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Pressure and Motion of Dry Sand – Translation of Hagen’s Paper from 1852

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Abstract In a remarkable paper from 1852, Gotthilf Heinrich Ludwig Hagen measured and explained two fundamental aspects of granular matter: The first effect is the saturation of pressure with depth in a static granular system confined by silo walls – generally known as the Janssen effect. The second part of his paper describes the dynamics observed during the flow out of the container – today often called the Beverloo law – and forms the foundation of the hourglass theory. The following is a translation of the original German paper from 1852.

1 Introduction

Gotthilf Heinrich Ludwig Hagen is most renowned for his contributions to the study on laminar flow in pipes; his measurements published in 1839 studied what is now well-known as the (Hagen-)Poiseuille law [1, 2]. Less well-known is Hagen’s work on granular systems. While Janssen, with his 1895 paper, typically receives credit for the saturation effect in granular silos [3, 4], it was Hagen in his paper Über den Druck und die Bewegung des trocknen Sandes [5] who measured this effect earlier – but also not for the first time, cf. [6] – and offered a first model that provided a qualitative understanding of the effect. Hagen proposes a quadratic law (with some cutoff) for the pressure instead of the exponential form put forward by Janssen more than 40 years later [7].

In addition to the discussion of a static pile of sand, Hagen examines in considerable detail the flow through an opening of the container, and discovers that the rate of discharge is proportional to the diameter of the opening raised to the power 5/2. This result is elegantly derived from dimensional analysis [7, ch 10.2]; but it fits the data best if instead of the real diameter some effective diameter is used – the resulting law is known as the Beverloo correlation [8]. Hagen finds an effective opening diameter that is smaller by twice the particle diameter, which is consistent with more recent measurements.

Hagen’s work lays out the basis of the so-called hourglass theory where the flow of granular material is found to be independent of the filling height in the container, thus allowing the measure of time with an hourglass [2, sec 10.4]. Later work both confirms and extends Hagen’s early analysis [9]. More recently, a lot of work has been devoted to understanding the fundamental differences between fluid and granular flows [10]. While on the level of the individual grains the probability of arching at the opening is a matter of current investigations [11], Hagen provides a successful route to a continuum description of the flow by the rescaling of the diameter of the opening.

2 Gotthilf Heinrich Ludwig Hagen, Pressure and Motion of Dry Sand, 19th January 1852

Consider a container with a horizontal bottom that includes a circular opening of radius $r$. In this opening is placed a disk which is easily movable but seals tightly; on top of it is an extended filling of sand up to a height $h$. As a result, there is a pressure exerted on the disk created by the weight of the cylinder of sand above it less the friction which is experienced by that cylinder from the sand surrounding it. The friction is proportional to the horizontal pressure, or the square of the height. Let $l$ denote a friction dependent constant and $\gamma$ be the weight of a unit volume of sand, then the pressure against the disk equals

$$r^2\gamma h - 2r\gamma lh^2.$$

For a growing $h$, this expression will increase in the beginning, reach a maximum, and subsequently decrease afterwards; it will become not only zero but even negative. However, the sand cylinder is not rigidly connected throughout, and therefore the axial pressure, which its lower part exerts on the bottom disk, cannot be compensated by the strong friction acting on the cylinder as a whole. Hence, the pressure on the disk in fact remains unchanged for fillings higher...
than that height at which the pressure on the disk reaches its maximum value.

For the case of pressures below the maximum, we seek to represent that pressure by the weight of a free-standing body of sand on the disk. The body is bounded by the surface of the filling. It is a conoid that is formed by the rotation of a parabola around its axis. Here the parameter of the parabola is $4rl$, while the height of the paraboloid is $r/(4l)$. The latter body joins the circumference of the opening.

To compare these results with the real phenomenon, I created openings of radii of 0.3791 inch$^1$ and 0.7271 inch, respectively, in two brass plates used as the the bottom of the sand-filled container; these openings were closed with suitable disks that were supported from underneath by hooks that were connected to one arm of a balance, while the other arm carried the counterweight. To reduce the counterweight to the pressure on the disk slowly and without concussion, sand was allowed to discharge through a small hole in the bottom of the plate. The constant error of this method could be found easily by measurement of the excess weight of the disk and the hook compared to the plate both by the discharge of sand and by direct weighing.

