Real Hypersurfaces in the Complex Hyperbolic Quadric with Reeb Parallel Structure Jacobi Operator

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Abstract
We introduce the notion of Reeb parallel structure Jacobi operator for real hypersurfaces in the complex hyperbolic quadric $Q^{*m} = SO_{2,m}^0 / SO_2 SO_m$, $m \geq 3$, and give a classification theorem for real hypersurfaces in $Q^{*m}$, $m \geq 3$, with Reeb parallel structure Jacobi operator.

Keywords
Complex hyperbolic quadric · Reeb parallel structure Jacobi operator · $\mathfrak{A}$-isotropic · $\mathfrak{A}$-principal · Complex structure · Real structure

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1 Introduction

As a dual space of $m$-dimensional complex quadric $Q^m$, we can consider a Hermitian symmetric space with rank 2 of noncompact type $Q^{*m}$, which is said to be complex hyperbolic quadric. Montiel and Romero [11] proved that the complex hyperbolic quadric $Q^{*m}$ can be immersed in the indefinite complex hyperbolic space $\mathbb{C}H_1^{m+1}(-c)$, $c > 0$, by interchanging the Kähler metric by its opposite as follows:

If we change the Kähler metric $g$ of $\mathbb{C}P_{m-s}^{m+1}$ by its opposite $g' = -g$, we have that $Q_{m-s}^m$ endowed with its opposite metric $g' = -g$ is also an Einstein hypersurface of $\mathbb{C}H_{s+1}^{m+1}(-c)$. When $s = 0$, we know that $(Q_m^m, g' = -g)$ can be regarded as

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$Q^{sm} = SO_{2,m}^0/SO_2SO_m$, which is immersed in $\mathbb{C}H_{1}^{m+1}(-c)$, $c > 0$ as a complex Einstein hypersurface. The complex hyperbolic quadric can be regarded as a kind of real Grassmann manifold of noncompact type with rank 2 (see [4, 10, 17, 18, 20], and [24]). Accordingly, $Q^{sm}$ admits two important geometric structures, a real structure $A$ and a complex structure $J$, which anti-commute with each other, that is, $AJ = -JA$. 

Then for $m \geq 2$ the triple $(Q^{sm}, J, g)$ is a Hermitian symmetric space of non-compact type with rank 2 and its minimal sectional curvature is equal to $-4$ (see [10] and [24]).

In addition to the complex structure $J$ there is another distinguished geometric structure on $Q^{sm}$, namely a parallel rank two vector bundle $\mathfrak{A}$ which contains an $S^1$-bundle of real structures, that is, complex conjugations $A$ on the tangent spaces of $Q^{sm}$. This geometric structure determines a maximal $\mathfrak{A}$-invariant subbundle $Q$ of the tangent bundle $TM$ of a real hypersurface $M$ in $Q^{sm}$, that is, for each point $p \in M$ we can define

$$Q_p = \{X \in T_pM \mid AX \in T_pM \text{ for all } A \in \mathfrak{A}_p\}.$$ 

Recall that a nonzero tangent vector $W \in T_pQ^{sm}$, $p \in Q^{sm}$, is called singular if it is tangent to more than one maximal flat in $Q^{sm}$. There are two types of singular tangent vectors for the complex hyperbolic quadric $Q^{sm}$:

- If there exists a conjugation $A \in \mathfrak{A}_p$ such that $W \in V(A) = \{X \in T_pQ^{sm} \mid AX = X\}$, then $W$ is singular. Such a singular tangent vector is called $\mathfrak{A}$-principal.
- If there exist a conjugation $A \in \mathfrak{A}_p$ and orthonormal vectors $Z_1, Z_2 \in V(A)$ such that $W/||W|| = (Z_1 + JZ_2)/\sqrt{2}$, then $W$ is singular. Such a singular tangent vector is called $\mathfrak{A}$-isotropic, where $V(A) = \{X \in T_pQ^{sm} \mid AX = X\}$ and $JV(A) = \{X \in T_pQ^{sm} \mid AX = -X\}$ are the $(+1)$-eigenspace and $(-1)$-eigenspace for the involution $A$ on $T_pQ^{sm}$, $p \in Q^{sm}$.

Now, let $M$ be a real hypersurface in a Kähler manifold $\tilde{M}$, and denote by $(\phi, \xi, \eta, g)$ the induced almost contact metric structure of $M$. As a typical classification theorem for such real hypersurface, many geometers considered the condition that a real hypersurface $M$ in $\tilde{M}$ has isometric Reeb flow, which means that the Riemannian metric is invariant along the Reeb direction $\xi = -JN$. Algebraically it is equivalent to the notion of commuting shape operator given by $S\phi = \phi S$, where $S$ is the shape operator of $M$ defined by $\nabla_XN = -SX, X \in TM$.

For instance, Okumura [13] proved that the Reeb flow on a real hypersurface in complex projective space $\mathbb{C}P^m = SU_{m+1}/S(U_1U_m)$ is isometric if and only if $M$ is an open part of a tube around a totally geodesic $\mathbb{C}P^k \subset \mathbb{C}P^m$ for some $k \in \{0, \ldots, m-1\}$. For the complex 2-plane Grassmannian $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_2U_m)$ a classification was obtained by Berndt and Suh [1]. The Reeb flow on a real hypersurface in $G_2(\mathbb{C}^{m+2})$ is isometric if and only if $M$ is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$. For the complex quadric $Q^m = SO_{m+2}/SO_2SO_m$, Berndt and Suh [2] have obtained the following result:
**Theorem A** Let $M$ be a real hypersurface of the complex quadric $Q^m$, $m \geq 3$. Then the Reeb flow on $M$ is isometric if and only if $m$ is even, say $m = 2k$, and $M$ is an open part of a tube around a totally geodesic $\mathbb{CP}^k \subset Q^{2k}$.

On the other hand, as non-compact type ambient spaces, for the complex hyperbolic space $\mathbb{CH}^m = SU_{1,m}/S(U_mU_1)$ a classification was obtained by Montiel and Romero [12]. They proved that the Reeb flow on a real hypersurface in $\mathbb{CH}^m$ is isometric if and only if $M$ is an open part of a tube around a totally geodesic $\mathbb{CH}^k$ in $\mathbb{CH}^m$ for some $k \in \{0, \ldots, m - 1\}$. For the complex hyperbolic 2-plane Grassmannian $G^*_2(\mathbb{C}^{m+2}) = SU_{2,m}/S(U_mU_2)$ the classification of isometric Reeb flow was obtained by Suh [22]. In this case, the Reeb flow on a real hypersurface in $G^*_2(\mathbb{C}^{m+2})$ is isometric if and only if $M$ is an open part of a tube around a totally geodesic $G^*_2(\mathbb{C}^{m+1}) \subset G^*_2(\mathbb{C}^{m+2})$ or a horosphere with singular normal $JN \in \mathfrak{J}N$. The geometric construction of horospheres in a non-compact manifold of negative curvature was mainly discussed in the book due to Eberlein [3].

In the paper due to Suh [23] he investigated this problem of isometric Reeb flow for the complex hyperbolic quadric $Q^{*m} = SO_{0,m}/SO_m \otimes SO_m$. In view of the previous results, naturally, we expected that the classification might include at least the totally geodesic $Q^{*m-1} \subset Q^{*m}$. But, the results are quite different from our expectations. The totally geodesic submanifolds of the above type are not included. Now compared to Theorem A, the classification is as follows:

**Theorem B** Let $M$ be a real hypersurface of the complex hyperbolic quadric $Q^{*m}$, $m \geq 3$. The Reeb flow on $M$ is isometric if and only if $m$ is even, say $m = 2k$, and $M$ is an open part of a tube around a totally geodesic $\mathbb{CH}^k \subset Q^{2k*}$ or a horosphere whose center at infinity is $\mathfrak{A}$-isotropic singular.

