ON IRREDUCIBLE SUPERSINGULAR REPRESENTATIONS OF
GL_2(F)

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ABSTRACT. Let $F$ be a non-archimedean local field of residual characteristic $p > 3$ and residue degree $f > 1$. We study a certain type of diagram, called cyclic diagrams, and use them to show that the universal supersingular modules of $GL_2(F)$ admit infinitely many non-isomorphic irreducible admissible quotients.

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INTRODUCTION

Let $F$ be a non-archimedean local field of residual characteristic $p$ and residue degree $f$. Fix a uniformizer $\varpi \in F$. The theory of smooth representations of reductive $F$-groups on $\mathbb{F}_p$-vector spaces has its origins in the paper [1] of Barthel and Livné in which they classify all smooth irreducible representations of $GL_2(F)$ with central characters except supersingular representations. The first examples of supersingular representations of $GL_2(F)$ were constructed by Paškūnas using equivariant coefficient systems on the Bruhat-Tits tree, or equivalently, using diagrams [7]. Let $K$, $Z$ and $N$ denote respectively the standard maximal compact subgroup, the center and the normalizer of the standard Iwahori subgroup $I$ of $GL_2(F)$ so that the stabilizer of the standard vertex of the tree is $KZ$ and that of the standard edge is $N$. A diagram is a finite data of a smooth $KZ$-representation $D_0$, a smooth $N$-representation $D_1$ and an $IZ$-equivariant map $D_1 \to D_0$. This data can be glued together (in a non-canonical way) to obtain smooth representations of $GL_2(F)$ inside some injective envelopes.

In [3], Breuil and Paškūnas develop the theory of diagrams further and construct irreducible supersingular representations of $GL_2(\mathbb{Q}_p^f)$ with prescribed $K$-socles from certain

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indecomposable (but not irreducible) diagrams. Here, \( \mathbb{Q}_p f \) is the degree \( f \) unramified extension of \( \mathbb{Q}_p \). Their results, in particular, imply that \( \text{GL}_2(\mathbb{Q}_p f) \) with \( f > 1 \) has infinitely many irreducible admissible supersingular representations on which \( p \) acts trivially, unlike \( \text{GL}_2(\mathbb{Q}_p) \) which has only finitely many such representations. Since the diagrams considered by them are not irreducible, the irreducibility of the corresponding representations of \( \text{GL}_2(\mathbb{Q}_p f) \) depends on certain computations with Witt vectors which do not extend to a ramified \( F \) or to an \( F \) of positive characteristic. In this note, we focus on irreducible diagrams in order to construct irreducible supersingular representations of \( \text{GL}_2(F) \) for all local fields \( F \).

The complexity of supersingular representations of \( \text{GL}_2(F) \) for \( f > 1 \) can already be seen in the complexity in classifying irreducible diagrams for \( f > 1 \). To this end, we consider a particular type of irreducible diagrams which are rigid enough. We call them cyclic diagrams. These are irreducible diagrams on direct sums of extensions of weights such that the action of \( \left( \begin{smallmatrix} 0 & 1 \\ \omega & 0 \end{smallmatrix} \right) \) permutes characters cyclically. We show that cyclic diagrams exist for all \( \text{GL}_2(F) \) and the \( D_0 \) of any cyclic diagram has more than 2 irreducible subquotients if \( f > 1 \) (see Theorem 1.6 and Remark 1.2). As a result, when \( f > 1 \), a family of cyclic diagrams parametrized by \( \mathbb{F}_p^* \) gives rise to infinitely many non-isomorphic irreducible admissible supersingular representations of \( \text{GL}_2(F) \) with trivial \( \omega \)-action (see Theorem 3.2). This implies that, for all local fields \( F \) with \( f > 1 \), the universal supersingular modules of \( \text{GL}_2(F) \) have infinitely many non-isomorphic irreducible admissible quotients (see Corollary 3.3). While Corollary 3.3 follows from the main results of [3] for \( F = \mathbb{Q}_p f \), it is a new result, to our knowledge, for \( F \) ramified over \( \mathbb{Q}_p \) and for \( F \) of positive characteristic.

We conclude by mentioning a recent note by Z. Wu in the similar spirit in which he gives a uniform proof of the fact that the universal supersingular modules of \( \text{GL}_2(F) \) are not admissible for any \( p \)-adic field \( F \neq \mathbb{Q}_p \) by showing that the supersingular representations are not of finite presentations [9].

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Notation and convention: Let \( p > 3 \) be a prime number. Let \( F \) be a non-archimedean local field of residual characteristic \( p \) and residue degree \( f \). Let \( \mathcal{O} \subseteq F \) be the valuation ring with a uniformizer \( \varpi \). Let \( \mathbb{F}_p \) be the algebraic closure of the finite field \( \mathbb{F}_{p^f} \) of size \( p^f \). Fix an embedding \( \mathbb{F}_{p^f} \hookrightarrow \mathbb{F}_p \). Let \( G = \text{GL}_2(F) \), \( K = \text{GL}_2(\mathcal{O}) \), \( \Gamma = \text{GL}_2(\mathbb{F}_p) \), and \( Z \) be the center of \( G \). Let \( B \) and \( U \) be the subgroups of \( \Gamma \) consisting of the upper triangular matrices and the upper triangular unipotent matrices respectively. Let \( I \) and \( I(1) \) be the preimages of \( B \) and \( U \) respectively under the reduction modulo \( \varpi \) map \( K \rightarrow \Gamma \). The subgroups \( I \) and \( I(1) \) of \( K \) are the Iwahori and the pro-\( p \) Iwahori subgroup of \( K \) respectively.
The normalizer $N$ of $I$ in $G$ is a subgroup generated by $I$ and $\Pi = (\frac{a}{0} \frac{1}{0})$. Note that $N$ is also the normalizer of $I(1)$ in $G$. Let $K(1)$ denote the kernel of the map $K \rightarrow \Gamma$, i.e., first principal congruence subgroup of $K$. Unless stated otherwise, all representations considered in this note are on $\mathbb{F}_p$-vector spaces.

