste} v_2^2 (\alpha + 1) - 3 2^{\alpha+1} v_2^{\alpha+2}}{\text{Ste} \left( v_2^2 (\alpha + 1) + 3 \right)},
\]

\[
B_2 = \frac{-3 \text{Ste} + 3 \text{Ste} v_2^2 (\alpha + 1) + 3 2^{\alpha+1} v_2^{\alpha+2}}{\text{Ste} \left( v_2^2 (\alpha + 1) + 3 \right)},
\]

and where \( v_2 \) is the unique positive solution to the equation

\[
z^{\alpha+4} 2^\alpha (\alpha + 1) + z^{\alpha+2} 3 2^{\alpha+1} + z^2 \text{Ste}(\alpha + 1) - 3 \text{Ste} = 0, \quad z > 0.
\]
Proof  Condition (1.1c) is clearly checked from the chosen temperature profile. From the Stefan condition (1.1d), we obtain

\[-kt^{\alpha/2}\theta_\infty \frac{A_2}{s_2(t)} = -\gamma s_2'(t)s_2(t).\]  \hspace{1cm} (3.24)

Therefore, it results that

\[s_2(t) = \left(\frac{\alpha + 2}{\alpha + 1}\right) \frac{k\theta_\infty}{\gamma} A_2 \left(\frac{\alpha + 1}{\alpha + 2}\right)^{1/(\alpha + 2)} \sqrt{t}.\]  \hspace{1cm} (3.25)

If we introduce the coefficient \(v_2\) such that \(v_2 = \frac{1}{2a} \left(\frac{\alpha + 2}{\alpha + 1}\right) \frac{k\theta_\infty}{\gamma} A_2 \left(\frac{\alpha + 1}{\alpha + 2}\right)^{1/(\alpha + 2)}\), the free boundary can be expressed as

\[s_2(t) = 2a v_2 \sqrt{t},\]  \hspace{1cm} (3.26)

where the following relation holds:

\[A_2 = \frac{2^{\alpha + 1} v_2^{\alpha + 2}}{\operatorname{Ste}}.\]  \hspace{1cm} (3.27)

Taking into account the boundary condition at the fixed face (1.1b), we get

\[A_2 + B_2 = 1.\]  \hspace{1cm} (3.28)

In addition, in virtue of equation (1.1a*), we get

\[A_2 \left((\alpha + 1)v_2^2 - 1\right) + B_2 \left(\frac{2}{3}(\alpha + 1)v_2^2 - 2\right) = \frac{-2^{\alpha + 1} v_2^{\alpha + 2}}{\operatorname{Ste}}.\]  \hspace{1cm} (3.29)

From equations (3.27), (3.28) and (3.29), we claim that \(A_2\) and \(B_2\) can be written in function of \(v_2\) through formulas (3.21) and (3.22), respectively. In addition, \(v_2\) must be a solution to equation (3.23). So that, to finish the proof, it remains to show that equation (3.23) has a unique positive
solution, i.e. the function defined by the left-hand side of this equation \( w_2 = w_2(z) \) has a unique positive root. This is easily checked by noting that

\[
    w_2(0) = -3\text{Ste} < 0, \quad w_2(+\infty) = +\infty, \quad \frac{dw_2}{dz}(z) > 0, \quad \forall z > 0. \tag{\ref{eq:1.1d}+}
\]

In Figure 4, as we did for the classical HBIM, we compare the coefficients \( \nu_2 \) (approximate) with \( \nu \) (exact) for different values of \( 0 < \text{Ste} < 1 \) and \( \alpha \).

The RIM intends to approximate the problem (P) through solving a problem (P_3) that consists in finding the free boundary \( s_3 = s_3(t) \) and the temperature \( T_3 = T_3(x, t) \) in \( 0 < x < s_3(t) \) such that equation (\ref{eq:1.1a}) and conditions (\ref{eq:1.1b}), (\ref{eq:1.1c}), (\ref{eq:1.1d}) and (\ref{eq:1.1e}) are satisfied.

Under the assumption that \( T_3 \) adopts a quadratic profile in space like (\ref{eq:2.2}), we can state the following result.

**Theorem 3.6** The unique solution to problem (P_3) is given by

\[
    T_3(x, t) = t^{\alpha/2} \left[ A_3\theta_\infty \left( 1 - \frac{x}{s_3(t)} \right) + B_3\theta_\infty \left( 1 - \frac{x}{s_3(t)} \right)^2 \right], \tag{3.30}
\]

\[
    s_3(t) = 2av_3^\alpha t, \tag{3.31}
\]

where the constants \( A_3 \) and \( B_3 \) are given by

\[
    A_3 = \frac{6\text{Ste} - 2\text{Ste} v_3^2(\alpha + 1) - 3 2^\alpha+1 v_3^{\alpha+2}}{\text{Ste} (v_3^2(\alpha + 1) + 3)}, \tag{3.32}
\]

\[
    B_3 = \frac{-3\text{Ste} + 3\text{Ste} v_3^2(\alpha + 1) + 3 2^\alpha+1 v_3^{\alpha+2}}{\text{Ste} (v_3^2(\alpha + 1) + 3)}, \tag{3.33}
\]

and where \( v_3 \) is the unique solution to equation

\[
    z^{\alpha+4} 2^{\alpha+1} + z^{\alpha+2} 2^{\alpha+2} + z^2\text{Ste}(2 + 3\alpha) - 6\text{Ste} = 0, \quad z > 0. \tag{3.34}
\]
Proof The proof is similar to the one of the Theorem 3.5. The only difference to take into account is the fact that equation (1.1a) is equivalent to

$$v_3^2 \left[ A_3 \left( \frac{1}{3} + \frac{2}{3} \alpha \right) + B_3 \left( \frac{1}{3} + \frac{\alpha}{2} \right) \right] = B_3.$$ (3.35)

In Figure 5, we compare graphically the coefficient $v_3$ that characterises the approximate free boundary $s_3$ with the coefficient $v$ that characterises the exact boundary $s$.

### 3.3 Comparisons between the approximate solutions and the exact one

In the previous section, we have applied three different methods to approximate the solution to the Stefan problem (P), with a Dirichlet condition at the fixed face and a variable latent heat.

For each method, we have stated a problem $(P_i)$, $i = 1, 2, 3$ and we have compared graphically the dimensionless coefficients $v_i$ that characterises their free boundaries $s_i$, with the coefficient $v$ that characterises the exact free boundary $s$.

Then the goal will be to compare numerically, for different Ste, the coefficient $v$ given by (3.5) with the approximate coefficients $v_1$, $v_2$ and $v_3$ defined by (3.13), (3.23) and (3.34), respectively.

In order that the comparisons be more representative, in Tables 1–3 we show the exact values obtained for $v$, the approximate value $v_i$ and percentage error committed in each case $E(v_i) = 100 \left| \frac{v - v_i}{v} \right|$, $i = 1, 2, 3$ for different values of Ste and $\alpha$.

From the tables, we can notice that for $\alpha = 0.5$, the error committed by each method is lower than for $\alpha = 0$ or $\alpha = 5$. In all cases, the method which shows the greatest accuracy is the modified integral balance method. In other words, the best approximate problem to (P) is given by problem $(P_2)$.

Besides, we can also provide an illustration at the exact temperature $T$ with the approximate temperatures $T_i$, $i = 1, 2, 3$, given by (3.9), (3.19) and (3.30), respectively. If we consider $\alpha = 5$, Ste = 0.5, $\theta_\infty = 30$ and $a = 1$, we obtain Figures 6–9.
### Table 1. Dimensionless coefficients of the free boundaries and their percentage relative error for $\alpha = 0$

| Ste | $\nu$ | $\nu_1$ | $E_{rel}(\nu_1)$ (%) | $\nu_2$ | $E_{rel}(\nu_2)$ (%) | $\nu_3$ | $E_{rel}(\nu_3)$ (%) |
|-----|-------|---------|----------------------|-------|----------------------|-------|----------------------|
| 0.1 | 0.2200 | 0.2232  | 1.4530               | 0.2209 | 0.3947               | 0.2218 | 0.7954               |
| 0.2 | 0.3064 | 0.3143  | 2.5729               | 0.3087 | 0.7499               | 0.3111 | 1.5213               |
| 0.3 | 0.3699 | 0.3827  | 3.4575               | 0.3738 | 1.0707               | 0.3780 | 2.1856               |
| 0.4 | 0.4212 | 0.4388  | 4.1687               | 0.4270 | 1.3618               | 0.4330 | 2.7953               |
| 0.5 | 0.4648 | 0.4869  | 4.7478               | 0.4723 | 1.6266               | 0.4804 | 3.3561               |
| 0.6 | 0.5028 | 0.5290  | 5.2236               | 0.5122 | 1.8683               | 0.5222 | 3.8729               |
| 0.7 | 0.5365 | 0.5666  | 5.6173               | 0.5477 | 2.0895               | 0.5599 | 4.3501               |
| 0.8 | 0.5669 | 0.6006  | 5.9443               | 0.5799 | 2.2923               | 0.5941 | 4.7913               |
| 0.9 | 0.5946 | 0.6316  | 6.2165               | 0.6094 | 2.4786               | 0.6255 | 5.1999               |
| 1.0 | 0.6201 | 0.6600  | 6.4342               | 0.6365 | 2.6500               | 0.6547 | 5.5786               |