Hereafter, we denote $(\mathcal{T}_A)$ and $(\mathcal{H}_A)$ such a tube and horosphere given in Theorem B, respectively. Then we see that $(\mathcal{T}_A)$ and $(\mathcal{H}_A)$ should be Hopf, that is, $S\xi = \alpha \xi$, and they have $\mathfrak{A}$-isotropic singular normal vector field.

On the other hand, Jacobi fields along geodesics of a given Riemannian manifold $(\tilde{M}, g)$ satisfy a well known differential equation. This equation naturally inspires the so-called Jacobi operator. That is, if $R$ denotes the curvature operator of $\tilde{M}$, and $X$ is a tangent vector field to $\tilde{M}$, then the Jacobi operator $R_X \in \text{End}(T_p\tilde{M})$ with respect to $X$ at $p \in \tilde{M}$, defined by $(R_X Y)(p) = (R(Y, X)X)(p)$ for any $Y \in T_p\tilde{M}$, becomes a self adjoint endomorphism of the tangent bundle $T\tilde{M}$ of $\tilde{M}$. Thus, each tangent vector field $X$ to $\tilde{M}$ provides a Jacobi operator $R_X$ with respect to $X$. In particular, for the Reeb vector field $\xi$, the Jacobi operator $R_\xi$ is said to be the structure Jacobi operator.

Actually, many geometers have considered the condition that a real hypersurface $M$ in Kähler manifolds has parallel structure Jacobi operator (or Reeb parallel structure Jacobi operator, respectively), that is, $\nabla_X R_\xi = 0$ for any tangent vector field $X$ on $M$ (or $\nabla_\xi R_\xi = 0$, respectively). In [7], Ki, Pérez, Santos and Suh have investigated the Reeb parallel structure Jacobi operator in the complex space form $M_m(c)$, $c \neq 0$, and have used it to study some principal curvatures for a tube over a totally
geodesic submanifold. On the other hand, Pérez et al. [16] have investigated Hopf real hypersurfaces $M$ in $G_2(\mathbb{C}^{m+2})$ with parallel structure Jacobi operator, that is, $\nabla_X R_\xi = 0$ for any tangent vector field $X$ on $M$. Jeong et al. [6] and Pérez and Santos [14] have generalized such a notion to the recurrent structure Jacobi operator, that is, $(\nabla_X R_\xi)Y = \beta(X)R_\xi Y$ for a certain 1-form $\beta$ and any vector fields $X, Y$ on $M$ in $G_2(\mathbb{C}^{m+2})$ or $\mathbb{C}P^m$. In [5], Jeong, Lee, and Suh have considered a Hopf real hypersurface with structure Jacobi operator of Codazzi type, $(\nabla_X R_\xi)Y = (\nabla_Y R_\xi)X$, in $G_2(\mathbb{C}^{m+2})$. Moreover, Pérez et al. [15] have further investigated the property of the Lie $\xi$-parallel structure Jacobi operator in complex projective space $\mathbb{C}P^m$, that is, $\mathcal{L}_\xi R_\xi = 0$. In [27] Suh, Pérez, and Woo investigated the parallelism property with respect to the structure Jacobi operator $R_\xi$ defined on $M$ in the complex hyperbolic quadric $Q^*_m = SO_{2,m}/SO_2SO_m$ and gave the following result.

Theorem C There does not exist a Hopf real hypersurface in the complex hyperbolic quadrics $Q^*_m$, $m \geq 3$, with parallel structure Jacobi operator, that is, $\nabla_X R_\xi = 0$ for any tangent vector field $X$ on $M$.

Motivated by these results, in this paper we consider the case when $R_\xi$ of $M$ in $Q^*_m$ is Reeb parallel, that is, $\nabla_\xi R_\xi = 0$, and first we prove the following:

Main Theorem 1 Let $M$ be a Hopf real hypersurface in $Q^*_m$, $m \geq 3$, with Reeb parallel structure Jacobi operator. Then the unit normal vector field $N$ is singular, that is, $N$ is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal.

On the other hand, in [26] we have considered the notion of Reeb parallel shape operator $S$ for a real hypersurface $M$ in $Q^*_m$, that is, $\nabla_\xi S = 0$, and have proved:

Theorem D Let $M$ be a Hopf real hypersurface in complex hyperbolic quadric $Q^*_m$, $m \geq 3$, with Reeb parallel shape operator and non-vanishing Reeb curvature. Then $M$ is an open part of the following:

1. a tube around the totally geodesic $\mathbb{C}H^k \subset Q^{*2k}$, where $m = 2k$,
2. a horosphere whose center at infinity is $\mathfrak{A}$-isotropic singular,
3. a tube around the totally geodesic Hermitian symmetric space $Q^{*m-1}$ embedded in $Q^*_m$,
4. a horosphere in $Q^*_m$ whose center at infinity is the equivalence class of an $\mathfrak{A}$-principal geodesic in $Q^*_m$,
5. a tube around the $m$-dimensional real hyperbolic space $\mathbb{R}H^m$ which is embedded in $Q^*_m$ as a real space form, or otherwise
6. $M$ has two distinct constant principal curvatures given by

$$\alpha, \quad \lambda = \frac{\alpha^2 - 2}{\alpha}$$

with multiplicities $m$ and $(m - 1)$, respectively.

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Using Main Theorem 1 and Theorem D, we give a classification for Hopf real hypersurfaces in the complex hyperbolic quadric \( Q^{*m} \) with Reeb parallel structure Jacobi operator as follows:

**Main Theorem 2** Let \( M \) be a Hopf real hypersurface in the complex hyperbolic quadric \( Q^{*m} \), \( m \geq 3 \), with Reeb parallel structure Jacobi operator. If the Reeb curvature function \( \alpha \) is non-vanishing, then \( M \) is locally congruent to the one of the following:

1. a tube around the totally geodesic \( CH^k \subset Q^{*2k} \), where \( m = 2k \),
2. a horosphere whose center at infinity is \( \mathbb{R} \)-isotropic singular.

### 2 The Complex Hyperbolic Quadric

In this section, let us introduce known results about the complex hyperbolic quadric \( Q^{*m} \) which are mentioned in [10, 25] and [26].

The \( m \)-dimensional complex hyperbolic quadric \( Q^{*m} \) is the non-compact dual of the \( m \)-dimensional complex quadric \( Q^m \), i.e. the simply connected Riemannian symmetric space whose curvature tensor is the negative of the curvature tensor of \( Q^m \).

The complex hyperbolic quadric \( Q^{*m} \) cannot be realized as a homogeneous complex hypersurface of the complex hyperbolic space \( CH^{m+1} \). In fact, Smyth [21, Theorem 3(ii)] has shown that every homogeneous complex hypersurface in \( CH^{m+1} \) is totally geodesic. This is in marked contrast to the situation for the complex quadric \( Q^m \), which can be realized as a homogeneous complex hypersurface of the complex projective space \( CP^{m+1} \) in such a way that the shape operator for any unit normal vector to \( Q^m \) is a real structure on the corresponding tangent space of \( Q^m \), (see [19]). Another related result by Smyth, [21, Theorem 1], which states that any complex hypersurface of \( CH^{m+1} \) for which the square of the shape operator has constant eigenvalues (counted with multiplicity) is totally geodesic, also precludes the possibility of a model of \( Q^{*m} \) as a complex hypersurface of \( CH^{m+1} \) with the analogous property for the shape operator.