A weight is an irreducible representation of $\Gamma$. Any weight is of the form of \[
\bigotimes_{j=0}^{f-1} \text{Sym}^r \mathbb{F}_p^2 \circ \Phi^j \otimes \det^m \] for some integers $0 \leq r_0, \ldots, r_{f-1} \leq p-1$ and $0 \leq m \leq p^f - 2$, where $\Phi : \Gamma \rightarrow \Gamma$ is the automorphism induced by the Frobenius map $\sigma \mapsto \sigma^p$ on $\mathbb{F}_p^f$ and $\det : \Gamma \rightarrow \mathbb{F}_p^\times$ is the determinant character. We denote such a weight by $r \otimes \det^m$ where $r$ is the $f$-tuple $(r_0, \ldots, r_{f-1})$ of integers. Let $\sigma = r \otimes \det^m$ be a weight; its subspace $\sigma^U$ of $U$-fixed vectors is 1-dimensional and stable under the action of $B$ because $B$ normalizes $U$. The resulting $B$-character, denoted by $\chi(\sigma)$, sends $(\frac{a}{b} \frac{c}{d}) \in B$ to $a^\sigma (ad)^m$ where $r = \sum_{j=0}^{f-1} r_j p^j$. Any $B$-character valued in $\mathbb{F}_p^\times$ factors through the quotient $B/U$ which is identified with the subgroup of diagonal matrices in $B$ by the section $B/U \rightarrow B$, $(\frac{a}{b} \frac{c}{d}) U \mapsto (\frac{a}{b} \frac{c}{d})$. For a $B$-character $\chi$, let $\chi^s$ be the inflation to $B$ of the conjugation-by-$s$ character $t \mapsto \chi(sts^{-1})$ on $B/U$ where $s = (\frac{1}{0} \frac{1}{1})$. We say that a weight is generic if it is not equal to $(0, 0, \ldots, 0) \otimes \det^m$ and $(p-1, p-1, \ldots, p-1) \otimes \det^m$ for any $m$. The map $\sigma \mapsto \chi(\sigma)$ gives a bijection from the set of generic weights to the set of $B$-characters $\chi$ such that $\chi \neq \chi^s$. If $\sigma$ is a generic weight, let us denote by $\sigma^{[s]}$ the generic weight corresponding to the character $\chi(\sigma)^s$. For $\sigma = r \otimes \det^m$, $\sigma^{[s]} = (p-1 - r_0, \ldots, p-1 - r_{f-1}) \otimes \det^{m+r}$. We refer the reader to [I §1] for all non-trivial assertions in this paragraph.

Given two weights $\sigma$ and $\tau$, let $E(\sigma, \tau)$ be the unique non-split $\Gamma$-extension $0 \rightarrow \sigma \rightarrow E(\sigma, \tau) \rightarrow \tau \rightarrow 0$ if it exists [3 Corollary 5.6]. We also denote $E(\sigma, \tau)$ by $\sigma \longrightarrow \tau$ . A finite-dimensional representation of $\Gamma$ is said to be multiplicity-free if its Jordan-Hölder factors are multiplicity-free. For any group $H$, the socle and the cosocle of an $H$-representation $\pi$ are denoted by $\text{soc}_H \pi$ and $\text{cosoc}_H \pi$ respectively.

Note that a weight is a smooth irreducible representation of $K$ (resp. of $KZ$) and a $B$-character is a smooth $I$-character (resp. $IZ$-character) via the map $K \rightarrow \Gamma$ (resp. $KZ \rightarrow \Gamma$). In fact, the weights exhaust all smooth irreducible representations of $K$ (resp. of $KZ$ such that $\varpi$ acts trivially).

1. Cyclic modules

We are interested in the following type of representations of $\Gamma$.

**Definition 1.1.** A finite-dimensional representation $D_0$ of $\Gamma$ is called a cyclic module of $\Gamma$ if there exists a finite set $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ of distinct generic weights such that $E(\sigma_i, \sigma_i^{[s]})$ exists for all $1 \leq i \leq n$, $D_0 = \bigoplus_{i=1}^n E(\sigma_i, \sigma_i^{[s]})$ and $D_0^U = \bigoplus_{i=1}^n E(\sigma_i, \sigma_i^{[s]})^U = \bigoplus_{i=1}^n \chi(\sigma_i) \oplus \chi(\sigma_i^{[s]})^s$ with the convention $\sigma_0 = \sigma_n$.