### Table 2. Dimensionless coefficients of the free boundaries and their percentage relative error for $\alpha = 0.5$

| Ste | $\nu$ | $\nu_1$ | $E_{rel}(\nu_1)$ (%) | $\nu_2$ | $E_{rel}(\nu_2)$ (%) | $\nu_3$ | $E_{rel}(\nu_3)$ (%) |
|-----|-------|---------|----------------------|-------|----------------------|-------|----------------------|
| 0.1 | 0.2569 | 0.2587  | 0.6956               | 0.2574 | 0.2001               | 0.2580 | 0.4012               |
| 0.2 | 0.3339 | 0.3372  | 0.9999               | 0.3349 | 0.3147               | 0.3360 | 0.6321               |
| 0.3 | 0.3876 | 0.3921  | 1.1718               | 0.3891 | 0.3974               | 0.3907 | 0.7995               |
| 0.4 | 0.4298 | 0.4353  | 1.2678               | 0.4318 | 0.4596               | 0.4338 | 0.9260               |
| 0.5 | 0.4650 | 0.4711  | 1.3143               | 0.4674 | 0.5067               | 0.4698 | 1.0225               |
| 0.6 | 0.4953 | 0.5018  | 1.3264               | 0.4980 | 0.5423               | 0.5007 | 1.0959               |
| 0.7 | 0.5220 | 0.5288  | 1.3133               | 0.5249 | 0.5684               | 0.5280 | 1.1508               |
| 0.8 | 0.5458 | 0.5528  | 1.2814               | 0.5491 | 0.5869               | 0.5523 | 1.1905               |
| 0.9 | 0.5675 | 0.5745  | 1.2352               | 0.5709 | 0.5989               | 0.5744 | 1.2173               |
| 1.0 | 0.5873 | 0.5943  | 1.1777               | 0.5909 | 0.6054               | 0.5946 | 1.2334               |

### Table 3. Dimensionless coefficients of the free boundaries and their percentage relative error for $\alpha = 5$

| Ste | $\nu$ | $\nu_1$ | $E_{rel}(\nu_1)$ (%) | $\nu_2$ | $E_{rel}(\nu_2)$ (%) | $\nu_3$ | $E_{rel}(\nu_3)$ (%) |
|-----|-------|---------|----------------------|-------|----------------------|-------|----------------------|
| 0.1 | 0.3793 | 0.3563  | 6.0700               | 0.3723 | 1.8469               | 0.3656 | 3.6135               |
| 0.2 | 0.4151 | 0.3849  | 7.2853               | 0.4055 | 2.3333               | 0.3963 | 4.5496               |
| 0.3 | 0.4374 | 0.4020  | 8.0816               | 0.4256 | 2.6810               | 0.4145 | 5.2154               |
| 0.4 | 0.4537 | 0.4143  | 8.6859               | 0.4403 | 2.9615               | 0.4276 | 5.7505               |
| 0.5 | 0.4667 | 0.4239  | 9.1776               | 0.4518 | 3.2010               | 0.4377 | 6.2058               |
| 0.6 | 0.4775 | 0.4317  | 9.5943               | 0.4612 | 3.4122               | 0.4460 | 6.6060               |
| 0.7 | 0.4869 | 0.4384  | 9.9572               | 0.4693 | 3.6025               | 0.4529 | 6.9656               |
| 0.8 | 0.4950 | 0.4442  | 10.2795              | 0.4763 | 3.7766               | 0.4589 | 7.2936               |
| 0.9 | 0.5023 | 0.4492  | 10.5699              | 0.4826 | 3.9376               | 0.4642 | 7.5962               |
| 1.0 | 0.5090 | 0.4538  | 10.8345              | 0.4881 | 4.0880               | 0.4689 | 7.8780               |
Figure 6. Colour map for $T$.

Figure 7. Colour map for $T_1$.

Figure 8. Colour map for $T_2$. 
In this section, we are going to present the exact solution of the problem with a Robin condition, then we will obtain different approximate solutions that will be compared and we will analyse their convergence when the coefficient that characterises the heat transfer at the fixed boundary goes to infinity.

4.1 Exact solution

We recall that the exact solution to problem (Pₘ) governed by equations (1.1a), (1.1b*) and (1.1c)–(1.1e) given in [3] can be written as

\[ Tₘ(x, t) = \frac{t^{α/2}}{Aₘ} \left[ Aₘ M \left( -\frac{α}{2}, \frac{1}{2}, -η^2 \right) + Bₘ M \left( -\frac{α}{2} + \frac{1}{2}, \frac{3}{2}, -η^2 \right) \right], \] (4.1)

\[ sₘ(t) = 2avₘ \sqrt{t}, \] (4.2)

where \( η = \frac{x}{2avₘ \sqrt{t}} \) is the similarity variable, the coefficients \( Aₘ \) and \( Bₘ \) are given by

\[ Aₘ = \frac{-vₘ M \left( -\frac{α}{2} + \frac{1}{2}, \frac{3}{2}, -vₘ^2 \right)}{M \left( -\frac{α}{2}, \frac{1}{2}, -vₘ^2 \right)} Bₘ, \] (4.3)

\[ Bₘ = -θ∞ M \left( -\frac{α}{2}, \frac{1}{2}, -vₘ^2 \right) \left[ \frac{1}{2Bi} M \left( -\frac{α}{2}, \frac{1}{2}, -vₘ^2 \right) + vₘ M \left( -\frac{α}{2} + \frac{1}{2}, \frac{3}{2}, -vₘ^2 \right) \right], \] (4.4)

and with \( vₘ \) defined as the unique solution to the following equation:

\[ \frac{Ste}{2^{α+1}} \left[ \frac{1}{f(z)} + \frac{1}{2Bi} M \left( \frac{α}{2}, \frac{1}{2}, \frac{1}{2}, z^2 \right) \right] = z^{α+1}, \quad z > 0, \] (4.5)

where \( Ste \) and \( f \) are given by (3.1) and (3.6), respectively, and where the Biot number (Bi) is defined by \( Bi = \frac{ah}{k} \).
In [3], it was also proved that the unique solution to the exact problem with convective condition \((P_1)\) converges pointwise to the unique solution to the problem with temperature condition \((P)\) when the \(h\) goes to infinity (i.e. \(h \to \infty\)).

### 4.2 Approximate solutions and convergence

As it was done for the problem \((P)\), we will now apply the classical integral balance method, the modified integral balance method and the RIM to the problem \((P_1)\). For each method, we will state an approximate problem \((P_{1h})\), \(i = 1, 2, 3\). Assuming a quadratic profile in space, we will obtain the solutions to the approximate problems. Finally, we will show that the solution of each problem \((P_{1h})\) converges to the solution of the problem \((P)\) defined in the previous section, when \(h \to \infty\). This fact is intuitively expected because the same happens to the exact problems \((P_h)\) and \((P)\).

We introduce an approximate problem \((P_{1h})\) that arises when applying the classical HBIM to the problem \((P_1)\). It consists in finding the free boundary \(s_{1h}(t)\) and the temperature \(T_{1h}(x, t)\) in \(0 < x < s_{1h}(t)\) such that conditions: (1.1a*), (1.1b*), (1.1c), (1.1d*) and (1.1e) are satisfied.

Provided that \(T_{1h}\) adopts a quadratic profile in space, like (2.2) we can prove the next result.