Therefore we realize the complex hyperbolic quadric \( Q^{*m} \) as the quotient manifold \( SO_{2,m}^0/SO_2SO_m \). As \( Q^{*1} \) is isomorphic to the real hyperbolic space \( RH^2 = SO_{1,2}^0/SO_2 \), and \( Q^{*2} \) is isomorphic to the Hermitian product of complex hyperbolic spaces \( CH^1 \times CH^1 \), we suppose \( m \geq 3 \) in the sequel and throughout this paper. Let \( G := SO_{2,m}^0 \) be the transvection group of \( Q^{*m} \) and \( K := SO_2SO_m \) be the isotropy group of \( Q^{*m} \) at the “origin” \( p_0 := eK \in Q^{*m} \). Then

\[
\sigma : G \to G, \quad g \mapsto sgs^{-1} \quad \text{with} \quad s := \begin{pmatrix}
-1 & -1 & \cdots & 0 \\
1 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}
\]

is an involutive Lie group automorphism of \( G \) with \( \text{Fix}(\sigma)_0 = K \), and therefore \( Q^{*m} = G/K \) is a Riemannian symmetric space. The center of the isotropy group \( K \) is isomorphic to \( SO_2 \), and therefore \( Q^{*m} \) is in fact a Hermitian symmetric space.
The Lie algebra $\mathfrak{g} := so_{2,m}$ of $G$ is given by

$$\mathfrak{g} = \{ X ∈ \mathfrak{gl}(m + 2, \mathbb{R}) | X^t \cdot s = -s \cdot X \}$$

(see [8, p. 59]). In the sequel we will write members of $\mathfrak{g}$ as block matrices with respect to the decomposition $\mathbb{R}^{m+2} = \mathbb{R}^2 \oplus \mathbb{R}^m$, i.e. in the form

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix},$$

where $X_{11}, X_{12}, X_{21}, X_{22}$ are real matrices of orders $2 \times 2, 2 \times m, m \times 2$ and $m \times m$, respectively. Then

$$\mathfrak{g} = \left\{ \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \mid X_{11}^t = -X_{11}, X_{12}^t = X_{21}, X_{22}^t = -X_{22} \right\}.$$

The linearization $\sigma_L = \text{Ad}(s) : \mathfrak{g} → \mathfrak{g}$ of the involutive Lie group automorphism $\sigma$ induces the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where the Lie subalgebra $\mathfrak{k} = \text{Eig}(\sigma_L, 1) = \{ X ∈ \mathfrak{g} | sXs^{-1} = X \}$

$$\cong so_2 \oplus so_m$$

is the Lie algebra of the isotropy group $K$, and the $2m$-dimensional linear subspace

$$\mathfrak{m} = \text{Eig}(\sigma_L, -1) = \{ X ∈ \mathfrak{g} | sXs^{-1} = -X \} = \left\{ \begin{pmatrix} 0 & X_{12} \\ X_{21} & 0 \end{pmatrix} \mid X_{12}^t = X_{21} \right\}$$

is canonically isomorphic to the tangent space $T_{p_0}Q^{*m}$. Under the identification $T_{p_0}Q^{*m} \cong \mathfrak{m}$, the Riemannian metric $g$ of $Q^{*m}$ (where the constant factor of the metric is chosen so that the formulae become as simple as possible) is given by

$$g(X, Y) = \frac{1}{2} \text{tr}(Y^t \cdot X) = \text{tr}(Y_{12} \cdot X_{21}) \quad \text{for} \quad X, Y ∈ \mathfrak{m}.$$ 

g is clearly $\text{Ad}(K)$-invariant, and therefore corresponds to an $G$-invariant Riemannian metric on $Q^{*m}$. The complex structure $J$ of the Hermitian symmetric space is given by

$$JX = \text{Ad}(j)X \quad \text{for} \quad X ∈ \mathfrak{m}, \quad \text{where} \quad j := \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ \vdots & \ddots & 1 \end{pmatrix} ∈ K.$$

Because $j$ is in the center of $K$, the orthogonal linear map $J$ is $\text{Ad}(K)$-invariant, and thus defines a $G$-invariant Hermitian structure on $Q^{*m}$. By identifying the multiplication with the unit complex number $i$ with the application of the linear map $J$, the tangent spaces of $Q^{*m}$ thus become $m$-dimensional complex linear spaces, and we will adopt this point of view in the sequel. 

For any $p ∈ Q^{*m}$ and $A ∈ \mathfrak{A}_p := \{ \lambda A_0 | \lambda ∈ S^1 \}$, the real structure $A$ induces a splitting

$$T_pQ^{*m} = V(A) ⊕ JV(A)$$

into two orthogonal, maximal totally real subspaces of the tangent space $T_pQ^{*m}$. Here $V(A)$ (resp., $JV(A)$) is the $(+1)$-eigenspace (resp., the $(-1)$-eigenspace) of
A. For every unit vector \( Z \in T_pQ^*m \) there exist \( t \in \left[ 0, \frac{\pi}{4} \right], A \in \mathfrak{A}_p \) and orthonormal vectors \( Z_1, Z_2 \in V(A) \) so that

\[
Z = \cos(t)Z_1 + \sin(t)JZ_2
\]

holds; see [19, Proposition 3]. Here \( t \) is uniquely determined by \( Z \). The vector \( Z \) is singular, i.e. contained in more than one Cartan subalgebra of \( m \), if and only if either \( t = 0 \) or \( t = \frac{\pi}{4} \) holds. The vectors with \( t = 0 \) are called \( \mathfrak{A} \)-principal, whereas the vectors with \( t = \frac{\pi}{4} \) are called \( \mathfrak{A} \)-isotropic. If \( Z \) is regular, i.e. \( 0 < t < \frac{\pi}{4} \) holds, then also \( A \) and \( Z_1, Z_2 \) are uniquely determined by \( Z \).

As for the complex quadric, the Riemannian curvature tensor \( \bar{R} \) of \( Q^*m \) can be fully described in terms of the “fundamental geometric structures” \( g, J \) and \( \mathfrak{A} \). In fact, under the correspondence \( T_{po}Q^*m \cong m \), the curvature \( \bar{R}(X, Y)Z \) corresponds to \(-[[X, Y], Z] \) for \( X, Y, Z \in m \), see [9, Chapter XI, Theorem 3.2(1)]. By evaluating the latter expression explicitly, one can show that one has

\[
\bar{R}(X, Y)Z = -g(Y, Z)X + g(X, Z)Y - g(JY, Z)JX + g(JX, Z)JY - 2g(JX, Y)JZ - g(AY, Z)AX + g(AX, Z)AY - g(JAY, Z)JAX + g(JAX, Z)JAY
\]

(2.1)

for arbitrary \( A \in \mathfrak{A}_p \). Therefore the curvature of \( Q^*m \) is the negative of that of the complex quadric \( Q^m \), compare [19, Theorem 1]. This confirms that the symmetric space \( Q^*m \) which we have constructed here is indeed the non-compact dual of the complex quadric.