If $D_0 = \bigoplus_{i=1}^n E(\sigma_i, \sigma_i^{[s]})$ is a cyclic module of $\Gamma$, then, by Frobenius reciprocity, there is a non-zero map $\text{Ind}_B^\Gamma \chi(\sigma_i^{[s]}) \rightarrow E(\sigma_i, \sigma_i^{[s]})$ for all $1 \leq i \leq n$. Since the principal series representation $\text{Ind}_B^\Gamma \chi(\sigma_i^{[s]})^s$ has cosocle $\sigma_i^{[s]}$, and $\sigma_i \neq \sigma_i^{[s]}$, the map
Ind_{Γ}^{B}χ(σ_i^{-1}) \to E(σ_i,σ_i^{-1}) is surjective, and hence σ_i belongs to the first graded piece gr_{cosoc}^1(Ind_{Γ}^{B}χ(σ_i^{-1})) of the cosocle filtration of Ind_{Γ}^{B}χ(σ_i^{-1}) for all 1 ≤ i ≤ n.

Remark 1.2. If D_0 = \bigoplus_{i=1}^{n} E(σ_i,σ_i^{-1}) is a cyclic module of Γ with n = 1, i.e., D_0 = E(σ,σ^{-1}), then the surjective map Ind_{Γ}^{B}χ(σ) \to E(σ,σ^{-1}) is actually an isomorphism: if the kernel is non-zero, then it has socle σ because soc_{Γ} Ind_{Γ}^{B}χ(σ) = σ. But σ also occurs in the image as a subquotient which contradicts the fact that a principal series is multiplicity-free \[3\] Lemma 2.2. Therefore Ind_{Γ}^{B}χ(σ) \cong E(σ,σ^{-1}), and this forces Γ = GL_2(F_p) by \[3\] Theorem 2.4. In fact, any cyclic module of GL_2(F_p) is a principal series representation. Indeed, if Γ = GL_2(F_p) and E(σ,τ) is a non-split Γ-extension between generic weights σ and τ such that E(σ,τ)^0 = χ(σ) ⊕ χ(τ) then τ = σ^{-1} and thus E(σ,τ) = Ind_{Γ}^{B}χ(σ)^{-1} \[3\] Corollary 5.6 (i) and Proposition 4.13 or Corollary 4.10.

In order to construct cyclic modules of Γ = GL_2(F_p) for f > 1, we take a closer look at the weights appearing in the first graded pieces of cosocle filtrations of principal series. Let x be a formal variable and let Z \pm x := \{n \pm x : n \in Z\} denote the set of linear polynomials in x having integral coefficients with leading coefficient ±1. Let (Z \pm x)^f be the set of f-tuples of polynomials in Z \pm x. For λ = (λ_0(x), ..., λ_{f-1}(x)) ∈ (Z \pm x)^f and r ∈ Z^f, let λ(r) := (λ_0(r_0), λ_1(r_1), ..., λ_{f-1}(r_{f-1})) ∈ Z^f. Recall the polynomial e(λ) ∈ Z \bigoplus_{j=0}^{f-1} Zx_j associated to λ ∈ (Z \pm x)^f in \[3\] §2:

\[ e(λ)(x_0, ..., x_{f-1}) := \begin{cases} \frac{1}{2} \left( \sum_{j=0}^{f-1} p^j(x_j - λ_j(x_j)) \right) & \text{if } λ_{f-1}(x_{f-1}) \in \{x_{f-1}, x_{f-1} - 1\}, \\ \frac{1}{2} \left( p^f - 1 + \sum_{j=0}^{f-1} p^j(x_j - λ_j(x_j)) \right) & \text{otherwise.} \end{cases} \]

For each f > 1, let μ \in (Z \pm x)^f be the f-tuple of polynomials defined by

\[ \begin{align*} μ_0(x) & := x - 1, \\ μ_1(x) & := p - 2 - x, \\ μ_j(x) & := p - 1 - x \text{ for } 2 \leq j \leq f - 1. \end{align*} \tag{1.3} \]

Let g \in S_f be the cyclic permutation of an f-tuple mapping its j-th entry to (j + 1)-th entry and the last entry to the first one. If σ = λ(μ) is a generic weight of Γ = GL_2(F_p) for some determinant-power character η and f > 1, then gr_{cosoc}^1(Ind_{Γ}^{B}χ(σ)) consists of f number of weights which can be described by the set

\[ \{(g^iμ)(λ(r)) \otimes \det e^{[σ^iμ]λ(r)}η : 0 \leq i \leq f - 1\} \]

(see \[3\] Theorem 2.4).

For λ = (λ_0(x), ..., λ_{f-1}(x)) and λ' = (λ'_0(x), ..., λ'_{f-1}(x)) ∈ (Z \pm x)^f, let

\[ λ \circ λ' := (λ_0(λ'_0(x)), ..., λ_{f-1}(λ'_{f-1}(x))) ∈ (Z \pm x)^f. \]
Define an integer \( l \) to be equal to \( f \) (resp. \( 2f \)) if \( f \) is odd (resp. even). Let
\[
\mu^{(0)} := (x, x, \ldots, x) \quad \text{and} \quad \mu^{(k)} := g^{k-1} \mu \circ g^{k-2} \mu \circ \ldots \circ g \mu \circ \mu \quad \text{for all} \quad 1 \leq k \leq l.
\]
For \( r \in \mathbb{Z}^l \), let
\[
e_0(r) := 0 \quad \text{and} \quad e_k(r) := \sum_{j=0}^{k-1} e(g^j \mu)(\mu^{(j)}(r)) \in \mathbb{Z} \quad \text{for all} \quad 1 \leq k \leq l.
\]

**Lemma 1.4.**
(1) We have \( \mu^{(l)} = \mu^{(0)} = (x, x, \ldots, x) \) in \( (\mathbb{Z} \pm x)^l \).
(2) The \( f \)-tuples \( \mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(l)} \) are all distinct.
(3) The integer \( e_l(r) \) is independent of \( r \) and is 0 modulo \( p^f - 1 \).