**Theorem 4.1** If \(0 < \text{Ste} < 1\), \(\alpha \geq 0\) and \(\text{Bi}\) is large enough, there exists at least one solution to problem \((P_{1h})\), which is given by

\[
T_{1h}(x, t) = \theta_{\infty} \left[ A_{1h} \left( 1 - \frac{x}{s_{1h}(t)} \right) + B_{1h} \left( 1 - \frac{x}{s_{1h}(t)} \right)^2 \right],
\]

\[
s_{1h}(t) = 2a v_{1h} \sqrt{t},
\]

where the constants \(A_{1h}\) and \(B_{1h}\) are defined as a function of \(v_{1h}\)

\[
A_{1h} = \frac{6 \text{Ste} - 2 \text{Ste}^2 v_{1h}^2 (\alpha + 1) - \frac{3}{\text{Bi}} 2^{\alpha+1} v_{1h}^2 (\alpha + 1) - 3 2^{\alpha+1} v_{1h}^2 (\alpha + 2)}{\text{Ste} \left[ v_{1h}^2 (\alpha + 1) + \frac{2}{\text{Bi}} v_{1h} (\alpha + 1) + 3 \right]},
\]

\[
B_{1h} = \frac{-3 \text{Ste} + 3 \text{Ste} v_{1h}^2 (\alpha + 1) + \frac{3}{\text{Bi}} 2^{\alpha+1} v_{1h}^2 (\alpha + 1) + 3 2^{\alpha+1} v_{1h}^2 (\alpha + 2)}{\text{Ste} \left[ v_{1h}^2 (\alpha + 1) + \frac{2}{\text{Bi}} v_{1h} (\alpha + 1) + 3 \right]},
\]

where \(v_{1h}\) is a solution to the following equation:

\[
z^{2\alpha+4}(-3)2^{\alpha+1}(\alpha - 2) + z^{2\alpha+3}(-3)^2 \frac{2^{\alpha}}{\text{Bi}}(5\alpha - 7) + z^{2\alpha+2}(-3)2^{\alpha+1} \left( \frac{\alpha - 2}{\text{Bi}^2} + 3 \right)
\]

\[
+ z^{2\alpha+1}(-9) \frac{2^{2\alpha}}{\text{Bi}} + z^{\alpha+4}(-3)2^{\alpha} \text{Ste}(\alpha - 3)(\alpha + 1) + z^{\alpha+3}(-3)2^{\alpha+1} \text{Bi} \text{Ste}(\alpha - 1)(\alpha + 1)
\]

\[
+ z^{\alpha+2}(-3)2^{\alpha+1} \text{Ste}(\alpha + 7) + z^{\alpha+3}2^{\alpha+1} \text{Bi} \text{Ste}(\alpha - 5) + z^{\alpha+9} 2^{\alpha} \text{Ste} + z^{\alpha+2} \text{Ste}^2(1 + \alpha)^2
\]

\[
+ z^2(12) \text{Ste}^2(\alpha + 1) + 18 \text{Ste}^2 = 0, \quad z > 0.
\]
Proof It can be easily checked that the chosen profile (4.6) verifies condition (1.1c). In addition, we have
\[
\frac{\partial T_{1h}(x,t)}{\partial x} = -t^{\alpha/2} \theta_\infty \left[ \frac{A_{1h}}{s_{1h}(t)} + \frac{2B_{1h}}{s_{1h}(t)} \left( 1 - \frac{x}{s_{1h}(t)} \right) \right],
\]
and
\[
\frac{\partial^2 T_{1h}(x,t)}{\partial x^2} = t^{\alpha/2} \frac{2B_{1h}}{s_{1h}(t)}.
\]
In virtue of condition (1.1d*), the following equality holds:
\[
\frac{k}{\gamma s_{1h}^2(t)} t^{\alpha/2} \frac{A_{1h}^2}{s_{1h}^3(t)} = a^2 t^{\alpha/2} \frac{2B_{1h}}{s_{1h}^2(t)}.
\]
Consequently,
\[
s_{1h}(t) = \left( \frac{A_{1h}^2 k \theta_\infty}{2B_{1h} \gamma a^2} \right)^{1/\alpha} \sqrt{t}.
\]
Defining \( \nu_{1h} \) such that
\[
\nu_{1h} \equiv \frac{1}{2a} \left( \frac{A_{1h}^2 k \theta_\infty}{2B_{1h} \gamma a^2} \right)^{1/\alpha},
\]
we conclude that
\[
s_{1h}(t) = 2a \nu_{1h} \sqrt{t}, \quad (4.11)
\]
where \( \nu_{1h} \) is an unknown that is related with \( A_{1h} \) and \( B_{1h} \) in the following way:
\[
A_{1h}^2 = \frac{2^{\alpha+1} \nu_{1h}}{\text{St}} B_{1h}. \quad (4.12)
\]
Then, condition (1.1a*) leads to
\[
A_{1h} \left[ (\alpha + 1) \nu_{1h}^2 - 1 \right] + B_{1h} \left[ \frac{2}{3} (\alpha + 1) \nu_{1h}^2 - 2 \right] = -\frac{2^{\alpha+1}}{\text{St}} \nu_{1h}. \quad (4.13)
\]
In addition, according to (1.1b*), we have
\[
A_{1h} \left( 1 + 2\text{Bi} \nu_{1h} \right) + 2B_{1h} \left( 1 + \text{Bi} \nu_{1h} \right) = 2\text{Bi} \nu_{1h}. \quad (4.14)
\]
Thus, we have obtained three equations (4.12), (4.13) and (4.14), for the three unknown coefficients \( A_{1h}, B_{1h} \) and \( \nu_{1h} \).
From (4.13) and (4.14), we obtain that \( A_{1h} \) and \( B_{1h} \) are given by (4.8) and (4.9), respectively.
Then, equation (4.12) leads to \( \nu_{1h} \) as a positive solution to equation (4.10). If we denote by \( \omega_{1h} = \omega_{1h}(z) \) the left-hand side of equation (4.10), we have
\[
\omega_{1h}(0) = 18 \text{Ste}^2 > 0 \quad (4.15)
\]
and
\[
\omega_{1h}(1) = -\alpha^2 \left( 3 \left( 2^{\alpha} - 2 \text{Ste} + \frac{3}{\text{Bi}} 2^{\alpha+1} \right) \text{Ste} - 2 \alpha \left( 3 \left( 4^{\alpha} + 4 \text{Ste}^2 + \frac{21}{\text{Bi}} 2^{\alpha-1} - \frac{3}{\text{Bi}} 2^{\alpha} \text{Ste} \right) - 2 \left( 3 \left( 4^{\alpha} + 3 \left( 2^{\alpha+3} \text{Ste} - 4 \text{Ste}^2 \right) + \frac{3}{\text{Bi}} \left( 2^{2\alpha+3} - 2^{3+\alpha} \text{Ste} \right). \quad (4.16)
\]
It can be noticed that if $0 < \text{Ste} < 1$ and $\alpha \geq 0$, we have
\[
3 \cdot 2^\alpha - 2\text{Ste} + \frac{3}{\text{Bi}} 2^{\alpha+1} > 0,
\]
\[
3 \cdot 4^\alpha + 3 \cdot 2^{2+\alpha} \text{Ste} - 4\text{Ste}^2 > 0,
\]
and
\[
3 \cdot 4^\alpha + 4\text{Ste}^2 + \frac{21}{\text{Bi}} 2^{\alpha-1} - \frac{3}{\text{Bi}} 2^\alpha \text{Ste} = 3 \cdot 4^\alpha + 4\text{Ste}^2 + \frac{3}{\text{Bi}} 2^\alpha \left( \frac{7}{2} - \text{Ste} \right) > 0.
\]
As $2^{2\alpha+3} - 2^{3+\alpha} \text{Ste} = 2^\alpha 2^3(2^\alpha - \text{Ste}) > 0$, there exists a large enough Bi that makes $\omega_{1h}(1) < 0$. In consequence, there will exists at least one solution to equation (4.10).

With the aim of testing the accuracy of the classical HBIM and taking into account that the exact free boundary $s_h(t) = 2a\nu_h \sqrt{t}$ and the approximate one is given by $s_{1h}(t) = 2a\nu_{1h} \sqrt{t}$ we are going to compare graphically only the coefficients $\nu_h$ with $\nu_{1h}$ for different values of Bi and $\alpha$, fixing Ste = 0.5 (see Figure 10).

The modified integral balance method defines a new approximated problem for $(P_h)$ that will be called as problem $(P_{2h})$ and which consists in finding the free boundary $s_{2h} = s_{2h}(t)$ and the temperature $T_{2h} = T_{2h}(x, t)$ in $0 < x < s_{2h}(t)$ such that equations (1.1a*), (1.1b*) and (1.1c)–(1.1e) are satisfied.

Once again assuming a quadratic profile in space as (2.2) for the temperature $T_{2h}$, we can state the following results.