### 3 Some General Equations

Let \( M \) be a real hypersurface in the complex hyperbolic quadric \( Q^*m \). Then, at each point \( p \in M \) we can choose \( A \in \mathfrak{A}_p \) such that

\[
N = \cos(t)Z_1 + \sin(t)JZ_2
\]

(3.1)

for some orthonormal vectors \( Z_1, Z_2 \in V(A) \) and \( 0 \leq t \leq \frac{\pi}{4} \) (see Proposition 3 in [19]). Note that \( t \) is a function on \( M \). In addition, for any vector field \( X \) on \( M \) in \( Q^*m \), we may decompose \( JX \) as

\[
JX = \phi X + \eta(X)N \]

where \( N \) denotes a unit normal vector field to \( M \). The vector field

\[
\xi = -JN
\]

is said to be the Reeb vector field, and the 1-form \( \eta \) is given by \( \eta(X) = g(\xi, X) \). Then naturally \( M \) admits an almost contact metric structure \((\phi, \xi, \eta, g)\) induced from the Kähler structure \((J, g)\) of \( Q^*m \) satisfying \( \phi^2 = -I + \eta \otimes \xi, \phi \xi = 0 \) and \( \eta(\xi) = 1 \). The tangent bundle \( TM \) of \( M \) splits orthogonally into \( TM = C \oplus \mathbb{R} \xi \), where \( C = \ker(\eta) \) is the maximal complex subbundle of \( TM \). The structure tensor field \( \phi \) restricted to \( C \) coincides with the complex structure \( J \) restricted to \( C \). Real hypersurfaces in a Kähler manifold for which the maximal complex subbundle is invariant under the shape operator are known as Hopf hypersurfaces. This condition is equivalent to the
Reeb flow on $M$, that is, the flow of the structure vector field $\xi$, to be geodesic. We assume now that $M$ is a Hopf hypersurface. Then the shape operator $S$ of $M$ in $Q^{*m}$ satisfies

$$S\xi = \alpha\xi$$

for the Reeb vector field $\xi$ and the smooth function $\alpha = g(S\xi, \xi)$ on $M$.

Moreover, since the ambient space $Q^{*m}$ has also the real structure $A$, we decompose $AX$ into its tangential and normal components for a fixed $A \in \mathfrak{A}_p$ and $X \in T_pM$, $p \in M$:

$$AY = BY + \rho(Y)N,$$

where $BY$ denotes the tangential component of $AY$ and $\rho(Y) = g(AY, N)$. Thus (3.1) gives us

$$\xi = \sin(t)Z_2 - \cos(t)JZ_1,$$

$$AN = \cos(t)Z_1 - \sin(t)JZ_2,$$

$$A\xi = \sin(t)Z_2 + \cos(t)JZ_1.$$

So, we get $g(A\xi, N) = g(AN, \xi) = 0$.

As the normal part of (2.1) we have the following equation, which is called the Codazzi equation,

$$g(\nabla X Y - \nabla Y X, Z) = \eta(X)g(\phi Y, Z) + \eta(Y)g(\phi X, Z) + 2\eta(Z)g(\phi X, Y)$$

$$- g(X, AN)g(AY, Z) + g(Y, AN)g(AX, Z)$$

$$- g(X, A\xi)g(JAY, Z) + g(Y, A\xi)g(JAX, Z). \quad (3.2)$$

By virtue of (3.2) we obtain the following lemma.

**Lemma 3.1** [26] Let $M$ be a Hopf real hypersurface in the complex hyperbolic quadric $Q^{*m}$ with (local) unit normal vector field $N$. For each point $p \in M$ we choose $A \in \mathfrak{A}_p$ such that $N_p = \cos(t)Z_1 + \sin(t)JZ_2$ holds for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 \leq t \leq \frac{\pi}{4}$. Then

$$Y\alpha = (\xi\alpha)\eta(Y) - 2g(Y, AN)g(\xi, A\xi). \quad (3.3)$$

and

$$2g(S\phi SX, Y) = \alpha g((\phi S + S\phi)X, Y) - 2g(\phi X, Y)$$

$$+ g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi)$$

$$- g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi)$$

$$+ 2g(\xi, AN)g(X, A\xi)\eta(Y) - 2g(X, AN)g(\xi, A\xi)\eta(Y)$$

$$- 2g(\xi, AN)g(Y, A\xi)\eta(X) + 2g(Y, AN)g(\xi, A\xi)\eta(X). \quad (3.4)$$

Moreover, by virtue of (3.3), we can assert:

**Lemma 3.2** Let $M$ be a Hopf real hypersurface in complex hyperbolic quadric $Q^{*m}$, $m \geq 3$. If the Reeb curvature function $\alpha = g(S\xi, \xi)$ is constant, then the normal vector field $N$ should be singular, that is, $N$ is either $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal.
Proof Assume the Reeb curvature function $\alpha = g(S\xi, \xi)$ is constant. From this, together with (3.3) and $g(A\xi, N) = 0$, it follows that

$$g(A\xi, \xi)g(Y, AN) = g(AN, Y) = 0$$

for any $Y \in T_p M$, $p \in M$. The first part $g(A\xi, \xi) = 0$ implies $N$ is $\mathfrak{A}$-isotropic. Now let us work on the open subset $\mathcal{U} = \{ p \in M | \beta(p) = g(A\xi, \xi)(p) \neq 0 \}$. Then it follows that $g(AN, Y) = 0$ for all $Y \in T_p M$, $p \in \mathcal{U}$. Then, for the orthonormal basis $\{ e_1, e_2, \cdots, e_{2m-1}, e_{2m} := N \}$ of $T_p Q^*m$, the tangent vector $AN \in T_p Q^*m$ given by

$$AN = \sum_{i=1}^{2m-1} g(AN, e_i)e_i + g(AN, N)N$$

(3.5)

Applying the complex conjugate $A$ to this equation and using $A^2 = I$ and (3.5) again, we get

$$N = A^2 N = g(AN, N)AN = g(AN, N)N$$

which means that $N$ is $\mathfrak{A}$-principal. In fact, from (3.1), we see that $g(AN, N) = \cot 2t$, $t \in [0, \frac{\pi}{4})$ on $\mathcal{U}$. So, $g(AN, N) = \pm 1$ leads to $t = 0$. This completes the proof of our Lemma.

Remark 3.1 By virtue of Lemma 3.2, we assert that if the Reeb function $\alpha$ is identically vanishing on $M$ then $N$ should be singular. Now let us denote by $\nabla$ and $\bar{\nabla}$ the covariant derivative of $M$ and the covariant derivative of $Q^*m$, respectively.

If $N$ is $\mathfrak{A}$-principal, that is, $A\xi = -\xi$ and $AN = N$, we have $\rho = 0$, because $\rho(Y) := g(Y, AN) = g(Y, N) = 0$ for any tangent vector field $Y$ on $M$ in $Q^*m$. So we have $AY = BY$ for any tangent vector field $Y$ on $M$ in $Q^*m$. Now we want to give some lemmas which will be useful to prove our main theorem as follows:

Lemma 3.3 [27] Let $M$ be a Hopf hypersurface in the complex hyperbolic quadric $Q^*m$, $m \geq 3$, such that the normal vector field $N$ is $\mathfrak{A}$-principal everywhere. Then the following statements hold:

(i) The Reeb curvature function $\alpha$ is constant.

(ii) If $X \in \mathcal{C}$ is a principal curvature vector of $M$ with principal curvature $\lambda$, then $\alpha = \pm 2$, $\lambda = \pm 1$ for $\alpha = 2\lambda$ or $\phi X$ is a principal curvature vector with principal curvature $\mu = \frac{\alpha\lambda - 2}{2\lambda - \alpha}$ for $\alpha \neq 2\lambda$.

(iii) $\bar{\nabla}X A = 0$ for any $X \in \mathcal{C}$.

(iv) $ASX = SX$ for any $X \in \mathcal{C}$.

Finally, let us induce the structure Jacobi operator $R_{\xi}$ of a Hopf real hypersurface $M$ in the complex hyperbolic quadric. As the tangential part of (2.1), the curvature tensor $R$ of $M$ in complex quadric $Q^*m$ is defined as follows. For any $A \in \mathfrak{A}$

$$R(X, Y)Z = -g(Y, Z)X + g(X, Z)Y - g(\phi Y, Z)\phi X + g(\phi X, Z)\phi Y + 2g(\phi X, Y)\phi Z - g(AY, Z)(AX)^T + g(AX, Z)(AY)^T - g(JAY, Z)(JAX)^T + g(JAX, Z)(JAY)^T + g(SY, Z)SX - g(SX, Z)SY,$$
where \((AX)^T\) and \(S\) denote the tangential component of the vector field \(AX\) and the shape operator of \(M\) in \(Q^{*m}\), respectively.