**Proof.** (1) It follows from the definition of \( \mu^{(k)} \) that \( \mu^{(k)} = g^{k-1} \mu \circ \mu^{(k-1)} \) for all \( 1 \leq k \leq l \). Hence, for \( 1 \leq k \leq l \),
\[
\mu_j^{(k)}(x) = \begin{cases} 
\mu_j^{(k-1)}(x) - 1 & \text{if } j \equiv 1 \mod f, \\
p - 2 - \mu_j^{(k-1)}(x) & \text{if } j \equiv 2 \mod f, \\
p - 1 - \mu_j^{(k-1)}(x) & \text{otherwise}. 
\end{cases}
\]
(1.5)

It is now easy to check using (1.5) that for each \( j \), \( \mu_j^{(l)}(x) = x \).
(2) Let us assign to \( \mu^{(k)} \) an element \( m^{(k)} \in (\mathbb{Z}/2\mathbb{Z})^f \) by the rule that its \( j \)-th entry \( m_j^{(k)} \) is 0 if and only if the sign of \( \mu_j^{(k)}(x) \) is +. Here, \( (\mathbb{Z}/2\mathbb{Z})^f \) is the direct sum of \( f \) copies of the group \( \mathbb{Z}/2\mathbb{Z} \) of order 2 and has a natural action of \( \langle g \rangle \) by group automorphisms. We show that the elements \( m^{(1)}, m^{(2)}, \ldots, m^{(l)} \) are all distinct in \( (\mathbb{Z}/2\mathbb{Z})^f \) which then implies part (2). We have \( m^{(1)} = (0, 1, 1, \ldots, 1) \) and \( m^{(k)} = g^{k-1} m^{(1)} + m^{(k-1)} \) for \( k > 1 \) because \( \mu^{(k)} = g^{k-1} \mu^{(1)} \circ \mu^{(k-1)} \). Suppose \( m^{(k_1)} = m^{(k_2)} \) for some \( 1 \leq k_1 < k_2 \leq l \). Then
\[
m^{(k_1)} = m^{(k_2)} = g^{k_2-k_1-1} m^{(1)} + g^{k_2-k_1-2} m^{(1)} + \ldots + g^{k_1} m^{(1)} + m^{(k_1)}.
\]
This gives that
\[
g^{k_1+(k_2-k_1)-1} m^{(1)} + g^{k_1+(k_2-k_1)-2} m^{(1)} + \ldots + g^{k_1} m^{(1)} = (0, 0, 0, \ldots, 0).
\]

The action of \( g^{-k_1} \) on both sides then gives
\[
g^{k_2-k_1-1} m^{(1)} + g^{k_2-k_1-2} m^{(1)} + \ldots + m^{(1)} = (0, 0, 0, \ldots, 0), \quad \text{i.e.,} \quad m^{(k_2-k_1)} = (0, 0, 0, \ldots, 0).
\]
This is a contradiction because \( k_2 - k_1 < l \) and for any \( l' < l \), \( m^{(l')} \neq (0, 0, 0, \ldots, 0) \). The latter fact can be easily checked by looking at \( m_0^{(l')} \) and \( m_1^{(l')} \). One has \( m_0^{(l')} \neq m_1^{(l')} \) for \( l' < l \) except when \( l = 2f \) and \( l' = f \) in which case \( m_0^{(l')} = m_1^{(l')} = 1 \).
(3) Let us first consider \( f \) to be odd (so \( l = f \)). Expanding the expression for \( e_l(r) \) and rearranging the terms, one gets
\[
e_l(r) = c + \sum_{k=1}^{f-1} \mu_0^{(k)}(r_0) + p \sum_{k=0}^{f-2} \mu_1^{(k)}(r_1) + p^2 \sum_{k=-1}^{f-3} \mu_2^{(k)}(r_2) + \cdots + p^{f-1} \sum_{k=-(f-2)}^{0} \mu_{f-1}^{(k)}(r_{f-1}),
\]
where $c$ is the constant term of the polynomial $e(g^{f-1} \mu) + e(g^{f-2} \mu) + \cdots + e(g \mu) + e(\mu)$, and $k = -n$ for positive $n$ means $k = f - n$ in the summation $\sum_k$. Using (1.5), one checks that each summand $\sum_k h_j^{(k)}(r_j)$ above (with appropriate lower and upper limit) is independent of $r_j$ and equals $\frac{f-1}{p-1} (p-1) - 1$. We leave it to the reader to check that $c \equiv \frac{f-1}{p-1} \mod p^f - 1$. Therefore, $e_l(r) = \frac{f-1}{p-1} (\frac{f-1}{p-1} (p-1)) \equiv 0 \mod p^f - 1$. The proof for even $f$ is similar. In this case, one gets $c \equiv 2 \left( \frac{f-1}{p-1} \right) \mod p^f - 1$, and $\sum_k \mu_j^{(k)}(r_j) = \frac{f-1}{2} (p-1) - 1$ for all $j$. Thus, $e_l(r)$ is again $0$ modulo $p^f - 1$.