While repeating measurements of the pressure due to the sand against the disks multiple times, considerable deviations among the measurements were observed; this was apparently due to different ways of settling. When the settling was as loose as possible, the weight of a cubic inch of sand, a crude ferrous grit, was 2.9 Loth$^2$. However, the weight increased to 3 Loth when there was modest agitation during filling and rose to 3.25 Loth as soon as a fairly compact settling was generated by severe agitation or by pushing a wire into the container. In so doing, the friction increased by even more than the specific weight. Hence, the pressure against the disk became remarkably smaller for the more compact settling.

With the larger disk the pressure maximum was reached at a filling height of around 1 inch: for larger height the pressure decreased somewhat, since despite great care the sand settled in a slightly denser state. For the loosest fillings I found $l = 0.154$ to 0.175. In contrast, the deviations were less when the sand was dropped from a height of several inches in a narrow stream, and when the sand flow was cone-shaped: the value for $l$ was limited to between 0.21 and 0.22.

The influence of different settling states was also clearly noticeable as the sand discharged through the openings of various radii. When the settled filling was somewhat denser, there was less sand flowing within a second. As the discharge duration increased, the sand became agitated, particularly in the vicinity of the opening, resulting in an increased sensitivity, which in turn led to a diminished flow. Incidentally, the height of the filling had no influence, as was already recognized by Huber-Burnand some time ago.

$^3$ To reduce the mentioned irregularities as much as possible, I limited the duration of each observation to 30 to 200 seconds and, additionally, tried to make the fillings rather uniform. To this end, the container was placed in a metal-sieve cylinder with a sieve-like bottom, filled with sand, and lifted slowly thereafter; in this way the sand poured into the container in several hundred thin streams, each from a very small height.

The openings at the bottom of the container, with their sharp edges always on the upper face, had the following radii:

| Opening | Radius (inches) |
|---------|----------------|
| 1       | 0.1677         |
| 2       | 0.1203         |
| 3       | 0.0986         |
| 4       | 0.0807         |
| 5       | 0.0549         |
| 6       | 0.0377         |

The amount of outflowing sand per second, averaged over six observations each, was:

- For the opening 1: 1.8995 Loth
- For the opening 2: 0.7596 Loth
- For the opening 3: 0.4330 Loth
- For the opening 4: 0.2481 Loth
- For the opening 5: 0.08242 Loth
- For the opening 6: 0.02676 Loth

Comparing these weights with the radii of the openings, the former seem to be approximately proportional to the third power of the latter. An attempt to represent them in the form

$$m = k r^3,$$

however, did not produce a satisfactory result; the remaining errors were rather significant and very regular, so they could not be interpreted as observational errors. In contrast, if the radius of the opening was reduced by some length, a nearly constant ratio was observed between the mass of sand and the $2.5^3$ power of the reduced radius. The reduction of the radius is justified by noting that the granules that touch the edge of the opening while falling lose their speed partially or completely and even disturb the motion of the neighboring granules when bouncing off. From repeated measurements it was found that, on average, 9 granules of sand constitute the length of a Rhineland line$^4$ hence the diameter of a single one equals 0.0093 inches.

Thus I compared the masses of the sand with the expression

$$m = k (r - x)^{\frac{3}{2}}$$

and found after introducing approximate values for $x$ according to the method of least squares from all six observations

$$k = 189.07$$

$$x = 0.00968.$$  

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1 trans. note: The German *Zoll* is translated as *inch*; however, it might deviate slightly from the value generally used today; most likely 1 Zoll = 26.15mm
2 trans. note: The German *Loth* (or *Lot*) is a unit of mass, 1 Loth = 1/32 Handelspfund (Berlin) ≈ 14.6g
3 trans. note: The reference was added for the translation. There are no references given in the original.
4 trans. note: The German Rheinhälsische Linie is a unit of length, typically 1/12 of a Zoll, hence 1 line = 2.18mm.
Using these values to calculate \( m \), the remaining errors are

\[
\begin{align*}
\text{for } & 1 & -0.0181 \\
\text{for } & 2 & +0.0094 \\
\text{for } & 3 & +0.0135 \\
\text{for } & 4 & +0.0075 \\
\text{for } & 5 & -0.00003 \\
\text{for } & 6 & -0.00142 \\
\end{align*}
\]

The first observation agrees the least with the fitted values, but is also in itself less accurate than the other observations for two reasons: for one part, its duration was the shortest, and for the other part, the sand was poured out vehemently and became heavily agitated, and the container was almost completely emptied. Therefore, I tried to determine the values of the two constants independently of the first observation, using only the last five. This gives

\[
m = 181.57(r - 0.00893)\,^\frac{4}{3}.
\]