From this, putting \(Y = Z = \xi\) and using \(g(A\xi, N) = 0\), the structure Jacobi operator is defined by

\[
R_{\xi}(X) = R(X, \xi)\xi
= -X + \eta(X)\xi - g(A\xi, \xi)(AX)^T + g(AX, \xi)A\xi
\]

\[
+ g(X, AN)(AN)^T + g(S\xi, \xi)SX - g(SX, \xi)S\xi.
\]

Then we may put the following

\[(AY)^T = AY - g(AY, N)N.\]

By using the Gauss and Weingarten formulas we obtain:

**Lemma 3.4** [27] Let \(M\) be a real hypersurface in the complex quadric \(Q^{*m}\). Then

\[
\nabla_X(AY)^T = q(X)JAY + A\nabla_X Y + g(SX, Y)AN
- g([q(X)JAY + A\nabla_X Y + g(SX, Y)AN], N)N + g(AY, SX)N + g(AY, N)SX - g(SX, AY)N,
\]

and

\[
\nabla_X(AN)^T = q(X)JAN - ASX
- g(q(X)JAN - ASX, N)N + g(AN, N)SX.
\]

where \(q\) denote a certain 1-form defined on \(T_pQ^{*m}\), \(p \in Q^{*m}\), satisfying \((\tilde{\nabla}_U A)V = q(U)JAV\) for any vector fields \(U, V \in T_pQ^{*m}\).

In particular, by putting \(Y = \xi\) in (3.6) and using \(g(A\xi, N) = 0\), we have

\[
\nabla_X(A\xi) = q(X)JA\xi + A\phi SX + \alpha \eta(X)AN
- [q(X)g(JA\xi, N) + g(A\phi SX, N) + \alpha \eta(X)g(AN, N)] N.
\]

Moreover, we know that

\[
X\beta = X(g(A\xi, \xi))
= g((\tilde{\nabla}_X A)\xi + A\tilde{\nabla}_X \xi, \xi) + g(A\xi, \tilde{\nabla}_X \xi)
= g(q(X)JA\xi + A\phi SX + g(SX, \xi)AN, \xi) + g(A\xi, \phi SX + g(SX, \xi)N)
= 2g(A\phi SX, \xi).
\]

### 4 Reeb Parallel Structure Jacobi Operator and Proof of Main Theorem 1

The curvature tensor \(R(X, Y)Z\) for a Hopf real hypersurface \(M\) in the complex hyperbolic quadric \(Q^{*m} = SO_{2,m}^0/\SO_2 \SO_m\) induced from the curvature tensor of \(Q^{*m}\)
is given in Section 3. Now the structure Jacobi operator $R_{\xi}$ from Section 3 can be rewritten as follows:

$$
R_{\xi}(Y) = R(Y, \xi) \xi
- Y + \eta(Y) \xi - \beta(AY)^T + g(AY, \xi) A\xi + g(AY, N)(AN)^T \\
+ \alpha SY - g(SY, \xi) S\xi,
$$

(4.1)

where we have put $\alpha = g(S\xi, \xi)$ and $\beta = g(A\xi, \xi)$, because we assume that $M$ is Hopf. The Reeb vector field $\xi = -JN$ and the anti-commuting property $AJ = -JA$ gives that the function $\beta$ becomes $\beta = -g(AN, N)$. When this function $\beta = g(A\xi, \xi)$ identically vanishes, we say that a real hypersurface $M$ in $Q^{*m}$ is $A$-isotropic as in Lemma 3.2.

Here we use the assumption of Reeb parallel structure Jacobi operator, that is, $\nabla_\xi R_{\xi} = 0$. Then (4.1), (3.6), and (3.7) give that

$$(\nabla_\xi R_{\xi})Y = \nabla_\xi (R_{\xi}Y) - R_{\xi}(\nabla_\xi Y)
= -((\xi \beta)g(AY, \xi) - \beta g(q(\xi)g(JAY, \xi) + g(\nabla_\xi Y, A\xi) + \alpha g(AY, N))
+ g(q(\xi)JAX + \alpha AN, Y)g(A\xi, \xi)
+ g(Y, AN)\{q(\xi)g(JAN, \xi) - \alpha A\xi + \alpha g(AN, N)\xi - g(q(\xi)JAN - \alpha A\xi, N)\}
+ (\xi \alpha)SY + \alpha(\nabla_\xi S)Y - ((\xi \alpha^2)\eta(Y)\xi,
$$

where we have used $g(A\xi, N) = 0$.

From this, by taking the inner product with the Reeb vector field $\xi$, and using $\nabla_\xi R_{\xi} = 0$, we have

$$
0 = -((\xi \beta)g(AY, \xi) - \beta g(q(\xi)g(JAY, \xi) + g(\nabla_\xi Y, A\xi) + \alpha g(AY, N))
+ g(q(\xi)JAX + \alpha AN, Y)g(A\xi, \xi)
+ g(Y, AN)\{q(\xi)g(JAN, \xi) - \alpha A\xi + \alpha g(AN, N)\xi - g(q(\xi)JAN - \alpha A\xi, N)\}
+ (\xi \alpha)SY + \alpha(\nabla_\xi S)Y - ((\xi \alpha^2)\eta(Y)\xi.
$$

(4.2)

Then first, using $g(A\xi, N) = 0$ and $(\xi \beta) = 0$ in (3.9), we have

$$
0 = \beta\{g(\nabla_\xi Y, A\xi) - (q(\xi) - 2\alpha)g(Y, AN)\}.
$$

(4.3)

From this we have $\beta = 0$ or $g(\nabla_\xi Y, A\xi) = (q(\xi) - 2\alpha)g(Y, AN)$. The first part $\beta = g(A\xi, \xi) = 0$ implies $N$ is $A$-isotropic. Now let us work on the open subset $U = \{p \in M | \beta(p) \neq 0 \}$. Then it follows that

$$
g(\nabla_\xi Y, A\xi) = (q(\xi) - 2\alpha)g(Y, AN)
$$

(4.4)
Then by putting \( Y = (AN)^T \) in (4.4), we have
\[
\begin{align*}
g(\nabla_\xi (AN)^T, A\xi) &= (q(\xi) - 2\alpha)g((AN)^T, AN) \\
&= (q(\xi) - 2\alpha)(1 - \beta^2) \\
&= q(\xi) - 2\alpha - q(\xi)\beta^2 + 2\alpha\beta^2
\end{align*}
\]
(4.5)

On the other hand, by (3.7) the left term of (4.5) becomes
\[
\begin{align*}
g(\nabla_\xi (AN)^T, A\xi) &= q(\xi)g(JAN, A\xi) - \alpha g(A\xi, A\xi) + \alpha g(AN, N)g(\xi, A\xi) \\
&= q(\xi) - \alpha - \alpha\beta^2.
\end{align*}
\]
(4.6)

Then from (4.5) and (4.6) it follows that
\[
\alpha + q(\xi)\beta^2 - 3\alpha\beta^2 = 0. \tag{4.7}
\]

So for any \( Y \) orthogonal to \( A\xi \) by (3.8), we have
\[
g(\nabla_\xi Y, A\xi) = -g(Y, \nabla_\xi A\xi) = (q(\xi) - \alpha)g(Y, AN).
\]
(4.8)

From this, comparing with (4.4), we have
\[
\alpha g(AN, Y) = 0 \tag{4.9}
\]
for all \( Y \) orthogonal to \( A\xi \).