**Theorem 1.6.** The group $\Gamma$ admits a multiplicity-free cyclic module $D_0$.

*Proof.* The case $f = 1$ is treated in Remark 1.2. Let $f > 1$. The proof is constructive. Start with a weight $\sigma_0 := r \otimes \eta$ of $\Gamma$ for some $1 \leq r_0, \ldots, r_{f-1} \leq p - 3$ and for some determinant-power character $\eta$. Observe that $\sigma_0 := \mu^{(0)}(r) \otimes \det^{e_0(r)}(r) \eta$. Let

$$\sigma_k := \mu^{(k)}(r) \otimes \det^{e_k(r)}(r) \eta$$

for all $1 \leq k \leq l$. We claim that the set $\{\sigma_1, \sigma_2, \ldots, \sigma_l\}$ is the required set to construct a cyclic module. Using (1.5), one checks that $\mu_j^{(k)}(x) \in \{x, x-1, x+1, p-2-x, p-3-x, p-1-x\}$ for all $1 \leq k \leq l$ and $0 \leq j \leq f - 1$. Since $p > 3$, this means that the weights $\sigma_1, \sigma_2, \ldots, \sigma_l$ are well-defined. Further, by Lemma 1.4 and its proof, one sees that the weights $\sigma_1, \sigma_2, \ldots, \sigma_l$ are all distinct generic weights and $\sigma_l = \sigma_0$. Now let $1 \leq k \leq l$. We know that the weights appearing in $\det^{e_k(r)}(r)$ are

$$\{(g^j \mu)(\mu^{(k-1)}(r)) \otimes \det^{e_k(r)}(\mu^{(k-1)}(r)) : 0 \leq i \leq f - 1\}.$$ 

In particular, $\det^{e_k(r)}(r)$ contains the weight

$$(g^{k-1} \mu)(\mu^{(k-1)}(r)) \otimes \det^{e_k(r)}(\mu^{(k-1)}(r)) \eta = \mu^{(k)}(r) \otimes \det^{e_k(r)}(r) \eta = \sigma_k.$$ 

Since $\det^{e_k(r)}(r)$ contains the unique quotient of $\mathcal{D}_B \chi(\sigma_{k-1})^s$ with socle $\sigma_k$. Since $\det^{e_k(r)}(r)$ contains the unique quotient of $\mathcal{D}_B \chi(\sigma_{k-1})^s$ with socle $\sigma_k$. Since $\mathcal{D}_B \chi(\sigma_{k-1})^s$ contains the unique quotient of $\mathcal{D}_B \chi(\sigma_{k-1})^s$ with socle $\sigma_k$. Since $\det^{e_k(r)}(r)$ contains the unique quotient of $\mathcal{D}_B \chi(\sigma_{k-1})^s$ with socle $\sigma_k$. Since $\det^{e_k(r)}(r)$ contains the unique quotient of $\mathcal{D}_B \chi(\sigma_{k-1})^s$ with socle $\sigma_k$. Since $\det^{e_k(r)}(r)$ contains the unique quotient of $\mathcal{D}_B \chi(\sigma_{k-1})^s$ with socle $\sigma_k$. Since $\det^{e_k(r)}(r)$ contains the unique quotient of $\mathcal{D}_B \chi(\sigma_{k-1})^s$ with socle $\sigma_k$.

It remains to show that $D_0$ is multiplicity-free. By definition, $\det^{e_k(r)}(r)$ is multiplicity-free. Thus also $\sigma_k \neq \sigma_k$ for any $k_1 \neq k_2$, $1 \leq k_1, k_2 \leq l$, because $(\sigma_k)[s] = \sigma$. Now, if $\sigma_k = \sigma_k[s]$, for some $1 \leq k_1, k_2 \leq l$, then there is a non-split $\Gamma$-extension between $\sigma_{k_1}$ and $\sigma_{k_2}$. Consider the elements $m^{(k_1)}$ and $m^{(k_2)}$ of $(\mathbb{Z}/2\mathbb{Z})^f$ assigned to $\mu^{(k_1)}$ and $\mu^{(k_2)}$ respectively in the proof of Lemma 1.4. By [3, Lemma 5.6(ii)], the number of $1$’s in $m^{(k_1)}$ and $m^{(k_2)}$ have different parity. However, if $f$ is odd, then one checks that the number of $1$’s in $m^{(k)}$ is always even for all $1 \leq k \leq l$ implying that $\sigma_k \neq \sigma_k[s]$, for any $1 \leq k_1, k_2 \leq l$. If $f$ is even, then it is not true that the number of $1$’s in $m^{(k)}$ is always even or odd, and it is a priori possible that $\sigma_k = \sigma_k[s]$, because $m^{(k)} + m^{(k+f)} = (1, 1, \ldots, 1)$. However, using (1.5), one explicitly checks that $\sigma_k \neq \sigma_k[s]$ for any $1 \leq k \leq l$. \qed
Remark 1.7. When $f$ is odd, the argument given in the proof Theorem of [10] shows that any cyclic module of $\Gamma$ is multiplicity-free. This is not true when $f$ is even (see the next remark). We further point out that the definition of the $f$-tuple $\mu$ is not canonical. Any other cyclic permutation of $\mu$ also gives rise to a cyclic module of $\Gamma$ by the same construction as above. We expect that all multiplicity-free cyclic modules of $\Gamma$ are obtained in this way, and thus any multiplicity-free cyclic module of $\Gamma$ has socle of length $l$.