**Theorem 4.2** Given $\text{Ste} > 0$ and $\alpha \geq 0$, there exists a unique solution to the problem $(P_{2h})$ which is given by

\[
T_{2h}(x, t) = \varphi^{\alpha/2} \left[ A_{2h} \theta_\infty \left( 1 - \frac{x}{s_{2h}(t)} \right) + B_{2h} \theta_\infty \left( 1 - \frac{x}{s_{2h}(t)} \right)^2 \right], \tag{4.17}
\]

\[
s_{2h}(t) = 2a\nu_{2h} \sqrt{t}, \tag{4.18}
\]
where the constants $A_{2h}$ and $B_{2h}$ are given by

$$A_{2h} = \frac{6 \text{Ste} - 2 \text{Ste} v_{2h}^2 (\alpha + 1) - \frac{3}{\text{Bi}} 2^{\alpha+1} \nu_2 \alpha^{\alpha+1} - 3 2^{\alpha+1} \nu_2^{\alpha+2}}{\text{Ste} \left[ v_{2h}^2 (\alpha + 1) + \frac{2}{\text{Bi}} v_{2h} (\alpha + 1) + 3 \right]}, \quad (4.19)$$

$$B_{2h} = \frac{-3 \text{Ste} + 3 \text{Ste} v_{2h}^2 (\alpha + 1) + \frac{3}{\text{Bi}} 2^{\alpha} \nu_2^{\alpha+1} + 3 2^{\alpha+1} \nu_2^{\alpha+2}}{\text{Ste} \left[ v_{2h}^2 (\alpha + 1) + \frac{2}{\text{Bi}} v_{2h} (\alpha + 1) + 3 \right]}, \quad (4.20)$$

and where the coefficient $v_{2h}$ is the unique solution to the following equation:

$$z^{\alpha+4} 2^{\alpha} (\alpha + 1) + z^{\alpha+3} \frac{2^{\alpha+1}}{\text{Bi}} (\alpha + 1) + z^{\alpha+2} 2^{\alpha+1}$$

$$+ z^{\alpha+1} \frac{2^{\alpha}}{\text{Bi}} + z^2 \text{Ste}(\alpha + 1) - 3 \text{Ste} = 0, \quad z > 0. \quad (4.21)$$

**Proof** It is clear immediate that the chosen profile temperature leads the condition (1.1c) to be automatically verified. From condition (1.1d), we obtain

$$-k t^{\alpha/2} \frac{\nu_{2h}}{s_{2h}(t)} = -\gamma s_{2h}'(t). \quad (4.22)$$

Therefore,

$$s_{2h}(t) = \left( \frac{(\alpha + 2) k \theta_{\infty}}{(\frac{\alpha}{2} + 1)} \right)^{1/(\alpha+2)} \sqrt{t}. \quad (4.23)$$

Introducing the new coefficient $v_{2h}$ such that $v_{2h} = \frac{1}{2a} \left( \frac{(\alpha + 2) k \theta_{\infty}}{(\frac{\alpha}{2} + 1)} \right)^{1/(\alpha+2)} A_{2h}$, the free boundary can be expressed as

$$s_{2h}(t) = 2a v_{2h} \sqrt{t}, \quad (4.24)$$

where the following equality holds:

$$A_{2h} = \frac{2^{\alpha+1} \nu_2^{\alpha+2}}{\text{Ste}}. \quad (4.25)$$

The convective boundary condition at $x = 0$, i.e. condition (1.1b*), leads to

$$A_{2h} (1 + 2 \text{Bi} v_{2h}) + 2 B_{2h} (1 + \text{Bi} v_{2h}) = 2 \text{Bi} v_{2h}. \quad (4.26)$$

In addition, from (1.1a*) it results that

$$A_{2h} \left( (\alpha + 1) v_{2h}^2 - 1 \right) + B_{2h} \left( \frac{2}{3} (\alpha + 1) v_{2h}^2 - 2 \right) = \frac{-2^{\alpha+1} \nu_2^{\alpha+2}}{\text{Ste}}. \quad (4.27)$$

Taking into account equations (4.25)–(4.27), we obtain that $A_{2h}$ and $B_{2h}$ can be given as functions of $v_{2h}$ through formulas (4.19) and (4.20), respectively. Moreover, we get that $v_{2h}$ must be a solution to equation (4.21). To finish the proof, it remains to show that We shall notice first that
(4.21) has a unique positive solution. If we define the function $w_{2h} = w_{2h}(z)$ as the left-hand side of equation (4.21), we have that

$$w_{2h}(0) = -3\text{Ste} < 0, \quad w_{2h}(+\infty) = +\infty, \quad \frac{dw_{2h}}{dz}(z) > 0, \quad \forall z > 0.$$ 

So we conclude that $w_{2h}$ has a unique positive root.

In what follows, we will show that the unique solution to the problem $(P_{2h})$ converges to the unique solution to the problem $(P_2)$ when $h \to \infty$.

**Theorem 4.3** The solution to problem $(P_{2h})$ given in Theorem 4.2 converges to the solution to problem $(P_2)$ given by Theorem 3.5 when the coefficient $h$, which characterises the heat transfer in the fixed boundary, goes to infinity.

**Proof** The free boundary of the problem $(P_{2h})$ is characterised by a dimensionless coefficient $v_{2h}$ which is the unique positive root of the function $\omega_{2h} = \omega_{2h}(z)$ defined as the left-hand side of equation (4.21). On the one hand, we can notice that if $h_1 < h_2$ then $\omega_{2h_1}(z) > \omega_{2h_2}(z)$ and consequently their unique positive root verify $v_{2h_1} < v_{2h_2}$.

On the other hand, if we define $\omega_2 = \omega_2(z)$ as the left-hand side of equation (3.23), we get

$$\omega_{2h}(z) - \omega_2(z) = z^{\alpha+3}2^{\alpha+1}\frac{\text{Bi}}{(\alpha + 1)} + z^{\alpha+3}\frac{2^\alpha}{\text{Bi}} > 0, \quad \forall z > 0.$$ 

Therefore, $\{v_{2h}\}_h$ is increasing and bounded from above by $v$.

In addition, it is easily seen that when $h \to \infty$, or equivalently when $\text{Bi} \to \infty$, we obtain $\omega_{2h} \to \omega_2$ and so $v_{2h} \to v_2$. Therefore, it is obtained that $s_{2h}(t) \to s_2(t)$, for every $t > 0$. Showing that $A_{2h} \to A_2$ and $B_{2h} \to B_2$ we get $T_{2h}(x,t) \to T_2(x,t)$ when $h \to \infty$ for every $t > 0$ and $0 < x < s_2(t)$.

In Figure 11, we compare graphically, for different values of $\text{Bi} > 1$, the coefficient $v_{2h}$ that characterises the free boundary $s_{2h}$ with the coefficient $v_h$ that characterises the exact free
boundary $s_h$, for different values of $\alpha$, fixing $\text{Ste} = 0.5$. We shall notice that when the $\text{Bi}$ increases then the value of $v_{2h}$ gets closer to the value of $v_2$.

Lastly, we will turn to the RIM applied to problem $(P_h)$. We define a new approximate problem $(P_{3h})$ which consists in finding the free boundary $s_{3h} = s_{3h}(t)$ and the temperature $T_{3h} = T_{3h}(x, t)$ in $0 < x < s_{3h}(t)$ such that equations (1.1a*), (1.1b*) and (1.1c)–(1.1e) are verified.

Provided that $T_{3h}$ adopts a profile like (2.2), we state the following theorem.

**Theorem 4.4** Let $0 < \text{Ste} < 1$, $\alpha \geq 0$ and $\text{Bi} \geq 0$, then there exists a unique solution to problem $(P_{3h})$ which is given by

$$T_{3h}(x, t) = t^{\alpha/2} \left[ A_{3h} \theta_{\infty} \left( 1 - \frac{x}{s_{3h}(t)} \right) + B_{3h} \theta_{\infty} \left( 1 - \frac{x}{s_{3h}(t)} \right)^2 \right], \quad (4.28)$$

$$s_{3h}(t) = 2a_{3h} \sqrt{t}, \quad (4.29)$$

where the constants $A_{3h}$ and $B_{3h}$ are defined by

$$A_{3h} = \frac{12 v_{3h} \left( 1 - v_{3h}^2 \left( \frac{\alpha}{2} + \frac{1}{2} \right) \right)}{2\alpha v_{3h}^3 + \left( \frac{5\alpha+2}{\text{Bi}} \right) v_{3h}^2 + \frac{6}{\text{Bi}} + 12v_{3h}}, \quad (4.30)$$

$$B_{3h} = \frac{12 v_{3h}^2 \left( \frac{\alpha}{3} + \frac{1}{3} \right)}{2\alpha v_{3h}^3 + \left( \frac{5\alpha+2}{\text{Bi}} \right) v_{3h}^2 + \frac{6}{\text{Bi}} + 12v_{3h}}, \quad (4.31)$$

and where $v_{3h}$ is the unique solution to the following equation:

$$z^{\alpha+4} 2^{\alpha+1} \alpha + z^{\alpha+3} \left( \frac{2^{\alpha+3} 2^{\alpha+2}}{\text{Bi}} \right) + z^{\alpha+2} 3^{\alpha+2} 2^{\alpha+1} \frac{3^{\alpha+1}}{\text{Bi}} + z^2 \text{Ste}(2 + 3\alpha) - 6\text{Ste} = 0, \quad z > 0. \quad (4.32)$$

**Proof** The proof is similar to the one given in Theorem 4.2. The only difference lies in the fact that equation (1.1a*) is equivalent to

$$v_{3h}^2 \left[ A_{3h} \left( \frac{1}{3} + \frac{\alpha}{3} \right) + B_{3h} \left( \frac{1}{3} + \frac{\alpha}{2} \right) \right] = B_{3h}. \quad (4.33)$$

The approximated problem $(P_{3h})$ obtained when applying the RIM verifies the same convergence property than the exact problem $(P_h)$.