By virtue of Lemma 3.2 in Section 3, if the Reeb curvature function \( \alpha = g(S\xi, \xi) \) is vanishing, then the unit normal vector field is singular. Thus now we only consider the case \( \alpha \neq 0 \) on \( U \). By (4.9), together with \( AN = AJ\xi = -JA\xi = -\phi A\xi - g(A\xi, \xi)N \), it follows that \( g(\phi A\xi, Y) = 0 \) for all \( Y \in \mathcal{F} \). Here we denote \( \mathcal{F} = \{ Y \in T_pM \mid Y \perp A\xi, \ p \in U \} \). Substituting \( Y = \phi A\xi(\in \mathcal{F}) \), we get \( 0 = 1 - g^2(A\xi, \xi) = 1 - \beta^2 \). This implies that the unit normal \( N \) is \( \mathfrak{A} \)-principal on \( U \). Together with Lemma 3.2 and this observation we give the following lemma.

**Lemma 4.1** Let \( M \) be a Hopf real hypersurface in the complex hyperbolic quadric \( Q^{*m}, m \geq 3 \), with Reeb parallel structure Jacobi operator. Then the unit normal vector field \( N \) is \( \mathfrak{A} \)-principal or \( \mathfrak{A} \)-isotropic.

By virtue of Lemma 3.2, we can consider two classes of real hypersurfaces in complex hyperbolic quadric \( Q^{*m} \) with Reeb parallel structure Jacobi operator: with \( \mathfrak{A} \)-principal unit normal vector field \( N \) or otherwise, with \( \mathfrak{A} \)-isotropic unit normal vector field \( N \). We will consider each case in Sections 5 and 6 respectively.

### 5 Reeb Parallel Structure Jacobi Operator with \( \mathfrak{A} \)-Isotropic Normal Vector Field

In this section we assume that the unit normal vector field \( N \) of a real hypersurface \( M \) in the complex hyperbolic quadric \( Q^{*m} = SO_{2,m}^0/SO_2SO_m \) is \( \mathfrak{A} \)-isotropic. Then the normal vector field \( N \) can be written as
\[
N = \frac{1}{\sqrt{2}}(Z_1 + JZ_2)
\]
for \( Z_1, Z_2 \in V(A) \), where \( V(A) \) denotes the \((+1)\)-eigenspace of the complex conjugation \( A \in \mathfrak{A} \). Here we note that \( Z_1 \) and \( Z_2 \) are orthonormal, i.e., we have \( \|Z_1\| = \|Z_2\| = 1 \) and \( Z_1 \perp Z_2 \). Then it follows that

\[
AN = \frac{1}{\sqrt{2}}(Z_1 - JZ_2), \quad AJN = -\frac{1}{\sqrt{2}}(JZ_1 + Z_2), \quad \text{and} \quad JN = \frac{1}{\sqrt{2}}(JZ_1 - Z_2).
\]

Then it gives that

\[
g(\xi, A\xi) = g(JN, AJN) = 0, \quad g(\xi, AN) = 0, \quad \text{and} \quad g(AN, N) = 0.
\]

By virtue of these formulas for \( \mathfrak{A} \)-isotropic unit normal vector field, the structure Jacobi operator is given by

\[
R_\xi(X) = R(X, \xi)\xi - X + \eta(X)\xi + g(AX, \xi)A\xi + g(JAX, \xi)JA\xi + g(S\xi, \xi)SX - g(SX, \xi)S\xi. \tag{5.1}
\]

On the other hand, we know that \( JA\xi = -JAJN = AJ^2N = -AN \), and \( g(JAX, \xi) = -g(AX, J\xi) = -g(AX, N) \). Now the structure Jacobi operator \( R_\xi \) can be rearranged as follows:

\[
R_\xi(X) = -X + \eta(X)\xi + g(AX, \xi)A\xi + g(X, AN)AN + \alpha SX - \alpha^2\eta(X)\xi. \tag{5.2}
\]

Differentiating (5.2) we obtain

\[
(\nabla Y R_\xi)X = \nabla_Y (R_\xi(X)) - R_\xi(\nabla_Y X)
\]

\[
= (\nabla_Y \eta)(X)\xi + \eta(X)\nabla_Y \xi + g(X, \nabla_Y (AX))A\xi + g(X, \nabla_Y (AN))AN + g(X, AN)\nabla_Y (AN)
\]

\[
+ (Y\alpha)SX + \alpha(\nabla_Y S)X - (Y\alpha^2)\eta(X)\xi - \alpha^2(\nabla_Y \eta)(X)\xi - \alpha^2\eta(X)\nabla_Y \xi. \tag{5.3}
\]

On the other hand, by virtue of Lemma 3.4, we obtain the following for a Hopf real hypersurface in \( Q^{*m} \) with \( \mathfrak{A} \)-isotropic unit normal as follows:

**Lemma 5.1** [26] Let \( M \) be a Hopf real hypersurface in the complex hyperbolic quadric \( Q^{*m} \), \( m \geq 3 \), with \( \mathfrak{A} \)-isotropic unit normal. Then

\[
SAN = 0, \quad \text{and} \quad SA\xi = 0.
\]

Substituting these formulas into (5.3) and using our assumption that \( M \) is a Hopf real hypersurface with \( \mathfrak{A} \)-isotropic singular normal vector \( N \) in \( Q^{*m} \), it yields

\[
(\nabla_Y R_\xi)X = g(X, \phi SY)\xi + \eta(X)\phi SY
\]

\[
- g(\phi SY, \xi)BX - g(AX, \phi SY)BX
\]

\[
+ \left\{ g(\phi SY, X) + \alpha \eta(Y)g(AN, X) \right\}A\xi
\]

\[
+ g(AX, X)\left\{ B\phi SY + \alpha \eta(Y)AN \right\}
\]

\[
- g(AX, SY)AN - g(AN, X)BSY
\]

\[
+ (Y\alpha)SX + \alpha(\nabla_Y S)X - 2\alpha(Y\alpha)\eta(X)\xi
\]

\[
- \alpha^2 g(X, \phi SY)\xi - \alpha^2\eta(X)\phi SY. \tag{5.4}
\]

\[\square\]
From this and using the assumption of Reeb parallel structure Jacobi operator, it follows:

\[(\xi \alpha)SX + \alpha(\nabla_{\xi} S)X - 2\alpha(\xi \alpha)\eta(X)\xi = 0.\]  

(5.5)

**Lemma 5.2** Let M be a real hypersurface in the complex hyperbolic quadric \(Q^m\), \(m \geq 3\), with Reeb parallel structure Jacobi operator and non-vanishing geodesic Reeb flow. If the unit normal vector field N of M is \(\mathfrak{A}\)-isotropic, then the Reeb curvature function \(\alpha\) is constant. Moreover, the shape operator S should be Reeb parallel, that is, the shape operator S satisfies the property \(\nabla_{\xi} S = 0\).