Example 1.8. The construction in the proof of Theorem [10] produces following multiplicity-free cyclic modules for $f = 2, 3$. The weights are written without their twists by determinant-power characters.

$f = 2$: $D_0 = (r_0 - 1, p - 2 - r_1) \oplus (p - 1 - r_0, p - 1 - r_1)$

$p - 1 - r_0, p - 3 - r_1 \oplus (p - r_0, r_1 + 1)$

$p - 2 - r_0, r_1 + 1 \oplus (r_0, r_1 + 2)$

$(r_0, r_1) \oplus (r_0 + 1, p - 2 - r_1)$.

$f = 3$: $D_0 = (r_0 - 1, p - 2 - r_1, p - 1 - r_2) \oplus (p - 1 - r_0, p - 1 - r_1, p - 1 - r_2)$

$(p - 1 - r_0, r_1 + 1, p - 2 - r_2) \oplus (p - r_0, r_1 + 1, r_2)$

$(r_0, r_1, r_2) \oplus (r_0, p - 2 - r_1, r_2 + 1)$.

Remark 1.9. Let $\mathbb{Q}_p$ denote the degree $f$ unramified extension of $\mathbb{Q}_p$. The multiplicity-free cyclic module of $GL_2(\mathbb{F}_{p^2})$ (resp. of $GL_2(\mathbb{F}_{p^3})$) in Example 1.8 occurs as a submodule of $D_0(\rho)$ of a Diamond diagram associated to an irreducible (resp. reducible split) generic Galois representation $\rho$ of $\mathbb{Q}_p$ (resp. of $\mathbb{Q}_p$) (see [3 §14]).

In [5], M. Schein constructs irreducible supersingular representations of $G = GL_2(F)$ with $K$-socles compatible with Serre’s weight conjecture for a ramified $p$-adic field $F$ of residue degree 2. His construction is based on constructing cyclic modules of $GL_2(\mathbb{F}_{p^2})$ with prescribed socles. The involved cyclic modules of $GL_2(\mathbb{F}_{p^3})$ have socles of lengths $> l$ and are not multiplicity-free (see [5] Example 3.9).

2. Cyclic diagrams

Recall from [3 §9] that a diagram (of $G$) is a data $(D_0, D_1, r)$ consisting of a smooth $KZ$-representation $D_0$, a smooth $N$-representation $D_1$ and an $IZ$-equivariant map $r : D_1 \to D_0$. A diagram $(D_0, D_1, r)$ is called a basic 0-diagram if $\varpi$ acts trivially on $D_0$ and $D_1$, and the map $r$ induces an isomorphism $D_1 \cong D_0^{(1)}$ of $IZ$-representations. Now, let $D_0 = \bigoplus_{i=1}^n E(\sigma_i, \sigma_i^{[i]})$ be a multiplicity-free cyclic module of $\Gamma$. Viewing $D_0$ as a smooth $KZ$-representation via $KZ \to \Gamma$ with trivial $\varpi$-action, $D_1 := D_0^{(1)} = D_0^U$ can be equipped with a smooth $N$-action by defining the action of $\Pi : \chi(\sigma_i) \to \chi(\sigma_i)^*$ to be the multiplication by a scalar $t_i \in \mathbb{F}_p^\times$ for all $i$ after choosing bases. This defines a unique $N$-action on $D_1$ such that $\varpi$-acts trivially and gives a basic 0-diagram $(D_0, D_1, \text{can})$ where can : $D_1 \to D_0$ is the canonical inclusion.
Proof.  (1) Let \( D \subseteq D_0 \) be a non-zero \( KZ \)-subrepresentation such that \( V^{I(t)} \) is stable under the action of \( N \). Then, for some \( 1 \leq i \leq n \), \( V \) contains \( \sigma_i \) and thus also contains \( \chi(\sigma_i) \). Since \( \Pi(\chi(\sigma_i)) = \chi(\sigma_i)^s \), \( V \) contains \( \chi(\sigma_i)^s \). By Frobenius reciprocity, there is a non-zero map \( \text{Ind}_E^D(\chi(\sigma_i)^s) \to V \) whose image is \( E(\sigma_i^{s+1}, \sigma_i^{[s]}) \). Thus \( E(\sigma_i^{s+1}, \sigma_i^{[s]}) \subseteq V \).

Continuing in this way, we get that \( D_0 = \bigoplus_{i=1}^n E(\sigma_i^{[s]} \sigma_i^{[s-1]} \cdots \sigma_i^1) \). Hence \( V = D_0 \).