**Theorem 4.5** The unique solution to problem $(P_{3h})$ given by Theorem 4.4 converges to the unique solution to problem $(P_h)$, given by Theorem 3.6, when the coefficient that characterises the heat transfer at the fixed face $h$ goes to infinity.

**Proof** The proof is analogous to the proof given in Theorem 4.3.
In Figure 12, we compare graphically, for different values of \( \text{Bi} > 1 \), the coefficient \( v_{3h} \) that characterises the approximate free boundary \( s_{3h} \) with the coefficient \( v_h \) corresponding to the exact free boundary \( s_h \), for different values of \( \alpha \) fixing \( \text{Ste} = 0.5 \). Once again, as \( \text{Bi} \) increases, the value \( v_{3h} \) becomes closer to the value \( v_h \).

### 4.3 Comparisons between the approximate solutions and the exact one

In this section, we are going to compare the exact solution to the problem with a convective condition at the fixed face (\( P_h \)) with the approximate solutions obtained by applying the integral balance methods proposed in the previous sections.

For each method, we have defined a new problem (\( P_{ih} \)), \( i = 1, 2, 3 \) and we have compared graphically the coefficient \( v_{ih} \) that characterises each free boundary \( s_{ih} \), with the coefficient \( v_h \) that corresponds to the exact free boundary \( s_h \).

The goal is to compare numerically the coefficient \( v_h \) given by (4.5) with the approximate coefficients \( v_{1h} \), \( v_{2h} \) and \( v_{3h} \) given by (4.10), (4.21) and (4.32), respectively.

In order that the comparisons be more representative, in Tables 4–6 we show the exact value \( v_h \), the approximate value \( v_{ih} \) and the percentage error committed in each case \( E(v_{ih}) = 100 \left| \frac{v_h - v_{ih}}{v_h} \right|, \ i = 1, 2, 3 \) for different values of \( \text{Bi} \) and \( \alpha \) fixing \( \text{Ste} = 0.5 \).

From the above tables, we can deduce that for \( \alpha = 0.5 \), the percentage error committed is smaller than for the other cases. In all cases, as it happened with the problem (P), the method with best accuracy for approximating the problem (\( P_h \)) is the modified integral method, i.e. the best approximate problem is given by (\( P_{2h} \)).

We can also compare the exact temperature \( T_h \) with the approximate ones \( T_{ih}, \ i = 1, 2, 3 \), given by (4.6), (4.17) and (4.28), respectively. In Figures 13–16, we show a colour map for \( \alpha = 5, \text{Ste} = 0.5, \theta_{\infty} = 30, a = 1 \).
Table 4. Dimensionless coefficients of the free boundaries and their percentage relative error for $\alpha = 0$ and $\text{Ste} = 0.5$

| Bi | $v_h$ | $v_{1h}$ | $E_{\text{rel}}(v_{1h})$ (%) | $v_{2h}$ | $E_{\text{rel}}(v_{2h})$ (%) | $v_{3h}$ | $E_{\text{rel}}(v_{3h})$ (%) |
|----|-------|----------|-----------------|-------|-----------------|-------|-----------------|
| 1  | 0.2926  | 0.2966  | 1.3828  | 0.2937  | 0.3939  | 0.2899  | 0.9103  |
| 10 | 0.4422  | 0.4681  | 5.8548  | 0.4484  | 1.4111  | 0.4545  | 2.7969  |
| 20 | 0.4533  | 0.4776  | 5.3525  | 0.4602  | 1.5151  | 0.4712  | 3.0744  |
| 30 | 0.4571  | 0.4807  | 5.1622  | 0.4642  | 1.5514  | 0.4716  | 3.1679  |
| 40 | 0.4590  | 0.4822  | 5.0628  | 0.4662  | 1.5699  | 0.4738  | 3.2148  |
| 50 | 0.4601  | 0.4832  | 5.0019  | 0.4674  | 1.5811  | 0.4751  | 3.2430  |
| 60 | 0.4609  | 0.4838  | 4.9606  | 0.4682  | 1.5886  | 0.4759  | 3.2618  |
| 70 | 0.4615  | 0.4842  | 4.9309  | 0.4688  | 1.5940  | 0.4771  | 3.2752  |
| 80 | 0.4619  | 0.4845  | 4.9085  | 0.4693  | 1.5980  | 0.4771  | 3.2853  |
| 90 | 0.4622  | 0.4848  | 4.8909  | 0.4696  | 1.6012  | 0.4774  | 3.2932  |
| 100| 0.4625  | 0.4850  | 4.8768  | 0.4699  | 1.6037  | 0.4777  | 3.2994  |

Table 5. Dimensionless coefficients of the free boundaries and their percentage relative error for $\alpha = 5$ and $\text{Ste} = 0.5$

| Bi | $v_h$ | $v_{1h}$ | $E_{\text{rel}}(v_{1h})$ (%) | $v_{2h}$ | $E_{\text{rel}}(v_{2h})$ (%) | $v_{3h}$ | $E_{\text{rel}}(v_{3h})$ (%) |
|----|-------|----------|-----------------|-------|-----------------|-------|-----------------|
| 1  | 0.3274  | 0.3293  | 0.5908  | 0.3280  | 0.1779  | 0.3160  | 3.4746  |
| 10 | 0.4459  | 0.4551  | 2.0484  | 0.4480  | 0.4543  | 0.4474  | 0.3370  |
| 20 | 0.4553  | 0.4631  | 1.7173  | 0.4574  | 0.4798  | 0.4583  | 0.6724  |
| 30 | 0.4585  | 0.4657  | 1.5912  | 0.4607  | 0.4886  | 0.4621  | 0.7874  |
| 40 | 0.4601  | 0.4671  | 1.5250  | 0.4623  | 0.4931  | 0.4640  | 0.8456  |
| 50 | 0.4610  | 0.4679  | 1.4844  | 0.4633  | 0.4958  | 0.4651  | 0.8807  |
| 60 | 0.4617  | 0.4684  | 1.4569  | 0.4640  | 0.4976  | 0.4659  | 0.9042  |
| 70 | 0.4622  | 0.4688  | 1.4370  | 0.4645  | 0.4989  | 0.4664  | 0.9210  |
| 80 | 0.4625  | 0.4691  | 1.4220  | 0.4648  | 0.4999  | 0.4668  | 0.9336  |
| 90 | 0.4628  | 0.4693  | 1.4103  | 0.4651  | 0.5006  | 0.4672  | 0.9434  |
| 100| 0.4630  | 0.4695  | 1.4009  | 0.4653  | 0.5012  | 0.4674  | 0.9513  |

Table 6. Dimensionless coefficients of the free boundaries and their percentage relative error for $\alpha = 0.5$ and $\text{Ste} = 0.5$

| Bi | $v_h$ | $v_{1h}$ | $E_{\text{rel}}(v_{1h})$ (%) | $v_{2h}$ | $E_{\text{rel}}(v_{2h})$ (%) | $v_{3h}$ | $E_{\text{rel}}(v_{3h})$ (%) |
|----|-------|----------|-----------------|-------|-----------------|-------|-----------------|
| 1  | 0.4073  | 0.3834  | 5.8702  | 0.4005  | 1.6647  | 0.3730  | 8.4069  |
| 10 | 0.4569  | 0.4170  | 8.7307  | 0.4437  | 2.8806  | 0.4259  | 6.7799  |
| 20 | 0.4616  | 0.4203  | 8.9507  | 0.4476  | 3.0301  | 0.4315  | 6.5196  |
| 30 | 0.4632  | 0.4214  | 9.0256  | 0.4489  | 3.0845  | 0.4335  | 6.4217  |
| 40 | 0.4641  | 0.4220  | 9.0633  | 0.4496  | 3.1126  | 0.4345  | 6.3703  |
| 50 | 0.4646  | 0.4224  | 9.0861  | 0.4501  | 3.1298  | 0.4351  | 6.3387  |
| 60 | 0.4649  | 0.4226  | 9.1012  | 0.4503  | 3.1414  | 0.4356  | 6.3173  |
| 70 | 0.4652  | 0.4228  | 9.1121  | 0.4505  | 3.1497  | 0.4359  | 6.3018  |
| 80 | 0.4654  | 0.4229  | 9.1203  | 0.4507  | 3.1560  | 0.4361  | 6.2901  |
| 90 | 0.4655  | 0.4230  | 9.1266  | 0.4508  | 3.1609  | 0.4363  | 6.2809  |
| 100| 0.4656  | 0.4231  | 9.1317  | 0.4509  | 3.1649  | 0.4364  | 6.2736  |
Figure 13. Colour map for $T_h$.