**Proof** By putting \(X = \xi\) in the equation of Codazzi in Section 3, we have

\[(\nabla_{\xi} S)Y = (\nabla_{Y} S)\xi - \phi Y + g(Y, AN)A\xi - g(Y, A\xi)AN = (Y\alpha)\xi + \alpha(\phi SX - \phi SY) - \phi Y + g(Y, AN)A\xi - g(Y, A\xi)AN.\]  

(5.6)

From this, together with (5.5), it follows that

\[(\xi \alpha)SX + \alpha(\xi \alpha)\eta(X)\xi + \alpha\left\{\alpha\phi SX - \phi SY - \phi Y + g(Y, AN)A\xi - g(Y, A\xi)AN\right\} = 0.\]  

(5.7)

Then by taking the inner product (5.7) with the Reeb vector field \(\xi\), we have \(\alpha X\alpha = \alpha(\xi \alpha)\eta(X)\). Then (5.7) gives

\[(\xi \alpha)SX - \alpha(\xi \alpha)\eta(X)\xi + \alpha\left\{\alpha\phi SX - \phi SY - \phi Y + g(X, AN)A\xi - g(X, A\xi)AN\right\} = 0.\]  

(5.8)

Since the unit normal vector field \(N\) is \(\mathfrak{A}\)-isotropic, Lemma 3.1 gives

\[S\phi SX - \alpha(\phi S + S\phi)X = -\phi X + g(X, AN)A\xi - g(X, A\xi)AN.\]

Substituting this one into (5.8), we have

\[2(\xi \alpha)SX - 2\alpha(\xi \alpha)\eta(X)\xi + \alpha^2(\phi S - S\phi)X = 0.\]  

(5.9)

On the other hand, by (3.4) in Lemma 3.1, when a unit normal vector field \(N\) of \(M\) is \(\mathfrak{A}\)-isotropic, we get

\[2S\phi SX - \alpha(\phi S + S\phi)X + 2\phi X - 2g(AN, X)A\xi + 2g(X, A\xi)AN = 0\]

for any \(X \in T_pM, p \in M\). For some \(X_0 \in Q := \{X \in TM \mid X \perp \xi, A\xi, AN\}\) such that \(SX_0 = \lambda X_0\), it becomes \((2\lambda - \alpha)S\phi X_0 = (\alpha \lambda - 2)\phi X_0\). Thus we obtain:

- if \(\alpha = 2\lambda\), then \(\lambda = \pm 1\). Moreover, \(\alpha = \pm 2\).
- if \(\alpha \neq 2\lambda\), then the vector \(\phi X_0\) is also principal with eigenvalue \(\mu\), where \(\mu = \frac{\alpha \lambda - 2}{2\lambda - \alpha}\).

From this, let us consider two cases as follows.
Case I. $\alpha = 2\lambda$

Since $S$ is symmetric, we can choose a basis $\{e_1 = \xi, e_2 = A\xi, e_3 = AN, e_4, \ldots, e_{2m-1}\}$ for $T_pM$ such that $Se_i = \lambda_i e_i$ (in particular, $\lambda_1 = \alpha, \lambda_2 = \lambda_3 = 0$). It follows that the expression of the shape operator $S$ becomes

$$S = \text{diag}(\alpha, 0, 0, \lambda_4, \ldots, \lambda_{2m-1})$$

$$= \text{diag}(\pm 2, 0, 0, \pm 1, \ldots, \pm 1),$$

where $\text{diag}(a_1, \ldots, a_n)$ denote a diagonal matrix whose diagonal entries starting in the upper left corner are $a_1, \ldots, a_n$. From this and (5.5), we see that $M$ becomes a Hopf real hypersurface in $Q^{*m}, m \geq 3$, with Reeb parallel shape operator, $\nabla_\xi S = 0$.

Case II. $\alpha \neq 2\lambda$

For some unit $X_0 \in Q$ such that $SX_0 = \lambda X_0$, we have $S\phi X_0 = \mu \phi X_0, \mu = \frac{\alpha^3 - 2}{2\lambda - \alpha}$. Then (5.9) gives

$$2(\xi \alpha)\lambda X_0 + \alpha^2(\lambda - \mu)\phi X_0 = 0. \quad (5.10)$$

From this, by taking the inner product with $X_0$ we have $(\xi \alpha)\lambda = 0$. Now let us consider an open subset $\mathcal{U} = \{p \in M \mid (\xi \alpha)(p) \neq 0\}$ in $M$. Then on such an open subset $\mathcal{U}$ the principal curvature $\lambda$ identically vanishes. Then (5.10) gives that the Reeb curvature function $\alpha$ identically vanishes on $\mathcal{U}$. This gives a contradiction. So such an open subset $\mathcal{U}$ can not exist. This means that $\xi \alpha = 0$ on $M$. That is, $X\alpha = 0$ for any $X$ on $M$ in $Q^{*m}$. From this and using our assumption, $\alpha \neq 0$, (5.5) implies that $M$ is a Hopf real hypersurface with Reeb parallel shape operator in $Q^{*m}, m \geq 3$.

Then by Theorem D in the introduction we can assert the following:

**Theorem 5.3** Let $M$ be a real hypersurface in the complex hyperbolic quadric $Q^{*m}, m \geq 3$, with Reeb parallel structure Jacobi operator. If the unit normal vector field $N$ is $A$-isotropic and $M$ has non-vanishing Reeb curvature, then $M$ is locally congruent to

1. a tube around the totally geodesic $\mathbb{CH}^k \subset Q^{*2k}$, where $m = 2k$,
2. a horosphere whose center at infinity is $A$-isotropic singular.

### 6 Reeb Parallel Structure Jacobi Operator with $A$-Principal Normal Vector Field

Let $M$ be a real hypersurface with non-vanishing geodesic Reeb flow, $\alpha \neq 0$, in the complex hyperbolic quadric $Q^{*m} = SO_{2,m}^0 / SO_2 SO_m, m \geq 3$. In addition, we assume that $M$ has Reeb parallel structure Jacobi operator and $A$-principal normal vector field. Then the unit normal vector field $N$ satisfies $AN = N$ for a complex conjugation $A \in A$. Then it follows that $A\xi = -\xi$ and $g(A\xi, \xi) = -1$. 

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By the assumption of Reeb parallel structure Jacobi operator, we have

\[
0 = (\nabla_\xi R_\xi)X = \left\{ q(\xi)JAX + g(SX, \xi)N - g(S_\xi, AX)N \right\} + (\xi \alpha)SX + \alpha(\nabla_\xi S)X - (\xi \alpha^2)\eta(X)\xi \quad (6.1)
\]

On the other hand, differentiating \( g(AN, JN) = 0 \) along any \( X \in T_pM, p \in M \), then it follows that

\[
q(X) = g(ASX, JN) + g(\xi, S_\xi),
\]

which implies that \( q(X) = 2\alpha \eta(X) \) if \( N \) is \( A \)-principal. From this, we know \( q(\xi) = 2\alpha \). By Lemma 3.3, the Reeb curvature function \( \alpha \) is constant on \( M \). So (6.1) reduces to the following

\[
\alpha(\nabla_\xi S)X = -2\alpha \phi AX.
\]

Since \( M \) has non-vanishing geodesic Reeb flow, that is, \( \alpha = g(S_\xi, \xi) \neq 0 \), we have

\[
(\nabla_\xi S)X = -2\phi AX. \quad (6.2)
\]

On the other hand, by using the equation of Codazzi in Section 3, we have

\[
g\left((\nabla_X S)\xi - (\nabla_\xi S)X, Z\right) = g(\phi X, Z) - g(X, AN)g(A_\xi, Z) - g(X, A_\xi)g(JA_\xi, Z) + g(\xi, A_\xi)g(JAX, Z) = g(\phi X, Z) - g(\phi AX, Z).
\]

In addition, since \( M \) is Hopf, it leads to

\[
(\nabla_\xi S)X = (\nabla_X S)\xi - \phi X + \phi AX = \alpha \phi SX - S\phi SX - \phi X + \phi AX
\]

From this, together with (6.2), it follows that

\[
\alpha \phi SX - S\phi SX - \phi X = -3\phi AX. \quad (6.3)
\]

By virtue of Lemma 3.1, for the \( A \)-principal unit normal vector field, we obtain

\[
2S\phi SX = \alpha(S\phi + \phi S)X - 2\phi X. \quad (6.4)
\]

Therefore, (6.3) can be written as

\[
\alpha(\phi S - S\phi)X = -6\phi AX. \quad (6.5)
\]

Inserting \( X = SY \) for \( Y \in \mathcal{C} \) into (6.5) and applying the structure tensor \( \phi \) leads to

\[
-\alpha S^2 Y - \alpha \phi S\phi SY = 6ASY,
\]

where \( \mathcal{C} = \ker \eta \) denotes the maximal complex subbundle of \( TM \).