(2) Let \( D = (D_0, D_1, \text{can}) \) and let \( D' \) be a diagram isomorphic to \( D \). Then \( D' \) is also a cyclic diagram on the cyclic module \( D_0 \). Let \( \Pi : \chi(\sigma_i) \to \chi(\sigma_i)^s \) in \( D' \) be given by the multiplication by scalar \( t_i' \in \mathbb{F}_p^\times \) for all \( 1 \leq i \leq n \). As the diagrams \( D \) and \( D' \) are isomorphic, there is an isomorphism \( \varphi : D_0 \to D_0 \) of \( KZ \)-representations such that \( \varphi(\Pi v) = \Pi(\varphi(v)) \) for all \( v \in D_1 \). Since \( D_0 \) is multiplicity-free, an easy application of Schur’s lemma gives

\[
\text{End}_{KZ}(D_0) = \text{End}_G(D_0) \cong \text{End}_G(E(\sigma_1, \sigma_0^{[s]})) \times \cdots \times \text{End}_G(E(\sigma_n, \sigma_{n-1}^{[s]})) \cong \mathbb{F}_p^n.
\]

So, if the isomorphism \( \varphi \) corresponds to \( (a_1, \ldots, a_n) \in \left( \mathbb{F}_p^\times \right)^n \), then \( (a_1, \ldots, a_n) \) satisfies \( a_i = a_{i-1}t_{i-1}^{-1}(t_{i-1})^{-1} \) for all \( 1 \leq i \leq n \). This implies that \( t_1t_2 \cdots t_n = t_1' \cdots t_n' \). On the other hand, if \( t_1t_2 \cdots t_n = t_1' \cdots t_n' \), then the scalar multiplications by \( a_i = \prod_{j=1}^{i-1} t_j' t_j^{-1} \) on \( E(\sigma_i, \sigma_{i-1}^{[s]} \cdots \sigma_1^1) \) with \( a_1 = 1 \) give an isomorphism of cyclic diagrams on \( D_0 \). See also [H Proposition 4.4].

For a cyclic diagram \( D = (D_0, D_1, \text{can}) \), we introduce the notation \( t(D) = t_1t_2 \cdots t_n \) for later use. With this notation, Lemma 2.2 (2) says that the map \( D \mapsto t(D) \) gives a bijection between the set of isomorphism classes of cyclic diagrams on \( D_0 \) and \( \mathbb{F}_p^\times \).

3. Supersingular representations

We now use cyclic diagrams to show that \( G = \text{GL}_2(F) \) admits infinitely many smooth admissible irreducible supersingular representations when \( F \) has residue degree \( f > 1 \). It uses the following key theorem of Breuil and Paškūnas.

**Theorem 3.1.** Let \( (D_0, D_1, r) \) be a basic 0-diagram such that \( D_0^{K(1)} \) is finite-dimensional. Then there exists a smooth admissible representation \( \pi \) of \( G \) on which \( \varpi \) acts trivially, and such that

\[
\sum_{i=1}^{d} \chi(\sigma_i) \in \mathbb{F}_p^\times.
\]
(1) one has the inclusion \((D_0, D_1, r) \subseteq (\pi|_{KZ}, \pi|_{N}, \text{id})\) of diagrams,
(2) \(\pi\) is generated as a \(G\)-representation by \(D_0\), and
(3) \(\text{soc}_T D_0 = \text{soc}_K D_0 = \text{soc}_K \pi\).

Moreover, if \((D_0, D_1, r)\) is irreducible, then any such \(G\)-representation \(\pi\) is irreducible.

Proof. The first part is essentially proved in \([3\text{ Theorem }9.8]\). See also the proof of \([3\text{ Theorem }19.8 \text{ (i)}]\). The proof of the irreducibility of \(\pi\) is given in unpublished lecture notes of Breuil \([2\text{ Proposition }5.11]\). We reproduce it here: let \(\pi' \subseteq \pi\) be a nonzero \(G\)-subrepresentation. Then \(\pi' \cap D_0\) is a non-zero \(KZ\)-subrepresentation of \(D_0\) by (3), and
\[
(\pi' \cap D_0)^{f(1)} = \pi' \cap D_1
\]
is stable under the action of \(\Pi\). Hence \((\pi' \cap D_0, (\pi' \cap D_0)^{f(1)}, \text{can})\) is a non-zero subdiagram of \((D_0, D_1, r)\). By irreducibility of \((D_0, D_1, r)\), we get \(\pi' \cap D_0 = D_0\). Hence, \(\pi' = \pi\) using (2).
\(\square\)

When \(F\) has residue degree 1, the cyclic diagrams are the basic 0-diagrams on principal series representations of \(GL_2(F)\) (Remark 1.6) and thus Theorem 3.1 applied to cyclic diagrams gives rise to irreducible (ramified) principal series representations of \(G\) as we shall see now. Recall from \([1]\) that a smooth irreducible representation \(\pi\) of \(G\) with central character is a quotient of \(\pi(\sigma, \lambda, \chi) := \text{ind}^G_{KZ} \sigma \otimes (\chi \circ \det)\) for some weight \(\sigma\), some \(\lambda \in \mathbb{F}_p^\times\) and some smooth character \(\chi : F^\times \rightarrow \mathbb{F}_p^\times\). Here, \(\text{ind}^G_{KZ} \sigma\) is the compactly induced representation with \(\varpi\) acting trivially on \(\sigma\), and \(T \in \text{End}_G(\text{ind}^G_{KZ} \sigma)\) is the distinguished Hecke operator. By definition, \(\pi\) is supersingular if it is a quotient of some \(\pi(\sigma, 0, \chi)\). The representations \(\pi(\sigma, 0, \chi)\) are called universal supersingular modules.