Figure 14. Colour map for $T_{1h}$.

Figure 15. Colour map for $T_{2h}$. 
5 Minimising the least-squares error in the HBIM

In this section, we are going to analyse the least-squares error that we commit when assuming a quadratic profile in space. If we have an approximate solution for the heat equation given by \( \hat{T} \), \( \hat{s} \) such that

\[
\hat{T}(x, t) = \frac{\alpha}{2} \theta_\infty \left[ \hat{A} \left(1 - \frac{x}{\hat{s}(t)}\right) + \hat{B} \left(1 - \frac{x}{\hat{s}(t)}\right)^2 \right],
\]  

(5.1)

with adequate coefficients \( \hat{A} \), \( \hat{B} \) and \( \hat{s} \), then we can measure how far we are from the heat equation by computing the least-squares error (see [19]) given by

\[
E = \int_0^{\hat{s}(t)} \left( \frac{\partial \hat{T}}{\partial t}(x, t) - \alpha^2 \frac{\partial^2 \hat{T}}{\partial x^2}(x, t) \right)^2 dx
\]

(5.2)

Taking into account that

\[
\frac{\partial \hat{T}}{\partial t}(x, t) = \frac{\alpha}{2} \theta_\infty \left[ \hat{A} \left(1 - \frac{x}{\hat{s}(t)}\right) + \hat{B} \left(1 - \frac{x}{\hat{s}(t)}\right)^2 \right] \\
+ \frac{\alpha^2}{2} \frac{\hat{s}(t)}{\hat{s}^2(t)} \hat{s}(t) \theta_\infty \left[ \hat{A} + 2\hat{B} \left(1 - \frac{x}{\hat{s}(t)}\right) \right]
\]

(5.3)

and

\[
\frac{\partial^2 \hat{T}}{\partial x^2}(x, t) = \frac{\alpha^2}{2} \frac{\hat{B} \theta_\infty}{\hat{s}^2(t)}
\]

(5.4)
we get
\[ E = \frac{\alpha^2}{4} \theta^2 \tau^{\alpha-2} \left( \frac{\tau^2}{3} + \frac{\dot{B}^2}{\tau^2} + \frac{\ddot{B}^2}{\tau^3} \right) + t^{\alpha^2} \theta^2 \tau^{\alpha-2} \left( \frac{\tau^2}{3} + \frac{\dot{B}^2}{\tau^2} + \frac{\ddot{B}^2}{\tau^3} \right) 
+ 4\alpha^4 \theta^2 \tau^{\alpha-2} \left( \frac{\tau^2}{3} + \frac{\dot{B}^2}{\tau^2} + \frac{\ddot{B}^2}{\tau^3} \right) - 2\alpha^2 \theta^2 \tau^{\alpha-1} \left( \frac{\tau}{3} + \frac{\ddot{B}}{\tau^2} \right) 
- 4\alpha^2 \theta^2 \tau^{\alpha} \left( \frac{\dot{B}}{\tau} \right) \right) . \tag{5.5} \]

In case that the free boundary \( \tilde{s}(t) = 2a\xi \sqrt{t} \) with \( \xi > 0 \), by simple computations, the least-squares error becomes \( E = E(\xi) \), given by the following expression:
\[ E(\xi) = \frac{\alpha^2}{4} \theta^2 \tau^{\alpha-2} \left[ \xi^4 \left( \frac{\tau^2}{3} + \frac{\dot{B}^2}{\tau^2} + \frac{\ddot{B}^2}{\tau^3} \right) + 2\alpha \left( \frac{\tau^2}{3} + \frac{\dot{B}^2}{\tau^2} + \frac{\ddot{B}^2}{\tau^3} \right) + \frac{\tau^2}{3} + \frac{\ddot{B}^2}{\tau^3} \right] 
- \frac{\xi^2}{2} \hat{B}(\alpha + 1) \left( \frac{\dot{B}}{\tau} + \frac{\ddot{B}}{\tau^2} \right) \right] . \tag{5.6} \]

Let us then define a new approximate problem \( (P_4) \) for the problem \( (P) \) that consists in finding the free boundary \( s_4 = s_4(t) \) and the temperature \( T_4 = T_4(x, t) \) in the domain \( 0 < x < s_4(t) \) given by the profile (5.1) such that they minimise the least-squares error (5.5) subject to the conditions (1.1b), (1.1c), (1.1d) and (1.1e).

**Theorem 5.1** If a free boundary \( s_4 \) and a temperature \( T_4 \) constitute a solution to problem \( (P_4) \) then they are given by the expressions
\[ T_4(x, t) = \rho^{\alpha/2} \tau^{\alpha} \left[ A_4 \left( \frac{1 - x}{s_4(t)} \right) + B_4 \left( \frac{1 - x}{s_4(t)} \right)^2 \right], \tag{5.7} \]
\[ s_4(t) = 2av_4 \sqrt{t}, \tag{5.8} \]
where the constants \( A_4 \) and \( B_4 \) are defined as a function of \( v_4 \) as
\[ A_4 = \frac{2^{\alpha+1} v_4^{\alpha+2}}{\sigma}, \quad B_4 = 1 - \frac{2^{\alpha+1} v_4^{\alpha+2}}{\sigma}, \tag{5.9} \]
and where \( v_4 > 0 \) must minimise for every \( t > 0 \), the function
\[ E(\xi) = \frac{\alpha^2}{4} \theta^2 \tau^{\alpha-2} \xi^4 p(\xi) \cdot \forall \xi > 0 \tag{5.10} \]
with
\[ p(\xi) = \xi^8 + 2^{\alpha+1} 2^{\alpha+1} (\alpha^2 + \alpha + 4) + 5 \xi^{2\alpha+6} 2^{\alpha+2} (1 + \alpha) + 15 \xi^{2\alpha+4} 2^{\alpha+2} + 6 \xi^{\alpha+6} 2^{\alpha+2} \sigma (2 + 3\alpha + 3\alpha^2) + 5 \xi^{\alpha+4} 2^{\alpha+1} \sigma (1 + \alpha) 
- 15 \xi^{2\alpha+2} 2^{\alpha+2} \sigma (2 + 3\alpha + 3\alpha^2) 
- 10 \xi^2 \sigma (1 + \alpha) + 15 \sigma^2 . \tag{5.11} \]

**Proof** Provided that \( T_4 \) adopts a quadratic profile in space given by (5.7), then the condition (1.1c) holds immediately and the Stefan condition (1.1d) becomes equivalent to
\[ -k\rho^{\alpha/2} \tau^{\alpha} \frac{A_4}{s_4(t)} = -\gamma s_4(t) \tilde{s}_4(t) . \tag{5.12} \]
Then
\[ s_4(t) = \left( \frac{(\alpha + 2) k\theta_{\infty}}{(\frac{\alpha}{2} + 1) \gamma} A_4 \right)^{1/(\alpha+2)} \sqrt{t}. \] (5.13)

Introducing \( v_4 = \frac{1}{2a} \left( \frac{(\alpha+2) k\theta_{\infty}}{(\frac{\alpha}{2} + 1) \gamma} A_4 \right)^{1/(\alpha+2)} \), the free boundary becomes
\[ s_4(t) = 2a v_4 \sqrt{t}, \] (5.14)
and
\[ A_4 = \frac{2^{\alpha+1} v_4^{\alpha+2}}{\text{Ste}}. \] (5.15)

In addition, from the boundary condition at the fixed face (1.1b) we get
\[ A_4 + B_4 = 1. \] (5.16)