By using (6.4) and Lemma 3.3 this equation gives

\[
\alpha^2 \phi S\phi Y = -2\alpha S^2 Y + \alpha^2 SY - 2\alpha Y - 12SY \quad (6.6)
\]

for all \( Y \in \mathcal{C} \).

On the other hand, in this section we have assumed that the normal vector field \( N \) of \( M \) is \( 2\xi \)-principal. So it follows that \( AX \in TM \) for all \( X \in TM \). From this, the anti-commuting property with respect to \( J \) and \( A \) implies \( \phi AX = -A\phi X \). Hence (6.5) can be expressed as

\[
\alpha(\phi S - S\phi)X = 6A\phi X. \quad (6.7)
\]
Putting $X = \phi Y$ into (6.7), it gives

\[ \alpha \phi S \phi Y = -\alpha SY - 6AY \]

for all $Y \in C$. Inserting this into (6.6), we get

\[ 3\alpha AY - \alpha S^2Y + \alpha^2 SY - \alpha Y - 6SY = 0. \]  \hspace{1cm} (6.8)

Applying the complex conjugate $A$ to (6.8) again and using the fourth equation in Lemma 3.3, we get

\[ 3\alpha Y - \alpha S^2Y + \alpha^2 SY - \alpha AY - 6SY = 0, \]  \hspace{1cm} (6.9)

for all $Y \in C$. Summing up (6.8) and (6.9), we have

\[ AY = Y \] for all $Y \in C$. This gives a contradiction. In fact, it is well known that the trace of the real structure $A$ is zero, that is, $\text{Tr}A = 0$ (see Lemma 1 in [20]). For an orthonormal basis $\{ e_1, e_2, \cdots, e_{2m-2}, e_{2m-1} = \xi, e_{2m} = N \}$ for $TQ^*m$, where $e_j \in C$ ($j = 1, 2, \cdots, 2m - 2$), the trace of $A$ is given by

\[ \text{Tr}A = \sum_{i=1}^{2m} g(Ae_i, e_i) \]
\[ = g(AN, N) + g(A\xi, \xi) + \sum_{i=1}^{2m-2} g(Ae_i, e_i) \]
\[ = 2m - 2. \]

It implies that $m = 1$. But we consider $m \geq 3$.

Consequently, this completes the proof that there does not exists a Hopf real hypersurface ($\alpha \neq 0$) in complex hyperbolic quadrics $Q^*m$, $m \geq 3$, with Reeb parallel structure Jacobi operator and $\phi$-principal normal vector field.

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References

1. Berndt, J., Suh, Y.J.: Real hypersurfaces with isometric Reeb flow in complex two-plane Grassmannians. Monatsh. Math. 137, 87–98 (2002)
2. Berndt, J., Suh, Y.J.: Real hypersurfaces with isometric Reeb flow in complex quadrics. Internat. J. Math. 24(1350050), 18 (2013)
3. Eberlein, P.B.: Geometry of Nonpositively Curved Manifolds. University of Chicago Press, Chicago (1996)
4. Helgason, S.: Differential geometry, Lie groups and symmetric spaces, Graduate Studies in Math. Amer. Math. Soc., p. 34 (2001)
5. Jeong, I., Lee, H., Suh, Y.J.: Real hypersurfaces in complex two-plane Grassmannians whose structure Jacobi operator is of Codazzi type. Acta Math. Hungar. 125, 141–160 (2009)
6. Jeong, I., Suh, Y.J., Woo, C.: Real hypersurfaces in complex two-plane Grassmannians with recurrent structure Jacobi operator. Real and Complex Submanifolds, Springer Proc. Math. Stat. Springer, Tokyo 106, 267–278 (2014)
7. Ki, U.-H., Pérez, J.D., Santos, F.G., Suh, Y.J.: Real hypersurfaces in complex space forms with $\xi$-parallel Ricci tensor and structure Jacobi operator. J. Korean Math. Soc. 44, 307–326 (2007)
8. Knapp, A.W.: Lie Groups Beyond an Introduction. Progress in Mathematics, 2nd edn. Birkhäuser, Boston (2002)
9. Kobayashi, S., Nomizu, K.: Foundations of Differential Geometry, vol. II. A Wiley-Interscience Publ., Wiley Classics Library Ed. (1996)
10. Klein, S., Suh, Y.J.: Contact real hypersurfaces in the complex hyperbolic quadric. Ann. Mat. Pura Appl. 198(4), 1481–1494 (2019)
11. Montiel, S., Romero, A.: CompleX Einstein hypersurfaces of indefinite complex space form. Math. Proc. Cambridge Philos. Soc. 94, 495–508 (1983)
12. Montiel, S., Romero, A.: On some real hypersurfaces of a complex hyperbolic space. Geom. Dedicata 20, 242–2615 (1986)
13. Okumura, M.: On some real hypersurfaces of a complex projective space. Trans. Amer. Math. Soc. 212, 355–364 (1975)
14. Pérez, J.D., Santos, F.G.: Real hypersurfaces in complex projective space with recurrent structure Jacobi operator. Differential Geom. Appl. 26, 218–223 (2008)
15. Pérez, J.D., Santos, F.G., Suh, Y.J.: Real hypersurfaces in complex projective space whose structure Jacobi operator is Lie $\xi$-parallel. Differential Geom. Appl. 22, 181–188 (2005)
16. Pérez, J.D., Jeong, I., Suh, Y.J.: Real hypersurfaces in complex two-plane Grassmannian with parallel structure Jacobi operator. Acta Math. Hungar. 122, 173–186 (2009)
17. Romero, A.: Some examples of indefinite complete complex Einstein hypersurfaces not locally symmetric. Proc. Amer. Math. Soc. 98, 283–286 (1986)
18. Romero, A.: On a certain class of complex Einstein hypersurfaces in indefinite complex space forms. Math. Z. 192, 627–635 (1986)
19. Reckziegel, H.: On the Geometry of the Complex Quadric, Geometry and Topology of Submanifolds. Lect. Notes in Math. VIII, pp. 302–315. World Sci. Publ., River Edge (1995)
20. Smyth, B.: Differential geometry of complex hypersurfaces. Ann. Math. 85, 246–266 (1967)
21. Smyth, B.: Homogeneous complex hypersurfaces. J. Math. Soc. Japan 20, 643–647 (1968)
22. Suh, Y.J.: Hypersurfaces with isometric Reeb flow in complex hyperbolic two-plane Grassmannians. Adv. Appl. Math. 50, 645–659 (2013)
23. Suh, Y.J.: Real hypersurfaces in the complex hyperbolic quadric with isometric Reeb flow. Commun. Contemp. Math. 20, 1750031 (2018). (20 pages)
24. Suh, Y.J.: Pseudo-anti commuting Ricci tensor for real hypersurfaces in the complex hyperbolic quadric. Sci. China Math. 62(4), 679–698 (2019)
25. Suh, Y.J.: Real hypersurfaces in the complex hyperbolic quadric and related topics. In: Proceedings of the 22nd International Workshop on Differential Geometry of Submanifolds in Symmetric Spaces and Related Problems, pp. 15–36, Daegu (2019)
26. Suh, Y.J., Hwang, D.H.: Real hypersurfaces in the complex hyperbolic quadric with Reeb parallel shape operator. Ann. Mat. Pura Appl. 196, 1307–1326 (2017)
27. Suh, Y.J., Pérez, J.D., Woo, C.: Real hypersurfaces in the complex hyperbolic quadric with parallel structure Jacobi operator. Publ. Math. Debrecen 94(1–2), 75–107 (2019)

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