**Theorem 3.2.** Let \(F\) be a non-archimedean local field of residue degree \(f > 1\). Then the group \(G\) admits infinitely many non-isomorphic smooth admissible irreducible supersingular representations on which \(\varpi\) acts trivially. Further, all these representations have the same \(K\)-socle.

**Proof.** We use the existence of multiplicity-free cyclic modules from Theorem 1.6 to construct a family of cyclic diagrams of \(G\). Let \(D_0\) be a multiplicity-free cyclic module constructed in Theorem 1.6 and for each \(t \in \mathbb{F}_p^\times\), let \(D(t) = (D_0, D_1, \text{can})\) be a cyclic diagram on \(D_0\) such that \(t(D(t)) = t\). By Theorem 3.1 there is a smooth admissible representation \(\pi(t)\) (fix one for each \(D(t)\)) of \(G\) with trivial action of \(\varpi\) such that \(D(t) \subseteq (\pi(t)|_{KZ}, \pi(t)|_{N}, \text{can})\), \(D_0\) generates \(\pi(t)\) as a \(G\)-representation, and \(\text{soc}_K D_0 = \text{soc}_K \pi(t)\). We claim that \(\{\pi(t)\}_{t \in \mathbb{F}_p^\times}\) is the desired family of representations of \(G\). By Lemma 2.2 (1) and Theorem 3.1 each \(\pi(t)\) is an irreducible \(G\)-representation.

Suppose there is an isomorphism \(\varphi : \pi(t) \cong \pi(t')\) of \(G\)-representations for \(t \neq t'\). It restricts to an isomorphism \(\varphi : D_0 \cong D_0\) of \(KZ\)-representations because \(\text{soc}_K D_0 = \text{soc}_K \pi(t) = \text{soc}_K \pi(t')\) and because \(D_0\) is multiplicity-free. This gives rise to an isomorphism \(D(t) \cong D(t')\) of cyclic diagrams which contradicts Lemma 2.2 (2). Thus \(\pi(t)\) and \(\pi(t')\) are not isomorphic for \(t \neq t'\).
It remains to show that each $\pi(t)$ is supersingular. Let $\sigma \in \text{soc}_K \pi(t)$. Then $\text{Hom}_G(\text{ind}_K^G \sigma, \pi(t)) = \text{Hom}_K(\sigma, \pi(t)^{K(1)})$ is a non-zero finite-dimensional $\mathbb{F}_p$-vector space because $\pi(t)$ is admissible. Hence $\text{Hom}_G(\text{ind}_K^G \sigma, \pi(t))$ contains a non-zero eigenvector for the action of Hecke operator $T$ with eigenvalue, let’s say, $\lambda$. As $\pi(t)$ is irreducible, it follows that $\pi(t)$ is a quotient of $\pi(\sigma, \lambda, 1)$. If $\lambda \neq 0$, then by [1, Lemma 28 and Theorem 33] we have $\dim_{\mathbb{F}_p} \pi(t)^{I(1)} \leq 2$. However, as $f > 1$, $\text{soc}_K D_0$ is not irreducible and thus $\dim_{\mathbb{F}_p} D_0^{I(1)} = \dim_{\mathbb{F}_p} D_1 \geq 4$ (in fact, $\dim_{\mathbb{F}_p} D_1 = 2l$). But this implies that $\dim_{\mathbb{F}_p} \pi(t)^{I(1)} > 2$ because $\pi(t)$ contains $D_0$. So we get a contradiction. Therefore $\lambda = 0$ and $\pi(t)$ is supersingular. 

Recall from [1, Corollary 31] that $\pi(\sigma, \lambda, \chi)$ has a unique (admissible) irreducible quotient for $\lambda \neq 0$. However for $\lambda = 0$, we have the following result as an immediate corollary of Theorem 3.2:

**Corollary 3.3.** Let $F$ be a non-archimedean local field of residue degree $f > 1$. Then the universal supersingular module $\pi(\sigma, 0, \chi)$ of $G$ has infinitely many non-isomorphic admissible irreducible quotients for any given weight $\sigma = r \otimes \eta$ with $1 \leq r_0, \ldots, r_{f-1} \leq p-3$ and any smooth character $\chi$.

**Proof.** As in the proof of Theorem 3.2, consider a family $\{D(t)\}_{t \in \mathbb{F}_p^*}$ of cyclic diagrams on a cyclic module $D_0$ from Theorem 1.6 whose socle contains the given weight $\sigma$, and let $\{\pi(t)\}_{t \in \mathbb{F}_p^*}$ be a corresponding family of smooth admissible irreducible supersingular $G$-representations. By the proof of Theorem 3.2, each $\pi(t)$ occurs as a quotient of $\pi(\sigma, 0, 1)$. So the corollary holds for $\pi(\sigma, 0, 1)$ and hence also for its smooth twist $\pi(\sigma, 0, \chi)$. 

**Remark 3.4.** If $F = \mathbb{Q}_p$ with $f > 1$, then the recent works of Le [6] and Ghate-Sheth [5] show that the universal supersingular modules of $G$ also admit infinitely many non-isomorphic non-admissible irreducible quotients.

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