Then we obtain formulas (5.9) for the coefficients \( A_4 \) and \( B_4 \). Finally, as the free boundary \( s_4 \) is defined by (5.14), we have to minimise the least-squares error \( E \) given by (5.6). In addition, replacing \( A_4 \) and \( B_4 \) by the formulas given in (5.9), we get that \( v_4 \) must minimise (5.10).

\[ \text{Corollary 1} \quad \text{For the classical Stefan problem, i.e. for the case } \alpha = 0, \text{ we get that problem } (P_4) \text{ has a unique solution given by} \]
\[ T_4^{(0)}(x, t) = \theta_{\infty} \left[ A_4^{(0)} \left( 1 - \frac{x}{s_4^{(0)}(t)} \right) + B_4^{(0)} \left( 1 - \frac{x}{s_4^{(0)}(t)} \right)^2 \right], \] (5.17)
\[ s_4^{(0)}(t) = 2av_4^{(0)} \sqrt{t}, \] (5.18)
where the superscript \((0)\) makes reference to the value of \( \alpha = 0 \) and the constants \( A_4^{(0)} \) and \( B_4^{(0)} \) are defined as a function of \( v_4^{(0)} \) as
\[ A_4^{(0)} = \frac{2(v_4^{(0)})^2}{\text{Ste}}, \quad B_4 = 1 - \frac{2(v_4^{(0)})^2}{\text{Ste}} \] (5.19)
being \( v_4^{(0)} > 0 \) the value where the function \( E^{(0)} \) attains its minimum
\[ E^{(0)}(\xi) = \frac{t^{-2} \theta_{\infty}^2 p^{(0)}(\xi)}{60\text{Ste}^2 \xi^4}, \quad \forall t > 0 \] (5.20)
with
\[ p^{(0)}(\xi) = 8\xi^8 + 2(10 + \text{Ste})\xi^6 + 2(30 + 5\text{Ste} + \text{Ste}^2)\xi^4 \]
\[ - 10\text{Ste}(6 + \text{Ste})\xi^2 + 15\text{Ste}^2 \] (5.21)

In addition, \( v_4^{(0)} \) can be obtained as the unique positive root of the following real polynomial
\[ r(\xi) = 32\xi^8 + 4(10 + \text{Ste})\xi^6 + 20\text{Ste}(6 + \text{Ste})\xi^4 - 60\text{Ste}^2. \] (5.22)
Table 7. Dimensionless coefficients of the free boundaries and their percentage relative error for \( \alpha = 0 \)

| Ste | \( \nu \)   | \( \nu_2 \) | \( E_{rel}(\nu_2) \) (%) | \( \nu_4 \)   | \( E_{rel}(\nu_4) \) (%) |
|-----|-------------|-------------|--------------------------|-------------|--------------------------|
| 0.1 | 0.2200      | 0.2209      | 0.3947                   | 0.2209      | 0.3855                   |
| 0.2 | 0.3064      | 0.3087      | 0.7499                   | 0.3086      | 0.7168                   |
| 0.3 | 0.3699      | 0.3738      | 1.0707                   | 0.3736      | 1.0040                   |
| 0.4 | 0.4212      | 0.4270      | 1.3618                   | 0.4265      | 1.2551                   |
| 0.5 | 0.4648      | 0.4723      | 1.6266                   | 0.4716      | 1.4762                   |
| 0.6 | 0.5028      | 0.5122      | 1.8683                   | 0.5112      | 1.6722                   |
| 0.7 | 0.5365      | 0.5477      | 2.0895                   | 0.5464      | 1.8470                   |
| 0.8 | 0.5669      | 0.5799      | 2.2923                   | 0.5783      | 2.0037                   |
| 0.9 | 0.5946      | 0.6094      | 2.4786                   | 0.6074      | 2.1449                   |
| 1.0 | 0.6201      | 0.6365      | 2.6500                   | 0.6342      | 2.2727                   |

Remark 5.2 Due to formula (5.20), we have that the error we commit when approximating with problem \( (P_4) \) for the case \( \alpha = 0 \) is inversely proportional to the square of time, i.e. \( E(0) \propto 1/t^2 \).

Proof From Theorem 5.1, we need to minimise the function \( E(\xi) \) given by (5.10) for the case \( \alpha = 0 \). So, it is clear evident that we need to minimise the function \( E^{(0)}(\xi) \) given by (5.20) which is equivalent to minimise the function \( F^{(0)}(\xi) = E^{(0)}(\xi)/\xi^4 \). Therefore, let us show that \( F^{(0)} \) has a unique positive value where the minimum is attained. Observe that \( F^{(0)} \) is a continuous function in \( \mathbb{R}^+ \). Moreover if we compute its derivative, we obtain

\[
F^{(0)}(\xi) = \frac{r(\xi)}{\xi^5}
\]

with \( r \) given by (5.22). As \( r \) is a polynomial that verifies \( r(0) = -60\text{Ste}^2 < 0, r(+\infty) = +\infty \), and \( r'(\xi) > 0, \forall \xi > 0 \), we obtain that there exists a unique value \( \xi_0 > 0 \) such that \( r(\xi_0) = 0 \). In addition, we can assure that \( r(\xi) < 0 \), for every \( \xi < \xi_0 \) and \( r(\xi) > 0 \), for every \( \xi > \xi_0 \). Consequently, we have

\[
F^{(0)}(\xi) < 0, \quad \forall \xi < \xi_0, \quad F^{(0)}(\xi_0) = 0, \quad F^{(0)}(\xi) > 0, \quad \forall \xi > \xi_0.
\]

We can conclude that \( F^{(0)} \) decreases in \( (0, \xi_0) \) and increases in \( (\xi_0, +\infty) \). This means that \( F^{(0)} \) has a unique minimum that is attained at \( \xi_0 \). Calling \( \nu_4^{(0)} = \xi_0 \), we get that \( \nu_4^{(0)} \) is the unique positive root of \( r \) and minimises the error function \( E^{(0)} \).

Taking into account the last result we show in Table 7 the coefficient \( \nu \) that characterises the exact free boundary of problem \( (P) \), the approximate coefficient \( \nu_2 \) obtained by the modified integral balance method (which until now was the most accurate technique) and the coefficient \( \nu_4 \) defined by the Corollary 1 for different values of Ste numbers. Computing also the percentage relative error committed in each case we assure that the approximate problem \( (P_4) \) is the best approximation we can obtain adopting a quadratic profile in space for the temperature.

In a similar way, we can define a new approximate problem \( (P_{4h}) \) for the problem \( (P_h) \) that consists in finding the free boundary \( s_{4h} = s_{4h}(t) \) and the temperature \( T_{4h} = T_{4h}(x, t) \) in \( 0 < x < \)
s_{4h}(t) given by the profile (5.1) such that they minimise the least-squares error (5.5) subject to the conditions (1.1b∗) and (1.1c)–(1.1e).

**Theorem 5.3** If a free boundary \( s_{4h} \) and a temperature \( T_{4h} \) constitute a solution to problem \((P_{4h})\) then they are given by the expressions:

\[
T_{4h}(x, t) = \rho^{\alpha/2} \theta_{\infty} \left[ A_{4h} \left( 1 - \frac{x}{s_{4h}(t)} \right) + B_{4h} \left( 1 - \frac{x}{s_{4h}(t)} \right) \right], \tag{5.23}
\]

\[
s_{4h}(t) = 2a \nu_{4h} \sqrt{t}, \tag{5.24}
\]

where the constants \( A_{4h} \) and \( B_{4h} \) are defined as a function of \( \nu_{4h} \) as

\[
A_{4h} = \frac{2^{\alpha+1} \nu_{4h}^{\alpha+2} \text{Ste}}{\text{Ste}}, \quad B_{4h} = \frac{2 \nu_{4h} - A_{4h} (1 + 2 \nu_{4h})}{2(1 + \nu_{4h})} \tag{5.25}
\]

and where \( \nu_{4h} > 0 \) must minimise for every \( t > 0 \), the real function:

\[
E_h(\xi) = \frac{e^{\alpha-\gamma_{\infty}^2}}{60 \text{Ste}^2 \left( \frac{1}{\text{Bi}} + \xi \right)^2} \cdot \left\{ p(\xi) + \frac{1}{\text{Bi}} \left[ 2^{2\alpha} (7\alpha^2 + 7\alpha + 18) x^{2\alpha+1} + 25 2^{2\alpha+1} (\alpha + 1) x^{2\alpha+5},
+ 2^{\alpha} (9\alpha^2 + 9\alpha + 6) \text{Ste} x^{\alpha+5} + 15 2^{\alpha+1} x^{2\alpha+3}
- 5 2^{\alpha+1} (\alpha + 1) \text{Ste} x^{\alpha+3} - 15 2^{\alpha+1} \text{Ste} x^{\alpha+1}
+ \frac{1}{\text{Bi}^2} \left[ 4^{\alpha+1} (2\alpha^2 + 2\alpha + 3) x^{2\alpha+6} + 5 4^{\alpha+1} (\alpha + 1) x^{2\alpha+4} + 15 4^{\alpha+1} x^{2\alpha+2} \right] \right\} \tag{5.26}
\]

with \( p(\xi) \) given by formula (5.11).