The deviations from the observed weights were thereupon

\[
\begin{align*}
\text{for } & 1 & -0.0712 \\
\text{for } & 2 & -0.0085 \\
\text{for } & 3 & +0.0050 \\
\text{for } & 4 & +0.0039 \\
\text{for } & 5 & +0.00002 \\
\text{for } & 6 & -0.00078 \\
\end{align*}
\]

The radius reduction \( x \) for the discharge opening is found to be close to the diameter of a grain of sand. From the constant \( k \) can be determined the average distance from the opening at which the sand begins its free fall. Assuming that the sand forms a compact mass until reaching the opening, the amount of sand discharged per second is

\[
m = 2\rho^2\pi f \sqrt{\gamma h}
\]

where \( h \) designates the mentioned height of fall and \( \rho \) the effective opening. On the other hand, the observations yield

\[
m = 181.57 \cdot \rho^\frac{4}{3}.
\]

Equating both expressions and setting \( \gamma \) to 2.93 as obtained from the average loose packings, one finds

\[
h = 0.5185 \cdot \rho.
\]

If, in contrast, one assumes that for each unit of time a layer of sand of the same vertical height separates from the whole inner area of the paraboloid mentioned above in a free fall, then one can easily find the average velocity of this layer while passing the opening, and from the latter the average height of fall of the entire mass. This height is

\[
h = \frac{r}{9l}.
\]

But from the data for loose packings it was found that

\[
l = 0.16
\]

and hence

\[
h = 0.6944 \cdot r.
\]

If instead one introduces the value

\[
l = 0.225,
\]

which is valid for packings where sand is flowing sideways, which really happens during the discharge, then

\[
h = 0.4938 \cdot r.
\]

The result derived from the observations is in between the two when \( r \) is interchanged with \( \rho \). This interchange is necessary because the sand only hits the edge of the opening when in motion; in contrast, when at rest all the sand grains encountering the movable disk also load it. This confirms the assumption from above that the free fall of the sand starts on the surface of the paraboloid; it also explains that the amount of sand flowing through the opening is proportional to the power \( \frac{4}{3} \) of the effective radius of the opening.

Finally, some comments should be made about the motion of the sand during discharge.

In the four inch wide and ten inch tall container, above the outlet, the entire surface of the sand packing subsided uniformly in the beginning. Only gradually did a dip form vertically above the outlet. The dip grew continually, and above its sides sand fell down. Concurrently, at the rim of the container a ring-shaped, almost horizontal surface remained. This surface also subsided, but without the granules of sand experiencing strong sideways motion. The flat ring gradually assumed a smaller width and disappeared completely when the funnel-shaped dip reached the outlet. From this it follows that the sand flows not only vertically towards the outlet but also along concentric inclined trajectories, and that the motion extends up to a slope, which a free surface of sand can exhibit.

The motion in the inner part of the sand mass revealed itself very explicitly when I filled sand in a container having side walls made from a glass panel. Since this glass panel touched the outlet, one could follow the motion of single grains of sand down to the opening. The strongest flow formed vertically above the outlet; in fact the sand granules approached it with increasing yet moderate speed until, directly above, they were accelerated in a way that they could no longer be seen. Nevertheless, the sand also flowed inwards from the side of the outlet, but this motion was interrupted frequently and only occurred periodically, presumably due to friction at the glass.

Underneath the opening, the stream of sand was not nearly as sharply bounded as a water-jet; rather, it was surrounded by single granules that from time to time departed as far as several lines\(^5\). Because of this the streams flowing from the larger outlets showed a significant reduction in their diameters, extending about 2 inches deep. In addition, the measurement showed that even immediately under the disk the stream is already much weaker than at the outlet. The outlet was 0.335 inch in diameter while the stream was only 0.29 inch at a distance of 14 lines, and contracted to 0.27 inch at greater depth. The reason for this effect is not the effective

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\(^5\) trans. note: *Linien*, plural as in *Rheinländische Linien*. 
reduction of the opening mentioned above, since this would only explain the weakening of the stream by about 2% of an inch; instead, the sand flowing sideways continues its motion towards the axis even after having passed the outlet, and the granules hitting the rim of the opening are also reflected towards the axis.

The stream of sand hence experiences a contraction similar to the stream of a liquid; and when one compares the diameter of the opening with the smallest diameter of the stream, the ratio appears as

$$1 : 0.806$$

or for the ratio of cross sections

$$1 : 0.650.$$  

This agrees closely with the known contraction ratios for liquid streams leaving openings in thin walls.

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