**Proof** It is clear immediate that the chosen profile temperature leads the condition (1.1c) to be automatically verified. From condition (1.1d), we obtain

\[
-k t^{\alpha/2} \theta_{\infty} A_{4h} \frac{s_{4h}(t)}{s_{4h}(t)} = -\gamma s_{4h}^\prime(t) s_{4h}^\prime(t). \tag{5.27}
\]

Therefore,

\[
s_{4h}(t) = \left( \frac{(\alpha + 2) \ k \theta_{\infty}}{\text{Ste}} A_{4h} \right)^{1/(\alpha+2)} \sqrt{t}. \tag{5.28}
\]

Introducing the new coefficient \( \nu_{4h} \) such that \( \nu_{4h} = \frac{1}{2a} \left( \frac{(\alpha + 2) \ k \theta_{\infty}}{\text{Ste}} A_{4h} \right)^{1/(\alpha+2)} \), the free boundary can be expressed as

\[
s_{4h}(t) = 2a \nu_{4h} \sqrt{t}, \tag{5.29}
\]

where the following equality holds:

\[
A_{4h} = \frac{2^{\alpha+1} \nu_{4h}^{\alpha+2}}{\text{Ste}}. \tag{5.30}
\]
The convective boundary condition at \( x = 0 \), i.e. condition (1.1b*), leads to

\[
A_{4h}(1 + 2\Bi \nu_{4h}) + 2B_{4h}(1 + \Bi \nu_{4h}) = 2\Bi \nu_{4h}. \tag{5.31}
\]

Therefore, we obtain the formulas given in (5.25). Replacing \( A_{4h}, B_{4h} \) and \( s_{4h} \) for their expressions in function of \( \nu_{4h} \), minimising the least-squares error (5.5) is equivalent to minimising (5.26) (obtained by Mathematica software).

**Corollary 2** For the classical Stefan problem, i.e. for the case \( \alpha = 0 \), we get that if \( \Bi > \frac{1}{\sqrt{12}} \) and \( \Ste < \frac{1}{2\Bi^2} \), then \((P_{4h})\) has a unique solution given by

\[
T^{(0)}_{4h}(x, t) = \theta_\infty \left[ A^{(0)}_{4h} \left(1 - \frac{x}{s_{(0)(t)}}\right) + B^{(0)}_{4h} \left(1 - \frac{x}{s_{(0)(t)}}\right)\right]^2, \tag{5.32}
\]

\[
s^{(0)}_{4h}(t) = 2a
\nu^{(0)}_{4h} \sqrt{t}, \tag{5.33}
\]

where the superscript \((0)\) makes reference to the value of \( \alpha = 0 \) and the constants \( A^{(0)}_{4h} \) and \( B^{(0)}_{4h} \) are defined as a function of \( \nu^{(0)}_{4h} \) as

\[
A^{(0)}_{4h} = \frac{2(\nu^{(0)}_{4h})^2}{\Ste}, \quad B^{(0)}_{4h} = \frac{2\Bi \nu^{(0)}_{4h} - A^{(0)}_{4h}(1 + 2\Bi \nu^{(0)}_{4h})}{2(1 + \nu^{(0)}_{4h} \Bi)}, \tag{5.34}
\]

being \( \nu^{(0)}_{4h} > 0 \) the value where the function \( E^{(0)}_{h} \) attains its minimum

\[
E^{(0)}_{h}(\xi) = \frac{\beta_\infty^2}{60 \Ste^2 x^2(\frac{1}{\Bi} + x)^2} \left\{ p^{(0)}(\xi) + \frac{1}{\Bi} \left[ 2x(9x^6 + (3\Ste + 25)x^4 + 5(6 - \Ste)x^2 - 15\Ste) + \frac{1}{\Bi^3} x^2(12x^4 + 20x^2 + 15) \right] \right\}, \tag{5.35}
\]

where \( p^{(0)} \) is given by (5.21). Moreover, \( \nu^{(0)}_{4h} \) can be obtained as the unique positive root of the following polynomial:

\[
r_{h}(\xi) = 16\Bi^3 \xi^9 + 51\Bi^2 \xi^8 + \xi^7 \left( 2\Bi^3 \Ste + 20\Bi^3 + 57\Bi \right)
+ \xi^6 \left( 7\Bi^2 \Ste^2 + 65\Bi^2 + 24 \right)
+ \Bi(3\Ste + 25)\xi^5 + \xi^4 \left( \Bi^2(2\Ste^2 + 15\Ste + 30) + 20 \right)
+ 5\Bi(3 + (-1 + 12\Bi^2)\Ste + 2\Bi^2 \Ste^2)\xi^3 + 45\Bi^2 \Ste^2 \xi^2
+ 15\Bi \Ste(1 - 2\Bi^2 \Ste)\xi - 15\Bi^2 \Ste^2. \tag{5.36}
\]

**Proof** When we replace \( \alpha = 0 \) in Theorem 5.3, we immediately obtain the formulas (5.34) and (5.35). In order to prove that there exists a unique value that minimises the least-squares error, we compute \( E^{(0)}_{h}(\xi) \) and we get that \( E^{(0)}_{h}(\xi) = \frac{\beta_\infty^2}{30 \Ste^2 x(\frac{1}{\Bi} + x)^2} r_{h}(\xi) \) with \( r_{h} \) given by (5.36). We can observe that \( r_{h}(0) < 0, r_{h}(+\infty) = +\infty, r'_{h} > 0 \) under the hypothesis that \( \Bi > \frac{1}{\sqrt{12}} \) and \( \Ste < \frac{1}{2\Bi^2} \).

Therefore, we can assure that there exists a unique \( \xi_{50} \) such that \( r_{h}(\xi_{50}) = 0 \). In addition, we have that \( r_{h}(\xi) < 0, \forall \xi < \xi_{50} \) and \( r_{h}(\xi) > 0, \forall \xi > \xi_{50} \). Then we get that \( E_{h}(\xi) \) decreases \( \forall \xi < \xi_{50} \).
and increases \( \forall \xi > \xi_{h0} \). Consequently, we obtain that \( \xi_{h0} \) constitutes the unique minimum of the least-squares error.

In view of the above result, in Table 8, for \( \alpha = 0 \) we compare the coefficient \( v_h \) that characterises the exact free boundary problem with the coefficient \( v_{2h} \) corresponding to the modified integral method, which was until now the most accurate, and we also compare with the coefficient \( v_{4h} \) obtained when minimising the least-squares error. We fix \( Ste = 0.02 \) and vary Bi between 1 and 5. The value of this parameters are chosen in order to verify the hypothesis of Corollary 2. By computing the percentage relative error of each method, we conclude that the approximate problem \( (P_{4h}) \) gives us the best approximate solution to problem \( (P_h) \).

In case we decide to use the formula (5.36) to compute \( v_{4h} \) without satisfying the hypothesis of the Corollary 2, fixing \( Ste = 0.5 \) and varying Bi from 1 to 100 we get the results shown in Table 9.

### 6 Conclusions

In this paper, we have studied different approximate methods for one-dimensional one-phase Stefan problems where the main feature consists in taking a space-dependent latent heat. We have considered two different problems that differ from each other in their boundary condition
at the fixed face \( x = 0 \): Dirichlet or Robin condition. We have implemented the classical HBIM, a modified integral method and the RIM. Exploiting the knowledge of the exact solution of both problems (available in literature), we have studied the accuracy of the different approximations obtained. All the analysis have been carried out using dimensionless parameters like \( \text{Ste} \) and \( \text{Bi} \). Furthermore, we have studied the case when \( \text{Bi} \) goes to infinity in the problem with a convective condition, recovering the approximate solutions when a temperature condition is imposed at the fixed face. We provided some numerical simulations and we have concluded that, in the majority of cases, the modified integral method is the most reliable in terms of accuracy. When approaching by the minimisation of the least-squares error, we get better approximations but only for the case \( \alpha = 0 \) (where we could prove existence and uniqueness of solution). The least accurate method was the classical HBIM, not only to the high percentage error committed but also because we could not obtain a result that assures uniqueness of the approximate solution.

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**Conflict of interest**

None.

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